n-APR TILTING AND $\tau$-MUTATIONS

Abstract. APR tilts for path algebra $kQ$ can be realized as the mutation of the quiver $Q$ in $\mathbb{Z}Q$ with respect to the translation. In this paper, we show that we have similar results for the quadratic dual of truncations of $n$-translation algebras, that is, under certain condition, the $n$-APR tilts of such algebras are realized as $\tau$-mutations. For the dual $\tau$-slice algebras with bound quiver $Q^\perp$, we show that their iterated $n$-APR tilts are realized by the iterated $\tau$-mutations in $\mathbb{Z}_{|n-1}Q^\perp$.

1. Introduction

Higher representation theory is developed by Iyama and his coauthors [18, 19, 20, 21, 10], and is widely used in representation theory of algebra and non-commutative geometry. We observed that graded self-injective algebra bear certain feature of higher representation theory [12, 8], and introduce $n$-translation algebra for studying higher representation theory related to self-injective algebras in [9].

$n$-APR tilts are introduced by Iyama and Oppermann in [21] as generalization of Bernstein-Gelfand-Ponomarev reflection functor and the APR tilts, which played very important role in studying representation finite hereditary algebras.

For a quiver $Q$ or the path algebra $kQ$ defined by the quiver, the Bernstein-Gelfand-Ponomarev reflection or APR tilts can be realized as follows. Embedding $Q$ as a slice in the translation quiver $\mathbb{Z}Q$ with $i$ a sink (respectively, a source), and take 'mutation' with respect to the translation $\tau$ by replace $i$ by $\tau i$ (respectively, $\tau^{-1} i$), the quiver obtained is exactly the one obtained by Bernstein-Gelfand-Ponomarev reflection or APR tilts on $Q$ at $i$. For the quadratic dual of algebras which are truncated from an $n$-translation algebra suitably, we show that the same thing holds for $n$-APR tilts and the $\tau$-mutation with respect to the $n$-translation in the $n$-translation quiver.

An $n$-translation quiver is a bound quiver on which an $n$-translation $\tau$ is defined, and an $n$-translation algebra $\Lambda$ is a $(n+1,q)$-Koszul algebra whose bound quiver is an $n$-translation quiver $\overline{Q}$ (see Section 2 for the definitions). An algebra $\Lambda$ whose bound quiver $Q$ is embedded in $\overline{Q}$ as a convex full subquiver is regarded a truncation of $\overline{Q}$. We consider the quadratic dual $\Gamma$ of such truncation $\Lambda$. In this paper, we show that the $n$-APR tilts for such algebra $\Gamma$ is obtained by applying the $\tau$-mutations on its bound quiver, that is, replacing a sink with its image under $\tau^{-1}$, or replacing a source with its image under $\tau$. In the case $Q$ is a complete $\tau$-slice of $\overline{Q}$, $n$-APR tilts of $\Gamma$ are realized by the $\tau$-mutations in $\mathbb{Z}_{|n-1}Q^\perp$, hence we obtain all the iterated $n$-APR tilted algebras of $\Gamma$ using $\tau$-mutations.

The paper is organized as follows. In Section 2 we recall concepts and results needed in this paper. In Section 3 we study the quadratic dual of the truncation of an $n$-translation algebra, and characterize its tilting module and tilted algebra.

1991 Mathematics Subject Classification. Primary 16G20; Secondary 16G70, 16S37.

Key words and phrases. $n$-translation quiver; dual $\tau$-slice algebra; $n$-translation algebra; $n$-APR tilting; $\tau$-mutation.
related to the Koszul complex given by a τ-hammock. In Section 4 we characterize the \(n\)-APR tilting module and \(n\)-APR tilted algebras for the removable sinks and sources, and prove that such \(n\)-APR tilts are realized by \(τ\)-mutations. Our result are applied in Section 5 for dual \(τ\)-slice algebras, we show that for dual \(τ\)-slice algebra with bound quiver \(Q^⊥\), its iterated \(n\)-APR tilts are realized by the iterated \(τ\)-mutations in \(Z_{1,n-1} Q^⊥\). We also present two examples in this section using \(τ\)-mutations. In one example, we recover the list of iterated 2-APR tilts in \([21]\) for 2-representation-finite Auslander algebra of type \(A_3\), and in the other example, we give a list of iterated 2-APR tilts of quasi 1-Fano algebras related to the McKay quiver of type \(D_4\) in \(SL(3)\).

This paper is a extended and generalized version of the results concerning \(n\)-APR tilts in \(τ\)-slice algebras of \(n\)-translation algebras and quasi \(n\)-Fano algebras, arXiv:1707.01393, which is discontinued.

2. Preliminary

Let \(k\) be a field, and let \(Λ = Λ_0 + Λ_1 + \cdots\) be a graded algebra over \(k\) with \(Λ_0\) direct sum of copies of \(k\) such that \(Λ\) is generated by \(Λ_0\) and \(Λ_1\). Such algebra is determined by a bound quiver \(Q = (Q_0, Q_1, ρ)\) \([9]\). A module means a left module in this paper, when not specialized.

Recall that a bound quiver \(Q = (Q_0, Q_1, ρ)\) is a quiver with \(Q_0\) the set of vertices, \(Q_1\) the set of arrows and \(ρ\) a set of relations. In this paper the vertex set \(Q_0\) may be infinite. The arrow set \(Q_1\) is as usual defined with two maps \(s, t\) from \(Q_1\) to \(Q_0\) to assign an arrow \(α\) its starting vertex \(s(α)\) and its ending vertex \(t(α)\). The arrow set \(Q_1\) in this paper is assumed to be locally finite in the sense that for each pair \(i, j \in Q_0\), the arrows \(α\) with \(s(α) = i\) and \(t(α) = j\) are finite. We also write \(s(ρ) = i, t(ρ) = j\) for a path in \(Q\) from \(i\) to \(j\). The relation set \(ρ\) is a set of linear combinations of paths of length \(≥ 2\), since we study graded algebra, it can be normalized such that each element in \(ρ\) is a linear combination of paths of the same length starting at the same vertex and ending at the same vertex.

Let \(Λ_0 = \bigoplus_{i \in Q_0} k_i = kQ_0\), with \(k_i \simeq k\) as algebras, and let \(e_i\) be the image of the identity of \(k\) under the canonical embedding of the \(k_i\) into \(kQ_0\). Then \(\{e_i | i \in Q_0\}\) is a complete set of orthogonal primitive idempotents in \(kQ_0\) and \(kQ_1 = Λ_1 = \bigoplus_{i,j \in Q_0} e_j Λ_1 e_i\) as \(kQ_0\)-bimodules. Fix a basis \(Q^1\) of \(e_j Λ_1 e_i\) for any pair \(i, j \in Q_0\), take the elements of \(Q^1\) as arrows from \(i\) to \(j\), and let \(Q_1 = \bigcup_{(i,j) \in Q_0 \times Q_0} Q^1\). Let \(Q_t\) be the set of the paths of length \(t\) in \(Q\) and let \(kQ_t\) be the \(k\)-space spanned by \(Q_t\).

There is canonical epimorphism from \(kQ\) to \(Λ\) with kernel \(I\) contained in \(kQ_2 + kQ_3 + \cdots\). Choose a generating set \(ρ\) of \(I\), whose elements are linear combinations of paths of length \(≥ 2\). Then we have \(I = (ρ)\) and \(Λ \simeq kQ/(ρ)\), a quotient algebra of the path algebra \(kQ\). \(Λ\) is also called the bound quiver algebra of the bound quiver \(Q = (Q_0, Q_1, ρ)\). A path \(p\) in \(Q\) is called bound path if its image in \(kQ/(ρ)\) is non-zero.

A quiver \(Q\) is called \textit{acyclic} if \(Q\) contains no oriented cycle, the algebra \(Λ\) of the bound quiver \(Q\) is called \textit{acyclic} if its quiver \(Q\) is acyclic.
Let $Q = (Q_0, Q_1, \rho)$ be a bound quiver with quadratics relations. In this case, the quotient algebra $\Lambda = kQ/(\rho)$ is called a quadratic algebra. Set $\delta_{x,y} = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$, by defining $e_i(e_j) = \delta_{i,j}$, and

$$
\gamma(\alpha) = \delta_{\alpha, \gamma} \quad \text{and} \quad \gamma_1 \cdots \gamma_1(\alpha_1 \cdots \alpha_1) = \gamma_1(\alpha_1) \cdots \gamma_1(\alpha_1),
$$

(1)

for arrows $\gamma$, $\alpha$ and paths $\gamma_1 \cdots \gamma_1(\alpha_1 \cdots \alpha_1)$. Then paths in $kQ_i$ are identified with the corresponding elements in dual basis in $DkQ_i = \text{Hom}_k(kQ_i, k)$. That is, $kQ_i$ is identified with $DkQ_i$ and paths form a dual basis of itself. If $\rho \subset kQ_2$ and $\rho$ spans the orthogonal subspace of $\rho$, the algebra $\Lambda = kQ/(\rho)$ is called the quadratic dual of $\Lambda$.

