A general comparison theorem for 1-dimensional anticipated BSDEs

Xiaoming Xu∗
School of Mathematical Sciences, Nanjing Normal University, Nanjing, 210046, China
School of Mathematics, Shandong University, Jinan, 250100, China

Abstract

Anticipated backward stochastic differential equation (ABSDE) studied the first time in 2007 is a new type of stochastic differential equations. In this paper, we establish a general comparison theorem for 1-dimensional ABSDEs with the generators depending on the anticipated term of $Z$.

Keywords: Anticipated backward stochastic differential equation, Backward stochastic differential equation, Comparison theorem

1 Introduction

Backward Stochastic Differential Equation (BSDE) of the following general form was considered the first time by Pardoux-Peng \cite{Pardoux} in 1990:

$$Y_t = \xi + \int_t^T g(s,Y_s,Z_s)ds - \int_t^T Z_s dB_s.$$  

Since then, the theory of BSDEs has been studied with great interest. One of the achievements of this theory is the comparison theorem, which is due to Peng \cite{Peng} and then generalized by Pardoux-Peng \cite{Pardoux}, El Karoui-Peng-Quenez \cite{ElKaroui} and Hu-Peng \cite{Hu}. It allows to compare the solutions of two BSDEs whenever we can compare the terminal conditions and the generators.

Recently, a new type of BSDE, called anticipated BSDE (ABSDE in short), was introduced by Peng-Yang \cite{PengYang} (see also Yang \cite{Yang}). The ABSDE is of the

∗E-mail: xmxu@mail.sdu.edu.cn
following form:

\[
\begin{aligned}
-\frac{dY_t}{\delta} &= f(t, Y_t, Z_t, Y_{t+\delta(t)}, Z_{t+\delta(t)})dt - Z_t dB_t, \quad t \in [0, T]; \\
Y_t &= \xi_t, \quad t \in [T, T + K]; \\
Z_t &= \eta_t, \quad t \in [T, T + K],
\end{aligned}
\]

where \(\delta(\cdot) : [0, T] \to \mathbb{R}^+ \setminus \{0\}\) and \(\zeta(\cdot) : [0, T] \to \mathbb{R}^+ \setminus \{0\}\) are continuous functions satisfying

(a1) there exists a constant \(K \geq 0\) such that for each \(s \in [0, T]\), \(s + \delta(s) \leq T + K\); \(s + \zeta(s) \leq T + K\);

(a2) there exists a constant \(M \geq 0\) such that for each \(t \in [0, T]\) and each nonnegative integrable function \(g(\cdot)\), \(\int_t^T g(s + \delta(s))ds \leq M \int_t^{T+K} g(s)ds, \int_t^T g(s + \zeta(s))ds \leq M \int_t^{T+K} g(s)ds\).

Peng and Yang proved in [6] that (1.1) has a unique adapted solution under proper assumptions, furthermore, they established a comparison theorem, which requires that the generators of the ABSDEs cannot depend on the anticipated term of \(Z\) and one of them must be increasing in the anticipated term of \(Y\).

The aim of this paper is to give a more general comparison theorem in which the generators of the ABSDEs break through the above restrictions. The main approach we adopt is to consider an ABSDE as a series of BSDEs and then apply the well-known comparison theorem for 1-dimensional BSDEs (see [4]).

The paper is organized as follows: in Section 2, we list some notations and some existing results. In Section 3, we mainly study the comparison theorem for ABSDEs.

## 2 Preliminaries

Let \(\{B_t; t \geq 0\}\) be a \(d\)-dimensional standard Brownian motion on a probability space \((\Omega, \mathcal{F}, P)\) and \(\{\mathcal{F}_t; t \geq 0\}\) be its natural filtration. Denote by \(|\cdot|\) the norm in \(\mathbb{R}^m\). Given \(T > 0\), we make the following notations:

\[
L^2(\mathcal{F}_T; \mathbb{R}^m) = \{\xi \in \mathbb{R}^m \mid \xi\text{ is an }\mathcal{F}_T\text{-measurable random variable such that }E|\xi|^2 < \infty\};
\]

\[
L^2_{P}(0, T; \mathbb{R}^m) = \{\varphi : \Omega \times [0, T] \to \mathbb{R}^m \mid \varphi\text{ is progressively measurable; }E\int_0^T |\varphi_t|^2dt < \infty\};
\]

\[
S^2_{P}(0, T; \mathbb{R}^m) = \{\psi : \Omega \times [0, T] \to \mathbb{R}^m \mid \psi\text{ is continuous and progressively measurable; }E[\sup_{0 \leq t \leq T} |\psi_t|^2] < \infty\}.
\]

