Nearly Divergence of Correlation Length and Perturbation Spectrum in String Gas Cosmology

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Recently, it has been shown in Ref. [1] that the string thermodynamic fluctuation may lead to a scale invariant spectrum of scalar metric perturbation. However, its realization is still in study. In this note we suppose that the correlation length of metric perturbation, which is proportional to the sound speed, might be nearly divergent at the critical point of phase transition. In this case we find that the string gas mechanism responsible for the generation of primordial perturbation may be applied well.

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In string gas cosmology, see Refs. [4] and [5] for recent reviews, it is assumed that the universe starts in a Hagedorn phase, in which the universe is initially in thermal equilibrium at a temperature close to the Hagedorn temperature, the limiting temperature of perturbative string theory. In this case, by using the tools of string thermodynamics it was showed that the nearly scale invariant spectrum of cosmological fluctuations responsible for the structure formation of observable universe may be obtained, see Ref. [6] for the tensor perturbation and also Refs. [7, 8] for more details.

However, to make this seeding mechanism feasible, it seems that the dilaton need to be fixed, which may be argued with the strong coupling Hagedorn phase [6]. If the dilaton is not fixed, the terms related to the dilaton running will dominate the (00) equation of metric perturbation \( \Phi \). In this case instead scale invariant spectrum one will obtain a Poisson spectrum \([9]\). When the dilaton is fixed, the perturbation equations are those of Einstein gravity. In this case, the perturbation equation \( k^2 \Phi \sim \delta \rho \) may be used only when \( k > h \), in which \( k \) and \( h \) are the comoving wave number and comoving Hubble parameter in the Einstein frame respectively. However, in Ref. [5], for fixed dilaton, it seems that the scales of metric perturbation interested today are generally super Hubble radius during Hagedorn phase, i.e. \( k < h \) [10].

The metric perturbation can be driven by the matter fluctuation only on scale smaller than the Hubble radius, while on super Hubble scale the metric perturbation is dominated since the matter oscillations have frozen out. Thus one is not able to compute the matter fluctuations on such scales and to then use them to induce the metric fluctuations. Thus in this case the calculations of metric perturbation seems be controversial [10]. However, one may relax these controversies by assuming that the strong Hagedorn phase lasts sufficiently long [9] or invoking a bounce cosmology [11] and also [12].

In this note, we propose a different possibility. We suppose that the correlation length \( l \sim c_s h^{-1} \) of metric perturbation might be nearly diverged at the critical point of phase transition, where \( c_s \) denotes the sound speed of metric perturbation. When \( T \rightarrow T_c \), where \( T_c \) is the critical temperature of phase transition, since the change of \( h \) is generally expected to be moderate during phase transition \(^1\), the divergence of correlation length of metric perturbation suggests that \( c_s(T) \) will rapidly increase to infinity with \( T \), see Fig.1. Based this supposition we find those controversial issues relevant to the string gas mechanism responsible for the generation of primordial perturbation can be solved well. For some ansatz of correlation length, we calculate the spectrum of scalar metric perturbation by using the tools of string thermodynamics, as in Ref. [1]. The spectral index generally has a moderate red tilt, which may be consistent with recent observations well. For an illustration we firstly assume that the background dilaton velocity is negligible. We will relax this assumption later. Besides, here we will not involve relevant contents with the stability of moduli, which may be seen in Refs. [11, 12] in the string gas cosmology.

When the dilaton is fixed, the background and perturbation equations are those of Einstein gravity. Though the nearly divergence of sound speed can hardly be understood in Einstein gravity and is generally expected to have a profound origin, for a phenomenological discussion we may implemented it by e.g. endowing the Einstein action with an added term

\[
\sim \int d^4x \sqrt{-g} c_s^2 R^{(3)},
\]

where \( R^{(3)} \) is the Ricci curvature described by the space component \( g_{ij} \) of metric \( g_{\mu \nu} \) and its corresponding affine connections. In this sense \( R^{(3)} \) is actually the intrinsic spatial curvature on constant time slicing. The factor \( c_s^2 \) only depends on the temperature, when \( T \sim T_c \), it is required to be very large, while \( T \ll T_c \) we expect that the Einstein gravity should be resumed, thus it should approach 0. It seems that this term breaks the Lorentz invariance, however, it can be recovered after the transition, i.e. \( T \ll T_c \), which is consistent with present

