FROM GORENSTEIN DERIVED EQUIVALENCES TO STABLE 
FUNCTIONS OF GORENSTEIN PROJECTIVE MODULES 

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Abstract. In the paper, we mainly connect the Gorenstein derived equivalence and stable functors of Gorenstein projective modules. Specially, we prove that a Gorenstein derived equivalence between CM-finite algebras $A$ and $B$ can induce a stable functor between the factor categories $A\text{-}\text{mod}/A\text{-}\text{Gproj}$ and $B\text{-}\text{mod}/B\text{-}\text{Gproj}$. Furthermore, the above stable functor is an equivalence when $A$ and $B$ are Gorenstein.

1. Introduction

Gorenstein derived categories and Gorenstein derived equivalences were introduced by Gao and Zhang [3] have some advantages in the study of Gorenstein homological algebras. A Gorenstein derived equivalence is a triangle equivalence between the Gorenstein derived categories over Artin algebras (see [3]). For Gorenstein derived equivalent Artin algebras, it is hard to directly compare the modules over them, since a Gorenstein derived equivalence typically takes modules of one algebra to complexes over the other algebra. It is a well-known result of Rickard [8] which says that a derived equivalence between two selfinjective algebras always induces a stable equivalence of Morita type. Recently, Hu and Pan [4] proved that a derived equivalence induces a stable equivalence of Gorenstein projective objects, where the nonnegative functor and the uniformly bounded functor between derived categories are introduced.

Let $A$ be an Artin algebra and $A\text{-}\text{mod}$ the category of finitely generated $A$-modules. Let $A\text{-}\text{Gproj}$ be the full subcategory consisting of Gorenstein projective $A$-modules. The Gorenstein stable category of $A$, denoted by $A\text{-}\text{mod}/A\text{-}\text{Gproj}$, is defined to be the additive quotient, where the objects are the same as those in $A\text{-}\text{mod}$ and the morphism space $\text{Hom}_{A\text{-}\text{mod}/A\text{-}\text{Gproj}}(X,Y)$ is the quotient space of $\text{Hom}_{A}(X,Y)$ modulo all morphisms factorizing through Gorenstein projective $A$-modules. Two algebras $A$ and $B$ are Gorenstein stably equivalent if there is an additive equivalence between $A\text{-}\text{mod}/A\text{-}\text{Gproj}$ and $B\text{-}\text{mod}/B\text{-}\text{Gproj}$.

For an arbitrary Gorenstein derived equivalence $F$ between Artin algebras, there is a basic question when $F$ can induce the Gorenstein stable functor $\overline{F}$ between the corresponding Gorenstein stable categories.

In this paper, we consider the nonnegative functor and the uniformly bounded functor between Gorenstein derived categories, respectively. We show that two
kinds of functors induce the stable functor between the corresponding Gorenstein stable categories (see Proposition 2.7). We prove that a Gorenstein derived equivalence is both the nonnegative functor and the uniformly bounded functor, and consequently, it can induce a Gorenstein stable equivalence (see Theorem 2.11).

In the following, we recall the basic notions which is necessary in the paper.

From [5, 6], let \( A \) be an algebra, \( G \in A\text{-mod} \) is called \textbf{Gorenstein projective} if there exists an exact complex

\[
\ldots \longrightarrow P^{-1} \longrightarrow P^0 \overset{d^0}{\longrightarrow} P^1 \longrightarrow \ldots
\]

of projective modules, which stays exact after applying \( \text{Hom}_A(\_, P) \) for all projective \( A \)-module \( P \), with \( G \cong \ker d^0 \).

Denote by \( A\text{-Gproj} \) the full subcategory of \( A\text{-mod} \) consists of all finitely generated Gorenstein projective modules, and \( K^b(A\text{-Gproj}) \) the full subcategory of \( K^b(A\text{-mod}) \) consists of all complexes \( P^\bullet \) with \( P^i \in A\text{-Gproj} \) for all \( i \in \mathbb{Z} \).

