DEFINING AND CLASSIFYING TQFT’S VIA HANDLE ATTACHMENTS

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Abstract. We describe a framework for defining and classifying TQFTs via handle attachments. Given a functor from the category of smooth manifolds and diffeomorphisms to finite-dimensional vector spaces, and maps induced by surgery along framed spheres, we give a set of axioms that allows one to assemble functorial cobordism maps. Using this, we reprove the correspondence between (1+1)-dimensional TQFTs and commutative Frobenius algebras, and classify (2+1)-dimensional TQFTs in terms of a new structure, namely split graded involutive nearly Frobenius algebras endowed with a certain mapping class group representation. The latter has not appeared in the literature even in conjectural form. This framework is also well-suited to defining natural cobordism maps in Heegaard Floer homology.

1. Introduction

Suppose we can assign a vector space $F(M)$ to every smooth $n$-manifold $M$ such that diffeomorphisms induce homomorphisms invariant up to isotopy. A framed sphere $S$ in $M$ is an embedding of $S^k \times D^{n-k}$ into $M$ for some $k$. Then we can perform surgery on $M$ along $S$ by removing the image of $S$ and gluing in $D^{k+1} \times S^{n-k-1}$ via $S|_{S^k \times S^{n-k-1}}$; after smoothing the corners we obtain the manifold $M(S)$. Assume we are given linear maps $F_{M,S}: F(M) \to F(M(S))$ induced by any such surgery. As every $(n+1)$-dimensional cobordism can be realized via a sequence of handle attachments, one can try to associate a linear map to a cobordism by composing the above surgery maps. We provide a set of axioms the above data has to satisfy for these maps to be independent of the choice of handle decomposition.

This provides an ideal method for classifying topological quantum field theories (TQFTs). As our first application, we reprove the folklore theorem claiming that (1+1)-dimensional TQFTs correspond to commutative Frobenius algebras. Then we proceed to obtain a complete classification of (2+1)-dimensional TQFTs. Such a classification has not appeared in the literature even in conjectural form. For the closest result, see the preprint of Kontsevich [8]. For the definition of split graded involutive nearly Frobenius algebras, cf. Definition 4.4, and for mapping class group representations on these, see Definition 4.12. Then our main result is the following, which answers [11, Problem 8.1].

Theorem 1.1. There is a one-to-one correspondence between (2+1)-dimensional TQFTs and split graded involutive nearly Frobenius algebras endowed with a mapping class group representation.

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However, our main motivation for this project was to define natural cobordism maps in the various flavors of Heegaard Floer homology, which is a package of (3+1)-dimensional theories. This was outlined by Ozsváth and Szabó [12], and natural 3-manifold invariants were obtained by Thurston and the author [5]. The latter constructs the Ozsváth-Szabó 3-manifold invariants in a way they become functorial under the diffeomorphism action, and the former outlines cobordism maps by assigning homomorphisms to handle attachments, and then checking invariance under Kirby moves. Instead of Kirby moves, we follow a slightly different path.

Gay, Wehrheim, and Woodward [3, 14] introduced the notion of Cerf decomposition to construct TQFTs by assigning maps to elementary cobordisms, and showing that any two decompositions of a cobordism into elementary pieces can be related by a short list of moves. An elementary cobordism is one that admits a Morse function with at most one interior critical point. Every cobordism can be decomposed into elementary cobordisms, and two decompositions can be related by critical point cancelations or creations, critical point reversals, and gluing or splitting cylinders. This is based on the work of Cerf [1].

However, Cerf decompositions do not keep track of the attaching spheres of the handles in the elementary cobordisms, which feature in the definition of cobordism maps in Heegaard Floer homology. Note that the natural definition of Heegaard Floer homology requires taking into account the embedding of the Heegaard surface into the 3-manifold, hence one has to be particularly careful with various identifications when defining the cobordism maps.

Let \( \text{Man}_n \) be the category whose objects are closed \( n \)-manifolds and whose morphisms are diffeomorphisms, and let \( \text{Cob}_n \) be the category of closed \( n \)-manifolds and equivalence classes of cobordisms. Furthermore, \( \text{Cob}'_n \) is the subcategory of \( \text{Cob}_n \) that does not contain the empty \( n \)-manifold, and such that each component of every cobordism has a non-empty incoming and outgoing end. We denote by \( \text{Cob}''_n \) the subcategory of \( \text{Cob}'_n \) where all objects (and hence cobordisms) are connected. Finally, \( \text{BSut}' \) is the category of balanced sutured manifolds and special cobordisms that are trivial along the boundary, cf. [4]. We denote by \( \text{Vect} \) the category of finite-dimensional vector spaces over some field \( \mathbb{F} \).

Let \( M \) be an \( n \)-manifold and \( \mathbb{S} \subset M \) a framed sphere of arbitrary dimension, or \( \mathbb{S} = \emptyset \). We denote by \( 0 \) the attaching sphere of a 0-handle to distinguish it from the empty set. As above, we write \( M(\mathbb{S}) \) for the result of surgery along \( \mathbb{S} \), where \( M(0) = M \sqcup S^n \) and \( M(\emptyset) = M \). If \( \mathbb{S} : S^k \times D^{n-k} \hookrightarrow M \) is a framed \( k \)-sphere for \( k < n \), let \( \mathbb{S} \) be the framed sphere defined by

\[
\mathbb{S}(\underline{x}, \underline{y}) = S \left( r_{k+1}(\underline{x}), r_{n-k}(\underline{y}) \right),
\]

where \( \underline{x} \in \mathbb{R}^{k+1}, \underline{y} \in \mathbb{R}^{n-k} \), and

\[
r_{k+1}(x_1, x_2, \ldots, x_{k+1}) = (-x_1, x_2, \ldots, x_{k+1}).
\]

The main technical result of this paper is the following.

**Theorem 1.2.** To define a functor \( F : \text{Cob}_n \to \text{Vect} \), it suffices to construct a symmetric monoidal functor \( F : \text{Man}_n \to \text{Vect} \), and for every \( n \)-manifold \( M \) and framed sphere \( \mathbb{S} \subset M \), a linear map \( F_{M,\mathbb{S}} : F(M) \to F(M(\mathbb{S})) \) that satisfy the following axioms:

1. We have \( F_{M,\emptyset} = \text{Id}_{F(M)} \), and if \( d \in \text{Diff}_0(M) \), then \( F(d) = \text{Id}_{F(M)} \).
(2) Given a diffeomorphism \(d: M \to M'\) between \(n\)-manifolds and a framed sphere \(S \subset M\), let \(S' = d(S)\), and let \(\hat{d}: M \to M'(S')\) be the induced diffeomorphism. Then the following diagram is commutative:

\[
\begin{array}{ccc}
F(M) & \xrightarrow{F_{M,S}} & F(M(S)) \\
\downarrow F(d) & & \downarrow F(\hat{d}) \\
F(M') & \xrightarrow{F_{M'(S')}} & F(M'(S')).
\end{array}
\]

(3) If \(M\) is an \(n\)-manifold and \(S\) and \(S'\) are disjoint framed spheres in \(M\), then \(M(S)(S') = M(S')(S)\), we denote this manifold by \(M(S,S')\). Then the following diagram is commutative:

\[
\begin{array}{ccc}
F(M) & \xrightarrow{F_{M,S}} & F(M(S)) \\
\downarrow F_{M,S'} & & \downarrow F_{M(S),S'} \\
F(M(S')) & \xrightarrow{F_{M(S'),S}} & F(M(S,S')).
\end{array}
\]

(4) If \(S' \subset M(S)\) intersects the belt sphere of the handle attached along \(S\) once transversely, then there is a diffeomorphism \(\varphi: M \to M(S)(S')\) (which is defined below; it is the identity on \(M \cap M(S)(S')\) and is unique up to isotopy), for which

\[F_{M(S),S'} \circ F_{M,S} = F(\varphi).\]

(5) \(F_{M,S} = F_{M,S'}\).

The functor \(F\) is a TQFT if and only if it is symmetric and monoidal. In the opposite direction, every functor \(F: \text{Cob}_n \to \text{Vect}\) arises in this way.

An analogous result holds for \(\text{Cob}'_n\), and we can avoid \(S = \emptyset\) and framed \(n\)-spheres. In the case of \(\text{Cob}'_n\) for \(n \geq 2\), we need to avoid \(S = \emptyset\) and \(n\)-spheres, together with separating \((n-1)\)-spheres. Finally, for \(\text{BSut}'\), we have a similar result, and we can avoid \(S = \emptyset\) and framed \(3\)-spheres.

It might come as a surprise that handleslide invariance does not feature among the above axioms. This is because the proof relies on proper and not self-indexing Morse functions, and a handleslide can be replaced by moving one of the corresponding critical points to a higher level, isotoping its attaching sphere, then moving it back to the same level. So handleslide invariance follows from axioms (2) and (3).

Hatcher proved that \(\text{Diff}(D^3, \partial D^3)\) is contractible, hence every diffeomorphism of a 3-manifold supported in a ball is isotopic to the identity. So, when \(n \leq 3\), in axiom (4), the diffeomorphism \(\varphi\) is uniquely characterized up to isotopy by the property that it fixes \(M \cap M(S)(S')\). In higher dimensions, \(\text{Diff}(D^n, \partial D^n)\) might be disconnected; we describe the diffeomorphism \(\varphi\) as follows.

Let \(W\) be the cobordism obtained by attaching a handle \(h\) to \(M \times I\) along \(S \times \{1\}\), followed by a handle \(h'\) attached along \(S'\). Let \(D = N(S) \cup (N(S') \cap M)\), this becomes diffeomorphic to a disk after smoothing its corners since \(S'\) intersects the belt sphere of \(h\) in a single point. Finally, let \(H = (D \times I) \cup h \cup h'\); this is diffeomorphic to \(D \times I\). Let \(F: M \times I \to W\) be a diffeomorphism such that \(F(x, 0) = (x, 0)\) for every \(x \in M\) and \(F(x, t) = (x, t)\) for every \(x \in M \setminus D\) and \(t \in I\). Then let \(\varphi = F|_{M \times \{1\}}\). To define \(F\), one only needs to choose a diffeomorphism \(D \times I \to H\) that is the identity along \((D \times \{0\}) \cup (\partial D \times I)\). If \(F'\) is another such map, then the induced \(\varphi'\) differs
from $\varphi$ by a pseudo-isotopy supported in the disk $H \cap M(S)(S')$. By Cerf \cite{cerf}, for $n \geq 5$, any diffeomorphism of $D^n$ that fixes $\partial D^n$ and is pseudo-isotopic to the identity is actually isotopic to the identity, as $D^n$ is simply-connected. The only case when we do not know whether $\varphi$ is well-defined up to isotopy is when $n = 4$.

The following construction works in all dimensions. Now let $W$ be the cobordism obtained by composing $W(S)$ and $W(S')$. By Lemma 2.13 there is a Morse function $f$ on $W$ and a gradient-like vector field $v$ that are compatible with the natural parameterized Cerf decomposition of $W$ (cf. Definition 2.7) with diffeomorphisms $\text{Id}_M(S)$ and $\text{Id}_M(S)(S')$. In particular, $f$ has exactly two critical points $p$ and $p'$ at the centers of $h$ and $h'$, respectively. Furthermore, the stable manifold $W^s(p)$ is the core of $h$ union $S \times I$, the unstable manifold $W^u(p) \cap W(S)$ is the co-core of $h$, and similarly, $W^s(p') \cap W(S')$ is the core of $h'$ union $S' \times I$, while $W^u(p')$ is the co-core of $h'$. There is a homotopically unique 1-parameter family \{ $f_t : t \in [-1,1]$ \} of smooth functions $(W, \partial W) \to (I, \partial I)$ such that $f_{-1} = f$, it has a single death bifurcation at $t = 0$, and the stable manifold of the larger critical point and the unstable manifold of the smaller critical point remain transverse for $t \in [-1,0)$. In the terminology of Cerf \cite{cerf} Proposition 2, Chapitre III, there is a ‘chemin élémentaire’; i.e., an elementary path canceling the two critical points that can be described in a local model in a neighborhood $U$ of $W^u(p) \cup W^u(p')$. Outside $U$, the family $f_t$ is constant. In particular, $f_1$ has no critical points, and according to Cerf \cite{cerf}, the space of such paths is connected. Hence, if $f_1$ and $f'_1$ are two different paths, then $f_1$ and $f'_1$ are homotopic through smooth functions with no critical points. The gradient flows of $f_1$ and $f'_1$ give rise to isotopic diffeomorphisms from $M$ to $M(S)(S')$, and changing the metric also preserves the isotopy class. It is important to note that keeping the ascending and descending manifolds of the canceling critical points transverse throughout (or equivalently, the pair of spheres obtained by intersecting them with $M(S)$) is what ensures the uniqueness. The space of ascending and descending manifolds intersecting in a single flow-line might have several components, each of which might result in different cancelations. Also see the First Cancelation Theorem of Morse in the book of Milnor \cite{milnor} Theorem 5.4.

In condition \ref{condition1}, it would suffice to assume that $F(d) = \text{Id}_{F(M)}$ whenever $d$ is isotopic to the identity and supported in a ball. However, according to the classical result of Palis and Smale \cite{palis}, such diffeomorphisms generate $\text{Diff}_0(M)$.

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## 2. Parameterized Cerf decompositions

### 2.1. Cobordism categories and TQFTs.

When talking about cobordism categories, it is important to keep the following definition in mind, see Milnor \cite{milnor}.

**Definition 2.1.** A *cobordism* from $M_0^n$ to $M_1^n$ is a 5-tuple $(W; V_0, V_1; h_0, h_1)$, where $W$ is a compact $(n + 1)$-manifold such that $\partial W$ is the disjoint union of $V_0$ and $V_1$, and $h_i : V_i \to M_i$ are diffeomorphisms for $i \in \{0, 1\}$.

If $M_0$ and $M_1$ are oriented, we require that $W$ be oriented as well, such that if $V_0$ and $V_1$ are given the boundary orientation, then $h_0$ is orientation reversing, while $h_1$ is orientation preserving.

Given cobordisms from $M_0$ to $M_1$ and $M_1$ to $M_2$, we can glue them together, but the smooth structure on the result is only well-defined up to diffeomorphism.
fixing the boundaries. Hence, to be able to define the composition of cobordisms, we consider the following equivalence relation.

**Definition 2.2.** The cobordisms \((W; V_0, V_1; h_0, h_1)\) and \((W'; V_0', V_1'; h_0', h_1')\) from \(M_0\) to \(M_1\) are equivalent if there is a diffeomorphism \(g: W \to W'\) such that \(g(V_i) = V_i'\) and \(h_i' \circ g|_{V_i} = h_i\) for \(i \in \{0, 1\}\).

The following definition is due to Eilenberg and Steenrod.

**Definition 2.3.** Let \(\text{Cob}_n\) be the category whose objects are closed \(n\)-manifolds, and whose morphisms are equivalence classes of cobordisms. For an \(n\)-manifold \(M\), the identity morphism \(i_M\) is the equivalence class of the tuple \((M \times I; M \times \{0\}, M \times \{1\}; p_0, p_1)\), where \(p_i: M \times \{i\} \to M\) is the map \(p_i(x, i) = x\).

The description of the identity morphism highlights the role of the parameterizations \(h_i\), as only using triads \((W; V_0, V_1)\), we would not have any morphisms from \(M\) to itself. Furthermore, we can assign a cobordism to any diffeomorphism as follows. Suppose that \(h: M \to M'\) is a diffeomorphism of \(n\)-manifolds. Then let \(c_h\) be the equivalence class of the tuple \((M \times I; M \times \{0\}, M \times \{1\}; p_0, h_1)\), where \(p_0\) is as above, and \(h_1\) is defined by the formula \(h_1(x, 0) = h(x)\). Recall that two diffeomorphisms \(h, h': M \to M'\) are pseudo-isotopic if there is a diffeomorphism \(g: M \times I \to M \times I\) such that \(g(x, i) = (h_i(x), i)\) for \(i \in \{0, 1\}\) and \(x \in M\). Note that \(g\) does not have to preserve level sets. Then \(c_{h_0} = c_{h_1}\) if and only if \(h_0\) and \(h_1\) are pseudo-isotopic. Furthermore, \(c_{h}c_{h'} = c_{h'h}\).

**Definition 2.4.** Let \(\text{Vect}\) be the category of vector spaces and linear maps over some field \(F\). An \((n+1)\)-dimensional topological quantum field theory is a functor 

\[ F: \text{Cob}_n \to \text{Vect}\]

such that for any two closed \(n\)-manifolds \(M\) and \(M'\), there is a functorial isomorphism \(F(M \sqcup M') \cong F(M) \otimes F(M')\) that makes the following diagram commutative:

\[
\begin{array}{ccc}
F(M \sqcup M') & \xrightarrow{F(c_h)} & F(M' \sqcup M) \\
\downarrow \cong & & \downarrow \cong \\
F(M) \otimes F(M') & \xrightarrow{r} & F(M') \otimes F(M),
\end{array}
\]

where \(s: M \sqcup M' \to M' \sqcup M\) is the diffeomorphism swapping the two factors, and \(r(x \otimes y) = y \otimes x\).

Similarly, a TQFT on the category of connected \(n\)-manifolds is a functor 

\[ F: \text{Cob}^0_n \to \text{Vect}\]

but in this case we drop the condition on disjoint unions. A TQFT on an oriented cobordism category has to satisfy functorial isomorphisms \(F(-M) \cong F(M)^*\).

Given a diffeomorphism \(h\), we denote the map \(F(c_h)\) by \(h_*\). We shall see in Lemma 2.22 that if \(F\) arises from a functor \(F: \text{Man}_n \to \text{Vect}\) and surgery maps \(F_{M,S}\) as in Theorem 1.2, then \(h_* = F(h)\). If \(h\) and \(h'\) are pseudo-isotopic, then \(c_h = c_{h'}\), hence \(h_* = h'_*\). Once we can associate cobordisms to diffeomorphisms, the following statement follows from the functoriality of \(F\).
Proposition 2.5. Let \( W = (W; V_0, V_1; h_0, h_1) \) be a cobordism from \( M_0 \) to \( M_1 \), and let \( W' = (W'; V'_0, V'_1; h'_0, h'_1) \) be a cobordisms from \( M'_0 \) to \( M'_1 \). If \( d : W \to W' \) is a diffeomorphism such that \( d(V_i) = V'_i \) for \( i \in \{0, 1\} \), we write
\[
d|_{M_i} := h'_i \circ d|_{V_i} \circ h_i^{-1} : M_i \to M'_i.
\]
Then the following diagram is commutative:
\[
\begin{array}{ccc}
F(M_0) & \xrightarrow{F(c)} & F(M_1) \\
\downarrow^{(d|_{M_0})_*} & & \downarrow^{(d|_{M_1})_*} \\
F(M'_0) & \xrightarrow{F(c')} & F(M'_1),
\end{array}
\]
where \( c \) is the equivalence class of \( W \) and \( c' \) is the equivalence class of \( W' \).

2.2. Parameterized Cerf decompositions. To simplify the notation, from now on, we will suppress the diffeomorphisms \( h_0 \) and \( h_1 \) and identify \( V_i \) and \( M_i \) when talking about cobordisms. So an oriented cobordism from \( M_0 \) to \( M_1 \) is viewed as a compact \((n+1)\)-manifold \( W \) with \( \partial W = -M_0 \cup M_1 \). With this convention, two cobordisms \( W \) and \( W' \) from \( M_0 \) to \( M_1 \) are equivalent if there is a diffeomorphism \( d : W \to W' \) that fixes the boundary pointwise. We say that \( f : W \to [a, b] \) is a Morse function if \( f^{-1}(a) = M_0, f^{-1}(b) = M_1 \), and \( f \) has only non-degenerate critical points, all lying in the interior of \( W \).

Given an \( n \)-manifold \( M \), a framed \( k \)-sphere \( S \subset M \) is an embedding of \( S^k \times D^{n-k} \) into \( M \), where we think of \( S \) as the image of \( S^k \times \{0\} \), together with a trivialization of its normal bundle. We write \( W(S) \) for the manifold obtained by attaching the handle \( D^{k+1} \times D^{n-k} \) to \( M \times I \) along \( S \times \{1\} \); this is a cobordism from \( M \) to the manifold \( M(S) \) obtained by surgery on \( M \) along \( S \). We recall the following definition from Milnor [10].

Definition 2.6. A cobordism \( W \) from \( M_0 \) to \( M_1 \) is elementary if there is a Morse function \( f : W \to [a, b] \) such that it has at most one critical point. An attaching sphere \( S \) for \( W \) is the empty-set if \( f \) has no critical points; otherwise, it is a framed sphere in \( M_0 \) such that there is a diffeomorphism \( D : W(S) \to W \) that is the identity along \( M_0 \) (where we identify \( M_0 \) with \( M_0 \times \{0\} \)).

It is a classical result of Morse theory that every elementary cobordism admits an attaching sphere in the above sense.

Definition 2.7. A parameterized Cerf decomposition of a cobordism \( W \) from \( M \) to \( M' \) consists of
- a Cerf decomposition
  \[
  W = W_0 \cup_{M_1} W_1 \cup_{M_2} \cdots \cup_{M_m} W_m
  \]
in the sense of Gay et al. [3]; i.e., each \( W_i \) is an elementary cobordism from \( M_i \) to \( M_{i+1} \) (where \( M_0 = M \) and \( M_{m+1} = M' \)),
- an attaching sphere \( S_i \subset M_i \) for \( W_i \) of dimension \( k_i \),
- a diffeomorphism \( d_i : M(S_i) \to M_i \), well-defined up to isotopy, such that there exists a diffeomorphism \( D_i : W(S_i) \to W_i \) with \( D_i|_{M_i \times \{0\}} = p_0 \) and \( D_i|_{M_i(S_i)} = d_i \).
Remark 2.8. The existence of the diffeomorphism $D_i$ ensures that the cobordism \\
$$(W(S_i); M_i \times \{0\}, M_i(S_i); p_0, d_i)$$

is equivalent to $(W_i; M_i, M_{i+1}; \text{Id}_M, \text{Id}_{M_{i+1}})$. So we are replacing each elementary component in the Cerf decomposition of $W$ by an equivalent handle cobordism. In particular, the composition of these handle cobordisms is equivalent to $(W; M, M'; \text{Id}_M, \text{Id}_{M'})$.

2.3. Morse data. The following definition is due to Milnor [10].

**Definition 2.9.** Let $f$ be a Morse function on the cobordism $W$. We say that the vector field $v$ on $W$ is gradient-like for $f$ if $v_p(f) > 0$ for every $p \in W \setminus \text{Crit}(f)$, and for every point $p \in \text{Crit}(f)$, there exists a local positively oriented coordinate system $(x_1, \ldots, x_{n+1})$ centered at $p$ in which
\\n\begin{equation}
 f = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_{n+1}^2,
\end{equation}

and where $v$ is the Euclidean gradient; i.e.,
\\n\begin{equation}
 v = 2 \left( -x_1 \frac{\partial}{\partial x_1} - \cdots - x_k \frac{\partial}{\partial x_k} + x_{k+1} \frac{\partial}{\partial x_{k+1}} + \cdots + x_{n+1} \frac{\partial}{\partial x_{n+1}} \right).
\end{equation}

The space of positive coordinate systems at a Morse critical point in which $f$ is of the normal form (2.1) is homotopy equivalent to $SO(k, n+1-k)$, and hence is connected for $k \in \{0, n+1\}$, and has two components otherwise; cf. Cerf [1]. However, the space of gradient vector fields $v$ induced by such coordinate systems is connected for every $k$. Indeed, if $k \notin \{0, n+1\}$ and $(x_1, \ldots, x_{n+1})$ is a positive coordinate system in which $f$ is of the form (2.1), then
\\n$$(-x_1, x_2, \ldots, x_n, -x_{n+1})$$

is also a positive coordinate system as in (2.1), but which lies in the opposite component since it reverses the orientation of both the positive and negative definite subspaces. In both coordinate systems $v$ is of the same form.

