Sandpile monomorphisms and harmonic functions

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Abstract

The abelian sandpile model is a cellular automaton defined on a finite convex domain $\Gamma \subset \mathbb{Z}^2$ of the standard square lattice $\mathbb{Z}^2$. Its recurrent configurations form an abelian group, the sandpile group. Little is known about the structure of this group, or the relationships between sandpile groups defined on different domains. In this article, we show that the sandpile group is isomorphic to the rational-valued discrete harmonic functions which take integer-values on the boundary $\partial(\mathbb{Z}^2 \setminus \Gamma)$ of the complement of the domain, modulo the integer-valued harmonic functions on the same domain. We use this isomorphism to derive unique coordinates for every recurrent sandpile configuration as well as an alternative formula for the order of the sandpile group. Furthermore, we derive several lemmata on the existence of cyclic subgroups for sandpile groups on $N \times N$ square domains. Finally, we show that there exist families of monomorphisms from each sandpile group on an $N \times M$ domain to all sandpile groups on $\psi(N + 1) - 1 \times \psi(M + 1) - 1$ domains, with $\psi \in \mathbb{N}$. The latter result can likely be extended to sandpile groups defined on certain other, non-rectangular domains of $\mathbb{Z}^2$, or on domains of higher dimensional lattices $\mathbb{Z}^k$, $k \in \mathbb{N}$. Our work proposes that injective limits of the sandpile group with respect to domains of certain shapes might exist.

1 Introduction

The abelian sandpile model was introduced by Bak, Tang and Wiesenfeld in 1987 [1] as the first example of a system showing self-organized criticality (SOC), a phenomenon which subsequently became important in several areas of physics, biology, geology and other fields (see [2] for a recent review). The sandpile model is a cellular automaton defined on a (typically rectangular) domain $\Gamma \subset \mathbb{Z}^2$ of the standard square lattice $\mathbb{Z}^2$. Every vertex of the domain can carry between

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zero and three particles (“grains of sand”), and the number of particles for all vertices of the domain is referred to as the configuration of the sandpile. Given an initial configuration, the automaton progresses by adding additional particles to vertices chosen at random. If, during this process, a vertex happens to carry four or more particles, it becomes unstable and “topples”: four particles are removed from the vertex, and one particle is added to each of its (four or less) direct neighbors. The toppling of one vertex can render other vertices unstable which are subsequently toppled in a process resembling an avalanche. Every avalanche eventually stops due to the loss of particles at the boundary of the domain [3], and the “relaxed” configuration which is reached after the last unstable vertex toppled is independent of the order of the topplings [3].

Shortly after the publication of the sandpile model, Dhar [4, 5, 6] and Creutz [3] laid the theoretical foundation for its mathematical analysis. Dhar distinguished between transient and recurrent configurations [4], i.e. between configurations appearing finitely and infinitely often, respectively, in the above described Markov process. Dhar then showed that the recurrent configurations form an abelian group, the sandpile group, which is isomorphic to \( \mathbb{Z}^\Gamma / \Delta \mathbb{Z}^\Gamma \) [4]. Furthermore, he introduced the “burning algorithm” which tests if a given configuration is recurrent [4]. Subsequently, this algorithm was used to define a bijection (in the category of sets) between the sandpile group and the set of spanning trees/forests on the same domain [5] (see also [7] for similar results). Creutz, on the other hand, was the first to analyze the identity of the sandpile group, and provided an algorithm for its construction [3, 8]. This sandpile identity is composed of self-similar fractal patterns [3]: since these patterns are remarkably similar on domains with the same shape, scaling limits have been proposed for the sandpile identity (see e.g. [9]). Creutz also showed that every recurrent configuration can be reached from every other configuration by only adding particles to the boundary of the domain and relaxing the sandpile [3].

Despite this remarkable initial progress and subsequent 30 years of intensive research, relatively little is known about the structure of the sandpile group as well as the relationship between sandpile groups defined on different domains. For example, while it is possible to calculate the order of the sandpile group on a given domain via the determinant of the reduced graph laplacian [4], this formula only provides indirect information, on a per domain basis, of the decomposition of the sandpile group into the direct sum of cyclic groups of prime-power order which should exist according to the fundamental theorem of finite abelian groups. Similarly, while numeric studies indicated for a long time that there might exist scaling limits for the sandpile identity on certain domains [3, 9], the orders of the sandpile groups on different domains were found to be in general “incompatible” in the sense that no group monomorphisms can exist between them. For example, the order of the sandpile group on a 3 × 3 square domain is 2^{11}7^2, while the order on a 5 × 5 domain is 2^{18}3^55^211^213^2. This lack of known relationships between the sandpile groups on different domains is in stark contrast to the role of the sandpile model as the archetypical example for self-organized criticality, given that the concept of criticality itself is based on the notion of scaling.
Recently, we introduced the extended sandpile group, a $|\partial \Gamma|$-dimensional Lie group with the topology of a torus [9], where $\partial \Gamma$ denotes the set of boundary vertices of the domain. The elements of this Lie group correspond to the recurrent configurations of an extended sandpile model obtained by allowing all vertices at the boundary of the domain to carry a real-valued number of particles, while all vertices in the interior of the domain can still only carry an integer-valued number [9]. We have shown that the extended sandpile group is isomorphic to the space of real-valued harmonic functions modulo the space of integer-valued ones. We have then defined a natural renormalization of the extended sandpile group which corresponds to epimorphisms mapping recurrent configurations on a given domain to recurrent configurations on one of its sub-domains [9]. Since the usual sandpile group forms a discrete subgroup of the extended one, also the former possesses a natural renormalization. However, on the level of the usual sandpile group, this renormalization is defined in the category of sets, and only “approximates” a group homeomorphism for sufficiently large domains [9]. Furthermore, the renormalization does not distinguish between domains of different shape, and thus e.g. allows to map configurations on square domains onto configurations on circular domains. In contrast, several properties of the sandpile group, including the conjectured scaling limit for the sandpile identity [9], are known or suspected to depend on the shape of the domain.

In this article, we study the relationship between sandpile groups and harmonic functions. We first derive a basis for the module of integer-valued harmonic functions on a given finite and convex domain, and then use this basis to define coordinates which uniquely identify each recurrent configuration. Based on this basis, we also derive an alternative formula for the order of the sandpile group. We then show that the sandpile group is isomorphic to the group of rational-valued harmonic functions which take integer-values on the boundary $\partial (\mathbb{Z}^2 \setminus \Gamma)$ of the complement of the domain, modulo the group of integer-valued harmonic functions on the same domain. We then formulate a problem statement concerning the extendability of such harmonic functions to larger domains, which directly asks for the existence of monomorphisms between the respective sandpile groups. Subsequently, we solve this problem statement for rectangular domains, and thereby directly construct families of monomorphisms from each sandpile group on a $N \times M$ domain to all sandpile groups on $\psi(N+1) - 1 \times \psi(M+1) - 1$ domains, with $\psi \in \mathbb{N}$. Finally, we derive several lemmata on the existence of cyclic subgroups of certain order for $N \times N$ domains, which explain several experimentally observed regularities in the factorization of the order of the sandpile group.

2 Results

2.1 Notation

Throughout this article, we denote by $\Gamma \subseteq \mathbb{Z}^2$ domains of the standard square lattice $\mathbb{Z}^2$, and by $\partial \Gamma$ and $\Gamma_0 = \Gamma \setminus \partial \Gamma$ their boundaries and interiors, respectively.
Two vertices \((x_1, y_1), (x_2, y_2) \in \mathbb{Z}^2\) are neighbors, denoted by \((x_1, y_1) \sim (x_2, y_2)\), if \(|x_1 - x_2| + |y_1 - y_2| = 1\). We say that a domain \(\Gamma \subseteq \mathbb{Z}^2\) is convex if there exists a convex domain \(\hat{\Gamma} \subseteq \mathbb{R}^2\) such that \(\Gamma = \Gamma \cap \mathbb{Z}^2\). The convex domains of \(\mathbb{Z}^2\) form the category \(\mathcal{D}^\infty\) with inclusions \(\Gamma_1 \subseteq \Gamma_2\) as morphisms. The finite convex domains form a full subcategory of \(\mathcal{D}^\infty\), which we denote by \(\mathcal{D}\).

For every commutative ring \(R\) and every domain \(D \supseteq \Gamma \in \mathcal{D}\), we define the Laplace operator \(\Delta^\Gamma : (D \rightarrow R) \rightarrow (\Gamma \rightarrow R)\) by

\[
\Delta^\Gamma F(x, y) = \tilde{F}(x + 1, y) + \tilde{F}(x - 1, y) + \tilde{F}(x, y + 1) + \tilde{F}(x, y - 1) - 4\tilde{F}(x, y),
\]

with

\[
\tilde{F}(x, y) = \begin{cases} 
F(x, y) & \text{if } (x, y) \in \Gamma \\
0 & \text{otherwise.}
\end{cases}
\]

We say that a function \(H : D \rightarrow R\) is harmonic (on \(\Gamma\)) if \(\Delta^\Gamma H(z) = 0\) for all \(z \in \Gamma_0\), and denote by \(X^\Gamma_H : \Gamma \rightarrow R\), \(X^\Gamma_H = -\Delta^\Gamma H\), the potential of \(H\) (on \(\Gamma\)) \[9\]. Finally, we denote by \(H^\Gamma_R\) the module of all harmonic functions \(H : \Gamma \rightarrow R\) over \(R\) (on \(\Gamma\)).

For a given domain \(\Gamma \in \mathcal{D}\), we denote by \(G^\Gamma\) the sandpile group (on \(\Gamma\)). Recall that \(G^\Gamma\) is isomorphic to \(\mathbb{Z}^\Gamma / \Delta^\Gamma\mathbb{Z}^\Gamma\) \[4\]. Usually, the relaxation operator \((.)^o : \mathbb{Z}^\Gamma_{\geq 0} \rightarrow \{0, \ldots, 3\}^\Gamma\) is defined to correspond to a series of (allowed) topplings which stabilizes a given configuration \[4, 3\]. While intuitive, this definition has the disadvantage to be only applicable to non-negative configurations, and to not necessarily result in a recurrent configuration. In this article, we thus redefine the relaxation operator \((.)^o : \mathbb{Z}^\Gamma \rightarrow G^\Gamma\) to map a (potentially negative) configuration to the respective recurrent configuration in the same equivalence class, according to the isomorphism \(G^\Gamma \cong \mathbb{Z}^\Gamma / \Delta^\Gamma\).