Recall from [2] and [1] that an algebra \( A \) is said to be \textbf{CM-finite} if there are only finite indecomposable Gorenstein projective modules in \( A\text{-mod} \), which are \( G_1, G_2, \ldots, G_n \), up to isomorphism. Denote by \( G_A = \bigoplus_{i=1}^n G_i \) the Gorenstein projective generator in \( A\text{-mod} \), and \( \mathcal{GP}(A) := (\text{End}_A(G))^{\text{op}} \). Recall from [7] that an Artin algebra \( A \) is \textbf{Gorenstein} if \( \text{inj}\dim_A A < \infty \) and \( \text{inj}\dim_A A < \infty \).

A complex \( X^\bullet \in C^0(\mathcal{GP}(A)) \) is called \( G \)-acyclic, if \( \text{Hom}_A(G, X^\bullet) \) is acyclic for each \( G \in A\text{-Gproj} \). For a complex \( X^\bullet \in K^b(A\text{-mod}) \), \( G_{X^\bullet} \in K^-(A\text{-Gproj}) \) is called a Gorenstein projective resolution of \( X^\bullet \), if there exists a \( G \)-quasi-isomorphism \( f^\bullet : G_{X^\bullet} \longrightarrow X^\bullet \).

Put \( K^\ast_{\text{gpac}}(A\text{-mod}) := \{ X^\bullet \in K^\ast(A\text{-mod})| X^\bullet \text{ is } G\text{-acyclic} \} \) with \( \ast \in \{ \text{blank}, -, b \} \). Then \( K^\ast_{\text{gpac}}(A\text{-mod}) \) is a thick triangulated subcategory of \( K^\ast(A\text{-mod}) \). Following [3], the Verdier quotient \( D^\ast_{\text{gp}}(A\text{-mod}) := K^\ast(A\text{-mod})/K^\ast_{\text{gpac}}(A\text{-mod}) \), which is called the \textbf{Gorenstein derived category}. Recall from [8] that two algebras \( A \) and \( B \) are \textbf{Gorenstein derived equivalent}, if there exists a triangle equivalence between \( D^b_{\text{gp}}(A\text{-mod}) \) and \( D^b_{\text{gp}}(B\text{-mod}) \).

Let \( A \) and \( B \) be CM-finite Gorenstein Artin algebras. Recall from [3, Remark] that \( A \) and \( B \) are Gorenstein derived equivalent if and only if there exists a complex \( E^\bullet \in K^b(A\text{-Gproj}) \) such that (1) \( \mathcal{GP}(A) \cong \text{End}_A(E^\bullet) \); (2) \( \text{Hom}_{K^b(A\text{-Gproj})}(E^\bullet, E^\bullet[i]) = 0, \forall i \neq 0 \); and (3) \( \text{add}(E^\bullet) \) generates \( K^b(A\text{-Gproj}) \) as a triangulated category, where \( \text{add}(E^\bullet) \) is the full subcategory of \( K^b(A\text{-Gproj}) \) consisting of direct summands of finite direct sums of \( E^\bullet \). In the following, we call \( E^\bullet \) the Gorenstein silting complex.

2. Gorenstein stable equivalences

In this section, we prove that a Gorenstein derived equivalence of two CM-finite algebras is both the nonnegative functor and the uniformly bounded functor, and consequently, it can induce a Gorenstein stable equivalence when above algebras are Gorenstein.

