Consistency tests of AMPCALCULATOR and chiral amplitudes in SU(3) Chiral Perturbation Theory: A tutorial based approach

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Abstract

Ampcalculator is a Mathematica\textsuperscript{©} based program that was made publicly available some time ago by Unterdorfer and Ecker. It enables the user to compute several processes at one-loop (upto $O(p^4)$) in $SU(3)$ chiral perturbation theory. They include computing matrix elements and form factors for strong and non-leptonic weak processes with at most six external states. It was used to compute some novel processes and was tested against well-known results by the original authors. Here we present the results of several thorough checks of the package. Exhaustive checks performed by the original authors are not publicly available, and hence the present effort. Some new results are obtained from the software especially in the kaon odd-intrinsic parity non-leptonic decay sector involving the coupling $G_{27}$. Another illustrative set of amplitudes at tree level we provide is in the context of $\tau$-decays with several mesons including quark mass effects, of use to the BELLE experiment. All eight meson-meson scattering amplitudes have been checked. Kaon-Compton amplitude has been checked and a minor error in published results has been pointed out. This exercise is a tutorial based one, wherein several input and output notebooks are also being made available as ancillary files on the arXiv. Some of the additional notebooks we provide contain explicit expressions that we have used for comparison with established results. The purpose is to encourage users to apply the software to suit their specific needs. An automatic amplitude generator of this type can provide error-free outputs that could be used as inputs for further simplification, and used in varied scenarios such as applications of chiral perturbation theory at finite temperature, density and volume. This can also be used by students as a learning aid in low-energy hadron dynamics.
1 Introduction

$SU(3)$ chiral perturbation theory (ChPT) is a mature subject and has been over the years tested in great detail. Since the pioneering work of Gasser and Leutwyler \([1, 2]\), many teams have worked hard and have produced a large body of work and have computed processes of interest to experiment and theory. The processes that have been computed include form factors and scattering amplitudes of importance to electromagnetic interactions and weak interactions. There are also odd-intrinsic parity processes which have been computed. Several non-leptonic decays of kaons have also been studied which deals with a near independent sector \([3]\). In general, results have appeared in the literature over the last couple of decades and virtually all processes that are tractable and of interest to phenomenology and experiment are now exhausted.

Some time ago, a very useful Mathematica\textsuperscript{©} based program that can compute amplitudes in $SU(3)$ ChPT in the even-intrinsic parity and odd-intrinsic parity (anomaly mediated) processes due to Unterdorfer and Ecker (UE)\([4]\) has been made publicly available. With the exception of certain anomalous processes, the program is capable of producing a representation for form factors and scattering amplitudes in the theory with user supplied input for the choice of particles and momenta for up to six external particles (with a photon and $W$-boson counting for 2 particles each). UE have developed the program for evaluating amplitudes for some hitherto unstudied processes and also to check amplitudes for known processes such as $e^+e^- \rightarrow 4\pi$ and a $\tau \rightarrow 4\pi$ \([5]\), of importance to, e.g, TAUOLA \([6, 7]\). More recently, Ampcalculator (AMPC) was used to look at the decay $K \rightarrow \pi l^+l^-$ \([8]\) and a missing $G_{27}$ piece in the $SU(3)$ one-loop amplitudes was found.

In the light of the above, it is perhaps a useful exercise to employ AMPC to try and build an exhaustive library of Mathematica\textsuperscript{©} based programs that can check the existing results in the literature and alternatively to use the published results to check the consistency of AMPC. Our aim is to provide a first attempt at such a compilation. In many cases, we also provide Mathematica\textsuperscript{©} input for each of the programs and the corresponding output notebooks obtained by us. It may be noted that AMPC may be sensitive to version of Mathematica\textsuperscript{©} used, as it was first written in Mathematica\textsuperscript{©} 5. Here we also provide a dictionary for translating the loop functions coming out of AMPC denoted by the $A$ and $B$ into more familiar functions. We also carry out some simple tree-level computations which are of importance to $\tau$-decays. Although the issue of the neglect of the quark masses was noted in Finkemeier et al.\([9]\), even today experiments appear to use the work of Aubrecht, et.al \([10]\), especially when $\eta$ mesons are in the final state \([11]\). In order to draw the attention of the community to this, we carry out tree-level computations of all the relevant processes using AMPC and provide a detailed comparison with the results of Ref. \([10]\), so that experimentalists...
may update their data bases using information that does not neglect quark masses. It is our belief that AMPC can provide readily accessible results also of importance to experimental efforts such as the BELLE.

The motivation of the present work is also to present a thorough comparison to the extent possible with amplitudes and form factors that are sufficiently simple. Amplitudes that involve a large number of particles gives rise to results that are not easily amenable to comparison. Examples include \( K \rightarrow 3\pi \), \( \tau \rightarrow 3\pi\nu_\tau \) decays. We do not provide a comparison with these amplitudes. However, it should be noted that as recently as two years ago, one of the AMPC accessible processes was computed in a heroic effort by Kaiser [12] who computed the amplitudes for the processes \( \pi^-\gamma \rightarrow 3\pi \) diagram by diagram. In future, AMPC could be employed for such practical needs.

Let us recall some essential facts. Some of the basic processes in one loop \( SU(3) \) ChPT that were first studied were form factors that enter into weak decays of mesons. These are readily produced by AMPC by providing as input the kaon, pion and the \( W \)-boson, and the kaon, \( \eta \) and the \( W \)-boson. These when properly normalized yield the \( K\pi \) and \( K\eta \) form factors. We have checked the amplitudes from AMPC and we present the results.

Of the basic meson processes, the earliest to have been computed are the \( K\pi \) [13] and \( \pi\eta \) scattering amplitudes [14]. The \( \pi\pi \) [15] amplitude was also computed by these authors. The two \( KK \) amplitudes were computed by Guererro and Oller [16]. These have all been collected by Gomez-Nicola and Pelaez (GNP) [17]. In addition they computed the three remaining amplitudes, the \( K\eta \) elastic, \( K\eta \rightarrow K\pi \), and \( \eta\eta \) scattering amplitudes. Here we explicitly provide notebooks that produce the results from AMPC. We have checked all the amplitudes in GNP and find complete agreement, when the Gell-Mann-Okubo (GMO) relation is used both in their results as well as in AMPC result.

Another process we have looked at is the Kaon-Compton process which was studied in [18], see also [19]. By fixing a factor of 4 in \( B(\tau\nu) \) in Ref. [18], we bring AMPC and [18] into agreement. The loop part agrees and we do not repeat it here. Our example is done setting the AMPC switch “onlytreep2 = 1”.

It is possible to employ AMPC to compute several amplitudes in the odd-intrinsic parity sector or the anomaly sector. We have carried out what we believe to be a comprehensive test of AMPC accessible amplitudes that are available in the literature. Of special interest are the non-leptonic kaon decays. We verify the results expressed in Table 1 of Ecker et al.[20] and provide the explicit contributions to the amplitudes. In addition, we have generated all the contributions from the 27-plet contributions.

It should be kept in mind that this report is to serve primarily as a user manual-cum-report on checks carried out. It is not meant to be a comprehensive review of existing results. We also provide references to those
published works with which our comparisons have been made, which are not often the first to report results. Earlier references may be traced from those.

2 Chiral amplitudes and Form Factors

In this section, we present various checks and results obtained with AMPC. Here all the external particles, including final states are treated as incoming and hence the signs of the momenta are labeled accordingly while writing out the momentum conservation for each process. We give all these specifications along with the associated scalar products explicitly in the input notebooks. As mentioned in Sec. 1, we caution the reader that AMPC has originally been written in Mathematica® 5 due to which the older subroutine gives null result for some processes owing to possible incompatibility of Mathematica® fonts. We deposit a new version\(^1\) of the subroutine which was made available later and have been added to the ancillary files of the arXiv submission. One of these two versions reproduce the result, for instance, in the case of the odd-parity sector \(\pi^+ \rightarrow l^+ \nu \gamma\), only the new subroutine reproduces the required result while the old one does not. It may thus be noted that in this comprehensive study which have been carried out, we have found one or another version that yields the results, although a priori we could not say which would work. In what follows, unless otherwise mentioned, we use the old subroutine for the various processes under study. To our knowledge both versions give identical in case where no Lev-Civita symbol is involved, i.e in pure even-intrinsic parity sector. We will indicate explicitly whenever we use the new subroutine. Further, we compare our AMPC results with those in the literature whenever available.

