PONCELET PROPELLERS: IN Variant TOTAL BLADE AREA

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Abstract. Given a triangle, a trio of circumellipses can be defined, each centered on an excenter. Over the family of Poncelet 3-periodics (triangles) in a concentric ellipse pair (axis-aligned or not), the trio resembles a rotating propeller, where each “blade” has variable area. Amazingly, their total area is invariant, even when the ellipse pair is not axis-aligned. We also prove a closely-related invariant involving the sum of blade-to-excircle area ratios.

1. Introduction

Invariants of Poncelet N-periodics in various ellipse pairs have been a recent focus of research [9, 6, 5]. For the confocal pair alone (elliptic billiard), 80+ invariants have been catalogued [10], and several proofs have ensued [1, 3, 4].

We start by describing an invariant manifested by Poncelet 3-periodics (triangles) inscribed in an ellipse and circumscribed about a concentric circle; see Figure 1. For such a pair to admit a 3-periodic family, its axes are constrained in a simple way, explained below.

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Figure 1. A Poncelet 3-periodic (blue) inscribed in an outer ellipse (black) and circumscribed about an inner concentric circle (brown). Over the family, the circumcircle (dashed blue) has constant circumradius \( R = (a + b)/2 \) [5, Thm 1]. Video, app
By definition, the incenter $X_1$ of this family is stationary at the common center and the inradius $r$ is fixed. Remarkably, the circumradius $R$ is also invariant, and this implies the sum of cosines of its internal angles is as well [5].

Referring to Figure 2, consider a circumellipse [11, Circumellipse] of a 3-periodic in the incircle pair centered on one of the excenters, i.e., a vertex of the excentral triangle [11, Excentral Triangle].

**Main Result.** Referring to Figure 3, over 3-periodics in the concentric ellipse pair with an incircle, the total area of the three excenter-centered circumellipses (blades of a “Poncelet propeller”) is invariant.

We then extend this property to a 3-periodic family interscribed in any pair of concentric ellipses, including non-axis-aligned; see Figure 4. The only proviso is that the circumellipses be centered on the vertices of the anticevian triangle with respect to the common center [11, Anticevian Triangle].

**Structure of Article.** In Section 2 we derive the areas for the 3 circumellipses centered on the vertices of the anticevian triangle with respect to some point $X$. In Section 3 we apply those formulas for 3-periodics in a pair with incircle. In Section 4 we generalize the result to an unaligned concentric ellipse pair. In Section 5 we prove a related invariant contributed by L. Gheorghe [8] involving area ratios of circumellipses and excircles. We encourage the reader to watch some of the videos mentioned herein listed in Section 6.
2. Areas of Anticevian Circumellipses

Let $\mathcal{T} = P_i, i = 1, 2, 3$ be a reference triangle. Let $\mathcal{C}_X$ denote the circumellipse [11, Circumellipse] centered on point $X$ interior to $\mathcal{T}$ and let $\Delta_X$ denote its area. Let the trilinear coordinates of $X$ be $[u, v, w]$ [11, Trilinear Coordinates]. Let $\mathcal{T}_X$ denote the anticevian triangle of $\mathcal{T}$ wrt $X$, and $P'_i$ its vertices [11, Anticevian Triangle]. Let $\mathcal{C}'_i$ denote the three circumellipses centered on $P'_i$. Let $\Delta_i$ denote their areas, respectively.
Lemma 1. The areas of the aforementioned circumellipses are given by:

\[
\Delta_X = \frac{z_1 z_2 z_3 u v w}{s_1 u + s_2 v + s_3 w}
\]

where \(a > b > r > 0\).

\[
\Delta_1 = \frac{z_1 z_2 z_3 u v w}{-s_1 u + s_2 v + s_3 w}
\]

\[
\Delta_2 = \frac{z_1 z_2 z_3 u v w}{s_1 u - s_2 v + s_3 w}
\]

\[
\Delta_3 = \frac{z_1 z_2 z_3 u v w}{s_1 u + s_2 v - s_3 w}
\]

\[
z = \pi \sqrt{\frac{(s_1 + s_2 + s_3)(-s_1 + s_2 + s_3)(s_1 - s_2 + s_3)(s_1 + s_2 - s_3)}{\prod (s_1 u + s_2 v + s_3 w) - (s_1 u + s_2 v + s_3 w) - (s_1 u - s_2 v + s_3 w) - (s_1 u + s_2 v - s_3 w)}}
\]

Proof. Let the trilinears of \(X\) be \([u : v : w]\). The trilinears of \(P_1^r\) are known to be \([-u : v : w], [u : -v : w]\) and \([u : v : -w]\), see [11, Anticevian Triangle]. The expressions are obtained from the formula of the area of a circumellipse centered on \(X = [u : v : w]\) [11, Circumellipse].

\[
\Delta_X = \frac{z_1 z_2 z_3 u v w}{s_1 u + s_2 v + s_3 w}
\]

\(\square\)

The above also implies:

**Corollary 1.** For any triangle the following relation holds:

\[
\frac{1}{\Delta_a} = \frac{1}{\Delta_1} + \frac{1}{\Delta_2} + \frac{1}{\Delta_3}
\]

Note: this is reminiscent of the well-known property \(1/r = \sum 1/r_i\) where \(r\) is the inradius and \(r_i\) are the exradii of a triangle [11, Excircles].

