Pricing Path-Independent Payoffs with Exotic Features in the Fractional Diffusion Model

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Abstract: We provide several practical formulas for pricing path-independent exotic instruments (log options and log contracts, digital options, gap options, power options with or without capped payoffs . . . ) in the context of the fractional diffusion model. This model combines a tail parameter governed by the space fractional derivative, and a subordination parameter governed by the time-fractional derivative. The pricing formulas we derive take the form of quickly convergent series of powers of the moneyness and of the convexity adjustment; they are obtained thanks to a factorized formula in the Mellin space valid for arbitrary payoffs, and by means of residue theory. We also discuss other aspects of option pricing such as volatility modeling, and provide comparisons of our results with other financial models.

Keywords: fractional diffusion equation; subordination; exotic options; volatility modeling

1. Introduction

Many financial models are based on the idea that the instantaneous variations (or log-returns) of market prices can be accurately described by some stochastic dynamics, resulting in the so-called class of exponential market models; this class includes, among others, the Black–Scholes model [1], the Heston model [2] (which features a supplementary dynamics for the volatility) or models based on Variance Gamma [3] or Normal inverse Gaussian [4] processes. When the stochastic dynamics is given by a Lévy process, one speaks of an exponential Lévy model (a comprehensive introduction to Lévy processes can be found in the classical reference [5], and a review of their applications to financial modeling is provided for example in [6]). Within this family, spectrally negative exponential Lévy models are of particular financial interest because their Lévy measure is supported by the real negative axis only, allowing for a polynomial decay of the distribution’s tail while preserving the finiteness of expectations; perhaps the most prominent model in this category is given by the so-called Finite Moment Log Stable (FMLS) process, introduced in [7].

From the point of view of financial modeling, the presence of a left fat tail in the FMLS model corresponds to the occurrence of negative jumps in the distribution of returns, i.e., to brutal drops in market prices, observed for instance in stressed market conditions or in case of a firm’s default (such events, sometimes called “black swans”, are described in detail in [8]). This model is very closely related to fractional calculus because its probability densities satisfy a space fractional diffusion equation, involving a Riesz–Feller derivative whose order governs the tail index of the distribution; when this fractional derivative coincides with the usual second derivative, then the model degenerates into the Black–Scholes model [1], whose densities are given by the usual Gaussian kernel. More details about fractional diffusion can be found e.g., in the recent books [9,10] and references therein; see also [11] for a review of economical applications of fractional calculus.

Recently, the FMLS model has been generalized by allowing the time derivative to be fractional as well. This model, introduced in [12,13] and that we shall refer to as the “fractional diffusion
model”, features two degrees of freedom: while the spatial parameter still controls the heavy tail of the distribution of returns, the temporal parameter acts as a subordination parameter allowing the capture of behavioral phenomena observable in financial time series, such as clustering or memory (see [14] for an empirical and theoretical overview of the subject; such phenomena are also discussed from the point of view of fractional calculus in [15]). In that extent, the fractional diffusion model can be regarded as an alternative to multifractal volatility models [16], subordinated random walks [17] or time-changed Lévy processes [18].

In a series of recent papers, practical pricing tools have been provided for European and binary options in the context of the fractional diffusion model [19–21], and the behavior of their risk sensitivities has also been discussed in [21]. The purpose of the present paper is to extend these pricing formulas to the case of commonly traded payoffs, that, like European options, are time independent, but feature more exotic attributes. We will be particularly interested in log options and log contracts [22], which arise in the construction of vega-neutral strategies (i.e., strategies that are independent of the market volatility) and in power options [23], which are a popular tool for increasing the leverage ratio of portfolios. We will also study the impact of a supplementary optionality, such as the presence of a cap/floor (typically introduced to avoid the payment of too large payoffs), and discuss some aspects of volatility modeling, notably by comparing the fractional diffusion model with another time-subordinated model (the Variance Gamma model). The validity of our results will be assessed by several numerical tests and by showing that they recover the formulas known in the Gaussian and FMLS cases.

The paper is organized as follows: after the present introduction Section 1, we introduce the fractional diffusion model and discuss its financial interpretations in Section 2. In this section, we also establish a pricing formula which holds in the Mellin space for an arbitrary payoff. This representation has a remarkable factorization property which allows us to derive several practical formula for pricing exotic payoffs: log options and log contracts in Section 3, digital, gap and European options (with applications to implied volatility) in Section 4, power options and cap/floor features in Section 5. Finally, Section 6 is devoted to concluding remarks.

2. The Fractional Diffusion Model and the Pricing of Contingent Claims

2.1. Exponential Market Models

Let \( T > 0 \) and \( S : t \in [0, T] \rightarrow S_t \) be the market price of some financial asset, seen as the realization of a time dependent random variable on the canonical space \( \Omega = \mathbb{R}_+ \) equipped with its natural filtration. One speaks of an exponential market model if there exist a stochastic process \( \{X(t)\}_{t \geq 0} \) and a measure \( Q \) (called the risk-neutral, or martingale measure) under which the terminal market value \( S_T \) can be written down as:

\[
S_T = S_t e^{(r-q)\tau + X(\tau)}
\]

for all \( t \in [0, T] \). In (1), \( r \) is the risk-free interest rate and \( q \) is the dividend yield (both continuously compounded), \( \tau := T - t \) is the time horizon and \( \mu \) is a convexity adjustment (also called martingale adjustment or compensator) determined by the martingale condition \( \mathbb{E}^Q[S_T] = e^r S_t \). If we define the Laplace exponent \( \Phi(p) := \log \mathbb{E}^P[e^{pX(1)}] \) where \( P \) is the physical measure, then the martingale condition is equivalent to \( \mu = -\Phi(1) \) that is, if the process \( \{X(t)\}_{t \geq 0} \) admits a density \( g(x, t) \), to:

\[
\mu = -\log \int_{-\infty}^{+\infty} e^x g(x, 1) \, dx.
\]

Particular cases of exponential market models include:
- The Black–Scholes model [1]: in this model, \( X(t) = \sigma W(t) \) where \( \{W(t)\}_{t \geq 0} \) is a standard Wiener process and where \( \sigma \) represents the market volatility. It is well-known that, in this case, the convexity adjustment is equal to \(-\frac{\sigma^2}{2}t\) and that the probability densities \( g_{BS}(x,t) \) of \( \{X(t)\}_{t \geq 0} \) satisfy the diffusion (heat) equation (see, e.g., [24]):

\[
\frac{\partial g_{BS}}{\partial t} - \frac{\sigma^2}{2} \frac{\partial^2 g_{BS}}{\partial x^2} = 0.
\] (3)

The Black–Scholes model is a Gaussian model with no heavy-tail behavior; in other words, it is driven by a Lévy process whose characteristic exponent involves only a diffusion (Brownian) part and no jump part. We may note, however, that it is also a limiting case of several non-Gaussian models, such as the exponential Normal inverse Gaussian model (in the large steepness limit), the exponential Variance Gamma model (in the low variance limit), or, as we shall see, the FMLS model.

