BORDERED KNOT ALGEBRAS WITH MATCHINGS

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Abstract. This paper generalizes the bordered-algebraic knot invariant introduced in an earlier paper, giving an invariant now with more algebraic structure. It also introduces signs to define these invariants with integral coefficients. We describe effective computations of the resulting invariant.

1. Introduction

The aim of the present paper is to generalize the bordered-algebraic knot invariant from [19]. This generalization gives a knot invariant with more algebraic structure than the earlier invariant. Another aim is to develop techniques that make both the new invariant and the one from [19] more readily computable. In fact, C++ code for computing knot Floer homology can be found here [13]. The constructions presented here are very similar to and build on the material from [19]. For other similar algebraic constructions, see [8, 5, 6, 10, 20, 24]. The identification between these constructions and constructions from the knot Floer homology of [16, 22] will be given in [12]. A further generalization of this algebra will be given in [14].

The basic set-up is the following. We start with a collection of $2n$ points on the real line, which we will think of as $\{1, \ldots, 2n\}$ equipped with a matching, which is a partition $M$ of $\{1, \ldots, 2n\}$ into pairs of points. To the collection of $2n$ points on the real line, equipped with a matching and an integer $k$ with $0 \leq k \leq 2n+1$, we associate an algebra. Like the algebras from [19], idempotents $I_x$ in the algebra correspond to idempotent states $x \subset \{0, \ldots, 2n\}$ with $|x| = k$. Additional generators for the algebra are elements $\{L_i\}_{i=1,\ldots,2n}$ and $\{R_i\}_{i=1,\ldots,2n}$, where

$$L_i = \sum_{\{x\} \in \mathbf{R}(x, i \in x, i-1 \notin x)} I_x \cdot L_i \cdot I(x \setminus \{i\}) \cup \{i-1\}$$

$$R_i = \sum_{\{x\} \in \mathbf{R}(x, i \in x, i-1 \notin x)} I_x \cdot R_i \cdot I(x \setminus \{i-1\}) \cup \{i\}$$

and $\{U_i\}_{i=1,\ldots,2n}$, which in certain idempotents can be expressed as $L_i \cdot R_i$ or $R_i \cdot L_i$.

The remaining generators for the algebra are supplied by the matching; for each $\{a, b\} \in M$, there is a central algebra element $C_{\{a, b\}}$ with $dC_{\{a, b\}} = U_a \cdot U_b$ and $C_{\{a, b\}}^2 = 0$. Denote this algebra $A(n, k, M)$. (For details on these constructions, see Section 2.)

PSO was supported by NSF grant number DMS-1405114.

ZSz was supported by NSF grant numbers DMS-1309152 and DMS-1606571.
We wish to construct a chain complex associated to a knot diagram $D$. As a preliminary point, observe that for a generic value of $t$, the slice in the $(x, y)$-plane with $y = t$ meets $D$ in a collection of $2n$-points; moreover, the arcs in the portion of the diagram with $y \geq t$ induce a matching on the $y = t$-points in the diagram. Thus, for generic $t$, there is an associated algebra for the $y = t$ slice of the diagram.

Now, slice up the diagram into strips; i.e. restrict the diagram to the strip in the $(x, y)$-plane with $t_i \leq y \leq t_{i+1}$, for a suitable increasing sequence $t_1, \ldots, t_m$. These pieces are called partial knot diagrams, and by using sufficiently thin slices, and assuming that the knot projection is in general position, we can assume that each partial knot diagram contains either one maximum, one minimum, or one crossing. To these basic pieces, we associate a $DA$ bimodule, whose incoming algebra is associated to the $y = t_{i+1}$-slice of the knot diagram, and whose outgoing algebra is associated to the $y = t_i$-slice.

Tensoring together $DA$ bimodules associated to partial knot diagrams, we obtain a chain complex $C(D)$ associated to an oriented knot diagram $D$ with a distinguished edge, which we think of as containing the unique global minimum. Generators of this chain complex correspond to Kauffman states for the Alexander polynomial as in [3].

The resulting complex has the following algebraic structure. Consider the bigraded ring $R = \mathbb{F}[U, V]/U \cdot V = 0$, equipped with a $\Delta$-grading with

$$\Delta(U) = \Delta(V) = -1$$

and an Alexander-grading $A$, determined by

$$A(U) = -1, \quad A(V) = +1.$$ 

An orientation on the knot gives the complex $C(D)$ the structure of a bigraded module over that ring; i.e. the complex is also equipped with two integer-valued gradings, called $\Delta$ and $A$, i.e.

$$C(D) = \bigoplus_{\delta, s} C_\delta(D, s),$$

and

$$U : C_\delta(D, s) \to C_{\delta-1}(D, s-1), \quad V : C_\delta(D, s) \to C_{\delta-1}(D, s+1).$$

$$\partial : C_\delta(D, s) \to C_{\delta-1}(D, s)$$

Recall in [19], we used a Maslov grading $M$; that is related to $\Delta$ and $A$ by the formula

$$\Delta = M - A.$$ 

While the local formulas take values in $\frac{1}{2}\mathbb{Z}$, for a knot, summing over the local contributions, the gradings of generators take values in $\mathbb{Z}$. (For example, the Alexander grading of a generator corresponding to a Kauffman state is the exponent of $t$ in the corresponding contribution to the Alexander polynomial in the state sum formula; see [3].)

Taking homology of this complex, we obtain a bigraded module $J(\vec{K}) = H(C(D))$ over $R$; i.e.

$$J(\vec{K}) = \bigoplus_{\delta, s} J_{\delta}(\vec{K}, s),$$

with

$$U : J_{\delta}(\vec{K}, s) \to J_{\delta-1}(\vec{K}, s-1), \quad V : J_{\delta}(\vec{K}, s) \to J_{\delta-1}(\vec{K}, s+1).$$

As the notation suggests, these modules are invariants of the knot:
Figure 1. Local Alexander and $\Delta$-contributions. The first and second rows illustrate the Alexander and $\Delta$-gradings of each quadrant, respectively.

Theorem 1.1. The bigraded module $\mathcal{J}(\vec{K})$ is an invariant of the oriented knot $\vec{K}$. The module itself is the homology of a chain complex whose generators correspond to Kauffman states.

Setting $U = V = 0$, and taking the homology of the resulting complex, we obtain a knot invariant whose bigraded Euler characteristic (using the Alexander and Maslov gradings) is the symmetrized Alexander polynomial. This is obvious from the local description of the bigradings from Figure 1; for more on this, see [19].

The complex $C(D)$ is constructed in Section 8, building on the work from earlier sections; Theorem 1.1 is also proved in that section, where the base algebra is taken with coefficients in $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, a simplification which we make for the rest of this introduction. The generalization of the construction with coefficients in $\mathbb{Z}$ is given in Section 12.

Given any bigraded $\mathcal{R}$-module $M$, there is a new bigraded module $S(M)$ whose underlying Abelian group is the same as that for $M$, equipped with a map

$$\sigma: M \to S(M)$$

that restricts to $\mathbb{F}$-isomorphisms $\sigma: M_{\delta,s} \to S(M)_{\delta,-s}$, and satisfies the rule

$$\sigma(U \cdot m) = V \cdot \sigma(m) \quad \text{and} \quad \sigma(V \cdot m) = U \cdot \sigma(m).$$

Our bordered knot invariants are symmetric in the following sense:

Proposition 1.2. The bigraded module $\mathcal{J}(\vec{K})$ is symmetric, in the sense that $\mathcal{J}(\vec{K}) \cong S(\mathcal{J}(\vec{K})).$

Let $C(\vec{K})$ be the bordered knot complex from [19]; which is a bigraded module over $\mathbb{F}[U]$. The relationship between the $C(\vec{K})$ from [19] and $C(\vec{K})$ is summarized as follows:

Proposition 1.3. There is a homotopy equivalence of bigraded chain complexes

$$\frac{C^-(\vec{K})}{U = 0} \cong \frac{C(\vec{K})}{U = V = 0}.$$

Moreover, $C(\vec{K})/(V = 0) \cong C^-(\vec{K})$.

See Section 9 for a proof of a more detailed version.
1.1. Numerical knot invariants. The above bigraded knot invariants naturally give rise to some numerical knot invariants. Their construction hinges on the behaviour of $\mathcal{J}(\vec{K})$ under crossing changes (see Proposition 11.1), which leads to the following structure theorem.

For the statement of the structure theorem, will consider the module $\mathcal{J}(\vec{K}) \otimes \mathbb{F}[U,U^{-1}]$, which can be thought of as obtained from $\mathcal{J}(\vec{K})$ by inverting $U$.

**Proposition 1.4.** For any oriented knot $\vec{K}$, we have an identification of $\mathcal{R}$-modules $\mathcal{J}(\vec{K}) \otimes \mathbb{F}[U,U^{-1}] \cong \mathbb{F}[U,U^{-1}]$, where on the right hand side, $\mathbb{F}[U,U^{-1}]$ is thought of as the $\mathcal{R}$ module for which $V$ acts as multiplication by 0.

It follows quickly from the above proposition (see Lemma 11.4) that we can make the following:

**Definition 1.5.** Given any oriented knot $\vec{K}$, let $\nu(\vec{K}) \in \mathbb{Z}$ be the knot invariant defined by

$$\nu(\vec{K}) = -\max\{s | U^d \cdot \mathcal{J}(\vec{K}, s) \neq 0 \forall d \geq 0\}.$$ 

The above defined integer $\nu(\vec{K})$ is clearly a knot invariant, since $\mathcal{J}(\vec{K})$ is. Consider $\mathcal{J}^U(\vec{K}) = H(C(D)/V = 0)$. (Recall that $\mathcal{J}^U(\vec{K}) = \mathcal{H}^{-}(-\vec{K})$, by Proposition 1.3.)

There is a canonical map $\mathcal{J}(\vec{K}) \to \mathcal{J}^U(\vec{K})$. Clearly, any $U$-nontorsion element in $\mathcal{J}(\vec{K})$ has $U$-non-torsion image in $\mathcal{J}^U(\vec{K})$. Moreover, for all sufficiently large $s$, $\mathcal{J}^U(\vec{K}, s)$ is zero. Thus, we have the following:

**Definition 1.6.** Given any oriented knot $\vec{K}$, let $\tau(\vec{K}) \in \mathbb{Z}$ be the knot invariant defined by

$$\tau(\vec{K}) = -\max\{s | U^d \cdot \mathcal{J}(\vec{K}, s) \neq 0 \forall d \geq 0\}.$$ 

Clearly, $\tau(\vec{K}) \leq \nu(\vec{K})$.

Our crossing change result stated in Section 11 implies the following inequalities:

**Proposition 1.7.** Let $\vec{K}_+$ and $\vec{K}_-$ be two oriented knots that differ in a single crossing, which is positive for $\vec{K}_+$ and negative for $\vec{K}_-$, then

$$0 \leq \tau(\vec{K}_+) - \tau(\vec{K}_-) \leq 1$$

$$0 \leq \nu(\vec{K}_+) - \nu(\vec{K}_-) \leq 1.$$ 

In particular, for any knot $\vec{K}$, the unknotting number $u(\vec{K})$ is bounded by:

$$|\tau(\vec{K})| \leq u(\vec{K}) \quad \text{and} \quad |\nu(\vec{K})| \leq u(\vec{K}).$$

1.2. Organization. This paper is organized as follows. In Section 2, we define the relevant algebras, and verify some of their basic formal properties. Indeed, in that section we define two algebras $\mathcal{A}$ and $\mathcal{A}'$ which are Koszul dual to one another; we prefer to work with $\mathcal{A}$ whenever possible. In Section 3 we construct the bimodules associated to a crossing (both of type $DD$ and of type $DA$, over $\mathcal{A}$), and verify that they satisfy the braid relations in Section 4. In Section 5, we define the bimodule associated to a local maximum. In Section 6, give DA bimodules associated to crossings over $\mathcal{A}'$ (instead of $\mathcal{A}$, as was constructed earlier). In Section 7 we construct the bimodule associated to a local minimum, using the
material from Section 6 to verify that this bimodule satisfies the “trident relation”. In Section 8, the pieces are assembled to define the knot invariant \( J(K) \). In that section, we also prove topological invariance of the construction. In Section 9, we identify specializations of these constructions with the invariants from [19]. In that section, we are also able to verify that the specialized knot invariants are multiplicative under connected sums. In Section 10, we verify some symmetries of our constructions. In Section 11, we investigate how the bordered invariants change under crossing changes, to verify Proposition 1.7. In Section 12, we explain how to lift the present invariant to coefficients in \( \mathbb{Z} \). Finally, in Section 13, we describe some methods for optimizing the computer computations of these invariants.

1.3. **Further remarks.** Note that there is an analogous construction of knot Floer homology with coefficients in \( \mathbb{R} \): the differential in that construction counts pseudo-holomorphic disks in a homotopy class \( \phi \), weighted by \( U^{n_u(\phi)} V^{n_z(\phi)} \). Since \( UV = 0 \), the differential is counting disks that can cross either one of the two base-points. This construction is sufficient for computing \( \hat{HF} \) for surgeries on \( K \) [17, 18], the concordance invariant \( \tau(K) \) [15], and indeed Hom’s countable collection of concordance homomorphisms [2] (defined using \( \tau \) and \( \nu \) from [18]). In forthcoming work [12], we identify the presently defined invariants with their knot-Floer homological analogues.

In view of that identification, the concordance invariants \( \tau(K) \) and \( \nu(K) \) agree \( \tau(K) \) and \( \nu(K) \) described above; and so we will be able to conclude Proposition 1.7 from corresponding properties of the holomorphically defined invariants. Nonetheless, it is instructive to have a self-contained proof of Proposition 1.7 within the present algebraic framework; compare [21], [23], and [11, Chapters 6 and 8].
Throughout most of this paper, we suppress signs, working with coefficients over \(\mathbb{Z}/2\mathbb{Z}\). See Section 12 for the generalization to coefficients in \(\mathbb{Z}\).

2. Bordered algebras

2.1. Previous bordered algebras for knot diagrams. We recall the construction of the algebras \(B_0(m, k)\) and \(B(m, k)\) from [19]. Fix integers \(k\) and \(m\) with \(0 \leq k \leq m + 1\). The algebra \(B_0(m, k)\) is an algebra over \(\mathbb{F}[U_1, \ldots, U_m]\), whose basic idempotents correspond to idempotent states, or \(I\)-states which are \(k\)-element subsets of \(\{0, \ldots, m\}\). Given an \(I\)-state \(x\), define its weight \(v^x \in \mathbb{Z}^m\) by

\[
v^x_i = \# \{x \in x | x \geq i\}.
\]

Given two \(I\)-states \(x\) and \(y\), define their minimal relative weight vector \(w^{x,y} \in (\frac{1}{2}\mathbb{Z})^m\) to be vector with components

\[
w^{x,y}_i = \frac{1}{2}|v^x_i - v^y_i|.
\]

\(B_0(m, k)\) is defined so that there is an identification of \(\mathbb{F}[U_1, \ldots, U_m]\)-modules

\[I_x \cdot B_0(m, k) \cdot I_y \cong \mathbb{F}[U_1, \ldots, U_m];\]

and denote the identification by

\[
\phi^{x,y}: \mathbb{F}[U_1, \ldots, U_m] \to I_x \cdot B_0(m, k) \cdot I_y.
\]

An element of \(B_0(m, k)\) is called pure if it is of the form \(\phi^{x,y}(U_1^{t_1} \cdots U_m^{t_m})\) for some non-negative sequence of integers \(t_1, \ldots, t_m\).

A grading by \((\frac{1}{2}\mathbb{Z})^m\) on \(I_x \cdot B_0(m, k) \cdot I_y\) is specified by its values on pure algebra elements:

\[
w(\phi^{x,y}(U_1^{t_1} \cdots U_m^{t_m})) = w^{x,y} + (t_1, \ldots, t_m).
\]

An element in \(B_0(m, k)\) is called homogeneous of degree \((w_1, \ldots, w_m)\) if it can be written as a sum of pure algebra elements, all of which have weight \((w_1, \ldots, w_m)\).

Multiplication

\[
(I_x \cdot B_0(m, k) \cdot I_y) \ast (I_y \cdot B_0(m, k) \cdot I_x) \to (I_x \cdot B_0(m, k) \cdot I_x)
\]

is the unique non-trivial, grading-preserving \(\mathbb{F}[U_1, \ldots, U_m]\)-equivariant map.

If \(x\) is an \(I\)-state with \(j - 1 \in x\) but \(j \notin x\), we can form a new \(I\)-state \(y = x \cup \{j\} \setminus \{j - 1\}\). Let \(R_j^x = \phi^{x,y}(1)\), \(L_j^y = \phi^{y,x}(1)\),

\[
R_j = \sum_{\{x | j-1 \in x, j \notin x\}} R_j^x \quad \text{and} \quad L_j = \sum_{\{y | j \in y, j-1 \notin y\}} L_j^y.
\]

The algebra \(B(m, k)\) (also denoted \(B(m, k, \emptyset)\) in [19]) is the quotient of \(B_0(m, k)\) by the relations

\[
L_{i+1} \cdot L_i = 0, \quad R_i \cdot R_{i+1} = 0;
\]

and also, if \(\{x_1, ..., x_k\} \cap \{j - 1, j\} = \emptyset\), then

\[
(I_x \cdot U_j) = 0.
\]

The grading on \(B_0(m, k)\) by \((\frac{1}{2}\mathbb{Z})^m\) descends to a grading on \(B(m, k)\). A non-zero element of \(B(m, k)\) is called pure if it is the image of a pure algebra element in \(B_0(m, k)\).
For more on the construction of $\mathcal{B}(m,k)$, see [19, Section 3].

2.2. Algebras associated to matchings. In [19], we constructed various algebras containing $\mathcal{B}(m,k)$, associated to orientations on the points $\{1,\ldots,m\}$ or, equivalently, a subset of the points $\{1,\ldots,m\}$. In the present paper, we associate instead an algebra containing $\mathcal{B}(2n,k)$, associated to matchings, as follows. Let $M$ be a matching, a partition of $\{1,\ldots,2n\}$ into $n$ two-element subsets. Fix any integer $0 \leq k \leq 2n + 1$. We will describe presently two algebras associated to matchings, $\mathcal{A}(n,k,M)$ and $\mathcal{A}'(n,k,M)$, both containing the algebra $\mathcal{B}(2n,k) = \mathcal{B}(2n,k,\emptyset)$ from above.

The algebra $\mathcal{A}(n,k,M)$ is obtained from $\mathcal{B}(2n,k)$ by including $n$ further central elements $C_{\{i,j\}}$ for each $\{i,j\} \in M$, satisfying the properties that

$$C_{\{i,j\}}^2 = 0 \quad dC_{\{i,j\}} = U_iU_j.$$

For $\mathcal{A}'(n,k,M)$, we start from $\mathcal{B}(2n,k)$, and include $2n$ further algebra elements $E_i$, one for each $i \in \{1,\ldots,2n\}$, so that $E_i^2 = 0$, $dE_i = U_i$, $E_i \cdot b = b \cdot E_i$ for all $b \in \mathcal{B}(2n,k)$; and

$$E_i \cdot E_j + E_j \cdot E_i = 0 \quad \text{if } \{i,j\} \notin M.$$

Observe that for each $\{i,j\} \in M$, there is an associated non-zero algebra element $[E_i,E_j] = E_iE_j + E_jE_i$,

which is closed and central.

There are two gradings on $\mathcal{A}(n,k,M)$: one is an Alexander multi-grading with values in $\mathbb{Q}^{2n}$. To define this, we say that an algebra element in $\mathcal{A}(n,k,M)$ is pure if it is the product of a pure algebra element $b \in \mathcal{B}(2n,k)$ with elements of the form $C_{\{i,j\}}$, with $\{i,j\} \in M$. The Alexander grading of such element is the grading of $b$ (an element of $\mathbb{Q}^{2n}$) plus, for each factor of $C_{\{i,j\}}$, the sum of the basis vectors $e_i + e_j$. A non-zero algebra element in $\mathcal{A}(n,k,M)$ is called homogeneous of degree $v \in \mathbb{Q}^{2n}$ if it can be written as a sum of pure algebra elements with multi-grading equal to $v$.

Equivalently, the Alexander grading is characterized by the following properties:

- Each idempotent is homogenous, with weight specified by the 0 vector in $\mathbb{Q}^{2n}$.
- The algebra elements $L_i$ and $R_i$ are both homogeneous, and their weight is half of the $i^{th}$ standard basis vector $e_i$ in $\mathbb{Q}^{2n}$.
- The weight of $U_i$ is $e_i$.
- The weight of $C_{i,j}$ is $e_i + e_j$.
- Weight is additive under multiplication; i.e. if $a$ and $b$ are homogeneous algebra elements whose product $a \cdot b$ is non-zero, then $a \cdot b$ is homogeneous, too; and

$$w_i(a \cdot b) = w_i(a) + w_i(b)$$

for all $i = 1,\ldots,2n$.

There is a second grading $\Delta$, specified by its values on pure algebra elements, by

$$\Delta(a) = \#(C_{i,j} \text{ that divide } a) - \sum_i w_i(a).$$
There are analogous gradings on $A'(n, k, M)$, as above. A pure algebra element is a product of a pure algebra element in $B(2n, k)$ and some word in the various $E_i$. The weight of $E_i$ is $e_i$, and

$$\Delta(a) = \#(E_j \text{ that divide } a) - \sum w_i(a),$$

where now the number of $E_j$ is counted with multiplicity. For example, if $\{1, 2\} \in M$, then $E_1 \cdot E_2 \cdot E_1 \neq 0$, and

$$\Delta(E_1 \cdot E_2 \cdot E_1) = 0 \quad \text{and} \quad \Delta(E_1 \cdot E_2 \cdot U_1) = -1.$$

2.3. Canonical $DD$-bimodules. Let

$$A = A(n, k_1, M), \quad A' = A'(n, k_2, M)$$

where $k_1 + k_2 = 2n + 1$.

Note that there is a natural one-to-one correspondence between the $I$-states for $A$ and those for $A'$: if $\mathbf{x} \subset \{0, \ldots, 2n\}$ is a $k_1$-element subset, then its complement $\mathbf{x}'$ is a $k_2$-element subset of $\{0, \ldots, 2n\}$. In this case, we say that $\mathbf{x}$ and $\mathbf{x}'$ are complementary $I$-states.

A $DD$ bimodule over $A - A'$ is specified as follows. Let $K$ be the $\mathbb{F}$-vector space whose generators $K_{\mathbf{x}}$ correspond to $I$-states for $A(n, k_1, M)$. We give $K$ the structure of a left module over $I(A) \otimes I(A')$, so that the action of $I(A) \otimes I(A')$ is specified by

$$(I_y \otimes I_w) \cdot K_{\mathbf{x}} = \begin{cases} K_{\mathbf{x}} & \text{if } \mathbf{x} = \mathbf{y} \text{ and } w \text{ is complementary to } \mathbf{x} \\ 0 & \text{otherwise.} \end{cases}$$

The algebra element

$$A = \sum_{i=1}^{2n} (L_i \otimes R_i + R_i \otimes L_i) + \sum_{i=1}^{2n} U_i \otimes E_i + \sum_{\{i,j\} \in M} C_{\{i,j\}} \otimes [E_i, E_j] \in A \otimes A'$$

specifies a map

$$\delta^1 : K \to A \otimes A' \otimes K.$$

by $\delta^1(v) = A \otimes v$ (where the tensor product is taken over $I(A) \otimes I(A')$).

Lemma 2.1. The map $\delta^1$ satisfies the type $DD$ structure relation.

Proof. This is equivalent to the statement that

$$dA + A \cdot A = 0,$$

thought of as an element of $A \otimes A'$. This is a straightforward verification; compare [19, Lemma 3.9].

The above defined type $DD$ bimodule $K$ is called the canonical type $DD$ bimodule over $A(n, k_1, M)$ and $A'(n, k_2, M)$.
2.4. Koszul duality. We consider the candidate inverse module
\[ Y_{A', A} = \text{Mor}^A(A', A') \otimes A' \, K, \, \text{Id}_A. \]

Note that the pure algebra elements give a basis for \( A' \). If \( a \in A' \) is a pure algebra element, let \( \pi \) denote the linear map \( A' \to \mathbb{Z}/2\mathbb{Z} \) which is non-trivial on \( a \) and vanishes on all other pure algebra elements.

As a vector space, \( Y \) is spanned by elements of the form \( (\pi|b) \), where \( a \in A' \) and \( b \in A \) are pure algebra elements, subject to the constraint that the left idempotent of \( \pi \) (i.e. the right idempotent of \( a \)) is complementary to the left idempotent of \( b \). Thus, the vector space \( Y \) is spanned by pairs of pure non-zero algebra elements \( a \in A' \) and \( b \in A \), with \( a = a \cdot I_x \), \( b = I_y \cdot b \), so that \( x \) and \( y \) are complementary idempotent states; the corresponding generator is denoted \( (\pi|b) \).

The differential on \( Y \) has the form
\[ \partial(\pi|b) = (\pi|db) + (d(\pi)|b) + \sum_i (L_i \cdot \pi|R_i \cdot b) + (R_i \cdot \pi|L_i \cdot b) + (E_i \cdot \pi|U_i \cdot b) + \sum_{\{i,j\} \in M} ([E_i, E_j] \cdot \pi|C_{i,j} \cdot b). \]

In fact, \( Y \) is naturally a right \( A' - A \) bimodule, with action specified by
\[ (\pi|b) \cdot (a_2 \otimes a_1) = (\xi \mapsto \pi(\xi \cdot b) \cdot (a_2 \otimes a_1) \]

**Theorem 2.2.** The canonical type \( DD \) bimodule \( A', K \) is invertible.

The above theorem is proved by reducing to a case considered in [19]. We review the necessary background, first.

