Determining symmetry groups on geometric builds using reflection and rotation tests

Rizki Kurniawan Rangkuti¹, Raden Sri Ayu Ramadhana², Wahyu Azhar Ritonga³, Muhammad Fauzi Romadhon Marpaung², Syabrina Rasyid Munthe² and Ronal Watrianthos²*

¹Department of Mathematics Education, Universitas Al-Washliyah Labuhanbatu, Indonesia.
²Department of Informatics, Universitas Al-Washliyah Labuhanbatu, Indonesia.
³
* ronal.watrianthos@gmail.com

Abstract. This research is motivated by the researcher's desire to see the folding and rotational symmetry in geometric shapes whether they meet the symmetries for certain aspects and form a group. The method used in testing the symmetry of a group in geometric shapes is by testing reflection and rotation with a combination of linear and abstract algebra using an ordered matrix, where the geometric shapes are traced on each particular aspect until they meet the symmetry. From the results of the method used, it is found that the triangles to the octagon groups are symmetrical through the test of rotation and reflection of the geometric shapes. Thus, it can be concluded that group symmetry occurs up to the order of eight in geometric shapes. Based on the pattern, starting from the pentagon it is not a symmetry group, the hexagon includes the symmetry group, the heptagon is not a symmetry group, the octagon includes a symmetry group, then the nonagon is not a symmetry group and the decagon belongs to the symmetry group

1. Introduction

This research is the development of the symmetry group sub-material. It contains a symmetric group that reflects a plane figure that meets rotational and folding symmetry by involving the permutation set of an ordo matrix using binary operations (*). It meets reclusive, associative, and there is an identity matrix with each permutation matrix that has the opposite [1]. When fulfilling some of these traits, it is said that a permutation set forms a group with a binary operation. The binary operation is intended to define the multiplication of the two sets in such a way that the results obtained remain on elements of that group. A permutation set is a matrix with a certain order that can be operated with other permutation sets. The plane figures tested show that each vertex in the rotational symmetry will meets each other vertex until it return to its original point. Furthermore, in the folding symmetry vertex meets each other vertex until there is no vertex that does not occupy another vertex. The concept built on symmetry begins with the theory of reflection and rotation. Geometry is an instrument to strengthen the concepts which connect various fields in mathematics [2]. The concept formed in geometry is an elaboration of various fields of mathematics, therefore to define a new theory it has references from the matrix, algebra, and plane figure. The results obtained from the proof by reflection and rotation
provide theoretical contributions to abstract algebra material which can then be used in various
disciplinary applications.

However, there has been previous research [2] which states that for the symmetrical set of a
triangular n-triangles with rotations of 120°0 and 240°0 are 2^(n+1)-1 cycle for n∈N while with a
rotation of 360°0 is 3(2^(n+1)-1) cycle for n∈N. In addition, reflection on the S_1, S_2, S_3 is the
3.2^n-2 cycle for n∈N. (2) The symmetry set of n rectangles with rotations of 90°0 and 270°0 are
1/2(3^(n+1)-1) cycle, for n∈N the rotation of 180°0 is 3^(n+1)-1 cycle for n∈N and with the rotation
of 360°0 is 2(3^(n+1)-1) cycle for n∈N. Whereas for reflection on the axis S_1, S_2 is 3^(n+1)-n-2
cycle for n∈N and for reflection on the S_3, S_4 is 3^(n+1)-1 cycle for n natural number [3].

2. Methodology
2.1 Group Definition
The G-blank set is said to be a group if in G there is a binary operation stated in " * ", as according to
Herstein[4]
1. For each a, b, c ∈ G result a * (b * c) = (a * b) * c (associative)
2. There is such an element e ∈ G therefore a * e = e * a = a for each a ∈ G (e is the identity
element in G).
3. For each a ∈ G, there is an element a^−1 ∈ G therefore a * a^−1 = a^−1 * a = e (a^−1 is the inverse
   of a in G).

2.2 Cayley Table
The concept of a set was first invented by a German mathematician, namely George Cantor (1918), in
the late 19th century[5]. In a group, it always involves only one specific operation. Defining
operations on a non-blank set is one of the terms to construct a group structure. Operation on a set can
be defined by drawing a Cayley table that contains the results of the operations for each of the two
elements in the set. The binary composition can be defined analytically through Cayley's table, the
Cayley table is a list designed by Arthur Cayley in the 19th century[6].

2.3 Rotation and reflection
Rotation is the process of rotating a geometry that is determined by the direction and angle of rotation,
while reflection is a reflection of geometry using the mirror image properties[7].

