SCOTT FUNCTIONS, THEIR REPRESENTATIONS ON DOMAINS, AND APPLICATIONS TO RANDOM SETS

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Abstract. Choquet theorems (1954) on integral representation for capacities are fundamental to probability theory. They inspired a growing body of research into different approaches and generalizations of Choquet’s results by many other researchers. Notably Mathérion’s work (1975) on distributions over the space of closed subsets has led to further advancements in the theory of random sets.

This paper was inspired by the work of Norberg (1989) who generalized Choquet’s results to distributions over domains. While Choquet’s original theorems were obtained for locally compact Hausdorff (LCH) spaces, both Mathérion’s and Norberg’s depend on the assumption of separability in their application of the Carathéodory’s method.

Our Radon measure approach differs from the work of Mathérion and Norberg, in that it does not require separability. This investigation naturally leads to the introduction of finite and locally finite valuations, which allows us to characterize finite and locally finite random sets in terms of capacities on the class of compact subsets.

Finally, the treatment of Lévy exponent by Mathérion and Norberg is revisited, and the notion of exponential valuation is proposed for the representation of general Poisson processes.

1. Introduction

1.1. Capacities and Choquet theorems. The theory of random sets forms an important area of probability that has generated a lot of research activity (see [13] for an extensive survey of the field and the references therein) dating back to the foundational work of Kolmogorov [9] in the 1930s. An important tool in the analysis of the distribution of a given random set is supplied by the capacity functional that translates the study of the (very rich) σ-algebra of random set into the investigation of measure-like functionals on the sets themselves.

The capacity functional is not a direct replacement of a measure due to its nonadditivity. Nonetheless, the classical results of Choquet show that a few natural algebraic properties are all that is required to establish a one-to-one correspondence between random set distributions and capacities.

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Let us introduce the notion of capacities over the class $K$ of compact subsets of a locally compact Hausdorff (LCH) space $R$. A set function $\Phi$ on $K$ is called continuous on the right in [3] if for any $E \in K$ and $\varepsilon > 0$ there is an open neighborhood $G$ of $E$ such that $|\Phi(F) - \Phi(E)| < \varepsilon$ whenever $E \subseteq F \subseteq G$. Call $\Phi$ a capacity if it is nonnegative, increasing [i.e. $\Phi(E) \leq \Phi(F)$ whenever $E \subseteq F$], and continuous on the right. We assume $\Phi(\emptyset) = 0$, although $\Phi$ may not be bounded in general. When it is bounded, it is often assumed that $\Phi$ is normalized [i.e. $\sup_{Q \in K} \Phi(Q) = 1$].

Define a difference $\nabla Q_1 \Phi(Q) = \Phi(Q) - \Phi(Q \cup Q_1)$, and the successive difference $\nabla Q_1, \ldots, Q_n \Phi(Q)$ recursively by $\nabla Q_1, \ldots, Q_n \Phi(Q) = \nabla Q_1, ..., Q_{n-1} \Phi(Q_n)$. A capacity $\Phi$ is called completely alternating if $\nabla Q_1, \ldots, Q_n \Phi(Q) \leq 0$ holds for any finite sequence $Q, Q_1, \ldots, Q_n$ of $K$.

The Fell topology on the space $F'$ of nonempty closed subsets in $R$ is LCH (see Example 2.3, or [12, 13]). Now a natural capacity can be defined as $\Phi(Q) = \lambda(\{F \in F' : Q \cap F \neq \emptyset\})$, $Q \in K$ (1.1) where $\lambda$ is a Radon measure on $F'$. It may be observed that $-\nabla Q_1, ..., Q_n \Phi(Q) = \lambda(\{F \in F' : Q \cap F = \emptyset, Q_i \cap F \neq \emptyset, i = 1, \ldots, n\})$, and that $\Phi$ as defined above is always completely alternating. Conversely, Choquet [3] has shown that this property is sufficient for a given capacity to be representable in the form above.

Theorem 1.1. If a capacity $\Phi$ is completely alternating then there exists a unique Radon measure $\lambda$ on $F'$ satisfying (1.1).

Theorem 1.1 is referred to as “Choquet theorem” in [1, 12, 13], and appears as Theorem 5.6 below. This result fundamentally characterizes the distribution of a closed random set on an LCH space $R$. That is, when $\Phi$ is normalized, the representation (1.1) is interpreted as the probability of a closed random set hitting the set $Q$; thus, $\Phi$ of Theorem 1.1 is called a hitting capacity.

Let $F := F' \cup \{\emptyset\}$ be the space of closed sets which can be viewed as the one-point compactification of $F'$. Choquet theorems over $F$ and $F'$ (Theorems 5.3 and 5.6 respectively) are discussed in Section 5. A nonnegative set function $\varphi$ on $K$ is called a conjugate functional (or, simply a conjugate) if it is decreasing and continuous on the right, and it is said to be completely monotone if $\nabla Q_1, ..., Q_n \varphi(Q) \geq 0$ holds for any $Q$ and $Q_i$'s. A completely monotone conjugate functional $\varphi$ uniquely corresponds to a Radon measure $\mu$ on $F$ (Theorem 5.3) that satisfies $\varphi(Q) = \mu(\{F \in F : Q \cap F = \emptyset\})$, $Q \in K$, (1.2) in which case $\varphi$ is known as an avoidance functional.

The conjugate $\varphi$ is bounded by $\varphi(\emptyset)$, so we may assume $\varphi(\emptyset) = 1$ without loss of generality. Then the corresponding $\mu$ in (1.2) is a probability measure, and it has the mass $\mu(\{\emptyset\}) = \inf_{Q \in K} \varphi(Q)$. We call $\varphi$ degenerate.
if \( \inf_{Q \in K} \varphi(Q) > 0 \). If an avoidance functional \( \varphi \) is nondegenerate then the measure \( \mu \) of (1.2) is identified with \( \lambda \) of Theorem 1.1 so that the hitting capacity \( \Phi \) of (1.2) is the complement of \( \varphi \) of (1.2) [i.e., \( \Phi(Q) = 1 - \varphi(Q) \)].

We define another difference operator \( \Delta_{Q_1, \ldots, Q_n} \Phi(Q) = \Phi(Q) - \Phi(Q \cap Q_1), \) and the successive difference \( \Delta_{Q_1, \ldots, Q_n} \Phi \) recursively. Then we call \( \Phi \) completely \( \cap \)-monotone if \( \Delta_{Q_1, \ldots, Q_n} \Phi(Q) \geq 0 \) holds for any finite sequence \( Q, Q_1, \ldots, Q_n \) of \( K \).

The definition was inspired by the pioneering work of Norberg [15] on the existence theorems for measures on domains that extend the results of Choquet's. In Section 2 we review domain theory, and explore the notion of Scott and Lawson topology for partially ordered sets (posets), in which the ordering of \( K \) is given by the reverse inclusion (Example 2.3). Consequently the conjugate \( \varphi \) is increasing on the poset \( K \), so \( \Phi \) becomes its conjugate, with both of them treated as continuous functions in the Scott topology on \( K \), and called \textit{Scott functions} (Definition 3.2).

In Section 3 we discuss a representation of Scott functions in terms of a Radon measure over the lattice \( S \) of nonempty Scott open sets (over the semilattice \( S' \)) in Proposition 3.4 (Proposition 3.8, respectively); in this paper \( S' \) indicates the removal of the top element from \( S \). A representation over \( S \) or \( S' \) has been suggested by Choquet [3], and explored by Murofushi and Sugeno [14] in their study of monotone set functions (Remark 3.5).

However, to our knowledge, this approach has never been applied to establish Choquet's theorems in the general case over domains.

One advantage of Radon measure approach over the technique of [15] is the lack of any separability conditions. Throughout Sections 5–6 we will employ approximation theorems of Section 4 extensively, and establish Theorem 1.1 and other flavors of Choquet theorem on domains without any appeal to separability. Although approximation theorems are standard techniques in harmonic analysis (cf. [1, 4]), Theorems 4.3 and 4.6 in this context are novel to the best of the authors’ knowledge. Let us outline the existence proof for the Choquet’s theorem over \( F \) (Theorem 5.3): By Theorem 2.4 the space \( F \) is homeomorphic to a compact subspace of \( S \). Once \( \varphi \) is approximated by Radon measures over \( F \) (see Definition 4.2 and the existence proof of Theorem 5.3), Theorem 4.3 guarantees the existence of a Radon measure \( \mu \) over \( F \) satisfying (1.2).

1.2. Finite and locally finite random sets and point processes. The continuous lattice and domain versions of Choquet’s theorems obtained in
this paper provided an inspiration for the following idea of a \( k \)-valuation which proved useful in the study of point-processes below.

A capacity \( \Phi \) is called a \textit{valuation} if it is completely alternating and completely \( \cap \)-monotone, or equivalently if it satisfies

\[
\Phi(Q_1) + \Phi(Q_2) = \Phi(Q_1 \cup Q_2) + \Phi(Q_1 \cap Q_2)
\]

for every pair \( Q_1, Q_2 \) of \( K \). We call \( \Phi \) a \( k \)-valuation if \( \Phi \) is completely alternating and it satisfies

\[
\nabla_{Q_1,\ldots,Q_{k+1}} \Phi\left( \bigcup_{i \neq j} Q_i \cap Q_j \right) = 0
\]

for every \((k+1)\)-tuple \( Q_1, \ldots, Q_{k+1} \) of \( K \). By \( \mathcal{P}_k' := \{ A \in \mathcal{F}' : 1 \leq |A| \leq k \} \) we denote the collection of non-empty finite subsets of at most size \( k \). To the best of our knowledge, the notion of a \( k \)-valuation and the corresponding representation theorem below have not been studied before.

**Theorem 1.2.** If \( \Phi \) is a \( k \)-valuation then the Radon measure \( \lambda \) of Theorem 1.1 is supported by \( \mathcal{P}_k' \).

This flavor of Choquet’s theorem is discussed in Section 7. We should note that the collection \( \mathcal{P}_1' = \{ \{a\} : a \in R \} \) is closed in \( \mathcal{F}' \), and is homeomorphic to \( R \). A Radon measure on the LCH space \( R \), viewed as a capacity on \( K \), is a valuation. The converse is also true, which is a special case of Theorem 1.2 when \( k = 1 \).

In the rest of Section 1.2 we assume further that the LCH space \( R \) is \( \sigma \)-compact. A nonnegative integer-valued random valuation \( N(Q) \) of \( Q \in K \) on some probability space \((\Omega, \mathcal{B}, P)\) is called a \textit{point process}, which is said to be \textit{simple} if \( N(\{a\}) \leq 1 \) for every \( a \in R \). A closed subset \( F \) is called \textit{locally finite} if \( F \cap Q \) is finite for every \( Q \in K \). By \( \mathcal{P}_f \) we denote the collection of locally finite sets.

Now a simple point process \( N \) is associated with a random closed set \( \xi \) taking its values on \( \mathcal{P}_f \), defined as \( N(Q) = |\xi \cap Q| \) where \( |\xi \cap Q| \) is the cardinality of the finite set \( \xi \cap Q \). Kurtz [10] studied simple point processes in terms of the avoidance functional \( \varphi \), and called \( \varphi \) a \textit{zero probability function} because it satisfies \( \varphi(Q) = P(N(Q) = 0) \).

In order to explore representations over \( \mathcal{P}_f \), he also introduced a natural extension of capacities over the Borel \( \sigma \)-algebra, and used arbitrary finite partitions of a compact set \( W \in K \) by means of Borel-measurable subsets. Since such an extension is not available for Scott functions, we replace partitions with opening-free partitions of subsemilattice antichain. A finite subset \( \mathcal{G} \) of \( K \) is called a \textit{subsemilattice covering} of \( W \) if (i) \( Q_1 \cap Q_2 \in \mathcal{G} \) whenever \( Q_1, Q_2 \in \mathcal{G} \), and (ii) \( \bigcup \mathcal{G} = W \), where the union is over all the elements of \( \mathcal{G} \). For any antichain \( Q_1, \ldots, Q_k \) of \( \mathcal{G}' = \mathcal{G} \setminus \{ \bigcap \mathcal{G} \} \) (i.e., \( Q_i \not\subseteq Q_j \) for any pair \((Q_i, Q_j)\)), we set

\[
O_{Q_1,\ldots,Q_k} = \bigcup\{ Q \in \mathcal{G} : Q_i \not\subseteq Q, \text{ for all } i = 1, \ldots, k \}
\]
if the above collection for the union contains at least one element \( Q \in G \); otherwise, put \( O_{Q_1,\ldots,Q_k} = \emptyset \). We call it an \textit{opening} if \( Q_i \nsubseteq O_{Q_1,\ldots,Q_k} \) for \( i = 1,\ldots,k \) so that the disjoint sequence 
\[
Q_1 \setminus O_{Q_1,\ldots,Q_k}, \ldots, Q_k \setminus O_{Q_1,\ldots,Q_k}
\]
partitions \( W \setminus O_{Q_1,\ldots,Q_k} \).

Call an avoidance functional \( \varphi \) a \textit{locally finite valuation} if for any \( \delta > 0 \) and \( W \) of \( K \) we can find a sufficiently large \( n \) such that for an arbitrary finite subsemilattice \( G \) covering \( W \),
\[
\varphi(W) + \sum \nabla_{Q_1,\ldots,Q_k} \varphi(O_{Q_1,\ldots,Q_k}) \geq 1 - \delta \tag{1.4}
\]
where the summation is over all antichains \( Q_1,\ldots,Q_k \) in \( G' \) such that \( O_{Q_1,\ldots,Q_k} \) is an opening and \( k \leq n \).

The condition (1.4) roughly corresponds to that of Theorem 2.13 of [10] in terms of avoidance functional. If \( \varphi \) is nondegenerate then (1.4) is equivalently expressed in terms of the hitting capacity \( \Phi \) by
\[
-\sum \nabla_{Q_1,\ldots,Q_k} \Phi(O_{Q_1,\ldots,Q_k}) \geq \Phi(W) - \delta \tag{1.5}
\]

In Section 8 we characterize the representation over \( \mathcal{P}_{lf} \) and obtain

**Theorem 1.3.** If a hitting capacity \( \Phi \) of Theorem 1.1 is a locally finite valuation then the corresponding Radon measure \( \lambda \) uniquely determines the distribution of a simple point process.

For the last part of Section 1.2 we generally assume that \( \Phi \) is unbounded. If \( \Phi \) is completely alternating then the conjugate \( \varphi(Q) = \exp[-\Phi(Q)] \) is completely monotone (Lemma 9.1). The converse is not always true even if \( \varphi \) is strictly positive. Hence, \( \Phi \) is called the \textit{Lévy exponent} if \( \Phi(Q) = -\log \varphi(Q) \) is completely alternating. In Section 9 we demonstrate that the Lévy exponent has a probabilistic interpretation analogous to that of Lévy-Khinchin formula (cf. [11, 2]), and that it is sufficiently characterized by the conjugate \( \varphi \) being infinitely divisible (Proposition 9.2).