Consider the algebras \( B_1 = B(2n, k, \emptyset) \) and \( B_2 = B(2n, 2n + 1 - k, \{1, \ldots, 2n\}) \). There is a type \( DD \) bimodule over \( B_1 \) and \( B_2 \), \( B_1, B_2 \), whose generators correspond to pairs of complementary idempotent states, and whose differential is specified by the algebra element
\[ \sum_{i=1}^{2n} (L_i \otimes R_i + R_i \otimes L_i) + \sum_{i=1}^{2n} U_i \otimes C_i \in B_1 \otimes B_2. \]

The candidate inverse to \( B_1, B_2 \) is given by
\[ Y_{B_1, B_2} = \text{Mor}^{B_1}(B_1, B_2) \otimes B_2 \, (B_1, B_2 \, K), \, \text{Id}_{B_1}. \]

(This is naturally a \( B_2 - B_1 \)-bimodule, with both actions on the right.) Explicitly, as a vector space, \( Y \) is spanned by elements of the form \( (\pi|b) \), where \( \pi \in B_2 \) and \( b \in B_1 \), where here \( B_2 \) is opposite bimodule to \( B \), subject the restriction that the left idempotent of \( \pi \) is complementary to the left idempotent of \( b \). Recall that \( B_2 \) is the \( B_2 \)-bimodule consisting of maps from \( B_2 \) to \( F \).

The differential on \( Y \) is given by
\[ \partial(\pi|b) = \sum_{i=1}^{2n} (L_i \cdot \pi|R_i \cdot b) + (R_i \cdot \pi|L_i \cdot b) + (C_i \cdot \pi|U_i \cdot b) + (d(\pi)|b) + (\pi|db). \]
The key step in showing that $B_1, B_2 K$ is invertible consists of verifying that the homology of $Y$ is generated by elements of the form $(1\gamma I_\chi)$, where $\chi$ and $\chi'$ are complementary idempotents. (This is proved in [19, Proposition 3.10].) This argument can be used to verify that $A, A' K$ is invertible, as well:

To relate $Y$ and $\mathcal{Y}$, we use the following maps. First, observe that there is a canonical map of differential graded algebras

$$\alpha: A' \to B_2$$

with

$$\alpha(I_\chi) = I_\chi, \quad \alpha(L_i) = L_i, \quad \alpha(R_i) = R_i, \quad \alpha(U_i) = U_i, \quad \alpha(E_i) = C_i.$$ 

This map is surjective, and so it has an injective dualization

$$\alpha: B_2 \to A'.$$

Also, there is a canonical inclusion of $B_1$ into $A$. These tensor together to give a map

$$\phi: B_2 \otimes B_1 \to A' \otimes A,$$

where the tensors are taken over the ring of idempotents over the rings $B_1, B_2, A, A'$ (all of which are canonically identified).

**Lemma 2.3.** For $\{i, j\} \in M$, let $E^{(i,j)} \subset A' \otimes A$ be the subset generated by pairs $(\pi|b)$ where $b$ is a pure algebra element of $A$ which is not divisible by $C_{(i,j)}$, and $[E_i, E_j] \cdot \pi = 0$.

(C-1) $E^{(i,j)}$ is a subcomplex of $\mathcal{Y}$.

(C-2) The above defined map $\phi$ gives an injective chain map from $Y$ to $\mathcal{Y}$, and hence realizing $Y$ as a subcomplex of $\mathcal{Y}$.

(C-3) The intersection $\bigcap_{p \in M} E^p$ is the image of $\phi$.

**Proof.** First, we note that $d[E_i, E_j] = 0$. It follows immediately that

$$[E_i, E_j] \cdot (\overline{\partial(\pi)}) = \overline{\partial([E_i, E_j] \cdot \pi)};$$

from which Condition (C-1) is an easy consequence.

Next, we verify Condition (C-2). It is easy to see that for $(\pi|b) \in B_2 \otimes B_1$,

$$\phi(\partial(\pi|b)) = \partial(\phi(\pi|b)) + \sum_{\{i,j\} \in M} ([E_i, E_j] \cdot \overline{\pi(\pi)})(C_{i,j} \cdot b).$$

It remains to check that

$$[E_i, E_j] \cdot \overline{\pi(\pi)} = 0$$

for all $\pi \in B_2$. To this end, note that $\alpha(E_i, E_j) = \alpha(E_j E_i) = C_i C_j$; and dually

$$\overline{\alpha}(C_i \cdot C_j) = \overline{E_i \cdot E_j} + \overline{E_j \cdot E_i}.$$

To verify Condition (C-3), observe that Equation (2.5) gives the containment

$$\text{Image}(\phi) \subseteq \bigcap_{p \in M} E^p.$$

For the other containment, a straightforward computation shows that $E^{(i,j)}$ is contained in the span of $(\pi|b)$, where $b$ is a pure algebra element not divisible by
Proof of Theorem 2.2. We wish to show that \( \phi \) induces an isomorphism on homology.

Fix any \( \{i, j\} \in M \) with \( i < j \), and consider the associated linear map \( h_{0}^{\{i,j\}} : \mathcal{A} \rightarrow \mathcal{A} \) characterized by its values on \( \pi \in \mathcal{A} \) dual to a pure algebra element \( a \), specified as

\[
h_{0}^{\{i,j\}}(\pi) = \begin{cases} [E_i, E_j] \cdot a & \text{if } E_i \cdot a = 0 \text{ or } E_j \cdot a = 0, \\ E_i \cdot E_j \cdot a & \text{otherwise.} \end{cases}
\]

The following identities are easily verified:

\[
(2.6) \quad [E_i, E_j] \cdot h_{0}^{\{ij\}}(\pi) = \pi
\]

\[
(2.7) \quad \pi(h_{0}^{\{i,j\}}(\pi)) = h_{0}^{\{i,j\}}(\pi) \pi
\]

\[
(2.8) \quad c \cdot h_{0}^{\{i,j\}}(\pi) = h_{0}^{\{i,j\}}(c \cdot \pi),
\]

for any pure algebra element \( c \) not divisible by \( E_i \) or \( E_j \).

Now, fix some \( \{i, j\} \in M \), and consider the operator on \( h^{\{i,j\}} : \mathcal{Y} \rightarrow \mathcal{Y} \) given by \( h^{\{i,j\}}(\pi C_{\{i,j\}} \cdot b) = (h_{0}^{\{i,j\}}(\pi)) b \). Let

\[
\Pi^{\{i,j\}} = \text{Id} + \partial \circ h^{\{i,j\}} + h^{\{i,j\}} \circ \partial.
\]

We argue that \( \Pi^{\{i,j\}} \) maps into \( \mathcal{E}^{\{i,j\}} \). First consider an element of the form \( (\pi C_{\{i,j\}} \cdot b) \). The component of \( \Pi^{\{i,j\}}(\pi C_{\{i,j\}} b) \) in \( (\pi C_{\{i,j\}} b) \) vanishes by Equation (2.6); and the other components vanish by Equations (2.7) and (2.8). Next, consider an element of the form \( (\pi b) \) where \( b \) is pure and \( C_{\{i,j\}} \) does not divide \( b \). Then,

\[
\Pi^{\{i,j\}}(\pi) = (\pi + h_{0}^{\{i,j\}}([E_i, E_j] \cdot \pi)) b.
\]

From Equation (2.6), it follows that

\[
[E_i, E_j] \left( \pi + h_{0}^{\{i,j\}}([E_i, E_j] \cdot \pi)) \right) = 0.
\]

We claim also that for all \( \{i', j'\} \in M \), the map \( \Pi^{\{i,j\}} \) maps \( \mathcal{E}^{\{i',j'\}} \) into itself. This follows from the fact that the elements \( [E_i, E_j] \) are both central and closed, and they commute with \( h_{0}^{\{i,j\}} \) (Equation (2.8)). Thus, the composition \( \Pi \) for all the \( \Pi^p \) for \( p \in M \) induces a quasi-isomorphism of \( \mathcal{A}' \otimes \mathcal{A} \) with its subcomplex \( \cap_p \mathcal{E}^p \); i.e. in view of Lemma 2.3, the inclusion map \( \phi \) is a quasi-isomorphism. \( \square \)

2.5. Symmetries. The algebras \( \mathcal{A} \) and \( \mathcal{A}' \) inherit symmetries from \( \mathcal{B} \); cf. [19, Section 3.6].

Consider the map \( \rho_n : \{0, \ldots, 2n\} \rightarrow \{0, \ldots, 2n\} \) with \( \rho_n(i) = 2n - i \); and let \( \rho'_n : \{1, \ldots, 2n\} \rightarrow \{1, \ldots, 2n\} \) be the map

\[
\rho'_n(i) = 2n + 1 - i.
\]
We will drop the subscript $n$ on $\rho$ and $\rho'$ when it is clear from the context. There is a map
\begin{equation}
(2.9)
\mathcal{R} : \mathcal{A}(n, k, M) \to \mathcal{A}(n, k, \rho'(M))
\end{equation}
characterized as follows. First,
$$\mathcal{R}(I_\chi) = I_{\rho(\chi)};$$
and if $a \in \mathcal{B}(2n, k)$ is non-zero and homogeneous with specified weights
$$(w_1(a), \ldots, w_{2n}(a)),$$
then $\mathcal{R}(a) = b$ is the non-zero element that is homogeneous with specified weights
$$w_i(b) = w_{\rho'(i)}(a).$$
This defines an isomorphism $\mathcal{R} : \mathcal{B}(2n, k) \to \mathcal{B}(2n, k)$.

We extend this to the desired map on $\mathcal{A}$ by requiring
$$\mathcal{R}(C_p \cdot a) = C_{\rho'(p)} \cdot \mathcal{R}(a).$$
Note that for all $i = 1, \ldots, 2n$
$$\mathcal{R}(L_i) = R_{\rho'(i)} \quad \mathcal{R}(R_i) = L_{\rho'(i)} \quad \mathcal{R}(U_i) = U_{\rho'(i)} \quad \mathcal{R}(C_{\{p, q\}}) = C_{\rho'(p), \rho'(q)}.$$ 
Clearly, this induced map $\mathcal{R}$ induces an isomorphism of differential graded algebras.

We can also extend $\mathcal{R}$ to an isomorphism $\mathcal{R}' : \mathcal{A}'(n, k, M) \to \mathcal{A}'(n, k, \rho'(M)).$

Another symmetry is constructed as follows. Recall that the map $o(I_\chi) = I_\chi$
extends to an isomorphism of differential graded algebras
\begin{equation}
(2.10)
o : \mathcal{B}(2n, k) \to \mathcal{B}(2n, k)^{op},
\end{equation}
with
$$o(L_i) = R_i \quad o(R_i) = L_i \quad o(U_i) = U_i$$
The map can be extended to either $\mathcal{A}(n, k, M)$ or $\mathcal{A}'(n, k, M)$; we denote these extensions also by $o$.

2.6. Relationship with knot diagrams. Recall that the algebras used here are associated to slices of a knot diagram: generic horizontal slices of the knot diagram give rise to an even number of points. The matchings correspond to pairs of points that are connected by arcs in the portion above the horizontal slice.

Moreover, if the knot diagram is oriented, the orientation can be used to choose a preferred section $\mathcal{S}$ of the matching: $\mathcal{S}$ consists of those points on the boundary where the knot is oriented to point into the upper part of the knot diagram.

In [19], we defined an Alexander grading, depending on the choice of $\mathcal{S}$, given by
$$Alex(a) = - \sum_{s \in \mathcal{S}} w_s(a) + \sum_{t \notin \mathcal{S}} w_t(a).$$

With the choice of orientation, we could now define a Maslov grading on the algebras $\mathcal{A}$ and $\mathcal{A}'$, defined by
$$\Delta = \text{Maslov} - \text{Alex}.$$
This grading generalizes the Maslov grading from [19]. For more on the relationship between this algebra and the one from [19], see Section 9.

2.7. Gradings. As in [19], the bimodules we will construct here will have a special grading set. Let \( W_1 \) be a disjoint union of finitely many intervals, equipped with a partition of its boundary \( \partial W_1 = Y_1 \cup Y_2 \) into two sets of points, where both \( |Y_1| \) and \( |Y_2| \) are even. In this case, we write \( W_1 : Y_1 \to Y_2 \).

Fix a matching \( M_1 \) on \( Y_1 \). There is an equivalence relation on points in \( Y = Y_1 \cup Y_2 \), generated by the relation \( y \sim y' \) if they are connected by an interval in \( W \), or \( y \sim y' \) if both are in \( Y_1 \), and \( \{y, y'\} \in M_1 \). It is easy to see that each equivalence class of points contains either zero or two points in \( Y_2 \); let \( M_2 \) denote the induced matching on \( Y_2 \).

**Definition 2.4.** The matching \( M_1 \) on \( Y_1 \) and the one-manifold \( W \) with \( \partial W_1 = Y_1 \cup Y_2 \) are called compatible if every point point \( y \in Y_1 \) is equivalent to some point \( y' \in Y_2 \); equivalently, if we think of \( M_1 \) as a union \( W_0 \) of abstract arcs joining the points in \( Y_1 \) that are matched in \( M_1 \), we say that \( W_1 \) is compatible if the one-manifold \( W_0 \cup W_1 \) has no closed components.

Let
\[
(2.11) \quad A_1 = A(|Y_1|/2, s + s_1, M_1) \quad \text{and} \quad A_2 = A(|Y_2|/2, s + s_2, M_2),
\]
where \( s_1 = s_1(W_1) \) is the number of components of \( W_1 \) that connect \( Y_1 \) to itself; \( s_2 = s_2(W_2) \) is the number of components of \( W_1 \) that connect \( Y_2 \) to itself, and \( 0 \leq s \leq s_0 + 1 \), where \( s_0 = s_0(W_1) \) is the number of intervals in \( W_1 \) that match some point in \( Y_1 \) to \( Y_2 \). We can think of the Alexander multi-grading on \( A_i \) as taking values in \( H^0(Y; \mathbb{Q}) \): the weights of algebra elements are functions on the points in \( Y_i \). The sum of grading groups \( H^0(Y_1; \mathbb{Q}) \oplus H^0(Y_2; \mathbb{Q}) \) act on \( H^1(W, \partial W; \mathbb{Q}) \), via the map
\[
H^0(Y_1) \oplus H^0(Y_2) \to H^1(W, \partial W)
\]
given by \( (y_1, y_2) \mapsto -d^0(y_1) + d^0(y_2) \).

**Definition 2.5.** Fix \( W, A_1 \), and \( A_2 \) as above. A type \( DA \) bimodule \( X = A_2 X_{A_1} \) is called adapted to \( W \) if the following conditions hold:

(Ad-1) \( X \) is \( \mathbb{Z} \)-graded (by \( \Delta \)) and multi-graded by \( H^1(W, \partial W) \), as described above,
(Ad-2) \( X \) is a finite dimensional \( \mathbb{F} \) vector space.

The above definition specializes readily to the case of \( D \) modules \( A_2 X : \) in this case, \( \partial W = Y_2 \) (i.e. \( W_1 = \emptyset \)). Similarly, if we let \( A'_1 = A(|Y_1|/2, |Y_1| - (s + s_1), M_1) \), a type \( DA \) bimodule \( X = A_2 A'_1 \) is called adapted to \( W \) if the above two properties hold.

**Example 2.6.** The type \( DD \) bimodule \( A_i A'_i \mathcal{K} \) is adapted to the one-manifold \( W = \{1, \ldots, 2n\} \times [0, 1] \).

Let \( A_1 \) and \( A_2 \) be as in Equation (2.11). In addition, choose a one-manifold \( W_2 \) connecting \( Y_2 \) to \( Y_3 \), and let
\[
(2.12) \quad A_3 = A(|Y_3|/2, s + s_2, M_3),
\]
where $M_3$ is the matching induced by $M_2$ and $W_2$, and $s$ is chosen so that $0 \leq s \leq s_0(W_1 \cup W_2) + 1$, where $s_0$ is the number of components of $W_1 \cup W_2$ connecting $Y_1$ to $Y_3$.

The following is an adaptation of [19, Proposition 3.19]:

**Proposition 2.7.** Choose $W_1 : Y_1 \to Y_2$, $W_2 : Y_2 \to Y_3$, $A_1$, $A_2$, and $A_3$ as above. Suppose moreover that $W_1 \cup W_2$ has no closed components, i.e. it is a disjoint union of finitely many intervals joining $Y_1$ to $Y_3$. Given any two bimodules $A_2 X^3_{A_3}$ and $A_1 X^2_{A_2}$ adapted to $W_1$ and $W_2$ respectively, we can form their tensor product $A_1 X^2_{A_2} \otimes A_2 X^3_{A_3}$ (i.e. only finitely many terms in the infinite sums in its definition are non-zero); and moreover, it is a bimodule that is adapted to $W = W_1 \cup W_2$.

**Proof.** Consider equivalence classes of points in $Y_1 \cup Y_2 \cup Y_3$, where two points are considered equivalent if they can be joined by arcs in $W$. Fix $x_1, x_1' \in X_1$, $x_2, x_2' \in X_2$, and a sequence $a_1, \ldots, a_\ell$ in $A_1$. Suppose that $(b_1 \otimes \cdots \otimes b_j) \otimes x_1'$ appears with non-zero multiplicity in $X_1 \delta_{\ell+1}^1(x_1, a_1, \ldots, a_\ell)$, and $c \otimes x_2'$ appears with non-zero multiplicity in $X_2 \delta_{\ell+1}^1(x_2, b_1, \ldots, b_j)$. Then,

$$\Delta(c) + \Delta(x_1') + \Delta(x_2') = \ell - 1 + \Delta(x_1) + \Delta(x_2) + \sum \Delta(a_i).$$

In $A_3$, the number of $C_p$ that divide any algebra element is universally bounded, so we obtain a universal bound on $|w_i(c)|$ for all $i = 1, \ldots, |Y_3|$ in terms of the input.

At points $i \in Y_2$ that are equivalent to points $Y_1$ in $W$, the grading assumptions bound $|w_i(\delta_{\ell+1}^1(x_1, a_1, \ldots, a_\ell))|$ in terms of the inputs $x_1$ and $a_1, \ldots, a_\ell$. In more detail, first suppose that some point in $Y_2$ is matched with a point $z \in Y_1$ by $W_1$. Let $\xi_i$ and $\xi'_i$ be the coefficient of $gr(x_1')$ and $gr(x_1)$ in the component of $W_1$ that contains $i \in Y_2$. By the grading hypotheses,

$$w_i(b_1 \otimes \cdots \otimes b_j) + \xi'_i = \xi_i + w_z(a_1 \otimes \cdots \otimes a_{\ell}).$$

The claimed bound on $|w_i(b_1 \otimes \cdots \otimes b_j)|$ in terms of the inputs (appearing on the right-hand-side) is immediate in this case. Next, suppose that $i, i' \in Y_2$ are matched by some component in $W_2$, let $\eta_i$ and $\eta'_i$ be the corresponding components of $gr(x_2)$ and $gr(x_2')$, then

$$w_i(b_1 \otimes \cdots \otimes b_j) + \eta_i = w_{i'}(b_1 \otimes \cdots \otimes b_j) + \eta'_i.$$  

Finally, if $i, i' \in Y_2$ are matched by some component in $W_1$, then

$$w_i(b_1 \otimes \cdots \otimes b_j) + \xi_i = w_{i'}(b_1 \otimes \cdots \otimes b_j) + \xi_{i'}.$$  

Putting together Equations (2.13), (2.14) and (2.15), and using the fact that the $\xi_i$, $\xi'_i$, and $\eta_i$ are universally bounded (since $X_1$ and $X_2$ are finite dimensional), we obtain a universal bound on $|w_i(b_1 \otimes \cdots \otimes b_j)|$ in terms of $w_z(a_1 \otimes \cdots \otimes a_{\ell})$ provided that $i \in Y_2$ is equivalent to $z \in Y_1$; and hence the claimed bound of $|w_i(\delta_{\ell+1}^1(x_1, a_1, \ldots, a_\ell))|$ in terms of the inputs.

For points $i \in Y_2$ that are equivalent to points $i' \in Y_3$, we obtain a similar bound of $|w_i(\delta_{\ell+1}^1(x_1, a_1, \ldots, a_\ell)) - w_{i'}(c)|$.

Since $W_2 \cup W_1$ has no closed components, for given input sequence $a_1, \ldots, a_{\ell}$, we have obtained an upper bound on $w_i(b_1 \otimes \cdots \otimes b_j)$ for all $i \in Y_2$, depending only
on the weights of the inputs $a_1, \ldots, a_k$. If $j$ could be arbitrarily large, we would be able to find arbitrarily large $k$ for which $\delta^k(x')$ contains some term of the form $y' \in I(A_2) \otimes X_1 \subset A_2 \otimes X_1$, so that $\Delta(y') = \Delta(x') - k$, violating the hypothesis that $X_1$ is finitely generated.

Having defined $A_2^2X_3^2 \otimes A_2^2X_3^1$, the verification that it is adapted to $W_1 \cup W_2$ is straightforward.

\begin{remark}
It is a key point in the above proof that in $A(n, k)$, the number of $C_p$ that divide any pure algebra element is universally bounded. The corresponding statement does not hold over $A'(n, k)$: if $i, j \in M$, the elements $[E_i, E_j]^m$ are non-zero for all $m \geq 0$; and correspondingly other boundedness criteria must be satisfied when forming $\otimes$ over $A'(n, k)$. (See for example Lemma 6.2.)
\end{remark}

Proposition 2.7 has the following statement for type $DD$ bimodules. Choose $W_1: Y_1 \to Y_2$, $W_2: Y_2 \to Y_3$, $A_2$ and $A_3$ as above; and let

$$A'_1 = A'([Y_1]/2, [Y_1] - (s + s_1), M_1).$$

Proposition 2.9. Fix $A_1$, $A_2$, and $A_3$ as in Equation (2.11) and (2.12). Let $W_0$ be the one-manifold with boundary consisting of intervals that connect pairs of points in $Y_1$ matched by $Y_1$. Suppose that $W_0 \cup W_1 \cup W_2$ has no closed components. Given any two bimodules $A_2^1, A_3^1X_1^1$ and $A_3^1X_2^1$ adapted to $W_1$ and $W_2$ respectively, we can form their tensor product $A_3^2X_3^2 \otimes A_3^2X_3^1$; and moreover, it is a bimodule that is adapted to $W = W_1 \cup W_2$.

**Proof.** Let $W$ be the grading manifold for $X$, with $\partial W = Y_1 \cup Y_2$. Fix $x_1, x_1' \in X_1$ and $x_2, x_2' \in X_2$. Suppose that $(b_1 \otimes b_1') \otimes \ldots (b_j \otimes b_j') \otimes x_1'$ appears with non-zero multiplicity in $X_1\delta_1(x_1')$ and $c \otimes x_2'$ appears with non-zero multiplicity in $X_2\delta_j(x_2', b_1, \ldots, b_j)$. To show that the sums defining $\otimes$ are finite amounts to establishing an upper bound on $j$.

Note first that

$$\Delta(c) + \Delta(c') + \Delta(x_1') + \Delta(x_2') = \Delta(x_1) + \Delta(x_2) - 1,$$

for $c' = b_1' \cdots b_j'$. Since $\Delta(c') \leq 0$ (in fact, this is true for any $c' \in A_1^1$), Equation (2.16) gives an upper bound on $w_i(c)$ for all $i \in Y_3$.

For all $i \in Y_2$ that are equivalent via $W_2$ to some $j \in Y_3$, the fact that $X$ finitely generated and graded by is standard gives an upper bound on $w_i(b_1 \otimes \cdots \otimes b_j)$. Moreover since $X^1$ is finitely generated and graded by $W_1$, every $i \in Y_2$ that is equivalent via $W_1$ to some $k \in Y_1$, we obtain a lower bound on $|w_i(b_1 \otimes \cdots \otimes b_j) - w_k(c')|$ for any $\{i, k\} \in M_1$. Since $W_0 \cup W_1 \cup W_2$ has no closed components, it follows that any $i \in Y_1 \cup Y_2$ is equivalent to some $k \in Y_3$, and hence the above reasoning gives bounds on $w_i((b_1 \otimes b_1') \otimes \cdots \otimes (b_j \otimes b_j'))$ for all $i \in Y_1 \cup Y_2$. The desired bound on $j$ now follows from the $\Delta$-grading on $X^1$, and the fact that it is finite dimensional, as before.

The fact that the tensor product is adapted to $W_1 \cup W_2$ follows easily. \qed
2.8. Standard type $D$ structures. A type $D$ structure $X$, specified by

$$\delta^1: X \to A(n, k, M) \otimes X,$$

is called standard if it is adapted to a one-manifold with boundary $W$, and it has the following form:

$$\delta^1_1(x) = \left( \sum_{p \in M} C_p \right) \otimes x + \epsilon^1(x),$$

where $\epsilon^1(x) \in B(2n, k) \otimes X \subset A(n, k, M) \otimes X$, for all $x \in X$.

A standard sequence $(a_1, \ldots, a_\ell)$ in $A(n, k, M)$ is a sequence of algebra elements for which each $a_i$ is either in $B(2n, k)$, or it is equal to some $C_p$ for $p \in M$.

A type $A$ module is characterized up to homotopy by its values on standard sequences. This follows from Lemma 2.12 below, stated in terms of the following generalization of type $D$ structures to bimodules.

For $i = 1, 2$, fix integers $n_i$ and $k_i$ with $0 \leq k_i \leq 2n_i + 1$; fix also matchings $M_i$ on $\{1, \ldots, 2n_i\}$. Let $B_i = B(2n_i, k_i) \subset A_i(n_i, k_i, M_i) = A_i$ for $i = 1, 2$. Let $C^i = \sum_{p \in M_i} C_p \in A_i$.

**Definition 2.10.** A type $DA$ bimodule $^A_2 A_1X_A$ is called standard if the following conditions hold:

(DA-1) The bimodule $X$ is adapted to some one-manifold $W$, in the sense of Definition 2.5.

(DA-2) The one-manifold $W$ is compatible with the matching $M_1$ used in the algebra $A_1$.

(DA-3) For any standard sequence of elements $a_1, \ldots, a_{\ell-1}$ with at least some $a_i \in B_1$, $\delta^1_1(x, a_1, \ldots, a_{\ell-1}) \in B_2 \otimes X$.

(DA-4) For any $x \in X$,

$$C^2 \otimes x + \sum_{\ell=0}^\infty \delta^1_{1+\ell}(x, C^1, \ldots, C^\ell) \in B_2 \otimes X.$$

**Lemma 2.11.** Let $A_i = A(n_i, k_i, M_i)$ for $i = 1, 2, 3$. Suppose that $^A_1 A_2X_A$ and $^A_1 A_3Y_A$ are standard type $DA$ bimodules adapted to one-manifolds $W_1$ and $W_2$, so that $W_1 \cup W_2$ contains no closed components. Then their product $X \boxtimes Y$ is standard, too.

