Holographic dictionary for generic asymptotically AdS black holes

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We propose a sequence of steps and a generic transformation for connecting common thermodynamic quantities considered in asymptotically anti-de Sitter black hole thermodynamics in the bulk and those that are appropriate for CFT thermodynamics in the boundary. We prove this “holographic dictionary” and demonstrate its usefulness by considering some examples.

I. INTRODUCTION

In 1997, Maldacena proposed a duality between string theories in an anti-de Sitter (AdS) spacetime (the bulk) and conformal field theories at its boundary [1], which could give insights, for instance, about a theory of quantum gravity [2]. Based on this idea, Witten related phase transitions in the bulk with the confinement/deconfinement transition of the gauge fields at the boundary [3]. The phase transitions considered by Witten are the so-called Hawking-Page phase transitions, originally the transition between a black hole in an anti-de Sitter space and pure radiation in this same space [4]. In thermodynamics, we say this is a first-order transition.

Witten’s analysis is based on traditional black hole thermodynamics, where the black hole’s mass, for example, corresponds to its internal energy. More recently, however, Kastor, Ray and Traschen have proposed a modification to this framework, equating the black hole’s mass not with its internal energy, but with its enthalpy [5]. With that, they were able to introduce a notion of a mechanical pressure for the black hole, as well as a thermodynamic volume.

In this new framework, Kubiznak and Mann showed that anti-de Sitter charged black holes exhibit other types of phase transitions, such as second-order phase transitions, similar to the phase transitions of usual matter, which often can be described by equations of state such as that of Van der Waals [6]. Because the black hole’s mass is equaled to its enthalpy, such a framework was called black hole chemistry, to differentiate it from the usual thermodynamics of black holes (for a review on the subject, see [7]). This new framework, however, requires extensions to the dictionary proposed by Witten. The reason for this is multiple. For instance, the mechanical pressure of the black hole (the pressure in the bulk) is not the same as the boundary pressure [8]. But what does this pressure mean then? And how to calculate the boundary pressure from the bulk pressure?

Another key point is the relationship between the size of the anti-de Sitter space and the number of colors, \( N \), of the dual Yang-Mills (Y-M) theory or degrees of freedom of the Conformal Field Theory (CFT) at the boundary. According to the usual dictionary, the latter is measured by the central charge, \( C \), as \( N^2 \propto C \propto L^{D-2}/G \), where \( L \) is the anti-de Sitter radius, \( G \) is the gravitational constant in \( D \) bulk dimensions [9, 10]. This means that, for a fixed \( G \), a change in the size of the anti-de Sitter space leads to a variation in the number of colors, i.e., we modify the Y-M theory at the boundary or takes us from a CFT to another.

On the other hand, the radius of the AdS space is related to the volume of the CFT as \( V = L^{D-2} \). Therefore if one aims to vary, in an independent way, both the volume and the central charge (or degrees of freedom) of the CFT, it becomes necessary to allow the gravitational constant to vary. Or if one aims to consider modifications of the CFT volume, but keeping the same theory, it becomes necessary to also vary simultaneously the gravitational constant and the AdS radius in order to keep the central charge unmodified (for further details, please refer to [9] and references therein).

The compatibility between the black hole thermodynamics valid in the bulk and the CFT at the boundary requires that a dictionary is constructed in a way that relates the “hair” parameters that define the black hole and the CFT thermodynamic quantities. Usually, this is done by identifying a map between the boundary-defined quantities and the bulk black hole ones in a way that preserves the first law and Smarr relations in both representations. Recently, such...
an approach has been carried out for cases with rotation [9, 10], electric charge with different types of electrodynamics (reflected in the black hole solutions considered) [11–13], and also for string theory corrections modeled by higher derivative terms [14] in a way that present, as a common point, the need for the aforementioned redefinition of variables for each case under scrutiny.