- The FMLS model [7]: here, the stochastic process \( \{X(t)\}_{t \geq 0} \) is a Lévy process whose distribution is given by the stable (or \( \alpha \)-stable) law \( L(\sigma^2, \beta) \) with stability parameter \( \alpha \in (1,2) \) and asymmetry parameter \( \beta \in [-1,1] \)—see the classical reference [25] for more details on stable distributions. For the process to be spectrally negative, one has to impose \( \beta = -1 \) (maximal negative asymmetry hypothesis); in that case, the distribution’s density \( g_{\alpha}(x,t) \) has a left fat tail decreasing in \( 1/|x|^{1+\alpha} \) for any \( \alpha \in (1,2) \) and satisfies the space fractional diffusion equation

\[
\frac{\partial g_{\alpha}}{\partial t} + \alpha \mu_{\alpha} \mathcal{D}^{\alpha}_{x} g_{\alpha}(x,t) = 0
\] (4)

where \( \mathcal{D}^{\alpha}_{x} \) stands for the Riesz–Feller derivative—see definition thereafter. When \( \alpha = 2 \), (4) degenerates into the classical diffusion (3) and the probability distribution degenerates into the centered normal law: \( L(\sigma^2, -1) = N(0, (\sigma \sqrt{2})^2) \). The spectral negativity condition is crucial to ensure that the bilateral Laplace transform (2) exists, and in this case its value is (see calculation details in [7,26])

\[
\mu_{\alpha} = \left( \frac{\sigma}{\sqrt{2}} \right)^{\alpha} \frac{1}{\alpha} \left( \frac{1 - \alpha}{2} \right) \Gamma \left( \frac{1 + \alpha}{2} \right) \left( \frac{\sigma}{\sqrt{2}} \right)^{\alpha},
\] (5)

where the \( \sqrt{2} \) normalization has been introduced to coincide with the Gaussian convexity adjustment when \( \alpha = 2 \); the second equality in (5) is a consequence of the reflection formula for the Gamma function.

Given a path-independent payoff function \( P \), i.e., a positive function depending only on the terminal value \( S_T \) of the market price and on some strike parameters \( K_1, \ldots, K_N > 0 \), then the value at time \( t \) of a contingent claim delivering a payoff \( P \) at maturity is equal to the risk-neutral expectation

\[
\mathcal{C} = E_{Q}^{\mathbb{F}}_{t} [ e^{-rT} P(S_T, K_1, \ldots, K_n) ] .
\] (6)

Using (1) and assuming that the process \( \{X(t)\}_{t \geq 0} \) admits a density \( g(x,t) \), then the expectation (6) can be obtained by integrating all possible realizations for the discounted terminal payoff over the martingale measure:

\[
\mathcal{C} = e^{-rT} \int_{-\infty}^{+\infty} P \left( S_t e^{(r-q+\mu)T+x}, K_1, \ldots, K_n \right) g(x, T) \, dx .
\] (7)
2.2. The Fractional Diffusion Model and Its Financial Interpretations

To generalize the classical diffusion (3) (Black–Scholes model) and the space fractional diffusion (4) (FMLS model), we consider the following space-time-fractional diffusion equation:

\[
\left( ^{\alpha}D_{t}^{\gamma} + \mu_{a,\beta,\gamma} \partial_{x}^{\theta} \right) g(x, t) = 0, \quad x \in \mathbb{R}, \ t > 0,
\]

where the parameters \( \alpha \) and \( \gamma \) are restricted as follows: \( \alpha \in (0, 2], \gamma \in (0, \alpha] \), and where the asymmetry parameter \( \theta \) belongs to the so-called Feller–Takayasu diamond \( |\theta| \leq \min \{ \alpha, 2 - \alpha \} \). The temporal Caputo derivative is defined by:

\[
^{\alpha}D_{t}^{\gamma} f(t) = \frac{1}{\Gamma(1 - \gamma)} \int_{0}^{t} (t - u)^{1 - \gamma} f^{(\gamma)}(u) \, du,
\]

and the spatial Riesz–Feller derivative is defined via its Fourier transform by:

\[
\hat{\partial}_{x}^{\theta} \hat{f}(k) = |k|^\theta \hat{f}(\text{sign} k) e^{\theta^2 k^2 / 2} \hat{f}(k)
\]

which is simply called Riesz derivative when \( \theta = 0 \) (as an operator inverse to the Riesz potential).

For more details about the properties of the fractional operators (9) and (10), we refer e.g., to [27] for the Caputo derivative, and to [28] for the Riesz–Feller derivative.

The fundamental solution, or Green function, of the fractional diffusion equation (8) (i.e., the solution satisfying \( g(x, 0) = \delta(x) \)) has been determined in [29] under the form of a Mellin–Barnes integral:

\[
g_{a,\beta,\gamma}(x, t) = \frac{1}{2\pi i} \int_{c_{1} - i\infty}^{c_{1} + i\infty} \frac{\Gamma\left(\frac{1}{\alpha}\right) \Gamma\left(1 - \frac{1}{\alpha}\right) \Gamma(1 + t)}{\Gamma\left(1 - \frac{\theta}{\alpha} t_{1}\right) \Gamma\left(1 - \frac{\theta}{\alpha} t_{1}\right) \Gamma\left(1 - \frac{\theta}{\alpha} t_{1}\right) \Gamma\left(1 - \frac{\theta}{\alpha} t_{1}\right)} \left(\frac{x}{\mu_{a,\beta,\gamma} t_{1}^{\gamma}}\right) \, dt_{1}.
\]

Equation (11) holds for \( x > 0 \) and extends to the negative axis via the symmetry relation

\[
g_{a,\beta,\gamma}(-x, t) = g_{a,\beta,\gamma}(x, t).
\]

with this parametrization, the maximal asymmetry hypothesis \( \beta = -1 \) is equivalent to \( \theta = \alpha - 2 \); the corresponding Green function and convexity adjustment will be denoted by \( g_{a,\gamma} := g_{a,\beta,\gamma} \) and \( \mu_{a,\gamma} := \mu_{a,\beta,\gamma} \), and, by analogy with (7), the price of a contingent claim in the fractional diffusion model will be defined as:

\[
C_{a,\gamma} = e^{-rt} \int_{-\infty}^{+\infty} \mathcal{P} \left( S_{t} e^{(r-q + \mu_{a,\gamma}) t + x}, K_{1}, \ldots, K_{n} \right) g_{a,\gamma}(x, \tau) \, dx.
\]

Let us discuss some financial implications of such a model, and what behavior it can capture. First, we should note that we can re-write the Green function (11) as a function of the re-scaled spatial variable only: \( g_{a,\gamma}(x, t) = C_{a,\gamma}(\frac{|x|}{(-\mu_{a,\gamma} t^{\gamma})}) \). This very interesting fact shows that the fractional diffusion model has a so-called self-similarity property: a re-scaling of time is actually equivalent to some suitable re-scaling of space. In other words, the introduction of the time-fractional parameter \( \gamma \) preserves the self-similarity property that holds for \( \alpha \)-stable laws (see more details in [30]), but modifies the re-scaling ratio.

This observation is coherent with the fact that the fractional diffusion model (13) extends the exponential market models previously discussed: when \( \gamma = 1 \) and \( 1 < \alpha < 2 \) we recover the FMLS price and, when \( \alpha = 2 \), the Black–Scholes price. The introduction of a time-fractional parameter, however, opens the way to the modeling of time related phenomena: in [17], the fractional diffusion equation is interpreted as a random walk subordinated to a renewal process, which allows for the
market time to differ from the physical time (such models have been tested on German and Italian bond markets in [31]). Please note that the idea of introducing a non-uniform business time is not novel (it was first introduced in by Clark in [32] to model the cotton futures markets) and is motivated by the fact that in capital markets, the volume of transactions is not constant over time, and that periods of high volatility alternate with periods of relative calm.

