HOCHSCHILD COHOMOLOGY OF TRIANGULAR MATRIX ALGEBRAS

Jorge A. Guccione and Juan J. Guccione

Abstract. Let $E = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ be a triangular algebra, where $A$ and $B$ are algebras over an arbitrary commutative ring $k$ and $M$ is an $(A, B)$-bimodule. We prove the existence of two long exact sequence of $k$-modules relating the Hochschild cohomology of $A$, $B$ and $E$. We also study the structure of the maps of the first of these exact sequences.

Introduction

Let $k$ be an arbitrary commutative ring with unit, $A$ and $B$ two $k$-algebras with unit, $M$ an $(A, B)$-bimodule, $E = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ the triangular algebra and $X$ an $E$-bimodule. Let $1_A$ and $1_B$ be the unit elements of $A$ and $B$ respectively. Let us write $X_{AA} = 1_A X_1 A$, $X_{AB} = 1_A X_1 B$, $X_{BA} = 1_B X_1 A$ and $X_{BB} = 1_B X_1 B$. For example, for $X = E$, we have $X_{AA} = A$, $X_{AB} = M$, $X_{BA} = 0$ and $X_{BB} = B$.

The purpose of this paper is to prove the following results:

Theorem 1. There exists a long exact sequence

\[ 0 \rightarrow H^0(E, X) \overset{j_0}{\rightarrow} H^0(A, X_{AA}) \oplus H^0(B, X_{BB}) \overset{\delta^0}{\rightarrow} \text{Ext}^0_{A \otimes B^{op}, k}(M, X_{AB}) \rightarrow \]

\[ \overset{\pi^0}{\rightarrow} H^1(E, X) \overset{j_1}{\rightarrow} H^1(A, X_{AA}) \oplus H^1(B, X_{BB}) \overset{\delta^1}{\rightarrow} \text{Ext}^1_{A \otimes B^{op}, k}(M, X_{AB}) \rightarrow \ldots, \]

where $\text{Ext}^n_{A \otimes B^{op}, k}(M, X_{AB})$ denotes the Ext groups of the $A \otimes B^{op}$-module $M$, relative to the family of the $A \otimes B^{op}$-linear epimorphisms which split as $k$-linear morphisms.

Let $\pi: E \rightarrow B$ the ring morphism defined by $\pi \left( \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \right) = b$. We let $B_E$ denote the ring $B$ consider as an $E$-bimodule via $\pi$.

Theorem 2. There exists a long exact sequence

\[ 0 \rightarrow \text{Ext}^0_{E \otimes B^{op}, k}(B_E, X_{1B}) \rightarrow H^0(E, X) \rightarrow H^0(A, X_{AA}) \rightarrow \]

\[ \rightarrow \text{Ext}^1_{E \otimes B^{op}, k}(B_E, X_{1B}) \rightarrow H^1(E, X) \rightarrow H^1(A, X_{AA}) \rightarrow \ldots, \]

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where \( \text{Ext}^E_{B^p, k}(B_E, X_{1B}) \) denotes the Ext groups of the \( E \otimes B^p \)-module \( B_E \), relative to the family of the \( E \otimes B^p \)-linear epimorphisms which split as \( k \)-linear morphisms.

Let \( C \) be a \( k \)-algebra. The complex \( \text{Hom}_{C^*}(\mathcal{M}^{*+2}, b'_r) \) is a differential graded algebra via the cup product

\[
(fg)(x_0 \otimes \cdots \otimes x_{r+s+1}) = f(x_0 \otimes \cdots \otimes x_r \otimes 1_E)g(1 \otimes x_{r+1} \otimes \cdots \otimes x_{r+s+1}),
\]

where \( f \in \text{Hom}_{C^*}(\mathcal{M}^{*+2}, C) \) and \( g \in \text{Hom}_{C^*}(\mathcal{M}^{*+2}, C) \). Hence \( H^*(E, E) \) becomes a graded associative algebra. As it is well known, \( H^*(E, E) \) is a graded commutative algebra. Similarly, for a left \( C \)-module \( Y \), the complex \( \text{Hom}_{C^*}(\mathcal{M}^{*+1} \otimes Y, b'_r) \) is a differential graded algebra via

\[
(fg)(x_0 \otimes \cdots \otimes x_{r+s} \otimes y) = f(x_0 \otimes \cdots \otimes x_r \otimes g(1 \otimes x_{r+1} \otimes \cdots \otimes x_{r+s} \otimes y),
\]

where \( f \in \text{Hom}_{C^*}(\mathcal{M}^{*+1} \otimes Y, Y) \) and \( g \in \text{Hom}_{C^*}(\mathcal{M}^{*+1} \otimes Y, Y) \). Hence \( \text{Ext}^*_A(B, B) \), \( H^*(E, E) \) and \( \text{Ext}^*_A(\mathcal{M}^p, k) \) equipped with these algebra structures.