**Proof.** The fact that $W_1 \cup W_2$ has no closed components ensures that the tensor product makes sense, according to Proposition 2.7. The fact that $X \boxtimes Y$ is standard is clear now from the definition of $\boxtimes$. \qed

**Lemma 2.12.** A standard type $DA$ bimodule is determined uniquely (up to homotopy) by its bimodule structure over the idempotent algebras, together with values on standard sequences.

**Proof.** Suppose that $^A_1 A_2X_A$ and $^A_1 A_3Y_A$ are two type $DA$ bimodule structures on the same underlying $I_A \otimes I_A$-bimodules, that have the same values on standard sequences. By Proposition 2.9, we can form the tensor products $^A_1 A_2X_A \boxtimes^A_1 A_1K$ and $^A_1 A_3Y_A \boxtimes^A_1 A_1K$. 

We claim that in this case, 

\[ A_2 X_{A_1} \boxtimes^{A_1 \cdot A'_1} K \cong A_2 Y_{A_1} \boxtimes^{A_1 \cdot A'_1} K, \]

because $K$ has the property that its outputs in the $A_1$ tensor factor are either in $B_1$ or they are algebra elements $C_p \in A_1$; and so the tensor product of $X$ (or $Y$) with $K$ is determined by the values of the $DA$ bimodule on standard sequences. By Theorem 2.2, we can conclude that $X \simeq Y$, as needed. \qed
3. BIMODULES ASSOCIATED TO CROSSINGS

Bimodules associated to crossings are defined very similarly to those from [19]. We start with the type $DD$ bimodules, and then describe the corresponding type $DA$ bimodules.

3.1. $DD$ BIMODULES. Fix $i$ with $1 \leq i \leq 2n-1$, fix a matching $M$ on $\{1, \ldots, 2n\}$, and fix an auxiliary integer $k$ with $0 \leq k \leq 2n+1$.

Let $\tau = \tau_i : \{1, \ldots, 2n\} \to \{1, \ldots, 2n\}$ be the transposition that switches $i$ and $i+1$, and let $\tau(M)$ be the induced matching that matches $\tau(\alpha)$ and $\tau(\beta)$ iff $\alpha$ and $\beta$ are matched in $M$. Let

$$(3.1) \quad A_1 = A(n,k,M) \quad \text{and} \quad A'_2 = A'(n,2n+1-k,\tau_i(M));$$

and correspondingly

$$(3.2) \quad I_1 = I(2n,k) \quad \text{and} \quad I'_2 = I(2n,2n+1-k).$$

We think of the algebra $A_1$ as being below the crossing and $A'_2$ as above the crossing.

![Figure 2. Positive crossing $DD$ bimodule generators. The four generator types are pictured to the right.](image)

As an $I_1 - I'_2$-bimodule, $\mathcal{P}_i$ is the submodule of $I_1 \otimes I'_2$ generated by elements $I_x \otimes I_y$ where $x \cap y = \emptyset$ or

$$\quad x \cap y = \{i\} \quad \text{and} \quad \{0, \ldots, 2n\} \setminus (x \cup y) = \{i-1\} \text{ or } \{i+1\}.$$

In a little more detail, generators correspond to certain pairs of idempotent states $x$ and $y$, where $|x| = k$ and $|y| = 2n+1-k$. They are further classified into four types, $N$, $S$, $W$, and $E$. For generators of type $N$ the subsets $x$ and $y$ are complementary subsets of $\{0, \ldots, 2n\}$ with $i \in x$. For generators of type $S$, $x$ and $y$ are complementary subsets of $\{0, \ldots, 2n\}$ with $i \in y$. For generators of type $W$, $i-1 \not\in x$ and $i-1 \not\in y$, and $x \cap y = \{i\}$. For generators of type $E$, $i+1 \not\in x$ and $i+1 \not\in y$, and $x \cap y = \{i\}$.

The differential has the following types of terms:

(P-1) $R_j \otimes L_j$ and $L_j \otimes R_j$ for all $j \in \{1, \ldots, 2n\} \setminus \{i,i+1\}$; these connect generators of the same type.

(P-2) $U_j \otimes E_{\tau(j)}$ for all $j = 1, \ldots, 2n$

(P-3) $C_{\{\alpha,\beta\}} \otimes [E_{\tau(\alpha)},E_{\tau(\beta)}]$, for all $\{\alpha,\beta\} \in M$; these connect generators of the same type.
(P-4) Terms in the diagram below that connect generators of different types:

![Diagram](image)

(3.3)

Note that for a generator of type $E$, the terms of Type (P-1) with $j = i + 2$ vanish; while for one of type $W$, the terms of Type (P-1) with $j = i - 1$ vanish.

The bimodules $P_i$ are graded by the set $S = \mathbb{Q}^{2n}$ as follows. Let $e_1, \ldots, e_{2n}$ be the standard basis for $\mathbb{Q}^{2n}$, and set

$$
\text{gr}(N) = \frac{e_i + e_{i+1}}{4} \quad \text{gr}(W) = \frac{e_i - e_{i+1}}{4} \quad \text{gr}(E) = \frac{-e_i + e_{i+1}}{4} \quad \text{gr}(S) = \frac{-e_i - e_{i+1}}{4}.
$$

**Proposition 3.1.** The bimodule $A_1, A_2'^2 P_i$ is a type $DD$ bimodule.

**Proof.** The proof is straightforward; compare [19, Proposition 4.1].

Taking opposite modules, we can form

$$
\overline{A_1, A_2'^2 P_i} = \overline{\mathcal{P}_i}^{A_1, A_2'^2} = \overline{\mathcal{A}_i}^{v_p, (A_2')^{v_p}} \mathcal{P}_i.
$$

Combining this with the identification $o$ of $A_1$ and $A_2'$ with their opposites, we arrive at a type $DD$ bimodule, denoted $A_1, A_2'^2 N_i$. In a little more detail, $N_i$ has generators of type $N, S, W, E$ as in the definition of $\mathcal{P}_i$. Differential are also as enumerated earlier (Types (P-1)-(P-3)), with those of Type (P-4) replaced by those specified in the following diagram:

![Diagram](image)

(3.5)

3.2. The $DA$ bimodules. In [19, Section 5], we associated a bimodule to crossing. Specifically, continuing notation from the previous section (for $i, n, k$), we constructed a bimodule $B(2n,k) \mathcal{P}_{B(2n,k)}$, corresponding to a partial knot diagram.
with a crossing in it. To assist in computations occurring later in this paper, we recall the construction presently.

As a bimodule over $I(2n,k)$-bimodule generators, $P^i$ is the submodule of $I(2n,k) \otimes_F I(2n,k)$, generated by of $I_x \otimes I_y$ where either $x = y$ or there is some $w \subset \{1, \ldots, i - 1, i + 1, \ldots, 2n\}$ with $x = w \cup \{i\}$ and $y = w \cup \{i - 1\}$ or $y = w \cup \{i + 1\}$. Thus, as in the case of the type $DD$ bimodules, there are once four types of generators, of type $N$, $S$, $W$, $E$, as pictured in Figure 3; i.e.

$$\sum_{i \in x} I_x \cdot N \cdot I_x = N, \quad \sum_{i \notin x} I_x \cdot S \cdot I_x = S,$$

$$\sum_{i \in x, i - 1 \notin x} I_x \cdot W \cdot I_{\{i-1\} \cup x \setminus \{i\}} = W, \quad \sum_{i \in x, i + 1 \notin x} I_x \cdot E \cdot I_{\{i+1\} \cup x \setminus \{i\}} = E.$$

In cases where $i = 1$, the $\delta_1^2$ and $\delta_2^2$ actions are specified by the following diagram:

(3.6)

These actions are further extended to the algebra with the following conventions. For any $X \in \{N, W, E, S\}$ and any pure algebra element $a \in B(2) = B(2, 0) \oplus B(2, 1) \oplus B(2, 2) \oplus B(2, 3)$,

(3.7) \[ \delta_1^2(X, U_1 U_2 \cdot a) = U_1 U_2 \cdot \delta_1^1(X, a). \]

and also:
• If $b \otimes Y$ appears with non-zero coefficient in $\delta_2^1(N, a)$, then $(b \cdot U_2) \otimes Y$ appears with non-zero coefficient in $\delta_2^1(N, a \cdot U_1)$ and $(b \cdot U_1) \otimes Y$ appears with non-zero coefficient in $\delta_2^1(N, a \cdot U_2)$.

• If $b \otimes Y$ appears with non-zero coefficient in $\delta_2^1(W, a)$, then $(U_2 \cdot b) \otimes Y$ appears with non-zero coefficient in $\delta_2^1(W, U_1 \cdot a)$.

• If $b \otimes Y$ appears with non-zero coefficient in $\delta_2^1(E, a)$, then $(U_1 \cdot b) \otimes Y$ appears with non-zero coefficient in $\delta_2^1(E, U_2 \cdot a)$.

Next we specify the actions $\delta_1^1$. To this end, an algebra element in $B(2)$ is called elementary if it is of the form $p \cdot e$, where $p$ is a monomial in $U_1$ and $U_2$, and

$$e \in \{ 1, L_1, R_1, L_2, R_2, L_1L_2, R_2R_1 \}.$$ 

So far, we have defined $\delta_1^1$ by specifying the $\delta_2^1$ actions of the form $\delta_2^1(X, a)$, where $X \in \{ N, S, W, E \}$ and $a$ is elementary.

We will now specify $\delta_2^1(X, a_1, a_2)$ where $a_1$ and $a_2$ are elementary. Suppose that $a_1 \otimes a_2 \neq 0$ (i.e. there is an idempotent state $x$ so that $a_1 \cdot I_x \neq 0$ and $I_x \cdot a_2 \neq 0$); and suppose moreover that $U_1 \cdot U_2$ does not divide either $a_1$ nor $a_2$. In this case, $\delta_3^1(S, a_1, a_2)$ is the sum of terms:

• $R_1U_1^n \otimes E$ if $(a_1, a_2) = (R_1, R_2U_2^n)$ and $n \geq 0$

• $L_2U_2^n \otimes W$ if $(a_1, a_2) = (L_2, L_1U_1^n)$ and $n \geq 0$

• $R_1U_1^n \otimes W$ if $(a_1, a_2) = (R_1, R_1U_2^n)$ and $n \geq 1$

• $L_2U_2^n \otimes N$ if $(a_1, a_2) = (L_2, L_1U_1^n)$ and $n \geq 0$

• $R_1U_1^n \otimes N$ if $(a_1, a_2) = (R_1, R_1U_2^n)$ and $n \geq 1$

• $L_2U_2^n \otimes N$ if $(a_1, a_2) = (L_2, L_1U_1^n)$ and $n \geq 0$

• $R_1U_1^n \otimes N$ if $(a_1, a_2) = (R_1, R_1U_2^n)$ and $n \geq 1$
These operations are generalized to the case of arbitrary \( i \), by first defining a map \( t \) from \( B(2n, k) \) to expressions in \( U_1, U_2, R_1, L_1, R_2, \) and \( L_2 \)

\[
t(3.8)
\]

\[
\begin{align*}
R_2R_1U_1^{w_1(a)} - \frac{1}{2} U_2^{w_1(a)} - \frac{3}{2} & \quad \text{if } w_i(a) \equiv w_{i+1}(a) \equiv \frac{1}{2} \pmod{Z} \\
& \quad \text{and } v_{i+1}^x < v_{i+1}^y \\
L_1L_2U_1^{w_1(a)} - \frac{1}{2} U_2^{w_1(a)} - \frac{3}{2} & \quad \text{if } w_i(a) \equiv w_{i+1}(a) \equiv \frac{1}{2} \pmod{Z} \\
& \quad \text{and } v_{i+1}^x > v_{i+1}^y \\
R_2U_1^{w_1(a)}U_2^{w_1(a)} - \frac{1}{2} & \quad \text{if } w_i(a) \in Z \text{ and } w_{i+1}(a) \equiv \frac{1}{2} \pmod{Z}, \\
& \quad \text{and } v_{i+1}^x < v_{i+1}^y \\
L_2U_1^{w_1(a)}U_2^{w_1(a)} - \frac{1}{2} & \quad \text{if } w_i(a) \in Z \text{ and } w_{i+1}(a) \equiv \frac{1}{2} \pmod{Z}, \\
& \quad \text{and } v_{i+1}^x > v_{i+1}^y \\
R_1U_1^{w_1(a)}U_2^{w_1(a)} - \frac{1}{2} & \quad \text{if } w_i(a) \equiv \frac{1}{2} \pmod{Z} \text{ and } w_{i+1}(a) \in Z, \\
& \quad \text{and } v_i^x < v_i^y \\
L_1U_1^{w_1(a)}U_2^{w_1(a)} - \frac{1}{2} & \quad \text{if } w_i(a) \equiv \frac{1}{2} \pmod{Z} \text{ and } w_{i+1}(a) \in Z, \\
& \quad \text{and } v_i^x > v_i^y \\
U_1^{w_1(a)}U_2^{w_1(a)} & \quad \text{if } w_i(a) \text{ and } w_{i+1}(a) \text{ are integers.}
\end{align*}
\]

Similarly, there is a map \( t \) from generators of \( P^i \) to the four generators of \( P \), that remembers only the type \( (N, S, W, E) \) of the generator of \( P^i \).

**Definition 3.2.** For \( X \in P^i \), an integer \( \ell \geq 1 \), and a sequence of algebra elements \( a_1, \ldots, a_{\ell-1} \) in \( B_0(m, k) \) with specified weights, so that there exists a sequence of idempotent states \( x_0, \ldots, x_\ell \) with

- \( X = I_{x_0} \cdot X \cdot I_{x_\ell} \)
- \( a_t = I_{x_t} \cdot a_t \cdot I_{x_{t+1}} \) for \( t = 1, \ldots, \ell - 1 \)
- \( x_t \) and \( x_{t+1} \) are close enough (for \( t = 0, \ldots, \ell - 1 \)),

define \( \delta_1^i(X, a_1, \ldots, a_{\ell-1}) \in B(2n, k) \otimes P^i \) as the sum of pairs \( b \otimes Y \) where \( b \in B(2n, k) \) and \( Y \) is a generator of \( P^i \), satisfying the following conditions:

- The weights of \( b \) and \( Y \) satisfy

\[
(3.9) \quad \text{gr}(X) + \tau_i^{gr}(\text{gr}(a_1) + \cdots + \text{gr}(a_{\ell-1})) = \text{gr}(b) + \text{gr}(Y),
\]

where here \( \text{gr}(a_t) \) and \( \text{gr}(b) \) denote the weight gradings on \( B_1 \) and \( B_2 \); \( \text{gr}(X) \) and \( \text{gr}(Y) \) are as in Equation (3.4), and \( \tau_i^{gr} : Q^{2n} \to Q^{2n} \) is the linear transformation which acts as \( \tau_i \) on the basis vectors for \( Q^{2n} \).

- There are generators \( X_0 \) and \( Y_0 \) with the same type (i.e. with the same label \( \{N, S, W, E\} \) ) as \( X \) and \( Y \) respectively, so that \( t(b) \otimes Y_0 \) appears with non-zero multiplicity in \( \delta_1^i(X_0, t(a_1), \ldots, t(a_{\ell-1})) \).

According to [19, Proposition 5.10], the above operations give \( P^i \) the structure of a type DA bimodule.

Let \( A_1 = A(n, k, M) \) and \( A_2 = A(n, k, \tau_i(M)) \). Our goal is to extend \( B_2P_{B_1} \) to a type DA bimodule \( A_2P_{A_1}^i \). For simplicity of notation, suppose that \( i = 1 \).
First, extend the actions coming from $B_1$, so that they commute with the actions of all $C_{\{\beta,\gamma\}}$ with $\{\beta,\gamma\} \in M$, in the sense that

\begin{align}
\delta_1^p(X, C_p \cdot a_1) &= C_{\tau(p)} \cdot \delta_1^p(X, a_1) \\
\delta_3^p(X, C_p \cdot a_1, a_2) &= \delta_3^p(X, C_p \cdot a_1, a_2) = C_{\tau(p)} \cdot \delta_3^p(X, a_1, a_2).
\end{align}

for all $p \in M$.

When $\{1, 2\} \not\in M$, we add the further terms in $\delta_2^p$ from $S$ to $\{N, W, E\}$, as follows. Choose $\alpha$ and $\beta$ so that $\{1, 2\}, \{1, 3\} \in M$, and write $C_1 = C_{\{1, \alpha\}}, C_2 = C_{\{2, \beta\}}$, $\tilde{C}_1 = C_{\{1, \beta\}}$, and $\tilde{C}_2 = C_{\{2, \alpha\}}$. The additional terms are of the form:

\begin{align}
\delta_2^1(S, C_2) &\to U_\beta R_1 \otimes W \\
\delta_2^1(S, C_1) &\to U_\alpha L_2 \otimes E \\
\delta_2^1(S, U_1 C_2) &\to U_\beta U_1 L_2 \otimes E \\
\delta_2^1(S, C_1 U_2) &\to U_\alpha R_1 U_2 \otimes W \\
\delta_2^1(S, R_1 C_2) &\to U_\beta R_1 \otimes N \\
\delta_2^1(S, L_2 C_1) &\to U_\alpha L_2 \otimes N \\
\delta_2^1(S, R_1 C_1) &\to U_\alpha R_1 U_2 \otimes N \\
\delta_2^1(S, U_1 L_2 C_2) &\to U_\beta U_1 L_2 \otimes C_2 \otimes N \\
\delta_2^1(S, U_1 L_2 C_2) &\to U_\beta U_1 L_2 \otimes C_2 \otimes N.
\end{align}

These are further extended to commute with multiplication by $U_1 U_2$, and multiplication by algebra elements with $w_1 = w_2 = 0$.

With the above definition, we have, for example:

$$\delta_1^3(S, C_2) = \tilde{C}_1 \otimes S + (U_\beta R_1) \otimes W.$$ 

For the case of general $i$, use $i$ and $i+1$ in place of 1 and 2 in the subscripts for the algebra elements above.

**Proposition 3.3.** The operations defined above give $P^i$ the structure of a type $DA$ bimodule, $A^{(n, k, \tau(M))} P^i_{A^{(n, k, M)}}$. Moreover, $P^i$ is standard in the sense of Definition 2.10.

**Proof.** There are two slightly different cases, depending on whether or not $i$ and $i+1$ are matched.

Suppose that they are matched. In [19, Proposition 5.10], we constructed a type $DA$ bimodule $B^{(2n, k)} P^i_{B^{(2n, k)}}$. Multiplication by $dC_p$ for all $p \in M$ commute with the actions on this bimodule. (This is true also for $dC^i_{\{i, i+1\}} = U_i U_{i+1}$ by the construction from [19].) It follows that the extension of $P^i$ to $A(n, k)$, extending so that all maps are $C_p$-equivariant, is a $DA$ bimodule, as well.

The case where $i$ and $i+1$ are not matched requires a little extra care, when considering multiplication by $C_{\{i, \alpha\}}$ and $C_{\{i+1, \beta\}}$. Recall from [19, Proposition 5.14] that the $DA$ bimodules for crossings are extended to an algebra containing elements $C_i$ with $dC_i = U_i$ and $dC_{i+1} = U_{i+1}$. Instead, here, we have here $dC_{\{i, \alpha\}} = U_\alpha U_i$ and $dC_{\{i+1, \beta\}} = U_\beta U_{i+1}$; and the formulas for the extension here have extra factors of $U_\alpha$ or $U_\beta$ in the output. Thus, the $DA$ bimodule relations in the present case follow from the same analysis as the $DA$ bimodule relations in that earlier case.
Next, we verify that $\mathcal{P}^i$ is standard. Property (DA-3) is clear from the definition, so we turn to Property (DA-4). Suppose that $\{i, i + 1\} \subseteq M$. Then, for all $X \in \mathcal{P}^i$ and $\{\alpha, \beta\} \in M$,

\begin{equation}
\delta^i_2(X, C_{\alpha, \beta}) = C_{\alpha, \beta} \otimes X;
\end{equation}

also, if $a_1, \ldots, a_m$ is any standard sequence with $m > 1$, so that $a_j = C_{\{\alpha, \beta\}}$ for some $j$, then

\begin{equation}
\delta^i_{m+1}(X, a_1, \ldots, a_m) = 0.
\end{equation}

Property (DA-4) follows readily.

When $\{i, i + 1\} \notin M$, Equation (3.12) is not true when $\{\alpha, \beta\}$ contains one of $i$ or $i + 1$; in those cases, Equation (3.10) gives

\[\delta^i_2(X, C_{\{i, i+1\}}) = C_{\{i, i+1\}} \otimes X + V \quad \text{and} \quad \delta^i_2(X, C_{\{i+1, \beta\}}) = C_{\{i, \beta\}} \otimes X + V',\]

for $V, V' \in B_2 \otimes \mathcal{P}^i$. It still follows that

\[\sum_{\{\alpha, \beta\} \in M} \delta^i_2(X, C_{\alpha, \beta}) = \left( \sum_{\{\alpha, \beta\} \in \tau(M)} C_{\alpha, \beta} \right) \otimes X + V'',\]

with $V'' \in B_2 \otimes \mathcal{P}^i$; which, together with Equation (3.13) verifies Property (DA-4). \qed

We have the following analogue of [19, Lemma 6.2]:

**Proposition 3.4.** Let

\[A_1 = A(n, k, M), \quad A_2 = A(n, k, \tau_i(M)), \quad A'_1 = A'(n, 2n + 1 - k, M).\]

$\mathcal{P}^i$ is dual to $\mathcal{P}_A$, in the sense that

\[A_2 \mathcal{P}^i_A \cong A_1.A'_1 \cong A_2.A'_1 \mathcal{P}_A.

**Proof.** There are two cases, according to whether or not $i$ and $i + 1$ are matched in $M$.

Consider the case where $M$ matches $i$ with $\alpha \neq i + 1$ and $i + 1$ with $\beta \neq i$. 
For notational simplicity, let $i = 1$. A straightforward computation shows that $\mathcal{P}^1 \boxtimes \mathcal{K}$ is given by

(3.14)

along with all the usual self-arrows $L_t \otimes R_t$, $R_t \otimes L_t$, $U_t \otimes E_t$ for $t \neq 1, 2$; and $C_{\{m, t\}} \otimes [E_{\tau(m)}, E_{\tau(t)}]$.

Consider the map $h^1: \mathcal{P}^1 \boxtimes \mathcal{K} \to \mathcal{P}_1$

$$h^1(X) = \begin{cases} S + (L_2 \otimes E_1) \cdot E + (R_1 \otimes E_2) \cdot W & \text{if } X = S \\ \text{otherwise.} \end{cases}$$

Let $g^1: \mathcal{P}_1 \to \mathcal{P}^1 \boxtimes \mathcal{K}$ be given by the same formula. It is easy to verify that $h^1$ and $g^1$ are homomorphisms of type $DD$ structures, $h^1 \circ g^1 = \text{Id}$, and $g^1 \circ h^1 = \text{Id}$.

When $\{i, i + 1\} \in M$, $\mathcal{P}^1 \boxtimes \mathcal{K}$ is as shown in Equation (3.14), except that in this case, the two terms involving $E_\alpha$ and $E_\beta$ are to be deleted. With that modification, the above computations hold. \hfill \Box

**Definition 3.5.** With $n$, $k$, $M$, and $\tau_i$ as before, take the opposite module of $\mathcal{A}(n, k, \tau_i(M)) \mathcal{P}^i \mathcal{A}(n, k, M)$ to get a module $\mathcal{A}(n, k, \tau_i(M)) \mathcal{P}^i \mathcal{A}(n, k, M)^{op}$, and use the identifications $\alpha: \mathcal{A}(n, k, M) \to \mathcal{A}(n, k, M)^{op}$ for both $M' = M$ and $M' = \tau_i(M)$ to define $\mathcal{A}(n, k, \tau_i(M)) \mathcal{N}^i \mathcal{A}(n, k, M)$, the bimodule associated to a negative crossing.
Since the opposite module of a standard module is clearly standard, and $P_i$ is standard, it follows that $N_i$ is standard, too. Also, the analogue of Proposition 3.4 holds for $N_i$, as well.
Theorem 4.1. Fix integers \( n, k \) with \( 0 \leq k \leq 2n + 1 \), and some \( i \) with \( 1 \leq i \leq 2n - 1 \), and \( M \) be a matching on \( \{1, \ldots, 2n\} \). Let \( A_1 = A(n, k, M) \) and \( A_2 = A(n, k, \tau_i(M)) \). Then,

\[
\begin{align*}
\tag{4.1} A_1 \mathcal{P}_A \otimes A_2 \mathcal{N}_A & \simeq A_1 \text{Id}_{A_1} \otimes A_2 \mathcal{P}_A, \\
\text{Given } j \neq i \text{ with } 1 \leq j \leq 2n - 1, \text{ let} \\
A_3 = A(n, k, \tau_j \tau_i(M)) \quad \text{and} \quad A_4 = A(n, k, \tau_j(M)).
\end{align*}
\]

If \( |i - j| > 1 \)

\[
\begin{align*}
\tag{4.2} A_3 \mathcal{P}_{A_3} \otimes A_2 \mathcal{P}_{A_1} & \simeq A_3 \mathcal{P}_{A_3} \otimes A_1 \mathcal{P}_{A_1}, \\
\text{while if } j = i + 1, \text{ let} \\
A_5 = A(n, k, \tau_i \tau_{i+1} \tau_i(M)) \quad \text{and} \quad A_6 = A(n, k, \tau_{i+1}(M));
\end{align*}
\]

then,

\[
\begin{align*}
\tag{4.3} A_3 \mathcal{P}_{A_3} \otimes A_2 \mathcal{P}_{A_2} \otimes A_1 \mathcal{P}_{A_1} & \simeq A_3 \mathcal{P}_{A_3} \otimes A_1 \mathcal{P}_{A_1}.
\end{align*}
\]

Analogous identities for the bimodules from \([19]\) were proved in \([19, \text{Theorem 6.1}]\), by direct computation. A similar direct computation can be used to verify the above theorem; we prefer instead to show that the identities from the earlier paper formally imply the identities here.

To this end, recall that in \([19]\), we constructed the Koszul dual algebra of \( \mathcal{B}(2n, k) = \mathcal{B}(2n, k, 0) \), which is an algebra \( \mathcal{B}(2n, 2n + 1 - k, \{1, \ldots, 2n\}) \). This is an algebra that contains new commuting elements \( C_1, \ldots, C_{2n} \) with \( dC_i = U_i \). (This algebra is similar to \( \mathcal{A}(n, 2n + 1 - k) \), except that in \( \mathcal{A}' \), some of the new elements \( E_i \) do not commute with one another.)