In this paper, we construct a general algorithm for identifying the necessary transformation between generic thermodynamic quantities that are considered in standard black hole thermodynamics in the bulk and those needed for making this thermodynamics compatible with the one carried out in the boundary CFT. The dictionary itself is written in definition II.1 and the proof of theorem II.1 (that states the compatibility between the black hole thermodynamics in the bulk and the one in the boundary) is done throughout section II. Using this algorithm (that is explicitly written in section III), we show its application by recovering some known results in the literature and we propose how it should be applied in a novel scenario. Finally, we draw our final remarks in section IV.

II. HOLOGRAPHIC DICTIONARY

The starting point of this derivation consists in identifying how the gravitational constant $G$ scales in $D$ spacetime dimensions. In fact, for simplicity, assuming units in which $c = \hbar = 1$ (which can be assumed since these quantities are held fixed), if $l$ is a fiducial quantity with length dimensions, we have that the dimension of $G$ behaves like $[G] = l^{D-2}$ [10]. Given a set of thermodynamic quantities $\tau_i$ ($i = 1, ..., n$) which are not coupled to the gravitational constant $G$, let us define a set of new thermodynamic variables $\eta_i = G^{\alpha_i} \tau_i$ (no sum over indices assumed here). The motivation for this definition stems from usual couplings of $\alpha_i$ powers of $G$ and $\tau_i$ in the metric functions of black holes, which will serve as the starting point of the holographic map that we shall propose.$^1$

The length dimension of the variables $\eta_i$ scale as $[\eta_i] = l^{\rho_i(D-2)}[\tau_i]$ and if $\tau_i$ scale as $[\tau_i] = l^{\beta_i}$, then we have

$$[\eta_i] = l^{\rho_i(D-2) + \beta_i}.$$  \hspace{1cm} (1)

The thermodynamic quantities $\eta_i$ have conjugate potentials $A_i$, which can also present factors of $G$ once one redefines the $\tau_i$ variables, but we shall not deal with them, because the roles of $\tau_i$ and $A_i$ only differ by a Legendre transformation.

At this point, one should notice that although, for simplicity, we are using an auxiliary quantity $l$ to describe length dimensions, what we have, in fact, is an implicit scaling of different thermodynamic quantities with $G$, since $l = [G]^{1/(D-2)}$.

Now, suppose that we consider a black hole whose mechanic/thermodynamic variables are the redefined mass $GM$, a term proportional to the area as $A/4$ (with conjugate variable proportional to the surface gravity as $\kappa/2\pi$), a negative cosmological constant $\Lambda/8\pi$ (with a potential $\Theta$) and a set of variables $G^{\alpha_i} \tau_i$ (and related potentials $A_i$).

Using the fact that these quantities scale as $[G] = l^{D-2}$, $[M] = l^{-1}$, $[A] = l^{D-2}$, $[\Lambda] = l^{-2}$ and $[G^{\alpha_i} \tau_i] = l^{(D-2)\alpha_i + \beta_i}$, we have the following first law of thermodynamics and Smarr relation, that follows from Euler’s theorem for homogeneous functions (where we have $\sum_i = \sum_i = 1$):

$$d(GM) = \frac{\kappa}{8\pi} dA + \frac{\Theta}{8\pi} d\Lambda + \sum_i A_i d(G^{\alpha_i} \tau_i),$$  \hspace{1cm} (2)

$$(D - 3)M = (D - 2) \frac{\kappa A}{8\pi G} - \frac{\Theta \Lambda}{4\pi G} + \sum_i [(D - 2)\alpha_i + \beta_i] G^{\alpha_i - 1} \tau_i A_i.$$  \hspace{1cm} (3)

These expressions are just the usual first law and Smarr relation of black hole thermodynamics and for quantities like entropy $S = A/4G$, temperature $T = \kappa/2\pi$, black hole’s chemistry pressure $P = -\Lambda/8\pi G$ and volume $V = -\Theta$. For example, by setting $n = 1$ and $\tau_1 = Q$, $A_1 = \sqrt{G}\Phi$, $\alpha_1 = 1/2$ and $\beta_1 = (D - 4)/2$, we are able to derive the usual Smarr relation of the Reissner-Nordström-anti-de Sitter (AdS) black hole in $D$-dimensions $(D - 3)M = (D - 2)TS + 2PV + (D - 3)Q\Phi$ (please, refer to Ref.[10, 15] to verify the scaling of these quantities.)

