Minimal diagrams of classical and virtual links

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To Lou Kauffman, on the occasion of his 60th birthday.

Abstract

We prove that a virtual link diagrams satisfying two conditions on the
Khovanov homology is minimal, that is, there is no virtual diagram repre-
senting the same link with smaller number of crossings. This approach
works for both classical and virtual links

For definitions of the Jones polynomial, Kauffman bracket, and the
Khovanov homology, we send the reader to the original papers [3, 1, 4].

The theory of virtual links was first proposed by Kauffman in 1996,
see [2].

We shall treat virtual links as a combinatorial generalisation of classical
links with a new crossing type, called virtual, allowed. The set of Reide-
meister moves is enlarged by a new move, called the detour move, which
means the following. Having a virtual diagram L with an arc AB con-
sisting of some consecutive virtual crossings (and no classical crossings),
one can remove this arc and draw it elsewhere as a curve connecting the
same endpoints, A and B, so that all new crossing are set to be virtual.
Thus, a virtual link is an equivalence class of virtual link diagrams modulo
Reidemeister moves and detour moves, for more details, see [2].

With any virtual link diagram L, one associates an atom, i.e. two-
dimensional surface together with a 4-graph embedded into and a checker-
board colouring of 2-cells.

The vertices of the atom come from the classical crossings of the di-
agrams, whence the rules for attaching the black cells come from over/
undercrossing structures. We decree “black angles” to be those lying be-
tween overcrossing and undercrossing such that while moving from an
undercrossing to an overcrossing clockwise, we sweep the black angle.

One recovers the initial diagram from the atom up to virtualisations
and detours. Denote the atom corresponding to the link L by V(L),
its Euler characteristic by χ(L), and its genus by g(L). Note that the
atom need not be orientable. Thus, e.g., the atom corresponding to the
virtual trefoil lives on the 2-dimensional projective plane. Also, the genus of the atom does not coincide with the underlying genus of thickened surface where the virtual knot lives, see, e.g. [5]. Indeed, the underlying surface need not admit any checkerboard colouring. By a virtualisation (see, e.g., [6]) we mean a replacement of a classical crossing by a (2-2)-tangle consisting of three consecutive crossings: a virtual one, the classical one (with the same writhe as the initial one) and one virtual crossing. The atom does not feel the virtualisation of the initial diagram whence the checkerboard surface does.

All necessary constructions concerning knots and atoms can be found in [6].

By span of a 1-variable Laurent polynomial we mean the difference between its leading degree and its lowest degree.

**Lemma 1 ([6])**. Given a virtual diagram $L$. Then $\text{span}(L) \leq 4n + 2(\chi(L) - 2)$.

Here $n$ is the number of classical vertices of the diagram $L$.

Indeed, this quantity $4n + 2(\chi(L) - 2)$ appears while considering the A-state and the B-state of the Kauffman state sum expansion. We should just take into account that (by definition) the number of white cells of the atom equals the number of curves in the A-state, and the number of black cells of the atom equals the number of curves in the B-state.

Let us say that a virtual diagram $L$ satisfies the first completeness condition if the inequality in Lemma 1 becomes a strict equality. In other words, we have the first completeness condition when neither the leading nor the lowest coefficient of the bracket polynomial vanishes.

A very important question is to classify all those links having a diagram satisfying the first completeness condition (briefly, 1-complete).

One can slightly generalise the first completeness equality in different ways. For instance, one can use the span of the $\Xi$-polynomial (see [7]) in the case of virtual knots and take span in the sense of leading and lowest degree of its coefficients. This polynomial has the same state sum expansions as the Kauffman bracket, the main difference being some new “geometric coefficients” at monomials. Thus, $\text{span}\Xi(L) \leq 4n + 2(\chi(L) - 2)$. In this case, the span might be larger than that of the Kauffman bracket. Also, using the Khovanov polynomial $K\text{h}(q,t)$, one can obtain the bracket polynomial after a variable change $t = -1$. Thus, it readily follows from the above discussion that $\text{span}_q(K\text{h}(q,t)) \leq 2n + \chi(L)$.

The Khovanov homology is well defined over arbitrary field of coefficients for the case of classical links. Note that the Khovanov homology can be used for the $\mathbb{Z}_2$ case for arbitrary virtual links link diagrams and with arbitrary field coefficients for oriented link diagrams (in the sense of atoms), [8].

