I. Introduction

While there is now a large and rapidly growing literature on the study of direct limits of subalgebras of finite dimensional C*-algebras, the focus in almost every paper on the subject has been on systems with embeddings which have *-extensions to the generated C*-algebra. One notable exception is the paper by Power [P1]. The purpose of this note is to extend the study of systems which are not *-extendible and to produce examples which illustrate some new phenomena.

When all the embeddings in a direct system are *-extendible, then the limit algebra is, in a natural way, a subalgebra of an AF C*-algebra. On the other hand, if the embeddings are not *-extendible, then it is no longer a priori obvious that the limit algebra is an operator algebra. (Initially, the direct limit must be taken within the category of Banach algebras.) Even if the limit algebra is an operator algebra, its image under some representations may generate a C*-algebra which is not approximately finite (as happens in the situation studied by Power). For systems with the type of embeddings which we consider in this paper, the limit algebra will always be an operator algebra, a fact which can easily be seen with the aid of the abstract characterization of operator algebras by Blecher, Ruan, and Sinclair [BRS]. We also produce a representation of the limit algebra which is “natural” in the sense that the C*-envelope of the image algebra (as defined by Hamana [H]) is isomorphic to the C*-algebra generated by the image algebra.

II. Compression Embeddings

The key observation behind the choice of embeddings under investigation is that if $A$ is a CSL-algebra and if $p$ is an interval from the lattice of invariant projections for $A$
A, then the mapping $x \mapsto pxp$ is an algebra homomorphism. Since we are interested in direct limits of finite dimensional algebras, we shall assume that every CSL-algebra is finite dimensional. This implies that, with respect to a suitable choice of matrix units, $A$ is a subalgebra of some full matrix algebra $M_n$ which contains the algebra $D_n$ of diagonal matrices. Such algebras go under a variety of names: poset algebras, incidence algebras, or digraph algebras; henceforth, we use the term digraph algebra.

The invariant projections of a digraph algebra are all projections in the diagonal $D_n$; in particular, they all commute with one another. If $e$ and $f$ are two invariant projections such that $e \leq f$, then $p = f - e$ is called an interval from the lattice. (The term semi-invariant projection is also sometimes used.)

Since we want our embeddings to be unital, a compression will be the mapping which takes $x$ to the restriction of $pxp$ to the range of $p$, where $p$ is an interval from the lattice. The image algebra under the compression will be viewed as a subalgebra of a full matrix algebra, usually of rank smaller than the rank of the original containing matrix algebra.

**Definition.** Let $A \subseteq M_n$ be a digraph algebra. A compression embedding is a direct sum of compression mappings on $A$ subject to the proviso that at least one summand is the identity mapping.

The purpose of the assumption that at least one summand is the identity mapping is to ensure that the embedding is completely isometric. (Compressions are completely contractive, but not completely isometric except in the trivial case of the identity mapping.)

**Definition.** A compression limit algebra is the limit of a direct system

$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \xrightarrow{\phi_3} \ldots \rightarrow A$$

in which each embedding is a unital compression embedding.

A compression embedding of $A_k$ into $A_{k+1}$ is a unital map of $A_k$ onto its range; we are assuming then that the unit of $A_{k+1}$ is equal to the unit of the subalgebra $\phi_k(A_k)$. (This avoids unwanted degeneracies.) The special case in which each summand is compression to the identity is just the familiar case of a standard embedding. In this special case, compression embeddings are *-extendible; in general they will not be.

In this note, we shall focus on a special class of compression algebras: direct limits of full upper triangular matrix algebras,

$$T_{n_1} \xrightarrow{\phi_1} T_{n_2} \xrightarrow{\phi_2} T_{n_3} \xrightarrow{\phi_3} \ldots \rightarrow A.$$
dimensional approximants. The systems studied here have one other new feature: the only constraint on $n_k$ and $n_{k+1}$ is that $n_k < n_{k+1}$; no divisibility is required.

For each $k$, one of the summands of $\phi_k$ is the identity compression; let $q_k$ denote the support interval for this summand from the nest of invariant projections for $T_{n_{k+1}}$. Also, let $\psi_k$ denote the compression of $T_{n_{k+1}}$ onto $q_k$. We may, in a natural way, view $\psi_k$ as an algebra homomorphism from $T_{n_{k+1}}$ onto $T_{n_k}$. It is clear that $\psi_k \circ \phi_k = id$, for all $k$. Thus compression embedding systems are structured in the sense introduced by Larson [L]; in particular, compression limit algebras are all structured Banach algebras.

Compression embeddings satisfy one additional property: they map matrix units to sums of matrix units. This implies that they are regular embeddings, in the sense used by Power [P2, section 4.9].

