RANK 1 PERTURBATIONS IN RANDOM MATRIX THEORY — A REVIEW OF EXACT RESULTS

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Abstract. A number of random matrix ensembles permitting exact determination of their eigenvalue and eigenvector statistics maintain this property under a rank 1 perturbation. Considered in this review are the additive rank 1 perturbation of the Hermitian Gaussian ensembles, the multiplicative rank 1 perturbation of the Wishart ensembles, and rank 1 perturbations of Hermitian and unitary matrices giving rise to a two-dimensional support for the eigenvalues. The focus throughout is on exact formulas, which are typically the result of various integrable structures. The simplest is that of a determinantal point process, with others relating to partial differential equations implied by a formulation in terms of certain random tridiagonal matrices. Attention is also given to eigenvector overlaps in the setting of a rank 1 perturbation.

1. Introduction

In the mid 90’s I was possession of early edition of Wolfram’s The Mathematic Book. The introductory gallery section contained the command, up to the accuracy of my memory

\begin{equation}
\text{ComplexListPlot[Eigenvalues[RandomVariate[UniformDistribution[],100,100]],}
\text{PlotRange-> All]}
\end{equation}

and the accompanying output reproduced below in Figure 1.1. Thus a random $100 \times 100$ matrix, with each entry identically and independently formed drawn from the uniform distribution on $[0,1]$, was formed, the eigenvalues were calculated, and these were plotted in the complex plane.

Now a random variable $u[0,1]$, uniformly chosen from the interval $[0,1]$ can be decomposed $\frac{1}{2} + u[-\frac{1}{2}, \frac{1}{2}]$, where $u[-\frac{1}{2}, \frac{1}{2}]$ is a uniform random variable with support $[-\frac{1}{2}, \frac{1}{2}]$. Hence the random matrix being formed in (1.1) can be written

\begin{equation}
X + \frac{1}{2}I_{N \times N} = X + \frac{N}{2} \hat{1}_N \hat{1}_N^T.
\end{equation}

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Here $I_{N \times N}$ denotes the $N \times N$ matrix with all entries 1, and $\hat{1}$ is the $N \times 1$ unit column vector with all entries equal to $1/\sqrt{N}$. Also, $X$ is the random matrix with all entries identically and independently distributed as uniform random variables on $[-1/2, 1/2]$, and thus having mean zero and standard deviation $1/\sqrt{12}$.

A celebrated result in random matrix theory, known as the circular law (see [18] for a review), tells us that an $N \times N$ random matrix with all entries identically and independently distributed with mean zero and standard deviation $1/\sqrt{N}$ has, for large $N$, its eigenvalues uniformly supported on the unit disk in the complex plane. Moreover, the probability that an eigenvalue is of order unity outside of this disk fall off exponentially fast in $N$. This suggests we write the second expression in (1.2) as

$$\sqrt{\frac{N}{12}} Y, \quad Y := \left( \tilde{X} + \sqrt{3N} \hat{1}_N \hat{1}_N^T \right), \tag{1.3}$$

where in the definition of $Y$ the $N \times N$ random matrix $\tilde{X}$ obeys the conditions required for the applicability of the circular law. The spectrum of $Y$ is obtained from the random matrix (1.2) by the simple scaling of dividing by $\sqrt{\frac{N}{12}}$.

In the setting of Figure 1.1, $\sqrt{\frac{N}{12}} \approx 2.9$. Dividing the scale of both axes by this value it is observed that all but one of the eigenvalues form a disk of radius 1, which to the naked eye has a uniform density. However, there is also an outlier eigenvalue, appearing on the real axis with value close to $\sqrt{3N} \big|_{N=100} \approx 17.3$. The latter is precisely the scalar of the perturbation of $\tilde{X}$ in (1.3) by the addition of a scalar times the rank 1 matrix formed by the unit vector $\hat{1}_N$. These are general features. Thus for $\alpha > 1$, $\psi$ a unit column vector and $\tilde{X}$ obeying the conditions required of the circular law, we have that for large $N$ the random matrix

$$\tilde{X} + \alpha \psi \psi^T \tag{1.4}$$

conforms to the circular law, with a single outlier eigenvalue on the real axis at $x = \alpha$, as established by Tao [92]. For applications of the matrix structure (1.4) — specifically the averaged absolute value of the corresponding determinant — to the stability and resilience of large complex systems, see [10] [33].
An even earlier numerical experiment relating to outlier eigenvalues in random matrix spectra was carried out by Porter in the earlier 1960's, as cited in [71]. The random matrices \{H\} say in this experiment were real symmetric, with Gaussian entries, distribution N[\mu, 1] on the diagonal and distribution N[\mu, 1/\sqrt{2}] off the diagonal. The joint distribution of all the independent entries gives for that the probability density of the matrices H are proportional to \( e^{-\text{Tr}(H-\mu I_N)^2/2} \). We can write
\[
H = G + \frac{\mu N}{2} \mathbf{1}_N \mathbf{1}_N^T,
\]
where G has a probability density proportional to \( e^{-\text{Tr}G^2/2} \). The latter is invariant under the mapping \( G \mapsto RGR^T \) for \( R \) real orthogonal, which gives rise to the name of the random matrices G as the Gaussian orthogonal ensemble (GOE); the random matrices H are examples of particular shifted GOE matrices. Let us scale H by multiplying by a factor of \( 1/\sqrt{2N} \), and let us replace \( \mu N/2\sqrt{2N} \) by \( \alpha \). This replaces (1.5) by
\[
\tilde{G} + \alpha \mathbf{1}_N \mathbf{1}_N^T,
\]
where \( \tilde{G} \) is \( 1/\sqrt{2N} \) times a GOE matrix. A classical result in random matrix theory (see [78]) tells us that the eigenvalues of the latter are to leading order supported on the interval \([-1, 1]\), with normalised density
\[
\rho^W(x) = \frac{2}{\sqrt{\pi}}(1-x^2)^{1/2},
\]
known as the Wigner semi-circle. As in the case of the circular law, for the scaled GOE matrices \( \tilde{G} \) the probability that an eigenvalue is of order unity outside of this interval falls off exponentially fast in \( N \) [13, 40]. The effect observed in Porter’s simulations — Figure 1.2 gives an example produced using modern software — is that for \( N \) large and \( \alpha > 1/2 \) there is a single outlier eigenvalue located at the value
\[
\alpha + 1/(4\alpha),
\]
with the Wigner semi-circle otherwise remaining unchanged; see also Section 2.1. To leading order in \( \alpha \) for \( \alpha \) large this was first explained theoretically in Lang [71], although as reviewed in Section 2.1 this exact value is now well understood theoretically.

The above examples have involved additive rank 1 perturbations of a random matrix. Closely related is a sequence of additive rank 1 perturbations — to be referred to as (additive) rank 1 updates — which in fact can be used to define a discrete random evolution. Consider for example the sequence of random matrices \{W_n\}_{n=1,2,\ldots} defined by
\[
W_n = \sum_{j=1}^n \mathbf{v}_j \mathbf{v}_j^T,
\]
where each $v_j$ is an independent $N \times 1$ standard Gaussian column vector, and calculate their eigenvalues. This is easy to simulate. An example is given in Figure 1.3. Here there is no weighting of the rank 1 matrices and thus no eigenvalue separation phenomenon, but visible is another feature of an Hermitian rank 1 perturbation of an Hermitian matrix, namely that of an interlacing of eigenvalues.

The purpose of this survey is to give an account of exact formulas, typically driven by underlying integrability, associated with rank 1 perturbations in random matrix theory. The topic of §2 is the additive rank 1 structure (1.6). Considered in this section are the derivation of the formula (1.8) for the location of the outlier, and generalisations; an explicit formula for the eigenvalue PDF from the viewpoints of an underlying tridiagonal matrix, and from a matrix integral related to Dyson Brownian motion; a characterisation of the distribution of the largest eigenvalue in the critical regime in terms of a partial differential equation; a $\beta$ generalisation of the latter and its solution for $\beta = 2,4$ in terms of Painlevé transcendent; and overlap properties of the eigenvector corresponding to the largest eigenvalue. The topic of §3 is a multiplicative rank 1 perturbation of a complex Wishart matrix. It is shown how such a multiplicative perturbation can be recast as an additive rank 1 perturbation, allowing the theory of subsection 2.1 to be used to determine the criteria and location of an outlier. Two derivations of the explicit formula for the eigenvalue PDF are given, one involving the HCIZ matrix integral, and the other computing first the joint distribution of the eigenvalues of the matrix involved in the equivalent additive rank 1 perturbation, and the perturbed matrix. The eigenvalues of the perturbed complex Wishart matrix under consideration form a determinantal point process, and the explicit form of the correlation kernel for the soft edge critical regime is revised in subsection 3.4. The final subsection relates to the hard edge critical regime for general $\beta > 0$. Rank 1 perturbations of Hermitian and unitary matrices giving rise to a two-dimensional support for the eigenvalues is the topic of §4. First considered in this section is an additive anti-symmetric perturbation for the GUE. For a scaling close to the origin of the real axis, this gives rise to a determinantal point process for the eigenvalues, with a simple functional form for the kernel. Next a multiplicative sub-unitary rank 1 perturbation of Haar distributed unitary matrices is considered. In a
scaling near the unit circle in the complex plane, the eigenvalue point process is identical to that of the previous subsection. However in the bulk of interior of the unit circle the eigenvalue point process is no longer determinantal, and in fact relates to the zeros of a certain class of random Laurent series, related to the limiting Kac polynomial. In the final subsection overlaps between the left and right eigenvectors of the setting of subsection 4.1 are considered.

2. An additive rank 1 perturbation for the GOE

2.1. Location of the separated eigenvalue for $\alpha > 1/2$. The formula (1.8) for the separated eigenvalue was given by Jones, Kosterlitz and Thouless in 1978 [64], and independently in more general setting by Furedi and Komlos in 1981 [49]. We follow the derivation of these works.

Proposition 2.1. Consider the particular shifted scaled GOE matrix (1.6), suppose $\alpha > 0$ and take the limit $N \to \infty$. For $\alpha \leq 1/2$ all eigenvalues are supported on $[-1, 1]$ and have density given by the Wigner semi-circle (1.7). For $\alpha > 1/2$ all but one of the eigenvalues are supported on $[-1, 1]$ and have density given by the Wigner semi-circle, with the separated eigenvalue located at the value (1.8).

Proof. The characteristic equation determining the eigenvalues of (1.6) is

$$0 = \det(\mu I_{N \times N} - \tilde{G} - \alpha \hat{1}_N \hat{1}_N^T) = \det \left( \mu I_{N \times N} - \tilde{G} - \alpha \hat{e}_N^{(1)} (\hat{e}_N^{(1)})^T \right),$$

Figure 1.3. Distinct eigenvalues of a sequence of random matrices (1.9) with $n = 1, 2, \ldots, 15$ reading from bottom to top and $N = 10$. 

where $\hat{e}_N^{(1)}$ denotes the unit column vector in $\mathbb{R}^N$, $(1,0,\ldots,0)$. Here the second equality follows from the fact, noted below (1.5), that the distribution of $\hat{G}$ is invariant under conjugation by a real orthogonal matrix, allowing the rank 1 matrix $\hat{1}_N\hat{1}_N^T$ to be replaced by its diagonal matrix of eigenvalues $\text{diag}(1,0,\ldots,0)$, then writing the latter as $\hat{e}_N^{(1)}\hat{e}_N^{(1)T}$.

Next introduce the spectral decomposition $(\lambda I_{N\times N} - \hat{G})^{-1} = U(\lambda I_{N\times N} - \Lambda)^{-1}U^T$, where $\Lambda = \text{diag}(\mu_1,\ldots,\mu_N)$ is the diagonal matrix of eigenvalues of $\hat{G}$, and $U = [u_k]_{k=1}^N$ is the corresponding real orthogonal matrix of eigenvectors. With this substituted in (2.1), the decisive step in the argument is to apply the general determinant identity (see e.g. [77, Corollary 2.1])

$$
\text{det}(I_{N\times N} - A_{N\times MB_{M\times N}}) = \text{det}(I_{M\times M} - B_{M\times N}A_{N\times M})
$$

(2.2) to the second determinant in (2.1), thereby reducing it to the scalar

$$
1 - \alpha \sum_{j=1}^N \frac{(u_1^{(j)})^2}{\lambda - \mu_j},
$$

(2.3) where $u_1^{(j)}$ denotes the $j$-th component of $u_1$. Since the first factor in (2.1) has zeros at the poles of (2.3), the eigenvalues as determined by (1.6) are seen to be given by the zeros of (2.3) as a function of $\mu$.

For $\{\mu_j\}_{j=1}^N$ distinct and ordered

$$
\mu_N < \mu_{N-1} < \cdots < \mu_2 < \mu_1,
$$

(2.4) a sketch of the graph of (2.3) under the assumption $\alpha > 0$ shows that its zeros, $\{\lambda_j\}_{j=1}^N$ say, ordered from biggest to smallest are also distinct, and interlace with (2.4) according to

$$
\mu_N < \lambda_N < \mu_{N-1} < \lambda_{N-1} < \cdots < \mu_2 < \lambda_2 < \mu_1 < \lambda_1.
$$

(2.5) For the scaled GOE matrix $\hat{G}$ we know the eigenvalues $\{\mu_j\}_{j=1}^N$ for large $N$ concentrate on $[-1,1]$, having density (1.7), and there are no outliers. It remains then to determine the location of the largest zero of (2.3) in this setting.

Thus we average (2.3) over the eigenvalues of the scaled GOE, and the components of the first eigenvector. We know that the eigenvectors are independent of the eigenvalues, and are distributed uniformly on the unit sphere in $\mathbb{R}^N$ (see [38, Eq. (1.11) in relation to the former point and Exercises 1.2 q.2 in relation to the latter]), telling us that each $(u_1^{(j)})^2$ can be replaced by $1/N$. The sum in the remaining quantity

$$
1 - \frac{\alpha}{N} \sum_{j=1}^N \frac{1}{\lambda - \mu_j},
$$

(2.6)
is a linear statistic, which when averaged over the eigenvalues can be written in terms of the eigenvalue density according to $\int_{\mathbb{R}} \rho(\lambda, N(\lambda)) / (\mu - \lambda) \, d\lambda$. For large $N$ we have that the normalised density $(1/N)\rho(\lambda, N(\lambda))$ tends to the Wigner semi-circle (1.7), implying that condition for a zero of (2.3) reduces to

$$0 = 1 - \frac{2\alpha}{\sqrt{\pi}} \int_{-1}^{1} \frac{(1 - \lambda^2)^{1/2}}{\mu - \lambda} \, d\lambda.$$  

We have the integral evaluation (see e.g. [38] Exercises 1.6 q.2(ii))

$$\frac{2}{\sqrt{\pi}} \int_{-1}^{1} \frac{(1 - \lambda^2)^{1/2}}{\mu - \lambda} \, d\lambda = \begin{cases} 
2\mu, & |\mu| \leq 1 \\
2\mu(1 - (1 - 1/\mu^2)^{1/2}), & |\mu| > 1.
\end{cases}$$  

We see by substituting (2.8) in (2.7) that the latter admits a solution with $\mu > 1$ only if $\alpha > 1/2$, in which case it is given by (1.8).

