Many-Body Entanglement: Permutations and Equivalence classes

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With an easily applicable criterion based on permutation symmetries of (identically prepared) replicas of quantum states we identify distinct entanglement classes in high-dimensional multipartite systems. The different symmetry properties of inequivalent states provide a rather intuitive picture of the otherwise very abstract classification of many-body entangled states.

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Many-body quantum coherence is responsible for many astonishing properties of composite quantum systems, what concerns their spectral, e.g., specifically ground state [1] properties but also their quantum dynamics: think of superconductivity where coherently composed quantum systems function as carriers that guarantee lossless charge transport [2], or of enhanced excitation transport in photodynamic processes, which is under debate to be of quantum origin [3]. The inherent many-body character of such features makes them difficult to understand in terms of single particle observables which are the categories of our classically trained intuition. Consequently, we need a better understanding of the nature of many-body quantum coherence, and, a fortiori of multi-partite entanglement.

So far, the structure of entangled states is still widely unexplored. Only for pure states of bipartite systems do we have an exhaustive understanding based on the Schmidt decomposition [4]. But in multipartite systems, where there is no comparably useful tool, we have only very partial knowledge. We know that there exist two inequivalent classes of entangled states in three qubit systems [5], the GHZ and W states. Four qubit systems reveal even a number of distinct families, i.e. continuously parametrized sets of classes, of entangled states [6]. Further efforts [7–11], still restrict to rather small and low-dimensional systems, which hitherto leaves us with a rather intransparent picture of the general structure of multipartite entanglement.

The defining property of entanglement is that it cannot be created by local manipulation. Formally, this is expressed in terms of a Kraus operator $K$ [12, 13]: if there is an operator $K$ such that $|\Psi\rangle = K|\Psi\rangle$, then there is a generalized measurement, also referred to as POVM [13], in which one of the outcomes is associated with the transition of the state $|\Psi\rangle$ to $|\Phi\rangle$. If the Kraus operator acting on an $N$-partite system is a simple direct product of local Krauss operators $F_i$, i.e. if $K = F_1 \otimes \ldots \otimes F_N$, then the measurement can be implemented in terms of local measurements on the $N$ subsystems.

In bipartite systems a state $|\Phi\rangle$ can be created from a state $|\Psi\rangle$ by local manipulation if and only if the Schmidt rank, i.e. the number of finite Schmidt coefficients of $|\Phi\rangle$, does not exceed that of $|\Psi\rangle$. Thus, starting out from a state with maximal Schmidt rank in a given finite dimensional bipartite system, any other state can be obtained through local manipulation only. This is fundamentally different in multipartite systems where there is in general no such so-called maximally entangled state, but there are distinct classes or families of multipartite states. Here, two states $|\Phi\rangle$ and $|\Psi\rangle$ define two different equivalence classes if there are no local Kraus operators $F_i (i = 1, \ldots, N)$ and $G_i (i = 1, \ldots, N)$ such that $|\Phi\rangle = F_1 \otimes \ldots \otimes F_N |\Psi\rangle$ and $|\Psi\rangle = G_1 \otimes \ldots \otimes G_N |\Phi\rangle$. In tripartite qubit systems there are the two classes of $|W\rangle = (|111\rangle + |121\rangle + |112\rangle)/\sqrt{3}$ and $(GHZ) = (|111\rangle + |222\rangle)/\sqrt{2}$ states [5], and in fourpartite qubit systems there are even continuous families of inequivalently entangled states [6]. Only if the dimension of one of the subsystems is as least as large as the product of the dimensions of all other subsystems there is a maximally entangled state in the above sense [14].

