FREE ROTA–BAXTER ALGEBRAS AND ROOTED TREES

KURUSCH Eebrahi-M-Fard AND LI GUO

ABSTRACT. A Rota–Baxter algebra, also known as a Baxter algebra, is an algebra with a linear operator satisfying a relation, called the Rota–Baxter relation, that generalizes the integration by parts formula. Most of the studies on Rota–Baxter algebras have been for commutative algebras. Two constructions of free commutative Rota–Baxter algebras were obtained by Rota and Cartier in the 1970s and a third one by Keigher and one of the authors in the 1990s in terms of mixable shuffles. Recently, noncommutative Rota–Baxter algebras have appeared both in physics in connection with the work of Connes and Kreimer on renormalization in perturbative quantum field theory, and in mathematics related to the work of Loday and Ronco on dendriform dialgebras and trialgebras.

This paper uses rooted trees and forests to give explicit constructions of free noncommutative Rota–Baxter algebras on modules and sets. This highlights the combinatorial nature of Rota–Baxter algebras and facilitates their further study. As an application, we obtain the unitarization of Rota–Baxter algebras.

1. Introduction

We construct the free Rota–Baxter algebra on a set \(X\) in terms of angularly decorated rooted trees with \(X\) as the decoration set. We also consider the more general case of free objects on modules. As an application, we prove the existence and uniqueness of the unitarization of Rota–Baxter algebras.

A Rota–Baxter algebra (also known as a Baxter algebra) is an associative algebra \(R\) with a linear endomorphism \(P\) satisfying the Rota–Baxter relation:

\[
P(x)P(y) = P(P(x)y + xP(y) + \lambda xy), \quad \forall \ x, y \in R.
\]

Here \(\lambda\) is a fixed element in the base ring and is sometimes denoted by \(-\theta\). The relation was introduced by the mathematician Glen E. Baxter \([7]\) in his probability study, and was popularized mainly by the work of G.-C. Rota \([54, 55, 56]\) and his school.

Note that the Rota–Baxter relation \([11]\) is defined even if the binary operation is not associative. In fact, such a relation for Lie algebras was introduced independently by Belavin and Drinfeld \([8]\), and Semenov-Tian-Shansky \([58]\) in the 1980s, under the disguise of \(r\)-matrices, of the (modified) classical Yang–Baxter equation, named after the physicists Chen-ning Yang and Rodney Baxter. Recently, there have been several interesting developments of Rota–Baxter algebras in theoretical physics and mathematics, including quantum field theory \([12, 13, 43, 44, 51]\), associative Yang–Baxter equations \([1, 2]\), shuffle products \([15, 36, 37]\), operads \([4, 15, 19, 45, 46]\), Hopf algebras \([6, 18]\), combinatorics \([33]\) and number theory \([24, 34, 39, 52]\). The most prominent of these is the work of Connes and Kreimer in their Hopf algebraic approach to renormalization theory in perturbative quantum field theory \([12, 13]\), continued in \([17, 23, 25, 26]\).

Our goal in this paper is to give an explicit construction of free noncommutative Rota–Baxter algebras in terms of rooted trees. To help put this study in perspective, we briefly review the interesting development of the commutative case. Cartier \([10]\) pointed out over
thirty years ago “The existence of free (Rota–)Baxter algebras follows from well-known arguments in universal algebra but remains quite immaterial as long as the corresponding word problem is not solved in an explicit way as Rota was the first to do.” Both Rota’s aforementioned construction [54] and the construction of Cartier himself [10] dealt with free commutative Rota–Baxter algebras. Later, a third construction was obtained by the second named author and Keigher [36, 37] as a generalization of shuffle product algebras.

These constructions of free commutative Rota–Baxter algebras have important implications. For example, Rota [55, 56] applied his construction to give a proof of the celebrated Spitzer identity [57, 26] by relating it to Waring’s identity, another basic formula in combinatorics. The product in Cartier’s paper [10] is readily seen to be the same as the one by Ehrenborg [27] for monomial quasi-symmetric functions and more recently by Bradley [9] to explicitly describe stuffles and q-stuffles for multiple zeta values. Furthermore, the mixable shuffle product in the construction of [36] appeared also in the work of Goncharov [30] to study motivic shuffle relations and the work of Hazewinkel [40] on overlapping shuffles. In [18], the mixable shuffle product is shown to be the same as Hoffman’s quasi-shuffle product [42] which has played a fundamental role in the study of algebraic relations among multiple zeta values. There is also a description [3, 28, 48] of quasi-shuffles in terms of piecewise linear paths (Delannoy paths).

Our consideration of the noncommutative case has motivations beyond a simple pursuit of generalization. In the algebraic framework of Connes and Kreimer [12, 13] for renormalization in quantum field theory, a regularized Feynman rule is viewed and studied as an algebra homomorphism from their Hopf algebra of Feynman diagrams to a Rota–Baxter algebra associated to the renormalization scheme. The renormalization and counter term for the Feynman rule are derived from the algebraic Birkhoff decomposition. In [26], the algebraic Birkhoff decomposition and the renormalization are shown to follow from the Atkinson decomposition and the Spitzer’s identity in a noncommutative Rota–Baxter algebra. Since the Rota-Baxter algebra varies with the choice of a quantum field theory and renormalization scheme, it is desirable to investigate universal or free Rota-Baxter algebras.

In a more theoretical context, there have been quite strong interests lately in possible noncommutative generalizations of shuffles and quasi-shuffles (that is, mixable shuffles). From the connection of these shuffles with free commutative Rota–Baxter algebras mentioned above, such noncommutative generalizations should be related to free noncommutative Rota–Baxter algebras. Indeed one such generalization is the Hopf algebra of planar rooted trees of Loday and Ronco [49] and we have shown in [21] that this algebra canonically embeds into a free noncommutative Rota–Baxter algebra. The tree construction of free Rota–Baxter algebras obtained in this paper make this embedding a even more natural tree-to-tree embedding. In fact, such an embedding has been our motivation to achieve a tree interpretation of free Rota–Baxter algebras.

It is also our hope that our explicit constructions of the free Rota–Baxter algebras here will lead to further studies of Rota–Baxter algebras. Indeed, some of such studies [5, 38] have already been carried out concurrently with the writing of this paper. To compare with these and other related papers [21, 22], we note that there are different types of free Rota–Baxter algebras obtained from the adjoint functors of the forgetful functors from the category of unitary Rota–Baxter algebras to the categories of sets, modules, and algebras. They give rise to free Rota–Baxter algebras generated by (or on) a set, a module or an algebra. Further, by replacing unitary algebras by nonunitary algebras, we get more forgetful
functors and their adjoint functors. We summarize these categories and forgetful functors in the following diagram.

Unitary Rota-Baxter algebras $\longrightarrow$ Unitary Algebras $\longrightarrow$ Modules $\longrightarrow$ Sets

Nonunitary Rota-Baxter algebras $\longrightarrow$ Nonunitary Algebras $\longrightarrow$ Modules $\longrightarrow$ Sets

The distinction between unitarity and nonunitarity for a Rota–Baxter algebra is more significant than for an associative algebra, because of the involvement of the Rota–Baxter operator. In fact, it is with the help of our constructions of unitary and nonunitary free Rota–Baxter algebras that we prove the existence of unitarization of Rota–Baxter algebras.

In [21] the first construction of free Rota–Baxter algebras on another algebra were obtained in terms of bracketed words (called Rota–Baxter words). In the present paper we consider free Rota–Baxter algebras on a module and on a set in terms of rooted trees and forests. In [5], free Rota–Baxter algebras are also constructed in terms of decorated rooted trees. It considers the singleton generating set while we consider any generating set. The forms of the types of trees and decorations in the two papers are different with [5] using rooted tree with numerical decorations on the vertices and angles while us using rooted forests with angles decorated by the generating set or module. Also in [5] the Rota–Baxter algebras are constructed on the decorated trees while in our paper Rota–Baxter algebras are defined on forests without decoration and then are extended to forests with decorations. Another related paper is [38] where enumeration, generating functions and algorithms of bracketed words in free Rota–Baxter algebras were studied. These aspects, in terms of trees and other combinatorial objects, were also considered in [31, 35].

This paper can be summarized by the following diagram of Rota–Baxter algebras.

In Section 2 we will consider the set of planar rooted forests $\mathcal{F}$ and its subset $\mathcal{F}^0$ of ladder-free forests, and the corresponding free $k$-modules $k\mathcal{F}$ and $k\mathcal{F}^0$ over a commutative unitary ring $k$. We equip these two modules with a Rota–Baxter algebra structure (Theorem 2.3 and Proposition 2.4). By decorating angles of the forests in these Rota–Baxter algebras by elements of a module $M$, we construct in Section 3 the free unitary (resp. nonunitary) Rota–Baxter algebra $\mathbb{III}^{NC}(M)$ (resp. $\mathbb{III}^{NC, 0}(M)$) on $M$ in Theorem 3.4 (resp. Theorem 3.6). By taking $M = kX$ for a set $X$, we obtain free Rota–Baxter algebras on a set $X$ in Section 3.4 and display a canonical basis in the form of angularly decorated forests (Theorem 3.8).

As an application of these free Rota–Baxter algebras, the unitarization of Rota–Baxter algebras is obtained in Section 4.

**Notations:** In this paper, $k$ is a commutative unitary ring. By a $k$-algebra we mean a unitary algebra over the base ring $k$ unless otherwise stated. The same applies to Rota–Baxter algebras. For a set $X$, let $kX$ be the free $k$-module $\bigoplus_{x \in X} kx$ generated by $X$. If $X$
is a semigroup (resp. monoid), $kX$ is equipped with the natural nonunitary (resp. unitary) $k$-algebra structure.

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2. **The Rota–Baxter algebra of planar rooted forests**

We first obtain a Rota–Baxter algebra structure on planar rooted forests and their various subsets. This allows us to give a uniform construction of free Rota–Baxter algebras in different settings in §3. For other variations of this construction, see [5, 22, 23].

2.1. **Planar rooted forests.** For the convenience of the reader and for fixing notations, we recall basic concepts and facts of planar rooted trees. For references, see [14, 59].

A free tree is an undirected graph that is connected and contains no cycles. A **rooted tree** is a free tree in which a particular vertex has been distinguished as the root. Such a distinguished vertex endows the tree with a directed graph structure when the edges of the tree are given the orientation of pointing away from the root. If two vertices of a rooted tree are connected by such an oriented edge, then the vertex on the side of the root is called the parent and the vertex on the opposite side of the root is called a child. A vertex with no children is called a leaf. By our convention, in a tree with only one vertex, this vertex is a leaf, as well as the root. The number of edges in a path connecting two vertices in a rooted tree is called the **length** of the path. The **depth** $d(T)$ (or **height**) of a rooted tree $T$ is the length of the longest path from its root to its leaves. A **planar rooted tree** is a rooted tree with a fixed embedding into the plane.

