On the Motion of Billiards in Ellipses

Hellmuth Stachel

Abstract For billiards in an ellipse with an ellipse as caustic, there exist canonical coordinates such that the billiard transformation from vertex to vertex is equivalent to a shift of coordinates. A kinematic analysis of billiard motions paves the way to an explicit canonical parametrization of the billiard and even of the associated Poncelet grid. This parametrization uses Jacobian elliptic functions to the numerical eccentricity of the caustic as modulus.

Keywords billiard · billiard motion · confocal conics, elliptic functions

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1 Introduction

Already for two centuries, billiards in ellipses have attracted the attention of mathematicians, beginning with J.-V. Poncelet and A. Cayley. The assertion that one periodic billiard inscribed in an ellipse $e$ and tangent to a confocal ellipse $c$ implies a one-parameter family of such polygons, is known as the standard example of a Poncelet porism. It was Cayley who derived analytical conditions for such a pair $(e, c)$ of ellipses.

In 2005 S. Tabachnikov published a book on billiards from the viewpoint of integrable systems [14]. The book [8] and various papers by V. Dragović and M. Radnović addressed billiards in conics and quadrics within the framework of dynamical systems.

Computer animations carried out by D. Reznik, stimulated a new vivid interest on this well studied topic, where algebraic and analytic methods are meeting. Originally, Reznik’s experiments focused on billiard motions in ellipses, i.e., on the variation of billiards with a fixed circumscribed ellipse $e$ and...
a fixed caustic $c$. He published a list of more than 80 numerically detected invariants in [11] and contributed, together with his coauthors R. Garcia, J. Koiller and M. Helman, several proofs. Other authors like A. Akopyan, M. Bialy, A. Chavez-Caliz, R. Schwartz, and S. Tabachnikov published several proofs and found more invariants (e.g., in [12,4]).

For a long time, at least since Jacobi’s proof of the Poncelet theorem on periodic billiards (see further references in [5, p. 320]), it has been wellknown that there is a tight connection between billiards and elliptic functions (note also [7, Sect. 11.2] and [3]). On the other hand, S. Tabachnikov proved in his book [14] the existence of canonical parameters on ellipses with the property that the billiard transformation between consecutive vertices $P_i \mapsto P_{i+1}$ of a billiard acts like a shift on the parameters.

The goal of this paper is to prove that Jacobian elliptic functions with the numerical eccentricity of the caustic $c$ as modulus pave the way to canonical coordinates on the ellipse $e$. This is a consequence of a kinematic analysis of the billiard motion. It yields an infinitesimal transformation in the plane which preserves a family of confocal ellipses while it permutes the confocal hyperbolas as well as the tangents of the caustic. Integration results in a group of transformations with a canonical parameter. In terms of elliptic functions, we also obtain a mapping that sends a square grid together with the diagonals to a Poncelet grid. The paper concludes with applying the results of the velocity analysis to a few invariants of periodic billiards. These invariants deal mainly with the distances which occur on each side between the contact point with the caustic and the endpoints.

2 Confocal conics and billiards

At the begin, we recall a few properties of confocal conics. A family of confocal central conics is given by

$$\frac{x^2}{a^2 + k} + \frac{y^2}{b^2 + k} = 1, \text{ where } k \in \mathbb{R} \setminus \{-a^2, -b^2\}$$

serves as a parameter in the family. All these conics share the focal points $F_{1,2} = (\pm d, 0)$, where $d^2 := a^2 - b^2$.

The confocal family sends through each point $P$ outside the common axes of symmetry two orthogonally intersecting conics, one ellipse and one hyperbola [8, p. 38]. The parameters $(k_e, k_h)$ of these two conics define the elliptic coordinates of $P$ with

$$-a^2 < k_h < -b^2 < k_e.$$  

If $(x, y)$ are the cartesian coordinates of $P$, then $(k_e, k_h)$ are the roots of the quadratic equation

$$k^2 + (a^2 + b^2 - x^2 - y^2)k + (a^2 b^2 - b^2 x^2 - a^2 y^2) = 0,$$  

1 Recently, parametrizations of confocal conics in terms of elliptic functions were also presented in [3], but not from the viewpoint of billiards.
while conversely
\[ x^2 = \frac{(a^2 + k_e)(a^2 + k_h)}{d^2}, \quad y^2 = -\frac{(b^2 + k_e)(b^2 + k_h)}{d^2}. \]  

Suppose that \((a, b)\) in (2.1) are the semiaxes \((a_c, b_c)\) of the ellipse \(e\) with \(k = 0\). Then, for points \(P\) on a confocal ellipse \(e\) with semiaxes \((a_e, b_e)\) and \(k = k_e > 0\), i.e., exterior to \(e\), the standard parametrization yields
\[ P = (x, y) = (a_c \cos t, b_c \sin t), \quad 0 \leq t < 2\pi, \]
with \(a_e^2 = a_c^2 + k_e\), \(b_e^2 = b_c^2 + k_e\).

For the elliptic coordinates \((k_e, k_h)\) of \(P\) follows from (2.2) that
\[ k_e + k_h = a_e^2 \cos^2 t + b_e^2 \sin^2 t - a_c^2 - b_c^2. \]

After introducing the respective tangent vectors of \(e\) and \(c\), namely
\[ t_e(t) := (-a_c \sin t, b_c \cos t), \quad t_c(t) := (-a_e \sin t, b_e \cos t), \]
where \(\|t_e\|^2 = \|t_c\|^2 + k_e\),
we obtain
\[ k_h = k_h(t) = -(a_e^2 \sin^2 t + b_e^2 \cos^2 t) = -\|t_e(t)\|^2 = -\|t_c(t)\|^2 + k_e \quad (2.6) \]
and \(\|t_e(t)\|^2 = k_e - k_h(t)\). Note that points on the confocal ellipses \(e\) and \(c\) with the same parameter \(t\) have the same coordinate \(k_h\). Consequently, they belong to the same confocal hyperbola (Figure 1). Conversely, points of \(e\) or \(c\) on this hyperbola have a parameter out of \(\{t, -t, \pi + t, \pi - t\}\).