\textbf{Lemma 2.1.} Let \( A \) and \( B \) be two CM-finite algebras such that \( G_A \) and \( G_B \) are the Gorenstein projective generators of \( A \) and \( B \), respectively. Assume that \( F : A \text{-mod} \rightarrow B \text{-mod} \) is a Gorenstein derived equivalence.
$D^b_{gr}(A{\text{-mod}}) \to D^b_{gr}(B{\text{-mod}})$ is the Gorenstein derived equivalence. Then $F(G_A)$ is isomorphic in $D^b_{gr}(B{\text{-mod}})$ to a complex $T^\bullet \in K^b(B{-}\text{Gproj})$ of the form

$$0 \to T^0 \to T^1 \to \ldots \to T^n \to 0$$

for some $n \geq 0$ if and only if $F^{-1}(G_B)$ is isomorphic in $D^b_{gr}(A{\text{-mod}})$ to a complex $T^\bullet \in K^b(A{-}\text{Gproj})$ of the form

$$0 \to T^{-n} \to \ldots \to T^{-1} \to T^0 \to 0.$$

Proof. ($\Rightarrow$) Note that

$$\text{Hom}_{K^b(A{-}\text{Gproj})}(G_A, T^\bullet[i]) \cong \text{Hom}_{K^b(B{-}\text{Gproj})}(T^\bullet, G_B[i]) = 0$$

for all $i > 0$. Then $H^n\text{Hom}_A(E, T^\bullet) = 0$ for all $n > 0$ and all $E \in A{-}\text{Gproj}$. This means that $T^\bullet$ splits in all positive degrees, i.e. the exact sequence

$$0 \to \text{Ker}(T^n \to T^{n+1}) \to T^n \to \text{Im}(T^n \to T^{n+1}) \to 0$$

splits for all positive $n$. Thus we may assume that $T^i = 0$ for all $i > 0$.

To prove that $T^\bullet$ is isomorphic to a complex in $K^b(A{-}\text{Gproj})$ with zero terms in all degrees $< -n$, it suffices to show that $\text{Hom}_{K^b(A{-}\text{Gproj})}(T^\bullet, E[i]) = 0$ for all $i > n$ and all $E \in A{-}\text{Gproj}$. Note that $F(E) \in \text{add}(T^\bullet)$, which is the smallest full subcategory of $K^b(B{-}\text{Gproj})$ closed under finite direct sums and direct summands, since $T^\bullet \cong F(G_A)$ and $G_A$ is the Gorenstein projective generator of $A$. Then we deduce that

$$\text{Hom}_{K^b(A{-}\text{Gproj})}(T^\bullet, E[i]) \cong \text{Hom}_{K^b(B{-}\text{Gproj})}(G_B, F(E)[i])$$

for all $i > n$.

($\Leftarrow$) The proof is similar to the above. \hfill $\square$

Let $A$ be an Artin algebra, let $Q : K(A{\text{-mod}}) \to D_{gr}(A{\text{-mod}})$ be the Verdier quotient functor. Consider the induced map

$$Q_{(X^\bullet, Y^\bullet)} : \text{Hom}_{K(A{\text{-mod}})}(X^\bullet, Y^\bullet) \to \text{Hom}_{D_{gr}(A{\text{-mod}})}(X^\bullet, Y^\bullet).$$

Define

$$\mathcal{U} Y^\bullet := \{ X^\bullet \in K(A{\text{-mod}}) \mid Q_{(X^\bullet, Y^\bullet[i])} \text{ is isomorphic for } i \leq 0, \text{ and is monic for } i = 1 \},$$

and for the full subcategory $X$ of $D_{gr}(A{\text{-mod}})$,

$$\mathcal{U} X := \{ Z^\bullet \in D_{gr}(A{\text{-mod}}) \mid \text{Hom}_{D_{gr}(A{\text{-mod}})}(Z^\bullet, X^\bullet[i]) = 0 \text{ for } i > 0 \text{ and } X^\bullet \in X \}. $$

$X^\perp$ is defined dually. Note that for full subcategories $X, Y$ of triangulated category $\mathcal{C}$, denote by

$$X^\perp Y := \{ Z \in \mathcal{C} \mid X \to Z \to Y \to X[1] \text{ is a triangle in } \mathcal{C} \text{ with } X \in X \text{ and } Y \in Y \}$$

**Lemma 2.2.** Let $A$ be an Artin algebra. Take $X \in A{\text{-mod}}$ and a bounded below complex $Y^\bullet$ over $A{-}\text{mod}$. Suppose that $Y^i \in X^\perp$ for all $i < m$. Then $X[i] \in \mathcal{U} Y^\bullet$ for all $i \geq -m$.