As described in [9] and in the introduction of our paper, the compatibility of these expressions with the holographic principle leads to a necessary variation of the gravitational constant. In this case, the above first law leads to

$$dM = \frac{\kappa}{8\pi} d\left(\frac{A}{G}\right) + \frac{\kappa A}{8\pi G} dG + \frac{\Theta}{8\pi G} d\Lambda + \sum_i A_i d\left(G^{\alpha_i} \tau_i \frac{dG}{G}\right) + \frac{A_i}{G} G^{\alpha_i} \tau_i \frac{dG}{G} - M \frac{dG}{G},$$  \hspace{1cm} (4)

$^1$ To see how this works, consider the Reissner-Nordström metric function [15] $f(r) = 1 - \frac{2GM}{r} + \frac{Q^2}{r^2}$, where $M$ and $Q$ are the black hole’s mass and charge. In this case, by choosing $\eta_1 = GM$ and $\tau_1 = M$, we have $\alpha_1 = 1$. Besides that, if $\eta_2 = \sqrt{G}Q$ and $\tau_2 = Q$, we have $\alpha_2 = 1/2$. This way, the metric becomes $f(r) = 1 - \frac{2\eta_1}{r^{D-3}} + \frac{\eta_2^2}{r^{2(D-3)}}$. 

where we integrated by parts the terms involving the area $A$ and the parameters $\tau_i$. Rearranging this expression, we can isolate the term that depends on the variation of $G$

$$dM = \frac{\kappa}{8\pi} d \left( \frac{A}{G} \right) + \frac{\Theta}{8\pi G} d\Lambda + \sum_i A_i d (G^{\alpha_i-1} \tau_i) + \frac{dG}{G} \left( -M + \frac{\kappa A}{8\pi G} + \sum_i A_i G^{\alpha_i-1} \tau_i \right).$$

(5)

The choice of the remaining term with $G^{\alpha_i-1} \tau_i$ will be important for a future simplification.

We now redefine the negative cosmological constant in terms of the AdS radius $L$, as follows $[10]$

$$\Lambda = -\frac{(D-1)(D-2)}{2L^2}.$$ 

(6)

Using the Smarr relation (3), we express $\Theta/(8\pi G)$ as a function of the other variables, which can be used in Eq.(5), along with the relation $d\Lambda/\Lambda = -2dL/L$ (which follows from (6)) to furnish

$$dM = \frac{\kappa}{8\pi} d \left( \frac{A}{G} \right) + \sum_i A_i d (G^{\alpha_i-1} \tau_i) + \frac{dG}{G} \left( -M + \frac{\kappa A}{8\pi G} + \sum_i A_i G^{\alpha_i-1} \tau_i \right) - \frac{dL}{L} \left[ -(D-3)M + (D-2) \frac{\kappa A}{8\pi G} + \sum_i [(D-2)\alpha_i + \beta_i] G^{\alpha_i-1} \tau_i A_i \right].$$

(7)

Now, we express the variation of $G$ in term of the variation of novel variables $L^{D-2}$ and $L^{D-2}/G$ (which will be related to the CFT volume and central charge). This gives

$$\frac{dG}{G} = \frac{dL^{D-2}}{L^{D-2}} - \frac{d(L^{D-2}/G)}{L^{D-2}/G}, \quad \frac{dL}{L} = \frac{1}{D-2} \frac{dL^{D-2}}{L^{D-2}}.$$ 

(8)

