Existence of breathers in nonlinear Klein-Gordon lattices

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We prove the existence of time-periodic and spatially localised solutions (breathers) in general nonlinear Klein-Gordon infinite lattices of weakly coupled oscillators by using Schauder’s fixed point theorem establishing that there are time-reversal initial conditions leading to breather solutions.

I. INTRODUCTION

Discrete breather solutions in nonlinear lattices have attracted significant interest recently, not least due to the important role they play in many physical realms where features of localisation in systems of coupled oscillators are involved (for a review see [1] and references therein), [2]-[21]. Proofs of existence and nonexistence of breathers, as spatially localised and time-periodically varying solutions, were provided in [22]-[26]. The exponential stability of breathers was proven in [27]. Analytical and numerical methods have been developed to continue breather solutions in conservative and dissipative systems starting from the anti-integrable limit [28]-[31].

Here we present a concise existence proof of breather solutions in general nonlinear Klein-Gordon infinite lattices of weakly coupled oscillators using methods alternative to the approaches in [22]-[26]. That is, we demonstrate that there exist initial conditions, complying with the time reversibility of the system, such that the ensuing evolution is characterised by spatially localised and time-periodic solutions. In order to prove the existence of such initial conditions, we formulate the problem in terms of an operator equation on a function space which is solved by virtue of Schauder’s Fixed Point Theorem: Let $S$ be a closed convex subset of the Banach space $X$. Suppose $g : S \to S$ is compact. Then $g$ has at least one fixed point in $S$ (see in [32]-[36]).

**Definition:** Let $X$ and $Y$ be normed spaces. The map $g : M \subseteq X \to Y$ is called compact iff
(i) $g$ is continuous, and
(ii) $g$ transforms bounded sets into relatively compact sets.

We study the dynamics of general nonlinear Klein-Gordon (KG) infinite lattice systems given by the following set of coupled equations

$$\frac{d^2 q_n}{dt^2} = -V'(q_n) + \kappa(q_{n+1} - 2q_n + q_{n-1}), \quad n \in \mathbb{Z}, \tag{1.1}$$

and the prime $'$ stands for the derivative with respect to $q_n$, the latter being the coordinate of the oscillator at site $n$ evolving in an anharmonic on-site potential $V(q_n)$. Each oscillator interacts with its neighbours to the left and right with strength that is regulated by the value of the non-negative parameter $\kappa$.

In what follows we differentiate between soft on-site potentials and hard on-site potentials. For the former (latter) the oscillation frequency of an oscillator moving in the on-site potential $V(q)$ decreases (increases) with increasing oscillation amplitude.

We make the following assumptions:

**Assumption I.1** The anharmonic on-site potentials $V(x)$ are analytic and have the following properties:

$$V(0) = V'(0) = 0, \quad V''(0) = \omega_0^2 > 0.$$

Soft on-site potentials $V(x)$ possess adjacent to the minimum at $x = 0$ inflection points at $x_i^-$ and maxima at $x_i^+$, respectively with $x_M^+ > x_M^- > 0$ and $x_M^- < x_i^- < 0$, and one has

$$V'(x_M^- < x < 0) < 0, \quad V'(x_M^+) = 0, \quad V'(0 < x < x_M^+) > 0,$$

$$V''(x_i^- < x < x_i^+) > 0, \quad V''(|x_i^-| < |x| < |x_M^+|) < 0.$$

(We remark that $V(x)$ can have more than one (local) minimum. An example is a periodic potential $V(x) = 1 - \cos(x)$. However, in the frame of the current study we are only interested in motion between the maxima, $x_M^+$,
adjacent to the minimum of $V(x)$ at $x = 0$. Furthermore, the case when a soft on-site potential has only one inflection point can be easily implemented. An example is $V(x) = (\omega_0^2/2)x^2 - ax^3$, $a > 0$.)