In [3], we also introduce $n$-translation quivers and $n$-translation algebras. Recall that a bound quiver $Q = (Q_0, Q_1, \rho)$, with $\rho$ homogeneous relations, is called an $n$-translation quiver if there is a bijective map $\tau : Q_0 \setminus \mathcal{P} \longrightarrow Q_0 \setminus \mathcal{I}$, called the $n$-translation of $Q$, for two subsets $\mathcal{P}$ and $\mathcal{I}$ of $Q_0$, whose elements are called projective vertices and respectively injective vertices, satisfying the following conditions for the quotient algebra $\Lambda = kQ/(\rho)$:

1. Any maximal bound path is of length $n + 1$ from $\tau i$ in $Q_0 \setminus \mathcal{I}$ to $i$, for some vertex $i$ in $Q_0 \setminus \mathcal{P}$.

2. Two bound paths of length $n + 1$ from $\tau i$ to $i$ are linearly dependent, for any $i \in Q_0 \setminus \mathcal{P}$.

3. For each $i \in Q_0 \setminus \mathcal{P}$ and each $j \in Q_0$ such that there is a bound path from $j$ to $i$ of length $t$ or there is a bound path from $\tau i$ to $j$, then the multiplication of the quotient algebra $\Lambda = kQ/(\rho)$ defines a non-degenerated bilinear from $e_i \Lambda_t e_j \times e_j \Lambda_{n+1-t} e_{\tau i}$ to $e_i \Lambda_{n+1} e_{\tau i}$, here $\Lambda_i$ is the subspaces spanned by the images of the paths of length $t$.

$Q$ is called stable if $\mathcal{I} = \mathcal{P} = \emptyset$, and this is the case if and only if $\Lambda$ is a self-injective algebra.

An algebra $\Lambda$ with an $n$-translation quiver $Q = (Q_0, Q_1, \rho)$ as its bound quiver is called an $n$-translation algebra, if there is a $q \in \mathbb{N} \cup \{\infty\}$ such that $\Lambda$ is $(n + 1, q)$-Koszul [9, 5].

$\tau$-hammocks are introduced in [3] for stable $n$-translation quivers, and are extended to $n$-translation quivers in [6], which are generalization of meshes in a translation quivers. Let $\overline{Q} = (\overline{Q}_0, \overline{Q}_1, \overline{\mathcal{P}})$ be an $n$-translation quiver with $n$-translation $\tau$ and let $\overline{\Lambda} = k\overline{Q}/(\overline{\mathcal{P}})$. For each non-injective vertex $i \in \overline{Q}_0$, $\tau^{-1}i$ is defined in $\overline{Q}$, the $\tau$-hammock $H_i = H_{\overline{Q}}$ ending at $i$ is defined as the quivers with the vertex set $H_{i,0} = \{(j,t)| j \in \overline{Q}_0, \exists p \in \overline{Q}_t, s(p) = j, t(p) = i, 0 \neq p \in \overline{\Lambda}\}$, the arrow set $H_{i,1} = \{(\alpha, t) : (j,t+1) \longrightarrow (j',t)| \alpha : j \longrightarrow j' \in \overline{Q}_1, \exists p \in \overline{Q}_t, s(p) = j', t(p) = i, 0 \neq \rho = \rho \in \overline{\Lambda}_{t+1}\}$, and a hammock function $\mu_i : H_{i,0} \longrightarrow \mathbb{Z}$ which is integral map on the vertices defined by $\mu_i((j,t)) = \dim_k e_j \overline{\Lambda}_{t} e_i$ for $(j,t) \in H_{i,0}$.

Similarly, for non-projective vertex $i$, $\tau i$ is defined in $\overline{Q}$, the $\tau$-hammock $H_i = H_{\overline{Q}}$ starting at $i$ as is the quivers with the vertex set $H_{i,0} = \{(j,t)| j \in \overline{Q}_0, \exists p \in \overline{Q}_t, s(p) \neq i, t(p) = j, 0 \neq p \in \overline{\Lambda}\}$, the arrow set $H_{i,1} = \{(\alpha, t) : (j,t) \longrightarrow (j,t+1)| \alpha : j \longrightarrow j' \in \overline{Q}_1, \exists p \in \overline{Q}_t, s(p) = i, t(p) = j, 0 \neq \rho = \rho \in \overline{\Lambda}_{t+1}\}$, and a hammock functions $\mu_i : H_{i,0} \longrightarrow \mathbb{Z}$ which is the integral maps on the vertices defined by $\mu_i((j,t)) = \dim_k e_j \overline{\Lambda}_{t} e_i$ for $(j,t) \in H_{i,0}$.
We also denote by $H^0$ and $H_{i.0}$ the sets \{ $j \in Q_0$ | $(j,t) \in H^0$ for some $t \in Z$ \} and \{ $j \in Q_0$ | $(j,t) \in H_{i.0}$ for some $t \in Z$ \}, respectively. When $Q$ is acyclic, the hammocks $H^*$ and $H_*$ is regarded as a sub-quiver of $\overline{Q}$ via projection to the first component of the vertices and arrows. In this case, $H_i = H^{*-i}$ when $\tau^{-i}t$ is defined and $H^* = H_{*i}$ when $\tau i$ is defined.

For the quadratic dual of an $n$-translation algebra, $n$-almost split sequences in the category of its finite generated projective modules are related to the Koszul complexes which are described using $\tau$-hammocks [9].

Let $\overline{\Lambda}$ be a $n$-translation algebra with bound quiver $\overline{Q} = (Q_0, Q_1, \overline{p})$, and let $\overline{\Gamma}$ be the quadratic dual of $\overline{\Lambda}$ with bound quiver $\overline{Q}^\perp = (Q_0, Q_1, \overline{p}^\perp)$. Let $\mathcal{G} = \text{add} (\overline{\Gamma})$ be the category of finite generated projective $\overline{\Gamma}$-modules. The Koszul complexes in $\overline{\Gamma}$ are the $n$-almost split sequences in $\mathcal{G}$ under certain conditions [9]. If $i$ is a non-injective vertex in $\overline{Q}_i$, (or $j = \tau^{-i}i$ is a non-projective vertex), by [4], we have a left Koszul complex,

$$
\overline{\Gamma} \otimes \overline{\Lambda}_{n+1} e_{r-i} \longrightarrow \overline{\Gamma} \otimes \overline{\Lambda}_n e_{r-i} \longrightarrow \cdots \longrightarrow \overline{\Gamma} \otimes \overline{\Lambda}_0 e_{r-i} \longrightarrow \overline{\Gamma} \otimes \overline{\Lambda}_n \longrightarrow \overline{\Gamma} \otimes \overline{\Lambda}_{n-1} e_{r-i} \longrightarrow \cdots
$$

By the definition of the hammocks, we get the following characterization of the Koszul complexes in $\overline{\Gamma}$.

**Proposition 2.1.** For each vertex $i \in Q_0$ such that $\tau^{-i}t$ is defined in $\overline{Q}$, we have a Koszul complex

$$
\xi_i : M_{n+1} = \overline{\Gamma} e_{r-i} \longrightarrow \cdots \longrightarrow M_t \longrightarrow \cdots \longrightarrow M_0 = \overline{\Gamma} e_{r-i}
$$

with $M_t = \bigoplus_{(j,n+1-t) \in H_{i.0}} (\overline{\Gamma} e_j)^{\mu(j,n+1-t)}$ for $0 \leq t \leq n$ and for each vertex $i \in Q_0$ such that $\tau i$ is defined in $\overline{Q}$, a Koszul complex

$$
\zeta_i : M_{n+1} = \overline{\Gamma} e_{r-i} \longrightarrow \cdots \longrightarrow M_t \longrightarrow \cdots \longrightarrow M_0 = \overline{\Gamma} e_{r-i}
$$

with $M_t = \bigoplus_{(j,t) \in H_{i.0}} (\overline{\Gamma} e_j)^{\mu(j,t)}$ for $0 \leq t \leq n$.

We have that $\xi_i = \xi_{r-i}$ and $\zeta_i = \zeta_{r-i}$. We also call [2] the Koszul complex related to the $\tau$-hammock $H_i = H^{*-i}$.

### 3. Tilting for the Truncation of $n$-Translation Algebras

We study the tilting for the algebras defined by a truncation of an $n$-translation algebra. We first introduce the truncation of a stable $n$-translation algebra.

