Form factor relations for heavy-to-light meson transitions: tests of the Quark Model predictions.

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Abstract
In the amplitudes for the weak semi-leptonic decays of mesons, the hadronic matrix elements are parametrized by form factors that describe the non-perturbative QCD effects and are a source of large theoretical uncertainties. In the case of heavy-to-light meson transitions, a Quark Model derivation leads to very general relations between those hadronic form factors, such that only two remain that are independent — one for pseudoscalar-to-pseudoscalar and one for pseudoscalar-to-vector meson transitions. Here, we investigate to what extent these form factor relations remain a good approximation, beyond the Quark Model.

In heavy-to-light pseudoscalar-to-vector meson transitions, a simple argument shows that the V–A structure of the weak interaction leads to a strong suppression of the helicity $\lambda = +1$ amplitude — an effect that has been confirmed experimentally by the CLEO Collaboration, with a full angular analysis of the $B \to K^* J/\psi$ decay. We show that the theoretical predictions, in terms of the hadronic form factors, can accommodate the suppression of the $\lambda = +1$ amplitude, only if the Quark Model relations are verified. Moreover, the form factor relations also allow us to predict the ratio of the two
remaining helicity amplitudes, with $\lambda = 0$ and $-1$; here too, there is excellent agreement with the CLEO data for the $B \to K^* J/\psi$ decay. In the future, similar experimental tests can be carried out, with a few advantages, using the semileptonic decay $B \to \rho l^- \bar{\nu}_l$.

The Quark Model relations can also be tested against the predictions for the hadronic form factors, from the more powerful theoretical methods of Lattice QCD and Light Cone Sum-Rules. The excellent agreement points once again to the validity of the form factor relations, beyond the Quark Model framework where they were derived.

PACS: 13.20.He, 13.20.Fc, 13.25.Ft, 13.25.Hw.
1 Introduction

The study of the weak decays of quarks is hampered by the presence of the long distance QCD effects that are responsible for the binding of the quarks into hadrons. These non-perturbative effects are hard to evaluate in a model independent way, and so tend to bring large uncertainties to the theoretical predictions. They appear in the matrix elements of the weak Hamiltonian operators, between the initial and final hadronic states:

\[ \langle X(p')|\overline{q}\Gamma|B(p)\rangle , \quad \Gamma \equiv \gamma^\mu, \gamma^\mu\gamma_5, i\sigma^{\mu\nu}(p-p')_\nu, i\sigma^{\mu\nu}(p-p')_\nu\gamma_5 . \]  

We are interested in the case of heavy-to-light transitions, where \( q \equiv u, d \) or \( s \) and the final state meson \( X \) is a light pseudoscalar \( P \equiv \pi, \ldots \), or a light vector meson \( V \equiv \rho, \ldots \); the initial state meson is a heavy pseudoscalar \( B \) meson, although the results may also be valid, to some degree, for the case of the lighter \( D \) meson. The hadronic matrix elements in Eq. (1) are parametrized in terms of Lorentz invariant form factors as follows:

\[ \langle P(p')|\overline{q}\gamma^\mu b|B(p)\rangle = (p+p')^\mu f_1(q^2) + \frac{m_B^2-m_P^2}{q^2} q^\mu \left[ f_0(q^2) - f_1(q^2) \right] , \]  

where \( f_1(0) = f_0(0) \);

\[ \langle P(p')|\overline{q}i\sigma^{\mu\nu}q_\nu b|B(p)\rangle = s(q^2) \left[ (p+p')^\mu q^2 - (m_B^2-m_P^2)q^\mu \right] ; \]  

\[ \langle V(p',\vec{\varepsilon})|\overline{q}\gamma^\mu b|B(p)\rangle = -\frac{1}{m_B+m_V} - 2ie^{\mu\alpha\beta\gamma}\varepsilon^*_{\alpha p}\delta_{p\gamma}V(q^2) ; \]  

\[ \langle V(p',\vec{\varepsilon})|\overline{q}\gamma^\mu\gamma_5 b|B(p)\rangle = (m_B+m_V)\varepsilon^* A_1(q^2) \]  

\[ - \frac{\varepsilon^* q}{m_B+m_V} (p+p')^\mu A_2(q^2) \]  

\[ - 2m_V \frac{\varepsilon^* q}{q^2} q^\mu \left[ A_3(q^2) - A_0(q^2) \right] , \]  

where \( 2m_V A_3(q^2) \equiv (m_B+m_V)A_1(q^2)-(m_B-m_V)A_2(q^2) \) and \( A_0(0) = A_3(0) \);
\[
\langle V(\vec{p}', \varepsilon) | \bar{q} i \sigma^{\mu\nu} q_{\nu} b | B(\vec{p}) \rangle = i \epsilon^{\mu\alpha\beta\gamma} \varepsilon_{\alpha}^{*} q_{\beta} p_{\gamma} F_1(q^2) ; \tag{6}
\]
\[
\langle V(\vec{p}', \varepsilon) | \bar{q} i \sigma^{\mu\nu} q_{\nu} \gamma_5 b | B(\vec{p}) \rangle = \left[ (m_B^2 - m_V^2) \varepsilon^\mu - \varepsilon^{*} q (p + p')^\mu \right] F_2(q^2)
+ \varepsilon^{*} q \left[ q^\mu - \frac{q^2}{m_B^2 - m_V^2} (p + p')^\mu \right] F_3(q^2) , \tag{7}
\]

