What can (partition) logic contribute to information theory?

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Abstract
Logical probability theory was developed as a quantitative measure based on Boole’s logic of subsets. But information theory was developed into a mature theory by Claude Shannon with no such connection to logic. A recent development in logic changes this situation. In category theory, the notion of a subset is dual to the notion of a quotient set or partition, and recently the logic of partitions has been developed in a parallel relationship to the Boolean logic of subsets (subset logic is usually mis-specified as the special case of propositional logic). What then is the quantitative measure based on partition logic in the same sense that logical probability theory is based on subset logic? It is a measure of information that is named “logical entropy” in view of that logical basis. This paper develops the notion of logical entropy and the basic notions of the resulting logical information theory. Then an extensive comparison is made with the corresponding notions based on Shannon entropy.

Key words: partition logic, logical entropy, Shannon entropy

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1 Introduction

This paper develops the application of the logic of partitions [15] to information theory. Partitions are dual (in a category-theoretic sense) to subsets. George Boole developed the notion of logical probability [7] as the normalized counting measure on subsets in his logic of subsets. This paper develops the normalized counting measure on partitions as the analogous quantitative treatment in the logic of partitions. The resulting measure is a new logical derivation of an old formula measuring diversity and distinctions, e.g., the Gini-Simpson index of diversity, that goes back to the early 20th century [19]. In view of the idea of information as being based on distinctions (see next section), I refer to this logical measure of distinctions as “logical entropy”.

This raises the question of the relationship of logical entropy to the standard notion of Shannon entropy. Firstly, logical entropy directly counts the distinctions (as defined in partition logic) whereas Shannon entropy, in effect, counts the minimum number of binary partitions (or yes/no questions) it takes, on average, to uniquely determine or designate the distinct entities. Since that gives a binary code for the distinct entities, the Shannon theory (unlike the logical theory) is perfectly adapted for the theory of coding and communications.

The second way to relate the logical theory and the Shannon theory is to consider the relationship between the compound notions (e.g., conditional entropy, joint entropy, and mutual information) in the two theories. Logical entropy is a measure in the mathematical sense, so as with any measure, the compound formulas satisfy the usual Venn-diagram relationships. The compound notions of Shannon entropy are defined so that they also satisfy similar Venn diagram relationships. However, as various information theorists, principally Lorne Campbell, have noted [9], Shannon entropy is not a measure (outside of the special case of $2^n$ equiprobable distinct entities where it is the count $n$ of the number of yes/no questions necessary to unique determine the distinct entities)–so one can conclude only that the ”analogies provide a convenient mnemonic” [9, p. 112] in terms of the usual Venn diagrams for measures. Campbell wondered if there might be a ”deeper foundation” [9, p. 112] to clarify how the Shannon formulas can defined to satisfy the measure-like relations in spite of not being a measure. That question is addressed in this paper by showing that there is a transformation of formulas that transforms each of the logical entropy compound formulas into the corresponding Shannon entropy compound formula, and the transform preserves the Venn diagram relationships that automatically hold for measures. This ”dit-bit transform” is heuristically motivated by showing how certain counts of distinctions ("dits") can be converted in counts of binary partitions ("bits").

Moreover, Campbell remarked that it would be "particularly interesting" and "quite significant" if there was an entropy measure of sets so that joint entropy corresponded to the measure of the
union of sets, conditional entropy to the difference of sets, and mutual information to the intersection of sets \[9\] p. 113. Logical entropy precisely satisfies those requirements, so we turn to the underlying idea of information as a measure of distinctions.

2 Logical information as the measure of distinctions

There is now a widespread view that information is fundamentally about differences, distinguishability, and distinctions. As Charles H. Bennett, one of the founders of quantum information theory, put it:

So information really is a very useful abstraction. It is the notion of distinguishability abstracted away from what we are distinguishing, or from the carrier of information. \[5\] p. 155

This view even has an interesting history. In James Gleick’s book, *The Information: A History, A Theory, A Flood*, he noted the focus on differences in the seventeenth century polymath, John Wilkins, who was a founder of the Royal Society. In 1641, the year before Newton was born, Wilkins published one of the earliest books on cryptography, *Mercury or the Secret and Swift Messenger*, which not only pointed out the fundamental role of differences but noted that any (finite) set of different things could be encoded by words in a binary code.

For in the general we must note, That whatever is capable of a competent Difference, perceptible to any Sense, may be a sufficient Means whereby to express the Cogitations. It is more convenient, indeed, that these Differences should be of as great Variety as the Letters of the Alphabet; but it is sufficient if they be but twofold, because Two alone may, with somewhat more Labour and Time, be well enough contrived to express all the rest. \[54\] Chap. XVII, p. 69

Wilkins explains that a five letter binary code would be sufficient to code the letters of the alphabet since \(2^5 = 32\).

Thus any two Letters or Numbers, suppose A.B. being transposed through five Places, will yield Thirty Two Differences, and so consequently will superabundantly serve for the Four and twenty Letters... \[54\] Chap. XVII, p. 69

As Gleick noted:

Any difference meant a binary choice. Any binary choice began the expressing of cogitations. Here, in this arcane and anonymous treatise of 1641, the essential idea of information theory poked to the surface of human thought, saw its shadow, and disappeared again for [three] hundred years. \[21\] p. 161

Thus *counting distinctions* \[13\] would seem the right way to measure information\[1\] and that is the measure that emerges naturally out of partition logic—just as finite logical probability emerges naturally as the measure of *counting elements* in Boole’s subset logic.

Although usually named after the special case of ‘propositional’ logic, the general case is Boole’s logic of subsets of a universe \(U\) (the special case of \(U = 1\) allows the propositional interpretation since the only subsets are 1 and \(\emptyset\) standing for truth and falsity). Category theory shows that is a

\[1\] This paper is about what Adriaans and Benthem call "Information B: Probabilistic, information-theoretic, measured quantitatively", not about "Information A: knowledge, logic, what is conveyed in informative answers" where the connection to philosophy and logic is built-in from the beginning. Likewise, the paper is not about Kolmogorov-style "Information C: Algorithmic, code compression, measured quantitatively." \[4\] p. 11
duality between sub-sets and quotient-sets (= partitions = equivalence relations), and that allowed the recent development of the dual logic of partitions ([14], [15]). As indicated in the title of his book, An Investigation of the Laws of Thought on which are founded the Mathematical Theories of Logic and Probabilities [7], Boole also developed the normalized counting measure on subsets of a finite universe \( U \) which was finite logical probability theory. When the same mathematical notion of the normalized counting measure is applied to the partitions on a finite universe set \( U \) (when the partition is represented as the complement of the corresponding equivalence relation on \( U \times U \)) then the result is the formula for logical entropy.

In addition to the philosophy of information literature [4], there is a whole sub-industry in mathematics concerned with different notions of ‘entropy’ or ‘information’ ([2]; see [52] for a recent ‘extensive’ analysis) that is long on formulas and ‘intuitive axioms’ but short on interpretations. Out of that plethora of definitions, logical entropy is the measure (in the technical sense of measure) of information that arises out of partition logic just as logical probability theory arises out of subset logic.

The logical notion of information-as-distinctions supports the view that the notion of information is a more primitive notion than probability and should be based on finite combinatorics. As Kolmogorov put it:

Information theory must precede probability theory, and not be based on it. By the very essence of this discipline, the foundations of information theory have a finite combinatorial character. [31, p. 39]

Logical information theory starts simply with a set of distinctions defined by a partition on \( U \), where a distinction is an ordered pair of elements of \( U \) in distinct blocks of the partition. Thus the set of distinctions (‘ditset’) or information set (‘infoset’) associated with the partition is just the complement of the equivalence relation associated with the partition. To get a quantitative measure of information, any probability distribution on \( U \) defines a product probability measure on \( U \times U \), and the logical entropy is simply that probability measure of the information set. In this manner, the logical theory of information-as-distinctions starts with the information set (set of distinctions) as a finite combinatorial object and then for any probability measure on the underlying set, the product probability measure on the information set gives the quantitative notion of logical entropy.

### 3 Duality of subsets and partitions

Logical entropy is to the logic of partitions as logical probability is to the Boolean logic of subsets. Hence we will start with a brief review of the relationship between these two dual forms of mathematical logic.

Modern category theory shows that the concept of a subset dualizes to the concept of a quotient set, equivalence relation, or partition. F. William Lawvere called a subset or, in general, a subobject a “part” and then noted: “The dual notion (obtained by reversing the arrows) of ‘part’ is the notion of partition.” [34, p. 85] That suggests that the Boolean logic of subsets (usually named after the special case of propositions as ‘propositional’ logic) should have a dual logic of partitions ([14], [15]).

A partition \( \pi = \{B_1, ..., B_m\} \) on \( U \) is a set of subsets, called cells or blocks, \( B_i \) that are mutually disjoint and jointly exhaustive (\( \cup_i B_i = U \)). In the duality between subset logic and partition logic, the dual to the notion of an ‘element’ of a subset is the notion of a ‘distinction’ of a partition, where \( (u, u') \in U \times U \) is a distinction or dit of \( \pi \) if the two elements are in different blocks, i.e., the ‘dits’ of a partition are dual to the ‘its’ (or elements) of a subset. Let dit \( (\pi) \subseteq U \times U \) be the set of distinctions or ditset of \( \pi \). Thus the information set or infoset associated with a partition \( \pi \) is ditset dit \( (\pi) \). Similarly an indistinction or indit of \( \pi \) is a pair \( (u, u') \in U \times U \) in the same block of \( \pi \). Let indit \( (\pi) \subseteq U \times U \) be the set of indistinctions or inditset of \( \pi \). Then indit \( (\pi) \) is the equivalence relation associated with \( \pi \) and dit \( (\pi) = U \times U - \text{indit} (\pi) \) is the complementary binary relation that might be called a partition relation or an apartness relation.
4 Classical subset logic and partition logic

The algebra associated with the subsets $S \subseteq U$ is, of course, the Boolean algebra $\wp(U)$ of subsets of $U$ with the partial order as the inclusion of elements. The corresponding algebra of partitions $\pi$ on $U$ is the partition algebra $\prod(U)$ defined as follows:

- the partial order $\sigma \leq \pi$ of partitions $\sigma = \{C, C', \ldots\}$ and $\pi = \{B, B', \ldots\}$ holds when $\pi$ refines $\sigma$ in the sense that for every block $B \in \pi$ there is a block $C \in \sigma$ such that $B \subseteq C$, or, equivalently, using the element-distinction pairing, the partial order is the inclusion of distinctions: $\sigma \leq \pi$ if and only if (iff) $\text{dit} (\sigma) \subseteq \text{dit} (\pi)$;

- the minimum or bottom partition is the indiscrete partition (or blob) $0 = \{U\}$ with one block consisting of all of $U$;

- the maximum or top partition is the discrete partition $1 = \{\{u_j\}\}_{j=1, \ldots, n}$ consisting of singleton blocks;

- the join $\pi \lor \sigma$ is the partition whose blocks are the non-empty intersections $B \cap C$ of blocks of $\pi$ and blocks of $\sigma$, or, equivalently, using the element-distinction pairing, $\text{dit} (\pi \lor \sigma) = \text{dit} (\pi) \cup \text{dit} (\sigma)$;

- the meet $\pi \land \sigma$ is the partition whose blocks are the equivalence classes for the equivalence relation generated by: $u_j \sim u_j'$ if $u_j \in B \in \pi$, $u_j' \in C \in \sigma$, and $B \cap C \neq \emptyset$; and

- $\sigma \Rightarrow \pi$ is the implication partition whose blocks are: (1) the singletons $\{u_j\}$ for $u_j \in B \in \pi$ if there is a $C \in \sigma$ such that $B \subseteq C$, or (2) just $B \in \pi$ if there is no $C \in \sigma$ with $B \subseteq C$, so that trivially: $\sigma \Rightarrow \pi = 1$ iff $\sigma \preceq \pi$.

The logical partition operations can also be defined in terms of the corresponding logical operations on subsets. A ditset $\text{dit}(\pi)$ of a partition on $U$ is a subset of $U \times U$ of a particular kind, namely the complement of an equivalence relation. An equivalence relation is reflexive, symmetric, and transitive. Hence the complement, i.e., a partition relation (or apartness relation), is a subset $P \subseteq U \times U$ that is:

1. irreflexive (or anti-reflexive), $P \cap \Delta = \emptyset$ (where $\Delta = \{(u, u) : u \in U\}$ is the diagonal);
2. symmetric, $(u, u') \in P$ implies $(u', u) \in P$; and
3. anti-transitive (or co-transitive), if $(u, u'') \in P$ then for any $u' \in U$, $(u, u') \in P$ or $(u', u'') \in P$.

Given any subset $S \subseteq U \times U$, the reflexive-symmetric-transitive (rst) closure $\overline{S}$ of the complement $S^c$ is the smallest equivalence relation containing $S^c$, so its complement is the largest partition relation contained in $S$, which is called the interior $\text{int}(S)$ of $S$. This usage is consistent with calling the subsets that equal their rst-closures closed subsets of $U \times U$ (so closed subsets = equivalence relations) so the complements are the open subsets (= partition relations). However it should be noted that the rst-closure is not a topological closure since the closure of a union is not necessarily the union of the closures, so the ‘open’ subsets do not form a topology on $U \times U$. Indeed, any two nonempty open sets have a nonempty intersection.

