GROMOV’S PINCHING CONSTANT.

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February 10, 2022

Abstract

In early 80’s M.Gromov showed that there exists a constant \( \varepsilon \) such that any compact Riemannian manifold \( M^n \) with \( |K|_{M^n} \cdot \text{diam}^2(M^n) \leq \varepsilon \) can be finitely covered by a nilmanifold. The present paper illustrates by an explicit example that the pinching constant \( \varepsilon \) depends on the dimension \( n \) of the manifold, in particular, it decreases with the dimension at least as \( \frac{12}{n^2} \).

1 Introduction

A compact Riemannian manifold \( M^n \) is called \( \varepsilon \)-flat if its curvature is bounded in terms of the diameter as follows:

\[ |K| \leq \varepsilon \cdot \text{diam}^{-2}(M), \]

where \( K \) denotes the sectional curvature and \( \text{diam}(M) \) the diameter of \( M \). If one scales an \( \varepsilon \)-flat metric it remains \( \varepsilon \)-flat.

By almost flat we mean that the manifold carries \( \varepsilon \)-flat metrics for arbitrary \( \varepsilon > 0 \).

A question may arise whether there exist really substantial examples of such manifolds. In [4] Gromov showed that any nilmanifold (= a compact quotient of a nilpotent Lie group) is almost flat. Moreover, he proved that those are up to finite quotients the only almost flat manifolds (cf. [1]). This is a remarkable result in the sense that it makes it possible to get information on the topology and algebraic structure of the manifold solely from assumptions on its curvature:

Theorem (Gromov)

Let \( M^n \) be an \( \varepsilon(n) \)-flat manifold, where
\[ \varepsilon(n) = \exp(-\exp(n^2)). \]  

Then \( M \) is finitely covered by a nilmanifold. More precisely:

(i) The fundamental group \( \pi_1(M) \) contains a torsion-free nilpotent normal subgroup \( \psi \) of rank \( n \),
(ii) The quotient \( G = \pi_1(M)/\psi \) has finite order and is isomorphic to a subgroup of \( O(n) \),
(iii) the finite covering of \( M \) with fundamental group \( \psi \) and deckgroup \( G \) is diffeomorphic to a nilmanifold \( N/\psi \),
(iv) The simply connected nilpotent group \( N \) is uniquely determined by \( \pi_1(M) \).

Gromov in [4] does not specify the restrictions upon \( \varepsilon(n) \). The constant (1) is given as in the proof of Buser and Karcher, [1]. It reflects for larger \( n \) what their proof can yield. It is clear that this constant may not be optimal: much better constants can be obtained for small \( n (= 2, 3, 4) \). In this context the problem of obtaining an effective pinching constant is, therefore, quite natural.

In particular, one may ask a question whether \( \varepsilon \) should necessarily depend on the dimension of the manifold. This question we can answer in the affirmative:

**Theorem (Main Result)**

In every dimension \( n \) there exists a manifold \((M^n, g)\) with

\[ |K|_{(M^n, g)} \cdot \text{diam}^2(M^n, g) < \frac{12}{n^2} \]

which can not be finitely covered by a nilmanifold.

2 An \( n \)-dimensional solvable Lie group with the sectional curvature bounded by \( \frac{12}{n^2} \).

Consider a Lie group \( S = \mathbb{R}^n \rtimes \mathbb{R} \) with the group operation \( L_{(v,t)}(w,s) = (v + h(t)w, t + s) \), where \( h(t) = \text{Exp}(tA) \), \( A \in \text{GL}(n, \mathbb{R}) \). As can be easily seen, \( S \) is solvable.

Indeed, take any two elements \((v, t), (w, s) \in S \). Direct computation shows that their commutator is equal to \([[(v, t), (w, s)], (w, s)] = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, (v, t), (w, s)] = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \) and the conclusion follows.

Describe a lattice \( \Gamma' \) in \( S \).

**Lemma 1** A matrix \( B \in \text{GL}(\mathbb{R}, n) \) preserves a lattice in \( \mathbb{R}^n \) \((B\Gamma = \Gamma)\) if and only if \( B \) is conjugate to a matrix in \( \text{GL}(\mathbb{Z}, n) \).
Proof
Note first that if a matrix $B$ preserves a lattice, then any conjugate $TBT^{-1}$ of $B$ also preserves a lattice. Indeed, if $\Gamma$ is a lattice, $T\Gamma$ is also a lattice and is preserved by $TBT^{-1}$. Let now $B$ preserves a canonical lattice (the one spanned to an orthonormal basis of vectors). Straightforward computation shows that in this case $B$ and $B^{-1}$ have got the determinant equal to 1 or -1 and integer entries, hence, $B$ belongs to $GL(\mathbb{Z}, n)$. And the other way round: direct computation shows that a matrix $B$ from $GL(\mathbb{Z}, n)$ preserves a canonical lattice, therefore, its conjugate preserves a given one.