where \( F_1(0) = 2F_2(0) \). In all of the above, \( q \equiv p - p' \). There are three form factors — \( f_{0,1} \) and \( s \) — for a \( B \to P \) transition, and seven form factors — \( V, A_{0,1,2} \) and \( F_{1,2,3} \) — for a \( B \to V \) transition. These form factors contain the long distance QCD effects and are therefore poorly known; relations between them, that will hold under certain conditions or approximations, can then be very useful: they will reduce the number of uncertain quantities, and improve the accuracy of the theoretical predictions. Moreover, they may help us understand better the general features of the underlying long distance QCD effects. In the case of the heavy-to-heavy transitions \( B \to D \) or \( D^* \), for example, all the form factors are related to a single function of \( q^2 \), as a result of the Heavy Quark Symmetry (HQS) of QCD, in the limit of heavy \( b \) and \( c \) quarks [1].

In the case of heavy-to-light transitions, in the limit of a heavy \( b \) quark, static in the \( B \) meson rest-frame, HQS leads to the model independent relations [2]

\[
2m_B s(q^2) = -f_1(q^2) + \frac{m_B^2 - m_{\bar{p}}^2}{q^2} \left[ f_0(q^2) - f_1(q^2) \right] , \tag{8}
\]
\[
F_1(q^2) = \frac{m_B - E'}{m_B + m_V} 2V(q^2) + \frac{m_B + m_V}{m_B} A_1(q^2) , \tag{9}
\]
\[
F_2(q^2) = \frac{2m_B |\vec{p}'|^2}{(m_B + m_V)(m_B^2 - m_V^2)} V(q^2) + \frac{m_B - E'}{m_B - m_V} A_1(q^2) , \tag{10}
\]
\[
F_3(q^2) = \frac{m_B E' + m_{\bar{p}}^2}{m_B(m_B + m_V)} V(q^2) - \frac{m_V}{m_B} A_3(q^2) \nonumber
- \frac{m_B^2 - m_{\bar{p}}^2 m_V}{q^2} \frac{m_B}{m_B} [A_3(q^2) - A_0(q^2)] . \tag{11}
\]

Note that these relations are valid in any reference frame; the energy \( E' \) and momentum \( |\vec{p}'| \) of the light recoiling meson, \( X = P \) or \( V \), in the rest frame of
the \( B \) meson, are used as an abbreviation for the more cumbersome invariant functions of \( q^2 \):

\[
E' = \frac{m_B^2 + m_X^2 - q^2}{2m_B}, \tag{12}
\]

\[
|\vec{p}'| = \sqrt{(m_B^2 + m_X^2 - q^2)^2 - 4m_B^2m_X^2} \over 2m_B. \tag{13}
\]

Unfortunately, not much more can be said about the form factors, that relies solely on the properties of QCD [3].

On the other hand, if one adopts the naive description of the hadronic transition that is provided by the Quark Model, additional relations exist between the form factors [4, 5, 6]:

- In Ref. [6], it was shown that, in the Quark Model, the \( B \rightarrow V \) form factors \( F_{1,2,3} \) are not independent form factors; instead, one has

\[
F_1(q^2) = 2A_0(q^2), \tag{14}
\]

\[
m_V(m_B - m_V)F_2(q^2) = (m_BE' - m_V^2)A_1(q^2) - \frac{2m_B^2|\vec{p}'|^2}{(m_B + m_V)^2}A_2(q^2), \tag{15}
\]

\[
(m_BE' + m_V^2)F_2(q^2) = 2\frac{m_B^2|\vec{p}'|^2}{m_B^2 - m_V^2}F_3(q^2)
= m_V(m_B + m_V)A_1(q^2). \tag{16}
\]

These relations are valid for any value of the meson masses. In the special case of heavy-to-light transitions, that concerns us here, further relations can be obtained:

- For a heavy \( b \) quark, static in the \( B \) meson rest frame \( (m_b \gg |\vec{p}_b|) \), \[3\]

\[
2m_Bs(q^2) = -f_1(q^2) + \frac{m_B^2 - m_P^2}{q^2} [f_0(q^2) - f_1(q^2)], \tag{17}
\]

\[
V(q^2) = \frac{(m_B + m_V)^3}{2m_Bm_V(E' + m_V)}A_1(q^2) - \frac{m_B}{m_V}A_2(q^2). \tag{18}
\]
\[
A_0(q^2) = \frac{m_B + m_V}{2m_V} \left[ -1 + \frac{(m_B + m_V)^2}{m_B(E' + m_V)} \right] A_1(q^2)
\]
\[
- \frac{m_B(m_B - E')}{m_V(m_B + m_V)} A_2(q^2) .
\] (19)

It is easy to see that, together with the relations in Eqs. 14–16, these Quark Model relations reproduce the model independent results of Eqs. 8–11 (and provide two additional relations between \(V, A_{0,1,2}\)).

