§1. Introduction.

The purpose of this short note is to disprove a conjecture made in [G], which gave a proof of Szemerédi’s theorem that generalized Roth’s analytic approach [R] to the case of progressions of length 3. Let us very briefly explain the motivation for the conjecture, and the reason that it seemed plausible: more details can be found in [G].

The main guiding principle behind the proof of [G], and indeed all other known proofs of Szemerédi’s theorem, is that a set that is sufficiently “random-like” contains about as many arithmetic progressions of length $k$ as a typical random set of the same density. However, it is not immediately obvious what should count as a “random-like” set. If one is interested in progressions of length 3, then a good definition turns out to come from Fourier analysis. First, one associates with a subset $A \subset \{1, 2, \ldots, N\}$ its characteristic function. Following [G], we shall write $A(x)$ rather than $\chi_A(x)$. Next, one regards this function as a function defined on $\mathbb{Z}_N \equiv \mathbb{Z}/N\mathbb{Z}$ so that there is a group structure appropriate for discrete Fourier analysis. Having done so, one defines a discrete Fourier transform as follows. If $f : \mathbb{Z}_N \to \mathbb{C}$ then

$$\hat{f}(r) = \sum_{x \in \mathbb{Z}_N} f(x) \omega^{rx},$$

where $\omega = \exp(2\pi i/N)$.

Notice that $\hat{f}(0) = \sum_x f(x)$, so if we return to our set $A$, then $\hat{A}(0) = |A|$. It turns out that an appropriate definition of quasirandomness for progressions of length three is that $\hat{A}(r)$ should be significantly smaller than this for all non-zero $r$. To be precise, one can say that $A$ is a $c$-uniform set if $|\hat{A}(r)| \leq cN$ for every non-zero $r$.

Let $A$ be a $c$-uniform subset of size $\delta N$. If $c = c(\delta)$ is sufficiently small, then $A$ must contain several arithmetic progressions of length 3, as the following calculation indicates. It doesn’t quite prove it because the sums below will be over $\mathbb{Z}_N$ and we shall count triples $(x, y, z)$ such that $x + z = 2y \mod N$. That is, we count “mod-$N$ progressions” rather than “genuine progressions.” However, this is a merely technical problem that is not too hard to deal with. The first equality below follows from the fact that $\sum_x \omega^{rx}$ equals $N$ if
\[ r = 0 \text{ and } 0 \text{ otherwise.} \]

\[
\sum_{x+z=2y} A(x)A(y)A(z) = N^{-1} \sum_{x,y,z} A(x)A(y)A(z) \sum_r \omega^r(x-2y+z)
\]

\[
= N^{-1} \sum_r \sum_x A(x) \omega^{rx} \sum_y \omega^{-2ry} \sum_z \omega^{rz}
\]

\[
= N^{-1} \sum_r \hat{A}(r)^2 \hat{A}(-2r)
\]

\[
= \delta^3 N^2 + N^{-1} \sum_{r \neq 0} \hat{A}(r)^2 \hat{A}(-2r)
\]

Now let us estimate the second term on the right hand side using the assumption that \( A \) is \( c \)-uniform.

\[
\left| \sum_{r \neq 0} \hat{A}(r)^2 \hat{A}(-2r) \right| \leq cN \sum_{r \neq 0} |\hat{A}(r)|^2
\]

\[
\leq cN \sum_r |\hat{A}(r)|^2
\]

\[
= c\delta N^3,
\]

where the last inequality follows from Parseval’s identity, which for this definition of the Fourier transform takes the form

\[
\sum_r |\hat{f}(r)|^2 = N \sum_x |f(x)|^2.
\]

Thus, if \( c \) is substantially smaller than \( \delta^2 \), the number of triples of the form \((x, x+d, x+2d)\) in \( \mathbb{Z}_N \) with all of \( x, x+d \) and \( x+2d \) lying in \( A \) is approximately \( \delta^3 N^2 \), as one would expect if \( A \) were a randomly chosen set of size \( \delta N \).

