LOW DISTORTION EMBEDDINGS BETWEEN $C(K)$ SPACES

ANTONÍN PROCHÁZKA† AND LUIS SÁNCHEZ-GONZÁLEZ‡

ABSTRACT. We show that, for each ordinal $\alpha < \omega_1$, the space $C([0, \omega^\alpha])$ does not embed into $C(K)$ with distortion strictly less than 2 unless $K^{(\alpha)} \neq \emptyset$.

1. Introduction

A mapping $f : M \to N$ between metric spaces $(M, d)$ and $(N, \rho)$ is called Lipschitz embedding if there are constants $C_1, C_2 > 0$ such that $C_1 d(x, y) \leq \rho(f(x), f(y)) \leq C_2 d(x, y)$ for all $x, y \in M$. The distortion dist$(f)$ of $f$ is defined as inf $\frac{D}{M}$ where the infimum is taken over all constants $C_1, C_2$ which satisfy the above inequality. We say that $M$ embeds into $N$ with distortion $D$ (in short $M \hookrightarrow_D N$) if there exists a Lipschitz embedding $f : M \to N$ with dist$(f) \leq D$ (in short $f : M \hookrightarrow_D N$). In this case, if the target space $N$ is a Banach space, we may always assume (by changing $f$) that $C_1 = 1$. The $N$-distortion of $M$ is defined as $c_N(M) := \inf \left\{ D : M \hookrightarrow_D N \right\}$.

The main result of this article (Theorem 1) implies in particular that, for every countable ordinal $\alpha$ and every $\beta < \omega^\alpha$, the space $C([0, \omega^\alpha])$ does not embed into the space $C([0, \beta])$ with distortion strictly less than 2. On the other hand, it has been shown by Kalton and Lancien [15] that every separable metric space embeds into $c_0$ with distortion 2. Since every $C(K)$ contains $c_0$ as a closed subspace, our result gives $c_{C([0, \beta])}(C([0, \omega^\alpha])) = 2$ if $\beta < \omega^\alpha$.

It is a well known theorem of Mazurkiewicz and Sierpiński (see [13, Theorem 2.56]) that every countable ordinal interval $[0, \beta]$ (every countable Hausdorff compact in fact) is homeomorphic to the interval $[0, \omega^\alpha \cdot n]$ where for some $\alpha < \omega_1$ and $1 \leq n < \omega$. Thus the corresponding spaces of continuous functions are isometrically isomorphic. We do not know, whether one has $c_{C([0, \omega^\alpha \cdot n])}(C([0, \omega^\alpha \cdot n])) = 2$ for $1 \leq m < n < \omega$ but we get as a byproduct of the proof of our main result that, for every $1 \leq D < 2$ and every $1 \leq m < \omega$, there is $1 \leq n < \omega$ such that for all $\alpha < \omega_1$ the space $C([0, \omega^\alpha \cdot n])$ does not embed into the space $C([0, \omega^\alpha \cdot m])$ with distortion $D$ (Proposition 13).

Metric spaces $M$ and $N$ are called Lipschitz homeomorphic if there is a surjective Lipschitz embedding from $M$ onto $N$. Such embedding is then called Lipschitz homeomorphism. The theorem of Amir [3] and Cambern [8] is the following generalization of Banach-Stone theorem: Let $K$ and $L$ be two compact spaces. If there exists a linear isomorphism $f : C(K) \to C(L)$ such that dist$(f) < 2$, then $K$ and $L$ are homeomorphic. A result of Cohen [9] shows that the constant 2 above is optimal but at the present it is not clear whether one could draw the same conclusion under the weaker hypothesis of $f : C(K) \to C(L)$ being a Lipschitz homeomorphism.

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such that \( \text{dist}(f) < 2 \). The results of Jarosz [14], resp. Dutrieux and Kalton [11], resp. Górał [12], show that \( K \) and \( L \) are homeomorphic if there is a Lipschitz homeomorphism \( f : C(K) \to C(L) \) such that \( \text{dist}(f) < 1 + \varepsilon \) (\( \varepsilon > 0 \) universal but small), resp. \( \text{dist}(f) < 17/16 \), resp. \( \text{dist}(f) < 6/5 \). The main result of this article also implies that if \( K \) and \( L \) are two countable compacts, then assuming the existence of a Lipschitz homeomorphism \( f : C(K) \to C(L) \) with \( \text{dist}(f) < 2 \) implies that \( K \) and \( L \) have the same height (Corollary 14).

The proof of the main result consists in identifying a uniformly discrete subset of \( C([0, \omega^n]) \) which does not embed into \( C([0, \beta]) \) with distortion strictly less than 2 if \( \beta < \omega^n \). The uniform discreteness allows to translate the above results into the language of uniform and net homeomorphisms (Corollary 14). Also, using Zippin’s lemma, it permits to prove that if the Szlenk index of a Banach space \( X \) satisfies \( Sz(X) \leq \omega^n \), then \( C([0, \omega^{\omega^n}]) \) does not embed into \( X \) with distortion strictly less than 2 (Theorem 15). This can be understood as a refinement of our result in [18] that \( C([0, 1]) \) does not embed into any Asplund space with distortion strictly less than 2. The underlying idea of the proof there is basically the same as here (and originates in fact in [1] and in [4]) but the proof is not obscured by the technicalities that are necessary in the present article.

Finally, let us recall the following result of Bessaga and Pełczyński [7, 13]: Let \( \omega \leq \alpha \leq \beta < \omega_1 \). Then \( C([0, \alpha]) \) is linearly isomorphic to \( C([0, \beta]) \) iff \( \beta < \alpha^{\omega} \). It is a longstanding open problem whether \( C([0, \beta]) \) can be Lipschitz homeomorphic to a subspace of \( C([0, \alpha]) \) if \( \beta > \alpha^{\omega} \). Our results described above imply that the distortion of any such Lipschitz homeomorphism onto a subspace must be at least 2.

