RIGHT-PERMUTATIVE CELLULAR AUTOMATA ON TOPOLOGICAL MARKOV CHAINS

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Abstract. In this paper we consider cellular automata \((\mathcal{G}, \Phi)\) with algebraic local rules and such that \(\mathcal{G}\) is a topological Markov chain which has a structure compatible to this local rule. We characterize such cellular automata and study the convergence of the Cesàro mean distribution of the iterates of any probability measure with complete connections and summable decay.

1. Introduction. Let \(G^Z\) be the two-sided full shift on the finite alphabet \(G\), and let \(\sigma : G^Z \to G^Z\) be the shift map. Suppose \(\mathcal{G} \subseteq G^Z\) is a topological Markov chain which, without loss of generality, we can consider to use all alphabet \(G\).

Consider the cellular automaton \((\mathcal{G}, \Phi)\) which has a local rule defined from some algebraic operation on \(G\). Motivated by their several applications in information theory, physics, and biological sciences, among others, the problem of characterizing and analyzing the dynamical behavior of such cellular automata has been widely investigated. More specifically, there are three important questions about \((\mathcal{G}, \Phi)\):

whether it is possible to recode it in the way to understand and to classify its dynamics (see [7] and [15]);
what \(\sigma\)-invariant probability measures are also \(\Phi\)-invariant (see [7], [19] and [23]);
and how \(\sigma\)-invariant probability measures evolve under the dynamics of \(\Phi\) (see [7], [13] and [15]).

When \(\mathcal{G} = G^Z\) and \((\mathcal{G}, \Phi)\) is a right-permutative \(\Psi\)-associative or \(N\)-scaling cellular automaton, Host-Maass-Martínez [7] proved that it is topologically conjugate to \((K^Z \times B^Z, \Phi_K \times G_B)\), which is the product of an affine cellular automaton with a translation map. Moreover, they showed sufficient conditions under which the unique shift-affine invariant measure is the maximum entropy measure (a property which is known as rigidity), and they studied the convergence of the Cesàro mean distribution of \(\sigma\)-invariant probability measures under the action of \(\Phi\). The results of [7] about rigidity were generalized by Pivato [19] for the case of bipermutative endomorphic cellular automata. In his work, Pivato also showed results about the characterization of the topological dynamics of bipermutative cellular automata.

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Later, the rigidity results of Pivato were generalized by Sablik [23], who also includes the case of $\mathcal{S}$ being a proper subgroup shift of $G^Z$.

Recently, Mass-Martínez-Sobottka [15] have showed that if $(\mathcal{S}, +)$ is an Abelian subgroup shift and a $p^s$-torsion for some prime number $p$, and $\Phi$ is an affine cellular automaton given by $\Phi := a \cdot id + b \cdot \sigma + c$, where $a, b \in \mathbb{N}$ are relatively prime to $p$ and $c \in \mathcal{S}$ is a constant sequence, then the Cesàro mean distribution of any measure with complete connections (compatible with $\mathcal{S}$) and summable decay under the action of $\Phi$ converges to the maximum entropy measure on $\mathcal{S}$. The proof of this result combines regeneration theory, combinatorics, and the recodification of $\mathcal{S}$. As a consequence of the convergence of the Cesàro mean distribution we get a rigidity property, namely: the unique $(\sigma, \Phi)$-invariant measure with complete connections (compatible with $\mathcal{S}$) and summable decay for that case is the maximum entropy measure.

This paper concentrates mainly on the first problem, characterizing the dynamical behavior of bipermutative and some right-permutative cellular automata defined on subshifts $\mathcal{S}$ which are not necessarily subgroup shifts but have some algebraic structure. As a direct application of these results we recuperate several results about rigidity and about the evolution of $\sigma$-invariant measures under the action of $\Phi$.

This paper is organized as follows. In §2 we develop the background. In §3, we define the class of structurally-compatible cellular automata and study the case of bipermutative cellular automata. In §4 we study the representation of right-permutative $\Psi$-associative or $N$-scaling cellular automata. In §5 we present some sufficient conditions under which a block code preserves the properties of complete connections and summable decay of a probability measure, and so in §6 we apply the results obtained in the previous sections to study the convergence of the Cesàro mean distribution. In §7 we gives some results about rigidity.

2. Background. Let $\mathcal{S} \subseteq G^Z$ be a subshift. Given $g \in \mathcal{S}$, and $m \leq n$, we denote by $g_m^n = (g_m, g_{m+1}, \ldots, g_n)$. For $k \geq 1$, denote by $\mathcal{S}_k$ the set of all allowed words with length $k$ in $\mathcal{S}$. Given $g \in \mathcal{S}_k$, $g = (g_1, \ldots, g_k)$, we write $F(g)$ as the follower set of $g$ in $\mathcal{S}$:

$$F(g) = \{ h \in G : (g_1, \ldots, g_k, h) \in \mathcal{S}_{k+1} \}.$$ 

In the same way, we define $P(g)$ as the set of predecessors of $g \in \mathcal{S}_k$ in $\mathcal{S}$.

We say a subshift $\mathcal{S}$ is a topological Markov chain if for any $k \geq 1$ and $g = (g_1, \ldots, g_k) \in \mathcal{S}_k$ we have $F(g) = F(g_k)$, which means $\mathcal{S}$ can be thought as generated by a bi-infinite walking on an oriented graph. A topological Markov chain $\mathcal{S}$ is irreducible if and only if for any $u, w \in G$ there exist $k \geq 1$ and $(v_1, \ldots, v_k) \in \mathcal{S}_k$ such that $(u, v_1, \ldots, v_k, w) \in \mathcal{S}_{k+2}$, and it is mixing if there exists $q \geq 1$ such that for any $k \geq q$ and $u, w \in G$ we can always find $(v_1, \ldots, v_k) \in \mathcal{S}_k$ such that $(u, v_1, \ldots, v_k, w) \in \mathcal{S}_{k+2}$.

Denote by $\mathcal{S}_-^-$ and $\mathcal{S}_+^+$ the projections of $\mathcal{S}$ on $G^{-N^*}$ and $G^N$, respectively. Given $w \in \mathcal{S}_-^-$, denote by $\mathcal{S}_w^+$ the projection on $\mathcal{S}_+^+$ of the set of all sequences $(g_i)_{i \in \mathbb{Z}} \in \mathcal{S}$, with $g_i = w_i$ for $i \leq -1$.

