power structures of directed spaces

Xiaolin Xie, Yuxu Chen*, Hui Kou*

Department of Mathematics, Sichuan University, Chengdu, 610064, China

Abstract

Powerdomains in domain theory plays an important role in modeling the semantics of nondeterministic functional programming languages. In this paper, we extend the notion of powerdomain to the category of directed spaces, which is equivalent to the notion of the $T_0$ monotone-determined space [4]. We define the notion of upper, lower and convex powerspace of a directed space by the way of free algebras. We show that the upper, lower and convex powerspace over any directed space exist and give their concrete structures. Generally, the upper, lower and convex powerspaces of a directed spaces are different from the upper, lower and convex powerdomains of a dcpos endowed with the Scott topology and the observationally-induced upper and lower powerspaces introduced by Battenfeld and Schöder in 2015.

Keywords: powerdomain, directed lower powerspace of directed spaces, directed upper powerspace of directed spaces, directed convex powerspace of directed spaces, observationally-induced lower powerspace, observationally-induced lower powerspace

Mathematics Subject Classification: 06B35; 54A20; 54B30; 54H10

1. Introduction

Powerdomain is one of the most important part of domain theory. Its purpose is to provide a mathematical model for the semantics of nondeterministic functional programming languages. There are three classical power structures in domain theory, they are the lower powerdomain (also known as the Hoare powerdomain), the upper powerdomain (also known as the Smyth powerdomain) and the convex powerdomain (also known as the Plotikin powerdomain). Moreover, each power structure has a standard topological representation. In recent years, papers [6, 38, 39, 40] have made a lot of generalizations on these power structures. Generally speaking, these power structures are free algebras generated by domain respect to some binary operation. In 2015, I. Battenfeld and M. Schöder ([2]) introduced power structure in general topological spaces, and in this paper, Boolean algebra 2 is endowed with Sierpinski topology as an observable structure, and then the upper power structure and lower power structure are defined, which is called the observationally-induced upper(lower) powerspace. The work of paper [2] makes it possible for the category of general topological space to be applied to express the nondeterministic semantics of functional programming languages. But the method of power structure induced by Boolean
algebra 2 is complicated and difficult to understand. In this paper, we consider a class of special topological spaces, the directed space (equivalent to those in [4]). It is proved in [37] that directed spaces contain the basic objects of domain theory, all directed complete posets endowed with the Scott topology, and continuous functions, which forms a cartesian closed category. Thus, the directed space is an extended framework of domain theory. In this paper, we are going to extend the structure of powerdomain to the category of directed spaces, and define the concept of directed upper (lower, convex) powerspace by means of free algebra. We show that the upper, lower and convex powerspace over any directed space exist and give their concrete structures. Generally, the upper, lower and convex powerspaces of a directed spaces are different from the upper, lower and convex powerdomains of a dcpo endowed with the Scott topology and the observationally-induced upper and lower powerspaces.

In 1991, Heckmann introduced an algebraic method and it does not rely on any explicit representations of the powerdomains [15]. In the last part of this article, we will discuss the commutativity of the directed upper and lower functors.

2. Preliminaries

Now, we introduce the concepts needed in this article. On domain theory, topology, and category theory, see [1, 7, 23]. Let P be a nonempty set. A relation ≤ on P is called a partial order, if ≤ satisfies reflexivity (x ≤ x), transitivity (x ≤ y & y ≤ z ⇒ x ≤ z) and antisymmetry (x ≤ y & y ≤ x ⇒ x = y). P is called a partial ordered set(poset) if P is endowed with some partial order ≤. Given A ⊆ P, denote ↓A = {x ∈ P : ∃a ∈ A, x ≤ a}, ↑A = {x ∈ P : ∃a ∈ A, a ≤ x}. We say A is a lower set (upper set) if A = ↓A (A = ↑A). A nonempty set D ⊆ P is called a directed set if each finite nonempty subset of D has upper bound in D. Particularly, a poset is called a directed complete poset if each directed subset has a supremum(denoted by ∨ D), abbreviated as dcpo. The subset U of poset P is called a Scott open set if U is an upper set and for each directed set D ⊆ P, which ∨ D exists and belongs to U, then U ∩ D ⊇ ∅. The set of all Scott open sets of poset P is a topology on P, which is called the Scott topology and denoted by σ(P). Suppose P, E are two posets, a function f : P → E is called Scott continuous if it is continuous respect to Scott topology σ(P) and σ(E).

All topological spaces in this paper are required T0 separation. A net of a topological space X is a map χ : J → X, here J is a directed set. Thus, each directed subset of a poset can be regarded as a net, and its index set is itself. Usually, we denote a net by (xj)j∈J or (xj). Let x ∈ X, saying (xj) converges to x, denote by (xj) → x or x = lim xj, if (xj) is eventually in every open neighborhood of x, that is, for each given open neighborhood U of x, there exists j0 ∈ J such that for every j ∈ J, j ≥ j0 ⇒ xj ∈ U.

Let X be a T0 topological space, its topology is denoted by O(X), the specialization order on X is defined as follows:

∀x, y ∈ X, x ⊥ y ⇔ x ∈ {y}

here, {y} means the closure of {y}. From now on, the order of a T0 topological space always indicates the specialization order "⊥". Here are some basic properties of specialization order.

Proposition 2.1. [1, 7] For a T0 topological space X, the following are always true:
1. For each open set $U \subseteq X$, $U = \uparrow U$;
2. For each closed set $A \subseteq X$, $A = \downarrow A$;
3. Suppose $Y$ is another $T_0$ topological space, and $f : X \to Y$ is a continuous function from $X$ to $Y$. Then for each $x, y \in X$, $x \subseteq y \Rightarrow f(x) \subseteq f(y)$, that is, every continuous function is monotone.

Suppose $X$ is a $T_0$ space, then every directed set $D \subseteq X$ can be regarded as a net of $X$, we use $D \to x$ or $x \equiv \lim D$ to represent $D$ converges to $x$. Define notation

$$D(X) = \{(D, x) : x \in X, D \text{ is a directed subset of } X \}.$$  

It is easy to verify that, for each $x, y \in X$, $x \sqsubseteq y \iff \{y\} \to x$. Therefore, if $x \sqsubseteq y$ then $\{(y), x\} \in D(X)$. Next, we give the concept of directed space.

**Definition 2.2.** Let $X$ be a $T_0$ space.

1. A subset $U$ of $X$ is called a directed open set if $\forall (D, x) \in D(X)$, $x \in U \Rightarrow D \cap U \neq \emptyset$.
2. $X$ is called a directed space if each directed open set of $X$ is an open set, that is, $d(X) = O(X)$.

**Remark 2.3.**

1. Each open set of a $T_0$ space is directed open, but the contrary is not necessarily true. For example, suppose $Y$ is a non-discrete $T_1$ topological space, its specialization order is diagonal, that is, $\forall x, y \in Y$, $x \subseteq y \iff x = y$. Thus, all subsets of $Y$ are directed open. We notice that $Y$ is non-discrete, at least one directed open set is not an open set.
2. The definition of directed space here is equivalent to the $T_0$ monotone determined space defined in [4].
3. Every poset endowed with the Scott topology is a directed space [21, 37], besides, each Alexandroff space is a directed space. Thus, the directed space extends the concept of the Scott topology.

Next, we introduce the directed continuous function.

**Definition 2.4.** Suppose $X, Y$ are two $T_0$ spaces. A function $f : X \to Y$ is called directed continuous if it is monotone and preserves all limits of directed set of $X$; that is, $(D, x) \in D(X) \Rightarrow (f(D), f(x)) \in D(Y)$.

Here are some characterizations of the directed continuous functions.

**Proposition 2.5.** Suppose $X, Y$ are two $T_0$ spaces. $f : X \to Y$ is a function between $X$ and $Y$.

1. $f$ is directed continuous if and only if $\forall U \in d(Y)$, $f^{-1}(U) \in d(X)$.
2. If $X, Y$ are directed spaces, then $f$ is continuous if and only if it is directed continuous.

Now we introduce the product of directed spaces.

Suppose $X, Y$ are two directed spaces. Let $X \times Y$ represents the cartesian product of $X$ and $Y$, then we have a natural partial order on it: $\forall (x_1, y_1), (x_2, y_2) \in X \times Y$,

$$(x_1, y_1) \leq (x_2, y_2) \iff x_1 \sqsubseteq x_2, y_1 \sqsubseteq y_2,$$

which is called the pointwise order on $X \times Y$. Now, we define a topological space $X \otimes Y$ as follows:
1. The underlying set of \( X \otimes Y \) is \( X \times Y \);
2. The topology on \( X \times Y \) is generated as follows: for each given \( \leq \) directed set \( D \subseteq X \times Y \) and \((x, y) \in X \times Y\),

\[ D \rightarrow (x, y) \in X \otimes Y \iff \pi_1D \rightarrow x \in X, \ \pi_2D \rightarrow y \in Y, \]

That is, a subset \( U \subseteq X \times Y \) is open if and only if for every directed limit defined as above \( D \rightarrow (x, y), \ (x, y) \in U \Rightarrow U \cap D \neq \emptyset \).

Theorem 2.6. \([37]\) Suppose \( X \) and \( Y \) are two directed spaces.

1. The topological space \( X \otimes Y \) defined as above is a directed space and satisfies the following properties: the specialization order on \( X \otimes Y \) equals to the pointwise order on \( X \times Y \), that is \( \sqsubseteq = \leq \).
2. Suppose \( Z \) is another directed space, then \( f : X \otimes Y \rightarrow Z \) is continuous if and only if it is continuous in each variable separately.

Denote the category of all directed spaces with continuous functions by \( \text{Dtop} \). It is proved in \([36, 37]\) that, \( \text{Dtop} \) contains all posets endowed with the Scott topology and \( \text{Dtop} \) is a cartesian closed category; specifically, the categorical products of two directed spaces \( X \) and \( Y \) are isomorphic to \( X \otimes Y \). So, the directed space is an extended framework of Domain Theory.

Let \( P \) be a dcpo, and \( x, y \in P \). We say \( x \) way below \( y \), if for each given directed set \( D \subseteq P \), \( y \leq \bigvee D \) implies there exists some \( d \in D \) such that \( x \leq d \). We write \( \downarrow x = \{ a \in P : a \ll x \} \), \( \uparrow x = \{ a \in P : x \ll a \} \).

Definition 2.7. A dcpo \( P \) is called a continuous domain if for each \( x \in P \), \( \downarrow x \) is directed and \( x = \bigvee \downarrow x \).

Theorem 2.8. \([7]\) Suppose \( P \) is a continuous domain. The followings hold:

1. \( \forall x, y \in P, \ x \ll y \Rightarrow \exists z \in P, \ x \ll z \ll y \).
2. \( \forall x \in P, \ \uparrow x \) is a Scott open set. Particularly, \( \{ \uparrow x : x \in P \} \) is a base of \((P, \sigma(P))\).

3. The directed lower powerspaces of directed spaces

As mentioned above, directed space is a extended framework of domain theory, just like the work done in article \([2]\), extending the powerdomain in the category of directed space is very meaningful. In this section, we will construct the directed lower powerspace of the directed space, which is a free algebra generated by the inflationary operation of the directed space.

Definition 3.1. \([34]\) Let \( X \) be a directed space.

1. A binary operation \( \oplus : X \otimes X \rightarrow X \) on \( X \) is called an inflationary operation if it is continuous and satisfies the following four conditions: \( \forall x, \ y, \ z \in X \),

(a) \( x \oplus x = x \),
(b) \( (x \oplus y) \oplus z = x \oplus (y \oplus z) \),
(c) \( x \oplus y = y \oplus x \),
(d) \( x \oplus y \geq x \).
If \( \oplus \) is a inflationary operation on \( X \), then \((X, \oplus)\) is called a directed inflationary semilattice, that is, directed inflationary semilattices are those directed spaces with inflationary operations.

By Theorem 2.6(2), the operation \( \oplus \) on a directed space \( X \) is continuous if and only if it is monotone and for each given \( x, y \in X \) and directed set \( D \subseteq X \), \( x \equiv \lim D \) implies \( x \oplus y \equiv \lim(D \oplus y) \). Here, \( D \oplus y = \{ d \oplus y : d \in D \} \).

Here are two examples.

**Example 3.2.**

1. Suppose \( P \) is a poset endowed with the Scott topology, and for each \( a, b \in P \), the supremum of \( a \) and \( b \) exists in \( P \) (denoted by \( a \lor b \)). Then \((P, \lor)\) is a directed inflationary semilattice.

2. Let \( I = [0, 1] \) (the unit interval), let \( A \) denote the topology generated by \( \{ [a, 1] : a \in I \} \), then \( A \) is the Alexandroff topology on \( I \). It is easy to check that \((I, A)\) is a directed space, and \((I, \max)\) is a directed inflationary semilattice endowed with \( A \).

**Definition 3.3.**

Suppose \((X, \oplus)\), \((Y, \sqcup)\) are two directed inflationary semilattices, \( f : (X, \oplus) \to (Y, \sqcup) \) is called an inflationary homomorphism between \( X \) and \( Y \), if \( f \) is continuous and \( f(x \oplus y) = f(x) \sqcup f(y) \) holds, \( \forall x, y \in X \).

Denote the category of all directed inflationary semilattices and inflationary homomorphisms by \( \text{Disl} \). Then \( \text{Disl} \) is a subcategory of \( \text{Dtop} \).

**Lemma 3.4.**

Suppose \((X, \oplus)\) is a directed inflationary semilattice, then \( \oplus = \lor \). Here, \( \forall x, y \in X, x \lor y \) means the supremum of \( x \) and \( y \) respect to the specialization order \( \sqsubset \) on \( X \) (calling it sup operation). Conversely, suppose \( X \) is a directed space such that for each \( x, y \in X \), \( x \lor y \) exists, and the continuity of \( \lor \) will naturally imply that \((X, \lor)\) is a directed inflationary semilattice.

**Proof** By definition 3.1, \( \forall x, y \in X, x \lor y \geq x, y \), that is \( x \lor y \) is a upper bound of \( \{x, y\} \). Suppose \( z \) is another arbitrary upper bound of \( \{x, y\} \). By Theorem 2.6(1), the pointwise order equals to the specialization order of \( X \otimes X \), then \( (x, y) \sqsubset (z, z) \). By the continuity and idempotence of the inflationary operation, we have \( x \otimes y \sqsubseteq z \otimes z = z \). That is \( x \otimes y \) is the supremum of \( \{x, y\} \), which means \( x \otimes y = x \lor y \). Conversely, continuous sup operation naturally satisfy all conditions in definition 5.1, thus, we get the conclusion. □

The above results show that a directed inflationary semilattice \((X, \oplus)\) is just a directed space with a continuous sup operation satisfy \( \oplus = \lor \). Since the order on the directed space in this paper is always the specialization order, we will use the symbol \( \lor \) instead of \( \lor \) in the following part. Therefore, a directed inflationary semilattice is always represented by a tuple of the form \((X, \lor)\), here \( X \) is a directed space, \( \lor \) represents the continuous sup operation on \( X \).

Next, we give the definition of directed lower powerspace.

**Definition 3.5.**

Suppose \( X \) is a directed space. A directed space \( Z \) is called the directed lower powerspace of \( X \) if and only if the following two conditions are satisfied:

1. \( Z \) is a directed inflationary semilattice, that is the sup operation \( \lor \) on \( Z \) exists and which is continuous,
(2) There is a continuous function \( i : X \rightarrow Z \) satisfying: for an arbitrary directed inflationary semilattice \((Y, \lor)\) and continuous function \( f : X \rightarrow Y \), there exists an unique inflationary homomorphism \( \bar{f} : (Z, \lor) \rightarrow (Y, \lor) \) such that \( f = \bar{f} \circ i \).

By the definition above, if directed inflationary semilattices \((Z_1, \lor)\) and \((Z_2, \lor)\) are both the directed lower powerspaces of \(X\), then there exists a topological homomorphism which is also an inflationary homomorphism \( g : Z_1 \rightarrow Z_2 \). Therefore, up to of order isomorphism and topological homomorphism, the directed lower powerspace of a directed space is unique. Particularly, we denote the directed lower powerspace of each directed space \(X\) by \( P_L(X) \).

Now, we will prove the existence of the directed lower powerspace of each directed space \(X\) by way of concrete construction.