- For a light \(q\) quark, ultra-relativistic in the \(B\) meson rest frame \((m_q \ll |\vec{p}_q|)\), [5]

\[
f_0(q^2) = \left(1 - \frac{q^2}{m_B^2 - m_B^2 - m_B - E' + |\vec{p}'|} \right) f_1(q^2) ,
\] (20)

\[
A_1(q^2) = \frac{2m_B|\vec{p}'|}{(m_B + m_V)^2} V(q^2)
\] (21)

\[
= \frac{2m_B|\vec{p}'|}{(m_B + m_V)(m_B E' - m_V^2)}
\]

\[
\times \left[ \frac{m_B|\vec{p}'|}{m_B + m_V} A_2(q^2) + m_V A_0(q^2) \right] .
\] (22)

- Finally, in the heavy-to-light case, when both \(m_b \gg |\vec{p}_b|\) and \(m_q \ll |\vec{p}_q|\) limits are considered, Eqs. 17–19 and Eqs. 20–22 lead to five independent relations:

\[
f_0(q^2) = \left(1 - \frac{q^2}{m_B^2 - m_B^2 - m_B - E' + |\vec{p}'|} \right) f_1(q^2) ,
\] (23)

\[
s(q^2) = -\frac{1}{m_B - E' + |\vec{p}'|} f_1(q^2) ,
\] (24)

\[
A_1(q^2) = \frac{2m_B|\vec{p}'|}{(m_B + m_V)^2} V(q^2) ,
\] (25)

\[
A_2(q^2) = \frac{(m_B + m_V)|\vec{p}'| - m_V(E' + m_V)}{m_B(E' + m_V)} V(q^2) ,
\] (26)

\[
A_0(q^2) = \frac{m_B - E' + |\vec{p}'|}{m_B + m_V} V(q^2) .
\] (27)
With the expressions for $F_{1,2,3}$, in Eqs. 14–16, we obtain the additional relations

\begin{align*}
F_1(q^2) &= \frac{m_B - E' + |\vec{p}'|}{m_B + m_V} 2V(q^2), \quad (28) \\
F_2(q^2) &= \frac{m_B - E' + |\vec{p}'|}{m_B - m_V} A_1(q^2), \quad (29) \\
F_3(q^2) &= \left[ \frac{m_B + E' - |\vec{p}'|}{m_B + m_V} ight. \\
&\quad \left. \left( 1 - \frac{m_V}{m_B} \right) \frac{m_V + E' - |\vec{p}'|}{E' + m_V} \right] V(q^2). \quad (30)
\end{align*}

This leaves us with two independent form factors, one for the case of a $B \to P$ transition and one for a $B \to V$ transition.

Although these results rely on the naive Quark Model picture, where mesons are viewed as simple $q\bar{q}$ bound states, they do not depend on the particular choice for the internal momentum wavefunction of the mesons, and so they are very general results of the Quark Model [4, 5, 6]. Our main concern in this paper is to probe to what extent these form factor relations remain a good approximation, beyond the Quark Model. To do so, we will test them against both experimental and theoretical results.

We have already pointed out that the model independent results of Eqs. 8–11, which follow from the HQS of QCD in the heavy $b$ quark limit, are reproduced correctly by the Quark Model relations in that same limit. This provides a first test of the Quark Model derivations [7]. In section 2, we test the Quark Model relations obtained in the light $q$ quark limit. We explore the simple but remarkable prediction that, in that limit and due to the $V-A$ structure of the weak interaction, the $\lambda = +1$ helicity amplitude in a $B \to V$ transition must be strongly suppressed. This effect has been confirmed experimentally by the full angular analysis of the $B \to K^*J/\psi$ decay, performed by the CLEO Collaboration. We show that the theoretical expression for the $\lambda = +1$ amplitude, written in terms of the usual hadronic form factors, is not suppressed, unless the form factors relations are verified. A further test is provided by the ratio of the two remaining helicity amplitudes, with $\lambda = 0$ and $-1$; using the Quark Model relations, that ratio can be predicted and it is in excellent agreement with the CLEO data for the $B \to K^*J/\psi$ decay. We also discuss analogous tests that can be carried out
for semileptonic decays, such as $B \to \rho l^-\bar{\nu}_l$, where strong interaction effects are less of a problem and a range of recoil momenta is available. In section 3, the Quark Model form factor relations are compared to Lattice QCD and Light Cone Sum-Rules predictions — two methods that provide model independent, albeit approximate, insights into the long distance QCD effects that are at play in the hadronic transitions. Here too the agreement with the Quark Model results is striking.

2 Helicity amplitudes in heavy-to-light transitions

2.1 $B \to K^* J/\psi$

2.1.1 Helicity $\lambda = +1$ amplitude

Let us consider heavy-to-light meson transitions where the underlying quark process is the weak decay of a heavy $b$ quark into a light quark $q = u, d$ or $s$ and a spin-1 particle. One such process is the inclusive $J/\psi$ production mechanism

$$b \to q + (c\bar{c})J/\psi,$$

where $q = s$ or $d$ and the $c\bar{c}$ pair hadronizes into a $J/\psi$. The weak charged current at the origin of the decay produces a light $q$ quark that is left-handed. In the limit $m_q \to 0$ (or, more precisely, in the ultra-relativistic limit $m_q \ll |\vec{p}_q|$), the left-handed light quark has negative helicity and so, from angular momentum conservation, the spin-1 state $J/\psi$ can have helicity $\lambda = 0$ or $-1$, but not $\lambda = +1$. The diagrams in Fig. 1 illustrate this simple but remarkable consequence of the V–A structure of the charged weak interaction.

