Analysis of a Predator-Prey Model with Distributed Delay

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1. Introduction

In applied engineering and complex system sciences, mathematical models that display deterministic chaotic dynamical behaviour are of interest. The majority of encounters in nature are admittedly delayed or isolated, as both predator and prey function stochastically in absorbing available resources. This can be used to share bandwidth and resources among network users at a bottleneck node or a leaky bucket used to track flows, for example. If we assume that network users’ behaviour is stochastic and that the accommodating segment has limited buffering space, then forwarding generated data packets can be compared to a predator-prey style interaction with limited resources characteristics during rush hours, when users interact intensively. One approach to examining a heterogeneous network susceptible to attack is modeling cyberspace as a predator-prey landscape. The predator-prey model of Gauss type is a well-known simple mathematical model describing the interaction between species. Its variations and extensions are studied in modern day population dynamics theory (see, for example, [1–14]). This model is based on the assumption that in real-world ecosystems prey populations do not grow exponentially in the absence of a predator, but rather their size is eventually limited by the absence of resources. Fan and Wolkowicz studied the effects of incorporating discrete delay in [15]. The delay corresponds to the time lag between predator capturing the prey and its conversion to biomass for predators. When the delay is absent, the model exhibits asymptotic convergence to an equilibrium. Therefore, any nonequilibrium dynamics in the model when the delay is included can be attributed to the delay’s inclusion. We assume that the delay is distributed and model the delay using integrodifferential equations. We established the well-posedness and basic properties of solutions of the model with nonspecified delay. Then, we analyzed the local and global dynamics as the mean delay varies.
of the system for the functional responses of Holling type I and Holling type II. They established the existence of stability switches due to Hopf bifurcations. These bifurcations occur in pairs that are connected and are nested. They have also shown that there is a range of parameters for which there exist two or more stable periodic solutions.

In nature, for each case, the processing delay rarely has the same duration, and instead follows a distribution of some mean value. Recently, Chaudhuri et al. [17] studied the following epidemic model consisting of four species, namely, sound prey, infected prey, sound predator, and infected predators.

\[
\frac{dX_1(t)}{dt} = X_1(r_1 - X_1 \left(\frac{X_2 + X_3}{k}\right) - \gamma X_1 X_2 - \beta X_1 X_3 - b_1 X_1 X_4 - b_2 X_1 X_4)
\]

\[
\frac{dX_2(t)}{dt} = X_2\tau_1 - X_2 v + \gamma X_2 X_3 - X_2 \left(\frac{X_4 + X_5}{k}\right) - b_3 X_2 X_3 - b_4 X_2 X_4 + \beta X_2 X_4
\]

\[
\frac{dX_3(t)}{dt} = X_3(-m - \alpha X_3 - \eta X_4 + d(b_1 X_1 + b_2 X_2))
\]

\[
\frac{dX_4(t)}{dt} = X_4(-m - \mu - \eta X_4 + d(b_1 X_1 + b_2 X_2) + aX_2 X_4).
\]

(1)

In [4], we have modified the system (1) with discrete delay.

\[
\frac{dX_1(t)}{dt} = X_1\tau_1 - X_1 \left(\frac{X_2 + X_3}{k}\right) - \gamma X_1 X_2 - \beta X_1 X_3 - b_1 X_1 X_4 - b_2 X_1 X_4
\]

\[
\frac{dX_2(t)}{dt} = X_2\tau_2 - X_2 v + \gamma X_2 X_3 - X_2 \left(\frac{X_4 + X_5}{k}\right) - b_3 X_2 X_3 - b_4 X_2 X_4 + \beta X_2 X_4
\]

\[
\frac{dX_3(t)}{dt} = X_3(-m - \alpha X_3 - \eta X_4 + d(b_1 X_1 + b_2 X_2))
\]

\[
\frac{dX_4(t)}{dt} = X_4(-m - \mu - \eta X_4 + d(b_1 X_1 + b_2 X_2) + aX_2 X_4).
\]

(2)

They investigated the stability properties and the existence of Hopf bifurcation. In this paper, we study the effects of incorporating distributed delay in the system (1) for infected predator-free equilibrium.

