Let $F$ be a totally real field. In Grobner–Raghuram [14] the special values of the standard $L$-function of a cuspidal automorphic representation $\pi$ of $GL(2n)/F$ of cohomological type and admitting a Shalika model were studied. As explained via examples in loc. cit., the geometric condition of being cohomological and the analytic condition of admitting a Shalika model are entirely independent of each other, and there is no a priori reason why such a $\pi$ should even exist. The purpose of this article is to address such existence questions; indeed, we prove that such a $\pi$ exists. More generally, let $G = \text{Res}_{F/Q}(GL_n)$ where $F$ now is any number field, and let $S^G_{K_f}$ denote an adelic locally symmetric space for some level structure $K_f$. Let $\mathcal{M}_{\mu,\mathbb{C}}$ be an algebraic irreducible representation of $G(\mathbb{R})$ and we let $\overline{\mathcal{M}}_{\mu,\mathbb{C}}$ denote the associated sheaf on $S^G_{K_f}$. The basic problem addressed in this paper is to classify the data $(F, n, \mu)$ for which cuspidal cohomology of $G$ with $\mu$-coefficients, denoted $H_{\text{cusp}}^\bullet(S^G_{K_f}, \overline{\mathcal{M}}_{\mu,\mathbb{C}})$, is nonzero for some $K_f$. We prove nonvanishing of cuspidal cohomology when $F$ is a totally real field or a totally imaginary quadratic extension of a totally real field, and also for a general number field but when $\mu$ is a parallel weight. The proof in the totally real case involves Arthur’s endoscopic classification [1] of the discrete spectrum for classical groups, together with Clozel’s results on globalizing discrete series representations via limit multiplicity arguments [8]; in the CM case we use Mok’s classification [23] of the discrete spectrum for unitary groups; and for the parallel weight case it is an easy generalization of the constructive proof in Clozel [10]. Finally, we add some remarks on an endoscopic stratification of inner cohomology suggested by results on arithmeticity as in Gan–Raghuram [12].

1. Statements of the main results

1.1. Cuspidal cohomology of $GL_n$. We need a somewhat lengthy notational preparation; the reader familiar with such objects can directly jump to (1.1.3). Let $F$ be a number field of degree $d = [F : \mathbb{Q}]$ with ring of integers $\mathcal{O}$. For any place $v$ we write $F_v$ for the topological completion of $F$ at $v$. Let $S_\infty(F) = S_\infty$ be the set of archimedean places of $F$. Let $S_\infty = S_r \cup S_c$, where $S_r$ (resp., $S_c$) is the set of real (resp., complex) places. Let $\mathcal{E}_F = \text{Hom}(F, \mathbb{C})$ be the set of all embeddings of $F$ as a field into $\mathbb{C}$. There is a canonical surjective map $\mathcal{E}_F \to S_\infty$, which is a bijection on the real embeddings and real places, and identifies a pair of complex conjugate embeddings $\{\iota_v, \overline{\iota}_v\}$ with the complex place $v$. For each $v \in S_r$, we fix an isomorphism $F_v \cong \mathbb{R}$ which is canonical. Similarly for $v \in S_c$, we fix $F_v \cong \mathbb{C}$ given by (say) $\iota_v$; this choice is not canonical. Let $r_1 = |S_r|$ and $r_2 = |S_c|$; hence $d = r_1 + 2r_2$. If $v \notin S_\infty$, we let $\mathcal{O}_v$ be the ring of integers at $v$. The reader may find it useful to familiarize himself with the following objects:

- The field $F_v$ for each place $v$.
- The set $S^G_{K_f}$ of adelic locally symmetric spaces.
- The representation $\mathcal{M}_{\mu,\mathbb{C}}$ of $G(\mathbb{R})$. 
- The sheaf $\overline{\mathcal{M}}_{\mu,\mathbb{C}}$ on $S^G_{K_f}$.

For the reader’s convenience, we list the notation used in this paper:

- $F$: A number field.
- $d$: The degree of $F$ over $\mathbb{Q}$.
- $\mathcal{O}$: The ring of integers of $F$.
- $v$: A place of $F$.
- $F_v$: The completion of $F$ at $v$.
- $S_\infty$: The set of archimedean places of $F$.
- $S_r$: The set of real places of $F$.
- $S_c$: The set of complex places of $F$.
- $S^G_{K_f}$: The adelic locally symmetric space for $G$ with level structure $K_f$.
- $\mathcal{M}_{\mu,\mathbb{C}}$: An algebraic irreducible representation of $G(\mathbb{R})$.
- $\overline{\mathcal{M}}_{\mu,\mathbb{C}}$: The associated sheaf on $S^G_{K_f}$.

The main results of this paper are as follows:

- For $F$ a totally real field, $G = \text{Res}_{F/Q}(GL_n)$, and $\mu$ a parallel weight, the cuspidal cohomology $H_{\text{cusp}}^\bullet(S^G_{K_f}, \overline{\mathcal{M}}_{\mu,\mathbb{C}})$ is nonzero for some $K_f$.
- For $F$ a totally imaginary quadratic extension of a totally real field, and $G = \text{Res}_{F/Q}(GL_n)$, the cuspidal cohomology $H_{\text{cusp}}^\bullet(S^G_{K_f}, \overline{\mathcal{M}}_{\mu,\mathbb{C}})$ is nonzero for some $K_f$.

The proof in the totally real case involves Arthur’s endoscopic classification [1] of the discrete spectrum for classical groups, together with Clozel’s results on globalizing discrete series representations via limit multiplicity arguments [8]; in the CM case we use Mok’s classification [23] of the discrete spectrum for unitary groups; and for the parallel weight case it is an easy generalization of the constructive proof in Clozel [10]. Finally, we add some remarks on an endoscopic stratification of inner cohomology suggested by results on arithmeticity as in Gan–Raghuram [12].
of integers of \( F_v \), and \( \varphi_v \) it’s unique maximal ideal. Moreover, \( A_F \) denotes the ring of adèles of \( F \) and \( A_{F,f} \) its finite part. The group of idèles of \( F \) will be denoted \( A_F^\times \) and similarly, \( A_{F,f}^\times \) is the group of finite idèles. We will drop the subscript \( F \) when talking about \( Q \). Hence, \( A \) is \( A_{Q,\cdot} \), etc.

The algebraic group \( GL_n / F \) will be denoted as \( G_n \), and we put \( G_n = \text{Res}_{F/Q}(G_n) \); an \( F \)-group will be denoted by an underline and the corresponding \( Q \)-group via Weil restriction of scalars will be denoted without the underline; hence for any \( Q \)-algebra \( A \), the group of \( A \)-points of \( G_n \) is \( G_n(A) = G_n(A \otimes_Q F) \). Let \( \overline{B}_n = T_n U_n \) stand for the standard Borel subgroup of \( G_n \) of all upper triangular matrices, where \( U_n \) is the unipotent radical of \( B_n \), and \( T_n \) the diagonal torus. The center of \( G_n \) will be denoted by \( Z_n \). These groups define the corresponding \( Q \)-groups \( G_n \supset B_n = T_n U_n > Z_n \). Observe that \( Z_n \) is not \( Q \)-split, and we let \( S_n \) be the maximal \( Q \)-split torus in \( Z_n \); we have \( S_n \cong G_m \) over \( Q \).

Note that

\[
G_{n,\infty} := G_n(\mathbb{R}) = G_n(F \otimes \mathbb{Q} \mathbb{R}) = \prod_{v \in S_\infty} GL_n(F_v) \cong \prod_{v \in S_F} GL_n(\mathbb{R}) \times \prod_{v \in S_C} GL_n(\mathbb{C}).
\]

We have \( Z_n(\mathbb{R}) \cong \prod_{v \in S_\infty} \mathbb{R}^\times \times \prod_{v \in S_C} \mathbb{C}^\times \). The subgroup \( S_n(\mathbb{R}) \) is \( \mathbb{R}^\times \) diagonally embedded in \( Z_n(\mathbb{R}) \). Let \( C_{n,\infty} = \prod_{v \in S_\infty} O(n) \times \prod_{v \in S_C} U(n) \) be the maximal compact subgroup of \( G_n(\mathbb{R}) \), and let \( K_{n,\infty} = Z_n(\mathbb{R}) C_{n,\infty} = Z_n(\mathbb{R})^0 C_{n,\infty} \). Let \( K_{n,\infty}^0 \) be the topological connected component of \( K_{n,\infty} \). For a real Lie group \( G \), we denote its Lie algebra by \( \mathfrak{g}^0 \) and the complexified Lie algebra by \( \mathfrak{g} \), i.e., \( \mathfrak{g} = \mathfrak{g}^0 \otimes \mathbb{C} \). If \( G = GL_n(\mathbb{R}) \) then \( \mathfrak{g}^0 = \mathfrak{gl}_n(\mathbb{R}) \) and \( \mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}) \), on the other hand, if \( G \) stands for the real Lie group \( GL_n(\mathbb{C}) \) then \( \mathfrak{g}^0 = \mathfrak{gl}_n(\mathbb{C}) \) as a Lie algebra over \( \mathbb{R} \), and \( \mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}) \otimes \mathbb{C} \). With this notational scheme, we have \( \mathfrak{g}_n \), \( \mathfrak{b}_n \), \( t_n \) and \( \mathfrak{t}_n \) denoting the complexified Lie algebras of \( G_n(\mathbb{R}) \), \( B_n(\mathbb{R}) \), \( T_n(\mathbb{R}) \) and \( K_{n,\infty}^0 \) respectively. For example, \( \mathfrak{g}_n = \prod_{v \in S_\infty} \mathfrak{g}_n(\mathbb{C}) \times \prod_{v \in S_C} (\mathfrak{gl}_n(\mathbb{C}) \otimes \mathbb{C}) \).