**Theorem 3.** Assume that \( X = E \). Then

1. The map \( H^*(E, E) \xrightarrow{\delta^*} H^*(A, A) \oplus H^*(B, B) \) is a morphism of graded rings.

2. The maps \( H^*(A, A) \hookrightarrow H^*(A, A) \oplus H^*(B, B) \xrightarrow{\delta^*} \text{Ext}^*_A(\mathcal{M}^p, k) \) and \( H^*(B, B) \hookrightarrow H^*(A, A) \oplus H^*(B, B) \xrightarrow{\delta^*} \text{Ext}^*_A(\mathcal{M}^p, k) \) are morphisms of graded rings.

3. The map \( \text{Ext}^*_A(\mathcal{M}^p, k) \xrightarrow{\delta^*} H^{*+1}(E, E) \) has image in the annihilator of \( \bigoplus_{n \geq 1} H^n(E, E) \).

Theorems 1 and 2, which generalize a previous result of [H], were established in [M-P] under the assumptions that \( k \) is a field, \( A \) and \( B \) are finite dimensional \( k \)-algebras, \( M \) is a finitely generated \( (A, B) \)-bimodule and \( X = E \). As was pointed out in [M-P], this version of Theorem 1, also follows from a result of [C]. Theorem 3 was proved in [G-M-S, Section 5], under the assumptions that \( k \) is a field, \( A \) is a finite dimensional \( k \)-algebra and \( E \) is a one point extension of \( A \).

Our proofs are elementary. The main tool that we use is the existence of a simple relative projective resolution of \( E \).

Next, we enunciate the homological versions of Theorems 1 and 2. Similar methods to the ones used to prove Theorems 1 and 2 work in the homological context. We left the task of giving the proofs to the reader.

**Theorem 1.** Let \( X_{BA} = 1_B X_{1A} \). There exists a long exact sequence

\[
\cdots \rightarrow \text{Tor}^1_{A \otimes B^p, k}(M, X_{BA}) \rightarrow H_1(A, X_{AA}) \oplus H_1(B, X_{BB}) \rightarrow H_1(E, X) \rightarrow \\
\rightarrow \text{Tor}^0_{A \otimes B^p, k}(M, X_{BA}) \rightarrow H_0(A, X_{AA}) \oplus H_0(B, X_{BB}) \rightarrow H_0(E, X) \rightarrow 0,
\]

where \( \text{Tor}^*_A(B^p, k)(M, X_{BA}) \) denotes the Tor groups of the \( A \otimes B^p \)-module \( M \), relative to the family of the \( A \otimes B^p \)-linear epimorphisms which split as \( k \)-linear morphisms.
Theorem 2'. There exists a long exact sequence
\[ \ldots \to H_1(A, X_{AA}) \to H_1(E, X) \to \text{Tor}^{E \otimes B^{op}, k}_1(B_E, 1_B X) \to \]
\[ \to H_0(A, X_{AA}) \to H_0(E, X) \to \text{Tor}^{E \otimes B^{op}, k}_0(B_E, 1_B X) \to 0, \]
where \( \text{Tor}^{E \otimes B^{op}, k}_k(B_E, 1_B X) \) denotes the Tor groups of the \( E \otimes B^{op} \)-module \( B_E \), relative to the family of the \( E \otimes B^{op} \)-linear epimorphisms which split as \( k \)-linear morphisms.

Remark. When the present paper was finished we learned that Theorem 1 was also obtained in [C-M-R-S] under the additional assumptions that \( k \) is a field, \( X = E \) and \( M \) is \( A \)-projective on the left or \( B \)-projective on the right.

