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BERRY-ESSEEN TYPE BOUND FOR FRACTIONAL
ORNSTEIN-UHLENBECK TYPE PROCESS DRIVEN BY
SUB-FRACTIONAL BROWNIAN MOTION

We obtain a Berry-Esseen type bound for the distribution of the maximum likelihood
estimator of the drift parameter for fractional Ornstein-Uhlenbeck type process driven
by sub-fractional Brownian motion.

1. Introduction

Statistical inference for fractional diffusion processes satisfying stochastic differential
equations driven by a fractional Brownian motion (fBm) has been studied earlier and
a comprehensive survey of various methods is given in Prakasa Rao [17]. There has
been a recent interest to study similar problems for stochastic processes driven by a
sub-fractional Brownian motion. Bojdecki et al. [2] introduced a centered Gaussian
process \( \zeta^H = \{ \zeta^H(t), t \geq 0 \} \) called sub-fractional Brownian motion (sub-fBm) with the
covariance function

\[
C_H(s, t) = s^{2H} + t^{2H} - \frac{1}{2}[(s + t)^{2H} + |s - t|^{2H}]
\]

where \( 0 < H < 1 \). The increments of this process are not stationary and are more
weakly correlated on non-overlapping intervals than those of a fBm. Tudor [25] intro-
duced a Wiener integral with respect to a sub-fBm. Tudor [22, 23, 24, 25] discussed
some properties related to sub-fBm and its corresponding stochastic calculus. By using
a fundamental martingale associated to sub-fBm, a Girsanov type theorem is obtained
in Tudor [25]. Diedhiou et al. [3] investigated parametric estimation for a stochastic dif-
fferential equation (SDE) driven by a sub-fBm. Mendy [13] studied parameter estimation
for the sub-fractional Ornstein-Uhlenbeck process defined by the stochastic differential
equation

\[
dX_t = \theta X_t dt + d\zeta^H(t), t \geq 0
\]

where \( H > \frac{1}{2} \). This is an analogue of the Ornstein-Uhlenbeck process, that is, a
continuous time first order autoregressive process \( X = \{ X_t, t \geq 0 \} \) which is the solution
of a one-dimensional homogeneous linear stochastic differential equation driven by a
sub-fBm \( \zeta^H = \{ \zeta^H_t, t \geq 0 \} \) with Hurst parameter \( H \). Mendy [13] proved that the least
squares estimator estimator \( \hat{\theta}_T \) is strongly consistent as \( T \to \infty \). Kuang and Xie [10]
studied properties of maximum likelihood estimator for sub-fBm through approximation
by a random walk. Kuang and Liu [9] discussed about the \( L^2 \)-consistency and strong
consistency of the maximum likelihood estimators for sub-fBm with drift based on
discrete observations. Yan et al. [26] obtained the Itô’s formula for sub-fractional Brown-
ian motion with Hurst index \( H > \frac{1}{2} \). Shen and Yan [21] studied estimation for the
drift of sub-fractional Brownian motion and constructed a class of biased estimators of
James-Stein type which dominate the maximum likelihood estimator under the quadratic
risk. El Machkouri et al. [5] investigated the asymptotic properties of the least squares

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estimator for non-ergodic Ornstein-Uhlenbeck process driven by Gaussian processes, in particular, sub-fractional Brownian motion. In a recent paper, we have investigated optimal estimation of a signal perturbed by a sub-fractional Brownian motion in Prakasa Rao [19]. Some maximal and integral inequalities for a sub-fBm were derived in Prakasa Rao [18]. Parametric estimation for linear stochastic differential equations driven by a sub-fractional Brownian motion is studied in Prakasa Rao [20]. We now obtain a Berry-Esseen type bound for the distribution of the maximum likelihood estimator for the drift parameter of a fractional Ornstein-Uhlenbeck type process driven by a sub-fractional Brownian motion.