This leads us to the following expressions for the 1st law, that we express in details below:

$$dM = \frac{\kappa}{8\pi} d \left( \frac{A}{G} \right) + \left( \frac{dL^{D-2}}{L^{D-2}} - \frac{d(L^{D-2}/G)}{L^{D-2}/G} \right) \left( -M + \frac{\kappa A}{8\pi G} + \sum_i A_i G^{\alpha_i-1} \tau_i \right) + \sum_i A_i d (G^{\alpha_i-1} \tau_i)$$

$$- \frac{1}{D-2} \frac{dL^{D-2}}{L^{D-2}} \left[ -(D-3)M + (D-2) \frac{\kappa A}{8\pi G} + \sum_i [(D-2)\alpha_i + \beta_i] G^{\alpha_i-1} \tau_i A_i \right]$$

$$= \frac{\kappa}{8\pi} d \left( \frac{A}{G} \right) + \sum_i A_i d (G^{\alpha_i-1} \tau_i) + \frac{d(L^{D-2}/G)}{L^{D-2}/G} \left( M - \frac{\kappa A}{8\pi G} - \sum_i A_i G^{\alpha_i-1} \tau_i \right)$$

$$+ \frac{dL^{D-2}}{L^{D-2}} \left( \frac{M}{D-2} + \sum_i \left( 1 - \alpha_i - \frac{\beta_i}{D-2} \right) G^{\alpha_i-1} A_i \tau_i \right)$$

$$= \frac{\kappa}{8\pi} d \left( \frac{A}{G} \right) + \sum_i A_i d (G^{\alpha_i-1} \tau_i) + \frac{d(L^{D-2}/G)}{L^{D-2}/G} \left( M - \frac{\kappa A}{8\pi G} - \sum_i A_i G^{\alpha_i-1} \tau_i \right)$$

$$- \frac{M}{D-2} \frac{dL^{D-2}}{L^{D-2}} - (D-2) \frac{dL}{L} \sum_i \left( 1 - \alpha_i - \frac{\beta_i}{D-2} \right) G^{\alpha_i-1} \tau_i A_i \right).$$

(9)

In the last step, we used again $dL^{D-2}/L^{D-2} = (D-2)dL/L$. We can rearrange again this equation in terms of new variables $\tau_i G^{\alpha_i-1} \tau_i^{(D-2)(1-\alpha_i)-\beta_i}$ and $A_i L^{(D-2)(1-\alpha_i)-\beta_i}$ by joining the second and the last terms of the above equation. This leads to

$$dM = \frac{\kappa}{8\pi} d \left( \frac{A}{G} \right) + \sum_i \frac{A_i}{L^{(D-2)(1-\alpha_i)-\beta_i}} d \left( \tau_i G^{\alpha_i-1} \tau_i^{(D-2)(1-\alpha_i)-\beta_i} \right) - \frac{M}{D-2} \frac{dL^{D-2}}{L^{D-2}}$$

$$+ \frac{d(L^{D-2}/G)}{L^{D-2}/G} \left( M - \frac{\kappa A}{8\pi G} - \sum_i \frac{A_i}{L^{(D-2)(1-\alpha_i)-\beta_i}} \tau_i G^{\alpha_i-1} \tau_i^{(D-2)(1-\alpha_i)-\beta_i} \right).$$

(10)

At this point, we are ready to derive the holographic dictionary for any set of variables.
Definition II.1 (Holographic dictionary) Let $M$, $A$, $\kappa$ and $L$ be the black hole’s mass, area, surface gravity and anti-de Sitter radius in $D$ spacetime dimensions. Consider a set of $n$ thermodynamic quantities, called charges, $\tau_i$ (where $n = 1, \ldots, n$), that scale with length dimensions $l^{\beta_i}$ and that couple to the $\alpha_i$-th power of the gravitational constant (which presents length dimension $l^{D-2}$) as $\tau_i G^{\alpha_i}$, where $l$ is fiducial length scale. The holographic dictionary consists in the transformation from the set of thermodynamic variables $(M, A, \kappa, L, \tau_i, A_i, G)$ to the set $(E, S, T, V, p, \nu_i, B_i, C, \mu)$ as follows

