Crossover of magnetoconductance autocorrelation 
for a ballistic chaotic quantum dot

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(February 21, 2018)

Abstract

The autocorrelation function $C_{\varphi,\varepsilon}(\Delta \varphi, \Delta \varepsilon) = \langle \delta g(\varphi, \varepsilon) \delta g(\varphi + \Delta \varphi, \varepsilon + \Delta \varepsilon) \rangle$ 
($\varphi$ and $\varepsilon$ are rescaled magnetic flux and energy) for the magnetoconductance of a ballistic chaotic quantum dot is calculated in the framework of the supersymmetric non-linear $\sigma$-model. The Hamiltonian of the quantum dot is modelled by a Gaussian random matrix. The particular form of the symmetry breaking matrix is found to be relevant for the autocorrelation function but not for the average conductance. Our results are valid for the complete crossover from orthogonal to unitary symmetry and their relation with semiclassical theory and an S-matrix Brownian motion ensemble is discussed.

PACS. 05.45, 72.10B, 72.15R

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GaAs/Al$_x$Ga$_{1-x}$As heterostructures provide useful experimental realizations\cite{1-3} of two-dimensional ballistic cavities known as quantum dots. Measuring the conductance of a quantum dot connected to electron reservoirs, one can study the quantum behavior of classically chaotic billiards. Apart from the channel number, the magnetic field $B$ and the gate voltage $E$ are the two important adjustable parameters. In experiments, the sample dependent conductance is measured as a function of these parameters. Theoretical approaches\cite{4-9} consider the weak localization peak (by calculating the average conductance as a function of magnetic field) and the conductance autocorrelation with respect to magnetic field and gate voltage. Applying an ergodicity argument, one may compare these quantities with experiment.

Chaotic ballistic systems were treated numerically and semiclassically in\cite{4-6}, the latter predicting an algebraic decay of the correlation functions and the weak localization peak. A random matrix approach for the unitary scattering matrix in terms of the circular Brownian motion ensemble\cite{7,8} yields an exponential decay of these quantities as a function of a fictitious Brownian motion time $t$. It is not obvious how to relate these results with the semiclassical theory. Furthermore, it is not entirely clear if the problematic diagonal approximation in the Gutzwiller trace formula, used in Refs.\cite{5,6}, plays an important role here.

Recently, the supersymmetric non-linear $\sigma$-model has been applied\cite{9} to quantum dots to calculate the weak localization peak, describing the crossover from orthogonal to unitary symmetry for the average conductance. In the unitary symmetry class where the magnetic field is sufficiently strong to completely break the time reversal invariance, the conductance autocorrelation function has been calculated\cite{10} in the limit of many channels.

Corresponding perturbative treatments of the $\sigma$-model which are predominantly concerned with the metallic diffusive case can be found in Ref.\cite{11} for the magnetic field correlations and in Ref.\cite{12} for the energy correlations, respectively.

The purpose of this letter is to generalize the treatment of Ref.\cite{10} to the complete crossover from orthogonal to unitary symmetry. In addition, we point out that the precise form of the symmetry breaking matrix is relevant for the autocorrelation function but not for the weak localization peak. This observation is very important to understand the Brownian motion results\cite{7,8}.

We are interested in the generic chaotic features of the quantum dot and model the Hamiltonian as a Gaussian random matrix of the form

$$H_\alpha = H^{(1)}_{\text{GOE}} + \alpha(\kappa H^{(2)}_{\text{GOE}} + iA)$$  \hspace{1cm} (1)$$

where $H^{(1,2)}_{\text{GOE}}$ are (independent) real symmetric $N\times N$-random matrices given by the Gaussian orthogonal ensemble\cite{13} with the variance $\lambda^2/N$ of the nondiagonal elements and $A$ is a real antisymmetric random matrix whose independent elements are normal distributed with zero mean and variance $\lambda^2/N$. The parameter $\alpha$ (which is proportional to the magnetic flux penetrating the quantum dot) describes the strength of the symmetry breaking. Concerning the parameter $\kappa$, which distinguishes between different types of symmetry breaking, we are mostly interested in the particular cases of a purely imaginary antisymmetric symmetry breaking ($\kappa = 0$) and a hermitian symmetry breaking ($\kappa = 1$). The first case yields the well known Pandey-Mehta Hamiltonian\cite{14} which is believed to correspond to the application
of a small magnetic field \[\mathcal{E}\]. The other case is rather directly related \[\mathcal{E}\] to the circular Brownian motion ensemble.