Besides this Introduction, the paper features two more sections. In Section 2 the main theorem is stated and proved. In Section 3 we state and prove its various consequences.

### 2. Main theorem

**Theorem 1.** For every ordinal \( \alpha < \omega_1 \) there exists a countable uniformly discrete metric space \( M_\alpha \subset C([0, \omega^n]) \) such that \( M_\alpha \) does not embed with distortion strictly less than 2 into \( C(K) \) if \( K^{\alpha} = \emptyset \).

We start by defining finite metric graphs that do not embed well into \( \ell^n \) if \( n \) is small. We then “glue” them together infinitely many times, via a relatively natural procedure that we call sup-amalgamation. The sup-amalgamation is done in a precise order which will be encoded by certain trees on \( \mathbb{N} \).

#### 2.1. Construction of 3-level metric graphs.

**Definition 2.** Let \( C_1, \ldots, C_h \) be pairwise disjoint sets. We put

\[
M(C_1, \ldots, C_h) = \{0\} \text{ first level} \cup \bigcup_{i=1}^{h} C_i \text{ second level} \cup F(C_1, \ldots, C_h) \text{ third level}
\]

where \( F(C_1, \ldots, C_h) = \{\{c_1, \ldots, c_h\} : c_i \in C_i\} \). We turn \( M(C_1, \ldots, C_h) \) into a graph by putting an edge between \( x, y \in M(C_1, \ldots, C_h) \) iff \( x = 0 \) and \( y \in \bigcup C_i \) or \( x \in \bigcup C_i \), \( y \in F(C_1, \ldots, C_h) \) and \( x \in y \). We consider the shortest path distance \( d \) on \( M(C_1, \ldots, C_h) \).
Lemma 3. Let \( C_1, \ldots, C_h \) be pairwise disjoint sets and let us assume that \( C_1 = \{1, 2\} \). We define first a mapping \( \ell_\infty(F) \) which satisfies

- \( f(0) = 0 \)
- \( f(x)(\beta) \in \{\pm 1\} \) for all \( x \in \bigcup_{i=1}^{h} C_i \) and all \( \beta \in F \)
- \( f(1)(\beta) = 1 \) and \( f(2)(\beta) = -1 \) for all \( \beta \in F \).

Proof. We define first a mapping \( g : M(C_1, \ldots, C_h) \to \ell_\infty(F) \) as \( g(x)(\beta) = d(x, \beta) - d(0, \beta) \).

It is clearly 1-Lipschitz. Given \( x, y \in M(C_1, \ldots, C_h) \), we can always find \( A, B \in F \) such that \( d(A, B) = d(A, x) + d(x, y) + d(y, B) \). We have \( g(x)(B) - g(y)(B) = d(x, B) - d(y, B) = d(x, B) + d(x, A) - d(A, B) + d(x, y) \geq g(x, y) \). So \( g \) is an isometry. Observe that \( g(0) = 0 \) and also \( g \) satisfies the second additional property. Now since \( 1 \in \beta \) iff \( 2 \notin \beta \) for any \( \beta \in F \), we have that \( g(1)(\beta) = +1 \) iff \( g(2)(\beta) = -1 \) for all \( \beta \in F \). We thus define

\[
    f(x)(\beta) := \begin{cases} 
        g(x)(\beta) & \text{if } g(1)(\beta) = 1 \\
        -g(x)(\beta) & \text{if } g(1)(\beta) = -1
    \end{cases}
\]

for all \( x \in M(C_1, \ldots, C_h) \).

\[ \square \]

2.2. Sup-amalgam of metric spaces.

Definition 4. Let \( ((M_i, d_i))_{i \in I} \) be a collection of metric spaces of uniformly bounded diameter. Let us assume that there is a set \( A \) and a distinguished point \( 0 \in A \) such that \( A \subset M_i \) for every \( i \in I \). A product \( (P, d) \) is the set \( P = \prod_{i \in I} M_i \) equipped with the metric \( d(x, y) = \sup_{i \in I} d_i(x(i), y(i)) \). The set \( M_A \subset P \) defined by \( x \in M_A \) iff either \( x(i) = x(j) \in A \) for all \( i, j \in I \) or there exists exactly one \( i \in I \) such that \( x(i) \notin A \) and for all \( j \neq i \) we have \( x(j) = 0 \), equipped with the metric \( d \), is called the sup-amalgam of \( (M_i) \) with respect to \( A \). We denote it \( (M_i)_{i \in I}/A \).

Standing assumption SA1: Even though the definition admits the possibility that \( d_i \) and \( d_j \) for \( i \neq j \) are different on \( A \), in what follows we will always assume that \( d_i \upharpoonright_{A \times A} = d_j \upharpoonright_{A \times A} \). In that case there is a canonical isometric copy of \( A \) in \( M_A \) which we will denote by \( A \) again.

Standing assumption SA2: We will also assume from now on that for each \( i \in I \) we have that \( d_i(x, y) \geq 1 \) for all \( x, y \in M_i \) and \( d_i(x, 0) \leq 1 \) for each \( x \in A \). Then, for each \( i \in I \), the canonical copy of \( M_i \) in \( M_A \) is isometric to \( M_i \).

Proof: We only need to show that for \( x \in A \) and \( y \in M_i \setminus A \) we have \( d_i(x, y) = d(x, y) \). This is equivalent to saying that \( d_j(x, 0) \leq d_i(x, y) \) for all \( j \in I \).