2. Preliminaries

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t), P)\) be a stochastic basis satisfying the usual conditions and the processes discussed in the following are \((\mathcal{F}_t)\)-adapted. Further the natural filtration of a process is understood as the \(P\)-completion of the filtration generated by this process.

Let \(\zeta^H = \{\zeta^H_t, t \geq 0\}\) be a normalized sub-fractional Brownian motion (sub-fBm) with Hurst parameter \(H \in (0, 1)\), that is, a Gaussian process with continuous sample paths such that \(\zeta^H_0 = 0, E(\zeta^H_t) = 0\) and

\[
E(\zeta^H_s \zeta^H_t) = t^{2H} + s^{2H} - \frac{1}{2}(s + t)^{2H} + |s - t|^{2H}, t \geq 0, s \geq 0.
\]

Bojdecki et al. [2] noted that the process

\[
\frac{1}{\sqrt{2}}[W^H(t) + W^H(-t)], t \geq 0,
\]

where \(\{W^H(t), -\infty < t < \infty\}\) is a fBm, is a centered Gaussian process with the same covariance function as that of a sub-fBm. This proves the existence of a sub-fBm. Let \(D_H(s,t)\) denote the covariance function of a standard fractional Brownian motion with Hurst index \(H\). Note that

\[
D_H(s,t) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}).
\]

Bojdecki et al. [2] proved the following result concerning properties of a sub-fBm.

**Theorem 2.1.** Let \(\zeta^H = \{\zeta^H_t, t \geq 0\}\) be a sub-fBm defined on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), P)\). Then the following properties hold:

(i) The process \(\zeta^H\) is self-similar, that is, for every \(a > 0\),

\[
\{\zeta^H(at), t \geq 0\} \overset{\Delta}{=} \{a^H \zeta^H(t), t \geq 0\}
\]

in the sense that the processes, on both the sides of the equality sign, have the same finite dimensional distributions.

(ii) The process \(\zeta^H\) is not Markov and it is not a semi-martingale.

(iii) For all \(s, t \geq 0\), the covariance function \(C_H(s, t)\) of the process \(\zeta^H\) is positive for all \(s > 0, t > 0\). Furthermore

\[
C_H(s, t) > D_H(s, t) \text{ if } H < \frac{1}{2}
\]

and

\[
C_H(s, t) < D_H(s, t) \text{ if } H > \frac{1}{2}.
\]

(iv) Let \(\beta_H = 2 - 2^{2H-1}\). For all \(s \geq 0, t \geq 0\),

\[
\beta_H(t - s)^{2H} \leq E[\zeta^H(t) - \zeta^H(s)]^2 \leq (t - s)^{2H}, \text{ if } H > \frac{1}{2}
\]
and
\[(t - s)^{2H} \leq E[|\zeta^H(t) - \zeta^H(s)|^2] \leq \beta_H(t - s)^{2H}, \text{ if } H < \frac{1}{2}\]
and the constants in the above inequalities are sharp.

(v) The process \(\zeta^H\) has continuous sample paths almost surely and, for each \(0 < \epsilon < H\) and \(T > 0\), there exists a random variable \(K_{\epsilon,T}\) such that
\[|\zeta^H(t) - \zeta^H(s)| \leq K_{\epsilon,T}|t - s|^{H - \epsilon}, 0 \leq s, t \leq T.\]

Let \(f : [0, T] \to \mathbb{R}\) be a measurable function and \(\alpha > 0\), and \(\sigma\) and \(\eta\) be real. Define the Erdelyi-Kober-type fractional integral
\[(2.2) \quad (\mathcal{I}_{T,\sigma,\eta}^\alpha f)(s) = \frac{\sigma^\alpha}{\Gamma(\alpha)} \int_s^T \frac{f(t)}{(t^\sigma - s^\sigma)^{1 - \alpha}} dt, s \in [0, T],\]
and the function
\[(2.3) \quad n_H(t, s) = \frac{\sqrt{\pi} I_{T,2}^{H - \frac{1}{2}}(u^{H - \frac{1}{2}})}{2^{2H - 2}} \frac{\Gamma(H - \frac{1}{2})}{2^H} s^{\frac{1}{2} - H} \int_0^t (x^2 - s^2)^{H - \frac{1}{2}} dx I_{(0,1)}(s).\]

The following theorem is due to Dzhaparidze and Van Zanten [4] (cf. Tudor [25]).