2.4 Library research method
The method in this study is the library-research method. It conducts research to obtain data and
information and objects used in the problem discussion. This research relates scientific reasoning
arguments to explain the results of research problems. This is in line with the opinion that literature
studies are theoretical studies, references, and scientific literature relating to cultures, values, and
norms that develop in the social situations studied [8].

3. Result and Discussion
Supposing that G is a non-blank set, the operation (*) on all G members is called binary operations if
every a*b∈G, then a*b∈G, it can also be said that the operation (*) on G is reclusive. The reclusive
characteristic here is intended that all the results of each member's binary operations remain with
the members themselves[9]. A permutation is a one-to-one mapping (bijective function) from set A to
the set itself [10]. If the set A={1,2,3,...,n} then the following function is a permutation.

\[
\begin{align*}
1 & \rightarrow f(1) = j_1 \\
2 & \rightarrow f(2) = j_2 \\
3 & \rightarrow f(3) = j_3 \\
\vdots & \text{ until } n & \rightarrow f(n) = j_n
\end{align*}
\]
The bijective function is an injective dan surjective function[11]. If $f$ bijective and for $j_i \in A \in 1,2,3,...,n$ then the permutation is given with the following two line notations:

$$\begin{bmatrix}
1 & 2 & 3 & \cdots & n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 \quad j_1 & j_2 & \cdots & j_n \\
\end{bmatrix}$$ (2)

If given $A=\{1,2,3\}$. The permutation can be indicated by:

$$
\begin{align*}
& f : 1 \mapsto f(1) = 1g : 1 \mapsto g(1) = 2 \\
& 2 \mapsto f(2) = 22 \mapsto g(2) = 3 \\
& 3 \mapsto f(3) = 33 \mapsto g(3) = 1
\end{align*}
$$ (3)

Written by $f = [1 \quad 2 \quad 3]$ Written by $g = [1 \quad 2 \quad 3]$ 

$$
\begin{align*}
& h : 1 \mapsto h(1) = 3 \\
& 2 \mapsto h(2) = 12 \mapsto i(1) = 1 \\
& 3 \mapsto h(3) = 23 \mapsto i(3) = 2
\end{align*}
$$ (4)

Written by $h = [1 \quad 2 \quad 3]$ Written by $i = [1 \quad 2 \quad 3]$ 

$$
\begin{align*}
& j : 1 \mapsto j(1) = 3 \\
& 2 \mapsto j(2) = 22 \mapsto k(1) = 2 \\
& 3 \mapsto j(3) = 13 \mapsto k(3) = 3
\end{align*}
$$ (5)

Written by $j = [1 \quad 2 \quad 3]$ Written by $k = [1 \quad 2 \quad 3]$ 

So there are six permutation sets of $A=\{1,2,3\}$, or it can be determined $P_3^3$ which states many sets of permutations. The permutation set can also be written in the form of a cycle.

$$f = [1 \quad 2 \quad 3] \quad g = [1 \quad 2 \quad 3]$$ (6)

Suppose the function $f : A \mapsto B$ determined by rule $f(a) = b$, while the function $f : B \mapsto C$ determined by rule $f(b) = c$. The composition function $g$ and $f$ written by $gof$ function defined by rule $(gof)(a) = g(f(a))$[12]. The permutation of $fog$ and $gof$ can be given by:

$$fog = [1 \quad 2 \quad 3] \quad gof = [1 \quad 2 \quad 3]$$ (7)

1. $f = [1 \quad 2 \quad 3]$ is the inverse of $f = [1 \quad 2 \quad 3]$ 
2. $g = [1 \quad 2 \quad 3]$ is the inverse of $g = [1 \quad 2 \quad 3]$ 
3. $h = [1 \quad 2 \quad 3]$ is the inverse of $h = [1 \quad 2 \quad 3]$ 

It is known that $A=\{1,2,3\}$ there are six permutations. The permutation set $A$ is $f, g, h, i, j, k$ whose inverse that:

$$
\begin{align*}
& f = [1 \quad 2 \quad 3] \\
& g = [1 \quad 2 \quad 3] \\
& h = [1 \quad 2 \quad 3]
\end{align*}
$$ (8)

Members of the set $A=\{1,2,3\}$ can be supposed as three dots on a triangle, we will test whether to form a group with multiplication operations ($\ast$) through reflection (frame folds) and rotation (permutations). We know that members of set $A$ are permutations so that we can write them down by:

$$
\begin{align*}
& f = [1 \quad 2 \quad 3] = (1)(2)(3) \\
& g = [1 \quad 2 \quad 3] = (1 \quad 2 \quad 3) \\
& h = [1 \quad 2 \quad 3] = (1 \quad 3 \quad 2)
\end{align*}
$$ (9)