A point process \( N \) is said to be a \textit{general Poisson process} if there exists a Radon measure \( \lambda \) on \( \mathbb{R} \) satisfying for any sequence \( Q_1,\ldots,Q_k \) of disjoint compact subsets of \( \mathbb{R} \)
\[
\mathbb{P}(N(Q_i) = n_i, i = 1,\ldots,k) = \prod_{i=1}^{k} e^{-\lambda(Q_i)} \frac{[\lambda(Q_i)]^{n_i}}{n_i!}, \tag{1.6}
\]
where \( \lambda \) is called a parameter measure; see [5]. It should be noted that the parameter measure \( \lambda \) could be atomic with positive measure \( \lambda\{a\} \) for some singleton \( \{a\} \), therefore the general Poisson process \( N \) may not be simple.

**Theorem 1.4.** A conjugate \( \varphi \) is a zero probability function of a general Poisson process if and only if it is strictly positive and satisfies
\[
\varphi(Q_1)\varphi(Q_2) = \varphi(Q_1 \cup Q_2)\varphi(Q_1 \cap Q_2)
\]
for every pair \( Q_1, Q_2 \) of \( K \).
We call a conjugate $\varphi$ of Theorem 1.4 an exponential valuation. The proof of Theorem 1.4 is presented at the end of Section 9.

1.3. Separability and the assertion of uniqueness. The following example is the classical case handled by Choquet’s original paper [3], and separability may not be an entirely natural restriction.

Example 1.5. Let $R_0 = [0, 1]$ be equipped with the discrete topology, and let $R_1 = [0, 1]$ be the standard Euclidean metric space. Then the product topology $R = R_0 \times R_1$ is LCH, but not second-countable; see Example 8.2. Let $\pi_1$ be the canonical projection from $R$ to $R_1$, and let $\nu$ be the standard Lebesgue measure on $R_1$. Introduce a capacity $\Phi$ over the family $K$ of compact subsets of $R$ by setting $\Phi(Q) = \nu(\pi_1(Q))$ for $Q \in K$. This capacity is normalized, satisfies

$$\nabla_{Q_1, \ldots, Q_n} \Phi(Q) = \nu(\pi_1(Q)) - \nu\left(\pi_1(Q) \cup \bigcap_{i=1}^n \pi_1(Q_i)\right);$$

and thus, completely alternating. Let $\mathcal{R}_1 = \{\pi_1^{-1}(\{x_1\}) : x_1 \in R_1\}$ be a collection of closed subsets in $R$. Then $\mathcal{R}_1$ can be shown to be a compact subspace of $\mathcal{F}'$ in the Fell topology, homeomorphic to $R_1$. We can introduce a measure $\lambda$ on $\mathcal{R}_1$ by setting

$$\lambda\left(\pi_1^{-1}(\{x_1\}) : x_1 \in B\right) = \nu(B)$$

for any Borel measurable subset $B$ of $R_1$, and view it as a measure on $\mathcal{F}'$ supported by $\mathcal{R}_1$. One can show that $\lambda$ is the Radon measure of Theorem 1.1, establishing that $\Phi$ represents a hitting capacity over $\mathcal{R}_1$.

The approach taken by Mathéron [12] and Norberg [15] used Carathéodory’s method of construction, and proved the unique existence of Borel measure $\lambda$ on $\mathcal{F}'$ satisfying (1.1) when the space $R$ is second-countable. Therefore, the existence of $\lambda$ in Example 1.5 does not follow from their versions of Choquet theorem. Ross [16] applied essentially the same approach without separability, and built a measure on $\mathcal{F}$ satisfying (1.1) for $Q \in \mathcal{V}$ with a different choice of space $\mathcal{V}$. Although the conditions for the pair of $\mathcal{V}$ and $\mathcal{F}$ are less restrictive, they require that $\mathcal{F}$ contain all the complements $Q^c$ for $Q \in \mathcal{V}$; thus, excluding the pair of $\mathcal{K}$ and $\mathcal{F}$ handled in Example 1.5.

Theorem 1.1 establishes the uniqueness of $\lambda$ over Radon measures, that is, over Borel measures which are inner regular on all open sets and outer regular on all Borel sets (see, e.g., Folland [7] in the setting of LCH spaces). If the space $R$ is second-countable, so is $\mathcal{F}'$ (Remark 5.1). Consequently our results include those of Matheron and Norberg, since Borel and Radon measures coincide for second-countable LCH spaces. The example below, however, demonstrates that the representation $\lambda$ of Theorem 1.1 may not be unique over Borel measures for an $R$ that is not second-countable.

Example 1.6. Let $R$ be the LCH space of Example 1.5 and let $C_c(R)$ be the space of continuous functions on $R$ with compact support. If $f \in$
\[ \begin{aligned} &C_c(R) \text{ then } f(x_0, \cdot) \equiv 0 \text{ for all but finitely many } x_0 \in R_0. \text{ By the Riesz theorem we can construct a Radon measure } \lambda \text{ on } R \text{ satisfying } \int f \, d\lambda = \sum_{x_0 \in R_0} \int_0^1 f(x_0, x_1) \, dx_1 \text{ for } f \in C_c(R), \text{ and define an unbounded capacity } \Phi \text{ by setting } \Phi(Q) = \lambda(Q) \text{ for } Q \in \mathcal{K}. \text{ This capacity is apparently a valuation, and it has the obvious representation } \lambda \text{ supported by } \mathcal{P}_1. \text{ We can choose an open subset } A = R_0 \times (R_1 \setminus \{0\}), \text{ and define a Borel measure } \tilde{\lambda} \text{ by setting } \\
&\tilde{\lambda}(B) = \lambda(A \cap B) \text{ for any Borel-measurable subset } B \text{ of } R. \text{ Then } \tilde{\lambda} \neq \lambda, \text{ and } \\
&\Phi(Q) = \tilde{\lambda}(Q) \text{ for every } Q \in \mathcal{K}. \text{ Thus, } \tilde{\lambda} \text{ is a distinct representation, though it is not a Radon measure; see, e.g., Folland [7].} \end{aligned} \]

It may be worth pointing out that one may naively hope to demonstrate nonuniqueness of such Borel measures by constructing a measure on \( R \) endowed with the discrete topology. Alas, no Borel measures exist on \( R \) with such a topology (at least assuming the Continuum Hypothesis (CH), see [6] for a simple proof).

The measure in the example above may be made finite (and the space compact) if one is willing to make some extra set-theoretic assumptions (such as the existence of measurable cardinals). While the construction is straightforward, we omit it here for the sake of brevity.

It should be emphasized that the original results of Choquet [3] asserted the unique existence of Radon measure, not requiring the space \( R \) to be separable. Our treatise of Choquet theorems over domains could lend further weight to the justification of unified Radon measure approach as Tjur presented so enthusiastically in his book [15].

2. Domains and topologies

Let \( \mathbb{L} \) be a poset equipped with a partial order \( \preceq \). A nonempty subset \( E \) of \( \mathbb{L} \) is called \emph{directed} if every pair of elements in \( E \) has an upper bound in \( E \), and \( \mathbb{L} \) is called a \emph{directed complete poset} (or, \textit{dcpo} for short) if \( \sup E \) exists for any directed subset \( E \). A subset \( E \) is called a \emph{lower set} if \( x \in E \) and \( y \leq x \) imply \( y \in E \), and a lower set \( E \) is called an \emph{ideal} if it is also directed. In particular, the ideal \( \{x\} := \{z : z \leq x\} \) generated by an element \( x \) is called \emph{principal}. Assuming that \( \mathbb{L} \) is a dcpo, an element \( x \) of \( \mathbb{L} \) is said to be “way below” \( y \), denoted by \( x \ll y \), if for every directed set \( E \subseteq \mathbb{L} \) that satisfies \( y \leq \sup E \) one can find \( w \in E \) such that \( x \leq w \). An element \( x \) is called isolated from below if \( x \ll x \). A dcpo \( \mathbb{L} \) is called a \emph{domain} if (i) \( \langle x \rangle := \{z : z \ll x\} \) is an ideal and (ii) it satisfies \( x = \sup \langle x \rangle \) for any \( x \in \mathbb{L} \). Every domain possesses the \emph{strong interpolation property}: If \( x \ll z \) and \( z \neq x \) then there exists some \( y \neq x \) interpolating \( x \ll y \ll z \).

Throughout this paper we frequently use [8] as a standard reference on domain theory and generally follow the notation introduced therein. One notable exception is our choice of notation \( \langle \cdot \rangle \) and \( \langle \cdot \rangle \) for generators (which in [8] are denoted as \( \downarrow \) and \( \uparrow \), respectively). A poset with the reversed (or “dual”) order relation \( \leq^* \) is referred to as a \emph{dual} poset, denoted by \( \mathbb{L}^* \). A subset is called \emph{filtered} in \( \mathbb{L} \) if it is directed in \( \mathbb{L}^* \). An \emph{upper set} of \( \mathbb{L} \)}
is dually defined as a lower set of $\mathbb{L}^*$, and a filtered upper set is simply
called a filter. In the analogous manner we write $\langle x \rangle^* := \{ z : x \leq z \}$
and $\langle x \rangle^* := \{ z : x \ll z \}$ (which are denoted $\uparrow x$ and $\uparrow^* x$, respectively, in
$[S]$). For a subset $A$ we can write $\langle A \rangle := \{ z : z \leq x \text{ for some } x \in A \}$ and
$\langle A \rangle := \{ z : z \ll x \text{ for some } x \in A \}$. Again, analogously we can define $\langle A \rangle^*$
and $\langle A \rangle^*$.

**Definition 2.1.** Let $\mathbb{L}$ be a domain. Then a subset $U$ of $\mathbb{L}$ is said to be
Scott open if (i) it is an upper set and (ii) $U \cap E \neq \emptyset$ holds whenever $E$
is a directed subset and satisfies $\sup E \in U$. A refinement can be made
by introducing additional closed upper sets $\langle x \rangle^*$, thus defining a Lawson
topology on $\mathbb{L}$. A subbase of the Lawson topology is formed by all the Scott
open subsets $U$ and all the lower subsets of the form $\mathbb{L} \setminus \langle x \rangle^*$.

Scott topology is $T_0$; specifically, if $x \not\leq y$ then there is a Scott open set $U$
such that $x \in U$ and $y \notin U$. On the other hand it is not Hausdorff in general
as the following quick example shows. The real line $(-\infty, \infty)$ is a domain
in which $x \ll y$ is equivalent to $x < y$, and the Scott topology consists of
open intervals $(x, \infty)$ unbounded above, while the Lawson topology is the
standard metric one.

A poset $\mathbb{L}$ is said to be a semilattice if $x \land y := \inf \{ x, y \}$, called the
meet, exists for every pair $\{ x, y \}$. Similarly we can define a sup-semilattice
if $x \lor y := \sup \{ x, y \}$, called the join, exists for every pair $\{ x, y \}$. A poset is
called a lattice if both the meet and the join exist for every pair, and it is
said to be a complete lattice if both the supremum and the infimum exist for
every subset of $\mathbb{L}$. A domain $\mathbb{L}$ is called a continuous sup-semilattice if it is
a sup-semilattice. It is called a continuous lattice if it is a complete lattice.
For example, a half-closed interval $(0, 1]$ is a continuous sup-semilattice. It
is also a lattice, but not a continuous lattice. By Scott$(\mathbb{L})$ we denote the
family of Scott open subsets in a domain $\mathbb{L}$. The poset Scott$(\mathbb{L})$ ordered by
inclusion is a continuous lattice, in which $U \ll V$ if $U \subseteq \langle A \rangle^*$ holds for some
finite subset $A$ of $V$.

If a sup-semilattice $\mathbb{L}$ is a dcpo then it would suffice to check $x = \sup \langle x \rangle$
in order to see whether it is a domain, or equivalently, to find some $z \ll x$
with $z \not\leq y$ whenever $x \not\leq y$. A continuous sup-semilattice $\mathbb{L}$ is unital,
containing the top element $1 := \sup \mathbb{L}$. If it also has the bottom element
inf$\mathbb{L}$ then it becomes a continuous lattice. Regardless of whether there
exists a bottom element or not, we can always form a continuous lattice,
denoted by $\mathbb{L} := \mathbb{L} \cup \{ 0 \}$, by adjoining a bottom element $0$. Equipped
with the Lawson topology, a continuous sup-semilattice $\mathbb{L}$ is LCH, and $\mathbb{L}$ can be
viewed as the one-point compactification of $\mathbb{L}$ (cf. Theorem III-1.9 of [S]).

By OFilt$(\mathbb{L})$ we denote the semilattice of Scott open filters ordered by
inclusion. Assuming that a domain $\mathbb{L}$ is a semilattice, the way-below relation
$\ll$ is said to be multiplicative if $a \land b \ll x \lor y$ holds whenever $a \ll x$ and $b \ll y$.
Lawson [11] proved that if a domain $\mathbb{L}$ is a semilattice with multiplicative
way-below relation that has a top element \( \hat{1} \) satisfying \( \hat{1} \ll \hat{1} \) then \( \text{OFilt}(L) \) is a continuous lattice with the bottom element \( \{\hat{1}\} \).

**Assumption 2.2.** In the rest of the paper we will consider a domain \( \mathbb{K} \) which is also a lattice with the top element \( \hat{1} \), and assume that the way-below relation satisfies Lawson’s conditions for the continuous lattice of open filters, that is, (i) it is multiplicative, and (ii) \( \hat{1} \ll \hat{1} \).

Now \( \mathbb{K} \) is a continuous sup-semilattice (although not necessarily a continuous lattice), and \( \text{OFilt}(\mathbb{K}) \) is a continuous lattice.

**Example 2.3.** The family \( \mathcal{K} \) of compact subsets of an LCH space \( R \) is a domain and a lattice with reverse inclusion. Here we have \( E \ll F \) if and only if \( F \subseteq \text{int}(E) \) (cf. Proposition I-1.24.2 of [8]), which is multiplicative. The top element \( \emptyset \) is isolated from below, and therefore, Assumption 2.2 holds for \( \mathcal{K} \). It should be noted that \( \mathcal{K} \) is a continuous lattice if the entire space \( R \) itself is compact, and that each connected compact component of \( R \), if any, is isolated from below. Let \( \mathcal{F} \) denote the class of closed sets in \( R \). The Lawson topology of \( \mathcal{K} \) is formed by a subbase consisting of \( K_F = \{Q \in \mathcal{K} : Q \cap F = \emptyset\}, F \in \mathcal{F} \), and \( \mathcal{K} \setminus \langle E \rangle^* = \{Q \in \mathcal{K} : Q \cap E^c \neq \emptyset\}, E \in \mathcal{K} \).

