REMARKS ON THE FRACTIONAL INHOMOGENEOUS HARTREE EQUATION

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Abstract. This paper studies the inhomogeneous fractional Schrödinger equation
\[ i\dot{u} - (-\Delta)^s u = \pm (I_\alpha \ast | \cdot |^b |u|^p |x|^b |u|^{p-2} u. \]
In the mass super-critical and energy sub-critical regimes, using a Gagliardo-Nirenberg adapted to the above problem, the standing waves give a sharp threshold of global existence versus finite time blow-up of solutions.

1. Introduction

It is the purpose of this note, to investigate the Cauchy problem for a fractional inhomogeneous Schrödinger equation of Choquard type
\[
\begin{aligned}
&i\dot{u} - (-\Delta)^s u = \epsilon (I_\alpha \ast | \cdot |^b |u|^p |x|^b |u|^{p-2} u; \\
&u(0,.) = u_0.
\end{aligned}
\] (1.1)
The fractional Schrödinger equation is a fundamental equation of fractional quantum mechanics [7]. In three space dimensions, if \( s = \frac{1}{2}, b = 0 \) and \( \alpha = p = 2 \), the above equation arises as an effective description of pseudo-relativistic Boson stars [9].

Here and hereafter \( u \) is a complex valued function of the variable \((t, x) \in \mathbb{R} \times \mathbb{R}^N\), for an integer \( N \geq 2 \). The real number \( \epsilon = \pm 1 \) refers to the defocusing versus focusing regime. The fractional Laplacian operator is
\[ (-\Delta)^s := \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F} f), \quad s \in (0, 1). \]
The unbounded inhomogeneous term is \(| \cdot |^b f\) for a real number \( b < 0 \) and the Riesz-potential is the radial function defined on \( \mathbb{R}^N \) as follows
\[ I_\alpha := \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{N}{2}) \pi^{\frac{N}{2}} 2^\alpha} \cdot |N-\alpha| := \frac{\mathcal{K}}{| \cdot |^{N-\alpha}}, \quad 0 < \alpha < N. \]
In all this note, one assumes the next restriction on the different parameters of the above problem,
\[
\min\{-b, \alpha, N - \alpha, N + b - s, 2s + 2b + \alpha, N + \alpha + 2b - 2s\} > 0. \quad (1.2)
\]

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If \( u \) a solution to the above problem, so is the scaled function
\[
u_\lambda = \lambda^{2s + 2b + \alpha \left(p - 1\right) / 2(p - 1)} u(\lambda^2, \lambda), \quad \lambda > 0.
\]

The critical exponent is the unique real number conserving the homogeneous Sobolev norm
\[
\|u_\lambda(t)\|_{H^{s_c}} = \|u(\lambda^2 t)\|_{H^{s_c}}, \quad s_c := \frac{N}{2} - \frac{2s + 2b + \alpha}{2(p - 1)}.
\]

In this note, one focus on the mass super-critical \((s_c > 0)\) and energy sub-critical \((s_c < 1)\) regimes.

The inhomogeneous fractional Schrödinger problem was considered recently [11]. Indeed, a sharp dichotomy of global existence versus finite time blow-up of solutions was obtained in the mass super-critical and energy sub-critical regimes. This study follows the ideas of [6]. It is the aim of this work to extend the previous results to the inhomogeneous fractional Choquard problem (1.1). This note is also a generalization of the previous paper [13].

It is the contribution of this manuscript, to overcome three difficulties. The first one is the existence of a fractional Schrödinger operator, which is partially resolved by considering the radial case. In order to obtain the existence of non-global solutions, one uses a localized variance identity [1]. The second one is the presence of an unbounded homogeneous term. The last one is the non-local source term.

The rest of this paper is organized as follows. The next section contains some technical tools needed in the sequel. The existence of ground states is proved in section three. Section 4 contains a proof of a sharp Gagliardo-Nirenberg type inequality. The existence of energy sub-critical solutions is established in the fifth section. In section six, a variance type estimate is obtained. A sharp dichotomy of global/non global existence of solutions is given in the last section.

We mention that \( C \) will denote a constant which may vary from line to line. If \( A \) and \( B \) are non-negative real numbers, \( A \lesssim B \) means that \( A \leq CB \).

Denote for simplicity the Lebesgue space \( L^r := L^r(\mathbb{R}^N) \) with the usual norm \( \| \cdot \|_r := \| \cdot \|_{L^r} \) and \( \| \cdot \| := \| \cdot \|_2 \). Take \( H^s := H^s(\mathbb{R}^N) \) be the usual inhomogeneous Sobolev space endowed with the complete norm
\[
\| \cdot \|_{H^s} := \left( \| \cdot \|^2 + \|(-\Delta)^{s} \cdot \|^2 \right)^{\frac{1}{2}}.
\]

If \( X \) is an abstract space \( C_T(X) := C([0, T], X) \) stands for the set of continuous functions valued in \( X \) and \( X_{rd} \) is the set of radial elements in \( X \), moreover for an eventual solution to (1.1), \( T^* > 0 \) denotes it’s lifespan. Finally, \( x^\pm \) are two real numbers near to \( x \) satisfying \( x^+ > x \) and \( x^- < x \).
2. Background Material

In this section, one gives the main results and some standard tools needed in the sequel.

2.1. Preliminary. The mass-critical and energy-critical exponents are

\[ p_* := 1 + \frac{\alpha + 2s + 2b}{N}, \quad p^* := \begin{cases} 1 + \frac{2s + 2b + \alpha}{N - 2s} & \text{if } N \geq 3; \\ \infty & \text{if } N = 1, 2. \end{cases} \]

Here and hereafter define the real numbers

\[ B := \frac{Np - N - \alpha - 2b}{s}, \quad A := 2p - B; \]
\[ \tilde{p} := 1 + \frac{2b + \alpha}{N}, \quad \bar{p} := 1 + \frac{2b + \alpha}{N - 2s}. \]

For \( u \in H^s \), take the quantities

\[ S(u) := \|u\|_{H^s}^2 - \frac{1}{p} \int_{\mathbb{R}^N} (I_{\alpha} \ast |\cdot|^b |u|^p) |x|^b |u|^p \, dx; \]
\[ K(u) := \frac{4s}{N} \left( \|(-\Delta)^s u\|^2 - \frac{B}{2p} \int_{\mathbb{R}^N} (I_{\alpha} \ast |\cdot|^b |u|^p) |x|^b |u|^p \, dx \right); \]
\[ H(u) := S(u) - \frac{N}{4s} K(u) = \|u\|^2 + \frac{B - 2}{2p} \int_{\mathbb{R}^N} (I_{\alpha} \ast |\cdot|^b |u|^p) |x|^b |u|^p \, dx. \]

**Definition 2.1.** A ground state of (1.1) is a solution to

\[ (-\Delta)^s \phi + \phi = (I_{\alpha} \ast |\cdot|^b |\phi|^p) |x|^b |\phi|^{p-2} \phi, \quad 0 \neq \phi \in H^s, \tag{2.3} \]

which minimizes the problem

\[ m := \inf_{0 \neq u \in H^s} \left\{ S(u) \quad \text{s.t.} \quad K(u) = 0 \right\}. \tag{2.4} \]

If \( \phi \) is a ground state to (1.1), the following scale invariant quantities describe the dichotomy of global/non-global existence of solutions [5].

\[ \mathcal{ME}(u) := \frac{E(u)^s_{sc} M(u)^{s-\frac{s}{sc}}}{E(\phi)^s_{sc} M(\phi)^{s-\frac{s}{sc}}} \quad \text{and} \quad \mathcal{G}(u) := \frac{\|(-\Delta)^s \phi\|_{sc}^s \|u\|_{sc}^{s-\frac{s}{sc}}}{\|(-\Delta)^s \phi\|_{sc}^{s-\frac{s}{sc}}}. \]

Take \( \psi \in C_0^\infty(\mathbb{R}^n) \) is a radial function satisfying \( \psi'' \leq 1 \) and

\[ \psi(x) = \begin{cases} \frac{1}{2} |x|^2, & |x| \leq 1; \\ 0, & |x| \geq 2. \end{cases} \]

Then, \( \psi_R := R^2 \psi(\frac{\cdot}{R}) \) satisfies

\[ \psi_R'' \leq 1, \quad \psi_R'(r) \leq r \quad \text{and} \quad \Delta \psi_R \leq N. \]

Denote the localized variance

\[ M_\psi[u] := 2\mathfrak{F} \int_{\mathbb{R}^N} u \nabla \psi \nabla u \, dx = 2\mathfrak{F} \int_{\mathbb{R}^N} u \partial_k \psi \partial_k u \, dx. \]
Finally, define the self-adjoint differential operator
\[ \Gamma_\psi := -i\left[ \nabla((\nabla \psi) \cdot) + \nabla \psi \nabla \cdot \right]. \]

The next sub-section contains the contribution of this manuscript.