**Remark 2.2.** 1. Consider the additive rank 1 perturbation (1.6), with $\tilde{G}$ belonging to a general ensemble of matrices with limiting normalised density $\rho(x)$, supported on $[a, b]$; references addressing this setting include [21, 14, 6, 85]. Define the Stieltjes transform of the latter

$$G^{(\tilde{G})}(y) = \int_{a}^{b} \frac{\rho(x)}{y - x} \, dx.$$  

According to the above proof, eigenvalue separation occurs whenever the equation

$$\frac{1}{\alpha} = G^{(\tilde{G})}(y)$$  

admits a solution with $y = y^* > b$. This will always be the case for $\alpha$ large enough since for $y$ large, it follows from the definition (2.9) and the normalisation of $\tilde{G}(x)$ that $G^{(\tilde{G})}(y) \sim 1/y$, telling us that $y^* \sim \alpha$. This latter conclusion is in keeping with the early finding of Lang [71].

2. Generalising (1.6) to include several additive multiple rank 1 perturbations of strengths $\alpha_i$, involving linearly independent unit vectors, the eigenvalue separation equation (2.10) applies to each such perturbation separately, with the other perturbations ignored; see the cited references from point 1 above.

3. Suppose the real symmetric matrix $\tilde{G}$ in (1.6) is replaced by $\tilde{G}_1 + t\tilde{G}_2$, where $\tilde{G}_1$ is real symmetric, $t \geq 0$ is a real parameter, and $\tilde{G}_2$ is real anti-symmetric. If the independent entries of each $\tilde{G}_i$ are identically and independently distributed with mean zero and unit variance, after scaling by dividing by $\sqrt{N}$ the eigenvalue density satisfies the elliptical law. The location of outliers due to low rank perturbations in this setting, which interpolates between Hermitian matrices satisfying the Wigner semi-circle law, and non-Hermitian
matrices satisfying the circular law (recall the third paragraph of the Introduction) have been studied in [75].

2.2. Joint eigenvalue probability density function. According to (2.1) the eigenvalues of the shifted scaled GOE matrix (1.6) are the same as for \( \tilde{G} \), but with \( \alpha \sqrt{2N} \) added to the entry in the top right corner. Thus, with \( \text{eig} A \) denoting the eigenvalues of \( A \), we have

\[
\text{eig} (\tilde{G} - \alpha \mathbf{1}_N \mathbf{1}_N^T) = \frac{1}{\sqrt{2N}} \text{eig} \left( G - \alpha \sqrt{2N} \text{diag} (1,0,\ldots,0) \right),
\]

where \( G \) is a GOE matrix (no scaling). It was observed by Trotter [94] that a sequence of Householder reflector transforms, \( R \) say, can be applied symmetrically to \( G \) to reduce it to the tridiagonal form

\[
T = RGR^T = A_0 + A_1 + \tilde{A}_1^T,
\]

where

\[
A_0 = \text{diag} \left( N \left[ 0, 1 \right], \ldots, N \left[ 0, 1 \right] \right), \quad A_1 = \text{diag}^+ \left( \tilde{\chi}_{N-1}, \tilde{\chi}_{N-2}, \ldots, \tilde{\chi}_1 \right),
\]

with \( \tilde{\chi}_k \) denoting the square root of the gamma distribution \( \Gamma \left[ k/2, 1 \right] \). In \( A_1 \) the notation \( \text{diag}^+ \) refers to a matrix with all entries equal to zero, except those on the diagonal immediately above the main diagonal, which take the values as listed. As is evident from the form of \( A_0 \), this transformation leaves the diagonal entries unchanged, and so [17]

\[
\text{eig} (\tilde{G} - \alpha \mathbf{1}_N \mathbf{1}_N^T) = \frac{1}{\sqrt{2N}} \text{eig} \left( T - \alpha \sqrt{2N} \text{diag} (1,0,\ldots,0) \right).
\]

Following working introduced in the context of determining the eigenvalue PDF for a multiplicative rank perturbation of Wishart matrices [41] (see also Section 3.5 as well as [74, 97]), the random tridiagonal matrix in (2.12) can be used to determine the joint eigenvalue probability density function (PDF) for the shifted scaled GOE matrix (1.6).

**Proposition 2.3.** Up to normalisation, the eigenvalue PDF of the random matrix (1.6), eigenvalues ordered \( \lambda_1 > \cdots > \lambda_N \) is proportional to

\[
\prod_{j=1}^N e^{-N\lambda_j^2} \prod_{1 \leq j < k \leq N} (\lambda_j - \lambda_k) \int_{-\infty}^{\infty-ic} e^{it} \prod_{j=1}^N \left( it - 2\alpha N \lambda_j \right)^{-1/2} dt,
\]

where \( c > 2\alpha N \lambda_1 \).

**Proof.** The matrix in brackets on the RHS of (2.12) say, \( \tilde{T} \) say, differs from \( T \) in the distribution of the top left entry. Thus for \( \tilde{T} \) this entry has distribution \( N \left[ \alpha \sqrt{2N}, 1 \right] \) instead of \( N \left[ 0, 1 \right] \) for \( T \).
Now denote the entries of $\tilde{T}$ by writing $\tilde{T} = \tilde{A}_0 + \tilde{A}_1 + \tilde{A}_1^T$, where

$$
\tilde{A}_0 = \text{diag} \left( a_N, a_{N-1}, \ldots, a_1 \right), \quad \tilde{A}_1 = \text{diag}^+ \left( b_{N-1}, b_{N-2}, \ldots, b_1 \right).
$$

It follows that up to normalisation the probability measure $P(\tilde{T})(d\tilde{T})$ associated with $\tilde{T}$ can be factored in terms of the probability measure $P(T)(dT)$ associated with $T$ to be given by

$$
P(T)(dT)e^{\alpha\sqrt{2N}a_N}.
$$

Next denote the eigenvalues of the tridiagonal matrix $\tilde{T}$ by $\{\lambda_j\}_{j=1}^N$, and denote the first component of the corresponding normalised eigenvector, which is required to be positive, by $\{q_j\}_{j=1}^N$. We know from the working in [29], or from [38] §1.9.2, that in terms of these variables and up to normalisation the probability measure $P(T)(dT)$ is proportional to

$$
\prod_{j=1}^N e^{-\lambda_j^2/2} \prod_{1 \leq j < k \leq N} (\lambda_j - \lambda_k) \delta \left( \sum_{j=1}^N q_j^2 - 1 \right) (d\tilde{\lambda})(d\tilde{q}),
$$

while

$$
e^{\alpha\sqrt{2N}a_N} = e^{\alpha\sqrt{2N}\sum_{j=1}^N q_j^2 \lambda_j}.
$$

Writing the Dirac delta function in (2.14) as a Fourier transform and supposing temporarily that each $\lambda_j < 0$ allows the integral over $d\tilde{\lambda}$ in the product of (2.15) and (2.14) to be computed explicitly, showing

$$
\int (\mathbb{R}^+)^N \delta \left( \sum_{j=1}^N q_j^2 - 1 \right) e^{\alpha\sqrt{2N} \sum_{j=1}^N q_j^2 \lambda_j} (d\tilde{q}) \propto \int_{-\infty}^{\infty} e^{it} \prod_{j=1}^N \left( it - \alpha \sqrt{2N} \lambda_j \right)^{-1/2} dt.
$$

Deforming the contour to again be parallel to the real axis but to pass through the imaginary axis at a point $-ic$ with $c > \alpha\sqrt{2N}a_N$ allows the assumption $\lambda_j < 0$ to be removed. Multiplying this modified integral by the eigenvalue dependent factors in (2.14) and scaling $\lambda_j \mapsto \sqrt{2N} \lambda_j$ to account for the scaling in (2.12) gives (2.13). 

2.3. A matrix integral over the orthogonal group. There is a matrix integral over the orthogonal group associated with (2.13). This is based on the fact that for a real symmetric matrix $\tilde{H}$ the product of differentials of independent elements $(d\tilde{H})$ decomposes in terms of the eigenvalues $\{\lambda_j\}$ and matrix of eigenvectors $R$ of $\tilde{H}$ according to [38] Eq. (1.11)]

$$
(d\tilde{H}) = \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j) d\tilde{\lambda} (R^T dR),
$$

where $(R^T dR)$ corresponds to the Haar measure on the orthogonal group. In the case that the distribution on the matrices $\tilde{H}$ is proportional to $e^{-N \text{Tr}(\tilde{H} - a \mathbb{1}N \mathbb{1}^T)^2}$, it follows that the
eigenvalue PDF is proportional to

\[
\prod_{1 \leq j < k \leq N} (\lambda_j - \lambda_k) \int_{\mathbb{R} \in O(N)} e^{-N\text{Tr}(\Lambda R^T - \alpha \mathbf{1}_N R^T)} (R^T dR)
\]

\[
\propto \prod_{j=1}^{N} e^{-N\lambda_j^2} \prod_{1 \leq j < k \leq N} (\lambda_j - \lambda_k) \int_{\mathbb{R} \in O(N)} e^{2N\text{Tr}(\Lambda R^T \mathbf{1}_N R^T - \alpha \mathbf{1}_N R^T)} (R^T dR),
\]

where \( \Lambda \) is the diagonal matrix of eigenvalues. Comparing (2.15) with (2.13) implies an evaluation of the matrix integral in the second line of the former \[68, 41, 68, 73\].

**Proposition 2.4.** We have

\[
\int_{\mathbb{R} \in O(N)} e^{2N\text{Tr}(\Lambda R^T \mathbf{1}_N R^T)} (R^T dR) \propto \int_{(\mathbb{R}^+)^N} e^{\sum_{j=1}^{N} q_j^2 - 1} e^{2N\sum_{j=1}^{N} q_j^2 \lambda_j} (dq)
\]

\[
\propto \int_{-\infty - i\epsilon}^{\infty - i\epsilon} e^{it \sum_{j=1}^{N} (it - 2\alpha N\lambda_j)^{1/2} dt,
\]

where \( c > 2\alpha N\lambda_1 \).

**Proof.** We have already explained how the first matrix integral has an evaluation, up to proportionality, given by the final of these integrals. In relation to the second integral, we have used the fact that

\[
\text{Tr}(\Lambda R^T \mathbf{1}_N R^T) = \mathbf{q}^T \Lambda \mathbf{q} = \sum_{j=1}^{N} q_j^2 \lambda_j.
\]

Here the first of these equalities follows from the cyclic property of the trace and the fact that for \( R \) a real orthogonal matrix chosen with Haar measure, \( \mathbf{1}_N R = \mathbf{q}^T \), where \( \mathbf{q} \) is uniformly distributed on the unit sphere; see e.g. \[25\]. \( \square \)

**Remark 2.5.** The second integral in (2.18) has the interpretation as the partition function of a spherical spin glass model \[68\]. In the circumstance that \( \{ \lambda_j \} \) result the eigenvalues of a GOE matrix, it is shown in \[68, 68, 73\] that the saddle point equation of the integrand in the evaluation given by the third integral in (2.18) relates to (2.7). This in turn implies that for large \( N \) the model undergoes a phase transition as a function of \( \alpha \).

Let us rescale and generalise the (un-normalised) matrix distribution \( e^{-N\text{Tr}(\hat{H} - \alpha \mathbf{1}_N \hat{1}_N^T)^2} \), by considering

\[
\frac{1}{t^{N(N+1)/2}} e^{-\frac{\text{Tr}(\hat{H} - \hat{H}_0)^2}{2t}}.
\]

(2.19)
Here the $t$-dependence of the normalisation has been made explicit for later purposes. Changing variables to the eigenvalues and eigenvectors of $\tilde{H}$, analogous to (2.17), one has that the eigenvalue PDF of $\tilde{H}$, $p_t(\lambda_1, \ldots, \lambda_N)$, is proportional to
\begin{equation}
\prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j| \frac{1}{tN(N+1)/2} \int_{R \in O(N)} e^{-\text{Tr}(R\Lambda R^T - \Lambda_0)^2/2t} (R^T dR).
\end{equation}

Implicit in the work of Dyson [31], on what is now known as Dyson Brownian motion, is that $p_t$ satisfies a particular Fokker-Planck equation.

**Proposition 2.6.** We have that $p_t$ satisfies the Fokker-Planck equation
\begin{equation}
2 \frac{\partial p_t}{\partial t} = \mathcal{L} p_t,
\end{equation}
where
\begin{equation}
\mathcal{L} = \sum_{j=1}^N \frac{\partial}{\partial \lambda_j} \left( \frac{\partial W}{\partial \lambda_j} + \frac{\partial}{\partial \lambda_j} \right), \quad W = -\sum_{1 \leq j < k \leq N} \log |\lambda_k - \lambda_j|
\end{equation}
and subject to the initial condition
\begin{equation}
ap_{t}(\lambda_1, \ldots, \lambda_N)|_{t=0} = \prod_{j=1}^N \delta(\lambda_j - \lambda_j^{(0)}).
\end{equation}

**Proof.** (Sketch) With respect to the independent entries of $\tilde{H}$, the matrix distribution (4.21) factorises to be proportional to
\begin{equation}
\prod_{j=1}^N e^{-(\tilde{H}_{jj} - (H_{jj}^{(0)})^2)/2t} \prod_{j<k} \sqrt{t} e^{-(\tilde{H}_{jk} - (H_{jk}^{(0)})^2)/2t},
\end{equation}
where the $t$-dependence of the normalisation has been made explicit. Denoting this by $P_t$, the fact that the functional form corresponding to each independent element satisfies a one-dimensional heat equation implies that $P_t$ satisfies the multidimensional heat equation
\begin{equation}
2 \frac{\partial P_t}{\partial \tau} = \sum_{\mu} D_{\mu} \frac{\partial^2 P_t}{\partial H_{\mu}^2}.
\end{equation}
Here the label $\mu$ ranges over the label of the independent diagonal and upper triangular entries, while $D_{\mu} = 1$ for the diagonal entries and $D_{\mu} = 1/2$ for the off diagonal entries.

An essential idea from here, see e.g. [38] §11.1 for details, is to observe that as a function of $\tilde{H} = R\Lambda R^T$, the PDF (2.20) must also satisfy (2.25), provided we change variables in the latter. The change of variables can be carried out using theory relating to the Laplacian associated with metric forms — specifically the RHS of (2.25) can be identified with the Laplacian operator on the space of real symmetric matrices — and (2.21) results. □
Remark 2.7. As emphasised in [31], the Fokker-Planck equation specified by (2.21) and (2.22) corresponds to a repelling $N$-particle system with a potential energy $W$, executing overdamped Brownian motion in a fictitious viscous fluid with friction coefficient $\gamma = 2$ at inverse temperature $\beta = 1$.

In the case of $\tilde{H}_0 = \alpha \hat{1}_N \hat{1}_N^T$, the initial condition (2.23) has

\begin{equation}
\lambda^{(0)}_1 = \alpha, \quad \lambda^{(0)}_j = 0 \ (j = 2, \ldots, N).
\end{equation}

With this initial condition the matrix distribution relating to (2.17), for which the density obeys the description of Proposition 2.1 when $N$ is large, results when $t = 1/(2N)$. The trajectories of the eigenvalues are easy to simulate by choosing a value of $\delta t = 1/(2NM)$ for some $M \gg N$, forming a sequence of random real symmetric matrices $\{\tilde{H}^{(j)}\}_{j=0,\ldots,M}$ by sampling the entries according to (2.24) with $H^{(0)} = \tilde{H}^{(0)}$ and $t \mapsto \delta t$, calculating their eigenvalues and forming paths. An example is given in Figure 2.1.