What makes it hard to prove Szemerédi’s theorem analytically is that a set can be \( c \)-uniform for a very small \( c \), even one that tends to zero quite fast as \( N \) tends to infinity, but yet not contain roughly the expected number of arithmetic progressions of length 4. An example was given in [G], which we shall discuss in a moment. However, a defect of the set defined in that example was that the number of progressions of length 4 was greater than one would expect for a random set, so it did not demonstrate in a wholly satisfactory way that \( c \)-uniformity could not be used. At one stage, I thought it would be routine to convert the example into one for which the number of progressions was smaller, but eventually Gil Kalai asked me how I proposed to do it, and I realized that it was not routine after all. In fact, all the obvious ideas seemed to fail, and the result was the
conjecture in [G], which says (in its mod-$N$ version) that if $A$ is a set of density $\delta$ that is $c$-uniform, then the number of quadruples in $A^4$ of the form $(x, x+d, x+2d, x+3d)$ is at least $(\delta^4 - c')N^2$, where $c'$ tends to zero as $c$ tends to zero. It is this conjecture that we shall disprove here.

The example from [G] that yields “too many” progressions of length 4 is easy to define. Fix a small constant $c$, such as $1/1000$, and let $A \subset \mathbb{Z}_N$ be the set of all $x \in \mathbb{Z}_N$ such that the least residue of $x^2 \mod N$ lies in the interval $[-cN, cN]$. It is not too hard to verify that $A$ is $N^{-1/2+\epsilon}$-uniform (which, since $\sum_r |\hat{A}(r)|^2 \leq N^2$ is almost as uniform as a set can possibly be) and has cardinality approximately $2cN$: proofs of these facts will be sketched later in this paper. Therefore, by the argument above, $A$ contains about $8c^3N^2$ triples $(x, x+d, x+2d)$. In fact, one can say a little more: triples of the form $(x^2, (x+d)^2, (x+2d)^2)$ are “approximately uniformly distributed” in the sense that, given any three reasonably long intervals $I, J$ and $K$, they contain about $N^{-1}|I||J||K|$ of them. Now we use this uniform distribution property, together with the identity

$$(x + 3d)^2 = x^2 - 3(x+d)^2 + 3(x+2d)^2$$

to conclude that, if $x$ and $d$ are chosen randomly, then the probability that $(x+3d)^2$ belongs to $[-cN, cN]$ given that $x^2, (x+d)^2$ and $(x+2d)^2$ all belong to $[-cN, cN]$ is significantly bigger than $2c$ (indeed, it will be approximately equal to an absolute constant that one could work out after a tedious calculation), and therefore that the number of quadruples $(x, x+d, x+2d, x+3d)$ in $A^4$ is significantly greater than $16c^4N^2$.

A natural way to try to generalize this example is by replacing the interval $[-cN, cN]$ with a more general set $B \subset \mathbb{Z}_N$. For the above argument to work, we would want $B$ to be a union of longish intervals, so that the uniform distribution property was still valid. Then we could define $A$ to be $\{x \in \mathbb{Z}_N : x^2 \in B\}$. If $B$ has size $\beta N$, then the uniform distribution property implies that the number of triples in $A$ of the form $(x, x+d, x+2d)$ is approximately $\beta^3N^2$, and that the corresponding triples $(x^2, (x+d)^2, (x+2d)^2)$ are “approximately uniformly distributed” in $B^3$. We now want to estimate the conditional probability that $(x+3d)^2 \in B$, given that $x^2, (x+d)^2$ and $(x+2d)^2$ all lie in $B$.

If the distribution were genuinely uniform, then, by the identity we used earlier, this conditional probability would be $|B|^{-3}$ times the number of quadruples $(x, y, z, w) \in B^4$ such that $w = x - 3y + 3z$. The notion of uniform distribution turns out to be strong enough for this to be a good estimate even in the situation we actually face, so let us think about how many such quadruples there are, by expressing the number in terms of Fourier
coefficients, just as we did when counting arithmetic progressions of length 3. We obtain
\[
\sum_{x-3y+3z-w=0} B(x)B(y)B(z)B(w) = N^{-1}\sum_{r} \sum_{x,y,z,w} B(x)B(y)B(z)B(w)\omega^{r(x-3y+3z-w)}
\]
\[
= N^{-1}\sum_{r} |\hat{B}(r)|^2|\hat{B}(3r)|^2
\]
\[
= N^{-1}|B|^4 + N^{-1}\sum_{r \neq 0} |\hat{B}(r)|^2|\hat{B}(3r)|^2.
\]

If we now multiply by $|B|^{-3}$ we find that the conditional probability in question is at least $|B|/N = \beta$, since the contribution from the non-zero $r$ is positive. Therefore, the number of progressions of length 4 in $A$ is bounded below (at least approximately) by $\beta^4N^2$. In other words, no $A$ constructed in this way can give an example of a set with “too few” progressions of length 4.

