A CRITERION FOR THE EXISTENCE OF RELAXATION OSCILLATIONS WITH APPLICATIONS TO PREDATOR-PREY SYSTEMS AND AN EPIDEMIC MODEL

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Abstract. We derive characteristic functions to determine the number and stability of relaxation oscillations for a class of planar systems. Applying our criterion, we give conditions under which the chemostat predator-prey system has a globally orbitally asymptotically stable limit cycle. Also we demonstrate that a prescribed number of relaxation oscillations can be constructed by varying the perturbation for an epidemic model studied by Li et al. [SIAM J. Appl. Math, 2016].

1. Introduction. Periodic orbits in ecological models are important because they can be used to explain oscillatory phenomena observed in real-world data. Relaxation oscillations are periodic orbits formed from slow and fast sections. In this paper, we extend the criterion for the existence of relaxation oscillations given by Hsu [11] to a class of planar systems. We apply our criterion to two ecological models: a predator-prey system in a chemostat and an epidemic model. Both systems are three-dimensional, and have two-dimensional invariant manifolds.

Predator-prey interaction in a well-stirred chemostat (see e.g. [23]) can be modeled by the following ordinary differential equations:

\[
\begin{align*}
\dot{S} &= (S^0 - S)\epsilon - \rho mSx, \\
\dot{x} &= x\left( -\epsilon + mS \right) - cyp(x), \\
\dot{y} &= y\left( -\epsilon + p(x) \right),
\end{align*}
\]

where \( \dot{\cdot} \) denotes \( \frac{d}{dt} \), \( S(t) \) is the concentration of the nutrient in the growth chamber at time \( t \), \( x(t) \) and \( y(t) \) are the density of prey (which feeds off this nutrient) and predator populations, respectively, \( S^0 \) denotes the concentration of the input nutrient, and \( \rho \) and \( c \) are constants related to the consumption of the nutrient by the prey population and the consumption of the prey by the predators, respectively.

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To ensure that the volume of this vessel remains constant, $\epsilon$ denotes both the rate of inflow from the nutrient reservoir to the growth chamber, as well as the rate of outflow from the growth chamber. The functional response $p(x)$, which describes the change in the density of the prey attacked per unit time per predator, is continuously differentiable and satisfies

$$p(0) = 0, \quad p'(0) > 0, \quad \text{and} \quad p(x) > 0 \quad \forall \ x > 0.$$  \hspace{1cm} (2)

It can be verified that system (1) under assumption (2) has a unique positive equilibrium for all small $\epsilon > 0$. From the equations in (1),

$$\frac{d}{dt} (S + \rho x + cpy) = -\epsilon \left( S + \rho x + cpy - S^0 \right),$$  \hspace{1cm} (3)

so system (1) has an invariant simplex

$$\Lambda = \{ (S, x, y) \in \mathbb{R}_+^3 : S + \rho x + cpy = S^0 \}$$  \hspace{1cm} (4)

that attracts all points in $\mathbb{R}_+^3$. For system (1) with $\epsilon = 0$, there is a continuous family of heteroclinic orbits on $\Lambda$ as illustrated in Figure 1 (see equation (51) in Section 3 and its succeeding paragraph for the limiting system on $\Lambda$). Each heteroclinic orbit connects two points on the boundary of $\Lambda$, where $x = 0$. On the other hand, the restriction of system (1) on the plane $\{x = 0\}$ is

$$\dot{S} = (S^0 - S)\epsilon, \quad \dot{x} = 0, \quad \dot{y} = -\epsilon y.$$  \hspace{1cm} (5)

Hence the segment $\Lambda \cap \{x = 0\}$ is a trajectory of system (1) approaching the positive $S$-axis. The heteroclinic orbits for the limiting system and the trajectory for (5) form a continuous family of closed loops. In Section 3, we use these loops to construct periodic orbits for the full system (see Theorem 3.1). Under certain conditions, we show that (1) has a globally asymptotically periodic orbit in $\mathbb{R}_+^3$ for all sufficiently small $\epsilon > 0$. Moreover, the minimal period of the periodic orbit is of order $1/\epsilon$ as $\epsilon \to 0$, and the trajectory converges to one of the closed loops described above. That is, this family of periodic orbits forms a relaxation oscillation.

![Figure 1. System (1) with $\epsilon = 0$ exhibits a family of heteroclinic orbits on $\Lambda$. The dynamics on the segment $\Lambda \cap \{x = 0\}$ is governed by system (5).](image-url)
We also study the epidemic model

\[
\begin{align*}
\dot{S} &= b(N) - g(S, N)I - (D + p)S, \\
\dot{I} &= g(S, N)I - (d + \gamma + \alpha)I, \\
\dot{R} &= pS + \gamma I - DR,
\end{align*}
\]

which was proposed and investigated by Graef et al. [8] and Li et al. [16]. Here \( N = S + I + R \) and \( b(N) = DN + \epsilon f(N) \), with

\[
\begin{align*}
f(N) &= rN \left(1 - \frac{N}{N_{\text{max}}} \right),
\end{align*}
\]

and

\[
\forall S \geq 0, \quad N \geq 0.
\]

Let \( a = d + \gamma + \alpha \). System (6) is equivalent to

\[
\begin{align*}
\dot{S} &= DN + \epsilon f(N) - g(S, N)I - (D + p)S, \\
\dot{I} &= g(S, N)I - aI, \\
\dot{N} &= \epsilon f(N) - aI.
\end{align*}
\]

Setting \( \epsilon = 0 \) in (8), we obtain the limiting system

\[
\begin{align*}
\dot{S} &= DN + g(S, N)I - (D + p)S, \\
\dot{I} &= g(S, N)I - aI, \\
\dot{N} &= -\alpha I.
\end{align*}
\]

Note that the line \( Z_0 \equiv \{(S, I, N) : I = 0, S = \frac{D}{D + p}N\} \) is a set of equilibria of (8) in the invariant plane \( \{I = 0\} \). It is known [8, 16] that \( Z_0 \) consists of the endpoints of a family of heteroclinic orbits (see Figure 2). The existence of periodic orbits of (8) was proved by Li et. al [16]. In Section 4, we demonstrate that a prescribed number of relaxation oscillations for system (8) can be obtained by varying the perturbation term \( \epsilon f(N) \).

![Figure 2. Trajectories for the limiting system (9) of (8) with \( \epsilon = 0 \).]
Singular perturbations in predator-prey systems were studied in various contexts. For a model of two predators competing for the same prey, when the prey population grows much faster than the predator populations, the existence of a relaxation oscillation was proved by Liu, Xiao and Yi [18]. For predator-prey systems with Holling type III or IV, when the death and the yield rates of the predator are small and proportional to each other, the canard phenomenon and the cyclicity of limit cycles were investigated by Li and Zhu [15]. For a class of the Holling-Tanner model, when the intrinsic growth rate of the predator is sufficiently small, the existence of a relaxation oscillation was proved by Ghazaryan, Manukian and Schecter [7]. For predator-prey models with eco-evolutionary dynamics in which ecological and evolutionary interactions occur on different time scales, relaxation oscillations were investigated by Piltz et al. [20] and Shen et al. [22]. For the classical predator-prey model with Monod functional response and small predator death rates, the unique periodic orbit, which was proved to exist by Liou and Cheng [17] (who corrected a flaw in the original proof of Cheng [3]) and Kuang and Freedman [13], was proved to form a relaxation oscillation by Hsu and Shi [9], Wang et al. [24], and Lundström and Söderbacka [19], and the cyclicity of the limit cycle was investigated by Huzak [21]. For general functional responses, the number of relaxation oscillations depending on the number of local extrema of the prey-isocline was studied by Hsu [11].

This paper is organized as follows. In Section 2, we state and prove criteria for the location and stability of relaxation oscillations in a class of planar systems. In Section 3 we investigate our criterion for the chemostat predator-prey system (1), and give conditions under which the system has exactly one or two periodic orbits. The epidemic model (8) is studied in Section 4, in which we compute the characteristic functions in terms of a parametrization of the center manifold, and we use numerical simulation to find the number of relaxation oscillations.

