On quantum Hall states at $\nu = 1/3$ and $\nu = 1/5$ for a spinless hollow-core interaction

Thierry Jolicoeur
Laboratoire de Physique Théorique et Modèles statistiques, CNRS,
Université Paris-Sud, Université Paris-Saclay, 91405 Orsay, France

(Dated: March, 2017)

We investigate fractional quantum Hall states for model interactions restricted to a repulsive hard-core. When the hard-core excludes relative angular momentum one between spinless electrons the ground state at Landau level filling factor $\nu = 1/3$ is exactly given by the Laughlin wavefunction. When we exclude relative angular momentum three only, Wojs Quinn and Yi have suggested the appearance of a liquid state with non-Laughlin correlations. We study this special hard-core interaction at filling factors 1/3 and 1/5 on the sphere and on the torus geometry. At $\nu = 1/3$ an analysis of the charged and neutral gaps on the sphere geometry points to a gapless state. On the torus geometry the projected static structure factor has a two-peak feature pointing to one-dimensional density ordering. We conclude that the ground state is likely a compressible stripe. At $\nu = 1/5$ we find that the torus spectrum has a ground state degeneracy which is extensive as in the case of the Haffnian state so is likely also a compressible state. For this filling factor on the sphere we show that there is a unique special polynomial with integer coefficients which is a unique zero-energy ground state. This polynomial obeys a dominance relation in the expansion onto Slater determinants with a special root configuration $110_3(10_4)_e0_611$.

I. INTRODUCTION

The quantum Hall effect is a striking phenomenon in condensed matter physics. It appears as a low-temperature anomaly in the transport properties of some two-dimensional electronic systems. For special values of an applied perpendicular magnetic field the longitudinal resistance goes to zero with an activated law as a function of the temperature and at the same time there is a plateau in the Hall resistance. The one-electron spectrum in these special circumstances consists of Landau levels with macroscopic degeneracy separated by the cyclotron energy. Coulomb interactions between electrons inside lowest-lying Landau levels give rise to a family of incompressible liquid states that are responsible for the fractional quantum Hall effect (FQHE). Theoretical understanding of the most prominent state at filling factor $\nu = 1/3$ of the lowest Landau level (LLL) is based on an explicit many-particle wavefunction due to Laughlin[1]. The composite fermion (CF) theory of Jain[2] is also based on explicit wavefunctions and capture successfully many physical properties of other FQHE states. These wavefunctions are not exact eigenstates of the Coulomb interaction Hamiltonian projected onto the LLL. In the case of the Laughlin state it is known that it is the exact ground state of a hard-core interaction that gives a nonzero energy to state with relative angular momentum one between spinless electrons[3]. The physical relevance of the Laughlin state means that one can adiabatically follow a path in Hamiltonian space between this special hard-core model and the complete Coulomb interaction. No such model is known for the Jain CF wavefunctions. The CF theory explains the appearance of the experimentally prominent series of FQHE states observed for filling factors $\nu = p/(2p \pm 1)$ with $p$ a positive integer. However this does not exhaust the observed incompressible states. For example in the LLL between $\nu = 1/3$ and $\nu = 2/5$ two-dimensional electron gases with high mobility exhibit additional fractions at $\nu = 5/13, 4/11$. There is also a fraction with even denominator in the second Landau level at $\nu = 5/2$ which may very well be the so-called Pfaffian state[3,15]. The Laughlin/CF states are built with some Jastrow-like correlation factors giving them the correct low-energy properties. The range of validity of these correlations is not yet known. Wojs and Quinn[21] have argued that the repulsive potential between electrons should have a special “super-harmonic” dependence on the relative angular momentum.

If we consider that FQHE states between $\nu = 1/3$ and $\nu = 2/5$ are due to condensation of quasiparticles or quasiholes emanating from the parent states then it is not clear what is the effective interaction between the quasiparticles/quasiholes. Notably it may be that they are not of the Laughlin/CF type. Wojs, Yi and Quinn (WYQ) in a series of work[20,23] have suggested that there is an incompressible state at filling factor $\nu = 1/3$ with non-Laughlin correlations. They consider a special hard-core Hamiltonian which gives nonzero energy only for two-body states with relative angular momentum (RAM) three. This may be called a “hollow-core” model since the most repulsive part of the interaction induced by the RAM 1 interaction is artificially set to zero. By use of extensive exact diagonalizations on the sphere geometry they have given evidence for a series of states that are fluid-like e.g. with a ground state with zero total angular momentum and that have several hallmarks of the FQHE states. This series appears for a specific relationship between the number of electrons $N_e$ and the number of flux quanta through the sphere $N_\phi$: $N_\phi = 3N_e - 7$. This is to be contrasted with the Laughlin state which happens for $N_\phi = 3N_e - 3$. The offset 7 vs 3 in the flux-number of particles is called the shift quantum number and is related to the topological properties of the state. This state which does not belong to the CF family has been proposed[12,19] as a candidate to describe some
of the weaker FQHE states at $\nu = 4/11, 5/13$. No candidate wavefunction is known for the WYQ state. Detailed study are required to understand if it is really a new type of FQHE or if it is a state breaking translation symmetry like a stripe or a bubble phase which are not easily discovered on the sphere geometry. It may also be related to quantum Hall nematic states as proposed in a recent study.

