Stress-Energy Tensor of the Quantized Massive Fields in Friedman-Robertson-Walker Spacetimes.

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Abstract

The approximate stress-energy tensor of the quantized massive scalar, spinor and vector fields in the spatially flat Friedman-Robertson-Walker universe is constructed. It is shown that for the scalar fields with arbitrary curvature coupling, $\xi$, the stress-energy tensor calculated within the framework of the Schwinger-DeWitt approach is identical to the analogous tensor constructed in the adiabatic vacuum. Similarly, the Schwinger-DeWitt stress-energy tensor for the fields of spin 1/2 and 1 coincides with the analogous result calculated by the Zeldovich-Starobinsky method. The stress-energy tensor thus obtained are subsequently used in the back reaction problem. It is shown that for pure semiclassical Einstein field equations with the vanishing cosmological constant and the source term consisting exclusively of its quantum part there are no self-consistent exponential solutions driven by the spinor and vector fields. A similar situation takes place for the scalar field if the coupling constant belongs to the interval $\xi > 0.1$. For a positive cosmological constant the expansion slows down for all considered types of massive fields except for minimally coupled scalar field. The perturbative approach to the problem is briefly discussed and possible generalizations of the stress-energy tensor are indicated.

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I. INTRODUCTION

In their recent paper [1], Kaya and Tarman constructed the stress-energy tensor, $T_{ab}$, of the massive quantized scalar field with the arbitrary curvature coupling, $\xi$, in the spatially flat ($k = 0$) Friedman-Robertson-Walker spacetime, described by the line element

$$ds^2 = a^2(\eta) \left(-d\eta^2 + dw^2 + dy^2 + dz^2\right)$$  (1)

within the framework of the adiabatic regularization [2–8]. Using sixth-order WKB approximation to the solutions of the covariant Klein-Gordon equation

$$-\square \phi^{(0)} + (m^2 + \xi R)\phi^{(0)} = 0$$  (2)

in the formally divergent expression for the $T_{ab}$, after adiabatic regularization, they calculated the leading term of the approximation to the components of the (covariantly) conserved stress-energy tensor. In the adiabatic regularization one has to subtract, mode by mode, the infinite terms of the adiabatic order 0, 2 and 4. To avoid ambiguities it is necessary to subtract all terms that contain at least one divergent part for arbitrary parameters of the theory [9].

Although the intermediate stages of the calculation are rather involved, the final result is surprisingly simple and can be schematically written in the form

$$\frac{1}{96\pi^2 m^2 a^2} \left(\sum A^{pq}_{ijk} [a^{(i)}]^p [a^{(j)}]^q [a^{(k)}]^s + Ba^3 \dot{a} \ddot{a} \dot{a}^{(3)}\right),$$  (3)

where $A$ and $B$ are numerical coefficients (possibly dependent on the coupling constant), the overdots denote differentiation with respect to the coordinate $\eta$ and

$$a^{(k)} = \frac{d^k a(\eta)}{d\eta^k} \quad \text{with} \quad a^{(0)} = a(\eta).$$  (4)

The summation is extended over all $i, ...$ and $p, ...$ satisfying $pi + qj + ks = 6$, Additionally, the number of appearances of the function $a(\eta)$ and its derivatives in each term is 6, i.e., $p + q + s = 6$. The same is true for the last term in Eq. (3).

Because of the spatial symmetries it suffices to calculate the energy density, $\rho = -T^0_0$, as the remaining components are easily calculated from the equations

$$T^x_x \equiv T^1_1 = T^2_2 = T^3_3$$  (5)
and
\[ T^x_x = T^0_0 + \frac{a}{3\dot{a}} \dot{T}^0_0. \] (6)

Alternatively, one can calculate the trace of the tensor, or, due to simplicity of the spatial mode functions for the \( k = 0 \) geometries, the component \( T^x_x \). Of course the first method is simplest and the remaining ones may serve as a useful check of the calculations.

It should be noted that being local, the adiabatic approximation does not give much information about the particle creation. On the other hand, it is very useful when the vacuum polarization effects dominate. In the adiabatic expansion (for \( k = 0 \)) the number of derivatives plays the role of the perturbation parameter. It is expected that this procedure gives reasonable results provided \( \frac{\dot{a}}{a}, \frac{\ddot{a}}{a}, \ldots \ll \Omega = (m^2 a^2 + \bar{k}^2)^{1/2} \), where \( 0 \leq \bar{k} < \infty \) [10].

In this paper we shall demonstrate that the adiabatic tensor of Ref. [1] can be obtained as the special case of the more general expression calculated within the framework of the Schwinger-DeWitt approach [11]. The generalization can be twofold: one can allow for nonvanishing \( k \) in the line element and consider the massive fields of spin \( 1/2 \) and \( 1 \). In what follows we shall restrict ourselves to the simplest case of the spatially flat Friedman-Robertson-Walker geometries, thus avoiding the subtleties connected with the Euclideanization of the line element. Like the adiabatic expansion, the Schwinger-DeWitt method is local and ignores nonlocal phenomena such as particle creation, and, consequently, its domain of applicability is limited. Nevertheless, it gives reasonable results if the Compton length, \( \lambda_C \), associated with the quantized field is much smaller than the characteristic radius of curvature.