Let \(X\) be a directed space. Set
\[
LX = \{ \downarrow F : F \subseteq_{\text{fin}} X \},
\]
here, \( F \subseteq_{\text{fin}} X \) is an arbitrary nonempty finite subset of \(X\). Define an order \( \leq_L \) on \(LX\) as follows:
\[
\downarrow F_1 \leq_L \downarrow F_2 \iff \downarrow F_1 \subseteq \downarrow F_2.
\]
Let \(D \subseteq LX\) be a directed set (respect to order \(\leq_L\)), \(\downarrow F \in LX\). Define a convergence notation \(D \Rightarrow_L \downarrow F\) as follows:
\[
D \Rightarrow_L \downarrow F \iff \forall a \in F, \text{ there exists a directed set } D_a \text{ of } X \text{ satisfying } D_a \subseteq \bigcup D \text{ and } D_a \rightarrow a.
\]
A subset \(U \subseteq LX\) is called a \(\Rightarrow_L\) convergence open set of \(LX\) if and only if for each directed subset \(D\) of \(LX\) and \(\downarrow F \in LX\), \(D \Rightarrow_L \downarrow F \in U\) implies \(D \cap U \neq \emptyset\). Denote all \(\Rightarrow_L\) convergence open sets of \(LX\) by \(O_{\Rightarrow_L}(LX)\).

**Proposition 3.6.** \[34\] Suppose \(X\) is a directed space, the following are true:

1. \((LX, O_{\Rightarrow_L}(LX))\) is a topological space, abbreviated as \(LX\).
2. The specialization order \(\subseteq\) on \((LX, O_{\Rightarrow_L}(LX))\) equals to \(\leq_L\).
3. \((LX, O_{\Rightarrow_L}(LX))\) is a directed space, that is \(O_{\Rightarrow_L}(LX) = d(LX)\).

**Proof** (1) Firstly \(\emptyset, \ LX \in O_{\Rightarrow_L}(LX)\). Suppose \(U \in O_{\Rightarrow_L}(LX), \downarrow F_1, \downarrow F_2 \in LX\). If \(\downarrow F_1 \leq \downarrow F_2\), then \(\{\downarrow F_2\} \Rightarrow_L \downarrow F_1\) is obviously hold. Therefore \(\downarrow F_1 \in U\) will imply \(\downarrow F_2 \in U\). This means \(U\) is an upper set respect to order \(\leq_L\) on \(LX\). Suppose we have \(U_1, U_2 \in O_{\Rightarrow_L}(LX)\), and \(D\) is a directed set in \(LX\), \(\downarrow F \in LX\) and \(D \Rightarrow \downarrow F \in U_1 \cap U_2\), there will exist \(\downarrow G_1 \in D \cap U_1\) and \(\downarrow G_2 \in D \cap U_2\). Since \(D\) is directed, there exists \(\downarrow G \in D\) such that \(\downarrow G_1, \downarrow G_2 \leq_L \downarrow G\), tauh is \(\downarrow G \in D \cap U_1 \cap U_2\). By the same way, \(O_{\Rightarrow_L}(LX)\) is closed under the arbitrary union. Thus, \(O_{\Rightarrow_L}(LX)\) is a topology on \(LX\).

(2) Let \(\downarrow F_1, \downarrow F_2 \in LX\). If \(\downarrow F_1 \leq_L \downarrow F_2\), according the proof of (1), every \(\Rightarrow_L\) convergence open set is an upper set respect to partial order \(\leq_L\), then \(\downarrow F_1 \in \{\downarrow F_2\}\), that is, \(\downarrow F_1 \subseteq \downarrow F_2\). On the other hand, let \(\downarrow F_1 \subseteq \downarrow F_2\). We want to prove \(\downarrow F_1 \leq_L \downarrow F_2\), that is \(\{\downarrow F \in LX : \downarrow F \subseteq \downarrow F_1\}\) is a closed set of \(LX\) respect to the topology \(O_{\Rightarrow_L}(LX)\), since \(\{\downarrow F \in LX : \downarrow F \subseteq \downarrow F_2\} \subseteq \{\downarrow F_2\}\), \(\{\downarrow F \in LX : \downarrow F \subseteq \downarrow F_2\}\) is a closed set of \(LX\) respect to the topology \(O_{\Rightarrow_L}(LX)\) will imply \(\downarrow F \in
According to (2), $D \cup \emptyset \subseteq \mathcal{O}$, where $\mathcal{O}$ is the directed open set, $D \Rightarrow \mathcal{O}$, thus $D$ is closed respect to topology $\mathcal{O}$. By Theorem 3.8, we can directly check that for arbitrary $D \subseteq \mathcal{L}, D \cup G \notin \mathcal{U}$, where $G = \{a_1, a_2, \ldots, a_k\}$. By the definition of $\Rightarrow_L$ convergence, we have finitary directed subsets $D_i \subseteq \bigcup \mathcal{D}$ in $X$ such that $D_i \rightarrow a_i$, $i = 1, 2, \ldots, k$. By contradiction, we suppose that $\mathcal{D} \cap \mathcal{U} = \emptyset$. Then $\bigcup \mathcal{D} \subseteq \downarrow F_2$, that is $D_i \subseteq \downarrow F_2$, $i = 1, 2, \ldots, k$. Since $\downarrow F_2$ is closed in $X$, the limit points of $D_i$ are in $\downarrow F_2$, that is $G = \{a_1, a_2, \ldots, a_k\} \subseteq \downarrow F_2$, and, $\downarrow G \leq_L \downarrow F_2$. This contradicts with $\downarrow G \notin \mathcal{U}$. Therefore, $\mathcal{U} = \mathcal{L} \setminus \{\downarrow F \in \mathcal{L} : \downarrow F \subseteq \downarrow F_2\}$ is a $\Rightarrow_L$ convergence open set in $\mathcal{L}$.

(3) For an arbitrary topological space $X$, we have $\mathcal{O}(X) \subseteq d(X)$, then $\Rightarrow_L(LX) \subseteq d(LX)$. On the other hand, according the definition of $\Rightarrow_L$ convergence, if directed set $\mathcal{D} \Rightarrow_L \downarrow F$ in $\mathcal{L}$, according to (2), $\mathcal{D}$ converges to $\downarrow F$ respect to the topology $\Rightarrow_L(LX)$. Then, by the definition of directed open set, $\mathcal{D} \Rightarrow_L \downarrow F \in \mathcal{U} \subset d(LX)$, will imply $\mathcal{U} \cap \mathcal{D} \neq \emptyset$. This means $\mathcal{U} \subset \Rightarrow_L(LX)$, thus $\Rightarrow_L(LX) = d(LX)$, that is, $(LX, \mathcal{O}_{\Rightarrow_L(LX)})$ is a directed space. $\square$

**Proposition 3.7.** Suppose $X, Y$ are two directed spaces. Then function $f : \mathcal{L}X \rightarrow Y$ is continuous if and only if for each directed set $\mathcal{D} \subseteq \mathcal{L}X$ and $\downarrow F \in \mathcal{L}X$, $\mathcal{D} \Rightarrow_L \downarrow F$ implies $f(\mathcal{D}) \rightarrow f(\downarrow F)$.

**Proof** Since $\Rightarrow_L$ convergence will lead to $\Rightarrow_L(LX)$ topological convergence, the necessity is obviously. We are going to prove the sufficiency. First to check that $f$ is monotone. If $\downarrow F_1, \downarrow F_2 \in \mathcal{L}X$ and $\downarrow F_1 \leq_L \downarrow F_2$, then $\{\downarrow F_2\} \Rightarrow_L \downarrow F_1$, by the given condition, $\{f(\downarrow F_2)\} \rightarrow f(\downarrow F_1)$, thus $f(\downarrow F_2) \subseteq f(\downarrow F_1)$. Suppose $\mathcal{U}$ is an open set of $Y$ and the directed set $\mathcal{D} \Rightarrow_L \downarrow F \in f^{-1}(\mathcal{U})$, then $f(\mathcal{D})$ is a directed set of $Y$ and $f(\mathcal{D}) \rightarrow f(\downarrow F) \in \mathcal{U}$, there exists $a \downarrow F \in \mathcal{D}$ such that $f(\downarrow F) \in \mathcal{U}$. Thus, $\downarrow F \subseteq \mathcal{D} \cap f^{-1}(\mathcal{U})$. According to the definition of $\Rightarrow_L$ convergence open set, $f^{-1}(\mathcal{U}) \subset \Rightarrow_L(LX)$, that is $f$ is continuous. $\square$

**Theorem 3.8.** Let $X$ be a directed space. Then $(\mathcal{L}X, \Rightarrow_L(LX))$ respect the set union operation $\cup$ is a directed inflationary semilattice.

**Proof** According to Proposition 3.6 $(\mathcal{L}X, \Rightarrow_L(LX))$ is a directed space. We will prove that $\cup$ is an inflationary operation. For arbitrary $\downarrow F_1, \downarrow F_2 \in \mathcal{L}X$, then $\downarrow F_1 \cup \downarrow F_2 = \downarrow (F_1 \cup F_2) \in \mathcal{L}X$. Obviously, $\cup$ satisfy the conditions (a), (b), (c), (d) in Definition 3.1, we only need to prove that, for each directed set $\mathcal{D} \subseteq \mathcal{L}X$ and $\downarrow F \in \mathcal{L}X$, $\mathcal{D} \Rightarrow_L \downarrow F$, will imply $G \cup \mathcal{D} \Rightarrow_L \downarrow (G \cup F) = \downarrow (G \cup F)$. Here, $G \cup \mathcal{D} = \{\downarrow (G \cup F') : \downarrow F' \in \mathcal{D}\}$ is a directed set. For each $a \in G \cup F$, if $a \in G$, then $\{a\} \rightarrow a$; if $a \in F$, since $\mathcal{D} \Rightarrow_L \downarrow F$, there exists $D \subseteq \mathcal{D}$ such that $D \rightarrow a$. According to the definition of $\Rightarrow_L$ convergence, we have $G \cup \mathcal{D} \Rightarrow_L \downarrow (G \cup F)$. $\square$

**Remark 3.9.** We can directly check that for arbitrary $\downarrow F_1, \downarrow F_2 \in \mathcal{L}X$, $\downarrow F_1 \cup \downarrow F_2 = \downarrow (F_1 \cup F_2)$ is well-defined although each $\downarrow F$ may generated by different $F$.

The following theorem is the main result in this section.

**Theorem 3.10.** Suppose $X$ is a directed space, then $(\mathcal{L}X, \Rightarrow_L(LX))$ is the lower powerspace of $X$, that is, endowed with topology $\Rightarrow_L(LX)$, $(\mathcal{L}X, \cup) \cong P_L(X)$.

**Proof** Define function $i : X \rightarrow \mathcal{L}X, \forall x \in X$, $i(x) = \downarrow x$. We prove the continuity of $i$. It is evident that $i$ is monotone. If $D \subseteq X$ and $x \in X$ satisfy $D \rightarrow x$. Let $\mathcal{D} = \{\downarrow d : d \in D\}$, then
$D$ is a directed set in $LX$ and $D \Rightarrow_L x$. But $i(D) = D$, so $i(D) \Rightarrow_L x = i(x)$. According to Proposition 3.7, $\bar{f}$ is continuous.

Let $(Y, \lor)$ be an arbitrary directed inflationary semilattice, $f : X \to Y$ is a continuous function. Define $\bar{f} : LX \to Y$ as follows: $\forall \downarrow F \in LX$ (let $F = \{a_1, a_2, \ldots, a_n\}$),

$$\bar{f}(\downarrow F) = f(a_1) \lor f(a_2) \lor \cdots \lor f(a_n) = \bigvee_{a \in F} f(a).$$

$\bar{f}$ is obvious well-defined. Especially, denote $\bar{f}(\downarrow F) = \lor f(F)$.

1. $f = \bar{f} \circ i$.

For arbitrary $x \in X$, $(f \circ i)(x) = f(i(x)) = f(\downarrow x) = f(x)$.

2. $\bar{f}$ is an inflationary homomorphism, that is, $\bar{f}$ is continuous and for arbitrary $\downarrow F_1, \downarrow F_2 \in LX$, $f(\downarrow F_1 \cup \downarrow F_2) = f(\downarrow F_1) \lor f(\downarrow F_2)$.

First, we prove that $\bar{f}$ preserves the union operation. Let $\downarrow F_1, \downarrow F_2 \in LX$. Then $\bar{f}(\downarrow F_1 \cup \downarrow F_2) = \bar{f}((\downarrow F_1 \cup \downarrow F_2)) = \lor f(F_1 \cup F_2) = (\lor f(F_1) \lor (\lor f(F_2))) = \lor f(F_1) \lor \bar{f}(\downarrow F_2)$. Next, we prove the continuity of $\bar{f}$. Since $\lor$ is the sup operation, $\bar{f}$ is monotone. Suppose $\mathcal{D} \subseteq LX$ and $\mathcal{D} \subseteq LX$ satisfy $\mathcal{D} \Rightarrow_L G$. Let $G = \{b_1, b_2, \ldots, b_k\}$. By the definition of $\Rightarrow_L$ convergence, for each $b_i \in G$, there exists a directed set $D_i \subseteq \bigcup \mathcal{D}$ such that $D_i \to b_i$. By the continuity of $f$, we have $f(D_i) \to f(b_i)$, $i = 1, 2, \ldots, k$. Since $\lor$ is continuous, the following convergence hold in $Y$:

$$f(D_1) \lor f(D_2) \lor \cdots \lor f(D_k) \to f(b_1) \lor \cdots \lor f(b_k).$$ (**)

Here,

$$f(D_1) \lor f(D_2) \lor \cdots \lor f(D_k) = \{f(d_1) \lor f(d_2) \lor \cdots \lor f(d_k) : (d_1, d_2, \ldots, d_k) \in \prod_{i=1}^{k} D_i\}.$$

Suppose $(d_1, d_2, \ldots, d_k) \in \prod_{i=1}^{k} D_i$. For arbitrary $i \in \{1, 2, \ldots, k\}$, there exists $\downarrow F_i \in \mathcal{D}$ such that $d_i \in \downarrow F_i$. Since $\mathcal{D}$ is directed, there exists some $\downarrow F \in \mathcal{D}$ such that $\downarrow F_i \subseteq \downarrow F$. Therefore $f(d_1) \lor \cdots \lor f(d_n) \subseteq (\lor f(F_1)) \lor \cdots \lor (\lor f(F_k)) \subseteq \lor f(F)$. This means that,

$$f(D_1) \lor \cdots \lor f(D_n) \subseteq \downarrow \{\lor f(F) : \downarrow F \in \mathcal{D}\}.$$ (***)

Let $U \subseteq Y$ be an open neighborhood of $f(b_1) \lor f(b_2) \lor \cdots \lor f(b_k)$. By (**), there exists some $(d_1, d_2, \ldots, d_k) \in \prod_{i=1}^{k} D_i$ such that $f(d_1) \lor f(d_2) \lor \cdots \lor f(d_k) \in U$. For each open set is an upper set, by (**), there exists some $\downarrow F \in \mathcal{D}$ such that $\lor f(F) \in U$. That is $\bar{f}(\mathcal{D}) \to \bar{f}(\downarrow G)$. Therefore, by Proposition 3.7, $\bar{f}$ is continuous.

3. Inflationary homomorphism $\bar{f}$ is unique.

Suppose there is an inflationary homomorphism $g : (LX, \cup) \to (Y, \lor)$ satisfy $f = g \circ i$. Then $g(\downarrow x) = f(x) = \bar{f}(\downarrow x)$. For each $\downarrow F \in LX$ (let $F = \{a_1, \ldots, a_n\}$),

$$g(\downarrow F) = g(\downarrow a_1 \cup \downarrow a_2 \cdots \cup \downarrow a_n)$$

$$= g(\downarrow a_1) \lor g(\downarrow a_2) \lor \cdots \lor g(\downarrow a_n)$$

$$= \bar{f}(\downarrow a_1) \lor \bar{f}(\downarrow a_2) \lor \cdots \lor \bar{f}(\downarrow a_n)$$

$$\bar{f}(\downarrow a_1 \cup \downarrow a_2 \cdots \cup \downarrow a_n)$$

$$\bar{f}(\downarrow F).$$
Thus $f$ is unique.

In conclusion, according to Definition 3.5, endowed with topology $O_{\Rightarrow_{\geq}}(LX)$, the directed inflationary semilattice $(LX, \cup)$ is the lower powerspace of $X$, that is, $PL(X) \cong (LX, \cup)$. □

The lower powerspace is unique in the sense of order isomorphism and topological homomorphism, so we can directly denote the lower powerspace of each directed space $X$ by $PL(X) = (LX, \cup)$.