**Theorem 2.2.** The following representation holds, in distribution, for a sub-fBm \(\zeta^H:\)
\[(2.4) \quad \zeta^H_s = c_H \int_0^t n_H(t, s) dW_s, 0 \leq t \leq T\]
where
\[(2.5) \quad c_H = \frac{\Gamma(2H + 1) \sin(\pi H)}{\pi}\]
and \([W_t, t \geq 0]\) is the standard Brownian motion.

Tudor [25] has defined integration of a non-random function \(f(t)\) with respect to a sub-fBm \(\zeta^H\) on an interval \([0, T]\) and obtained a representation of this integral as a Wiener integral for a suitable transformed function \(\phi_f(t)\) depending on \(H\) and \(T\). For details, see Theorem 3.2 in Tudor [25].

Tudor [23] (cf. Tudor [25], p. 467) obtained the prediction formula for a sub-fBm. For any \(0 < H < 1\), and \(0 < a < t,\)
\[(2.6) \quad E[|\zeta^H_s|_a, 0 \leq s \leq a] = \zeta^H_a + \int_0^a \psi_{a,t}(u) d\zeta^H_u\]
where
\[(2.7) \quad \psi_{a,t}(u) = \frac{2 \sin(\pi (H - \frac{1}{2}))}{\pi} a^{H - \frac{1}{2}} u^{\frac{1}{2} - H} \int_a^t \frac{(z^2 - a^2)^{H - \frac{1}{2}}}{z^2 - u^2} dz.\]

Let
\[(2.8) \quad M^H_t = d_H \int_0^t s^{\frac{1}{2} - H} dW_s = \int_0^t k_H(t, s) ds^H\]
where
\[(2.9) \quad d_H = \frac{2^{H - \frac{1}{2}}}{c_H \Gamma(\frac{3}{2} - H) \sqrt{\pi}},\]
\[(2.10) \quad k_H(t, s) = d_H s^{\frac{1}{2} - H} \psi_H(t, s),\]
and

\[
\psi_H(t, s) = \frac{s^{H-\frac{1}{2}}}{\Gamma(\frac{1}{2} - H)} [1^{H-\frac{1}{2}}(t^2 - s^2)^{\frac{1}{2}-H} - (H - \frac{3}{2}) \int_s^t (x^2 - s^2)^{\frac{1}{2}-H} x^{H-\frac{1}{2}} dx] I_{(0,t)}(s).
\]

It can be shown that the process \( M^H = \{M^H_t, 0 \leq t \leq T\} \) is a Gaussian martingale (cf. Tudor [25], Diedhiou et al. [3]) and is called the sub-fractional fundamental martingale. The filtration generated by this martingale is the same as the filtration \( \{\mathcal{F}_t, t \geq 0\} \) generated by the sub-fBm \( \zeta^H \) and the quadratic variation \(<M^H>_s >_s \) of the martingale \( M^H \) over the interval \([0, s]\) is equal to \( w^H_s = \frac{d^2}{2H} s^{2-2H} = \lambda_H s^{2-2H} \) (say). For any measurable function \( f : [0, T] \to \mathbb{R} \) with \( \int_0^T f^2(s) s^{1-2H} ds < \infty \), define the probability measure \( Q_f \) by

\[
\frac{dQ_f}{dP} |_{\mathcal{F}_t} = \exp\left( \int_0^t f(s) dM^H_s - \frac{1}{2} \int_0^t f^2(s) ds < M^H >_s (s) \right)
\]

where \( P \) is the underlying probability measure. Let

\[
(\psi_H f)(s) = \frac{1}{\Gamma(\frac{1}{2} - H)} [1^{H-\frac{1}{2}} s^{\frac{1}{2}-H} f(s)
\]

where, for \( \alpha > 0 \),

\[
(I^\alpha_{0, \sigma, \eta, f})(s) = \frac{\sigma s^{-\frac{\alpha+\eta}{\alpha}}}{\Gamma(\alpha)} \int_0^s \frac{1}{(t^\sigma - s^\sigma)^{1-\alpha}} f(t) dt, s \in [0, T].
\]

Then the following Girsanov type theorem holds for the sub-fBm process (Tudor [25]).