Let \(\theta_i/2\) denote the angle between the tangents drawn from any point 
\(P_i \in e\) to \(c\) and the tangent to \(e\) at \(P_i\) (Figures 2 or 3). Then we obtain for
\(P_i = (a_c \cos t_i, b_c \sin t_i)\) with elliptic coordinates \((k_e, k_h(t_i))\)
\[ \sin^2 \frac{\theta_i}{2} = \frac{k_e}{\|t_c(t_i)\|^2} = \frac{k_e}{k_e - k_h(t_i)}, \quad \tan \frac{\theta_i}{2} = \pm \frac{k_e}{k_h(t_i)} \]
and \(\sin \theta_i = \pm 2\sqrt{-k_e k_h(t_i)} = \pm 2\|t_c(t_i)\|\sqrt{k_e(\|t_c(t_i)\|^2)}\).

For a proof see [13]. We can assume a counter-clockwise order of the billiard.
Hence, all exterior angles are positive.

From (2.6) follows
\[ k_h = -\frac{a_e^2 \tan^2 t + b_e^2}{1 + \tan^2 t}, \quad \text{hence} \quad \tan^2 t(a_e^2 + k_h) = -b_e^2 - k_h \]
and furthermore
\[ \sin t \cos t = \frac{\tan t}{1 + \tan^2 t} = \sqrt{-\frac{(b_e^2 + k_h)(a_e^2 + k_h)}{a_e^2 - b_e^2}} = \frac{a_h b_h}{d^2}. \]  

(2.8)
with $a_h$ and $b_h$ as semiaxes of the hyperbola corresponding to the parameter $t$, i.e., $a_h^2 = a_e^2 + k_h$ and $b_h^2 = -(b_e^2 + k_h)$.

Let $\ldots P_1P_2P_3\ldots$ be a billiard in the ellipse $e$. Then the extended sides intersect at points

$$S_i^{(j)} := \begin{cases} [P_{i-k-1}, P_{i-k}] \cap [P_{i+k}, P_{i+k+1}] & \text{for } j = 2k, \\ [P_{i-k}, P_{i-k+1}] \cap [P_{i+k}, P_{i+k+1}] & \text{for } j = 2k-1, \end{cases} \quad (2.9)$$

where $i = \ldots, 1, 2, 3, \ldots$ and $j = 1, 2, \ldots$. These points are distributed on different confocal conics: For fixed $j$, there are ellipses $e^{(j)}$ passing through the points $S_i^{(j)}$. On the other hand, the points $S_i^{(2)}$, $S_i^{(4)}$, $\ldots$ are located on the confocal hyperbola through $P_i$, while $S_i^{(1)}$, $S_i^{(3)}$, $\ldots$ belong to the confocal hyperbola through the contact point $Q_i$ between the side $P_iP_{i+1}$ and the caustic $c$. This configuration is called the associated Poncelet grid (Figure 1).

For periodic billiards the sets of points $S_i^{(j)}$ and associated conics are finite. The turning number $\tau$ of a periodic billiard in $e$ with an ellipse as caustic counts how often one period of the billiard surrounds the center $O$ of $e$ (note Figure 1).

For each billiard $P_1P_2\ldots$ in $e$ with caustic $c$ there exists a conjugate billiard $P'_1P'_2\ldots$ in $e$ with the same caustic. The axial scaling $c \rightarrow e$ maps the contact point $Q_i$ of $P_iP_{i+1}$ to $P'_i$ while the inverse scaling brings $P_i$ to the contact point $Q'_{i-1}$ of $P'_{i-1}P'_i$ with the caustic. The relation between these billiards is symmetric. For further details see [13] Sect. 3.2.
3 Velocity analysis

Let the first vertex of a billiard $P_1P_2\ldots$ move smoothly along the circumscribed ellipse $e$. Then this induces a continuous variation of all other vertices along $e$ and also of the intersection points $S_1^{(j)}$ along $e^{(j)}$. We call this a billiard motion, though it neither preserves angles or distances nor is an affine or projective motion.

According to Graves’s construction [8, p. 47], we can conceive the periodic billiard $P_1P_2\ldots P_N$ as a flexible chain of fixed total length $L_e$ and the caustic $c$ as a fixed non-circular chain wheel. The vertices $P_1, P_2, \ldots$ move along $e$ and relative to the chain such that they keep the chain strengthened, while the chain contacts $c$ only at the single points $Q_1, Q_2, \ldots, Q_N$.

Let us pick out a single vertex $P_2$ (see Figure 2). In the language of kinematics, the line spanned by the straight segment $Q_1P_2$ rolls at $Q_1$ on $c$ (= fixed polode) while point $P_2$ moves along the line (= moving polode) with the velocity vector $v_t\,_1$. The instantaneous rotation about $Q_1$ with the angular velocity $\omega_1$ assigns to $P_2$ a velocity vector $v_n\,_1$ orthogonal to $Q_1P_2$ in order to keep the vector of absolute velocity of $P_2$, namely $v_2 = v_t\,_1 + v_n\,_1$, tangent to the ellipse $e$.

![Figure 2](image)

**Fig. 2** Graves’s string construction of an ellipse $e$ confocal to $c$.

Similarly, we have a second decomposition $v_2 = v_t\,_2 + v_n\,_2$, since at the same time the line $[Q_2, P_2]$ rotates about $Q_2$ with the angular velocity $\omega_2$, while $P_2$ moves relative to this line. Due to the constant length of the chain, the tangential components in these two decompositions must be of equal lengths $\|v_{t2}\| = \|v_{t1}\|$. Since the tangent $t_P$ to $e$ at $P_2$ bisects the exterior angle of $Q_1P_2Q_2$, the second decomposition is symmetric w.r.t. $t_P$ to the first one.
From $\|v_{n2}\| = \|v_{n1}\|$ follows for the distances $r_2 := P_2Q_1$ and $l_2 := P_2Q_2$

$$l_2 \omega_2 = r_2 \omega_1, \quad \text{or} \quad \frac{\omega_1}{\omega_2} = \frac{l_2}{r_2},$$

(3.1)

and similarly for all other vertices. If the billiard is $N$-periodic, then the product of all ratios $l_i/r_i$ for $i = 1, \ldots, N$ yields

$$\frac{l_1}{r_1} \cdot \frac{l_2}{r_2} \cdots \frac{l_N}{r_N} = \frac{\omega_N}{\omega_1} \cdot \frac{\omega_1}{\omega_2} \cdots \frac{\omega_{N-1}}{\omega_N} = 1,$$

which results in the equation

$$l_1 l_2 \cdots l_N = r_1 r_2 \cdots r_N.$$  

(3.2)

listed as k116 in Table 2.