I. REPLICAS OF QUANTUM SYSTEMS

The criterion [15] that we exploit here to identify structure of many-body entangled states is based on permutation symmetries of multiple replicas of a quantum state. ‘Multiple’ in the present context means that rather than considering a quantum state $|\Phi\rangle$ of an $N$-partite quantum system, we consider $M$ replicas of this state vector, i.e. the $M$-fold tensor product of the state vector $|\Phi\rangle$ with itself. Such permutation symmetries must not be mistaken with symmetries under permutations of subsystems which are frequently considered [16,19]. In order to avoid such confusion between the different subsystems and different replicas of a quantum state, we will adopt the convention of [20] to write the different replicas of a quantum state in different rows. All terms associated with a specific subsystem will be within a column. That is, if we consider a bipartite quan-
tum state \(|\Psi\rangle = \sum_{ij} \Psi_{ij} |i\rangle \otimes |j\rangle\), then two replicas read
\(|\Psi\rangle \otimes 2 = \sum_{i_1,i_2,j_1,j_2} \Psi_{i_1,j_1} \Psi_{i_2,j_2} |i_1\rangle \otimes |i_2\rangle \otimes |j_1\rangle \otimes |j_2\rangle\), and we will denote the four-component state vector as
\[
\begin{align*}
| i_1 \rangle \otimes | j_1 \rangle & \in \mathcal{H}_1 \otimes \mathcal{H}_2 \\
| i_2 \rangle \otimes | j_2 \rangle & \in \mathcal{H}_1 \otimes \mathcal{H}_2 \\
\mathcal{H}_1 & \otimes \mathcal{H}_2 \\
\mathcal{H}_1 & \otimes \mathcal{H}_2,
\end{align*}
\]
and similarly for a permutation on the second subsystem. Consequently, the projector \(P_+ = (I - \Pi)/2\) onto the antisymmetric component of \(\mathcal{H}_1 \otimes \mathcal{H}_1\) (or analogously of \(\mathcal{H}_2 \otimes \mathcal{H}_2\)) can now be used to probe the invariance of any given duplicate state \(|\Psi\rangle \otimes |\Phi\rangle\) under \(\Pi\): any separable state is invariant under \(\Pi\) and is thus mapped on the zero-vector
\[
(P_- \otimes I)|\Psi\rangle \otimes 2 = 0 ,
\]
whereas any entangled state remains finite:
\[
(P_- \otimes I)|\Psi\rangle \otimes 2 = \sqrt{\lambda_1 \lambda_2} \left( \begin{array}{c}
|1/2\rangle \otimes |1/2\rangle \\
|2/1\rangle \otimes |1/2\rangle \\
|2/1\rangle \otimes |2/1\rangle
\end{array} \right) .
\]
Later-on, we will see that similar distinctions also exist for different classes of multipartite entangled states. However, before doing so, let us formulate our criterion to identify inequivalent entanglement classes.

### A. Permutations

As we will discuss in the remainder of this article we can use permutations on several replicas of a subsystem to distinguish different classes of entangled states. In the case of two replicas, the (unique) permutation operator \(\Pi\) associated with one subsystem is defined via
\[
\Pi \left| k_1 \right\rangle \left\langle k_2 \right| = \left| k_2 \right\rangle \left\langle k_1 \right| ,
\]
and, in the case of \(M\) replicas there are \(M!\) inequivalent permutations [21] per subsystem that are defined similarly to Eq. (3). In the case of \(M\) there are two distinct permutations possible: one that interchanges \(|i_1\rangle\) and \(|i_2\rangle\) on subsystems one and another one that interchanges \(|j_1\rangle\) and \(|j_2\rangle\) on the subsystems two. Any separable state \(|\Psi_1\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle\) is invariant under either of these permutations:
\[
|\Psi_1\rangle \otimes 2 = \begin{array}{c}
|\varphi_1\rangle \\
|\varphi_2\rangle
\end{array} \otimes \begin{array}{c}
|\varphi_2\rangle \\
|\varphi_1\rangle
\end{array} = \Pi \begin{array}{c}
|\varphi_1\rangle \\
|\varphi_2\rangle
\end{array} \otimes \begin{array}{c}
|\varphi_2\rangle \\
|\varphi_1\rangle
\end{array} = \begin{array}{c}
|\varphi_1\rangle \\
|\varphi_2\rangle
\end{array} \otimes \Pi \begin{array}{c}
|\varphi_2\rangle \\
|\varphi_1\rangle
\end{array} .
\]

This is different for an entangled state, that we can write in its Schmidt decomposition as \(|\Psi_2\rangle = \sqrt{\lambda_1}|11\rangle + \sqrt{\lambda_2}|22\rangle\). Its duplicate version reads
\[
|\Psi_2\rangle \otimes 2 = \lambda_1 \begin{array}{c}
|1/1\rangle \\
|2/2\rangle
\end{array} \otimes \begin{array}{c}
|1/1\rangle \\
|2/2\rangle
\end{array} + \lambda_2 \begin{array}{c}
|2/1\rangle \\
|1/2\rangle
\end{array} \otimes \begin{array}{c}
|2/1\rangle \\
|1/2\rangle
\end{array} + \sqrt{\lambda_1 \lambda_2} \left( \begin{array}{c}
|1/2\rangle \otimes |1/2\rangle \\
|2/1\rangle \otimes |2/1\rangle
\end{array} \right) .
\]
This object is actually altered by a permutation on the first subsystem:
\[
|\Psi_2\rangle \otimes 2 \neq \lambda_1 \begin{array}{c}
|1/1\rangle \\
|2/2\rangle
\end{array} \otimes \begin{array}{c}
|1/1\rangle \\
|2/2\rangle
\end{array} + \lambda_2 \begin{array}{c}
|2/1\rangle \\
|1/2\rangle
\end{array} \otimes \begin{array}{c}
|2/1\rangle \\
|1/2\rangle
\end{array} + \sqrt{\lambda_1 \lambda_2} \left( \begin{array}{c}
|2/1\rangle \otimes |1/2\rangle \\
|1/2\rangle \otimes |2/1\rangle
\end{array} \right) .
\]