**Theorem 2.3.** The process

\[
\zeta^H_t - \int_0^t (\psi_H f)(s) ds, 0 \leq t \leq T
\]

is a sub-fBm with respect to the probability measure \( Q_f \). In particular, choosing the function \( f \equiv a \in \mathbb{R} \), it follows that the process \( \{\zeta^H_t - at, 0 \leq t \leq T\} \) is a sub-fBm under the probability measure \( Q_f \) with \( f \equiv a \in \mathbb{R} \).

Let \( Y = \{Y_t, t \geq 0\} \) be a stochastic process defined on the filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), P)\) and suppose the process \( Y \) satisfies the stochastic differential equation

\[
dY_t = C(t) dt + d\zeta^H_t, t \geq 0
\]

where the process \( \{C(t), t \geq 0\} \), adapted to the filtration \( \{\mathcal{F}_t, t \geq 0\} \), such that the process

\[
R_H(t) = \frac{d}{dt} \int_0^t k_H(t, s) C(s) ds, t \geq 0
\]

is well-defined and the derivative is understood in the sense of absolute continuity with respect to the measure generated by the function \( w_H \). Differentiation with respect to \( w^H_t \) is understood in the sense:

\[
dw^H_t = \lambda_H (2 - 2H) t^{1-2H} dt
\]
and \[ \frac{df(t)}{dw_t^H} = \frac{df(t)}{dt}/dw_t^H. \]

Suppose the process \( \{R_H(t), 0 \leq t \leq T\} \), defined over the interval \([0, T]\) belongs to the space \( L^2([0, T], dw_t^H) \). Define

\[ (2.15) \quad \Lambda_H(t) = \exp\{\int_0^t R_H(s) dM_s^H - \frac{1}{2} \int_0^t [R_H(s)]^2 dw_s^H\} \]

with \( E[\Lambda_H(T)] = 1 \) and the distribution of the process \( \{Y_t, 0 \leq t \leq T\} \) with respect to the measure \( P^Y = \Lambda_H(t) P \) coincides with the distribution of the process \( \{\zeta_t^H, 0 \leq t \leq T\} \) with respect to the measure \( P \).

We call the process \( \Lambda^H \) as the likelihood process or the Radon-Nikodym derivative \( \frac{dP^Y}{dP} \) of the measure \( P^Y \) with respect to the measure \( P \).

Tudor [25] derived the following Girsanov type formula.

**Theorem 2.4.** Suppose the assumptions of Theorem 2.2 hold. Define

\[ (2.16) \quad \Lambda_H(T) = \exp\{\int_0^T R_H(t) dM_t^H - \frac{1}{2} \int_0^T R_H^2(t) dw_t^H\}. \]

Suppose that \( E[\Lambda_H(T)] = 1 \). Then the measure \( P^* = \Lambda_H(T) P \) is a probability measure and the probability measure of the process \( Y \) under \( P^* \) is the same as that of the process \( V \) defined by

\[ (2.17) \quad V_t = \int_0^t d\zeta_s^H, 0 \leq t \leq T. \]

3. **Main Results**

Let us consider the stochastic differential equation

\[ (3.1) \quad dX(t) = \theta X(t) dt + d\zeta_s^H, X(0) = 0, t \geq 0 \]

where \( \theta \in \Theta \subset R, \zeta_s^H = \{\zeta_t^H, t \geq 0\} \) is a sub-fractional Brownian motion with known Hurst parameter \( H \). In other words \( X = \{X(t), t \geq 0\} \) is a stochastic process satisfying the stochastic integral equation

\[ (3.2) \quad X(t) = \theta \int_0^t X(s) ds + \int_0^t d\zeta_s^H, t \geq 0. \]

We call such a process as fractional Ornstein-Uhlenbeck type process driven by sub-fractional Brownian motion. Diedhiou et al. [3] and Mendy [13] investigated parametric estimation for such a stochastic differential equation driven by a sub-fBm. We will now obtain a Berry-Esseen type bound for the distribution of the maximum likelihood estimator for the drift parameter for such processes.