The interior operation $\text{int} : \wp(U \times U) \rightarrow \wp(U \times U)$ provides a universal way to define logical operations on partitions from the corresponding logical subset operations in Boolean logic:

\footnote{There is a general method to define operations on partitions corresponding to operations on subsets \cite{14, 15} but the lattice operations of join and meet, and the implication operation are sufficient to define a partition algebra $\prod(U)$ parallel to the familiar powerset Boolean algebra $\wp(U)$.}
apply the subset operation to the ditsets and then, if necessary, take the interior to obtain the ditset of the partition operation.

Since the same operations can be defined for subsets and partitions, one can interpret a formula $\Phi(\pi, \sigma, \ldots)$ either way as a subset or a partition. Given either subsets on or partitions of $U$ substituted for the variables $\pi, \sigma, \ldots$, one can apply, respectively, subset or partition operations to evaluate the whole formula. Since $\Phi(\pi, \sigma, \ldots)$ is either a subset or a partition, the corresponding proposition is “$u$ is an element of $\Phi(\pi, \sigma, \ldots)$” or “$(u, u')$ is a distinction of $\Phi(\pi, \sigma, \ldots)$”. And then the definitions of a valid formula are also parallel, namely, no matter what is substituted for the variables, the whole formula evaluates to the top of the algebra. In that case, the subset $\Phi(\pi, \sigma, \ldots)$ contains all elements of $U$, i.e., $\Phi(\pi, \sigma, \ldots) = U$, or the partition $\Phi(\pi, \sigma, \ldots)$ distinguishes all pairs $(u, u')$ for distinct elements of $U$, i.e., $\Phi(\pi, \sigma, \ldots) = 1$. The parallelism between the dual logics is summarized in the following table 1.

| Table 1: Duality between subset logic and partition logic |
|-----------------------------------------------------------|
| **‘Elements’ (its or dit)*** | **Subset logic** | **Partition logic** |
| Elements $u$ of $S$ | $\text{Dits } (u, u')$ of $\pi$ | |
| **Inclusion of ‘elements’*** | $\text{Inclusion } S \subseteq T$ | $\text{Refinement: dit } (\sigma) \subseteq \text{dit } (\pi)$ |
| **Top of order = all ‘elements’*** | $\text{U all elements}$ | $\text{dit } (1) = U^2 - \Delta$, all dits |
| **Bottom of order = no ‘elements’*** | $\emptyset$ no elements | $\text{dit } (0) = \emptyset$, no dits |
| **Variables in formulas*** | **Subsets $S$ of $U$** | **Partitions $\pi$ on $U$** |
| **Operations: $\lor, \land, \Rightarrow, \ldots$*** | **Subset ops.** | **Partition ops.** |
| **Formula $\Phi(x, y, \ldots)$ holds*** | $u$ element of $\Phi(S, T, \ldots)$ | $(u, u')$ dit of $\Phi(\pi, \sigma, \ldots)$ |
| **Valid formula*** | $\Phi(S, T, \ldots) = U, \forall S, T, \ldots$ | $\Phi(\pi, \sigma, \ldots) = 1, \forall \pi, \sigma, \ldots$ |

### 5 Classical logical probability and logical entropy

George Boole [7] extended his logic of subsets to finite logical probability theory where, in the equiprobable case, the probability of a subset $S$ (event) of a finite universe set (outcome set or sample space) $U = \{u_1, \ldots, u_n\}$ was the number of elements in $S$ over the total number of elements: $\Pr(S) = \frac{|S|}{|U|} = \sum_{u_i \in S} \frac{1}{|U|}$. Laplace’s classical finite probability theory [33] also dealt with the case where the outcomes were assigned real point probabilities $p = \{p_1, \ldots, p_n\}$ (where $p_j \geq 0$ and $\sum_j p_j = 1$) so rather than summing the equal probabilities $\frac{1}{|U|}$ the point probabilities of the elements were summed: $\Pr(S) = \sum_{u_i \in S} p_j = p(S)$–where the equiprobable formula is for $p_j = \frac{1}{|U|}$ for $j = 1, \ldots, n$. The conditional probability of an event $T \subseteq U$ given an event $S$ is $\Pr(T|S) = \frac{p(T \cap S)}{p(S)}$.

Then we may mimic Boole’s move going from the logic of subsets to the finite logical probabilities of subsets by starting with the logic of partitions and using the dual relation between elements and distinctions. The dual notion to probability turns out to be ‘information content’ or ‘entropy’ so we define the logical entropy of $\pi = \{B_1, \ldots, B_m\}$, denoted $h(\pi)$, as the size of the ditset $\text{dit } (\pi) \subseteq U \times U$ normalized by the size of $U \times U$:

$$h(\pi) = \frac{|\text{dit}(\pi)|}{|U \times U|} = \sum_{(u_j, u_k) \in \text{dit}(\pi)} \frac{1}{|U|} \frac{1}{|U|}$$

Logical entropy of $\pi$ (equiprobable case).

This is just the product probability measure of the equiprobable or uniform probability distribution on $U$ applied to the information set or ditset $\text{dit } (\pi)$. The indit of $\pi$ is $\text{indit } (\pi) = \cup_{i=1}^m (B_i \times B_i)$ so where $p(B_i) = \frac{|B_i|}{|U|}$ in the equiprobable case, we have:

$$h(\pi) = \frac{|\text{dit}(\pi)|}{|U \times U|} = \frac{|U \times U| - \sum_{i=1}^m |B_i \times B_i|}{|U \times U|} = 1 - \sum_{i=1}^m \left( \frac{|B_i|}{|U|} \right)^2 = 1 - \sum_{i=1}^m p(B_i)^2.$$
The corresponding definition for the case of point probabilities \( p = \{p_1, \ldots, p_n\} \) is to just add up the probabilities of getting a particular distinction:

\[
h_p(\pi) = \sum_{(u_i, u_k) \in \text{dit}(\pi)} p_j p_k
\]

Logical entropy of \( \pi \) with point probabilities \( p \).

Taking \( p(B_i) = \sum_{u_j \in B_i} p_j \), the logical entropy with point probabilities is:

\[
h_p(\pi) = \sum_{(u_i, u_k) \in \text{dit}(\pi)} p_j p_k = \sum_{i \neq j} p(B_i) p(B_j) = 2 \sum_{i < j} p(B_i) p(B_j) = 1 - \sum_{i=1}^m p(B_i)^2.
\]

Instead of being given a partition \( \pi = \{B_1, \ldots, B_m\} \) on \( U \) with point probabilities \( p_j \) defining the finite probability distribution of block probabilities \( \{p(B_i)\}_i \), one might be given only a finite probability distribution \( p = \{p_1, \ldots, p_m\} \). Then substituting \( p_i \) for \( p(B_i) \) gives the:

\[
h(p) = 1 - \sum_{i=1}^m p_i^2 = \sum_{i \neq j} p_i p_j
\]

Logical entropy of a finite probability distribution.

Since \( 1 = (\sum_{i=1}^n p_i)^2 = \sum_i p_i^2 + \sum_{i \neq j} p_i p_j \), we again have the logical entropy \( h(p) \) as the probability \( \sum_{i \neq j} p_i p_j \) of drawing a distinction in two independent samplings of the probability distribution \( p \).

That two-draw probability interpretation follows from the important fact that logical entropy is always the value of a probability measure. The product probability measure on the subsets \( S \subseteq U \times U \) is:

\[
\mu(S) = \sum \{p_i p_j : (u_i, u_j) \in S\}
\]

Product measure on \( U \times U \).

Then the logical entropy \( h(p) = \mu(\text{dit}(1_U)) \) is just the product measure of the information set or ditset \( \text{dit}(1_U) = U \times U - \Delta \) of the discrete partition \( 1_U \) on \( U \).

There are also parallel “element ↔ distinction” probabilistic interpretations:

- \( \Pr(S) = p_S \) is the probability that a single draw, sample, or experiment with \( U \) gives a element \( u_j \) of \( S \), and
- \( h_p(\pi) = \mu(\text{dit}(\pi)) = \sum_{(u_i, u_k) \in \text{dit}(\pi)} p_j p_k = \sum_{i \neq j} p(B_i) p(B_j) = 1 - \sum_i p(B_i)^2 \) is the probability that two independent (with replacement) draws, samples, or experiments with \( U \) gives a distinction \( (u_j, u_k) \) of \( \pi \), or if we interpret the independent experiments as sampling from the set of blocks \( \pi = \{B_i\} \), then it is the probability of getting distinct blocks.

In probability theory, when a random draw gives an outcome \( u_j \) in the subset or event \( S \), we say the event \( S \) occurs, and in logical information theory, when the random draw of a pair \( (u_j, u_k) \) gives a distinction of \( \pi \), we say the partition \( \pi \) distinguishes.

The parallelism or duality between logical probabilities and logical entropies based on the parallel roles of ‘dits’ (elements of subsets) and ‘ditsets’ (distinctions of partitions) is summarized in Table 2.

| Table 2 | Logical Probability Theory | Logical Information Theory |
|---------|-----------------------------|----------------------------|
| ‘Outcomes’ | Elements \( u \in U \) finite | Ditsets \( (u, u') \in U \times U \) finite |
| ‘Events’ | Subsets \( S \subseteq U \) | Ditsets \( \text{dit}(\pi) \subseteq U \times U \) |
| Equiprobable points | \( \Pr(S) = \frac{|S|}{|U|} \) | \( h(\pi) = \frac{\text{dit}(\pi)}{|U \times U|} \) |
| Point probabilities | \( \Pr(S) = \sum \{p_j : u_j \in S\} \) | \( h(\pi) = \sum \{p_j p_k : (u_j, u_k) \in \text{dit}(\pi)\} \) |

**Table 2:** Classical logical probability theory and classical logical information theory

This concludes the argument that logical information theory arises out of partition logic just as logical probability theory arises out of subset logic. Now we turn to the formulas of logical information theory and the comparison to the formulas of Shannon information theory.
6 History of logical entropy formula

The formula for logical entropy is not new. Given a finite probability distribution \( p = (p_1, ..., p_n) \), the formula \( h(p) = 1 - \sum_{i=1}^{n} p_i^2 \) was used by Gini in 1912 ([19] reprinted in [20, p. 369]) as a measure of “mutability” or diversity. What is new here is not the formula, but the derivation from partition logic.

As befits the logical origin of the formula, it occurs in a variety of fields. The formula in the complementary form, \( \sum_i p_i = 1 - h(p) \), was developed early in the 20th century in cryptography. The American cryptologist, William F. Friedman, devoted a 1922 book ([18]) to the index of coincidence (i.e., \( \sum p_i^2 \)). Solomon Kullback worked as an assistant to Friedman and wrote a book on cryptography which used the index. [22] During World War II, Alan M. Turing worked for a time in the Government Code and Cypher School at the Bletchley Park facility in England. Probably unaware of the earlier work, Turing used \( \rho = \sum p_i^2 \) in his cryptoanalysis work and called it the repeat rate since it is the probability of a repeat in a pair of independent draws from a population with those probabilities.

After the war, Edward H. Simpson, a British statistician, proposed \( \sum_{B \in \pi} p_B^2 \) as a measure of species concentration (the opposite of diversity) where \( \pi = \{B, B', ..., \} \) is the partition of animals or plants according to species and where each animal or plant is considered as equiprobable so \( p_B = \frac{|B|}{|U|} \). And Simpson gave the interpretation of this homogeneity measure as “the probability that two individuals chosen at random and independently from the population will be found to belong to the same group.” [43, p. 688] Hence \( 1 - \sum_{B \in \pi} p_B^2 \) is the probability that a random ordered pair will belong to different species, i.e., will be distinguished by the species partition. In the biodiversity literature [44], the formula \( 1 - \sum_{B \in \pi} p_B^2 \) is known as Simpson’s index of diversity or sometimes, the Gini-Simpson index [42].