In the next section, an analysis of infected predator-free equilibrium of (1) is presented. In Section 3, we established the well posedness and basic properties of the model. We investigated the stability properties for different equilibriums in Section 4. Section 5 with conclusions completes the paper.

2. Infected Predator-Free Equilibrium

Consider (1)

\[
\frac{dX_1(t)}{dt} = X_1\tau_1 - X_1 \left(\frac{X_2 + X_3}{k}\right) - \gamma X_1 X_2 - \beta X_1 X_3 - b_1 X_1 X_4 - b_2 X_1 X_4
\]

\[
\frac{dX_2(t)}{dt} = X_2\tau_2 - X_2 v + \gamma X_2 X_3 - X_2 \left(\frac{X_4 + X_5}{k}\right) - b_3 X_2 X_3 - b_4 X_2 X_4 + \beta X_2 X_4
\]

\[
\frac{dX_3(t)}{dt} = X_3(-m - \alpha X_3 - \eta X_4 + c(b_1 X_1 + b_2 X_2))
\]

\[
\frac{dX_4(t)}{dt} = X_4(-m - \mu - \eta X_4 + c(b_1 X_1 + b_2 X_2)) + aX_2 X_4).
\]

(3)

By introducing scaling variables \(x_i(t) = \theta X_i(x), x_i(t) = \psi X_i(x), x_i(t) = \omega X_i(x), t = \sigma t\) where \(\sigma = m, \phi = \gamma = \omega = \beta/m, \theta = (eb_1)/m, \psi = b_1/m\).

Let \(A = \gamma/eb_1, B = 1/k\gamma, C = b_0/\beta, E = \alpha/\gamma, F = \eta/\beta, G = b_1/b_0, X = eb_2/\gamma, R_1 = r_1/m, R_2 = r_2 - v/m, M = \mu/m\). We obtain

\[
\dot{x}_1 = x_1 R_1 - ABx_1^2 - Bx_2 x_1 - x_2 x_1 - x_3 x_4 - Cx_1 x_4
\]

\[
\dot{x}_2 = x_2 R_2 - ABx_2 x_2 - Ax_1 x_2 - Ax_2 x_3 - Dx_2 x_3 + Ax_3 x_4
\]

\[
\dot{x}_3 = -x_3 + x_1 x_3 + x_2 x_3 - Ex_2 x_3 - Fx_3 x_4
\]

\[
\dot{x}_4 = -x_4 - Mx_4 + Gx_1 x_4 + \frac{GD}{A}x_2 x_4 + \frac{GFx_2 x_4}{C} + \frac{EGx_2 x_3}{C}.
\]

(4)

Now assume that the predator becomes disease free and for simplicity let us consider \(X = E = 1\). Then, (4) becomes

\[
\dot{x}_1 = x_1 R_1 - ABx_1^2 - Bx_2 x_1 - x_2 x_1 - x_3 x_3 - Cx_1 x_4
\]

\[
\dot{x}_2 = x_2 R_2 - ABx_2 x_2 - Ax_1 x_2 - Ax_2 x_3 - Ax_3 x_3
\]

(5)

Now, we introduce distributed delay to (5)

\[
\dot{x}_1 = x_1 (R_1 - ABx_1) - (B + 1)x_1 x_4 - x_1 x_3
\]

\[
\dot{x}_2 = x_2 (R_2 - Bx_2) - x_2 x_1 (AB - A) - Ax_2 x_3
\]

(6)

\[
\dot{x}_3 = -x_3 + \int_0^\infty x_1(t - u)x_3(t - u)e^{-\theta u}h(u)du.
\]

Here, the function \(h(u)\) is the kernel of the distributed delay with the following properties

\[
\int_0^\infty h(u)du = 1
\]

\[
\int_0^\infty uh(u)du = \varphi,
\]

where \(\varphi\) is the mean delay between the capture of the prey to the conversion into the biomass of the predator.

Denote by \(i^3\), the Banach space of bounded continuous functions mapping from \((\infty, 0]\) into \(R^2\) fitted with the uniform norm. We consider initial data \(\Phi = (\Phi_1, \Phi_2, \Phi_3)\in i^3\) (\(\Phi(t) : \Phi(\theta) = \Phi(t)\geq 0, i = 1, 2, 3, \theta \leq 0\)). Define int \(i^3\). \(i^3\) \(\Phi(\theta) > 0, i = 1, 2, 3, \theta \leq 0\). The solutions of (6) with initial data \(\Phi(t) \in i^3\) at time \(t\) by \(I(\Phi, t)\) when they exist. Hence, for mentioning the positive solutions, we are referring to the solutions \(I(\Phi, t)\) with \(\Phi \in \text{int } i^3\). Later, we show that each component is positive for all \(t > 0\) in this case.