Let \( \Gamma \) be a disjoint union of \( 2n \) intervals (called “strands”), equipped with a partition of its boundary \( \partial \Gamma = Y_1 \cup Y_2 \), where \( Y_1 \cong Y_2 \cong \{1, \ldots, 2n\} \). Suppose moreover that each interval in \( \Gamma \) connects a point in \( Y_1 \) with a point in \( Y_2 \). There is an induced one-to-one correspondence \( f: \{1, \ldots, 2n\} \to \{1, \ldots, 2n\} \) between the incoming strands and the out-going ones.

Definition 4.2. Choose any \( 0 \leq k \leq 2n + 1 \). Let

\[
B_1 = \mathcal{B}(2n, 2n + 1 - k) \quad B_2 = \mathcal{B}(2n, k, \{1, \ldots, 2n\}) \quad B_2 = \mathcal{B}(2n, k, 0) \subset B_2,
\]

Fix also \( \Gamma \) as above, and let \( X = \mathcal{A}_2 \mathcal{B}_1 \) be a type \( DD \) bimodule adapted to \( \Gamma \) as in Definition 2.5. We say \( X \) is special if its differential

\[
\delta^1: X \to (B_2 \otimes B_1) \otimes X
\]

has a decomposition \( \delta^1 = \epsilon + d_0 \), where

\[
d_0: X \to (B_2 \otimes B_1) \otimes X
\]

and

\[
\epsilon(x) = \left( \sum_i C_{f(i)} \otimes U_i \right) \otimes x.
\]
Suppose $X$ is special. Consider the map 
\[(\mu_{B_2 \otimes B_1}^* \otimes \text{Id}_X) \circ (\text{Id}_{B_2 \otimes B_1} \otimes d_0) \circ d_0 + (\mu_{B_1 \otimes B_1}^* \otimes \text{Id}_X) \circ d_0 : X \to (B_2 \otimes B_1) \otimes X.\]
(Note in fact that $\mu_{B_1 \otimes B_1}^* = 0$.) The type $DD$ bimodule structure relation and this hypothesis ensures that this map is multiplication by $\sum_{i=1}^{2n} U_{f(i)} \otimes U_i$. This allows us to make the following definition:

**Definition 4.3.** Choose $B_1$ and $B_2'$ as in Definition 4.2, and let $B'_2 \otimes B'_1 X$ be a special type $DD$ bimodule. Let
\[
\mathcal{A}'_1 = \mathcal{A}(n, 2n + 1 - k, M) \quad \text{and} \quad \mathcal{A}_2 = \mathcal{A}(n, k, f \circ M),
\]
there is an associated type $DD$ bimodule $F(X, f) = \mathcal{A}_2 \cdot \mathcal{A}_1'$, which is defined by the differential
\[
\delta^1(x) = d_0(x) + \left( \sum_i U_{f(i)} \otimes E_i + \sum_i C_{(f(i), f(i))} \otimes [E_i, E_j] \right) \otimes x.
\]

It follows immediately from the above remarks that $\mathcal{A}_2 \cdot \mathcal{A}_1' F(X, f)$ is indeed a type $DD$ bimodule.

Examples of special bimodules are abundant. Let
\[
\mathcal{B} = \mathcal{B}(2n, k, 0) \quad \mathcal{B}' = \mathcal{B}(2n, 2n + 1 - k, \{1, \ldots, 2n\})
\]
In [19, Section 3.7], we defined a canonical type $DD$ bimodule $\mathcal{B}' \cdot \mathcal{B}_K$. We also defined bimodules associated to crossings, including $\mathcal{B}' \cdot \mathcal{B}_P$ and $\mathcal{B}' \cdot \mathcal{B}_N_i$. All of these are special.

**Lemma 4.4.** Choose any matching $M$ on $\{1, \ldots, 2n\}$, and $i = 1, \ldots, 2n - 1$. The type $DD$ bimodules $\mathcal{B}' \cdot \mathcal{B}_K$, $\mathcal{B}' \cdot \mathcal{B}_P$, and $\mathcal{B}' \cdot \mathcal{B}_N_i$ are special, and
\[
F(\mathcal{B}' \cdot \mathcal{B}_K, \text{Id}) = \mathcal{A}_2 \cdot \mathcal{A}_1' \mathcal{K} \quad F(\mathcal{B}' \cdot \mathcal{B}_P, \tau_i) = \mathcal{A}_2 \cdot \mathcal{A}_1' \mathcal{P}_i \quad F(\mathcal{B}' \cdot \mathcal{B}_N_i, \tau_i) = \mathcal{A}_2 \cdot \mathcal{A}_1' \mathcal{N}_i;
\]
where here $\text{Id}$ denotes the identity map from $\{1, \ldots, 2n\}$ to itself.

**Proof.** All of these statements are immediate from the definitions of the various bimodules (see [19, Sections 3.7 and 4] for the canonical $DD$ bimodule and the crossing bimodules respectively). \hfill \square

**Lemma 4.5.** Choose any $0 \leq k \leq 2n + 1$, and a braid $\Gamma$ with $2n$ strands inducing the correspondence $f$. Choose further any $i = 1, \ldots, 2n - 1$, and a matching $M$ on $\{1, \ldots, 2n\}$. Let
\[
\mathcal{B}_1 = \mathcal{B}(2n, 2n + 1 - k), \quad \mathcal{B}_2' = \mathcal{B}(2n, k, \{1, \ldots, 2n\})
\]
\[
\mathcal{A}_1' = \mathcal{A}'(n, 2n + 1 - k, M), \quad \mathcal{A}_2 = \mathcal{A}(n, k, f(M))
\]
Suppose that $B'_2 \otimes B'_1 X$ be an special type $DD$ with one-manifold $\Gamma$ and matching $f$. Then, for the bimodule $B'_2 \cdot \mathcal{B}_1$ from [19], we have that $B'_2 \cdot \mathcal{B}_1$ is also special; and there is a homotopy equivalence
\[
\mathcal{A}_2 \cdot \mathcal{A}_1' F(X, f) \simeq F(\mathcal{B}_2' \cdot \mathcal{B}_1, X, \tau_i \circ f).
\]
**Proof.** For notational simplicity we assume $i = 1$; the general case follows with minor notational modifications. There are two cases, according to whether or not 1 and 2 are matched in $f(M)$. Assume first that 1 and 2 are not matched.

The fact that $\mathcal{B}_3 \mathcal{P}_{\mathcal{B}_2}^1 \boxtimes \mathcal{B}_1 \mathcal{B}_1 X$ is special (with correspondence $\tau_1 \circ f$) is a straightforward consequence of the bimodule $Q = \mathcal{B}_3 \mathcal{P}_{\mathcal{B}_2}^1$, which has the property that

$$\delta_2(X, C_j) - C_{\tau_1(j)} \otimes X \in \mathcal{B}_3 \otimes Q \subset \mathcal{B}_3' \otimes Q.$$ 

We will abbreviate $P' = \mathcal{B}_3 \mathcal{P}_{\mathcal{B}_2}^1, P = \mathcal{A}_3 \mathcal{A}_2^1.$ Consider the linear map

$$h_1 : F(P' \boxtimes \mathcal{B}_1 \mathcal{B}_1 X, \tau_1 \circ f) \to P \otimes \mathcal{A}_1 \mathcal{A}_2^1 \mathcal{F}(X, f).$$

with $h_1(T \otimes X) = T \otimes X$ for $T \in \{N, W, E\}$ and

$$h_1(S \otimes X) = S \otimes X + (R_1 \otimes E_{g(2)}) \otimes W \otimes X + (L_2 \otimes E_{g(1)}) \otimes E \otimes X,$$

where $g = f^{-1}$.

We claim that $h_1$ is a type $DD$ bimodule homomorphism; i.e.

$$(\mu_2 \otimes Id) \circ (Id \otimes \delta_1) \circ h_1 + (\mu_2 \otimes Id) \circ (Id \otimes h_1) \circ \delta_1 + (\mu_1 \otimes Id_X) \circ h_1 = 0,$$

where $\mu_1$ and $\mu_2$ are computed in the algebra $\mathcal{A}_3 \otimes \mathcal{A}_1^1$. There are various components of this map, decomposed according to how many $E_{g(1)}$ and $E_{g(2)}$ appear in the $\mathcal{A}_1^1$ factor. If there are none, then we have cancellation of terms

And a similar cancellation for the term involving $U_{g(2)}$. 

\[0 = \begin{array}{|c|c|}
\hline
\mathcal{P} & X \\
\hline
\vdots & \vdots \\
\delta_2 & \delta_1 \\
\hline
h_1 & h_1 \\
\hline
\mathcal{E} & \mathcal{E} \\
\hline
\end{array} + \begin{array}{|c|c|}
\hline
\mathcal{P} & X \\
\hline
\vdots & \vdots \\
\delta_1 & h_1 \\
\hline
h_1 & h_1 \\
\hline
\mathcal{E} & \mathcal{E} \\
\hline
\end{array} \begin{array}{|c|c|}
\hline
h_1 & h_1 \\
\hline
\mu_1 & \mu_1 \\
\hline
\end{array} \begin{array}{|c|c|}
\hline
U_{g(2)} & U_{g(2)} \\
\hline
\end{array} \begin{array}{|c|c|}
\hline
\mu_1 & \mu_1 \\
\hline
\end{array} \begin{array}{|c|c|}
\hline
U_{g(2)} & U_{g(2)} \\
\hline
\end{array} \begin{array}{|c|c|}
\hline
\mu_1 & \mu_1 \\
\hline
\end{array} \begin{array}{|c|c|}
\hline
U_{g(2)} & U_{g(2)} \\
\hline
\end{array} \begin{array}{|c|c|}
\hline
\mu_1 & \mu_1 \\
\hline
\end{array} \begin{array}{|c|c|}
\hline
U_{g(2)} & U_{g(2)} \\
\hline
\end{array} \begin{array}{|c|c|}
\hline
\mu_1 & \mu_1 \\
\hline
\end{array} \begin{array}{|c|c|}
\hline
U_{g(2)} & U_{g(2)} \\
\hline
\end{array}
\end{array}
For terms of the form \( a \otimes b \cdot E_{g(1)} \) with \( b \in \mathcal{B}_1 \), we have the following cancellations. First, for \( T \in \{N, W\} \), we have:

\[
0 = \sum h_1 \delta_1 P' X + \sum h_1 h_1 \delta_1 \eta \eta E_{g(1)} \eta \eta
\]

Further cancellations involving \( \delta^1_\ell \) with \( \ell \leq 2 \) in \( \mathcal{P}' \) are:

\[
0 = \sum \delta^1 \mu_2 \eta \eta E_{g(1)} \eta \eta + \sum \delta^1 \eta \eta \eta \eta
\]

Cancellations using \( \delta^1_3 \) in \( \mathcal{P}' \) involve some term \( (a \otimes b) \otimes X \) appearing in \( \delta^1(X) \), where \( a \in \mathcal{B}'_2 \) will have its type (as in Equation (3.8)) specified below (it will be
$U_2^1$ or $R_1U_2^1$ or $L_2U_2^1$); and $b \in B$. Terms that land in $E \otimes X$ cancel as below:

\[
0 = \delta_1 \mu_2 \delta_1^2 \delta_1 \mu_2 + \delta_1 \mu_2 \delta_1^2 \delta_1 \mu_2
\]

For terms that land in $W$, we have similar cancellations. For terms that land in $N$, for $t > 0$ we have cancellations

\[
0 = \delta_1 \mu_2 \delta_1^2 \delta_1 \mu_2 + \delta_1 \mu_2 \delta_1^2 \delta_1 \mu_2
\]

The analogous cancellation when $t = 0$ has the form:

\[
0 = \delta_1 \mu_2 \delta_1^2 \delta_1 \mu_2 + \delta_1 \mu_2 \delta_1^2 \delta_1 \mu_2
\]

There are similar cancellations when the output is $(R_1U_1^1 \otimes E_{g(1)} \cdot b) \otimes N \otimes y$. There are also analogous cancellations for terms of the form $a \otimes b \cdot E_{g(2)}$.

Cancellation of terms involving pairs $E_i$ and $E_j$ (with $\{i, j\} \cap \{g(1), g(2)\} = \emptyset$) are straightforward. For terms involving $E_{g(1)}$ and $E_{g(\alpha)}$ landing in $E$, we have
cancellation:

\[(4.4)\]

\[
0 = \left( P' \begin{array}{c} X \end{array} \right)_{s} + \left( P' \begin{array}{c} X \end{array} \right)_{s} + \left( P' \begin{array}{c} X \end{array} \right)_{s}
\]

For terms landing in \( S \), the cancellation is easier to see. There is an analogous cancellation involving \( E_{g(2)} \) and \( E_{g(3)} \).

Finally, for terms involving \( E_{g(1)} \) and \( E_{g(2)} \), we have the following:

\[(4.5)\]

\[
0 = \left( P' \begin{array}{c} X \end{array} \right)_{s} + \left( P' \begin{array}{c} X \end{array} \right)_{s} + \left( P' \begin{array}{c} X \end{array} \right)_{s}
\]

There is a similar cancellation for terms involving \( E_{g(1)} \cdot E_{g(2)} \) with output in \( E \).

This completes the verification when 1 and 2 are not matched. When they are matched, the cancellations involving products of \( E_{g(1)} \) and \( E_{g(2)} \) work a little differently. Instead of the cancellation from Equation (4.4), we have the cancellations from Equation (4.5), noting that in this case there are two canceling terms outputting \( E_{g(1)} \cdot E_{g(2)} \) and two outputting \( E_{g(2)} \cdot E_{g(1)} \). The other cancellations are as above.
Lemma 4.6. Let $B'_2, B_1, Y$ and $B'_2, B_1, Z$ be two type DD bimodules adapted to the same braid, and $\phi^1 \in \text{Mor}^{B'_2, B_1}(Y, Z)$ is a homomorphism with

$$\phi^1 : Y \to (B_2 \otimes B_1) \otimes Z \subset (B'_2 \otimes B_1) \otimes Z,$$

then $\phi$ induces a morphism

$$F(\phi^1) : F(Y, f) \to F(Z, f).$$

Proof. The formula for $F(\phi^1)$ is the same as the formula for $\phi^1$. The structure equation is obviously still satisfied. \qed

With these pieces in place, the proof of Theorem 4.1 reduces quickly to its analogue, from [19, Theorem 6.1]:

Proof of Theorem 4.1. The theorem is now an easy consequence of the braid relations for the modules from [19].

Since $(P^i \otimes N^i) \otimes K \simeq P^i \otimes (N^i \otimes K) \simeq P^i \otimes N_i$ (by associativity of $\otimes$ and Proposition 3.4), the verification that $P^i \otimes N^i \simeq \text{Id}_{A_i}$, will follow from the identity

$$P^i \otimes N_i \simeq K,$$

In more detail, we wish to show that

$$A_1 P_{A_2} \otimes A^1 \otimes A^i \otimes N \simeq A_1 \otimes A^i \otimes K.$$

In [19, Section 6], we verified that

$$(4.7) \quad B^i P_{B^i} \otimes B^i \otimes K \simeq B^i \otimes K,$$

(i.e. Equation (4.6) for the previous algebras) using a homotopy equivalence $\phi^1$ satisfying the hypothesis of Lemma 4.6. (In fact, in this case,

$$\phi^1 : B^i P_{B^i} \otimes B^i \otimes N \to B^i \otimes K$$

is the identity map.) Combining Lemma 4.4, 4.5, with the the above result, we see that

$$A_1 P_{A_2} \otimes A \otimes A^i \otimes N = A_1 P_{A_2} \otimes F(B^i, B_i),$$

Equations (4.2) and (4.3) follow from the same logic. \qed
5. Bimodule associated to a maximum

Fix \( M \) a matching on \( \{1, \ldots, 2n\} \), \( k \) an integer with \( 0 \leq k \leq 2n+1 \), and \( c \) an integer with \( 1 \leq c \leq 2n+1 \).

Let \( \phi_c : \{1, \ldots, 2n\} \to \{1, \ldots, 2n+2\} \) be the map
\[
(5.1) \quad \phi_c(j) = \begin{cases} 
  j & \text{if } j < c \\
  j + 2 & \text{if } j \geq c.
\end{cases}
\]

Let
\[
(5.2) \quad \mathcal{A}_1 = \mathcal{A}(n, k, M) \quad \text{and} \quad \mathcal{A}_2 = \mathcal{A}(n+1, k+1, \phi_c(M) \cup \{c, c+1\})
\]

We will define the bimodule associated to the partial knot diagram containing a single local maximum connecting the strands \( c \) and \( c+1 \) (in the output), denoted \( \mathcal{A}_2 \Omega_{\mathcal{A}_1} \). Before doing this, we describe its dual type \( DD \) bimodule.

5.1. The type \( DD \) bimodule. Let
\[
(5.3) \quad \mathcal{A}'_1 = \mathcal{A}'(n, 2n+1-k, M)
\]

We define a type \( DD \) bimodule \( \mathcal{A}_2 - \mathcal{A}'_1 \Omega_c \), as follows.

To describe the underlying vector space for \( \Omega_c \), we proceed as follows. We call an idempotent state \( y \) for \( \mathcal{A}_2 \) an allowed idempotent state if
\[
(5.4) \quad c \in y \quad \text{and} \quad |y \cap \{c-1, c+1\}| \leq 1.
\]

Consider the map \( \psi' \) from allowed idempotent states \( y \) for \( \mathcal{A}_2 \) to idempotent states for \( \mathcal{A}'_1 \), where \( x = \psi'(y) \subset \{0, \ldots, 2n\} \) is characterized by the property that
\[
(5.5) \quad |y \cap \{c-1, c+1\}| + |x \cap \{c-1\}| = 2 \quad \text{and} \quad \phi_c(x) \cap y = \emptyset.
\]

As a vector space, \( \Omega_c \) is spanned by vectors that are in one-to-one correspondence with allowed idempotent states for \( \mathcal{A}_2 \). The module structure over the ring of idempotents \( I(\mathcal{A}_2) \otimes I(\mathcal{A}'_1) \), is specified as follows. If \( P = P_y \) is the generator associated to the idempotent state \( y \), then
\[
(I_y \otimes I_{\psi'(y)}) \cdot P_y = P_y.
\]

To specify the differential, consider the element \( A \in \mathcal{A}_2 \otimes \mathcal{A}'_1 \)
\[
A = (L_c L_{c+1} \otimes 1) + (R_{c+1} R_c \otimes 1) + \sum_{i=1}^{2n} L_{\phi(i)} \otimes R_i + R_{\phi(i)} \otimes L_i
\]
\[
+ C_{\{c,c+1\}} \otimes 1 + \sum_{i=1}^{2n} U_{\phi(i)} \otimes E_i + \sum_{(i,j) \in M} C_{\{\phi(i), \phi(j)\}} \otimes [E_i, E_j]
\]

where we have dropped the subscript \( c \) from \( \phi_c = \phi \). Let
\[
\delta^1(P_y) = (I_y \otimes I_{\psi'(y)}) \cdot A \otimes \sum_{z} P_z,
\]

where the latter sum is taken over all allowed idempotent states \( z \) for \( \mathcal{A}_2 \).

**Lemma 5.1.** The space \( \mathcal{A}_2 - \mathcal{A}'_1 \Omega_c \) defined above, and equipped with the map
\[
\delta^1 : \Omega_c \to (\mathcal{A}_2 \otimes \mathcal{A}'_1) \otimes \Omega_c,
\]

specified above, is a type \( DD \) bimodule over \( \mathcal{A}_2 \) and \( \mathcal{A}'_1 \).
**Proof.** The proof is a straightforward adaptation of Lemma 2.1.

It is helpful to understand $\Omega_c$ a little more explicitly. To this end, we classify the allowed idempotents for $A_2$ into three types, labeled $X$, $Y$, and $Z$:

- $y$ is of type $X$ if $y \cap \{c - 1, c, c + 1\} = \{c - 1, c\}$,
- $y$ is of type $Y$ if $y \cap \{c - 1, c, c + 1\} = \{c, c + 1\}$,
- $y$ is of type $Z$ if $y \cap \{c - 1, c, c + 1\} = \{c\}$.

There is a corresponding classification of the generators $P_y$ into $X$, $Y$, and $Z$, according to the type of $y$; see Figure 4.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig4.png}
\caption{DD bimodule of a maximum. Three generator types are illustrated.}
\end{figure}

With respect to this decomposition, terms in the differential are of the following types:

\begin{enumerate}
\item[(Ω-1)] $R_{\phi(j)} \otimes L_j$ and $L_{\phi(j)} \otimes R_j$ for all $j \in \{1, \ldots, 2n\} \setminus \{c - 1, c\}$; these connect generators of the same type.
\item[(Ω-2)] $U_{\phi(i)} \otimes E_i$ for $i = 1, \ldots, 2n$
\item[(Ω-3)] $C_{\{\phi(i), \phi(j)\}} \otimes [E_i, E_j]$ for all $\{i, j\} \in M$;
\item[(Ω-4)] $C_{\{c, c+1\}} \otimes 1$
\item[(Ω-5)] Terms in the diagram below connect generators of different types.
\end{enumerate}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig5.png}
\caption{5.6 Terms in the diagram below connect generators of different types.}
\end{figure}

With the understanding that if $c = 1$, then the terms containing $L_{c-1}$ or $R_{c-1}$ are missing; similarly, if $c = 2n + 1$, the terms containing $R_{c+2}$ and $L_{c+2}$ are missing.

5.2. **The $DA$ bimodule of a maximum.** We will describe now the type $DA$ bimodule $A_2^c \Omega_{A_1}$ promised in the beginning of the section (where $A_1$ and $A_2$ are specified in Equation (5.2)). This is the bimodule associated to a region in the knot diagram where there are no crossings and a single local maximum, which connects the $c^{th}$ and the $(c + 1)^{st}$ outgoing strand; see Figure 5.
Recall the definition of allowed idempotent states for \( A_2 \), specified in Equation (5.4). There is a map \( \psi \) from allowed idempotent states for \( A_2 \) to idempotent states for \( A_1 \), given by

\[
\psi(x) = \begin{cases} 
\phi^{-1}(y) & \text{if } c + 1 \not\in y \\
\phi^{-1}(y) \cup \{c - 1\} & \text{if } c + 1 \in y
\end{cases}
\]

Observe that \( \psi(y) = \{0, \ldots, 2n\} \setminus \psi'(y) \), where \( \psi' \) is the map specified in Equation (7.2).

A basis for the underlying vector space of \( A_2 \Omega_c \) is specified by the allowed idempotent states for \( A_2 \). The bimodule structure, over the rings of idempotents \( I(A_1) \) and \( I(A_2) \), is specified as follows. If \( Q_y \) is the generator associated to the allowed idempotent state \( y \), then

\[
I_y \cdot Q_y \cdot I_{\psi(y)} = Q_y.
\]

The map

\[
\delta_1^1: A_2 \Omega_c \rightarrow A_2 \otimes I(A_2) A_2 \Omega_c
\]

is given by

\[
Q_y \mapsto I_y \cdot (R_{c+1}R_c + L_cL_{c+1} + C_{(c,c)}}) \otimes \sum_{z} Q_z,
\]

where the sum is taken over all allowed idempotents \( z \) for \( A_2 \).

Split the bimodule

\[
A_2 \Omega_c \cong X \oplus Y \oplus Z
\]

according to the types of the corresponding idempotents as defined in Section 5.1; i.e.

\[
X = \bigoplus_{\{y \mid y \cap \{c-1,c,c+1\} = \{c-1,c\}\}} Q_y
\]

\[
Y = \bigoplus_{\{y \mid y \cap \{c-1,c,c+1\} = \{c,c+1\}\}} Q_y
\]

\[
Z = \bigoplus_{\{y \mid y \cap \{c-1,c,c+1\} = \{c\}\}} Q_y
\]

With respect to this splitting, \( \delta_1^1 \) can be expressed as

\[
\delta_1^1(X) = C_{(c,c+1)} \otimes X + R_{c+1}R_c \otimes Y
\]

(5.7)

\[
\delta_1^1(Y) = C_{(c,c+1)} \otimes Y + L_cL_{c+1} \otimes X
\]

\[
\delta_1^1(Z) = C_{(c,c+1)} \otimes Z;
\]
Moreover, the state \( z \) is uniquely characterized by the existence of such a map \( \Phi_x \).

If \( x \) is an allowed idempotent state for \( A_2 \) and \( a = I_{\psi(x)} \cdot a \cdot I_y \in A_1 \) is a non-zero algebra element, let

\[
\delta^1_2(Q_x,a) = \Phi_x(a) \otimes Q_z,
\]

where \( z \) is as in Lemma 5.2.

**Theorem 5.3.** The above specified actions \( \delta^1_2 \) and \( \delta^1_1 \) (and \( \delta^1_0 = 0 \) for all \( \ell > 2 \)) give \( A_2 \Omega_{A_1}^c \) the structure of a DA bimodule. Moreover, \( A_2 \Omega_{A_1}^c \) is standard, in the sense of Definition 2.10.

**Proof.** This is straightforward; compare [19, Theorem 8.3]. \( \square \)

The type DA bimodule associated to a local maximum is dual to the type DD bimodule of a local maximum, in the following sense.

**Proposition 5.4.** Fix \( A_1, A_2, \) and \( A_1' \) as in Equations (5.2) and (5.3), and \( c \) with \( 1 \leq c \leq 2n + 1 \). There is an identification

\[
A_2 \Omega_{A_1}^c \cong A_1 \cdot A_1'. \Omega_c.
\]

**Proof.** This is a straightforward computation. \( \square \)

5.3. **The trident relation.** As in [19], an important ingredient in the invariance proof is the following relation between bimodules for critical points and crossings.