\(^1\) see Refs. [11, 12] for a different case in which the bounce is introduced during phase transition, which actually also corresponds to one of the cases with diverged correlation length since \( h = 0 \) at some epoch of the bounce.
We may define the effective comoving wave number as $c_s k$, which is Poisson like, but is in relativistic sense. We can obtain Eq. (3) completely contributes to the introduction of added term to Einstein action, since it brings a nearly diverged sound speed when the temperature approaches the critical point. Thus we have

$$P_\Phi(k) \simeq \frac{a^2 G^2}{c_s k} P_{\delta \rho}(k) = \frac{a^2 G^2}{c_s k} \left( \frac{\delta \rho}{c_s} \right)^2 > R = a/(c_s k),$$

where $R$ is the size of region in which the fluctuations are calculated. In Eq. (4), it is obviously seen that we replace $R = a/k$ used in Ref. [1] with $R = a/(c_s k)$. The reason is in the following. What we suppose here is only the nearly divergence of correlation length of metric perturbation. That of matter fluctuation is still limited by Hubble scale during phase transition. Thus in principle one need to calculate the fluctuations of the energy momentum tensor of stringy matter on various length $R$, up to the physical Hubble scale $a/h$. However, here though the physical wavelength $a/k$ of matter fluctuations required to induce the metric perturbation at present observable scale are sub sound horizon scale, they are generally super Hubble scale, i.e. $k < h$, see ‘A’ mode denoted by the red solid lines in Fig.1, thus in this case we can not deduce the metric perturbation by calculating the matter fluctuation, since in super Hubble scale the oscillating of matter fluctuations are freeze. However, after we introduce the nearly divergent sound speed, the effective physical wavelength of metric perturbation will obtain a strong suppression $\sim 1/c_s$ and become $a/(c_s k)$, which may be sub Hubble scale well, see ‘A’ mode denoted by the red dashed lines in Fig.1. Thus in this case we may take $R = a/(c_s k)$ as length scale to calculate the matter fluctuations and then use them to deduce the metric perturbations by Eq. (4), up to $a/h$.

This can also be explained in another perspective. We may define the effective comoving wave number as $\omega = c_s k$, which is rapidly decreased during phase transition due to the change of $c_s$. Thus the evolution of comoving wavelength $\sim 1/c_s$ of corresponding mode can be much faster than that of the Hubble radius, which will make relevant mode be able to leaving the horizon during phase transition, see ‘A’ mode in Fig.1. In this case initially the modes with the effective comoving wavelength $1/\omega$ are well in the Hubble horizon, thus we may calculate the matter fluctuation with length scale $R = a/\omega = a/(c_s k)$ and then deduce the metric perturbation by Eq. (4). This explanation correspond to that in the perspective of the effective comoving wave number. The similar analysis has appeared in Ref. [17]. While in the perspective of the correlation length or sound horizon in last paragraph, the comoving wavelength of ‘A’ mode in Fig.1 is constant during phase transition, with the decreasing of sound speed it will leave the sound horizon and become the primordial perturbation. Note that the time when ‘A’ mode leaves the sound horizon, i.e. $k = h/c_s$, corresponds to that when ‘A’ mode leaves the Hubble horizon, i.e. $c_s k = h$, thus in this sense both perspectives are actually equivalent, which can be seen in Fig.1.
During radiation domination, the energy is in the radiative degrees of freedom, which correspond to the momentum modes of strings. But when the temperature is about $T_c$, it may be expected that the oscillatory and winding modes of strings can be excited, which will contribute most of energy in the string gas. Thus in principle one is able to compute the spectrum of matter fluctuation by using the result of closed string thermodynamics, which is required to source the metric perturbation by Eq. (4). Following [1], the perturbation of energy density of closed string modes in the thermal equilibrium inside an arbitrary volume $\sim R^3$ is given by

$$<\delta \rho^2> = \frac{R^2}{R_p^6}c_V,$$  \hspace{1cm} (5)

where the specific heat

$$c_V \simeq \frac{R^2T^4}{T(T_c - T)},$$  \hspace{1cm} (6)

which scales as $R^2$. Eq. (4) indicates that the fluctuation of stringy matter is a Poisson spectrum $\sim k^4$. This result is a key feature of string thermodynamics, which was derived in Ref. [18] and holds in the case of three large dimensions with the topology of a torus. We substitute Eq. (5) into Eq. (4), and then can obtain