Proof of the results

Let \((E^{*+2}, b'_i)\) be the canonical resolution of \( E \) and let \((X_*, b'_*)\) be the \( E \)-bimodule subcomplex of \((E^{*+2}, b'_i)\), defined by
\[ X_n = A^{n+2} \oplus B^{n+2} \oplus \bigoplus_{i=0}^{n+1} A^i \otimes M \otimes B^{n+1-i}. \]
It is easy to see that \((X_*, b'_*)\) is a direct summand of \((E^{*+2}, b'_i)\) as an \( E \)-bimodule complex. Moreover, the complex
\[ E \xleftarrow{b'_0} X_0 \xleftarrow{b'_1} X_1 \xleftarrow{b'_2} X_2 \xleftarrow{b'_3} X_3 \xleftarrow{b'_4} X_4 \xleftarrow{b'_5} X_5 \xleftarrow{b'_6} X_6 \xleftarrow{b'_7} X_7 \xleftarrow{b'_8} \ldots \]
is contractible as a right \( E \)-module complex. Hence, \((X_*, b'_*)\) is a projective resolution of the \( E^* \)-module \( E \), relative to the family of the \( E^* \)-linear epimorphisms which split as \( k \)-linear morphisms.

Let \((X_0^A, b'_0)\) and \((X_0^B, b'_0)\) be the subcomplexes of \((X_*, b'_*)\), defined by \( X_0^A = A^{n+1} \otimes (A \oplus M) \) and \( X_0^B = (B \oplus M) \otimes B^{n+1} \). It is easy to see that \((X_0^A, b'_0)\) and \((X_0^B, b'_0)\) are projective resolutions of the \( E^* \)-modules \( 1_A E \) and \( 1_B E \) respectively, relative to the family of the \( E^* \)-linear epimorphisms which split as \( k \)-linear morphisms. We have the following:

Lemma 3. Let \( \mu : \frac{X}{X_{n+1} \oplus X_n} \to M \) be the map defined by \( \mu(a \otimes m \otimes b) = amb \), for \( a \in A \), \( b \in B \) and \( m \in M \). The complex
\[(*) \quad M \xleftarrow{\mu} \frac{X}{X_{1}^{A} \oplus X_{1}^{B}} \xleftarrow{\beta'_1} \frac{X}{X_{2}^{A} \oplus X_{2}^{B}} \xleftarrow{\beta'_2} \frac{X}{X_{3}^{A} \oplus X_{3}^{B}} \xleftarrow{\beta'_3} \frac{X}{X_{4}^{A} \oplus X_{4}^{B}} \xleftarrow{\beta'_4} \ldots, \]
is a relative projective resolution of \( M \) as an \( E \)-bimodule. A contracting homotopy of \((*)\) as a complex of \( k \)-modules is the the family \( \sigma_1 : M \to \frac{X}{X_{n+1} \oplus X_{n+1}} \) and \( \sigma_{n+1} : \frac{X}{X_{n+1} \oplus X_{n+1}} \to \frac{X}{X_{n+1} \oplus X_{n+1}} \) \((n \geq 1)\), defined by:
\[ \sigma_1(m) = 1_A \otimes m \otimes 1_B, \]
\[ \sigma_{n+1}(a_0 \otimes m \otimes b_{2,n+1}) = 1_A \otimes a_0 \otimes m \otimes b_{2,n+1} + (-1)^n 1_A \otimes a_0 m \otimes b_{2,n+1} \otimes 1_B, \]
\[ \sigma_{n+1}(a_{0,i} \otimes m \otimes b_{i+2,n+1}) = 1_A \otimes a_{0,i} \otimes m \otimes b_{i+2,n+1} \text{ for } i > 0, \]
where \( a_0 = a_0 \otimes \cdots \otimes a_i \) and \( b_{i+2,n+1} = b_{i+2} \otimes \cdots \otimes b_{n+1} \).

Proof. It follows by a direct computation. \( \square \)
Lemma 4. We have

\[ \text{Hom}_{A^*}(A^{n+2}, b'_n), X_{AA}) \cong \text{Hom}_{E^*}(X^A_n, b'_n), X), \]
\[ \text{Hom}_{B^*}(B^{n+2}, b'_n), X_{BB}) \cong \text{Hom}_{E^*}(X^B_n, b'_n), X). \]