In this paper we study the WYQ hard-core model by exact diagonalizations using sphere and torus geometry. We concentrate on filling factors $\nu = 1/3$ and $\nu = 1/5$. For filling factor $\nu = 1/3$ we show that the values of gaps extracted from neutral excitations extrapolate smoothly to zero in the thermodynamic limit from sphere calculations if we stick to the special WYQ shift $N_\phi = 3N_e - 7$. On the torus geometry there is no shift and it possible to compare directly with the Laughlin-like physics by varying the so-called Haldane pseudopotentials. We find a phase transition between the Laughlin state and the WYQ state. This is coherent with recent findings on a related model. The LLL-projected static structure factor of the WYQ state has several peaks indicating the tendency to spontaneous breakdown of translation symmetry as is observed in the bubble phase in higher Landau levels.

For filling factor $\nu = 1/5$ we find evidence for a series of states with $N_\phi = 5N_e - 9$ starting at $N_e = 5$ up to $N_e = 12$ which has an isotropic ground state with zero angular momentum. The shift is again different with that of the Laughlin state at $\nu = 1/5$. In fact for the WYQ hard-core Hamiltonian there is a multiply degenerate ground state with zero energy with the Laughlin shift while if we reduce the flux down to the relation $N_\phi = 5N_e - 9$ the ground state becomes unique and still has full rotational invariance. This special state when expressed in the standard Fock space basis has only integer coefficients for all the accessible sizes we could reach. This does not happen for the WYQ state which has always nonzero energy. The property of integer coefficients is reminiscent of the Jack polynomial, that describe several special states many of them being critical with zero gap in the thermodynamic limit. In the torus geometry the WYQ hard-core Hamiltonian has a set of zero-energy ground states that grows with the system size. This unphysical behavior is the same as the Haffnian state of Read and Green which is thought to be a critical state.

In section II we give the basic formalism needed for the numerical studies. Section III is devoted to the study of the fate of the fraction $\nu = 1/3$ for the pure RAM 3 pure on the sphere and on the torus geometry. Section IV gives our findings for the state realized for $N_\phi = 5N_e - 9$ hence at $\nu = 1/5$. Finally section V presents our conclusions.

II. PSEUDOPOTENTIALS AND POLYNOMIALS

In this work we consider only spin-polarized electronic systems. In the symmetric gauge defined by the vector potential $A = 1/2(\mathbf{B} \times \mathbf{r})$ the LLL basis states can be written as:

$$\phi_m(z) = \frac{1}{\sqrt{2^m m! \pi}} z^m e^{-|z|^2/4 \ell^2},$$

(1)

where $m$ is a positive integer which is disk angular momentum of the state and $\ell = \sqrt{\hbar/eB}$ is the magnetic length. A generic many body state for $N$ electrons is thus of the form:

$$\Psi(z_1, \ldots, z_N) = P(z_1, \ldots, z_N) e^{-\sum_i |z_i|^2/4 \ell^2},$$

(2)

where $P$ is an antisymmetric polynomial. Since the exponential factor is universal i.e. does not depend of the precise state we will omit it in what follows. If we fill all orbitals exactly without any hole using states Eq.(1) for $m = 0, 1, 2, \ldots, N - 1$ we obtain the Vandermonde determinant which can be written as:

$$\Psi_V(z_1, \ldots, z_N) = \prod_{i<j} (z_i - z_j)$$

(3)

If we consider a spinless quantum state in the LLL it is completely antisymmetric and as a consequence one can factor out a Vandermonde factor to obtain an associated symmetric polynomial:

$$\Psi(z_1, \ldots, z_N) = \Psi_V(z_1, \ldots, z_N) \times S(z_1, \ldots, z_N).$$

(4)

The Laughlin wavefunction is defined as power of the Vandermonde determinant:

$$\Psi_L^{(m)} = \Psi_V^m = \prod_{i<j} (z_i - z_j)^m.$$  

(5)

It describes successfully the FQHE state at $\nu = 1/3$ (resp. $\nu = 1/5$) for $m = 3$ (resp. $m = 5$).
A generic two-body interaction Hamiltonian projected onto the LLL can be written as a sum of projectors onto states of definite relative angular momentum $m$:

$$
\mathcal{H} = \sum_{i<j} \sum_{m} V_{m} \hat{\mathcal{D}}_{ij}^{(m)},
$$

where $m$ is a non-negative integer and the coefficients $V_{m}$ are the so-called Haldane pseudopotentials. Spinless fermions are sensitive only to odd values of $m$. The set of pseudopotentials $\{V_{m}\}$ thus completely characterizes the projected interactions. For the physically relevant case of the Coulomb interaction the $V_{m}$ are monotonous and decreasing with large $m$ as $\sim m^{-1/2}$. It has been argued by Wojs and Quinn that FQHE liquids form only when the decrease of the $V_{m}$s with $m$ is quick enough. If we consider the hard-core Hamiltonian $\mathcal{H}_{1}$ with $V_{1} = 1$ and all other pseudopotentials set to zero $V_{m} = 0$, $m > 1$ then $\mathcal{H}_{1}$ has many zero energy eigenstates but the densest such state corresponding to a polynomial of smallest total degree is unique and is given precisely by the Laughlin wavefunction for $m = 3$ : $\Psi_{L}^{(3)}$. Similarly we can construct an Hamiltonian with $\Psi_{L}^{(3)}$ as its exact densest zero-energy state by taking $\mathcal{H}_{1} + \mathcal{H}_{3}$ where $\mathcal{H}_{3}$ has only $V_{3}$ nonzero pseudopotential. In fact any linear combination with positive coefficients of $\mathcal{H}_{1}$ and $\mathcal{H}_{3}$ has this property.