Figure 3 shows a graphical velocity analysis for the billiard motion of a 5-sided periodic billiard in $e$. We can begin this analysis by choosing an arbitrary length for the arrow representing the velocity vector $v_2$ of $P_2$. This defines the two components $v_{t_2}$ and $v_{n_2}$, where the latter determines the angular velocity $\omega_2$ of the side $P_2P_1$ and furtheron the absolute velocity $v_3$ of $P_3$. This can be continued. From now on, we denote the norms $\|v_{t_i}\| = \|v_{t_2}\|$ and $\|v_{n_i}\| = \|v_{n_2}\|$ of the respective components of the velocity vector $v_i$ of $P_i$ with $v_{t_i}$ and $v_{n_i}$.

Fig. 3 Velocities of the vertices $P_1, P_2, \ldots, P_5$ of a periodic billiard in the ellipse $e$ with the caustic $c$. 
In terms of the exterior angles $\theta_1, \ldots, \theta_N$ of the billiard, we obtain from (3.1)

$$\sin \frac{\theta_2}{2} = \frac{l_2 \omega_2}{v_2} = \frac{r_2 \omega_1}{v_2}$$

and

$$\cos \frac{\theta_2}{2} = \frac{v_{1|2}}{v_2}$$

where $v_2 := \|v_2\|$. (3.3)

Let $R_i$ denote the pole of the line $[P_i, P_{i+1}]$ with respect to (w.r.t. in brief) $e$. Since the poles of a line $\ell$ w.r.t. confocal conics lie on a line orthogonal to $\ell$, the side $P_1P_2$ is orthogonal to $[Q_1, R_1]$ (Figure 3), which means

$$R_1Q_1 = l_1 \tan \frac{\theta_1}{2} = r_2 \tan \frac{\theta_2}{2}. \quad (3.4)$$

From (3.4) and (3.3) follows

$$l_1 \tan \frac{\theta_1}{2} = l_1 \frac{\omega_1}{v_{1|1}} = r_2 \tan \frac{\theta_2}{2}$$

and

$$v_{1|1} \tan^2 \frac{\theta_1}{2} = v_{1|2} \tan^2 \frac{\theta_2}{2} = \cdots = v_{1|N} \tan^2 \frac{\theta_N}{2} = v_{1|i} \left\| \frac{k_e}{t_{c|i}} \right\|^2 \quad (3.5)$$

for $i = 1, 2, \ldots, N$ are equal along the billiard. We denote this quantity with $C$. Instead of a free choice of $v_2$, it means no restriction of generality to set $C = k_e$. Then we obtain for the point $P_i = (a_i \cos t_i, b_i \sin t_i)$ of the ellipse $e$, by virtue of (2.7),

$$v_{1|i} = \|t_c\|^2 = -k_h, \quad v_{n|i} = v_i \sin \theta_i = \|t_c\| \sqrt{k_e} = \sqrt{-k_e k_h}$$

$$v_i = \left\| \frac{t_c}{\cos \theta_i} \right\|^2 = \left\| \frac{t_c}{\|t_c\|} \right\| = \sqrt{k_h(k_h - k_e)} \text{ for } t = t_i \text{ and } k_h = k_h(t_i). \quad (3.6)$$

4 Billiard motion and the underlying Lie group

Our specification of the quantity $C$ assigns to the vertex $P_i \in e$ with parameter $t_i$ a non-vanishing velocity vector $v_i = \|t_c(t_i)\| t_c(t_i)$. This assignment can immediately be extended to all points of $e$. There exists a parameter $u$ on $e$ such that the differentiation by $u$ gives the said velocity vector. Let a dot indicate this differentiation. Then

$$v(t) = \dot{\mathbf{p}}(t) = \frac{d\mathbf{p}(t)}{du} = \|t_c(t)\| t_c(t) = \sqrt{a_e^2 \sin^2 t + b_e^2 \cos^2 t} \mathbf{t}_e(t). \quad (4.1)$$

We can even extend this to all confocal ellipses of $c$. The assignment of a velocity vector

$$v(t) = \|t_c(t)\| \mathbf{t}_c(t) = \dot{t}_c(t)$$
to each point \( P = (a_c \cos t, b_c \sin t) \) with \( a_c^2 - b_c^2 = a_c^2 - b_c^2 \) defines an ‘instant motion’ of the plane, where

\[
\dot{t} = \frac{dt}{du} = \|t_c(t)\| = \sqrt{-k_h(t)} = \sqrt{a_c^2 \sin^2 t + b_c^2 \cos^2 t}.
\] (4.2)

We prove below, that this ‘instant motion’ is compatible with the billiard and the associated Poncelet grid. This means in particular, that the velocity \( \dot{t} \) is also valid for all points \( S_j(i) \in e^{(j)} \).

Figure 4 shows a portion of the Poncelet grid and the velocity vectors of a couple of points, each represented by a scaled arrow. As indicated, for all points on the tangent \( t_Q \) to \( c \) at any point \( Q \in c \), the normal components \( \|v_n\| \) of the respective velocity vectors \( v \) are proportional to the distance to \( Q \).

On the other hand, all points on any confocal hyperbola share the tangential component \( \|v_t\| \).

**Theorem 1.** Let the billiard \( P_1 P_2 \ldots \) with the ellipse \( c \) as caustic be moving along the circumscribed ellipse \( e \). Then the motion is the action of a one-parameter Lie group \( \Gamma \). Each transformation \( \gamma(u) \in \Gamma \) preserves the confocal ellipses and permutes the confocal hyperbolas as well as the tangents to \( c \).

1. If \( (a_c, b_c) \) are the semiaxes of the caustic \( c \) with the tangent vectors \( t_c(t) = (-a_c \sin t, b_c \cos t) \), then for all confocal ellipses \( e \) with semiaxes \( (a_e, b_e) \) the \( \Gamma \) generating instant motion is defined, up to a scalar, by the vector field

\[
(x, y) = (a_c \cos t, b_c \sin t) \mapsto \|t_c\| \frac{d}{dt} = \sqrt{-k_h(t)} \left( \frac{a_c y}{b_c} \frac{b_c x}{a_c} \right)
\] (4.3)

with \( a_e^2 - b_e^2 = a_c^2 - b_c^2 = d^2 \) and \( a_c^2 - a_e^2 \geq 0 \).