The class \( \mathcal{F} \) of closed subsets of \( R \) is a continuous lattice with reverse inclusion, in which \( E \ll F \) if and only if there exists \( Q \in \mathcal{K} \) such that \( E \cup Q = R \) and \( F \cap Q = \emptyset \) (cf. Section III-1 of [8]). The Lawson topology on \( \mathcal{F} \) is also known as the Fell topology (see [8, 12]), with a subbase consisting of \( F_Q = \{F \in \mathcal{F} : F \cap Q = \emptyset\}, Q \in \mathcal{K} \), and \( \mathcal{F} \setminus \langle E \rangle^* = \{F \in \mathcal{F} : F \cap E^c \neq \emptyset\}, E \in \mathcal{F} \). The next result is a version of Hofmann-Mislove theorem; see [8].

**Theorem 2.4.** The map \( \Psi(F) = K_F \) is a homeomorphism from \( F \in \mathcal{F} \) to \( \Psi(F) \in \text{OFilt}(\mathcal{K}) \).

**Proof.** The map \( \Psi \) is clearly injective. Let \( V \in \text{OFilt}(\mathcal{K}) \) be fixed arbitrarily, and let \( F = R \setminus \cup_{Q \in V} Q \). Since \( \cup_{Q \in V} Q = \cup_{Q \in \text{int}(Q)} \), we find \( F \in \mathcal{F} \). If \( Q \in V \) then \( Q \cap F = \emptyset \). If \( E \in \Psi(F) \) then \( E \subseteq \cup_{Q \in \text{int}(Q)} \), therefore \( E \subseteq Q \) for some \( Q \in V \). Thus we obtain \( V = \Psi(F) \), and consequently, \( \Psi \) is bijective.

Moreover, \( \Psi(F_Q) = \{V \in \text{OFilt}(\mathcal{K}) : Q \subseteq V\}, Q \in \mathcal{K} \), and \( \Psi(F \setminus \langle E \rangle^*) = \text{OFilt}(\mathcal{K}) \setminus \{V \in \text{OFilt}(\mathcal{K}) : \Psi(E) \subseteq V\}, E \in \mathcal{F} \), form a subbase for \( \text{OFilt}(\mathcal{K}) \), which implies that \( \Psi \) is a homeomorphism.

**Remark 2.5.** The top element \( \emptyset \) of the continuous lattice \( \mathcal{F} \) is not isolated from below unless \( R \) is compact, and therefore, the domain \( \mathcal{F} \) does not satisfy Assumption 2.2.

**Definition 2.6.** By \( \mathcal{S} \) we denote the lattice \( \text{Scott}(\mathbb{K}) \setminus \{\emptyset\} \) of nonempty Scott open subsets in \( \mathbb{K} \) ordered by inclusion. We view \( \text{Scott}(\mathbb{K}) \) as an extension of \( \mathcal{S} \) by adjoining the bottom element \( \emptyset \), and denote it by \( \check{\mathcal{S}} \). By \( \mathcal{F} \) we denote the lattice \( \text{OFilt}(\mathbb{K}) \) of Scott open filters; there should be no confusion with the class \( \mathcal{F} \) of closed sets in light of Theorem 2.4.
The lattice $\mathcal{S}$ itself becomes a continuous lattice with the bottom element \{1\}, and it becomes a compact Hausdorff space when equipped with the Lawson topology. For $x \in \mathbb{K}$ we can define a filter $\mathcal{S}_x := \{U \in \mathcal{S} : x \in U\}$. Given a directed subset $\mathcal{E}$ of $\mathcal{S}$ satisfying $\sup \mathcal{E} \in \mathcal{S}_x$, we can find some $U \in \mathcal{E}$ which contains an $x$ such that $U \in \mathcal{S}_x$; thus, $\mathcal{S}_x$ is Scott-open. The collection of $\mathcal{S}_x$, $x \in \mathbb{K}$, becomes an open subbase for the Scott topology of $\mathcal{S}$; in fact, it forms a base since $\mathbb{K}$ is sup-semilattice.

We can view $\mathcal{F}$ as a base for the Scott topology of $\mathbb{K}$, therefore we can express a principal filter $\langle U \rangle^\ast := \{W \in \mathcal{S} : U \subseteq W\}$. Thus, the Lawson topology of $\mathcal{S}$ is formed by a subbase consisting of $\mathcal{S}_x$, $x \in \mathbb{K}$, and $\mathcal{S} \setminus \langle \langle V \rangle^\ast \rangle^\ast$, $V \in \mathcal{F}$.

**Lemma 2.7.** $\mathcal{F}$ is a closed subset of the compact Hausdorff space $\mathcal{S}$.

**Proof.** A subbase of the Lawson topology of $\mathcal{F}$ consists of $\mathcal{F}_x = \{V \in \mathcal{F} : x \in V\}$, $x \in \mathbb{K}$, and $\mathcal{F} \setminus \langle \langle V \rangle^\ast \rangle^\ast$, $V \in \mathcal{F}$, which coincides with the topology induced by the Lawson topology of $\mathcal{S}$. Since $\mathcal{F}$ is a continuous lattice, $\mathcal{F}$ is compact in the Lawson topology, and therefore, it is closed in $\mathcal{S}$. □

### 3. Scott functions and their representations

A map $f$ from a domain $\mathbb{L}$ to another domain $\mathbb{L}$ is called **Scott-continuous** if it is continuous in their respective Scott topologies. Recall that the domains of interest satisfy Assumption 2.2 and that their Lawson topologies are LCH. Thus, we call $f$ simply “continuous” (not Lawson-continuous) if $f$ is continuous with respect to the Lawson topology. Scott-continuity implies monotonicity, and the following equivalent conditions hold (Proposition II-2.1 of [8]).

**Proposition 3.1.** Suppose $f$ maps some domain $\mathbb{L}$ to another domain $\mathbb{L}$. Then the following statements are equivalent:

(i) $f$ is Scott-continuous;

(ii) $f(x) = \sup f(E)$ whenever $E$ is a directed subset of $\mathbb{L}$ converging to $x$;

(iii) $f(x) = \sup f(\langle \langle x \rangle \rangle)$ for every $x \in \mathbb{L}$.

When a real-valued function $f$ on a domain $\mathbb{L}$ is considered, $f$ is Scott-continuous if and only if $f$ is increasing [i.e. $f(x) \leq f(y)$ whenever $x \leq y$] and lower semi-continuous (l.s.c) on $\mathbb{L}$ equipped with the Scott topology.

**Definition 3.2.** Let $\varphi$ and $\Phi$ be nonnegative functions on $\mathbb{K}$. Then we call $\varphi$ a **Scott function** if $\varphi$ is Scott-continuous from $\mathbb{K}$ to $[0, \infty)$, and call $\Phi$ a **conjugate Scott function** (or, simply a **conjugate**) if $\Phi$ is Scott-continuous from $\mathbb{K}$ to $[0, \infty)^\ast$ (i.e., to the dual poset of $[0, \infty)$).

A Scott function $\varphi$ is increasing and bounded, while a conjugate $\Phi$ is decreasing, and not necessarily bounded. Without loss of generality we set $\varphi(\hat{1}) = 1$ and $\Phi(\hat{1}) = 0$ throughout this paper. Observe that an increasing (or a decreasing) nonnegative function $\psi$ is a Scott function (a conjugate...
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Scott function, respectively) if for any \( x \in \mathbb{K} \) and \( \varepsilon > 0 \) there exists some element \( z \ll x \) such that \( |\psi(x) - \psi(y)| < \varepsilon \) whenever \( z \ll y \leq x \). In Section 4.3, a conjugate functional \( \varphi \) over \( \mathcal{K} \) corresponds to a Scott function on the domain \( \mathcal{K} \) of Example 2.3 and a capacity \( \Phi \) over \( \mathcal{K} \) to a conjugate Scott function on \( \mathcal{K} \).

Let \( \xi \) be a Scott-continuous map from \((0, 1]\) to the continuous lattice \( \mathcal{S} \). Then \( \xi \) is a Borel-measurable map from \((0, 1]\) to the compact Hausdorff space \( \mathcal{S} \). By the Riesz theorem there exists a unique Radon measure \( \mu \) on \( \mathcal{S} \) which corresponds to the positive functional

\[
I(f) = \int_0^1 f(\xi(r))dr = \int f \, d\mu \tag{3.1}
\]

over the space \( C(\mathcal{S}) \) of continuous functions on \( \mathcal{S} \).

**Lemma 3.3.** Let \( \nu \) be the standard Lebesgue measure on \((0, 1]\), and let \( \mathcal{U} \) be a Scott open subset of \( \mathcal{S} \). Then the Radon measure \( \mu \) of \( \mathcal{U} \) satisfies \( \mu(\mathcal{U}) = \nu(\xi^{-1}(\mathcal{U})) \).

**Proof.** By the Riesz representation we have

\[
\mu(\mathcal{U}) = \sup\{I(f) : f \in C(\mathcal{S}), 0 \leq f \leq 1, \text{supp} f \subseteq \mathcal{U}\},
\]

where \( \text{supp} f \) denotes the support of \( f \). Thus, we find \( \mu(\mathcal{U}) \leq \nu(\xi^{-1}(\mathcal{U})) \).

Assuming \( \xi^{-1}(\mathcal{U}) \neq \emptyset \), we can write \( \xi^{-1}(\mathcal{U}) = (r, 1] \) for some \( 0 \leq r < 1 \). By choosing an arbitrary \( 0 < \varepsilon < 1 - r \), we can construct an \( f \in C(\mathcal{S}) \) such that \( f \equiv 1 \) on the compact subset \( (\xi(r + \varepsilon))^\ast \) of \( \mathcal{U} \) and \( \text{supp} f \subseteq \mathcal{U} \) using a locally compact version of Urysohn’s lemma (cf. Section 4.5 of [7]). By the Riesz representation we obtain \( \mu(\mathcal{U}) \geq 1 - r - \varepsilon \), which completes the proof. \( \square \)

Let \( \varphi \) be a Scott function. Then we can construct a Scott-continuous map \( \xi(r) = \{z \in \mathbb{K} : \varphi(z) > 1 - r\} \) from \((0, 1]\) to \( \mathcal{S} \), where the Scott-continuity of \( \xi \) is implied by \( \xi^{-1}(\mathcal{S}_x) = (1 - \varphi(x), 1] \). By Lemma 3.3 the corresponding Radon measure \( \mu \) of \( \mathcal{S}_x \) on \( \mathcal{S} \) satisfies \( \mu(\mathcal{S}_x) = \nu(\xi^{-1}(\mathcal{S}_x)) = \varphi(x) \), which shows the existence of \( \mu \) for the following proposition.

**Proposition 3.4.** There exists a Radon measure \( \mu \) on \( \mathcal{S} \) that satisfies \( \varphi(x) = \mu(\mathcal{S}_x) \) if and only if \( \varphi \) is a Scott function.

**Proof.** It suffices to show that \( \varphi(x) = \mu(\mathcal{S}_x) \) is Scott-continuous. Suppose that \( E \) is a directed subset of \( \mathbb{K} \) converging to \( x = \sup E \). Then for any \( U \in \mathcal{S} \) satisfying \( \delta_x(U) = 1 \) there exists some \( z \in E \) such that \( \delta_z(U) = 1 \). Thus, we can view \( \{\delta_z\}_{z \in E} \) as an increasing net of l.s.c. functions on \( \mathcal{S} \) converging to \( \delta_x = \sup_{z \in E} \delta_z \). By applying the monotone convergence theorem for nets (MCT for short; Proposition 7.12 of [7]), we obtain

\[
\sup_{z \in E} \varphi(z) = \sup_{z \in E} \int \delta_z \, d\mu = \int \delta_x \, d\mu = \varphi(x),
\]

which implies that \( \varphi \) is Scott-continuous. \( \square \)
Remark 3.5. Proposition 3.4 provides a representation of a Scott function on the compact Hausdorff space $S$. In the context of Krein-Milman theorem the collection $C_1$ of Scott functions can be viewed as a compact convex space. For each $U \subseteq S$ the indicator function $I_U$ [i.e., $I_U(x) = 1$ if $x \in U$; otherwise, $I_U(x) = 0$] is a Scott function. Moreover, $S$ is naturally embedded onto $C_1$ as the collection $\text{ex}(C_1)$ of extreme points in $C_1$. Choquet and others (e.g., [1]) showed that $C_1$ is the closure of convex hull of $\text{ex}(C_1)$, therefore an element $\varphi \in C_1$ is represented by the integral $\ell(\varphi) = \int_{\text{ex}(C_1)} \ell(\rho)d\mu(\rho)$ for any continuous linear functional $\ell$ on $C_1$. In particular, the measure $\mu$ on $S$ satisfies $\varphi(x) = \mu(S_x)$, $x \in K$. In contrast the construction of $\mu$ in Proposition 3.4 is elementary, and was first presented by Murofushi and Sugeno in [14].

Lemma 3.6. Let $\langle\langle x\rangle\rangle^*_S = \{V \subseteq S : \langle\langle x\rangle\rangle^* \subseteq V\}$ be the closed upper subset of $S$ generated by $\langle\langle x\rangle\rangle^* \subseteq S$, and let $S' = S \setminus \{K\}$. Then $S \setminus \langle\langle x\rangle\rangle^*_S$, $x \in K$, cover the LCH space $S'$.

Proof. Let $K^*$ be the dual poset of $K$. Then the net of the closed sets $\langle\langle x\rangle\rangle^*_S$ indexed by $x \in K^*$ converges to the singleton $\{K\}$. Thus, their complements cover $S'$.

Let $\mu$ be the Radon measure of Proposition 3.4. In the proof of Lemma 3.6 for any $x \in K$ we can find some $y \ll x$ so that $\langle\langle y\rangle\rangle^*_S \subseteq S_x \subseteq \langle\langle x\rangle\rangle^*_S$, therefore we obtain $\inf_{x \in K} \mu(\langle\langle x\rangle\rangle^*_S) = \inf_{x \in K} \mu(S_x)$. By MCT we have $\inf_{x \in K} \varphi(x) = \inf_{x \in K} \mu(\langle\langle x\rangle\rangle^*_S) = \mu(\{K\})$; thus, $\mu$ is non-atomic at $K$ whenever $\inf_{x \in K} \varphi(x) = 0$.

Definition 3.7. We call a Scott function $\varphi$ degenerate if $\inf_{x \in K} \varphi(x) > 0$. When $\varphi$ is nondegenerate, the representation $\mu$ of Proposition 3.4 can be viewed as a Radon measure on $S'$.