### B. Criterion for the classification of many-body entanglement

The central property of any permutation operator \(\Pi\) acting on \(M\) replicas of a single-particle Hilbert space is that it commutes with the \(M\)-fold tensor product of any operator \(F\):
\[
[\Pi, F^{\otimes M}] = 0 .
\]
Permutation operators are ideally suited to distinguish different classes of entangled states, since this relation naturally generalizes to \(M\) replicas of an \(N\)-partite system, if we introduce introduce the operator
\[
A = \sum_{i_1...i_N} \eta_{i_1...i_N}^{j_1...j_N} \Pi_{i_1}^{(j_1)} \otimes ... \otimes \Pi_{i_N}^{(j_N)} .
\]

A is composed of permutations \(\Pi_{i_1}^{(j_1)}\), where ‘\(j\)’ denotes the subsystem that the permutation is associated with, ‘\(i\)’ labels different permutations, and \(\eta_{i_1...i_N}^{j_1...j_N}\) are complex numbers. Any such operator \(A\) commutes with \(M\) replicas of a local Kraus operator
\[
[A, (F_1 \otimes ... \otimes F_N)^{\otimes M}] = 0 .
\]
With this at hand, we can rephrase the criterion for the classification of many-body entanglement [15] that we will exploit in the following:

**If an operator \(A\) as specified in Eq. (10) exists such that**
\[
A|\Psi\rangle^{\otimes M} = 0 , \quad A|\Phi\rangle^{\otimes M} \neq 0 ,
\]
**then the state \(|\Psi\rangle\) cannot be obtained from \(|\Phi\rangle\) through local manipulation.**
The proof of this assertion follows by contradiction:

\[ 0 \neq A\phi^{\otimes M} = A((F_1 \otimes \ldots \otimes F_N)\Psi)\otimes M \]
\[ = (F_1 \otimes \ldots \otimes F_N)A\Psi \otimes M = 0 , \]

as a direct consequence of Eqs. (11) and (12).

There are cases in which a state |Φ⟩ can be created through local action on a state |Ψ⟩, but where the inverse is not possible. In these cases it is meaningful to assert that |Ψ⟩ carries entanglement of superior type than the the state |Φ⟩. However, as it is the case for example with the GHZ and the W state, there are also different classes of entangled states and neither can one state of the former class be obtained through local manipulation of a state of the latter, nor is the reverse possible. To show such distinctness for two states |Φ⟩ and |Ψ⟩ with the help of Eq. (12), two operators A₁ and A₂ are required with

\[ A_1 |Φ⟩ = 0 \text{ and } A_1 |Ψ⟩ \neq 0, \text{ but } \]
\[ A_2 |Ψ⟩ \neq 0 \text{ and } A_2 |Φ⟩ = 0 . \]

However, if the ranks of all single-subsystem reduced density matrices of both states |Φ⟩ and |Ψ⟩ coincide for any subsystem, Eq. (12) is already sufficient for distinctiveness: in this case all local Kraus operators can be assumed to be invertible, without loss of generality [5], so that the non-existence of local operators F_i (i = 1,...,n) with |Φ⟩ = F_i ⊗ \ldots ⊗ F_n |Ψ⟩ implies the non-existence of local operators G_i (i = 1,...,n) with |Ψ⟩ = G_i ⊗ \ldots ⊗ G_n |Φ⟩.

II. CLASSIFICATION OF ENTANGLED MANY-BODY STATES

Before using the above formalism to identify new structures within the set of multi-partite states, let us first discuss how prior knowledge can be reproduced, in order to gain some insight in this technique.

A. The bipartite case revisited

With Eq. (8) we have verified that no entangled biqubit states |Ψ_2⟩ can be created by local action on a separable state |Ψ_1⟩. To show that in a system of higher dimensional constituents a state |Ψ_3⟩ = \sqrt{λ_1}|11⟩ + \sqrt{λ_2}|22⟩ + \sqrt{λ_3}|33⟩ with three finite Schmidt coefficients can not be obtained from a state with only two finite Schmidt coefficient, i.e. a state with a Schmidt decomposition as |Ψ_2⟩, we have to consider three replicas of this quantum state. In this framework we can define the projector A_3 onto the completely antisymmetric (fermionic) subspace of a triplicate single-particle Hilbert space, i.e.

\[ A_3 = \frac{1}{3!} \sum_{i=1}^{3!} \text{sgn}(\Pi_i) \Pi_i . \]

