Photoinduced tunable anomalous Hall and Nernst effects in multi-Weyl semimetals

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We discuss the circularly polarized light (of amplitude $A_0$ and frequency $\omega$) driven thermal transport properties of type-I and type-II multi-Weyl semimetals (mWSMs) in the high frequency limit. Considering the low energy model, we employ the Floquet-Kubo formalism to compute the thermal Hall and Nernst conductivities for both types of mWSMs. We show that the anisotropic nature of the dispersion for arbitrary integer monopole charge $n > 1$ plays an important role in determining the effective Fermi surface behavior; interestingly, one can observe momentum dependent corrections in Floquet mWSMs in addition to momentum independent contribution as observed for Floquet single WSMs. Apart from the non-trivial tuning of the Weyl node separation $Q \rightarrow Q + A_0^n/\omega$, our study reveals that the momentum independent terms result in leading order contribution in the conductivity tensor which has the form of $n \times$ the single WSMs results with $\mu \rightarrow \mu - A_0^n/\omega$. On the other hand, momentum dependent correction leads to sub-leading order term which are algebraic function of $\mu$ as $\mu^{(n)}$ and can only be present for $n > 1$. Remarkably, this further allows us to distinguish type-I mWSMs from their type-II counterparts. For type-II mWSMs, we find that the transport coefficients for $n \geq 2$ exhibit algebraic dependences on the momentum cutoff in addition to the weak logarithmic dependence as observed for $n = 1$ WSMs. We demonstrate the variation and qualitative differences of transport coefficients as a function of external driving parameter $\omega$.

I. INTRODUCTION

Recent years have witnessed Weyl Semimetals (WSMs) as a focus of research attraction due to their exotic properties. The upsurge of recent attention on this new class of quantum materials is due to its unusual Fermi arc surface states and chiral anomaly that is intimately related to topological order\textsuperscript{1,2}. In WSMs, the bulk band gap closes at an even number of discrete points in the Brillouin zone, these special gap closing points, protected by some crystalline symmetry, are referred as Weyl nodes\textsuperscript{3}. Weyl nodes act as monopoles or anti-monopoles of Berry curvature characterized by monopole charge $n$. Weyl nodes are located at different momenta for time reversal broken system while inversion symmetry breaking leads to the energy separated Weyl points\textsuperscript{4}. The existence of Fermi arc surface states, chiral-anomaly related negative magnetoresistance, the quantum anomalous Hall effect are the direct consequences of topological nature of WSMs\textsuperscript{5-7}. As compared to the conventional Weyl semimetals with $n = 1$, reported in TaP, TaAs, NbAs\textsuperscript{8-10}, it has been recently shown that in general $n$ can be greater than one, with the crystalline symmetries bounding its maximum value to three\textsuperscript{11-13}. These are called multi WSMs (mWSMs); interestingly, the single-WSM with $n = 1$ can be considered as 3D analogue of graphene whereas the double-WSM (triple-WSM) with $n = 2$ ($n = 3$) can be represented as 3D counterparts of bilayer (ABC-stacked trilayer) graphene\textsuperscript{14-16}. Close to the Weyl points, mWSMs host low-energy quasiparticles with the dispersion which is, in general, linear only in one direction signifying anomalous features in the transport properties\textsuperscript{17-23}. Ideal WSM has a conical spectrum and a point-like Fermi surface at the Weyl point. An interesting situation arises when large tilting of the Weyl cones results in a Lifshitz transition. This leads to a new class of materials called type-II WSMs, where the Fermi surface is no longer point-like\textsuperscript{2,24-28}. The existence of type-II WSM has been experimentally demonstrated\textsuperscript{29,30} while theoretical prediction shows that type II WSM can be engineered by applying strain or chemical doping to the original type I WSM\textsuperscript{31}. The type II WSM is characterized by a different class of Weyl fermion manifesting the violation of the Lorentz symmetry. The type II WSM can yield intriguing electronic transport properties due to a markedly different density of states at the Fermi level\textsuperscript{32-35}. In addition to electric transport, thermal responses also carry signatures of the exotic physics of WSMs which have been studied theoretically\textsuperscript{36-39} and experimentally\textsuperscript{40-43}. At the same time, optical conductivity of WSMs have been extensively studied along with other characteristic signatures\textsuperscript{44-49}. While a lot of progress has been made experimentally and theoretically in identifying $n = 1$ type-I and type-II WSMs, the experimental discovery of mWSMs with $n \geq 2$ is yet to be made; however, using density functional theory calculation is conjectured to host Weyl nodes with monopole charges $n = 2, 3$\textsuperscript{11,50-52}. This further motivates the theoretical search for finding additional tools to identify these exotic phases with higher monopole charge. For example, mWSMs, in general, can exhibit a smoothly deformed conical spectrum and a point-like Fermi surface at the Weyl point. Interestingly, for type II mWSM these features are expected to change and might lead to distinct transport behavior compared to type I mWSM.
On the other hand periodically driven Floquet systems, where the static Hamiltonian is perturbed with a time-periodic drive, have attracted a lot of interest recently. Floquet systems can host unique phases which have no counterparts in equilibrium systems, such as anomalous Floquet topological phases, dynamical freezing, many-body energy localization, dynamical llocalisation, and Floquet higher order topological phases and dynamical generation of edge Majorana modes. It has been shown that circularly polarized light can be employed to switch between Weyl semimetal, Dirac semimetal and topological insulator states in a prototypical three-dimensional (3D) Dirac material Na$_3$Bi. In general, D.C. transport is expected to be drastically modified under such irradiation. Interestingly, linearly polarized light can lead a band insulator to a WSM phase where the relative separation of Weyl points can be controlled: similarly, circularly polarized light drives a nodal line semimetals into Weyl semimetals.

In the high frequency driving limit, the system does not absorb energy via electronic transitions, resulting in a non-equilibrium steady state. At this limit an effective static Sambe space Hamiltonian picture successfully describes the non-trivial outcome.

Given these background on the generation and optical manipulation of Weyl nodes, our aim here is to study the thermo-electric transport properties of mWSMs when it is driven by a circularly polarized source in the high frequency limit. A recent study using non-equilibrium Kubo formalism has revealed that the thermo-electric response of type-I WSMs can be distinguished from type-II WSMs under the application of light. Motivated by this study, the question we ask here is “how do the thermal Hall conductivity and Nernst conductivity of the type-I phase differ qualitatively and quantitatively from the type-II phase when the underlying Weyl Hamiltonian supports higher topological charge $n > 1$?" We find for $n > 1$ ($n = 1$) that the position of the Weyl point can be tuned in a non-trivial (trivial) manner and the Fermi surface gets renormalized with both momentum dependent and independent (only momentum independent) terms. These additional interesting features in $n > 1$ heavily influence the subsequent transport. The momentum independent term gives a single Weyl results for conductivity tensor in its leading order while momentum dependent terms can lead to sub-leading correction in the conductivity tensor. Our study further suggests that the vacuum contribution becomes cut-off dependent, unlike the $n = 1$ case, due to the coupling of the $U(1)$ gauge field to the anisotropic dispersion, containing higher momentum modes. In addition to the logarithmic cut-off dependence in Fermi surface contribution for type-II $n = 1$ WSM, we find a rather strong algebraic cut-off dependence for $n \geq 2$. Interestingly, the Fermi surface contribution for type-I mWSM continues to show a cut-off independent response, similar to the observation for $n = 1$ mWSMs. The type I mWSM behaves in a dissimilar manner as compared to type II mWSM as far as the chemical potential behavior is concerned; this is very clearly visible when the Nernst conductivity is investigated.

The paper is organized as follows: Sec. II discusses the equilibrium and non-equilibrium low-energy model Hamiltonian and compares it with the single WSM case. We then study the Berry curvature and anomalous Hall conductivity in detail in Sec. III. Next in Sec. IV, we present our analytical results for optical conductivity using Floquet-Kubo formalism. We here extensively analyze the vacuum and fermi surface contribution and pictorially represent the distinctive behavior of type I and type II mWSM. This section is adequately supplemented in Appendix A and B. We finally conclude in Sec. V.

