Age Problem in Lemaître-Tolman-Bondi Void Models

Xiao-Peng Yan,1 De-Zi Liu,2 and Hao Wei1

1School of Physics, Beijing Institute of Technology, Beijing 100081, China
2Department of Astronomy, Peking University, Beijing 100871, China

ABSTRACT

As is well known, one can explain the current cosmic acceleration by considering an inhomogenous and/or anisotropic universe (which violates the cosmological principle), without invoking dark energy or modified gravity. The well-known one of this kind of models is the so-called Lemaître-Tolman-Bondi (LTB) void model, in which the universe is spherically symmetric and radially inhomogenous, and we are living in a locally underdense void centered nearby our location. In the present work, we test various LTB void models with some old high redshift objects (OHROs). Obviously, the universe cannot be younger than its constituents. We find that these OHROs bring a serious crisis to the LTB void models.

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1. INTRODUCTION

Since the discovery of the current accelerated expansion of the universe [1–6], various models have been proposed to explain this mysterious phenomenon. As is well known, the modern cosmology is based on general relativity and the cosmological principle. The well-known Einstein field equations read

$$G_{\mu\nu} = 8\pi G T_{\mu\nu},$$

where $G_{\mu\nu}$ and $T_{\mu\nu}$ are the Einstein tensor and the stress-energy tensor respectively, and we set the speed of light $c = 1$ throughout this work. According to the pillars of modern cosmology, these theoretical models can be categorized into the following three major types.

The first one is to modify the right hand side of Einstein field equations. That is, one can introduce an exotic energy component, namely dark energy with negative pressure [3–5], while general relativity still holds. The simplest candidate of dark energy is a tiny cosmological constant [10, 11] introduced by Einstein himself in 1917. As is well known, it seriously suffers from the fine-turning problem and the cosmological coincidence problem [11–14]. To alleviate these problems, various dynamical models of dark energy were proposed, such as quintessence [15–17], phantom [18, 19], $k$-essence [20, 22], quintom [23], Chaplygin gas [24, 25], vector-like dark energy [26, 28], holographic dark energy [29], (new) agegraphic dark energy [30, 32], hessence [33, 34], spinor dark energy [35, 37], and so on.

The second one is to modify the left hand side of Einstein field equations, namely to modify general relativity on cosmological scale. Einstein’s general relativity is checked to hold in the range from large scales like the solar system to small scales in the order of millimeter. However, there is no a priori reason to believe that general relativity cannot be modified on cosmological scales. In the literature, various modified gravity theories were proposed to account for the cosmic acceleration, for instance, $f(R)$ theory [38–40], scalar-tensor theory [40, 41], Dvali-Gabadadze-Porrati (DGP) model [42, 43], Galileon gravity [44, 47], Gauss-Bonnet gravity [48, 49], massive gravity [50, 51], and so on.

The third one is to give up the cosmological principle, and consider an inhomogenous and/or anisotropic universe, without invoking dark energy or modified gravity. As a tenet, the cosmological principle is known to be partly satisfied on large scales. However, it has not been proven on cosmic scales $\gtrsim 1$ Gpc [51]. Obviously, our local universe is inhomogenous and anisotropic on small scales. On the other hand, the nearby sample has been examined for evidence of a local “Hubble Bubble” [52]. It is reasonable to imagine that we are living in a locally underdense void. If the cosmological principle is relaxed, it is possible to explain the apparent cosmic acceleration in terms of a peculiar distribution of matter centered upon our location [53–55]. In the literature, the cosmological principle has been tested by using e.g. type Ia supernovae (SNIa) [56, 57], cosmic microwave background (CMB) [55, 56, 57], time drift of cosmological redshifts [58], baryon acoustic oscillations (BAO) [59, 60], integrated Sachs-Wolfe effect [61], galaxy surveys [62], kinetic Sunyaev Zel’dovich effect [63, 64], observational H(z) data [65, 66], so on. It is found that the violation of cosmological principle can be consistent with these observations (in fact few observations slightly favor the violation of cosmological principle). Therefore, it is reasonable to consider an inhomogenous and/or anisotropic universe.

In the literature, the well-known models violating cosmological principle are the so-called Lemaître-Tolman-Bondi (LTB) void models [54, 55]. In LTB void models, the universe is spherically symmetric and radially inhomogenous, and we are living in a locally underdense void centered nearby our location. The Hubble diagram inferred from lines-of-sight originating at the center of the void might be misinterpreted to indicate cosmic acceleration. In fact, LTB void models can be consistent with (even slightly favored by) the observations mentioned above.