Proof. For $i \geq m$, we have $-m \geq -i$, and $X[-m] \in \mathcal{U} Y^\bullet[-i]$. For $i < m$, since $Y^i \in X^\perp$, we have $X[-m] \in \mathcal{U} Y^\bullet[-i]$. It follows that $X[-m] \in \mathcal{U} Y^\bullet[-i]$ for all $i \in \mathbb{Z}$. Note that there is some integer $n < m$ such that $Y^i = 0$ for all $i < n$, since $Y^\bullet$ is bounded below. Then $\sigma_{m+1} Y^\bullet$, the left brutal truncation of $Y^\bullet$ at degree $m+1$, is in $\{ Y^{m+1}[-m-1] \} \ast \ldots \ast \{ Y^m[-n] \}$, and thus $X[-m] \in \mathcal{U} \sigma_{m+1} Y^\bullet$. 


Now it is clear for all $i \leq 1$ that
\[ \text{Hom}_{K(A\text{-mod})}(X[-m], (\sigma_{>m+1}X^\bullet)[i]) = 0 \]
and so $Q_i(X[-m], (\sigma_{>m+1}Y^\bullet)[i])$ is an isomorphism for all $i \leq 1$. This establishes that $X[-m] \in \mathcal{U}_{\sigma_{>m+1}Y^\bullet}$. Since $Y^\bullet$ is in $\{\sigma_{>m+1}Y^\bullet\} \ast \{\sigma_{\leq m+1}Y^\bullet\}$, we deduce that $X[-m] \in \mathcal{U}_{Y^\bullet}$. Therefore, $X[i] \in \mathcal{U}_{Y^\bullet}$ for all $i \geq -m$. □

**Proposition 2.3.** Let $A$ be an Artin algebra and $X^\bullet$ and $Y^\bullet$ bounded above and bounded below complexes over $A$-mod, respectively. Suppose that $X^i \in \mathcal{U}_{Y^\bullet}$ for all integers $j < i$. Then $X^\bullet \in \mathcal{U}_{Y^\bullet}$.

**Proof.** We have $X^i[-j] \in \mathcal{U}_{Y^\bullet}$ for all $i \in \mathbb{Z}$ by Lemma 2.2. Note that there is an integer $n$ such that $X^0 = 0$ for all $i > n$, since $X^\bullet$ is bounded above. Thus for each integer $m < n$, the complex $\sigma_{\geq m}X^\bullet$ belongs to $\{X^m[-n]\} \ast \ldots \ast \{X^m[-m]\}$, and is consequently in $\mathcal{U}_{Y^\bullet}$. Taking $m$ to be sufficiently small such that $Y^j = 0$ for all $j < m + 1$. Then for each integer $i \leq 1$, both $\text{Hom}_{K(A)}(\sigma_{<m}X^\bullet, Y^\bullet[i])$ and $\text{Hom}_{D_{\text{proj}}(A)}(\sigma_{<m}X^\bullet, Y^\bullet[i])$ vanish. Hence $Q_i(\sigma_{<m}X^\bullet, Y^\bullet[i])$ is an isomorphism for all $i \leq 1$, and consequently $\sigma_{<m}X^\bullet \in \mathcal{U}_{Y^\bullet}$. Note that $X^\bullet \in \{\sigma_{\geq m}X^\bullet\} \ast \{\sigma_{<m}X^\bullet\}$. It follows that $X^\bullet \in \mathcal{U}_{Y^\bullet}$.

**Corollary 2.4.** Let $A$ be an Artin algebra and let $f : X \to Y$ be a homomorphism in $A$-mod. Suppose that $Z^\bullet$ is a bounded complex over $A$-mod such that $Z^i \in X^{\perp_{<0}}$ for all $i < 0$ and that $Z^i \in \mathcal{U}_{Y^\bullet}$ for all $i > 0$. If $f$ factors through $Z^\bullet$ in $D_{\text{proj}}(A)$, then $f$ factors through $Z^0$ in $A$-mod.