2.1 Odd-intrinsic parity sector

2.1.1 \(\pi^0 \rightarrow \gamma \gamma\)

The AMPC input for this process is given as

\[
\{\pi_0 (p_1), \gamma (k_1), \gamma (k_2)\}
\]

The anomalous term contributing to the total amplitude for this process reads

\[
- \frac{e^2 (k_1)_{\epsilon}(p_1)_{\mu} \epsilon \epsilon_{\sigma \tau} \epsilon (k_1)_{\sigma} \epsilon (k_2)_{\tau}}{4\pi^2 F_\pi}
\]

The AMPC Mathematica® notebooks containing the above expressions are given in \texttt{Ip01.nb} and \texttt{Op01.nb}. We have checked our result with the expression given in Eq. (5.1), section VI of Donoghue et.al \([21]\) and Eq. (160)

\(^1\)We thank Gerhard Ecker for providing us both version. At present, the new version is posted on his home page.
of Ref. [22]. We caution the reader there is a missing factor of $i$ in the AMPC result compared to that of the established result found in [21].

2.1.2 $\eta_8 \rightarrow \gamma\gamma^*$

The AMPC input for this process is given as

$$\{\eta_8(p_1), \gamma(p_3), \gamma(p_2)\}$$

The anomalous term contributing to the total amplitude for this process reads

$$-\frac{e^2(p_1)\xi(p_3)\mu\epsilon(p_2)\sigma\epsilon(p_3)\tau}{4\sqrt{3}\pi^2 F_\pi}$$ (2)

The AMPC Mathematica© notebooks containing the above expressions are given in Ip02.nb and Op02.nb. We have checked our result with the expression given in Eq. (160) of ref. [22]. Here also, there is a missing factor of $i$ in the AMPC result compared to that of the established result found in [22].

2.1.3 $\pi^+ \rightarrow l^+\nu\gamma$

The AMPC input for this process is given as

$$\{\pi^+(k), W_-(Q), \gamma(r)\}$$

The anomalous term contributing to the total amplitude for this process reads

$$\frac{eG_Fk_\mu Q_\nu\bar{V}_{ud}\epsilon(r)\tau}{8\pi^2 F_\pi}$$ (3)

The AMPC Mathematica© notebooks containing the above expressions are given in Ip03.nb and Op03.nb. We have checked our result against the expression given in Eq. (7) of ref. [23]. We find that our result agrees except for the fact that we need to use the newer version of the AMPC subroutine which is given. For comparison purpose, we simplify our results by replacing $Q$ by $-q$ so that the lepton pairs are outgoing and also making the replacement $f = \sqrt{2}F_\pi$ where $f$ is the pion decay constant $f_\pi = f = 132\text{MeV}(= f_K$, at lowest order). We have added these remarks to assist the reader with differing conventions.

2.1.4 $K^+ \rightarrow l^+\nu\gamma$

The AMPC input for this process is given as

$$\{K_+(k), W_-(Q), \gamma(r)\}$$
The anomalous term contributing to the total amplitude for this process reads
\[
\frac{\epsilon G_F k \xi \rho \eta \sigma \mu \nu \sigma \tau}{8 \pi^2 F_\pi}
\]  
(4)

The AMPC Mathematica® notebooks containing the above expressions are given in Ip04.nb and Op04.nb. We have checked our result with the expression given in Eq. (8) of Ref. [23]. We find that our result agrees except for a factor of $m_K/m_\pi$, which therefore limits the use of AMPC in this and related process. Here also, we obtain the result only with the newer version of the AMPC subroutine. As in the previous case, we replace $Q$ by $-q$ so that the lepton pairs are outgoing and also make the replacement $f = \sqrt{2} F_\pi$.

2.1.5 $\eta_8 \to \pi^+ \pi^- \gamma$

The AMPC input for this process is given as
\[ \{ \eta_8(p_3), \pi^-(p_1), \pi^+(p_2), \gamma(q) \} \]

The anomalous term contributing to the total amplitude for this process reads
\[
\frac{\epsilon(p_1) \xi(p_2) \rho(p_3) \sigma \epsilon(q) \tau}{4 \sqrt{3} \pi^2 F_\pi^3}
\]  
(5)

The AMPC Mathematica® notebooks containing the above expressions are given in Ip05.nb and Op05.nb. We have checked our result against the expression given in Eq. (2) of Ref.[24]. Our results agree.

2.1.6 $\tau^- \to \eta_8 \pi^- \pi^0 \pi^0 \nu$

The AMPC input for this process is given as
\[ \{ W_-(q), \eta_8(k), \pi^+(p_1), \pi_0(p_2), \pi_0(p_3) \} \]

The anomalous term contributing to the total amplitude for this process reads
\[
\frac{i G_F k \xi \mu (p_1) \rho q \nu \hat{V}_{ud} \xi \rho \sigma \mu \nu}{4 \sqrt{3} \pi^2 F_\pi^4}
\]  
(6)

The AMPC Mathematica® notebooks containing the above expressions are given in Ip06.nb and Op06.nb.

2.1.7 $\tau^- \to \eta_8 \pi^- \pi^0 \nu$

The AMPC input for this process is given as
\[ \{ W_-(p), \eta_8(q_1), \pi^+(q_2), \pi_0(q_3) \} \]
The anomalous term contributing to the total amplitude for this process reads

$$\frac{-G_{F}l_{\mu}p_{\xi}(q_{1})_{\rho}(q_{2})_{\sigma}\tilde{V}_{ud}\epsilon^{\rho\sigma\mu}}{4\sqrt{3}\pi^{2}F_{3}^{3}}$$  \hspace{1cm} (7)$$

The AMPC Mathematica© notebooks containing the above expressions are given in \textbf{Ip07.nb} and \textbf{Op07.nb}. We obtain this result with the newer version of the AMPC subroutine.

**2.1.8 \quad \tau^{-}\rightarrow K^{-}\pi^{-}K^{+}\nu**

The AMPC input for this process is given as

$$\{W_{-}(q), K_{+}(k_{1}), \pi_{+}(p), K_{-}(k_{2})\}$$

The anomalous term contributing to the total amplitude for this process reads

$$\frac{-G_{F}(k_{1})_{\alpha}l_{\mu}p_{\xi}q_{\rho}\tilde{V}_{ud}\epsilon^{\rho\sigma\mu}}{4\pi^{2}F_{3}^{3}}$$  \hspace{1cm} (8)$$

The AMPC Mathematica© notebooks containing the above expressions are given in \textbf{Ip08.nb} and \textbf{Op08.nb}. Here also, we obtain the results with the newer version of the AMPC subroutine.

**2.2 \quad \gamma\pi^{-}\rightarrow \pi^{-}\pi^{0}**

The AMPC input for this process is given as

$$\{\gamma(A), \pi_{-}(p_{1}), \pi_{+}(p_{2}), \pi_{0}(p_{0})\}$$

The anomalous term contributing to the total amplitude for this process reads

$$\frac{-eA_{\xi}(p_{1})_{\alpha}(p_{2})_{\rho}\epsilon^{\rho\sigma\tau}\epsilon(A)_{\tau}}{4\pi^{2}F_{3}^{3}F_{\pi}}$$  \hspace{1cm} (9)$$

The AMPC Mathematica© notebooks containing the above expressions are given in \textbf{Ip09.nb} and \textbf{Op09.nb}. We have checked our result against the expression given in Eq. (202) of [22]. We caution the reader there is a missing factor of $i$ in the AMPC result compared to that of the established result found in [22].

**2.3 \quad K_{l4} \: decay.**

We investigated the anomalous part of the $K_{l4}$ decay $K^{+}(p)\rightarrow l(q_{1})\nu(q_{0})\pi^{-}(q_{2})\pi^{+}(q_{1})$ in AMPC. The AMPC input for the process is

$$\{K_{+}(p), W_{-}(q), \pi_{+}(q_{2}), \pi_{-}(q_{1})\}$$
where, $q = q_l + q_\nu$. The anomalous part obtained from AMPC is

$$\frac{G_F V_{ub} \epsilon^{\rho\sigma\mu\nu} p_\rho q_\nu q_{2\sigma} l_{\mu}}{4\pi^2 F_\pi^3}.$$  \hspace{1cm} (10)

where $l_{\mu}$ is the leptonic part of the amplitude. This result can be compared with Eq.(8) of [23].

It may be noted that the old version of the AMPC does not give the anomalous part, and it can be only obtained in the new version of the AMPC. The input and output notebook for this process are Ip10.nb and Op10.nb respectively. This notebook also produces the even-intrinsic part which agrees with [23]. Note here that the pole part is correctly reproduced by AMPC.

