K-THEORY FOR CUNTZ-KRIEGER ALGEBRAS ARISING FROM REAL QUADRATIC MAPS

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Abstract. We compute the K-groups for the Cuntz-Krieger algebras $\mathcal{O}_{A_K(f_\mu)}$, where $A_K(f_\mu)$ is the Markov transition matrix arising from the kneading sequence $K(f_\mu)$ of the one-parameter family of real quadratic maps $f_\mu$.

Consider the one-parameter family of real quadratic maps $f_\mu : [0, 1] \to [0, 1]$ defined by $f_\mu(x) = \mu x (1-x)$, with $\mu \in [0, 4]$. Using Milnor-Thurston’s kneading theory [14], J. Guckenheimer [5] has classified up to topological conjugacy, a certain class of maps which includes the quadratic family. The idea of kneading theory is to encode information about the orbits of a map in terms of infinite sequences of symbols and to exploit the natural order of the interval to establish topological properties of the map. In the following, $I$ will denote the unit interval $[0,1]$ and $c$ the unique turning point of $f_\mu$.

For $x \in I$, let

$$
\varepsilon_n(x) = \begin{cases} 
-1 & \text{if } f_\mu^n(x) > c, \\
0 & \text{if } f_\mu^n(x) = c, \\
+1 & \text{if } f_\mu^n(x) < c.
\end{cases}
$$

The sequence $\varepsilon(x) = (\varepsilon_n(x))_{n=0}^\infty$ is called the itinerary of $x$. The itinerary of $f_\mu(c)$ is called the kneading sequence of $f_\mu$ and will be denoted by $K(f_\mu)$. Observe that $\varepsilon_n(f_\mu(x)) = \varepsilon_{n+1}(x)$, i.e. $\varepsilon(f_\mu(x)) = \sigma \varepsilon(x)$ where $\sigma$ is the shift map. Let $\Sigma = \{-1,0,+1\}$ be the alphabet set. The sequences on $\Sigma^\mathbb{N}$ are ordered lexicographically. However, this ordering is not reflected by the mapping $x \to \varepsilon(x)$, because the map $f_\mu$ reverses orientation on $[c,1]$. To take this into account, for a sequence $\varepsilon = (\varepsilon_n)_{n=0}^\infty$ of the symbols $-1,0,+1$, another sequence $\theta = (\theta_n)_{n=0}^\infty$ is defined by $\theta_n = \prod_{i=0}^{n} \varepsilon_i$. If $\varepsilon = \varepsilon(x)$ is the itinerary of a point $x \in I$ then $\theta = \theta(x)$ is called the invariant coordinate of

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The fundamental observation of J. Milnor and W. Thurston [14] is the monotonicity of the invariant coordinates:

\[ x < y \Rightarrow \theta(x) \leq \theta(y). \]

Let us now consider only those kneading sequences that are periodic, i.e.

\[ K(f_\mu) = \varepsilon_0(f_\mu(c)) \ldots \varepsilon_{n-1}(f_\mu(c)) \varepsilon_0(f_\mu(c)) \ldots \varepsilon_{n-1}(f_\mu(c)) \ldots = (\varepsilon_0(f_\mu(c)) \ldots \varepsilon_{n-1}(f_\mu(c)))^\infty \equiv (\varepsilon_1(c) \ldots \varepsilon_n(c))^\infty \]

for some \( n \in \mathbb{N} \). The sequences \( \sigma^i(K(f_\mu)) = \varepsilon_{i+1}(c)\varepsilon_{i+2}(c), \ldots \) \( i = 0, 1, 2, \ldots \), will then determine a Markov partition of \( BF \) produced the group readily verified by the matrices \( A \) of topological Markov subshifts determined by \( A \) we have

Theorem 1.

\[ A_K(f_\mu) := (a_{ij}) \text{ with } a_{ij} = \begin{cases} 1 & \text{if } f_\mu(\text{int } I_i) \supseteq \text{int } I_j \\ 0 & \text{otherwise.} \end{cases} \]

It is easy to see that this matrix \( A_K(f_\mu) \) is not a permutation matrix and no row or column of \( A_K(f_\mu) \) is zero. Thus, for each one of these matrices and following the work of J. Cuntz and W. Krieger [2], one can construct the Cuntz-Krieger algebra \( O_{A_K(f_\mu)} \). In [3], J. Cuntz proved that

\[ K_0(O_A) \cong \mathbb{Z}^r / (1 - A^T)\mathbb{Z}^r \text{ and } K_1(O_A) \cong \ker(I - A^t : \mathbb{Z}^r \to \mathbb{Z}^r). \]

for a \( r \times r \) matrix \( A \) that satisfies a certain condition (I) (see [2]), which is readily verified by the matrices \( A_K(f_\mu) \). In [1] R. Bowen and J. Franks introduced the group \( BF(A) := \mathbb{Z}^r / (1 - A)\mathbb{Z}^r \) as an invariant for flow equivalence of topological Markov subshifts determined by \( A \).