The adiabatic calculations are based on the WKB approximation to the mode functions and their derivatives and subsequent integration (summation) of the functions thus constructed. On the other hand, the Schwinger-deWitt approach may be considered as purely geometrical. Moreover, it can be demonstrated that for the massive spinor and vector fields the method of Zeldovich and Starobinsky [10, 12] and Schwinger-DeWitt give the same results. (All fields considered in this paper are neutral). The equality of the results obtained from both methods is impressive. It should be noted that such equality must not be taken for granted: Indeed, it is expected that discussed methods give (approximate) Green functions with the same structure of singularities as \( x' \rightarrow x \). However, this does not mean the functions are the same.

The paper is organized as follows. In Section II, after giving a brief sketch of the
Schwinger-DeWitt method, we construct the renormalized stress-energy tensor of the quantized massive fields in a large mass limit in the spatially-flat Friedman-Robertson-Walker universe. The result (in a scalar case) is subsequently compared with the analogous tensor obtained within the framework of the adiabatic method. We also briefly report on our calculations of the stress-energy tensor of the quantized massive s=1/2 and s=1 fields using the Zeldovich-Starobinsky method. This section contains the core results of the paper. In Section III, to illustrate some possible applications of the constructed tensors, two families of solutions to the semiclassical Einstein field equations (both exact and approximate) are constructed. Using the Routh-Hurwitz criterion we perform stability analysis. It is pointed out that the perturbative method adapted in this paper is, in this context, equivalent to the reduction of order used by Parker and Simon [13] Finally, in Sec. IV, among other things, we indicate possible generalizations of the results presented in this paper. Throughout the paper we use the MTW conventions [14].

II. THE SCHWINGER-DEWITT APPROACH

The one-loop approximation to the effective action of the quantized massive fields in a large mass limit can be constructed within the framework of the Schwinger-DeWitt method. The effective action of the scalar field described by Eq.(2) can be obtained from the coincidence limit of the Hadamard-DeWitt coefficient $a_3$. Unfortunately, neither the covariant Dirac equation

$$\left(\gamma^a \nabla_a + m\right)\phi^{(1/2)} = 0,$$

nor the equation describing the massive vector field

$$\left(\delta^a_b \square - \nabla_b \nabla^a - R^a_b - \delta^a_b m^2\right)\phi^{(1)} = 0,$$

have the form required by the Schwinger-DeWitt method. In the first case one can introduce a new spinor, $\psi^{(1/2)}$ defined as $\phi^{(1/2)} = \left(\gamma^a \nabla_a - m\right)\psi^{(1/2)}$. This, after commuting covariant derivatives and making use of the elementary properties of the $\gamma$ matrices, results in the second-order equation:

$$\left(\nabla_a \nabla^a - \frac{1}{4} R - m^2\right)\psi^{(1/2)} = 0.$$

To eliminate the nondiagonal differential operator $\nabla_b \nabla^a$ in Eq. (8) one can employ the method of Barvinsky and Vilkovisky [15, 16] and demonstrate that the effective action...
of the massive vector field equals the effective action calculated for Eq. (8) without the
nondiagonal term minus the effective action of the minimally coupled scalar field. The
approximate effective action of the quantized spinor, scalar and vector fields can be written
as [17, 18]

$$W_{\text{ren}}^{(1)} = \frac{1}{192\pi^2 m^2} \int d^4 x g^{1/2} \left( \alpha_1^{(s)} R \Box R + \alpha_2^{(s)} R_{ab} \Box R^{ab} + \alpha_3^{(s)} R^3 + \alpha_4^{(s)} R R_{ab} R^{ab} \\
+ \alpha_5^{(s)} R R_{abcd} R^{abcd} + \alpha_6^{(s)} R_a R^a R_b R^b_c R^c + \alpha_7^{(s)} R_a R^a R^c R^a_d R^d + \alpha_8^{(s)} R_a R^a R R_{abcd} R^{abcd} \\
+ \alpha_9^{(s)} R_{cd} R_{ab} e_{bc} R_{cd} + \alpha_{10}^{(s)} R_{cd} R_{ab} e_{bc} R_{abcd} \right)$$

$$= \frac{1}{192\pi^2 m^2} \int d^4 x g^{1/2} \sum_{i}^{10} \alpha_i \text{Inv}_i,$$  (10)

where $m$ is the mass of the field and the numerical coefficients depending on the spin of the
field are given in a Table I.

|   | $s = 0$ | $s = 1/2$ | $s = 1$ |
|---|--------|--------|--------|
| $\alpha_1^{(s)}$ | $\frac{1}{2} \xi^2 - \frac{1}{5} \xi + \frac{1}{56}$ | $-\frac{3}{280}$ | $-\frac{27}{280}$ |
| $\alpha_2^{(s)}$ | $\frac{1}{140}$ | $\frac{1}{28}$ | $\frac{9}{28}$ |
| $\alpha_3^{(s)}$ | $\left( \frac{1}{6} - \xi \right)^3$ | $\frac{1}{804}$ | $-\frac{5}{72}$ |
| $\alpha_4^{(s)}$ | $-\frac{1}{30} \left( \frac{1}{6} - \xi \right)$ | $-\frac{1}{1400}$ | $\frac{31}{60}$ |
| $\alpha_5^{(s)}$ | $\frac{1}{30} \left( \frac{1}{6} - \xi \right)$ | $-\frac{7}{1440}$ | $-\frac{1}{10}$ |
| $\alpha_6^{(s)}$ | $-\frac{8}{945}$ | $-\frac{25}{756}$ | $-\frac{52}{63}$ |
| $\alpha_7^{(s)}$ | $\frac{2}{315}$ | $\frac{47}{1260}$ | $-\frac{19}{105}$ |
| $\alpha_8^{(s)}$ | $\frac{1}{1260}$ | $\frac{19}{1260}$ | $\frac{61}{140}$ |
| $\alpha_9^{(s)}$ | $\frac{17}{7560}$ | $\frac{29}{7560}$ | $-\frac{67}{2520}$ |
| $\alpha_{10}^{(s)}$ | $-\frac{1}{270}$ | $-\frac{1}{108}$ | $\frac{1}{18}$ |