Thus, we get some conditions slightly weaker than the first completeness conditions. We say that a virtual link diagram is $1$-complete in a broad sense if either $\text{span}\Xi(L) = 4n + 2(\chi(L) - 2)$ or $\text{span}_q(K(q,t)) \leq 2n + \chi(L)$ for Khovanov homologies over some ring of coefficients.
This leads us to examples where the first completeness condition fails. Thus, for instance, there exist a link $L$ for which the leading term of the Khovanov polynomial (with respect to $q$) is some polynomial $P(t)$ such that $P(-1) = 0$, whereas $P(-1)$ coincides with the leading (or lowest, which does not matter) coefficient of the Kauffman polynomial.

It would be very interesting to classify all those (classical and virtual) links not admitting 1-complete diagrams in the broad sense.

This problem remains actual if we allow the virtualisation move: this move does not change the Kauffman bracket, it does not change the Khovanov homology either. Thus we get a weaker equivalence on virtual knot diagrams, which is definitely worth studying, especially, in view of minimality problems.

From the above discussion, we obtain the following:

**Theorem 1** If a virtual link diagram $L'$ is 1-complete (in a broad sense) then we can decrease the number of its vertices only at the expense of the genus.

In other words, if a diagram $L'$ (of a link $L$) has $n$ classical vertices and genus $g$ and $L$ has a diagram $L''$ with $n' < n$ classical vertices then $g(L'') < g(L')$.

An immediate corollary is the Kauffman-Murasugi theorem on alternating links: they live on the sphere, thus having maximal possible Euler characteristic equal to 2 (resp., minimal possible genus). In this case, the 1-completeness yields the minimality.

The question now is **how to handle the genus**?

From now on, for the sake of simplicity, we deal only with orientable link diagrams (in the sense of atoms). On one hand, the case of non-orientable diagrams can be handled by means of double-coverings [8]. On the other hand, all arguments below work in the unorientable case, see Remark 2.

Here we have some “grading” on the set of (orientable) virtual link diagrams: the lowest level is represented by alternating classical knots and quasialternating knots (obtained from former ones by virtualisations and detours), the next levels are regulated by the genus of the corresponding atom.

It is conceptually important that classical knots should not be considered separately from virtual knots; all results work in both categories.

We shall use the following result.

**Theorem 2** Let $L$ be a virtual link diagram. Then the (unnormalised) Khovanov complex for $L$ is quasi-isomorphic to the complex of the form

$$\sum_{s \in K_1(L)} A[r(s)][r(s)][w(L_s)](2w(L_s)), \text{ where } A \text{ is the two-term complex with terms } v_{\pm} \text{ of grading } (0, \pm 1).$$

Herewith, we use the spanning tree model for the Khovanov homology. Here $K_1$ is the set of states each of which having precisely one circle. For any $s$ we take some diagram $L_s$ obtained from $L$ by smoothing some crossings in such a way that: we can get to the state $s$ by smoothing the
remaining circles of \( L_s \) and the diagram \( L_s \) can be unknotted only by using the first Reidemeister move, for more details, see [10]. Also, \( r(s) \) is the number of \( B \)-type smoothings of the diagram \( s \) and \( w(s) \) is minus the writhe number (in Wehrli’s setting).

This theorem was proved by Stephan Wehrli [10] for the case of classical links. The proof in the virtual case belongs to the author of the paper. Namely, one should take the construction from [8] and repeat Wehrli’s proof. The only thing to mention is that in the case of unorientable virtual knot diagrams, one can use only \( \mathbb{Z}_2 \)-coefficients. However, we shall deal only with oriented diagrams and a field of coefficients. The proof of this generalisation is literally the same; one should just accurately use the Khovanov homology construction proposed in [8] and check all steps of the proof.

**Remark 1** As shown in my work [8], Khovanov homology for orientable links is invariant with coefficients in a given field (without torsions in homology), the main difficulty being the Künneth formula for the “tensor square”. Perhaps, one can make a sharper statement about arbitrary coefficients, but from now on, we deal only with coefficients in a field.

The Khovanov homology lies on several diagonals, i.e. lines \( t - 2q = \text{const} \). The thickness of the Khovanov homology is the number of diagonals between the two extreme ones, i.e., \( (t - 2q)_{\text{max}} - (t - 2q)_{\text{min}})/2 + 1 \).