Consider \( T_n(\mathbb{R}) = \overline{T}_n(F \otimes \mathbb{R}) \cong \prod_{v \in S_{\infty}} T_n(F_v) \). We let \( X^+(T_n \times \mathbb{C}) \) stand for the group of all algebraic characters of \( T_n \times \mathbb{C} \), and let \( X^+(T_n \times \mathbb{C}) \) stand for all those characters in \( X^+(T_n \times \mathbb{C}) \) which are dominant with respect to \( B_n \). A weight \( \mu \in X^+(T_n \times \mathbb{C}) \) may be described as follows: \( \mu = (\mu_v^{(i)})_{v \in S_F} \), also sometimes written as \( \mu = (\mu_v^{(i)})_{v \in S_{\infty}} \), where \( \mu_v^{(i)} = (\mu_v^{(i)^{\text{even}}}, \mu_v^{(i)^{\text{odd}}}) \) for \( v \in S_C \), and furthermore

- for \( v \in S_F \), we have \( \mu_v^{(i)} = (\mu_v^{(i)^{\text{even}}}, \mu_v^{(i)^{\text{odd}}}) \), \( \mu_v^{(i)} \in \mathbb{Z}, \mu_v^{(i)^{\text{even}}} \geq \cdots \geq \mu_v^{(i)^{\text{odd}}} \), and the character \( \mu_v^{(i)} \) sends \( t = \text{diag}(t_1, \ldots, t_n) \in T_n(F_v) \) to \( \prod_i t_i^{\mu_v^{(i)^{\text{even}}} \mu_v^{(i)^{\text{odd}}}} \), and

- if \( v \in S_C \) then \( \mu_v^{(i)} \) is a pair \( (\mu_v^{(i)^{\text{even}}}, \mu_v^{(i)^{\text{odd}}}) \), with \( \mu_v^{(i)^{\text{even}}} = (\mu_v^{(i)^{\text{even}}}^{(1)}, \ldots, \mu_v^{(i)^{\text{even}}}^{(n)}) \), \( \mu_v^{(i)^{\text{odd}}} \in \mathbb{Z}, \mu_v^{(i)^{\text{even}}} \geq \cdots \geq \mu_v^{(i)^{\text{odd}}} \); likewise \( \mu_v^{(i)^{\text{even}}} = (\mu_v^{(i)^{\text{even}}}^{(1)}, \ldots, \mu_v^{(i)^{\text{even}}}^{(n)}) \) and \( \mu_v^{(i)^{\text{odd}}} \geq \cdots \geq \mu_v^{(i)^{\text{odd}}} \); the character \( \mu_v^{(i)} \) is given by sending \( t = \text{diag}(z_1, \ldots, z_n) \in T_n(F_v) \) to \( \prod_{i=1}^n z_i^{\mu_v^{(i)^{\text{even}}} \mu_v^{(i)^{\text{odd}}}} z_i^{\mu_v^{(i)^{\text{odd}}}} \), where \( z_i \) is the conjugate of \( z_i \).

For \( \mu \in X^+(T_n \times \mathbb{C}) \), define a finite-dimensional complex representation \((\rho_\mu, \mathcal{M}_\mu, \mathcal{C})\) of \( G_n(\mathbb{R}) \) as follows: For \( v \in S_\infty \), let \((\rho_\mu^{(i)}, \mathcal{M}_\mu^{(i)}, \mathcal{C})\) be the irreducible complex representation of \( G_n(F_v) = GL_n(\mathbb{R}) \) with highest weight \( \mu_v^{(i)} \). For \( v \in S_\infty \), let \((\rho_\mu^{(i)}, \mathcal{M}_\mu^{(i)}, \mathcal{C})\) be the complex representation of the real algebraic group \( G(F_v) = GL_n(\mathbb{C}) \) defined as \( \rho_\mu^{(i)}(g) = \rho_\mu^{(i)^{\text{even}}}(g) \otimes \rho_\mu^{(i)^{\text{odd}}}(g) \); here \( \rho_\mu^{(i)^{\text{even}}} \) (resp., \( \rho_\mu^{(i)^{\text{odd}}} \)) is the irreducible representation of the complex group \( GL_n(\mathbb{C}) \) with highest weight \( \mu_v^{(i)^{\text{even}}} \) (resp., \( \mu_v^{(i)^{\text{odd}}} \)). Now we let \( \rho_\mu = \bigotimes_{v \in S_{\infty}} \rho_\mu^{(i)^{\text{even}}} \) which acts on \( \mathcal{M}_\mu, \mathcal{C} = \bigotimes_{v \in S_{\infty}} \mathcal{M}_\mu^{(i)}, \mathcal{C} \).
Let $K_f$ be an open compact subgroup of $G_n(\mathbb{A}_f) = GL_n(\mathbb{A}_{F,f})$. Let us write $K_f = \prod_p K_p$ where each $K_p$ is an open compact subgroup of $G_n(\mathbb{Q}_p)$ and for almost all $p$ we have $K_p = \prod_{v|p} GL_n(\mathcal{O}_v)$. Define the double-coset space
\[ S^{G_n}_{K_f} = \frac{G_n(\mathbb{Q}) \backslash G_n(\mathbb{A})}{K_{n,\infty}^0 K_f} = \frac{GL_n(F) \backslash GL_n(\mathbb{A}_F)}{K_{n,\infty}^0 K_f}. \]

Given a dominant-integral weight $\mu \in X^+(T_n \times \mathbb{C})$ we get a sheaf $\widetilde{\mathcal{M}}_{\mu,\mathbb{C}}$ of $\mathbb{C}$-vector spaces on $S^{G_n}_{K_f}$. (See, for example, [25, Sect. 2.3.3].) We are interested in the sheaf cohomology groups $H^*(S^{G_n}_{K_f}, \widetilde{\mathcal{M}}^\vee_{\mu,\mathbb{C}})$. Here $\widetilde{\mathcal{M}}^\vee_{\mu,\mathbb{C}}$ is the sheaf attached to the contragredient representation $\mathcal{M}^\vee_{\mu,\mathbb{C}}$ of $\mathcal{M}_{\mu,\mathbb{C}}$. This dualizing is only for convenience. Let $\omega_{\rho_\mu}$ be the central character of $\rho_\mu$. For the sheaf $\widetilde{\mathcal{M}}^\vee_{\mu,\mathbb{C}}$ or $\mathcal{M}_{\mu,\mathbb{C}}$ to be nonzero we need the condition:

\[(1.1.1) \quad \text{"The central character } \omega_{\rho_\mu} \text{ is trivial on } Z_n(\mathbb{Q}) \cap K_{n,\infty}^0 K_f."\]

Henceforth, we will assume this condition on $\mu$ and $K_f$. (This can be a nontrivial condition even in simple situations: take, for example, $F = \mathbb{Q}$, $n = 2$, $K_f = \prod_p GL_2(\mathbb{Z}_p)$, and $\mu = (a,b)$ with integers $a \geq b$; then this condition boils down to $(-1)^{a+b} = 1$; but by taking $K_f$ slightly deep enough we can ensure $Z_2(\mathbb{Q}) \cap K_{2,\infty}^0 K_f$ is trivial and so the condition vacuously holds.) For more details, the reader is referred to Harder [16, (1.1.3)]. Passing to the limit over all open compact subgroups $K_f$ and let $H^*(S^{G_n}_{K_f}, \widetilde{\mathcal{M}}^\vee_{\mu,\mathbb{C}}) := \lim_{\longrightarrow K_f} H^*(S^{G_n}_{K_f}, \widetilde{\mathcal{M}}^\vee_{\mu,\mathbb{C}})$. There is an action of $\pi_0(G_{n,\infty}) \times G_n(\mathbb{A}_f)$ on $H^*(S^{G_n}_{K_f}, \widetilde{\mathcal{M}}^\vee_{\mu,\mathbb{C}})$, and the cohomology of $S^{G_n}_{K_f}$ is obtained by taking invariants under $K_f$, i.e., $H^*(S^{G_n}_{K_f}, \widetilde{\mathcal{M}}^\vee_{\mu,\mathbb{C}}) = H^*(S^{G_n}_{K_f}, \widetilde{\mathcal{M}}^\vee_{\mu,\mathbb{C}})^{K_f}$. We can compute the above sheaf cohomology via the de Rham complex, and then reinterpret the de Rham complex in terms of the complex computing relative Lie algebra cohomology, we get the isomorphism:

\[ H^*(S^{G_n}_{K_f}, \widetilde{\mathcal{M}}^\vee_{\mu,\mathbb{C}}) \simeq H^*(\mathfrak{g}_n, K_{n,\infty}^0; C^\infty(G_n(\mathbb{Q}) \backslash G_n(\mathbb{A})) \otimes \mathcal{M}_{\mu,\mathbb{C}}). \]