TABLE I. The coefficients $\alpha_i^{(s)}$ for the massive scalar with arbitrary curvature coupling $\xi$, spinor,
and vector field

The (renormalized) stress-energy tensor can be calculated from the standard relation

$$T^{ab} = \frac{2}{\sqrt{g}} \frac{\delta W_{\text{ren}}^{(1)}}{\delta g_{ab}}$$  (11)

and consists of the purely geometric terms constructed from the Riemann tensor, its covari-
ant derivatives and contractions. The type of the field enters through the coefficients $\alpha_i$. 
Additionally, each term

\[
\frac{2}{\sqrt{g}} \frac{\delta}{\delta g_{ab}} \int d^4x g^{1/2} \text{Inv}_i
\]  

(12)
is covariantly conserved. In a general D-dimensional manifold the invariants \( \text{Inv}_i \) are independent and this is the reason why we prefer to work with this form of the action [19]. It should be noted, however, that the Riemann tensor satisfies numerous dimension-dependent identities and some invariants in a specific dimension, say, \( D = 4 \), are not independent. We shall briefly discuss this problem at the end of this section. It should be stressed that regardless of the particular representation of the effective action the stress-energy tensor for a given background and field should give unique results.

In Refs. [20, 21] the most general form of the stress-energy tensor has been calculated. It consists of almost 100 terms of the background dimensionality 6 and, for obvious reason, it will be not presented here. In spite of the limitations mentioned above it is still the most general result. Moreover, by construction, it depends functionally on the metric tensor allowing, in principle, to solve the semiclassical Einstein field equations in a self-consistent way.

Defining \( \tilde{\alpha}_i = \alpha_i / 192\pi^2 m^2 \) in (10) and treating them as free parameters one arrives at the action functional considered by Lu and Wise [22] in the black hole context\(^1\) On the other hand, by retaining only \( \alpha_9 \) term in (10) one obtains the Goroff-Sagnotti effective action [23] studied by Dobato and Maroto [24].

For the line element (1) the “time” component of the stress-energy tensor has the form

\[
T^0_0 = \frac{1}{32\pi^2 m^2 a^{12}} \left[ (22\alpha_{10} - 72\alpha_2 - 84\alpha_4 - 168\alpha_5 - 85\alpha_6 - 13\alpha_7 - 70\alpha_8 - 84\alpha_9) \dot{a}^6 \\
+ 12(36\alpha_1 - 3\alpha_{10} + 22\alpha_2 + 12\alpha_4 + 24\alpha_5 + 12\alpha_6 + 2\alpha_7 + 10\alpha_8 + 12\alpha_9) a \dot{a}^4 \dot{\bar{a}} \\
- 2(156\alpha_1 - 3\alpha_{10} + 62\alpha_2 + 12\alpha_4 + 24\alpha_5 + 12\alpha_6 + 2\alpha_7 + 10\alpha_8 + 12\alpha_9) a^2 \dot{a}^3 \dot{\bar{a}}^{(3)} \\
- (306\alpha_1 - 3\alpha_2 + 142\alpha_3 + 540\alpha_4 + 192\alpha_5 + 204\alpha_6 + 87\alpha_7 + 47\alpha_8 + 70\alpha_9) a^2 \dot{a} \dot{\bar{a}}^{(2)} \\
- 2(18\alpha_1 + 8\alpha_2 + 36\alpha_3 + 12\alpha_4 + 12\alpha_5 + 5\alpha_6 + 3\alpha_7 + 4\alpha_8 + 4\alpha_9) a^3 \dot{a}^{(2)} \\
- 4(3\alpha_1 + \alpha_2) a^4 \dot{a} \dot{\bar{a}}^{(5)} + 2(6\alpha_1 + 2\alpha_2) a^4 \dot{\bar{a}}^{(4)} - 2(3\alpha_1 + \alpha_2) a^4 \dot{a}^{(3)} - 2(3\alpha_1 + \alpha_2) a^4 \dot{a}^{(2)} \\
+ 6(18\alpha_1 + 8\alpha_2 + 36\alpha_3 + 12\alpha_4 + 12\alpha_5 + 5\alpha_6 + 3\alpha_7 + 4\alpha_8 + 4\alpha_9) a^3 \dot{\bar{a}}^{(3)} \right],
\]  

(13)

where, for typographical reasons, the superscripts describing spin of the field have been omitted. It can be shown that substituting the \( \alpha^{(0)} \) parameters into the \( T_0^0 \) component

\(^1\) Actually, Lu and Wise consider the action functional with \( \tilde{a}_1 = \tilde{a}_2 = 0 \) and a different organization of indices in the last term term, which, however, is equal to the last term of (10) plus 1/4 the \( \tilde{a}_9 \) term.
of the stress-energy tensor one gets precisely the result obtained by Kaya and Tarman. Further, making use of (6) and (13) one obtains the spatial components which are identical to those obtained within the framework of the Schwinger-DeWitt method. The Mathematica notebooks with the details of this calculation for results that are identical to those calculated within the framework of the Schwinger-DeWitt approach \[10, 12, 25\]. Performing integrations over momenta and summations over spin states we obtained, after much algebra, the results for the spinor and vector fields reported here are new.