We can check that an explicit calculation of the $b \to q + J/\psi$ decay rate, for the different helicity states of the $J/\psi$, leads to the same conclusion. The decay amplitude can be written in the form of an effective $bqJ/\psi$ weak vertex, where one must be careful to take into account the effects of the strong interaction. In particular, the exchange of gluons can generate terms with Lorentz structures that are not present in the weak vertex, in the absence of QCD. For that reason, we write the effective $bqJ/\psi$ weak vertex in its most
Figure 1: Allowed helicity amplitudes in $b \to q + J/\psi$

general form \[ \Lambda_{bqJ/\psi}^{\mu} = \frac{G_F}{\sqrt{2}} V_{cb} V^*_{cq} \frac{f_{J/\psi}}{m_{J/\psi}} \{G_0 k^\mu \vec{k}(1 - \gamma_5) + G_1 (k^2 g^{\mu\nu} - k^\mu k^\nu)\gamma_\nu(1 - \gamma_5) + G_2 i\sigma_{\mu\nu} k^\nu [m_b(1 + \gamma_5) + m_q(1 - \gamma_5)]\} b, \]

where $k \equiv p_b - p_q = p_{J/\psi}$. Each term in the vertex is multiplied by a coefficient $G_i$ ($i = 0, 1, 2$), that includes both perturbative and non-perturbative QCD effects, but whose value is not of immediate importance for our discussion. Notice that the $G_0$ term, proportional to $k^\mu$, does not contribute to the decay amplitude, and it is only included for completeness. Notice also the important fact that only left-handed quark fields participate in the interaction (in the $G_2$ term, we have used the equations of motion to write that term in a more familiar form). The inclusive $b \to q + J/\psi$ decay rate that follows is

$$\Gamma_\lambda = \frac{1}{4\pi} G_F^2 |V_{cb} V^*_{cq}|^2 \left| \vec{p}_q \right| \frac{f_{J/\psi}^2}{m_b m_{J/\psi}^2}$$
\[
\left\{ \begin{array}{l}
m_{J/\psi}^2 |G_1 - G_2|^2 \left[ (m_b^2 + m_q^2) E_q - 2m_b m_q^2 \right] \\
\left( E_q \mp |\vec{p}_q| \right) \left| G_1 m_{J/\psi}^2 - G_2 (m_{J/\psi}^2 \mp 2m_b |\vec{p}_q|) \right|^2
\end{array} \right\} \quad \lambda = 0
\]

\[
\lambda = \pm 1
\]

where \( E_q \) and \( |\vec{p}_q| \) are the energy and momentum of the \( q \) quark in the \( b \) rest-frame. We see that the decay rate for a \( J/\psi \) with helicity \( \lambda = +1 \) does indeed vanish, in the limit \( m_q \to 0 \).

The conclusion that the \( \lambda = +1 \) decay rate is strongly suppressed must also apply to each one of the exclusive decay channels that add up to the inclusive \( b \to q + J/\psi \) process. In particular, when the light quark \( q = s, d \) hadronizes into a vector meson \( V = K^*, \rho, \omega \) or \( \phi \) (for a \( B_s \) decay), the two body decay \( B \to V J/\psi \) could \textit{a priori} have three helicity final states \( \lambda = 0, \pm 1 \) (the two vector mesons must have the same helicity \( \lambda \)); but the argument above tells us that the \( \lambda = +1 \) amplitude is very suppressed, and vanishes in the limit \( m_q \to 0 \).

Contrary to the case of the inclusive process, the suppression of the \( \lambda = +1 \) amplitude in the exclusive decay \( B \to V J/\psi \) can be tested experimentally: by analyzing the angular distribution of the decay products of the \( J/\psi \) and of the vector meson \( V \), one can determine the relative sizes and phases of all three helicity amplitudes. This has been done recently for the decay

\[
B \to K^* J/\psi , \quad \begin{array}{c} \uparrow \downarrow \\
K \pi \\
l^+l^-
\end{array}
\]

by the CLEO Collaboration [9]. The CLEO analysis was able to determine the differential decay rate

\[
\frac{1}{\Gamma} \frac{d^3 \Gamma}{d \cos \theta_K \cdot d \cos \theta_\psi \cdot d\chi} = \frac{9}{16\pi} \left\{ \sin^2 \theta_\psi \cos^2 \theta_K^* |H_0|^2 + \frac{1}{4} (1 + \cos^2 \theta_\psi) \sin^2 \theta_K^* (|H_+|^2 + |H_-|^2) - \frac{1}{2} \sin^2 \theta_\psi \sin^2 \theta_K^* \left[ \cos \chi Re(H_+ H^*_+) - \sin \chi Im(H_+ H^*_+) \right] - \frac{1}{4} \sin 2\theta_\psi \sin 2\theta_K^* \left[ \cos \chi Re(H_+ H^*_0 + H_- H^*_0) - \sin \chi Im(H_+ H^*_0 - H_- H^*_0) \right] \right\},
\]