Let $\sigma : \mathcal{S} \to \mathcal{S}$ be the shift map, which is defined for every $g \in \mathcal{S}$ and $n \in \mathbb{Z}$ as $(\sigma(g))_n = g_{n+1}$.
We say a map $\Theta : \Lambda \rightarrow \Lambda'$ between two topological Markov chains is a $(\ell + r + 1)$-block code if it has a local rule $\theta : \Lambda_{\ell+r+1} \rightarrow \Lambda'_1$ such that for any $x = (x_i)_{i \in \mathbb{Z}} \in \Lambda$ and $j \in \mathbb{Z}$ it follows that $(\Theta(x))_j = \Theta(x_{j-\ell}, \ldots, x_{j+r})$. Under these notations, we say $\Theta$ has memory $\ell$ and anticipation $r$. We recall a map $\Theta$ is a block code if and only if it is continuous and commutes with the shift map.

$$x = (\ldots, x_{j-\ell}, \ldots, x_j, \ldots, x_{j+r}, \ldots)$$

$$\Theta(x) = (\ldots, \Theta(x)_j, \ldots)$$

A cellular automaton (c.a.) is a pair $(\mathcal{G}, \Phi)$, where $\Phi : \mathcal{G} \rightarrow \mathcal{G}$ is a $(\ell + r + 1)$-block code. Without loss of generality we always can consider $\ell = 0$ and so say the c.a. has radio $r$.

A c.a. with radio $r$ is said right permutative, if its local rule $\phi$ verifies for any fixed word $(w_0, \ldots, w_{r-1}) \in \mathcal{G}$, that the map $g \mapsto \phi(w_0, \ldots, w_{r-1}, g)$ is a permutation on $G$. In the analogous way we define left permutativity. When a c.a. is right and left permutative, we will say it is bipermutative. From now on, we will consider that $\Phi : \mathcal{G} \rightarrow \mathcal{G}$ is a restriction on $\mathcal{G}$ of some c.a. $\hat{\Phi} : G^2 \rightarrow G^2$. It is equivalent to say that there exists a map $\hat{\phi} : G^{r+1} \rightarrow G$ such that the local rule of $\Phi$ is $\phi = \hat{\phi}|_{G^{r+1}}$.

Let us define three types of cellular automata which are fundamental in this work: translations: $(\mathcal{G}, g)$ is a translation if $g = s \circ \sigma$, where $s : G^2 \rightarrow G^2$ is a 1-block code with local rule $s : G \rightarrow G$, which is a permutation on $G$; affine c.a.: $(\mathcal{G}, \Phi)$ is an affine c.a. if its local rule is given by $\phi(a, b) = \eta(a) + \rho(b) + c$, where $+$ is an Abelian group operation on $G$, $\eta : G \rightarrow G$ and $\rho : G \rightarrow G$ are two commuting automorphisms (that is, $\eta \circ \rho = \rho \circ \eta$), and $c \in G$; and group c.a.: $(\mathcal{G}, \Phi)$ is a group c.a. if its local rule is given by $\phi(a, b) = a + b$, where $+$ is an Abelian group operation on $G$.

We say a binary operation $\ast$ on $\mathcal{G}$ is $(\ell + r + 1)$-block if the map $(x, y) \in \mathcal{G} \times \mathcal{G} \rightarrow x \ast y \in \mathcal{G}$ is a $(\ell + r + 1)$-block code. When $\ast$ is a (quasi) group operation, then we say $(\mathcal{G}, \ast)$ is a (quasi) group shift.

Let $\mu$ be any $\sigma$-invariant probability measure on $\mathcal{G}$. For a past $w \in \mathcal{G}^-$, $w = (\ldots, w_{-2}, w_{-1})$, let $\mu_w$ be the probability measure on $\mathcal{G}_w^+$ obtained for $\mu$ conditioning to the past $w$.

We say $\mu$ has complete connections (compatible with $\mathcal{G}$) if given $a \in G$, for all $w \in \mathcal{G}^-$ such that $a \in \mathcal{F}(w_{-1})$, one has $\mu_w(a) > 0$.

If $\mu$ is a probability measure with complete connections, we define the quantities $\gamma_m$, for $m \geq 1$, by

$$\gamma_m := \sup \left\{ \left| \frac{\mu_v(a)}{\mu_w(a)} - 1 \right| : \begin{array}{l} v, w \in \mathcal{G}^-; \quad v_{-i} = w_{-i}, \quad 1 \leq i \leq m; \\ a \in \mathcal{F}(v_{-1}) = \mathcal{F}(w_{-1}) \end{array} \right\}.$$  

When $\sum_{m \geq 1} \gamma_m < \infty$, we say $\mu$ has summable decay.
3. Cellular automata with algebraic local rules. In this section we shall define the class of STRUCTURALLY-COMPATIBLE cellular automata, which is the subject of this work. Moreover, we will study the case of structurally-compatible bipermutative c.a.

**Definition 3.1.** We say a cellular automaton \((G, \Phi)\) with radio 1 is structurally compatible (SC) if it verifies the following property:

\[
(x_i)_{i \in \mathbb{Z}}, (y_i)_{i \in \mathbb{Z}} \in G \implies (\phi(x_i, y_i))_{i \in \mathbb{Z}} \in G,
\]

where \(\phi\) denotes the local rule of \(\Phi\).

Define \(\cdot\) as the binary operation on \(G\) given for all \(a, b \in G\) by \(a \cdot b := \phi(a, b)\). The structural compatibility implies we can consider the componentwise operation \(*\) on \(G\):

\[
\forall (x_i)_{i \in \mathbb{Z}}, (y_i)_{i \in \mathbb{Z}} \in G, (x_i)_{i \in \mathbb{Z}} * (y_i)_{i \in \mathbb{Z}} := (x_i \cdot y_i)_{i \in \mathbb{Z}}.
\]

Notice neither \(\cdot\) nor \(*\) are necessarily algebraic operations on \(G\) and \(G\), respectively. However, the c.a. is left permutative (as well right permutative or bipermutative) if and only if \(\cdot\) (and so \(*\)) is a left cancellable operation (as well right cancellable or a quasi-group operation, respectively). We recall that an operation which is simultaneously left and right cancellable is called a quasi-group operation.

In terms of \(*\), the map \(\Phi\) can be written as

\[
\Phi = id * \sigma.
\]

**Example 3.2.** Let \(\cdot\) be the quasi-group operation on \(G = \{a_i, b_i, c_i, d_i : i = 1, 2, 3\}\), given by the following Latin square:

| \(\cdot\) | \(a_1\) | \(b_1\) | \(c_1\) | \(d_1\) | \(a_2\) | \(b_2\) | \(c_2\) | \(d_2\) | \(a_3\) | \(b_3\) | \(c_3\) | \(d_3\) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \(a_1\) | \(a_3\) | \(b_3\) | \(c_3\) | \(d_3\) | \(a_2\) | \(b_2\) | \(c_2\) | \(d_2\) | \(a_1\) | \(b_1\) | \(c_1\) | \(d_1\) |
| \(b_1\) | \(b_3\) | \(a_3\) | \(d_3\) | \(c_3\) | \(b_2\) | \(a_2\) | \(d_2\) | \(c_2\) | \(b_1\) | \(a_1\) | \(d_1\) | \(c_1\) |
| \(c_1\) | \(c_3\) | \(d_3\) | \(a_3\) | \(b_3\) | \(c_2\) | \(d_2\) | \(a_2\) | \(b_2\) | \(c_1\) | \(d_1\) | \(a_1\) | \(b_1\) |
| \(d_1\) | \(d_3\) | \(c_3\) | \(b_3\) | \(a_3\) | \(d_2\) | \(c_2\) | \(b_2\) | \(a_2\) | \(d_1\) | \(c_1\) | \(b_1\) | \(a_1\) |
| \(a_2\) | \(a_1\) | \(b_1\) | \(c_1\) | \(d_1\) | \(a_3\) | \(b_3\) | \(c_3\) | \(d_3\) | \(a_2\) | \(b_2\) | \(c_2\) | \(d_2\) |
| \(b_2\) | \(b_1\) | \(c_1\) | \(d_1\) | \(a_1\) | \(b_3\) | \(a_3\) | \(c_3\) | \(d_3\) | \(b_2\) | \(a_2\) | \(c_2\) | \(d_2\) |
| \(c_2\) | \(c_1\) | \(d_1\) | \(a_1\) | \(b_1\) | \(c_3\) | \(d_3\) | \(a_3\) | \(b_3\) | \(c_2\) | \(d_2\) | \(a_2\) | \(b_2\) |
| \(d_2\) | \(d_1\) | \(a_1\) | \(c_1\) | \(b_1\) | \(d_3\) | \(c_3\) | \(a_3\) | \(b_3\) | \(d_2\) | \(a_2\) | \(c_2\) | \(b_2\) |

Denote as \(*\) the 1-block operation induced by \(\cdot\) on \(G^\mathbb{Z}\). Let \(G \subset G^\mathbb{Z}\) be the topological Markov chain defined by the oriented graph of Figure 1. We have that \((G, *\)) is an irreducible quasi-group shift.

Define the bipermutative cellular automaton \((G, \Phi)\), where \(\Phi := id * \sigma\). It follows that \(\Phi\) verifies the property (3.1), and so it is structurally compatible. Moreover, since \(\cdot\) has the medial property:

\[
\forall a, b, c, d \in G, \quad (a \cdot b) \cdot (c \cdot d) = (a \cdot c) \cdot (b \cdot d),
\]
it follows, from ([3], Theorem 2.2.2, p.70), that there exist an Abelian group operation on \( G \), \( \eta \) and \( \rho \) commuting automorphisms, and \( c \in G \) such that \( a \bullet b = \eta(a) + \rho(b) + c \). Therefore, \((\mathfrak{G}, \Phi)\) is an affine c.a.

The next proposition gives a characterization of SC bipermutative cellular automata.

**Proposition 3.3.** Let \((\mathfrak{G}, \Phi)\) be a SC bipermutative c.a. Then,

i. \((\mathfrak{G}, \Phi)\) is topologically conjugate to \((\mathcal{H}, \Phi_H)\) through a 1-block code, where \( \mathcal{H} = F \times \Sigma_n \), \( F \) is finite, \( \Sigma_n \) is a full n shift, and \( \Phi_H = \text{id}_H \otimes \sigma_H \), where \( \otimes \) is a \( k \)-block quasi-group operation on \( \mathcal{H} \).

ii. \( h(\mathfrak{G}) = 0 \) (the topological entropy of the shift is zero) if and only if \( \Sigma_n = \{(..., a, a, a, ...\} \) (that is, the full shift is trivial).

iii. \( \mathfrak{G} \) is irreducible and has constant sequence if and only if \( F = \{e\} \) (that is, \( F \) is unitary).

**Proof.**

i. Let \((\mathfrak{G}, \Phi)\) be a bipermutative c.a. with radio 1, which verifies (3.1). As before, for \( a, b \in G \), denote \( a \bullet b = \phi(a, b) \), which is a quasi-group operation on \( G \). Thus, \( \Phi = \text{id} \ast \sigma \), where \( \ast \) is the componentwise quasi-group operation on \( \mathfrak{G} \) induced from \( \bullet \).

From Theorem 4.25 and Remark 4.28 in [26], the quasi group \((\mathfrak{G}, \ast)\) is isomorphic to a quasi group \((F \times \Sigma_n, \otimes)\), where \( \otimes \) is a \( k \)-block operation, with anticipation \( k-1 \). We denote \( \mathcal{H} := F \times \Sigma_n \), as \( \phi : \mathfrak{G} \to \mathcal{H} \) the isomorphism between the quasi groups, and \( \Phi_H := \text{id}_H \otimes \sigma_H \). It follows that

\[
\varphi \circ \Phi = (\varphi \circ (\text{id}_\mathfrak{G} \ast \sigma_\mathfrak{G})) = (\varphi \circ \text{id}_\mathfrak{G}) \otimes (\varphi \circ \sigma_\mathfrak{G}) \quad \overset{(a)}{=} \quad (\text{id}_\mathcal{H} \circ \varphi) \otimes (\sigma_\mathcal{H} \circ \varphi) = (\text{id}_\mathcal{H} \otimes \sigma_\mathcal{H}) \circ \varphi = \Phi_H \circ \varphi,
\]
where \( = (a) \) comes from the fact that \( \varphi \) is an isomorphism between \((\mathcal{G}, \ast)\) and \((\mathcal{H}, \otimes)\), and \( = (b) \) is due the fact that \( \varphi \) is a 1-block code (see Theorem 4.25 in [26]) and so it commutes with the shift map.

Since \( \otimes \) is a \( k \)-block quasi-group operation (with memory 0 and anticipation \( k - 1 \)), we have that \( \Phi_{\mathcal{H}} \) has radio \( k \).

ii. and iii. They follow directly from Theorem 4.25 of [26]. \( \square \)

**Remark 3.4.** From Theorem 4.25 in [26] we could get an analogous result, but with \( \otimes \) being an operation with memory \( k - 1 \) and anticipation 0. Therefore, \( \Phi_{\mathcal{H}} \) would have memory \( k \) and anticipation 0.

We notice \((\mathcal{H}, \Phi_{\mathcal{H}})\) in the previous theorem is not necessarily a bipermutative c.a. For instance, if \((\mathcal{G}, \Phi)\) is a group c.a. (which means \((G, \bullet)\) is a group) such that (3.1) holds, then it verifies all hypotheses of Theorem 3.3, but \( \otimes \) can be a \( k \)-block group operation with memory 0 and anticipation \( k - 1 \), for some \( k > 1 \). Thus, \((\mathcal{H}, \Phi_{\mathcal{H}})\) can not be right permutative. In fact, since \( \rho : \mathcal{H}_k \times \mathcal{H}_k \to \mathcal{H}_1 \) the local rule of \( \otimes \), then since \( \sigma_{\mathcal{H}} \) is an automorphism to the group \((\mathcal{H}, \otimes)\), it follows that the identity element \( e \) of the group is such that \( \sigma_{\mathcal{H}}(e) = e \), which implies \( e = (\ldots, e, e, e, \ldots) \).

Therefore, taking \( w \in \mathcal{H}_k \), \( w = (e, e, \ldots, e) \), we have for all \( a \in \mathcal{H}_1 \):

\[
\phi_{\mathcal{H}}(wa) = \phi_{\mathcal{H}}((e, e, \ldots, e, a)) = \rho((e, e, \ldots, e), (e, e, \ldots, a)) = e.
\]

4. **Right-permutative cellular automata.** In this section we shall study two types of cellular automata: \( N \)-scaling and \( \Psi \)-associative.