Suppose $X$, $Y$ are two directed spaces, $f : X \to Y$ is a continuous function. Define map $PL(f) : PL(X) \to PL(Y)$ as follows: $\forall \downarrow F \in LX$,

$$PL(f)(\downarrow F) = \downarrow f(F).$$

it is evident that $PL(f)$ is well-defined and order preserving. According to 3.10, it is easy to check that $PL(f)$ is an inflationary homomorphism between these two lower powerspaces. If $id_X$ is the identity function and $g : Y \to Z$ is an arbitrary continuous function from $Y$ to a directed space $Z$, then, $PL(id_X) = id_{PL(X)}$, $PL(g \circ f) = PL(g) \circ PL(f)$. Thus, $PL : Dtop \to Disl$ is a functor from $Dtop$ to $Disl$. Let $U : Disl \to Dtop$ be the forgetful functor, by Theorem 3.10, we have the following result.

**Corollary 3.11.** 3.10 $PL$ is a left adjoint of the forgetful functor $U$, that is, $Disl$ is a reflective subcategory of $Dtop$.

4. Relations Between Lower Powerspaces

In this section, we will discuss the relation between the lower powerdomain of dcpo, observationally-induced lower powerspaces and directed lower powerspaces.

Suppose $X$ is a topological space (denote the topology by $O(X)$). Let $C(X)$ be the set of all nonempty closed sets of $X$. Obviously, $C(X)$ is closed under the set operation $\cup$. For each $U \in O(X)$, let

$$\langle U \rangle = \{ A \in C(X) : A \cap U \neq \emptyset \}.$$ 

It is easy to check that, $\{ \langle U \rangle : U \in O(X) \}$ is closed under finite intersection and arbitrary union, it forms a topology on $C(X)$, called the lower Vietoris topology of $C(X)$ and denoted by $VL(C(X))$. Besides, $C(X)$ is a dcpo respect to the inclusion order of set. Denote the Scott topology on $C(X)$ by $\sigma(C(X))$. We can check that, the lower Vietoris topology $VL(C(X))$ is contained in the Scott topology $\sigma(C(X))$, that is, $VL(C(X)) \subseteq \sigma(C(X))$. Particularly, write $P_{OL}(X) = (C(X), \cup)$ endowed with the lower Vietoris topology $VL(C(X))$, write $H(X) = (C(X), \cup)$ endowed with the Scott topology $\sigma(C(X))$.

**Theorem 4.1.** 7. Corollary IV-8.6] Let $P$ be a dcpo endowed with the Scott topology $\sigma(P)$. Then $H(P)$ isomorphic to the lower powerdomain of $P$ (which is also called Hoare powerdomain), that is, $H(P)$ is free dcpo sup semilattice of generated by $P$.

**Theorem 4.2.** 2. Theorem 3.8] Let $X$ be a topological space. Then $P_{OL}(X)$ is isomorphic to the observationally-induced lower powerspace over $X$. 

9
Let $P$ be a dcpo, endowed with the Scott topology, then $(P, \sigma(P))$ is also a directed space. By Theorem 3.10 the directed lower powerspace $P_L(P) = (LP, \cup)$, endowed with topology $O_{\Rightarrow_L}(LP)$, compared with the lower powerdomain and the observationally-induced lower powerspace over $P$, in which the operation are both the union of sets $\cup$, $LP$ is a subset of $C(P)$, and in general, $LP \neq C(P)$. For the topological structure, $P_L(P)$ is neither necessarily a subspace of lower powerdomain $H(P)$, nor a subspace of the observationally-induced lower powerspace $P_{OL}(P)$.

The following example tells us that, in general, a dcpo $P$, endowed with Scott topology, its lower powerdomain is not equal to its observationally-induced lower powerspace, that is, $H(P) \neq P_{OL}(P)$.

**Example 4.3.** Let $P = (\mathbb{N} \times \mathbb{N}) \cup \{\infty\}$, here $\mathbb{N}$ is the set of natural numbers. Define a partial order on $P$ as follows: $\forall x, y \in P, x \leq y$ if and only if one of the following conditions is true:

- $y = \infty$,
- $\exists n_0 \in \mathbb{N}, x = (m, n_0), y = (m', n_0)$ and $m' - m \geq 0$.

Evidently, $P$ is a dcpo. And for each nonempty subset $A$, $A$ is a Scott closed set of $P$ if and only if $A = P$ or there exists an antichain $B$ of $P$ such that $A = \downarrow B$. Then, we have $(\mathbb{N} \times \{n\}) \cup \{\infty\}$ is in $C(P)$ but not in $LP$, that is $LP \neq C(P)$. For each $n \in \mathbb{N}$, let $D_n = \mathbb{N} \times \{n\}$, then $D_n$ is a directed set and $\bigvee (\mathbb{N} \times \{n\}) = \infty$. For each $U \subseteq P$, $U$ is a Scott open subset if and only if $U = \emptyset$ or $U \cap D_n \neq \emptyset$. So, each nonempty Scott open set of $P$ equals to the upper set of its minimal elements $\min U$. Let $U \subseteq C(P)$, then $U$ is a open set respect to the lower Vietoris topology if and only if there exists some $U \in \sigma(P)$ such that $U = \{B \in C(P) : B \cap \min U \neq \emptyset\}$. Let

$$V = \{A \in C(P) : \exists n \in \mathbb{N}, (5, n) \in A\} \cup \{A \in C(P) : \exists n \in \mathbb{N}, (4, n), (4, n + 1) \in A\}.$$ 

we can easily check that, $V$ is a Scott open set of $C(P)$, and for arbitrary $V \in \sigma(P)$, $V \neq \langle V \rangle$. Thus, $V$ is not an open set in lower Vietoris topology. That is, $H(P) \neq P_{OL}(P)$.

Next, we are going to consider the relation between the lower powerdomains and the directed lower powerspaces.

Let $P$ be a dcpo endowed with Scott topology $\sigma(P)$. Write

$$\sigma(C(P))|_{LP} = \{U \cap LP : U \in \sigma(C(P))\}$$

to denote the relative topology on $LP$ from the Scott topology on $C(P)$. For arbitrary $V \in O_{\Rightarrow_L}(LP)$, let

$$\uparrow_{C(P)} V = \{A \in C(P) : \exists \downarrow F \in V, \downarrow F \subseteq A\}.$$

**Proposition 4.4.** Let $P$ be a dcpo endowed with Scott topology $\sigma(P)$. Then $\sigma(C(P))|_{LP} \subseteq O_{\Rightarrow_L}(LP)$. Particularly, $\sigma(C(P))|_{LP} = O_{\Rightarrow_L}(LP)$ if and only if for each $V \in O_{\Rightarrow_L}(LP)$, $\uparrow_{C(P)} V \in \sigma(C(P))$.

**Proof** Let $U \in \sigma(C(P))$, and a directed set $D \subseteq LP$ satisfy $D \Rightarrow_L \downarrow F \in U \cap LP$. Then for each $a \in F$, there exists a directed set $D_a \subseteq \bigcup D$ such that $D_a \rightarrow a$. Thus, $\downarrow F \subseteq \bigcup D$, Here
\( \bigcup D \) means the Scott closure of \( \bigcup D \). But \( \bigcup D \) is just the supremum of \( \bigcup D \) in \( C(P) \). Then \( D \cap U \cap LP = \bar{D} \cap U \neq \emptyset \). It follows that \( U \cap LP \in O_{\Rightarrow_L}(LP) \), that is, \( \sigma(C(P))\big|_{LP} \subseteq O_{\Rightarrow_L}(LP) \).

On the other hand, for arbitrary Scott closed set \( A \) with \( A \in U \), here \( U \) is a Scott open set of \( C(P) \). Let \( F(A) = \{ \downarrow F : F \subseteq \text{fin} \ A \} \), then \( F(A) \) is a directed set of \( LP \) and \( A = \bigcup F(A) \). We have a nonempty finite set \( F \) of \( A \) such that \( \downarrow F \in U \). It means that \( U = \uparrow_{C(P)}(U \cap LP) \). Thus, \( \sigma(C(P))\big|_{LP} = O_{\Rightarrow_L}(LP) \) if and only if \( \forall V \in O_{\Rightarrow_L}(LP) \), \( \uparrow_{C(P)} V \in \sigma(C(P)) \). \( \square \)

Suppose \( A \) is an arbitrary subset of dcpo \( P \). Let \( A^1 = \{ x \in P : \exists \text{ directed set } D \subseteq A, \forall D = x \} \).

**Lemma 4.5.** [3] Let \( P \) be a quasi-continuous domain and \( A \subseteq P \). Then, the Scott closure of \( A \) equals to \( A^1 \), that is \( \overline{A} = A^1 \).

**Corollary 4.6.** [34] Let \( P \) be a continuous or quasi-continuous domain. Then \( \sigma(C(X))\big|_{LP} = O_{\Rightarrow_L}(LP) \), that is, endowed with the Scott topology, \( P \) is a directed space, the directed lower powerspace \( P_L(X) \) is a subspace of the lower powerdomain \( H(P) \).

**Proof** By Proposition [4.3] we only need to prove that, \( \forall V \in O_{\Rightarrow_L}(LP), \uparrow_{C(P)} V \in \sigma(C(P)) \). Suppose \( D \subseteq C(P) \) and \( \bigcup D \in \uparrow_{C(P)} V \). Thus we have some \( \downarrow F \in V \) such that \( \downarrow F \subseteq \bigcup D \). Let \( D' = \{ \downarrow F : F \subseteq \text{fin} \bigcup D \} \), then \( D' \) is a directed set of \( LP \) and \( \bigcup D' = \bigcup D \). By Lemma 4.5 \( \forall a \in F, \exists \text{ a directed set } D_a \subseteq \bigcup D' \) such that \( D_a \rightarrow a \). Thus, \( D' \Rightarrow_L \downarrow F \in V \). Noticed that \( V \in O_{\Rightarrow_L}(LP) \), so we have some \( \downarrow G \in D' \cap V \). For \( D \) is directed, there exists \( A \in D \) such that \( \downarrow G \subseteq A \). Thus \( A \in \uparrow_{C(P)} V \). That is, \( \uparrow_{C(P)} V \in \sigma(C(P)) \). \( \square \)

5. The Directed Upper Powerspaces of Directed Spaces

In this section, we will construct the directed upper powerspace of the directed space, which is a free algebra generated by the directed deflationary operation of the directed space.

**Definition 5.1.** Let \( X \) be a directed space.

1. A binary operation \( \oplus : X \otimes X \rightarrow X \) on \( X \) is called a deflationary operation if it is continuous and satisfy the following four conditions: \( \forall x, y, z \in X \),
   - (a) \( x \oplus x = x \),
   - (b) \( x \oplus (y \oplus z) = x \oplus (y \oplus z) \),
   - (c) \( x \oplus y = y \oplus x \),
   - (d) \( x \oplus y \leq x \).
2. If \( \oplus \) is a deflationary operation on \( X \), then \( (X, \oplus) \) is called a directed deflationary semilattice, that is, directed deflationary semilattices are those directed spaces with deflationary operations.

By Theorem [2.6 2], the operation \( \oplus \) on a directed space \( X \) is continuous if and only if it is monotone and for each given \( x, y \in X \) and directed set \( D \subseteq X \), \( D \rightarrow x \) implies \( (D \oplus y) \rightarrow x \oplus y \). Here, \( D \oplus y = \{ d \oplus y : d \in D \} \).

Here are two examples.

**Example 5.2.**
(1) Suppose $P$ is a poset endowed with the Scott topology, and for each $a, b \in P$, the infimum of $a$ and $b$ exists in $P$ (denote by $a \land b$). Then $(P, \land)$ is a directed deflationary semilattice.

(2) Let $I = [0, 1]$ (the unit interval), let $T$ denote the topology generated by $\{[0, a]: a \in I\}$. It is easy to check that $(I, T)$ is a directed space, and $(I, \min)$ is a directed deflationary semilattice endowed with $T$.

**Definition 5.3.** Suppose $(X, \oplus)$, $(Y, \sqcup)$ are two directed deflationary semilattices, function $f : (X, \oplus) \rightarrow (Y, \sqcup)$ is called a deflationary homomorphism between $X$ and $Y$, if $f$ is continuous and $f(x \oplus y) = f(x) \sqcup f(y)$ holds for $\forall x, y \in X$.

Denote the category of all directed deflationary semilattices and deflationary homomorphisms by $\text{Ddssl}$. Then $\text{Ddssl}$ is a subcategory of $\text{Dtop}$.

**Lemma 5.4.** Suppose $(X, \oplus)$ is a directed deflationary semilattice, then $\oplus = \land_{\leq}$. Here, $\forall x, y \in X$, $x \land_{\leq} y$ means the infimum of $x$ and $y$ respect to the specialization order $\leq$ on $X$ (calling it meet operation). Conversely, suppose $X$ is a directed space and for each $x, y \in X$, $x \land_{\leq} y$ exists, the continuity of $\land_{\leq}$ will naturally imply that $(X, \land_{\leq})$ is a directed deflationary semilattice.

**Proof** By Definition 5.3.1 $\forall x, y \in X$, $x \oplus y \leq x, y$, that is $x \oplus y$ is a lower bound of $\{x, y\}$. Suppose $z$ is another arbitrary lower bound of $\{x, y\}$, by Theorem 2.6(1), the pointwise order equals to the specialization order of $X \times X$, then $(x, y) \leq (z, z)$. By the continuity and idempotence of the deflationary operation, we have $z \oplus z = z \leq x \oplus y$. That is $x \oplus y$ is the infimum of $\{x, y\}$, which means $x \oplus y = x \land_{\leq} y$. Conversely, continuous meet operation naturally satisfy all conditions in Definition 5.3.1 thus, we get the conclusion. □

The above results show that a directed deflationary semilattice $(X, \oplus)$ is just a directed space with a continuous meet operation $\land_{\leq}$ satisfying $\oplus = \land_{\leq}$. Since the order on the directed space in this paper is always the specialization order, we will use the symbol $\land$ instead of $\land_{\leq}$ in the following part. Therefore, a directed deflationary semilattice is always represented by a tuple of the form $(X, \land)$, here $X$ is a directed space, $\land$ represents the continuous meet operation on $X$.

Next, we give the definition of directed upper powerspace.

**Definition 5.5.** Suppose $X$ is a directed space. A directed space $Z$ is called the directed upper powerspace over $X$ if and only if the following two conditions are satisfied:

1. $Z$ is a directed deflationary semilattice, that is the meet operation $\land$ on $Z$ exists and which is continuous,

2. There is a continuous function $i : X \rightarrow Z$ satisfy: for an arbitrary directed deflationary semilattice $(Y, \land)$ and continuous function $f : X \rightarrow Y$, there exists a unique deflationary homomorphism $f : (Z, \land) \rightarrow (Y, \land)$ such that $f = f \circ i$.

By the definition above, if directed deflationary semilattices $(Z_{1}, \land)$ and $(Z_{2}, \land)$ are both the directed upper powerspaces of $X$, then there exists a topological homomorphism which is also a deflationary homomorphism $g : Z_{1} \rightarrow Z_{2}$. Therefore, up to order isomorphism and topological homomorphism, the directed upper powerspace of a directed space is unique. Particularly, we denote the directed upper powerspace of each directed space $X$ by $P_U(X)$. 12
Now, we will prove the existence of the directed upper powerspace of each directed space \( X \) by way of concrete construction.

Let \( X \) be a directed space. Denote

\[
UX = \{ \uparrow F : F \subseteq_{\text{fin}} X \},
\]

here, \( F \subseteq_{\text{fin}} X \) is an arbitrary nonempty finitary subset of \( X \). Define an order \( \leq_U \) on \( UX : \)

\[
\uparrow F_1 \leq_U \uparrow F_2 \iff \uparrow F_2 \subseteq \uparrow F_1.
\]

Let \( \mathcal{F} \subseteq UX \) be a directed set (respect to order \( \leq_U \)) and \( \uparrow F \in UX \). Define a convergence notation \( \mathcal{F} \Rightarrow_U \neg \neg F \iff \) there exists finite directed sets \( D_1, \ldots, D_n \subseteq X \) such that

1. \( F \cap \lim n D_i \neq \emptyset, \forall D_i; \)
2. \( F \subseteq \bigcup_{i=1}^n \lim D_i; \)
3. \( \forall (d_1, \ldots, d_n) \in \prod_{i=1}^n D_i, \) there exists some \( \uparrow F' \in \mathcal{F}, \) such that \( \uparrow F' \subseteq \bigcup_{i=1}^n \uparrow d_i. \)

A subset \( \mathcal{U} \subseteq UX \) is called a \( \Rightarrow U \) convergence open set of \( UX \) if and only if for each directed subset \( \mathcal{F} \) of \( UX \) and \( \uparrow F \in UX, \mathcal{F} \Rightarrow_U \uparrow F \in \mathcal{U} \) implies \( \mathcal{F} \cap \mathcal{U} \neq \emptyset. \) Denote all \( \Rightarrow_U \) convergence open set of \( UX \) by \( O_{\Rightarrow U}(UX) \).