2. A criterion for relaxation oscillations in planar systems. To determine the location and stability of relaxation oscillations for predator-prey systems with small predator death, a criterion was given in [11]. In this section we extend that criteria by considering planar systems of the form

\[ \dot{a} = \epsilon f(a, b, \epsilon) + bh(a, b, \epsilon), \]
\[ \dot{b} = bg(a, b, \epsilon), \] (10)

where \( f, g \) and \( h \) are smooth functions satisfying, for some numbers \(-\infty < a_{\text{min}} < \bar{a} < a_{\text{max}} \leq \infty,\)

\[ f(a, 0, 0) > 0 \quad \forall a \in (a_{\text{min}}, a_{\text{max}}), \] (11)

\[ h(a, b, 0) < 0 \quad \forall a \in (a_{\text{min}}, a_{\text{max}}), b \geq 0, \] (12)

and

\[ g(a, 0, 0) \begin{cases} < 0, & \text{if } a \in (a_{\text{min}}, \bar{a}), \\ > 0, & \text{if } a \in (\bar{a}, a_{\text{max}}). \end{cases} \] (13)

Setting \( \epsilon = 0 \) in (10), we obtain the limiting system

\[ \dot{a} = bh(a, b, 0), \]
\[ \dot{b} = bg(a, b, 0). \] (14)

We assume the following condition (see Figure 3):
(H) There exists a nonempty open interval $I$ and smooth functions $a_\alpha : I \to (\bar{a}, a_{\max})$ and $a_\omega : I \to (a_{\min}, \bar{a})$, such that, for each $s \in I$, the points $(a_\alpha(s), 0)$ and $(a_\omega(s), 0)$ are the alpha- and omega-limit points, respectively, of a trajectory of (14).

\[ \gamma(s) = \int_{a_\omega(s)}^{a_\alpha(s)} \frac{g(a, 0, 0)}{f(a, 0, 0)} \, da \quad (15) \]

and

\[ \lambda(s) = \max \left\{ \int_{a_\omega(s)}^{a_\alpha(s)} \frac{\partial_t h(a, b, 0)}{h(a, b, 0)} \, da + \int_{\gamma(s)} \frac{\partial_b g(a, b, 0)}{g(a, b, 0)} \, db \right\} \quad (16) \]

If $s_0 \in I$ satisfies $\chi(s_0) = 0$ and $\lambda(s_0) \neq 0$, then for all sufficiently small $\epsilon > 0$, there is a periodic orbit $\ell_\epsilon$ of (10) in a $O(\epsilon)$-neighborhood of $\Gamma(s_0)$. The minimal period of $\ell_\epsilon$, denoted by $T_\epsilon$, satisfies

\[ T_\epsilon = \frac{1}{\epsilon} \left( \int_{a_\omega(s_0)}^{a_\alpha(s_0)} \frac{1}{f(a, 0, 0)} \, da + o(1) \right) \quad \text{as } \epsilon \to 0. \quad (17) \]

Moreover, $\ell_\epsilon$ is locally orbitally asymptotically stable if $\lambda(s_0) < 0$, and is orbitally unstable if $\lambda(s_0) > 0$. Conversely, if $\chi(s_0) \neq 0$, then for any point $z_1$ in the interior of the trajectory $\gamma(s_0)$, there is a neighborhood $U$ of $z_1$ such that no periodic orbit of (10) intersects $U$ for any sufficiently small $\epsilon > 0$.

**Remark 1.** Condition (12) is not essential and was only provided for convenience. Without assuming the positivity of $h$, the results in Theorem 2.1 still hold with $\lambda$ defined using integrals in terms of the time variable $t$. Also note that all integrals in (15) and (16) exist and are finite under the assumption that $f$ and $h$ remain

**Figure 3.** A family of heteroclinic orbits, parameterized by $\gamma(s)$, of limiting system (14).
nonzero. One way to see this for the last integral of (16) is to use the relation 
\[ \frac{db}{da} = \frac{g(a, b, 0)}{h(a, b, 0)} \] 
from (14) to obtain
\[ \int_{\gamma(s)} \frac{\partial_b g(a, b, 0)}{g(a, b, 0)} \, db = \int_{\gamma(s)} \frac{\partial_b g(a, b, 0)}{h(a, b, 0)} \, da. \]
This integral is finite since the integrand in the last expression is bounded.

**Remark 2.** The function \( \chi \) defined by (15) is related to the *slow divergence integral* studied by De Maesschalck and Dumortier [4]. In their work, the cyclicity of a relaxation oscillation near \( \gamma(s_0) \) is bounded by an algebraic expression of the multiplicity of \( s_0 \) as a zero of \( \chi(s) \). The function \( \lambda(s) \) in the present work is not necessarily equivalent to \( \chi'(s) \) (in the sense of multiplication by a positive function). When \( \chi(s_0) = 0 \), it can be shown that \( \lambda(s_0) \) and \( \chi'(s_0) \) have the same sign. Therefore, the function \( \lambda \) provides an essentially different approach than that in [4] for determining the sign of \( \chi'(s_0) \).

**Remark 3.** The function \( \chi(s) \) defined by (15) can be expressed in terms of integrals on \( \gamma(s) \): For any fixed \( s \in I \), let \( \gamma(s) = (A(t), B(t)) \) be a solution of (14) with trajectory \( \gamma(s) \). Then the relations \( A(\alpha) = a_\omega(s) \) and \( A(-\alpha) = a_\alpha(s) \) yield
\[ \int_{a_\omega(s)}^{a_\omega(s)} \frac{g(a, 0, 0)}{f(a, 0, 0)} \, da = \int_{-\infty}^{\infty} \frac{g(a, 0, 0)}{f(a, 0, 0)} \, A'(t) \, dt = \int_{-\infty}^{\infty} \frac{g(a, 0, 0)}{f(a, 0, 0)} \, A'(t) \, B'(t) \, dt, \]
and therefore
\[ \chi(s) = \int_{\gamma(s)} \frac{g(a, 0, 0)}{f(a, 0, 0)} \, da = \int_{\gamma(s)} \frac{g(a, 0, 0)}{f(a, 0, 0)} \frac{g(a, b, 0)}{h(a, b, 0)} \, db. \]
These last two identities sometimes can be advantageous for determining the sign of \( \chi(s) \) (see Theorem 3.3).

**Remark 4.** For the perspective of modeling, if a planar system of the form (14) possesses a line of equilibria and a family of heteroclinic orbits connecting points on this line, then by a suitable choice of \( g(a, b) \) to control the signs of \( \chi \) and \( \lambda \) in (15) and (16), the perturbed system (10) can have a relaxation oscillation at a prescribed location with a prescribed local stability (see Example 4.2).

**Remark 5.** In the case that \( \chi(s_0) = 0 \) and \( \lambda(s_0) = 0 \), Theorem 2.1 does not guarantee the existence of a periodic orbit near \( \gamma(s_0) \). However, in the region filled by the family of heteroclinic orbits given in (H), by Theorem 2.1 all possible periodic orbits must lie in an arbitrarily small neighborhood of \( \bigcup_{\{s_0: \chi(s_0) = 0\}} \gamma(s_0) \) regardless of the sign of \( \lambda(s_0) \) for all sufficiently small \( \epsilon > 0 \).

**Proposition 2.2.** Assume the functions \( f(a, b, \epsilon) \) and \( h(a, b, \epsilon) \) in (10) are separable functions of the form
\[ f(a, b, \epsilon) = a \tilde{f}(b, \epsilon) \quad \text{and} \quad h(a, b, \epsilon) = a \tilde{h}(b, \epsilon). \]
Then \( \lambda(s) \) defined in (16) equals
\[ \lambda(s) = \int_{\gamma(s)} \frac{\partial_b g(a, b, 0)}{g(a, b, 0)} \, db. \]

**Proof.** Under condition (19),
\[ \ln \frac{f(a_\omega(s), 0, 0)}{f(a_\omega(s), 0, 0)} = \ln \frac{a_\alpha(s)}{a_\omega(s)} \]
(21)
Theorem 2.5

If \( \text{Remark 6.} \) Proposition 2.2 is the case considered in [11]. Assume that the function \( h(a, b, \epsilon) \) in (10) is independent of \( a \), that is, 
\[
h(a, b, \epsilon) = \hat{h}(b, \epsilon).
\] Then \( \lambda(s) \) defined in (16) equals
\[
\lambda(a_0) = \ln \left( \frac{f(a_0(s), 0, 0)}{g(a_0(s), 0, 0)} \right) + \int_{\gamma(a_0)} \frac{\tilde{\partial}_a h(a, b, 0)}{h(a, b, 0)} \, da + \int_{\gamma(a_0)} \frac{\tilde{\partial}_b g(a, b, 0)}{G(a, b, 0)} \, db.
\] (24)

Proof. The partial derivative \( \tilde{\partial}_a h \) is identically zero under assumption (23). Hence the second integral in (16) is zero. \( \Box \)

Proposition 2.4. If \( g(a, b, \epsilon) = \varphi(b)G(a, b, \epsilon) \) for some smooth functions \( \varphi \) and \( G \) with \( \varphi(0) \neq 0 \), then \( \lambda \) defined in (16) is equal to
\[
\lambda(s) = \ln \left( \frac{f(a_\alpha(s), 0, 0)}{f(a_\omega(s), 0, 0)} \right) + \int_{\gamma(a_\alpha)} \frac{\tilde{\partial}_a h(a, b, 0)}{h(a, b, 0)} \, da + \int_{\gamma(a_\alpha)} \frac{\tilde{\partial}_b G(a, b, 0)}{G(a, b, 0)} \, db.
\] (25)

