Unified Description of Macroscopic Quantum Forces

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Abstract

We introduce a unified description of Casimir and Casimir-Polder forces between classical objects in quantum field theory (QFT). We focus on interactions mediated by a scalar field. We first show that the quantum work felt by an arbitrary (either rigid or deformable) classical body is finite upon requiring conservation of matter. Using our formulation, we explicitly show how the complete QFT prediction for the quantum pressure inside the Dirichlet sphere is finite, thereby solving a long-standing problem. We then show that our general result interpolates between Casimir and Casimir-Polder forces for arbitrary rigid bodies. We provide the expressions for the generalised Casimir force in the simple cases of plate-plate and plate-point geometries. In the latter geometry, we show how to compute phase shifts observable in atomic interferometry, induced by the generalized quantum force.

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1 Introduction

Why another paper on Casimir forces? Remarkably, even though the seminal works are from the nineteen fifties [1, 2], the topic of Casimir forces is still very much active and outstanding questions remain so far unanswered (see e.g. [3] for a review).

In this work, we address the following two problems. The first one is the description of the Casimir and Casimir-Polder forces in a unified framework. The two effects can be defined as follows. The Casimir force is the force between extended bodies induced in the presence of the quantum vacuum. The Casimir-Polder force is the force induced by a quantum loop exchanged between two point sources. Both effects are purely quantum and relativistic in essence. Many results are known for both phenomena: the Casimir force has been evaluated for a variety of geometries, fields, and dimensions (see e.g. [4] for a review). The Casimir-Polder force has been evaluated for a variety of fields and effective coupling to the point sources (see e.g. [1, 5–9]). But what is the precise relation between these two effects?

By integrating both point sources of the Casimir-Polder force over extended regions, one might naively expect to recover the corresponding Casimir force between the corresponding bodies — but this expectation fails. We can understand such a mismatch as a signal that the Casimir and Casimir-Polder forces, as defined above, must arise as limits taken from a more fundamental object. In this paper we provide, in the case of a scalar theory, the master formula that unifies Casimir and Casimir-Polder forces.

The second outstanding problem is the presence of divergences in calculations of Casimir forces in certain geometries. The presence of divergences in QFT calculations is commonplace, although they usually signal the need for renormalization by local counterterms, and ultimately that the coupling constants of the theory become scale-dependent. In contrast, in Casimir calculations, seemingly “unremovable” divergences have been consistently reported in the predictions of physical quantities which should be finite. Such divergences are not removable using the usual introduction of local counterterms. The presence of such divergences has led to various partial solutions, of both conceptual and technical nature, and sometimes an in-depth questioning of standard QFT calculations [3, 10–13]. In this work we claim that we bring a new and satisfactory answer to the puzzles raised by the existence of these divergences, using only conventional quantum field theory techniques.

The paper is arranged as follows. Section 2 presents the most general framework for quantum forces, giving a formula for the quantum work valid at the non-perturbative level for arbitrary deformable sources. Section 3 specialises to the weak coupling case (which includes effective field theory). The quantum work in the limiting case of thin-shell geometry is further evaluated. Section 4 specialises to two rigid bodies, showing that Casimir and Casimir-Polder forces are asymptotically recovered as limits of our unifying

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Our calculations bear some technical similarities with those presented in these references. For example the source we consider has finite density, and the Dirichlet limit is recovered by sending the density to infinity. A similar approach was taken in [13]. The point-splitting regularisation method is used in [3]. However, both our general approach and the subsequent conclusions differ from these references.
formula for quantum forces. The Casimir pressure on the Dirichlet sphere is revisited and shown to be finite in section 5. The generalized Casimir forces in plane-plane and point-plane geometry are respectively computed in sections 6, 7. In an appendix, we compare a computation of the Casimir-Polder interaction from first principle and our derivation, showing explicitly that they coincide.

Definitions and Conventions

We assume \( d + 1 \)-dimensional Minkowski spacetime \( \mathcal{M}_{d+1} \) with mostly-minus signature \((+,−,\ldots,−)\). The \( d + 1 \) Cartesian coordinates are denoted by \( x_\mu \), spatial coordinates are denoted by \( x_i \equiv x \). We will be considering a source \( J(x) \) with arbitrary shape and dimension. The support of the source is described by the indicator function \( 1_{J}(x) \) or equivalently using a continuous support function \( l(x) \) which is positive where \( J \) is supported and negative where it is not, with \( 1_{J}(x) = \Theta(l(x)) \) where \( \Theta \) is the Heaviside distribution.

The boundary of the source is denoted by \( \partial J = \{x \in \mathcal{M}_{d+1} | l(x) = 0\} \). Integration over the support of the source \( J \) is denoted \( \int_{J} dx^{d+1} \). Integration over the support of the boundary \( \partial J \) is denoted \( \int_{\partial J} d\sigma(x) \).

2 Unified Description of Quantum Forces

In this section we compute the quantum work felt by a source bilinearly coupled to a quantum field under an arbitrary deformation of the source. We focus on a scalar field \( \Phi \) for simplicity — generalization to spinning fields is identical although more technical. The bodies subject to the Casimir forces are assumed to be classical and static. The set of bodies is collectively represented in the partition function by a static source term \( J(x) \).

More precisely, the \( J(x) \) distribution corresponds to the expectation of the density operator \( \hat{n}(x) \) in the presence of matter, \( J(x) = \langle \Omega | \hat{n}(x) | \Omega \rangle \).

2.1 Action and Quantum Vacuum Energy

We consider the fundamental Lagrangian

\[
\mathcal{L}[\Phi] = \frac{1}{2} (\partial_{M} \Phi)^2 - \frac{1}{2} m^2 \Phi^2 + \ldots .
\]  

(2.1)

The ellipses include possible interactions of \( \Phi \), which do not need to be specified. The interacting theory for \( \Phi \) can either be renormalizable — with either weak or strong coupling, or may also be an effective field theory (EFT) involving a series of operators of arbitrary dimension. In this latter case the theory is weakly coupled below the EFT cutoff scale on short distances larger than \( \Delta x \sim \frac{1}{\Lambda} \). The scale \( \Lambda \) is the energy cut-off of the theory.

We consider the partition function in Minkowski spacetime

\[
Z[J] = \int D\Phi e^{i(S[\Phi]−\int d^{d+1}x \mathcal{L}(\Phi)J(x))}
\]  

(2.2)

\[\text{We call the generating functional } Z[J] \text{ the partition function in analogy to the Euclidean case.}\]
where $\mathcal{B}[\Phi]$ is a bilinear operator in $\Phi$. This operator can encode an arbitrary number of field derivatives. We distinguish two cases. If the bilinear operator has no derivative, then the scalar theory can be renormalizable. In this case we write the operator as

$$\mathcal{B}_m[\Phi] = \frac{1}{2\Lambda} \Phi^2$$  \hspace{1cm} (2.3)

If the bilinear operator has derivatives, then the scalar theory is an EFT and in general contains a whole series of higher dimensional operators. In this case, including an arbitrary number of such terms, we can write the operator as

$$\mathcal{B}_{\text{EFT}}[\Phi] = \sum_{n>1,i} \frac{1}{2\Lambda^n} (D_{1,n,i}\Phi)(D_{2,n,i}\Phi)$$  \hspace{1cm} (2.4)

where the $D_{1,2}$ are combinations of derivatives and of d’Alembertians.

Since the source is static, the partition function takes the form

$$Z[J] = e^{-iE[J]T}$$  \hspace{1cm} (2.5)

where $E[J]$ is referred to as the quantum vacuum energy and $T$ is an arbitrary time interval specified in evaluating the time integrals. This time scale will drops from the subsequent calculations. In general, we can set $J$ as an abstract quantity that can be used to generate the correlators of the theory. Taking functional derivatives of $E[J]$ in $J$ generates the connected correlators — here built from correlators involving the composite operator $\mathcal{B}$. In this work, we consider that $J$ represents a physical distribution of matter, i.e. $J(x)$ is taken to be the expectation value of the density operator $\hat{n}(x)$ in the presence of matter, $J(x) = \langle \Omega | \hat{n}(x) | \Omega \rangle$. For concreteness, one can for instance think of a nonrelativistic fermion density, appearing for example via $\bar{\psi}\psi = n(x)$ or $\bar{\psi}\gamma^\mu\psi = \delta^{\mu0} n(x)$ in the relativistic formulation.

### 2.2 The Source and its Deformation

The source is parametrized by

$$J(x) = n(x) 1_J(x)$$  \hspace{1cm} (2.6)

and corresponds to a particle number distribution of mass dimension $d$. The support of this distribution is encoded in $1_J(x) = \Theta(l(x))$ where the continuous function $l(x)$ is positive where $J$ is supported and negative where it is not. The number density $n(x)$ is in general an arbitrary distribution over the support. The integral $N_J = \int d\mu_i J(\mu_i)$ amounts to the total particle number of the source.