Now consider the ABSDE (1.1). First for the generator \(f(\omega, s, y, z, \theta, \phi) : \Omega \times [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times S^2_{P}(s, T + K; \mathbb{R}^m) \times L^2_{P}(s, T + K; \mathbb{R}^{m \times d}) \to L^2(\mathcal{F}_s; \mathbb{R}^m)\), we use two hypotheses:
Theorem 2.1 Assume that $f$ satisfies (H1) and (H2), $\delta, \zeta$ satisfy (a1) and (a2), then for arbitrary given terminal conditions $(\xi, \eta) \in S^2_T(T, T + K; \mathbb{R}^m) \times L^2_T(T, T + K; \mathbb{R}^{m \times d})$, the ABSDE (1.1) has a unique solution, i.e., there exists a unique pair of $\mathcal{F}_t$-adapted processes $(Y, Z) \in S^2_T(0, T + K; \mathbb{R}^m) \times L^2_T(0, T + K; \mathbb{R}^{m \times d})$ satisfying (1.1).

Next we will recall the comparison theorem from [6]. Let $(Y^{(j)}, Z^{(j)})$ $(j = 1, 2)$ be solutions of the following 1-dimensional ABSDEs respectively:

$$
\begin{align*}
-dY^{(j)}_t &= f_j(t, Y^{(j)}_t, Z^{(j)}_t, Y^{(j)}_{t+\delta(t)}, Z^{(j)}_{t+\zeta(t)})dt - Z^{(j)}_t dB_t, & t \in [0, T]; \\
Y^{(j)}_t &= \xi^{(j)}_t, & t \in [T, T + K].
\end{align*}
$$

Theorem 2.2 Assume that $f_1, f_2$ satisfy (H1) and (H2), $\xi^{(1)}, \xi^{(2)} \in S^2_T(T, T + K; \mathbb{R})$, $\delta$ satisfies (a1), (a2), and for each $t \in [0, T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $f_2(t, y, z, \cdot) \geq f_2(t, y, z, \theta')$, if $\theta_r \geq \theta_r'$, $\theta, \theta' \in L^2_T(t, T + K; \mathbb{R}), r \in [t, T + K]$. If $\xi^{(1)} \geq \xi^{(2)}$, $s \in [T, T + K]$ and $f_1(t, y, z, \theta_r) \geq f_2(t, y, z, \theta_r), t \in [0, T], y \in \mathbb{R}, z \in \mathbb{R}^d, \theta \in L^2_T(t, T + K; \mathbb{R}), r \in [t, T + K]$, then $Y^{(1)}_t \geq Y^{(2)}_t$, a.e., a.s..

3 Comparison Theorem for Anticipated BSDEs

Consider the following 1-dimensional ABSDEs:

$$
\begin{align*}
-dY^{(j)}_t &= f_j(t, Y^{(j)}_t, Z^{(j)}_t, Y^{(j)}_{t+\delta(t)}, Z^{(j)}_{t+\zeta(t)})dt - Z^{(j)}_t dB_t, & t \in [0, T]; \\
Y^{(j)}_t &= \xi^{(j)}_t, & t \in [T, T + K]; \\
Z^{(j)}_t &= \eta^{(j)}_t, & t \in [T, T + K],
\end{align*}
$$

where $j = 1, 2$, $f_j$ satisfies (H1), (H2), $(\xi^{(j)}, \eta^{(j)}) \in S^2_T(T, T + K; \mathbb{R}) \times L^2_T(T, T + K; \mathbb{R}^d)$, $\delta, \zeta$ satisfy (a1) and (a2). By Theorem 2.1 either of the above ABSDEs has a unique adapted solution.

