Integrability properties of minimal surfaces in hyperbolic space

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Abstract. We present a method, based on integrability, to find minimal area surfaces in hyperbolic space ending on a given contour at the boundary. The problem has physical interest since the AdS/CFT correspondence relates the area of the minimal surface to the expectation value of the Wilson loop defined by the boundary contour. We give particular solutions using the Mathieu functions and more general results using a numerical method.

1. Introduction
According to the AdS/CFT correspondence [1, 2, 3], Wilson loops in $\mathcal{N} = 4$ SYM can be calculated holographically from the area of the minimal surface ending on the boundary of $AdS_5$ [4, 5] at a contour defined by the Wilson loop. Much work has been done to calculate Wilson loops of different shapes. In Euclidean signature, examples include the circular Wilson loop [6, 7], wavy lines [8, 9] and solutions constructed using Riemann-theta function [10, 11]. In Minkowski signature, the most interesting example is the light-like cusp [12] due to its relation to scattering amplitudes [13, 14, 15]. In this paper, we study the integrability structure of a general smooth Wilson loop on the boundary of Euclidean $AdS_3$ using the Pohlmeyer reduction [16]. We construct analytical solutions in terms of Mathieu functions and also provide a numerical recipe for studying a Wilson loop of a general shape.

The paper is organized as follows: in section 2, we review the general setup for studying minimal surfaces in Euclidean $AdS_3$. Section 3 describes the Schrödinger equation defined on the boundary contour. In section 4, we construct analytical solutions using Mathieu functions. In section 5 we briefly describe a method to find the contour and area associated with a given analytic function $f(z)$ appearing in the Pohlmeyer reduction. Finally, we provide a numerical method for solving a general boundary contour in section 6 where we also give some results we found by implementing such numerical method.

2. General setup
Euclidean $AdS_3$ is embedded as a hyperboloid in $\mathbb{R}^{1,3}$:

$$-X_0^2 + X_1^2 + X_2^2 + X_3^2 = -1.$$  (1)

The Poincaré coordinates are given by

$$Z = \frac{1}{X_0 - X_3}, \quad X = \frac{X_1 + iX_2}{X_0 - X_3}, \quad \bar{X} = \frac{X_1 - iX_2}{X_0 - X_3},$$  (2)
with the metric
\[ ds^2 = \frac{dZ^2 + dX d\bar{X}}{Z^2}. \] (3)

The string worldsheet can be conformally mapped to the unit disk \((|z| \leq 1)\) on the complex plane \(z = \sigma + i\tau = re^{i\theta}\) so the Poincaré coordinates are parametrized as \(X(r, \theta)\) and \(Z(r, \theta)\). Near the boundary, i.e., \(r \to 1\), one has
\[ Z(r = 1, \theta) = 0, \quad X(r = 1, \theta) = X(s(\theta)). \] (4)

\(X(s)\) is a given contour defined on the boundary of Euclidean \(AdS_3\) parametrized by \(s\), which is related to the conformal angle \(\theta\) by an unknown reparametrization \(s(\theta)\).

The string action is
\[ S = \frac{1}{2} \int d\sigma d\tau (\partial X_\mu \bar{\partial} X^\mu + \Lambda(X_\mu X^\mu + 1)), \] (5)
where \(\Lambda\) is a Lagrange multiplier. The classical string solution satisfies the equation of motion
\[ \partial \bar{\partial} X - \Lambda X = 0, \quad \Lambda = \bar{\partial} X \cdot \partial X. \] (6)

This should be supplemented by the Virasoro constraints
\[ \bar{\partial} X \cdot \partial X = 0 = \partial X \cdot \bar{\partial} X. \] (7)

Writing \(X = X_0 + X_i \sigma^i\), the embedding condition, the equation of motion and the Virasoro constraints become
\[ \det X = 1, \quad \bar{\partial} \bar{\partial} \bar{X} = \Lambda \bar{X}, \quad \det(\bar{\partial} \bar{X}) = \det(\partial X) = 0, \] (8)
where we have the reality condition \(X^\dagger = \bar{X}\). This condition can be solved by writing
\[ X = \bar{\mathbb{A}} \mathbb{A}^\dagger, \] (9)
with
\[ \det \mathbb{A} = 1, \quad \mathbb{A} \in SL(2, \mathbb{C}). \] (10)