Now, by substituting \(\alpha_s^{(1/2)}\) and \(\alpha_s^{(1)}\) into Eqs. (13) and (14) one obtains the renormalized stress-energy tensor of the spinor and vector fields, respectively. To the best of our knowledge the results for the spinor and vector fields reported here are new.

Having established that the two different approaches give the same result for the scalar field, the natural question is whether it is also true for the fields of spin 1/2 and 1. In order to answer this question we have used asymptotic expansion of the Green function constructed within the framework of the Zeldovich-Starobinsky approach \[10, 12, 25\]. Performing integrations over momenta and summations over spin states we obtained, after much algebra, the results that are identical to those calculated within the framework of the Schwinger-DeWitt method. The Mathematica notebooks with the details of this calculation for \(s = 0, 1/2\) and 1 fields are available upon request from the first author.

It is of some interest, especially when one wants to address the back reaction problem perturbatively, to restore the physical constants. In that case the multiplicative factor

\[
T^x_x = \frac{1}{96\pi^2 m^2 a^{12}} \left[ -9 (22\alpha_{10} - 72\alpha_2 - 84\alpha_4 - 168\alpha_5 - 85\alpha_6 - 13\alpha_7 - 70\alpha_8 - 84\alpha_9) a^6 
- 6 (576\alpha_1 - 70\alpha_{10} + 424\alpha_2 + 276\alpha_4 + 552\alpha_5 + 277\alpha_6 + 45\alpha_7 + 230\alpha_8 + 276\alpha_9) aa^4\dot{a}^2 
- 2 (1308\alpha_1 - 39\alpha_{10} + 566\alpha_2 + 156\alpha_4 + 312\alpha_5 + 26\alpha_7 + 130\alpha_8 + 156\alpha_9) a^2 \dot{a}^3 a^{(3)} 
- 2 (444\alpha_1 - 3\alpha_{10} + 158\alpha_2 + 12\alpha_4 + 24\alpha_5 + 12\alpha_6 + 2\alpha_7 + 10\alpha_8 + 12\alpha_9) a^3 \dot{a}^2 a^{(4)} 
+ 2 (120\alpha_1 + 46\alpha_2 + 108\alpha_3 + 36\alpha_4 + 6\alpha_5 + 1\alpha_6 + 9\alpha_7 + 12\alpha_8 + 12\alpha_9) a^4 \dot{a} a^{(4)} 
- 2 (198\alpha_1 - 3\alpha_{10} + 94\alpha_2 + 324\alpha_3 + 120\alpha_4 + 132\alpha_5 + 57\alpha_6 + 29\alpha_7 + 46\alpha_8 + 48\alpha_9) a^3 \dot{a}^3 
+ 2 (69\alpha_1 + 29\alpha_2 + 108\alpha_3 + 36\alpha_4 + 36\alpha_5 + 1\alpha_6 + 9\alpha_7 + 12\alpha_8 + 12\alpha_9) a^4 a^{(3)} \right].
\]

Now, by substituting \(\alpha_s^{(1/2)}\) and \(\alpha_s^{(1)}\) into Eqs. (13) and (14) one obtains the renormalized stress-energy tensor of the massive scalar field using the adiabatic approach. There are two typographical errors in the expression for spatial components of the stress-tensor as presented in \[1\]. With these terms corrected, the adiabatic tensor is identical to the tensor constructed within the framework of the Schwinger-DeWitt method.
standing in front of the integral (10) is \( \hbar \lambda /192\pi^2 \) and the integrand has the dimension Length\(^{-6}\). In order keep control of the order of terms in complicated expansions we shall introduce, whenever useful, a small parameter \( \varepsilon \) which should be set to 1 at the end of the calculations.

We conclude this section with some remarks about the approximate action (10). In four dimensions there are two additional identities [26, 27], which can be directly obtained from

\[
R^{cd}_{[ab} R_{cd}^{e]} = 0
\]

and

\[
R^{cd}_{[ab} R_{cd}^{e]} R^{ji}_{el} = 0.
\]

It means that the 8th and 10th term in the effective action can be expressed as linear combinations of the remaining ones. Consequently, the same result can be obtained simply by substituting \( \sum_{i=1}^{10} \alpha_i \text{In} v_i \) in Eq. (10) by \( \sum_{i=1}^{10} \bar{\beta}_i \text{In} v_i \), where \( \bar{\beta}_1 = \alpha_1, \bar{\beta}_2 = \alpha_2, \bar{\beta}_3 = \alpha_3 + 1/4\alpha_8 - 5/8\alpha_{10}, \bar{\beta}_4 = \alpha_4 - 2\alpha_8 + 9/2\alpha_{10}, \bar{\beta}_5 = \alpha_5 + 1/4\alpha_8 - 3/8\alpha_{10}, \bar{\beta}_6 = \alpha_6 + 2\alpha_8 - 4\alpha_{10}, \bar{\beta}_7 = \alpha_7 + 2\alpha_8 - 3\alpha_{10}, \bar{\beta}_8 = 0, \bar{\beta}_9 = \alpha_9 + 1/2\alpha_{10} \) and \( \bar{\beta}_{10} = 0 \). Moreover, one can further simplify the effective action of the quantized fields in a Weyl-flat \( (C_{abcd} = 0) \) spacetime making use of the invariants constructed from the Weyl tensor. It is possible provided the functional derivative of such invariant with respect to the metric tensor vanishes. For example functionals constructed from \( C_{abcd} C_{ij}^{cd} C^{ij}_{ab}, R C_{abcd} C^{abcd} \) and \( C_{abcd} \Box C^{abcd} \) have the desired property.