2.4 Chiral anomaly in nonleptonic radiative kaon decays.

The chiral anomaly in the non-leptonic radiative kaon decays is discussed in detail in [20]. Such decays can be described by the $\Delta S = 1$ weak Hamiltonian

$$\mathcal{H}^{\Delta S=1} = \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \Sigma_i C_i Q_i + hc,$$ \hspace{1cm} (11)

where $Q_i$ are the four quark operators and $C_i$ Wilson coefficients. The Lagrangian (11) has two parts, one that transforms as an octet and another as a 27-plet under the chiral transformation. The corresponding coupling constants are $G_8$ and $G_{27}$. The chiral anomaly contributes to the coefficients $N_{28}, N_{29}, N_{30}, N_{31}$ of the octet operator and to the coefficients $R_{21}, R_{22}, R_{23}$ of the 27-plet operator. The anomaly contribution coming from the octet and 27-plet part of the amplitude are isolated by separating the coefficients of $N$ and $R$ respectively. It may be noted that in AMPC the coupling constant for the octet and the 27-plet part for $K^0$ decay are $G_8$ and $G_{27}$, and for a $K^0$ decay they are called $\hat{G}_8$ and $\hat{G}_{27}$. However in the limit of $CP$ conservation $\hat{G}_8 \rightarrow G_8$ and $\hat{G}_{27} \rightarrow G_{27}$. Using AMPC, we calculate the amplitude of $K^{+0}$ decay into two and three pions and a photon and the anomaly part of the amplitude is checked against the Table 1 of [20]. It may be noted that the explicit amplitudes of all the AMPC accessible decays considered here are not given in [20]. However we find agreement with [20] regarding the anomaly contributions coming from different octet and 27-plet operators.
2.4.1 Anomaly contribution to $K^+ \rightarrow \pi^+\pi^0\gamma$ decay.

We consider the process $K^+(k) \rightarrow \pi^+(p_1)\pi^0(p_2)\gamma(q)$. The anomaly contribution coming from the octet part of the contribution is

$$A^8 = \frac{8ieG_8 k_2 (N_{30}^r - 3N_{20}^r) p_{1\rho}p_{2\sigma}\epsilon^{\xi\rho\sigma\tau}\epsilon(q)_\tau}{F_\pi}$$

(12)

The anomaly coming from the 27-plet part of the amplitude is given by

$$A^{27} = \frac{2eG_{27} k_2 p_{1\rho}p_{2\sigma} (5R_{22}^r - 3R_{23}^r) \epsilon^{\xi\rho\sigma\tau}\epsilon(q)_\tau}{3F_\pi}$$

(13)

The input and output notebooks for this process are given in Ip11.nb Op11.nb respectively.

2.4.2 Anomaly contribution to $K^+ \rightarrow \pi^+\pi^0\pi^0\gamma$ decay.

The anomaly contribution coming from the octet part of the Lagrangian for the process $K^+(p) \rightarrow \pi^+(q)\pi^0(r)\pi^0(s)\gamma(t)$ is

$$A^8 = \frac{4ieG_8 (N_{s0}^r - 3N_{r0}^r) p_{\xi\rho} (r_\sigma + s_\sigma) \epsilon(t)_\tau}{F_\pi^2}$$

(14)

and the contribution coming from the 27-plet part is

$$A^{27} = \frac{ieG_{27} p_{\xi\rho} (5R_{22}^r - 3R_{23}^r) \epsilon^{\xi\rho\sigma\tau}\epsilon(t)_\tau}{3F_\pi^2}$$

(15)

The input and the output notebooks for this process are given in Ip12.nb Op12.nb respectively.

2.4.3 Anomaly contribution to $K^+ \rightarrow \pi^+\pi^+\pi^-\gamma$ decay.

We considered the process $K^+(p_1) \rightarrow \pi^+(p_2)\pi^+(p_3)\pi^-(p_4)\gamma(q)$. The anomaly contribution coming from the octet part of the Lagrangian is

$$A^8 = \frac{16ieG_8 (N_{20}^r + N_{31}^r) p_{1\xi} (p_{2\rho} + p_{3\rho}) p_{4\sigma}\epsilon^{\xi\rho\sigma\tau}\epsilon(q)_\tau}{F_\pi^2}$$

(16)

The anomaly contribution from the 27-plet part is

$$A^{27} = \frac{8ieG_{27} p_{1\xi} (p_{2\rho} + p_{3\rho}) p_{4\sigma} (R_{22}^r - 3R_{23}^r) \epsilon^{\xi\rho\sigma\tau}\epsilon(q)_\tau}{3F_\pi^2}$$

(17)

The input and the output notebooks for this process are given in Ip13.nb Op13.nb respectively.
2.4.4 Anomaly contribution to $K_{L,S} \to \pi^+\pi^-\pi^0\gamma$ decay.

In the limit of $CP$ conservation we can write the $K_L$ and $K_S$ as

\[
K_L = \frac{1}{\sqrt{2}}(K^0 + \overline{K}^0)
\]

\[
K_S = \frac{1}{\sqrt{2}}(K^0 - \overline{K}^0)
\]

Using AMPC the decay $K_0(k), \overline{K}_0(k) \to \pi^+(-p)\pi^-(-p_2)\pi^0(-p_3)\gamma(-q)$, are calculated and the anomaly part of the amplitude is separated. By adding and subtracting, we get respectively the anomaly contributions of the amplitude in $K_L$ and $K_S$ decays.

\[
A_{K_L}^{G_{27}} = \frac{4i\sqrt{2}\alpha G_{27}k_\xi \left((p_1)_\rho + (p_2)_\rho\right)(p_3)_\sigma R_{21}^{\rho} \epsilon^{\xi\rho\sigma\tau} \epsilon(q)_\tau}{3F_\pi^2} + \frac{i\sqrt{2}\alpha G_{27}k_\xi R_{23}^{\rho} \epsilon^{\xi\rho\sigma\tau} \epsilon(q)_\tau (2(p_2)_\sigma - 7(p_3)_\sigma) - 9(p_2)_\rho(p_3)_\sigma}{3F_\pi^2} \epsilon(q)_\tau - \frac{i\sqrt{2}\alpha G_{27}k_\xi R_{22}^{\rho} \epsilon^{\xi\rho\sigma\tau} \epsilon(q)_\tau (25(p_2)_\rho(p_3)_\rho + (p_1)_\rho(6(p_2)_\rho + 31(p_3)_\rho)} {3F_\pi^2} \epsilon(q)_\tau
\]

\[
A_{K_S}^{G_{27}} = -\frac{4i\sqrt{2}\alpha G_{27}k_\xi \left((p_1)_\rho + (p_2)_\rho\right)(p_3)_\sigma R_{20}^{\rho} \epsilon^{\xi\rho\sigma\tau} \epsilon(q)_\tau}{3F_\pi^2} \frac{i\sqrt{2}\alpha G_{27}(p_3)_\sigma R_{23}^{\rho} \epsilon^{\xi\rho\sigma\tau} \epsilon(q)_\tau (5k_\xi(p_1)_\rho + (p_2)_\rho(9k_\xi + 4(p_1)_\xi))}{3F_\pi^2} \frac{i\sqrt{2}\alpha G_{27}(p_3)_\sigma R_{22}^{\rho} \epsilon^{\xi\rho\sigma\tau} \epsilon(q)_\tau (8k_\xi(p_1)_\rho + (p_2)_\rho(19k_\xi(p_1)_\rho + (p_2)_\rho(7k_\xi - 4(p_1)_\xi)))}{3F_\pi^2} \epsilon(q)_\tau
\]

The input and output notebooks for the processes $K_0 \to \pi^+\pi^-\pi^0\gamma$ and $\overline{K}_0 \to \pi^+\pi^-\pi^0\gamma$ can be found in Op14.nb and Op14.nb respectively. We
have extracted the anomalous parts of the amplitude \( K_{L,S} \rightarrow \pi^+\pi^-\pi^0\gamma \) in the same notebook.