\[
E = M, \quad S = \frac{A}{4G}, \quad T = \frac{\kappa}{2\pi}, \quad \text{(11)}
\]
\[
V = L^{D-2}, \quad p = \frac{E}{(D-2)V}, \quad \text{(12)}
\]
\[
\nu_i = \tau_i G^{\alpha_i-1} l^{(D-2)(1-\alpha_i)-\beta_i}, \quad \text{(13)}
\]
\[
B_i = \frac{A_i}{L^{(D-2)(1-\alpha_i)-\beta_i}}, \quad \text{(14)}
\]
\[
C = \frac{k}{16\pi G} L^{D-2}, \quad \text{(15)}
\]
\[
\mu = \frac{1}{C}(E - TS - \sum_i B_i \nu_i), \quad \text{(16)}
\]

where $k$ is an arbitrary constant that labels different holographic systems. The set $(E, S, T, V, p, C, \mu)$ corresponds to variables known as internal energy, entropy, temperature, volume, pressure, central charge and chemical potential, respectively. The quantities $(\nu_i, B_i)$ are the redefined charges and potentials.

The substitution of Eqs.(11)-(16) into equation (10) constitutes a proof of the following

Theorem II.1 Let $M$, $A$, $\kappa$, $\Lambda = -(D-1)(D-2)/(2L^2)$ and $\Theta$ be the black hole’s mass, area, surface gravity, cosmological constant and the conjugate variable to $\Lambda$ in $D$ spacetime dimensions, respectively (where $L$ is the anti-de Sitter radius). Consider a set of $n$ thermodynamic quantities $\tau_i$ (where $n = 1, \ldots, n$) that scale with length dimensions $l^{\beta_i}$ and that couple to the $\alpha_i$-th power of the gravitational constant as $\tau_i G^{\alpha_i}$. If $G$ is a variable gravitational constant (that presents length dimension $l^{D-2}$), such that the black hole’s first law and Smarr relation read

\[
d(GM) = \frac{\kappa}{8\pi} dA + \frac{\Theta}{8\pi} d\Lambda + \sum_i A_i d(G^{\alpha_i} \tau_i), \quad \text{(17)}
\]
\[
(D - 3)M = (D - 2) \frac{\kappa A}{8\pi G} - \frac{\Theta \Lambda}{4\pi G} + \sum_i [(D - 2)\alpha_i + \beta_i] G^{\alpha_i-1} \tau_i A_i, \quad \text{(18)}
\]

then the Holographic Dictionary consists in a map between these expressions and the following holographic first law and holographic Smarr relation

\[
dE = TdS + \sum_i B_i d\nu_i - p dV + \mu dC, \quad \text{(19)}
\]
\[
E = TS + \sum_i B_i \nu_i + \mu C. \quad \text{(20)}
\]

This way, from the holographic first law, one sees that the quantity $E$ is the equivalent of an internal energy, $S$ and $T$ are the entropy and temperature, $p$ and $V$ are the pressure and volume. Since the central charge $C = k L^{D-2}/(16\pi G)$ measures the number of degrees of freedom of a corresponding CFT in the AdS/CFT correspondence, then the quantity $\mu$ should be interpreted as a chemical potential in this analogy. The rest of the holographic dictionary, i.e., that defines $\nu_i$ and $B_i$, corresponds to the appropriate choice of variables to extend the holographic principle to black hole thermodynamics.

III. SOME EXAMPLES

A general algorithm for applying the holographic dictionary would be the following:

1. Identify the relevant charges, $\tau_i$, in which the standard black hole thermodynamics would be formulated.
2. Explicitly write $G$ in the equations of the gravitational system (for example, in the metric), in order to identify the power, $\alpha_i$, in which it couples to $\tau_i$ for finding the quantity $\eta_i = G^\alpha_i \tau_i$.