Another famous time-subordinated model is provided by the Variance Gamma model [3], where the stochastic process in (1) is a Brownian motion whose time follows a Gamma process \( \alpha \) (it was first introduced in by Clark in [32] to model the cotton futures markets) and is motivated by the presence of a fat tail parameter \( \alpha \) allows for a supplementary degree of freedom by assuming that the return’s distribution can be leptokurtic (and not only Gaussian, like in the Variance Gamma model). Interestingly, calibrations made, e.g., in [12] on index options markets have shown that \( \alpha \) could be quite different from 2 (typically, \( \alpha \simeq 1.6–1.7 \)) and that both fractional parameters \( \alpha \) and \( \gamma \) appeared to vary simultaneously and in the same direction, leaving the diffusion scaling exponent \( \gamma \) relatively stable. Last, let us also mention that the impact of \( \alpha \) on long term volatility behavior has been discussed in full in [7], and that the impact of \( \gamma \) on the construction of volatility surface has been studied in [20] in the case of a time-fractional Black–Scholes setup.

### 2.3. Fully Factorized Pricing Formula

In all of the following, we will restrict ourselves to the case \( \alpha \in (1,2) \) corresponding to the so-called stable Pareto distributions, because this configuration is known to be the most financially relevant (see initial calibrations by Mandelbrot [33] and subsequent works by Mittnik, Rachev, etc. [34]; this hypothesis is also made by Carr and Wu for the FMLS model). To simplify the notations, we also assume that there is no dividend yield (\( q = 0 \)). Last, let us mention that the convexity adjustment \( \mu_{\alpha,\gamma} \) for the fractional diffusion model has been determined in [19,20]:

\[
\mu_{\alpha,\gamma} = -\log \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(1+an)}{\Gamma(1+\gamma an)} \mu_n^n
\]  

(14)

where \( \mu_n \) is the FMLS convexity adjustment (5) and where the series (14) is valid for \( \gamma \in (1 - \frac{1}{\alpha}, \alpha) \). Using a first order Taylor expansion for \( \log(1+u) \) and the expression (5), we get the useful approximation:

\[
\mu_{\alpha,\gamma} = \frac{1}{\pi} \frac{\Gamma(1+\alpha)\Gamma\left(\frac{1-a}{\alpha}\right)}{\Gamma(1+\gamma a)} \left(\frac{\sigma}{\sqrt{2}}\right)^{\alpha} + O\left(\sigma^{2\alpha}\right).
\]  

(15)

Let us now establish a factorized formula for the price of a contingent claim; it follows from (11) and (12) that introducing the Mellin transforms:

\[
G^+(t_1) := \frac{\Gamma\left(1-t_1\right)}{\Gamma\left(1-\gamma \frac{t_1}{\alpha}\right)} \quad G^-(t_1) := \frac{\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(1-\frac{1}{\alpha}\right)\Gamma(1-t_1)}{\Gamma\left(1-\frac{2}{\alpha}t_1\right)\Gamma\left(\frac{2-1}{\alpha}t_1\right)}
\]  

(16)

then the Green function reads:

\[
g_{\alpha,\gamma}(x,t) = \frac{1}{\alpha x} \int_{c_1-i\infty}^{c_1+i\infty} \left(G^+(t_1)\mathbb{1}_{\{x>0\}} + G^-(t_1)\mathbb{1}_{\{x<0\}}\right) \left(\frac{|x|}{(-\mu_{\alpha,\gamma}^{-1}t)^{\frac{1}{\alpha}}}\right)^{t_1} \frac{dt_1}{2i\pi}
\]  

(17)

where \( c_1 \) belongs to the region of convergence of the Mellin transforms, i.e., the region where the Gamma function are analytic: if \( G^+(t_1) \) is considered, one has to impose \( c_1 \in (-\infty, 1) \) and, if \( G^-(t_1) \) is considered, one has to impose \( c_1 \in (0, 1) \) (because \( \alpha > 1 \)). Let us consider payoff functions taking values on the positive real axis only (this corresponds to call-like options, where the payoff...
we will adopt the standard financial notation \( X \) price and the payoff function:

\[
P^s(t_1) := \int_0^\infty \mathcal{P} \left( S_t e^{(r+\mu_\gamma)\tau + x}, K_1, \ldots, K_n \right) x^{t_1-1} \, dx.
\] (18)

If we assume that this transform converges for \( \text{Re}(t_1) \in (c_-, c_+) \) for some real numbers \( c_- < c_+ \), then, putting together (13), (17) and (18), we have proved:

**Proposition 1** (Factorized pricing formula). Let \( c_1 \in (-\infty, 1) \cap (c_-, c_+) \), where the intersection is assumed to be nonempty. Then the price of the contingent claim admits a fully factorized form in the Mellin space:

\[
C_{a,\gamma} = \frac{e^{-rt}}{a} \int_{c_1-\infty}^{c_1+\infty} G^+_s(t_1) P^s(t_1) \left( -a_{\gamma} \tau^r \right)^{-\frac{t_1}{\pi}} \frac{df_1}{2\pi i}.
\] (19)

The factorized formula (19) is a very powerful tool for establishing practical expressions for various option prices. Indeed, as an integral over a vertical line in the complex plane, it can be conveniently expressed as a sum of residues by right or left closing the contour; this procedure results in an expression of the form

\[
C_{a,\gamma} = \frac{e^{-rt}}{a} \times \sum \left[ \text{residues of } G^+_s(t_1) P^s(t_1) \times \text{powers of } \left( -a_{\gamma} \tau^r \right)^{\frac{t_1}{\pi}} \right].
\] (20)

The only technical difficulty lies in the computation of the Mellin transform for the payoff function \( P^s(t_1) \); as we shall see in the examples to come, it can necessitate the introduction of a second complex variable \( t_2 \), and in that case the residue summation (20) becomes a double summation. Last, let us mention that the result is of course also valid for payoffs taking non-null value on the negative axis, and in this case it suffices to replace \( G^+_s(t_1) \) by \( G^-_s(t_1) \) in the factorized formula (19) and to replace \( x \) by \( |x| \) in the Mellin transform (18).

We are now ready to establish pricing formulas for several path-independent payoffs featuring some exotic features (log, power, capped options etc.); to that extent, let us define the forward strike price \( F \) and the log-forward moneyness \( k \):

\[
F := Ke^{-rt} \quad k := \log \frac{S_t}{F} = \log \frac{S_t}{K} + rt.
\] (21)

As we will be dealing extensively with the Gamma function, let us recall that \( \Gamma(z) \) is analytic in the right half-plane \( \text{Re}(z) > 0 \) and that it can be extended to the left half-plane thanks to the functional relation \( \Gamma(1 + z) = z\Gamma(z) \); this continuation is singular at every negative integer \( -n, n \in \mathbb{N} \), and the associated residues are \( \frac{(-1)^n}{n!} \) (see [35] or any other classical monograph on special functions). Last, we will adopt the standard financial notation \( X^+ \) := \( X \mathbb{1}_{\{X > 0\}} \).