The above Lemma shows that we can choose a lattice in $S$ in the form $\Gamma' = \Gamma \rtimes \mathbb{Z}$, provided that $\Gamma$ is a lattice in $\mathbb{R}^n$ invariant under $ExpA$. Notice also that if $\Gamma' = \Gamma \rtimes \mathbb{Z}$ is a lattice, $\Gamma'_h = \frac{1}{h}\Gamma \times \mathbb{Z}$ is also a lattice for $h \neq 0$.

Lemma 2 Suppose that $ExpA$ has eigenvalues with absolute value different from 1. A quotient manifold of $S$ by a uniform discrete subgroup (a lattice) $\Gamma' = \Gamma \rtimes \mathbb{Z}$ can not be covered by a nilmanifold.

Proof
Suppose, there exists a covering of $M' = S/\Gamma'$ by a nilmanifold $N'$. Then, by Gromov’s theorem, without loss of generality, we can regard $N'$ as $N' = N/\Psi$, where $\Psi$ is a lattice in a simply connected nilmanifold $N$ and $\Psi \subset \pi_1(M')$ as a normal subgroup of finite index $k$. Since $S$ is simply connected, $\pi_1(M') \cong \Gamma'$. So, according to our assumption, the nilpotent group $\Psi$ is contained in the solvable group $\Gamma'$ as a normal subgroup of finite index $k$.

It means that $\Gamma^k = \langle \gamma^k | \gamma \in \Gamma' \rangle \subset \Psi$ is nilpotent and the element $(0, k)$ is contained in $\Gamma^k$. Moreover, $\Psi' := \Psi \cap \Gamma^k$ is contained in $\Psi$ as a subgroup of finite index. Since $\Psi'$ is normal in $\Gamma^k$ and $\Gamma^k$ is nilpotent we can find normal subgroups $\Psi_1 \subset \ldots \subset \Psi_d = \Psi'$ such that $\Psi_i/\Psi_{i-1}$ is in the center of the group $\Gamma^k/\Psi_{i-1}$. This in turn implies that the subgroups $\Psi_i$ are invariant under the map

$$c : \Psi' \rightarrow \Psi'$$

$$x \rightarrow (0, k)x(0, -k)$$

In addition we know that $c$ induces the identity on the quotient group $\Psi_i/\Psi_{i-1}$. Notice that $c$ is a linear map on the lattice $\Psi'$. The above properties clearly imply that the eigenvalues of this map are 1. On the other hand the eigenvalues are given by the eigenvalues of $Exp(kA)$. Thus the eigenvalues of $ExpA$ are roots of unity. A contradiction.

The next aim is to estimate the sectional curvature of the Lie group $S$ endowed with a suitable left-invariant metric.

We consider a matrix $A$ which is given as the sum of the following two matrices:
\[
\begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_1 \\
\vdots & \\
0 & \lambda_l \\
0 & \lambda_{l+1} \\
0 & \lambda_m
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 \\
0 \\
\end{pmatrix}
\]

\[
+ 
\begin{pmatrix}
0 & \varphi_1 \\
-\varphi_1 & 0 \\
\vdots & \\
0 & \varphi_l \\
0 & -\varphi_1 \\
0 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
\end{pmatrix}
\]

with \( \lambda_i \) real.

Then \( \text{Exp}(tA) = \)

\[
\begin{pmatrix}
e^{t\lambda_1} \begin{pmatrix}
cos(t\varphi_1) & sin(t\varphi_1) \\
-sin(t\varphi_1) & cos(t\varphi_1)
\end{pmatrix} & 0 \\
0 & \vdots \\
0 & e^{t\lambda_l} \begin{pmatrix}
cos(t\varphi_l) & sin(t\varphi_l) \\
-sin(t\varphi_l) & cos(t\varphi_l)
\end{pmatrix} & 0 \\
0 & e^{t\lambda_{l+1}} & 0 \\
0 & 0 & e^{t\lambda_m}
\end{pmatrix}
\]