2.2. Main results. First, one considers the stationary problem associated to (1.1) and investigates the existence of ground states.

**Theorem 2.2.** Take \( N \geq 2 \), \( s \in (0, 1) \), \( b, \alpha \) satisfying (1.2) and \( p_* < p < p^* \). Then, there is a ground state solution to (2.3)-(2.4).

The next inhomogeneous Gagliardo-Nirenberg type inequality adapted to the problem (1.1), will be useful in this note.

**Theorem 2.3.** Let \( N \geq 2 \), \( s \in (0, 1) \), \( b, \alpha \) satisfying (1.2) and \( \tilde{p} < p < p^* \). Then,

1. there exists \( C(N, p, b, \alpha, s) > 0 \), such that for any \( u \in H^s \),
   \[ \frac{1}{C(N, p, b, \alpha, s)} = \inf \left\{ J(u) : \frac{\|u\|^4 \|(-\Delta)^{\frac{s}{2}} u\|^B}{\int_{\mathbb{R}^N} (I_\alpha * | \cdot |^b |u|^p) |x|^b |u|^p \, dx}, \quad 0 \neq u \in H^s \right\} \]
   such that \( C(N, p, b, \alpha, s) = \int_{\mathbb{R}^N} (I_\alpha * | \cdot |^b |\psi|^p) |x|^b |\psi|^p \, dx \)
   and
   \[ B(-\Delta)^{\frac{s}{2}} \psi + A\psi - \frac{2p}{C(N, p, b, \alpha, s)} (I_\alpha * | \cdot |^b |\psi|^p) |x|^b |\psi|^{p-2}\psi = 0; \]
2. if \( \phi \) is a ground state to (1.1), then
   \[ C(N, p, b, \alpha, s) = \frac{2p}{A}(\frac{A}{B})^\frac{2}{p} \|\phi\|^{-2(p-1)}. \]

Second, one considers the evolution problem (1.1) and obtains a local well-posed result in the energy space.

**Theorem 2.4.** Let \( N \geq 2 \), \( \frac{N}{2N-1} \leq s < 1 \), \( b, \alpha \) satisfying (1.2) and \( N < 4s + \alpha + 2b \), \( u_0 \in H^s \) and \( \max \{2, \tilde{p} \} < p < p^* \). Then, there exists \( T^* = T^*(\|u_0\|_{H^s}) \) such that (1.1) admits a unique maximal solution
\[ u \in C_{T^*}(H^s), \]
which satisfies the conservation laws
\[ \text{Mass} := M(u(t)) := \int_{\mathbb{R}^N} |u(t,x)|^2 \, dx = M(u_0); \]
\[ \text{Energy} := E(u(t)) := \|(-\Delta)^{\frac{s}{2}} u(t)\|^2 + \frac{\epsilon}{p} \int_{\mathbb{R}^N} |x|^b (I_\alpha * | \cdot |^b |u(t)|^p) |u(t)|^p \, dx = E(u_0). \]

**Remarks 2.5.**
1. \( u \in L^q_t((0, T^*), W^{1,r}) \) for any admissible pair \((q, r)\);
2. \( T^* = \infty \) in the defocusing case or mass-sub-critical case;
3. the previous theorem seems to hold for \( \tilde{p} < p \leq p^* \).
The next localized variance identity, will be needed to obtain the existence of non-global solutions to (1.1).

**Proposition 2.6.** Let \( N \geq 2, \ s \in (\frac{1}{2}, 1), \ b, \alpha \) satisfying (1.2) and \( \bar{p} < p < p^\star \). Take \( u \in C_T(H^s_{rd}) \) be a local solution of (1.1). Then, for any \( R > 0 \) and \( \varepsilon > 0 \) near to zero, holds on \([0, T)\),

\[
\frac{d}{dt} M_{\psi R}[u] \leq \ 2sBE - 2s(B - 2)\|(-\Delta)^{\frac{s}{2}} u\|^2 + \frac{C}{R^{2s}}
+ \frac{C}{R^{(N-1-s-2b)(p-1-\frac{1}{N})}} \|(-\Delta)^{\frac{s}{2}} u\|^{2(p-1-\frac{1}{N})}.
\]

The standing waves give a threshold of global existence versus finite time blow-up of solutions.

**Theorem 2.7.** Let \( N \geq 2, \ \frac{N}{2N-1} \leq s < 1, \ b, \alpha \) satisfying (1.2), \( u_0 \in H^s, \ 0 < s_c < s, \phi \) be a ground state solution to (2.3) satisfying

\[
\mathcal{ME}(u_0) < 1. \tag{2.8}
\]

Take a maximal solution \( u \in C_T(H^s) \) of (1.1). Thus,

1. if \( \mathcal{G}(u_0) < 1 \), then, \( u \) is global;
2. if \( p < 1 + \frac{\alpha}{N} + 2s \) and \( \mathcal{G}(u_0) > 1 \), then, \( u \) blows-up in finite time.

**Remark 2.8.**
1. The scattering of global solutions to (1.1) is considered in a paper in progress;
2. the condition \( p < 1 + \frac{\alpha}{N} + 2s \) is due to the localized variance identity.

The next sub-section contains some standard tools.

### 2.3. Useful estimates

First, recall a Hardy-Littlewood-Sobolev inequality [8].

**Lemma 2.9.** Let \( N \geq 1, \ 0 < \lambda < N \) and \( 1 < s, r < \infty \) be such that \( \frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N} = 2 \). Then,

\[
\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{f(x)g(y)}{|x - y|^{\lambda}} \ dx \ dy \leq C(N, s, \lambda)\|f\|_r\|g\|_s, \quad \forall f \in L^r, \forall g \in L^s.
\]

The next consequence [12], is adapted to the Choquard problem.

**Corollary 2.10.** Let \( N \geq 1, \ 0 < \lambda < N \) and \( 1 < s, r, q < \infty \) be such that \( \frac{1}{q} + \frac{1}{r} + \frac{1}{s} = 1 + \frac{\lambda}{N} \). Then,

\[
\|(I_\alpha * f)g\|_r \leq C(N, s, \alpha)\|f\|_s\|g\|_q, \quad \forall f \in L^s, \forall g \in L^q.
\]

Sobolev injections [2] give a meaning to several computations done in this note.

**Lemma 2.11.** Let \( N \geq 2, \ p \in (1, \infty) \) and \( s \in (0, 1) \), then

1. \( H^s \hookrightarrow L^q \) for any \( q \in [2, \frac{2N}{N-2s}] \);
2. the following injection $H^s_{rd} \hookrightarrow L^q$ is compact for any $q \in (2, \frac{2N}{N-2s})$;
3. for all $\frac{1}{2} < \mu < \frac{N}{2}$,

$$
\sup_{x \neq 0} |x|^{\frac{s}{2} - \mu} |u(x)| \leq C(N, \mu) \|(-\Delta)^{\frac{s}{2}} u\|, \quad \forall u \in H^\mu_{rd}(\mathbb{R}^N).
$$

The following fractional Gagliardo-Nirenberg inequality will be useful [10].

**Lemma 2.12.** Let $0 < 2s < N$. Then, there exists a positive constant $C := C(N, s) > 0$ such that,

$$
\| \cdot \|_p \leq C \| \cdot \|^1 - \frac{N}{2} \left(\frac{1}{2} - \frac{1}{q}\right) \|(-\Delta)^{\frac{s}{2}} u\|^{\frac{N}{2}} \left(\frac{1}{2} - \frac{1}{q}\right).
$$

One will use the next fractional chain rule [3].