\section{Effective Floquet Hamiltonian}

The low energy Hamiltonian for a multinode WSM of monopole charge $n$ near each Weyl point is given by

\begin{equation}
H^e_k = \hbar C_s(k_z - sQ) + sh\alpha_n \sigma \cdot (n_k - se) \tag{1}
\end{equation}

Here, $s = \pm$ indicates the chirality of nodes, $n_k = [k^x_s \cos(n\phi_k), k^y_s \sin(n\phi_k), \frac{s k^z_s}{\sin(\phi_k)}]$, $e = (0, 0, Q)$, and $Q$ is the separation between two Weyl nodes. We define the $x-y$ plane azimuthal angle $\phi_k = \arctan(\frac{k^y_s}{k^x_s})$, and the in-plane momentum $k_\perp = \sqrt{k^2_x + k^2_y}$. The Hamiltonian (1) represents the two Weyl nodes separated by a distance $2Q$, while $C_s$ indicates the tilt parameter associated with $s$ Weyl node. Type-I mWSMs corresponds to $|C_s|/v \ll 1$ while for type-II mWSMs we have $|C_s|/v \gg 1$. We restrict to the inversion symmetric tilt given by $sC_s = C$. We cast the above Hamiltonian in matrix notation:

\begin{equation}
H^e_k = \begin{bmatrix}
\hbar C_s(k_z - sQ) + sh\alpha_n(k_z - sQ) \\
h\alpha_n(k_x - ik_y)^n \\
hC_s(k_z - sQ) - sh\alpha_n(k_x - ik_y)^n
\end{bmatrix} \tag{2}
\end{equation}

Hereafter, we use Natural units and set $\hbar = c = k_B = 1$. We now examine the effect of circularly polarized light on the mWSM. Under the influence of a periodic optical driving with electric field of frequency
\( \omega, \mathbf{E}(t) = E_0(-\cos \omega t, \sin \omega t, 0) \), the Hamiltonian transforms via the Pierel’s substitution \( k_x \to k_x' = k_x - A_0 \sin \omega t \), where the vector potential is given by \( \mathbf{A}(t) = \frac{E_0}{\omega} (\sin \omega t, \cos \omega t, 0) \), in the Landau gauge. The gauge dependent momenta transform as \( k_x \to k_x' = k_x - A_0 \sin \omega t, k_x \to k_x'' = k_y + A_0 \cos \omega t \), and \( k_x \to k_x''' = k_z \). The driving amplitude of the vector potential is related to the amplitude of the electric field by \( A_0 = \frac{E_0}{\omega} \). Considering the fact that

\[ (k_x' \pm ik_y')^n = \sum_{m=1}^{n} (k_x e^{\pm i\theta})^{n-m} (A_0)^m e^{\pm i m(k \cdot \mathbf{A} - \omega t)} nC_m, \]

where \( nC_m = (\frac{n!}{m!(n-m)!}) \) represents the combinatorial operator, the time dependent Hamiltonian takes the form

\[ H_k^F(\mathbf{A}, t) = s\sigma_+ (k_x' + ik_y')^n + s\sigma_- (k_x' - ik_y')^n + C(k_z - sQ) + i (k_z - sQ) \sigma_z, \]

where \( s\sigma_+ \) and \( s\sigma_- \) are given by

\[ s\sigma_+ = \begin{pmatrix} 0 & e^{-i[(n-m)\phi + m\theta]}/\delta_{m,n} \\ e^{i[(n-m)\phi - m\theta]}/\delta_{m,n} & 0 \end{pmatrix} \]

Solving the problem with a time-dependent potential may be out of the reach of analytical tractability. Instead, we resort to using Floquet’s theorem and the extracting the sub-leading order term in the high frequency expansion (HFE, van-Vleck expansion), to obtain a closed form expression for the effective Hamiltonian \( H_k^F \). In this limit, one can describe the dynamics of the driven system over a period \( T \) in terms of the effective Floquet Hamiltonian: \( H_k^F \). The contribution to order \( 1/\omega \) in the HFE is given by \( H_k^F \approx H_k^F + V_k^s \), where \( V_k^s \) represents perturbative driving term. We restrict to contributions of order \( 1/\omega \) throughout the manuscript, and the form of \( V_k^s \) is given by

\[ V_k^s = \sum_{p=1}^{\infty} \frac{|V_{p-1}^s V_p^s|}{p\omega}, \]

with \( V_p = \frac{1}{T} \int_0^T H_k^F(\mathbf{A}, t) e^{i\omega t} dt \) and \( \omega = \frac{2\pi}{T} \). Evaluating \( V_k^s \) for our system, we arrive at

\[ V_p = s\alpha_0 \sum_{m=1}^{n} (k_\perp)^{n-m} (-A_0)^m nC_m \]

\[ = \begin{cases} 0 & \text{if } m = n \\ e^{-i[(n-m)\phi + m\theta]}/\delta_{m,n} & \text{otherwise} \end{cases} \]

Using the result in (5) and evaluating the commutator in (4), we find that the effective Floquet Hamiltonian takes the form

\[ H_k^F = H_k^F + V_k^s \]

\[ = C_s (k_z - sQ) + s\alpha_0 s\sigma_+ (\mathbf{n}_k - sQ\hat{e}_z) \]

\[ + \frac{\alpha_0^2}{\omega} \sum_{p=1}^{n} \frac{1}{p} (C_p A_0^n) k^{2n-2p} \sigma_z \]

\[ = C_s (k_z - sQ) + s\alpha_0 s\sigma_+ (\mathbf{n}_k - sQ\hat{e}_z) \cdot \sigma \]

with \( \mathbf{n}_k = (k_x' \cos(n\phi_k), k_y' \sin(n\phi_k), T_k/\alpha_n) \). In all subsequent analysis we define \( T_k = vk_z + \frac{\alpha_0^2}{\omega} \sum_{p=1}^{n} \frac{1}{p} k^{2n-2p} \)

\[ \approx \Delta_n + T_k', \]

with \( T_k' = vk_z + \frac{\alpha_0^2}{\omega} \sum_{p=1}^{n} \frac{1}{p} k^{2n-2p} \approx \Delta_n + T_k' \).

### III. BERRY CURVATURE

It is very important to study the Berry curvature in any topological system as the anomalous response function is directly given by the Berry curvature. Here our aim would be to investigate the effect of the driving on the Berry curvature and subsequently on the anomalous transport. Before going into detail, we begin by defining the Berry curvature associated with the Floquet Hamiltonian \( H_k^F \). The Berry curvature of the \( m \)th band for a Bloch Hamiltonian \( H(k) \), defined as the Berry phase per
unit area in the $k$ space, is given by \[^71\]

$$
\Omega^m_n(k) = (-1)^m \frac{1}{4|n_k|^2} \epsilon_{abc} n_k \cdot \left( \frac{\partial n_k}{\partial k_b} \times \frac{\partial n_k}{\partial k_c} \right). \quad (8)
$$

The explicit form of the Berry curvature associated with the Weyl node as obtained from Floquet effective Hamiltonian (6) is given by

$$
\Omega^\pm_F(k) = \pm \frac{1}{2|E_F|^2} \left( n v a_n^2 k_{\perp}^{2n-1} \cos \phi_k, n v a_n^2 k_{\perp}^{2n-1} \sin \phi_k, -n \beta_k a_n^2 k_{\perp}^{2n} + T k n^2 a_n^2 k_{\perp}^{2n-2} \right), \quad (9)
$$

with $\beta_k = \frac{\omega}{\nu} \sum_{p=1}^n (2n-2p) \beta_p k_{\perp}^{2n-2p-2}$. One can obtain regular static Berry curvature when $A_0 = 0$, $\beta_k = 0$ and $T = v k_z$. The static Berry curvature using Hamiltonian (1) becomes

$$
\Omega^\pm_0(k) = \pm \frac{1}{2|E_0|^2} \left( n v a_n^2 k_{\perp}^{2n-1} \cos \phi_k, n v a_n^2 k_{\perp}^{2n-1} \sin \phi_k, n^2 v a_n^2 k_{\perp}^{2n-2} k_z \right), \quad (10)
$$

Therefore, one can observe that $\Omega_0(z)$ would modify due to the driving while the remaining two components of $\Omega^\pm_F(k)$ receives the correction from the effective energy $E_F$ appearing in the denominator. This suggests that anomalous conductivity $\sigma_{xy}$ would be heavily modified due to the driving as compared to $\sigma_{xx}$ and $\sigma_{yy}$. We shall extensively analyze this below.

Now, turning to $n = 1$ case, the Berry curvature for driven single Weyl case is given by

$$
\Omega^\pm_F(k, n = 1) = (k_z, k_y, k_z + v^2 \Delta_1)/|E_F(n = 1)|^3.
$$

One can clearly observe that for driven mWSM all components of $\Omega(k)$ depend on $k_\perp$ while for driven single Weyl case individual components are comprised of separate momentum. The anisotropy of mWSMs imprints its effect which is absent for single Weyl case. Most importantly, even for $\Omega_0(k)$ in single WSM, the momentum independent term $\Delta_1 \sim A_0^2$ bears the signature of driving. For $n > 1$, the topological charge gets coupled with the driving parameter which leads to a much complicated form of $\Omega(z(k))$ as compared to $n = 1$.

We shall compute the anomalous Hall conductivity $\sigma_{F,xy}$, considering the effective Floquet Hamiltonian, from the $z$-component of Berry curvature in Eq. (10). In order to obtain a closed form results in the leading order, we neglect $\beta_k$ as $\omega \to \infty$ as the effective energy in the denominator bears the correction terms due to driving as shown in Eq. (7). We, on the other hand, consider the effect of the Floquet driving on the cut-off limit of $k_z$ integration. In particular, $z_1 = -\Lambda - SQ \to z_1'$ and $z_n = \Lambda - SQ \to z_n'$ with $z_1' = -\Lambda - SQ + S \Delta_n$ and $z_n' = \Lambda - SQ + S \Delta_n$. Therefore, one can safely consider the static energy in the denominator. We shall motivate this assumption extensively while discussing the vacuum contribution Sec. IV A. The anomalous contribution in its leading order is thus given by

$$
\sigma_{F,xy}^\nu = -e^2 \int \frac{dk}{4\pi^2} \sum_{s} \Omega_{F}^{s}(k) \sim -\frac{ne^2}{4\pi^2} \int \frac{d k_z}{z_1'} \int_{0}^{\infty} dk_{\perp} \frac{k_{\perp} k_{\perp}}{(k_z'^2 + k_{\perp}^2)^{3/2}} \quad (11)
$$

We here consider cylindrical polar co-ordinate for the convenience of the integration along with the following rescaling: $k_z \to k_z/v$ and $k_\perp \to k_\perp/\alpha_{\perp}^{-1/n}$. It is noteworthy that this anomalous Hall coefficient has a topological property due to the appearance of the monopole charge. For the static system, it is just given by $-\frac{ne^2}{2\pi^2} Q$.