In the present work, we try to test LTB void models with the age of the universe. Obviously, the universe cannot be younger than its constituents. In history, the age problem played an important role in the cosmology for many times. However, we should clarify the two meanings of age problem. The first meaning is that the total age of the universe (namely the age measured at present day, or, redshift $z = 0$) cannot be smaller than the age of the oldest known objects (e.g. globular clusters, galaxies, quasars) in our universe. Historically, the matter-dominated Friedmann-Robertson-Walker (FRW) model without cosmological constant can be ruled out [57] because its total age is smaller than the ages inferred from old globular clusters, unless the Hubble constant is extremely low or the universe is extremely open. In the literature, one might consider a variant of this type of age problem. For instance, the authors of [58, 59] reconstructed LTB model from ΛCDM model by requiring they share the same expansion
history (luminosity distance, light-cone mass density, angular diameter distance \( d_A(z) \), Hubble parameter \( H(z) \)), and found that the total age of the universe inferred from LTB model is much smaller than the one inferred from \( \Lambda \)CDM model \((t_{\Lambda \text{CDM}} - t_{\text{LTB}} \sim 2.4 \text{Gyr})\). However, strictly speaking, this variant of age problem is not the real age problem, since LTB model is the reconstructed one, and the total age of the universe is not compared with the real age of old objects (e.g. globular clusters, galaxies, quasars). So, we do not consider this kind of age problem in the present work.

Instead, here we consider the second meaning of age problem, namely the age of the universe at any high redshift \( z > 0 \) (rather than the total age at present day, \( z = 0 \)) cannot be younger than its constituents at the same redshift. Obviously, in this case the age problem becomes more serious than the first one. There are some old high redshift objects (OHROs) considered extensively in the literature, for instance, the 3.5 Gyr old galaxy LBDS 53W091 at redshift \( z = 1.55 \) \([90, 91]\), the 4.0 Gyr old galaxy LBDS 53W069 at redshift \( z = 1.43 \) \([92]\). In addition, the old quasar APM 08279+5255 at redshift \( z = 3.91 \) \([93, 94]\) is also used extensively. Its age is estimated to be \( 2.0 - 3.0 \) Gyr \([93, 94]\). In \([95]\), by using a different method, its age is reevaluated to be 2.1 Gyr. To assure the robustness of our analysis, we use the most conservative lower age estimate 2.0 Gyr for the old quasar APM 08279+5255 at redshift \( z = 3.91 \) throughout the present work. In the literature, these three OHROs have been extensively used to test various dark energy models (see e.g. \([87, 95–102]\)) and modified gravity models (see e.g. \([103–107]\)). In the present work, we will use them to test various LTB void models.

The rest of this paper is organized as followings. In Sec. II, we briefly review the main points of LTB model. In Sec. III, we test various LTB void models with OHROs. In Sec. IV, we give the brief conclusion and discussion.

II. THE LTB MODEL

In the LTB void model, the universe is spherically symmetric and radially inhomogenous, and we are living in a locally underdense void centered nearby our location. The dynamic of a spherically symmetric dust universe is described by the LTB solution to Einstein field equations. It was firstly proposed by Lemaitre \([84]\), then was further discussed by Tolman \([85]\) and Bondi \([86]\). The LTB metric, in comoving coordinates \((r, \theta, \phi)\) and synchronous time \( t \), is given by \([84–86]\) (see also e.g. \([81, 82, 110]\))

\[
ds^2 = -dt^2 + A^2(r, t) (1 - k(r)) dr^2 + A^2(r, t) d\Omega^2, \tag{1}
\]

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \); a prime denotes a derivative with respect to \( r \), and \( k(r) \) is an arbitrary function of \( r \), playing the role of spatial curvature. Note that it reduces to the well-known FRW metric if \( A(r, t) = a(t) r \) and \( k(r) = k r^2 \). The stress-energy tensor of the mass source is given by

\[
T^\nu_\mu = -\rho_M(r, t) \delta^\nu_0 \delta^\mu_0, \tag{2}
\]

where \( \rho_M \) is the energy density of dust matter. The Einstein field equations read \([82, 108–110, 113–115]\)

\[
H^2 + 2H_\perp H_\parallel + \frac{k(r)}{A^2} + \frac{k'(r)}{AA'} = 8\pi G \rho_M, \tag{3}
\]

\[
\dot{A}^2 + 2\dot{A}A + k(r) = 0, \tag{4}
\]

where a dot denotes a derivative with respect to \( t \), and

\[
H_\perp(r, t) = \frac{\dot{A}(r, t)}{A(r, t)}, \tag{5}
\]

\[
H_\parallel(r, t) = \frac{\dot{A}'(r, t)}{A(r, t)}, \tag{6}
\]

