On a certain subclass of analytic functions defined by a differential operator

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Abstract
In this paper, we introduce and study a new subclass of analytic functions which are defined by means of a new differential operator. Some results connected to coefficient estimates, growth and distortion theorems, radii of starlikeness, convexity close-to-convexity and integral means inequalities related to the subclass is obtained.

Keywords
Analytic functions, differential operator, coefficient estimates.

AMS Subject Classification
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1. Introduction

Let \( A \) denote the class of functions \( u \) of the form

\[
u(z) = z + \sum_{\eta=2}^{\infty} a_{\eta}z^{\eta}, \quad (a_{\eta} \geq 0) \tag{1.3}\]

which are analytic in the open unit disc \( E = \{z \in \mathbb{C} : |z| < 1\} \).

A function \( u \) in the class \( A \) is said to be in the class \( ST(\alpha) \) of starlike functions of order \( \alpha \) in \( E \), if it satisfy the inequality

\[
\Re \left\{ \frac{zu'(z)}{u(z)} \right\} > \alpha, \quad (0 \leq \alpha < 1), z \in E \tag{1.2}\]

Note that \( ST(0) = ST \) is the class of Starlike functions. Denote by \( T \) the subclass of \( A \) consisting of functions \( u \) of the form

\[
u(z) = z + \sum_{\eta=2}^{\infty} a_{\eta}z^{\eta}, \quad (a_{\eta} \geq 0) \tag{1.3}\]

This subclass was introduced and extensively studied by Silvermann[4].

Let \( u \) be a function in the class \( A \). We define the following differential operator introduced by Deniz and Ozkan [1].

\[
D^0_\lambda u(z) = u(z)
\]
\[
D^1_\lambda u(z) = D_\lambda u(z) = \lambda z^3 (u(z))^m + (2\lambda + 1)z^2 (u(z))^{\eta} + zu'(z)
\]
\[
D^2_\lambda u(z) = D_\lambda (D^1_\lambda u(z))
\]
\[
\vdots
\]
\[
D^m_\lambda u(z) = D_\lambda (D^{m-1}_\lambda u(z))
\]

where \( \lambda \geq 0 \) and \( m \in N_0 = N \cup \{0\} \). If \( u \) is given by (1.1), then from the definition of the operator \( D^m_\lambda u(z) \), it is to see that

\[
D^m_\lambda u(z) = z + \sum_{\eta=2}^{\infty} \phi^m(\lambda, \eta)a_{\eta}z^{\eta} \tag{1.4}\]

where

\[
\phi^m(\lambda, \eta) = \eta^{2m}[(\lambda(\eta - 1) + 1)^m \tag{1.5}\]

\[
\Re \left\{ \frac{zu'(z)}{u(z)} \right\} > \alpha, \quad (0 \leq \alpha < 1), z \in E \tag{1.2}\]

\[
D^m_\lambda u(z) = z + \sum_{\eta=2}^{\infty} \phi^m(\lambda, \eta)a_{\eta}z^{\eta} \tag{1.4}\]

where

\[
\phi^m(\lambda, \eta) = \eta^{2m}[(\lambda(\eta - 1) + 1)^m \tag{1.5}\]
Further we define \( T \{ S \} \phi \) class \( S \)

Now we define the following new subclass motivated by Murugusunderamoothy and Magesh [3]

**Definition 1.1.** The function \( u(z) \) of the form (1.1) is in the class \( S^m_\alpha(\mu, \gamma) \), if it satisfies the inequality

\[
\Re \left\{ \frac{z(D^n_\alpha u(z))'}{(1 - \mu)z + \mu D^n_\alpha u(z)} - \alpha \right\} > \left| \frac{z(D^n_\alpha u(z))'}{(1 - \mu)z + \mu D^n_\alpha u(z)} - 1 \right|
\]

for \( 0 \leq \lambda \leq 1, 0 \leq \gamma \leq 1 \)

