The sharp lifespan estimate for semilinear damped wave equation with Fujita critical power in higher dimensions

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Abstract

This paper is concerned with the lifespan estimate of classical solutions with small initial data to the Cauchy problem of semilinear damped wave equations with the Fujita critical exponent. We establish the following sharp upper bound of the lifespan

$$T(\varepsilon) \leq \exp\left(C\varepsilon^{-\frac{2}{n+2}}\right)$$

in higher dimensions ($n \geq 4$), by using the heat kernel as the test function. Then, together with the previous results, a complete result on the sharp lower and upper bound estimates is obtained in this case.

Résumé

Cet article porte sur l’estimation de la durée de vie des solutions classiques avec petites données initiales pour le problème de Cauchy de l’équation des ondes dissipative semi-linéaire avec l’exposant critique de Fujita. Nous établissons la borne supérieure de la durée de vie suivante

$$T(\varepsilon) \leq \exp\left(C\varepsilon^{-\frac{2}{n+2}}\right)$$

en dimension supérieure ($n \geq 4$), en utilisant le noyau de la chaleur comme fonction de test. Donc, avec les résultats précédents, nous obtenons un résultat complet sur les estimations optimale de borne supérieure et inférieure de la durée de vie dans ce cas.

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1. Introduction

We consider the Cauchy problem of semilinear damped wave equations with Fujita critical power in $n$ dimensions:

\[
\begin{align*}
  & u_{tt} - \Delta u + u_t = |u|^p, \quad (t, x) \in [0, T(\varepsilon)) \times \mathbb{R}^n, \\
  & u(0, x) = \varepsilon f(x), \quad u_t(0, x) = \varepsilon g(x), \quad x \in \mathbb{R}^n,
\end{align*}
\]

where $p = 1 + \frac{2}{n}$, $n \geq 4$ and $\varepsilon > 0$ is a parameter which represents the smallness of the initial data. We are devoted to establish the lifespan estimate from above in the form $T(\varepsilon) \leq \exp(C\varepsilon^{-\frac{2}{n}})$, where $C$ is a positive constant independent of $\varepsilon$, then by combining the result about the lifespan estimate from below obtained in Li [7] and Ikeda and Ogawa [3], we get the sharpness of the lifespan estimate to Cauchy problem (1).

The study on Cauchy problem (1) with small initial data has a long history. In 1995, Li and Zhou [5] studied Cauchy problem (1) with $p$ satisfying $1 < p \leq 1 + \frac{2}{n}$ in lower dimensional cases, i.e., $n = 1, 2$, and established the sharp upper bound of the lifespan in the form

\[
T(\varepsilon) \leq \begin{cases} 
\exp(C\varepsilon^{-\frac{2}{n}}), & p = 1 + \frac{2}{n}, \\
C\varepsilon^{-\frac{1}{p(n-p)}}, & 1 < p < 1 + \frac{2}{n},
\end{cases}
\]

where $C = C(n, p, f, g)$ is a positive constant independent of $\varepsilon$. One year later Li [7] considered the lower bound of lifespan estimate for Cauchy problem (1) for integer $p$ and $n \geq 1$ and obtained the following result by using the global iteration method introduced by Li and Yu [4] (see also Li and Zhou [6]):

\[
T(\varepsilon) \geq \begin{cases} 
+ \infty, & p > 1 + \frac{2}{n}, \\
\exp(C\varepsilon^{-\frac{2}{n}}), & p = 1 + \frac{2}{n}, \\
C\varepsilon^{-\frac{1}{p(n-p)}}, & 1 < p < 1 + \frac{2}{n},
\end{cases}
\]
where $C = C(n, p, f, g)$ is a positive constant independent of $\varepsilon$. Then Todorova and Yordanov \cite{9} found that Cauchy problem \cite{1} with small initial data admits a critical power $p = 1 + \frac{2}{n}$ such that Cauchy problem \cite{1} has global solutions if $p > 1 + \frac{2}{n}$, while the solutions blow up in a finite time if $1 < p < 1 + \frac{2}{n}$. It is interesting to see that this critical power is exactly the same as the Fujita critical power for the corresponding semilinear heat equations $v_t - \Delta v = |v|^p$, see \cite{2} for details. Later, Zhang \cite{11} showed that the solutions also blow up in a finite time when $p = 1 + \frac{2}{n}$, by using the test function method. Nishihara \cite{8} studied the 3-D case and established the sharp upper bound of the lifespan estimate. Up to now, the sharp lifespan estimate from above for $p = 1 + \frac{2}{n}$ in higher dimensional spaces($n \geq 4$) is still open. Both the methods used in \cite{5} and \cite{8} do not work in this case, since their proofs are based on the explicit formula of the solutions and the positivity of the fundamental solution to the wave operator in lower dimensions $n = 1, 2, 3$. Recently, Ikeda and Ogawa \cite{3} established the lifespan estimate for Cauchy problem \cite{1} for $n \geq 1$ in the form

$$\exp\left(c\varepsilon^{-(p-1)}\right) \leq T(\varepsilon) \leq \exp\left(C\varepsilon^{-p}\right),$$

obviously, there is a gap between the upper and lower bound of the lifespan estimate.