### 2.2 Motivation

Our initial interest in the research which lead to this article was sparked by several regularities we observed during a numeric study of the orders of the sandpile groups on different \(N \times N\) square domains \(\Gamma \in \mathcal{D}\) (Supplementary Table S1). In the factorizations of these orders, we found that most factors appeared with even multiplicities, and only few with odd. Furthermore, the factor 2 always appeared with high multiplicities, while most other factors had low multiplicities–if they appeared at all. Our by far most surprising observation, however, was that the order of the sandpile group on a given \(N \times N\) domain appeared to always divide the order of all sandpile groups on \(\psi(N + 1) - 1 \times \psi(N + 1) - 1\) domains, with \(\psi \in \mathbb{N}\) (Figure 1A). We took the latter as an indication that monomorphisms between the respective sandpile groups might exist.

Back then, the only handle we had to mathematically explain these phenomena was an observation we made during the writeup of our previous article, where we analyzed the continuous sandpile dynamics induced by integer-valued harmonic functions \[9\]. For a given finite convex domain \(\Gamma \in \mathcal{D}\) and a given
integer-valued harmonic function \( H \in \mathcal{H}_Z^g \), these harmonic sandpile dynamics are defined by

\[
\Delta^g H(t) = (I - [t \Delta^g H])^\circ,
\]

with \( t \geq 0 \) denoting the time, \( I \) the sandpile identity, and \([.\] the element-wise floor function [9]. Already while working on this article, we asked ourselves under which conditions configurations appear “exactly” in these harmonic dynamics, in the sense that, for a given harmonic function \( H \in \mathcal{H}_Z^g \), there exists a
time $0 < t < 1$ such that $\lfloor t \Delta^\Gamma H \rfloor = t \Delta^\Gamma H$ in (1). If we denote by $C^\Gamma_1 = D^\Gamma_H(t)$ such a configuration appearing exactly at time $t$, also at multiples $kt$, $k \in \mathbb{N}$, of this time, other configurations $C^\Gamma_k = D^\Gamma_H(kt) = (kC^\Gamma_1)^k$ will appear exactly. Assuming that the values of $H$ are coprime, it is easy to see that a configuration can only appear exactly at time $0 < t < 1$ if $t = \frac{t_D}{t_D} \in \mathbb{Q}$ is rational, and if the potential $X^\Gamma_H = -\Delta^\Gamma H$ of the harmonic function is divisible by $t_D$ (Figure 1B). Since the harmonic dynamics are cyclic with period 1 (i.e. $D^\Gamma_H(t) = D^\Gamma_H(t + n)$, $n \in \mathbb{N}$, see Lemma 1 in [9]), it directly follows that

$$\left\{ I = D^\Gamma_H(0), D^\Gamma_H \left( \frac{1}{t_D} \right), \ldots, D^\Gamma_H \left( \frac{t_D - 1}{t_D} \right) \right\}$$

forms a cyclic subgroup of the sandpile group with order $t_D$ (Figure 1C).

For example, consider the harmonic dynamics induced by $H = xy$ on an $N \times N$ domain $\Gamma \in \mathcal{D}$, with $N = 2n + 1$ and $n \in \mathbb{N}$. Since $H$ is defined on the whole of $\mathbb{Z}^2$, the potential $X^\Gamma_H = -\Delta^\Gamma H$ at vertex $v \in \Gamma$ can be equivalently expressed as

$$X^\Gamma_H(v) = \sum_{w \sim v \in \partial (\mathbb{Z}^2 \setminus \Gamma)} H(w), \quad (2)$$

with $w \sim v$ if vertex $w$ is a direct neighbor of $v$. It is easy to see that all values of $H = xy$ on the boundary $\partial (\mathbb{Z}^2 \setminus \Gamma)$ of the complement of the domain $\Gamma$ are divisible by $\frac{N + 1}{2}$, and, thus, that also $X^\Gamma_H$ is divisible by $\frac{N + 1}{2}$ (Figure 1B). Since $H = xy$ is coprime on $\Gamma$, this directly proves our first lemma:

**Lemma 1** Assume that $\Gamma \in \mathcal{D}$ is an $N \times N$ domain, with $N = 2n - 1$ and $n \in \mathbb{N}$. Then, the sandpile group $G^\Gamma$ on $\Gamma$ has a cyclic subgroup $\mathbb{Z}/\frac{N + 1}{2}\mathbb{Z} < G^\Gamma$ of order $\frac{N + 1}{2}$.

We note that for $N = 1$, this subgroup is the trivial group.

Our initial hope was that, by “guessing” other integer-valued harmonic functions defining non-trivial cyclic subgroups, we would eventually discover sufficient mathematical structure to derive a closed formula for the decomposition of the sandpile group into the direct sum of cyclic groups of prime-power order. While we ultimately failed in achieving this goal, the framework we developed, and explain in detail below, was at least powerful enough to explain (nearly) all of our initial observations. We provide this background on the history of our article in the hope that others might take up our work and succeed in deriving a closed formula for the decomposition.

### 2.3 A basis for integer-valued harmonic functions

On a finite convex domain $\Gamma \in \mathcal{D}$, the real-valued harmonic functions $\mathcal{H}_R^\Gamma$ form a vector space. From the existence and uniqueness of solutions of the Dirichlet problem on $\Gamma$, it then follows that every subset $\mathcal{B}_R^\Gamma = \{ B_1, \ldots, B_{|\partial \Gamma|} \} \subseteq \mathcal{H}_R^\Gamma$ of $|\partial \Gamma|$ linearly independent harmonic functions forms a basis of $\mathcal{H}_R^\Gamma$. In contrast,
Figure 2: Harmonic basis functions used in this article, and construction of a basis for the module of integer-valued harmonic functions $\mathcal{H}_2^\Gamma$ on a finite convex domain $\Gamma \in \mathcal{D}$. A&B) Two examples of diagonal harmonic basis functions which are zero for all vertices above the diagonal $d_i^+$ (A), respectively below $d_i^{-1}$ (B). Note that both harmonic functions only have values in $\{+1, -1\}$ on their defining diagonals, and that $B_{d_i^{-1}}^{-}$ is symmetric with respect to $d_i^+$ (B), while $B_{d_i^+}^+$ is anti-symmetric with respect to $d_i^-$ (A). C) Iterative construction of a basis for $\mathcal{H}_2^\Gamma$ on a $4 \times 4$ square domain. Light-gray backgrounds denote those vertices already belonging to the growing domain at the beginning of the respective iteration step, and dark-gray backgrounds those vertices added during the step. The numbers denote the values of the harmonic function added to the basis in the respective step.

The integer-valued harmonic functions $\mathcal{H}_2^\Gamma$ on $\Gamma$ only form a module. While also $\mathcal{H}_2^\Gamma$ possesses a basis $\mathcal{B}_2^\Gamma$ with $|\mathcal{B}_2^\Gamma| = |\partial \Gamma|$ (see below), the selection of appropriate basis functions is non-trivial. In this section, we thus provide an algorithm for the construction of such a basis.

We first define four families of harmonic functions from which the basis functions in $\mathcal{B}_2^\Gamma$ will be drawn (Figure 2A&B). We denote by $d_i^+ = \{(x, y) \in$
Corollary 5

Let \(|\Gamma|\) the reverse is also true, which implies that we can map bases between every two domains \(\Gamma_1, \Gamma_2 \in \mathcal{D}\) with \(\text{diam}(\Gamma_1) = \text{diam}(\Gamma_2)\).
**Definition 6 (Line-segment)** A vertex \( v \in \Gamma \) is a line-segment in \( \Gamma \in D \) if it has exactly two direct neighbors in \( \Gamma \), and if all three vertices lie on the same line. We denote by \( \text{lines}(\Gamma) \subseteq \Gamma \) the set of all line-segments in \( \Gamma \).

With these two definitions, we can finally state our algorithm for the construction of a basis \( B_\Gamma^\Gamma \) for the module \( H_\Gamma^\Gamma \) of integer-valued harmonic functions on a finite convex domain \( \Gamma \in D \) (Figure 2C):

**Input:** A finite convex domain \( \Gamma \in D \).

**Output:** A basis \( B_\Gamma^\Gamma \) for \( H_\Gamma^\Gamma \).

**begin**

Set \( \Gamma_0 := \{\} \), \( B_0 := \{\} \), \( s := 0 \)

**while** \( \Gamma_s \neq \Gamma \) **do**

\[ s := s + 1 \]

Choose \( v_s \in \Gamma \setminus \Gamma_{s-1} \) such that \( \Gamma_{s-1} \cup \{v_s\} \) is convex and \( \text{lines}(\Gamma_{s-1} \cup \{v_s\}) \subseteq \text{lines}(\Gamma) \)

Determine \( i \) and \( j \) such that \( d_i^+ \cap d_j^- = \{v_s\} \)

Choose \( B_s \in \{B_{\geq i}^+, B_{\leq i}^+, B_{\geq j}^-, B_{\leq j}^-\} \) such that \( B_s|_{\Gamma_{s-1}} = 0 \)

Set \( B_s := B_{s-1} \cup \{B_s\} \)

Set \( \Gamma_s = \text{diam}(\Gamma_{s-1} \cup \{v_s\}) \cap \Gamma \)

**end**

**return** \( B_s \)

**end**

**Function** Basis(\( \Gamma \))

**Lemma 7** For every finite convex domain \( \Gamma \in D \), the algorithm terminates and returns a basis \( B_\Gamma^\Gamma \) for the module \( H_\Gamma^\Gamma \).