**Definition 2.10.** A *Morse datum* [10] for the cobordism $W$ is a pair $(f, b)$, where
\\n$$b = (b_0, \ldots, b_{m+1}) \in \mathbb{R}^{m+2}$$

is an ordered tuple; i.e., $b_0 < b_1 < \cdots < b_{m+1}$, and $f: M \to [b_0, b_{m+1}]$ is a proper Morse function such that each $b_i$ is a regular value of $f$, and $f$ has at most one critical value in each interval $(b_{i-1}, b_i)$. We will also call a triple $(f, b, v)$ a Morse datum, where $(f, b)$ is as above, and $v$ is a gradient-like vector field for $f$.

A Morse datum $(f, b)$ induces a Cerf decomposition $C(f, b)$ of $W$ by taking $W_i = f^{-1}([b_i, b_{i+1}])$ and $M_i = f^{-1}(b_i)$. As we shall now see, a triple $(f, b, v)$ induces a parameterized Cerf decomposition of $W$.

Suppose that $W$ is an elementary cobordism from $M$ to $M'$, together with a Morse function $f$ and gradient-like vector field $v$. If $f$ has no critical points, then one obtains a diffeomorphism $d_v: M \to M'$ by flowing along $w = v/v(f)$. When $f$ has one critical point $p$ of index $k$, then we obtain a framed sphere $S \subset M$, and a diffeomorphism $d_v: M(S) \to M'$, well-defined up to isotopy, as follows.

Let $W^*(p)$ be the stable manifold of $p$. The sphere $S$ will be $W^*(p) \cap M$, with the following framing. As in Milnor [9] p16, choose a positive coordinate system
\\n$$(x_1, \ldots, x_{n+1}): U \to \mathbb{R}^{n+1}$$

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centered at \( p \) in which \( f \) is of the form (2.1), and let \( \varepsilon \) be so small that the image of \((x_1, \ldots, x_{n+1})\) contains a ball of radius \( \sqrt{2\varepsilon} \) centered at the origin. Let \( c = f(p) \), and consider the level sets \( f^{-1}(c - \varepsilon) \) and \( f^{-1}(c + \varepsilon) \). Define the cell \( e \) to be the subset of \( U \) where \( x_1^2 + \cdots + x_k^2 \leq \varepsilon \) and \( x_{k+1} = \cdots = x_{n+1} = 0 \). Furthermore, let \( E \) be a regular neighborhood of \( e \) of width \( \varepsilon/2 \), extending all the way to \( f^{-1}(c - \varepsilon) \), this can canonically be identified with the \( k \)-handle \( D^k \times D^{n-k+1} \). It is straightforward to check that \( v \) is transverse to \( \partial E \setminus f^{-1}(c - \varepsilon) \). The framing of \( S \subset M \) is given by flowing \( E \cap f^{-1}(c - \varepsilon) \) along \(-w\), giving a regular neighborhood \( N(S) \) of \( S \). The diffeomorphism \( d_v \) is defined by flowing \( M \setminus N(S) \) along \( v/v(f) \) to \( f^{-1}(c - \varepsilon) \setminus E \), and identifying the part \( D^k \times S^{n-k} \) of \( M(S) \) with \( E \setminus f^{-1}(c - \varepsilon) \), then flowing again along \( v/v(f) \) to \( M' \) (as we are not flowing from a level set, for different points, we need to flow for a different amount of time to reach \( M' \)). Note that \( d_v|_{M \setminus S} \) is simply given by the flow of \( v \). It is easy to see that \( d_v \) extends to a diffeomorphism from \( W(S) \) to \( W \) that is the identity on \( M(S) \).

Remark 2.11. The above construction depends on the choice of \( \varepsilon \) and local coordinate system, but different choices give isotopic framings and diffeomorphisms. Furthermore, \( S \) and \( d_v \) depend on \( v \) only up to isotopy, since the space of gradient-like vector fields \( v \) compatible with a given Morse function \( f \) is connected. The only caveat is that when \( k \notin \{0, n+1\} \), the space of coordinate systems is homotopy equivalent to \( SO(k, n+1-k) \), which has two components. The two components correspond to non-isotopic framed spheres. If \( S \) is one, then \( S \) represents the other isotopy class, cf. axiom (5) in Theorem 1.2.

Definition 2.12. Let \( W \) be a cobordism from \( M \) to \( M' \). We say that the Morse datum \((f, b, v)\) induces the parameterized Cerf decomposition \( C \) if \((f, b)\) is the Cerf decomposition underlying \( C \), and for every component \( W_i \), the attaching sphere \( S_i \) and the diffeomorphism \( d_i : M_i(S_i) \to M_{i+1} \) are obtained as above for some choice of compatible local coordinate systems and radii \( \varepsilon_i \) at the critical points.

Hence, the Morse datum \((f, b, v)\) gives rise to a well-defined parameterized Cerf decomposition that we denote by \( C(f, b, v) \), up to possibly replacing a framed sphere \( S \) with \( \overline{S} \). The following result states that this assignment is surjective.

Lemma 2.13. Let \( C \) be a parameterized Cerf decomposition of the cobordism \( W \). Then there exists a Morse datum \((f, b, v)\) inducing \( C \).

Proof. Recall that each diffeomorphism \( d_i : M_i(S_i) \to M_{i+1} \) extends to a diffeomorphism \( D_i : W(S_i) \to W_i \). We claim that there is a Morse function \( f_i' : W(S_i) \to \mathbb{R} \) and a gradient-like vector field \( v_i' \) on \( W(S_i) \) such that \( f_i' \) has a single critical point in the handle if \( S_i \neq \emptyset \), and the diffeomorphism induced by \( f_i' \) and \( v_i' \) on \( W(S_i) \) is \( \text{Id}_{M_i(S_i)} \). If \( S_i = \emptyset \), then we take \( f_i' \) to be the projection \( p_2 : M_i \times I \to I \) and \( v_i' \) to be \( \partial/\partial t \).

If \( S_i \neq \emptyset \) is a \((k-1)\)-sphere, then consider the functions
\[
s(x_1, \ldots, x_{n+1}) = 1/2 - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_{n+1}^2 \quad \text{and} \quad u(x_1, \ldots, x_{n+1}) = \sqrt{(x_1^2 + \cdots + x_k^2)(x_{k+1}^2 + \cdots + x_{n+1}^2)}
\]
on \( \mathbb{R}^{n+1} \). Let
\[
H = \{ x \in \mathbb{R}^{n+1} : 0 \leq s(x) \leq 1, u(x) \leq 1 \}.
\]
If $N(S_i)$ is the regular neighborhood of $S_i$, identified with $S^{k-1} \times D^{n-k+1}$ via the framing, then

$$G = (N(S_i) \times 1) \cup (D^k \times D^{n-k+1}) \subset W(S_i)$$

is diffeomorphic to $H$ if we smooth the corners after attaching the handle. We choose a diffeomorphism $\phi: G \to H$ such that it maps $M_i \times \{0\}$ to $H \cap \{s = 0\}$ and $\partial D^k \times D^{n-k+1}$ to $H \cap \{s = 1\}$, while there is a small $\nu \in \mathbb{R}_+$ such that for any $t \in (0,1)$ if $s(x) = t$ and $u(x) \in [1 - \nu, 1]$, then $\phi^{-1}(x) \in M_i \times \{t\}$. For $y \in (M_i \times I) \setminus G$, we let $f_i^t(y) = p_2(y)$, while for $y \in G$, let $f_i^t(y) = s(\phi(y))$. This is a smooth function by construction. The gradient-like vector field $v'_i$ on $W(S_i)$ is defined on $G$ by pulling back the Euclidean gradient of $s$ on $H$ via $\phi$. We extend this to $(M_i \times I) \setminus G$ via $\partial/\partial t$. It is now straightforward to check that the function $f_i$ and the gradient-like vector field $v'_i$ induce the identity diffeomorphism from $M_i(S_i)$ to itself for $\varepsilon = 1$.

Let $a_i: I \to [b_{i-1}, b_i]$ be the affine equivalence $a_i(t) = b_{i-1}(1-t) + b_i t$, and we set $f_i := a_i \circ f_i \circ D_i^{-1}$. By [3, Lemma 2.6], we can modify the $f_i$ by an ambient isotopy on a collar neighborhood of $M_i$ such that they patch together to a Morse function $f$. If $v_i = D_i(f'_i)$ possibly modified on a collar of $M_i$ so that for different $i$ they fit together to a smooth vector field $v$, then the induced diffeomorphism from $M_i(S_i)$ to $M_{i+1}$ will be isotopic to $d_i$. □

**Lemma 2.14.** Let $C$ be a Cerf decomposition of the cobordism $W$. Suppose that the Morse data $(f, b, v)$ and $(f', b', v')$ both induce $C$, in the sense that for given local coordinate systems about the critical points and radii the framings of the attaching spheres and the diffeomorphisms $d_i$ coincide. Then there exist diffeomorphisms $D: W \to W$ and $\phi: \mathbb{R} \to \mathbb{R}$ such that

1. $b' = \phi(b)$,
2. $f' = \phi \circ f \circ D^{-1}$,
3. $v \cdot v' = D_s(v)$ for some positive function $v \in C^\infty(W, \mathbb{R}_+)$, and
4. $D|_{M_i} = I d_{M_i}$.

**Proof.** First, suppose that $W$ is an elementary cobordism, $b = b'$, and $|b| = |b'| = 2$. Let the critical points of $f$ and $f'$ be $p$ and $p'$ with values $c$ and $c'$, respectively. Choose coordinate charts $\mathcal{G}: U \to \mathbb{R}^{n+1}$ and $\mathcal{G}' : U' \to \mathbb{R}^{n+1}$ about $p$ and $p'$, respectively, such that their images coincide with the disc $D(0, \sqrt{2 \varepsilon})$, and in which $f$ and $f'$ have the normal form of equation (2.1), while $v$ and $v'$ have the normal form (2.2). Furthermore, we write $K_p = W^s(p) \cup W^u(p)$ and $K_{p'} = W^s(p') \cup W^u(p')$.

Let $\phi_0 : [b_0, b_1] \to [b_0, b_1]$ be a diffeomorphism such that $\phi_0(b_i) = b_i$ for $i \in \{0, 1\}$, and such that $\phi_0(t) = c' - c + t$ for $t \in [c - 2 \varepsilon, c + 2 \varepsilon]$. Then $v$ is also a gradient-like vector field for $\phi \circ f$; moreover, $\phi_0 \circ f = f(p')$, and the Morse datum $(\phi_0 \circ f, b', v)$ induces the same parameterized Cerf decomposition $C$. Hence, we can assume that $f(p) = f(p') = c$.

Let $\gamma : Z \subset W \times \mathbb{R} \to W$ and $\gamma' : Z' \subset W \times \mathbb{R} \to W$ be the flows of $v$ and $v'$, respectively. For $x \in W$, the set $(\{x\} \times \mathbb{R}) \cap Z$ is a closed interval $\{x\} \times [-\alpha(x), \omega(x)]$ when $x \not\in K_p$, a half-interval $\{x\} \times [-\alpha(x), \infty)$ when $x \in W^s(p)$, and a half-interval $\{x\} \times (-\infty, \omega(x)]$ for $x \in W^u(p)$. Using $Z'$, we obtain the functions $\alpha'$ and $\omega'$ in an analogous way.

Let $D(p) = p'$. We define the diffeomorphism $D$ on $W \setminus \{p\}$ as follows. If $x \in M \cup (W^u(p) \cap M')$ and $t \in (\{x\} \times \mathbb{R}) \cap Z$, then let

$$D(\gamma(x,t)) = \gamma'(x, h(x,t)),$$
where $h(x, t) \in (\{x\} \times \mathbb{R}) \cap \mathbb{Z}'$ is the unique parameter value for which

$$f(\gamma'(x, h(x, t))) = f(\gamma(x, t)).$$

It is clear that $D$ restricts to a diffeomorphism

$$W \setminus W^u(p) \to W \setminus W^u(p')$$

that fixes $\partial W \setminus W^u(p) = \partial W \setminus W^u(p')$ pointwise. Indeed, for $x \in M \setminus S$, we have $\gamma(x, \omega(x)) = \gamma'(x, \omega'(x))$ since the Morse data $(f, b, v)$ and $(f', b', v')$ induce the same diffeomorphism $d: M(S) \to M'$ in $\mathcal{C}$.

Recall that $E$ is a subset of $\mathbb{R}^{n+1}$ diffeomorphic to the $k$-handle $D^k \times D^{n-k+1}$. We denote by $\partial_\pm E$ the part of $\partial E$ corresponding to $S^{k-1} \times D^{n-k+1}$, and by $\partial_+ E$ the part corresponding to $D^k \times S^{n-k}$. Let $F$ be the smallest subset of $W$ that contains $\mathcal{E} = \partial_+ E$ and is saturated under the flow of $v$, and we define $F'$ containing $\mathcal{E}' = (\mathcal{E}')^{-1}(E)$ analogously. Note that $F$ is a regular neighborhood of $K_p$ and $F'$ is a regular neighborhood of $K_p'$. Furthermore, let $\partial_+ \mathcal{E} = \partial_+ (\partial_+ E)$, and $\partial_+ \mathcal{E}'$ is the same map $\partial_+ \mathcal{E}'$ as $(\partial_+ E')$ since the Morse data $(f, b, v)$ is compatible with $\mathcal{C}$, by definition, the flow of $v$ from

$$\mathcal{E} \cap f^{-1}(c - \epsilon) = \partial_+ \mathcal{E} \approx S^{k-1} \times D^{n-k+1}$$

gives the framing of $S$. Similarly, the flow of $v'$ from $\mathcal{E}' \cap (f')^{-1}(c - \epsilon) = \partial_+ \mathcal{E}'$ gives the framing of $S'$ as $(f', b', v')$ also induces $\mathcal{C}$. If $H$ denotes the handle part of $M(S)$, which is diffeomorphic to $D^k \times S^{n-k}$, then $d: M(S) \to M'$ restricts to a map $d|_H$ that gives a framing of $W^u(p) \cap M' = W^u(p') \cap M'$ that is given by either flowing from $\partial_+ \mathcal{E}$ along $v$ to $M'$, or from $\partial_+ \mathcal{E}'$ along $v'$ to $M'$.

We claim that

$$(2.3) \quad D|_{\mathcal{E}'} = (\mathcal{g}')^{-1} \circ \mathcal{g} : \mathcal{E} \to \mathcal{E}'$$

To see this, it suffices to show that for any point $e \in \partial \mathcal{E}$, we have

$$\mathcal{g}'(D(e)) = \mathcal{g}(e) \in \partial \mathcal{E}.$$ 

Indeed, if $e \in \mathcal{E} \setminus W^u(p)$, then there is a unique $t \in \mathbb{R}_{\geq 0}$ for which $\gamma(e, t) \in \partial_+ \mathcal{E}$, we write $e_- = \gamma(e, t)$. By definition, $D(e)$ is given by flowing back to $M$ along $v$, and then forward along $v'$ until the value of $f'$ agrees with $f(e)$. We obtain the same point by flowing back along $v$ to $e_-$ in $\partial_+ \mathcal{E}$, then forward along $v'$ from $D(e_-) = (\mathcal{g}')^{-1} \circ \mathcal{g}(e_-)$ until $f'$ becomes $f(e)$. Since $(\mathcal{g}')^{-1} \circ \mathcal{g}$ takes $v$ to $v'$ and $f$ to $f'$ as they are in normal form in $\mathcal{g}$ and $\mathcal{g}'$, respectively, we see that $D(e) = (\mathcal{g}')^{-1} \circ \mathcal{g}(e)$. If $e \in W^u(p) \setminus \{p\}$, then there is a unique $t \in \mathbb{R}_{\geq 0}$ for which $\gamma(e, t) \in \partial_+ E$, let $e_+ = \gamma(e, t)$. In this case, we get $D(e)$ by flowing forward to $M'$ along $v$, then back along $v'$ until the value of $f'$ becomes $f(e)$. We get the same point by flowing back from $D(e_+) = (\mathcal{g}')^{-1} \circ \mathcal{g}(e_+)$. Just like in the previous case, it follows that $D(e) = (\mathcal{g}')^{-1} \circ \mathcal{g}(e)$.

We now prove $(2.3)$. Let $r \in \partial_+ E$. Since $v$ and $v'$ both give the same framed sphere $S$, we get the same point $m \in M$ if we flow back along $v$ from $(\mathcal{g}')^{-1}(r) \in \partial_+ E$ or if we flow back along $v'$ from $(\mathcal{g}')^{-1}(r)$. But $f((\mathcal{g}')^{-1}(r)) = f((\mathcal{g}')^{-1}(r)) = c - \epsilon$, hence $D((\mathcal{g}')^{-1}(r)) = (\mathcal{g}')^{-1}(r)$. Now let $r \in S^{n-k} := \partial_+ E \cap \{x_1 = \cdots = x_k = 0\}$.

Flowing forward along $v$ from $\mathcal{g}(S^{n-k})$ to $M'$, or along $v'$ from $\mathcal{g}'(S^{n-k})$ to $M'$ give the same parametrization of $W^u(p) \cap M' = W^u(p') \cap M'$. Indeed, they induce the same map $M(S) \to M'$, and the handle part of $M(S)$ is identified with $\partial_+ E$. So if we flow forward from $\mathcal{g}(r)$ to $M'$ along $v$ and then back along $v'$ to $\partial_+ \mathcal{E}'$, we
get \( \frac{d}{dr}(r) \). However, \( f(\frac{d}{dr}(r)) = f'(\frac{d}{dr}(r)) \), hence \( D(\frac{d}{dr}(r)) = \frac{d}{dr}(r) \). This concludes the proof of (2.3).

It follows that \( D \) is smooth in \( \mathcal{E} \). To see that it is smooth along \( W^u(p) \), note that if \( x \in W \) and there is a \( t \in \mathbb{R}_{\leq 0} \) for which \( \gamma(x, t) \in \partial_x \mathcal{E} \), then \( D(x) \) can also be obtained by flowing forward from \( D(\gamma(x, t)) \) along \( v' \) until the value of \( f' \) becomes \( f(x) \), together with equation (2.3), which implies that \( D \) smoothly maps \( \partial_x \mathcal{E} \) to \( \partial_x \mathcal{E}' \). This follows from the fact that \( D \) maps flow-lines of \( v \) to flow-lines of \( v' \).

The fact that \( D|_M = \text{Id}_M \) follows from the definition of \( D \). To see that \( D|_{M'} = \text{Id}_{M'} \), note that \( v \) and \( v' \) induce the same diffeomorphisms \( M(\mathcal{S}) \to M' \). Hence, for every \( x \in M \setminus \mathcal{S} \), the flow-lines of \( v \) and \( v' \) starting at \( x \) end at the same point of \( M' \). Furthermore, for every \( r \in \partial_x E \), the flow-line of \( v \) starting at \( x(r) \) and the flow-line of \( v' \) starting at \( x'(r) \) end at the same point of \( M' \). This concludes the proof when the cobordism is elementary and \( b = b' \).

Now we consider the case of a general Cerf decomposition \( \mathcal{C} \). Choose a diffeomorphism \( \phi: \mathbb{R} \to \mathbb{R} \) such that \( \phi(b) = b' \) and such that \( \phi \) is linear in a neighborhood of each critical value of \( f \) (the latter is to ensure that \( v \) is also gradient-like at the critical points of \( \phi \circ f \)). We can then apply the previous argument to each elementary piece \( W_i \) with Morse data \((\phi \circ f|_{W_i}, (b_{i-1}^l, b_i^l), v|_{W_i}) \) and \((f'|_{W_i}, (b_{i-1}^r, b_i^r), v'|_{W_i}) \) to obtain diffeomorphisms \( D_i: W_i \to W_i \) that piece together to a diffeomorphism \( D: W \to W \) with the required properties. \( \square \)

Next, we describe some moves on parameterized Morse data. We show that any two Morse data can be connected by a sequence of such moves, and describe what happens to the induced Cerf decompositions. In the following, let \( \mathcal{M} = (f, b, v) \) and \( \mathcal{M}' = (f', b', v') \) be Morse data on the cobordism \( W \), and let \( \mathcal{C} = \mathcal{C}(\mathcal{M}) \) and \( \mathcal{C}' = \mathcal{C}(\mathcal{M}') \) be the induced Cerf decompositions. Furthermore, we denote by \( p_i \) the critical point of \( f \) in \( W_i \), assuming \( W_i \) is not cylindrical.

We say that \( \mathcal{M} \) and \( \mathcal{M}' \) are related by a critical point cancelation (cf. the analogous move of [3], Definition 2.8]) if there exists a one-parameter family

\[
\{(f_t, b_t, v_t): t \in [-1, 1]\}
\]

of triples such that

- \((f_{-1}, b_{-1}, v_{-1}) = \mathcal{M}, (f_1, b_1, v_1) = \mathcal{M}'\),
- \(f_t\) is a family of smooth functions and \(v_t\) is a family of smooth vector fields,
- \((f_t, b_t, v_t)\) is a Morse datum for every \( t \in [-1, 1] \) \setminus \{0\},
- \(b_t = b_t\) for \( t \in [-1, 0) \), and there is a \( j \) such that \( b_t = b_0 \setminus \{b_j\} \) for \( t \in (0, 1] \),
- the critical points \( p_{j-1}(t) \) and \( p_j(t) \) cancel at \( t = 0 \), and \( f_t \) has no critical values in \([b_{j-1}(t), b_{j+1}(t)]\) for \( t > 0 \),
- \(W^u(p_{j-1}(t))\) and \(W^s(p_j(t))\) are transverse and intersect in a single flow-line for every \( t \in (-1, 0] \),
- \(\{f_t: t \in [-1, 1]\}\) is a ‘chemin élémentaire de mort’ with support a small neighborhood \( U \) of

\[
(W^u(p_{j-1}(t)) \cup W^s(p_j(t))) \cap f^{-1}[b_{j-1}(t), b_{j+1}(t)],
\]

see Cerf [11], Section 2.3, p.71]. Inside \( U \), the path \( f_t \) is of normal form, while outside both \( f_t \) and \( v_t \) are constant.

Cerf [11], Chapter II.2] proved that, given a pair of ascending and descending manifolds for a pair of consecutive critical points that intersect in a single flow-line,
the space of standard neighborhoods is connected, and hence any two ‘chemin élémentaire de mort’ starting at \( f \) compatible with this stable and unstable manifold are homotopic. A critical point creation is the reverse of a critical point cancelation.

**Lemma 2.15.** Suppose that the Morse data \( \mathcal{M} = (f, \mathbf{b}, \mathbf{v}) \) and \( \mathcal{M}' = (f', \mathbf{b}', \mathbf{v}') \) are related by a critical point cancelation. Then the corresponding parameterized Cerf decompositions \( \mathcal{C} = \mathcal{C}(\mathcal{M}) \) and \( \mathcal{C}' = \mathcal{C}(\mathcal{M}') \) are related as follows.

The sphere \( S_{j+1} \) intersects \( d_j(\{0\} \times S^{n-k_j}) \) in a single point, where \( \{0\} \times S^{n-k_j} \subset D^{k_j} \times D^{n-k_j+1} \) is the belt circle of the handle in \( W_j(S_j) \). The cobordism \( W_j \cup W_{j+1} \) is cylindrical. We obtain \( \mathcal{C}' \) from \( \mathcal{C} \) by removing \( M_{j+1} \), more precisely,

\[
M'_i = \begin{cases} 
M_i & \text{if } i < j + 1, \\
M_{j+1} & \text{otherwise}.
\end{cases}
\]

We obtain the attaching spheres \( S'_i \) and the diffeomorphisms \( d'_i \) for \( i \neq j \) analogously. We have \( S'_j = \emptyset \), and let \( S_{j+1} = d_j^{-1}(S_{j+1}) \subset M_j(S_j) \). To determine \( d'_j: M'_j(S'_j) = M_j \to M'_{j+1} = M_{j+2} \), note that there is a diffeomorphism \( \varphi: M_j \to M_j(S_j)(S_{j+1}) \)

defined as in property (1) of Theorem 1.2. Furthermore, \( d_j \) induces a diffeomorphism

\[
d'_j^{S_{j+1}}: M_j(S_j)(S_{j+1}) \to M_{j+1}(S_{j+1}).
\]

Then

\[
d'_j \approx d_{j+1} \circ d_j^{S_{j+1}} \circ \varphi,
\]

where ‘\( \approx \)’ means ‘isotopic to.’