An arbitrary LLL state can be expanded in powers of the relative coordinates for any pair of particles $i, j$:

$$
\phi(z_{1}, \cdots, z_{N}) = \sum_{n,m} A^{(ij)}(\{z_{k}\})(z_{i} + z_{j})^{n}(z_{i} - z_{j})^{m},
$$

where $n, m$ are non-negative integers and the coefficients $A^{(ij)}$ are polynomials in all other variables $\{z_{k}\}$ with $k \neq i, j$. The condition of zero-energy for a hard-core model with a single nonzero pseudopotential $V_{m_{0}}$ means that one has:

$$
\sum_{i<j} \sum_{n} A^{(ij)}(\{z_{k}\})(z_{i} + z_{j})^{n}(z_{i} - z_{j})^{m_{0}} = 0.
$$

Another way to write this condition is to translate it with derivative operators:

$$
\sum_{i<j} \left\{ (\partial_{z_{i}} - \partial_{z_{j}})^{m_{0}} \phi \right\}(z_{1}, \cdots, \frac{z_{i} + z_{j}}{2}, \cdots, \frac{z_{i} + z_{j}}{2}, \cdots, z_{N}) = 0,
$$

where we do the substitution $z_{i}, z_{j} \rightarrow \frac{z_{i} + z_{j}}{2}$ after acting with the derivatives. While theoretically simple this is not the easiest way to compute exact eigenstates. For this purpose we use the sphere geometry and work in second-quantization. Electrons are constrained to move at the surface of a sphere of radius $R = \sqrt{S}$ with $S = N_{\phi}/2$ and the LLL basis is given by:

$$
\Psi_{M}^{(S)} = \sqrt{\frac{2S + 1}{4\pi}} \left( \frac{2S}{S + M} \right) u^{S+M} v^{S-M}, \quad M = -S, \ldots, +S.
$$

where we have introduced the elementary spinors:

$$
u = \cos(\theta/2) e^{i\phi/2}, \quad v = \sin(\theta/2) e^{-i\phi/2}.
$$

The basis states form a multiplet of angular momentum $L = S$. In this geometry the Laughlin wavefunction can be written as:

$$
\Psi_{L}^{(m)} = \prod_{i<j} (u_{i}v_{j} - u_{j}v_{i})^{m}.
$$

When expanded by brute force we find that all coefficients are integers times powers of spinors $u$ and $v$ as is the case of the disk Laughlin state. However this property of integer coefficients is not obvious in terms of the basis states of Eq.10 : one has to remove the square root of the binomial coefficients to find the integers. This is what we do in section IV. The wavefunction Eq.(12) is a singlet of total orbital angular momentum since it involves only combinations of factors $u_{i}v_{j} - u_{j}v_{i}$ which are themselves singlets. We use the spherical geometry in some of our exact diagonalization studies. Hence the eigenstates can be classified by their total angular momentum.

We have performed exact diagonalizations on the torus geometry using the algebra of magnetic translations which allows us to factor out the overall translation invariance. Eigenstates can be classified by two conserved quantum numbers $s, t = 0, \ldots, N$ where $N$ is the GCD of $N_{e}$ and $N_{ph}\hbar$. They correspond to the two-dimensional moment.
FIG. 1: The gaps for the WYQ sequence of states with $N_\phi = 3N_e - 7$ vs. inverse number of particles. Lower graph gives the neutral excitation gaps defined without change of the flux and the excited state may have any orbital angular momentum. Upper graph gives the quasiparticle-quasihole gap defined through addition/removal of one flux quantum. Sizes range from $N_e = 8$ to 14 in the the neutral case and up to $N_e = 13$ in the charged case.