The inclusion $C^\infty_{\text{cusp}}(G_n(\mathbb{Q}) \backslash G_n(\mathbb{A})) \hookrightarrow C^\infty(G_n(\mathbb{Q}) \backslash G_n(\mathbb{A}))$ of the space of smooth cusp forms in the space of all smooth functions induces, via results of Borel [5], an injection in cohomology; this defines cuspidal cohomology:

\[ H^*(S^{G_n}_{K_f}, \widetilde{\mathcal{M}}^\vee_{\mu,\mathbb{C}}) \simeq H^*(\mathfrak{g}_n, K_{n,\infty}^0; C^\infty_{\text{cusp}}(G_n(\mathbb{Q}) \backslash G_n(\mathbb{A})) \otimes \mathcal{M}_{\mu,\mathbb{C}}). \]

With level structure $K_f$ we have:

\[ H^*_{\text{cusp}}(S^{G_n}_{K_f}, \widetilde{\mathcal{M}}^\vee_{\mu,\mathbb{C}}) \simeq H^*(\mathfrak{g}_n, K_{n,\infty}^0; C^\infty_{\text{cusp}}(G_n(\mathbb{Q}) \backslash G_n(\mathbb{A}))^{K_f} \otimes \mathcal{M}_{\mu,\mathbb{C}}). \]

A fundamental problem concerning cuspidal cohomology for $GL_n/F$ is:

**Problem 1.1.4.** Classify all $(F, n, \mu, K_f)$, subject to (1.1.1), for which $H^*_{\text{cusp}}(S^{G_n}_{K_f}, \widetilde{\mathcal{M}}^\vee_{\mu,\mathbb{C}}) \neq 0$.

This problem includes classical situations such as: take $F = \mathbb{Q}$ and $n = 2$, and given integers $N \geq 1$ and $k \geq 2$ is there a holomorphic cusp form of weight $k$ for $\Gamma_1(N) \subset SL_2(\mathbb{Z})$? One may relax the dependence on an explicit level structure $K_f$ and ask for a solution of the weaker
Problem 1.1.5. Classify all \((F, n, \mu)\) for which \(H^\bullet_{\text{cusp}}(S^G_n, \mathcal{M}^\nu_{\mu, \mathbb{C}}) \neq 0\) for some \(K_f\).

The purpose of this article is to provide a solution of the weaker problem for \(F\) a totally real field, or a totally imaginary quadratic extension of a totally real field, or for a general number field \(F\) and for a parallel weight \(\mu\); see below for a definition of a parallel weight.

1.2. Necessary conditions for nonvanishing of cuspidal cohomology. Using the usual decomposition of the space of cusp forms into a direct sum of cuspidal automorphic representations, we get the following fundamental decomposition of \(\pi_0(G_n(\mathbb{R})) \times G_n(\mathbb{A}_f)\)-modules:

\[
H^\bullet_{\text{cusp}}(S^G_n, \mathcal{M}^\nu_{\mu, \mathbb{C}}) = \bigoplus \Pi H^\bullet(\mathfrak{g}_n, K^0_{n, \infty}; \Pi_{\infty} \otimes \mathcal{M}^\nu_{\mu, \mathbb{C}}) \otimes \Pi_f.
\]

We say that \(\Pi\) contributes to the cuspidal cohomology of \(G_n\) with coefficients in \(\mathcal{M}^\nu_{\mu, \mathbb{C}}\), and we write \(\Pi \in \text{Coh}(G_n, \mu^\nu)\), if \(\Pi\) has a nonzero contribution to the above decomposition. Equivalently, if \(\Pi\) is a cuspidal automorphic representation whose representation at infinity \(\Pi_{\infty}\) after twisting by \(\mathcal{M}^\nu_{\mu, \mathbb{C}}\) has nontrivial relative Lie algebra cohomology. With a level structure \(K_f\), (1.2.1) takes the form:

\[
H^\bullet_{\text{cusp}}(S^G_n, \mathcal{M}^\nu_{\mu, \mathbb{C}}) = \bigoplus \Pi H^\bullet(\mathfrak{g}_n, K^0_{n, \infty}; \Pi_{\infty} \otimes \mathcal{M}^\nu_{\mu, \mathbb{C}}) \otimes \Pi^K_f.
\]

We write \(\Pi \in \text{Coh}(G_n, \mu^\nu, K_f)\) if a cuspidal automorphic representation \(\Pi\) contributes nontrivially to (1.2.2). Note that each \(\Pi\) \(\in\) \(\text{Coh}(G_n, \mu^\nu)\) is a finite set, and clearly, \(\text{Coh}(G_n, \mu^\nu) = \cup_{K_f} \text{Coh}(G_n, \mu^\nu, K_f)\).

Suppose \(\Pi \in \text{Coh}(G_n, \mu^\nu)\). The fact that \(\mu^\nu\) supports cuspidal cohomology places some restrictions on \(\mu\). First of all, essential unitarity of \(\Pi\), and in particular of \(\Pi_{\infty}\) gives, via Wigner’s Lemma, essential self-duality of \(\mu\): there is an integer \(w(\mu)\) such that

1. For \(v \in S_r\) and \(1 \leq i \leq n\) we have \(\mu_{i^v} + \mu_{n+i-1}^v = w(\mu)\);

2. For \(v \in S_c\) and \(1 \leq i \leq n\) we have \(\mu_{i^c} + \mu_{n+i-1}^c = w(\mu)\).

We will call such a weight \(\mu\) as a pure weight and call \(w(\mu)\) the purity weight of \(\mu\). Let \(X^+_0(T_n)\) denote the set of dominant integral pure weights. Applying Clozel [9, Theorem 3.13], we get

\[\Pi \in \text{Coh}(G_n, \mu^\nu) \implies \sigma\Pi \in \text{Coh}(G_n, \sigma^\nu\mu), \quad \forall \sigma \in \text{Aut}(\mathbb{C}),\]

where, \(\sigma\mu \in X^*(T_n \times \mathbb{C})\) is defined as: \(\sigma\mu = (\sigma\mu^i)_{i \in \mathcal{E}_F}\) with \(\sigma\mu^i := \mu^{\sigma^{-1} a_i}\). The reader is referred to [9] or [25] for a definition of \(\sigma\Pi\). In particular, \(\sigma\mu\) also satisfies the purity conditions (1) and (2) above. Note that \(w(\mu) = w(\sigma\mu)\).

Definition 1.2.3. Let \(\mu \in X^+_0(T_n)\) be a pure dominant integral weight. We say \(\mu\) is strongly pure if \(\sigma\mu\) is pure for all \(\sigma \in \text{Aut}(\mathbb{C})\). Let \(X^+_0(T_n)\) stand for the set of dominant integral strongly pure weights.

For any \(F\), we have the following inclusions \(X^+_{00}(T_n) \subset X^+_0(T_n) \subset X^+(T_n) \subset X^*(T_n)\) and in general they are all strict inclusions. If \(F\) is a totally real field or a CM field (totally imaginary quadratic extension of a totally real field) then \(\mu\) is pure if and only if \(\mu\) is strongly pure. For any number field, one may see that there are strongly pure weights: take an integer \(b\) and integers \(a_1 \geq a_2 \geq \cdots \geq a_n\) such that \(a_j + a_{n-j+1} = b\); now for each \(i \in \mathcal{E}_F\) put...
µ^t = (a_1, \ldots, a_n); then µ is strongly pure with w(µ) = b; such a weight may be called a parallel weight. We formulate the following conjecture as a possible answer to Problem 1.1.5.

