ON STOCHASTIC CONTINUITY OF GENERALIZED DIFFUSION PROCESSES CONSTRUCTED AS THE STRONG SOLUTION TO AN SDE

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Abstract. The comparison theorem for skew Brownian motions is proved. As the corollary we get the estimate on $L_1$–distance between two skew Brownian motions started from different points. Using this result we prove the continuous dependence on starting point of one class of generalized diffusion processes constructed as the strong solution to an SDE.

Introduction

The problem we consider in this paper is estimation of the distance between two strong solutions to SDE with singular coefficients. Considered processes belong to the class of generalized diffusion processes, their drift vectors and diffusion matrices include delta-function concentrated on a hyperplane.

The class of generalized diffusion processes was introduced by Portenko M.I. (see [1]). One of the most known representative of this class is skew Brownian motion. Firstly it appears in monograph by Itô K. and McKean H.P. (see [2], Section 4.2, Problem 1), then it is constructed by Portenko M.I. as a generalized diffusion processes in [1] and by Walsh J.B. [3] in the terms of its scale function and speed measure. Harrison J.M. and Shepp L.A. prove (see [4]) that skew Brownian motion can be constructed as the strong solution to an SDE.

This fact allows us to consider a family of such processes indexed by starting points or skewing parameters on the same probability space. It occurs that this family has new properties in comparison with solutions of standard SDE’s or processes with reflection. For example, using the Itô formula for these classic processes one can obtain the estimate for $L_p$–distance between such processes starting from different points for all $p \geq 1$. However, one can obtain as the corollary of the results of Burdzy K. and Kaspi H. (see [5]) that skew Brownian motion is not continuous function of the starting point. This means that there does not exist good estimate on $L_p$–distance between two skew Brownian motions starting from different points for $p > 2$. Therefore the estimation of distance between two skew Brownian motions and, moreover, between two strong solutions to SDE with singular coefficients is non-trivial problem which demands new technique to deal with. We will use the results of this paper in our next paper devoted to the Markov property of solutions to SDE with singular coefficients.

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The plan of the paper is the following one. In Section 1 we prove the comparison theorem for skew Brownian motions, the simple corollary of this theorem will be the estimate for $L_1$—distance between two skew Brownian motions. We use approximation approach for proving this result. It is known that skew Brownian motion can be constructed (see [6], p.111) as the weak limit of an appropriate sequence of diffusion processes. We prove that a pair of skew Brownian motions constructed as the functional of the same Wiener process can be approximated by a pair of diffusion process. This result together with the known comparison theorem (see [7], Section VI, Theorem 1.1) for diffusion processes gives us the required result. In Section 2 we use the estimate for distance between skew Brownian motions to prove that solutions to SDE with singular coefficients depend continuously on starting point.

1. The comparison theorem for skew Brownian motions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Consider one-dimensional Wiener process $\{w(t)\}$ started from 0 and filtration $\mathcal{F}_t^w = \sigma\{w(u), 0 \leq u \leq t\}, t \geq 0$. For given parameters $q \in (-1, 1)$ and $x_0 \in \mathbb{R}$ one can construct (see [4]) a pair of $\{\mathcal{F}_t^w\}$-adapted processes $\{(x(t), \eta_t)\}$ such that $\{\eta_t\}$ is the local time in 0 for $\{x(t)\}$ and the equality $x(t) = x_0 + qt \eta_t + w(t), t \geq 0$ is true. The process $\{x(t)\}$ is called skew Brownian motion.

For $i = 1, 2$, for given parameters $q_i \in [-1, 1)$ and $x_0^i \in \mathbb{R}$ let us construct a pair of skew Brownian motions as the functional of the one Wiener process $\{w(t)\}$ in such a way: $x^i(t) = x_0^i + q_i \eta^i_t + w(t), t \geq 0$.

**Theorem 1.** (Comparison theorem for skew Brownian motions) Let $q_1, q_2 \in (-1, 1)$ and

1) $x_0^1 \leq x_0^2$;  2) $q_1 \leq q_2$.

Then $x_1(t) \leq x_2(t)$ for all $t \geq 0$ a.s.

The proof is based on an appropriate approximation procedure for the processes $\{x_1(t)\}, \{x_2(t)\}$ by diffusion processes. In a sequel we denote by the symbol $\overset{W}{\longrightarrow}$ the weak convergence of sequences of distributions of the processes, considered as random elements of $C([0, +\infty), \mathbb{X})$, where $\mathbb{X}$ is equal $\mathbb{R}^1, \mathbb{R}^2$ or $\mathbb{R}^3$ according to context. The following limit theorem for one skew Brownian motion is known (see [6], p.111).

**Proposition 1.** Consider a sequence of diffusion processes in $\mathbb{R}$:

$$x_n(t) = x_0 + \int_0^t a_n(x_n(\tau))d\tau + w(t), \ t \geq 0, n \geq 1,$$

where $a_n(x) = na(nx), x \in \mathbb{R}, n \geq 1$, the function $a : \mathbb{R} \to \mathbb{R}$ satisfies conditions

1) $\int_\mathbb{R}|a(x)|dx < +\infty, 2)|a(x) - a(y)| < K|x - y|, x, y \in \mathbb{R}$, for some $K > 0$. Then $x_n(\cdot) \overset{W}{\longrightarrow} x(\cdot), n \to +\infty$, where the process $\{x(t)\}$ is skew Brownian motion with skewing parameter $q = \text{th} A, A = \int_\mathbb{R} a(x)dx$.

The idea of proof of Theorem 1 is to approximate a pair of skew Brownian motions by a pair of diffusion processes and then apply the comparison theorem for diffusion processes. We arrange the approximation procedure in two steps.

**Lemma 1.** In a situation of Proposition 1 we have

$$\bar{x}_n(t) = \left(\frac{x^1_n(t)}{x^2_n(t)}\right) = \left(\frac{x_0 + \int_0^t a_n(x_n^1(\tau))d\tau + w(t)}{w(t)}\right) \overset{W}{\longrightarrow} \bar{x}(\cdot) = \left(\frac{x_0 + q\eta + \tilde{w}(\cdot)}{\tilde{w}(\cdot)}\right)$$
when \( n \to +\infty \), \( \{\tilde{w}(t)\} \) is a Wiener process.