We say a cellular automaton \((\mathcal{G}, \Phi)\) with radius 1 is a \( N \)-scaling c.a. for some \( N \geq 2 \) if its local rule \( \phi : G \times G \to G \) is such that for any \( x = (x_i)_{i \in \mathbb{Z}} \in \mathcal{G} \),

\[
(\Phi^N(x))_0 = x_0 \bullet x_N.
\]

On the other hand, \((\mathcal{G}, \Phi)\) is said to be \( \Psi \)-associative if there exists a permutation \( \Psi : G \to G \) such that for any \( a, b, c \in G \), we have

\[
(a \bullet b) \bullet c = \Psi(a \bullet (b \bullet c)).
\]

When \( \mathcal{G} = G^Z \), Host-Maass-Martínez [7] have proved that every right-permutative \( N \)-scaling c.a. \((\mathcal{G}, \Phi)\) is topologically conjugate to the product of an affine c.a. with a translation, while every right-permutative \( \Psi \)-associative is topologically conjugate to the product of a group c.a. with a translation.

Theorems 4.2 and 4.3 below reproduce those results for the general case of cellular automata defined on topological Markov chains. To prove these theorems we shall make remarks on some basics on these types of cellular automata.

**Remark 4.1.**

- If \((G^Z, \Phi)\) is a right-permutative \( \psi \)-associative c.a., from Theorem 6 in [7], we get that there exists a 1-block code \( u : G^Z \to K^Z \times B^Z \), which is a topological conjugacy between \((G^Z, \Phi)\) and \((K^Z \times B^Z, \Phi_K \times g_B)\), where \( B \subseteq G \) and \( K \) are two finite alphabets, \( \Phi_K \) is a group c.a., and \( g_B \) is a translation.

We recall \( g_B = s_B \circ \sigma_B \), where \( s_B : B^Z \to B^Z \) is a 1-block code with local rule \( s_B : B \to B \), which is a permutation on \( B \). Moreover, [7] gives that \( s_B : B \to B \) is defined for all \( e' \in B \) by \( s_B(e') = e' \bullet e' \), where \( e'' \in B \) is any element.
Furthermore, $u$ has local rule $u : G \to K \times B$ which is a bijection and is given for any $a \in G$ by

$$u(a) = (\hat{a}, e_a),$$

where $\hat{a}$ is the equivalent class of $a \in G$ to the equivalence relation,

$$a \sim b \iff \forall c \in G, \ a \cdot c = b \cdot c,$$

and $e_a$ is the unique element of $G$ for which $a \cdot e_a = a$. We notice that for all $a \in G$ and $e \in B$, we have $a \cdot e \sim a$. Moreover, the following property holds:

$$e_{a \cdot b} = e_a \cdot e_b = s_B(e_b).$$

Finally, since $\Phi_K$ is a group, its local rule defines a group operation on $K$:

$$\forall \hat{a}, \hat{b} \in K, \hat{a} \hat{b} := \phi_K(\hat{a}, \hat{b}).$$

- From Theorem 8 in [7], if $(G^Z, \Phi)$ is a $N$-scaling c.a., then the above statements hold, but $\Phi_K$ will be an affine c.a. and in the code $u(a) = (\hat{a}, e_a)$, $e_a$ will be defined as the unique element of $B$ for which the equation $e_a = x \cdot a$ has solution.

**Theorem 4.2.** Let $(\mathcal{G}, \Phi)$ be a SC right-permutative $\Psi$-associative c.a. Then, $(\mathcal{G}, \Phi)$ is topologically conjugate through a 1-block code to $(\mathcal{K}, \Phi_K \times g_B)$, where $\mathcal{K}$ and $\mathcal{B}$ are topological Markov chains, $(\mathcal{K}, \Phi_K)$ is a group c.a., and $(\mathcal{B}, g_B)$ is a translation.

**Proof.**

**Step 1:** Since $(\mathcal{G}, \Phi)$ has radio 1 and verifies (3.1), we can consider that $\Phi : \mathcal{G} \to \mathcal{G}$ is a restriction on $\mathcal{G}$ of some right-permutative $\Psi$-associative c.a. $(G^Z, \tilde{\Phi})$ which has the same local rule $\phi : G \times G \to G$.

Let $(K^Z \times B^Z, \Phi_K \times g_B)$ be the group-translation and $u : G^Z \to K^Z \times B^Z$ be the topological conjugacy presented in Remark 4.1.

We consider $K \times B$ the right-permutative operation also denoted as $\cdot$ and induced from the local rule of $\Phi_K \times g_B$: given $(\hat{a}_1, e_1), (\hat{a}_2, e_2) \in K \times B$ define

$$(\hat{a}_1, e_1) \cdot (\hat{a}_2, e_2) = (\hat{a}_1 \hat{a}_2, s_B(e_2)).$$

Notice that $u : G \to K \times B$ is an isomorphism between $(G, \cdot)$ and $(K \times B, \cdot)$. In fact, $u$ is bijective and

$$u(a \cdot c) = (\hat{a} \cdot c, e_a \cdot e_c) = (\hat{a} \hat{c}, e_a \cdot e_c) = (\hat{a} \hat{c}, s_B(e_c)) = (\hat{a}, e_a) \cdot (\hat{c}, e_c) = u(a) \cdot u(c),$$

where $=(a)$ comes from Theorem 6 of [7].

The operation $\cdot$ on $K \times B$ induces the componentwise operation also denoted as $*$ on $K^Z \times B^Z$. Thus, $u : G^Z \to K^Z \times B^Z$ is an isomorphism between $(G^Z, \cdot)$ and $(K^Z \times B^Z, \cdot)$.

Define $\Lambda := u(\mathcal{G}) \subseteq K^Z \times B^Z$. Since $u$ is topological conjugacy between $\Phi$ and $\Phi_K \times g_B$, it follows that $\Phi_K \times g_B(\Lambda) = \Lambda$. Therefore, we have that the cellular automaton $(\Lambda, \Phi_K \times g_B)$ is well defined and $*$ is closed on $\Lambda$. Moreover, $u|_{\mathcal{G}}$ is a topological conjugacy between $(\mathcal{G}, \Phi)$ and $(\Lambda, \Phi_K \times g_B)$, and an isomorphism between $(\mathcal{G}, \cdot)$ and $(\Lambda, \cdot)$. 


step 2: We will show that there exists $M \geq 1$ such that for all $e \in B$ we have $s_B^M(e) = e$.
Since $s_B$ is a permutation on $B$, it follows that for all $e \in B$ there exists $M_e \geq 1$ such that $s_B^{M_e}(e) = e$. Because $B$ is a finite alphabet, we can take $M$ as a multiple of all periods of each element of $B$. Then, the result follows.

step 3: Let us to prove that $\Lambda = \mathcal{A} \times \mathcal{B}$, where $\mathcal{A} \subseteq K^\mathbb{Z}$ and $\mathcal{B} \subseteq B^\mathbb{Z}$ are both topological Markov chains.