III. THE WOJS-YI-QUINN SERIES $N_\phi = 3N_e - 7$

A. Sphere study

On the sphere geometry incompressible FQHE states have distinct characteristic features. Notably the ground state is a singlet of total orbital angular momentum and there is a gap above this ground state which is large with respect to the finite-size spacing typical of higher-lying levels. This is at least the case for the standard Laughlin state at $\nu = 1/3$ in the LLL and many other FQHE states. In the case of the $\nu = 1/3$ state if we add one flux quantum then the ground state becomes an isolated multiplet with $L = N_e/2$ which is the quasihole. Similarly the quasielectron state involves removal of one flux quantum with respect to the fiducial state following $N_\phi = 3(N_e - 1)$. Wojs, Yi and Quinn have shown by exact diagonalizations up to 12 electrons for the $H_3$ model i.e. with only nonzero pseudopotential $V_3$ that there is also a series of states with essentially the same spectral signatures as the $\nu = 1/3$ FQHE state but with a distinct relation between flux and number of electrons given by $N_\phi = 3N_e - 7$. Even if there is no clear collective mode resembling the magnetoroton, the ground state is well separated from higher lying continuum for all accessible sizes. Taken at face value these results imply the existence of a FQHE state at $\nu = 1/3$ which is topologically distinct from the Laughlin fluid. However one has to check the convergence to the thermodynamic limit. Here we have studied the gap of this system as a function of the number of electrons. The first gap one can define is the lowest excitation energy at $N_\phi = 3N_e - 7$ irrespective of its quantum number. In the standard Laughlin case it is the gap to the minimum of the magnetoroton branch. This neutral gap is displayed in the lower part Fig. (1). One can also define a gap to the creation of a quasielectron-quasihole pair by:

$$\Delta_N = E_0(N_\phi + 1) + E_0(N_\phi - 1) - 2E_0(N_\phi),$$

where $E_0(N_\phi)$ is the ground state of the system with $N$ electrons at flux $N_\phi$. This gap when nonzero in the thermodynamic limit signals a cusp in the energy as a function of density. It is given by the upper curve in Fig. (1).

To write down an explicit trial wavefunction for the WYQ state one has to remove Jastrow-type factors out of the Laughlin wavefunction changing the shift but without changing the total filling factor. A way to do this can be found in the CF construction of wavefunctions. In this theory a composite fermion is a bound state of an electron and two quantized vortices. The vortex attachment reduces the flux felt by the electron and we have $N^*_\phi = N_\phi - 2(N_e - 1)$ on the sphere. The particles now occupy effective Landau levels called ΛLLs and not simply the LLL because they feel this reduced effective magnetic flux. Filling an integer number $p$ of these ΛLLs leads to the FQHE states at electron filling factor $\nu = p/(2p + 1)$. Implicit in this reasoning is the minimization of some kind of mean-field energy given by the effective cyclotron energy governing the spacing between the ΛLLs. If we relax this mean-field type of reasoning and just consider the algebraic machinery alone it is possible to fill only an excited Λ level with CFs and leaving empty the lower-lying ΛLLs. Certainly this does not lead to low-energy states when using the Coulomb interaction.
FIG. 2: The pair correlation function obtained from the WYQ state by exact diagonalization with $N_e = 10$ electrons is displayed in red. A composite fermion trial wavefunction with the correct shift and filling factor gives a very different type of correlations blue curve. Both calculations are performed on the sphere geometry.

However it is not immediately clear what happens with the hard-core $\mathcal{H}_3$ interaction. This procedure of filling only one higher-lying ALL indeed changes the shift but not the filling factor. If we fill only the second ALL one has a state with shift 5 and filling only the third ALL gives the WYQ shift of 7. Such states are by construction orbital singlets. So we consider a trial wavefunction:

$$\Psi_t = P_{\Lambda LLL} \Phi_2 J^2,$$

where the Jastrow correlation factor $J$ is the Vandermonde determinant on the sphere:

$$J = \prod_{i<j} (u_i v_i - u_j v_j).$$

In this equation $\Phi_2$ is a Slater determinant for the $n = 2$ ALL only and $P_{\Lambda LLL}$ is the projection operator onto the LLL. To perform this projection in an efficient way we have used the technique introduced by Jain and Kamilla. By construction this state is an orbital singlet with the WYQ relation between flux and number of particles. We have computed the pair correlation function of this state:

$$g(\vec{r}) = \frac{1}{\rho N_e} \sum_{i \neq j} \delta^{(2)}(\vec{r} - \vec{r}_i + \vec{r}_j)$$

where $\rho$ is the density. It may be evaluated by Monte-Carlo sampling. The result is given by the blue curve in Fig. 2. The same pair correlation function for the WYQ state obtained from direct exact diagonalization is given by the red curve in the same figure. While they have both a complex structure they are very different. So we conclude that it is unlikely that the CF wavefunctions can be used to describe the WYQ state at $\nu = 1/3$.

B. Torus study

The geometry used in ED calculations introduces a bias on the states that can be studied. Notably states with broken space symmetries are frustrated on the sphere and are revealed more clearly on the torus. This is known to be the case for the stripe states that appear for half-filling in the $N=2$ Landau level and also for the bubble phase for quarter-filling of $N=2$ also. They are identified by a set of quasi-degenerate ground states that form a one-dimensional lattice in momentum space for stripe phases or a 2D lattice for bubble phases. We have performed ED studies on the torus up to $N_e = 12$ electrons for the $\mathcal{H}_3$ model. In the case of a rectangular unit cell by varying the aspect ratio $a_0$ it is possible to favor states breaking translation invariance. For $0.3 \leq a_0 \leq 1$ there is no evidence for quasi-degenerate states. The ground state remains at $K = 0$ and as in the sphere case there is no well-defined collective excitation mode before reaching a higher-lying continuum of excited states. In addition to spectral signature
FIG. 3: The projected structure factor $S_0[\mathbf{q}]$ drawn above the basal plane $(q_x, q_y)$ for $N_e = 12$ electrons. The unit cell is hexagonal and the ground state is the WYQ state at filling factor 1/3. The correlations have a double ring structure with a small modulation of sixfold symmetry due to the choice of the unit cell. This shape of the guiding center structure factor is resilient to deformation in a rectangular cell till an aspect ratio $a_0 \approx 0.5$.