Proof. Since, for every \( f \in \text{Hom}_{A^*}(A^{n+2}, X), \)
\[ f(a_0 \otimes \cdots \otimes a_{n+1}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} f(a_0 \otimes \cdots \otimes a_{n+1}) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in X_{AA}, \]
the canonical inclusion \( i_n : \text{Hom}_{A^*}(A^{n+2}, X_{AA}) \to \text{Hom}_{A^*}(A^{n+2}, X) \) is an isomorphism. Let \( \theta^A_n : \text{Hom}_{A^*}(A^{n+2}, X) \to \text{Hom}_{E^*}(X^A_n, X) \) be the map defined by
\[ \theta^A_n(f)(a_0 \otimes \cdots \otimes a_{n+1}) = f(a_0 \otimes \cdots \otimes a_{n+1}) \quad \text{for } a_i \in A, \]
\[ \theta^A_n(f)(a_0 \otimes \cdots \otimes a_n \otimes m) = f(a_0 \otimes \cdots \otimes a_n \otimes 1_A) \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \quad \text{for } a_i \in A \text{ and } m \in M, \]
and let \( \varphi^A_n : \text{Hom}_{E^*}(X^A_n, X) \to \text{Hom}_{A^*}(A^{n+2}, X) \) be the map defined by restriction. Clearly \( \varphi^A_n \circ \theta^A_n = \text{id} \). Let us see that \( \theta^A_n \circ \varphi^A_n = \text{id} \). Let \( \varphi \in \text{Hom}_{E^*}(X^A_n, X) \). It is clear that \( \theta^A_n \circ \varphi^A_n(\varphi)(a_0 \otimes \cdots \otimes a_{n+1}) = \varphi(a_0 \otimes \cdots \otimes a_{n+1}) \) for all \( a_0, \ldots, a_{n+1} \in A \). Since
\[ \varphi(a_0 \otimes \cdots \otimes a_n \otimes m) = \varphi(a_0 \otimes \cdots \otimes a_n \otimes 1_A) \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \]
\[ = \theta^A_n(\varphi^A_n(\varphi))(a_0 \otimes \cdots \otimes a_n \otimes 1_A) \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \]
\[ = \theta^A_n(\varphi^A_n(\varphi))(a_0 \otimes \cdots \otimes a_n \otimes m), \]
for all \( a_0, \ldots, a_n \in A \) and \( m \in M \), we have that \( \theta^A_n \circ \varphi^A_n = \varphi \). As the family \( \theta_n \circ i_n \) is a map of complexes, the first assertion holds. The proof of the second one is similar. □

Lemma 5. We have

\[ \text{Hom}_{A \otimes B^*}(X_{nA}^* \oplus X_{nB}^*), X_{AB}) \cong \text{Hom}_{E^*}(X_n^* \oplus X_n^*), X_{AB}). \]

Proof. Since, for every \( f \in \text{Hom}_{E^*}(X_n^* \oplus X_n^*), X), \)
\[ f(x_0 \otimes \cdots \otimes x_{n+1}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} f(x_0 \otimes \cdots \otimes x_{n+1}) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in X_{AB}, \]
the canonical inclusion \( \text{Hom}_{E^*}(X_n^* \oplus X_n^*, X_{AB}) \to \text{Hom}_{E^*}(X_n^* \oplus X_n^*, X) \) is an isomorphism. To end the proof it suffices to observe that
\[ \text{Hom}_{E^*}(X_n^* \oplus X_n^*, X_{AB}) \cong \text{Hom}_{A \otimes B^*}(X_n^* \oplus X_n^*, X_{AB}). \] □
Proof of Theorem 1. Because of Lemma 3, the short exact sequence
\[ 0 \to (X^A, b'_*) \oplus (X^B, b'_*) \to (X_*, b'_*) \to \left( \frac{X_*}{X_4^A \oplus X_4^B}, b'_* \right) \to 0, \]
gives rise to the long exact sequence
\[ 0 \to \operatorname{Ext}^0_{E^*} (E, X) \to \operatorname{Ext}^0_{E^*} (1_A E \oplus E1_B, X) \to \operatorname{Ext}^0_{E^*} (M, X) \to \]
\[ \rightarrow \operatorname{Ext}^1_{E^*} (E, X) \to \operatorname{Ext}^1_{E^*} (1_A E \oplus E1_B, X) \to \operatorname{Ext}^1_{E^*} (M, X) \to \ldots. \]
To end the proof it suffices to apply Lemmas 4 and 5. □