Proposition 3.1 Putting $t_0 = T$, we define by iteration
\[ t_i := \min\{t \in [0,T] : \min\{s + \delta(s), s + \zeta(s)\} \geq t_{i-1}, \text{ for all } s \in [t,T]\}, \quad i \geq 1. \]

Set \( N := \max\{i : t_{i-1} > 0\}. \) Then \( N \) is finite, \( t_N = 0 \) and
\[
[0, T] = [0, t_{N-1}] \cup [t_{N-1}, t_{n-2}] \cup \cdots \cup [t_2, t_1] \cup [t_1, T].
\]

**Proof.** Let us first prove that \( N \) is finite. For this purpose, we apply the method of reduction to absurdity. Suppose \( N \) is infinite. From the definition of \( \{t_i\}_{i=1}^{+\infty} \), we know
\[
\min\{t_i + \delta(t_i), t_i + \zeta(t_i)\} = t_{i-1}, \quad i = 1, 2, \ldots. \tag{3.2}
\]
Since \( \delta(\cdot) \) and \( \zeta(\cdot) \) are continuous and positive, thus obviously we have \( t_i < t_{i-1} \) \((i = 1, 2, \ldots)\). Therefore \( \{t_i\}_{i=1}^{+\infty} \) converges as a strictly monotone and bounded series. Denote its limit by \( \bar{t} \). Letting \( i \rightarrow +\infty \) on both sides of (3.2), we get
\[
\min\{\bar{t} + \delta(\bar{t}), \bar{t} + \zeta(\bar{t})\} = \bar{t}.
\]
Hence \( \delta(\bar{t}) = 0 \) or \( \zeta(\bar{t}) = 0 \), which is just a contradiction since both \( \delta \) and \( \zeta \) are positive.

Next we will show that \( t_N = 0 \). In fact, the following holds obviously:
\[
\min\{t_N + \delta(t_N), t_N + \zeta(t_N)\} > t_N,
\]
which implies \( t_N = 0 \), or else we can find a \( \bar{t} \in [0, t_N) \) due to the continuity of \( \delta(\cdot) \) and \( \zeta(\cdot) \) such that
\[
\min\{s + \delta(s), s + \zeta(s)\} \geq t_N, \quad \text{for all } s \in [\bar{t}, T],
\]
from which we know that \( \bar{t} \) is an element of the series as well. \( \square \)

**Proposition 3.2** Suppose \((Y^{(j)}, Z^{(j)}) \ (j = 1, 2)\) are the solutions of \( \text{ABSDEs (3.1)} \) respectively. Then for fixed \( i \in \{1, 2, \ldots, N\} \), over time interval \([t_i, t_{i-1}]\), \( \text{ABSDEs (3.1)} \) are equivalent to the following \( \text{ABSDEs} \):
\[
\begin{align*}
- dY_t^{(j)} &= f_j(t, Y_t^{(j)}, Z_t^{(j)}, Y_{t+\delta(t)}, Z_{t+\zeta(t)}) dt - Z_t^{(j)} dB_t, \quad t \in [t_i, t_{i-1}]; \\
\bar{Y}_t^{(j)} &= \bar{Y}_t^{(j)}, \quad t \in [t_{i-1}, T + K]; \\
\bar{Z}_t^{(j)} &= \bar{Z}_t^{(j)}, \quad t \in [t_{i-1}, T + K],
\end{align*}
\]
(3.3)
which are also equivalent to the following \( \text{BSDEs with terminal conditions } Y_{t_{i-1}}^{(j)} \) respectively:
\[
\bar{Y}_t^{(j)} = Y_{t_{i-1}}^{(j)} + \int_t^{t_{i-1}} f_j(s, \bar{Y}_s^{(j)}, \bar{Z}_s^{(j)}, Y_s^{(j)}, Z_s^{(j)}, Y_{s+\delta(s)}, Z_{s+\zeta(s)}) ds - \int_t^{t_{i-1}} \bar{Z}_s^{(j)} dB_s. \tag{3.4}
\]