are the expansion rates at the transverse and longitudinal directions, respectively. Integrating Eq. \ref{4}, we obtain \([108–110, 113–115]\)

\[
\dot{A}^2(r, t) = \frac{2M(r)}{A(r, t)} - k(r), \tag{7}
\]
where \( M(r) \) is an arbitrary function (the factor 2 is introduced just for convenience; one should be aware of the different symbol conventions in the relevant references). If \( M(r) \) and \( k(r) \) are given, one can obtain \( A(r, t) \) by directly solving Eq. (7). For convenience, we instead try to find the parametric solutions for it. Following e.g. [82, 110, 112–114], we recast Eq. (7) as

\[
\frac{\dot{A}^2(r, t)}{|k(r)|} = -\tilde{k} + \frac{2M(r)}{A(r, t)|k(r)|},
\]

to normalize \( \tilde{k} \equiv k(r)/|k(r)| = +1, -1, 0 \) for \( k(r) > 0, k(r) < 0, k(r) = 0 \), respectively. The solutions of Eq. (8) can be written implicitly in terms of an auxiliary variable \( \eta \) as [111]

\[
A(r, t) = \frac{M(r)}{|k(r)|} \frac{ds(\eta)}{d\eta}, \quad \text{with} \quad t - t_B(r) = \frac{M(r) s(\eta)}{|k(r)|^{3/2}},
\]

where \( t_B(r) \) is actually a “constant” of integration. Therefore, Eq. (8) becomes an ordinary differential equation of the function \( s(\eta) \),

\[
\left[ \frac{d^2 s(\eta)}{d\eta^2} \right] = -\tilde{k} \left[ \frac{ds(\eta)}{d\eta} \right]^2 + 2 \frac{ds(\eta)}{d\eta},
\]

whose solutions are given by [111]

\[
s(\eta) = \begin{cases} 
\eta - \sin \eta & \text{for } \tilde{k} = +1, \\
\sinh \eta - \eta & \text{for } \tilde{k} = -1, \\
\eta^3/6 & \text{for } \tilde{k} = 0.
\end{cases}
\]

Substituting Eq. (11) into Eq. (9), the parametric solutions of Eq. (7) read (see e.g. 82, 110, 112–114)

\[
A(r, t) = \frac{M(r)}{|k(r)|} (1 - \cosh \eta), \quad t - t_B(r) = \frac{M(r)}{|-k(r)|^{3/2}} (\sinh \eta - \eta) \quad \text{for } k(r) < 0,
\]

\[
A(r, t) = \frac{M(r)}{|k(r)|} (1 - \cos \eta), \quad t - t_B(r) = \frac{M(r)}{|k(r)|^{3/2}} (\eta - \sin \eta) \quad \text{for } k(r) > 0,
\]

\[
A(r, t) = \left[ \frac{9M(r)}{2} \right]^{1/3} [t - t_B(r)]^{2/3} \quad \text{for } k(r) = 0,
\]

where \( t_B(r) \) is an arbitrary function of \( r \), usually interpreted as the “bang time” due to singularity behavior at \( t = t_B \). Substituting Eq. (7) into Eq. (3), we have [108, 110, 113, 115]

\[
\frac{2M'(r)}{A'A^2} = 8\pi G \rho_M.
\]

Considering Eq. (7) at the present day (\( t = t_0 \)), it can be recast as

\[
1 = \frac{2M(r)}{H_{\perp 0}^2 r A_0(r)^3} - \frac{k(r)}{H_{\perp 0}^2 r A_0(r)^2} \equiv \Omega_M(r) + \Omega_K(r),
\]

where the subscript “0” indicates the present value of corresponding quantity, i.e., \( A_0(r) = A(r, t = t_0), H_{\perp 0}(r) = H_{\perp}(r, t = t_0) \). Therefore, we can parameterize the functions \( M(r) \) and \( k(r) \) as [108, 110]

\[
2M(r) = H_{\perp 0}^2(r) \Omega_M(r) A_0^3(r),
\]

\[
-k(r) = H_{\perp 0}^2(r) \Omega_K(r) A_0^2(r),
\]