Let

\[ (3.3) \quad C(\theta, t) = \theta X(t), t \geq 0 \]

and assume that the sample paths of the process \( \{C(\theta, t), t \geq 0\} \) are smooth enough so that the process

\[ (3.4) \quad R_{H, \theta}(t) = \theta \frac{d}{dw_t^H} \int_0^t k_H(t, s) X(s) ds, t \geq 0 \]

is well-defined where \( w_t^H \) and \( k_H(t, s) \) are as defined in Section 2. Suppose the sample paths of the process \( \{R_{H, \theta}(t), 0 \leq t \leq T\} \) belong almost surely to \( L^2([0, T], dw_t^H) \). Define

\[ (3.5) \quad Z_t = \int_0^t k_H(t, s) dX_s, t \geq 0. \]
Then the process $Z = \{Z_t, t \geq 0\}$ is an $(\mathcal{F}_t)$-semimartingale with the decomposition

$$Z_t = \int_0^t R_{H,\theta}(s)dw^H_s + M^H_t, \quad t \geq 0$$

(3.6)

where $M^H$ is the fundamental martingale defined by the equation (2.8) and the process $X$ admits the representation

$$X_t = \int_0^t K_H(t,s)dZ_s$$

(3.7)

where the function

$$K_H(t,s) = \frac{e^H}{d^H} s^{H/2} n_H(t,s).$$

Let $P^T_\theta$ be the measure induced by the process $\{X_t, 0 \leq t \leq T\}$ when $\theta$ is the true parameter. Following Theorem 2.4, we get that the Radon-Nikodym derivative of $P^T_\theta$ with respect to $P^T_0$ is given by

$$\frac{dP^T_\theta}{dP^T_0} = \exp\left[\int_0^T R_{H,\theta}(s)dZ_s - \frac{1}{2} \int_0^T R_{H,\theta}(s)dM^H_s\right].$$

(3.8)

Maximum likelihood estimation

We now consider the problem of estimation of the parameter $\theta$ based on the observation of the process $X = \{X_t, 0 \leq t \leq T\}$ and study its asymptotic properties as $T \to \infty$.

**Strong consistency:**

Let $L_T(\theta)$ denote the Radon-Nikodym derivative $\frac{dP^T_\theta}{dP^T_0}$. The maximum likelihood estimator (MLE) is defined by the relation

$$L_T(\hat{\theta}_T) = \sup_{\theta \in \Theta} L_T(\theta).$$

(3.9)

We assume that there exists a measurable maximum likelihood estimator. Sufficient conditions can be given for the existence of such an estimator (cf. Lemma 3.1.2, Prakasa Rao [15]). Note that

$$R_{H,\theta}(t) = \theta \frac{d}{dw^H_t} \int_0^t k_H(t,s)X(s)ds = \theta J(t). \text{(say)}$$

(3.10)

Then

$$\log L_T(\theta) = \theta \int_0^T J(t)dZ_t - \frac{1}{2} \theta^2 \int_0^T J^2(t)dw^H_t$$

(3.11)

and the likelihood equation is given by

$$\int_0^T J(t)dZ_t - \theta \int_0^T J^2(t)dw^H_t = 0.$$

(3.12)

Hence the MLE $\hat{\theta}_T$ of $\theta$ is given by

$$\hat{\theta}_T = \frac{\int_0^T J(t)dZ(t)}{\int_0^T J^2(t)dw^H_t}.$$

(3.13)