However, Simpson along with I. J. Good worked at Bletchley Park during WWII, and, according to Good, “E. H. Simpson and I both obtained the notion [the repeat rate] from Turing.” [22, p. 395] When Simpson published the index in 1948, he (again, according to Good) did not acknowledge Turing “fearing that to acknowledge him would be regarded as a breach of security.” [23, p. 562] Since for many purposes logical entropy offers an alternative to Shannon entropy ([46], [47]) in classical information theory, and the quantum version of logical entropy offers an alternative to von Neumann entropy [39] in quantum information theory, it might be useful to call it ‘Turing entropy’ to have a competitive ‘famous name’ label. But even before the logical derivation of the formula, I. J. Good pointed out a certain naturalness:

If \( p_1, ..., p_t \) are the probabilities of \( t \) mutually exclusive and exhaustive events, any statistician of this century who wanted a measure of homogeneity would have take about two seconds to suggest \( \sum p_i^2 \) which I shall call \( \rho \). [23, p. 561]

In view of the frequent and independent discovery and rediscovery of the formula \( \rho = \sum p_i^2 \) or its complement \( h(p) = 1 - \sum p_i^2 \) by Gini, Friedman, Turing, and many others [e.g., the Hirschman-Herfindahl index of industrial concentration in economics ([28, 27])], I. J. Good wisely advises that “it is unjust to associate \( \rho \) with any one person.” [23, p. 562]

7 Entropy as a measure of information

For a partition \( \pi = \{B_1, ..., B_m\} \) with block probabilities \( p(B_i) \) (obtained using equiprobable points or with point probabilities), the Shannon entropy of the partition (using natural logs) is:

\[
H(\pi) = -\sum_{i=1}^{m} p(B_i) \ln(p(B_i)).
\]

Or if given a finite probability distribution \( p = \{p_1, ..., p_m\} \), the Shannon entropy of the probability distribution is:
\[ H(p) = -\sum_{i=1}^{n} p_i \ln (p_i). \]

Shannon entropy and the many other suggested ‘entropies’ are routinely called “measures of information” [2]. The formulas for mutual information, joint entropy, and conditional entropy are defined so these Shannon entropies satisfy Venn diagram formulas ([1] p. 109; [39] p. 508) that would follow automatically if Shannon entropy were a measure in the technical sense. As Lorne Campbell put it:

Certain analogies between entropy and measure have been noted by various authors. These analogies provide a convenient mnemonic for the various relations between entropy, conditional entropy, joint entropy, and mutual information. It is interesting to speculate whether these analogies have a deeper foundation. It would seem to be quite significant if entropy did admit an interpretation as the measure of some set. [9, p. 112]

For any finite set \( X \), a measure \( \mu \) is a function \( \mu: \wp(X) \to \mathbb{R} \) such that:

1. \( \mu(\emptyset) = 0 \),
2. for any \( E \subseteq X \), \( \mu(E) \geq 0 \), and
3. for any disjoint subsets \( E_1 \) and \( E_2 \), \( \mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2) \).

Considerable effort has been expended to try to find a framework in which Shannon entropy would be a measure in this technical sense and thus would satisfy the desiderata:

that \( H(\alpha) \) and \( H(\beta) \) are measures of sets, that \( H(\alpha, \beta) \) is the measure of their union, that \( I(\alpha, \beta) \) is the measure of their intersection, and that \( H(\alpha|\beta) \) is the measure of their difference. The possibility that \( I(\alpha, \beta) \) is the entropy of the “intersection” of two partitions is particularly interesting. This “intersection,” if it existed, would presumably contain the information common to the partitions \( \alpha \) and \( \beta \).[9, p. 113]

But these efforts have not been successful beyond special cases such as \( 2^n \) equiprobable elements where, as Campbell notes, the Shannon entropy is just the counting measure \( n \) of the minimum number of binary partitions it takes to distinguish all the elements. In general, Shannon entropy is not a measure.

In contrast, it is “quite significant” that logical entropy is a measure, the normalized counting measure on the ditset \( \text{dit}(\pi) \) representation of a partition \( \pi \) as a subset of the set \( U \times U \). Thus all of Campbell’s desiderata are true when:

- “sets” = ditsets, the set of distinctions of partitions (or, in general, information sets or infosets), and
- “entropies” = normalized counting measure of the ditsets (or, in general, product probability measure on the infosets), i.e., the logical entropies.

The compound Shannon entropy notions satisfy the measure-like formulas, e.g., \( H(\alpha, \beta) = H(\alpha) + H(\beta) - I(\alpha, \beta) \), not because Shannon entropy is a “measure of some set” but because logical entropy is such a measure and all the Shannon compound entropy notions result from the corresponding logical entropy compound notions by a “dit-bit transform” that preserves those formulas.
8 The dit-bit transform

The logical entropy formulas for various compound notions (e.g., conditional entropy, mutual information, and joint entropy) stand in certain Venn diagram relationships because logical entropy is a measure. The Shannon entropy formulas for these compound notions are defined so as to satisfy the Venn diagram relationships as if Shannon entropy was a measure when it is not. How can that be? Perhaps there is some “deeper foundation” \[9, p. 112\] to explain why the Shannon formulas still satisfy those measure-like Venn diagram relationships.

Indeed, there is such a connection, the dit-bit transform. This transform can be heuristically motivated by considering two ways to treat the set \(U_n\) of \(n\) elements with the equal probabilities \(p_0 = \frac{1}{n}\). In that basic case of an equiprobable set, we can derive the dit-bit connection, and then by using a probabilistic average, we can develop the Shannon entropy, expressed in terms of bits, from the logical entropy, expressed in terms of (normalized) dits, or vice-versa.

Given \(U_n\) with \(n\) equiprobable elements, the number of dits (of the discrete partition on \(U_n\)) is \(n^2 - n\) so the normalized dit count is:

\[
h(p_0) = h\left(\frac{1}{n}\right) = 1 - p_0 = 1 - \frac{1}{n}\text{ normalized dits.}
\]

That is the dit-count or logical measure of the information in a set of \(n\) distinct elements (think of it as the logical entropy of the discrete partition on \(U_n\) with equiprobable elements).

But we can also measure the information in the set by the number of binary partitions it takes (on average) to distinguish the elements, and that bit-count is \[25\]:

\[
H(p_0) = H\left(\frac{1}{n}\right) = \log\left(\frac{1}{p_0}\right) = \log(n) \text{ bits.}
\]

**Shannon-Hartley entropy for an equiprobable set** \(U\) of \(n\) elements

The dit-bit connection is that the Shannon-Hartley entropy \(H(p_0) = \log\left(\frac{1}{p_0}\right)\) will play the same role in the Shannon formulas that \(h(p_0) = 1 - p_0\) plays in the logical entropy formulas—when both are formulated as probabilistic averages.

The common thing being measured is an equiprobable \(U_n\) where \(n = \frac{1}{p_0}\). The dit-count for \(U_n\) is \(h(p_0) = 1 - p_0\) and the bit-count for \(U\) is \(H(p_0) = \log\left(\frac{1}{p_0}\right)\). and the dit-bit transform converts one count into the other. Using this dit-bit transform between the two different ways to quantify the ‘information’ in \(U_n\), each entropy can be developed from the other. Nevertheless, this dit-bit connection should not be interpreted as if there is one thing ‘information’ that can be measured on different scales—like measuring a length using inches or centimeters. Indeed, the (average) bit-count is a “coarser-grid” that loses some information in comparison to the (exact) dit-count as shown by the analysis (below) of mutual information. There is no bit-count mutual information between independent probability distributions but there is always dit-count information even between two (non-trivial) independent distributions (see the proposition that nonempty ditsets always intersect).

We start with the logical entropy of a probability distribution \(p = (p_1, ..., p_n)\):

\[
h(p) = \sum_{i=1}^{n} p_i h(p_i) = \sum_{i} p_i (1 - p_i).
\]

It is expressed as the probabilistic average of the dit-counts or logical entropies of the sets \(U_1/p_i\) with \(\frac{1}{p_i}\) equiprobable elements. But if we switch to the binary-partition bit-counts of the information content of those same sets \(U_1/p_i\) of \(\frac{1}{p_i}\) equiprobable elements, then the bit-counts are \(H(p_i) = \log\left(\frac{1}{p_i}\right)\)

\[\text{Note that } n = 1/p_0 \text{ need not be an integer. We are following the usual practice in information theory where an implicit “on average” interpretation is assumed since actual “binary partitions” or “binary digits” (or “bits”) only come in integral units. The “on average” provisos are justified by the “noiseless coding theorem” covered in the later section on the statistical interpretation of Shannon entropy.}\]
and the probabilistic average is the Shannon entropy: \( H(p) = \sum_{i=1}^{n} p_i H(p_i) = \sum_{i} p_i \log \left( \frac{1}{p_i} \right) \). Both entropies have the mathematical form as a probabilistic average or expectation:

\[
\sum_{i} p_i \text{(amount of ‘information’ in } U_{1/p_i})
\]

and differ by using either the dit-count or bit-count conception of information in \( U_{1/p_i} \).

The dit-bit connection carries over to all the compound notions of entropy so that the Shannon notions of conditional entropy, mutual information, cross-entropy, and divergence can all be developed from the corresponding notions for logical entropy. Since the logical notions are the values of a probability measure, the compound notions of logical entropy have the usual Venn diagram relationships. And then by the dit-bit transform, those Venn diagram relationships carry over to the compound Shannon formulas since the dit-bit transform preserves sums and differences (i.e., is, in that sense, linear). \textit{That is} why the Shannon formulas satisfy the Venn diagram relationships even though Shannon entropy is not a measure.\(^5\)

Note that while the logical entropy formula \( h(p) = \sum_{i} p_i (1 - p_i) \) (and the corresponding compound formulas) are put into that form of an average or expectation to apply the dit-bit transform, logical entropy is the exact measure of the subset \( S_p = \{(i, i') : i \neq i'\} \subseteq \{1, ..., n\} \times \{1, ..., n\} \) for the product probability measure \( \mu : \{1, ..., n\}^2 \to [0, 1] \) where for \( S \subseteq \{1, ..., n\}^2, \mu(S) = \sum \{p_i p_{i'} : (i, i') \in S\} \), i.e., \( h(p) = \mu(S_p) \).

9 Conditional entropies

9.1 Logical conditional entropy

All the compound notions for Shannon and logical entropy could be developed using either partitions (with point probabilities) or probability distributions of random variables as the given data. Since the treatment of Shannon entropy is most often in terms of probability distributions, we will stick to that case for both types of entropy. The formula for the compound notion of logical entropy will be developed first, and then the formula for the corresponding Shannon compound entropy will be obtained by the dit-bit transform.

The general idea of a conditional entropy of a random variable \( X \) given a random variable \( Y \) is to measure the additional information in \( X \) when we take away the information contained in \( Y \).

Consider a joint probability distribution \( \{p(x, y)\} \) on the finite sample space \( X \times Y \), with the marginal distributions \( \{p(x)\} \) and \( \{p(y)\} \) where \( p(x) = \sum_{y \in Y} p(x, y) \) and \( p(y) = \sum_{x \in X} p(x, y) \). For notational simplicity, the entropies can be considered as functions of the random variables or of their probability distributions, e.g., \( h(\{p(x, y)\}) = h(X, Y) \), \( h(\{p(x)\}) = h(X) \), and \( h(\{p(y)\}) = h(Y) \). For the joint distribution, we have the:

\[
h(X, Y) = \sum_{x \in X, y \in Y} p(x, y) [1 - p(x, y)] = 1 - \sum_{x,y} p(x, y)^2
\]

Logical entropy of the joint distribution

which is the probability that two samplings of the joint distribution will yield a pair of \textit{distinct} ordered pairs \( (x, y), (x', y') \in X \times Y \), i.e., with an \( X \)-distinction \( x \neq x' \) or a \( Y \)-distinction \( y \neq y' \)(since ordered pairs are distinct if distinct on one of the coordinates). The logical entropy notions for the probability distribution \( \{p(x, y)\} \) on \( X \times Y \) are all product probability measures \( \mu(S) \) of certain subsets \( S \subseteq (X \times Y)^2 \). For the logical entropies defined so far, the infosets are:

\(^5\)Perhaps, one should say that Shannon entropy is not the measure of any independently defined set. The fact that the Shannon formulas ‘act like a measure’ can, of course, be formalized by formally associating an (indefinite) ‘set’ with each random variable \( X \) and then \textit{defining} the measure value on the ‘set’ as \( H(X) \). Since this ‘measure’ is defined by the Shannon entropy values, nothing is added to the already-known fact that the Shannon entropies act like a measure in the Venn diagram relationships. This formalization seems to have been first carried out by Hu \cite{29} but was also used by Csiszar and Körner \cite{12}, and by Yeung \cite{53, 55}.\)
Then the Shannon conditional entropy is defined as the average of the conditional entropy of each function:

\[ S_X = \{(x, y), (x', y') : x \neq x'\} \text{ where } h(X) = \mu(S_X); \]
\[ S_Y = \{(x, y), (x', y') : y \neq y'\} \text{ where } h(Y) = \mu(S_Y); \text{ and} \]
\[ S_{X \lor Y} = \{(x, y), (x', y') : x \neq x' \lor y \neq y'\} = S_X \cup S_Y \text{ where } h(X, Y) = \mu(S_{X \lor Y}) = \mu(S_X \cup S_Y). \]

The infosets \( S_X \) and \( S_Y \), as well as their complements \( S_{\sim X} = \{(x, y), (x', y') : x = x'\} \) and \( S_{\sim Y} = \{(x, y), (x', y') : y = y'\} \), generate a Boolean subalgebra \( \mathcal{I}(X \times Y) \) of \( \wp((X \times Y) \times (X \times Y)) \) which might be called the *information algebra of* \( X \times Y \). It is defined independently of any probability measure \( \{p(x, y)\} \) on \( X \times Y \), and any such measure defines the product measure \( \mu \) on \( (X \times Y) \times (X \times Y) \), and the corresponding logical entropies are the product measures on the infosets in \( \mathcal{I}(X \times Y) \).

