THE TWO PHASE PARABOLIC SIGNORINI PROBLEM

MARK ALLEN AND WENHUI SHI

ABSTRACT. We study solutions to a variational equation that models heat control on the boundary. This problem can be thought of as the two-phase parabolic Signorini problem. Specifically, we study variational solutions to the equation

\[
\int_{Q^+} \partial_t u(w - u) + \langle \nabla u, \nabla (w - u) \rangle \\
+ \int_{Q'} \lambda_+ (w^+ - u^+) + \lambda_- (w^- - u^-) \, dH^{n-1} \, dt \geq 0
\]

without any sign restriction on the function \( u \). We show that when the solution \( u \) has a sign on \( Q' \), then \( u \) is a solution to the parabolic Signorini problem modulo a linear factor. The main result states that the two free boundaries

\[ \Gamma^+ = \partial \{ u(\cdot, 0, t) > 0 \} \quad \text{and} \quad \Gamma^- = \partial \{ u(\cdot, 0, t) < 0 \} \]

cannot touch, i.e. \( \Gamma^+ \cap \Gamma^- = \emptyset \) therefore reducing the study of the free boundary to the parabolic Signorini problem. The separation also allows us to show the optimal regularity of the solutions.

1. INTRODUCTION

In this paper we study what can be termed the “two phase parabolic Signorini problem”. To formulate this problem we first consider a smooth domain \( \Omega \) in \( \mathbb{R}^n \) that is even with respect to the \( x_n \) variable. We denote the several pieces of the parabolic cylinder by

\[
\begin{align*}
\Omega^+ & := \{ x \in \Omega \mid x_n > 0 \} \\
\Omega_T^+ & := \Omega^+ \times (0, T] \\
\Omega' & := \Omega \cap (\mathbb{R}^{n-1} \times \{0\}) \\
\Omega_T' & := \Omega' \times (0, T]
\end{align*}
\]

The variational formulation of this problem is given by finding a solution \( u \) with \( u(\cdot, 0) = \phi_0 \) and \( u = g \) on \( (\partial \Omega)^+ \times (0, T] \) to the variational inequality

\[
\int_{\Omega_T^+} \partial_t u(w - u) + \langle \nabla u, \nabla (w - u) \rangle \\
+ \int_{\Omega_T'} \lambda_+ (w^+ - u^+) + \lambda_- (w^- - u^-) \, dH^{n-1} \, dt \geq 0
\]

for every \( w \in \mathcal{R}_g = \{ w \in W^{1,0}_2(\Omega_T) \mid w = g \text{ on } (\partial \Omega)^+ \times (0, T] \} \). Throughout the paper \( \lambda_+ > 0 \).

This problem arises as a limiting case in modeling heat control on the boundary. Specifically, consider the problem of fixing the temperature on one portion of the boundary \( \partial \Omega^+ \) - for instance \( (\partial \Omega)^+ \). On \( \Omega' \) - the other portion of \( \partial \Omega^+ \) - we

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seek to maintain the temperature in the fixed interval \([h_1, h_2]\). If the temperature rises above \(h_2\) or below \(h_1\) at \(x \in \Omega'\), we cool or heat the domain by injecting a fixed quantity of heat at \(x\). A similar problem was recently studied in [3]. Our problem formulated in (1.1) corresponds to the limiting case when we try to maintain the temperature in the fixed interval \([h_1, h_2]\). If the temperature on \(\Omega'\) drops above or below a fixed constant temperature, a fixed quantity of heat is injected. For more information on the modeling of the problem see the book [8]. There are several relevant reasons for studying the specific limiting case in which \(h_1 = h_2\). The problem in (1.1) can be seen as the “thin” version of the two phase parabolic obstacle problem which is formulated by replacing the domains of integration \(\Omega_T^+\) and \(\Omega_T^-\) in (1.1) with the solid cylinder \(\Omega_T\). The two phase parabolic obstacle problem arises as a limiting case for a model for heat control regulated through the interior of the domain [5]. Then in both mathematical formulation and application, (1.1) can be considered the “parabolic two phase thin obstacle problem”. We can also illustrate the connection between this two phase thin obstacle problem and the one phase thin obstacle problem which is also commonly referred to as the Signorini problem. The (zero obstacle) parabolic Signorini problem has the variational formulation

\[(1.2) \quad \int_{\Omega_T^+} \partial_t u (w-u) + \langle \nabla u, \nabla (w-u) \rangle \geq 0\]

for every \(w \in \{w \in W^{1,0}_2(\Omega_T^+) \mid w = g\ on\ (\partial \Omega)^+ \times (0,T)\ and\ w \geq 0\ on\ \Omega_T'\}.

We have the following

**Proposition 1.1.** Let \(u\) be a solution to (1.1) such that \(u \geq 0\ (\leq 0)\ on\ \Omega_T\). Then \(u - \lambda x_n\ (-u - \lambda x_n)\) is a solution to (1.2).

**Proof.** Let \(u\) be a solution to (1.1). Without loss of generality we assume \(u \geq 0\ on\ \Omega_T'.\) Assume \(w = u - \lambda x_n\ on\ (\partial \Omega)^+ \times (0,T)\ and\ w \geq 0\ on\ \Omega_T'.\) Then define \(v := w + \lambda x_n.\) Then

\[
\int_{\Omega_T^+} \partial_t (u - \lambda x_n)(w-u + \lambda x_n) + \langle \nabla (u - \lambda x_n), \nabla (w-u + \lambda x_n) \rangle \\
= \int_{\Omega_T^+} \partial_t (v - \lambda x_n - u + \lambda x_n) + \langle \nabla (v - \lambda x_n), \nabla (v - \lambda x_n - u + \lambda x_n) \rangle \\
= \int_{\Omega_T^+} \partial_t (v - u) + \langle \nabla u, \nabla (v-u) \rangle - \lambda \partial_n x_n (v-u) \\
= \int_{\Omega_T^+} \partial_t u (v-u) + \langle \nabla u, \nabla (v-u) \rangle + \int_{\Omega^-} \lambda v - \lambda u \\
= \int_{\Omega_T^+} \partial_t u (v-u) + \langle \nabla u, \nabla (v-u) \rangle + \int_{\Omega^+} \lambda v - \lambda u \geq 0
\]

The last equality is true because \(u, v \geq 0\ on\ \Omega_T'\) by hypothesis. The case in which \(u \leq 0\ on\ \Omega_T'\) is proven similarly. \(\square\)

One may very well ask about the other direction. If \(v\) is a solution to (1.2) is \(u := v + \lambda x_n^+\ a\ solution\ to\ (1.1)\)? If \(v\) is a solution to (1.2), then \(\nabla v \in H^{1,0/2}\ [11][7].\) It then follows that one may choose \(\lambda\) large enough so that \(v + \lambda x_n^+ \geq 0\ in\ a\ local\ parabolic\ neighborhood\ \Omega_T^+.\) Then

\[
\partial_{x_n} u(x', 0, t) = \lambda_+ \quad \text{if} \ u(x',0, t) > 0 \\
0 \leq \partial_{x_n} u(x', 0, t) \leq \lambda_+ \quad \text{if} \ u(x',0, t) = 0.
\]
It then follows that \( u = v + \lambda \pm x_n \) is a solution to (1.1) in \( Q^+_r \).