Proof. Without loss of generality we can assume that \( x_0 = 0 \). Let us apply so-called "drift eliminating" transformation to the first component of \( \{\tilde{x}_n(t)\} \) (see [6], p.111):

\[
\bar{y}_n(t) = \left( \frac{y_1^n(t)}{y^n_2(t)} \right) = \left( \frac{S_n(x_1^n(t))}{x_2^n(t)} \right) = \left( \int_0^t \sigma_n(y^n_1(\tau))d\tilde{w}(\tau) \right),
\]

where

\[
S_n(x) = c \int_0^x \exp\{-2(A(nu) - A(0))\}du, \quad A(x) = \int_{-\infty}^x a(z)dz, \quad c = \frac{\exp\{-2A(0)\}}{1 + \exp\{-2A\}},
\]

\[
\sigma_n(x) = S_n'(S_n^{-1}(x)) \quad \text{for all} \ x \in \mathbb{R}. \quad \text{Then} \quad y_1^n(\cdot) \xrightarrow{W} y(\cdot), \ n \to +\infty, \ \text{where} \ \{y(t)\} \ \text{is the solution of the following SDE:} \ dy(t) = \sigma(y(t))d\tilde{w}(t), \ t \geq 0, \ \sigma(x) = \frac{1}{2}(1 - q \text{sign} x), \ \{\tilde{w}(t)\} \ \text{is a Wiener process.}
\]

The sequence \( \{\bar{y}_n(t)\}, n \geq 1 \) is weakly compact because each component of this sequence is weakly compact. Therefore we prove the lemma if we show the uniqueness of the limit point. If we prove that the equality \( y(t) = \int_0^t \sigma(y(\tau))d\tilde{w}(\tau) \) is valid for every limit point of \( \{\bar{y}_n(t)\} \) then the needed uniqueness follows from Nakao pathwise uniqueness theorem (see [8]). Note that \( \sigma(\cdot) \) is separate from 0 and has bounded variation, i.e. Nakao theorem can be applied here.

Let \( \{\bar{y}_{n_k}(t)\} \) be a convergent subsequence (we denote it by \( \{\bar{y}_k(t)\} \)):

\[
\bar{y}_k(t) \xrightarrow{W} \bar{y}(t) \equiv \left( \frac{y(t)}{\tilde{w}(t)} \right), \quad k \to +\infty.
\]

Further we show that \( y(t) = \int_0^t \sigma(y(\tau))d\tilde{w}(\tau) \). For some \( m \geq 1 \) we denote by \( \lambda_m \) the partition of the segment \([0, t] : \lambda_m = \{0 = t_0 < t_1 < \ldots < t_m = t\} \), where \( t_j = t_j/m \). Then

\[
\mathbb{E} \left| y(t) - \int_0^t \sigma(y(\tau))d\tilde{w}(\tau) \right|^2 \leq 2\mathbb{E} \left( \left| y(t) - \sum_{j=0}^{m-1} \sigma(y(t_j))\Delta \tilde{w}^j \right|^2 + \right.
\]

\[
+ 2\mathbb{E} \left| \int_0^t \sigma(y(\tau))d\tilde{w}(\tau) - \sum_{j=0}^{m-1} \sigma(y(t_j))\Delta \tilde{w}^j \right|^2,
\]

(1.1)

where \( \Delta \tilde{w}^j = \tilde{w}(t_{j+1}) - \tilde{w}(t_j) \).

Consider the first summand. Since \( \sigma(\cdot) \) has jump only at one point and the process \( \{y(t)\} \) has transition probability density the mapping \( \Phi : (y_0, w_0) \to y(t) - \sum_{j=0}^{m-1} \sigma(y(t_j))\Delta w^j \) is continuous a.s. (we denote by \( \gamma_0 \) the trajectory of \( \gamma \) on the segment \([0, t]\)). One can observe also that \( \mathbb{E} \left| y_k(t) - \sum_{j=0}^{m-1} \sigma(y(t_j))\Delta w^j \right|^4 \leq 48t^2 \). Therefore using the theorems 5.1 and 5.4, [9] we get the equality:

(1.2)

\[
\mathbb{E} \left| y(t) - \sum_{j=0}^{m-1} \sigma(y(t_j))\Delta \tilde{w}^j \right|^2 \leq \lim_{k \to +\infty} \mathbb{E} \left| y_k(t) - \sum_{j=0}^{m-1} \sigma(y(t_j))\Delta w^j \right|^2.
\]
Let us denote \( \hat{y}_k(t) = y_k(t_j), \) \( t \in [t_j, t_{j+1}) \). We obtain

\[
\mathbb{E} \left| y_k(t) - \sum_{j=0}^{m-1} \sigma(y_k(t_j)) \Delta w^j \right|^2 = \mathbb{E} \left| \int_0^t \left[ \sigma_k(y_k(\tau)) - \sigma(\hat{y}_k(\tau)) \right] d\tau \right|^2 = \\
= \mathbb{E} \int_0^t [\sigma_k(y_k(\tau)) - \sigma(\hat{y}_k(\tau))]^2 d\tau 
\leq 2 \mathbb{E} \int_0^t [\sigma_k(y_k(\tau)) - \sigma(y_k(\tau))]^2 d\tau + \\
+ 2 \mathbb{E} \int_0^t [\sigma(\hat{y}_k(\tau)) - \sigma(\hat{y}_k(\tau))]^2 d\tau.
\]

(1.3)

Let us estimate the first summand in (1.3). Put \( \tau_R = \inf\{t \geq 0 : |y_k(t)| \geq R\} \), one has

\[
\mathbb{E} \int_0^t [\sigma_k(y_k(\tau)) - \sigma(y_k(\tau))]^2 d\tau = \mathbb{E} \mathbb{I}_{\{\tau_R \leq t\}} \int_0^t [\sigma_k(y_k(\tau)) - \sigma(y_k(\tau))]^2 d\tau + \\
\mathbb{E} \mathbb{I}_{\{\tau_R > t\}} \int_0^t [\sigma_k(y_k(\tau)) - \sigma(y_k(\tau))]^2 d\tau 
\leq t \max_{x,y \in \mathbb{R}} [\sigma_k(x) - \sigma(y)]^2 \mathbb{P}\{\tau_R \leq t\} + \\
+ \mathbb{E} \int_0^{t \wedge \tau_R} [\sigma_k(y_k(\tau)) - \sigma(y_k(\tau))]^2 d\tau.
\]

(1.4)

For some \( \delta > 0 \) take \( R = R_{\delta} = \sqrt{\frac{4t}{\delta}} \) such that the following inequality holds

\[
\mathbb{P}\{\tau_{R_{\delta}} < t\} = \mathbb{P}\{\max_{0 \leq s \leq t} |y_k(t)| \geq R_{\delta}\} \leq \frac{1}{R_{\delta}^2} \mathbb{E} \left[ \max_{0 \leq s \leq t} |y_k(s)| \right]^2 
\leq \\
\leq \frac{4t}{R_{\delta}^2} \mathbb{E} \int_0^t \sigma_k^2(y_k(s)) ds \leq \frac{4t}{R_{\delta}^2} = \delta.
\]