an important diagnostic quantity is the LLL-projected static structure factor $S_0(\mathbf{q})$ which can be defined through the guiding center coordinates $\mathbf{R}_i$:

$$S_0[\mathbf{q}] = \frac{1}{N_e} \sum_{i \neq j} \langle \exp i\mathbf{q}(\mathbf{R}_i - \mathbf{R}_j) \rangle.$$  \hspace{1cm} (17)

When evaluated for the stripe or bubble phases it has well-defined peaks in reciprocal space corresponding to the ordering wavevectors. The sensitivity to changes in the shape of the unit is also an indication that the state is compressible. This is what we observe in the case of the WYQ state. The projected structure factor computed in the highly symmetric hexagonal cell is given in Fig.(3). It has a prominent two-ring structure. This is very different from the Laughlin state which has strongly damped oscillation beyond a single central ring surrounding the correlation hole. If we distort the cell to a rectangle with aspect ratio 0.4 we find that there are now two well-defined peaks hinting at some form of one-dimensional ordering: see Fig.(4). They persist for $0.3 \lesssim a_0 \lesssim 0.5$ and correspond to ordering wavevectors $\mathbf{q}^*\ell = (3.2, 0)$.

These findings are consistent with a compressible stripe state as the ground state of the WYQ model for filling $\nu = 1/3$. This identification would be complete with the observation of an associated manifold of quasi-degenerate states. Stripe states\cite{20,21,22} have been proposed as solutions of the Hartree-Fock approximation for half-filled Landau levels with Coulomb interactions with Landau level index at least 2. The characteristic wavevector of the stripe then decreases with the LL index. If extrapolated to the LLL the wavevector of the CDW is $q^* \approx 2.33\ell^{-1}$ while our value from exact diagonalization of the hard-core model is $q^* \approx 3.2\ell^{-1}$.

**IV. THE $H_3$ MODEL SERIES OF STATES AT $N_\phi = 5N_e - 9$**

**A. Sphere study**

The electron pairs in the Laughlin wavefunction for $m = 5$ describing the filling factor $\nu = 1/5$ have at least RAM 5 hence $\Psi_L^{(5)}$ is a zero-energy eigenstate of $H_1 + H_3$. For this special Hamiltonian it is the unique zero-energy ground state for total disk angular momentum $L_z = 5N_e(N_e - 1)/2$. However if we consider only the Hamiltonian $H_3$ we are guaranteed that $\Psi_L^{(5)}$ is still a zero-energy eigenstate but there is no reason why it should be unique and in addition it may be also that there are more compact states i.e. with smaller total angular momentum that have also zero energy since one can pile up electrons with RAM 1 and still keep energy to zero. This is exactly what we observe by exact diagonalization. For the Laughlin shift 5 i.e. when $N_\phi = 5(N_e - 1)$ the ground state has zero energy but has a
FIG. 4: The projected structure factor in a rectangular unit cell with aspect ratio \( a_0 = 0.4 \) computed for \( N_e = 12 \) electrons. The two sharp peaks have \( q^*\ell = (3.2, 0) \).

...multiplicity that grows with the number of particles. When reducing the number of flux quanta with respect to this fiducial value we find that the number of zero-energy state decreases till there is a unique such state: see Table[I].

| \( N_\phi \) | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15* |
|------------|----|----|----|----|----|----|----|-----|
| \( N_e=4 \) |    |    |    |    |    |    | 4  |     |
| \( N_e=5 \) | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20* |
| \( N_e=6 \) |    |    |    |    | 1  | 3  | 6  | > 10 |
| \( N_e=7 \) | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25* |
| \( N_e=8 \) |    |    |    |    | 1  | 5  | 11 | > 30 |
| \( N_e=9 \) | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30* |
| \( N_e=10 \) |    |    |    |    | 1  | 5  | 18 | > 40 |
| \( N_e=11 \) | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35* |
| \( N_e=12 \) |    |    |    |    | 1  | 7  | 29 | > 40 |
| \( N_e=13 \) | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40* |
| \( N_e=14 \) |    |    |    |    | 1  | 7  | 39 | > 40 |

TABLE I: The number of zero-energy eigenstates for the \( V_3 \) model as a function of the number of particles and number of flux quanta on the sphere. A dash “−” means that the lowest energy is nonzero. When \( N_\phi = 5(N - 1) \) we find the \( m = 5 \) Laughlin states which is degenerate. The corresponding flux are indicated by a star on the value of \( N_\phi \).