First, notice that, because $\bullet$ is a quasi-group operation, there exists $L \in \mathbb{N}$ such that for all $\tilde{a}, \tilde{c} \in K$:

$$\left(\left(\left(\left(\left(\tilde{c} \bullet \tilde{a}\right) \cdots \tilde{a}\right) \bullet \tilde{a}\right) \cdots \tilde{a}\right) \bullet \tilde{a}\right) \cdots \tilde{a}\right) = \tilde{c}.$$

$\tilde{c}$ multiplied $L$ times by $\tilde{a}$ for the right side

Denote $\pi_K : \Lambda \to K^\mathbb{Z}$ and $\pi_B : \Lambda \to B^\mathbb{Z}$ as the canonical projections on the first and second coordinates, respectively.

It is straightforward that $\Lambda \subseteq \pi_K(\Lambda) \times \pi_B(\Lambda)$. So, we only need to show that $\pi_K(\Lambda) \times \pi_B(\Lambda) \subseteq \Lambda$.

In fact, given $(\tilde{c}_i)_{i \in \mathbb{Z}} \in \pi_K(\Lambda)$ and $(e_i)_{i \in \mathbb{Z}} \in \pi_B(\Lambda)$ there must exist $(\tilde{a}_i, e_i)_{i \in \mathbb{Z}} \in \Lambda$, and so

$$\left(\left(\left(\left(\left(\tilde{c}_i, e_i\right)_{i \in \mathbb{Z}} * \tilde{a}_i, e_i\right)_{i \in \mathbb{Z}}\right) \cdots \tilde{a}_i, e_i\right)_{i \in \mathbb{Z}}\right) = \left(\left(\left(\left(\left(\tilde{c}_i, e_i\right)_{i \in \mathbb{Z}} \bullet \tilde{a}_i, e_i\right)_{i \in \mathbb{Z}}\right) \cdots \tilde{a}_i, e_i\right)_{i \in \mathbb{Z}}\right)$$

$\tilde{c}$ multiplied $L$ times by $\tilde{a}$ for the right side

multiplying $(\tilde{c}_i, e_i)_{i \in \mathbb{Z}} L$ times by $(\tilde{a}_i, e_i)_{i \in \mathbb{Z}}$ for the right side

$$\left(\left(\left(\left(\left(\tilde{c}_i, e_i\right)_{i \in \mathbb{Z}} \bullet \tilde{a}_i, e_i\right)_{i \in \mathbb{Z}}\right) \cdots \tilde{a}_i, e_i\right)_{i \in \mathbb{Z}}\right) = \left(\left(\left(\left(\left(\tilde{c}_i, e_i\right)_{i \in \mathbb{Z}} s_B(e_i) \bullet \tilde{a}_i, e_i\right)_{i \in \mathbb{Z}}\right) \cdots \tilde{a}_i, e_i\right)_{i \in \mathbb{Z}}\right)$$

$\tilde{c}$ multiplied $L$ times by $\tilde{a}$ for the right side

multiplying $(\tilde{c}_i, s_B(e_i))_{i \in \mathbb{Z}} L$ times by itself for the right side, and so we get $(\tilde{c}_i, s_B(e_i))_{i \in \mathbb{Z}} \in \Lambda$. By induction, we can obtain that for all $m \geq 0$, $(\tilde{c}_i, s_B^m(e_i))_{i \in \mathbb{Z}} \in \Lambda$. From Step 2 there exists $M \geq 1$ such that for all $i \in \mathbb{Z}$ we have $s_B^M(e_i) = e_i$. Therefore, we get $(\tilde{c}_i, e_i)_{i \in \mathbb{Z}} \in \Lambda$, which allows us to deduce that $\Lambda = \pi_K(\Lambda) \times \pi_B(\Lambda)$.

Notice that $u_{\mathcal{G}}$ is a 1-block code from $\mathcal{G}$ to $\Lambda$ such that its inverse is also a 1-block code. Thus, since $\mathcal{G}$ is a topological Markov chain, it follows that $\Lambda$ is also a topological Markov chain. Finally, since $\Lambda = \pi_K(\Lambda) \times \pi_B(\Lambda)$ we have that $\pi_K(\Lambda)$ and $\pi_B(\Lambda)$ are also both topological Markov chains, and denoting $\mathcal{R} := \pi_K(\Lambda)$, $\mathcal{B} := \pi_B(\Lambda)$, $\Phi_\mathcal{R} := \Phi_{\mathcal{K}}|_\mathcal{R}$ and $\Phi_\mathcal{B} := \Phi_{\mathcal{B}}|_\mathcal{B}$, we finish the proof. □

**Theorem 4.3.** Let $(\mathcal{G}, \Phi)$ be a SC right-permutative $N$-scaling c.a. If its extension $(\mathcal{G}^\mathbb{Z}, \hat{\Phi})$ is also a $N$-scaling c.a., then $(\mathcal{G}, \Phi)$ is topologically conjugate through a 1-block code to $(\mathcal{R} \times \mathcal{B}, \Phi_\mathcal{R} \times \Phi_\mathcal{B})$, where $\mathcal{R}$ and $\mathcal{B}$ are topological Markov chains, $(\mathcal{R}, \Phi_\mathcal{R})$ is an affine c.a., and $(\mathcal{B}, \Phi_\mathcal{B})$ is a translation.

**Proof.** Since $(\mathcal{G}, \Phi)$ is the restriction on $\mathcal{G}$ of a $N$-scaling c.a. $(\mathcal{G}^\mathbb{Z}, \hat{\Phi})$, we can apply a reasoning similar to Theorem 4.2. □
Corollary 4.4. Let \((\mathfrak{G}, \Phi)\) be a SC right-permutative N-scaling c.a. If \(\mathfrak{G}\) is mixing, then \((\mathfrak{G}, \Phi)\) is topologically conjugate through a 1-block code to \((\mathfrak{R} \times \mathfrak{B}, \Phi_R \times \Phi_B)\), where \(\mathfrak{R}\) and \(\mathfrak{B}\) are topological Markov chains, \((\mathfrak{R}, \Phi_R)\) is an affine c.a., and \((\mathfrak{B}, \Phi_B)\) is a translation.