**Proof.** We prove equation (2.4), the rest of the statement is straightforward. Let \( \mathbf{W} \) be the cobordism obtained by gluing \( W(S_j) \) and \( W(S_{j+1}) \) along \( M(S_j) \). This carries a parameterized Cerf decomposition \( \mathcal{C} \), with diffeomorphisms \( \text{Id}_{M(S_j)} \) and \( \text{Id}_{M(S_{j+1})} \). According to Lemma 2.13 there exists a Morse datum \( (f, \mathbf{b}, \mathbf{v}) \) inducing \( \mathcal{C} \).

Next, we construct a diffeomorphism \( G: \mathbf{W} \to W_j \cup W_{j+1} \). Choose an extension \( D_i: W_i(S_i) \to W_i \) of \( d_i \) for \( i \in \{j, j+1\} \). Then \( D_j \) and \( D_{j+1} \) glue together to a diffeomorphism \( G_0: W(S_j) \cup d_j W(S_{j+1}) \to W_j \cup W_{j+1} \).

Furthermore, we can glue together \( \text{Id}_{W(S_j)} \) and \( D_j^* \) to a diffeomorphism \( G_1: \mathbf{W} \to W(S_j) \cup d_j W(S_{j+1}) \). Then we set \( G = G_0 \circ G_1 \).

The Morse datum \( (f \circ G, (b_{j-1}, b_j, b_{j+1}), G^*(v)) \) on \( \mathbf{W} \) also induces the parameterized Cerf decomposition \( \mathcal{C} \). Hence, by Lemma 2.14, there exists a diffeomorphism \( D: \mathbf{W} \to \mathbf{W} \) that fixes \( M_j(S_j) \), and \( M(S_j)(S_{j+1}) \) pointwise, and such that \( f \circ G \circ D = f \) and \( (G \circ D)^*(v) = v \cdot \mathbf{v} \). In particular, \( f_t \circ G \circ D \) for \( t \in [-1, 1] \) is a ‘chemin élémentaire de mort’ starting from \( f \) and ending at a function \( f_1 \circ G \circ D \) with no critical points that induces the diffeomorphism \( \varphi: M_j \to M_j(S_j)(S_{j+1}) \), up to isotopy. Indeed, by Cerf [11] Chapter 2.3, the space of ‘chemin élémentaire’ starting at a given Morse function that cancel two consecutive critical points with
a single flow-line between them, and which is supported in a neighborhood of their stable and unstable manifolds where it is in normal form is connected, and so their endpoints can be connected through Morse functions with no critical points. So for any choice of gradient-like vector fields, the endpoints induce isotopic diffeomorphisms. Hence \( f_1 \) on \( W_j \cup W_{j+1} \) induces a diffeomorphism \( d_{j+1} : M_j \to M_{j+2} \) that is conjugate to \( \varphi \) along \( G \). As \( G|M = \text{Id}_M \) and \( G|_{M(S_j)(S_{j+1})} = d_{j+1} \circ d_{j+1}^S \), we obtain equation (2.4).

We say that \( \mathcal{M} \) and \( \mathcal{M}' \) are related by a critical point switch if there exists a one-parameter family

\[ \{(f_t, b_t, v_t) : t \in [-1, 1]\} \]

of triples such that

- \((f_{-1}, b_{-1}, v_{-1}) = \mathcal{M}, (f_1, b_1, v_1) = \mathcal{M}', \)
- \(f_t\) is a family of smooth functions and \(v_t\) is a family of smooth vector fields,
- \((f_t, b_t, v_t)\) is a Morse datum for every \(t \in [-1, 1] \setminus \{0\},\)
- there is an \(j\) such that \(b \setminus \{b_{j+1}(t)\}\) is independent of \(t,\)
- two critical values cross each other; i.e., \(f_t(p_j) < f_t(p_{j+1})\) for \(t < 0\) and \(f_t(p_j) > f_t(p_{j+1})\) for \(t > 0,\) with equality for \(t = 0,\)
- \(W^s(p_j) \cap W^u(p_{j+1}) = \emptyset\) for every \(t \in [-1, 1],\)
- \(\{f_t : t \in [-1, 1]\}\) is a ‘chemin élémentaire de croisement ascendante or descendente’ with support in a small neighborhood \(U\) of \(W^s(p_j) \cap f^{-1}[b_j(t), b_{j+2}(t)]\)

or in

\(W^s(p_{j+1}) \cap f^{-1}[b_j(t), b_{j+2}(t)],\)

see Cerf [1] Chapter II, p.40. Inside \(U,\) the path \(f_t\) is of normal form, while outside both \(f_t\) and \(v_t\) are constant.

Lemma 2.16. Suppose that the Morse data \( \mathcal{M} \) and \( \mathcal{M}' \) are related by a critical point switch, and consider the induced parameterized Cerf decompositions \( \mathcal{C} \) and \( \mathcal{C}' \). Then these satisfy the following properties:

1. in \( \mathcal{C}, \) the part \( \ldots W_j \cup_{M_{j+1}} W_{j+1} \ldots \) is replaced by \( \ldots W'_j \cup_{M'_{j+1}} W'_{j+1} \ldots,\)
   the rest of the decomposition is unchanged,
2. \(S_{j+1} \cap d_j(D^{k_j} \times S^{n-k_j}) = \emptyset,\)
3. \(d'_j(S_j) = S'_{j+1}\) and \(d_j(S'_j) = S_{j+1},\) and
4. the following diagram is commutative up to isotopy:

\[
\begin{align*}
M_j(S_j, S'_j) \xrightarrow{(d_j)_{j'}} M_{j+1}(S_{j+1}) \\
\downarrow \quad \quad \quad \quad \quad \downarrow d_{j+1} \\
M'_{j+1}(S'_{j+1}) \xrightarrow{d'_{j+1}} M'_{j+2}.
\end{align*}
\]

Proof. Without loss of generality, suppose we are dealing with a descending path; i.e., the critical value \(f_t(p_j)\) decreases until it gets below \(f(p_{j-1}).\) The deformation of \((f_t, v_t)\) is supported in a saturated neighborhood \(U\) of \(W^s(p_j) \cap f^{-1}[b_j, \infty)\). To see [1], note that if \(i \notin \{j, j+1\},\) then on \(W_i\) the function and the vector field remain unchanged, and so do the regular values \(b_{i-1}\) and \(b_i.\) The deformation is supported...
inside \( W_j \cup W_{j+1} \), and \( b_{j+1}(t) \) stays between the critical values \( f_i(p_j) \) and \( f_i(p_{j+1}) \) for every \( t \in [-1, 1] \). Part (2) follows from the fact that \( W^u(p_j) \cap W^s(p_{j+1}) \cap M_{j+1} = \emptyset \).

To prove (3), recall that \( S_j \) is given by \( W^s(p_j) \cap M_j \), with framing coming from a local normal form of \( f \) about \( p_j \). Along an elementary path, this local form remains the same except for a constant shift. In particular, \( W^s(p_j) \) intersects \( M_j \) in \( S_j \) with the same framing, and \( M_{j+1}^f \) in \( S_{j+1}^f \). Hence, if we flow from \( S_j \) along \( v_1 \) to \( M_{j+1}^f \), we obtain \( d'_j(S_j) = S_{j+1}^f \) as \( S_j \cap S_{j+1}^f = \emptyset \). Similarly, \( W^s(p_{j+1}) \) intersects \( M_j \) in \( S'_j \) and \( M_{j+1} \) in \( S_{j+1} \), so flowing along \( v = v_{-1} \) we see that \( d_j(S'_j) = S_{j+1} \).

Finally, we show part (4): i.e., that

\[
d_{j+1} \circ d_{j}^{\varphi \cdot v}(x) = d_{j+1}^{\varphi \cdot v}(x)
\]

for every \( x \in M_j(S_j, S'_j) \). Since the deformation \((f_t, v_t)\) is supported in a neighborhood of \( W^s(p_j) \), for every \( x \in M_j \setminus (S_j \cup S'_j) \) this is clear since both compositions are induced by flowing along \( v \) from \( M_j \) to \( M_{j+2} \). When \( x \) is in the handle part of \( M_j(S_j, S'_j) \) corresponding to \( S'_j \), both compositions are obtained by flowing along \( v \) from the corresponding point of a standard neighborhood of \( p_{j+1} \) to \( M_{j+2} \). In the handle part corresponding to \( S_j \), since for an elementary deformation \( f_t - f \) is constant near \( p_j \) and \( v_t \) is the Euclidean gradient, flowing up to \( M_{j+2} \) along \( v \) or \( v' \) give isotopic diffeomorphisms.

We say that \( \mathcal{M} \) and \( \mathcal{M}' \) are related by an isotopy of the gradient if \( f = f' \) and \( \bar{b} = \bar{b}' \). Given a parameterized Cerf decomposition \( \mathcal{C} \), an isotopy of an attaching sphere is a move described as follows. Let \( \varphi_t : M_j \to M_j \) for \( t \in I \) be an ambient isotopy of the attaching sphere \( S_j \), and let \( S_j' = \varphi_1(S_j) \). There is an induced map

\[
\varphi'_1 = (\varphi_1)^{S_j} : M_j(S_j) \to M_j(S_j'),
\]

and we let \( d'_j := d_j \circ (\varphi'_1)^{-1} \). It is easy to see that \( d'_j \) extends to a diffeomorphism \( D'_j : W(S'_j) \to W_j \) via the formula

\[
D'_j(x, t) = (D_j \circ \varphi_t^{-1}(x), t)
\]

for \((x, t) \in M_j \times I\), and extending to the handle in the natural way.

**Lemma 2.17.** Let \((f, b)\) be a Morse datum for the cobordism \( W \). If \( \mathcal{C} \) and \( \mathcal{C}' \) are parameterized Cerf decompositions induced by the triples \((f, b, v)\) and \((f, b, v')\), respectively, then they are related by isotopies of the attaching spheres \( S_i \) and of the diffeomorphisms \( d_i \), and possibly by reversing framed spheres.

**Proof.** This is a direct consequence of Remark [2211](#).

The Morse data \( \mathcal{M} \) and \( \mathcal{M}' \) are related by adding or removing a regular value if \( [b, b'] \) = 1. In this case, there is an \( i \) for which either \([b_i, b_{i+1}]\) contains no critical value of \( f \), or \([b'_i, b'_{i+1}]\) contains no critical value of \( f' \). Then the corresponding parameterized Cerf decompositions are related by merging or splitting a product: Suppose that one of \( W_j \) and \( W_{j+1} \) is cylindrical; i.e., \( S_j \) or \( S_{j+1} \) is empty. We describe the case when \( S_j = \emptyset \), the other case is analogous. Then we remove \( M_{j+1} \) and merge \( W_j \) and \( W_{j+1} \). We set \( S_j' = d_j^{-1}(S_{j+1}) \) and

\[
d'_j = d_{j+1} \circ (d_j)^{S_j' : M_j(S_j') \to M_{j+2}},
\]
where \((d_j)_{j'}: M_j(S_j') \to M_{j+1}(S_{j+1})\) is the diffeomorphism induced by \(d_j: M_j \to M_{j+1}\). Splitting a product is the reverse of the above move. In general, we have the following result for changing \(b\).

**Lemma 2.18.** Suppose that \((f, b, v)\) and \((f, b', v')\) are Morse data for the cobordism \(W\), and let \(C\) and \(C'\) be the corresponding parameterized Cerf decompositions. Then \((f, b \cup b', v)\) is also a Morse datum for \(W\), and if \(C''\) denotes the induced parameterized Cerf decomposition, then \(C''\) can be obtained from both \(C\) and \(C'\) by splitting products. In particular, one can get from \(C\) to \(C'\) by splitting then merging products.

Finally, \(M\) and \(M'\) are related by a left-right equivalence if there are diffeomorphisms \(\Phi: W \to W\) and \(\varphi: \mathbb{R} \to \mathbb{R}\) such that \(f' = \varphi \circ f \circ \Phi^{-1}, b' = \varphi(b)\), \(v' = \Phi(v), \Phi|M': M \to M\) is isotopic to \(\text{Id}_M\), and \(\Phi|M'|: M' \to M'\) is isotopic to \(\text{Id}_M\). Then we obtain \(C(M')\) from \(C(M)\) by a diffeomorphism equivalence; i.e., setting \(W'_i = \Phi(W_i), S'_i = \Phi(S_i)\), and

\[
d'_i = \Phi_{i+1} \circ d_i \circ (\Phi_{i+1}^{-1}),
\]

where \(\Phi_i = \Phi|_{M_i}\).

The content of the following lemma is that an isotopy of one of the \(d_j\) can be written in terms of the above moves on Cerf decompositions.

**Lemma 2.19.** Suppose that the Cerf decomposition \(C'\) is obtained from \(C\) by replacing one of the diffeomorphisms \(d_j\) by a diffeomorphism \(d'_j = \phi \circ d_j\), where \(\phi: M_{j+1} \to M_{j+1}\) is isotopic to \(\text{Id}_{M_{j+1}}\). If we extend \(\phi\) to a diffeomorphism \(\Phi: W \to W\) isotopic to \(\text{Id}_W\) and supported in a collar neighborhood of \(M_{j+1}\), then \(C'\) can also be obtained from \(C\) by performing the diffeomorphism equivalence corresponding to \(\Phi\), and then isotoping \(\phi(S_{j+1})\) back to \(S_{j+1}\).

**Proof.** It is clear that \(W_i = W'_i, M_i = M'_i\), and \(S_i = S'_i\) for any \(i\) in \(C\) and \(C'\). What we do need to check is that \(d_j = d'_j\) and \(d_{j+1} = d'_{j+1}\). If we use the notation \(\Phi_i = \Phi|_{M_i}\), then \(\Phi_i = \text{Id}_{M_i}\) unless \(i = j+1\). Hence, the diffeomorphism equivalence replaces \(d_j\) by \(\Phi_{j+1} \circ d_j\) and \(d_{j+1}\) by \(d_{j+1} \circ (\Phi_{j+1}^{-1})^{-1}\). Then isotoping \(\phi(S_{j+1})\) back to \(S_{j+1}\) replaces \(d_{j+1} \circ (\phi_{j+1}^{-1})^{-1}\) by

\[
d_{j+1} \circ (\phi_{j+1}^{-1})^{-1} = d_{j+1}.
\]

**Theorem 2.20.** Let \(M = (f, b, v)\) and \(M' = (f', b', v')\) be Morse data on the cobordism \(W\). Then they can be connected by a sequence of critical point creations and cancelations, critical point switches, isotopies of the gradient, adding or removing regular values, and left-right equivalences.

Furthermore, if the ends of each component of the cobordism \(W\) are non-empty, then we can avoid index \(0\) and \(n+1\) critical points throughout. If, in addition, we assume that \(n \geq 2\), and the cobordism \(W\) and each level set \(f^{-1}(b_i)\) and \((f')^{-1}(b'_j)\) is connected, then we can choose the above sequence such that in the corresponding Cerf decompositions all level sets are connected. In particular, there are no index \(0\) or \(n+1\) critical points throughout, and no index \(n\) critical points with separating attaching spheres.
Proof. Connect $f$ and $f'$ by a generic one-parameter family $\{f_s : s \in [0,1]\}$ of smooth functions. This family fails to be a proper Morse function at the parameter values $c_1, \ldots, c_l$, where either we have a birth-death singularity, or two critical points have the same value. We also choose the parameter values $s_0, \ldots, s_{2l+1}$ such that

$$0 = s_0 < s_1 < c_1 < s_2 < c_2 < \cdots < s_{2l} < c_l < s_{2l+1} = 1,$$

and $s_{2i-1}$ and $s_{2i}$ are close to $c_i$ in a sense to be specified below. For every $i \in \{0, \ldots, 2l+1\}$, let $v_i$ be a gradient-like vector field for $f_s$. Furthermore, for every $i \in \{0, \ldots, l\}$, if we choose the ordered tuples $b_{2i}$ and $b_{2i+1}$ such that they can be connected by a continuous path of tuples $b(s)$ for $s \in [s_{2i}, s_{2i+1}]$, then by Lemma 3.1, the Morse data $M_{2i} = (f_{2i}, b_{2i}, v_{2i})$ and $M_{2i+1} = (f_{2i+1}, b_{2i+1}, v_{2i+1})$ are related by a left-right equivalence and an isotopy of the gradient. Clearly, one can choose $b_{2i}$ and $b_{2i+1}$ in this way. Furthermore, by Lemma 2.13 different choices of $b$ give decompositions related by adding and removing regular values.

It remains to prove that $M_{2i-1}$ and $M_{2i}$ are related by the moves listed in the statement. To simplify the notation, let $M_- = M_{2i-1}$, $M_+ = M_{2i}$, $s_- = s_{2i-1}$, $s_+ = s_{2i}$, $f_p = f_{s_+}$, $v_+ = v_{s_+}$, and $c = c_i$. Choose an ordered tuple $b$ such that there is exactly one element of $b$ between any two consecutive critical points of $f_c$.

First, suppose that the function $f_s$ has a death singularity at $p \in W$ with $f_c(p) \in (b_j, b_{j+1})$. According to Cerf [1, p.71, Proposition 2], we can modifying the family $f_s$ such that it becomes a ‘chemin élémentaire de mort.’ In particular, it is constant in $s$ outside a ball $B \subset f_c^{-1}([b_j, b_{j+1}])$ containing $p$ for $s \in [s_-, s_+]$, if $s_\pm$ are very close to $c$. Furthermore, there is a coordinate system about $p$ in which

$$f_s(x) = f_c(p) + x_1^2 + sx_1 - x_2^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_{n+1}^2.$$

Let $v_-$ and $v_+$ be gradient-like vector fields for $f_-$ and $f_+$, respectively, that coincide outside $B$. Notice that $f_c(p)$ lies between the values of the two critical points that cancel for $s < 0$, hence $(f_-, b_-)$ is a Morse datum for $b_- = b \cup \{f_c(p)\}$. Then $(f_-, b_-, v_-)$ and $(f_+, b_+, v_+)$ are Morse data for $W$. It follows from the above construction that in $M_-$ the attaching sphere and the belt sphere of the canceling pair of critical points intersect in a single point. So $M_-$ and $M_+$ are related by a critical point cancellation.

Now consider the case when $f_c$ has two critical points at $p$ and $q$ such that

$$f_c(p) = f_c(q) \in [b_j, b_{j+1}].$$

Then we can modify the family $f_s$ in the interval $[s_-, s_+]$ such that it becomes a ‘chemin élémentaire de 1-croisement,’ this is possible by Cerf [1, p.49, Proposition 2]. In particular, $f_s$ is independent of $s$ outside a neighborhood $N$ of either $W^s(p)$ or $W^s(q)$, and the points $p$ and $q$ remain critical throughout. Furthermore, for $s \in [s_-, c]$, we have $f_s(p) < f_s(q)$, while for $s \in (c, s_+]$, we have $f_s(q) < f_s(p)$. In fact, we can arrange that a fixed vector field $v$ on $W$ remains gradient-like for every $f_s$. If we set $b'_- = b \cup \{f_c(p) = f_c(q)\}$, then $(f_-, b'_-, v)$ and $(f_+, b'_+, v)$ are Morse data. Then we can get from $M_-$ to $M_+$ by a critical point switch and isotopies of the gradient.

When each component of the cobordism $W$ has non-empty ends, then we can avoid index 0 and $n + 1$ critical points using Cerf theory as in Kirby [10]. The statement on connected Cerf decompositions follows from [3, Theorem 3.6].
2.4. Constructing TQFTs. In this section, we describe how Theorem 2.20 together with the lemmas of the previous section, imply Theorem 1.2. So suppose that $F: \text{Man}_n \to \text{Vect}$ is a functor, and we are given maps $F_{M,S}$ that satisfy all the properties listed in Theorem 1.2. Now suppose that $W$ is a cobordism from $M$ to $M'$. Choose a parameterized Cerf decomposition $C$, consisting of a decomposition

$$W = W_0 \cup_{M_1} W_1 \cup_{M_2} \cdots \cup_{M_m} W_m,$$

together with attaching spheres $S_i$ and diffeomorphisms $d_i: M_i(S_i) \to M_{i+1}$. When $n \geq 2$ and $W$, $M$, and $M'$ are all connected, we can assume that each $M_i$ is connected as well by [3 Lemma 2.5]. Then we define

$$F(W,C) = \prod_{i=0}^m (F(d_i) \circ F_{M_i,S_i}) : F(M) \to F(M').$$

The content of Theorem 2.20 is that the map $F(W,C)$ is independent of the choice of Cerf decomposition $C$; we denote it by $F(W)$.

Remark 2.21. To illustrate why working with Cerf decompositions without the parameterization is insufficient to define the cobordism map $F(W)$, consider the simplest possible case when $W$ itself is diffeomorphic to $M \times I$. Then this is a Cerf decomposition with a single component. Given a diffeomorphism $D: M \times I \to W$, let $d_i = D|_{M \times \{i\}}$; then it is natural to define $F(W)$ as $F(d_1 \circ d_0^{-1})$. However, $D$ is not unique, and for different choices we only know that the corresponding $d_1 \circ d_0^{-1}$ are pseudo-isotopic, not necessarily isotopic, and hence a priori might induce different homomorphisms via $F$. Consequently, we identify each component $W_i$ of the Cerf decomposition with a concrete handle cobordism $W(S_i)$, and once we know this induces a TQFT, we obtain as a corollary that pseudo-isotopic diffeomorphisms induce the same homomorphism. When $W$ is cylindrical, one might have to pass through a sequence of moves to get from one parameterization as a product to another.

Given a cobordism $W'$ from $M'$ to $M''$ and a parameterized Cerf decomposition $C'$ of $W'$, we get a Cerf decomposition $CC'$ of the cobordism $WW'$ obtained by gluing (and with smooth structure unique up to diffeomorphism fixing $M$, $M'$, and $M''$). If follows from the definition that

$$F(W',C') \circ F(W,C) = F(WW',CC'),$$

hence $F(W') \circ F(W) = F(WW')$.

Next, suppose that $W$ and $W'$ are equivalent cobordisms from $M$ to $M'$, with equivalence given by the diffeomorphism $h: W \to W'$ fixing $M$ and $M'$ pointwise. Let $C$ be a parameterized Cerf decomposition of $W$, as above. Then $h$ induces a parameterized Cerf decomposition $C'$ of $W'$ by setting $W'_i = h(W_i)$, $S'_i = d(S_i)$, and

$$d'_i = h_{i+1} \circ d_i \circ \left(h_i^{-1}\right)^{-1} : M'_i(S'_i) \to M'_{i+1}.$$ 

We claim that

$$F(W,C) = F(W',C').$$
Indeed, consider the diagram

\[
\begin{array}{ccc}
F(M_i) & \xrightarrow{F_{M_i}} & F(M_i(S_i)) \\
\downarrow F(h_i) & & \downarrow F(h_i) \\
F(M'_i) & \xrightarrow{F_{M'_i}} & F(M'_i(S'_i)) \\
\end{array}
\]

where \(h_i = h |_{M_i}\). The rectangle on the left is commutative because of property (2) of Theorem 1.2 while the rectangle on the right commutes by the above definition of \(d'_j\) and the functoriality of \(F\) under the composition of diffeomorphisms. Putting the above rectangles together for \(i = 0, \ldots, m\), and using the property that \(h_0 = Id_M\) and \(h_{m+1} = Id_{M'}\), the claim follows.