**Proposition 5.6.** Suppose \( X \) is a directed space, the following are true:

1. \( (UX, O_{\Rightarrow U}(UX)) \) is a topological space, abbreviated as \( UX. \)
2. The specialization order \( \subseteq \) of \( (UX, O_{\Rightarrow U}(UX)) \) equals to \( \leq_U. \)
3. \( (UX, O_{\Rightarrow U}(UX)) \) is a directed space, that is \( O_{\Rightarrow U}(UX) = d(UX). \)

**Proof** (1) Obviously we have \( \emptyset, UX \in O_{\Rightarrow U}(UX). \) If \( \mathcal{U} \in O_{\Rightarrow U}(UX), \) and \( \uparrow F_1 \leq_U \uparrow F_2, \uparrow F_1 \in \mathcal{U}, \) \( F_1 = \{a_1, \ldots, a_n\}. \) Then it is evident that \( \{\uparrow F_2\} \Rightarrow_U \uparrow F_1, \) since we only need to take \( D_i = \{a_i\}, i = 1, \ldots, n. \) Then, \( \{\uparrow F_2\} \cap \mathcal{U} \neq \emptyset, \) this means \( \uparrow F_2 \in \mathcal{U}, \) and \( \mathcal{U} \) is an upper set respect to order \( \leq_U, \mathcal{U} = \uparrow \mathcal{U}. \)

Let \( \mathcal{U}_1, \mathcal{U}_2 \in O_{\Rightarrow U}(UX), \) and a directed set \( \mathcal{F} \subseteq UX \) with \( \mathcal{F} \Rightarrow_U \uparrow F \in \mathcal{U}_1 \cap \mathcal{U}_2, \) then, there exists \( \uparrow F_1 \in \mathcal{F} \cap \mathcal{U}_1 \) and \( \uparrow F_2 \in \mathcal{F} \cap \mathcal{U}_2, \) but \( \mathcal{F} \) is directed, we have \( \uparrow F_3 \in \mathcal{F}, \uparrow F_3 \subseteq \uparrow F_2 \) and \( \uparrow F_2. \) Then, \( \uparrow F_3 \in \mathcal{F} \cap \mathcal{U}_1 \cap \mathcal{U}_2. \) By the same way, we can prove that \( O_{\Rightarrow U}(UX) \) is closed under arbitrary union. It follows that \( O_{\Rightarrow U}(UX) \) is a topology.

(2) Let \( \uparrow F_1, \uparrow F_2 \in UX. \) If \( \uparrow F_1 \leq_L \uparrow F_2, \) By the proof of (1), each \( \Rightarrow_U \) convergence open set is an upper set respect to \( \leq_U, \) then \( \uparrow F_1 \in \{\uparrow F_2\}, \) that is, \( \uparrow F_1 \subseteq \uparrow F_2. \)

On the other hand, suppose \( \uparrow F_1 \subseteq \uparrow F_2. \) We need to prove that \( \uparrow F_1 \leq_U \uparrow F_2, \) that is to prove that \( \{\uparrow F \in UX : \uparrow F_2 \subseteq \uparrow F\} \) is a closed set in \( UX \) respect to topology \( O_{\Rightarrow U}(UX), \) since \( \{\uparrow F \in UX : \uparrow F_2 \subseteq \uparrow F\} = \{\uparrow F : \uparrow F \leq_U \uparrow F_2\} \subseteq \{\uparrow F : \uparrow F \subseteq \uparrow F_2\} = \{\uparrow F_2\}, \) then, \( \{\uparrow F \in UX : \uparrow F_2 \subseteq \uparrow F\} \) is a closed set in \( UX \) respect to \( O_{\Rightarrow U}(UX) \) implies \( \{\uparrow F \in UX : \uparrow F \subseteq \uparrow F_2\} = \{\uparrow F_2\}, \) that is \( \uparrow F_1 \leq_U \uparrow F_2. \) Now, we prove that \( \{\uparrow F \in UX : \uparrow F_2 \subseteq \uparrow F\} \) is closed in \( UX \) respect to \( O_{\Rightarrow U}(UX), \) equivalently, \( \mathcal{U} = UX \setminus \{\uparrow F \in UX : \uparrow F_2 \subseteq \uparrow F\} \) is a \( \Rightarrow_U \) convergence open set in \( UX. \)

By contradiction, suppose \( \mathcal{U} \) is not a \( \Rightarrow_U \) convergence open set. Then there exists a directed set \( \mathcal{F} \) of \( UX \) with \( \mathcal{F} \Rightarrow_U \uparrow F \in \mathcal{U} \) but \( \mathcal{U} \cap \mathcal{F} = \emptyset. \) According to the definition of \( \Rightarrow_U \) convergence, there exists finite directed sets \( D_1, \ldots, D_n \subseteq X \) such that
1. $F \cap \lim D_i \neq \emptyset$, $i = 1, 2, \ldots, n$;
2. $F \subseteq \bigcup_{i=1}^{n} \lim D_i$;
3. $\forall (d_1, \ldots, d_n) \in \prod_{i=1}^{n} D_i$, there exists some $\uparrow F' \in \mathcal{F}$, such that $\uparrow F' \subseteq \bigcup_{i=1}^{n} \uparrow d_i$.

Since $\uparrow F \in \mathcal{U}$, then $\uparrow F_2 \nsubseteq \uparrow F$, there exists some $a \in F_2$ with $a \notin \uparrow F$, then $F \subseteq X \setminus \downarrow a$. According to 1, 2 in the definition above, for arbitrary $i \in \{1, 2, \ldots, n\}$, $D_i \cap (X \setminus \downarrow a) \neq \emptyset$. For each $i$, pick $d_i \in D_i \cap (X \setminus \downarrow a)$. Then $(d_1, d_2, \ldots, d_n) \in \prod_{i=1}^{n} D_i$ and $a \notin \bigcup_{i=1}^{n} \uparrow d_i$. Since $F \cap \mathcal{U} = \emptyset$, we have $\forall F' \in \mathcal{F}$, $\uparrow F_2 \subseteq \uparrow F'$, thus $\uparrow F' \subseteq \bigcup_{i=1}^{n} \uparrow d_i$, this contradicts with 3 in the definition above. Therefore, $\mathcal{U}$ is a $\Rightarrow_U$ convergence open set in $UX$.

(3) For an arbitrary topological space $X$, $O(X) \subseteq d(X)$ holds, then $O_{\Rightarrow_U} (UX) \subseteq d(UX)$. On the other hand, according to the definition of $\Rightarrow_U$ convergence topology, if directed set $\mathcal{F} \subseteq UX$ with $\mathcal{F} \Rightarrow_U \uparrow F$, then $\mathcal{F}$ convergents to $\uparrow F$ respect to $O_{\Rightarrow_U} (UX)$. Thus, by the definition of directed open set, $\Rightarrow_U \uparrow F \in \mathcal{U} \subseteq d(UX)$ will imply $\mathcal{U} \cap \mathcal{F} \neq \emptyset$. Then, $\mathcal{U} \in O_{\Rightarrow_U} (UX)$, it follows that $O_{\Rightarrow_U} (UX) = d(UX)$, that is, $(UX, O_{\Rightarrow_U}(UX))$ is a directed space. □

**Proposition 5.7.** Suppose $X, Y$ are two directed spaces. Then function $f: (UX, O_{\Rightarrow_U}(UX)) \rightarrow Y$ is continuous if and only if for each directed set $\mathcal{F} \subseteq UX$ and $\uparrow F \in UX$, $\mathcal{F} \Rightarrow_U \uparrow F$ implies $f(\mathcal{F}) \rightarrow f(\uparrow F)$.

**Proof** Since $\Rightarrow_U$ convergence will lead to $O_{\Rightarrow_U} (UX)$ topological convergence, the necessity is obvious. We are going to prove the sufficiency. Firstly, we check that $f$ is monotone. If $\uparrow F_1, \uparrow F_2 \in UX$ and $\uparrow F_1 \subseteq \uparrow F_2$, then $\{\uparrow F_2\} \Rightarrow_U \uparrow F_1$, by the hypothesis, $\{f(\uparrow F_2)\} \rightarrow f(\uparrow F_1)$, thus $f(\uparrow F_2) \subseteq f(\uparrow F_1)$. Suppose $U$ is an open set of $Y$ and the directed set $\mathcal{F} \Rightarrow_U \uparrow F \in f^{-1}(U)$, then $f(\mathcal{F})$ is a directed set of $Y$ and $f(\mathcal{F}) \rightarrow f(\uparrow F) \in U$, thus $\exists \mathcal{F} \subseteq D$ such that $f(\uparrow F) \in U$. That is, $\uparrow F \in \mathcal{F} \cap f^{-1}(U)$. According to the definition of $\Rightarrow_U$ convergence open set, $f^{-1}(U) \in O_{\Rightarrow_U} (UX)$, that is $f$ is continuous. □

Define a binary operation $\cup$ on $UX : \forall \uparrow F_1, \uparrow F_2 \in UX$, $\uparrow F_1 \cup \uparrow F_2 = \uparrow (F_1 \cup F_2)$. Suppose we have $\uparrow F_1 = \uparrow F_2$, $\uparrow G_1 = \uparrow G_2$ with $F_1 \neq F_2$, $G_1 \neq G_2$, $F_1 \cup G_1 \subseteq \uparrow (F_2 \cup G_2)$ implies $\uparrow (F_1 \cup G_1) \subseteq \uparrow (F_2 \cup G_2)$. Similarly, we have the opposite containment, thus, $\cup$ is well-defined.

**Theorem 5.8.** Let $X$ be a directed space. Then $(UX, O_{\Rightarrow_U}(UX))$ respect to the set union operation $\cup$ is a directed deflationary semilattice.

**Proof** By Proposition 5.6 $(UX, O_{\Rightarrow_U}(UX))$ is a directed space. We will prove that $\cup$ is a deflationary operation. For arbitrary $\uparrow F_1, \uparrow F_2 \in UX$, $\uparrow F_1 \cup \uparrow F_2 = \uparrow (F_1 \cup F_2) \in UX$. Obviously, $\cup$ satisfy the conditions (a), (b), (c), (d) in Definition 5.1 we now prove the continuity of $\cup$. The monotonicity of $\cup$ is evident. By Theorem 2.6 (2) and Proposition 5.7, we only need to prove that, for each directed set $\mathcal{F} \subseteq UX$ and $\uparrow F$, $\uparrow G \in UX$, $\mathcal{F} \Rightarrow_U \uparrow F$ will imply $G \cup \mathcal{F} \Rightarrow_U \uparrow G \cup \uparrow F = \uparrow (G \cup F)$. Here, $G \cup \mathcal{F} = \{\uparrow (G \cup F') : \uparrow F' \in \mathcal{F}\}$ is still a directed set. According to the definition of $\Rightarrow_U$ convergence, there exists finite directed sets $D_1, \ldots, D_k \subseteq X$, satisfy the conditions such that $\mathcal{F} \Rightarrow_U \uparrow F$. Let $G = \{a_1, \ldots, a_n\}$, and $D_{k+1} = \{a_1\}$, $D_{k+2} = \{a_2\}$, $\ldots$, $D_{k+n} = \{a_n\}$. It is straightforward to verify that, $D_1$, $D_2$, $\ldots$, $D_k$, $D_{k+1}$, $\ldots$, $D_{k+n}$ satisfy all the conditions such that $G \cup \mathcal{F} \Rightarrow_U \uparrow (G \cup \uparrow F)$. It follows that, $(UX, \cup)$ is a directed deflationary semilattice. □
The following theorem is the main result in this paper.

**Theorem 5.9.** Suppose $X$ is a directed space, then $(UX, \triangleright_U(UX))$ is the directed upper powerspace over $X$, that is, endowed with topology $\triangleright_U(UX), (UX, \cup) \cong P_U(X)$.

**Proof** Define function $i : X \to UX$: $\forall x \in X, i(x) = \uparrow x$. We prove the continuity of $i$. It is evident that $i$ is monotone. Suppose we have directed set $D \subseteq X$ and $x \in X$ satisfy $D \to x$. Let $\mathcal{D} = \{\uparrow d : d \in D\}$, then $\mathcal{D}$ is a directed set in $UX$ and $\mathcal{D} \Rightarrow_U \uparrow x$. Notice that $i(D) = \mathcal{D}$, so $i(D) \Rightarrow_U \uparrow x = i(x)$. By Proposition 2.5, $i$ is continuous.

Let $(Y, \wedge)$ be an arbitrary directed deflationary semilattice, $f : X \to Y$ is a continuous function. Define $\bar{f} : UX \to Y$ as follows: $\forall \uparrow F \in UX$ (let $F = \{a_1, a_2, \ldots, a_n\}$),

$$\bar{f}(\uparrow F) = f(a_1) \wedge f(a_2) \wedge \cdots \wedge f(a_n) = \bigwedge_{a \in F} f(a).$$

Particularly, we write $\bar{f}(\uparrow F) = \wedge f(F)$. $\bar{f}$ is well-defined, since if we have $\uparrow F = \uparrow G$ with $F \neq G$, then $f(F) \subseteq \uparrow f(G)$ implies $\wedge f(G) \leq \wedge f(F)$, that is $\bar{f}(\uparrow G) \leq \bar{f}(\uparrow F)$. Similarly, we have $\bar{f}(\uparrow F) \leq \bar{f}(\uparrow G)$. Thus, $\bar{f}$ is well-defined.

(1) $\bar{f} = \bar{f} \circ i$.

For arbitrary $x \in X$, $(\bar{f} \circ i)(x) = \bar{f}(i(x)) = \bar{f}(\uparrow x) = f(x)$.

(2) $\bar{f}$ is a deflationary homomorphism, that is, $\bar{f}$ is continuous and for arbitrary $\uparrow F_1, \uparrow F_2 \in UX$, $\bar{f}(\uparrow F_1 \cup \uparrow F_2) = \bar{f}(\uparrow F_1) \wedge \bar{f}(\uparrow F_2)$.

First, we prove that $\bar{f}$ preserves the union operation. Suppose $\uparrow F_1, \uparrow F_2 \in UX$. Then $\bar{f}(\uparrow F_1 \cup \uparrow F_2) = \bar{f}(\uparrow (F_1 \cup F_2)) = \wedge f(F_1 \cup F_2) = (\wedge f(F_1) \wedge (\wedge f(F_2))) = \bar{f}(\uparrow F_1) \wedge \bar{f}(\uparrow F_2)$. Next, we prove the continuity of $\bar{f}$. Notice that $\wedge$ is the meet operation, $\bar{f}$ is evidently monotone. Suppose $\mathcal{F} \subseteq UX$ is a directed set and $\uparrow F \in UX$ satisfy $\mathcal{F} \Rightarrow_U \uparrow F$. By the definition of $\Rightarrow_U$, there exists finite directed sets $D_1, \ldots, D_n \subseteq X$ such that

1. $F \cap \lim D_i \neq \emptyset$, $i = 1, 2, \ldots, n$;
2. $F \subseteq \bigcup_{i=1}^n \lim D_i$;
3. $\forall(d_1, \ldots, d_n) \in \prod_{i=1}^n D_i$, there exists some $\uparrow F' \in \mathcal{F}$, such that $\uparrow F' \leq \bigcup_{i=1}^n \uparrow d_i$.