Let $\Phi$ be a conjugate Scott function, and let $S^x = \{U \subseteq S : x \notin U\}$ be the complement of $S_x$. Observe that $S_x$ is an open neighborhood of $K$ in $S$, and therefore, that $S^x$ is a compact subset of $S'$. If $\sup_{x \in K} \Phi(z) < \infty$, we may normalize $\Phi$ [i.e. require that $\sup_{x \in K} \Phi(z) = 1$]. Then $\varphi(z) = 1 - \Phi(z)$ is a nondegenerate Scott function, and its representation $\mu$ of Proposition 3.4 is a Radon measure on $S'$ satisfying $\Phi(x) = \mu(S^x)$. Suppose that $\sup_{x \in K} \Phi(z) = \infty$. Then the map $\xi(r) = \{z \in K : \Phi(z) < r\}$ is Scott-continuous from $(0, \infty)$ to $S'$. By the Riesz theorem there is a Radon measure $\lambda$ on $S'$ satisfying

$$\int_0^\infty f(\xi(r))dr = \int f d\lambda, \quad f \in C_c(S'),$$

where $C_c(S')$ denotes the space of continuous functions on $S'$ with compact support. Similar to the proof of Lemma 3.3 and Proposition 3.4 we can show that $\lambda(S^x) = \nu(\xi^{-1}(S^x)) = \nu([0, \Phi(x)]) = \Phi(x)$, and that $-\Phi(x) = -\lambda(S^x)$ is Scott-continuous; thus, establishing...
**Proposition 3.8.** There exists a Radon measure \( \lambda \) on \( S' \) that satisfies \( \Phi(x) = \lambda(Sx) \) if and only if \( \Phi \) is a conjugate Scott function on \( K \).

4. Approximation theorems

As introduced in Section 2, the Lawson topology of \( S \) is compact Hausdorff, and therefore, normal. Let \( C(S) \) be the space of continuous functions on \( S \), and let \( I_{Sx}(U) \) be the indicator function of a Scott-open subset \( S_x \) over \( U \in S \). By applying Urysohn’s lemma we obtain the following result.

**Lemma 4.1.** If \( a \ll x \) then there exists \( f \in C(S) \) such that \( I_{Sx}(U) \leq f(U) \leq I_{Sa}(U) \) for every \( U \in S \).

**Proof.** Observe that the closed set \( \langle \langle a \rangle \rangle^*_S \) satisfies \( S_a \subseteq \langle \langle a \rangle \rangle^*_S \subseteq S_x \). By Urysohn’s lemma there exists \( f \in C(S) \) such that \( 0 \leq f \leq 1 \), \( f(U) = 0 \) for \( U \not\in S_x \), and \( f(U) = 1 \) for \( U \in \langle \langle a \rangle \rangle^*_S \), as desired. \( \Box \)

We consider the collection \( M^+(S) \) of Radon measures on \( S \), and equip it with the weak* topology, in which the convergence \( \mu = \lim_{\alpha} \mu_\alpha \) of Radon measures is characterized by \( \int f d\mu = \lim_{\alpha} \int f d\mu_\alpha \) for every \( f \in C(S) \). It should be noted that \( M^+(S) \) is complete, that is, that every Cauchy net \( \{\mu_\alpha\} \) has the limit \( \mu \) in \( M^+(S) \) (cf. Section 12 of [4]), and that \( M_1^+(S) = \{\mu \in M^+(S) : \mu(S) = 1\} \) is compact (cf. Corollary 12.7 of [4]).

**Definition 4.2.** Let \( \varphi \) be a Scott function on \( K \), and let \( \mathcal{H} \) be a closed subset of \( S \). A net \( \{\mu_F\} \) of \( M^+(S) \) indexed by finite subsets \( F \) of \( K \) is said to approximate \( \varphi \) over \( \mathcal{H} \) if each \( \mu_F \) is supported by \( \mathcal{H} \) [i.e. \( \mu_F(S \setminus \mathcal{H}) = 0 \)] and satisfies \( \varphi(x) = \mu_F(S_x) \) for all \( x \in F \).

A Radon measure \( \mu \) on \( S \) is said to represent \( \varphi \) over \( \mathcal{H} \) if \( \mu \) is supported by \( \mathcal{H} \) and satisfies \( \varphi(x) = \mu(H_x), x \in K \), where we simply write \( H_x = \mathcal{H} \cap S_x \).

**Theorem 4.3.** If a Scott function \( \varphi \) is approximated over \( \mathcal{H} \) then there exists some \( \mu \in M_1^+(S) \) that represents \( \varphi \) over \( \mathcal{H} \).

To prove Theorem 4.3 we first need the following lemma.

**Lemma 4.4.** Let \( \{\mu_\alpha\} \) be a net converging to \( \mu \) in \( M^+(S) \). Then \( \mu(\mathcal{U}) \leq \lim\inf_\alpha \mu_\alpha(\mathcal{U}) \) for any open subset \( \mathcal{U} \) of \( S \).

**Proof.** Let \( \mathcal{U} \) be an open subset of \( S \), and let \( \varepsilon > 0 \) be arbitrary. Since \( \mu \) is a Radon measure, there exists a compact subset \( \mathcal{V} \subseteq \mathcal{U} \) such that \( \mu(\mathcal{U}) - \varepsilon < \mu(\mathcal{V}) \). By Urysohn’s lemma we can find \( f \in C(S) \) such that \( I_V \leq f \leq I_U \), and obtain

\[
\mu(\mathcal{U}) - \varepsilon < \mu(\mathcal{V}) \leq \int f d\mu = \lim_\alpha \int f d\mu_\alpha \leq \lim\inf_\alpha \mu_\alpha(\mathcal{U})
\]

which completes the proof. \( \Box \)

**Proof of Theorem 4.3.** Let \( \{\mu_F\} \) be an approximating net of \( M_1^+(S) \). Without loss of generality assume \( 1 \in F \). Since \( \mu_F(S) = \mu_F(S_1) = \varphi(1) = 1 \), the net \( \{\mu_F\} \) is a subset of \( M_1^+(S) \), therefore it has a subnet \( \{\mu_{F'}\} \) converging to
some $\mu \in M_+(S)$. Let $x \in K$ and fix an arbitrary $\varepsilon > 0$. By Lemma 4.4 we can observe that $\mu(S) \leq \lim \inf_{F'} \mu_{F'}(S) \leq \varphi(x)$. By the Scott-continuity of $\varphi$ we can find some $a \ll x$ satisfying $\varphi(x) - \varepsilon < \varphi(a)$. By Lemma 4.1 there exists $f \in C(S)$ such that $I_{S_n} \leq f \leq I_{S_x}$. Together we obtain

$$\varphi(x) - \varepsilon < \varphi(a) \leq \lim \sup_{F'} \mu_{F'}(S) \leq \lim \int f d\mu F' = \int f d\mu \leq \mu(S),$$

which implies that $\varphi(x) - \varepsilon < \varphi(a)$. Again by Lemma 4.4 we have $0 \leq \mu(S \setminus H) \leq \lim \inf_{F'} \mu_{F'}(S \setminus H) \leq 0$; thus, $\mu$ is also supported by $H$. □

We set $\mathcal{S}' = S \setminus \{K\}$, and view it as an LCH space. Then $\mathcal{S}'$ is a compact subset of $\mathcal{S}$.

**Definition 4.5.** Let $\Phi$ be a conjugate Scott function on $K$, and let $\mathcal{H}' = \mathcal{H} \setminus \{K\}$ be a closed subset of the LCH space $\mathcal{S}'$. A Radon measure $\lambda$ on $\mathcal{S}'$ is said to represent $\Phi$ over $\mathcal{H}'$ if $\lambda$ is supported by $\mathcal{H}'$ and satisfies $\Phi(x) = \lambda(\mathcal{H}')$, $x \in K$, where we customarily write $\mathcal{H}' = \mathcal{H}' \cap \mathcal{S}$. When $\lambda(\mathcal{S}') < \infty$, we identify a measure $\lambda$ on $\mathcal{S}'$ interchangeably as a measure on $\mathcal{S}$ with $\lambda(\{K\}) = 0$.

**Theorem 4.6.** Let $\{\lambda_\alpha\}$ be a net of Radon measures on $\mathcal{S}'$ supported by $\mathcal{H}'$. Suppose that $\lambda_\alpha(\mathcal{S}') \leq \Phi(x)$ for any $x \in K$, and that for each $x \in K$ there is some $\beta$ such that $\lambda_\alpha(\mathcal{H}') = \Phi(x)$ for all $\alpha > \beta$. Then there exists some $\lambda \in M^+(\mathcal{S}')$ that represents $\Phi$ over $\mathcal{H}'$.

**Proof.** Let $V$ be a compact subset of $\mathcal{H}'$. By Lemma 3.6 we can find a finite sequence $x_1, \ldots, x_n \in K$ and some $z \leq x_i$ for all $i$ so that $V \subseteq \bigcup_{i=1}^n S \setminus \langle \langle x_i \rangle \rangle^*_z \subseteq S$. Thus, the net satisfies $\lambda_\alpha(V) \leq \lambda_\alpha(\mathcal{S}') \leq \Phi(z)$ for each $\alpha$, therefore it is relatively compact in $M^+(\mathcal{S}')$ (see Section 12 of [4]). Let $\{\lambda_\alpha\}$ be a converging subnet, $\lambda$ be the limit of the subnet, and let $\varepsilon > 0$ be arbitrary. Since $\Phi(x)$ and $\lambda(\mathcal{S}')$ are conjugate Scott functions, we can find some $a \ll x$ satisfying $\Phi(a) < \Phi(x) + \varepsilon$ and $\lambda(\mathcal{S}) < \lambda(\mathcal{S}') + \varepsilon$. Similarly to Lemma 4.1 (but applying the locally compact version of Urysohn’s lemma) we can find $g \in C_c(\mathcal{S'})$ such that $I_{S'} \leq g \leq I_{S''}$. Thus, we obtain

$$\lambda(\mathcal{S}') \leq \int g d\lambda = \lim_{\alpha'} \int g d\lambda_\alpha' \leq \lim \inf_{\alpha'} \lambda_\alpha'(\mathcal{S}') \leq \Phi(a) < \Phi(x) + \varepsilon,$$

and

$$\Phi(x) \leq \lim \sup_{\alpha'} \lambda_\alpha'(\mathcal{S}') \leq \lim_{\alpha'} \int g d\lambda_\alpha' = \int g d\lambda \leq \lambda(\mathcal{S}') < \lambda(\mathcal{S}') + \varepsilon;$$

thus, the equality holds. Similarly to the proof of Theorem 4.3 we can observe that Lemma 4.4 holds for a Radon measure $\lambda$ on $\mathcal{S}'$ thus $0 \leq \lambda(\mathcal{S}' \setminus \mathcal{H}') \leq \lim \inf_{\alpha'} \lambda_\alpha'(\mathcal{S}' \setminus \mathcal{H}') = 0$. □

**Remark 4.7.** Theorem 4.6 provides an alternative construction of Radon measure of Proposition 3.8 when $\sup_{x \in K} \Phi(x) = \infty$. For each $a \in K^*$ one can define the Scott function $\psi_\alpha(x) = \max(\Phi(a) - \Phi(x), 0)$, which is nondegenerate with $\psi_\alpha(1) = \Phi(a)$. By Proposition 3.4 we have a representation
\[ \lambda_a \text{ on } S' \text{ for } \psi_a. \text{ Observe that } \lambda_a(S') = \Phi(a) - \lambda_a(S_x) \leq \Phi(x), \text{ so the equality holds if } \Phi(a) \geq \Phi(x). \text{ Hence, we can apply Theorem 4.6 and show the existence of a Radon measure } \lambda \text{ on } S' \text{ satisfying } \Phi(x) = \lambda(S^*) \]

5. Choquet theorems on domains

Let \( F \) be a semilattice, and let \( \phi \) be a function on \( F \). Then \( F \) can be defined by \( \nabla_z \phi(x) = \phi(x) - \phi(x \wedge z) \), and the successive difference operator \( \nabla_{z_1, \ldots, z_n} \) recursively by \( \nabla_{z_1, \ldots, z_n} \phi = \nabla_{z_n}(\nabla_{z_1, \ldots, z_{n-1}} \phi) \) for \( n = 2, 3, \ldots \). The operator \( \nabla_{z_1, \ldots, z_n} \) does not depend on the order of \( z_i \)'s, nor a repetition of elements, and therefore, it is denoted by \( \nabla_A \) for a finite subset \( A = \{z_1, \ldots, z_n\} \).

**Definition 5.1.** An increasing function \( \phi \) is called completely monotone if \( \nabla_A \phi \geq 0 \) holds for every nonempty finite subset \( A \). We can call similarly a decreasing function \( \phi \) completely alternating if \( \nabla_A \phi \leq 0 \) for each \( A \).

**Proposition 5.2.** Suppose \( m \) is a finite measure on \( F \), and \( \langle x \rangle \) is measurable for each \( x \in F \). Then

\[ \phi(x) = m(\langle x \rangle) \quad (5.1) \]

is nonnegative and completely monotone.

**Proof.** For any nonempty finite subset \( A \) we can express

\[ \nabla_A \phi(x) = \sum_{B \subseteq A} (-1)^{|B|} \phi(\bigwedge B \wedge x), \quad (5.2) \]

where \( \bigwedge B \wedge x \) denotes the greatest lower bound of \( B \cup \{x\} \). By applying the inclusion-exclusion principle we can also show that

\[ m \left( \bigcup_{z \in A} \langle z \rangle \cap \langle x \rangle \right) = \sum_{B \subseteq A, B \neq \emptyset} (-1)^{|B|+1} m \left( \bigcap_{z \in B} \langle z \rangle \cap \langle x \rangle \right) = \sum_{B \subseteq A, B \neq \emptyset} (-1)^{|B|+1} \phi(\bigwedge B \wedge x) \]

Comparing the sums above, we obtain

\[ \nabla_A \phi(x) = m(\langle x \rangle) - m(\langle x \rangle \cap \langle A \rangle) = m(\langle x \rangle \setminus \langle A \rangle), \quad (5.3) \]

which immediately implies \( \nabla_A \phi \geq 0 \).

The converse of Proposition 5.2 is also true if \( F \) is finite. We say “\( x \) covers \( z \)” in \( F \) if \( z < x \) and there is no other element of \( F \) between \( z \) and \( x \). We set \( r(x) = \phi(x) \) for the bottom element \( x = \bigwedge F \), and \( r(x) = \nabla_A \phi(x) \) with where \( A \) is the collection of all the elements covered by \( x \) when \( x > \bigwedge F \). Then we can construct a measure \( m(E) = \sum_{x \in E} r(x) \) for each \( E \subseteq F \). Using the inclusion-exclusion principle, we can show by induction on each element \( x \) of a linear extension of \( F \) that \( r(x) = \nabla_A \phi(x) = m(\langle x \rangle) - m(\langle A \rangle) \) for the collection \( A \) of elements covered by \( x \); thus we obtain (5.1). Even if \( \phi \) is neither nonnegative nor completely monotone, the above construction of
Theorem 5.3. There exists a unique Radon measure \( \mu \) on \( \mathcal{F} \) that represents \( \varphi \) if and only if \( \varphi \) is a completely monotone Scott function on \( \mathbb{K} \).

Theorem 5.3 is the first in a series of results, which we collectively call “Choquet theorems on domains.” The existence of \( \mu \) in Theorem 5.3 will be proved first, followed by the proof of Lemma 5.4 for the uniqueness.

