Research Article

New Estimates of Solution to Coupled System of Damped Wave Equations with Logarithmic External Forces

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In the present paper, we consider an initial boundary value problem with damping terms and logarithmic sources, for $x \in \mathbb{R}^n$, $t > 0$

$$\begin{align*}
\partial_t^2 v_1 + b v_1 &= \phi(x) \Delta v_1 - \int_0^t \omega_1(t-p)v_1(p)dp + kv_1 \ln |v_1|, \\
\partial_t^2 v_2 + b v_1 &= \phi(x) \Delta v_2 - \int_0^t \omega_2(t-p)v_2(p)dp + kv_2 \ln |v_2|, \\
v_1(x,0) &= v_{10}(x), v_2(x,0) = v_{20}(x), \\
\partial_t v_1(x,0) &= v_{11}(x), \partial_t v_2(x,0) = v_{21}(x),
\end{align*}$$

(1)

where $b > 0$, $n \geq 3$, and $k$ is a small positive real number. The density function $\rho(x) > 0$, for all $x \in \mathbb{R}^n$, where $(\phi(x))^{-1} = 1/\phi(x) \equiv \rho(x)$, under homogeneous Dirichlet boundary conditions.

A related initial boundary value problem was considered by Han in [1]:

$$\begin{align*}
\partial_t^2 u + \Delta u &= \mu u, \quad x \in \Omega, t \in [0,T), \\
u(x,0) &= u_0(x)u_1(x,0) = u_1(x), \quad x \in \Omega, \\
u(x,t) &= 0, \quad x \in \partial \Omega, t \in [0,T),
\end{align*}$$

(2)

and the global existence of weak solutions was proved, for all $(u_0, u_1) \in H_0^1 \times L^2$ in $\mathbb{R}^3$. The weak and strong damping terms in logarithmic wave equation

$$\begin{align*}
\partial_t^2 u + \Delta u - \omega \partial_t u &= u \ln |u|, \quad x \in \Omega, t \in (0,\infty), \\
u(x,0) &= u_0(x)u_1(x,0) = u_1(x), \quad x \in \Omega, \\
u(x,t) &= 0, \quad x \in \partial \Omega, t \in (0,\infty)
\end{align*}$$

(3)

were introduced by Lian and Xu [2]. The global existence, asymptotic behavior, and blowup at three different initial energy levels (subcritical energy $E(0) < d$, critical initial energy $E(0) = d$, and the arbitrary high initial energy $E(0) > 0(\omega = 0)$) were proved. In [3], Al-ghanbari established explicit and general energy decay results for the problem

$$\begin{align*}
\partial_t^2 u + \Delta^2 u + u - \int_0^t g(t-s)\Delta^2 uds &= ku \ln |u|, \quad x \in \Omega, t \in (0,\infty), \\
u(x,0) &= u_0(x)u_1(x,0) = u_1(x), \quad x \in \Omega, \\
u(x,t) &= \frac{\partial u}{\partial n} = 0, \quad x \in \partial \Omega, t \in (0,\infty).
\end{align*}$$

(4)
When the density $\phi(x) \neq 1$, Papadopoulos and Stavrakakis [4] considered the following semilinear hyperbolic initial value problem:

$$ u_{tt} + \phi(x) \Delta u + \delta u_t + \lambda f(u) = \eta(x), (x, t) \in \mathbb{R}^n \times \mathbb{R}^+. \quad (5) $$

The authors proved local existence of solutions and established the existence of a global attractor in the energy space $\mathcal{G}^{1,2}(\mathbb{R}^n) \times L^2_2(\mathbb{R}^n)$, where $(\phi(x))^{-1} = g(x)$. Miyasita and Zennir [5] proved the global existence of the following viscoelastic wave equation:

$$
\begin{cases}
  u_{tt} + \mathcal{A}u + \omega \Delta u_t - \int_0^t g(t-s)\Delta u(s)\,d\sigma = u|u|^{p-1}, & x \in \mathbb{R}^n, t > 0, \\
  u(x, 0) = u_0(x), & x \in \mathbb{R}^n, \\
  u_t(x, 0) = u_1(x), & x \in \mathbb{R}^n.
\end{cases}
$$

