Memory Effects in Turbulent Dynamo: Generation and Propagation of Large Scale Magnetic Field

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Abstract.

We are concerned with large scale magnetic field dynamo generation and propagation of magnetic fronts in turbulent electrically conducting fluids. An effective equation for the large scale magnetic field is developed here that takes into account the finite correlation times of the turbulent flow. This equation involves the memory integrals corresponding to the dynamo source term describing the alpha-effect and turbulent transport of magnetic field. We find that the memory effects can drastically change the dynamo growth rate, in particular, non-local turbulent transport might increase the growth rate several times compared to the conventional gradient transport expression. Moreover, the integral turbulent transport term leads to a large decrease of the speed of magnetic front propagation.
The problem of the generation and propagation of a magnetic field in turbulent electrically conducting fluids is of fundamental importance due to various applications in plasma physics, astrophysics, geophysics, etc. [1]-[5]. It has attracted much attention since the late 60s when it was realized that the conditions for the occurrence of a large scale magnetic field can be found by applying a simple technique involving the mean helicity of the turbulence [3]. This technique is based on the effective macroscopic equation governing the large scale magnetic field \( B \). The standard form of the mean field dynamo equation is

\[
\frac{\partial B}{\partial t} = \text{rot}(\alpha B) + \beta \Delta B + \text{rot}[u \times B] \tag{1}
\]

where \( u \) is the mean velocity field, \( \alpha \) is the helicity and \( \beta \) is the turbulent diffusivity.

This equation is a common starting point for analyzing the generation of the large-scale magnetic field [1]- [8]. It has been also used for the analysis of the propagation of magnetic fronts in spiral galaxies [9]- [11].

The main disadvantage of equation (1) is that it has been derived for turbulent flow involving only two separated length scales for the velocity field - the integral length scale and the small turbulent scale [3]. It is clear that the assumption of two separated scales is rather unrealistic for fully developed turbulent flow, which involves a continuous range of spatial and temporal scales [6]. Thus, the purely local equation (1) is applicable only under the assumption of a clear-cut separation between macroscopic behavior of the averaged magnetic field and the turbulent fluctuations at the ”microscopic level”.

It should be noted that the equation (1) is similar to the convection-diffusion-reaction equations [12, 13]. In fact, it can be reduced to a famous Fisher-Kolmogorov-Petrovskii-Piskunov (FKPP) equation [5, 9], which has become a basic mathematical tool in the theory of propagating fronts travelling into the unstable state of the reaction-diffusion systems. There has been an increased interest in this topic, because of the large number
of physical, chemical and biological problems that can be treated in terms of the FKPP equation (see, for example, \[13\]-\[16\]). Recently, there has been a tremendous activity to extend this analysis by introducing more realistic description of the transport processes. The main motivation for this is that the diffusion approximation for transport admits an infinite speed of propagation. Due to this nonphysical property of the diffusion solution, the FKPP equation yields an overestimation of the propagation speed of travelling fronts \[14\]-\[16\]. We expect that a similar situation might take place in magnetohydrodynamics. The mean-field dynamo equation (1) also admits an infinite speed of a magnetic field propagation. It is clearly a nonphysical property, because the speed of magnetic field propagation can not exceed the velocity of the largest eddy of turbulent flow. The origin of this contradiction lies in the $\delta$–correlated-in-time approximations for the turbulent velocity field (see below). In reality these correlations have finite times of relaxation, and what is more these might be of the same order as the characteristic times for the growth rate of the large scale magnetic field since the physical origin of these correlations and the magnetic field generation are the same, namely, the turbulent fluctuations.

It is the aim of this Letter to extend mean field dynamo equation (1) to the case when long range in time correlations of turbulent flow are taken into account and find out how the non-local in time effect might influence the critical conditions for the generation of magnetic field and its spatial propagation.