Let $\theta_0$ be the true parameter. Using the fact that

$$dZ_t = \theta_0 J(t)dw^H_t + dM^H_t,$$

(3.14)
it can be shown that
\[
\frac{dP_T}{dP_{\theta_0}} = \exp[(\theta - \theta_0) \int_0^T J(t) dM_t^H - \frac{1}{2} (\theta - \theta_0)^2 \int_0^T J^2(t) dw_t^H].
\]

Following this representation of the Radon-Nikodym derivative, we obtain that
\[
\hat{\theta}_T - \theta_0 = \frac{\int_0^T J(t) dM_t^H}{\int_0^T J^2(t) dw_t^H}.
\]

We now discuss the problem of estimation of the parameter $\theta$ on the basis of the observation of the process $X$ or equivalently the process $Z$ on the interval $[0,T]$.

**Theorem 3.1.** The maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent, that is,
\[
\hat{\theta}_T \to \theta_0 \quad a.s \ [P_{\theta_0}] \quad \text{as} \quad T \to \infty
\]
provided
\[
\int_0^T J^2(t) dw_t^H \to \infty \quad a.s \ [P_{\theta_0}] \quad \text{as} \quad T \to \infty.
\]

**Proof.** This theorem follows by observing that the process
\[
\gamma_T = \int_0^T J(t) dM_t^H, \ t \geq 0
\]
is a local continuous martingale with the quadratic variation process
\[
<\gamma>_T = \int_0^T J^2(t) dw_t^H
\]
and applying the Strong law of large numbers (cf. Liptser [11]; Liptser and Shiryayev [12]; Prakasa Rao [16], p. 61) under the condition (3.18) stated above. \hfill \Box

**Remark:** For the case of sub-fractional Ornstein-Uhlenbeck process investigated here and in Mendy [13], it can be checked that the condition stated in equation (3.18) holds and hence the maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent as $T \to \infty$.

**Limiting distribution:**
We now discuss the limiting distribution of the MLE $\hat{\theta}_T$ as $T \to \infty$.

**Theorem 3.2.** Suppose there exists a norming function $I_t$, $t \geq 0$ such that
\[
I_T^2 < \gamma_T >= I_T^2 \int_0^T J^2(t) dw_t^H \to \eta^2 \quad \text{in probability} \quad \text{as} \quad T \to \infty
\]
where $I_T \to 0$ as $T \to \infty$ and $\eta$ is a random variable such that $P(\eta > 0) = 1$. Then
\[
(I_T \gamma_T, I_T^2 < \gamma_T >) \to (\eta Z, \eta^2) \quad \text{in law} \quad \text{as} \quad T \to \infty
\]
where the random variable $Z$ has the standard normal distribution and the random variables $Z$ and $\eta$ are independent.

**Proof.** This theorem follows as a consequence of the central limit theorem for martingales (cf. Theorem 1.49 ; Remark 1.47 , Prakasa Rao [16], p. 65). \hfill \Box

Observe that
\[
I_T^{-1}(\hat{\theta}_T - \theta_0) = \frac{I_T \gamma_T}{I_T^2 < \gamma_T >}
\]
Applying the Theorem 3.2, we obtain the following result.
Theorem 3.3. Suppose the conditions stated in the Theorem 3.2 hold. Then

\[ I_T^{-1}(\hat{\theta}_T - \theta_0) \rightarrow Z \] in law as \( t \rightarrow \infty \)

where the random variable \( Z \) has the standard normal distribution and the random variables \( Z \) and \( \eta \) are independent.

Remarks: If the random variable \( \eta \) is a constant with probability one, then the limiting distribution of the maximum likelihood estimator is normal with mean 0 and variance \( \eta^{-2} \). Otherwise it is a mixture of the normal distributions with mean zero and variance \( \eta^{-2} \) with the mixing distribution as that of \( \eta \).