We then introduce a deformation of the source. We assume that matter is deformable i.e. both the support and the number density can vary under the deformation. We will see that such a generalization from rigid to deformable matter is necessary in order to ensure that the calculation is well-defined and that no infinities show up.

\[\text{In the sections on plate-plate and plate-point interaction, we will take the source as the matter density. This case will be treated explicitly where needed.}\]
The infinitesimal deformation of the source is parametrized by a scalar parameter $\lambda$. Under our assumptions the source depends on the deformation parameter as $J_\lambda(x) = n_\lambda(x)\Theta[l_\lambda(x)]$. The deformed source takes the form $J_{\lambda+d\lambda}(x) = n_{\lambda+d\lambda}(x)\Theta[l_{\lambda+d\lambda}(x)]$. The deformation of the support of the source is parametrized by

$$l_{\lambda+d\lambda}(x) = l_\lambda(x - L(x)d\lambda) \quad (2.7)$$

where the $L$ vector is the deformation flow. Defining $\frac{\partial}{\partial \lambda} \equiv \partial$ the variation of the source under the $\lambda$ deformation is then given by $\partial_\lambda J_\lambda(x) = \partial_\lambda n_\lambda \Theta[l_\lambda(x)] - n_\lambda L \cdot \partial l_\lambda \delta[l_\lambda(x)]$. An arbitrary deformation of a generic source is pictured in Fig. 1.

Let us make a critical observation. We assume that $J(x)$ is made out of classical matter and is not a completely abstract distribution. As the source is made of classical matter, then its local number density must be conserved. Any deformation of the source must be subject to the conservation of the number density. The local conservation equation under the deformation parametrized by $\lambda$ is

$$\partial_\lambda n_\lambda + \partial \cdot (n_\lambda L) = 0 \quad (2.8)$$

It implies the integral form

$$\partial_\lambda \int_{J_\lambda} d^d x \ n_\lambda(x) = \int d^d x \ \partial_\lambda J_\lambda(x) = 0 \quad (2.9)$$

where the second integral is over all space. The case of a rigid source is recovered for $n$ constant in $\lambda$ and $x$, in which case Eq. (2.8) reduces to the condition of an incompressible deformation flow, $\partial \cdot L = 0$.

### 2.3 Quantum Work

We now study how the quantum system evolves upon a general, infinitesimal deformation of the source, $\lambda \to \lambda + d\lambda$. To proceed we introduce the quantum work under variation in

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5The premise of identifying the source as physical matter was also formulated in [13]. However in this work conservation of matter was not taken into account and the subsequent conclusions drawn in this reference differ from those of our work.
\[ W_\lambda = -\partial_\lambda E[J_\lambda]. \] (2.10)

In the particular case of the source being a rigid body and if the deformation field can be factored out, then a quantum force can be defined as \( W_\lambda = \mathbf{L} \cdot \mathbf{F} \). However this is not possible in general, and the most fundamental quantity to consider is the quantum work.

The quantum vacuum energy \( E[J] \) is a formally divergent quantity. However, if one varies it with respect to a physical parameter, the resulting variation is a physical observable and thus must be finite. Hence even though \( W_\lambda \) encodes all the quantum effects felt by the source(s), the only divergences remaining in this quantity are those with a physical meaning, i.e. the ones which must be treated in the framework of renormalization. One origin for such divergences are the field interactions in perturbative calculations. Another origin for divergences can be the curvature of spacetime, as pointed out in [14] in AdS. Thus, as we will work in Minkowski spacetime at the 1-loop level involving only diagrams with no interactions, no renormalization will be necessary and the quantum work given by Eq. (2.10) will be finite. We explicitly prove this in the next subsection.

Using the definition of Eqs. (2.2), (2.5), the quantum work is given by

\[ W_\lambda = -\int d^dx \langle B \rangle_{J_\lambda}(x,x) \partial_\lambda J_\lambda(x) \] (2.11)

where \( \langle B \rangle_{J_\lambda} \) is the time-ordered quantum average of \( B \) in the presence of the source \( J_\lambda \). Here and throughout we use the shortcut notation \( W_\lambda = W_\lambda, \langle B \rangle_{J_\lambda} = \langle B \rangle_{\lambda} \). While all quantities depend on \( \lambda \), the \( \lambda \) dependence is relevant only under the \( \partial_\lambda \) variation and is often dropped elsewhere. The two-point function in presence of the source at coinciding points will be sometimes denoted by \( \langle B \rangle_J \) in the rest of the paper.

Using the general form of the bilinear coupling, \( B \) is expressed in terms of the two point function of \( \Phi \) in the presence of \( J \), \( \langle T \Phi(x)\Phi(y) \rangle_J \), evaluated at coinciding points. We assume that the classical value of \( \Phi \) is zero.\(^6\) Hence the disconnected part of the two-point function vanishes and the two-point function reduces to \( \langle T \Phi(x)\Phi(y) \rangle_J = \Delta_J(x,y) \), where \( \Delta_J(x,y) \) is the Feynman propagator in the presence of the source \( J \). The quantum average of \( B \) is then expressed in terms of the Feynman propagator \( \Delta_J \) as

\[ \langle B_m \rangle_J = \frac{1}{2\Lambda} \Delta_J(x_1,x_2) \] (2.12)

\[ \langle B_{\text{EFT}} \rangle_J = \sum_{n>1,i} \frac{1}{2\Lambda^n} D_{1,n,i} D_{2,n,i} \Delta_J(x_1,x_2) \] (2.13)

The general formula for the quantum work presented in Eq. (2.11) is valid at the non-perturbative level.

\(^6\)This can be the consequence of a \( Z_2 \) symmetry, enforcing that \( \Phi \) only appears bilinearly in the action such that \( \langle \Phi \rangle = 0 \). In the presence of nonzero \( \langle \Phi \rangle \), the possible \( Z_2 \) symmetry is broken, implying that at weak coupling the fluctuation of \( \Phi \) over the \( \langle \Phi \rangle \) background has a linear coupling to the source. As a result a classical force is also present in addition to the quantum force. This case does not need to be investigated in the present paper. This extra classical force appearing in the presence of \( \langle \Phi \rangle \neq 0 \) does not automatically dominate over the quantum force — instead, various regimes arise as a function of the geometry. These aspects have been partly investigated in [15].
2.4 $W_\lambda$ is Finite (Renormalizable Case)

The quantity $\langle B \rangle_J(x,x)$ is formally divergent since it contains the propagator at coinciding points. It is thus not obvious why the quantum work $W_\lambda$ should be finite. Throughout this subsection, we regulate the divergence in $\langle B \rangle_J$ by introducing a small splitting of the endpoints, $\langle B \rangle_J(x,x) \equiv \langle B \rangle_J^\epsilon(x,x)\vert_{\epsilon \to 0}$ where we defined $x_\epsilon = x + \epsilon$.\footnote{The analogous regularization in Fourier space is momentum cutoff, $p < \frac{1}{\epsilon}$. These regularizations admit a physical meaning. $\epsilon$ can be thought as the distance scale below which the description of the classical matter as a continuous distribution breaks down. In this view, the cutoff length $\epsilon$ has a physically meaningful value, and the statement of existence of a divergence turns into the statement of the $\epsilon$-dependence of the result.}

We consider the renormalisable case i.e. the $\langle B \rangle_m$ operator (the EFT case is addressed in section 3.1). We assume that $n(x)$ is finite for any $x$ over the support of $J$ (the $n \to \infty$ limit is discussed in section 3.2). No assumption on the interactions of $\Phi$ is necessary thus $\Phi$ can be strongly coupled.

Under these conditions we prove finiteness of the quantum work as follows. In the $\epsilon \to 0$ limit, we can decompose the expectation value as the sum of i) the $\epsilon$-dependent, would-be divergent term, and ii) a finite term in which the $\epsilon$ dependence amounts to $O(\epsilon)$ corrections that can be neglected for $\epsilon \to 0$. This gives

$$\langle B \rangle_J^\epsilon(x,x) = \langle B \rangle_J^\text{div}^\epsilon(x,x) + \langle B \rangle_J^\text{fin}^\epsilon(x,x) + O(\epsilon). \quad (2.14)$$

We then use the assumption that the number density is finite. It implies that the effective mass of $\Phi$ inside the source, $m^2 + \frac{n(x)}{\Lambda}$, is finite. The divergence arises from the short distance propagator contained in $\langle B \rangle(x,x_\epsilon)$. As the mass term is a relevant operator with finite value, it is negligible in the short distance limit of the propagator.

Since the source $J$ influences the propagator only via the effective mass term, we conclude that the divergent term is independent of the source. Furthermore, in that short distance limit the propagator is Lorentz invariant and we conclude that the divergent piece in Eq. (2.14) is independent of $x$,

$$\langle B \rangle_J^\epsilon(x,x_\epsilon)\vert_{\text{small }\epsilon} = \langle B \rangle_J^\text{div}^\epsilon(0,0) \equiv \langle B \rangle_J^\epsilon. \quad (2.15)$$

where the result is only dependent on $\epsilon$ and diverges in the $\epsilon \to 0$ limit. Using the decomposition Eq. (2.14) and the definition of the quantum work, we obtain the decomposition

$$W_\lambda = W_\lambda^\text{fin} + W_\lambda^\text{div} \quad (2.16)$$

with

$$W_\lambda^\text{fin,div} = -\int d^d x \langle B \rangle_J^\text{fin,div}(x,x)\partial_\lambda J_\lambda(x). \quad (2.17)$$

In the divergent piece, $\langle B \rangle_J^\epsilon$ factors out of the integral because it is independent of $x$. This gives

$$W_\lambda^\text{div} = -\langle B \rangle_J^\epsilon \int d^d x \partial_\lambda J_\lambda(x). \quad (2.18)$$
The remaining integral corresponds exactly to the variation of the total density of the source under the deformation, appearing in the integral form of the conservation equation Eq. (2.9). Thus if the equation of conservation Eq. (2.9) is satisfied, then Eq. (2.18) vanishes.

