Regularities and Exponential Ergodicity in Entropy for SDEs Driven by Distribution Dependent Noise

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Abstract
As two crucial tools characterizing regularity properties of stochastic systems, the log-Harnack inequality and Bismut formula have been intensively studied for distribution dependent (McKean-Vlasov) SDEs. However, due to technical difficulties, existing results mainly focus on the case with distribution free noise. In this paper, we introduce a noise decomposition argument to establish the log-Harnack inequality and Bismut formula for SDEs with distribution dependent noise, in both non-degenerate and degenerate situations. As application, the exponential ergodicity in entropy is investigated.

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1 Introduction

Let \( \mathcal{P}(\mathbb{R}^d) \) be the space of all probability measures on \( \mathbb{R}^d \) equipped with the weak topology. Consider the following distribution dependent SDE on \( \mathbb{R}^d \):

\[
\begin{equation}
    dX_t = b_t(X_t, \mathcal{L}_{X_t})dt + \sigma_t(X_t, \mathcal{L}_{X_t})dB_t, \quad t \in [0, T],
\end{equation}
\]

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where \( T > 0 \) is a fixed time, \( \mathcal{L}_{X_t} \) is the distribution of \( X_t \),

\[
b : [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d, \quad \sigma : [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d \otimes \mathbb{R}^d
\]

are measurable, and \( B_t \) is a \( d \)-dimensional Brownian motion on a complete filtration probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P}) \).

We investigate the regularity in initial distributions for solutions to (1.1). More precisely, for \( k > 1 \) let

\[
\mathcal{P}_k(\mathbb{R}^d) := \{ \mu \in \mathcal{P}(\mathbb{R}^d) : \|\mu\|_k := \mu(|\cdot|^k)^{\frac{1}{k}} < \infty \},
\]

which is a Polish space under the \( L^k \)-Wasserstein distance

\[
\mathbb{W}_k(\mu, \nu) = \inf_{\pi \in \mathcal{C}(\mu, \nu)} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^k \pi(dx, dy) \right)^{\frac{1}{k}}, \quad \mu, \nu \in \mathcal{P}_k(\mathbb{R}^d),
\]

where \( \mathcal{C}(\mu, \nu) \) is the set of all couplings of \( \mu \) and \( \nu \). When (1.1) is well-posed for distributions in \( \mathcal{P}_k(\mathbb{R}^d) \), i.e. for any \( \mathcal{F}_0 \)-measurable initial value \( X_0 \) with \( \mathcal{L}_{X_0} \in \mathcal{P}_k(\mathbb{R}^d) \) (correspondingly, any initial distribution \( \mu \in \mathcal{P}_k(\mathbb{R}^d) \)), the SDE (1.1) has a unique solution (correspondingly, a unique weak solution) with \( \mathcal{L}_{X_t} \in C([0, T], \mathcal{P}_k(\mathbb{R}^d)) \), we consider the regularity of the maps

\[
\mathcal{P}_k(\mathbb{R}^d) \ni \mu \mapsto P_t^* \mu := \mathcal{L}_{X_t} \text{ for } \mathcal{L}_{X_0} = \mu, \quad t \in (0, T].
\]

Since \( P_t^* \mu \) is uniquely determined by

\[
(1.2) \quad P_t f(\mu) := \int_{\mathbb{R}^d} f(P_t^* \mu), \quad f \in \mathcal{B}_b(\mathbb{R}^d),
\]

where \( \mathcal{B}_b(\mathbb{R}^d) \) is the space of bounded measurable functions on \( \mathbb{R}^d \), we study the regularity of functionals

\[
\mathcal{P}_k(\mathbb{R}^d) \ni \mu \mapsto P_t f(\mu), \quad t \in (0, T], f \in \mathcal{B}_b(\mathbb{R}^d).
\]

When the noise is distribution free, i.e. \( \sigma_t(x, \mu) = \sigma_t(x) \) does not depend on the distribution argument \( \mu \), the log-Harnack inequality

\[
(1.3) \quad P_t \log f(\mu) \leq \log P_t f(\nu) + \frac{c}{t} \mathbb{W}_2(\mu, \nu)^2, \quad 0 < f \in \mathcal{B}_b(\mathbb{R}^d), t \in (0, T], \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d),
\]

for some constant \( c > 0 \) has been established in \([14, 15, 19, 28]\) under different conditions, see also \([12, 13]\) for extensions to the infinite-dimensional case. A crucial application of this inequality is that it is equivalent to the entropy-cost estimate

\[
\text{Ent}(P_t^* \nu | P_t^* \mu) \leq \frac{c}{t} \mathbb{W}_2(\mu, \nu)^2, \quad t \in (0, T], \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d),
\]

2
where $\text{Ent}(\nu|\mu)$ is the relative entropy of $\nu$ with respect to $\mu$. With this estimate, the exponential ergodicity of $P_t^\nu$ in entropy is proved in [19] for a class of time-homogeneous distribution dependent SDEs. The study of (1.3) goes back to [24, 25] where the family of dimension-free Harnack inequalities is introduced, see [26] for various applications of this type inequalities. We emphasize that arguments used in the above mentioned references do not apply to distribution dependent noise. The only known log-Harnack inequality for distribution dependent noise is established in [2] for Ornstein-Ulenbeck type SDEs whose solutions are Gaussian processes and thus easy to manage.

Another crucial tool characterizing the regularity of $\mu \mapsto P_t^\mu$ is the following Bismut type formula for the intrinsic derivative $D^f$ in $\mu \in \mathcal{P}_k(\mathbb{R}^d)$ (see Definition 2.1 below):

$$D^f_P f(\mu) = \mathbb{E} \left[ f(X_t^\mu) \int_0^t \langle M^\mu_{s,\phi}, dB_s^\mu \rangle \right],$$  

$t \in (0, T)$, $f \in \mathcal{B}_b(\mathbb{R}^d)$, $\phi \in L^k(\mathbb{R}^d \rightarrow \mathbb{R}^d; \mu)$,

where $\int_0^t \langle M^\mu_{s,\phi}, dB_s^\mu \rangle$ is a martingale depending on $\mu$ and $\phi$ and $X_t^\mu$ solves (1.1) from initial distribution $\mu$.

Bismut formula was first established in [6] for the derivative formula of diffusion semigroups on Riemannian manifolds by using Malliavin calculus, which is also called Bismut-Elworthy-Li formula due to [9] where the martingale method is developed. When $\sigma_t(x, \mu) = \sigma_t(x)$ is distribution free, this type formulas have been established in [4, 5, 15, 18, 29] under different conditions.

In the distribution dependent setting, Bismut formula is studied in [8] for the decoupled SDEs with fixed distribution parameter, while (1.4) is derived in [3] for the Dirac measure $\mu = \delta_x, x \in \mathbb{R}^d$. An implicit Bismut formula is presented in [22, 23] where the noise is allowed to be distribution dependent. So far, an explicit Bismut formula is still open for distribution dependent noise. Nevertheless, intrinsic derivative estimates have been presented for a class of SDEs with distribution dependent noise, see [16] and references therein. This convinces us of establishing the log-Harnack inequality and explicit Bismut formula for SDEs with distribution dependent noise.

In this paper, we propose a noise decomposition argument which reduces the study of distribution dependent noise to distribution free noise. For simplicity, we only explain here the idea on establishing the log-Harnack inequality for the following distribution dependent SDE:

$$dX_t = b_t(X_t, \mathcal{L}X_t)dt + \sigma_t(\mathcal{L}X_t)dB_t, \ t \in [0, T].$$

Assume that $\sigma_t$ is bounded and Lipschitz continuous on $\mathcal{P}_2(\mathbb{R}^d)$, such that

$$\langle \sigma_t \sigma_t^* \rangle(\gamma) \geq 2\lambda^2 I_d, \ \gamma \in \mathcal{P}_2(\mathbb{R}^d)$$

holds for some constant $\lambda > 0$, where $I_d$ is the $d \times d$ identity matrix. We take

$$\bar{\sigma}_t(\gamma) := \sqrt{\langle \sigma_t \sigma_t^* \rangle(\gamma) - \lambda^2 I_d}.$$
Then $\tilde{\sigma}_t(\gamma) \geq \lambda I_d$, and [7, Lemma 3.3] implies that $\tilde{\sigma}_t(\gamma)$ is Lipschitz continuous in $\gamma \in \mathcal{P}_2(\mathbb{R}^d)$ as well. Moreover, for two independent $d$-dimensional Brownian motions $W_t$ and $\tilde{W}_t$,

$$dB_t := \sigma_t(\mathcal{L}_{X_t})^{-1}\{\lambda dW_t + \tilde{\sigma}_t(\mathcal{L}_{X_t})d\tilde{W}_t\}$$

is a $d$-dimensional Brownian motion, so that (1.5) is reduced to

(1.6) $dX_t = b_i(t, X_i^t)dt + \lambda dW_t + \tilde{\sigma}_t(\mathcal{L}_{X_t})d\tilde{W}_t, \quad t \in [0, T].$

Thus, by the well-posedness, (1.5) and (1.6) provide the same operator $P_t$. Now, consider the conditional probability $\mathbb{P}^W$ given $\tilde{W}$, under which $\int_0^t \tilde{\sigma}_s(\mathcal{L}_{X_s})d\tilde{W}_s$ is deterministic so that (1.6) becomes an SDE with constant noise $\lambda dW_t$, and hence its log-Harnack inequality follows from existing arguments developed for distribution free noise.

However, this noise decomposition argument is hard to extend to spatial-distribution dependent noise. So, in the following we only consider (1.5) or (1.6), rather than (1.1).