where \( \Omega_K(r) = 1 - \Omega_M(r) \). Noting Eq. (15), it is easy to see that \( \Omega_M \) and \( \Omega_K \) defined in Eqs. (17) and (18) can reduce to the present fractional densities of FRW cosmology if \( A(r, t) = a(t) r \) and \( k(r) = k r^2 \) while
\( H_{\perp 0} \) and \( \Omega_M \) are spatially homogeneous. So, the above parameterizations are justified. Substituting Eqs. (17), (18) into Eqs. (12)—(14), we obtain the total cosmic age as a function of \( r \) \[82\], namely
\[
t_0 - t_B(r) = \frac{\mathcal{F}(\Omega_M)}{H_{\perp 0}(r)},
\]
where
\[
\mathcal{F}(x) \equiv \begin{cases} 
- \sqrt{x - 1} + x \sin^{-1} \sqrt{\frac{x - 1}{x}} & \text{for } x > 1 \\
\frac{2}{3} & \text{for } x = 1 \\
\sqrt{1 - x} - x \sinh^{-1} \sqrt{\frac{1 - x}{x}} & \text{for } x < 1 .
\end{cases}
\]
Furthermore, to compare our theoretical models with observations, we need to associate the coordinates with redshift \( z \). For an observer located at the center \( r = 0 \), by symmetry, incoming light travels along radial null geodesics, \( ds^2 = d\Omega^2 = 0 \), and hence we have \[110\]
\[
dt \frac{dr}{dr} = - \frac{\dot{A}'(r, t)}{\sqrt{1 - k(r)}},
\]
where the minus sign is due to \( dt/dr < 0 \), namely time decreases when going away. Together with the redshift equation \[108–110, 115\]
\[
d\ln(1 + z) \frac{dr}{dr} = \frac{\dot{A}'(r, t)}{\sqrt{1 - k(r)}},
\]
we can write a parametric set of differential equations \[110\]
\[
\frac{dt}{d\ln(1 + z)} = - \frac{\dot{A}'(r, t)}{A'(r, t)},
\]
\[
\frac{dr}{d\ln(1 + z)} = \frac{\sqrt{1 - k(r)}}{A'(r, t)}.
\]
Once the functions \( \Omega_M(r) \) and \( H_{\perp 0}(r) \) characterizing LTB model are given, substituting Eqs. (17) and (18) into Eq. (7), the scale function \( A(r, t) \) can be found by solving the resulting differential equation. Then, one can obtain \( t(z) \) and \( r(z) \) as functions of redshift \( z \) from Eqs. (23) and (24) with the initial conditions \( r(z = 0) = 0 \) and \( t(z = 0) = t_0 \). Note that in solving Eq. (22), the parametric solutions given in Eqs. (12)—(14) are useful. One can do this numerically using a modified version of the code easyLTB \[110\] (see e.g. \[82\] for a brief technical illustration; however, one should be careful of the typos in \[82\], and the different symbol conventions in the relevant references, e.g. \[82, 108–110, 112–115\], as well as the difference between the relevant references and the code easyLTB \[110\]). It is worth noting that the present scale function \( A_0(r) = A(r, t = t_0) \) of LTB model can be chosen to be any smooth and invertible positive function. Following \[82, 108, 110\], we choose the conventional gauge \( A_0(r) = A(r, t = t_0) = r \), which actually corresponds to set the present scale factor \( a_0 = a(t = t_0) = 1 \) in FRW cosmology.

### III. TESTING VARIOUS LTB VOID MODELS WITH OHROS

In the LTB void models, we are living at a special space point, which is close to the center of a large local underdense region of the universe \[84–86, 115–117\]. At very large distances from the observer, the inhomogeneous LTB region goes to an external FRW space. Obviously, it violates the Copernican principle that states we do not occupy any special place in the universe. In the literature, it is found that
FIG. 1: The 3D plot of the allowed parameter space of the Gaussian model for (a) OHRO at $z = 1.43$, (b) OHRO at $z = 1.55$, (c) OHRO at $z = 3.91$, respectively. The blue contours indicate the model parameters making the theoretical cosmic age equal to the age of OHRO at the same redshift. The allowed parameter spaces are the upper regions of these contours. Note that $r_0$ is in units of Gpc. See the text for details.

the LTB void models can be consistent with (even slightly favored by) various observations mentioned in Sec. I. Here, we try to test various LTB void models with three OHROs mentioned in Sec. I namely the 3.5 Gyr old galaxy LBDS 53W091 at redshift $z = 1.55$ [90, 91], the 4.0 Gyr old galaxy LBDS 53W069 at redshift $z = 1.43$ [92], and the 2.0 Gyr old quasar APM 08279+5255 at redshift $z = 3.91$ [93, 94].

A. The Gaussian model

The gradient in the bang time $t_B(r)$ corresponds to a currently non-vanishing decaying mode [118, 119], which might imply an inhomogeneous early universe that violates inflation, and lead to inhomogeneities in the galaxy formation time. To be simple, one might assume that the big bang is spatially homogeneous, namely $t_B$ is a constant. Following e.g. [82, 110], we can set $t_B = 0$ for convenience. In this case, Eq. (19) becomes

$$H_{\perp 0}(r) = H_0 F(\Omega_M),$$

(25)
where the function $F$ is given in Eq. (20), and
\begin{equation}
H_0 \equiv 1/t_0 .
\end{equation}
So, in this case, one only needs to specify $\Omega_M(r)$, and then $H_\perp(r)$ can be found from Eq. (25).