**Hard on-site potentials** $V(x)$ are characterised by

$$
V'(x < 0) < 0, \; V'(x > 0) > 0,
$$

$$
V''(x) > 0, \; \forall x \in \mathbb{R}.
$$

For soft and hard on-site potentials $V'(x)$ can be expressed as

$$
V'(x) = \omega_0^2 x + W'(x),
$$

so that

$$
V(x) = \frac{1}{2}\omega_0^2 x^2 + W(x),
$$

with the anharmonic part $W$.

Further, we assume that some positive constants $\alpha, \beta, K_\alpha$ and $K_\beta$ the condition

$$
|W'(x)| \leq K_\alpha |x|^\alpha, \quad |W'(x) - W'(y)| \leq K_\beta (|x|^\beta + |y|^\beta)|x - y|
$$

(1.2)

for all $x \in \mathbb{R}$ for hard on-site potentials, and for $x^-_M \leq x \leq x^+_M$ for soft on-site potentials, is fulfilled.

The system of equations (1.1) has an energy integral

$$
E = \sum_{n \in \mathbb{Z}} \left[ \frac{1}{2}q_n^2 + V(q_n) \right] + \frac{\kappa}{2} \sum_{n \in \mathbb{Z}} (q_{n+1} - q_n)^2,
$$

relating it to a Hamiltonian structure, associated with the Hamiltonian,

$$
H = \sum_{n \in \mathbb{Z}} \left[ \frac{1}{2}p_n^2 + V(q_n) \right] + \frac{\kappa}{2} \sum_{n \in \mathbb{Z}} (q_{n+1} - q_n)^2,
$$

(1.3)

where $p_n$ and $q_n$ are canonically conjugate momentum and coordinate variables. Denoting $p = (p_1, p_2, ...)$ and $q = (q_1, q_2, ...)$, we note that the Hamiltonian system is time-reversible with respect to the involution $p \mapsto -p$.

For systems (1.1) with a hard on-site potential for any finite initial data the solutions are always bounded, that is

$$
||q(t)||_t \leq q_E < \infty, \; \forall t > 0,
$$

(1.4)

where $q_E$ depends on the energy level $E > 0$ in (1.3), because for all values of the total energy $0 < E < \infty$ there exists a closed maximum equipotential surface

$$
\Sigma : \sum_{n \in \mathbb{Z}} V(q_n) + \frac{\kappa}{2} \sum_{n \in \mathbb{Z}} (q_{n+1} - q_n)^2 = E,
$$

(1.5)

with $p_n = 0$ for all $n \in \mathbb{Z}$. As trajectories cannot penetrate the maximum equipotential surface all coordinates $q(t)$ are bounded to perform motions about the only equilibrium position in configuration space at $q = 0$. In contrast, for systems with a soft on-site potential even for finite initial data (respectively sufficiently high values of $E$) unbounded solutions may ensue. An example for such a potential is $V(x) = (\omega_0^2/2)x^2 - ax^4$, $a > 0$. Being interested only in bounded motion, we make the following assumption:

**Assumption I.2** For systems with a soft on-site potential assume that $E$ is chosen such that a closed maximum equipotential surface $\Sigma$ exists bounding all motions about the equilibrium position in configuration space situated at $q = 0$ such that

$$
||q(t)||_t \leq q_E \leq q_M < \infty, \; \forall t > 0,
$$

(1.6)
The level of $E$ is unrestricted for hard on-site potentials and is restricted to $(0, E_0)$ for some $E_0 < \infty$ for soft on-site potentials.

We study the existence of solutions to the system (1.1) that are time-periodic and spatially localised satisfying

$$q_n(t + T) = q_n(t), \quad \lim_{n \to -\infty} |q_n| = 0,$$

with period $T$.