### 2.4.5 Anomaly contribution in \( K_{L,S} \rightarrow \gamma\pi^+\pi^- \)

We calculate the anomaly contribution in \( K_{L,S} \rightarrow \gamma\pi^+\pi^- \) in the same way as we did in the previous sections. Using AMPC we calculate the decay \( K_0(p) \rightarrow \gamma(-q)\pi^-(-p_1)\pi^+(-p_2) \), and \( \overline{K}_0(p) \rightarrow \gamma(-q)\pi^-(-p_1)\pi^+(-p_2) \) and extract the anomaly parts coming from the octet and 27-plet part of the Lagrangian. We finally add and subtract these anomaly parts to obtain the anomaly contribution to \( K_{L,S} \rightarrow \gamma\pi^+\pi^- \) decay.

\[
\begin{align*}
A_{K_L}^{G_8} &= 16\sqrt{2}eG_8N_{29\rho}(p_1)_\rho q_\mu \epsilon^{\rho\sigma\tau\nu} \epsilon(q)_\tau F_\pi + 16\sqrt{2}eG_8N_{31\rho}(p_1)_\rho q_\mu \epsilon^{\rho\sigma\tau\nu} \epsilon(q)_\tau F_\pi \\
A_{K_S}^{G_8} &= 0 \\
A_{K_L}^{G_{27}} &= \frac{16\sqrt{2}eG_{27}\rho}(p_1)_\rho q_\mu R_{22}^{\rho}(\epsilon^{\rho\sigma\tau\nu} \epsilon(q)_\tau) F_\pi \\
A_{K_S}^{G_{27}} &= \frac{4\sqrt{2}eG_{27}\rho}(p_1)_\rho q_\mu R_{22}^{\rho}(\epsilon^{\rho\sigma\tau\nu} \epsilon(q)_\tau) F_\pi - \frac{4\sqrt{2}eG_{27}\rho}(p_1)_\rho q_\mu R_{23}^{\rho}(\epsilon^{\rho\sigma\tau\nu} \epsilon(q)_\tau) F_\pi
\end{align*}
\]

The input and output notebooks for the processes \( K_0 \rightarrow \gamma\pi^+\pi^- \) and \( \overline{K}_0 \rightarrow \gamma\pi^+\pi^- \) can be found in \texttt{Ip15.nb} and \texttt{Op15.nb} respectively. We have extracted the anomalous parts of the amplitude \( K_{L,S} \rightarrow \gamma\pi^+\pi^- \) in the same notebook.

### 2.5 Form Factors results from AMPC

We present a check for the \( \pi^+\pi^+ \), \( K^+K^- \), \( K^0\overline{K}^0 \), \( K^+\eta \) and the \( K^+\pi^0 \) form factors given in Eq. (2.1) of Gasser et al., [2]. The matrix elements are defined below

\[
\begin{align*}
\langle \pi^+|j_\mu|\pi^+ \rangle &= (p'_\mu + p_\mu)F_{V}(t), \\
\langle K^+|j_\mu|K^+ \rangle &= (p'_\mu + p_\mu)F_{V}(K^+)(t), \\
\langle K^0|j_\mu|K^0 \rangle &= (p'_\mu + p_\mu)F_{V}(K^0)(t), \\
\langle K^+|\overline{\psi}\gamma_\mu s|\eta \rangle &= \sqrt{3}2[(p'_\mu + p_\mu)f_+^{K^+\eta}(t) + (p'_\mu - p_\mu)f_-^{K^+\eta}(t)], \\
\langle K^+|\overline{\psi}\gamma_\mu s|\pi^0 \rangle &= \sqrt{1}2[(p'_\mu + p_\mu)f_+^{K^+\pi^0}(t) + (p'_\mu - p_\mu)f_-^{K^+\pi^0}(t)]
\end{align*}
\]

(18)
The above processes are AMPC accessible since they appear in semi-leptonic weak decays. The form factors appear in the above matrix elements which are denoted by \( f^+_{K\pi}(t), f^0_{K\pi}(t) \) and \( f^K_{\eta}(t), f^K_{\eta}(t), f^K_{\eta}(t) \) respectively for the \( K^+\pi^0 \) and the \( K^+\eta \). We have extracted the coefficients for these vector currents from our AMPC amplitudes for the given processes with appropriate matching of various functions appearing in both the AMPC and the the results in [2]. We check explicitly against the various form factor definitions given in Eq.(2.4) of [2]. The AMPC results are given in terms of the \( \bar{A} \) and \( \bar{B} \) functions where

\[
\bar{A}[M^2] = -M^2/(4\pi^2)\ln[M^2/\mu^2], \quad \bar{B}[t, M_P^2, M_Q^2] = \bar{J}[t, M_P^2, M_Q^2] \tag{19}
\]

with the \( \bar{J}[t, M_P^2, M_Q^2] \) as given in [1]. We present the various definitions found in [1, 2] required for evaluating the form factors.

\[
H_{PQ}(t) = \frac{1}{F_0}(tM^r(t) - L(t)) + \frac{2}{3F_0}L_0 t, \tag{20}
\]

\[
M^r(t) = \frac{1}{12t}(t - 2\Sigma)\bar{J}(t) + \frac{\Delta^2}{3t^2}\bar{J}(t) - \frac{1}{6}k + \frac{1}{288\pi^2}, \tag{21}
\]

where,

\[
\Sigma = M_P^2 + M_Q^2, \quad \Delta = M_P^2 - M_Q^2, \quad L(t) = \frac{\Delta^2}{4t}\bar{J} \tag{22}
\]

Two cases arise where the form factors could be for the equal mass like in the case of the \( \pi\pi \) and \( KK \) while they could be for the unequal mass case like in the \( K^+\pi^0 \) and the \( K^+\eta \).

**Case 1:** \((M_P = M_Q = M)\),

\[
\bar{J}(t) = \frac{1}{16\pi^2} \left( \sigma(t)\ln\frac{\sigma(t)}{\sigma(t) + 1} + 2 \right), \tag{23}
\]

where,

\[
\sigma(t) = \sqrt{1 - \frac{4M^2}{t}}, \tag{24}
\]

Also,

\[
\bar{J}'(0) = \frac{1}{96\pi^2 M^2}, \tag{26}
\]

---

\(^2\)The standard loop functions can be found in [1, 2]. They are given here in the interest of making this paper fully self-contained. The reader may always consult primary references for further clarification if required.
\[ k = \frac{1}{32\pi^2} \left( \ln \frac{M^2}{\mu^2} + 1 \right) \]  
\tag{27}

**Case 2:** \((M_P \neq M_Q)\),

\[ \bar{J}(t) = \frac{1}{32\pi^2} \left( 2 + \frac{\Delta \ln M_Q^2}{t M_P^2} - \sum \frac{M_Q^2}{M_P^2} \ln \frac{M_Q^2}{M_P^2} - \frac{\nu}{t} \ln \frac{(t + \nu)^2 - \Delta^2}{(t - \nu)^2} \right) \]  
\tag{28}

where,

\[ \nu^2 = t - (M_P + M_Q)^2(t - (M_P - M_Q)^2) \]  
\tag{29}

Also,

\[ \bar{J}'(0) = \frac{1}{32\pi^2} \left( \sum \frac{M_Q^4}{M_P^4} - \frac{2}{\Delta^4} \ln \frac{M_Q^2}{M_P^2} - \frac{\nu}{t} \ln \frac{(t + \nu)^2 - \Delta^2}{(t - \nu)^2} \right) \]  
\tag{30}

\[ k = \frac{1}{32\pi^2} \left( M_P^2 \ln \frac{M_P^2}{\mu^2} - M_Q^2 \ln \frac{M_Q^2}{\mu^2} \right) \frac{1}{\Delta} \]  
\tag{31}

We recall that the matrix element for the weak decay is given by

\[ \mathcal{M} = -\frac{G_F}{\sqrt{2}} V_{CKM} j_\mu J^\mu \]  
\tag{32}

where, \(j_\mu\) and \(J^\mu\) are the leptonic and hadronic currents respectively, and \(V_{CKM}\) is the CKM elements. In AMPC the weak decay amplitude is written with a vertex factor \(-G_F\). To match the AMPC conventions with that of in literatures, we have multiplied the AMPC results with \(\sqrt{2}\).

The input and output notebooks for the form factors of different processes considered are tabulated in Table 1.