We now connect our finding to the transport in the mWSMs. It has been shown that there exist $n$ number of Fermi arc for a mWSM with topological charge $n$\[^72\]. As we know the transport is mainly governed by the surface state present in the Fermi arc for WSM. Interestingly, driving shifts the position of Weyl points $\pm Q \to \pm Q \pm \Delta_n$: this results in an enhancement of the length of Fermi arc. As a result, the transport receives an additional correction from driving. It has been shown that Fermi arc can be tuned using Floquet replica technique when a WSM is irradiated with circularly polarized light\[^73\]. The factor $n$ in front of Eq. (11) signifies that effective Floquet Hamiltonian still supports $n$ number of Fermi arc. We here mention in passing that the neglected $\beta_k$ term would give rise to sub-leading non-topological contribution. Since we here probe the question of transport due to laser driving, it would be appropriate to investigate the optical conductivity using Floquet-Kubo formalism. However, we note at the outset that one can find similar expression as given in Eq. (11) while calculating the vacuum contribution of optical conductivity. Therefore, the Berry curvature induced anomalous Hall conductivity can indeed become very informative to understand the results obtained using Floquet-Kubo formalism which we discuss below.

### IV. CONDUCTIVITY TENSOR

Having derived the Berry curvature induced anomalous Hall conductivity, we shall now systematically formulate the conductivity tensor using the current-current correlation function. This is constructed using the Matsubara Greens function method. The current-current correlation is written as

$$
\Pi_{\mu\nu}(\Omega, \mathbf{k}) = T \sum_{\omega_n} \sum_{\mathbf{q}} \int \frac{d^3 k}{(2\pi)^3} J^\nu(s) G_s(i\omega_n, \mathbf{k}) J^\nu(s) G_s(i\omega_n - i\Omega_m, \mathbf{k} - \mathbf{q})|_{\Omega_m \to \Omega + i\delta} \quad (12)
$$

Here, $\mu, \nu = \{x, y, z\}$, $T$ is the temperature, $\omega_n$ and $\Omega_n$ are the fermionic and bosonic Matsubara frequencies and
$G$ is the single particle Green’s function. The Hall conductivity can now be derived from the zero frequency \( \Omega \to 0 \) and zero wave-vector limit.

Using the current-current correlation (12), one can define the static conductivity tensor \( \sigma^0_{ab} \). We here use the form of the time-averaged conductivity tensor \( \sigma^F_{ab} \) in the form of the Kubo formula, modified for the Floquet states as

\[
\sigma^F_{ab} = i \int \frac{d^3k}{(2\pi)^3} \sum_{\alpha \neq \beta} f_\beta(k) - f_\alpha(k) \langle \Phi_\beta \psi_\alpha \rangle (k) \epsilon_\beta(k) - \epsilon_\alpha(k)
\]

which resembles exactly the standard form of the Kubo formula where \( J_{0(ab)} \) represents the current operator, the \( \langle \Phi_\alpha \psi_\beta \rangle(k) \) represents the states of the effective Floquet Hamiltonian (6), and \( \epsilon_\alpha \) represent the corresponding quasi-energies. The \( f_\alpha \) represent the occupations which in general could be non-universal in systems which are out of equilibrium. In this case, the steady-state occupations can be regarded as the Fermi-Dirac distribution function associated with the quasi-energies of the Floquet states.

One can start from Luttinger’s phenomenological transport equations\(^{14}\) for the electric and energy DC currents. The energy current is originated from the combination of heat current \( J_Q \) and energy transported by the electric current \( J_E \) in presence of electromagnetic field while the underlying system is characterized by a finite chemical potential \( \mu \) and temperature \( T \). Within the Fermi liquid limit \( k_B T \ll |\mu| \), the Mott rule and the Wiedemann-Franz law relate the thermopower \( \alpha \) and thermal conductivity \( K \), respectively, to the electric conductivity \( \sigma^{75-77} \):

\[
\alpha_{ab} = cLT \frac{d\sigma_{ab}}{dm}, \quad K_{ab} = LT \sigma_{ab}.
\]

Here, \( \alpha_{ab} \) is the Nernst conductivity and \( K_{ab} \) is the thermal Hall conductivity and \( L = \pi^2 k_B^2 / 3e^2 \) is the Lorentz number. These formulas are valid for effective time independent Floquet Hamiltonian picture. We shall below investigate these quantities under the influence of laser field using the Floquet effective Hamiltonian.

One can define the current operator from the effective Floquet Hamiltonian \( H^F_k \) Eq. (6)

\[
J_\mu = c \frac{\partial H^F_k}{\partial k_\mu}
\]

In order to derive \( J_\mu \), we consider the leading order term neglecting \( \partial T_k / \partial k_\mu \) term as it contains \( 1/\omega \) factor. We note that the current operator obtained in this manner would be the same as the static current operator for mWSM Hamiltonian. This leading order term can be further confirmed by the zero-th order of Fourier component of current as shown in the Appendix A. However, one can indeed consider the full current operator with \( \partial T_k / \partial k_\mu \) to obtain the higher order contribution. The effect of \( T_k \) term is also encoded in the single particle Greens function \( G_\alpha \). We compute the optical conductivity by using the complete expression of \( G \) and approximated current operator.

In terms of \( \sigma \)’s, we can write upto leading order as

\[
J_x \approx e s n_\alpha k_\perp^{n-1} \cos((n-1)\phi_k) \sigma_x + \sin((n-1)\phi_k) \sigma_y
\]

\[
J_y \approx e s n_\alpha k_\perp^{n-1} \cos((n-1)\phi_k) \sigma_y - \sin((n-1)\phi_k) \sigma_x
\]

The point to note here is that \( J_x \) and \( J_y \) both depend on \( k_x \) and \( k_y \) which is in contrast to the single WSM case where \( J_x \sim k_x \sigma_x \). The anisotropic nature of dispersion of the mWSM Hamiltonian thus imprts its effect on the current operator.

Employing the current-current correlation and performing a detailed calculation (see Appendix B), we arrive at the conductivity tensor as

\[
\sigma_{xy} = \frac{e^2 n^2 \alpha^2}{4\pi^2} \sum_{s=\pm} \int_0^\infty dk_\perp k_\perp^{2n-1} \int_{-\Lambda}^{\Lambda} dk_z \frac{sv(k_z - Q) + \frac{s^2}{\omega} \sum_{p=1}^{n} \beta_p k_\perp^{2(n-p)} [\frac{s^2}{\omega} \sum_{q=1}^{n} \beta_q k_\perp^{2(n-q)} + sv(k_z - sQ)]^2 + \alpha^2 k_\perp^{2n} [\frac{s^2}{\omega} \sum_{q=1}^{n} \beta_q k_\perp^{2(n-q)}]^2}{\Lambda - \sigma Q + \frac{s^2}{\omega} \sum_{p=1}^{n} \beta_p k_\perp^{2(n-p)} (\Lambda - \sigma Q + \frac{s^2}{\omega} \sum_{q=1}^{n} \beta_q k_\perp^{2(n-q)})}
\]

where \( \Lambda \) is the ultra-violet cut-off of \( k_\perp \) integral, \( n_F(E_k^-) = 1 / [e(sE_k^- + 1)] \) is the Fermi-Dirac distribution function, and \( \beta = 1/T \) is inverse temperature.

## A. Vacuum contribution

We here investigate the vacuum contribution which is obtained in the limit \( [n_F(E_k^-) - n_F(E_k^+)] \to 1 \). Physically this means that valence (conduction) band is completely filled (empty). This vacuum contribution amounts for an intrinsic contribution that remains finite for \( \mu \to 0 \) Now in terms of technicality, this refers to the situation where the limits \( k_\perp \) is considered to be \( \infty \). With suitable redefinitions and linear integration variable shifts, we arrive at

\[
\sigma_{xy}^{vac} = \frac{e^2 n^2 \alpha^2}{4\pi^2} \sum_{s=\pm} \int_0^\infty dk_\perp k_\perp^{2n-1} \int_{-\Lambda}^{\Lambda} \frac{[sv(k_z) + \frac{s^2}{\omega} \sum_{p=1}^{n} \beta_p k_\perp^{2(n-p)}]}{\Lambda - \sigma Q + \frac{s^2}{\omega} \sum_{q=1}^{n} \beta_q k_\perp^{2(n-q)}} ^2 + \alpha^2 k_\perp^{2n} [\frac{s^2}{\omega} \sum_{q=1}^{n} \beta_q k_\perp^{2(n-q)}]^2}
\]

We will compute the vacuum contribution using two separate procedures involving suitable approximations and then compare the results obtained.
1. Coordinate Transformation Method