**Proposition 5.5.** Fix integers \( 0 \leq k \leq 2n + 1 \), and a matching \( M_1 \) on \( \{1, \ldots, 2n\} \).

Let

\[
\begin{align*}
M_2 &= \phi_{c+1}(M_1) \cup \{c + 1, c + 2\} \quad M_3 = \tau_c(M_2) \\
M_4 &= \tau_{c+1}(M_3) \\
A_1' &= \mathcal{A}(n, 2n + 1 - k, M_1), \quad A_2 = \mathcal{A}(n + 1, k + 1, M_2) \\
A_3 &= \mathcal{A}(n + 1, k + 1, M_3) \quad A_4 = \mathcal{A}(n + 1, k + 1, M_4)
\end{align*}
\]

There is homotopy equivalence of graded bimodules:

\[
A_3 \Omega_{A_2}^c \boxtimes A_2 \cdot A_1' \Omega_{A_1}^{c+1} \cong A_3 \mathcal{P}_{A_4}^{c+1} \boxtimes A_1 \cdot A_1' \Omega_c.
\]
Proof. For notational simplicity, suppose that $c = 1$. A straightforward computation shows that $\mathcal{A}_3 \mathcal{P}_{\mathcal{A}_4}^2 \otimes \mathcal{A}_2 \mathcal{A}_1^c \Omega_1$ is the bimodule generated by $N$, $W$, $S$, $E$, and with arrows:

\[
\begin{align*}
N & \xrightarrow{U_2 \otimes E_1} W \xleftarrow{L_3 \otimes 1} S \xrightarrow{L_2 \otimes 1} E \xrightarrow{R_3 \otimes 1} S \xrightarrow{R_2 \otimes E_1} W \xrightarrow{L_1 \otimes 1} S \xrightarrow{L_1 \otimes 1} E \xrightarrow{R_1 \otimes 1} S \xrightarrow{R_1 \otimes 1} W \xrightarrow{U_2 \otimes E_1} N
\end{align*}
\]

(5.9)

along with further self-arrows of the form $C_{\{1,3\}} \otimes 1$ and self-arrows $L_{j+2} \otimes R_j$, $R_{j+2} \otimes L_j$, $U_{j+2} \otimes E_j$ for $j = 2, \ldots, 2n$; and $C_{\{m+2, \ell+2\}} \otimes [E_m, E_\ell]$ for those $\{m, \ell\} \in M_1$ with $1 \not\in \{m, \ell\}$, and $C_{\{2, \alpha+2\}} \otimes [E_1, E_\alpha]$ with $\{1, \alpha\} \in M_1$.

After making the substitution $S' = S + (R_2 \otimes E_1) \otimes W$, we arrive at the more symmetric version:

\[
\begin{align*}
N & \xrightarrow{U_2 \otimes E_1} W \xleftarrow{L_3 \otimes 1} S \xrightarrow{L_2 \otimes 1} E \xrightarrow{R_3 \otimes 1} S \xrightarrow{R_2 \otimes E_1} W \xrightarrow{L_1 \otimes 1} S \xrightarrow{L_1 \otimes 1} E \xrightarrow{R_1 \otimes 1} S \xrightarrow{R_1 \otimes 1} W \xrightarrow{U_2 \otimes E_1} N
\end{align*}
\]

(5.10)

along with the same self-arrows as before, and the further self-arrow of the form $U_2 \otimes E_1$ (now on all four generator types rather than on only two, as before).

A symmetric computation reduces $\mathcal{A}_3 \mathcal{N}_{\mathcal{A}_2}^c \otimes \mathcal{A}_2 \mathcal{A}_1^c \Omega_{c+1}$ to the same bimodule. □
6. Working in the dual algebra

For some purposes (especially in Section 7), it will be convenient to work in the algebra \( \mathcal{A}' \) and with bimodules defined over this algebra. Many of our previous constructions have a straightforward adaption to this case.

### 6.1. Bimodules associated to positive crossings

We construct the type \( DA \) bimodule of a positive crossing \( \mathcal{A}(n,k,M) / \mathcal{P}^i_{\mathcal{A}(n,k,M)} \).

The construction starts from \( \mathcal{B}(2n,k) \mathcal{P}^i_{\mathcal{B}(2n,k)} \) from [19] and recalled in Section 3.2.

In Section 3.2, we explained how to extend the bimodules from \( \mathcal{B}(2n,k) \) to \( \mathcal{A}(n,k) \).

The extension to \( \mathcal{A}'(n,k) \) is done similarly, as follows. We spell out the case where \( i = 1 \); the general case follows with straightforward notational changes. There are two subcases, according to whether or not 1 and 2 are matched.

First, we extend all the actions coming from \( \mathcal{B}_1 = \mathcal{B}(2n,k) \) so that they commute with the action of the \( E_j \) with \( j = 1, \ldots, 2n \), so that the following identities hold:

\[
\begin{align*}
\delta_1^3(X, E_j \cdot a_1) &= E_{\tau(j)} \cdot \delta_2^1(X, a_1) \\
\delta_3^1(X, E_j \cdot a_1, a_2) &= E_{\tau(j)} \cdot \delta_3^1(X, a_1, a_2) \\
\delta_3^1(X, a_1 \cdot E_j, a_2) &= \delta_3^1(X, a_1, E_j \cdot a_2) \\
\delta_3^1(X, a_1, a_2 \cdot E_j) &= \delta_3^1(X, a_1, a_2) \cdot E_{\tau(j)},
\end{align*}
\]

where \( a_1, a_2 \in \mathcal{A}'_1 \) are arbitrary algebra elements. Note that these extension rules are slightly more complicated than the corresponding rules for the bimodules over \( \mathcal{A} \) because the new variables \( E_j \) here are not commutative. In particular, the last identity should be interpreted as follows: multiplication by \( E_{\tau(j)} \) takes place on the right of the algebra output of \( \delta_3^1(X, a_1, a_2) \).

Note that any algebra element in \( \mathcal{A}' \) can be written as the product of an element of \( \mathcal{B}_1 \) with a word in the various \( E_j \). Thus, the first three relations can be regarded as definitions of the actions, inductively defined in terms of the length of the words in the \( E_j \). Taking the first three relations as the definition, Equation (6.2) follows easily.

There are two subcases, according to whether \( \{1,2\} \notin M \). Consider first the case where \( \{1,2\} \notin M \). In this case, we must add more terms in the definition, as follows. Fix \( \alpha \) and \( \beta \) so that \( \{1,\alpha\}, \{2,\beta\} \in M \). We add further terms in \( \delta_3^1 \) from \( S \) to \( \{N, W, E\} \), as follows:

\[
\begin{align*}
(S, E_2) &\rightarrow R_1 \otimes W \\
(S, E_1) &\rightarrow L_2 \otimes E & (S, E_1 E_2) &\rightarrow E_2 R_1 \otimes W + E_1 L_2 \otimes E \\
(S, U_1 E_2) &\rightarrow U_1 L_2 \otimes E & (S, U_1 E_1 E_2) &\rightarrow U_1 E_2 L_2 \otimes E \\
(S, E_1 U_2) &\rightarrow R_1 U_2 \otimes W & (S, E_1 E_2 U_2) &\rightarrow E_1 R_1 U_2 \otimes W \\
(S, R_1 E_2) &\rightarrow R_1 \otimes N & (S, R_1 E_1 E_2) &\rightarrow R_1 E_2 \otimes N \\
(S, L_2 E_1) &\rightarrow L_2 \otimes N & (S, E_1 L_2 E_2) &\rightarrow E_1 L_2 \otimes N \\
(S, R_1 E_1 U_2) &\rightarrow R_1 U_2 \otimes N & (S, R_1 E_1 U_2 E_2) &\rightarrow R_1 E_1 U_2 \otimes N \\
(S, U_1 L_2 E_2) &\rightarrow L_2 U_1 \otimes N & (S, U_1 E_1 L_2 E_2) &\rightarrow U_1 L_2 E_2 \otimes N
\end{align*}
\]
These further terms are extended as follows. Suppose we have a term \((X, a) \to b \otimes Y\) as above. Then for any words \(c_1\) and \(c_2\) in the \(E_k\) with \(k \not\in \{1, 2\}\) so that \(c_1ac_2 \neq 0\), we have a further term \((X, c_1ac_2) \to c_1bc_2 \otimes Y\).

For example,

\[
\delta_2^1(S, E_1 E_\alpha) = E_2 E_\alpha \otimes S + L_2 E_\alpha \otimes E
\]
\[
\delta_2^1(S, E_\alpha E_1) = E_\alpha E_2 \otimes S + L_2 E_\alpha \otimes E.
\]

New terms are added so that the following relations hold:

\[
\delta_2^1(X, [E_1, E_\alpha] \cdot a) = [E_2, E_\alpha] \cdot \delta_2^1(X, a)
\]
\[
\delta_2^1(X, [E_2, E_\beta] \cdot a) = [E_1, E_\beta] \cdot \delta_2^1(X, a)
\]

for all \(a \in \mathcal{A}'\). These new terms in \(\delta_2^1\) are further extended to commute with multiplication by \(U_1 U_2\); and then they are extended to commute with multiplication by algebra elements with \(w_1 = w_2 = 0\).

For example, since \(E_1 \cdot E_\alpha \cdot E_1 = [E_1, E_\alpha] \cdot E_1\), it follows that

\[
\delta_2^1(S, E_1 \cdot E_\alpha \cdot E_1) = [E_2, E_\alpha] \cdot L_2 \otimes E + E_2 \cdot E_\alpha \cdot E_2 \otimes S.
\]

Note also that

\[
\delta_3^1(X, [E_1, E_\alpha] \cdot a_1, a_2) = \delta_3^1(X, a_1, [E_1, E_\alpha] \cdot a_2) = [E_2, E_\alpha] \cdot \delta_3^1(X, a_1, a_2)
\]
\[
\delta_3^1(X, [E_2, E_\beta] \cdot a_1, a_2) = \delta_3^1(X, a_1, [E_2, E_\beta] \cdot a_2) = [E_1, E_\beta] \cdot \delta_3^1(X, a_1, a_2).
\]

When \(\{1, 2\} \in M\), we add terms as in Equation (6.3), further adding terms in the second column, as well, with the order of \(E_1\) and \(E_2\) exchanged. For example, we add the term

\[(S, E_2 E_1) \to E_2 R_1 \otimes W + E_1 L_2 \otimes E.
\]

These are extended over monomials in \(E_k \not\in \{1, 2\}\) as before. Finally, we extend all actions so that

\[
\delta_2^1(X, [E_1, E_2] \cdot a) = [E_1, E_2] \cdot \delta_2^1(X, a)
\]

for all \(a \in \mathcal{A}'\).

The resulting bimodule has a straightforward generalizations when \(i \neq 1\).

**Proposition 6.1.** The above operations give \(\mathcal{A}'^{(n,k,\tau(M))}\mathcal{P}_j\mathcal{A}'^{(n,k,M)}\) the structure of a type DA bimodule over the specified algebras.

**Proof.** The proof is similar to the proof of Proposition 3.3. □

We have the following analogue of Proposition 3.4. For statement below, the order of two algebras appearing in the type \(DD\) bimodules \(\mathcal{K}\) and \(\mathcal{P}_j\) are opposite to what they were in Sections 2.3 and 3.1. Moreover, the tensor product in the statement can be formed because \(\mathcal{A}'^{(n,k,M)}\mathcal{P}_j\mathcal{A}'^{(n,k,M)}\) is suitably bounded; indeed, by the definition of the bimodule given above, it is clear that \(\delta_2^k = 0\) for all \(k > 3\).
Lemma 6.2. Let
\[ A'_1 = A'(n, k, M), \quad A'_2 = A'(n, k, \tau_i(M)), \quad A_1 = A'(n, 2n + 1 - k, M). \]
\( P^i \) is dual to \( P_i \), in the sense that
\[ A'_2 P^i_{A'_1} \cong A'_1 A_1 K \cong A'_2 A_1 P_i. \]

Proof. This is a straightforward computation in the spirit of Proposition 3.4. Indeed, the computation is slightly easier in that there is no need to introduce a homotopy equivalence \( h^i \) in fact, the computation verifies that \( A'_2 P^i_{A'_1} \cong A'_1 A_1 K \cong A'_2 A_1 P_i. \) (This is analogous to the case of Lemma [19, Lemma 6.2] where the input algebra has both \( C_i \) and \( C_{i+1} \) in it.)

We also have the following:

Proposition 6.3. Let
\[ A_1 = A(n, k, M), \quad A_2 = A(n, k, \tau_i(M)) \]
\[ A'_1 = A(n, 2n + 1 - k, M), \quad A'_2 = A(n, 2n + 1 - k, \tau_i(M)). \]
The DA bimodules of a crossing from Section 3.2 are related to those here by the homotopy equivalence:
\[ A'_2 P^i_{A'_1} \cong A'_1 A_1 K \cong A'_2 A_1 P_i. \]

Proof. This is an immediate consequence of the invertability of \( K \) (Theorem 2.2), Proposition 3.4, and Lemma 6.2.

6.2. Bimodules associated to negative crossings. The bimodules associated to negative crossings can be derived from the positive case.

With \( n, k, M \), and \( \tau_i \) as before, take the opposite module of \( A'(n, k, \tau_i(M)) P^i_{A'(n, k, M)} \), to get a module
\[ A'(n, k, \tau_i(M)) P^i_{A'(n, k, M)} \cong A'(n, k, M)^{\text{op}} P^i_{A'(n, k, \tau_i(M))} \]
and use natural identifications \( o: A'(n, k, M') \to A'(n, k, M')^{\text{op}} \) for both \( M' = M \) and \( M' = \tau_i(M) \) to define \( A'(n, k, \tau_i(M)) N^i_{A'_1} A'(n, k, M) \), the bimodule over \( A' \) associated to a negative crossing.

Lemma 6.4. For
\[ A'_1 = A'(n, k, M) \quad A'_2 = A'(n, k, \tau_i(M)), \]
we have that
\[ A'_1 \text{Id}_{A'_1} \cong A'_1 P^i_{A'_2} N^i_{A'_1} \cong A'_1 N^i_{A'_2} P^i_{A'_1}. \]

Proof. From Lemma 6.2 and Proposition 3.4, it follows that
\[ A'_2 P^i_{A'_1} \cong A'_1 A'_1, K \cong A'_2 P^i_{A'_2} \cong A'_2 A'_2, K. \]
Taking opposites we also find that
\[ A'_1 N^i_{A'_2} \cong A'_2 N^i_{A'_1} \cong A'_2 A'_1, K. \]
Combining with Equation (4.1), we conclude that
\[ A_1' \mathcal{N}_{A_2'} \boxtimes A_2' P_i A_1' \mathcal{K} \simeq A_1' P_{A_2} \boxtimes A_2 \mathcal{N}_{A_1'} \boxtimes A_1' \mathcal{K} \]

The desired relation
\[ A_1' \text{Id}_{A_1'} \simeq A_1' \mathcal{N}_{A_2'} \boxtimes A_2' P_i \]

now follows from Theorem 2.2. The other equivalence (reversing the order of $P_i$ and $\mathcal{N}_i$) follows similarly. \[ \square \]
Consider a partial knot diagram that contains exactly one local minimum in it. Our aim here is to associate a type DA bimodule to this object. Before doing that, we start with the more easily defined type DD bimodule.

7.1. The type DD bimodule of a minimum. Fix $1 \leq c \leq 2n + 1$. Let
\[
\phi_c : \{1, \ldots, 2n\} \to \{1, \ldots, 2n + 2\}
\]
be the function defined in Equation (5.1). Fix an integer $k$ with $0 \leq k \leq 2n + 1$. Fix a matching $M_1$ on $\{1, \ldots, 2n + 2\}$ that does not match $c$ and $c + 1$, so we have $\alpha, \beta \in \{1, \ldots, 2n\}$ so that $\{c, \phi_c(\alpha)\}, \{c + 1, \phi_c(\beta)\} \in M_1$. There is an induced matching $M_2$ on $\{1, \ldots, 2n\}$ consisting of $\{\alpha, \beta\}$, and all pairs $\{i, j\}$ with $\{i, j\} \cap \{\alpha, \beta\} = \emptyset$ so that $\phi_c(i)$ and $\phi_c(j)$ are matched in $M_1$. Let
\[
\mathcal{A}_1' = \mathcal{A}(n + 1, 2n + 2 - k, M_1) \quad \text{and} \quad \mathcal{A}_2 = \mathcal{A}(n, k, M_2).
\]
Let $\mathcal{O}_c = \mathcal{A}_2 \cdot \mathcal{A}_1'$, $\mathcal{O}_c$ be the bimodule defined as follows.

We call an idempotent state $y$ for $\mathcal{A}_1'$ an allowed idempotent state for $\mathcal{A}_1'$ if
\[
|y \cap \{c - 1, c, c + 1\}| \leq 2 \quad \text{and} \quad c \in y.
\]
There is a map $\psi'$ from allowed idempotent states $y$ for $\mathcal{A}_1'$ to idempotent states for $\mathcal{A}_2$, where $x = \psi'(y) \subset \{0, \ldots, 2n\}$ is characterized by
\[
|y \cap \{c - 1, c, c + 1\}| + |x \cap \{c - 1\}| = 2 \quad \text{and} \quad \phi_c(x) \cap y = \emptyset.
\]
As a vector space, $\mathcal{O}_c$ is spanned by vectors $P_y$ that are in one-to-one correspondence with allowed idempotent states $y$ for $\mathcal{A}_1'$. The bimodule structure, over the rings of idempotents $I(\mathcal{A}_2) \otimes I(\mathcal{A}_1')$, is specified by
\[
(I_{\psi'(y)} \otimes I_y) \cdot P_y = P_y.
\]
To specify the differential, consider the element $A \in \mathcal{A}_2 \otimes \mathcal{A}_1'$
\[
A = (1 \otimes L_c L_{c+1}) + (1 \otimes R_{c+1} R_c) + \sum_{j=1}^{2n} R_j \otimes L_{\phi(j)} + L_j \otimes R_{\phi(j)} + U_j \otimes E_{\phi(j)} + 1 \otimes E_c U_{c+1} + U_a \otimes [E_{\phi(a)}, E_c] E_{c+1} + C_{i,j} \otimes [E_{\phi(i)}, E_{\phi(j)}],
\]
where we have dropped the subscript $c$ from $\phi_c = \phi$. Let
\[
\delta^1(P_y) = (I_{\psi'(y)} \otimes I_y) \cdot A \otimes \sum_z P_z,
\]
where the latter sum is taken over all allowed idempotent states $z$ for $\mathcal{A}_1'$.

Lemma 7.1. The space $\mathcal{A}_2 \cdot \mathcal{A}_1' \cdot \mathcal{O}_c$ defined above, equipped with the map
\[
\delta^1 : \mathcal{O}_c \to (\mathcal{A}_2 \otimes \mathcal{A}_1') \otimes \mathcal{O}_c,
\]
specified above, is a type DD bimodule over $\mathcal{A}_2$ and $\mathcal{A}_1'$. 

Proof. The proof is a straightforward adaptation of Lemma 2.1. □

There is a symmetric version of this bimodule, exchanging the roles of \( c \) and \( c + 1 \) in \( \delta^1 \). The map \( \phi^1(x) = x + (1 \otimes E_c E_{c+1})x \) is easily seen to give an isomorphism between these two bimodules.

7.2. The type \( DA \) bimodule of a minimum when \( c = 1 \). We will construct a type \( DA \) bimodule over the algebra that is dual to the bimodule of a minimum described above. Specifically, we continue with the notation at the beginning of Section 7.1, further letting \( A_1 = \mathcal{A}(n + 1, k + 1, M_1) \). We start by describing the case where \( c = 1 \), returning to the general case in Section 7.5.

A preferred idempotent state for \( A_1 = \mathcal{A}(n + 1, k + 1, M_1) \) is an idempotent state \( x \) with \( x \cap \{0, 1, 2\} \in \{\{0\}, \{2\}, \{0, 2\}\} \).

We define a map \( \psi \) from preferred idempotent states of \( A_1 \) to idempotent states of \( A_2 \), as follows. Given preferred idempotent state \( x \) for \( A_1 \), order the components \( x = \{x_1, \ldots, x_{k+1}\} \) so that \( x_1 < \cdots < x_{k+1} \). Define

\[
\psi(x) = \begin{cases} 
\{0, x_3 - 2, \ldots, x_{k+1} - 2\} & \text{if } |x \cap \{0, 1, 2\}| = 2 \\
\{x_2 - 2, \ldots, x_{k+1} - 2\} & \text{if } |x \cap \{0, 1, 2\}| = 1
\end{cases}
\]

Generators of the \( DA \) bimodule \( \mathcal{B}_1 = A_2 \mathcal{B}_1 A_1 \) correspond to preferred idempotent states; and its bimodule structure over the idempotent algebras is specified by the property that if \( x \) is such a preferred idempotent state, then the corresponding generator \( T_x \) satisfies

\[
I_{\psi(x)} \cdot T_x \cdot I_x = T_x.
\]

Operations are described in terms of the following graph \( \Gamma \) with four vertices, \( X L_1, Y R_2, X, \) and \( Y \); and the following families of edges (each indexed by integers \( m \geq 0 \)):

The labels on the edges consist of some element in \( B_1 = B(2n + 2, k + 1) \) tensored with some tensor powers of \( C_1 = C_{\{a+2,1\}} \) and \( C_2 = C_{\{2,\beta+2\}} \). We call the element of \( B_1 \) the \( B\)-label.

Definition 7.2. Recall that a standard sequence \((a_1, \ldots, a_m)\) for \( A_1 \) has a subsequence \( a_{k_1}, \ldots, a_{k_t} \) in \( B_1 \), whose complement contains elements of the form \( C_p \). A standard sequence is a preferred sequence if it satisfies the following properties:
Definition 7.3. For a preferred sequence, there is at most one pure non-zero algebra element \( b \) homotopy by the following properties: If \( x \) is any preferred sequence, then there is a unique pure non-zero algebra element \( b \) defined as above; and for all other standard sequences consisting of \( m \) elements labeling the edge.

\( \delta \) (pure) The relationship between the two bimodules associated to a minimum is given in the following sense. Given \( A_1, A'_1, \) and \( A_2, A'_2, \) as above, the following identity holds:

\[
A_2 \otimes A'_1, K \cong A_2, A'_2, \tilde{\mathcal{U}}_1.
\]

Proof. We compute \( A_2, A'_1, K \) and the actions from Equation (7.4). We obtain a bimodule whose generators correspond to allowed idempotent states.
y as in Equation (7.1). The differential does not have exactly the form as it did for $\hat{U}_1$; rather, it is described as follows:

\[
\begin{align*}
1 \otimes U_1E_2 + U_\beta \otimes [E_2, E_{\beta+2}]E_1 + C_{\alpha, \beta} \otimes [E_1, E_{\alpha+2}][E_2, E_{\beta+2}] \\
1 \otimes L_1L_2 + 1 \otimes L_1L_2E_1E_2 \\
1 \otimes R_2R_1 + 1 \otimes R_2R_1E_1E_2 \\
1 \otimes U_2E_1 + U_\alpha \otimes [E_1, E_{\alpha+2}]E_2 + C_{\alpha, \beta} \otimes [E_1, E_{\alpha+2}][E_2, E_{\beta+2}]
\end{align*}
\]

in addition to the usual terms

\[
\left( \sum_{i=1}^{2n} L_i \otimes R_{i+2} + R_i \otimes L_{i+2} + U_i \otimes E_{i+2} \right) + \sum_{\{i,j\} \in M_2-\{\alpha, \beta\}} C_{\{i,j\}} \otimes [E_{i+2}, E_{j+2}]
\]

that connect $X$ to itself and $Y$ to itself. (Note that $E_1$ and $E_2$ commute with each other.) The map sending $X$ to $X + (1 \otimes E_1E_2)X$ and $Y$ to $Y$ identifies this bimodule with $\hat{U}_1$.

To verify Theorem 7.4, we give an alternative construction of the type $DA$ bimodule in the next section.

7.3. An alternative construction. Continuing notation on $A_1, A_2, M_1, M_2$ from the previous section, we give an alternative construction of the bimodule promised in Theorem 7.4, which makes the proof transparent.

Consider the idempotent in $A_1$

\[
I = \sum_{\{x | x_1=1\}} I_x.
\]

Let $B$ be the subalgebra of $A(n+1, k+1, M_1)$ with $C_{\{\alpha+2, 1\}}$ and $C_{\{2, \beta+2\}}$ (and their multiples) removed from it.

Consider the right $B$-module

\[
M = \frac{I \cdot B}{L_1L_2B} \oplus \frac{I \cdot B}{L_1L_2B^*}.
\]

$M$ can also be viewed as a left module over the subalgebra of $I \cdot B \cdot I$ consisting of elements $w_1 = w_2 = 0$, which in turn can be identified with the subalgebra $B_2$ of $A(n, M_2)$ with $C_{\{\alpha, \beta\}}$ removed. Denote this identification

\[
\phi : (B_2 \subset A(n, M_2)) \rightarrow I \cdot B \cdot I.
\]

Let $X$ and $Y$ be the generators of the two summands of $M$. Let

\[
m_{1|1|0}(b_2, X \cdot b_1) = X \cdot \phi(b_2) \cdot b_1 \quad m_{1|1|0}(b_2, Y \cdot b_1) = Y \cdot \phi(b_2) \cdot b_1,
\]

where $b_2 \in B_2$ and $b_1 \in \frac{1}{L_1L_2B^*}$. Equip $M$ with the differential

\[
m_{0|1|0}(X) = Y \cdot U_2 \\
m_{0|1|0}(Y) = X \cdot U_1.
\]

Think of the right $B$-action as inducing further operations

\[
m_{0|1|1}(X \cdot b_1, b_1') = X \cdot (b_1 \cdot b_1') \\
m_{0|1|1}(Y \cdot b_1, b_1') = Y \cdot (b_1 \cdot b_1').
\]

All the operations described above give $M$ the structure of a $B_2 - B$ bimodule, written $\mathcal{B}_2 \mathcal{M}_B$. 