**Remark 1.1.** The test function method introduced in \cite{11} is very useful for proving blow up result for both semilinear heat and damped wave equation, also it is powerful for obtaining the sharp upper bound of lifespan for the corresponding semilinear problems with subcritical power. However, it doesn’t work for the critical power. In \cite{3}, in additional to the test function method, they also used a more delicate argument, by using the support property of the test function to separate the time interval into small pieces $\sim 2^{j/2}$. Unfortunately, it seems we can’t improve the upper bound of the lifespan estimate by the same argument.

In this paper we are devoted to filling the gap between the upper bound and the lower bound of the lifespan estimate for problem \cite{1} in higher dimensional spaces($n \geq 4$). As mentioned above, we can not use the positivity of the
fundamental solution to the wave operator in higher dimensional case. However, we may use the idea of test function method for semilinear wave equations $u_{tt} - \Delta u = |u|^p$, which was introduced by Yordanov and Zhang \cite{10} and Zhou \cite{12}. The key ingredient now is to find an appropriate test function. Since the linear damped wave equation $u_{tt} - \Delta u + u_t = 0$ has the so-called diffusion phenomenon and its solution behaves like that of the corresponding linear heat equation $v_t - \Delta v = 0$ as $t \to \infty$, it will be useful to use the heat kernel, the fundamental solution of the heat equation, as the test function. Then we treat the equation in (1) as a heat equation, and hence we may get the expression of the solution in terms of the initial data and the source terms $-u_{tt}$ and $|u|^p$, by Duhamel’s principle. After that we can establish an ordinary differential inequality for an appropriate functional to get the desired lifespan estimate. The essence is that we use the semigroup property of the heat kernel to deal with the source term $u_{tt}$.

**Theorem 1.2.** Let $f, g \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and $\text{supp}(f, g) \subset \{x \in \mathbb{R}^n : |x| \leq 1\}$. Assume furthermore that $f(x), g(x) \geq 0$ are non-trivial. Let $u(t, x)$ solve the Cauchy problem (1) with $p = 1 + \frac{2}{n}$ on $[0, T(\varepsilon))$. Then we have

$$T(\varepsilon) \leq \exp(C\varepsilon^{-\frac{2}{n}}),$$

where $C = C(n, f, g)$ is a positive constant independent of $\varepsilon$.

**Remark 1.3.** Our method also works for the lower dimensional cases $n = 1, 2, 3$. The methods used in \cite{5} for $n = 1, 2$ and \cite{8} for $n = 3$ include sophisticated analysis for the asymptotic behavior of some special functions, i.e., the modified Bessel function and hyperbolic cosine function.

### 2. Heat Kernel

In this section we make a brief introduction to the heat kernel. In the $n$ dimensional case, the heat kernel in $\mathbb{R}^n$ is given by

$$E(t, x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}.$$
Lemma 2.1 (Evans [1]). For any given time $t > 0$, the heat kernel satisfies

$$\int_{\mathbb{R}^n} E(t, x) dx = 1,$$

(7)

and

$$\begin{cases} E_t - \Delta E = 0, & (t, x) \in (0, \infty) \times \mathbb{R}^n, \\ E = \delta_O, & \text{on } \{t = 0\} \times \mathbb{R}^n, \end{cases}$$

(8)

where $\delta_O$ denotes the Dirac measure on $\mathbb{R}^n$, giving unit mass to the original point $O$.

Lemma 2.2 (Semigroup property). For any given $t, s > 0$ and $x \in \mathbb{R}^n$, the heat kernel $E(t, x)$ satisfies

$$E(t, x) \ast E(s, x) = E(t + s, x),$$

(9)

where $\ast$ denotes the convolution.

Proof. By direction computation we know the Fourier transformation of $E(t, x)$ is given by

$$\hat{E}(t, \xi) = e^{-|\xi|^2},$$

(10)

and then we have

$$\hat{E}(t, \xi) \cdot \hat{E}(s, \xi) = e^{-(t+s)|\xi|^2} = \hat{E}(t+s, \xi),$$

(11)

which implies $\Box$ in Lemma 2.2.