$$\mathcal{P}_\Phi \simeq \frac{T^4}{m_p^4} \left( \frac{T}{T_c - T} \right),$$  \hspace{1cm} (7)

where $G = m_p$ has been used and the numerical factor $\mathcal{O}(1 - 100)$, which depends on the rescale of Planck scale $m_p$, has been neglected. Thus one can see that the Poisson spectrum of stringy matter induces a scale invariant spectrum of metric perturbation $\Phi$ by Eq. (4), which is the same as the result obtained in Ref. [1]. The difference is that here the metric perturbations calculated are in super Hubble scale, i.e. $k < h$, but sub sound horizon scale, i.e. $k > h/c_s$. During the transition in which the winding modes of strings decay into radiation, we expect that $c_s$ decreases rapidly, as is illustrated in Fig. 1. In this case the metric perturbation can exit the sound horizon, and after its leaving, it will quickly freeze, since when $c_s k < h$, which means that the effective wavelength of metric perturbation in perspective of effective comoving wave number is larger than the horizon, the oscillating of matter fluctuations has frozen. The spectrum of curvature perturbation $\xi$ in comoving supersurface $P_\xi \simeq P_\Phi$ up to a factor with order one, which is constant in the super horizon scale. Thus the spectrum of the comoving curvature perturbation can be nearly scale invariant and its amplitude can be calculated at the time when the perturbation exits the sound horizon, i.e. $k = h/c_s$, which gives the value of $T$ in Eq. (7) at the sound horizon crossing.

The phase transition is not generally instantaneous, which will lead to a tilt in the spectrum. This tilt depends on the change of sound speed, since the change of sound speed determines the evolution of sound horizon and so the sound horizon crossing time of perturbation with some given wavelength $\sim k^{-1}$. In principle to obtain the tilt of spectrum we need to know the detailed evolution of sound speed with the temperature. However, since we lack for the detailed knowledge of phase transition, here we will appeal to some interesting ansatz of $c_s$, which in some sense might also help to our understanding for phase transition. We firstly take

$$c_s = \left( \frac{T_c}{T_c - T} \right)^p$$  \hspace{1cm} (8)

as an attempt, where $p > 0$ is a constant, and $p \to \infty$ corresponds to an instantaneous transition. This ansatz in some sense may be analogous to the case in condensed matter physics, in which when the temperature approaches the critical point of phase transition the correlation length of order parameter diverges. For Eq. (8), when $T \to T_c$, the sound speed diverges, while $T \sim T_c \ll T_c$, where ‘e’ denotes the value at the end of transition, we have $c_s = 1$, which means the end of transition. In this time the perturbation equation is that with the Einstein gravity. The sound horizon crossing requires $k = h/c_s$, thus we can obtain

$$\frac{k}{k_s} \simeq \left( \frac{T_c}{T_c - T} \right)^{-p},$$  \hspace{1cm} (9)

where we have neglected the change of $h$ during phase transition, since the change of $h$ is much smaller than that of $c_s$. When we include the change of $h$, where we take $a \sim t^n$ and $n \sim \mathcal{O}(1/2)$ is a constant, there will be a factor $(T_c/T)^{(n-1)/n}$ before the right hand term of Eq. (9), which is negligible when being compared to that of $c_s$. We define

$$N \equiv \ln \left( \frac{k_s}{k} \right),$$  \hspace{1cm} (10)

which measures the efolding number of mode with some scale $\sim k^{-1}$ which leaves the sound horizon before the end of transition, and thus $k_s$ means the last mode to be generated, see Fig. 1. When taking the comoving Hubble parameter $h = h_0$, where the subscript ‘0’ denotes the present time, we generally have $N \sim 50$, which is required by observable cosmology. From Eq. (9), we can see that when $T \to T_c$, $k_s/k$ nearly approaches to infinity, thus the efolding number is actually always enough as long as the initial volume of early universe is large enough. By using Eq. (9), the spectrum of metric perturbation can be rewritten as

$$\mathcal{P}_\Phi \simeq \frac{T^4}{m_p^4} \left( \frac{k}{k_s} \right)^{-\frac{n}{p}} \left[ 1 - \left( \frac{k}{k_s} \right)^{\frac{n}{p}} \right]^{-1}.$$  \hspace{1cm} (11)

Thus the amplitude is approximately $T_c^4/m_p^4$ and the spectral index is given by

$$n_s - 1 = -\frac{1}{p} \cdot \frac{1}{1 - e^{-N/p}},$$  \hspace{1cm} (12)
where Eq. (10) has been used. From Eq. (12), we can see that for fixed efolding number the spectral index is only determined by the critical exponent $p$. To obtain the red tilt required by the observations [19], i.e. $n_s - 1 \simeq -0.05$, for $N \simeq 50$, it seems that we need $p \simeq 20$. In fact we also calculate the factor which is relevant to the change of $h$ and has been neglected in Eq. (9), and find that it only contributes a term $\sim 1/p^2$, which is second effect. Thus the approximation done in Eq. (10) is consistent.