Proof. From the condition \( g(a, b, \epsilon) = \varphi(b)G(a, b, \epsilon) \),
\[
\int_{\gamma(a_\alpha)} \frac{\tilde{\partial}_b g(a, b, 0)}{g(a, b, 0)} \, db = \int_{\gamma(a_\alpha)} \frac{\tilde{\partial}_b \varphi(b)G(a, b, 0)}{\varphi(b)G(a, b, 0)} \, db
\]
\[
= \int_{\gamma(a_\alpha)} \frac{\varphi'(b)G(a, b, 0) + \varphi(b) \tilde{\partial}_b G(a, b, 0)}{\varphi(b)G(a, b, 0)} \, db
\]
\[
= \int_{\gamma(a_\alpha)} \frac{\varphi'(b)}{\varphi(b)} \, db + \int_{\gamma(a_\alpha)} \frac{\tilde{\partial}_b G(a, b, 0)}{G(a, b, 0)} \, db.
\]
Note that the \( b \)-coordinates of the endpoints of \( \gamma(s) \) are 0. Since \( \varphi(0) \neq 0 \),
\[
\int_{\gamma(a_\alpha)} \frac{\varphi'(b)}{\varphi(b)} \, db = \ln \left( \frac{\varphi(b)}{\varphi(b)} \right)_{b=0}^{b=0} = 0
\] and consequently
\[
\int_{\gamma(a_\alpha)} \frac{\tilde{\partial}_b g(a, b, 0)}{g(a, b, 0)} \, db = \int_{\gamma(a_\alpha)} \frac{\tilde{\partial}_b G(a, b, 0)}{G(a, b, 0)} \, db.
\]
Therefore (16) yields (25). \( \Box \)

To prove Theorem 2.1, we recall the following two theorems from [11].

Theorem 2.5 (Theorem 5.1 in [11]). Consider system (10), where \( f, g \) and \( h \) are \( C^{r+1} \) functions, \( r \in \mathbb{N} \), that satisfy (11)–(13). Assume that \( a_0 < a_1 \) satisfies relation
\[
\int_{a_0}^{a_{\infty}} \frac{g(a, 0, 0)}{f(a, 0, 0)} \, da = 0
\] (26)
and that there exist trajectories \( \gamma_1 \) and \( \gamma_2 \) of the limiting system
\[
\dot{a} = b h(a, b, 0), \quad \dot{b} = b g(a, b, 0),
\] (27)
such that \( (a_1, 0) \) is the omega-limit point of \( \gamma_1 \) and \( (a_2, 0) \) is the alpha-limit point of \( \gamma_2 \). Then for all sufficiently small \( \delta_1 > 0 \) and \( \delta_2 > 0 \), there exists \( \epsilon_0 > 0 \) such that the following holds. Let
\[
(a_{\infty}, \delta_1) = \gamma_1 \cap \{ b = \delta_1 \} \quad \text{and} \quad (a_{\infty}, \delta_1) = \gamma_2 \cap \{ b = \delta_1 \}.
\] (28)
Let
\[ \Sigma^{\text{in}} = \{(a, \delta_1) : |a - a^{\text{in}}| < \delta_2\} \quad \text{and} \quad \Sigma^{\text{out}} = \{(a, \delta_1) : |a - a^{\text{out}}| < \frac{|a_1|}{2}\}, \] (29)

Then the transition mapping from \( \Sigma^{\text{in}} \) to \( \Sigma^{\text{out}} \) of (10) is well-defined for \( \epsilon \in (0, \epsilon_0] \), and is \( C^r \) up to \( \epsilon = 0 \). That is, there exists a \( C^r \) function
\[ \pi_{\epsilon}(z) : \Sigma^{\text{in}} \times [0, \epsilon_0] \to \Sigma^{\text{out}} \]
such that, for each \( z \in \Sigma^{\text{in}} \) and \( \epsilon \in (0, \epsilon_0] \), \( z \) and \( \pi_{\epsilon}(z) \) are connected by a trajectory of (10), and
\[ \pi_{\epsilon}(z_0) = \pi_{\epsilon}(z_1) \quad \text{for} \quad z_0 = (a_0, \delta) \quad \text{and} \quad z_1 = (a_1, \delta) \quad \text{satisfies} \quad (26). \]
The time span \( T_{\epsilon, \delta_1} \) of the trajectory \( \sigma_{\epsilon} \) connecting \( z \in \Sigma^{\text{in}} \) and \( \pi_{\epsilon}(z) \) satisfies
\[ T_{\epsilon, \delta_1} = \frac{1}{\epsilon} \left( \int_{a_0}^{a_1} \frac{1}{f(a,0,0)} \, da + o(1) \right) \quad \text{as} \quad \epsilon \to 0. \] (30)
Moreover, there exists \( M > 0 \) such that for each \( \Delta \in (0, \delta_1) \), if we parameterize \( \sigma_{\epsilon} \cap \{b < \Delta\} \) by \((a_{\epsilon}(t), b_{\epsilon}(t))\), \( t \in [0, T_{\epsilon, \Delta}] \), then there exists \( \epsilon_\Delta > 0 \) satisfying
\[ \int_0^{T_{\epsilon, \Delta}} b_{\epsilon}(t) \, dt \leq M \Delta \quad \forall \epsilon \in (0, \epsilon_\Delta]. \] (31)

The next theorem is the variation of Floquet Theory.

**Theorem 2.6** (Theorem 5.2 in [11]). Consider systems in \( \mathbb{R}^N \), \( N \geq 2 \), of the form
\[ \dot{z} = h_{\epsilon}(z), \] (32)
where \( h_{\epsilon}(z) = h(z, \epsilon) \) is a \( C^2 \) function of \((z, \epsilon) \in \mathbb{R}^N \times [0, \epsilon_0]\). Let \( z_0 \in \mathbb{R}^N \) with \( h(z_0,0) \neq 0 \), and let \( \Sigma \) be a cross section of \( z_0 \) transversal to \( h_0(z_0) \). Assume that there exist \( \epsilon_0 > 0 \) and a neighborhood \( \Sigma^{(1)} \subset \Sigma \) of \( z_0 \) such that the return map from \( \Sigma^{(1)} \) to \( \Sigma \) is well-defined for \( \epsilon \in (0, \epsilon_0] \) and is \( C^1 \) up to \( \epsilon = 0 \). That is, there is a \( C^1 \) function
\[ P_{\epsilon}(z) : \Sigma^{(1)} \times [0, \epsilon_0] \to \Sigma \]
such that for any \( z \in \Sigma^{(1)} \) and \( \epsilon \in (0, \epsilon_0] \) there is a trajectory of (32) that starts at \( z \in \Sigma^{(1)} \) and returns to \( \Sigma \) at \( P_{\epsilon}(z) \).
Let \( \zeta_{\epsilon}(t), 0 \leq t \leq T_{\epsilon}, \) be a trajectory of (32) that starts at \( z_0 \) and ends at \( P_{\epsilon}(z_0) \). Assume that
\[ \int_0^{T_{\epsilon}} \text{div}(h_{\epsilon}(\zeta_{\epsilon}(t))) \, dt \to \lambda_0 \quad \text{as} \quad \epsilon \to 0 \]
for some \( \lambda_0 \in \mathbb{R} \). Then
\[ \det (DP_{\epsilon}(u_0)) = \exp(\lambda_0), \] (33)
where \( DP_{\epsilon}(u_0) \) is regarded as a linear transform on the tangent space \( T_{z_0} \Sigma \).

Now we use Theorems 2.5 and 2.6 to prove Theorem 2.1.

**Proof of Theorem 2.1.** The proof for the case with \( \chi(s_0) \neq 0 \) is similar to that in [11, Theorem 2.1], so we omit it here. In this case, there is a neighborhood \( U \) of \( z_1 \) such that no periodic orbit of (10) intersects \( U \) for any sufficiently small \( \epsilon > 0 \).
Assume \( \chi(s_0) = 0 \) and \( \lambda(s_0) \neq 0 \). Let \( a_0 = a_{\chi}(s_0) \) and \( a_1 = a_{\chi}(s_0) \). The condition \( \chi(a_0) = 0 \) means that (26) holds. Let \((a^{\text{in}}, \delta_1), (a^{\text{out}}, \delta_1), \Sigma^{\text{in}} \) and \( \Sigma^{\text{out}} \) be the points and segments defined in (28) and (29). By Theorem 2.5, The transition map \( \pi^{(1)}_{\epsilon} : \Sigma^{\text{in}} \to \Sigma^{\text{out}}, \epsilon \in (0, \epsilon_0] \) is well defined and is \( C^1 \) up to \( \epsilon = 0 \). Let
\[ \Sigma^{\text{in}} = \{(a, \delta_1) : a \in (-\infty, \tilde{a})\}. \]
Since (10) is a regular perturbation of (14) in the region \{(a, b) : b \geq \delta_1\}, the transition map \(\pi_e^{(2)} : \Sigma^{\text{out}} \rightarrow \Sigma^{\text{in}}\) for system (10) is well-defined and is smooth for \(\epsilon \in [0, \epsilon_0]\). Since \((a^{\text{in}}, \delta), (a^{\text{out}}, \delta) \in \gamma(s_0)\), we have \(\pi_0^{(2)}(a^{\text{out}}, \delta_1) = (a^{\text{in}}, \delta_1)\).