Thus we observe that there is a unique zero-energy ground state for the \( V_3 \) model satisfying \( N_\phi = 5N_e - 9 \) starting from \( N_e = 5 \). We note that multiplication by a Vandermonde square factor lead to a state which has exactly the WYQ relation between flux and number of particles. This state is also an orbital singlet as expected for a fluid state without ordering. This feature exists at least till \( N_e = 11 \). Contrary to the case of Laughlin wavefunction this peculiar state is not flanked by zero-energy quasiholes when adding one extra flux quantum. Adding one flux quantum leads to a state with \( L = N_e/2 \) and a very small but nonzero gap. In fact one needs two additional flux quanta to obtain new states with zero-energy which are now degenerate. For even number of electrons these states are grouped in orbital multiplets with \( L = N_e, N_e - 2, N_e - 4, \ldots, 0 \) each multiplet appearing exactly once and there are extra states with \( L = 0 \). Apart from the extra singlet states this is what one expect from two-particle states built with elementary quasiholes having \( L = N_e/2 \). When \( N_e \) is odd the pattern of states is identical and the lowest total angular momentum of the set of states is now \( L = 1 \).
We now focus on the properties of the unique zero-energy state at \( N_\phi = 5N_e - 9 \). The component of the corresponding eigenvector are all integers after removing the normalization factors of the spherical basis and writing the state in terms of the disk states Eq. (1). In fact the property of having integer coefficients is true also for small size systems \( N_e = 4, N_\phi = 9, 12 \) but the Fock spaces are of very small sizes and it may happen that eigenstates are simple. For example the \( L_{tot} = 0 \) subspaces for \( N_e = 4 \) in Table (I) have dimension only 2. On the contrary the states satisfying \( N_\phi = 5N_e - 9 \) quickly involve huge Fock spaces with growing number of particles and thus the integer decomposition is a non-trivial property. The statement of integer coefficients is quickly limited by the machine precision used in exact diagonalization. In fact to obtain all integers coefficients one has to use quadruple precision already for the state at \( N_e = 6 \) and \( N_\phi = 21 \) which lies in a space of \( L_z \) dimension 2137. In Table (II) we give the first coefficients of the state with \( N_e = 5 \). The left column gives the integers while the right column contains the binary representation of the occupation numbers in the Slater determinant. Since we are dealing with spinless fermions the occupations numbers are only 0 or 1.

Starting from \( N_e = 6 \) particles we find that the polynomial associated to the special state has a dominance property i.e. not all possible occupation number configuration appear in the expansion. Indeed those with nonzero coefficients can be deduced from a root configuration by successive squeezing operations as happens in many known multivariate special polynomials like the Jack polynomials [29–32]. The squeezing operation moves a given particle from angular momentum \( m_1 \) to \( m'_1 \), another particle from \( m_2 \) to \( m'_2 \) with \( m_1 \leq m'_1 < m'_2 < m_2 \) and keeping the “center of mass” intact \( m_1 + m_2 = m'_1 + m'_2 \). The root configuration is 11000000(10000) that we note as 1101000010100011 in a chemistry-like notation. The Laughlin wavefunction at the same filling factor has also a root configuration but which is (10000)1. This root is non-trivial only starting from \( N_e = 6 \) because for smaller number of particles there are no constraints on the configurations apart from the \( L_z \) angular momentum. We have obtained evidence up to \( N_e = 11 \) a value beyond which the zero coefficients starts to be numerically indistinguishable from the nonzero ones.

To discuss the special dominance structure it is convenient to use also a bosonic wavefunction obtained by factoring out a Vandermonde determinant since the state is antisymmetric:

\[
P(z_1, \ldots, z_N) = \prod_{i < j} (z_i - z_j) \times S(z_1, \ldots, z_N) \tag{18}
\]

The antisymmetric \( N \)-body state is expanded on a Slater determinant basis:

\[
P(z_1, \ldots, z_N) = \prod_{i < j} (z_i - z_j) \times S(z_1, \ldots, z_N) = \sum_{\{n_i\}} I_{\{n_i\}} \det\{[z_j^{n_i}]\}. \tag{19}
\]

If we divide out both sides by the Vandermonde determinant we see that the coefficients \( I_{\{n_i\}} \) determine the expansion.
of the symmetric polynomial $S$ onto the Schur basis. The root partition for the bosonic polynomial $S$ is given by:

$$20000100010001 \ldots 10002$$

This is in fact a partitioning of the total degree of the polynomial. For $N_e = 5$ the root function contains the monomial $x_1^{0}x_2^{0}x_3^{6}x_4^{12}x_5^{12}$ and the total degree is $30 = 12 + 12 + 6$. So we have the following partitions:

$$N_e = 5: \ [6, 12, 12] \equiv 30$$

$$N_e = 6: \ [6, 10, 16, 16] \equiv 48$$

$$N_e = 7: \ [6, 10, 14, 20, 20] \equiv 70$$

The total degree of $S$ is given by $\frac{1}{2} N_e (4 N_e - 8)$. It would be interesting to obtain a closed analytic formula for this special state. The numerous known examples suggest that this may be possible. However it is not a bona fide Jack polynomial since it is known that the rotational invariance of the state constrains both the root partition as well as the parameter defining the Jack.