There are two ways to draw planar rooted trees. In one drawing all vertices are represented by a dot and the root is usually at the top of the tree. The following list shows the first few of them.

\[
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\]

Note that we distinguish the sides of the trees, so the trees are planar. The tree $\bullet$ with only the root is called the **empty tree**. This drawing is used, for example, in the above reference [14, 59] of trees and in the Hopf algebra of non-planar rooted trees of Connes and Kreimer [11, 12].

In the second drawing the leaf vertices are removed with only the edges leading to them left, and the root, placed at the bottom in opposite to the first drawing, gets an extra edge pointing down. The following list shows the first few of them.

\[
\searrow \quad \searrow \quad \searrow \quad \searrow \quad \searrow \quad \searrow \quad \searrow
\]

This is used, for example in the Hopf algebra of planar rooted trees of Loday and Ronco [47, 49] and noncommutative variation of the Connes-Kreimer Hopf algebra [11, 29]. In the following we will mostly use the first drawing.
Let \( \mathcal{T} \) be the set of planar rooted trees and let \( \mathcal{F} \) be the free semigroup generated by \( \mathcal{T} \) in which the product is denoted by \( \sqcup \), called the concatenation. Thus each element in \( \mathcal{F} \) is a noncommutative product \( T_1 \sqcup \cdots \sqcup T_n \) consisting of trees \( T_1, \ldots, T_n \in \mathcal{T} \), called a planar rooted forest. We also use the abbreviation

\[
T^{\sqcup n} = T \sqcup \cdots \sqcup T_{n \text{ terms}}.
\]

Remark 2.1. For the rest of this paper, a tree or forest means a planar rooted one unless otherwise specified.

We use the (grafting) brackets \( [T_1 \sqcup \cdots \sqcup T_n] \) to denote the tree obtained by grafting, that is, by adding a new root together with an edge from the new root to the root of each of the trees \( T_1, \ldots, T_n \). This is the \( B^+ \) operator in the work of Connes and Kreimer [12]. The operation is also denoted by \( T_1 \lor \cdots \lor T_n \) in some other literatures, such as in Loday and Ronco [47, 49]. Note that our operation \( \sqcup \) is different from \( \lor \). Their relation is

\[
[T_1 \sqcup \cdots \sqcup T_n] = T_1 \lor \cdots \lor T_n.
\]

See [35] for a general framework to view such algebraic structures with operators.

The depth of a forest \( F \) is the maximal depth \( d = d(F) \) of trees in \( F \). Clearly, \( d([F]) = d(F) + 1 \). The trees in a forest \( F \) are called root branches of \([F]\). Furthermore, for a forest \( F = T_1 \sqcup \cdots \sqcup T_b \) with trees \( T_1, \ldots, T_b \), we define \( b = b(F) \) to be the breadth of \( F \). Let \( \ell(F) \) be the number of leafs of \( F \). Then

\[
\ell(F) = \sum_{i=1}^b \ell(T_i).
\]

We will often use the following recursive structure on forests. For any subset \( X \) of \( \mathcal{T} \), let \( \langle X \rangle \) be the sub-semigroup of \( \mathcal{T} \) generated by \( X \). Let \( \mathcal{F}_0 = \langle \bullet \rangle \), consisting of forests \( \bullet^{\sqcup n}, n \geq 0 \). These are also the forests of depth zero. Then recursively define

\[
\mathcal{F}_n = \langle \{\bullet\} \cup [\mathcal{F}_{n-1}] \rangle.
\]

It is clear that \( \mathcal{F}_n \) is the set of forests with depth less or equal to \( n \). From this observation, we see that \( \mathcal{F}_n \) form a linear ordered direct system: \( \mathcal{F}_n \supseteq \mathcal{F}_{n-1} \), and

\[
\mathcal{F} = \bigcup_{n \geq 0} \mathcal{F}_n = \lim_{\rightarrow} \mathcal{F}_n.
\]

2.2. Rota-Baxter operator on rooted forests. We note that \( k\mathcal{F} \) with the product \( \sqcup \) is also the free noncommutative nonunitary \( k \)-algebra on the alphabet set \( \mathcal{T} \). We are going to define, for each fixed \( \lambda \in k \), another product \( \circ = \circ_{\lambda} \) on \( k\mathcal{F} \), making it into a unitary Rota–Baxter algebra (of weight \( \lambda \)). To ease notation, we will suppress \( \lambda \).

We define \( \circ \) by giving a set map

\[
\circ : \mathcal{F} \times \mathcal{F} \to k\mathcal{F}
\]

and then extending it bilinearly. For this, we use the depth filtration \( \mathcal{F} = \bigcup_{n \geq 0} \mathcal{F}_n \) in Eq. (5) and apply induction on \( i + j \) to define

\[
\circ : \mathcal{F}_i \times \mathcal{F}_j \to k\mathcal{F}.
\]

When \( i + j = 0 \), we have \( \mathcal{F}_0 = \mathcal{F}_0 = \langle \bullet \rangle \). With the notation in Eq. (2), we define

\[
\circ : \mathcal{F}_0 \times \mathcal{F}_0 \to k\mathcal{F}, \quad \bullet^{\sqcup m} \circ \bullet^{\sqcup n} := \bullet^{\sqcup (m+n-1)}.
\]
Lemma 2.2. Let $\mathcal{F}$ be a forest.

First assume that $F$ and $F'$ are trees. Note that a tree is either $\bullet$ or is of the form $[\overline{F}]$ for a forest $\overline{F}$ of smaller depth. Thus we can define

$$F \odot F' = \begin{cases} F, & \text{if } F' = \bullet, \\ F', & \text{if } F = \bullet, \\ \lfloor F \rfloor \odot \lfloor F' \rfloor + \lfloor F \odot \lfloor F' \rfloor \rfloor + \lambda \lfloor \overline{F} \odot \overline{F'} \rfloor, & \text{if } F = [\overline{F}], F' = [\overline{F'}], \\ \end{cases}$$

since for the three products on the right hand of the third equation, the sums

$$d(\lfloor F \rfloor) + d(\lfloor F' \rfloor), \quad d(\lfloor F \rfloor) + d(\lfloor F' \rfloor), \quad d(\overline{F}) + d(\overline{F'})$$

are all less than or equal to $k$. Note that in either case, $F \odot F'$ is a tree or a sum of trees.

Now consider arbitrary forests $F = T_1 \sqcup \cdots \sqcup T_b$ and $F' = T'_1 \sqcup \cdots \sqcup T'_{b'}$ with $d(F) + d(F') = k + 1$. We then define

$$F \odot F' = T_1 \sqcup \cdots \sqcup T_{b-1} \sqcup (T_b \odot T'_1) \sqcup T'_2 \cdots \sqcup T'_{b'}$$

where $T_b \odot T'_1$ is defined by Eq. (7). By the remark after Eq. (8), $F \odot F'$ is in $k \mathcal{F}$. This completes the definition of the set map $\odot$ on $\mathcal{F} \times \mathcal{F}$.

As an example, we have

$$\Lambda \odot 1 = [\cdot \sqcup \cdot] \odot [\cdot] = \left[(\cdot \sqcup \cdot) \odot [\cdot] \right] + \left[[\cdot \sqcup \cdot] \odot [\cdot] \right] + \lambda \left[(\cdot \sqcup \cdot) \odot [\cdot] \right] = \Lambda \left[\Lambda \right] + \Lambda \left[\cdot\right] + \lambda \left[\Lambda \right].$$

We record the following simple properties of $\odot$ for later applications.

Lemma 2.2. Let $F, F', F''$ be forests.

(a) $(F \sqcup F') \odot F'' = F \sqcup (F' \odot F'')$, \quad $F'' \odot (F \sqcup F') = (F'' \odot F) \sqcup F'$.

(b) $\ell(F \odot F') = \ell(F') + \ell(F) - 1$.

So $k \mathcal{F}$ with the operations $\sqcup$ and $\odot$ forms a 2-associative algebra in the sense of [50, 53].

Proof. (a). Let $F = T_1 \sqcup \cdots \sqcup T_b$, $F' = T'_1 \sqcup \cdots \sqcup T'_{b'}$ and $F'' = T''_1 \sqcup \cdots \sqcup T''_{b''}$ be the decomposition of the forests into trees. Since $\sqcup$ is an associative product, by Eq. (9) we have,

$$(F \sqcup F') \odot F'' = (T_1 \sqcup \cdots \sqcup T_b \sqcup T'_1 \sqcup \cdots \sqcup T'_{b'}) \odot (T''_1 \sqcup T''_2 \sqcup \cdots \sqcup T''_{b''})$$

$$= T_1 \sqcup \cdots \sqcup T_b \sqcup T'_1 \sqcup \cdots \sqcup T'_{b'} \sqcup (T''_1 \odot T''_1) \sqcup T''_2 \sqcup \cdots \sqcup T''_{b''}$$

$$= (T_1 \sqcup \cdots \sqcup T_b) \sqcup (T'_1 \sqcup \cdots \sqcup T'_{b'}) \sqcup (T''_1 \odot T''_1) \sqcup T''_2 \sqcup \cdots \sqcup T''_{b''})$$

$$= F \sqcup (F' \odot F'').$$

The proof of the second equation is the same.

We prove by induction on the sum $m := d(F) + d(F')$. When $m = 0$, it follows from Eq. (9). Assume that the equation holds for all $F$ and $F'$ with $m \leq k$ and consider $F$ and $F'$ with $d(F) + d(F') = k + 1$. If $F$ and $F'$ are trees, then the equation holds by Eq. (7), the induction hypothesis and the fact that $\ell([\overline{F}]) = \ell(\overline{F})$ for a forest $\overline{F}$. Then for forests $F$ and $F'$, the equation follows from Eq. (9) and Eq. (3) \hfill $\Box$

Extending $\odot$ bilinearly, we obtain a binary operation

$$\odot : k \mathcal{F} \otimes k \mathcal{F} \rightarrow k \mathcal{F}. $$
For $F \in \mathcal{F}$, we use the grafting operation to define
\begin{equation}
(11) \quad P_\mathcal{F}(F) = [F].
\end{equation}
Then $P_\mathcal{F}$ extends to a linear operator on $k\mathcal{F}$.

The following is our first main result and will be proved in the next subsection.