When considering the temperature control in the interval \([h_1, h_2]\), an important aspect in the study of these problems relates to the free boundaries

\[
\Gamma^+_h := \partial\{u(x', 0, t) > h\} \quad \Gamma^-_h := \partial\{u(x', 0, t) < h\}
\]

For the free boundary we assume \( t > 0 \) for \( \Omega_T \) or \( t > t_0 - r^2 \) when working over the parabolic cylinder \( Q_r(x_0, t_0) \). When \( h_1 < h_2 \) the continuity of solutions follows from utilizing the penalization method used in Section 3. It is then clear from continuity of solutions that the two free boundaries cannot intersect, i.e. \( \Gamma^+_h \cap \Gamma^-_h = \emptyset \). One may ask if this separation is preserved in the limiting case when \( h_1 = h_2 \). When \( h_1 = h_2 = 0 \) we denote the free boundaries by \( \Gamma^+ \) and \( \Gamma^- \). When the phases separate, one may reduce the study of the free boundaries in (1.1) to the study of the free boundary in the parabolic Signorini problem since addition of \( \pm \lambda \pm x_n \) leaves the free boundary invariant. The main question therefore becomes: is \( \Gamma^+ \cap \Gamma^- = \emptyset \)?

Recently, the elliptic (time-independent) version of this problem was studied in [1]. The authors in [1] showed that indeed the separation \( \Gamma^+ \cap \Gamma^- \) is preserved in the limiting case. The main result in this paper is the parabolic analogue

**Theorem 1.2.** Let \( u \) be a solution to (1.1) in \( \Omega_T \). Then \( \Gamma^+ \cap \Gamma^- = \emptyset \).

If the positivity and negativity phases touch in the initial condition, then the above Theorem states that there is an immediate separation of the free boundaries. As previously noted, the separation of the free boundaries makes the problem locally equivalent to the parabolic Signorini problem. Therefore, as corollaries to Theorem 1.2, we obtain the optimal regularity (Corollary 5.6) of the solutions to (1.1) as well as regularity results for the free boundaries.

The outline of this paper is as follows.

- In section 2 we provide the notation that will be used throughout the paper. We also include a few preliminary results.
- In section 3 we prove the H"older and Lipschitz regularity of solutions as well as the existence of solutions.
- In section 4 we prove a nondegeneracy estimate that states that solutions must grow by a certain factor away from free boundary points.
- In section 5 we use the results from the previous sections in combination with a monotonicity formula to provide a simple proof of Theorem 1.2. We also state as a consequence results involving the optimal regularity and the free boundary.

2. Notation and Preliminaries

Since the results in this paper are local in nature, for simplicity we will work over Euclidean balls and parabolic cylinders centered on the thin space \( \mathbb{R}^{n-1} \times \{0\} \). We will follow the notation used in [7]. For a point \( x \in \mathbb{R}^n \) we denote \( x = (x', x_n) \)
where \( x' = (x_1, \ldots, x_{n-1}) \). For \( r > 0, x_0 \in \mathbb{R}^n, t_0 \in \mathbb{R} \) we let

\[
B_r(x_0) = \{ x \in \mathbb{R}^n \mid |x| < r \} \quad \text{(Euclidean ball)}
\]

\[
B^\pm_r(x_0) = B_r(x_0) \cap \mathbb{R}^n_+ \quad \text{(Euclidean halfball)}
\]

\[
B'_r(x_0) = B_r(x) \cap \mathbb{R}^{n-1} \quad \text{("thin" ball)}
\]

\[
Q_r(x_0, t_0) = B_r(x_0) \times (t_0 - r^2, t_0] \quad \text{(parabolic cylinder)}
\]

\[
Q'_r(x_0, t_0) = B'_r(x_0) \times (t_0 - r^2, t_0] \quad \text{("thin" parabolic cylinder)}
\]

\[
Q^\pm_r(x_0, t_0) = B^\pm_r(x_0) \times (t_0 - r^2, t_0] \quad \text{(parabolic halfcylinders)}
\]

\[
\bar{Q}_r(x_0, t_0) = B_r(x_0) \times (t_0 - r^2, t_0 + r^2) \quad \text{(full parabolic cylinder)}
\]

\[
\bar{Q}'_r(x_0, t_0) = B'_r(x_0) \times (t_0 - r^2, t_0 + r^2) \quad \text{(full "thin" parabolic cylinder)}
\]

For simplicity we write \( Q, Q^+ \) and \( Q' \) instead if the center is on \( \mathbb{R}^{n-1} \times \{0\} \times \mathbb{R} \) and the radius is not specified.

For \( \Omega_T = \Omega \times (0, T] \) with \( \Omega \) a bounded open subset in \( \mathbb{R}^n \) and \( T > 0, \partial_p \Omega_T = (\partial \Omega \times (0, T]) \cup (\Omega \times \{0\}) \) is the parabolic boundary of \( \Omega_T \). For \( (x, t), (y, s) \in \mathbb{R}^{n+1}, d_p((x, t), (y, s)) = \max \{|x - y|, |t - s|^{\frac{1}{2}}\} \) is the parabolic distance; \( d_p(E_1, E_2) = \inf_{(x, t) \in E_1, (y, s) \in E_2} d_p((x, t), (y, s)) \) is the parabolic distance between two subsets \( E_1, E_2 \subset \mathbb{R}^{n+1} \).