**Lemma 2.13.** Let $s \in (0, 1]$ and $1 < p, p_i, q_i < \infty$ satisfying $\frac{1}{p} = \frac{1}{p_i} + \frac{1}{q_i}$. Then,

1. if $G \in C^1(\mathbb{C})$, then

$$
\|(-\Delta)^{\frac{s}{2}} [G(u)]\|_p \lesssim \|G'(u)\|_{p_i} \|(-\Delta)^{\frac{s}{2}} u\|_{q_i};
$$

2. \(|(-\Delta)^{\frac{s}{2}} (uv)|\|_p \lesssim \|(-\Delta)^{\frac{s}{2}} u\|_{p_i} \|v\|_{q_i} + \|(-\Delta)^{\frac{s}{2}} v\|_{p_2} \|u\|_{q_2}.
$$

**Definition 2.14.** A couple of real numbers $(q, r)$ is said to be admissible if

$$
q \geq 2, \quad r \in [2, \infty), \quad (q, r) \neq (2, \frac{4N - 2}{4N - 3}) \quad \text{and} \quad N(\frac{1}{2} - \frac{1}{r}) = \frac{2s}{q}.
$$

Denote the set of admissible pairs by $\Gamma$ and $(q, r) \in \Gamma'$ if $(q', r') \in \Gamma$.

The so-called radial Strichartz estimate [4] ends this section.

**Proposition 2.15.** Let $N \geq 2, \frac{N}{2N-1} \leq s < 1$ and $u_0 \in L^2_{rd}$. Then

$$
\|u\|_{L^2_t(L^r')} \lesssim \|u_0\| + \|i u - (-\Delta)^s u\|_{L^q_t(L^r')}.
$$

if $(q, r)$ and $(\tilde{q}, \tilde{r})$ are admissible pairs.

3. Existence of ground states

This section is devoted to prove Theorem 2.2 about the existence of ground states.

Let us give some intermediary results.

**Lemma 3.1.** For any $u \in H^s$ and $\lambda > 0$

$$
\min\{H(u), \partial_\lambda H(u^\lambda)\} > 0.
$$

**Proof.** A direct computation gives

$$
H(u) = \|u\|^2 + \frac{B - 2}{2p} \int_{\mathbb{R}^N} (I_\alpha * | \cdot |^b |u|^p) |x|^b |u|^p dx;
$$

$$
H(u^\lambda) = \|u\|^2 + \frac{B - 2}{2p} \lambda^{\frac{2b}{N}} \int_{\mathbb{R}^N} (I_\alpha * | \cdot |^b |u|^p) |x|^b |u|^p dx;
$$

$$
\partial_\lambda H(u^\lambda) = \frac{B(B - 2)}{Np} \lambda^{\frac{2b}{N} - 1} \int_{\mathbb{R}^N} (I_\alpha * | \cdot |^b |u|^p) |x|^b |u|^p dx.
$$
The constraint is positive if its quadratic part vanishes.

**Lemma 3.2.** Let $0 \neq u_n$ be a bounded sequence of $H^s$ such that
\[
\lim_{n} \|(-\Delta)^{\frac{s}{2}} u_n\| = 0.
\]
Then, there exists $n_0 \in \mathbb{N}$ such that $K(u_n) > 0$ for all $n \geq n_0$.

**Proof.** Since $B > 2$,
\[
\int_{\mathbb{R}^N} (I_\alpha * | \cdot |^b |u_n|^p) |x|^b |u_n|^p \, dx \leq C \|u_n\|^A \|(-\Delta)^{\frac{s}{2}} u_n\|^B = o\left(\|(-\Delta)^{\frac{s}{2}} u_n\|^2\right).
\]
Thus, when $n \to \infty$,
\[
K(u_n) \simeq \frac{4s}{N} \|(-\Delta)^{\frac{s}{2}} u_n\|^2 > 0.
\]

The minimizing problem (2.4) can be expressed with a negative constraint.

**Lemma 3.3.** One has
\[
m = \inf_{0 \neq u \in H^s} \{H(u) \text{ s. t. } K(u) \leq 0\}.
\]

**Proof.** Denoting by $r$ the right hand side of the previous equality, it is sufficient to prove that $m_{a,c} \leq r$. Take $u \in H^s$ such that $K(u) < 0$. Because $\lim_{\lambda \to 0} \|(-\Delta)^{\frac{s}{2}} u^\lambda\| = 0$, by the previous Lemma, there exists $\lambda \in (0,1)$ such that $K(u^\lambda) > 0$. With a continuity argument there exists $\lambda_0 \in (0,1)$ such that $K(u^{\lambda_0}) = 0$, then since $\lambda \mapsto H(u^\lambda)$ is increasing, one gets
\[
m \leq H(u^{\lambda_0}) \leq H(u).
\]
This closes the proof.

**Proof of theorem 2.2.** Let $(\phi_n)$ be a minimizing sequence, namely
\[
0 \neq \phi_n \in H^s, \quad K(\phi_n) = 0 \quad \text{and} \quad \lim_{n} H(\phi_n) = \lim_{n} S(\phi_n) = m. \tag{3.12}
\]
With a rearrangement argument via Lemma 3.3, we can assume that $(\phi_n)$ is radial decreasing.

- $(\phi_n)$ is bounded in $H^s$.

Since
\[
H(\phi_n) = \|\phi_n\|^2 + \frac{B - 2}{2p} \int_{\mathbb{R}^N} (I_\alpha * | \cdot |^b |\phi_n|^p) |x|^b |\phi_n|^p \, dx \to m,
\]
it follows that
\[
\sup_n \int_{\mathbb{R}^N} (I_\alpha * | \cdot |^b |\phi_n|^p) |x|^b |\phi_n|^p \, dx < \infty.
\]
Then, because
\[
S(\phi_n) = \|\phi_n\|_{H^s}^2 - \frac{1}{p} \int_{\mathbb{R}^N} (I_\alpha * | \cdot |^b |\phi_n|^p) |x|^b |\phi_n|^p \, dx \to m,
\]
one gets 
\[ \sup_n \| \phi_n \|_{H^s} < \infty. \]

- The limit of \((\phi_n)\) is nonzero and \(m > 0\).

Taking account of the compact injection in Lemma 2.11, take 
\[ \phi_n \rightharpoonup \phi \quad \text{in} \quad H^s \]
and for all \(2 < p < \frac{2N}{N-2s} \), 
\[ \phi_n \to \phi \quad \text{in} \quad L^p. \]

The equality \(K(\phi_n) = 0\) implies that 
\[ \|(-\Delta)^{\frac{s}{2}} \phi_n\|^2 = \frac{B}{2p} \int_{\mathbb{R}^N} (I_\alpha * |\cdot|^b|\phi_n|^p)|x|^b|\phi_n|^p \, dx. \]

Assume that \(\phi = 0\). Thanks to Hardy-Littlewood-Paley inequality, one gets 
\[ \int_{\mathbb{R}^N} (I_\alpha * |\cdot|^b|\phi_n|^p)|x|^b|\phi_n|^p \, dx \lesssim \|x|^{b}\phi_n^p\|^{2}_{\frac{2N}{a+b}}. \]

Take \(\rho := (\frac{N}{b})^-\) and \(r := \frac{2Np}{\alpha+N-2|b|-\epsilon}\). Using Hölder inequality, write 
\[ \|x|^{b}\phi_n^p\|^{2N}_{\frac{2N}{a+b}}(\{x|<1\}) \lesssim \|x|^{b}\|^{p}_{L^p(\{x|<1\})}\|\phi_n^p\|^{p}_{r} \lesssim \|\phi_n^p\|^{p}_{r}. \]

Since \(\tilde{p} < p < p^*\), taking \(\epsilon \to 0\), it follows that \(2 < r < \frac{2N}{N-2s}\). Then, 
\[ \|x|^{b}\phi_n^p\|^{2N}_{\frac{2N}{a+b}}(\{x|<1\}) \to 0, \quad \text{as} \quad n \to \infty. \]

Similarly, one obtains 
\[ \|x|^{b}\phi_n^p\|^{2N}_{\frac{2N}{a+b}}(\{x|>1\}) \to 0, \quad \text{as} \quad n \to \infty. \]

Thus, 
\[ \int_{\mathbb{R}^N} (I_\alpha * |\cdot|^b|\phi_n|^p)|x|^b|\phi_n|^p \, dx \to 0, \quad \text{as} \quad n \to \infty. \]

Now, by Lemma 3.2 yields \(K(\phi_n) > 0\) for large \(n\). This contradiction implies that 
\(\phi \neq 0\).