The method prescribed in this section relies on the fact that while several quantities are set to infinity in a computation, in order to get physically plausible answers one might need to define the order in which the limits are taken. For computation of the integrals, the following coordinate map $M : \mathbb{R}^2 \to \mathbb{R}^2$ is prescribed with the action $k_\perp \to k_\perp' = k_\perp^{\pm} \alpha_0 \frac{\omega}{2}$, and $k_x \to k_x$. With this coordinate transformation the vacuum contribution of the conductivity tensor looks like

$$\sigma_{xy}^{\text{vac}} = -\frac{e^2 n \alpha_n}{4\pi^2} \sum_{s=\pm} \int_{z_l}^{z_u} \int_{x_l}^{x_u} \frac{k_\perp T_k}{(k_\perp^2 + T_k^2)^{3/2}} dk_\perp dk_z$$

(20)

Here, the upper and lower limits of the integrals has been determined with physical justifications (see Appendix B):

$$x_l = 0, \quad x_u = \Lambda_{\perp}$$

$$z_u = v(\Lambda - sQ) + s \left( \Delta_n + \frac{\alpha_n^2}{\omega} \sum_{p=1}^{n-1} \beta_p \omega_n \frac{2(n-p)}{2(n-p)} \frac{\Delta_n}{\Lambda_{\perp}^2} \right)$$

$$z_l = v(-\Lambda - sQ) + s \left( \Delta_n + \frac{\alpha_n^2}{\omega} \sum_{p=1}^{n-1} \beta_p \omega_n \frac{2(n-p)}{2(n-p)} \frac{\Delta_n}{\Lambda_{\perp}^2} \right)$$

(21)

$\Lambda_{\perp}$ is the cut-off for $k_\perp$ integral. One can segregate $z_{l,u}$ from $\Lambda_{\perp}$: $z_{l,u} = z_{l,u}' + s \frac{\alpha_n^2}{\omega} X_{\Lambda_{\perp}}$ with $X_{\Lambda_{\perp}} = \sum_{p=1}^{n-1} \beta_p \omega_n \frac{2(n-p)}{2(n-p)} \frac{\Delta_n}{\Lambda_{\perp}^2}$ and $z_{l,u}' = v(\pm \Lambda - sQ) + s \Delta_n$. For the mWSM case, anisotropy appears in $k_\perp$ as it depends on $k_x$ and $k_y$. Hence one has to handle this cut-off with care. We note that while solving $k_\perp$ integral, $\Lambda_{\perp}$ is considered to be $\infty$. On the other hand, we keep $\Lambda_{\perp}$ while solving $k_x$ integral. This precisely the order that we want to maintain here to solve the integral. We are allowed to continue with these integrals as $Q \ll \Lambda_{\perp}$ and $\Lambda_{\perp}/\omega$ remains finite considering $\omega \to \infty$. We justify the above assumption for the low energy model, the band-width can be assumed to be $\infty$, at the same time, high frequency Floquet effective Hamiltonian $H_{kF}^n$ is valid when $\omega$ is kept at least above the band-width.

Finally, we obtain for the type-I mWSM,

$$\sigma_{xy}^{\text{vac}}(n = 2) = -\frac{e^2 n \alpha_n}{4\pi^2} \sum_{s=\pm} \int_{z_l}^{z_u}' \int_0^\infty \frac{k_\perp T_k}{(k_\perp^2 + T_k^2)^{3/2}} dk_\perp dk_z$$

$$\approx -\frac{e^2 n \alpha_n^2}{4\pi^2} \sum_{s=\pm} \int_{z_l}^{z_u}' \int_0^\infty \left( v(z_l + z_u) + v^2 \omega_n \beta_1 \frac{z_l - z_u}{\omega} \right)$$

(22)

In this derivation, we ignore the divergent contribution coming from the integrals having higher powers of $k_\perp$ in the numerator. This type of terms, being an artifact of the underlying low-energy model, do not appear in the lattice model. In order to obtain $\Lambda_{\perp}$, we equate the coefficient of $1/\omega$ from Eq. 25 and Eq. 22. We find $\Lambda_{\perp}$ linearly depends on $\Lambda': \Lambda_{\perp} = 2e^2 \Lambda'$. For $n = 3$, we find

$$\sigma_{xy}^{\text{vac}}(n = 3) = -\frac{e^2 n \alpha_n}{4\pi^2} \left[ v(-2Q + 2\Delta_n) - \frac{2e^2 \omega_n^2 \beta_1}{\omega \sqrt{\pi}} \Gamma \left( \frac{5}{6} \right) \Gamma \left( \frac{5}{3} \right) \left( |z_l| \frac{\pi}{2} + z_u \frac{4}{3} \right) \right]$$

$$- \frac{2 \omega_n^2 \beta_1}{\omega \sqrt{\pi}} \Gamma \left( \frac{5}{6} \right) \Gamma \left( \frac{5}{3} \right) \left( |z_l| \frac{\pi}{2} + z_u \frac{4}{3} \right)$$

$$+ \frac{\alpha_n^2}{\sqrt{\pi}} \Gamma \left( \frac{1}{6} \right) \Gamma \left( \frac{4}{3} \right) \left( |z_l| \frac{\pi}{2} + z_u \frac{4}{3} \right)$$

(23)
It is noted that contrary to the $n = 2$ case, $\Lambda_\perp$ is non-linearly related to $\Lambda_\parallel$ for $n = 3$: $\Lambda_\perp^2 = \eta_1|z|^{4/3} + z_4^{2/3}$ and $\Lambda_\parallel^2 = \eta_2|z|^{2/3} + z_4^{2/3}$ where $\eta_{1,2}$ can be obtained by matching the coefficient of $\beta_1^2/\omega$ and $\beta_2^2/\omega$.

**B. Fermi-surface contribution**

We take note of the point that for the calculation of the Fermi surface contribution, one has to consider the finite upper limit in the $k_\perp$ integral as $b$. The Fermi surface contribution for a given $n$ becomes

$$
\sigma_{xy}^{FS}(n) = n\alpha_n^{2-2/n} \sum_s \int_{z_i}^{z_n} dk_\perp \int_0^b \frac{k_\perp T_k}{(k^2_\perp + T^2_k)^{3/2}} dk_\perp \times \{\theta(v^2 k^2 + (C k z + s C \Delta_n - \mu)^2) - 1\} \quad (27)
$$

It is then more convenient to write $T_k$ explicitly for $n = 2$ as $T_k = v k + \beta_1^2 \alpha_n^{-2/n} k_\perp^2$ and for $n = 3$ as $T_k = v k + \beta_1^2 \alpha_n^{-2/n} k_\perp^2$. In a more compact notation, for $n = 3$, we define $\beta_1^2 = \beta_2^2 \alpha^{-2/n}$, $\beta_1^3 = \beta_2^2 \alpha^{2/n}$. In $n = 2$, we define $\beta_1^2 = \beta_2^2 \alpha^{-2/n}$ and $\beta_2^3 = 0$. On the other hand, $b = (C k z + s C \beta_1 - \mu)^2 - v^2 k_\perp^2 \beta_3^2$. Below we shall express all our findings in terms of $\beta_1^2$ and $\beta_3^2$ for a given $n$.

In the leading order approximation, $\beta_1 = O(1)$, $k_\perp \to 0$ and $\mu = \text{finite}$. We shall consider the cases for type-I and type-II cases separately; $|C| > \mu$, $\mu = C k z$, and $|C| \ll v, b = (\mu - v^2 k_\perp^2)^{1/2}$. We again make resort to leading order method where we only allow $O(1/\omega)$ order term.

$$
\sigma_{xy}(n) = \frac{n\alpha_n^{2-2/n}}{v} \int_{z_i}^{z_n} dk_\perp \int_0^b dk_\perp k_\perp (F_{k,1} + F_{k,2} + F_{k,3}) \quad (28)
$$

with $F_{k,1} = \frac{k_\perp}{E_k^2}, F_{k,2} = \frac{\beta_1^2 k_\perp^{2/3}}{E_k^{2/3}}, F_{k,3} = -\frac{3\beta_2^2 k_\perp^2}{E_k^4}$. We note that in (28) the leading order term $F_{k,1}$ is also present for the $n = 1$ Weyl node case. Similar to the vacuum contribution of optical conductivity, here also multi Weyl nature appears through a multiplicative factor $n\alpha_n^{2-2/n}$. The additional anisotropic and band bending corrections appear in terms of $1/\omega$ in $F_{k,2}$ and $F_{k,3}$. To obtain a minimal expression, the above derivation is simplified by neglecting the term $F_{k,3}$ as $k_\perp^2 / E_k^4 \to 0$ for $k_\perp \to 0$ considered for low-energy model.