Notice that the above argument depends on the positivity of the Fourier expression, which in turn depends on the fact that the set $1, -3, 3, -1$ can be partitioned into pairs of the form $\{a, -a\}$. Thus, the fact that 4 is an even number is highly relevant. Indeed, this obstruction does not exist for odd $k$: it is possible to choose a union of intervals $B$ of density $\beta$ such that the number of quintuples $(v, w, x, y, z) \in B^5$ with $v - 4w + 6x - 4y + z = 0$ is significantly less than $\beta^5N^5$. Then one can define $A$ to be $\{x \in \mathbb{Z}_N : x^3 \in B\}$. It turns out that the uniform distribution property applies to quadruples of the form $(x^3, (x+d)^3, (x+2d)^3, (x+3d)^3)$, from which it follows that $A$ contains significantly fewer than $\beta^5N^2$ quintuples of the form $(x, x+d, x+2d, x+3d, x+4d)$. The details of this argument were worked out in a conversation with Ben Green, but here we leave them as an exercise for the interested reader and return to progressions of length 4.

We have now given the main motivation for the false conjecture in [G]: there seems to be a clear difference between 4 and 5, arising from their differing parities; for progressions of length 4, there is a certain positivity phenomenon that stops examples of a certain kind from having too few progressions of length 4. However, with a little more ingenuity one can after all produce an example that works for progressions of length 4, as we shall now show.

2. Construction of a uniform set with few progressions of length 4.

Lemma 1. There is a function $f: \mathbb{Z} \to \{-1, 0, 1\}$ such that $f(n) = 0$ unless $1 \leq n \leq 300$ and such that
\[
\sum_{x,d \in \mathbb{Z}} f(x)f(x+d)f(x+2d)f(x+3d) < 0.
\]
Proof. First let us define a function \( g : \mathbb{Z}^3 \to \{−1, 0, 1\} \) with the required property (where now \( x \) and \( d \) are elements of \( \mathbb{Z}^3 \)) that is supported in the set \( \{1, 2, 3, 4\}^3 \). Given \((a, b, c) \in \{1, 2, 3, 4\}^3 \) we let \( g(a, b, c) = 1 \) unless \((a, b, c)\) is one of the following sixteen triples, in which case we set \( g(a, b, c) = −1 \): 
\[
113, 121, 132, 144, 212, 224, 233, 241, 314, 322, 331, 343, 411, 423, 434, 442.
\]
These triples are chosen to have the following property: every line that goes through four points of the grid \( \{1, 2, 3, 4\}^3 \) and is not one of the four main diagonals contains precisely one of these points. (The search for such a system of points was not very difficult, since in each horizontal plane one had to take exactly one point in each row and column, and this had to be done in a disjoint way for the four planes. This was already enough of a restriction to make aesthetically driven trial and error a feasible method.)

Each pair \((x, d)\) that contributes to the sum above, with \( g \) instead of \( f \), is either degenerate, in the sense that \( d = 0 \), or it corresponds to a geometrical line. Note also that each non-degenerate line is counted twice, since the line for \( x \) and \( d \) is the same as the line for \( x + 3d \) and \( −d \). The contribution to the sum from each degenerate pairs is 1, and there are \( 4^3 = 64 \) of them. The contribution from each line that is parallel to one of the three coordinate axes is \( −2 \) and there are \( 3 \times 16 = 48 \) of these. The contribution from each non-main diagonal is \( −2 \) and there are \( 3 \times 8 = 24 \) of those. Finally, each of the 4 main diagonals contributes 2. Therefore, the sum is \( 64 − 96 − 48 + 8 = −72 \).