(35)
where $\theta_\psi$ [resp. $\theta_{K^*}$] is the angle between the $l^+$ [resp. $K$] and the $J/\psi$ [resp. $K^*$] momenta, in the rest-frame of the vector meson, and $\chi$ is the azimuthal angle between the decay planes of the $J/\psi$ and the $K^*$. The coefficients $H_{0,\pm}$ are proportional to the helicity amplitudes $A_{0,\pm}$ for the $B \rightarrow K^* J/\psi$ decay:

$$H_i = \frac{A_i}{|A_+|^2 + |A_-|^2 + |A_0|^2} \quad (i = 0, \pm). \quad (36)$$

The experimental analysis of Ref. [9] determined the magnitude and phase of the ratios

$$r_\pm \equiv \frac{H_\pm}{H_0} = \frac{A_\pm}{A_0}; \quad (37)$$

they are (after translating the experimental results from the transversity basis to the helicity basis used in here)

$$|r_+|^2 = 0.03 \pm 0.07 \quad (38)$$
$$|r_-|^2 = 0.90 \pm 0.28 \quad (39)$$
$$\text{Arg}(r_+) = 2.9 \pm 1.7 \quad (40)$$
$$\text{Arg}(r_-) = 3.01 \pm 0.29. \quad (41)$$

The size of the $\lambda = +1$ amplitude, $A_+$, is consistent with zero, and much smaller than the other two helicity amplitudes — this is the effect that was predicted above. The relative phases $\phi_\pm$ between $A_\pm$ and $A_0$ are consistent with $\pi$ (the large error in the phase of $r_+$ is due to the small size of the $\lambda = +1$ amplitude), and show no signs of imaginary parts in the decay amplitudes. This is important for our discussion, as the presence of imaginary terms would signal the existence of significant final state interaction effects. It would then be possible for the $J/\psi$ to scatter from one helicity state to another invalidating, or at least weakening, our argument for the suppression of the $\lambda = +1$ amplitude. Later on, we will consider other decays where this potential problem is not a concern.

With the simple argument for the suppression of the $\lambda = +1$ amplitude in $B \rightarrow V J/\psi$ confirmed by experiment, we want to see how that suppression appears explicitly in the theoretical expressions for the helicity amplitudes, $A_\lambda$. These amplitudes are obtained from the matrix element of the effective $b q J/\psi$ vertex of Eq. [32] between the $B$ and $V$ meson states; we use the form factors of Section 1 to parametrize the different terms in the expression. The
results are

\[ A_0 = \frac{G_F}{\sqrt{2}} V_{cb} V_{cq}^{*} \frac{f_{J/\psi}}{m_{J/\psi}} \frac{m_{J/\psi}}{2m_V} \]

\[ \times \left\{ G_1 \left[ (m_B + m_V)(m_B^2 - m_V^2 - m_{J/\psi}^2)A_1 - \frac{4m_B^2 |\vec{p_V}|^2}{m_B + m_V} A_2 \right] + G_2 (m_b - m_q) \left[ -(m_B^2 + 3m_V^2 - m_{J/\psi}^2)F_2 + \frac{4m_B^2 |\vec{p_V}|^2}{m_B^2 - m_V^2} F_3 \right] \right\} \]

(42)

for the longitudinal amplitude, and

\[ A_\pm = -\frac{G_F}{\sqrt{2}} V_{cb} V_{cq}^{*} \frac{f_{J/\psi}}{m_{J/\psi}} \frac{m_{J/\psi}}{2m_V} \]

\[ \times \left\{ G_1 m_{J/\psi}^2 \left[ -(m_B + m_V)A_1 \pm \frac{2m_B |\vec{p_V}|}{m_B + m_V} V \right] + G_2 m_b \left[ \mp m_B |\vec{p_V}| F_1 + (m_B^2 - m_V^2)F_2 \right] + G_2 m_q \left[ \mp m_B |\vec{p_V}| F_1 - (m_B^2 - m_V^2)F_2 \right] \right\} \]

(43)

for the transversal amplitudes; the form factors are evaluated at \( q^2 = m_{J/\psi}^2 \).

It is easy to check that the \( \lambda = +1 \) amplitude will vanish, in the limit \( m_q \to 0 \), provided the form factors obey precisely the Quark Model relations of Eqs. 20-22 (together with the relations in Eqs. 14-16) that are valid in that same limit [10]. Without those relations, the form factors that appear in the helicity amplitudes are independent of each other, and the large suppression in \( A_+ \), that is expected and which has been confirmed experimentally in the case of \( B \to K^*J/\psi \), cannot be accounted for.