For the definition of the conditional entropy \( h(X|Y) \), we simply take the product measure of the set of pairs \((x, y)\) and \((x', y')\) that give an \( X \)-distinction but not a \( Y \)-distinction. Hence we use the inequation \( x \neq x' \) for the \( X \)-distinction and negate the \( Y \)-distinction \( y \neq y' \) to get the infoset that is the difference of the infosets for \( X \) and \( Y \):

\[ S_{X \land Y} = \{(x, y), (x', y') : x \neq x' \land y = y'\} = S_X - S_Y \text{ so} \]
\[ h(X|Y) = \mu(S_{X \land Y}) = \mu(S_X - S_Y). \]

Since \( S_{X \lor Y} = S_{X \land Y} \cup S_Y \) and the union is disjoint, we have for the measure \( \mu \):

\[ h(X, Y) = \mu(S_{X \lor Y}) = \mu(S_{X \land Y}) + \mu(S_Y) = h(X|Y) + h(Y), \]

which is illustrated in the Venn diagram Figure 1.

![Figure 1: h(X, Y) = h(X|Y) + h(Y)](image)

In terms of the probabilities:

\[ h(X|Y) = h(X, Y) - h(Y) = \sum_{x, y} p(x, y) (1 - p(x, y)) - \sum_y p(y) (1 - p(y)) \]
\[ \sum_{x, y} p(x, y) [(1 - p(x, y)) - (1 - p(y))] \]

Logical conditional entropy of \( X \) given \( Y \).

### 9.2 Shannon conditional entropy

Given the joint distribution \( \{p(x, y)\} \) on \( X \times Y \), the conditional probability distribution for a specific \( y_0 \in Y \) is \( p(x|y_0) = \frac{p(x, y_0)}{p(y_0)} \) which has the Shannon entropy: \( H(X|y_0) = \sum_x p(x|y_0) \log \left( \frac{1}{p(x|y_0)} \right) \).

Then the Shannon conditional entropy is defined as the average of these entropies:

\[ H(X|Y) = \sum_y p(y) \sum_x \frac{p(x, y)}{p(y)} \log \left( \frac{p(y)}{p(x, y)} \right) = \sum_{x, y} p(x, y) \log \left( \frac{p(y)}{p(x, y)} \right) \]

Shannon conditional entropy of \( X \) given \( Y \).
All the Shannon notions can be obtained by the dit-bit transform of the corresponding logical notions. Applying the transform $1 - p \rightsquigarrow \log \left( \frac{1}{p} \right)$ to the logical conditional entropy expressed as an average of “$1 - p$” expressions:

$$h(X|Y) = \sum_{x,y} p(x,y) \left[ (1 - p(x,y)) - (1 - p(y)) \right],$$

yields the Shannon conditional entropy:

$$H(X|Y) = \sum_{x,y} p(x,y) \left[ \log \left( \frac{1}{p(x,y)} \right) - \log \left( \frac{1}{p(y)} \right) \right] = \sum_{x,y} p(x,y) \log \left( \frac{p(y)}{p(x,y)} \right).$$

Since the dit-bit transform preserves sums and differences, we will have the same sort of Venn diagram formula for the Shannon entropies (even though the Shannon notions are not the values of a measure) and this can be illustrated in a similar “mnemonic” Venn diagram.

Figure 2: $H(X|Y) = H(X,Y) - H(Y)$.

10 Mutual information

10.1 Logical mutual information

Intuitively, the mutual logical information $m(X,Y)$ in the joint distribution $\{p(x,y)\}$ would be the probability that a sampled pair of pairs $(x, y)$ and $(x', y')$ would be distinguished in both coordinates, i.e., a distinction $x \neq x'$ of $p(x)$ and a distinction $y \neq y'$ of $p(y)$. In terms of subsets, the subset for the mutual information is intersection of infosets for $X$ and $Y$:

$$S_{X \land Y} = S_X \cap S_Y \text{ so } m(X,Y) = \mu(S_{X \land Y}) = \mu(S_X \cap S_Y).$$

In terms of disjoint unions of subsets:

$$S_{X \lor Y} = S_{X \land Y} \uplus S_{Y \land \neg X} \uplus S_{X \land Y}$$

so

$$h(X,Y) = \mu(S_{X \lor Y}) = \mu(S_{X \land Y}) + \mu(S_{Y \land \neg X}) + \mu(S_{X \land Y}) = h(X|Y) + h(Y|X) + m(X,Y) \text{ (as in Figure 3)},$$

or:

$$m(X,Y) = h(X) + h(Y) - h(X,Y).$$
corresponding 'ditsets' for \( X \) and similarly \( \mu \)
the sets \( S \).

**Proposition 1 (Nonempty ditsets always intersect)**

If \( h(X) > 0 \), then \( m(X, Y) > 0 \).

Proof: Since \( \text{dit}(X) \) is nonempty, there are two pairs \((x, y)\) and \((x', y')\) such that \( x \neq x' \) and \( p(x, y) p(x', y') > 0 \). If \( y \neq y' \) then \( ((x, y), (x', y')) \in \text{dit}(Y) \) as well and we are finished, i.e., \( \text{dit}(X) \cap \text{dit}(Y) \neq 0 \). Hence assume \( y = y' \). Since \( \text{dit}(Y) \) is also nonempty and thus \( p(y) \neq 1 \), there is another \( y'' \) such that for some \( x'' \), \( p(x'', y'') > 0 \). Since \( x'' \) can’t be equal to both \( x \) and \( x' \), at least one of the pairs \(((x, y), (x'', y''))\) or \(((x', y), (x'', y''))\) is in both \( \text{dit}(X) \) and \( \text{dit}(Y) \), and thus the product measure on \( S_{X \wedge Y} = \{(x, y), (x', y') : x \neq x' \wedge y \neq y'\} \) is positive, i.e., \( m(X, Y) > 0 \). \( \square \)

**Corollary 1** Nonempty infosets \( S_X \) and \( S_Y \) always intersect.

Proof: For the uniform distribution on \( X \times Y \), \( \text{dit}(X) = S_X \) and \( \text{dit}(Y) = S_Y \). \( \square \)

Note that compound infosets like \( S_{X \wedge \neg Y} \) and \( S_{X \wedge Y} \) do not intersect.
10.2 Shannon mutual information

Applying the dit-bit transform $1 - p \sim \log \left( \frac{1}{p} \right)$ to the logical mutual information formula

$$m(X, Y) = \sum_{x, y} p(x, y) \left[ [1 - p(x)] + [1 - p(y)] - [1 - p(x, y)] \right]$$

expressed in terms of probability averages gives the corresponding Shannon notion:

$$I(X, Y) = \sum_{x, y} p(x, y) \left[ \log \left( \frac{1}{p(x)} \right) + \log \left( \frac{1}{p(y)} \right) - \log \left( \frac{1}{p(x, y)} \right) \right]$$

Shannon mutual information in a joint probability distribution.

Since the dit-bit transform preserves sums and differences, the logical formulas for the measures gives the mnemonic Figure 4:

$$I(X, Y) = H(X) + H(Y) - H(X, Y) = H(X, Y) - H(X|Y) - H(Y|X).$$

Figure 4: $H(X, Y) = H(X|Y) + H(Y|X) + I(X, Y)$.

This is the usual Venn diagram for the Shannon entropy notions that needs to be explained—since the Shannon entropies are not measures. Of course, one could just say the relationship holds for the Shannon entropies because that’s how they were defined. It may seem a happy accident that the Shannon definitions all satisfy the measure-like Venn diagram formulas, but as one author put it: “Shannon carefully contrived for this ‘accident’ to occur” [45, p. 153]. As noted above, Campbell asked if “these analogies have a deeper foundation” [9, p. 112] and the dit-bit transform answers that question.

11 Independent Joint Distributions

A joint probability distribution \( \{p(x, y)\} \) on \( X \times Y \) is independent if each value is the product of the marginals: \( p(x, y) = p(x)p(y) \).

For an independent distribution, the Shannon mutual information

$$I(X, Y) = \sum_{x \in X, y \in Y} p(x, y) \log \left( \frac{p(x, y)}{p(x)p(y)} \right)$$

is immediately seen to be zero so we have:

$$H(X, Y) = H(X) + H(Y)$$

Shannon entropies for independent \( \{p(x, y)\} \).
For the logical mutual information $m(X, Y)$, independence gives:

$$m(X, Y) = \sum_{x,y} p(x,y) [1 - p(x) - p(y) + p(x,y)]$$

$$= \sum_{x,y} p(x)p(y) [1 - p(x) - p(y) + p(x)p(y)]$$

$$= \sum_x p(x) [1 - p(x)] \sum_y p(y) [1 - p(y)]$$

$$= h(X) h(Y)$$

Logical entropies for independent \{p(x,y)\}.

The logical conditional entropy $h(X|Y) = h(X,Y) - h(Y) = h(X) - m(X,Y)$ is the probability that a random pair of pairs $(x,y)$ and $(x',y')$ is a distinction $x \neq x'$ for \{p(x)\} but not a distinction $y \neq y'$ of \{p(y)\}. Under independence, that logical conditional entropy is $h(X|Y) = h(X) (1 - h(Y))$ which is the probability of randomly drawing a distinction from the marginal distribution \{p(x)\} times the probability of randomly drawing an indistinction from the other marginal distribution \{p(y)\}.

The nonempty-ditsets-always-intersect proposition shows that $h(X) h(Y) > 0$, and thus that logical mutual information $m(X,Y)$ is still positive for independent distributions when $h(X) h(Y) > 0$, in which case $m(X,Y) = h(X) h(Y)$. This is a striking difference between the average bit-count Shannon entropy and the dit-count logical entropy. Aside from the waste case where $h(X) h(Y) = 0$, there are always positive probability mutual distinctions for $X$ and $Y$, and that dit-count information is not recognized by the average bit-count Shannon entropy.

## 12 Cross-entropies and divergences

Given two probability distributions $p = (p_1, ..., p_n)$ and $q = (q_1, ..., q_n)$ on the same sample space \{1, ..., n\}, we can again consider the drawing of a pair of points but where the first drawing is according to $p$ and the second drawing according to $q$. The probability that the points are distinct would be a natural and more general notion of logical entropy that would be the:

$$h(p||q) = \sum_i p_i (1 - q_i) = 1 - \sum_i p_i q_i$$

Logical cross entropy of $p$ and $q$

which is symmetric. The logical cross entropy is the same as the logical entropy when the distributions are the same, i.e., if $p = q$, then $h(p||q) = h(p)$.

Although the logical cross entropy formula is symmetrical in $p$ and $q$, there are two different ways to express it as an average in order to apply the dit-bit transform: $\sum_i p_i (1 - q_i)$ and $\sum_i q_i (1 - p_i)$. The two transforms are the two asymmetrical versions of Shannon cross entropy:

$$H(p||q) = \sum_i p_i \log \left( \frac{1}{q_i} \right) \text{ and } H(q||p) = \sum_i q_i \log \left( \frac{1}{p_i} \right).$$

which is not symmetrical due to the asymmetric role of the logarithm, although if $p = q$, then $H(p||q) = H(p)$. When the logical cross entropy is expressed as an average in a symmetrical way: $h(p||q) = \frac{1}{2} [\sum_i p_i (1 - q_i) + \sum_i q_i (1 - p_i)]$, then the dit-bit transform is the symmetrized Shannon cross entropy:

$$H_s(p||q) = \frac{1}{2} [H(p||q) + H(q||p)].$$

The Kullback-Leibler divergence (or relative entropy) $D(p||q) = \sum_i p_i \log \left( \frac{p_i}{q_i} \right)$ is defined as a measure of the distance or divergence between the two distributions where $D(p||q) = H(p||q) - H(p)$. A basic result is the:
The symmetrized Kullback-Leibler divergence is:

\[ D_s(p||q) = \frac{1}{2} \left[ D(p||q) + D(q||p) \right] = H_s(p||q) - \frac{H(p) + H(q)}{2}. \]

But starting afresh, one might ask: “What is the natural measure of the difference or distance between two probability distributions \( p = (p_1, ..., p_n) \) and \( q = (q_1, ..., q_n) \) that would always be non-negative, and would be zero if and only if they are equal?” The (Euclidean) distance between the two points in \( \mathbb{R}^n \) would seem to be the logical answer—so we take that distance (squared with a scale factor) as the definition of the:

\[ d(p||q) = \frac{1}{2} \sum_i (p_i - q_i)^2 \]

Logical divergence (or logical relative entropy) \( ^5 \)

which is symmetric and we trivially have:

\[ d(p||q) \geq 0 \text{ with equality iff } p = q \]

Logical information inequality.