**Proof.** It is easy to see that, in each step \( s \) of the algorithm, there is at least one vertex \( v_s \) such that \( \Gamma_{s-1} \cup \{v_s\} \) is convex and \( \text{lines}(\Gamma_{s-1} \cup \{v_s\}) \subseteq \text{lines}(\Gamma) \). Since \( \Gamma \) is finite, all we have to show to prove termination is thus that, independent of the choice of \( v_s \), at least one of the harmonic functions \( B_{\geq i}^+, B_{\leq i}^+, B_{\geq j}^-, B_{\leq j}^- \) is zero on \( \Gamma_{s-1} \). For \( s = 1 \), this is trivially true. For \( s > 1 \), \( v_s \) has at least one neighbor in \( \Gamma_{s-1} \). Denote this vertex by \( v_N \), and, w.l.o.g., assume that it is to the bottom of \( v_s \). In \( \Gamma_{s-1} \), \( v_N \) must have at most two neighbors, since otherwise \( v_s \in \text{diam}(\Gamma_{s-1}) \). Furthermore, \( v_N \) must not have both a neighbor to the right and to the left in \( \Gamma_{s-1} \), since this would constitute a line segment in \( \Gamma_{s-1} \) which is not a line-segment in \( \Gamma \). W.l.o.g., assume that the vertex to the right of \( v_N \) is not in \( \Gamma_{s-1} \), and denote this vertex by \( v_R \). Furthermore, let \( d_i^+ \) be the diagonal going through \( v_s \) and \( v_R \). No vertex on this diagonal or to the right of this diagonal can be an element of \( \Gamma_{s-1} \), since otherwise either \( \Gamma_{s-1} \cup \{v_s\} \) would not be convex, or \( v_s \in \Gamma_{s-1} \). Thus, \( B_s = B_{\geq i}^+ \) is a valid choice at step \( s \). To show that \( B_\Gamma^\Gamma \) is a basis for \( H_\Gamma^\Gamma \), note that \( B_1 \) is the only harmonic function in \( B_\Gamma^\Gamma \) which is non-zero at \( v_1 \). By definition, \( B_1 \) takes the value \( \pm 1 \) at \( v_1 \), and thus only linear combination of \( B_\Gamma^\Gamma \) can be integer-valued for which the coefficient
corresponding to \( B_1 \) is integer-valued. Assume that, in step \( s \), this is true for all harmonic functions in \( \mathcal{B}_{s-1} \). Then, since \( B_s \) takes the value \( \pm 1 \) at \( v_s \), this is also true for \( B_s \). This shows that the harmonic functions in \( \mathcal{B}_s \) are linearly independent. Since \(|B_s| = |\partial \Gamma_s| = s, |\mathcal{B}_s| = |\partial \Gamma|\). Thus, \( \mathcal{B}_s \) is a basis for \( \mathcal{H}_s \)

We conclude our proof by noting that \( \mathcal{H}_s \subset \mathcal{H}_s \).

**Corollary 8** Let \( \Gamma \in \mathcal{D} \) be an \( N \times N \) square domain. Assume that \( d_{s_0}^+ \) and \( d_{s_0}^- \) correspond to the main diagonals of the domain. Then, (i) for \( N = 1 \), \( \{B_{s_0}^+\} \) is a basis for \( \mathcal{H}_s \); (ii) for \( N = 2n \), \( n \in \mathbb{N} \), \( \{B_{s_0}^+, B_{s_0}^-, B_{s_0}^+_{\leq i}, B_{s_0}^-_{\leq i}, B_{s_0}^-_{\geq i}, B_{s_0}^+_{\geq i}\}_{i=1,...,N-1} \) is a basis; and (iii) for \( N = 2n + 1 \), \( \{B_{s_0}^+, B_{s_0}^-, B_{s_0}^+_{\leq i}, B_{s_0}^-_{\leq i}, B_{s_0}^+_{\geq i}, B_{s_0}^-_{\geq i}, B_{s_0}^-_{\leq i}, B_{s_0}^+_{\geq i}, B_{s_0}^+_{\leq i}, B_{s_0}^-_{\leq i}\}_{i=2,...,N-1} \) is a basis for \( \mathcal{H}_s \).

**Proof.** See construction in Figure 2C.

Since the potential of a harmonic function only has support on the boundary of the domain, we may, by a slight abuse of notation, interpret the potential \( X_B \) as a column vector over the boundary vertices \( \partial \Gamma \). The potentials of all basis functions in \( \mathcal{B}_s \) then form the matrix \( \Delta \mathcal{B}_s = (\Delta B_1, \ldots, \Delta B_{|\partial \Gamma|}) \in \mathbb{Z}^{|\partial \Gamma|} \times |\partial \Gamma| \), to which we refer as the potential matrix of \( \mathcal{B}_s \).

**Corollary 9** Let \( \mathcal{B}_s = \{B_1, \ldots, B_i, \ldots, B_{|\partial \Gamma|}\} \) be a basis for the module \( \mathcal{H}_s \) of integer-valued harmonic functions on a finite convex domain \( \Gamma \in \mathcal{D} \) with potential matrix \( \Delta \mathcal{B}_s = (\Delta B_1, \ldots, \Delta B_i, \ldots, \Delta B_{|\partial \Gamma|}) \). Then, for every \( z \in \mathbb{Z} \) and every \( j \neq i \), \( \mathcal{B}_s = \{B_1, \ldots, B_i + zB_j, \ldots, B_{|\partial \Gamma|}\} \) is a basis for \( \mathcal{H}_s \), too, and the potential matrix with respect to this basis is given by \( \Delta \mathcal{B}_s = (\Delta B_1, \ldots, \Delta B_i + z\Delta B_j, \ldots, \Delta B_{|\partial \Gamma|}) \).

Note that elementary properties of matrix determinants directly imply that \( |\det(\Delta \mathcal{B}_s)| = |\det(\Delta \mathcal{B}_s)| \). This is not a coincident, as will become clear at the end of next section.

### 2.4 Unique coordinates for configurations and order of the sandpile group

In the previous section, we have constructed a basis \( \mathcal{B}_s \) for every module \( \mathcal{H}_s \) of integer-valued harmonic functions on a finite convex domain \( \Gamma \in \mathcal{D} \). In this section, we use this basis to define unique coordinates for each recurrent configuration, as well as to derive an alternative formula for the order of the sandpile group.

**Theorem 10** Let \( \Gamma \in \mathcal{D} \) be a finite convex domain, and \( \mathcal{B}_s \) be a basis for the module \( \mathcal{H}_s \) of integer-valued harmonic functions on \( \Gamma \). Then, for every recurrent configuration \( C \in \mathcal{G}_\Gamma \) of the sandpile group \( \mathcal{G}_\Gamma \) on \( \Gamma \), there exists unique coordinates \( s = \sigma(C) \in (\mathbb{Q} / \mathbb{Z})^{\partial \Gamma} \) with respect to \( \mathcal{B}_s \) such that

\[
C = \left( I - \sum_{i=1}^{\partial \Gamma} s_i \Delta_i \right) \circ \ .
\]
Proof. Recall that every recurrent configuration can be reached from the identity by only adding particles to boundary vertices of $\Gamma$. Thus, for every $C \in G^\Gamma$, there exists a $X_C \in \mathbb{Z}^\Gamma$ with $X_C|_{\Gamma_0} = 0$ and $(I + X_C)^\circ = C$. Since the basis functions in $B^\Gamma_{\mathbb{Z}}$ are linearly independent, $\Delta B^\Gamma_{\mathbb{Z}}$ has full rank, and thus there exists at least one (rational-valued) solution $s = -(\Delta B^\Gamma_{\mathbb{Z}})^{-1} X_C$ for $[3]$. Let $\tilde{s} = s + z$ with $z \in \mathbb{Z}_{\partial \Gamma}$. Then, $\tilde{C} = (I - \sum_{i=1}^{[\partial \Gamma]} (s_i + z_i) \Delta^I B_i)^\circ = ((I - \sum_{i=1}^{[\partial \Gamma]} z_i \Delta^I B_i) - \sum_{i=1}^{[\partial \Gamma]} s_i \Delta^I B_i)^\circ = (I - \sum_{i=1}^{[\partial \Gamma]} s_i \Delta^I B_i)^\circ = C$; justifying to reinterpret $s$ to lie in $(\mathbb{Q}/\mathbb{Z})^{[\partial \Gamma]}$. Finally, to show uniqueness of $s$ in $(\mathbb{Q}/\mathbb{Z})^{[\partial \Gamma]}$, it is sufficient to show uniqueness for the coordinates $s = 0$ of the identity, since every other configuration can be brought to the origin by an appropriate coordinate transformation. Assume the opposite, i.e. that the identity is also described by some coordinates $\tilde{s} \in (\mathbb{Q}/\mathbb{Z})^{[\partial \Gamma]}$ with at least one $\tilde{s}_i \neq 0$. With $C = I$ in (3), $\sum_{i=1}^{[\partial \Gamma]} s_i \Delta I B_i$ must be the potential of some integer-valued harmonic function $\tilde{H}$ $= \sum_i \tilde{s}_i H_i \in \mathcal{H}^\Gamma_{\mathbb{Z}}$. However, since $B^\Gamma_{\mathbb{Z}}$ is a basis for $\mathcal{H}^\Gamma_{\mathbb{Z}}$, there also exists a $z \in \mathbb{Z}_{\partial \Gamma}$ such that $\tilde{H} = \sum_i \tilde{s}_i H_i$. Due to the linear independence of the harmonic functions in $B^\Gamma_{\mathbb{Z}}$, we get that $\tilde{s} = z \mod 1 = 0$, which is a contradiction.

If we denote by $s_a, s_b \in (\mathbb{Q}/\mathbb{Z})^{[\partial \Gamma]}$ the unique coordinates of two recurrent configurations $C_a, C_b \in G^\Gamma$, it directly follows from (4) that $s_a + s_b$ are the unique coordinates of the recurrent configuration $(C_a + C_b)^\circ$. Thus, Theorem 10 implies that $\sigma^\Gamma : G^\Gamma \to (\mathbb{Q}/\mathbb{Z})^{[\partial \Gamma]}$ is a group monomorphism, which corresponds to the exact sequence

$$0 \longrightarrow G^\Gamma \overset{\sigma^\Gamma}{\longrightarrow} (\mathbb{Q}/\mathbb{Z})^{[\partial \Gamma]}.$$  \hspace{1cm} (4)

This monomorphism cannot be surjective since the sandpile group on $\Gamma \in \mathcal{D}$ is finite but $(\mathbb{Q}/\mathbb{Z})^{[\partial \Gamma]}$ is not. However, every coordinate $s \in (\mathbb{Q}/\mathbb{Z})^{[\partial \Gamma]}$ also directly defines a harmonic function $\phi^\Gamma(s) = \sum_{i=1}^{[\partial \Gamma]} s_i B_i \in \mathcal{H}^\Gamma_{\mathbb{Q}}/\mathcal{H}^\Gamma_{\mathbb{Z}}$ (and vice versa), which corresponds to the exact sequence

$$0 \longrightarrow (\mathbb{Q}/\mathbb{Z})^{[\partial \Gamma]} \overset{\phi^\Gamma}{\longrightarrow} \mathcal{H}^\Gamma_{\mathbb{Q}}/\mathcal{H}^\Gamma_{\mathbb{Z}} \longrightarrow 0.$$  \hspace{1cm} (5)

From (3), it follows that the image of $\sigma^\Gamma$ contains exactly those coordinates $s = (\mathbb{Q}/\mathbb{Z})^{[\partial \Gamma]}$ for which $\Delta^\Gamma \phi^\Gamma(s) \in \mathbb{Z}^\Gamma$. This implies that, for a finite convex domain $\Gamma \in \mathcal{D}$, $\phi^\Gamma \circ \sigma^\Gamma$ is an isomorphism between the sandpile group $G^\Gamma$ and the subgroup of $\mathcal{H}^\Gamma_{\mathbb{Q}}/\mathcal{H}^\Gamma_{\mathbb{Z}}$ containing those harmonic functions whose potential $\Delta^\Gamma H$ is integer-valued. This isomorphism corresponds to the following exact sequence

$$0 \longrightarrow G^\Gamma \overset{\phi^\Gamma \circ \sigma^\Gamma}{\longrightarrow} \mathcal{H}^\Gamma_{\mathbb{Q}}/\mathcal{H}^\Gamma_{\mathbb{Z}} \overset{\Delta^\Gamma(\cdot)|_{[\partial \Gamma]}}{\longrightarrow} (\mathbb{Q}/\mathbb{Z})^{[\partial \Gamma]} \longrightarrow 0.$$  \hspace{1cm} (5)