### III. THE BACK REACTION

In the purest form of the semiclassical approximation, the gravitational field is treated classically, but it is driven by the mean value of \( T^b_a \). Having constructed the leading term of the renormalized stress-energy tensor which depends functionally on a generic metric, one can analyze the semiclassical Einstein field equations with the total stress-energy tensor consisting of the classical and quantum parts. It should be emphasized once more that accepting the approximation (10) we ignore particle creation, which is a nonlocal effect and concentrate on the vacuum polarization effects.

Further, to simplify our discussion we shall assume that the renormalized coupling pa-
rameters $k_1$ and $k_2$ in the quadratic part of the total action

$$S_q = \int d^4x \sqrt{-g} \left( k_1 R_{ab} R^{ab} + k_2 R^2 \right), \quad (17)$$

identically vanish. In the case on hand the tensors obtained by functional differentiation of the integrated $R^2$ and $R_{ab} R^{ab}$ with respect to the metric tensor are not independent but there is an additional term

$$k_3 \left( -\frac{1}{12} R^2 g_{ab} + R^{cd} R_{cdab} \right) \quad (18)$$

which can be added to the left hand side of the equations. In the latter we put $k_3 = 0$. The semiclassical Einstein field equations have, therefore, a standard form

$$G_{ab}[^g] + \Lambda g_{ab} = 8\pi T_{ab}^{(total)}[^g], \quad (19)$$

where $T_{ab}^{(total)} = T_{ab}^{(class)} + T_{ab}$, i.e., the right hand side of (19) is a sum of the classical and quantum stress-energy tensor.

Let us postpone the further discussion of the semiclassical equations for a while and return to the quadratic terms. It should be noted that such terms appear in a natural way as a low-energy limit of the string action. Indeed, they appear as the first-order correction to the classical action in $\alpha'$ expansion ($\alpha' = 2\pi \lambda_S^2$ where $\lambda_S$ is a fundamental length). In a general parametrization in D-dimensions the action should be supplemented by a Kretschmann scalar, $R_{abcd} R^{abcd}$. Functionally differentiating the total action with respect to the metric tensor and taking into account the $(0,0)$ component of the thus obtained tensor equation, which usually is the simplest one to analyze, it can be demonstrated [28] that for a constant dilaton field there is no self-consistent exponential solution in $D = 2$ and $D = 4$. The latter result, that is of immediate relevance here, is a simple consequence of the fact that the quadratic terms vanish for the spatially-flat Friedman-Robertson-Walker spacetime with the exponential scale factor whereas the Einstein tensor does not.

If the total stress-energy tensor is known, one can, in principle, construct the self-consistent solution of the system (19). It should be noted however, that since the general stress-energy tensor, $T^b_a$, is constructed from the coincidence limit of $a_3(x,x')$, it contains the terms with higher-order derivatives of $g_{ab}$, and, consequently, there is a real danger that the semiclassical equations may lead to physically unacceptable solutions. Moreover, the tensor $T^b_a$ and the resulting equations are rather complicated and it is natural that one is
forced to look for simplifications. First, let us consider the function $a(t)$ which also satisfies the additional relation [24]

$$a'(t) = c_1^{1/2} a(t), \quad (20)$$

where $t$ is “ordinary” time coordinate related to $\eta$ by $dt = a(\eta) d\eta$, $c_1^{1/2}$ is some constant and a prime denotes differentiation with respect to $t$. Transforming the semiclassical equations (19) to $(t, w, y, z)$ coordinates and making use of (20) one obtains

$$-3c_1 + 3\tau c_1^3 + \Lambda = -8\pi \rho \quad (21)$$

and the similar equation with the right hand side substituted by $8\pi \rho$, where

$$\tau = \frac{1}{12\pi m^2} (144\alpha_3 + 36\alpha_4 + 24\alpha_5 + 9\alpha_6 + 9\alpha_7 + 6\alpha_8 + 4\alpha_9 + 2\alpha_{10}). \quad (22)$$

From the semiclassical equations one has either $p = \sigma \rho$ with $\sigma = -1$ and $\rho \neq 0$ (which is the equation of state for the cosmological constant) or $\rho = p = 0$. Since $\sigma = -1$ the only nondegenerate equation of state compatible with the simplified semiclassical equations one can consider an effective cosmological constant $\Lambda_{eff} = \Lambda + 8\pi \rho$.

In the absence of both $\Lambda$ and $\rho$ (or $\Lambda_{eff}$) one has either $c_1 = 0$ or $c_1^2 = 1/\tau$. The right hand side of the above equation is negative for spinors and vectors, and, consequently, there are no exponential solutions driven by massive spinors and vectors in this simple model. The scalar case is slightly more complicated because of the coupling constant $\xi$. Indeed, inspection of (22) shows that it is positive for $\xi < \xi_{crit} = 0.1023$ and hence there is no solution for the conformally coupled scalar field. The temporal evolution of the model is governed by (20). For $c_1^{1/2} > 0$ one has

$$a(t) = a(t_0) \exp(c_1^{1/2}(t - t_0)), \quad (23)$$

whereas for $c_1 = 0$ one has static universe with the constant scale factor. A few words of comment are in order. First observe that the quantum part of the tensor can be made arbitrarily large simply by taking large number of fields. Indeed, for $N$ fields of a given spin, $s$ with masses $m_i$ the renormalized effective action is still of the form (10) with