With lower semi continuity of \(\| \cdot \|_{H^s}\), one has 
\[ 0 = \liminf_n K(\phi_n) \]
\[ \geq \frac{4s}{N} \liminf_n \|(-\Delta)^{\frac{s}{2}} \phi_n\|^2 - \frac{2sB}{Np} \int_{\mathbb{R}^N} (I_\alpha * |\cdot|^b|\phi|^p)|x|^b|\phi|^p \, dx \]
\[ \geq K(\phi). \]

Similarly, \(H(\phi) \leq m\). Moreover, with Lemma 3.3, one can assume that 
\[ K(\phi) = 0 \quad \text{and} \quad S(\phi) = H(\phi) \leq m. \]

So, \(\phi\) is a minimizer satisfying (3.12). Thus, 
\[ m = H(\phi) > 0. \]
• The limit \( \phi \) is a solution to (2.3).

There is a Lagrange multiplier \( \eta \in \mathbb{R} \) such that \( S'(\phi) = \eta K'(\phi) \). Thus,

\[
0 = K(\phi) = \partial_\lambda (S(\phi^\lambda))|_{\lambda=1} \\
= \langle S'(\phi), \partial_\lambda (\phi^\lambda)|_{\lambda=1} \rangle \\
= \langle K'(\phi), \partial_\lambda (\phi^\lambda)|_{\lambda=1} \rangle \\
= \eta \partial_\lambda (K(\phi^\lambda))|_{\lambda=1}.
\]

Compute

\[
K(\phi^\lambda) = \frac{4s}{N} \left( \lambda \frac{4}{N} \right) \|(-\Delta)^{\frac{s}{2}} \phi \|^2 \\
- \frac{B\lambda^{\frac{2s}{N}}}{2p} \int_{\mathbb{R}^N} (I_\alpha \ast |\cdot|^b |\phi_n|^p |x|^b |\phi_n|^p \, dx);
\]

\[
\partial_\lambda (K(\phi^\lambda))|_{\lambda=1} = \left( \frac{4s}{N} \right)^2 \left( \left\|(-\Delta)^{\frac{s}{2}} \phi \right\|^2 \\
- \frac{B^2}{4p} \int_{\mathbb{R}^N} (I_\alpha \ast |\cdot|^b |\phi_n|^p |x|^b |\phi_n|^p \, dx) \right); \]

Since \( K(\phi) = 0 \), one gets

\[
\partial_\lambda (K(\phi^\lambda))|_{\lambda=1} = \left( \frac{4s}{N} \right)^2 \frac{B(2-B)}{4p} \int_{\mathbb{R}^N} (I_\alpha \ast |\cdot|^b |\phi_n|^p |x|^b |\phi_n|^p \, dx).
\]

Thus, \( \eta = 0 \) and \( S'(\phi) = 0 \). So, \( \phi \) is a ground state.

\[\blacksquare\]

4. GAGLIARDO-NIRENBERG INEQUALITY

This section, one establishes a sharp Gagliardo-Nirenberg type inequality related to the Choquard problem (1.1). The proof follows by [3] ideas. Let us prove (2.6). Thanks to a Schwartz symmetrization argument, take a minimizing sequence

\[
\beta := \frac{1}{C(N, p, b, \alpha, s)} = \lim_n J(v_n), \quad v_n \in H^s_{rd}.
\]

Define the scaling \( u^{\lambda, \mu} := \lambda u(\mu), \lambda, \mu \in \mathbb{R} \). Then,

\[
\left\|(-\Delta)^{\frac{s}{2}} u^{\lambda, \mu} \right\|^2 = \lambda^2 \mu^{2s-N} \left\|(-\Delta)^{\frac{s}{2}} u \right\|^2; \\
\left\|u^{\lambda, \mu} \right\|^2 = \lambda^2 \mu^{-N} \left\|u \right\|^2; \]

\[
\int_{\mathbb{R}^N} (I_\alpha \ast |\cdot|^b |u^{\lambda, \mu}|^p |x|^b |u^{\lambda, \mu}|^p \, dx = \lambda^{2p} \mu^{-N-a-2b} \int_{\mathbb{R}^N} (I_\alpha \ast |\cdot|^b |u|^p |x|^b |u|^p \, dx.
\]

It follows that

\[
J(u^{\lambda, \mu}) = J(u).
\]

Take \( \psi_n := v_n^{\lambda_n, \mu_n} \), where

\[
\mu_n := \left( \frac{\|v_n\|}{\|(-\Delta)^{\frac{s}{2}} v_n\|} \right)^\frac{1}{s} \quad \text{and} \quad \lambda_n := \frac{\|v_n\|^{N-1}}{\|(-\Delta)^{\frac{s}{2}} v_n\|^{\frac{N-1}{2}}}.
\]

So,

\[
\|\psi_n\| = \|(-\Delta)^{\frac{s}{2}} \psi_n\| = 1 \quad \text{and} \quad \beta = \lim_n J(\psi_n).
\]
Thanks to Lemma 2.9 and Sobolev embedding,

\[
(A_n) := \int_{\mathbb{R}^N} |x|^b (I_\alpha \ast |\cdot|^b|\psi_n|^p)|\psi_n|^p - (I_\alpha \ast |\cdot|^b|\psi|^p)|\psi|^p \, dx
\]

\[
\leq \int_{\mathbb{R}^N} |x|^b (I_\alpha \ast |\cdot|^b(|\psi_n|^p - |\psi|^p))|\psi|^p - (I_\alpha \ast |\cdot|^b|\psi_n|^p)(|\psi|^p - |\psi_n|^p) \, dx
\]

\[
\lesssim (|||x|^b\psi_p||_{L^{\frac{2N}{\alpha+N}}_{\alpha+N}} + |||x|^b\psi_{p,n}||_{L^{\frac{2N}{\alpha+N}}_{\alpha+N}})|||x|^b(|\psi|^p - |\psi_n|^p)||_{L^{\frac{2N}{\alpha+N}}_{\alpha+N}}.
\]

Take \( \rho := \left(\frac{N}{b}\right)^{-} \) and \( r := \frac{2N}{\alpha+N-\frac{2N}{p}} \). Because \( \bar{p} < p < p^* \), it follows that \( 2 < rp < \frac{2N}{N-2s} \). So, by compact Sobolev injections via Hölder inequality, one gets

\[
|||x|^b(|\psi|^p - |\psi_n|^p)||_{L^{\frac{2N}{\alpha+N}}_{\alpha+N}(|x|<1)} \leq |||x|^b||_{L^r(|x|<1)}|||\psi|^p - |\psi_n|^p||_{L^{r'}(|x|<1)}
\]

\[
\lesssim |||\psi|^p - |\psi_n|^p||_{L^r(|x|<1)}
\]

\[
\lesssim |||\psi_n| - |\psi|||\sum_{k=1}^{p-1} |\psi_n|^k|\psi|^{p-k-1}||_{L^r}
\]

\[
\lesssim |||\psi_n| - |\psi||_{pr} \sum_{k=1}^{p-1} |||\psi_n|^k||_{pr}|||\psi|^{p-k-1}||_{pr} \to 0.
\]

Similarly, one estimates the previous integrals on \( \{|x| > 1\} \). Then,

\[
\lim_n(A_n) = 0.
\]

Thus,

\[
J(\psi_n) = \frac{1}{\int_{\mathbb{R}^N} (I_\alpha \ast |\cdot|^b|\psi_n|^p)|\psi_n|^p \, dx} \to \frac{1}{\int_{\mathbb{R}^N} (I_\alpha \ast |\cdot|^b|\psi|^p)|\psi|^p \, dx}.
\]

Thanks to the lower semi continuity of \( \| \cdot \|_{H^s} \),

\[
|||\psi||| \leq 1 \quad \text{and} \quad \|(-\Delta)^\frac{s}{2}\psi\| \leq 1.
\]

So, \( J(\psi) \leq \beta \) if \( |||\psi|||\|(-\Delta)^\frac{s}{2}\psi\| < 1 \), which gives

\[
|||\psi||| = 1 \quad \text{and} \quad \|(-\Delta)^\frac{s}{2}\psi\| = 1.
\]