Lemma 6. Let \( \frac{X_0}{X_A} \to B_E \) be the map defined by \( \mu(b_0 \otimes b_1 + m \otimes b) = b_0 b_1 \), for \( b, b_0, b_1 \in B \) and \( m \in M \). The complex
\[(*) \quad B_E \xleftarrow{\mu} \frac{X_0}{X_A} \xleftarrow{\psi_1} \frac{X_1}{X_A^2} \xleftarrow{\psi_2} \frac{X_2}{X_A^3} \xleftarrow{\psi_3} \frac{X_3}{X_A^4} \xleftarrow{\psi_4} \frac{X_4}{X_A^5} \xleftarrow{\psi_5} \frac{X_5}{X_A^6} \xleftarrow{\psi_6} \ldots, \]
is a relative projective resolution of \( B_E \) as an \( E \)-bimodule. A contracting homotopy of \( (*) \) as a complex of \( k \)-modules is the family \( \sigma_0 : B_E \to \frac{X_0}{X_A} \) and \( \sigma_{n+1} : \frac{X_n}{X_A^n} \to \frac{X_{n+1}}{X_A^{n+1}} \) \((n \geq 0)\), defined by:
\[ \sigma_{n+1}(x_0 \otimes \cdots \otimes x_n) = 1_A \otimes x_0 \otimes \cdots \otimes x_n. \]
Proof. It follows by a direct computation. □

Lemma 7. We have \( \operatorname{Hom}_{E \otimes B^\vee} \left( \left( \frac{X_*, b'_*}, X1_B \right) \right) \simeq \operatorname{Hom}_{E^*} \left( \left( \frac{X_*, b'_*}, X \right) \right). \)
Proof. Since, for every \( f \in \operatorname{Hom}_{E^*} \left( \frac{X_*, b'_*}{X2_A}, X \right) \),
\[ f(x_0 \otimes \cdots \otimes x_{n+1}) = f(x_0 \otimes \cdots \otimes x_{n+1}) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in X1_B, \]
the canonical inclusion \( \operatorname{Hom}_{E^*} \left( \frac{X_*, b'_*}{X2_A}, X1_B \right) \to \operatorname{Hom}_{E^*} \left( \frac{X_*, b'_*}{X2_A}, X \right) \) is an isomorphism.
To end the proof it suffices to observe that
\[ \operatorname{Hom}_{E \otimes B^\vee} \left( \frac{X_*, b'_*}{X2_A}, X1_B \right) \simeq \operatorname{Hom}_{E^*} \left( \frac{X_*, b'_*}{X2_A}, X1_B \right). \] □

Proof of Theorem 2. Because of Lemma 6, the short exact sequence
\[ 0 \to (X^A, b'_*) \to (X_*, b'_*) \to \left( \frac{X_*}{X_A^*, b'_*} \right) \to 0, \]
gives rise to the long exact sequence
\[ 0 \to \operatorname{Ext}^0_{E^*} (B_E, X) \to \operatorname{Ext}^0_{E^*} (E, X) \to \operatorname{Ext}^0_{E^*} (1_A E, X) \to \]
\[ \rightarrow \operatorname{Ext}^1_{E^*} (B_E, X) \to \operatorname{Ext}^1_{E^*} (E, X) \to \operatorname{Ext}^1_{E^*} (1_A E, X) \to \ldots. \]
To end the proof it suffices to apply Lemmas 4 and 7. □
Lemma 8. Let \((A \otimes \text{Bop})^{n+1} \otimes M, b'_n\) be the bar resolution of \(M\) as a left \(A \otimes \text{Bop}\)-module. There is a map of resolutions \(\gamma_n: ((A \otimes \text{Bop})^{n+1} \otimes M, b'_n) \rightarrow \left(\frac{X_{n+1}}{X_{n+1} \otimes X_{n+1}}, b'_{n+1}\right)\), defined by

\[
\gamma_n((a_0 \otimes b_0) \otimes \cdots \otimes (a_n \otimes b_n) \otimes m) = \sum_{i=0}^{n} (-1)^{\binom{n-i}{2}} a_{0,i} \otimes a_{i+1,n} m \otimes b_{n,i+1} \otimes b_{i,0},
\]

where

\[
\begin{align*}
a_{0,i} &= a_0 \otimes a_1 \otimes \cdots \otimes a_i, & a_{i+1,n} &= a_{i+1}a_{i+2} \cdots a_n, \\
b_{n,i+1} &= b_n \otimes b_{n-1} \otimes \cdots \otimes b_{i+1}, & b_{i,0} &= b_i b_{i-1} \cdots b_0.
\end{align*}
\]

