QUANTUM $E(2)$ GROUPS
AND LIE BIALGEBRA STRUCTURES

J. Sobczyk*
Max-Planck-Institut für Physik, Werner Heisenberg Institute
Föhringer Ring 6, D-80805 Munich, Germany

ABSTRACT

Lie bialgebra structures on $e(2)$ are classified. For two Lie bialgebra structures which are not coboundaries (i.e. which are not determined by a classical $r$-matrix) we solve the cocycle condition, find the Lie-Poisson brackets and obtain quantum group relations. There is one to one correspondence between Lie bialgebra structures on $e(2)$ and possible quantum deformations of $U(e(2))$ and $E(2)$.

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* On leave on absence from Institute for Theoretical Physics, Wroclaw University, Poland, E-mail: jsobczyk@proton.ift.uni.wroc.pl
1. Quantum deformations \[1\] \[2\] of the $D = 2$ Euclidean group $E(2)$ and its universal enveloping algebra $U(e(2))$ turn out to be a useful laboratory to study various aspects of quantum groups \[3\] - \[5\]. It is one of the simplest examples of non-simple Lie group and there is no canonical way to introduce its deformation. During last five years many approaches have been developed \[6\] - \[7\] to construct such deformations. The study of $E_q(2)$ is interesting for itself (one can ask questions about how many different quantum deformations exist in this case, about classical $r$ and quantum $R$ matrices, differential calculi, representations, bicrosproduct structures etc.) but is also useful in order to understand properties of quantum deformations of $D = 4$ Poincaré group \[8\]. The structure of $D = 4$ quantum groups are important to explore if one wishes to examine possible implications of quantum groups ideas in physics. Recently interesting results were obtained in this direction including a classification of possible deformations of $D = 4$ Poincaré group \[9\].

The aim of this paper is to argue that all the possible quantum deformations of $E(2)$ can be deduced from the analysis of Lie bialgebra structures on $e(2)$. The paper is organized as follows. In the chapter 2 we review obtained so far in the literature quantum deformations of $E(2)$ and $U(e(2))$. In chapter 3 we present the classification of the Lie bialgebra structures for $e(2)$. We obtain one one-parameter family of Lie bialgebra structures and three separate ”points”. Some of them turn out to be coboundaries (i.e. they are determined by classical $r$-matrices) but some are not of that kind. We show that quantum deformations described in the chapter 2 give rise to all of them except one case. The missing quantum deformation of $E(2)$ turns out to be a simplest one and will be discussed in the chapter 4. In this chapter we describe in detail how to derive Lie-Poisson brackets corresponding to Lie bialgebra structures which are not coboundaries. In the new case of Lie Poisson brackets which did not yet appear in the literature we obtain quantum group relations by changing Poisson brackets into commutators. We also calculate by duality the corresponding quantum deformation of $U(e(2))$. In chapter 5 we conclude the paper with some final remarks.

In our presentation we will concentrate on algebra and coalgebra structures of $E_q(2)$ and $U_q(e(2))$. It is a trivial exercise to guess what is a form of antipode and counit which make them Hopf algebras.

2. The first papers in the interesting us domain were dedicated to quantum deformations of enveloping algebra $U(e(2))$ \[3\]. These deformations were obtained by applying the technique of contraction from the standard deformation of $U(sl(2))$. It turns out that there are two different quantum contractions \[10\]:
The deformation parameters are $q$ in the case (A) and $\kappa$ in the case (B) and the classical limits are $q \to 1$ and $\kappa \to \infty$. It should be perhaps mentioned that till now no general theory of contractions of quantum groups exist. Some recent papers investigate such contractions by analysing the Lie bialgebra level: [11] - [12].