Now we transfer this example to \( \mathbb{Z} \) by a suitable projection. Define a map \( \phi : \{1, 2, 3, 4\}^3 \to \mathbb{Z} \) by \( \phi(a, b, c) = a + 8b + 64c \). This map is easily checked to be a Freiman homomorphism: that is, \( \phi(x) − \phi(y) = \phi(z) − \phi(w) \) if and only if \( x − y = z − w \). Indeed, if
\[
a_1 + 8b_1 + 64c_1 − a_2 − 8b_2 − 64c_2 = a_3 + 8b_3 + 64c_3 − a_4 − 8b_4 − 64c_4,
\]
then
\[
(a_1 − a_2 − a_3 + a_4) + 8(b_1 − b_2 − b_3 + b_4) + 64(c_1 − c_2 − c_3 + c_4) = 0.
\]
But \( a_1 − a_2 − a_3 + a_4, b_1 − b_2 − b_3 + b_4 \) and \( c_1 − c_2 − c_3 + c_4 \) all lie between \(-6\) and \(6\), from which it follows easily that they are all zero.

A quadruple of the form \( (x, x + d, x + 2d, x + 3d) \) is the same as a quadruple \( (x, y, z, w) \) such that \( y − x = z − y \) and \( z − y = w − z \). Therefore, \( (\phi x, \phi y, \phi z, \phi w) \) is such a quadruple if and only if \( (x, y, z, w) \) is. It follows that if we define \( f(\phi(a, b, c)) \) to be \( g(a, b, c) \) and \( f(x) \) to be 0 if \( x \) is not in the image of \( \phi \), then
\[
\sum_{x, d} f(x)f(x + d)f(x + 2d)f(x + 3d) = −72.
\]
This proves the lemma.

Corollary 2. There exists an absolute constant $c > 0$ such that for every sufficiently large $N$ there is a function $F : \mathbb{Z}_N \to \{-1, 0, 1\}$ with the following properties. First, $F$ is a $\pm 1$-combination of characteristic functions of disjoint intervals of size at least $N/1500$, and, secondly,

$$\sum_{x,d \in \mathbb{Z}_N} F(x)F(x + d)F(x + 2d)F(x + 3d) \leq -cN^2 .$$

Proof. Let $t$ be a positive integer between $N/1500$ and $N/1200$, and for $1 \leq k \leq 300$ let $I_k$ be the interval $\{(2k - 1)t + 1, (2k - 1)t + 2, \ldots, 2kt\}$. Then an argument similar to the proof that the function $\phi$ in Lemma 1 was a Freiman homomorphism shows that if $x$ and $d$ are elements of $\mathbb{Z}_N$ and $x \in I_k$, $x + d \in I_l$, $x + 2d \in I_m$ and $x + 3d \in I_n$, then $l - k = m - l = n - m$. In other words, an arithmetic progression can lie in the union of the intervals $I_k$ only if the corresponding intervals themselves lie in an arithmetic progression (which may be degenerate).

Suppose that $(k, l, m, n)$ is an arithmetic progression. Then the number of arithmetic progressions $(x, y, z, w) \in I_k \times I_l \times I_m \times I_n$ is the same as the number of such progressions in $\{1, 2, \ldots, t\}^4$, since the map

$$(x, y, z, w) \mapsto (x - (2k - 1)t, y - (2l - 1)t, z - (2m - 1)t, w - (2n - 1)t)$$

is a bijection between the two sets of progressions in question. Notice also that every mod-$N$ progression that lives in the union of the $I_k$ must in fact correspond to a genuine arithmetic progression in the set $\{1, 2, \ldots, N\}$, since the union does not contain any numbers between $N/2$ and $N$.

Let $p$ be the number of progressions in $I_k^4$. Then $p$ is bounded below by $at^2$ for an absolute constant $a > 0$. It is easy, for example, to obtain a lower bound of $t^2/9$, if by “progression” we mean “quadruple of the form $(x, x + d, x + 2d, x + 3d)$” and allow $d$ to be any element of $\mathbb{Z}_N$.