3. Proof of the Main Theorem

First the equation in Cauchy problem [11] can be regarded as the heat equation with source term

$$u_t - \Delta u = |u|^p - u_{tt},$$

then by Duhamel’s principle and using the first initial data in (11) we have

$$u(t, x) = \varepsilon E(t) \ast f + \int_0^t E(t - \tau) \ast |u(\tau)|^p d\tau - \int_0^t E(t - \tau) \ast \partial_{x\tau} u(\tau) d\tau.$$
By integration by parts and using the second initial data in (1) the last term can be rewritten as
\[ \int_0^t E(t - \tau) \ast \partial_\tau u(\tau)d\tau \]
\[ = \int_{\mathbb{R}^n} \int_0^t E(t - \tau, x - y) \partial_\tau u(\tau, y)d\tau dy \]
\[ = \int_{\mathbb{R}^n} E(0, x - y) u_t(x) dy - \int_{\mathbb{R}^n} E(t, x - y) u_t(0, y) dy \]
\[ - \int_{\mathbb{R}^n} \int_0^t \partial_\tau E(t - \tau, x - y) \partial_\tau u(\tau, y)d\tau dy \]
\[ = u_t(x) - \varepsilon E(t) \ast g - \int_{\mathbb{R}^n} \int_0^t \partial_\tau E(t - \tau, x - y) \partial_\tau u(\tau, y)d\tau dy, \]
in which we used the second equality of (8) in Lemma 2.1. Plugging (13) into (12) we have
\[ u(t, x) + u_t(t, x) - \int_{\mathbb{R}^n} \int_0^t \partial_\tau E(t - \tau, x - y) \partial_\tau u(\tau, y)d\tau dy \]
\[ = -\varepsilon E(t) \ast f + \varepsilon E(t) \ast g + \int_0^t E(t - \tau) \ast |u|^p d\tau. \] (14)

Multiplying the both sides of (14) with \((4\pi(t+1))^{\frac{n}{2}} E(t+1)\) and then integrating over \(\mathbb{R}^n\), we have
\[ \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} u_t(x) dx + \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} u_t(t, x) dx \]
\[ - (4\pi(t+1))^{\frac{n}{2}} \int_{\mathbb{R}^n} \int_0^t \int_{\mathbb{R}^n} E(t + 1, x) \partial_\tau E(t, x - y) \]
\[ \cdot \partial_\tau u(\tau, y) dy d\tau dx \]
\[ = \varepsilon (4\pi(t+1))^{\frac{n}{2}} \int_{\mathbb{R}^n} E(2t + 1, x) (f(x) + g(x)) dx \]
\[ + (4\pi(t+1))^{\frac{n}{2}} \int_0^t \int_{\mathbb{R}^n} E(2t + 1 - \tau, x) |u|^p(\tau, x) dx d\tau, \] (15)
in which we used the semigroup property of \(E(t, x)\) given in Lemma 2.2.

Setting
\[ G(t) = \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} u(t, x) dx, \]
\[ F(t) = \left( \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} |u(t, x)|^p dx \right)^\frac{1}{p} (t + 1)^{\frac{n}{2p} - 1} \]
\[ = \left( \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} |u(t, x)|^p dx \right)^\frac{1}{p} (t + 1)^{\frac{n}{2}} \] (16)
and

\[ A(t) = \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} u_t(t, x) dx, \]

\[ B(t) = -\left(4\pi(t+1)\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \int_0^t \int_{\mathbb{R}^n} E(t+1, x) \partial_\tau E(t-\tau, x-y) \cdot \partial_\tau u(\tau, y) dy \, d\tau \, dx, \]

\[ D(t) = \left(4\pi(t+1)\right)^{\frac{n}{2}} \int_0^t \int_{\mathbb{R}^n} E(2t+1-\tau, x) |u|^p(\tau, x) dx \, d\tau, \]

by Hölder inequality we have

\[ G(t) = \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} u(t, x) dx \]

\[ \leq \left(\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} |u(t, x)|^p dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} dx\right)^{\frac{1}{p'}} \]

\[ \leq C \left(\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} |u(t, x)|^p dx\right)^{\frac{1}{p}} (t+1)^{\frac{np-1}{2p}} \left(\int_{\mathbb{R}^n} e^{-|y|^2} dy\right)^{\frac{1}{p'}} \]

\[ \leq CF(t). \]  

Here and hereafter, \(C\) denotes a positive constant which is independent of \(\varepsilon\) and may change from line to line. As to \(A(t)\), a direct computation leads to

\[ A(t) = \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} u_t(t, x) dx \]

\[ = \frac{d}{dt} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} u(t, x) dx - \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} u(t, x) \frac{|x|^2}{4(t+1)^2} dx \]  

(19)

with

\[ \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} u(t, x) \frac{|x|^2}{4(t+1)^2} dx \]

\[ \leq \left(\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} |u(t, x)|^p dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} \left(\frac{|x|^2}{4(t+1)^2}\right)^p dx\right)^{\frac{1}{p'}} \]  

\[ \leq C(t+1)^{-1} F(t), \]