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Lemma 7.6. There is a type $DA$ bimodule $B^2 \Theta_B$, generated as a right $B$-module by two generators $X$ and $Y$ so that $B_2 M_B = B_2 (B_2 \otimes B_2 \Theta_B)$. As a type $D$ structure, $B^2 \Theta_B$ is generated by elements of the form $X \cdot a$ or $Y \cdot a$, where $a$ is chosen from:

\[ (7.5) \quad \Gamma = \{ R^2 U^n, \quad U^n, \quad U^1 \}_{n \geq 0, n > 0}. \]

Proof. As a left module over $I \cdot B \cdot I \cong B_2$, $M$ is generated by the above generating set. Moreover, each of $B_2$-modules $B_2 \cdot a_i$, where $a_i$ is chosen from the above generating set, is isomorphic to $B_2 \cdot I_B$, where $a = I_B \cdot a$. Both statements are proved in [19, Lemma 9.3]. The existence of $B^2 \Theta_B$ with the stated generating set is a formal consequence of these facts. The fact that it has the given two generators $X$ and $Y$ as a right module is clear from the construction of $M$. \hfill $\Box$

Let abbreviate $C_{(\alpha, \beta)} \in A_2$ by $C$; and $C_{(\alpha+2, 1)}$ and $C_{(2, \beta+2)}$ by $C_1$ and $C_2$ respectively. To promote the right action on $B^2 \Theta_B$ to an action by $A_1$, we introduce the following actions:

\[ \begin{align*}
\delta^1_1(X, C_1) &= U_\alpha \otimes Y \quad \delta^1_1(Y, C_2) = U_\beta \otimes X \\
\delta^1_2(X, C_1 \cdot C_2) &= C \otimes Y \cdot U_2 \quad \delta^1_2(Y, C_1 \cdot C_2) = C \otimes X \cdot U_1 \\
\delta^1_3(X, C_1, C_2) &= C \otimes X \quad \delta^1_3(Y, C_2, C_1) = C \otimes Y.
\end{align*} \]

These actions are illustrated in the following picture:

\[ (7.6) \]

(Here the arrow labels with $U_2$ and $U_1$ alone represent $\delta^1_1$, actions where the outgoing algebra element is 1.)

We denote the result by $A^2 \Theta_{A_1}$.

Lemma 7.7. The operations described above make $A^2 \Theta_{A_1}$ into a type $DA$ bimodule, with the generating set described in Lemma 7.6

Proof. This is straightforward. \hfill $\Box$

Consider the map $h^1 : \Theta \to \Theta$ determined on the generating set $X \cdot a$ and $Y \cdot a$ for $a \in \Gamma$) by

\[ \begin{align*}
h^1(Xa) &= \begin{cases} 
Y a' & \text{if there is a } a' \in \Gamma \text{ with } a = U_1 a' \\
0 & \text{otherwise}
\end{cases} \\
h^1(Ya) &= \begin{cases} 
X a' & \text{if there is a pure } a' \in \Gamma \text{ with } a = U_2 a' \\
0 & \text{otherwise}
\end{cases}
\]

Consider $Q \subset \Theta$, the two-dimensional vector space spanned by $XL_1$ and $YR_2$. Observe that $\delta^1_1|_Q$ on $Q$ is identically 0; so we have an inclusion of type $D$ structures $i^1 : Q \to \Theta \subset A_2 \otimes \Theta$, and a projection $\pi^1 : \Theta \to Q \subset A_2 \otimes Q$. 
Lemma 7.8. For the above operators, we have the identities:

\[(\pi^1 \circ i^1) = \text{Id}_Q, \quad i^1 \circ \pi^1 = \text{Id}_\Theta + dh^1, \quad h^1 \circ h^1 = 0, \quad h^1 \circ i^1 = 0, \quad \pi^1 \circ h^1 = 0.\]

Proof. This is a straightforward verification. 

Proof of Theorem 7.4. Apply homological perturbation theory and Lemma 7.8 to give \(XL_1 \oplus YR_2\) the structure of a type DA bimodule. It is straightforward to verify that induced structure is the one stated in Theorem 7.4. To this end, it is helpful to draw the following diagram, where we have written \(C_{\{1, \alpha + 2\}}, C_{\{2, \beta + 2\}}\) and \(C_{\{\alpha, \beta\}}\) respectively:

The horizontal arrows here indicate the map \(h^1\), and other arrows indicate the actions on \(A_1 \Theta A_1\). To keep the picture clean, we have suppressed some of the further arrows that follow from the shown ones by right translation in \(A_1\). For example, the label on the arrow \(Y \to X\) by \(U_\beta \otimes C_2\) also implies an algebra action.
By homological perturbation theory, the $A_\infty$ operations induced on the $XL_1\oplus YR_2$ are specified by closed paths starting and ending at $XL_1$ and $YR_2$, composed of arrows that alternate between algebra operations and the homotopy operator.

In the statement of Theorem 7.4, we described only the $A_\infty$ operations formed by standard sequences; but there are others, such as $\delta^2_1(XL_1, R_1, C_2L_1) = U_\beta \otimes XL_1$.

The above homological perturbation theory in fact gives all $A_\infty$ operations, though the statement is a little complicated, and unnecessary for our present purposes; we leave it to the interested reader to work out.

7.4. Another trident relation. We will give a trident relation analogous to the one from Section 5.3; compare Proposition 5.5. Whereas this relation involves the type $DD$ bimodule of the minimum from Section 7.1 and the crossing bimodules defined over the dual matched algebras from Section 6, it plays an important role in the construction of the general $DA$ bimodule of a minimum, Section 7.5.

Lemma 7.9. Fix integers $0 \leq k \leq 2n + 1$, and a matching $M_1$ on $\{1, \ldots, 2n\}$.

Let

\[ M_2 = \phi_{c+1}(M_1) \cup \{c + 1, c + 2\} \quad M_3 = \tau_c(M_2) \]

\[ M_4 = \tau_{c+1}(M_3) \]

\[ A_1 = A(n, k, M_1), \quad A_1' = A'(n + 1, 2n + 2 - k, M_2) \]

\[ A_3 = A'(n + 1, 2n + 2 - k, M_3) \quad A_3' = A'(n + 1, 2n + 2 - k, M_4) \]

There is homotopy equivalence of graded bimodules:

\[ A_1^c N_{A_2^c} \boxtimes A_2^c \cdot A_1 U_{c+1} \simeq A_4^c \mathcal{P}^{c+1}_{A_4^c} \boxtimes A_1^c U_c \]

Proof. For notational simplicity, suppose that $c = 1$. To ease computation, we use the homotopy equivalent model for $A_4 \cdot A_1 \cdot U_1$ containing the term $U_1 E_2 \otimes 1$ rather than $E_1 U_2 \otimes 1$ suggested by Equation (7.3). (Note also that we have reversed the two tensor factors from Equation (7.3).)

A straightforward computation shows that $A_4^c \mathcal{P}_A^2 \boxtimes A_1^c \cdot U_1$ is the bimodule whose arrows are as given in Equation 5.10, along with self-arrows of the $U_1 E_2 \otimes 1$, $E_2 \otimes U_1$, $E_{j+2} \otimes R_j$ for $j = 1, \ldots, 2n$, $E_{j+2} \otimes U_j$ for $j = 1, \ldots, 2n$, $[E_m, E_{\ell+2}] \otimes C_{m, \ell}$ for all $\{m, \ell\} \in M_1$; and additional self-arrows of the form $[E_{\beta+2}, E_1] \cdot U_1$ and $[E_1, E_{n+2}] \cdot [E_3, E_{\beta+2}] \otimes C_{\alpha, \beta}$.

A symmetric computation reduces $A_1^c N_{A_2^c} \boxtimes A_2^c \cdot A_1 U_{c+1}$ to the same bimodule. \hfill \square

7.5. The general case. We have so far defined $U^c$ for $c = 1$. We can define $U^c$ in general by the following inductive procedure. We begin by specifying the algebras. Fix integers $n$ and $k$ with $0 \leq k \leq 2n + 1$. As before, fix a matching $M_1$ on $\{1, \ldots, 2n + 2\}$ that does not match $c$ and $c + 1$, so we have $\alpha, \beta \in \{1, \ldots, 2n\}$ so that $\{c, \phi_c(\alpha)\}, \{c + 1, \phi_c(\beta)\} \in M_1$. There is an induced matching $M_2$ on $\{1, \ldots, 2n\}$ consisting of $\{\alpha, \beta\}$, and all pairs $\{i, j\}$ with $\{i, j\} \cap \{\alpha, \beta\} = \emptyset$ so that $\phi_c(i)$ and $\phi_c(j)$ are matched in $M_1$. Let

\[ A_2 = A(n, k, M_2) \quad A_3 = A(n + 1, k + 1, \tau_{c-1}(M_1)) \quad A_4 = A(n + 1, k + 1, \tau_c \circ \tau_{c-1}(M_1)), \]
as indicated in Figure 6.

![Figure 6. Algebras used in the construction of $\mathcal{U}^c$ from $\mathcal{U}^{c-1}$.](image)

With the algebras in place, the bimodule is defined inductively by the formula:

$$A_2 \mathcal{U}_{A_1}^c = A_2 \mathcal{U}_{A_1}^{c-1} \bigotimes A_4 \mathcal{P}_{A_4}^c \bigotimes A_3 \mathcal{P}_{A_3}^{c-1}.$$  

The $DA$ bimodule of a minimum defined as above corresponds to the $DD$ bimodule of a minimum defined in Section 7.1, according to the following result. Note that both the statement and proof follow analogously to [19, Proposition 9.5]).

**Theorem 7.10.** Fix integers $n$ and $k$ with $0 \leq k \leq 2n+1$, and $c \in 1, \ldots, 2n-1$, and let $A_1 = A(n+1, k+1, M_1)$, $A_2 = A(n, k, M_2)$, where $M_1$ does not match $c$ and $c+1$, and $M_2$ is the induced matching, as in the beginning of Section 7.1. The above defined type $DA$ bimodule $\mathcal{U}^c$ is standard, and it is is dual to $\mathcal{U}_c$, in the sense that

$$(7.8) \quad A_2 \mathcal{U}_{A_1}^c \otimes A_1.A_1 \mathcal{K} \simeq A_2.A_1 \mathcal{U}_c.$$  

**Proof.** The verification of Equation (7.8) proceeds by induction on $c$, and the basic case $c = 1$ is Lemma 7.5. For the inductive step, with algebras chosen as in Figure 6 (choosing $A_i'$ to be the algebra connected to $A_i$ via the canonical $DD$ bimodule), we compute:

$$A_2 \mathcal{U}_{A_1}^c \otimes A_1.A_1' \mathcal{K} \simeq A_2 \mathcal{U}_{A_1}^{c-1} \bigotimes A_4 \mathcal{P}_{A_4}^c \bigotimes \left( A_3 \mathcal{P}_{A_3}^{c-1} \bigotimes A_1.A_1' \mathcal{K} \right)$$

$$\simeq A_2 \mathcal{U}_{A_1}^{c-1} \bigotimes A_4 \mathcal{P}_{A_4}^c \bigotimes \left( A_1'.\mathcal{P}_{A_1}^{c-1} \bigotimes A_3.A_3 \mathcal{K} \right)$$

$$\simeq A_2 \mathcal{U}_{A_1}^{c-1} \bigotimes A_1'.\mathcal{P}_{A_1}^{c-1} \bigotimes \left( A_3 \mathcal{P}_{A_3}^c \bigotimes A_3.A_1' \mathcal{K} \right)$$

$$\simeq A_2 \mathcal{U}_{A_1}^{c-1} \bigotimes A_1'.\mathcal{P}_{A_1}^{c-1} \bigotimes \left( A_4 \mathcal{P}_{A_4}^c \bigotimes A_3.A_1' \mathcal{K} \right)$$

$$\simeq A_1'.\mathcal{P}_{A_1}^{c-1} \bigotimes A_3 \mathcal{P}_{A_3}^c \bigotimes \left( A_2 \mathcal{U}_{A_1}^{c-1} \bigotimes A_1.A_1' \mathcal{K} \right)$$

$$\simeq A_1'.\mathcal{P}_{A_1}^{c-1} \bigotimes A_3 \mathcal{P}_{A_3}^c \bigotimes \left( A_1'.A_2 \mathcal{U}_{A_1}^{c-1} \bigotimes \mathcal{U}_c \right)$$

$$\simeq A_1'.A_2 \mathcal{U}_c,$$

using associativity of $\bigotimes$ (bearing in mind that the bimodules associated to a crossing are always bounded), Proposition 6.3, the trident relation (Lemma 7.9), the
inductive hypothesis, and the fact that $\mathcal{P}$ and $\mathcal{N}$ are inverses (Lemma 6.4); see Figure 7

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{Verifying the inductive step in Equation (7.8). Boxed components correspond to type $DD$ bimodules, and unboxed ones correspond to type $DA$ bimodules; algebras above the box are of the form $\mathcal{A}'$ while those below are of the form $\mathcal{A}$. Since the order of tensor products is not indicated in these pictures, we have skipped those steps that correspond to associating in different order from the picture.}
\end{figure}

It is clear from its description in Theorem 7.4 that $\mathcal{U}^1$ is a standard type $DA$ bimodule. Since the bimodules for crossings are also standard, it follows from the inductive definition of $\mathcal{U}^c$ and the fact that this property is preserved by tensor products (Lemma 2.11) that $\mathcal{U}^c$ is also standard. $\square$
We give now the construction of the knot invariant, and verify its invariance properties. The basic logic follows as in [19, Section 11].

8. The knot invariant

8.1. A symmetry. Before proceeding to the invariance proof, we establish symmetries in the bimodules constructed earlier will be useful in the invariance proof.

Lemma 8.1. Fix integers $c$, $k$, and $n$, with $1 \leq c \leq 2n + 1$ and $0 \leq k \leq 2n + 1$; and fix a matching $M_1$ on $\{1, \ldots, 2n\}$, and let $M_2 = \phi_c(M_1) \cup \{c, c + 1\}$ (as in Definition 5.1). Let

\[
A_1 = A(n, k, M_1), \quad A_2 = A(n, k, \rho'_c(M_1)) \\
A_3 = A(n + 1, k + 1, M_2), \quad A_4 = A(n + 1, k + 1, \rho'_{n+1}(M_2)),
\]

where $\rho'$ (and $R$ below) is as in Section 10. The following identities hold:

\begin{align}
(8.1) & \ A_1[A_R]_{A_1} \otimes A_1^* \Omega_{A_1} \simeq A_1^* [A_R]_{A_1} \otimes A_1 \Omega_{A_1} \\
(8.2) & \ A_2[A_R]_{A_1} \otimes A_1^* \Omega_{A_1} \simeq A_2^* [A_R]_{A_1} \otimes A_1 \Omega_{A_1}
\end{align}

Also, for any $i = 1, \ldots, 2n - 1$, let $A_5 = A(n, k, \rho'_c(M_1))$ and $A_6 = A(n, k, \tau_i(M_1))$.

We have identities

\begin{align}
(8.3) & \ A_5[A_R]_{A_6} \otimes A_6^* \mathcal{P}_{A_6} \simeq A_5^* [A_R]_{A_6} \otimes A_6 \mathcal{P}_{A_6} \\
(8.4) & \ A_4[A_R]_{A_6} \otimes A_6^* \mathcal{N}_{A_6} \simeq A_4^* [A_R]_{A_6} \otimes A_6 \mathcal{N}_{A_6}.
\end{align}

Proof. It is straightforward to see that

\[
A_1[A_R]_{A_1} \otimes A_1^* \mathcal{K} \simeq A_1^* [A_R]_{A_1} \otimes A_1 \mathcal{K}.
\]

We now verify the identities from the lemma by converting $DD$ bimodules, and appealing to the fact that $\mathcal{K}$ is invertible (Theorem 2.2).

For example, if we let

\[
A_1' = A'(n, 2n + 1 - k, M_1) \quad A_2' = A'(n, 2n + 1 - k, \rho'_c(M_1))),
\]

then it is evident from the description of the type $DD$ bimodule of a positive crossing that it has the symmetry

\[
A_1'[A_R']_{A_1'} \otimes A_1'^* \mathcal{K} = A_1'[A_R]_{A_1} \otimes A_1^* \mathcal{K}.
\]

Equation (8.3) now follows from Proposition 3.4 and Theorem 2.2. Equation (8.4) follows similarly.

Similarly, there obvious symmetry of the type $DD$ bimodule of a maximum

\[
A_4[A_R]_{A_6} \otimes A_6^* [A_R]_{A_4} \simeq A_4^* [A_R]_{A_6} \otimes A_6 \Omega_{A_6}.
\]

Equation (8.1) follows from this symmetry, Proposition 5.4, and Theorem 2.2.

The symmetry of type $DD$ bimodules that gives Equation (8.2),

\[
A_2[A_R]_{A_1} \otimes A_1^* \Omega_{A_1} \simeq A_2^* [A_R]_{A_1} \otimes A_1 \Omega_{A_1}.
\]

is supplied by the map $x \rightarrow (1 + (E_{2n+2-c}E_{2n+3-c} \otimes 1)) \otimes x$. \hfill $\Box$
8.2. Invariants associated to partial knot diagrams. Let $D$ be a planar knot diagram, thought of as lying in the $(x,y)$ plane. Let $D_t$ denote the $y = t$ slice of the diagram. Similarly, given $t_1 < t_2$, let $D_{[t_1,t_2]}$ denote the portion of $D$ with $t_1 \leq y \leq t_2$.

Consider a knot diagram in the plane with a distinguished point. Recall that the diagram is said to be in bridge position if for the projection to the $y$ axis, all critical points are non-degenerate, and all minima, maxima, and crossings project to distinct points on the $y$ axis, and the global minimum is the marked point.

Let $D$ be a planar knot diagram in bridge position. Fix some generic $t \in \mathbb{R}$. Then $D_t$ consists of $2n$ points, which we think of as the points $\{1,\ldots,2n\}$. There is a naturally associated matching $M$ on these points by the rule that $i$ and $j$ are matched if they are connected by an arc in $D \cap (y \geq t)$. Thus, there is a natural algebra associated to the $t$-slice, $A(n,n,M)$, which we denote $A(D_t)$.

Choose a knot planar diagram $D$ in bridge position, and slice it up into pieces $t_1 < \cdots < t_k$ so that the following conditions hold:

- for $i = 1, \ldots, k - 1$, the interval $[t_i, t_{i+1}]$ contains the projection onto the $y$ axis of exactly one crossing or critical point
- for $i = 1, \ldots, k$, $t_i$ is not the projection of any crossing or critical point
- there are no crossings or critical points whose $y$ value is greater than $t_k$ (and so $[t_{k-1}, t_k]$ contains the global maximum)
- there are no crossings below $t_1$, and the only critical point whose $y$ value is smaller than $t_1$ is the global minimum.

Thus, each piece $D \cap (y \in [t_i, t_{i+1}])$ is either a maximum, a minimum, or a crossing; and hence in Sections 5, 7, and 3 we explained how to associate to it a type $DA$ bimodule, with incoming algebra associated to the $t_{i+1}$-slice of the diagram and outgoing algebra associated to the $t_i$-slice of the diagram, for $i \geq 1$. We denote all of these bimodules by $Q(D_{[t_i, t_{i+1}]})$. More generally, for any $1 \leq j \leq m$, define the invariant associate to the partial knot diagram $D_{[j,m]}$ to be the tensor product of the bimodules associated to the various basic pieces, $Q(D_{[i,i+1]})$ with $j \leq i < i+1 \leq m$.

Up to homotopy, this type $DA$ bimodule is independent of the order in which the tensor product is taken.

We can think of $D_{[1,k]}$ as a partial knot diagram with the global minimum removed. Thus, $Q(D_{[1,k]})$ is a type $D$ structure over $A(1,1,\{1,2\})$, which gets a grading from the orientation on the knot.

Let $S = F[u,v]/uv = 0$. The closed knot invariant is constructed via a type $DA$ bimodule $\tilde{t}t\tilde{\Omega}_{A(1,1,\{1,2\})}$, as follows. The bimodule $\tilde{t}t\tilde{\Omega}$ has three generators $X$, $Y$, and $Z$, with

$$ X \cdot I_{(0)} = X, \quad Y \cdot I_{(1)} = Y, \quad Z \cdot I_{(2)} = Z. $$

When $1$ is oriented upwards, $\tilde{t}t\tilde{\Omega}$ is the $DA$ bimodule with $\delta_k^1 = 0$ for $k \neq 2$, and all $\delta_k^2$ are determined by

$$
\begin{align*}
\delta_1^2(Y, L_1) &= u \otimes X, & \delta_1^2(X, R_1) &= u \otimes Y, \\
\delta_1^2(Y, R_2) &= v \otimes Z, & \delta_1^2(X, L_2) &= v \otimes Y, \\
\delta_2^2(X, C_{\{1,2\}}) &= \delta_2^2(Y, C_{\{1,2\}}) = \delta_2^2(Z, C_{\{1,2\}}) = 0.
\end{align*}
$$
When 1 is oriented downwards, we define the actions as above, exchanging the roles of $u$ and $v$.

**Proposition 8.2.** The idempotent of the type $D$ invariant of knot diagram with the minimum removed is restricted by

$$Q(D_{[1,k]}) = I_{(1)} \cdot Q(D_{[1,k]}).$$

**Proof.** Given any generic slice $t$ of a knot diagram, and consider the subalgebra

$$A^{loc}(D_t) = \left( \sum_{\{x|\text{0,2n} \cap x=\emptyset\}} I_x \right) \cdot A(D_t) \cdot \left( \sum_{\{x|\text{0,2n} \cap x=\emptyset\}} I_x \right).$$

We claim that if we restrict the input algebra of $Q(D_{[t_1,t_2]})$ to $A^{loc}(D_{t_2})$, then the output algebra is contained in $A^{loc}(D_{t_1}) \subset A(D_{t_1})$. This is a straightforward verification for the bimodules of a crossing, a maximum, and a minimum defined in the earlier sections; and it clearly, is a property that is preserved under tensor product. Specializing to the case where the partial knot diagram is missing only its global minimum, we arrive at the statement of the proposition. □

For a knot diagram, consider $t \mathcal{U} \otimes Q(D_{[1,k]})$. Restricting the input algebra of $t \mathcal{U}$ to the subalgebra $A^{loc} = A^{loc} \subset A(1,1,\{1,2\}) = A$, we find that the output algebra is contained in the subalgebra $R \subset S$ generated by $U = u^2$ and $V = v^2$. Thus, in view of Proposition 8.2, we can consider the chain complex over $R = \mathbb{F}[U,V]/UV = 0$ defined by

$$C(D) = R R \otimes R t \mathcal{U} A^{loc} \otimes A^{loc} \cdot Q(D_{[1,k]}).$$

Let $J(D)$ denote its homology, which is also a module over $R$. We can now prove that $J(D)$ is an invariant of the underlying oriented knot $\vec{K}$:

**Proof of Theorem 1.1.** To check that $J$ is a knot invariant, we must check that it is invariant under bridge moves and Reidemeister moves. To recall, bridge moves can be classified into the following:

1. Commutations of distant crossings
2. Trident moves
3. Critical points commute with distant crossings
4. Commuting distant critical points
5. Pair creation and annihilation

Invariance under these various moves is verified as it was in [19], by comparing type $DD$ bimodules for partial knot diagrams, and appealing to the invertibility of $K$ (Theorem 2.2).

In fact, commutations of distant crossings was already verified in the braid relations. (See especially Equation (4.2).)

Trident moves correspond to passing a local minimum or maximum through a crossing. For the maximum, this follows from the trident relation for its type $DA$ bimodule (Proposition 5.5) and the fact that the type $DA$ bimodule of a maximum
is dual to its type $DD$ bimodule (Proposition 5.4). For the minimum, we follow a similar logic; we spell out the details presently. First note that
\[
A_3 \cong A_2 \cong A_2 \bigoplus A_1 \cong A_1 \bigoplus A_1,
\]
Similarly,
\[
A_3 \cong A_2 \bigoplus A_1 \cong A_1 \bigoplus A_1.
\]
Thus, the desired trident relation
\[
A_3 \cong A_2 \bigoplus A_1 \cong A_1 \bigoplus A_1,
\]
Commutations between distant crossings and critical points are also straightforward; the identity
\[
P \phi_c(i) \bigotimes \Omega \cong \Omega \bigotimes P^i
\]
is easy to verify on the type $DD$ level (i.e. after tensoring with $\mathcal{K}$); as is the identity
\[
P^i \bigotimes \mathcal{U}\cong \mathcal{U}\bigotimes P^{\phi_c(i)}.
\]
Commutations between distant critical points is also mostly straightforward: for $i < j$, we claim that
\[
\Omega^i \bigotimes \Omega^{-1} \cong \Omega^{j+1} \bigotimes \Omega^i,
\]
\[
\mathcal{U}^i \bigotimes \mathcal{U}^{-1} \cong \mathcal{U}^{j+1} \bigotimes \mathcal{U}^i,
\]
\[
\Omega^{-1} \bigotimes \mathcal{U}^i \cong \mathcal{U}^{-1} \bigotimes \Omega^{j+1},
\]
\[
\mathcal{U}^{j+1} \bigotimes \Omega^i \cong \Omega^i \bigotimes \mathcal{U}^{j+1}.
\]
The first two are clear by tensoring with $\mathcal{K}$. The third identity can be established similarly when $i = 1$, using the explicit form of $\mathcal{U}^1$ from Section 7.2. For general $i$, it follows from the definition of $\mathcal{U}^i$, and commutation of local maxima with crossings. The fourth follows from the third using the symmetry Lemma 8.1.

Arbitrary pair creation and annihilations can be reduced to the case that
\[
(8.6) \quad \mathcal{U}^1 \bigotimes \Omega^2 \simeq \text{Id},
\]
using braidlike Reidemeister 2 moves (Equation (4.1)), and possible reflections (Lemma 8.1). The verification of Equation (8.6) follows from
\[
\mathcal{U}^1 \bigotimes \Omega_2 \simeq \mathcal{K},
\]
which is a straightforward computation. (Compare 11.11.)

Correspondence between the generators of $\mathcal{C}(\mathcal{D})$ and Kauffman states follow from local considerations: the bimodules are all associated to partial Kauffman states, and the tensor over the idempotent ring corresponds to extending partial Kauffman states. This is exactly as in [19].
Before concluding this section, we note the following result, which is both convenient for computations, and will also be conceptually useful. (See the proof of Proposition 9.4.)

**Proposition 8.3.** For any partial knot diagram $\mathcal{D}$, the associated type $DA$ bimodule (or type $D$ structure, in cases where the top of the diagram is empty) $Q(\mathcal{D})$ is a standard type $DA$ bimodule.

**Proof.** The $DA$ bimodules for crossings, maxima, and minima are all standard (Proposition 3.3, Theorem 5.3, and Theorem 7.4). Since tensor products of standard type $DA$ bimodules are standard (Lemma 2.11), the proposition follows. $\square$
Our aim here is to relate the knot invariant constructed here with the ones from [19], to verify the following more detailed version of Proposition 1.3:

**Proposition 9.1.** The complex $C(\bar{K})/V$ is bigraded homotopy equivalent to the complex $C^-(\bar{K})$ from [19, Section 11.4], thought of as a module over $\mathbb{F}[U]$.

The proof compares the two constructions one bimodule at a time.