3. Well Posedness and Basic Properties of the Model

Define \(L > 0\) and assume that \(g(s) = 0\) for all \(s \in \text{int } L^\infty\). We allow \(L = \infty\).
**Theorem 1.** Solutions of (6) exist, with initial data in $i_2^+$, and for all $t > 0$, they are unique and remain in $i_2^+$.

**Proof.** For each bounded functions $\Phi \in i_2^+$, there exists a unique solution of (6), $\Pi(\Phi, t)$ such that $\Pi(\Phi, t)|_{(0,\infty]} = \Phi(t)$.

For all $t \geq 0$, $x_1(t) = x_2(t) = 0$ if $x_1(0) = x_2(0) = 0$. If $x_1(0) > 0$, then $x_1(t)$ will remain positive. Similarly, if $x_2(0) > 0$, $x_2(t)$ will remain positive. Hence, for all $t > 0$, there exists a unique solution.

Finally, as $x_3(t) \geq 0$ on $[-L, 0]$, $x_3(t) \geq -x_3(0)$ for all $0 \leq t \leq L$. Hence, $x_3(t) \geq 0$ for all $-L \leq t \leq L$. By induction, $x_3(t)$ remains positive, for all $t \in [L(n-1), Ln]$ and $n \in N, n > 0$. Hence, for all $t > 0$, $x_3(t) \geq 0$.

**Proposition 2.** Solutions of (6) with positive initial conditions remain positive for all $t > 0$.

**Proof.** By the previous theorem, $x_1(0) > 0$, then $x_1(t) > 0$ for all $t > 0$ and $x_2(0) > 0$, then $x_2(t) > 0$ for all $t > 0$. Assume that $x_3(t) = 0$ at $t$. This implies that $x_3(t) = 0$. Then $x_3(t) = x_1(t) + x_3(t) - x_3(t)$ which is positive, a contradiction.

**Lemma 3.** Solutions of (6) are bounded and limsup $x_3(t) \leq R_1/AB$, limsup $x_3(t) \leq R_2/AB$ and $x_1(t) \leq 1 + R_1^2/4R_1$ $e^{-\mu t}h(u)du$.

**Proof.** Note that $x_1(t) \leq x_1(t)(R_1 - ABx_1(t))$. Also, $x_1(0) \geq 0$, given $\epsilon > 0$ for $\forall \epsilon > 0$. Then $x_1(t) < (R_1/AB) + \epsilon$ for all $t \in T$. Therefore, limsup $x_3(t) \leq R_1/AB$. Similarly, limsup $x_2(t) \leq R_2/AB$.

Consider

$$z(t) = x_3(t) + \int_0^\infty x_1(t-u)e^{-\mu t}h(u)du.$$  \hspace{1cm} (8)

The derivative of $z(t)$ with respect to $t$,

$$\dot{z}(t) = \dot{x}_3(t) + \int_0^\infty \dot{x}_1(t-u)e^{-\mu t}h(u)du.$$  \hspace{1cm} (9)

Now,

$$\dot{z}(t) = -x_3(t) + \int_0^\infty x_1(t-u)x_2(t-u)e^{-\mu t}h(u)du + e^{-\mu t}h(u)du$$

$$+ e^{-\mu t}h(u)du \left(\left\{x_2(t-u)R_1 - ABx_1(t-u) - x_1(t-u)x_2(t-u)(B+1) - x_1(t-u)x_2(t-u)\right\}\right).$$

$$\dot{z}(t) = -x_3(t) + \int_0^\infty x_1(t-u)e^{-\mu t}h(u)du + 1 + R_1 - ABx_1(t-u) - (B+1)x_2(t-u)).$$

$$\dot{z}(t) = -x_3(t) + \int_0^\infty x_1(t-u)e^{-\mu t}h(u)du + \dot{x}_1(t-u)x_2(t-u) + (B+1)x_2(t-u)).$$

$$\dot{z}(t) = -x_3(t) + \int_0^\infty x_1(t-u)e^{-\mu t}h(u)du + \dot{x}_1(t-u)x_2(t-u) + (B+1)x_2(t-u)).$$

$$\dot{z}(t) = -x_3(t) + \int_0^\infty x_1(t-u)e^{-\mu t}h(u)du + \dot{x}_1(t-u)x_2(t-u) + (B+1)x_2(t-u)).$$