3. Log Options and Log Contracts

Log options (see [24]) are options on the rate of return of the underlying asset \( S \); the payoff of a call option is:

\[
\mathcal{P}_{\log \text{call}}(S_T, K) := [\log S_T - \log K]^+
\] (22)

and the payoff of a put options is:

\[
\mathcal{P}_{\log \text{put}}(S_T, K) := [\log K - \log S_T]^+.
\] (23)
Log contracts were introduced by Neuberger in [22] for volatility hedging purposes, and can be synthetically obtained by being long of a log call and short of a log put:

$$P_{\text{log}}(S_T, K) := P_{\text{log call}}(S_T, K) - P_{\text{log put}}(S_T, K) = \log \frac{S_T}{K}. \quad (24)$$

**Formula 1** (Log call option). The price of the log call option is:

$$C_{\alpha, \gamma}^{(\text{log call})} = e^{-rT} \sum_{n=0}^{\infty} \frac{1}{n! \Gamma \left(1 + \frac{1-n}{\alpha} \right)} (k + \mu_{\alpha, \gamma} \tau)^n (-\mu_{\alpha, \gamma} \tau)^{\frac{1-n}{\alpha}}. \quad (25)$$

**Proof.** With the notations (21), we can write

$$P_{\text{log call}}(S_T e^{(r+\mu_{\alpha, \gamma})T+x}, K) = [k + \mu_{\alpha, \gamma} \tau + x]^+ \quad (26)$$

therefore, the Mellin transform (18) for the payoff function can be computed directly:

$$P_{\text{log call}}^\prime(t_1) = \int_{-k-\mu_{\alpha, \gamma} \tau}^{\infty} (k + \mu_{\alpha, \gamma} \tau + x)^{t_1-1} dx = \frac{(-k - \mu_{\alpha, \gamma} \tau)^{1+t_1}}{t_1(1+t_1)}. \quad (27)$$

Using Proposition 1 and the functional relation $\Gamma(1 - t_1) = -t_1 \Gamma(-t_1)$, the option price is:

$$C_{\alpha, \gamma}^{(\text{log call})} = e^{-rT} \frac{c_1^{\gamma} \Gamma(-t_1)}{c_1^{\alpha}} = \frac{\Gamma(-t_1)}{t_1(1+t_1)\Gamma(1-\gamma \frac{n}{\alpha})} (-k - \mu_{\alpha, \gamma} \tau)^{1+t_1} (-\mu_{\alpha, \gamma} \tau)^{-\frac{1}{\alpha} - n} \frac{dt_1}{2i\pi} \quad (28)$$

which converges for $Re(t_1) < -1$. The analytic continuation of the integrand in the right half-plane has:

- a simple pole in $t_1 = -1$ with residue
  $$\frac{(-\mu_{\alpha, \gamma} \tau)^{\frac{1}{\alpha}}}{\Gamma(1+\frac{1}{\alpha})}, \quad (29)$$
  - a series of poles at every positive integer $t_1 = n$ with residues:
  $$\frac{(-1)^n}{(n+1)! \Gamma(1-\gamma \frac{n}{\alpha})} (-k - \mu_{\alpha, \gamma} \tau)^{1+n} (-\mu_{\alpha, \gamma} \tau)^{-\frac{1}{\alpha} - n}. \quad (30)$$

Summing the residues (29) and (30) for all $n$ and re-ordering yields (25). \( \square \)

It is interesting to note that Formula 1 extends the formula obtained in [26] for the FMLS model ($\gamma = 1$)

$$C_{\alpha}^{(\text{log call})} = e^{-rT} \sum_{n=0}^{\infty} \frac{1}{n! \Gamma \left(1 + \frac{1-n}{\alpha} \right)} (k + \mu_{\alpha} \tau)^n (-\mu_{\alpha} \tau)^{\frac{1-n}{\alpha}}$$

(31)

to the more general case $\gamma \neq 1$; note also that in the case of the Black–Scholes model ($\alpha = 2, \gamma = 1$) a closed pricing formula has been derived by Haug (see [36]):

$$C_2^{(\text{log})}(S, K, r, \sigma, \tau) = e^{-r\tau} \sigma \sqrt{\tau} \left[ n(d_2) + d_2 N(d_2) \right], \quad d_2 := \frac{k - \sigma^2 \tau}{\sigma \sqrt{\tau}} \quad (32)$$

where $n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ is the Gaussian density and $N(x)$ is the Normal cumulative distribution function. This case is also recovered by Formula 1, as demonstrated in Table 1.
Table 1. Price of a log call option given by Formula 1 truncated at \( n = n_{max} \), compared with previously known formula for particular cases (FMLS and Black–Scholes model) and several market situations. We observe that the convergence is extremely fast: typically, 5 to 10 terms are enough to get a precision of \( 10^{-4} \), and even less when the asset is at the money. Choice of parameters: \( K = 4000, \sigma = 20\%, r = 1\% \) and \( \tau = 2 \) years.

| In the money (\( S_t = 5000 \)) | \( n_{max} = 3 \) | \( n_{max} = 5 \) | \( n_{max} = 10 \) | Aguilar (31) | Haug (32) |
|---------------------------------|-----------------|-----------------|-----------------|-------------|----------|
| \( a = 1.7, \gamma = 0.8 \)    | 0.2527          | 0.2525          | 0.2525          | -           | -        |
| \( a = 1.7, \gamma = 1 \)      | 0.2550          | 0.2548          | 0.2548          | -           | -        |
| \( a = 2, \gamma = 1 \)        | 0.2387          | 0.2375          | 0.2375          | 0.2375      | 0.2375   |

| At the money forward (\( S_t = F = 3920.79 \)) | \( n_{max} = 3 \) | \( n_{max} = 5 \) | \( n_{max} = 10 \) | Aguilar (31) | Haug (32) |
|-----------------------------------------------|-----------------|-----------------|-----------------|-------------|----------|
| \( a = 1.7, \gamma = 0.8 \)                  | 0.1061          | 0.1061          | 0.1061          | -           | -        |
| \( a = 1.7, \gamma = 1 \)                    | 0.1045          | 0.1045          | 0.1045          | 0.1045      | -        |
| \( a = 2, \gamma = 1 \)                       | 0.0922          | 0.0921          | 0.0921          | 0.0921      | 0.0921   |

| Out of the money (\( S_t = 3000 \)) | \( n_{max} = 3 \) | \( n_{max} = 5 \) | \( n_{max} = 10 \) | Aguilar (31) | Haug (32) |
|------------------------------------|-----------------|-----------------|-----------------|-------------|----------|
| \( a = 1.7, \gamma = 0.8 \)        | 0.0368          | 0.0292          | 0.0299          | -           | -        |
| \( a = 1.7, \gamma = 1 \)          | 0.0297          | 0.0206          | 0.0214          | 0.0214      | -        |
| \( a = 2, \gamma = 1 \)            | 0.0253          | 0.0188          | 0.0195          | 0.0195      | 0.0195   |

To illustrate an application of Proposition 1 for payoffs taking non-null values on the negative axis, we now provide a pricing formula for a log put option; for the sake of simplicity, we restrict ourselves to the space fractional model (i.e., \( \gamma = 1 \)).

**Formula 2 (Log put option).** The price of the log put option (in the space fractional model) is:

\[
P_{\log \text{put}}(t_1) = e^{-rt_1} \left[ -(k + \mu_{\alpha})^{1+\gamma} \sum_{n=0}^{\infty} \frac{1}{n!\Gamma\left(1 + \frac{1}{\alpha}\right)} (k + \mu_{\alpha})^{\frac{1}{\alpha}} \right]. \quad (33)
\]

**Proof.** The Mellin transform (18) now reads:

\[
P_{\log \text{put}}(t_1) = \int_{-\infty}^{-k+\mu_{\alpha}t} (k + \mu_{\alpha}t + x)|\gamma|^{\frac{1}{\gamma}} \, dx = -\frac{(k + \mu_{\alpha}t)^{1+\gamma}t_1}{t_1(1+t_1)} \quad (34)
\]

and consequently, the put price reads

\[
P_{\log \text{put}} = e^{-rt_1} \int_{c_1}^{c_1+i\infty} \frac{\Gamma\left(\frac{t_1}{\alpha}\right)}{(1+t_1)^{\frac{\alpha}{\alpha}} \Gamma\left(\frac{\alpha-1}{\alpha}t_1\right)}(k + \mu_{\alpha}t)^{1+\gamma}(-\mu_{\alpha}t)^{-\frac{\alpha}{\alpha}} \, d\gamma \quad (35)
\]

As for Formula 1, the analytic continuation of the integrand in the half-plane \( Re(t_1) \geq -1 \) admits several poles:

- a simple pole in \( t_1 = -1 \) with residue
  \[
  -\frac{\Gamma\left(-\frac{1}{\alpha}\right)}{(1+1)^{\frac{\alpha}{\alpha}}}(-\mu_{\alpha}t)^\frac{1}{\alpha} \quad (36)
  \]
  where the simplification is a direct consequence of the functional relation \( \Gamma(1+z) = z\Gamma(z) \);

- a series of poles at every positive integer \( t_1 = n \) with residues:
  \[
  -\frac{(1)^n}{(1+n)^{\frac{n}{\alpha}}} \frac{\Gamma\left(\frac{n}{\alpha}\right)}{(1+n)^{\frac{\alpha-1}{\alpha}} \Gamma(\frac{n}{\alpha})} (k + \mu_{\alpha}t)^{1+\gamma}(-\mu_{\alpha}t)^{-\frac{n}{\alpha}}. \quad (37)
  \]
The residue for \( n = 0 \) requires special attention, because, although both numerator and denominator are singular, the ratio is not:

\[
\frac{\Gamma\left(\frac{n}{\alpha}\right)}{\Gamma\left(\frac{n-1}{\alpha}\right)} \sim n \to 0 \quad \frac{\frac{n}{\alpha}}{(n-1)n} = \alpha - 1,
\]

and therefore the residue for \( n = 0 \) is:

\[
-(\alpha - 1)(k + \mu_\alpha \tau).
\]

The other residues (37) for \( n \geq 1 \) can be simplified thanks to the functional relation \( \Gamma(1 + z) = z\Gamma(z) \), resulting in:

\[
\frac{1}{(n+1)!} \frac{1}{\Gamma(1 - \frac{n}{\alpha})} (k + \mu_\alpha \tau)^{1+\eta}(-\mu_\alpha \tau)^{-\eta}.
\]

Summing (36), (39) and (40) and re-organizing the terms yields (33).

As a direct consequence of Formulas 1 and 2, we obtain the following formula for the price of the log contract:

**Formula 3 (Log contract).** The price of the log contract (in the space fractional model) is:

\[
C^{(\text{log})}_a = e^{-r\tau} (k + \mu_\alpha \tau).
\]

Please note that taking \( \alpha = 2 \) in Formula 3, we have \( \mu_2 = -\sigma^2/2 \) from the definition (5) and therefore we recover the formula by Neuberger [22] for the price of a log contract in the Black–Scholes model:

\[
C^{(\text{log})}_{B-S} = e^{-r\tau} \left[ \log \frac{S_T}{K} + \left( r - \frac{\sigma^2}{2} \right) \tau \right].
\]

4. Digital and Related Options (European, Gap)

Digital (also called binary) call options are of two kinds: cash-or-nothing (C/N) call options, paying a constant amount if the option terminates in the money, and asset-or-nothing (A/N) call options paying the value of the underlying asset if the option terminates in the money:

\[
P_{C/N}(S_T, K) := 1_{\{S_T > K\}} \quad P_{A/N}(S_T, K) := S_T \times 1_{\{S_T > K\}}.
\]

A gap call option (also called pay-later option) offers a nonzero payoff on the condition that some trigger price is attained at \( t = T \). More precisely, the payoff depends on the terminal price \( S_T \), on a strike price \( K_1 \) and on a trigger price \( K_2 \):

\[
P_{\text{gap}}(S_T, K_1, K_2) := P_{A/N}(S_T, K_2) - K_1 P_{C/N}(S_T, K_2).
\]

A European call option is a gap option where trigger and strike prices coincide; from definitions (43) and (44), we have:

\[
P_{\text{eur}}(S_T, K) := P_{\text{gap}}(S_T, K, K) = [S_T - K]^+.
\]

Pricing formulas for the digital call options have been derived in [21]:

**Formula 4** (Digital options). The price of the cash-or-nothing call option is:

\[
C^{(C/N)}_{a,\gamma} = e^{-r\tau} \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(1 - \frac{2}{\alpha n})} \left( k + \mu_{a,\gamma} \tau \right)^n (-\mu_{a,\gamma} \tau)^{-\eta}.
\]
and the price of the asset-or-nothing call option is:

\[ C_{A/N}^{(A/N)}(\alpha, \gamma) = \frac{Ke^{-rT}}{\alpha} \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(1 + \gamma \frac{m-n}{\alpha})} (k + \mu_{\alpha, \gamma} \tau)^n (-\mu_{\alpha, \gamma} \tau)^{\frac{m-n}{\alpha}}. \] (47)

**Proof.** See [21]. \( \square \)

The value of the gap call option can be easily obtained from (46) and (47):

\[ C_{\text{gap}}^{(A/N)}(\alpha, \gamma) = C_{A/N}^{(A/N)}(\alpha, \gamma) - K_1 C_{A/N}^{(C/N)}(\alpha, \gamma). \] (48)

where the moneyness \( k \) here is to be understood in terms of the trigger price \( K_2 \), i.e., \( k = \log \frac{S}{K_2} + r \tau \). We can also note that the series (46) is actually a particular case of the series (47) for \( m = 0 \); therefore, it follows immediately from definition (45) that the value of an European option is:

**Formula 5** (European option). The price of the European call option is:

\[ C_{\text{eur}}(\alpha, \gamma) = \frac{Ke^{-rT}}{\alpha} \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(1 + \gamma \frac{m-n}{\alpha})} (k + \mu_{\alpha, \gamma} \tau)^n (-\mu_{\alpha, \gamma} \tau)^{\frac{m-n}{\alpha}}. \] (49)

Particular interest should be paid to the case where the asset is at the money forward, i.e., \( S_t = F \) (or, equivalently, \( k = 0 \)), because it is the starting point for building volatility surfaces. When \( k = 0 \), the series (49) becomes

\[ C_{\text{eur atm}}^{(A)} = \frac{S}{\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1 + \gamma \frac{m-n}{\alpha})} (-\mu_{\alpha, \gamma} \tau)^{\frac{m-n}{\alpha}} + O(\mu_{\alpha, \gamma} \tau). \] (50)

Now, let us choose \( \alpha = 2 \) and let us call this configuration the “subordinated Black–Scholes model” (that is, a model with a tail index \( \alpha = 2 \) like in the Black–Scholes case, but with a time subordination parameter given by the order of the time-fractional derivative \( \gamma \)). In this configuration, the formula for the convexity adjustment (15) becomes:

\[ \mu_{\gamma} = -\frac{\sigma^2}{\Gamma(1 + 2\gamma)} + O(\sigma^4) \] (51)

and the ATM price (50) becomes

\[ C_{\text{eur atm}}^{(eur atm)} = \frac{S}{2} \sigma \sqrt{\frac{\tau^2}{\Gamma(1 + 2\gamma)\Gamma(1 + \frac{\gamma}{2})}} + O(\sigma^2 \tau). \] (52)

Formula (52) allows for a practical estimation of implied volatility (at first order); if we denote by \( C_0 \) the ATM market price of an European call, then the ATM implied volatility \( \sigma_I \) in the subordinated Black–Scholes model is given by:

\[ \sigma_I = \frac{2\sqrt{\Gamma(1 + 2\gamma)\Gamma(1 + \frac{\gamma}{2})}}{\Gamma(1 + \frac{\gamma}{2})} \frac{C_0}{S}. \] (53)
Taking $\alpha = 2$ in (52), we also recover the celebrated Brenner-Subramanyam approximation [37] for the ATM Black–Scholes price:

$$C_{\text{eur atm}}^{(B-S)} \simeq \frac{S}{\sqrt{2\pi}} \sigma \sqrt{\tau},$$

(54)

that market practitioners often write $C \simeq 0.4 \sigma \sqrt{\tau}$ because $1/\sqrt{2\pi} \simeq 0.399$. Inverting (54), we obtain the Black–Scholes ATM volatility:

$$\sigma_l = \sqrt{\frac{2\pi}{\tau}} \frac{C_0}{S},$$

(55)