We have component-wise:

\[ 0 \leq (p_i - q_i)^2 = p_i^2 - 2p_iq_i + q_i^2 = 2 \left( \frac{1}{n} - p_iq_i \right) - \left( \frac{1}{n} - p_i^2 \right) - \left( \frac{1}{n} - q_i^2 \right) \]

so that taking the sum for \( i = 1, ..., n \) gives:

\[ d(p||q) = \frac{1}{2} \sum_i (p_i - q_i)^2 = \left[ 1 - \sum_i p_iq_i \right] - \frac{1}{2} \left[ (1 - \sum_i p_i^2) + (1 - \sum_i q_i^2) \right] = h(p||q) - \frac{h(p) + h(q)}{2}. \]

Logical divergence = Jensen difference \( ^{17} \) between probability distributions.

Then the information inequality implies that the logical cross-entropy is greater than or equal to the average of the logical entropies:

\[ h(p||q) \geq \frac{h(p) + h(q)}{2} \text{ with equality iff } p = q. \]

The half-and-half probability distribution \( \frac{p+q}{2} \) that mixes \( p \) and \( q \) has the logical entropy of

\[ h\left( \frac{p+q}{2} \right) = \frac{h(p) + h(q)}{2} = \frac{1}{2} \left[ h(p||q) + \frac{h(p) + h(q)}{2} \right] \]

so that:

\[ h(p||q) \geq h\left( \frac{p+q}{2} \right) \geq \frac{h(p) + h(q)}{2} \text{ with equality iff } p = q. \]

Mixing different \( p \) and \( q \) increases logical entropy.

\(^5\)In \( ^{15} \), this definition was given without the useful scale factor of 1/2.
The logical divergence can be expressed in the proper symmetrical form of averages to apply the dit-bit transform:

\[ d(p||q) = \frac{1}{2} \left[ \sum_i p_i (1-q_i) + \sum_i q_i (1-p_i) \right] - \frac{1}{2} \left[ (\sum_i p_i (1-p_i)) + (\sum_i q_i (1-q_i)) \right] \]

so the transform is:

\[
\begin{align*}
\frac{1}{2} \left[ \sum_i p_i \log \left( \frac{1}{q_i} \right) + \sum_i q_i \log \left( \frac{1}{p_i} \right) \right] & - \frac{1}{2} \left[ \sum_i p_i \log \left( \frac{1}{p_i} \right) - \sum_i q_i \log \left( \frac{1}{q_i} \right) \right] \\
= \frac{1}{2} \left[ \sum_i p_i \log \left( \frac{p_i}{q_i} \right) + \sum_i q_i \log \left( \frac{q_i}{p_i} \right) \right] & = \frac{1}{2} \left[ D(p||q) + D(q||p) \right] \\
& = D_s(p||q).
\end{align*}
\]

Since the logical divergence \( d(p||q) \) is symmetrical, it develops via the dit-bit transform to the symmetrized version \( D_s(p||q) \) of the Kullback-Leibler divergence.

13 Summary of formulas and dit-bit transforms

The following table 3 summarizes the concepts for the Shannon and logical entropies. We use the abbreviations \( p_{xy} = p(x,y) \), \( p_x = p(x) \), and \( p_y = p(y) \).

| Table 3 | Shannon Entropy | Logical Entropy |
|---------|----------------|----------------|
| Entropy | \( H(p) = \sum p_i \log(1/p_i) \) | \( h(p) = \sum p_i (1-p_i) \) |
| Mutual Info. | \( I(X,Y) = H(X) + H(Y) - H(X,Y) \) | \( m(X,Y) = h(X) + h(Y) - h(X,Y) \) |
| Cond. entropy | \( H(X|Y) = H(X) - I(X,Y) \) | \( h(X|Y) = h(X) - m(X,Y) \) |
| Independence | \( I(X,Y) = 0 \) | \( m(X,Y) = h(X) h(Y) \) |
| Indep. Relations | \( H(X|Y) = H(X) \) | \( h(X|Y) = h(X) (1-h(Y)) \) |
| Cross entropy | \( D(p||q) = \sum p_i \log \left( \frac{p_i}{q_i} \right) \) | \( h(p||q) = \sum p_i (1-q_i) \) |
| Divergence | \( D(p||q) = H(p|q) - H(p) \) | \( d(p||q) = \frac{1}{2} \sum_i (p_i - q_i)^2 \) |
| Relationships | \( D(p||q) \geq 0 \) with \( = \) iff \( p = q \) | \( d(p||q) \geq 0 \) with \( = \) iff \( p = q \) |

Table 3: Comparisons between Shannon and logical entropy formulas

The following table 4 summarizes the dit-bit transforms.

| Table 4 | The Dit-Bit Transform: \( 1-p_i \rightarrow \log \left( \frac{1}{p_i} \right) \) |
|---------|------------------------------------------------------------------|
| \( h(p) \) = | \( \sum p_i (1-p_i) \) |
| \( H(p) = \) | \( \sum p_i \log(1/p_i) \) |
| \( h(X|Y) = \) | \( \sum_{x,y} p(x,y) \left[ (1-p(x,y)) - (1-p(y)) \right] \) |
| \( H(X|Y) = \) | \( \sum_{x,y} p(x,y) \log \left( \frac{1}{p(y)} \right) - \log \left( \frac{1}{p(x,y)} \right) \) |
| \( m(X,Y) = \) | \( \sum_{x,y} p(x,y) \left[ (1-p(x)) + (1-p(y)) - (1-p(x,y)) \right] \) |
| \( I(X,Y) = \) | \( \sum_{x,y} p(x,y) \log \left( \frac{1}{p(x)} \right) + \log \left( \frac{1}{p(y)} \right) - \log \left( \frac{1}{p(x,y)} \right) \) |
| \( h(p||q) = \) | \( \frac{1}{2} \left[ \sum_i p_i (1-q_i) + \sum_i q_i (1-p_i) \right] \) |
| \( H_s(p||q) = \) | \( \frac{1}{2} \sum_i p_i \log \left( \frac{1}{p_i} \right) + \sum_i q_i \log \left( \frac{1}{q_i} \right) \) |
| \( d(p||q) = \) | \( \frac{1}{2} \sum_i p_i - \frac{1}{2} \sum_i p_i (1-p_i) + \frac{1}{2} \sum_i q_i (1-q_i) \) |
| \( D_s(p||q) = \) | \( H_s(p||q) - \frac{1}{2} \sum_i p_i \log \left( \frac{1}{p_i} \right) - \frac{1}{2} \sum_i q_i \log \left( \frac{1}{q_i} \right) \) |

Table 4: The dit-bit transform from logical entropy to Shannon entropy
14 Entropies for multivariate joint distributions

Let \( \{p(x_1,\ldots,x_n)\} \) be a probability distribution on \( X_1 \times \ldots \times X_n \) for finite \( X_i \)'s. Let \( S \) be a subset of \( (X_1 \times \ldots \times X_n)^2 \) consisting of certain ordered pairs of ordered \( n \)-tuples \( ((x_1,\ldots,x_n),(x_1',\ldots,x_n')) \) so the product probability measure on \( S \) is:

\[
\mu(S) = \sum \{p(x_1,\ldots,x_n)p(x_1',\ldots,x_n') : ((x_1,\ldots,x_n),(x_1',\ldots,x_n')) \in S\}.
\]

Then all the logical entropies for this \( n \)-variable case are given as the product measure of certain infosets \( S \). Let \( I, J \subseteq N \) be subsets of the set of all variables \( N = \{X_1,\ldots,X_n\} \) and let \( x = (x_1,\ldots,x_n) \) and similarly for \( x' \).

The joint logical entropy of all the variables is: \( h(X_1,\ldots,X_n) = \mu(S_N) \) where:

\[
S_N = \left\{ (x,x') : \bigvee_{i=1}^n x_i \neq x'_i \right\} = \bigcup \{S_{X_i} : X_i \in N\}
\]

(where \( \bigvee \) represents the disjunction of statements). For a non-empty \( I \subseteq N \), the joint logical entropy of the variables in \( I \) could be represented as \( h(I) = \mu(S_I) \) where:

\[
S_I = \{ (x,x') : \bigvee x_i \neq x'_i \text{ for } X_i \in I \} = \bigcup \{S_{X_i} : X_i \in I\}
\]

so that \( h(X_1,\ldots,X_n) = h(N) \).

As before, the information algebra \( \mathcal{I}(X_1 \times \ldots \times X_n) \) is the Boolean subalgebra of \( \mathcal{P}(X_1 \times \ldots \times X_n)^2 \) generated by the infosets \( S_{X_i} \) for the variables and their complements \( S_{\neg X_i} \).

For the conditional logical entropies, let \( I, J \subseteq N \) be two non-empty disjoint subsets of \( N \). The idea for the conditional entropy \( h(I,J) \) is to represent the information in the variables \( I \) given by the defining condition: \( \bigvee x_i \neq x'_i \) for \( X_i \in I \), after taking away the information in the variables \( J \) which is defined by the condition: \( \bigvee x_j \neq x'_j \) for \( X_j \in J \). Hence we negate that condition for \( J \) and add it to the condition for \( I \) to obtain the conditional logical entropy as \( h(I,J) = \mu(S_{I\setminus J}^{\perp}) \) where:

\[
S_{I\setminus J}^{\perp} = \{ (x,x') : \bigvee x_i \neq x'_i \text{ for } X_i \in I \text{ and } \bigwedge x_j = x'_j \text{ for } X_j \in J\}
\]

(\( \bigwedge \) represents the conjunction of statements).

For the mutual logical information of a nonempty set of variables \( I \), \( m(I) = \mu(S_I) \) where:

\[
S_I = \{ (x,x') : \bigwedge x_i \neq x'_i \text{ for } X_i \in I\}.
\]

For the conditional mutual logical information, let \( I, J \subseteq N \) be two non-empty disjoint subsets of \( N \) so that \( m(I,J) = \mu(S_{I\setminus J}^{\perp}) \) where:

\[
S_{I\setminus J}^{\perp} = \{ (x,x') : \bigwedge x_i \neq x'_i \text{ for } X_i \in I \text{ and } \bigwedge x_j = x'_j \text{ for } X_j \in J\}.
\]

And finally by expressing the logical entropy formulas as averages, the dit-bit transform will give the corresponding versions of Shannon entropy.

Consider an example of a joint distribution \( \{p(x,y,z)\} \) on \( X \times Y \times Z \). The mutual logical information \( m(X,Y,Z) = \mu(S_{X,Y,Z}) \) where:

\[
S_{X,Y,Z} = \{ (((x,y,z),(x',y',z')) : x \neq x' \land y \neq y' \land z \neq z' \}.
\]
From the Venn diagram for $h(X, Y, Z)$, we have (using a variation on the inclusion-exclusion principle): \[ m(X, Y, Z) = h(X) + h(Y) + h(Z) - h(X, Y) - h(X, Z) - h(Y, Z) + h(X, Y, Z). \]

Substituting the averaging formulas for the logical entropies gives:

\[
\sum_{x,y,z} p(x, y, z) \left[ 1 - p(x) - p(y) - p(z) + p(x, y) + p(x, z) + p(y, z) - p(x, y, z) \right] = m(X, Y, Z).
\]

Then applying the dit-bit transform gives the corresponding formula for the multivariate Shannon mutual information:

\[
I(X, Y, Z) = \sum_{x,y,z} p(x, y, z) \left[ \log \left( \frac{1}{p(x)} \right) + \log \left( \frac{1}{p(y)} \right) + \log \left( \frac{1}{p(z)} \right) - \log \left( \frac{1}{p(x, y)} \right) - \log \left( \frac{1}{p(x, z)} \right) - \log \left( \frac{1}{p(y, z)} \right) + \log \left( \frac{1}{p(x, y, z)} \right) \right] .
\]

To emphasize that Venn-like diagrams are only a mnemonic analogy, Abramson gives an example \[ \text{where the Shannon mutual information of three variables is negative.}^8 \]

Consider the joint distribution \( \{ p(x, y, z) \} \) on \( X \times Y \times Z \) where \( X = Y = Z = \{0, 1\} \).

| $X$ | $Y$ | $Z$ | $p(x, y, z)$ | $p(x, y)$, $p(x, z)$, $p(y, z)$ | $p(x)$, $p(y)$, $p(z)$ |
|-----|-----|-----|--------------|-------------------------------|-----------------|
| 0   | 0   | 0   | 1/8          | 1/8, 1/8, 1/8                | 1/8, 1/8, 1/8   |
| 0   | 0   | 1   | 1/8          | 1/8, 1/8, 1/8                | 1/8, 1/8, 1/8   |
| 0   | 1   | 0   | 1/8          | 1/8, 1/8, 1/8                | 1/8, 1/8, 1/8   |
| 0   | 1   | 1   | 1/8          | 1/8, 1/8, 1/8                | 1/8, 1/8, 1/8   |
| 1   | 0   | 0   | 1/8          | 1/8, 1/8, 1/8                | 1/8, 1/8, 1/8   |
| 1   | 0   | 1   | 1/8          | 1/8, 1/8, 1/8                | 1/8, 1/8, 1/8   |
| 1   | 1   | 0   | 1/8          | 1/8, 1/8, 1/8                | 1/8, 1/8, 1/8   |
| 1   | 1   | 1   | 1/8          | 1/8, 1/8, 1/8                | 1/8, 1/8, 1/8   |

Table 5: Abramson’s example giving negative Shannon mutual information $I(X, Y, Z)$.