Note that $(1 + R_1^2 - (ABx_1(t-u) + (B+1)x_2(t-u)))^2 \geq 0$. Therefore,

$$\dot{z}(t) \leq -z(t) + \frac{(1 + R_1^2 - (ABx_1(t-u) + (B+1)x_2(t-u)))^2}{4}.$$  \hspace{1cm} (11)

Also,

$$\dot{w}(t) = -w(t) + \frac{(1 + R_1^2 - (ABx_1(t-u) + (B+1)x_2(t-u)))^2}{4}.$$  \hspace{1cm} (12)

has a solution

$$w(t) = w(0)e^{-t} + \frac{(1 + R_1^2 - (ABx_1(t-u) + (B+1)x_2(t-u)))^2}{4}.$$  \hspace{1cm} (13)

For each $w(t)$ with $w(0) > 0$ and for every $\epsilon > 0, T > 0$

$$w(t) \leq (1 + R_1^2 - (ABx_1(t-u) + (B+1)x_2(t-u)))^2 \leq 0, \forall t > T.$$  \hspace{1cm} (14)

Consider $x_1(t) = x_2(t) = x_3(t)$ to be the solution of (6) with the initial data $\Phi(t)$. Then $\exists t_1, t_2 \in R$ such that $x_3(t) > 0$ and $\exists \Omega \in [t_1, t_2]$.

Consider the case $\Phi(t) = 0$. Since $\Phi(t)$ and $\Phi(t)$ are continuous functions, then there exists $t_1, t_2 \in R$ such that $x_3(T) > 0$ and $\exists \Omega \in [t_1, t_2]$.
\( \varepsilon > 0 \) such that \( x_3(t) > 0 \forall t \in (T, T + \varepsilon) \). Then, as \( x_3(t) \geq -x_3(t), x_3(t) > 0 \forall t > T, \) it follows that \( \Theta_0 \in [t_1, t_2] \subset [t_1, T] \).

**Lemma 5.** Sets \( X^0, X_1, \) and \( X_2 \) are positively invariant under \( T(t) \).

**Proof.** For \( X^0 \), the result is true under \( T(t) \) by Lemma 4. If the solution has initial conditions in \( X_1 \), then, by (6), \( x_3(t) = 0 \forall t > 0 \). By Theorem 1, \( x_1(t) \geq 0 \forall t > 0 \). Now, the solutions with initial conditions in \( X_2 \) are taken into consideration. As \( x_3(t) = 0 \) and since \( x_2(t) > 0 \) is a solution of (6) with \( x_3(t) = 0 \) and \( \Phi_2(\Theta) = 0 \forall \Theta \in [-L, 0], \) then by the uniqueness of solutions, \( x_3(t) = 0 \forall t > 0 \). Hence, \( \int_0^\infty x_1(t-u)x_3(t-u)e^{-\varepsilon u}h(u)du > 0 \forall t > 0 \). Now, from Theorem 1, \( x_1(t) \geq 0 \) and \( x_2(t) > 0 \forall t > 0 \).

### 4. Stability Results with General Delay

Consider three equilibria of (6), \( E_0 = (0, 0, 0), E_1 = ((R_1/AB), 0), \) and \( E_2 = (0, 0, 0) \). Then \( E_1 = (0, 0, 0) \). Hence, \( \int_0^\infty x_1(t-u)x_3(t-u)e^{-\varepsilon u}h(u)du > 0 \forall t > 0 \). Now, from Theorem 1, \( x_1(t) \geq 0 \) and \( x_2(t) > 0 \forall t > 0 \).