(20)

hence

\[ A(t) \leq G'(t) + C(t+1)^{-1} F(t). \]  

(21)

It is easy to see that \(2(t+1) \geq 2t + 1 - \tau \geq \tau + 1\) for \(0 \leq \tau \leq t\), then the
nonlinear term \( D(t) \) can be estimated by

\[
D(t) = \left(4\pi(t + 1)\right)^{\frac{1}{2}} \int_0^t \int_{\mathbb{R}^n} E(2t + 1 - \tau, x)|u|^p(\tau, x)dx d\tau
\]

\[
= \int_0^t \left(4\pi(t + 1)\right)^{\frac{1}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t + 1 - \tau)}} |u|^p dx d\tau
\]

\[
\geq 2^{-\frac{n}{2}} \int_0^t \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t + 1 - \tau)}} |u|^p dx d\tau
\]

\[
\geq C \int_0^t \frac{F^p(\tau)}{\tau + 1} d\tau,
\]

in which we used the definition of \( F(\tau) \) given by \((16)\). Therefore, the combination of \((15)-(17)\) and \((21)\) yields

\[
G'(t) + G(t) + \frac{CF(t)}{t + 1} + B(t)
\]

\[
\geq \varepsilon \left(4\pi(t + 1)^{\frac{1}{2}} \int_0^t \int_{\mathbb{R}^n} E(2t + 1, x)(f(x) + g(x)) dx + D(t)\right)
\]

\[
= \varepsilon \left(\frac{t + 1}{2t + 1}\right)^{\frac{1}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t + 1 - \tau)}} (f(x) + g(x)) dx + D(t).
\]

We now estimate the term \( B(t) \). By integration by parts and using Lemma \((2.2)\) we have

\[
B(t)
\]

\[
= -\left(4\pi(t + 1)\right)^{\frac{1}{2}} \int_{\mathbb{R}^n} \int_0^t \int_{\mathbb{R}^n} E(t + 1, x) \partial_\tau E(t - \tau, x - y) \cdot \partial_\tau u(\tau, y) dy d\tau dx
\]

\[
\geq \varepsilon \left(4\pi(t + 1)\right)^{\frac{1}{2}} \int_{\mathbb{R}^n} \int_0^t \int_{\mathbb{R}^n} E(t + 1, x) E(t - \tau, x - y) dx \cdot \partial_\tau u(\tau, y) dy d\tau
\]

\[
\geq \varepsilon \left(\frac{t + 1}{2t + 1}\right)^{\frac{1}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t + 1 - \tau)}} (f(x) + g(x)) dx + D(t).
\]

\[
\triangleq B_1(t) + B_2(t).
\]
There are four terms in $B_1(t)$:

$$B_1(t) = - (4\pi(t + 1))^\frac{n}{2} \int_{\mathbb{R}^n} \frac{n}{2} (4\pi(t + 1))^{-\frac{1}{2}} e^{-\frac{|x|^2}{4(t + 1)^2}} u(t, x) dx$$

$$- \int_{\mathbb{R}^n} (4\pi(t + 1))^{-\frac{1}{2}} e^{-\frac{|x|^2}{4(t + 1)^2}} u(t, x) \frac{|x|^2}{4(t + 1)^2} dx$$

$$+ (4\pi(t + 1))^\frac{n}{2} \left[ \int_{\mathbb{R}^n} (4\pi(2t + 1))^{-\frac{1}{2}} e^{-\frac{|x|^2}{4(2t + 1)^2}} u(0, x) dx \right]$$

$$- \int_{\mathbb{R}^n} (4\pi(2t + 1))^{-\frac{1}{2}} e^{-\frac{|x|^2}{4(2t + 1)^2}} u(0, x) \frac{|x|^2}{4(2t + 1)^2} dx$$

$$\triangleq B_{11}(t) + B_{12}(t) + B_{13}(t) + B_{14}(t).$$

By (18) and (20) it is easy to get

$$B_{11}(t) = -(4\pi(t + 1))^\frac{n}{2} \int_{\mathbb{R}^n} \frac{n}{2} (4\pi(t + 1))^{-\frac{1}{2}} e^{-\frac{|x|^2}{4(t + 1)^2}} u(t, x) dx$$

$$\leq C(t + 1)^{-1} |G(t)|$$

$$\leq C(t + 1)^{-1} F(t),$$

$$B_{12}(t) = \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t + 1)^2}} u(t, x) \frac{|x|^2}{4(t + 1)^2} dx$$

$$\leq C(t + 1)^{-1} F(t).$$

By (26) and (27) it is easy to get

$$B_{13}(t) = \varepsilon \frac{1}{2} \frac{1}{4\pi(2t + 1)} (t + 1)^\frac{n}{2} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t + 1)^2}} f(x) dx,$$

$$B_{14}(t) = -\varepsilon \frac{1}{2} (t + 1) \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t + 1)^2}} f(x) \frac{|x|^2}{4(2t + 1)^2} dx.$$