The parabolic Sobolev spaces \( W^{2m, m}_q(\Omega_T), \) \( m \in \mathbb{Z}^+ \), are the Banach spaces of functions with a generalized derivative \( \partial_x^\alpha \partial_t^j u \in L_q(\Omega_T) \) for \( |\alpha| + 2j \leq 2m \) and the norm

\[
\|u\|_{W^{2m, m}_q(\Omega_T)} = \sum_{|\alpha| + 2j \leq 2m} \|\partial_x^\alpha \partial_t^j u\|_{L_q(\Omega_T)}.
\]

Parabolic Sobolev spaces \( W^{1, 0}_q(\Omega_T), W^{1, 1}_q(\Omega_T) \) are the Banach spaces with the norms

\[
\|u\|_{W^{1, 0, 0}_q(\Omega_T)} = \|u\|_{L_q(\Omega_T)} + \|\nabla u\|_{L_q(\Omega_T)}
\]

\[
\|u\|_{W^{1, 1}_q(\Omega_T)} = \|u\|_{L_q(\Omega_T)} + \|\nabla u\|_{L_q(\Omega_T)} + \|\partial_t u\|_{L_q(\Omega_T)}.
\]

Parabolic Hölder spaces \( H^{\alpha, \alpha/2}_q(\Omega_T) \) for \( \alpha \in (0, 1] \) is the set of continuous functions which are Hölder-\( \alpha \) in space and Hölder-\( \alpha/2 \) in time.

We denote the free boundaries of a solution \( u \) to (\ref{eq:Q}) by

\[
\Gamma^+ = \partial \{ u(x', 0, t) > 0 \} \quad \Gamma^- = \partial \{ u(x', 0, t) < 0 \}
\]

for \( t > t_0 \) where the initial condition \( \phi_0 \) is given at time \( t_0. \) The boundary here is in the sense of the topology of \( \mathbb{R}^{n-1} \times \{0\} \times \mathbb{R}. \)

Since our solutions are defined on the half cylinder \( Q^+ \) we may evenly reflect with respect to the \( x_n \) variable to obtain a function on \( Q. \) Whenever a solution \( u \) is considered on \( Q \) we mean that \( u \) is defined by even reflection unless otherwise specified. This evenly reflected \( u \) will satisfy (\ref{eq:Q'}) with \( Q^+ \) replaced by \( Q \) and \( \lambda_\pm \) replaced by \( 2\lambda_\pm. \) Sometimes it will be more convenient to work over \( Q. \)

The following rescaling property of solutions combined with the translation invariance of our equation in the \( x' \) and \( t \) variables will allow us to assume without loss of generality in proving most of our results that our cylinder is \( Q_1(0, 0). \)
Proposition 2.1. Let \( u \) be a solution to (1.1) in \( Q_1(0,0) \). Define
\[
    u_r(x,t) := \frac{u(rx,r^2t)}{r}
\]
Then \( u_r \) is a solution in \( Q_{1/r}(0,0) \).

It is clear from the variational equation that

Proposition 2.2. Let \( u \) be a solution to (1.1) on \( Q^+ \). Then
\[
    \partial_t u(x,t) - \Delta u(x,t) = 0
\]
for all \((x,t) \in Q^+\). Also if \( u > 0 \) (\( u < 0 \)) in a neighborhood \( Q \) of \((x',0,t)\), then
\[
    \partial_{x_n} u = \lambda_+ \quad (-\lambda_-) \text{ on } Q'
\]

Remark 2.3. (2.1) follows from the continuity of solutions which is proved in the next section.

Proposition 2.4 (Uniqueness). Let \( u_1, u_2 \) be two solutions to (1.1) with \( u_1 = u_2 \) on \((\partial_p Q_1)^+\). Then \( u_1 \equiv u_2 \).

Proof. Since \( u_i \) are both solutions with the same boundary data we obtain for \(-1 < s \leq 0\)
\[
    \int_{B^+_1 \times [-1,s]} \partial_t u_1 (u_2 - u_1) + \langle \nabla u_1, \nabla (u_2 - u_1) \rangle 
    + \int_{B'_1 \times [-1,s]} \lambda_+ (u_2^+ - u_1^+) + \lambda_- (u_2^- - u_1^-) \, d\mathcal{H}^{n-1} \, dt \geq 0
\]
and
\[
    \int_{B^+_1 \times [-1,s]} \partial_t u_2 (u_1 - u_2) + \langle \nabla u_2, \nabla (u_1 - u_2) \rangle 
    + \int_{B'_1 \times [-1,s]} \lambda_+ (u_1^+ - u_2^+) + \lambda_- (u_1^- - u_2^-) \, d\mathcal{H}^{n-1} \, dt \geq 0
\]
we add the two inequalities above to obtain
\[
    -\int_{B^+_1} \int_{-1}^s \partial_t \left[ \frac{1}{2} (u_1 - u_2)^2 \right] \geq \int_{B^+_1 \times [-1,s]} \| \nabla (u_1 - u_2) \|_{L^2(B^+_1)}^2 \geq 0
\]
since \( u_1 = u_2 \) on \( B^+_1 \times \{-1\} \) we obtain
\[
    \int_{B^+_1 \times \{s\}} (u_1 - u_2)^2 \leq 0
\]
This holds for any \(-1 < s \leq 0\), so \( u_1 \equiv u_2 \). \( \square \)

3. Existence and Lipschitz regularity

In this section we prove the existence of the solution to the variational inequality (1.1) and its local Lipschitz continuity. For simplicity we show it for \( Q^+_1 \cup Q'_1 \).

Given
\[
    g \in W^{1,1}_2(Q_1^+), \partial_t g \in W^{1,0}_2(Q_1^+), f \in L_2(Q_1^+) \text{ and } \varphi_0 \in W^{1}_2(B_1^+),
\]
where $g$ and $\varphi_0$ satisfy the compatibility condition on $B^+_1 \times \{-1\}$, we want to find a function $u \in W^{1,1}_2(Q^+_1)$, such that $u$ solves the variational inequality

$$\Delta u - \partial_t u = f \text{ in } Q^+_1$$

$$\partial_n u = \beta_\epsilon(u) \text{ on } Q^+_1$$

$$u = g \text{ a.e. on } (\partial B_1)^+ \times (-1,0), \quad u(-1) = \varphi_0$$

where $\beta_\epsilon = \beta_\epsilon^\prime \in C^\infty(\mathbb{R})$. It is easy to verify that

$$\beta_\epsilon(s) = \lambda_+ \text{ if } s \geq \epsilon; \quad \beta_\epsilon(s) = -\lambda_- \text{ if } s \leq -\epsilon; \quad \beta_\epsilon^\prime(s) \geq 0.$$

There exists a unique variational solution to $(P)$, assuming $u = g \text{ a.e. on } (\partial B_1)^+ \times (-1,0)$, and $u_\epsilon(-1) = \varphi_0$.