Let \(P_1 = \pi_e^{(2)} \circ \pi_e^{(1)}\). Then \(P_0(a^{\text{in}}, \delta_1) = \pi_0^{(2)}(a^{\text{out}}, \delta_1)\). That is, \((a^{\text{in}}, \delta_1)\) is a fixed point of \(P_0\). To show that this fixed point persists for small \(\epsilon > 0\), by the implicit function theorem it suffices to show that the Jacobian matrix of the return map \(P_0\) evaluated at \((a^{\text{in}}, \delta)\) is non-singular. Let

\[
\mu_\epsilon = \int_{C_\epsilon} \text{div} \left( \frac{\epsilon f(a, b, \epsilon) + bh(a, b, \epsilon)}{b g(a, b, \epsilon)} \right) dt,
\]

where \(C_\epsilon\) is the trajectory of (10) that starts at \((a^{\text{in}}, \delta)\) and ends at \(P_\epsilon(a^{\text{in}}, \delta)\). By Theorem 2.6, it suffices to show that \(\mu_\epsilon\) approaches a nonzero value as \(\epsilon \to 0\).

Using the equation for \(\dot{a}\) in (10),

\[
\int_{C_\epsilon} \frac{\partial_a \left( \frac{\epsilon f(a, b, \epsilon) + bh(a, b, \epsilon)}{b g(a, b, \epsilon)} \right)}{\epsilon f(a, b, \epsilon) + bh(a, b, \epsilon)} \, da = \int_{\gamma(s_0) \cap \{b > \Delta\}} \frac{\partial_a h(a, b, 0)}{h(a, b, 0)} \, da.
\]

For any fixed \(\Delta \in (0, \delta_1)\), since \(bh(a, b, 0)\) is bounded away from zero on \(\zeta_\epsilon \cap \{b > \Delta\}\), by regular perturbation theory we have

\[
\lim_{\epsilon \to 0} \int_{\zeta_\epsilon \cap \{b > \Delta\}} \frac{\partial_a \left( \frac{\epsilon f(a, b, \epsilon) + bh(a, b, \epsilon)}{b g(a, b, \epsilon)} \right)}{\epsilon f(a, b, \epsilon) + bh(a, b, \epsilon)} \, da = \int_{\gamma(s_0) \cap \{b > \Delta\}} \frac{\partial_a h(a, b, 0)}{h(a, b, 0)} \, da.
\]

Hence

\[
\lim_{\epsilon \to 0} \int_{\zeta_\epsilon \cap \{b > \Delta\}} \frac{\partial_a \left( \frac{\epsilon f(a, b, \epsilon) + bh(a, b, \epsilon)}{b g(a, b, \epsilon)} \right)}{\epsilon f(a, b, \epsilon) + bh(a, b, \epsilon)} \, da - \int_{\gamma(s_0) \cap \{b > \Delta\}} \frac{\partial_a h(a, b, 0)}{h(a, b, 0)} \, da \leq C_\Delta.
\]

Here and in the rest of the proof we use \(C\) to denote constants independent of \(\epsilon\) and \(\Delta\). Next we claim that

\[
\lim_{\epsilon \to 0} \sup \int_{\zeta_\epsilon \cap \{b < \Delta\}} \frac{\epsilon \partial_a f(a, b, \epsilon) + bh(a, b, \epsilon)}{\epsilon f(a, b, \epsilon) + bh(a, b, \epsilon)} \, da - \ln \frac{f(a, 0, 0)}{f(a, 0, 0)} \leq C_\Delta
\]

and that

\[
\lim_{\epsilon \to 0} \sup \int_{\zeta_\epsilon \cap \{b < \Delta\}} \frac{b \partial_a h(a, b, \epsilon)}{\epsilon f(a, b, \epsilon) + bh(a, b, \epsilon)} \, da \leq C_\Delta.
\]

We write \(\zeta_\epsilon \cap \{b < \Delta\} = \zeta_\epsilon^{(1)}_\Delta \cup \zeta_\epsilon^{(2)}_\Delta \cup \zeta_\epsilon^{(3)}_\Delta\) by

\[
\zeta_\epsilon^{(1)}_\Delta = \zeta_\epsilon \cap \{b < \Delta\} \cap \{a < a_0 + \Delta\},
\]

\[
\zeta_\epsilon^{(2)}_\Delta = \zeta_\epsilon \cap \{b < \Delta\} \cap \{a_0 + \Delta < a < a_1 + \Delta\},
\]

\[
\zeta_\epsilon^{(3)}_\Delta = \zeta_\epsilon \cap \{b < \Delta\} \cap \{a > a_1 - \Delta\}.
\]

It can be shown (from the proof of [11, Theorem 2.1]) that for some \(K = K(\Delta) > 0\) independent of \(\epsilon\),

\[
b < e^{-K/\epsilon} \quad \text{on} \quad \zeta^{(2)}_\epsilon_\Delta.
\]
Since $f(a,0,0)$ is bounded away from zero, (39) gives
\[
\lim_{\epsilon \to 0} \int_{\zeta^{(1)} - \epsilon}^{\zeta^{(1)} + \epsilon} \frac{b \varphi_{a} h(a,b,\epsilon)}{f(a,b,\epsilon) + b h(a,b,\epsilon)} \, da = \lim_{\epsilon \to 0} \int_{\zeta^{(1)} - \epsilon}^{\zeta^{(1)} + \epsilon} \frac{\partial_{a} f(a,b,\epsilon)}{f(a,b,\epsilon) + O(e^{-K/\epsilon})} \, da.
\]
Hence
\[
\lim_{\epsilon \to 0} \int_{\zeta^{(1)} - \epsilon}^{\zeta^{(1)} + \epsilon} \frac{b \varphi_{a} h(a,b,\epsilon)}{f(a,b,\epsilon) + b h(a,b,\epsilon)} \, da = \ln \frac{f(a_{1} - \Delta,0,0)}{f(a_{0} + \Delta,0,0)}.
\]
Similarly, from (40), (41) and (42), we obtain (37).

Combining estimates (40), (41) and (42), we obtain (37).

Since $f(a,0,0)$ is bounded away from zero, (39) gives
\[
\int_{\zeta^{(1)} - \epsilon}^{\zeta^{(1)} + \epsilon} \frac{b \varphi_{a} h(a,b,\epsilon)}{f(a,b,\epsilon) + b h(a,b,\epsilon)} \, da = \int_{\zeta^{(1)} - \epsilon}^{\zeta^{(1)} + \epsilon} \frac{O(e^{-K/\epsilon})}{f(a,b,\epsilon) + O(e^{-K/\epsilon})} \, da \to 0
\]
as $\epsilon \to 0$. On the other hand, by the equations for $\dot{a}$ and $\dot{b}$ in (14),
\[
\int_{\zeta^{(1)} - \epsilon}^{\zeta^{(1)} + \epsilon} \frac{b \varphi_{a} h(a,b,\epsilon)}{f(a,b,\epsilon) + b h(a,b,\epsilon)} \, da = \int_{\zeta^{(1)} - \epsilon}^{\zeta^{(1)} + \epsilon} \frac{b \varphi_{a} h(a,b,\epsilon)}{b g(a,b,\epsilon)} \, db = \int_{\zeta^{(1)} - \epsilon}^{\zeta^{(1)} + \epsilon} \frac{\partial_{a} h(a,b,\epsilon)}{g(a,b,\epsilon)} \, db.
\]
Since $g(a_{0},0,0) \neq 0$ and length$(\zeta^{(1)}) \leq C \Delta$, it follows that
\[
\left| \int_{\zeta^{(1)} - \epsilon}^{\zeta^{(1)} + \epsilon} \frac{b \varphi_{a} h(a,b,\epsilon)}{f(a,b,\epsilon) + b h(a,b,\epsilon)} \, da \right| \leq C \Delta.
\]
Similarly, from $g(a_{1},0,0) \neq 0$,
\[
\left| \int_{\zeta^{(1)} - \epsilon}^{\zeta^{(1)} + \epsilon} \frac{b \varphi_{a} h(a,b,\epsilon)}{f(a,b,\epsilon) + b h(a,b,\epsilon)} \, da \right| \leq C \Delta.
\]
Combining (43), (44) and (45), we obtain (38).