**Conjecture 1.2.4.** Let \( \mu \in X^+(T_n \times \mathbb{C}) \) be a dominant integral weight. Then
\[
H^\bullet\text{cusp}(S_{G_n}^{F_0}, \widetilde{M}_{\mu,\mathbb{C}}') \neq 0 \text{ for some } K_f \iff \mu \in X^+_0(T_n \times \mathbb{C}).
\]

We give a brief glimpse, without pretending to be exhaustive, into available results in the literature. Borel, Labesse and Schwermer [7, Theorem 11.3] proved nonvanishing of cuspidal cohomology for \( G = \text{SL}_n \) over any number field which is an extension via cyclic prime degree extensions of a totally real number field, and for the trivial coefficient system, i.e., when \( \mu = 0 \); their proof involved studying Lefschetz numbers for rational automorphisms of \( G \). Barbasch and Speh [3, §XI] considered \( \text{GL}_n/\mathbb{Q} \) and a coefficient system with certain technical restrictions on the weight \( \mu \) (which in fact exclude many pure weights \( \mu \)) and proved nonvanishing of cuspidal cohomology via an application of trace formula for certain Lefschetz functions. Clozel [10, Theorem 4] gave a constructive proof of the nonvanishing of cuspidal cohomology for \( \text{GL}_{2n} \) over any number field for the trivial coefficient system using automorphic induction.

### 1.3. The main results of this article on cuspidal cohomology of \( \text{GL}(n) \).

**Theorem 1.3.1.** Take an integer \( N \geq 2 \). Let \( F_0 \) be a totally real field extension of \( \mathbb{Q} \), and we take an extension \( F/F_0 \) to be either

1. \( F = F_0 \), i.e., \( F \) itself is a totally real field; or
2. \( F \) is a totally imaginary quadratic extension over \( F_0 \).

Let \( G = G_N = \text{Res}_{F/\mathbb{Q}}(\text{GL}_N/F) \). Let \( \mu \in X^+_0(T_N \times \mathbb{C}) \) be a pure dominant integral weight with purity \( w(\mu) = 0 \). In case (2), assume furthermore that \( \mu \) is trivial on the center \( Z_N \), i.e., for all \( \iota \in \mathbb{E}_F \) suppose that
\[
\mu_1 + \cdots + \mu_N = 0.
\]
Then
\[
H^\bullet\text{cusp}(S^G, \widetilde{M}_{\mu',\mathbb{C}}) \neq 0.
\]

The proof involves endoscopic transfer from certain classical groups and breaks up into the following sub-cases:

1. \( F \) is totally real, and \( N = 2n + 1 \) is odd;
2. \( F \) is totally real, and \( N = 2n \) is even;
3. \( F \) is a totally imaginary quadratic extension of a totally real \( F_0 \).

In case (1a), we transfer from the group \( G' = \text{Sp}(2n)/F \). To begin, we transfer the weight \( \mu \) to a weight \( \mu' \) on \( G' \). Then, using results on limit multiplicities due to Clozel [8], we produce a cuspidal representation \( \pi' \) of \( G' \) with a discrete series representation at infinity which is cohomological with respect to \( \mu' \). Furthermore, one can arrange for \( \pi' \) to have the Steinberg representation at some finite place. Arthur’s results ensure then that \( \pi' \) corresponds to an Arthur parameter \( \pi \) on \( G \) which is cuspidal; the cuspidality of \( \pi \) uses an observation of Magaard and Savin [22], and then one checks that indeed \( \pi \) contributes to the above cohomology group.

For the proof in case (1b) (resp., case (2)) we transfer from the split group \( G' = \text{SO}(2n + 1) \) (resp., \( G' \) a unitary group in \( N \) variables). In case (1a) it might be possible to use the results of Weselmann’s papers [33] and [34].
We may draw several inferences from the above proof. As summarized in [14, Section 3.1], a cuspidal representation \( \pi \) of \( \text{GL}_{2n}/F \) is a transfer from \( \text{SO}(2n+1) \) if and only if a partial exterior-square \( L \)-function \( L^S(s, \wedge^2, \pi) \) has a pole at \( s = 1 \) and this is so if and only if \( \pi \) admits a Shalika model, which gives us

**Corollary 1.3.2.** Let \( F \) be a totally real field, and take \( G = \text{GL}(2n)/F \). Let \( \mu \) be a pure weight with purity 0. Then there exists a cuspidal automorphic representation \( \pi \) of \( G \) such that

1. \( \pi \in \text{Coh}(G, \mu) \), i.e., it is cohomological with respect to \( \mu \), and
2. \( \pi \) has a Shalika model, or equivalently, a partial exterior-square \( L \)-function \( L^S(s, \wedge^2, \pi) \) has a pole at \( s = 1 \).

A similar characterization for a \( \pi \) on \( \text{GL}_{2n+1}/F \) being a transfer from \( \text{Sp}(2n) \) if and only if a partial symmetric-square \( L \)-function \( L^S(s, \text{Sym}^2, \pi) \) having a pole at \( s = 1 \) gives us

**Corollary 1.3.3.** Let \( F \) be a totally real field, and take \( G = \text{GL}(2n+1)/F \). Let \( \mu \) be a pure weight with purity 0. Then there exists a cuspidal automorphic representation \( \pi \) of \( G \) such that

1. \( \pi \in \text{Coh}(G, \mu) \), i.e., it is cohomological with respect to \( \mu \), and
2. a partial symmetric-square \( L \)-function \( L^S(s, \text{Sym}^2, \pi) \) has a pole at \( s = 1 \).

Using an idea in Labesse and Schwermer [21], that at a finite place the Steinberg representation retains the property of being Steinberg upon base change, and at archimedean places the property of being cohomological is preserved under base-change, we get the following

**Corollary 1.3.4.** Let \( F \) be a totally real field and suppose that \( \tilde{F}/F \) is a finite extension that is filtered by cyclic extensions of prime degrees. Let \( \mu \) be a pure weight for \( G = \text{GL}(N)/\tilde{F} \) with purity 0. Define a weight \( \tilde{\mu} \) for \( \tilde{G} = \text{GL}(N)/\tilde{F} \) as for any \( \tilde{\iota}: \tilde{F} \to \mathbb{C} \), we let \( \tilde{\mu}^{\tilde{\iota}} = \mu^{\iota} \) where \( \iota = \tilde{\iota}|_F \). (If \( \mu \) is a parallel weight then so is \( \tilde{\mu} \).) Then

\[
H^\bullet_{\text{cusp}}(S^G, M_{\mu, \mathbb{C}}) \neq 0.
\]

The conditions on \( \tilde{F}/F \) are dictated by the main theorem on base change for \( \text{GL}_N \) due to Arthur and Clozel [2].

Next, we consider a general number field \( F \). The following theorem is a generalization of the main theorem of Clozel [10] which is proved by constructing cohomological cuspidal representations via automorphic induction. See also Ramakrishnan–Wang [30, Appendix].

**Theorem 1.3.5.** Let \( F \) be any number field, and take \( G = \text{GL}(2n)/F \). Let \( \mu \) be a parallel weight with purity \( w(\mu) = 0 \). Then

\[
H^\bullet_{\text{cusp}}(S^G, M_{\mu, \mathbb{C}}) \neq 0.
\]

2. Proof of Theorem 1.3.1

2.1. Archimedean preliminaries. Case (1a) \( (F \text{ is totally real, and } N = 2n+1 \text{ is odd.}) \)

Fix a real place \( v \) of \( F \), and since \( \mu \) is a pure weight with purity 0, we have

\[
\mu_1^v \geq \mu_2^v \geq \cdots \geq \mu_n^v \geq 0 \geq -\mu_n^v \geq \cdots \geq -\mu_2^v \geq -\mu_1^v.
\]
The place $v$ of $F$ is fixed, and for brevity, we will drop $v$ from the notation for $\mu^v$.

Consider the endoscopy group $G' = \text{Sp}(2n)/F$ defined so that upper-triangular subgroup $B'$ in $G'$ is a Borel subgroup. The connected component of the $L$-group of $G'$ is $L_{G'^0} = \text{SO}(2n + 1, \mathbb{C})$. The maximal compact subgroup $K'$ of $G'(\mathbb{R}) = \text{Sp}(2n, \mathbb{R})$ is isomorphic to $\text{U}(n)$. Define a dominant integral weight $\mu'$ for $G'$ which at the place $v$ (dropped from the notation) is given by:

$$\mu' := (\mu_1, \mu_2, \ldots, \mu_n) = \sum_{i=1}^n \mu_i e_i,$$

where $e_i$ gives the $i$-th coordinate of a diagonal matrix. Let $\rho'$ be the half-sum of positive roots for $G'$, which is written as $\rho' = \sum_{j=1}^n (n+1-j)e_j = (n, n-1, \ldots, 1)$. Let

$$\Lambda' = \mu' + \rho' = (\mu_1 + n, \mu_2 + n - 1, \ldots, \mu_{n-1} + 1, \mu_n).$$

Thus $\Lambda'$ is a regular weight and using Harish-Chandra’s classification theorem of discrete series representations (see, for example, Knapp [20]), there exists a discrete series representation $\pi' = \pi_{\Lambda'}$ of $G'(\mathbb{R})$ whose infinitesimal character is $\chi_{\Lambda'}$. Let $M_{\mu', \mathbb{C}}$ be the algebraic irreducible representation of $G'(\mathbb{C})$ with highest weight $\mu'$. From the well-known results on the cohomology of discrete series representations (see, for example, Borel-Wallach [6, Theorem II.5.3]), one knows that $\pi'$ is cohomological with respect to the coefficient system $M_{\mu', \mathbb{C}}$ of $G'$, i.e., the relative Lie algebra cohomology $H^*(g'_{\infty}, K', \pi' \otimes M_{\mu'})$ is non-zero; in fact it is nonzero only in the middle degree $\bullet = \frac{1}{2} \dim(G'(\mathbb{R}))/\dim(K')$.

