ON SOME NEW INTEGRAL INEQUALITIES FOR $K^2_s$

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Abstract. In this paper we establish some new inequalities of Hadamard-type for product of convex and $s-$convex functions in the second sense.

1. Introduction

A largely applied inequality for convex functions, due to its geometrical significance, is Hadamard’s inequality (see [2], [3] or [6]) which has generated a wide range of directions for extension and a rich mathematical literature. The following definitions are well known in the mathematical literature: a function $f : I \rightarrow \mathbb{R}, \emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on $I$ if inequality

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$. Geometrically, this means that if $P, Q$ and $R$ are three distinct points on the graph of $f$ with $Q$ between $P$ and $R$, then $Q$ is on or below chord $PR$.

In the paper [4] Hudzik and Maligranda considered, among others, the class of functions which are $s-$convex in the second sense. This class is defined in the following way: [1] A function $f : [0, \infty) \rightarrow \mathbb{R}$ is said to be $s-$convex in the second sense if

$$f(tx + (1 - t)y) \leq tsf(x) + (1 - t)sf(y)$$

holds for all $x, y \in [0, \infty), t \in [0, 1]$ and for some fixed $s \in (0, 1]$. The class of $s-$convex functions in the second sense is usually denoted with $K^2_s$.

It can be easily seen that for $s = 1$, $s-$convexity reduces to ordinary convexity of functions defined on $[0, \infty)$.

In the same paper [4] Hudzik and Maligranda proved that if $s \in (0, 1), f \in K^2_s$ implies $f([0, \infty)) \subseteq [0, \infty), i.e., they proved that all functions from $K^2_s, s \in (0, 1)$, are nonnegative.

Example 1. [4]. Let $s \in (0, 1)$ and $a, b, c \in \mathbb{R}$. We define function $f : [0, \infty) \rightarrow \mathbb{R}$ as

$$f(t) = \begin{cases} a, & t = 0, \\ bt^s + c, & t > 0. \end{cases}$$

(1.3)

It can be easily checked that

(1) If $b \geq 0$ and $0 \leq c \leq a$, then $f \in K^2_s$

(2) If $b > 0$ and $c < 0$, then $f \notin K^2_s$

Many important inequalities are established for the class of convex functions, but one of the most famous is so called Hermite-Hadamard inequality (or Hadamard’s inequality). This double inequality is stated as follows (see for example [7, p.137]): let $f$ be a convex function on $[a, b] \subseteq \mathbb{R}$, where $a \neq b$. Then

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}. \tag{1.4}$$

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Then to the case in which \( f, g \) is not an integer.

**Theorem 2.** Let \( f, g : [a, b] \to \mathbb{R} \) be two convex functions and \( fg \in L^1 ([a, b]) \). Then,

\[
\frac{1}{(b-a)^2} \int_a^b (b-x)(f(a)g(x)+g(a)f(x)) \, dx \\
+ \frac{1}{(b-a)^2} \int_a^b (x-a)(f(b)g(x)+g(b)f(x)) \, dx \\
\leq \frac{1}{b-a} \int_a^b f(x)g(x) \, dx + \frac{M(a,b)}{3} + \frac{N(a,b)}{6},
\]

where \( M(a,b) = f(a)g(a) + f(b)g(b) \), \( N(a,b) = f(a)g(b) + f(b)g(a) \).

The main purpose of this paper is to establish new inequalities as given in Theorem 1, but now for the class of \( s \)-convex functions in the second sense by using the elementary inequalities.

**2. Main Results**

In the our next theorems we will also make use of Beta function of Euler type, which is for \( u, v > 0 \) defined as

\[
\beta (u, v) = \int_0^1 t^{u-1}(1-t)^{v-1} \, dt = \frac{\Gamma (u) \Gamma (v)}{\Gamma (u+v)}
\]

and

\[
\beta (u, v) = \beta (v, u),
\]

where the gamma function, denoted by \( \Gamma (x) \), provides a generalization of factorial \( n \) to the case in which \( n \) is not an integer.