Lemma 5. a) Let \( A \) be a finite set. Let \( (M_n)_{n \in \mathbb{N}} \) be a sequence of metric spaces of uniformly bounded diameter such that \( 0 \in A \subset M_n \) for every \( n \in \mathbb{N} \). We assume SA1 and SA2. If for each \( n \in \mathbb{N} \) there is an ordinal \( \alpha_n < \omega_1 \) and an isometric embedding \( f_n : M_n \to C([0, \alpha_n]) \) so that for each \( n \in \mathbb{N} \) we have

- \( f_n(0) = 0 \)
- \( f_n(x)(\beta) \in \{\pm 1\} \) for each \( x \in A \setminus \{0\} =: A_\ast \) and \( \beta \in [0, \alpha_n] \),
then there are \( N \leq 2^{|A^*|} \) and an isometric embedding \( f : M_A \to C([0, (\sum_{n=1}^{\infty} \alpha_n) \cdot N]) \) such that \( f(0) = 0 \) and \( f(x)(\beta) \in \{\pm 1\} \) for each \( x \in A^* \) and \( \beta \in [0, (\sum_{n=1}^{\infty} \alpha_n) \cdot N] \).

b) Let us assume moreover that \( 1, 2 \in A \) and that for every \( n \in \mathbb{N} \) we have \( f_n(1)(\beta) = 1 \) and \( f_n(2)(\beta) = -1 \) for all \( \beta \in [0, \alpha_n] \). Then we have \( f(1)(\beta) = 1 \) and \( f(2)(\beta) = -1 \) for all \( \beta \in [0, (\sum_{n=1}^{\infty} \alpha_n) \cdot N] \).

c) Finally, assume moreover that \( A = \{0, 1, 2\} \). Then \( N = 1 \).

**Proof.** Let us consider the restriction \( g \) of the product mapping

\[
\prod_{n=1}^{\infty} M_n \ni x \mapsto (f_n(x(n)))_{n=1}^{\infty} \in \left( \bigoplus_{n=1}^{\infty} C([0, \alpha_n]) \right)_{\infty}
\]

to the set \( M_A \). The mapping \( g \) is clearly an isometry, \( g(0) = 0 \) and for each \( x \in M_A \setminus A \) we have \( g(x) \in C([0, (\sum_{n=1}^{\infty} \alpha_n)]) \) as it has exactly one non-zero entry. Notice that \( g(a) \in C([0, (\sum_{n=1}^{\infty} \alpha_n)]) \) for \( a \in A \). Now, for each \( \varepsilon \in \{\pm 1\}^{A^*} \) we consider the set \( T_\varepsilon = \bigcap_{a \in A^*} \left\{ \beta \in \left[0, (\sum_{n=1}^{\infty} \alpha_n)\right] : g(a)(\beta) = \varepsilon(a) \right\} \).

Let us consider \( I = \{ \varepsilon \in \{\pm 1\}^{A^*} : T_\varepsilon \neq \emptyset \} \). The sets \( (T_\varepsilon)_{\varepsilon \in I} \) are a disjoint cover of \( \left[0, \sum_{n=1}^{\infty} \alpha_n\right] \).

The set \( T_\varepsilon \cap [0, \eta] \) is clopen for each \( \eta < \sum \alpha_n \) and each \( \varepsilon \in I \). Let \( (J_\varepsilon)_{\varepsilon \in I} \) be mutually disjoint copies of \( [0, \sum_{n=1}^{\infty} \alpha_n] \), say \( J_\varepsilon = [0, \sum_{n=1}^{\infty} \alpha_n] \times \{\varepsilon\} \). If \( x \in A^* \), we define \( f(x) \) as the continuous function on \( \bigcup_{\varepsilon \in I} J_\varepsilon \) such that \( f(x)(\beta) = \varepsilon(x) \) when \( \beta \in J_\varepsilon \). If \( x \in M_A \setminus A^* \) we choose \( \eta < \sum \alpha_n \) such that \( g(x)(\gamma) = 0 \) for \( \gamma > \eta \) and we define \( f(x) \) as the continuous function on \( \bigcup_{\varepsilon \in I} J_\varepsilon \) such that for each \( \varepsilon \in I \) we have

\[
f(x)((\beta, \varepsilon)) = \begin{cases} g(x)(\beta) & \text{when } \beta \in T_\varepsilon \cap [0, \eta] \\ 0 & \text{otherwise.} \end{cases}
\]

Notice that we have \( f(0) = 0 \) and \( f(x)(\beta) \in \{\pm 1\} \) for \( x \in A^* \) and \( \beta \in \bigcup_{\varepsilon \in I} J_\varepsilon \). Let us check that \( f \) is an isometry. Using that \( g \) is an isometry and the definition of \( f \) and \( T_\varepsilon \), it is obvious that

\[
d(x, y) = \|g(x) - g(y)\| = \sup_{\varepsilon \in I} \left\{ |f(x)(\beta) - f(y)(\beta)| : \beta \in \bigcup_{\varepsilon \in I} T_\varepsilon \times \{\varepsilon\} \right\}.
\]

On the other hand, checking the four possibilities \( (x \in A^* \text{ or } x \notin A^*) \) and \( (y \in A^* \text{ or } y \notin A^*) \), and remembering SA2, we see that \( |f(x)(\beta) - f(y)(\beta)| \leq d(x, y) \) if \( \beta \notin \bigcup_{\varepsilon \in I} T_\varepsilon \times \{\varepsilon\} \).
Thus, $f$ is an isometry from $M_A$ into $C \left( \bigcup_{\epsilon \in I} J_{\epsilon} \right)$. It is clear that $\bigcup_{\epsilon \in I} J_{\epsilon}$ is homeomorphic to $[0, (\sum_{n=1}^{\infty} \alpha_n) \cdot N]$ where $N := |I|$. Hence $f$ maps isometrically $M_A$ into $C([0, (\sum_{n=1}^{\infty} \alpha_n) \cdot N])$.

The hypothesis in b) means that $\epsilon(1) = 1$ and $\epsilon(2) = -1$ for every $\epsilon \in I$. Now the definition of $f$ on $A_*$ gives the conclusion of b).