**Theorem 2.3.**

(a) The pair $(k\mathcal{F}, \diamond)$ is a unitary associative algebra.

(b) The triple $(k\mathcal{F}, \diamond, P_\mathcal{F})$ is a unitary Rota–Baxter algebra of weight $\lambda$.

We next construct a nonunitary sub-Rota–Baxter algebra in $k\mathcal{F}$.

Let $\mathcal{F}^0$ be the subset of $\mathcal{F}$ consisting of forests that are not $\bullet$ and do not contain any subtree $[\bullet] = 1$. For example,
\[
\begin{array}{c}
\wedge \\
\wedge \\
\wedge
\end{array}
\]
are in $\mathcal{F}^0$ while
\[
\begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\wedge \\
\wedge
\end{array}
\]
are not in $\mathcal{F}^0$. Forests in $\mathcal{F}^0$ will be called the **ladder-free forests**.

**Proposition 2.4.** The submodule $k\mathcal{F}^0$ of $k\mathcal{F}$ is a nonunitary Rota–Baxter subalgebra of $k\mathcal{F}$ under the product $\diamond$.

**Proof.** We only need to check that $k\mathcal{F}^0$ is closed under $\diamond$ and $P_\mathcal{F} = [\ ]$. The following lemma shows that $k\mathcal{F}^0$ is closed under the Rota–Baxter operator $P_\mathcal{F}$.

**Lemma 2.5.** If $F$ is in $\mathcal{F}^0$, then $[F]$ does not contain $[\bullet]$ and hence is in $\mathcal{F}^0$.

**Proof.** Let $F$ be in $\mathcal{F}^0$. Then $F$ does not contain $[\bullet]$. In other words, none of the brackets $[B]$ in $F$ is of the form $[\bullet]$. The only other brackets in $[F]$ is $[F]$ itself. So suppose $[F]$ contains a $[\bullet]$, then we must have $[F] = [\bullet]$, implying $F = \bullet$. This is a contradiction. So we have $[F] \in \mathcal{F}^0$. \hfill $\square$

To prove that $k\mathcal{F}^0$ is closed under the multiplication $\diamond$, consider $F$ and $F'$ in $\mathcal{F}^0$. Since none of $F$ or $F'$ is $\bullet$, we have $F \diamond F' \neq \bullet$. So the following lemma completes the proof of Proposition 2.4. \hfill $\square$

**Lemma 2.6.** If $F$ and $F'$ are in $\mathcal{F}^0$, then $F \diamond F'$ is either a forest that does not contain $[\bullet]$ or is a linear combination of forests that do not contain $[\bullet]$.

**Proof.** Let $F = T_1 \sqcup \cdots \sqcup T_b$ and $F' = T'_1 \sqcup \cdots \sqcup T'_{b'}$. We will prove the lemma using induction on $n := d(T_b) + d(T'_1)$.

When $n = 0$, we have $T_b = T'_1 = \bullet$. Since none of $F$ or $F'$ is $\bullet$, we have $b > 1$ and $b' > 1$. So by Eq. (17),
\[
F \diamond F' = T_1 \sqcup \cdots \sqcup T_{b-1} \sqcup \bullet \sqcup T'_1 \sqcup \cdots \sqcup T'_{b'}.
\]
Since neither $F$ nor $F'$ contains $[\bullet]$, none of $T_i$ or $T'_j$ contains $[\bullet]$. Then none of the trees on the right hand side contains $[\bullet]$. So the right hand side does not contain $[\bullet]$, as needed.

Let $k \geq 0$. Assume that the claim has been proved for $n \leq k$ and let $F$ and $F'$ be in $\mathcal{F}^0$ with $n = k + 1$. Then $n \geq 1$. So at least one of $d(T_b)$ and $d(T'_1)$ is not zero. If one of them is zero, then the same argument as in the $n = 0$ case works using the first two cases of Eq. (17). If none of them is zero, then by the third case of Eq. (17), we have $T_b = [\mathcal{F}_b]$, $T'_1 = [\mathcal{F}'_1]$ and
\[
T_b \diamond T'_{b'} = [\mathcal{F}_b] \diamond [\mathcal{F}'_1] \circ [\mathcal{F}_b \diamond [\mathcal{F}'_1]] + \lambda [\mathcal{F}_b \diamond [\mathcal{F}'_1]].
\]
Since \( T_b \) does not contain \([\bullet]\), \( T_b \) is not \( \cdot \) and does not contain \([\bullet]\). So \( T_b \) is in \( T^0 \). Similarly, \( T_1 \)' is in \( T^0 \). By the induction hypothesis, none of the terms \([T_b] \cdot T_1', T_b \cdot [T_1'], T_b \cdot T_1' \) contains \([\bullet]\). Thus they are in \( k T^0 \). By Lemma 2.3, the terms on the right hand side themselves do not contain \([\bullet]\). Therefore \( T_i \cdot T_j' \) is a linear combination of terms that do not contain \([\bullet]\). Since \( F \) and \( F' \) do not contain \([\bullet]\), none of \( T_i \) and \( T_j' \) contains \([\bullet]\). By Eq. (9), we have

\[
F \odot F' = T_1 \sqcup \cdots \sqcup T_{b-1} \sqcup (T_b \cdot T_1') \sqcup T_2' \cdots \sqcup T_{b'}.
\]

Then \( F \odot F' \) is a linear combination of terms that do not contain \([\bullet]\). This completes the induction. \( \square \)

2.3. The proof of Theorem 2.3.

Proof. (a). By Definition (7), \( \bullet \) is the identity under the product \( \odot \). So we just need to verify the associativity. For this we only need to verify

\[
(F \odot F') \odot F'' = F \odot (F' \odot F'')
\]

for forests \( F, F', F'' \in T \). We will accomplish this by induction on the sum of the depths \( n := d(F) + d(F') + d(F'') \). If \( n = 0 \), then all of \( F, F', F'' \) have depth zero and so are in \( T^0 = \langle \bullet \rangle \), the sub-semigroup of \( T \) generated by \( \bullet \). Then we have \( F = \bullet \odot \), \( F' = \bullet \odot \) and \( F'' = \bullet \odot \), for \( i, i', i'' \geq 1 \). Then the associativity follows from Eq. (6) since both sides of Eq. (12) is \( \bullet \odot(i+i'+i''-2) \) in this case.

Let \( k \geq 0 \). Assume Eq. (12) holds for \( n \leq k \) and assume that \( F, F', F'' \in T \) satisfy \( n = d(F) + d(F') + d(F'') = k + 1 \). We next reduce the breadths of the forests.

Lemma 2.7. If the associativity

\[
(F \odot F') \odot F'' = F \odot (F' \odot F'')
\]

holds when \( F, F' \) and \( F'' \) are trees, then it holds when they are forests.

Proof. We use induction on the sum of breadths \( m := b(F) + b(F') + b(F'') \). Then \( m \geq 3 \). The case when \( m = 3 \) is the assumption of the lemma. Assume the associativity holds for \( 3 \leq m \leq j \) and take \( F, F', F'' \in T \) with \( m = j + 1 \). Then \( j + 1 \geq 4 \). So at least one of \( F, F', F'' \) has breadth greater than or equal to 2.

First assume \( b(F) \geq 2 \). Then \( F = F_1 \sqcup F_2 \) with \( F_1, F_2 \in T \). Thus by Lemma 2.2

\[
(F \odot F') \odot F'' = ((F_1 \sqcup F_2) \odot F') \odot F'' = (F_1 \sqcup (F_2 \odot F')) \odot F'' = F_1 \sqcup ((F_2 \odot F') \odot F'').
\]

Similarly,

\[
F \odot (F' \odot F'') = (F_1 \sqcup F_2) \odot (F' \odot F'') = F_1 \sqcup (F_2 \odot (F' \odot F'')).
\]

Thus

\[
(F \odot F') \odot F'' = F \odot (F' \odot F'')
\]

whenever

\[
(F_2 \odot F') \odot F'' = F_2 \odot (F' \odot F'')
\]

which follows from the induction hypothesis. A similar proof works if \( b(F'') \geq 2 \).

Finally if \( b(F') \geq 2 \), then \( F' = F_1' \sqcup F_2' \) with \( F_1', F_2' \in T \). Using Lemma 2.2 repeatedly, we have

\[
(F \odot F') \odot F'' = (F \odot (F_1' \sqcup F_2')) \odot F'' = ((F \odot F_1') \sqcup F_2') \odot F'' = (F \odot F_1') \sqcup (F_2' \odot F'').
\]
In the same way, we have $F \odot (F' \odot F'') = (F \odot F'_1) \sqcup (F'_2 \odot F'')$. This again proves the associativity.

To summarize, our proof of the associativity \((12)\) has been reduced to the special case when the forests $F, F', F'' \in \mathcal{F}$ are chosen such that

(a) $n := d(F) + d(F') + d(F'') = k + 1 \geq 1$ with the assumption that the associativity holds when $n \leq k$, and

(b) the forests are of breadth one, that is, they are trees.

If either one of the trees is $\bullet$, the identity under the product $\odot$, then the associativity is clear. So it remains to consider the case when $F, F', F''$ are all in $[\mathcal{F}]$. Then $F = [\mathcal{F}], F' = [\mathcal{F}'], F'' = [\mathcal{F}'']$ with $\mathcal{F}, \mathcal{F}', \mathcal{F}'' \in \mathcal{F}$. To deal with this case, we prove the following general fact on Rota–Baxter operators on not necessarily associative algebras.

**Lemma 2.8.** Let $R$ be a $k$-module with a multiplication $\cdot$ that is not necessarily associative. Let $[\cdot]_R : R \to R$ be a $k$-linear map such that the Rota–Baxter identity holds:

\[(13) \quad [x]_R \cdot [x']_R = [x \cdot [x']_R]_R + [x]_R \cdot x' + \lambda [x \cdot x']_R, \quad \forall x, x' \in R.\]

Let $x, x'$ and $x''$ be in $R$. If
\[(x \cdot x') \cdot x'' = x \cdot (x' \cdot x''),\]
then we say that $(x, x', x'')$ is an **associative triple** for the product $\cdot$. For any $y, y', y'' \in R$, if all the triples
\[(14) \quad (y, y', y''), \quad ([y]_R, y', y''), \quad (y, [y']_R, y''), \quad (y, y', [y'']_R), \quad ([y]_R, y', y''), \quad ([y]_R, [y']_R, y''), \quad (y, [y']_R, [y'']_R) \]
are associative triples for $\cdot$, then $([y]_R, [y']_R, [y'']_R)$ is an associative triple for $\cdot$.