Let $Q = (Q_0, Q_1, \overline{p})$ be a bound quiver, and let $\Lambda \simeq k\overline{Q}/(\overline{p})$. Let $Q = (Q_0, Q_1)$ be a finite full sub-quiver of $\overline{Q}$, that is, $Q_0$ and $Q_1$ are subsets of $\overline{Q}_0$ and $\overline{Q}_1$, respectively, such that all the arrows of $\overline{Q}$ from $i$ to $j$ are in $Q_1$ whenever $i$ and $j$ are both in $Q_0$. If $\Lambda$ is quadratic, write $\Lambda = \Lambda^{op}$ for its quadratic dual. For a full subquiver $Q = (Q_0, Q_1)$ of $\overline{Q}$, we have two algebras associate to it, that is, the subalgebra $\Lambda(Q) = e\overline{\Lambda}e$ with $e = \sum e_j$ and the quotient algebra $\Lambda'(Q) = \Lambda/(\{e_j | j \in Q_0 \setminus Q_0\})$. Write $L(Q) = \bigoplus_{j \in Q_0 \setminus Q} \Gamma e_j$ and write $L^{op}(Q) = \bigoplus_{j \in Q_0 \setminus Q} \Gamma e_j$. For an elements $x = \sum_{p \in \cup_{j \in Q_0 \setminus Q} a_p p}$ in $k\overline{Q}$, write $x_Q = \sum_{p \in \cup_{j \in Q_0 \setminus Q} a_p p}$ and let $\rho = \{x_Q | x \in \overline{p}\}$, then we have $\Lambda'(Q) \simeq k\overline{Q}/(\rho)$. If $\overline{Q}$ is quadratic, $\Lambda'(Q)$ is quadratic and write $\Gamma'(Q) = (\Lambda'(Q))^{op}$, the quadratic dual of $\Lambda'(Q)$.
We say that $Q$ is a convex truncation of $\overline{Q}$ if $Q$ is convex in the sense that if a path $p$ from $i$ to $j$ in $Q$, then all the paths from $i$ to $j$ are in $Q$. In this case, $\overline{p}$ can be normalized such that $\rho = \{e_j xe_i | x \in p, i,j \in Q_0\}$ is a subset of $\overline{p}$. We have the following proposition.

**Proposition 3.1.** Assume that $Q = (Q_0, Q_1, \rho)$ is a convex truncation of a bound quiver $\overline{Q} = (\overline{Q}_0, \overline{Q}_1, \overline{\rho})$. Then $\Lambda(Q) \simeq \Lambda'(Q)$ is both a subalgebra and a quotient algebra of $\overline{\Lambda}$.

**Proof.** Clearly, $\Lambda(Q) = e\overline{\Lambda}e \simeq (\text{End}_{\overline{\Lambda}}^\text{op}(e\overline{\Lambda}e))^{\text{op}} = \text{End}^{\text{op}}_{\overline{\Lambda}}(\bigoplus_{j \in Q_0} e\overline{\Lambda}e_j)$ is a subalgebra of $\overline{\Lambda}$ generated by $e_j | j \in Q_0\}$, and $e\overline{e}_j e \not\in \{0\}$, so $\Lambda \simeq kQ/J$ for some ideal $J$ of $kQ$. Clearly, $J \subseteq kQ \cap (\overline{\rho}) = kQ \cap (\rho) = (\rho)$. If $z \in (\rho) = kQ \cap (\overline{\rho})$, then the image of $z$ in $\overline{\Lambda}$ is zero, hence its image in $\Lambda(Q)$ is zero, too. This shows that $J = (\rho)$, and $\Lambda(Q) = e\overline{\Lambda}e \simeq kQ/(\rho) \simeq \Lambda'(Q)$.

We call such algebra $\Lambda = \Lambda(Q)$ a truncation of $\overline{\Lambda}$ if $Q$ is a convex truncation. If $\overline{\Lambda}$ is quadratic, so is $\Lambda$, and we call its quadratic dual $\Gamma' = \Gamma(\Lambda)$, $\Gamma$-modules.

Assume that $\overline{\Lambda}$ is an acyclic $n$-translation algebra with bound quiver $\overline{Q} = (\overline{Q}_0, \overline{Q}_1, \overline{\rho})$. Recall that if $\overline{\Lambda}$ is an $(n+1,q)$-Koszul with finite $q$, then its quadratic dual $\Gamma = (q,n+1)$-Koszul [5] and $\Gamma$ is a $(q-1)$-translation algebra [9], write its $(q-1)$-translation as $\tau_\Gamma$. If $\overline{\Lambda}$ is Koszul, this include the case $q = \infty$, then $\tau_\Gamma$ is not defined, and we conventionally assume that $\tau_{\Gamma}^i i \not\in \overline{Q}_0$ for any $i \in \overline{Q}_0$ and $t \in \mathbb{Z}$.

If $Q$ is finite truncation of $\overline{Q}$, set $L = L_Q = \bigoplus_{j \in Q_0} e\overline{\Gamma}e_j, L^{(i)} = \bigoplus_{j \in Q_0 \setminus \{i\}} e\overline{\Gamma}e_j$. We have

**Lemma 3.2.** If $i$ is a forward movable vertex of $Q$, then

$\text{coKer }\text{Hom}_G(L, \phi) \simeq \text{Hom}_G(L, \overline{\Gamma} \otimes \overline{\Lambda}_0 e_{\tau-1} i) = \text{Hom}_G(L, \overline{\Gamma} e_{\tau-1} i)$.

If $\tau^{\sigma-1} i$ is backward movable vertex of $Q$, then

$\text{coKer }\text{Hom}_G(\psi, L) \simeq \text{Hom}_G(\psi, \overline{\Gamma} \otimes \overline{\Lambda}_0, L) = \text{Hom}_G(\psi, \overline{\Gamma}, L)$.

**Proof.** Note that $\text{Hom}_G(\overline{\Gamma} e_j, \overline{\Gamma} e_i)/J_G(\overline{\Gamma} e_j, \overline{\Gamma} e_i) = 0$ for $i \neq j$ since by our assumption, $\tau_{\overline{\Gamma}}^{-1} i \not\in Q_0$ and the Koszul complex [9] is an $n$-almost split sequence in $G$ [9].

So $\text{Hom}_G(\overline{\Gamma} e_j, \overline{\Gamma} e_i)/J_G(\overline{\Gamma} e_j, \overline{\Gamma} e_i) = 0$, and we have exact sequence

$\text{Hom}_G(L, \overline{\Gamma} \otimes \overline{\Lambda}_0 e_{\tau-1} i) \xrightarrow{\text{Hom}_G(L, \phi)} \text{Hom}_G(L, \overline{\Gamma} \otimes \overline{\Lambda}_0 e_{\tau-1} i) \xrightarrow{\text{Hom}_G(L, \psi)} \text{Hom}_G(L, \overline{\Gamma} \otimes \overline{\Lambda}_0 e_{\tau-1} i) \xrightarrow{0}$.

This proves the first equality.

The second equality is proven dually.
Recall that a module $T$ over an algebra $\Gamma$ is called a tilting module if (1). $T$ has a finite projective resolution $0 \to P_n \to \cdots \to P_0 \to T \to 0$, with each $P_i$ finitely generated projective $\Gamma$-module; (2). $\text{Ext}_1^\Gamma(T,T) = 0$ for all $t > 0$; (3). there is an exact sequence $0 \to \Gamma \to T_0 \to \cdots \to T_m \to 0$ of $\Gamma$-modules with each $T_i$ in add $(T)$. A cotilting module is defined dually.

Now we study the tilts and cotilts or $\Gamma$ related to the Koszul complex \(2\). We first prove the following theorem.

**Theorem 3.3.** Let $\overline{\Gamma}$ be an acyclic $n$-translation algebra with $n$-translation quiver $\overline{Q}$ and $n$-translation relation $\tau$, let $\overline{\Gamma} \simeq k\overline{Q}/(\overline{\tau}^n)$ be its quadratic dual. Let $Q = (Q_0, Q_1, \rho)$ be a convex truncation of $\overline{Q}$, $L = L(Q)$ and $\Gamma = \Gamma(Q)$.

(1). If $i$ is a forward movable vertex in $Q$, set

$$T = \text{Hom}_\Gamma(L, L^{(i)}(Q)) \oplus \text{coKer} \text{Hom}_\Gamma(L, \phi),$$

where $\phi : \overline{\Gamma} \otimes \overline{\Lambda}_e e_{\tau^{-1}i} \to \overline{\Gamma} \otimes \overline{\Lambda}_e e_{\tau^{-1}i}$ is the map in the Koszul complex \(2\). Then $T$ is a tilting $\Gamma$-module of projective dimension at most $n$, and $\text{End}_\Gamma^T T \simeq \text{End}_{\Gamma^{op}}^n L(s_i^1 Q)$.