The hypothesis in b) and c) mean that $I = \{\epsilon\}$ where $\epsilon(1) = 1$, $\epsilon(2) = -1$. So $N = 1$. \hfill \square

Given a metric space $M$, a mapping $f : M \to C(K)$, points $a, b \in M$ and a constant $1 \leq D < 2$ we denote

$$X_{a,b}^f := \{x^* \in K : |\langle x^*, f(a) - f(b) \rangle| \geq 4 - 2D\}.$$

The duality above means the evaluation at the point $x^* \in K$. We do not indicate the dependence on $D$ since it will always be clear from the context, which $D$ we have in mind.

**Lemma 6.** Let $C_1, \ldots, C_h$ be pairwise disjoint finite sets, we denote $A = \{0\} \cup \bigcup_{i=1}^{h} C_i$. Let $(M_n)_{n \in \mathbb{N}}$ be a sequence of metric spaces of uniformly bounded diameter such that $A \subseteq M_n$ for every $n \in \mathbb{N}$ (and they satisfy SA1 and SA2). Let $1 \leq D < 2$ and let $K$ be a Hausdorff compact. We assume that for each $n \in \mathbb{N}$ there is an ordinal $\alpha_n < \omega_1$ and an integer $\beta_n \in \mathbb{N}$ such that every $f_n : M_n \xrightarrow{D} C(K)$ satisfies

$$\left|\bigcap_{i=1}^{h} X_{a_i,b_i}^{f_n} \cap K^{(\alpha_n)}\right| \geq \beta_n$$

for all $a_i \neq b_i \in C_i \subset M_n$. Then the sup-amalgam $M_A = (M_n)_{n=1}^{\infty}/A$ satisfies for every $f : M_A \xrightarrow{D} C(K)$,

a) if $\alpha_m < \lim_n \alpha_n = \alpha$ for all $m \in \mathbb{N}$, then

$$\bigcap_{i=1}^{h} X_{a_i,b_i}^{f} \cap K^{(\alpha)} \neq \emptyset$$

for all $a_i \neq b_i \in C_i \subset M_A$.

b) if $\alpha_n = \alpha$ for all $n \in \mathbb{N}$, then

$$\left|\bigcap_{i=1}^{h} X_{a_i,b_i}^{f} \cap K^{(\alpha)}\right| \geq \sup \beta_n$$

for all $a_i \neq b_i \in C_i \subset M_A$.

**Proof.** Let $f : M_A \xrightarrow{D} C(K)$. Then $f_n := f |_{M_n}$ is an embedding of $M_n$ into $C(K)$ with $\text{dist}(f_n) \leq D$, and the conclusion follows. \hfill \square

Clearly, we do not cover all the possibilities in the above lemma, but these two are the interesting for us.
2.3. Iterative construction of $M_\alpha$. We use trees here in a very basic fashion as index sets. For the notation, check [13]. For two finite sequences $\overline{m} = (m_1, \ldots , m_h)$ and $\overline{n} = (n_1, \ldots , n_l)$, we write $\overline{m} \cdot \overline{n} = (m_1, \ldots , m_h, n_1, \ldots , n_l)$ for their concatenation. We omit the parentheses for the sequences of length one, thus $(n)$ is written as $n$. Let us construct for every ordinal $\alpha < \omega_1$ the tree $T_{\alpha+1}$ on $\mathbb{N}$ as follows:

- $T_1 = \mathbb{N}$
- if $\alpha$ is non-limit, we put $T_{\alpha+1} = \mathbb{N} \cup \bigcup_{n=1}^{\infty} n^\alpha T_\alpha$
- if $\alpha$ is limit, we choose some $\alpha_n \prec \alpha$ and put $T_{\alpha+1} = \mathbb{N} \cup \bigcup_{n=1}^{\infty} n^\alpha T_{\alpha_n+1}$

where $n^\alpha T_\alpha = \{n^\alpha \cdot \overline{m} : \overline{m} \in T_\alpha\}$. Clearly, for each $\alpha$, the tree $T_{\alpha+1}$ is well founded.

Further, let us define a derivation on trees as follows:

$$T' = \{(n_1, \ldots , n_h) \in T : (n_1, \ldots , n_{h-1}, k) \text{ is not maximal in } T \text{ for some } k \in \mathbb{N}\}$$

There is an index $\beta$ naturally tied to this derivation. We put $T^{(0)} = T$, $T^{(\alpha+1)} = (T^{(\alpha)})'$ and $T^{(\alpha)} = \bigcap_{\beta < \alpha} T^{(\beta)}$ whenever $\alpha$ is a limit ordinal. We put $o(T) = \inf \{\alpha : T^{(\alpha)} = \emptyset\}$ if the set is nonempty, otherwise $o(T) = \infty$. We do not intend to characterize the trees for which $o(T) < \infty$. Instead, we will compute the index of the trees $T_{\alpha+1}$ above. It is worth noting that for every $\alpha, \beta$ and $\overline{n} \in T^{(\beta)}_{\alpha+1}$ the set of successors of $\overline{n}$ in $T^{(\beta)}_{\alpha+1}$ is either empty or equal to $\{\overline{n} \cdot \overline{m} : k \in \mathbb{N}\}$.

**Lemma 7.** For each $\alpha < \omega_1$ we have $o(T_{\alpha+1}) = \alpha + 1$.

**Proof.** We have clearly $o(T_1) = 1$ since all the nodes are maximal, and are mutual siblings. Assume the claim to be true for all $\beta < \alpha$, we try to prove it for $\alpha$. If $\alpha = \beta + 1$ is non-limit, we have $T_{\alpha+1} = \mathbb{N} \cup \bigcup_{n=1}^{\infty} n^\alpha T_{\beta+1}$. It is thus clear that $T^{(\beta+1)}_{\alpha+1} = T_1$ and so $o(T_{\alpha+1}) = \beta + 1 + 1 = \alpha + 1$. Finally assume that $\alpha$ is limit, we have $T_{\alpha+1} = \mathbb{N} \cup \bigcup_{n=1}^{\infty} n^\alpha T_{\alpha_n+1}$. It is thus clear that $T^{(\alpha_{n+1})}_{\alpha+1} = \mathbb{N} \cup \bigcup_{n=m+1}^{\infty} n^\alpha T^{(\alpha_{n+1})}_{\alpha_n+1}$ for all $m \in \mathbb{N}$. Hence $T^{(\alpha)}_{\alpha+1} = T_1$, and so $o(T_{\alpha+1}) = \alpha + 1$. □

For each node of a tree $T_{\alpha+1}$ it will be important to know how many derivations it takes till the node becomes maximal.