### B. Torus study

If now we study the $H_3$ model on a torus at filling $1/5$ then we know already that there will be at least one zero-energy state at the center of the Brillouin zone $K = 0$ which is the non-degenerate Laughlin state. However we find more zero-energy states in a complex pattern. For $N_e = 5$ we find that there is a threefold degenerate state at the center of the Brillouin zone and also additional zero-energy states at the zone boundaries: see Table (III).

| $K$       | $(0,0)$ | $(0,N/2)$ | $(N/2,0)$ | $(N/2,N/2)$ |
|-----------|---------|-----------|-----------|-------------|
| deg.      | x3      | x1        | x1        | x1          |

TABLE III: The quantum numbers of zero-energy eigenstates for the $H_3$ model. Here we display the case of $N_e = 5$ particles. The two components of the wavevector $K$ are given on the first line in units of $2\pi/L_{x,y}$. The calculation has been done in a rectangular unit cell and is insensitive to the aspect ratio.

| $K$       | $(0,0)$ | $(0,N/2)$ | $(N/2,0)$ | $(N/2,N/2)$ |
|-----------|---------|-----------|-----------|-------------|
| deg.      | x4      | x1        | x1        | x1          |

TABLE IV: location of the zero-energy states for $N_e = 6$ electrons. Same definitions as in Table (III).

| $K$       | $(0,0)$ | $(0,N/2)$ | $(N/2,0)$ | $(N/2,N/2)$ | $(0,N/3)$ | $(N/3,0)$ | $(N/3,N/3)$ |
|-----------|---------|-----------|-----------|-------------|-----------|-----------|-------------|
| deg.      | x7      | x1        | x1        | x1          | x1        | x1        | x1          |

TABLE V: location of the zero-energy states for $N_e = 6$ electrons. Same definitions as in Table (III). There are now states inside the Brillouin zone with zero-energy.

The number of zero-energy states is growing with the number particles in a manner reminiscent of the Haffnian state. This special wavefunction is related to a non-unitary conformal field theory and is presumably gapless. There are several quantum Hall states including the Haldane-Rezayi spin-singlet state, the Gaffnian state, that share these properties. They are probably critical points describing quantum phase transitions between different states of matter.
V. CONCLUSIONS

We have studied the hard-core \( H_3 \) model with repulsive interactions only in the RAM 3 channel at the two filling fractions \( \nu = 1/3 \) and \( \nu = 1/5 \). In the former case it has been shown by WYQ that many spectral signatures of an incompressible state are present on the sphere geometry when the shift is taken to be 7. However the scaling of gaps calculated on the corresponding both to neutral and charged excitations points to a compressible state in the thermodynamic limit. This is in agreement with recent calculations on the torus geometry\(^{22}\). To clarify the nature of this state we have computed the correlations of the WYQ state on the torus by using the projected static structure factor. For a hexagonal or square unit cell this quantity has a double ring ring structure unlike that of the Laughlin liquid at the same filling factor. When tuning the aspect ratio of a rectangular cell in the range 0.3 \( \lesssim a_0 \lesssim 0.5 \) the structure factor develops two very sharp peaks indicating the presence of a one-dimensional ordering pattern as is the case of half-filled Landau levels of indices \( N \geq 2 \) for the Coulomb interaction. The characteristic wavevector \( q^* \) is close to the extrapolation of Hartree-Fock results to the LLL case.\(^{20,25}\) This is evidence for the stripe phase proposed recently in a torus study.\(^{20}\) However there is no evidence for the corresponding low-lying manifold of quasi-degenerate states in the spectrum due to translation symmetry breaking. So the situation is not as satisfactory as in the case of stripe phases for half-filled higher Landau levels.\(^{21}\)

In the case of the fraction \( \nu = 1/5 \) we have shown that the sphere geometry has a family of zero-energy states with \( N_\phi = 5N_e - 9 \) defining a set of polynomial wavefunctions with integer coefficients and a dominance relation on these coefficients with a bosonic root configuration given by \( 20_3(10)_3,2_0.2 \). This is reminiscent of special FQHE states based on Jack polynomials. However while this state is likely to be gapless it is not given by a simple Jack polynomial. It is natural to conjecture that it is member of a new family of states generalizing existing known states. Notably it does not belong to the family of states generated by using two-body and three-body projection operators. It is the fact that one allows a hollow-core in the pseudopotentials by setting \( V_1 = 0 \) that leads to the appearance of this new special state. If it can be derived from a conformal field theory then presumably this is a non-unitary theory.\(^{20,23}\)

Acknowledgments

We acknowledge discussions with O. Golinelli and J.-G. Luque. Thanks are due to T. Mizusaki for help with quadruple precision calculations. Numerical calculations were performed under the project allocation 100383 from GENCI-IDRIS-CNRS.