Let $F = \{b_1, b_2, \ldots, b_k\}$. By 1, for each $1 \leq i \leq n$, we have some $b_i \in F$ such that $D_i \to b_i$. If $F \setminus \{b_1, \ldots, b_n\} \neq \emptyset$, which is denoted by $G = \{a_1, a_2, \ldots, a_s\}$. By 2, For each $a_j \in G$, we have $1 \leq i_j \leq n$ such that $D_{i_j} \to a_j$. By the continuity of $f$, then $f(D_{i_j}) \to f(b_i)$, $i = 1, \ldots, n$ and, $f(D_{i_j}) \to f(a_j)$, $j = 1, 2, \ldots, s$. Since the meet operation $\wedge$ on $Y$ is continuous, the following convergence holds

$$f(D_1) \wedge \cdots \wedge f(D_n) \to f(b_1) \wedge \cdots \wedge f(b_n) \wedge f(a_{i_1}) \wedge \cdots \wedge f(a_{i_s}).$$

Here, $f(D_1) \wedge \cdots \wedge f(D_n) \wedge f(D_{i_1}) \wedge \cdots \wedge f(D_{i_s}) = \{f(d_1) \wedge \cdots \wedge f(d_n) \wedge f(d_{i_1}) \wedge \cdots \wedge f(d_{i_s}) : (d_1, \ldots, d_k, d_{i_1}, \ldots, d_{i_s}) \in (\prod_{i=1}^n D_i) \times (\prod_{j=1}^s D_{i_j})\}$. Let $U$ be an arbitrary open neighborhood of $\wedge F$, by $(*)$, there exists $(d_1, \ldots, d_n, d_{i_1}, \ldots, d_{i_s}) \in (\prod_{i=1}^n D_i) \times (\prod_{j=1}^s D_{i_j})$ such that $f(d_1) \wedge \cdots \wedge f(d_n) \wedge f(d_{i_1}) \wedge \cdots \wedge f(d_{i_s}) \in U$. For each $D_{i_j}$ repeats $D_i$ and each $D_i$ is directed, we
have \((d_1', d_2', \ldots, d_n') \in \prod_{i=1}^{n} D_i\) such that \(f(d_1') \land \cdots \land f(d_n') \supseteq f(d_1) \land \cdots \land f(d_n) \land f(d_{i_1}) \land \cdots \land f(d_{i_s})\).

By 3, there exists some \(\uparrow F' \in \mathcal{F}\) such that \(\uparrow F' \subseteq \bigcup_{i=1}^{n} \uparrow d_i'\). Thus \(\bar{f}(\uparrow F') = \bigwedge f(F') \supseteq f(d_1') \land \cdots \land f(d_n')\). But \(U\) is an upper set, it follows that \(\bigwedge f(F') = \bar{f}(\uparrow F') \in U\), then \(\bar{f}(F) = \{ \bigwedge f(F') : \uparrow F' \in \mathcal{F} \} \to \bigwedge f(F)\). By Proposition 5.7, function \(\bar{f}\) is continuous.

(3) Homomorphism \(\bar{f}\) is unique.

Suppose we have a deflationary homomorphism \(g : (UX, \uplus) \to (Y, \land)\) such that \(f = g \circ i\), then \(g(\uparrow x) = f(x) = \bar{f}(\uparrow x)\). For each \(\uparrow F \in UX\) (let \(F = (a_1, \ldots, a_n)\)),

\[
g(\uparrow F) = g(\uparrow a_1 \cup \uparrow a_2 \cdots \cup \uparrow a_n) = g(\uparrow a_1) \land g(\uparrow a_2) \land \cdots \land g(\uparrow a_n) = \bar{f}(\uparrow a_1) \land \bar{f}(\uparrow a_2) \land \cdots \land \bar{f}(\uparrow a_n) = \bar{f}(\uparrow a_1 \cup \uparrow a_2 \cdots \cup \uparrow a_n).
\]

Thus \(\bar{f}\) is unique.

In conclusion, according to definition 5.1, endowed with topology \(O_{\Rightarrow_U}(UX)\), the directed deflationary semilattice \((UX, \cup)\) is the directed upper powerspace of \(X\), that is, \(P_U(X) \cong (UX, \cup)\). □

The directed upper powerspace is unique in the sense of order isomorphism and topological homomorphism, so we can directly denote the directed upper powerspace by \(P_U(X) = (UX, \cup)\) of each directed space \(X\).

Suppose \(X, Y\) are two directed spaces, \(f : X \to Y\) is a continuous function. Define function \(P_U(f) : P_U(X) \to P_U(Y)\) as follows: \(\forall \uparrow F \in UX,\)

\[
P_U(f)(\uparrow F) = \uparrow f(F).
\]

We can check that, \(P_U(f)\) is well-defined and order preserving. It is easy to check that, \(P_U(f)\) is a deflationary homomorphism between these two lower powerspaces. If \(id_X\) is the identity function and \(g : Y \to Z\) is an arbitrary continuous function from \(Y\) to a directed space \(Z\), then, \(P_U(id_X) = id_{P_U(X)}, P_U(g \circ f) = P_U(g) \circ P_U(f)\). Thus, \(P_U : Dtop \to Ddsl\) is a functor from \(Dtop\) to \(Ddsl\). Let \(U : Ddsl \to Dtop\) be the forgetful functor. By Theorem 5.9 we have the following result.

**Proposition 5.10.** \(P_U\) is a deflationary homomorphism between \(P_U(X)\) and \(P_U(Y)\) for each directed spaces \(X\) and \(Y\).

**Proof** It is directed to check that \(P_U\) preserves deflationary operation between \(P_U(X)\) and \(P_U(Y)\) for each directed spaces \(X\) and \(Y\). We only to check that \(P_U\) is continuous.

Let \(X\) and \(Y\) be two directed spaces, let \(D = \{f_i\}_{i \in I}\) be a directed set in \([X \to Y]\) and convergent to \(f \in [X \to Y]\) pointwise. Let \(\uparrow F \in P_U(X)\) and \(F = \{a_1, \ldots, a_n\}\), we shall verify that \(\{P_U(f_i)(\uparrow F)\}_{i \in I} \to P_U(f)(\uparrow F)\), that is \(\{\uparrow f_i(F)\}_{i \in I} \to \uparrow f(F)\) in \(P_U(Y)\). According to the definition of \(\Rightarrow_U\) convergence, firstly, we have \(D_a = \{f_i(a)\}_{i \in I} \to f(a)\) in \(Y\) for each \(a \in F\). Secondly, for each \(f_{i_1}(a_1) \in D_{a_1}, \ldots, f_{i_n}(a_n) \in D_{a_n}\), since each \(a \in F, \ D_a\) is directed, we have some \(\uparrow f_{i_1}(F) \in \{\uparrow f_i(F)\}_{i \in I}\) such that \(\uparrow f_{i_1}(F) \subseteq \uparrow (f_{i_1}(a_1), \ldots, f_{i_n}(a_n))\). Thus \(\{\uparrow f_i(F)\}_{i \in I} \Rightarrow_U f(F), P_U\) is continuous.
Corollary 5.11. \( P_U \) is a left adjoint of the forgetful functor \( U \), that is, \( \text{Ddsl} \) is a reflective subcategory of \( \text{Dtop} \).

6. Relations Between Upper Powerspace

In this section, we will discuss the relations between the upper powerdomains of dcpos, the observationally-induced upper powerspaces and the directed upper powerspaces.

According to the results in the last section, for an arbitrary directed space, the directed lower-space is the set of \( UX \), endowed with the \( \Rightarrow_U \) convergence topology, the deflationary operation equals to the union operation of sets. In general, for an arbitrary topological space and arbitrary dcpo, although the observationally-induced upper powerspace and the upper powerdomain exist, their concrete structure cannot be expressed (see article [2, 13, 14]).

Let \( (X, O(X)) \) be a topological space. We say that an nonempty set \( A \subseteq X \) is a saturated set, if \( A = \bigcap \{ U \in O(X) : A \subseteq U \} \). Denote all nonempty compact saturated sets of \( X \) by \( Q(X) \).

For each \( U \in O(X) \), let \( [U] = \{ K \in Q(X) : K \subseteq U \} \). Denote \( B_X = \{ [U] : U \in O(X) \} \). The upper Vietoris topology of \( Q(X) \) is generated by subbase \( B_X \), denote by \( V_U(Q(X)) \). Particularly, for a dcpo, endowed with the Scott topology, the compact saturated sets are just the compact upper sets.

Theorem 6.1. \[2\] Suppose \( X \) is a sober and locally compact space, \( (Q(X), V_U(Q(X))) \) is order isomorphic and topological homomorphic to the observationally-induced upper space of \( X \) respect to the union operation of sets. Under this condition, we have \( V_U(Q(X)) = \sigma(Q(X)) \). Here, \( Q(X) \) with the order reverse to containment, \( \sigma(Q(X)) \) denotes the Scott topology.

Theorem 6.2. \[7\] Let \( P \) be a continuous domain. Then \( (Q(P), \supseteq) \), endowed with the Scott topology and the order reverse to containment, isomorphic to the upper powerdomain over \( P \) (which is also called Smyth powerdomain) \( S(P) \), besides, the following holds:

1. \( (Q(P), \supseteq) \) is a continuous meet semilattice,
2. \( \forall K \in Q(P), K = \bigcap \{ \uparrow F : 1 \leq |F| < \omega \& K \subseteq (\uparrow F)^\circ \} \).

According to these two theorems above, for each continuous domain with the Scott topology, its observationally-induced upper powerspace isomorphic to the upper powerdomain. Next, we discuss the directed upper powerspace of continuous domain.

Let \( X \) be a continuous domain. Then \( (X, \sigma(X)) \) is a directed space, and each upper set of nonempty finitary elements is a compact saturated set. Thus, \( UX \subseteq Q(X) \). By \( \sigma(Q(X))|_{UX} \) denote the relative topology from the Scott topology on \( Q(X) \).

Proposition 6.3. Suppose \( X \) is a continuous domain which is endowed with Scott topology \( \sigma(X) \).

1. For each given directed set \( F \subseteq UX \) and \( \uparrow F \in UX \), \( F \Rightarrow_U \uparrow F \Leftrightarrow \bigcap \{ \uparrow G : \uparrow G \in F \} \subseteq \uparrow F \).
2. \( O_{\Rightarrow_U}(UX) = \sigma(Q(X))|_{UX} \).

Proof (1) Suppose \( F \subseteq UX \) is a directed set of \( UX \), \( \uparrow F \in UX \), and \( F \Rightarrow_U \uparrow F \). By definition, there exist finitary directed sets \( D_1, \ldots, D_n \subseteq X \) such that

1. \( F \cap \lim D_i \neq \emptyset, i = 1, 2, \ldots, n; \)
2. \( F \subseteq \bigcup_{i=1}^{n} \lim D_i \);

3. \( \forall (d_1, \ldots, d_n) \in \prod_{i=1}^{n} D_i \), there exists some \( \uparrow F' \in F \), such that \( \uparrow F' \subseteq \bigcup_{i=1}^{n} \uparrow d_i \).

By contradiction, suppose \( \bigcap \{ \uparrow G : \uparrow G \in F \} \not\subseteq \uparrow F \). Then there exists some \( a \in \bigcap \{ \uparrow G : \uparrow G \in F \} \) such that \( a \not\in \uparrow F \). Thus, \( F \subseteq X \setminus a \). By 1 and 2, for each \( i \), we have \( d_i \in D_i \) such that \( d_i \in X \setminus a \).

By 3, there exists some \( \uparrow F' \in F \) such that \( \uparrow F' \subseteq \bigcup_{i=1}^{n} \uparrow d_i \), this contradicts with \( a \not\in \bigcup_{i=1}^{n} \uparrow d_i \). That is \( \bigcap \{ \uparrow G : \uparrow G \in F \} \not\subseteq \uparrow F \).

On the other hand, suppose \( \bigcap \{ \uparrow G : \uparrow G \in F \} \subseteq \uparrow F \). Let \( F = \{ a_1, a_2, \ldots, a_n \} \). For \( X \) is a continuous domain, then each \( \downarrow a_i \) is directed and \( a_i = \bigvee \downarrow a_i \), \( i = 1, 2, \ldots, n \). Let \( D_i = \downarrow a_i \), \( D_i \to a_i \), then each \( D_i \) satisfy 1 and 2 in the definition of \( \Rightarrow_U \). For arbitrary \( (d_1, d_2, \ldots, d_n) \in \prod_{i=1}^{n} D_i \), we have \( \uparrow F \subseteq \bigcup_{i=1}^{n} \uparrow d_i = (\bigcup_{i=1}^{n} \uparrow d_i)^\circ \). Since each continuous domain is well-filtered, it follows that there exists some \( \uparrow G \in F \) such that \( \uparrow G \subseteq (\bigcup_{i=1}^{n} \uparrow d_i)^\circ \), then 3 in the definition of \( \Rightarrow_U \) holds. In conclusion, \( F \Rightarrow_U \uparrow F \).

(2) Suppose \( U \in O_{\Rightarrow_U}(UX) \). Let \( U_Q = \{ K \in Q(X) : \exists \uparrow F \in U, K \subseteq \uparrow F \} \). Obviously, \( U = U_Q \cap Q(X) \). Let \( K \subseteq Q(X) \) be a directed set respect to the order reverse to containment and \( \bigcap \{ K : K \in \mathcal{K} \} \in U_Q \). There exists some \( \uparrow F \in U \) such that \( \bigcap \{ K : K \in \mathcal{K} \} \subseteq \uparrow F \). Let \( F = \{ a_1, a_2, \ldots, a_n \} \). Since \( X \) is a continuous domain, each \( \downarrow a_i \) is directed and \( a_i = \bigvee \downarrow a_i \), \( i = 1, 2, \ldots, n \). Let \( F = \{ \bigcup_{i=1}^{n} \uparrow d_i : d_i \ll a_i, \ i = 1, 2, \ldots, n \} \). For \( X \) is a continuous domain, then \( Q(X) \) is a continuous domain, we have \( \uparrow F \subseteq (\bigcup_{i=1}^{n} \uparrow d_i)^\circ \subseteq \bigcup_{i=1}^{n} \uparrow d_i, \ i = 1, 2, \ldots, n, \) besides, \( \cap F = \uparrow F \). By (1), \( F \Rightarrow_U \uparrow F \). Thus, we have \( (d_1, d_2, \ldots, d_n) \in \prod_{i=1}^{n} \downarrow a_i \) such that \( \bigcup_{i=1}^{n} \uparrow d_i \in U \). Notice that \( \bigcap \{ K : K \in \mathcal{K} \} \subseteq \uparrow F \subseteq (\bigcup_{i=1}^{n} \uparrow d_i)^\circ \), there exists some \( K \in \mathcal{K} \) such that \( K \subseteq (\bigcup_{i=1}^{n} \uparrow d_i)^\circ \). By the definition of \( U_Q \), \( K \in U \), that is, \( U_Q \) is a Scott open set in \( Q(X) \). Therefore, \( O_{\Rightarrow_U}(UX) \subseteq \sigma(Q(X))(UX) \).

On the other hand, let \( V \in \sigma(Q(X)), \) and \( F \subseteq UX \) is a directed set with \( F \Rightarrow_U \uparrow F \in V \cap UX \).

By (1), \( \bigcap \{ \uparrow G : \uparrow G \in F \} \in V \), there exists some \( \uparrow G \in F \cap V \), that is, \( F \cap V \cap UX \neq \emptyset \). Thus, \( \sigma(Q(X))(UX) \subseteq O_{\Rightarrow_U}(UX) \). We get the conclusion. \( \square \)

**Example 6.4.** Let \( X = \mathbb{R}^n \) be the \( n \)-dimensional Euclidean space. Then \( X \) is locally compact \( T_2 \) space, naturally sober and locally compact. Denote the observationally-induced upper space over \( X \) by \( P_0(X) \). By Theorem 6.1 \( P_0(X) = \{ K \subseteq X : K \text{ is a nonempty compact set of } X \} \) and the topology of which is the Scott topology. It is easy to check that, \( P_0(X) \) is a continuous domain and for each nonempty compact set \( K \subseteq X \), \( \{ a \} : a \in K \} \) is a compact saturated set of \( P_0(X) \).

It follows that, for directed space \( P_0(X) \), its directed upper powerspace \( P_U(P_0(X)) \neq Q(P_0(X)) \).

7. The Directed Convex Powerspace

In this section, we will construct the directed convex powerspace of the directed space, which is a free algebra generated by the directed semilattice operation of the directed space.

**Definition 7.1.** Let \( X \) be a directed space.
A binary operation $+ : X \otimes X \to X$ on $X$ is called a semilattice operation if it is continuous and satisfy the following three conditions: \( \forall x, y, z \in X, \)

(a) \( x + x = x, \)
(b) \( (x + y) + z = x + (y + z), \)
(c) \( x + y = y + x. \)

(2) If $+$ is a directed semilattice operation on $X$, then $(X, +)$ is called a directed semilattice, that is, directed semilattices are those directed spaces with semilattice operations.

By Theorem 2.6(2), the operation $+$ on $X$ is continuous if and only if it is monotone and for each given $x, y \in X$ and directed set $D \subseteq X$, $D \to x$ implies $(D + y) \to x + y$. Here, $D + y = \{d + y : d \in D\}$.

Example 7.2.

(1) Suppose $(P, +)$ is a directed inflationary (or deflationary) semilattice, then, $(P, +)$ is a directed semilattice.