**Proof.** Using Eq. \((13)\) and bilinearity of the product $\cdot$, we have
\[
([y]_R \cdot [y']_R) \cdot [y'']_R = ([y]_R \cdot [y']_R) \cdot [y'']_R + y \cdot [y]_R \cdot [y']_R + \lambda y \cdot [y]_R \cdot [y']_R
\]

Applying the associativity of the second triple in Eq. \((15)\) to $y \cdot [y']_R \cdot [y'']_R$ in the fifth term above and then using Eq. \((13)\) again, we have
\[
([y]_R \cdot [y']_R) \cdot [y'']_R
\]

By a similar calculation, we have
\[
[y]_R \cdot ([y']_R \cdot [y'']_R) = ([y]_R \cdot [y']_R) \cdot [y'']_R + y \cdot [y]_R \cdot [y']_R
\]

\[
\]
\[ +[[y]_{R} \cdot (y' \cdot [y'']_{R})]_{R} \cdot y' \cdot [y'']_{R}]_{R} \cdot y + \lambda[y \cdot (y' \cdot [y'']_{R})]_{R} \\
+ \lambda[[y]_{R} \cdot (y' \cdot [y'']_{R})]_{R} \cdot [y \cdot (y' \cdot [y'']_{R})]_{R} + \lambda^{2}[y \cdot (y' \cdot [y'']_{R})]_{R}. \]

Now by the associativity of the triples in Eq. (14), the \(i\)-th term in the expansion of \([y]_{R} \cdot (y')_{R} \cdot [y'']_{R}\) matches with the \(\sigma(i)\)-th term in the expansion of \([y]_{R} \cdot (y')_{R} \cdot [y'']_{R}\). Here the permutation \(\sigma \in \Sigma_{11}\) is

\[
\begin{pmatrix}
\sigma(i)
\end{pmatrix} = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11
\end{pmatrix}.
\]

This proves the lemma. \(\square\)

To continue the proof of Theorem 2.3 we apply Lemma 2.8 to the situation where \(R\) is \(k\mathcal{F}\) with the multiplication \(\cdot = \circ\), the Rota–Baxter operator \([\ ]_{R} = [\ ]\) and the triple \((y, y', y'') = (F, F', F'')\). By the induction hypothesis on \(n\), all the triples in Eq. (14) and (13) are associative for \(\circ\). So by Lemma 2.8 the triple \((F, F', F'')\) is associative for \(\circ\). This completes the induction and therefore the proof of the first part of Theorem 2.3.

We just need to prove that \(P_{\mathcal{F}}(F) = [F]\) is a Rota–Baxter operator of weight \(\lambda\). This is immediate from Eq. (7). \(\square\)

## 3. Free Rota–Baxter algebras on a module or a set

We will construct the free unitary Rota–Baxter algebra on a \(k\)-module or on a set by expressing elements in the Rota–Baxter algebra in terms of forests from Section 2, in addition with angles decorated by elements from the \(k\)-module or set. These decorated forests will be introduced in Section 3.1. The free unitary Rota–Baxter algebra will be constructed in Section 3.2. In Section 3.3 we also give a similar construction of free nonunitary Rota–Baxter algebra on a \(k\)-module in terms of the ladder-free forests introduced in Proposition 2.4. When the \(k\)-module is taken to be free on a set, we obtain the free unitary Rota–Baxter algebra on the set. This will be discussed in Section 3.4.

### 3.1. Rooted forests with angular decoration by a module

Let \(M\) be a non-zero \(k\)-module. Let \(F\) be in \(\mathcal{F}\) with \(\ell\) leafs. We let \(M^{\otimes F}\) denote the tensor power \(M^{\otimes(\ell-1)}\) labeled by \(F\). In other words,

\[
M^{\otimes F} = \{(F; m) \mid m \in M^{\otimes(\ell-1)}\}
\]

with the \(k\)-module structure coming from the second component and with the convention that \(M^{\otimes 0} = k\). We can think of \(M^{\otimes F}\) as the tensor power of \(M\) with exponent \(F\) with the usual tensor power \(M^{\otimes n}, n \geq 0\), corresponding to \(M^{\otimes F}\) when \(F\) is the forest \(\bullet^{\ell(n+1)}\).

**Definition 3.1.** We call \(M^{\otimes F}\) the module of the forest \(F\) with angular decoration by \(M\), and call \((F; m)\), for \(m \in M^{\otimes(\ell(F)-1)}\), an angularly decorated forest \(F\) with the decoration tensor \(m\).

Also define the depth and breadth of \((F; m)\) by

\[
d(F; m) = d(F), \quad b(F; m) = b(F).
\]

Definition 3.1 is justified by the following tree interpretation of \(M^{\otimes F}\). Let \((F; m)\) be an angularly decorated forest with a pure tensor \(m = a_1 \otimes \cdots \otimes a_{\ell-1} \in M^{\otimes(\ell-1)}, \ell \geq 2\). We picture \((F; m)\) as the forest \(F\) with its angles between adjacent leafs (either from the same tree or from adjacent trees) decorated by \(a_1, \cdots, a_{\ell-1}\) from the left most angle to the right...
most angle. If $\ell(F) = 1$, so $F$ is a ladder tree with only one leaf, then $(F; a), a \in k$, is interpreted as the multiple $aF$ of the ladder tree $F$. For example, we have

\[
\left(1; x\right) = \begin{array}{c}1 \\ x\end{array}, \quad \left(\begin{array}{c}1 \\ \begin{array}{c}1 \\ x\end{array}\end{array}; x \otimes y\right) = \begin{array}{c}1 \\ \begin{array}{c}1 \\ x\end{array}\end{array}, \quad \left(\begin{array}{c}1 \\ \begin{array}{c}1 \\ x\end{array}\end{array}; x \otimes y\right) = \begin{array}{c}1 \\ \begin{array}{c}1 \\ x\end{array}\end{array}, \quad \left(\begin{array}{c}1 \\ \begin{array}{c}1 \\ x\end{array}\end{array}; x \otimes y\right) = \begin{array}{c}1 \\ \begin{array}{c}1 \\ x\end{array}\end{array}, \quad (\begin{array}{c}1 \\ a\end{array}; a) = a\cdot a.
\]

When $m = \sum_i m_i$ is not a pure tensor, but a sum of pure tensors $m_i$ in $M^{\otimes (\ell-1)}$, we can picture $(F; m)$ as a sum $\sum_i (F; m_i)$ of the forest $F$ with decorations from the pure tensors. Likewise, if $F$ is a linear combination $\sum_i c_i F_i$ of forests $F_i$ with the same number of leaves $\ell$ and if $m = a_1 \otimes \cdots \otimes a_{\ell-1} \in M^{\otimes (\ell-1)}$, we also use $(F; m)$ to denote the linear combination $\sum_i c_i (F_i; m)$. For example,

\[
\left(\begin{array}{c}1 \\ \begin{array}{c}1 \\ x\end{array}\end{array} + \begin{array}{c}1 \\ \begin{array}{c}1 \\ x\end{array}\end{array}; x \otimes y\right) = \begin{array}{c}1 \\ \begin{array}{c}1 \\ x\end{array}\end{array} + \begin{array}{c}1 \\ \begin{array}{c}1 \\ x\end{array}\end{array}
\]

Let $(F; m)$ be an angular decoration of the forest $F$ by a pure tensor $m$. Let $F = T_1 \sqcup \cdots \sqcup T_b$ be the decomposition of $F$ into trees. We consider the corresponding decomposition of decorated forests. If $b = 1$, then $F$ is a tree and $(F; m)$ has no further decompositions. If $b > 1$, then there is the relation

\[
\ell(F) = \ell(T_1) + \cdots + \ell(T_b).
\]

Denote $\ell_i = \ell(T_i), 1 \leq i \leq b$. Then

\[
(T_1; a_1 \otimes \cdots \otimes a_{\ell_1}, (T_2; a_{\ell_1+1} \otimes \cdots \otimes a_{\ell_1+\ell_2-1}), \cdots, (T_b; a_{\ell_1+\cdots+\ell_{b-1}+1} \otimes \cdots \otimes a_{\ell_1+\cdots+\ell_b}))
\]

are well-defined angularly decorated trees for the trees $T_i$ with $\ell(T_i) > 1$. If $\ell(T_i) = 1$, then $a_{\ell_i+\ell_{i-1}} = a_{\ell_{i-1}}$ and we use the convention $(T_i; a_{\ell_{i-1}+\ell_{i-1}}) = (T_i; 1)$. With this convention, we have,

\[
(F; a_1 \otimes \cdots \otimes a_{\ell-1}) = (T_1; a_1 \otimes \cdots \otimes a_{\ell_1-1}) \sqcup (T_2; a_{\ell_1+1} \otimes \cdots \otimes a_{\ell_1+\ell_2-1}) \sqcup (T_3; a_{\ell_1+\ell_2} \otimes \cdots \otimes a_{\ell_1+\cdots+\ell_{b-1}})
\]

We call this the **standard decomposition** of $(F; m)$ and abbreviate it as

\[(F; m) = (T_1; m_1) \sqcup (T_2; m_2) \sqcup \cdots \sqcup (T_b; m_b).
\]

In other words,

\[
(T_i; m_i) = \begin{cases} (T_i; a_{\ell_1+\cdots+\ell_{i-1}+1} \otimes \cdots \otimes a_{\ell_1+\cdots+\ell_i}), & \ell_i < 1, i < b, \\ (T_i; a_{\ell_1+\cdots+\ell_{i-1}+1} \otimes \cdots \otimes a_{\ell_1+\cdots+\ell_i}, (T_i; 1), & \ell_i > 1, i = b, \\ (T_i; 1), & \ell_i = 1
\end{cases}
\]

and $u_i = a_{\ell_1+\cdots+\ell_i}$. For example,

\[
(\begin{array}{c}1 \\ \begin{array}{c}1 \\ x\end{array}\end{array}; v \otimes x \otimes w \otimes y) = (\begin{array}{c}1 \\ \begin{array}{c}1 \\ x\end{array}\end{array}) \sqcup (\begin{array}{c}1 \\ \begin{array}{c}1 \\ x\end{array}\end{array}) \sqcup (\begin{array}{c}1 \\ \begin{array}{c}1 \\ x\end{array}\end{array}) \sqcup (\begin{array}{c}1 \\ \begin{array}{c}1 \\ x\end{array}\end{array}) = \begin{array}{c}1 \\ \begin{array}{c}1 \\ x\end{array}\end{array} \sqcup \begin{array}{c}1 \\ \begin{array}{c}1 \\ x\end{array}\end{array} \sqcup \begin{array}{c}1 \\ \begin{array}{c}1 \\ x\end{array}\end{array} \sqcup \begin{array}{c}1 \\ \begin{array}{c}1 \\ x\end{array}\end{array}
\]

We display the following simple property for later applications.