It follows then from (26) and (27) that

$$B_1(t) \leq C(t + 1)^{-1} F(t) + \varepsilon \frac{n}{2} \frac{1}{4\pi(2t + 1)} (t + 1)^\frac{n}{2} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t + 1)^2}} f(x) dx$$

$$- \varepsilon \left( \frac{t + 1}{2t + 1} \right) \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t + 1)^2}} f(x) \frac{|x|^2}{4(2t + 1)^2} dx.$$
There are also four terms in $B_2(t)$:

\[
B_2(t)
\]

\[
= (4\pi(t+1))^{\frac{n}{2}} \int_0^t \int_{\mathbb{R}^n} \partial_{\tau} E(2t+1-\tau, x) u(\tau, x) d\tau d\tau
\]

\[
= (4\pi(t+1))^{\frac{n}{2}} \left[ \int_0^t \int_{\mathbb{R}^n} \frac{n}{2} \left( \frac{n}{2} + 1 \right) (4\pi(2t+1-\tau))^{-\frac{n}{2}-2}
\]

\[
\cdot e^{-\frac{|x|^2}{4(2t+1-\tau)}} |u(\tau, x)| d\tau d\tau
\]

\[
- \int_0^t \int_{\mathbb{R}^n} (4\pi(2t+1-\tau))^{-\frac{n}{2}-1} e^{-\frac{|x|^2}{4(2t+1-\tau)}} |u(\tau, x)| d\tau d\tau
\]

\[
+ \int_0^t \int_{\mathbb{R}^n} (4\pi(2t+1-\tau))^{-\frac{n}{2}-1} e^{-\frac{|x|^2}{4(2t+1-\tau)}} |u(\tau, x)| d\tau d\tau
\]

\[
- \frac{1}{8} \left( 4\pi(t+1) \right)^{\frac{n}{2}-2} \left( \int_0^t \int_{\mathbb{R}^n} |u|^p(\tau, x) d\tau d\tau \right)^{\frac{1}{p}}
\]

\[
\cdot \left( \int_0^t \int_{\mathbb{R}^n} E(2t+1-\tau, x) d\tau d\tau \right)^{\frac{1}{p}}
\]

\[
\leq C(t+1)^{\frac{n}{2}-2} \int_0^t \int_{\mathbb{R}^n} E(2t+1-\tau, x) u(\tau, x) d\tau d\tau
\]

\[
\leq C(t+1)^{\frac{n}{2}-2} \left( \int_0^t \int_{\mathbb{R}^n} E(2t+1-\tau, x) |u|^p(\tau, x) d\tau d\tau \right)^{\frac{1}{p}}
\]

\[
\leq \frac{1}{8} \left( 4\pi(t+1) \right)^{\frac{n}{2}-2} \left( \int_0^t \int_{\mathbb{R}^n} E(2t+1-\tau, x) |u|^p(\tau, x) d\tau d\tau \right)^{\frac{1}{p}}
\]

\[
+ C(t+1)^{-\frac{n}{2}-1}
\]

\[
= \frac{1}{8} D(t) + C(t+1)^{-\frac{n}{2}-1}
\]

(29)

Since we have $2t+1-\tau \geq t+1$ for $0 \leq \tau \leq t$, by Hölder inequality and Yong inequality we get

\[
B_{21}(t) \leq C(t+1)^{\frac{n}{2}-2} \int_0^t \int_{\mathbb{R}^n} E(2t+1-\tau, x) u(\tau, x) d\tau d\tau
\]

(30)
and

\[ B_{22}(t) \leq C(t+1)^{\frac{n}{2}-1} \left( \int_0^t \int_{\mathbb{R}^n} E(2t+1-\tau, x) \frac{|x|^2}{4(2t+1-\tau)^2} u(\tau, x) dx d\tau \right)^{\frac{1}{p'}} \]

\[ \leq C(t+1)^{\frac{n}{2}-1} \left( \int_0^t \int_{\mathbb{R}^n} E(2t+1-\tau, x)|u|^p(\tau, x) dx d\tau \right)^{\frac{1}{p'}} \]

\[ \cdot \left( \int_0^t \int_{\mathbb{R}^n} E(2t+1-\tau, x) \left( \frac{|x|^2}{4(2t+1-\tau)^2} \right)^{p'} dx d\tau \right)^{\frac{1}{p}} \]

\[ \leq C(t+1)^{\frac{n}{2}-1} \left( \int_0^t \int_{\mathbb{R}^n} E(2t+1-\tau, x)|u|^p(\tau, x) dx d\tau \right)^{\frac{1}{p'}} \]

\[ \cdot \left( \int_0^t \frac{1}{(2t+1-\tau)^{p'}} \int_{\mathbb{R}^n} e^{-|y|^2/4(2t+1-\tau)} dy d\tau \right)^{\frac{1}{p}} \]

\[ \leq \frac{1}{8} D(t) + C(t+1)^{-\frac{n}{2}-1}, \]  