(1.5)

It follows from Krylov’s inequality (see, for example, [10], lemma 1, p.562) that there exists a constant \( q_{\delta,t} \) such that the following estimate holds

\[
\mathbb{E} \int_0^{t \wedge \tau_{R_{\delta}}} [\sigma_k(y_k(\tau)) - \sigma(y_k(\tau))]^2 d\tau \leq q_{\delta,t} \left[ \int_0^t d\tau \int_{|x| \leq R_{\delta}} [\sigma_k(x) - \sigma(x)]^4 dx \right]^{1/2}.
\]

Let \( k \to +\infty \). Using the Lebesgue’s majorized convergence theorem we see that the second summand in (1.4) tends to 0. Therefore, we get

\[
\limsup_{k \to +\infty} \mathbb{E} \int_0^t [\sigma_k(y_k(\tau)) - \sigma(y_k(\tau))]^2 d\tau \leq 4\delta.
\]

Then we proceed to the limit as \( \delta \to 0 \) and obtain that the first summand in (1.3) tends to 0 when \( k \to +\infty \).
Consider the second summand on the right hand side of (1.3). Using the explicit form of the function \( \sigma(\cdot) \) we get
\[
\mathbb{E} \int_0^t [\sigma(y_k(\tau)) - \sigma(\hat{y}_k(\tau))]^2 d\tau = \frac{q^2}{4} \mathbb{E} \int_0^t [\text{sign } y_k(s) - \text{sign } \hat{y}_k(s)]^2 ds = \\
= \frac{q^2}{4} \sum_{j=0}^{m-1} \mathbb{E} \int_{t_j}^{t_{j+1}} [\text{sign } y_k(s) - \text{sign } y_k(t_j)]^2 ds = \\
= q^2 \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \mathbb{P} \{ \text{sign } y_k(s) \neq \text{sign } y_k(t_j) \} ds \leq q^2 \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \mathbb{P} \{|y_k(s)| \leq \varepsilon\} ds + \\
+ q^2 \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \mathbb{P} \{ \text{sign } y_k(s) \neq \text{sign } y_k(t_j), |y_k(s)| > \varepsilon \} ds,
\]
where some \( \varepsilon > 0 \) is fixed. Observing that \( \{ \text{sign } y_k(s) \neq \text{sign } y_k(t_j), |y_k(s)| > \varepsilon \} \subseteq \{ \max_{t_j \leq s \leq t_{j+1}} |y_k(s) - y_k(t_j)| > \varepsilon \} \) we estimate the second summand in (1.6) in the following way:
\[
\mathbb{E} \int_{t_j}^{t_{j+1}} \mathbb{P} \{ \text{sign } y_k(s) \neq \text{sign } y_k(t_j), |y_k(s)| > \varepsilon \} ds \leq \frac{t}{m} \mathbb{P} \{ \max_{t_j \leq s \leq t_{j+1}} |y_k(s) - y_k(t_j)| > \varepsilon \} \leq \\
\leq \frac{t}{m} \mathbb{E} \left[ \max_{t_j \leq s \leq t_{j+1}} |y_k(s) - y_k(t_j)| \right]^2 \leq \frac{4t}{me^2} \mathbb{E} \int_{t_j}^{t_{j+1}} \sigma_k^2(y_k(s)) ds \leq \frac{4t^2}{m^2 \varepsilon^2}.
\]
Consider the first summand on the right hand side of (1.6). We have
\[
\sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \mathbb{P} \{ |y_k(s)| \leq \varepsilon \} ds = \int_0^t \mathbb{P} \{ |y_k(s)| \leq \varepsilon \} ds = \int_0^t \mathbb{P} \{ |y_k(s)| \leq \varepsilon, \tau_R < t \} ds + \\
+ \int_0^t \mathbb{P} \{ |y_k(s)| \leq \varepsilon, \tau_R \geq t \} ds \leq t \mathbb{P} \{ \tau_R < t \} + \mathbb{E} \int_0^{\tau \wedge \tau_R} \mathbb{I}_{\{|y_k(s)| \leq \varepsilon\}} ds.
\]
Let \( R_0 \) be as in previous one. It follows from Krylov’s inequality that there exists a constant \( q_{t, \delta} \) such that the following estimate holds
\[
\mathbb{E} \int_0^{\tau \wedge \tau_R} \mathbb{I}_{\{|y_k(s)| \leq \varepsilon\}} ds \leq q_{t, \delta} \left[ \int_0^t ds \int_{|x| < R_0} \mathbb{I}_{\{|x| \leq \varepsilon\}} dx \right]^{1/2} = 2q_{t, \delta} \sqrt{t(\varepsilon \wedge R_0)}.
\]
For the second summand in (1.1) one can write estimates analogous to (1.6)-(1.9) and obtain inequality
\[
\mathbb{E} \int_0^t [\sigma(y(\tau)) - \sigma(\hat{y}(\tau))]^2 d\tau \leq \frac{4t^2q^2}{me^2} + t\delta + 2q_{t, \delta} \sqrt{t(\varepsilon \wedge R_0)},
\]
with the same \( \varepsilon, \delta, q_{t, \delta}, R_0 \). \( \hat{y}(t) = y(t_j), t \in [t_j, t_{j+1}) \).

Finally, from (1.1), (1.5)-(1.10) we get
\[
\mathbb{E} \left[ y(t) - \int_0^t \sigma(y(\tau)) d\tau \right]^2 \leq \frac{24t^2q^2}{me^2} + 6t\delta + 12q_{t, \delta} \sqrt{t(\varepsilon \wedge R_0)}.
\]
Proceeding first \( m \to +\infty \), then \( \varepsilon \to 0 \) and, at last, \( \delta \to 0 \), we obtain the required result. The lemma is proved.
Lemma 2. Consider the sequence of processes in \( \mathbb{R}^2 :\)

\[
\vec{x}_n(t) = \left( \begin{array}{c} x^1_n(t) \\ x^2_n(t) \end{array} \right) = \left( \begin{array}{c} x^1_0 + \int_0^t a^1_n(x^1_n(\tau))d\tau + w(t) \\ x^2_0 + \int_0^t a^2_n(x^2_n(\tau))d\tau + w(t) \end{array} \right),
\]

where each component \( \{x^i_n(t)\}, n \geq 1, i = 1, 2 \) are defined as in Proposition 1. Then

\[
\vec{x}_n(\cdot) \xrightarrow{W} \vec{x}(\cdot) = \left( \begin{array}{c} x^1(\cdot) \\ x^2(\cdot) \end{array} \right) = \left( \begin{array}{c} x^1_0 + q^1 \eta^1 + \tilde{\omega}(\cdot) \\ x^2_0 + q^2 \eta^2 + \tilde{\omega}(\cdot) \end{array} \right), \quad n \to +\infty,
\]

\( \{x^i(t)\}, i = 1, 2 \) are skew Brownian motions constructed as the functional of the same Wiener process \( \{\tilde{\omega}(t)\} \) and \( q^1, \{\eta^1\}, q^2, \{\eta^2\} \) defined as in Proposition 1.