**Remark.** Compression embeddings can be placed into a broader context. If $C = [c_{ij}]$ is an $n \times n$ matrix, the mapping $\phi_C : M_n \to M_n$ given by $\phi_C[a_{ij}] = [c_{ij}a_{ij}]$ is known as a Schur mapping. It is easy to check that $\phi_C$ is an algebra homomorphism if, and only if, $C$ satisfies the cocycle condition: $c_{ik} = c_{ij}c_{jk}$. Furthermore, $\phi_C$ is unital if, and only if, all $c_{ii} = 1$. By taking a direct sum of algebra homomorphisms each of which is a compression composed with a Schur mapping, we can obtain embeddings more general than compression embeddings. (Compression embeddings arise by insisting that appropriate $c_{ij} = 1$.) We defer the study of these more general systems to another time.

### III. Representations

Let

$$ T_{n_1} \xrightarrow{\phi_1} T_{n_2} \xrightarrow{\phi_2} T_{n_3} \xrightarrow{\phi_3} \ldots \to A $$

be a direct system with compression embeddings. Since each $\phi_k$ is a unital complete isometry, we may identify each $T_{n_k}$ with a unital subalgebra of $A$; the subalgebras so obtained are nested and the closure of their union is $A$. Each subalgebra carries a matricial norm and the sequence of norms is compatible with the nesting; consequently, $A$ has a matricial norm. It is easy to see that this matricial norm satisfies the axioms of Blecher, Ruan and Sinclair [BRS], so $A$ is an (abstract) operator algebra, i.e., there is a Hilbert space $\mathcal{H}$ and a completely isometric unital algebra homomorphism $\rho$ mapping $A$ into $\mathcal{B}(\mathcal{H})$. This argument remains valid for systems of digraph algebras and for systems with the more general Schur embeddings, provided that the embeddings are complete isometries.

Our primary interest, however, is to obtain an explicit representation which is, in an appropriate sense, canonical. To that end, let $\mathcal{H}$ denote a Hilbert space with basis $\{e_n\}$. This index set will depend on the specific direct system, but will always be either $\mathbb{Z}$ or $\{n : n \leq b\}$ for some integer $b \geq 0$ or $\{n : n \geq a\}$ for some integer $a \leq 0$. For each $k$, let $p_k$ denote the projection on the closed linear span of $\{e_n : n \leq k\}$ and let $\mathcal{N}$ denote the nest consisting of the projections $p_k$ together with $0$ and $I$. $\text{Alg}(\mathcal{N})$ denotes the nest
algebra associated with \( \mathcal{N} \). The representation which we will construct will map \( A \) to a weakly dense subalgebra of \( \text{Alg}(\mathcal{N}) \).

For each \( k \), we may write \( \phi_k = \psi^k_a \oplus \cdots \oplus \psi^k_b \), where each \( \psi^k_j \) is a compression map. At least one of these compression maps must be the identity; in order to keep track of a selection of identity compressions, arrange the indexing so that \( \psi^k_0 = \text{id} \) for all \( k \). Thus, we will always have that the first index \( a \) is non-positive and the last index \( b \) is non-negative.

Now suppose that \( \phi : T_{n_k} \to T_{n_k+1} \) is a compression embedding and that \( \psi \) is a single compression to an interval from the invariant projection lattice of \( T_{n_k+1} \). It is easy to check that \( \psi \circ \phi \) is a sum of compressions to intervals from the lattice of \( T_{n_k} \). Note that the intervals associated with \( \psi \circ \phi \) may include some which were not associated with \( \phi \) itself.

A consequence of the observation above is that a composition of two compression embeddings is again a compression embedding. Since we need to consider multiple compositions, let \( \phi_{j,k} \) denote the composition \( \phi_k \circ \cdots \circ \phi_j \). Then \( \phi_{j,k} \) can be written in the form
\[
\sum_{a}^{b} \oplus \eta_i,
\]
where each \( \eta_i \) is a simple compression. Furthermore, since \( \psi^n_0 = \text{id} \) for each \( n \), we may arrange the indexing so that \( \eta_0 = \text{id} \). Thus, \( a \leq 0 \) and \( b \geq 0 \). The indices \( a \) and \( b \) depend, of course, on \( j \) and \( k \), but that dependence is suppressed in the notation.

Observe also that the string of \( \eta \)'s in the summation for \( \phi_{j,k} \) appears as a substring in the expression for \( \phi_{j,k+1} \). The reason, of course, is that \( \psi_{k+1}^0 = \text{id} \). Furthermore, this substring includes the \( \eta_0 = \text{id} \) term. Thus, as we increase \( k \), we change the summation for \( \phi_{j,k} \) by adding terms at one or both ends of the sum. By increasing \( k \) indefinitely, we obtain a mapping \( \rho_j = \sum \oplus \eta_i \). The index set for this sum will either be \( \mathbb{Z} \) or a set of the form \( \{ i : i \geq a \} \) or of the form \( \{ i : i \leq b \} \) for some \( a \leq 0 \) or \( b \geq 0 \).

An index set consisting of integers greater than or equal to some \( a \) will occur precisely when all but finitely many of the embeddings \( \phi_k \) have \( \phi^k_0 = \text{id} \) as the first term; the other singly infinite index set arises when the distinguished identity summand occurs as the last term in all but finitely many embeddings. In either case, we can change the presentation by deleting finitely many embeddings to arrange that the index set for the \( \eta \)'s is either the non-negative integers or the non-positive integers, as appropriate. In either of these cases, we may now view \( \rho_j \) as a representation on the Hilbert space \( \mathcal{H} \) with respect to the appropriate basis. The doubly infinite case still presents an ambiguity: we need to “anchor” the representation \( \rho_j \) with respect to the basis \( \{ e_n : n \in \mathbb{Z} \} \).