We conclude that, upon conservation of matter in the source, for any deformation and finite \( n \) the quantum work is finite:

\[
W_{\lambda}^{\text{div}} = 0 \quad \text{(Finiteness Theorem)} \quad (2.19)
\]

This is true at the nonperturbative level.

The finite part of the quantum work can be put in the useful alternative form by evaluating the integrand, using the divergence theorem and using the conservation equation,

\[
W_{\lambda}^{\text{fin}} = -\int d^d x \ n_\lambda(x) \mathbf{L} \cdot \partial [(\mathcal{B})\lambda(x, x)] . \quad (2.20)
\]

This is another way to verify that any constant piece of \( \langle \mathcal{B} \rangle(x, x) \) drops from the quantum work since it is under a gradient.

In the particular case of a rigid source, Eq. (2.18) reduces to

\[
W_{\lambda}^{\text{div}}|_{\text{rigid}} = -n \langle \mathcal{B} \rangle^{\text{div}} \int_J d^d x \partial \cdot L(x) . \quad (2.21)
\]

We conclude that, upon conservation of matter in the source, for any divergent-free deformation flow and finite \( n \), the quantum work is finite:

\[
W_{\lambda}^{\text{div}}|_{\text{rigid}} = 0 \quad \text{(Finiteness Theorem, Rigid case)} \quad (2.22)
\]

### 3 Weak Coupling: Finite Quantum Work and Thin Shell Limit

At weak coupling the \( \Phi \) field has an equation of motion (EOM) that we can use to evaluate the quantum work. We introduce the bilinear operator \( \mathcal{B}'' \), defined by

\[
\mathcal{B} = \frac{1}{2} \Phi(x) \mathcal{B}'' \Phi(x) . \quad (3.1)
\]

This is the operator that appears in the EOM. For example, when applied to \( \mathcal{B}_m \) this is \( \mathcal{B}_m'' = \frac{1}{m} \). In the EFT case \( \mathcal{B}'' \) is the differential operator appearing in Eq. (2.4).

At weak coupling, at leading order in the perturbative expansion, the \( \Delta_J(x, x') \) propagator satisfies the equation of motion

\[
\mathcal{D}_x \Delta_J(x, x') + \mathcal{B}'' J(x) \Delta_J(x, x') = -i\delta^{d+1}(x - x') \quad (3.2)
\]

where \( \mathcal{D} = \Box + m^2 \) is the wave operator and \( \Box \) is the scalar d’Alembertian. The solution to Eq. (3.2) is a Born series that describes the bare propagator \( \Delta_0 \) (i.e. \( \Delta_J|_{J \rightarrow 0} \)) dressed by insertions of \( \mathcal{B}'' J \). For convenience we define the insertion

\[
\Sigma(x, y) = -i\mathcal{B}'' J(x)\delta^{d+1}(x - y) \quad (3.3)
\]
and we use the inner product $f \ast g = \int d^{d+1}u f(u)g(u)$. With these definitions the dressed propagator is given by

$$
\Delta J(x, x') = \sum_{q=0}^{\infty} \Delta_0 [\ast \Sigma \ast \Delta_0]^q (x, x') (3.4)
$$

$$
= \Delta_0(x, x') - \int d^{d+1}u \Delta_0(x, u)iB''J(u)\Delta_0(u, x') + \ldots \quad (3.5)
$$

Putting this result back into Eq. (2.11) provides the leading, one-loop contribution to the quantum work,

$$
W^{1-\text{loop}}_\lambda = -\frac{1}{2\Lambda} \int d^d xB'' \sum_{q=0}^{\infty} \Delta_0 [\ast \Sigma \ast \Delta_0]^q (x, x) \partial_\lambda J(x) \quad (3.6)
$$

This is valid for both $B_m$ and $B_{\text{EFT}}$ insertions. In terms of Feynman diagrams Eq. (3.6) is simply a loop with an arbitrary number of insertions of $B''J$ and one insertion of $\partial_\lambda J$. A term of the series is represented (without the $\partial_\lambda$ variation) in Fig. 2.

### 3.1 $W_\lambda$ is Finite (EFT Case)

Our proof of the finitess theorem Eq. (2.19) uses that the effective mass is a relevant operator that is negligible at short distances. In contrast, the insertions from $B_{\text{EFT}}$ correspond to irrelevant operators hence the same reasoning cannot apply — the operators become more important at short distance. The solution to this apparent puzzle is that the EFT in its domain of validity is necessarily weakly coupled, hence instead of using a non-perturbative argument one can use the series representation Eq. (3.6) to prove finiteness.

We will show finitess term-by-term. We single out a term from Eq. (3.6) and introduce point-splitting, $B''\Delta_0 [\ast \Sigma \ast \Delta_0]^q (x, x_{\epsilon})$, with $x_{\epsilon} = x + \epsilon$. Our goal is to show that the divergent piece in this quantity is independent of $x$. To proceed, we first rewrite the source
as \( J(x) = \int d^d \mu J(\mu) \delta^d(\mu - x) \), such that the term is reexpressed as a loop of the propagator dressed by point sources convoluted with the \( J(\mu) \) distributions,

\[
B'' \Delta_0 [\ast \Sigma \ast \Delta_0]^q (x, x_\epsilon) = (-i)^q \left( \prod_{i=1}^q \int d^d \mu_i J(\mu_i) \int dt_i \right) \prod_{i=0}^q B'' \Delta_0(\mu_i, \mu_{i+1}) \bigg|_{\mu_q=x, \mu_{q+1}=x_\epsilon}
\]

(3.7)

It is understood that one of the block of derivatives in \( B'' \) acts to the left and the other acts to the right. From Eq. (3.7) we isolate the one-loop diagram

\[
L_q[\mu_0, \mu_1, \ldots, \mu_q, \mu_{q+1}] = \left( \prod_{i=0}^q \int dt_i \right) \prod_{i=0}^q B'' \Delta_0(\mu_i, \mu_{i+1}).
\]

(3.8)

This diagram has \( q \) insertions at \( q \) definite spacetime coordinates, while the associated time coordinates are integrated over.

Each of the insertions of the diagram can be located at any value on the support of \( J \).

The diagram diverges when all the endpoints coincide in space — corresponding to infinite loop momentum in Fourier space. The (regularized) divergent piece of \( L_q \) is thus given by

\[
L^\text{div}_q(\mu, \mu_\epsilon) \equiv L[\mu, \mu, \ldots, \mu, \mu_\epsilon].
\]

(3.9)

We then use that \( L_q \) is built from free propagators \( \Delta_0 \), which are Lorentz invariant, hence \( L_q \) is Lorentz invariant. This implies that the divergent piece \( L^\text{div}_q \) is coordinate independent, \( L^\text{div}_q(\mu, \mu_\epsilon) = L^\text{div}_q(0, \epsilon) \equiv L^\text{div}_q \). A divergent loop is pictured in Fig. 3.

Putting back this piece of the loop into the definition of the quantum work Eq. (3.6) gives the divergent piece of the quantum work

\[
W^\text{1-loop, div}_\lambda = -\frac{1}{2\Lambda} \sum_{q=0}^\infty (-i N_J)^q L^\text{div}_q \int d^d x \partial_\lambda J
\]

(3.10)

where we introduced the integrated number density of the source \( i.e. \) the total particle number \( N_J = \int d\mu_i J(\mu_i) \). The remaining integral corresponds exactly to the variation of
the total density of the source under the deformation, appearing in the integral form of the conservation equation Eq. (2.9). Thus if the equation of conservation Eq. (2.9) is satisfied, then Eq. (3.10) vanishes.

We conclude that, upon conservation of matter in the source, for any deformation and finite \( n \) the quantum work is finite:

\[
W_{\lambda}^{1-\text{loop}, \text{div}} = 0 \quad (\text{One-loop Finiteness Theorem}) \tag{3.11}
\]

The rigid version of this finiteness theorem trivially follows, like for Eq. (2.22).

### 3.2 The Thin Shell Limit

So far we have considered a generic source as an arbitrary volume in \( d \)-dimensional space. Here we investigate a subset of sources for which the support is a thin shell approaching a codimension-one hypersurface.

We denote the source by \( J_\eta = n(x)\mathbf{1}_{S_\eta,\lambda}(x) \) where \( \eta \) parametrizes the small width of the shell. For \( \eta \to 0 \) the support of the shell tends to a hypersurface denoted by \( S \). The volume element can be split as

\[
\int_{S_\eta} d^d x = \int_S d\sigma(x) \int_{\text{width}} dx_\perp \tag{3.12}
\]

where the \( x_\perp \) coordinate parametrizes the direction normal to \( S \). The boundary of \( S_\eta \) can also be decomposed as

\[
\partial S_{\eta \to 0} = S_{\text{in}} \cup S_{\text{out}} \tag{3.13}
\]

where \( S_{\text{in}}, S_{\text{out}} \) are the two hypersurfaces bounding the volume enclosed by \( S_\eta \) in the limit \( \eta \to 0 \). The propagator in the presence of the thin shell is denoted by \( \Delta_S(x, x') \). The density can be chosen to scale with \( \eta \) such that it remains finite for \( \eta \to 0 \). This happens if the density scales as \( \eta^n = \text{cst} \). In the following, the deformation of the source is kept arbitrary. All quantities depend on the deformation parameter \( \lambda \). We will drop the \( \lambda \) index when appropriate.