Table 1: Input and output notebooks for the form factors.

| Process | Input Notebook | Output Notebook |
|---------|----------------|----------------|
| \(\pi^+\pi^+\) | Ip16a.nb | Op16a.nb |
| \(K^+K^-\) | Ip16b.nb | Op16b.nb |
| \(K^0\bar{K}^0\) | Ip16c.nb | Op16c.nb |
| \(k^+\eta\) | Ip16d.nb | Op16d.nb |
| \(K^+\pi^0\) | Ip16e.nb | Op16e.nb |
Using the above expressions for the equal mass case and doing the necessary simplifications, we find that our AMPC result agrees for the equal mass case. The comparison and simplification is given in detail in the AMPC notebook Op16equal.nb. Coming to the unequal mass case, the $f_+(t)$ agrees for both the $K\pi$ and the $K\eta$ for the expression given in [2]. The comparison is given in detail in the AMPC notebook Op16unequal.nb, Op16d1.nb, Op16e1.nb. We check $f_-(t)$ for the $K\pi$ form factor using the expression given in Eq. (4.4) of [25] against our result and find that they agree. As a check we also do the calculation for $f_-(t)$ given in Eq.(4.3) and find agreement. For details of comparison, see the AMPC notebook Op16e2.nb.

In doing this calculation, the expressions for the various loop functions are taken from [4] which are introduced below.

$$\bar{B}_{20}(t) = \frac{-t - 3M_P^2 - 3M_Q^2}{288\pi^2} + \frac{\bar{A}(M_Q^2) + 2\bar{B}(t)M_P^2 - (M_P^2 - M_Q^2 + t)\bar{B}_{11}(t)}{6}$$

$$\bar{B}_{22}(t) = \frac{t - 3M_P^2 - 3M_Q^2}{288\pi^2} + \frac{\bar{A}(M_Q^2) - \bar{B}(t)M_P^2 + 2(M_P^2 - M_Q^2 + t)\bar{B}_{11}(t)}{3t}$$

$$\bar{B}_{11}(t) = \frac{-\bar{A}(M_P^2) + \bar{A}(M_Q^2) + \bar{B}(t)(M_P^2 - M_Q^2 + t)}{2t}$$

where,$$
\bar{B}(t) = \bar{B}(t) + B(0)$$

$$(33)$$

$$B(t) = \frac{1}{32\pi^2}(2 + \frac{M_P^2 - M_Q^2}{t})\ln \frac{M_P^2}{M_P^2 - M_Q^2} - \frac{M_P^2 + M_Q^2}{M_P^2 - M_Q^2} \ln \frac{M_P^2}{M_P^2 - M_Q^2} - \frac{\sqrt{\lambda(t, M_P^2, M_Q^2)}}{t}$$

$$\ln \left(\frac{(t + \sqrt{\lambda(t, M_P^2, M_Q^2)})^2 - (M_P^2 - M_Q^2)^2}{(t - \sqrt{\lambda(t, M_P^2, M_Q^2)})^2 - (M_P^2 - M_Q^2)^2}\right)$$

$$(34)$$

$$B(0) = \frac{\bar{A}(M_P^2) - \bar{A}(M_Q^2)}{M_P^2 - M_Q^2}$$

$$(35)$$

It may be noted that the notations in [25] are different from that of [4]. Specifically we give the relations, $\bar{B}_{22}[25] = \bar{B}_{20}[4]$, $\bar{B}_{21}[25] = \bar{B}_{22}[4]$, $\bar{B}[25] = B[4]$. This is done by comparing the Lorentz structures in the expressions (B.1) of [26] with eqn (B.5) of [4].

2.6 Amplitude for Kaon polarizability $\gamma K^+ \rightarrow \gamma K^+$

One of the AMPC applications is of special interest to studying the Compton amplitudes. The pion amplitude was computed by Bijnens et.al.,[27], while the kaon analog was computed by Guererro and Prades [18] and later by Fuchs et.al.,[19]. The amplitude for the process $\gamma(q_1)K^+(p_1) \rightarrow$
\[ \gamma(p_2)K^+(p_2) \] is given in terms of \( A(t, \nu) \) and \( B(t, \nu) \). The tree level expressions of \( A(t, \nu) \) and \( B(t, \nu) \) at \( \mathcal{O}(p^4) \) are given

\[
A(t, \nu) = \frac{2}{t - \nu} + \frac{2}{t + \nu}
\]
\[
B(t, \nu) = \frac{1}{t} \left( \frac{1}{t - \nu} + \frac{1}{t + \nu} \right),
\]

where \( t \) and \( \nu \) are kinematics variables defined in Ref. [18].

The amplitude can be generated with the attached input file \textbf{Ip17.nb}. The output can be found in the attached output file \textbf{Op17.nb}, where we have shown that the tree level amplitude of [18] does not match with that generated by AMPC, unless the \( B(t, \nu) \) is multiplied by factor 4. The correct expression is given below -

\[
B(t, \nu) = 4 \left( \frac{1}{t - \nu} + \frac{1}{t + \nu} \right).
\]

It may be noted that in the attached input and output notebooks we have considered the process \( \gamma(k_1)K^+(p_1) \to \gamma(k_2)K^+(p_2) \).

We have also compared the tree level amplitude of \( \gamma K \to \gamma K \) generated from AMPC against that given in Ref. [19] and the results agree providing a further check to the new expression for \( B(t, \nu) \) given above.

### 2.7 Scattering amplitudes at 1-loop

As mentioned in Sec. 1, we have checked all the processes given in GMO [17], against our AMPC results and we find that they agree to the best of our knowledge. All the notation in [17] except for the \( \mu_\pi, \mu_K, \mu_\eta \) function agrees with the ones present in AMPC results which are the expressions already introduced. One crucial simplification needs to be done for the expression of \( \mathcal{J}(0) \). This is as follows -

\[
\mathcal{J}(0) = \frac{1}{32\pi^2} \left[ \sum \Delta \right] + 3 \frac{M^2_{\pi} M^2_{Q}}{\Delta^3} \ln \left( \frac{M^2_{Q}}{M^2_{P}} \right)
\]

\[
= \frac{1}{32\pi^2} \sum \Delta \left( \mathcal{A} [M^2_{Q}] + \frac{M^2_{Q} \mathcal{A} [M^2_{P}]}{\Delta^3} \right)
\]

\[
\mathcal{A} [M^2_{i}] = -2F^2_{i} \mu_i \quad \mu_i = \frac{M^2_i}{32\pi^2 F^2_{i}} \log \frac{M^2_{Q}}{\mu^2} \quad i = \pi, K, \eta
\]

As an example we demonstrate our comparison for one of the processes. See attached notebook \textbf{Op18ccheck.nb}. In the results to follow, we use the GMO mass formula,

\[
3M^2_\eta = 4M^2_K - M^2_\pi
\]
as well as appropriate s,t,u relations wherever necessary.

The input and output notebooks for different scattering processes are tabulated in Table 2.

Table 2: Input and output notebooks for various scattering processes.

| Process                | Input Notebook | Output Notebook |
|------------------------|----------------|-----------------|
| $\eta\eta \rightarrow \eta\eta$ | Ip18a.nb       | Op18a.nb        |
| $\overline{K}^0\eta \rightarrow \overline{K}^0\eta$ | Ip18b.nb       | Op18b.nb        |
| $\overline{K}^0\eta \rightarrow \overline{K}\pi^0$ | Ip18c.nb       | Op18c.nb        |
| $\overline{K}^0K^0 \rightarrow K^+K^-$ | Ip18d.nb       | Op18d.nb        |
| $K^+\pi^+ \rightarrow K^+\pi^+$ | Ip18e.nb       | Op18e.nb        |
| $\pi^0\eta \rightarrow \pi^0\eta$ | Ip18f.nb       | Op18f.nb        |

2.8 Application of Chiral Dynamics in $\tau$ decays.

In Ref. [10] the tree level amplitudes for $\tau$ decays to multi-meson states are obtained using $SU(3) \times SU(3)$ Lagrangian in the limit of vanishing quark mass for one, two and three meson final states involving $\pi$, $K$ and $\eta$. In this section we compare the AMPC generated amplitudes with that given in [10]. A few points are in order regarding the comparisons. We define the weak matrix element which is given in Eq. (32). Here the results are presented up to an overall factor of $G_FV_{CKM}$.

For hadronic matrix elements involving three final states hadrons, we have simplified the Lorentz structures and compared the coefficients of the momentum vectors. In [10] the authors neglect the quark masses in the Lagrangian. However the meson masses are retained in the propagator. The denominator of the AMPC results match with that of [10], and the numerator match when the meson masses are neglected. We have used the GMO relation to simplify the numerator in few cases. Let us again emphasize that the results presented here supersede that of Ref. [10] when quark masses are no longer neglected.