3.1 Triangle

Triangle shown in figure 1 below that has points A,B,C can be inserted frame in 6 ways, which case it can be said that there are 6 transformations so that the triangle is the same side through reflection there are 3 ways and rotation there are 3 ways. In this case, the vertices of triangle can be written by A,B,C which is in the member of our permutation set with 1,2,3.
The three reflections in Figure 1 are the reflections on the AX, BY, dan CZ. Reflection AX, point A is shrimp but B and C shipment each other, the permutations that occur are:

\[ i = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} = (2\ 3) \]

(10)

Reflection BY, B is shimpit but A and C shimpit each other, the permutations that occur are:

\[ j = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix} = (1\ 3) \]

(11)

Reflection CY, C shimpit but A and B shimpit each other, then the permutations that occur are:

\[ k = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} = (1\ 3) \]

(12)

The three rotations (permutations) are the rotation of the field with the center of the O and the direction of the rotation in the opposite direction of the clock rotation. Rotation with an angle 360°, A rotates back at its, B and C, then the permutations that occur are:

\[ f = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} \]

(13)

Rotation with an angle 120°, A rotates one-third of the angle of one full round, B and C, so the permutations that occur are:

\[ g = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} = (1\ 2\ 3) \]

(14)

Rotation with an angle 240°, A rotates two-thirds of the angle of one full round, B and C, so the permutations that occur are:

\[ h = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = (1\ 3\ 2) \]

(15)

Multiplication \( A \ast A \) can be shown in cayley table below:

|   | f | g | h | i | j | k |
|---|---|---|---|---|---|---|
| f | f | g | h | i | j | k |
| g | g | h | f | k | i | j |
| h | h | f | g | j | k | i |
| i | i | j | k | f | g | h |
| j | j | k | i | h | f | g |
| k | k | i | j | g | h | f |

From the table above can be proven \( A \ast A \) which is empty space forming a group[13]:

- reclusive, indicated from \( A \ast A \) included \( A \)
- associative, indicated by taking section \( A \) where the result of the operation shows the right field the same as the left

\[
f \ast (g \ast h) = (f \ast g) \ast (i \ast k) = (i \ast j) \ast k
\]

\[
f \ast f = g \ast h \ast g = g \ast k
\]

\[
f = f = j
\]

- the element of identity \( \forall g \in A \exists f \in A \exists g \ast f = g \)

(17)

- invers \( \exists g^{-1} \in A \forall g \in A \exists g^{-1} \ast g = f \)

(18)

The fulfillment of \( A \ast A \) reclusive, associative, an element of identity and inverse, then \( (A \ast) \) is a group. Because the ABC triangle meets rotational symmetry and reflection, the triangle is a symmetry group.
3.2 Rectangle

Rectangle with the same side (equilateral rectangle) of A, B, C, D as seen in figure 2 can be inserted in the frame in 8 ways, in this case, it can be said that there are 8 transformations so that the rectangle through reflection there are 4 ways and the rotation there are 4 ways. Using Cayley table for \( G \times G \) multiplication:

### Table 2. Multiplication Cayley Table of Permutation Seton Rectangle

| (G,*) | I | R | R² | R³ | A | B | C | D |
|-------|---|---|----|----|---|---|---|---|
| I     | I | R | R² | R³ | A | B | C | D |
| R     | R | R² | R³ | I  | C | D | B | A |
| R²    | R²| I  | R  | B  | A | D | C |   |
| R³    | R³| I  | R² | D  | C | A | B |   |
| A     | A | D | B  | C  | I | R²| R³| R |
| B     | B | C | A  | D  | R²| I | R | R³|
| C     | C | A | D  | B  | R | R³| I | R²|
| D     | D | B | C  | A  | R³| R | R²| I |

From the table above can be proven whether \( G \times G \) forms a group because

- reclusive, indicated from \( G \times G \) included \( G \)
- associative, indicated by taking section \( G \) where the result of the operation shows the right field the same as the left
  \[
  R \ast (R^3 \ast D) = (R \ast R^3) \ast DR \ast (A \ast C) = (R \ast A) \ast C
  \]
  \[
  R \ast B = I \ast DR \ast R^3 = C \ast C
  \]
  \[
  D = DI = I
  \]

- the element of identity
  \[
  \forall R^2 \in G \ \exists I \in G \ \forall R^2 \ast I = R^2
  \]
- invers
  \[
  \exists C^{-1} \in G \ \forall C \in G \ \exists C^{-1} \ast C = I
  \]
  \[
  \exists C^{-1} \in G \ \forall C \in G \ \forall C \ast C = I
  \]

The fulfillment of \( G \times G \) reclusive, associative, element of identity, and inverse, then \((G*)\) is a group. Therefore the ABCD rectangle is a symmetry group indicated by rotation and reflection.