From direct calculations:

**Lemma 2.** Let \(\Sigma_X\) denote the sum \(\Delta_1 + \Delta_2 + \Delta_3\). This is given by:

\[
\Sigma_X = \frac{(s_1^2 u^2 + s_2^2 v^2 + s_3^2 w^2 - 2s_1 s_2 u v - 2s_1 s_3 u w - 2s_2 s_3 v w) s_1 s_2 s_3 u v w z}{(s_1 u + s_2 v - s_3 w)(s_1 u - s_2 v + s_3 w)(s_1 u - s_2 v - s_3 w)}
\]

### 3. Poncelet Family with Incircle

Let \(\mathcal{F}\) denote the 1d Poncelet family of 3-periodics inscribed in an outer ellipse \(\mathcal{E}\) with semi-axes \(a, b\) and circumscribed about a concentric circle \(\mathcal{E}'\) with radius \(r\). Assume \(a > b > r > 0\). Let \(O\) denote the common center.

Recall Cayley’s condition for the existence of a 3-periodic family interscribed between two concentric, axis-aligned ellipses [7]:

\[
\frac{a_c}{a} + \frac{b_c}{b} = 1
\]

where \(a_c\) and \(b_c\) are the major and minor semi-axes of the inscribed ellipse corresponding to the caustic of the Poncelet family. For \(\mathcal{F}\), \(a_c = b_c = r\), then \(\frac{a}{r} + \frac{b}{r} = 1\), i.e.:

\[
r = \frac{ab}{a + b}
\]
By definition, the incenter $X_1$ of $\mathcal{F}$ triangles is stationary at $O$. Also by definition, the inradius $r$ is fixed. Also known is the fact that the circumradius $R$ of $\mathcal{F}$ triangles is invariant and given by [5]:

$$R = \frac{ab}{2r} = \frac{a+b}{2}.$$ 

Let $\rho = \frac{r}{R}$ denote the ratio $r/R$. From the above it is invariant and given by:

$$\rho = \frac{r}{R} = \frac{2r^2}{ab}.$$ 

Taking $X = O$, note that the area $\Delta_o = \pi ab$ of $\mathcal{E}$ is by definition, invariant. Also note the anticevian $\mathcal{T}_o$ in this case is the excentral triangle [11, Excentral Triangle].

**Lemma 3.** Over $\mathcal{F}$, $\Sigma_o$ is invariant and given by:

$$\Sigma_o = \left(1 + \frac{4}{\rho}\right) \Delta_o$$

**Proof.** By using well-known formulas for $r$ and $R$ [11, Inradius,Circumradius], one can express $\rho$ in terms of the sidelengths:

$$\rho = \frac{(s_1 + s_2 - s_3)(s_1 - s_2 + s_3)(-s_1 + s_2 + s_3)}{2s_1 s_2 s_3}.$$ 

Using the trilinears for the incenter $O = X_1 = [u, v, w] = [1 : 1 : 1]$ in (2), direct calculations yield the claim. \hfill \square

4. Generalizing the Result

It turns out Lemma 3 can be generalized to a larger class of Poncelet 3-periodic families.

Let $(\mathcal{E}, \mathcal{E}')$ be a generic pair of concentric ellipses (axis-aligned or not), admitting a Poncelet 3-periodic family. Let $O$ be their common center. Let $C'_i$ be the circum-ellipses centered on the vertices of the anticevian with respect to $O$. Referring to Figure 4:

**Theorem 1.** Over the 3-periodic family interscribed in $(\mathcal{E}, \mathcal{E}')$, $\Sigma_o$ is invariant.

**Proof.** $(\mathcal{E}, \mathcal{E}')$ can be regarded as an affine image of the original family $\mathcal{F}$ (with incircle). Let matrix $A$ represent the required affine transform. Since conics are equivariant under affine transformations\(^1\) of the 5 constraints that define them (in our case, passing through the vertices of the 3-periodic and being centered on an anticevian vertex) [2], the area of each circumellipse will scale by $\det(A)$, so by Lemma 3 the result follows. \hfill \square

In Figure 5, a few other concentric, axis-aligned pairs are shown which illustrate the above corollary.

**Observation 1.** If $(\mathcal{E}, \mathcal{E}')$ are homothetic and concentric, then each of the $\Delta_i$ are constant.

\(^1\)In fact, they are equivariant under projective transformations [2].
Figure 4. Consider Poncelet 3-periodics (blue) interscribed in a concentric, pair of ellipses which in general is not axis-aligned. The total area of the three circumellipses (orange, light blue, pink) centered on the vertices of the O-anticevian triangle (green) is invariant. Video

This arises from the fact that this pair is affinely-related to a pair of concentric circles. The associated Poncelet 3-periodic is then given by an equilateral triangle where \( \Delta_1 = \Delta_2 = \Delta_3 \).