Here ‘sgn’ denotes the signature, i.e. the function that yields \text{sgn}(\Pi_e) = 1 for any permutation \Pi_e that can be decomposed into an even number of pairwise permutations (transpositions), and \text{sgn}(\Pi_o) = -1 for any permutation \Pi_o that can be decomposed into an odd number of pairwise permutations. Let us inspect the action of A_3 on a three-replica state-vector. If the components of the three replicas are pairwise different, then the corresponding state is mapped on the completely anti-symmetric state

\[ |A⟩ = \frac{1}{\sqrt{6}} \left( |1⟩ - |2⟩ - |3⟩ + |1⟩ + |2⟩ + |3⟩ \right) \]

by A_3, i.e.

\[ A_3 |i⟩ = \frac{1}{\sqrt{6}} \text{sgn}(ijk)|A⟩ , \]

where the signature sgn of a string of numbers is defined similarly to the signature of a permutation above, i.e. \text{sgn}(123) = \text{sgn}(231) = 1 and \text{sgn}(213) = \text{sgn}(132) = -1. If i, j and k are not pairwise different, then the state-vector is mapped on the zero-vector by A_3.

Since three replicas of a state |Ψ_2⟩ contain no term with three pairwise different contributions per subsystem, the three replicas |Ψ_3⟩ are necessarily mapped on the zero-vector by the application of A_3 on either subsystem. But, the three-level entangled state |Ψ_3⟩ behaves differently: three replicas contain the component

\[ \sqrt{λ_1λ_2λ_3} |i⟩ \otimes |j⟩ \otimes |k⟩ , \]

with i ≠ j ≠ k ≠ i, that survive the application of A_3, such that

\[ A_3 \otimes I |Ψ_3⟩ \otimes = \sqrt{λ_1λ_2λ_3} \sum_{ijk} \frac{\text{sgn}(ijk)}{\sqrt{6}} |A⟩ \otimes |i⟩ \otimes |j⟩ \otimes |k⟩ \neq 0 . \]

Together with Eq. (12) this verifies that |Ψ_3⟩ cannot be obtained from |Ψ_2⟩ through local manipulation. With exactly the same argument, one can also show that a state |Ψ_N⟩ with N finite Schmidt coefficients can not be obtained from states with fewer finite Schmidt coefficients, simply invoking the projector

\[ A_N = \frac{1}{N!} \sum_{i=1}^{N!} \text{sgn}(\Pi_i) \Pi_i , \]

onto the completely antisymmetric part of \mathcal{H}_1⊗N. Doing so, one recovers the full classification of bipartite entanglement in terms of the Schmidt rank [22].
B. GHZ states vs. W states

The inequivalence of GHZ and W states can be shown in a similar manner. Rephrasing the three-qubit tangle \( \tau \) [23] in terms of permutation operators, one finds that

\[
\tau(\Psi) = 16 \sqrt{\langle \Psi | A_\tau | \Psi \rangle}^4
\]

with

\[
A_\tau = \sum_{ij} \frac{\rho^{(12)} \otimes \rho^{(34)} \otimes \rho^{(13)} \otimes \rho^{(24)}}{\mathcal{H}^{44} \mathcal{H}^{34} \mathcal{H}^{24}}
\]

(22)

where two consecutive projectors act on the four replicas of a subsystem as depicted above. The fact that \( A_\tau |\text{GHZ}\rangle^\otimes^4 \) is finite is implied by the finite tangle of the GHZ state, but, one also readily convinces oneself that

\[
A_\tau |\text{GHZ}\rangle^\otimes^4 = \frac{1}{4} |\xi_\tau\rangle
\]

(23)

where \( |\xi_\tau\rangle \) is the unique eigenvector of \( A_\tau \) associated with the unit eigenvalue. For this purpose, let us take a look at those terms in \( |\text{GHZ}\rangle^\otimes^4 \) that are not mapped on the zero vector by \( A_\tau \). The projection \( \rho^{(12)} \otimes \rho^{(34)} \) on the first subsystem requires the terms associated with the first and second replica as well as those associated with the third and fourth replica to be different. In turn, \( \rho^{(13)} \otimes \rho^{(24)} \) on the third subsystem enforces terms associated with the first (second) and third (fourth) replica to be different. The only terms in \( |\text{GHZ}\rangle^\otimes^4 \) that satisfy these requirements simultaneously are of the form

\[
|\kappa_{ij}\rangle = \left| \begin{array}{cc}
i \\ j \\
i \\ j \\
i \\ j \end{array} \right\rangle \otimes \left| \begin{array}{cc}
i \\ j \\
i \\ j \\
i \\ j \end{array} \right\rangle \otimes \left| \begin{array}{cc}
i \\ j \\
i \\ j \\
i \\ j \end{array} \right\rangle \otimes \left| \begin{array}{cc}
i \\ j \\
i \\ j \\
i \\ j \end{array} \right\rangle
\]