**Lemma 8.** For each $\alpha < \omega_1$ and each $\overline{n} \in T_{\alpha+1}$ the ordinal $r_{\alpha+1}(\overline{n}) := \inf \{\beta : \overline{n} \in \max(T^{(\beta)}_{\alpha+1})\}$ is isolated (0 or successor).

**Proof.** Induction on $\alpha$. Clearly true for $\alpha = 0$ (we have $r_1(\overline{n}) = 0$ for each $\overline{n} \in T_1$). Assume the claim to be true for $\alpha$. Then $r_{\alpha+1}(n) = o(T_{\alpha+1}) = \alpha + 1$ for all $n \in \mathbb{N}$ by the construction of the tree $T_{\alpha+2}$ and by Lemma 7. Further, for each $n \in \mathbb{N}$ and each $\overline{n} \in T_{\alpha+1}$, the ordinal $r_{\alpha+1}(n^\alpha \overline{n}) = r_{\alpha+1}(\overline{n})$ is isolated by the inductive hypothesis. In the case when $\alpha$ is limit and the claim has been proved for all $\beta < \alpha$, we have $r_{\alpha+1}(\overline{n}) = o(T_{\alpha_n+1}) = \alpha_n + 1$ for all $n \in \mathbb{N}$ by the construction of the tree $T_{\alpha+1}$ and by Lemma 7. Further, for each $n \in \mathbb{N}$ and each $\overline{n} \in T_{\alpha_n+1}$, the ordinal $r_{\alpha+1}(\overline{n}^\alpha) = r_{\alpha_n+1}(\overline{n})$ is isolated by the inductive hypothesis. □
Definition 9. Let \((A_i)_{i=0}^{\infty}\) be a sequence of countably infinite pairwise disjoint sets which will be fixed from now on. Suppose that \(A_i = \{a_{i}^{1}, a_{i}^{2}, \ldots\}\) and denote \(A_{i}^{k} = \{a_{i}^{1}, \ldots, a_{i}^{n}\}\) for all \(i \geq 0\) and \(k \geq 1\). We assume that \(A_{0}^{2} = \{1, 2\}\). To shorten the notation we put
\[
M_{\overline{n}} = M(A_{0}^{2}, A_{1}^{n_{1}}, \ldots, A_{h}^{n_{h}})
\]
for each \(\overline{n} = (n_{1}, \ldots, n_{h}) \in \mathbb{N}^{h}\).

Let us fix \(\mu < \omega_{1}\) from now on. The tree \(T = T_{\mu+1}\) encodes a construction of a metric space. First we consider all the maximal elements \(\max T\) of \(T\). We thus have a collection \(\mathcal{M}_{0} = \{M_{\overline{n}} : \overline{n} \in \max T\}\). Each space in this collection is finite. Once \(\mathcal{M}_{\alpha} = \{M_{\overline{n}} : \overline{n} \in \max T^{(\alpha)}\}\) has been defined, we pass to the collection \(\mathcal{M}_{\alpha+1} = \{M_{\overline{n}}^{\alpha+1} : \overline{n} \in \max T^{(\alpha+1)}\}\) as follows. For every \(\overline{n} = (n_{1}, \ldots, n_{h}) \in \max T^{(\alpha+1)}\) we put
\[
M_{\overline{n}}^{\alpha+1} = \begin{cases} 
(M_{\overline{n}}^{\alpha})_{k=1}^{\infty}/(\{0\} \cup A_{0}^{2} \cup A_{1}^{n_{1}} \cup \ldots \cup A_{h}^{n_{h}}) & \text{if } \overline{n} \in \max T^{(\alpha+1)} \setminus \max T^{(\alpha)} \\
M_{\overline{n}}^{\alpha} & \text{if } \overline{n} \in \max T^{(\alpha+1)} \cap \max T^{(\alpha)}.
\end{cases}
\]
When \(\alpha < \mu\) is a limit ordinal, we define the elements of \(\mathcal{M}_{\alpha} = \{M_{\overline{n}}^{\alpha} : \overline{n} \in \max T^{(\alpha)}\}\) as \(M_{\overline{n}}^{\alpha} = \lim_{\beta<\alpha} M_{\overline{n}}^{\beta}\). This definition makes sense since for every \(\overline{n} \in \max T^{(\alpha)}\) there is \(\beta_{0} < \alpha\) such that \(\overline{n} \in \max T^{(\beta)}\) for all \(\beta_{0} \leq \beta < \alpha\). Indeed, by Lemma 8, we may take \(\beta_{0} = r_{\mu+1}(\overline{n})\). It is isolated, so \(\beta_{0} < \alpha\).

At the end of the day we have \(\mathcal{M}_{\mu} = \{M_{n}^{\mu} : n \in \mathbb{N}\}\) which we glue into a single space \(M_{\emptyset}^{\mu+1} = (M_{n}^{\mu})_{n=1}^{\infty}/(\{0\} \cup A_{0}^{2})\).

Lemma 10. For every \(1 \leq \alpha \leq \mu+1\) for every \(\overline{n} = (n_{1}, \ldots, n_{h}) \in \max T^{(\alpha)}\) (or rather \(\overline{n} = \emptyset\) in the case \(\alpha = \mu+1\)) there are \(N \leq 2^{2+\alpha+\cdots+n_{h}}\) and an isometric embedding \(f : M_{\overline{n}}^{\alpha} \to C([0, \omega^{\alpha} \cdot N])\) such that \(f(0) = 0\), \(f(1) \equiv 1\) and \(f(2) \equiv -1\). When \(\alpha = \mu+1\), we have \(N = 1\).