The linearization of the system (6) around an equilibrium \( E_+ = (x_1^*, x_2^*, x_3^*) \) is given by

\[
\dot{X}(t) = AX(t) + B\int_0^\infty e^{-u}h(u)X(t-u)du.
\]

Here,

\[
A = \begin{bmatrix}
R_1 - 2ABx_1^* + x_1^*(B + 1) & -x_1^* & -x_1^*\\
0 & A(1-B)x_2^* + R_1 - 2Bx_2^* + x_2^*(A - AB) - Ax_2^* - Ax_2^* & 0 & 0 \end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
x_3^* & 0 & x_3^* & 0
\end{bmatrix}
\]

#### 4.1. Stability at \( E_0 \). At \( E_0 \) (15) becomes

\[
\begin{align*}
\dot{x}_1(t) &= R_1 0 0 x_1(t) \\
\dot{x}_2(t) &= 0 R_2 0 x_2(t) \\
\dot{x}_3(t) &= 0 0 -1 x_3(t)
\end{align*}
\]

Since two of the eigenvalues are positive \( E_0 \) is an unstable saddle point.

**Lemma 6.** \( E_0 \) is globally asymptotically stable with initial data in \( X_1 \).

**Proof.** We know that \( x_1(t) \) and \( x_2(t) \) is equal to 0 for all \( t > 0 \). Now, consider \( L = \infty \). If \( \Phi_2(\Theta) = 0 \forall \Theta \in [-L, 0] \) then the function \( \Phi_2(\Theta) = 0 \forall \Theta \in [-L, 0] \). Hence, \( \dot{x}_3(t) = -x_3(t) \forall t > 0 \). Hence, \( \lim_{t \to \infty} x_3(t) = 0 \).

If \( \exists \Theta \in [-L, 0] \exists \Phi_2(\Theta) = 0 \forall t > 0 \), then \( \exists \Theta > 0 \), \( \exists \int_0^\infty x_1(t-u)x_3(t-u)e^{\varepsilon u}h(u)du > 0 \forall t > 0 \). Also, as \( x_1(t) = 0 \forall t > 0 \), hence,

\[
\begin{align*}
\dot{x}_3(t) &= -x_3(t) + \int_0^\infty x_1(t-u)x_3(t-u)e^{\varepsilon u}h(u)du \\
&+ \int_0^\infty x_1(t-u)x_3(t-u)e^{\varepsilon u}h(u)du
\end{align*}
\]

The limit of \( \int_0^\infty x_1(t-u)x_3(t-u)e^{\varepsilon u}h(u)du \) as \( t \to \infty \) is 0. Therefore, \( \lim_{\Theta \to \infty} x_3(t) = 0 \). When \( L < \infty \), as \( x_1(t) = 0 \) for \( t > 0 \), then \( \int_0^\infty x_1(t-u)x_3(t-u)e^{\varepsilon u}h(u)du = 0 \forall t > L \). Then, \( x_3(t) = -x_3(t) \forall t > L \).

#### 4.2. Stability at \( E_1 \). The linearization around \( E_1 \) takes the form

\[
\begin{align*}
\dot{x}_1(t) &= -R_1 \frac{R_1}{AB} x_1(t) - R_1 x_1(t) - x_1(t) \\
\dot{x}_2(t) &= 0 + R_1 - R_1 x_2(t) - x_2(t) \\
\dot{x}_3(t) &= 0 0 -1 x_3(t)
\end{align*}
\]

The characteristic equation takes the form

\[
\lambda^2 + \lambda (R_2 + R_1 AB (A-AB)) = 0.
\]

**Theorem 7.** \( E_1 \) is locally asymptotically stable if \( \int_0^\infty e^{\varepsilon u}h(u)du < 1 \) and unstable if either inequality is reversed.

**Proof.** The term in the square brackets has roots \(-R_2 + R_1 A B (A-AB), -R_1 \), which are both negative if \( R_2 > R_1 A B (A-AB) \). The stability of \( E_1 \) is determined by the sign of the real parts of the roots of \( m(\lambda) = \lambda + 1 - \int_0^\infty e^{\varepsilon u}h(u)du \).