Hence, once we show that \(F(M, C)\) is independent of \(C\), we do obtain a functor \(F: \text{Cob}_n \rightarrow \text{Vect}\). Let \(C\) and \(C'\) be parameterized Cerf decompositions of \(W\). By Lemma 2.15 there exist Morse data \(\mathcal{M} = (f, b, v)\) and \(\mathcal{M}' = (f', b', v')\) inducing \(C\) and \(C'\), respectively. It suffices to prove that \(F(W, C) = F(W, C')\) when \(\mathcal{M}'\) is obtained from \(\mathcal{M}\) by one of the moves listed in Theorem 2.20 since any two Morse data can be connected by a sequence of such moves.

First, suppose that \(\mathcal{M}'\) is obtained from \(\mathcal{M}\) by a critical point cancelation. Then what we need to show is that

\[
(2.5) \quad F(d_{j+1}) \circ F_{M_{j+1}, S_{j+1}} \circ F(d_j) \circ F_{M_j, S_j} = F(d'_j).
\]

By Lemma 2.15 \(d'_j = d_{j+1} \circ d_{j+1}^{-1} \circ \varphi\), where \(S_{j+1} = d_{j+1}^{-1}(S_{j+1})\). Hence, using the functoriality of \(F\), equation (2.5) reduces to

\[
F_{M_{j+1}, S_{j+1}} \circ F(d_j) \circ F_{M_j, S_j} = F \left( d_{j+1}^{-1} \right) \circ F(\varphi).
\]

By property (1) in Theorem 1.2, we have

\[
F(\varphi) = F_{M_j(S_j), S_{j+1}} \circ F_{M_j, S_j}.
\]

Now, according to property (2),

\[
F \left( d_{j+1}^{-1} \right) \circ F_{M_{j+1}, S_{j+1}} = F_{M_{j+1}, S_{j+1}} \circ F(d_j),
\]

and the result follows. The case of a critical point creation follows by reversing the roles of \(\mathcal{M}\) and \(\mathcal{M}'\).

Now assume that \(\mathcal{M}\) and \(\mathcal{M}'\) are related by a critical point switch. Then we will show that

\[
F(d_{j+1}) \circ F_{M_{j+1}, S_{j+1}} \circ F(d_j) \circ F_{M_j, S_j} = F(d'_{j+1}) \circ F_{M'_{j+1}, S'_{j+1}} \circ F(d'_j) \circ F_{M_j, S'_j}.
\]

Using property (2),

\[
F_{M_{j+1}, S_{j+1}} \circ F(d_j) = F \left( (d_j)^{S_j} \right) \circ F_{M_j(S_j), S'_j},
\]

and similarly,

\[
F_{M'_{j+1}, S'_{j+1}} \circ F(d'_j) = F \left( (d'_j)^{S'_j} \right) \circ F_{M'_j(S'_j), S'_j}.
\]

Substitute these into the above equation, and notice that, by property (3), we have

\[
F_{M_j(S_j), S'_j} \circ F_{M_j, S_j} = F_{M'_j(S'_j), S'_j} \circ F_{M_j, S'_j},
\]

so it suffices to prove that

\[
F(d_{j+1}) \circ F \left( (d_j)^{S_j} \right) = F(d'_{j+1}) \circ F \left( (d'_j)^{S'_j} \right).
\]
But this follows from part (1) of Lemma 2.16 and the functoriality of \( F \).

Assume now that \( \mathcal{M}' \) is obtained from \( \mathcal{M} \) via an isotopy of the gradient \( v \). By Lemma 2.17, the induced parameterized Cerf decompositions \( C \) and \( C' \) are related by a sequence of isotopies of the attaching spheres \( S_i \) and of the diffeomorphisms \( d_i \), and reversing framed 0-spheres. First suppose that \( C \) and \( C' \) are related by an isotopy of \( S_j \). More precisely, let \( \varphi_i \) be an ambient isotopy of the attaching sphere \( S_j \). Recall that \( d_j' := d_j \circ (\varphi_i')^{-1} \), where \( \varphi_i' = (\varphi_i)^{S_j} \), everything else remains the same. By property (2),

\[
F_{M_j, S_j'} \circ F(\varphi_1) = F(\varphi_1') \circ F_{M_j, S_j},
\]

However, \( \varphi_1 \) is isotopic to the identity, hence \( F(\varphi_1) = \text{Id}_{F(M_j)} \). Using the functoriality of \( F \),

\[
F(d_j') \circ F_{M_j, S_j'} = F(d_j) \circ F(\varphi_1')^{-1} \circ F_{M_j, S_j'} = F(d_j) \circ F_{M_j, S_j},
\]

hence \( F(W, C) = F(W, C') \). If \( C \) and \( C' \) are related by an isotopy of one of the diffeomorphisms \( d_j \), then invariance follows from assumption (1) of Theorem 1.2. The map is also unchanged by reversing a framed sphere by axiom (3).

Now consider the case when \( \mathcal{M}' \) is obtained from \( \mathcal{M} \) by adding or removing a regular value. Then \( C' \) is obtained from \( C \) by merging or splitting a product. Without loss of generality, suppose we are merging the cylindrical \( W_j \) to \( W_{j+1} \). The cases when \( W_{j+1} \) is cylindrical and when we are splitting a product are analogous. Recall that \( S_j' = d_j^{-1}(S_{j+1}) \) and \( d_{j}' = d_{j+1} \circ (d_j)^{S_j'} \). Then

\[
F(d_j') \circ F_{M_j, S_j'} = F(d_{j+1}) \circ F((d_j)^{S_j'}) \circ F_{M_j, S_j'}.
\]

According to property (2), applied to \( d_j : (M_j, S_j) \to (M_{j+1}, S_{j+1}) \), we have

\[
F((d_j)^{S_j'}) \circ F_{M_j, S_j'} = F_{M_{j+1}, S_{j+1}} \circ F(d_{j+1}).
\]

Hence, as \( F_{M_j, 0} = \text{Id}_{F(M_j)} \),

\[
F(d_j') \circ F_{M_j, S_j'} = F(d_{j+1}) \circ F_{M_{j+1}, S_{j+1}} \circ F(d_j) \circ F_{M_j, 0},
\]

and the result follows for merging a product.

Finally, suppose that \( \mathcal{M}' \) is obtained from \( \mathcal{M} \) by a left-right equivalence. In this case, \( C \) and \( C' \) are related by a diffeomorphism equivalence \( \Phi : W \to W \). Then, by the definition of \( d_i \),

\[
F(W, C') = \prod_{i=0}^{m} \left( F(\Phi_{i+1}) \circ F(d_i) \circ F(\Phi_{i}^{S_i})^{-1} \circ F_{M_i', S_i'} \right).
\]

If we apply property (2) to the diffeomorphism \( \Phi_i : (M_i, S_i) \to (M_i', S_i') \), we obtain that

\[
F(\Phi_i^{S_i})^{-1} \circ F_{M_i', S_i'} = F_{M_i, S_i} \circ F(\Phi_i)^{-1}.
\]

Substituting this into the previous formula, and using the fact that \( F(\Phi_0) = \text{Id}_{F(M)} \) and \( F(\Phi_m) = \text{Id}_{F(M')} \), we obtain that \( F(W, C) = F(W, C') \).

**Lemma 2.22.** Suppose that \( F \) arises from a functor \( F : \text{Man}_n \to \text{Vect} \) and surgery maps \( F_{M,S} \) as in Theorem 1.2. Then for any diffeomorphism \( h : M \to M' \), we have

\[
F(h) = h_*.\]
Proof. Recall that \( h_n \) is defined as \( F(c_h) \), where \( c_h \) is the cylindrical cobordism \((M \times I; M \times \{0\}, M \times \{1\}; p_0, h_1)\). Then this is in itself a parameterized Cerf decomposition \( C \) of a single level, and so \( F(c_h, C) = F(h) \circ F_{M,0} = F(h) \). \( \square \)

In the opposite direction, given a functor \( F: \text{Cob}_n \rightarrow \text{Vect} \), we let \( F(h) = F(c_h) \) for a diffeomorphism \( h: M \rightarrow M' \), and given a framed sphere \( S \) in \( M \), we define \( F_{M,S}: F(M) \rightarrow F(M(S)) \) to be \( F_{W(S)} \). These clearly satisfy the properties listed in Theorem 1.2. The correspondence is one-to-one by Lemma 2.22. This concludes the proof of Theorem 1.2 in case of the category \( \text{Cob}_n \). For \( \text{Cob}^+_n \), \( \text{Cob}^+_n \), and \( \text{BSut}' \), we apply the second paragraph of Theorem 2.20.

3. Classifying \((1 + 1)\)-dimensional TQFTs

Recall that a Frobenius algebra is a finite-dimensional unital associative \( \mathbb{F} \)-algebra \( A \) with multiplication \( \mu: A \otimes A \rightarrow A \) and a trace functional \( \theta: A \rightarrow \mathbb{F} \) such that \( \ker(\theta) \) contains no non-zero left ideal of \( A \). Then \( \sigma(a, b) = \theta(ab) \) is a non-degenerate bilinear form. In particular, \( \sigma \) sets up an isomorphism between \( A \) and \( A^* \). Dualizing the algebra structure, we also get a coalgebra structure on \( A \) with counit; we denote the coproduct by \( \delta: A \rightarrow A \otimes A \). Note that \( \delta \) is obtained by dualizing the product \( A \otimes A \rightarrow A \), and using the fact that \( (A \otimes A)^* \approx A^* \otimes A^* \) since \( A \) is finite-dimensional. The Frobenius algebra \( A \) is called commutative if the product \( \mu \) is commutative and the coproduct \( \delta \) is cocommutative.

In this section, we reprove the following folklore result on the classification of \((1 + 1)\)-dimensional TQFTs using Theorem 1.2, cf. [7]. This can be viewed as a warm up for the following section, where we will classify \((2 + 1)\)-dimensional TQFTs. Here all 1-manifolds and cobordisms are assumed to be oriented.

**Theorem 3.1.** There is a one-to-one correspondence between \((1 + 1)\)-dimensional TQFTs and finite-dimensional commutative Frobenius algebras.

**Proof.** It is straightforward to see that a \((1 + 1)\)-dimensional TQFT

\[
F: \text{Cob}_2 \rightarrow \text{Vect}_\mathbb{F}
\]

gives rise to a Frobenius algebra. Indeed, let \( A := F(S^1) \). If \( S \) is a pair-of-pants cobordism from \( S^1 \sqcup S^1 \) to \( S^1 \), then the multiplication is given by

\[
F(S): F(S^1 \sqcup S^1) \cong F(S^1) \otimes F(S^1) = A \otimes A \rightarrow F(S^1) = A.
\]

If \( D \) denotes the cobordism from \( S^1 \) to \( \emptyset \) given by a disk, then \( \emptyset := F(D) \). If we turn \( D \) upside-down and reverse its orientation, we obtain a cobordism \( -D \) from \( \emptyset \) to \( S^1 \). Then \( F(-D)(1) \in A \) is the unit. It is now straightforward to check that these satisfy the Frobenius algebra axioms.

The non-trivial direction is associating a TQFT to a Frobenius algebra. Given a Frobenius algebra \( A \), we describe the ingredients of Theorem 1.2 needed to define a TQFT, namely, a functor \( F: \text{Man}_1 \rightarrow \text{Vect}_\mathbb{F} \) and maps induced by framed spheres that satisfy the required properties.

Throughout this paper, for oriented manifolds \( X, Y \), we denote by \( \text{Diff}(X, Y) \) the set of orientation preserving diffeomorphisms from \( X \) to \( Y \), and we write \( \text{Diff}(X) := \text{Diff}(X, X) \). Furthermore,

\[
\text{MCG}(X) = \text{Diff}(X)/\text{Diff}_0(X)
\]
is the oriented mapping class group of $X$. The group $\text{Diff}(Y)$ acts on $\text{Diff}(X, Y)$ by composition. By slight abuse of notation, we write

$$\text{MCG}(X, Y) := \text{Diff}(X, Y)/\text{Diff}_0(Y),$$

even though this is not actually a group, only an affine copy of $\text{MCG}(X)$ if $X$ and $Y$ are diffeomorphic, and the empty set otherwise.

Let $C_k = S^1 \times \{1, \ldots, k\}$; i.e., the disjoint union of $k$ copies of $S^1$. Given a closed 1-manifold $M$ of $k$ components, note that $\text{MCG}(C_k, M)$ is an affine copy of $S_k$. An element of $\text{MCG}(C_k, M)$ can be thought of as a labeling of the components of $M$ by the integers $1, \ldots, k$. Given diffeomorphisms $\phi, \phi' \in \text{MCG}(C_k, M)$, their difference $(\phi')^{-1} \circ \phi$ is an element $\sigma(\phi, \phi')$ of $\text{MCG}(C_k, C_k)$, which is canonically isomorphic to $S_k$.

For a closed 1-manifold $M$, let $F(M)$ be the set of those elements $a$ of

$$\prod_{\phi \in \text{MCG}(C_k, M)} A^{\otimes k}$$

such that for any $\phi, \phi' \in \text{MCG}(C_k, M)$ the coordinates $a(\phi)$ and $a(\phi')$ in $A^{\otimes k}$ differ by the permutation of factors given by $\sigma(\phi, \phi') \in S_k$. Notice that the function $a$ is uniquely determined by its value $a(\phi)$ for any $\phi \in \text{MCG}(C_k, M)$; i.e., for any labeling of the components of $M$ by the numbers $1, \ldots, k$.

Suppose that $M$ and $M'$ are diffeomorphic 1-manifolds; i.e., they have the same number of components $k$, and let $d \in \text{MCG}(M, M')$. Given an element $a \in F(M)$ and $\phi \in \text{MCG}(C_k, M)$, we define

$$(F(d)(a))(d \circ \phi) = a(\phi).$$

A framed 0-sphere in a closed 1-manifold $M$ of $k$ components is given by an embedding

$$\mathbb{S} : S^0 \times D^1 = \{-1, 1\} \times [-1, 1] \to M.$$ 

Since we only consider oriented cobordisms, the framing should be orientation preserving, and is hence unique up to isotopy. So $\mathbb{S}$ is completely determined by a pair of points $\mathbb{S} = \{s_-, s_+\}$. If $s_-$ and $s_+$ lie in different components $M_-$ and $M_+$ of $M$, respectively, then we define the map

$$F_{M, \mathbb{S}} : F(M) \to F(M(\mathbb{S}))$$

as follows. Let $a \in F(M)$, and let $\phi \in \text{MCG}(C_k, M)$ correspond to a labeling of the components of $M$ such that $M_-$ is labeled $k - 1$ and $M_+$ is labeled $k$. This gives rise to a labeling $\phi_\mathbb{S}$ of the components of $M(\mathbb{S})$, where the component arising from surgery on $M_-$ and $M_+$ is labeled $k - 1$, while every other component is unchanged and retains its label. Then $F_{M, \mathbb{S}}(a)$ is the element of $F(M(\mathbb{S}))$ for which $F_{M, \mathbb{S}}(a)(\phi_\mathbb{S})$ is the image of $a(\phi)$ under the map

$$A^{\otimes (k-2)} \otimes A \otimes A \to A^{\otimes (k-2)} \otimes A$$

that multiplies the last two factors using the algebra product of $A$; i.e., takes $a_1 \otimes \cdots \otimes a_{k-2} \otimes a_{k-1} \otimes a_k$ to $a_1 \otimes \cdots \otimes a_{k-2} \otimes (a_{k-1}a_k)$. It is straightforward to see that the above definition of $F_{M, \mathbb{S}}(a)$ is independent of the choice of the choice of $\phi$. Indeed, if $\phi'$ is another labeling such that $M_-$ is labeled $k - 1$ and $M_+$ is labeled $k$, then $F_{M, \mathbb{S}}(a)(\phi_\mathbb{S})$ and $F_{M, \mathbb{S}}(a)(\phi'_\mathbb{S})$ differ by the action of the permutation $\sigma(\phi_\mathbb{S}, \phi'_\mathbb{S})$ that fixes $k - 1$, and maps to $\sigma(\phi, \phi')$ under the embedding $S_{k-1} \to S_k$. So, by definition, these two elements of $A^{\otimes (k-1)}$ define the same element $F_{M, \mathbb{S}}(a)$ of $F(M(\mathbb{S}))$. 

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Now suppose that $s_-$ and $s_+$ lie in the same component $M_s$ of $M$. Then $M_S$ has $k+1$ components. The component $M_s$ splits into a component $M_-$ corresponding to the arc of $M_s \setminus S$ going from $s_-$ to $s_+$, and a component $M_+$ corresponding to the arc of $M_s \setminus S$ going from $s_+$ to $s_-$. Let $\phi$ be a labeling of the components of $M$ such that $M_+$ is labeled $k$. Then we denote by $\phi_S$ the labeling of the components of $M_S$ where each component of $M \setminus M_s$ retains its label, $M_-$ is labeled $k$, and $M_+$ is labeled $k + 1$. Given $a \in F(M)$, we define $F_{M,S}(a)(\phi_S) \in A^\otimes(k+1)$ by applying to $a(\phi) \in A^\otimes k$ the map $A^\otimes k \to A^\otimes(k+1)$ that sends $a_1 \otimes \cdots \otimes a_{k-1} \otimes a_k$ to $a_1 \otimes \cdots \otimes a_{k-1} \otimes \delta(a_k)$, where $\delta$ is the coproduct of the Frobenius algebra $A$. As in the previous case, $F_{M,S}(a)$ is independent of the choice of $\phi$.

Surgery along the attaching sphere of a 0-handle results in the manifold $M(0) = M \sqcup S^1$. Chose an arbitrary labeling $\phi$ of the components of $M$ with the numbers $1, \ldots, k$. We obtain the labeling $\phi_0$ of the components of $M(0)$ by labeling the new $S^1$ component $k + 1$. Let $\iota_k : A^\otimes k \to A^\otimes(k+1)$ be the map $\iota_k(x) = x \otimes 1$, where 1 is the unit of $A$. For $a \in F(M)$, we define $F_{M,0}(a)(\phi_0) = \iota_k(a(\phi))$; the map $F_{M,0}$ is independent of the choice of $\phi$.

Finally, a framed 1-sphere in a 1-manifold $M$ of $k$ components is simply an embedding $S : S^1 \to M$. Let $S$ be the image of $S$, then $M(S) = M \setminus S$. Let $\phi$ be a labeling of the components of $M$ such that $S$ is given the label $k$, and let $\phi_S$ be the corresponding labeling of $M(S)$. Let $\iota_k : A^\otimes k \to A^\otimes(k-1)$ be the map given by extending linearly

$$t_k(a_1 \otimes \cdots \otimes a_{k-1} \otimes a_k) = \theta(a_k) \cdot a_1 \otimes \cdots \otimes a_{k-1}.$$

For $a \in F(M)$, let $F_{M,S}(a)(\phi_S) = \iota_k(a(\phi))$. Again, this gives a well-defined map $F_{M,S}$ independent of the choice of labeling $\phi$.

Now all we need to check is that axioms (1)-(3) of Theorem 12 hold for the data defined above. We only give an outline here and leave the details to the reader. Axiom (1) is straightforward, as if $d \in \text{Diff}_0(M)$, then $d \circ \phi = \phi \in \text{MCG}(M)$, and $(F(d)(a))(\phi) = (F(d)(a))(d \circ \phi) = a(\phi)$; i.e., $F(d) = \text{Id}_{F(M)}$.

Now consider axiom (2), naturality. We check this in the case where $S = \{s_-, s_+\}$ is a framed 0-sphere with $s_-$ and $s_+$ lying in different components $M_-$ and $M_+$ of $M$, respectively; the other cases are similar. Choose a labeling $\phi$ of the components of $M$ such that $M_-$ is labeled $k-1$ and $M_+$ is labeled $k$. For $a_1, \ldots, a_k \in A$, let $a$ be the element of $F(M)$ for which $a(\phi) = a_1 \otimes \cdots \otimes a_k$. Then, by definition,

$$F_{M,S}(a)(\phi_S) = a_1 \otimes \cdots \otimes a_{k-2} \otimes (a_{k-1}a_k).$$

Given a diffeomorphism $d : M \to M'$, this induces a labeling $d \circ \phi$ of $M'$. Then $(F(d)(a))(d \circ \phi) = a(\phi) = a_1 \otimes \cdots \otimes a_k$. Consider $S' = \{d(s_-), d(s_+)\}$. Under $d \circ \phi$, the component $M'_-$ of $M'$ containing $d(s_-)$ is labeled $k-1$ and the component $M'_+$ containing $d(s_+)$ is labeled $k$. Hence, we can use the labeling $d \circ \phi$ of $M'$ to compute the map $F_{M,S'}$. This induces the labeling $(d \circ \phi)_{S'}$ where the component obtained by taking the connected sum of $M'_-$ and $M'_+$ is labeled $k-1$ and every other component retains its label. With this notation in place,

$$[F_{M',S'} \circ F(d)(a)]((d \circ \phi)_{S'}) = a_1 \otimes \cdots \otimes a_{k-2} \otimes (a_{k-1}a_k).$$

The diffeomorphism $d^S$ maps $M_- \# M_+$ to $M'_- \# M'_+$, and on the other components it acts just like $d$. It follows that $d^S \circ \phi_S = (d \circ \phi)_{S'}$. Furthermore,

$$[F(d^S) \circ F_{M,S}(a)](d^S \circ \phi_S) = F_{M,S}(a)(\phi_S) = a_1 \otimes \cdots \otimes a_{k-2} \otimes (a_{k-1}a_k).$$
This establishes the commutativity of the diagram in axiom (3).

Now consider axiom (3); i.e., that

\[(3.1) \quad F_{M(S),S'} \circ F_{M,S} = F_{M(S'),S} \circ F_{M,S'}.
\]

Here we have several cases depending on the dimensions of the attaching spheres. This is obviously true when \( S = S' = 0 \). When \( S \) and \( S' \) are framed 1-spheres glued along distinct components \( S \) and \( S' \) of \( M \), then let \( \phi \) be a labeling of \( M \) such that \( S \) is labeled \( k \) and \( S' \) is labeled \( k - 1 \). As above, let \( a \in F(M) \) be such that \( a(\phi) = a_1 \otimes \cdots \otimes a_k \). Then

\[ [F_{M(S),S'} \circ F_{M,S}(a)](\phi_{S',S}) = \theta(a_{k-1})\theta(a_k) \cdot a_1 \otimes \cdots \otimes a_{k-2}. \]

On the other hand, let \( \phi' \) be the labeling of the components of \( M \) where \( S \) is labeled \( k - 1 \) and \( S' \) is labeled \( k \), otherwise it agrees with \( \phi \). The permutation \( \sigma(\phi, \phi') \in S_k \) is the transposition of \( k - 1 \) and \( k \), and so

\[ a(\phi') = a_1 \otimes \cdots \otimes a_{k-2} \otimes a_k \otimes a_{k-1}. \]

It follows that

\[ [F_{M(S'),S} \circ F_{M,S}(a)](\phi',\phi) = \theta(a_k)\theta(a_k-1) \cdot a_1 \otimes \cdots \otimes a_{k-2}. \]

Since \( \phi_{S,S'} = \phi_{S',S} \), the result follows from the commutativity of \( \mathbb{F} \) in this case.

When \( S' = 0 \) and \( S \) is a 1-sphere in a component \( S \) of \( M \), then choose a labeling \( \phi \) such that \( S \) is labeled \( k \). Then

\[ [F_{M(S),0} \circ F_{M,S}(a)](\phi_{S,0}) = \theta(a_k) \cdot a_1 \otimes \cdots \otimes a_{k-1} \otimes 1, \]

where \( \phi_{S,0} \) labels the components of \( M \setminus S \) just like \( \phi \), and the new \( S' \)-component is labeled \( k \). To compute \( F_{M(S),0}(a) \), first note that \( F_{M,0}(a)(\phi_0) = a_1 \otimes \cdots \otimes a_k \otimes 1 \). If \( \tau \) is the transposition of \( k \) and \( k + 1 \), then

\[ F_{M,0}(a)(\tau \circ \phi_0) = a_1 \otimes \cdots \otimes a_{k-1} \otimes 1 \otimes a_k. \]

As \( \tau \circ \phi_0 \) labels \( S \) with \( k + 1 \),

\[ [F_{M(S),0} \circ F_{M,0}(a)]((\tau \circ \phi_0)_0) = \theta(a_k) \cdot a_1 \otimes \cdots \otimes a_{k-1} \otimes 1, \]

and \( (\tau \circ \phi_0)_0 = \phi_{S,0} \), which proves equation (3.1) in this case.