Now let $F$ be defined as follows. If $x \in I_k$ then $F(x) = f(k)$. For all other $x$, $F(x) = 0$. The remarks we have just made show that

$$\sum_{x,d \in \mathbb{Z}_N} F(x)F(x + d)F(x + 2d)F(x + 3d) = p \sum_{x,d \in \mathbb{Z}} f(x)f(x + d)f(x + 2d)f(x + 3d) .$$

But $p \geq t^2/9$, $t \geq N/1500$ and $\sum_{x,d \in \mathbb{Z}} f(x)f(x + d)f(x + 2d)f(x + 3d) = -72$. Therefore, the lemma is proved, with $c = 72/9(1500)^2$, which is greater than $1/300000$. □
Corollary 2 may seem rather pointless, since the function $F$ is composed of characteristic functions of intervals and therefore has large Fourier coefficients. However, the main observation that underlies the construction is that such an example can be converted into a uniform set in a rather simple way, while keeping its property of having a negative sum over progressions of length 4.

The trick is to multiply $F$ pointwise by a small linear combination of “quadratic phase functions,” that is, functions of the form $\omega^q(x)$ for a quadratic polynomial $q$. These have two properties that make them useful. First, quadratic functions satisfy the identity

$$q(x) - 3q(x + d) + 3q(x + 2d) - q(x + 3d) = 0,$$

as we have already noted. Secondly, functions of the form $\omega^q(x)$ are highly uniform – indeed, so uniform that they remain highly uniform even when multiplied by characteristic functions of intervals. Let us prove the second of these facts, which will allow us to show that the function we create is uniform. The proof is standard, but is included for the convenience of readers not familiar with this kind of argument. Once we have established the second fact, we can show why the first fact helps us to obtain a good estimate for sums of products over arithmetic progressions.

**Lemma 3.** Let $I$ be an interval of length $t$ in $\mathbb{Z}_N$ and let $q$ be a quadratic polynomial. Then no Fourier coefficient of the product $I(x)\omega^q(x)$ is greater than $2N^{1/2}\log N$.

**Proof.** First we show that all Fourier coefficients of function $Q(x) = \omega^q(x)$ have size $N^{1/2}$. Indeed, let $q(x) = ax^2 + bx + c$, with $a \neq 0$. Then

$$\left| \sum_x Q(x)\omega^{rx} \right|^2 = \sum_{x,y} \omega^{q(x) - q(y) + r(x - y)} = \sum_{x,y} \omega^{(ax + ay + b + r)(x - y)} = \sum_z \sum_x \omega^{(2ax - az + b + r)z}.$$

For each fixed $z$, the sum over $x$ is zero, except when $z = 0$, in which case it is $N$. Therefore, the sum equals $N$, which proves the claim.

Now the discrete Fourier transform we have defined satisfies, as it should, a convolution identity. Here we shall use the fact that the $r$th Fourier coefficient of a pointwise product $f(x)g(x)$ is $N^{-1}\sum_{u+v=r} \hat{f}(u)\hat{g}(v)$. In particular, if we take $f(x) = Q(x)$ and $g(x) = I(x)$, then the modulus of the $r$th Fourier coefficient of $fg$ is at most $N^{-1/2} \sum_r |\hat{I}(r)|$. 

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Suppose that \( I \) is the interval \( \{s, s + 1, \ldots, s + t - 1\} \). Then \( \hat{I}(r) = \omega^k \sum_{q=0}^{t-1} \omega^{rq} \), which equals \( \omega^k (1 - \omega^t) / (1 - \omega^r) \), except that when \( r = 0 \) it equals \( t \).

Now, for \( -N/2 < r \leq N/2 \) it is easy to show that \( |1 - \omega^r| \geq 4r/N \), and hence that \( |\hat{I}(r)| \leq N/2r \). It follows that \( \sum_r |\hat{I}(r)| \) is at most \( N \left( 1 + \sum_{r \leq N/2} r^{-1} \right) \), which is at most \( 2N \log N \). The result follows.