2. If we parametrize the quadrant \( x, y > 0 \) by elliptic coordinates as \( X(k_e, k_h) \), then the vector field can be expressed as

\[
X(k_e, k_h) \mapsto -2\sqrt{k_h(a_e^2 + k_h)(b_e^2 + k_h)} \frac{\partial X}{\partial k_h}.
\] (4.4)
The vector field defines a canonical parameter $u$ for the one-parameter Lie group $\Gamma$, i.e., for transformations $\gamma(u) \in \Gamma$ holds $\gamma(u_2) \circ \gamma(u_1) = \gamma(u_1 + u_2)$. At the same time, $u$ provides canonical coordinates\footnote{Of course, we still obtain canonical coordinates on the ellipses, when all velocity vectors are multiplied with any constant $\lambda \in \mathbb{R} \setminus \{0\}$.} on each confocal ellipse with the property

$$\gamma(2\Delta u): P_i \mapsto P_{i+1}, \ S_i^{(j)} \mapsto S_{i+1}^{(j)}.$$  

**Proof.** 1. The first derivative $\dot{t}$ in (4.2) is independent of the choice of the ellipse $e$. Therefore $\gamma(u)$ permutes the confocal hyperboloids. On the other hand, the representation $v = \| \mathbf{t}_e \| \mathbf{t}_e$ reveals that all confocal ellipses remain fixed. Furthermore, we verify that the position of any point $P$ on the tangent $t_Q$ to $e$ at $Q$ (see Figure 4) is preserved under the infinitesimal motion:

Given $P = (a_c \cos t, b_e \sin t) \in e$ and $Q = (a_c \cos t', b_e \sin t')$, the point $P$ lies on $t_Q$ if and only if

$$b_e a_c \cos t' \cos t + a_e b_e \sin t' \sin t = a_e b_e.$$  

(4.5)

This is preserved under the infinitesimal motion if differentiation by $u$ based on (2.9) yields an identity, namely

$$\| \mathbf{t}_e(t') \| \left( -b_e a_c \sin t' \cos t + a_e b_e \cos t' \sin t \right) = -\| \mathbf{t}_e(t) \| \left( -b_e a_c \cos t' \sin t + a_e b_e \sin t' \cos t \right).$$

(4.6)

In order to verify this, we square both sides and substitute from the squared equation (4.5) the mixed term $2a_e b_e a_c b_e \sin t' \cos t \sin t \cos t$. After some computations, this yields for both sides

$$d^2 \left( \sin^2 t - \sin^2 t' \right) \left( a_e^2 b_e^2 \sin^2 t' \sin^2 t + b_e^2 a_e^2 \cos^2 t' \cos^2 t - a_e^2 b_e^2 \right).$$

The velocity analysis in (3.6) for the particular ellipse $e$ confirms, that also the signs of both sides in (4.6) are equal.

2. From (2.3) follows for $(x, y) = \mathbf{X}(k_e, k_h)$

$$2x \frac{\partial x}{\partial k_h} = \frac{a_e^2 + k_e}{d^2}, \quad 2y \frac{\partial y}{\partial k_h} = -\frac{b_e^2 + k_e}{d^2}$$

and therefore

$$\mathbf{X}_{kh} = \frac{\partial \mathbf{X}}{\partial k_h} = \frac{1}{2d^2} \left( \frac{a_e^2 + k_e}{x}, -\frac{b_e^2 + k_e}{y} \right) = \frac{-1}{2d^2 \sin t \cos t} \mathbf{t}_e.$$  

This implies by (2.6) and (2.8)

$$\| \mathbf{t}_e \| \mathbf{t}_e = \lambda \mathbf{X}_{kh}, \ \text{with} \ \lambda = -2a_e b_h \sqrt{-k_h} = -2 \sqrt{k_h(a_e^2 + k_h)(b_e^2 + k_h)},$$

which confirms the claim in (4.1).
In order to express the action of the transformation $\gamma(u) \in \Gamma$ on an initial point \((a_c \cos t, b_c \sin t)\), we integrate the differential equation

$$i = \frac{dt}{du} = \sqrt{a_c^2 \sin^2 t + (a_c^2 - d^2) \cos^2 t} = a_c \sqrt{1 - m^2 \cos^2 t}$$

with $m := d/a_c < 1$ as numerical eccentricity of the caustic $c$. The substitution

$$\varphi := t - \frac{\pi}{2}$$

results in

$$\frac{d\varphi}{\sqrt{1 - m^2 \sin^2 \varphi}} = a_c \, du .$$

The initial condition $\varphi = 0$ for $u = 0$ yields the unique solution

$$a_c u(\varphi) = F(\varphi, m) = \int_{0}^{\varphi} \frac{d\varphi}{\sqrt{1 - m^2 \sin^2 \varphi}}$$

with $F(\varphi, m)$ as the elliptic integral of the first kind with the *modulus* $m$. The equation (4.7) shows the canonical coordinate $u$ in terms of $\varphi$ with the *quarter period*

$$K := a_c u \left( \frac{\pi}{2} \right) = \int_{0}^{\pi/2} \frac{d\varphi}{\sqrt{1 - m^2 \sin^2 \varphi}}$$

For the sake of simplicity, we introduce

$$\tilde{u}(\varphi) := a_c u(\varphi)$$

(4.8)

as a new canonical coordinate.