The shape of the Langlands parameter, denoted $\tau_{\Lambda'}$, of the discrete series representation $\pi_{\Lambda'}$ of $G'(\mathbb{R})$ is well-known; we can deduce the following from [4, Example 10.5]:

$$\tau_{\Lambda'} = \text{Ind}_{C^\times}^{W_\mathbb{R}}(\chi_{\ell_1}) \oplus \text{Ind}_{C^\times}^{W_\mathbb{R}}(\chi_{\ell_2}) \oplus \cdots \oplus \text{Ind}_{C^\times}^{W_\mathbb{R}}(\chi_{\ell_n}) \oplus \text{sgn}^z,$$

where $\ell_1, \ldots, \ell_n$ are positive integers and the first $n$-summands are irreducible 2-dimensional representations of the Weil group $W_\mathbb{R}$ of $\mathbb{R}$ induced from characters of $C^\times$ with the character $\chi_{\ell_j}$ sending $z = re^{i\theta} \in C^\times$ to $r^\ell_j e^{i\theta} = (z/\bar{z})^{\ell_j/2}$ with the proviso that each summand be of orthogonal type forcing each $\ell_j$ to be even, and the determinant of the parameter is 1 so as to have image inside $L_{G'^0}$ forcing the last summand to be $\text{sgn}^z$. Furthermore, the relation between the integers $\ell_j$ and the weight $\Lambda'$ is captured by

$$\tau_{\Lambda'}|_{C^\times} = z^{\Lambda'} \bar{z}^{-\Lambda'},$$

(Here we have tacitly used that $\Lambda'$, which is a character of a maximal torus $T'$ of $G'$, is also, by definition, a co-character of the dual $L_{T'^0} \subset L_{G'^0}$ justifying the above notation.) If we put $\ell = (\ell_1, \ldots, \ell_n)$ then we get $\ell = 2\Lambda'$, i.e.,

$$(\ell_1, \ldots, \ell_n) = (2\mu_1 + 2n, 2\mu_2 + 2n - 2, \ldots, 2\mu_{n-1} + 2, 2\mu_n).$$

Let $\pi_\mu$ denote the Langlands transfer of $\pi'$ to an irreducible representation of $GL_{2n+1}(\mathbb{R})$, where the transfer is mitigated by the Langlands parameter of $\pi'$ being that of $\pi_\mu$ via the standard embedding $L_{G'^0} = \text{SO}(2n + 1, \mathbb{C}) \subset GL(2n + 1, \mathbb{C}) = L_{G^0}$. Using the local Langlands correspondence for $GL_N(\mathbb{R})$ (see, for example, Knapp [19]), we can deduce

$$\pi_\mu = \text{Ind}_{P(2,2,\ldots,2)}^G(D_{\ell_1} \otimes D_{\ell_2} \otimes \cdots \otimes D_{\ell_n} \otimes \text{sgn}^z),$$
where, for any integer \( l \), we denote by \( D_l \) the discrete series representation of \( \text{GL}_2(\mathbb{R}) \) as normalized in [27, 3.1.3]. It is well-known ([9, Lemme 3.14]) that
\[
H^\bullet(\mathfrak{g}l_N, \mathbb{R}^\times \text{SO}(N); \pi_\mu \otimes \mathcal{M}_{\mu, \mathbb{C}}) \neq 0.
\]

Case (1b) \( (F \text{ is totally real, and } N = 2n \text{ is even.}) \) In this case we take \( G' = \text{SO}(2n+1)/F \) which is the split orthogonal group in \( 2n + 1 \) variables. We have \( G'(\mathbb{R}) = \text{SO}(n, n + 1) \). The maximal compact subgroup \( K' \) of \( G'(\mathbb{R}) \) is isomorphic to \( \text{SO}(n) \times \text{O}(n + 1) \). The connected component of the \( L \)-group is \( LG'' = \text{Sp}(2n, \mathbb{C}) \). Fix a real place \( v \) of \( F \) and we drop it from the notations. We have
\[
\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq -\mu_n \geq \cdots \geq -\mu_2 \geq -\mu_1)
\]
be the given dominant integral weight with purity weight 0. Following the argument as in Case (1) above, we put
\[
\mu' = (\mu_1, \mu_2, \ldots, \mu_n), \text{ and } \\
\Lambda' = \mu' + \rho' = (\mu_1 + n - \frac{1}{2}, \mu_2 + n - \frac{3}{2}, \cdots, \mu_{n-1} + \frac{3}{2}, \mu_n + \frac{1}{2}).
\]

Consider the discrete series representation \( \pi' = \pi_{\Lambda'} \) with infinitesimal character given by \( \Lambda' \). The Langlands parameter \( \tau_{\Lambda'} \) of \( \pi_{\Lambda'} \) has the form
\[
\tau_{\Lambda'} = \text{Ind}^{W_{\mathbb{C}}}_{\mathbb{C}^\times}(\chi_{\ell_1}) \oplus \text{Ind}^{W_{\mathbb{C}}}_{\mathbb{C}^\times}(\chi_{\ell_2}) \oplus \cdots \oplus \text{Ind}^{W_{\mathbb{C}}}_{\mathbb{C}^\times}(\chi_{\ell_n}),
\]
with all the \( \ell_j \) being odd positive integers. The infinitesimal character of the discrete series is seen in terms of the exponents of the parameter restricted to \( \mathbb{C}^\times \) giving us \( \ell = 2\Lambda' \), or that
\[
(\ell_1, \ldots, \ell_n) = (2\mu_1 + 2n - 1, 2\mu_2 + 2n - 3, \cdots, 2\mu_{n-1} + 3, 2\mu_n + 1).
\]

Via the local Langlands correspondence for \( \text{GL}_{2n}(\mathbb{R}) \) we get that \( \pi' \) transfers to \( \pi_{\mu} \) given by
\[
\pi_{\mu} = \text{Ind}^{G'}_{\text{P}(2,2,\ldots,2)}(D_{\ell_1} \otimes D_{\ell_2} \otimes \cdots \otimes D_{\ell_n}),
\]
which has the property that \( H^\bullet(\mathfrak{g}l_N, \mathbb{R}^\times \text{SO}(N); \pi \otimes \mathcal{M}_{\mu, \mathbb{C}}) \neq 0. \)

Case (2) \( (F_0 \text{ is totally real and } F \text{ a totally imaginary quadratic extension of } F_0.) \)

Let \( \mu \) be a pure dominant integral weight for \( G \) with purity weight 0. For a (complex) place \( v \) of \( F \), we have \( \mu^v = (\mu_1^v, \mu_2^v) \). The place \( v \) is fixed and we drop it from the notations. We have
\[
\mu^v = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n) \text{ and } \mu^v = (\mu_1^v \geq \mu_2^v \geq \cdots \geq \mu_n^v) \text{ where purity, with the purity-weight being 0, implies } \mu_j^v = -\mu_{n-j+1}.
\]

In this case, we let \( G' \) be the quasi-split unitary group over \( F \); more precisely, define the matrix
\[
[\Phi] = \Phi_{i,j}, \text{ where } \Phi_{i,j} = (-1)^{i+j}(j, n-i+1).
\]

This gives a Hermitian form \( \Phi \) on an \( n \)-dimensional \( F \)-vector space (with respect to a fixed basis) via \( \Phi(x, y) = \frac{i}{2}x \cdot [\Phi] \cdot \bar{y} \), for \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in F^n \) and \( \bar{y} = (\bar{y}_1, \ldots, \bar{y}_n) \) being induced by the nontrivial Galois automorphism \( F/F_0 \). Denote \( U(\Phi) \) the corresponding unitary group over \( F_0 \), whose \( F_0 \)-points consists of all \( g \in \text{GL}_n(F) \) such that \( g \cdot [\Phi] \cdot \bar{g} = [\Phi] \).