(2). If $\tau^{-1}i$ is a backward movable vertex in $Q$, set

$$T' = \text{Hom}_\Gamma(L, (\tau^{-1})^{(i)}(Q)), L \oplus \text{coKer} \text{Hom}_\Gamma(\psi, L),$$

where $\psi : \overline{\Gamma} \otimes \overline{\Lambda}_e e_{\tau^{-1}i} \to \overline{\Gamma} \otimes \overline{\Lambda}_e e_{\tau^{-1}i}$ is the map in the Koszul complex \(2\). Then $T'$ is a cotilting $\Gamma$-module of injective dimension at most $n$, and $\text{End}_{\Gamma^{op}}^n L(s_i^1 Q)$.

We call the tilting module $T$ in (1) (respectively, $T'$ in (2)) of Theorem 3.3 the tilting (respectively, cotilting) $\Gamma$-module related to Koszul complex \(2\) for vertex $i$ (respectively, for $\tau^{-1}i$).

**Proof.** We prove the first case, the second case is proven dually.

Assume that $i$ is forward movable in $Q$ and let $H_i = H_{\overline{\tau}_i}$ be the hammock ending at $i$ with hammock function $\mu_i$. \(2\) is a truncation of the projective resolution of the simple $\overline{\Gamma}$-module $\overline{\Gamma}_0 e_{\tau^{-1}i}$.

By Proposition 2.1 \(2\) $\overline{\Gamma} \otimes \overline{\Lambda}_e e_{\tau^{-1}i} \simeq \bigoplus_{(j,n+1-t) \in H_{i,o}} (\overline{\Gamma}_t e_j)^{\mu(j,n+1-t)}$. Let $M = L^{(i)}(Q)$ and let $M_i = \bigoplus_{(j,n+1-t) \in H_{i,o}} (\overline{\Gamma}_t e_j)^{\mu(j,n+1-t)}$. Let $X = \overline{\Gamma}_e$ and let $Y = \overline{\Gamma}_e e_{\tau^{-1}i}$, the Koszul complex \(2\) becomes

$$X \xrightarrow{f} M_1 \to \cdots \to M_n \xrightarrow{g} Y. \tag{3}$$

By Lemma 7.1 of \(2\), $f$ is a left add $(M)$-approximation and $g : M_n \to Y$ is a right add $(M)$-approximation.

Let $V = X \oplus M$ and $W = Y \oplus M$. If $\overline{\Lambda}$ is Koszul, so is $\overline{\Gamma}$ and Ker $f = 0$, otherwise, $\overline{\Gamma}$ is almost Koszul and Ker $f \simeq \overline{\Gamma}_0 e_{\tau^{-1}i}$. Thus $\text{Hom}_\Gamma(V, \text{Ker } f) = 0$ since $i$ is forward movable in $Q$. So we get an exact sequence

$$0 \to \text{Hom}_\Gamma(V, X) \to \text{Hom}_\Gamma(V, M_1) \to \cdots \to \text{Hom}_\Gamma(V, M_n) \to \text{Hom}_\Gamma(V, Y).$$

Note that coKer $g \simeq \overline{\Gamma}_0 e_{\tau^{-1}i}$. Since $Q$ is convex and $\tau^{-1}i \notin Q_0$ and $\tau^{-1}i^- \subseteq Q_0$, thus $\text{Hom}_\Gamma(\text{coKer } g, W) = 0$. So we have an exact sequence

$$0 \to \text{Hom}_\Gamma(Y, W) \to \text{Hom}_\Gamma(M_n, W) \to \cdots \to \text{Hom}_\Gamma(M_1, W) \to \text{Hom}_\Gamma(X, W).$$
Note $T = \text{Hom}_\Gamma(L, L^{(i)}) \oplus \text{coKer} \text{Hom}_\Gamma(L, \phi)$, where $\phi : \Gamma \otimes \overline{X}_2 e_{\tau - 1} \rightarrow \Gamma \otimes \overline{X}_1 e_{\tau - 1}$ is the map in the Koszul complex. So by Proposition 3.4 of [27], $T$ is a tilting $\Gamma$-module of projective dimension at most $n$.

By Lemma 3.3, $\text{coKer} \text{Hom}_\Gamma(L, \phi) \cong \text{Hom}_\Gamma(L, \overline{\Gamma} e_{\tau - 1} i)$, thus

$$\text{End}_{\Gamma}^D T = \text{Hom}_\Gamma(\text{Hom}_\Gamma(L, L^{(i)}(Q)) \oplus \text{coKer} \text{Hom}_\Gamma(L, \phi),$$

$$\text{Hom}_\Gamma(L, L^{(i)}(Q))) \oplus \text{coKer} \text{Hom}_\Gamma(L, \phi)) = \text{Hom}_\Gamma(\text{Hom}_\Gamma(L, L^{(i)}(Q)), \text{Hom}_\Gamma(L, L^{(i)}(Q)))$$

$$\oplus \text{Hom}_\Gamma(\text{coKer} \text{Hom}_\Gamma(L, \phi), \text{Hom}_\Gamma(L, L^{(i)}))$$

$$\oplus \text{Hom}_\Gamma(\text{coKer} \text{Hom}_\Gamma(L, \phi), \text{coKer} \text{Hom}_\Gamma(L, \phi)) = \text{Hom}_\Gamma(L^{(i)}(Q)) \oplus \text{Hom}_\Gamma(L^{(i)}(Q), \overline{\Gamma} e_{\tau - 1} i)$$

$$\oplus \text{Hom}_\Gamma(\overline{\Gamma} e_{\tau - 1}, L^{(i)}(Q)) \oplus \text{Hom}_\Gamma(\overline{\Gamma} e_{\tau - 1}, \overline{\Gamma} e_{\tau - 1} i)$$

$$\cong \text{Hom}_{Q}(\bigoplus_{j \in (s^+ Q)} \overline{\Gamma} e_j, \bigoplus_{j \in (s^- Q)} \overline{\Gamma} e_j) = \text{End}_{\Gamma}^D \bigoplus_{j \in (s^+ Q)} \overline{\Gamma} e_j$$

$$= \text{End}_{\Gamma}^D L(s^+ Q).$$

This proves the Theorem. □

We remark that for a convex $Q$ in $\overline{Q}$, $s^+ Q$ (respectively, $s^- Q$) may not be convex in general, and $s^- Q$ (respectively, $s^+ Q$) is convex if $i$ is a sink (respectively, source).

4. $n$-APR Tilts and $\tau$-Mutations

Let $\Gamma$ be a finitely dimensional algebra, recall that $n$-Auslander-Reiten translations are defined by $\tau_n = DT \Omega^{n-1}$ and $\tau_n^{-1} = TR \Omega^{-(n-1)}$ [20]. For a simple projective $\Gamma$-module $P$ satisfying $\Gamma = P \oplus Q$, $\tau_n^{-1} P \oplus Q$ is called an $n$-APR tilting module associated to $P$, if $\text{id} P = n$ and $\text{Ext}_F^1(D \Gamma, \Gamma e_i) = 0$ for $0 \leq t < n$. An $n$-APR cotilting module is defined dually [21].

Let $\overline{\Gamma}$ be an acyclic $n$-translation algebra with $n$-translation quiver $\overline{Q}$ and $n$-translation $\tau$, let $\Gamma \simeq k\overline{\Gamma}/(\overline{Q}^2)$ be its quadratic dual. Let $Q = (Q_0, Q_1, \rho)$ be a convex truncation of $\overline{Q}$, we have the following Proposition for the forward movable sources and backward movable sinks.

**Proposition 4.1.** If $i$ is a forwards movable sink in $Q_0$, then $\tau_n^{-1} \Gamma e_i \oplus \Gamma (1 - e_i)$ is an $n$-APR tilting module of $\Gamma$.

If $\tau^{-1}i$ is a backwards movable source in $Q_0$, then $\tau_n De_{\tau - 1} \Gamma \oplus D(1 - e_i)\Gamma$ is an $n$-APR cotilting module of $\Gamma$.