By (36), (37) and (38),
\[
\lim_{\epsilon \to 0} \int_{\zeta^{(1)}} \varphi_{a} \left( ef + bh \right) \, dt - \ln \frac{f(a_{1},0,0)}{f(a_{0},0,0)} - \int_{\gamma(s_{0})} \frac{\partial_{a} h(a,b,0)}{h(a,b,0)} \, da \leq C \Delta,
\]
where $(ef + bh)$ is evaluated at $(a,b,\epsilon)$. Since $\Delta$ can be arbitrarily small, we obtain
\[
\lim_{\epsilon \to 0} \int_{\zeta^{(1)}} \varphi_{a} \left( ef(a,b,\epsilon) + bh(a,b,\epsilon) \right) \, dt = \ln \frac{f(a_{1},0,0)}{f(a_{0},0,0)} + \int_{\gamma(s_{0})} \frac{\partial_{a} h(a,b,0)}{h(a,b,0)} \, da.
\]
A similar calculation gives

$$\lim_{\epsilon \to 0} \int_t^{t+\epsilon} \hat{c}_s(b \hat{g}(a, b, \epsilon)) \, dt = \int_{\gamma(s_0)} \frac{\hat{c}_s g(a, b, 0)}{h(a, b, 0)} \, da. \quad (47)$$

By (46) and (47) we conclude that the number $\mu_\epsilon$ defined by (34) satisfies

$$\lim_{\epsilon \to 0} \mu_\epsilon = \ln \frac{f(a_1, 0, 0)}{f(a_0, 0, 0)} + \int_{\gamma(s_0)} \frac{\hat{c}_s h(a, b, 0) + \hat{c}_s g(a, b, 0)}{h(a, b, 0)} \, da.$$  

Using the relation $dh/da = g(a, b, 0)/h(a, b, 0)$ from (14), it follows that $\lim_{\epsilon \to 0} \mu_\epsilon = \lambda(s_0)$ with $\lambda$ defined by (16).

Since we assume $\lambda(s_0) \neq 0$, by Theorem 2.6 the return map $P_\epsilon$ has a unique fixed point near $(a_0, \delta_1)$ for all small $\epsilon > 0$, and $P_\epsilon$ is a contraction if $\lambda(s_0) < 0$, and an expansion if $\lambda(s_0) > 0$. Hence there is a unique periodic orbit $\ell_\epsilon$ of (10) near $\gamma(s_0)$ for every small $\epsilon > 0$. This periodic orbit is locally orbitally asymptotically stable if $\lambda(s_0) < 0$, and is unstable if $\lambda(s_0) > 0$. The estimate (17) follows from (30). \hfill \Box

3. The chemostat predator-prey system. The restriction of system (1) on the invariant plane $\Lambda$ is governed by $S = S^0 - \rho x - cp y$ and

$$\dot{x} = x(-\epsilon + m(S^0 - \rho x - cp y)) - cy p(x)$$

$$= c(\rho mx + p(x)) \left(F_\epsilon(x) - y\right),$$

$$\dot{y} = y(-\epsilon + p(x)),$$

where

$$F_\epsilon(x) = \frac{x(-\epsilon + mS^0 - \rho mx)}{c(\rho mx + p(x))}. \quad (49)$$

Define $F(x) = F_0(x)$, that is

$$F(x) = \frac{x(mS^0 - \rho mx)}{c(\rho mx + p(x))},$$

and define

$$\gamma = F(0) = \lim_{x \to 0} F(x) = \frac{mS^0}{c(\rho mx + p(0))} > 0.$$  

The last inequality follows from $p'(0) > 0$ in condition (2). Note that $F_\epsilon(x)$ is continuous at $(\epsilon, x) = (0, 0)$ by setting $F_\epsilon(0) = -\frac{\epsilon + mS^0}{c(\rho mx + p'(0))}$. Also note that $F(x) > 0$ for all $x \in (0, S^0/\rho)$, and that $F(S^0/\rho) = 0$.

When $\epsilon = 0$, system (48) reduces to

$$\dot{x} = c(\rho mx + p(x)) \left(F(x) - y\right),$$

$$\dot{y} = y p(x). \quad (51)$$

Since $p(0) = 0$ and $F(S^0/\rho) = 0$, system (51) has a line of equilibria $\{x = 0\}$ and an isolated saddle equilibrium $E_1 = (S^0/\rho, 0)$. The unstable manifold of $E_1$ is the portion of the line $\{x + cy = S^0/\rho\}$. Given any $x_0 \in (0, S^0/\rho)$, since $p(x) > 0$ for all $x \in (0, S^0/\rho)$, both the forward and backward trajectories starting at $(x, y) = (x_0, F(x_0))$ approach points on the boundary of $\Lambda$ (see Figure 1). This gives a continuous family of heteroclinic orbits parameterized by $x_0 \in (0, S^0/\rho)$ that fills the region in the positive quadrant bounded below by the unstable manifold of $E_1$.

Denote the trajectory passing through $(x_0, F(x_0))$ by $\gamma(x_0)$, and the $y$-values

at the omega- and alpha-limit points of $\gamma(x_0)$ by $y_\omega(x_0)$ and $y_\alpha(x_0)$, respectively.
Denote the segment connecting the two endpoints of $\gamma(x_0)$ by $\sigma(x_0)$. Then $\gamma(x_0) \cup \sigma(x_0)$ is a singular closed orbit for each $x_0 \in (0, S^0/\rho)$.

Applying Theorem 2.1 to system (48) with $(-y, x)$ playing the role of $(a, b)$, the formula (15) gives, up to multiplication by positive constants,

$$\chi(x_0) = \int_{y_\alpha(x_0)}^{y_\omega(x_0)} \frac{y - \bar{y}}{y} dy. \quad (52)$$

By Propositions 2.2 and 2.3, the formula (16) gives, up to multiplication by positive constants,

$$\lambda(x_0) = \int_{\gamma(x_0)} \frac{\partial_x [F(x) - y]}{F(x) - y} dy = \int_{\gamma(x_0)} \frac{F'(x)}{F(x) - y} dy. \quad (53)$$

**Theorem 3.1.** Assume that $x_0 \in (0, S^0/\rho)$ satisfies $\chi(x_0) = 0$ and $\lambda(x_0) \neq 0$, where $\chi$ and $\lambda$ are defined in (52) and (53). Then for any sufficiently small $\epsilon > 0$, there is a periodic orbit $\ell_\epsilon$ of (1) in a $O(\epsilon)$-neighborhood of $\Gamma(x_0) = \gamma(x_0) \cup \sigma(x_0)$. The minimal period of $\ell_\epsilon$, denoted by $T_\epsilon$, satisfies

$$T_\epsilon = \frac{1}{\epsilon} \ln \left( \frac{y_\omega(x_0)}{y_\alpha(x_0)} \right) + o(1) \quad \text{as } \epsilon \to 0. \quad (54)$$

Moreover, $\ell_\epsilon$ is locally orbitally asymptotically stable if $\lambda(x_0) < 0$, and is orbitally unstable if $\lambda(x_0) > 0$. Conversely, if $\chi(x_0) \neq 0$, then for any point $z_1$ in the interior of the trajectory $\gamma(x_0)$, there is a neighborhood $U$ of $z_1$ such that no periodic orbit of (1) intersects $U$ for any sufficiently small $\epsilon > 0$.

**Proof.** If $\chi(x_0) = 0$ and $\lambda(x_0) \neq 0$, then by Theorem 2.1, for every small $\epsilon > 0$, system (48) has unique periodic orbit $\ell_\epsilon$ near $\gamma(x_0) \subset \Lambda$. Since $\Lambda$ is invariant under (48), $\ell_\epsilon$ is also a periodic orbit for (1). If $\lambda(x_0) > 0$, then $\ell_\epsilon$ is orbitally unstable for (48), and therefore $\ell_\epsilon$ is orbitally unstable for (1). If $\lambda(x_0) < 0$, then $\ell_\epsilon$ is locally orbitally asymptotically stable for (48). Since $\Lambda$ is a hyperbolic attractor, it follows that $\ell_\epsilon$ is locally orbitally asymptotically stable for (1).

On the other hand, if $\chi(x_0) \neq 0$ and $\lambda(x_0) \neq 0$, then by Theorem 2.1, for every small $\epsilon > 0$, there is no periodic orbit of (48) near $\gamma(x_0) \subset \Lambda$. Since $\Lambda$ is a global attractor, it follows that there is no periodic orbit of (1) near $\gamma(x_0)$.

The function $\chi(x_0)$ defined by (52) can be expressed as a line integral along the trajectory $\gamma(x_0)$ as follows.