3.3 Pentagon and Hexagon

By following the pattern of symmetry through reflection and rotation tests, the same hexagon side A, B, C, D, E cannot be inserted in the frame, as some points cannot be reflected the other point, meaning it is not filled with folding symmetry.

Based on figure 2 when point A reflected C, but E and D do not attach each other. The evidence shows that the hexagon is not a symmetry group. The hexagon sides of A, B, C, D, E, F as seen in figure 2 can be inserted in the frame, meaning that symmetry is filled. As seen in figure 4. If point A is reflected B, C with F, and D enclosed with E. If A reflected C, then D enclosed with F then B and E are attached.

![Figure 2](image)

(a) Pentagon, (b) Hexagon

3.4 Heptagon and Octagon

By following the symmetry through reflection and rotation tests, then the octagon A,B,C,D,E,F,G cannot be inserted in the frame, since some points cannot be reflected another point, meaning that the symmetry is not filled.
From Figure 5 it can be seen when point A is reflected to point C, point E, and point D are not attached to each other. The evidence can already show that the octagon is not a symmetry group. In figure 6, the octagon A, B, C, D, E, F, G, H can be inserted into the frame, meaning that the folded symmetry is filled, and an octagonal point equal to the side A, B, C, D, E, F, G, H can be rotated and occupy the other points on the octagon on the same side. It means the rotation symmetry is filled. Thus the octagon A, B, C, D, E, F, G, H is defined as a symmetry group.

4. Conclusion
Based on the pattern, starting from the pentagon it is not a symmetry group, the hexagon includes the symmetry group, the heptagon is not a symmetry group, the octagon includes a symmetry group, the nonagon is not a symmetry group and the decagon belongs to the symmetry group. Finally, it can be concluded that the n-facets of a plane figure with n>3 and n odd numbers are not symmetry groups and n-facets in a plane figure with n>3 and n even numbers are symmetry groups.

References
[1] Syahrir, “Pengembangan Pola Berpikir Siswa Kelas Xi Tentang Pengoprasian Dasar Pada Matriks Identitas melalui metode pembelajaran Tanya jawab,” *J. Ilmu Sos. dan Pendidik.*, vol. 2, no. 1, 2018.
[2] K. Safrina, M. Ikhsan, and A. Ahmad, “Peningkatan Kemampuan Pemecahan Masalah Geometri melalui Pembelajaran Kooperatif Berbasis Teori Van Hiele,” *J. Didakt. Mat.*, vol. 1, no. 1, 2014.
[3] J. Wang, X. G. Wen, and E. Witten, “Symmetric Gapped Interfaces of SPT and SET States: Systematic Constructions,” *Phys. Rev. X*, 2018, doi: 10.1103/PhysRevX.8.031048.
[4] N. R. Dewi, N. Eliyati, and O. H. Marbun, “Kajian Struktur Aljabar Grup pada Himpunan Matriks yang Invertibel,” *J. Penelit. Sains*, vol. 14, no. 1, 2011.
[5] K. W. Johnson and J. D. H. Smith, “On the category of weak Cayley table morphisms between groups,” *Sel. Math.*, vol. 13, no. 1, pp. 57–67, Jun. 2007, doi: 10.1007/s00029-007-0032-x.
[6] G. Cooperman and L. Finkelstein, “New methods for using Cayley graphs in interconnection networks,” *Discret. Appl. Math.*, 1992, doi: 10.1016/0166-218X(92)90127-V.
[7] D. Setyawan, “Eksplorasi Proses Konstruksi Pengetahuan Materi Bangun Ruang Siswa Dengan Gaya Berpikir Acak Dan Kemampuan Keruangan Level Rotasi Mental,” *Ecosystem*, vol. 17, no. 1, 2017.
[8] Sugiyono, *Memahami Penelitian Kualitatif*. Bandung: Alfabeta, 2012.
[9] R. S. Wasikotingtyas, *MENGENAL MATEMATIKA DISKRIT*. Cirebon: Nusa Litera Inspirasi, 2019.
[10] R. T. Damayanti, “Automorfisme Graf Bintang Dan Graf Lintasan,” *Xcauchy*, vol. 2, no. 1, 2011.
[11] 020 Umy Zahroh, “Kecenderungan Gaya Belajar Mahasiswa dalam Menyelesaikan Masalah Fungsi Bijektif,” *J. Kebijak. dan Pengemb. Pendidik.*, 2014.
[12] Siswanto & Supraptinah, *Matematika Inovatif 2; Konsep dan Aplikasinya*. Pusat Perbukuan Depdiknas Tahun 2009, 2009.
[13] A. J. Noor and N. Hijriati, “GRUP RING,” *J. Mat. Murni dan Terap.*, vol. 4, no. 1, 2010.