The inverse function of $\tilde{u} = F(\varphi, m)$, namely the Jacobian *amplitude* $\varphi = \text{am}(\tilde{u})$ leads to the Jacobian elliptic functions, the *elliptic sine*

$$\text{sn} \, \tilde{u} = \sin(\text{am}(\tilde{u})) = \sin \varphi = - \cos t$$

with $\text{sn}(-\tilde{u}) = - \text{sn} \, \tilde{u}$, the *elliptic cosine*

$$\text{cn} \, \tilde{u} = \cos(\text{am}(\tilde{u})) = \cos \varphi = \sin t$$

with $\text{cn}(-\tilde{u}) = \text{cn} \, \tilde{u}$, and the *delta amplitude*

$$\text{dn} \, \tilde{u} = \sqrt{1 - m^2 \text{sn}^2 \tilde{u}}$$

with $\text{dn}(-\tilde{u}) = \text{dn} \, \tilde{u}$ as the third elliptic base function [9]. Moreover, for $k \in \mathbb{Z}$ holds

$$\text{sn}(\tilde{u} + 2kK) = (-1)^k \text{sn} \, \tilde{u} , \quad \text{cn}(\tilde{u} + 2kK) = (-1)^k \text{cn} \, \tilde{u} , \quad \text{dn}(\tilde{u} + 2kK) = (-1)^k \text{dn} \, \tilde{u} .$$

This gives rise to the canonical parametrization of the ellipse $e$ with semi-axes $(a_c, b_c)$ as

$$(-a_c \text{sn} \, \tilde{u}, b_c \text{cn} \, \tilde{u}) \quad \text{for} \quad 0 \leq \tilde{u} < 4K = 4\tilde{u} \left( \frac{\pi}{2} \right) ,$$
As an alternative, we can proceed with elliptic coordinates. From (4.4) and
\[
\frac{dX}{du} = \dot{k}_e \frac{\partial X}{\partial k_e} + \dot{k}_h \frac{\partial X}{\partial k_h} = -2 \sqrt{k_h(a_e^2 + k_h)(b_e^2 + k_h)} \frac{\partial X}{\partial k_h}
\]
follows for the orbits of the Lie group \( \dot{k}_e = 0 \) and
\[
\dot{k}_h = -2 \sqrt{k_h(a_e^2 + k_h)(b_e^2 + k_h)}.
\]
As expected, the orbits are confocal ellipses. In order to express the action of \( \gamma(u) \in \Gamma \) on an initial point \( X(k_{e|0}, k_{h|0}) \), we have to integrate the differential equation
\[
\frac{dk_h}{\sqrt{k_h(a_e^2 + k_h)(b_e^2 + k_h)}} = -2 du
\]
with any initial condition. Again, we face an elliptic integral, this time in the so-called Riemannian form.

**Theorem 2.** 1. Let \( c \) be the ellipse \( c \) with semiaxes \( (a_c, b_c) \) and linear eccentricity \( d = \sqrt{a_c^2 - b_c^2} \). Then for all confocal ellipses \( e \) with semiaxes \( (a_e, b_e) \), the inscribed billiards with the caustic \( c \) can be canonically parametrized using the Jacobian elliptic functions to the modulus \( m = d/a_c (= \text{numerical eccentricity of } c) \) as
\[
(-a_c \text{sn} \tilde{u}, b_c \text{cn} \tilde{u}).
\]
This means that, if \( b_c = b_c \text{cn}(\Delta \tilde{u}) \), then the vertices of the billiards in \( e \) have the canonical parameters \( \tilde{u} = (\tilde{u}_0 + 2k\Delta \tilde{u}) \) for \( k \in \mathbb{Z} \) and any given initial \( \tilde{u}_0 \).

2. Conversely, we obtain an ellipse \( e \) for which the billiards with caustic \( c \) are \( N \)-periodic with turning number \( \tau \), where \( \gcd(N, \tau) = 1 \), by the choice
\[
\Delta \tilde{u} = \frac{2\tau K}{N}
\]
with \( K \) as the complete elliptic integral of the first kind to the modulus \( m \), provided that
\[
a_e = \frac{a_c \text{dn}(\Delta \tilde{u})}{\text{cn}(\Delta \tilde{u})} \quad \text{and} \quad b_e = \frac{b_c}{\text{cn}(\Delta \tilde{u})}.
\]

**Fig. 5** Dependence between the minor semiaxes \( b_e, b_c \) and the intervall \( \Delta \tilde{u} \).
Proof. If the first vertex $P_1 \in e$ of the billiard is chosen on the positive $y$-axis, i.e., with canonical parameter $\tilde{u} = 0$ (see Figure 5), then the first contact point $Q_1$ has the parameter $\Delta \tilde{u}$, and the tangent to $e$ at $Q_1$ passes through $P_1 = (0, b_e)$. Hence, the points $P_1$ and $Q_1$ are conjugate w.r.t. $e$, which means by (4.5) that the product of the respective $y$-coordinates $b_e$ and $b_e \text{cn}(\Delta \tilde{u})$ equals $b_e^2$.

In view of the major semiaxis $a_e$ follows from $dn^2(\Delta \tilde{u}) = 1 - m^2 sn^2(\Delta \tilde{u})$ and $sn^2(\Delta \tilde{u}) + cn^2(\Delta \tilde{u}) = 1$ that $dn(\Delta \tilde{u}) = a_e \text{cn}(\Delta \tilde{u})/a_e$.

**Corollary 3.** If in the ellipse $e$ with semiaxes $(a_e, b_e)$ the billiard with caustic $c$ is $N$-periodic with turning number $\tau = 1$ and $\Delta \tilde{u} = \frac{2k\pi}{N}$, then the associated Poncelet grid contains the ellipses $e^{(1)}, e^{(2)}, \ldots, e^{(k)}$, $k = \left[ \frac{N-3}{2} \right]$, with respective semiaxes

\[
\begin{align*}
a_{e|1} &= a_e \frac{dn(2\Delta \tilde{u})}{cn(2\Delta \tilde{u})}, & b_{e|1} &= b_e \frac{cn(2\Delta \tilde{u})}{cn(2\Delta \tilde{u})}, & a_{e|2} &= a_e \frac{dn(3\Delta \tilde{u})}{cn(3\Delta \tilde{u})}, & b_{e|2} &= b_e \frac{cn(3\Delta \tilde{u})}{cn(3\Delta \tilde{u})}, \\
& \quad \ldots, & a_{e|k} &= a_e \frac{dn(k\Delta \tilde{u})}{cn(k\Delta \tilde{u})}, & b_{e|k} &= b_e \frac{cn(k\Delta \tilde{u})}{cn(k\Delta \tilde{u})}.
\end{align*}
\]

Fig. 6 Canonical coordinates $u$ (blue) for the confocal hyperbolas and $v$ (red) for the confocal ellipses exterior to the caustic $c$ such that $u \pm v =$ const. represent the tangents of the caustic.