We evaluate the quantum work for this specific class of sources, taking \( \eta \) small but finite. For simplicity we consider the coupling to the source induced via the \( B_m \) operator. Starting from the general expression of the quantum work Eq. (2.11), we evaluate the \( \partial_\lambda J_\lambda \) variation. We use \( \frac{\partial}{\partial \eta} = n_{\text{in}} \) with \( n_{\text{in}} \) the inward-pointing normal vector, then use the divergence theorem \( \int_{\partial S} d\sigma(x)n_{\text{out}} \cdot f(x) = \int_S d^d x \partial \left[ f(x) \right] \) with \( n_{\text{out}} = -n_{\text{in}} \). We obtain

\[
W_{S_\eta,\lambda} = -\frac{1}{2\Lambda} \int_{S_\eta} d^d x \Delta_S(x, x) \partial_\lambda n_\lambda(x) - \frac{1}{2\Lambda} \int_{S_\eta} d^d x \partial \left[ \mathbf{L} n_\lambda(x) \Delta_S(x, x) \right]. \tag{3.14}
\]

We will further simplify the second term by observing that it can be related to discontinuities determined by the equation of motion. In order to proceed we introduce the notation for derivatives acting on either the first or second argument of the propagator, \( \partial_1 \Delta(x, x') \equiv \partial_{x''} \Delta(x'', x') \big|_{x''=x} \), \( \partial_2 \Delta(x, x') \equiv \partial_{x'} \Delta(x, x'') \big|_{x''=x'} \). The coincident-point propagator is regularized via point-splitting using \( x_\epsilon = x + \epsilon \) where the shifted point is taken to belong to \( S \) when \( x \in S \).
In the thin shell limit, the second term in Eq. (3.14) takes the form

$$- \frac{1}{2\Lambda} \eta \int_S d\sigma(x) \left( \partial_1[L(x)n(x)\Delta_S(x,x_e)] + L(x)n(x)\partial_2[\Delta_S(x,x_e)] \right) (1 + O(\eta)) \quad (3.15)$$

We used the volume element Eq. (3.12) and used that the integrand is continuous over \( S_\eta \). In the first term of Eq. (3.15) the vector \( L \) is kept inside the derivative for further convenience.

We will then use the EOM

$$D_x \Delta_S(x,x') + \frac{1}{\Lambda} J_\eta(x) \Delta_S(x,x') = -i\delta^{d+1}(x-x') \quad (3.16)$$

where \( D_x = \Box_x + m^2 \). Each of the terms in Eq. (3.15) can be expressed using an appropriate derivative of the EOM, with the remaining endpoint set to an appropriate value. The first term in Eq. (3.15) is obtained by multiplying the EOM with \( L(x) \) then applying \( \partial_x \).

We then integrate across the normal coordinate and set the remaining endpoint of the propagator \( x' \) to coincide with \( x \) in the transverse coordinates. This gives the identity

$$\int_{\text{width}} dx_\perp \partial_x \left[ L(x)D_x \Delta_S(x,x') \right] \bigg|_{x' - x_e}^{x' \rightarrow x_e} = -\frac{\eta}{\Lambda} \partial_1 \left[ Ln\Delta_S(x,x_e) \right] (1 + O(\eta)) \quad (3.17)$$

After integrating over the transverse coordinates (i.e. applying \( \int_S d\sigma(x) \)), the r.h.s of Eq. (3.17) coincides with the first term of Eq. (3.15). The second term in Eq. (3.15) is obtained by applying \( \partial_{x'} \) to the EOM then contracting with \( L(x') \). The subsequent steps are the same as above. The result is

$$\int_{\text{width}} dx_\perp L(x') \partial_{x'} \left[ D_x \Delta_S(x,x') \right] \bigg|_{x' - x_e}^{x' \rightarrow x_e} = -\frac{\eta}{\Lambda} Ln \partial_2 \left[ \Delta_S(x,x_e) \right] (1 + O(\eta)). \quad (3.18)$$

Finally we use the divergence theorem in the right hand side of both Eqs. (3.17) and (3.18). Replacing these results in Eq. (3.14) we obtain the final form for the quantum work on a thin shell,

$$W_\lambda^S = -\frac{1}{2} \int_{S_{\text{in}} \cup S_{\text{out}}} d\sigma(x) \left( n_i L_j \partial_i^1 \partial_2^j \Delta_S(x,x_e) + n.L D_1 \Delta_S(x,x_e) \right)$$

$$- \frac{1}{2\Lambda} \int_{S_\eta} d^d x \Delta_S(x,x_e) \partial_\lambda n_\lambda(x) + O(\eta) \quad (3.19)$$

The volume term in the second line encodes the variation of the number density of the source under the deformation. We have not used the conservation equation in our derivation of Eq. (3.19). If we assume that the conservation equation is satisfied, then the presence of the density variation term must contribute to ensuring that the quantum work is finite, along the line of the finiteness theorem (2.19). This will be exemplified in Sec. 5.

### 3.3 The Dirichlet Limit

The number density on the hypersurface, \( \eta n = n_S \), can take any value. In the limit \( n_S \rightarrow \infty \), the propagator satisfies Dirichlet boundary conditions on \( S \) obtained when
\( \eta \to 0 \). In this limit the propagator vanishes on \( S \), \( \Delta_S(x,x') = 0 \) for any \( x' \) or \( x \in S \) — while its derivatives do not. This implies that the second surface term in the first line of Eq. (3.19) vanishes in the Dirichlet limit since the second endpoint belongs to the boundary and has no derivative acting on it. In contrast the first surface term involves derivatives on both endpoints and thus does not vanish in the Dirichlet limit. We recognize in this remaining surface term a piece of the scalar stress-energy tensor corresponding to the vacuum expectation value of the momentum flux obtained from \( T^{ij} = \frac{1}{2} \partial^i \phi \partial^j \phi \) across the surface \( S \) with normal vector \( n_i \) in the direction parameterized by \( L_j \). This is, of course, not a coincidence. This contribution to the quantum work reproduces exactly the difference of stress-energy tensors that is used to compute the Casimir forces or pressures on thin shells, see e.g. [10]. Here we have obtained the generalised version of this surface term, which emerges naturally from the quantum work formalism.

3.4 Why is the Casimir pressure on the Dirichlet sphere divergent?

In the literature, the QFT predictions of Casimir forces and pressures sometimes feature divergences while such quantities must be finite since they are physical observables. Attempts to remove such divergences include analytical continuation [10] and renormalization of the energy density inside the source or at its surface [11–13]. In any case, some seemingly “unremovable” divergences remain, and some of the proposed arguments seem somewhat ad hoc. As an example of geometry for which divergences have been reported in the literature, we focus on the much-studied case of the Dirichlet sphere. An expression for the Casimir pressure for a scalar field inside a \( d-1 \)-sphere with Dirichlet boundary conditions on the sphere has been computed from QFT in [10].

Using our analysis of Sec. 2, we can immediately see why the result obtained in [10] contains an infinite piece. On the one hand the sphere is implicitly assumed to be rigid \( i.e. \) its density does not change under deformation. On the other hand the deformation flow describing the radial deformation of the sphere is not divergence-free, \( \partial \cdot L_{\text{Sphere}} \neq 0 \) for arbitrary spacetime dimension \( d \). As a result matter in the source is not conserved under deformations. Hence none of the finiteness theorems, \( i.e. \) (2.19) and (2.22) apply. Accordingly, the obtained expression for the quantum work is infinite for arbitrary \( d \). In the particular case \( d = 1 \), the geometry of the sphere reduces to two point particles in \( 1+1 \) spacetime. In this particular case the deformation flow is divergent-free, \( \partial \cdot L_{\text{Sphere}}|_{d=1} = 0 \). Therefore Th. (2.22) applies and the quantum work must be finite. Consistently with this claim, the infinite piece obtained in [10] vanishes when \( d = 1 \). This is, in a sense, a hint that the “unremovable” divergence arising in [10] has to do with the sources being extended objects.

A meaningful result for the quantum pressure in the \( d-1 \)-sphere cannot be obtained by considering the sphere as rigid \( i.e. \) instead, we must allow for the density to vary such that Th. (2.19) applies. Namely, while the radius of the sphere gets infinitesimally changed, the density has to change accordingly such that the conservation of the number density is ensured. The fact that the number density must change is obscured by taking Dirichlet boundary conditions, which formally amounts to assuming an infinite number density. The key point is that the density, even if taken very large, must be allowed to vary in order to
satisfy the conservation equation. This is physically meaningful in the sense that in the physical world there is no infinite matter density. A convenient way to proceed consistently in order to compute the force is to assume a sphere with finite density, let the density vary appropriately under the deformation and take the large \( n \) limit at the end of the calculation. This is done in section 5 for the sphere, using the results from section 3.2.