**Theorem 2.** Let \( f, g : I \to \mathbb{R} \), \( I \subset [0, \infty) \), \( a, b \in I \), with \( a < b \) be functions such that \( f, g \) and \( fg \) are in \( L^1 ([a, b]) \). \( f \) is convex and \( g \) is \( s \)-convex function in the second sense on \([a, b] \), for some \( s \in (0, 1] \), then

\[
\frac{f(a)}{(b-a)^2} \int_a^b (b-x)g(x) \, dx + \frac{f(b)}{(b-a)^2} \int_a^b (x-a)g(x) \, dx \\
+ \frac{g(a)}{(b-a)^{s+1}} \int_a^b (b-x)^s f(x) \, dx + \frac{g(b)}{(b-a)^{s+1}} \int_a^b (x-a)^s f(x) \, dx \\
\leq \frac{1}{b-a} \int_a^b f(x)g(x) \, dx + \frac{M(a,b)}{s+2} + \frac{N(a,b)}{(s+1)(s+2)}
\]

where \( M(a,b) = f(a)g(a) + f(b)g(b) \) and \( N(a,b) = f(a)g(b) + f(b)g(a) \).

**Proof.** Since \( f \) is convex and \( g \) is \( s \)-convex on \([a, b] \), we have

\[
f(ta+(1-t)b) \leq tf(a) + (1-t)f(b) \\
g(ta+(1-t)b) \leq tg(a) + (1-t)g(b)
\]
Integral inequalities for $K^2$ for all $t \in [0, 1]$. Now, using the elementary inequality [5, p.4] $(a - b)(c - d) \geq 0$ $(a, b, c, d \in \mathbb{R}$ and $a < b, c < d)$, we get inequality:

$$
\begin{align*}
& tf(a) g(ta + (1-t)b) + (1-t) f(b) g(ta + (1-t)b) \\
& + t^s g(a) f(ta + (1-t)b) + (1-t)^s g(b) f(ta + (1-t)b) \\
& \leq f(ta + (1-t)b) g(ta + (1-t)b) + t^{s+1} f(a) g(a) \\
& + t(1-t)^s f(a) g(b) + t^s (1-t) f(b) g(a) \\
& + (1-t)^{s+1} f(b) g(b)
\end{align*}
$$

Integrating this inequality over $t$ on $[0, 1]$, we deduce that

$$
\begin{align*}
& f(a) \int_0^1 tg(ta + (1-t)b) dt + f(b) \int_0^1 (1-t) g(ta + (1-t)b) dt \\
& + g(a) \int_0^1 t^s f(ta + (1-t)b) dt + g(b) \int_0^1 (1-t)^s f(ta + (1-t)b) dt \\
& \leq \int_0^1 f(ta + (1-t)b) g(ta + (1-t)b) dt \\
& + f(a) g(a) \int_0^1 t^{s+1} dt + f(a) g(b) \int_0^1 t (1-t)^s dt \\
& + f(b) g(a) \int_0^1 t^s (1-t) dt + f(b) g(b) \int_0^1 (1-t)^{s+1} dt
\end{align*}
$$
By substituting \( ta + (1 - t) b = x, (a - b) \, dt = dx \), we obtain
\[
\begin{align*}
& f(a) \int_0^1 t g(ta + (1 - t)b) \, dt + f(b) \int_0^1 (1 - t) g(ta + (1 - t)b) \, dt \\
& + g(a) \int_0^1 t^s f(ta + (1 - t)b) \, dt + g(b) \int_0^1 (1 - t)^s f(ta + (1 - t)b) \, dt \\
& = \frac{f(a)}{(b - a)^2} \int_a^b (b - x) g(x) \, dx + \frac{f(b)}{(b - a)^2} \int_a^b (x - a) g(x) \, dx \\
& + \frac{g(a)}{(b - a)^{s+1}} \int_a^b (b - x)^s f(x) \, dx + \frac{g(b)}{(b - a)^{s+1}} \int_a^b (x - a)^s f(x) \, dx \\
& \leq \int_0^1 f(ta + (1 - t)b) g(ta + (1 - t)b) \, dt \\
& + f(a) g(a) \int_0^1 t^{s+1} dt + f(a) g(b) \int_0^1 t (1 - t)^s dt \\
& + f(b) g(a) \int_0^1 t^s (1 - t) dt + f(b) g(b) \int_0^1 (1 - t)^{s+1} dt \\
& = \frac{1}{b-a} \int_a^b f(x) g(x) \, dx + \frac{f(a) g(a) + f(b) g(b)}{s+2} \\
& + f(a) g(b) \beta(2, s + 1) + f(b) g(a) \beta(s + 1, 2) \\
& = \frac{1}{b-a} \int_a^b f(x) g(x) \, dx + \frac{M(a, b)}{s+2} \\
& + f(a) g(b) \beta(2, s + 1) + f(b) g(a) \beta(2, s + 1) \\
& = \frac{1}{b-a} \int_a^b f(x) g(x) \, dx + \frac{M(a, b)}{s+2} + \beta(2, s + 1) [f(a) g(b) + f(b) g(a)] \\
& = \frac{1}{b-a} \int_a^b f(x) g(x) \, dx + \frac{M(a, b)}{s+2} + \frac{\Gamma(2) \Gamma(s + 1)}{\Gamma(s + 3)} N(a, b) \\
& = \frac{1}{b-a} \int_a^b f(x) g(x) \, dx + \frac{M(a, b)}{s+2} + \frac{\Gamma(s + 1)}{\Gamma(s + 3)} N(a, b) \\
& = \frac{1}{b-a} \int_a^b f(x) g(x) \, dx + \frac{M(a, b)}{s+2} + \frac{N(a, b)}{(s + 1)(s + 2)} \\
\end{align*}
\]
which completes the proof. \( \square \)