Proof. By the definition of the spaces in \(\mathcal{M}_{0}\), we get the claim for \(\alpha = 1\) using Lemma 3 and Lemma 5 a)-b) (here the ordinals \(\alpha_{n}\) are finite so the spaces \(C([0, \alpha_{n}])\) and \(\ell_{\infty}(\alpha_{n})\) are isometric). For \(\alpha > 1\), the proof is a standard transfinite induction argument exploiting Lemma 5 a)-b). Finally, when \(\alpha = \mu+1\), the definition of \(M_{\emptyset}^{\mu+1}\) shows that also the part c) of Lemma 5 applies.

Lemma 11. (i) For each \(1 \leq D < 2\) there exists a constant \(C_{D} = (\log(\lfloor \frac{D}{2-D} \rfloor + 1))^{-1} > 0\) such that for every \(\alpha \leq \mu\) and every \(\overline{n} = (n_{1}, \ldots, n_{h}) \in \max T^{(\alpha)}\) and every \(f : M_{\overline{n}}^{\alpha} \to C(K)\) we have
\[
\left| X_{1,2}^{f} \cap \bigcap_{i=1}^{h-1} X_{a_{i},b_{i}}^{f} \cap K^{(r_{\mu+1}(\overline{n}))} \right| \geq C_{D} \log(n_{h})
\]
for all \(a_{i} \neq b_{i} \in A_{i}^{n_{i}}, 1 \leq i \leq h-1\) (with the obvious meaning when \(h = 1\)).

(ii) For each \(1 \leq D < 2\) and every \(f : M_{\emptyset}^{\mu+1} \to C(K)\) we have
\[
X_{1,2}^{f} \cap K^{(\mu+1)} \neq \emptyset.
\]

Proof. We will proceed by a transfinite induction on \(\alpha\). We assume the result to be true for every \(\beta < \alpha\) and we want to prove it for \(\alpha\). Clearly it is enough to prove the claim for
\( \alpha = r_{\mu+1}(\pi) \). Indeed, if \( \alpha > r_{\mu+1}(\pi) \) then \( M^\alpha_\pi \) and \( M^{r_{\mu+1}(\pi)}_\pi \) are the same, and the result follows by the inductive hypothesis.

In the case \( 0 < \alpha = r_{\mu+1}(\pi) \), the node \( \pi \) just became maximal, i.e. \( \pi \in \max T(\alpha) \setminus \max T(\beta) \) where \( \alpha = \beta + 1 \) (remember Lemma 8). This means that the immediate successors \( \{ \pi : k : k \in \mathbb{N} \} \) of \( \pi \) in \( T(\beta) \) are all maximal in \( T(\beta) \) and

\[
M^\alpha_\pi = (M^{\beta}_\pi)^{\infty}_{k=1}/(0) \cup A_0^2 \cup A_1^1 \cup \ldots \cup A_h^n.
\]

By the inductive hypothesis we have for every \( k \in \mathbb{N} \) and every \( f_k : M^{\beta}_\pi \to C(K) \) that

\[
(1) \quad \left| X_{a_i}^{f_k} \cap \bigcap_{i=1}^h X_{a_i}^{f_k} \cap K^{(r_{\mu+1}(\pi)-k)} \right| \geq C_D \log(k)
\]

for every choice \( a_i \neq b_i \in A_i^n \), \( 1 \leq i \leq h \).

By the construction of \( T_{\mu+1} \) it is clear that we have two types of non-maximal nodes \( \pi \in T_{\mu+1} \):

a) those for which \( (r_{\mu+1}(\pi)-k) \) is a strictly increasing sequence of ordinals and

b) those for which it is a constant sequence.

The case a) means that \( \beta \) is a limit ordinal and \( \sup \{ r_{\mu+1}(\pi)-k : k \in \mathbb{N} \} + 1 = r_{\mu+1}(\pi) \) by the definition of \( r_{\mu+1}(\pi) \). The case b) means that \( \beta \) is a successor and for all \( k \in \mathbb{N} \) we have

\( r_{\mu+1}(\pi)-k + 1 = r_{\mu+1}(\pi) \).

Let \( f \) be an embedding such that \( d(x, y) \leq \| f(x) - f(y) \| \leq D d(x, y) \). Remembering (1), we apply Lemma 6 to get

\[
X_{a_i}^{f_k} \cap \bigcap_{i=1}^h X_{a_i}^{f_k} \cap K^{(r_{\mu+1}(\pi)-k)} \neq \emptyset
\]

for every choice \( a_i \neq b_i \in A_i^n \), \( 1 \leq i \leq h \). Notice that since \( \sup \{ r_{\mu+1}(n) : n \in \mathbb{N} \} = \mu \), we can put formally \( r_{\mu+1}(\emptyset) = \mu + 1 \), and so the above also proves (ii).

Now we need to pass from “non-empty” to “larger than \( C_D \log(n_h) \)”. We may assume that \( f(\emptyset) = 0 \). Let \( a_i \neq b_i \in A_i^n \) be fixed for \( 1 \leq i \leq h - 1 \). By the above there exists

\[
x_{a,b}^* \in X_{a_i}^{f_k} \cap \bigcap_{i=1}^{h-1} X_{a_i}^{f_k} \cap X_{a_i}^{f_k} \cap K^{(r_{\mu+1}(\pi))}
\]

for each \( a \neq b \in A_h^n \). We set \( \Gamma = \{ x_{a,b}^* : a, b \in A_h^n, a \neq b \} \). Now

\[
\{(f(a), \gamma) : a \in A_h^n \} \subset B_{\ell_{\infty}(\Gamma)}(0, D)
\]

is \((4 - 2D)\)-separated set of cardinality \( n_h \). We thus get that \( |\Gamma| \geq C_D \log(n_h) \).