Closely related to the log-Harnack inequality, a very nice entropy estimate has been derived in [7] for two SDEs with different noise coefficients. Consider, for instance, the following SDEs on $\mathbb{R}^d$ for $i = 1, 2$:

$$dX^i_t = b_i(t, X^i_t)dt + \sqrt{a_i(t)}dB_t, \quad X^i_0 = x \in \mathbb{R}^d, \quad t \geq 0,$$

where $a_i(t)$ is positive definite, and for some constant $K > 1$,

$$|b_i(t, x) - b_i(t, y)| \leq K|x - y|, \quad K^{-1}I_d \leq a_i(t) \leq KI_d, \quad x, y \in \mathbb{R}^d, \quad t \geq 0.$$

Then [7, Theorem 1.1] gives the entropy estimate

$$\text{Ent}(\mathcal{L}_{X^2_t}|\mathcal{L}_{X^1_t}) \leq \frac{1}{2} \int_0^t ds \int_{\mathbb{R}^d} |a_1(s)^{-\frac{1}{2}}\Phi(s, y)|^2 \rho_2(s, y)dy,$$

$$\Phi(s, y) := (a_1(s) - a_2(s))\nabla \log \rho_2(s, y) + b_2(s, y) - b_1(s, y), \quad s > 0, y \in \mathbb{R}^d,$$

where $\rho_2(s, y) := \frac{\mathcal{L}_{X^2_t}(dy)}{dy}$ is the distribution density function of $X^2_s$. Since for elliptic diffusion processes

$$\int_{\mathbb{R}^d} |\nabla \log \rho_2(s, y)|^2 \rho_2(s, y)dy\]

behaves like $\frac{1}{s}$ for some constant $c > 0$ and small $s > 0$, to derive finite entropy upper bound from this estimate one may assume

(1.7) $\int_0^1 \frac{||a_1(s) - a_2(s)||^2}{s}ds < \infty,$

where $||\cdot||$ is the operator norm of matrices. To bound $\text{Ent}(P_t^*\nu|P_t^*\mu)$ for (1.5), we take

$$a_1(s) := (\sigma_s\sigma^*_s)(P^*_s\mu), \quad a_2(s) := (\sigma_s\sigma^*_s)(P^*_s\nu).$$
But (1.7) fails when \( \| (\sigma_s \sigma_s^*)(P_s^* \mu) - (\sigma_s \sigma_s^*)(P_s^* \nu) \| \) is uniformly positive for small \( s \).

The remainder of the paper is organized as follows. In Section 2 and Section 3, we establish the log-Harnack inequality and Bismut formula for the non-degenerate case and degenerate cases respectively. In Section 4 we apply the log-Harnack inequality to study the exponential ergodicity in entropy.

2 Non-degenerate case

In this part, we establish the log-Harnack inequality and Bismut formula for \( P_t f \) defined in (1.2), where \( P_t^* \mu := \mathcal{L}_{X_t^\mu} \) for \( X_t^\mu \) solving (1.6) with initial distribution \( \mu \).

2.1 Log-Harnack inequality

To establish the log-Harnack inequality, we make the following assumption.

(A) \( \lambda > 0 \) is a constant, and there exists \( 0 \leq K \in L^1([0, T]) \) such that

\[
|b_t(x, \mu) - b_t(y, \nu)|^2 + \| \tilde{\sigma}_t(\mu) - \tilde{\sigma}_t(\nu) \|^2 \leq K_t(|x - y|^2 + \mathbb{W}_2(\mu, \nu)^2),
\]
\[
|b_t(0, \delta_0)| + \| \tilde{\sigma}_t(\delta_0) \|^2 \leq K_t, \quad t \in [0, T], \ x, y \in \mathbb{R}^d, \ \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d).
\]

By [11, Theorems 2.1 and 3.3] or [28, Theorem 2.1], assumption (A) implies that the SDE (1.6) is well-posed for distributions in \( \mathcal{P}_2(\mathbb{R}^d) \), and there exists a constant \( c > 0 \) such that

\[
\mathbb{W}_2(P_t^* \nu, P_t^* \mu) \leq c \mathbb{W}_2(\nu, \mu), \ \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d), \ t \in [0, T].
\]

Theorem 2.1. Assume (A) and let \( P_t \) be defined in (1.2) for the SDE (1.6). Then there exists a constant \( c > 0 \) such that

\[
P_t \log f(\nu) \leq \log P_t f(\mu) + \frac{c}{t} \mathbb{W}_2(\mu, \nu)^2, \quad 0 < f \in \mathcal{B}_b(\mathbb{R}^d), \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d), \ t \in (0, T].
\]

Equivalently,

\[
\text{Ent}(P_t^* \nu | P_t^* \mu) \leq \frac{c}{t} \mathbb{W}_2(\mu, \nu)^2, \quad \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d), \ t \in (0, T].
\]

Proof. As explained in Introduction, we will use coupling by change of measure under the conditional expectation given \( \tilde{W} \), which will be enough for the proof of the log-Harnack inequality. But for the study of Bismut formula later on, we will use the conditional probability and the conditional expectation given both \( \tilde{W} \) and \( \mathcal{F}_0 \):

\[
P^{\tilde{W}, 0} := \mathbb{P}(\cdot | \tilde{W}, \mathcal{F}_0), \quad \mathbb{E}^{\tilde{W}, 0} := \mathbb{E}(\cdot | \tilde{W}, \mathcal{F}_0).
\]
(a) For any \( t \in [0, T], \mu \in \mathcal{P}_2(\mathbb{R}^d) \) and \( f \in \mathcal{B}(\mathbb{R}^d) \), let
\[
P_t^{\bar{W},0} f(X_0^\mu) := \mathbb{E}[\bar{W}, f(X_t^\mu)] = \mathbb{E}[f(X_t^\mu) | \bar{W}, \mathcal{F}_0],
\]
where \( X_t^\mu \) solves (1.6) with \( \mathcal{L}_{X_0^\mu} = \mu \). By (1.2),
\[
P_t f(\mu) = \mathbb{E}[P_t^{\bar{W},0} f(X_0^\mu)], \quad t \in [0, T], \mu \in \mathcal{P}_2(\mathbb{R}^d), f \in \mathcal{B}(\mathbb{R}^d).
\]
Next, let
\[
(2.3) \quad \xi_t^\mu := \int_0^t \bar{\sigma}_s(P_s^* \mu) d\bar{W}_s, \quad t \in [0, T], \mu \in \mathcal{P}_2(\mathbb{R}^d).
\]
By (A), BDG’s inequality and (2.1), we find constants \( C_1, C_2 > 0 \) such that
\[
(2.4) \quad \mathbb{E} \left[ \sup_{t \in [0,T]} |\xi_t^\mu - \xi_t^\nu|^2 \right] \leq C_1 \mathbb{W}_2(\mu, \nu)^2 \int_0^T K_s ds \leq C_2 \mathbb{W}_2(\mu, \nu)^2, \quad \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d).
\]
(b) For fixed \( \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d) \), we take \( \mathcal{F}_0 \)-measurable \( X_0^\mu \) and \( X_0^\nu \) such that
\[
(2.5) \quad \mathcal{L}_{X_0^\mu} = \mu, \quad \mathcal{L}_{X_0^\nu} = \nu, \quad \mathbb{E} [|X_0^\mu - X_0^\nu|^2] = \mathbb{W}_2(\mu, \nu)^2.
\]
Since \( X_t^\mu \) solves (1.6) with \( \mathcal{L}_{X_0^\mu} = \mu \), we have \( \mathcal{L}_{X_t^\mu} = P_t^* \mu \) and the SDE becomes
\[
(2.6) \quad dX_t^\mu = b_t(X_t^\mu, P_t^* \mu) dt + \lambda dW_t + \tilde{\sigma}_t(P_t^* \mu) d\tilde{W}_t, \quad t \in [0, T].
\]
For fixed \( t_0 \in (0, T] \), consider the following SDE:
\[
(2.7) \quad dY_t = \left\{ b_t(X_t^\mu, P_t^* \mu) + \frac{1}{t_0} [\xi_t^\mu - \xi_t^\nu + X_t^\mu - X_t^\nu] \right\} dt + \lambda dW_t + \tilde{\sigma}_t(P_t^* \nu) d\tilde{W}_t,
\]
\[ \quad t \in [0, t_0], \quad Y_0 = X_0^\nu. \]
By (2.3), (2.6) and (2.7), we obtain
\[
(2.8) \quad Y_t - X_t^\mu = \frac{t_0 - t}{t_0} (X_{t_0}^\nu - X_0^\mu) + \frac{t}{t_0} (\xi_t^\mu - \xi_{t_0}^\nu) + \xi_t^\nu - \xi_t^\mu, \quad t \in [0, t_0].
\]
To formulate \( P_{t_0} f(\nu) \) using \( Y_{t_0} \), we make Girsanov’s transform as follows. Let
\[
(2.9) \quad \eta_t := b_t(Y_t, P_t^* \nu) - b_t(X_t^\mu, P_t^* \mu) + \frac{1}{t_0} [\xi_t^\nu - \xi_t^\mu + X_t^\nu - X_t^\mu], \quad t \in [0, t_0].
\]
By (A) and (2.1), we find a constant \( c_1 > 0 \) such that
\[
|\eta_t|^2 \leq c_1 K_t (\mathbb{W}_2(\mu, \nu)^2 + |\xi_t^\nu - \xi_t^\mu|^2)
+ c_1 \left( \frac{t^2 K_t + 1}{t_0^2} |\xi_{t_0}^\nu - \xi_t^\mu|^2 + \frac{1}{t_0^2} |X_{t_0}^\nu - X_0^\mu|^2 \right), \quad t \in [0, t_0].
\]
Since \( \int_0^T K_t dt < \infty \), we find a constant \( c_2 > 0 \) uniform in \( t_0 \in (0, T] \), such that
\[
\frac{1}{2 \lambda^2} \int_0^{t_0} |\eta_t|^2 dt \leq c_2 \mathbb{W}_2(\mu, \nu)^2 + \frac{c_2}{t_0} \left( |X_0^\mu - X_0^\nu|^2 + \sup_{t \in [0, t_0]} |\xi_t^\mu - \xi_t^\nu|^2 \right).
\]
Let \( dQ^{\tilde{W}, 0} := R^{\tilde{W}, 0} dP^{\tilde{W}, 0} \), where
\[
R^{\tilde{W}, 0} := e^{\int_0^{t_0} \frac{1}{2} \eta_s dW_s - \frac{1}{2} \int_0^{t_0} |\frac{1}{2} \eta_s|^2 ds}.
\]
By Girsanov’s theorem, under the weighted conditional probability \( Q^{\tilde{W}, 0} \),
\[
\tilde{W}_t := W_t - \int_0^t \frac{1}{\lambda} \eta_s ds, \quad t \in [0, t_0]
\]
is a \( d \)-dimensional Brownian motion. By (2.7), \( \tilde{Y}_t := Y_t - \xi_t^\nu \) solves the SDE
\[
d\tilde{Y}_t = b_t(\tilde{Y}_t + \xi_t^\nu, P_t^\nu) dt + \lambda d\tilde{W}_t, \quad t \in [0, t_0], \tilde{Y}_0 = X_0^\nu.
\]
On the other hand, let \( X_t^\nu \) solve (1.6) with initial value \( X_0^\nu \). Then
\[
\hat{X}_t^\nu := X_t^\nu - \xi_t^\nu, \quad t \in [0, t_0]
\]
solves the same SDE as \( \tilde{Y}_t \) for \( W \) replacing \( \tilde{W} \). Then the weak uniqueness of this equation ensured by (A) implies
\[
\mathcal{L}_{\tilde{Y}_0 | Q^{\tilde{W}, 0}} = \mathcal{L}_{\hat{X}_0^\nu | P^{\tilde{W}, 0}},
\]
where \( \mathcal{L}_{\tilde{Y}_0 | Q^{\tilde{W}, 0}} \) is the law of \( \tilde{Y}_0 \) under \( Q^{\tilde{W}, 0} \), while \( \mathcal{L}_{\hat{X}_0^\nu | P^{\tilde{W}, 0}} \) is the law of \( \hat{X}_0^\nu \) under \( P^{\tilde{W}, 0} \).
Since \( \xi_t^\nu \) is deterministic given \( \tilde{W} \), it follows that
\[
\mathcal{L}_{\tilde{Y}_0 | Q^{\tilde{W}, 0}} = \mathcal{L}_{\tilde{Y}_0 + \xi_0^\nu | Q^{\tilde{W}, 0}} = \mathcal{L}_{\hat{X}_0^\nu + \xi_0^\nu | P^{\tilde{W}, 0}} = \mathcal{L}_{X_0^\nu | P^{\tilde{W}, 0}}.
\]
Combining this with \( X_0^\mu = Y_0 \) due to (2.8), we obtain
\[
(2.12) \quad P_{t_0}^{\tilde{W}, 0} f(X_0^\nu) := \mathbb{E}^{\tilde{W}, 0}[f(X_0^\nu)] = \mathbb{E}_{Q^{\tilde{W}, 0}}[f(Y_0)] = \mathbb{E}^{\tilde{W}, 0}[R^{\tilde{W}, 0} f(X_0^\mu)], \quad f \in \mathcal{B}_b(\mathbb{R}^d).
\]
By Young’s inequality [1, Lemma 2.4], we derive
\[
P_{t_0}^{\tilde{W}, 0} \log f(X_0^\nu) := \mathbb{E}^{\tilde{W}, 0}[\log f(X_0^\nu)] = \mathbb{E}_{Q^{\tilde{W}, 0}}[\log f(Y_0)]
\]
\[
= \mathbb{E}^{\tilde{W}, 0}[R^{\tilde{W}, 0} \log f(X_0^\mu)] \leq \log \mathbb{E}^{\tilde{W}, 0}[f(X_0^\mu)] + \mathbb{E}^{\tilde{W}, 0}[R^{\tilde{W}, 0} \log R^{\tilde{W}, 0}]
\]
\[
= \log P_{t_0}^{\tilde{W}, 0} f(X_0^\mu) + \frac{1}{2} \int_0^{t_0} \frac{1}{\lambda^2} \mathbb{E}_{Q^{\tilde{W}, 0}}[|\eta_t|^2] dt, \quad 0 < f \in \mathcal{B}_b(\mathbb{R}^d).
\]
This together with (2.10) gives
\[(2.13) \quad P_{t_0}^W \log f(X_t^\nu) \leq \log P_{t_0}^W f(X_0^\nu) + c_2 \mathbb{W}_2(\mu, \nu)^2 + \frac{c_2}{t_0} \left( |X_0^\nu - X_t^\nu|^2 + \sup_{t \in [0,t_0]} |\xi_t^\nu - \xi_t^\nu|^2 \right).\]