Then,

\[
\psi_n \to \psi \quad \text{in} \quad H^s; \quad \beta = J(\psi) = \frac{1}{\int_{\mathbb{R}^N} (I_\alpha \ast |\cdot|^b|\psi|^p)|\psi|^p \, dx}.
\]

Finally, \( \psi \) satisfies (2.6) because the minimizer satisfies the Euler equation

\[
\partial_x J(\psi + \varepsilon \eta)|_{\varepsilon=0} = 0, \quad \forall \eta \in C^\infty_0 \cap H^s.
\]

Eventually, one proves (2.7). Let \( \psi \) satisfying (2.6) and

\[
C(N, p, b, \alpha, s) = \frac{1}{\beta} = \int_{\mathbb{R}^N} (I_\alpha \ast |\cdot|^b|\psi|^p)|\psi|^p \, dx.
\]
The scaled function
\[ \psi = \phi^{\lambda \mu} := \lambda \phi(\mu), \quad \mu = \left( \frac{A}{B} \right)^{\frac{1}{2p}}, \quad \lambda = \left( \frac{A}{B} \right)^{\frac{1}{2p} - \frac{1}{2}}, \]
satisfies
\[ (-\Delta)^s \phi + \phi - (I_{\alpha} \ast |\cdot|^b|\phi|^p)|x|^b|\phi|^{p-2}\phi = 0. \]
Thus, the equalities
\[ \|\psi\| = 1 = \lambda \mu^{\frac{N}{r}} \|\phi\|, \]
give
\[ \beta = \frac{A}{2p} \left( \frac{A}{B} \right)^{\frac{1}{2p} - \frac{1}{2}} \|\phi\|^{2(p-1)}. \]
This finishes the proof.

5. Existence of solutions

This section is devoted to prove the existence and uniqueness of energy solutions to the non-linear Schrödinger problem \((1.1)\). A standard fixed point argument is used. Take \(u, v\) in the space
\[ B_T(R) := \{ w \in \cap_{(q, r) \in \Gamma} L^q_T(W^{s, r}) \quad \text{s. t.} \quad \sup_{(q, r) \in \Gamma} \|w\|_{L^q_T(W^{s, r})} \leq R \}, \]
equipped with the complete distance
\[ d(u, v) := \|u\|_{S_T(\mathbb{R}^N)} := \sup_{(q, r) \in \Gamma} \|u - v\|_{L^q_T(L^r(\mathbb{R}^N))}. \]
Define the function
\[ \phi(u) := e^{-i(-\Delta)^s u_0 - \int_0^T e^{-i(-\gamma - \gamma')(-\Delta)^s}} [(I_{\alpha} \ast |\cdot|^b|u|^p)|x|^b|u|^{p-2}u] \, d\tau. \]
Using Strichartz estimate and Corollary 2.10, one has
\[ d(\phi(u), \phi(v)) \lesssim \|(I_{\alpha} \ast |\cdot|^b|u|^p - |\cdot|^b|v|^p)|x|^b|u|^{p-2}u\|_{L^q_T(L^{r'}(|x|<1))} \]
\[ + \|(I_{\alpha} \ast |\cdot|^b|v|^p)(|x|^b|u|^{p-2}u - |x|^b|v|^{p-2}v)\|_{L^q_T(L^{r'}(|x|<1))} \]
\[ + \|(I_{\alpha} \ast |\cdot|^b|u|^p - |\cdot|^b|v|^p)|x|^b|u|^{p-2}u\|_{L^q_T(L^{r'}(|x|>1))} \]
\[ + \|(I_{\alpha} \ast |\cdot|^b|v|^p)(|x|^b|u|^{p-2}u - |x|^b|v|^{p-2}v)\|_{L^q_T(L^{r'}(|x|>1))} \]
\[ \lesssim (I) + (II) + (III) + (IV), \]
where \((q, r), (q_1, r_1) \in \Gamma\). Take \(\mu := (\frac{N}{b})^-\) and \(r := \frac{2Np}{\alpha + N - \frac{2p}{b}}\). Then, \(1 + \frac{\mu}{N} = \frac{2}{\mu} + \frac{2p}{r}\) and using Hölder and Hardy-Littlewood-Paley inequalities, one gets
\[ (II) \quad \|[(I_{\alpha} \ast |\cdot|^b|v|^p)(|x|^b|u|^{p-2}u - |x|^b|v|^{p-2}v)]\|_{L^q_T(L^{r'}(|x|<1))} \]
\[ \lesssim \|(I_{\alpha} \ast |\cdot|^b|v|^p)|x|^b(|u|^{p-2} + |v|^{p-2})(u - v)\|_{L^q_T(L^{r'}(|x|<1))} \]
\[ \lesssim \|(|x|^b|v|^2_{2^*(|x|<1)})\|_r \|(u\|_r^{2(p-1)} + \|v\|_r^{2(p-1)})\|_{L^q_T(0, T)} \]
\[ \lesssim \|(u\|_r^{2(p-1)} + \|v\|_r^{2(p-1)})\|_T \|u - v\|_r \|L^q_T(0, T). \]
Because $p < p_\ast$, there exists $\delta > 0$ such that $\frac{1}{q} = \frac{2p-1}{q} + \frac{1}{\delta}$. Then, taking account of Sobolev embeddings and H"older inequality, one obtains

\[
(II) \lesssim T^\frac{1}{2} \left( \|u\|_{L^q_y(L^r)}^{2(p-1)} + \|v\|_{L^q_y(L^r)}^{2(p-1)} \right) d(u, v)
\lesssim T^\frac{1}{2} R^{2(p-1)} d(u, v).
\]

Similarly, one estimates $(I)$. Taking in the previous computation, $\mu := \left( \frac{N}{s-b} \right)^+$, one controls integrals on $\{ |x| > 1 \}$ and gets

\[
(III) + (IV) \lesssim T^\frac{1}{2} R^{2(p-1)} d(u, v).
\]

Thus,

\[d(\phi(u), \phi(v)) \lesssim T^\frac{1}{2} R^{2(p-1)} d(u, v).\]

Moreover, taking $v = 0$ in the previous estimate, one gets

\[\|\phi(u)\|_{ST(\mathbb{R}^N)} \leq C \|u_0\| + CT^\frac{1}{2} R^{2p-1}.\]

It remains to estimate $\|(-\Delta)^{\frac{1}{2}} |\phi(u)|\|_{ST(\mathbb{R}^N)}$. Taking account of Fourrier transform, one can check that

\[(-\Delta)^{\frac{1}{2}} |x|^b = C_{N,b} |x|^{b-s}.
\]

Using the chain rules in Lemma 2.13 via Corollary 2.10,

\[
(A) := \|(-\Delta)^{\frac{1}{2}} [\phi(u) - e^{-it(-\Delta)^s} u_0] \|_{L^q_y(L^r)} \lesssim \| (I_\alpha * (-\Delta)^{\frac{1}{2}} (|b| |u|)) |x|^b |u|^{p-2} u + (I_\alpha * |b| |u|) (-\Delta)^{\frac{1}{2}} (|x|^b |u|^{p-2} u) \|_{L^q_y(L^r)}
\lesssim \| (I_\alpha * |b| (-\Delta)^{\frac{1}{2}} (|u|)) |x|^b |u|^{p-2} u + (I_\alpha * |b| |u|) |x|^b (-\Delta)^{\frac{1}{2}} (|u|^{p-2} u) \|_{L^q_y(L^r)} + \| (I_\alpha * (|b|^{b-s} |u|)) |x|^b |u|^{p-2} u + (I_\alpha * |b| |u|) (|x|^b |u|^{p-2} u) \|_{L^q_y(L^r)}.
\]