Further we define \( T S^m_\alpha(\mu, \gamma) = S^m_\alpha(\mu, \gamma) \cap T \)

The aim of present paper is to study the coefficient bounds, radii of close-to-convex and starlikness convex linear combinations and integral means inequalities of the \( T S^m_\alpha(\mu, \gamma) \)

## 2. Coefficient bounds

**Theorem 2.1.** A function \( u(z) \) of the form (1.1) is in \( S^m_\alpha(\mu, \gamma) \), then

\[
\sum_{\eta = 2}^{\infty} [2\eta - \mu(\gamma + 1)]\phi^m(\lambda, \eta)|a_\eta| \leq 1 - \gamma \quad (2.1)
\]

where \( 0 \leq \mu \leq 1, 0 \leq \gamma \leq 1 \) and \( \phi^m(\lambda, \eta) \) is given by (1.5)

**Proof.** It suffices to show that

\[
\left| \frac{z(D^n_\alpha u(z))'}{(1 - \mu)z + \mu D^n_\alpha u(z)} - 1 \right| - \Re \left\{ \frac{z(D^n_\alpha u(z))'}{(1 - \mu)z + \mu D^n_\alpha u(z)} - 1 \right\} \leq 1 - \gamma
\]

We have

\[
\leq 2 \left| \frac{z(D^n_\alpha u(z))'}{(1 - \mu)z + \mu D^n_\alpha u(z)} - 1 \right| - \Re \left\{ \frac{z(D^n_\alpha u(z))'}{(1 - \mu)z + \mu D^n_\alpha u(z)} - 1 \right\}
\]

\[
\leq 2 \sum_{\eta = 2}^{\infty} (\eta - \mu)\phi^m(\lambda, \eta)|a_\eta| |z|^{\eta - 1}
\]

and the proof is complete.

**Theorem 2.2.** Let \( 0 \leq \mu \leq 1, 0 \leq \gamma \leq 1 \), then a function \( u \) of the form (1.3) to be in the class \( T S^m_\alpha(\mu, \gamma) \) if and only if

\[
\sum_{\eta = 2}^{\infty} \frac{[2\eta - \mu(\gamma + 1)]\phi^m(\lambda, \eta)|a_\eta|}{2(1 - \gamma)} \leq 1 - \eta \quad (2.2)
\]

where \( \phi^m(\lambda, \eta) \) is given by (1.5)

**Proof.** In view of Theorem (2.1) we need only to prove the necessity. If \( u \in T S^m_\alpha(\mu, \gamma) \) and \( z \) is real, then

\[
\Re \left\{ \frac{1 - \sum_{\eta = 2}^{\infty} \eta\phi^m(\lambda, \eta)|a_\eta| z^{\eta - 1}}{1 - \sum_{\eta = 2}^{\infty} \mu\phi^m(\lambda, \eta)|a_\eta| z^{\eta - 1}} \right\} \geq \frac{\sum_{\eta = 2}^{\infty} (\eta - \mu)\phi^m(\lambda, \eta)|a_\eta| z^{\eta - 1}}{1 - \sum_{\eta = 2}^{\infty} \mu\phi^m(\lambda, \eta)|a_\eta| z^{\eta - 1}}
\]

Letting \( z \to 1 \) along the real axis, we obtain the desired inequality

\[
\sum_{\eta = 2}^{\infty} [2\eta - \mu(\gamma + 1)]\phi^m(\lambda, \eta)|a_\eta| \leq 1 - \gamma
\]

where \( 0 \leq \mu \leq 1, 0 \leq \gamma \leq 1 \) and \( \phi^m(\lambda, \eta) \) is given by (1.5)

**Corollary 2.3.** If \( u(z) \in T S^m_\alpha(\mu, \gamma) \), then

\[
|a_\eta| \leq \frac{1 - \gamma}{[2\eta - \mu(\gamma + 1)]\phi^m(\lambda, \eta)} \quad (2.3)
\]

where \( 0 \leq \mu \leq 1, 0 \leq \gamma \leq 1 \) and \( \phi^m(\lambda, \eta) \) is given by (1.5).