2.4. The soft edge critical regime. In §2.2 a random tridiagonal matrix was specified which has the same eigenvalue probability density function as the rank 1 perturbed scaled GOE matrix (1.6). One application, due to Bloemendal and Virág [16], of this reduction has been to characterise the distribution of the largest eigenvalue $\lambda_1$ in a so-called critical regime. The latter is specified by the large $N$ scalings

\begin{equation}
2N^{2/3}(\lambda_1 - 1) \to x_1, \quad N^{1/3}(1 - 2\alpha) \to w,
\end{equation}

Figure 2.1. Sample trajectories corresponding to the Dyson Brownian motion underlying (2.17) with $N = 30$ and $\alpha = 1.2$. 
which have the feature that the distribution function then tends to a non-trivial limit dependent on $w$. To gain insight, a parameter $\beta > 0$ referred to as the Dyson index can be introduced so that under consideration is the random tridiagonal matrix

\begin{equation}
T_\beta := \frac{1}{\sqrt{2\beta N}}(C_0 + C_1 + C_1^T),
\end{equation}

where

\[ C_0 = \text{diag} \left( N[\sqrt{2\beta N}, 1], N[0, 1], \ldots, N[0, 1] \right), \quad C_1 = \text{diag}^+ \left( \tilde{\lambda}_{\beta(N-1)}, \tilde{\lambda}_{\beta(N-2)}, \ldots, \tilde{\lambda}_{\beta} \right). \]

We can verify the non-random limits

\begin{equation}
\lim_{\beta \to \infty} \frac{1}{\sqrt{\beta}} C_0 = \text{diag} \left( \sqrt{2N}, 0, \ldots, 0 \right), \quad \lim_{\beta \to \infty} \frac{1}{\sqrt{\beta}} C_1 = \text{diag}^+ \left( \sqrt{\frac{N-1}{2}}, \sqrt{\frac{N-2}{2}}, \ldots, \sqrt{\frac{1}{2}} \right).
\end{equation}

As a minor modification of working in [29], these facts can be used to show that the large $\beta$ form of $T_\beta$ relates to a discretisation of the differential operator

\begin{equation}
-\frac{d^2}{dx^2} + x,
\end{equation}

subject to the boundary conditions

\begin{equation}
\frac{\psi'(0)}{\psi(0)} = w, \quad \psi(x) \to 0 \text{ as } x \to \infty.
\end{equation}

**Proposition 2.8.** Write $T_\beta = [t_{ij}]_{i,j=1,\ldots,N}$ and set

\[ D = \text{diag} \left( (N/2)^{-i+1/2} \prod_{k=1}^{i-1} t_{k,k+1} \right). \]

We have

\begin{equation}
\lim_{\beta \to \infty} DT_\beta D^{-1} - \mathbb{I}_N = -\frac{1}{2N^{2/3}} \left( N^{2/3} \Delta_N + N^{-1/3} \tilde{J}_N \right),
\end{equation}

where

\[ \Delta_N = \text{diag} (2, 2, \ldots, 2) + \text{diag}^+ (-1, -1, \ldots, -1) + \text{diag}^- (-1, -1, \ldots, -1), \]
\[ \tilde{J}_N = \text{diag}^- (1, 2, \ldots, N - 1). \]

Furthermore the eigenvectors of (2.32) are given by $Dx$ where $x = [x_i]_{i=1,\ldots,N}$ with

\[ x_{N-n} = \left( \frac{1}{\sqrt{n!}2^n} \right)^{1/2} e^{-\lambda^2/2} H_n(\lambda), \]
where $H_n(\lambda)$ denotes the Hermite polynomial of degree $n$, and $\lambda$ is required to be such that

$$2\alpha = \frac{x_0}{x_1}. \tag{2.33}$$

Proof. The normalised Hermite polynomials as a function of $\lambda$ and multiplied by $e^{-\lambda^2/2}$, $\phi_n(\lambda)$ say, satisfy the recurrence

$$\lambda \phi_n(\lambda) = \sqrt{n/2} \phi_{n-1}(\lambda) + \sqrt{(n+1)/2} \phi_{n+1}(\lambda). \tag{2.34}$$

The structure of this recurrence is identical to that for the components of the eigen-equation for (2.28), eigenvalue $\lambda$ and eigenvectors $[\phi_{N-1-j}(x)]_{j=0}^{N-1}$, except in the first row. The latter requires

$$\alpha \sqrt{2N} \phi_{N-1}(\lambda) + \sqrt{\frac{N-1}{2}} \phi_{N-2}(\lambda) = \lambda \phi_{N-1}(x),$$

which upon use of (2.34) with $n = N - 1$ implies the restriction on $\lambda$ (2.33). $\square$

Taking $N$ large in (2.32) we recognise the right hand side as a discretisation of (2.30) with lattice spacing $N^{-1/3}$. With this latter value, upon rewriting (2.33) to read

$$N^{1/3}(1 - 2\alpha) = \frac{1}{x_1} \left( \frac{x_1 - x_0}{N^{1/3}} \right),$$

we see from (2.27) that the first of the boundary conditions in (2.31) results.

As explained in (2.29) and is readily verified, taking $N$ large with $\beta$ fixed, the appropriate modification of (2.32) is that

$$DT_\beta D^{-1} - \mathbb{I}_N = -\frac{1}{2N^{2/3}} \left( N^{2/3} \Delta_N + N^{-1/3} J_N + \frac{2}{\sqrt{\beta}} W \right), \tag{2.35}$$

where $W$ is the bidiagonal random matrix

$$W = -\frac{N^{1/6}}{\sqrt{2}} \begin{bmatrix} N[0,1] & b_{(N-1)\beta} & N[0,1] \\ b_{(N-2)\beta} & N[0,1] & \ddots \\ \vdots & \ddots & \ddots \\ b_\beta & \cdots & N[0,1] \end{bmatrix},$$

with $b_{(N-j)\beta} = (2\chi^2_{(N-j)\beta} - (N - j)\beta) / \sqrt{2\beta N}$. A direct calculation shows $b_{(N-j)\beta}$ has mean zero and variance $1 - j/N$, and so each element of $W$ has mean zero, and to leading in $N$ for $j$ fixed has standard deviation $N^{1/6}$. This is consistent with a discretisation, lattice spacing $h = N^{-1/3}$, of a Brownian motion process which has mean zero and standard deviation...
\( \sqrt{h} \) over an interval \((x, x + h]\). Recalling that \((2.32)\) is a discretisation of \((2.30)\), these facts suggest that \((2.35)\) is a discretisation of the stochastic Airy operator

\[
- \frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} B'(x),
\]

where \(B(x)\) defines a standard Brownian path.

For \(\alpha = 0\) the above reasoning was made rigorous in the work \([83]\) with the boundary condition \(\psi(0) = 0\), and subsequently extended in \([16]\) to the case of nonzero \(\alpha\) with the scaling as in \((2.27)\) and boundary condition \((2.31)\); in relation to the latter see too \([58, 72]\).

Note in particular that the ground state eigenvalue \(\Lambda_0\) of \((2.36)\) with boundary conditions \((2.31)\) corresponds to \(-x_1\) as specified by the scaling of the largest eigenvalue \(\lambda_1\) for the tridiagonal matrix \((2.28)\). Transforming \((2.36)\) to a stochastic diffusion equation using a simple Ricatti change of independent function \(p(x) = \frac{d}{dx} \psi(x)\) allows theory relating to Kolmogorov’s backward equation to be invoked. With \(F_{\beta,w}(x)\) the cumulative distribution function of \(-\Lambda_0\), this implies the partial differential equation \([16]\)

\[
\frac{\partial F}{\partial x} + \frac{2}{\beta} \frac{\partial^2 F}{\partial w^2} + (x - w^2) \frac{\partial F}{\partial w} = 0,
\]

subject to the boundary conditions that \(F(x, w) \to 1\) as \(x, w \to \infty\) simultaneously, and \(F(x, w) \to 0\) as \(w \to -\infty\) with \(x\) bounded above.

With regards to graphing the PDF of the largest eigenvalue in the critical regime, use of \((2.37)\) is yet to demonstrate a numerical scheme with guaranteed accuracy. Instead, following a suggestion in Edelman and Rao \([32]\) an accurate and efficient Monte Carlo procedure can be based on \((2.28)\). First, it is argued that with respect to the largest eigenvalue and for \(N\) large, truncating the \(N \times N\) tridiagonal matrix to an \(N_0 \times N_0\) tridiagonal matrix with \(N_0 \approx 10N^{1/3}\) does not cause appreciable error. Moreover, the operations of storing a sparse (tridiagonal) matrix and computing the largest eigenvalue, knowing that it is near 1 are all highly efficient with modern software. Finally, scaling the largest eigenvalue as required by \((2.27)\), and repeating \(M \gg 1\) times with \(M\) large allows for a histogram approximating the PDF to be obtained for a given value of \(w\). An example is given in Figure 2.2

Remark 2.9. Combining a number of ideas, in particular that of a Pfaffian point process (see Remark 3.11 below) and that of a Riemann-Hilbert characterisation of certain transcendents, a special function evaluation of \(F_{1,w}(x)\) has been given in \([74]\). However the complexity of this expression is such that it is yet to be used for further analysis of properties \(F_{1,w}(x)\), nor specifically for the determination of its numerical values.
2.5 General $\beta > 0$. In the previous section the Dyson index $\beta$ was introduced into the tridiagonal matrix formulation of the rank 1 perturbed GOE (1.6) for convenience of motivating the ensuing working. As known from the pioneering studies on random matrix theory undertaken by Dyson in the early 1960’s, the first being [30], there is a special significance in the three values $\beta = 1, 2$ and 4. We know already that $\beta = 1$ corresponds to the GOE. The values $\beta = 2$ and 4 correspond to the Gaussian unitary ensemble (GUE) and Gaussian symplectic ensemble (GSE), for which the joint PDF of the elements of the corresponding Hermitian matrices $G$ is again $e^{-\text{Tr} G^2/2}$. For the GUE the off-diagonal entries are complex, while for the GSE they are quaternion; see [38], §1.3. With this choice of joint element PDF, upon scaling by $1/\sqrt{2N\beta}$ the eigenvalues to leading order are supported on the interval $[-1,1]$ and have limiting normalised density given by the Wigner semi-circle (1.7) independent of $\beta$; see [78]. From the discussion of Remark 2.2.1, the corresponding rank 1 perturbation (1.6) then has eigenvalue separation properties as specified by Proposition 2.1.

An extension of the tridiagonalisation procedure of Trotter [94], as discussed in §2.2, gives for each of the real, complex and quaternion cases of (1.6) the tridiagonal matrix $T_\beta$ (2.28). In fact the tridiagonal model allows Proposition 2.3 to be extended to general $\beta > 0$.

**Proposition 2.10.** Consider the rank 1 perturbation (1.6) in the case of the GOE ($\beta = 1$), GUE ($\beta = 2$) and GSE ($\beta = 4$). Extend this to general $\beta > 0$ according to the tridiagonal matrix (2.28).
Up to normalisation, the eigenvalue PDF is proportional to
\[
\prod_{j=1}^{N} e^{-\beta N \lambda_j^2} \prod_{1 \leq j < k \leq N} |\lambda_j - \lambda_k|^\beta \int_{-\infty - i c}^{\infty - i c} e^{it} \prod_{j=1}^{N} (it - 2\alpha \beta N \lambda_j)^{-\beta/2} dt,
\]
where \( c > 2\alpha \beta N \lambda_1 \).

Proof. This requires only minor modification of the proof of Proposition 2.1. Specifically, from (2.39) we know that the \( \beta \) generalisation of (2.14) is to include a factor of \( \prod_{l=1}^{N} q_l^{-1} \). Including this factor in the integrand on the LHS of (2.16) and evaluating implies the integral in (2.38). \( \square \)

The considerations of §2.3 also admit \( \beta \) generalisations. First, the matrix integral evaluation (2.18) extends to a matrix integral over the Haar measure for unitary matrices and unitary symplectic matrices, these diagonalising GUE and GSE matrices respectively. Of particular interest for the future working of Section 3.2 is the unitary matrix integral
\[
\int_U e^{i \alpha N \text{Tr}(U \Lambda U^\dagger 1_N 1_N^\dagger)} (U^t dU) \propto \int_{-\infty - i c}^{\infty - i c} e^{it} \prod_{j=1}^{N} (it - 4\alpha N \lambda_j)^{-1} dt.
\]

Generalising (4.21) to the shifted GUE and GSE ensembles gives a characterisation of the corresponding parameter dependent probability density function \( p_t \) as satisfying the Fokker-Planck equation (2.21) with the Dyson index \( \beta \) appearing as a factor of the second appearance of the partial derivative with respect to \( \lambda_j \) through a multiplication by \( 1/\beta \); see [58] §11.1. In the theory of the Fokker-Planck equation, \( \beta \) then has the interpretation as the inverse temperature, which is a prevalent point in the writings of Dyson on random matrices, beginning with [30]. A conjugation of the Fokker-Planck operator gives a Schrödinger operator of Calogero-Sutherland type in imaginary time — see [38] §11.3 — with the case \( \beta = 2 \) then corresponding to free fermions. The latter also admits an interpretation as non-intersecting Brownian walkers [66], for a study of outliers in this context see [11].

In the theory of the scaled distribution of the largest eigenvalue for the Gaussian ensembles, as implied by the joint PDF (2.38) in the case \( \alpha = 0 \), well known results due to Tracy and Widom (see the review [48]) give evaluations in terms of a particular Painlevé II transcendent. This transcendent is the Hasting-Macleod solution of the Painlevé II equation, specified as satisfying
\[
q'' = sq + 2q^3, \quad q(s) \sim_{s \to \infty} \text{Ai}(s),
\]
where \( \text{Ai}(s) \) denotes the Airy function. Let \( p_{\beta}^{\text{soft}}(s) \) denote the PDF of the scaled largest eigenvalue, with the scaling defined by the first formula in (2.27) for \( \beta = 1, 2 \) and by that
formula with \( N \mapsto N/2 \) for \( \beta = 4 \). Denote the corresponding cumulative distribution by \( E^{\text{soft}}_{\beta}(s) \) so that \( p^{\text{soft}}_{\beta}(s) = -\frac{d}{ds} E^{\text{soft}}_{\beta}(s) \). The results of Tracy and Widom give

\[
E^{\text{soft}}_{2}(s) = \exp \left( - \int_{s}^{\infty} (x - s) q^{2}(x) \, dx \right)
\]

and

\[
E^{\text{soft}}_{1}(s) = (E^{\text{soft}}_{2}(s))^{1/2} \exp \left( \frac{1}{2} \int_{s}^{\infty} q(x) \, dx \right), \quad E^{\text{soft}}_{4}(s) = (E^{\text{soft}}_{2}(s))^{1/2} \cosh \left( \frac{1}{2} \int_{s}^{\infty} q(x) \, dx \right).
\]

For the general critical regime scaling, define the scaled variable \( w \) as in (2.27) for \( \beta = 1, 2 \) and by that formula with \( N \mapsto N/2 \) for \( \beta = 4 \), in keeping with the prescription relating to the scaling of the largest eigenvalue. In the cases \( \beta = 2, 4 \) it turns out that formulas in terms of the transcendent \( q(s) \) again hold true [7, 87, 96, 16, 41], although \( q(s) \) must be supplemented by two functions \( f = f(s, w), g = g(s, w) \) satisfying

\[
\frac{\partial}{\partial w} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} q^{2} & -wq - q' \\ -wq + q' & w^{2} - s - q^{2} \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix},
\]

subject to the initial conditions

\[
f(s, 0) = g(s, 0) = E(s), \quad E(s) := \exp \left( - \int_{s}^{\infty} q(t) \, dt \right).
\]

The equation (2.43) is known in the theory of Painlevé II as one member of the Lax pair for \( q(s) \), first considered in [24].