Taking expectation for both sides, by (2.2), (2.4), (2.5) and Jensen’s inequality, we find a constant \(c > 0\) such that
\[
P_{t_0} \log f(\nu) = \mathbb{E}[P_{t_0}^W \log f(X_t^\nu)] \leq \mathbb{E}[\log P_{t_0}^W f(X_0^\nu)] + \frac{c}{t_0} \mathbb{W}_2(\mu, \nu)^2 \leq \log P_{t_0} f(\mu) + \frac{c}{t_0} \mathbb{W}_2(\mu, \nu)^2, \quad t_0 \in (0, T], \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d).
\]

\[
\square
\]

### 2.2 Bismut formula

We aim to establish the Bismut type formula (1.4) for the intrinsic derivative of \(P_t f\). To this end, we first recall the definition of intrinsic derivative, see [20] for historical remarks on this derivative and links to other derivatives for functions of measures.

**Definition 2.1.** Let \(k \in (1, \infty)\).

1. A continuous function \(f\) on \(\mathcal{P}_k(\mathbb{R}^d)\) is called intrinsically differentiable, if for any \(\mu \in \mathcal{P}_k(\mathbb{R}^d)\),

\[
T_{\mu,k}(\mathbb{R}^d) := L^k(\mathbb{R}^d \to \mathbb{R}; \mu) \ni \phi \mapsto D^I f(\mu) := \lim_{\varepsilon \downarrow 0} \frac{f(\mu \circ (id + \varepsilon \phi)^{-1}) - f(\mu)}{\varepsilon} \in \mathbb{R}
\]
is a well defined bounded linear operator. In this case, the norm of the intrinsic derivative \(D^I f(\mu)\) is given by

\[
\|D^I f(\mu)\|_{L^k(\mu)} := \sup_{\|\phi\|_{L^k(\mu)} \leq 1} |D^I f(\mu)|.
\]

2. \(f\) is called \(L\)-differentiable on \(\mathcal{P}_k(\mathbb{R}^d)\), if it is intrinsically differentiable and

\[
\lim_{\|\phi\|_{T_{\mu,k}(\mathbb{R}^d)} \downarrow 0} \frac{|f(\mu \circ (id + \phi)^{-1}) - f(\mu) - D^I f(\mu)|}{\|\phi\|_{T_{\mu,k}(\mathbb{R}^d)}} = 0, \quad \mu \in \mathcal{P}_k(\mathbb{R}^d).
\]

We denote \(f \in C^1(\mathcal{P}_k(\mathbb{R}^d))\), if it is \(L\)-differentiable such that \(D^I f(\mu)(x)\) has a jointly continuous version in \((x, \mu) \in \mathbb{R}^d \times \mathcal{P}_k(\mathbb{R}^d)\).

3. We denote \(g \in C^{1,1}(\mathbb{R}^d \times \mathcal{P}_k(\mathbb{R}^d))\), if \(g : \mathbb{R}^d \times \mathcal{P}_k(\mathbb{R}^d) \to \mathbb{R}\) is \(C^1\) in \(x \in \mathbb{R}^d\) and \(\mu \in \mathcal{P}_k(\mathbb{R}^d)\) respectively, such that

\[
\nabla g(x, \mu) := \nabla \{g(\cdot, \mu)\}(x), \quad D^I g(x, \mu)(y) := D^I \{g(x, \cdot)\}(\mu)(y)
\]
are jointly continuous in \((x, y, \mu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}_k(\mathbb{R}^d)\).
In this part, we consider (1.6) with coefficients
\[ \tilde{\sigma} : [0, T] \times \mathcal{P}_k(\mathbb{R}^d) \to \mathbb{R}^d \otimes \mathbb{R}^d, \quad b : [0, T] \times \mathbb{R}^d \times \mathcal{P}_k(\mathbb{R}^d) \to \mathbb{R}^d \]
satisfying the following assumption.

(B) \( \lambda > 0 \) and \( k \in (1, \infty) \) are constants, denote \( k^* := \frac{k}{k-1} \). For any \( t \in [0, T], \) \( b_t \in C_{\text{Lip}}(\mathbb{R}^d \times \mathcal{P}_k(\mathbb{R}^d)), \) \( \tilde{\sigma}_t \in C^1(\mathcal{P}_k(\mathbb{R}^d)), \) and there exists \( 0 \leq K \in L^1([0, T]) \) such that
\[
|D^I b_t(x, \cdot)(\mu)(y)| + \|D^I \tilde{\sigma}_t(\mu)(y)\| \leq \sqrt{K_t} (1 + |y|^{k-1}),
\]
\[
|b_t(0, \delta_0)| + |\nabla b_t(\cdot, \mu)(x)| \leq \sqrt{K_t}, \quad (t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_k(\mathbb{R}^d), \quad y \in \mathbb{R}^d.
\]

By [29, Lemma 3.1], (B) implies (A) for \((\mathcal{P}_k(\mathbb{R}^d), \mathcal{W}_k)\) replacing \((\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_2)\). So, according to [11, Theorem 3.3], the SDE (1.6) is well-posed for distributions in \( \mathcal{P}_k(\mathbb{R}^d) \), and there exists a constant \( c > 0 \) such that
\[
(2.14) \quad \mathcal{W}_k(P_t^\mu \mu, P_t^\nu \nu) \leq c \mathcal{W}_k(\mu, \nu), \quad \mu, \nu \in \mathcal{P}_k(\mathbb{R}^d), t \in [0, T].
\]