The two and three meson final states are accessible in AMPC. In the attached notebook Op19.nb, the AMPC generated output for each of the processes are shown and simplifications are done using the FeynCalc [28]. Also provided are two three input notebooks Ip19a.nb ($J_\mu(\pi^+\pi^0)$), Ip19b.nb ($J_\mu(\pi^+K^+K^-)$), Ip19c.nb ($J_\mu(\eta_1\eta_2K^+)$).
2.8.1 Hadron Current matrix elements in two mesons final state

Table 3: Comparison of hadron matrix element from Ref. [10] and that obtained from AMPC for two mesons in the final state.

| Process          | Ref. [10] | AMPC                  |
|------------------|-----------|-----------------------|
| $J_\mu(\pi^+\pi^0)$ | $\sqrt{2}(p_+ - p)_\mu$ | $\sqrt{2}(p_+ - p)_\mu$ |
| $J_\mu(K^+K^0)$   | $-(k_+ - k)_\mu$ | $-(k_+ - k)_\mu$ |
| $J_\mu(\pi^+K^0)$ | $\frac{1}{\sqrt{2}}(k_+ - p)_\mu$ | $\frac{1}{\sqrt{2}}(k_+ - p)_\mu$ |
| $J_\mu(\pi^+ K^0)$ | $(k - p_+)_\mu$ | $(k - p_+)_\mu$ |
| $J_\mu(K^+\eta_8)$ | $\frac{1}{\sqrt{2}}(k_+ - \eta)_\mu$ | $\frac{1}{\sqrt{2}}(k_+ - \eta)_\mu$ |

2.8.2 $J_\mu(\pi^+(p_1)\pi^+(p_2)\pi^-(p_-))$

Table 4: Comparison of the coefficients of external hadron momentum from Ref. [10] and from AMPC for the hadron current $J_\mu(\pi^+(p_1)\pi^+(p_2)\pi^-(p_-))$

| Coefficients | Ref. [10] | AMPC |
|--------------|-----------|------|
| $p_\mu^-$    | $\sqrt{2}(-4M_0^2 - 3p_- p_1 - 3p_- p_2 - 6p_1 p_2)$ | $\sqrt{2}(M_0^2 + p_- p_1 + p_- p_2 + 2p_1 p_2)$ |
| $p_\mu^+$    | $\sqrt{2}(2M_0^2 + 3p_- p_1 + 3p_- p_2)$ | $\sqrt{2}(2M_0^2 + p_- p_1 + p_- p_2 + 2p_1 p_2)$ |
| $p_\mu^-$    | $\sqrt{2}(2M_0^2 + 3p_- p_1 + 3p_- p_2)$ | $\sqrt{2}(2M_0^2 + p_- p_1 + p_- p_2 + 2p_1 p_2)$ |

2.8.3 $J_\mu(\pi^0(p_1)\pi^0(p_2)\pi^+(p_+))$

Table 5: Comparison of the coefficients of external hadron momentum from Ref. [10] and from AMPC for the hadron current $J_\mu(\pi^0(p_1)\pi^0(p_2)\pi^+(p_+))$

| Coefficients | Ref. [10] | AMPC |
|--------------|-----------|------|
| $p_\mu^-$    | $\sqrt{2}(-2M_0^2 - 3p_+ p_1 - 3p_+ p_2)$ | $-M_0^2 - 2p_+ p_1 - 2p_+ p_2$ |
| $p_\mu^+$    | $\sqrt{2}(2M_0^2 + 3p_+ p_1 + 3p_+ p_2)$ | $\sqrt{2}(2M_0^2 + p_+ p_1 + p_+ p_2 + 2p_1 p_2)$ |
| $p_\mu^-$    | $\sqrt{2}(2M_0^2 + 3p_+ p_1 + 3p_+ p_2)$ | $\sqrt{2}(2M_0^2 + p_+ p_1 + p_+ p_2 + 2p_1 p_2)$ |
| $p_\mu^+$    | $\sqrt{2}(2M_0^2 + 3p_+ p_1 + 3p_+ p_2)$ | $\sqrt{2}(2M_0^2 + p_+ p_1 + p_+ p_2 + 2p_1 p_2)$ |

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2.8.4 $J_\mu(\pi^+(p_+)^+K^+(k_+)^-K^-(k_-))$

Table 6: Comparison of the coefficients of external hadron momentum from Ref. [10] and from AMPC for the hadron current $J_\mu(\pi^+(p_+)^+K^+(k_+)^-K^-(k_-))$

| Coefficients | Ref. [10] | AMPC |
|--------------|-----------|------|
| $p^\mu_+$    | $3k_p+3k_{k_+}+3M_{p_k}-M_k^2$ | $-k_+p_+k_+k_+M_k^2$ |
| $k^\mu_+$    | $3\sqrt{2}F\{k_{p_+}+k_{k_+}+k_{p_-}+k_{k_-}+k_{p_+}+k_{k_-}+k_{p_-}+k_{k_+}+M_k^2\}$ | $-k_+p_+k_+k_+M_k^2$ |
| $k^\mu_-$    | $-3k_-p_+6k_{p_+}+3k_-k_+M_k^2-3M_{k_+}^2-M_{k_+}^2$ | $3\sqrt{2}F\{k_{p_+}+k_{k_+}+k_{p_-}+k_{k_-}+k_{p_+}+k_{k_-}+k_{p_-}+k_{k_+}+M_k^2\}$ |

2.8.5 $J_\mu(\pi^+(p_+)^0(k)K^0(\bar{k}))$

Table 7: Comparison of the coefficients of external hadron momentum from Ref. [10] and from AMPC for the hadron current $J_\mu(\pi^+(p_+)^0(k)K^0(\bar{k}))$

| Coefficients | Ref. [10] | AMPC |
|--------------|-----------|------|
| $p^\mu_+$    | $3k_k+3k_{p_+}+3M_{p_k}^0-M_k^2$ | $-k_+p_+k_+k_+M_k^2$ |
| $k^\mu_+$    | $3\sqrt{2}F\{k_{p_+}+k_{k_+}+k_{p_-}+k_{k_-}+k_{p_+}+k_{k_-}+k_{p_-}+k_{k_+}+M_k^2\}$ | $-k_+p_+k_+k_+M_k^2$ |
| $k^\mu_-$    | $3\sqrt{2}F\{k_{p_+}+k_{k_+}+k_{p_-}+k_{k_-}+k_{p_+}+k_{k_-}+k_{p_-}+k_{k_+}+M_k^2\}$ | $-k_+p_+k_+k_+M_k^2$ |

2.8.6 $J_\mu(\pi^0(p)\overline{K}^0(\bar{k})K^+(k_+))$

Table 8: Comparison of the coefficients of external hadron momentum from Ref. [10] and from AMPC for the hadron current $J_\mu(\pi^0(p)\overline{K}^0(\bar{k})K^+(k_+))$

| Coefficients | Ref. [10] | AMPC |
|--------------|-----------|------|
| $p^\mu_+$    | $2F\{p+k_k_k_+p_k_++k_+M_k^2\}$ | $2F\{p+k_k_k_+p_k_++k_+M_k^2\}$ |
| $k^\mu_+$    | $2F\{p+k_k_k_+p_k_++k_+M_k^2\}$ | $2F\{p+k_k_k_+p_k_++k_+M_k^2\}$ |
| $k^\mu_-$    | $2F\{p+k_k_k_+p_k_++k_+M_k^2\}$ | $2F\{p+k_k_k_+p_k_++k_+M_k^2\}$ |
| $p^\mu_+$    | $2F\{p+k_k_k_+p_k_++k_+M_k^2\}$ | $2F\{p+k_k_k_+p_k_++k_+M_k^2\}$ |
| $k^\mu_+$    | $2F\{p+k_k_k_+p_k_++k_+M_k^2\}$ | $2F\{p+k_k_k_+p_k_++k_+M_k^2\}$ |
| $k^\mu_-$    | $2F\{p+k_k_k_+p_k_++k_+M_k^2\}$ | $2F\{p+k_k_k_+p_k_++k_+M_k^2\}$ |

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2.8.7 \( J_\mu(\eta_8)K^0(\bar{k})K^+(k_+) \)

Table 9: Comparison of the coefficients of external hadron momentum from Ref. [10] and from AMPC for the hadron current \( J_\mu(\eta_8)K^0(\bar{k})K^+(k_+) \)

| Coefficients | Ref. [10] | AMPC |
|--------------|-----------|------|
| \( \eta^\mu \) | \( 3\eta k+6k_k+3\eta k_++6M_{K_2}^2-2M_Z^2 \) | \( 9\eta k+18k_k+9\eta k_+-12M_{K_2}^2-6M_Z^2 \) |
| \( \bar{k}^\mu \) | \( \sqrt{3F_8(2\eta k+2k_k+2\eta k_++2M_{K_2}^2+M_Z^2-M_T^2}) \) | \( -9\eta k-9k_k-12M_{K_2}^2+6M_Z^2 \) |
| \( k^\mu_+ \) | \( \sqrt{3F_8(2\eta k+2k_k+2\eta k_++2M_{K_2}^2+M_Z^2-M_T^2}) \) | \( -9\eta k-9k_k-12M_{K_2}^2+6M_Z^2 \) |

In the above table the numerator of each expression is further simplified using GMO relation, and presented below.