The novelty of our work lies primarily in the use of a new condition between the weights of damping the external forces, where we outline the effects of the damping term with less conditions on the viscoelastic terms. We also propose logarithmic nonlinearities in sources and used classical arguments to estimate them. These nonlinearities make the problem very interesting in the application point of view. In order to compensate for the lack of classical Poincaré’s inequality in $\mathbb{R}^n$, we use the weighted function to use generalized Poincaré’s one. The main contribution of this paper is introduced in Theorem 8, where we obtain decay estimates with positive initial energy under a general assumption on the kernel. The rest of the paper is outlined as follows. In Section 2, we give some preliminaries and our main results. In Section 3, we will prove the general decay of energy to the problem.

2. Preliminaries and Main Results

We state some assumptions and definitions that will be useful in this paper. With respect to the relaxation functions $\omega_1, \omega_2$, we assume for $i = 1, 2$,

1. $\omega_1, \omega_2 \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfy for any $t \geq 0, \omega_i(0) > 0, \int_0^\infty \omega_i(p)\,dp = l_{i0} < \infty, 1 - \int_0^\infty \omega_i(p)\,dp = l_i > 0 \quad (7)$

2. There exist nonincreasing differentiable functions $\xi_1, \xi_2 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ that satisfy

$$ \xi_i(t) > 0, \omega_i(t) \leq -\xi_i(t)\omega_i(t) \text{ for } t \geq 0 \quad (8) $$

3. The function $\rho : \mathbb{R}^n \rightarrow \mathbb{R}^*_+, \rho(x) \in C^{\gamma}(\mathbb{R}^n)$ with $\gamma \in (0, 1)$ and $\rho \in L^s(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$, where $s = 2n/2n - qn + 2q$

Definition 1 (see [4]). We define the function spaces of our problem and their norms as follows:

$$ \mathcal{H} = \left\{ v \in L^{2n/(n-2)}(\mathbb{R}^n), \nabla v \in \left( L^2(\mathbb{R}^n) \right)^n \right\}. \quad (9) $$

Let the function spaces $\mathcal{H}$ as the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\| v \|_{\mathcal{H}} = (v, v)^{1/2}_{\mathcal{H}}$ for the inner product:

$$ (v, w)_{\mathcal{H}} = \int_{\mathbb{R}^n} \nabla v \cdot \nabla w \,dx, \quad (10) $$

and $L^2_\rho(\mathbb{R}^n)$ be defined with the norm $\| v \|_{L^2_\rho} = (v, v)^{1/2}_{L^2_\rho}$ for

$$ (v, w)_{L^2_\rho} = \int_{\mathbb{R}^n} \rho \| v \|_q \,dx. \quad (11) $$

For general $q \in [1, +\infty)$, $L^q_\rho(\mathbb{R}^n)$ is the weighted $L^q$ space under a weighted norm

$$ \| v \|_{L^q_\rho} = \left( \int_{\mathbb{R}^n} \rho |v|^q \,dx \right)^{1/q}. \quad (12) $$

To distinguish the usual $L^q$ space from the weighted one, we denote the standard $L^q$ norm by

$$ \| v \|_q = \left( \int_{\mathbb{R}^n} |v|^q \,dx \right)^{1/q}. \quad (13) $$

We denote an eigenpair $\{(\lambda_j, w_j)\}_{j \in \mathbb{N}} \subset \mathbb{R} \times \mathcal{H}$ by

$$ -\phi(x)\Delta w_j = \lambda_j w_j, x \in \mathbb{R}^n \quad (14) $$

for any $j \in \mathbb{N}$. Then, according to [4],

$$ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots \uparrow +\infty \quad (15) $$

holds and $\{w_j\}$ is a complete orthonormal system in $\mathcal{H}$.

Now, we introduce Sobolev embedding and generalized Poincaré’s inequalities.