The following lemma shows the global uniform estimates of $\|u_\epsilon\|_{W^{1,1}_2(Q^+_1)}$.

**Lemma 3.1.** Assume $u_\epsilon \in W^{2,1}_2(Q^+_1)$ is a solution to the above approximation problem $(P)$. Assume $f, g$ and $\varphi_0$ satisfy (3.1). Then there exists a constant $C = C(n, \lambda_\pm, f, g, \varphi_0)$ independent of $\epsilon$ such that

$$\|u_\epsilon\|_{W^{1,1}_2(Q^+_1)} \leq C.$$
Proof. Consider the variational equation of $u_\epsilon$:

\begin{equation}
\int_{B_1^+ \times (t_1, t_2]} \partial_t u_\epsilon \eta + \langle \nabla u_\epsilon, \nabla \eta \rangle + f \eta + \int_{B_1^+ \times (t_1, t_2]} \beta_\epsilon(u_\epsilon) \eta = 0
\end{equation}

for any $\eta \in W^{1,0}_2(Q_1^+)$ vanishing a.e. on $(\partial B_1^+) \times (-1, 0]$ and $(t_1, t_2) \subset (-1, 0]$.

Plugging in $\eta = u_\epsilon - g$ in (3.10) and using the fact that $\beta_\epsilon(u_\epsilon) \geq 0$ a.e., we obtain

\begin{equation}
\sup_{t \in (-1, 0]} \|u_\epsilon(\cdot, t)\|_{L_2(B_1^+)}^2 + \|\nabla u_\epsilon\|_{L_2(Q_1^+)}^2
\end{equation}

\begin{equation}
\leq C_n \left( \|\varphi_0\|_{L_2(B_1^+)}^2 + \|g\|_{W^{2,1}_1(Q_1^+)}^2 + \|f\|_{L_2(Q_1^+)} \right).
\end{equation}

Plugging in $\eta = \partial_t (u_\epsilon - g)$ in (3.10) and using the fact that $B_\epsilon(s) \geq 0$, we obtain

\begin{equation}
\sup_{t \in (-1, 0]} \|\nabla u_\epsilon(\cdot, t)\|_{L_2(B_1^+)}^2 + \|\partial_t u_\epsilon\|_{L_2(Q_1^+)}^2
\end{equation}

\begin{equation}
\leq C_n, \lambda_\pm \left( \|\nabla \varphi_0\|_{L_2(B_1^+)}^2 + \|\nabla g\|_{W^{1,0}_1(Q_1^+)}^2 + \|\nabla g\|_{L_2(Q_1^+)} + \|f\|_{L_2(Q_1^+)} \right).
\end{equation}

Thus, $u_\epsilon$ is uniformly bounded in $W^{1,1}_2(Q_1^+)$. \hfill \square

From the global uniform estimate we can deduce the existence of solutions to the variational inequality (3.2).

**Lemma 3.2** (Existence). Let $g$, $f$ and $\varphi_0$ satisfy the condition (3.3), then there exists a solution $u$ to the variational inequality (3.2) and (3.3).

**Proof.** Let $u_\epsilon$ be the solution to $(P_\epsilon)$, then from the arguments above, $u_\epsilon$ is uniformly bounded in $W^{1,1}_2(Q_1^+)$ and satisfies (3.11) and (3.12).

Up to a subsequence $\epsilon_j \to 0$, $u_\epsilon_j \to u$ in $W^{1,1}_2(Q_1^+)$ and $u_\epsilon_j \to u$ in $L_2(Q_1^+)$. We claim that $u$ satisfies the variational inequality (3.2).

To show this, we take $\eta = w - u_\epsilon_j$ in (3.10) with $w \in \mathcal{R}_g$. Since $\beta_\epsilon(s) \geq 0$, we have

\begin{equation}
\int_{Q_1^+} \partial_t u_\epsilon_j (w - u_\epsilon_j) + \langle \nabla u_\epsilon_j, \nabla (w - u_\epsilon_j) \rangle + f(w - u_\epsilon_j)
\end{equation}

\begin{equation}
+ \int_{Q_1^+} B_\epsilon_j (w) - B_\epsilon_j (u_\epsilon_j) \geq 0, \quad \forall v \in \mathcal{R}_g.
\end{equation}

We notice that

\begin{equation}
|B_\epsilon_j(s) - B(s)| \leq \lambda_\pm s \chi_{(-\epsilon_j, \epsilon_j)}
\end{equation}

and $u_\epsilon_j \to u$ in $L_2(Q_1^+)$ by the trace theorem (e.g., 1.5.3 in [3]). Then passing to the limit $\epsilon_j \to 0$ and arguing as in section 5.6.1 in [3], we obtain (3.2). The boundary condition (3.3) follows from the trace theorem. \hfill \square

Next we show the local uniform estimates for $u_\epsilon$.

**Lemma 3.3.** Assume $\|u_\epsilon\|_{W^{1,1}_2(Q_1^+)} \leq M$ and $u_\epsilon$ satisfies (3.5), (3.6) with $f \in L\infty(Q_1^+)$ and $\beta_\epsilon$ satisfying (3.3). Then for $0 < r < 1/2$ there exists a constant $C = C(n, \lambda_\pm, r, M, \|f\|_{L\infty(Q_1^+)})$ independent of $\epsilon$ such that

\begin{equation}
\|D^2 u_\epsilon\|_{L_2(Q_1^+)} \leq C,
\end{equation}

\begin{equation}
\|\nabla u_\epsilon\|_{L_\infty(Q_1^+ \cup Q_1^+)} \leq C.
\end{equation}
Here \( \partial_{x_i} u_i \) on \( Q' \) is understood as the normal derivative from the \( Q_r^+ \) side.

Proof. First we plug in \( \eta = \partial_{x_i}[(\partial_{x_i} u_i)\xi^2] \), \( i = 1, \ldots, n - 1 \) in \((3.10)\), where \( \xi \in C^\infty_0(B_r) \) is a cut-off function with \( \xi = 1 \) in \( B_r \) and \( |\nabla \xi| \leq 1/(1 - r) \). Then we apply integration by parts and make use of the fact \( \beta^\prime_{i} \geq 0 \) to obtain

\[
(3.13) \quad \|\partial_{x_i}^2 u_i\|_{L^2(Q^+_{r})} \leq C_{n,r} \left( \|\nabla u_i\|_{L^2(Q^+_{r})} + \|\partial_t u_i\|_{L^2(Q^+_{r})} \right),
\]

for \( i = 1, \ldots, n, j = 1, \ldots, n - 1 \).