**Lemma 3.2.** Let $F \neq \cdot$. In the standard decomposition (18) of $(F; m)$, if $T_i = \cdot$ for some $1 \leq i \leq b$, then $b > 1$ and the corresponding factor $(T_i; m_i)$ is $(T_i; 1)$.

**Proof.** Let $F \neq \cdot$ and let $F = T_1 \sqcup \cdots \sqcup T_b$ be its standard decomposition. Suppose $T_i = \cdot$ for some $1 \leq i \leq b$ and $b = 1$. Then $F = T_i = \cdot$, a contradiction. So $b > 1$, and by our convention, $(T_i; m_i) = (T_i; \cdot)$. \qed
3.2. Free Rota–Baxter algebra on a module as decorated forests. We define the k-module

\[ \mathbb{III}^{NC}(M) = \bigoplus_{F \in \mathcal{F}} M^{\otimes F}. \]

and define a product \( \bar{\otimes} \) on \( \mathbb{III}^{NC}(M) \) by using the product \( \circ \) on \( \mathcal{F} \) in Section 2.2.

Let \( T(M) = \oplus_{n \geq 0} M^{\otimes n} \) be the tensor algebra and let \( \otimes \) be its product, so for \( m \in M^{\otimes n} \) and \( m' \in M^{\otimes n'} \), we have

\[
\otimes m' = \begin{cases} 
    m \otimes m' \in M^{\otimes n+n'}, & \text{if } n > 0, n' > 0, \\
    mm' \in M^{\otimes n'}, & \text{if } n = 0, n' > 0, \\
    m'm \in M^{\otimes n}, & \text{if } n > 0, n' = 0, \\
    m'm \in \mathbf{k}, & \text{if } n = n' = 0.
\end{cases}
\]

Here the products in the second and third case are scalar product and in the fourth case is the product in \( \mathbf{k} \). In other words, \( \otimes \) identifies \( \mathbf{k} \otimes M \) and \( M \otimes \mathbf{k} \) with \( M \) by the structure maps \( \mathbf{k} \otimes M \to M \) and \( M \otimes \mathbf{k} \to M \) of the \( \mathbf{k} \)-module \( M \).

**Definition 3.3.** For tensors \( D = (F; m) \in M^{\otimes F} \) and \( D' = (F'; m') \in M^{\otimes F'} \), define

\[
D \bar{\otimes} D' = (F \circ F'; \mbox{\textit{m}} \bar{\otimes} \mbox{\textit{m}}').
\]

The right hand side is well-defined since \( \mbox{\textit{m}} \bar{\otimes} \mbox{\textit{m}}' \) has tensor degree

\[
\deg(\mbox{\textit{m}} \bar{\otimes} \mbox{\textit{m}}') = \deg(\mbox{\textit{m}}) + \deg(\mbox{\textit{m}}') = \ell(F) - 1 + \ell(F') - 1
\]

which equals \( \ell(F \circ F') - 1 \) by Lemma 2.2(i). For example, from Eq. (10) we have

\[
\bigotimes 1 = \bigotimes + \lambda \bigotimes .
\]

By Eq. (6) – (9), we have a more explicit expression.

\[
D \bar{\otimes} D' = \begin{cases} 
    (\bullet; cc'), & \text{if } D = (\bullet; c), D' = (\bullet; c'), \\
    (F; c'm), & \text{if } D' = (\bullet; c'), F \neq \bullet, \\
    (F'; cm'), & \text{if } D = (\bullet; c), F' \neq \bullet, \\
    (F \circ F'; m \otimes m'), & \text{if } F \neq \bullet, F' \neq \bullet.
\end{cases}
\]

We can describe \( \bar{\otimes} \) even more explicitly in terms of the standard decompositions in Eq. (15) of \( D = (F; m) \) and \( D' = (F'; m') \) for pure tensors \( m \) and \( m' \):

\[
D = (F; m) = (T_1; m_1) \sqcup (T_2; m_2) \sqcup \cdots \sqcup (T_b; m_b),
\]

\[
D' = (F'; m') = (T'_1; m'_1) \sqcup (T'_2; m'_2) \sqcup \cdots \sqcup (T'_{b'}; m'_{b'}).
\]

Then by Eq. (6) – (9) and Eq. (21) – (22), it is easy to see that the product \( \bar{\otimes} \) can be defined by induction on the sum of the depths \( d = d(F) \) and \( d' = d(F') \) as follows: If \( d + d' = 0 \), then \( F = \bullet^i \) and \( F' = \bullet^j \) for \( i, j \geq 1 \). If \( i = 1 \), then \( D = (F; m) = (\bullet; c) = c(\bullet; 1) \) and we define \( D \bar{\otimes} D' = cD' = (F'; cm') \). Similarly define \( D \bar{\otimes} D' \) if \( j = 1 \). If \( i > 1 \) and \( j > 1 \), then \( (F; m) = (\bullet; 1) \sqcup u_1 \cdots \sqcup u_{b-1} (\bullet; 1) \) with \( u_1, \cdots , u_{b-1} \in M \). Similarly, \( (F'; m') = (\bullet; 1) \sqcup u'_1 \cdots \sqcup u'_{b'-1} (\bullet; 1) \). Then define

\[
(F; m) \bar{\otimes} (F'; m') = (\bullet; 1) \sqcup u_1 \cdots \sqcup u_{b-1} (\bullet; 1) \sqcup u'_1 \cdots \sqcup u'_{b'-1} (\bullet; 1).
\]
Suppose $D \boxtimes D'$ has been defined for all $D = (F; m)$ and $D' = (F'; m')$ with $d(F) + d(F') \leq k$ and consider $D$ and $D'$ with $d(F) + d(F') = k + 1$. Then we define

$$D \boxtimes D' = (T_1; m_1) \sqcup \cdots \sqcup u_{k-1} (T_b; m_b) \boxtimes (T'_1; m'_1) \sqcup v_{k'} \cdots \sqcup v_{k'-1} (T'_{b'}; m'_{b'})$$

where

$$\begin{align*}
(T_b; m_b) \boxtimes (T'_1; m'_1) &= \left\{ \begin{array}{ll}
(\bullet; 1), & \text{if } T_b = T'_1 = \bullet \text{ (so } m_b = m'_1 = 1), \\
(T_b, m_b), & \text{if } T'_1 = \bullet, T_b \neq \bullet, \\
(T'_1, m'_1), & \text{if } T'_1 \neq \bullet, T_b = \bullet, \\
[(T_b; m) \boxtimes (T'_1; m')] + [(T_b; m) \boxtimes (T'_1; m')] & + \lambda [(T_b; m) \boxtimes (T'_1; m')], & \text{if } T'_1 = [F'_1] \neq \bullet, T_b = [F_b] \neq \bullet.
\end{array} \right.
\end{align*}$$

In the last case, we have applied the induction hypothesis on $d(F) + d(F')$ to define the terms in the brackets on the right hand side. Further, for $(F; m) \in M^{\otimes F}$, define $[(F; m)] = ([F]; m)$. This is well-defined since $\ell(F) = \ell([F])$.

The product $\boxtimes$ is clearly bilinear. So extending it biadditively, we obtain a binary operation

$$\boxtimes : \III_{NC}(M) \otimes \III_{NC}(M) \to \III_{NC}(M).$$

For $(F; m) \in (F; M)$, define

$$P_M(F; m) = [(F; m)] = ([F]; m) \in ([F]; M).$$

As commented above, this is well-defined. Thus $P_M$ defines a linear operator on $\III_{NC}(M)$. Note that the right hand side is also $(P_M(F; m))$ with $P_M$ defined in Eq. (11). Let

$$j_M : M \to \III_{NC}(M)$$

be the $k$-module map sending $a \in M$ to $(\bullet \sqcup \bullet; a)$.

**Theorem 3.4.** Let $M$ be a $k$-module.

(a) The pair $(\III_{NC}(M), \boxtimes)$ is a unitary associative algebra.

(b) The triple $(\III_{NC}(M), \boxtimes, P_M)$ is a unitary Rota–Baxter algebra of weight $\lambda$.

(c) The quadruple $(\III_{NC}(M), \boxtimes, P_M, j_M)$ is the free unitary Rota–Baxter algebra of weight $\lambda$ on the module $M$. More precisely, for any unitary Rota–Baxter algebra $(R, P)$ and module morphism $f : M \to R$, there is a unique unitary Rota–Baxter algebra morphism $\bar{f} : \III_{NC}(M) \to R$ such that $f = \bar{f} \circ j_M$.

**Proof.** (a) By definition, $(\bullet, 1)$ is the unit of the multiplication $\boxtimes$. For the associativity of $\boxtimes$ on $\III_{NC}(M)$ we only need to prove

$$(D \boxtimes D') \boxtimes D'' = D \boxtimes (D' \boxtimes D'')$$

for any angularly decorated forests $D = (F; m) \in M^{\otimes F}, D' = (F'; m') \in M^{\otimes F'}$ and $D'' = (F''; m'') \in M^{\otimes F''}$. Then by Eq. (24), we have

$$\begin{align*}
(D \boxtimes D') \boxtimes D'' &= \left( (F \circ F') \circ F''; (m \boxtimes m') \boxtimes m'' \right), \\
D \boxtimes (D' \boxtimes D'') &= \left( F \circ (F' \circ F''); m \boxtimes (m' \boxtimes m'') \right).
\end{align*}$$

The first components of the two right hand sides agree since the product $\circ$ is associative by Theorem 2.3. The second component of the two right hand sides agree because the product
\(\boxtimes\) in Eq. (20) for the tensor algebra \(T(M) := \bigoplus_{n \geq 0} M^\otimes n\) is also associative. This proves the associativity of \(\boxtimes\).

(1). The Rota–Baxter relation of \(\Box\) on \(\III_{NC}(M)\) follows from the Rota–Baxter relation of \(\Box\) on \(k \mathcal{F}\) in Theorem 2.3. More specifically, it is the last equation in (24).

(2). Let \((R, P)\) be a unitary Rota–Baxter algebra of weight \(\lambda\). Let \(*\) be the multiplication in \(R\) and let \(1_R\) be its unit. Let \(f : M \to R\) be a \(k\)-module map. We will construct a \(k\)-linear map \(\bar{f} : \III_{NC}(M) \to R\) by defining \(\bar{f}(D)\) for \(D = (F; m) \in M^\otimes F\). We will achieve this by induction on the depth \(d(F)\) of \(F\).