(24)

with \( i \neq j \) (i.e. all other terms vanish one by one under the application of \( A_\tau \)). What remains to be verified is that the term with \( i = 1, j = 2 \) does not cancel the term with \( i = 2, j = 1 \), i.e.

\[
\sum_{ij} A_\tau |\kappa_{ij}\rangle \neq 0
\]

(25)

Both state-vectors \( \left| \begin{array}{c}1 \\ 2 \\
i \\ j \end{array} \right\rangle \) and \( \left| \begin{array}{c}1 \\ 2 \\
i \\ j \end{array} \right\rangle \) are mapped on the singlet state by \( P_- \), but with different phases:

\[
P_- \left| \begin{array}{c}1 \\ 2 \\
i \\ j \end{array} \right\rangle = - P_- \left| \begin{array}{c}1 \\ 2 \\
i \\ j \end{array} \right\rangle = \frac{1}{2} \left( \left| \begin{array}{c}1 \\ 2 \\
i \\ j \end{array} \right\rangle - \left| \begin{array}{c}2 \\ 1 \\
i \\ j \end{array} \right\rangle \right) = \frac{1}{\sqrt{2}} |\chi\rangle
\]

(26)

In Eq. (24), both of the terms \( \left| \begin{array}{c}i \\ j \\
i \\ j \end{array} \right\rangle \) and \( \left| \begin{array}{c}j \\ i \\
i \\ j \end{array} \right\rangle \) appear threefold: \( \left| \begin{array}{c}i \\ j \\
i \\ j \end{array} \right\rangle \) appears in the first and second replica component of the first and second subsystem, and in the first and third replica component of the third subsystem; \( \left| \begin{array}{c}j \\ i \\
i \\ j \end{array} \right\rangle \) appears in the third and fourth replica component of the first and second subsystem, and in the second and fourth replica component of the third subsystem. That is, in both of the two cases \( i = 1, j = 2 \) and \( i = 2, j = 1 \), the application of \( A_\tau \), which is composed of 6 terms \( P_- \), yields a sixfold tensor product of \( |\chi\rangle \) with three times the negative prefactor \(-1/\sqrt{2}\), and three times the positive prefactor \(1/\sqrt{2}\). That is, in both of these cases, the same resulting prefactor \(-1/8\) is obtained, so that these two terms add up constructively, i.e. do not cancel each other.

As we discuss in the following, the \( |W\rangle \) state behaves differently, i.e. \( A_\tau |W\rangle^\otimes^4 = 0 \). To verify this, let us introduce the short hand notations \( |\omega_1\rangle = |211\rangle \), \( |\omega_2\rangle = |121\rangle \) and \( |\omega_3\rangle = |112\rangle \). Now, the quadruplication of \( |W\rangle \) is comprised of \( 3^4 \) terms, but we can easily verify that each of these terms is mapped on the zero-vector by \( A_\tau \). The projector \( \rho^{(12)} \otimes \rho^{(34)} \) acting on the replicas of the first subsystem, maps all those terms onto the zero-vector that do not contain \( |\omega_1\rangle \) exactly once in the first two replicas and exactly once in both the third and fourth replicas. Similarly, the projector \( \rho^{(13)} \otimes \rho^{(24)} \) acting on the second subsystem, maps all those terms onto the zero-vector that do not contain \( |\omega_2\rangle \) exactly once in the first two replicas and exactly once in both the third and fourth replicas. Thus, the only terms that survive the application of the first– and second– subsystem component of \( A_\tau \) read

\[
|\omega_1\rangle, |\omega_2\rangle, |\omega_1\rangle, |\omega_2\rangle \text{ and } |\omega_1\rangle, |\omega_2\rangle
\]

(27)

However, all these terms are completely symmetric in the third subsystem components, and, therefore, are mapped onto the zero-vector by \( \rho^{(13)} \otimes \rho^{(24)} \) on \( \mathcal{H}^\otimes^4 \). That is, all-together, we have verified that

\[
A_\tau |W\rangle^\otimes^4 = 0
\]

(28)

what, together with Eq. (23), verifies that GHZ states cannot be generated through local manipulation of W states. Since all reduced density matrices of both the GHZ and the W state have full rank, this, in turn, implies the in-equivalence of these two states.