Lemma 2. Let $\rho$ satisfy (H3). Then, there are positive constants $C_{\xi} > 0$ and $C_{\rho} > 0$ that depend only on $n$ and $\rho$ such
for \( v \in \mathcal{H} \).

**Lemma 3** (see Lemma 2.2 in [6]). Let \( p \) satisfy (H3). Then, we have

\[
\|v\|_{L^q_p} \leq C_q \|v\|_{\mathcal{H}},
\]

\[
C_q = C_{\omega}\|p\|_{L^q_p}^{1/q}
\]

for \( v \in \mathcal{H} \), where \( s = 2n/(2n - qn + 2q) \) for \( 1 \leq q \leq 2n/(n - 2) \).

The energy functional associated to problem (1) is given by

\[
\mathcal{E}(t) = \frac{1}{2} \sum_{i=1}^{2} \|\partial_1 v_i(t)\|_{L^2_p}^2 + \frac{1}{2} \sum_{i=1}^{2} \left( 1 - \int_0^t \omega_i(p)dp \right) \|\nabla v_i(t)\|^2 + \|\partial_2 \omega \|_{L^2_p} \|v(t)\|^2,
\]

\[
\omega(t) \leq c \left( 1 + \int_0^t \gamma(p) \omega(p) \ln \gamma(p) dp \right), \quad 0 \leq t \leq T,
\]

then

\[
\omega(t) \leq c \exp \left( c \int_0^t \gamma(p) dp \right), \quad 0 \leq t \leq T.
\]

We define the following functionals

\[
J(t) = \frac{1}{2} \sum_{i=1}^{2} \left( 1 - \int_0^t \omega_i(p)dp \right) \|\nabla v_i(t)\|^2 + \frac{1}{2} \sum_{i=1}^{2} (\omega_i \nabla v_i)(t) - \frac{k^2}{2} \sum_{i=1}^{2} \int_{\mathbb{R}^n} \rho(x)v_i^2 \ln |v_i| dx + \frac{k}{4} \sum_{i=1}^{2} \|v_i\|_{L^2_p}^2,
\]

\[
I(t) = \frac{1}{2} \sum_{i=1}^{2} \left( 1 - \int_0^t \omega_i(p)dp \right) \|\nabla v_i(t)\|^2 + \frac{1}{2} \sum_{i=1}^{2} (\omega_i \nabla v_i)(t) - \frac{k^2}{2} \sum_{i=1}^{2} \int_{\mathbb{R}^n} \rho(x)v_i^2 \ln |v_i| dx.
\]

Then, we introduce

\[
W = \{(v_1, v_2) : v_1, v_2 \in \mathcal{H} : I(t) > 0, J(t) < d \} \cup \{0\}. \quad \sum_{i=1}^{2} \|v_i\|^2 < 4d \text{ for all } t \in [0, T).
\]

**Lemma 4** (see [7]) (logarithmic Sobolev inequality). Let \( u \) be any function in \( H^1_0(\Omega) \) and \( a > 0 \) be any number. Then,

\[
\int_{\Omega} v^2 \ln |v| dx \leq \frac{1}{2} \|v\|_2^2 \ln \|v\|^2 + \frac{a^2}{2n} \|\nabla v\|^2 - (1 + \ln a) \|v\|_2^2.
\]

**Lemma 5** (see [8]) (logarithmic Gronwall inequality). Let \( c > 0, \gamma \in L^1(0, T ; \mathbb{R}^+), \) and assume that the function \( \omega : [0, T] \to [1, \infty) \) satisfies

\[
\omega(t) \leq c \left( 1 + \int_0^t \gamma(p) \omega(p) \ln \omega(p) dp \right), \quad 0 \leq t \leq T,
\]

then

\[
\omega(t) \leq c \exp \left( c \int_0^t \gamma(p) dp \right), \quad 0 \leq t \leq T.
\]