Thanks to the Chain rule in Lemma 2.13 and arguing as previously, one gets

\[
(I_1) + (II_1) := \| (I_\alpha * |b| (-\Delta)^{\frac{1}{2}} (|u|)) |x|^b |u|^{p-2} u \|_{L^q_y(L^r'(|x|<1))} + \| (I_\alpha * |b| |u|) |x|^b (-\Delta)^{\frac{1}{2}} (|u|^{p-2} u) \|_{L^q_y(L^r'(|x|<1))}
\lesssim \| |x|^b \|_{L^q_y(L^r'(|x|<1))} \left[ \| (-\Delta)^{\frac{1}{2}} (|u|) \|_{L^p} \| u \|_{L^p}^{p-1} + \| u \|_{L^p} \| (-\Delta)^{\frac{1}{2}} (|u|^{p-2} u) \|_{L^p} \right] \|_{L^q_y}
\lesssim \| |u|^{2p-2} \|_{L^{2^p-2}_y(H^s)} \| u \|_{L^q_y(W^{s,r})}
\lesssim \| |u|^{2p-2} \|_{L^{2^p-2}_y(H^s)} \| u \|_{L^q_y(W^{s,r})}
\lesssim T^{1-\frac{2}{p}} |u|^{2p-2} \| u \|_{L^q_y(H^s)} \| u \|_{L^q_y(W^{s,r})}
\lesssim T^{1-\frac{2}{p}} R^{2p-1}.
\]

Take the choice $\rho := \left( \frac{N}{s-b} \right)^-$, $r_1 := \frac{2Np}{N + \alpha + 2b + 2s(p-1) - \epsilon}$ and $a := \left( \frac{N}{b} \right)^-$, one gets

\[1 + \frac{\alpha}{N} = \frac{1}{r_1} + \frac{1}{\rho} + \frac{1}{a} + (2p-1)(\frac{1}{r_1} - \frac{s}{N}).\]
Since \( p < p^* \), it follows that \((2p - 1)q_1' < q_1\) and there exists a positive number denoted also \( \delta > 0 \) such that \( \frac{1}{q_1'} = \frac{2p-1}{q_1} + \frac{1}{\delta} \). Then, taking account of Hardy-Littlewood-Paley and Hölder inequalities and to the Chain rule in Lemma 2.13, one gets

\[
(III_1) + (IV_1) := \| (I_\alpha * (| \cdot |^{b-s}|u|^p)) |x|^b |u|^{p-2} u \|_{L^\frac{q_1}{r}(|x|<1)} + \| (I_\alpha * | \cdot |^{b}|u|^p) |x|^{b-s}|u|^{p-2} u \|_{L^\frac{q_1}{r}(|x|<1)} \\
\lesssim \| |x|^b \|_{L^\alpha(|x|<1)} \| |x|^{b-s} \|_{L^\beta(|x|<1)} \| u \|_{L^{\frac{2p-1}{\alpha}}(0,T)}^{2p-1} \lesssim T^{\frac{1}{2}} \| u \|_{L^\frac{3}{2}(W^{s, r_1})}^{2p-1} \lesssim T^{\frac{1}{2}} R^{2p-1}.
\]

The conditions \( N + \alpha + 2b - 2s > 0 \) and \( p > \bar{p} \) imply that \( 2 < r_1 < N \). So by Sobolev injections, one obtains

\[
(III_1) + (IV_1) \lesssim T^{\frac{1}{2}} \| u \|_{L^\frac{3}{2}(W^{s, r_1})}^{2p-1} \lesssim T^{\frac{1}{2}} R^{2p-1}.
\]

Similarly, one estimates the integrals on \( \{|x| > 1\} \). Taking \( R > C\| u_0 \|_{H^s} \), it follows that \( \phi \) is a contraction of \( B_T(R) \) for some \( T > 0 \) small enough. The fix point is a solution to (1.1). The uniqueness is a consequence of the previous computations via a translation argument.

### 6. Variance Type Identity

This section is devoted to prove Theorem 2.6. By taking the time derivative and using (1.1), one gets

\[
\frac{d}{dt} M_\psi[u(t)] = < u(t), (\Delta) \gamma, i \Gamma_\psi]u(t) > + < u(t), [- (I_\alpha * | \cdot |^{b}|u|^p)|x|^b |u|^{p-2}, i \Gamma_\psi]u(t) >,
\]

where \([X, Y] := XY - YX\) denotes the commutator of \( X \) and \( Y \). According to computation done in [1], one have

\[
< u(t), (\Delta) \gamma, i \Gamma_\psi]u(t) \leq 4s \| (\Delta) \frac{1}{2} u(t) \|^2 + R^{-2s}.
\]

Let us treat the nonlinear term

\[
(N) := < u(t), [- (I_\alpha * | \cdot |^{b}|u|^p)|x|^b |u|^{p-2}, i \Gamma_\psi]u(t) > \\
= - < u(t), (I_\alpha * | \cdot |^{b}|u|^p)|x|^b |u|^{p-2} \partial_\psi \partial_\psi u > - < u(t), (I_\alpha * | \cdot |^{b}|u|^p)|x|^b |u|^{p-2} \partial_\psi \partial_\psi u > + < u(t), \partial_\psi \partial_\psi (I_\alpha * | \cdot |^{b}|u|^p)|x|^b |u|^{p-2} u(t) > \\
= -2 < u(t), (I_\alpha * | \cdot |^{b}|u|^p)|x|^b |u|^{p-2} \partial_\psi \partial_\psi u > + 2 < u(t), \partial_\psi \partial_\psi (I_\alpha * | \cdot |^{b}|u|^p)|x|^b |u|^{p-2} u(t) > \\
= 2 \int_{\mathbb{R}^N} |u|^2 \partial_\psi \partial_\psi (I_\alpha * | \cdot |^{b}|u|^p)|x|^b |u|^{p-2} |dx.\]

By integration by parts

\[(N) = 2 \int_{\mathbb{R}^N} |u|^2 \nabla \psi_R \nabla [(I_{\alpha} \ast |b|^p)|x|^b|u|^{p-2}] \, dx\]

\[= -2 \int_{\mathbb{R}^N} \nabla (|u|^2) \nabla \psi_R (I_{\alpha} \ast |b|^p)|x|^b|u|^{p-2} \, dx - 2 \int_{\mathbb{R}^N} \Delta \psi_R (I_{\alpha} \ast |b|^p)|x|^b|u|^p \, dx\]

\[= -\frac{4}{p} \int_{\mathbb{R}^N} (I_{\alpha} \ast |b|^p)|x|^b \nabla \psi_R \nabla (|u|^p) \, dx - 2 \int_{\mathbb{R}^N} \Delta \psi_R (I_{\alpha} \ast |b|^p)|x|^b|u|^p \, dx\]

\[= \frac{4}{p} \int_{\mathbb{R}^N} \nabla (I_{\alpha} \ast |b|^p) \nabla \psi_R |x|^b|u|^p \, dx + 2(\frac{2}{p} - 1) \int_{\mathbb{R}^N} \Delta \psi_R (I_{\alpha} \ast |b|^p)|x|^b|u|^p \, dx\]

\[+ \frac{4b}{p} \int_{\mathbb{R}^N} (I_{\alpha} \ast |b|^p)x. \nabla \psi_R |x|^b-2|u|^p \, dx.\]

Thanks to the properties of \(\psi\), it follows that

\[(L) := \int_{\mathbb{R}^N} (I_{\alpha} \ast |b|^p)x. \nabla \psi_R |x|^b-2|u|^p \, dx\]

\[= \int_{|x|<R} (I_{\alpha} \ast |b|^p)|x|^b|u|^p \, dx + O \left( \int_{|x|>R} (I_{\alpha} \ast |b|^p)|x|^b|u|^p \, dx \right)\]

\[= \int_{\mathbb{R}^N} (I_{\alpha} \ast |b|^p)|x|^b|u|^p \, dx + O \left( \int_{|x|>R} (I_{\alpha} \ast |b|^p)|x|^b|u|^p \, dx \right).\]

Using the symmetry of \(I_{\alpha}\), yield

\[(M) := \frac{4}{p} \int_{\mathbb{R}^N} \nabla (I_{\alpha} \ast |b|^p) \nabla \psi_R |x|^b|u|^p \, dx\]

\[= -2 \frac{N - \alpha}{p} \mathcal{K} \int_{\mathbb{R}^N} (\nabla \psi_R (x) - \nabla \psi_R (y)) \frac{x - y}{|x - y|^{N - \alpha + 2}} |y|^b|u(y)|^p |x|^b|u(x)|^p \, dy \, dx.\]