Proof. The sequence of the processes \( \vec{x}_n(\cdot) \) is weakly compact. This means that the sequence \( \vec{x}_n(\cdot) \) has a limit point. If we show the uniqueness of this point then we prove this lemma. We prove the uniqueness by contradiction. Let \( \vec{x}_{n_1}(\cdot) \) and \( \vec{x}_{n_2}(\cdot) \) be converged subsequences of \( \vec{x}_n(\cdot) \) with different limit points. Consider sequences in \( \mathbb{R}^3 :\)

\[
\vec{X}^1_{n_1}(\cdot) = \left( \begin{array}{c} x^1_{n_1}(\cdot) \\ x^2_{n_1}(\cdot) \\ \tilde{\omega}(\cdot) \end{array} \right), \quad \vec{X}^2_{n_2}(\cdot) = \left( \begin{array}{c} x^1_{n_2}(\cdot) \\ x^2_{n_2}(\cdot) \\ \tilde{\omega}(\cdot) \end{array} \right).
\]

Analogously to previous consideration these sequences are weakly compact. Let \( \{\vec{X}^1_{m_1}\} \) and \( \{\vec{X}^2_{m_2}\} \) be some convergent subsequences of the sequences \( \{\vec{X}^1_{n_1}\} \) and \( \{\vec{X}^2_{n_2}\} \):

\[
\vec{X}^i_{m_k}(\cdot) = \left( \begin{array}{c} x^1_{m_k}(\cdot) \\ x^2_{m_k}(\cdot) \\ \tilde{\omega}(\cdot) \end{array} \right) \xrightarrow{W} \vec{X}_i(\cdot) = \left( \begin{array}{c} x^1_i(\cdot) \\ x^2_i(\cdot) \\ \tilde{\omega}(\cdot) \end{array} \right), \quad i = 1, 2.
\]

Consider the process \( \{\vec{X}_1(\cdot)\} \). According to Lemma 1 the first component \( x^1_1(\cdot) \) is skew Brownian motion constructed as the functional of the Wiener process \( x^1_0(\cdot), i.e. \)

there exists a measurable functional \( \Phi_{q^1,x^1_0} : C[0, +\infty) \to C[0, +\infty) \) such that \( x^1_1(\cdot) = \Phi_{q^1,x^1_0}(x^1_0(\cdot)) \). The second component is the same functional of the \( x^1_i(\cdot) : x^2_i(\cdot) = \Phi_{q^2,x^2_0}(x^2_0(\cdot)) \). Therefore the distribution of the process \( \{\vec{X}_1(\cdot)\} \) is the image of the Wiener measure under the mapping \( \Psi : C[0, +\infty) \to C([0, +\infty), \mathbb{R}^3) \), where

\[
\Psi : y(\cdot) \to \left( \begin{array}{c} \Phi_{q^1,x^1_0}(y(\cdot)) \\ \Phi_{q^2,x^2_0}(y(\cdot)) \\ y(\cdot) \end{array} \right).
\]

The same arguments are valid for the process \( \{\vec{X}_2(\cdot)\} \). This means that the distributions of the processes \( \{\vec{X}_1(\cdot)\} \) and \( \{\vec{X}_2(\cdot)\} \) coincide, that gives contradiction. The lemma is proved.

Proof of Theorem 1. Let \( a_1(\cdot) \) be a function satisfying the conditions of Proposition 1 and let \( a^2(x) = a^1(x) + \frac{A_i}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), x \in \mathbb{R} \), th \( A_i = q_i, i = 1, 2 \). One can see that \( a^2(\cdot) \) satisfies the conditions of Proposition 1 too and \( a^2_n(\cdot) \geq a^1_n(\cdot) \) for all \( n \geq 1 \), where \( a^1_n(\cdot) = na_n(n). \) For \( a^1_n(\cdot), a^2_n(\cdot) \) consider the sequence of diffusion processes \( \{\vec{x}_n(\cdot)\} = \{\{x^1_n(\cdot), x^2_n(\cdot)\}\} \) defined as in Lemma 2. These processes satisfies the conditions of the comparison theorem for diffusion processes (see, for example, [3], Section VI, Theorem 1.1), i.e. \( \mathbb{P}\{x^1_n(t) \leq x^2_n(t), \forall t \geq 0\} = 1, \quad n \geq 1. \)

The set \( A = \{(\varphi_1, \varphi_2) \in C([0, +\infty), \mathbb{R}^2) : \varphi_1(t) \leq \varphi_2(t), \forall t \geq 0\} \) is closed in \( C([0, +\infty), \mathbb{R}^2) \). Therefore from the properties of weak convergence we have \( \mathbb{P}\{x_1(t) \leq x_2(t), \forall t \geq 0\} \geq \lim \sup_{n \to +\infty} \mathbb{P}\{x^1_n(t) \leq x^2_n(t), \forall t \geq 0\} = 1. \) The theorem is proved.
Corollary 1. Consider a pair of skew Brownian motions \( \{x^1(t)\}, \{x^2(t)\} \) constructed as the functional of the Wiener process \( \{w(t)\} \) with different skewing parameters \( q_1, q_2 \in (-1,1) \) and started from the same point \( x \in \mathbb{R} \). Then the equality

\[
\mathbb{E}|x^1(t) - x^2(t)| = |q_1 - q_2|I_t(x), \quad t \geq 0,
\]

holds with \( I_t(x) = \mathbb{E}_x \eta_1^t = \mathbb{E}_x \eta_2^t = \int_0^t \frac{1}{\sqrt{2\pi\tau}} \exp \left\{ -\frac{x^2}{2\tau} \right\} d\tau \).

Corollary 2. Consider a pair of skew Brownian motions \( \{x^1(t)\}, \{x^2(t)\} \) constructed as the functional of the Wiener process \( \{w(t)\} \) with the same skewing parameters \( q \in (-1, 0) \cup (0,1) \) and started from the different points \( x^1_0, x^2_0 \in \mathbb{R} \). Then the inequalities

\[
\mathbb{E}|x^1(t) - x^2(t)| \leq |x^1_0 - x^2_0| + |q||I_t(x^1_0) - I_t(x^2_0)|,
\]

(1.11) \[ \mathbb{E}||\eta^1_t - \eta^2_t|| \leq \frac{1}{|q|} |x^1_0 - x^2_0| + |I_t(x^1_0) - I_t(x^2_0)|, \quad t \geq 0, \]

hold, where the function \( I_t(\cdot), t \geq 0 \) is defined in the Corollary 1.