**Lemma 6.** Let \((v_{10}, v_{11}), (v_{20}, v_{21}) \in \mathcal{H} \times L^2_p(\mathbb{R}^n)\) such that \( 0 < \mathcal{E}(0) < d \) and \( I(t_0) > 0 \). Then, we have

\[
(v_1, v_2) \in W,
\]

**Theorem 7** (see [5]). Let \((v_{10}, v_{11}), (v_{20}, v_{21}) \in \mathcal{H} \times L^2_p(\mathbb{R}^n).\) Under the assumptions (H1)–(H3). Then, problem (1) has a global weak solution \( u \) in the space

\[
(v_1, v_2) \in C([0, \infty) ; \mathcal{H}) \cap C^1(\{0, \infty\) ; \(L_p^2(\mathbb{R}^n)^2\)).
\]
Then, the main result in this paper is the general decay of energy to problem (1) which is given in the following theorem.

**Theorem 8.** Assume the assumptions (H1)–(H3) hold and 0 < $\varepsilon(0) < d$. Let $(v_1, v_2)$ be the weak solution of problem (1) with the initial data $(v_{10}, v_{20}, v_{11}, v_{21}) \in \mathcal{H}(\mathbb{R}^n) \times L^2_{2}\mathbb{R}^n$. Then, there exist constant $\beta > 0$ such that the energy $\mathcal{E}(t)$ defined by (18) satisfies for all $t > 0$,

$$
\mathcal{E}(t) \leq \beta \left( 1 + \int_0^t \xi \varepsilon^{1/2}(p) dp \right)^{-1/2}, \quad \varepsilon \in (0, 1). \tag{28}
$$

**3. Asymptotic Behavior for $\varepsilon(0) < d$**

The following technical lemmas are useful to prove the general decay of energy to problem (1).

**Lemma 9.** Under the assumptions in Theorem 8, then the functional $\Phi(t)$ defined by

$$
\Phi(t) = \int_{\mathbb{R}^n} \rho(x) (v_1(t) \partial_i v_1 + v_2(t) \partial_i v_2)(t) dx \tag{29}
$$

satisfies for any $t \geq 0$,

$$
\Phi'(t) \leq \sum_{i=1}^2 \|\partial_i v_i(t)\|_{L^2_{2}}^2 - \frac{1}{2} \sum_{i=1}^2 l_i \|\nabla v_i(t)\|^2
+ \sum_{i=1}^2 \frac{1-l_i}{4\varepsilon} \langle \partial_i \varepsilon \nabla v_i \rangle(t) \tag{30}
$$

$$
+ \kappa \sum_{i=1}^2 \int_{\mathbb{R}^n} \rho(x) v_i^2 \ln |v_i| \ dx.
$$

**Proof.** We differentiate $\Phi(t)$, using (1), we can get

$$
\Phi'(t) = \sum_{i=1}^2 \|\partial_i v_i(t)\|_{L^2_{2}}^2 - \sum_{i=1}^2 \|\nabla v_i\|^2
+ \sum_{i=1}^2 \int_{\mathbb{R}^n} \nabla v_i(t) \cdot \int_0^t \partial_i (t-p) \nabla v_1(p) dp \ dx
$$

$$
- 2b \int_{\mathbb{R}^n} \rho(x) v_1 v_2 \ dx + \kappa \sum_{i=1}^2 \int_{\mathbb{R}^n} \rho(x) v_i^2 \ln |v_i| \ dx. \tag{31}
$$

It follows from Young and Poincaré’s inequality that for any $\varepsilon > 0$,

$$
\int_{\mathbb{R}^n} \partial_i v_1(t) \cdot \int_0^t \partial_i (t-p) \nabla v_1(p) dp \ dx
$$

$$
= \int_{\mathbb{R}^n} \nabla v_1(t) \cdot \int_0^t \partial_i (t-p) \nabla v_1(p) \nabla v_1(p) \ dp \ dx
+ \int_0^t \partial_i (p) dp \|\nabla v_1(t)\|^2
$$

$$
\leq (1-l_i) \|\nabla v_1\|^2 + \varepsilon \|\nabla v_1\|^2 + \frac{1}{4\varepsilon} \int_{\mathbb{R}^n} (\partial_i \varepsilon \nabla v_i)(t). \tag{32}
$$