Equality holds for the function

\[
u(z) = z - \frac{1 - \gamma}{[2\eta - \mu(\gamma + 1)]\phi^m(\lambda, \eta)} z^\eta
\]

**Theorem 2.4.** Let \( u_1(z) = z \) and

\[
u_w(z) = z - \frac{1 - \gamma}{[2\eta - \mu(\gamma + 1)]\phi^m(\lambda, \eta)} z^\eta, \quad \eta \geq 2 \quad (2.5)
\]

Then \( u(z) \in T S^m_\alpha(\mu, \gamma) \), if and only if, it can be expressed in the form

\[
u(z) = \sum_{\eta = 1}^{\infty} w_\eta u_\eta(z), \quad w_\eta \geq 0, \quad \sum_{\eta = 1}^{\infty} w_\eta = 1 \quad (2.6)
\]

**Proof.** Suppose \( u(z) \) can be written as in (2.6), then

\[
u(z) = z - \sum_{\eta = 2}^{\infty} w_\eta \frac{1 - \gamma}{[2\eta - \mu(\gamma + 1)]\phi^m(\lambda, \eta)} z^\eta
\]

Now,

\[
\sum_{\eta = 2}^{\infty} w_\eta \frac{(1 - \gamma)[2\eta - \mu(\gamma + 1)]\phi^m(\lambda, \eta)}{[2\eta - \mu(\gamma + 1)]\phi^m(\lambda, \eta)} = \sum_{\eta = 2}^{\infty} \frac{1}{1 - w_1} \leq 1
\]
Thus \( u(z) \in TS^m_\lambda(\mu, \gamma) \).

Conversely, let \( u(z) \in TS^m_\lambda(\mu, \gamma) \), then by using (2.3), we get

\[
 w_\eta = \frac{[2\eta - \mu(\gamma + 1)]\phi^m(\lambda, \eta)}{(1 - \gamma)} a_\eta, \quad \eta \geq 2
\]

and \( w_1 = 1 - \sum_{\eta=2}^\infty w_\eta \). Then we have \( u(z) = \sum_{\eta=1}^\infty w_\eta u_\eta(z) \) and hence this completes the proof of Theorem.

**Theorem 2.5.** The class \( TS^m_\lambda(\mu, \gamma) \) is a convex set.

**Proof.** Let the function

\[
 u_j(z) = z - \sum_{\eta=2}^\infty a_{\eta,j}z^\eta, \quad a_{\eta,j} \geq 0, \quad j = 1, 2
\]

be in the class \( TS^m_\lambda(\mu, \gamma) \). It is sufficient to show that the function \( h(z) \) defined by

\[
 h(z) = \xi u_1(z) + (1 - \xi)u_2(z), \quad 0 \leq \xi < 1,
\]

in the class \( TS^m_\lambda(\mu, \gamma) \). Since

\[
 h(z) = z - \sum_{\eta=2}^\infty [\xi a_{\eta,1} + (1 - \xi) a_{\eta,2}]z^\eta,
\]

An easy computation with the aid of Theorem (2.2) gives

\[
 \sum_{\eta=2}^\infty [2\eta - \mu(\gamma + 1)]\xi \phi^m(\lambda, \eta)a_{\eta,1} + \\
 \sum_{\eta=2}^\infty [2\eta - \mu(\gamma + 1)](1 - \xi)\phi^m(\lambda, \eta)a_{\eta,2} \\
 \leq \xi(1 - \gamma) + (1 - \xi)(1 - \gamma) \\
 \leq (1 - \gamma)
\]

which implies that \( h \in TS^m_\lambda(\mu, \gamma) \).

Hence \( TS^m_\lambda(\mu, \gamma) \) is convex.