(2) Let $X = \{0, 1, 2\}$ and $0 \leq 1 \leq 2$, define $0 + 0 = 0$, $1 + 1 = 1$, $2 + 2 = 2$, $0 + 1 = 1$, $1 + 2 = 1$, $0 + 2 = 1$. Then $(X, +)$, endowed with the Scott topology, is a directed semilattice.

Definition 7.3. Suppose $(X, \oplus)$, $(Y, \uplus)$ are two directed semilattice, function $f : (X, \oplus) \to (Y, \uplus)$ is called a directed semilattice homomorphism between these two directed space and $f(x \oplus y) = f(x) \uplus f(y)$ holds, $\forall x, y \in X$.

Denote the category of all directed semilattice and directed homomorphism by $Dsl$. Then $Dsl$ is a subcategory of the category of $Dtop$.

Next, we give the definition of directed convex powerspace.

Definition 7.4. Suppose $X$ is a directed space. A directed space $Z$ is called the directed convex powerspace over $X$ if and only if the following two conditions are satisfied:

(1) $Z$ is a directed semilattice, that is the semilattice operation $+$ on $Z$ exists and which is continuous,

(2) There is a continuous function $i : X \to Z$ satisfy: for an arbitrary directed semilattice $(Y, +)$ and continuous function $f : X \to Y$, there exists an unique directed semilattice homomorphism $\bar{f} : (Z, +) \to (Y, +)$ such that $f = \bar{f} \circ i$.

By the definition above, if directed semilattices $(Z_1, +)$ and $(Z_2, +)$ are both the directed convex powerspaces of $X$, then, there exists a topological homomorphism which is still a directed semilattice homomorphism $g : Z_1 \to Z_2$. Therefore, up to order isomorphism and topological homomorphism, the directed convex powerspace of a directed space is unique. Particularly, we denote the directed convex powerspace of each directed space $X$ by $P_p(X)$.

Now, we will prove the existence of the directed convex powerspace of each directed space $X$ by way of concrete construction.

Let $X$ be a directed space, for each compact set $A \subseteq X$, $\bar{A}$ denotes the closure of $A$, and $sat(A)$ the saturation of $A$, that is $sat(A) = \cap\{U \in O(X) : A \subseteq U\}$. Thus, for each finite subset $F$, it follows that $\bar{F} = \downarrow F$, $sat(F) = \uparrow F$ respect to the specialization order of $X$. Denote $\hat{A} = (\bar{A}, sat(A))$, $PX = \{\bar{F} : F \subseteq_{fin} X\}$,
Here, \( F \subseteq_{\text{fin}} X \) is an arbitrary nonempty finitary subset of \( X \). Define an order \( \leq_p \) on \( PX \) as follows:

\[
\widehat{F}_1 \leq_p \widehat{F}_2 \iff \downarrow F_1 \subseteq \downarrow F_2 \text{ and } \uparrow F_2 \subseteq \uparrow F_1.
\]

Let \( D = \{ \widehat{F}_i \}_{i \in I} \subseteq PX \) be a directed set (respect to the order \( \leq_p \)), and \( \widehat{F} \in PX \). Define a convergence notation \( D \Rightarrow_p \widehat{F} \) as follows:

\[
D \Rightarrow_p \widehat{F} \iff \text{there exists finite directed sets } D_1, \ldots, D_k \text{ of } X \text{ satisfy the following four conditions:}
\]

1. \( D_i \subseteq \bigcup_{i \in I} \downarrow F_i, \ i = 1, \ldots, k; \)
2. \( \forall i = 1, \ldots, k, \ F \cap \lim D_i \neq \emptyset; \)
3. \( F \subseteq \bigcup_{i = 1}^k \lim D_i; \)
4. \( \forall (d_1, \ldots, d_k) \in \prod_{i = 1}^k D_i, \ \exists \hat{F}' \in D, \text{ such that } \uparrow F' \subseteq \uparrow (d_1, \ldots, d_k). \)

A subset \( U \subseteq PX \) is called a \( \Rightarrow_p \) convergence open set of \( PX \) if and only if for each directed subset \( D = \{ \widehat{F}_i \}_{i \in I} \) of \( PX \) and \( \widehat{F} \in PX, \ D \Rightarrow_p \widehat{F} \in U \) will imply \( D \cap U \neq \emptyset \). Denote all \( \Rightarrow_p \) convergence open set of \( PX \) by \( O_{\Rightarrow_p}(PX) \).

**Proposition 7.5.** Suppose \( X \) is a directed space, the following are true:

1. \( (PX, O_{\Rightarrow_p}(PX)) \) is a topological space, abbreviated as \( PX \).
2. The specialization order \( \subseteq \) on \( (PX, O_{\Rightarrow_p}(PX)) \) equals to \( \leq_p \).
3. \( (PX, O_{\Rightarrow_p}(PX)) \) is a directed space, that is \( O_{\Rightarrow_p}(PX) = d(PX) \).

**Proof** (1) Firstly, \( \emptyset, \ PX \in O_{\Rightarrow_p}(PX) \). Suppose \( U \in O_{\Rightarrow_p}(PX), \ \widehat{F}_1, \ \widehat{F}_2 \in PX, \ \text{Let } F_1 = \{ a_1, \ldots, a_n \}. \ If \ \widehat{F}_1 \leq_p \widehat{F}_2, \ then \ \{ \widehat{F}_2 \} \Rightarrow_p \widehat{F}_1 \) holds, since the finite directed sets \( \{ a_1 \}, \ldots, \{ a_n \} \) is sufficient to satisfy the definition of \( \{ \widehat{F}_2 \} \Rightarrow_p \widehat{F}_1 \), then \( \widehat{F}_1 \in U \) implies \( \widehat{F}_2 \in U \). It follows that \( U \) is an upper set of \( PX \) respect to \( \leq_p \). Suppose \( U_1, U_2 \in O_{\Rightarrow_p}(PX) \), \( D \) is a directed set in \( PX, \ \widehat{F} \in PX \) and satisfy \( D \Rightarrow_p \widehat{F} \in U_1 \cap U_2 \), then there exists some \( \widehat{G}_1 \in D \cap U_1 \) and \( \widehat{G}_2 \in D \cap U_2 \). Since \( D \) is directed, we have \( \widehat{G} \in D \) such that \( \widehat{G}_1, \widehat{G}_2 \leq_p \widehat{G} \), then \( \widehat{G} \in D \cap U_1 \cap U_2 \). By the same way, we can evidently prove that \( O_{\Rightarrow_p}(PX) \) is closed under arbitrary union. That is, \( O_{\Rightarrow_p}(PX) \) is a topology on \( PX \).

(2) Let \( \widehat{F}_1, \widehat{F}_2 \in PX \). If \( \widehat{F}_1 \leq_p \widehat{F}_2 \), according the proof of (1), each \( \Rightarrow_p \) convergence open set is an upper set respect to partial order \( \leq_p \), then \( \widehat{F}_1 \in \{ \widehat{F}_2 \} \), that is, \( \widehat{F}_1 \subseteq \widehat{F}_2 \). On the other hand, suppose \( \widehat{F}_1 \subseteq \widehat{F}_2 \). We need to prove that \( \{ \widehat{F} \in PX : \widehat{F} \leq_p \widehat{F}_2 \} \) is a closed set in \( PX \) respect to \( O_{\Rightarrow_p}(PX) \), by the proof of (1), \( \{ \widehat{F} \in PX : \widehat{F} \leq_p \widehat{F}_2 \} \) is closed in \( PX \) respect to \( O_{\Rightarrow_p}(PX) \), or equivalently, \( U = PX \setminus \{ \widehat{F} \in PX : \widehat{F} \leq_p \widehat{F}_2 \} \) is a \( \Rightarrow_p \) convergence open set. Suppose \( D = \{ \widehat{F}_i \}_{i \in I} \subseteq PX \) is a \( \leq_p \) - directed set and \( D \Rightarrow_p \widehat{G} \in U \). Let \( G = \{ a_1, a_2, \ldots, a_k \} \). By the definition of \( \Rightarrow_p \) convergence, there exists finite directed sets \( D_1, \ldots, D_n \subseteq X \) such that \( D_1, \ldots, D_n \subseteq \bigcup_{i \in I} \{ F_i \} \). For each \( a_i \in G \), there is one of the above directed set \( D_i \ldots, D_1 \) such that \( D_i \to a_i, \ i = 1, 2, \ldots, k \). By contradiction, suppose \( D \cap U = \emptyset \), then, by the definition of order \( \leq_p \),

20
∀ \hat{F}_i \in \mathcal{D}, \downarrow F_i \subseteq \downarrow F_2, thus \bigcup \downarrow F_i \subseteq \downarrow F_2, it follows that D_i \subseteq \downarrow F_2, i = 1, 2, \ldots, k. But \downarrow F_2 is a closed set of X, then, all limit points of D_i belongs to \downarrow F_2, it follows that G = \{a_1, a_2, \ldots, a_k\} \subseteq \downarrow F_2. However, according to the hypothesis, \hat{G} \in \mathcal{U}, that is \hat{G} \not\subseteq \hat{F}_2, then, it follows that \uparrow F_2 \not\subseteq \uparrow G, that is \exists a \in F_2 such that G \subseteq X \downarrow a. By the definition of \Rightarrow_p convergence, each D_i converges to the points in G, and X \downarrow a is an open set of X, then D_i \cap (X \downarrow a) \neq \emptyset. For each i = 1, \ldots, n, pick d_i = D_i \cap (X \downarrow a). We have (d_1, \ldots, d_n) \in \prod_{i=1}^n D_i and a \notin \{d_1, \ldots, d_n\}. By the hypothesis, \mathcal{D} \cap \mathcal{U} = \emptyset, it follows that \forall \hat{F}_i \in \mathcal{D}, \uparrow F_2 \subseteq \uparrow F_i. Then \forall i \in I, a \notin \uparrow F_i, \uparrow F_i \not\subseteq \{d_1, \ldots, d_n\}. This contradicts with (4) in the definition of \Rightarrow_p convergence. Therefore, \mathcal{U} is a \Rightarrow_p convergence open set in PX.

(3) For an arbitrary topological space X, O(X) \subseteq d(X) holds, then O_{\Rightarrow_p}(PX) \subseteq d(PX). On the other hand, according to the definition of \Rightarrow_p convergence topology, if directed set \mathcal{D} \subseteq PX with \mathcal{D} \Rightarrow_p \hat{F}, then \mathcal{D} converges to \hat{F} respect to O_{\Rightarrow_p}(PX). Thus, by the definition of directed open set, \mathcal{D} \Rightarrow_p \hat{F} \in U \subseteq d(PX) implies U \cap \mathcal{D} \neq \emptyset. Then, U \in O_{\Rightarrow_p}(PX), it follows that O_{\Rightarrow_p}(PX) = d(PX), that is (PX, O_{\Rightarrow_p}(PX)) is a directed space. □

Proposition 7.6. Suppose X, Y are two directed spaces. Then function \( f : (PX, O_{\Rightarrow_p}(PX)) \to Y \) is continuous if and only if for each directed set \mathcal{D} \subseteq PX and \hat{F} \in PX, \mathcal{D} \Rightarrow_p \hat{F} implies \( f(\mathcal{D}) \to f(\hat{F}) \).

Proof Since \Rightarrow_p convergence will lead to O_{\Rightarrow_p}(PX) topological convergence, the necessity is obvious. We are going to prove the sufficiency. Firstly, we check that f is monotone. If \hat{F}_1, \hat{F}_2 \in PX and \hat{F}_1 \leq_p \hat{F}_2, then \{\hat{F}_2\} \Rightarrow_p \hat{F}_1. By the hypothesis, \{f(\hat{F}_2)\} \to f(\hat{F}_1), it follows that f(\hat{F}_2) \subseteq f(\hat{F}_1). Suppose U is an open set of Y and there is a directed set \mathcal{D} \Rightarrow_p \hat{F} \in f^{-1}(U), then f(\mathcal{D}) is a directed set of Y and f(\mathcal{D}) \to f(\hat{F}) \in U, thus, there exists some \hat{F} \in \mathcal{D} such that f(\hat{F}) \in U, that is, \hat{F} \in \mathcal{D} \cap f^{-1}(U). According to the definition of \Rightarrow_p convergence open set, f^{-1}(U) \in O_{\Rightarrow_p}(PX), that is f is continuous. □

Define a binary operation \( \oplus : PX \times PX \to PX \) on PX : \( \forall \hat{F}_1, \hat{F}_2 \in PX, \hat{F}_1 \oplus \hat{F}_2 = \hat{F}_1 \cup \hat{F}_2 \). For arbitrary \( \hat{F}_1 = \hat{F}_3, G_1 = \hat{G}_2 \) with \( F_1 \neq F_2, G_1 \neq G_2, F_1 \cup G_1 \subseteq \downarrow (F_2 \cup G_2) \) and \( F_1 \cup G_1 \subseteq \uparrow (F_2 \cup G_2) \) imply \( \hat{F}_1 \cup \hat{G}_1 \leq_p \hat{F}_2 \cup \hat{G}_2 \). Similarly, we have the opposite inequality, thus, \( \oplus \) is well-defined.

Theorem 7.7. Let X be a directed space. Then (PX, \( \oplus \)) is a directed semilattice.

Proof By Proposition 7.5 (PX, O_{\Rightarrow_p}(PX)) is a directed space. We will prove that \( \oplus \) is a directed semilattice operation. It is evidently to check that \( \oplus \) satisfy (a), (b), (c) in definition 7.1, we now prove the continuity of \( \oplus \). \( \oplus \) is obviously monotone. By Theorem 2.6(2) and proposition 7.6, we only need to prove that, for each directed set \mathcal{D} = \hat{F}_1 \subseteq PX and \hat{F}, \hat{G} \in PX, \mathcal{D} \Rightarrow_p \hat{F} will imply \( \hat{G} \oplus \mathcal{D} \Rightarrow_p (\hat{G} \oplus \hat{F}) = (\hat{G} \cup \hat{F}) \). Here, \( \mathcal{G} \cup \mathcal{D} = \{\hat{G} \oplus \hat{F}_i : \hat{F}_i \in \mathcal{D}\} \) is also a directed set. By the definition of \Rightarrow_p convergence, there exists finite directed sets D_1, \ldots, D_n \subseteq X such that \mathcal{D} \Rightarrow_p \hat{F}. Let \( G = \{a_1, \ldots, a_k\} \), take \( D_{n+1} = \{a_1\}, \ldots, D_{n+k} = \{a_k\} \). We can evidently verify that \( D_1, \ldots, D_{n+k} \) is sufficient to satisfy the definition of \( \hat{G} \oplus \mathcal{D} \Rightarrow_p \hat{G} \cup \hat{F} \). Therefore, (PX, \( \oplus \)) is a directed semilattice. □

The following theorem is the main result of this section.

21
Theorem 7.8. Suppose \( X \) is a directed space, then \((PX, O_p(PX))\) is the directed convex powerspace over \( X \), that is, endowed with topology \( O_p(PX), (PX, \oplus) \cong P_p(X)\).

Proof Define function \( i : X \rightarrow PX \) as follows: \( \forall x \in X, i(x) = \hat{x} \). We prove the continuity of \( i \). \( i \) is evidently monotone. Suppose we have directed set \( D \subseteq X \) and \( x \in X \) satisfy \( D \rightarrow x \). Let \( \mathcal{D} = \{D = \{d : d \in D\} \), then \( \mathcal{D} \) is a directed set in \( PX \) and \( \mathcal{D} \rightarrow_p \hat{x} \). Since \( \mathcal{D} \) satisfy the definition such that \( \mathcal{D} \Rightarrow p \hat{x} \). By Proposition 2.5 \( i \) is continuous.

Let \((Y, \oplus)\) be a directed semilattice, \( f : X \rightarrow Y \) is a continuous function. Define \( \tilde{f} : PX \rightarrow Y \) as follows: \( \forall \bar{F} \in PX \) (let \( F = \{a_1, a_2, \ldots, a_n\} \)),

\[
\tilde{f}(\bar{F}) = f(a_1) \oplus f(a_2) \oplus \cdots \oplus f(a_n) = \sum_{a \in F} f(a)
\]

Particularly, we write \( \tilde{f} = \sum f(F) \).