Now let us define a function \( G \) by

\[
G(x) = F(x)(\omega x^2 + \omega^{-x^2} + \omega^{3x^2} + \omega^{-3x^2})
\]

Then, since \( F \) is a \( \pm 1 \)-combination of characteristic functions of 64 intervals, Lemma 3 implies that no Fourier coefficient of \( G \) exceeds \( 4 \times 64 \times 2N^{1/2} \log N = 512N^{1/2} \log N \). Notice also that \( G \) is real valued and that its values all belong to the interval \([-4, 4]\).

**Lemma 4.** Let \( G \) be as just defined. Then

\[
\left| \sum_{x,d} G(x)G(x+d)G(x+2d)G(x+3d) - 2 \sum_{x,d} F(x)F(x+d)F(x+2d)F(x+3d) \right|
\]

is at most \( 2^{18}N^{3/2} \log N \).

**Proof.** Let us write \( G(x) = F(x)U(x) \). Then \( G(x)G(x+d)G(x+2d)G(x+3d) \) splits into a sum of 256 terms of the form \( \omega^{px^2+q(x+d)^2+r(x+2d)^2+s(x+3d)^2} \), where \( p, q, r \) and \( s \) belong to the set \( \{-3, -1, 1, 3\} \). Let us fix a choice of \( p, q, r \) and \( s \), set \( q(x,d) = px^2 + q(x+d)^2 + r(x+2d)^2 + s(x+3d)^2 \), and estimate the sum

\[
\sum_{x,d} F(x)F(x+d)F(x+2d)F(x+3d)\omega^{q(x,d)}
\]

We can write \( q(x,d) = ux^2 + v(x+d) + wd^2 \). Let us suppose that \( u \neq 0 \) and consider the sum for a fixed value of \( d \), writing \( F_d(x) \) for \( F(x)F(x+d)F(x+2d)F(x+3d) \). The number of intervals \( I_k \) on which \( F \) is non-zero is 64, so as \( x \) increases, the number of times at least one of \( x, x+d, x+2d \) and \( x+3d \) changes from not belonging to a certain \( I_k \) to belonging to it is at most \( 4 \times 64 = 256 \). It follows that \( F_d \) is a \( \pm 1 \)-sum of characteristic functions of at most 256 disjoint intervals. Lemma 3 (in the case of the Fourier coefficient at zero) then implies that \( \sum_{x} F_d(x)\omega^{q(x,d)} \) has modulus at most \( 512N^{1/2} \log N \). If we now sum over \( d \) we find that

\[
\sum_{x,d} F(x)F(x+d)F(x+2d)F(x+3d)\omega^{q(x,d)} \leq 512N^{3/2} \log N.
\]
Now let us consider what happens if \( w \neq 0 \). The argument is very similar. This time we shall fix \( x \) and define \( F_x(d) \) to be \( F(x)F(x+d)F(x+2d)F(x+3d) \). (Notice that we are using the same notation for a different definition in this case.) As \( x \) increases, the number of times \( 3x \) can change from not belonging to \( I_k \) to belonging to \( I_k \) is at most 3. Therefore, \( F_x \) is a \( \pm 1 \)-sum of characteristic functions of at most \((1 + 3 + 3 + 1) \times 64 = 512\) disjoint intervals. So in this case we obtain the estimate

\[
\sum_{x,d} F(x)F(x+d)F(x+2d)F(x+3d)\omega^{q(x,d)} \leq 1024N^{3/2} \log N.
\]

If \( u = 0 \) and \( w = 0 \) then \( p + q + r + s = 0 \) and \( q + 4r + 9s = 0 \). Because \( p, q, r \) and \( s \) all lie in the set \( \{-3, -1, 1, 3\} \), it is not possible for \( q + 4r + 9s \) to equal 0 unless \( s = \pm 1 \). Since we can multiply any solution by \(-1\) and obtain another solution, let us suppose that \( s = -1 \). Then \( q + 4r = 9 \), which, it is simple to check, can happen only if \( q = -3 \) and \( r = 3 \). But then \( p = 1 \). It follows that the only two solutions are \((1, -3, 3, -1)\) and \((-1, 3, -3, 1)\).

