FURTHER RIGID TRIPLES OF CLASSES IN $G_2$

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Abstract. We establish the existence of two rigid triples of conjugacy classes in the algebraic group $G_2$ in characteristic 5, extending results of the second author with Liebeck and Marion. As a corollary, the finite groups $G_2(5^n)$ are not $(2, 4, 5)$-generated; this reproves a case of a conjecture of Marion.

1. Introduction

Let $G$ be a connected simple algebraic group over an algebraically closed field $K$, and let $C_1, \ldots, C_s$ be conjugacy classes of $G$. Following [16], we say the $s$-tuple $C = (C_1, \ldots, C_s)$ is rigid in $G$ if the set

$$C_0 \overset{\text{def}}{=} \{(x_1, \ldots, x_s) \in C_1 \times \ldots \times C_s : x_1x_2 \ldots x_s = 1\}$$

is non-empty and forms a single orbit under the action of $G$ by simultaneous conjugation.

Some well-known examples of rigid tuples of classes in simple algebraic groups are the Belyi triples and Thompson tuples, defined in [18]. Other rigid triples are known, see for instance [2, 3, 5, 9, 17]. Rigid tuples of classes are interesting in the context of the inverse Galois problem [12], and also arise naturally in the theory of ordinary differential equations [7].

Recall that a group is $(a, b, c)$-generated if it is generated by elements $x, y$ and $z$, of respective orders $a, b$ and $c$, such that $xyz = 1$. The group is then called an $(a, b, c)$-group, and the triple $(x, y, z)$ is called an $(a, b, c)$-triple of the group. The theory of $(a, b, c)$-generation of finite groups has close connections to rigidity, for instance it is a basic observation that given a rigid tuple $C$ of classes of $G$, all subgroups $\langle x_1, \ldots, x_s \rangle$ for $(x_1, \ldots, x_s) \in C_0$ are conjugate in $G$, so that there is at most one $r > 0$ such that the finite subgroup $G(5^r)$ is generated by elements in such an $s$-tuple.

Let $K = \bar{\mathbb{F}}_5$ be the algebraic closure of the field of five elements. In [8] it is shown that the simple algebraic group $G = G_2(K)$ has a rigid triple of conjugacy classes of elements of orders 2, 5 and 5, and any triple of elements $(x_1, x_2, x_3)$ in the corresponding set $C_0$ generates a copy of $A_5$. This is then used to show that none of the groups $G_2(5^n)$, $SL_3(5^n)$ or $SU_3(5^n)$ is a $(2, 5, 5)$-group.

Here we produce two further rigid triples of classes in $G = G_2(K)$, closely related to the triple above. Recall from [1] that $G$ has a unique class of involutions, with representative $t$, say, and $C_G(t) = A_1 \bar{A}_1$ is a central product of two subgroups $SL_2(K)$, where $A_1$ (resp. $\bar{A}_1$) is generated by a long (resp. short) root subgroup of $G$. There also exist two classes of elements of order 4, with representatives $s_1$ and $s_2$, such that $C_G(s_1) = A_1 T'$ and $C_G(s_2) = \bar{A}_1 T''$, where $T'$ and $T''$ are 1-dimensional tori. Finally, recall from [8] that $G$ has three classes of unipotent elements of order 5: the long and short root elements, and the class labelled $G_2(a_1)$,
with representative \( u = x_\beta(1)x_{3\alpha+\beta}(1) \), where \( \alpha \) (resp. \( \beta \)) is the short (resp. long) simple root of \( G \). From [11] Table 22.1.5, the centraliser \( C_G(u) = U_4.\text{Sym}_3 \), where \( U_4 \) is a 4-dimensional connected unipotent group.

**Theorem 1.**

(i) The triples of classes \( C = (t^G, s^G_t, u^G) \) and \( D = (t^G, s^G_2, u^G) \) are rigid in \( G = G_2(K) \).

(ii) Every triple of elements in \( C_0 \) or \( D_0 \) generates a subgroup isomorphic to the symmetric group \( \text{Sym}_5 \).

(iii) None of the groups \( G_2(5^n) \) are a \((2, 4, 5)\)-group for any \( n \). Neither are the groups \( \text{SL}_3(5^n) \) or \( \text{SU}_3(5^n) \).

**Remarks.**

(1) Each subgroup \( \text{Sym}_5 \) in part (ii) here contains a subgroup \( \text{Alt}_5 \) arising from [9] Theorem 1(ii)].