**Proposition 2.11.** Specify \( F_{\beta,w}(s) \) as the critical regime scaling generalisation of \( E_{\beta}^{\text{soft}}(s) \). We have

\[
F_{2,w}(s) = f(s; w)E^{\text{soft}}_{2}(s),
\]

\[
F_{4,w}(s) = \frac{1}{2} \left( (f(s; w) + g(s; w))(E(s))^{-1/2} + (f(s; w) - g(s; w))(E(s))^{1/2} \right) (E^{\text{soft}}_{2}(s))^{1/2}.
\]

**Proof.** Bloemendal and Virág [16] have noted that the validity of these formulas can be established by directly checking the characterisation (2.37) of \( F_{\beta,w}(x) \). \( \square \)

**Remark 2.12.** 1. Substituting \( w = 0 \) in (2.46) and comparing with the first formula in (2.42) shows \( F_{4,w}(s)|_{w=0} = E_{1}^{\text{soft}}(s) \), which in fact can be anticipated [96, 41].

2. Rumanov [87, 88] has initiated a program of study on Lax pairs associated with (2.37) for general even \( \beta \), with concrete results obtained for \( \beta = 6 \). The latter have been further
refined in [59].
3. For $s \to -\infty$ it is known [225] (see also the review [42, Eq. (3.33)])
\begin{equation}
E_2^{\text{soft}}(s) \sim e^{-|s|^3/12-(1/8)\log |s|}
\end{equation}
and [8]
\begin{equation}
f(s, w) \sim e^{-|s|^{3/2}/6+|x|w/2-w^2|x|^{1/2}},
\end{equation}
where in both formulas the exponents have been truncated at the constant term (i.e. term independent of $s$). Substituting in (2.45) gives the left tail asymptotics of $F_{2, w}(s)$. Note that the resulting formula is consistent with (2.37).

2.6. Eigenvector overlap. For a GOE matrix the eigenvectors $\hat{\mathbf{v}}$ say are distributed uniformly on the unit sphere in $\mathbb{R}^N$. One consequence is that if we take a particular direction, say $\hat{\mathbf{1}}$, and form
\begin{equation}
(\hat{\mathbf{1}}_N \cdot \hat{\mathbf{v}})^2,
\end{equation}
then averaging over the eigenvectors gives zero. Moreover, for large $N$ this is the value of (2.49) almost surely. Of interest is the value of (2.49) in relation to the eigenvector corresponding to the largest eigenvalue of the random matrix (1.6). This was first determined by Benaych-Georges and Nadakuditi [14].

**Proposition 2.13.** Denote the unit eigenvector corresponding to the largest eigenvalue of (1.6) by $\hat{\mathbf{v}}$. For $N \to \infty$ we have almost surely
\begin{equation}
(\hat{\mathbf{1}}_N \cdot \hat{\mathbf{v}})^2 \to \begin{cases} 0, & 0 \leq \alpha \leq 1 \\ 1 - 1/\alpha^2, & \alpha > 1. \end{cases}
\end{equation}

**Proof.** (Outline) Let $\mu$ denote the largest eigenvalue of (1.6) with corresponding unit eigenvector $\hat{\mathbf{v}}$. Rearranging the eigen-equation shows
\[(\mu \mathbb{I}_N - \hat{\mathbf{X}})\hat{\mathbf{v}} = c \hat{\mathbf{1}}_N \hat{\mathbf{v}}, \quad c = (\hat{\mathbf{1}}_N^T(\mu \mathbb{I}_N - \hat{\mathbf{X}})^{-1}\hat{\mathbf{1}}_N)^{1/2}.\]
where the second equality follows from the fact that $\hat{\mathbf{1}}_N^T\hat{\mathbf{v}}$ is a scalar. This implies $\hat{\mathbf{v}}$ is proportional to $(\mu \mathbb{I}_N - \hat{\mathbf{X}})^{-1}\hat{\mathbf{1}}_N$. Moreover, the proportionality can be specified by the fact that $\hat{\mathbf{v}}$ is a unit vector. Thus
\[\hat{\mathbf{v}} = \frac{1}{c}(\mu \mathbb{I}_N - \hat{\mathbf{X}})^{-1}\hat{\mathbf{1}}_N, \quad c = (\hat{\mathbf{1}}_N^T(\mu \mathbb{I}_N - \hat{\mathbf{X}})^{-1}\hat{\mathbf{1}}_N)^{1/2}.
\]

Next diagonalise the GOE matrix $\hat{\mathbf{X}}$ using $\hat{\mathbf{X}} = U\Lambda U^T$ as in the proof of Proposition 2.1. With $\hat{\mathbf{w}} = U^T\hat{\mathbf{1}}_N$ this shows
\[\hat{\mathbf{v}} = U(\mu \mathbb{I}_N - \hat{\mathbf{X}})^{-1}\hat{\mathbf{w}}, \quad c = (\hat{\mathbf{w}}^T(\mu \mathbb{I}_N - \Lambda)^{-2}\hat{\mathbf{w}})^{1/2}.
\]
It then follows
\[(\hat{\mathbf{1}} \cdot \hat{\mathbf{v}})^2 = \frac{1}{c^2} (\hat{\mathbf{w}}^T (\mu \mathbb{I}_N - \Lambda)^{-1} \hat{\mathbf{w}})^2.\]

To close out the proof from here we need the fact that for \( N \) large, and \( \rho_W(x) \) denoting the Wigner semi-circle (1.7), almost surely
\[\hat{\mathbf{w}}^T (\mu \mathbb{I}_N - \Lambda)^{-1} \hat{\mathbf{w}} \rightarrow \int_{-1}^{1} \frac{\rho_W(x)}{\mu - x} \, dx.\]

Note that in the proof of Proposition 2.1 is established upon averaging. The proportionality \( c \) is just minus the derivative with respect to \( \mu \) of this for \( \mu > 1 \). Recalling now (2.7) gives (3.9) for \( \mu \geq 1 \), with the cases \( 0 \leq \alpha \leq 1 \) obtained by taking \( \lim_{\mu \rightarrow 1^+} \).

Remark 2.14. Recently Bao and Wang [11] have studied the eigenvector overlap for the random matrix (1.6), with \( \tilde{G} \) therein a scaled GUE matrix, in the critical regime as specified by the scaling (2.27). Specifically, they studied the first component \( x_j^{(1)} \) of the eigenvector \( x_j \) corresponding to the \( j \)-th largest eigenvalue. With \( \{\sigma_j\}_{j=1}^{N} \) the eigenvalues of (1.6) and \( \{\mu_j\}_{j=1}^{N-1} \) the eigenvalues of this same random matrix with the first row and column removed, the starting point of their analysis is the identity
\[(2.51) \quad |x_j^{(1)}|^2 = \prod_{l=1}^{j-1} \sigma_l - \mu_j \prod_{l=j+1}^{N} \sigma_l - \mu_{i-1} \prod_{l=j}^{i-1} \sigma_l.\]

Their analysis reveals that for \( j \) fixed and in the critical regime, \( N^{1/3} |x_j^{(1)}|^2 \) has a well defined limit proportional in distribution to the RHS of (2.51) with the eigenvalues therein replaced by their scaled critical regime counterparts.

3. A multiplicative rank 1 perturbation for the LUE

3.1. Reduction to an additive rank 1 perturbation. Let \( X \) be a standard complex Gaussian matrix of size \( n \times N \), \( n \geq N \), and form the matrix \( W = X^* X \). The matrices \( \{W\} \) are said to be particular complex Wishart matrices (specifically such that \( X \) has mean zero, and the covariance matrix associated with \( W \) is the identity) and are also matrix realisations of the Laguerre unitary ensemble (LUE) in the case of the Laguerre parameter \( a = n - N \). For general Laguerre parameter \( a > -1 \), the LUE can be specified by the eigenvalue PDF proportional to
\[(3.1) \quad \prod_{l=1}^{N} x_l^a e^{-x_l} \prod_{1 \leq j < k \leq N} (x_k - x_j)^2, \quad x_l \in \mathbb{R}^+.\]
It follows that for $a = n - N$ a viewpoint on \((3.1)\) is as the PDF for the squared singular values of $X$.

For $\Sigma$ and $N \times N$ positive definite matrix, construct from $X$ a correlated complex Gaussian matrix

\[(3.2) \quad \tilde{X} = X\Sigma^{1/2},\]

and use it in turn to construct a correlation complex Wishart matrix

\[(3.3) \quad \tilde{W} = \Sigma^{1/2}X^\dagger X\Sigma^{1/2}.\]

We see that the choice

\[(3.4) \quad \Sigma = \text{diag} \left( b, 1, \ldots, 1 \right), \quad b > 0,\]

corresponds to a rank 1 multiplicative perturbation of $X$. In fact from the viewpoint of eigenvalues, the corresponding multiplicative perturbation of $\tilde{W}$ can be written as a rank 1 additive perturbation. To see this we note that the nonzero eigenvalues of $\Sigma^{1/2}X^\dagger X\Sigma^{1/2}$ are the same as those for $X\Sigma X^\dagger$ (more generally $AB$ and $BA$ have the same nonzero eigenvalues). But with $\Sigma$ given by \((3.4)\),

\[(3.5) \quad X\Sigma X^\dagger = X_1 X_1^\dagger + bxx^\dagger,\]

where $X_1$ refers to $X$ with the first column deleted, and $x$ denotes the first column of $X$. A strategy analogous to that used in the proof of Proposition \(2.1\) now suffices to specify an eigenvalue separation effect as a function of $b$ \([4, 5, 12]\).

Proposition 3.1. Consider the particular correlated complex Wishart matrix specified by \((3.3)\) and \((3.4)\), and scale the eigenvalues by dividing by $n$. In the limit $n \to \infty$ with $n/N := \gamma \geq 1$ fixed. Provided $b > 1 + \sqrt{\gamma}$, a single eigenvalue separates from the upper endpoint of the support $(1 + 1/\sqrt{\gamma})^2$, and occurs at the point

\[(3.6) \quad b \left( 1 + \frac{\gamma^{-1}}{b - 1} \right).\]

Proof. (Sketch) When divided by $n$, replacing the Wigner semi-circle \((1.7)\) for the normalised density of $X_1 X_1^\dagger$ in \((3.5)\) is the Marčenko–Pastur functional form \([78]\)

\[(3.7) \quad \left( 1 - \frac{1}{\gamma} \right) \delta(x) + \frac{\sqrt{(x - c)(d - x)}}{2\pi\gamma x} \delta_{c < x < d},\]

where $c = (1 - \sqrt{\gamma})^2$ and $d = (1 + \sqrt{\gamma})^2$. Here the delta function is in keeping with the fraction of zero eigenvalues of $X_1 X_1^\dagger$ equalling $(1 - 1/\gamma)$. The argument of the working of
the proof of Proposition 2.1 gives that the secular equation for the eigenvalues of the RHS of (3.5), divided by $n$, reads

$$1 = \frac{b(1 - 1/\gamma)}{\lambda} + b \int_c^d \frac{\tilde{\rho}_\text{MP}(x)}{\lambda - x} \, dx,$$

where $\tilde{\rho}_\text{MP}(x)$ denotes the second term in (3.7); cf. (2.7). For the integral in (3.8) we have the evaluation (see e.g. \[12], Eq. (2.24))

$$\frac{1}{2\gamma} \left( 1 - \frac{\gamma - 1}{z} - \left( 1 - \frac{2(\gamma + 1)}{z} + \frac{(\gamma - 1)^2}{z^2} \right)^{1/2} \right).$$

Substituting in (3.8), and observing that both terms therein are decreasing functions of $\lambda$ and so take their maximum value when $\lambda = d$ gives the stated condition for eigenvalue separation, while solving for $\lambda$ under this condition gives the value (3.6). \hfill $\Box$

An illustration of the prediction of Proposition 3.1 is given in Figure 3.1. A comprehensive study of this phase transition effect, extended to (3.3) with the parameter $b$ repeated $r$ times down the diagonal of $\Sigma$ and including the critical regime (see subsection 3.4 below) was undertaken by Baik, Ben Arous and Péche \[4\]. Subsequently, it has been customary to use the term BBP transition in this context.

Remark 3.2. 1. (Rank 1 update) Generalise (3.4) so that $\Sigma = \text{diag} \, (b_1, b_2, \ldots, b_N)$ with each $b_i > 0$. Specify $X$ as in (3.5), denote by $X_j$ the matrix obtained from $X$ by deleting the first $j$ columns $x_1, \ldots, x_j$, and denote by $\Sigma_j$ the $(N - j) \times (N - j)$ matrix $\text{diag} \, (b_{j+1}, \ldots, b_N)$. In keeping with (3.5) we have

$$X_{j-1} \Sigma_{j-1} X_{j-1}^\dagger = X_j \Sigma_j X_j^\dagger + b_j x_j x_j^\dagger \quad (j = 1, \ldots, N).$$

Iterating this backwards, $j = N, N - 1, \ldots, 1$ gives a rank 1 update construction of $X \Sigma X^\dagger$; recall the paragraph including (1.9).

2. (Note on eigenvector overlap) Before the derivation of Proposition 2.13 the eigenvalue overlap associated with the rank 1 perturbation (3.5) was calculated by Paul \[79\]. This was in the case that $X$ therein has real rather than complex entries, but this has no effect on the result. The method of the proof of Proposition 2.13 carries over, with the role of (2.7) now...
played by (3.8). To state the result, denote the unit eigenvector corresponding to the largest
eigenvalue of (3.5) by $\tilde{\psi}$. Then for $N \to \infty$ we have the almost surely convergence
\begin{equation}
|\tilde{1}_N \cdot \tilde{\psi}|^2 \to \begin{cases} 
\frac{(b - 1)^2 - \gamma}{(b - 1)^2 + \gamma (b - 1)}, & b > 1 + \sqrt{\gamma} \\
0, & \text{otherwise.}
\end{cases}
\end{equation}
(3.9)

The recent work [24] considers this overlap in the case of complex entries for the eigenvector
corresponding to the smallest eigenvalue, and shows that when multiplied by $N$ it has the
limiting distribution $\chi_2^2 / (2b)$, with $\chi_2^2$ the chi-square random variable with two degrees of
freedom.

3.2. An application of the HCIZ matrix integral. It follows from (3.3) and the definition of
$X$ therein that the correlated complex Wishart matrix is specified by a PDF proportional to
\begin{equation}
\left( \frac{1}{\text{det} \Sigma} \right)^{1/2} e^{-\text{Tr}(X^* \tilde{\Sigma}^{-1})}.
\end{equation}
(3.10)

A fundamental change of variables in random matrix theory, see e.g. [38 Eq. (2.32)], gives that
$\tilde{W} = \tilde{X}^* \tilde{X}$ then has PDF proportional to
\begin{equation}
\left( \frac{1}{\text{det} \Sigma} \right)^{1/2} \left( \text{det} \tilde{W} \right)^{(n-N)} e^{-\text{Tr}(\tilde{W} \Sigma^{-1})}.
\end{equation}
(3.11)

Diagonalising the complex Hermitian matrix $\tilde{W}$ according to $\tilde{W} = U \Lambda U^*$ for $U \in U(N)$
and making use of the corresponding change of variables formula (see [38 Prop. 1.3.4])
gives that the eigenvalue PDF of $\tilde{W}$ is proportional to
\begin{equation}
\left( \frac{1}{\text{det} \Sigma} \right)^{1/2} \prod_{l=1}^{N} \lambda_l^{n-N} e^{-\lambda_l} \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2 \int_{U \in U(N)} e^{-\text{Tr}(U \Lambda U^*(\Sigma^{-1} - 1_N))} (U^* dU).
\end{equation}
(3.12)

For $A, B$ Hermitian matrices of size $N \times N$, the HCIZ matrix integral (named after
Harish-Chandra [62], and Itzykson and Zuber [63])
\begin{equation}
\int_{U} \exp(U^* A U B) [U^* dU] = \prod_{j=1}^{N} \Gamma(j) \frac{\text{det}[e^{i\bar{b}_j}]_{1 \leq j \leq N}}{\Delta_N(a) \Delta_N(b)},
\end{equation}
(3.13)

where $[U^* dU]$ denotes the normalised Haar measure for $U(n)$, and for an array $x =
(x_1, \ldots, x_N)$, $\Delta_N(x) := \prod_{1 \leq j < k \leq N} (x_k - x_j)$. Application of (3.12) allows the matrix integral
in (3.11) to be calculated.