Let \( \bar{G}' := U(\Phi) \). Note that \( \bar{G}'(\bar{F}_0) \otimes \mathbb{R} \) is a product of copies of \( U(\frac{n+1}{2}, \frac{n-1}{2}) \) if \( n \) is even, and is a product of copies of \( U(\frac{n+1}{2}, \frac{n-1}{2}) \) if \( n \) is odd. At \( v \) (dropped from the notation) we put
\[
\mu' = (\mu_1, \mu_2, \ldots, \mu_n).
\]
The half sum $\rho'$ of positive roots is given by

$$\rho' = \frac{1}{2} \sum_{1 \leq i < j \leq n} (e_i - e_j) = \left(\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{n-2i+1}{2}, \ldots, \frac{1-n}{2}\right).$$

Once again define

$$\Lambda' := \mu' + \rho' = \left(\mu_1 + \frac{n-1}{2}, \mu_2 + \frac{n-3}{2}, \ldots, \mu_n + \frac{1-n}{2}\right),$$

which parametrizes a discrete series representation $\pi' = \pi_{\Lambda'}$ of $U([\frac{n}{2}, \frac{n}{2}])$. One may see, either from [13, Section 2] or from [4, Example 10.5], that the Langlands parameter $\tau_{\Lambda'}$ of $\pi_{\Lambda'}$ has the form

$$\tau_{\Lambda'} = z^{a_1}z^{-a_1} \oplus z^{a_2}z^{-a_2} \oplus \cdots \oplus z^{a_n}z^{-a_n},$$

where, once again, comparing the infinitesimal character with the exponents of the parameter we get $(a_1, \ldots, a_n) = 2\Lambda'$, i.e.,

$$(a_1, \ldots, a_n) = (2\mu_1 + n - 1, 2\mu_2 + n - 3, \ldots, 2\mu_n + 1 - n).$$

The transfer $\pi_\mu$ of $\pi'$ to a representation $\text{GL}(n, \mathbb{C})$ is given by:

$$\pi_\mu = \text{Ind}_{\mathcal{B}(\mathcal{C})}^{\text{GL}(n, \mathbb{C})} (z^{a_1}z^{-a_1} \otimes \cdots \otimes z^{a_n}z^{-a_n}),$$

which has the property that $H^\bullet(\mathfrak{g}_\mu, \mathcal{C}^\times \text{U}(n); \pi_\mu \otimes \mathcal{M}_{\mu, \mathbb{C}}) \neq 0$.

The above discussion in all the three cases, together with K"unneth theorem for relative Lie algebra cohomology, gives the following

**Proposition 2.1.1.** Let $F$ and $G$ be as in Theorem 1.3.1. Let $\mu$ be a pure dominant integral weight with purity $0$. Define $G'$ as:

1. (a) if $F = F_0$ is totally real, and $N = 2n + 1$ is odd; take $G' = \text{Sp}(2n)/F$;
2. (b) if $F = F_0$ is totally real, and $N = 2n$ is even; take $G' = \text{SO}(2n+1)/F$ (the split group);
3. if $F$ is a totally imaginary quadratic extension of a totally real $F_0$; take $G' = U(\Phi)/F_0$.

In each case, we define a dominant integral weight $\mu'$ for $G'$, and put $\Lambda' = \mu' + \rho'$. The discrete series representation $\pi_{\Lambda'}$ of $G'(F_0 \otimes \mathbb{R})$ with infinitesimal character given by $\Lambda'$ has the property that it transfers to a representation $\pi_\mu$ of $G(\mathbb{R}) = \text{GL}_N(F \otimes \mathbb{R})$ that has nontrivial relative Lie algebra cohomology after twisting by $\mathcal{M}_{\mu, \mathbb{C}}$.

2.2. Consequences of embedding theorems and global transfer.

2.2.1. Clozel’s result on globalizing local discrete series representations. In this section we describe a result of Clozel ([8, Section 4.3]) on limit multiplicity of discrete series. Let $G$ be a semi-simple connected group defined over a number field $F$. Let $v_0$ be a place such that $G_{v_0}$ has supercuspidal representations. Let $S$ be a finite set of places containing the archimedean ones and such that $v_0 \notin S$. Let $S'$ be a finite set of finite places disjoint from $S$. Let $K_{S'}$ be a compact open subgroup of $G_{S'} = \prod_{v \in S'} G_v$. Fix $K$, a compact open subgroup of $\prod_{p \in S \cup S' \cup p_0} G_v$ and let $L^{K_0 \times K_{S'} \times K}$ be the $K_0 \times K_{S'} \times K$-invariant functions in the space $L^2_{\text{cusp}} (G(\mathbb{Q}) \backslash G(\mathbb{A}))$ of cusp forms on $G$. Let $\delta_S$ be a discrete series representation of the group $G_S$. 
Theorem 2.2.1.\[ \liminf_{K_S' \to 1} v(K_{S'}) \ \text{mult} \ (\delta_{S}, \mathcal{L}^{K_0 \times K_{S'} \times K}) \geq c \]

where \( c \) is a positive real number.

The above theorem is applicable to the simple groups \( \text{SO}(2n+1) \) or \( \text{Sp}(2n) \), however, it is not directly applicable to \( \text{U}(\Phi) \). It is for this reason, that in Theorem 1.3.1, in Case (2), we take \( \mu \) to be trivial on the center, and so the discrete series representation \( \pi_{\Lambda'} \) has trivial central character and so gives a representation of \( \text{PU}(\Phi)_{\infty} \); and the above theorem is applicable to \( \text{PU}(\Phi) \). (One expects, although we have not carried this out, that Clozel’s theorem should apply to reductive groups as well and so in particular to \( \text{U}(\Phi) \).) Having gotten a representation of \( \text{PU}(\Phi) \) we will think of it as a representation of \( \text{U}(\Phi) \) with trivial central character. We get the following consequence of Clozel’s theorem:

Proposition 2.2.2. Let \( F, G \) and \( \mu \) be as in Theorem 1.3.1. Fix two distinct finite places \( v \) and \( w \) of \( F_0 \). Then there exists a cuspidal automorphic representation \( \pi' \) of \( G'/F_0 \) such that

- \( \pi'_\infty = \pi_{\Lambda'} \), the discrete series representation of \( G'_\infty = G'(F_0 \otimes \mathbb{R}) \) as in Proposition 2.1.1,
- \( \pi'_v \) is a supercuspidal representation of \( G'(F_0,v) \), and furthermore
- in Cases (1a) and (1b): \( \pi'_w \) is the Steinberg representation of \( G'(F_0,w) \), whereas,
- in Case (2): \( v \) is chosen such that \( \text{U}(\Phi) \times F_{0,v} \cong \text{GL}(n)/F_{0,v} \), and we impose no condition on \( w \).

In Case (2), the representation \( \pi' \) has trivial central character.

2.2.2. Consequences of the classification of the discrete spectrum of classical groups.

Let’s begin by recalling the following result due to Magaard and Savin [22, Proposition 8.2]:

Proposition 2.2.3. Let \( \sigma \) be a cuspidal automorphic representation on \( \text{Sp}(2n) \) over a number field \( F \), such that for some finite place \( v \) of \( F \) the local component \( \sigma_v \) is the Steinberg representation. Let \( \pi \) be the automorphic representation of \( \text{GL}(2n+1)/F \) which is the lift of \( \sigma \) as in [1, Theorem 1.5.2]. Then \( \pi_v \) is the Steinberg representation and \( \pi \) is cuspidal.

The key point in the proof of the above proposition in loc. cit., is to show that \( \pi_v \) is the Steinberg representation (cuspidality of \( \pi \) then easily follows). But, it is not a priori guaranteed from Arthur’s work whether the Steinberg representation \( \sigma_v \) transfers to the Steinberg representation. However, Arthur transfer is well-behaved with respect to the Aubert involution, and this involution switches the Steinberg representation with the trivial representation, and one observes using strong approximation for \( \text{Sp}(2n) \) that an automorphic representation containing the trivial representation as some local component is itself the trivial representation. Let’s remark that this proof goes through mutatis mutandis for the split odd orthogonal group after one makes the same observation concerning the trivial representation for the split odd orthogonal group: let \( \tau \) be an automorphic representation of \( \text{SO}(2n+1) \) with a trivial local component, say at \( v \); now inflate \( \tau \) to an automorphic representation \( \tilde{\tau} \) of \( \text{Spin}(2n+1) \) which also has a trivial component at \( v \); applying strong approximation which is known for the almost simple simply-connected group \( \text{Spin}(2n+1) \) (see, for example, Platonov–Rapinchuk [24]) we conclude that \( \tilde{\tau} \) is trivial, a fortiori, \( \tau \) is trivial. We record the analogue of the above proposition for odd orthogonal group as:
Proposition 2.2.4. Let $\sigma$ be a cuspidal automorphic representation of the split $SO(2n+1)$ over a number field $F$, such that for some finite place $v$ of $F$ the local component $\tau_v$ is the Steinberg representation. Let $\pi$ be the automorphic representation of $GL(2n+1)/F$ which is the lift of $\sigma$ as in [1, Theorem 1.5.2]. Then $\pi_v$ is the Steinberg representation and $\pi$ is cuspidal.