This “anchoring” may be done as follows. Let the index set for the matrices in \( T_{n_1} \) be \( \{ 0,1,\ldots,b \} \), so that \( f^1_{00} \) is the matrix unit which has a 1 in the upper left corner and zeros elsewhere. Identify \( \psi^1_0 \) with the mapping \( 0 \oplus \cdots \oplus 0 \oplus \psi^1_0 \oplus 0 \oplus \cdots \oplus 0 \), a mapping of \( T_{n_1} \) into \( T_{n_2} \). (In other words, replace all terms in \( \phi_1 \) except for the distinguished copy of the identity by a 0 mapping of appropriate dimension.) The image of \( f^1_{00} \) under \( \psi^1_0 \) is now a matrix unit in \( T_{n_2} \); arrange the index set for \( T_{n_2} \) so that this matrix unit is
By iterating this procedure, we can index all the \( T_{n_k} \) so that the successive images of the matrix unit \( f_{00}^1 \) under the distinguished identity compressions will be \( f_{00}^k \). In the doubly infinite case, the indexing of the \( T_{n_k} \)'s will begin with negative integers and end with positive integers each of arbitrarily large magnitude for sufficiently large \( k \).

It is now clear how to define the representation \( \rho_k \): the image of the distinguished matrix unit \( f_{00}^k \) in the component \( \eta_0 \) should be the one dimensional projection onto the span of the basis vector \( e_0 \). For each \( k \), \( \rho_k \) is a completely isometric representation of \( T_{n_k} \) into \( \text{Alg}(\mathcal{N}) \) acting on the Hilbert space \( \mathcal{H} \). Furthermore, the following diagram commutes:

\[
\begin{array}{ccc}
T_{n_k} & \xrightarrow{\phi_k} & T_{n_k+1} \\
\downarrow \rho_k & & \downarrow \rho_{k+1} \\
\text{Alg}(\mathcal{N}) & \xrightarrow{id} & \text{Alg}(\mathcal{N})
\end{array}
\]

This system of representations induces a completely isometric representation \( \rho \) from the direct limit \( A \) into \( \text{Alg}(\mathcal{N}) \).

**Remark.** Observe that the image \( \rho(A) \) is weakly (or \( \sigma \)-weakly) dense in \( \text{Alg}(\mathcal{N}) \). To see this it is enough to note that, given \( a \leq 0 \) and \( b \geq 0 \), the image contains matrices with arbitrarily specified entries in the locations \( ij \) for \( a \leq i \leq j \leq b \). To accomplish this, choose \( k \) sufficiently large, select an appropriate matrix in \( T_{n_k} \), and inspect the image of that matrix under the representation \( \rho \).

In the special case in which each \( \phi_k \) is a direct sum of identity mappings, i.e., \( \phi_k \) is a standard embedding, the representation which we have constructed is just the representation of the standard limit algebra introduced by Roger Smith. For a description of this representation, see [OP]. Depending on the locations of the distinguished copy of the identity, the representation will act on a Hilbert space with a basis indexed by the integers, by the non-negative integers, or by the non-positive integers. The choice here is arbitrary, but since the representations in this special case are all \( \ast \)-extendible, the \( \text{C}^\ast \)-algebras generated by the images are all isomorphic. (Indeed, they are all the UHF algebra associated with the appropriate supernatural number.)

In the general case, it is possible to have two completely isometric representations, \( \rho \) and \( \sigma \), of \( A \) with \( \text{C}^\ast(\rho(A)) \) not isomorphic to \( \text{C}^\ast(\sigma(A)) \). It is desirable, then, to find a representation for which the \( \text{C}^\ast \)-algebra generated by the image is the \( \text{C}^\ast \)-envelope in the sense of Hamana. We shall show below that \( \rho \), as constructed above, has just this property. It is in this sense that \( \rho \) is canonical.

First, we provide a brief review of the idea of a \( \text{C}^\ast \)-envelope. The appropriate setting for this is actually the category of unital operator spaces with unital complete order injections, but we only need to deal with operator algebras so we will restrict the discussion to that domain.
If \( A \) is an operator algebra, then a C*-extension of \( A \) is a C*-algebra \( B \) together with a unital complete order injection \( \rho \) of \( A \) into \( B \) such that \( C^*(\rho(A)) = B \). A C*-extension \( B \) is a C*-envelope of \( A \) provided that, given any operator system, \( C \), and any unital completely positive map \( \tau: B \to C \), \( \tau \) is a complete order injection whenever \( \tau \circ \rho \) is. Hamana [H] proves the existence and uniqueness (up to a suitable notion of equivalence) of C*-envelopes; further, he shows that the C*-envelope of \( A \) is a minimal C*-extension in the family of all C*-extensions of \( A \).