The solutions of the system obtained when linearising equations (1.1) around the equilibrium $q_n = 0$, are superpositions of plane wave solutions (phonons)

$$q_n(t) = \exp(i(kn - \omega_0 t)), $$

with frequencies

$$\omega_0^2 (k) = \omega_0^2 + 4 \kappa \sin^2 \left( \frac{k}{2} \right), \quad k \in [-\pi, \pi].$$

These (extended) states disperse. Therefore, the frequency $\omega_b$ of a localised time-periodic solution must satisfy the non-resonance condition $\omega_b \neq |\omega_0(k)|/m$ for any integer $m \geq 1$. This requires $\omega_b^2 > \omega_0^2 + 4 \kappa \omega_0^2 < \omega_0^2$ for hard (soft) on-site potentials as a necessary condition for the existence of localised time-periodic solutions of system (1.1).

The system (1.1) with its associated Hamiltonian (1.3) belongs to the class of Hamiltonian systems that are even.

**Lemma I.1** Consider the system (1.1) on the infinite lattice $n \in \mathbb{Z}$. Let assumption I.1 hold. In addition, for systems with soft on-site potential let assumptions I.2 hold. Further, let $(p_n(t), q_n(t))_{n \in \mathbb{Z}}$, be the solution to the time-reversible system

$$\dot{p}_n = -\frac{\partial H}{\partial q_n}, \quad \dot{q}_n = \frac{\partial H}{\partial p_n}, \quad n \in \mathbb{Z},$$

with $H$ given in (1.3) and with initial data

$$(p_n(0))_{n \in \mathbb{Z}} = 0, \ (q_n(0))_{n \in \mathbb{Z}} \neq 0,$$  

and $(q_n)_{n \in \mathbb{Z}} \in l^1$. If for some $\tilde{t} > 0$ one has

$$p_n(\tilde{t}) = p_n(0) = 0, \quad n \in \mathbb{Z},$$

then the data $(p_n(0), q_n(0))_{n \in \mathbb{Z}}$ belong to a periodic orbit with period $2\tilde{t}$.

**Proof:** Note that as a consequence of the assumptions on the initial data (1.8) in conjunction with the conservation of energy, $E \neq 0$, the attributed solution $(p_n(t), q_n(t))_{n \in \mathbb{Z}}$ to system (1.7) is non-trivial. Suppose there is a $\tilde{t} > 0$ such that for initial data (1.8) the solution satisfies the conditions in (1.9). Then the time-reversibility symmetry $p \rightarrow -p$ implies that the solution for the initial data (1.8) possesses the following symmetry features

$$p_n(\tilde{t} + t) = -p_n(\tilde{t} - t), \quad q_n(\tilde{t} + t) = q_n(\tilde{t} - t), \quad 0 \leq t \leq \tilde{t}, \quad n \in \mathbb{Z}.$$ 

Therefore,

$$q_n(0) = q_n(2\tilde{t}), \quad p_n(0) = -p_n(2\tilde{t}) = 0, \quad n \in \mathbb{Z},$$

verifying that the solution is periodic with period $2\tilde{t}$ and the proof is complete. 

□
II. LOCALISED SOLUTIONS FOR AN INFINITE LATTICE OF COUPLED NONLINEAR OSCILLATORS

In the following we prove the existence of spatially localised and time-periodic solutions for the system (1.1). Using Duhamel’s principle, the solution of system (1.1) with initial data

\[(q_n(0))_{n \in \mathbb{Z}} = (A_n)_{n \in \mathbb{Z}}, \quad (\dot{q}_n(0))_{n \in \mathbb{Z}} = 0,\]  

(2.10)
can be expressed as a system of integral equations

\[q_n(t) = A_n \cos(\omega_0 t) + \frac{1}{\omega_0} \int_0^t \sin[\omega_0(t - \tau)]f(q_{n-1}(\tau), q_n(\tau), q_{n+1}(\tau))d\tau, \quad n \in \mathbb{Z}\]  

(2.11)
where \(f(q_{n-1}, q_n(\tau), q_{n+1}) = -W'(q_n) + \kappa(q_{n+1} + q_{n-1} - 2q_n)\) and we assume

\[A = (A_n)_{n \in \mathbb{Z}} \in l^1.\]