2.2.3. Conclusion of the proof. Now consider the situation of Proposition 2.2.2. The cuspidal automorphic representation $\pi'_{G'}$ transfers to an automorphic representation of $G = GL_N/F$; this is via Arthur [1, Theorem 1.5.2] in cases (1a) and (1b) and by Mok [23, Theorem 2.5.2] in case (2). In cases (1a) and (1b), Propositions 2.2.3 and 2.2.4 ensure that $\pi$ is cuspidal. In case (2) it is clear since the place $v$ is taken so that the unitary group splits and has a supercuspidal local component guaranteeing cuspidality of $\pi$. Now by Proposition 2.1.1 we know that $\pi$ is cohomological with respect to the given weight $\mu$.

3. Proof of Theorem 1.3.5

We give a proof of Theorem 1.3.5 generalizing Clozel’s construction in [10] using automorphic induction. Let $F$ be the given number field and $\mu$ be the given parallel weight assumed to have purity weight $w = 0$. So $\mu$ is of the form $\mu = (\mu_1 \geq \cdots \geq \mu_{2n})$ with $\mu_j \in \mathbb{Z}$ and such that $\mu_j = -\mu_{2n+1-j}$, $\forall 1 \leq j \leq n$. Let $\rho = (n - \frac{1}{2}, n - \frac{3}{2}, \ldots, -n + \frac{1}{2})$ be the half sum of positive roots for $gl(2n, \mathbb{C})$. Put $\ell = (\ell_1, \ldots, \ell_{2n}) = 2\mu + 2\rho$. Note that $\ell_{2n-i+1} = -\ell_i$. We proceed as in [10]. Choose a totally real number field $K_1$, which is cyclic over $\mathbb{Q}$ and linearly disjoint with $F$ over $\mathbb{Q}$; let $\text{Gal}(K_1|\mathbb{Q}) = \langle \sigma \rangle \cong \mathbb{Z}/n\mathbb{Z}$. We also choose a totally imaginary quadratic extension $L_1$ of $K_1$ which is also linearly disjoint with $F$ over $\mathbb{Q}$. Let $L = FL_1$ and $K = FK_1$ be the corresponding compositum fields as shown in the diagram below.

```
      L
     /\  \\
    K \  / \ L_1
   / \  /    \\
F  \ / \ 2
   \ /  \\
      Q
```

Let $\mathbb{I}_{L_1} = L_1^\times \backslash A_{L_1}^\times$ be the idèle class group of the number field $L_1$. Let $\chi_1 \in \text{Hom}(\mathbb{I}_{L_1}, S^1)$ be a unitary algebraic Hecke character of $\mathbb{I}_{L_1}$. From [32], one knows that the infinity type of $\chi_1$, i.e., its restriction to the group $\prod_{v \in S_{\infty}(L_1)} L_{1,v}^\times \cong (\mathbb{C}^\times)^n$, is of the form:

$$\chi_1((a_v)) = \prod_{v \in S_{\infty}(L_1)} (a_v/|a_v|)^{-f_v},$$

for integers $f_v$. Conversely, from [32], given any set of integers $\{f_v\}$, there is a Hecke character with infinity type as above; this is because $L_1$ is a totally imaginary quadratic extension of a
totally real number field $K_1$. Fix an ordering on $S_\infty(L_1)$ and take $\{f_v\} = (\ell_1, \ldots, \ell_n)$, to get a character $\chi_1$ with infinity type:

$$(\chi_1)(z_1, z_2, \ldots, z_n) = \prod_{j=1}^n (z_j/\bar{z}_j)^{\ell_j/2} = \prod_{j=1}^n (z_j/|z_j|)^{\ell_j},$$

where, $(z_1, \ldots, z_n) = (z_v)_{v \in S_\infty(L_1)}$.

Now we briefly explain the construction of a representation $\pi(\chi)$ of $GL(2n, \mathbb{A}_F)$ obtained by automorphic induction (in two steps) from $\chi = \chi_1 \circ N(L/L_1)$. Following Jacquet–Langlands [18], one constructs a cuspidal representation $\pi_K(\chi)$ of $GL(2, \mathbb{A}_K)$ via automorphic induction across $L/K$. Next, we let $\Pi_K$ be the representation of $GL(2n, \mathbb{A}_K)$ obtained by inducing

$$\pi_K(\chi) \times \pi_K(\chi)^\sigma \times \cdots \times \pi_K(\chi)^{\sigma^{n-1}}$$

from the Levi component $GL(2, \mathbb{A}_K) \times GL(2, \mathbb{A}_K) \times \cdots \times GL(2, \mathbb{A}_K)$ of the parabolic subgroup $P(2, 2, \ldots, 2)$. Since $\Pi_K$ is stable under $Gal(K_1/Q)$ action, it descends to a unique cuspidal representation $\pi(\chi)$ of $GL(2n, \mathbb{A}_F)$ such that for every archimedean place $v$ of $F$, the local component $\pi_v$ is described as follows:

- For a real place $v$ of $F$,
  $$\pi_v = J_v(\mu) := \text{Ind}_{P(2, 2, \ldots, 2)}^{GL(2n)}(D(\ell_1) \otimes D(\ell_2) \otimes \cdots \otimes D(\ell_n)).$$

- For a complex place $v$ of $F$,
  $$\pi_v = J_v(\mu) := \text{Ind}_{\text{Diag}(\mathbb{C})}^{GL(2n)(\mathbb{C})} \left( z^{a_1} \bar{z}^{b_1} \otimes \cdots \otimes z^{a_{2n}} \bar{z}^{b_{2n}} \right),$$

  where $a := \mu + \rho$, $b := -\mu - \rho$.

As in [10], from the results of Speh [31] and Enright [11], one knows that in both the above cases, $J_v(\mu)$ is the only generic representation of $GL(2n, F_v)$ which is cohomological with respect to the given parallel weight $\mu$. The proof of Theorem 1.3.5 follows as in [10] using the above observations and (1.2.1).

4. Remarks

4.1. Transferring from even orthogonal groups. One may ask what sort of representations of $GL_n$ are obtained by transferring from even orthogonal groups in as much as cohomological properties are concerned. The basic problem is that this transfer does not preserve “algebraicity.” This is already seen in Gan–Raghuram [12, Lemma 9.2 (2)] for local unramified representations. We begin by illustrating the issue if we use Ramakrishnan’s transfer [28] from $GL(2) \times GL(2)$ to $GL(4)$, which under a condition on central characters of the representations on $GL(2) \times GL(2)$, is the same as transferring from $SGO(4)$ to $GL_4$; see, for example, Ramakrishnan [29, Section 2].

4.1.1. On Ramakrishnan’s transfer from $GL(2) \times GL(2)$ to $GL(4)$. For $j = 1, 2$, let $\varphi_j$ be a holomorphic cuspidal normalized eigen-newform of weight $k_j$. Let $\pi(\varphi_j)$ be the associated cuspidal automorphic representation of $GL_2(\mathbb{A})$. Recall that the representation at infinity is $\pi(\varphi_j)_{\infty} = D_{k_j-1}$. Choose $\epsilon_j \in \{0, 1\}$ such that $k_j \equiv \epsilon_j \pmod{2}$. Then $\pi_j = \pi(\varphi_j) \otimes |\cdot|^{\epsilon_j/2}$ is a cohomological cuspidal representation. (See [26, Theorem 5.5 (2)] and [27, Theorem 1.4].)
Consider the representation $\Pi = \pi_1 \boxtimes \pi_2$ as constructed in [28]. The representation at infinity is given by

$$\Pi_\infty = D_{k_1-1}(\frac{\epsilon_1}{2}) \boxtimes D_{k_2-1}(\frac{\epsilon_2}{2}) = (D_{k_1-1} \boxtimes D_{k_2-1})(\frac{\epsilon_1+\epsilon_2}{2}) = (D_{k_1+k_2-2} \times D_{k_1-k_2})(\frac{\epsilon_1+\epsilon_2}{2}),$$

where the right-most representation is obtained by parabolic induction from the $(2,2)$-parabolic subgroup in $GL_4(\mathbb{R})$. Now suppose there is a dominant integral pure weight $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$, with $\mu_1 + \mu_4 = \mu_2 + \mu_3 = w$ such that $\Pi$ is cohomological with respect to $M'_\mu$ then:

$$\Pi_\infty = D_{2\mu_1+3-w}(\frac{\epsilon_1}{2}) \times D_{2\mu_2+1-w}(\frac{\epsilon_2}{2}).$$