By this estimate and (A) for \((\mathcal{P}_k(\mathbb{R}^d), \mathcal{W}_k)\) replacing \((\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_2)\), the argument leading to (2.13) yields that there exists a constant \( c > 0 \) such that for any \( t \in (0, T), 0 < f \in \mathcal{B}_b(\mathbb{R}^d), \)
\[
(2.15) \quad P_t^0 \log f(X_0^\mu) \leq \log P_t^0 f(X_0^\mu) + c \mathcal{W}_k(\mu, \nu)^2 + \frac{c}{t} \left( |X_0^\mu - X_0^\nu|^2 + \sup_{s \in [0,t]} |\xi_s^\mu - \xi_s^\nu|^2 \right).
\]

To calculate \( D^I \sigma_t f(\mu) \) for \( \mu \in \mathcal{P}_k(\mathbb{R}^d) \) and \( \phi \in T_{\mu,k}(\mathbb{R}^d), \) let \( X_t^\mu \) be \( \mathcal{F}_0 \)-measurable such that \( \mathcal{L} X_0^\mu = \mu. \) Then
\[
\mathcal{L} X_0^\mu + \varepsilon \phi(X_0^\mu) = \mu^\varepsilon := \mu \circ (id + \varepsilon \phi)^{-1}, \quad \varepsilon \in [0, 1].
\]

For any \( \varepsilon \in [0, 1], \) let \( X_t^{\mu^\varepsilon} \) solve (1.6) with \( X_0^{\mu^\varepsilon} = X_0^\mu + \varepsilon \phi(X_0^\mu), \) i.e.
\[
dX_t^{\mu^\varepsilon} = b_t(X_t^{\mu^\varepsilon}, P_t^{* \mu^\varepsilon}) dt + \lambda dW_t + \tilde{\sigma}_t(P_t^{* \mu^\varepsilon}) d\tilde{W}_t,
\]
\[
X_t^{\mu^\varepsilon} = X_0^\mu + \varepsilon \phi(X_0^\mu), t \in [0, T], \varepsilon \in [0, 1].
\]

Consider the spatial derivative of \( X_t^\mu \) along \( \phi: \)
\[
\nabla \phi X_t^\mu := \lim_{\varepsilon \downarrow 0} \frac{X_t^{\mu^\varepsilon} - X_t^\mu}{\varepsilon}, \quad t \in [0, T], \phi \in T_{\mu,k}(\mathbb{R}^d).
\]

For any \( 0 \leq s < t \leq T, \) define
\[
N^{\mu,\phi}_{s,t} := \frac{t - s}{t} \phi(X_0^\mu) + \int_0^s \left\langle \mathbb{E} \left[ (D^I \tilde{\sigma}_r(P_r^\mu)(X_r^\mu), \nabla \phi X_r^\mu) \right] , d\tilde{W}_r \right\rangle.
\]
Proof. The first assertion follows from \cite[Lemma 5.2]{4}. By the first assertion, let the definition of $\mathcal{N}$, \Theoremref{2.1} up to (2.12) still applies. For fixed $t$, \Theoremref{2.2}.

The main result in this part is the following.

\textbf{Theorem 2.2.} Assume (B).

(1) For any $\mu \in \mathcal{P}_k(\mathbb{R}^d)$ and $\phi \in T_{\mu,k}(\mathbb{R}^d)$, $(\nabla \phi X_t^\mu)$ exists in $L^k(\Omega \to C([0,T], \mathbb{R}^d), \mathbb{P})$ such that for some constant $c > 0$,

\[ \mathbb{E}\left[ \sup_{t \in [0,T]} |\nabla \phi X_t^\mu|^k \right] \leq c ||\phi||_L^k(\mu), \quad \mu \in \mathcal{P}_k(\mathbb{R}^d), \phi \in T_{\mu,k}(\mathbb{R}^d). \]

(2) For any $f \in \mathcal{B}_b(\mathbb{R}^d)$, $t \in (0,T]$, $\mu \in \mathcal{P}_k(\mathbb{R}^d)$ and $\phi \in T_{\mu,k}(\mathbb{R}^d)$, $D_t^f P_t f(\mu)$ exists and satisfies

\[ D_t^f P_t f(\mu) = \frac{1}{\mathcal{X}} \mathbb{E}\left[ f(X_t^\mu) \int_0^t \left\langle \nabla_{N_{\mu,t}} b_s(\cdot), P_s^\phi \right\rangle (X_t^\mu) + M_{s,t}^{\mu,\phi}, \, dW_s \right]. \]

Consequently, $P_t f$ is intrinsically differentiable and for some constant $c > 0$,

\[ ||D_t^f P_t f(\mu)||_{L^{k*}(\mu)} \leq \frac{c}{\sqrt{t}} (P_t |f|^{k*}(\mu))^\frac{1}{k*}, \]

\[ \quad f \in \mathcal{B}_b(\mathbb{R}^d), \mu \in \mathcal{P}_k(\mathbb{R}^d), t \in (0,T]. \]

\textbf{Proof.} The first assertion follows from \cite[Lemma 5.2]{4}. By the first assertion, (B) and the definition of $(N_{\mu,t}^{\mu,\phi}, M_{s,t}^{\mu,\phi})$, we deduce (2.17) from (2.16). So, it remains to prove (2.16).

(a) Since (B) implies (A) for $\mathcal{P}_k(\mathbb{R}^d)$ replacing $\mathcal{P}_2(\mathbb{R}^d)$, the argument in the proof of \Theoremref{2.1} up to (2.12) still applies. For fixed $t_0 \in (0,T]$, $\mu \in \mathcal{P}_k(\mathbb{R}^d)$ and $\phi \in T_{\mu,k}(\mathbb{R}^d)$, let $X_0^\mu$ solve (2.6). Next, for any $\varepsilon \in (0,1]$, let $Y_t^\varepsilon$ solve (2.7) for

\[ \nu = \mu^\varepsilon, \quad Y_0 = Y_0^\varepsilon := X_0^\mu + \varepsilon \phi(X_0^\mu). \]

Then (2.8) with $(Y_t, \nu) = (Y_t^\varepsilon, \mu^\varepsilon)$ becomes

\[ Y_t^\varepsilon - X_t^\mu = \frac{t_0 - t}{t_0} \varepsilon \phi(X_0^\mu) + \frac{t}{t_0} (\xi_0^\mu - \xi_0^\varepsilon) + \xi_0^t - \xi_t^\mu, \quad t \in [0,t_0]. \]

Let

\[ H_t := \int_0^t \left\langle \mathbb{E}[\langle D_t^f \tilde{\sigma}(P_s^\phi \mu)(X_s^\mu), \nabla \phi X_s^\mu \rangle], \, d\tilde{W}_s \right\rangle, \quad t \in [0,T]. \]
By (B) and (2.14), we obtain

\[(2.19) \quad \|\bar{s}_s(P^\varepsilon_s\mu_s) - \bar{s}_s(P^\varepsilon_s\mu_s)\|^2 \leq \varepsilon^2 \varepsilon^2 K_s\|\phi\|^2_{L^k(\mu)}, \quad \varepsilon \in [0, 1], s \in [0, T].\]

So, by (B), the chain rule in [4, Theorem 2.1(1)], (2.3), BDG’s inequality and the dominated convergence theorem, we obtain

\[(2.20) \quad \lim_{\varepsilon \to 0} \mathbb{E} \left[ \sup_{t \in [0, T]} \left| \frac{\xi^\varepsilon_t - \xi^\mu_t}{\varepsilon} - H_t \right|^2 \right] = 0.\]

Let \((\eta^\varepsilon_t, R^\varepsilon_t) = (\eta_t, R^\varepsilon_{\tilde{W},0})\) be defined in (2.9) and (2.11) for \((Y_t, \nu) = (Y^\varepsilon_t, \mu^\varepsilon).\) By (B) and (2.18), we find a constant \(\kappa > 0\) such that

\[\left| \frac{\eta^\varepsilon_t}{\varepsilon} \right|^2 \leq \kappa K_s \left( \|\phi\|^2_{L^k(\mu)} + \|\phi(X^\mu_0)\|^2 + \sup_{t \in [0, t_0]} \left| \frac{\xi^\varepsilon_t - \xi^\mu_t}{\varepsilon} \right|^2 \right) =: \Lambda_s,\]

\[\lim_{\varepsilon \to 0} \frac{\eta^\varepsilon_t}{\varepsilon} = \nabla_{N^\mu_s,0} b_s(\cdot, P^\varepsilon_s\mu_s)(X^\mu_s) + M^\mu_s,0, \quad s \in [0, t_0].\]

Since \(\Lambda_s\) is deterministic given \(\tilde{W}\) and \(\mathcal{F}_0\), this together with (2.12) and the dominated convergence theorem yields

\[(2.21) \quad \lim_{\varepsilon \to 0} \frac{P^\varepsilon_{t_0} f(X^\varepsilon_0) - P^\varepsilon_{t_0} f(X^\mu_0)}{\varepsilon} = \lim_{\varepsilon \to 0} \mathbb{E}^\varepsilon_{\tilde{W},0} \left[ f(X^\mu_{t_0}) \frac{R^\varepsilon - 1}{\varepsilon} \right]
= \frac{1}{\lambda} \mathbb{E}^\varepsilon_{\tilde{W},0} \left[ f(X^\mu_{t_0}) \int_0^{t_0} \left\langle \nabla_{N^\mu_s,0} b_s(\cdot, P^\varepsilon_s\mu)(X^\mu_s) + M^\mu_s,0, dW_s \right\rangle \right].\]