But in this case, \( q(x, d) = 0 \), by the identity used earlier, so the sum becomes \( \sum_{x,d} F(x)F(x+d)F(x+2d)F(x+3d) \). It follows that the difference between the sum with \( G \) and the sum with \( F \) is at most \( 256 \times 1024N^{3/2} \log N \), which is what the lemma states.

We are almost done. \( G \) is a function taking values in \([-4, 4]\) with a “negative number of progressions of length 4.” All that remains is to convert it into a set. This we do by first setting \( P(x) \) to be \((G(x) + 4)/8\) and then choosing a set \( A \) by letting \( x \) belong to \( A \) with probability \( P(x) \). Here are the details.

**Corollary 5.** There exists an absolute constant \( c > 0 \) such that for every sufficiently large \( N \) there is a function \( P : \mathbb{Z}_N \to [0, 1] \) such that \( \left| \sum_x P(x) - N/2 \right| \leq 64N^{1/2} \log N \), such that \( |\hat{P}(r)| \leq 64N^{1/2} \log N \) for every \( r \neq 0 \) and such that

\[
\sum_{x,d} P(x)P(x+d)P(x+2d)P(x+3d) \leq (1/16 - c)N^2.
\]

**Proof.** As we have already said, let \( P(x) \) equal \((G(x) + 4)/8\). Then \( \sum_x P(x) - N/2 = \sum_x G(x)/8 \), which is at most \( 64N^{1/2} \log N \), since it is \( \hat{G}(0)/8 \) and all Fourier coefficients of \( G \) are at most \( 512N^{1/2} \log N \). Similarly, when \( r \neq 0 \), \( \hat{P}(r) = \hat{G}(r)/8 \), since adding a constant to a function does not alter its Fourier coefficients for \( r \neq 0 \).
To prove the last property, write the product \( P(x)P(x + d)P(x + 2d)P(x + 3d) \) as

\[
2^{-12}(4 + G(x))(4 + G(x + d))(4 + G(x + 2d))(4 + G(x + 3d)) .
\]

This product splits into 16 parts, each of which we shall sum separately.

If we choose 4 from every bracket, then we obtain \( 2^{8-12} = 1/16 \). Summing over \( x \) and \( d \) gives us \( N^2/16 \). If we choose \( G \) from every bracket then we are estimating

\[
2^{-12} \sum_{x,d} G(x)G(x+d)G(x+2d)G(x+3d) ,
\]

which, by Lemmas 2 (with the quantitative estimate at the end) and 4 is at most \(-2^{-18}N^2 + 2^6N^{3/2}\log N\).

If we choose \( G \) from precisely one bracket, then we obtain \( 2^{-6}N \sum_x G(x) \), after a suitable change of variables. We have already remarked that \( |\sum_x G(x)| \leq 512N1/2\log N \), so this is at most \( 8N^{3/2}\log N \).

If we choose \( G \) from precisely two brackets, then we obtain \( 2^{-8} (\sum_x G(x)) \), again after a suitable change of variables. This has modulus at most \( 1024N(\log N)^2 \).

If we choose \( G \) from three brackets, then we must do a calculation similar to the one that proves that a uniform set contains many arithmetic progressions of length 3. We must estimate a sum such as

\[
2^{-10} \sum_{x,d} G(x)G(x+d)G(x+3d) .
\]

To do this, note that triples of the form \((x, x+d, x+3d)\) are precisely triples \((x, y, z)\) for which \(2x + z = 3y\). So

\[
\sum_{x,d} G(x)G(x+d)G(x+3d) = N^{-1} \sum_r \sum_{x,y,z} G(x)G(y)G(z)\omega^{r(2x-3y+z)}
\]

\[
= N^{-1} \sum_r \hat{G}(2r)\hat{G}(-3r)\hat{G}(r) .
\]

But \( |\hat{G}(r)| \leq 512N^{1/2}\log N \) for every \( r \). Using this fact and the Cauchy-Schwarz inequality, the last expression can be bounded above in modulus by

\[
512N^{-1/2}\log N \left( \sum_r |\hat{G}(2r)|^2 \right)^{1/2} \left( \sum_r |\hat{G}(-3r)|^2 \right)^{1/2} .
\]

Since \( N \) is prime, the product of the last two brackets is just \( \sum_r |\hat{G}(r)|^2 \), which is \( N \sum_x |G(x)|^2 \), which is at most \( 16N^2 \). Therefore, the whole sum is at most \( 2^{13}N^{3/2}\log N \).
After multiplying by $2^{-10}$, as we need to, we have shown that the contribution from this term is at most $4N^{3/2}\log N$. The estimates for the other terms with $G$ chosen 3 times are proved in just the same way, except that the linear combinations that characterize the relevant triples are different.