Note that by continuous embedding \(A \in l^1 \subset l^2\) and \(\|A\|_{l^2} \leq \|A\|_{l^1}\). To show the existence of a unique local solution of (2.11) we consider \(q = (q_n)_{n \in \mathbb{Z}} \in C^1([0,s];l^2)\), where \(s\) is a fixed positive number. Consider the Banach space

\[\mathcal{B} = \{ q \in C^1([0,s];l^2) \}\]

with norm

\[\| q \|_{\mathcal{B}} = \max_{t \in [0,s]} \{ \| q(t) \|_{l^2}, \| \dot{q}(t) \|_{l^2} \} \]

We define a subset \(S\) of \(\mathcal{B}\) as

\[S = \{ q \in \mathcal{B} : \| q \|_{\mathcal{B}} \leq R \} .\]

Clearly, \(S\) is a closed, bounded and convex set.

Related to (2.11) we define an operator on \(S\) as follows:

\[U(q) = A \cos(\omega_0 t) + \frac{1}{\omega_0} \int_0^t \sin[\omega_0(t - \tau)]f(q(\tau))d\tau.\]

Next we show that the IVP (1.1), (2.10) admits a unique global solution.

Lemma II.1 For every \(A \in l^1\) there exists a unique global solution \(q(t)\) of the system (1.1) in \(C^2([0,\infty),l^2)\) such that \(q(0) = A\) and \(\dot{q}(0) = 0\).

Proof: Local well-posedness of the initial value problem (IVP) (1.1), (2.10) and differentiability of the local solution \(q\) with respect to \(t\) is shown using the contraction mapping principle applied to the integral representation (2.11). Using the Banach algebra property of \(l^2\), i.e. \(\|xy\|_{l^2} \leq \|x\|_{l^2}\|y\|_{l^2}\) for all \(x,y \in l^2\), we derive for the following upper bound

\[\|U(q)\|_{l^2} \leq \max \{1, \omega_0\} \left( R_0 + \frac{s}{\omega_0} (K_\alpha R^\alpha + 4\kappa R) \right), \quad R_0 = \|A\|_{l^2}, \quad \forall q \in S.\]

We may choose \(R_0 < R/2\) and \(s \leq \max \{1, \omega_0\} \cdot R\omega_0/(2(K_\alpha R^\alpha + 4\kappa R))\), so that \(U : S \rightarrow S\). Now for every \(x, y \in S\) one has

\[(U(x))_n(t) - (U(y))_n(t) = \frac{1}{\omega_0} \int_0^t \sin[\omega_0(t - \tau)](f(x_{n-1}(\tau), x_n(\tau), x_{n+1}(\tau)) - f(y_{n-1}(\tau), y_n(\tau), y_{n+1}(\tau)))d\tau \]

\[= \frac{1}{\omega_0} \int_0^t \sin[\omega_0(t - \tau)]((\Delta x)_n(\tau) - (\Delta y)_n(\tau)) - (W'(x_n(\tau)) - W'(y_n(\tau)))d\tau,\]

and we used the notation \((\Delta x)_n = x_{n+1} + x_{n-1} - 2x_n\). We derive the estimate

\[\|U(x) - U(y)\|_{l^2} \leq \frac{s}{\omega_0} \left( 4\kappa R + 2K_\beta R^2 \right) \|x - y\|_{l^2}.\]
Hence, by taking

\[ s < \min \left\{ \frac{\omega_0}{4\kappa R + 2K_\beta R^2}, \frac{R\omega_0}{2(K_\alpha R^\alpha + 4\kappa R)} \right\}, \]

one concludes that \( U \) constitutes a contraction mapping on \( S \) so that by the Banach Fixed Point Theorem there exists a unique fixed point of \( U \) in \( S \) which is a unique local solution of (2.11).

A maximal solution can be constructed by repeated application of the procedure above with initial data \( A(s - s_0) \) for some \( 0 < s_0 < s \) where we exploit the uniqueness of the solution to continue the latter.