3. Discover the power length dimension, $\beta_i$, of $\tau_i$ as $[\tau_i] = l^{\beta_i}$.

4. Express the first law of black hole thermodynamics through variations of $GM$, $A$, $\Lambda$ and $G^\alpha \tau_i$ as

\[
d(GM) = \frac{\kappa}{8\pi} dA + \frac{\Theta}{8\pi} d\Lambda + \sum_i A_i d(G^\alpha \tau_i) .
\]

(21)

5. From the expected form of the Smarr relation in standard black hole thermodynamics

\[
(D-3)M = (D-2) \frac{\kappa A}{8\pi G} - \frac{\Theta \Lambda}{4\pi G} + \sum_i [(D-2)\alpha_i + \beta_i] G^{\alpha_i} \tau_i A_i .
\]

(22)

one can find the form of the potentials $A_i$ as given by the product of $\gamma_i$-powers of $G$ and the usual potentials $\Phi_i$ that are conjugate to $\tau_i$ in standard black hole thermodynamics (the one in which $G$ was held fixed)

\[
(D-3)M = (D-2) \frac{\kappa A}{8\pi G} - \frac{\Theta \Lambda}{4\pi G} + \sum_i [(D-2)\alpha_i + \beta_i] G^{\alpha_i} \tau_i \Phi_i ,
\]

(23)

where $A_i = G^{\gamma_i-\alpha_i+1} \Phi_i$.

6. Endowed with these quantities, one uses the holographic dictionary of II.1, in order to find the transformation to the holographic first law and Smarr relation given by Eqs.(19) and (20).

We shall employ this algorithm to the following examples.

### A. Charged, rotating black hole

Consider the charged, rotating black hole, such that the charges considered are the electric charge $\tau_1 = Q$ and angular momentum $\tau_2 = J$. By referring to the Kerr-Newman metric \[16\], we see that the electric charge couples to the gravitational constant as $GQ^2$, which means that from the definition of the quantity $\eta_1$ (referred in the second item of the above algorithm) it would be natural to define $\eta_1 = \sqrt{G} Q$. From this prescription, one sees that the power of the gravitational constant coupling is $\alpha_1 = 1/2$. On the other hand, the electric charge itself has length dimensions $[Q] = l^{(D-4)/2}$, which means that $\beta_1 = (D-4)/2$ (this behavior was verified in \[10\] for the Kerr-Newman case and in \[15\] for the Reissner-Nordström one).

For the angular momentum, to find its length dimension $l$, we could rely on the Kerr parameter $a = J/M$ that obeys $[a] = l$. Since $[M] = l^{-1}$, we should have $[J] = l^{l}$, thus giving $\beta_2 = 0$. To verify its $G$-coupling, we recover $G$ in the thermodynamic quantities of the Kerr parameter $a = GJ/(GM)$. Therefore, the quantity $\eta_2 = GJ$, which gives $\alpha_2 = 1$.

The form of the potentials $A_i$ can be seen from the form of the first law of black hole mechanics and Smarr relation for the quantities $\eta_i$ (21), (22) and (23). This leads to

\[
d(GM) = \frac{\kappa}{8\pi} dA + \frac{\Theta}{8\pi} d\Lambda + \sqrt{G} \Phi d\left(\sqrt{G} Q\right) + \Omega d(GJ) ,
\]

(24)

\[
(D-3)M = (D-2) \frac{\kappa A}{8\pi G} - \frac{\Theta \Lambda}{4\pi G} + (D-3)Q \Phi + (D-2)J \Omega .
\]

(25)

Notice that since $\sqrt{G}$ is absorbed into the electric charge variation, one must have $A_1 = \sqrt{G} \Phi$, where $\Phi$ is the electric potential. And since $G$ is coupled to $J$ in the variation equation, we must have $A_2 = \Omega$, the angular velocity. With this quantities at hand, we can use the holographic dictionary II.1 to find the redefinition (13), (14)

\[
\nu_1 = \tilde{Q} = Q G^{-1/2} L , \quad B_1 = \tilde{\Phi} = \sqrt{G} \Phi / L ,
\]

(26)

\[
\nu_2 = J , \quad B_2 = \Omega .
\]

(27)

Therefore, from II.1, the holographic first law and Smarr relation read

\[
dE = T dS + \tilde{\Phi} d\tilde{Q} + \Omega dJ - pdV + \mu dC ,
\]