Exploit Young and Poincaré’s inequalities to estimate

$$
2b \int_{\mathbb{R}^n} \rho(x) v_1 v_2 \ dx \leq \varepsilon \|\nabla v_1\|^2 + \varepsilon \|\nabla v_2\|^2. \tag{33}
$$

Inserting (32)–(33) into (31) yields for any $\varepsilon > 0$,

$$
\Phi'(t) \leq \sum_{i=1}^2 \|\partial_i v_i(t)\|_{L^2_{2}}^2 - \sum_{i=1}^2 (1-\varepsilon) \|\nabla v_i(t)\|^2
$$

$$
+ \kappa \sum_{i=1}^2 \frac{1-l_i}{4\varepsilon} \langle \partial_i \varepsilon \nabla v_i \rangle(t) \tag{34}
$$

+ k \sum_{i=1}^2 \int_{\mathbb{R}^n} \rho(x) v_i^2 \ln |v_i| \ dx.
$$

Taking $\varepsilon > 0$ small enough in (34) such that

$$
l_i - \varepsilon \varepsilon \varepsilon > \frac{1}{2}. \tag{35}
$$

The proof is hence complete.

**Lemma 10.** Under the assumptions in Theorem 8, then the functional $\psi(t)$ defined by

$$
\psi(t) = - \int_{\mathbb{R}^n} \rho(x) \partial_i v_1(t) \int_0^t \partial_i (t-p) (v_1(t) - v_1(p)) dp \ dx
$$

$$
- \int_{\mathbb{R}^n} \rho(x) \partial_i v_2(t) \int_0^t \partial_i (t-p) (v_2(t) - v_2(p)) dp \ dx \tag{36}
$$
satisfies for any $\delta > 0$,

$$
\psi'(t) \leq \sum_{i=1}^{2} \delta ((1 - t)^2 + 1 + b c_{i}) \|\nabla v_i(t)\|^2 - 2 \delta \|\partial_{t} v_i(t)\|_{L^2}^2 + C \sum_{i=1}^{2} \left( \int_{0}^{t} \alpha_i(s)\,ds \right) \left( \partial_{o}^* \nabla v_i(t) \right) + C_{2} \sum_{i=1}^{2} \left( \partial_{o}^* \Delta v_i(t) \right) + 2 \varepsilon \left( \partial_{o}^* \nabla v_i(t) \right).
$$

Proof. Taking the derivative of $\psi(t)$ and using (1), we conclude that

$$
\psi'(t) = \sum_{i=1}^{2} \int_{\mathbb{R}} \nabla v_i(t) \int_{0}^{t} \alpha_i(t - p) (\nabla v_i(t) - \nabla v_i(p)) \,dp\,dx - \int_{\mathbb{R}} \nabla \psi(t) \int_{0}^{t} \alpha_i(t - p) (\nabla v_i(t) - \nabla v_i(p)) \,dp\,dx - \int_{\mathbb{R}} \nabla \psi(t) \int_{0}^{t} \alpha_i(t - p) (\nabla v_i(t) - \nabla v_i(p)) \,dp\,dx - \int_{\mathbb{R}} \nabla \psi(t) \int_{0}^{t} \alpha_i(t - p) (\nabla v_i(t) - \nabla v_i(p)) \,dp\,dx + b \int_{\mathbb{R}} \rho(x) v_i \int_{0}^{t} \alpha_i(t - p) (v_i(t) - v_i(p)) \,dp\,dx + b \int_{\mathbb{R}} \rho(x) v_i \int_{0}^{t} \alpha_i(t - p) (v_i(t) - v_i(p)) \,dp\,dx - k \sum_{i=1}^{2} \int_{\mathbb{R}} \rho(x) v_i \ln |v_i| \int_{0}^{t} \alpha_i(t - p) (v_i(t) - v_i(p)) \,dp\,dx - \sum_{i=1}^{2} \int_{\mathbb{R}} \rho(x) \partial_{t} v_i \int_{0}^{t} \alpha_i(t - p) (v_i(t) - v_i(p)) \,dp\,dx - \sum_{i=1}^{2} \int_{\mathbb{R}} \rho(x) \partial_{t} v_i \int_{0}^{t} \alpha_i(t - p) (v_i(t) - v_i(p)) \,dp\,dx.
$$