2.1.2 Helicity \( \lambda = -1 \) amplitude

If, in addition to the limit of an ultra-relativistic light \( q \) quark \( (m_q \ll |\vec{p_q}|) \), we adopt the limit of a static heavy \( b \) quark \( (m_b \gg |\vec{p_b}|) \), the set of Quark Model relations is extended to those in Eqs. 23-30. It is then possible to predict, in addition to \( A_+/A_0 = 0 \), the value of the ratio \( A_-/A_0 \). In fact, applying the additional form factor relations to the amplitudes in Eqs. 12 and
we can write $A_0$ and $A_-$ in terms of a single form factor; the dependence on that form factor can then be eliminated by considering the ratio

$$\frac{A_-}{A_0} = -\frac{2m_{J/\psi}}{m_B - E_V + |\vec{p}_V|} \left(1 - \frac{G_2}{G_1} m_b (m_B - E_V + |\vec{p}_V|)/m_{J/\psi}^2\right).$$

(44)

In order to compare this prediction to the experimental result from the CLEO analysis, we must first determine the ratio $G_2/G_1$ of the parameters that appear in the effective $bqJ/\psi$ vertex of Eq. 32.

If not for the QCD corrections to the weak $bqJ/\psi$ vertex, $G_2/G_1 = 0$. On the other hand, if the predominant QCD corrections were short-distance in nature, this ratio could be calculated perturbatively from the Feynman diagrams for the vertex, with the appropriate gluon exchanges [11]. However, the large discrepancy between the perturbative calculation for the inclusive $b \rightarrow q + J/\psi$ decay rate and the experimental result tells us that large non-perturbative QCD effects are at play, and so a theoretical estimate of $G_2/G_1$ becomes very hard to obtain. Instead, we can derive this ratio from the measurement of the $J/\psi$ polarization, $P \equiv \Gamma_L/\Gamma$, in the inclusive decay. From Eq. 33 with $m_q = 0$, we obtain

$$P = \left[1 + \frac{2|m_{J/\psi}/m_b^2 - G_2/G_1|^2}{m_{J/\psi}/m_b^2[1 - G_2/G_1]^2}\right]^{-1}.\quad (45)$$

On the other hand, the experimental result for the polarization is [12]

$$P' = 0.59 \pm 0.15,\quad (46)$$

where the prime reminds us that the measurement is contaminated by the contribution from decays of $B$ mesons into higher charmonium states, that in turn decay to $J/\psi$. These cascade decays account for the substantial difference between the inclusive $B \rightarrow J/\psi + X$ branching ratio $B' = (1.13 \pm 0.06)%$ [13], and the branching ratio for the direct decay $b \rightarrow q + J/\psi$, $B = (0.80 \pm 0.08)%$ [13]. In order to obtain a value for the polarization of the $J/\psi$ in the direct decay, we assume (to lack of a better estimate) that the cascade decays produce unpolarized $J/\psi$s. Then, the experimental result of Eq. 16 translates into

$$P = 0.70 \pm 0.22.\quad (47)$$
Comparing with the theoretical prediction of Eq. 45 and taking $m_b = 5.0$ GeV, we obtain

$$\frac{G_2}{G_1} = 0.13 \pm 0.18 \ .$$

(There is a second, larger, solution for $G_2/G_1$ that can be discarded, as discussed in Ref. [8]). Notice that the error is very large, and does not include the uncertainty in the polarization of the $J/\psi$ from the cascade decays.

Applying this estimate of $G_2/G_1$ to Eq. 44, we obtain the following prediction for the ratio of the $\lambda = 0$ and $-1$ helicity amplitudes in $B \to K^*J/\psi$:

$$\frac{A_-}{A_0} = -0.93 \pm 0.48 \ .$$

This is to be compared with the experimental results for the magnitude and phase of $r_-$, in Eqs. 39-41. Despite the large error in the theoretical prediction, it is clear that it gives the correct relative sign between the two amplitudes. The central value in the theoretical estimate for the magnitude of $A_-/A_0$ is also in excellent agreement with the experimental result, $|r_-| = 0.95 \pm 0.15$.

2.2 $B \to \rho l^- \overline{\nu}_l$

Similar tests of the Quark Model relations can be performed with semileptonic decays of the type

$$b \to u + W^* \ ,$$

with the light $u$ quark hadronizing into a vector meson $V = \rho, \omega$ or $K^*$ (for a $B_s$ decay). As before, the V–A structure of the charged weak interaction produces a left-handed $u$ quark; in the limit $m_u \ll |\vec{p}_u|$, the ultra-relativistic quark has negative helicity and so the virtual $W$ cannot have helicity $\lambda = +1$. In the case of the exclusive decays, where both the vector meson and the virtual $W$ have the same helicity, this translates into the cancellation of the $\lambda = +1$ amplitude, in the limit $m_u \to 0$.