4. Berry-Esseen Type Bound

Let \( \theta_0 \) be the true parameter. In addition to the conditions stated in Section 3, suppose that the random variable \( \eta \) is a positive constant with probability one under \( P_{\theta_0} \)-measure. Theorem 3.3 implies that

\[ I_T^{-1}(\hat{\theta}_T - \theta_0) \rightarrow N(0, \eta^{-2}) \] in law as \( T \rightarrow \infty \) under \( P_{\theta_0} \)-measure where \( N(0, \sigma^2) \) denoted the Gaussian distribution with mean zero and variance \( \sigma^2 \).

We would now like to obtain the rate of convergence in this limit leading to a Berry-Esseen type bound.

Suppose there exists non-random positive functions \( \delta_T \) and \( \epsilon_T \) decreasing to zero as \( T \rightarrow \infty \) such that

\[ \delta_T^{-1} \epsilon_T^2 \rightarrow \infty \] as \( T \rightarrow \infty \)

and

\[ \sup_{\theta \in \Theta} P_{\theta}(|\gamma_T \delta_T^{-1/2} - 1| \geq \epsilon_T) = O(\epsilon_T^{1/2}) \]

where the process \( \{\gamma_T, T \geq 0\} \) is as defined by equation (3.19). Note that the process \( \{\gamma_T, T \geq 0\} \) is a locally square integrable continuous martingale. From the results on the representation of locally square integrable continuous martingales (cf. Ikeda and Watanabe [8], Chapter II, Theorem 7.2), it follows that there exists a standard Wiener process \( \{B(t), t \geq 0\} \) adapted to \( (F_t) \) such that \( \gamma_T = B(\gamma_T) \), \( t \geq 0 \). In particular

\[ \gamma_T \delta_T^{1/2} = B(\gamma_T \delta_T) \] a.s. \( [P_{\theta_0}] \)

for all \( T \geq 0 \).

We use the following lemmas in the sequel.

Lemma 4.1. Let \( (\Omega, \mathcal{F}, P) \) be a probability space and \( f \) and \( g \) be \( \mathcal{F} \)-measurable functions. Then, for any \( \epsilon > 0 \),

\[ \sup_x |P(\omega : f(\omega) \leq x) - \Phi(x)| \leq \epsilon + \sup_y |P(\omega : f(\omega) \leq y) - \Phi(x)| + P(\omega : |g(\omega) - 1| > \epsilon) + \epsilon \]

where \( \Phi(x) \) is the distribution function of the standard Gaussian distribution.

Proof. See Michael and Pfanzagl [14].

Lemma 4.2. Let \( \{B(t), t \geq 0\} \) be a standard Wiener process and \( V \) be a nonnegative random variable. Then, for every \( x \in \mathbb{R} \) and \( \epsilon > 0 \),

\[ |P(B(V) \leq x) - \Phi(x)| \leq (2\epsilon)^{1/2} + P(|V - 1| > \epsilon) \]

Proof. See Hall and Heyde [7], p.85.
Let us fix $\theta \in \Theta$. It is clear from the earlier remarks that
\begin{equation}
\gamma_T = \gamma > T \quad \text{under } P_\theta\text{-measure. Then it follows, from the Lemmas 4.1 and 4.2, that}
\end{equation}
\begin{equation}
P_\theta[\delta_T^{1/2} I_T^{-1}(\hat{\theta}_T - \theta) \leq x] = \Phi(x)
\end{equation}
\begin{equation}
\leq |P_\theta[\gamma_T \leq x] - \Phi(x)|
\end{equation}
\begin{equation}
= |P_\theta[\gamma_T \leq x] - \Phi(x)|
\end{equation}
\begin{equation}
= |P_\theta[\gamma_T / \delta_T^{1/2} \leq x] - \Phi(x)|
\end{equation}
\begin{equation}
\leq \sup_x |P_\theta[\gamma_T / \delta_T^{1/2} \leq x] - \Phi(x)|
\end{equation}
\begin{equation}
+ \sup_y P_\theta[|\delta_T < \gamma > T - 1| \geq \varepsilon_T] + \varepsilon_T
\end{equation}
\begin{equation}
= \sup_y |P(B(<\gamma > T \delta_T) \leq y) - \Phi(y)| + \sup_y P_\theta[|\delta_T < \gamma > T - 1| \geq \varepsilon_T] + \varepsilon_T
\end{equation}
\begin{equation}
\leq (2\varepsilon_T)^{1/2} + 2P_\theta[|\delta_T < \gamma > T - 1| \geq \varepsilon_T] + \varepsilon_T
\end{equation}