If \(d(F) = 0\), then \(F = (\bullet; j)\) for some \(i \geq 1\). If \(i = 1\), then \(D = (\bullet; c)\), \(c \in k\). Define \(\bar{f}(D) = c 1_R\). In particular, define \(\bar{f}(\bullet; 1) = 1_R\). Then \(\bar{f}\) sends the unit to the unit. If \(i \geq 2\), then \(D = (F; m)\) with \(m = a_1 \otimes \cdots \otimes a_n \in M^\otimes n\) where \(n + 1\) is the number of leaves \(\ell(F)\).

Then we define \(\bar{f}(a) = f(a_1) \ast \cdots \ast f(a_n)\). In particular, \(\bar{f} \circ j_M = f\).

Assume that \(\bar{f}(D)\) has been defined for all \(D = (F; m)\) with \(d(F) \leq k\) and let \(D = (F; m)\) with \(d(F) = k + 1\). So \(F \neq \bullet\). Let \(D = (T_1; m_1) \sqcup \cdots \sqcup (T_b; m_b)\) be the standard decomposition of \(D\) given in Eq. (18). For each \(1 \leq i \leq b\), \(T_i\) is a tree, so it is either \(\bullet\) or is of the form \([\mathcal{F}_i]\) for another forest \(\mathcal{F}_i\). By Lemma 3.2 if \(T_i = \bullet\), then \(b > 1\) and \(m_i = 1\).

We accordingly define

\[
\bar{f}(T_i; m_i) = \begin{cases} 
1_R, & \text{if } T_i = \bullet; \\
P(\bar{f}(\mathcal{F}_i; m_i)), & \text{if } T_i = [\mathcal{F}_i].
\end{cases}
\]

In the later case, \((\mathcal{F}_i; m_i)\) is a well-defined angularly decorated forest since \(\mathcal{F}_i\) has the same number of leaves as the number of leaves of \(T_i\), and then \(\bar{f}(\mathcal{F}_i; m_i)\) is defined by the induction hypothesis since \(d(\mathcal{F}_i) = d(T_i) - 1 \leq k\). Therefore we can define

\[
\bar{f}(D) = \bar{f}(T_1; m_1) \ast f(u_1) \ast \cdots \ast f(u_{b-1}) \ast \bar{f}(T_b; m_b).
\]

For any \(D = (F; m) \in M^\otimes F\), we have \(P_M(D) = ([F]; m) \in \III_{NC}(M)\), and by the definition of \(\bar{f}\) in Eq. (27) and (28), we have

\[
\bar{f}(\left[ D \right]) = P(\bar{f}(D)).
\]

So \(\bar{f}\) commutes with the Rota–Baxter operators.

Further, Eq. (27) and (28) are clearly the only way to define \(\bar{f}\) in order for \(\bar{f}\) to be a Rota–Baxter algebra homomorphism that extends \(f\).

It remains to prove that the map \(\bar{f}\) defined in Eq. (28) is indeed an algebra homomorphism. For this we only need to check the multiplicativity

\[
\bar{f}(D \otimes D') = \bar{f}(D) \ast \bar{f}(D')
\]

for all angularly decorated forests \(D = (F; m), D' = (F'; m')\) with pure tensors \(m\) and \(m'\). Let

\[
(F; m) = (T_1; m_1) \sqcup (T_2; m_2) \sqcup \cdots \sqcup (T_b; m_b)
\]

and

\[
(F'; m') = (T'_1; m'_1) \sqcup (T'_2; m'_2) \sqcup \cdots \sqcup (T'_b; m'_b)
\]

be their standard decompositions.

We first note that, since \(\bar{f}\) sends the identity \((\bullet; 1)\) of \(\III_{NC}(M)\) to the identity \(1_R\) of \(R\), the multiplicativity is clear if either one of \(D\) or \(D'\) is in \((\bullet; k)\), that is, if either one of \(F\) or \(F'\) is \(\bullet\). So we only need to verify the multiplicativity when \(F \neq \bullet\) and \(F' \neq \bullet\).
We further make the following reduction. By Eq. \((28)\) and Eq. \((23)\), we have
\[
\tilde{f}(D \circ D') = \tilde{f}(T_1; m_1) * f(u_1) * \cdots * f(u_{b-1})
\]
\[
* \tilde{f}((T_b; m_b) \circ (T_1'; m_1')) * f(u_1') * \cdots * f(u_{b'-1}') * \tilde{f}(T_{b'}'; m_{b'}')
\]
and
\[
\tilde{f}(D) * \tilde{f}(D') = \tilde{f}(T_1; m_1) * f(u_1) * \cdots * f(u_{b-1})
\]
\[
* \tilde{f}(T_b; m_b) * \tilde{f}(T_1'; m_1') * f(u_1') * \cdots * f(u_{b'-1}') * \tilde{f}(T_{b'}'; m_{b'}').
\]
We thus have
\[
\tilde{f}((D; m) \circ (D'; m')) = \tilde{f}(D; m) * \tilde{f}(D'; m')
\]
if and only if
\[
\tilde{f}((T_b; m_b) \circ (T_1'; m_1')) = \tilde{f}(T_b; m_b) * \tilde{f}(T_1'; m_1').
\]
So we only need to prove Eq. \((32)\). For this we use induction on the sum of depths \(d(T_b) + d(T_1')\) of \(T_b\) and \(T_1'\). Then \(n \geq 0\). When \(n = 0\), we have \(T_b = T_1' = \bullet\). So by Lemma \(3.2\) we have \(b > 1, b' > 1\), and
\[
(T_b; m_b) = (T_1'; m_1') = (T_b; m_b) \circ (T_1'; m_1') = (\bullet; 1).
\]
Then
\[
\tilde{f}(T_b; m_b) = \tilde{f}(T_1'; m_1') = \tilde{f}((T_b; m_b) \circ (T_1'; m_1')) = 1_R.
\]
Thus Eq. \((32)\) and hence Eq. \((31)\) holds.

Assume that the multiplicativity holds for \(D\) and \(D'\) in \(M^{\otimes \mathcal{F}}\) with \(n = d(T_b) + d(T_1') \leq k\) and take \(D, D' \in M^{\otimes \mathcal{F}}\) with \(n = k + 1\). So \(n \geq 1\). Then at least one of \(d(T_b)\) and \(d(T_1')\) is not zero. If exactly one of them is zero, so exactly one of \(T_b\) and \(T_1'\) is \(\bullet\), then by Eq. \((24)\),
\[
(T_b; m_b) \circ (T_1'; m_1') = \begin{cases} (T_b; m_b), & \text{if } T_1' = \bullet, T_b \neq \bullet, \\ (T_1'; m_1'), & \text{if } T_1' \neq \bullet, T_b = \bullet. \end{cases}
\]
Then
\[
\tilde{f}((T_b; m_b) \circ (T_1'; m_1')) = \begin{cases} \tilde{f}(T_b; m_b), & \text{if } T_1' = \bullet, T_b \neq \bullet, \\ \tilde{f}(T_1'; m_1'), & \text{if } T_1' \neq \bullet, T_b = \bullet. \end{cases}
\]
Then Eq. \((32)\) and hence \((31)\) holds since one factor in \(\tilde{f}(T_b; m_b) * \tilde{f}(T_1'; m_1')\) is \(1_R\).

If neither \(d(T_b)\) nor \(d(T_1')\) is zero, then \(T_b = [\mathcal{F}_b]\) and \(T_1' = [\mathcal{F}_1']\) for some forests \(\mathcal{F}_b\) and \(\mathcal{F}_1'\) in \(\mathcal{F}\). Then \((T_b; m_b) = [(\mathcal{F}_b; m_b)]\) and \((T_1'; m_1') = [(\mathcal{F}_1'; m_1')]\). We will take care of this case by the following lemma.

**Lemma 3.5.** Let \((R_1, P_1)\) and \((R_2, P_2)\) be not necessarily associative \(k\)-algebras \(R_1\) and \(R_2\) together with \(k\)-linear endomorphisms \(P_1\) and \(P_2\) that each satisfies the Rota–Baxter identity in Eq. \((1)\). Let \(g : R_1 \to R_2\) be a \(k\)-linear map such that
\[
g \circ P_1 = P_2 \circ g.
\]
Let \(x, y \in R_1\) be such that
\[
g(xP_1(y)) = g(x) \cdot g(P_1(y)), \quad g(P_1(x)y) = g(P_1(x)) \cdot g(y), \quad g(xy) = g(x) \cdot g(y).
\]
Here we have suppressed the product in \(R_1\) and denote the product in \(R_2\) by \(\cdot\). Then
\[
g(P_1(x)P_1(y)) = g(P_1(x)) \cdot g(P_1(y)).
\]
Proof. By the Rota–Baxter relations of \( P_1 \) and \( P_2 \), Eq. \( (33) \) and Eq. \( (34) \), we have
\[
g(P_1(x)P_1(y)) = g(P_1(P_1(xy)) + P_1(xP_1(y)) + \lambda P_1(xy))
\]
\[
= g(P_1(x)P_1(y)) + g(P_1(xP_1(y))) + g(\lambda P_1(xy))
\]
\[
= P_2(g(P_1(xy)) + P_2(g(xP_1(y))) + \lambda P_2(g(xy))
\]
\[
= P_2(P_2(g(x) \cdot y) + P_2(g(x) \cdot P_1(y)) + \lambda P_2(g(x) \cdot y))
\]
\[
= P_2(g(x) \cdot P_2(g(y))
\]
\[
= g(P_1(x)) \cdot P_2(g(y)).
\]

Now we apply Lemma \( 3.5 \) to our proof with \((R_1, P_1) = (\mathfrak{III}^{\text{NC}}(M), [\cdot])\), \((R_2, P_2) = (R, P)\) and \( g = f \). By the induction hypothesis, Eq. \( (34) \) holds for \( x = (F_k; m_k) \) and \( y = (F'_1; m'_1) \).

Therefore by Lemma \( 3.5 \) holds for \( n = k + 1 \). This completes the induction and the proof of Theorem 3.4. \( \square \)

3.3. Free nonunitary Rota–Baxter algebra on a module. We now modify the construction of free unitary Rota–Baxter algebras in Section 3.2 to obtain free nonunitary Rota–Baxter algebras. Since the constructions are quite similar, we will be brief for most parts except for the differences.