For type-I mWSM, one can keep in mind the fact that $b$ remains always positive. The total contribution from the Fermi surface is given by

$$
\sigma_{xy}^{FS}(t) = -\frac{e^2 n\alpha_n^{2-2/n}}{4\pi^2} \left[ (\mu - C \Delta_n) \left( \frac{v + C}{\omega^2} \ln \left( \frac{v + C}{\omega - C} \right) - 2 \right) + \beta_2^{2} a(M) \left( \frac{2\mu}{n - 3} \right)^{2/3} \right] + \beta_3^{2} (M \to 2M)
$$

with $M = 1$. Therefore, the leading contribution is not just given by $n\alpha_n^{2-2/n}$ multiplied to $n = 1$ contribution. In this first term $\mu$ gets renormalized by $\mu - C \Delta_n$ while $\Delta_\perp$ depends on the multi Weyl nature. The other subleading order terms are of order $1/\omega$. The multi Weyl nature thus imprints its effect on the Fermi surface part of the optical conductivity. We can write a closed form expression for $v \gg |C|$ as follows,

$$
\sigma_{xy}^{FS}(t) = -\frac{e^2 n\alpha_n^{2-2/n}}{4\pi^2} \left[ C(\mu - C \Delta_n) \left( \frac{1}{\omega^2} \ln \left( \frac{v + C}{\omega - C} \right) - 2 \right) + \beta_2^{2} a(M) \left( \frac{2\mu}{n - 3} \right)^{2/3} \right] + \beta_3^{2} (M \to 2M)
$$

Therefore, total conductivity of type-I mWSM for a given $n$ is expressed as

$$
\sigma_{xy}(n) = \frac{\alpha_n^{2-2/n}}{4\pi^2} \left[ (Q + \Delta_n) + C \left( \frac{1}{\omega^2} \ln \left( \frac{v + C}{\omega - C} \right) - 2 \right) + \beta_2^{2} a(M) \left( \frac{2\mu}{n - 3} \right)^{2/3} \right] + \beta_3^{2} (M \to 2M)
$$

with $\beta_2^{2} = \beta_2^2 \left( \frac{2\mu}{n - 3} \right)^{2/3}$ and $\beta_3^{2} = \beta_3^2 \left( \frac{2\mu}{n - 3} \right)^{2/3}$. This helps us to write the anomalous thermal Hall conductivity $K_{xy}^I$ and Nernst conductivity $\alpha_{xy}^I$ respectively for the type-I mWSMs as,

$$
K_{xy}^I(n) = \frac{\pi^2}{3e^2 k_B^2} \frac{\alpha_n^{2-2/n}}{v} \left[ (Q + \Delta_n) + C \left( \frac{1}{\omega^2} \ln \left( \frac{v + C}{\omega - C} \right) - 2 \right) + \beta_2^{2} a(M) \left( \frac{2\mu}{n - 3} \right)^{2/3} \right] + \beta_3^{2} (M \to 2M)
$$

(33)
One can find
\[
\alpha_{xy}^I(n) = \frac{\pi^2}{3e^2} k_B^2 T \frac{d\sigma_{xy}^I}{d\mu} = n \frac{e k_B^2}{12} \alpha_n^{2-2/n} \left[ -\frac{C}{6\pi^2} + 4\beta_n^2 a(M) \left( \frac{2}{n} - 2 \right) \cdot \mu^{2/n-3} + 4\beta_n^2 a(2M) \left( \frac{4}{n} - 2 \right) \mu^{4/n-3} \right]
\]
(34)

One can now easily derive the expressions for \(\sigma_{xy}^I\), \(K_{xy}^I\), and \(\alpha_{xy}^I\) for \(n = 2\) by considering \(\beta_3^2 = 0\). Comments on the new results for \(n = 2\) and \(n = 3\) and its characteristic dissimilarity from \(n = 1\) case are in order now. In general, non-linear \(\mu\) dependence comes from \(1/\omega\) order term in multi Weyl case while the linear \(\mu\) dependence term only appear for \(n = 1\).

Let us now explore the thermal responses for the type-II case of mWSM where sign of \(k_\perp\) momentum cut-off \(b\) depends on \(k_\perp\). Handling of the \(k_\perp\) integral requires extra care as \(\text{sgn}(b)\) becomes \(+\) \((-\)) depending on \(k_\perp\) being \(-(+), |C| \gg v\), refers to the fact \(v^2 k_\perp^2 - (C k_\perp + sC\Delta_n - \mu)^2 < 0\) Keeping in mind the \(k_\perp\) integral properly, we find

\[
\sigma_{xy}^{II}(n) = n \frac{e^2}{4\pi^2} \alpha_n^{2-2/n} \left[ (\Delta_n + Q) \left( -1 + \frac{v}{C} \right) \cdot \frac{v(C\Delta_n - \mu)}{C^2} \ln \frac{C^2\Lambda}{v(C\Delta_n - \mu)} + \beta_n^2 a(M) \cdot \left( \frac{2\mu^{2/n-3}v}{C} - 2Q \right) \cdot \left( \frac{2M}{n} - 3 \right) C^2 \mu^{2M/n-4} \right]
\]
\[
\frac{4\Delta_n^2v}{C} - 2\Delta_n - 2Q^2 \right) + \beta_3^2 \{ M \rightarrow 2M \}
\]
(35)

with \(M = 1\). The remarkable point to note here is that the momentum cut-off \(\Lambda\) algebraically shows up in the Fermi surface contribution. However, this is accompanied with the sub-leading term \(O(1/\omega)\). This is indeed a new feature for the anisotropic character of the dispersion of mWSM. In type II single WSM, the momentum cut-off can appear logaritthmically.

This gives us the anomalous thermal Hall conductivity:

\[
K_{xy}^{II}(n) = nT \frac{k_B^2}{12} \alpha_n^{2-2/n} \left[ (\Delta_n + Q) \left( \frac{v}{C} - 1 \right) \cdot \frac{v(C\Delta_n - \mu)}{C^2} \ln \frac{C^2\Lambda}{v(C\Delta_n - \mu)} + \beta_n^2 a(M) \cdot \left( \frac{2\mu^{2M/n-3}a_2(M) + 2M/n-4a_3(M) \right)
\]
\[
+ \beta_3^2 \{ M \rightarrow 2M \}
\]
(36)

with \(a_2(M) = (2\Delta_n v/C - 2Q), a_3(M) = C(2M/n - 3)(4\Delta_n^2 v/C - 2\Delta_n - 2Q^2)\) and \(M = 1\). On the other hand, the Nernst conductivity is given by

\[
\alpha_{xy}^{II}(n) = ne \frac{k_B^2}{12} \alpha_n^{2-2/n} \left[ \frac{1}{C^2} - 1 + \frac{C^2\Lambda}{v(C\Delta_n - \mu)} + \beta_n^2 a(M) \cdot \left( \frac{2M}{n} - 3 \right) C^2 \mu^{2M/n-4} a_2(M) \right]
\]
\[
+ \beta_3^2 \{ M \rightarrow 2M \}
\]
(37)

One can easily obtain the \(n = 2\) results by considering \(\beta_3^2 = 0\).

We now discuss some important aspects of our findings on type I and II mWSMs. It is to be noted that \(\Lambda^2\) is associated with \(\mu^{2M/n-4} (\mu^{2M/n-5})\), \(M = 1, 2\) for optical Hall conductivity, and Nernst conductivity in case of type II mWSMs, respectively. Therefore, the transport is heavily influenced by \(\mu\) and \(\Lambda\). This is in contrary to the type I mWSM where only \(\mu\) can affect the transport in addition to the driving field. Even for type I single WSM, logarithmic cut-off dependence is only observed. Hence, the anisotropy in the tilted dispersion would non-trivially couples with the field parameter to generate all these extraordinary characteristics. The shape of the Fermi pocket for type II mWSM is very different from type I mWSM as it evident from the cut-off dependence of transport coefficients. In case of irradiated tilted mWSMs, the topological charge imprints its effect not only in a simple multiplicative way but also in a much more fundamental way by the tilt dependent effective chemical potential where \(\Lambda\) appears algebraically. However, this algebraic cut-off dependent term is associated with the additional correction of \(O(1/\omega)\). The leading order correction here is given by \(n\times\) the single Weyl result; anisotropic nature of the dispersion is encoded in the renormalized \(\mu \rightarrow \mu - C\Delta_n\), where \(\Delta_n = O(A_0^2/\omega)\). The effective chemical potential is also dependent on the frequency of the driving potential and the monopole charge.

Having investigated the implication of cut-off, we here investigate the non-linearity of \(\mu\) that arises besides the effective \(\mu\). In type I mWSM, considering \(v \gg |C|\), the vacuum contribution \(\sigma_{xy}^I\) associated with \(\beta_{2,3}^2\) term becomes increasing function of \(\mu\) for both for \(n = 2\) and \(n = 3\); \(\beta_n^2\) term decays inversely with \(\mu\) for \(n = 2\) (as \(\mu^{-1}\)) and \(\beta_{2,3,4}^2\) decays non-linearly \(\mu^{-4}(3)(-3/3)\) with \(\mu\) for \(n = 3\). Nernst conductivity on the other hand, goes as \(\mu^{-2}\) for \(n = 2\); for \(n = 3\), it becomes decreasing function of \(\mu\) (as \(\mu^{-7/3}\) and \(\mu^{-5/3}\)). In type II mWSM, considering \(|C| \gg v\), the vacuum contribution \(\sigma_{xy}^{II}\) associated with \(\beta_{2,3,4}^2\) becomes decreasing function of \(\mu\) for both for \(n = 2\) and \(n = 3\). We note that the sub-leading correction decays more rapidly with \(\mu\) for type II as compared to type I mWSM. In particular, \(\beta_{2,3,4,5}^2\) term varies as \(\mu^{-2}\) and \(\mu^{-3}\) for \(n = 2\). While for \(n = 3\), \(\beta_{2,3,4,5,6}^2\) goes as \(\mu^{-7/3}\) and \(\mu^{-7/3}\) \((\mu^{-5/3}\) and \(\mu^{-8/3}\)). Nernst conductivity, on the other hand, becomes strongly decreasing
function of $\mu$ for both $n = 2$ and $n = 3$ with the lowest power as $\mu^{-3}$ and $\mu^{-8/3}$, respectively.