As with the case of the hadronic decay, the effect can be seen explicitly in the theoretical expression for the inclusive semi-leptonic decay rate. For
$b \rightarrow u + W^* \rightarrow u + l^- \nu_l$, the differential decay rate is
\[
\frac{d\Gamma_\lambda}{dk^2/m_b^2} = \frac{1}{24\pi^3} G_F^2 |V_{ub}|^2 m_b |\vec{p}_u| \times \begin{cases} 2m_b|\vec{p}_u|^2 + E_u k^2 & \lambda = 0 \\ (E_u \mp |\vec{p}_u|)k^2 & \lambda = \pm 1 \end{cases},
\]
(51)
where $k \equiv p_b - p_u$, and $E_u$ and $|\vec{p}_u|$ are the energy and momentum of the $u$ quark, in the $b$ rest-frame. As expected, the decay rate for the $\lambda = +1$ helicity of the virtual $W$ vanishes, in the limit $m_u \rightarrow 0$. On the other hand, for the exclusive semi-leptonic decays $B \rightarrow V + W^* \rightarrow V + l^- \nu_l$, with the $B \rightarrow V$ hadronic matrix element parametrized by the form factors of Section 1, the differential decay rate is
\[
\frac{d\Gamma_\lambda}{dq^2/m_B^2} = \frac{1}{96\pi^3} G_F^2 |V_{ub}|^2 |\vec{p}_V| q^2 |H_\lambda(q^2)|^2,
\]
(52)
where
\[
2m_V \sqrt{q^2} H_0(q^2) = \left( m_B + m_V \right) \left( m_B^2 - m_V^2 - q^2 \right) A_1(q^2) - \frac{4m_B^2 |\vec{p}_V|^2}{m_B + m_V} A_2(q^2),
\]
(53)
\[
H_\pm(q^2) = \frac{2m_B |\vec{p}_V|}{m_B + m_V} V(q^2) \mp (m_B + m_V) A_1(q^2)
\]
(54)
and $q \equiv p_B - p_V$. As in the case of the hadronic decay, we find that we need the form factor relations of Eqs. 20-22 in order for $H_{+1}$ to vanish, when $m_u \rightarrow 0$.

In the limit of a static heavy $b$ quark and an ultra-relativistic light $u$ quark, in addition to $|H_+|^2/|H_0|^2 = 0$, the extended form factor relations of Eqs. 23-30 lead to the prediction that
\[
\frac{|H_-|^2}{|H_0|^2} = \frac{4q^2}{(m_B - E_V + |\vec{p}_V|)^2}.
\]
(55)
This provides another possible test of the form factor relations, in analogy to the case of the hadronic $B \rightarrow V J/\psi$ decay.
At present, there is no experimental data to compare these predictions to. In fact, using the semileptonic decay \( B \to \rho l^- \nu_l \) to test the Quark Model form factor relations, instead of the \( B \to K^* J/\psi \) decay, requires a substantial experimental effort, due to the additional Cabibbo suppression. There are however significant advantages to using the semileptonic decay. The first one is that no strong final state interactions are present that could scatter the vector meson between states of different helicities. This makes the argument for the suppression of the \( \lambda = +1 \) amplitude much stronger than in the case of an hadronic decay. The other advantage is that the magnitude of the vector meson momentum is not fixed in the three body decay. This allows testing the form factor relations throughout the entire range of \( q^2 \), from \( q^2 = 0 \) (maximum recoil) to \( q^2 = q_{\text{max}}^2 = (m_B - m_V)^2 \) (zero recoil).

### 2.3 \( B \to K^* \gamma \)

Other well known decays, of the same type as those discussed in here, are the inclusive radiative decay \( b \to q + \gamma \), with \( q = s \) or \( d \), and the corresponding exclusive decays \( B \to V \gamma \), with \( V = K^*, \rho, \omega \) or \( \phi \) (for a \( B_s \) decay). Unfortunately, this special case of the \( B \to V \) transition, with \( q^2 = (p_B - p_V)^2 = 0 \), cannot be used to test the form factor relations.

The discussion for the radiative decay is analogous to that for the hadronic decay into \( J/\psi \). The \( bq \gamma \) vertex is similar to the \( bq J/\psi \) vertex of Eq. [32], with the CKM factor \( |V_{tb} V_{tq}| \) replacing \( |V_{cb} V_{cq}| \) and the \( G_0 \) term omitted, to preserve gauge invariance:

\[
\Lambda_{bq\gamma}^\mu = -\frac{G_F}{\sqrt{2}} V_{tb} V_{tq}^{*} q \left\{ G'_1 \left( k^2 g^{\mu\nu} - k^\mu k^\nu \right) \gamma_\nu (1 - \gamma_5) \\
+ G'_2 i \sigma_{\mu\nu} k^\nu \left[ m_b (1 + \gamma_5) + m_q (1 - \gamma_5) \right] \right\} b .
\]

Using \( k^2 = 0 \) to simplify our expressions, the decay rate that follows is

\[
\Gamma_\lambda = \frac{1}{8\pi} G_F^2 |V_{tb} V_{tq}^{*}|^2 \left( 1 - \frac{m_q^2}{m_b^2} \right) |G'_2|^2 m_b^3 \left\{ \begin{array}{c} m_b^2 \\ m_q^2 \end{array} \right\} \left\{ \begin{array}{c} \lambda = -1 \\ \lambda = +1 \end{array} \right\} .
\]

Using \( m_q \to 0 \), the \( \lambda = +1 \) rate is very suppressed and vanishes in the limit. However, and contrary to the more general case of the \( b \to q + J/\psi \) decay, that is a trivial result, since the only contribution to the \( \lambda = +1 \) rate comes from
the term proportional to $G'_2m_q$, in the effective vertex. The same conclusion applies to the decay amplitude for the exclusive decay $B \to V\gamma$,

$$A_{\pm} = -\frac{G_F}{\sqrt{2}} V_{tb} V_{tq}^* (m_B^2 - m_V^2) G'_2 F_2(0) \begin{cases} m_b \quad \lambda = -1 \\ (-)m_q \quad \lambda = +1 \end{cases} .$$ (58)

The $\lambda = +1$ amplitude only has contributions from the term proportional to $G'_2m_q$ in the vertex, and it vanishes automatically in the limit $m_q \to 0$. There is no need to impose any constraint on the form factors and so the radiative decays cannot be used as a test of the Quark Model relations.