Comparing (53) and (55) clearly demonstrates the difference between the usual Black–Scholes model and the subordinated one. In (55), the implied volatility does not evolve in time (as a function of the time horizon), the reason being that the underlying stochastic process in the Black–Scholes model is a Brownian motion, which is a stationary process (or, more precisely, has stationary increments). On the contrary, thanks to the supplementary degree of freedom $\gamma$, (53) allows the volatility to be non-constant and to vary over time, and therefore to capture complex time dependent phenomena that the Black–Scholes model fails to reproduce. Let us put these results into perspective with another subordinated model, namely the Variance Gamma (VG) model. In the (symmetric) VG model, the stochastic process driving the exponential process (1) is the time-changed Wiener process $X_t := W_{\gamma(t,1,\nu)}$, where $\gamma(t,1,\nu)$ is a process whose increments follow a Gamma distribution with mean 1 and variance $\nu$; in [38], it has been proved that when the asset is at the money forward, the VG European call price is

$$C_{\text{eur atm}}^{(VG)} = \frac{S}{\sqrt{2\pi}} \frac{\Gamma\left(\frac{1}{2} + \frac{\nu}{\nu'}\right)}{\Gamma\left(\frac{1}{2}\right)} \sigma \sqrt{\nu'},$$

(56)

Please note that it follows from Stirling’s approximation that

$$\frac{\Gamma\left(\frac{1}{2} + \frac{\nu}{\nu'}\right)}{\Gamma\left(\frac{1}{2}\right)} \sim \sqrt{\frac{\nu'}{\nu}},$$

(57)

and therefore, the VG price (56) also recovers the Black–Scholes price (54) in the low variance limit. From (56) we also deduce the implied ATM volatility

$$\sigma_l = \sqrt{\frac{2\pi}{\nu'}} \frac{C_0}{S},$$

(58)

whose structure presents obvious similarities with the subordinated Black–Scholes volatility (53): both volatilities are linear functions of $\frac{C_0}{S}$ and non-linear functions of their subordination parameter. Notable difference is the dependence on the horizon $\tau$: in the subordinated Black–Scholes model, the volatility is a function of $\tau^2$ for all maturities, while in the Variance Gamma model we can Taylor expand (58) for $\tau \to 0$ with the result:

$$\sigma_l = \sqrt{\frac{2\pi}{\nu}} \frac{C_0}{S} + O(1).$$

(59)

In other words, the sort term behavior of the implied volatility depends exponentially on $\gamma$ in the subordinated Black–Scholes model, but polynomially on $\nu$ in the VG model. These observations are recorded in Table 2.
Table 2. Volatility modeling for the subordinated Black–Scholes model (fractional diffusion model with \( \alpha = 2 \)) and the symmetric Variance Gamma model. In both cases the non-linear dependence in the subordination parameter allows for a richer structure than in the Black–Scholes framework, which is recovered in particular regimes.

| Subordinated Black–Scholes Model | Symmetric Variance Gamma Model |
|----------------------------------|--------------------------------|
| ATM price \( \frac{S}{\sigma \sqrt{T}} \) | \( \frac{S}{\sigma \sqrt{T}} \) |
| Non-fractional regime (\( \gamma \to 1 \)) | \( \frac{S}{\sigma \sqrt{T}} \) |
| ATM vol \( \frac{2 \sqrt{T}}{\Gamma(1+\frac{2\gamma}{\tau})} \frac{C_0}{\tau^2} \) | \( \sigma_t = \frac{2 \sqrt{T}}{\Gamma(1+\frac{2\gamma}{\tau})} \frac{C_0}{\tau^2} \) |
| Non-fractional regime (\( \gamma \to 1 \)) | \( \sigma_t = \frac{2 \sqrt{T}}{\Gamma(1+\frac{2\gamma}{\tau})} \frac{C_0}{\tau^2} \) |
| Short-term volatility (\( \tau \to 0 \)) | Short-term volatility (\( \tau \to 0 \)) |
| “exponential subordination” | “polynomial subordination” |

5. Power Options

Power options are a popular tool for delivering a non-linear payoff and therefore for increasing the leverage ratio of trading strategies. The payoff of a power cash-or-nothing and of a power asset-or-nothing call are:

\[
P_{power C/N}(S_T, K) := \mathbb{I}_{\{S_T^\alpha > K\}} \quad P_{power A/N}(S_T, K) := S_T^\alpha \times \mathbb{I}_{\{S_T^\alpha > K\}} \tag{60}
\]

where \( u > 0 \). The power European call is the difference between the power asset-or-nothing call and the power cash-or-nothing call:

\[
P_{power eur}(S_T, K) := P_{power A/N}(S_T, K) - K P_{power C/N}(S_T, K) = [S_T^\alpha - K]^+. \tag{61}
\]

Let us introduce the modified log-forward moneyness

\[
k_u := \log \frac{S_t}{K^\alpha} + rt \tag{62}
\]

which coincide with the usual definition of moneyness (21) when \( u = 1 \) (i.e., \( k_1 = k \)).

**Formula 6** (Power option: cash-or-nothing). The *price of the power cash-or-nothing call option* is:

\[
C_{u,\gamma}^{power C/N}(t) = \frac{e^{-rt}}{\alpha} \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(1 - \gamma \frac{n}{\alpha})} (k_u + \mu_{\alpha,\gamma} \tau)^n (-\mu_{\alpha,\gamma} \tau)^{-\frac{n}{\alpha}}. \tag{63}
\]

**Proof.** From definitions (60) and (62), we can write

\[
P_{power C/N}(S_T e^{(\tau + \mu_{\alpha,\gamma})r}, K) = \mathbb{I}_{\{x > -k_u - \mu_{\alpha,\gamma} \tau\}} \tag{64}
\]

therefore, the Mellin transform (18) for the payoff function is:

\[
P_{log call}(t_1) = \int_{-k_u - \mu_{\alpha,\gamma} \tau}^{\infty} x^{t_1-1} dx = -\frac{(-k_u - \mu_{\alpha,\gamma} \tau)^{t_1}}{t_1}. \tag{65}
\]
Using Proposition 1 and the functional relation \( \Gamma(1 - t_1) = -t_1 \Gamma(-t_1) \), the option price is:

\[
C_{\text{log call}}^{(\log\, call)} = \frac{e^{-\mu t}}{\alpha} \int_{c_1-i\infty}^{c_1+i\infty} \Gamma(-t_1) \left( -k_u - \mu a, \gamma \right)^{t_1} \left( -\mu a, \gamma \right)^{1-t_1} \frac{dt_1}{2\pi i} \tag{66}
\]

which converges for \( \text{Re}(t_1) < 0 \). The analytic continuation of the integrand in the right half-plane has poles induced by the singularities of the \( \Gamma(-t_1) \) function at every positive integer \( t_1 = n \in \mathbb{N} \), with residue:

\[
\frac{(-1)^n}{n!} \frac{1}{\Gamma(1 - \gamma \frac{u}{t})} \left( -k_u - \mu a, \gamma \right)^n \left( -\mu a, \gamma \right)^{-\frac{n}{2}}. \tag{67}
\]

Simplifying and summing the residues for all \( n \) yields (63). \( \square \)

In Table 3 we study the influence of the power \( u \) and of the time-fractional parameter \( \gamma \) on the speed of convergence of the series (63). As expected, the convergence is slower when \( u \) grows but is accelerated in the small \( \gamma \) regime: for instance, when \( \gamma = 0.6 \), only 5 terms are enough to get a precision of \( 10^{-3} \) when \( u = 1 \) and only 10 terms to get a precision \( 10^{-2} \) when \( u = 1.1 \).