**3. Radii of Close-to-Convexity, Starlikeness and Convexity**

In this section, we obtain the radii of close-to-convexity, starlikeness and convexity for the class \( TS^m_\lambda(\mu, \gamma) \).

**Theorem 3.1.** Let the function \( u(z) \) defined by (1.3) belong to the class \( TS^m_\lambda(\mu, \gamma) \). Then \( u(z) \) is close-to-convex of order \( \delta(0 \leq \delta < 1) \) in the disc \( |z| < r_1 \), where

\[
 r_1 = \inf_{\eta \geq 2} \left[ \frac{(1 - \delta) \sum_{\eta=2}^\infty [2\eta - \mu(\gamma + 1)]\phi^m(\lambda, \eta)}{\eta(1 - \gamma)} \right]^{1/\eta-1}, \eta \geq 2
\]

The result is sharp, with the external function \( u(z) \) is given by (2.5)

**Proof.** Given \( u \in T \) and \( u \) is close-to-convex of order \( \delta \), we have

\[
 |f'(z) - 1| < 1 - \delta
\]

(3.2)

For the left hand side of (3.2), we have

\[
 |u'(z) - 1| \leq \sum_{\eta=2}^\infty \eta a_\eta |z|^{\eta-1}
\]

The last expression is less than \( 1 - \delta \)

\[
 \sum_{\eta=2}^\infty \frac{\eta - \delta}{1 - \delta} a_\eta |z|^{\eta-1} < 1
\]

\( \square \)

**Theorem 3.2.** Let the function \( u(z) \) defined by (1.3) belong to the class \( TS^m_\lambda(\mu, \gamma) \). Then \( u(z) \) is starlike of order \( \delta(0 \leq \delta < 1) \) in the disc \( |z| < r_2 \), where

\[
 r_2 = \inf_{\eta \geq 2} \left[ \frac{(1 - \delta) \sum_{\eta=2}^\infty [2\eta - \mu(\gamma + 1)]\phi^m(\lambda, \eta)}{(\eta - \delta)(1 - \gamma)} \right]^{1/\eta-1}
\]

(3.3)

The result is sharp, with external function \( u(z) \) is given by (2.5)

**Proof.** Given \( u \in T \) and \( u \) is starlike of order \( \delta \), we have

\[
 \left| \frac{zu'(z)}{u(z)} - 1 \right| < 1 - \delta
\]

(3.4)

For the left hand side of (3.4), we have

\[
 \left| \frac{zu'(z)}{u(z)} - 1 \right| \leq \sum_{\eta=2}^\infty \frac{(\eta - 1)a_\eta |z|^{\eta-1}}{\sum_{\eta=2}^\infty a_\eta |z|^{\eta-1}}
\]

The last expression is less than \( 1 - \delta \) if

\[
 \sum_{\eta=2}^\infty \frac{\eta - \delta}{1 - \delta} a_\eta |z|^{\eta-1} < 1
\]

\( \square \)
Using the fact that \( u(z) \in TS^m_\lambda(\mu, \gamma) \) if and only if
\[
\sum_{\eta=2}^{\infty} \frac{[2\eta - \mu(\gamma + 1)]\phi^m(\lambda, \eta)}{1 - \gamma} a_\eta \leq 1
\]
We can say (3.4) is true, if
\[
\sum_{\eta=2}^{\infty} \frac{\eta - \delta}{\eta - \delta} |z|^{\eta-1} \leq \frac{[2\eta - \mu(\gamma + 1)]\phi^m(\lambda, \eta)}{(\eta - \delta)(1 - \gamma)}
\]
or equivalently
\[
|z|^{\eta-1} \leq \frac{(1 - \delta)[2\eta - \mu(\gamma + 1)]\phi^m(\lambda, \eta)}{(\eta - \delta)(1 - \gamma)}
\]
which yields the starlikeness of the family.