By using the equation, \((3.13)\) and \((3.12)\) we get a similar estimate as \((3.13)\) for \( \partial_{x_i} u_i \).

Let \( w_{\epsilon,i} = \partial_{x_i} u_i, i = 1, \ldots, n \). Then by \((3.12)\) and \((3.13)\) we have \( w_{\epsilon,i} \in W^{1,0}_2(Q_r^+) \). Moreover, it is not hard to see that \( w_{\epsilon,i} \) solves

\[
\begin{cases}
\Delta w_{\epsilon,i} - \partial_i w_{\epsilon,i} = \partial_i f & \text{in } Q^+_r, \\
\partial_{x_i} w_{\epsilon,i} = \beta^\prime_{i}(u_i) w_{\epsilon,i} & \text{on } Q^+_r,
\end{cases}
\]

if \( i = 1, \ldots, n - 1 \), and

\[
\begin{cases}
\Delta w_{\epsilon,n} - \partial_n w_{\epsilon,n} = \partial_n f & \text{in } Q^+_r, \\
w_{\epsilon,n} = \beta_{\epsilon}(u_i) & \text{on } Q^+_r.
\end{cases}
\]

Since \( \beta^\prime_{i}(u_i) \geq 0 \), then by the local \( L^\infty \) estimate to \( w_{\epsilon,i} \) with \( i = 1, \ldots, n - 1 \) (Theorem 6.41 in \cite{10}) we have

\[
\sup_{Q^+_r \cup Q'_r} w_{\epsilon,i} \leq C_{n,r,f} \|w_{\epsilon,i}\|_{L^2(Q^+_r)}.
\]

Similar estimate applied to \( -w_{\epsilon,i} \) yields

\[
\inf_{Q^+_r \cup Q'_r} w_{\epsilon,i} \geq -C_{n,r,f} \|w_{\epsilon,i}\|_{L^2(Q^+_r)}.
\]

To estimate \( w_{\epsilon,n} \), we notice that \( |\beta_{\epsilon}(u_i)| \leq \max\{\lambda_+, \lambda_-\} \). Hence by the local estimates at the boundary (Theorem 6.30 in \cite{10}) we obtain

\[
\sup_{Q^+_r \cup Q'_r} |w_{\epsilon,n}| \leq C_{n,r,f} \left( \|w_{\epsilon,n}\|_{L^2(Q^+_r)} + \lambda_+ + \lambda_- \right).
\]

The above estimates together with \((3.11)\) give

\[
\|\nabla u_i\|_{L^\infty(Q^+_r \cup Q'_r)} \leq C(n, r, \lambda_\pm, \varphi_0, g, f).
\]

\(\square\)

To show the uniform Hölder-1/2 estimate in \( t \), we use the maximum principle argument of Gilding, which for the Dirichlet boundary datum can be found in Chapter II of \cite{10}.

Assume

\[
\|\nabla u_i\|_{L^\infty(Q^+_r)} \leq L.
\]

For \( (x_0, t_0) \in \{x_n = 0\} \) and \( 0 < R < r \), let

\[
\hat{Q} = B_R(x_0) \times (t_0, t_0 + R^2/4n), \\
\hat{Q}' = B'_R(x_0) \times (t_0, t_0 + R^2/4n), \\
\hat{Q}^+ = \hat{Q} \cap \{x_n > 0\}.
\]
We construct the upper and lower barrier functions
\[ v^\pm_c = \left( \sup_{\hat{Q}^+} |f| + \frac{2sn}{R^2} \right) (t - t_0) + \frac{s}{R^2} |x - x_0|^2 + (L + \Lambda)R \pm (u_c - u_c(x_0,t_0)) - \Lambda x_n, \]
where
\[ s = \sup_{t \in (t_0,t_0+R^2/4n)} |u_c(x_0,t) - u_c(x_0,t_0)|, \quad \Lambda = \max\{\lambda_+,\lambda_-\}. \]
It is not hard to verify that
\[ \Delta v_c^\pm - \partial_t v_c^\pm \leq 0 \text{ in } \hat{Q}^+, \]
\[ \partial_{x_n} v_c^\pm \leq 0 \text{ on } \hat{Q}', \]
\[ v_c^\pm \geq 0 \text{ on } \partial_p \hat{Q} \cap \{x_n \geq 0\}. \]
Hence by the maximum principle we have \( v_c^\pm \geq 0 \) in \( \hat{Q}^+ \). Evaluating the inequality at \( x = x_0 \) and taking the supremum over all \( t \) gives
\[ s \leq \left( \sup_{\hat{Q}^+} |f| + \frac{2sn}{R^2} \right) \frac{R^2}{4n} + (L + \Lambda)R, \]
which yields
\[
(3.14) \quad s \leq \sup_{\hat{Q}^+} |f| \frac{R^2}{2n} + 2(L + \Lambda)R.
\]
Now let \( R = (t_1 - t_0)^{1/2} \), where \( t_1 \in (t_0,t_0 + r^2) \). Then (3.14) implies
\[ |u_c(x_0,t) - u_c(x_0,t_0)| \leq C(n,f,R,L) |t_1 - t_0|^{1/2}, \quad x_0 \in \mathbb{R}^{n-1} \times \{0\}. \]
Then by a standard argument using the representation formula for the heat equation (see chapter IV in [9]), one can show that \( u_c \) is uniform Hölder 1/2 in \( t \) in \( Q_t^+ \) for \( r \in (0,1) \).

We summarize the estimates above in the following lemma.