This would entail the following equalities: $w = \epsilon_1 + \epsilon_2, 2\mu_1 + 3 - w = k_1 + k_2 - 2$, and $2\mu_2 + 1 - w = k_1 - k_2$, which is impossible for parity reasons. A possible remedy is suggested by Clozel [9] by considering appropriate half-integral Tate twists. Define

$$\Pi^T := \pi_1 \boxtimes \pi_2 := (\pi_1(-\frac{1}{2}) \boxtimes \pi_1(-\frac{1}{2}))\left(\frac{1}{2}\right) = (\pi_1 \boxtimes \pi_2)\left(\frac{1}{2}\right).$$

Now, we would have

$$\Pi^T_\infty = (D_{k_1+k_2-2} \times D_{k_1-k_2})(\frac{\epsilon_1+\epsilon_2+1}{2}) = D_{2\mu_1+3-w}(\frac{\epsilon_1}{2}) \times D_{2\mu_2+1-w}(\frac{\epsilon_2}{2}).$$

This is possible by taking $w = \epsilon_1 + \epsilon_2 + \frac{1}{2}, \mu_1 = (k_1 + k_2 + \epsilon_1 + \epsilon_2)/2 - 2$ and $\mu_2 = (k_1 - k_2 + \epsilon_1 + \epsilon_2)/2$. There are similar difficulties in the general case of transferring from even orthogonal groups for which there seems to be no obvious remedy.

### 4.1.2. Difficulties in the general case.

Let $N = 2n$, and for simplicity take $F = \mathbb{Q}$. Suppose that we want to construct a representation as in Case (1b) of the proof of Theorem 1.3.1 using an even orthogonal group as our endoscopy group. We follow [15]: take an even orthogonal group $G'/\mathbb{Q}$ such that if $n$ is even then $G'(\mathbb{R}) = SO(n, n)$, and if $n$ is odd $G'(\mathbb{R}) = SO(n-1, n+1)$. In both cases $G'(\mathbb{R})$ has discrete series representations. Let’s denote $G' = SO(2n)/\mathbb{Q}$ (resp., $G' = SO'(2n)/\mathbb{Q}$) if $n$ is even (resp., odd). Let $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq -\mu_n \geq \cdots \geq -\mu_2 \geq -\mu_1)$ be the given dominant integral weight for $GL_N$ with purity weight 0. Let $\mu', \rho', \Lambda'$ and $\ell'$ be defined as follows:

$$\mu' = (\mu_1, \mu_2, \ldots, \mu_n);$$
$$\rho' = (n-1, n-2, \ldots, 1, 0);$$
$$\Lambda' = \mu' + \rho' = (\mu_1 + n - 1, \mu_2 + n - 2, \ldots, \mu_{n-1} + 1, \mu_n);$$
$$\ell' = (\ell'_i)_{i=1}^n := 2\Lambda' = (2\mu_1 + 2n - 2, 2\mu_2 + 2n - 4, \ldots, 2\mu_{n-1} + 2, 2\mu_n).$$

If we use a similar argument, we get a representation $\pi_\infty$ of $GL_{2n}(\mathbb{R})$ given by:

$$\pi_\infty = \text{Ind}_{F(2,2,\ldots,2)}^{GL_N(\mathbb{R})} \left( D_\ell' \otimes D_{\ell_2} \otimes \cdots \otimes D_{\ell_n} \right),$$

which is in fact not cohomological with respect to the weight $\mu$. The generic representation of $GL_{2n}(\mathbb{R})$ that has nonzero cohomology with respect to $\mu^\times$ is given by inducing from $D_{\ell_1} \otimes D_{\ell_2} \otimes \cdots \otimes D_{\ell_n}$, where $(\ell_1, \ldots, \ell_{2n}) = 2\mu + 2\rho$ with $\rho$ being the half sum of positive roots for $GL_N$. This gives $(\ell_1, \ldots, \ell_{2n}) = (2\mu_1 + 2n - 1, 2\mu_2 + 2n - 3, \ldots, 2\mu_n + 1)$. Observe that $(\ell_1, \ldots, \ell_n)$ and $(\ell'_1, \ldots, \ell'_n)$ differ by 1 at every coordinate. The remedy of taking a half-integral Tate twist in the previous subsection will not work here. It seems (to the authors) that there is no easy way to resolve this difficulty; the question being how one might modify the Langlands transfer from $SO(2n)$ or $SO'(2n)$ so as to preserve the property of being cohomological.
4.2. An endoscopic stratification of inner cohomology. Let’s recall the definition of inner or interior cohomology. Take a field $E$ that is Galois over $\mathbb{Q}$ and containing a copy of $F$. Refining the notations as in Section 1.1, consider a dominant integral weight $\mu \in X^+(T \times E)$, giving us a rational finite-dimensional representation $M_{\mu,E}$ of $G \times E$, which in turn gives a sheaf $\tilde{M}_{\mu,E}$ of $E$-vector spaces on $S^G_{K_f}$. Inner cohomology is the image of cohomology with compact supports in global cohomology:

$$H^*_c(S^G_{K_f}, \tilde{M}_{\mu,E}^\vee) := \text{Image} \left( H^*_c(S^G_{K_f}, \tilde{M}_{\mu,E}^\vee) \to H^*(S^G_{K_f}, \tilde{M}_{\mu,E}^\vee) \right).$$

If we pass to a transcendental situation via any embedding $\iota : E \to \mathbb{C}$, then it is well-known

$$H^*_\text{cusp}(S^G_{K_f}, \tilde{M}_{\mu,\mathbb{C}}^\vee) \subset H^*_c(S^G_{K_f}, \tilde{M}_{\mu,E}^\vee) \otimes_{E,\iota} \mathbb{C}.$$  

(See, for example, [9] or [17].) Next, let’s recall arithmeticity for Shalika models ([14, Appendix]): if $\Pi$ is a cuspidal representation of $\text{GL}_{2n}$ over a totally real $F$, and suppose that $\Pi$ is cohomological and has a Shalika model, then for $\sigma \in \text{Aut}(\mathbb{C})$, the representation $\sigma \Pi$ also has a Shalika model. In other words, if $\Pi$ is cohomological and is a transfer from $\text{SO}(2n+1)$ then so is any conjugate of $\Pi$. The above considerations gives the following

**Corollary 4.2.1.** Let $F$ be a totally real field, and take $G = \text{GL}(2n)/F$. Let $\mu$ be a parallel weight with purity 0. Let $E$ be a Galois extension of $\mathbb{Q}$ that contains a copy of $F$. There exists a nontrivial $E$-subspace

$$H^*_\text{symp}(S^G_{K_f}, \tilde{M}_{\mu,E}^\vee) \subset H^*_c(S^G_{K_f}, \tilde{M}_{\mu,E}^\vee)$$

stable under all Hecke operators, such that $H^*_\text{symp}(S^G_{K_f}, \tilde{M}_{\mu,E}^\vee) \otimes_{E,\iota} \mathbb{C}$ is spanned by cuspidal representations of $G$ that are all transfers from $\text{SO}(2n+1)$.

Similarly, another result on arithmeticity ([12, Remark 5.5]), says that if a cuspidal representation $\Pi$ of $\text{GL}_{2n+1}$ over a totally real field is such that a partial symmetric square $L$-function has a pole at $s = 1$ (a property that characterizes $\Pi$ being a transfer from $\text{Sp}(2n)$) then the same is true of the representation $\sigma \Pi$ for $\sigma \in \text{Aut}(\mathbb{C})$. This gives us

**Corollary 4.2.2.** Let $F$ be a totally real field, and take $G = \text{GL}(2n+1)/F$. Let $\mu$ be a parallel weight with purity 0. Let $E$ be a Galois extension of $\mathbb{Q}$ that contains a copy of $F$. There exists a nontrivial $E$-subspace

$$H^*_\text{orth}(S^G_{K_f}, \tilde{M}_{\mu,E}^\vee) \subset H^*_c(S^G_{K_f}, \tilde{M}_{\mu,E}^\vee)$$

stable under all Hecke operators, such that $H^*_\text{orth}(S^G_{K_f}, \tilde{M}_{\mu,E}^\vee) \otimes_{E,\iota} \mathbb{C}$ is spanned by cuspidal representations of $G$ that are all transfers from $\text{Sp}(2n)$.

The notation $H^*_\text{symp}$ (resp., $H^*_\text{orth}$) is to suggest that we are looking at the contribution of cuspidal representations with ‘symplectic’ (resp., ‘orthogonal’) parameters.

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