**Proof.** Suppose that $f = gh$ for $g \in \text{Hom}_{D_{\text{proj}}(A)}(X, Z^\bullet)$ and $h \in \text{Hom}_{D_{\text{proj}}(A)}(Z^\bullet, Y)$. By Proposition 2.3, both $g$ and $h$ can be presented by a chain map. Namely, $g = g^\bullet$ and $h = h^\bullet$ in $D_{\text{proj}}(A)$ for some chain maps $g^\bullet : X \to Z^\bullet$ and $h^\bullet : Z^\bullet \to Y$. Hence $f = g^\bullet h^\bullet = g^0 h^0$ in $D_{\text{proj}}(A)$, and consequently, $f = g^0 h^0$ since $A \to D_{\text{proj}}(A)$ is a fully faithful embedding. □

The following definition is a Gorenstein version of [4, Definition 4.1].

**Definition 2.5.** A triangle functor $F : D_{\text{proj}}^b(A)$-mod $\to D_{\text{proj}}^b(B)$-mod is called uniformly bounded if there are integers $r < s$ such that $F(X) \in D_{\text{proj}}^{[r,s]}(B)$-mod, i.e. $F(X)^i$ vanishes for all $i < r$ and $i > s$, for all $X \in A$-mod.

**Definition 2.6.** A triangle functor $F : D_{\text{proj}}^b(A)$-mod $\to D_{\text{proj}}^b(B)$-mod is called nonnegative if it satisfies the following conditions:

1. $H^i \text{Hom}_{D_{\text{proj}}^b(A)}(G_B, F(X)) = 0$ for all $i < 0$ and $X \in A$-mod;
2. $F(G)$ is isomorphic to a complex in $K^b(B$-Gproj) with zero terms in all negative degrees for all $G \in A$-Gproj.

**Proposition 2.7.** Let $F : D_{\text{proj}}^b(A)$-mod $\to D_{\text{proj}}^b(B)$-mod be a Gorenstein derived equivalence between two CM-finite algebras $A$ and $B$. Then

1. $F$ is uniformly bounded.
2. $F$ is nonnegative if and only if the Gorenstein tilting complex associated to $F$ is isomorphic in $K^b(A$-Gproj) to a complex with zero terms in all positive degrees.
Proof. By [3, Proposition 4.2] we see that the functor $F$ induces a triangle equivalence between $K^b(A\text{-Gproj})$ and $K^b(B\text{-Gproj})$. By Lemma 2.1, we may assume that $E^\bullet$ is a complex associated to $F$ such that $E^i = 0$ for all $i > 0$ and $i < -n$ and $F(E^\bullet) \cong G_B$.

(1) Let $X$ be in $A\text{-mod}$. Then for any integer $i$,
\[ H^i(\text{Hom}_B(G_B, F(X))) \cong \text{Hom}_{D^b_{gp}(B\text{-mod})}(G_B, F(X)[i]) \cong \text{Hom}_{K^b(A\text{-mod})}(E^\bullet, X[i]). \]
So $H^i(\text{Hom}_B(G_B, F(X))) = 0$ for all $i > n$ and $i < 0$. This proves that $F$ is uniformly bounded.

(2) By Lemma 2.1, $F(G_A)$ is isomorphic to a complex $E^\bullet \in K^{[0,n]}(B\text{-Gproj})$ for some nonnegative integer $n$. As an equivalence, $F$ preserves coproducts. This means that $F(G) \subseteq K^{[0,n]}(B\text{-Gproj})$ for any $G \in A\text{-Gproj}$. Let $X$ be an $A\text{-module}$. Since
\[ \text{Hom}_{D^b_{gp}(B\text{-mod})}(G_B, F(X)[i]) \cong \text{Hom}_{D^b_{gp}(A\text{-mod})}(E^\bullet, X[i]) = 0 \]
for all $i < 0$, it follows that $H^i(\text{Hom}_B(G_B, F(X))) = 0$ for all $X \in A\text{-mod}$ and all $i < 0$, i.e., $F$ is nonnegative.