In [19], we defined an algebra $B(m, k, S)$, where $S \subset \{1, \ldots, m\}$. There is a corresponding knot invariant, defined in $B(2n, n, S)$, where $|S| = n$. Let $S$ be a section of $M$; i.e. each element of $p \in M$, $p \cap S$ consists of one element. There is a map

$$\phi: A(n, k, M) \to B(2n, k, S),$$

with

$$\phi(C_{\{i,j\}}) = U_i \cdot C_j,$$

if $j \in S$. There is an associated type $DA$ bimodule $B_{[\phi], A}$.

There is also an algebra map

$$\psi: B(2n, k, S) \to A'(n, k, M),$$

that induces the identity map on the subalgebra $B(2n, k)$ of both $B(2n, k, S)$ and $A'(n, k, M)$, and that has $\psi(C_j) = E_j$ for all $j \in S$. This gives a bimodule $A'[\psi]_B$.

The bimodules $[\phi]$ and $[\psi]$ can be used to express identities between the various type $DD$ bimodules from [19] with their analogues in the present work. We start with the simplest case, the canonical type $DD$ bimodules:

**Lemma 9.2.** Let $B = B(2n, k, S)$, $B' = B(2n, 2n + 1 - k, \{1, \ldots, 2n\} \setminus S)$, $A = A(n, k, M)$, and $A' = A'(n, 2n + 1 - k, M)$. There is an isomorphism of type $DD$ bimodules:

$$A'[\psi]_B \boxtimes B', B \cong B_{[\phi], A} \boxtimes A', A'.$$

**Proof.** The generators of $B, A'X = A'[\psi]_B \boxtimes B', B$ and $B, A'Y = B_{[\phi], A} \boxtimes A', A'$ are identified, so there are maps in both directions. These identifications are not chain maps, though. The differential in $X$ contains the usual terms $L_i \otimes R_i$ and $R_i \otimes L_i$, and

$$\sum_{j \in S} U_j \otimes E_j + \sum_{j \in S} C_j \otimes U_j;$$

and the differential in $Y$ contains terms of the form

$$\sum_{j=1}^{2n} U_j \otimes E_j + \sum_{j \in S, \{i,j\} \in M} U_i C_j \otimes [E_i, E_j].$$

Consider maps

$$\Phi: B, A'X \to B, A'Y \quad \text{and} \quad \Psi: B, A'Y \to B, A'X$$
given by the formulas
\[
\Phi(x) = x + (\sum_{j \in S} C_j \otimes E_j) \otimes x
\]
\[
\Psi(x) = x + (\sum_{j \in S} C_j \otimes E_j) \otimes x
\]

It is easy to see that both $\Phi$ and $\Psi$ are $DD$-bimodule homomorphisms; and that $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are the identity maps.

Let $D$ be a partial knot diagram with two boundaries $\partial_1 D$ and $\partial_2 D$. Let $B^2 C^-(D)_{B_1}$ be the associated type $DA$ bimodule from [19]; and let $A_2 Q(D)_{A_1}$ denote the bimodules constructed in the present work, as in Section 8.

**Proposition 9.3.** For any partial knot diagram, we have that
\[
B_2[\phi]_{A_2} \boxtimes A_2 Q(D)_{A_1} \simeq B_2 C^-(D)_{B_1} \boxtimes B_1 [\phi]_{A_1}
\]

**Proof.** By associativity of tensor products, it suffices to check Equation 9.1 for the elementary bimodules associated to a crossing, a maximum, and a minimum.

We start with a positive crossing. Recall that in [19], we defined the bimodule $B_1 B_1 P_i$ of a positive crossing, where $B_1 = B(2n, k, S)$ and $B_2 = B(2n, 2n + 1 - k, \{1, \ldots, 2n\} \setminus \tau(S))$. The generators over the idempotents are the same as for the $DD$-bimodule of a positive $P_i$ defined over the matched algebras from Section 3.1.

In fact, the terms of Types (P-1) and (P-4) are the same as those for the $DD$ in Section 3.1; there are no terms of Type (P-3), and instead of terms of Type (P-2), we have terms for each $j = 1, \ldots, 2n$ of the form $C_j \otimes U_{\tau(j)}$ if $j \in S$ and $U_j \otimes C_{\tau(j)}$ if $j \notin S$.

To verify
\[
B_2[\phi]_{A_2} \boxtimes A_2 P_i \simeq B_2 P_i \boxtimes B_1 [\phi]_{A_1},
\]
we tensor both sides with the canonical type $DD$ bimodule $A_1 A_1^i K$, where $A_1^i = A'(n, 2n + 1 - k, M)$, and appeal to Theorem 2.2. By Proposition 3.4, the left side gives
\[
B_2[\phi]_{A_2} \boxtimes A_2 A_i^i P_i
\]
Using Lemma 9.2 and the fact that $P_i$ is also dual to $P_i$ for the previous algebras ([19, Lemma 6.2]), we can identify the right hand side with
\[
B_2 P_i \boxtimes B_1 [\phi]_{A_1} \boxtimes A_1 A_i^i K \simeq B_2 P_i \boxtimes A'_i [\psi]_{B_1} \boxtimes B'_i B_i K
\]
\[
\simeq A'_i [\psi]_{B_1} \boxtimes (B_2 P_i \boxtimes B'_i B_i K)
\]
\[
\simeq A'_i [\psi]_{B_1} \boxtimes B'_i B_2 P_i
\]
Thus, Equation (9.2) is reduced to the identification
\[
B_2[\phi]_{A_2} \boxtimes A_2 A_i^i P_i \simeq A'_i [\psi]_{B_1} \boxtimes B'_i B_2 P_i
\]
To verify that identification, note that the generators of $B_2[\phi]_{A_2} \boxtimes A_2 A_i^i P_i$ correspond to those of $P_i$, with terms in the differential of Types (P-1), (P-2), and (P-4) as before; the terms of Type (P-3) are replaced by terms of the form $U_{\tau(\alpha)} C_{\tau(\beta)} \otimes \sum E_{\alpha, \beta}$ for all $\{\alpha, \beta\} \in M$, with $\beta \in S$. Similarly, the generators for $A'_i [\psi]_{B_1} \boxtimes B'_i B_2 P_i$...
are the same, with differentials of Types (P-1) and (P-4) as before; the only terms
\( U_{\tau(j)} \otimes E_j \) of Type (P-2) appearing now are those for which \( j \notin S \). Also, there
are no terms of Type (P-3). As in the proof of Lemma 9.2, there is an isomorphism \( X \to Y \) (and back) obtained by adding to the identity map the terms
\( \sum_{j \in S} C_{\tau(j)} \otimes E_j \).

The result for \( \mathcal{N}^i \) follows formally, since \( \mathcal{N}^i \) is the inverse of \( \mathcal{P}^i \).

Consider next Equation (9.1) in the case where \( \mathcal{D} \) consists of a single local maximum, i.e. where
\[
C^- (\mathcal{D}) = B_1 \otimes_{B_1} \Omega^c_{B_1} \quad \text{and} \quad Q(\mathcal{D}) = A_2 \otimes_{A_2} \Omega^c_{A_2},
\]
where
\[
B_1 = B(2n, k, S) \quad \text{and} \quad B_2 = B(2n + 2, k + 1, S_2),
\]
for
\[
S_1 = \phi(S) \quad \text{or} \quad S_2 = S_1 \cup \{c + 1\},
\]
where \( S_1 = \phi(S) \). Let \( B'_1 = B(2n, 2n + 1 - k, \{1, \ldots, 2n\} \setminus S) \). Recall that in [19, Section 7], we defined a bimodule \( B_2 \otimes_{B_1} E_c \) with the property that
\[
B_2 \otimes_{B_1} E_c \cong B_2 \otimes_{B_1} E_c.
\]
The generators of \( E_c \) are the same as those for \( \Omega_c \); and the terms in the differential
of \( E_c \) are the terms of Type (Ω-1) and (Ω-5); Instead of terms of Type (Ω-2), we have terms for \( j = 1, \ldots, 2n \) of type \( C_{\phi(j)} \otimes U_j \) if \( j \in S_2 \) or \( U_{\phi(j)} \otimes C_j \) if \( j \notin S_2 \); and instead of the term of Type (Ω-3), we have \( C_c U_{c+1} \otimes 1 \) if \( c \in S_2 \) and \( U_c U_{c+1} \otimes 1 \) if \( c \in S_2 \). Equation (9.1) for \( \Omega^c \) reduces to verifying
\[
B_2 \otimes_{B_1} \Omega^c_{B_1} \cong B_2 \otimes_{B_1} \Omega^c_{B_1}.
\]
This is shown by the usual identification of generators, added to \( C_{\phi(j)} \otimes E_j \) for all \( j \in S_1 \).

Similarly, to verify Equation (9.1) for \( C(\mathcal{D}) = \overline{\mathcal{U}}^c \), we reduce to the identity
\[
B_2 \otimes_{A_2} \Omega^c_{A_2} \cong A_2 \otimes_{A_2} \Omega^c_{A_2}
\]
(where now \( B_2 = B(2n, k, S_1) \) and \( B_1 = B(2n + 2, k + 1, S_1 \cup \{c\}) \) or \( B(2n + 2, k + 1, S_1 \cup \{c + 1\}) \)), which is verified in the same way. \( \square \)

**Proof of Proposition 9.1.** Let \( K \) be a knot and let \( \mathcal{D} \) be the partial knot diagram
with the global minimum removed. When the strand 2 is oriented upwards, \( C^- (\mathcal{D}) \)
is a type \( D \) structure over the algebra
\[
\mathcal{B} = \mathbf{I}_{(1)} \cdot \mathcal{B}(2, 1, \{2\}) \cdot \mathbf{I}_{(1)} \cong \mathbb{F}[U_1, U_2, C_2] / (C_2^2, U_1 U_2)
\]
with \( dC_2 = U_2 \). We can think of \( \mathbb{F}[U] \) as a bimodule \( \mathbb{F}[U] \mathcal{B} \), where the action
by \( U_2 \) and \( C_2 \) are 0, and \( U_1 \) acts on the right as as multiplication by \( U \). By
construction,
\[
C^- (\overline{K}) = \mathbb{F}[U] \mathcal{B} \mathcal{B} C^- (\mathcal{D}).
\]
Similarly, by Proposition 8.2, \( Q(\mathcal{D}) \) is a type \( D \) structure over the algebra
\[
\mathcal{A}^{\text{loc}} = \mathbf{I}_{(1)} \cdot \mathcal{A}(2, 1) \cdot \mathbf{I}_{(1)} \cong \mathbb{F}[U, V, C] / (U V, C^2).
equipped with trivial differential. The complex $\mathcal{RC}(\tilde{K})$ is obtained from $A^{\text{loc}} \otimes Q(D)$ by tensoring with $\mathcal{R} \otimes t \mathcal{R} \otimes \mathcal{A}^{\text{loc}}$, where $\mathcal{R} = \frac{U}{I(U)}$, and the right action by $C$ is defined to vanish, and $U_1$ acts as multiplication by $U$; i.e.

$$
\mathcal{RC}(K) = \mathcal{R} \otimes \mathcal{A}^{\text{loc}} \otimes Q(D).
$$

But

$$
V \mathcal{RC}(\tilde{K}) = \mathcal{F}[U] \otimes A^{\text{loc}} \otimes Q(D)
$$

$$
= \mathcal{F}[U] \otimes A^{\text{loc}} \otimes Q(D)
$$

$$
\simeq \mathcal{F}[U] \otimes \mathcal{C}^{-}(D) = \mathcal{F}[U] \otimes \mathcal{C}^{-}(\tilde{K})
$$

where the last homotopy equivalence uses Proposition 9.3.

9.1. Connected sums. The previous result can be used to verify the following K"unneth property for $H^{-}$ under connected sums. Specifically, we have the following:

**Proposition 9.4.** Let $\tilde{K}_1$ and $\tilde{K}_2$ be two oriented knots. The invariant of their connected sum is given by

$$
\mathcal{C}^{-}(\tilde{K}_1 \# \tilde{K}_2) \cong \mathcal{F}[U] \otimes \mathcal{C}^{-}(\tilde{K}_1) \otimes \mathcal{C}^{-}(\tilde{K}_2);
$$

and hence

$$
H^{-}(\tilde{K}_1 \# \tilde{K}_2) \cong (H^{-}(\tilde{K}_1) \otimes \mathcal{F}[U] H^{-}(\tilde{K}_2)) \oplus \text{Tor}_{\mathcal{F}[U]}(H^{-}(\tilde{K}_1), H^{-}(\tilde{K}_2));
$$

where the Tor appearing above is equipped with its natural shift in bigrading.

**Proof.** Consider a connected sum diagram for $K_1$ and $K_2$, where the connected sum region is taken to be the global minimum and the next minimum above it, as pictured in Figure 8.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{connected_sum_diagram.png}
\caption{Connected sums. Form the disjoint union of the two diagrams on the left; and then form the connected sum as shown.}
\end{figure}

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be disjoint partial knot diagrams for $K_1$ and $K_2$ with both global minima removed. We can form the type $D$ structure $Q(\mathcal{D}_1 \cup \mathcal{D}_2)$ over $\mathcal{A}(2, 2, \{1, 2\}, \{3, 4\})$. By Proposition 8.2, the output algebra of this $D$ module is contained in $I_{\{1,3\}} \cdot \mathcal{A}(2, 2, \{1, 2\}, \{3, 4\}) \cdot I_{\{1,3\}}$. There is a natural identification

$$
(I_{\{1\}} \cdot \mathcal{A}(1, 1, \{1, 2\}) \cdot I_{\{1\}}) \otimes \mathcal{F} (I_{\{1\}} \cdot \mathcal{A}(1, 1, \{1, 2\}) \cdot I_{\{1\}})
$$

$$
\cong I_{\{1,3\}} \cdot \mathcal{A}(2, 2, \{1, 2\}, \{3, 4\}) \cdot I_{\{1,3\}}.
$$

Under this identification,

$$
Q(\mathcal{D}_1 \cup \mathcal{D}_2) = Q(\mathcal{D}_1) \otimes \mathcal{F} Q(\mathcal{D}_2).
$$
This can be seen by drawing $D_2$ so that all its crossings and critical points occur above $D_1$ (at which point there are simply two strands extending from $D_2$).

Consider the type $DA$ bimodule $\mathcal{B}^2$ associated to the picture on the left in Figure 9, with input algebra restricted to

$$A = I_{\{1,3\}} \cdot A(2, 2, \{\{1, 2\}, \{3, 4\}\}) \cdot I_{\{1,3\}} \cong \frac{F[U_1, U_2, U_3, U_4, C_{\{1,2\}}, C_{\{3,4\}}]}{U_1U_2 = U_3U_4 = C_{\{1,2\}}^2 = C_{\{3,4\}}^2 = 0}.$$

This bimodule is defined to be the tensor product $\mathcal{B}^1 \boxtimes P^2 \boxtimes P^1 = Q(D_3)$, as shown on the right on the same picture.

![Figure 9. Bimodule for a minimum.](image)

The bimodule associated to the diagram on the left is a tensor product of the bimodules associated to the diagram on the right.

To understand the knot invariant for $K_1 \# K_2$, we must understand how $Q(D_3)$ acts on the standard type $D$ structure $Q(D_1 \cup D_2)$. In fact, since we consider only $C^-(K_1 \# K_2) \simeq C(\bar{K}_1 \# \bar{K}_2)/V$ (in view of Proposition 9.1), it suffices to project $Q(D_3)$ to the part of the output algebra with $w_1 = 0$, and restrict this to above algebra $A$. Split $Q(D_3) = Q \boxtimes P$, where $P$ is the bimodule corresponding to the two crossing regions. The actions in $P$ (with the algebras restricted as above, and $w_3 = 0$ in the output) are all specified in the following diagram:

![Diagram](image)

It follows readily from the description of the bimodule associated to a minimum that the tensor product of this with $Q$ has one generator (coming from $\text{N W}$) and actions

$$U_2^k \otimes U_4^l + U_2^{k+l} \otimes (U_2^k U_4^l, C_{34}^{\otimes k}).$$

Since $Q(D_1 \cup D_2)$ is standard, it follows that an output of $U_2^k$ from $Q(D_1 \cup D_2)$ is converted by this bimodule to an output $U_2^k$, and an output of $U_2^k$ from $Q(D_1 \cup D_2)$ (followed by a string of $C_{34}$ outputs, which exist since the bimodule is standard) is
converted to an output of $U_2^k$; see for example the following diagram when $k = 1$:

(9.3)

It now follows from the above computation that

$$(\mathcal{R}/V) \otimes_{\mathcal{R}} \mathcal{C}(\tilde{K}_1 \# \tilde{K}_2) = \mathcal{F}[U, A(1,1,\{1,2\})] \otimes \mathcal{U}^2 \otimes (\mathcal{Q}(\mathcal{D}_1) \otimes \mathcal{Q}(\mathcal{D}_2)) \cong (\mathcal{C}(\mathcal{D}_1)/V) \otimes_{\mathcal{F}[U]} (\mathcal{C}(\mathcal{D}_1)/V).$$

The conclusion about $\mathcal{C}^-$ now follows from Proposition 9.1. The statement on the homological level follows from the universal coefficient theorem. $\square$
10. Symmetries

We describe some natural symmetries of our knot invariant.

Let $S$ and $R$ be as in Section 1.

**Proposition 10.1.** Let $\tilde{K}$ be an oriented knot. Then, there is an isomorphism of bigraded $R$-modules $S(J(\tilde{K})) \cong J(-\tilde{K})$.

**Proof.** On $C(\tilde{K})$, the local Alexander gradings (cf. Figure 1) all change by $-1$ when we reverse orientation. Moreover, the roles of the $U$ and $V$ variables are exchanged. The complexes otherwise remain the same. Taking homology gives the claimed symmetry. □

Taking mirrors also gives a symmetry. We begin with the following observation:

**Proposition 10.2.** Under the identifications of algebras

$$A(n,k_1,M) \cong A(n,k_1,M)^{op} \quad \text{and} \quad A'(n,k_2,M) \cong A'(n,k_2,M)^{op}$$

(cf. Equation (2.10)), there is a corresponding identification of type $DD$ bimodules

$$K^{op} \cong K \quad \Omega^op_c \cong \Omega_c \quad \tilde{U}_c^{op} \cong \tilde{U}_c \quad (P_i)^{op} \cong N_i \quad (N_i)^{op} \cong P_i$$

Similarly, there is an identification of $DA$ bimodules

$$\Omega^c \cong (\Omega)^c \quad \tilde{U}_c^{op} \cong \tilde{U}_c \quad (P_i)^{op} \cong P_i \quad (N_i)^{op} \cong P_i$$

**Proof.** The identification for $K$ is clear.

The identification $\Omega^op_c \cong \Omega_c$ switches the roles of $X$ and $Y$, and fixes $Z$. For each pair of generators in Equation (5.6), there are two arrows, and the symmetry switches those two arrows; observe that if one arrow is labeled by $a \otimes b$, then the other is labeled by $o(a) \otimes o(b)$.

There is identification $\tilde{U}_c^{op} \cong \tilde{U}_c$ is defined similarly.

The identity $(P_i)^{op} \cong N_i$ follows from the definition of $N_i$ from Section 3.1. This completes the verification of Equation (10.1).

Equation (10.2) is now a formal consequence of these identities. □

The above proposition has the following immediate consequence:

**Proposition 10.3.** If $\tilde{K}$ is an oriented knot and $m(\tilde{K})$ denotes its mirror, obtained by reversing all of the crossings in a diagram for $\tilde{K}$. Then, $C(m(\tilde{K})) \cong \text{Hom}(C(\tilde{K}), R)$. □
11. On the module structure of the knot invariant

We turn our attention now to verifying the claims from Section 1.1. A key step is a crossing change morphism established in Section 11.1. Consequences are derived in Section 11.2.

11.1. Crossing change morphisms.

Proposition 11.1. Let \( K_+ \) and \( K_- \) be two knots represented by knot projections \( \mathcal{D}_+ \) and \( \mathcal{D}_- \), that differ in a single crossing, which is positive in \( \mathcal{D}_+ \) and negative in \( \mathcal{D}_- \). Then, there are maps

\[
c_+: \mathcal{J}(K_-) \to \mathcal{J}(K_+) \quad \text{and} \quad c_-: \mathcal{J}(K_+) \to \mathcal{J}(K_-),
\]

where \( c_- \) preserves bidegree and \( c_+ \) is of degree \((-1, -1)\), so that \( c_+ \circ c_- = U \) and \( c_- \circ c_+ = U \).

The above proposition hinges on the following local result, stated in terms of the type DD bimodules \( P_i = A_2 \times A'_1 \) and \( N_i = A_2 \times A'_1 \) in the notation of Section 3.1. For any \( j \in \{1, \ldots, 2n\} \), \( U_j \otimes 1 \) times the identity map gives bimodule homomorphisms

\[
U_j \otimes 1: P_i \to P_i \quad \text{and} \quad U_j \otimes 1: N_i \to N_i.
\]

Lemma 11.2. There are bimodule homomorphisms

\[
\phi_-: P_i \to N_i \quad \text{and} \quad \phi_+: N_i \to P_i,
\]

so that \( \phi_+ \circ \phi_- \) and \( \phi_- \circ \phi_+ \), thought of as endomorphisms of \( P_i \) and \( N_i \) respectively, are homotopic to \( U_{i+1} \otimes 1 \) times the corresponding identity maps.

Proof. We write the formulas when \( i = 1 \); the general case is obtained with minor notational changes. In the diagram below, \( P_1 \) is represented by the top row, \( N_1 \) by the bottom row; arrows within each row represent the differentials (suppressing all self-loops within each generator type; i.e. terms of Type (P-1), (P-2), and (P-3), in the notation of Section 3.1); and the maps from the top row to the bottom represent components of a map \( \phi_-: P_1 \to N_1 \).

(Both terms labelled \( W_+ \) are identified in the above diagram as are both terms labelled \( W_- \).) In the next diagram, \( N_1 \) is represented by the top row, \( P_1 \) by the...
where the first equation is to be taken as endomorphisms of \( \phi \) of a map bottom row, and the maps from the top row to the bottom represent components of a map \( \phi_+ : N_1 \to P_1 \).

Finally, define

\[
\begin{align*}
  h_+(N_+) &= h_+(W_+) = h_+(E_+) = 0 \\
  h_+(S_+) &= (L_2 \otimes 1) \otimes E_+
\end{align*}
\]

and

\[
\begin{align*}
  h_-(N_-) &= h_-(W_-) = h_-(S_-) = 0 \\
  h_-(E_-) &= (R_2 \otimes 1) \otimes S_-
\end{align*}
\]

It is straightforward to verify that

\[
\begin{align*}
  dh_+ &= \phi_+ \circ \phi_- + (U_2 \otimes 1) \text{Id}_{P_1} \\
  dh_- &= \phi_- \circ \phi_+ + (U_2 \otimes 1) \text{Id}_{N_1},
\end{align*}
\]

where the first equation is to be taken as endomorphisms of \( P_1 \) and the second as endomorphisms of \( N_1 \).

\begin{remark}
Observe that there is another map \( \phi_- : P_1 \to N_1 \) with the property that \( \phi_+ \circ \phi_- \) and \( \phi_- \circ \phi_+ \) are chain homotopic to \( U_1 \otimes 1 \) times the corresponding identity maps. This map is obtained by modifying the definition of \( \phi_- \), switching the roles of \( W_+ \) and \( E_+ \), the strands 1 and 2, and \( L \) and \( R \).
\end{remark}

\begin{proof}[Proof of Proposition 11.1]
Note that Lemma 11.2 is stated for type \( DD \) bi-modules; but tensoring with the inverse of \( K \) gives the corresponding maps

\[
\phi^+ : N^i \to P^i \quad \phi^- : P^i \to N^i
\]

so that \( \phi^+ \circ \phi^- \) and \( \phi^- \circ \phi^+ \) are homotopic to the bimodule endomorphism of \( P^i \) (and \( N^i \)), \( t : P^i \to P^i \) (and \( t : N^i \to N^i \)) with \( t_1^j(x) = U_{i+1} \otimes x \) and \( t_2^j = 0 \) for all \( j > 1 \), which we call simply “multiplication by \( U_{i+1} \)”, and denote \( T^{U_{i+1}} \).

For simplicity, we draw the diagram for \( D_+ \) so that the distinguished crossing feeds into a local minimum, which is to the lower left of the crossing, and to the global minimum, which is to the lower right; and so that that both strands through the distinguished crossing are oriented upwards (i.e. so that the invariant for \( D_+ \) uses the bimodule \( P^i \) where \( D_- \) uses \( N^i \)), as shown in Figure 10.

Thus,

\[
\mathcal{C}(D_+) = t\bar{U} \bar{\otimes} U^1 \otimes P^2 \otimes Q(D) \quad \text{and} \quad \mathcal{C}(D_-) = t\bar{U} \otimes U^1 \otimes N^2 \otimes Q(D).
\]
Moreover, for the given crossing change, the variable $U_{i+1}$ corresponds to $U_3$ in the output algebra $\mathcal{P}^2$ (resp. $\mathcal{N}^2$), which is connected to $U_1$ in the output algebra for $\mathfrak{U}^1$, which in turn corresponds to the variable $U$ for $\mathcal{Q}(\mathcal{D}_+)$ (resp. $\mathcal{Q}(\mathcal{D}_-)$).

We claim that the bimodule endomorphisms of $\mathfrak{U}^1 \boxtimes \mathcal{P}^2$ given by $\operatorname{Id}_{\mathfrak{U}^1} \boxtimes T_{\mathcal{P}^2}^{U_3}$ coincides with the endomorphism $T_{\mathfrak{U}^1 \boxtimes \mathcal{P}^2}^{U_1}$. To see this, recall (see for example [9, Figure 5]) that $\operatorname{Id}_{\mathfrak{U}^1} \boxtimes T_{\mathcal{P}^2}^{U_3}$ is defined by

By the construction of $\mathfrak{U}^1$, the only non-zero operation $\delta^1_1(x, a_1, \ldots, a_{\ell-1})$ with some $a_i = U_3$ is $\delta^1_2(x, U_3) = U_1 \otimes x$; i.e. the only non-zero term of the above type is

verifying the claim.
Feeding the $U_1$ action into $t\Theta$ gives the algebra element $U$.