The existence part of Theorem 5.3. Let \( \mathcal{F} \) be a finite subsemilattice of \( \mathbb{K} \) (possibly generated by a finite subset of \( \mathbb{K} \)), and let \( r \) be the Möbius inverse of \( \varphi \) restricted on \( \mathcal{F} \). Since \( \mathcal{F} \) is a Scott open base for \( \mathbb{K} \), we can select a collection of distinct elements \( V_i \)'s from \( \mathcal{F} \) so that \( F \cap V_i = F \cap (z)_* \) for each \( z \in F \). Then we can define a discrete measure \( \mu \) on \( \mathcal{F} \) by setting \( \mu(\{ V_i \}) = r(z), z \in F \), and obtain \( \mu(\mathcal{F}_x) = \sum_{z \leq x} r(z) = \varphi(x) \) for each \( x \in F \). Hence, \( \varphi \) can be approximated over \( \mathcal{F} \), and \( \mu \) is obtained by Theorem 4.3. \( \square \)

Assuming the representation of \( \mu \) in Theorem 5.3 above we can obtain

\[
\nabla_{z_1,\ldots,z_n} \varphi(x) = \mu(\mathcal{F}_{x}^{z_1,\ldots,z_n}),
\]

where

\[
\mathcal{F}_{x}^{z_1,\ldots,z_n} := \{ V \in \mathcal{F} : x \in V, z_i \not\in V, i = 1,\ldots,n \}.
\]

Representation (5.4) implies the necessity of complete monotonicity of \( \varphi \) in Theorem 5.3. The uniqueness of \( \mu \) is implied by (5.4) and \( \varphi(x) = \mu(\mathcal{F}_x) \), which is the immediate consequence of the lemma below.

Lemma 5.4. A Radon measure \( \mu \) on \( \mathcal{F} \) is uniquely determined by the measure on the collection of subsets \( \mathcal{F}_x \) and \( \mathcal{F}_{x}^{z_1,\ldots,z_n} \).

Proof. The open filters \( \mathcal{F}_x, x \in \mathbb{K} \), and the closed upper subsets

\[
\langle V_1,\ldots,V_n \rangle^*_{\mathcal{F}} := \{ V \in \mathcal{F} : V_i \subseteq V \text{ for some } i \}, \quad V_1,\ldots,V_n \in \mathcal{F},
\]

generate an open base that consists of subsets of the form \( \mathcal{F}_x \) or \( \mathcal{F}_x \setminus \langle V_1,\ldots,V_n \rangle^*_{\mathcal{F}} \) with \( n \geq 1 \). Consider a net \( \{ (\langle z_1 \rangle^*,\ldots,\langle z_n \rangle^*)^*_{\mathcal{F}} \} \) of closed upper subsets of \( \mathcal{F} \) indexed by \( (z_1,\ldots,z_n) \in V_1 \times \cdots \times V_n \), where the indices are inversely ordered coordinate-wise [i.e., \( (z_1,\ldots,z_n) \preceq (z_1',\ldots,z_n') \) if \( z_i \geq z_i' \) for each \( i \)]. Then the net is decreasing and converges to \( \langle V_1,\ldots,V_n \rangle^*_{\mathcal{F}} \). By MCT we obtain

\[
\mu(\mathcal{F}_x \setminus \langle V_1,\ldots,V_n \rangle^*_{\mathcal{F}}) = \sup \mu(\mathcal{F}_x \setminus \langle \langle z_1 \rangle^*,\ldots,\langle z_n \rangle^* \rangle^*_{\mathcal{F}}),
\]

where the supremum is over \( (z_1,\ldots,z_n) \in V_1 \times \cdots \times V_n \), in fact, (5.6) equals the supremum of \( \mu(\mathcal{F}_{x}^{z_1,\ldots,z_n}) \) over the same range.

The measure \( \mu \) of the intersection

\[
(\mathcal{F}_x \setminus \langle V_1,\ldots,V_n \rangle^*_{\mathcal{F}}) \cap (\mathcal{F}_y \setminus \langle U_1,\ldots,U_m \rangle^*_{\mathcal{F}}) = \mathcal{F}_{x \land y} \setminus \langle V_1,\ldots,V_n,U_1,\ldots,U_m \rangle^*_{\mathcal{F}}
\]

is determined by (5.6), and so is that of the finite union of the form \( \mathcal{F}_x \setminus \langle V_1,\ldots,V_n \rangle^*_{\mathcal{F}} \) by the inclusion-exclusion principle. An open subset of \( \mathcal{F} \) can
be expressed as a union of these open subsets, and is uniquely determined by MCT.

The following characterization is known in the general semi-group setting (see, e.g., [1]).

**Lemma 5.5.** $\Phi$ is a completely alternating conjugate if and only if $\varphi_a(x) = \Phi(x \land a) - \Phi(x)$ is a completely monotone Scott function for every $a \in \mathbb{K}$.

**Proof.** Assuming that $\Phi$ is a completely alternating conjugate, $\varphi_a = -\nabla_a \Phi$ is clearly a completely monotone Scott function. Conversely, suppose that $\varphi_a$ is a completely monotone Scott function. Then we find $\nabla_B \Phi(x) = -\nabla_B \varphi_a(x)$ if $a \leq \bigwedge B \land x$, implying that $\Phi$ is a completely alternating conjugate. □

Let $\Phi$ be a completely alternating conjugate Scott function. By Theorem 5.3 we can construct a representation $\lambda_a \in M^+(F)$ for the completely monotone Scott function $\varphi_a$ of Lemma 5.5. Since $\varphi_a$ is nondegenerate, $\lambda_a$ can be viewed as a Radon measure on the LCH space $F' = F \setminus \{\mathbb{K}\}$ (see Definition 3.7). Observe that

$$\lambda_a(F^x) = \varphi_a(\hat{1}) - \varphi_a(x) = \Phi(a) + \Phi(x) - \Phi(x \land a) \leq \Phi(x),$$

where the equality holds if $a \geq^* x$. Thus, Theorem 4.6 assures the existence of the measure $\lambda$ for the following version of Choquet theorem.

**Theorem 5.6.** There exists a unique Radon measure $\lambda$ on $F'$ such that $\Phi(x) = \lambda(F^x)$ for any $x \in \mathbb{K}$ if and only if $\Phi$ is a completely alternating conjugate.

**Proof.** If such a measure $\lambda$ exists, it is easily verified that

$$\nabla_{z_1, \ldots, z_n} \Phi(x) = -\lambda(F_{x}^{z_1, \ldots, z_n}),$$

which implies that $\Phi$ is complete alternating. An open base for the LCH $F'$ consists of open subsets of the form $F_x \setminus \langle V_1, \ldots, V_n \rangle_F$ with $n \geq 1$, and similarly to the proof of Lemma 5.3 the uniqueness of $\lambda$ is implied by (5.8). □

**Remark 5.7.** When $\Phi$ is normalized [i.e., $\sup_{x \in \mathbb{K}} \Phi(x) = 1$], we can introduce a nondegenerate completely monotone Scott function $\varphi(x) = 1 - \Phi(x)$, $x \in \mathbb{K}$. Then the representation $\mu$ of Theorem 5.3 for $\varphi$ satisfies $\mu(\{\mathbb{K}\}) = 0$, and coincides with $\lambda$ of Theorem 5.6 for $\Phi$ by uniqueness. The normalization implies that $\lambda(F') = 1$; in general, we have $\lambda(F') = \sup_{x \in \mathbb{K}} \Phi(x)$.

Recall the class $\mathcal{K}$ of compact sets in Example 2.3. By means of Theorem 2.4 we can identify $F'$ with the class of nonempty closed subsets of $R$. Thus, the representation $\lambda$ of $\Phi$ in Theorem 5.6 corresponds to that of Theorem 1.1.
6. Choquet theorems for sup-difference operators

Recall that \( \hat{\mathbb{K}} = \mathbb{K} \cup \{ \hat{0} \} \) is the one-point compactification of \( \mathbb{K} \), and that \( \hat{\mathcal{S}} = \text{Scott}(\mathbb{K}) \) is a continuous lattice with the bottom element \( \emptyset \). In the representation theorems below we consider the subset \( \bar{Q} := \{ \hat{\mathbb{K}} \setminus \langle z \rangle : z \in \hat{\mathbb{K}} \} \) of \( \hat{\mathcal{S}} \), and introduce an open subbase for \( \bar{Q} \) consisting of \( \bar{Q}_x = \{ \hat{\mathbb{K}} \setminus \langle z \rangle : x \leq z \}, x \in \mathbb{K} \), and \( \bar{Q} \setminus \langle U \rangle^* \), \( U \in \mathcal{S} \).

**Lemma 6.1.** The map \( \Psi(x) = \hat{\mathbb{K}} \setminus \langle x \rangle \) is a homeomorphism between \( \hat{\mathbb{K}} \) and \( \bar{Q} \).

**Proof.** Given any \( U \in \mathcal{S} \), we have \( z \in U \) if and only if \( U \not\subseteq \hat{\mathbb{K}} \setminus \langle z \rangle \), or equivalently, \( \Psi(U) = \bar{Q} \setminus \langle U \rangle^* \). Similarly for any \( x \in \mathbb{K} \) we have \( z \in \hat{\mathbb{K}} \setminus \langle x \rangle^* \) if and only if \( x \in \hat{\mathbb{K}} \setminus \langle z \rangle \), which implies that \( \Psi(\hat{\mathbb{K}} \setminus \langle x \rangle^*) = \bar{Q}_x \). Together we have shown that \( \Psi \) is a homeomorphism. \( \square \)

We dually define the operator \( \Delta_A \) on a \( \vee \)-semilattice, and call it a successive \( \vee \)-difference. It can be constructed with the \( \vee \)-difference operator \( \Delta_{z_1} \phi(x) = \phi(x) - \phi(x \vee z_1) \), and recursively by \( \Delta_{z_1,\ldots,z_n} \phi = \Delta_z(\Delta_{z_1,\ldots,z_{n-1}} \phi) \) for \( n = 2, 3, \ldots \)

**Definition 6.2.** An increasing function \( \phi \) is called completely \( \vee \)-alternating if \( \Delta_A \phi \leq 0 \) holds for any nonempty finite subset \( A \). Similarly a decreasing function \( \phi \) is called completely \( \vee \)-monotone if \( \Delta_A \phi \geq 0 \) holds.

Consider the complete semilattice \( \hat{\mathbb{K}}' := \hat{\mathbb{K}} \setminus \{ \hat{1} \} \). Then the Lawson topology of \( \hat{\mathbb{K}}' \) is compact since \( \{ \hat{1} \} \) is an open subset of \( \hat{\mathbb{K}} \). In fact, any continuous complete semilattice (i.e., complete semilattice which is also a domain) is compact Hausdorff in the Lawson topology (cf. Section III-1 of [8]). The map \( \Psi \) in Lemma 6.1 is homeomorphic from \( \hat{\mathbb{K}}' \) to the compact subset \( \bar{Q} := \bar{Q} \setminus \{ \emptyset \} \) of \( \mathcal{S} \).

**Lemma 6.3.** A completely \( \vee \)-alternating Scott function \( \varphi \) on \( \mathbb{K} \) has a representation \( \mu \) on \( \bar{Q} \).

**Proof.** We can naturally extend \( \varphi \) to the Scott function \( \hat{\varphi} \) on \( \hat{\mathbb{K}} \) by setting \( \hat{\varphi}(\hat{0}) = 0 \). For any nonempty finite subset \( A \) of \( \mathbb{K} \) we can observe that \( \Delta_A \hat{\varphi}(\hat{0}) \leq \Delta_A \varphi(\wedge A) \leq 0 \) where \( \wedge A \) is the greatest lower bound of \( A \); thus \( \hat{\varphi} \) is also completely \( \vee \)-alternating. Let \( \hat{F} \) be a finite sup-subsemilattice of \( \hat{\mathbb{K}} \), and let \( \hat{\varphi}^*(x) = 1 - \hat{\varphi}(x) \) be a completely monotone function on the dual poset \( \hat{F}^* \). Hence we can introduce the Möbius inverse \( r^* \) of \( \hat{\varphi}^* \) on \( \hat{F}^* \). Without loss of generality we assume \( \hat{0}, \hat{1} \in \hat{F} \), and note that \( r^*(\hat{1}) = 0 \).

For \( F = \hat{F} \setminus \{ \hat{0} \} \), we can construct a discrete measure \( \mu_F \) on \( \bar{Q} \) by setting \( \mu_F(\{ \hat{\mathbb{K}} \setminus \langle z \rangle \}) = r^*(z) \) for each \( z \in \hat{F} \setminus \{ \hat{1} \} \). Then we obtain

\[
\mu_F(\bar{Q}_x) = \sum_{z \in \hat{F} \setminus \langle x \rangle^*} r^*(z) = \hat{\varphi}^*(\hat{0}) - \hat{\varphi}^*(x) = \varphi(x)
\]

for each \( x \in F \). Hence, \( \varphi \) is approximated over \( \bar{Q} \), and it has a representation \( \mu \) on \( \bar{Q} \) by Theorem 4.3. \( \square \)
Let $Φ$ be a completely $∨$-monotone conjugate on $K$. Then $φ_a(x) = Φ(a) − Φ(a∧x)$ is a completely $∨$-alternating Scott function for each $a ∈ K^*$, and the corresponding representation $λ_a$ of $φ_a$ over $Q$ may be viewed as a measure on the LCH $Q' := Q \setminus \{K\}$ since $φ_a$ is nondegenerate (see Definition 3.7). Similarly to (5.7) we can observe that $λ_a(Q^x) = Φ(a∧x) ≤ Φ(x)$, for which the equality holds if $a ≥^* x$. Hence, by Theorem 4.6 we have established

**Corollary 6.4.** For any completely $∨$-monotone conjugate $Φ$ on $K$, there exists some $λ ∈ M^+(Q')$ such that $Φ(x) = λ(Q^x)$ for any $x ∈ K$.

Let $Ψ$ be the homeomorphism of Lemma 6.1. Then the measure $μ$ of Lemma 6.3 induces a Radon measure $Λ$ on the compact space $K'$ satisfying $Λ(λ)$ exists some for any completely $∨$-monotone conjugate $Φ$ on $K$. Theorem 6.4. In what follows we only present the proof of (i), the proof of (ii) being similar.

**Lemma 6.5.** (i) A Radon measure $Λ$ on the compact space $K'$ is uniquely determined by $λ$ on the collection of subsets $K \setminus \langle x \rangle^*$ and $\langle x \rangle^* \setminus \langle A \rangle^*$ where $x ∈ K$ and $A$ is a nonempty finite subset of $K$.

(ii) A Radon measure $Λ$ on the LCH space $K'$ is uniquely determined by the values of $Λ$ on the collection of subsets $\langle x \rangle^* \setminus \langle A \rangle^*$ where $x ∈ K$ and $A$ is a nonempty finite subset of $K$.