(2) Keeping track of details in the proof in [9] shows that \( G_2(K) \) has a unique class of subgroups \( \text{Alt}_5 \). These subgroups have centraliser \( \text{Sym}_3 \), and by Lang’s theorem these split into three classes in \( G_2(5^r) \), with centraliser orders 6, 3 and 2. Similarly, if \( S \) and \( S' \) are representatives of the two subgroup classes in part (ii) here, then \( C_G(S) \cong \text{Sym}_5 \), while \( C_G(S') \) is cyclic of order 2. It follows that the class of \( S \) (resp. \( S' \)) splits into 3 (resp. 2) classes of subgroups in \( G_2(5^r) \), with centralisers of order 6, 3, 2 (resp. 2 and 2).

(3) A conjecture of Marion [14] states that, for a simple algebraic group \( G \) in characteristic \( p \), if \( \delta_i \) denotes the dimension of the variety of elements of \( G \) of order \( i \) and if \( \delta_a + \delta_b + \delta_c = 2 \dim(G) \), then at most finitely many of the finite groups \( G(p^r) \) are \((a, b, c)\)-groups. For \( G = G_2 \), this criterion holds precisely when \((a, b, c) = (2, 4, 5) \) or \((2, 5, 5) \). Hence part (iii), together with [9] Theorem 1(iii)], verifies the conjecture for \( G = G_2 \) in characteristic 5. An alternative proof of this fact is given in [6] Proposition 3.7(i)], where it is shown that every \((2, 4, 5)\)-subgroup and \((2, 5, 5)\)-subgroup of \( G_2(K) \) is reducible on the natural 7-dimensional module, by considering the dimensions of \( \text{SL}_7(K) \)-conjugacy classes of elements in the relevant \((a, b, c)\)-triples.

2. **Proof of the Theorem**

We proceed in the manner of [9]. Let \( G = G_2(K) \) and \( t, u, s_1, s_2 \in G \) as above. If \( \sigma \) is a Frobenius morphism of \( G \) induced from the field map \( x \mapsto x^5 \) of \( K \), then

\[
G = \bigcup_{n=1}^{\infty} G_\sigma^n = \bigcup_{n=1}^{\infty} G_2(5^n).
\]

The element \( u = x_\beta(1)x_{3\alpha+\beta}(1) \) is a regular unipotent element in a subgroup \( A_2 = \text{SL}_3(K) \) of \( G \) generated by long root groups, and therefore lies in a subgroup \( \Omega_3(5) \cong \text{Alt}_5 \) of \( G \), which we denote by \( A \). Now, let \( S = N_{A_2}(A) = \text{SO}_4(5) \cong \text{Sym}_5 \). Following the proof given in [9] we find that \( N_G(A) = S \times C_G(A) \) and \( C_G(A) = \langle z, \tau \rangle \cong \text{Sym}_3 \), where \( \langle z \rangle \) is the centre of \( A_2 \) and \( \tau \) is an outer involution in \( N_G(A_2) = A_2.2 \). Note that \( C_{A_2}(\tau) = SO_3(K) \), so \( \tau \in C_G(S) \).

Let \( v \) be an involution in \( S \setminus A \), so that \( S = \langle A, v \rangle \), and define \( S' = \langle A, v\tau \rangle \), so that \( S' \cong \text{Sym}_5 \) also. Then \( C_G(S), C_G(S') \leq C_G(A) = \langle z, \tau \rangle \) and therefore

(1) \( C_G(S) = \langle z, \tau \rangle \),

(2) \( C_G(S') = \langle \tau \rangle \).

In particular \( S \) and \( S' \) are not conjugate in \( G \).
Next consider the set of $(2, 4, 5)$-triples of $\text{Sym}_5$. It is straightforward to show that there are exactly 120 such triples, and that $\text{Sym}_5$ acts transitively on these by simultaneous conjugation.