**Proposition 3.3.** In the case that $\Sigma$ is given by (3.4) the PDF (3.11) of $\tilde{W}$ simplifies to be proportional to
\begin{equation}
\prod_{l=1}^{N} \lambda_l^{n-N} e^{-\lambda_l} \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j) \text{det} \left[ [\lambda_j^{k-1}]_{k=1, \ldots, N;j=1, \ldots, N} \right]_{j=1}^{N},
\end{equation}
(3.13)
where \( c := 1 - 1/b \).

**Proof.** Write the eigenvalues of \((\Sigma^{-1} - \mathbb{I}_N)\) as \( c_1, c_2, \ldots, c_N \). We then see that (3.13) follows from (3.11) by making use of (3.12) and the limit formula

\[
\lim_{c_1, \ldots, c_{N-1} \to 0} \frac{\det[e^{\lambda/c}]_{j,k=1}^N}{\Delta_n(c)} \propto \frac{1}{c_{N-1}} \det \left[ \frac{[\lambda_j^{k-1}]}{[e^{\lambda_j/c}]_{j=1}^N} \right]_{k=1}^N N_{k=1}^N.
\]

This limit formula in turn follows by taking the limits in order \( c_k \to 0 \), \((k = 1, \ldots, N - 1)\) applied to column \( k \) of the determinant, after first subtracting multiples of the limiting value of the earlier columns so that the first \((k-1)\) terms of the power series expansion have been eliminated. \( \square \)

**Remark 3.4.** The above working implies that for \( B \) of rank 1, with its nonzero eigenvalue equal to \( b \),

\[
(3.14) \quad \int \exp(U^* A U) \mid U^* dU \propto \frac{\det[[a_j^{k-1}]]_{j=1}^N N_{k=1}^N}{\Delta_n(a) b^{N-1}}.
\]

On the other hand (2.39) tells us that this same matrix integral is proportional to

\[
(3.15) \quad \int_{-\infty - ic}^{\infty - ic} e^{it} \prod_{j=1}^N (it - ba_j) \frac{1}{dt}.
\]

Indeed one can check that computing this contour integral using residues gives the same functional form as expanding the determinant in (3.14) by the final column and simplifying using the Vandermonde determinant formula. It follows that in the case \( \beta = 2 \) the eigenvalue PDF (2.38) for the perturbed GUE can also be written, up to proportionality, as

\[
(3.16) \quad \prod_{j=1}^N e^{-\beta N a_j^2} \prod_{1 \leq j < k \leq N} (\lambda_j - \lambda_k) \det \left[ \frac{[\lambda_j^{k-1}]}{[\lambda_j^{k-1}]}_{j=1}^N \right]_{k=1}^N.
\]

### 3.3. A joint eigenvalue PDF

Starting from (3.5), it is possible to compute the joint eigenvalue PDF for the eigenvalues of both \( X_1 X_1^\dagger \) and \( X \Sigma X^\dagger \), following [41] §3.1. Denoting the \( N - 1 \) nonzero eigenvalues of the \( n \times n \) matrix \( X_1 X_1^\dagger \) by \( \{\mu_j\}_{j=1}^{N-1} \) and ordered as in (2.4), from the derivation of (2.3) we have that the equation determining the eigenvalues of \( X \Sigma X^\dagger \) is

\[
(3.17) \quad 0 = 1 - b \left( \frac{u_0}{\lambda} + \sum_{j=1}^{N-1} \frac{u_j}{\lambda - \mu_j} \right), \quad u_0 = \sum_{j=N}^{n-N-1} |x^{(j)}|^2, \quad u_j = |x^{(j)}|^2.
\]
Since \( x \) is a standard complex vector, we have that \( u_0 \) is distributed as \( \Gamma[n - N - 1, 1] \), and each \( u_j \) as \( \Gamma[1, 1] \). The joint distribution of \( \{u_j\}_{j=0}^{N-1} \) is therefore proportional to

\[
(3.18) \quad u_0^{n-N-1} e^{-u_0} \prod_{l=1}^{N-1} e^{-u_l}.
\]

Regarding these variables as the residues in the random rational function specified by the RHS of (3.17), we have that \( \{\mu_j\}_{j=1}^{N-1} \) are the poles, while the eigenvalues of \( X\Sigma X^\dagger \) are the zeros. Let the latter be denoted \( \{\lambda_j\}_{j=1}^N \), which we know must be interlaced as in (2.5). In terms of the zeros and the poles we have

\[
(3.19) \quad 1 - b \left( \frac{u_0}{\lambda} + \sum_{j=1}^{N-1} \frac{u_j}{\lambda - \mu_j} \right) = \frac{\prod_{j=1}^{N} (\lambda - \lambda_j)}{\lambda \prod_{j=1}^{N-1} (\lambda - \mu_j)}.
\]

For given \( \mu_j \), computing the Jacobian for the change of variables from residues to the zeros gives a particular conditional PDF.

**Proposition 3.5.** Let the PDF for \( \{u_j\} \) to be given by (3.18). The PDF for \( \{\lambda_j\} \) with \( \{\mu_j\} \) given is proportional to

\[
(3.20) \quad \prod_{l=1}^{N-1} \mu_l^{n+N-1} e^{u_l/b} \prod_{k=1}^{N} \lambda_k^{n-N} e^{-\lambda_k/b} \prod_{j=1}^{N} \prod_{k=1<j<k}^{N} (\lambda_j - \lambda_k) \prod_{1<j<k<N} (\mu_j - \mu_k)
\]

supported on (2.5) with \( \mu_N = 0 \).

**Proof.** Expanding both sides of (3.19) in powers of \( 1/\lambda \) and equating the coefficient of \( 1/\lambda \) shows

\[
b \sum_{j=0}^{N-1} u_j = \sum_{j=1}^{N} \lambda_j - \sum_{j=1}^{N-1} \mu_j,
\]

which when substituted in (3.18) accounts for the exponential term in (3.20). It remains to compute the Jacobian. For this purpose, note from (3.19) by computing residues that

\[
b u_0 = \frac{\prod_{j=1}^{N} \lambda_j}{\prod_{j=1}^{N-1} \mu_j}, \quad -b u_j = \frac{\prod_{j=1}^{N} (\mu_j - \lambda_j)}{\prod_{j=1, j \neq i}^{N} (\mu_j - \mu_i)}.
\]

This shows that up to a possible sign, and with \( \lambda_N = 0 \),

\[
(3.21) \quad \det \left[ \frac{\partial u_{j-1}}{\partial \lambda_k} \right]_{j,k=1}^{N} = \frac{1}{b^N \prod_{j=1}^{N} (\mu_j - \mu_i)} \det \left[ \frac{1}{\mu_j - \lambda_{k}} \right]_{j,k=1}^{N}.
\]

The determinant on the RHS is known of the Cauchy double alternant and has an evaluation in terms of products (see e.g. [58 Eq. (4.33)]) which implies the remaining terms in (3.20). 
\[ \square \]
In the setting of (3.5) the given eigenvalues \( \{\mu_j\} \) in (3.20) have the PDF (3.1) with \( N \) replaced by \( N - 1 \) and \( n = n - N + 1 \). Hence the joint PDF of the eigenvalues of the matrices \( X_1 X_1^\dagger \) and \( X \Sigma X^\dagger \) in (3.5) is proportional to

\[
\prod_{l=1}^{N-1} e^{-\mu_l(1-1/b)} \prod_{k=1}^N \lambda_k^{n-N} e^{-\lambda_k/b} \prod_{1 \leq j < k \leq N} (\lambda_j - \lambda_k) \prod_{1 \leq j < k \leq N-1} (\mu_j - \mu_k)
\]

(3.22) again with the requirement of the interlacing (2.5) with \( \lambda_N = 0 \). The latter the ordering of \( \{\mu_j\}_{j=1}^N \) (2.4), the function of \( \{\lambda_j\}_{j=1}^N \) defined by the interlacing when viewed as an indicator function has the determinantal form

\[
\det |\chi_{\lambda_j - \mu_k > 0}|_{j,k=1}^N \chi A = \begin{cases} 1 & A \text{ true} \\ 0 & A \text{ false.} \end{cases}
\]

(3.23) Including this as a factor in (3.22) allows the PDF for \( \{\mu_j\} \) to be computed by integrating each \( \mu_j \) over \( \mathbb{R}^+ \). These integrations can be done can be done with the aid of a minor variant of Andréief’s identity (see [43]) which shows

\[
\frac{1}{(N-1)!} \int_0^\infty \! d\mu_1 \cdots \int_0^\infty \! d\mu_{N-1} \prod_{l=1}^{N-1} e^{-(1-1/b)\mu_l} \prod_{1 \leq j < k \leq N-1} (\mu_k - \mu_j) \det |\chi_{\lambda_j - \mu_k > 0}|_{j,k=1}^N \\
= \det \left[ \int_0^{\lambda_j} \mu^{1-1/b} e^{-\mu(1-1/b)} d\mu \right]_{j=1,\ldots,N}^{k=1,\ldots,N-1} [1]_{j=1,\ldots,N} \\
= \prod_{j=1}^N e^{-\mu(1-1/b)\lambda_j} \det \left[ \lambda_j^{-1} \chi_{\lambda_j - \mu_k > 0} \right]_{j=1,\ldots,N}^{k=1,\ldots,N-1} [e^{\mu(1-1/b)\lambda_j}]_{j=1,\ldots,N},
\]

where the final expression follows using integration by parts and elementary column operations. Replacing the terms dependent on \( \{\mu_j\} \) in (3.22) by the final expression in (3.24) reclaims (3.13) for the marginal PDF of \( \{\lambda_j\} \).

Remark 3.6. Consider the setting of (3.5) with \( X \) of size \( N \times (N+1) \), \( \Sigma \) of size \( (N+1) \times (N+1) \) and \( X_1 \) of size \( N \times N \). Denote the eigenvalues of \( X \Sigma X^\dagger (X_1 X_1^\dagger) \) by \( \{\lambda_j\} \) and \( \{\{\mu_j\}\} \). Repeating the considerations which lead to (3.22) shows that the joint eigenvalue PDF is proportional to

\[
\prod_{l=1}^{N-1} e^{-\mu_l(1-1/b)} \prod_{k=1}^N e^{-\lambda_k/b} \prod_{1 \leq j < k \leq N} (\lambda_j - \lambda_k)(\mu_j - \mu_k)
\]

subject to the interlacing (2.5). This PDF first appeared in the study of probabilistic models related to the longest increasing subsequence of a random permutation [7]; see also [47].
3.4. Correlation kernel for the soft edge critical regime. Let $p_N(x_1, \ldots, x_N)$ denote an eigenvalue PDF supported on $I \subset \mathbb{R}$. The $k$-point correlation function $\rho_{(k)}(x_1, \ldots, x_k)$ is specified in terms of $p_N$ by

$$\rho_{(k)}(x_1, \ldots, x_k) = \frac{N!}{(N-k)!} \int_I dx_{k+1} \cdots \int_I dx_N p_N(x_1, \ldots, x_N).$$

Note that the case $k = 1$ corresponds to the eigenvalue density. The eigenvalue PDFs (3.13) and (3.16) correspond to a determinantal point processes. This means that $\rho_{(k)}$ can be expressed in determinant form

$$\rho_{(k)}(x_1, \ldots, x_k) = \det[K_N(x_j, x_i)]_{j, l=1, \ldots, k},$$

where $K_N(x, y)$ — referred to as the correlation kernel — can be expressed in terms of certain orthogonal polynomials and special functions. In keeping with the focus of this section on perturbation of the LUE, we consider (3.13). Relevant for this is the particular non-symmetric Laguerre polynomial kernel

$$K^a_n(x, y) = y^a e^{-y} \sum_{p=0}^{n-1} \frac{(p + a)!}{p!} L^a_p(x) L^a_p(y).$$

It is a standard result in random matrix theory that substituting (3.27) with $n = N$ in (3.26) gives $\rho_{(k)}$ for the unperturbed LUE (3.1); see [38, §5.1.2]. Also relevant are the so-called incomplete multiple Laguerre functions of type I and II [15, 23]

$$\tilde{\Lambda}^{(1)}(x) = \int_{C_{(0,-c)}} e^{-xz} \frac{(1 + z)^{N+a}}{z^{N-1}(z + c)} \frac{dz}{2\pi i}, \quad \Lambda^{(1)}(x) = \int_{C_{(-1)}} e^{xzN-r} \frac{(1 + z)^{N+a}}{2\pi i} dz.$$

Here $C_{(0,-c)}, C_{(-1)}$ are simple contours encircling the points $\{0, -c\}$ and $\{-1\}$ respectively.

**Proposition 3.7.** The $k$-point correlation function for the PDF (3.13) with $n - N = a$ is given by (3.26) with

$$K_N(x, y) = K^a_{N-1}(x, y) + \tilde{\Lambda}^{(1)}(x) \Lambda^{(1)}(y).$$

**Remark 3.8.** This kernel is a special case of the correlation kernel for the PDF (3.13) with the determinant factor therein replaced by $\det[e^{c_k A_k}]_{k=1, \ldots, N}$. A double contour form of the determinant in this more general case was first given in [14], and rederived in the context of multiple orthogonal polynomials in [23]. Choosing all but $r$ of the $\{c_k\}$ equal to zero was then shown in the latter reference to allow an evaluation in terms of the nonsymmetric Laguerre kernel $K^{a/r}_{N-r}(x, y)$, plus a sum of $r$ terms involving incomplete multiple Laguerre functions of type I and II, which for $r = 1$ is (3.29).
Common to both (3.13) and (3.16) is that there is a tuning of the parameters $c$ and $\alpha$ respectively so that the statistical state corresponding to the critical regime — recall §2.4 — is identical for both. For (3.16) the required scaling is given by (2.27) while $w, x_1$ for (3.13) with $n, N \to \infty, n/N = \gamma \geq 1$ fixed are specified by

$$
N^{1/3} \left( 1 + \frac{\gamma}{c+1} \right) \to w, \quad N^{2/3} \frac{\sqrt{1/\gamma}}{(1 + \sqrt{1/\gamma})^{4/3}} \left( \frac{\lambda_1}{N} - (1 + \sqrt{\gamma})^2 \right) \to x_1.
$$

Applying the scaling (3.30) to (3.29), or the scaling (2.27) to the analogue of (3.29) for the PDF (3.16) gives a functional form involving Airy functions \[23, 4\].