From (1.11) we get

\[ \sup_{t \in [0, s]} ||\dot{q}(t)||_2 \leq (K_\alpha R^{\alpha - 1} + 4\kappa)R, \]

assuring that the solution belongs to \( C^2([0, s], l^2) \).

Global continuation of the solution is guaranteed by the energy method leading to the constraints (1.4) and (1.6) for hard and soft on-site potentials, respectively, from which follows that there is no blow-up of the solutions in finite time.

The following lemma on the smooth dependence of the solutions on the initial conditions will be made use of for the existence proof of breather solutions.

**Lemma II.2** Let \( \phi(t, 0, A) \) be the solutions to the IVP determined by (1.1) and (2.10) with \( \sup_{t \in [0, \infty)} ||\phi(t, 0, A)||_1 \leq M < \infty \). Then for all \( A \in l^1 \) it holds

\[ ||\phi(t, 0, A)||_1 \leq ||A||_1 \exp\left[ \frac{K_\alpha M^{\alpha - 1} + 4\kappa}{\omega_0} t \right], \quad \forall t \in [0, \infty). \]  

\[ (2.12) \]

**Proof:** Using (1.2) and (2.11) we derive

\[ ||\phi(t, 0, A)||_1 = \sum_{n \in \mathbb{Z}} |A_n \cos(\omega_0 t) + \frac{1}{\omega_0} \int_0^t \sin(\omega_0(t - \tau)) f(\phi_{n-1}(\tau, 0, A), \phi_n(\tau, 0, A), \phi_{n+1}(\tau, 0, A)) d\tau| \]

\[ \leq ||A||_1 + \frac{1}{\omega_0} \int_0^t \sum_{n \in \mathbb{Z}} | - W'(\phi_n(\tau, 0, A)) + \kappa[\phi_{n+1}(\tau, 0, A) - 2\phi_n(\tau, 0, A) + \phi_{n-1}(\tau, 0, A)]| d\tau \]

\[ \leq ||A||_1 + \frac{K_\alpha M^{\alpha - 1} + 4\kappa}{\omega_0} \int_0^t ||\phi(\tau, 0, A)||_1 d\tau, \quad \forall t \in [0, \infty). \]

Facilitating Gronwall’s inequality we get (2.12) concluding the proof.

Furthermore, we need the following result for compact operators on infinite dimensional Banach spaces.

**Lemma II.3** Let \( X \) be an infinite dimensional Banach space and \( g : X \mapsto X \) compact. Then \( g \) is not surjective, i.e. there is \( y \in X \) such that \( g(x) = y \) has no solution.

**Proof:** Denote by \( B(0, r) = \{ x \in X : ||x||_X < r \} \) the open ball in \( X \) of center 0 and radius \( r \). For a contradiction, suppose that \( g \) is surjective. By the open mapping theorem it follows that \( g \) is an open operator. Particularly, \( g(B(0, 1)) \subseteq X \) is an open set. That is, there exists \( \epsilon > 0 \) such that

\[ B(0, \epsilon) \subseteq g(B(0, 1)). \]  

\[ (2.13) \]

\( g \) being compact, implies \( g(B(0, 1)) \) is relatively compact. Then due to (2.13), it follows that \( B(0, \epsilon) \) is relatively compact. Furthermore, there exists \( \tilde{\epsilon} < \epsilon \), such that the closed ball \( \overline{B}(0, \tilde{\epsilon}) \subseteq g(B(0, 1)) \) and \( \overline{B}(0, \tilde{\epsilon}) \) is compact. However, if \( \overline{B}(0, \tilde{\epsilon}) \) is compact, then by the Riesz theorem \( X \) must be finite dimensional, contradicting the hypothesis that \( X \) is infinite dimensional. Therefore, \( g \) is not surjective.