The most general phenomenological formulation of the dynamo problem that is considered here is represented by the equations for the average magnetic field $B$ \[3\]:

$$\frac{\partial B}{\partial t} = \text{rot} \mathcal{E} + \text{rot} [\mathbf{u} \times B], \quad \mathcal{E} = \langle \mathbf{u}' \times B' \rangle, \quad (2)$$

where primes denote the turbulent fluctuations and the angular brackets denote an ensemble averaging over the turbulent pulsations. The main closure problem here is to express the turbulent electromotive force $\mathcal{E}$ in terms of the average field $B$. The classical expression leading to (1) is $\mathcal{E} = \alpha B - \beta \text{rot} B$, where $\alpha$ and $\beta$ are the statistical characteristics.
of the turbulent velocity \( u' \). However, under the assumptions of infinite conductivity and weak turbulence the electromotive force \( \mathcal{E} \) can be written as [3]:

\[
\mathcal{E}(x, t) = -\frac{1}{3} \int_{-\infty}^{t} \langle u'(x, t) \cdot \text{rot} u'(x, s) \rangle B(x, s) \, ds - \frac{1}{3} \int_{-\infty}^{t} \langle u'(x, t) \cdot u'(x, s) \rangle \text{rot} B(x, s) \, ds .
\]

The local mean field equation (1) can be derived from (3) under the assumptions that the correlations appearing in (3) are approximated by the delta-functions in time:

\[
\langle u'(x, t) \cdot \text{rot} u'(x, s) \rangle = -3\alpha \delta(t - s), \quad \langle u'(x, t) \cdot u'(x, s) \rangle = 3\beta \delta(t - s).
\]

However, as was mentioned in [4] (p. 136) "... the assumption of instantaneous correlations seems to be a serious restriction to the theory...", since in real turbulence the characteristic times of these correlations are finite. It follows from (3) that in the case of finite correlation times the electromotive force should contain the integrals over the history of \( B \) and \( \text{rot} B \).

In this Letter we suggest the following general form for the electromotive force \( \mathcal{E} \) in the limit of infinite conductivity

\[
\mathcal{E} = \alpha(x) \int_{-\infty}^{t} G_{\alpha} \left( \frac{t - s}{\tau_{\alpha}} \right) B(x, s) \, ds - \beta(x) \int_{-\infty}^{t} G_{\beta} \left( \frac{t - s}{\tau_{\beta}} \right) \text{rot} B(x, s) \, ds ,
\]

where \( G_{\alpha}(y) \) and \( G_{\beta}(y) \) are positive, decreasing functions that tend to zero as \( y \to \infty \). The parameters \( \tau_{\alpha} \) and \( \tau_{\beta} \) control the time correlations in the random velocity field at a fixed space position. By inserting (4) into (3) one can get the non-local in time mean field dynamo equation:

\[
\frac{\partial B}{\partial t} = \int_{-\infty}^{t} G_{\alpha} \left( \frac{t - s}{\tau_{\alpha}} \right) \text{rot} [\alpha B(s)] \, ds - \int_{-\infty}^{t} G_{\beta} \left( \frac{t - s}{\tau_{\beta}} \right) \text{rot} [\beta \text{rot} B(s)] \, ds + \text{rot} [\mathbf{u} \times \mathbf{B}] .
\]
It should be noted that this equation is valid only in the limit of infinitely large conductivity. This case is of specific interest for magnetohydrodynamics of plasma and many problems of astrophysics and geophysics [3]. Many different constitutive models might arise from different choices of $G_\alpha(y)$ and $G_\beta(y)$. The equation (1) can be considered as a limiting case of (5) when $G_{\alpha,\beta}(t-s) = \delta(t-s) \left( \tau_{\alpha,\beta} \to 0 \right)$. The transport memory kernel $G_\beta$ ensures the finite velocity of propagation of a magnetic field, which is determined by the rate of turbulent pulsations [3]. The relaxation time $\tau_\beta$ can be determined from the root mean square velocity $u_{rms} = \sqrt{\beta/\tau_\beta}$ of turbulent fluctuations. The integral kernel $G_\alpha$ is used to express the fact that the dynamo growth of a magnetic field at a time $t$ in the local vicinity of the spatial point $x$ is determined by the past values of $\text{rot} B$.

Although there has been a large number of studies of the generation of macroscopic magnetic fields, the implications of finite correlations for the generation and propagation of the large scale magnetic field have not been studied. The implications of our results for dynamo theory are twofold. First, the non-locality in time greatly influences the dynamo growth rate of a magnetic field, however, the generation conditions are not sensitive to the addition of the memory effect due to non-zero time correlations: they are robust to the addition of non-local terms for the mean field dynamo equation (5). Secondly, the memory effects introduce drastic changes to the front dynamics of a magnetic field.