---

6. The usual version of the inclusion-exclusion principle would be: $h(X, Y, Z) = h(X) + h(Y) + h(Z) - h(X, Y) - h(X, Z) - h(Y, Z) + h(X, Y, Z)$.  
7. The multivariate generalization of the Shannon mutual information was developed by William J. McGill and Robert M. Fano at MIT in the early 50’s and independently by Nelson M. Blachman. The criterion for it being the ‘correct’ generalization seems to be that it satisfied the generalized inclusion-exclusion formulas (which generalize the two-variable Venn diagram) that are automatically satisfied by any measure and are thus also obtained from the multivariate logical mutual information using the dit-bit transform.  
8. Fano had earlier noted that for three or more variables, the Shannon mutual information could be negative. [17] p. 58
Since the logical mutual information \( m(X, Y, Z) \) is the measure \( \mu(S_{\wedge\{X, Y, Z\}}) \), it is always non-negative and in this case is 0:

\[
m(X, Y, Z) = h(X) + h(Y) + h(Z) - h(X, Y) - h(X, Z) - h(Y, Z) + h(X, Y, Z)
= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} - \frac{3}{4} - \frac{3}{4} + \frac{3}{4} = \frac{3}{4} = 0.
\]

All the compound notions of logical entropy have a direct interpretation as a two-draw probability. The logical mutual information \( m(X, Y, Z) \) is the probability that in two independent samples of \( X \times Y \times Z \), the outcomes would differ in all coordinates. This means the two draws would have the form \((x, y, z)\) and \((1-x, 1-y, 1-z)\) for the binary variables, but it is easily seen by inspection that \( p(x, y, z) = 0 \) or \( p(1-x, 1-y, 1-z) = 0 \), so the products are 0 as computed.

The Venn-diagram-like formula for \( m(X, Y, Z) \) carries over to \( I(X, Y, Z) \) by the dit-bit transform (since it preserves sums and differences), but the “area” \( I(X, Y, Z) \) is negative:

\[
I(X, Y, Z) = H(X) + H(Y) + H(Z) - H(X, Y) - H(X, Z) - H(Y, Z) + H(X, Y, Z)
= 1 + 1 + 1 - 2 - 2 - 2 + 2 = 3 - 4 = -1.
\]

It is unclear how that can be interpreted as the mutual information contained in the three variables or how the corresponding “Venn diagram” (Figure 6) can be anything more than a mnemonic for a formula. Indeed, as Csiszar and Körner remark:

The set-function analogy might suggest to introduce further information quantities corresponding to arbitrary Boolean expressions of sets. E.g., the “information quantity” corresponding to \( \mu(A \cap B \cap C) = \mu(A \cap B) - \mu((A \cap B) - C) \) would be \( I(X, Y) - I(X, Y|Z) \); this quantity has, however, no natural intuitive meaning. [12, pp. 53-4]

Of course, all this works perfectly well in logical information theory for “arbitrary Boolean expressions of sets” in the information algebra \( I(X \times Y \times Z) \), e.g., \( m(X, Y, Z) = \mu(S_X \cap S_Y \cap S_Z) = \mu(S_X \cap S_Y) - \mu((S_X \cap S_Y) - S_Z) = m(X, Y) - m(X, Y|Z) \), which also as a (two-draw) probability measure is always non-negative.

![Figure 6: Negative \( I(X, Y, Z) \) in ‘Venn diagram.’](image)

Note how the ‘intuitiveness’ of independent random variables giving disjoint Venn diagram circles comes back in a strange form in the multivariate case since the three variables \( X, Y, \) and \( Z \) in the example are pairwise independent but not mutually independent (since any two determines the third). Hence the circles for, say, \( H(X) \) and \( H(Y) \) ‘intersect’ but the lense-shaped intersection is \( I(X, Y) = I(X, Y|Z) + I(X, Y, Z) = +1 - 1 = 0. \)

The dit-bit transform turns the formula for \( m(X, Y, Z) \) into the formula for \( I(X, Y, Z) \), and it preserves the Venn-diagram relationships but it does not preserve non-negativity.
15 Logical entropy as so-called ‘linear entropy’

The Taylor series for $\ln(x+1)$ around $x = 0$ is:

$$
\ln(x+1) = \ln(1) + x - \frac{x^2}{2} + \frac{x^3}{3} (x+1)^{-3} - \ldots = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots
$$

so substituting $x = p_i - 1$ (with $p_i > 0$) gives a version of the Newton-Mercator series:

$$
-\ln(p_i) = \ln \left( \frac{1}{p_i} \right) = 1 - p_i + \frac{(p_i-1)^2}{2} - \frac{(p_i-1)^3}{3} + \ldots
$$

Then multiplying by $p_i$ and summing yields:

$$
H_e(p) = -\sum_i p_i \ln(p_i) = \sum_i p_i (1-p_i) + \sum_i \frac{p_i(p_i-1)^2}{2} - \ldots
$$

A similar relationship holds in the quantum case between the von Neumann entropy $S(\rho) = -\text{tr}[\rho \ln(\rho)]$ and the quantum logical entropy $h(\rho) = \text{tr}[\rho (1-\rho)] = 1 - \text{tr}[\rho^2]$ which is defined by having a density matrix $\rho$ replace the probability distribution $p$ and the trace replace the sum.

Quantum logical entropy is beyond the scope of this paper but it might be noted that some quantum information theorists have been using that concept to rederive results previously derived using the von Neumann entropy such as the Klein inequality, concavity, and a Holevo-type bound for Hilbert-Schmidt distance ([49], [50]). There are many older results derived under the misnomer “linear entropy” or derived for the quadratic special case of the Tsallis-Havrda-Charvat entropy ([26], [51], [52]). Moreover the logical derivation of the logical entropy formulas using the notion of “linear entropy” or derived for the quadratic special case of the Tsaliss-Havrda-Charvat entropy ([26], [51], [52]).

We find this framework of partitions and distinction most suitable (at least conceptually) for describing the problems of quantum state discrimination, quantum cryptography and in general, for discussing quantum channel capacity. In these problems, we are basically interested in a distance measure between such sets of states, and this is exactly the kind of distinctions gives a certain naturalness to the notion of quantum logical entropy.

The relationship between the Shannon/von Neumann entropies and the logical entropies in the classical and quantum cases is responsible for presenting the logical entropy as a ‘linear’ approximation to the Shannon or von Neumann entropies since $1 - p_i$ is the linear term in the series for $-\ln(p_i)$ [before the multiplication by $p_i$ to make the term quadratic]! And $h(p) = 1 - \sum_i p_i^2$ or quantum counterpart $h(\rho) = 1 - \text{tr}[\rho^2]$ are even called “linear entropy” (e.g., [8] or [41]) even though the formulas are obviously quadratic! Another name for the quantum logical entropy found in the literature is “mixedness” ([30], p. 5) which at least doesn’t call a quadratic formula ‘linear.’ It is even called “impurity” since the complement $1 - h(\rho) = \text{tr}[\rho^2]$ (i.e., the quantum version of Turing’s repeat rate $\sum_i p_i^2$) is called the “purity.” And as noted above, the formula for logical entropy occurs as the quadratic special case of the Tsallis-Havrda-Charvat entropy. Those parameterized families of entropy formulas are sometimes criticized for lacking a convincing interpretation, but we have seen that the quadratic case is based on partition logic dual to Boole’s subset logic. In terms of the
duality between elements of a subset (its) and distinctions of a partition (dits), the two measures are based on the normalized counting measures of ‘its’ and ‘dits’.

In accordance with its quadratic nature, logical entropy is the logical special case of C. R. Rao’s quadratic entropy [42]. Two elements from \( U = \{ u_1, \ldots, u_n \} \) are either identical or distinct. Gini [19] introduced \( d_{ij} \) as the ‘distance’ between the \( i^{th} \) and \( j^{th} \) elements where \( d_{ij} = 1 \) for \( i \neq j \) and \( d_{ii} = 0 \)–which might be considered the ‘logical distance function’ \( d_{ij} = 1 - \delta_{ij} \), the complement of the Kronecker delta. Since \( 1 = (p_1 + \ldots + p_n)(p_1 + \ldots + p_n) = \sum_ip_i^2 + \sum_{i \neq j}p_ip_j \), the logical entropy, i.e., Gini’s index of mutability, \( h(p) = 1 - \sum_ip_i^2 = \sum_{i \neq j}p_ip_j \), is the average logical distance between distinct elements. But one might generalize by allowing other distances \( d_{ij} = d_{ji} \) for \( i \neq j \) (but always \( d_{ii} = 0 \)) so that \( Q = \sum_{i \neq j}d_{ij}p_ip_j \) would be the average distance between distinct elements from \( U \).

In 1982, C. R. Rao introduced this concept as quadratic entropy [42].

Rao’s treatment also includes (and generalizes) the natural extension of logical entropy to continuous (square-integrable) probability density functions \( f(x) \) for a random variable \( X \): \( h(X) = 1 - \int f(x)^2 \, dx \). It might be noted that the natural extension of Shannon entropy to continuous probability density functions \( f(x) \) through the limit of discrete approximations contains terms \( 1/\log(\Delta x_i) \) that blow up as the mesh size \( \Delta x_i \) goes to zero (see [38] pp. 34-38). Hence the definition of Shannon entropy in the continuous case is not defined by the limit of the discrete formula but by the analogous formula \( H(X) = -\int f(x) \log(f(x)) \, dx \) which, as McEliece points out, “is not in any sense a measure of the randomness of \( X \” [38] p. 38] in addition to possibly having negative values.

16 On ‘intuitions’ about information

Lacking an immediate and convincing interpretation for an entropy formula, one might produce a number of axioms about a ‘measure of information’ where each axiom is more or less intuitive. One supposed intuition about ‘information’ is that the information in independent random variables should be additive (unlike probabilities \( p(x, y) = p(x)p(y) \)) or that the ‘information’ in one variable conditional on a second variable should be the same as the ‘information’ in the first variable alone when the variables are independent (like probabilities \( p(x|y) = p(x) \)).

Another intuition is that the information gathered from the occurrence of an event is inversely related to the probability of the event. For instance, if the probability of an outcome is \( p_i \), then \( 1/p_i \) is a good indicator of the surprise-value information gained by the occurrence of the event. Very well; let us follow out that intuition to construct a ‘surprise-value entropy.’ We need to average the surprise-values across the probability distribution \( p = \{ p_i \} = \{ p_1, \ldots, p_n \} \), and since the surprise-value is the multiplicative inverse of the \( p_i \), the natural notion of average is the multiplicative (or geometric) average:

\[
E(p) = \prod_{i=1}^{n} \left( \frac{1}{p_i} \right)^{p_i}.
\]

Surprise-value entropy of a probability distribution \( p = \{ p_i \} = \{ p_1, \ldots, p_n \} \).

How do the surprise-value intuitions square with intuitions about additive information content for independent events? Given a joint probability distribution \( p_{xy} = p(x, y) \) on \( X \times Y \), the two marginal distributions are \( p_x = \sum_y p_{xy} \) and \( p_y = \sum_x p_{xy} \). Then we showed previously that if the joint distribution was independent, i.e., \( p_{xy} = p_xp_y \), then the Shannon entropies were additive (unlike probabilities):

\[
H(x, y) = H(x) + H(y)
\]

Shannon entropies under independence.

11 For expository purposes, we have restricted the treatment to finite sample spaces \( U \). For some countable discrete probability distributions, the Shannon entropy blows up to infinity [38] Example 2.46, p. 30], while the logical infosets are always well-defined and the logical entropy is always in the half-open interval \([0, 1)\).
This is in accordance with one ‘intuition’ about independence.

But the surprise-value entropy is also based on intuitions so we need to check if it is also additive for an independent joint distribution so that the intuitions would be consistent. The surprise-value entropy of the independent joint distribution \( \{p_{xy}\} \) is:

\[
E(\{p_{xy}\}) = \prod_{x,y} (\frac{1}{p_{xy}})^{p_{xy}} = \prod_{x,y} \left( \frac{1}{p_x p_y} \right)^{p_{xy}} = \prod_x \prod_y \left( \frac{1}{p_x p_y} \right)^{p_{xy}} = \prod_x \prod_y \left( \frac{1}{p_x} \right)^{p_{xy}} \prod_x \prod_y \left( \frac{1}{p_y} \right)^{p_{xy}} \\
= \left[ \prod_x \left( \frac{1}{p_x} \right)^{p_{x}} \right] \left[ \prod_y \left( \frac{1}{p_y} \right)^{p_{y}} \right] = E(\{p_x\}) E(\{p_y\})
\]

so the surprise-value of an independent joint distribution is the product of the surprise-value entropies of the marginal distributions (like probabilities). The derivation used the fact that the multiplicative average of a constant is, of course, that constant, e.g., \( \prod_y e^{p_y} = e^{\sum_y p_y} = c \).