(b) Let \(\mathcal{L}_{\xi|\tilde{W},0}\) be the conditional distribution of a random variable \(\xi\) under \(\mathbb{P}_{\tilde{W},0}\). By Pinsker’s inequality and (2.15), we have

\[\sup_{\|f\|_{\infty} \leq 1} \left| P^\varepsilon_{t_0} f(X^\varepsilon_0) - P^\varepsilon_{t_0} f(X^\mu_0) \right|^2 \leq 2 \text{Ent} \left( \mathcal{L}_{X^\mu_{t_0}|\tilde{W},0} \right) \mathcal{L}_{X^\mu_{t_0}|\tilde{W},0} \]

\[\leq c \mathbb{W}_k(\mu^\varepsilon, \mu)^2 + \frac{c}{t_0} \left( \varepsilon^2 \|\phi(X^\mu_0)\|^2 + \sup_{t \in [0, t_0]} \left| \frac{\xi^\varepsilon_t - \xi^\mu_t}{\varepsilon} \right| \right)^2.\]

This together with \(\mathbb{W}_k(\mu^\varepsilon, \mu) \leq \varepsilon \|\phi\|_{L^k(\mu)}\) implies that for some constant \(c(t_0) > 0\),

\[\frac{P^\varepsilon_{t_0} f(X^\varepsilon_0) - P^\varepsilon_{t_0} f(X^\mu_0)}{\varepsilon} \]

\[\leq \|f\|_{\infty} c(t_0) \left( \|\phi\|_{L^k(\mu)} + \|\phi(X^\mu_0)\| + \sup_{t \in [0, t_0]} \left| \frac{\xi^\varepsilon_t - \xi^\mu_t}{\varepsilon} \right| \right), \quad \varepsilon \in (0, 1].\]
Combining this with (2.4) and (2.19), we may apply the dominated convergence theorem to (2.21) to derive

\[ D^I_t \varphi_{P_0} (\mu) := \lim_{\varepsilon \downarrow 0} E \left[ \frac{P^\varepsilon_{t_0} W, f(X^\mu_{t_0}) - P^\varepsilon_{t_0} W, f(X^\mu_{t_0})}{\varepsilon} \right] = E \left[ \lim_{\varepsilon \downarrow 0} \frac{P^\varepsilon_{t_0} W, f(X^\mu_{t_0}) - P^\varepsilon_{t_0} W, f(X^\mu_{t_0})}{\varepsilon} \right] = E \left[ \lim_{\varepsilon \downarrow 0} \frac{P^\varepsilon_{t_0} W, f(X^\mu_{t_0}) - P^\varepsilon_{t_0} W, f(X^\mu_{t_0})}{\varepsilon} \right] = \lambda E \left[ \int_0^T \nabla \mathcal{L}^\mu_{s,t_0} (\cdot, P^*_{s_0} \mu) (X^\mu_{s,t_0}) + M^\mu_{s,t_0} (\cdot, dW_s) \right]. \]

3 Degenerate case

Consider the following distribution dependent stochastic Hamiltonian system for \( X_t = (X_t^{(1)}, X_t^{(2)}) \in \mathbb{R}^{m+d} \):

\[
\begin{cases}
\mathrm{d}X_t^{(1)} &= \{ AX_t^{(1)} + MX_t^{(2)} \} \mathrm{d}t, \\
\mathrm{d}X_t^{(2)} &= b_t(X_t, \mathcal{L}X_t) \mathrm{d}t + \sigma_t(\mathcal{L}X_t) \mathrm{d}B_t, \quad t \in [0, T],
\end{cases}
\]

where \( B = (B_t)_{t \in [0,T]} \) is a \( d \)-dimensional standard Brownian motion, \( A \) is an \( m \times m \) and \( M \) is an \( m \times d \) matrix, and

\[
\sigma : [0, T] \times \mathcal{P}(\mathbb{R}^{m+d}) \to \mathbb{R}^d \otimes \mathbb{R}^d, \quad b : [0, T] \times \mathbb{R}^{m+d} \times \mathcal{P}(\mathbb{R}^{m+d}) \to \mathbb{R}^d
\]

are measurable, where \( \mathcal{P}(\mathbb{R}^{m+d}) \) is the space of probability measures on \( \mathbb{R}^{m+d} \) equipped with the weak topology. In [10, 30], where the coefficients are distribution independent, the Bismut formula is derived for stochastic Hamiltonian system.

For any \( k \geq 1 \), let

\[
\mathcal{P}_k(\mathbb{R}^{m+d}) := \{ \mu \in \mathcal{P}(\mathbb{R}^{m+d}) : \| \mu \|_k := \mu(\| \cdot \|_k^{1/2}) < \infty \},
\]

which is a Polish space under the \( L^k \)-Wasserstein distance \( W_k \). When (3.1) is well-posed for distributions in \( \mathcal{P}_k(\mathbb{R}^{m+d}) \), let \( P^*_{t_0} \mu = \mathcal{L}X_t \) for the solution with initial distribution \( \mu \in \mathcal{P}_k(\mathbb{R}^{m+d}) \). We aim to establish the log-Harnack inequality and Bismut formula for

\[
P_t f(\mu) := \int_{\mathbb{R}^{m+d}} f(P^*_{t_0} \mu) \, dP^*_{t_0} \mu, \quad f \in \mathcal{B}_b(\mathbb{R}^{m+d}).
\]

By the same reason reformulating (1.5) as (1.6), instead of (3.1) we consider

\[
\begin{cases}
\mathrm{d}X_t^{(1)} &= \{ AX_t^{(1)} + MX_t^{(2)} \} \mathrm{d}t, \\
\mathrm{d}X_t^{(2)} &= b_t(X_t, \mathcal{L}X_t) \mathrm{d}t + \lambda \mathrm{d}W_t + \tilde{\sigma}_t(\mathcal{L}X_t) \mathrm{d}\tilde{W}_t, \quad t \in [0, T],
\end{cases}
\]

where \( W_t, \tilde{W}_t \) are two independent \( d \)-dimensional Brownian motions, and

\[
\tilde{\sigma} : [0, T] \times \mathcal{P}(\mathbb{R}^{m+d}) \to \mathbb{R}^d \otimes \mathbb{R}^d
\]

are measurable.
3.1 Log-Harnack inequality

To establish the log-Harnack inequality, we make the following assumption.

(C) $\lambda > 0$ is a constant, $(\tilde{\sigma}, b)$ satisfies conditions in (A) for $(x, \mu) \in \mathbb{R}^{m+d} \times \mathcal{P}_2(\mathbb{R}^{m+d})$, and the following Kalman’s rank condition holds for some integer $1 \leq l \leq m$:

\begin{equation}
\text{Rank}[A^i M, 0 \leq i \leq l-1] = m,
\end{equation}

where $A^0 := I_m$ is the $m \times m$-identity matrix.

By [28, Theorem 2.1], (C) implies that (3.2) is well-posed for distributions in $\mathcal{P}_2(\mathbb{R}^{m+d})$, and there exists a constant $c > 0$ such that

\begin{equation}
W_2^2(P^* t \mu, P^* t \nu) \leq c W_2^2(\mu, \nu), \quad \mu, \nu \in \mathcal{P}_2(\mathbb{R}^{m+d}), t \in [0, T].
\end{equation}

So, as in (2.4), we find a constant $C > 0$ such that

\begin{equation}
\mathbb{E} \left[ \sup_{t \in [0, T]} |\xi^\mu_t - \xi^\nu_t|^2 \right] \leq C W_2^2(\mu, \nu)^2, \quad \mu, \nu \in \mathcal{P}_2(\mathbb{R}^{m+d}).
\end{equation}

To distinguish the singularity of $P_{t}$ in the degenerate component $x^{(1)}$ and the non-degenerate one $x^{(2)}$, for any $t > 0$ we consider the modified distance

\[ \rho_t(x, y) := \sqrt{t^{-2} |x^{(1)}(1) - y^{(1)}(1)|^2 + |x^{(2)}(1) - y^{(2)}(1)|^2}, \quad x, y \in \mathbb{R}^{m+d}, \]

and define the associated $L^2$-Wasserstein distance

\[ \mathbb{W}_{2,t}(\mu, \nu) := \inf_{\pi \in \mathcal{C}(\mu, \nu)} \left( \int_{\mathbb{R}^{m+d} \times \mathbb{R}^{m+d}} \rho_t(x, y)^2 \pi(dx, dy) \right)^{1/2}. \]

It is clear that

\begin{equation}
\frac{1}{T^2 \vee 1} \mathbb{W}_2^2 \leq \mathbb{W}_{2,t}^2 \leq \frac{1 \vee T^2}{t^2} \mathbb{W}_2^2, \quad t \in (0, T].
\end{equation}

For $t \in (0, T]$, let

\[ Q_t := \int_0^t \frac{s(t - s)}{t^2} e^{-sA} M M^* e^{-sA^*} ds. \]

According to [21], see also [30, Proof of Theorem 4.2(1)], the rank condition (3.3) implies

\begin{equation}
\|Q_t^{-1}\| \leq c_0 t^{1-2l}, \quad t \in (0, T]
\end{equation}

for some constant $c_0 > 0$. 