In what follows, we study the influence of the memory effects on the dynamo generation and propagation by using an important example of a thin turbulent slab of thickness $2h$ and radius $R \ (R \gg h)$, which rotates with the angular velocity $\omega(r)$. This is a standard model for disc-like galaxies. We neglect the effects of compressibility, diamagnetism and deviations from the axial symmetry. We use the standard approximation of constant turbulent diffusion coefficient $\beta$ [4, 3, 9, 10, 11]. We restrict our analysis to the kinematical aspects of the problem, neglecting the influence of the magnetic field on the turbulent flow, i.e. the dependencies of $\alpha, \beta, \tau_\alpha$ and $\tau_\beta$ on the magnetic field $B$. With these simplifications, the equations for the mean axisymmetric magnetic field in polar cylindrical...
coordinates \((r, \varphi, z)\) with z-axis coincident with the rotation axis follow from the basic equation (5):

\[
\frac{\partial B_r}{\partial t} = - \int_{-\infty}^{t} G_\alpha \left( \frac{t-s}{\tau_\alpha} \right) \frac{\partial}{\partial z} (\alpha B_\varphi) \, ds + \\
+ \beta \int_{-\infty}^{t} G_\beta \left( \frac{t-s}{\tau_\beta} \right) \left\{ \frac{\partial^2 B_r}{\partial z^2} + \frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial}{\partial r} (r B_r) \right] \right\} \, ds ,
\]

\[
\frac{\partial B_\varphi}{\partial t} = g \, B_r + \int_{-\infty}^{t} G_\alpha \left( \frac{t-s}{\tau_\alpha} \right) \frac{\partial}{\partial z} (\alpha B_r) \, ds + \\
+ \beta \int_{-\infty}^{t} G_\beta \left( \frac{t-s}{\tau_\beta} \right) \left\{ \frac{\partial^2 B_\varphi}{\partial z^2} + \frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial}{\partial r} (r B_\varphi) \right] \right\} \, ds .
\]

Here \( g = rd\omega/dr \) is the measure of differential rotation, and we are interested only in \( B_r \) and \( B_\varphi \) components of a magnetic field, because \( B_z/B_{r,\varphi} = O(h/R) \)[4, 5]. These components obey the vacuum boundary conditions on the thin disc surfaces [4, 5]:

\( B_{r,\varphi}(z = \pm h) = 0 \). The main goal of our analysis is to find the local growth rate of the magnetic field and its spatial propagation, taking into account the memory effects. Thus, considering the field generation in the usual way [4, 5, 9, 10], we neglect first the radial derivatives in equations (6), (7). We represent the components of the magnetic field as follows \( B_{r,\varphi}(z, t) = b_{r,\varphi}(z) \exp(\gamma t) \), where \( b_{r,\varphi}(z) \) have to be found from the eigenvalue problem

\[(\tilde{\gamma} + \frac{\partial^2}{\partial z^2}) b_r = -\tilde{R}_\alpha \frac{\partial (\alpha b_\varphi)}{\partial z} , \quad (\tilde{\gamma} + \frac{\partial^2}{\partial z^2}) b_\varphi = \tilde{R}_w b_r + \tilde{R}_\alpha \frac{\partial (\alpha b_r)}{\partial z} , \quad b_{r,\varphi}(z = \pm 1) = 0 ,\]