Having investigated the transport behavior analytically, we below illustrate them to analyze some salient qualitative features. We note that our aim is to pictorially differentiate the type I from type II mWSM based on our low-energy model. Hence, at the outset, we confess that the true lattice effect might not be possible to capture following our analysis. However, at least, our study indicates some trends which we believe can be probed in real materials.

We now discuss the transport coefficients for both type-I and type-II mWSMs as shown in Fig. 1(a) for thermal Hall conductivity and Fig. 1(b) for Nernst conductivity. We here depict the high frequency behavior of $K_{xy}$ and $\alpha_{xy}$, calculated using Eq. (33) and Eq. (34), respectively. Noticeably the response from the external field for a general $n$ mWSM is not a simple multiplicative factor of the response of the $n = 1$ WSM. This is also very clearly evident from the variation of $K_{xy}$ and $\alpha_{xy}$ with driving frequency $\omega$. The sub-leading term plays an important role with the chemical potential $\mu$ as the frequency dependent factors in $\beta_2'$, $\beta_3'$ are coupled to the terms which are $f(\mu, n)$. The important point to note here is that $K_{xy}$ decreases and eventually saturates with optical frequency $\omega$; while $\alpha_{xy}$ remains unchanged with $\omega$ except for $n = 3$ which increases followed by a saturation. Similarly, for type II mWSM, we depict the behavior of $K_{xy}$, obtained from Eq. (36), in Fig. 2 (a) and $\alpha_{xy}$, obtained from Eq. (37), in Fig. 2 (b), respectively. One can find here for type II mWSM, unlike the type I mWSM, that $K_{xy}$ and $\alpha_{xy}$ both exhibit decreas-
ing tendency with $\omega$. This might be due to the fact that they carry the quadratic momentum cutoff $\Lambda^2$ dependent sub-leading term in addition to the term containing the function $f(\mu, \omega, n)$. We additionally note that $n = 3$ acquires higher value for type I as compared to $n = 1, 2$, while this feature is reversed for type II mWSM. Having investigated anomalous thermal Hall and Nernst coefficients for a range of physically viable parameter such as $\omega$, we can convey type I and type II mWSM can be qualitatively distinguished in terms of their transport behavior.

Now we shall focus on the role of $n$ in different transport properties. For that, we plot $K_{xy}$ as a function of $\alpha_{xy}$ for $n = 2$ in Fig. 3(a), and for $n = 3$ in Fig. 3(b). It is known that $K_{xy}$ and $\alpha_{xy}$ share a linear relationship for $n = 1$, and we notice that this holds for $n = 2$: This can be attributed to the fact the sub-leading order term remains small for a given chemical potential. This no longer holds for $n = 3$ as is evident from Fig. 3(b) where the sub-leading order term and the additional $\Lambda^2$ dependence play crucial role. Therefore, one can indeed find a qualitative change in the transport character with $n$ as the degree of anisotropy enhances.

We shall now demonstrate a relevant experimental set up where our proposal can be tested. One can have double (HgCr$_2$Se$_4$) and triple WSM (Rb(MoTe)$_3$) material as the samples. The Floquet driving can be realized by the pump (strong beam)-probe (weak beam) optical set up where ultrafast electronic dynamics of the sample is observed as a function of time delay between the arrival of pump and probe pulses. Recently, using polarized photons at mid-infrared wavelengths, Floquet-Bloch states and photo-induced band gaps are clearly visible in time- and angle-resolved photoemission spectroscopy. We believe using the similar arrangement with suitably chosen frequency of pump laser, one experimentally measure the transport properties in the following way. One can also consider a non-optical substrate-terminal based closed circuit measurement of Nernst conductivity and thermal Hall conductivity. The electric and heat current can be measured considering a mutually perpendicular arrangement of DC power source and thermocouple, respectively.

V. CONCLUSION

We here investigate the circularly polarized light (of amplitude $A_0$) induced contributions to the thermo-electric transport coefficients in type-I and type-II mWSM with topological charge $n > 1$ considering a low energy dispersion of the model. Using the high frequency expansion (driving frequency $\omega \to \infty$) and appropriately employing the non-equilibrium Floquet-Kubo formalism, where the energy and state of the Hamiltonian are replaced with the quasi-energy and quasi-states of the effective Hamiltonian, we study the anomalous thermal Hall conductivity and Nernst conductivity. The effective Floquet Hamiltonian suggests that the Weyl nodes, separated by $Q$ in the momentum space for the static case, are further displaced by a distance $2\Delta_n \sim A_0^2/\omega$. Most importantly, the low energy Hamiltonian of Floquet mWSMs receives momentum dependent corrections in addition to the constant $A_0^2$ shift Floquet single WSM (as characterized by $n = 1$). This results in a change in the effective Fermi surface which in turn leads to an array of non-trivial consequences for the transport coefficients. The leading order contribution varies linearly with the topological charge with the renormalized chemical potential as $\mu - C\Delta_n$. Therefore, the for type-I, and type-II mWSM, the light induced transport phenomena become significantly different. In particular, one can show that optical conductivity increases with $A_0$ for type-I mWSM while it decreases with $A_0$ in the case of type-II mWSM.
We additionally estimate the in-plane momentum cut-off from the non-topological vacuum contribution of optical conductivity. However, the leading order vacuum contribution remain topological which we further verified using Berry curvature induced anomalous Hall conductivity.

Going beyond the leading order contribution, we compute the effect of the momentum dependent correction term in the Fermi surface effect to conductivity tensor. We find Floquet driving induced sub-leading contribution can show non-trivial algebraic dependence on the chemical potential $\mu$ as $\mu^{l(n)}$. Most surprisingly, unlike to the case of type-II single WSM, for type-II $m$WSM, Nernst and thermal Hall conductivity depends algebraically on the momentum cut-off parameter. However, for type-I $m$WSM, the Fermi surface part remain cut-off independent. On the other hand, the above part, interestingly, decays slowly for type I $m$WSM as compared to type II $m$WSM. Therefore, unlike the type-I single WSM, the Nernst conductivity for type-I $m$WSM depends on $\mu$. Combining all these, we investigate the total thermal Hall and Nernst conductivity as a function of optical driving frequency by numerically evaluating the analytical expression to obtain a clear picture. Our study indeed suggests that type I and type II $m$WSM exhibit distinct behavior while the multi Weyl nature can also be captured vividly. We also discuss about the possible experimental measurement of our analytical findings.

VI. ACKNOWLEDGEMENT

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**Appendix A: Alternative definition of current operator**

One can alternatively obtain the above current operator by using the Floquet-Kubo formula. We use Fourier component $J_\mu$ instead of the real time version $J_\mu(A,t)$. Here, $J_\mu(A,t)$ is defined from $H_k^\mu(A,t)$ in Eq. 4

$$J_\mu(A,t) = \frac{\partial H_k^\mu(A,t)}{\partial k_\mu}$$

We define the Fourier expansion of $\mathbf{J}$ in frequency space as $J_\mu(A,t) = \sum_\Omega e^{-i\Omega t} J_\mu$. We get

$$J_x(A,t) = \text{sen} \alpha_n \sum_{q=0}^{n-1} \left[ C_q(ieA_0 e^{-i\omega t})^n d q (k_\perp e^{-i\phi})^q \right]$$

$$J_y(A,t) = \text{sen} \alpha_n \sum_{q=0}^{n-1} \left[ C_q(-ieA_0 e^{-i\omega t})^n d q (k_\perp e^{i\phi})^q \right]$$

The time averaged current operators are given by

$$\langle \langle \Phi_\alpha(k)|J_\mu|\Phi_\beta(k) \langle \rangle \rangle = \frac{1}{T} \int_0^\infty dt \sum_{m,n,l} e^{-i\Omega(m+n-l)t} \langle u_\alpha^m | J_\mu | u_\beta^n \rangle \approx \langle u_\alpha^0 | J_\mu^0 | u_\beta^0 \rangle$$

Here, we take into account that the leading order contribution can only come from the zeroth level Floquet states $|u_\alpha^0\rangle$ as $|u_\beta^0\rangle \sim \mathcal{O}(\omega^{-n})$.