4 Unified Description of Casimir Forces

In this section we choose a specific shape for the source and the deformation. The source is assumed to be the compound of two rigid bodies \( J = J_1 + J_2 \) with number densities \( n_{1,2} \). We assume that the \( J_2 \) source moves rigidly with respect to \( J_1 \). The deformation flow \( \mathbf{L} \) thus reduces to a constant vector over \( J_2 \) and vanishes elsewhere. In this particular case the \( \mathbf{L} \) factors out in \( W_\lambda \) and we can talk about the quantum force \( \mathbf{F}_{1\rightarrow 2} \) between \( J_1 \) and \( J_2 \). Using Eq. (3.6) the general expression for the quantum work is expressed as

\[
W_\lambda^{1\text{-loop}} = \mathbf{L} \cdot \mathbf{F}_{1\rightarrow 2} = -\frac{1}{2\Lambda} \int d^d x \sum_{q=0}^\infty \Delta_0 [\ast \Sigma \ast \Delta_0]^q (x, x) \mathbf{L} \cdot \partial J_2(x) \tag{4.1}
\]

where \( \Sigma = -\frac{i}{\Lambda} (J_1 + J_2) \delta^{d+1}(x - y) \). We will then evaluate the general formula Eq. (4.1) in specific limits. An arbitrary term of the series is represented in Fig. 4i.
4.1 Vanishing of tadpoles

In Eq.(4.1), each insertion of Σ contains both \( J_1 \) and \( J_2 \). Let us focus on subterms involving only \( J_2 \). Such terms amount to the generalization of “tadpole” diagrams for an extended source, here \( J_2 \). Using integrations by parts and the fact that the propagators \( \Delta_0 \) in empty spacetime are Lorentz invariant, \( i.e \) are functions of \( u-v \) only, one can check that any such tadpole term is equal to minus itself and thus vanishes. This makes sense since such terms do not involve the \( J_1 \) source at all and should not contribute to the force between the two sources. A tadpole diagram is represented in Fig. 4ii. These diagrams contain divergent contributions to the quantum work (see Fig. 3). Therefore the vanishing of the tadpole diagrams ensures that the perturbative finiteness theorem Eq. (3.11) is satisfied.

4.2 The Casimir-Polder Limit

Let us assume that the values of \( \frac{n_1^2}{\Lambda} \) are small enough such that the leading contributions come from the first terms in the series. The first term of the series has \( q = 0 \). This term amounts to a tadpole diagram, hence it vanishes as shown in Sec. 4.1. We thus turn to the \( q = 1 \) term. This term is

\[
W_{q=1}^{1\text{-loop}} = \frac{i}{2\Lambda^2} \int d^d u \, d^{d+1} v \, \Delta_0(u,v)J(v)\Delta_0(v,u)\mathbf{L} \cdot \partial J_2(u). \tag{4.2}
\]

We then decompose \( J(u) = J_1(u) + J_2(u) \). The \( \int J_2 \Delta_0^2 \partial J_2 \) piece is again a tadpole and thus vanishes as shown in Sec. 4.1 — using integration by parts, one can check that it is equal to minus itself. The remaining term is

\[
W_{q=1}^{1\text{-loop}} = \frac{i}{2\Lambda^2} \int d^d u \, d^{d+1} v \, \Delta_0(u,v)J_1(v)\Delta_0(v,u)\mathbf{L} \cdot \partial J_2(u). \tag{4.3}
\]

Upon integrating by parts (or evaluating \( \partial J_2 \) and using the divergence theorem), we recognize the variation of a bubble diagram that corresponds precisely to the definition of the Casimir-Polder potential \( V_{CP}(R) \) between two point sources. Namely,

\[
W_{q=1}^{1\text{-loop}} = -n_1 n_2 \int d^d u d^d v \mathbf{L} \cdot \partial V_{CP}(u-v) \tag{4.4}
\]

with the potential

\[
V_{CP}(r) = -\frac{i}{2} \int dt \, (\Delta_0(0; r, t))^2 = -\frac{1}{32\pi^3\Lambda^2} \frac{m}{r^2} K_1(2mr). \tag{4.5}
\]

Details of the explicit evaluation in the last step can be found in \( e.g. \) Ref. [9]. What we have obtained in Eq. (4.4) also amounts to the integral over the \( J_{1,2} \) supports of the quantum work between two point sources generated by the directional derivative of the Casimir potential, \( W_{n=1}^{1\text{-loop}} = n_1 n_2 \int d^d u d^d v W_{CP} \) with \( W_{CP} = -\mathbf{L} \cdot \partial V_{CP} \). A diagram in the Casimir-Polder limit is shown in Fig. 4iv.
4.3 The Casimir Limit

A different limit is obtained when the effective mass inside the sources, \( m^2 + \frac{n_{1,2}(x)}{A} \), is large enough for the dressed propagator to be repelled from the sources. This occurs whenever \( D_x \Delta(x, x') \ll J(x) \Delta_J(x, x') \) for any \( x, x' \). The EOM Eq. (3.2) is then \( J(x) \Delta_J(x, x') \approx 0 \), which enforces \( \Delta_J(x, x') \approx 0 \) for any \( x \) in the source and any \( x' \) in the whole space. The propagator vanishes on the boundary, \( \Delta_J(x \in \partial J, x') \approx 0 \), by continuity. Therefore the propagator has Dirichlet boundary conditions in this regime. We refer to this limit as the Casimir limit since it reproduces a Dirichlet problem for which the quantum force is usually referred to as Casimir. A sample diagram from the Casimir limit is shown in Fig. 4iii.

Summarizing, we have shown that our general formula for the one-loop quantum work Eq.(4.1) interpolates between the Casimir-Polder force and the Casimir force. The two limits are realized in different physical regimes, which essentially depend on the competition between the magnitudes of the effective mass and of the d’Alembertian. Qualitatively, we can say that for fixed fundamental parameters and densities, the Casimir-Polder limit emerges in the short separation regime while the Casimir limit emerges in the large separation regime.

5 Example 1: The Dirichlet Sphere, Revisited

The derivation of the Casimir pressure on a spherical shell in the presence of a scalar field with Dirichlet boundary condition is a long-standing problem. An early attempt is [16] and an expression describing the Casimir pressure on a \( d-1 \)-sphere with Dirichlet boundary conditions has been derived in [10], which will be our main reference. The proposed expression features a divergence.

We have explained in section 3.4 why obtaining such a divergence is actually expected in the light of Ths. (2.19), (2.22). Namely, in this calculation the sphere is implicitly assumed to be rigid while the deformation is not divergence-free. Matter in the source is thus not conserved under the deformation, and therefore neither (2.19) nor (2.22) apply. As a result the expression found in [10] cannot be finite unless if \( d = 1 \). In this section we revisit the quantum work on the \( d-1 \)-sphere using our general formalism, obtain a finite result and show how it relates to the result from [10].

5.1 A Review

We first review the result from [10]. The radius of the sphere is denoted by \( a \). The pressure on the sphere obtained in this reference can be put in the form

\[
\frac{F_{S_{d-1},[10]}}{A_{d-1}} = \frac{F_{S_{d-1},[10]}^\text{fin}}{A_{d-1}} + \frac{F_{S_{d-1},[10]}^\text{div}}{A_{d-1}}
\]  

(5.1)
where
\[
\begin{align*}
\frac{F_{S_{d-1}}^{\text{fin}}}{A_{d-1}} &= i \frac{1}{a^d} \sum_{h=0}^{\infty} c_h \int_{-\infty}^{\infty} d\omega \omega \frac{d}{d\omega} \log \left( \omega a J_{h-1+\frac{d}{2}} \left( |\omega| a H_{h-1+\frac{d}{2}}^1 \left( |\omega| \right) \right) \right) \\
\frac{F_{S_{d-1}}^{\text{div}}}{A_{d-1}} &= i \frac{1-d}{a^d} \sum_{h=0}^{\infty} c_h \int_{-\infty}^{\infty} d\omega \\
c_h &= \frac{(h-1+\frac{d}{2}) \Gamma(h+d-2)}{2^{d+1} \pi^{d/2} h! \Gamma(d+\frac{d-2}{2})} 
\end{align*}
\] (5.2)

The $F_{S_{d-1}}^{\text{fin}}$ term is finite for any $d$ while $F_{S_{d-1}}^{\text{div}}$ is infinite except if $d = 1$. $A_{d-1}$ is the area of the $d-1$-sphere with radius $a$. All the terms are real upon rotation to Euclidian time, here for convenience we keep the Lorentzian integrals. The result Eq. (5.1) is obtained based on the difference between the radial component of the stress tensor on each side of the sphere.

In our language, this is equivalently obtained by considering the sphere as a rigid source with infinite density and deforming it along the radial flow $L = e_r$.

5.2 The quantum work on the Dirichlet $d-1$-Sphere

We now proceed with our calculation of the quantum work on the Dirichlet sphere with radial deformation. We first define the source and the deformation. We consider a source with the geometry of a spherical shell of width $\eta$ and with a finite number density $n$ in the $\eta \to 0$ limit,
\[
J_\eta(x) = \frac{n_\lambda}{\eta} 1_{a-\frac{\eta}{2} < r < a+\frac{\eta}{2}}(x)
\] (5.3)

The propagator in the presence of this source is denoted by $\Delta_S(x,x')$. The boundary of the shell for small $\eta$ is identified with $\partial S_{\eta \to 0} = S_{\text{in}} \cup S_{\text{out}}$ where $S_{\text{in/out}}$ are the $d-1$-spheres with respective radii $r = a-, a+$. The deformation of the source is parametrized by $\lambda$ and changes the sphere radius such that $a_\lambda = a_\lambda + Ld\lambda$. Equivalently, in terms of the support function, the deformation is $l_\lambda(r) = l_\lambda(r - Ld\lambda)$. The deformation flow vector is thus $L = Le_r$. Using the conservation equation Eq. (2.8) we can easily derive the variation of density corresponding to such a deformation. We find
\[
\partial_\lambda n = -\frac{d-1}{a} nL.
\] (5.4)

We can now compute the quantum work. Since we are interested in a sphere we can readily use the general formula for the quantum work on a thin shell Eq. (3.19). In the Dirichlet limit the quantum work reads
\[
\begin{align*}
W_{S_{d-1}} &= -\frac{1}{2} A_{d-1} L \left[ \partial_{r_i} \partial_{r_j} \Delta_S(r', t; r'', t)|_{r'=r'' \in S_{\text{out}}} - \partial_{r_i} \partial_{r_j} \Delta_S(r', t; r'', t)|_{r'=r'' \in S_{\text{in}}} \right] \\
&= -\frac{1}{2\lambda} \int_S d^{d}x \Delta_S(x, x) \partial_\lambda n|_{\eta \to \infty}.
\end{align*}
\] (5.5)
The first term in Eq. (5.5) matches precisely the quantity computed in [10]. Namely we find
\[ W_{S_{d-1}} = L \left( F_{S_{d-1},[10]}^{\text{fin}} + F_{S_{d-1},[10]}^{\text{div}} \right) - \frac{1}{2\Lambda} \int_S d\sigma(x) \Delta_S(x,x) \partial_\lambda n_\lambda \bigg|_{n \to \infty}, \] (5.6)
where the components are defined in Eqs. (5.2).