It remains to prove what happens if \( \alpha = 0 \). So let \( f : M^{\emptyset}_\pi \to C(K) \) and let \( a_i \neq b_i \in A_i^n \), \( 1 \leq i \leq h \), be given. We put \( A = \{1\} \cup \{a_i : 1 \leq i \leq h \} \) and \( B = \{2\} \cup \{b_i : 1 \leq i \leq h \} \). We get by the triangle inequality that each \( x^* \in K \) such that \( \| (x^*, f(A) - f(B)) \| = \| f(A) - f(B) \| \) satisfies

\[
x^* \in X_{1,2}^{f_k} \cap \bigcap_{i=1}^h X_{a_i}^{f_k}.
\]

Now by the same argument as above we get the desired inequality. \( \square \)
Proof of Theorem 1. Let $\alpha < \omega_1$ be given. If it is isolated, say $\alpha = \mu + 1$, we put $M_\alpha := M^{\mu+1}_\emptyset$. This space embeds isometrically into $C([0, \omega^\alpha])$ by Lemma 10. By Lemma 11 (ii) we see that if $M^{\mu+1}_\emptyset \hookrightarrow C(K)$, $D < 2$, then $K(\alpha) \neq \emptyset$.

Finally, if $\alpha$ is a limit ordinal we choose $\mu_n \not\nearrow \alpha$ and we put

$$M_\alpha = (M^{\mu_n+1}_\emptyset)_{n=1}^{\infty}/(\{0\} \cup A^0_0).$$

Since $\sum \mu_n = \alpha$ and since $M^{\mu_n+1}_\emptyset$ embeds isometrically into $C([0, \omega^{\mu_n+1}])$ for every $n \in \mathbb{N}$, Lemma 5 shows that $M_\alpha$ embeds isometrically into $C([0, \omega^\alpha])$. If $f : M_\alpha \hookrightarrow C(K)$, $D < 2$, then $X_{1,2}^f \cap K(\mu_n+1) \neq \emptyset$ for each $n \in \mathbb{N}$. This is a decreasing set of compact sets. Hence $X_{1,2}^f \cap K(\alpha) \neq \emptyset$. $\square$

3. Concluding remarks

Remark 12. Let $\alpha < \omega_1$ and let $K$ be a Hausdorff compact space such that $K(\alpha) \neq \emptyset$. We denote by $C_0(K)$ the closed subspace of $C(K)$ of the functions whose restrictions on $K(\alpha)$ are identically zero. An inspection of above proof shows that $C([0, \omega^\alpha])$ does not embed with distortion strictly less than 2 into $C_0(K)$. In particular, $C([0, \omega^\alpha])$ does not embed with distortion strictly less than 2 into $C_0([0, \omega^\alpha])$. For $\alpha = 1$ the last statement means that $c$ does not embed with distortion strictly less than 2 into $c_0$. This also follows from [15, Proposition 3.1] as an easy but entertaining exercise.

Proposition 13. Let $1 \leq D < 2$ be given. Then for every $1 \leq m < \omega$ there is $1 \leq n < \omega$ such that for all $\alpha < \omega_1$ the space $C([0, \omega^\alpha \cdot n])$ does not embed into the space $C([0, \omega^\alpha \cdot m])$ with distortion $D$.

Proof. We find $k \in \mathbb{N}$ such that $C_D \log(k) > m$ and we put $n = 2^{2+k}$. Suppose first that $\alpha$ is a successor ordinal. Notice that if we consider $k$ as an element of $\max T_{\alpha+1}^{(\alpha)}$, we have $r_{\alpha+1}(k) = \alpha$. By Lemma 11, the space $M^k_\emptyset$ as defined in Definition 9 does not embed with distortion $D$ into $C([0, \omega^\alpha \cdot m])$. On the other hand, it embeds isometrically into $C([0, \omega^\alpha \cdot m])$ by Lemma 10.

If $\alpha$ is a limit ordinal, we replace $A^0_0$ by $A^0_\emptyset$ in Definition 9 and we consider the space

$$\hat{M}_\alpha := (M^{\mu_{\alpha+1}}_\emptyset)_{\alpha=1}^{\infty}/(\{0\} \cup A^0_\emptyset).$$

As above, we see that $\hat{M}_\alpha$ embeds isometrically into $C([0, \omega^\alpha \cdot n])$. By repeating the proof of Lemma 11 we get that $\hat{M}_\alpha$ does not embed with distortion $D$ into $C([0, \omega^\alpha \cdot m])$. We leave the details to the reader. $\square$

We recall that if $X$ and $Y$ are Banach spaces and $u : X \to Y$ is uniformly continuous, the following Lipschitz constant of $u$ at infinity

$$l_\infty(u) = \inf_{\eta > 0} \sup_{||x - x'|| \geq \eta} \frac{||u(x) - u(x')||}{||x - x'||}$$

is finite (sometimes called Corson-Klee lemma, see [5, Proposition 1.11]). The uniform distance between $X$ and $Y$ is $d_U(X, Y) = \inf l_\infty(u) \cdot l_\infty(u^{-1})$, where the infimum is taken over all uniform homeomorphisms between $X$ and $Y$.

A net in a Banach space $X$ is a subset $\mathcal{N}$ of $X$ such that there exist $a, b > 0$ which satisfy
\begin{itemize}
\item for any \(x, x' \in \mathcal{N}\) with \(x \neq x'\), we have \(\|x - x'\| \geq a\) and,
\item for any \(x \in X\), there exists \(y \in \mathcal{N}\) with \(\|x - y\| \leq b\).
\end{itemize}

We say that two Banach spaces are \textit{net-equivalent} when they have Lipschitz homeomorphic nets. The \textit{net distance} between \(X\) and \(Y\) is the number \(d_N(X, Y) = \inf \text{dist}(f)\) where the infimum is taken over all mappings \(f : \mathcal{N} \rightarrow \mathcal{M}\) with \(\mathcal{N} \subset X\) and \(\mathcal{M} \subset Y\) being nets. Finally \(d_L(X, Y) = \inf \text{dist}(f)\) where the infimum is taken over all Lipschitz homeomorphisms \(f : X \rightarrow Y\). It is well known and easy to see that for any couple of Banach spaces \(X\) and \(Y\) we have

\[ d_N(X, Y) \leq d_U(X, Y) \leq d_L(X, Y). \]

**Corollary 14.** Let \(\gamma \neq \alpha < \omega_1\) and \(n, m \in \mathbb{N}\). Then \(d_N(C([0, \omega^n \cdot n]), C([0, \omega^\alpha \cdot m])) \geq 2\).