**Table 3.** Price of the power cash-or-nothing call given by formula 6 truncated at \( n = n_{\text{max}} \), for various powers \( u \). The convergence is less rapid when \( u \) grows, but is accelerated for small time-fractional parameter \( \gamma \). Choice of parameters: \( S = 4200, K = 4000, \sigma = 20\%, r = 1\%, \tau = 2 \) years and \( \alpha = 2 \).

| \( n_{\text{max}} \) | 5 | 10 | 50 | 100 | 150 |
|---|---|---|---|---|---|
| \( u = 1 \) | | | | | |
| \( \gamma = 0.6 \) | 0.481 | 0.481 | 0.481 | 0.481 | 0.481 |
| \( \gamma = 0.8 \) | 0.509 | 0.509 | 0.509 | 0.509 | 0.509 |
| \( \gamma = 1 \) | 0.530 | 0.530 | 0.530 | 0.530 | 0.530 |
| \( u = 1.1 \) | | | | | |
| \( \gamma = 0.6 \) | 2.340 | 2.316 | 2.312 | 2.312 | 2.312 |
| \( \gamma = 0.8 \) | 0.896 | 1.294 | 1.328 | 1.328 | 1.328 |
| \( \gamma = 1 \) | 1.778 | 1.409 | 0.977 | 0.977 | 0.977 |
| \( u = 1.2 \) | | | | | |
| \( \gamma = 0.6 \) | -3.397 | 4.026 | 0.761 | 0.761 | 0.761 |
| \( \gamma = 0.8 \) | -7.488 | -16.760 | 1.224 | 1.224 | 1.224 |
| \( \gamma = 1 \) | 24.567 | 151.718 | 4.973 | 0.980 | 0.980 |

Let us now study how cap/floor features can affect the price of options in the fractional diffusion model. When power options are involved, payoffs can become catastrophically huge: think, for instance, of a call option struck at \( K = 4000 \), which would end in the money at \( S_T = 5000 \); an European call would deliver a payoff of \( 5000 - 4000 = 1000 \), but a “square” European call (that is, an European power call with \( u = 2 \)) would deliver an enormous payoff of \( 5000^2 - 4000 = 24,996,000 \). To avoid such a situation, it is common to introduce a cap, i.e., an upper bound that considerably limits the exercise range of the option. For instance, a capped power cash-or-nothing call has the following payoff:

\[
P_{\text{capped power}} C/N(S_T, K_+, K_-) = \mathbb{I}_{\{K_- < S_T < K_+\}} \tag{68}
\]

and, when \( K_+ \to +\infty \), it degenerates into a usual power cash-or-nothing call with strike \( K = K_- \). If we introduce the notation

\[
k_u^\pm := \log \frac{S_t}{K_u^\pm} + r \tau, \tag{69}
\]

then the following pricing formula holds:
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Formula 7 (Power option: capped cash-or-nothing). The price of the capped power cash-or-nothing call option is:

\[
C_{\text{capped power } C/N}^{\alpha, \gamma} = \frac{e^{-rt}}{\alpha} \sum_{n=0}^{\infty} \frac{1}{n! \Gamma \left(1 - \frac{\gamma}{\alpha} \right)} \left( (k_u^{+} + \mu_{a, \gamma} \tau)^{n} - (k_u^{-} + \mu_{a, \gamma} \tau)^{n} \right) (-\mu_{a, \gamma} \tau)^{\frac{n}{\alpha}}. \tag{70}
\]

Proof. It suffices to note that we can write:

\[
P_{\text{capped power } C/N}(S_{t}e^{(r+\mu_{a, \gamma})\tau + x}, K_{+}, K_{-}) = \mathbb{I}_{\{k_{u}^{-} - \mu_{a, \gamma} \tau < x < k_{u}^{+} - \mu_{a, \gamma} \tau\}} \tag{71}
\]

and therefore, that the Mellin transform (18) for the payoff function is:

\[
P_{\text{capped power } C/N}(t_{1}) = \int_{-k_{u}^{-} - \mu_{a, \gamma} \tau}^{-k_{u}^{+} - \mu_{a, \gamma} \tau} x^{t_{1} - 1} dx = \frac{(-k_{u}^{+} - \mu_{a, \gamma} \tau)^{t_{1}} - (-k_{u}^{-} - \mu_{a, \gamma} \tau)^{t_{1}}}{t_{1}}. \tag{72}
\]

The proof is straightforward to conclude, by using Proposition 1 and by summing all residues arising at positive integers \(t_{1} = n \in \mathbb{N}\). \(\square\)

It is interesting to determine at which speed the capped option (70) converges to the uncapped one (63); it follows from formula 7 and from the definition of the moneyness (69) that this speed is:

\[
\frac{\partial C_{\text{capped power } C/N}}{\partial K_{+}} = \frac{e^{-rt}}{u \alpha K_{+}} \sum_{n=1}^{\infty} \frac{1}{(n-1)! \Gamma \left(1 - \frac{\gamma}{\alpha} \right)} \left( (k_u^{+} + \mu_{a, \gamma} \tau)^{n-1} - (k_u^{-} + \mu_{a, \gamma} \tau)^{n-1} \right) (-\mu_{a, \gamma} \tau)^{\frac{n}{\alpha}}. \tag{73}
\]

It is clear from (73) that the capped power option converges to the uncapped one slower when the power \(u\) or the cap \(K_{+}\) increase. To study the impact of the fractional parameters, let us consider the case of the subordinated Black–Scholes model (\(\alpha = 2\); inserting the representation (51) for the convexity adjustment into (73), we get, at first order:

\[
\frac{\partial C_{\gamma}^{\text{capped power } C/N}}{\partial K_{+}} = \frac{e^{-rt}}{2u K_{+}} \sqrt{\frac{\Gamma\left(1 + 2\gamma\right)}{\Gamma\left(1 - \frac{\gamma}{2}\right)}} \frac{1}{\sigma^{\frac{\gamma}{2}}}. \tag{74}
\]

The function \(\gamma \rightarrow \frac{\sqrt{\Gamma\left(1 + 2\gamma\right)}}{\Gamma\left(1 - \frac{\gamma}{2}\right)}\) is decreasing on \([0, 2]\); its maximal value is 1 (attained for \(\gamma = 0\)) and its minimal value is 0 (attained for \(\gamma = 2\)). It appears therefore that the capped power price converges to the uncapped one faster in the small \(\gamma\) regime; when \(\gamma = 1\), we recover the first order speed in the Black–Scholes model (recall that \(\Gamma\left(1/2\right) = \sqrt{\pi}\)):

\[
\frac{\partial C_{\text{B–S}}^{\text{capped power } C/N}}{\partial K_{+}} = \frac{e^{-rt}}{u K_{+} \sigma \sqrt{2\pi \tau}}. \tag{75}
\]

Let us now establish a formula that calls for the introduction of a second Mellin variable.