Quantum $E_q(2)$ group has been discussed by many authors from different points of view [3] - [6]. In order to fix the notation let us introduce the following matrix representation of elements of $E(2)$:

$$g(c, a, b) = \begin{pmatrix} \cos(c) & \sin(c) & a \\ -\sin(c) & \cos(c) & b \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix multiplication defines coproduct and antipode for $a, b$ and $c$. It turns out to be convenient to introduce the complex notation:

$$\eta = a + ib, \quad \bar{\eta} = a - ib \quad \text{and} \quad e^{ic}$$

The coproducts take the form:

$$\Delta(\eta) = e^{-ic} \otimes \eta + \eta \otimes 1$$

$$\Delta(\bar{\eta}) = e^{ic} \otimes \bar{\eta} + \bar{\eta} \otimes 1$$

$$\Delta(e^{ic}) = e^{ic} \otimes e^{ic}$$
There are many approaches to obtain quantum group relations for $E(2)$. They lead to two sets of relations:

(A')

\[
\eta \bar{\eta} = q^{2} \bar{\eta} \eta \\
\eta e^{i \epsilon} = q^{2} e^{i \epsilon} \eta \\
\bar{\eta} e^{i \epsilon} = q^{2} e^{i \epsilon} \bar{\eta}
\]  

or

(B')

\[
[e^{i \epsilon}, \eta] = \frac{1}{\kappa} (1 - e^{i \epsilon}) \\
[e^{i \epsilon}, \bar{\eta}] = \frac{1}{\kappa} (e^{2 i \epsilon} - e^{i \epsilon}) \\
[\eta, \bar{\eta}] = \frac{1}{\kappa} (\bar{\eta} + \eta)
\]

Coproducts for $\eta, \bar{\eta}$ and $e^{i \epsilon}$ are given in (11)-(13).

It should be mentioned that the full Hopf algebra duality has been demonstrated for two discussed so far deformations of the group $E(2)$ and $U(e(2))$. In the case (A) and (A') it was shown in [5] and in the case (B) and (B') in [6].

There exists still another approach to the quantization of $E(2)$. The starting point is the nonstandard (sometimes called Jordanian) quantum deformation of $U(sl(2))$. Following the general ideas it is possible to perform the contraction from $U_{q}(sl(2))$ to $U_{\mu}(e(2))$ [13]. The new deformation parameter is called $\mu$.

(C)

\[
[P_{+}, P_{-}] = 0, \quad [J, P_{\pm}] = \mu sh \frac{P_{\pm}}{\mu}
\]

\[
[J, P_{-}] = -P_{-} ch \frac{P_{+}}{\mu}
\]

\[
\Delta(P_{+}) = P_{+} \otimes 1 + 1 \otimes P_{+}
\]

\[
\Delta(J) = J \otimes e^{P_{+}/\mu} + e^{-P_{+}/\mu} \otimes J
\]

\[
\Delta(P_{-}) = P_{-} \otimes e^{P_{+}/\mu} + e^{-P_{+}/\mu} \otimes P_{-}
\]

The bad feature of this deformation of $U(e(2))$ is that it is strictly speaking a deformation of the complex $e(2)$ algebra. This is seen in the formulas (20)-(24). The operation $J^{*} = J$, $P_{\pm}^{*} = P_{\mp}$ is not a star operation in the Hopf algebra $U_{\mu}(e(2))$. 

3. It is possible to give a complete classification of Lie bialgebra structures for \( e(2) \).

Let us introduce \( e(2) \) Lie algebra with generators \( P_1, P_2 \) and \( J \) satisfying relations:

\[
[P_1, P_2] = 0, \quad [J, P_1] = iP_2 \quad [J, P_2] = -iP_1.
\]  

(25)

In the classification of Lie-bialgebra structures for \( e(2) \) one should take into account its invariance under the following transformations: (i) \( J \to J + \mu P_1 + \nu P_2 \); (ii) \( P_1 \to \cos \beta P_1 + \sin \beta P_2 \), \( P_2 \to -\sin \beta P_1 + \cos \beta P_2 \); (iii) \( P_1 \to \lambda P_1 \), \( P_2 \to \lambda P_2 \). The complete list of Lie bialgebra structures consist from:

\[
\delta_1(P_1) = sP_1 \wedge J, \quad \delta_1(P_2) = sP_2 \wedge J, \quad \delta_1(J) = 0
\]  

(26)

\[
\delta_2(J) = P_1 \wedge P_2, \quad \delta_2(P_1) = \delta_2(P_2) = 0
\]  