We note that this isomorphism does not depend on the choice of the basis $B^\Gamma_{\mathbb{Z}}$. It can also be directly derived by combining the facts that every recurrent configuration $C \in G^\Gamma$ can be reached from the identity $I$ by only adding particles to the
boundary of the domain [3]; that, by the existence and uniqueness of solutions of the Dirichlet problem, for every such \(X_C \in \mathbb{Z}^\Gamma\), \(X_C|_{\partial \Gamma} = 0\), \((I + X_C)^o = C\), there exists a unique harmonic function \(H \in \mathcal{H}_Q^\Gamma\) with potential \(-\Delta^\Gamma H = X_C\); and that, by \(G^\Gamma \cong \mathbb{Z}^\Gamma / \Delta^\Gamma \mathbb{Z}^\Gamma\) [4], every two harmonic functions \(H_1, H_2 \in \mathcal{H}_Q^\Gamma\), \(\Delta^\Gamma H_1, \Delta^\Gamma H_2 \in \mathbb{Z}^\Gamma\), which only differ by an integer-valued harmonic function correspond to the same recurrent configuration.

Due to this alternative derivation, the map from \(\{H \in \mathcal{H}_Q^\Gamma / \mathcal{H}_Z^\Gamma \mid \Delta^\Gamma H \in \mathbb{Z}^\Gamma\}\) to \(G^\Gamma\) of this isomorphism is given by \(C = (I - \Delta^\Gamma H)^o\). To construct \(\sigma^\Gamma\) for the inverse map \(\phi^\Gamma \circ \sigma^\Gamma\) [3], let \(X_C \in \mathbb{Z}^\Gamma\), \(X_C|_{\partial \Gamma} = 0\), encode (any possible) set of particle additions to the boundary such that \((I + X_C)^o = C\) (see [3] for the construction of \(X_C\)). The map \(\sigma^\Gamma\) is then given by

\[
\sigma^\Gamma(C) = - (\Delta B_2^\Gamma)^{-1} X_C \pmod{1}. \tag{6}
\]

We conclude this section by stating an alternative formula for the order of the sandpile group, which is based on the isomorphism (5):

**Theorem 11** Let \(\Gamma \in \mathcal{D}\) be a finite convex domain, and \(B_2^\Gamma\) a basis for the module of integer-valued harmonic functions \(\mathcal{H}_Z^\Gamma\) on \(\Gamma\). Then, the order of the sandpile group \(G^\Gamma\) on \(\Gamma\) is given by

\[
|G^\Gamma| = |\det(\Delta B_2^\Gamma)|.
\]

**Proof.** Define \(f : (\mathbb{R}/\mathbb{Z})^{\partial \Gamma} \to G^\Gamma\), \(f(s) = (I - \left[\sum_{i=1}^{\partial \Gamma} s_i \Delta B_i\right])^o\), which extends (3) to all \(s \in (\mathbb{R}/\mathbb{Z})^{\partial \Gamma}\). From the properties of the floor function, it follows that the pre-image \(f^{-1}(C)\) of a recurrent configuration \(C \in G^\Gamma\) under this map is connected. Denote by \(\text{vol}(f^{-1}(C))\) the volume of this pre-image, with \(\text{vol}((\mathbb{R}/\mathbb{Z})^{\partial \Gamma}) = 1\). Since, we can always make a coordinate transformations \(s \mapsto \tilde{s}\) such that a given configuration \(C\) has coordinates \(\tilde{s} = 0\), we get that \(\text{vol}(f^{-1}(C)) = \text{vol}(f^{-1}(I)) = \frac{1}{|\Delta B_2^\Gamma|}\) for all \(C \in G^\Gamma\). It is easy to see that \(f^{-1}(I)\) forms a \(|\partial \Gamma|\)-parallelotope with edges \(g_i\) given by \((\Delta B_2^\Gamma)g_i = e_i\), with \((e_i)_j = \delta_{ij}\) the \(i^{th}\) unit vector and \(\delta_{ij}\) the Kronecker delta. The volume of this parallelotope is \(\text{vol}(f^{-1}(I)) = |\det(\Delta B_2^\Gamma)^{-1}| = |G^\Gamma|^{-1}\). \(\blacksquare\)

**Corollary 12** Let \(\Gamma \in \mathcal{D}\) be a finite convex domain. Then,

\[
|G(\Gamma)| = |\det(D^\Gamma)| = |\det(\Delta B_2^\Gamma)|,
\]

with \(D^\Gamma\) the reduced graph Laplacian of \(\Gamma\).

**Proof.** See [3] for \(|G^\Gamma| = |\det(D^\Gamma)|\). \(\blacksquare\)

This corollary is mainly stated to make the reader aware that this is not the first formula for the order of the sandpile group.

### 2.5 Extensions of harmonic functions

In the last section, we have shown that the sandpile group \(G^\Gamma\) on a finite convex domain \(\Gamma \in \mathcal{D}\) is isomorphic to the space of harmonic functions \(H \in \mathcal{H}_Q^\Gamma / \mathcal{H}_Z^\Gamma\).
whose potential $\chi_H^{\Gamma_1} = -\Delta^{\Gamma_1} H$ is integer-valued. Furthermore, we have shown that each of these harmonic functions can be uniquely identified by its coordinates $s \in (\mathbb{Q}/\mathbb{Z})^{\partial_1}$. In this section, we introduce a category of harmonic functions, and formulate a problem statement asking for which subcategories $\mathcal{D}_{\text{sub}}$ of domains $\mathcal{D}$ the map from $\mathcal{D}_{\text{sub}}$ to this category of harmonic functions becomes functorial. As we will see, such functors directly correspond to families of monomorphisms between sandpile groups, and we will use this formalism in the next section to construct such sandpile monomorphisms for rectangular domains.

We begin our analysis by extending a harmonic function $H = \phi^F(\sigma^F(C)) \in \mathcal{H}_Q^F/\mathcal{H}_Z^F$ corresponding to a recurrent configuration $C \in G^F$ (5) from $\Gamma$ to the extended domain $\Gamma_{\text{ext}} = \Gamma \cup \partial(\mathbb{Z}^2 \setminus \Gamma)$, which is the result of augmenting $\Gamma$ at the boundary by one vertex in each direction. We assume that the bases $\mathcal{B}_Z^G$ and $\mathcal{B}_Z^{\Gamma_{\text{ext}}}$ for the modules of integer-valued harmonic functions on $\Gamma$ and $\Gamma_{\text{ext}}$ are chosen such that $\mathcal{B}_Z^G \subset \mathcal{B}_Z^{\Gamma_{\text{ext}}}$ and $B \in \mathcal{B}_Z^{\Gamma_{\text{ext}}} \setminus \mathcal{B}_Z^G \Rightarrow B|_\Gamma = 0$, which is always possible (see proof of Lemma 7). When $s \in (\mathbb{Q}/\mathbb{Z})^{\partial_1}$ and $s_{\text{ext}} \in (\mathbb{Q}/\mathbb{Z})^{\partial_1_{\text{ext}}}$ denote the coordinates of the harmonic function $H = \phi^F(s)$ and of its extension $H_{\text{ext}} = \phi^{F_{\text{ext}}}(s_{\text{ext}})$, respectively, it then directly follows that $s_{\text{ext},i} = s_i$ for all coordinates corresponding to basis functions in $\mathcal{B}_Z^{\Gamma_{\text{ext}}} \cap \mathcal{B}_Z^G$. Note that, without further requirements, the extension of $H$ to $\Gamma_{\text{ext}}$ is not unique, and that thus also the extended coordinates corresponding to basis functions in $\mathcal{B}_Z^{\Gamma_{\text{ext}}} \setminus \mathcal{B}_Z^G$ are not yet determined.

To make the extension of $H$ to $\Gamma_{\text{ext}}$ unique, recall that, by Corollary 5 all possible extension $H_{\text{ext}}$ of a given harmonic function $H$ from $\Gamma$ to $\Gamma_{\text{ext}}$ have the same values on $\Gamma_{\text{ext}} \cap \text{diam}(\Gamma)$. Furthermore, since $\Delta^{\Gamma} H$ is integer-valued, all values of $H_{\text{ext}}$ on $\partial \Gamma_{\text{ext}} \cap \text{diam}(\Gamma)$ are necessarily integer-valued, too. With this in mind, the discussion below (5), the following lemma becomes trivial:

**Lemma 13** Let $\Gamma \in \mathcal{D}$ be a finite convex domain, and $\Gamma_{\text{ext}}$ its extension as described above. Denote by $H = \phi^F(\sigma^F(C)) \in \mathcal{H}_Q^F/\mathcal{H}_Z^F$, $\Delta^{\Gamma} H \in \mathbb{Z}$, the harmonic function corresponding to a given recurrent configuration $C \in G^F$. Then, there exists a unique harmonic function $H_{\text{ext}} \in \mathcal{H}_Q^{\Gamma_{\text{ext}}}/\mathcal{H}_Z^{\Gamma_{\text{ext}}}$ such that $H_{\text{ext}}|_\Gamma = H$ and $H_{\text{ext}}|_{\partial \Gamma_{\text{ext}}} \in \mathbb{Z}$.