\( \tilde{f} \) is well-defined, since if we have \( \bar{F}_1, \bar{F}_2 \in PX \) and \( \bar{F}_1 = \bar{F}_2 \) but with \( F_1 \neq F_2 \). Let \( F_1 = \{a_1, \ldots, a_k\} \), \( F_2 = \{b_1, \ldots, b_k\} \). For \( \downarrow F_1 \subseteq \downarrow F_2 \), we have \( b_1, \ldots, b_k \in F_2 \) such that \( a_1 \leq b_1, \ldots, a_k \leq b_k \). If \( F_2 \setminus \{b_1, \ldots, b_k\} = \emptyset \), then \( \Sigma f(F_1) = \Sigma f(F_2) = \Sigma f(\bar{F}_1) \).

If not, suppose \( F_2 \setminus \{b_1, \ldots, b_k\} = \{b_{k+1}, \ldots, b_is\} \), but \( \uparrow F_2 \subseteq \uparrow F_1 \), we have \( a_{i_1}, \ldots, a_{i_s} \in F_1 \) such that \( a_{i_1} \leq b_{i_1}, \ldots, a_{i_s} \leq b_{i_s} \). Notice that \( a_{i_1}, \ldots, a_{i_s} \) repeat \( \{a_1, \ldots, a_k\} \), it follows that \( \tilde{f}(\bar{F}_1) = f(a_1) \oplus \cdots \oplus f(a_n) \oplus f(a_{i_1}) \oplus \cdots \oplus f(a_{i_s}) \leq f(b_1) \oplus \cdots \oplus f(b_n) \oplus f(b_{i_1}) \oplus \cdots \oplus f(b_{i_s}) = \tilde{f}(\bar{F}_2) \).

Similarly, we have \( \tilde{f}(\bar{F}_2) \leq \tilde{f}(\bar{F}_1) \). Thus, \( \tilde{f} \) is well-defined.

(1) \( f = f \circ i \).

For arbitrary \( x \in X \), \( (f \circ i)(x) = f(i(x)) = \hat{f}(\hat{x}) = f(x) \).

(2) \( \tilde{f} \) is a directed semilattice homomorphism, that is, \( \tilde{f} \) is continuous and for arbitrary \( \bar{F}_1, \bar{F}_2 \in PX \), \( \tilde{f}(\bar{F}_1 \oplus \bar{F}_2) = \tilde{f}(\bar{F}_1 \cup \bar{F}_2) = \Sigma (\tilde{f}(\bar{F}_1 \cup \bar{F}_2)) = \sum (\Sigma f(F_1) + (\Sigma f(F_2))) = \tilde{f}(\bar{F}_1) + \tilde{f}(\bar{F}_2) \).

Firstly, we prove that \( \tilde{f} \) preserves directed semilattice operation. Suppose \( \bar{F}_1, \bar{F}_2 \in PX \). Then \( \tilde{f}(\bar{F}_1 \oplus \bar{F}_2) = \tilde{f}(\bar{F}_1 \cup \bar{F}_2) = \Sigma (\tilde{f}(\bar{F}_1 \cup \bar{F}_2)) = \Sigma f(\bar{F}_1) + (\Sigma f(F_2)) = \tilde{f}(\bar{F}_1) + \tilde{f}(\bar{F}_2) \). Next, we prove the continuity of \( \tilde{f} \). We check that \( \tilde{f} \) is monotone. Suppose \( \bar{F}_1, \bar{F}_2 \in PX \). Let \( F_1 = \{a_1, \ldots, a_k\} \), \( F_2 = \{b_1, \ldots, b_k\} \), and \( \bar{F}_1 \leq_p \bar{F}_2 \), that is \( \downarrow F_1 \subseteq \downarrow F_2 \) & \( \uparrow F_2 \subseteq \uparrow F_1 \). For \( \downarrow F_1 \subseteq \downarrow F_2 \), we have \( b_1, \ldots, b_k \in F_2 \) such that \( a_1 \leq b_1, \ldots, a_k \leq b_k \). If \( F_2 \setminus \{b_1, \ldots, b_k\} = \emptyset \), then \( \Sigma f(F_1) = \tilde{f}(\bar{F}_1) \leq \Sigma f(F_2) = \tilde{f}(\bar{F}_2) \).

If not, suppose \( F_2 \setminus \{b_1, \ldots, b_k\} = \{b_{k+1}, \ldots, b_is\} \), but \( \uparrow F_2 \subseteq \uparrow F_1 \), we have \( a_{i_1}, \ldots, a_{i_s} \in F_1 \) such that \( a_{i_1} \leq b_{i_1}, \ldots, a_{i_s} \leq b_{i_s} \), now, we have \( a_1 \leq b_1, \ldots, a_k \leq b_k, a_{i_1} \leq b_{i_1}, \ldots, a_{i_s} \leq b_{i_s} \).

Notice that \( a_{i_1}, \ldots, a_{i_s} \) repeat \( \{a_1, \ldots, a_k\} \), it follows that \( \tilde{f}(\bar{F}_1) = f(a_1) \oplus \cdots \oplus f(a_n) \oplus f(a_{i_1}) \oplus \cdots \oplus f(a_{i_s}) \leq f(b_1) \oplus \cdots \oplus f(b_n) \oplus f(b_{i_1}) \oplus \cdots \oplus f(b_{i_s}) = \tilde{f}(\bar{F}_2) \).

In conclusion, \( \bar{F}_1 \leq_p \bar{F}_2 \), \( \tilde{f}(\bar{F}_1) \leq \tilde{f}(\bar{F}_2) \), that is, \( \tilde{f} \) is monotone.

Finally, we verify the continuity of \( \tilde{f} \). Suppose \( \mathcal{D} \subseteq PX \) is a directed set and \( \mathcal{D} \Rightarrow p \bar{F} \in PX \), there exists finite directed sets \( D_1, \ldots, D_n \subseteq X \) satisfy definition of \( \Rightarrow p \) convergence. Let \( F = \{b_1, b_2, \ldots, b_k\} \). By (2) in the definition, for each \( D_i, 1 \leq i \leq n \), there exists some \( b_i \in F \) such that \( D_i \rightarrow \hat{b}_i \). If \( F \setminus \{b_1, \ldots, b_n\} \neq \emptyset \), which is denoted by \( \bar{G} = \{a_1, a_2, \ldots, a_s\} \). By (3) in the definition, for each \( a_j \in \bar{G} \), there exists \( 1 \leq j \leq n \) such that \( D_{i_j} \rightarrow \hat{a}_j \). By the continuity of \( f \), we have \( f(D_i) \rightarrow f(b_i), i = 1, \ldots, n \), and \( f(D_{i_j}) \rightarrow f(a_j), j = 1, 2, \ldots, s \). Since the directed semilattice operation \( \oplus \) on \( Y \) is continuous, the following convergence holds

\[
f(D_1) \oplus \cdots \oplus f(D_n) + f(D_{i_1}) \oplus \cdots \oplus f(D_{i_s}) \rightarrow f(b_1) \oplus \cdots \oplus f(b_n) + f(a_{i_1}) \oplus \cdots \oplus f(a_{i_s}). (*)
\]
Here, \( f(D_1) + \cdots + f(D_n) + f(D_i_1) + \cdots + f(D_i_s) = \{ f(d_1) + \cdots + f(d_n) + f(d_i_1) + \cdots + f(d_i_s) : (d_1, \ldots, d_k, d_i_1, \ldots, d_s) \in (\prod_{i=1}^n D_i) \times (\prod_{j=1}^s D_{i_j}) \} \). Let \( U \) be an arbitrary open neighborhood of \( \Sigma f(F) \), by (\(*\)), there exists some \( (d_1, \ldots, d_n, d_i_1, \ldots, d_s) \in (\prod_{i=1}^n D_i) \times (\prod_{j=1}^s D_{i_j}) \) such that \( f(d_1) + \cdots + f(d_n) + f(d_i_1) + \cdots + f(d_i_s) \in U \). Since each \( D_{i_j} \) repeats \( D_i \), there exists \( (d'_i, d'_2, \ldots, d'_n) \in \prod_{i=1}^n D_i \) such that \( f(d'_i) + \cdots + f(d'_n) \subseteq f(d_1) + \cdots + f(d_n) + f(d_i_1) + \cdots + f(d_i_s) \). By (4) in the definition, there exists some \( \hat{F}_1 \in \mathcal{D} \) such that \( \uparrow F_1 \subseteq (d'_1, \ldots, d'_n) \). On the other hand, by (1) in definition of \( \Rightarrow \_P \) convergence, for each \( d'_i \), there exists some \( \hat{F}_1 \in \mathcal{D} \) such that \( d'_i \subseteq \downarrow \hat{F}_1 \). Since \( \mathcal{D} \) is directed, we have \( \hat{F}_2 \in \mathcal{D} \) with \( \hat{F}_1 \leq_P \hat{F}_2 \), \( i = 1, \ldots, n \). Again, by the directness of \( \mathcal{D} \), we obtain \( \hat{F}_1 \in \mathcal{D} \) such that \( \hat{F}_1 \leq_P \hat{F}_2 \). Now, we have \( \downarrow (d'_1, \ldots, d'_n) \subseteq \downarrow \hat{F}_2 \subseteq \downarrow \hat{F}' \subseteq \downarrow \hat{F}_1 \subseteq \downarrow (d'_1, \ldots, d'_n) \), that is \( \{ d'_1, \ldots, d'_n \} \leq_P \hat{F}' \), since \( f \) is monotone and \( U \) is an upper set in \( Y \), we got \( \check{f}(\hat{F}') \in U \). It follows that \( \hat{f}(\mathcal{D}) = \{ \Sigma f(F) : \hat{F}_1 \in \mathcal{D} \} \to \Sigma f(F) \). By Proposition 7.4, \( \hat{f} \) is continuous.

(3) \( \hat{f} \) is unique.

Suppose we have a directed semilattice homomorphism \( g : (PX, \oplus) \to (Y, +) \) such that \( f = g \circ i \). Then \( g(\check{x}) = f(x) = \hat{f}(\check{x}) \). For each \( \hat{F} \in PX \) (Let \( F = \{ a_1, \ldots, a_n \} \) ),

\[
g(\hat{F}) = g(\check{a}_1 \oplus \check{a}_2 \oplus \cdots \oplus \check{a}_n) = g(\check{a}_1) \oplus g(\check{a}_2) \oplus \cdots \oplus g(\check{a}_n) = \hat{f}(\check{a}_1) \oplus \hat{f}(\check{a}_2) \oplus \cdots \oplus \hat{f}(\check{a}_n) = \hat{f}(\check{a}_1 \oplus \check{a}_2 \oplus \cdots \oplus \check{a}_n) = \hat{f}(\hat{F}).
\]

That is \( \hat{f} \) is unique.

In conclusion, by Definition 7.4 endowed with the topology \( O_{\Rightarrow_P}(PX) \), the directed semilattice \( (PX, \oplus) \) is a directed convex powerspace over \( X \), that is, \( P_P(X) \cong (PX, \oplus) \). \( \square \)

The directed convex powerspace is unique in the sense of order isomorphism and topological homomorphism, so we can denote the directed convex powerspace of each \( X \) by \( P_P(X) = (PX, \oplus) \).

Suppose \( X, Y \) are two directed spaces, \( f : X \to Y \) is a continuous function. Define map \( P_P(f) : P_P(X) \to P_P(Y) \) as follows: \( \forall \hat{F} \in PX \),

\[
P_P(f)(\downarrow \hat{F}) = \check{f}(\hat{F}).
\]

It is evident that, \( P_P(f) \) is well-defined and order preserving. According to the proof of the theorem above, it is easy to check that \( P_P(f) \) is a directed semilattice homomorphism between these two directed convex powerspaces. If \( id_X \) is the identity function and \( g : Y \to Z \) is an arbitrary continuous function from \( Y \) to a directed space \( Z \), then, \( P_P(id_X) = id_{P_P(X)} \), \( P_P(g \circ f) = P_P(g) \circ_P P_P(f) \). Thus, \( P_P : \textbf{Dtop} \to \textbf{Dsl} \) is a functor from \( \textbf{Dtop} \) to \( \textbf{Dsl} \). Let \( U : \textbf{Dsl} \to \textbf{Dtop} \) be the forgetful functor. By Theorem 7.8 we have the following results.

**Corollary 7.9.** \( P_P \) is a left adjoint of the forgetful functor \( U \), that is, \( \textbf{Dsl} \) is a reflective subcategory of \( \textbf{Dtop} \).
8. Relations Between Convex Powerspaces

In this section, we will discuss the relations between the convex powerdomain of dcpo and its directed lower powerspace.

Let $L$ be a domain equipped with the Scott topology. A nonempty subset $A$ is a lens if $A$ can be written as the intersection of a closed set and a compact saturated set. A lens $A = C \cap K$ has a canonical representation of the form $A \cap \uparrow A$ (since $A \subseteq A \cap \uparrow A \subseteq C \cap K = A$); note that $\uparrow A$ is compact since $A$ is. A pair $(C, K) \in C(L) \times Q(L)$ is called a lens factorization if $C = A$ and $K = \uparrow A$ for some lens $A$. We denote by $\text{Lens} L$ the set of all lenses.

**Theorem 8.1.** [5] Let $L$ be a domain equipped with the Scott topology. If $L$ is coherent, resp. countably based, then the convex powerdomain $P(L)$ (which is also called the Plotkin powerdomain) may be identified with $\text{Lens} L$ equipped with the Egli-Milner order, resp. topological Egli-Milner order.

**Proposition 8.2.** Let $X$ be a continuous domain endowed with Scott topology, then $\sigma(P(X))|_{P X} = O_{\Rightarrow P}(P X)$.

**Proof** By proposition 4.4 and proposition 6.3(1), it is not hard to check that $\sigma(P(X))|_{P X} \subseteq O_{\Rightarrow P}(P X)$.

On the other hand, for each $U \in O_{\Rightarrow P}(P X)$, let

$$\uparrow_{P(X)} U = \{(C, D) \in C(X) \times Q(X) : \exists \hat{F} \in P X \text{ such that } \downarrow F \subseteq C \& D \subseteq \uparrow F\}.$$ 

Then, it is obviously that $U = \uparrow_{P(X)} U \cap P X$, it is sufficient to prove that $\uparrow_{P(X)} U \in \sigma(P(X))$. Let $D = \{(C_i, D_i)\}_{i \in I} \subseteq P(X)$ be a directed set with $\forall D = (\bigcup_{i \in I} C_i, \bigcap_{i \in I} D_i) \in \uparrow_{P(X)} U$. By the definition of $\uparrow_{P(X)} U$ we have some $\hat{F} \in U$ such that $\downarrow F \subseteq C \& D \subseteq \uparrow F$. Set $F = \{a_1, \ldots, a_n\}$. Let $D_i = \downarrow a_i$, $1 \leq i \leq n$, $F = \{d_1, \ldots, d_n\} : (d_1, \ldots, d_n) \in \prod_{i=1}^{n} D_i$. Since $X$ is a continuous domain and by the proof of proposition 6.3(1), we can directly check that $F \Rightarrow_{p} \hat{F}$; it follows that there exists some $\{d_1, \ldots, d_n\} \in F \cap U$. Since $F \subseteq \bigcup_{i \in I} C_i$ and $X$ is continuous, we can infer that each $D_i \subseteq \downarrow \bigcup_{i \in I} C_i = \bigcup_{i \in I} C_i$. For $D$ is directed, we have some $(C', D') \in D$ such that $\downarrow\{d_1, \ldots, d_n\} \subseteq C'$. Besides, $\bigcap_{i \in I} D_i \subseteq \downarrow F \subseteq (\bigcup_{i=1}^{n} \downarrow d_i)^\circ$, since $X$ is a continuous domain endowed with the Scott topology, hence well-filtered, we have some $(C'', D'') \in D$ such that $D'' \subseteq (\bigcup_{i=1}^{n} \downarrow d_i)^\circ \subseteq \downarrow\{d_1, \ldots, d_n\}$. Again, by the directness of $D$, choose $(C, D) \in D$ with $C', C'' \subseteq C \& D \subseteq D', D''$. Then, $\downarrow\{d_1, \ldots, d_n\} \subseteq C' \subseteq C \& D \subseteq D'' \subseteq \downarrow\{d_1, \ldots, d_n\}$, that is $(C, D) \in \uparrow_{P(X)} U$ and $\uparrow_{P(X)} U$ is a Scott open set. □

**Example 8.3.** Let $X$ be the same as in example 6.4 by example 6.4 we can directly infer that, $P_P(P_D(X)) \neq P(P_D(X))$, that is, in general, the directed convex powerspace over a dcpo endowed with the Scott topology is not agree with its convex powerdomain.