(27)

\[
\delta_3(P_2) = P_1 \wedge P_2, \quad \delta_3(J) = P_1 \wedge J, \quad \delta_3(P_1) = 0
\]  

(28)

\[
\delta_4(P_1) = -iP_1 \wedge P_2, \quad \delta_4(P_2) = P_1 \wedge P_2, \quad \delta_4(J) = P_1 \wedge J + iP_2 \wedge J
\]  

(29)

In the case (26) there is a one-parameter (\( s \)) family of Lie bialgebras. Out of above four possibilities only the last two are coboundaries with the classical \( r \)-matrices:

\[
r_3 = J \wedge P_2
\]  

(30)

\[
r_4 = J \wedge P_1 + iJ \wedge P_2
\]  

(31)

One could also write down more general form of the classical \( r \)-matrices by adding terms \( \tau P_1 \wedge P_2 + \gamma (P_1 \otimes P_1 + P_2 \otimes P_2) \). This generalization will however turn out to be inessential if one deduces the form of Lie-Poisson brackets out of \( r \). We find two Lie bialgebra structures which are not a coboundary. It is interesting to stress that the case of \( D = 2 \) is very particular one. It was shown that for \( D \geq 3 \) all the Lie bialgebra structures of homogeneous groups built from space-time rotations (with arbitrary signature) and translations are coboundaries [14].

It is easy to observe that the deformation (A) of chapter 2 corresponds to \( \delta_1 \), (B) corresponds to \( \delta_3 \) and (C) to \( \delta_4 \). On the other hand Lie bialgebra \( \delta_2 \) does not have yet its quantum group counterpart.

4. When Lie bialgebra is not a coboundary the computation of Lie-Poisson brackets is not straightforward. The problem is to solve the cocycle equation for \( \phi : G \to \mathcal{G} \wedge \mathcal{G} \) where \( G \) and \( \mathcal{G} \) denote Lie group and its Lie algebra [15]:

\[
\phi(gh) = \phi(g) + g\phi(h)g^{-1}
\]  

(32)
Let us discuss first the case of Lie bialgebra structure $\delta_1$. The "initial" conditions are ($\epsilon$ is an infinitesimal parameter)

$$\phi(1 + \epsilon P_1 + ...) = \epsilon s P_1 \wedge J + ...$$  \hspace{1cm} (33)

$$\phi(1 + \epsilon P_2 + ...) = \epsilon s P_2 \wedge J + ...$$  \hspace{1cm} (34)

$$\phi(1 + \epsilon J + ...) = 0$$  \hspace{1cm} (35)

It is understood that elements of $e(2)$ and $E(2)$ are given in the 3-dimensional representation and:

$$P_1 = i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad J = i \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$  \hspace{1cm} (36)

The strategy is to find first $\phi$ on 1-parameter subgroups generated by $P_1$, $P_2$ and $J$. Let

$$\phi(e^{-iaP_1}) = A(a)P_1 \wedge J + B(a)P_2 \wedge J + C(a)P_1 \wedge P_2$$  \hspace{1cm} (37)

Cocycle equation (32) gives rise to the following set of algebraic equations

$$2A(a) = A(2a)$$  \hspace{1cm} (38)

$$2B(a) = B(2a)$$  \hspace{1cm} (39)

$$2C(a) - aA(a) = C(2a)$$  \hspace{1cm} (40)

We assume further functions $A(a)$, $B(a)$ and $C(a)$ to be analytic in $a$. Taking into the account (33) one obtains

$$\phi(e^{-iaP_1}) = is \left(-aP_1 \wedge J + \frac{a^2}{2} P_1 \wedge P_2\right)$$  \hspace{1cm} (41)

Using the same methods one calculates also:

$$\phi(e^{-ibP_2}) = is \left(-bP_2 \wedge J + \frac{b^2}{2} P_1 \wedge P_2\right)$$  \hspace{1cm} (42)

$$\phi(e^{icJ}) = 0$$  \hspace{1cm} (43)