In the large \(N\) limit, \(\lambda\) is expressed in terms of the average level spacing \(\Delta_0\) (at energy \(E = 0\)) \[\mathcal{E}\] by \(\lambda = N \Delta_0 / \pi\) and the proper crossover-parameter \[\mathcal{E}\] to describe the symmetry breaking is just \(\sqrt{N}\). Keeping this quantity finite as \(N \to \infty\) the value of \(\kappa\) does not affect \(S\)-matrix averages because the additional contribution of \(H_{\text{GOE}}^{(2)}\) can be taken into account by an infinitesimal rescaling of the variance of \(H_{\text{GOE}}^{(1)}\). On the other hand, correlations between different values of \(\alpha\) depend on the choice of \(\kappa\).

We follow the treatment of Refs. \[\mathcal{E}\] and use for the \(M \times M\)-scattering matrix the expression \(S = 1 - 2\pi i W^\dagger (E - H_\alpha + i \pi W W^\dagger)^{-1} W\) where \(W\) is a \(N \times M\)-matrix describing the coupling of the scattering channels with the states of the quantum dot. As in Ref. \[\mathcal{E}\], we use the choice of \(W\) that corresponds to the ideal coupling of the wires characterized \[\mathcal{E}\] by a vanishing average \(\langle S \rangle = 0\) and leading to equivalent scattering channels. Then, the distribution of the scattering matrix (for \(\alpha = 0\)) is \[\mathcal{E}\] equivalent to the circular orthogonal ensemble \[\mathcal{E}\]. We assume the dimension \(M\) of the \(S\)-matrix to be even and express the conductance by the Landauer formula \(g = \text{Tr}(S_{12}^1 S_{12})\) where \(S_{12}\) is the \(M/2\)-dimensional \((1, 2)\)-block of \(S\).

We apply the supersymmetric approach as described in Refs. \[\mathcal{E}\] to calculate the conductance autocorrelation function at two different energies \(E \pm \Delta E / 2\) and two different values \(\alpha \pm \Delta \alpha / 2\) of the symmetry breaking parameter. We assume that in the large \(N\)-limit the quantities \(N \alpha^2\), \(N \Delta \alpha^2\), \(E / \Delta_0\), and \(\Delta E / \Delta_0\) remain finite. The product of the two conductance contributions is expressed as a Gaussian integral over a \(16N\)-dimensional supervector with the \(16N\)-dimensional supermatrix

\[
\mathcal{H} = \Lambda \left( E + \frac{\Delta E}{2} \Sigma_3 - \left( H_{\text{GOE}}^{(1)} + (\alpha + \frac{\Delta \alpha}{2} \Sigma_3) (\kappa H_{\text{GOE}}^{(2)} + i A \tau_3) \right) + i \pi W W^\dagger \right)
\]

We adopt here the notational conventions of Ref. \[\mathcal{E}\]. The supermatrices \(\Lambda\), \(\tau_3\) and \(\Sigma_3\) are diagonal matrices with an equal number of diagonal entries \(+1\) and \(−1\), respectively, defining different types of gradings. \(\Lambda\) describes the grading imposed by the advanced and retarded Green’s functions and the \(\tau_3\)-grading is determined by the time reversal transformation. \(\Sigma_3\) corresponds to the additional grading for the two conductance contributions at different magnetic fields and energies. As in \[\mathcal{E}\] we denote by \(L_g\) the supermatrix describing the decomposition in bosonic and fermionic sub-blocks.