After proving the existence of C*-envelopes, Hamana then uses this to prove the existence of a Šilov boundary for \( A \). The Šilov boundary is a generalization to operator spaces of the usual notion of Šilov boundary from function spaces; it was first developed by Arveson in [A1]. Here is the appropriate definition. Let \( B \) be a C*-algebra and let \( A \) be a unital subalgebra such that \( B = C^*(A) \). An ideal \( J \) in \( B \) is called a boundary ideal for \( A \) if the canonical quotient map \( B \to B/J \) is completely isometric on \( A \). A boundary ideal which contains every other boundary ideal is called the Šilov boundary for \( A \).

Hamana shows that if \( B \) is a C*-extension for \( A \), then the C*-envelope of \( A \) is isomorphic to \( B/J \), where \( J \) is the Šilov boundary for \( A \). In particular, \( B \) is the C*-envelope for \( A \) if, and only if, the Šilov boundary is 0. This last fact is the one which we shall use to show that \( C^*(\rho(A)) \) is the C*-envelope of the image \( \rho(A) \) for the representation \( \rho \) defined above.

The first step needed to accomplish this goal is to compute the C*-algebra generated by each \( \rho_k(T_{nk}) \). The following simple lemma is helpful:

**Lemma.** Let \( T_n \) be the \( n \times n \) upper triangular matrices acting on \( \mathbb{C}^n \) and let \( p \) and \( q \) be distinct intervals from the nest of invariant projections for \( T_n \). Then

\[
C^*(\{\text{pap} \oplus \text{qaq}: a \in T_n\}) = \mathcal{B}(p\mathbb{C}^n) \oplus \mathcal{B}(q\mathbb{C}^n).
\]

**Proof.** Here, we view \( \text{pap} \) and \( \text{qaq} \) as being restricted to the ranges \( p\mathbb{C}^n \) and \( q\mathbb{C}^n \) of \( p \) and \( q \) respectively. Since \( p \) and \( q \) are distinct, there is an element \( a \) of \( T_n \) such that one of \( \text{pap} \) and \( \text{qaq} \), say \( \text{pap} \), is non-zero while the other is 0. So we have an element of the form \( b \oplus 0 \) in \( C^*(\{\text{pap} \oplus \text{qaq}: a \in T_n\}) \). From this and the fact that the C*-algebra generated by \( T_n \) is \( M_n \), it follows that \( \mathcal{B}(p\mathbb{C}^n) \oplus 0 \subseteq C^*(\{\text{pap} \oplus \text{qaq}: a \in T_n\}) \). This in turn implies that \( 0 \oplus \mathcal{B}(q\mathbb{C}^n) \subseteq C^*(\{\text{pap} \oplus \text{qaq}: a \in T_n\}) \) and hence that \( \mathcal{B}(p\mathbb{C}^n) \oplus \mathcal{B}(q\mathbb{C}^n) \subseteq C^*(\{\text{pap} \oplus \text{qaq}: a \in T_n\}) \). The reverse containment is evident.

By using the obvious extension of this lemma to multiple direct sums (including countable direct sums), we can describe the C*-algebra, \( C^*(\rho_k(T_{nk})) \), both as a subalgebra of \( \mathcal{B}(\mathcal{H}) \) and as an abstract (finite-dimensional) C*-algebra. Indeed, write \( \rho_k = \sum \eta_i \) and, for each \( i \), let \( q_i \) be the interval in the nest for \( T_{nk} \) to which \( \eta_i \) is a compression. Then

\[
C^*(\rho_k(T_{nk})) = \{ (b_j) \in \bigoplus \mathcal{B}(q_j \mathbb{C}^{nk}) : b_i = b_j \text{ whenever } q_i = q_j \}.
\]
If \( r_1, \ldots, r_m \) is the list of distinct intervals to which the \( \eta_i \) are compressions, then

\[
C^*(\rho_k(T_{n_k})) \cong \bigoplus_{j=1}^m \mathcal{B}(r_j \mathbb{C}^{n_k}).
\]

We shall need both the isomorphism class of \( C^*(\rho_k(T_{n_k})) \) and its actual expression as an operator algebra acting on \( \mathcal{H} \). Also, since \( C^*(\rho(A)) \) is the closure of the union of the \( C^*(\rho_k(T_{n_k})) \), we have proven the following:

**Proposition.** Let \( A \) be the direct limit of a system of full upper triangular matrix algebras with compression embeddings and let \( \rho \) be the representation of \( A \) defined above. Then \( C^*(\rho(A)) \) is an AF \( C^* \)-algebra.

Our main result is the following theorem.

**Theorem.** Let \( A \) be the direct limit of a system of full upper triangular matrix algebras with compression embeddings and let \( \rho \) be the representation of \( A \) defined above. Then \( C^*(\rho(A)) \) is the \( C^* \)-envelope of \( \rho(A) \).