Table 10: Comparison of the coefficients of external hadron momentum from Ref. [10] and from AMPC for the hadron current \( J_\mu(\eta_8)K^0(\bar{k})K^+(k_+) \)

| Coefficients | Ref. [10] | AMPC |
|--------------|-----------|------|
| \( \eta^\mu \) | \( 3\eta k+6k_k+3\eta k_++6M_{K_2}^2-2M_Z^2 \) | \( 9\eta k+18k_k+9\eta k_+-12M_{K_2}^2-6M_Z^2 \) |
| \( \bar{k}^\mu \) | \( \sqrt{3F_8(2\eta k+2k_k+2\eta k_++2M_{K_2}^2+M_Z^2-M_T^2}) \) | \( -9\eta k-9k_k-12M_{K_2}^2+6M_Z^2 \) |
| \( k^\mu_+ \) | \( \sqrt{3F_8(2\eta k+2k_k+2\eta k_++2M_{K_2}^2+M_Z^2-M_T^2}) \) | \( -9\eta k-9k_k-12M_{K_2}^2+6M_Z^2 \) |

\( 2.8.8 \ J_\mu(K_1^+(k_1)K_2^+(k_2)K^-(k_-)) \)

Table 11: Comparison of the coefficients of external hadron momentum from Ref. [10] and from AMPC for the hadron current \( J_\mu(K_1^+(k_1)K_2^+(k_2)K^-(k_-)) \)

| Coefficients | Ref. [10] | AMPC |
|--------------|-----------|------|
| \( \bar{k}_1^\mu \) | \( \sqrt{3F_8(-k_-k_+k_-k_2+2M_{K_2}^2)} \) | \( \sqrt{3F_8(-k_-k_+k_-k_2+2M_{K_2}^2)} \) |
| \( \bar{k}_2^\mu \) | \( \sqrt{3F_8(-k_-k_+k_-k_2+2M_{K_2}^2)} \) | \( \sqrt{3F_8(-k_-k_+k_-k_2+2M_{K_2}^2)} \) |
| \( \bar{k}^\mu_+ \) | \( \sqrt{3F_8(-k_-k_+k_-k_2+2M_{K_2}^2)} \) | \( \sqrt{3F_8(-k_-k_+k_-k_2+2M_{K_2}^2)} \) |
2.8.9 \( J_\mu(K^0(k)\bar{K}^0(\bar{k})K^+(k_+)) \)

Table 12: Comparison of the coefficients of external hadron momentum from Ref. [10] and from AMPC for the hadron current \( J_\mu(K^0(k)\bar{K}^0(\bar{k})K^+(k_+)) \)

| Coefficients | Ref. [10] | AMPC |
|--------------|-----------|------|
| \( k^\mu \)  | \( 3k - k + k_+ + 2M_K^2 \) | \( k - k + k_+ + M_K^2 \) |
| \( \bar{k}^\mu \) | \( 3\sqrt{2}F_x(k + k + k_+ + M_K^2) \) | \( \sqrt{2}F_x(k + k + k_+ + M_K^2) \) |
| \( k^\mu_+ \)  | \( 3\sqrt{2}F_x(k + k + k_+ + M_K^2) \) | \( \sqrt{2}F_x(k + k + k_+ + M_K^2) \) |

2.8.10 \( J_\mu(\pi^0(p_1)\pi^0(p_2)K^+(k_+)) \)

Table 13: Comparison of the coefficients of external hadron momentum from Ref. [10] and from AMPC for the hadron current \( J_\mu(\pi^0(p_1)\pi^0(p_2)K^+(k_+)) \)

| Coefficients | Ref. [10] | AMPC |
|--------------|-----------|------|
| \( p_1^\mu \)  | \( -5k - p_1 - 3k - p_2 - 2M_K^2 \) | \( -k - p_1 - k_+ \) |
| \( p_2^\mu \)  | \( 6\sqrt{2}F_x(k + p_1 + k_+ + p_2 + M_K^2 + p_2) \) | \( 2\sqrt{2}F_x(k + k_+ + p_1 + p_2 + M_K^2 + p_1 + p_2) \) |
| \( k^\mu_+ \)  | \( 6\sqrt{2}F_x(k + p_1 + k_+ + p_2 + M_K^2 + p_1 + p_2) \) | \( 2\sqrt{2}F_x(k + k_+ + p_1 + p_2 + M_K^2 + p_1 + p_2) \) |

2.8.11 \( J_\mu(\pi^+(p_+)(-\pi^-)(-p_-)K^+(k_+)) \)

Table 14: Comparison of the coefficients of external hadron momentum from Ref. [10] and from AMPC for the hadron current \( J_\mu(\pi^+(p_+)(-\pi^-)(-p_-)K^+(k_+)) \)

| Coefficients | Ref. [10] | AMPC |
|--------------|-----------|------|
| \( p^\mu_+ \)  | \( 3p_+ - k_+ - M_K^2 + 3M_K^2 + p_+ \) | \( -p_+ - k_+ - M_K^2 - p_+ \) |
| \( p^\mu_- \)  | \( 3\sqrt{2}F_x(p_+ - k_+ + p_1 + M_K^2 + p_1 + p_2) \) | \( \sqrt{2}F_x(p_+ - k_+ + p_1 + M_K^2 + p_1 + p_2) \) |
| \( k^\mu_+ \)  | \( 3\sqrt{2}F_x(p_+ - k_+ + p_1 + M_K^2 + p_1 + p_2) \) | \( \sqrt{2}F_x(p_+ - k_+ + p_1 + M_K^2 + p_1 + p_2) \) |

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2.8.12 $J_\mu(\pi^0(p)\pi^+(p_+))K^0(k_0)$

Table 15: Comparison of the coefficients of external hadron momentum from Ref. [10] and from AMPC for the hadron current $J_\mu(\pi^0(p)\pi^+(p_+)K^0(k_0))$

| Coefficients | Ref. [10] | AMPC |
|--------------|-----------|------|
| $p_+^\mu$    | $-3p_+k_0-k_0-2M_0^2-2p_{p_+}$ | $-3p_+k_0-k_0-2M_0^2-2p_{p_+}$ |
|              | $2F_+(p_0+k_0+k_0+M_0^2+p_{p_+})$ | $2F_+(p_0+k_0+k_0+M_0^2+p_{p_+})$ |
| $\eta_{\mu}$ | $\sqrt{3}F_+(2k_+p_++2k_+\eta+M_0^2+M_0^2+2\eta p_{p_+})$ | $\sqrt{3}F_+(2k_+p_++2k_+\eta+M_0^2+M_0^2+2\eta p_{p_+})$ |
| $k_{\mu}$    | $\sqrt{3}F_+(2k_+p_++2k_+\eta+M_0^2+M_0^2+2\eta p_{p_+})$ | $\sqrt{3}F_+(2k_+p_++2k_+\eta+M_0^2+M_0^2+2\eta p_{p_+})$ |

2.8.13 $J_\mu(\eta_8(\eta)\pi^+(p_+)K^0(k))$

Table 16: Comparison of the coefficients of external hadron momentum from Ref. [10] and from AMPC for the hadron current $J_\mu(\eta_8(\eta)\pi^+(p_+)K^0(k))$

| Coefficients | Ref. [10] | AMPC |
|--------------|-----------|------|
| $\eta^\mu$   | $-3k_{p_+}-3k_{\eta}-2M_0^2$ | $-18k_{p_+}-18k_{\eta}-8M_K^2-3M_0^2-M_0^2$ |
|              | $\sqrt{3}F_+(2k_+p_++2k_+\eta+M_0^2+M_0^2+2\eta p_{p_+})$ | $6\sqrt{3}F_+(2k_+p_++2k_+\eta+M_0^2+M_0^2+2\eta p_{p_+})$ |
| $p_+^\mu$    | $-3k_{p_+}-3k_{\eta}-2M_0^2$ | $-18k_{p_+}-18k_{\eta}-8M_K^2-3M_0^2-M_0^2$ |
|              | $\sqrt{3}F_+(2k_+p_++2k_+\eta+M_0^2+M_0^2+2\eta p_{p_+})$ | $6\sqrt{3}F_+(2k_+p_++2k_+\eta+M_0^2+M_0^2+2\eta p_{p_+})$ |
| $\eta_{\mu}$ | $\sqrt{3}F_+(2k_+p_++2k_+\eta+M_0^2+M_0^2+2\eta p_{p_+})$ | $18k_{p_+}+18k_{\eta}-8M_K^2+15M_0^2+17M_0^2+36\eta p_{p_+}$ |
| $k_{\mu}$    | $\sqrt{3}F_+(2k_+p_++2k_+\eta+M_0^2+M_0^2+2\eta p_{p_+})$ | $6\sqrt{3}F_+(2k_+p_++2k_+\eta+M_0^2+M_0^2+2\eta p_{p_+})$ |

In the above table the numerator of each expression is further simplified using GMO relation, and presented below.