Performing the usual steps \[\mathcal{E}\] and using the rescaled quantities \(\varphi = (8N\alpha^2/(M + 1))^{1/2}\) and \(\varepsilon = 2\pi E / [(M + 1)\Delta_0]\) for the symmetry breaking parameter and the energy, respectively, we arrive at the \(Q\)-integral

\[
\langle g(\varphi - \frac{1}{2}\Delta \varphi, \varepsilon - \frac{1}{2}\Delta \varepsilon) g(\varphi + \frac{1}{2}\Delta \varphi, \varepsilon + \frac{1}{2}\Delta \varepsilon) \rangle
= \left( \frac{M^2}{32} \right)^2 \int dQ \prod_{\nu=1,2} \left[ \text{Str}((1 + QA)^{-1} Q A J_\nu) \right]^2 e^{-\mathcal{L}(Q)}
\]

with the action

\[
\mathcal{L}(Q) = \frac{M}{2} \text{Str} \ln(1 + QA) + \frac{M + 1}{32} \left( 4i \Delta \varepsilon \text{Str}(Q \Sigma_3) + \sum_{\nu=1,2} \text{Str}((Q \gamma^\nu)^2) \right)
\]
and the matrices \( \gamma_1 = (\varphi + \frac{i}{2} \Delta \varphi \Sigma_3) \tau_3 \), \( \gamma_2 = \frac{i}{2} \Delta \varphi \Sigma_3 \), \( J_{1,2} = (1 + L_g) (1 + \Sigma_3) \sigma_1 (\Lambda)/4 \) \((\sigma_1(\Lambda) is the first Pauli matrix in the \( \Lambda \)-grading). The integration variable \( Q \) is a 16 \times 16-supermatrix which belongs to the coset-space defining the non-linear \( \sigma \)-model \([10,21,23]\) for the orthogonal symmetry class.

Now, we consider the limit \( M \gg 1 \) and perform an expansion in terms of the small parameter \( x = (1 + M)^{-1} \). In typical experiments with ballistic quantum dots, the number of channels is about 3-6 and correspondingly \( x = 1/7-1/13 \). Using the well known square root parametrization \([10,22]\), the \( Q \)-matrix can be written as \( Q = T^{-1} \Lambda T \), \( T = \sqrt{1 + D^2} + D \). \( D \) is a supermatrix which has only non diagonal entries in the \( \Lambda \)-grading: \( D_{11} = D_{22} = 0 \) and \( D_{21} = L_g D_{12} \). The integration measure \([22]\) is \( dQ = \exp \left( -\frac{1}{4} \text{Str} \ln(1 + D^2) \right) d\mu(D_{12}) \) where \( d\mu(D_{12}) \) is the flat measure of the 8 supermatrix \( D_{12} \). The Jacobian factor of this measure modifies the action in Eq. (4), i.e. in the first contribution \( M \) is be replaced by \( M + 1 \). The \( Q \)-integral can then be evaluated perturbatively \([22]\) where the (modified) action has to be expanded in even powers of \( x \). \( \mathcal{L}(D) = \mathcal{L}_2(D) + \mathcal{L}_4(D) + \mathcal{L}_6(D) + \ldots \). The expansion of the source term contributions and of the exponential of \( \mathcal{L}_4(D) + \mathcal{L}_6(D) \) up to three leading orders in \( x \) yields a Gaussian integral with the action \( \mathcal{L}_2(D) \). The supermatrix \( D_{12} \) has to be decomposed into the 16 blocks corresponding to the \( \tau_3 \) and \( \Sigma_3 \)-grading. Each of these blocks is associated with a particular propagator determined by \( \mathcal{L}_2(D) \). Due to the large number of different terms in the higher order supertraces, we applied a computer program, which was developed for this purpose in \( \text{C++} \), for the final evaluation of the integral. To get the autocorrelation, one has to subtract from (3) the product of the two independently averaged conductance contributions which have been calculated similarly.