We denote by $\sigma^{\Gamma_{\text{ext}}}: G^F \to (\mathbb{Q}/\mathbb{Z})^{\partial_1_{\text{ext}}}$ the function mapping a given recurrent configuration to its respective extended coordinates defined such that, for every $C \in G^F$, $H_{\text{ext}} = \phi^{\Gamma_{\text{ext}}}(\sigma^{\Gamma_{\text{ext}}}(C))$ is in the group

$$\Pi^\Gamma = \{ H \in \mathcal{H}_Q^{\Gamma_{\text{ext}}}/\mathcal{H}_Z^{\Gamma_{\text{ext}}} | H|_{\partial \Gamma_{\text{ext}}} \in \mathbb{Z} \}$$

of rational-valued harmonic functions on $\Gamma_{\text{ext}}$ with integer-values on the boundary $\partial \Gamma_{\text{ext}}$, modulo the integer-valued harmonic functions on the same domain. Because $\sigma^{\Gamma_{\text{ext}}}(C_1 + C_2) = \sigma^{\Gamma_{\text{ext}}}(C_1) + \sigma^{\Gamma_{\text{ext}}}(C_2)$, the map $\sigma^{\Gamma_{\text{ext}}}$ is a group monomorphism, and, by (5), $\phi^{\Gamma_{\text{ext}}} \circ \sigma^{\Gamma_{\text{ext}}}$ is a group isomorphism. The latter corresponds to the exact sequence

$$0 \longrightarrow G^F \xrightarrow{\phi^{\Gamma_{\text{ext}}} \circ \sigma^{\Gamma_{\text{ext}}}} \Pi^\Gamma \longrightarrow 0,$$  \hspace{1cm} (7)
Remark 14 The extended coordinates of a given recurrent configuration $C \in G^t$ can be written as the product $s_{\text{ext}} = \frac{s_{\text{num}}}{s_{\text{denom}}}$. For the time $0 \leq t = \frac{t_{\text{num}}}{t_{\text{denom}}} = \gcd(s_{\text{ext}}) < 1$ and an integer-valued set of coordinates $0 \leq (s_{\text{ext}})_i < t_{\text{denom}}$. The latter defines an integer-valued harmonic function $\phi_{t_{\text{ext}}}(s_{\text{ext}}) \in H_{t_{\text{ext}}}$ whose values at the boundary $\partial \Gamma_{t_{\text{ext}}}$ are divisible by $t_{\text{denom}}$ (compare Section 2.2). In the harmonic dynamics induced by $\phi_{t_{\text{ext}}}(s_{\text{ext}})$, configurations exactly appear at multiples of $t = \frac{1}{t_{\text{denom}}}$. Thus, $s_{\text{ext}}$ may be thought of as the coordinates of the cyclic subgroup of group homeomorphisms extending the harmonic functions, i.e. group homeomorphisms $\Gamma_{t_{\text{denom}}}$. The latter defines an integer-valued harmonic function $\phi_{t_{\text{ext}}}(s_{\text{ext}})$ in this subgroup.

In the following, we denote by $\mathcal{P}$ the category with objects $\text{obj}(\mathcal{P}) = \{\Pi^i \}_{i \in \mathcal{D}}$. The morphisms $\pi^{i_1 \rightarrow i_2} : \Pi^{i_1} \rightarrow \Pi^{i_2}$, $\Gamma_1 \subseteq \Gamma_2$ in this category correspond to extensions of harmonic functions, i.e. group homeomorphisms satisfying $\pi^{i_1 \rightarrow i_2}(H)|_{\Gamma_{ext,1}} = H$. Note that the latter requirement implies that all morphisms in $\mathcal{P}$ are monomorphisms.

Problem Statement 15 For which full subcategories $\mathcal{D}^{\text{sub}}$ of $\mathcal{D}$, the map $\Gamma \mapsto \Pi^\Gamma$ becomes functorial, i.e. the following commutative diagram exists for every sequence of domains $\Gamma_i \in \mathcal{D}^{\text{sub}}$:

\[
\begin{array}{ccccccc}
\cdots & \cdots & \Gamma_i & \cdots & \cdots & \cdots & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Pi^{\Gamma_i} & \pi^{\Gamma_i \rightarrow \Gamma_{i+1}} & \Pi^{\Gamma_{i+1}} & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

By $[\mathcal{I}]$, the maps $\pi^{\Gamma_i \rightarrow \Gamma_{i+1}}$ in Problem Statement 15 directly correspond to monomorphisms $G^{\Gamma_i} \rightarrow G^{\Gamma_{i+1}}$ between the respective sandpile groups. However, not every map $\pi^{\Gamma_i \rightarrow \Gamma_{i+1}}$ corresponding to a monomorphism $G^{\Gamma_i} \rightarrow G^{\Gamma_{i+1}}$ satisfies $\pi^{\Gamma_i \rightarrow \Gamma_{i+1}}(H)|_{\Gamma_{ext,1}} = H$ and is thus part of the category $\mathcal{P}$. For example, the endomorphism corresponding to the rotation of recurrent configurations on an $N \times N$ domain by $90^\circ$ is not in $\mathcal{P}$.

2.6 A family of sandpile monomorphisms

In the following, we answer Problem Statement 15 for the full subcategories $\mathcal{D}^{N_1,M_1,\psi}$ of $\mathcal{D}$ containing all $N_k \times M_k$ domains of size $N_k = \psi^{k-1}(N_1 + 1) - 1$ and $M_k = \psi^{k-1}(M_1 + 1) - 1$, with $N_1, M_1, \psi, k \in \mathbb{N}$. For example, $\mathcal{D}^{1,1,2}$ contains all $N \times N$ domains with $N = 1, 3, 7, 15, 31, \ldots$, $\mathcal{D}^{2,2,3}$ contains all $N \times N$ domains with $N = 2, 8, 26, 80, 242, \ldots$, and $\mathcal{D}^{2,3,3}$ contains all $N \times M$ domains with $N = 2, 14, 74, 374, 1874, \ldots$ and $M = 3, 19, 99, 499, 2499, \ldots$ (compare Figure 1A). For these subcategories of domains, we explicitly construct the homomorphisms $\pi^{\Gamma_i \rightarrow \Gamma_{i+1}}$ and thus prove the following theorem:

Theorem 16 For every subcategory $\mathcal{D}^{N_1,M_1,\psi}$ of $\mathcal{D}$, the map $\Gamma \mapsto \Pi^\Gamma$ becomes functorial, i.e. for every $\Gamma_1, \Gamma_2 \in \mathcal{D}^{N_1,M_1,\psi}$, $\Gamma_1 \subseteq \Gamma_2$, there exists a homomorphism $\pi^{\Gamma_1 \rightarrow \Gamma_{2}} : \Pi^{\Gamma_1} \rightarrow \Pi^{\Gamma_{2}}$ satisfying $\pi^{\Gamma_1 \rightarrow \Gamma_{2}}(H)|_{\Gamma_{ext,1}} = H$. 
Figure 3: Monomorphisms from the sandpile group on a $2 \times 2$ domain to the respective groups on $8 \times 8$ and $14 \times 14$ domains. For a given configuration $C \in G^{2 \times 2}$ on the $2 \times 2$ domain, we first determine its extended coordinates $s_{ext} = (s_{ext}^+, s_{ext}^-) = \sigma^{2 \times 2}(C)$, and then its corresponding harmonic function $\phi^{2 \times 2}(s_{ext}) \in \Pi^{2 \times 2}$ (red vertices). We then use the morphisms $\pi^{2 \times 2 \rightarrow 8 \times 8}$ and $\pi^{2 \times 2 \rightarrow 14 \times 14}$ to extend the harmonic function to the domains $\Gamma_{8 \times 8}$ (green) and $\Gamma_{14 \times 14}$ (blue). Finally, we determine the recurrent configurations on $\Gamma_{8 \times 8}$ (green) and $\Gamma_{14 \times 14}$ onto which C is mapped via $(I - \Delta^{8 \times 8} H)^o$ and $(I - \Delta^{14 \times 14} H)^o$, where $H$ denotes the respective extended harmonic function.

We note that, by [5], these morphisms $\pi^{\Gamma_1 \rightarrow \Gamma_2}$ directly define families of monomorphisms from each sandpile group on a given $N \times M$ domain to all sandpile groups.
on \( \psi(N+1) - 1 \times \psi(M+1) - 1 \) domains, with \( \psi \in \mathbb{N} \) (Figure 3).

For the subcategories \( \mathcal{D}^{2n,2n,2t+1} \), \( n, t \in \mathbb{N} \), with odd scaling factors \( \psi = 2t + 1 \) and only even sized domains, these monomorphisms take an especially simple form. Assume that \( \mathcal{B}_{\mathcal{G}}^{ext} \), \( \Gamma \in \mathcal{D}^{2n,2n,2t+1} \) denotes the basis from from Corollary 8 with basis functions defined as described in Remark 2 and shown in Figure 2A&B. We then represent the extended coordinates \( s_{ext} \in (\mathbb{Q}/\mathbb{Z})^{\partial \Gamma, ext} \) of a recurrent configuration in \( \Gamma^t \) as \( s_{ext} = \left( s_{ext}^+, s_{ext}^- \right) \), where

\[
s_{ext}^+ = \left( s_{ext,-N}^+, \ldots, s_{ext,-1}^+, 0, s_{ext,1}^+, \ldots, s_{ext,N}^+ \right)
\]
\[
s_{ext}^- = \left( s_{ext,-N}^-, \ldots, s_{ext,-1}^-, 0, s_{ext,1}^-, \ldots, s_{ext,N}^- \right)
\]

denote the coordinates corresponding to the basis functions \( \{B_{\leq-N}^+, \ldots, B_{\leq-1}^+, 0, B_{\geq 1}^+, \ldots, B_{\leq N}^+ \} \), respectively \( \{B_{\leq-N}^-, \ldots, B_{\leq-1}^-, 0, B_{\geq 1}^-, \ldots, B_{\leq N}^- \} \). The additional coordinate with value zero in the middle of \( s_{ext}^+ \) and \( s_{ext}^- \) is added solely to simplify the following definitions. We then define the conjugates \( (s_{ext}^+)^* \) and \( (s_{ext}^-)^* \) by \( s^* = \left( (-1)^{k+1} s_k \right)_{k=-N \ldots N} \), and the scaling \( \Psi_{2t+1} : (\mathbb{Q}/\mathbb{Z})^{2(N+1)} \rightarrow (\mathbb{Q}/\mathbb{Z})^{2(2t+1)(N+1)-1} \)

\[
\Psi_{2t+1}(s) = \left( \ldots, s, 0, s^*, 0, s, 0, s^*, 0, s, \ldots \right).
\]

For example, for \( t = 1 \), we get \( \Psi_3(s) = (s^*, 0, s, 0, s^*) \).

**Corollary 17** Let \( \Gamma_1, \Gamma_2 \in \mathcal{D} \), denote two \( N \times N \) domains with sizes \( N = 2n \) and \( N = (2t+1)(2n+1) - 1 \), respectively, with \( n, t \in \mathbb{N} \). Assume that the basis functions in \( \mathcal{B}_{\mathcal{G}}^{ext,1} \) and \( \mathcal{B}_{\mathcal{G}}^{ext,2} \) are defined as described above, and that \( s_{ext} = (s_{ext}^+, s_{ext}^-) \) are the extended coordinates corresponding to a harmonic function \( H_1 = \partial \Gamma, ext \cdot (s_{ext}) \in \Gamma^{\Pi_1} \). Then, the harmonic function \( H_2 = \pi^{\partial \Gamma_1, ext} (H_1) \in \Pi^{\Gamma_2} \) has extended coordinates \( (\Psi_{2t+1}(s_{ext}^+), \Psi_{2t+1}(s_{ext}^-)) \).