Since the two intuitions give conflicting results, which, if either, is ‘correct’? Should ‘entropy’ be additive or multiplicative for independent distributions? At this point, it is helpful to step back and note that in statistics, for example, any product of random variables \( XY \) can sometimes, with advantage, be analyzed using the sum of log-variables, \( \log (XY) = \log (X) + \log (Y) \). It is best seen as a question of convenience rather than ‘truth’ whether to use the product or the log of the product.

In the case at hand, the notion of surprise-value entropy, which is multiplicative for independent distributions, can trivially be turned into an expression that is additive for independent distributions by taking logarithms to some base:

\[
\log E(\{p_{xy}\}) = \log E(\{p_x\}) + \log E(\{p_y\}).
\]

Is the original surprise-value formula \( E(p) \) or the log-of-surprise-value formula \( \log E(p) \) the ‘true’ measure? And, in the case at hand, the point is that the log-of-surprise-value formula is the Shannon entropy:

\[
\log E(p) = H(p) \text{ or } E(p) = 2^{H(p)}.
\]

Some authors have even suggested that the surprise-value formula is more intuitive than the log-formula. To understand this intuition, we need to develop another interpretation of the surprise-value formula. When an event or outcome has a probability \( p_i \), it is intuitive to think of it as being drawn from a set of \( \frac{1}{p_i} \) equiprobable elements (particularly when \( \frac{1}{p_i} \) is an integer) so \( \frac{1}{p_i} \) is called the numbers-equivalent [3] of the probability \( p_i \). Hence the multiplicative average of the numbers-equivalents for a probability distribution \( p = (p_1, \ldots, p_n) \) is \( E(p) \), which thus will now be called the numbers-equivalent entropy (also called “exponential entropy” [10]). This approach also supplies an interpretation: Sampling a probability distribution \( p \) is like, on average, sampling from a distribution with \( E(p) \) equiprobable outcomes.

In the biodiversity literature, the situation is that each animal (in a certain territory) is considered to be equiprobable to be sampled and the partition of the animals is by species. Taking \( p = (p_1, \ldots, p_n) \) as the probability distribution of the \( n \) species, the numbers-equivalent entropy \( E(p) \) is the measure of biodiversity that says sampling the population is like sampling a population of \( E(p) \) equally common species. The mathematical biologist Robert H. MacArthur finds this much more intuitive than Shannon entropy.

Returning to the example of a census of 99 individuals of one species and 1 of a second, we calculate \( H = \ldots = 0.0560 \) [as the Shannon entropy using natural logs]. For a census of fifty individuals of each of the two species we would get \( H = \ldots = 0.693 \). To convert these back to ‘equally common species’, we take \( e^{0.0560} = 1.057 \) for the first census and \( e^{0.693} = 2.000 \) for the second. These numbers, 1.057 and 2, accord much more closely with our intuition of how diverse the areas actually are,.... [35] p. 514]
MacArthur’s interpretation is “that \([E(p)]\) equally common species would have the same diversity as the \([n]\) equally common species in our census.” [35] p. 514

The point is that ‘intuitions’ differ even between Shannon entropy \(H(p)\) and its base-free anti-log \(E(p)\), not to mention between other approaches to entropy. There is now in the literature a ‘veritable plethora’ of entropy definitions [2; 52] each with its ‘intuitive axioms.’ Surely there are better criteria for entropy concepts that differing subjective intuitions.

17 The connection with entropy in statistical mechanics

Shannon entropy is sometimes referred to as “Boltzmann-Shannon entropy” or “Boltzmann-Gibbs-Shannon entropy” since the Shannon formula supposedly has the same functional form as Boltzmann entropy which even motivated the name “entropy.” The name “entropy” is here to stay, but no one would suggest using that “more accurate” entropy formula in information theory or dream of calling it the “Boltzmann-Shannon entropy.” Shannon’s formula should be justified and understood on its own terms (see next section), and not by over-interpreting the numerically approximate relationship with entropy in statistical mechanics.

\[ S = \frac{1}{N} \ln W = \frac{1}{N} \ln \left( \prod_{i=1}^{n} \frac{N!}{n!} \right) = \frac{1}{N} \left[ \ln(N!) - \sum_i \ln(N_i!) \right] \]

can then be developed using the first two terms in the Stirling approximation

\[
\frac{1}{N} \ln W \approx \frac{1}{N} \left[ N \ln N - N - \sum_i N_i \ln N_i \right] = \frac{1}{N} \left[ N \ln N - \sum_i N_i \ln(N_i) - \sum_i N_i \ln(N) \right] = \sum_i \frac{N_i}{N} \ln \left( \frac{N_i}{N} \right) = \sum_i p_i \ln \left( \frac{1}{p_i} \right) = H_e(p)
\]

where \(p_i = \frac{N_i}{N}\) (and where the formula with logs to the base \(e\) only differs from the usual base 2 formula by a scaling factor). Shannon’s entropy \(H_e(p)\) is in fact an excellent numerical approximation to Boltzmann entropy \(S = \frac{1}{N} \ln W\) for large \(N\) (e.g., in statistical mechanics). But that does not justify using expressions like “Boltzmann-Shannon entropy” as if the log of the combinatorial formula \(W\) involving factorials was the same as the two-term Stirling approximation.

The common claim that Shannon’s entropy has the same functional form as entropy in statistical mechanics is simply false. If we use a three-term Stirling approximation, then we obtain an even better numerical approximation[12]

\[
S = \frac{1}{N} \ln W \approx H_e(p) + \frac{1}{2N} \ln \left( \frac{2\pi N^n}{(2\pi)^n N^N} \right)
\]

but no one would suggest using that “more accurate” entropy formula in information theory or dream of calling it the “Boltzmann-Shannon entropy.” Shannon’s formula should be justified and understood on its own terms (see next section), and not by over-interpreting the numerically approximate relationship with entropy in statistical mechanics.

[12] For the case \(n = 2\), MacKay [36] p. 2] also uses the next term in the Stirling’s approximation to give a “more accurate approximation” to the entropy of statistical mechanics than the Shannon entropy (the two-term approximation).
The statistical interpretation of Shannon entropy

Shannon, like Ralph Hartley before him, starts with the question of how much ‘information’ is required to single out a designated element from a set \( U \) of equiprobable elements. Renyi formulated this in terms of the search for a hidden element like the answer in a Twenty Questions game or the sent message in a communication. But being able to always find the designated element is equivalent to being able to distinguish all elements from one another.

One might quantify ‘information’ as the minimum number of yes-or-no questions in a game of Twenty Questions that it would take in general to distinguish all the possible “answers” (or “messages” in the context of communications). This is readily seen in the simple case where \(|U| = n = 2^m\), i.e., the size of the set of equiprobable elements is a power of 2. Then following the lead of Wilkins over three centuries earlier, the \( 2^m \) elements could be encoded using words of length \( m \) in a binary code such as the digits \{0, 1\} of binary arithmetic (or \{A, B\} in the case of Wilkins). Then an efficient or minimum set of yes-or-no questions needed to single out the hidden element is the set of \( m \) questions:

“Is the \( j \)th digit in the binary code for the hidden element a 1?”

for \( j = 1, \ldots, m \). Each element is distinguished from any other element by their binary codes differing in at least one digit. The information gained in finding the outcome of an equiprobable binary trial, like flipping a fair coin, is what Shannon calls a bit (derived from “binary digit”). Hence the information gained in distinguishing all the elements out of \( 2^m \) equiprobable elements is:

\[
m = \log_2 (2^m) = \log_2 (|U|) = \log_2 \left( \frac{1}{p_0} \right) \text{ bits}
\]

where \( p_0 = \frac{1}{2^m} \) is the probability of any given element (henceforth all logs to base 2).\(^{13}\)

In the more general case where \(|U| = n\) is not a power of 2, Shannon and Hartley extrapolate to the definition of \( H(p_0) \) where \( p_0 = \frac{1}{n} \) as:

\[
H(p_0) = \log \left( \frac{1}{p_0} \right) = \log (n)
\]

Shannon-Hartley entropy for an equiprobable set \( U \) of \( n \) elements.

The Shannon formula then extrapolates further to the case of different probabilities \( p = (p_1, \ldots, p_n) \) by taking the average:

\[
H(p) = \sum_{i=1}^{n} p_i \log_2 \left( \frac{1}{p_i} \right).
\]

Shannon entropy for a probability distribution \( p = (p_1, \ldots, p_n) \).

How can that extrapolation and averaging be made rigorous to offer a more convincing interpretation? Shannon uses the law of large numbers. Suppose that we have a three-letter alphabet \{a, b, c\} where each letter was equiprobable, \( p_a = p_b = p_c = \frac{1}{3} \), in a multi-letter message. Then a one-letter or two-letter message cannot be exactly coded with a binary 0, 1 code with equiprobable 0’s and 1’s. But any probability can be better and better approximated by longer and longer representations in the binary number system. Hence we can consider longer and longer messages of \( N \) letters along with better and better approximations with binary codes. The long run behavior of messages \( u_1u_2\ldots u_N \) where \( u_i \in \{a, b, c\} \) is modeled by the law of large numbers so that the letter \( a \) on average occurs \( p_aN = \frac{1}{3}N \) times and similarly for \( b \) and \( c \). Such a message is called typical.

The probability of any one of those typical messages is:

\[
p_a^N p_b^N p_c^N = \left[p_a^N p_b^N p_c^N \right]^{\frac{1}{N}}
\]

\(^{13}\)This is the special case where Campbell noted that Shannon entropy acted as a measure to count that number of binary partitions.
or, in this case,
\[
\left[ \left( \frac{1}{3} \right)^{1/3} \left( \frac{1}{2} \right)^{1/3} \left( \frac{1}{4} \right)^{1/3} \right]^N = \left( \frac{1}{3} \right)^N.
\]
Hence the number of such typical messages is \(3^N\).

If each message was assigned a unique binary code, then the number of 0,1’s in the code would have to be \(X\) where \(2^X = 3^N\) or \(X = \log(3^N) = N \log(3)\). Hence the number of equiprobable binary questions or bits needed per letter (i.e., to distinguish each letter) of a typical message is:

\[
N \log(3)/N = \log(3) = 3 \times \log \left( \frac{1}{1/3} \right) = H(p).
\]

This example shows the general pattern.

In the general case, let \(p = (p_1, ..., p_n)\) be the probabilities over a \(n\)-letter alphabet \(A = \{a_1, ..., a_n\}\). In an \(N\)-letter message, the probability of a particular message \(u_1u_2...u_N = \Pi_{i=1}^{N} \Pr(u_i)\) where \(u_i\) could be any of the symbols in the alphabet so if \(u_i = a_j\) then \(\Pr(u_i) = p_j\).

In a typical message, the \(i^{th}\) symbol will occur \(p_iN\) times (law of large numbers) so the probability of a typical message is (note change of indices to the letters of the alphabet):

\[
\Pi_{k=1}^{n} \prod_{i=1}^{P} p_{k}^{p_{k}}N = \left[ \Pi_{k=1}^{n} \prod_{i=1}^{P} p_{k}^{p_{k}} \right]^{N}.
\]

Thus the probability of a typical message is \(P^N\) where it is as if each letter in a typical message was equiprobable with probability \(P = \Pi_{k=1}^{n} \prod_{i=1}^{P} p_{k}^{p_{k}}\). No logs have been introduced into the argument yet, so we have an interpretation of the base-free numbers-equivalent entropy \(E(p) = P^{-1}\): it is as if each letter in a typical message is being draw from an alphabet with \(P^{-1} = \Pi_{k=1}^{n} \prod_{i=1}^{P} p_{k}^{p_{k}}\) equiprobable letters. Hence the number of \(N\)-letter messages from the equiprobable alphabet is then \(\left[ \Pi_{k=1}^{n} \prod_{i=1}^{P} p_{k}^{p_{k}} \right]^{N}\).

The choice of base 2 means assigning a unique binary code to each typical message requires \(X\) bits where \(2^X = \left[ \Pi_{k=1}^{n} \prod_{i=1}^{P} p_{k}^{p_{k}} \right]^{N}\) where:

\[
X = \log \left\{ \left[ \Pi_{k=1}^{n} \prod_{i=1}^{P} p_{k}^{p_{k}} \right]^{N} \right\} = N \log \left[ \Pi_{k=1}^{n} \prod_{i=1}^{P} p_{k}^{p_{k}} \right] = N \sum_{k=1}^{n} \log \left( p_{k}^{p_{k}} \right) = N \sum_{k=1}^{n} \log(p_{k}) = N \sum_{k=1}^{n} p_{k} \log \left( \frac{1}{p_{k}} \right) = NH(p).
\]

Dividing by the number \(N\) of letters gives the average bit-count interpretation of the Shannon entropy; \(H(p) = \sum_{k=1}^{n} p_{k} \log \left( \frac{1}{p_{k}} \right)\) is the average number of bits necessary to distinguish each letter in a typical message.