Take the sets

\[\Omega := \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N, \ \text{s.t.} \ R < |x| < 2R \ \text{or} \ R < |y| < 2R\};\]

\[\Omega' := \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N, \ \text{s.t.} \ |x| > 2R, |y| < R \ \text{or} \ |x| < R, |y| > 2R\}.\]
Then, by the properties of $\psi$, it follows that

$$\frac{p}{2(N-\alpha)}(M) = \mathcal{K} \int_{\{x < R, |y| < R\}} \frac{|y|^b|u(y)|^p|x|^b|u(x)|^p}{|x-y|^{N-\alpha}} dx dy$$

$$+ \mathcal{K} \int_{\Omega \setminus \Omega'} (\nabla \psi_R(x) - \nabla \psi_R(y)) \frac{x-y}{|x-y|^{N+\alpha}} |y|^b|u(y)|^p|x|^b|u(x)|^p dx dy$$

$$= \mathcal{K} \int_{\mathbb{R}^N \setminus \mathbb{R}^N} \frac{|y|^b|u(y)|^p|x|^b|u(x)|^p}{|x-y|^{N-\alpha}} dx dy$$

$$+ O\left(\int_{\{x > R\} \times \mathbb{R}^N} \frac{|y|^b|u(y)|^p|x|^b|u(x)|^p}{|x-y|^{N-\alpha}} dx dy + \int_{\mathbb{R}^N \setminus \{y > R\}} \frac{|y|^b|u(y)|^p|x|^b|u(x)|^p}{|x-y|^{N-\alpha}} dx dy\right)$$

$$+ O\left(\int_{\{x > R\} \times \{y < R\} \times \frac{|x-y|}{2}} (\nabla \psi_R(x) - \nabla \psi_R(y)) \frac{x-y}{|x-y|^{N+\alpha}} |y|^b|u(y)|^p|x|^b|u(x)|^p dx dy\right)$$

$$+ O\left(\int_{\{x > R\} \times \{y < R\} \times \frac{|x-y|}{2}} (\nabla \psi_R(x) - \nabla \psi_R(y)) \frac{x-y}{|x-y|^{N+\alpha}} |y|^b|u(y)|^p|x|^b|u(x)|^p dx dy\right)$$

$$= \int_{\mathbb{R}^N} (I_\alpha * |\cdot|^b|u|^p)|x|^b|u(x)|^p dx + O\left(\int_{\{x > R\}} (I_\alpha * |\cdot|^b|u|^p)|x|^b|u(x)|^p dx\right)$$

Regrouping previous computations, and using that $\Delta \psi_R(r) = N$ for $r \leq R$,

$$(N) = -\frac{2(N-\alpha)}{p} \int_{\mathbb{R}^N} (I_\alpha * |\cdot|^b|u|^p)|x|^b|u(x)|^p dx + O\left(\int_{\{x > R\}} (I_\alpha * |\cdot|^b|u|^p)|x|^b|u(x)|^p dx\right)$$

$$+ 2\left(\frac{2}{p} - 1\right) \int_{\mathbb{R}^N} \Delta \psi_R(I_\alpha * |\cdot|^b|u|^p)|x|^b|u(x)|^p dx + \frac{4b}{p}(L)$$

$$= -2\left(\frac{N-\alpha}{p} + N(1 - \frac{2}{p})\right) \int_{\mathbb{R}^N} (I_\alpha * |\cdot|^b|u|^p)|x|^b|u(x)|^p dx + O\left(\int_{\{x > R\}} (I_\alpha * |u|^p)|u|^p dx\right)$$

$$+ 2\left(\frac{2}{p} - 1\right) \int_{\{x > R\}} (\Delta \psi_R - N)(I_\alpha * |\cdot|^b|u|^p)|x|^b|u(x)|^p dx + \frac{4b}{p} \int_{\mathbb{R}^N} (I_\alpha * |\cdot|^b|u|^p)|x|^b|u(x)|^p dx$$

$$= -2\frac{Np - N - \alpha - 2b}{p} \int_{\mathbb{R}^N} (I_\alpha * |\cdot|^b|u|^p)|x|^b|u(x)|^p dx + O\left(\int_{\{x > R\}} (I_\alpha * |\cdot|^b|u|^p)|x|^b|u(x)|^p dx\right)$$

$$= -\frac{2sB}{p} \int_{\mathbb{R}^N} (I_\alpha * |\cdot|^b|u|^p)|x|^b|u(x)|^p dx + O\left(\int_{\{x > R\}} (I_\alpha * |\cdot|^b|u|^p)|x|^b|u(x)|^p dx\right).$$

Taking account of Hardy-Littlewood-Sobolev inequality,

$$\begin{aligned}
(I) &:= \int_{\{x > R\}} (I_\alpha * |\cdot|^b|u|^p)|x|^b|u(x)|^p dx \\
&\lesssim ||| \cdot|^b|u|^p||^2_{L^{2N/(N-\alpha)}(\{x > R\})}^{-\alpha N} \\
&\lesssim \left(||| \cdot|^p|u|^p||^{p-\alpha}_{L^\infty(\{x > R\})}||u||^{\alpha N}_{L^\infty(\{x > R\})}\right)^2.
\end{aligned}$$
For $\frac{1}{2} < \mu := \frac{1+\varepsilon}{2} < s < \frac{N}{2}$, by (2.11), one gets

$$
(I) \lesssim \| \cdot \|^2_{L^\infty(\cdot > R)} \| (\nabla)^s u \|^2_{L^\infty(\cdot > R)} \\
\lesssim \left( R^{-N/2 + \mu + b} \| (\nabla)^s u \| \right)^{2(1-1/\mu)} \\
\lesssim \frac{1}{R^{(N-1-2b-\varepsilon)(p-1-2\mu)}} \| (-\Delta)^{\frac{s}{2}} u \|^2_{L^\infty(\cdot > R)} \\
\lesssim \frac{1}{R^{(N-1-2b-\varepsilon)(p-1-2\mu)}} \left( \| u \|^{1-\mu \frac{s}{2}} \| (-\Delta)^{\frac{s}{2}} u \|^{\frac{s}{2}} \right)^{2(1-1/\mu)} \\
\lesssim \frac{1}{R^{(N-1-2b-\varepsilon)(p-1-2\mu)}} \| (-\Delta)^{\frac{s}{2}} u \|^{1+\mu \frac{s}{2}(p-1-\frac{N}{2})},
$$

where we used (2.11) in the second inequality and an interpolation estimate in the fourth one. In summary, one have has

$$
\frac{d}{dt} M_{\psi_R}[u] = <u, [(-\Delta)^s, i\Gamma_\psi] u> + <u, [(I_\alpha * | \cdot | |b||u|^p)|x|^b|u|^{p-2}, i\Gamma_\psi] u> \\
\leq 4s \| (-\Delta)^{\frac{s}{2}} u \|^2 + CR^{-2s} - \frac{2sB}{p} \int_{\mathbb{R}^N} (I_\alpha * | \cdot | |b||u|^p)|x|^b|u(x)|^p \, dx \\
+ O \left( \int_{\{x > R\}} (I_\alpha * | \cdot | |b||u|^p)|x|^b|u|^p \, dx \right) \\
\leq 2sBE - 2s(B-2) \| (-\Delta)^{\frac{s}{2}} u \|^2 + \frac{C}{R^{2s}} \\
+ \frac{C}{R^{(N-1-\varepsilon-2b)(p-1-\frac{N}{2})}} \| (-\Delta)^{\frac{s}{2}} u \|^\frac{1+\mu \frac{s}{2}(p-1-\frac{N}{2})}. 
$$

7. Global/non-global existence of solutions

In this section, a sharp criteria of finite time blow-up/global well-posedness is given. Let us start with an auxiliary result.

**Lemma 7.1.** The following conditions are invariant under the flow of (1.1),

1. (2.8) and (2.9);
2. (2.8) and (2.10).