**Proposition 3.9.** The k-point correlation function for the PDF (3.13) with the scaling (3.30) is given by (3.26) with correlation kernel

$$
K^{\text{soft,c}}(x,y;w) = K^{\text{soft}}(x,y) + \text{Ai}(y) \int_{-\infty}^{x} e^{-w(x-t)} \text{Ai}(t) \, dt,
$$

where

$$
K^{\text{soft}}(x,y) = \frac{\text{Ai}(x) \text{Ai}'(y) - \text{Ai}(y) \text{Ai}'(x)}{x-y}.
$$

**Remark 3.10.** 1. The correlation kernel (3.32) is well known in random matrix theory as specifying the scaled state in the neighbourhood of the largest eigenvalue for the GUE and LUE [35], and in fact for a much broader class of random matrices relating to Hermitian random matrices with complex entries [78]. Note that $K^{\text{soft,c}}(x,y;w) \to K^{\text{soft}}(x,y)$ as $w \to \infty$. 2. The density is given by setting $x = y$ in $K^{\text{soft,c}}(x,y;w)$. Using the integral $\int_{-\infty}^{\infty} e^{wt} \text{Ai}(t) \, dt = e^{w^3/3}$ shows

$$
\rho^{\text{soft}}_{(1)}(x) \sim \text{Ai}(x) e^{-wx+w^3/3},
$$

which from general considerations (see [42, §3.5]) coincides with the right tail of the PDF corresponding to $F_{2,w}(x)$.

The fact that the statistical state of the critical regime is a determinantal point process implies a formula for the cumulative distribution function $F^{2,w}(s)$ in terms of a Fredholm determinant,

$$
F^{2,w}(s) = \det(I - K^{2,w}_2);
$$

see e.g. [38, §9.1] for the general theory. Here $K^{2,w}_2$ is the integral operator on $(s,\infty)$ with kernel $K^{\text{soft,c}}(x,y;w)$. As made explicit by Bornemann [19], there are advantages in using
the Fredholm determinant for a numerical tabulation rather than the Painlevé expression \((2.45)\). An exception is the case \(w = 0\). The, according to \((2.45)\), \((2.44)\) and \((2.42)\)

\[
F^{2,\nu}(s)
\]

\[
\bigg|_{w=0} = \left(E_{\text{soft}}(s)\right)^2,
\]

which can be anticipated already at the finite \(N\) level \([46, \text{Eq. (5.8)}]\). The significance of this is that \(E_{\text{soft}}(s)\) and the corresponding PDF, which correspond to Tracy–Widom \(\beta = 1\), are now part of standard software. A tabulation of the PDF corresponding to \((3.35)\), compared against a simulation based on \((2.28)\), is given in Figure 3.2.

Remark 3.11. The statistical state of the \(\beta = 4\) soft edge statistical regime is known to form a Pfaffian point process \([46, \text{Prop. 16}]\), meaning that the general \(k\)-point correlation function can be written in the form

\[
\rho_{(k)}(x_1, \ldots, x_k) = \text{Pf}(AZ_{2k}), \quad A = \begin{bmatrix} f^{11}(x_j, x_l) & f^{12}(x_j, x_l) \\ f^{21}(x_j, x_l) & f^{11}(x_l, x_j) \end{bmatrix}_{j,l=1,\ldots,k},
\]

Here the functions \(f^{11}, f^{12}, f^{21}\) can each be expressed in terms of \(K_{\text{soft}}(X, Y)\), and \(Z_{2k}\) is the elementary \(2k \times 2k\) anti-symmetric matrix \(I_k \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\). Explicitly, in the simplest case
k = 1, we have [41] Eq. (4.15) with \( w \mapsto -w \)

\[
(3.37) \quad \rho_{(1)}(X) = \frac{1}{2} k^{\text{soft}}(X, X) - \frac{1}{2} \int_{-\infty}^{X} e^{-w(x-t)/2} \frac{\partial}{\partial X} K^{\text{soft}}(t, X) \, dt + \frac{w}{4} \int_{-\infty}^{X} dt \, e^{-w(X-t)/2} \int_{-\infty}^{\infty} du \left( \frac{\partial}{\partial t} K^{\text{soft}}(u, t) \right).
\]

3.5. Characterisation of the hard edge critical regime for general \( \beta > 0 \). Analogous to the tridiagonal reduction of (1.6) introduced in §2.2 and \( \beta \) generalised in §2.4, the Wishart matrix \((X)\) with correlation matrix \( \Sigma \) can, by the application of Householder transformations, be reduced to a tridiagonal form which allows for a \( \beta \) generalisation. This follows by first applying Householder transformations to reduce \( X^\dagger \) to the \( N \times N \) bidiagonal form [89] \( \begin{bmatrix}
\sqrt{\hat{\Gamma}} & & & \\
\hat{\Gamma} \beta(n-2) & \hat{\Gamma} \beta(n-1) & \hat{\Gamma} \beta(n-1) & \\
\hat{\Gamma} \beta(n-1) & \hat{\Gamma} \beta(n-1) & \hat{\Gamma} \beta(n-2) & \\
\hat{\Gamma} \beta(n-1) & \hat{\Gamma} \beta(n-2) & \hat{\Gamma} \beta(n-2) & \\
\hat{\Gamma} \beta & \hat{\Gamma} \beta(n-2) & \hat{\Gamma} \beta(n-2) & \\
\hat{\Gamma} \beta & \hat{\Gamma} \beta(n-1) & \hat{\Gamma} \beta(n-1) & \\
\hat{\Gamma} \beta & \hat{\Gamma} \beta(n-1) & \hat{\Gamma} \beta(n-2) & \\
\hat{\Gamma} \beta & \hat{\Gamma} \beta(n-1) & \hat{\Gamma} \beta(n-2) & \hat{\Gamma} \beta(n-2) & \\
\hat{\Gamma} \beta(n+1) & \hat{\Gamma} \beta(n+1) & \hat{\Gamma} \beta(n+1) & \\
\hat{\Gamma} \beta(n+1) & \hat{\Gamma} \beta(n+1) & \hat{\Gamma} \beta(n+1) & \\
\hat{\Gamma} \beta(n+1) & \hat{\Gamma} \beta(n+1) & \hat{\Gamma} \beta(n+1)
\end{bmatrix}
\]

with \( \beta = 2 \). Here \( \chi_p^2 \) refers to the particular gamma distribution \( \Gamma(p/2, 2) \), and \( n - N \) zero columns which do not effect the non-zero eigenvalues of \( X^\dagger X \) have been removed. We know from [41] that for general \( \beta > 0 \), \( B_\beta^\dagger B_\beta \) has eigenvalue PDF proportional to

\[
(3.39) \quad \prod_{j=1}^{N} \lambda_j^{\beta(n-N+1)/2-1} e^{-\beta \lambda_j/2} \prod_{1 \leq j < k \leq N} \lambda_j - \lambda_k |^\beta \int_{-\infty + ic}^{\infty + ic} e^{it} \prod_{j=1}^{N} \left( it - \frac{b-1}{2b} \beta \lambda_j \right)^{-\beta/2} \, dt,
\]

(cf. (2.38)) which for \( \beta = 2 \) is consistent with (3.11) and the evaluation of the matrix integral as implied by Remark 3.4.

Since Wishart matrices are positive definite, the smallest eigenvalue is the eigenvalue closest to the origin. With \( \Sigma = I \) in (3.3), a well defined statistical state in the neighbourhood of the origin — refereed to as the hard edge — results from scaling the eigenvalues \( \lambda_j \mapsto \lambda_j/N \). In this scaling, the eigenvalues about the origin are spaced of order unity apart, and the Laguerre parameter \( \beta(n - N + 1)/2 - 1 = a \) (this is the exponent in the first term in (3.39)) is fixed. Introducing now the covariance matrix (3.4), it was shown in [23] by explicit calculation of the correlation functions for \( \beta = 2 \) that scaling \( b = c/N \) leads to a well defined hard edge critical regime dependent on \( c \). Subsequently these scalings applied to \( B_\beta^\dagger B_\beta \) were shown to extend the meaning of this regime to general \( \beta > 0 \). Moreover, with \( 1 - F_{\beta,c}(x) \) denoting the cumulative distribution of the smallest scaled eigenvalue, and thus \( F_{\beta,c}(x) \) equal to the probability that the interval \((0, x)\) is free of eigenvalues, ideas relating
to the derivation of \((2.37)\) as applies at the soft edge critical regime were adapted to obtain an analogous characterisation at the hard edge \([84]\).

**Proposition 3.12.** The hard edge scaled distribution \(F_{\beta,c}(x)\) satisfies the partial differential equation

\[
- \frac{x}{\beta} \frac{dF}{dx} + \frac{2}{\beta^2} \frac{d^2F}{dc^2} + \left( \frac{2}{\beta}(a + 2) - c^2 - x \right) \frac{dF}{dc} = 0,
\]

subject to the boundary conditions

\[
F_{\beta,c}(0) = 1, \quad \lim_{x \to \infty} F_{\beta,c}(x) = 0, \quad \lim_{c \to 0^+} F_{\beta,c}(x) = 0.
\]

In the special case \(a = 0\), it is known from \([41\) Eq. (3.24)] that for finite \(N\) and with \(\Sigma = \text{diag}(b_1, \ldots, b_N)\) that the probability of no eigenvalues in the interval \((0, s)\) for the \(\beta\) generalisation of \((3.3)\) has the simple functional form \(e^{-s \sum_{i=1}^{N} (1/2b_i)}\) and hence

\[
F_{\beta,c}(x) \bigg|_{a=0} = \exp \left( -\frac{\beta x}{2} \left( \frac{1}{c} + 1 \right) \right).
\]

It is a simple task to show that \((3.42)\) is consistent with Proposition 3.12. Note the large \(c\) limiting behaviour

\[
\lim_{c \to \infty} F_{\beta,c}(x) \bigg|_{a=0} = F_{\text{hard}}^{\beta}(s;a) \bigg|_{a=0},
\]

where \(F_{\text{hard}}^{\beta}(s;a)\) denotes the probability of no eigenvalues in \((0, s)\) of the scaled hard state for \(\Sigma = I\); this relies on knowledge of the formula \(F_{\text{hard}}^{\beta}(s;a) \big|_{a=0} = e^{-\beta s/2}\ [36\]. The formula \((3.43)\) is to be expected for general \(a > -1\).

Rumanov \([86\] has found a Lax pair solution of \((3.40), (3.41)\) in the cases \(\beta = 2, 4\) analogous to that of Proposition 2.11 for the distribution of the soft edge critical state. These Lax pair solutions now involve particular Painlevé III transcendent, and are more complicated than for the soft edge critical state. Nonetheless, it is shown in \([86\] that in the limit \(a \to \infty\), and with suitable scaling of \(c\) and \(x\) that the results for the latter can be reclaimed. Previously Painlevé III transcendent evaluations were known for \(E_{\beta}^{\text{hard}}(s;a)\) in the case \(\beta = 2\ [93\] and \(\beta = 1, 4\ [37\].

**Remark 3.13.** For \(\beta = 2\) the hard edge critical state is a determinantal point process. The explicit form of the correlation kernel is given in \([23\] and \([38\) §7.2.4].

4. Rank 1 perturbations with two-dimensional support

4.1. An additive rank 1 anti-Hermitian perturbation for the GUE. Let \(A\) be an Hermitian matrix with fixed eigenvalues \(\{\mu_j\}_{j=1}^{N}\) ordered as in \((2.4)\), and let \(\hat{\psi}\) be an \(N \times 1\) column
vector chosen uniformly on the sphere in \( \mathbb{C}^N \). Form the projection matrix \( \hat{\mathbf{v}} \hat{\mathbf{v}}^\dagger \), and use it to create the additive rank 1 anti-Hermitian perturbation of \( A \),

\[
A + ia\hat{\mathbf{v}} \hat{\mathbf{v}}^\dagger, \quad a > 0.
\]

This model, in the case of \( A \) is random from the GOE, was first considered by Ullah \[95\] in the context of resonances in scattering processes. The working leading to (2.3) tells us that the eigenvalues of (4.1) are determined by the solution of the equation, in the variable \( z \),

\[
0 = 1 - ia \sum_{j=1}^{N} |v(j)|^2 \left( \frac{1}{z - \mu_j} \right).
\]

As noticed in \[60, 91\], for \( a \to \infty \) this implies that the \( N - 1 \) of the eigenvalues, which are in general complex, will approach the real axis and interlace with the sequence (2.4); in fact \( \text{Re} z \) is between \( \mu_N \) and \( \mu_1 \) for each solution \[28\]. The remaining eigenvalue in this limit can be read off from (4.2) by searching for a solution with \( |\mu| \) large. This gives, after averaging over the components of \( \mathbf{v} \),

\[
z \sim ia.
\]

The large \( z \) form of (4.2) implies a sum rule constraining the eigenvalues for fixed \( a \). Thus, with \( \{z_l\} \) the eigenvalues, partial fractions give that the rational function in (4.2) can be written as \( \prod_{l=1}^{N} (z - z_l) \) divided by \( \prod_{l=1}^{N} (z - \mu_l) \). Equating the coefficient of \( 1/z \) in the large \( z \) expansion of both expressions and taking imaginary parts shows

\[
\alpha = \sum_{l=1}^{N} \text{Im} z_l.
\]

Note the similarity between (4.3) and the \( a \to \infty \) form the outlier (1.8) for (1.6), as well as the corresponding interlacing in this limit. A further known general property of (4.2) is that the roots \( \{\mu_j\} \) all have positive imaginary parts \[76, 69, 28\]. One way to see this is, in keeping with the Schur decomposition discussed in the text including (3.10) below, to conjugate (4.1) by a unitary matrix, bring it to triangular form with diagonal entries equal to the eigenvalues. By inspection the diagonal entries on the RHS have positive imaginary part for \( a > 0 \).

Specialise now, as in the references \[91, 52\], to the case that \( A \) is random from the GUE, which in the physics application corresponds to a broken time reversal symmetry. Since such matrices are unchanged by conjugation with unitary matrices, the eigenvalues of the perturbed matrix (4.1) are the same as for

\[
A + ia\mathbf{1} \mathbf{1}^\dagger \quad \text{or} \quad A + i\text{diag} (1, 0, \ldots, 0).
\]
In the case that $\alpha$ itself is a random variable with distribution $\Gamma[N - 1, \alpha_0]$, which is equivalent to replacing $\hat{\psi}$ in (4.1) by a standard complex Gaussian vector, and setting $\alpha = \alpha_0$, the exact form of the joint eigenvalue PDF was first calculated in [38]. A different working was later given by Fyodorov and Khoruzhenko [52], allowing the eigenvalue PDF of (4.5) to be determined directly.

**Proposition 4.1.** The eigenvalue PDF of the random matrices (4.5) in the case that $A$ is a chosen from the GUE, with the eigenvalues denoted $\{z_l = x_l + iy_l\}$, is proportional to

$$
\prod_{l=1}^{N} e^{-(x_l^2 - y_l^2)} \prod_{1 \leq j < k \leq N} |z_k - z_j|^2 \frac{e^{-\alpha^2}}{\alpha^{N-1}} \delta(\alpha - \sum_{l=1}^{N} y_l),
$$

supported on $y_l > 0, (l = 1, \ldots, N)$.