Proof. We must prove that \(\mu \circ \gamma_0 = b'_0\), where \(\mu\) is the map introduced in Lemma 3 and that \(\gamma_{n-1} \circ b'_n = b'_n \circ \gamma_n\) for all \(n \geq 1\). The first assertion is evident. Let us check the second one. We have

\[
\begin{align*}
\gamma_{n-1} \circ b'_n &((a_0 \otimes b_0) \otimes \cdots \otimes (a_n \otimes b_n) \otimes m) \\
 &= \gamma_{n-1} \left(\sum_{j=0}^{n-1} (-1)^j (a_0 \otimes b_0) \otimes \cdots \otimes (a_j a_{j+1} \otimes b_{j+1} b_j) \otimes \cdots \otimes (a_n \otimes b_n) \otimes m\right) \\
&+ (-1)^n \gamma_{n-1} ((a_0 \otimes b_0) \otimes \cdots \otimes (a_{n-1} \otimes b_{n-1}) \otimes a_n m b_n) \\
&= \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} (-1)^{i+j} \binom{n-i-1}{2} a_{0,j-1} \otimes a_j a_{j+1} \otimes a_{j+2,i+1} \otimes a_{i+2,n} m \otimes b_{n,i+2} \otimes b_{i+1,0} \\
&+ \sum_{j=0}^{n-1} \sum_{i=0}^{j-1} (-1)^{i+j} \binom{n-i-1}{2} a_{0,i} \otimes a_{i+1,n} m \otimes b_{n,j+2} \otimes b_{j+1,b_j} \otimes b_{j-1,i-1} \otimes b_{i,0} \\
&+ \sum_{i=0}^{n-1} (-1)^{n+i} \binom{n-i-1}{2} a_{0,i} \otimes a_{i+1,n} m b_n \otimes b_{n-1,i+1} \otimes b_{i,0} \\
&= \sum_{j=1}^{n-1} (-1)^{n+j} a_{0,j-1} \otimes a_j a_{j+1} \otimes a_{j+2,i} \otimes a_{i+1,n} m \otimes b_{n,i+1} \otimes b_{i,0} \\
&+ \sum_{j=0}^{n-2} \sum_{i=j+1}^{n-1} (-1)^{n-i+j} a_{0,i} \otimes a_{i+1,n} m \otimes b_{n,j+2} \otimes b_{j+1,b_j} \otimes b_{j-1,i} \otimes b_{i-1,0} \\
&+ \sum_{i=0}^{n-1} (-1)^{n+i} a_{0,i} \otimes a_{i+1,n} m b_n \otimes b_{n-1,i+1} \otimes b_{i,0} \\
&= b'_n \left(\sum_{i=0}^{n} (-1)^{n-i} a_{0,i} \otimes a_{i+1,n} m \otimes b_{n,i+1} \otimes b_{i,0}\right) \\
&= b'_n \circ \gamma_n((a_0 \otimes b_0) \otimes \cdots \otimes (a_n \otimes b_n) \otimes m). \quad \blacksquare
\]
Proof of Theorem 3. 1) It is easy to see that $H^*(E, E) \rightarrow H^*(A, A) \oplus H^*(B, B)$ is induced by the canonical restriction

$$ \text{Hom}_{E^*}((E^*+2, b'_s), E) \rightarrow \text{Hom}_{A^*}((A^*+2, b'_s), A) \oplus \text{Hom}_{B^*}((B^*+2, b'_s), B). $$

From this fact follows immediately that $j^*$ is a map of graded rings.

2) We prove the second assertion. The first one follows similarly. It is easy to see that $H^*(B, B) \rightarrow \text{Ext}^*_{A \otimes B^*}(M, M)$ is induced by the map of complexes

$$ \text{Hom}_{B^*}((B^*+2, b'_s), B) \xrightarrow{\gamma_1} \text{Hom}_{A \otimes B^*}(\frac{X_{s+1}}{X_{s+1} \oplus X_{s+2}}, -b'_{s+1}), M), $$

defined by

$$ \tilde{\gamma}_n(f)(a_{0,i} \otimes m \otimes b_{i+1,n+1}) = \begin{cases} (-1)^{n}f(a_{0,n} \otimes 1_A)mb_{n+1} & \text{if } i = n, \\ 0 & \text{in other cases,} \end{cases} $$