C. Beyond qubits

Having identified the distinctness of GHZ and W states, we can now leave the realm of two-level systems and identify in a similar fashion inequivalent states in higher-dimensional systems. Natural generalizations of these states to three-level systems with full rank of all reduced density matrices are

\[
|GZH_3\rangle = \frac{1}{\sqrt{3}} \left( |000\rangle + |111\rangle + |222\rangle \right)
\]

(29)
such a term in the six-fold $W_3$-state, $|W_3\rangle^\otimes 6$, it necessarily would need to contain each of the terms [1], [2] and [3] exactly twice per subsystem. The only terms in $|W_3\rangle$ that contain the term [3] are [322], [232] and [223]. If a term in $|W_3\rangle^\otimes 6$ is to contain [3] exactly twice per subsystem, i.e. twice in each column in Fig. 1, it must solely consist of a six-fold tensor product of the terms [322], [232] and [223], each entering twice. Therefore any such term in $|W_3\rangle^\otimes 6$ does not contain the term [1], that is, contains no contribution of the completely anti-symmetric three-level state, and therefore is necessarily mapped to the zero vector by $A_{3\tau}$.

The sixfold $|GHZ_3\rangle$ reads $|\Psi\rangle^\otimes 6 = \sum_{i,j,k,p,q,l=1}^3 |iii\rangle \otimes |jjj\rangle \otimes |kkk\rangle \otimes |lll\rangle \otimes |ppp\rangle \otimes |qqq\rangle$, but as we discuss in the following, only a few of the overall $3^6$ terms yield a finite contribution after the application of $A_{3\tau}$. Due to $A^{(123)}$ acting on the first subsystem the first three replicas need to contain three pairwise different components, that is, we label $i$, $j$ and $k$. Because of $A^{(125)}$ acting on the second subsystem, the fifth replica needs to be in the state $|k\rangle$ since the first replicas are in the states $|i\rangle$ and $|j\rangle$. Similarly, the fourth replica needs to be in $|j\rangle$, due to the action of $A^{(134)}$ on the third subsystem. Finally, the sixth replica needs to be in state $|i\rangle$, since $A^{(256)}$ acts on the first subsystems. Consequently, all terms in $|GHZ_3\rangle^\otimes 6$ that are not mapped on the zero vector by $A_{3\tau}$, are of the form

$$ |W_3\rangle = \frac{1}{\sqrt{6}} \left( |211\rangle + |121\rangle + |112\rangle + |322\rangle + |232\rangle + |223\rangle \right). $$

The operator $A_{3\tau}$ as defined in Eq. (22) maps the quadruplication of both of these states on something finite, i.e. it is not suited to show the inequivalence of these two states. To find a suitable operator, we can resort to the projector $A_3$ onto the completely anti-symmetric component of a triplicate Hilbert space defined above in Eq. (13) in the context of bipartite systems. In terms of this projector – that we will simply denote $A$ in the remainder – we can define the generalization of $A_{3\tau}$ to

$$ A_{3\tau} = A^{(123)} \otimes A^{(456)} \otimes A^{(125)} \otimes A^{(346)} \otimes A^{(134)} \otimes A^{(256)}, $$

where the indices $ijk$ of $A^{(ijk)}$ denote the $i$-th, $j$-th and $k$-th replica of the respective single-particle Hilbert space as also illustrated in Fig. 1.

With this operator, we can explicitly verify that

$$ A_{3\tau} |W_3\rangle^\otimes 6 = 0, \text{ but } A_{3\tau} |GHZ_3\rangle^\otimes 6 = \frac{1}{4 \cdot 3^5} |\xi\rangle, $$

where $|\xi\rangle$ is the unique eigenvector associated with the unit eigenvalue of $A_{3\tau}$.

As discussed in the context of Eq. (18), only three-replica state-vectors with pairwise different components survive the application of $A$. Consequently, if there was

$$ |\kappa_1\rangle = \begin{pmatrix} i \\ j \\ k \end{pmatrix}, \text{ and } |\kappa_2\rangle = \begin{pmatrix} j \\ k \\ i \end{pmatrix}. $$

In the second subsystem, $A$ acts on replicas 1, 2 and 5 and on replicas 3, 4 and 6. In these two terms, we have

$$ |\kappa_3\rangle = \begin{pmatrix} i \\ j \\ k \end{pmatrix}, \text{ and } |\kappa_4\rangle = \begin{pmatrix} k \\ j \\ i \end{pmatrix}. $$

In the third subsystem, $A$ acts on replicas 1, 3 and 4 and on replicas 2, 5 and 6. In these two terms, we have

$$ |\kappa_5\rangle = \begin{pmatrix} i \\ j \\ k \end{pmatrix}, \text{ and } |\kappa_6\rangle = \begin{pmatrix} j \\ k \\ i \end{pmatrix}. $$
The states |$\kappa_1$⟩, |$\kappa_2$⟩, |$\kappa_3$⟩ and |$\kappa_6$⟩ have the same signature, whereas the states |$\kappa_4$⟩ and |$\kappa_5$⟩ have the opposite signature. But, since both signatures appear an even number of times, the overall prefactor is always positive, so that the different terms of Eq. (33) add up constructively after the application of $A_{3\tau}$. With this, we have shown the inequivalence of the $|W_3⟩$ and the $|GHZ_3⟩$ state.