**Proof.** In order to prove that \( C^*(\rho(A)) \) is the \( C^* \)-envelope of \( \rho(A) \), it is sufficient to show that the Šilov boundary of \( \rho(A) \) is 0. Since the Šilov boundary is the largest boundary ideal, we need merely show that if \( J \) is a non-zero ideal then \( J \) is not a boundary ideal. This requires proving that the canonical quotient map \( C^*(\rho(A)) \rightarrow C^*(\rho(A))/J \) is not completely isometric when restricted to \( \rho(A) \). We shall, in fact, prove that \( \rho(A) \cap J \neq \emptyset \); the quotient map is not even isometric.

For convenience, let \( B_k = C^*(\rho_k(T_{n_k})) \) and \( B = C^*(\rho(A)) \). Each \( B_k \) is isomorphic to a direct sum of full matrix algebras, one for each compression map which appears in the expression for \( \rho_k \). Since the identity must appear, one summand must be \( M_{n_k} \). This summand is the largest rank summand and it appears one time only. \( M_{n_k-1} \) appears at most two times, reflecting the fact that the nest for \( T_{n_k} \) has exactly two interval projections of rank \( n_k - 1 \). More generally, \( M_{n_k-j} \) appears at most \( j + 1 \) times in the expression for \( B_k \).

Thus, the isomorphism class of \( B_k \) is \( M_{n_k} \oplus M_{t_1} \oplus \cdots \oplus M_{t_s} \), where \( t_1, \ldots, t_s \) are integers less than \( n_k \). In the Bratteli diagram for \( B \), the \( k \)th level has a node for each summand in the isomorphism class of \( B_k \). Next, we need the following observation: given \( k \) and a summand \( M_d \) in the isomorphism class of \( B_k \), there is an integer \( j > k \) such that \( M_d \) partially embeds into the summand \( M_{n_j} \) in the isomorphism class of \( B_j \).

The observation is verified by inspecting the chain of finite dimensional \( C^* \)-algebras acting on \( \mathcal{H} \). Now, \( \rho_k(T_{n_k}) \) is an infinite sum whose terms are selected from finitely many compressions of \( T_{n_k} \). Consequently, there exist integers \( a \leq 0 \) and \( b \geq 0 \) such that if \( p \) is the projection onto the linear span of \( \{ e_i : a \leq i \leq b \} \), then \( p \) is reducing for \( C^*(\rho_k(T_{n_k})) \)
and every compression, and in particular the one corresponding to $M_d$, appears in the restriction of $C^*(\rho_k(T_{n_k}))$ to $p$. Then, by the way in which the representations are constructed, there is an integer $j > k$ so that the $\eta_0 = id$ term in $\rho_j$ acts on a subspace of $H$ which includes the range of $p$. (This was the reason for “anchoring” the $\rho_j$ so that the $\eta_0$ terms act on the vector $e_0$ and that the support space for the $\eta_0$ terms “grows away” from $e_0$.) This shows that in the abstract Bratteli diagram, each node eventually partially embeds into a node corresponding to the identity summand (the $M_{n_j}$ node).

Now let $J$ be a non-zero closed two-sided ideal in $B$. By a result in [B], $J$ is the closure of the union of the $J \cap B_k$. In particular, there is a positive integer $k$ such that $J \cap B_k \neq 0$. Since $B_k$ is isomorphic to a finite direct sum of full matrix algebras, $J \cap B_k$ is isomorphic to a direct sum of some of those algebras, with $0$’s as the remaining summands. Let $M_d$ be one of the non-zero summands appearing in $J \cap B_k$. Let $j > k$ be an integer such that the $M_d$ term partially embeds into $M_{n_j}$. It now follows that $M_{n_j}$ is one of the non-zero summands for $J \cap B_j$. By utilizing the support projection for the $M_{n_j}$ term (a reducing projection for $C^*(\rho_j(T_{n_j}))$), we see that the subalgebra of $C^*(\rho_j(T_{n_j}))$ isomorphic to $M_{n_j}$ is contained in $J$. This subalgebra consists of all sequences $(b_i)$ in $C^*(\rho_j(T_{n_j}))$ for which $b_i = 0$ whenever the corresponding interval $q_i$ is not the identity and the remaining $b_i$ are all equal.

The proof is now completed by observing that there is an element of this subalgebra which lies in $\rho_j(T_{n_j})$. Indeed, let $v$ be the matrix unit in $T_{n_j}$ which has a 1 in the extreme upper right corner and zeros elsewhere. Then the compression of $v$ to any interval $q$ other than the identity is zero. Thus, $\rho_j(v) \in J \cap \rho(A)$, and the proof is complete.

Remark. We have shown that the $C^*$-envelope of a direct limit of full upper triangular matrix algebras with compression embeddings is an AF $C^*$-algebra. If, on the other hand, $A$ is the direct limit of full upper triangular matrix algebras with $^*$-extendible embeddings, then $A$ can be represented as a generating subalgebra of a UHF algebra. Since UHF algebras are simple, it is immediate that the generated $C^*$-algebra is the $C^*$-envelope. More is true. It is easy to show that if $A$ is a generating subalgebra of an AF $C^*$-algebra $B$ and if $A$ contains a Stratila-Voiculescu masa, then $B$ is the $C^*$-envelope of $A$. Thus, if $A$ is the direct limit of a system of digraph algebras with $^*$-extendible embeddings, then the $C^*$-envelope of $A$ is an AF algebra.