(28)

\[
E = TS + \tilde{\Phi} \tilde{Q} + \Omega J + \mu dC ,
\]

(29)

which recovers the results of \[10\].
B. Charged black hole with an alternative convention for the charge coupling

If one considers different conventions for the bulk charges, i.e., such that they grow differently with $G$, the dictionary would simply imply in a different translation. For instance, considering the convention of [11] (see Eqs. (2.3) and (2.5), and notice that $d = D - 1$) one has $q \propto \eta_1 = GQ$, which implies $\tau_1 = Q$, and $\alpha_1 = 1$. By referring to the metric function (2.3) of [11] and that $[G] = l^{D-2}$, we see that $[\eta_1] = l^{D-3} \Rightarrow [Q] = l^{-1}$, i.e., $\beta_1 = -1$. For this reason, since $G$ couples to $Q$ linearly, we identify the shape of the potential $A_1 = \Phi$. In fact, we check it by looking at the first law and Smarr relation with variable $G$

\[ d(GM) = \frac{\kappa}{8\pi} dA + \frac{\Theta}{8\pi} d\Lambda + \Phi d(GQ), \]
\[ (D-3)M = (D-2) \frac{\kappa A}{8\pi G} - \frac{\Theta \Lambda}{4\pi G} + (D-3)Q\Phi. \]

From these parameters ($\alpha_1 = 1$, $\beta_1 = -1$), the use of the holographic dictionary II.1 leads to the following redefinition, followed by the holographic first law and Smarr relation:

\[ v_1 = \tilde{Q} = QL, \quad B_1 = \frac{\Phi}{L}, \]
\[ dE = TdS + \tilde{\Phi} d\tilde{Q} - pdV + \mu dC, \]
\[ E = TS + \tilde{\Phi} \tilde{Q} + \mu dC, \]

which coincides with the results of [11], given by Eqs. (2.20)-(2.23) (for $R = L$, where $R$ is the radius of the sphere in which the CFT is resides).

C. Kiselev black hole

Consider the case of the Kiselev black hole [15, 17], whose static metric function is given by

\[ f(r) = 1 - \frac{16\pi GM}{(D-2)\Omega_D - 2r^2 D^{-3}} - \frac{2Ar^2}{(D-1)(D-2)} - \frac{Gb}{r^{(D-1)\omega + D-3}}. \]

In this case, we have $\eta_1 = Gb$ and $\tau_1 = b$, such that $[\eta_1] = l^{D-1}\omega + D-3$. This means that the dimension of $\tau_1$ is $[\tau_1] = l^{D-1}\omega - 1$ (since $[G] = l^{D-2}$). Therefore, we have $\alpha_1 = 1$, $\beta_1 = (D-1)\omega - 1$. From this, the first law and Smarr relation read

\[ d(GM) = \frac{\kappa}{8\pi} dA + \frac{\Theta}{8\pi} d\Lambda + Bd(Gb), \]
\[ (D-3)M = (D-2)TS - \frac{\Theta \Lambda}{4\pi G} + [\omega(D-1) + D-3]Bb. \]

Therefore, from $A_1 = B$, and we can apply the holographic dictionary II.1 to redefine the thermodynamic quantities and find the holographic first law and Smarr relation as follows

\[ v_1 = \tilde{b} = bL^{1-(D-1)\omega}, \quad B_1 = \tilde{B} = BL^{(D-1)\omega - 1}, \]
\[ dE = TdS + \tilde{B} \tilde{d}b - pdV + \mu dC, \]
\[ E = TS + \tilde{B} \tilde{d}b + \mu dC. \]

In fact, whenever a quantity couples linearly to $G$, such that $\eta_i = G\tau_i$, i.e., if $\alpha_i = 1$, then, the holographic dictionary simply gives $\nu_i = \tau_i L^{-\beta_i}$ and $B_i = A_i L^{\beta_i}$, such that $\nu_i$ becomes dimensionless.