Table 17: Comparison of the coefficients of external hadron momentum from Ref. [10] and from AMPC for the hadron current $J_\mu(\eta_8(\eta)\pi^+(p_+)K^0(k))$

| Coefficients | Ref. [10] | AMPC |
|--------------|-----------|------|
| $\eta^\mu$   | $-3k_{p_+}-3k_{\eta}-2M_0^2$ | $-18k_{p_+}-18k_{\eta}-12M_K^2$ |
|              | $\sqrt{3}F_+(2k_+p_++2k_+\eta+M_0^2+M_0^2+2\eta p_{p_+})$ | $6\sqrt{3}F_+(2k_+p_++2k_+\eta+M_0^2+M_0^2+2\eta p_{p_+})$ |
| $p_+^\mu$    | $-3k_{p_+}-3k_{\eta}-2M_0^2$ | $-18k_{p_+}-18k_{\eta}-12M_K^2$ |
|              | $\sqrt{3}F_+(2k_+p_++2k_+\eta+M_0^2+M_0^2+2\eta p_{p_+})$ | $6\sqrt{3}F_+(2k_+p_++2k_+\eta+M_0^2+M_0^2+2\eta p_{p_+})$ |
| $\eta_{\mu}$ | $\sqrt{3}F_+(2k_+p_++2k_+\eta+M_0^2+M_0^2+2\eta p_{p_+})$ | $18k_{p_+}+18k_{\eta}+12M_K^2+12M_0^2+12M_0^2+36\eta p_{p_+}$ |
| $k_{\mu}$    | $\sqrt{3}F_+(2k_+p_++2k_+\eta+M_0^2+M_0^2+2\eta p_{p_+})$ | $6\sqrt{3}F_+(2k_+p_++2k_+\eta+M_0^2+M_0^2+2\eta p_{p_+})$ |
2.8.14 \( J_\mu (\eta_S(\eta)\pi^0(p)K^+(k_+)) \)

Table 18: Comparison of the coefficients of external hadron momentum from Ref. [10] and from AMPC for the hadron current \( J_\mu (\eta_S(\eta)\pi^0(p)K^+(k_+)) \)

| Coefficients | Ref. [10] | AMPC |
|--------------|-----------|------|
| \( \eta^\mu \) | \(-3p_k+3q_k-2M_k^2\) | \(-3p_k-3q_k-2M_k^2\) |
| \( p^\mu \) | \(\sqrt[6]{F_6(2p_k+2q_k+M_k^2+2p_N)}\) | \(\sqrt[6]{F_6(2p_k+2q_k+M_k^2+2p_N)}\) |
| \( k_+^\mu \) | \(\sqrt[6]{F_6(2p_k+2q_k+M_k^2+2p_N)}\) | \(\sqrt[6]{F_6(2p_k+2q_k+M_k^2+2p_N)}\) |

In the above table the numerator of each expression is further simplified using GMO relation, and presented below.

Table 19: Comparison of the coefficients of external hadron momentum from Ref. [10] and from AMPC for the hadron current \( J_\mu (\eta_S(\eta)\pi^0(p)K^+(k_+)) \)

| Coefficients | Ref. [10] | AMPC |
|--------------|-----------|------|
| \( \eta^\mu \) | \(-3p_k+3q_k-2M_k^2\) | \(-18p_k+18q_k-12M_k^2\) |
| \( p^\mu \) | \(\sqrt[6]{F_6(2p_k+2q_k+M_k^2+2p_N)}\) | \(\sqrt[6]{F_6(2p_k+2q_k+M_k^2+2p_N)}\) |
| \( k_+^\mu \) | \(\sqrt[6]{F_6(2p_k+2q_k+M_k^2+2p_N)}\) | \(\sqrt[6]{F_6(2p_k+2q_k+M_k^2+2p_N)}\) |

2.8.15 \( J_\mu (\eta_S(\eta_1)\eta_S(\eta_2)K^+(k_+)) \)

Table 20: Comparison of the coefficients of external hadron momentum from Ref. [10] and from AMPC for the hadron current \( J_\mu (\eta_S(\eta_1)\eta_S(\eta_2)K^+(k_+)) \)

| Coefficients | Ref. [10] | AMPC |
|--------------|-----------|------|
| \( \eta_1^\mu \) | \(-3k_+\eta_1-3k_+\eta_2-2M_k^2\) | \(-9k_+\eta_1-9k_+\eta_2-3M_k^2-M_2^2\) |
| \( \eta_2^\mu \) | \(\sqrt[2]{F_2(2k_+\eta_1+2k_+\eta_2+M_2^2+\eta_1\eta_2)}\) | \(\sqrt[2]{F_2(2k_+\eta_1+2k_+\eta_2+M_2^2+\eta_1\eta_2)}\) |
| \( k_+^\mu \) | \(\sqrt[2]{F_2(2k_+\eta_1+2k_+\eta_2+M_2^2+\eta_1\eta_2)}\) | \(\sqrt[2]{F_2(2k_+\eta_1+2k_+\eta_2+M_2^2+\eta_1\eta_2)}\) |

In the above table the numerator of each expression is further simplified using GMO relation, and presented below.
Table 21: Comparison of the coefficients of external hadron momentum from Ref. [10] and from AMPC for the hadronic current $J_{\mu}(\eta_8(\eta_1)\eta_8(\eta_2)K^+(k_+))$

| Coefficients | Ref. [10]                                      | AMPC                                      |
|--------------|------------------------------------------------|-------------------------------------------|
| $\eta_1^{\mu}$ | $-3k_+\cdot \eta_1 - 3k_+\cdot \eta_2 - 2M_K^2$ | $-2\sqrt{2}F_\pi(k_+\cdot \eta_1 + k_+\cdot \eta_2 + M_K^2 + \eta_1\cdot \eta_2)$ |
| $\eta_2^{\mu}$ | $2\sqrt{2}F_\pi(k_+\cdot \eta_1 + k_+\cdot \eta_2 + M_K^2 + \eta_1\cdot \eta_2)$ | $-9k_+\cdot \eta_1 - 9k_+\cdot \eta_2 - 4M_K^2$ |
| $k_+^{\mu}$   | $3k_+\cdot \eta_1 + 3k_+\cdot \eta_2 + 6M_K^2 - 2M_K^2 + \eta_1\cdot \eta_2$ | $9k_+\cdot \eta_1 + 9k_+\cdot \eta_2 + 20M_K^2 - 6M_K^2 + 10\eta_1\cdot \eta_2$ |

3 Summary

In this report, we have presented results of our checks of the consistency of AMPC and established results. In the meson sector, we have analyzed accessible form factors and scattering amplitudes including the kaon-Compton process. As long as the number of particles is manageable, explicit checks were tractable. Large number of notebooks are provided. This work was spurred by our recent investigations of the $K \rightarrow \pi l^+ l^-$ process where we discovered that the $G_{27}$ piece was not published in the literature. Since AMPC is very versatile, we have used it in the non-leptonic kaon decay sector to isolate the contributions of the odd-intrinsic parity sector that also involves $G_{27}$ and the higher order pieces as well. All the details of the notations are given in [4]. Another application is to the tree-level chiral processes appearing in $\tau$-decays. As recently as [11], BELLE was using the results of [10] which neglected the quark masses. We give the quark mass corrected results here. It is our belief that AMPC can be used to obtain amplitudes such as $\pi\gamma \rightarrow \pi\pi$ and others of importance to the COMPASS experiment as well. By providing explicit notebooks, we believe we have provided a service to the community which can also be used as a learning aid.

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