In the following proof of Theorem 16 we exemplary describe in detail the construction of the monomorphisms \( \pi^{\partial \Gamma_1, ext} \) for a given subcategory \( \mathcal{D}^{N_1, M_1, 3} \) with scaling factor \( \psi = 3 \). As we shortly describe at the end of the proof, the construction of the corresponding monomorphisms for subcategories with other scaling factors is accordingly.

**Proof.** Let \( \Gamma_1, \Gamma_2 \in \mathcal{D}^{N_1, M_1, 3} \) be an \( N \times M \), respectively a \( 3(N+1) - 1 \times 3(M+1) - 1 \) domain, and let the harmonic function \( H_1 \in \Pi^{\Gamma_1} \) be given by

\[
H_1 = \begin{bmatrix}
h_{0,0} & \cdots & h_{0,M} \\
h_{1,0} & \cdots & h_{1,M} & h_{1,M+1} \\
\vdots & \ddots & \vdots & \vdots \\
h_{N,0} & \cdots & h_{N,M} & h_{N,M+1} \\
h_{N+1,0} & \cdots & h_{N+1,M} & h_{N+1,M+1}
\end{bmatrix}
\]

with vertices in \( \partial \Gamma_{1, ext} \) highlighted by a gray background. Note that \( H_1 \), by definition, takes integer-values at these boundary vertices.
Now, consider the following function defined on $\Gamma_{2,\mathrm{ext}}$:

$$
\hat{H}_2 = \begin{bmatrix}
  h_{N,M+1} & h_{N+1,1} & \cdots & h_{N+1,1} & 0 & \cdots & h_{N+1,M} & 0 & \cdots & h_{N+1,1} \\
  h_{N,M} & h_{N+1,1} & \cdots & h_{N+1,1} & 0 & \cdots & h_{N+1,M} & 0 & \cdots & h_{N+1,1} \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  h_{1,M+1} & h_{1,1} & \cdots & h_{1,1} & 0 & \cdots & h_{1,M} & 0 & \cdots & h_{1,1} \\
  0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  -h_{1,M+1} & -h_{1,1} & \cdots & -h_{1,1} & 0 & \cdots & -h_{1,M} & 0 & \cdots & -h_{1,1} \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  -h_{N+1,N} & -h_{N,1} & \cdots & -h_{N,1} & 0 & \cdots & -h_{N,M} & 0 & \cdots & -h_{N,1} \\
  0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
  h_{N,M+1} & h_{N,1} & \cdots & h_{N+1,1} & 0 & \cdots & -h_{N+1,M} & 0 & \cdots & h_{N+1,1} \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  h_{1,M+1} & h_{1,1} & \cdots & h_{1,1} & 0 & \cdots & h_{1,M} & 0 & \cdots & h_{1,1} \\
  h_{0,M} & h_{0,1} & \cdots & h_{0,1} & 0 & \cdots & h_{0,M} & 0 & \cdots & h_{0,1} \\
\end{bmatrix}
$$

This function is harmonic everywhere except on the vertices directly next to one of the four “internal boundaries” at columns $M+2$ and $2M+3$, and rows $N+2$ and $2N+3$, which are also highlighted by a gray background.

We can “cure” the vertices to the left of the internal boundary at column $2M+3$ by adding the matrix

$$
\hat{C}^v_{2M+2} = \begin{bmatrix}
  0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
  0 & 0 & \cdots & 0 & -h_{N,M+1} & -4h_{N,1} - h_{N-1,M+1} & \cdots & X X \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 0 & -h_{1,1} & -4h_{1,1} + h_{2,M+1} & \cdots & X X \\
  0 & 0 & \cdots & 0 & 0 & 0 & \cdots & X X \\
  0 & 0 & \cdots & 0 & h_{1,1} + 4h_{1,1} - h_{2,M+1} & \cdots & X X \\
  0 & 0 & \cdots & 0 & 0 & 0 & \cdots & X X \\
  0 & 0 & \cdots & 0 & -h_{N,1} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
  0 & 0 & \cdots & 0 & -h_{1,1} & \cdots & 0 \\
  0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\end{bmatrix}
$$

This integer-valued matrix is harmonic everywhere except on column $2M+2$, and defined as follows: $\hat{C}^v_{2M+2}$ is zero for all vertices left of column $2M+3$, as well as for all vertices in the first and the last row. On column $2M+3$, it takes the values of the last column of $\hat{H}_1$ which correspond to the respective elements in column $2M+2$ directly next to them. Finally, the values of $\hat{C}^v_{2M+2}$ in columns $j > 2M+3$ are defined recursively by the formula $(\hat{C}^v_{2M+2})_{i,j} = 4(\hat{C}^h_{2M+2})_{i,j-1} - (\hat{C}^v_{2M+2})_{i+1,j-1} - (\hat{C}^v_{2M+2})_{i-1,j-1} - (\hat{C}^v_{2M+2})_{i,j-2}$.

The vertices on the right side (column $2M+4$) of the internal boundary at column $2M+3$ can be cured by adding the matrix $\hat{C}^h_{2M+4}$, which is obtained by shifting $\hat{C}^v_{2M+2}$ by two vertices to the right (padding zeros at the left), and multiplying the result by $-1$. In this way, it is possible to cure all vertices of $\hat{H}_2$. With $\hat{C}^v_{N+1}$, $\hat{C}^v_{N+3}$, $\hat{C}^h_{N+1}$, $\hat{C}^h_{N+3}$, $\hat{C}^h_{2N+2}$, and $\hat{C}^h_{2N+4}$ the corresponding matrices defined to cure the vertices next to the remaining three internal boundaries at column $N+2$, row $N+2$ and row $2N+2$, we obtain the function

$$
H_2 = \hat{H}_2 + \hat{C}^v_{N+1} + \hat{C}^v_{N+3} + \hat{C}^v_{2N+2} + \hat{C}^v_{2N+4} + \hat{C}^h_{N+1} + \hat{C}^h_{N+3} + \hat{C}^h_{2N+2} + \hat{C}^h_{2N+4}
$$
which is harmonic everywhere. Since all matrices which we added to \( \hat{H}_2 \) were integer-valued, it follows that \( H_2 \in \Gamma^2 \). If we chose the “direction” of each of these matrices such that they are zero in the central \( N \times N \) square of the domain, it follows that \( H_2|_{\Gamma_{ext}} = H_1 \). It is then easy to see that \( \pi_{\Gamma_1 \rightarrow \Gamma_2}(H_1) = H_2 \) is a group monomorphism from \( \Pi_{\Gamma_1} \) to \( \Pi_{\Gamma_2} \). The other morphisms in the category \( \mathcal{D}^{N_1,M_1,3} \) can be obtained by appropriate function compositions \( \pi_{\Gamma_A \rightarrow \Gamma_B} = \pi_{\Gamma_A \rightarrow \Gamma_B} \circ \ldots \circ \pi_{\Gamma_A \rightarrow \Gamma_1} \). The morphisms \( \pi_{\Gamma_1 \rightarrow \Gamma_2} \) for all other subcategories \( \mathcal{D}^{N_1,M_1,\psi} \) with \( \psi \neq 3 \) can be constructed in a similar way, or by restricting the result of appropriate morphisms \( \pi_{\Gamma_1 \rightarrow \Gamma_3} \), \( \Gamma_3 \in \mathcal{D}^{N,3} \), \( \Gamma_2 \subset \Gamma_3 \), to \( \Gamma_2 \).

2.7 More on cyclic subgroups

In the previous section, we have constructed families of monomorphisms from each sandpile group on an \( N \times M \) domain to all sandpile groups on \( \psi(N+1) - 1 \times \psi(M+1) - 1 \) domains, which explains our initial observation that the orders of the latter groups are divisible by the order of the former (Figure 1). In this section, we discuss our two other initial observations (Section 2.2), i.e. that the factor 2 appears with high multiplicities in the factorization of the order of all sandpile groups on \( N \times N \) domains, and that most factors appear with even multiplicities (Supplementary Table S5).

Recall that, in the definition of a basis \( B_{\psi}^{\Gamma_{ext}} \) for the module of integer-valued harmonic functions on the extension \( \Gamma_{ext} \) of a given domain \( \Gamma \in \mathcal{D} \), we only required that the harmonic functions \( B_{\psi}^{\Gamma_{ext}} \) are zero below their defining diagonal \( d_i^+ \), and that they take the values \( \pm 1 \) on \( d_i^+ \) (Section 2.3). On every diagonal \( d_i^+ \), \( j > i \), we thus still have the possibility to freely choose the value of \( B_{\psi}^{\Gamma_{ext}} \) for one vertex, which then determines the value of all other vertices on the same diagonal [10]. If we consistently choose these “free values” to be zero (for an arbitrary vertex on the respective diagonal), it is easy to see that the harmonic function \( B_{\psi}^{\Gamma_{ext}} \) becomes divisible by four on all vertices except for the ones on the diagonal \( d_i^+ \) (Figure 4A). This specifically holds for the boundary, i.e. all vertices of \( B_{\psi}^{\Gamma_{ext}} \) on \( \partial \Gamma_{ext} \) are divisible by four, except for two vertices for which \( B_{\psi}^{\Gamma_{ext}} \) takes the values \( \pm 1 \) (Figure 4A). Furthermore, for each vertex \( v \in \partial \Gamma_{ext} \) of the boundary, there exist exactly two harmonic basis functions in \( B_{\psi}^{\Gamma_{ext}} \) which take the values \( \pm 1 \). Four each \( i = 1, \ldots, N \), we can then define the harmonic function \( H_i^{\psi} = B_{\psi}^{\Gamma_{ext}} \pm B_{\psi}^{\Gamma_{ext}} \pm B_{\psi}^{\Gamma_{ext}} \pm B_{\psi}^{\Gamma_{ext}} \), where the signs are chosen such that the values of the basis functions on the boundary which are \( \pm 1 \) cancel each other out (Figure 4A). With this construction, all values of \( H_i^{\psi} \) on \( \partial \Gamma_{ext} \) and thus also all values of \( \Delta^F H_i^{\psi} \) are divisible by four.

Lemma 18 Let \( G^\psi \) be the sandpile group on an \( N \times N \) domain \( \Gamma \). Then, \( G^\psi \) has a subgroup \( S^{4N} \leq G^{N \times N} \) isomorphic to the direct sum \( S^{4N} \cong \bigoplus_{i=1}^{N}(\mathbb{Z}/4\mathbb{Z}) \) of \( N \) cyclic groups of order four.