This result, usually called the noiseless coding theorem, allows us to conceptually relate the logical and Shannon entropies (the dit-bit transform gives the quantitative relationship). In terms of the simplest case for partitions, the Shannon entropy \(H(\pi) = \sum_{B \in \pi} p_B \log_2 (1/p_B) = -\sum_{B \in \pi} p_B \log_2 (p_B)\) is a requantification of the logical measure of information \(h(\pi) = \frac{\text{dit}(\pi)}{\text{dit}(\pi)} = 1 - \sum_{B \in \pi} p_B\). Instead of directly counting the distinctions of \(\pi\), the idea behind Shannon entropy is to count the (minimum) number of binary partitions needed to make all the distinctions of \(\pi\). In the special case of \(\pi\) having \(2^m\) equiprobable blocks, the number of binary partitions \(\beta_i\) needed to make the distinctions \(\text{dit}(\pi)\) of \(\pi\) is \(m\). Represent each block by an \(m\)-digit binary number so the \(i^{th}\) binary partition \(\beta_i\) just distinguishes those blocks with \(i^{th}\) digit 0 from those with \(i^{th}\) digit 1. Thus there are \(m\) binary partitions \(\beta_i\) such that \(\bigvee_{i=1}^{m} \beta_i = \pi\) or, equivalently, \(\bigvee_{i=1}^{m} \text{dit}(\beta_i) = \text{dit}(\pi)\). Thus \(m\) is the exact number of binary partitions it takes to make the distinctions of \(\pi\). In the general case,

\[^{14}\text{Thus as noted by John Wilkins in 1641, five letter words in a two-letter code would suffice to distinguished }2^5 = 32\text{ distinct entities. }\]
Shannon gives the above statistical interpretation so that $H(\pi)$ is the minimum average number of binary partitions or bits needed to make the distinctions of $\pi$.

Note the difference in emphasis. Logical information theory is only concerned with counting the distinctions between distinct elements, not with uniquely designating the distinct entities. By requantifying to count the number of binary partitions it takes to make the same distinctions, the emphasis shifts to the length of the binary code necessary to uniquely designate the distinct elements. Thus the Shannon information theory perfectly dovetails into coding theory and is often presented today as the unified theory of information and coding (e.g., [38] or [24]). It is that shift to not only making distinctions but uniquely coding the distinct outcomes that gives the Shannon theory of information, coding, and communication such importance in applications.

19 Concluding remarks

The answer to the title question is that partition logic gives a derivation of the (old) formula $h(\pi) = 1 - \sum_i p_{B_i}^2$ for partitions as the normalized counting measure on the distinctions (‘dits’) of a partition $\pi = (B_1, ..., B_m)$ that is the analogue of the Boolean subset logic derivation of logical probability as the normalized counting measure on the elements (‘its’) of a subset. In short, logical information is the quantitative measure built on top of partition logic just as logical probability is the quantitative measure built on top of ordinary subset logic which might be symbolized as:

$$\frac{\text{logical information}}{\text{partition logic}} = \frac{\text{logical probability}}{\text{subset logic}}.$$ 

Since conventional information theory has heretofore been focused on the original notion of Shannon entropy (and quantum information theory on the corresponding notion of von Neumann entropy), much of the paper has compared the logical entropy notions to the corresponding Shannon entropy notions.

Logical entropy, like logical probability, is a measure, while Shannon entropy is not. The compound Shannon entropy concepts nevertheless satisfy the measure-like Venn diagram relationships that are automatically satisfied by a measure. This can be explained by the dit-bit transform so that by putting a logical entropy notion into the proper form as an average of dit-counts, one can replace a dit-count by a bit-count and obtain the corresponding Shannon entropy notion—which shows why the latter concepts satisfy the same Venn diagram relationships.

Other comparisons were made in terms of the various ‘intuitions’ expressed in axioms, on the alleged identity in functional form between Shannon entropy and entropy in statistical mechanics, and on the statistical interpretation of Shannon entropy and its base-free antilog, the numbers-equivalent entropy $E(p) = 2^{H(p)}$.

The basic idea of information is distinctions, and distinctions have a precise definition (dits) in partition logic. Prior to using any probabilities, logical information theory defines the information sets (i.e., sets of distinctions) which for partitions are the ditsets. Given a probability distribution on a set $U$, the product probability measure on $U \times U$ applied to the information sets gives the quantitative notion of logical entropy. Information sets and logical entropy give the basic combinatorial and quantitative notions of information-as-distinctions. Shannon entropy is a requantification (well-adapted for the theory of coding and communication) that counts the minimum number of binary partitions (bits) that are required, on average, to make all the same distinctions, i.e., to encode the distinguished elements.

References

[1] Abramson, Norman 1963. Information Theory and Coding. New York: McGraw-Hill.
2 Aczel, J., and Z. Daroczy. 1975. *On Measures of Information and Their Characterization*. New York: Academic Press.

3 Adelman, M. A. 1969. Comment on the H Concentration Measure as a Numbers-Equivalent. *Review of Economics and Statistics*. 51: 99-101.

4 Adriaans, Pieter, and Johan van Benthem, eds. 2008. *Philosophy of Information*. Vol. 8. Hand- book of the Philosophy of Science. Amsterdam: North-Holland.

5 Bennett, Charles H. 2003. Quantum Information: Qubits and Quantum Error Correction. *International Journal of Theoretical Physics* 42 (2 February): 153–76.

6 Blachman, Nelson M. 1961. A Generalization of Mutual Information. *Proc. IRE* 49 (8 August): 1331–32.

7 Boole, George 1854. *An Investigation of the Laws of Thought on which are founded the Mathematical Theories of Logic and Probabilities*. Cambridge: Macmillan and Co.

8 Buscemi, Fabrizio, Paolo Bordone, and Andrea Bertoni. 2007. Linear Entropy as an Entanglement Measure in Two-Fermion Systems. *ArXiv.org*. March 2. http://arxiv.org/abs/quant-ph/0611223v2

9 Campbell, L. Lorne 1965. Entropy as a Measure. *IEEE Trans. on Information Theory*, IT-11 (January): 112-114.

10 Campbell, L. Lorne. 1966. Exponential Entropy as a Measure of the Extent of a Distribution. *Zeitschrift Für Wahrscheinlichkeitstheorie Und Verwandte Gebiete* 5: 217–25.

11 Cover, Thomas and Joy Thomas 1991. *Elements of Information Theory*. New York: John Wiley.

12 Csiszar, Imre, and Janos Körner. 1981. *Information Theory: Coding Theorems for Discrete Memoryless Systems*. New York: Academic Press.

13 Ellerman, David. 2009. Counting Distinctions: On the Conceptual Foundations of Shannon’s Information Theory. *Synthese* 168 (1 May): 119–49.

14 Ellerman, David 2010. The Logic of Partitions: Introduction to the Dual of the Logic of Subsets. *Review of Symbolic Logic*. 3 (2 June): 287-350.

15 Ellerman, David 2014. An Introduction of Partition Logic. *Logic Journal of the IGPL*. 22, no. 1: 94–125.

16 Fano, Robert M. 1950. The Transmission of Information II. *Research Laboratory of Electronics Report* 149. Cambridge MA: MIT.

17 Fano, Robert M. 1961. *Transmission of Information*. Cambridge MA: MIT Press.

18 Friedman, William F. 1922. *The Index of Coincidence and Its Applications in Cryptography*. Geneva IL: Riverbank Laboratories.

19 Gini, Corrado 1912. Variabilità e mutabilità. Bologna: Tipografia di Paolo Cuppini.

20 Gini, Corrado 1955. Variabilità e mutabilità. In *Memorie di metodologica statistica*. E. Pizetti and T. Salvemini eds., Rome: Libreria Eredi Virgilio Veschi.

21 Gleick, James 2011. *The Information: A History, A Theory, A Flood*. New York: Pantheon.

22 Good, I. J. 1979. A.M. Turing’s statistical work in World War II. *Biometrika*. 66 (2): 393-6.
[23] Good, I. J. 1982. Comment (on Patil and Taillie: Diversity as a Concept and its Measurement). *Journal of the American Statistical Association*. 77 (379): 561-3.

[24] Hamming, Richard W. 1980. *Coding and Information Theory*. Englewood Cliffs, NJ: Prentice-Hall.

[25] Hartley, Ralph V. L. 1928. Transmission of information. *Bell System Technical Journal*. 7 (3, July): 535-63.

[26] Havrda, Jan, and Frantisek Charvat. 1967. Quantification Methods of Classification Processes: Concept of Structural α-Entropy. *Kybernetika* (Prague) 3: 30–35.

[27] Herfindahl, Orris C. 1950. *Concentration in the U.S. Steel Industry*. Unpublished doctoral dissertation, Columbia University.

[28] Hirschman, Albert O. 1964. The Paternity of an Index. *American Economic Review*. 54 (5): 761-2.

[29] Hu, Guo Ding. 1962. On the Amount of Information (in Russian). *Teor. Veroyatnost. I Primenen*. 4: 447–55.

[30] Jaeger, Gregg. 2007. *Quantum Information: An Overview*. New York: Springer Science+Business Media.

[31] Kolmogorov, A. N. 1983. Combinatorial Foundations of Information Theory and the Calculus of Probabilities. *Russian Math. Surveys* 38 (4): 29–40.

[32] Kullback, Solomon 1976. *Statistical Methods in Cryptanalysis*. Walnut Creek CA: Aegean Park Press.

[33] Laplace, Pierre-Simon. 1995 (1825). *Philosophical Essay on Probabilities*. Translated by A. I. Dale. New York: Springer Verlag.

[34] Lawvere, F. William and Robert Rosebrugh 2003. *Sets for Mathematics*. Cambridge: Cambridge University Press.

[35] MacArthur, Robert H. 1965. Patterns of Species Diversity. *Biol. Rev.* 40: 510–33.

[36] MacKay, David J. C. 2003. *Information Theory, Inference, and Learning Algorithms*. Cambridge UK: Cambridge University Press.

[37] McGill, William J. 1954. Multivariate Information Transmission. *Psychometrika* 19 (2 June): 97–116.

[38] McEliece, R. J. 1977. *The Theory of Information and Coding: A Mathematical Framework for Communication* (Encyclopedia of Mathematics and Its Applications, Vol. 3). Reading MA: Addison-Wesley.

[39] Nielsen, M., and I. Chuang. 2000. *Quantum Computation and Quantum Information*. Cambridge: Cambridge University Press.

[40] Patil, G. P. and C. Taillie 1982. Diversity as a Concept and its Measurement. *Journal of the American Statistical Association*. 77 (379): 548-61.

[41] Peters, Nicholas A., Tzu-Chieh Wei, and Paul G. Kwiat. 2004. Mixed State Sensitivity of Several Quantum Information Benchmarks. *ArXiv.org*. October 22. [http://arxiv.org/abs/quant-ph/0407172v2](http://arxiv.org/abs/quant-ph/0407172v2)
[42] Rao, C. R. 1982. Diversity and Dissimilarity Coefficients: A Unified Approach. *Theoretical Population Biology*. 21: 24-43.

[43] Rényi, Alfréd 1970. *Probability Theory*. Laszlo Vekerdi (trans.), Amsterdam: North-Holland.

[44] Ricotta, Carlo and Laszlo Szeidl 2006. Towards a unifying approach to diversity measures: Bridging the gap between the Shannon entropy and Rao’s quadratic index. *Theoretical Population Biology*. 70: 237-43.

[45] Rozeboom, William W. 1968. The Theory of Abstract Partials: An Introduction. *Psychometrika* 33 (2 June): 133–67.

[46] Shannon, Claude E. 1948. A Mathematical Theory of Communication. *Bell System Technical Journal*. 27: 379-423; 623-56.

[47] Shannon, Claude E. and Warren Weaver 1964. *The Mathematical Theory of Communication*. Urbana: University of Illinois Press.

[48] Simpson, Edward Hugh 1949. Measurement of Diversity. *Nature*. 163: 688.

[49] Tamir, Boaz, and Eliahu Cohen. 2014. Logical Entropy for Quantum States. ArXiv.org. December. http://de.arxiv.org/abs/1412.0616v2

[50] Tamir, Boaz, and Eliahu Cohen. 2015. A Holevo-Type Bound for a Hilbert Schmidt Distance Measure. *Journal of Quantum Information Science* 5: 127–33.

[51] Tsallis, Constantino 1988. Possible Generalization for Boltzmann-Gibbs Statistics. *J. Stat. Physics* 52: 479–87.

[52] Tsallis, Constantino. 2009. *Introduction to Nonextensive Statistical Mechanics*. New York: Springer Science+Business Media.

[53] Uffink, Jos. 1990. *Measures of Uncertainty and the Uncertainty Principle* (PhD thesis). Utrecht Netherlands: University of Utrecht.

[54] Wilkins, John 1707 (1641). *Mercury or the Secret and Swift Messenger*. London.

[55] Yeung, Raymond W. 1991. A New Outlook on Shannon’s Information Measures. *IEEE Trans. on Information Theory* 37 (3): 466–74.

[56] Yeung, Raymond W. 2002. *A First Course in Information Theory*. New York: Springer Science+Business Media.

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