At first, we consider the simplest Gaussian LTB void model \[65, 82\], in which the matter density function $\Omega_M(r)$ has a Gaussian profile, namely
\begin{equation}
\Omega_M(r) = 1 + (\Omega_{in} - 1) \exp \left(\frac{-r^2}{2r_0^2}\right),
\end{equation}
where $\Omega_{in}$ is the matter density at the center of the void, and $r_0$ describes the size of the void. In this work, we only consider the case of $\Omega_{in} < 1$. From Eq. (25), it is easy to obtain
\begin{equation}
H_\perp(r) = H_0 \sqrt{\frac{\Omega_K(r) - \Omega_M(r)}{\Omega_K(r)}} \sinh^{-1} \left(\frac{\Omega_K(r)}{H_0 r_0}\right),
\end{equation}
where $\Omega_K(r) = 1 - \Omega_M(r)$, and $H_0$ actually plays the role of Hubble constant. So, there are three free model parameters, namely $\Omega_{in}$, $r_0$, and $h$ (which is the Hubble constant $H_0$ in units of 100 km/s/Mpc).

To test the Gaussian model with the three OHROs at redshift $z = 1.43, 1.55, 3.91$, we scan a fairly wide parameter space $0.01 \leq \Omega_{in} \leq 0.99$, $1.0 \text{Gpc} \leq r_0 \leq 501.0 \text{Gpc}$, and $0.4 \leq h \leq 1.0$. At every point, we numerically calculate the theoretical cosmic age at redshift $z = 1.43, 1.55, 3.91$ for the Gaussian model with the corresponding parameters $\Omega_{in}$, $r_0$, and $h$. Then, we obtain three contours which indicate the model parameters making the theoretical cosmic age equal to the age of OHRO at the same redshift $z = 1.43, 1.55, 3.91$. We present them in Fig. 1. Only the parameters corresponding to a theoretical cosmic age larger than (or equal to) the age of OHRO at the same redshift are allowed. In fact, the allowed parameter spaces are the upper regions of the contours shown in Fig. 1. From Fig. 1 it is easy to see

FIG. 2: The 2D plot of the allowed parameter space of the Gaussian model with fixed $h = 0.738$ (black contour lines), $0.673$ (red contour lines), $0.623$ (green contour lines) for OHROs at $z = 1.43$ (solid contour lines), $z = 1.55$ (dashed contour lines), $z = 3.91$ (dash-dotted contour lines), respectively. The contour lines indicate the model parameters making the theoretical cosmic age equal to the age of OHRO at the same redshift. The allowed parameter spaces are the upper regions of these contour lines. Note that $r_0$ is in units of Gpc. See the text for details.
FIG. 3: Cosmic age as function of redshift $z$ for the parameter space $0.01 \leq \Omega_{\text{m}} \leq 0.99$, $0.1 \text{ Gpc} \leq r_0 \leq 25 \text{ Gpc}$ of the Gaussian model with fixed $h = 0.738$ (cyan), 0.673 (red), 0.623 (green). The three OHROs at redshift $z = 1.43, 1.55, 3.91$ are also indicated by black stars. See the text for details.