\[
(8)
\]
where $\tilde{\gamma} = \gamma / f_\beta(\gamma T_\beta)$, $\tilde{R}_w = R_w / f_\beta(\gamma T_\beta)$, $\tilde{R}_\alpha = R_\alpha f_\alpha(\gamma T_\alpha / R_\alpha) / f_\beta(\gamma T_\beta)$, $f_{\alpha,\beta}(\gamma \theta_{\alpha,\beta}) = \int_0^\infty G_{\alpha,\beta}(\xi / \theta_{\alpha,\beta}) \exp(-\gamma \xi) \, d\xi$, $\theta_\alpha = T_\alpha / R_\alpha$, $\theta_\beta = T_\beta$. Here we use the dimensionless variables $z \to z / h$, $t \to \beta t / h^2$, $\alpha \to \alpha h(z)$, and dimensional parameters $T_\alpha = \alpha_0 \tau_\alpha / h$, $T_\beta = \beta \tau_\beta / h^2$, $R_\alpha = \alpha_0 h / \beta$, $R_w = gh^2 / \beta$. The parameter $\gamma$ describes the growth rate of the magnetic field. The eigenvalue problem (8) reduces to the well-known form \cite{4, 5, 9, 10} in the case when $T_{\alpha,\beta} = 0$. Otherwise, the eigenvalue $\gamma$ becomes a function of these correlation times, and this function is determined from the problem (8) using the renormalized parameters $\tilde{\gamma}$, $\tilde{R}_\alpha$, $\tilde{R}_w$.

The problem (8) coincides with the generation equations in the local mean-field dynamo theory \cite{4, 5, 9, 10}. The only difference is that the renormalized parameters $\tilde{\gamma}$, $\tilde{R}_\alpha$ and $\tilde{R}_w$ are dependent on the increment $\gamma$ of the time growth of the magnetic field. This means that the generation equations (8) are universal and do not depend on the specific choice of the memory kernels $G_{\alpha,\beta}$. The main physical conclusion from this universality is that since physically meaningful memory kernels must satisfy the normalization condition $f_{\alpha,\beta}(\gamma = 0) = 1$, then the threshold combinations of dynamo parameters $R_\alpha$ and $R_\beta$ at a given $\alpha(z)$, providing the critical point of instability ($Re \gamma = 0$), are the same for all memory kernels. These critical parameters can be determined from the local mean-field dynamo theory. To determine the growth rate $\gamma$ in the generation region, the eigenvalue problem (8) should be solved as $\tilde{\gamma} = \tilde{\gamma}(\tilde{R}_\alpha, \tilde{R}_w)$. This relation with the definitions of the renormalized parameters $\tilde{\gamma}$, $\tilde{R}_\alpha$, $\tilde{R}_w$ leads to a transcendental equation for $\gamma$. The specific dependence of $\gamma$ on the relaxation times $T_{\alpha,\beta}$ and dynamo parameters $R_{\alpha,\beta}$ has to be determined by the specific forms of the memory kernels $G_{\alpha,\beta}$.

Let us illustrate the general results by using the important example of $\alpha \omega$ dynamo ($R_\alpha \ll |R_w|$), which is of specific interest for astrophysical problems \cite{3}-\cite{5}, \cite{9}-\cite{11}. We use the exponential relaxation form

$$G_{\alpha,\beta}(y) = \exp(-y / \tau_{\alpha,\beta}) / \tau_{\alpha,\beta}. \tag{9}$$
The solution of the eigenvalue problem (8) depends on the form of the function $\alpha(z)$. The $\alpha\omega$ approximation requires the anti-symmetric character $\alpha(z) = -\alpha(-z)$. Following the asymptotic method of solution of the eigenvalue problem (8) [4, 5, 10], for the model case $\alpha(z) = z$, the increment of maximal growth satisfies the algebraic equation:

$$\gamma(1 + \gamma T_\beta) = -\frac{\pi^2}{4} + \sqrt{D} \frac{1 + \gamma T_\beta}{\sqrt{1 + \gamma T_\alpha/R_\alpha}}, \quad D = -R_\alpha R_w. \tag{10}$$

The growth rate $\gamma$ is shown in Fig. 1 as a function of dimensionless correlation times $T_\alpha$ and $T_\beta$. Clearly, the non-local terms corresponding to the turbulent helicity and the turbulent transport influence the growth rate $\gamma$ in opposing way. The turbulent $\alpha$-source non-locality leads to a slower field growth (curve 1) due to the physical nature of the process: the integral representation of the $\alpha$-term in basic equations (4), (5) introduces the time memory of excitation. The non-locality of turbulent $\beta$-transport reduces the energy loss out of the disc and, hence, results in an increase of the field growth rate (curve 2). Moreover, for dynamo numbers $D$, close to the critical value $D_{cr}$, the influence of the transport non-locality becomes very significant, the growth rate $\gamma$ increases drastically with $T_\beta$. It is worth noting that the critical value $D_{cr} \simeq \pi^4/16$ of the dynamo number $D$, at which the generation commences ($\gamma = 0$), does not depend on the memory effects; this result is in agreement with the general conclusion made above.