Finally, for the existence of breathers in system (1.1) we have the following statement:
Theorem II.1 Let \( \mu = \max_{x \in I} |\sin(x)|/x = |\sin(u)|/u \) with \( I = [0, \pi] \) (\( I = [\pi, 2\pi] \)) for hard (soft) on-site potentials, where \( u \) solves \( u = \tan(u) \) in the respective interval.

Consider the solutions \( \phi(t, 0, A) \) to the IVP determined by (1.14) and (2.10). Take initial data \( A \in S_A \subseteq l^1 \), where \( S_A \) is the closed ball centered at 0 of radius \( R_A \) of \( l^1 \),

\[
S_A = \left\{ A \in l^1 : \| A \|_1 \leq R_A < \left( \frac{\omega_0^2 M}{K_\alpha} \right)^{1/(\alpha - 1)} \right\}. \quad (2.14)
\]

Then for \( \kappa \) small enough there exists a breather solution.

**Proof:** We have

\[
\dot{\phi}_n(t, 0, A) = -\omega_0 A_n \sin(\omega_0 t) + \int_0^t \cos[\omega_0(t - \tau)] f(\phi_{n-1}(\tau, 0, A), \phi_n(\tau, 0, A), \phi_{n+1}(\tau, 0, A)) d\tau, \quad n \in \mathbb{Z}.
\]

The relation \( \dot{\phi}_n(\tilde{t}) = 0 \) for all \( n \in \mathbb{Z} \), for some \( \tilde{t} \) with \( 0 < \tilde{t} < \pi/\sqrt{\omega_0^2 + 4\kappa} \) for hard on-site potentials and \( \pi/\omega_0 < \tilde{t} < 2\pi/\omega_0 \), for soft on-site potentials, is equivalent to

\[
A_n = -\frac{1}{\omega_0 \sin(\omega_0 t)} \int_0^{\tilde{t}} \cos[\omega_0(\tilde{t} - \tau)] f(\phi_{n-1}(\tau, 0, A), \phi_n(\tau, 0, A), \phi_{n+1}(\tau, 0, A)) d\tau, \quad n \in \mathbb{Z}. \quad (2.15)
\]

The right side of system (2.15) constitutes a mapping \( g : l^1 \to l^1 \), with

\[
(g(A))_n = g_n(A) = -\frac{1}{\omega_0 \sin(\omega_0 t)} \int_0^{\tilde{t}} \cos[\omega_0(\tilde{t} - \tau)] f(\phi_{n-1}(\tau, 0, A), \phi_n(\tau, 0, A), \phi_{n+1}(\tau, 0, A)) d\tau, \quad n \in \mathbb{Z}.
\]

We show that \( g \) is sequentially continuous on \( l^1 \). Let \( (A_k)_{k \in \mathbb{N}} \subseteq l^1 \) be such that \( A_k \to A \) as \( k \to \infty \) in \( l^1 \). By the continuous dependence of the solutions on the initial conditions (by virtue of Lemma II.2), it holds that \( \phi_i(\tau, 0, A_k^i) \to \phi_i(\tau, 0, A) \) as \( k \to \infty \) for all \( i \in \mathbb{Z} \) and \( \tau \in [0, \tilde{t}] \). Furthermore, as \( W'(\phi_i(\tau, 0, A^i)) \to W'(\phi_i(\tau, 0, A)) \) and \( (\Delta \phi_i(\tau, 0, A^i))_i \to (\Delta \phi_i(\tau, 0, A))_i \) as \( k \to \infty \) for all \( i \in \mathbb{Z} \) and \( \tau \in [0, \tilde{t}] \), one has \( f(\phi_{i-1}(\tau, 0, A^i), \phi_i(\tau, 0, A^i), \phi_{i+1}(\tau, 0, A^i)) \to f(\phi_{i-1}(\tau, 0, A), \phi_i(\tau, 0, A), \phi_{i+1}(\tau, 0, A)) \) as \( k \to \infty \) for all \( i \in \mathbb{Z} \) and \( \tau \in [0, \tilde{t}] \). Hence,

\[
g_i(A_k^i) \to g_i(A) \quad \text{as} \quad k \to \infty, \quad \forall i \in \mathbb{Z},
\]

i.e. \( g \) is sequentially continuous on \( l^1 \), and thus, as sequential continuity is equivalent to continuity in normed spaces, \( g \) is continuous.