$$\frac{1}{m^2} \rightarrow \sum_{i=1}^{N} \frac{1}{m_i^2}. \quad (24)$$

Moreover, even if the $c_1^2$ is negative for some fields it does not mean that the field should be excluded from the model. It can contribute to the total stress-energy tensor, which yields a proper overall characteristics.
Let us return to the semiclassical equations (21) and (22) with $\Lambda_{\text{eff}} \neq 0$. The third order equation has exact solutions, but they are not particularly illuminating. It is of some interest to consider them in some important regimes. Since the present value of the cosmological constant is extremely small one has

$$H_0^2 \equiv c_1 \approx \frac{\Lambda_{\text{eff}}}{3} \left( 1 + \frac{\Lambda_{\text{eff}}^2 \tau}{9} \right),$$

(25)

$$H_0^2 \equiv c_1 \approx \frac{1}{\sqrt{\tau}} - \frac{\Lambda_{\text{eff}}}{6}$$

(26)

and

$$H_0^2 \equiv c_1 \approx -\frac{1}{\sqrt{\tau}} - \frac{\Lambda_{\text{eff}}}{6}.$$  

(27)

Following [31] we shall call (25) the classical de Sitter (or to be more precise the quantum-corrected classical) solution whereas (26) is the quantum solution. Now, a natural question arises about the stability of (25) and (26) against small perturbations. To answer this question let us express the full set of the semiclassical equations (19) in terms of $H(t) = a'(t)/a(t)$, insert $H(t) = H_0 + \delta H(t)$ (where $\delta H(t)$ is small perturbation) and finally linearize the thus constructed equations. Assuming $\delta H(t) = \exp(\lambda t)$ one obtains the characteristic polynomial(s) $F(\lambda)$ with the real coefficients. The solution is stable under small perturbations if all roots of the characteristic polynomial lie in the left complex half-plane. A useful criterion has been given by Routh and Hurwitz: If the determinants of all upper left-corner minors of the Hurwitz matrix are strictly positive then all roots of the polynomial $F(\lambda)$ have negative real parts. Sometimes it is more efficient to use the necessary condition which requires all the coefficients of $F(\lambda)$ to be positive [30]. Of course, if the order of the characteristic polynomial is low one can solve the equation exactly and check if the conditions $\text{Re}(\lambda_i) < 0$ are satisfied.

Let us return to our case. It is advantageous not to specify the values of $H_0$ and analyze the relations among the characteristic polynomials $F_t(\lambda), F_x(\lambda)$ and $F(\lambda)$, where $F_t$ is obtained from the $(0,0)$ component of the semiclassical equations, $F_x$ from the spatial component and $F = F_x + \sigma F_t$, i.e., is linear combination of the former two. It should be noted that the zeroth-order equations coincide with (21) and (22) and their only acceptable solutions are with $p = -\rho$ or without the matter. Simple manipulations give

$$F_x(\lambda) = \frac{3H_0 + \lambda}{3H_0} F_t(\lambda)$$

(28)
and for $\sigma \neq -1$

$$F(\lambda) = \left( \sigma + \frac{3H_0 + \lambda}{3H_0} \right) F_t(\lambda),$$

where $\sigma$ is the coefficient of the linear equation of state $p = \sigma\rho$. It should be noted that for $H_0 > 0$ the polynomial $F_x(\lambda)$ does not give more information than $F_t(\lambda)$. From Eq. (29) one has $\lambda = -3H_0(1 + \sigma)$, which is negative for nontachyonic matter. The problem reduces to the analysis of the roots of the equation

$$F_t(\lambda) = \sum_{k=0}^{4} b_{4-k}\lambda^k = 0,$$

where the real coefficients $b_k = b_k(a_1, ..., a_{10}, m, H_0)$ depend on type of the field and $H_0$. It can be shown by a direct calculation that taking $b_0$ to be positive the coefficient $b_3$ is always negative for spinor and vector field. The massive scalar field is more complicated, nevertheless, it is impossible to have all the coefficients simultaneously positive. It can be seen that regardless of the exact value of $H_0$ there is no stable solutions. Consequently, one can draw a conclusion that there are no stable solutions for the massive spinors, vectors and scalars with arbitrary coupling parameter $\xi$.\(^3\)

Although our analyses are devoted solely to the massive fields, it is instructive, for a comparison, to consider quantized massless conformally invariant fields. This case has been studied in Ref. [31] to which the reader is referred for details. The trace of the stress-energy tensor considered in Ref. [31] has the form [32, 33]

$$T_a^a = bF + b'(E - \frac{2}{3}\Box R) + b''\Box R,$$

where $F = C_{abcd}C^{abcd}$,

$$E = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2,$$

is the Gauss-Bonnet invariant and the numeric coefficients $b$, $b'$ and $b''$ depend on the type and number of fields of a given spin.