Now we are in a position to discuss the problem of magnetic front propagation. If the dynamo excitation takes place within a certain radius (say, $r < r_0$) then the magnetic field can propagate in the form of travelling fronts [11]. According to the local dynamo theory the speed $v$ of a propagating magnetic front is proportional to $\sqrt{\beta_\gamma}$ due to the FKPP type of the mean field equation (1). The problem is that in the local mean-field description of magnetic fields, turbulent diffusion induces an infinitely long Gaussian tail ahead of magnetic front which leads to the overestimation of the propagation rate. A
qualitatively different situation appears when the memory effects are taken into account. This means that the speed $v$ can not exceed the maximum possible velocity $\sqrt{\beta/\tau_\beta}$ however large the growth rate $\gamma$ is.

Now let us turn to the problem of front propagation in the case when the dynamo equation with memory (5) is considered in the thin disk approximation. Assume that the initial distribution of magnetic field satisfies: $B = B_0 = \text{const}$ if $r < r_0$ and $B = 0$ otherwise. The main quantity of interest is the speed $v$ at which magnetic field propagates in the form of a self-similar wave for the large values of $r$ and $t$. In the large-distance limit ($r \to \infty$) the radial Laplace operator $\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} r \right)$ in equations (6) and (7) can be approximated by the second derivative $\frac{\partial^2}{\partial r^2}$ and we may find the solution of these equations in the form of Fourier modes $B_{r,\varphi}(r, z, t) = b_{r,\varphi}(z) \exp(\gamma t) \exp(ikr)$. Substitution of this expression into equations (6) and (7) leads to the problem for eigenfunctions $b_{r,\varphi}$ from which one can find the equation for the exponent $\gamma(k)$ as a function of the wave number $k$. The general theory of front propagation in non-local reaction-diffusion media \[15, 16\], based on the saddle-point method of calculation of the inverse Fourier integrals, leads to the following equation for the propagating velocity $v = \gamma(\lambda)/\lambda$, where $\lambda = ik$ satisfies the equation $\gamma(\lambda)/\lambda = d\gamma(\lambda)/d\lambda$. We found that the front velocity $v$ is a monotonically decreasing function of both correlation times $\tau_{\alpha,\beta}$. In particular, it depends weakly on $\tau_{\alpha}$ (see Fig. 2), but decreases significantly (up to 2-3 times) with growing $\tau_{\beta}$. Therefore, the use of FKPP-like estimation for the travelling wave velocity $v \sim \sqrt{\beta \gamma}$ for systems with memory leads to significant overestimations. This general conclusion is of great importance not only for magnetic dynamo front propagation but for a wide class of excitable media.

Basically, we have extended the classical mean field dynamo theory to the case when the memory effects are taken into account. We have suggested the effective equation for the large scale magnetic field that takes into account the finite correlation times of the turbulent flow. This equation involves memory integrals corresponding to the dynamo
source term describing the alpha-effect and turbulent transport of magnetic field. We have found that memory effects can drastically change the dynamo growth rate, in particular, the finite turbulent transport involving memory might increase the growth rate several times. We have also found that memory effects lead to the essential decrease of the speed of magnetic front propagation.

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Figure 1: Dependencies of the dimensionless growth rate $\gamma$ on the turbulent $\alpha$-source relaxation time $T_\alpha$ at $T_\beta = 0$ (curve 1) as well as on the transport time $T_\beta$ when $T_\alpha = 0$ (curve 2). The memory kernels $G_{\alpha,\beta}$ are chosen to have form (9); the dimensionless dynamo parameters of the system are: $R_\alpha = 1, R_w = 6.1$.

Figure 2: Dependence of the dimensionless front velocity $\bar{v} = v(T_\alpha, T_\beta)/v(0, 0)$ on the relaxation time $T_\beta$ at $T_\alpha = 0$ (curve 1); 0.3 (curve 2) and 0.9 (curve 3). The dynamo parameters are $R_\alpha = 2$ and $R_w = -40$