Corollary 3 reveals that $\Delta \tilde{u}$ serves as a canonical coordinate for confocal ellipses in the exterior of $c$. If $\Delta \tilde{u}$ corresponds by (4.10) to the ellipse $e$ with semiaxes $(a_e, b_e)$, then $2\Delta \tilde{u}$ is the shift for the billiards in $e$ with caustic $c$. If these billiards have the turning number 1, then increasing the shift by $\Delta \tilde{u}$ means to increase the turning number of the billiard in a confocal ellipse by 1, while the caustic $c$ remains fixed (Figure 3). The billiard $P_1 P_2 \ldots$ and its conjugate $P_1' P_2' \ldots$ in $e$ (cf. [13, Sect. 3.2]) intersect each other along the ellipse with the canonical coordinate $\Delta \tilde{u}/2$. Note that the ellipses $e^{(j)}$ and $e^{(N-2-j)}$ coincide while the corresponding $\Delta \tilde{u}$’s differ in their signs. For even $N$, the
points $S^{(N/2-1)}_i$ are at infinity, and the line at infinity as a limit of a confocal ellipse corresponds to $\Delta \tilde{u} = K$.

The following formulas express the elliptic coordinates $(k_c, k_h)$ of the point $P = (-a_c \text{sn} \tilde{u}, b_c \text{cn} \tilde{u})$ of $e$ in terms of the canonical coordinate $\tilde{u}$ on $e$ and the shift $\Delta \tilde{u}$ corresponding to $e$.

\[ k_c = k_c(\Delta \tilde{u}) = \frac{a_c^2 \text{sn}^2 \Delta \tilde{u}}{\text{cn}^2 \Delta \tilde{u}}(1 - m^2), \quad k_h = k_h(\tilde{u}) = -a_c^2 \text{dn}^2 \tilde{u}. \quad (4.11) \]

This follows from

\[ k_c = a_c^2 - a_c^2 = a_c^2 \frac{\text{dn}^2 \Delta \tilde{u} - \text{cn}^2 \Delta \tilde{u}}{\text{cn}^2 \Delta \tilde{u}} = a_c^2(1 - m^2) \frac{\text{sn}^2 \Delta \tilde{u}}{\text{cn}^2 \Delta \tilde{u}} \]

and

\[ k_h = -a_c^2 \text{cn}^2 \tilde{u} - b_c^2 \text{sn}^2 \tilde{u} = -a_c^2 + d^2 \text{sn}^2 \tilde{u} = -a_c^2 + m^2 a_c^2 \text{sn}^2 \tilde{u}. \]

Note that $k_h = k_h(\tilde{u})$ is a solution of (4.9).

In [10], an unordered pair of coordinates $(r, s)$ is proposed for each point $P$ in the exterior of $c$, namely with $r$ and $s$ as canonical coordinates of the tangency points for the tangent lines from $P$ to $c$ (see also [12] p. 358). This means for $P = (-a_c \text{sn} \tilde{u}, b_c \text{cn} \tilde{u})$ that

\[ r = \tilde{u} - \Delta \tilde{u}, \quad s = \tilde{u} + \Delta \tilde{u}, \]

where $\Delta \tilde{u}$ corresponds to $e$ according to (4.10).

If we keep the sum $\tilde{u} + \Delta \tilde{u}$ or the difference $\tilde{u} - \Delta \tilde{u}$ constant, the corresponding point $P$ runs along a tangent of the caustic $c$ (compare with [3] Prop. 8.3]). For the sake of simplicity, we replace in the summary below $\Delta \tilde{u}$ by $\tilde{u}$.

**Fig. 7** The injective mapping $Y$ sends the square grid of points $Q_i, P_i$ and $S^{(i)}, i = 1, \ldots, 9, j = 1, \ldots, 3$, to the vertices and the diagonals to the confocal conics of the Poncelet grid depicted in Figure 4.
Theorem 4. Referring to the notation in Theorem 3, the injective mapping

\[ Y : U \times V \to \mathbb{R}^2, \quad (\tilde{u}, \tilde{v}) \mapsto \left( -a_c \frac{\text{sn} \tilde{v}}{\text{cn} \tilde{u}}, \frac{b_c \text{cn} \tilde{u}}{\text{sn} \tilde{v}} \right) \]

for \( U := \{ \tilde{u} | 0 \leq \tilde{u} < 4K \}, \ V := \{ \tilde{v} | 0 \leq \tilde{v} < K \} \)

parametrizes the exterior of the caustic \( c \) with semiaxes \( (a_c, b_c) \) in such a way, that the lines \( \tilde{u} = \text{const} \) are branches of confocal hyperbolas; \( \tilde{v} = \text{const} \) are confocal ellipses and \( \tilde{u} \pm \tilde{v} = \text{const} \) are confocal tangents of \( c \).

The domain of the mapping \( Y \) (Figure 7) can be extended to \( \mathbb{R}^2 \) and satisfies

\[ Y((\tilde{u} + 4K), \tilde{v}) = Y(\tilde{u}, (\tilde{v} + 2K)) = Y(\tilde{u}, -\tilde{v}) = Y(\tilde{u}, \tilde{v}) \]

and therefore \( Y(\tilde{u}, (K + \tilde{v})) = Y(\tilde{u}, (K - \tilde{v})) \) (Figure 7). The Lie group \( \Gamma \) mentioned in Theorem 4 is the \( Y \)-transform of the group of translations along the \( \tilde{u} \)-axis.

5 More about invariants of periodic billiards

In this section we study how the infinitesimal motion induced by the vector field \( \Gamma \) affects distances and angles of periodic billiards. As before, the dot means differentiation by the canonical parameter \( u \), and we call the billiard motion canonical when it is parametrized by a canonical parameter like \( u \).