The remaining task is to evaluate the term from the variation of the density. To this end we first have to evaluate the propagator in the presence of the sphere with finite density. This is done by recomputing the propagator in [10], replacing the two Dirichlet boundary conditions on \( S \) by two boundary conditions obtained from integrating the EOM on a shell enclosing \( r = a \) and using the divergence theorem. Introducing the Fourier transform in time \( \Delta(x,x) = \int d\omega \frac{2}{\pi} \Delta_\omega(x,x) \), we find for the propagator at coinciding points
\[ \Delta_\omega(a,a) \xrightarrow{\text{large } n} i \frac{\Lambda}{na^{d-1}} \sum_{h=0}^{\infty} \frac{(h-1+\frac{d}{2})\Gamma(h+d-2)}{2^{d-2}\pi}\frac{1}{h!}\frac{1}{\Gamma\left(h+\frac{1}{2}\right)} = i \frac{\Lambda}{na^{d-1}} \sum_{h=0}^{\infty} 4\pi c_h \] (5.7)
with \( c_h \) defined in Eq. (5.2). Using the variation of density dictated by matter conservation Eq. (5.4) we then obtain
\[ \frac{1}{2\Lambda} \int_S d\sigma(x) \Delta_S(x,x) \partial_\lambda n_\lambda \bigg|_{n \to \infty} = i \frac{1-d}{2a^d} L \sum_{h=0}^{\infty} c_h \int_{-\infty}^{\infty} d\omega = LF_{S_{d-1},[10]}^{\text{div}} \] (5.8)
We see that this contribution from the variation of density exactly cancels the divergent piece in Eq. (5.6). It follows that our final result for the quantum work on the Dirichlet sphere amounts to the finite part of the result from [10], namely
\[ W_{S_{d-1}} = LF_{S_{d-1},[10]}^{\text{fin}}. \] (5.9)

The fact that the term from the variation of density cancels the \( F_{S_{d-1},[10]}^{\text{div}} \) divergence upon requirement of matter conservation is a non trivial feature. This cancellation provides a check for our expression for the quantum work on a thin shell Eq. (3.19) and shows the finiteness theorem at work. Summarizing, our result removes the seemingly “unremovable” divergence \( F_{S_{d-1},[10]}^{\text{div}} \), thereby solving a long-standing puzzle in QFT.

6 Example 2: Force Between two Plates

As a second application we focus on the classic Casimir setup with two plates facing each other and separated by a distance \( \ell \) along the \( z \) axis. The deformation we consider amounts to a variation of \( \ell \).

The quantum field \( \Phi \) is described by the Lagrangian
\[ \mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{m^2}{2} \Phi^2 - \frac{1}{2\Lambda^2} \Phi^2 J(x). \] (6.1)
In this application, it is convenient to define \( J \) to be the mass density distribution of the sources. In this setting there are five regions along \( z \), with the plates supported on region 1 and 3. The source \( J \) is defined as
\[ J(z) = \rho_1 \Theta(-z_\infty < z < 0) + \rho_3 \Theta(\ell < z < z_\infty) \] (6.2)
The width of the plates is taken to be much larger than the separation, $z_\infty \gg \ell$. The fact that the plates actually end instead of continuing to infinity is crucial in order to ensure matter conservation, and thus that the finiteness theorem applies. The effective mass can be written as

$$m^2(z) = m^2_\infty \Theta(z < -z_\infty) + m^2_1 \Theta(-z_\infty < z < 0) + m^2_2 \Theta(0 < z < \ell)$$

$$+ m^2_3 \Theta(\ell < z < z_\infty) + m^2_\infty \Theta(z > z_\infty), \quad (6.3)$$

where

$$m^2_{\infty,2} = \rho_1 \Lambda + m^2, \quad (6.4)$$

and

$$m^2_{1,3} = \rho \Lambda + m^2. \quad (6.5)$$

In Eq. (6.1) the boundary operator amounts to

$$B_m(\Phi) = \frac{\Phi^2}{2\Lambda}. \quad (6.6)$$

Derivative operators such as $B(\Phi) = \frac{(\partial \Phi)^2}{2\Lambda}$ could be treated along the same lines.

The deformation of the source that we consider amounts to shifting the right plate (i.e. region 3, the second term in Eq. (6.2). This corresponds to an infinitesimal shift of $\ell$ and $z_\infty$, $\ell_\lambda + d\lambda = \ell_\lambda + Ld\lambda$, $z_\infty, \lambda + d\lambda = z_\infty, \lambda + Ld\lambda$. Equivalently, in terms of the support function of the right plate, this is described by $l_\lambda + d\lambda(z) = l_\lambda(z - Ld\lambda)$. Since the source moves rigidly, the formula for the quantum work Eq. (4.1) applies and a quantum force can be defined out from the quantum work, $W^{1-loop} = LF_{quant}$. In the following we determine $F_{quant}$.

**Propagator**

The Feynman propagator in position-momentum space, defined by $\Delta(x, x') = \int \frac{d^3p}{(2\pi)^3} e^{ip\alpha(x-x')} \Delta_p(z, z')$ with $(p^\alpha, z)$, $\alpha = (0, 1, 2)$, has been calculated in the presence of a piecewise constant mass in [17]. Defining the $z$-dependent momentum with the Feynman $\epsilon$-prescription

$$\omega(z) = \sqrt{(p_0)^2 - (p_1)^2 - (p_2)^2 + i\epsilon - m^2(z)}, \quad (6.7)$$

the homogeneous equation of motion becomes

$$(\partial_z^2 + \omega^2(z))\Phi(z) = 0 \quad (6.8)$$

whose solutions in a given region $i$ are simply $e^{\pm i\omega_i z}$. The solution everywhere can be found by continuity of the solution and its derivative at each of the interfaces. The propagator is obtained by solving the equations of motion in the five regions and matching them at the boundary. Details can be found in the appendix of [17]. For the calculation of the quantum force, we will only need to evaluate the propagator at coinciding points on the two boundaries of the right-hand plate, $z = \ell$ and $z = z_\infty$. 

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Quantum Force

The deformation of the source term is found to be
\[
\partial_\lambda J = -(\rho_3 - \rho_2) L \left( \delta(z - \ell) - \delta(z - z_\infty) \right).
\] (6.9)

Putting this variation back into the definition of the quantum work, we can factor out the \( L \) term and obtain the quantum force as defined in Eq. (4.1). As a result the quantum force is given by
\[
F_{\text{quant}} = \frac{1}{2} (m_3^2 - m_2^2) \int d^2x_\parallel \left( \Delta(x^\alpha, \ell; x^\alpha) - \Delta(x^\alpha, z_\infty; x^\alpha, z_\infty) \right)
\]
\[
= \frac{1}{2} (m_3^2 - m_2^2) \int d^2x_\parallel \int \frac{d^3p}{(2\pi)^3} (\Delta_p(\ell, \ell) - \Delta_p(z_\infty, z_\infty))
\] (6.10)

with \( x_\parallel = (x_1, x_2) \). The cancellation between the divergent parts of the two propagators at coinciding points is evident in Eq. (6.10), using the fact that the divergence is location-independent. In the second line we have introduced the propagator in position-momentum space. Here we have explicitly
\[
\Delta_p(\ell, \ell) - \Delta(z_\infty, z_\infty) = \frac{(\omega_1 + \omega_2) + e^{2i\ell\omega_2}(\omega_2 - \omega_1)}{(\omega_1 + \omega_2)(\omega_2 + \omega_3) - e^{2i\ell\omega_2}(\omega_2 - \omega_1)(\omega_2 - \omega_3)} - \frac{1}{\omega_2 + \omega_3}. \quad (6.11)
\]

with \( \omega_i = \sqrt{(p_\alpha)^2 + i\epsilon - m_i^2} \).

The surface integral \( \int d^2x_\parallel = S \) is factored out hence defining a pressure. The final expression for the quantum pressure between the two plates (i.e. regions 1 and 3) is then
\[
\frac{F_{\text{quant}}}{S} = \frac{\int_0^\infty dp_\rho^2}{2\pi^2} \frac{\gamma_2(\gamma_2 - \gamma_1)(\gamma_2 - \gamma_3)}{(\gamma_2 - \gamma_1)(\gamma_2 - \gamma_3) - e^{2\gamma_2}(\gamma_1 + \gamma_2)(\gamma_2 + \gamma_3)}
\] (6.12)

after Wick’s rotation with \( \gamma_i = i\gamma_1 = i\sqrt{\rho^2 + m_i^2} \). This is the general expression of the plate-plate quantum pressure in the presence of a scalar coupled quadratically to matter.