**Proof.** Observe that $U \setminus \langle A \rangle^*$ with open filter $U$ of $K$ (respectively, of $K'$) and nonempty finite subset $A$ of $K$ forms an open base for the Lawson topology of $K'$ (respectively, of $K'$). In what follows we only present the proof of (i), the proof of (ii) being similar.

If $U = K$ then after setting $x = \bigwedge A$ the set $K \setminus \langle A \rangle^*$ is a disjoint union of $K \setminus \langle x \rangle^*$ and $\langle x \rangle^* \setminus \langle A \rangle^*$. Otherwise, we can assume $U ≠ K$, and consider a net $\{\langle z \rangle^* \setminus \langle A \rangle^*\}$ of open subsets of $K$ indexed by $z ∈ U$, where the index set $U$ is equipped with the dual order $≤^*$. Then the net is increasing, and converges to $U \setminus \langle A \rangle^*$. By MCT we can show that $Λ(U \setminus \langle A \rangle^*) = \sup_{z ∈ U} Λ(\langle z \rangle^* \setminus \langle A \rangle^*)$, which is equal to $\sup_{z ∈ U} Λ(\langle z \rangle^* \setminus \langle A \rangle^*)$. Thus, the measure $Λ$ on the open subset $U \setminus \langle A \rangle^*$ is uniquely determined as in (i). By the inclusion-exclusion principle, the measure $Λ$ of the finite union $\bigcup_{i=1}^n U_i \setminus \langle A_i \rangle^*$ can be expressed in terms of the intersection $\bigcap_{i=1}^n U_i \setminus \langle A_i \rangle^* = (\bigcap_{i=1}^n U_i) \setminus \langle \bigcup_{i=1}^n A_i \rangle^*$ with open filter $\bigcap_{i=1}^n U_i$. Hence, a Radon measure $Λ$ is uniquely extended to the collection of open subsets of $K'$ by MCT. □
Lemma 6.5 together with Lemma 6.3 and Corollary 6.4 completes the proof of Theorem 6.6. The version of Choquet theorem for completely \(\cap\)-monotone capacities in Section 1.1 is a special case of Theorem 6.6(ii) where the lattice \(\mathbb{K}\) of Example 2.3 is considered.

**Theorem 6.6.** (i) There exists a unique Radon measure \(\Lambda\) on \(\check{\mathbb{K}}'\) such that
\[
\varphi(x) = \Lambda(\check{\mathbb{K}} \setminus \langle x \rangle^*), \quad x \in \mathbb{K},
\]
if and only if \(\varphi\) is a completely \(\lor\)-alternating Scott function.

(ii) There exists a unique Radon measure \(\Lambda\) on \(\mathbb{K}'\) such that
\[
\Phi(x) = \Lambda(\langle x \rangle^*), \quad x \in \mathbb{K},
\]
if and only if \(\Phi\) is a completely \(\lor\)-monotone conjugate.

**Remark 6.7.** A decreasing net \(\check{\mathbb{K}} \setminus \langle \langle z \rangle \rangle^*, \quad z \in \mathbb{K}^*\), of closed sets converges to \(\{\hat{0}\}\). Thus, a Radon measure \(\Lambda\) satisfies
\[
\Lambda(\{\hat{0}\}) = \inf_{z \in \mathbb{K}^*} \Lambda(\check{\mathbb{K}} \setminus \langle z \rangle^*),
\]
which equals \(\inf_{z \in \mathbb{K}^*} \varphi(z)\) in the context of Theorem 6.6(i). Consequently, \(\Lambda\) is nonatomic at \(\hat{0}\) if \(\varphi\) is nondegenerate. Similarly in Theorem 6.6(ii) we can show that \(\Lambda(\mathbb{K}') = \sup_{z \in \mathbb{K}} \Phi(z)\).

7. Valuations on a distributive lattice

A lattice \(\mathbb{L}\) is distributive if for any nonempty finite subset \(A\) of \(\mathbb{L}\) we have
\[
x \land \lor A = \lor(x \land A), \quad (7.1)
\]
where \(\lor A\) denotes the least upper bound of \(A\) and \(x \lor A := \{x \lor z : z \in A\}\).

The distributivity of \(7.1\) is dually characterized by \(x \lor \land A = \land(x \lor A)\).

Given any finite subset \(G\) of \(\mathbb{L}\) the distributivity allows us to construct the finite distributive sublattice \(H\) by first generating the sup-subsemilattice \(H\) by all elements of the form \(\lor A\), \(\emptyset \neq A \subseteq G\), then extending \(H\) to the sublattice \(F\) which consists of all elements of the form \(\land B\), \(\emptyset \neq B \subseteq H\).

In a distributive lattice \(\mathbb{L}\) the following statements are equivalent for \(z \neq \hat{1}\) (cf. Section I-3 of [8]).

(a) \(\mathbb{L} \setminus \langle z \rangle\) is a filter;
(b) \(z = x \land y\) implies \(z = x\) or \(z = y\);
(c) \(z\) is maximal in \(\mathbb{L} \setminus U\) with some open filter \(U\).

An element \(z\) is called prime [or, \(\land\)-irreducible] if it satisfies either (a) or (c) [respectively, if it satisfies (b)]. The top element \(\hat{1}\) satisfies neither (a) nor (c) while (b) holds for \(z = \hat{1}\); thus, there is a subtle distinction between prime and \(\land\)-irreducible elements. The following result is a straightforward consequence of (c) along with the \(T_0\)-property of Scott topology of a domain (cf. Theorem I-3.7 of [8] for the proof).

**Proposition 7.1.** Assume that \(\mathbb{L}\) is a domain. Then if \(x \not\leq y\) then there exists some prime element \(z\) of \(\mathbb{L}\) such that \(x \not\leq z\) and \(y \leq z\).

In the rest of our investigation we extend Assumption 2.2 for the domain \(\mathbb{K}\) to be distributive. The lattice \(\mathcal{K}\) of Example 2.3 is distributive, and moreover, the following characterization of prime elements of \(\mathcal{K}\) follows (cf. Example I-3.14 of [8]).
Proposition 7.2. A compact set $Q$ is prime in $K$ if and only if $Q$ is a singleton.

Proof. A singleton is obviously prime. Conversely, suppose $Q$ is prime in $K$. Because $\mathcal{U} = \{E \in K : Q \setminus E \neq \emptyset\}$ is a filter in $K$, the corresponding filter base $\mathcal{B} = \{Q \setminus E : E \in \mathcal{U}\}$ on the compact Hausdorff space $Q$ has a cluster point, say $a \in Q$. If $E$ is a compact neighborhood of $a$ in the LCH space $R$ then $Q \subseteq E$; otherwise, $Q \setminus E \in \mathcal{B}$, and the set of cluster points must be contained in $Q \setminus \text{int}(E)$, which contradicts $a \not\in Q \setminus \text{int}(E)$. Thus, we conclude that $Q = \{a\}$. \qed

By $P$ we denote the collection of prime elements in $\hat{K}$. The continuous lattice $\hat{K}$ is also multiplicative and distributive, and $\hat{P} := P \cup \{\emptyset\}$ is the corresponding collection of prime elements in $\hat{K}$. We can observe that

$$\mathcal{P} := Q \cap \mathcal{F} = \{\hat{K} \setminus \langle z \rangle : z \in \hat{P}\}$$

is a compact subset of $\mathcal{S}$, and isomorphic to $\hat{P}$. For any positive integer $k$ we can introduce a continuous map $\Pi_k$ from $(V_1, \ldots, V_k) \in \mathcal{P}^k$ to $\Pi_k(V_1, \ldots, V_k) := \bigcap_{i=1}^k V_i \in \mathcal{F}$. Since the product space $\mathcal{P}^k$ is compact, so is the image

$$\Pi_k \left( \mathcal{P}^k \right) = \{\hat{K} \setminus \langle A \rangle : A \subseteq \hat{P}, 1 \leq |A| \leq k\},$$

which we denote by $\mathcal{P}_k$. It should be noted that $K = \hat{K} \setminus \langle \emptyset \rangle \in \mathcal{P}_k$.

Recall the notation of (5.5); for any $w \in \hat{K}$ and any finite subset $B$ of $\hat{K}$, we will write

$$\mathcal{F}_B^w := \{V \in \mathcal{F} : w \in V, V \cap B = \emptyset\},$$

in which $\mathcal{F}_B^w = \mathcal{F}_x$ if $B = \emptyset$. Then we can observe the following property.

Lemma 7.3. If a finite subset $B$ of $\hat{K}$ satisfies $|B| \geq k+1$ then $\mathcal{P}_k \cap \mathcal{F}_B^w = \emptyset$ whenever $w \leq \bigwedge_{(x,y) \subseteq B} x \lor y$.

Proof. Suppose that there is some $A \subseteq \hat{P}$ such that $1 \leq |A| \leq k$ and $\hat{K} \setminus \langle A \rangle \in \mathcal{F}_B^w$. Since $B \subseteq \langle A \rangle$ and $|A| < |B|$, there exists a pair $(x, y) \subseteq B$ such that $x, y \leq z$ for some $z \in A$. But it implies that $w \leq x \lor y \leq z$, and therefore, that $\hat{K} \setminus \langle A \rangle \not\in \mathcal{F}_B^w$, which is a contradiction. \qed

Remark 7.4. Recall that in the setting of Theorem 2.4 the lattice $\mathcal{F}$ of closed sets is homeomorphic to $\text{OFilt}(K)$, and that the element $\emptyset$ of $\mathcal{F}$ corresponds to the top element $\hat{K}$ of $\text{OFilt}(K)$. In the context of Section 1.2 the collection $\{A \in C : 0 \leq |A| \leq k\}$ of finite subsets of at most size $k$ in $R$ is isomorphic to the compact subset $\mathcal{P}_k$ of $\text{OFilt}(K)$, and it will be referred to by the same symbol $\mathcal{P}_k$. In particular, $\mathcal{P} (= \mathcal{P}_1)$ consists of all the singletons and the empty set $\emptyset$, and it is isomorphic to the one-point compactification of $R$. For any finite subset $B = \{Q_1, \ldots, Q_n\}$ of $K$ and $W \in K$ satisfying $W \supseteq \bigcup_{1 \leq i < j \leq n} Q_i \cap Q_j$, we can express (7.2) by

$$\mathcal{F}_W^B = \{F \in \mathcal{F} : F \cap W = \emptyset, F \cap Q_i \neq \emptyset \text{ for all } i = 1, \ldots, n\}.$$
Thus, if \( F \in \mathcal{F}_W \) then \( F \) must contain at least one point for each disjoint sequence \( Q_1 \setminus W, \ldots, Q_n \setminus W \), therefore \( F \not\in \mathcal{P}_k \) for \( k < n \). This validates Lemma 7.3.

**Definition 7.5.** For any positive integer \( k \), a completely monotone Scott function \( \varphi \) on \( K \) is called a \( k \)-valuation if

\[
\nabla_B \varphi \left( \bigwedge_{\{x,y\} \subseteq B} x \lor y \right) = 0 \tag{7.3}
\]

holds for any \((k + 1)\)-element subset \( B \) of \( K \). Similarly, a completely alternating conjugate \( \Phi \) is called a conjugate \( k \)-valuation if \( 7.3 \) holds for any \((k + 1)\)-element subset \( B \) of \( K \).

**Remark 7.6.** We note in \( 7.3 \) that the greatest lower bound

\[
o_B = \bigwedge_{\{x,y\} \subseteq B} x \lor y
\]

is considered for all the joins \((x \lor y)\)’s of distinct pair \( \{x, y\} \) from \( B \). If \( B \) contains some comparable pair, say \( x_1 < y_1 \), then we have \( o_B \leq y_1 \) and by setting \( B_1 = B \setminus \{y_1\} \) we can find \( \nabla_B \varphi(o_B) = \nabla_{B_1} \varphi(o_B) - \nabla_{B_1} \varphi(o_B \lor y_1) = 0 \). Thus, it suffices to check \( 7.3 \) only for antichains \( B \)’s [i.e., \( B \)’s consisting of pairwise incomparable elements].

**Proposition 7.7.** Let \( \phi \) be an increasing or a decreasing function on \( K \). The 1-valuation condition

\[
\phi(x) + \phi(y) = \phi(x \land y) + \phi(x \lor y), \quad x, y \in K, \tag{7.4}
\]

is equivalent to:

(i) \( \phi \) is completely monotone and completely \( \lor \)-alternating if it is increasing; or

(ii) \( \phi \) is completely alternating and completely \( \lor \)-monotone if it is decreasing.

**Proof.** We can observe that

\[
\phi(x \land y) + \phi(x \lor y) - \phi(x) - \phi(y) = \nabla_{x,y} \phi(x \lor y) = \Delta_{x,y} \phi(x \land y).
\]

Then (i) implies that \( \nabla_{x,y} \phi(x \lor y) \geq 0 \) and \( \Delta_{x,y} \phi(x \land y) \leq 0 \), and therefore, that \( \phi \) satisfies \( 7.4 \). Similarly (ii) implies \( 7.4 \). Conversely, suppose that \( 7.4 \) holds. Then for any nonempty finite subset \( A \) of \( K \) we can deduce \( \nabla_A \phi(x) = \phi(x) - \phi(x \lor \bigvee A) \) and \( \Delta_A \phi(x) = \phi(x) - \phi(x \land \bigwedge A) \) by induction, which implies either (i) or (ii). \( \square \)

When \( \varphi \) is a 1-valuation in Definition 7.5 we simply call it valuation (or module). By Proposition 7.7 Scott function \( \varphi \) or conjugate Scott function \( \Phi \) is a valuation or a conjugate valuation respectively if \( 7.4 \) holds for \( \varphi \) or \( \Phi \); there is no need to check complete monotonicity or completely alternating property for 1-valuation.
Proposition 7.8. A $k$-valuation $\varphi$ is also $k'$-valuation for every $k' > k$.

Proof. Let $B'$ be an arbitrary $(k' + 1)$-element antichain, and let $F$ be a finite sublattice generated by $B'$. Since $\varphi$ is completely monotone, we can construct a measure $m(A) = \sum_{x \in A} r(x)$, $A \subseteq F$, with the Möbius inverse $r$ of the restriction of $\varphi$ to $F$. Choose any $(k + 1)$-element subset $B$ of $B'$, and observe that $z = \bigwedge_{\{x,y\} \subseteq B} x \vee y \geq z' = \bigwedge_{\{x,y\} \subseteq B'} x \vee y$. By applying (5.3), we obtain

$$0 \leq \nabla_{B'} \varphi(z') = m(\langle z' \rangle \setminus \langle B' \rangle) \leq m(\langle z \rangle \setminus \langle B \rangle) = \nabla_B \varphi(z) = 0,$$

where $\langle z \rangle$ and $\langle B \rangle$ are viewed as the lower subsets of $F$ generated by $z$ and $B$. Thus, $\varphi$ is a $k'$-valuation. \hfill \Box

Suppose that $\varphi$ is represented by $\mu$ over $P_k$. Recalling (5.4) from Section 5, we find (7.3) by Lemma 7.3 which leads to the following version of Choquet theorem.