Proof. Since \(\mathfrak{G}\) is mixing, there exists \(q \geq 1\) such that for any \(k \geq q\) and \(u, w \in G\), we can always find \((v_1, \ldots, v_k) \in \mathfrak{G}_k\) such that \((u, v_1, \ldots, v_k, w) \in \mathfrak{G}_{k+2}\). Without loss of generality, we can consider \(N \geq q\), because if \((\mathfrak{G}, \Phi)\) is N-scaling, then it is also \(N^m\)-scaling for any \(m \geq 1\). We will show that \((G^G, \tilde{\Phi})\) is also N-scaling:

Given a sequence \(x = (x_i)_{i \in \mathbb{Z}} \in G\), due the fact that \(\mathfrak{G}\) is mixing and \(N \geq q\), we can find a sequence \(y = (y_i)_{i \in \mathbb{Z}} \in \mathfrak{G}\) such that \(y_jN = x_j\) for all \(i \in \mathbb{Z}\). Thus,

\[
\tilde{\Phi}(x_i) = x_i \ast x_{i+1} = y_jN \ast y_{j+1}N = (\Phi_N^N(y))_{jN},
\]

and by induction we get that for any \(k \geq 1\), \(\tilde{\Phi}^k(x_i) = (\Phi_N^k(y))_{jN}\). Therefore,

\[
\tilde{\Phi}^N(x_i) = (\Phi_N^N)(y)_{jN} = y_j \ast y_{j+N^2} = y_jN \ast y_{(j+N)^N} = x_j \ast x_{j+N}.
\]

Now, since \(\tilde{\Phi}\) is a N-scaling c.a., we can apply Theorem 4.3 to conclude the proof. □

Notice that \((\mathfrak{R}, \Phi_K)\) obtained in the previous theorems is a group c.a. (or an affine c.a.) which is also structurally compatible. Thus, since \((\mathfrak{R}, \Phi_K)\) is bipermutative, we can apply Proposition 3.3 to get that it is topologically conjugate through a 1-block code to \((\tilde{\mathfrak{G}}, \tilde{\Phi})\), where \(G = \mathbb{F} \times \Sigma_n\) with \(\mathbb{F}\) is finite and \(\Sigma_n\) is a full \(n\)-shift, and \(\Phi_G\) is a group c.a. (or an affine c.a.) with radio \(k\).

5. Projections of measures with complete connections and summable decay. In this section we shall present sufficient conditions to reproduce results about the convergence of the Cesàro mean distribution ([7], [15]) to the more general case of \(\mathfrak{G}\) being neither a full shift nor a groupshift, but \((\mathfrak{G}, \Phi)\) being structurally compatible.

Lemma 5.1. Let \(\Lambda\) and let \(\Lambda'\) be two topological Markov chains, and \(\Theta : \Lambda \rightarrow \Lambda'\) be an invertible 1-block code which is constant on the predecessor sets. Suppose \(\Theta^{-1}\) has memory 1 and anticipation 0. If \(\mu\) is a \(\sigma\)-invariant probability measure on \(\Lambda\) with complete connections (compatible with \(\Lambda\)) and summable decay, then \(\mu' = \mu \circ \Theta^{-1}\) also has complete connections (compatible with \(\Lambda'\)) and summable decay.

Proof. Let \(C'\) be a cylinder of \(\Lambda'\) defined by the coordinates \(i = 0, \ldots, m\) with \(m \geq 1\), that is, \(C' = \left[c'_0, \ldots, c'_m\right]\). We will show that \(C := \Theta^{-1}(C')\) is a cylinder of \(\Lambda\) defined by the coordinates \(i = 1, \ldots, m\), that is, \(C = \left[c_1, \ldots, c_m\right]\).

Denote as \(\theta\) the local rule of \(\Theta\). Notice that for all \(1 \leq i \leq m\), \(c_i \in \Lambda_1\) is well defined by \(c_i := \theta^{-1}(c'_{i-1}, c'_i)\). Therefore,

\[
\Theta^{-1}(C') = \bigcup_{c_0 \in \mathcal{P}(c_1)} \left[c_0, c_1, \ldots, c_m\right] = \bigcup_{c_0 \in \mathcal{P}(c_1)} \left[c_0, c_1, \ldots, c_m\right] = \left[c_1, \ldots, c_m\right].
\]
Through the use of a similar reasoning and because $\Theta^{-1}$ has anticipation 0, we get that for any $v', w' \in \Lambda^-$, we can define $v := \Theta^{-1}(v')$ and $w := \Theta^{-1}(w')$, which are both pasts belonging to $\Lambda^-$. In particular, if $v_i = w_i$ for $1 \leq i \leq m$, with $m \geq 2$, then $v_i = w_i$ for $1 \leq i \leq m - 1$.

On the other hand, since $\mu$ has complete connections (compatible with $\Lambda$), given $w' \in \Lambda^-$ and $a' \in F(w_{-1})$ there exist unique $w \in \Lambda$ and $a \in F(v_{-1})$ such that $\mu'_{w'}(a') = \mu_w(a) > 0$. It means $\mu'$ also has complete connections (compatible with $\Lambda'$). Moreover, for $m \geq 2$, it follows that

$$
\gamma'_m = \sup \left\{ \left| \frac{\mu'_{w'}(a')}{\mu'_{w'}(a')} - 1 \right| : v', w' \in \Lambda^-; \quad v'_{i-1} = w'_{i-1}, \quad 1 \leq i \leq m; \quad a' \in F(v'_{-1}) = F(w'_{-1}) \right\}
$$

$$
= \sup \left\{ \left| \frac{\mu_v(a)}{\mu_w(a)} - 1 \right| : v, w \in \Lambda^-; \quad v_i = w_i, \quad 1 \leq i \leq m - 1; \quad a \in F(v_{-1}) = F(w_{-1}) \right\} = \gamma_{m-1},
$$

which means $\mu'$ has summable decay. 

In an analogous way, we can prove the following Lemma.

**Lemma 5.2.** Let $\Lambda$ and $\Lambda'$ be two topological Markov chains, and let $\varphi : \Lambda \times \Sigma \to \Lambda' \times \Sigma$ be block code defined by $\varphi := \Theta \times id$, where $\Theta : \Lambda \to \Lambda'$ is an invertible 1-block code which is constant on the predecessor sets. Suppose $\Theta^{-1}$ has memory 1 and anticipation 0. If $\mu$ is $\sigma$-invariant probability measure on $\Lambda$ with complete connections (compatible with $\Lambda$) and summable decay, then $\mu' = \mu \circ \Theta^{-1}$ also has complete connections (compatible with $\Lambda'$) and summable decay.

Now, consider $(\mathfrak{S}, \Phi)$ being a SC bipermutable c.a. Let $\varphi : \mathfrak{S} \to G$ be the topological conjugacy between $(\mathfrak{S}, \Phi)$ and $(\mathfrak{g}, \Phi_{\mathfrak{g}})$, where $\mathfrak{g} = \mathfrak{f} \times \Sigma_n$, given by Proposition 3.3. From Remark 3.4 we can suppose that $\varphi$ has memory $k$ and anticipation 0. With this notations, we have the following:

**Proposition 5.3.** If $(\mathfrak{S}, \Phi)$ is a SC bipermutable c.a., and $\mu$ is a probability measure with complete connections (compatible with $\mathfrak{S}$) and summable decay, then $\mu \circ \varphi^{-1}$ is a probability measure on $\mathfrak{g} = \mathfrak{f} \times \Sigma_n$ which also has complete connections and summable decay.