As in Proposition 2.4, we let \( \mathcal{F}^0 \) be the subset of \( \mathcal{F} \setminus \{\bullet\} \) consisting of forests that do not contain any \([\bullet] = 1\). For any \( k \)-module \( M \), define the \( k \)-submodule
\[
\mathfrak{III}^{\text{NC}, 0}(M) = \bigoplus_{F \in \mathcal{F}^0} M^{\otimes F}
\]
of \( \mathfrak{III}^{\text{NC}}(M) \). We define a product \( \varpi \) on \( \mathfrak{III}^{\text{NC}, 0}(M) \) to be the restriction of \( \varpi \) on \( \mathfrak{III}^{\text{NC}}(M) \). This product is well-defined since for \( D = (F; m) \) and \( D' = (F'; m') \) with \( F, F' \in \mathcal{F}^0 \), \( F \circ F' \) is in \( k \mathcal{F}^0 \) by Proposition 2.4. Thus by Eq. \( (21) \), \( D \varpi D' = (F \circ F'; m \otimes m') \) is in \( \mathfrak{III}^{\text{NC}, 0}(M) \).

Also define \([\cdot] : \mathfrak{III}^{\text{NC}, 0}(M) \to \mathfrak{III}^{\text{NC}, 0}(M)\) to be the restriction of \([\cdot]\) on \( \mathfrak{III}^{\text{NC}}(M) \). This again is well-defined since by Proposition 2.4 \([\mathcal{F}^0] \subseteq \mathcal{F}^0 \). Then adapting the notation and proof of Theorem 3.4, we obtain

Theorem 3.6. Let \( M \) be a \( k \)-module.

(a) The pair \( (\mathfrak{III}^{\text{NC}, 0}(M), \varpi) \) is a nonunitary associative algebra.
(b) The triple \( (\mathfrak{III}^{\text{NC}, 0}(M), \varpi, P_M) \) is a nonunitary Rota–Baxter algebra of weight \( \lambda \).
(c) The quadruple \( (\mathfrak{III}^{\text{NC}, 0}(M), \varpi, P_M, j_M) \) is the free nonunitary Rota–Baxter algebra of weight \( \lambda \) on the \( k \)-module \( M \).

Proof. (a) and (b) are clear from (a) and (b) of Theorem 3.4.

Part (c) is proved in the same way as (c) of Theorem 3.4 with the following modification. Let \((R, \ast, P)\) be a nonunitary Rota–Baxter algebra. In the recursive definition of \( \bar{f} \) in Eq. \( (28) \), when \( (T_i; m_i) = (\bullet; 1) \), simply delete the factor \( \bar{f}(T_i; m_i) \) instead of letting it be \( 1_R \) which is not defined. Alternatively, augment \( R \) to a unitary \( k \)-algebra \( \bar{R} = k1_R \oplus R \) with unit \( 1_R \). Of course \( \bar{R} \) can not be expected to be a Rota–Baxter algebra. But it does not matter since we only need the algebra structure on \( \bar{R} \) to obtain a Rota–Baxter algebra structure on \( R \). For \( D = (F; m) \in M^{\otimes F} \) with \( F \in \mathcal{F}^0 \), just define \( \bar{f}(D) \) as in Eq. \( (28) \). Note
that \( F \) has at least two leafs, so \( m \) is in \( M^\otimes r \) with \( r \geq 1 \). Then it follows by induction that \( f(D) \) is always in \( R \). Then the rest of the proof goes through. \( \square \)

3.4. Free Rota–Baxter algebra on a set. Here we use the tree construction of free Rota–Baxter algebra on a module above to obtain a similar construction of a free Rota–Baxter algebra on a set and display a canonical basis of the free Rota–Baxter algebra in terms of forests decorated by the set.

Remark 3.7. Either by the general principle of forgetful functors or by an easy direct check, the free Rota–Baxter algebra on a set \( X \) is the free Rota–Baxter algebra on the free \( k \)-module \( M = kX \). Thus we can easily obtain a construction of the free Rota–Baxter algebra on \( X \) by decorated forests from the construction of \( \mathbb{W}^{NC}(M) \) in Section 3.2.

For any \( n \geq 1 \), the tensor power \( M^\otimes n \) has a natural basis \( X^n = \{(x_1, \cdots, x_n) \mid x_i \in X, \ 1 \leq i \leq n\} \). Accordingly, for any rooted forest \( F \in \mathcal{F} \), with \( \ell = \ell(F) \geq 2 \), the set

\[
X^F := \{(F; (x_1, \cdots, x_{\ell-1})) := (F; x_1 \otimes \cdots \otimes x_{\ell-1}) \mid x_i \in X, \ 1 \leq i \leq \ell - 1\}
\]

form a basis of \( M^\otimes F \) defined in Eq. (17). Note that when \( \ell(F) = 1 \), \( M^\otimes F = kF \) has a basis \( X^F := \{(F; 1)\} \). In summary, every \( M^\otimes F, F \in \mathcal{F} \), has a basis

\[
X^F := \{(F; \overline{x}) \mid \overline{x} \in X^{\ell(F) - 1}\}
\]

with the convention that \( X^0 = \{1\} \). Thus the disjoint union

\[
X^\mathcal{F} := \bigsqcup_{F \in \mathcal{F}} X^F.
\]

forms a basis of

\[
\mathbb{W}^{NC}(X) := \mathbb{W}^{NC}(M).
\]

We call \( X^\mathcal{F} \) the set of **angularly decorated rooted forests with decoration set** \( X \). As in Section 3.1, they can be pictured as rooted forests with adjacent leafs decorated by elements from \( X \).

Likewise, for \( (F; \overline{x}) \in X^\mathcal{F} \), the decomposition (18) gives the **standard decomposition**

\[
(F; \overline{x}) = (T_1; \overline{x}_1) \sqcup u_1 (T_2; \overline{x}_2) \sqcup u_2 \cdots \sqcup u_{b-1} (T_b; \overline{x}_b)
\]

where \( F = T_1 \sqcup \cdots \sqcup T_b \) is the decomposition of \( F \) into trees and \( \overline{x} \) is the vector concatenation of the elements of \( \overline{x}_1, u_1, \overline{x}_2, \cdots, u_{b-1}, \overline{x}_b \) which are not the unit \( 1 \). As a corollary of Theorem 3.4 we have

**Theorem 3.8.** For \( D = (F; (x_1, \cdots, x_b)) \), \( D' = (F'; (x_1', \cdots, x_{b'})) \) in \( X^\mathcal{F} \), define

\[
D \odot D' = \begin{cases} 
(\bullet; 1), & \text{if } F = F' = \bullet, \\
D, & \text{if } F' = \bullet, F \neq \bullet, \\
D', & \text{if } F = \bullet, F' \neq \bullet, \\
(F \odot F'; (x_1, \cdots, x_b, x_1', \cdots, x_{b'})), & \text{if } F \neq \bullet, F' \neq \bullet,
\end{cases}
\]

where \( \odot \) is defined in Eq. (7) and (9). Define

\[
P_X : \mathbb{W}^{NC}(X) \to \mathbb{W}^{NC}(X), \quad P_X(F; (x_1, \cdots, x_b)) = [F]; (x_1, \cdots, x_b),
\]

and

\[
j_X : X \to \mathbb{W}^{NC}(X), \quad j_X(x) = (\bullet \sqcup \bullet; (x)), \quad x \in X.
\]

Then the quadruple \( (\mathbb{W}^{NC}(X), \odot, P_X, j_X) \) is the free Rota–Baxter algebra on \( X \).
Proof. The product $\overline{\circ}$ in Eq. $(38)$ is defined to be the restriction of the product $\overline{\circ}$ in Eq. $(22)$ to $X^F$. Since $X^F$ is a basis of $\mathcal{III}^{NC}(X)$, the two products coincide. So $\mathcal{III}^{NC}(X)$ and $\mathcal{III}^{NC}(M)$ are the same as Rota-Baxter algebras. Then as commented in Remark 3.7, $\mathcal{III}^{NC}(X)$ is the free Rota–Baxter algebra on $X$.

As with Theorem 3.6, the same proof there also gives

**Theorem 3.9.** The subalgebra $\mathcal{III}^{NC,0}(X)$ of $\mathcal{III}^{NC}(X)$ generated by the $k$-basis $X^{j0} := \cup_{F \in p} X^F$, with the same product $\overline{\circ}$, Rota–Baxter operator $P_X$ and set map $j_X$, is the free nonunitary Rota–Baxter algebra on $X$.

4. Unitarization of Rota–Baxter algebras

For any nonunitary algebra $A$ (even if $A$ does have an identity), define $\tilde{A} := k \oplus A$ with component wise addition and with product defined by

$$(a, x)(b, y) = (ab, ay + bx + xy).$$

As is well-known, the unitarization of $A$ is $\tilde{A}$ together with the natural embedding

$$u_A : A \rightarrow \tilde{A}, \ x \mapsto (0, x).$$

To generalize this process to Rota–Baxter algebras turns out to be much more involved since, after formally adding a unit $1$ to a nonunitary Rota–Baxter algebra $(A, P)$, we also need to add its images under the Rota–Baxter operator $P$ and its iterations, such as $P(1)$ and $P(P(1))$. Then it is not clear in general how these new elements should fit together to form a Rota–Baxter algebra, except possibly in special cases (see Proposition 4.4 below). We will start with the unitarization of free Rota–Baxter algebras and then take care of the case of a general Rota–Baxter algebra by regarding it as a quotient of a free Rota–Baxter algebra. Let us first give the definition.

**Definition 4.1.** Let $(A, P)$ be a nonunitary Rota–Baxter $k$-algebra. A unitarization of $A$ is a unitary Rota–Baxter algebra $(\tilde{A}, \tilde{P})$ with a nonunitary Rota-Baxter algebra homomorphism $u_A : A \rightarrow \tilde{A}$ such that for any unitary Rota–Baxter algebra $B$ and a homomorphism $f : A \rightarrow B$ of nonunitary Rota–Baxter algebras, there is a unique homomorphism $\tilde{f} : \tilde{A} \rightarrow B$ of unitary Rota-Baxter algebras such that $f = \tilde{f} \circ u_A$.

4.1. Unitarization of free Rota–Baxter algebras. Let $X$ be a set. Let $\mathcal{III}^{NC}(X)$ and $\mathcal{III}^{NC,0}(X)$ be the free unitary and nonunitary Rota–Baxter algebras in Theorem 3.8 and Theorem 3.9. Let $\tilde{j}_X : X \rightarrow \mathcal{III}^{NC}(X)$ and $j_X : X \rightarrow \mathcal{III}^{NC,0}(X)$ be the canonical embeddings. Regarding $\mathcal{III}^{NC}(X)$ as a nonunitary Rota–Baxter algebra, then by the universal property of the free nonunitary Rota–Baxter algebra $\mathcal{III}^{NC,0}(X)$, there is a unique homomorphism $u_X : \mathcal{III}^{NC,0}(X) \rightarrow \mathcal{III}^{NC}(X)$ of nonunitary Rota–Baxter algebras such that $\tilde{j}_X = u_X \circ j_X$.