Now suppose that \( S = \{s, s'\} \) is a framed 0-sphere in \( M \). The cases when \( S' = 0 \) or when \( S' \) is a 1-sphere disjoint from \( S \) are similar to the previous one. When \( S' \) is a 0-sphere, we have four cases depending on whether \( S \cup S' \) intersects \( M \) in 1, 2, 3, or 4 components. The case \( c = 1 \) splits into two subcases depending on whether \( S \) and \( S' \) are linked. When they are linked, both sides of equation (3.1) will be of the form \( a_1 \otimes \cdots \otimes a_{k-1} \otimes (\mu \circ \delta(a_k)) \), where \( \mu \) is the product and \( \delta \) is the coproduct of \( A \). When \( S \) and \( S' \) are unlinked, then one side becomes

\[ a_1 \otimes \cdots \otimes a_{k-1} \otimes (\delta \otimes \text{Id}_A)(\delta(a_k)), \]

while the other side is

\[ a_1 \otimes \cdots \otimes a_{k-1} \otimes (\text{Id}_A \otimes \delta)(\delta(a_k)). \]

The two coincide by the coassociativity of the coalgebra \((A, \delta)\). When \( c = 2 \) and one of \( S \) and \( S' \) lies in a single component \( M_s \) of \( M \), while the other one intersects \( M_s \) in one point, then the equality boils down to the fact that \( \delta \) is a left and right \( A \)-module homomorphism; i.e.,

\[ (\mu \otimes \text{Id}_A)(a_{k-1} \otimes \delta(a_k)) = (\delta \circ \mu)(a_{k-1} \otimes a_k) = (\text{Id}_A \otimes \mu)(\delta(a_{k-1}) \otimes a_k). \]
If $c = 2$ and $S, S'$ both intersect the same two components of $M$, then both sides of equation (3.1) become $a_1 \otimes \cdots \otimes a_{k-2} \otimes (\delta \circ \mu(a_{k-1}, a_k))$. When $c = 2$ and $S$ and $S'$ lie in two distinct components of $M$, then the result is clear as we have two coproduct maps acting on distinct components of $M$. When $c = 3$ and $S$ and $S'$ share a component, then the result follows from the associativity of the algebra $(A, \mu)$. When $c = 3$ and $S$ occupies two components and $S'$ a third, then we have a non-interacting product and coproduct. The case $c = 4$ is also straightforward as we are dealing with two non-interacting product maps.

We now check axiom 4. When $S = 0$ and $S' \subset M(0)$ is a 1-sphere that intersects the new $S^1$ component in one point, then the result follows from the fact that $1$ is a left and right unit of $A$. Now suppose that $S$ is a 0-sphere and $S' \subset M(S)$ is a 1-sphere that intersects the co-core of the handle attached along $S$ in one point. Then $S$ has to occupy a single component of $M$ that splits into the components $M_-$ and $M_+$ when we perform surgery along $S$, and $S'$ maps to either $M_-$ or $M_+$. The result follows from the fact that $\theta$ is a left and right counit of the coalgebra $(A, \delta)$; i.e., that

$$(\theta \otimes \text{Id}_A) \circ \delta = \text{Id}_A = (\text{Id}_A \otimes \theta) \circ \delta.$$

Finally, consider axiom 5. If $S = \{s_-, s_+\}$ and $s_-$ and $s_+$ lie in different components of $M$, then $F_{M,S}(a)(\theta) = a_1 \otimes \cdots \otimes a_{k-2} \otimes a_k$. In $S$ we reverse $s_-$ and $s_+$, and so $F_{M,S}(a)(\delta) = a_1 \otimes \cdots \otimes a_{k-2} \otimes a_k$. These coincide as the Frobenius algebra is commutative. When $s_-$ and $s_+$ occupy the same component of $M$, then $F_{M,S} = F_{M,S}'$ follows from cocommutativity. This proves the proof of Theorem 3.1.

\[ \square \]

4. (2+1)-dimensional TQFTs

Kontsevich [8] outlined a correspondence between (1+1+1)-dimensional TQFTs and modular functors. In this section, we apply Theorem 1.2 to the study of (2+1)-dimensional TQFTs.

For every $g \geq 0$, let $\Sigma_g$ be a fixed oriented surface of genus $g$ obtained as the connected sum $\#S^1 \times S^1$, and let $M_g = \text{MCG}(\Sigma_g)$. The connected sums are taken at the point $(1, 1)$ of component $i$ and the point $(-1, 1)$ of component $(i+1)$. Let $l_i = (S^1 \times \{-1\})$, be a longitude of summand $i$, while $m_0 = (\{-1\} \times S^1)$ is a meridian of the first summand. $m_g = (\{1\} \times S^1)$ a meridian of the last summand. Furthermore, for $i \in \{1, \ldots, g - 1\}$, consider the curves $m_i = (\{1\} \times S^1)_i \# (\{-1\} \times S^1)_{i+1}$.

The curves $m_i^\pm = (\{\pm\sqrt{-1}\} \times S^1)$, are meridians of the $i$-th $S^1 \times S^1$ summand of $\Sigma_g$. All the above curves are oriented coherently with $S^1$. If $j \in \{1, \ldots, g - 1\}$, we write $s_j$ for the connected sum curve between the $j$-th and $(j+1)$-st $S^1 \times S^1$ summands, oriented as the boundary of the $j$-th $S^1 \times S^1$ summand. Finally, let $s_0$ be an inessential curve in the first summand and $s_g$ and inessential curve in the last summand, both oriented from the left. For an illustration, see Figure 4.

Suppose that the functor $F : \text{Cob}_2 \to \text{Vect}_F$ is a TQFT. Then we write $V_g = F(\Sigma_g)$, this vector space comes equipped with a representation $\rho_g : M_g \to \text{End}(V_g)$. The orientation of $\Sigma_g$ gives a natural framing of the curves $m_i$, $l_i$, and $s_j$, so we can also view them as framed circles. The parametrization by $S^1$ is unique up to isotopy as each curve is oriented and $\text{MCG}(S^1) = 1$. There is a natural
identification between $\Sigma_g(l_g)$ and $\Sigma_{g-1}$, and so we can view $W(l_g)$, the trace of the surgery along $l_g$, as a cobordism from $\Sigma_g$ to $\Sigma_{g-1}$. We write

$$\alpha_g := F_{\Sigma_g,l_g} : V_g \to V_{g-1}.\] Let $\text{MCG}(\Sigma_g,l_g)$ be the mapping class group of diffeomorphisms that fix the oriented curve $l_g$. There is a forgetful map $\text{MCG}(\Sigma_g,l_g) \to \mathcal{M}_g$ and a destabilization map $\text{MCG}(\Sigma_g,l_g) \to \mathcal{M}_{g-1}$. It follows from Proposition 2.5 that $\alpha_g$ is $\text{MCG}(\Sigma_g,l_g)$-equivariant.

Similarly, we can identify $\Sigma_g(s_j)$ with $\Sigma_j \sqcup \Sigma_{g-j}$, and hence we obtain a map

$$\delta_{j,g-j} := F_{\Sigma_g,s_j} : V_g \to V_j \otimes V_{g-j}$$
for every $j \in \{0,\ldots,g\}$. As above, $\text{MCG}(\Sigma_g,s_j)$ acts on both $V_g$ and $V_j \otimes V_{g-j}$, and the map $\delta_{j,g-j}$ is equivariant under these actions.

Let $p_g, q_g \in \Sigma_g$ be points in the first and last $S^1 \times S^1$ summand of $\Sigma_g$, respectively. For $i, j \in \mathbb{Z}_{\geq 0}$, let

$$P_{i,j} = \{q_i, p_j\} \in \Sigma_i \sqcup \Sigma_j,$$
this is a framed 0-sphere with the framing given by the orientation. We can canonically identify $(\Sigma_i \sqcup \Sigma_j) (P_{i,j})$ with $\Sigma_{i+j}$, hence we obtain a map

$$\mu_{i,j} := F_{\Sigma_i \sqcup \Sigma_j, P_{i,j}} : V_i \otimes V_j \to V_{i+j}.$$ This map is $\text{MCG}(\Sigma_i \sqcup \Sigma_j, P_{i,j}) \cong \text{MCG}(\Sigma_{i+j}, s_i)$ equivariant.

Furthermore, for every $g \in \mathbb{Z}_{\geq 0}$, let $P_g$ be the framed sphere given by two points very close to $q_g$, both lying on $l_g$. Then $\Sigma_g(P_g)$ is canonically diffeomorphic to $\Sigma_{g+1}$; hence we obtain a map

$$\omega_g := F_{\Sigma_g, P_g} : V_g \to V_{g+1}.$$ This map is $\text{MCG}(\Sigma_g, P_g) \cong \text{MCG}(\Sigma_{g+1}, m_{g+1})$ equivariant.

A framed 2-sphere in $\Sigma_0 = S^2$ gives rise to a map $\tau : V_0 \to \mathbb{F}$, while the framed sphere 0 corresponding to a 0-handle attachment gives a map $\varepsilon : \mathbb{F} \to V_0$.

Note that the vector space $V_0$, together with the product $\mu_{0,0}$, coproduct $\delta_{0,0}$, trace $\tau$, and unit $\varepsilon$ form a commutative Frobenius algebra.

As shall now see, the $\mathbb{F}[\mathcal{M}_g]$-modules $V_g$, together with the operations $\alpha_g, \omega_g, \delta_{j,g-j}, \mu_{i,j}, \tau, \varepsilon$ completely determine the functor $F$, up to natural isomorphism. By Theorem 1.2 it suffices to construct $F(M)$ for an arbitrary surface $M$ and maps $F_{M,S}$ for any framed sphere $S$ in $M$. The following constructions are all determined by the naturality of the TQFT under diffeomorphisms. After constructing the groups $F(M)$ and the surgery maps $F_{M,S}$, we check what algebraic properties axioms (1)–(5) of Theorem 1.2 translate to.

First, we construct $F(M)$ for a surface $M$ with $k$ components of genera $g_1 > \cdots > g_r$ with multiplicities $n_1, \ldots, n_r$, respectively. In particular, $n_1 + \cdots + n_r = k$,
and we denote the vector \((g_1, \ldots, g_1, \ldots, g_r, \ldots, g_r)\) of genera by \(\underline{g}\). Let

\[
\Sigma_{\underline{g}} = \prod_{i=1}^{r} \prod_{j=1}^{n_i} \Sigma_{g_i}.
\]

We follow the same scheme as one dimension lower. In particular, let

\[
V_{\underline{g}} = V_{g_1}^{\otimes n_1} \otimes \cdots \otimes V_{g_r}^{\otimes n_r},
\]

and \(F(M)\) is defined to be the set of those elements \(v\) of

\[
\prod_{\phi \in \text{Diff}(\Sigma_{\underline{g}}, M)} V_{\underline{g}}
\]

for which \(v(\phi') = ((\phi')^{-1} \circ \phi) \cdot v(\phi)\) for every \(\phi, \phi' \in \text{Diff}(\Sigma, M)\). Note that here \((\phi')^{-1} \circ \phi \in \text{Diff}(\Sigma_{\underline{g}})\), which acts on \(V_{\underline{g}}\) via the representations \(\rho_i\) and permuting the factors with the same genus. More precisely, the action of \(\text{Diff}(\Sigma_{\underline{g}})\) on \(V_{\underline{g}}\) factors through the action of

\[
\text{MCG}(\Sigma_{\underline{g}}) \cong \prod_{i=1}^{r} \mathcal{M}_{g_i} \times S_{n_i},
\]

where the group \(\mathcal{M}_{g_i}\) acts on \(V_{g_i}\) via \(\rho_{g_i}\), while \(S_{n_i}\) permutes the factors of \(V_{g_i}^{\otimes n_i}\).

Suppose that \(M\) and \(M'\) are diffeomorphic surfaces; i.e., they have the same number of components \(k\) with genera \(g_1 = g_i'\) and multiplicities \(n_i = n_i'\) for every \(i \in \{1, \ldots, r\}\), and let \(d \in \text{Diff}(M, M')\). Given an element \(v \in F(M)\) and \(\phi \in \text{Diff}(\Sigma_{\underline{g}}, M)\), we let

\[
[F(d)(v)](d \circ \phi) = v(\phi).
\]

We now define the surgery maps \(F_{M, \mathcal{S}}\) for a surface \(M\) of diffeomorphism type \(\Sigma_{\underline{g}}\), equipped with a framed sphere \(\mathcal{S} \subset M\).

First, suppose that \(\mathcal{S} = 0\); then \(M(\mathcal{S}) = M \sqcup S^2\). Given \(\phi \in \text{Diff}(\Sigma_{\underline{g}}, M)\), let \(\phi_0 = \phi \sqcup \text{Id}_{S^2} \in \text{Diff}(\Sigma \sqcup S^2, M(\mathcal{S}))\). For \(v \in F(M)\), we let

\[
F_{M, 0}(v)(\phi_0) = v(\phi) \otimes 1 \in V_{\underline{g}} \otimes V_0,
\]

where \(1 \in V_0\) is the image of \(1 \in F\) under the map \(\varepsilon\). The element \(F_{M, 0}(v)\) is independent of the choice of \(\phi\).

Now suppose that \(\mathcal{S} : S^2 \hookrightarrow M\) is a framed 2-sphere with image \(S \subset M\). Then \(M(\mathcal{S}) = M \setminus S\). Choose a parametrization \(\phi \in \text{Diff}(\Sigma_{\underline{g}}, M)\) such that \(\phi|_{\Sigma_{\underline{g}} \times \{r\}} = \mathcal{S}\), and let \(\phi_{\mathcal{S}} = \phi|_{\Sigma_{\underline{g}'}, S'}\), where \(\underline{g}' = \underline{g} \setminus \{(g_r, n_r)\}\). Consider the map

\[
t_{\underline{g}} : V_{\underline{g}} \rightarrow V_{\underline{g}'}
\]

defined on monomials by

\[
t_{\underline{g}}(v_1 \otimes \cdots \otimes v_k) = \tau(v_k) \cdot v_1 \otimes \cdots \otimes v_{k-1},
\]

and extending linearly. For \(v \in F(M)\), let

\[
F_{M, \mathcal{S}}(v)(\phi_{\mathcal{S}}) = t_{\underline{g}}(v(\phi)).
\]

Again, this is well-defined; i.e., independent of the choice of \(\phi\).

Assume that \(\mathcal{S} = \{s_-, s_+\}\) is a framed 0-sphere. If \(s_-\) and \(s_+\) lie in different components \(M_-\) and \(M_+\) of \(M\) of genera \(g_a\) and \(g_b\), respectively, then let

\[
q_- = (q_{ga}, n_a) \in \Sigma_- := \Sigma_{ga} \times \{n_a\}, \quad \text{and}
\]

\[
p_+ = (p_{gb}, n_b) \in \Sigma_+ := \Sigma_{gb} \times \{n_b\}.
\]
Choose a parametrization \( \phi \in \text{Diff}(\Sigma_g, M) \) such that \( \phi(q_-) = s_- \) and \( \phi(p_+) = s_+ \). Let \( \Sigma_g(q_-, p_+) \) be the result of surgery along the 0-sphere \( \{q_-, p_+\} \). If \( n_{a,b} \) is the multiplicity of \( g_a + g_b \) in \( \Sigma_g \), then we can identify \( \Sigma_g(q_-, p_+) \) with the canonical surface \( \Sigma_{g'} \) for

\[
g' = g \setminus \{ (g_a, n_a), (g_b, n_b) \} \cup \{ (g_a + g_b, n_{a,b} + 1) \}.
\]

There is an induced parametrization \( \phi_S : \Sigma_g(q_-, p_+) = \Sigma_g' \to M(S) \) that is the connected sum \((\phi|_{\Sigma_-}) \# (\phi|_{\Sigma_+})\) on \( \Sigma_- \# \Sigma_+ \), and agrees with \( \phi \) on all the other components. If \( v \in F(M) \) is an element such that \( v(\phi) \) is a monomial

\[
\otimes_{i=1}^{r'} \otimes_{j=1}^{n'_i} v(i,j),
\]

the integer \( n'_i \) is the multiplicity of \( g_i \) in \( g' \) for \( i \in \{1, \ldots, r'\} \), and \( c \) is such that \( g'_c = g_a + g_b \), then we define \( F_{M,S}(v)(\phi_S) \) as

\[
\left( \otimes_{i=1}^{c-1} \otimes_{j=1}^{n_i} v(i,j) \right) \otimes \left( \otimes_{j=1}^{n'_i} v(c,j) \otimes \mu_{g_a,g_b}(v(a,n_a), v(b,n_b)) \right) \otimes \left( \otimes_{i=c+1}^{r'} \otimes_{j=1}^{n'_i} v(i,j) \right).
\]

In other words, we omit \( v(a,n_a) \) and \( v(b,n_b) \) from \( v(\phi) \), and insert their \( \mu_{g_a,g_b} \)-product in position \( n'_1 + \cdots + n'_c \). The element \( F_{M,S}(v) \) defined above is independent of the choice of \( \phi_S \) since \( \mu_{g_a,g_b} \) is MCG-\((\Sigma_g \sqcup \Sigma_g, \mathbb{P}_{g_a,g_b})\)-equivariant.

If \( s_- \) and \( s_+ \) lie in the same component \( M_s \) of \( M \), then let \( g_a = g(M_s) \). Consider the framed 0-sphere \( \mathbb{P} = \mathbb{P}_{g_a} \times \{ n_a \} \subset \Sigma_g \times \{ n_a \} \), and choose a parametrization \( \phi \in \text{Diff}(\Sigma_g, M) \) such that \( \phi(\mathbb{P}) = S \). The surgered manifold \( M(S) \) is diffeomorphic to \( \Sigma_g' \), which is in turn can be canonically identified with \( \Sigma_g' \) obtained from \( g \) by removing a copy of \( g_a \) and inserting \( g_a + 1 \). By surgery, we obtain the parametrization

\[
\phi_S := \phi^\mathbb{P} : \Sigma_g' \approx \Sigma_g(\mathbb{P}) \to M(S).
\]

Given an element \( v \in F(M) \) such that \( v(\phi) = \otimes_{i=1}^{r} \otimes_{j=1}^{n_i} v(i,j) \), the element \( F_{M,S}(v)(\phi_S) \) is obtained by applying \( \omega_{g_a} \) to \( v(a,n_a) \). The element \( F_{M,S}(v) \) is independent of the choice of \( \phi_S \) since \( \omega_{g_a} \) is MCG-\((\Sigma_g, \mathbb{P}_{g_a})\)-equivariant.

Now suppose that \( S \) is a framed 1-sphere in \( M \), lying in a component \( M_s \) of genus \( g_a \in q_0 \). If \( S \) is non-separating, consider the curve \( l = l_{g_a} \times \{ n_a \} \subset \Sigma_g \). Then there is a diffeomorphism \( \phi : \Sigma_g \to M \) such that \( \phi|_l = S \). This is possible since any two non-separating simple closed curves on a connected surface are ambient diffeomorphic. We obtain \( g' \) by removing a copy of \( g_a \) and replacing it by \( g_a - 1 \). The surgered manifold \( M(S) \) is diffeomorphic to \( \Sigma_g(\mathbb{l}) \), which is canonically identified with \( \Sigma_g' \). Then let

\[
\phi_S := \phi : \Sigma_g' \approx \Sigma_g(\mathbb{l}) \to M(S).
\]

If \( v \in F(M) \) is such that \( v(\phi) \) is of the form \( \otimes_{i=1}^{r} \otimes_{j=1}^{n_i} v(i,j) \), then we obtain \( F_{M,S}(v)(\phi_S) \) by applying \( \alpha_{g_a} \) to the factor \( v_{g_a,n_a} \). The map \( F_{M,S} \) is independent of the choice of \( \phi_S \) since \( \alpha_{g_a} \) is MCG-\((\Sigma_g, l_{g_a})\)-equivariant.

Finally, suppose that \( S \) separates \( M_s \) into pieces of genera \( g_- \) on the negative side and \( g_+ \) on the positive side (in particular, \( g_a = g_- + g_+ \)). Consider the curve \( c = s_{g_-} \times \{ n_a \} \subset \Sigma_g \times \{ n_a \} \). Then there is a diffeomorphism \( \phi : \Sigma_g \to M \) such that \( \phi|_c = S \). Let \( g' \) be the vector obtained from \( g \) by removing \( g_a \) and inserting \( g_- \) and \( g_+ \) to keep the sequence of coordinates decreasing. There is a canonical diffeomorphism \( d_c : \Sigma_g(\mathbb{g}) \to \Sigma_g' \) that maps the components of \( \Sigma_g \times \{ n_a \} \) to the last components of \( \Sigma_g \) of genus \( g_- \) and \( g_+ \), respectively. If \( g_- = g_+ \), then we map the part coming from the negative side of \( c \) as the last but one such
component, and the part coming from the positive side of \( c \) as the last component of the appropriate genus. We define the map

\[
\phi_\mathcal{S} := \phi \circ (d_c)^{-1} : \Sigma_{g'} \to M(\mathcal{S}).
\]

If \( v(\phi) \) is of the form \( \otimes_{i=1}^{n} \otimes_{j=1}^{n_\mathcal{S}} v_{(i,j)} \), then \( F_{M,\mathcal{S}}(v)(\phi_\mathcal{S}) \) is obtained by applying the map \( \delta_{g_{-},g_{+}} \) to \( v_{g_{n_{\mathcal{S}}}} \), and then permuting the factors according to the diffeomorphism \( d_c \). In this case, \( F_{M,\mathcal{S}}(v) \) is independent of the choice of \( \phi \) since \( \delta_{g_{-},g_{+}} \) is \( \text{MCG}(\Sigma_{g_{a}}, \Sigma_{g_{+}}) \)-equivariant.

This concludes the construction of the vector spaces \( F(M) \) and maps \( F_{M,\mathcal{S}} \). By Theorem 1.2 these completely determine the \( (2 + 1) \)-dimensional TQFT \( F \), assuming they satisfy axioms (1)–(5). We check these next.

Axiom (1) follows analogously to the \((1 + 1)\)-dimensional case and the fact that the \( \text{Diff}(\Sigma_g) \)-action on \( V_g \) factors through a \( \text{MCG}(\Sigma_g) \)-action, and it does not impose any additional algebraic restrictions.

Axiom (2) also follows analogously to the \((1 + 1)\)-dimensional case, and requires no additional assumptions. As an illustration, we check axiom (2) when \( M \) is a connected surface of genus \( g \), and \( \mathcal{S} \) is a non-separating 1-sphere. In particular, let \( g = (g) \). Choose a parametrization \( \phi \in \text{Diff}(\Sigma_g, M) \) for which \( \phi|_\mathcal{S} = \mathcal{S} \), and let \( \phi_\mathcal{S} \in \text{Diff}(\Sigma_{g+1}, M(\mathcal{S})) \) be the induced parametrization. Let \( d : M \to M' \) be a diffeomorphism, \( \mathcal{S}' = d(\mathcal{S}) \), and choose an element \( v \in F(M) \). Then \( v(\phi) \in V_g \), and, by the definition of \( F(d) \),

\[
[F(d) \circ F_{M,\mathcal{S}}(v)](d \circ \phi_{\mathcal{S}'}) = F_{M,\mathcal{S}}(v)(\phi_\mathcal{S}) = \alpha_g(v(\phi)) \in V_{g-1}.
\]

On the other hand,

\[
[F_{M',\mathcal{S}'} \circ F(d)(v)]((d \circ \phi)'_{\mathcal{S}'}) = \alpha_g([F(d)(v)](d \circ \phi)) = \alpha_g(v(\phi)).
\]

The result follows once we observe that \( d \circ \phi_\mathcal{S} = (d \circ \phi)'_{\mathcal{S}'} \).