Remark 1. In Theorem 3, if we choose \( s = 1 \), then (2.1) reduces to (1.3).

**Theorem 3.** Let \( f, g : I \to \mathbb{R}, I \subseteq [0, \infty), a, b \in I, a < b \) be functions such that \( f, g \) and \( fg \) are in \( L^1([a, b]) \). If \( f \) is \( s_1 \)-convex and \( g \) is \( s_2 \)-convex in the second sense on \([a, b]\) for some \( s_1, s_2 \in (0, 1] \), then
\[
\frac{f(a)}{(b - a)^{s_1+1}} \int_a^b (b - x)^{s_1} g(x) \, dx + \frac{f(b)}{(b - a)^{s_1+1}} \int_a^b (x - a)^{s_1} g(x) \, dx \\
+ \frac{g(a)}{(b - a)^{s_2+1}} \int_a^b (b - x)^{s_2} f(x) \, dx + \frac{g(b)}{(b - a)^{s_2+1}} \int_a^b (x - a)^{s_2} f(x) \, dx \\
\leq \frac{1}{b-a} \int_a^b f(x) g(x) \, dx + \frac{1}{s_1 + s_2 + 1} \left[ M(a, b) + s_1 s_2 \frac{\Gamma(s_1) \Gamma(s_2)}{\Gamma(s_1 + s_2 + 1)} N(a, b) \right],
\]
where \( M(a, b) = f(a) g(a) + f(b) g(b) \) and \( N(a, b) = f(a) g(b) + f(b) g(a) \).
Proof. Since \( f \) is \( s_1 \)-convex and \( g \) is \( s_2 \)-convex on \([a, b]\), we have

\[
\begin{align*}
  f (ta + (1 - t)b) & \leq t^{s_1} f(a) + (1 - t)^{s_1} f(b) \\
  g (ta + (1 - t)b) & \leq t^{s_2} g(a) + (1 - t)^{s_2} g(b)
\end{align*}
\]

for all \( a, b \in I \) and \( t \in [0, 1] \). Now, using the elementary inequality [(5, p.4)]
\((a - b)(c - d) \geq 0 \) \((a, b, c, d \in \mathbb{R} \) and \( a < b, c < d \)\), we get inequality:

\[
\begin{align*}
  t^{s_1} f(a) g(ta + (1 - t)b) + (1 - t)^{s_1} f(b) g(ta + (1 - t)b) \\
+ t^{s_2} g(a) f(ta + (1 - t)b) + (1 - t)^{s_2} g(b) f(ta + (1 - t)b) \\
\leq f(ta + (1 - t)b) g(ta + (1 - t)b) + t^{s_1 + s_2} f(a) g(a) \\
+ t^{s_1} (1 - t)^{s_2} f(a) g(b) + t^{s_2} (1 - t)^{s_1} f(b) g(a) \\
+ (1 - t)^{s_1 + s_2} f(b) g(b)
\end{align*}
\]