The proofs of Corollary 1 and Corollary 2 are easy and omitted.

Remark 1. Consider the case of \( |q| = 1 \). We assume that the phase space is equal \([0, +\infty)\) when \( q = 1 \) and \((-\infty, 0)\) when \( q = -1 \). The estimate similar to (1.11) in this situation also holds true (see [11]):

\[
\mathbb{E}|x_1(t) - x_2(t)|^2 \leq |x^1_0 - x^2_0|^2.
\]

Remark 2. In the case of \( q = 0 \) the inequality

\[
\mathbb{E}||\eta^1_t - \eta^2_t||^2 \leq 16|x^1_0 - x^2_0|^2 + \frac{8\sqrt{t}}{\sqrt{n}}|x^1_0 - x^2_0|
\]

holds. The proof of this remark is easy corollary of Tanaka’s formula for local time of Wiener process.

2. ON STOCHASTIC CONTINUITY OF STRONG SOLUTION TO SDE WITH SINGULAR COEFFICIENTS

Let \( S \) be a hyperplane in \( \mathbb{R}^d \) orthogonal to the fixed ort \( \nu \in \mathbb{R}^d \). We denote by \( \pi_S \) the operator of orthogonal projection on \( S \). For a pair of independent Wiener processes \( \{w(t)\} \) and \( \{\tilde{w}(t)\} \) in \( \mathbb{R}^d \) and \( S \) respectively, for given parameters \( x_0 \in \mathbb{R}^d, q \in [-1,1] \), given measurable function \( \alpha : S \rightarrow S \) and operator \( \beta : S \rightarrow \mathcal{L}_+(S) \) (\( \mathcal{L}_+(S) \) is the space of all linear symmetric nonnegative operators on \( S \)) we consider the following stochastic equation in \( \mathbb{R}^d \)

\[
x(t) = x_0 + \int_0^t (q\nu + \alpha(x^S(\tau))) \, d\eta_\tau + \int_0^t \tilde{\beta}(x^S(\tau)) \, d\tilde{\omega}(\eta_\tau) + w(t)
\]

where \( \tilde{\beta}(\cdot) = \beta^{1/2}(\cdot), x^S(\cdot) = \pi_S x^S(\cdot) \). It is proved in [12] that under the following assumptions on the coefficients

\[
(1) \sup_{x \in S}(|\alpha(x)| + \|\tilde{\beta}(x)\|) \leq K,
\]

\[
(2) |\alpha(x) - \alpha(y)|^2 + \|\tilde{\beta}(x) - \tilde{\beta}(y)\|^2 \leq K|x - y|^2, \text{ for all } x, y \in S
\]

for some \( K > 0 \) the solution to the equation (2.1) exists and is unique. In the next theorem we prove that this solution continuously depends on the starting point.
Lemma 3. For all \( P \).

Firstly we prove that

\[
\text{Proof. We construct the processes } P \text{ in the case of }\]

Let \( \rho \{ \cdot \} \rightarrow \mathbb{R} \) started from \( \infty \) where \( \{ x(t) \} \) is the solution to (2.1) started from \( x \).

Consider a new process \( \rho \{ \cdot \} \) instead of \( t \) in equation (2.1) and set \( u = \eta_r \) in all integrals in (2.1). Then we obtain

\[
(2.2) \quad x(\rho_t) = x + \int_0^t (qu + \alpha(x^S(\rho_u))) \, du + \int_0^t \beta(x^S(\rho_u)) \, dw(u) + w(\rho_t)
\]

We construct the processes \( \{ \rho_n \}, \{ x_n(\rho_n) \} \) in the same way.

Lemma 3. For all \( t \geq 0 : \rho_n \xrightarrow{P} \rho_t \) when \( n \rightarrow +\infty \).

Proof. Firstly we prove that \( \mathbb{P}(\rho_n - \rho_t > \varepsilon) \rightarrow 0, n \rightarrow +\infty \). We can write

\[
\mathbb{P}(\rho_n - \rho_t > \varepsilon) = \mathbb{P}(\rho_n > \varepsilon + \rho_t) \leq \sum_{k \geq 0} \mathbb{P} \left( \rho_n > \varepsilon + \rho_t, \rho_n \in \left( \frac{(k+1)\varepsilon}{2}, \frac{(k+2)\varepsilon}{2} \right) \right) \leq \sum_{k \geq 0} \mathbb{P} \left( \eta_{k\varepsilon/2} > t, \eta_{(k+1)\varepsilon/2} < t \right).
\]

Consider the k-th summand. For some \( \delta > 0 \)

\[
\mathbb{P} \left( \eta_{k\varepsilon/2} > t, \eta_{(k+1)\varepsilon/2} < t \right) = \mathbb{P} \left( \eta_{k\varepsilon/2} > t + \delta, \eta_{(k+1)\varepsilon/2} < t \right) + \mathbb{P} \left( \eta_{k\varepsilon/2} > t, \eta_{(k+1)\varepsilon/2} < t \right) + \mathbb{P} \left( \eta_{(k+1)\varepsilon/2} > t, \eta_{(k+1)\varepsilon/2} < t \right) \leq \mathbb{P} \left( \eta_{(k+1)\varepsilon/2} > t + \delta, \eta_{(k+1)\varepsilon/2} < t \right).
\]

We use the distribution of \( \{ \eta_t \} \) for estimating the second summand in (2.3). We remind that \( \{ \eta_t \} \) has the same distribution with the local time in 0 of a Wiener process in \( \mathbb{R} \) started from \( x^\nu = (x, \nu) \):

\[
\mathbb{P}(\eta^\nu_t < a) = \left( 1 - 2 \int_{a}^{+\infty} \frac{\exp \left\{ -u^2/2 \right\}}{\sqrt{2\pi}} \, du \right) \mathbb{1}_{\{a > 0\}}.
\]

Thus we have

\[
(2.4) \quad \mathbb{P} \left( t < \eta_{\varepsilon/2} \leq t + \delta \right) = 2 \int_{\varepsilon/2}^{+\infty} \frac{\exp \left\{ -u^2/2 \right\}}{\sqrt{2\pi}} \, du \leq 2 \sqrt{\frac{2}{\pi \varepsilon \delta}}.
\]