**Lemma 3.4** (Interior regularity). Let \( u \in W^{1,1}_2(Q_t^+) \) solve the variational inequality \( (2.9) \) and \( (2.10) \) with \( g \in W^{2,1}_2(Q_t^+) \), \( \partial_t g \in W^{1,0}_2(Q_t^+) \), \( f \in L_\infty(Q_t^+) \) and \( \varphi_0 \in W^{1,0}_2(B_1^+) \). Then \( u \in W^{2,1}_2(Q_t^+) \) and \( u \in H^{1,1/2}(Q_t^+ \cup Q_r^+) \) for \( r \in (0,1) \). More precisely,
\[
\|u\|_{W^{2,1}_2(Q_t^+)} \leq C_{n,r} \left( \|\varphi_0\|_{W^{1,2}_2(B_1^+)} + \|\partial_t g\|_{W^{1,0}_2(Q_t^+)} + \|\partial_t g\|_{L_2(Q_t^+)} + \|f\|_{L_2(Q_t^+)} \right),
\]
and
\[
\|u\|_{H^{1,1/2}(Q_t^+ \cup Q_r^+)} \leq C(n,r,\lambda_\pm,\varphi_0,g,f).
\]

From the regularity we obtain the immediate corollaries

**Corollary 3.5.** For a domain \( \Omega \), let \( u_k \) be a sequence of solutions to (1.1) on \( \Omega_T \).
Let \( D \Subset \Omega \). Assume that
\[
\|u_k\|_{H^{1,1/2}(Q_t^+ \cup Q_r^+)} \leq C.
\]
Then there exists a convergent subsequence \( u_k \to u_0 \) such that
1. \( u_k \to u_0 \) in \( H^{1,1/2}(\Omega_T) \)
2. \( u_k \to u_0 \) in \( H^{\alpha,\alpha/2}(D_T) \) for every \( \alpha < 1 \)
3. \( u_0 \in H^{1,1/2}(\Omega_T) \) and \( u_0 \) is a solution to (1.1) in \( D_T \)

This next corollary will provide us with the existence of so called “blow-ups”.

**Corollary 3.6.** Let $u$ be a solution to (1.1) with $u(0,0) = 0$. Let $u_r$ be defined as in Proposition 2.1. Then there exists a subsequence $u_{r_k} \to u_0$ such that

1. $u_0 \in H^{1,1/2}(Q_R)$ for every $R > 0$
2. $u_0$ is a solution to (1.1) in $Q_R$ for every $R > 0$

4. **Nondegeneracy**

This section is devoted to proving a nondegeneracy property. This result states that the sup (inf) of a solution must grow linearly from a free boundary point of $\Gamma^+$ ($\Gamma^-$). In this section it will be convenient to work with the solution over all of $Q$ obtained by even reflection.

We begin with the following comparison principle.

**Proposition 4.1 (Comparison Principle).** Let $u, v$ be two solutions with $u \leq v$ on $\partial p Q$. Then $u \leq v$ in $Q$.

**Proof.** Let $w_1 = \max\{u, v\}$ and $w_2 = \min\{u, v\}$. Then

\begin{align*}
\int_{Q^+} \partial_t v(w_1 - v) + \langle \nabla v, \nabla (w_1 - v) \rangle + \int_{Q^+} \mathcal{B}(w_1) - \mathcal{B}(v) \geq 0
\end{align*}

Taking $w = w_1$ in (1.1) for $v$ and taking $w = w_2$ in (1.1) for $u$, we have

\begin{align*}
\int_{Q^+} \partial_t u(w_2 - u) + \langle \nabla u, \nabla (w_2 - u) \rangle + \int_{Q^+} \mathcal{B}(w_2) - \mathcal{B}(u) \geq 0.
\end{align*}

Let $\theta = u - v$. Note that $w_1 - v = \theta^+$ and $w_2 - u = -\theta^+$. Adding the above two inequalities we have

\begin{align*}
\int_{Q^+} - (\partial_t \theta)\theta^+ + \langle \nabla \theta, \nabla \theta^+ \rangle + \int_{Q^+} \mathcal{B}(w_1) + \mathcal{B}(w_2) - \mathcal{B}(v) - \mathcal{B}(u) \geq 0,
\end{align*}

which taking account (3.1) yields

\begin{align*}
\int_{Q^+} (\partial_t \theta)\theta^+ + \langle \nabla \theta, \nabla \theta^+ \rangle \leq 0.
\end{align*}

Hence

\begin{align*}
\frac{1}{2} \int_{Q^+} \partial_t [(\theta^+)^2] + |\nabla \theta^+|^2 \leq 0.
\end{align*}

□

**Lemma 4.2.** There exists $\delta > 0$ depending only on $\lambda_\pm$ and $n$ such that if $u_\delta$ is the solution with constant boundary data $\delta$ on $\partial_p Q_1$, then

\begin{align*}
u_\delta(x',0,t) = 0 \quad \text{for all } (x',0,t) \in Q'_1/2
\end{align*}

**Proof.** Suppose by way of contradiction that there exists $\delta_k \to 0$ and points $(x'_k,0,t_k) \in Q'_1/2$ such that

\begin{align*}
u_\delta(x'_k,0,t_k) > 0
\end{align*}

By Proposition 4.1 since $u_\delta = \delta$ on $\partial_p Q_1$, then $u_\delta \geq 0$ in $Q_1$. Then $u_\delta - \lambda_\pm x_0^+$ is a solution to the parabolic Signorini problem (1.2) on $Q^+_1$. By $H^{\alpha,\alpha/2}$ estimates
in $Q_{3/4}$ for $\nabla (u_\delta - \lambda_+ x_+^\delta)$ independent of $\delta$, we obtain that $u_\delta \to u_0$ and $\nabla u_\delta \to \nabla u_0$ in $H^{\alpha, \alpha/2}$ in $Q_{3/4}$ for $\alpha < 1/2$. It is clear $u_0 \equiv 0$. But

$$\partial_x u_\delta(x_k', 0, t_k) = \lambda_+$$

and for a subsequence $(x_k', 0, t_k) \to (x_0', 0, t_0) \in \overline{Q}_{1/2}$, so that

$$\partial_x u_0(x_0', 0, t_0) = \lambda_+$$

which is a contradiction since $u_0 \equiv 0$. □

**Theorem 4.3** (Nondegeneracy). Let $u$ be a variational solution to (1.1). There exists $\delta > 0$ with $\delta$ depending only on $\lambda_\pm$ and $n$ such that if $u|_{\partial \mathcal{Q}_r} \leq \delta r$ ($u|_{\partial \mathcal{Q}_r} \geq -\delta r$) then $u(x) \leq 0$ ($u(x) \geq 0$) for $x \in Q'_r$.

**Proof.** First we note that by rescaling we only need to prove Theorem 4.3 on $Q_1$. By Lemma 4.2 $u_\delta = 0$ in $Q'_{1/2}$ for $\delta$ sufficiently small. By Proposition 4.1 if $u \leq \delta$ on $\partial \mathcal{Q}_1$, then $u \leq u_\delta$. The case for which $u \geq -\delta$ is proven similarly. □

From Theorem 4.3 we immediately obtain the following corollary.