It is clear that the bound obtained above is of the order $O(\varepsilon_T^{1/2})$ under the condition (4.3) and it is uniform in $\theta \in \Theta$. Hence we have the following result giving a Berry-Esseen type bound for the distribution of the MLE.

**Theorem 4.3.** Under the conditions (4.2) and (4.3),
\begin{equation}
\sup_{\theta \in \Theta} \sup_{x \in \mathbb{R}} |P_\theta[\delta_T^{1/2} I_T^{-1}(\hat{\theta}_T - \theta) \leq x] - \Phi(x)|
\end{equation}
\begin{equation}
\leq (2\varepsilon_T)^{1/2} + 2P_\theta[|\delta_T < \gamma > T - 1| \geq \varepsilon_T] + \varepsilon_T = O(\varepsilon_T^{1/2}).
\end{equation}

As a consequence of this result, we have the following theorem giving the rate of convergence of the MLE $\hat{\theta}_T$.

**Theorem 4.4.** Suppose the conditions (4.2) and (4.3) hold. Then there exists a constant $c > 0$ such that for every $d > 0$,
\begin{equation}
\sup_{\theta \in \Theta} P_\theta[I_T^{-1}(\hat{\theta}_T - \theta) \geq d] \leq c\varepsilon_T^{1/2} + 2P_\theta[|\delta_T < \gamma > T - 1| \geq \varepsilon_T] = O(\varepsilon_T^{1/2}).
\end{equation}

**Proof.** Observe that
\begin{equation}
\sup_{\theta \in \Theta} P_\theta[I_T^{-1}(\hat{\theta}_T - \theta) \geq d]
\end{equation}
\begin{equation}
\leq \sup_{\theta \in \Theta} |P_\theta[\delta_T^{1/2} I_T^{-1}(\hat{\theta}_T - \theta) \geq d\delta_T^{-1/2}] - 2(1 - \Phi(d\delta_T^{-1/2}))|
\end{equation}
\begin{equation}
+ 2(1 - \Phi(d\delta_T^{-1/2}))
\end{equation}
\begin{equation}
\leq (2\varepsilon_T)^{1/2} + 2\sup_{\theta \in \Theta} P_\theta[|\delta_T < \gamma > T - 1| \geq \varepsilon_T] + \varepsilon_T
\end{equation}
\begin{equation}
+ 2d^{-1}\delta_T^{1/2}(2\pi)^{-1/2}\exp\left[-\frac{1}{2}\delta_T^{-1}d^2\right]
\end{equation}
by Theorem 4.3 and the inequality
\begin{equation}
1 - \Phi(x) < \frac{1}{x\sqrt{2\pi}} \exp\left[-\frac{1}{2}x^2\right]
\end{equation}
for all $x > 0$ (cf. Feller [6], p.175). Since
\begin{equation}
\delta_T^{-1/2} \to \infty \quad \text{as} \quad T \to \infty
\end{equation}
by the condition (4.2), it follows that
\begin{equation}
\sup_{\theta \in \Theta} P_\theta[I_T^{-1}(\hat{\theta}_T - \theta) \geq d] \leq c\varepsilon_T^{1/2} + 2\sup_{\theta \in \Theta} P_\theta[|\delta_T < R > T - 1| \geq \varepsilon_T]
for some constant $c > 0$ and the last term is of the order $O(\varepsilon^{1/2} / T)$ by the condition (4.3). This proves Theorem 4.4.

\begin{flushright}
\square
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