The simple proof of this last fact does not apply to compression limits. Examples in the next section will show that the image of the limit algebra under the representation $\rho$ need not contain a masa in the generated $C^*$-algebra. (The natural diagonal in $\rho(A)$ need not be a masa in $C^*(\rho(A))$.) The following problem is suggested.

Problem. Is the $C^*$-envelope of a direct limit of digraph algebras with compression embeddings an AF $C^*$-algebra?

Remark. In [P1], Power studies a direct limit of digraph algebras with embeddings which
are neither $^*$-extendible nor compression embeddings. For systems of tri-diagonal algebras with certain natural non-$^*$-extendible embeddings, Power shows that the limit algebra is completely isometrically isomorphic to a generating subalgebra of an appropriate Bunce-Deddens algebra. Since this latter algebra is simple, it is the $C^*$-envelope of the limit algebra for the tri-diagonal system. In particular, this provides an example of a system of digraph algebras whose limit algebra has a $C^*$-envelope which is not AF.

**Remark.** Since $\rho(A)$ is weakly dense in $\mathcal{Alg}(\mathcal{N})$, $C^*(\rho(A))$ is an irreducible $C^*$-algebra. If it happens that $\rho(A)$ contains a non-zero compact operator, then Arveson’s boundary theorem [A2] immediately implies that the Silov boundary for $\rho(A)$ is 0. The boundary theorem is a deep theorem, so the argument above could be considered more elementary.

In most of the examples in the next section, $\rho(A)$ contains non-zero compact operators. Here is how to determine in general if $\rho(A)$ contains compact operators. Each representation $\rho_k$ of $T_{n_k}$ can be written as an infinite direct sum $\sum \oplus \eta_i$, where the $\eta_i$’s are compressions. Let $z_k$ be the number of times that $\eta_i = id$ in the expression for $\rho_k$. Then $z_k \in \{1, 2, \ldots, \infty\}$ and the sequence $z_k$ is decreasing. Thus there are two possibilities: all $z_k = \infty$, or there is a finite integer $y$ such that $z_k = y$ for all large $k$. In the first case $\rho(A)$ will contain no non-zero compact operators. In the second case, $\rho(A)$ contains finite rank operators and $C^*(\rho(A))$ contains all compact operators.

For the first assertion, for any $a \in T_{n_k}$, $\|a\| = \|\rho_k(a)\| = \|\rho_k(a)\|_{\text{ess}}$. It follows from the density of $\bigcup \rho_k(T_{n_k})$ that $\|\rho(a)\| = \|\rho(a)\|_{\text{ess}}$ for all $a \in A$. Thus, $\rho(A)$ contains no non-zero compact operators. As for the second assertion, choose $k$ so that $z_k$ is finite. Let $v$ be the matrix unit in $T_{n_k}$ which has a 1 in the extreme upper right hand corner and zeros elsewhere. Then $\rho_k(v)$ has finite rank and lies in $\rho(A)$. Since $\rho(A)$ is weakly dense in $\mathcal{Alg}(\mathcal{N})$, it has no non-trivial reducing subspaces. Consequently, $C^*(\rho(A))$ is irreducible and contains a non-zero compact operator, which implies that it contains all compact operators.

**IV. Some Examples**

**A.** Fix an integer $i$ between 1 and $n$ and consider the system:

$$
T_n \xrightarrow{\phi_1} T_{n+k_1} \xrightarrow{\phi_2} T_{n+k_1+k_2} \xrightarrow{\phi_3} \ldots \to A_i.
$$

Each embedding $\phi_n$ is given by

$$
a \mapsto a \oplus a_{ii} I_{k_n},
$$
where $I_k$ is the $k \times k$ identity matrix. For example,

\[
\begin{pmatrix}
a & a_{ii} \\
a & a_{ii}
\end{pmatrix} \mapsto \begin{pmatrix}
a & a_{ii} \\
a_{ii} & a_{ii} \\
\end{pmatrix} \mapsto \begin{pmatrix}
a & a_{ii} \\
a_{ii} & a_{ii} \\
\end{pmatrix}.
\]

The representation $\rho$ of the limit algebra $A_i$ will act on a Hilbert space $H$ with orthonormal basis $\{e_j\}$ indexed by $\mathbb{N} \cup \{0\}$. Let $p_k$ denote the projection on the linear span of $\{e_0, \ldots, e_k\}$ and $N$ the nest consisting of the $p_k$'s. Also, let $\mathcal{K}_N$ be the algebra of compact operators which leave $N$ invariant.

It is easy to check that for each $j$, the image of any matrix under $\rho_j$ is the sum of a finite rank matrix and a scalar multiple of the identity $I$. Consequently, $\rho(A_i) \subseteq \mathcal{K}_N + CI$. The containment will, in fact, be proper.