We define a standard left-invariant metric on \( S \):

\[
\langle v_1, v_2 \rangle_{(w,t)} = \langle (dL_{(w,t)}^{-1}) v_1, (dL_{(w,t)}^{-1}) v_2 \rangle_{(0,0)} = \langle \tilde{h}(-t)v_1, \tilde{h}(-t)v_2 \rangle_{(0,0)} \] (3)

where

\[
\tilde{h}(t) = \begin{pmatrix} h(t) & 0 \\ 0 & 1 \end{pmatrix},
\]

\( \langle \cdot, \cdot \rangle_{(0,0)} \) is a standard Euclidean scalar product on \( s \cong \mathbb{R}^{n+1} \) for \( s \) - the Lie algebra of \( S \) and \( v_1, v_2 \in s \). From the explicit expression for this metric follows the obvious

**Lemma 3** \( S \) is isometric to \( \tilde{S} \), where \( \tilde{S} \) is a Lie group corresponding to the matrix \( A \) equal to
Lemma 4 If one puts $\lambda_{\max} = \max \{ |\lambda_i|, i = 1, \ldots, m \}$, the sectional curvature of $S$ is bounded from above by $|K|_S \leq \frac{1}{4} \lambda_{\max}^2$

Proof
The curvature of a left-invariant metric on a Lie group is given by

$$<R(X,Y)Y,X> = \frac{1}{4} \| (ad_X)^*(Y) + (ad_Y)^*(X) \|^2 - <(ad_X)^*(X), (ad_Y)^*(Y) >$$

$$- \frac{3}{4} \|[X,Y]\|^2 - \frac{1}{2} <[X,Y],X > - \frac{1}{2} <[Y,X],Y >$$

where $X,Y$ are left-invariant vector fields,

$X_{(v,t)} = (dL_{(v,t)})X_e = \tilde{h}(t)X_e,$

$Y_{(v,t)} = (dL_{(v,t)})Y_e = \tilde{h}(t)Y_e$

(cf., for example, [2]).

To simplify the computations, for metrical estimates we can use the group $\tilde{S}$.

The Lie bracket for $\tilde{S}$ is given by

$$[X,Y] = x_0AY' - y_0AX'$$

where

$$X = (X', x_0)$$

$$Y = (Y', y_0)$$

$X', Y' \in \mathbb{R}^n, x_0, y_0 \in \mathbb{R}$.

So,

$$\|ad_XY\| = \|[X,Y]\| \leq \max_i |\lambda_i| \|X\| \|Y\|$$

and the same estimation holds for the matrix of the adjoint operator $(ad_X)^*$. Indeed, take $(ad_X)^*Y \neq 0$ and put $Z = \frac{(ad_X)^*Y}{\|(ad_X)^*Y\|}$. Then

$$\|(ad_X)^*Y\| = <(ad_X)^*Y,Z> = <Y, ad_XZ > \leq \|ad_XZ\| \|Y\| \leq \max_i |\lambda_i| \|X\| \|Y\|$$

Hence, finally, $|K| \leq \frac{11}{4} \lambda_{\max}^2$.

So we see that the sectional curvature of $S$ is controlled by the eigenvalues of the matrix $A$.  

5
**Remark 5** If the eigenvalues of $\text{Exp}A$ satisfy the equation
\[ x^n + 1 = 0 \]
it corresponds to the case when $S$ is flat.

Consider the equation
\[ x^{2k} + 3x^k + 1 = 0 \]  \hspace{1cm} (4)

**Lemma 6** The left-hand side of the equation (4) is the characteristic polynomial of a matrix $T' \in GL(\mathbb{Z}, n)$ if $n = 2k$.

**Proof**

Let $T' =
\begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & 1 \\
-1 & 0 & 0 & \ldots & 0 & 0 \\
0 & -1 & 0 & \ldots & 0 & 0 \\
& & & \ddots & & \\
0 & 0 & 0 & \ldots & 0 & a \\
& & & & \ddots & -1 & 0
\end{pmatrix}^k,

where $a = (-1)^k \cdot 3$.

Direct computation shows that this matrix is indeed in $GL(\mathbb{Z}, n)$. From the explicit form of $T'$, the characteristic polynomial of $T'$ is exactly the polynomial on the left-hand side of the equation (4).