Obviously, \( S_A \) is a closed, bounded and convex set of \( l^1 \) and we consider the map \( g : S_A \to S_A \). A solution of the system of equations (2.15) is then a fixed point of the operator equation

\[
A = g(A). \quad (2.16)
\]

In order to apply Schauder’s Fixed Point Theorem to show the existence of at least one solution to (2.16), one needs to verify that the operator \( g \) maps \( S_A \) into itself and is compact on \( S_A \).

We introduce \( M = \max_{t \in [0, \tilde{t}]} \| \phi(t, 0, A) \|_{l^1} \), and show that for initial data \( A \in S_A = B_{R_A} \subseteq l^1 \), with \( R_A \) constrained by (2.14), and coupling strength \( \kappa \) small enough, the relation

\[
\frac{K_\alpha M_\alpha + 4\kappa M}{\omega_0 \sin(\omega_0 t)} \tilde{t} < R_A \quad (2.17)
\]

is fulfilled. Note that for the solutions, \( \phi(t, 0, A) \), to the IVP determined by (1.14) and (2.10) it holds for \( \kappa = 0 \) that \( \max_{t \in [0, \tilde{t}]} \| \phi(t, 0, A) \|_{l^1} = \| A \|_{l^1} \), and thus, \( M = \max_{t \in [0, \tilde{t}]} \| \phi(t, 0, A) \|_{l^1} = R_A \). Therefore, when \( \kappa = 0 \), (2.17) is satisfied if

\[
\frac{K_\alpha R_A^\alpha}{\omega_0 \sin(\omega_0 t)} \tilde{t} < R_A,
\]
yielding (2.14), that is

$$ R_A < \left( \frac{\omega_0 |\sin(\omega_0 t)|}{K_\alpha t} \right)^{1/(\alpha - 1)} \leq \left( \frac{\omega_0^2 \mu}{K_\alpha} \right)^{1/(\alpha - 1)} . \quad (2.18) $$

By the continuous dependence of the solutions on the parameter $\kappa$, there is a sufficiently small $\kappa_0$ such that for $0 < \kappa < \kappa_0$ the relation (2.17) holds for all $A \in B_{R_A}$.

Using (2.17) we estimate

$$ ||g(A)||_{l^1} = \sum_{n \in \mathbb{Z}} |(g(A))_n| = \sum_{n \in \mathbb{Z}} \left| \frac{1}{\omega_0 \sin(\omega_0 t)} \int_0^t \cos(\omega_0 (\tilde{t} - \tau)) f(\phi_{n-1}(\tau, 0, A), \phi_n(\tau, 0, A), \phi_{n+1}(\tau, 0, A)) d\tau \right| $$

$$ \leq \frac{1}{\omega_0 |\sin(\omega_0 t)|} \int_0^t \sum_{n \in \mathbb{Z}} | - W'(\phi_n(\tau, 0, A)) + \kappa [\phi_{n+1}(\tau, 0, A) - 2\phi_n(\tau, 0, A) + \phi_{n-1}(\tau, 0, A)] | d\tau $$

$$ \leq \frac{1}{\omega_0 |\sin(\omega_0 t)|} (K_\alpha M^{\alpha - 1} + 4\kappa) \int_0^t \sum_{n \in \mathbb{Z}} |\phi_n(\tau, 0, A)| |d\tau| $$

$$ = \frac{1}{\omega_0 |\sin(\omega_0 t)|} (K_\alpha M^{\alpha - 1} + 4\kappa) \int_0^t ||\phi(\tau, 0, A)||_{l^1} |\tau| $$

$$ \leq \frac{K_\alpha M^{\alpha} + 4\kappa M_0}{\omega_0 |\sin(\omega_0 t)|} \tilde{t} < R_A, $$

showing that for $\kappa$ sufficiently small indeed $g(S_A) \subseteq S_A$.