13
Theorem 3.1. Assume (C) and let \( P_t^* \) be associated with the degenerate SDE (3.2). Then there exists a constant \( c > 0 \) such that

\[
(3.7) \quad P_t \log f(\nu) - \log P_t f(\mu) \leq \frac{c}{t^{d-3}} \mathbb{W}_2, t(\mu, \nu)^2 \leq \frac{c(1 \lor T^2)}{t^{d-1}} \mathbb{W}_2(\mu, \nu)^2,
\]

\( t \in (0, T], \mu, \nu \in \mathcal{P}_2(\mathbb{R}^{m+d}), 0 < f \in \mathcal{B}(\mathbb{R}^{m+d}). \)

Equivalently, for any \( t \in (0, T] \) and \( \mu, \nu \in \mathcal{P}_2(\mathbb{R}^{m+d}), \)

\[
\text{Ent}(P^*_t\nu | P^*_t\mu) \leq \frac{c}{t^{d-3}} \mathbb{W}_2, t(\mu, \nu)^2 \leq \frac{c(1 \lor T^2)}{t^{d-1}} \mathbb{W}_2(\mu, \nu)^2.
\]

Proof. For any \( t_0 \in (0, T] \) and \( \mu, \nu \in \mathcal{P}_2(\mathbb{R}^{m+d}), \) let \( X_0, Y_0 \) be \( \mathcal{F}_0 \)-measurable such that

\[
(3.8) \quad \mathcal{L}_{X_0} = \mu, \quad \mathcal{L}_{Y_0} = \nu, \quad \mathbb{E}[\mu_0(X_0, Y_0)^2] = \mathbb{W}_2(t_0, \mu, \nu)^2.
\]

Let \( X_t \) solve (3.2) with initial value \( X_0 \), we have \( P_t^*\mu = \mathcal{L}_{X_t} \). Let

\[
(3.9) \quad \nu = (v^{(1)}, v^{(2)}) := (Y_0^{(1)} - X_0^{(1)}, Y_0^{(2)} - X_0^{(2)}) = Y_0 - X_0.
\]

For fixed \( t_0 \in (0, T] \), let

\[
(3.10) \quad \alpha_{t_0}(s) := \frac{s}{t_0} \left( \xi^{\mu} - \xi^{\nu} - v^{(2)} \right) - \frac{s(t_0 - s)}{t_0^2} M^* e^{-sA^*} Q_{t_0}^{-1} (v^{(1)} + V_{t_0}^{\mu, \nu}),
\]

\[
V_{t_0}^{\mu, \nu} := \int_0^{t_0} e^{-rA} M \left\{ \frac{t_0 - r}{t_0} v^{(2)} + \frac{r}{t_0} (\xi^{\mu} - \xi^{\nu}) + \xi^{\nu} - \xi^{\mu} \right\} dr.
\]

By (3.6), we find a constant \( c_1 \) independent of \( t_0 \) such that

\[
(3.11) \quad \sup_{t \in [0, t_0]} \left\{ |\alpha_{t_0}'(t)| + |\alpha_{t_0}(t)| \right\} \leq \frac{c_1}{t_0^{d-1}} \left( t_0^{-1} |v^{(1)}| + |v^{(2)}| + \sup_{t \in [0, t_0]} |\xi^{\mu} - \xi^{\nu}| \right).
\]

Let \( Y_t \) solve the SDE with initial value \( Y_0 \):

\[
(3.12) \quad \begin{cases} dY_t^{(1)} = \{ AY_t^{(1)} + MY_t^{(2)} \} dt, \\ dY_t^{(2)} = \{ b_t(X_t, P_t^*\mu) + \alpha_{t_0}(t) \} dt + \lambda dW_t + \sigma_t(P_t^*\nu)d\tilde{W}_t, \quad t \in [0, t_0]. \end{cases}
\]

This and (3.2) imply

\[
(3.13) \quad \begin{aligned} Y_t^{(2)} - X_t^{(2)} &= \alpha_{t_0}(t) + v^{(2)} + \xi^{\nu} - \xi^{\mu}, \\ Y_t^{(1)} - X_t^{(1)} &= e^{tA}v^{(1)} + \int_0^t e^{(t-s)A} M \left\{ \alpha_{t_0}(s) + v^{(2)} + \xi^{\nu} - \xi^{\mu} \right\} ds, \quad t \in [0, t_0]. \end{aligned}
\]

Consequently,

\[
Y_{t_0}^{(2)} - X_{t_0}^{(2)} = \xi^{\mu} - \xi^{\nu} - v^{(2)} + v^{(2)} + \xi^{\nu} - \xi^{\mu} = 0.
\]
By Girsanov’s theorem, let $\tilde{\xi}$ be a constant vector such that
\[
\begin{align*}
Y_{t_0}^{(1)} - X_{t_0}^{(1)} &= e^{t_0 A} v^{(1)} + \int_0^{t_0} e^{(t_0-s)A} M \left\{ \frac{s}{t_0} \left( \xi_{t_0}^{\mu} - \xi_{t_0}^{\nu} - v^{(2)} + \xi_{r}^{\nu} - \xi_{r}^{\mu} \right) \right\} ds \\
&\quad - e^{t_0 A} Q_{t_0} Q_{t_0}^{-1} \left( v^{(1)} + \int_0^{t_0} e^{-rA} M \left\{ \frac{t_0 - r}{t_0} v^{(2)} + \frac{r}{t_0} \left( \xi_{t_0}^{\mu} - \xi_{t_0}^{\nu} \right) + \xi_{r}^{\nu} - \xi_{r}^{\mu} \right\} dr \right) \\
&= 0,
\end{align*}
\]
so that
\[
Y_{t_0} = X_{t_0}.
\]
On the other hand, by (3.13) and (3.11) we find a constant $c_2 > 0$ uniform in $t_0 \in (0, T]$ such that
\[
\sup_{t \in [0, t_0]} \left| Y_t - X_t \right|^2 \leq \frac{c_2}{t_0^{2(4l-1)}} \left\{ t_0^{-2} \left| v^{(1)} \right|^2 + \sup_{t \in [0, t_0]} \left| \xi_t^{\mu} - \xi_t^{\nu} \right|^2 \right\}
\]
To formulate the equation of $Y_t$ as (3.2), let
\[
\eta_s := \frac{1}{\lambda} \left\{ b_s(Y_s, P_s^{\ast} \nu) - b_s(X_s, P_s^{\ast} \mu) - \alpha_t(s) \right\}, \quad s \in [0, t_0] \tag{3.15}
\]
By (C), (3.11) and (3.15), we find a constant $c_3 > 0$ uniformly in $t_0 \in (0, T]$ such that
\[
\begin{align*}
\left| \eta_s \right|^2 \leq c_3 K_s \left\{ \nu \mathbb{W}_2(\mu, \nu)^2 + t_0^{4(1-l)} \rho_0(X_0, Y_0)^2 + t_0^{4(1-l)} \sup_{t \in [0, t_0]} \left| \xi_t^{\mu} - \xi_t^{\nu} \right|^2 \right\} \\
&\quad + c_3 t_0^{2-4l} \left( \rho_0(X_0, Y_0)^2 + \sup_{t \in [0, t_0]} \left| \xi_t^{\nu} - \xi_t^{\mu} \right|^2 \right) \tag{3.16}
\end{align*}
\]
By Girsanov’s theorem,
\[
\hat{W}_t := W_t - \int_0^t \eta_s ds, \quad t \in [0, t_0]
\]
is a $d$-dimensional Brownian motion under the weighted conditional probability measure $d\hat{Q}^\hat{W}_0 := R^\hat{W}_0 d\hat{P}^\hat{W}_0$, where
\[
R^\hat{W}_0 := e^{\int_0^{t_0} \langle \eta_s, dW_s \rangle} - \frac{1}{2} \int_0^{t_0} |\eta_s|^2 ds.
\]
Let $\hat{\xi}_t^{\nu} = (0, \xi_t^{\nu})$. By (3.12), $\hat{Y}_t := Y_t - \hat{\xi}_t^{\nu}$ solves the SDE
\[
\begin{align*}
\begin{cases}
\begin{aligned}
\frac{d\hat{Y}_t}{t}^{(1)} &= \left\{ A \hat{Y}_t^{(1)} + M \hat{Y}_t^{(2)} + \hat{M} \hat{\xi}_t^{\nu} \right\} dt, \\
\frac{d\hat{Y}_t}{t}^{(2)} &= b_t(\hat{Y}_t + \hat{\xi}_t^{\nu}, P_t^{\ast} \nu) dt + \lambda d\hat{W}_t, \quad t \in [0, t_0], \quad \hat{Y}_0 = Y_0.
\end{aligned}
\end{cases}
\end{align*}
\]
Letting $X_t^\nu$ solve (3.2) with $X_0^\nu = Y_0$, we see that $\tilde{X}_t^\nu := X_t^\nu - \tilde{\zeta}_t^\nu$ solves the same equation as $Y_t$ for $W_t$ replacing $\tilde{W}_t$. By the weak uniqueness and (3.14), (2.12) holds for $\mathbb{R}^{m+d}$ replacing $\mathbb{R}^d$, i.e. for any $f \in \mathcal{B}_b(\mathbb{R}^{m+d})$,

\begin{equation}
  (3.18) \quad P_t^{\tilde{W},0} f(X_0^\nu) := \mathbb{E}^{\tilde{W},0}[f(X_t^\nu)] = \mathbb{E}^{\tilde{W},0}[R^{\tilde{W},0} f(Y_0)] = \mathbb{E}^{\tilde{W},0}[R^{\tilde{W},0} f(X_0)].
\end{equation}

Combining this with Young’s inequality and (3.17), we find constants $c_4 > 0$ such that

\begin{equation}
  (3.19) \quad P_t^{\tilde{W},0} \log f(X_0^\nu) - \log P_t^{\tilde{W},0} f(X_0^\mu) \leq \mathbb{E}^{\tilde{W},0}[R^{\tilde{W},0} \log R^{\tilde{W},0}] = \frac{1}{2} \mathbb{E} \mathbb{Q}^{\tilde{W},0} \int_0^t |\eta|^2 \, dt
  \leq c_4 \left\{ \mathbb{W}_2(\mu, \nu)^2 + t_0^{3-4l} \rho_0(X_0, Y_0)^2 + t_0^{3-4l} \sup_{t \in [0, t_0]} |\xi_t - \zeta_t|^2 \right\}.
\end{equation}

By taking expectation, using Jensen’s inequality, (3.4), (3.5) and (3.8), we prove (3.7). \qed

### 3.2 Bismut formula

We will use Definition 2.1 for $\mathbb{R}^{m+d}$ replacing $\mathbb{R}^d$. The following assumption is parallel to (B) with an additional rank condition.

**(D)** $(\tilde{\sigma}, b)$ satisfies (B) for $\mathbb{R}^{m+d}$ replacing $\mathbb{R}^d$, and the rank condition (3.3) holds for some $1 \leq l \leq m$.