**Proof.** We prove the first assertion, the second is proven dually.

Note that $\overline{\Gamma}$ is also right $n$-translation algebra with the $n$-translation $\tau^n = \tau^{-1}$. We have a right Koszul complex

$$e_{\tau - 1} \Gamma \rightarrow \overline{M}_1 \rightarrow \cdots \rightarrow \overline{M}_t \rightarrow \cdots \rightarrow \overline{M}_n \rightarrow e_i \Gamma$$

which is the projective resolution of the right simple $e_i \Gamma_0$, and we have $\overline{M}_t = \bigoplus_{(j,n+1-t) \in H_{i,0}} e_j \Gamma_{n+1-t}$. This induces a complex of right projective $\Gamma$-modules which is a projective resolution of the right simple $\Gamma$ module $S_i = e_i \Gamma_0$.

$$e_{\tau^n - 1} \Gamma e \rightarrow M_1 \rightarrow \cdots \rightarrow M_t \rightarrow \cdots \rightarrow M_n \rightarrow e_i \Gamma,$$
and $M_t = \bigoplus_{(j,n+1-t) \in H_{n,0}} e_j \Gamma^{\mu_j(j,n+1-t)}$. Apply the duality $D = \text{Hom}_k(A, k)$, one get a injective resolution of left simple $\Gamma$-module $S_i = \Gamma_0 e_i$.

$$
D e_i \Gamma \rightarrow DM_n \rightarrow \cdots \rightarrow DM_1 \rightarrow D e_{\tau^{op} \cdot i} \Gamma e. \quad (4)
$$

Apply $\text{Hom}_\Gamma(D \Gamma, -)$, one gets

$$
0 \rightarrow \text{Hom}_\Gamma(D \Gamma, S_i) \rightarrow \text{Hom}_\Gamma(D \Gamma, D e_i \Gamma) \rightarrow \cdots \rightarrow \text{Hom}_\Gamma(D \Gamma, D M_i) \rightarrow \cdots \rightarrow \text{Hom}_\Gamma(D \Gamma, D e_{\tau^{op} \cdot i} \Gamma e)
$$

Write $\varphi(-,-)$ for $\text{Hom}_\Gamma(-, -)$ and $\varphi(-,$) for $\text{Hom}_\Gamma(-, -)$, we have the following commutative diagram with isomorphisms between rows:

$$
\begin{array}{cccccccc}
\varphi(D \Gamma, D e_i \Gamma) & \rightarrow & \cdots & \rightarrow & \varphi(D \Gamma, \bigoplus_j \Gamma^{\mu_j(j,i)}) & \rightarrow & \cdots & \rightarrow & \varphi(D \Gamma, \bigoplus_j \Gamma^{\mu_j(j,i)}) \\
\downarrow \cong & & & & \downarrow \cong & & & & \downarrow \cong \\
\varphi(e \Gamma, \Gamma) & \rightarrow & \cdots & \rightarrow & \bigoplus_j \varphi(e \Gamma^{\mu_j(j,i)}, \Gamma) & \rightarrow & \cdots & \rightarrow & \bigoplus_j \varphi(e \Gamma^{\mu_j(j,i)}, \Gamma) \\
\downarrow \cong & & & & \downarrow \cong & & & & \downarrow \cong \\
\Gamma e_i & \rightarrow & \cdots & \rightarrow & \bigoplus_j \varphi(\Gamma e_j^{\mu_j(j,i)}) & \rightarrow & \cdots & \rightarrow & \bigoplus_j \varphi(\Gamma e_j^{\mu_j(j,i)}) \\
\downarrow \cong & & & & \downarrow \cong & & & & \downarrow \cong \\
\varphi(e \Gamma e_i) & \rightarrow & \cdots & \rightarrow & \bigoplus_j \varphi(e \Gamma e_j^{\mu_j(j,i)}) & \rightarrow & \cdots & \rightarrow & \bigoplus_j \varphi(e \Gamma e_j^{\mu_j(j,i)}) \\
\end{array}
$$

Since $\tau^{op}_{-1}i = \tau i$ is not a vertex in $Q$, none of the vertex prior it is in $Q$ since $\overline{Q}$ is acyclic and $Q$ is convex in $\overline{Q}$. So $\varphi(\Gamma e_i, \Gamma e_{\tau i}) = 0$ and we have an exact sequence

$$
0 \rightarrow \varphi(\Gamma e_i, \Gamma e_i) \rightarrow \cdots \rightarrow \bigoplus_j \varphi(\Gamma e_j^{\mu_j(j,i)}) \\
\rightarrow \bigoplus_j \varphi(\Gamma e_i, \Gamma e_j^{\mu_j(j,i)}) \rightarrow \varphi(\Gamma e_i, \Gamma e_{\tau i}) = 0
$$

for the lowest row. So

$$
0 \rightarrow \varphi(D \Gamma, D e_i \Gamma) \rightarrow \cdots \rightarrow \varphi(D \Gamma, \bigoplus_j \Gamma^{\mu_j(j,i)}) \\
\rightarrow \cdots \rightarrow \varphi(D \Gamma, \bigoplus_j \Gamma^{\mu_j(j,i)}) \rightarrow 0
$$

is exact and thus $\text{Ext}_t^\Gamma(D \Gamma, \Gamma e_i) = 0$ for $0 \leq t < n$. On the other hand, $e_{\tau^{op} \cdot -1} \Gamma e = \varphi(\Gamma e_i, \Gamma e_{\tau i}) = 0$, and we have that $\text{id} S_i = n$ from $\varphi$. This proves that $\tau^{n-1} \Gamma e_i \oplus \Gamma(1 - e_i)$ is an n-APR tilting module of $\Gamma$. □

Let $\overline{Q}$ be an $n$-translation quiver with $n$-translation $\tau$. Let $Q$ be a convex full subquiver in $\overline{Q}$. If $i$ is a forward movable sink of $Q$, we define the $\tau$-mutation $s^{-}_i(Q)$ of $Q$ at $i$ as the full bound sub-quiver of $\overline{Q}$ obtained by replacing the vertex $i$ by its inverse $n$-translation $\tau^{-1}i$. If $i$ is a backward movable source of $Q$, define the $\tau$-mutation $s^{+}_i(Q)$ of $Q$ at $i$ as the full bound sub-quiver in $\overline{Q}$ obtained by replacing the vertex $i$ by its $n$-translation $\tau i$.

**Proposition 4.2.** If $i$ is a forward movable sink of $Q$, then $s^{-}_i Q$ is convex in $\overline{Q}$ and $s^{+}_i s^{-}_i Q = Q$.

If $i$ is a backward movable source of $Q$, then $s^{+}_i Q$ is convex in $\overline{Q}$ and $s^{-}_i s^{+}_i Q = Q$. 
Proof. We prove the first assertion, the second follows dually. We need only to prove that $s_i^- Q$ is convex. Let $p$ be a path in $\overline{Q}$ from $j$ to $j'$ with $j, j'$ in $s_i^- Q$. If $j' \neq \tau^{-1} i$, then $j, j'$ are both in then full subquiver $Q'$ obtained from $Q$ by removing $i$, so $p$ is also in $s_i^- Q$ since $i$ is a source. Hence $p$ is in $s_i^- Q$ since $Q'$ is also a full subquiver of it. If $j' = \tau^{-1} i$, then $p = \alpha q$ for an arrow $\alpha$ in $Q$ from $j''$ to $j'$ and a path $q$ in $\overline{Q}$ from $j$ to $j''$. By definition, $j''$ is in $H^i = H_{\tau^{-1} i}$, hence in $Q$ since $i$ is forward removable. Thus $q$ is in $s_i^- Q$, as is proved above and $p$ is also in $s_i^- Q$. □

If $\Lambda$ is the algebra defined by the convex subquiver $Q$ and $\Gamma$ is the quadratic dual of $\Lambda$. If $i$ is a forward movable sink (respectively, a backward movable source), then $s_i^- Q$ (respectively, $s_i^+ Q$) is convex so we may regard it as a bound quiver with natural relations induced from $\overline{Q}$. The algebra of the convex bound subquiver $s_i^- Q$ (respectively, $s_i^+ Q$) is called the $\tau$-mutation of $\Lambda$ at $i$, and is denoted as $s_i^- \Lambda$ (respectively, $s_i^+ \Lambda$). The quadratic dual of $s_i^+ \Lambda$ is called the $\tau$-mutation of $\Gamma$ at $i$, and is denoted by $s_i^- \Gamma$ (respectively, $s_i^+ \Gamma$), when $\Lambda$ is quadratic.

Now we show that $n$-APR tilts (respectively, cotilts) for a dual truncation of an acyclic $n$-translation algebra are realized by $\tau$-mutation when the vertex is a forward movable sinks (respectively, a backward movable sources).

**Theorem 4.3.** Let $\overline{X}$ be an acyclic $n$-translation algebra with $n$-translation quiver $\overline{Q}$ and $n$-translation $\overline{\tau}$, let $\overline{\Gamma} \simeq k\overline{Q}/(\overline{\tau}^n)$ be its quadratic dual. Assume that $Q = (Q_0, Q_1, \rho)$ is a convex truncation of $\overline{Q}$, then

(1). If $i$ is a forward movable sink of $Q$, let $T$ be the tilting module of $\Gamma$ related to the Koszul complex (6). Then $T$ is the $n$-APR tilting module of $\Gamma$ at $i$ and $\text{End}_{\Gamma}^{op} T \simeq s_i^- \Gamma$.