Substituting \( \lambda = \beta + iy, \gamma \geq 0 \) in \( m(\lambda) \) and separating real and imaginary parts, we obtain

\[
L(\beta) = (\beta + 1) + \int_0^\infty e^{\varepsilon u}h(u)du = R(\beta).
\]

First, we show that if \( \int_0^\infty e^{\varepsilon u}h(u)du > 1 \), then \( m(\lambda) \) has a positive real root. Note that if \( y = 0 \), then (22) is satisfied. In this case in (21), \( L(0) < R(0) \), \( R(\beta) \) is a decreasing function of \( \beta \) and \( \lim_{\beta \to \infty} L(\beta) = +\infty \). Therefore, \( m(\lambda) \) has a real root which is positive and \( E_1 \) is unstable. Also, if \( \int_0^\infty e^{\varepsilon u}h(u)du < 1, L(0) > R(0), R(\beta) \) is decreasing and \( L(\beta) \) is increasing, and (21) can never be satisfied for \( \beta > 0 \). Hence, \( E_1 \) is locally asymptotically stable.
Lemma 8. $E_+$ is globally asymptotically stable with initial data in $X_2$.

Proof. We know that $x_3(t) = 0$ for $t > 0$. Then, (6) becomes an ODE model. By Lemma 6 in [9], this lemma is true.

4.3. Properties of the Model when $E_+$. Exists. The characteristic equation around $E_+ = (x_1^*, x_2^*, x_3^*)$ is

$$\lambda^3 + A\lambda^2 + B\lambda + C + \int_0^\infty e^{-(1+\lambda)u} h(u) du (D\lambda^2 + \lambda E + F) = 0,$$

where $A = 1 - x_1^* A + x_2^* + 3x_1^* AB + 3x_2^* B + x_3^* + Ax_1^* - R_1 - R_2$,

$$B = -x_1^* A + x_2^* + 2x_1^* AB - 2x_1^2 A^2 B + 3x_2^* B + 2x_1^2 A^2 B^2 + 4x_1^* A x_2^* B + x_1^* AB + x_1^* + Ax_1^* - x_1^* x_1^* A + x_1^* x_1^*$$

$$+ x_1^* x_1^* AB + 2x_1^* x_2^* A^2 B + 2x_1^* x_1^* B + 2A x_1^* x_1^* + Ax_1^* - R_1$$

$$+ x_1^* AR_1 - x_1^* AB R_1 - 2x_2^* BR_1 - Ax_1^* R_1 - R_2 - x_2^* R_2$$

$$- 2x_1^* R_2 - x_2^* BR_2 - x_2^* R_2 + R_1 R_2),$$

$$C = -2x_1^* A^2 B + 2x_1^* AB + 2x_1^* A^2 B^2 + 4x_1^* x_2^* AB^2 + 2x_1^* x_2^* B^2$$

$$- x_1^* x_2^* A + x_1^* x_2^* A + 2x_1^* x_2^* AB + 2x_1^* A^2 B^2 + 4x_1^* x_2^* AB^2 + 2x_1^* x_2^* B^2$$

$$+ AB x_1^* x_2^* + Ax_1^* x_2^* AR_1 - x_1^* AR_1 R_1 - 2x_2^* BR_1 - Ax_1^* R_1 - x_1^* x_2^* R_1$$

$$- x_1^* R_2 - 2x_1^* BR_2 - x_2^* R_2 + R_1 R_2),$$

$$D = -x_1^*, E = x_1^* A - x_2^* x_1^* + 3x_1^* AB - 3x_1^* x_2^* B - Ax_1^* x_1^*$$

$$+ R_1 x_1^* + R_2 x_1^*,$$

$$F = 2x_1^* A^2 B - 2x_1^* B - 2x_1^* A^2 B - 4x_1^* x_2^* AB^2 - 2x_1^* x_2^* B^2$$

$$- 2A x_1^* x_2^* - 2x_1^* x_2^* A^2 B - 2A x_1^* x_2^* x_2^* - x_1^* AR_1$$

$$+ x_1^* x_2^* A R_1 + 2x_1^* x_2^* B R_1 + Ax_1^* x_1^* R_1 + x_1^* x_1^* R_2 + 2x_1^* x_2^* AB R_1$$

$$+ x_1^* R_2 R_1 - x_1^* R_2 R_2.$$  

Simplify (25) to the following equation

$$\lambda^3 + (A + D)\lambda^2 + (B + E)\lambda + (C + F) = 0.$$  

By Routh hurwitz criterion if $(A + D)(B + E) - (C + F) > 0$, $E_+$ is locally asymptotically stable.