The first summand in (2.3) is estimated by using Chebyshev’s inequality and Corollary 2 in the case of \( q \in (-1, 0) \cup (0, 1) \):

\[
\mathbb{P} \left( |\eta_{(k+1)\varepsilon/2} - \eta_{(k+1)\varepsilon/2}^n| > \delta \right) \leq \frac{1}{\delta} \mathbb{E}|\eta_{(k+1)\varepsilon/2} - \eta_{(k+1)\varepsilon/2}^n| \leq \text{estimation}
\]

\[
\mathbb{P} \left( \eta_{(k+1)\varepsilon/2} - \eta_{(k+1)\varepsilon/2}^n > \delta \right) \leq \frac{1}{\delta} \mathbb{E}|\eta_{(k+1)\varepsilon/2} - \eta_{(k+1)\varepsilon/2}^n| \leq \text{estimation}
\]

\[
\mathbb{P} \left( \eta_{(k+1)\varepsilon/2} - \eta_{(k+1)\varepsilon/2}^n < -\delta \right) \leq \frac{1}{\delta} \mathbb{E}|\eta_{(k+1)\varepsilon/2} - \eta_{(k+1)\varepsilon/2}^n| \leq \text{estimation}
\]
Finally we obtain from (2.3)-(2.5) that
\[
\mathbb{P}\left\{ \rho_t^n > \varepsilon + \rho_t, \rho_t^n \in \left( \frac{(k+1)\varepsilon}{2}, \frac{(k+2)\varepsilon}{2} \right) \right\} \leq \left( 2\sqrt{\frac{2}{\pi k\varepsilon}} \right)^2 + \frac{1}{\delta} c_1 q |x_n - x| + c_2 q |I_{(k+1)\varepsilon/2}(x_n) - I_{(k+1)\varepsilon/2}(x)|.
\]

Taking into account that
\[
\sum_{k \geq 0} \mathbb{P}\left\{ \rho_t^n > \varepsilon + \rho_t, \rho_t^n \in \left( \frac{(k+1)\varepsilon}{2}, \frac{(k+2)\varepsilon}{2} \right) \right\} \leq \mathbb{P}\{ \rho_t^n \geq 0 \} = 1
\]
we see that conditions of Lebesgue’s majorized convergence theorem is satisfied. Therefore \( \mathbb{P}\{ \rho_t^n - \rho_t > \varepsilon \} \rightarrow 0, n \rightarrow +\infty \). The same arguments can be made in the case \( |q| = 1 \) (by using Remark 1), \( q = 0 \) (by using Remark 2). We prove that \( \mathbb{P}\{ \rho_t - \rho_t^n > \varepsilon \} \rightarrow 0, n \rightarrow +\infty \) in the same way. The lemma is proved.

**Lemma 4.** For all \( t \geq 0 \) : \( |x_n(\rho_t^n) - x(\rho_t)| \xrightarrow{P} 0, n \rightarrow +\infty \).

**Remark 3.** One can see that \( \mathbb{E}|x_n(\rho_t^n) - x(\rho_t)|^p = +\infty, p \geq 1 \) because \( \mathbb{E}\rho_t = +\infty \). Thus we cannot apply here standard technique such as using martingale inequalities.

**Proof.** For given \( N > 0, C > 0, n \geq 1 \) we consider the random set
\[
A_{N,C}^{n,t} = \left\{ \omega \in \Omega : \rho_t < N, \rho_t^n < N, \|w\|_{\text{Hol}_{1/4}([0,N])} \leq C \right\},
\]
where \( \|w\|_{\text{Hol}_{1/4}([0,N])} \) is the H"older norm with parameter 1/4 on the segment \([0, N] \).

Note that \( A_{N,t}^{n,t} \subseteq \mathcal{F}_N^n \) and \( A_{N,C}^{n,t} \subseteq A_{N,C}^{n,s} \) when \( s \leq t \).

Then for all \( \varepsilon > 0, t \in [0, T] \) we get
\[
\mathbb{P}\{ |x_n(\rho_t^n) - x(\rho_t)| > \varepsilon \} \leq \mathbb{P}\{ |x_n(\rho_t^n) - x(\rho_t)| > \varepsilon \} \cap A_{N,C}^{n,t} \} + \mathbb{P}\{ \rho_t \geq N \} + \mathbb{P}\{ \|w\|_{\text{Hol}_{1/4}([0,N])} > C \}
\]

(2.6)

Let us estimate the second moment of the process \( \{x_n(\rho_t^n) - x(\rho_t)\}^2 \mathbb{I}_{A_{N,C}^{n,t}} \}:
\[
\mathbb{E}|x_n(\rho_t^n) - x(\rho_t)|^2 \mathbb{I}_{A_{N,C}^{n,t}} \leq 4|x_n - x|^2 + 4t \int_0^t \mathbb{E}[\alpha(x_n^S(\rho_u^n)) - \alpha(x^S(\rho_u))]|^2 \mathbb{I}_{A_{N,C}^{n,t}} du
\]
\[\begin{align*}
&+ 4E \left[ \int_0^t (\tilde{\beta}(x_n^S(\rho^u_t)) - \tilde{\beta}(x^S(\rho^u_t))) d\tilde{w}(u) \right]^2 \mathbb{I}_{A_{N,C}^{n,t}} + 4E|w(\rho^u_t) - w(\rho_t)|^2 \mathbb{I}_{A_{N,C}^{n,t}} \leq \\
&\leq 4|x_n - x|^2 + 4KTE \int_0^t |x_n^S(\rho^u_t) - x^S(\rho_t)|^2 \mathbb{I}_{A_{N,C}^{n,t}} du + \\
&4CE|\rho^u_t - \rho_t|^{1/2} \mathbb{I}_{A_{N,C}^{n,t}}. \\
(2.7) \quad &+ 4E \left[ \int_0^t (\tilde{\beta}(x_n^S(\rho^u_t)) - \tilde{\beta}(x^S(\rho_t))) d\tilde{w}(u) \right]^2 \mathbb{I}_{A_{N,C}^{n,t}} + 4CE|\rho^u_t - \rho_t|^{1/2} \mathbb{I}_{A_{N,C}^{n,t}}.
\end{align*}\]