In the limit of large density-induced effective mass \( m_{1,3} \to \infty \), the general expression Eq. (6.12) becomes
\[
\frac{F_{\text{quant}}}{S} = \int_0^\infty dp_\rho^2 \frac{\gamma_2}{2\pi^2 \left(1 - e^{2\gamma_2}\right)}.
\] (6.13)

In this limit the effective mass in the plates become so large that the field obeys Dirichlet boundary conditions. Accordingly, Eq. (6.13) matches exactly the Casimir pressure from a massive scalar. For a massless scalar the integral can be explicitly performed and we retrieve
\[
\frac{F_{\text{quant}}}{S} = -\frac{\pi^2}{480\ell^2}. \quad (6.14)
\]

The is the classic Casimir pressure for a massless scalar field.

In the limit of small density-induced effective mass defined as \( (m_{1,3}^2 - m_2^2)/m_3^2 \ll 1 \), i.e. when the contribution of the density to the effective mass is small with respect to the fundamental mass, the pressure becomes
\[
\frac{F_{\text{quant}}}{S} = -(m_1^2 - m_2^2)(m_3^2 - m_2^2) \int_0^\infty dp_\rho^2 \frac{e^{-2\rho \gamma_2}}{2\pi^2 \left(16\gamma_2\right)^3}.
\] (6.15)
We checked that this corresponds exactly to the Casimir-Polder force integrated over regions 1 and 3. We will see how this calculation can be cross-checked in the next example.

In summary, we have verified that both the Casimir and Casimir-Polder pressures are recovered as limits of the more general expression of the plate-plate quantum pressure Eq. (6.12). Qualitative considerations are given in the next example.

7 Example 3: Force Between a Plate and a Point Source

As a third application of our formalism that we focus on the interaction between a point particle and a plate. As before we assume the Lagrangian

\[ \mathcal{L} = \frac{1}{2} (\partial_{\mu} \Phi)^2 - \frac{m^2}{2} \Phi^2 - \frac{1}{2 \lambda^2} \Phi^2 J(x). \]  

(7.1)

The plate is supported on \( z < 0 \) and has mass density \( \rho_1 \). The source is taken to be

\[ J(x) = \rho_1 \Theta(-z_{\infty} < z < 0) + m_N \delta^2(x ||) \delta(z - \ell). \]  

(7.2)

The mass of the point particle is \( m_N \). We define the effective mass of \( \Phi \) in the plate as

\[ m_1^2 = \frac{\rho_1}{\lambda^2} + m^2 \]  

(7.3)

which depends on the coupling to matter, \( \frac{1}{\lambda^2} \). The effective mass of \( \Phi \) is then piecewise constant,

\[ m^2(z) = \begin{cases} m_1^2 \Theta(-z_{\infty} < z < 0) + m_2^2 \Theta(z > 0) \end{cases} \]  

(7.4)

with \( m_2 = m \).

Propagator

The Feynman propagator in position-momentum space \((p^\alpha, z, \alpha = (0, 1, 2))\) in the presence of a piecewise constant mass has been calculated in [17]. The effect of the point source on the propagation is negligible.\(^8\) For the present calculation, if the frame is chosen such that the deformation changes the position of the point source and not of the plate, one can safely ignore the region at \(-z_{\infty}\) and thus consider the propagator over two regions, with \( m^2(z) = m_1^2 \Theta(z < z_{12}) + m_2^2 \Theta(z > z_{12}) \). The propagator is found to be

\[ \Delta_p(z, z') = \begin{cases} e^{i \omega_2 (z - z_{12})} E_2(z_{\infty}) & z_{12} < z_{\infty} < z < z_{12} < z > \\ e^{i \omega_1 (z - z_{12})} E_1(z_{12}) & z < z_{12} < z > \\ \end{cases} \]  

(7.5)

where

\[ E_1(z) = 1 + e^{i(2(z_{12} - z))} \omega_1 - \frac{\omega_1 - \omega_2}{\omega_1 + \omega_2} \]  

\[ E_2(z) = 1 + e^{i(2(z - z_{12}))} \omega_2 - \frac{\omega_2 - \omega_1}{\omega_1 + \omega_2}. \]  

(7.6)

\(^8\)This can be checked by evaluating the dressed propagator in energy-position space \((p_0, x)\). In the resummed propagator, the effect of the insertion is small within the EFT validity range, leaving the term with one point source insertion as the main non-vanishing contribution to the quantum work.
We have defined \( z_\prec = \min(z, z') \) and \( z_\succ = \max(z, z') \). We have introduced \( \omega_i = \sqrt{(p_\alpha)^2 - m_i^2 + i\epsilon} \). The \( \epsilon \) prescription guarantees that the propagators decay at infinity. The \( E_1, E_2 \) functions essentially describe how the presence of the boundary affects the propagator with both endpoints in the same region. When the boundary \( z_{12} \) is rejected to infinity, one recovers the usual expression for fully homogeneous space.

### Quantum Force

The deformation of the source is
\[
\partial_\lambda J = -m_N L \delta^2(x_\parallel) \partial_z \delta(z - \ell) .
\] (7.7)

Using this expression into the quantum work, one obtains after one integration by parts the quantum force
\[
F_{\text{quant}} = -\frac{1}{2} \frac{m_N}{\Lambda^2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega_2} \partial_z E_2(z) |_{z=\ell} = -\frac{1}{2} \frac{m_N}{\Lambda^2} \int \frac{d^3 p}{(2\pi)^3} \partial_z \Delta_J(x^\alpha, z; x^\alpha, z) |_{z=\ell}
\] (7.8)

In the last line we have introduced the position-momentum space propagator.

Using Eq. (7.5) with \( z_{12} = 0 \) since the plate is placed at the origin, we have
\[
F_{\text{quant}} = -\frac{1}{2} \frac{m_N}{\Lambda^2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega_2} \partial_z E_2(z) |_{z=\ell} = -\frac{1}{2} \frac{m_N}{\Lambda^2} \frac{1}{4\pi^2} \int d\rho \rho^2 e^{-2\ell\gamma_2} \] (7.9)

where we have performed a Wick rotation and introduced \( \gamma_i = \sqrt{\rho^2 + m_i^2} \). This is the general expression of the plate-point quantum force in the presence of a scalar coupled quadratically to matter. Diagrammatically, the particle will interact with the plate via loops starting at the point particle, going into the plate and coming back to the point particle.

In the limit of large density-induced effective mass \( m_1 \to \infty \), the force takes the form
\[
F_{\text{quant}} = -\frac{1}{2} \frac{m_N}{\Lambda^2} \frac{1}{4\pi^2} \int d\rho \rho^2 e^{-2\ell\gamma_2} .
\] (7.10)

This is the limit for which the density is so large that the field is repelled by the plate and the field obeys Dirichlet boundary conditions at the boundary of the plate. We refer to this case as the Casimir limit. In the massless case we obtain
\[
F_{\text{quant}} = -\frac{1}{2} \frac{m_N}{\Lambda^2} \frac{1}{16\pi^2} \ell^3 .
\] (7.11)

In the limit of small density-induced effective mass we expand in \( (m_1^2 - m_2^2) \ll m_1^2 \) and obtain
\[
F_{\text{quant}} = -(m_1^2 - m_2^2) m_N \frac{1}{4\pi^2} \int d\rho \rho^2 e^{-2\ell\gamma_2} \frac{1}{4\gamma_2} .
\] (7.12)
In the massless case we obtain
\[ F_{\text{quant}} = -\frac{m_1^2 m_N}{16\pi^2 \Lambda^2} \frac{1}{\ell^4}. \] (7.13)

This is the Casimir-Polder limit. As a cross check, in the Appendix we show that this limit is exactly recovered by integrating the point-point Casimir-Polder potential over the extended source.

In summary, we have verified that both the Casimir and Casimir-Polder pressures are recovered as limits of the more general expression of the plate-point quantum pressure Eq. (6.12). The two limits of the massless formula Eqs. (7.11) and (7.13) render transparent that there is a transition between the two regimes as a function of the separation \( \ell \). Namely, the Casimir regime occurs for \( \ell \gg m_1^{-1} \) while the Casimir-Polder regime occurs for \( \ell \ll m_1^{-1} \), with \( m_1 = \frac{m_1}{m} \). One way to think about this phenomenon is that, while at large distance the plate behaves as a mirror, leading to a Casimir force, at short distance the quantum fluctuations start penetrating the mirror. As a result the behaviour of the Casimir force gets softened into the the Casimir-Polder one at short distance. This behaviour is confirmed numerically.

7.1 Application to Interferometry

Here we briefly illustrate the importance of our general formula for a realistic observable. Our focus is on interferometry.

Atomic or neutron interferometry uses the difference of phase of two non-relativistic particles following two different paths. We denote these paths as \( ACB \) and \( ADB \) which are considered to be two broken straight lines with a change of direction at \( C \) and \( D \) respectively. Experimentally, these changes of direction of the wave packets are induced by the interaction with a laser beam. Along the trajectories and using the WKB approximation, the leading phase shift on a particle induced by the potential \( V(z) \) is simply
\[ \delta \phi = -\int_{\Gamma_0} dt \, V(z(t)) \] (7.14)
where \( \Gamma_0 \) is the closed path \( ACBD \) and \( z(t) \) the corresponding classical trajectory. Along the unperturbed paths, the trajectories are straight
\[ z(t) = z_i + v(t - t_i). \] (7.15)
where the initial points corresponds to the four corners of the path \( \Gamma_0 \).