Theorem 10. As $\varphi$ increases from zero, if a root appears on or crosses the imaginary axis as $\varphi$ increases from 0, then the number of roots of (23) with a positive real part can change.

Proof. For $g(\lambda) = A\lambda^2 + B\lambda + C + \int_0^\infty e^{-(1+\lambda)u} h(u) du (D\lambda^2 + \lambda E + F)$, it is easy that $\limsup_{\lambda\rightarrow0}\lambda^{-3}g(\lambda) = 0 < 1$.

Hence, none of the root of (23) with positive real part can enter from $\infty$ as $\varphi$ bifurcates from 0. As Lemma 6 holds, the result follows.

Also, if $\int_0^\infty e^{-\varphi h(u) du} > 1$, then $E_+$ is locally asymptotically stable and if $\int_0^\infty e^{-\varphi h(u) du} > 1$, then (23) has no positive real roots.

4.3.1. Global Dynamics

Lemma 11. If $\int_0^\infty e^{-\varphi h(u) du} > 1$ and $\varphi = 0$, then $E_+$ is globally asymptotically stable with respect to the solutions of (6) with $x_1(0) > 0, x_2(0) > 0$ and $x_3(0) > 0$.

Proof. Since $\varphi = 0$, then $h(u) = \delta_0(u)$, and therefore, system (6) reduces to its ODE prototype

$$\dot{x}_1(t) = x_1(R_1 - ABx_1 - x_2(B + 1) - x_3)$$

$$\dot{x}_2(t) = x_2(R_2 - Bx_2 - x_1(AB - A) - Ax_3)$$

$$\dot{x}_3(t) = -x_3 + x_1 x_3.$$  

Solutions with positive initial conditions will remain positive for all $t > 0$. Using the Dulac criterion, we observe that there are no periodic solutions lying in int $R_2^2$. Observe that the only solutions with initial conditions on the $y$-axis converge to $E_0$, while solutions on the $x$-axis, not including the origin, converge to $E_1$. On the other hand, $E_1$ repels the solutions with initial data not on the $x$-axis. Using straightforward phase plane argument, one can see that neither $E_0$ nor $E_1$ is in the $\omega$-limit set of solutions with initial data in int $R_2^2$. Then, by the Poincare-Bendixson theorem [10], $E_+$ is globally asymptotically stable.

Lemma 12. Consider the solutions of (6) with initial data in $X_0 \cup X_2$. There is no positive monotonically increasing sequence $\{t_n\}$, with $t_n \rightarrow \infty$ as $n \rightarrow \infty$ such that $(x_1(t_n), x_2(t_n), x_3(t_n))$ converges to $E_0$. Here, for every solution with initial data in $X_0 \cup X_2, x(t) > 0 \forall t \geq 0$. We prove this theorem by contradiction.

Assume that there exists a monotonic increasing sequence $\{t_n\}$ which is positive, with $t_n \rightarrow \infty$ such that $\dot{x}(t_n) \leq 0$ and $(x_1(t_n), x_2(t_n), x_3(t_n))$ converges to $E_0$ as $n \rightarrow \infty$. For every $\epsilon > 0$, $\exists \delta > 0$ such that $x_1(t_n) < \epsilon, x_2(t_n) < \epsilon$ and $x_3(t_n) < \epsilon$, for all $t_n > T$. Then, by the Poincare-Bendixson theorem [10], $E_+$ is globally asymptotically stable.
5. Conclusion

Through evolution, nature has developed natural propensities in complex systems (including animalia and plants) that enable survival through adaptation. Malicious agents, such as viruses, worms, and denial-of-service attacks, plague the Internet and the vast array of networks and applications that link to it. For example, using the Internet as an environment, the malicious attacks described above (viruses) can be viewed as predators, with their interactions with the ecosystem (servers) resembling a predator-prey relationship. A predator-prey model with distributed delay is considered in this paper. For infected predator-free equilibrium, we established properties for global asymptotic stability of some equilibria for the model with distributed delay. We were particularly interested in the dynamics when \( E_u \) exists. We showed that solutions with positive initial data remain positive for all time. Moreover, we determined the set of initial data such that the solutions eventually become positive.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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