Theorem 7.9. There exists a unique representation $\mu$ of $\varphi$ over $P_k$ if and only if $\varphi$ is a $k$-valuation.

Lemma 7.10. Let $F$ be a finite sublattice of $\mathbb{K}$, and let $J_F^{\uparrow}$ be the collection of maximal elements of $F \setminus \langle x \rangle^*$ where $x \neq \check{0}_F$; set $B_F^0 = \emptyset$. Then the collection $\{F_x^{B_F^x}\}_{x \in F}$ partitions $F$.

We note that $B_F^x \subseteq J_F^{\uparrow}$ is an antichain, and for every $x \in F$ the correspondence $x \mapsto B_F^x \subseteq J_F^{\uparrow}$ is one-to-one. Conversely, any antichain $B$ of $J_F^{\uparrow}$ (which is possibly the empty set) corresponds uniquely to $x_B := \bigwedge(F \setminus \langle B \rangle)$ in such a way that $B = B_F^{x_B}$, where we set $\langle B \rangle = \emptyset$ if $B = \emptyset$ for convenience.

Proof of Lemma 7.10. For any $V \in F$ and any $x \in F$, we have $V \in F_x^{B_F^x}$ if and only if $V \cap \langle B_F^x \rangle = \emptyset$ and $J_F \setminus \langle x \rangle^* \subseteq V$. Since $\langle B_F^x \rangle \cap J_F = J_F \setminus \langle x \rangle^*$, we can find $V \in F_x^{B_F^x}$ if and only if $x = \bigwedge(J_F \cap V)$.

In the setting of Lemma 7.10 we can observe that

$$x \leq \bigwedge_{\{y,z\} \subseteq B_F^x} y \vee z, \quad (7.5)$$
and by Lemmas 7.3 that $\mathcal{P}_k \cap J_{F}^{B_\varphi} = \emptyset$ if $|B_\varphi| \geq k + 1$. In what follows we set $\nabla_{B_\varphi} \varphi(x) = \varphi(x)$ if $B = \emptyset$ for convenience, and obtain the following corollary to Lemma 7.10

**Corollary 7.11.** Let $\varphi$ be a completely monotone Scott function on $\mathbb{K}$, and let $F$ be a finite sublattice of $\mathbb{K}$. Then the representation $\mu$ for $\varphi$ satisfies

$$
\mu(\mathcal{P}_k) \leq \sum \nabla_{B_\varphi} \varphi(x),
$$

where the summation is over all $x \in F$ satisfying $|B_\varphi| \leq k$.

In the following lemma we present a construction of approximation of $\varphi$ over $\mathcal{P}_k$. The proof of Lemma 7.12 is preceded by the construction of antichains of $P$, and followed by the proof of Theorem 7.9

**Lemma 7.12.** For $\varphi$ and $F$ of Corollary 7.11 we can construct $\mu_F \in M^+(\mathcal{F})$ so that it satisfies $\varphi(x) = \mu_F(\mathcal{F}_x)$, $x \in F$, and

$$
\mu_F(\mathcal{F} \setminus \mathcal{P}_k) \leq \sum \nabla_{B_\varphi} \varphi \left( \bigwedge_{\{x,y\} \subseteq B} x \lor y \right),
$$

(7.6)

where the summation is over antichains $B$ of $J'_F$ with $|B| = k + 1$.

By $\langle x \rangle_F$ and $\langle x \rangle^*_F$ we denote the principal lower and upper set in the lattice $F$, respectively. For each $q \in J'_F$ the coprime $\bar{q} := \bigwedge(F \setminus \langle q \rangle_F)$ satisfies $\langle \bar{q} \rangle_F = F \setminus \langle q \rangle_F$. By Proposition 7.1 we can choose $z(q) \in P$ satisfying $\bar{q} \nleq z(q)$ and $q \leq z(q)$. For each element $x \in F \setminus \{0_F\}$ the corresponding antichain $B_\varphi(x)$ satisfies $x = \bigwedge(F \setminus \langle B_\varphi(x) \rangle)$. Suppose $\{q_1, q_2\} \subseteq B_\varphi(x)$. Then we have $q_i \leq x$ and $x \nleq z(q_i)$ for $i = 1, 2$. Since $x \leq q_1 \lor q_2 \leq z(q_1) \lor z(q_2)$, $z(q_1)$ and $z(q_2)$ are not comparable. Thus, $A_x = \{z(q) : q \in B_\varphi(x)\} \subseteq P$ is an antichain if $x \neq 0_F$; set $A_{0_F} = \{0\}$.

**Proof of Lemma 7.12.** Let $r$ be the Möbius inverse of $\varphi$ on $F$, and let $m(C) = \sum_{x \in C} r(x)$ be the corresponding measure on $F$. Similarly to the proof of Theorem 5.3 we can construct a discrete measure $\mu_F$ of $\varphi$ on $\mathcal{F}$ by setting $\mu_F(\{K \setminus \langle A_x \rangle\}) = r(x)$, $x \in F$. Then we have

$$
\mu_F(\mathcal{F} \setminus \mathcal{P}_k) \leq \sum \mu_F(\{K \setminus \langle A_x \rangle\}) = m(C),
$$

where the summation is over the subset $C = \{x \in F : |B_\varphi(x)| \geq k + 1\}$.

Observe that $C$ is covered by the collection of the subsets $E_B = \{x \in F : B \subseteq B_\varphi(x)\}$ indexed by the antichains $B \subseteq J'_F$ such that $|B| = k + 1$. Let $B$ be such an antichain, and let $b = \bigwedge_{\{x,y\} \subseteq B} x \lor y$. For $x \in E_B$ we can observe that $x \nleq \langle B_\varphi(x) \rangle$, and (7.5) implies $x \leq b$; thus we have $x \in (b)_F \setminus \langle B_\varphi(x) \rangle \subseteq (b)_F \setminus \langle B \rangle$. Since $m(E_B) \leq m((b)_F \setminus \langle B \rangle) = \nabla_{B_\varphi}(b)$, the upper bound of (7.6) holds.

**Proof of Theorem 7.9.** Let $\varphi$ be a $k$-valuation, and let $\{\mu_F\}_{F \in \mathcal{F}}$ be the net of measures $\mu_F$ of Lemma 7.12 which approximates $\varphi$ over $\mathcal{P}_k$. Then the
resulting representation $\mu$ of Theorem 4.3 is that of Theorem 5.3 by uniqueness.

The proof of the Choquet theorem for conjugate $k$-valuations parallels that of Theorem 5.6. Once we obtain Lemma 7.13, we can observe that the Radon measure $\lambda$ of Theorem 5.6 represents $\Phi$ over the LCH $\mathcal{P}_k' := \mathcal{P}_k \setminus \{K\}$; thus, establishing Theorem 7.14.

Lemma 7.13. $\Phi$ is a conjugate $k$-valuation if and only if $\varphi_a$ of Lemma 5.5 is a $k$-valuation for every $a \in \mathbb{K}$.

Proof. By the expansion formula of (5.2) we obtain

$$\nabla_{z_1, \ldots, z_k} \varphi_a \left( \bigwedge_{i \neq j} z_i \lor z_j \right)$$

$$= \nabla_{z_1 \land a, \ldots, z_k \land a} \Phi \left( \bigwedge_{i \neq j} (z_i \land a) \lor (z_j \land a) \right) - \nabla_{z_1, \ldots, z_k} \Phi \left( \bigwedge_{i \neq j} z_i \lor z_j \right).$$

which implies the $k$-valuation of $\varphi_a$ if $\Phi$ is a conjugate $k$-valuation. As in Remark 7.6 the above successive difference with $z_1 \land a, \ldots, z_k \land a$ vanishes if it contains a comparable pair; for example, if $a \leq z_1 \land z_2$. Thus the $k$-valuation of $\varphi_a$ implies that of $\Phi$. 

Theorem 7.14. There exists a unique Radon measure $\lambda$ on $\mathcal{P}_k'$ such that $\Phi(x) = \lambda(\mathcal{P}_k')$ for any $x \in \mathbb{K}$ if and only if $\Phi$ is a conjugate $k$-valuation.

In the context of Section 1.2 (see Remark 7.4) the LCH $\mathcal{P}_k' = \mathcal{P}_k \setminus \{\emptyset\}$ is the collection of non-empty finite subsets of at most size $k$ in $R$, so Theorem 1.2 is an immediate corollary to Theorem 7.14.

8. Locally finite valuations

We say that an open filter $V$ in a domain $\mathbb{L}$ is $\sigma$-compact if there is a countable set $\{w_i\}$ of $V$ such that $V = \bigcup_{i=1}^{\infty} \langle w_i \rangle^*$. In what follows we assume that $\mathbb{K}$ is $\sigma$-compact, and fix a sequence $\{w_i\} \subseteq \mathbb{K}$ that satisfies $\mathbb{K} = \bigcup_{i=1}^{\infty} \langle w_i \rangle^*$ and $w_{i+1} \ll w_i$ for $i = 1, 2, \ldots$. Then we can introduce a continuous map $\Xi_i$ from $F \in \mathcal{F}$ to $\Xi_i(F) := F \cap \langle w_i \rangle^*$, and the compact subset

$$\Xi_i(\mathcal{P}_k) = \{\langle w_i \rangle^* \setminus \langle A \rangle : A \subseteq P, 0 \leq |A| \leq k\}$$

do $\mathcal{F}$. Thus, we can define the $F_\sigma$-set $\bigcup_{i=1}^{\infty} \Xi_i^{-1}(\Xi_i(\mathcal{P}_k))$, and the $F_{\sigma\delta}$-set

$$\mathcal{P}_M := \bigcap_{i=1}^{\infty} \bigcup_{k=1}^{\infty} \Xi_i^{-1}(\Xi_i(\mathcal{P}_k)) \quad (8.1)$$

In Example 2.3 we can further assume that the LCH space $R$ is $\sigma$-compact. Then we can construct a sequence $W_i$ of compact subsets of $R$ so that $R = \bigcup_{i=1}^{\infty} \text{int}(W_i)$ and $W_i \subseteq \text{int}(W_{i+1})$ for $i = 1, 2, \ldots$, therefore the domain $\mathcal{K}$ is $\sigma$-compact. In light of Theorem 2.4 we can view $\Xi_i$ as a map from
Suppose that \( F \in \mathcal{F} \) to the closed set \( F \cap \text{int}(W_i) \) on the space of \( \text{int}(W_i) \). Thus, the closed subset \( \Xi_i^{-1}(\Xi_i(P_k)) \) of \( F \) represents the collection of the closed subsets \( F \cup A \) satisfying \( F \cap \text{int}(W_i) = \emptyset \) and \( A \subseteq \text{int}(W_i) \) with \( |A| \leq k \). Hence, we can identify the \( F_{ag} \)-set \( P_{lf} \) of \( F \) with the collection \( P_{lf} \) of locally finite subsets in Section 1.2.

**Remark 8.1.** Norberg [15] in his proof of Choquet theorems on domains assumed the following property for a domain \( \mathcal{K} \) satisfying \( \Xi \). If \( x \ll y \) implies \( x \leq q \leq y \) for some \( q \in Q \). It is easily observed that if \( Q \) is a separating subset of \( \mathcal{L} \) then an open filter \( V \) of \( \mathcal{L} \) can be expressed as \( V = \bigcup_{q \in Q} \langle q \rangle^* \). Therefore, if a domain \( \mathcal{L} \) has a countable separating subset then any open filter of \( \mathcal{L} \) is \( \sigma \)-compact. In Example 2.3 the domain \( \mathcal{K} \) has a countable separating subset if \( \mathcal{R} \) is second-countable. In the setting of Definition 2.6 the lattice \( \mathcal{F} \) is second-countable if \( \mathcal{K} \) has a countable separating subset (Theorem 3.1 of [15]).

**Example 8.2.** Here we continue Example 1.5. Let \( Q_1 \) be the collection of rational numbers on \( R_1 \). Then \( R \) is not second-countable, having an uncountable open base consisting of \( G_{x_0,r_1,s_1} = \{(x_0,x_1) : x_1 \in (r_1-s_1,r_1+s_1) \cap R_1 \} \) with \( x_0 \in R_0 \) and \( r_1,s_1 \in Q_1 \). In order to separate each pair \( \{(x_0,x_1) : a_1 \leq x_1 \leq b_1 \} \ll \{(x_0,x_1) : a_2 \leq x_1 \leq b_2 \} \) of compact subsets in \( R \), we must have an uncountable separating subset of \( \mathcal{K} \).

In the proof of Lemma 7.12 we can observe that \( r(x) = m(\langle x \rangle \setminus \langle B^*_F \rangle) = \nabla B^*_F \varphi(x) \) for \( x \in F \), and that \( \mu_F(P_k) = \sum r(x) \), where the summation is over \( \{x \in F : 0 \leq |B^*_F| \leq k \} \). This observation leads to the following corollary to Lemma 7.12.

**Corollary 8.3.** The discrete measure \( \mu_F \) of Lemma 7.12 satisfies

\[
\mu_F(\Xi_i^{-1}(\Xi_i(P_k))) \geq \sum \nabla B^*_F \varphi(x),
\]

where the summation is over all \( x \in F \) satisfying \( 0 \leq |B^*_F| \leq k \). The equality in (8.2) holds if \( w_i \ll 0_F \).

**Definition 8.4.** A completely monotone Scott function \( \varphi \) is called a *locally finite valuation* if for any \( w \in \mathcal{K} \) and \( \delta > 0 \) we can find some positive integer \( k \) so that the right-hand side of (8.2) has the lower bound \( (1-\delta) \) whenever a finite sublattice \( F \) satisfies \( w \leq 0_F \).

**Theorem 8.5.** Let \( \varphi \) be a completely monotone Scott function on \( \mathcal{K} \), and let \( \mu \) be the representation of \( \varphi \) over \( \mathcal{F} \). Then \( \varphi \) is a locally finite valuation if and only if \( \mu \) satisfies \( \mu(P_{lf}) = 1 \).

**Proof.** Suppose \( \mu(P_{lf}) = 1 \). Then we have \( \mu(\bigcup_{k=1}^\infty \Xi_i^{-1}(\Xi_i(P_k))) = 1 \) for every \( i \). Let \( w \in \mathcal{K} \) and \( \delta > 0 \) be fixed. Then we can find \( w_i \ll w \) and choose \( k \) so that \( \mu(\Xi_i^{-1}(\Xi_i(P_k))) \geq 1-\delta \). Let \( F \) be a finite sublattice of \( \mathcal{K} \) satisfying \( w \leq 0_F \). If \( |B^*_F| \geq k + 1 \) then we have \( F_{ag} \subseteq \Xi_i^{-1}(\Xi_i(P_k)) = \Xi_i^{-1}(\Xi_i(F_{ag}) \cap P_k) = \emptyset \). As we demonstrated in Corollary 7.11 we can see...
that \( \mu(\bar{\Xi}_i^{-1}(\eta_i(P_k))) \) is the lower bound for the right-hand side of (8.2), so \( \varphi \) is a locally finite valuation.