\section{4. Integral Means Inequalities}

In [4], Silverman found that the function \( u_2(z) = z - \frac{z^2}{2} \) is often extremal over the family T. He applied this function to resolve his integral means inequality conjunctured [5] and settled in [6], that

\[
\int_0^{2\pi} |u(re^{i\theta})|^\tau d\phi \leq \int_0^{2\pi} |u_2(re^{i\theta})|^\tau d\phi
\]
for all \( u \in T \), \( \tau > 0 \) and \( 0 < r < 1 \). In [6], he also proved his conjuncture for the subclasses \( T^*(\alpha) \) and \( C(\alpha) \) of T.

Now, we prove Silverman’s conjecture for the class of functions \( TS^m_\lambda(\mu, \gamma) \). We need the concept of subordination between analytic functions and a subordination theorem of Littlewood [2].

Two functions \( u \) and \( v \), which are analytic in E, the function \( u \) is said to be subordinate to \( v \) in E, if there exists a function \( w \) analytic in E with \( w(0) = 0 \), \( |w(z)| < 1 \), \( z \in E \) such that \( u(z) = v(w(z)) \), \( z \in E \). We denote this subordination by \( u(z) \prec v(z) \). (\( \prec \) denote subordination)

\textbf{Lemma 4.1.} 
If the function \( u \) and \( v \) are analytic in E with \( u(z) \prec v(z) \), then for \( \tau > 0 \) and \( z = re^{i\theta} \), \( 0 < r < 1 \)
\[
\int_0^{2\pi} |v(re^{i\theta})|^\tau d\phi \leq \int_0^{2\pi} |u(re^{i\theta})|^\tau d\phi
\]

Now, we discuss the integral means inequalities for functions \( u \) in \( TS^m_\lambda(\mu, \gamma) \).

\textbf{Theorem 4.2.} \( u \in TS^m_\lambda(\mu, \gamma) \), \( 0 \leq \mu < 1 \), \( 0 \leq \gamma < 1 \) and \( u_2(z) \) be defined by
\[
u_2(z) = z - \frac{1 - \gamma}{\phi_2(\lambda, \gamma)} z^2 \quad (4.1)
\]

\textbf{Proof.} For \( u(z) = z - \sum_{\eta=2}^{\infty} a_\eta z^{\eta} \), (4.1) is equivalent to
\[
\frac{2\pi}{0} \left| 1 - \sum_{\eta=2}^{\infty} a_\eta z^{\eta-1} \right|^\tau d\varphi \leq \frac{2\pi}{0} \left| 1 - \frac{1 - \gamma}{\phi_2(\lambda, \gamma)} z \right|^\tau d\varphi
\]
By Lemma (4.1), it is enough to prove that
\[
1 - \sum_{\eta=2}^{\infty} a_\eta z^{\eta-1} \prec 1 - \frac{1 - \gamma}{\phi_2(\lambda, \gamma)} w(z),
\]
Assuming
\[
1 - \sum_{\eta=2}^{\infty} a_\eta z^{\eta-1} \prec 1 - \frac{1 - \gamma}{\phi_2(\lambda, \gamma)} w(z),
\]
and using (2.2), we obtain
\[
|w(z)| = \left| \sum_{\eta=2}^{\infty} \frac{\phi_2(\lambda, \gamma)}{1 - \gamma} a_\eta z^{\eta-1} \right| \leq \sum_{\eta=2}^{\infty} \frac{\phi_2(\lambda, \gamma)}{1 - \gamma} a_\eta \leq |z|
\]
where
\[
\phi_2(\lambda, \gamma) = [2\eta - \mu(\gamma + 1)]\phi^m(\lambda, \eta)
\]
This completes the proof.

\section{5. Conclusion}

This research has introduced a new linear differential operator related to Analytic function and studied some basic properties of geometric function theory. Accordingly, some results related to closure theorems have also been considered, inviting future research for this field of study.

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