**Proof.** From Theorem 4.25 of [26], $\varphi$ is given by the following composition:

$$
\varphi = \varphi_n \circ \eta_n \circ \varphi_{n-1} \circ \eta_{n-1} \circ \ldots \circ \varphi_1 \circ \eta_1,
$$

where for all $i = 1, \ldots, n$, $\varphi_i = \Theta_i \times id$ is a block code as in Lemma 5.2, and $\eta_i$ is an invertible 1-block code whose inverse is also a 1-block code. Thus, for each $i \leq n$ we have that $\eta_i$ and $\varphi_i$ preserve the properties of complete connections and summable decay of the measure, which concludes the proof.

6. Cesàro mean convergence of measures with complete connections and summable decay. In this section we shall present some results about the convergence of the Cesàro mean distribution of probability measures under the action of
cellular automata; namely, we study the following limit:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu \circ \Phi^{-n}.$$ 

The essential tools that we will use to study the convergence of the Cesàro mean distribution are propositions 3.3 and 5.3, and Corollary 29 of [7].

**Definition 6.1.** Given a SC bipermutative c.a. (Φ, F), we say it is regular if the quasi-group (F × Σn, ⊗) set in Theorem 3.3 is such that ⊗ = ⊗F × ⊗Σn, where (F, ⊗F) and (Σn, ⊗Σn) are both quasi-groups. Furthermore, if ⊗Σn is a 1-block operation, then we say (Φ, F) is simple.

**Example 6.2.** If Φ is irreducible or h(Φ) = 0, then Φ is regular due Theorem 4.25(ii,iii) of [26]. If h(Φ) = p, where p is a prime number, then Φ is simple due to Theorem 4.26 of [26].

**Theorem 6.3.** Let (Φ, F) be a SC cellular automaton, where Φ is not necessarily irreducible. Denote as (G², Φ̂) the extension of (Φ, F) to the full shift, and suppose µ is a probability measure on Φ with complete connections (compatible with Φ) and summable decay. Then:

i. If (Φ, F) is an affine c.a. which is regular and simple, then the Cesàro mean distribution of µ under the action of Φ converges to a maximum entropy measure. In particular, if Φ is irreducible and has a constant sequence, then the Cesàro mean distribution of µ under the action of Φ converges to the unique maximum entropy measure (the Parry measure);

ii. If (Φ, F) is a right-permutative Ψ-associative c.a. and the group c.a. associate to it (see Theorem 4.2) is Abelian, regular and simple, then the Cesàro mean distribution of µ under the action of Φ converges;

iii. If (G², Φ̂) is right permutative and N-scaling and the affine c.a. associate to it (see Theorem 4.3) is regular and simple, then the Cesàro mean distribution of µ under the action of Φ̂ converges.

**Proof.**

i. Let (F × Σn, ΦF×Σn) and ϕ : Φ → F × Σn be the cellular automaton and the topological conjugacy given by Proposition 3.3. From Proposition 5.3, we have that µ′ = µ ◦ ϕ⁻¹ is a probability measure on F × Σn with complete connections and summable decay. Moreover, since (Φ, F) is regular and simple, it follows that ΦF×Σn = ΦF × ΦΣn, where (Σn, ΦΣn) is an affine c.a. In fact, (F × Σn, ⊗F×Σn) = (F × Σn, ⊗F × ⊗Σn) has the medial property, thus (Σn, ⊗Σn) also has the medial property and we can apply ([3], Theorem 2.2.2, p.70) in the same way as in Theorem 7.1, which allows us to deduce that ΦΣn is an affine c.a.

Furthermore, since F is a finite set, we get that (F, ΦF) is equicontinuous. Therefore, from Corollary 29 in [7], it follows that the Cesàro mean of µ′ under the action of ΦF×Σn converges to a probability measure µ′F × ν, where µ′F is a ΦF-invariant probability measure on F and ν is the Parry measure on Σn (that is, the uniform Bernoulli measure). Since (Φ, F) is topologically conjugate to (F × Σn, ΦF×Σn), we conclude that
which is a maximum entropy measure since it is the projection of a measure on a finite set $F$ product the Parry measure on the full $n$-shift.

In particular, when $F$ is not unitary (and since it is finite and hence has zero entropy), we conclude that there could exist more than one maximum entropy measure for $\Phi$ (\(6.1\) is one of them). On the other hand, if $\mathfrak{S}$ is irreducible and has a constant sequence, then, from Proposition 3.3(iii), $F$ is unitary and the limit measure is exactly the projection of the Parry measure on the full $n$-shift. In such a case the limit measure is the Parry measure on $\mathfrak{S}$.

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu \circ \Phi^{-n} = (\mu'_{\mathfrak{F}} \times \nu) \circ \varphi, \quad (6.1) \]

ii. From Theorem 4.2 and Proposition 3.3, and using ([3], Theorem 2.2.2, p.70) in the same way as in the proof of Theorem 7.1, we deduce that $(\mathfrak{S}, \Phi)$ can be represented as $(\mathfrak{B} \times F \times \Sigma_n, g_B \times \Phi_F \times \Phi_{\Sigma_n})$, that is, a translation on a topological Markov chain, product a group c.a. on a finite set, product a group c.a. on a full shift. Thus, by Corollary 29 of [7], we conclude

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu \circ \Phi^{-n} = \mu_{\mathfrak{B}} \times (\mu'_{\mathfrak{F}} \times \nu) \circ \varphi, \quad (6.2) \]

where $\mu_{\mathfrak{B}}$ is a $g_B$-invariant probability measure on $\mathfrak{B}$, $\mu'_{\mathfrak{F}}$ is a $\Phi_F$-invariant probability measure on $\mathfrak{F}$, and $\nu$ is the Parry measure on $\Sigma_n$ (that is, the uniform Bernoulli measure).

iii. This proof is analogous to the part (ii), but it uses Theorem 4.3 instead of Theorem 4.2.

Example 6.4. The affine c.a. of Example 3.2 is regular and simple, because applying the reasoning presented in the proof of Proposition 3.3 we deduce that it is topologically conjugate to $(\mathfrak{B} \times \mathbb{Z}_2^n, \Phi_{\mathfrak{B}} \times \Phi_{\mathbb{Z}^n})$, where $\mathfrak{F} = \{\ldots, 0, 1, 2, 0, 1, 2, \ldots\}$, $\Phi_{\mathfrak{F}} = \sigma_F$ and $\Phi_{\mathbb{Z}_2^3} = id + \sigma$. Therefore, the Cesàro mean of any probability measure on $\mathfrak{S}$ with complete connections summable decay converges under the action of $(\mathfrak{S}, \Phi)$.