(2). If $\tau^{-1} i$ is a backward movable source of $Q$, let $T'$ be the cotilting module of $\Gamma$ related to the Koszul complex (6). Then $T'$ is the $n$-APR cotilting module of $\Gamma$ at $\tau^{-1} i$ and $\text{End}_{\Gamma}^{op} T' \simeq s_i^+ \Gamma$.

Proof. We prove the first assertion, the second is proven dually.

By Proposition 3.1, $\Gamma \simeq \text{End}_{\overline{\Gamma}}^{op} L$.

Note that for the $n$-translation algebra $\overline{X}$ with $n$-translation $\overline{\tau}$, $\overline{X}^{op}$ is an $n$-translation algebra with $n$-translation $\overline{\tau}^{-1}$. So the Koszul complex of right $\Gamma$-modules

$$0 \longrightarrow e_i \Lambda_0 \otimes \Gamma \longrightarrow e_i \Lambda_{n-1} \otimes \Gamma \xrightarrow{\xi} \cdots \longrightarrow e_i \Lambda_1 \otimes \Gamma \longrightarrow e_i \Lambda_0 \otimes \Gamma = e_i \Gamma \longrightarrow e_i \Gamma_0 \longrightarrow 0,$$

(5)

is the projective resolution of the simple right $\Gamma$-module $e_i \Gamma_0$. Note that $i$ is a sink of $Q$, thus by Proposition 2.1, $e_i \Lambda_0 \neq 0$. Apply $D$, we get and injective resolution of $\Gamma_0 e_i$:

$$0 \longrightarrow \Gamma_0 e_i = D(e_i \Gamma_0) \rightarrow D(e_i \Lambda_0 \otimes \Gamma) \rightarrow \cdots \rightarrow D(e_i \Lambda_{n-1} \otimes \Gamma) \rightarrow D(e_i \Lambda_n \otimes \Gamma) \rightarrow 0.$$

This is an injective resolution of the simple $\Gamma$-module $S(i) \simeq \Gamma_0 e_i$, and (5) is the projective resolution of $D(S_i) \simeq e_i \Gamma_0$. Applying $\text{Hom}_\Gamma(\cdot, \Gamma)$ to (5), one gets:

$$0 \longrightarrow \text{Hom}_\Gamma(e_i \Gamma_0, \Gamma) \rightarrow \text{Hom}_\Gamma(e_i \Lambda_0 \otimes \Gamma, \Gamma) = \text{Hom}_\Gamma(e_i \Gamma, \Gamma) \rightarrow \text{Hom}_\Gamma(e_i \Lambda_1 \otimes \Gamma, \Gamma) \rightarrow \cdots \rightarrow \text{Hom}_\Gamma(e_i \Lambda_{n-2} \otimes \Gamma, \Gamma) \xrightarrow{\xi^*} \text{Hom}_\Gamma(e_i \Lambda_{n-1} \otimes \Gamma, \Gamma).$$

Thus $\text{coKer} \xi^* \simeq \tau_{-n}^{-1} S_i$.
On the other hand, we have $e_i \Lambda \oplus \Gamma \simeq \bigoplus_{(j,n) \in H^0_T} e_j \Gamma$. So by Lemma 3.2, we have the following commutative diagram with isomorphisms between rows:

\[
\begin{array}{ccc}
\text{Hom}_\Gamma(\bigoplus_{(j,n) \in H^0_T} e_j \Gamma, \Gamma) & \xrightarrow{\xi^*} & \text{Hom}_\Gamma(\bigoplus_{(j,n) \in H^0_T} e_j \Gamma, \Gamma) \\
\downarrow{\cong} & & \downarrow{\cong} \\
\bigoplus_{(j,n) \in H^0_T} \text{re} \Gamma e_j & \xrightarrow{\xi^*} & \bigoplus_{(j,n) \in H^0_T} \text{re} \Gamma e_j \\
\downarrow{\cong} & & \downarrow{\cong} \\
\bigoplus_{(j,n) \in H^0_T} e \text{re} \Gamma e_j & \xrightarrow{\xi^*} & \bigoplus_{(j,n) \in H^0_T} e \text{re} \Gamma e_j \\
\end{array}
\]

Thus $\tau^{-1}_n S_i \simeq \text{Hom}_G(e_{-1,i} \Gamma, e \Gamma) \simeq e \text{re} \Gamma e_{-1,i}$.

So $T \simeq \text{coKer} \xi^* \oplus L(i)Q = L(s_i^- Q)$ is the $n$-APR-tilting module for the simple left $\Gamma$-projective module $S_i$. By Proposition 4.2, $s_i^- Q$ is convex, so we have that

\[
\text{End}_{\Gamma}^{op} T \simeq \text{End}_{\Gamma}^{op} L(s_i^- Q) = \text{End}_{\Gamma}^{op} \left( \bigoplus_{j \in (s_i^- Q)_0} \text{re} \Gamma e_j \right) \simeq k(s_i^- Q)/(s_i^- \rho^+) = s_i^- \Gamma,
\]

by Lemma 5.1. □

This shows that for a dual truncation algebra of an acyclic $n$-translation quiver, the $n$-APR tilts for a forward movable sink (respectively, cotilts for a backward removable source) is realized by the $\tau$-mutation of its bound quiver.

5. Application to dual $\tau$-slice algebras

In [21], $n$-APR tilting complexes of an $n$-representation finite algebra are related to the slices and mutations in $n$-cluster tilting subcategory of the derived category. In [8], we introduce the $\tau$-slice algebras of a given graded self-injective algebra and showed that they are derived equivalent by shown that they have isomorphic trivial extensions. Using our results in the previous section, we also have such equivalences for dual $\tau$-slice algebras, and the $\tau$-mutation is explained as $n$-APR tilts here.

Let $Q = (Q_0, Q_1, \rho)$ be a acyclic stable $n$-translation quiver with $n$-translation $\tau$, and assume that $Q$ has only finite many $\tau$-orbits. Let $Q$ be a full sub-quiver of $Q$. $Q$ is called a complete $\tau$-slice of $Q$ if it is convex (path complete in [8]) and for each vertex $v$ of $Q$, the intersection of the $\tau$-orbit of $v$ and the vertex set of $Q$ is a single-point set. When normalizing the relations such that they are linear combinations of paths with the same starting vertex and the same ending vertex, then $\rho = \{ x = \sum a_p \rho \in \overline{Q} \text{ s}(p), t(p) \in Q_0 \} \subseteq \overline{p}$. So we also regard a complete $\tau$-slice $Q$ as the bound quiver $Q = (Q_0, Q_1, \rho)$.

The algebra $\Lambda$ defined by a complete $\tau$-slice $Q$ in $Q$ is called a $\tau$-slice algebra of the bound quiver $Q$. If $Q$ is the bound quiver of an algebra $X$, we also say that $\Lambda$ is a $\tau$-slice algebra of $X$. If $\overline{X}$ is a $n$-translation algebra, $\Lambda$ is a quadratic algebra, and we call its quadratic dual $\Gamma = \Lambda^{op}$ a dual $\tau$-slice algebra.

Obviously, we have the following consequences.

**Lemma 5.1.** Let $\overline{Q}$ be a stable $n$-translation quiver, then its complete $\tau$-slices are convex truncations.
Lemma 5.2. If $Q$ is a complete $\tau$-slice in a stable $n$-translation quiver $\overline{Q}$, then its sinks are forward movable and its sources are back forward movable.

We have shown in [8] that $\tau$-slices are related by $\tau$-mutations, as in the following lemma.

Proposition 5.3. Let $Q$ be a complete $\tau$-slice of a stable $n$-translation quiver $\overline{Q}$.

If $i$ is a sink of $Q$, then $s_i^- Q$ is a complete $\tau$-slice of $\overline{Q}$.

If $i$ is a source of $Q$, then $s_i^+ Q$ is a complete $\tau$-slice of $\overline{Q}$.

If $Q, Q'$ are two complete $\tau$-slices in $\overline{Q}$, then there is a sequence $s_i^{1+}, \ldots, s_i^{r+}$, where $i_t$ are vertices in $\overline{Q}$ and $s_i \in \{+,-\}$, such that $Q' = s_i^{r+} \cdots s_i^{1+} Q$.

We remark that in the Proposition 5.3, we may take all the vertices $i_1, \ldots, i_r$ to be the sinks in the corresponding quivers, and the mutations as $s_i^{1+}, \ldots, s_i^{r+}$, or all to be the sources and the mutations as $s_i^{-1}, \ldots, s_i^{-r}$.