**Proposition 3.2.** The function $\chi$ defined by (52) satisfies

$$\chi(x_0) = \int_{\gamma(x_0)} \frac{p(x)}{\rho mx + p(x)} \frac{F(x) - F(0)}{F(x) - y} dx. \quad (55)$$

**Proof.** Fix any $x_0 \in (0, S^0/\rho)$. Let $(x(t), y(t))$ be the solution of (51) with trajectory $\gamma(x_0)$. Recall that $(0, y_\alpha(x_0))$ and $(0, y_\omega(x_0))$ are the alpha- and omega-limit point, respectively, of the trajectory $\gamma(x_0)$. From the equation for $\dot{y}$ in (51), equation (52) can be written as

$$\chi(x_0) = \int_{-\infty}^{\infty} (y - \bar{y}) p(x) dt, \quad (56)$$

where $\bar{y} = F(0)$. On the other hand, from the equation for $\dot{x}$ in (51),

$$\int_{-\infty}^{\infty} (\rho mx + p(x)) (F(x) - y) dt = 0. \quad (57)$$
From (56) and (57), it follows that

\[ \chi(x_0) = \int_{-\infty}^{\infty} p(x)(F(x) - \bar{y}) \, dt + \int_{-\infty}^{\infty} pmx(F(x) - y) \, dt. \]  

(58)

Note that

\[ \int_{-\infty}^{\infty} pmx(F(x) - y) \, dt = \int_{-\infty}^{\infty} \frac{pmx}{pmx + p(x)} x'(t) \, dt \]

\[ = \eta(x(t))_{t=-\infty}^{\infty}, \]

where \( \eta(u) = \int_{0}^{\infty} \frac{pmx}{pmx + p(x)} \, dx \), which is continuous at \( u = 0 \) because \( p(0) = 0 \) and \( p'(0) > 0 \). Since \( x(\infty) = x(-\infty) = 0 \), it follows that

\[ \int_{-\infty}^{\infty} pmx(F(x) - y) \, dt = 0. \]  

(59)

Equations (58) and (59) give

\[ \chi(x_0) = \int_{-\infty}^{\infty} p(x)(F(x) - \bar{y}) \, dt. \]

Therefore, from the equation for \( \dot{x} \) in (51), equation (55) follows.

Next we assume that the function \( F(x) \) in (1) satisfies the one-hump condition: For some \( \hat{x} \in (0, S^0/\rho) \),

\[ F'(x) > 0 \quad \forall \; x \in (0, \hat{x}) \quad \text{and} \quad F'(x) < 0 \quad \forall \; x \in (\hat{x}, S^0/\rho). \]  

(60)

The following result is similar to [11, Theorem 3.1].

**Theorem 3.3.** Assume (60) holds for \( F(x) \) defined by (50). Then system (1) has a globally orbitally asymptotically stable (with respect to all positive non-stationary solutions) periodic orbit for all small \( \epsilon > 0 \).

We will use the following two lemmas.

**Lemma 3.4.** Assume (60). Then the function \( \chi \) defined by (52) has a unique root \( x_0 \) in \((0, S^0/\rho)\), and it satisfies \( \lambda(x_0) < 0 \).

**Proof.** Note that condition (60) implies that there exists a unique value \( \bar{x} \in (\hat{\chi}, S^0/\rho) \) such that \( F(\bar{x}) = F(0) \).

First we claim that

\[ \chi(x_0) > 0 \; \text{for all} \; x_0 \in (0, \bar{x}]. \]  

(61)

Fix any \( x_0 \in (0, \bar{x}) \). We parameterize \( \chi(x_0) \) by

\[ \gamma(x_0) = \{(x, Y_-(x)) : x \in (0, x_0]\} \cup \{(x, Y_+(x)) : x \in (0, x_0]\} \]

(62)

with

\[ Y_-(x) < F(x) \quad \text{and} \quad Y_+(x) > F(x) \quad \forall \; x \in (0, x_0]. \]  

(63)

Then equation (55) in Proposition 3.2 yields

\[ \chi(x_0) = \int_{0}^{x_0} \frac{p(x)}{pmx + p(x)} \left[ \frac{F(x) - F(0)}{F(x) - Y_-(x)} - \frac{F(x) - F(0)}{F(x) - Y_+(x)} \right] \, dx. \]

Since \( F(x) - F(0) > 0 \) for \( 0 < x < x_0 < \bar{x} \), by (63) it follows that \( \chi(x_0) > 0 \).

Next we claim that

\[ \lambda(x_0) < 0 \; \text{for all} \; x_0 \in [\bar{x}, S^0/\rho]. \]  

(64)
Fix any \( x_0 \in [\bar{x}, S^0/\rho) \). From the definition of \( \lambda(x_0) \) in (53), using (62) we have
\[
\lambda(x_0) = \int_0^{x_0} F'(x) \left( \frac{1}{F(x) - Y_-(x)} + \frac{1}{Y_+(x) - F(x)} \right) dx.
\] (65)

Since \( F'(x) \) for \( x \in (\bar{x}, x_0) \), by (63) we obtain
\[
\lambda(x_0) < \int_0^{\bar{x}} F'(x) \left( \frac{1}{F(x) - Y_-(x)} + \frac{1}{Y_+(x) - F(x)} \right) dx.
\]

Since \( Y_+(x) \) is decreasing and \( Y_-(x) \) is increasing, condition (60) yields
\[
\lambda(x_0) < \int_0^{\bar{x}} F'(x) \left( \frac{1}{F(x) - Y_-(\bar{x})} + \frac{1}{Y_+(\bar{x}) - F(x)} \right) dx
\]
\[
= \ln \left( \frac{F(x) - Y_-(\bar{x})}{Y_+(\bar{x}) - F(x)} \right) \bigg|_{x=0}^{x=\bar{x}} = 0.
\]
The last equality follows from the condition \( F(\bar{x}) = F(0) \). Hence \( \lambda(x_0) < 0 \).

Finally, we claim that
\[
\lim_{x_0 \to S^0/\rho} \chi(x_0) = -\infty.
\] (66)

Note that the expression (52) of \( \chi \) can be written as
\[
\chi(x) = \psi(y_\omega(x)) - \psi(y_\alpha(x)),
\] (67)
where
\[
\psi(y) = y - \bar{y} - \ln(y/\bar{y}).
\]

Note also that the functions \( y_\alpha(x) \) and \( y_\omega(x) \) satisfy
\[
\lim_{x \to K^-} y_\alpha(x) = 0 \quad \text{and} \quad \lim_{x \to K^-} y_\omega(x) \quad \text{exists and is finite.}
\] (68)

Since \( \lim_{y \to 0^+} \psi(y) = \infty \), (66) follows from (67) and (68).

Since \( \chi(x) > 0 \) for \( x \in (0, \bar{x}) \) and \( \lim_{x \to S^0/\rho} \chi(x) = -\infty \), the continuous function \( \chi(x) \) has at least one root in \((\bar{x}, S^0/\rho)\). Suppose for contradiction that \( \chi \) has two distinct roots, say \( x_0 < x_1 \). By (61) and (64), we have \( \lambda(x_0) < 0 \) and \( \lambda(x_1) < 0 \).

By Theorem 2.1 there are locally orbitally asymptotically stable periodic orbits \( \ell_\epsilon^{(0)} \) and \( \ell_\epsilon^{(1)} \) near \( \Gamma(x_0) \) and \( \Gamma(x_1) \), respectively. Note that \( \Gamma(x_0) \) is enclosed by \( \Gamma(x_1) \). By (64) and Theorem 2.1 there is no unstable periodic orbit between \( \ell_\epsilon^{(0)} \) and \( \ell_\epsilon^{(1)} \). Also note that no equilibrium lies between \( \ell_\epsilon^{(0)} \) and \( \ell_\epsilon^{(1)} \). This contradicts the Poincaré-Bendixson Theorem. Therefore \( \chi \) has exactly one root \( x_0 \) in \((0, S^0/\rho)\). By (61) and (64), this root satisfies \( \bar{x} < x_0 < S^0/\rho \), and therefore \( \lambda(x_0) < 0 \).

The next lemma was derived by Wolkowicz [25], and we omit its proof here.

**Lemma 3.5.** If \( F'(x) > 0 \) on \((0, \bar{x}] \) or \( F'(x) < 0 \) on \((0, \bar{x}] \) for some \( \bar{x} > 0 \), then no periodic orbit of (48) lies entirely in the strip \( \{(x, y) : 0 < x < \bar{x} \} \) for any sufficiently small \( \epsilon > 0 \).

**Proof of Theorem 3.3.** By Lemma 3.4, the function \( \chi \) has a unique root \( x_0 \) in the interval \((0, S^0/\rho)\), and \( \lambda(x_0) < 0 \). From Theorem 2.1 and Lemma 3.5, it follows that system (48) has a unique periodic orbit \( \ell_\epsilon \) in \( \Lambda \) for all small \( \epsilon > 0 \). It can be shown by the Butler-McGehee Lemma [2] that the flow (48) is persistent in the sense that the omega-limit set of any point in \( \Lambda \) does not intersect the boundary of \( \Lambda \). Therefore, by the Poincaré-Bendixson Theorem, the periodic orbit \( \ell_\epsilon \) attracts all non-stationary points in \( \Lambda \).
Since $\ell_\epsilon$ is locally orbitally asymptotically stable in $\Lambda$ for (48), by (3) it follows that $\ell_\epsilon$ is locally orbitally asymptotically stable in $\mathbb{R}^3$ for (1). Given any solution $(S(t), x(t), y(t))$ of (1) with a positive initial value, it can be shown by the Butler-McGehee Lemma that the flow (1) is persistent in the sense that the omega-limit set of any point in $\mathbb{R}_+^3$ does not intersect the boundary of $\Lambda$. Let $\Omega$ be the omega-limit set of $(S(t), x(t), y(t))$. Then $\Omega \subset \Lambda$ since $\Lambda$ is a global attractor of (1) are persistent. By the positive invariance of omega-limit sets, $\Omega \nmid \ell_\epsilon = \emptyset$. Hence $\Omega = \ell_\epsilon$. This implies that the trajectory converges to $\ell_\epsilon$. 