For each $k$, let $f_k = p_k - p_{k-1}$. Let

\[
B_i = \{ s \in \mathcal{K}_N + CI : \lim_{k \to \infty} f_k s f_k = f_i s f_i \}.
\]

The image of each $\rho_k$ is clearly contained in $B_i$, whence $\rho(A_i) \subseteq B_i$. To see the reverse containment, let $b \in B_i$. There is a compact operator $c$ in $\mathcal{K}_N$ and a scalar $\alpha$ such that $b = c + \alpha I$. Observe that

\[
b_{ii} = f_i b f_i = \lim_{k \to \infty} f_k b f_k = \lim_{k \to \infty} (c_{kk} + \alpha) = \alpha.
\]

(Strictly speaking, this is abuse of notation, since, for example, $f_i b f_i$ is not a scalar but an infinite matrix with a single non-zero entry, viz., $b_{ii}$ in the $i$th position on the diagonal. In cases like this we identify the two and the meaning should be clear.)

Now, fix an integer $k > i$ such that $b_{kk}$ is close to $\alpha$ and $p_k c p_k$ is close to $c$. Then $a = p_k b p_k + \alpha p_k^\perp$ is in $\rho(A_i)$ and is close to $b$. Thus, $\rho(A_i)$ is dense in, and hence equal to $B_i$.

**Fact.** The family of algebras $A_i$ are pairwise non-isomorphic.

Indeed, the identity and the $p_k$ are the only projections in $\mathcal{K}_N + CI$ (or in $\mathcal{B}(H)$, for that matter) which are invariant under $\rho(A_i)$. But $p_k \in \rho(A_i)$ if, and only if, $k < i$. Thus, keeping in mind that the indexing starts with 0, there are exactly $i + 1$ invariant projections in $A_i$. This establishes the claim. Note also that the $C^*$-envelope for each $A_i$ is $\mathcal{K} + CI$. The “linking condition” that the $i$th diagonal element is equal to the limit of the diagonal elements disappears in the passage to the generated $C^*$-algebra. This is an example in which the diagonal of $\rho(A)$ is not a masa in $C^*(\rho(A))$. 

B. This time consider the system

\[ T_n \xrightarrow{\phi_1} T_{n+k_1} \xrightarrow{\phi_2} T_{n+k_1+k_2} \xrightarrow{\phi_3} \ldots \rightarrow A \]

with the embeddings \( \phi_j \) defined by

\[ \phi_j(a) = a \oplus a_{pp}I_{k_j}, \]

where \( p = n+k_1+\cdots+k_{j-1} \), i.e., \( a_{pp} \) is the last entry on the diagonal of the matrix \( a \).

The Hilbert space \( H \) and the nest \( N \) are as before. This time, however, \( \rho(A) = \mathcal{K}_N + \mathbb{C}I \); the arguments are much the same as in the previous example. The limit algebra \( A \) now has infinitely many invariant projections, and so is not isomorphic to any of the \( A_i \) from the previous example. Also, the \( \mathcal{C}^* \)-envelope of \( A \) is evidently once again \( \mathcal{K} + \mathbb{C}I \).

C. For this example, it is most convenient if the indices for each matrix algebra \( T_m \) are drawn from the set \( \{ -m+1, \ldots, -1, 0 \} \). The basis for \( H \) is indexed by the non-positive integers and the \( p_k \) will be the obvious infinite rank projections. Fix an integer \( i \) such that \( -n+1 \leq i \leq 0 \). The embeddings in the system

\[ T_n \xrightarrow{\phi_1} T_{n+k_1} \xrightarrow{\phi_2} T_{n+k_1+k_2} \xrightarrow{\phi_3} \ldots \rightarrow B_i \]

are given by

\[ a \xrightarrow{\phi_j} a_{ii}I_{k_j} \oplus a. \]

Arguments analogous to the one in example A show that

\[ \rho(B_i) = \{ s \in \mathcal{K}_N + \mathbb{C}I : \lim_{k \to -\infty} f_k s f_k = f_i s f_i \}. \]

Once again, \( B_i \cong B_j \) if, and only if \( i = j \), and the \( \mathcal{C}^* \)-envelope of each \( B_i \) is \( \mathcal{K} + \mathbb{C}I \). This is another example in which the diagonal of \( \rho(A) \) is not a masa in the \( \mathcal{C}^* \)-envelope.

The limit algebras in examples A and C are all non-isomorphic. All that needs to be checked is that \( B_i \not\cong A_i \) for each \( i \), since when \( i \neq j \), \( B_i \) and \( A_j \) have a different number of invariant projections. Observe that if \( p \) is a projection in \( \rho(A_i) \cap \rho(A_i)^* \) which is invariant under \( \rho(A_i) \) and is not the identity, then for any sequence \( x_j \) in \( A_i \), the set \( \{ x_j p \} \) is linearly dependent. On the other hand, if \( p \neq 0 \) is an invariant projection in \( \rho(B_i) \cap \rho(B_i)^* \), then there exist operators \( x_1, x_2, \ldots \in \rho(B_i) \) such that the set \( \{ x_j p \} \) is linearly independent.