Consider the third summand. Let us put \( \tilde{F}_t = F^w_t \vee \bar{F}^w_t \). One can observe that the process \( \{\tilde{\beta}(x_n^S(\rho^u_t)) - \tilde{\beta}(x^S(\rho_t))\} \) is \( \{\tilde{F}_t\} \)-adapted, the process \( \{\bar{w}(t)\} \) is the Wiener process w.r.t. \( \{\tilde{F}_t\} \) and \( A_{N,C}^{n,t} \in \tilde{F}_t \) for all \( t \geq 0 \). Therefore the equality

\[ E \int_0^t (\tilde{\beta}(x_n^S(\rho^u_t)) - \tilde{\beta}(x^S(\rho_t))) d\tilde{w}(u) \mathbb{I}_{A_{N,C}^{n,t}} = E \int_0^t (\bar{\beta}(x_n^S(\rho^u_t)) - \bar{\beta}(x^S(\rho_t))) \mathbb{I}_{A_{N,C}^{n,t}} d\bar{w}(u) \]

holds. Thus we obtain

\[ E \left[ \int_0^t (\tilde{\beta}(x_n^S(\rho^u_t)) - \tilde{\beta}(x^S(\rho_t))) \mathbb{I}_{A_{N,C}^{n,t}} d\tilde{w}(u) \right]^2 = \\
= \int_0^t E \left\| \tilde{\beta}(x_n^S(\rho^u_t)) - \tilde{\beta}(x^S(\rho_t)) \right\|^2 \mathbb{I}_{A_{N,C}^{n,t}} du \leq K \int_0^t E |x_n^S(\rho^u_t) - x^S(\rho_t)|^2 \mathbb{I}_{A_{N,C}^{n,t}} du.
\]

It follows from (2.7) and (2.8) that

\[ E|x_n(\rho^u_t) - x(\rho_t)|^2 \mathbb{I}_{A_{N,C}^{n,t}} \leq 4|x_n - x|^2 + \\
+ 4K(T + 1)E \int_0^t |x_n^S(\rho^u_t) - x^S(\rho_t)|^2 \mathbb{I}_{A_{N,C}^{n,t}} du + 4CE|\rho^u_t - \rho_t|^{1/2} \mathbb{I}_{A_{N,C}^{n,t}}.
\]

Using the Gronwall-Bellman inequality we obtain

\[ E|x_n(\rho^u_t) - x(\rho_t)|^2 \mathbb{I}_{A_{N,C}^{n,t}} \leq 4|x_n - x|^2 + 4CE|\rho^u_t - \rho_t|^{1/2} \mathbb{I}_{A_{N,C}^{n,t}} + \\
(2.9) + 4K(T + 1) \int_0^t \exp \{4K(T + 1)(t - u)\} \left(4|x_n - x|^2 + 4CE|\rho^u_t - \rho_u|^{1/2} \mathbb{I}_{A_{N,C}^{n,t}}\right) du
\]

for all \( t \in [0,T] \).

It follows from Lemma 3 and from the fact that the processes \( \{\rho^u_t \mathbb{I}_{A_{N,C}^{n,t}}\}, \{\rho_t \mathbb{I}_{A_{N,C}^{n,t}}\} \)
are bounded for fixed \( N,C,t \) that \( E|\rho^u_t - \rho_t|^{1/2} \mathbb{I}_{A_{N,C}^{n,t}} \rightarrow 0, n \rightarrow +\infty \). From (2.9) we see that for all \( \delta > 0 \) there exists \( n_0 > 0 \) such that \( E|x_n(\rho^u_t) - x(\rho_t)|^2 \mathbb{I}_{A_{N,C}^{n,t}} < \delta \) for all \( n \geq n_0 \). Using the Chebyshev’s inequality for the first summand of (2.6) we obtain that

\[ P\{|x_n(\rho^u_t) - x(\rho_t)| > \varepsilon\} \leq \frac{\delta}{\varepsilon^2} + P\{|\rho^u_t \geq N\} + \sup_{t \in [0,T]} P\{|\rho^u_t \geq N\} + P\{|w|_{\text{Hol}_{1/4}(\{0,N\})} > C\}
\]
for all $n \geq n_0$. Proceeding first $\delta \to 0$, then $C \to +\infty$ and, at last, $N \to +\infty$, we obtain that $\mathbb{P}\{|x_n(\rho^n_t) - x(\rho_t)| > \varepsilon\} \to 0, n \to +\infty$. Note that $\mathbb{P}\{\rho_t \geq N\} \leq \frac{2N}{\sqrt{2\pi N}} \to 0, N \to +\infty$. Lemma is proved.

Let us return to the proof of the Theorem 2. We have

$$\mathbb{P}\{|x_n(t) - x(t)| > \varepsilon\} \leq \mathbb{P}\{|x_n(t) - x_n(\rho^n_t) + x(\rho_t)| > \varepsilon/2\} +$$

$$\mathbb{P}\{|x_n(\rho^n_t) - x(\rho_t)| > \varepsilon/2\}$$

(2.10)

It follows from definition of $\{\rho_t\}$ that $\rho_{n_t} \leq t$. One can observe that $\eta_s = const, s \in [\rho_{n_t}, t]$, thus $x(s) = x_{0 + w(s)}$, $s \in [\rho_{n_t}, t]$ and $|x_n(t) - x(t) - x_n(\rho^n_{n_t}) + x(\rho_t)| = |w(\rho^n_{n_t}) - w(\rho_{n_t})|$

Consider the second summand in (2.11). We can write

$$|x_n(\rho^n_{n_t}) - x(\rho_t)| \leq |x_n - x| + \int_{n_t}^{n_{n_t}} (\eta + \alpha(x_n(\rho^n_{n_t}))) du + \int_{n_t}^{n_{n_t}} (\alpha(x_n(\rho^n_{n_t}) - \alpha(x^n(\rho^n_{n_t}))) du +$$

(2.11)

$$+ \int_{n_t}^{n_{n_t}} \beta(x_n(\rho^n_{n_t})) \tilde{w}(u) + \int_{n_t}^{n_{n_t}} (\beta(x_n(\rho^n_{n_t}) - \beta(x^n(\rho^n_{n_t}) \tilde{w}(u) + |w(\rho^n_{n_t}) - w(\rho_{n_t})|$$

Let us estimate the second moment of the fourth and fifth summands in (2.11) (the first moment of the second and third summands can be estimated in the same way). Using the fact that the processes $\{\eta_t\}$ and $\{\tilde{w}(t)\}$ are independent we obtain

$$\mathbb{E}\left[\int_{n_t}^{n_{n_t}} \beta(x_n(\rho^n_{n_t})) \tilde{w}(u) \right]^2 = \mathbb{E}\left[\int_{n_t}^{n_{n_t}} \beta(x_n(\rho^n_{n_t})) \tilde{w}(u) \right]^2 \bigg/ \mathcal{F}^w \infty$$

$$= \mathbb{E}\left[\mathbb{E}\left[\int_{n_t}^{n_{n_t}} \beta(x_n(\rho^n_{n_t})) \tilde{w}(u) ds \right]^2 \bigg/ \mathcal{F}^w \infty \right] \leq K^2 \mathbb{E}|\eta_t - \eta_t^n| \to 0, n \to +\infty.$$

Calculating in the same way the second moment we get

$$\mathbb{E}\left[\int_0^{n_{n_t}} (\beta(x_n(\rho^n_{n_t})) - \beta(x^n(\rho^n_{n_t})) \tilde{w}(u) \right]^2 = \mathbb{E}\int_0^{n_{n_t}} \left\|\beta(x_n(\rho^n_{n_t})) - \beta(x^n(\rho^n_{n_t}))\right\|^2 du \to 0$$

as $n \to +\infty$. We take into account the following arguments: $\beta(x_n(\rho^n_{n_t})) \to \beta(x^n(\rho^n_{n_t}))$, $n \to +\infty$ (from Lemma 4) and $\left\|\beta(x_n(\rho^n_{n_t})) - \beta(x^n(\rho^n_{n_t}))\right\| \leq 2K$.