Lemma 5. Let \( P_1 P_2 \ldots \) be a billiard in the ellipse \( c \) with \( Q_1, Q_2, \ldots \) as contact points with its caustic, the ellipse \( c \). If \( t_i \) is the parameter of \( P_i \) and \( t'_i \) that of \( Q_i \), then

\[ r_i = \frac{Q_{i-1} P_i}{a_c b_c} = \frac{\| t_c(t'_i) \|}{\sqrt{k_c}}, \quad l_i = \frac{P_i Q_i}{a_c b_c} = \frac{\| t_c(t_i) \|}{\sqrt{k_c}}. \]

The canonical motion of the billiard induces for the side \( P_i P_{i+1} \) the instant angular velocity

\[ \omega_i = \frac{a_c b_c}{\| t_c(t'_i) \|} = \frac{a_c b_c}{\sqrt{-k_c(t'_i)}}. \]

Proof. Referring to Figure 8 if the tangent \( [Q_1, P_2] \) rolls on \( c \), then the vertex \( P_2 \) receives the velocity vector \( v_{Q_2} \) satisfying (3.6), while the point of contact \( Q_1 \) moves with the velocity \( v_c(t'_i) \) along \( c \). We can express this velocity in terms of the radius of curvature \( \rho_c(t'_i) \) of \( c \) as

\[ v_c(t'_i) = \omega_1 \rho_c(t'_i), \]

where \( \rho_c(t) = \| t_c(t) \|^3 / a_c b_c \) by [8, p. 79]. On the other hand, from \( v_c = i t_c \) and (3.2) follows \( v_c(t'_i) = \| t_c(t'_i) \|^2. \) This yields

\[ \omega_1 = \frac{v_c(t'_i)}{\rho_c(t'_i)} = \frac{\| t_c(t'_i) \|^2 a_c b_c}{\| t_c(t'_i) \|^3} = \frac{a_c b_c}{\| t_c(t'_i) \|^2} = \frac{a_c b_c}{\sqrt{-k_c(t'_i)}}. \]
Similarly follows from this means by Lemma 5 of the contact points \( Q \) (with \( Q^* \) as respective centers of curvature), and of vertices \( P_i^* \) of the conjugate billiard, \( i = 1, 2 \).

by (2.6). Thus, we obtain for the velocity \( v_{n|2} \) of \( P_2 \) by (3.6)

\[
r_2 \omega_1 = v_{n|2} = \| \mathbf{t}_c(t_2) \| \sqrt{k_e}, \quad \text{hence} \quad r_2 = \frac{\| \mathbf{t}_c(t_2) \| \sqrt{k_e}}{a_e b_e} \| \mathbf{t}_c(t_1^*) \|.
\]

Similarly follows from \( l_2 \omega_2 = r_{n|2} \) the stated expression for the distance \( l_2 \).

Note that \( t_1, t_1^*, t_2, t_2^*, t_3, \ldots \) is the sequence of consecutive parameters of the points \( P_1, Q_1, P_2, Q_2, P_3, \ldots \) The formulas for \( r_i \) and \( l_i \) reveal again that the same distances appear at the conjugate billiard.

The angular velocity of the tangent to \( e \) at \( P_2 \) equals the arithmetic mean \( (\omega_1 + \omega_2)/2 \) (Figure 8). On the other hand, it is defined by the radius of curvature \( \rho_e \) of \( e \) at \( P_2 \) and the velocity \( v_2 \) by (3.6), since

\[
v_2 = \rho_e(t_2) \frac{\omega_1 + \omega_2}{2}.
\]

This means by Lemma 5

\[
\| \mathbf{t}_c(t_2) \| \| \mathbf{t}_c(t_2) \| = \frac{\| \mathbf{t}_c(t_2) \|^3}{a_e b_e} \left( \frac{1}{\| \mathbf{t}_c(t_1^*) \|} + \frac{1}{\| \mathbf{t}_c(t_2^*) \|} \right)
\]

and results by (2.7) in

\[
\frac{1}{\| \mathbf{t}_c(t_1^*) \|} + \frac{1}{\| \mathbf{t}_c(t_2^*) \|} = \frac{2 a_e b_e}{a_e b_e \sqrt{k_e}} \frac{\| \mathbf{t}_c(t_2) \|}{\| \mathbf{t}_c(t_2) \|^2} = \frac{a_e b_e}{a_e b_e \sqrt{k_e}} \sin \theta_2. \quad (5.1)
\]

**Theorem 6.** The exterior angles \( \theta_i \) of an \( N \)-periodic billiard in an ellipse and with an ellipse as caustic satisfy for \( N \equiv 0 \) (mod 2)

\[
\sum_{i=1}^{N} (-1)^i \sin \theta_i = 0 \quad \text{and for} \quad N \equiv 0 \, (\text{mod} \, 4) \sum_{i=1}^{N/2} (-1)^i \sin \theta_i = 0.
\]
Proof. By virtue of Corollary 4.2, periodic billiards with even $N = 2n$ are centrally symmetric, which implies $\theta_i = \theta_{i+n}$. For $N \equiv 2 \pmod{4}$ the sum from 1 to $N$ must vanish since $(-1)^i = -(-1)^{(i+n)}$.

In the remaining case $N \equiv 0 \pmod{4}$ follows from (5.1)

$$\sin \theta_i = \frac{a_c b_c \sqrt{k_c}}{a_c b_c} \left( \frac{1}{\|t_c(t_{i-1}^c)} \right) + \frac{1}{\|t_c(t_i^c)} \right)$$

and further

$$\sum_{i=1}^{N/2} \sin \theta_i = \frac{a_c b_c \sqrt{k_c}}{a_c b_c} \left( \frac{1}{\|t_c(t_{i}^c)} \right) - \frac{1}{\|t_c(t_i^c)} \right) - \frac{1}{\|t_c(t_{i-1}^c)} \right) - \cdots - \frac{1}{\|t_c(t_{i}^c)} \right).$$

This sum vanishes, since $t_c(t_n^c) = -t_c(t_{n}^c)$, due to the odd turning number $\tau$ because of $\gcd(N, \tau) = 1$.

At the same token, from

$$\hat{\theta}_i = \omega_i - \omega_{i-1} = \frac{a_c b_c}{\|t_c(t_{i}^c)} \right) - \frac{1}{\|t_c(t_i^c)} \right)$$

and (5.1) follows

$$\frac{d}{du} \cos \theta_i = -\hat{\theta}_i \sin \theta_i = \frac{a_c^2 b_c^2 \sqrt{k_c}}{a_c b_c} \left( \frac{1}{\|t_c(t_{i-1}^c)} \right) - \frac{1}{\|t_c(t_i^c)} \right) \right).$$

This shows that $\frac{d}{du} \left( \sum_{i=1}^{N} \cos \theta_i \right)$ vanishes and, therefore, $\sum_{i=1}^{N} \cos \theta_i$ is invariant against billiard motions, which was first proved in [1].