**Corollary 4.4.** If $u$ is a solution and $0 \in \Gamma^+$ ($0 \in \Gamma^-$), then

$$\sup_{\partial \mathcal{Q}_r} u \geq Cr \left( \inf_{\partial \mathcal{Q}_r} u \leq -Cr \right)$$

Where $C$ depends only on $\lambda_+, \lambda_-$ and $n$.

**Remark 4.5.** All of the results in this Section may be restated with $Q_r$ replaced by the full cylinder $\tilde{Q}_r$. The proofs will be identical.

5. The Separation

We begin this section by stating a monotonicity formula for parabolic equations that first appeared in [3]. Let

$$G(x, t) := \frac{1}{(4\pi t)^n/2} e^{-|x|^2/4t} \text{ for } (x, t) \in \mathbb{R}^n \times (0, \infty)$$

be the heat kernel. Then for a function $v$ and any $t > 0$ define

$$I(t, v) = \int_0^t \int_{\mathbb{R}^n} |\nabla v(x, s)|^2 G(x, -s) dx ds$$

**Theorem 5.1.** Let $u_1$ and $u_2$ satisfy the following conditions in the strip $\mathbb{R}^n \times [-1, 0)$

(a) $\Delta u_i - \partial_t u_i \geq 0$

(b) $u_1 \cdot u_2 = 0$

(c) $u_1(0, 0) = u_2(0, 0) = 0$

Assume also that the $u_i$ have moderate growth at infinity, for instance

$$\int_{B_R} u_i^2(x, -1) \, dx \leq C e^{\frac{|x|^2}{1+\epsilon}}$$

For $R$ large and some $\epsilon > 0$. Then

$$\Phi(t; u_1, u_2) := \frac{1}{t^2} I(t; u_1) I(t; u_2)$$

is monotone increasing for $0 < t \leq 1$. 

Proposition 5.3. Let functions, i.e., after a rotation \( u_1 = \alpha x_n^+ \) and \( u_2 = \beta x_n^- \) where \( \alpha, \beta \geq 0 \) are constants.

Remark 5.2. If \( u_r \) is defined as in Proposition 5.1 then
\[
\Phi(tr^2; u_1, u_2) = \Phi(t; (u_1)_r, (u_2)_r).
\]

We will also utilize the case of equality for the formula in Theorem 5.1.

Proposition 5.3. Let \( u_1, u_2 \) satisfy the assumptions in Theorem 5.1. Then \( \phi(t; u_1, u_2) \) is constant if and only if the \( u_i \) are two complementary linear functions, i.e., after a rotation \( u_1 = \alpha x_n^+ \) and \( u_2 = \beta x_n^- \) where \( \alpha, \beta \geq 0 \) are constants.

Proof. The case of equality is determined by replacing all the inequalities with equalities in the proof of Theorem 5.1. The fundamental inequality in the proof of Theorem 5.1 relies on a convexity property of eigenvalues. By a result of Beckner-Kenig-Pipher [4], equality is achieved in that instance when \( u_1 \) and \( u_2 \) are two complementary half planes passing through the origin (see discussion in [9]). Thus if \( \phi(t; u_1, u_2) \) is constant, then on each time slice \( \mathbb{R}^n \times \{ t \} \), \( u_i \) are two complementary linear functions. In the case of equality, each \( u_i \) will also solve the heat equation when positive. Then each \( u_i \) is time independent, and the \( u_i \) are therefore two complementary linear functions.

We now proceed with the proof of Theorem 1.2.

Theorem 1.2. Let \( u \) be a solution to (1.1) in \( Q_1^+ \). As discussed in Section 2, we may evenly reflect \( u \) across the thin space \( \mathbb{R}^{n-1} \times \{ 0 \} \) and consider the solution in all of \( Q \). Suppose by way of contradiction that there exists a point \( (x_0^t, 0, t_0) \in \Gamma^+ \cap \Gamma^- \) with \( t > -1 \). Since our solutions are translation invariant in the \( x' \) and \( t \) variables, we may translate our solution, so that \( (0, 0, 0) \in \Gamma^+ \cap \Gamma^- \). If before translating our solution, \( t_0 < 0 \), we will use the results from Sections 3 and 4 as stated for \( \tilde{Q} \) and the estimates that follow will be over \( \tilde{Q} \). If \( t_0 = 0 \) we will use the same results as stated for \( Q \) and the estimates that follow would be stated for \( Q \). We then proceed with the so called “blow-up” procedure. We consider the rescalings \( u_r \) as defined in Proposition 2.1. By Corollary 3.6 we obtain a subsequence \( u_r \to u_0 \) where \( u_0 \) is a solution to (1.1) on every compact set. We will relabel \( u_0 = v \). By Corollary 4.4 and Remark 4.5

\[
\sup_{\partial_r \tilde{Q}_R} u \geq CR \quad \text{and} \quad \inf_{\partial_r \tilde{Q}_R} u \leq -CR \tag{5.1}
\]

for small \( R \). Then in the limit, (5.1) will also hold for \( v \) for every \( 0 < R < \infty \). Also by Lemma 5.3 and Corollary 5.6 we have

\[
|v(x, t)| \leq C(|x| + |t|^{1/2}). \tag{5.2}
\]

\( v^\pm \) will satisfy the hypotheses for Theorem 5.1. Next we perform a blow-up on \( v \). That is we consider the rescalings \( v_r \) of \( v \) with again \( r \to 0 \) and obtain a convergent subsequence \( v_r \to v_0 \). \( v_0 \) will be a solution to (1.1) on every compact set and (5.1) will hold for \( v_0 \). By Theorem 5.1

\[
\Phi(t, v^+, v^-)
\]

is monotone increasing for \( 0 < t \leq 1 \). Then \( \Phi(0+, v^+, v^-) \) is well defined and finite. By Remark 5.2 we note that for \( 0 < t \leq 1 \)

\[
\Phi(t, v_0^+, v_0^-) = \lim_{r \to 0} \Phi(t, v_r^+, v_r^-) = \lim_{r \to 0} \Phi(tr^2, v^+, v^-) = \Phi(0+, v^+, v^-).
\]
Thus \( \Phi(t, v_0^+, v_0^-) \) is constant. By Proposition 5.3 we conclude that \( v_0^\pm \) are complementary linear functions. Since \( v_0 \) is even in the \( x_n \) variable and \( v_0(x',0,t) \) satisfies (5.1) when \( v(x',0,t) \neq 0 \), it follows that \( v_0 = c|x_n| \) for \( t \leq 0 \).