However, these two states certainly do not define the only classes of entanglement in tripartite three-level systems. For example, there is the Aharonov state $|\chi⟩ = 1/\sqrt{6}((|123⟩ + |231⟩ + |312⟩) - (|132⟩ + |213⟩ + |321⟩))$ [24], that also survives $A_{3\tau}$

$$A_{3\tau}|\chi⟩^{\otimes6} = \frac{1}{2 \cdot 6^3} |\xi⟩.$$ (37)

That is, whereas this shows the inequivalence of $|W_3⟩$ and Aharonov states, we need to invoke another symmetry operation to show the inequivalence of $|GHZ_3⟩$ and Aharonov states. For this purpose, we use the projector

$$S = \frac{1}{3!} \sum_{i=1}^{3!} \Pi_i$$ (38)
on to the completely symmetric (bosonic) part of a triplicate single particle Hilbert space. Similarly to Eq. (18) above, $S$ maps a tri-replica state-vector with components |1⟩, |2⟩ and |3⟩ on a completely symmetric state

$$|S⟩ = \frac{1}{\sqrt{6}} \left(\left|\begin{array}{ccc} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{array}\right| + \left|\begin{array}{ccc} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{array}\right| + \left|\begin{array}{ccc} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{array}\right| + \left|\begin{array}{ccc} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{array}\right| + \left|\begin{array}{ccc} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{array}\right| \right)$$ (39)

With the help of the operators $A$ and $S$, each acting on three replicas of one subsystem, we can now define the operator

$$\frac{A}{n_{1^{3}}} \otimes \frac{A}{n_{2^{3}}} \otimes \frac{S}{n_{3^{3}}}.$$ (40)

that allows the discrimination of $|GHZ_3⟩$ and Aharonov states. The triplication of the $|GHZ_3⟩$ state reads $|GHZ_3⟩^{\otimes3} = \sum_{i,j,k=1}^{6} |iij⟩ \otimes |jjj⟩ \otimes |kkk⟩$. All terms in this sum with $i$, $j$, and $k$ not mutually different are mapped on the zero-vector by $A$, since there is no completely anti-symmetric three-particle state in less than three dimensions. The only terms that are not mapped to zero are of the form

$$|\begin{array}{c} i \\ j \\ k \end{array}| \otimes |\begin{array}{c} i \\ j \\ k \end{array}| \otimes |\begin{array}{c} i \\ j \\ k \end{array}|,$$ (41)

with $i, j, k$ pairwise different. The application of $A \otimes A \otimes S$ on these terms yields

$$A \otimes A \otimes S |\begin{array}{c} i \\ j \\ k \end{array}| \otimes |\begin{array}{c} i \\ j \\ k \end{array}| \otimes |\begin{array}{c} i \\ j \\ k \end{array}| = \frac{1}{\sqrt{6}} \text{sgn}^2(ijk) |A⟩ \otimes |A⟩ \otimes |S⟩.$$ (42)

Since all terms enter with a positive prefactor, i.e. add constructively, the application of $A \otimes A \otimes S$ on the triplicated $|GHZ_3⟩$ state results in something finite:

$$A \otimes A \otimes S |GHZ_3⟩^{\otimes3} = \frac{1}{9\sqrt{2}} |A⟩ \otimes |A⟩ \otimes |S⟩.$$ (43)

This is different for the Aharonov state: let us review the terms of $|\chi⟩^{\otimes3}$ in which the terms associated with the three replicas of the first and second subsystems simultaneously have pairwise different terms, since only those survive the projection $A \otimes A$ onto the completely anti-symmetric parts. Labelling the replica components of the first subsystems $|i⟩$, $|j⟩$ and $|k⟩$ (pairwise different), we obtain the two different terms:

$$|\chi_a⟩ = |\begin{array}{c} i \\ j \\ k \end{array}| \otimes |\begin{array}{c} j \\ i \\ k \end{array}| \otimes |\begin{array}{c} k \\ i \\ j \end{array}|,$$ (44)

and

$$|\chi_b⟩ = |\begin{array}{c} i \\ j \\ k \end{array}| \otimes |\begin{array}{c} k \\ i \\ j \end{array}| \otimes |\begin{array}{c} j \\ i \\ k \end{array}|.$$ (45)