Proof. Following [52] we adopt the viewpoint that the sum of an Hermitian matrix $H$ and anti-Hermitian matrix $i\Gamma$ is a complex matrix $J = H + i\Gamma$. With $H$ chosen from the GUE and $\Gamma$ fixed, the distribution on $J$ is proportional to

$$
e^{-\text{Tr}(\text{Re}J)^2} \delta(\Gamma - \text{Im} J), \quad \text{Im} J := \frac{1}{2i}(J - J^\dagger), \quad \text{Re} J := \frac{1}{2}(J + J^\dagger).$$

The next step is to write $J$ in terms of its Schur decomposition, $J = UTU^\dagger$. Here $U$ is a unitary matrix, unique up to the phase of each column, and $T$ is an upper triangular matrix with the elements on the diagonal the eigenvalues. The Jacobian for the change of variables is $\prod_{1 \leq j < k \leq N} |z_k - z_j|^2$; see e.g. [38], Eq. (15.9).

Regarding the matrix delta function in (4.7), we have the matrix integral form over Hermitian matrices $A$ [38, Eq. (3.27)]

$$
\delta(\Gamma - \text{Im} J) \propto \int e^{i\text{Tr}(A(\Gamma - \text{Im} J))} (dA) = \int e^{-i\text{Tr}(A(\text{Adiag}(y_1, \ldots, y_N)))} e^{-i\text{Im} \text{Tr}(A\tilde{T})} e^{i\text{Tr}(AU^\dagger U)} (dA),
$$

where $\tilde{T}$ refers to the strictly upper triangular portion of $T$. In obtaining the final expression, use has been made of the invariance of the distribution of $A$ upon the mapping $A \mapsto U^\dagger AU$. We also have

$$
e^{-\text{Tr}((J + J^\dagger)/2)^2} = e^{-\sum_{l=1}^{N} x_l^2} e^{-\text{Tr}(\tilde{T}\tilde{T})/2}.$$

Multiplying the RHS of (4.9) with the RHS of (4.8) and observing by completing the square that

$$
\int e^{-\text{Tr}(\tilde{T}\tilde{T})/2} e^{-i\text{Im} \text{Tr}(A\tilde{T})} (d\tilde{T}) \propto e^{-\text{Tr}(\tilde{A}\tilde{A})/2},
$$
where $\tilde{A}$ denotes the strictly upper triangular portion of $A$, we are left with

$$
\int e^{-i\text{Tr}(A\text{diag}(y_1,...,y_N))} e^{i\text{Tr}(AU^\dagger U)} e^{-\text{Tr}(\tilde{A}A^\dagger)/2} \, (dA).
$$

Integrating over the independent elements of $\tilde{A}$ by completing the square gives the term $e^{-2\text{Tr} \Gamma t}$ in analogy with (4.10). This leaves an integration over the diagonal entries of $A$. Writing $K = \text{diag}(a_{11},...,a_{NN})$, and integrating too over the invariant measure $[U^\dagger dU]$ of the unitary matrices $U$ in the Schur decomposition gives for the eigenvalue PDF, up to proportionality,

$$
(4.11) \quad e^{-\sum_{l=1}^N (x_l^2-y_l^2)} e^{-\text{Tr} \Gamma^2} \prod_{1 \leq j < k \leq N} |z_k - z_j|^2 \int (dK) e^{-i\text{Tr} K \text{diag}(y_1,...,y_N)} \int [U^\dagger dU] e^{i\text{Tr}(KU^\dagger U)}.
$$

Here we have also used the fact that $e^{-2\text{Tr} \Gamma t} = e^{\sum_{l=1}^N y_l^2} e^{-\text{Tr} \Gamma^2}$; cf. (4.9).

All the above working holds for general $\Gamma$. We now specialise to the rank 1 case as implied by (4.5). The matrix integral over $[U^\dagger dU]$ is then the rank 1 HCIZ integral and so evaluates to a single contour integral. Substituting in (4.11) then shows the integrals over the diagonal matrices $K$ factorise as a product of $N$ independent contour integrals, each of which can be evaluated by closing the contour and computing the residue. The final integral over $t$ is then a delta function, accounting for all terms in (4.6).

\[ \text{Rem 4.2.} \] A method of proof of (4.6), together with a $\beta$ generalisation has been given by Kohzhan [69], which is based on a finite $N$ tridiagonal formalism; see also [2].

The bulk scaling of the eigenvalues at the origin for the GUE is specified by $x_j \mapsto X_j/\sqrt{2N}$ to give an expected density of $1/\pi$; see e.g. [38] §7.1.1. This scaling can be carried out in (4.5) by multiplying the matrix sum by $\sqrt{2N}$, which in turn requires that in (4.6) the imaginary part of the eigenvalues be similarly scaled $y_j \mapsto Y_j/\sqrt{2N}$. If we further scale $\alpha \mapsto \sqrt{N/2}\alpha_0$, then the delta function constraint in (4.6) tells us that $\frac{1}{N} \sum_{j=1}^N Y_j = \alpha_0$, and thus on average each $Y_j$ is of order unity. It was shown in [72] that this scaled limit gives rise to a determinantal point process and the explicit form of the correlation kernel was computed.

\[ \text{Prop 4.3.} \] Consider the scalings of (4.5) and the corresponding eigenvalues as specified in the above paragraph. The correlations have the determinantal form (3.26) with correlation kernel

$$
(4.12) \quad K(Z_j, Z_k) = e^{-g(Y_j+Y_k)} \int_{-1}^{1} (g + s)e^{is(Z_j-Z_k)} \, ds,
$$

where $Z = X + iY$ and $g = (1/2)(\alpha_0 + 1/\alpha_0)$. Here the normalisation has been chosen so that $\int_0^\infty K(Z, Z) \, dY = 1$. 

Figure 4.1. Simulated histogram of the PDF for the scaled imaginary part of (4.1) with $\alpha_0 = 2$ and thus $g = 5/2$, plotted against (4.14). In the simulations the scaled version of (4.1) was sampled 5,000 times, and the imaginary part of the 8 eigenvalues with real part closest to the origin were recorded each time.

Remark 4.4. 1. For finite $N$ the delta function constraint in (4.6) prohibits a determinantal form for the correlations. The starting point of the calculation in [52] is to write the $k$-point correlation function in terms of a certain product of determinants averaged over the GUE. 2. The reproducing property of the kernel

$$
\int_{-\infty}^{\infty} dX_2 \int_0^\infty dY_2 K(Z_1, Z_2)K(Z_2, Z_3) = K(Z_1, Z_3),
$$

which is associated with perfect screening (see [38, §14.1]), is readily verified.
3. The normalised density profile in the $Y$ direction, $\rho_{(1)}(Y)$, is obtained by setting $Z_j = Z_k = Z$ in (4.12), which shows

$$
\rho_{(1)}(Y) = e^{-2gY} \left( g \frac{\sinh 2Y}{Y} - \frac{\partial}{\partial Y} \frac{\sinh 2Y}{2Y} \right).
$$

In Figure 4.1 this profile is compared against a simulation for a particular $g$. For the random matrices (4.5) with $A$ chosen from the GOE, upon the scaling as used in Proposition 4.3 an exact evaluation of $\rho_{(1)}(Y)$ is also known [90], while unlike when $A$ chosen from the GUE, the corresponding higher order correlations remain unknown.
4. Take the viewpoint that in (4.5) one realisation of a GUE matrix is chosen, and $\alpha \geq 0$ is a continuous parameter. From the discussion at the beginning of this section we know that for
α > 0 the eigenvalues have a positive imaginary part, although that as α → ∞ all but one eigenvalue — which can be considered as an outlier — returns to the real axis. With the GUE matrix scaled by dividing by √N/2 so the leading order support is (−2, 2), in the recent work [28] it has been proved that with high probability the outlier can distinguished for all α > 1 + N^{−1/3+ε}, ε > 0; see Figure 4.2 for an illustration. Moreover, it was emphasised that the exponent −1/3 is identical to that giving rise to the soft edge critical regime of the additive perturbed GOE and GUE, as displayed in the second scaling relation of (2.27).

4.2. A multiplicative sub-unitary rank 1 perturbation for the CUE. Closely related to the anti-Hermitian additive rank 1 perturbation for the GUE of the previous section is a particular multiplicative rank 1 perturbation of CUE matrices. Here CUE stands for the circular unitary ensemble, this being terminology introduced by Dyson [30], which is realised by the set of complex unitary matrices distributed according to Haar measure. Let $A = \text{diag}(a, 1, \ldots, 1)$, with |a| < 1, and consider the multiplicative rank 1 perturbation of $U \in$ CUE define by $UA$. The joint eigenvalue PDF was shown by Fyodorov [50] (see also the review [57]), to be proportional to

$$
(1 − |a|^2)^{1−N} \delta \left( |a|^2 − \prod_{l=1}^{N} |z_l|^2 \right) \prod_{1 \leq j < k \leq N} |z_j − z_k|^2, \tag{4.15}
$$

Figure 4.2. Trajectories of the eigenvalues of the matrix (4.5) with one realisation of $A$ as a 100 × 100 scaled GUE matrix, and $a$ varying from 0 to 1.5 in intervals of 1/60. The outlier is clearly visible.
and supported on \(|z_l| < 1\).

As for the PDF \((4.16)\), the delta function constraint prohibits a determinantal form of the correlations for finite \(N\). Nonetheless, in distinction to \((4.6)\), an exact finite \(N\) expression is still possible \([50]\). To present this result, define

\[
q_j(z_1, \ldots, z_k) = [s^j] \det \left[ \left( s + x \frac{d}{dx} \right) \frac{x^N - 1}{x - 1} \right]_{x = z_i, i,j = 1}'
\]

where \([s^j]\) denotes the coefficient of \(s^j\) in the expression that follows.

**Proposition 4.5.** Require that \(|z_l| < 1, (l = 1, \ldots, k)\) and \(\prod_{l=1}^k |z_l|^2 \geq |a|^2\) we have

\[
\rho_k(z_1, \ldots, z_k) = \frac{1}{\pi^k} (1 - |a|^2)^{1-N} \sum_{l=0}^k q_l(z_1, \ldots, z_k) \left( \frac{d}{dx} \right)^l \left[ \frac{1}{x} \left( 1 - \frac{|a|^2}{x} \right)^{N-1} \right]_{x = \prod_{l=1}^k |z_l|^2}.
\]

**Remark 4.6.** 1. Considering the case \(a = 0\). Then the only nonzero term in \((4.17)\) is \(l = 0\), implying

\[
\rho_k(z_1, \ldots, z_k) \bigg|_{a=0} = \frac{1}{\pi^k} \det \left[ \frac{d}{dx} \frac{x^N - 1}{x - 1} \right]_{x = z_i, i,j = 1}'
\]

On the other hand setting \(a = 0\) in the definition of \(A\) and forming \(UA\) shows the resulting matrix has the first column of \(A\) replaced by a column of zeros. Hence there is one zero eigenvalue, with the remaining eigenvalues the same as those of the \((N-1) \times (N-1)\) submatrix of \(U\) obtained by deleting the first row and the first column. For this ensemble the joint eigenvalue PDF was first derived in \([99]\) to be proportional to \(\prod_{1 \leq j < k \leq N-1} |z_j - z_k|^2\), supported on \(|z_l| < 1\). Without any delta function constraint, this corresponds to a determinantal point process, and the \(k\)-point correlation \((4.18)\) was obtained in \([99]\).

2. The density formula \(k = 1\) of \((4.17)\) is a special case of a formula for the eigenvalue density of \(U \sqrt{G}, G = \text{diag} (g_1, \ldots, g_N)\), each \(g_i \geq 0\), obtained in \([98]\).

3. In analogy with Remark 4.4, point 4, varying the parameter \(a\) in \(A\) from 1 to 0 with \(U\) a single sample gives rise to an eigenvalue process where all eigenvalues begin on the unit circle for \(a = 1\), and as \(a\) varies to 0 exactly one eigenvalue ends at \(z = 0\). However unlike the setting for the random matrices \((4.5)\), the remaining eigenvalues do not return to the unit circle, although in a qualitative sense most do remain close to the unit circle; see Figure 4.3 for an illustration.

The finite \(N\) result \((4.17)\) admits two distinct scaling limits. One is to expand the coordinates in the neighbourhood of the boundary of the unit circle. This can be done by writing \(z_j = (1 + (ix_j - y_j) / N + O(1/N^2))\). Now taking \(N \to \infty\) reduces \((4.17)\) to the form \((3.26)\) for a determinantal point process \([50]\). In fact this form is precisely \((4.12)\) as found in
relation to the bulk scaling limit of \((4.5)\), with the identification \(g = 2/(1 - |a|^2) - 1\). The second scaling limit scales the coupling \(a\) but leaves the coordinates in the disk unchanged \([43]\). Then the finite \(N\) structure \((4.17)\) is conserved, and thus the statistical state is not a determinantal point process.

**Proposition 4.7.** Consider \((4.17)\) with the substitution \((4.16)\). Scale the parameter \(a\) to depend on \(N\) according to \(a = 1/(\sqrt{\mu}N)\), and define

\[
Q_j(z_1, \ldots, z_k) = [s^j] \det \left[ (s + x \frac{d}{dx}) \frac{1}{1-x} \right]_{x = z_l, \bar{z}_l} \quad (4.19)
\]

(cf. \((4.16)\)). In the limit \(N \to \infty\) the general \(k\)-point correlation is specified by

\[
\rho_{(k)}(z_1, \ldots, z_k) = \frac{e^{1/|\mu|^2}}{\pi^k} \sum_{l=0}^{k} Q_l(z_1, \ldots, z_k) \left( \frac{d}{dx} \right)_l \left( \frac{1}{x} e^{-1/(|\mu|^2 x)} \right) \bigg|_{x = \Pi_{l=1}^k |z_l|^2} \quad (4.20)
\]

restricted to \(|z_j| < 1\) \((j = 1, \ldots, k)\).

**Remark 4.8.** Setting \(k = 1\) in \((4.17)\) and simplifying shows

\[
\rho_{(1)}(z) = \frac{1}{\pi} \frac{1}{(1 - |z|^2)^2} \exp \left( \frac{1}{|\mu|^2} \left( 1 - \frac{1}{|z|^2} \right) \right) \left( 1 + \frac{1}{|\mu|^2 |z|^4} (1 - |z|^2) \right) \quad (4.21)
\]

This exhibits an essential singularity as \(|z| \to 0^+\).
4.3. **Relationship to the limiting Kac polynomial.** In the theory of random polynomials the Kac polynomial refers to the $N$-th degree polynomial $\sum_{n=0}^{N} c_n z^n$, where each coefficient $c_n$ is an independent standard real Gaussian [65]. Subsequently the complex version, where the vector of coefficients is specified to have a vector complex Gaussian distribution with general covariance matrix, was considered by Hammersley [61]. Some years later special properties of this complex version, in the Kac setting where each coefficient is independent and identically distributed with mean zero, and extended to an analytic function by taking $N \to \infty$, where identified [80] [70].

First, in [80] it was shown in this setting the statistical state is a determinantal point process, with the general $k$-point correlation supported on $|z| < 1$ and correlation kernel

$$\rho_k(z_1, \ldots, z_k) = \frac{1}{\pi^k} \det \left[ \frac{1}{1 - z_i \bar{z}_j} \right]_{i,j=1}^k.$$  

Comparison of (4.22) with (4.18) shows that the latter, in the limit $N \to \infty$, coincides with the former. In fact in [70] it was proved directly that the characteristic polynomial for the eigenvalues of a random matrix from the CUE, with one row and one column deleted is for $N \to \infty$ given by $\sum_{n=0}^{\infty} c_n z^n$, where each coefficient $c_n$ is an independent standard complex Gaussian. There is a simple extension of this result in relation to the characteristic polynomial of $UA$ as considered in the previous subsection, with the parameter $A$ scaled as in Proposition 4.7 [44].