Conversely, suppose that $F$ is a nonnegative Gorenstein derived equivalence. Then $F(G_A)$ is isomorphic to a bounded complex $Q^\bullet \in K^{\geq 0}(B\text{-Gproj})$. Note that for all $i > 0$,
\[ \text{Hom}_{K^b(A\text{-Gproj})}(G_A, E^\bullet[i]) \cong \text{Hom}_{K^b(B\text{-Gproj})}(F(G_A), G_B[i]) = 0. \]
This means that $H^i\text{Hom}_A(G, E^\bullet) = 0$ for all $G \in A\text{-Gproj}$ and $i > 0$. This shows that $E^\bullet$ splits in all positive degrees and thus is isomorphic to a complex in $K^b(A\text{-Gproj})$ with zero terms in all positive degrees. \hfill \qed

Lemma 2.8. Let $F: D^b_{gp}(A\text{-mod}) \rightarrow D^b_{gp}(B\text{-mod})$ be a uniformly bounded, nonnegative triangle functor. Suppose that there exists an integer $n > 0$ such that $F(X) \subseteq D^{[0,n]}_{gp}(B\text{-mod})$ for any $X \in A\text{-mod}$. Then the following statements hold.

(1) If $F$ admits a right adjoint $H$, then $H$ is uniformly bounded and $H(Y) \subseteq D^{[-n,0]}_{gp}(A\text{-mod})$.

(2) If $F$ admits a left adjoint $L$, then $L(Q) \in D^{[-n,0]}_{gp}(A\text{-mod})$ for any $Q \in B\text{-Gproj}$.

(3) If $H$ is both a left and right adjoint of $F$, then $H[-n]$ is uniformly bounded and nonnegative.

Proof. (1) Let $P$ be a Gorenstein projective $A\text{-module}$. Then for any $B\text{-module} Y$,
\[ \text{Hom}_{D^b_{gp}(A\text{-mod})}(P, H(Y)[i]) \cong \text{Hom}_{D^b_{gp}(B\text{-mod})}(F(P), Y[i]) = 0 \]
for all $i > 0$ and $i < -n$ since $F(P) \in K^{[0,n]}(B\text{-Gproj})$. It follows that $H(Y) \in D^{[-n,0]}_{gp}(A\text{-mod})$.

(2) Let $Q \in B\text{-Gproj}$ and $X \in A\text{-mod}$. Then
\[ \text{Hom}_{D^b_{gp}(A\text{-mod})}(L(Q), X[i]) \cong \text{Hom}_{D^b_{gp}(B\text{-mod})}(Q, F(X)[i]) = 0 \]
for all $i > n$ and $i < 0$. This implies that $L(Q) \in K^{[-n,0]}(A\text{-Gproj})$.

(3) It follows from (1) and (2) immediately. \hfill \qed
Lemma 2.9. Let $F : D^b_{gp}(A\text{-mod}) \to D^b_{gp}(B\text{-mod})$ be a uniformly bounded, non-negative triangle functor. Then for any $A$-module $X$, there is a triangle

$$U_X^* \xrightarrow{\iota_X} F(X) \xrightarrow{\pi_X} M_X \xrightarrow{\mu_X} U_X^*[1]$$

in $D^b_{gp}(B\text{-mod})$ with $M_X \in B\text{-mod}$ and $U_X^* \in D_{[1,n_X]}^{b}(B\text{-Gproj})$ for some $n_X > 0$.

Proof. Let $U_X^*$ be the Gorenstein projective resolution of $F(X)$, and then do good truncation of degree zero.