The matrix \(\mathbb{A}\) is the solution to the linear problem
\[ \partial \mathbb{A} = \mathbb{A} J, \quad \bar{\partial} \mathbb{A} = \mathbb{A} \bar{J}, \] (11)
where
\[ J = \begin{pmatrix} -\frac{1}{2} \partial \alpha & f e^{-\alpha} \\ \lambda e^{\alpha} & \frac{1}{2} \partial \alpha \end{pmatrix}, \quad \bar{J} = \begin{pmatrix} \frac{1}{2} \bar{\partial} \alpha & \frac{1}{\lambda} e^{\alpha} \\ -\bar{f} e^{-\alpha} & -\frac{1}{2} \bar{\partial} \alpha \end{pmatrix}. \] (12)

The consistency condition requires \(f\) to be holomorphic and \(\bar{f}\) anti-holomorphic, and \(\alpha(z, \bar{z})\) to satisfy the generalized cosh-Gordon equation:
\[ \partial \bar{\partial} \alpha = e^{2\alpha} + f \bar{f} e^{-2\alpha}. \] (13)

If \(|\lambda| = 1\) in \(J\) and \(\bar{J}\), one obtains a family of minimal surfaces ending on different boundary contours with the same area. This reflects the integrability of the problem and has been studied recently in [17, 18].

The metric induced on the minimal surface is
\[ ds^2 = 4e^{2\alpha} dz d\bar{z} \] (14)
so the area is calculated using
\[ A = 4 \int e^{2\alpha} d\sigma d\tau. \] (15)
After regularization, the finite area is
\[ A_f = -2\pi - \int f \bar{f} e^{2\alpha} d\sigma d\tau. \] (16)
3. Linear problem on the boundary

Near the boundary, $\alpha(z, \bar{z})$ has the expansion

$$
\alpha(\xi, \theta) \simeq -\ln \xi + \beta_2(\theta)(1 + \xi)\xi^2 + O(\xi^4),
$$

(17)

where $\xi = 1 - r^2$ and $\beta_2(\theta)$ can be defined as

$$
\beta_2(\theta) = \frac{1}{6} e^{2i\theta} \left( \partial^2 \alpha - (\partial \alpha)^2 \right)|_{r \to 1}.
$$

(18)

All the higher order coefficients in (17) are fixed by $\beta_2(\theta)$ and $f(\theta) = f(e^{i\theta})$.

The solution to the linear problem (11) can be written in terms of two linearly independent row vectors as

$$
\mathbf{A} = \left( \begin{array}{cc}
\psi_1 & \tilde{\psi}_1 \\
\psi_2 & \tilde{\psi}_2
\end{array} \right).
$$

(19)

Then the linear equations (11) are reduced to

$$
\partial \psi = \psi J, \quad \bar{\partial} \psi = \psi \bar{J},
$$

(20)

and the same equations for $\tilde{\psi}$.

Taking this linear problem to the boundary, one gets a Schrödinger-like equation:\[19\]:

$$
-\partial^2_\theta \chi(\theta) + V^\lambda(\theta) \chi(\theta) = 0
$$

(21)

where the potential is given by the Schwarzian derivative of the contour $X^\lambda(\theta)$:

$$
V^\lambda(\theta) = -\frac{1}{2} \{X^\lambda(\theta), \theta\} = -\frac{1}{4} + 6\beta_2(\theta) + \lambda f(\theta)e^{2i\theta} - \frac{1}{\lambda} \tilde{f}(\theta)e^{-2i\theta},
$$

(22)

and the contour $X^\lambda(\theta)$ can be constructed with two anti-periodic linearly independent solutions as

$$
X^\lambda(\theta) = \frac{\tilde{\chi}^\lambda}{\chi^\lambda}.
$$

(23)