There are 16 terms, and the worst error term (when $N$ is sufficiently large) has modulus at most $8N^{3/2}\log N$. Therefore, $\sum_{x,d} P(x)P(x+d)P(x+2d)P(x+3d)$ is at most $N^2(2^{-14} - 2^{-19}) + (2^6 + 2^7)N^{3/2}\log N$. For sufficiently large $N$, this proves the corollary.

\[ \text{Theorem 6.} \quad \text{There exist absolute constants } c > 0 \text{ and } C \text{ such that for all sufficiently large } N \text{ there is a } CN^{-1/2}\log N\text{-uniform subset } A \text{ of } \mathbb{Z}_N \text{ of cardinality } N/2 + o(N), \text{ for which} \]

\[ \sum_{x,d} A(x)A(x + d)A(x + 2d)A(x + 3d) \leq (2^{-4} - c)N^2. \]

\[ \text{Proof.} \quad \text{Choose } A \text{ randomly as follows. For every } x \in \mathbb{Z}_N, \text{ let } x \text{ belong to } A \text{ with probability } P(x), \text{ with all choices made independently. Then } A(x) - P(x) \text{ is a random variable of mean zero that is equal to either } 1 - P(x) \text{ or } -P(x). \text{ In particular, it always has modulus at most 1.} \]

Now let us consider the difference between the Fourier coefficients of $A$ and those of $P$. We have

\[ \hat{A}(r) - \hat{P}(r) = \sum_x (A(x) - P(x))\omega^{rx}. \]

This is a sum of $N$ random variables, each of mean zero and bounded above in modulus by 1. It follows immediately from Azuma’s inequality that the probability that $|\hat{A}(r) - \hat{P}(r)|$ is greater than $t\sqrt{N}$ is at most $2e^{-t^2/8}$. If we take $t$ to equal $\log N$, then this probability is, for $N$ sufficiently large, significantly smaller than $1/N^2$. It follows that with probability at least $1 - 1/N$ we have $|\hat{A}(r) - \hat{P}(r)| \leq N^{1/2}\log N$ for every $r$. This implies both the statement about the cardinality of $A$ and the claim about its uniformity.

For each non-zero $d$ the expectation of the sum $\sum_x A(x)A(x + d)A(x + 2d)A(x + 3d)$ is equal to $\sum_x P(x)P(x + d)P(x + 2d)P(x + 3d)$, and when $d = 0$ it is at most $N$. Therefore, the expectation of $\sum_x A(x)A(x + d)A(x + 2d)A(x + 3d)$ is at most $(1/16 - c)N^2 + N$, where $c$ is the absolute constant obtained in Lemma 5. Therefore, the probability that $\sum_x A(x)A(x + d)A(x + 2d)A(x + 3d)$ is at most $(1/16 - c/2)N^2$ is at least $8c$. Since this is bigger than $N^{-1}$ when $N$ is sufficiently large, we can find a choice of $A$ that has all the properties claimed. \[ \Box \]
The main result of this paper answers the conjecture in [G] but it certainly does not answer every question one might wish to ask. In particular, it is noticeable that the number of progressions of length 4 in $A$ was not that much less than it would be for a random set. Ruzsa has asked the following question (which we state in a slightly imprecise way).

**Problem.** Let $A$ be a uniform subset of $\mathbb{Z}_N$ of density $\alpha$. Must $A$ contain at least $\alpha^{1000}N^2$ arithmetic progressions of length 4?

**References.**

[G] W. T. Gowers, *A new proof of Szemerédi’s theorem*, GAFA, Geom. Funct. Anal. 11 (2001), 465-588.

[R] K. F. Roth, *On certain sets of integers*, J. London Math. Soc. 28 (1953), 245-252.