**Appendix B: Hall Conductivity Computation using modified Kubo Formalism**

The modified form of the Kubo formula as applicable to Floquet states of a strong and periodically driven system (derived in appendix B) is used in this appendix to compute the analytical form of the zero-temperature time-averaged components of the conductivity tensor. The time-averaged anomalous Hall conductivity for the tilted WSM under the action of the circularly polarized light may now be derived from the zero frequency and zero wave-vector limit (i.e.
the limit of an infinitesimal d.c. bias) of the current-current correlation function, constructed using the Matsubara Green’s function method (with \( h = 1 \)):

\[
\Pi_{ij}(\Omega, \mathbf{q}) = T \sum_{\omega_p} \sum_{s=\pm} \left. \frac{d^3k}{(2\pi)^3} J_i^{(s)}(\omega_p, \mathbf{k}) J_j^{(s)}(\omega_p - i\Omega_m, \mathbf{k} - \mathbf{q}) \right|_{i\Omega_m \to \Omega + i\delta},
\]

where \( i, j = \{x, y, z\} \), \( T \) is the temperature (setting the Boltzmann constant as unity) and \( \omega_p(\Omega_m) \) are the fermionic (bosonic) Matsubara frequencies. Here \( G_s(\omega_p, \mathbf{k}) \) is the single particle Green’s function of the electron and \( J_i^{(s)} = \cos(\mathbf{n}_\mathbf{k} \cdot \mathbf{s} \mathbf{\sigma}) \) is the current operator with \( i, j = \{x, y\} \). For \( \text{mWSM} \), \( J_{x,y} \) both depends on \( \sigma \) unlike the \( n = 1 \) case. Using the short hand notation \( J_{x(y)} = \cos(\mathbf{n}_\mathbf{k} \cdot \mathbf{s} \mathbf{\sigma}) \), one can find \( J_{z,1} = \cos(\mathbf{n}_\mathbf{k} \cdot \mathbf{s} \mathbf{\sigma}) \), \( J_{z,2} = \sin(\mathbf{n}_\mathbf{k} \cdot \mathbf{s} \mathbf{\sigma}) \), \( J_{y,1} = -\sin(\mathbf{n}_\mathbf{k} \cdot \mathbf{s} \mathbf{\sigma}) \), \( J_{y,2} = \cos(\mathbf{n}_\mathbf{k} \cdot \mathbf{s} \mathbf{\sigma}) \). One can relate the Hall conductivity to the current-current correlation function as follows,

\[
\sigma_{xy} = -\lim_{\Omega \to 0} \frac{\Pi_{xy}(\Omega, 0)}{i\Omega}.
\]

The one-particle Green functions have the following form

\[
G_s(\omega_p, \mathbf{k}) = \frac{1}{2} \sum_{t=\pm} \frac{1 - t_\mathbf{n}_k' / |t_\mathbf{n}_k'|}{\lambda - t_\mathbf{n}_k'}
\]

\[
= \frac{1}{2} \sum_{t=\pm} \frac{1 - s t \sigma \cdot t_\mathbf{n}_k' - s(\mathbf{Q} + \Delta_n) |e_\mathbf{z}|}{|t_\mathbf{n}_k'| - s(\mathbf{Q} + \Delta_n) |e_\mathbf{z}|} - C_s(k_z - s(\mathbf{Q} + \Delta_n) + \lambda)
\]

where \( \mu \) is the chemical potential. Here, \( t_\mathbf{n}_k' = (\alpha_n, \beta_p, \gamma_m) \), \( \sigma \) is the Pauli matrix. We refer, \( T_k' = v k_z + \frac{\alpha_n^2}{\omega_p} \sum_{p=1}^{n-1} \frac{k_z k_{p-1}^2}{2} \) and \( \Delta_n = \frac{\alpha_n}{\omega_p} A_0^2 \). Here, \( T_k' = (\kappa, \kappa, \kappa) \) and \( k_z = \omega_p + \mu - C_s(k_z - s(\mathbf{Q} + \Delta_n)) \). We note that in the main text, we refer \( T_k = T_k' + \Delta_n \) and \( \mathbf{n}_k' = (\alpha_n, \beta_p, \gamma_m) \). Since \( \Delta_n \) is independent of \( k_z \), we absorb \( \Delta_n \) in the \( k_z \) momentum cut off \( \Lambda \rightarrow \Lambda - \Delta_n \). We note here that we shall use the cylindrical co-ordinate \( f d^3k \rightarrow f k_z dk_{\perp} f dz f \phi \). After a few steps of detail calculation, considering the fact \( \int_0^{2\pi} d\phi \sin(M \phi) = \int_0^{2\pi} d\phi \cos(M \phi) = 0 \) with \( M \geq 1 \), the \( \Pi_{xy}(\Omega, 0) \) becomes

\[
\Pi_{xy}(\Omega, 0) = T \frac{e^2 n^2 \alpha_n^2}{4\pi^2} \sum_{s, t, u} \int_0^{\infty} dk_z k_{\perp}^{2n-1} \int_{-\Lambda-s(Q+\Delta_n)}^{\Lambda-s(Q+\Delta_n)} dk_z \int_0^{2\pi} d\phi \frac{\sin(M \phi)}{\lambda - t_\mathbf{n}_k' |(\lambda - t_\mathbf{n}_k')|}
\]

with \( J_{i,j} \) the \( j \)-th component of current \( J_i \). Now using the Matsubara Fermionic sum, one can show

\[
T \sum_{\omega_p} \frac{1}{\lambda - t_\mathbf{n}_k' |(\lambda - t_\mathbf{n}_k')|} = \frac{n_F(E_n') - n_F(E_n^u)}{E_n - E_n^u}
\]

We get the finite contribution only from \( u = -t \) and \( u = \pm \).

We sum over the Matsubara fermion frequencies and trace over Pauli \( \sigma \)-matrices to obtain the following form

\[
\Pi_{xy}(\Omega, 0) = \Pi_{xy}^{(+)}(\Omega, 0) + \Pi_{xy}^{(-)}(\Omega, 0),
\]

where we have separated the contributions from the two Weyl cones

\[
\Pi_{xy}^{(+)}(\Omega, 0) = \Pi_{xy}^{(-)}(\Omega, 0) + \Pi_{xy}^{(s)}(\Omega, 0),
\]

\[
\Pi_{xy}^{(+)}(\Omega, 0) = -se^2 n^2 \alpha_n^2 \int_{-\Lambda-s(Q+\Delta_n)}^{\Lambda-s(Q+\Delta_n)} \frac{dk_z}{2\pi} \int_{0}^{\infty} \frac{2\pi}{2\pi} \frac{k_{\perp}^{2n-1}dk_{\perp}}{\Omega_n^2 + 4|t_\mathbf{n}_k'|^2} \times \frac{T_k'}{|t_\mathbf{n}_k'|},
\]

\[
\Pi_{xy}^{(s)}(\Omega, 0) = se^2 n^2 \alpha_n^2 \int_{-\Lambda-s(Q+\Delta_n)}^{\Lambda-s(Q+\Delta_n)} \frac{dk_z}{2\pi} \int_{0}^{\infty} \frac{2\pi}{2\pi} \frac{k_{\perp}^{2n-1}dk_{\perp}}{\Omega_n^2 + 4|t_\mathbf{n}_k'|^2} \times \frac{T_k'}{|t_\mathbf{n}_k'|} \left( n_F(C_s k_z + |t_\mathbf{n}_k'| - \mu + sC_s \Delta_n) - n_F(C_s k_z - |t_\mathbf{n}_k'| - \mu + sC_s \Delta_n) + \frac{1}{2} \right) \right|_{i\Omega_m \to \Omega + i\delta}.
\]
Here, $\Lambda_1$ is cut-off considered for $k_\perp$ integral. $\Pi_0$ denotes the vacuum contribution for $\mu = 0$, whereas $\Pi_{FS}$ is the contribution of the states at the Fermi surface. $n_{F}(E) = \left(e^{(E - \mu)/T} + 1\right)^{-1}$ is the Fermi distribution function and $|n'_k| = \sqrt{(T_{k}'^2)^2 + \alpha^2_n k_{\perp}^2}$. The cut-off $\Lambda_0$, which is introduced in the $k_z$ integral, is known not to affect the vacuum contribution to the Hall conductivity. However, the other cutoff in $\Pi'_{FS}$, which is denoted as $\Lambda$, is crucial for finite Fermi surface effects in both the type-I and type-II regime.

Using eqn. (B8), we have

$$
\sigma_{xy}^{(s)} = \sigma_0^{(s)} + \sigma_{FS}^{(s)},
$$

$$
\sigma_0^{(s)} = -e^2 n^2 a_n^2 \int_{-\Lambda_0-s(Q+\Delta_n)}^\Lambda d k_z \int_0^{\Lambda_1-\infty} \frac{k_{\perp}^{2n-1} dk_{\perp}}{2\pi} \frac{2T_{k}'}{|n'_k|^3},
$$

$$
\sigma_{FS}^{(s)} = e^2 n^2 a_n^2 \int_{-\Lambda_0-s(Q+\Delta_n)}^\Lambda d k_z \int_0^{\Lambda_1-\infty} \frac{k_{\perp}^{2n-1} dk_{\perp}}{2\pi} \times \frac{sT_{k}'}{2|n'_k|^3} \left[ n_F(C_s k_z + |n'_k| - \mu + s C_s \Delta_n) - n_F(C_s k_z - |n'_k| - \mu + s C_s \Delta_n) + 1 \right].
$$

(B10)

(B11)

(B12)