All factors associated with a single replica component in $|\chi_a⟩$ have the same signature, and the same also holds for all single replica components in $|\chi_b⟩$. However, the signature of the $|\chi_a⟩$-components is different from the signature of the $|\chi_b⟩$-components, and since the number of replicas is odd, $|\chi_a⟩$ enters $|\chi⟩^{\otimes3}$ with a different sign than $|\chi_b⟩$. Similarly, the first subsystem-components of $|\chi_a⟩$ have the same signature as the second subsystem components, and the same holds true for $|\chi_b⟩$. That is, independently of what this signature is, the contribution from the application of $A$ on the first subsystem is canceled by the contribution of $A$ on the second subsystem, i.e.

$$A |\begin{array}{c} i \\ j \\ k \end{array}| \otimes A |\begin{array}{c} j \\ i \\ k \end{array}| = A |\begin{array}{c} i \\ j \\ k \end{array}| \otimes A |\begin{array}{c} k \\ i \\ j \end{array}| = \frac{1}{6} |A⟩ \otimes |A⟩.$$ (46)

Since, however, $|\chi_a⟩$ and $|\chi_b⟩$ enter $|\chi⟩^{\otimes3}$ with a different sign, and the third subsystem components are always mapped on the symmetric state $|S⟩$, these terms cancel after the application of $A \otimes A \otimes S$, so that

$$A \otimes A \otimes S |\chi⟩^{\otimes3} = 0.$$ (47)

Quite surprisingly, we can also see that a biseparable state $|Ψ_{BS}⟩ = \sum_{i=1}^{6} \sqrt{λ_i} |i⟩ |ϕ⟩$, with a bipartite entangled component of full Schmidt rank, can not be obtained from the Aharonov state, since the application of $A \otimes A \otimes S$ yields

$$A \otimes A \otimes S |Ψ_{BS}⟩^{\otimes3} = (λ_1 λ_2 λ_3)^{3/2} \frac{1}{6\sqrt{6}} |A⟩ \otimes |A⟩ \otimes |ϕ⟩.$$ (48)
i.e. something finite. If $\lambda_3$ vanishes, i.e. if the state exhibits only two-qubit entanglement, this component vanishes, and then, indeed, the biseparable state can be obtained from the Aharonov state. That is, whereas in the case of qubits tripartite entanglement is in general superior to bipartite entanglement, i.e. all bipartite qubit entangled states can be obtained from both W and GHZ entangled states, this is no longer true for higher dimensional systems.

III. OUTLOOK

The above exemplary cases of inequivalent tri-partite states of three-level systems and the reproduction of prior classification with one and the same recipe, raise confidence that Eq. (12), as originally established in [15], will eventually provide a systematic understanding of the general structure of multi-partite entanglement. Whereas we do not rigorously know whether Eq. (12) is sufficient to provide an exhaustive characterization of all inequivalent classes of multi-partite entangled states, the following arguments suggest that this indeed might be the case. On the one hand, the rapid growth of the number of different permutations with the number $M$ of a state’s multiplicity, which comes in hand with a growing variety of independent terms in Eq. (10), results in an enhanced freedom to build suitable operators $A$. On the other hand, any quantitative entanglement property of a quantum state, i.e. an entanglement measure, can be expressed in form of expectation values of these operators $A$ as well [21]. Whereas we focussed here exclusively on the classification of multi-partite entanglement, our presently described framework bears the potential for much broader applications. Consider for example the question of filtering [23] or entanglement distillation [26], where local Kraus operators $F_i$ that map a mixed state $\Sigma$ to a given pure state $|\Psi\rangle$, i.e. $F_1 \otimes \ldots \otimes F_N |\Psi\rangle = |\Psi\rangle$ are sought. Here, $\Sigma$ can be the state of an $N$-partite quantum system, in the case of filtering, or, it can be a multiple tensor product of the to-be-distilled state $\rho$, i.e. $\Sigma = \rho^\otimes N$, where $q$ subsystems define a single logical subsystem as in regular distillation. In this context, a relation $A\Sigma^\otimes N A^\dagger = 0$, but $A|\Psi\rangle^\otimes N \neq 0$ proves the non-existence of a filtering- or distillation-protocol.

Very naturally, the present framework also facilitates the construction of entanglement measures that target very specific entanglement properties: considering expectation values of the symmetry operations $A$ defined here, one arrives at scalar functions – just like the tangle [23] – that can be defined in terms of Eq. (22), or, generalized concurrences [27], that can be defined in terms of $A$. Once verified the equality $A|\Phi\rangle^\otimes N = 0$, one assures that the function $\langle \Psi| ^\otimes N A |\Psi\rangle$ is completely insensitive to entanglement of the class of $|\Phi\rangle$, what enables the construction of entanglement measures that access specific entanglement properties.

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