In order to prove that $g$ is compact we consider a sequence $(A_k)_{k \in \mathbb{N}} \subseteq S_A \subseteq l^1$. Since for every $k \in \mathbb{N}$ one has $||g(A_k)||_{l^1} \leq R_A$, the sequence $g(A_k)$ is bounded. Hence, $g(A_k)$ possesses a weakly convergent subsequence (not relabeled) that converges to a $B \in l^1$ ($g(A_k) \rightharpoonup B$ as $k \to \infty$). Since $l^1$ possesses the Schur property, i.e. weak and norm sequential convergence coincide in $l^1$, $g(A^k)$ possesses a subsequence that strongly (norm) converges in $l^1$. Hence, $g$ maps bounded subsets of $l^1$ into relatively compact subsets of $l^1$ and therefore, is compact. By Schauder’s Fixed Point Theorem the system (2.11) has at least one solution $A \in l^1$.

It remains to verify that there is at least one nontrivial fixed point solution. By contradiction: To this end assume that for every $A \in S_A \setminus \{0\}$ there exists a $B \in l^1$, $B \neq 0$ solving the inhomogeneous system

$$ A - g(A) = B \neq 0. $$

Suppose that the kernel of the operator $g - I$ is trivial. Then, for every $A \in S_A \setminus \{0\}$, there is $B \in l^1 \setminus \{0\}$ solving the inhomogeneous system

$$ g(A) - A = B \neq 0. \quad (2.19) $$

This is equivalent to $g(A) = A + B$ for all $A \in S_A \setminus \{0\}$. Since $||g(A)||_{l^1} = ||A + B||_{l^1} \leq R_A$ and $g : S_A \subseteq l^1 \mapsto S_A \subseteq l^1$, we have that $A + B$ in $S_A$, requiring that $g$ is surjective. But as $S_A$ is infinite dimensional and $g : S_A \subseteq l^1 \mapsto S_A \subseteq l^1$ is compact, from Lemma 11.3 follows that $g$ is not surjective. That is, there is $A + B$ for which $g(A) = A + B$ has no solution which contradicts (2.19). Conclusively, the kernel of $g - I$ is not trivial so that there is $A \neq 0$ solving $g(A) - A = 0$. Hence, the fixed point equation (2.16) possesses at least one non-trivial solution.

In conclusion, for $t > 0$, satisfying (2.17) and initial conditions $A \in l^1$, the existence of solutions for which

$$ \dot{\phi}_n(t, 0, A) = 0, \forall n \in \mathbb{Z}, $$

is verified. Periodicity of $\phi(t, 0, A)$ with $\phi(0, 0, A) = \phi(2\tilde{t}, 0, A)$, is a consequence of the assertions of Lemma 11.1.

Since $\phi(t, 0, A) \in \mathcal{C}^2([0, \infty); l^2)$ the localised solutions on the infinite lattice $\mathbb{Z}$ are represented by (infinite) square-summable sequences, viz. decay of the states for $|n| \to \infty$ takes place in the sense of the $l^2$ norm. In order to establish the existence of single-site breathers employing the above fixed point method, appropriately (e.g. exponentially) weighted function spaces can be used (see in 37).

\[ \square \]

Final remarks: Notice that (2.18) implies, when the frequency of the breather, $\omega_b = 4\pi/\tilde{t}$, approaches the edge of the continuous (phonon) spectrum, that is $\omega_b \to \sqrt{\omega_0^2 + 4\kappa}$ and small $\kappa$ for hard on-site potentials, and $\omega_b \to \omega_0^-$.
for soft on-site potentials, then $R_A \to 0$, that is the amplitude of the breather goes to zero. It is certainly of interest to extend the fixed point method to prove the existence of breathers also in higher dimensional lattices $\mathbb{Z}^d > 1$.

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