9. The Commutativity of The Directed Upper and Lower Functors

In domain theory, the Hoare and Smyth powerdomain constructors were firstly proved to be commuted under composition in [5] in 1990. In 1991, Heckmann gave an algebraic methods and
does not rely on any explicit representations of the powerdomains [12]. In this section, we will
discuss the commutativity of the directed upper and lower functors.

In this paper, for convenience, a directed inflationary semilattice is abbritted as \( \lor \)- directed
space, a directed deflationary semilattice is abbritted as \( \land \)- directed space. A morphism between
two directed deflationary semilattices is called a \( \lor \)- morphism, a morphism between two directed
inflationary semilattices is called a \( \land \)- morphism.

**Definition 9.1.**

1. A \( \lor \)- \( \land \)- directed space \( X \) is a directed space that is both a \( \lor \)- directed space and \( \land \)-
directed space that the distributive law \( a \land (b \lor c) = (a \land b) \lor (a \land c) \), \( \forall a, b, c \in X \).
2. A \( \lor \)- \( \land \)- morphism between two \( \lor \)- \( \land \)- directed spaces that is both a \( \lor \)- morphism and
a \( \land \)- morphism.

Denote the category of all \( \lor \)- \( \land \)- directed spaces together with \( \lor \)- \( \land \)- morphisms by \( \text{Ddl} \).
For all directed spaces \( X \), we then show that both \( P_U(P_L(X)) \) and \( P_L(P_U(X)) \) are free \( \lor \)- \( \land \)-
directed spaces over \( X \), whence they are isomorphic.

We will discuss some properties of functor \( P_L \) and functor \( P_U \) is analogously. Theorem 3.10
indicates that for each directed space \( X \), \( P_L(X) \) is an \( \land \)- directed space. The following proposition
tell us that if \( X \) is a \( \land \)- directed space, then \( P_L(X) \) is actually a \( \lor \)- \( \land \)- directed space.

**Theorem 9.2.**

1. If \( X \) is a \( \land \)- directed space, then \( P_L(X) \) is a \( \lor \)- \( \land \)- directed space, and \( \eta_L : X \to P_L(X) \) is
a \( \land \)- morphism, here \( \eta_L(x) = \downarrow x, \forall x \in X \).
2. If \( X \) is a \( \lor \)- directed space, then \( P_U(X) \) is a \( \lor \)- \( \land \)- directed space, and \( \eta_L : X \to P_U(X) \) is
a \( \lor \)- morphism, here \( \eta_L(x) = \uparrow x, \forall x \in X \).

**Proof** We only prove (1). Let \( X \) be a \( \land \)- directed space, then \( P_L(X) \) is a \( \lor \)- directed space,
we have to construct the operation \( \land \) in \( P_L(X) \). To this end, for each \( \downarrow F_1, \downarrow F_2 \in P_L(X) \), let
\( f_a(x) = \eta_L(a \land x) \) for each \( a \in X \). By Theorem 3.10 there exist a unique morphism \( \bar{f}_a : P_L(X) \to P_L(X) \) such that \( \bar{f}_a(\downarrow F) = \downarrow f_a(F) \) for each \( \downarrow F \in P_L(X) \). Let \( g(a) = \bar{f}_a(\downarrow F_2) = \downarrow \{a \land b : b \in F_2\} \),
again by Theorem 3.10 there exist a unique morphism \( \bar{g} : P_L(X) \to P_L(X) \) such that \( \bar{g}(\downarrow F) = \downarrow \{a \land b : a \in F, b \in F_2\} \). Thus, we have a continuous operation in \( P_L(X) \) defined as follows
\[
\downarrow F_1 \sqcap \downarrow F_2 = \{\eta_L(a \land b) : a \in F_1, b \in F_2\} = \downarrow \{a \land b : a \in F_1, b \in F_2\}
\]

It is straightly to check that \( \sqcap \) satisfy all the conditions in Definition 3.1.

1. \( \downarrow F \sqcap \downarrow F = \downarrow \{a \land a : a \in F\} = \downarrow F, \forall \downarrow F \in P_L(X) \).
2. \( (\downarrow F_1 \sqcap \downarrow F_2) \sqcap \downarrow F_3 = \downarrow \{a \land b : a \in F_1, b \in F_2\} \sqcap \downarrow F_3 = \downarrow \{a \land b \land c : a \in F_1, b \in F_2, c \in F_3\} = \downarrow F_1 \sqcap \{b \land c : b \in F_2, c \in F_3\} \sqcap \downarrow F_1, \downarrow F_2, \downarrow F_3 \in P_L(X) \).
3. \( \downarrow F_1 \sqcap \downarrow F_2 = \downarrow \{a \land b : a \in F_1, b \in F_2\} = \downarrow F_2 \sqcap \downarrow F_1, \forall \downarrow F_1, \downarrow F_2 \in P_L(X) \).
4. \( \downarrow F_1 \sqcap \downarrow F_2 = \downarrow \{a \land b : a \in F_1, b \in F_2\} \leq \downarrow F_1, \forall \downarrow F_1, \downarrow F_2 \in P_L(X) \).
Next, we shall check that the distributivity law for $\cup$ and $\cap$. Suppose we have $F_1 = \{a_1, \ldots, a_n\}$, $F_2 = \{b_1, \ldots, b_m\}$, $F_3 = \{c_1, \ldots, c_k\}$.

(1) $F_1 \cap (F_2 \cup F_3) = (F_1 \cap F_2) \cup (F_1 \cap F_3)$.

(2) $F_1 \cup (F_2 \cap F_3) = (F_1 \cup F_2) \cap (F_1 \cup F_3)$.

Hence, $(P_L(X), \vee, \cap)$ is a $\lor - \land$- directed space. Last, we shall prove that $\eta_L$ is a $\land$- morphism. For each $x, y \in X, \eta_L(x \land y) = \eta_L(x) \cap \eta_L(y) \cap \eta_L(y) \cap \eta_L(y)$.

In the pervious part, we showed that the functor $P_L$ does not destroy the $\land$ operator, now we prove that $P_L$ has the analogous property for $\land$- morphism.

**Theorem 9.3.**

1. Let $X$ be a $\land$- directed space, $Y$ a $\lor - \land$- directed space, and $f : X \rightarrow Y$ a $\land$- morphism. Then its unique extension $P_L(f) : P_L(X) \rightarrow Y$ is a $\lor - \land$- morphism.

2. Let $X$ be a $\lor - \land$- directed space, $Y$ a $\lor - \land$- directed space, and $f : X \rightarrow Y$ a $\lor$- morphism. Then its unique extension $P_U(f) : P_U(X) \rightarrow Y$ is a $\lor - \land$- morphism.

**Proof** We only prove (1). Only need to check that $P_L(f)$ is a $\land$- morphism. Suppose we have $F_1 = a_1, \ldots, a_n$, $F_2 = \{b_1, \ldots, b_m\}$.

$$P_L(f)(\downarrow F_1 \cap \downarrow F_2) = P_L(f)(\downarrow \{a \land b : a \in F_1, b \in F_2\})$$

$$= \bigvee_{a \in F_1, b \in F_2} f(a \land b)$$

$$= \bigvee_{a \in F_1, b \in F_2} f(a) \land f(b)$$

On the other hand,

$$P_L(f)(\downarrow F_1) \land P_L(f)(\downarrow F_2) = (\lor f(F_1)) \land (\lor f(F_2))$$

$$= (\lor f(F_1) \land f(b_1)) \lor (\lor f(F_1) \land f(b_2)) \lor \cdots \lor (\lor f(F_1) \land f(b_m))$$

$$= (f(a_1) \land f(b_1)) \lor \cdots \lor (f(a_n) \land f(b_1)) \lor \cdots \lor (f(a_n) \land f(b_m))$$

$$= \bigvee_{a \in F_1, b \in F_2} f(a) \land f(b)$$

Thus, $P_L(f)$ is a $\lor - \land$- morphism. □

Now, we will show the main theorem of this section.

**Theorem 9.4.**
(1) For each directed space $X$, each $\land - \land -$ directed space $Y$, and every morphism $f : X \to Y$, there is a unique $\land - \land -$ morphism $F$ from $P_L(P_U(X))$ to $Y$ such that $F \circ (\eta_\lor \circ \eta_\land) = f$. $F$ is given by $P_L(P_U(f))$. Thus, $P_L(P_U(X))$ is the free $\lor - \land -$ directed space over the directed space $X$.

(2) For each directed space $X$, each $\land - \land -$ directed space $Y$, and every morphism $f : X \to Y$, there is a unique $\land - \land -$ morphism $F$ from $P_U(P_L(X))$ to $Y$ such that $F \circ (\eta_\land \circ \eta_\lor) = f$. $F$ is given by $P_U(P_L(f))$. Thus, $P_U(P_L(X))$ is the free $\lor - \land -$ directed space over the directed space $X$.

(3) For each directed space $X$, $P_L(P_U(X))$ and $P_U(P_L(X))$ are isomorphic by a $\land - \land -$ morphism, which maps $\uparrow (\downarrow x)$ to $\downarrow (\uparrow x)$.

Proof We prove (1) and (2) is analogous. Suppose $X$ is a directed space, then $P_U(X)$ is a $\land -$ directed space and $P_U(f)$ is a $\land -$ morphism from $P_U(X)$ to $Y$. By Theorem 9.2 and 9.3 $P_L(P_U(X))$ is a $\lor - \land -$ directed space and $P_L(P_U(f))$ is a $\lor - \land -$ morphism. $f = P_U(f) \circ \eta_\land = (P_L(P_U(f)) \circ \eta_\lor) \circ \eta_\land = F \circ (\eta_\lor \circ \eta_\land)$. We remain to prove that $F$ is unique. Let $F_1, F_2$ be two such $\lor - \land -$ morphisms, i.e. $F_i \circ \eta_\lor \circ \eta_\land = f, i = 1, 2$ holds. We have to show $F_1 = F_2$. Let $F_i = F_i \circ \eta_\land, i = 1, 2$. Since $F_i \circ \eta_\land = f$, by the uniqueness of $P_U(f)$, $F_1 \circ \eta_\lor = F_2 \circ \eta_\lor$. Since the extension of $P_L(P_U(X))$ is unique, we have $F_1 = F_2$. Thus, $F$ is unique, and $P_L(P_U(X))$ is the free $\lor - \land -$ directed space over $X$.

(3) an immediate conclusion by (1) and (2). □

data availability statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflict of interest

The authors declare that they have no conflict of interest.

Reference

References

[1] Abramsky, S., Jung, A.: domain theory. In: Abramsky, S., Gabbay, D.M., Maibaum, T.S.E.(eds.), Semantic Structures. In: Handbook of Logic in Computer Science, vol.3, pp.1-168, Clarendon Press, Oxford (1994)

[2] Battenfeld, I., Schöder, M.:Observationally-induced lower and upper powerspace constructions. Journal of Logical and Algebraic Methods in Programming, 84, 668-682 (2015)

[3] Erne E. The ABC of order and topology. In: H. Herrlich, H. E. Porst (des.) Category Theory at Work, pp. 57-83, Heldermann, Berlin (1991)

[4] Erné, M.: Infinite distributive laws versus local connectedness and compactness properties. Topology and its Applications. 156, 2054-2069 (2009)
[5] Flannery, K.E, Martin,J.J.: The Hoare and Smyth power domain constructors commute under composition. Journal of Computer and System Sciences, 1990, 40(2): 125-135.

[6] Geng J., Kou H.: Consistent Hoare powerdomains over dcpos, Topology and its Applications. 232, 169-175 (2017)

[7] Gierz, G. et al.: Continuous Lattices and domains. Cambridge University Press, Cambridge (2003).

[8] Goubault-Larrecq, J., Jung, A.: QRB, QFS, and the Probabilistic Powerdomain. Electronic Notes in Theoretical Computer Science. 308, 167-182 (2014)

[9] Goubault-Larrecq, J.: Non-Hausdorff topology and domain theory: Selected topics in point-set topology. Cambridge University Press, Cambridge (2013)

[10] Graham, S.K.: Closure properties of a probabilistic domain construction. Springer, Berlin, Heidelberg, 213-233 (1987)

[11] Heckmann, R.: Probabilistic power domains, information systems, and locales. Springer, Berlin, Heidelberg. 410-437 (1993)

[12] Heckmann, R.: Spaces of valuations. In Andima, S., Flagg, R.C., Itzkowitz, G., Misra, P., Kong, Y. and Kopperman, R., editors, Papers on General Topology and Applications: Eleventh Summer Conference at the University of Southern Maine, volume 806 of Annals of the New York Academy of Sciences. 174-200 (1996)

[13] Heckmann, R.: Power domain constructions. Sci. Comput. Program. 17, 77-117 (1991)

[14] Heckmann, R.: An upper power domain construction in terms of strongly compact sets, Lect. Notes Comput. Sci. 598, 272-293 (1992)

[15] Heckmann, R.: Lower and upper power domain constructions commute on all cpos. Information Processing Letters, 1991, 40(1): 7-11.

[16] Heckmann, R., Keimel, K.: Quasicontinuous domains and the Smyth powerdomain, Electron. Notes Theor. Comput. Sci. 298, 215-232 (2013)

[17] Jones, C.G., Plotkin, D.: A probabilistic powerdomain of evaluations. Fourth Annual Symposium on Logic in Computer Science. IEEE Computer Society. 186-195 (1989)

[18] Jones, C.: Probabilistic Non-Determinism, Ph.D. Thesis, University of Edinburgh, Report ECS-LFCS-90-105(1990)

[19] Jung, A., Tix, R.: The troublesome probabilistic powerdomain. Electronic Notes in Theoretical Computer Science. 13, 70-91 (1998)

[20] Keimel,K.: Topological cones: functional analysis in a $T_0$-setting, Springer-Verlag. 77(1), 109-142 (2008)

[21] Kou, H.: Directed spaces: An extended framework for domain theory, 1th Pan Pacific International Conference on Topology and Applications.£–Min Nan Normal University, Zhangzhou City(2015). 11. 25-30
[22] Lyu Z., Kou H.: The probabilistic powerdomain from a topological viewpoint. Topology and its Applications. 237, 26-36. (2018)

[23] MacLane, S.: Categories for the Working Mathematician. Springer-Verlag, New York (1971)

[24] Mislove, M.: Generalizing domain theory. International Conference on Foundations of Software Science and Computation Structure. Springer, Berlin, Heidelberg. 1-19 (1998)

[25] Mislove, M.: Topology, domain theory and theoretical computer science. Topology and its Applications. 89(1-2), 3-59 (1998)

[26] Mislove, M.: On the Smyth power domain. Lect. Notes Comput. Sci. 298, 161-172 (1988)

[27] Plotkin, G.D.: A powerdomain construction. SIAM J. Comput. 5, 452-487 (1976)

[28] Saheb-Djahromi, N.: CPO’s of measures for nondeterminism. Theoretical Computer Science. 12(1), 19-37 (1980)

[29] Scott, D. S.: Outline of a mathematical theory of computation. In 4th Annual Princeton Conference on Information Sciences and Systems. (1970)

[30] Scott, D. S.: Continuous Lattices: Toposes, Algebraic Geometry and Logic. Springer Lecture Notes in Mathematics. 274, 97-136 (1972)

[31] Scott, D. S. Lectures on a mathematical theory of computation, In M. Broy and G. Schmidt, editors, Theoretical Foundations of Programming Methodology. 145-292 (1982)

[32] Smyth, M. B.: Power domains and predicate transformers: A topological view. Springer, Berlin, Heidelberg. 662-675 (1983)

[33] Tix, R., Keimel, K., Plotkin, G.: Semantic domains for combining probability and non-determinism. Electronic Notes in Theoretical Computer Science. 222, 3-99 (2009)

[34] Xie, X., Kou H.: Lower power structures of directed spaces (Chinese). Journal of Sichuan University (Natural Science Edition). 57, 211-217 (2020)

[35] Xu X.: Order and Topology (Chinese). Science Press, Beijing (2016)

[36] Yu, Y., Kou, H.: On directed Space (Chinese). Journal of Sichuan University (Natural Science Edition) (2014)

[37] Yu Y., Kou, H.: Directed spaces defined through $T_0$ spaces with specialization order (Chinese), Journal of Sichuan University (Natural Science Edition). 52(2), 217-222 (2015)

[38] Yuan, Y., Kou, H.: Consistent Smyth Powerdomains. Topology and its Applications. 173, 264-275 (2014)

[39] Yuan, Y., Kou, H.: Consistent Hoare Powerdomains. Topology and its Applications. 178, 40-45 (2014)

[40] Yuan, Y., Kou, H.: Consistent Plotkin powerdomains. Topology and its Applications. 178, 339-344 (2014)