Because of the high symmetry of the Friedman-Robertson-Walker spacetime the conservation equation can be integrated and the stress-energy tensor of the quantized fields can be constructed. Now, setting all time derivatives of $H(t)$ in the semiclassical equations to zero one obtains two equations:

$$-3H_0^2 - 48\pi b' H_0^4 + \Lambda = -8\pi \rho,$$

\(^3\) We confirmed this result solving the fourth-order equation (30).
\[-3H_0^2 - 48\pi b'H_0^4 + \Lambda = 8\pi p. \quad (34)\]

Note that the remarks made earlier when discussing the solutions of Eqs. (21) and (22) are also relevant here. The stability analysis leads to the equations similar to (28) and (29) with only one difference: Now the characteristic polynomial \( F_t(\lambda) \) has the form

\[
F_t(\lambda) = -8\pi (3b'' - 2b')\lambda^2 - 24\pi H_0(3b'' - 2b')\lambda + 3 + 96\pi b'H_0^2. \quad (35)
\]

Upon dividing the characteristic equation by \( b_0 \) one concludes that \( b'' - 2/3b' \) and \( 1 + 32\pi b'H_0^2 \) should have the opposite signs. A closer analysis carried out for a small cosmological constant indicates that when \( b'' - 2/3b' < 0 \) the classical attractor is unstable whereas the quantum attractor is stable. Similarly, when \( b'' - 2/3b' > 0 \) the classical attractor is stable and the quantum attractor is unstable. The attractors considered here are the solutions of the zeroth-order equations (33) and (34).

The characteristic polynomial (35) contains, as a special case, the results of Ref. [34]. Indeed, putting \( \tilde{b} = b' \) and \( \tilde{c} = b'' - 2/3b' \) and \( \Lambda = 0 \) after some simplifications and rearrangements, one obtains

\[
\lambda^2 + 3H_0^2\lambda + \frac{1}{8\pi\tilde{c}} = 0 \quad (36)
\]

and the behavior of system is governed by the sign of \( \tilde{c} \), which should be positive.\(^4\) In simplifying (35) to (36) we used the vacuum solution of (33)

\[
H_0^2 = -\frac{1}{16\pi\tilde{b}}. \quad (37)
\]

For the massless (conformally invariant) quantum fields one can use the trace equation simply because the trace is given by the linear combinations of the curvature invariants. Although this method does not work for the massive fields it is possible to use a procedure called conformization [29]. The resulting conformal anomaly tensor is composed of the Gauss-Bonnet topological invariant, the square of the Weyl tensor, \( \Box R \) and the auxiliary field \( \chi \) introduced to make the action invariant under the conformal transformation:

\[
T_a^a = \bar{w}F + \bar{b}E + \bar{c}\Box R + \bar{f} \left( R\chi^2 + 6(\partial\chi)^2 \right), \quad (38)
\]

where \( \bar{w}, \bar{b}, \bar{c} \) are the standard coefficients (see, e.g., Ref. [29]) and \( \bar{f} \) depends on the number of massive spinor fields and their masses. Subsequently, after construction of the anomaly-induced effective action one can construct equations of motion. Now, treating the terms

\(^4\) Note that the authors of Ref. [34] used the value of \( \bar{c} \) as predicted by \( \zeta \)-function or point-splitting regularization rather than the value predicted by the dimensional regularization.
proportional to $\tilde{f}$ as small perturbation one concludes that fermion anomaly-induced inflation automatically slows down [29]. In our next example we shall show that exponential expansion also slows down for $\tau < 0$.

Let us consider the semicalssical equations (19) with $\Lambda = 0$ and the classical matter characterized by the equation of state of the form $p = \sigma \rho$. The matter density of the classical matter evolves according to the formula

$$\rho(t) = \frac{C_1}{a(t)^{3(1+\sigma)}}$$

(39)

We shall treat the quantum part of the right hand side of the semiclassical Einstein field equations as small perturbation. Now, inserting the parameter $\varepsilon$ in front of the quantum part of the total stress-energy tensor and expanding the scale factor as

$$a(t) = a_{(0)}(t) + \varepsilon a_{(1)}(t) + \mathcal{O}(\varepsilon^2)$$

(40)

one obtains two simple differential equations. The first one

$$a'_{(0)} - \omega a_{(0)} = 0$$

(41)

can easily be solved: for $\tilde{\sigma} \neq -1$ one has

$$a_{(0)}(t) = \left[ (\tilde{\sigma} + 1)(\omega t + C_2) \right]^{-\frac{1}{\tilde{\sigma}}},$$

(42)

whereas for $\tilde{\sigma} = -1$ one has the exponential law of the evolution of the scale factor

$$a_{(0)}(t) = C_3 \exp(\omega t),$$

(43)

where $\tilde{\sigma} = \frac{1}{2}(3\sigma + 1)$, $\omega^2 = 8\pi C_1/3$, and $C_2$ and $C_3$ are the integration constants. The second equation is slightly more complicated and can be written in the form:

$$a'_{(1)} + \omega \tilde{\sigma} a_{(0)}^{-1+\tilde{\sigma}} a_{(1)} - \frac{1}{\pi m_2^2} \omega^2 a_{(0)}^{-4+5\tilde{\sigma}} \sum_{n=0}^{4} B_n \tilde{\sigma}^n = 0,$$

(44)

where the coefficients $B_n$ which depend on the spin of the field are tabulated in Table II. It can be solved to yield

$$a_{(1)} = (\omega t + C_2)^{-\tilde{\sigma}} C_4 - (1 + \tilde{\sigma})^{-\frac{4+5\tilde{\sigma}}{1+\tilde{\sigma}}} (\omega t + C_2)^{-\frac{4+5\tilde{\sigma}}{3+4\tilde{\sigma}}} \frac{\omega^4}{3\pi m_2^2} \sum_{n=0}^{4} B_n \tilde{\sigma}^n,$$