If $Q$ is a complete $\tau$-slice of $\overline{Q}$ and $i$ a sink (respectively, $\tau^{-1} i$ a source) of $Q$, we know that the algebra $\Lambda(Q) \simeq k\overline{Q}/\rho$ and its $\tau$-mutation $\Lambda(s_i^- Q) \simeq ks_i^- Q/(s_i^- \rho)$ (respectively, $\Lambda(s_i^+ Q) \simeq ks_i^+ Q/(s_i^+ \rho)$) are derived equivalent [8]. Using Lemma 4.1, Lemma 5.2 and Proposition 5.3 we have the following refinement of Theorem 5.3 for a dual $\tau$-slice algebra.

Corollary 5.4. Assume that $\overline{Q}$ is an $n$-translation algebra with bound quiver $\overline{Q}$, and $Q$ is a complete $\tau$-slice. Let $\Lambda = \Lambda(Q), \Gamma = \Gamma(Q)$. Then

1. If $i$ is a sink of $Q$, let $T$ be the tilting module of $\Gamma$ related to the Koszul complex $\overline{Q}$, then $T$ is an $n$-APR tilting module, $\text{End}_T^\tau T$ is a dual $\tau$-slice algebra and

   $$\text{End}_T^\tau T \simeq s_i^+ \Gamma.$$ 

2. If $\tau^{-1} i$ is a source of $Q$, let $T'$ be the cotilting module of $\Gamma$ related to the Koszul complex $\overline{Q}$, then $T'$ is an $n$-APR cotilting module, $\text{End}_{T'}^\tau T$ is a dual $\tau$-slice algebra and

   $$\text{End}_{T'}^\tau T \simeq s_i^- \Gamma.$$ 

So we see for the dual $\tau$-slice algebra of an acyclic stable $n$-translation algebra, $n$-APR tilts and cotilts are realized by $\tau$-mutations, and verse visa.

Let $\Lambda$ be a $\tau$-slice algebra with bound quiver $Q = (Q_0, Q_1, \rho)$ which is a $\tau$-slice of a stable $n$-translation quiver $\overline{Q}$. Now we show that $\overline{Q}$ can be take as the quiver $\mathbb{Z}_{n-1} Q$ defined in [9].

It is easy to see that maximal bound path of $Q$ have the same length $n$ (see also Lemma 6.1 of [10]). Let $M$ be a set of linearly independent maximal bound paths. Such quiver is called $n$-homogeneous there. Define returning arrow quiver $\tilde{Q} = (\tilde{Q}_0, \tilde{Q}_1, \tilde{\rho})$ with $\tilde{Q}_0 = Q_0$, $\tilde{Q}_1 = Q_1 \cup Q_{1,M}$, with $Q_{1,M} = \{ \beta_p : i(p) \to s(p) | p \in M \}$. So $\tilde{Q}$ is obtained by adding a arrow in the reversed direction to each maximal bound path in $Q$.

Denote $\Delta \Lambda = \Lambda \ltimes DA$ the trivial extension of $\Lambda$, the returning arrow quiver of $Q$ is exact the bound quiver of $\Delta \Lambda = \Lambda \ltimes DA$ (Proposition 2.2 of [15], see also Proposition 3.1 and Lemma 3.2 of [10]).

Proposition 5.5. If $Q = (Q_0, Q_1, \rho)$ is the bound quiver of a $\tau$-slice algebra $\Lambda$ such, then there is a relation set $\tilde{\rho}$ such that $\tilde{Q}^\rho = (\tilde{Q}_0, \tilde{Q}_1, \tilde{\rho})$ is the bound quiver of the trivial extension $\Delta \Lambda$ of $\Lambda$.

$\tilde{\rho}$ is quadratic if $\rho$ is so.
For a complete \( \tau \)-slice \( Q \), Recall that we can constructed the \( Z|_{n-1}Q \) for \( Q \) in \([9]\). Take vertex set \((Z|_{n-1}Q)_0 = \{(i,t)|i \in Q_0, t \in \mathbb{Z}\}\), arrow set \((Z|_{n-1}Q)_1 = \mathbb{Z} \times Q_1 \cup \mathbb{Z} \times \mathcal{M}^{op} = \{((\alpha, t) : (i, t) \rightarrow (j, t)) \alpha : i \rightarrow j \in Q, t \in \mathbb{Z}\} \cup \{((\beta_p, t) : (j, t) \rightarrow (i, t+1)|p \in M, s(p) = i, t(p) = j\}\) and relation set \( p_{Z|_{n-1}Q} = \mathbb{Z}p \cup \mathbb{Z}\rho M \cup \mathbb{Z}\rho_0\), where \( Zp = (\sum_s a_s(\alpha, t) \otimes (\alpha', t)) \sum_s a_s \alpha_s \otimes \alpha' \in p, t \in \mathbb{Z}\), \( Z\rho M = \{(\beta_{p', t+1}) \otimes \beta_p \in \rho M, t \in \mathbb{Z}\}\) and \( Z\rho_0 = \{\sum_s a_s(\beta_{p', t+1}) (\alpha', t) + \sum_s b_s(\alpha, t) \otimes (\beta_{p, t}) \sum_s a_s' \beta_{p', t} \otimes \alpha' + \sum_s b_s \alpha_s t \otimes \beta_{p, t} \in p_0, t \in \mathbb{Z}\}\).

Similar to Proposition 5.5 of \([9]\), we have the following realization of \( Z|_{n-1}Q \).

**Proposition 5.6.** Let \( \Lambda \) be an algebra as defined in Proposition 5.3 such that \( \Delta \Lambda \) is quadratic. Then the smash product \( \Delta \Lambda \# k\mathbb{Z}^* \) is a self-injective algebra with bound quiver \( Z|_{n-1}Q \), where \( \Delta \Lambda \) is graded by taking elements in the dual basis of \( M \) in \( D\Lambda_n \) as degree 1 generators.

Since \( Q \) is acyclic, it is a complete \( \tau \)-slice in \( Z|_{n-1}Q \), so \( \Lambda \) is a \( \tau \)-slice algebra of \( \Delta \Lambda \# k\mathbb{Z}^* \).

As a corollary of Corollary 5.4 and Proposition 5.3 we have the following corollary.

**Corollary 5.7.** Let \( \Gamma \) be a finite dimensional connected dual \( \tau \)-slice algebra of an acyclic stable \( n \)-translation algebra, there are only finitely many algebras obtained from \( \Gamma \) using iterated \( n \)-\( \text{APR} \) tilts and cotilts.

**Proof.** Assume that the bound quiver of \( \Gamma \) is \( Q^\perp \), then \( Q \) is a \( \tau \)-slice of \( Z|_{n-1}Q \) and the iterated \( n \)-\( \text{APR} \) tilts and cotilts are dual \( \tau \)-slice algebras of \( Z|_{n-1}Q \), by Corollary 5.4. But \( \tau \)-slices of \( Z|_{n-1}Q \) are connected and convex, so there are only finitely many up to shifted by the \( n \)-translation \( \tau \). This shows that up to isomorphism, there are only finitely many quiver algebras obtained from \( \Gamma \) using iterated \( n \)-\( \text{APR} \) tilts and cotilts.

Now we have the following algorithm to construction \( n \)-\( \text{APR} \) tilts and cotilts for dual \( \tau \)-slice algebras. Let \( \Gamma \) be a dual \( \tau \)-slice algebra with bound quiver \( Q^\perp = (Q_0, Q_1, \rho^\perp) \). Let \( p \) be a basis of the orthogonal subspace of \( p^\perp \) in \( kQ_2 \), then \( Q = (Q_0, Q_1, \rho) \) is the bound quiver of \( \tau \)-slice algebra. Construct \( Z|_{n-1}Q \) as above, \( Z|_{n-1}Q \) is the bound quiver of a stable \( n \)-translation algebra \( \Delta \). \( Q \) is a \( \tau \)-slice in \( Z|_{n-1}Q \), we in fact recovered the stable \( n \)-translation quiver \( Q \). For each sink \( i \) in \( Q \), take the \( \tau \)-mutation \( s_i^\perp Q \) in \( Z|_{n-1}Q \), we obtained the \( n \)-\( \text{APR} \) tilts of \( \Gamma \) with respect to the simple projective \( \Gamma \)-module \( \Gamma_0 e_i \). For each source \( i \) in \( Q \), take the \( \tau \)-mutation \( s_i^\perp Q \) in \( Z|_{n-1}Q \), we obtained the \( n \)-\( \text{APR} \) cotilts of \( \Gamma \) with respect to the simple injective \( \Gamma \)-module \( \Gamma_0 e_i \).