4
That is to say,
\[ Y_t^{(j)} = \tilde{Y}_t^{(j)} = \tilde{Y}_t^{(j)} = Z_t^{(j)} = \tilde{Z}_t^{(j)} = \frac{d(\tilde{Y}_t^{(j)}, B)_t}{dt}, \quad t \in [t_i, t_{i-1}], \quad j = 1, 2, \]
where \((\tilde{Y}_t^{(j)}, B)\) is the variation process generated by \(\tilde{Y}_t^{(j)}\) and the Brownian motion \(B\).

**Proof.** We only need to prove the equivalence between ABSDE (3.3) and BSDE (3.4). It is obvious that for each \(s \in [t_i, t_{i-1}], \quad s + \delta(s) \geq t_{i-1}, \quad s + \zeta(s) \geq t_{i-1}, \)
thus \((\tilde{Y}_{t+\delta(t)}^{(j)}, \tilde{Z}_{t+\zeta(t)}^{(j)}) = (Y_{t+\delta(t)}^{(j)}, Z_{t+\zeta(t)}^{(j)})\) in the generator of ABSDE (3.3). Clearly \(f_j(\cdot, \cdot, \cdot, \cdot)\) satisfies the Lipschitz condition as well as the integrable condition since \(f_j\) satisfies (H1), (H2). Thus BSDE (3.4) has a unique adapted solution.

Moreover, it is obvious that \((Y_t^{(j)}, Z_t^{(j)}), t \in [t_i, t_{i-1}]\) satisfies both ABSDE (3.3) and BSDE (3.4). Then from the uniqueness of ABSDE’s solution and that of BSDE’s, we can easily obtain the desired equalities. \(\square\)

**Theorem 3.1** Let \((Y_t^{(j)}, Z_t^{(j)}) \in S_T^2(0, T + K; \mathbb{R}) \times L_T^2(0, T + K; \mathbb{R}^d) \quad (j = 1, 2)\) be the unique solutions to ABSDEs (3.1) respectively. If

(i) \(\xi_s^{(1)} \geq \xi_s^{(2)}, \quad s \in [T, T + K], \text{a.e., a.s.};\)

(ii) for all \(t \in [0, T], \quad (y, z) \in \mathbb{R} \times \mathbb{R}^d, \quad \theta^{(1)} \in S_T^2(t, T + K; \mathbb{R})\) \((j = 1, 2)\) such that \(\theta^{(1)} \geq \theta^{(2)}, \{\theta_r^{(1)}\}_{r \in [t, T]}\) is a continuous semimartingale and \((\theta_r^{(1)}\}_{r \in [t, T + K]} = (\xi_r^{(1)})_{r \in [t, T + K]},\)

\[ f_1(t, y, z, \theta_t^{(1)} + \delta(t), \eta_t^{(1)} + \zeta(t)) \geq f_2(t, y, z, \theta_t^{(2)} + \delta(t), \eta_t^{(2)} + \zeta(t)), \quad \text{a.e., a.s.,} \quad (3.5) \]

\[ f_1(t, y, z, \theta_t^{(1)} + \delta(t), \frac{d(\theta_t^{(1)} + \delta(t))_r}{dr} |_{r = t + \zeta(t)}) \geq f_2(t, y, z, \theta_t^{(2)} + \delta(t), \frac{d(\theta_t^{(2)} + \delta(t))_r}{dr} |_{r = t + \zeta(t)}), \quad \text{a.e., a.s.,} \quad (3.6) \]

\[ f_1(t, y, z, \xi_t^{(1)} + \delta(t), \frac{d(\theta_t^{(1)} + \delta(t))_r}{dr} |_{r = t + \zeta(t)}) \geq f_2(t, y, z, \xi_t^{(2)} + \delta(t), \frac{d(\theta_t^{(2)} + \delta(t))_r}{dr} |_{r = t + \zeta(t)}), \quad \text{a.e., a.s.,} \quad (3.7) \]

then \(Y_t^{(1)} \geq Y_t^{(2)}, \text{a.e., a.s.} \).