We can now state and prove the following.

**Theorem 1.** Let \( K(f_\mu) = (\varepsilon_1(c)\varepsilon_2(c)\ldots\varepsilon_n(c))^\infty \), for some \( n \in \mathbb{N} \setminus \{1\} \). Thus, we have

\[ K_0(O_{A_{K(f_\mu)}}) \cong \mathbb{Z}_a \text{ with } a = \left| 1 + \sum_{i=1}^{n-1} \prod_{l=1}^{l} \varepsilon_i(c) \right| \]

and

\[ K_1(O_{A_{K(f_\mu)}}) \cong \begin{cases} \{0\} & \text{if } a \neq 0 \\ \mathbb{Z} & \text{if } a = 0. \end{cases} \]

**Proof.** Set \( z_i = \varepsilon_i(c)\varepsilon_{i+1}(c) \ldots \) for \( i = 1, 2, \ldots \). Let \( z'_i = f_\mu^i(c) \) be the point on the unit interval \([0, 1]\) represented by the sequence \( z_i \) for \( i = 1, 2, \ldots \). We have \( \sigma(z_i) = z_{i+1} \) for \( i = 1, \ldots, n - 1 \) and \( \sigma(z_n) = z_1 \). Denote by \( \omega \) the \( n \times n \)
matrix representing the shift map $\sigma$. Let $C_0$ be the vector space spanned by the formal basis $\{z'_1, \ldots, z'_n\}$. Now, let $\rho$ be the permutation of the set $\{1, \ldots, n\}$, which allows us to order the points $z'_1, \ldots, z'_n$ on the unit interval $[0, 1]$, i.e.

$$0 < z'_{\rho(1)} < z'_{\rho(2)} < \cdots < z'_{\rho(n)} < 1.$$ 

Set $x_i := z'_{\rho(i)}$ with $i = 1, \ldots, n$ and let $\pi$ denote the permutation matrix which takes the formal basis $\{z'_1, \ldots, z'_n\}$ to the formal basis $\{x_1, \ldots, x_n\}$. We will denote by $C_1$ the $n-1$ dimensional vector space spanned by the formal basis $\{x_{i+1} - x_i : i = 1, \ldots, n-1\}$. Set

$$I_i := [x_i, x_{i+1}], \text{ for } i = 1, \ldots, n-1.$$ 

Thus, we can define the Markov transition matrix $A_{K(f_\mu)}$ as above. Let $\varphi$ denote the incidence matrix that takes the formal basis $\{x_1, \ldots, x_n\}$ of $C_0$ to the formal basis $\{x_2 - x_1, \ldots, x_n - x_{n-1}\}$ of $C_1$. Put $\eta := \varphi \pi$. As in [7] and [8], we obtain an endomorphism $\alpha$ of $C_1$, that makes the following diagram commutative.

$$
\xymatrix{
C_0 \ar[r]^\eta \ar[d]_{\omega} & C_1 \\
C_0 \ar[r]_\eta & C_1 \\
C_0 \ar[r]^\alpha}
$$

We have $\alpha = \eta \omega \eta^T (\eta \eta^T)^{-1}$. Remark that if we neglect the negative signs on the matrix $\alpha$ then we will obtain precisely the Markov transition matrix $A_{K(f_\mu)}$. In fact, consider the $(n-1) \times (n-1)$ matrix

$$\beta := \begin{bmatrix}
1_{n_L} & 0 \\
0 & -1_{n_R}
\end{bmatrix}$$

where $1_{n_L}$ and $1_{n_R}$ are the identity matrices of rank $n_L$ and $n_R$ respectively, with $n_L$ ($n_R$) being the number of intervals $I_i$ of the Markov partition placed on the left (right) hand side of the turning point of $f_\mu$. Therefore, we have

$$A_{K(f_\mu)} = \beta \alpha.$$ 

Now, consider the following matrix defined by

$$\gamma_{K(f_\mu)} := (\gamma_{ij}) \text{ with } \left\{
\begin{array}{ll}
\gamma_{ii} = \varepsilon_i(c), & i = 1, \ldots, n \\
\gamma_{in} = -\varepsilon_i(c), & i = 1, \ldots, n \\
\gamma_{ij} = 0, & \text{otherwise}.
\end{array}
\right.$$
The matrix $\gamma_{K(f_\mu)}$ makes the diagram

$$
\begin{array}{c}
C_0 \xrightarrow{\eta} C_1 \\
\downarrow \gamma_{K(f_\mu)} \downarrow \beta \\
C_0 \xrightarrow{\eta} C_1
\end{array}
$$