that a large $r_0$ is required to accommodate the three OHROs. In particular, from the panel (c) of Fig. 1 we find that $r_0 > 20$ Gpc is required to accommodate OHRO at redshift $z = 3.91$. To see this clearer, in Fig. 2 we show the 2D slices of the allowed parameter space with fixed $h = 0.738, 0.673, 0.623$. Note that $h = 0.738$ is the best-fit value of the Hubble constant from SHOES SNIa project [120]. $h = 0.673$ is the one from Planck CMB data [121]. On the other hand, Sandage et al. advocated a lower Hubble constant from HST SNIa, and the best-fit value of their final result is $h = 0.623$ [122]. The allowed parameter spaces are the upper regions of the contour lines in Fig. 2. Note that the absence of green-solid contour line in Fig. 2 means that the entire plotted parameter space of the Gaussian model with a fixed $h = 0.623$ is allowed for OHRO at $z = 1.43$. This fact can be seen clearly from Fig. 3, in which we scan the parameter space $0.01 \leq \Omega_{\text{m}} \leq 0.99$, $0.1 \text{ Gpc} \leq r_0 \leq 25 \text{ Gpc}$ with fixed $h = 0.738, 0.673, 0.623$, and plot cosmic age as function of redshift $z$. It is clear that OHRO at $z = 1.43$ is below the lower boundary of the green region, and hence all the parameter space is allowed in this case. Similarly, the absence of red-dashed and green-dashed contour lines in Fig. 2 means that the entire plotted parameter space of the Gaussian model with fixed $h = 0.673, 0.623$ is allowed for OHRO at $z = 1.55$, and it can also be seen clearly from Fig. 3 since OHRO at $z = 1.55$ is below the lower boundaries of both the red and green regions. On the other hand, from Fig. 3 one can also see that at least $r_0 > 25$ Gpc is required to accommodate OHRO at $z = 3.91$, since it is above all the upper boundaries of the cyan, red, and green regions. In fact, from the small panel in Fig. 2 at least $r_0 > 30$ Gpc is required to accommodate OHRO at $z = 3.91$. The unusually large $r_0$ brings a serious crisis to the Gaussian LTB void model. There is a serious tension between this unusually large $r_0$ and the much lower $r_0$ of order 1.0 Gpc inferred from other observations (e.g. SNIa, CMB and so on) mentioned in Sec. I. If the Gaussian LTB void model can be consistent with other observations (e.g. SNIa, CMB and so on), it cannot accommodate OHROs. However, this is not the worst. As is well known, the Hubble radius (Hubble horizon) $H_0^{-1} \approx 3.0 \ h^{-1} \text{ Gpc}$ characterizes the size of our observable universe. The size of the void should be much larger than the size of our observable universe to accommodate OHROs. This makes the Gaussian LTB void model invalid in fact. Of course, it is known that the lower Hubble constant, the larger cosmic age is. However, as shown in the panel (c) of Fig. 1 $r_0 > 20$ Gpc is still required even for a very low $h = 0.4$. So, this serious crisis cannot be alleviated with a lower Hubble constant.
B. The CGBH model

Next, we consider a simplified version of the so-called Garcia-Bellido-Haugbølle (GBH) model [110], namely the constrained GBH (CGBH) model [110] (see also e.g. [82]). In CGBH model, one also assume that the big bang is spatially homogeneous, namely $t_B$ is a constant which can be set to zero. So, Eq. (25) is valid in the CGBH model. The matter density function $\Omega_M(r)$ is given by [110] (see also e.g. [82])

$$\Omega_M(r) = 1 + (\Omega_{in} - 1) \left\{ \frac{1 - \tanh[(r - r_0)/2\Delta r]}{1 + \tanh(r_0/2\Delta r)} \right\},$$

(29)

where $\Omega_{in}$ is the matter density at the center of the void; $r_0$ describes the size of the void; $\Delta r$ characterizes the transition to uniformity. In this work, we only consider the case of $\Omega_{in} < 1$. From Eq. (25), we get

$$H_{\perp 0}(r) = H_0 \frac{\sqrt{\Omega_K(r)} - \Omega_M(r) \sinh^{-1} \frac{\Omega_K(r)}{H_M(r)}}{[\Omega_K(r)]^{3/2}},$$

(30)

where $\Omega_K(r) = 1 - \Omega_M(r)$, and $H_0$ actually plays the role of Hubble constant. So, there are four free model parameters, namely $\Omega_{in}$, $r_0$, $h$ (which is the Hubble constant $H_0$ in units of 100 km/s/Mpc), and $\delta r \equiv \Delta r/r_0$ (which is equivalent to $\Delta r$ in fact).