(31)
in which we used the variable transformation \( x = 2\sqrt{2t+1-\tau} y \) in the third inequality of (31). By a similar way to that of \( B_{22} \), we have

\[ B_{23}(t) \leq \frac{1}{8} D(t) + C(t+1)^{-\frac{n}{2}-1}, \]

\[ B_{24}(t) \leq \frac{1}{8} D(t) + C(t+1)^{-\frac{n}{2}-1}. \]  

(32)

By combining (30)-(32) we get

\[ B_2(t) \leq \frac{1}{2} D(t) + C(t+1)^{-\frac{n}{2}-1}, \]  

(33)

which together with (28) gives

\[ B(t) = B_1(t) + B_2(t) \]

\[ \leq C(t+1)^{-1} F(t) + \frac{\varepsilon n}{24\pi(2t+1)} \left( \frac{t+1}{2t+1} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t+1)}} f(x) dx \]

\[ - \varepsilon \left( \frac{t+1}{2t+1} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t+1)}} f(x) \frac{|x|^2}{4(2t+1)^2} dx \]

\[ + \frac{1}{2} D(t) + C_0 (t+1)^{-\frac{n}{2}-1}, \]  

where \( C_0 > 0 \) is a constant depending only on \((\pi, n)\). By (22) \& (23) and (34) we
get

\[ G'(t) + G(t) + \frac{CF(t)}{t + 1} \]

\[ \geq \frac{1}{2} D(t) + (1 - \frac{n}{2 \cdot 4\pi(2t + 1)}) \varepsilon \left( \frac{t + 1}{2t + 1} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t + 1)^2}} f(x) \, dx \]

\[ + \varepsilon \left( \frac{t + 1}{2t + 1} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t + 1)^2}} f(x) \, dx \]

\[ + \varepsilon \left( \frac{t + 1}{2t + 1} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t + 1)^2}} g(x) \, dx \]

\[ \geq C_1 \int_0^t \frac{F_p(\tau)}{\tau + 1} \, d\tau + (1 - \frac{n}{2 \cdot 4\pi(2t + 1)}) \varepsilon \left( \frac{t + 1}{2t + 1} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t + 1)^2}} f(x) \, dx \]

\[ + \varepsilon \left( \frac{t + 1}{2t + 1} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t + 1)^2}} f(x) \, dx \]

\[ + \varepsilon \left( \frac{t + 1}{2t + 1} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t + 1)^2}} g(x) \, dx \]

\[ - C_0 (t + 1)^{-\frac{n}{2} - 1} \]

where \( C_1 > 0 \) is a constant depending only on \( n \) as it comes from (22).

Setting

\[ H(t) = \left( 1 - \frac{n}{2 \cdot 4\pi(2t + 1)} \right) \left( \frac{t + 1}{2t + 1} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t + 1)^2}} f(x) \, dx \]

\[ + \left( \frac{t + 1}{2t + 1} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t + 1)^2}} f(x) \, dx \]

\[ + \left( \frac{t + 1}{2t + 1} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t + 1)^2}} g(x) \, dx \]

and

\[ I(t) = C_0 (t + 1)^{-\frac{n}{2} - 1}. \]

Multiplying (35) with \( e^t \) and then integrating it over \([0, t]\), we have

\[ e^t F(t) + \int_0^t e^\tau F(\tau) \frac{d\tau}{\tau + 1} \geq e^t G(t) + \int_0^t e^\tau F(\tau) \frac{d\tau}{\tau + 1} \]

\[ \geq G(0) + C_1 \int_0^t \frac{(e^\tau - e^\tau) F_p(\tau)}{\tau + 1} \, d\tau \]

\[ + \int_0^t e^\tau (\varepsilon H(\tau) - I(\tau)) \, d\tau. \]

We claim that

\[ J(t) := \int_0^t e^\tau (\varepsilon H(\tau) - I(\tau)) \, d\tau \geq C_2 \varepsilon e^t \]

(37)
for some constant $C_2 > 0$ depending only on $(f, g, n)$ and $t$ large enough. Actually, since we assume that the initial data $f(x), g(x) \in L^2(\mathbb{R}^n)$ and have compact support, we have $f(x), g(x) \in L^1(\mathbb{R}^n)$.