where $f \in \text{Hom}_{B^*}(B^{n+2}, B)$, $a_{0,i} = a_0 \otimes \cdots \otimes a_i$ and $b_{i+1,n+1} = b_{i+1} \otimes \cdots \otimes b_{n+1}$. Let us consider the morphism

$$ \text{Hom}_{A \otimes B^*}(\frac{X_{s+1}}{X_{s+1} \oplus X_{s+2}}, -b'_{s+1}), M) \xrightarrow{\gamma_2} \text{Hom}_{A \otimes B^*}(((A \otimes B^{op})^{s+1} \otimes M), b'_s), M) $$

induced by the map $\gamma_2$ of Lemma 8. Let $\phi^* = \tilde{\gamma}_1 \circ \tilde{\gamma}_2$. It is immediate that

$$ \phi^*(f)g(a_0 \otimes b_0) \otimes \cdots \otimes (a_n \otimes b_n) \otimes m = (-1)^{\frac{n}{2}}a_0 \otimes \cdots \otimes a_n \otimes f(b_0 \otimes b_0) \otimes \cdots \otimes b_n). $$

Hence, for $f \in \text{Hom}_{B^*}(B^r+2, B)$, $g \in \text{Hom}_{B^*}(B^{s+2}, B)$ and $x_0 \otimes \cdots \otimes x_{r+s} = (a_0 \otimes b_0) \otimes \cdots \otimes (a_{r+s} \otimes b_{r+s})$, we have

$$ \phi^*(f)g(x_0 \otimes \cdots \otimes x_{r+s} \otimes m) $$

$$ = \phi^*(f)(x_0 \otimes \cdots \otimes x_r \otimes \phi^*(g)((1_A \otimes 1_B) \otimes x_{r+1} \otimes \cdots \otimes x_{r+s} \otimes m) $$

$$ = (-1)^{\frac{n}{2}}(1_A \otimes 1_B) \otimes x_{r+1} \otimes \cdots \otimes x_{r+s} \otimes m $$

$$ = (-1)^{r+s+1}a_0 \otimes \cdots \otimes a_{r+s} \otimes mg(1_B \otimes b_{r+s} \otimes \cdots \otimes b_{r+1} \otimes 1_B) $$

$$ = (-1)^{r+s+1}a_0 \otimes \cdots \otimes a_{r+s} \otimes mf(1_B \otimes b_{r+s} \otimes \cdots \otimes b_{r+1} \otimes 1_B) $$

$$ = (-1)^{r+s+1}(1_B \otimes b_{r+s} \otimes \cdots \otimes b_0) $$

$$ = \phi^*(f)(x_0 \otimes \cdots \otimes x_{r+s} \otimes m). $$

This finished the proof, since $H^*(B, B)$ is graded commutative.

3) The complex $\text{Hom}_{E^*}(\frac{X_s, b'_s}, E)$ is a differential graded algebra with the product defined by $(fg)(x_0 \otimes \cdots \otimes x_{r+s+1}) = f(x_0 \otimes \cdots \otimes x_r \otimes 1_E)g(1_E \otimes x_{r+1} \otimes \cdots \otimes x_{r+s+1})$, for $f \in \text{Hom}_{E^*}(X_r, E)$ and $g \in \text{Hom}_{E^*}(X_s, E)$. It is immediate that this product induces the cup product in $H^*(E, E)$. Assume that $s \geq 1$ and that $f$ belongs to the image of $\text{Hom}_{E^*}(X_r, E)$. Let $g' \in \text{Hom}_{E^*}(X_s, E)$ the cocycle defined by

$$ g'(x_0 \otimes \cdots \otimes x_r) = \begin{cases} g(x_0 \otimes \cdots \otimes x_r) & \text{if } x_0 \otimes \cdots \otimes x_r \in A^* \otimes (A \oplus M) \\ 0 & \text{in other case} \end{cases} $$

It is easy to check that $gf = g'f$ and that $fg' = 0$. Since $gf$ is homologous to $(-1)^{rs}fg'$ we have that the class of $gf$ in $H^{r+s}(E, E)$ is zero. □
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Jorge Alberto Guccione, Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Pabellón 1 - Ciudad Universitaria, (1428) Buenos Aires, Argentina.

E-mail address: vander@dm.uba.ar

Juan José Guccione, Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Pabellón 1 - Ciudad Universitaria, (1428) Buenos Aires, Argentina.

E-mail address: jjgucci@dm.uba.ar