**Proposition 4.9.** Consider the random matrix $UA$ as specified in the previous subsection, and set $a = 1/(\mu \sqrt{N})$. In the limit $N \to \infty$ the eigenvalues of $UA$ are given by the zeros of the random Laurent series

$$\frac{1}{\mu} - \sum_{j=1}^{\infty} \frac{c_j}{z^j}$$

in the variable $\lambda = 1/z$, $|\lambda| < 1$, with each $c_j$ an independent standard complex Gaussian.

**Proof.** (Sketch) The first step is to manipulate the characteristic polynomial for $UA$ to conclude that the condition for an eigenvalue $\lambda$ of $UA$ can be written

$$0 = \det \left( I_N - \frac{\lambda}{a} (I_N - \lambda U^{\dagger}I_N')^{-1} U^{\dagger} e_N^{(1)} (e_N^{(1)})^T \right)$$

$$= 1 - \frac{\lambda}{a} (e_N^{(1)})^T (I_N - \lambda U^{\dagger}I_N')^{-1} U^{\dagger} e_N^{(1)}$$

$$= 1 - \frac{\lambda}{a} \sum_{k=0}^{\infty} \lambda^k (e_N^{(1)})^T (U^{\dagger}I_N')^k U^{\dagger} e_N^{(1)}.$$  

(4.24)
Here $I_N'$ denotes the identity matrix with the first diagonal replaced by 0, the second line is obtained by using the determinant identity (2.2), and the final line is obtained by using the geometric series to expand the matrix inverse, which is valid for $|\lambda| < 1$.

It was established in [70] that for $V \in U(N)$ chosen with Haar measure, in the limit $N \to \infty$

\begin{equation}
\sqrt{N}((e_N^{(1)})^T V e_N^{(1)}, (e_N^{(1)})^T V^2 e_N^{(1)}, (e_N^{(1)})^T V^3 e_N^{(1)}, \ldots) \overset{d}{=} (a_1, a_2, a_3, \ldots),
\end{equation}

where each $a_i$ is an independent standard complex Gaussian. The stated result now follows by noting that for $a = 1/(\mu \sqrt{N})$,

\begin{equation}
(e_N^{(1)})^T (U^\dagger I_N')^k U^\dagger e_N^{(1)} = (e_N^{(1)})^T (U^\dagger)^{k+1} e_N^{(1)} \left(1 + O(k/N)\right).
\end{equation}

\[\square\]

Remark 4.10. 1. It has been noted in the Introduction that for a real matrix $\tilde{X}$ satisfying the circular law, the real symmetric perturbation (4.4) results in a single outlier for $a > 1$. This conclusion holds true for $\tilde{X}$ complex and (4.4) replaced by $\tilde{X} + a \tilde{v} \tilde{v}^\dagger$, for $\tilde{v}$ a complex unit vector [92]. On the other hand, considering instead the particular non-Hermitian rank 1 perturbation

\begin{equation}
\tilde{X} + \alpha \mathbf{1}_N \tilde{v} \tilde{v}^\dagger,
\end{equation}

with $\tilde{v}$ chosen randomly, the situation is very different — an important structural feature here is that the rank 1 term averages to zero, in contrast to the case of this term equalling $a \tilde{v} \tilde{v}^\dagger$. Set $\alpha = \mu / \sqrt{N}$ and take the limit $N \to \infty$. It was proved in [92] that the eigenvalues of (4.27) are given by the zeros of (4.23) with respect to the variable $z$, $|z| > 1$. Thus according to Proposition 4.9 and recalling that $\lambda = 1/z$, this characterisation is identical to that for the eigenvalues of the scaled random matrix $UA$. Note from steps analogous to the derivation of (4.24) that this would follow from the characteristic equation for (4.27) if it could be established that (4.25) holds true with each $(e_N^{(1)})^T V/ e_N^{(1)}$ replaced by $(e_N^{(1)})^T \tilde{X} \tilde{v}$. This is precisely what is established in [92].

2. Consider the random matrix $RA$ with $R \in O(N)$ chosen with Haar measure and $A = \text{diag}(a, 1, \ldots, 1)$, $|a| < 1$. The proof of Proposition 4.9 can be modified to lead to the conclusion that with $a = 1/(\mu \sqrt{N})$, the eigenvalues are given by the zeros of (4.23) in the variable $\lambda = 1/z$, $|\lambda| < 1$, with each $c_j$ an independent standard real Gaussian. On the other hand, consider the random matrix $\tilde{X} + \alpha \mathbf{1}_N \tilde{v} \tilde{v}^\dagger$, where $\tilde{X}$ is a real matrix obeying the circular law and $\tilde{v}$ is a random real unit vector. For $\alpha = \mu / \sqrt{N}$ and $N \to \infty$ it is proved in [92] that this same limiting Laurent polynomial — which is equivalent to the original Kac random polynomial as defined at the beginning of this subsection — specifies the eigenvalue
distribution in the region $|z| > 1$.

3. The eigenvalues of the case $a = 0$ of the random matrix $RA$ — this corresponding to deleting one row and column of $R$ — are known to form a Pfaffian point process \[67, 39\]. This point process can be considered to consist of two species, the real eigenvalues, and the complex eigenvalues. Statistics associated with the number of real eigenvalues \[45\] have been shown recently to relate to the persistence exponent for two-dimensional diffusion with random initial conditions \[81\].

4.4. **Left and right eigenvector statistics.** From matrix theory we know that eigenvectors of a matrix $X$ form an orthonormal set iff $[X, X^\dagger] = 0$. If $X$ is random, this will not be the case unless $X$ is Hermitian. Instead, for non-normal matrices orthonormality only shows itself when considering both the eigenvectors $\{|R_j\rangle\}_{j=1}^N$ of $X$ and the eigenvectors $\{|L_j\rangle\}_{j=1}^N$ of $X^\dagger$. These are referred to as the right (R) and left (L) eigenvectors respectively, with the distinction conveniently indicated symbolically in the bra-ket notation. Thus the left and right eigenvectors can be chosen so that $\langle L_i | R_j \rangle = \delta_{ij}$, which in words says that they form a bi-orthogonal family. Note that with respect to this condition, $|R_j\rangle$ can be multiplied by the scalar $\alpha$ provided $|L_j\rangle$ is multiplied by $1/\bar{\alpha}$. Invariant under such scaling is the so-called overlaps $O_{mn} = \langle L_m|L_n\rangle\langle R_m|R_n\rangle$, with the diagonal overlaps $O_{nn} = ||L_n||^2||R_n||^2$ of particular importance for their role as squared eigenvalue condition numbers; see \[27\] and references therein. The overlaps $O_{nn}$ were studied in the context of the additive rank 1 anti-Hermitian GUE perturbation (4.1) by Fyodorov and Mehlig \[53\]. Very recent work of Fyodorov and Osman \[54, 55\] has advanced knowledge of the diagonal overlaps $O_{nn}$ from the evaluation of their limiting mean in the sense of the quantity $O(z)$ defined in (4.33) below, to the evaluation of their limiting distribution $\mathcal{P}(t; z)$ as specified by (4.35). Moreover, this quantity was also calculated for the GOE analogue of (4.1). In another recent development, the overlaps $O_{nm,n}$, together with certain generalisations referred to as $q$-overlaps, were studied for the multiplicative sub-unitary rank 1 CUE perturbation of subsection 4.2 in the case $a = 0$ by Dubach \[27\].

Here we will focus attention on results from \[53\]. First, following \[56\], consider the scalar

\[
(4.28) \quad s = s(E) := \frac{1 + iK}{1 - iK} = 1 + 2 \frac{iK}{1 - iK}, \quad K = \alpha v^\dagger (E\mathbb{I}_N - A)^{-1}v.
\]

Here $E$ is an in general complex parameter, $\alpha$ is a positive real scalar, $A$ is an $N \times N$ Hermitian matrix and $v$ is an $N \times 1$ complex column vector. Hence, with $\tilde{K}$ the $N \times N$
matrix $\tilde{K} = \alpha (E_\text{IN} - A)^{-1}v v^\dagger$, we can expand

$$\frac{iK}{1 - iK} = \sum_{p=0}^{\infty} (iK)^{p+1} = i\alpha v^\dagger (\sum_{p=0}^{\infty} (i\tilde{K})^p) (E_\text{IN} - A)^{-1}v$$

$$= i\alpha v^\dagger (E_\text{IN} - i\tilde{K})^{-1} (E_\text{IN} - A)^{-1}v = i\alpha v^\dagger (E_\text{IN} - A - i\alpha vv^\dagger)^{-1}v.$$

It follows from (4.29) that the poles of (4.28) as a function of $E$ occurs at the eigenvalues $\{z_j\}$ of $\mathcal{H} = A + i\alpha vv^\dagger$, which is the matrix (4.1) with $\mathcal{V}$ replaced by $t v$. Moreover, $s(E)$ has unit modulus for $E$ real and so has the rational function form

$$s(E) = \prod_{j=1}^{N} \frac{E - z_j}{E - \bar{z}_j}.$$ 

Making use of this allows a formula for the overlaps $O_{nm}$ of $\mathcal{H}$ to be computed [53, 57].

**Proposition 4.11.** Consider the non-Hermitian matrix $\mathcal{H}$ defined in the above paragraph. Choose the left and right eigenvectors to form an orthonormal set. The overlaps of these eigenvectors are given in terms of the eigenvalues by

$$O_{nm} = \frac{(z_n - z_m)(z_m - z_n)}{(z_n - z_m)^2} \prod_{k=1}^{N} \frac{z_n - z_k}{z_m - z_k} \prod_{k=1}^{N} \frac{z_m - z_k}{z_n - z_k}.$$

**Proof.** Let the matrix of eigenvectors of $\mathcal{H}$ be denoted $P$, so that $\mathcal{H} = P Z P^{-1}$, where $Z$ is the diagonal matrix of eigenvalues. The overlaps can be expressed in terms of $V$ according to

$$O_{nm} = (P^\dagger P)_{nm} (P^{-1}(P^{-1})^\dagger)_{nm},$$

where on the RHS the subscripts indicate the positions in the corresponding matrix. On the other hand, we observe

$$(P^\dagger (\mathcal{H} - \mathcal{H}^\dagger) P)_{nm} = (z_n - z_m)(P^\dagger P)_{nm}$$

$$(P^{-1}(\mathcal{H}^\dagger - \mathcal{H})(P^{-1})^\dagger)_{nm} = (z_m - z_n)(P^{-1}(P^{-1})^\dagger)_{nm}.$$ 

Hence

$$O_{nm} = \frac{4\alpha^2}{(z_n - z_m)(z_m - z_n)} (P^\dagger v v^\dagger P)_{mn} (P^{-1}v v^\dagger(P^{-1})^\dagger)_{mn}.$$ 

It follows from (4.28) and (4.29) that

$$s(E) = 1 + 2i\alpha v^\dagger P(E_\text{IN} - Z)^{-1}P^{-1}v.$$
Consider now the leading term for \( E \to z_n \), which effectively replaces \((E \mathbb{1}_N - z)^{-1}\) by 
\((e_n e_n^\dagger (E - z_n)^{-1}\), where \( e_n \) is the \( n \)-th standard basis vector in \( \mathbb{R}^N \). We then have
\[
s(E) \sim \frac{2i\alpha}{E - z_n} (v^\dagger P)_{n} (P^{-1} v)_{n}.
\]
Taking the complex conjugate transpose of both sides shows that for \( z \to z_m \)
\[
s(E) \sim \frac{2i\alpha}{E - z_m} (p^\dagger v)_{m} (p^\dagger v)_{m}.
\]
Multiplying together these latter two equations shows that for \( E_1 \to z_n \) and \( E_2 \to z_m \) we have
\[
s(E_1)s(E_2) \sim \frac{4\alpha^2}{(E_1 - z_n)(E_2 - z_m)} (p^\dagger vv^\dagger P)_{nm} (p^{-1} v^\dagger (P^{-1})^\dagger)_{nm}.
\]
Making use of (4.30) allows this asymptotic equality to be turned into an identity, with the LHS expressed as a product over the eigenvalues. Substituting the resulting formula in (4.32) gives (4.31).

Associated with the overlaps \( O_{mn} \) are ensemble averages
\[
(4.33) \quad O(z) = \left\langle \frac{1}{N} \sum_n O_{nn} \delta(z - z_n) \right\rangle, \quad O(z, z') = \left\langle \frac{1}{N} \sum_{m \neq n} O_{mn} \delta(z - z_m) \delta(z - z_n) \right\rangle.
\]
For the class of non-Hermitian rank 1 perturbations (4.5), with \( N \) fixed these are given by substituting (4.31) for \( O_{nn}, O_{mn} \) and integrating against the functional form (4.6) for the eigenvalue PDF. With the scalings as specified in the paragraph above Proposition 4.3, the large \( N \) forms of both the averages in (4.33) were computed by Fyodorov and Mehlig \[53\].

With \( Y = \text{Im} z \) we record the expression for the limiting form of \( O(z) \),
\[
(4.34) \quad O(Y) = e^{-4\alpha Y} \frac{d}{dY} \left( e^{2\alpha Y} \frac{\sinh 2Y}{2Y} \right);
\]
cf. (4.14) and note in particular the normalisation \( O(Y)|_{Y=0} = \rho_{(1)}(Y)|_{Y=0} \). Although we don’t present an example, analogous to Figure 4.1 this functional form for particular parameters can be compared against numerical simulations; see \[53\] Fig. 1. Thus from (4.31) we have
\[
O_{nn} = \prod_{k=1}^{N} \left| \frac{z_n - z_k}{z_n - z_k} \right|^2,
\]
which when calculated at the eigenvalues used to generate the histogram in Figure 4.1 and averaged within each bin gives rise to a scale factor which is to multiply the existing heights.
Generalising $O(z)$ in (4.33) is the distribution function

$$\mathcal{P}(t; z) := \left\langle \frac{1}{N} \sum_n \delta(O_{nn} - 1 - t) \delta(z - z_n) \right\rangle.$$  

As commented in the introductory paragraph to this subsection, the scaled limiting form of this quantity, $\mathcal{P}_Y^{(\beta)}(t)$ say, has recently been calculated for both the ensemble (4.1) (the case $\beta = 2$) and its GOE analogue (the case $\beta = 1$) [54]. In particular, it was shown

$$\mathcal{P}_Y^{(2)}(t) = \frac{16}{3^3} e^{-2gY} I_2 e^{-2gY(1+2/t)} I_0 \left( \frac{4Y}{t} \sqrt{(g^2 - 1)(1 + t)} \right),$$

where $I_0(z)$ is a modified Bessel function and $I_2$ is the differential operator acting on smooth functions $f(y)$ according to

$$I_2 f(Y) = \left( 1 + \left( \frac{\sinh 2Y}{2Y} \right)^2 + \frac{1}{2Y} \left( 1 - \frac{\sinh 4Y}{4Y} \right) \frac{d}{dy} + \frac{1}{4} \left( \left( \frac{\sinh 2Y}{2Y} \right)^2 - 1 \right) \frac{d^2}{dy^2} \right) Y^2 f(Y).$$

As emphasised in [54], a noteworthy feature of (4.36) is heavy tail decay, specifically like $1/t^3$ as $t \to \infty$. This implies that all the moments $\langle O_{nn}^k \rangle$ diverge for $k \geq 2$, as is known for the analogous quantity in the case of Ginibre type ensembles [20] [51] [26]. Moreover computing the moment for $k = 1$ reclaims (4.34).

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