Example 6.5. Let $\bullet$ be a binary operation defined on the set $G = \{a, b, c, d, e, f, g, h\}$ by the following table:

| $\bullet$ | a | b | c | d | e | f | g | h |
|--------|---|---|---|---|---|---|---|---|
| a      | b | a | d | c | f | e | h | g |
| b      | b | a | d | c | f | e | h | g |
| c      | d | c | b | a | h | g | f | e |
| d      | d | c | b | a | h | g | f | e |
| e      | f | e | h | g | b | a | d | c |
| f      | f | e | h | g | b | a | d | c |
| g      | h | g | f | e | d | c | b | a |
| h      | h | g | f | e | d | c | b | a |

Let $\Lambda \subset G^n$ be the topological Markov chain defined by the oriented graph presented in Figure 2.
We define the map \( \phi : G \times G \to G \) given by \( \phi(u, v) = u \cdot v \), and we consider the cellular automaton \( \Phi : \Lambda \to \Lambda \) with radio 1, which the local rule is \( \phi \).

It is easy to check that \( (\Lambda, \Phi) \) is a SC right-permutative and \( \Psi \)-associative c.a., where \( \Psi \) is given by \( \Psi(x) := \phi(a, x) \).

From the algorithm developed in the proof of Theorem 4.2, we get that \( (\Lambda, \Phi) \) is topologically conjugate to \( (\Sigma \times \{0, 1\}^2, \Phi_{\Sigma} \times \phi) \), where: \( \Sigma \subset (\mathbb{Z}_2 \oplus \mathbb{Z}_2)^2 \); \( \Phi_{\Sigma} \) is a group c.a.; and \( \phi \) is a translation on \( \{0, 1\}^2 \) given by \( \phi((x_i)_{i \in \mathbb{Z}}) = (\Psi(x_{i+1}))_{i \in \mathbb{Z}} \).

Using the algorithm developed in the proof of Proposition 3.3, we find that \( (\Sigma, \Phi_{\Sigma}) \) is topologically conjugate to the group c.a. \( (\mathbb{Z}_2 \times \mathbb{Z}_2, \Phi_{\mathbb{Z}_2}) \), and so it is regular and simple. Therefore, Theorem 6.3 guarantees the convergence of the Cesàro mean of any probability measure on \( \Lambda \) with complete connections summable decay under the action of \( (\Lambda, \Phi) \).

\section*{7. Invariant measures for cellular automata on topological Markov chains}

Let \( (\mathcal{G}, \Phi) \) be a SC cellular automaton and suppose that \( \mathcal{G} \) is irreducible.

An important problem is to characterize probability measures on \( \mathcal{G} \) which are invariant for the \( \mathbb{Z}^2 \)-action defined on \( \mathcal{G} \) by \( (\Phi, \sigma) \). Several works ([7], [19], [23]) have studied this problem and for many cases have showed that the Parry measure (the unique maximum entropy measure for \( (\mathcal{G}, \sigma) \)) is the unique \( (\sigma, \Phi) \)-invariant measure.

We can deduce results about \( (\Phi, \sigma) \)-invariant measures through the use of the topological conjugacies presented previously. When \( \mathcal{G} \) is a group shift, then the following theorems are particular cases of the results presented by Sablik [23].

\textbf{Theorem 7.1.} Let \( (\mathcal{G}, \Phi) \) be a SC affine c.a., with \( \mathcal{G} \) being irreducible and \( h(\mathcal{G}) = \log p \), where \( p \) is a prime number. Let \( \mu \) be a \( (\Phi, \sigma) \)-invariant probability measure on \( \mathcal{G} \). If \( \mu \) is ergodic to \( \sigma \) and has positive entropy to \( \Phi \), then \( \mu \) is the Parry measure.

\textit{Proof.} Since \( (\mathcal{G}, \Phi) \) is an affine c.a. it follows that it is bipermutative, which implies that the operation \( \bullet \), defined by \( a \bullet b := \phi(a, b) \), is a quasi-group operation on \( G \).

On the other hand, from definition of affine c.a. there exists an Abelian group operation on \( G \), \( \eta \) and \( \rho \) commuting automorphism and \( k \in G \), such that \( \phi(a, b) = \eta(a) + \rho(b) + k \). It implies that \( \bullet \) has the medial property. Thus, the componentwise quasi-group operation \( * \) induced from \( \bullet \) on \( \mathcal{G} \) also has the medial property.
From Proposition 3.3, \((\mathfrak{G}, \Phi)\) is topologically conjugate to \((K^\mathbb{Z}, \Phi_K)\) through a 1-block code, where \(\Phi_K\) is given by \(\Phi_K = id \otimes \sigma\). Moreover the same code is an isomorphism between \((\mathfrak{G}, *)\) and \((K^\mathbb{Z}, \otimes)\). Therefore, \(\otimes\) is also a quasi-group operation which has the medial property.

Since \(h(\mathfrak{G}) = \log p\), with \(p\) being a prime number, from Theorem 4.26 of \([26]\) it gives that \(|K| = p\) and \(\otimes\) is a 1-block operation. Thus, there exists a quasi-group operation \(\otimes\) on \(K\), which induces the operation \(\otimes\). Notice that the local rule of \(\Phi_K\) is given by \(\phi_K(a', b') = a' \otimes b'\).

Hence, \(\otimes\) also has the medial property, and so from ([3], Theorem 2.2.2, p.70) there exist an Abelian group operation \(\oplus\) on \(K\), two commuting automorphisms \(\eta'\) and \(\rho'\), and \(c' \in K\), such that \(a' \otimes b' = \eta'(a') \oplus \rho'(b') \oplus c'\). In other words, \((K^\mathbb{Z}, \Phi_K)\) is an affine c.a.

Now, defining \(\mu' := \mu \circ \varphi^{-1}\), we have that \((K^\mathbb{Z}, \mu)\) and \(\mu'\) verify all hypotheses of Theorem 12 in \([7]\), which implies \(\mu'\) is the uniform Bernoulli measure on \(K^\mathbb{Z}\), i.e., the maximum entropy measure for the full shift. Therefore, we conclude that \(\mu\) is the maximum entropy measure on \(\mathfrak{G}\).

The following theorem has a proof analogous to the previous one, but it uses Theorem 13 instead Theorem 12 of \([7]\).

**Theorem 7.2.** Let \((\mathfrak{G}, \Phi)\) be a SC affine c.a., such that \(\mathfrak{G}\) is irreducible and \(h(\mathfrak{G}) = \log p\), where \(p\) is a prime number. Let \(\mu\) be \((\Phi, \sigma)\)-invariant probability measure on \(\mathfrak{G}\). Suppose that

i. \(\mu\) is ergodic for the action \((\Phi, \sigma)\);
ii. \(\mu\) has positive entropy for \(\Phi\);
iii. the sigma-algebra of the \((p-1)p\)-invariant sets coincides mod \(\mu\) to the sigma-algebra of the \(\sigma\)-invariant sets.

Then, \(\mu\) is the Parry measure.

**Remark 7.3.** Given a SC bipermutative c.a. \((\mathfrak{G}, \Phi)\), the c.a. \((K^\mathbb{Z}, \Phi_K)\) obtained from Proposition 3.3 would not be necessarily bipermutative. For the cases when \((K^\mathbb{Z}, \Phi_K)\) is bipermutative, we can use Proposition 5.3 to extend for \((\mathfrak{G}, \Phi)\) the results about invariant measures set out by Pivato \([19]\).

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