---

1. R. B. Laughlin, Phys. Rev. Lett. 50, 1395 (1983).
2. J. K. Jain, “Composite Fermions” (Cambridge University Press, Cambridge, England, 2007).
3. J. K. Jain, “Composite Fermions” (Cambridge University Press, Cambridge, England, 2007).
4. F. D. M. Haldane, in “The Quantum Hall Effect”, edited by Prange and Girvin 1990.
5. G. Moore and N. Read, Nucl. Phys. B360, 362 (1991).
6. K. Pakrouski, M. R. Peterson, Th. Jolicoeur, V. W. Scarola, C. Nayak, M. Troyer, Phys. Rev. X5, 021004 (2015).
7. A. Wojs and J. J. Quinn, Phys. Rev. B61, 2846 (2000).
8. A. Wojs and J. J. Quinn, Phys. Rev. B66, 045323 (2002).
9. A. Wojs and J. J. Quinn, Phys. Rev. B69, 205322 (2004).
10. A. Wojs, K.-S. Yi, and J. J. Quinn, Phys. Rev. B69, 205322 (2004).
11. A. Wojs, D. Wodziniski, and J. J. Quinn, Phys. Rev. B71, 245331 (2005).
12. John J. Quinn, Arkadiusz Wys, Kyung-Soo Yi, and George Simion, Physics Reports 481, 29 (2009).
13. George E. Simion and John J. Quinn, Physica E: Low-dimensional Systems and Nanostructures, 41, 1 (2008).
14. S. Mukherjee, S. S. Mandal, Y.-H. Wu, A. Wojs, and J. K. Jain, Phys. Rev. Lett. 112, 016801 (2014).
15. S. Mukherjee, S. S. Mandal, A. Wojs, and J. K. Jain, Phys. Rev. Lett. 109, 256801 (2012).
16. S. Mukherjee, J. K. Jain, and S. S. Mandal, Phys. Rev. B90, 121305 (2014).
17. S. Mukherjee and S. S. Mandal, Phys. Rev. B92, 235302 (2015).
18. A. C. Balram, C. Töke, A. Wojs, and J. K. Jain, Phys. Rev. B91, 045109 (2015).
19. A. C. Balram, C. Töke, A. Wojs, and J. K. Jain, Phys. Rev. B92, 205120 (2015).
20. A. C. Balram, Y.-H. Wu, G. J. Sreejith, A. Wojs, and J. K. Jain, Phys. Rev. Lett. 110, 186801 (2013).
21. J. A. Hutasoit, A. C. Balram, S. Mukherjee, S. S. Mandal, A. Wojs, V. Cheianov, and J. K. Jain, e-print arXiv:1605.07324
22. A. A. Koulakov, M. M. Fogler, and B. I. Shklovskii, Phys. Rev. Lett. 76, 499 (1996).
23. M. M. Fogler, A. A. Koulakov, and B. I. Shklovskii, Phys. Rev. B54, 1853 (1996).
24. M. M. Fogler and A. A. Koulakov, Phys. Rev. B55, 9326 (1997).
25. R. Moessner and J. T. Chalker, Phys. Rev. B54, 5006 (1996).
26. S. A. Kivelson, E. Fradkin, and V. J. Emery, Nature (London) 393, 550 (1998).
27. E. Fradkin and S. A. Kivelson, Phys. Rev. B59, 8065 (1999).
N. Regnault, J. Maciejko, S. A. Kivelson, and S. L. Sondhi, e-print arXiv:1607.02178.

G. Fano, F. Ortolani, and E. Colombo, Phys. Rev. B34, 2670 (1986).

F. D. M. Haldane, Phys. Rev. Lett. 55, 2095 (1985).

B. A. Bernevig and F. D. M. Haldane, Phys. Rev. Lett. 100, 246802 (2008).

B. A. Bernevig and F. D. M. Haldane, Phys. Rev. B77, 184502 (2008).

B. A. Bernevig and F. D. M. Haldane, Phys. Rev. Lett. 101, 246806 (2008).

B. A. Bernevig and F. D. M. Haldane, Phys. Rev. Lett. 102, 066802 (2009).

D. Green, Strongly Correlated States in Low Dimensions, Ph.D. thesis, Yale University, New Haven (2001); arXiv:cond-mat/0202455.

N. Read, D. Green, Phys. Rev. B 61 10267 (2000).

M. Hermanns, N. Regnault, B. A. Bernevig, and E. Ardonne, Phys. Rev. B83, 241302 (2011).

N. Read, Phys. Rev. B79, 245304 (2009).

N. Read, Phys. Rev. B79, 045308 (2009).

J. K. Jain and R. Kamilla, Int. J. Mod. Phys. B11, 2621 (1997).

J. K. Jain and R. Kamilla, Phys. Rev. B55, 4895(R) (1997).

F. D. M. Haldane, E. H. Rezayi, and K. Yang, Phys. Rev. Lett. 85, 5396 (2000).

E. H. Rezayi, F. D. M. Haldane, and K. Yang, Phys. Rev. Lett. 83, 1219 (1999).

K. Yang, F. D. M. Haldane, and E. H. Rezayi, Phys. Rev. B64, 081301(R) (2001).

B. A. Bernevig, V. Gurarie, and S. H. Simon, J. Phys. A: Math. Theor. 42, 245206 (2009).

F. D. M. Haldane, E. H. Rezayi, Phys. Rev. Lett. 60, 956 (1988).

V. Gurarie, M. Flohr, and C. Nayak, Nucl. Phys. B498, 513 (1997).

S. Simon, E. H. Rezayi, and N. R. Cooper, Phys. Rev. B75, 075318 (2007).