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action rate of string winding modes with the running of dilaton is generally more rapidly changed than the physical Hubble parameter $c_s^2$. This means that when the dilaton is running, the gas of winding strings will quickly fall out of the equilibrium. However, here due to the introduction of the term $\sim c_s^2$, the causal structure of spacetime has actually been changed. Thus in this case it seems be required to rephrase the thermal equilibrium condition. This is beyond the scope of this note, however, which is worthy of further study.

In conclusion, we show that when the correlation length of metric perturbation, which is proportional to the sound speed, at the critical point of phase transition is nearly divergence, the string gas mechanism of the generation of primordial perturbation may be applied well. Though here we are constrained to the case with the added term (1), our work may be more general. The only requirement is that the correlation length of metric perturbation is nearly diverged at the critical temperature, which is significant to obtain a causal structure responsible for the generation of primordial perturbation and a Poisson like equation of metric perturbation. In principle, this work may be applied to any cases in which the Poisson like matter or energy fluctuation, like Eq. (5), is required to induce the scale invariant spectrum of metric perturbation by a Poisson like equation of metric perturbation, like Eq. (3). Finally, it should be pointed out that the phase transition that the string winding modes become dominated is still a subject in development and only partially understood at present. Thus this study seems be slightly speculative, however, it might be helpful for understanding the generation of primordial perturbation based on the string thermodynamics and also the physics of phase transition.

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[1] A. Nayeri, R.H. Brandenberger, C. Vafa, Phys. Rev. Lett. 97 (2006) 021302.
[2] R. Brandenberger, C. Vafa, Nucl. Phys. B316, 391 (1989).
[3] A.A. Tseytlin, C. Vafa, Nucl. Phys. B372, 443 (1992).
[4] T. Battefeld, S. Watson, hep-th/0510022.
[5] R.H. Brandenberger, hep-th/0701111, hep-th/0701157.
[6] R.H. Brandenberger, A. Nayeri, S.P. Patil, C. Vafa, hep-th/0604126.
[7] A. Nayeri, hep-th/0607073.
[8] R.H. Brandenberger, A. Nayeri, S.P. Patil, C. Vafa, hep-th/0608121.
[9] R.H. Brandenberger, S. Kanno, J. Soda, D.A. Easson, J. Khoury, P. Martineau, A. Nayeri, S. Patil, hep-th/0608186.
[10] N. Kaloper, L. Kofman, A. Linde, V. Mukhanov, JCAP 0610 (2006) 006.
[11] T. Biswas, R. Brandenberger, A. Mazumdar, W. Siegel, hep-th/0610274.
[12] S. Arapoglu, A. Karakci, A. Kaya, hep-th/0611193.
[13] T. Biswas, A. Mazumdar and W. Siegel, JCAP 0603, 009 (2006).
[14] S.P. Patil, R.H. Brandenberger, hep-th/0401037, hep-th/0502069.
[15] R.H. Brandenberger, Y.K. Cheung, S. Watson, hep-th/0501032.
[16] V.F. Mukhanov, H.A. Feldman, R.H. Brandenberger, Phys. Rept. 215, 203 (1992).
[17] Y.S. Piao, gr-qc/0609071.
[18] N. Deo, S. Jain, O. Narayan, C.I. Tan, Phys. Rev. D45, 3641 (1992).
[19] D.N. Spergel et al. (WMAP Collaboration), astro-ph/0603449.
[20] S. Watson, R.H. Brandenberger, hep-th/0312097.
[21] R. Danos, A.R. Frey, A. Mazumdar, Phys. Rev. D70, 106010 (2004).
[22] R. Easther, B.R. Greene, M.G. Jackson, D. Kabat, JCAP 0502, 009 (2005).
[23] Y.I. Takamizu, H. Kudoh, hep-th/0607231.