This corollary answers partially Problem 2 in [12].

**Proof.** In fact, we are going to prove the stronger claim that for \(\beta < \omega^\alpha\) and for every net \(\mathcal{N}\) in \(C([0, \omega^\alpha])\) there is no Lipschitz embedding \(f : \mathcal{N} \rightarrow C([0, \beta])\) such that \(\text{dist}(f) < 2\).

Let us suppose that such \(f\) and \(\mathcal{N}\) exist. Assume that \(\mathcal{N}\) is an \((a, b)\)-net and consider a mapping \(\pi : C([0, \omega^\alpha]) \rightarrow \mathcal{N}\) such that \(\|x - \pi(x)\| \leq b\). Let us consider \(M_\alpha\) as a subset of \(C([0, \omega^\alpha])\) which we can by Theorem 1. Since \(d(x, y) \geq 1\) for all \(x \neq y \in M_\alpha\), we have, for \(\lambda > 2b\), that \(\text{dist}(\pi |_{M_\alpha}) \leq (1 + \frac{2b}{\lambda}) (1 + \frac{2b}{\lambda - 2b})\). Thus it is clear that for \(\lambda\) large enough we have \(\text{dist}(g) < 2\) for the embedding \(g : M_\alpha \rightarrow C([0, \beta])\) defined as \(g(x) = \frac{1}{\lambda} f(\pi(\lambda x))\) for \(x \in M_\alpha\). According to Theorem 1, such embedding cannot exist. Contradiction. \(\square\)

Finally, we will give a lower bound on the Szlenk index of a Banach space \(X\) that admits a certain \(M_\alpha\) with distortion strictly less than 2 (for the definition and properties of the Szlenk index, the reader can consult [13, 16]).

**Theorem 15.** Let \(X\) be an Asplund space and assume that \(M_\omega\) embeds into \(X\) with distortion strictly less than 2 for an ordinal \(\alpha < \omega_1\). Then \(\text{Sz}(X) \geq \omega^{\alpha + 1}\).

For the proof we will need the following version of Zippin’s lemma as presented in [6, page 27], see also [19, Lemma 5.11].

**Zippin’s lemma.** Let \(X\) be a separable Banach space with separable dual and let \(\frac{1}{2} > \varepsilon > 0\). Then there exist a compact \(K\), an ordinal \(\beta < \omega^{\text{Sz}(X, \frac{4}{\varepsilon}) + 1}\), a subspace \(Y\) of \(C(K)\), isometric to \(C([0, \beta])\) and an embedding \(i : X \rightarrow C(K)\) with \(\|i\| \|i^{-1}\| < 1 + \varepsilon\) such that for any \(x \in X\) we have

\[ \text{dist}(i(x), Y) \leq 2\varepsilon \|i(x)\|. \]

**Proof.** Let us assume that \(M_\omega \hookrightarrow D X\) with \(D < 2\). Let \(\varepsilon > 0\) be small enough so that \(D' = D(1 + \varepsilon) < 2\) and also that for \(\eta := 2\varepsilon D'\) we have \(\frac{1 + 4\varepsilon}{1 - 2\eta} D' < 2\). Let \(K\) and \(\beta < \omega^{\text{Sz}(X, \frac{4}{\varepsilon}) + 1}\) be as in Zippin’s lemma. Then \(M_\omega\) embeds into \(C(K)\) with distortion \(D' < 2\) via some embedding \(g\) such that \(d'(x, y) \leq \|g(x) - g(y)\| \leq D'd(x, y)\) and, without loss of generality, that \(g(0) = 0\). Thus for every \(x \in M_\omega\) we have \(\|g(x)\| \leq 2D'\). We know that for each \(x \in M_\omega\) there is \(f(x) \in C([0, \beta])\) such that \(\|g(x) - f(x)\| \leq \eta\). This implies that \(\|g(x) - g(y)\| - 2\eta \leq \|f(x) - f(y)\| \leq \|g(x) - g(y)\| + 2\eta\). Now since \(1 \leq d(x, y)\) we have

\[ d(x, y)(1 - 2\eta) \leq \|f(x) - f(y)\| \leq d(x, y)D'(1 + 4\varepsilon). \]
This proves that $f$ is a Lipschitz embedding of $M_{\omega^\alpha}$ into $C([0, \beta])$ with distortion strictly less than 2 and so, according to Theorem 1, we have $\beta \geq \omega^{\omega^\alpha}$. This implies that $Sz(X) > \omega^\alpha$ and so $Sz(X) \geq \omega^{\alpha+1}$ by [13, Theorem 2.43].

An interesting immediate consequence of the above theorem is the fact that, for every $\gamma < \alpha < \omega_1$ and for every equivalent norm $|\cdot|$ on $C([0, \omega^\alpha])$, the space $M_{\omega^\alpha}$ does not embed with distortion strictly less than 2 into $(C([0, \omega^\alpha]), |\cdot|)$.

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† Université Franche-Comté, Laboratoire de Mathématiques UMR 6623, 16 route de Gray, 25030 Besançon Cedex, France
E-mail address: antonin.prochazka@univ-fcomte.fr
‡ Departamento de Ingeniería Matemática, Facultad de CC. Físicas y Matemáticas, Universidad de Concepción, Casilla 160-C, Concepción, Chile
E-mail address: lsanchez@ing-mat.udec.cl