commutative. Finally, set $\theta_{K(f_\mu)} := \gamma_{K(f_\mu)} \omega$. Then, the following diagram

$$
\begin{array}{c}
C_0 \xrightarrow{\eta} C_1 \\
\downarrow \theta_{K(f_\mu)} \downarrow \beta \\
C_0 \xrightarrow{\eta} C_1
\end{array}
$$

is also commutative. Now, notice that the transpose of $\eta$ has the following factorization

$$
\eta^T = Y i X,
$$

where $Y$ is an invertible (over $\mathbb{Z}$) $n \times n$ integer matrix given by

$$
Y := \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots \\
0 & 0 & \cdots & 0 & 1 & 0 \\
-1 & -1 & \cdots & -1 & 1
\end{pmatrix},
$$

$i$ is the inclusion $C_1 \hookrightarrow C_0$ given by

$$
i := \begin{pmatrix}
1 & 0 & 0 \\
0 & \ddots & \ddots \\
\vdots & \ddots & 0 \\
0 & \cdots & 1 \\
0 & \cdots & 0
\end{pmatrix},
$$

and $X$ is an invertible (over $\mathbb{Z}$) $(n-1) \times (n-1)$ integer matrix obtained from the $(n-1) \times n$ matrix $\eta^T$ by removing the $n$-th row of $\eta^T$. Thus, from the commutative diagram

$$
\begin{array}{c}
C_1 \xrightarrow{\eta^T} C_0 \\
\downarrow A_{K(f_\mu)}^T \downarrow \theta_{K(f_\mu)}^T \\
C_1 \xrightarrow{\eta^T} C_0
\end{array}
$$
we will have the following commutative diagram with short exact rows

$$
\begin{align*}
0 & \rightarrow C_1 & p & \rightarrow C_0/C_1 & \rightarrow 0 \\
0 & \rightarrow C_1 & \theta' & \rightarrow C_0/C_1 & \rightarrow 0 \\
i & \rightarrow C_1 & p & & \\
\end{align*}
$$

where the map $p$ is represented by the $1 \times n$ matrix $[0 \ldots 0 1]$ and

$$A' = X A_{K(f_{\mu})}^T X^{-1} \text{ and } \theta' = Y^{-1} \theta_{K(f_{\mu})}^T Y$$

i.e., $A'$ is similar to $A_{K(f_{\mu})}^T$ over $\mathbb{Z}$ and $\theta'$ is similar to $\theta_{K(f_{\mu})}^T$ over $\mathbb{Z}$. Hence, for example by [10], we obtain respectively

$$\mathbb{Z}^{n-1}/(1 - A') \mathbb{Z}^{n-1} \cong \mathbb{Z}^{n-1}/(1 - A_{K(f_{\mu})}) \mathbb{Z}^{n-1} \text{ and}$$

$$\mathbb{Z}^{n}/(1 - \theta') \mathbb{Z}^{n} \cong \mathbb{Z}^{n}/(1 - \theta_{K(f_{\mu})}) \mathbb{Z}^{n}.$$ 

Now, from the last diagram we have, for example by [9],

$$\theta' = \begin{bmatrix} A' & * \\ 0 & 0 \end{bmatrix}.$$ 

Therefore,

$$\mathbb{Z}^{n-1}/(1 - A') \mathbb{Z}^{n-1} \cong \mathbb{Z}^{n}/(1 - \theta') \mathbb{Z}^{n}$$

and

$$\mathbb{Z}^{n-1}/(1 - A_{K(f_{\mu})}) \mathbb{Z}^{n-1} \cong \mathbb{Z}^{n}/(1 - \theta_{K(f_{\mu})}) \mathbb{Z}^{n}.$$ 

Next, we will compute $\mathbb{Z}^{n}/(1 - \theta_{K(f_{\mu})}) \mathbb{Z}^{n}$. From the previous discussions and notations, the $n \times n$ matrix $\theta_{K(f_{\mu})}$ is explicitly given by

$$\theta_{K(f_{\mu})} := \begin{pmatrix} -\varepsilon_1(c) & \varepsilon_1(c) & 0 & \cdots & 0 \\
\vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & 0 \\
-\varepsilon_{n-1}(c) & \varepsilon_{n-1}(c) & \varepsilon_{n-1}(c) \\
0 & 0 & \cdots & 0 \\
\end{pmatrix}.$$ 

Notice that this matrix $\theta_{K(f_{\mu})}$ completely describes the dynamics of $f_{\mu}$. Finally, using row and column elementary operations over $\mathbb{Z}$, we can find
invertible (over \( \mathbb{Z} \)) matrices \( U_1 \) and \( U_2 \) with integer entries such that

\[
1 - \theta_{\mathcal{K}(f_\mu)} = U_1 \begin{pmatrix} \sum_{l=1}^{n-1} \prod_{i=1}^{l} \varepsilon_i(c) & 1 & \cdots & 1 \\ \\ 1 & \end{pmatrix} U_2.
\]

Thus, we obtain

\[
K_0(\mathcal{O}_{A_{\mathcal{K}(f_\mu)}}) \cong \mathbb{Z}_{2n-1}/(1 - A_{\mathcal{K}(f_\mu)}^T)^{2n-1} \cong \mathbb{Z}_a,
\]

where \( a = 1 + \sum_{l=1}^{n-1} \prod_{i=1}^{l} \varepsilon_i(c) \) and \( n \in \mathbb{N} \setminus \{1\} \).