Proof. For \( N = 1 \), \( S^{4N} = G^\psi \cong \mathbb{Z}/4\mathbb{Z} \). For \( N = 2n, n \in \mathbb{N} \), by Corollaries 9 and 8 we can form a basis containing all harmonic functions \( H_i^{\psi} \) with \( i = 1, \ldots, N \). Since \( 4|\Delta^F H_i^{\psi} \), the lemma directly follows. For \( N = 2n + 1 \), we can only form
in this way a basis containing \( N - 1 \) harmonic functions \( H^\pi \). However, we can choose this basis to additionally contain \( B^+_{2,0} \) (Corollary 8), and \( 4|\Delta \Gamma B^+_{2,0} \cdot \)

For our next lemma, consider an \( N \times N \) domain \( \Gamma \in \mathcal{D} \) with \( N = 2n, n \in \mathbb{N} \), and let \( B^+_2 \) denote the basis for \( H^\pi \) from Corollary 8 with symmetric and anti-symmetric basis functions as described in Remark 2 and shown in Figure 2A&B. Then, define the harmonic function (Figure 4B)

\[
H^\pi = \sum_{i=0}^{\infty} (-1)^i (B^+_{2i+1} + B^+_{-2i+1} - B^-_{2i+1} - B^-_{-2i+1}).
\]

On the boundary \( \partial \Gamma_{ext} \) of the extension of an \( N \times N \) domain \( \Gamma \) with \( N = 2, 4 \) and 6, the values of \( H^\pi \) are divisible by \( N + 1 = 3, 5 \) and 7, respectively (blue, green and red vertices in Figure 4B). Since the values of \( H^\pi \) are coprime, this implies that the corresponding sandpile groups on \( \Gamma \) possess cyclic subgroups of order \( N + 1 \). However, this pattern breaks down for \( N = 8 \), since \( N + 1 = 9 \) is not prime (yellow vertices in Figure 4C).

**Lemma 19** Let \( \Gamma \in \mathcal{D} \) be an \( N \times N \) domain with \( N \in \mathbb{N} \). Then, if \( N + 1 \) is prime, the sandpile group \( G^\Gamma \) possesses a cyclic subgroup \( \mathbb{Z}/(N + 1)\mathbb{Z} \leq G^\Gamma \) of order \( N + 1 \).

**Proof.** For \( N = 1 \), \( G^\Gamma \cong \mathbb{Z}/4\mathbb{Z} > \mathbb{Z}/2\mathbb{Z} \). In the following, we thus assume that \( N = 2n, n \in \mathbb{N} \), is even. The harmonic function \( H^\pi \) is symmetric with respect to the main diagonals \( d^+_0 \) and \( d^-_0 \) of the domain. Thus, w.l.o.g. only consider the quadrant of the domain below \( d^+_0 \) and above \( d^-_0 \), and label the vertices in this quadrant by coordinates \((x, y)\), with \( x = 1, 3, 5, \ldots \) and \( y = \pm 1, \pm 3, \ldots, \pm x \), as shown in Figure 4C. In this quadrant, \( H^\pi = H^+ + H^- \) with \( H^+ = \sum_{i=0}^{\infty} (-1)^i B^+_{2i+1} \) and \( H^- = \sum_{i=0}^{\infty} (-1)^{i+1} B^-_{2i+1} \) (Figure 4C). We claim that, in this quadrant, \((x-y)H^+(x,y) = -(x+y)H^+(x,-y)\). Since \( H^+(x,y) = -H^-(x,-y) \), this implies that

\[
H^\pi = H^+(x,y) + H^-(x,y) = H^+(x,y) - H^+(x,-y)
\]

\[
= H^+(x,y) + \frac{x-y}{x+y} H^+(x,y) = \frac{2x}{x+y} H^+(x,y).
\]

For \( |y| < x \) and \( x \) prime, since \( y \neq 0 \) and \( H^+(x,y) \) is integer-valued, \( H^\pi(x,y) \) is thus divisible by \( x = N + 1 \).

For small enough values of \( x \), it is possible to directly check our claim (Figure 4). Furthermore, by induction, it is easy to validate, in this order, that \( H^+(x,x) = \pm 1, H^+(x,-x) = 0, H^+(x,x-2) = \mp(x-1), H^+(x,-x+2) = \pm 1, H^+(x,x-4) = \pm(x-2)^2, \) and \( H^+(x,-x+4) = \mp(2x-4) \), such that our claim also holds close to the diagonals \( d^+_0 \) and \( d^-_0 \). Now, assume that our claim holds
Figure 4: Harmonic functions used to prove the existence of cyclic subgroups of certain order. A) The harmonic function $H^\gamma_\alpha$ corresponds to the sum of the four harmonic basis functions depicted in blue, green, red, and yellow, such that the values of $H^\gamma_\alpha$ on $\partial\Gamma_{ext}$ are all divisible by four. B) All values of the harmonic function $H^\pi$ are divisible by three on vertices with a blue background, by five on vertices with a green background, and by seven on vertices with a red background, respectively. C) In each quadrant of the domain, $H^\pi = H^+ + H^-$ is the sum of two harmonic functions $H^+$ and $H^-$. Our proof of Lemma 19 is based on showing that $(x-y)H^+(x,y) = -(x+y)H^+(x,-y)$.

for all $\hat{x} \leq x$. Then,

$$H^+(x+2,y) = 4H^+(x,y) - H^+(x,y-2) - H^+(x,y+2) - H^+(x-2,y)$$

$$= -4\frac{x+y}{x-y}H^+(x,y) + \frac{x+y-2}{x-y+2}H^+(x,y-2) + \frac{x+y-2}{x-y+2}H^+(x,y+2) + \frac{x+y+2}{x-y-2}H^+(x,y-2) + \frac{x+y-2}{x-y-2}H^+(x,y+2)$$

$$= -\frac{x+y+2}{x-y+2}H^+(x+2,-y) + \frac{4}{x-y+2}c(x,-y),$$

1

20
Eventually existing harmonic functions of this type with orders that don’t divide 2 monic functions $H$ which concludes our proof. $H$ eliminates 4 factors of $N$. We note that, on our numeric results indicate that these might be the only factors which can appear with odd multiplicities (Supplementary Table S1). We then get that $\epsilon(x,y) = -\frac{4y}{x-y}H^+(x, y) - H^+(x, -y + 2) + \frac{x+y+2}{x-y-2}H^+(x, -y - 2) + \frac{2y}{x-y-2}H^+(x-2, -y)
+ \frac{2y}{x+y-2}H^+(x, y - 2)
- H^+(x, y + 2) - H^+(x-2, y),

To prove that $\epsilon(x,y) = 0$ is “a bit tedious”, and can best be done with the help of a computer algebra system (Supplementary File 1). In short, we utilize that $H^+$ is harmonic to replace $H^+(x,y)$ by $4H^+(x-2,y) - H^+(x-2,y+2) - H^+(x-2,y-2) - H^+(x-4,y)$, and similarly for $H^+(x,y-2)$ and $H^+(x,y-4)$. We then utilize that, by the inductive assumption, $\epsilon(\hat{x},y) = 0$ for all $\hat{x} < x$ to replace $H^+(x-2,y+4)$ by $\frac{4y}{x+y}H^+(x-2,y+2) + \frac{x+y+2}{x+y-2}H^+(x-2,y) - \frac{2y+4}{x+y-2}H^+(x-4,y+2)$, and similarly for $H^+(x-2,y+2)$ and $H^+(x-2,y-4)$, which also eliminates $H^+(x-2,y-2)$. The only remaining term in column $x-2$ is then in $H^+(x-2,y)$, which we replace again by $4H^+(x-4,y) - H^+(x-2,y+2) - H^+(x-4,y-2) - H^+(x-6,y)$. We then get that $\epsilon(x,y) = -\frac{x+y+6}{x+y-2}\epsilon(x-4,y) = 0$, which concludes our proof.

If $\Gamma$ is an $N \times N$ domain for which $N+1$ is not prime, Theorem 16 implies that there exist group monomorphisms from every sandpile group on an $N_1 \times N_2$ domain to the sandpile group on $\Gamma$ for which $N_1 + 1$ divides $N + 1$. It directly follows that the sandpile group on every $N \times N$ domain (independently if $N + 1$ is prime or not) possesses a cyclic subgroup with an order given by the product of the factors of $N+1$ (with all multiplicities being one). The only open question is thus if also sandpile groups on $N \times N$ square domains, for which one or more factors of $N + 1$ appear with a multiplicity greater than one, possess cyclic subgroups of order $N+1$. Our numeric results suggest that this is likely the case (Supplementary Table S1).

Finally, we note that the harmonic function $H^\pi$ is rotationally (anti-)symmetric, and that the same holds for every $\pi^{\Gamma_1 \rightarrow \Gamma_2}(H^\pi|_{\Gamma_1})$ with $\Gamma_1, \Gamma_2 \in \mathcal{D}^{N_1,N_2,\pi}$. This symmetry trivially implies that $H^\pi$ and the harmonic functions obtained by rotating $H^\pi$ by 90°, 180° or 270° identify the same cyclic subgroup in the sense of Remark 14. Due to this rotational (anti-)symmetry, it is not surprising that the factors corresponding to numbers which divide $N + 1$ often appear with odd multiplicities in the factorizations of the order of the sandpile group. Indeed, our numeric results indicate that these might be the only factors which can appear with odd multiplicities (Supplementary Table S1). We note that, on $N \times N$ domains with $N = 2n + 1$, $n \in \mathbb{N}$, it is possible to define one of the harmonic functions $H^\pi_4$ such that it is rotationally (anti-)symmetric. However, the corresponding subgroup has order $4 = 2^2$, and similar might hold for all other, eventually existing harmonic functions of this type with orders that don’t divide $N + 1$. 21
3 Discussion

In this article, we have first derived a basis for the module of integer-valued harmonic functions on a given finite convex domain \( \Gamma \in \mathcal{D} \). We have then used this basis to define unique coordinates for every recurrent configuration of the sandpile group on \( \Gamma \), and we have shown that the order of this group is given by the determinant of the potential matrix of this basis. Since the potential matrix has dimension \( |\partial \Gamma| \times |\partial \Gamma| \), this formula might be numerically more efficient than the “conventional formula” via the determinant of the reduced graph laplacian \( \mathbf{L} \), a matrix of dimension \( |\Gamma| \times |\Gamma| \). Subsequently, we have constructed a family of group monomorphisms from each sandpile group on an \( N \times M \) domains to all sandpile groups on \( \psi(N+1) - 1 \times \psi(M+1) - 1 \) domains, with \( \psi, N, M \in \mathbb{N} \). Finally, we have proven the existence of several cyclic subgroups for the sandpile groups on \( N \times N \) square domains, which explain several regularities experimentally observed in the factorization of the respective group orders.