Before we discuss the final result, we mention that the action (3) also describes diffusive metallic systems \([20,21]\) provided that all relevant energy scales, in particular the level broadening \( M \cdot \Delta_0 \), are smaller than the Thouless energy. The perturbation theory applied here corresponds exactly \([20]\) to the standard diagrammatic approach in terms of diffusion and cooperon modes. The relevant contributions for the conductance autocorrelation are the diagrams \([23]\) consisting of two connected conductance loops. In the \( \sigma \)-model, they correspond \([22,11,12]\) to the contributions of \( D_{12} \) that are non diagonal in the \( \Sigma_3 \)-grading and are associated with two propagators: the cooperon mode \( P_C = [(M+1)(1+\varphi^2+\frac{\kappa^2}{4} \Delta \varphi^2 + i \Delta \varepsilon)]^{-1} \) (non diagonal in the \( \tau_3 \)-grading) and the diffuson mode \( P_D = [(M+1)(1+\frac{\kappa^2}{4} \Delta \varphi^2 + i \Delta \varepsilon)]^{-1} \) (diagonal in the \( \tau_3 \)-grading). The conductance autocorrelation is estimated \([23]\) as \( \sim M^2 |P_D|^2 + |P_C|^2 \) which is indeed confirmed by the explicit computer calculation. We obtain (after the shifts \( \varphi \rightarrow \varphi + \frac{1}{2} \Delta \varphi \), \( \varepsilon \rightarrow \varepsilon + \frac{1}{2} \Delta \varepsilon \)) for the average conductance \( \langle g(\varphi, \varepsilon) \rangle \) and the autocorrelation function \( C_{\varphi,\varepsilon}(\Delta \varphi, \Delta \varepsilon) = \langle \delta g(\varphi, \varepsilon) \delta g(\varphi + \Delta \varphi, \varepsilon + \Delta \varepsilon) \rangle \) (\( \delta g = g - \langle g \rangle \))

the results

\[
\langle g(\varphi, \varepsilon) \rangle = \frac{M}{4} - \frac{M}{4(M+1)} \frac{1}{1+\varphi^2},
\]

\[
C_{\varphi,\varepsilon}(\Delta \varphi, \Delta \varepsilon) = \frac{1}{16} \left( \frac{1}{1 + (\varphi + \frac{1}{2} \Delta \varphi)^2 + \frac{\kappa^2}{4} \Delta \varphi^2} + \Delta \varepsilon^2 \right),
\]

which are correct up to terms of order \( \mathcal{O}(x^2) \) for (3) and of order \( \mathcal{O}(x) \) for (3).

The quantities \( \varphi = (8N\alpha^2/(M+1))^{1/2} \), and \( \Delta \varepsilon = 2\pi E/[(M+1)\Delta_0] \), given in terms of the original model parameters, are of course related to the corresponding physical quantities in
a ballistic quantum dot. The energy correlations decay on a scale \( \gamma = \Delta_0 (M + 1) / \pi \) which is just the level broadening (or inverse life time) due to the coupling with the channels \([22]\). \( \Delta_0 \) has to be adjusted to the level spacing in the quantum dot at the Fermi energy. The relation between the parameter \( \sqrt{N} \alpha \) and the magnetic flux \( \Phi \) applied on the dot was studied in \([15,7]\) and is given by \( \sqrt{N} \alpha = \text{const.} \sqrt{\hbar v_F / (L \Delta_0)} (\Phi / \Phi_0) \) where \( v_F \) is the Fermi velocity, \( L \) a typical diameter of the quantum dot and \( \Phi_0 = \hbar / e \) the flux quantum. The numerical constant is of the order of unity and depends on geometrical details. For a particular model, a circle (radius \( L \)) with a very rough surface, it takes the value \( \sqrt{4/3} \approx 1.15 \) \([7]\). To establish the connection with experiment one should therefore set \( \varphi = \text{const.} (8 \hbar v_F / L \gamma)^{1/2} (\Phi / \Phi_0) \) and \( \Delta \varepsilon = \Delta E / (2 \gamma) \) in Eqs. \((3)\) and \((4)\). The scale dependence on the channel number \( M \) (via \( \gamma \)) can be understood in terms of semiclassical trajectories. With increasing number of channels (decreasing life time \( \hbar / \gamma \)) the typical length of the trajectories (and therefore the enclosed area) decreases so that the critical value for the energy (or the magnetic flux) to break the phase-coherence of the electrons is increased.