Conversely, suppose \( \varphi \) is a locally finite valuation. For each \( w_i \) we can find a subnet \( \{\mu_{F'}\} \) of \( \mu_F \)'s constructed in Lemma 7.12 with the index of subnet consisting of sublattices \( F' \) satisfying \( w_i \ll \bigwedge F' \). Let \( \mu \) be a limit of the net \( \{\mu_{F'}\} \), and let \( \mu_i \) be a limit of the subnet \( \{\mu_{F'}\} \). Then we can construct a Radon measure \( \tilde{\mu}_i(\mathcal{U}) = \mu_i(\Xi_i^{-1}(\mathcal{U})) \) on Borel-measurable subsets \( \mathcal{U} \) of the compact space \( \text{OFilt}(\langle \langle \eta_i \rangle \rangle) = \Xi_i(\mathcal{F}) \), which represents \( \varphi \) restricted to the domain \( \langle \langle w_i \rangle \rangle^* \). Since \( \tilde{\mu}_i \) must be uniquely determined by Theorem 5.3, we must have \( \mu_i(\Xi_i^{-1}(\mathcal{U})) = \mu(\Xi_i^{-1}(\mathcal{U})) \).

Let \( \delta > 0 \) be arbitrary. By Corollary 8.3 and Definition 8.4 we can find a sufficiently large \( k \) so that \( \tilde{\mu}_i(\Xi_i(P_k)) \geq \sup_{F'} \mu_{F'}(\Xi_i^{-1}(\eta_i(P_k))) \geq 1 - \delta \). By applying the Lebesgue’s convergence theorem we obtain

\[
\mu \left( \bigcup_{k=1}^{\infty} \Xi_i^{-1}(\eta_i(P_k)) \right) = \tilde{\mu}_i \left( \bigcup_{k=1}^{\infty} \Xi_i(P_k) \right) = \lim_{k \to \infty} \tilde{\mu}_i(\Xi_i(P_k)) = 1
\]

and \( \mu(P_H) = \lim_{i \to \infty} \mu \left( \bigcup_{k=1}^{\infty} \Xi_i^{-1}(\eta_i(P_k)) \right) = 1. \)

To construct a finite sublattice \( F \) of \( K \), we may start with a finite sup-subsemilattice \( G \) of \( K \), and generate the sublattice \( F \). Then the collection \( J_F' \) of prime elements of \( F \) is contained in \( G' = G \setminus \{\top G\} \) (although not necessarily equal to it). Then the summation of (8.2) can be somewhat simplified as follows.

**Proposition 8.6.** Let \( G \) be a finite sup-subsemilattice of \( K \), and let \( F \) be the sublattice generated by \( G \). Then the right-hand side of (8.2) is equal to

\[
\sum \nabla_{B\varphi} \left( \bigwedge (G \setminus \langle B \rangle) \right)
\]

where the summation is over all the antichains \( B \)'s in \( G' \) satisfying \( 0 \leq |B| \leq k \).

**Proof.** Let \( B \) be an antichain in \( G' \). If \( B \subseteq J_F \) then we can find \( B = B_F^x \), for which we find \( \bigwedge (G \setminus \langle B \rangle) = x \). Otherwise, some element of \( B \) is not \( \land \)-irreducible, and \( F \setminus \langle B \rangle \) must have at least two distinct minimal elements, say \( y_1 \) and \( y_2 \), so that \( y_1 \land y_2 \in \langle B \rangle \). Since \( \bigwedge (G \setminus \langle B \rangle) \leq y_1 \land y_2 \), we must have \( \nabla_{B\varphi} \left( \bigwedge (G \setminus \langle B \rangle) \right) = 0. \)

In Proposition 8.6 we set \( o_B = \bigwedge (G \setminus \langle B \rangle) \), and call it an **opening** if \( o_B \notin \langle B \rangle \). If \( o_B \in \langle B \rangle \) then \( \nabla_{B\varphi}(o_B) = 0 \); thus, the summation of (8.3) is over those antichains \( B \)'s for which \( o_B \) is an opening, which is equal to the left-hand side of (1.4) in Section 1.2. Hence, we have established Theorem 1.3 by considering the nondegenerate locally finite valuation \( \varphi(x) = 1 - \Phi(x) \) in Theorem 8.5.
9. Lévy Exponents

Throughout this section we assume that $\mathbb{K}$ has a countable separating subset of Remark 8.1. Then we can consider an open filter $\mathcal{V}$ of $\mathbb{K}$, and set it as the $\sigma$-compact domain. Let $\Phi$ be a completely alternating conjugate on the domain $\mathcal{V}$, and let $\{v_i\}_{i=1}^{\infty}$ be a chain in $\mathcal{V}$ so that $\mathcal{V} = \bigcup_{i=1}^{\infty} \langle v_i \rangle$; set $v_0 = 1$ for convenience. By Theorem 5.6 we can find a unique representation $\lambda$ of $\Phi$ over $\mathcal{V}' = \text{OFilt}(\mathcal{V}) \setminus \{\mathcal{V}\}$. Without loss of generality we may assume that $\lambda_i = \Phi(v_i) - \Phi(v_{i-1}) > 0$, and partition $\mathcal{V}'$ into $\mathcal{V}_{v_i} = \{W \in \mathcal{V}' : v_i \in W, v_i \notin W\}$ for $i = 1, 2, \ldots$. It should be noted that $\sigma$-compactness is not required when $\Phi$ is bounded, in which case the sequence of constant values $\lambda_i$ is replaced by a single normalizing constant $\lambda(\mathcal{V}') = \sup_{x \in \mathcal{V}} \Phi(x)$; see Remark 5.7. Also, the case of $\lambda(\mathcal{V}') = 0$ [i.e., $\Phi \equiv 0$] is allowed in the following discussion.

Consider some probability space $(\Omega, \mathcal{B}, \mathbb{P})$. Then we can introduce a Poisson random variable $N_i$ with parameter $\lambda_i$ for each $i = 1, 2, \ldots$, and conditionally given $N_i = n_i$ we can independently sample random sets $\xi_{i,1}, \ldots, \xi_{i,n_i}$ from the probability measure $\lambda(\cdot)/\lambda_i$ normalized and restricted on $\mathcal{V}_{v_i}$. This collectively forms a collection of Poisson events $\xi_{i,j}$’s on the space $\mathcal{V}$. We can construct $\zeta = \bigcap_{i=1}^{\infty} \bigcap_{j=1}^{N_i} \xi_{i,j}$, where we set $\bigcap_{j=1}^{N_i} \xi_{i,j} = \emptyset$ if $N_i = 0$. Note that we have $\zeta = \emptyset$ with probability $e^{-\lambda(\emptyset)}$ if $\Phi$ is bounded. Suppose $x, y \in \zeta(\omega)$ at $\omega \in \Omega$. Then we can find some $v_k$ satisfying $v_k \ll x \wedge y$, and some $v_k \ll z \ll x \wedge y$ satisfying $\langle z \rangle^* \subset \bigcap_{i=1}^{k} \bigcap_{j=1}^{N_i(\omega)} \xi_{i,j}(\omega)$; thus $\langle z \rangle^* \subset \zeta(\omega)$, implying that $\zeta(\omega)$ is an open filter. Thus we obtain a $\mathcal{V}$-valued random variable $\zeta$, and call $\zeta$ a $\cap$-compound of Poisson events generated by $\Phi$.

**Lemma 9.1.** Let $\Phi$ be a completely alternating conjugate on the domain $\mathcal{V}$, and let $\varphi(x) = \exp[-\Phi(x)]$ if $x \in \mathcal{V}$; otherwise, $\varphi(x) = 0$. Then the $\cap$-compound $\zeta$ of Poisson events generated by $\Phi$ represents $\varphi$.

**Proof.** If $x \notin \mathcal{V}$ we see trivially $\mathbb{P}(x \in \zeta) = 0$. If $x \in \mathcal{V}$ then we obtain

$$
\mathbb{P}(x \in \zeta) = \prod_{i=1}^{\infty} \mathbb{E} \left[ \prod_{j=1}^{N_i} \mathbb{P}(x \in \xi_{i,j}) \right] = \prod_{i=1}^{\infty} \mathbb{E} \left[ \left( \frac{\lambda(\mathcal{V}_{v_i} \setminus \mathcal{V}_x)}{\lambda_i} \right)^{N_i} \right] = \prod_{i=1}^{\infty} \exp[-\lambda(\mathcal{V}_{v_i} \setminus \mathcal{V}_x)] = \exp[-\lambda(\mathcal{V}_x)] = \exp[-\Phi(x)],
$$

(9.1)

where $\mathbb{E}[X]$ denotes the expectation of a real-valued random variable $X$ on the probability space $(\Omega, \mathcal{B}, \mathbb{P})$. \hfill \Box

We call $\Phi$ the Lévy exponent of $\varphi$ if $\varphi$ of Lemma 9.1 is obtained from a completely alternating conjugate $\Phi$. This probabilistic interpretation of $\Phi$ is largely due to Mathéron [12] and Norberg [13] who found the formulation of $\varphi$ in Lemma 9.1 analogous to Lévy-Khinchin formula (cf. [1, 2]).
Consider independent and identically distributed (iid) $F$-valued random variables $\xi_1, \ldots, \xi_n$ whose distribution is determined by $\varphi_n(x) = \mathbb{P}(x \in \xi_i)$. Then the $F$-valued random variable $\bigcap_{i=1}^n \xi_i$ is distributed as $\mathbb{P}(x \in \bigcap_{i=1}^n \xi_i) = \prod_{i=1}^n \mathbb{P}(x \in \xi_i) = (\varphi_n(x))^n$. Conversely, if for any integer $n$ there exists a completely monotone Scott function $\varphi_n$ such that $\varphi = (\varphi_n)^n$ then $\varphi$ is called infinitely divisible.

**Proposition 9.2.** A Scott function $\varphi$ is infinitely divisible if and only if $\varphi$ has a Lévy exponent $\Phi$.

**Proof.** Suppose that $\varphi$ is infinitely divisible. Then we can claim that the Scott open subset $V = \{x \in K : \varphi(x) > 0\}$ is a filter. If not, we should find a pair $x, y \in V$ satisfying $\varphi(x) > 0, \varphi(y) > 0$, and $\varphi(x \land y) = 0$, for which the complete monotonicity of $\varphi_n$ implies $1 \geq \varphi_n(x \lor y) \geq \varphi_n(x) + \varphi_n(y) - \varphi_n(x \land y) = \varphi(x)^{1/n} + \varphi(y)^{1/n}$. But this cannot hold for sufficiently large $n$, drawing a contradiction. Thus, we see that

$$\Phi(x) = -\ln \varphi(x) = \lim_{n \to \infty} n[1 - \varphi(x)^{1/n}] = \lim_{n \to \infty} n[1 - \varphi_n(x)]$$

is a completely alternating conjugate on the domain $V$.

Conversely suppose that $\Phi$ is the Lévy exponent of $\varphi$ on some open filter $V$. Then $\Phi/n$ generates a $\cap$-compound $\zeta_n$ of Poisson events, and it is the Lévy exponent of $\varphi_n(x) = \exp[-\Phi(x)/n]$ if $x \in V$; otherwise, $\varphi_n(x) = 0$. Thus, we find $\varphi = (\varphi_n)^n$. \hfill \Box

In what follows we assume that $K$ is distributive. A Scott function $\varphi$ is called an exponential valuation if $\varphi(x)\varphi(y) = \varphi(x \land y)\varphi(x \lor y)$ for every pair $x, y \in K$. Then we can immediately observe the following characterization of exponential valuation.

**Proposition 9.3.** A Scott function $\varphi$ is a strictly positive exponential valuation if and only if $\varphi$ has a Lévy exponent $\Phi$ which is a conjugate valuation on $K$.

**Proof.** If $\varphi$ is an exponential valuation then $\Phi(x) = -\ln \varphi(x)$ satisfies $\mathcal{L}$; thus, it is a conjugate valuation on the domain $K$ by Proposition 7.7. The converse is obvious. \hfill \Box

Consider the Lévy exponent $\Phi$ of Proposition 9.3 and choose a sequence $\{v_i\}$ satisfying $K = \bigcup_{i=1}^\infty \langle v_i \rangle^*$ and $v_{i+1} \ll v_i$ for $i = 1, 2, \ldots$. By Theorem 7.13 we can find that the representation $\lambda$ of $\Phi$ is a Radon measure on $\mathcal{P}_1$. As demonstrated in Lemma 9.1 we can construct a $\cap$-compound $\zeta = \bigcap_{i=1}^\infty \bigcap_{j=1}^{N_i} \xi_{i,j}$ of Poisson events $\xi_{i,j}'s$ on some probability space $(\Omega, \mathcal{B}, \mathbb{P})$. Since each $\xi_{i,j}$ takes values on $\mathcal{P}_{\mathbb{R}} = \{K \setminus \{z : v_i \leq z, v_{i-1} \not\leq z, z \in P\}\}$, the $\cap$-compound $\zeta$ takes values on $\mathcal{P}_f$. By Theorem 8.5 we obtain the following corollary to Proposition 9.3.

**Corollary 9.4.** If a Scott function $\varphi$ is a strictly positive exponential valuation then it is a locally finite valuation.
In the context of Theorem 1.4 we can assume that \( R \) is \( \sigma \)-compact. Then we find an increasing sequence \( W_i \) of compact subsets of \( R \) satisfying \( R = \bigcup_{i=1}^{\infty} W_i \), and set \( W_0 = \emptyset \). Let \( \varphi \) be an exponential valuation continuous on the right over the class \( K \) of compact subsets of \( R \). By Theorem 1.2 (or equivalently by Theorem 7.14) the Lévy exponent \( \Phi \) of Proposition 9.3 is uniquely represented by a Radon measure \( \lambda \) on \( R \). We can generate Poisson events \( \xi_{i,j} \)'s of a singleton from the normalized measure \( \lambda \) each restricted on \( W_i \setminus W_{i-1} \), and define the number \( N(Q) \) of Poisson events \( \xi_{i,j} \)'s satisfying \( \xi_{i,j} \in Q \). Then it becomes a routine argument to show that \( N(Q) \) has a Poisson distribution with parameter \( \lambda(Q) \), and that \( N(Q_1), \ldots, N(Q_k) \) are independent for any disjoint sequence of \( Q_1, \ldots, Q_k \). Therefore, we have constructed a general Poisson process. Conversely, the zero probability function \( \varphi(Q) = \mathbb{P}(N(Q) = 0) = e^{-\lambda(Q)} \) is a strictly positive exponential valuation, which completes the proof of Theorem 1.4.

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