□

Lemma 2.10. Assume that $U^*_i \xrightarrow{\alpha_i} X^*_i \xrightarrow{\beta_i} M_i \xrightarrow{\gamma_i} U^*_i[1], i = 1, 2$, are triangles in $D^b_{gp}(B\text{-mod})$ such that $M_1, M_2$ are in $B\text{-mod}$ and $U^*_1, U^*_2 \in K^{[1,n]}(B\text{-Gproj})$. Then, for each morphism $f : X^*_1 \to X^*_2$ in $D^b_{gp}(B\text{-mod})$, there is a morphism $b : M_1 \to M_2$ in $B\text{-mod}$ and a morphism $a : U^*_1 \to U^*_2$ in $D^b_{gp}(B\text{-mod})$ such that the diagram

$$
\begin{array}{ccc}
U^*_1 & \xrightarrow{\alpha_1} & X^*_1 \\
\downarrow a & & \downarrow f \\
U^*_2 & \xrightarrow{\alpha_2} & X^*_2
\end{array} \quad \begin{array}{ccc}
M_1 & \xrightarrow{\beta_1} & U^*_1[1] \\
\downarrow b & & \downarrow a[1] \\
M_2 & \xrightarrow{\beta_2} & U^*_2[1]
\end{array}
$$

commutes. Moreover, if $f$ is an isomorphism in $D^b_{gp}(B\text{-mod})$, then $b$ is an isomorphism in $B\text{-mod}/B\text{-Gproj}$.

Proof. Since

$$\text{Hom}_{D^b_{gp}(B\text{-mod})}(U^*_1, M_2) \cong \text{Hom}_{K^{[1,n]}(B\text{-mod})}(U^*_1, M_2) = 0,$$

we see that $\alpha_1 f \beta_2$ is zero, and so $a$ and $b$ exist.

Now assume that $f$ is an isomorphism in $D^b_{gp}(B\text{-mod})$. Namely, there is an isomorphism $g : X^*_1 \to X^*_2$ in $D^b_{gp}(B\text{-mod})$ such that $f g = \text{Id}_{X^*_2}$ and $g f = \text{Id}_{X^*_1}$. By the above similar discussion, there is a morphism $c : M_2 \to M_1$, such that $c \beta_2 = \beta_1 g$. Then

$$\beta_1 - c b \beta_1 = \beta_1 - c \beta_2 f = \beta_1 - \beta_1 g f = 0$$

and $\text{Id}_{M_1} - c b$ factors through $U^*_1[1]$. Then $\text{Id}_{M_1} - c b$ factors through a Gorenstein projective $B$-module, and hence $c b = \text{Id}_{M_1}$ in $B\text{-mod}/B\text{-Gproj}$. Similarly we have $b c = \text{Id}_{M_2}$, and therefore $b : M_1 \to M_2$ is an isomorphism in $B\text{-mod}/B\text{-Gproj}$. □

Next we define the functor $\overline{F} : A\text{-mod}/A\text{-Gproj} \to B\text{-mod}/B\text{-Gproj}$ as follows. We fix a triangle

$$\xi_X : U_X^* \xrightarrow{\iota_X} F(X) \xrightarrow{\pi_X} M_X \xrightarrow{\mu_X} U_X^*[1]$$

in $D^b_{gp}(B\text{-mod})$ with $M_X \in B\text{-mod}$ and $U_X^*$ a complex in $K^{[1,n_X]}(B\text{-Gproj})$ for some $n_X > 0$. For each morphism $f : X \to Y$ in $A\text{-mod}$, we can form the following diagram in $D^b_{gp}(B\text{-mod})$:

$$
\begin{array}{ccc}
U_X^* & \xrightarrow{\iota_X} & F(X) \\
\downarrow a_f & & \downarrow F(f) \\
U_Y^* & \xrightarrow{\iota_Y} & F(Y)
\end{array} \quad \begin{array}{ccc}
M_X & \xrightarrow{\mu_X} & U_X^*[1] \\
\downarrow b_f & & \downarrow a_f[1] \\
M_Y & \xrightarrow{\mu_Y} & U_Y^*[1]
\end{array}
$$