For estimating the last summand in (2.11) and the first summand in (2.10) we need the following result.

**Lemma 5.** For all $t \geq 0$ : $\rho^n_{n_t} \xrightarrow{p} \rho_t$, when $n \to +\infty$.

**Proof.** Similarly to Lemma 2 we get for $\varepsilon > 0$:

$$\mathbb{P}\left\{\rho_t - \rho^n_{n_t} > \varepsilon\right\} = \sum_{k \geq 0} \mathbb{P}\left\{\rho_t > \varepsilon + \rho^n_{n_t}, \rho_t \in \left[\frac{(k + 1)\varepsilon}{2}, \frac{(k + 2)\varepsilon}{2}\right]\right\} \leq$$
\begin{equation}
\sum_{k\geq 0} \mathbb{P} \left\{ \lambda_k^n > \frac{k\varepsilon}{2}, \lambda_k \leq \frac{(k+1)\varepsilon}{2} \right\} = \sum_{k\geq 0} \mathbb{P} \left\{ \lambda_k^n \leq \lambda_{k+1}^n, \lambda_k > \lambda_{k+1}^n \right\}.
\end{equation}

Consider the \( k \)-th summand in (2.12). For some \( \delta > 0 \) we have
\[ \mathbb{P} \left\{ \lambda_k^n \leq \lambda_{k+1}^n, \lambda_k - \delta \geq \lambda_{k+1}^n, \lambda_k - \delta \leq \lambda_{k+1}^n \right\} + \]
\[ + \mathbb{P} \left\{ \lambda_k - \delta < \lambda_{k+1}^n < \lambda_k \right\} \leq \mathbb{P} \left\{ \lambda_k^n \leq \lambda_{k+1}^n, \lambda_k - \delta \geq \lambda_{k+1}^n, \lambda_k - \delta \leq \lambda_{k+1}^n \right\} + \]
\[ + \mathbb{P} \left\{ |\lambda_k^n - \lambda_k| \geq \frac{\delta}{2} \right\} + \mathbb{P} \left\{ \lambda_k - \delta < \lambda_{k+1}^n < \lambda_k \right\} \leq \mathbb{P} \left\{ \lambda_{k+1}^n - \lambda_{k+1}^n > \frac{\delta}{2} \right\} + \]
\[ + \mathbb{P} \left\{ \lambda_k^n - \lambda_k \geq \frac{\delta}{2} \right\} + \mathbb{P} \left\{ \lambda_k - \delta < \lambda_{k+1}^n < \lambda_k \right\}. \tag{2.13} \]

In the last inequality we use that
\[ \left\{ \lambda_k^n \leq \lambda_{k+1}^n, \lambda_k - \delta \geq \lambda_{k+1}^n, \lambda_k - \delta \leq \lambda_{k+1}^n \right\} \subseteq \left\{ \lambda_{k+1}^n - \lambda_{k+1}^n > \frac{\delta}{2} \right\}. \]

Proceeding \( n \to +\infty \) and using Corollary 2 (when \( q \in (-1,0) \cup (0,1) \)) or Remark 1 (when \( |q| = 1 \)) or Remark 2 (when \( q = 0 \)) we see that the first and second summands are equal to 0. Consider the third summand. Note that \{\( \eta_k \)\} is additive functional of the Markov process \{\( x^n(t) \)\} and for all \( a \geq 0 \): \( \mathbb{P}_x \{ \eta_t < a \} \leq \frac{a\sqrt{2}}{\sqrt{\pi t}} \) (\( \mathbb{P}_x \) is standard notation for \( \mathbb{P}_\{-/x_\nu(0) = x\} \)). Also \( \mathbb{P} \left\{ \lambda_k - \delta < \lambda_{k+1}^n < \lambda_k \right\} = 0 \) when \( t \geq \frac{(k+1)\varepsilon}{2} \). Therefore for all \( t < \frac{(k+1)\varepsilon}{2} \) we get
\[ \mathbb{P} \left\{ \lambda_k - \delta < \lambda_{k+1}^n < \lambda_k \right\} = \mathbb{E}\mathbb{P}_x \left\{ \lambda_{k+1}^n < \frac{(k+1)\varepsilon}{2} \right\} \left\{ 0 < \lambda_{k+1}^n < \frac{(k+1)\varepsilon}{2} \right\} \leq \frac{\sqrt{2}}{\sqrt{\pi (t - \frac{(k+1)\varepsilon}{2})}} \]

when \( \delta \to 0 \). In the same way one can prove that \( \mathbb{P} \left\{ \rho_k^n - \rho_{k+1} > \varepsilon \right\} \to 0, n \to +\infty \). The lemma is proved.

Due to Lemma 5 for the last summand in (2.11) and the first summand in (2.10), for some \( N > 0 \) we have
\[ \mathbb{P} \left\{ \|w(\rho_k^n) - w(\rho_{k+1})\| > \varepsilon \right\} \leq \mathbb{P} \left\{ \|w\|_{\text{Hol}_{1/4}(0,t)} \|\rho_k^n - \rho_{k+1}\|^{1/4} > \varepsilon, \|w\|_{\text{Hol}_{1/4}(0,t)} < N \right\} + \]
\[ + \mathbb{P} \left\{ \|w\|_{\text{Hol}_{1/4}(0,t)} \geq N \right\} \leq \mathbb{P} \left\{ \|\rho_k^n - \rho_{k+1}\|^{1/4} > \frac{\varepsilon}{N} \right\} + \mathbb{P} \left\{ \|w\|_{\text{Hol}_{1/4}(0,t)} \geq N \right\}. \tag{2.14} \]

Let \( n \to +\infty \), then the first summand in (2.14) tends to 0. Then let \( N \to +\infty \). We obtain that the last summand in (2.12) and the first summand in (2.10) tend to 0. This completes the proof of Theorem 2.

**Corollary 3.** It follows from Theorem 2 that solution of (2.1), considered as a random function on \( \mathbb{R}^+ \times \mathbb{R}^d \), has a measurable modification.
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