IV. CONCLUDING REMARKS

We defined a general sequence of steps with which one can translate quantities that are usually considered in standard asymptotically AdS black hole thermodynamics to those that are considered in CFTs in the boundary of the AdS space. Such a construction aims to serve as a guide for future analyses carried out in different scenarios on the recently considered proposal, for instance described in [9–11], to make compatible the black hole thermodynamics in AdS space and the thermodynamics defined in its boundary in light of the holographic principle. Our goal is that the algorithm proposed in this paper can be applied for any asymptotically anti-de Sitter black hole, thus working as a shortcut when one aims to investigate this subject in any scenario.
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[1] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2 (1998) 231–252, arXiv:hep-th/9711200.

[2] A. Addazi et al., “Quantum gravity phenomenology at the dawn of the multi-messenger era—A review,” Prog. Part. Nucl. Phys. 125 (2022) 103948, arXiv:2111.05659 [hep-ph].

[3] E. Witten, “Anti-de Sitter space, thermal phase transition, and confinement in gauge theories,” Adv. Theor. Math. Phys. 2 (1998) 505–532, arXiv:hep-th/9803131.

[4] S. W. Hawking and D. N. Page, “Thermodynamics of Black Holes in anti-De Sitter Space,” Commun. Math. Phys. 87 (1983) 577.

[5] D. Kastor, S. Ray, and J. Traschen, “Enthalpy and the Mechanics of AdS Black Holes,” Class. Quant. Grav. 26 (2009) 195011, arXiv:0904.2765 [hep-th].

[6] D. Kubiznak and R. B. Mann, “P-V criticality of charged AdS black holes,” JHEP 07 (2012) 033, arXiv:1205.0559 [hep-th].

[7] D. Kubiznak, R. B. Mann, and M. Teo, “Black hole chemistry: thermodynamics with Lambda,” Class. Quant. Grav. 34 (2017) no. 6, 063001, arXiv:1608.06147 [hep-th].

[8] C. V. Johnson, “Holographic Heat Engines,” Class. Quant. Grav. 31 (2014) 205002, arXiv:1404.5982 [hep-th].

[9] M. R. Visser, “Holographic thermodynamics requires a chemical potential for color,” Phys. Rev. D 105 (2022) no. 10, 106014, arXiv:2101.04145 [hep-th].

[10] W. Cong, D. Kubiznak, and R. B. Mann, “Thermodynamics of AdS Black Holes: Critical Behavior of the Central Charge,” Phys. Rev. Lett. 127 (2021) no. 9, 091301, arXiv:2105.02223 [hep-th].

[11] W. Cong, D. Kubiznak, R. Mann, and M. Visser, “Holographic CFT Phase Transitions and Criticality for Charged AdS Black Holes,” arXiv:2112.14848 [hep-th].

[12] N. Kumar, S. Sen, and S. Gangopadhyay, “Phase transition structure and breaking of universal nature of central charge criticality in a Born-Infeld AdS black hole,” arXiv:2206.00440 [gr-qc].

[13] M. Rafiee, S. A. H. Mansoori, S.-W. Wei, and R. B. Mann, “Universal criticality of thermodynamic geometry for boundary conformal field theories in gauge/gravity duality,” Phys. Rev. D 105 (2022) no. 2, 024058, arXiv:2107.08883 [gr-qc].

[14] S. Dutta and G. S. Punia, “String Theory Corrections to Holographic Black Hole Chemistry,” arXiv:2205.15593 [hep-th].

[15] R. B. Alfaià, I. P. Lobo, and L. C. T. Brito, “Central charge criticality of charged AdS black hole surrounded by different fluids,” Eur. Phys. J. Plus 137 (2022) no. 3, 402, arXiv:2109.06599 [hep-th].

[16] E. Poisson, A Relativist’s Toolkit: The Mathematics of Black-Hole Mechanics. Cambridge University Press, 12, 2009.

[17] V. V. Kiselev, “Quintessence and black holes,” Class. Quant. Grav. 20 (2003) 1187–1198, arXiv:gr-qc/0210040.