Similar to the previous subsection, we firstly scan the full parameter space to test this model with OHROs. However, since there are four free parameters in the CGBH model, it is difficult to plot a 4D parameter space. Instead, we consider the 3D plot of the allowed parameter space of the CGBH model with a fixed $h = 0.673$ coming from Planck CMB data [121], and we present it in Fig. 4. Note that the wide parameter ranges we scanned are $0.01 \leq \Omega_{in} \leq 0.99$, $1.0 \text{Gpc} \leq r_0 \leq 501.0 \text{Gpc}$, and $0.1 \leq \delta r \leq 0.9$. The absence of plot for OHRO at redshift $z = 1.55$ in Fig. 4 means that the entire plotted parameter space of the CGBH model with a fixed $h = 0.673$ is allowed for OHRO at $z = 1.55$. This can be seen clearly from Fig. 5 in which OHRO at $z = 1.55$ is below the lower boundary of the red region, and hence the entire parameter space is allowed in this case. Note that from the panel (b) in Fig. 4 a large $r_0 > 10 \text{Gpc}$ is required to accommodate OHRO at $z = 3.91$. Also, in Fig. 6 we show the 2D slices of the allowed parameter space with fixed $h = 0.623, 0.673, 0.738$, and $\delta r = 0.40, 0.64, 0.80$. Note that $\delta r = 0.64$ is
the best-fit value from SNIa, CMB and BAO \[110\], and $\delta r = 0.40$ and 0.80 are close to the edges of its 2σ confidence region. The absence of the contour lines for OHROs at $z = 1.43$ and 1.55 in the left panel of Fig. 3 means that the entire plotted parameter space of the CGBH model with a fixed $h = 0.623$ is allowed for these two OHROs. And the absence of the contour lines for OHRO at $z = 1.55$ in the middle panel of Fig. 3 means that the entire plotted parameter space of the CGBH model with a fixed $h = 0.673$ is allowed for OHRO at $z = 1.55$. This can be seen clearly from Fig. 5 in which OHRO at $z = 1.43$ is below the lower boundary of the green region, and OHRO at $z = 1.55$ is below the lower boundaries of both the red and green regions. From the three small panels in Fig. 5 and the panel (b) in Fig. 4 we see that at least $r_0 > 10$ Gpc is required to accommodate OHRO at $z = 3.91$. Therefore, the same crisis also exists in the CGBH model. There is a serious tension between this unusually large $r_0$ and the much lower $r_0$ of order 1.0 Gpc inferred from other observations (e.g. SNIa, CMB and so on) mentioned in Sec. I. If the CGBH LTB void model can be consistent with other observations, it cannot accommodate OHROs. Worst of all, the size of the void should be much larger than the size of our observable universe (characterized by the Hubble radius/horizon $H_0^{-1} \approx 3.0 h^{-1}$ Gpc \[123\]) to accommodate OHROs. This makes the CGBH LTB void model invalid in fact.

C. The GBH model

Finally, we consider the original version of GBH model \[110\], in which one does not assume that the big bang is spatially homogeneous. Therefore, Eqs. (25) and (26) are invalid, and hence $\Omega_M(r)$ and $H_\perp(r)$ should be specified independently. In GBH model, they are given by \[110\]

$$\Omega_M(r) = \Omega_{out} + (\Omega_{in} - \Omega_{out}) \begin{array}{ll}1 - \tanh[(r - r_0)/2\Delta r] \end{array} \begin{array}{ll}1 + \tanh(r_0/2\Delta r) \end{array},$$

$$H_{\perp}(r) = H_{out} + (H_{in} - H_{out}) \begin{array}{ll}1 - \tanh[(r - r_0)/2\Delta r] \end{array} \begin{array}{ll}1 + \tanh(r_0/2\Delta r) \end{array},$$

Fig. 5: The same as in Fig. 3 except for the CGBH model with an additional parameter $0.2 \leq \delta r \leq 0.9$. See the text for details.
where $\Omega_{\text{out}}$ is the asymptotic value of the matter density; $\Omega_{\text{in}}$ is the matter density at the center of the void; $H_{\text{out}}$ and $H_{\text{in}}$ describe the Hubble expansion rate outside and inside the void, respectively; $r_0$ describes the size of the void; $\Delta r$ characterizes the transition to uniformity. Following [110], we fix $\Omega_{\text{out}} = 1$. So, there are five free model parameters, namely $\Omega_{\text{in}}, r_0, \delta r \equiv \Delta r/r_0$ (which is equivalent to $\Delta r$ in fact), $h_{\text{in}}$ and $h_{\text{out}}$ (which are $H_{\text{in}}$ and $H_{\text{out}}$ in units of 100 km/s/Mpc).