Integrating both sides of the above inequality over \([0, 1]\), we deduce that:

\[
\begin{align*}
  f(a) \int_0^1 t^{s_1} g(ta + (1 - t)b) \, dt + f(b) \int_0^1 (1 - t)^{s_1} g(ta + (1 - t)b) \, dt \\
+ g(a) \int_0^1 t^{s_2} f(ta + (1 - t)b) \, dt + g(b) \int_0^1 (1 - t)^{s_2} f(ta + (1 - t)b) \, dt \\
\leq \int_0^1 f(ta + (1 - t)b) g(ta + (1 - t)b) \, dt \\
+ f(a) g(a) \int_0^1 t^{s_1 + s_2} \, dt + f(b) g(b) \int_0^1 (1 - t)^{s_1 + s_2} \, dt \\
+ f(b) g(a) \int_0^1 t^{s_2} (1 - t)^{s_1} \, dt + f(b) g(b) \int_0^1 (1 - t)^{s_1 + s_2} \, dt
\end{align*}
\]
By substituting $ta + (1-t)b = x$, $(a-b) dt = dx$, we obtain

$$f(a)\int_0^1 t^{s_2}g(ta + (1-t)b)\,dt + f(b)\int_0^1 (1-t)^{s_2}g(ta + (1-t)b)\,dt$$

$$+ g(a)\int_0^1 t^{s_2}f(ta + (1-t)b)\,dt + g(b)\int_0^1 (1-t)^{s_2}f(ta + (1-t)b)\,dt$$

$$= \frac{f(a)}{(b-a)^{s_1+1}} \int_a^b (b-x)^{s_1}g(x)\,dx + \frac{f(b)}{(b-a)^{s_1+1}} \int_a^b (a-x)^{s_1}g(x)\,dx$$

$$+ \frac{g(a)}{(b-a)^{s_2+1}} \int_a^b (b-x)^{s_2}f(x)\,dx + \frac{g(b)}{(b-a)^{s_2+1}} \int_a^b (a-x)^{s_2}f(x)\,dx$$

$$\leq \int_0^1 f(ta + (1-t)b)g(ta + (1-t)b)\,dt$$

$$+ f(a)g(a)\int_0^1 t^{s_1+s_2}dt + f(a)g(b)\int_0^1 t^{s_1}dt$$

$$+ f(b)g(a)\int_0^1 t^{s_2}dt + f(b)g(b)\int_0^1 dt$$

$$= \frac{1}{b-a} \int_a^b f(x)g(x)\,dx + f(a)g(a) + f(b)g(b)$$

$$+ f(a)g(b)\beta(s_1 + 1, s_2 + 1) + f(b)g(a)\beta(s_2 + 1, s_1 + 1)$$

$$= \frac{1}{b-a} \int_a^b f(x)g(x)\,dx + \frac{M(a,b)}{s_1 + s_2 + 1}$$

$$+ f(a)g(b)\beta(s_1 + 1, s_2 + 1) + f(b)g(a)\beta(s_2 + 1, s_1 + 1)$$

$$= \frac{1}{b-a} \int_a^b f(x)g(x)\,dx + \frac{M(a,b)}{s_1 + s_2 + 1}$$

$$+ f(a)g(b)\beta(s_1 + 1, s_2 + 1) + f(b)g(a)\beta(s_2 + 1, s_1 + 1)$$

which completes the proof.

Remark 2. In Theorem 3 if we choose $s_1 = s_2 = 1$, then (2.2) reduces to (1.5).

Corollary 1. With the above assumptions and under the conditions that $s_1 = s_2 = 1$ and $x = \frac{a+b}{2}$, the following inequality will be obtained

$$\frac{f(a) + f(b)}{2}g\left(\frac{a+b}{2}\right) + \frac{g(a) + g(b)}{2}f\left(\frac{a+b}{2}\right)$$

$$\leq f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) + \frac{M(a,b)}{3} + \frac{N(a,b)}{6}. \tag{2.3}$$

Remark 3. Similarly to Hadamard’s inequality applications, some applications to special means can be deduced by the above obtained two new theorems.
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