**Remark 7** If $n = 2k + 1$ we consider the polynomial
\[ (x + 1)(x^{2k} + 3x^k + 1) = 0 \]  \hspace{1cm} (5)

and the corresponding matrix $T'' \in GL(\mathbb{Z}, n)$

\[ T'' =
\begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & 1 \\
-1 & 0 & 0 & \ldots & 0 & 0 \\
0 & -1 & 0 & \ldots & 0 & 0 \\
& & & \ddots & & \\
0 & 0 & 0 & \ldots & 0 & a_1 \\
& & & & \ddots & 0 & a_2 \\
0 & 0 & 0 & \ldots & -1 & 0 \\
0 & 0 & 0 & \ldots & 0 & -1
\end{pmatrix}^{k+1},

where $a_1 = (-1)^k \cdot 3, a_2 = (-1)^{k+1} \cdot 3$.

Note also that the matrix $T'(T'')$ is semisimple (cf., for example, W. Greub, [3]), hence each of its invariant subspaces has a complement invariant subspace, therefore, $T'$ can be decomposed over the reals into $2 \times 2$ blocks. In particular, $T''$ is conjugate to...
\[ U' = \begin{pmatrix} e^{\lambda_1} & \begin{pmatrix} \cos \varphi_1 & \sin \varphi_1 \\ -\sin \varphi_1 & \cos \varphi_1 \end{pmatrix} & 0 \\ \vdots & \ddots & \ddots \\ 0 & \begin{pmatrix} \cos \varphi_l & \sin \varphi_l \\ -\sin \varphi_l & \cos \varphi_l \end{pmatrix} & e^{\lambda_l} \end{pmatrix} \]

for \( \lambda_i = \ln |r_i| \), where \( r_i \) are the roots of the characteristic equation (4), and corresponding \( \varphi \)'s.

Straightforward computation shows that for any \( i = 1, \ldots, n \), \( \ln |r_i| < \frac{2}{\pi} \).

Hence, \( \lambda_{\max} < \frac{2}{\pi} \).

Now consider the matrix \( A \) corresponding to these values of \( \lambda_i \) and \( \varphi_i \) and let \( S \) denote the corresponding group. By construction \( A \) is conjugate to a matrix in \( \text{Gl}(\mathbb{Z}, n) \). Therefore \( A \) preserves a lattice \( \Gamma \) and hence \( S \) contains the lattice of the form \( \Gamma' := \Gamma \rtimes \mathbb{Z} \). Clearly, \( \Gamma'_h = \frac{1}{n} \Gamma \rtimes \mathbb{Z} \) is lattice as well.

The next step is to estimate the diameter of the quotient manifold \( S/\Gamma'_h \):

**Lemma 8**

\[ \lim_{h \to \infty} \text{diam}(S/\Gamma'_h) = \text{diam}(\mathbb{R}/\mathbb{Z}) = 1. \]  

**Proof**

First note that \( S/\left(\frac{1}{h} \Gamma \times \mathbb{Z}\right) \) fibers over \( S/\left(\mathbb{R}^n \times \mathbb{Z}\right) \). The natural projection

\[ \text{pr} : S/\left(\frac{1}{h} \Gamma \times \mathbb{Z}\right) \to S/\left(\mathbb{R}^n \times \mathbb{Z}\right) = S^1 \]

is a Riemannian submersion (a maximal rank surjective map, preserving the lengths of vectors orthogonal to the fiber.) It is easy to see, that the diameter of the fiber tends to zero. Thus

\[ \text{diam}M^1 \to \text{diam}S^1 = 1 \]

\[ \blacksquare \]

### 3 Proof of the Main Result

In any dimension \( n \) take \((M^n, g)\) equal to \((S/\Gamma'_h, g')\), where \( S/\Gamma'_h \) is a quotient of an \( n \)-dimensional solvable Lie group described in Section 2 and \( g' \) is a left-invariant metric on \( S \) as in (3). From Lemma 8 the estimation for the maximal eigenvalue of \( A \) in the definition of \( S \) is \( \lambda_{\max} \leq \frac{2}{\pi} \).

Now we can use Lemma 4 to estimate the sectional curvature of \((S/\Gamma'_h, g')\):

\[ |K|_{(S/\Gamma'_h, g')} \leq \frac{11}{n^2}. \]  

(7)
From Lemma 8 we can choose an $h$ so that

$$diam(S/(\Gamma \cdot \mathbb{Z})) < \sqrt{\frac{12}{11}}.$$ (8)

For this $h$

$$diam^2(S/\Gamma_h') \cdot |K(S/\Gamma_h')| \leq \frac{12}{n^2}.$$

Recall that, from Lemma 2, $S/\Gamma_h'$ cannot be covered by a nilmanifold. So, we can conclude that the pinching constant in the Gromov’s Theorem decreases with the dimension at least as $\frac{12}{n^2}$.

References

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