Now consider axiom (3). In particular, let \( \mathcal{S} \) and \( \mathcal{S}' \) be disjoint framed spheres in the surface \( M \). The role of \( \mathcal{S} \) and \( \mathcal{S}' \) are symmetric, and – as in the \((1 + 1)\)-dimensional case – it is straightforward to check the axiom when \( \mathcal{S} = 0 \) or \( \mathcal{S} \) is a 2-sphere. This leaves us with three cases depending on the dimensions of the two spheres.

First, suppose that both \( \mathcal{S} \) and \( \mathcal{S}' \) are 0-spheres. The axiom is true if they occupy distinct components of \( M \). There are now four subcases:

1. \( \mathcal{S} \) and \( \mathcal{S}' \) occupy the same component \( M_s \),
2. \( \mathcal{S} \) intersects both \( M_s \) and another component \( M'_s \), and \( \mathcal{S}' \) lies in \( M'_s \),
3. both \( \mathcal{S} \) and \( \mathcal{S}' \) intersect two components that coincide, namely \( M_s \) and \( M'_s \),
4. \( \mathcal{S} \) intersects \( M_s \) and \( M'_s \), while \( \mathcal{S}' \) intersects \( M'_s \) and \( M''_s \).

Consider case (1). Without loss of generality, we can assume that \( M \) is connected, as we can deal with multiple components similarly to the \((1 + 1)\)-dimensional case. Let \( C \) and \( C' \) be the belt circles of the handles attached along \( \mathcal{S} \) and \( \mathcal{S}' \), respectively. Choose parameterizations \( \phi \), \( \phi' \in \text{Diff}(\Sigma_{g+2}, M(S, S')) \) such that \( \phi(m_{g+1}) = C \), \( \phi'(m_{g+1}) = C' \), \( \phi'(m_{g+2}) = C' \), \( \phi(m_{g+2}) = C \), and such that \( \psi := \phi_{m_{g+1},m_{g+2}} \) and \( \psi' := (\phi')_{m_{g+1},m_{g+2}} \) are isotopic in \( \text{Diff}(\Sigma_g, M) \). Furthermore, let \( v \in F(M) \). Note that \( \psi_{\mathcal{S},\mathcal{S}'} = \phi \), hence

\[
F_{M(\mathcal{S}),\mathcal{S}'} \circ F_{M,\mathcal{S}}(v)(\phi) = \omega_{g+1} \circ F_{M,\mathcal{S}}(v)(\psi_{\mathcal{S}}) = \omega_{g+1} \circ \omega_{g}(v(\psi))_.
\]

Similarly, \( (\psi')_{\mathcal{S}',\mathcal{S}} = \phi' \), hence

\[
F_{M(\mathcal{S}),\mathcal{S}'} \circ F_{M,\mathcal{S}}(v)(\phi') = \omega_{g+1} \circ \omega_{g}(v(\psi')).
\]
Since \( \psi \) and \( \psi' \) are isotopic, \( v(\psi) = v(\psi') \). Finally,
\[
F_{M(\mathcal{S}),S'} \circ F_{M,S}(v)(\phi') = \rho_{g+2}((\phi')^{-1} \circ \phi) \circ F_{M(\mathcal{S}),S'} \circ F_{M,S}(v)(\phi).
\]

As \( v \) can be an arbitrary element of \( F(M) \), so \( v(\phi) \) is an arbitrary element of \( V_g \).
Furthermore, \( d = (\phi')^{-1} \circ \phi \) is an automorphism of \( \Sigma_{g+2} \) that swaps \( m_{g+1} \) and \( m_{g+2} \), and for which \( dm_{g+1},m_{g+2} \) is isotopic to \( Id_{\Sigma_g} \). Hence, axiom 3 holds in case 11 if and only if for any diffeomorphism \( d \in \text{Diff}(\Sigma_{g+2}) \) that swaps \( m_{g+1} \) and \( m_{g+2} \), and for which \( d(m_{g+1},m_{g+2}) \in \text{Diff}(\Sigma_g) \) is isotopic to \( Id_{\Sigma_g} \), the automorphism \( \rho_{g+2}(d) \) of \( V_{g+2} \) is the identity on \( \text{Im}(\omega_{g+1} \circ \omega_g) \); i.e.,
\[
(4.1) \quad \rho_{g+2}(d) \circ \omega_{g+1} \circ \omega_g = \omega_{g+1} \circ \omega_g.
\]

**Remark 4.1.** As \( d^2 \) fixes both \( m_{g+1} \) and \( m_{g+2} \), by the \( \text{MCG}(\Sigma_{g+1},m_{g+1}) \)-equivariance of \( \omega_g \) and the \( \text{MCG}(\Sigma_{g+2},m_{g+2}) \)-equivariance of \( \omega_{g+1} \), we obtain that
\[
\rho_{g+2}(d^2) \circ \omega_{g+1} \circ \omega_g = \omega_{g+1} \circ \omega_g \circ \rho_g \left( (d(m_{g+1},m_{g+2}))^2 \right) = \omega_{g+1} \circ \omega_g.
\]
Hence \( \rho_{g+2}(d)^2 \) is automatically the identity on the image of \( \omega_{g+1} \circ \omega_g \), the additional assumption is that \( \rho_{g+2}(d) \) also satisfies this property.

Now consider case 2. Again, without loss of generality, assume that \( M \) has only two components, namely \( M_0 \) of genus \( g \) and \( M'_0 \) of genus \( g' \). Furthermore, by axiom 3 (which we will check later), we can replace \( S \) by \( \overline{S} \) if necessary to ensure that \( S(-1) \in M_0 \) and \( S(1) \in M'_0 \). Similarly to the previous case, one can deduce that commutativity of the two surgery maps holds if and only if
\[
(4.2) \quad \mu_{g,g'}(\text{Id}_{V_g} \circ \omega_{g'}) = \omega_{g+g'} \circ \mu_{g,g'}.
\]

Case 3 is similar to case 11. Without loss of generality, we can assume that \( M \) consists of only two components of genera \( g \) and \( g' \), respectively. Furthermore, by naturality and axiom 3, we can suppose that \( M = \Sigma_g \cup \Sigma_g', S = \mathbb{P}_{g,g'} \), and \( S' \) is a small translate of \( \mathbb{P}_{g,g'} \). Then we can canonically identify \( M(\mathcal{S}) \) with \( \Sigma_{g+g'} \), and the belt circle of the handle attached along \( S \) corresponds to \( s_g \). Furthermore, \( M(\mathcal{S},S') \) can be identified with \( \Sigma_{g+g'+1} \), where the belt circles of the handles attached along \( S \) and \( S' \) correspond to \( m_{g+1} \) and \( m_{g+1}' \), respectively. In particular, \( F_{M(\mathcal{S}),S} = \mu_{g,g'} \). To compute \( F_{M(\mathcal{S}),S'} \), choose a diffeomorphism \( d \in \text{Diff}(\Sigma_{g+g'}) \) with \( d(\mathbb{P}_{g,g'}) = \mathbb{P}_{g,g'} \).

By naturality,
\[
F_{M(\mathcal{S}),S'} = \rho_{g+g'+1}(\overline{d}) \circ \omega_{g+g'} \circ \rho_{g+g'}(d)^{-1},
\]
where
\[
\overline{d} := d^2 : \Sigma_{g+g'}(S') = \Sigma_{g+g'+1} \to \Sigma_{g+g'}(\mathbb{P}_{g,g'}) = \Sigma_{g+g'+1}.
\]
As \( d \) is isotopic to the identity, \( \rho_{g+g'}(d) = \text{Id}_{V_{g+g'}} \). We conclude that
\[
F_{M(\mathcal{S}),S'} \circ F_{M,S} = \rho_{g+g'+1}(\overline{d}) \circ \omega_{g+g'} \circ \mu_{g,g'}.
\]
The map \( \overline{d} \) can be characterized by the property that \( \overline{d}(m_{g+1}^+) = m_{g+g'}^+ + 1 \) and \( \overline{d}(m_{g+1}^-) = m_{g+g'}^- + 1 \), where \( m_{g+1}^+, m_{g+1}^- \) is isotopic to \( Id_{\Sigma_g} \) after the appropriate identifications. This leads to the restriction that if \( f \in \text{Diff}(\Sigma_{g+g'+1}) \) swaps \( m_+ := m_{g+1}^+ \) and \( m_- := m_{g+1}^- \), and if \( m_+ \circ m_- \in \text{Diff}(\Sigma_g \cup \Sigma_g) \) (it swaps \( S \) and \( S' \)), then \( \rho_{g+g'+1}(f) \) is the identity on
\[
\text{Im} \left( \rho_{g+g'+1}(\overline{d}) \circ \omega_{g+g'} \circ \mu_{g,g'} \right).
\]
We can restate this as
\[
(4.3) \quad \rho_{g+g'+1}(h) \circ \omega_{g+g'} \circ \mu_{g,g'} = \omega_{g+g'} \circ \mu_{g,g'}.
\]
where $h = \overline{d}^{-1} \circ f \circ \overline{d}$. As $f$ is an involution that swaps $m_-$ and $m_+$, its conjugate $h$ is also an involution that swaps $\overline{d}^{-1}(m_-) = s_{g+1} \# m_{g+g'}+1$ and $\overline{d}^{-1}(m_+) = m_{g+g'+1}$, and such that $h^s_{g+1} \# m_{g+g'}+1 \# m_{g+g'+1}$ is isotopic to the identity.

Finally, in case (4), we obtain the associativity relation
\begin{equation}
\mu_{g+g',g''} \circ (\mu_{g,g'} \otimes \text{Id}_{V_{g''}}) = \mu_{g,g'+g''} \circ (\text{Id}_{V_g} \otimes \mu_{g',g''}).
\end{equation}

We now study axiom (3) when both $S$ and $S'$ are framed 1-spheres. The axiom is straightforward if $S$ and $S'$ occupy different components of $M$. Hence, without loss of generality, we can assume that $M$ is connected of genus $g$. Then we have the following three cases:

1. Both $S$ and $S'$ are non-separating. There are two subcases depending on whether $S \cup S'$ is separating or not.

2. $S$ separates $M$ into components of genera $j$ and $g-j$, and $S'$ is non-separating. By axiom (5), we can assume that $S'$ lies on the positive side of $S$.

3. Both $S$ and $S'$ are separating. By symmetry, we can assume that $S'$ lies on the positive side of $S$, and by axiom (3) that $S$ is on the negative side of $S'$.

They divide $M$ into pieces of genera $i$, $j$, and $k$.

First, consider case (1), and suppose that $S \cup S'$ is non-separating. Then we can choose parameterizations $\phi, \phi' \in \text{Diff}(\Sigma_g, M)$ for which $\phi|_{l_g} = S$, $\phi|_{l_{g-1}} = S'$, $\phi'|_{l_g} = S'$, and $\phi'|_{l_{g-1}} = S$, and such that $\phi'|_{l_{g-1}} = (\phi')_{l_{g-1}}$ and $(\phi')_{l_{g-1}}$ are isotopic. Furthermore, let $v \in F(M)$. Then, by definition,
\begin{equation}
F_{M(S),S'} \circ F_{M,S}(v)(\phi_{S,S'}) = \alpha_{g-1} \circ \alpha_g(v(\phi)),
\end{equation}
and, symmetrically,
\begin{equation}
F_{M(S'),S} \circ F_{M,S'}(v)(\phi'_{S',S}) = \alpha_{g-1} \circ \alpha_g(v(\phi')).
\end{equation}

Since $\phi_{S,S'} = \phi|_{l_{g-1}}$ and $\phi'_{S',S} = (\phi')|_{l_{g-1}}$ are isotopic, the left-hand sides above are equal. Furthermore, $v(\phi') = \rho_g((\phi')^{-1} \circ \phi)(v(\phi))$. Hence axiom (3) holds in this case if and only if for any diffeomorphism $d \in \text{Diff}(\Sigma_g)$ that swaps $l_g$ and $l_{g-1}$ and for which $d_{l_{g-1}} \in \text{Diff}(\Sigma_{g-2})$, we have
\begin{equation}
\alpha_{g-1} \circ \alpha_g \circ \rho_g(d) = \alpha_{g-1} \circ \alpha_g.
\end{equation}

If in case (1) the union $S \cup S'$ separates $M$ into pieces of genera $i$ and $j$, respectively, then $g = i + j + 1$. The model case is when $M = \Sigma_g$, $S = m_i^{-1}$, and $S' = m_i^{+1}$. Similarly to equation (4.3), we obtain the following relation:
\begin{equation}
\delta_{i,j} \circ \alpha_g \circ \rho_g(u) = \delta_{i,j} \circ \alpha_g,
\end{equation}
where $u \in \text{Diff}(\Sigma_g)$ swaps $s_{i+1} \# l_g$ and $l_g$, and such that $u_{i+1} \# l_g$ is isotopic to the identity.

Now consider case (2). This leads to the relation
\begin{equation}
\delta_{j,g-1} \circ \alpha_g = (\text{Id}_{V_j} \otimes \alpha_{g-j}) \circ \delta_{j,g-j}.
\end{equation}
Case (3) leads to the following coassociativity relation:
\begin{equation}
(\text{Id}_{V_i} \otimes \delta_{j,k}) \circ \delta_{i,j+k} = (\delta_{i,j} \otimes \text{Id}_{V_k}) \circ \delta_{i,j+k}.
\end{equation}

Finally, we look at axiom (3) when $S$ is a framed 0-sphere and $S'$ is a framed 1-sphere. Without loss of generality, we can assume that $S$ intersects the component of $M$ that $S'$ occupies. Here we distinguish the following cases:
(1) $S$ lies in a single component $M_s$ and $S' \subset M_s$ is non-separating.
(2) $S$ lies in a single component $M_i$ and $S'$ separates $M_i$ into pieces of genera $i$ and $g-i$. There are three subcases depending on whether $S$ lies completely to the left of $S'$, on both sides, or completely to the right.
(3) $S$ occupies the components $M_s$ and $M'_s$, and $S' \subset M'_s$ is non-separating.
(4) $S$ occupies the components $M_s$ and $M'_s$, and $S'$ separates $M'_s$ into components of genera $i$ and $g'-i$. There are two subcases depending on whether the point of $S$ in $M'_s$ lies to the left or to the right of $S'$.

By the assumption, we can assume that $M = \Sigma_g$, $S = P_g$, and $S' = l_g$ (or, more precisely, we work with a parametrization $\phi \in \text{Diff}(\Sigma_g, M)$ such that $\phi(P'_g) = S$ and $\phi(l_g) = S'$, where $P'_g$ is small translate of $P_g$ disjoint from $l_g$).

Let $d \in \text{Diff}(\Sigma_{g+1})$ be such that $d(l_g) = l_{g+1}$, and $d^s = \text{Id}_{\Sigma_g}$ after the natural identifications of $\Sigma_{g+1}(l_g)$ and $\Sigma_{g+1}(l_{g+1})$ with $\Sigma_g$. As we already know the surgery maps are natural, the following diagram is commutative:

$$
\begin{array}{ccc}
F(\Sigma_{g+1}) = V_{g+1} & \xrightarrow{\alpha_{g+1}} & F(\Sigma_g) = V_g \\
\rho_{g+1}(d) \downarrow & & \downarrow & \text{Id}_{V_g} \\
V_{g+1} & \xrightarrow{F_{\Sigma_{g+1}, l_g}} & F(\Sigma_{g+1}(l_g)) \cong V_g.
\end{array}
$$

By the assumption, $d^s$ is isotopic to $\text{Id}_{\Sigma_g}$, so $F(d^s) = \text{Id}_{V_g}$, and

$$
F_{\Sigma_{g+1}, l_g} = \alpha_{g+1} \circ \rho_{g+1}(d).
$$

Hence, we obtain the relation

$$
\omega_{g-1} \circ \alpha_g = \alpha_{g+1} \circ \rho_{g+1}(d) \circ \omega_g,
$$

where $d \in \text{Diff}(\Sigma_{g+1})$ is such that $d(l_g) = l_{g+1}$, and $d^s = \text{Diff}_0(\Sigma_g)$ after the natural identifications of $\Sigma_{g+1}(l_g)$ and $\Sigma_{g+1}(l_{g+1})$ with $\Sigma_g$. Notice that the diffeomorphism $d$ coincides with the diffeomorphism of equation (4.9) acting on $\Sigma_{g+1}$ and interchanging $l_g$ and $l_{g+1}$.

In case (2), when $S$ lies to the left of $S'$, we replace $S'$ by $\overline{S'}$ and apply axiom (5). The other two cases lead to the relations

$$
\delta_{i,j+1} \circ \omega_g = (\text{Id}_{V'} \otimes \omega_j) \circ \delta_{i,j}
$$

and

$$
\alpha_{g+1} \circ \rho_{g+1}(d) \circ \omega_g = \mu_{i,g-i} \circ \delta_{i,g-i},
$$

where in the second equation, $d \in \text{Diff}(\Sigma_{g+1})$ is such that $d(s_i \# m_{g+1}) = l_{g+1}$, and $d^{s_i \# m_{g+1}}: \Sigma_{g+1}(s_i \# m_{g+1}) \to \Sigma_{g+1}(l_{g+1})$ is isotopic to the identity after we identify the source and the target with $\Sigma_g$ in a natural way. We explain this in more detail. Without loss of generality, we can assume that $M$ is connected, and by naturality, that $M = \Sigma_g$, $S = P_g$, and $S'$ is the curve obtained from $s_i$ by isotoping it via a finger move across one of the points of $P_g$ (so that there is exactly one point of $P_g$ on each side of $S'$). More precisely, the finger move induces a diffeomorphism $\nu$ of $\Sigma_g$ that maps a pair of points on the two sides of $s_i$ to $P_g$. There is a natural identification between $\Sigma_g(P_g)$ and $\Sigma_{g+1}$ under which $S'$ corresponds to the connected sum $s_i \# m_{g+1}$. Furthermore, via the diffeomorphism $(\nu^{-1})_{P_g,s_i \# m_{g+1}}$, we can identify $\Sigma_{g+1}(s_i \# m_{g+1})$ and $\Sigma_g$. Let $b \subset \Sigma_{g+1}(s_i \# m_{g+1}) \approx \Sigma_g$ be the belt circle of the handle attached to $\Sigma_{g+1}$ along $s_i \# m_{g+1}$; this is a pair of points.
Furthermore, let \( b' \subset \Sigma_{g+1}(l_{g+1}) \approx \Sigma_g \) be the belt circle of the handle attached to \( \Sigma_{g+1} \) along \( l_{g+1} \). By the homogeneity of \( \Sigma_g \), there is a diffeomorphism \( d_0 \) isotopic to \( \text{Id}_{\Sigma_g} \) that takes \( b \) to \( b' \). Then \( d := d_0^{-1} \in \text{Diff}(\Sigma_{g+1}) \) satisfies \( d(s_i \# m_{g+1}) = l_{g+1} \), and such that \( d^* s_i \# m_{g+1} = d_0 \) is isotopic to \( \text{Id}_{\Sigma_g} \). Hence, by naturality,

\[
F_{\Sigma_{g+1},s_i \# m_{g+1}} = \alpha_{g+1} \circ \rho_{g+1}(d).
\]

Consequently, surgery along \( S \), followed by surgery along \( S' \) induces the map

\[
F_{M(S),S'} \circ F_{M,S} = \alpha_{g+1} \circ \rho_{g+1}(d) \circ \omega_g.
\]

In case \( \text{(2)} \), the necessary and sufficient condition for axiom \( \text{(4)} \) to hold is

\[
(4.11) \quad \alpha_{g+g'} \circ \mu_{g,g'} = \mu_{g,g'-1} \circ (\text{Id}_{V_g} \otimes \alpha_{g'}). 
\]

There is a corresponding relation if \( S' \) lies on the other side of \( S \), but that follows from this one by axiom \( \text{(3)} \).

Finally, in case \( \text{(3)} \), we obtain

\[
(4.12) \quad \delta_{g+i,g'-i} \circ \mu_{g,g'} = (\mu_{g,i} \otimes \text{Id}_{V_{g'-i}}) \circ (\text{Id}_{V_g} \otimes \delta_{i,g'-i}).
\]

We now consider axiom \( \text{(5)} \); i.e., where \( S' \subset M(S) \) intersects the belt sphere of \( S \) once. If \( S = 0 \) and \( S' \) is a 0-sphere that has one point on the new \( S^2 \) component and another point on a component of \( M \) of genus \( g \), then we can assume \( S'(-1) \in S^2 \) by axiom \( \text{(6)} \). This leads to the relation

\[
(4.13) \quad \mu_{0,g} \circ (\varepsilon \otimes \text{Id}_{V_g}) = \text{Id}_{V_g};
\]

i.e., that \( 1 = \varepsilon(1) \) is a left unit for \( \mu \). If \( S \) is a 0-sphere, it has to lie in a single component of \( M \). Then we obtain the relation

\[
(4.14) \quad \alpha_{g+1} \circ \omega_g = \text{Id}_{V_g}.
\]

If \( S \) is a 1-sphere, then it has to be inessential, and \( S' \) is the 2-sphere split off by \( S \). By axiom \( \text{(7)} \), we can assume this 2-sphere lies on the negative side of \( S \). We obtain the relation

\[
(4.15) \quad (\tau \otimes \text{Id}_{V_g}) \circ \delta_{0,g} = \text{Id}_{V_g}
\]

i.e., that \( \tau \) is a left counit for the coproduct \( \delta \).

Finally, consider axiom \( \text{(3)} \). Think of \( \Sigma_g \) as being standardly embedded in \( \mathbb{R}^3 \) with center lying at the origin, and such that the \( x \)-axis intersects it in the points \( p_g \) and \( q_g \). Let \( \iota_g \in \text{Diff}(\Sigma_g) \) be the involution of \( \Sigma_g \) that is a \( \pi \)-rotation about the \( y \)-axis and swaps the \( i \)-th and \( (g - i) \)-th \( S^1 \times S^2 \) factor of \( \Sigma_g \). The \( y \)-axis passes through \( s_{g/2} \) if \( g \) is even, and through the hole of the \((g+1)/2\)-th \( S^1 \times S^2 \) summand when \( g \) is odd. This has the property that \( \iota_g(s_i) = s_{g-i} \) for every \( i \in \{0, \ldots, g\} \).

First, suppose that \( S \) is a 0-sphere that occupies two components of \( M \). Then the model scenario is \( M = \Sigma_i \sqcup \Sigma_j \) and \( S = P_{i,j} \). If \( \sigma: \Sigma_i \sqcup \Sigma_j \to \Sigma_j \sqcup \Sigma_i \) is the diffeomorphism that swaps the two components of \( \Sigma_i \sqcup \Sigma_j \), then acts via \( \iota_i \sqcup \iota_j \), satisfies \( \sigma(P_{i,j}) = P_{j,i} \). Furthermore, \( \sigma^* = \iota_{i+j} \). Hence, using that \( F_{M,S} = F_{M,S}^* \) and the naturality of the surgery maps, axiom \( \text{(9)} \) amounts to the relation

\[
\rho(\iota_{i+j}) \circ \mu_{i,j}(x \otimes y) = \mu_{j,i}(\rho(\iota_j)(y) \otimes \rho(\iota_i)(x))
\]

for every \( x \in V_i \) and \( y \in V_j \). After introducing the notation \( x^* := \rho(\iota_i)(x) \) for every \( i \in \mathbb{Z}_{\geq 0} \) and \( x \in V_i \), we can rewrite this relation as

\[
(4.16) \quad \mu_{i+j}(x, y)^* = \mu_{j,i}(y^* \otimes x^*).
\]
As \( \iota_i \) is an involution, \( x^{**} = x \). Furthermore, since \( \iota_i \in \text{Diff}_0(\Sigma_0) \) for \( i \in \{0,1\} \), we see that \( x^* = x \) for \( x \in V_0 \cup V_1 \). Sometimes we will also use the notation \( *_i \) for \( \rho(\iota_i) \).