We then use Young and Poincaré's inequalities; we can get for any $\delta > 0$,

$$
\int_{\mathbb{R}} \nabla v_i(t) \int_{0}^{t} \alpha_i(t - p) (\nabla v_i(t) - \nabla v_i(p)) \,dp\,dx \leq \delta \|\nabla v_i\|^2 + \frac{1}{4\delta} \left( \int_{0}^{t} \alpha_i(p)\,dp \right) (\partial_{o}^* \nabla v_i(t)).
$$

The second and third terms can be treated as

$$
\int_{\mathbb{R}} (\int_{0}^{t} \alpha_i(t - p) (\nabla v_i(t) - \nabla v_i(p)) \,dp) \right) (\nabla v_i(t)) = \delta (1 - t)^2 \|\nabla v_i\|^2 + \left( 1 + \frac{1}{4\delta} \right) \left( \int_{0}^{t} \alpha_i(p)\,dp \right) (\partial_{o}^* \nabla v_i(t)).
$$

The fourth and fifth terms will be estimated by

$$
\int_{\mathbb{R}} \rho(x) v_i \int_{0}^{t} \alpha_i(t - p) (v_i(t) - v_i(p)) \,dp\,dx \leq \delta \|\nabla v_i\|^2 + \frac{c_{2}}{4\delta} \left( \int_{0}^{t} \alpha_i(p)\,dp \right) (\partial_{o}^* \nabla v_i(t)).
$$

respectively.

For the last term, we have

$$
\int_{\mathbb{R}} \rho(x) \partial_{t} v_i \int_{0}^{t} \alpha_i(t - p) (v_i(t) - v_i(p)) \,dp\,dx \leq \delta \|\partial_{t} v_i\|_{L^2}^2 + \frac{c_{3}}{4\delta} \left( \int_{0}^{t} \alpha_i(p)\,dp \right) (\partial_{o}^* \nabla v_i(t)).
$$

Let $\epsilon_{0} \in (0, 1)$ and $g(s) = s^{\epsilon}_{0}(\ln |s| - s)$. Notice that $g$ is continuous on $(0, \infty)$, its limit at 0 is 0, and its limit at $\infty$ is $-\infty$. Then, $g$ has a maximum $m_{\epsilon_{0}}$ on $(0, \infty)$, so the following inequality holds

$$
|s| \ln |s| \leq s^{2} + m_{\epsilon_{0}} s^{1 - \epsilon_{0}}, \quad \text{for all } s > 0.
$$

Using the Cauchy-Schwarz's inequality and applying (43), yields, for any $\delta > 0$,

$$
k \int_{\mathbb{R}} \rho(x) v_i \ln |v_i| \int_{0}^{t} \alpha_i(t - p) (v_i(t) - v_i(p)) \,dp\,dx \leq k \int_{\mathbb{R}} \rho(x) \left( v_i^2 + m_{\epsilon_{0}} |v_i|^{1 - \epsilon_{0}} \right) \left( \int_{0}^{t} \alpha_i(t - p) (v_i(t) - v_i(p)) \,dp\,dx \right) \leq \int_{\mathbb{R}} \rho(x) v_i \left( \int_{0}^{t} \alpha_i(t - p) (v_i(t) - v_i(p)) \,dp\,dx \right) \leq \frac{\|\nabla v_i\|_{L^2}^2}{\delta} + \frac{1}{4\delta} (\partial_{o}^* \nabla v_i(t)) + c_{4} (\partial_{o}^* \nabla v_i(t))^{2(1 + \epsilon_{0})}.
$$
Combining (39)–(44) with (39) gives us (37) with

$$C = \frac{bc_\epsilon + 2}{4\delta} + 2\delta. \quad (45)$$

Therefore, the proof is complete.