If before translating our solution, \( (x_0',0,t_0) \in \Gamma^+ \cap \Gamma^- \) was such that \( t_0 < 0 \), we must also show \( v_0 = c|x_n| \) for \( t > 0 \). Since \( v_0 \) is a solution to (1.1) it follows that \( -\lambda_- \leq c \leq \lambda_+ \). Consider \( w_1 = (v_0 - c|x_n|)^+ \) and \( w_2 = -(v_0 - c|x_n|)^- \). Now \( v_0 + c|x_n| \) is a solution to the heat equation when \( x_n \neq 0 \). Also since \( -\lambda_- \leq c \leq \lambda_+ \)

\[
\partial_{x_n} w_1(x',0,t) \geq 0.
\]

Then by even reflection each \( w_i \) is a subsolution to the heat equation with initial condition \( w_i(x',x_n,0) = 0 \). Also each \( w_i \) will also satisfy the growth estimate (5.2). It follows from the usual proofs of Tychonoff’s theorem (or by bounding subsolutions from above by solutions and applying Tychonoff’s theorem) that \( w_i \equiv 0 \), and so \( v_0 \equiv c|x_n| \). This is a contradiction to \( v_0 \) satisfying (5.1). □

**Remark 5.4.** In the above proof we actually showed that if \( u \) is a solution to (1.1), then there is no point \( (x_0',0,t_0) \) such that

\[
(5.3) \quad \sup_{\partial_p Q,(x_0,t_0)} v \geq Cr \quad \text{and} \quad \inf_{\partial_p Q,(x_0,t_0)} v \leq -Cr
\]

for every \( 0 < r < r_0 \) for some fixed \( r_0 \).

As a consequence of Theorem 1.2 we may obtain a uniform separation of the free boundaries based on a compactness argument.

**Theorem 5.5.** Let \( u \) be a solution to (1.1) in \( Q_1 \) with

\[
\|u\|_{H^{1/2}(Q_1)} \leq C
\]

Then there exists \( d > 0 \) depending on \( C \) such that

\[
d_p((\Gamma^+ \cap Q_{1/2}), (\Gamma^- \cap Q_{1/2})) \geq d
\]

**Proof.** Suppose by way of contradiction that there exists a sequence of solutions \( u_k \) with

\[
d_p((\Gamma^+(u_k) \cap Q_{1/2}), (\Gamma^-(u_k) \cap Q_{1/2})) \to 0
\]

By Corollary 3.5 we have a subsequence \( u_k \to u_0 \) with \( u_0 \) a solution to (1.1). Furthermore, as a consequence of Corollary 4.4 it is clear that there would exist a point \( (x_0,t_0) \in \overline{Q}_{1/2} \) and \( (x_0,t_0) \) would be a point satisfying (5.3) which is a contradiction to Remark 5.4. □

Because of the uniform separation of the free boundaries, we are able to transfer known results for solutions of (1.2) to solutions of (1.1). In particular we may state results about the optimal regularity of solutions as well as the regularity of the free boundaries.

**Corollary 5.6.** Let \( u \) be a solution to (1.1) with

\[
\|u\|_{H^{1/2}(Q_1)} \leq C
\]

Then

\[
\|u\|_{H^{1/2}((Q^+_{1/2}) \cup Q^-_{1/2})} \leq C_1
\]

where \( C_1 \) is dependent on \( C, \lambda_{\pm} \).
Remark 5.7. \(H^{\alpha,\alpha/2}\) with \(\alpha > 1\) denotes the parabolic Hölder space as defined in [10] and [7].

Proof. We begin first by defining the coincidence set
\[
\Lambda(u) := \{(x', 0, t) \mid u(x', 0, t) = 0\}
\]
We now consider a point \((x, t) \in Q_{1/2}\). Let \(d = d_p((\Gamma^+ \cap Q_{1/2}), (\Gamma^- \cap Q_{1/2}))\). If the distance from \((x, t)\) to a free boundary in \(Q_{1/2}\) is greater than \(d/4\) and \((x, t) \notin \Lambda(u)\), then one may use regular interior estimates for solutions to the heat equation to obtain the bound in (5.4) for \(u\) in the cylinder \(Q_{d/8}(x, t)\). If \((x, t) \in \Lambda(u)\), then if we perform an odd reflection on \(u\) across the thin space \(\mathbb{R}^{n-1} \times \{0\}\), then the reflected function \(\tilde{u}\) will be a solution to the heat equation in the cylinder \(Q_{d/4}\), and so again we obtain (5.4) for \(u\) in the half cylinder \(Q_{d/8}^+(x, t) \cup Q_{d/8}^-(x, t)\). If the distance from \((x, t)\) to a free boundary is less than or equal to \(d/4\), then either \(u \mp \lambda_+ x_n^+\) is a solution to (1.2) in \(Q_{d/4}(x, t)\) and we utilize the optimal regularity result in [7] to conclude (5.4) for \(u\) in \(Q_{d/8}^+(x, t) \cup Q_{d/8}^-(x, t)\). Then by a covering argument we may conclude the result.

The above regularity result is optimal since \(\text{Re}(x_{n-1} + ix_n)^{3/2} + \lambda_+ x_n\) is a time-independent solution to (1.1) in \(Q_r(0, 0)\) for \(r\) sufficiently small and \(\lambda_+\) sufficiently large. See discussion in Section 4 following Proposition 4.1.

We remark here that it may be possible to use the approach in [2] to obtain priori \(H^{\alpha,\alpha/2}\) estimates with \(\alpha < 3/2\). The separation of the free boundaries could then be obtained as a consequence of the \(H^{\alpha,\alpha/2}\) regularity. Since this paper focuses on the interaction of the free boundaries, we chose to first prove the separation of the free boundaries. Our approach is relatively short and does not require many technical computations or difficulties.

Because of the separation of the free boundaries, the study of the local properties of the free boundaries is completely reduced to studying the free boundary in the parabolic Signorini problem (1.2). The regularity of the regular set of the free boundary as well as the structure of the singular set for the parabolic Signorini problem was recently studied in [7].

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Department of Mathematics, Purdue University, West Lafayette, IN 47907
*E-mail address:* allenma@math.purdue.edu

Department of Mathematics, Purdue University, West Lafayette, IN 47907
*E-mail address:* wshi@math.purdue.edu