**Example 3.1.** Consider the Holling type II functional response,

$$p(x) = \frac{bx}{a + x}.$$  \hspace{1cm} (69)

It was shown by Bolger et al. [1] that the function $F_\epsilon(x)$ defined by (49) is concave-down. Hence condition (60) is satisfied, and the results in Theorem 3.3 hold.

Numerical simulations are shown in Figures 4 and 5. In the simulations, the parameters are $(S^0, m, \gamma, c) = (10, 1, 1, 1)$ for (1), and $(a, b) = (1.5, 3)$ and $p(x)$ in (69). Figure 4 shows that the function $\chi$ has a root $x_0 \approx 6.92$, and it satisfies $\lambda(x_0) < 0$. Hence $\gamma(x_0)$ corresponds to a stable relaxation oscillation formed by the globally asymptotically stable periodic orbit of (1) as $\epsilon \to 0$. Figure 5(A) shows the periodic orbit $\ell_\epsilon$ for (1) with $\epsilon = 0.5$, and Figure 5(B) illustrates the trajectory $\gamma(x_0)$ for (51). The simulation confirms that the location of $\ell_\epsilon$ is close to $\gamma(x_0)$.

**Remark 7.** The analysis of (48) in this section can more generally be applied to systems of the form

$$\begin{align*}
\dot{x} &= q(x)(F_\epsilon(x) - y), \\
\dot{y} &= y(p(x) - \epsilon),
\end{align*}$$  \hspace{1cm} (70)

that satisfy condition (2), $q(x) > 0$ for $x > 0$, $\frac{p(x)}{q(x)}$ is continuous at $x = 0$, and

$$F_0(x) \begin{cases} > 0, & \text{if } 0 \leq x < K, \\
< 0, & \text{if } K < x,
\end{cases}$$

for some positive constant $K$. Similar to Theorem 3.3, a unique locally orbitally asymptotic stable relaxation oscillation for system (70) in the region filled by a family of heteroclinic orbits can be obtained for all small $\epsilon > 0$ under assumption (60) with $F(x) = F_0(x)$ and $S^0/\rho$ replaced by $K$.

4. **The epidemic model.** When $\epsilon = 0$, system (8) reduces to system (9). The line $Z_0 = \{(S, I, N) : I = 0, S = \frac{D}{D+\rho} N\}$ is a set of equilibria of (8) in the invariant plane $\{I = 0\}$. Let $N_0$ be the unique value that satisfies $h(\frac{D}{D+\rho} N, N) = a$. It is known [8, 16] that each point on the segment $Z_0 \cap \{0 < N < N_0\}$ is connected to a unique point on the segment $Z_0 \cap \{N_0 < N < N_{\text{max}}\}$ by a heteroclinic orbit of (9) (see Figure 2). We define

$$\omega : (N_0, N_{\text{max}}) \to (0, N_0)$$

such that the point $\left(\frac{D}{D+\rho}\omega(N_1), \omega(N_1), 0\right)$ is the omega-limit point of the heteroclinic orbit, denoted by $\gamma(N_1)$, of (9) starting from $\left(\frac{D}{D+\rho} N_1, N_1, 0\right)$.

It is also known [16] that the invariant manifold $Z_0$ and the center manifold $W^c_0(Z_0)$ of system (9) still exist for the perturbed system (8), and that the perturbed center manifold is a global attractor for (8) for each small $\epsilon > 0$. Denote the
Figure 4. Numerical simulations of $\chi$ and $\lambda$ for Example 3.1. The function $\chi$ has a single root $x_1 \approx 6.92$, with $\lambda(x_1) < 0$.

Figure 5. (A) The trajectory of system (1) for Example 3.1 with $\epsilon = 0.5$ and initial point $(S, x, y)(0) = (6, 1, 10)$ converges to a periodic orbit $\ell_\epsilon$. (B) The trajectory $\gamma(x_0)$ of (51), where $x_0$ is a root of $\chi$. The simulation shows that $\ell_\epsilon$ is close to $\gamma(x_0)$.

perturbed manifolds by $Z_\epsilon$ and $W^c_\epsilon(Z_\epsilon)$. We parameterize the center manifold $W^c_\epsilon(Z_\epsilon)$ by $S = \tilde{S}(I, N)$, and define $\tilde{S} = \tilde{S}_0$. Then the restriction of system (8) on $W^c_\epsilon(Z_\epsilon)$ can be written as

$$I' = (g(\tilde{S}(I, N), N) - a)I,$$

$$N' = \epsilon f(N) - \alpha I. \tag{71}$$

With $(N, I)$ in (71) playing the role of $(a, b)$ in (10), and $N_0$ playing the role of $\bar{a}$, the function $\chi : [N_0, N_{\text{max}}) \to \mathbb{R}$ defined by (15) is equal to

$$\chi(N_1) = \int_{\omega(N_1)}^{N_1} \frac{g(N, N) - a}{f(N)} dN. \tag{72}$$

By (24) in Proposition 2.3 and (18) in Remark 1, the function $\lambda : [N_0, N_{\text{max}}) \to \mathbb{R}$ defined by (16) is equal to
\[ \lambda(N_1) = \ln \frac{f(N_1)}{f(\omega(N_1))} + \int_{\gamma(N_1)} \frac{\partial I(\tilde{S}(I, N), N)}{-\alpha} \, dN. \]

That is,

\[ \lambda(N_1) = \ln \frac{f(N_1)}{f(\omega(N_1))} - \frac{1}{\alpha} \int_{\gamma(N_1)} \partial S g(\tilde{S}(I, N), N) \partial I(\tilde{S}(I, N)) \, dN. \] (73)

**Remark 8.** The function \( \chi \) in (72) is equivalent to the function \( \hat{F} \) in [16] in the sense that \( \chi(N) > 0 \) if and only if \( \hat{F}(N) > N \).

**Theorem 4.1.** Assume that \( N_1 \in (N_0, N_{\text{max}}) \) satisfies \( \chi(N_1) = 0 \) and \( \lambda(N_1) \neq 0 \), where \( \chi(N) \) and \( \lambda(x) \) are defined in (72) and (73), respectively. Then for any sufficiently small \( \epsilon > 0 \), there is a unique periodic orbit \( \ell_\epsilon \) of (8) in an \( O(\epsilon) \)-neighborhood of \( \gamma(N_1) \). The minimal period of \( \ell_\epsilon \), denoted by \( T_\epsilon \), satisfies

\[ T_\epsilon = \frac{1}{\epsilon} \left( \int_{\omega(N_1)} \frac{1}{f(N)} \, dN + o(1) \right) \quad \text{as } \epsilon \to 0. \] (74)

Moreover, \( \ell_\epsilon \) is locally orbitally asymptotically stable if \( \lambda(N_1) < 0 \), and is orbitally unstable if \( \lambda(N_1) > 0 \). Conversely, if \( \chi(N_1) \neq 0 \), then for any point \( z_1 \) in the interior of the trajectory \( \gamma(N_1) \), there is a neighborhood \( U \) of \( z_1 \) such that no periodic orbit of (10) intersects \( U \) for any sufficiently small \( \epsilon > 0 \).

**Proof.** If \( \chi(N_1) \neq 0 \), then by Theorem 2.1, system (9) has no periodic orbit near \( \gamma(N_1) \in W^r_c(Z_\epsilon) \) for any small \( \epsilon > 0 \). Since \( W^r_c(Z_\epsilon) \) is a global attractor, it follows that system (8) has no periodic orbit near \( \gamma(N_1) \) in \( \mathbb{R}^2_+ \).

Next assume that \( \chi(N_1) \neq 0 \). Then by Theorem 2.1, system (9) has a unique periodic orbit \( \ell_\epsilon \) in an \( O(\epsilon) \)-neighborhood of \( \gamma(N_1) \) in \( W^r_c(Z_\epsilon) \) for every small \( \epsilon > 0 \). If \( \lambda(N_1) > 0 \), then \( \ell_\epsilon \) is unstable for (9), and therefore is unstable for (8). If \( \lambda(N_1) < 0 \), then \( \ell_\epsilon \) is locally orbitally asymptotically stable for (9). Since \( W^r_c(Z_\epsilon) \) is a hyperbolic attractor, \( \ell_\epsilon \) is locally orbitally asymptotically stable for (8). \[ \square \]

**Remark 9.** The period \( T_\epsilon \) of the limit cycle \( \ell_\epsilon \) is referred to as the *interepidemic period* (IEP) in [16], where it was shown numerically that \( T_\epsilon \) is proportional to \( 1/\epsilon \). Their observation is consistent with the asymptotic formula (74).