Having obtained these equation, we now have to approximate the expression $\frac{sT_{k}'}{2|n'_k|^3}$, considering the leading order contribution around the bare term $vk_z/E_k^3$ with $E_k = k^{1/2}_\perp + \sqrt{2} k^{1/2}_z$. Before that, in order to simply the calculation, we use the following change of variable $k_\perp \rightarrow k^{1/2}_\perp \alpha^{-1/2}_n$. Under these transformation: $T_{k}' = vk_z + n^2 a_n^2 \sum_{p=1}^{n-1} \beta_p^2 \alpha_n^{2(p-n)} k_\perp^{2n-p}$ with $\beta_p = (\pi C_p A_0^2)^p / p$ and $\alpha_n^2 k_\perp^{2n} = k_\perp^{2n}$. In the following approximation, we consider the fact $\omega \rightarrow 0$ due to high frequency expansion of the Floquet effective Hamiltonian and hence, $O(\omega^{-q})$, $q > 1$ terms are neglected. Therefore the denominator $|n'_k| = \sqrt{k^2_\perp + (T_{k}')^2}$ becomes

$$
k^2_\perp + (T_{k}')^2 \approx E_k^2 + \frac{2vk_z a_n^2}{\omega} \sum_{p=1}^{n-1} \beta_p^2 \alpha_n^{2(p-n)} k_\perp^{2n-p}.
$$

(B13)

Now, the main integrand is thus given by

$$
\frac{T_k}{(k^2_\perp + (T_{k}')^2)^{3/2}} \approx \frac{1}{E_k^3} (vk_z - \frac{3\sqrt{2} a_n^2}{E_k^2 \omega} X_k + \frac{\alpha_n^2}{\omega} X_k (1 - \frac{3\sqrt{2} a_n^2}{E_k^2 \omega} X_k))
$$

(B14)

with $X_k = \sum_{p=1}^{n-1} \beta_p^2 \alpha_n^{2(p-n)} k_\perp^{2n-p}$ for $n = 2$, $X_k = \beta_1^2 \alpha^{-2} k_\perp^2/2$ and for $n = 3$, $X_k = \beta_1^2 \alpha^{-4} k_\perp^4/2 + \beta_2^2 \alpha^{-2} k_\perp^2/2$. It is then more convenient to write $T_k$ explicitly for $n = 2$ as $T_k = vk_z + \beta_1^2 \alpha^{-2} k_\perp^2/2$ and for $n = 3$ as $T_k = vk_z + \beta_1^2 \alpha^{-4} k_\perp^4/2 + \beta_2^2 \alpha^{-2} k_\perp^2/2$. Therefore, the vacuum contribution becomes

$$
\sigma_{xy}^{vac} = \sigma_0^{(+)} + \sigma_0^{(-)},
$$

(B15)

$$
\sigma_0^{(s)} \approx -se^2 n a_n^{2-2/n} \int_{-\Lambda_0-s(Q+\Delta_n)}^\Lambda d k_z \int_0^{\Lambda_1-\infty} \frac{k_{\perp}^{1-2} dk_{\perp}}{2\pi} (F_{k,1} + F_{k,2} + F_{k,3}),
$$

(B16)

On the other hand, the Fermi surface contribution becomes

$$
\sigma_{xy}^{FS} = \sigma_{FS}^{(+)} + \sigma_{FS}^{(-)},
$$

(B17)

$$
\sigma_{FS}^{(s)} \approx -se^2 n a_n^{2-2/n} \int_{-\Lambda_0-s(Q+\Delta_n)}^\Lambda d k_z \int_0^{b} \frac{k_{\perp}^{1-2} dk_{\perp}}{2\pi} \left( F_{k,1} + F_{k,2} + F_{k,3} \right) \times \{ \theta(v^2 k_z^2 + (C k_z + s C \Delta_n - \mu)^2) - 1 \}
$$

(B18)
with
\[
F_{k,1} = \frac{k_x}{E_k^{(1)}}
\]
\[
F_{k,2} = \frac{\beta_2 n \alpha_n k_x^2}{E_k^{(2)}}
\]
\[
F_{k,3} = -\frac{3k_z^2 F_{k,2}}{E_k^{(3)}}
\]
for \( n = 2 \)
\[
F_{k,1} = \frac{k_z}{E_k^{(1)}}
\]
\[
F_{k,2} = \frac{\beta_2 n \alpha_n k_x^2}{E_k^{(2)}}
\]
\[
F_{k,3} = -\frac{3k_z^2 F_{k,2}}{E_k^{(3)}}
\]
for \( n = 3 \)

Importantly, one should note that \( s k_z / 2k^3 \) is the \( z \)-component of the Berry curvature of the Weyl cone with chirality \( s \). Here, \( \Theta(x) \) is the Heaviside function. Here, \( b = \sqrt{(Ck_z + sC\Delta_n - \mu)^2 - v^2k_z^2} \). Now we are in a position to treat type I and type II nWMS differently. We know \( \Delta_n \sim O(1/\omega) \) and \( \mu \) is an externally tunable parameter. Now for small \( k_z \) as considered in the low-energy model: \( b \) acquires the form \( b \approx \sqrt{\mu^2 - v^2k_z^2} \) when \( |C| \gg v \) for type I, \( b \approx \mu - ck_z \) when \( |C| \ll v \) for type II. Therefore, the point to note here is that \( \text{sgn}(b) \) is always positive of small \( k_z \) (under \( k_z \) integral) while \( \text{sgn}(b) \) can be positive and negative depending on the \( \text{sgn}(k_z) \).

We shall now explicitly write the vacuum and Fermi surface contribution for \( n = 3 \). Taking the \( T \to 0 \), and performing the \( k_\perp \) integration, we get
\[
\sigma_0 = -\frac{se^2 n \alpha_n^2}{4\pi^2} \int_{-\infty}^{\infty} dk_z \left[ \frac{v \text{sgn}(k_z)}{(C_k - \mu + sC\Delta_n)} + \frac{\beta n}{\omega \sqrt{\pi}} \Gamma\left(\frac{1}{3}\right) k_z^{1/3} \text{sgn}(k_z) \right]
\]
\[
+ \frac{\beta n}{\omega} \left[ \frac{3v^2 n \alpha_n^2}{2\sqrt{\pi}} \Gamma\left(\frac{7}{6}\right) \Gamma\left(\frac{1}{3}\right) k_z^{1/3} \text{sgn}(k_z) + \frac{\alpha_n^4}{\omega \sqrt{\pi}} \Gamma\left(\frac{4}{3}\right) \Gamma\left(\frac{1}{3}\right) k_z^{1/3} \text{sgn}(k_z) \right]
\]
(B20)

Now, while calculating the Fermi surface contribution, we consider \( k_z \to 0 \) and \( b \neq 0 \). As a result, \( k_\perp \) integral approximated by only \( b \). The Fermi surface contribution for \( n = 3 \) is thus given by
\[
d_{FS}^{(n)} \approx -\frac{se^2 n \alpha_n^2 - 2/n}{8\pi^2} \int_{-\infty}^{\infty} dk_z \left[ \frac{v k_z}{(C_k - \mu + sC\Delta_n)} + \frac{2M - 1}{\omega \sqrt{\pi}} \Gamma\left(\frac{14}{3}\right) \Gamma\left(\frac{11}{3}\right) \Gamma\left(\frac{2}{3}\right) \right]
\]
\[
\times \left[ \Theta\left(\nu^2 k_z^2 - (C_k - \mu + sC\Delta_n)^2 \right) - 1 \right],
\]
(B22)

with \( a(M) = \frac{\Gamma\left(\frac{4}{3}\right) + 2}{(2\nu + 2)(\nu^2 - 1)\Gamma\left(\frac{2}{3}\right)} \) with \( M = 1 \).

Now, the leading order contribution for type I with \( n = 3 \), \( \sigma_{xy}^{(I)} \) is given by
\[
\sigma_{xy}^{(I)} = \sigma_0 + \sigma_{FS}
\]
(B23)
\[
\sigma_0 = \frac{e^2 n \alpha_n^2 - 2/n}{2\pi^2} \left( Q + \Delta_n + \mathcal{O}\left(\frac{\beta n}{\omega}\right) + \mathcal{O}\left(\frac{2n}{\omega}\right) \right)
\]
(B24)
\[
\sigma_{FS} = n \frac{\alpha_n^2 - 2/n}{v} \left[ \frac{e^2}{4\pi^2} \left( C\mu - C\Delta_n \right) + \mathcal{O}\left(\frac{\beta n}{\omega}\right) + \mathcal{O}\left(\frac{2n}{\omega}\right) \right]
\]
(B25)

Now, the leading order contribution for type I with \( n = 3 \), \( \sigma_{xy}^{(II)} \) is given by
\[
\sigma_{xy}^{(II)} = \sigma_0 + \sigma_{FS}
\]
(B26)
\[
\sigma_0 = \frac{e^2 n \alpha_n^2 - 2/n}{2\pi^2} \left( Q + \Delta_n + \mathcal{O}\left(\frac{\beta n}{\omega}\right) + \mathcal{O}\left(\frac{2n}{\omega}\right) \right)
\]
(B27)
\[
\sigma_{FS} = n \frac{e^2 \alpha_n^2 - 2/n}{v} \left[ -\frac{v(c\beta_1 - \mu)}{C^2} \ln\left(\frac{C^2 \Delta}{v(c\beta_1 - \mu)}\right) + \mathcal{O}\left(\frac{\beta n}{\omega}\right) + \mathcal{O}\left(\frac{2n}{\omega}\right) \right]
\]
(B28)

\[\text{with} \quad 1. \quad \text{M. Z. Hasan, S.-Y. Xu, I. Belopolsky, and S.-M. Huang, Annu. Rev. Condens. Matter Phys. 8, 289 (2017).} \]
\[\text{2. N. P. Armitage, E. J. Mele, and A. Vishwanath, Rev. Mod. }\]