**Notation:** \( T(L) \)

Let us prove the following important result.

**Theorem 3** \( T(L) \leq g(L) + 2 \).

Indeed, the diagonals correspond to the values of \( r(s) \), i.e., numbers of \( B \)-smoothings in \( I \)-states. In the case of alternating links (genus zero) it is known that \( r(s) \) is the same for all states \( s \in K_1 \). In the general case, we have to estimate the amplitude of \( r(s) \) for different \( s \) from \( K_1 \).

**Remark 2** For unorientable link diagrams (and Khovanov homologies with \( \mathbb{Z}_2 \)-coefficients, we have indeed the same formula allowing the genus \( g(L) \) to be half-integer. All arguments remain the same; the \( q \)-grading of the Khovanov homologies do not have the same parity any more, for detailed description see [8].

The remaining part of Theorem 3 follows from

**Lemma 2**. For an orientable link of genus \( g \), the maximal possible value of \( r(s) \) and minimal possible value (for all states from \( K_1 \)) is equal to \( 2g \). Namely, it can be equal to \( k, k - 2, \ldots, k - 2g \).

The proof of this lemma goes as follows. We have some number of curves \( x \) in the \( A \)-state and some number of circles \( y \) in the \( B \)-state. To get to \( K_1 \) from the \( A \)-state, one should switch at least \( x - 1 \) crossings; also, to get to \( K_1 \) from the \( B \)-state, one should switch at least \( y - 1 \) crossings. Thus, the values of \( r(s) \) are in between \( x - 1 \) and \( n + 1 - y \). Now, one should just recall the definition of the atom genus.
So, having two diagonals in Wehrli’s complex for the case of (quasi)alternating links, this number increases by one together with the genus of the atom.

Thus, we know, where the chains of Wehrli’s complex live. This tells us where to look for homologies of Wehrli’s complex which coincide with Khovanov’s homologies (unnormalised). From this we deduce Theorem 3.

Having this, we define a(n orientable virtual) link to be 2-complete if the number of diagonals is as large as it should be, i.e., \( g + 2 \), where \( g \) is the genus of the link. In other words, one should just take care that any of the two extreme diagonals in Wehrli’s complex have at least one homology.

From what above, we obtain the following

**Theorem 4** *(THE MINIMALITY THEOREM).* Suppose an orientable virtual link \( L \) is 1-complete and 2-complete. Then this diagram is minimal.

Indeed, from 2-completeness we see that we can not reduce the genus. Together with 1-completeness this implies that we cannot decrease the number of crossings.

**Remark 3** The minimality theorem remains true if we understand the 1-completeness condition in the broad sense.

**Remark 4** Note that the minimality theorem for classical link \( L \) says that if two completeness conditions hold then for the link \( L \) there is neither classical diagrams nor virtual diagrams with strictly smaller number of classical crossings.

In the general case, the question whether a minimal classical diagram is minimal in virtual category, is still open.

A very important question is to study the interrelation between the atom genus (that we have used) and the underlying genus and their minimalities. They are not the same. They become the same if we admit virtualisation.

A fair question to ask is how large is the set described in minimality theorem?

In my opinion, this question belongs either to philosophy or to measure theory. What I can say at least is that it is rather large for virtual knots and wider than just the class of alternating links in the classical case.

Definitely, both 1-completeness and 2-completeness are two very important properties to study, and I think, the set of minimal diagram detected by the Minimality Theorem can be enlarged by some things like coverings (which can make) or almost-complete diagrams with some estimates of the leading coefficient, and so on.

To support the Minimality Theorem, let us consider the knot 13n3663 from Shumakovitch’s paper [9]. Suppose we do not know how this knot (taken from some table) looks like but now only the \( \mathbb{Q} \)-Khovanov homology, see below.
This information is sufficient to prove that this diagram is minimal. Indeed, it has 13 crossings and 4 diagonals. Thus, its genus cannot be less than 2, and the span of the Kauffman polynomial can not exceed $52 - 8 = 44$, that is, the span of the Khovanov homology (with respect to $q$) cannot exceed 24. But it equals $24 = 2 \cdot 13 + \chi = 26 - 2$: it occupies places between $-11$ and $+13$. So, this knot diagram is 1-complete and 2-complete. Thus, the diagram is minimal by the minimality theorem.
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