We are not able to determine the signs of \( \chi \) and \( \lambda \) in (72) and (73) analytically. Nonetheless, we are able to compute \( \chi \) and \( \lambda \) numerically. A numerical difficulty in the computation is to approximate \( \partial I \tilde{S} \), since there is no explicit formula for \( \tilde{S} \). We implement the following algorithm to compute \( \partial I \tilde{S} \): Fix a small number \( \delta > 0 \) and large integers \( T \) and \( M \). For \( N_1 \in (N_0, N_{\text{max}}) \), denote by \( (S, I, N)(t; N_1) \) the solution of (9) with initial value \( (S, I, N)(0) = (\frac{D}{\partial T^p} N_1, 0, N_1) + \delta \tilde{v} \), where \( \tilde{v} \) is an eigenvector corresponding to the unstable eigenvalue of the linearization of (9) at \( (\frac{D}{\partial T^p} N_1, 0, N_1) \), so that the forward trajectory of this solution is near the heteroclinic orbit with the alpha-limit point \( (\frac{D}{\partial T^p} N_1, 0, N_1) \). Let \( N_k, k = 1, \ldots, M \), be a grid of the interval \( [N_0 + \delta, N_{\text{max}}] \), and let \( t_j, j = 0, 1, \ldots, TM \), be a grid of the interval \( [0, T] \). Define

\[ u_k^j = u(t_j; N_k) \quad \text{for } 1 \leq k \leq M, \ 1 \leq j \leq TM, \ u = S, I, N, \]
Define $\Delta_x u_k^j = u_{k+1}^j - u_k^j$ and $\Delta_t u_k^j = u_{k+1}^j - u_k^j$. Then the numerical approximations of $\partial_t \tilde{S}$ and $\partial_N \tilde{S}$ are

$$ \left[ (\partial_t \tilde{S})_k \quad (\partial_N \tilde{S})_k^T \right] = \left[ \Delta_t S_k^j \quad \Delta_x S_k^j \right] \left[ \Delta_t I_k^j \quad \Delta_x I_k^j \right]^{-1}. \quad (75) $$

We use this approximation for $\partial_t \tilde{S}$ to evaluate formula (73) for $\lambda$.

**Example 4.1.** Following Li et al. [16, Section 5.1, Case 1], we consider $g(S, N) = \frac{dS}{dt}$ and parameters $D = 0.2$, $p = 0.01$, $\alpha = 0.048$, $\beta = 1$, $\gamma = 0.75$, $m = 0.1$ and $N_{\max} = 400$. It was proved in [16] that, for small $\epsilon > 0$, system (8) has a stable periodic orbit and the positive equilibrium is unstable. Our numerical simulation, illustrated in Figure 6(A), shows that $\chi$ defined by (72) has a root $N_1 \approx 377.01$ with $\lambda(N_1) \approx -4.11 < 0$. Hence $\gamma(N_1)$ corresponds to a stable relaxation oscillation. A trajectory with initial data $(S, I, N) = (60, 2, 120)$ and $\epsilon = 10^{-5}$ is shown in Figure 6(B). The trajectory first is attracted by the slow manifold $W^c_\epsilon(Z_e)$, and then follows the dynamics on $W^c_\epsilon(Z_e)$ to approach the periodic orbit near $\gamma(N_1)$.

In the next example we demonstrate that by varying the perturbation term $\epsilon f(N)$, system (8) can have two relaxation oscillations. The idea is that the positive function $f(N)$ effects the magnitude of the integrand in formula (72) of $\chi(N)$, and hence the function $\chi(N)$ can gain extra roots by deforming $f(N)$. This method conceptually can be used to construct an arbitrary number of relaxation oscillations.

**Example 4.2.** We replace $f(N)$ defined by (7) with

$$ f_1(N) = f(N) - c_1 \exp(-c_2(N - c_3)) $$

with $(c_1, c_2, c_3) = (60, 0.04, 90)$. The function $f_1(N)$ differs from $f(N)$ essentially only in a small interval to the right of $N_0$ (see Figure 7). From our numerical simulation, as shown in Figure 8(A), the function $\chi$ defined by (72) with $f$ replaced by $f_1$ has two roots, $N_1 \approx 156.89$ and $N_2 \approx 342.18$, with $\lambda(N_1) \approx 1.06 > 0$ and $\lambda(N_2) \approx -2.48 < 0$. Hence $\gamma(N_1)$ corresponds to an unstable relaxation oscillation for system (8) with $f$ replaced by $f_1$, and $\gamma(N_2)$ corresponds to a stable relaxation oscillation. Some trajectories for the system with $\epsilon = 10^{-5}$ are shown in Figure 8(B).

**Remark 10.** Example 4.2 is consistent with Li et al. [16, Section 5.1, Case 2], in which two periodic orbits were observed. It was proved in [16] that, for some parameters, system (8) exhibits a stable periodic orbit while the positive equilibrium is asymptotically stable, which implies that there must be at least one unstable periodic orbit. It was commented in [16, Section 4.3] that, in general, the unstable periodic orbit is not necessarily a relaxation oscillation but a small periodic orbit obtained through a subcritical Hopf bifurcation. Using Theorem 4.1, we are able to confirm that there is an isolated unstable periodic orbit for small $\epsilon > 0$ that forms a relaxation oscillation.

5. Discussion. In this paper we derived a criterion to determine the location and stability of relaxation oscillations for the planar system (10) under assumption (H). In Theorem 2.1, characteristic functions $\chi(s)$ and $\lambda(s)$ were obtained from the trajectory $\gamma(s)$ of the limiting system with $\epsilon = 0$ and satisfy

(i) $\chi(s_0) \neq 0 \Rightarrow$ no periodic orbit passes near $\gamma(s_0)$ for all $0 < \epsilon < 1$.

(ii) $\chi(s_0) = 0$ and $\lambda(s_0) \neq 0 \Rightarrow \gamma(s_0)$ admits a relaxation oscillation as $\epsilon \to 0$. 

Figure 6. (A) For Example 4.1, the function $\chi$ has a root $N_1 \approx 377.01$ with $\lambda(N_1) \approx -4.11 < 0$. (B) The dashed curve is a trajectory for the system with $\epsilon = 10^{-5}$ with the initial condition $(S, I, N)(0) = (60, 2, 120)$, and the solid curve is the singular orbit $\gamma(N_1)$. The simulation shows that the trajectory approaches a periodic orbit near $\gamma(N_1)$.

Figure 7. The perturbation term $\epsilon f(N)$ with $f(N) = N(1 - N_{\text{max}})$ in Example 4.1 is replaced by $\epsilon f_1(N)$ with $f_1(N) = f(N) - c_1 \exp(-c_2(N - c_3))$ in Example 4.2. Essentially $f_1$ is obtained by dropping the value of $f$ in a small interval right to $N_0$.

In the latter case, the sign of $\lambda(s_0)$ determines the stability of the limit cycles. The proof of the theorem involves geometric singular perturbation theory and a variation of Floquet Theory. In Propositions 2.2, 2.3 and 2.4, simplified expressions of $\chi$ and $\lambda$ were provided for the case that some terms in the system satisfy certain forms.

We applied this criterion to two systems using two different techniques. By relating $\chi(s)$ with a line integral on the heteroclinic orbit, in Theorem 3.3 we derived a condition under which the chemostat predator-prey system (1) has a unique limit cycle. By varying the perturbation term, we demonstrate in Examples 4.1 and 4.2 that the epidemic model (6) can have a prescribed number of relaxation oscillations.
Figure 8. (A) For Example 4.2, the function $\chi$ has two roots, $N_1 \approx 156.89$ and $N_2 \approx 342.18$, with $\lambda(N_1) \approx 1.06 > 0$ and $\lambda(N_2) \approx -2.48 < 0$. (B) The dashed curves are trajectories for the system with $\epsilon = 10^{-5}$, and the solid curves are the singular orbits $\gamma(N_1)$ and $\gamma(N_2)$. The simulation shows that the trajectory with the initial condition $(S, I, N)(0) = (40, 2.5, 80)$ approaches a periodic orbit near $\gamma(N_2)$ while trajectory with the initial condition $(S, I, N)(0) = (40, 1.3, 80)$ approaches the interior equilibrium. Near $\gamma(N_1)$ is an unstable periodic orbit.

In general, it is a numerical challenge to compute unstable periodic orbits in three-dimensional systems. Our characteristic functions provide a way to locate both stable and unstable periodic orbits.

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