In the limiting cases of orthogonal \( (\beta = 1, \varphi = 0) \) and unitary \( (\beta = 2, \varphi \to \infty) \) symmetry, the correlation function becomes a squared Lorentzian in \( \Delta \varphi \) and a simple Lorentzian in \( \Delta \varepsilon \)

\[
C_{\varphi, \varepsilon}(\Delta \varphi, \Delta \varepsilon) = \frac{1}{8 \beta} \frac{1}{\left(1 + \frac{1 + \kappa^2}{\Delta \varphi^2}\right)^2 + \Delta \varepsilon^2}.
\]

(7)

For \( \kappa = 0, \beta = 2 \) this expression is consistent with the semiclassical approach \([2]\) and the result of Ref. \([10]\), provided the scales of the flux are properly translated. The prefactor in \((3)\) exhibits the \( \beta \)-dependence of the universal conductance fluctuations for a ballistic cavity obtained by the random matrix approach for the scattering matrix \([24,25]\). The expression \((7)\) for the average conductance is included here for completeness and agrees very well with the (numerical fit of the) precise result of Ref. \([4]\).

The semiclassical calculations of Refs. \([3,4]\) do not reproduce the correct amplitude but they predict the same functional dependence of the weak localization peak and the autocorrelation function on the magnetic field as \((3)\) and \((7)\), provided the imaginary antisymmetric symmetry breaking \( (\kappa = 0) \) is considered. We also recover the simple Lorentzian for the energy correlation \((4)\).

In Refs. \([24,25]\) the \( S \)-matrix of the quantum dot was described by one of the circular ensembles \([19]\) for orthogonal or unitary symmetry. The crossover between these cases can be modelled by a so-called Brownian motion ensemble \([18]\) where the \( S \)-distribution depends on a fictitious time \( t \) corresponding for \( t = 0 \) to the circular orthogonal ensemble and “diffusing” for \( t \to \infty \) to the circular unitary ensemble as stationary distribution. The average conductance and the autocorrelation function obtained in this approach are given by \([3,4]\):

\[
\langle g(t) \rangle = \frac{M}{4} - \frac{M}{4(M + 1)} e^{-t/t_c}, \quad \langle \delta g(t) \delta g(t + \Delta t) \rangle = \frac{1}{16} e^{-\Delta t/t_c} \left(1 + e^{-2t/t_c}\right) + \mathcal{O}(M^{-1})
\]

(8)

where \( t_c \) is the critical time that determines the crossover scale. The comparison of the average conductance expressions \((8)\) and \((3)\) suggests the identification \( t = t_c \ln(1 + \varphi^2) \)
between the Brownian motion time $t$ and the parameter $\varphi$. Then, the conductance fluctuations given by (6) ($\Delta \varphi = \Delta \varepsilon = 0$) and (8) ($\Delta t = 0$) coincide. Concerning the correlations, we obtain from (7) (for the limiting cases $\varphi = 0$ or $\varphi \to \infty$) in lowest order in $\Delta \varphi^2$:

$$C_{\varphi, \varepsilon}(\Delta \varphi, 0) \simeq \frac{1}{8\beta}(1 - \frac{1}{2}(1 + \kappa^2)\Delta \varphi^2),$$

which coincides with the Brownian motion expression (with $\Delta t/t_c \simeq \Delta \varphi^2$) only for the case of hermitian symmetry breaking ($\kappa = 1$). It is indeed possible [7] to relate the Brownian motion approach directly (for small $\varphi$, $\Delta \varphi$, large $M$) to the model considered here ($\kappa = 1$), giving the correct identification $\Delta \varphi^2 = \Delta t/t_c$. In the crossover regime $\varphi \sim 1$, this identification and the mapping onto the Brownian motion approach do not work since the correlation function (6) is not even in $\Delta \varphi$.

In summary, we have calculated the conductance autocorrelation function (with respect to magnetic field and energy) of a chaotic quantum dot for the complete crossover from orthogonal to unitary symmetry. The perturbative approach seems to work rather well, since our expression (5) for the weak localization peak agrees very well with the precise result of Ref. [9]. In addition, we have found that different types of symmetry breaking lead to different scales in the autocorrelation function explaining the inconsistencies between semiclassical theory ($\kappa = 0$) and the Brownian motion approach ($\kappa = 1$).

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The author thanks J.-L. Pichard, A. Müller-Groeling, H. A. Weidenmüller, and C. W. J. Beenakker for helpful discussions and acknowledges the D.F.G. for a post-doctoral fellowship.
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