Moreover, the following holds:

\[ Y_0^{(1)} = Y_0^{(2)} \iff \left\{ \begin{array}{l}
\xi_T^{(1)} = \xi_T^{(2)}; \\
f_1(t, Y_t^{(2)}, Z_t^{(2)}, Y_{t+\delta(t)}^{(2)}, Z_{t+\zeta(t)}^{(2)}) = f_2(t, Y_t^{(2)}, Z_t^{(2)}, Y_{t+\delta(t)}^{(2)}, Z_{t+\zeta(t)}^{(2)}), \quad t \in [0, T].
\end{array} \right. \]
Proof. Consider the ABSDE (3.1) one time interval by one time interval. For the first step, we consider the case when \( t \in [t_1, T] \). According to Proposition 3.2 we can equivalently consider the following BSDE:

\[
\tilde{Y}_t^{(j)} = \xi_T^{(j)} + \int_t^T f_j(s, \tilde{Y}_s^{(j)}, \tilde{Z}_s^{(j)}, \xi_s^{(j)}, \eta_s^{(j)})ds - \int_t^T \tilde{Z}_s^{(j)}dB_s,
\]

from which we have

\[
\tilde{Z}_t^{(j)} = \frac{d(\tilde{Y}_t^{(j)}, B_t)}{dt}, \quad t \in [t_1, T].
\]

Noticing that \( \xi^{(j)} \in S_F^2(T, T + K; \mathbb{R}) \) (\( j = 1, 2 \)) and \( \xi^{(1)} \geq \xi^{(2)} \), from (3.5) in (ii), we can get, for \( s \in [t_1, T] \), \( y \in \mathbb{R} \), \( z \in \mathbb{R}^d \),

\[
f_1(s, y, z, \xi^{(1)}_{s+\delta(s)}, \eta^{(1)}_{s+\zeta(s)}) \geq f_2(s, y, z, \xi^{(2)}_{s+\delta(s)}, \eta^{(2)}_{s+\zeta(s)}).
\]

According to the comparison theorem for 1-dimensional BSDEs, we can get

\[
\tilde{Y}_t^{(1)} \geq \tilde{Y}_t^{(2)}, \quad t \in [t_1, T], \text{ a.e., a.s.}
\]

as well as

\[
Y_{t_1}^{(1)} = Y_{t_1}^{(2)} \Leftrightarrow \begin{cases}
\xi_{T_1}^{(1)} = \xi_{T_1}^{(2)}; \\
f_1(t, \tilde{Y}_t^{(2)}, \tilde{Z}_t^{(2)}, \xi_t^{(2)}, \eta_t^{(2)}) = f_2(t, \tilde{Y}_t^{(2)}, \tilde{Z}_t^{(2)}, \xi_t^{(2)}, \eta_t^{(2)}), \quad t \in [t_1, T].
\end{cases}
\]

Consequently,

\[
Y_t^{(1)} \geq Y_t^{(2)}, \quad t \in [t_1, T + K], \text{ a.e., a.s..} \quad (3.9)
\]

For the second step, we consider the case when \( t \in [t_2, t_1] \). Similarly, according to Proposition 3.2 we can consider the following BSDE equivalently:

\[
\tilde{Y}_t^{(j)} = Y_{t_1}^{(j)} + \int_{t_1}^t f_j(s, \tilde{Y}_s^{(j)}, \tilde{Z}_s^{(j)}, Y_{s+\delta(s)}, Z_{s+\zeta(s)})ds - \int_{t_1}^t \tilde{Z}_s^{(j)}dB_s,
\]

from which we have

\[
\tilde{Z}_t^{(j)} = \frac{d(\tilde{Y}_t^{(j)}, B_t)}{dt}, \quad t \in [t_2, t_1].
\]