Group elements of $E(2)$ can be parameterized by

$$g(a, b, c) = e^{-iaP_1} e^{-ibP_2} e^{icJ} = \begin{pmatrix} \cos c & \sin c & a \\ -\sin c & \cos c & b \\ 0 & 0 & 1 \end{pmatrix}$$  \hspace{1cm} (44)
for which using (32) one calculates

\[
\phi(g(a, b, c)) = is \left(-a P_1 \wedge J - b P_2 \wedge J + \frac{a^2 + b^2}{2} P_1 \wedge P_2\right)
\]

(45)

Since one knows from the general theory that [13]:

\[
\{f, g\} = \phi^{ab} \partial_a f \partial_b g
\]

(46)

the Lie-Poisson brackets for \(a, b, c\) follow

\[
\{a, b\} = \frac{is}{2} (a^2 + b^2)
\]

(47)

\[
\{a, \cos c\} = ias \sin c
\]

(48)

\[
\{b, \cos c\} = ibs \sin c
\]

(49)

or, in the complex notation

\[
\{\eta, \bar{\eta}\} = s\eta\bar{\eta}
\]

(50)

\[
\{\eta, e^{ic}\} = s\eta e^{ic}
\]

(51)

\[
\{\bar{\eta}, e^{ic}\} = s\bar{\eta} e^{ic}
\]

(52)

These expressions should be compared with (14)-(16).

Let us apply the same method to the second non-coboundary Lie bialgebra structure \(\delta_2\):

\[
\delta_2(J) = P_1 \wedge P_2, \quad \delta_2(P_1) = \delta_2(P_2) = 0
\]

(53)

Using once more the technique described above one obtains:

\[
\phi(g(a, b, c)) = ic P_1 \wedge P_2
\]

(54)

After calculating Lie-Poisson brackets it turns out that the only non-vanishing bracket is:

\[
\{\eta, \bar{\eta}\} = -2c
\]

(55)

One can check explicitly that it in fact satisfies the required condition

\[
\{\Delta(\eta), \Delta(\bar{\eta})\} = \Delta(\{\eta, \bar{\eta}\}) = \Delta(-2c) = -2(c \otimes 1 + 1 \otimes c)
\]

(56)
Naive quantization seems applicable in this case so that one obtains as the quantum relations

$$\left[ \eta, \bar{\eta} \right] = ihc, \quad \left[ \eta, c \right] = \left[ \bar{\eta}, c \right] = 0$$ (57)

where by $h$ we denoted the deformation (quantization) parameter. It is instructive to find by duality the corresponding quantum deformation of $U(e(2))$. After short computations one arrives at the following structure

(D)

$$\left[ J, P_1 \right] = iP_2, \quad \left[ J, P_2 \right] = -iP_1, \quad \left[ P_1, P_2 \right] = 0$$ (58)

$$\Delta(P_1) = P_1 \otimes 1 + 1 \otimes P_1 \quad \Delta(P_2) = P_2 \otimes 1 + 1 \otimes P_2$$ (59)

$$\Delta(J) = J \otimes 1 + 1 \otimes J + h(P_1 \otimes P_2 - P_2 \otimes P_1)$$ (60)

This is in fact the simplest possible quantum deformation of $U(e(2))$. The antipode and counit are as in the undeformed case.

5. We conclude that a theory of quantum deformations of the $D = 2$ Euclidean group seems to be almost complete. All the Lie bialgebra structures on $e(2)$ can be quantized to Hopf algebras $U_q(e(2))$. There is however still one interesting unsolved problem. It is unknown whether for the quantum deformation (1)-(8) universal $R$-matrix exists. In many cases contraction prescription can be applied to $R$-matrix giving rise to a finite (usually after some manipulations) result [16] - [17]. In the case (A) discussed above the classical $r$-matrix does not exist and the same must be true for the universal $R$-matrix. In the case (B) the situation is unclear. Contraction of the universal $R$-matrix for $U_q(sl(2))$ leads to divergent expressions. Direct computation shows however that, at least up to terms $\frac{1}{\kappa^{10}}$, an expression for $R$ can be found [18]. Such $R$ satisfies the condition $R\Delta(a) = \Delta'(a)R$ for all the elements $a$ but it cannot satisfy Yang-Baxter equation as the classical $r$-matrix satisfies only modified classical Yang-Baxter equation.

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