Now consider the case when \( S \) is a 0-sphere in a single component of \( M \). Then the model case is \( M = \Sigma_g \) and \( S = \mathbb{P}g \). Let \( t_g \in \text{Diff}(\Sigma_g) \) be the diffeomorphism that is characterized by \( t_g(m_g) = -m_g \) and \( t_g^{\ast} \in \text{Diff}_0(\Sigma_g-1) \). Then axiom (15) in this case is equivalent to the relation

\[
\rho_{g+1}(t_{g+1}) \circ \omega_g = \omega_g.
\]

Applied to separating 1-spheres, we obtain the relation

\[
T_{i,j} \circ \delta_{i,j}(x) = \delta_{j,i}(x^*),
\]

where \( T_{i,j} : V_i \otimes V_j \to V_j \otimes V_i \) is given by \( T_{i,j}(v \otimes w) = w^* \otimes v^* \).

When \( S \) is a non-separating 1-sphere, we obtain that

\[
\alpha_g = \alpha_g \circ \rho(r_g),
\]

where \( r_g \in \text{Diff}(\Sigma_g) \) is characterized by \( r_g(l_g) = -l_g \) and \( (r_g)^{\ast} \in \text{Diff}_0(\Sigma_g-1) \).

### 4.1. The algebra

Having obtained a necessary and sufficient set of relations that our data has to satisfy, we synthesize this into a nice algebraic structure. First, we summarize what we have obtained so far. So (2 + 1)-dimensional TQFTs correspond to the following structures: We are given a sequence of finite-dimensional \( \mathbb{F} \)-vector spaces \( V_i \) for \( i \in \mathbb{N} \), together with products \( \mu_{i,j} : V_i \otimes V_j \to V_{i+j} \), coproducts \( \delta_{i,j} : V_{i+j} \to V_i \otimes V_j \), a left unit \( \varepsilon : \mathbb{F} \to V_0 \), a left counit \( \tau : V_0 \to \mathbb{F} \), embeddings \( \omega_i : V_i \to V_{i+1} \), projections \( \alpha_i : V_i \to V_{i-1} \), and representations \( \rho_i : \mathcal{M}_i \to \text{End}(V_i) \). These satisfy the following properties:

By equations (4.14) and (4.13), the product \( \mu \) is associative with left unit \( \varepsilon \):

\[
\mu_{i,j+k} \circ (\mu_{i,j} \otimes \text{Id}V_k) = \mu_{i,j+k} \circ (\text{Id}V_i \otimes \mu_{j,k}),
\]

\[
\mu_{0,j} \circ (\varepsilon \otimes \text{Id}V_j) = \text{Id}V_j.
\]

Equations (4.16) and (4.12) state that the coproduct \( \delta \) is coassociative with left counit \( \tau \):

\[
(\text{Id}V_i \otimes \delta_{j,k}) \circ \delta_{i,j+k} = (\delta_{i,j} \otimes \text{Id}V_k) \circ \delta_{i,j+k},
\]

\[
(\tau \otimes \text{Id}V_j) \circ \delta_{0,j} = \text{Id}V_j.
\]

Furthermore, according to equation (4.12), \( \mu \) and \( \delta \) satisfy the Frobenius condition

\[
\delta_{i,j+k} \circ \mu_{i,j+k} = (\mu_{i,j} \otimes \text{Id}V_k) \circ (\text{Id}V_i \otimes \delta_{j,k}).
\]

By (4.16) and (4.18), the operation \( * \) is an anti-automorphism:

\[
\mu_{i,j}(x^* \otimes y^*) = \mu_{j,i}(y \otimes x)^*,
\]

\[
T_{i,j} \circ \delta_{i,j}(x) = \delta_{j,i}(x^*),
\]

where \( T_{i,j} : V_i \otimes V_j \to V_j \otimes V_i \) is given by \( T_{i,j}(x \otimes y) = y^* \otimes x^* \). Furthermore, \( * \) is involutive, and is the identity on \( V_0 \) and \( V_1 \).

By equation (4.14), we have

\[
\alpha_{i+1} \circ \omega_i = \text{Id}V_i,
\]
graded involutive version of this notion. A non-degenerate pairing forces every Frobenius algebra to be finite dimensional, while a degenerate bilinear pairing identifies the algebra with its dual. Note that Frobenius algebras, but without the trace functional, and hence lack the non-degenerate pairing 

\[ \alpha \]

The map \( \alpha \) is \( \text{MCG}(\Sigma_i,l_i) \)-equivariant, \( \omega_i \) is \( \text{MCG}(\Sigma_i,\mathbb{P}_i) \)-equivariant, \( \mu_{i,j} \) is \( \text{MCG}(\Sigma_i \sqcup \Sigma_j, \mathbb{P}_{i,j}) \)-equivariant, and \( \delta_{i,j} \) is \( \text{MCG}(\Sigma_{i+j}, s_i) \)-equivariant. In addition, \( |V_i| = \rho(I_i) \), and the representations \( \rho_i \) satisfy the following conditions according to equations \( (4.1), (4.5), (4.10), (4.13), (4.16), (4.17), \) and \( (4.19) \), respectively:

\[ \rho_{i+2}(S_{i+2}) \circ \omega_{i+1} \circ \omega_i = \omega_{i+1} \circ \omega_i, \]
\[ \alpha_{i-1} \circ \alpha_i \circ \rho_1(L_i) = \alpha_{i-1} \circ \alpha_i, \]
\[ \alpha_{i+1} \circ \rho_{i+1}(L_{i+1}) \circ \omega_i = \omega_{i+1} \circ \alpha_i, \]
\[ \alpha_{n+1} \circ \rho_{n+1}(\sigma_{n+1,i}) \circ \omega_n = \mu_{i,j} \circ \sigma_{n+1,i}, \]
\[ \rho_{n+1}(h_{n+1,i+1}) \circ \omega_n \circ \mu_{i,j} = \omega_n \circ \mu_{i,j}, \]
\[ \delta_{i,j} \circ \alpha_n \circ \rho_n(u_{n,j+1}) = \delta_{i,j} \circ \alpha_n, \]
\[ \rho_{i+1}(l_{i+1}) \circ \omega_i = \omega_i, \]
\[ \alpha_i \circ \rho(r_i) = \alpha_i, \]

where \( n = i + j \), and:

- \( S_{i+2} \in \text{Diff}(\Sigma_{i+2}) \) swaps \( m_{i+1} \) and \( m_{i+2} \), and \( S_{i+2}^{m_{i+1},m_{i+2}} \in \text{Diff}(\Sigma_{i+1}) \),
- \( \alpha_{i+1} \circ \rho_{i+1}(L_{i+1}) \circ \omega_i = \omega_{i+1} \circ \alpha_i, \)
- \( \alpha_{i-1} \circ \alpha_i \circ \rho_1(L_i) = \alpha_{i-1} \circ \alpha_i, \)
- \( \alpha_{i+1} \circ \rho_{i+1}(L_{i+1}) \circ \omega_i = \omega_{i+1} \circ \alpha_i, \)
- \( \alpha_{n+1} \circ \rho_{n+1}(\sigma_{n+1,i}) \circ \omega_n = \mu_{i,j} \circ \sigma_{n+1,i}, \)
- \( \rho_{n+1}(h_{n+1,i+1}) \circ \omega_n \circ \mu_{i,j} = \omega_n \circ \mu_{i,j}, \)
- \( \delta_{i,j} \circ \alpha_n \circ \rho_n(u_{n,j+1}) = \delta_{i,j} \circ \alpha_n, \)
- \( \rho_{i+1}(l_{i+1}) \circ \omega_i = \omega_i, \)
- \( \alpha_i \circ \rho(r_i) = \alpha_i, \)

Definition 4.2. A \( (2+1) \)-algebra is a sequence of vector spaces \( V_i \) for \( i \in \mathbb{N} \), together with maps \( \mu_{i,j}, \delta_{i,j}, \varepsilon, \tau, \omega_i, \alpha_i, * \) as above. A \( (2+1) \)-representation is a sequence of homomorphisms \( \rho_i : M_i \to \text{End}(V_i) \) satisfying the above properties.

Using this terminology, we have obtained the following intermediate result.

Theorem 4.3. There is a bijective correspondence between \( (2+1) \)-dimensional TQFTs and \( (2+1) \)-algebras endowed with \((2+1)\)-representations.

Nearly Frobenius algebras were introduced by Cohen and Godin [2]. They are like Frobenius algebras, but without the trace functional, and hence lack the non-degenerate bilinear pairing that identifies the algebra with its dual. Note that a non-degenerate pairing forces every Frobenius algebra to be finite dimensional, whereas this is not the case for nearly Frobenius algebras. Next, we introduce a graded involutive version of this notion.
**Definition 4.4.** A graded involutive nearly Frobenius algebra (or GNF*-algebra for short) is a tuple $\mathcal{A} = (A, \mu, \delta, \varepsilon, \tau, \ast)$, where

$$A = \bigoplus_{i=0}^{\infty} A_i$$

is an $\mathbb{N}$-graded $\mathbb{F}$-vector space such that each $A_i$ is finite dimensional. Furthermore,

1. $\mu : A \otimes A \to A$ is a graded linear map, where $A \otimes A$ is the graded tensor product; i.e.,

$$\left( A \otimes A \right)_n = \bigoplus_{i=0}^{\infty} A_i \otimes A_{n-i} \leq A \otimes_{\mathbb{F}} A,$$

2. $\mu$ is associative and $\varepsilon : \mathbb{F} \to A_0$ is a left unit for $\mu$,

3. $\delta : A \to A \otimes A$ is coassociative and $\tau : A_0 \to \mathbb{F}$ is a partial left counit for $\delta$ in the sense that $(\tau \otimes \text{Id}_{A_j}) \circ \delta_{0,j} = \text{Id}_{A_j}$, where $\delta_{i,j} = \pi_{i,j} \circ \delta$ and $\pi_{i,j} : A \to A_i \otimes A_j$ is the projection.

4. $\delta$ is a left $(A, \mu)$-module homomorphism; i.e., the following diagram is commutative:

$$\begin{array}{ccc}
A \otimes A & \xrightarrow{\text{Id}_A \otimes \delta} & A \otimes A \\
\mu \downarrow & & \mu \downarrow \\
A & \xrightarrow{\delta} & A \otimes A,
\end{array}$$

5. $\ast : A \to A$ is a grading-preserving involution that is an antiautomorphism of $(A, \mu, \delta)$, and such that it is the identity on $A_0$ and $A_1$. More concretely,

$$\ast \circ \mu = \mu \circ T,$$

$$\delta \circ \ast = T \circ \delta,$$

where $T = \bigoplus_{i,j=0}^{\infty} T_{i,j}$, and $T_{i,j}(x \otimes y) = y^* \otimes x^*$ for $x \in A_i$ and $y \in A_j$.

A modular splitting of the GNF*-algebra $\mathcal{A}$ consists of a degree one endomorphism $\omega : A \to A$ and a degree $-1$ endomorphism $\alpha : A \to A$ such that they are both left $(A, \mu)$-module and $(A, \delta)$-comodule homomorphisms, and such that $\alpha \circ \omega = \text{Id}_A$.

We call the triple $(\mathcal{A}, \alpha, \omega)$ a split GNF*-algebra.

**Lemma 4.5.** If $\mathcal{A}$ is a GNF*-algebra, then $\varepsilon$ is also a right unit, $\tau$ is a partial right counit, and $\delta$ is a right $(A, \mu)$-module homomorphism.

If $(\alpha, \omega)$ is a modular splitting of $\mathcal{A}$, then $\mathcal{A} = \ker(\alpha) \oplus \text{Im}(\omega)$, both summands are left $(A, \mu, \delta)$-submodules, and $\omega \circ \alpha$ is projection onto $\text{Im}(\omega)$ along $\ker(\alpha)$.

**Proof.** By applying $\ast$ to the equation $\mu(\varepsilon(t) \otimes a) = a$ for $t \in \mathbb{F}$ and $a \in A$, we obtain that $\mu(a^* \otimes \varepsilon(t)) = a^*$, as $\varepsilon(t) \in A_0$ on which $\ast$ acts as the identity, and hence $\mu(a \otimes \varepsilon(t)) = a$ for every $a \in A$.

Similarly, since $\delta_{0,j} \circ \ast = T_{j,0} \circ \delta_{j,0}$,

$$\ast = (\tau \otimes \text{Id}_{A_j}) \circ \delta_{0,j} \circ \ast = (\text{Id}_{A_j} \otimes \tau) \circ T_{j,0} \circ \delta_{j,0} = (\ast \otimes \tau) \circ \delta_{j,0},$$

as $\tau \circ \ast = \tau$ since $\ast$ acts as the identity on $A_0$. Applying $\ast$ to both sides,

$$(\text{Id}_{A_j} \otimes \tau) \circ \delta_{j,0} = \text{Id}_{A_j}.$$
To prove $\delta$ is a right $(A, \mu)$-module homomorphism, we use the sumless Sweedler notation $\delta(x) = x(1) \otimes x(2)$ for $a \in A$. Then the fact that $\delta$ is a left $(A, \mu)$-module homomorphism can be written as

$$\mu(a \otimes b(1)) \otimes b(2) = \mu(a, b)(1) \otimes \mu(a, b)(2).$$

Applying $T$ to both sides,

$$\delta_{(2)}^* \otimes \mu(a \otimes b(1))^* = (\mu(a, b)^{(2)})^*(1) \otimes (\mu(a, b)^{(1)})^*(2).$$

Since $\ast$ is an $(A, \delta)$-antihomomorphism, $(\delta^*)_{(1)}^* \otimes (\delta^*)_{(2)}^* = (\delta_{(2)})^* \otimes (\delta_{(1)})^*$, hence

$$(\delta^*)_{(1)}^* \otimes \mu((\delta^*)_{(2)}^* \otimes a^*) = (\mu(a, b)^{(1)} (1) \otimes (\mu(a, b)^{(1)} (2) = \mu(b^*, a^*) (1) \otimes \mu(b^*, a^*) (2)).$$

Replacing $b^*$ by $a$ and $a^*$ by $b$, we get that the following diagram is commutative:

$$\begin{array}{ccc}
A \otimes A & \xrightarrow{\delta \otimes \text{Id}_A} & A \otimes A \\
\mu & & \text{Id}_A \otimes \mu \\
A & \xrightarrow{\delta} & A \otimes A.
\end{array}$$

For the last part, $\ker(\alpha)$ and $\text{Im}(\omega)$ are left $(A, \mu)$-submodules and left $(A, \delta)$-submodules since $\alpha$ and $\omega$ are left $(A, \mu)$-module and left $(A, \delta)$-comodule homomorphisms. Since $\alpha \circ \omega = \text{Id}_A$, we see that $\alpha$ is surjective and $\omega$ is injective. Furthermore, the endomorphism $\omega \circ \alpha$ is a projection since $(\omega \circ \alpha)(\omega \circ \alpha) = \omega \circ \alpha$. As $\alpha$ is onto, $\text{Im}(\omega \circ \alpha) = \text{Im}(\omega)$, and since $\omega$ is injective, $\ker(\omega \circ \alpha) = \ker(\alpha)$. It follows that $A = \ker(\alpha) \oplus \text{Im}(\omega)$, and that $\omega \circ \alpha$ is projection onto $\text{Im}(\omega)$ along $\ker(\alpha)$. □

**Remark 4.6.** Since $\omega$ is not necessarily $\ast$-invariant, the splitting $A = \ker(\alpha) \oplus \text{Im}(\omega)$ is not $\ast$-invariant in general. If we introduce the notation $\overline{\omega}(a) = \omega(a^*)^*$, then

$$\mu(\overline{\omega}(a) \otimes b) = \mu(b^* \otimes \omega(a^*))^* = (\omega \circ \mu(b^* \otimes a^*))^* = \overline{\omega} \circ \mu(a, b).$$

So, instead of $\omega$, it is $\overline{\omega}$ that is a right $(A, \mu)$-module homomorphism, and similarly for $(A, \delta)$.

**Remark 4.7.** Given a GNF$^*$-algebra, consider the direct system of vector spaces

$$\omega_{i,j} := \omega_{j-1} \circ \cdots \circ \omega_i : V_i \to V_j$$

for $i \leq j$, and let

$$M = \lim_{\longrightarrow} V_i = \prod_{i=0}^{\infty} V_i / \sim,$$

where $x_i \sim x_j$ for $x_i \in V_i$ and $x_j \in V_j$ if and only if there is some $k \geq i, j$ for which $\omega_{i,k}(x_i) = \omega_{j,k}(x_j)$. Since each $\omega_i$ is injective, we can choose $k = \max\{i, j\}$. Furthermore, we can canonically identify $V_i$ with a subspace $M_i$ of $M$, under which $\omega_i$ becomes the embedding $M_i \hookrightarrow M_{i+1}$. For simplicity, we also use the notation $\omega_i$ for this embedding. Using the same identification, $\alpha_i$ descends to a map $\alpha_i : M_i \to M_{i-1}$, which we also denote by $\alpha_i$. Since $\alpha_i \circ \omega_{i-1} = \text{Id}_{M_{i-1}}$, we have $\alpha_i(x) = x$ for every $x \in M_{i-1}$; i.e., $\omega_{i-1} \circ \alpha_i : M_i \to M_i$ is a projection onto $M_{i-1}$.

Next, we show that the $\mu_{i,j}$ descend to a well-defined product $\mu_i : A_i \otimes M \to M$. Given $m \in M$, we define $\mu(a, m)$ for $a \in A_i$ by taking an arbitrary representative $x \in V_j$ of $m$, and we let $\mu(a, b) = \mu_{i,j}(a, x)$. The equivalence class of this product is
indeed, given two representative $x \sim x'$ such that $x \in V_j, x' \in V_k$, and $\omega_{j,k}(x) = x'$, we have

$$\mu_{i,k}(a, \omega_{j,k}(x)) = \omega_{i+j,i+k} \circ \mu_{i,j}(a, x) \sim \mu_{i,j}(a, x)$$

as $\omega$ is a left $(A, \mu)$-module homomorphism.

Similarly, the maps $\delta_{i,j}$ descend to a map $\delta_i : M \to A_i \otimes M$ as $\omega$ is a left $(A, \delta)$-comodule homomorphism. In particular, for $m \in M$, we define $\delta_i(m)$ to be $\delta_{i,n-i}(x)$ for some representative $x \in V_n$ of $m$. We now show this is independent of the choice of $x$. Indeed,

$$\delta_{i,n-i}(x) \sim (\text{Id}_{V_i} \otimes \omega_{n-i}) \circ \delta_i(x) = \delta_{i,n-i} \circ \omega_n(x).$$

It follows that $M$ is a left $A$-module.

By taking the direct limit of $V_i$ along the maps $\overrightarrow{\pi_i}$, we get a right $A$-module $\overrightarrow{M}$. It follows from the previous remark that $*$ provides an anti-isomorphism between $M$ and $\overrightarrow{M}$; in particular, $\overrightarrow{M} \cong M^{\text{op}}$.

**Proposition 4.8.** There is a one-to-one correspondence between $(2 + 1)$-algebras and split $\text{GNF}^*$-algebras.

**Proof.** Given a $(2 + 1)$-algebra consisting of $V_i$, $\mu_{i,j}$, $\delta_{i,j}$, $\varepsilon$, $\tau$, $\alpha_i \omega_i$, * for $i, j \in \mathbb{N}$, let $A_i = V_i, A = \bigoplus_{i \in \mathbb{N}} A_i, \mu = \bigoplus_{i,j \in \mathbb{N}} \mu_{i,j}$, $\delta = \bigoplus_{i,j \in \mathbb{N}} \delta_{i,j}, \alpha = \bigoplus_{i \in \mathbb{N}} \alpha_i$, and $\omega = \bigoplus_{i \in \mathbb{N}} \omega_i$. It is straightforward to check that these satisfy the properties required for a split $\text{GNF}^*$-algebra.

In the opposite direction, suppose we are given a split $\text{GNF}^*$-algebra $(A, \alpha, \omega$).

Let $\pi_{i,j} : A \otimes A \to A_i \otimes A_j$ be the projection. Then we obtain a $(2 + 1)$-algebra by setting $V_i = A_i, \mu_{i,j} = |A_i \otimes A_j|, \delta_{i,j} = \pi_{i,j} \circ \delta, \alpha_i = \alpha|A_i$, and $\omega_i = \omega|A_i.
Furthermore, $\omega$ is a left $(A, \delta)$-comodule homomorphism as $\delta$ is a right $(A, \mu)$-module homomorphism according to Lemma 4.3

$$\delta_{i,j+1} \circ \omega_{i+j} = \delta_{i,j+1} \circ \mu_{i,j+1}(x, w) = (\text{Id}_A \otimes \mu_{i,1}) \circ (\delta_{i,j} \otimes \text{Id}_A) (x \otimes w) = (\text{Id}_A \otimes \omega_j) \circ \delta_{i,j}(x).$$

Similarly, if we are given the splitting $(\alpha, \omega)$ and let $\lambda = \tau \circ \alpha_1$, then

$$(\text{Id}_{A_{i-1}} \otimes \lambda) \circ \delta_{i-1,1} = (\text{Id}_{A_{i-1}} \otimes \tau) \circ (\text{Id}_{A_{i-1}} \otimes \alpha_1) \circ \delta_{i-1,1} = (\text{Id}_{A_{i-1}} \otimes \tau) \circ \delta_{i-1,0} \circ \alpha_i = \alpha_i$$

as $\alpha$ is a left $(A, \delta)$-comodule homomorphism and $\tau$ is a counit. So $\lambda \in A_i^+$ completely determines $\alpha_i$ for every $i \in \mathbb{N}$ via the formula

$$\alpha_i(x) := (\text{Id}_{A_{i-1}} \otimes \lambda) \circ \delta_{i-1,1}.$$

The $\alpha$ defined this way is a left $(A, \mu)$-module homomorphism by the Frobenius condition:

$$\alpha_{i+1} \circ \mu_{i,j} = (\text{Id}_{A_{i+1-1}} \otimes \lambda) \circ \delta_{i+1,j} \circ \mu_{i,j} = (\text{Id}_{A_{i+1-1}} \otimes \lambda) \circ (\mu_{i,j-1} \otimes \text{Id}_A) \circ (\text{Id}_A \otimes \delta_{j-1,1}) = \mu_{i,j-1} \circ (\text{Id}_A \otimes \alpha_j).$$

Similarly, $\alpha$ is a left $(A, \delta)$-comodule homomorphism by the coassociativity of $\delta$:

$$\delta_{i,j-1} \circ \alpha_{i,j} = \delta_{i,j-1} \circ (\text{Id}_{A_{i+j-1-1}} \otimes \lambda) \circ \delta_{i+1,j-1} = (\text{Id}_A \otimes \text{Id}_{A_{j-1}} \otimes \lambda) \circ (\delta_{i,j-1} \otimes \text{Id}_A) \circ \delta_{i+1,j-1} = (\text{Id}_A \otimes \text{Id}_{A_{j-1}} \otimes \lambda) \circ (\text{Id}_{A_i} \otimes \delta_{j-1,1}) \circ \delta_{i,j} = (\text{Id}_{A_i} \otimes \alpha_j) \circ \delta_{i,j}.$$

Finally, consider the condition $\alpha_{i+1} \circ \omega_i = \text{Id}_{A_i}$. Since

$$\alpha_i \circ \omega_i(x) = \alpha_i \circ \mu_{i,1}(x \otimes w) = \mu_{i,0}(x \otimes \alpha_1(w)),$$

this is equivalent to having $\mu_{i,0}(x \otimes \alpha_1(w)) = x$ for every $i \in \mathbb{N}$ and $x \in A_i$. In particular, if we set $i = 0$ and $x = 1$, we must have $\alpha_1(w) = 1$, and clearly this is also sufficient. But $\alpha_1(w) = (\text{Id}_{A_0} \otimes \lambda) \circ \delta_{0,1}(w)$, the condition $\alpha_{i+1} \circ \omega_i = \text{Id}_{A_i}$ is equivalent to

$$(\text{Id}_{A_0} \otimes \lambda) \circ \delta_{0,1}(w) = 1.$$ 

This concludes the proof of the lemma.