Noticing (3.8) and (3.9), according to (ii), we have, for \( s \in [t_2, t_1] \), \( y \in \mathbb{R} \), \( z \in \mathbb{R}^d \),

\[
f_1(s, y, z, Y_{s+\delta(s)}, Z_{s+\zeta(s)}) \geq f_2(s, y, z, Y_{s+\delta(s)}, Z_{s+\zeta(s)}).
\]

Applying the comparison theorem for BSDEs again, we can finally get

\[
Y_t^{(1)} \geq Y_t^{(2)}, \quad t \in [t_2, t_1], \text{ a.e., a.s.}
\]

as well as

\[
Y_{t_2}^{(1)} = Y_{t_2}^{(2)} \Leftrightarrow \begin{cases}
Y_{t_1}^{(1)} = Y_{t_1}^{(2}); \\
f_1(t, \tilde{Y}_t^{(2)}, \tilde{Z}_t^{(2)}, Y_{t+\delta(t)}, Z_{t+\zeta(t)}) = f_2(t, \tilde{Y}_t^{(2)}, \tilde{Z}_t^{(2)}, Y_{t+\delta(t)}, Z_{t+\zeta(t)}), \quad t \in [t_2, t_1].
\end{cases}
\]

Similarly to the above steps, we can give the proofs for the other cases when \( t \in [t_3, t_2], [t_4, t_3], \ldots, [t_N, t_{N-1}] \). \( \square \)
Example 3.1 Now suppose that we are facing with the following two ABSDEs:

\[
\begin{cases}
-dY_t^{(1)} = E^{\mathcal{F}_t}[Y_{t+\delta(t)}^{(1)} + \sin(2Y_{t+\delta(t)}^{(1)}) + |Z_{t+\zeta(t)}^{(1)}| + 2]dt - Z_t^{(1)}dB_t, & t \in [0,T]; \\
Y_t^{(1)} = \xi_t^{(1)}, & t \in [T,T+K]; \\
Z_t^{(1)} = \eta_t^{(1)}, & t \in [T,T+K],
\end{cases}
\]

\[
\begin{cases}
-dY_t^{(2)} = E^{\mathcal{F}_t}[Y_{t+\delta(t)}^{(2)} + 2|\cos Y_{t+\delta(t)}^{(2)}| + \sin Z_{t+\zeta(t)}^{(2)} - 2]dt - Z_t^{(2)}dB_t, & t \in [0,T]; \\
Y_t^{(2)} = \xi_t^{(2)}, & t \in [T,T+K]; \\
Z_t^{(2)} = \eta_t^{(2)}, & t \in [T,T+K],
\end{cases}
\]

where \(\xi_t^{(1)} \geq \xi_t^{(2)}, t \in [T,T+K]\).

As both the generators depend on the anticipated term of \(Z\) and neither of them is increasing in the anticipated term of \(Y\), we cannot apply Peng, Yang’s comparison theorem to compare \(Y^{(1)}\) and \(Y^{(2)}\). While the following holds true:

\[
x + \sin(2x) + |u| + 2 \geq y + 2|\cos y| + \sin v - 2, \text{ for all } x \geq y, x, y \in \mathbb{R}, u, v \in \mathbb{R}^d,
\]

which implies (3.5)-(3.7), then according to Theorem 3.1, we get \(Y_t^{(1)} \geq Y_t^{(2)}, \text{ a.e., a.s.}\).

Remark 3.1 By the same way, for the case when \(\delta = \zeta\), (3.5)-(3.7) can be replaced by (3.6) together with

\[
f_1(t,y,z,\xi_{t+\delta(t)}^{(1)},\eta_{t+\zeta(t)}^{(1)}) \geq f_2(t,y,z,\xi_{t+\delta(t)}^{(1)},\eta_{t+\zeta(t)}^{(1)}), \text{ a.e., a.s.}
\]

Remark 3.2 If \(f_1\) and \(f_2\) are independent of the anticipated term of \(Z\), then (3.5)-(3.7) reduces to

\[
f_1(t,y,z,\theta_{t+\delta(t)}^{(1)}) \geq f_2(t,y,z,\theta_{t+\delta(t)}^{(2)}). \tag{3.10}
\]

Note that this conclusion is just with respect to the ABSDEs (2.1).