Now, we define a Lyapunov functional $\mathcal{L}(t)$ by

$$\mathcal{L}(t) = M\mathcal{E}(t) + \varepsilon_1 \Phi(t) + \varepsilon_2 \psi(t), \quad (46)$$

where $M$, $\varepsilon_1$, and $\varepsilon_2$ are positive constants which will be taken later.

It is easy to see that $\mathcal{L}(t)$ and $\mathcal{E}(t)$ are equivalent in the sense that there exist two positive constants $\beta_1$ and $\beta_2$ such that

$$\beta_1 \mathcal{E}(t) \leq \mathcal{L}(t) \leq \beta_2 \mathcal{E}(t). \quad (47)$$

**Remark 11** (see [3]). Since $\zeta$ is nonincreasing, we have

$$\zeta_i(t) (\partial_0 \nabla \mathcal{V}_i) \frac{1}{1 + \varepsilon_0} \leq C \left( -\mathcal{E}'(t) \right)^{1/(1 + \varepsilon_0)}. \quad (48)$$

**Proof of Theorem 8.** For any fixed $t_0 > 0$, we have for any $t \geq t_0$,

$$\int_{t_0}^{t} \partial_0(p) dp \geq \int_{t_0}^{t} \partial_0(p) dp = \partial_0. \quad (49)$$

It follows from (37), (30), and (20) that

$$\mathcal{L}'(t) = M\mathcal{E}'(t) + \varepsilon_1 \Phi'(t) + \psi'(t)$$

$$\leq -\sum_{i=1}^{2} (\partial_0 - 2\delta - \varepsilon_1) \|\partial_i \mathcal{V}_i(t)\|_{L^2}^2$$

$$- \sum_{i=1}^{2} \left[ \frac{l}{2} \varepsilon_1 - \delta \left( (1 - l_i)^2 + 1 + bc_\epsilon \right) \right] \|\mathcal{V}_i(t)\|_{L^2}^2$$

$$+ \sum_{i=1}^{2} \left[ C_1 \varepsilon_1 + CL_i (\partial_0 \nabla \mathcal{V}_i)(t) - \frac{M}{2} \sum_{i=1}^{2} \partial_0(t) \|\mathcal{V}_i(t)\|_{L^2}^2$$

$$+ \varepsilon_1 k \sum_{i=1}^{2} \int_{\mathbb{R}^m} \rho(x) x_i^2 \ln |x_i| dx + \varepsilon_1 c_\epsilon \sum_{i=1}^{2} \|\partial_0 \nabla \mathcal{V}_i\|_{L^{1/(1 + \varepsilon_0)}}^{1/(1 + \varepsilon_0)}$$

$$+ C_3 \sum_{i=1}^{2} (\partial_0 \nabla \mathcal{V}_i)(t). \quad (50)$$

Using the logarithmic Sobolev inequality, we have

$$\mathcal{L}'(t) \leq -\sum_{i=1}^{2} (\partial_0 - 2\delta - \varepsilon_1) \|\partial_i \mathcal{V}_i(t)\|_{L^2}^2$$

$$+ C_3 \sum_{i=1}^{2} \|\partial_0 \nabla \mathcal{V}_i\|(t)$$

$$- \sum_{i=1}^{2} \left[ \frac{l}{2} \varepsilon_1 - \delta \left( (1 - l_i)^2 + 1 + bc_\epsilon \right) \right] \|\mathcal{V}_i(t)\|_{L^2}^2$$

$$- \sum_{i=1}^{2} \left[ C_1 \varepsilon_1 + CL_i (\partial_0 \nabla \mathcal{V}_i)(t) - \frac{M}{2} \sum_{i=1}^{2} \partial_0(t) \|\mathcal{V}_i(t)\|_{L^2}^2$$

$$- \varepsilon_1 k (1 + \ln \alpha) \sum_{i=1}^{2} \|\mathcal{V}_i(t)\|_{L^2}^2 + \varepsilon_1 c_\epsilon \sum_{i=1}^{2} (\partial_0 \nabla \mathcal{V}_i)^{1/(1 + \varepsilon_0)}. \quad (51)$$