While the focus of this article was on sandpile groups defined on rectangular domains of the standard square lattice \( \mathbb{Z}^2 \), our proof for the existence of sandpile monomorphisms (Section 2.6) is rather generic and can likely be extended to other families of domains of \( \mathbb{Z}^2 \) having non-rectangular shapes, as well as of higher dimensional lattices \( \mathbb{Z}^k \), \( k \in \mathbb{N} \). Specifically, we expect that this is possible whenever a given domain can be assembled from smaller copies of itself in a similar manner as shown in the proof of Theorem 16. For example, a quick numeric analysis of the orders of the sandpile groups on diamond-shaped domains (Supplementary Figure S1), triangular-shaped domains (Supplementary Figure S2) and arrow-shaped domains (Supplementary Figure S3) showed similar regularities as observed for square domains (Figure 1A), directly indicating the existence of sandpile monomorphisms for such domains.

The formulation of Problem Statement 15 directly proposes that it might be possible to use such families of sandpile monomorphisms to derive injective limits for the sandpile group with respect to certain subcategories of domains, e.g. with respect to \( N_k \times N_k \) domains with \( N_k = 3^k - 1 \). Different to the limit for the extended sandpile group which we proposed recently [9], the respective limiting process would explicitly depend on the shape of the domains (in the respective subcategory \( \mathcal{D}_{\text{sub}} \)), and thus the resulting limits might be different for domains with different shapes, too. Furthermore, the dependence of the limiting process on the shape of the domains might allow to not only derive a limit for the respective sandpile groups, but also for every recurrent configuration in these groups. For this, recall that numeric studies indicate that the sandpile identity on an sufficiently large \( N \times N \) domains possesses a central square-shaped region containing only (if \( N = 2n, n \in \mathbb{N} \)) or nearly only (if \( N = 2n+1 \)) vertices carrying two particles. If we consider that the harmonic functions corresponding to recurrent configurations are conceptually closely related to toppling functions (see e.g. [11, 12]), and that the map \( \pi_{\Gamma_1 \to \Gamma_2} \) satisfies that \( \pi_{\Gamma_1 \to \Gamma_2}(H)|_{\Gamma_1, \Gamma_2} = H \) (Problem Statement 15), we might expect that, for \( \mathcal{D}^{2n,2n,2n+1} \), \( n, s \in \mathbb{N} \), such a limit might correspond to a vertex-wise convergence of particle numbers. Since our current computational implementation of the morphisms \( \pi_{\Gamma_1 \to \Gamma_2} \) does not
scale well with the domain size and since thus our experimental analysis is restricted to rather small domains, it is however not clear to which extend experimental data (Figure 3, top) supports this expectation.

Finally, we want to state some observations we made during our experimental study of the order of the sandpile group on different domains. That we were able to computationally determine the factorizations of these orders for relatively large $N \times N$ square domains (Supplementary Table S1) already indicates that the involved factors are rather small. We observed similarly small factors for domains for which we also expect sandpile monomorphisms to exist (i.e. the ones in Supplementary Figures S1–S3), but we quickly run into computational problems when we tried to calculate the factorizations for other domains. These computational problems indicate that the respective factorizations contain some rather large factors, in agreement with our observations for those (smaller) domains for which a factorization was computationally still feasible. The apparent “avoidance” of large factors for $N \times N$ domains can only partially be explained by the constraints on the factorizations imposed by the respective sandpile monomorphisms (Theorem 16), since we also observed rather small factors when $N + 1$ was prime, i.e. when no such (known) constraints existed (e.g. $N = 30$ in Supplementary Table S1). We consider this as an indication that it might be possible to derive several other proofs for the existence of cyclic subgroups of certain order, similar to Lemmata 1, 18 and 19. Eventually, this might lead to a full characterization of the sandpile group on such domains in terms of a closed formula for their decomposition into the direct sums of cyclic groups of prime-power order.

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## Supplementary Information

| \( N \) | Order | Factorization |
|-------|-------|---------------|
| 1     | 4     | 2             |
| 2     | 192   | 2^{6}1       |
| 3     | 100352| 2^{11}72    |
| 4     | 5.5757 \times 10^{-10} | 2^{192}3^{3}5^{11}2 |
| 5     | 3.2566 \times 10^{-10} | 2^{192}3^{5}11^{18}2 |
| 6     | 1.9872 \times 10^{-10} | 2^{192}3^{12}18^{47}2 |
| 7     | 1.2623 \times 10^{-10} | 2^{192}3^{2}4^{2}7^{9}9^{2}72 |
| 8     | 8.3266 \times 10^{-10} | 2^{192}3^{17}10^{4}3^{17}5^{3}109^{2} |
| 9     | 5.6943 \times 10^{-10} | 2^{192}3^{5}11^{19}41^{5}99^{10}2^{18}281^{2} |
| 10    | 4.0325 \times 10^{-10} | 2^{192}3^{5}7^{11}13^{1}15^{2}23^{1}37^{3}197^{3}263^{2} |
| 11    | 2.2382 \times 10^{-10} | 2^{192}3^{6}5^{11}3^{11}3^{1}15^{2}51^{1}371^{1}1171^{2}1327^{1}1873^{2} |
| 12    | 1.7522 \times 10^{-10} | 2^{192}5^{6}7^{13}29^{1}41^{4}43^{1}71^{4}83^{11}123^{1}3767^{1}9289^{2}2521^{2}3529^{2} |
| 13    | 1.4710 \times 10^{-10} | 2^{192}11^{12}11^{6}19^{2}29^{3}161^{1}151^{1}81^{1}241^{2}271^{2}421^{1}631^{2} |
| 14    | 1.1833 \times 10^{-11} | 2^{192}7^{17}2^{31}4^{17}5^{19}7^{19}193^{2}253^{2}257^{2}577^{1}607^{1}1217^{2}2113^{2}2207^{2}793^{12}1251^{2} |
| 15    | 1.0201 \times 10^{-11} | 2^{192}13^{17}101^{3}13^{17}157^{1}201^{2}2857^{1}3571^{2}5849^{2}9929^{10}1033^{11}1089^{2} |
| 16    | 1.0011 \times 10^{-11} | 2^{192}151^{19}3^{2}163^{1}261^{2}263^{2}337^{3}341^{2}1971^{2}3069^{2}13863^{2} |
| 17    | 0.9774 \times 10^{-11} | 2^{192}3^{24}2^{11}13^{17}19^{5}7^{17}53^{11}7^{10}169^{2}467^{2}514^{1}757^{1}937^{12}1009^{2} |
| 18    | 8.3533 \times 10^{-11} | 2^{192}19^{37}11^{3}192^{1}229^{1}457^{1}192^{7}561^{2}8741^{3}9349^{3}20553^{1}30817^{3}3629^{2} |
| 19    | 7.8971 \times 10^{-11} | 2^{192}3^{15}7^{2}11^{3}19^{29}31^{4}5^{9}71^{3}101^{1}181^{24}2^{8}501^{6}919^{18}261^{1}2 |
| 20    | 7.1718 \times 10^{-11} | 2^{192}15^{17}11^{12}3^{9}29^{4}41^{10}13^{7}127^{1}139^{1}211^{1}33^{1}41^{2}547^{6}3^{1}75^{1}1009^{1}1303^{1} |
| 21    | 7.7821 \times 10^{-11} | 2^{192}17^{6}2^{33}4^{11}5^{9}8^{9}10^{9}13^{1}19^{2}21^{4}30^{7}2^{11}11^{1}3^{1}4^{1}51^{1}7^{2}143^{2}1079^{2} |
| 22    | 8.0926 \times 10^{-12} | 2^{192}3^{13}7^{2}17^{3}9^{3}3^{3}3^{7}2^{17}13^{1}17^{1}19^{3}27^{2}61^{1}69^{1}91^{3}193^{3}271^{3}649^{2}1108^{2}1297^{2}26867^{2} |
| 23    | 8.6792 \times 10^{-12} | 2^{192}3^{5}7^{2}17^{1}19^{2}23^{1}2^{17}5^{1}2^{1}3^{1}11^{3}37^{2}5^{2}3^{2}7^{2}9^{2}13^{1}19^{3}23^{9}2^{3}3^{3}57^{2}7^{2}3^{2}11 |
| 24    | 9.5991 \times 10^{-12} | 2^{192}3^{5}7^{2}17^{2}11^{1}10^{1}12^{1}14^{9}1^{1}2^{3}1^{1}2^{4}1^{1}4^{2}49^{1}7^{1}8^{1}12^{1}1^{1}2^{1}3^{1}5^{2}6^{1}3^{1}3^{1}6^{1}2^{1}1^{1}2^{1}4^{1}2^{1}1^{1}2^{1}4^{1}2^{1}2^{1}4^{1}2^{1}2^{1}1^{1}2^{1}3^{1}2^{1}2^{1}4^{1}2^{1}1^{1}2^{1}4^{1}2^{1}7^{1}1^{1}2^{1}3^{1}2^{1}4^{1}2^{1}1^{1}2^{1}3^{1}2^{1} 

### Supplementary Table S1: Order of the sandpile groups on \( \mathbb{Z} \times \mathbb{Z} \) square domains.
Supplementary Figure S1: Divisibility of the order of the sandpile group on diamond-shaped domains. A&B) Two examples of diamond-shaped domains with heights $N = 5$ (A) and a $N = 6$ (B), respectively. C) A green plus indicates that the order of the sandpile group on a diamond-shaped domain with height $N$, corresponding to the rows of the table, is divisible by the order of the respective group on a domain with height $M$, corresponding to the columns (compare Figure 1A).
Supplementary Figure S2: Divisibility of the order of the sandpile group on triangular-shaped domains. A&B) Two examples of triangular-shaped domains with heights $N = 5$ (A) and a $N = 6$ (B), respectively. C) A green plus indicates that the order of the sandpile group on a triangular-shaped domain with height $N$, corresponding to the rows of the table, is divisible by the order of the respective group on a domain with height $M$, corresponding to the columns (compare Figure 1A).
Supplementary Figure S3: Divisibility of the order of the sandpile group on arrow-shaped domains. A&B) Two examples of arrow-shaped domains with heights $N = 5$ (A) and a $N = 6$ (B), respectively. C) A green plus indicates that the order of the sandpile group on a arrow-shaped domain with height $N$, corresponding to the rows of the table, is divisible by the order of the respective group on a domain with height $M$, corresponding to the columns (compare Figure 1A).
Supplementary File 1

Detailed version of the steps in the proof of Lemma 19 showing that $\epsilon(x, y) = 0$ using the Symbolic Toolbox of Matlab (The MathWorks, Inc., Natick, MA) as a computer algebraic program.