Now let $C = \langle C^G, s^G_1, u^G \rangle$ and $D = \langle C \cap G_2(q)^3, D(q) = D \cap G_2(q)^3 \rangle$. We now show that $|C(q)| = |D(q)| = |G_2(q)|$. For this we require the character table of $G_2(q)$, given in [1] and available in the CHEVIE [4] computational package. Since $C_G(u)/C_G(u)^5 = S_3$, an application of Lang’s theorem [13, Theorem 21.11] shows that $u^G \cap G_2(q)$ splits into three classes, with representatives denoted in [11] by $u_3, u_4$ and $u_5$, and respective centraliser orders $6q^4, 3q^4$ and $2q^4$. For $x, y, z \in G_2(q)$ let $a_{xyz}$ be the corresponding class algebra constant. Calculations with the character table show that

$$a_{tsu_{ij}} = \begin{cases} q^4 & \text{if } i = 1, j \in \{3, 4, 5\} \text{ or } i = 2, j = 4, \\ 3q^4 & \text{if } i = 2, j = 3, \\ 0 & \text{if } i = 2, j = 5. \end{cases}$$

and it follows that

$$|C(q)| = \sum_{j=4}^5 |u_j^{G_2(q)}| a_{tsu_{ij}} = |G_2(q)| \left( \frac{q^4}{6q^4} + \frac{q^4}{3q^4} + \frac{q^4}{2q^4} \right) = |G_2(q)|,$$

$$|D(q)| = \sum_{j=4}^5 |u_j^{G_2(q)}| a_{tsu_{ij}} = |G_2(q)| \left( \frac{3q^4}{6q^4} + \frac{q^4}{2q^4} \right) = |G_2(q)|.$$

Now let $E$ denote (resp. $E'$) denote the set of triples $(x_1, x_2, x_3) \in C_0 \cup D_0$ which generate a conjugate of $S$ (resp. a conjugate of $S'$). Then $G$ is transitive on both $E$ and $E'$, since if $(x_1, x_2, x_3) = (y_1, y_2, y_3)^g$ are each isomorphic to $\text{Sym}_5$, then $(x^g_1, x^g_2, x^g_3)$ and $(y_1, y_2, y_3)$ are $(2, 4, 5)$-triples in a fixed copy of $\text{Sym}_5$, hence conjugate in $\text{Sym}_5$ by the observation above. Moreover both $E$ and $E'$ are non-empty, since $S$ and $S'$ each contain $(2, 4, 5)$-triples and a unique conjugacy class of unipotent elements, whose elements are conjugate to an element of $A$ and therefore are conjugate to $u$. By [1] and [2] the stabiliser of a point in $E$ is isomorphic to $\text{Sym}_3$, and the stabiliser of a point in $E'$ is cyclic of order 2. Hence applying Lang’s theorem shows that $E(q) = E \cap G_2(q)^3$ splits into three $G_2(q)$-orbits, of orders $|G_2(q)/r|$ for $r = 2, 3, 6$, and similarly $E'(q) = E' \cap G_2(q)^3$ splits into two orbits, each of order $|G_2(q)|/2$. Therefore

$$|E(q)| + |E'(q)| = |G_2(q)| \left( \frac{1}{6} + \frac{1}{3} + \frac{1}{2} + \frac{1}{2} \right) = |C(q)| + |D(q)|$$

and it follows that $C(q) \cup D(q) = E(q) \cup E'(q)$ for each $q$. Therefore

$$C \cup D = \bigcup_{n=1}^{\infty} C(5^n) \cup D(5^n) = \bigcup_{n=1}^{\infty} E(5^n) \cup E'(5^n) = E \cup E'$$

Hence $G$ has exactly two orbits on $C \cup D$. A triple in $C$ cannot lie in the same orbit as a triple in $D$ since the corresponding elements of order 4 are not $G$-conjugate, and it follows that the two $G$-orbits are $C$ and $D$.

This proves parts (i) and (ii) of the Theorem. For part (iii), suppose that $G_2(5^n)$, $SL_3(5^n)$ or $SU_3(5^n)$ is a $(2, 4, 5)$-group, with corresponding set of generators $x_1, x_2, x_3$. Since $L(G_2) \downarrow A_2$ is a direct sum of $L(A_2)$ and two 3-dimensional irreducible $A_2$-modules (cf. [10 Table 8.5]), it follows that $C_{L(G_2)}(x_1, x_2, x_3) = 0$. An application of a result of Scott [15] to the module $L(G)$, as in the proof of [16, Corollary 3.2], then yields

$$\dim(x_1^{G}) + \dim(x_2^{G}) + \dim(x_3^{G}) \geq 2 \dim(G) = 28,$$

implying $(x_1^{G}, x_2^{G}, x_3^{G}) = C$ or $D$, which contradicts part (ii) of the Theorem. \qed
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