**Example 1.** Set

\[
\mathcal{K}(f_\mu) = (RLLRRC)^\infty,
\]

where \( R = -1, \ L = +1, \ C = 0 \). Thus, we can construct the \( 5 \times 5 \) Markov transition matrix \( A_{\mathcal{K}(f_\mu)} \) and the matrices \( \theta_{\mathcal{K}(f_\mu)}, \omega, \varphi, \) and \( \pi \).

\[
A_{\mathcal{K}(f_\mu)} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad \theta_{\mathcal{K}(f_\mu)} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
\omega = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \varphi = \begin{pmatrix} -1 & 1 & \cdots & \cdots & \cdots & \cdots \\ -1 & 1 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & \cdots & \cdots & -1 \end{pmatrix},
\]

\[
\pi = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
\]

We have

\[
K_0(\mathcal{O}_{A_{\mathcal{K}(f_\mu)}}) \cong \mathbb{Z}_2 \quad \text{and} \quad K_1(\mathcal{O}_{A_{\mathcal{K}(f_\mu)}}) \cong \{0\}.
\]
Remark 1. In the statement of Theorem 1 the case $a = 0$ may occur. This happens when we have a star product factorizable kneading sequence [4]. In this case the correspondent Markov transition matrix is reducible.

Remark 2. In [6] the authors have constructed a class of $C^*$-algebras from the $\beta$-expansions of real numbers. In fact, by considering a semiconjugacy from the real quadratic map to the tent map [14], we can also obtain Theorem 1 using [6] and the $\lambda$-expansions of real numbers introduced in [4].

Remark 3. In [12] (see also [11]) and [13] the BF-groups are explicitly calculated with respect to other kind of maps on the interval.

References

[1] R. Bowen and J. Franks. Homology for zero dimensional basic sets. Ann. Math. (2) 106 (1977), 73-92.
[2] J. Cuntz and W. Krieger. A class of $C^*$-algebras and topological Markov chains. Invent. Math. 56 (1980), 251-268.
[3] J. Cuntz. A class of $C^*$-algebras and topological Markov chains II: Reducible chains and the Ext-functor for $C^*$-algebras. Invent. Math. 63 (1981), 25-40.
[4] B. Derrida, A. Gervois and Y. Pomeau. Iteration of endomorphisms on the real axis and representations of numbers. Ann. Inst. H. Poincaré Sect. A (N.S.) 29 (1978), 305-356.
[5] J. Guckenheimer. Sensitive Dependence to Initial Conditions for One Dimensional Maps. Comm. Math. Phys. 70 (1979), 133-160.
[6] Y. Katayama, K. Matsumoto and Y. Watatani. Simple $C^*$-algebras arising from $\beta$-expansion of real numbers. Ergodic Theory Dynam. Systems 18 (1998), 937-962.
[7] J. P. Lampreia and J. Sousa Ramos. Trimodal maps. Internat. J. Bifur. Chaos Appl. Sci. Engrg. 3 (1993), 1607-1617.
[8] J. P. Lampreia and J. Sousa Ramos. Symbolic dynamics for bimodal maps. Portugal. Math. 54 (1997), 1-18.
[9] S. Lang. Algebra. Addison-Wesley. 1965.
[10] D. Lind and B. Marcus. An Introduction to Symbolic Dynamics and Coding. Cambridge University Press. 1995.
[11] N. Martins and J. Sousa Ramos. Cuntz-Krieger algebras arising from linear mod one transformations. Fields Inst. Commun. 31 (2002), 265-273.
[12] N. Martins and J. Sousa Ramos. Bowen-Franks groups associated with linear mod one transformations. To appear in Internat. J. Bifur. Chaos Appl. Sci. Engrg. (2003).
[13] N. Martins, R. Severino and J. Sousa Ramos. Bowen-Franks groups for bimodal matrices. To appear in J. Differ. Equations Appl.
[14] J. Milnor and W. Thurston. On iterated maps of the interval. Lecture Notes in Math. 1342 (1988), 465-563.
[15] P. Stefan. A theorem of Sharkovsky on the existence of periodic orbits of continuous endomorphisms of the real line. Comm. Math. Phys. 54 (1977), 237-248.