5. Circumellipses Meet Excircles

Referring to Figure 6, L. Gheorghe detected experimentally that the sum of area ratios of excircles to excentral circumellipses is invariant for two Poncelet families (see below) [8]. Below we prove this and derive explicit values for the invariants. As before, let \( \Delta_i \) denote the area of a circumellipse centered on the \( i \)th excenter. Let \( \Omega_i \) denote the area of the \( i \)th excircle.

Theorem 2. Over the 3-periodic family \( F \) (with incircle):

\[
\frac{\Delta_1}{\Omega_1} + \frac{\Delta_2}{\Omega_2} + \frac{\Delta_3}{\Omega_3} = \frac{2}{\rho}
\]

Proof. The exradii are given by [11, Excircles]:

\[
r_1 = \frac{S}{s-s_1}, \quad r_2 = \frac{S}{s-s_2}, \quad r_3 = \frac{S}{s-s_3}
\]

where \( s = \frac{s_1+s_2+s_3}{2} \) and \( S \) are semi-perimeter and area of triangle \( P_1P_2P_3 \), respectively. From them obtain the excircle areas \( \Omega_i \):
Figure 5. A picture of four 3-periodic families. The circumellipses are centered on vertices of the anticevian with respect to the center. Since each case is affinely related to $F$, the total circumellipse area is invariant. For the homothetic pair (bottom left), the area of each circumellipse is invariant.

Video

Figure 6. Left: The pair with incircle centered on $X_1$. Shown also are a 3-periodic (blue), the excentral triangle (solid green), the excircles (dashed green), and the three excentral circumellipses (magenta, light blue, and orange). Right: the same arrangement for the confocal pair, centered on $X_9$. 
\[ \Omega_1 = \pi \frac{s(s - s_2)(s - s_3)}{s - s_1} \]
\[ \Omega_2 = \pi \frac{s(s - s_3)(s - s_1)}{s - s_2} \]
\[ \Omega_3 = \pi \frac{s(s - s_1)(s - s_2)}{s - s_3} \]

For \( i = 1, 2, 3 \), the following can be derived:

\[ \frac{\Delta_i}{\Omega_i} = \mu(s - s_i) \]

with \( \mu = \frac{s_1 s_2 s_3}{2s(s-s_1)(s-s_2)(s-s_3)} \). Therefore:

\[ \sum_{i=1}^{3} \frac{\Delta_i}{\Omega_i} = \mu s = \frac{2}{\rho} \]

Recall \( \rho \) is constant for family \( \mathcal{F} \) so the result for the pair incircle follows.

Using an analogous proof method:

**Theorem 3.** Over the 3-periodic family interscribed in the confocal the following quantity is also invariant:

\[ \frac{\Delta_1}{\Omega_1} + \frac{\Delta_2}{\Omega_2} + \frac{\Delta_3}{\Omega_3} = \frac{2}{\rho} \]

**Observation 2.** Although the pair with incircle and the confocal pair are affinely-related, neither Theorem 2 nor 3 for any other ellipse pair in the affine continuum.

6. List of Videos

Animations illustrating some of the above phenomena are listed on Table 1.

| id | Title | youtu.be/ |
|----|-------|----------|
| 01 | 3-Periodic incircle family has invariant \( R \) | eIxblso60Ro |
| 02 | Family of incircle 3-periodics and one excentral circumellipse | JUCmAMsfdkI |
| 03 | Three excentral circumellipses with invariant total area | ub4wAv8Hgb0 |
| 04 | 3-periodics in 5 concentric, axis-aligned ellipse pairs | ShkeksAsx0E |
| 05 | Incircle family | tHUDfx9o0Wg |
| 06 | Four Poncelet families | crXxP9J9ZDx |
| 07 | Non-concentric ellipse pair | FJXmpUcs1aA |

Table 1. Videos of some focus-inversive phenomena. The last column is clickable and provides the YouTube code.

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APPENDIX A. Table of Symbols

| symbol | meaning |
|--------|---------|
| $\mathcal{E}, \mathcal{E}_c$ | outer and inner ellipses |
| $a, b$ | outer ellipse semi-axes’ lengths |
| $a_c, b_c$ | inner ellipse semi-axes’ lengths |
| $O$ | common center of ellipses |
| $P_1, P_2, P_3$ | 3-periodic vertices |
| $s_1, s_2, s_3$ | 3-periodic sidelengths |
| $P'_1, P'_2, P'_3$ | vertices of the anticevian wrt $O$ |
| $C_X, \Delta_X$ | $X$-centered circumellipse and its area |
| $C'_i, \Delta_i$ | $P'_i$-centered circumellipse and its area |
| $T_X$ | anticevian wrt $X$ |
| $\Sigma_X$ | area sum of 3 circumellipses centered on the vertices of $T_X$ |
| $r, R$ | 3-periodic inradius and circumradius |
| $\rho$ | ratio $r/R$ |
| $[u : v : w]$ | trilinear coordinates |
| $\Omega_i$ | area of the excircle corresponding to $P_i$ |

Table 2. Symbols of euclidean geometry used

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