**Example 5.8.** In \([21]\), iterated \( n \)-\( \text{APR} \) tilts of an \( n \)-representation-finite algebra of type \( A \) are characterization using mutations on cuts. Now we show by example how we get the iterated \( 2 \)-\( \text{APR} \) tilts of a \( 2 \)-representation-finite algebra of type \( A \) using \( \tau \)-mutations on \( \tau \)-slices.
The Auslander algebra $\Gamma = \Gamma(2)$ of the path algebra $\Gamma(1)$ of type $A_3$ with linear orientation, is a 2-representation-finite algebra, given by the quiver $Q^\perp(2)$:

```
1 ◦ → 2 ◦ → 3 ◦
  ↓     ↓     ↓
  4 ◦ → 5 ◦ → 6 ◦
```

with the returning arrow quiver $\tilde{Q}^\perp(2)$,

```
 ◦ ➔ ◦ ➔ ◦ ➔
 ◦ ➔ ◦ ➔ ◦ ➔
 ◦ ➔ ◦ ➔ ◦ ➔
```

This is also the quiver of also the (twisted) preprojective algebra $\Gamma(2)$. The quiver $\mathbb{Z}_1 Q^\perp$ is as following

```
- - -
- - -
- - -
```

The hammocks

```
 ◦ ➔ ◦ ➔ ◦ ➔
 ◦ ➔ ◦ ➔ ◦ ➔
 ◦ ➔ ◦ ➔ ◦ ➔
```

The complete $\tau$-slices, or the quivers iterated 2-APR tilts and cotilts of $\Gamma(2)$ obtained by iterated $\tau$-mutations. The $\tau$-mutation $\tau^+_i$ with respect to a source $i$ is obtained by removing the source of hammock $H^i$ and adding the sink $\tau^{-1}i$ with the
arrows to $\tau^{-1}i$, as is shown below.

\[ \Gamma \]
\[ \Gamma_1 = s_1^+ \Gamma \]
\[ \Gamma_2 = s_2^+ \Gamma_1 \]
\[ \Gamma_3 = s_3^+ \Gamma_2 \simeq \Gamma_1 \]
\[ \Gamma_4 = s_4^+ \Gamma_2 \]
\[ \Gamma_5 = s_5^+ \Gamma_4 \simeq \Gamma \]

These are exactly the quivers listed in Table 1 of [21].

**Example 5.9.** By [22], the McKay quiver of a finite subgroup of $\text{SL}(\mathbb{C}^2)$ is a double quiver of extended Dynkin diagram. Fix $G$ with McKay quiver

![McKay quiver](image)

Embedding $G$ in $\text{SL}_3(\mathbb{C})$ in a natural way, the new McKay quiver for $G$ in 3-dimensional space is the returning arrow quiver $\tilde{Q}$ (see [7]).
with relations as described in [14]. By [11], this is a stable $n$-translation quiver with trivial extension, associated to a Koszul $n$-translation algebra $\Lambda(G)$.

We can construct a Koszul $n$-translation algebra $\Lambda(G)$, with acyclic stable $n$-translation quiver $\overline{Q}$:

$$\begin{align*}
\text{The hammocks are of the form}
\end{align*}$$

By [8], we have the following complete $\tau$-slice $Q$ of $\overline{Q}$.

Let $\Lambda$ be the $\tau$-slice algebra and let $\Gamma$ be the dual $\tau$-slice algebra, that is, the quadratic dual of $\Lambda$. Then by [8], $\Gamma$ is a quasi 2-Fano algebra. In $Q$, write $\alpha_{(i,t)}$ for the arrow from $(1,t)$ to $(i,t+1)$, $\beta_{(i,t)}$ for the arrow from $(i,t)$ to $(1,t+1)$ for $2 \leq i \leq 5$, and $\gamma_{(i,t)}$ for the arrow from $(i,t)$ to $(i,t+1)$ for $2 \leq i \leq 5$, $\Gamma$ is defined by the quiver $Q^\perp$ relations

$$\rho^\perp = \{ \sum_{i=1}^{4} \beta_{(i,1)} \alpha_{(i,0)} \} \cup \{ \gamma_{(1,t+1)} \beta_{(1,t)} + \beta_{(i,t)} \gamma_{(i,t)} | 2 \leq i \leq 5, t = 0, 1 \} \cup \{ \alpha_{(i,1)} \beta_{(i,0)} | 2 \leq i \leq 4 \} \cup \{ \gamma_{(i,t+1)} \alpha_{(i,t)} + \alpha_{(i,t)} \gamma_{(1,1)} | 2 \leq i \leq 5, t = 0, 1 \}. $$

Using $\tau$-mutations on $Q$, we get all the non-isomorphic complete $\tau$-slices. The dual $\tau$-slice algebras of these complete $\tau$-slices are all the iterated 2-APR tilting and
cotilting algebras obtained from $\Gamma$.

\[ \begin{array}{c}
\end{array} \]

**References**

[1] Auslander, M., Reiten, I., and Smalø, S.: *Representation theory of artin algebras*, Cambridge Studies in Advanced Math. 36, Cambridge University Press Cambridge (1995)

[2] Auslander, M., Platzeck, M. I., Reiten, I.: Coxeter functors without diagrams. Trans. Amer. Math. Soc. 250, 1-46 (1979)

[3] Bernstein, I. N., Gelfand, I. M., Ponomarev, V. A.: Coxeter functors, andGabriels theorem, Uspehi Mat. Nauk 28, no. 2(170), 19(33) (1973)

[4] Beilinson, A., Ginsberg, V., Soergel, W.: Koszul duality patterns in Representation theory, J. Amer. Math. Soc. 9, 473-527 (1996)

[5] Dlab, V., Ringel, C. M.: Indecomposable representations of graphs and algebras, Mem. Amer. Math. Soc. 6 (1976)

[7] Guo, J. Y.: On McKay quivers and covering spaces (in Chinese). Sci. Sin. Math. 41, 393-402 (2011)

[8] Guo, J. Y.: Coverings and Truncations of Graded Self-injective Algebras, J. Algebra 355, 9-34 (2012)

[9] Guo, J. Y.: On $n$-translation algebras. J. Algebra 433, 400-428 (2016)

[10] Guo, J. Y.: On Trivial extensions and higher preprojective algebras.

[11] Guo, J. Y., Martínez-Villa, R.: Algebra pairs associated to McKay quivers. Comm. Algebra 30, 1017-1032 (2002)

[12] Guo, J. Y., Wu, Q.: Loewy matrix, Koszul cone and applications, Comm. in algebra, 28(2000) 925-941.

[13] Guo, J. Y., Wan, Q.: $n$-translation Algebras, Trivial Extensions and Preprojective Algebras (in Chinese), Sci. Sin. Math. 48, 1681-1698 (2018)

[14] Guo, J. Y., Yin, Y., Zhu, C.: Returning Arrows for Self-injective Algebras and Artin-Schelter Regular Algebras, J. Algebra 397, 365C378 (2014)

[15] Fernández, E. A., Platzeck, M. I.: Presentations of Trivial Extensions of Finite Dimensional Algebras and a Theorem of Sheila Brenner, J. Algebra 249, 326-344 (2002)

[16] Herschend, M., Iyama, O., Oppermann, S.: $n$-Representation infinite algebras. Adv. Math. 252, 292-342 (2014)

[17] Hu, W., Xi, C.: D-split sequences and derived equivalences. Adv. Math. 227, 292-318 (2011)

[18] Iyama, O.: Higher-dimensional Auslander-Reiten theory on maximal orthogonal subcategories. Adv. Math. 210, 22-50 (2007)

[19] Iyama, O.: Auslander correspondence. Adv. Math. 210, 51-82 (2007)
[20] Iyama, O.: Cluster tilting for higher Auslander algebras. Adv. Math. 226, 1-61 (2011)
[21] Iyama, O., Oppermann, S.: n-representation-finite algebras and n-APR tilting, Trans. Amer. Math. Soc. 363, 6575-6614 (2011)
[22] McKay, J.: Graphs, singularities and finite groups. In Proceedings of Symposia in Pure Mathematics 37, 183-186, (1980)
[23] Minamoto, H., Mori, I.: Structures of AS-regular algebras. Adv. Math. 226 4061-4095 (2011)
[24] Ringel C.M. Tame algebra and integral quadratic forms. Lecture Note in Mathematics no.1099 Springer Verlag, Berlin (1984)

JIN YUN GUO, CONG XIAO, LCSM( MINISTRY OF EDUCATION), SCHOOL OF MATHEMATICS AND STATISTICS, HUNAN NORMAL UNIVERSITY, CHANGSHA, HUNAN 410081, P. R. CHINA
E-mail address: gjy@hunnu.edu.cn, 785519703@qq.com

1This work is supported by Natural Science Foundation of China #11271119, #11671126, and the Construct Program of the Key Discipline in Hunan Province