Remark 4.10. From now on, we use the notation $(\alpha, \omega)$ and $(w, \lambda)$ interchangeably for a modular splitting. Notice that the polynomial algebra $\mathbb{F}[w]$ is a subalgebra of $(A, \mu)$, and $\mathbb{F}[\lambda]$ is a subalgebra of $(A^*, \delta^*)$.

It is also worth noting that if the GNF*-algebra $A$ arises from a TQFT $F$, while $\omega_i$ geometrically corresponds to adding a 1-handle to $\Sigma_i$ along $P_i$, the operation $\mu_{i,1}$ amounts to connected summing $\Sigma_i$ with $T^2$, and $w \in F(T^2)$.

A corollary of Proposition 4.8 is that, given a $(2 + 1)$-representation

$$\{ \rho_i : \mathcal{M}_i \to \text{End}(V_i) \mid i \in \mathbb{N} \}$$

on a $(2 + 1)$-algebra $\mathcal{A}$, we can instead view it as a sequence of representations

$$\{ \rho_i : \mathcal{M}_i \to \text{End}(A_i) \mid i \in \mathbb{N} \}$$

for the split GNF*-algebra $(A, \alpha, \omega)$ corresponding to $\mathcal{A}$. Our next goal is to translate the $(2 + 1)$-representation axioms to this setting. Lemma 4.9 allows us to simplify and localize some of the conditions. In particular, let $(w, \lambda) \in A_1 \times A_1^*$ be the pair corresponding to the splitting $(\alpha, \omega)$. 
First, recall that the map $\alpha_i$ is $\text{MCG}(\Sigma_i, l_i)$-equivariant, $\omega_i$ is $\text{MCG}(\Sigma_i, \mathbb{P}_i) \cong \text{MCG}(\Sigma_{i+1}, m_{i+1})$-equivariant, $\mu_{i,j}$ is $\text{MCG}(\Sigma_i \sqcup \Sigma_j, \mathbb{P}_{i,j}) \cong \text{MCG}(\Sigma_{i+j}, s_i)$-equivariant, and $\delta_{i,j}$ is $\text{MCG}(\Sigma_{i+j}, s_i)$-equivariant.

**Remark 4.11.** The $\text{MCG}(\Sigma_{i+1}, m_{i+1})$-equivariance of $\omega_i$ does not follow from the $\text{MCG}(\Sigma_{i+1}, s_i)$-equivariance of $\mu_{i,1}$, even though $\omega_i(x) = \mu_{i,1}(x, w)$. Indeed, if $d \in \text{Diff}(\Sigma_{i+1}, m_{i+1})$, then $d^{m_{i+1}} \in \text{Diff}(\Sigma_i, \mathbb{P}_i)$, and the $\text{MCG}(\Sigma_{i+1}, m_{i+1})$-equivariance of $\omega_i$ translates to

$$\rho_{i+1}(d) \circ \mu_{i,1}(x, w) = \mu_{i,1}(\rho_i(x, w)).$$

But $d$ does not necessarily fix the isotopy class of $s_i$, and hence we cannot apply the appropriate invariance property of $\mu_{i,1}$. Consequently, we need to keep the equivariance assumptions on $\omega_i$ and $\alpha_i$.

Consider equation (4.17); i.e.,

$$\rho_{i+1}(t_{i+1}) \circ \omega_i = \omega_i,$$

where $t_{i+1}(m_{i+1}) = -m_{i+1}$ and $t_{i+1}^{m_{i+1}} \in \text{Diff}_0(\Sigma_i)$. Since $t_{i+1}$ fixes $s_i$ and $t_{i+1}^{m_{i+1}}$ is isotopic to $\text{Id}_{\Sigma_i} \cup t_1$, we can apply the $\text{MCG}(\Sigma_{i+1}, s_i)$-equivariance of $\mu_{i,1}$ to obtain

$$\rho_{i+1}(t_{i+1}) \circ \omega_i(x) = \rho_{i+1}(t_{i+1}) \circ \mu_{i,1}(x, w) = \mu_{i,1}(x, \rho_{i}(t_1)(w)).$$

In particular, equation (4.17) is equivalent to

$$\mu_{i,1}(x, \rho_{i}(t_1)(w)) = \mu_{i,1}(x, w)$$

for every $i \in \mathbb{N}$ and $x \in A_i$. In particular, if we take $i = 0$ and $x = 1$, it is necessary to have

$$\rho_{1}(t_1)(w) = w,$$

and clearly this is also sufficient, hence equivalent to equation (4.17).

Now look at equation (4.19); i.e.,

$$\alpha_i \circ \rho_{i}(r_i) = \alpha_i,$$

where $r_{i}(l_i) = -l_i$ and $r_{i}^{l_i} \in \text{Diff}_0(\Sigma_{i-1})$. Using the definition of $\alpha_i$, this is equivalent to

$$\text{Id}_{A_{i-1}} \otimes \lambda \circ \delta_{i-1,1} \circ \rho_{i}(r_i) = \text{Id}_{A_{i-1}} \otimes \lambda \circ \delta_{i-1,1}.$$

Using the $\text{MCG}(\Sigma_{i-1}, s_{i-1})$-equivariance of $\delta_{i-1,1}$ and that $r_{i}^{l_i} \approx \text{Id}_{\Sigma_{i-1}} \# r_1$, this is further equivalent to

$$\text{Id}_{A_{i-1}} \otimes (\lambda \circ \rho_{i}(r_1)) \circ \delta_{i-1,1} = \text{Id}_{A_{i-1}} \otimes \lambda \circ \delta_{i-1,1}.$$

Notice that $r_1 = t_1$. If we set $i = 1$ and apply $\tau \otimes \text{Id}_{A_1}$ to both sides, we obtain the necessary and sufficient condition

$$\lambda \circ \rho_{1}(t_1) = \lambda.$$

Next, consider condition (4.1); i.e.,

$$\rho_{i+2}(S_{i+2}) \circ \omega_{i+1} \circ \omega_i = \omega_{i+1} \circ \omega_i,$$

where $S_{i+2}$ swaps $m_{i+1}$ and $m_{i+2}$, and $S_{i+2}^{m_{i+1}, m_{i+2}} \in \text{Diff}_0(\Sigma_i)$. By Lemma 4.9 and the associativity of $\mu$, this is equivalent to

$$\rho_{i+2}(S_{i+2}) \circ \mu_{i+1,1}(\mu_{i,1}(x, w), w) = \rho_{i+2}(S_{i+2}) \circ \mu_{i,2}(x, \mu_{i,1}(w, w))$$

$$= \mu_{i,2}(x, \mu_{i,1}(w, w)).$$
for every \(x \in A_i\). Since \(\mu_{i,2}\) is \(\text{MCG}(\Sigma_{i+2}, s_i)\)-equivariant and \(S_{i+2}\) fixes \(s_i\) point-wise, in fact, \(S_{i+2}^\ast = \text{Id}_{\Sigma_i} \cup S_2\), this condition can be expressed as

\[
\mu_{i,2}(x, \rho_2(S_2) \circ \mu_{1,1}(w, w)) = \mu_{i,2}(x, \mu_{1,1}(w, w))
\]

for every \(i \in \mathbb{N}\) and \(x \in A_i\). In particular, if we set \(i = 0\) and \(x = 1\), it is necessary to have

\[
\rho_2(S_2) \circ \mu_{1,1}(w, w) = \mu_{1,1}(w, w),
\]

but this is also clearly sufficient. Now consider the diffeomorphism \(d := \iota_2 \circ S_2 \circ \iota_2\), this swaps the meridians \(m_0\) and \(m_1\) of \(\Sigma_2\), but fixes \(m_2\), hence lies in \(\text{Diff}(\Sigma_2, m_2)\). Furthermore, \(d^\ast\) is the automorphism \(t_1\) of the torus, and we have already seen that \(\rho_1(t_1)(w) = w\). Hence,

\[
\rho_2(d) \circ \omega_1(w) = \omega_1(\rho_1(t_1)(w)) = \omega_1(w) = \mu_{1,1}(w, w).
\]

On the other hand, \(\rho_2(d) = *_2 \circ \rho_2(S_2) \circ *_2\), hence the left-hand side of the above equation is \(*_2 \circ \rho_2(S_2) \circ *_2 \circ \omega_1(w)\). But \(*_1 = \text{Id}_{A_1}\), since \(\iota_1\) is isotopic to \(\text{Id}_{T^2}\), hence

\[
*_2 \circ \omega_1(w) = *_2 \circ \mu_{1,1}(w, w) = \mu_{1,1}(w^*, w^*) = \mu_{1,1}(w, w).
\]

It follows that

\[
\rho_2(S_2) \circ \mu_{1,1}(w, w) = *_2 \circ \mu_{1,1}(w, w) = \mu_{1,1}(w, w),
\]

and property (\ref{4.1}) is redundant.

Similarly, we can simplify condition (\ref{4.5}); i.e.,

\[
\alpha_{i-1} \circ \alpha_i \circ \rho_i(L_i) = \alpha_{i-1} \circ \alpha_i,
\]

where \(L_i\) swaps \(l_{i-1}\) and \(l_i\), and \(L_i^{l_{i-1}, l_i} \in \text{Diff}_0(\Sigma_{i-2})\). By the coassociativity of \(\delta\), and since \(\delta_{i-2,2}\) is \(\text{MCG}(\Sigma_i, s_{i-2})\)-equivariant and \(L_i^{s_{i-2}} = \text{Id}_{\Sigma_{i-2}} \cup L_2\), the left-hand side is

\[
(\text{Id}_{A_{i-2}} \otimes \lambda) \circ \delta_{i-2,1} \circ (\text{Id}_{A_{i-2}} \otimes \lambda) \circ \delta_{i-1,1} \circ \rho_i(L_i) =
(\text{Id}_{A_{i-2}} \otimes \lambda) \circ (\text{Id}_{A_{i-2}} \otimes \text{Id}_{A_1} \otimes \lambda) \circ \delta_{i-2,1} \circ \delta_{i-1,1} \circ \rho_i(L_i) =
(\text{Id}_{A_{i-2}} \otimes \lambda \otimes \lambda) \circ (\text{Id}_{A_{i-2}} \otimes \delta_{1,1}) \circ \delta_{i-2,2} \circ \rho_i(L_i) =
(\text{Id}_{A_{i-2}} \otimes \lambda \otimes \lambda) \circ (\text{Id}_{A_{i-2}} \otimes \delta_{1,1}) \circ (\text{Id}_{A_{i-2}} \otimes \rho_2(L_2)) \circ \delta_{i-2,2}.
\]

Since \(L_2\) is isotopic to \(\iota_2\), we have \(\rho_2(L_2) = *_2\), and condition (\ref{4.5}) is equivalent to

\[
(\text{Id}_{A_{i-2}} \otimes \lambda \otimes \lambda) \circ (\text{Id}_{A_{i-2}} \otimes \delta_{1,1}) \circ (\text{Id}_{A_{i-2}} \otimes *_2) \circ \delta_{i-2,2} =
(\text{Id}_{A_{i-2}} \otimes \lambda \otimes \lambda) \circ (\text{Id}_{A_{i-2}} \otimes \delta_{1,1}) \circ \delta_{i-2,2}.
\]

In particular, if we set \(i = 2\) and apply \(\tau \otimes \text{Id}_{A_2}\) to both sides, we get the necessary and sufficient condition

\[
(\lambda \otimes \lambda) \circ \delta_{1,1} \circ *_2 = (\lambda \otimes \lambda) \circ \delta_{1,1}.
\]

However, since \(\delta_{1,1} \circ *_2 = T \circ \delta_{1,1}\), and because \(*_1 = \text{Id}_{A_1}\), as \(\iota_1\) is isotopic to the identity, the above equation automatically follows from the GNF*-algebra axioms, and from the \(\text{MCG}(\Sigma_i, s_{i-2})\)-equivariance of \(\delta_{i-2,2}\).

Equation (\ref{4.0}) is dual to equation (\ref{4.5}). It states that

\[
\delta_{i,j} \circ \alpha_n \circ \rho_n(u_{n,i+1}) = \delta_{i,j} \circ \alpha_n,
\]
where \( u_{n,i+1} \in \text{Diff}(\Sigma_n) \) swaps \( s_{i+1} \# l_n \) and \( l_n \), and \( u^{s_{i+1} \# l_n} \) is isotopic to the identity. Using the coassociativity of \( \delta \), the left-hand side becomes

\[
\delta_{i,j} \circ (\text{Id}_{A_{i+j}} \otimes \lambda) \circ \delta_{i+j,1} \circ \rho_n(u_{n,i+1}) = (\text{Id}_{A_i} \otimes \text{Id}_{A_j} \otimes \lambda) \circ \delta_{i,j} \circ \rho_n(u_{n,i+1}) = (\text{Id}_{A_i} \otimes \text{Id}_{A_j} \otimes \lambda) \circ (\text{Id}_{A_i} \otimes \delta_{j,1} \circ \rho_n(u_{n,i+1})).
\]

Note that \( u_{n,i+1} \) fixes \( s_i \) and \( u_{n,i+1}^{s_i} = \text{Id}_{\Sigma_i} \sqcup u_{j+1,1}. \) Hence, by the MCG(\( \Sigma_n, s_i \))-equivariance of \( \delta_{i,j+1} \), the left-hand side further equals

\[
(\text{Id}_{A_i} \otimes \delta_{j,1} \circ \rho_{j+1}(u_{j+1,1})) \circ \delta_{i,j+1} = (\text{Id}_{A_i} \otimes \delta_{j,1} \circ \rho_{j+1}(u_{j+1,1})) \circ \delta_{i,j+1}.
\]

If we set \( i = 0 \), and apply \( \tau \otimes \text{Id}_{A_j} \) to both sides, we obtain the necessary and sufficient condition \( \alpha_{j+1} \circ \rho_{j+1}(u_{j+1,1}) = \alpha_{j+1} \), or equivalently,

\[
\alpha_i \circ \rho_i(u_{i,1}) = \alpha_i
\]

for \( i > 1 \). However, the \( i = 2 \) case holds automatically. Indeed, as \( u_{2,1} \) fixes \( s_i \) and \( u_{2,1}^{s_i} \approx \text{Id}_{\Sigma_i} \sqcup \text{Id}_{\Sigma_i} \), the MCG(\( \Sigma_2, s_i \))-equivariance of \( \delta_{1,1} \) implies that

\[
(\text{Id}_{A_i} \otimes \lambda) \circ \delta_{1,1} \circ \rho_{2}(u_{2,1}) = (\text{Id}_{A_i} \otimes \lambda) \circ \delta_{1,1}.
\]

We now simplify equation (4.23), i.e.,

\[
\rho_{n+1}(h_{n+1,i+1}) \circ \omega_n \circ \mu_{i,j}(x, y) = \omega_n \circ \mu_{i,j}(x, y),
\]

where \( h_{n+1,i+1} \) swaps \( s_{i+1} \# m_{n+1} \) with \( m_{n+1} \), and \( h_{n+1,i+1}^{s_{i+1} \# m_{n+1}} \) is isotopic to the identity. Using our formula for \( \omega_n \), the above equation becomes equivalent to

\[
\rho_{n+1}(h_{n+1,i+1}) \circ \mu_{n,1}(\mu_{i,j}(x, y), w) = \mu_{n,1}(\mu_{i,j}(x, y), w).
\]

As \( h_{n+1,i+1} \) fixes \( s_i \), using the associativity of \( \mu \) and the MCG(\( \Sigma_{i+1}, s_i \))-equivariance of \( \mu_{i,j+1} \), this is further equivalent to

\[
\mu_{i,j+1}(x, \rho_{j+1}(h_{j+1,1}) \circ \mu_{j,1}(y, w)) = \mu_{i,j+1}(x, \mu_{j,1}(y, w)).
\]

In particular, if we set \( i = 0 \) and \( x = 1 \), we obtain the necessary and sufficient condition

\[
\rho_{j+1}(h_{j+1,1}) \circ \mu_{j,1}(y, w) = \mu_{j,1}(y, w),
\]

or equivalently,

\[
\rho_{j+1}(h_{j+1,1}) \circ \omega_j = \omega_j
\]

for every \( j \in \mathbb{N} \). Note that this automatically holds on \( \text{Im}(\mu_{1,j-1}) \subset A_j \). Indeed, if \( y = \mu_{1,j-1}(a, b) \), then

\[
\rho_{j+1}(h_{j+1,1}) \circ \mu_{j,1}(y, w) = \rho_{j+1}(h_{j+1,1}) \circ \mu_{1,j}(a, \mu_{j-1,1}(b, w)) = \mu_{j,1}(a, \rho_{j}(h_{j,0}) \circ \mu_{j-1,1}(b, w)) = \mu_{j,1}(y, w)
\]

by the associativity of \( \mu \), together with the fact that \( h_{j+1,1} \) fixes the curve \( s_1 \) and \( h_{j+1,1}^{s_1} = \text{Id}_{\Sigma_j} \sqcup h_{j,0} \), where \( h_{j,0} \) is isotopic to \( \text{Id}_{\Sigma_j} \), and since \( \mu_{1,j} \) is MCG(\( \Sigma_{j+1}, s_1 \))-equivariant. As \( A \) is unital, \( \text{Im}(\mu_{1,j}) = A_j \), and condition (1.22) automatically holds for \( j = 1 \).

Finally, consider equation (4.9), i.e.,

\[
\alpha_{i+1} \circ \rho_{i+1}(L_{i+1}) \circ \omega_i = \omega_{i-1} \circ \alpha_i.
\]

We first remark that if we apply \( \alpha_i \) to both sides, the resulting equation follows from the existing properties by equation (4.5). Secondly, we prove that this automatically
holds on $\text{Im}(\mu_{i-1,1})$, and hence for $i = 1$ as in the previous case. Indeed, suppose that $x = \mu_{i-1,1}(a, b)$. Then

$$\begin{align*}
\alpha_{i+1} \circ \rho_{i+1}(L_{i+1}) \circ \omega_i(x) &= \alpha_{i+1} \circ \mu_{i-1,2}(a, \rho_2(L_2) \circ \mu_{1,1}(b, w)) = \\
\alpha_{i+1} &\circ 
(\text{Id}_{A_i} \otimes \lambda) \circ \delta_{i-1,1} \circ \mu_{i-1,2}(a, \mu_{1,1}(w, b)) = \\
(\text{Id}_{A_i} \otimes \lambda) &\circ \mu_{i-1,1}(a, \mu_{1,1}(w, b)) = \\
(\mu_{i-1,1} \otimes \lambda) &\circ (a \otimes [\delta_{1,1} \circ \mu_{1,1}(w, b)]) = \\
(\mu_{i-1,1} \otimes \lambda) &\circ (a \otimes (b_1 \otimes b_2)) = \\
(\mu_{i-1,1} \otimes \lambda) &\circ (a \otimes b_1) \circ b_2 = \\
\lambda(b_2)(a \cdot w \cdot b(1)).
\end{align*}$$

Here we used that $\mu_{i-1,2}$ is $\text{MCG}(\Sigma_{i+1}, s_{i-1})$-equivariant, $L_{i+1}^\mu = \text{Id}_{\Sigma_{i-1}} \sqcup L_2$, that $\rho_2(L_2) = *_2$, and the Frobenius condition twice. Furthermore, $\delta_{0,1}(b) = b(1) \otimes b(2)$ in sumless Sweedler notation, and $\cdot$ stands for the algebra multiplication $\mu$.

On the other hand, the right-hand side of equation (4.9) becomes

$$\begin{align*}
\omega_{i-1} \circ \alpha_i \circ \mu_{i-1,1}(a, b) &= \\
\omega_{i-1} &\circ (\text{Id}_{A_{i-1}} \otimes \lambda) \circ \delta_{i-1,1} \circ \mu_{i-1,1}(a, b) = \\
\omega_{i-1} \circ (\text{Id}_{A_{i-1}} \otimes \lambda) &\circ (\mu_{i-1,0} \otimes \text{Id}_{A_1}) \circ (\text{Id}_{A_{i-1}} \otimes \delta_{0,1})(a, b) = \\
\omega_{i-1} &\circ (\mu_{i-1,0} \otimes \lambda) \circ (a \otimes \delta_{0,1}(b)) = \\
[(\mu_{i-1,1} \otimes \lambda) &\circ (a \otimes b_1) \otimes b_2]) \cdot w = \\
\lambda(b_2)(a \cdot b(1) \cdot w).
\end{align*}$$

The claim follows once we observe that $b(1) \cdot w \in A_1$, hence $b(1) \cdot w = (b(1) \cdot w)^* = w^* \cdot b(1) = w \cdot b(1)$ since $*_0 = \text{Id}_{A_0}$ and $*_1 = \text{Id}_{A_1}$.

**Definition 4.12.** Let $(\mathcal{A}, \alpha, \omega)$ be a split GNF*-algebra. Then a sequence of homomorphisms

$$\{ \rho_i : \mathcal{M}_i \to \text{End}(A_i) \mid i \in \mathbb{N} \}$$

is called a mapping class group representation on $\mathcal{A}$ if it satisfies the following properties:

The map $\alpha_i$ is $\text{MCG}(\Sigma_i, l_i)$-equivariant, $\omega_i$ is $\text{MCG}(\Sigma_i, \overline{P}_i)$-equivariant, $\mu_{i,j}$ is $\text{MCG}(\Sigma_i \sqcup \Sigma_j, \overline{P}_{i,j})$-equivariant, and $\delta_{i,j}$ is $\text{MCG}(\Sigma_{i+j}, s_i)$-equivariant. In addition, $*_i | A_i = \rho(i)$, and the representations $\rho_i$ satisfy the following conditions:

$$\begin{align*}
\rho_1(t_1)(w) &= w, \\
\lambda \circ \rho_1(t_1) &= \lambda, \\
\rho_{i+1}(h_{i+1,1}) \circ \omega_i &= \omega_i \text{ for } i > 1, \\
\alpha_i \circ \rho_{i}(u_{i,1}) &= \alpha_i \text{ for } i > 2, \\
\alpha_{i+1} \circ \rho_{i+1}(L_{i+1}) \circ \omega_i &= \omega_{i-1} \circ \alpha_i \text{ for } i > 1, \\
\alpha_{n+1} \circ \rho_{n+1}(\sigma_{n+1,i}) \circ \omega_n &= \mu_{i,n-i} \circ \delta_{i,n-i} \text{ for } n \in \mathbb{N} \text{ and } 0 \leq i \leq n,
\end{align*}$$

where $w = \alpha_1(1)$ and $\lambda = \tau \circ \alpha_1$ are as in Lemma 4.9 and

- $t_i$ is $\pi$-rotation of the standard $\Sigma_i$ in $\mathbb{R}^3$ with center at $0$ about the $z$-axis,
- $t_1$ is $\pi$-rotation of the standard torus in $\mathbb{R}^3$ about the $x$-axis,
• \( h_{i+1,1} \in \text{Diff}(\Sigma_{i+1}) \) swaps \( s_1 \# m_{i+1} \) and \( m_{i+1} \), and \( h_{i+1,1}^{s_1 \# m_{i+1}, m_{i+1}} \) is isotopic to the identity,
• \( u_{i,1} \in \text{Diff}(\Sigma_i) \) swaps \( s_1 \# l_i \) and \( l_i \), and \( u_{i,1}^{s_1 \# l_i, l_i} \) is isotopic to the identity,
• \( L_i \in \text{Diff}(\Sigma_i) \) swaps \( l_i \) and \( l_{i-1} \), and \( L_i^{l_i, l_{i-1}} \in \text{Diff}_0(\Sigma_{i-1}) \),
• \( \sigma_{n+1,i} \in \text{Diff}(\Sigma_{n+1}) \) satisfies \( \sigma_{n+1,i}(s_1 \# m_{n+1}) = l_{n+1} \), and \( \sigma_{n+1,i}^{s_1 \# m_{n+1}} \in \text{Diff}_0(\Sigma_n) \).

With these definitions in place, the classification of (2 + 1)-dimensional TQFTs becomes the following, which is Theorem 1.1 from the introduction.

**Theorem.** There is a bijective correspondence between (2 + 1)-dimensional TQFTs and split GNF*-algebras endowed with a mapping class group representation.

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