For the sake of completeness, we also focus on the variation of the side lengths under the canonical billiard motion. We obtain

$$\frac{d}{du} \sum_{i=1}^{N} \left| t_{i+1} - t_i \right| = \left| t_{i+1} - t_i \right| = \|t_c(t_{i+1})\|^2 - \|t_c(t_i)\|^2 = d^2(\sin^2 t_{i+1} - \sin^2 t_i).$$

The vanishing sum over all $i$ confirms again the constant perimeter. We like to recall that already in [2] some proofs for invariants were based on differentiation.

Finally we concentrate on the effects showing up when the vertex $P_i$ traverses a quarter of the full period along the billiard.

Lemma 7. As before, let $t_1, t_1', t_2, t_2', \ldots, t_N'$ be the sequence of parameters of an $N$-periodic billiard in an ellipse $e$ with an ellipse $c$ as caustic. Then holds for $N \equiv 0 \pmod{2}$:

- if $N = 4n$: $\|t_c(t_i)\| \|t_c(t_{i+n})\| = \sqrt{k_c(t_i) k_c(t_{i+n})}$
- if $N = 4n+2$: $\|t_c(t_i)\| \|t_c(t_{i+n})\| = \sqrt{k_c(t_i) k_c(t_{i+n})}$

and the same after the parameter shift $t_i \mapsto t_i'$ and $t_i' \mapsto t_{i+1}'$. 

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Proof. Based on the canonical parametrization by \( \tilde{u} \), a quarter of the period 4\( K \) corresponds to a shift by \( K \). In the case \( N = 4n \) this shift effects \( t_i \mapsto t_{i+n} \) and \( t'_i \mapsto t'_{i+n} \). If \( N = 4n + 2 \), then \( t_i \mapsto t'_{i+n} \) and \( t'_i \mapsto t'_{i+n+1} \).

According to (4.11) holds \( k_h = -a_c^2 \, dn(\tilde{u}) \) and by (2.6) \( ||t_c(t)|| = \sqrt{-k_h(t)} = a_c \, dn(\tilde{u}) \).

The identity
\[
\frac{dn(\tilde{u} + K)}{dn(\tilde{u})} = \sqrt{1 - n^2}
\]
implies
\[
dn(\tilde{u}) \cdot dn(\tilde{u} + K) = \frac{b_c}{a_c}, \text{ hence } \sqrt{k_h(\tilde{u}) \cdot k_h(\tilde{u} + K)} = a_c \, b_c.
\]

This confirms the claim. \( \Box \)

**Theorem 8.** If an \( N \)-periodic billiard in an ellipse \( e \) with an ellipse as caustic is given with \( N \equiv 0 \) (mod \( 2 \)) then the distances \( r_i = \frac{Q_i-1}{P_i} \) and \( l_i = \frac{P_i}{Q_i} \) satisfy
\[
\begin{align*}
\text{for } N = 4n : & \quad r_i \cdot r_{i+n} = l_i \cdot l_{i+n} \\
\text{for } N = 4n + 2 : & \quad r_i \cdot l_{i+n} = l_i \cdot r_{i+n+1}
\end{align*}
\]

Proof. From the expressions for \( r_i \) and \( l_i \) in Lemma \( 5 \) follows, by virtue of Lemma \( 7 \) for \( N = 4n \)
\[
r_i \cdot r_{i+n} = \frac{k_e}{a_c^2 b_c} \, ||t_c(t_{i-1})|| \, ||t_c(t_i)|| \, ||t_c(t_{i+n-1})|| \, ||t_c(t_{i+n})|| = k_e
\]

and the same result for \( l_i \cdot l_{i+n} \). In the case \( N = 4n + 2 \) we obtain similarly
\[
r_i \cdot l_{i+n} = l_i \cdot r_{i+n+1} = k_e,
\]
as stated. \( \Box \)

The following corollary is an immediate consequence of Theorem \( 8 \)

**Corollary 9.** Let \( s_i = \frac{P_i P_{i+1}}{P_{i+1}} = l_i + r_{i+1} \) for \( i = 1, \ldots, N \) be the side lengths of the \( N \)-periodic billiard with even \( N \) and \( s'_i = \frac{P_i P'_{i+1}}{P'_{i+1}} = r_{i+1} + l_i \) that of the conjugate billiard. Then,
\[
\begin{align*}
\text{for } N = 4n : & \quad \frac{s_{i+n}}{s_i} = \frac{l_{i+n}}{r_{i+1}} = \frac{r_{i+n+1}}{l_i} \\
\text{for } N = 4n + 2 : & \quad \frac{s_{i+n}}{s_{i-1}} = \frac{l_{i+n}}{l_i} = \frac{r_{i+n+1}}{r_i}
\end{align*}
\]

Finally we prove the invariance of \( k^{117} \) in \( [11] \) Table 2.

**Theorem 10.** Referring to the notation in Lemma \( 5 \) for even \( N \) the products
\[
r_1 r_2 \ldots r_N = l_1 l_2 \ldots l_N = k_e \frac{N}{2}
\]
are invariant against billiard motions. For \( N \equiv 0 \) (mod \( 4 \)) this is already true for the products
\[
r_1 r_2 \ldots r_{N/2} = l_1 l_2 \ldots l_{N/2} = k_e \frac{N}{4}.
\]
Proof. For \( N \equiv 0 \pmod{4} \) the statements are a direct consequence of Theorem 8 and the central symmetry of the billiard which exchanges \( r_i \) with \( r_{i+N/2} \) and \( l_i \) with \( l_{i+N/2} \).

In the remaining case \( N = 2n + 2 \) we note that by (3.2) \( k(u) := r_1 r_2 \ldots r_N = l_1 l_2 \ldots l_N \). Hence, by virtue of Theorem 8,

\[
R^2(u) = \prod_{i=1}^{N} (r_i l_{i+n}) = k_N^N,
\]

which yields the stated result. \( \square \)

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