Let $X_t^\mu$ be $\mathcal{F}_0$-measurable such that $\mathcal{L}_{X_0^\mu} = \mu \in \mathcal{P}_k(\mathbb{R}^{m+d})$, and let $X_t^\mu$ solve (3.2) with initial value $X_0^\mu$. For any $\varepsilon \geq 0$, denote

$$ \mu^\varepsilon := \mu \circ (id + \varepsilon \phi)^{-1}, \quad X_t^\mu = X_0^\mu + \varepsilon \phi(X_t^\mu). $$

Let $X_t^{\mu^\varepsilon}$ solve (3.2) with initial value $X_0^{\mu^\varepsilon}$. So,

$$ X_t^\mu = X_t^{\mu^0}, \quad P_t^\mu X_t^\mu = \mathcal{L}_{X_t^{\mu^\varepsilon}}, \quad t \in [0, T], \varepsilon \geq 0. $$

By [4, Lemma 5.2], for any $\phi = (\phi^{(1)}, \phi^{(2)}) \in T_{\mu,k}(\mathbb{R}^{m+d})$, (D) implies that

$$ \nabla_\phi X_t^\mu := \lim_{\varepsilon \downarrow 0} \frac{X_t^{\mu^\varepsilon} - X_t^\mu}{\varepsilon} $$

exists in $L^k(\Omega \rightarrow C([0, T]; \mathbb{R}^{m+d}); \mathbb{P})$, and there exists a constant $c > 0$ such that

$$ \mathbb{E} \left[ \sup_{t \in [0, T]} |\nabla_\phi X_t^\mu|^k \right] \leq c \|\phi\|_{L^k(\mu)}^k, \quad \mu \in \mathcal{P}_k(\mathbb{R}^{m+d}), \phi \in T_{\mu,k}(\mathbb{R}^{m+d}). $$

Finally, for any $t \in (0, T]$ and $s \in [0, t]$, let

$$ \gamma_{t,s}^{\mu, \phi} := \int_0^t \mathbb{E} \left[ \langle D^s \tilde{\sigma}_r(P_r^\mu)(X_r^\mu), \nabla_\phi X_r^\mu \rangle \right] \, d\tilde{W}_r. $$
\[ V'^{\mu,\phi}_t := \int_0^t e^{-rA}M\left\{ \frac{t-r}{t} \phi^{(2)}(X_0^\mu) - \frac{r}{t}\gamma^{\mu,\phi}_t + \gamma^{\mu,\phi}_r \right\} dr, \]
\[ \alpha^{\mu,\phi}_t(s) := -s t \{ \phi^{(2)}(X_0^\mu) + \gamma^{\mu,\phi}_t \} - \frac{s(t-s)}{t^2}M e^{-sA}Q^{-1}_t \{ \phi^{(1)}(X_0^\mu) + V'_t^{\mu,\phi} \}, \]
and define
\[ N_{s,t}^{(1)} := e^{sA}\phi^{(1)}(X_0^\mu) + \int_s^t e^{(s-r)A}M\left\{ \alpha^{\mu,\phi}(r) + \phi^{(2)}(X_0^\mu) + \gamma^{\mu,\phi}_r \right\} dr, \]
\[ N_{s,t}^{(2)} := \alpha^{\mu,\phi}_t(s) + \phi^{(2)}(X_0^\mu) + \gamma^{\mu,\phi}_s, \]
\[ M^{\mu,\phi}_{s,t} := E[(D_0^t b_t(z, \cdot)(P_t^\mu)(X^\mu), \nabla \phi X^\mu)]_{z=X^\nu} - (\alpha^{\mu,\phi}_t)'(s). \]

Then we have the following result.

**Theorem 3.2.** Assume (D) and let \( N^{\mu,\phi}_{s,t} := (N_{s,t}^{(1)}, N_{s,t}^{(2)}) \in \mathbb{R}^{m+d}, 0 \leq s \leq t. \) For any \( t \in (0, T], \mu \in \mathcal{P}_k(\mathbb{R}^{m+d}), \phi \in T_{\mu,k}(\mathbb{R}^{m+d}) \) and \( f \in \mathcal{B}_b(\mathbb{R}^{m+d}), \)
\[ D_0^t P_t f(\mu) = \frac{1}{\lambda} E \left[ f(X^\mu) \int_0^t \left\langle \nabla N^{\mu,\phi}_{s,t} b_t(\cdot, P_t^\mu)(X^\mu) + M^{\mu,\phi}_{s,t}, dW_s \right\rangle \right]. \]

Consequently, \( P_t f \) is intrinsically differentiable, and there exists a constant \( c > 0 \) such that
\[ \| D_0^t P_t f(\mu) \|_{L^{k^*}(\mu)} \leq \frac{c}{t^{2l-\frac{1}{2}}}(P_t|f|^{k^*}(\mu))^{\frac{1}{k^*}}, \quad t \in (0, T], \quad f \in \mathcal{B}_b(\mathbb{R}^{m+d}). \]

**Proof.** It is easy to see that under (D), (3.21) follows from (3.20). So, it suffices to prove (3.20).

Let \( X_t^\mu \) solve (3.2) with initial value \( X_0^\mu \), and for any \( \varepsilon \in (0, 1] \), let \( Y_t^\varepsilon \) solve (3.12) for \( Y_0^\varepsilon := X_0^\mu + \varepsilon \phi(X_0^\mu) \) and \( \nu = \mu^\varepsilon \). Then
\[ \mathcal{L}_{Y_0} = \mathcal{L}_{Y_0} = \mu^\varepsilon. \]

Let \( \alpha^\varepsilon_{t_0}(s) \) be defined in (3.10) for \( \nu = \mu^\varepsilon \). By (2.20) and (3.9), we have
\[ \lim_{\varepsilon \downarrow 0} \varepsilon \alpha^\varepsilon_{t_0}(s) = \alpha^{\mu,\phi}_{t_0}(s), \quad s \in [0, t_0], \]
while (3.13) and (3.16) reduces to
\[ (Y_t^\varepsilon(2) - X_t^\mu(2)) = \alpha^\varepsilon_{t_0}(t) + \varepsilon \phi^{(2)}(X_0^\mu) + \xi^\mu_t, \]
\[ (Y_t^\varepsilon(1) - X_t^\mu(1)) = \varepsilon e^{tA}\phi^{(1)}(X_0^\mu) + \int_0^t e^{(t-s)A}M\left\{ \alpha^\varepsilon_{t_0}(s) + \varepsilon \phi^{(2)}(X_0^\mu) + \xi^\mu_s - \xi^\mu_t \right\} ds, \]
and
\[ \gamma^\varepsilon_t = \frac{1}{\lambda} \left\{ b_t(Y_t^\varepsilon, P_t^\mu \mu) - b_t(X_t^\mu, P_t^\mu \mu) - \{ \alpha^\varepsilon_{t_0} \}'(t) \right\}, \quad t \in [0, t_0]. \]
Finally, similarly to the proof of (2.15), since 

\[
(3.24) \quad R^\varepsilon := e^{\int_0^\varepsilon \langle \eta^\mu, dW_t \rangle - \frac{1}{2} \int_0^\varepsilon |\eta^\mu|^2 dt}.
\]

By (3.18), we obtain

\[
P_t^{\hat{W}, 0} f(X^\mu_0) := \mathbb{E}^{\hat{W}, 0}[f(X^\mu_t)], \quad f \in \mathcal{B}(\mathbb{R}^{m+d}).
\]

As in (2.21), by (D), (3.23) and (2.20), we derive

\[
\lim_{\varepsilon \downarrow 0} \frac{P_t^{\hat{W}, 0} f(X^\mu_0) - P_t^{\hat{W}, 0} f(X^\mu_t)}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \mathbb{E}^{\hat{W}, 0} \left[ f(X^\mu_t) \frac{R^\varepsilon - 1}{\varepsilon} \right]
\]

(3.24)

\[
= \frac{1}{\lambda} \mathbb{E}^{\hat{W}, 0} \left[ f(X^\mu_t) \int_0^\varepsilon \left\langle \nabla N_{\xi^\mu, \nu, 0} b_s(\cdot, P^*_s \mu)(X^\mu_s) + M_{\xi^\mu, \nu, 0}, dW_s \right\rangle \right].
\]

Finally, similarly to the proof of (2.15), since (D) implies (C) for \((\mathcal{P}_k(\mathbb{R}^{m+d}), \mathbb{W}_k)\) replacing \((\mathcal{P}_2(\mathbb{R}^{m+d}), \mathbb{W}_2)\), the argument leading to (3.19) implies

\[
P_t^{\hat{W}, 0} \log f(X^\nu_0) - log P_t^{\hat{W}, 0} f(X^\mu_0) \leq c(t_0) \left\{ \mathbb{W}_k(\mu, \nu)^2 + \rho_{t_0}(X^\mu_0, X^\nu_0)^2 + \sup_{t \in [0, t_0]} |\xi^\nu_t - \xi^\mu_t|^2 \right\}
\]

for some constant \(c(t_0) > 0\). Therefore, as shown in step (b) of the proof of Theorem 2.2, this enables us to apply the dominated convergence theorem with (3.24) to derive

\[
D^1_\phi P_t f(\mu) = \lim_{\varepsilon \downarrow 0} \mathbb{E} \left[ P_t^{\hat{W}, 0} f(X^\mu_0) - P_t^{\hat{W}, 0} f(X^\mu_t) \right] = \mathbb{E} \left\{ \lim_{\varepsilon \downarrow 0} \mathbb{E}^{\hat{W}, 0} \left[ f(X^\mu_t) \frac{R^\varepsilon - 1}{\varepsilon} \right] \right\}
\]

\[
= \frac{1}{\lambda} \mathbb{E} \left[ f(X^\mu_t) \int_0^\varepsilon \left\langle \nabla N_{\xi^\mu, \nu, 0} b_s(\cdot, P^*_s \mu)(X^\mu_s) + M_{\xi^\mu, \nu, 0}, dW_s \right\rangle \right].
\]

\[\Box\]