Remark 3.3 The generators \(f_1\) and \(f_2\) will satisfy (3.10), if for all \(t \in [0,T], \ y \in \mathbb{R}, \ z \in \mathbb{R}^d, \ \theta \in L^2_{\mathcal{F}}(t,T+K;\mathbb{R}), \ r \in [t,T+K], \ f_1(t,y,z,\theta_r) \geq f_2(t,y,z,\theta_r), \) together with one of the following:

(i) for all \(t \in [0,T], \ y \in \mathbb{R}, \ z \in \mathbb{R}^d, \ f_1(t,y,z,\cdot) \) is increasing, i.e., \(f_1(t,y,z,\theta_r) \geq f_1(t,y,z,\theta_r'), \) if \(\theta \geq \theta'\), \(\theta, \theta' \in L^2_{\mathcal{F}}(t,T+K;\mathbb{R}), \ r \in [t,T+K]\);

(ii) for all \(t \in [0,T], \ y \in \mathbb{R}, \ z \in \mathbb{R}^d, \ f_2(t,y,z,\cdot) \) is increasing, i.e., \(f_2(t,y,z,\theta_r) \geq f_2(t,y,z,\theta_r'), \) if \(\theta \geq \theta'\), \(\theta, \theta' \in L^2_{\mathcal{F}}(t,T+K;\mathbb{R}), \ r \in [t,T+K]\).

Note that the latter is just the case that Peng-Yang [6] discussed (see Theorem 2.2).
Remark 3.4 The generators $f_1$ and $f_2$ will satisfy (3.10), if

$$f_1(t, y, z, \theta_r) \geq \tilde{f}(t, y, z, \theta_r) \geq f_2(t, y, z, \theta_r),$$

for all $t \in [0, T], y \in \mathbb{R}, z \in \mathbb{R}^d, \theta \in L^2_F(t, T + K; \mathbb{R}), r \in [t, T + K]$. Here the function $\tilde{f}(t, y, z, \cdot)$ is increasing, for all $t \in [0, T], y \in \mathbb{R}, z \in \mathbb{R}^d$, i.e.,

$$\tilde{f}(t, y, z, \theta_r) \geq \tilde{f}(t, y, z, \theta'_r), \text{ if } \theta_r \geq \theta'_r, \theta, \theta' \in L^2_F(t, T + K; \mathbb{R}), r \in [t, T + K].$$

Example 3.2 The following three functions satisfy the conditions in Remark 3.4:

$$f_1(t, y, z, \theta_r) = E^{F_t} [\theta_r + 2 \cos \theta_r + 1], \quad \tilde{f}(t, y, z, \theta_r) = E^{F_t} [\theta_r + \cos \theta_r], \quad f_2(t, y, z, \theta_r) = E^{F_t} [\theta_r + \sin(2\theta_r) - 2].$$

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References

[1] N.El Karoui, S.Peng, M.C.Quenez, Backward stochastic differential equations in finance, Math Finance 7(1) (1997) 1-71.

[2] Y.Hu, S.Peng, On the comparison theorem for multidimensional BSDEs, C R Acad Sci Paris, Ser.I 343 (2006) 135-140.

[3] E.Pardoux, S.Peng, Adapted solution of a backward stochastic differential equation, Systems and Control Letters 14 (1990) 55-61.

[4] E.Pardoux, S.Peng, Backward stochastic differential equations and quasilinear parabolic partial differential equations, In: Stochastic partial differential equations and their applications, Lect Notes Control Inf Sci, 176. Berlin: Springer, 1992, 200-217.

[5] S.Peng, A generalized dynamic programming principle and Hamilton-Jacobi-Bellman equation, Stochastics and Stochastics Reports 38 (1992) 119-134.

[6] S.Peng, Z.Yang, Anticipated backward stochastic differential equations, The Annals of Probability 37(3) (2009) 877-902.

[7] Z.Yang, Anticipated BSDEs and related results in SDEs, Doctoral Dissertation. Jinan: Shandong University, 2007.