Recalling (18) and $\mathcal{E}(t) \leq \mathcal{E}(0) < d$, we get

$$\ln \|\mathcal{V}\|_{L^2}^2 < \ln \left( \frac{4}{k} \mathcal{E}(t) \right) < \ln \left( \frac{4}{k} \mathcal{E}(0) \right) < \ln \left( \frac{4}{k} d \right). \quad (52)$$

Now, we take $\varepsilon_1 > 0$ small enough so that

$$\partial_0 - 2\delta - \varepsilon_1 > 0. \quad (53)$$

For any fixed $\varepsilon_1 > 0$, we pick $\delta > 0$ so small that

$$\frac{l}{2} \varepsilon_1 - \delta \left( (1 - l_i)^2 + 1 \right) > \frac{l}{4} \varepsilon_1. \quad (54)$$

On the other hand, we choose $M > 0$ large enough so that (47) holds, and further

$$C_3 = \frac{M}{2} - \frac{\partial_0(0)}{4\delta} > 0. \quad (55)$$

We can conclude that there exist two positive constant $m$ and $C$ such that

$$\mathcal{L}'(t) \leq -m\mathcal{E}(t) + \sum_{i=1}^{2} (\partial_0 \nabla \mathcal{V}_i)(t) + \varepsilon_1 c_\epsilon \sum_{i=1}^{2} (\partial_0 \nabla \mathcal{V}_i)^{1/(1 + \varepsilon_0)}. \quad (56)$$

Multiplying (56) by $\zeta(t) = \min \{ \zeta_1, \zeta_2 \}$ by (H2) and use the fact that

$$(\partial_0 \nabla \mathcal{V}_i)(t) \leq C (\partial_0 \nabla \mathcal{V}_i)^{1/(1 + \varepsilon_0)}(t), \quad (57)$$

and (48), we get

$$\zeta(t) \mathcal{L}'(t) \leq -m \zeta(t) \mathcal{E}(t) + \zeta \left( -\mathcal{E}'(t) \right)^{1/(1 + \varepsilon_0)}. \quad (58)$$
Multiply (58) by \( \xi(t) \) and recall that \( \xi(t) \leq 0 \) to obtain

\[
\zeta(t) \xi(t)(t) \leq -m \xi(t) \xi(t) + c(\xi(t) \xi(t) - c \xi(t))
\]

Using Young’s inequality, for any \( \delta > 0 \),

\[
\zeta(t) \xi(t)(t) \leq -m \xi(t) \xi(t) + c(\delta \xi(t) \xi(t) - c \xi(t))
\]

which implies

\[
\left( \zeta(t) \xi(t) \right)(t) \leq -(m - c) \xi(t) \xi(t) - c \xi(t).
\]

It is clear that to get

\[
L_1(t) = \left( \xi(t) \xi(t) + c \xi(t) \right) \sim \xi(t).
\]

By using (61) and \( \xi(t) \leq 0 \), we arrive at

\[
L_1(t) = \left( \xi(t) \xi(t) + c \xi(t) \right) \leq -m \xi(t) \xi(t).
\]

Integration over \((t_0, t)\) leads to for some constant \( m' > 0 \) such that

\[
L_1(t) \leq m' \left( 1 + \int_{t_0}^{t} \xi(p) dp \right)^{-1/\alpha}.
\]

The equivalence of \( L_1(t) \) and \( \xi \) completes Proof of Theorem 8.

Remark 12.

1. We mention here that we have coupled our system without the classical way, i.e., our idea is not to couple equations in the logarithmic nonlinear terms

2. Most contribution here is to obtain our nonexistence result with less conditions on the viscoelastic terms

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that they have no competing interests.

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