**Theorem 4.2.** The unitary Rota–Baxter algebra $\mathcal{III}^{NC}(X)$, with the homomorphism $u_X : \mathcal{III}^{NC,0}(X) \rightarrow \mathcal{III}^{NC}(X)$, is the unitarization of the nonunitary Rota–Baxter algebra $\mathcal{III}^{NC,0}(X)$.

**Proof.** Let $(B, Q)$ be a unitary Rota–Baxter algebra and let $f : \mathcal{III}^{NC,0}(X) \rightarrow B$ be a homomorphism of nonunitary Rota–Baxter algebras. Let $f' = f \circ j_X : X \rightarrow B$, then by the freeness of the unitary Rota–Baxter algebra $\mathcal{III}^{NC}(X)$, there is a unique homomorphism $\tilde{f}' : \mathcal{III}^{NC}(X) \rightarrow B$ of unitary Rota–Baxter algebras such that $f' = \tilde{f}' \circ \tilde{j}_X$. 

We have
\[ \bar{f}' \circ u_X \circ j_X = \bar{f}' \circ \tilde{j}_X = f' = f \circ j_X. \]
By the freeness of \( \mathfrak{III}^{NC,0}(X) \), we have \( \bar{f}' \circ u_X = f \). Suppose there is another unitary Rota–Baxter algebra homomorphism \( g : \mathfrak{III}^{NC}(X) \to B \) such that \( g \circ u_X = f \). Then
\[ g \circ \tilde{j}_X = g \circ u_X \circ j_X = f \circ j_X = f' = \bar{f}' \circ \tilde{j}_X. \]
So \( g = \bar{f}' \) by the universal property of the free unitary Rota–Baxter algebra \( \mathfrak{III}^{NC}(X) \).

4.2. Unitarization of Rota–Baxter algebras. We now construct the unitarization of any given nonunitary Rota–Baxter algebra \( A \). We use the following diagram to keep track of the maps that we will introduced below.

(39)

Let \( X \) be a generating set of \( A \) as a nonunitary Rota–Baxter algebra with \( g : X \hookrightarrow A \) being the inclusion map. Let \( \mathfrak{III}^{NC,0}(X) \) be the free nonunitary Rota–Baxter algebra on \( X \) with the canonical embedding \( j_X : X \to \mathfrak{III}^{NC,0}(X) \). Then there is a unique nonunitary Rota–Baxter algebra homomorphism \( \tilde{g} : \mathfrak{III}^{NC,0}(X) \to A \) such that \( g = \tilde{g} \circ j_X \). Since \( X \) is a generating set of \( A \), \( \tilde{g} \) is surjective. So \( A \cong \mathfrak{III}^{NC,0}(X)/J \) where \( J \) is the kernel of \( \tilde{g} \) and is a Rota–Baxter ideal of \( \mathfrak{III}^{NC,0}(X) \). Recall from Theorem 4.2 that we have the unitarization \( u_X : \mathfrak{III}^{NC,0}(X) \to \mathfrak{III}^{NC}(X) \). Let \( \tilde{J} \) be the Rota–Baxter ideal of \( \mathfrak{III}^{NC}(X) \) generated by \( u_X(J) \), and define
\[ \tilde{A} = \mathfrak{III}^{NC}(X)/\tilde{J}. \]
with $\bar{g}: \mathfrak{III}^{NC}(X) \to \tilde{A}$ being the quotient Rota–Baxter homomorphism. Let $\bar{g} = \tilde{g} \circ \tilde{j}_X$. Then $\bar{g}: \mathfrak{III}^{NC}(X) \to \tilde{A}$ is the unique unitary Rota–Baxter algebra homomorphism induced from the set map $\tilde{g}$. So the notation $\bar{g}$ is justified.

Now since $u_X(J) \subseteq \tilde{J}$, we have $(\tilde{g} \circ u_X)(J) = 0$. Thus $\ker(\tilde{g} \circ u_X) \supseteq J$. Therefore, there is a unique homomorphism $u_A : A \cong \mathfrak{III}^{NC,0}(X)/J \to \tilde{A} = \mathfrak{III}^{NC}(X)/\tilde{J}$ of nonunitary Rota–Baxter algebras such that $$u_A \circ \bar{g} = \tilde{g} \circ u_X.$$ 

**Theorem 4.3.** With the above notations, the nonunitary Rota–Baxter algebra homomorphism $u_A : A \to \tilde{A}$ gives the unitarization of $A$.

By the uniqueness of the Rota–Baxter algebra unitarization, for a different choices of the generating set $X$ of $A$, the unitarization we obtain are isomorphic.

**Proof.** Let $B$ be a unitary Rota–Baxter algebra and let let $f : A \to B$ be a nonunitary Rota–Baxter algebra homomorphism. Let $h = f \circ \tilde{g}$. By Theorem 4.2 there is a unique unitary Rota–Baxter algebra homomorphism $\tilde{h} : \mathfrak{III}^{NC}(X) \to B$ such that $\tilde{h} \circ u_X = h$. Then $\ker \tilde{h} \supseteq u_X(\ker h) \supseteq u_X(\ker \tilde{g}) = J$.

Since $\tilde{h}$ is a Rota–Baxter ideal of $\mathfrak{III}^{NC}(X)$ and $\tilde{J}$ is the Rota–Baxter ideal of $\mathfrak{III}^{NC}(X)$ generated by $J$, we must have $\ker \tilde{h} \supseteq \tilde{J}$. Therefore, there is a unique $\tilde{f} : \tilde{A} \to B$ such that $\tilde{h} = \tilde{g} \circ \tilde{f}$. Now $$\tilde{f} \circ u_A \circ \tilde{g} = \tilde{f} \circ \tilde{g} \circ u_X = \tilde{h} \circ u_X = h = f \circ \tilde{g}.$$ Since $\tilde{g}$ is surjective, we have $\tilde{f} \circ u_A = f$. So the existence of $\tilde{f}$ in Definition 4.1 is proved.

To prove the uniqueness of $\tilde{f}$, suppose there is also a unitary Rota–Baxter algebra homomorphism $\tilde{f} : \tilde{A} \to B$ such that $\tilde{f} \circ u_A = f$. Then we have

$$\tilde{f} \circ \tilde{g} \circ u_X = \tilde{f} \circ u_A \circ \tilde{g} = f \circ \tilde{g} = \tilde{f} \circ u_A \circ \tilde{g} = \tilde{f} \circ \tilde{g} \circ u_X = \tilde{h} \circ u_X = h.$$ 

So $\tilde{f} \circ \tilde{g} : \mathfrak{III}^{NC,0}(X) \to B$, as well as $\tilde{h}$ is the unitarization of $h : \mathfrak{III}^{NC,0}(X) \to B$. By the uniqueness of this unitarization, proved in Theorem 4.2, we have $$\tilde{f} \circ \tilde{g} = \tilde{h} = f \circ \tilde{g}.$$ 

Since $\tilde{g}$ is surjective, we have $\tilde{f} = \tilde{f}$, as needed. 

4.3. **Unitarization with idempotent Rota–Baxter operators.** We end our discussion on unitariness of Rota–Baxter algebras with a simple case.

**Proposition 4.4.** Let $(R, P)$ be a Rota–Baxter algebra of weight $\lambda$ such that $P^2 = -\lambda P$. The unitarization $\tilde{R} := k1 \oplus R$ of $R$ together with the extension of $P$ to $\tilde{P} : \tilde{R} \to \tilde{R}$,

$$\tilde{P}(m, a) := (-\lambda m, P(a)), \quad \forall m \in k, \ a \in R,$$

forms a unitary Rota–Baxter $k$-algebra of weight $\lambda$ such that $\tilde{P}^2 = -\lambda \tilde{P}$.
Other results on such Rota–Baxter operators can be found in [5] where they are called pseudo-idempotent.

**Proof.** We first show that \( \tilde{P} : \tilde{R} \to \tilde{R} \) satisfies the Rota–Baxter relation of weight \( \lambda \)

\[
(40) \quad \tilde{P}(m, a) \tilde{P}(n, b) = \tilde{P}((m, a) \tilde{P}(n, b)) + \tilde{P}(\tilde{P}(m, a)(n, b)) + \lambda \tilde{P}((m, a)(n, b))
\]

for \((m, a), (n, b) \in \tilde{R} \). For the left hand side, we have

\[
\tilde{P}(m, a) \tilde{P}(n, b) = ( - \lambda m, P(a)) ( - \lambda n, P(b)) = (\lambda^2 mn, -\lambda mP(b) - \lambda nP(a) + P(a)P(b))
\]

\[
= (\lambda^2 mn, -\lambda mP(b) - \lambda nP(a) + P(a)P(b)) + P(P(a)b + \lambda P(ab)).
\]

For the right hand side, we have

\[
\tilde{P}((m, a) \tilde{P}(n, b)) = \tilde{P}(-\lambda mn, mP(b) - \lambda na + aP(b)) = (\lambda^2 mn, mP^2(b) - \lambda nP(a) + P(aP(b)))
\]

\[
= (\lambda^2 mn, -\lambda mP(b) - \lambda nP(a) + P(aP(b))),
\]

where we have used idempotency of \( P \) in the second equality. For the other terms we similarly find

\[
\tilde{P}(\tilde{P}(m, a)(n, b)) = (\lambda^2 mn, -\lambda mnP(b) - \lambda nP(a) + P(aP(b))),
\]

\[
\tilde{P}((m, a)(n, b)) = ( - \lambda mn, mP(b) + nP(a) + P(ab)).
\]

From these equations, Eq. (40) is immediately verified. Finally,

\[
\tilde{P}^2(m, a) = \tilde{P}(-\lambda m; P(a)) = ((-\lambda)^2 m; P^2(a)) = (\lambda^2 m; -\lambda P(a)) = -\lambda \tilde{P}(m, a).
\]

\[\square\]

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I.H.É.S. Le Bois-Marie, 35, Route de Chartres, F-91440 Bures-sur-Yvette, France
E-mail address: kurusch@ihes.fr

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, RUTGERS UNIVERSITY, NEWARK, NJ 07102
E-mail address: liguo@newark.rutgers.edu