If $b_f$ is another morphism such that $b_f \pi_X = \pi_Y F(f)$, then $(b_f - b'_f) \pi_X = 0$ and $b_f - b'_f$ factors through $U^*_1$ which is Gorenstein projective, so $b_f = b'_f \in \text{Hom}_{B\text{-mod}/B\text{-Gproj}}(M_X, M_Y)$. Moreover, if $f$ factors through a Gorenstein projective $A$-module $P$, say $f = h g$ for $g : X \to P$ and $h : P \to Y$. Then
(b_f - b_h b_g)\pi_X = \pi_Y F(f) - \pi_Y F(h)F(g) = 0. Hence b_f - b_h b_g factors through \text{U}^1_X, and then b_f factors through P \oplus \text{U}^1_X which is Gorenstein projective. Hence b_f = 0. Thus we get a well-defined map:

$$\phi : \text{Hom}_{A\text{-mod}/A\text{-Gproj}}(X, Y) \to \text{Hom}_{B\text{-mod}/B\text{-Gproj}}(MX, MY), \ f \mapsto b_f.$$ 

Define \(F := MX\text{ for each }X \in A\text{-mod}/A\text{-Gproj\ and }F(f) := \phi(f)\text{ for each morphism }f\text{ in }A\text{-mod},\) we get a functor \(F : A\text{-mod}/A\text{-Gproj} \to B\text{-mod}/B\text{-Gproj}.)

**Theorem 2.11.** Let \(A\) and \(B\) be CM-finite algebras, and \(F : D^b_{\text{sgp}}(A\text{-mod}) \to D^b_{\text{sgp}}(B\text{-mod})\) be a Gorenstein derived equivalence with the quasi-inverse \(G.\) Then the induced stable functors

\(\text{\(F : A\text{-mod}/A\text{-Gproj} \to B\text{-mod}/B\text{-Gproj}\)}\)

and

\(\text{\(G : B\text{-mod}/B\text{-Gproj} \to A\text{-mod}/A\text{-Gproj}\})\)

satisfy

\(\text{\(F \circ G[-n] \cong [-n] \cong \Omega^n_B,\text{ and } G[-n] \circ F \cong [-n] \cong \Omega^n_A,\text{ where }\Omega^n_A\text{ and }\Omega^n_B\text{ are the }n\text{-th syzygy functor.}}\)

Consequently, if furthermore \(A\) and \(B\) are Gorenstein algebras, then \(A\) and \(B\) are Gorenstein stably equivalent.

**Proof.** Since \(G\) is both a left adjoint and right adjoint of \(F,\) there exists some integer \(n > 0\) such that \(G[-n]\) is nonnegative by Lemma 2.8. Moreover, \(F[n]\) is a right adjoint of \(G[-n],\) and sends Gorenstein projective \(A\text{-modules to complexes in }K^b(B\text{-Gproj}).\)

Note that there is the following diagram of functors as follows:

\[
\begin{array}{ccc}
A\text{-mod}/A\text{-Gproj} & \xrightarrow{\text{can}} & D^b_{\text{sgp}}(A\text{-mod})/K^b(A\text{-Gproj}) \\
\text{\(F \downarrow\)} & & \text{\(F \downarrow\)} \\
B\text{-mod}/B\text{-Gproj} & \xrightarrow{\text{can}} & D^b_{\text{sgp}}(B\text{-mod})/K^b(B\text{-Gproj})
\end{array}
\]

Thus, we have isomorphisms of functors

\(\text{\(F \circ G[-n] \cong [-n] \cong \Omega^n_B,\text{ and } G[-n] \circ F \cong [-n] \cong \Omega^n_A.\)}\)

If \(A\) and \(B\) are Gorenstein, then from above equalities \(A\) and \(B\) are Gorenstein stably equivalent. \(\square\)

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