Formula 8 (Power option: asset-or-nothing). The price of the power asset-or-nothing call option is:

\[
C_{\alpha, \gamma}^{\text{power } A/N} = \frac{Ke^{-rt}}{\alpha} \sum_{n=0}^{\infty} \frac{1}{n! \Gamma \left(1 + \frac{\gamma m - n}{\alpha}\right)} u^{m} (k_{u}^{+} + \mu_{a, \gamma} \tau)^{n} (-\mu_{a, \gamma} \tau)^{\frac{m-n}{\alpha}}. \tag{76}
\]
Proof. From definitions (60) and (62), we can write:

\[ P_{\text{power } A/N}(S_t e^{(r+\mu_{\gamma})T+x}, K) = Ke^{(k_u + \mu_{\gamma}T + x)} \mathbb{1}_{\{x > -k_u - \mu_{\gamma}T\}}. \]  

(77)

We cannot compute the Mellin transform (18) at this stage because it would result in incomplete Gamma functions instead of Gamma functions, and therefore the residue method would not be applicable. However, we can introduce a supplementary representation for the exponential term (see e.g., [39] or any other monograph on integral transforms):

\[ e^{\alpha(k_u + \mu_{\gamma}T + x)} = \int_{c_2 - i\infty}^{c_2 + i\infty} (-1)^{-t_2} u^{-t_2} \Gamma(t_2)(k_u + \mu_{\gamma}T + x)^{-t_2} \frac{dt_2}{2i\pi} \]  

(78)

for some \( c_2 > 0 \). Inserting it into the payoff function (77) and integrating over the \( x \) variable as requested by definition (18) results into:

\[ P^*_{\text{power } A/N}(t_1) = K \int_{c_2 - i\infty}^{c_2 + i\infty} (-1)^{-t_2} u^{-t_2} \frac{\Gamma(t_2)(1 - t_2)\Gamma(-t_1 + t_2)}{\Gamma(1 - t_1)} (k_u - \mu_{\gamma}T)^{1 - t_2} \frac{dt_2}{2i\pi}. \]  

(79)

Using Proposition 1, the option price reads:

\[ C^{(\text{power } A/N)}_{a,\gamma} = \frac{Ke^{-rT}}{\alpha} \times \int_{c_1 - i\infty}^{c_1 + i\infty} \int_{c_2 - i\infty}^{c_2 + i\infty} (-1)^{-t_2} \frac{\Gamma(t_2)(1 - t_2)\Gamma(-t_1 + t_2)}{\Gamma(1 - \gamma \frac{t_2}{\alpha})} u^{-t_2} (k_u - \mu_{\gamma}T)^{1 - t_2} \frac{1}{2\pi} \frac{dt_2}{(2i\pi)^2}. \]  

(80)

where the integral exists in the triangle (\( 0 < \text{Re}(t_2) < 1, \text{Re}(-t_1 + t_2) > 0 \)). Poles occur when \( \Gamma(t_2) \) and \( \Gamma(-t_1 + t_2) \) are singular; performing the change of variables (\( -t_1 + t_2 \to U, t_2 \to V \)) allows computation of the associated residues:

\[ (-1)^m \left( \frac{-1}{n!} \right)^m \frac{\Gamma(1 + m)}{\Gamma(1 + \gamma \frac{m - n}{\alpha})} u^m (k_u - \mu_{\gamma}T)^n (-\mu_{\gamma}T)^{\frac{m - n}{\alpha}}. \]  

(81)

Simplifying and summing all residues for \( n, m \in \mathbb{N} \) yields the series (76). □

From definition (61) and from Formulas 6 and 8, we immediately obtain the price of the European power call:

**Formula 9 (Power option: European).** The price of the power European call option is:

\[ C^{(\text{power eur})}_{a,\gamma} = \frac{Ke^{-rT}}{\alpha} \sum_{m=0}^{\infty} \frac{1}{n!} \frac{\Gamma(1 + m)}{\Gamma(1 + \gamma \frac{m - n}{\alpha})} u^m (k_u + \mu_{\gamma}T)^n (-\mu_{\gamma}T)^{\frac{m - n}{\alpha}}. \]  

(82)

Taking \( \gamma = 1 \) in Formula 9 allows recovery of the formula obtained in [26] for the power European call in the FMLS model:

\[ C^{(\text{power eur})}_{a} = \frac{Ke^{-rT}}{\alpha} \sum_{m=0}^{\infty} \frac{1}{n!} \frac{\Gamma(1 + m)}{\Gamma(1 + \gamma \frac{m - n}{\alpha})} u^m (k_u + \mu T)^n (-\mu T)^{\frac{m - n}{\alpha}}. \]  

(83)
Formula (9) also extends the formula initially obtained by Heynen and Kat in [23] for the Black–Scholes model ($\alpha = 2, \gamma = 1$):

$$C^{\text{power eur}}_{B - S} = S^u e^{(u-1)(r + \frac{\sigma^2}{2})\tau} N(d_1) - Ke^{-r\tau} N(d_2),$$

where

$$d_1 := \frac{ku + (u - \frac{1}{2})\sigma^2\tau}{\sigma\sqrt{\tau}}, \quad d_2 := d_1 - u\sigma\sqrt{\tau}. \tag{85}$$

Moreover, taking $u = 1$, we recover the Black–Scholes formula [1]:

$$C_{B - S}^{\text{eur}} = S N(d_1) - Ke^{-r\tau} N(d_2). \tag{86}$$

Table 4 illustrates the fact that the formulas (83), (84) and (86) are particular cases of Formula 9; note that when $u > 1$, the convergence can become slightly slower because of the presence of the $u^m$ powers.

| $u = 1$ | $a = 1.7, \gamma = 0.8$ | $a = 1.7, \gamma = 1$ | $a = 2, \gamma = 1$ |
|--------|------------------------|------------------------|------------------------|
| Formula 9 | 702.15 | 681.57 | 609.59 |
| Aguilar [83] | - | 681.57 | 609.59 |
| Heynen & Kat [84] | - | - | 609.59 |
| Black & Scholes [86] | - | - | - |

| $u = 1.05$ | $a = 1.7, \gamma = 0.8$ | $a = 1.7, \gamma = 1$ | $a = 2, \gamma = 1$ |
|------------|------------------------|------------------------|------------------------|
| Formula 9 | 2732.51 | 2602.19 | 2503.67 |
| Aguilar [83] | - | 2602.19 | 2503.67 |
| Heynen & Kat [84] | - | - | - |
| Black & Scholes [86] | - | - | - |

| $u = 1.1$ | $a = 1.7, \gamma = 0.8$ | $a = 1.7, \gamma = 1$ | $a = 2, \gamma = 1$ |
|------------|------------------------|------------------------|------------------------|
| Formula 9 | 6558.22 | 5889.60 | 5814.54 |
| Aguilar [83] | - | 5889.60 | 5814.54 |
| Heynen & Kat [84] | - | - | - |
| Black & Scholes [86] | - | - | - |

6. Concluding Remarks

In this article, we have started by proving a factorized pricing formula in the Mellin space, associating the Mellin transform of the Green function for the fractional diffusion model and the Mellin transform of an arbitrary payoff. Evaluating the Mellin integral by means of residue summation, we have obtained a collection of practical pricing formulas for several exotic payoffs. These formulas take the form of quickly convergent series of powers of the log-forward moneyness and of the model’s convexity adjustment. We have provided both quantitative and qualitative analysis on pricing-related topics (volatility modeling, presence of a supplementary optional condition), and compared the results with other financial models. The validity of the results has been assessed by showing that they recovered multiple formulas known in particular cases of the double fractional model (Black–Scholes model, FMLS model).

Future work should include the study of path dependent payoffs, such as American or Asian options, or of path-independent payoffs on several assets, such as correlation or spread options. Particular interest should also be paid to the case of barrier options, whose risk-neutral evaluation relies on a supremum law for the underlying process. This law is well-known in the Gaussian case and can be extended to the case of stable distributions (space fractional diffusion); it would be an interesting
question to determine whether it can also be extended to the case of a generic space–time-fractional
diffusion.

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