### 3 Heavy-to-light Form Factors in Lattice QCD and Light Cone Sum-Rules

A comparison to the predictions of Lattice QCD and of Light Cone Sum-Rules (LCSR) provides another useful test of the Quark Model form factor relations. In Figs. 2 and 3, we plotted the ratios of form factors obtained from Eqs. 23-30, for the case of $B \to \pi$ and $B \to \rho$ transitions. Also shown in the same figures are the results obtained from the recent LCSR calculations of Ref. [14] and from the Lattice results of Ref. [15]. The LCSR predictions are valid in a sizable range of $q^2$ that only excludes the low recoil (high $q^2$) region; conversely, that is the region where it is possible to obtain reliable Lattice results, for heavy-to-light transitions. The LCSR curves should be understood as accompanied by an error estimated at about 20% [14], that is not shown in the plots. Also not shown is the systematic error in the Lattice data points; the error bars correspond to the statistical error only. The agreement between the Quark Model predictions and both the Lattice QCD and the LCSR results is remarkable. It is worth pointing out, in particular, how such small effects as the predicted deviations from unity, in the ratios $A_0/V$ and $2V/F_1$ (an effect of the order of $m_V/m_B$), are well supported by the Lattice results.

We have plotted the Quark Model form factor ratios for the entire range of $q^2$. However, the results near $q^2 = 0$ (maximum recoil) and $q^2 = q^2_{\text{max}}$ (zero recoil) must be viewed with care, since the static $b$ quark and the ultra-relativistic $q$ quark limits may not be entirely justified in these regions. In order to estimate at what point, and by how much, these approximations
may fail, we would need some information about the quark dynamics inside the mesons (in particular, we would need to estimate the ratios $m_b/|\vec{p}_b|$ and $|\vec{p}_q|/m_q$ as a function of $q^2$). In Ref. [5], a few simple scenarios for the internal momentum wavefunctions of the mesons were discussed, that led to estimates of the quark momenta relative to their masses. These, however, are rather simplistic and model dependent results, that can serve an illustrative purpose, but cannot provide a reliable estimate of the range of $q^2$ where the form factor relations are valid. Instead, our rationale in here is to determine that range of validity through a comparison to experimental data or model independent theoretical methods, such as Lattice QCD and LCSR. Surprisingly, no significant discrepancies can be found, at the present level of accuracy, for any of the tests that were performed, and at any value of $q^2$.

4 Conclusion

We have compared the Quark Model form factor relations of Refs. [4, 5, 6], to both experimental results and the theoretical predictions of Lattice QCD.
Figure 3: $B \rightarrow \rho$ form factor ratios in the Quark Model (full lines), Light Cone Sum-Rules (dashed lines) and Lattice QCD (data points).
and Light Cone Sum-Rules. In every case, the agreement is impressive and suggests that the relations remain a good approximation, beyond the Quark Model. If that proves to be the case, then large theoretical uncertainties associated with hadronic form factors can be avoided, in the study of heavy-to-light weak transitions. For example, one important application would be obtaining a measurement of $|V_{ub}|$ that is independent of hadronic form factors, as suggested in Refs. [6] or [16]. But beforehand, one should try to improve the tests of the form factor relations that were discussed in here. In the $B \to K^*J/\psi$ analysis, it would be interesting to improve the precision in the measurement of $|A_+/A_0|$, to the point were the deviation from an exact cancellation of $A_+$ can be detected. By itself, such a measurement could not be easily interpreted: a non-vanishing $A_+$ amplitude could be due to corrections to the Quark Model relations, but it could also be due to corrections from a non-vanishing light quark mass. To help desintangle both effects, the analysis can be repeated for the analogous, but Cabibbo suppressed, $B \to \rho J/\psi$ decay. There, the light quark is $q = u$ and one is likely to be much closer to the $m_q = 0$ limit where the Quark Model derivation applies. Even better would be to determine the helicity amplitudes in the semileptonic decay $B \to \rho l^- \bar{\nu}_l$, and repeat the test of the Quark Model relations, at different values of $q^2$, and without the added uncertainty of the effects of strong final state interactions. Another area for improvement is in the measurement of the $J/\psi$ polarization in $b \to q + J/\psi$. As we have seen, together with the measurement of the $b \to q + J/\psi$ branching ratio, that would fully determine the effective $bqJ/\psi$ vertex. From there, we could better predict the ratio $A_-/A_0$ in $B \to VJ/\psi$, and this additional test of the Quark Model relations could be improved. Again, the same test can be performed with the semileptonic decay, without any of the theoretical uncertainties regarding the form of the weak vertex.

Acknowledgments

This research was funded by a grant from the National Science Foundation.

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