**Proof.** Using the conservation laws via the sharp Gagliardo-Nirenberg inequality (2.5), one have

$$
E = \| (-\Delta)^{\frac{s}{2}} u \|^2 - \frac{1}{p} \int_{\mathbb{R}^N} (I_\alpha * | \cdot | |b||u|^p)|x|^b|u|^p \, dx \\
\geq \| (-\Delta)^{\frac{s}{2}} u \|^2 - \frac{C_{N,p,b,\alpha,s}}{p} \| u_0 \|^A \| (-\Delta)^{\frac{s}{2}} u \|^B.
$$

Define the quantities

$$
X(t) := \| (-\Delta)^{\frac{s}{2}} u(t) \|^2 \quad \text{and} \quad D := \frac{C_{N,p,b,\alpha,s}}{p} \| u_0 \|^A.
$$
Then,
\[ X - D X^B \leq E, \quad \text{on} \quad [0, T^*). \] (7.13)

The real function defined on \( \mathbb{R}^+ \) by \( f(x) := x - D x^B \), has a local maximum at
\[ x_1 := \left( \frac{2}{DB} \right)^{\frac{2}{B-2}} \]
with a maximum value
\[ f(x_1) = \left( \frac{2}{DB} \right)^{\frac{2}{B-2}} \left( 1 - \frac{2}{B} \right). \]

Taking account of Pohozaev identities,
\[ \|(-\Delta)^{\frac{s}{2}} \phi\|^2 = \frac{B}{A} \| \phi \|^2 \quad \text{and} \quad \int_{\mathbb{R}^N} (I_\alpha * | \cdot |^b | \phi|^p) |x|^b |\phi|^p \, dx = \frac{2p}{B} \|(-\Delta)^{\frac{s}{2}} \phi\|^2. \]

Then,
\[ E(\phi) = \frac{B}{A} \|(-\Delta)^{\frac{s}{2}} \phi\|^2 = \frac{B}{A} \| \phi \|^2. \]

Using the previous relation, the condition (2.8) is equivalent to
\[ E(u_0) < \frac{B}{A} M(\phi)^{\frac{s}{sc}} M(u_0)^{\frac{s-\frac{s}{sc}}{sc}}. \] (7.14)

Moreover, by (2.7),
\[
\begin{align*}
    f(x_1) &= \left( \frac{2}{BD} \right)^{\frac{2}{B-2}} \left( 1 - \frac{2}{B} \right) \\
    &= \left( \frac{2p}{C_{N,p,b,\alpha,s} M(u_0)^\frac{1}{2} B} \right)^{\frac{2}{B-2}} \left( 1 - \frac{2}{B} \right) \\
    &= \left( \frac{A}{B} \right)^{1-\frac{B}{2}} (M(\phi))^{p-1} (M(u_0))^{-\frac{4}{B}} \right)^{\frac{2}{B-2}} \left( 1 - \frac{2}{B} \right) \\
    &= \frac{B}{A} (M(\phi))^{\frac{s}{sc}} (M(u_0))^{\frac{s-\frac{s}{sc}}{sc}}. \quad (7.15)
\end{align*}
\]

The relations (7.14) and (7.15) imply that
\[ E(u_0) < f(x_1), \]
by the previous inequality and (7.13), one has
\[ f(\|(-\Delta)^{\frac{s}{2}} u(t)\|^2) \leq E(u_0) < f(x_1). \] (7.16)

Next, taking account of the previous computations,
\[ x_1 = \frac{B}{A} (M(\phi))^{\frac{s}{sc}} (M(u_0))^{\frac{s-\frac{s}{sc}}{sc}}. \]
1. The condition (2.9) is equivalent to
\[ \|(−Δ)\dot{u}_0\|^2 < \|(−Δ)\dot{ϕ}\|^2 \left( \frac{M(ϕ)}{M(u_0)} \right)^\frac{2−s}{s} \]
\[ < \frac{B}{A} \frac{M(ϕ)}{M(u_0)} \left( \frac{2−s}{s} \right) \]
\[ < x_1. \]

Then by (7.16) and the continuity of \( t \rightarrow \|(−Δ)\dot{u}(t)\| \), one gets
\[ \|(−Δ)\dot{u}(t)\|^2 < x_1 \]
for all time \( t \in [0, T^*] \) which gives (2.9). Thus, the conditions (2.8) and (2.9) are invariant under the flow of (1.1).

2. The condition (2.10) is equivalent to
\[ \|(−Δ)\dot{u}_0\|^2 > \|(−Δ)\dot{ϕ}\|^2 \left( \frac{M(ϕ)}{M(u_0)} \right)^\frac{2−s}{s} \]
\[ > \frac{B}{A} \frac{M(ϕ)}{M(u_0)} \left( \frac{2−s}{s} \right) \]
\[ > x_1. \]

Then by (7.16) and the continuity of \( t \rightarrow \|(−Δ)\dot{u}(t)\| \), one gets
\[ \|(−Δ)\dot{u}(t)\|^2 > x_1 \text{ for all time } t \in [0, T^*] \]
Thus, the conditions (2.8) and (2.10) are invariant under the flow of (1.1).

7.1. **Global well-posedness.** The first part of Theorem 2.7 is a direct consequence of Lemma 7.1. Indeed, in such a case, \( \sup_{t \in [0, T^*]} \|u(t)\|_{H^s} < \infty \).

7.2. **Blow-up.** Let us give an intermediate result which follows as in [12].

**Lemma 7.2.** Assume that \( \frac{1}{2} < s < 1 \), \( E(u_0) \neq 0 \) and there exist \( t_0 > 0 \) and \( \delta > 0 \) such that
\[ M_{ϕ_R}[u(t)] \leq −\delta \int_{t_0}^{t} \|(−Δ)\dot{u}(τ)\|^2 dτ, \quad \forall t \geq t_0. \]
Then, \( T^* < \infty \).

Let us discuss two cases.

1. Case 1: \( E(u_0) < 0 \).

Thanks to the variance identity in Proposition 2.6, for large \( R > 0 \),
\[ \frac{d}{dt} M_{ϕ_R}[u(t)] \leq 2sB E(u_0) − 2s(B − 2)\|(−Δ)\dot{ϕ}\|^2 \]
\[ + C \left( \frac{1}{R^{2s}} + \frac{1}{R^{(N−1−ε−2b)(p−1−\frac{α}{N})}} \right) \|(−Δ)\dot{u}(t)\|^{\frac{1+ε}{2}(p−1−\frac{α}{N})} \]
\[ \leq sBE(u_0) − s(B − 2)\|(−Δ)\dot{ϕ}\|^2, \]
where we discuss \( \| (\Delta)^{\frac{s}{2}} u(t) \| \leq 1 \) or \( \| (\Delta)^{\frac{s}{2}} u(t) \| > 1 \). Integrating in time the previous inequality, \( M_{\psi R}[u(t)] < 0 \), for large time. Thus, integrating in time, one gets

\[
M_{\psi R}[u(t)] \leq -\delta \int_{t_0}^{t} \| (\Delta)^{\frac{s}{2}} u(\tau) \|^2 d\tau, \quad \forall t \geq t_0.
\]

The previous Lemma closes the proof.

2. Case 2: Assume that (2.8)-(2.9) are satisfied.

Take \( \eta > 0 \) such that

\[
E(u_0)^{s c} M(u_0)^{s-s c} \leq [(1-\eta)E(\phi)]^{s c} M(\phi)^{s-s c}
\]

With a direct computation

\[
(1-\eta)(B-2)\| (\Delta)^{\frac{s}{2}} u(t) \|^2 > BE(u_0).
\]

By Proposition 2.6, for \( O_R(1) \to 0 \) uniformly in time,

\[
\frac{d}{dt} M_{\psi R}[u(t)] \leq 2s BE - 2s(B-2)\| (\Delta)^{\frac{s}{2}} u \|^2 + \frac{C}{R^{2s}}
\]

\[
+ \frac{1}{R^{(N-1-\frac{4p-2}{p-1})}} \| (\Delta)^{\frac{s}{2}} u \|^{\frac{1+p}{p}}(p-1-\frac{2}{N})
\]

\[
\leq -2s\eta(B-2)\| (\Delta)^{\frac{s}{2}} u \|^2
\]

\[
+ \frac{1}{R^{(N-1-\frac{4p-2}{p-1})}} \| (\Delta)^{\frac{s}{2}} u \|^{\frac{1+p}{p}}(p-1-\frac{2}{N}) + O_R(1)
\]

\[
\leq \left[ -2s\eta(B-2) + O_R(1) \right] \| (\Delta)^{\frac{s}{2}} u(t) \|^2 + O_R(1)
\]

\[
\leq -s\eta(B-2)\| (\Delta)^{\frac{s}{2}} u(t) \|^2.
\]

The proof follows by Lemma 7.2 via the fact that \( p < 1 + \frac{4}{N} + 2s \).

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