Therefore, knowing the original boundary contour in terms of the conformal angle $X(\theta)$, one can calculate the Schwarzian derivative $\{X(\theta), \theta\}$. The $\lambda$-deformed contours can be easily obtained by solving eq.(21) with the deformed potential according to eq.(22). Furthermore, one can extract $f(\theta)$ from the imaginary part of $\{X(\theta), \theta\}$, analytically continue it inside the unit disk, and plug it into eq.(13) to solve for $\alpha(z, \bar{z})$. With $f$, $\tilde{f}$ and $\alpha$, the area can be calculated using eq.(16). It is worth mentioning that in the case where $f(z)$ has no zeros inside the unit disk, the area can be written in terms of the Schwarzian derivative

$$
A_f = -2\pi \pm \frac{i}{2} \oint d\theta \frac{\text{Re}\{X(\theta), \theta\} - \{\omega, \theta\}}{\partial_\theta \ln \omega},
$$

(24)

with

$$
\omega(\theta) = i \int^\theta d\theta' \sqrt{f(\theta')e^{2i\theta'}}.
$$

(25)

This area formula is manifestly global conformal invariant and does not require solving eq. (13).
4. Mathieu solutions

From the previous section, we see that the minimal surfaces ending on a boundary contour can be studied by considering a Schrödinger equation with periodic potential and the contour can be constructed from two anti-periodic linearly independent solutions. Given a boundary contour $X(s)$, one can rewrite the Schrödinger equation using the parameter $s$

$$-\partial_s^2 \chi + V(\lambda, s) \chi = 0,$$

(26)

with $V(\lambda = 1, s) = -\frac{1}{2}\{X(s), s\}$. The prototypical Schrödinger equation with periodic potential is the Mathieu equation

$$\partial_u^2 \chi(u) + (a - 2q \cos 2u) \chi(u) = 0.$$

(27)

Here $q$ is a parameter and $a$ is the energy eigenvalue. Writing the potential as

$$V(\lambda, s) = -\frac{1}{4} + 6\beta_2 - \lambda f_0 e^{i(n+2)s} + \frac{f_0}{\lambda} e^{-i(n+2)s},$$

(28)

with $\lambda = e^{i\phi}$, the Schrödinger equation (26) becomes the Mathieu equation if we make the following identifications [20]:

$$a = \frac{1 - 24\beta_2}{(n+2)^2}, \quad q = \frac{4if_0}{(n+2)^2},$$

(29)

with the change of variable

$$u(s) = \frac{(n+2)s + \phi}{2} + \frac{\pi}{4}.$$

(30)

We look for Floquet solutions of Mathieu equation with characteristic exponent $\nu$

$$\chi_{\nu}(u + \pi) = e^{i\pi\nu} \chi_{\nu}(u),$$

$$\chi_{\nu}(u) = e^{ivu} p_{\nu}(q, u).$$

(31)

To guarantee the anti-periodicity of the solution, we take $\nu = \frac{2k+1}{n+2}$, with $k \in \mathbb{Z}$. The eigenvalue $a_{\nu}(q)$ can be determined given $\nu$ and the constant $\beta_2$ has the form

$$\beta_2 = \frac{1}{24}(1 - (n+2)^2 a_{\nu}(q)).$$

(32)

The boundary contour can be found from

$$X(u) = \frac{\chi_{\nu}(u)}{\chi_{\nu}(-u)} = e^{2i\nu u} \frac{p_{\nu}(q, u)}{p_{\nu}(q, -u)},$$

(33)

and the area is given by

$$A_f = -2\pi + \frac{\pi}{n+2} - (n+2)\pi a_{\nu}(q).$$

(34)

It should be noted that the case $n = 0$ has already been studied by J. Toledo in [21] using a different method. The present construction allows for a large class of new solutions where the contours are given in terms of Mathieu functions, the area by the eigenvalues of the Mathieu equation, and the shape of the surface by a numerical solution to the linear problem (11). An example is given in fig.1.
5. Numerical method for solving the cosh-Gordon equation