So by the dominated convergence theorem we have

$$\lim_{t \to \infty} H(t) = 2^{-\frac{n}{2}} \int_{\mathbb{R}^n} (f(x) + g(x)) \, dx = H_0 > 0,$$

and then there exists $t_1 = t_1(n, f, g)$ independent of $\varepsilon$, such that

$$H(t) > \frac{H_0}{2} > 0, \quad \text{for } t > t_1. \tag{39}$$

Let $t_2 = (\frac{H_0}{4C_0})^{-\frac{2}{n+2}} \varepsilon^{-\frac{2}{n+2}} - 1$. We get

$$I(t) \leq \frac{H_0 \varepsilon}{4}, \quad \text{for } t \geq t_2. \tag{40}$$

Setting $t_3 = \max\{t_1, t_2\}$, since $H(t)$ is continuous and has a limit as $t \to \infty$, $H(t)$ is bounded over $[0, \infty)$, and then we may set $M_H = \max_{t \in [0, \infty)} |H(t)| < \infty$. On the other hand, it is easy to get $M_I = \max_{t \in [0, \infty)} I(t) = C_0 > 0$. Hence we have

$$J(t) := \int_0^t e^\tau (\varepsilon H(\tau) - I(\tau)) \, d\tau$$

$$= \int_0^{t_3} e^\tau (\varepsilon H(\tau) - I(\tau)) \, d\tau + \int_{t_3}^t e^\tau (\varepsilon H(\tau) - I(\tau)) \, d\tau$$

$$\geq \int_0^{t_3} e^\tau (-\varepsilon M_H - C_0) \, d\tau + \frac{H_0 \varepsilon}{4} (e^t - e^{t_3})$$

$$\geq \frac{H_0 \varepsilon}{8} e^t + \frac{H_0 \varepsilon}{4} e^{t_3} - \frac{H_0 \varepsilon}{4} M_H e + C_0 e^t + M_H \varepsilon + C_0.$$

Hence, if $t \geq t_4 = \ln \left( 2 + \frac{8M_H}{H_0} + \frac{8C_0}{H_0} \varepsilon^{-1} \right) + t_3$, then we get

$$J(t) \geq \frac{H_0 \varepsilon}{8} e^t + M_H \varepsilon + C_0 \geq C_2 \varepsilon e^t, \tag{42}$$

which proves (37). With this claim in mind, it is easy to get from (36) that

$$e^t F(t) + \int_0^t \frac{e^\tau F(\tau)}{\tau + 1} \, d\tau \geq C_2 \varepsilon e^t + C_1 \int_0^t \frac{(e^\tau - e^\tau F(\tau)}{\tau + 1} \, d\tau \quad \text{for } t \geq t_4, \tag{43}$$

which can be rewritten as

$$\left( (1 + t) \int_0^t \frac{e^\tau F(\tau)}{\tau + 1} \, d\tau \right)' \geq C_2 \varepsilon e^t + C_1 \int_0^t \frac{(e^\tau - e^\tau F(\tau)}{\tau + 1} \, d\tau \quad \text{for } t \geq t_4. \tag{44}$$
Setting $t_0 = t_4 + \ln 2$ and

$$
\alpha(t) = (1 + t) \int_0^t \frac{e^\tau F(\tau)}{\tau + 1} d\tau,
$$

$$
\beta(t) = \int_{t_0}^t \int_0^s \frac{(e^s - e^\tau)F_p(\tau)}{1 + \tau} d\tau ds.
$$

Hence integrating (44) over $[t_4, t]$, we obtain

$$
\alpha(t) - \alpha(t_4) \geq C_2 \varepsilon (e^t - e^{t_4}) + C_1 (\beta(t) - \beta(t_4)), \quad t \geq t_4. \tag{45}
$$

Noting that $\alpha(t_4) \geq 0$ and $\beta(t_4) \leq 0$, we then get from the above inequality that

$$
\alpha(t) \geq \frac{1}{2} C_2 \varepsilon e^t + \frac{1}{2} C_2 \varepsilon e^{t_4} - C_2 \varepsilon e^{t_4} + C_1 \beta(t)
$$

$$
\geq \frac{1}{2} C_2 \varepsilon e^t + C_1 \beta(t) \quad \text{for } t \geq t_0. \tag{46}
$$

Also from the definition of $\beta(t)$ we have

$$
\beta'(t) = \int_0^t \frac{(e^\tau - e^\tau)F_p(\tau)}{\tau + 1} d\tau,
$$

then

$$
\beta''(t) - \beta'(t) = \int_0^t \frac{e^\tau F_p(\tau)}{\tau + 1} d\tau. \tag{47}
$$