We assume that the interferometry experiment is carried out over a plane located at \( z = 0 \). Experimentally, the plane is often a large ball whose radius is assumed to be much larger than the distance between the ball and the path \( \Gamma_0 \). Given this geometry, the potential \( V(z) \) corresponds to the plane-point quantum force derived in this section. The plane-point force (7.9) derives from a potential \( F(\ell) = -\frac{\partial V(\ell)}{\partial \ell} \)
\[ V(\ell) = -(m_1^2 - m_2^2) \frac{m_N}{\Lambda^2} \frac{1}{4\pi^2} \int d\rho \rho^2 \frac{e^{-2\ell \gamma_2}}{2\gamma_2(\gamma_1 + \gamma_2)\sqrt{2}} \] (7.16)
where $\ell$ is the distance from the particle to the plate, here taken to be along the $z$ direction.

Let us consider one arm of the path $\Gamma_0$ where the particle evolves between times $t_a$ and $t_b$. The associated phase shift is

$$
\delta \phi_{ab} = \int_{t_a}^{t_b} dt \left( m_1^2 - m_2^2 \right) \frac{m_N}{\Lambda^2} \frac{1}{4\pi^2} \int dp \rho^2 e^{-2(z_a + v(t-t_a))\gamma_2} \frac{\gamma_2}{2\gamma_2(\gamma_1 + \gamma_2)^2} \int d\rho \rho^2 e^{-2z_b\gamma_2} - e^{-2z_a\gamma_2} \frac{\gamma_2}{4\gamma_2(\gamma_1 + \gamma_2)^2} \right)
$$

We notice that the calculation of the phase shift induced by the exact quantum force can be done as easily as the one from e.g. the Casimir or Casimir-Polder forces.

Since in this experiment the separation between the point source and the plane greatly vary, using either the Casimir (large $\ell$) or the Casimir-Polder (small $\ell$) approximation of the force in the computation of the phase shift would be likely to give an erroneous result. Hence, here, we exhibit an example of realistic observable for which using the generalized Casimir force result is likely to be mandatory to obtain meaningful conclusions. In a forthcoming paper, we will analyse the behaviour of this phase shift in realistic situations.

8 Conclusion

How does a classical body respond to an arbitrary deformation in quantum vacuum? Or presented differently, how can one generalize the notion of Casimir force to arbitrary deformable bodies? This can be tackled by introducing the notion of quantum work. The quantum work $W$ is an observable quantity which goes beyond the notion of quantum force $F$ and reduces to $W = F \cdot L$ in specific cases when the deformation flow $L$ is simple enough to factor out. For example, the quantum work encodes the usual Casimir pressure.

The (set of) classical body(ies), referred to as the “source”, could formally be a completely abstract distribution. However, since they describe classical matter, their number densities must satisfy the local conservation of matter equation, which puts a constraint on the set of allowed deformations of the source.

We have shown that, upon requesting conservation of matter in the source, the quantum work must be finite. This result applies to any shape and geometry, either rigid or deformable. This is shown both for a renormalizable, possibly strongly-coupled, theory, and in a more general effective field theory setup allowing for higher derivative interactions.

The finiteness of the quantum work readily explains why earlier attempts to compute quantum forces featured seemingly “unremovable” divergences, although only for certain geometries. A key example is the quantum work felt by a Dirichlet sphere under a radial deformation. The radial deformation flow is not divergence-free and thus the sphere density must vary to ensure matter conservation. Not taking this into account implies that our finiteness theorem does not apply and as a result there is a divergence. We have explicitly verified that, in our formulation, the density variation removes the seemingly unremovable divergence found in earlier calculations. Therefore we have obtained the final answer for the quantum pressure on a $d - 1$-Dirichlet sphere.
When specialising to rigid bodies, the quantum work leads to a quantum force that reduces to the Casimir and Casimir-Polder forces as special limits. Hence we have obtained a master formula which unifies Casimir and Casimir-Polder forces. There is a clear diagrammatic understanding of this interpolation. In the short distance regime, the main contribution comes from the loop with only one coupling to each body, which corresponds to Casimir-Polder. In the long distance regime, loops with arbitrary number of insertions contribute, but their resummation amounts to having a Dirichlet condition on the boundary of the bodies, which corresponds to the traditional Casimir setup.

We have computed the generalized quantum force in plate-plate and plate-point geometries. For the plate-plate geometry, the force behaves as $\frac{1}{\ell^4}$ at long distance (Casimir) and as $\frac{1}{\ell^2}$ at short distance (Casimir-Polder). For plate-point geometry the force behaves as $\frac{1}{\ell^3}$ at long distance (Casimir) and as $\frac{1}{\ell}$ at short distance (Casimir-Polder). We also briefly illustrated that our general results can be easily implemented in the calculation of phase shift for cold atom interferometry. We point out that, at least for this type of observable, using our general result for the quantum force is likely to be mandatory to obtain correct predictions.

The examples displayed throughout this work make clear that our results can have concrete implications for simple existing physical observables. In future work we will investigate/revisit fifth forces constraints using the unified description of quantum forces obtained here, hence consistently taking into account both Casimir and Casimir-Polder regimes in the comparison with experimental results.

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A Derivation of the Plate-Point Casimir-Polder Potential

The Casimir-Polder potential between a particle and a plate can be obtained by direct calculation in the weak coupling regime (i.e. the limit of small density in the plate). We start by computing the potential between two point sources. The corresponding source term is

$$\mathcal{L} \supset -\frac{1}{2} \Phi^2 \left( \frac{m_a}{\Lambda^2} \delta^3(x - x_a) + \frac{m_N}{\Lambda^2} \delta^3(x - x_b) \right).$$

We remind that this is the non-relativistic approximation of the 4-point interaction

$$\mathcal{L} \supset -\frac{1}{2} \Phi^2 \left( \frac{m_a}{\Lambda^2} \bar{\psi}_a \psi_a + \frac{m_N}{\Lambda^2} \bar{\psi}_b \psi_b \right)$$

between the scalar and two fermion species. The scattering amplitude with two insertions of two particle-anti particles pairs leads to a bubble diagram which reads

$$i \mathcal{M} = -\frac{m_N m_a}{\Lambda^4} 4 m_N m_a \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \frac{e^{i \omega_2 |z_1 - z_2|}}{2 \omega_2} \frac{e^{i \omega_2 |z_1 - z_2|}}{2 \omega_2}$$

(A.3)
where \( \omega_2 = \sqrt{k^2 - m_2^2} \), \( \omega'_2 = \sqrt{(k + p)^2 - m_2^2} \). The factor of \( \frac{1}{2} \) is a symmetry factor and the external fermions are such that their nonrelativistic wavefunctions are normalised as \( \bar{u}_a u_a = 2m_a, \quad \bar{u}_b u_b = 2m_N \). The non-relativistic scattering potential is given by

\[
\tilde{V}(p, z_1 - z_2) = -\frac{M}{4m_a m_N},
\]

\[
= -i \frac{m_a m_N}{\Lambda^4} \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\omega_2|z_1 - z_2|}}{2\omega_2} \frac{e^{i\omega'_2|z_1 - z_2|}}{2\omega'_2} .
\]  

(A.4)

The spatial potential is then obtained from the Fourier transform of \( \tilde{V} \),

\[
V \left( \sqrt{(z_1 - z_2)^2 + x_\parallel^2} \right) = \int \frac{d^2p_\parallel}{(2\pi)^2} \tilde{V}(p_\parallel, z_1, z_2) e^{ip_\parallel \cdot x_\parallel},
\]  

(A.5)

where \( x_\parallel = (x_1, x_2) \).

We now consider an ensemble of \( N_1 \) particle of the species \( a \) in a volume \( V_1 \) with a number density \( n_1 = \frac{N_1}{V_1} \). We average the potential over the plate with a separation \( \ell \) to the point particle as

\[
V(\ell) = n_1 \int d^2x_\parallel \int_{-\infty}^{0} dz_1 \int \frac{d^2p_\parallel}{(2\pi)^2} \tilde{V}(p_\parallel, z_1, \ell) e^{ip_\parallel \cdot x_\parallel}.
\]  

(A.6)

The transverse integrals simplify and the potential becomes simply

\[
V(\ell) = n_1 \int_{-\infty}^{0} dz_1 \tilde{V}(0, z_1, \ell) = -n_1 \frac{m_a m_N}{\Lambda^4} \int \frac{d^3k E}{(2\pi)^3} \frac{e^{-2\gamma\ell}}{16(\gamma_2)^3}.
\]  

(A.7)

after Wick’s rotation. Using \( m_1^2 - m_2^2 = n_1 m_a / \Lambda^2 \), we find

\[
V(\ell) = -(m_1^2 - m_2^2) \frac{m_N}{\Lambda^2} \int \frac{d^3k E}{(2\pi)^3} \frac{e^{-2\gamma\ell}}{16(\gamma_2)^3}.
\]  

(A.8)

Finally the force is obtained by taking the derivative with respect to \( \ell \)

\[
F = -\partial_\ell V = -(m_1^2 - m_2^2) \frac{m_N}{\Lambda^2} \int \frac{d^3k E}{(2\pi)^3} \frac{e^{-2\gamma\ell}}{8(\gamma_2)^2}.
\]  

(A.9)

This reproduces Eq. (7.12).

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