4 Exponential ergodicity in entropy

Following the line of (2.11), we may use the log-Harnack inequality to study the exponential ergodicity in entropy. To this end, we consider the time homogeneous equation on \(\mathbb{R}^d\)

\[
(4.1) \quad dX_t = b(X_t, \mathcal{L}X_t)dt + \lambda dW_t + \tilde{\sigma}(\mathcal{L}X_t)d\tilde{W}_t, \quad t \geq 0,
\]

and the degenerate model on \(\mathbb{R}^{m+d}\)

\[
(4.2) \quad \begin{cases}
    dX^{(1)}_t = \{AX^{(1)}_t + MX^{(2)}_t\}dt, \\
    dX^{(2)}_t = b(X_t, \mathcal{L}X_t)dt + \lambda dW_t + \tilde{\sigma}(\mathcal{L}X_t)d\tilde{W}_t, \quad t \geq 0,
\end{cases}
\]

where \(\lambda > 0\) is a constant.
4.1 Non-degenerate case

(E) There exist constants $K, \theta_1, \theta_2 > 0$ with $\theta := \theta_2 - \theta_1 > 0$, such that for any $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and $x, y \in \mathbb{R}^d$,

$$|b(x, \mu) - b(y, \nu)| + |\dot{\sigma}(\mu) - \dot{\sigma}(\nu)| \leq K(|x - y| + \mathbb{W}_2(\mu, \nu)),
$$

$$2(b(x, \mu) - b(y, \nu), x - y) + \|\sigma(\mu) - \sigma(\nu)\|^2_{HS} \leq -\theta_2|x - y|^2 + \theta_1 \mathbb{W}_2(\mu, \nu)^2,$$

where $\| \cdot \|_{HS}$ is the Hilbert-Schmidt norm.

By [28, Theorem 2.1], this assumption implies that (4.1) is well-posed for distributions in $\mathcal{P}_2$, and $P^*_t$ has a unique invariant probability measure $\bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$ such that

$$\mathbb{W}_2(P^*_t \mu, \bar{\mu})^2 \leq e^{-\theta t} \mathbb{W}_2(\mu, \bar{\mu})^2, \quad t \geq 0. \tag{4.3}$$

The following result ensures the exponential convergence in entropy.

**Theorem 4.1.** Assume (E) and let $P^*_t$ be associated with (4.1). Then there exists a constant $c > 0$ such that

$$\max \{ \mathbb{W}_2(P^*_t \mu, \bar{\mu})^2, \mathcal{E}(P^*_t \mu | \bar{\mu}) \} \leq ce^{-\theta t} \min \{ \mathbb{W}_2(\mu, \bar{\mu})^2, \mathcal{E}(\mu | \bar{\mu}) \}, \quad \mu \in \mathcal{P}_2(\mathbb{R}^d), \quad t \geq 1.$$

**Proof.** According to the proof of [19, Theorem 2.3], (E) implies the Talagrand inequality

$$\mathbb{W}_2(\mu, \bar{\mu})^2 \leq c_1 \mathcal{E}(\mu | \bar{\mu}), \quad \mu \in \mathcal{P}_2(\mathbb{R}^d)$$

for some constant $c_1 > 0$. According to [19, Theorem 2.1], this together with (4.3) and Theorem 2.1 implies the desired assertion. \(\square\)

4.2 Degenerate case

To study the exponential ergodicity for the degenerate model (4.2), we extend the assumption (A1)-(A3) in [27] to the present distribution dependent case.

(F) $\bar{\sigma}$ and $b$ are Lipschitz continuous on $\mathcal{P}_2(\mathbb{R}^{m+d})$ and $\mathbb{R}^{m+d} \times \mathcal{P}_2(\mathbb{R}^{m+d})$ respectively. (A, M) satisfies the rank condition (3.3) for some $1 \leq l \leq m$, and there exist constants $r > 0, \theta_2 > \theta_1 \geq 0$ and $r_0 \in (-\|M\|^{-1}, \|M\|^{-1})$ such that

$$\frac{1}{2} \|\bar{\sigma}(\mu) - \bar{\sigma}(\nu)\|^2_{HS} + \langle b(x, \mu) - b(y, \nu), \ x^{(2)} - y^{(2)} + r_0 M^*(x^{(1)} - y^{(1)}) \rangle
$$

$$+ \langle r^2(x^{(1)} - y^{(1)}) + r_0 M(x^{(2)} - y^{(2)}), \ A(x^{(1)} - y^{(1)}) + M(x^{(2)} - y^{(2)}) \rangle
$$

$$\leq \theta_1 \mathbb{W}_2(\mu, \nu)^2 - \theta_2|x - y|^2, \quad x, y \in \mathbb{R}^{m+d}, \quad \mu, \nu \in \mathcal{P}_2(\mathbb{R}^{m+d}).$$
In the distribution free case, some examples are presented in [27, Section 5], which can be extended to the present setting if the Lipschitz constant of \( \tilde{\sigma}(\mu) \) and \( b(x, \mu) \) in \( \mu \in \mathcal{P}_2(\mathbb{R}^{m+d}) \) is small enough.

**Theorem 4.2.** Assume (F). Then \( P^*_t \) associated with (4.2) has a unique invariant probability measure \( \bar{\mu} \), and there exist constants \( c, \lambda > 0 \) such that

\[
\max \left\{ \text{Ent}(P^*_t \mu | \bar{\mu}), \mathcal{W}_2(P^*_t \mu, \bar{\mu})^2 \right\} \leq ce^{-\lambda t} \mathcal{W}_2(\mu, \bar{\mu})^2, \quad t \geq 1, \mu \in \mathcal{P}_2(\mathbb{R}^{m+d}).
\]

**Proof.** Let

\[
\rho(x) := \frac{r^2}{2} |x^{(1)}|^2 + \frac{1}{2} |x^{(2)}|^2 + r r_0 \langle x^{(1)}, M x^{(2)} \rangle, \quad x = (x^{(1)}, x^{(2)}) \in \mathbb{R}^{m+d}.
\]

By \( r_0 \| M \| < 1 \) and \( r > 0 \), we find a constant \( c_0 \in (0, 1) \) such that

\[(4.4) \quad c_0 |x|^2 \leq \rho(x) \leq c_0^{-1} |x|^2, \quad x \in \mathbb{R}^{m+d}.
\]

Let \( X_t \) and \( Y_t \) solve (4.2) with initial values

\[(4.5) \quad \mathcal{L}_X = \mu, \quad \mathcal{L}_Y = \nu, \quad \mathcal{W}_2(\mu, \nu)^2 = \mathbb{E}[|X_0 - Y_0|^2].
\]

By (F) and Itô’s formula, we obtain

\[(4.6) \quad d\rho(X_t - Y_t) \leq \left\{ \theta_1 \mathcal{W}_2(P^*_t \mu, P^*_t \nu)^2 - \theta_2 |X_t - Y_t|^2 \right\} dt + dM_t
\]

for some martingale \( M_t \), and

\[
\rho(X_t) \leq \left\{ \theta_1 \mathbb{E}[|X_t|^2] - \theta_2 |X_t|^2 + C |X_t| \right\} dt + d\tilde{M}_t
\]

for some martingale \( \tilde{M}_t \) and constant \( C > 0 \). In particular, by (4.4), the latter implies

\[(4.7) \quad \sup_{t \geq 0} \mathbb{E}[|X_t|^2] < \infty.
\]

Since

\[(4.8) \quad \mathcal{W}_2(P^*_t \mu, P^*_t \nu)^2 \leq \mathbb{E}[|X_t - Y_t|^2],
\]

(4.6) and (4.4) imply

\[
\mathbb{E}[\rho(X_t - Y_t)] - \mathbb{E}[\rho(X_s - Y_s)] \leq -c_0 (\theta_2 - \theta_1) \int_s^t \mathbb{E}[\rho(X_r - Y_r)] dr, \quad t \geq s \geq 0.
\]

By Gronwall’s inequality, we derive

\[
\mathbb{E}[\rho(X_t - Y_t)] \leq e^{-c_0 (\theta_2 - \theta_1)t} \mathbb{E}[\rho(X_0 - Y_0)], \quad t \geq 0.
\]
This together with (4.4), (4.5) and (4.8) yields
\[
\mathbb{W}_2(P_t^* \mu, P_t^* \nu)^2 \leq \mathbb{E}[|X_t - Y_t|^2] \leq c_0^{-1} \mathbb{E}[d_t(X_t - Y_t)] \leq c_0^{-1} e^{-c_0(t_1 - t)} \mathbb{E}[\rho(X_0 - Y_0)]
\]
\[
\leq c_0^{-2} e^{-c_0(t_2 - t)} \mathbb{E}[|X_0 - Y_0|^2] = c_0^{-2} e^{-c_0(t_2 - t)} \mathbb{W}_2(\mu, \nu)^2, \quad t \geq 0, \mu, \nu \in \mathcal{P}_2(\mathbb{R}^{m+d}).
\]

As shown in [28, Proof of Theorem 3.1(2)], this together with (4.7) implies that \( P_t^* \) has a unique invariant probability measure \( \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d) \), and
\[
(4.9) \quad \mathbb{W}_2(P_t^* \mu, \bar{\mu})^2 \leq c_0^{-2} e^{-c_0(t_2 - t)} \mathbb{W}_2(\mu, \bar{\mu})^2, \quad t \geq 0, \mu \in \mathcal{P}_2(\mathbb{R}^{m+d}).
\]

Finally, by the log-Harnack inequality (3.7), there exists a constant \( c_1 > 0 \) such that
\[
\text{Ent}(P_t^* \mu | \bar{\mu}) \leq c_1 \mathbb{W}_2(\mu, \bar{\mu})^2.
\]

Combining this with (4.9) and using the semigroup property \( P_t^* = P_{t-1}^* P_1^* \) for \( t \geq 1 \), we finish the proof.

When \( b \) is of a gradient type (induced by \( \sigma \)) as in [19, (2.21)] such that the invariant probability measure \( \bar{\mu} \) is explicitly given and satisfies the Talagrand inequality, we may also derive the stronger upper bound as in Theorem 4.1. We skip the details.

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