Before going on, we have another claim that

$$
\int_0^t \frac{e^\tau}{1 + \tau} d\tau \leq C_3 \frac{e^t}{1 + t}, \quad \text{for } t \geq 0, \tag{48}
$$

where $C_3$ denotes a positive constant independent of $\varepsilon$. For this claim, first it is easy to get that there exists a positive constant $C_3$ independent of $\varepsilon$ such that

$$
\int_0^t \frac{e^\tau}{1 + \tau} d\tau \leq \int_0^t \frac{e^\tau}{1 + \tau} d\tau \leq C_3 \frac{e^t}{1 + t}, \quad \text{for } t \geq 0.
$$

Since the integrand $K(t) = \frac{e^\tau}{1 + \tau}$ is an increasing function of $t$, then we have

$$
\int_0^t \frac{e^\tau}{1 + \tau} d\tau \leq \int_0^t \frac{e^\tau}{1 + \tau} d\tau \leq C_3 \frac{e^t}{1 + t}, \quad \text{for } t \geq 0,
$$

hence we prove claim (48) by setting $C_3 = 2C_3$. By Hölder inequality and (48), we obtain that

$$
\alpha(t) \leq (1 + t) \left( \int_0^t \frac{e^\tau F_p(\tau)}{\tau + 1} d\tau \right)^{\frac{1}{p}} \left( \int_0^t \frac{e^\tau}{\tau + 1} d\tau \right)^{\frac{1}{q}}
$$

$$
\leq C_4 e^t (1 + t)^{\frac{1}{p}} \left( \int_0^t \frac{e^\tau F_p(\tau)}{\tau + 1} d\tau \right)^{\frac{1}{p}}, \tag{49}
$$
where $C_4 := (C_3)^{1/p'}$. Thus, combining (46)-(47) and (49) we have
\[
\beta''(t) - \beta'(t) \geq \frac{C_5 \alpha^p(t)e^{-(p-1)t}}{1 + t} \geq \frac{C_6 \left(\varepsilon e^t + \beta(t)\right)^p e^{-(p-1)t}}{1 + t}, \quad t \geq t_0.
\]
where $C_5 := (C_4)^{-p}$ and
\[
C_6 := C_5 \left(\min\left\{\frac{1}{2}C_2, C_1\right\}\right)^p.
\]
Setting $\beta(t) = e^t \gamma(t)$ and plugging it into (50), we get
\[
\gamma''(t) + \gamma'(t) \geq \frac{C_6 (\varepsilon + \gamma(t))^p}{1 + t}, \quad t \geq t_0.
\]
Then, setting
\[
\tilde{\gamma}(t) = \varepsilon + \gamma(t) = \varepsilon + e^{-t} \beta(t) = \varepsilon + e^{-t} \int_{t_0}^{t} \int_{s}^{t} \frac{(e^s - e^\tau)F(p(\tau))}{1 + \tau} d\tau ds
\]
with
\[
\tilde{\gamma}(t_0) = \varepsilon > 0
\]
and
\[
\tilde{\gamma}'(t_0) > 0,
\]
it follows from (51) that
\[
\tilde{\gamma}''(t) + \tilde{\gamma}'(t) \geq \frac{C_6 \tilde{\gamma}^p(t)}{1 + t}, \quad t \geq t_0.
\]
We have

**Lemma 3.1 (Theorem 3.1 of Li and Zhou [5]).** Suppose that $I(t)$ satisfies
\[
I''(t) + I'(t) \geq C_0 \frac{I^{1+\alpha}(t)}{(1 + t)^{\beta}}, \quad (C_0 > 0, \text{constant})
\]
and
\[
I(0) > 0, \quad I'(0) \geq 0,
\]
where $\alpha > 0$. Then, when $0 \leq \beta \leq 1$, $I = I(t)$ must blow up in a finite time.
Moreover, if $I(0) = \varepsilon$, where $\varepsilon > 0$ is a small parameter, then the lifespan $T(\varepsilon)$ of $I = I(t)$ has the following upper bound:
\[
T(\varepsilon) \leq \left\{ \begin{array}{ll}
\exp(a \varepsilon^{-\alpha}), & \beta = 1, \\
\varepsilon^{-\frac{1}{p-1}}, & 0 \leq \beta < 1,
\end{array} \right.
\]
where $a$ is a constant.
where $a$ and $b$ are positive constants independent of $\varepsilon$.

Applying Lemma 3.1 to $I(t) = \tilde{\gamma}(t)$ with $p = 1 + \alpha = 1 + \frac{2}{n}$ and taking $\beta = 1$ and the initial time $t = t_0$, we then get the desired sharp upper bound of the lifespan given in Theorem 1.2.

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