D. In this example, \( H \) is a Hilbert space with basis indexed by \( \mathbb{N} \cup \{ 0 \} \), \( N = \{ p_k \} \) and \( \mathcal{K}_N \) are as before, and \( D \) denotes the algebra of diagonal matrices. Also, \( d \) will denote the conditional expectation from \( T_n \) onto \( D_n \); in other words, if \( a \) is an upper triangular matrix, \( d(a) \) is the diagonal part of \( a \). Note that \( d \) is a direct sum of rank one compressions.
Now consider the stationary system

\[ T_2 \xrightarrow{\phi_1} T_4 \xrightarrow{\phi_2} T_8 \xrightarrow{\phi_3} \ldots \rightarrow A \]

in which \( \phi_k : T_{2^k} \rightarrow T_{2^{k+1}} \) is given by \( \phi_k(a) = a \oplus d(a) \).

The representations \( \rho_k : T_{2^k} \rightarrow \mathcal{K}_N + \mathcal{D} \) are given by \( \rho_k(a) = a \oplus d(a) \oplus d(a) \oplus \cdots = a \oplus \sum \oplus d(a) \). The image \( \rho(A) \) of the limit algebra under the canonical representation is thus a subalgebra of \( \mathcal{K}_N + \mathcal{D} \). Once again, the image will be a proper subalgebra.

**Definition.** We say that an element \( b \in \mathcal{D} \) is periodic with period \( p \) if \( b_{m+p} = b_m \), for all \( m \). Here, \( b_m \) denotes the matrix entry \( b_{mm} \). We say \( b \) is dyadic periodic if the period \( p \) is a power of 2. Let \( \mathcal{D}_{AP}(2^\infty) \) denote the closure in the norm topology of the dyadic periodic elements of \( \mathcal{D} \).

Let \( \mathcal{K}_N^0 = \{ a \in \mathcal{K}_N : d(a) = 0 \} \). Clearly, \( \rho_k(T_{2^k}) \subseteq \mathcal{K}_N^0 + \mathcal{D}_{AP}(2^\infty) \), for each \( k \). Thus, \( \rho(A) \subseteq \mathcal{K}_N^0 + \mathcal{D}_{AP}(2^\infty) \). Furthermore, an argument analogous to the one used in example A shows that we have equality: \( \rho(A) = \mathcal{K}_N^0 + \mathcal{D}_{AP}(2^\infty) \). The C*-envelope of \( A \) is therefore \( \mathcal{K} + \mathcal{D}_{AP}(2^\infty) \).

**E.** The notation is the same as in the previous example, but the embedding is changed to

\[ a \mapsto a \oplus dlh(a) \oplus dlh(a) \]

where \( dlh(a) \) is the diagonal last half of \( a \). (When \( a \) is a \( 2^k \times 2^k \) matrix, \( dlh(a) \) is a \( 2^{k-1} \times 2^{k-1} \) matrix.)

If \( a \) is a matrix in \( T_{2^k} \), then the first \( 2^{k-1} \) diagonal terms in \( \rho_k(a) \) bear no relation to the remaining diagonal terms in \( \rho_k(a) \). This difference in behaviour compared with the last example results in a slightly different image for the limit algebra under the canonical representation: \( \rho(A) = \mathcal{K}_N + \mathcal{D}_{AP}(2^\infty) \). The C*-envelope is unchanged.

**F.** Consider the stationary system

\[ T_2 \xrightarrow{\phi_1} T_4 \xrightarrow{\phi_2} T_8 \xrightarrow{\phi_3} \ldots \rightarrow A \]

in which \( \phi_k : T_{2^k} \rightarrow T_{2^{k+1}} \) is given by \( \phi_k(a) = a \oplus lh(a) \oplus lh(a) \). Here \( lh(a) \) denotes compression to the last half of \( a \), resulting in a \( 2^{k-1} \times 2^{k-1} \) matrix when \( a \) is \( 2^k \times 2^k \). Let \( \mathcal{S}(2^\infty) \) denote the norm closure of all the dyadic periodic matrices. (A matrix on \( \mathcal{H} \) is periodic with period \( p \) if it has the form \( a \oplus a \oplus \cdots \), where \( a \) is a \( p \times p \) matrix.) The intersection of \( \mathcal{S}(2^\infty) \) and \( \mathcal{Alg}(N) \) will be denoted by \( \mathcal{S}_N(2^\infty) \). With this notation, the image of the limit algebra \( A \) under the canonical representation \( \rho \) is \( \mathcal{K}_N + \mathcal{S}_N(2^\infty) \) and the C*-envelope is \( \mathcal{K} + \mathcal{S}(2^\infty) \).
G. We conclude by mentioning an old example, the system

\[ T_2 \xrightarrow{\phi_1} T_4 \xrightarrow{\phi_2} T_8 \xrightarrow{\phi_3} \ldots \rightarrow A \]

with the standard embeddings of multiplicity 2: \( \phi_k(a) = a \oplus a \). The canonical representation is the Smith representation described in [OP]; \( \rho(A) = S_N(2^\infty) \) and the \( C^* \)-envelope is the UHF algebra \( S(2^\infty) \).

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