In the next section we describe a numerical method to solve the problem of finding the reparameterization $s(\theta)$ necessary to obtain the minimal area surface ending on a given contour. As a preliminary step it is necessary to implement a method to solve the cosh-Gordon equation (13) for a given function $f(z)$. This is useful in its own right since we can propose an arbitrary $f(z)$ and obtain the contour on which the surface ends together with the area of the surface. The equation can be solved by a simple collocation method as described in [22]. As an example, in figure 2, we plot the contour associated with the function

$$f(z) = 2(z - 0.3)(z - \frac{i}{2})(z + 0.7)(z + \frac{i}{2})$$

We just chose a polynomial function with four zeros inside the unit disk at arbitrary positions.

Figure 2: Contour associated with the function $f(z) = 2(z - 0.3)(z - \frac{i}{2})(z + 0.7)(z + \frac{i}{2})$. The area of the corresponding minimal surface is $A_f = -6.99563$. 

Figure 1: Example of Mathieu surface with $n = 2$. The area is $A_f = -6.660397$.
6. Numerical method for finding $s(\theta)$
For a general given contour $X(s)$, the problem remains of how to find the conformal reparametrization $s(\theta)$. Using the property of the Schwarzian derivative
\begin{equation}
\{F, \theta\} = \{s, \theta\} + (\partial_\theta s)^2 \{F, s\},
\end{equation}
eq. (22) becomes
\begin{align}
\{s, \theta\} + (\partial_\theta s)^2 &\text{Re}\{X(s), s\} = \frac{1}{2} - 12\beta_2(\theta), \\
(\partial_\theta s)^2 &\text{Im}\{X(s), s\} = -4\text{Im}(e^{2\theta}f(\theta)).
\end{align}
(37)
For a general contour, one can propose an arbitrary reparametrization $s(\theta)$, extract a $f(\theta)$ from the second equation in (37), analytically continue inside the unit disk to solve for $\alpha(z, \bar{z})$ and calculate $\beta_2(\theta)$ according to eq. (18), which we call $\tilde{\beta}_2(\theta)$. One can then compare this $\tilde{\beta}_2$ with the $\beta_2$ calculated from the first equation in (37). If the $s(\theta)$ is the conformal reparametrization, then we should have $\tilde{\beta}_2 = \beta_2$. We therefore perform a minimization procedure on the difference between $\tilde{\beta}_2$ and $\beta_2$ to search for the correct $s(\theta)$. For details of the numerics, see [23]. The procedure is summarized in Fig.3.

This method was used in [9] to analytically study the perturbations from the circular contour. We perform the numerical implementation which does not depend on a perturbative expansion, to study various contours and find their conformal reparametrization $s(\theta)$ and area beyond the regime where the perturbative expansion converges. In Fig. 4, we plot the calculated area compared to the perturbative calculation for symmetric contours
\begin{equation}
X(s) = e^{is + \alpha \sin(ps)}.
\end{equation}
(38)
Figure 4: The areas of the symmetric contours with $p = 2$ and $p = 13$. We plot a few partial sums (continuous curves) of the perturbative calculations as well as the results from our calculation (dots), which goes beyond the range of $a$ where the perturbative series converges.

Once the reparametrization is found, it is easy to perform the $\lambda$-deformation of the contour by solving eq. (21). In Figs. 5 and 6, we plot the $\lambda$-deformation for symmetric contours with $p = 2$ and $p = 13$.

Figure 5: The $\lambda$-deformation of symmetric contour with $p = 2$ and $a = 1$. From the left to the right we take $\lambda = 1, e^{i\frac{\pi}{4}}, e^{i\frac{\pi}{2}}, e^{i\frac{3\pi}{4}}, e^{i\pi}$.

Figure 6: The $\lambda$-deformation of symmetric contour with $p = 13$ and $a = 0.12$. From the left to the right we take $\lambda = 1, e^{i\frac{\pi}{2}}, e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{4}}, e^{i\pi}$.

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