Similar to the previous subsections, we try to scan the full parameter space to test this model with OHROs. However, since there are five free parameters in the GBH model, it is very difficult to plot a 5D parameter space. Instead, in Fig. 6 we show the 2D slices of the allowed parameter space with fixed $h_{\text{in}} = 0.50, 0.58, 0.70,$ and $h_{\text{out}} = 0.60, 0.49, 0.40,$ as well as $\delta r = 0.40, 0.62, 0.80$. Note that $h_{\text{in}} = 0.58, h_{\text{out}} = 0.49, \delta r = 0.62$ are the best-fit values from SNIa, CMB and BAO [110], and we appropriately vary these parameters to see their effect on the allowed parameter space. From Fig. 6, it is easy to see that the parameters $\delta r$ and $h_{\text{out}}$ have fairly minor effects on the allowed parameter space. On the other hand, comparing the three columns of Fig. 6 we find that the parameter $h_{\text{in}}$ plays a considerable role. The smaller $h_{\text{in}}$, the wider parameter space can be allowed. From the middle and right columns of Fig. 6 (especially from the small panels), it is easy to see that for $h_{\text{in}} \gtrsim 0.58$, a large $r_0 > 10$ Gpc is required to accommodate OHRO at $z = 3.91$. In this case, as in the previous two LTB void models, the serious crisis also exists in the GBH LTB void model. That is, there exists a serious tension between this unusually large $r_0 > 10$ Gpc and the much lower $r_0$ inferred from other observations (e.g. SNIa, CMB and so on) mentioned in Sec. II the size of the void should be much larger than the size of our observable universe to accommodate OHROs, and this makes the GBH LTB void model invalid in fact. However, for a very low $h_{\text{in}} = 0.50$, the required $r_0$ can be in a lower range $\sim 4 - 6$ Gpc to accommodate OHROs, as shown in the left column of Fig. 6. Note that the Hubble radius/horizon $H_0^{-1} \sim 3.0 h^{-1}$ Gpc $\sim 6$ Gpc for a very low $h \sim 0.50$. So, in this case, it is possible to accommodate OHROs while the size of the void is smaller than the size of our observable universe. However, if we further consider the constraints from other observations (e.g. SNIa, CMB and so on), the situation becomes subtle. In [110], the best-fit parameters with 2$\sigma$ uncertainties from SNIa, CMB and BAO are given by $h_{\text{in}} = 0.58 \pm 0.03, h_{\text{out}} = 0.49 \pm 0.2, \Omega_{\text{in}} = 0.13 \pm 0.06, r_0 = 2.3 \pm 0.9$ Gpc, $\delta r = 0.62 \pm (0.20)$. There exists still a remarkable tension far beyond 2$\sigma$ between OHROs and other observations, because $h_{\text{in}} = 0.50$ and $r_0 \sim 4 - 6$ Gpc can be excluded by other observations far beyond 2$\sigma$ regions. Even worse, the effect of $h_{\text{in}}$ is in contrast to the
one of $r_0$ actually. If we increase $h_{in}$, the required $r_0$ to accommodate OHRO at $z = 3.91$ will increase correspondingly, as shown in Fig. 7. Therefore, it is very difficult to conciliate both $h_{in}$ and $r_0$ with the higher $h_{in} = 0.58 \pm 0.03$ and the smaller $r_0 = 2.3 \pm 0.9$ from other observations (e.g. SNIa, CMB and BAO) at the same time, while we can still make the size of the void smaller than the size of our observable universe. In fact, we are in a serious dilemma. So, the age crisis cannot be completely alleviated in the GBH LTB void model, although it is in a situation slightly better than the Gaussian model and the CGBH model (but at the price of having more model parameters).

FIG. 7: The same as in Fig. 2 except for the GBH model with fixed $h_{in} = 0.50$ (left column), 0.58 (middle column), 0.70 (right column), $h_{out} = 0.60$ (top row), 0.49 (middle row), 0.40 (bottom row), $\delta r = 0.40$ (black contour lines), 0.62 (red contour lines), 0.80 (green contour lines), for OHROs at $z = 1.43$ (solid contour lines), $z = 1.55$ (dashed contour lines), $z = 3.91$ (dash-dotted contour lines). See the text for details.
IV. CONCLUSION AND DISCUSSION

As is well known, one can explain the current cosmic acceleration by considering an inhomogeneous and/or anisotropic universe (which violates the cosmological principle), without invoking dark energy or modified gravity. The well-known one of this kind of models is the so-called Lemaître-Tolman-Bondi (LTB) void model, in which the universe is spherically symmetric and radially inhomogeneous, and we are living in a locally underdense void centered nearby our location. In the present work, we test various LTB void models with some old high redshift objects (OHROs). Obviously, the universe cannot be younger than its constituents. We find that these OHROs bring a serious crisis to the LTB void models.

It is worth noting that in addition to the three OHROs used in this work, there are other OHROs in the literature, for instance, the 4.0 Gyr old radio galaxy 3C 65 at \( z = 1.175 \) \[124\], and the high redshift quasar B1422+231 at \( z = 3.62 \) whose best-fit age is 1.5 Gyr with a lower bound of 1.3 Gyr \[125\]. However, they cannot be used to constrain the models as restrictive as the three OHROs used in this work. So, we do not consider them here. On the other hand, 9 extremely old globular clusters in M31 galaxy \[126, 127\] were considered in \[102\]. However, their ages are estimated to be in the range \( 14 - 16 \) Gyr \[126, 127\], which is much larger than the total age of the universe \( \sim 13.8 \) Gyr inferred from the CMB observations (e.g. WMAP \[3\] and Planck \[121\]). This makes these 9 extremely old globular clusters not so reliable. Therefore, they have not been used in most of the relevant works.

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