(45)

where $C_4$ is another integration constant. The $\tilde{\sigma} = -1$ case has to be treated separately and after massive simplifications in (44) one obtains

$$a_1(t) = C_5 \exp(\omega t) + \frac{1}{2} C_3 \omega^5 \tau t \exp(\omega t),$$

(46)
TABLE II. The coefficients $B_n$ calculated for the massive scalar field with minimal and conformal curvature coupling, massive spinor and massive vector fields

| $\xi$  | $B_0$ | $B_1$  | $B_2$  | $B_3$  | $B_4$  |
|--------|-------|--------|--------|--------|--------|
| $\xi = 0$ | 13/144 | -131/420 | -1343/5040 | 2371/7560 | 17/84 |
| $\xi = 1/6$ | 0 | -2/315 | -1/252 | 41/3780 | 1/126 |
| $s = 1/2$ | 0 | -17/2520 | -11/10080 | 281/15120 | 1/84 |
| $s = 1$ | 1/144 | -101/1260 | -151/5040 | 439/2520 | 3/28 |

where $\tau$ is given by (22). Further, introducing new constant $\tilde{C}_3$ by means of the finite renormalization $C_3 \rightarrow \tilde{C}_3 = C_3 + \varepsilon C_5$ one gets

$$a(t) = \tilde{C}_3 \exp(t\omega) + \frac{1}{2}\varepsilon\tilde{C}_3\omega^5\tau t \exp(\omega t),$$

(47)

where $\tau$ is given as before by (22). The constant $\tilde{C}_3$ may be related to the quantum-corrected “observed” scale factor $a(t_0)$. Simple manipulations give

$$a(t) = a(t_0) \exp(\omega (t - t_0)) \left[1 + \frac{1}{2}\varepsilon\omega^5\tau(t - t_0)\right] + \mathcal{O}(\varepsilon^2)$$

(48)

and within the accuracy of our calculations, $\mathcal{O}(\varepsilon^2)$, the term in the square brackets is equal

$$\exp\left(\frac{1}{2}\varepsilon\omega^5\tau(t - t_0)\right).$$

(49)

Inspection of Eq. (22) shows that the quantum effects tend to decrease the rate of expansion for the massive spinors and vectors as well as the massive scalar fields with $\xi > \xi_{\text{crit}}$. For the massive scalar field with $\xi < \xi_{\text{crit}}$ the rate of the expansion is increased.

There are other families of (approximate) solutions which can easily be constructed simply by retaining the cosmological term or allowing the renormalized coupling constants to be nonzero or accepting some nonstandard equations of state, but we shall not dwell on them here. We only mention that the procedure adopted in the second example is equivalent to the approach of Ref. [13]. This can readily be verified by a direct computation. Indeed, setting the quantum part of the stress-energy tensor to zero while retaining the quadratic terms given by (17) and (18), expanding the thus obtained equations and collecting the terms with the like powers of $\varepsilon$, and finally constructing solutions one gets precisely the results obtained in Ref. [13].
IV. FINAL REMARKS

In this paper the renormalized stress-energy tensor of the quantized massive scalar, spinor and vector fields in the spatially flat Friedman-Robertson-Walker universe has been constructed within the framework of the Schwinger-DeWitt technique. For the scalar field (with an arbitrary curvature coupling) it reduces to the tensor constructed using the adiabatic regularization. There is a clear correspondence between the both methods: To calculate the first-order approximation of the tensor one needs the 6-order WKB approximation of the mode functions in the adiabatic method, or, in the Schwinger-DeWitt method, the functional derivative of the effective action constructed from coincidence limit of the Hadamard-DeWitt coefficient \[ a_3 \] with respect to the metric tensor. \(^5\) Such coincidences have been found earlier in the black hole physics \[37\]. For the \( s = 1/2 \) and \( s = 1 \) field we have found an agreement between the stress-energy tensor obtained using the Zeldovich-Starobinsky method and Eqs. (13) and (14) of the present paper.

With the stress-energy tensor which functionally depends on the scale factor \( a(t) \) one can attempt to solve the semi-classical Einstein field equations in a self-consistent way. We have illustrated the procedure in the two interesting and important cases constructing particular class of the exact solution of the semi-classical equations and treating the problem perturbatively. We have preformed stability analysis of our solutions and, for a comparison, we briefly discussed earlier results.

Finally, it should be noted, that the adiabatic calculations can be extended to both \( k = -1 \) and \( k = 1 \) cases (in the \( k = 1 \) case one has summation of the mode functions instead of integration which is an obstacle in constructing the final compact expressions). Once again it can be demonstrated that the Schwinger-DeWitt and the adiabatic methods yield identical results. We shall present and analyze this group of problems in a separate publication.

We conclude this paper with the observation that the Schwinger-DeWitt method, al-

\(^5\) We have gone a step further and calculated (for \( k = 0 \) and \( k = \pm 1 \)) the next-to-leading term of the renormalized stress-energy tensor using the effective action constructed from the coincidence limit of the coefficient \( a_4 \) on the one hand, and the 8-th order adiabatic approximation to the mode function, on the other. Both methods yield identical results. An interesting lesson from this calculations is that the adiabatic method was less time-consuming, at least in our implementation of the both algorithms. Since the calculations have been carried out for massive scalars only they are somewhat beyond the scope of the present paper and we shall present the details of the calculations in a separate publication. The next-to-leading term of the stress-energy tensor that has been used in the calculations is given in Refs. \[35, 36\].
though invented in the mid-1960s, still goes strong, ranging its domain of applicability from black hole physics to cosmology, and, despite its limitations it is still the best general approximation available on the market.

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