It follows from these considerations that the map

$$c_- = \text{Id}_{\mathcal{D}} \boxtimes \text{Id}_{\mathcal{D}} \boxtimes \phi_- \boxtimes \text{Id}_{\mathbb{Q}} : \mathcal{C}(\mathcal{D}_+) \to \mathcal{C}(\mathcal{D}_-),$$

composed with

$$c_+ = \text{Id}_{\mathcal{D}} \boxtimes \text{Id}_{\mathcal{D}} \boxtimes \phi_+ \boxtimes \text{Id}_{\mathbb{Q}} : \mathcal{C}(\mathcal{D}_-) \to \mathcal{C}(\mathcal{D}_+)$$

is chain homotopic to multiplication by $U$.

The grading properties of the maps follow immediately from the local descriptions of the maps $\phi_+$ and $\phi_-$ from Lemma 11.2 and the local formulas for the gradings from Figure 1.

11.2. Extracting numerical knot invariants. The chain complexes $\mathcal{C}(\mathcal{D})$ lead to natural numerical knot invariants in the same way that knot Floer homology gives rise to numerical invariants; see for example [15] and [22]. We recall the construction (standard in knot Floer homology), reformulated slightly to fit with the perspective of chain complexes over $\mathcal{R}$ stated in the introduction.

If $W$ is any $\mathcal{R}$-module, and let $W = \bigoplus_{s \in \mathbb{Z}} W_s$ be its splitting according to its Alexander grading, we can invert $U$ to form a new module, which we can think of as $W \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}]$, equipped with the left $\mathcal{R}$ action. Note that the action by $V$ on $W \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}]$ is trivial.

**Lemma 11.4.** If $W$ is a finitely generated, bigraded $\mathcal{R}$-module, with

$$W \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}] \cong \mathbb{F}[U, U^{-1}],$$

then for all sufficiently large $s$, $W_s$ is $U$-torsion; and for all sufficiently small $s$, there is an element of $W_s$ that is not $U$-torsion.

**Proof.** Fix a finite graded generating set for $W$, and choose $s_0$ greater than $A(x)$ for any $x$ in that generating set. Clearly, for any $s \geq s_0$, $W_s \subset V \cdot W_{s-1}$, and therefore $W_s$ contains only $U$-torsion elements.

By classification of finitely generated modules over the PID $\mathbb{F}[U]$, and Equation (11.1), we can conclude that $W$ has a single free summand $\mathbb{F}[U]$, so from our grading conventions, it follows that $W_s$ is non-torsion in all sufficiently negative grading.

By Lemma 11.4, we can make the following definition:

**Definition 11.5.** Let $W$ be a finitely generated $\mathcal{R}$-module, with the property that $W \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}] \cong \mathbb{F}[U, U^{-1}]$. The $\nu$-invariant of $W$ is the integer

$$\nu_0(W) = -\max\{s \mid U^d \cdot W_s \neq 0 \forall d \geq 0\}.$$ 

**Lemma 11.6.** If $W^1$ and $W^2$ are two finitely generated $\mathcal{R}$-modules with $W^1 \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}] \cong \mathbb{F}[U, U^{-1}]$, and $\phi : W^1 \to W^2$ is a grading-preserving $\mathcal{R}$-module homomorphism that induces an isomorphism after tensoring with $\mathbb{F}[U, U^{-1}]$, then $\nu_0(W^1) \geq \nu_0(W^2)$.

**Proof.** This is an immediate consequence of the fact that a homogeneous, $U$-non-torsion element in $W^1$ has $U$-non-torsion image. \qed
If $C_\ast$ is a finitely generated chain complex over $\mathcal{R}$, then since $\mathcal{R}$ is Noetherian, it follows that $H(C_\ast)$ is also finitely generated as well. If also $H(C_\ast) \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}] \cong \mathbb{F}[U, U^{-1}]$, we can let $\nu(C_\ast) = \nu_0(H(C_\ast))$ and $\tau(C_\ast) = \nu_0(H(C_\ast/V))$. Moreover, if $C^\ast = \text{Mor}(C, R)$, we can form

$$
\nu'(C) = \nu(C^\ast), \quad \tau'(C) = \tau(C^\ast).
$$

Obviously, if $C^1$ and $C^2$ are quasi-isomorphic as bigraded chain complexes over $\mathcal{R}$, then $\nu(C^1) = \nu(C^2)$, $\tau(C^1) = \tau(C^2)$, $\nu'(C^1) = \nu'(C^2)$, and $\tau'(C^1) = \tau'(C^2)$.

Observe also that $\tau(C) \leq \nu(C)$, since the $\mathcal{R}$-module chain map $C \to C/V$ induces a map on homology satisfying the hypotheses of Lemma 11.6, so that

$$
\nu(C) = \nu_0(H(C)) \geq \nu_0(H(C/V)) = \tau(C).
$$

Following [1], let

$$
\epsilon(C) = (\tau(C) - \nu(C)) - (\tau'(C) - \nu'(C)).
$$

Using this function applied to the chain complex computing knot Floer homology, Hom constructs in [2] infinitely many linearly independent smooth concordance homomorphisms to $\mathbb{Z}$ that are non-trivial on topologically slice knots.

**Proof of Proposition 1.4.** According to Proposition 11.1, $J(K) \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}]$ is independent of the choice of knot $K$. For $K$ the unknot, it is straightforward to see that $J(K) \cong R$. □

By Proposition 1.4, we can associate the invariants $\nu$ and $\tau$ to the complex $C(\mathcal{D})$, to extract the corresponding numerical invariants denoted $\nu'(K)$ and $\tau'(K)$ in the introduction. We can also define $\nu(K)$, as in Equation (11.2), to get bordered analogues of Hom’s invariants [1].

**Proposition 1.7** is an easy consequence of Proposition 11.1; compare [11, Chapter 6]:

**Proof of Proposition 1.7.** Since $c_+$ from Proposition 11.1 is a bigraded $\mathcal{R}$-module homomorphism inducing an isomorphism after we tensor with $\mathbb{F}[U, U^{-1}]$, Lemma 11.6 shows that $\nu(K_+) \geq \nu(K_-)$. The map $c_-$ is a bigraded homomorphism, too, after introducing a suitable grading shift. This gives the inequality $\nu(K_-) \geq \nu(K_+) - 1$. Specializing the complexes to $V = 0$ gives the analogous bounds for $\nu'$. □
12. Signs

Our aim here is to define a $\mathbb{Z}$ lift of the constructions from this paper. First, we define the algebras over $\mathbb{Z}$; then we recall some generalities on $\mathcal{A}_\infty$ algebras and bimodules with signs. The basic bimodules associated to crossings, maxima, and minima are then defined over $\mathbb{Z}$. Equipped with these definitions, we define $C_2(D)$ as we defined $C(D)$ before, only now using the modules defined over $\mathbb{Z}$; and we can use it as before to define the invariant of an oriented knot $\tilde{K}$; see Theorem 12.12 below.

12.1. Signs in the algebra. We define a $\mathbb{Z}/2\mathbb{Z}$ grading on all of the algebras, induced by a function on homogeneous generators $a$. We call the variables $C_{(i,j)} \in \mathcal{A}$ for $\{i, j\} \in \mathcal{M}$; and $C_i \in \mathcal{B}(m, k, S)$ for $i \in \mathcal{S}$, and $E_i \in \mathcal{A}'$ for $i \in \{1, \ldots, 2n\}$ exterior variables. All three algebras have a mod 2 grading, called the exterior grading, which, for pure algebra elements $a$, counts the number of exterior variable factors in $a$. Given $t \in \mathbb{Z}/2\mathbb{Z}$, an algebra element is called homogeneous if it can be written as a sum of pure algebra elements, each of which has exterior grading equal to $t$.

The algebras are graded by this exterior grading. This means that multiplication respects the mod 2 grading; $d$ reverses it; and multiplication and differentiation satisfy the Leibniz rule

$$d(a \cdot b) = (da) \cdot b + (-1)^{|a|} a \cdot (db).$$

Thus, the basic idempotents, and the algebra generators $R_i$ and $L_i$ have even exterior grading; while the exterior grading of $C_p$ or $E_i$ (in $\mathcal{A}$ or $\mathcal{A}'$ respectively) is odd.

In particular, the algebra $\mathcal{B}(2n, k)$ (now defined over $\mathbb{Z}$) is in exterior grading 0. This algebra in turn is the quotient of an algebra $\mathcal{B}_0(2n, k)$, defined exactly as in [19, Section 3.1], except now over the base ring $\mathbb{Z}[U_1, \ldots, U_{2n}]$. Note that the generators $L_i$ and $R_j$ for those algebras satisfy the following commutation relations for $|i - j| > 1$:

$$L_i \cdot L_j = L_j \cdot L_i, \quad L_i \cdot R_j = R_j \cdot L_i, \quad R_i \cdot R_j = R_j \cdot R_i.$$

We will also have some use for alternative generators $L'_i$ and $R'_j$ satisfying the following anti-commutation relations for all $i, j \in \{1, \ldots, 2n\}$ with $|i - j| > 1$:

$$L'_i \cdot L'_j = -L'_j \cdot L'_i, \quad L'_i \cdot R'_j = -R'_j \cdot L'_i, \quad R'_i \cdot R'_j = -R'_j \cdot R'_i.$$

Letting

$$f(i, x) = \sum_{\{x \in \mathbb{R}|x < i-1\}} x,$$

these generators are related by

$$I_x \cdot L'_i = (-1)^{f(i,x)} I_x \cdot L_i,$$

$$I_x \cdot R'_i = (-1)^{f(i,x)} I_x \cdot R_i.$$

Note that $L_i \cdot R_i = L'_i \cdot R'_i$ and $R_i \cdot L_i = R'_i \cdot L'_i$.

With these choices, we also have the commutation relations:

$$U_i \cdot a = a \cdot U_i.$$
for all \( a \in \mathcal{A} \). Further commutation relations in \( \mathcal{A} \) are:

\[
C_{i,j} \cdot a = (-1)^{|a|} a \cdot C_{i,j}.
\]

Similarly, for \( \mathcal{B}(m,k,S) \) (which will make a brief appearance in this section),

\[
C_i \cdot a = (-1)^{|a|} a \cdot C_i.
\]

Finally, in \( \mathcal{A}' \),

\[
E_i \cdot E_j = - E_j \cdot E_i
\]

if \( \{i,j\} \notin M \); and

\[
E_i \cdot b = b \cdot E_i
\]

if \( b \in \mathcal{B}(2n,k) \).

The differential in \( \mathcal{B}(m,k,S) \) satisfies \( dC_i = U_i \), and differential in \( \mathcal{A} \) satisfies \( dC_{i,j} = U_i U_j \), and the differential in \( \mathcal{A}' \) satisfies \( dE_i = U_i \).

12.2. **Sign conventions for tensor products.** If \( X, X', Y, \) and \( Y' \) are \( \mathbb{Z}/2\mathbb{Z} \)-graded Abelian groups and \( f: X \to X' \) and \( g: Y \to Y' \) are homomorphisms of Abelian groups with \( \mathbb{Z}/2\mathbb{Z} \)-grading \( |f| \) and \( |g| \) respectively, then we think of their tensor product \( f \otimes g: X \otimes Y \to X' \otimes Y' \) as defined by

\[
(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y),
\]

if \( x \in X \) is homogeneous of degree \( |x| \). With this convention, if \( (X, \partial_X) \) and \( (Y, \partial_Y) \) are chain complexes, the endomorphism \( \partial_X \otimes \text{Id}_Y + \text{Id}_X \otimes \partial_Y \) gives \( X \otimes Y \) the structure of a chain complex.

12.3. **Sign conventions on type \( D \) structures.** A graded \( D \) module is also equipped with a \( \mathbb{Z}/2\mathbb{Z} \) grading, with the property that the map

\[
\delta^1: M \to (\mathcal{A} \otimes M)_{t-1},
\]

satisfies the usual the structure relation

\[
(\mu_1 \otimes \text{Id}_M) \circ \delta^1 + (\mu_2 \otimes \text{Id}_M) \circ (\text{Id} \otimes \delta^1) \circ \delta^1 = 0.
\]

Note that certain signs are introduced by convention from Equation (12.3). Specifically, if \( \{x_i\}_{i=1}^n \) are generators for the type \( D \) module, and

\[
\delta^1(x_i) = \sum_j a_{i,j} \otimes x_j,
\]

then the structure relations state that

\[
da_{i,k} + \sum_j (-1)^{|a_{i,j}|} a_{i,j} \cdot a_{j,k} = 0.
\]

for all pairs \( i,k \in \{1,\ldots,n\} \). This map can be iterated to give a map

\[
\delta: M \to \mathcal{T}^* \mathcal{A} \otimes M;
\]

where here \( \mathcal{T}^* \mathcal{A} = \bigoplus_{i=0}^{\infty} \mathcal{A}^\otimes i \). Explicitly, letting

\[
f(a_1, \ldots, a_\ell) = \sum_{i=1}^{\ell-1} (\ell - i) \cdot |a_i|,
\]

we have that

\[
\delta(x_i) = \sum_{j_1, \ldots, j_\ell} (-1)^{f(a_{i,j_1}, \ldots, a_{i,j_{\ell-1},j_\ell})} a_{i,j_1} \otimes \cdots \otimes a_{j_{\ell-1},j_\ell} \otimes x_{j_\ell},
\]
where the sum is taken over all sequences $j_1, \ldots, j_\ell$ of elements of $\{1, \ldots, n\}$ (including the empty sequence, of length $\ell = 0$, whose contribution is simply $x_i$). Note that the sum defining $\delta$ might have infinitely many non-zero terms; in that case, the map $\delta$ is to be thought of as a map $\delta : M \to \mathcal{T}^\ast(A) \otimes M$, where $\mathcal{T}^\ast(A)$ is the completion of the tensor algebra. We will not make this notational distinction in the following discussion.

Let $A^M$ and $A^N$ be two type $D$ structures. A morphism from $M$ to $N$ is a map $h^1 : M_t \to (A \otimes N)_t$.

The morphism space is equipped with a differential

$$dh^1 = (\mu_1 \otimes \text{Id}_N) \circ h^1 + (\mu_2 \otimes \text{Id}_N) \circ (\text{Id} \otimes \delta^1) \circ h^1 - (\mu_2 \otimes \text{Id}_N) \circ (\text{Id} \otimes h^1) \circ \delta^1,$$

with sign conventions from Equation (12.3). A homomorphism is a map $h^1 : M \to A \otimes N$ as above, with $dh^1 = 0$.

12.4. Sign conventions on $A_\infty$ algebras. For an $A_\infty$ algebra, there is a $\mathbb{Z}/2\mathbb{Z}$ grading so that the maps satisfy

$$\mu_\ell : (A \otimes \cdots \otimes A)_t \to A_{t+\ell-2}.$$

The $A_\infty$ equation with $n \geq 1$ inputs takes the form

$$(12.4) \sum_{n=r+s+t} (-1)^{r+s} \mu_{r+1+t}(\text{Id} \otimes \mu_s \otimes \text{Id}^\otimes t) = 0.$$

The $A_\infty$ relation with $n = 2$ inputs takes the form

$$\mu_1 \mu_2 = \mu_2 (\mu_1 \otimes \text{Id} + \text{Id} \otimes \mu_1);$$

and with $n = 3$ inputs it takes the form

$$\mu_2 (\text{Id} \otimes \mu_2 - \mu_2 \otimes \text{Id}) = \mu_1 \mu_3 + \mu_3 (\mu_1 \otimes \text{Id} \otimes \text{Id} + \text{Id} \otimes \mu_1 \otimes \text{Id} + \text{Id} \otimes \text{Id} \otimes \mu_1).$$

It is easy to check that, with these conventions, the map $\mu^\prime : \mathcal{T}^\ast(A) \to \mathcal{T}^\ast(A)$ defined by

$$\mu^\prime_n(a_1, \ldots, a_n) = \sum_{n=r+s+t} (-1)^{r+s} \text{Id}^{\otimes r} \otimes \mu_s \otimes \text{Id}^{\otimes t}$$

is a differential; i.e.

$$\mu^\prime \circ \mu^\prime = 0.$$

Of course, the formulas are simpler for a $DG$ algebra, where the only non-zero contributions come from $s = 1$ and $2$.

12.5. Sign conventions on $A_\infty$ modules. The $A_\infty$ relation (for $A_\infty$ algebras) has a straightforward generalization to $A_\infty$ modules. The maps

$$m_s : M \otimes A \otimes \cdots \otimes A \to M$$

give a map

$$m : M \otimes \mathcal{T}^\ast(A) \to M.$$

Consider the map

$$m^\prime : M \otimes \mathcal{T}^\ast(A) \otimes \mathcal{T}^\ast(A) \to M \otimes \mathcal{T}^\ast(A)$$
defined by
\[(12.5) \quad m'(x, (a_1 \otimes \cdots \otimes a_{s-1}) \otimes (a_s \otimes \cdots a_{s+t-1})) = (-1)^{st} m_s(x, a_1 \otimes \cdots \otimes a_{s-1}) \otimes (a_s \otimes \cdots a_{s+t-1}).\]

The $A_\infty$ relation can be phrased now as the following relation between maps $M \otimes T^*(A) \to M$:
\[m \circ m' \circ (\text{Id}_X \otimes \Delta) = m \circ (\text{Id}_M \otimes \mu'),\]
where $\Delta: T^*(A) \to T^*(A) \otimes T^*(A)$ is the map induced by
\[\Delta(a_1 \otimes \cdots \otimes a_i) = \sum_{j=0}^{i} (a_1 \otimes \cdots \otimes a_j) \otimes (a_{j+1} \otimes \cdots \otimes a_i).\]

The $A_\infty$ condition is equivalent to the condition that the endomorphism
\[(12.6) \quad \partial_M = m'_M \circ (\text{Id}_M \otimes \Delta) - \text{Id}_M \otimes \mu'\]
is a differential on $M \otimes T^*(A)$.

Concretely, if $M$ is an $A_\infty$ module over a DG algebra (i.e. with $\mu_n = 0$ for all $n \geq 3$), the operations
\[m_t: (M \otimes A^{\otimes t-1})_t \to M_{t+t-2} = M_{t+t} \]
satisfy the following $A_\infty$ relations (for integers $n \geq 1$ and arbitrary homogeneous elements $x \in M$ and homogeneous $a_1, \ldots, a_{n-1} \in A$):
\[0 = \sum_{i=0}^{n-1} (-1)^i m_{i+1}(m_{n-i}(x, a_1, \ldots, a_{n-i-1}), a_{n-i}, \ldots, a_{n-1}) \]
\[+ \sum_{i=1}^{n-2} (-1)^i m_{n-1}(x, a_1, \ldots, \mu_2(a_i, a_{i+1}), a_{i+2}, \ldots, a_{n-1}) \]
\[+ (-1)^{n-1} \sum_{j=1}^{n-1} (-1)^{|x|+|a_1|+\cdots+|a_{j-1}|} m_n(x, a_1, \ldots, a_{j-1}, \mu_1(a_i), a_{i+1}, \ldots, a_{n-1}).\]

Signs for morphisms of $A_\infty$-modules are constructed similarly. Fix an $A_\infty$-algebra $A$, and $A_\infty$ modules $M_A$ and $N_A$. A homomorphism $f: M_A \to N_A$ is a collection of maps $\{f_i: M \otimes A^{\otimes t-1} \to N\}_{t=1}^n$ satisfying $A_\infty$ relations. To phrase those relations, view $f$ as giving rise to a map $f: M \otimes T^*(A) \to M$, and define
\[f': M \otimes T^*(A) \otimes T^*(A) \to N \otimes T^*(A)\]
by
\[f'(x, (a_1 \otimes \cdots \otimes a_{s-1}) \otimes (a_s \otimes \cdots a_{s+t-1})) = (-1)^{st} f_s(x, a_1, \ldots, a_{s-1}) \otimes (a_s \otimes \cdots a_{s+t-1}).\]

The $A_\infty$ relation takes the form
\[(f \circ m'_M) \circ (\text{Id}_X \otimes \Delta) + (m_N \circ f') \circ (\text{Id}_X \otimes \Delta) - f \circ (\text{Id}_X \otimes \mu') = 0;\]
i.e.

\[ \begin{align*}
\Delta m' + f' - f & = 0; \\
\end{align*} \]

or, equivalently, the map

\[ f' \circ (\text{Id} \otimes \Delta) : (M \otimes T^*(A), \partial_M) \to (N \otimes T^*(A), \partial_N) \]

is a chain map.

12.6. The \( A_\infty \) tensor product with signs. Given an \( A_\infty \) module \( X_A \) and a type D structure \( {}^A Y \), we can form their tensor product \( X_A \otimes^A Y \), defining (as in the unsigned case, bearing in mind the conventions from Equation (12.3))

\[ \partial(x \otimes y) = (m \otimes \text{Id}_Y) \circ (\text{Id}_X \otimes \delta). \]

As in the unsigned case, a little care must be taken: the \textit{a priori} infinite sums implicit in the above definition must be guaranteed to be finite by some boundedness hypothesis.

To make Equation (12.7) more concrete, suppose that \( {}^A Y \) is given with a basis \( \{ y_i \}_{i=1}^{n} \), so that

\[ \delta^1(y_i) = \sum_j a_{i,j} \otimes y_j, \]

and let

\[ e(x, a_1, \ldots, a_\ell) = \ell|x| + \sum_{i=1}^{\ell} (\ell - i)|a_i|. \]

Then,

\[ \partial(x \otimes y_i) = \sum_{j_1, \ldots, j_\ell} (-1)^{e(x,a_{i,j_1},\ldots,a_{i,j_{\ell-1}},j_\ell)} m_{\ell+1}(x, a_{i,j_1}, a_{j_1,j_2}, \ldots, a_{j_{\ell-1},j_\ell}) \otimes y_{j_\ell}; \]

where the sum is taken over all sequences \( j_1, \ldots, j_\ell \) of elements of \( \{1, \ldots, n\} \) (including the empty sequence, of length \( \ell = 0 \), which contributes \( m_1(x) \otimes y_i \)).

**Proposition 12.1.** If \( X_A \) is an \( A_\infty \) module and \( {}^A Y \) a type D structure satisfying the needed boundedness conditions (e.g. as in Proposition 2.7), then the map \( \partial \) from Equation 12.7 is a differential.

**Proof.** The proof that Equation (12.7) specifies a differential follows as in [19]. For the reader’s convenience, we recall the argument here, with a few remarks concerning signs.

From its construction, \( \delta \) satisfies the structure equation

\[ (\text{Id}_{T^*(A)} \otimes \delta) \circ \delta + (\Delta \otimes \text{Id}_Y) \circ \delta = 0; \]
i.e. $\delta$ is a comodule over the coalgebra $T^*(\mathcal{A})$. The structure equation for a type $D$ structure is equivalent to the condition
\[
(\mu' \otimes \text{Id}_Y) \circ \delta = 0;
\]
i.e. $\delta$ is a differential comodule. This latter verification makes a straightforward usage of the signs appearing in both $\mu'$ and $\delta$. It is similarly easy to verify that:
\[
(Id_X \otimes \delta') \circ (m_s \otimes \text{Id}_Y)(x \otimes (a_1 \otimes \cdots \otimes a_{s-1}) \otimes y)
= (-1)^{st}(m_s \otimes \text{Id}_Y) \circ (Id_{X \otimes \mathcal{A}^{\otimes (s-1)}} \otimes \delta')(x \otimes (a_1 \otimes \cdots \otimes a_{s-1}) \otimes y).
\]
i.e.
\[
(Id_X \otimes \delta) \circ (m \otimes \text{Id}_Y) = (m' \otimes \text{Id}_Y) \circ (Id_{X \otimes T^*(\mathcal{A})} \circ \delta).
\]

Thus,
\[
(m \otimes \text{Id}_Y) \circ (Id_X \otimes \delta) \circ (m \otimes \text{Id}_Y) \circ (Id_X \otimes \delta)
= (m \otimes \text{Id}_Y) \circ (m' \otimes \text{Id}_Y) \circ (Id_{X \otimes T^*(\mathcal{A})} \otimes \delta) \circ (Id_X \otimes \delta)
= (m \otimes \text{Id}_Y) \circ (m' \otimes \text{Id}_Y) \circ (Id_X \otimes \Delta \otimes \text{Id}_Y) \circ (Id_X \otimes \delta)
= \left((m \circ m') \otimes (Id_X \otimes \Delta)\right) \otimes \text{Id}_Y \circ (Id_X \otimes \delta)
= ((m \circ \mu') \otimes \text{Id}_Y) \circ (Id_X \otimes \delta)
= (m \otimes \text{Id}_Y) \circ (Id_X \otimes \mu' \otimes \text{Id}_Y) \circ (Id_X \otimes \delta)
= 0.
\]

\[\square\]

12.7. Bimodules over $\mathbb{Z}$. A type $DA$ bimodule $^A X_B$ is a bimodule over the idempotent rings, equipped with a map $\delta^1_j: X \otimes \mathcal{A}^{\otimes j-1} \to B \otimes X$, which we add up to get a map $\delta^1: X \otimes T^*(\mathcal{A}) \to B \otimes X$, satisfying certain $\mathcal{A}_\infty$ relations, which we state now with sign.

Consider the maps $X \otimes (\mathcal{A}^{\otimes (s-1)}) \otimes (\mathcal{A}^{\otimes t}) \to B \otimes X \otimes \mathcal{A}^{\otimes t}$ defined by
\[
x \otimes (a_1 \otimes \cdots \otimes a_{s-1}) \otimes (a_s \otimes \cdots a_{s+t-1}) \to (-1)^{st}\delta^1_s(a_1 \otimes \cdots \otimes a_{s-1}) \otimes (a_s \otimes \cdots a_{s+t-1}).
\]
These can be added together to give a map
\[
(\delta^1)': X \otimes T^*(\mathcal{A}) \otimes T^*(\mathcal{A}) \to B \otimes X \otimes T^*(\mathcal{A}).
\]
The $\mathcal{A}_\infty$ relations now read:
\[
(\mu_1 \otimes \text{Id}_X) \circ \delta^1 - \delta^1 \circ (Id_X \otimes (\mu^A)') + (\mu_2^B \otimes \text{Id}_X) \circ (Id_B \otimes \delta) \circ (\delta^1)' \circ (Id_X \otimes \Delta) = 0,
\]
as maps from $X \otimes T^*(\mathcal{A}) \to B \otimes X$. 
Concretely, for any integer $i \geq 1$, the $DA$ bimodule relation takes the form

$$0 = (\mu_{1}^{A} \otimes \text{Id}_{X}) \circ \delta_{i}^{1}$$

$$+ (-1)^{i-1} \sum_{j=1}^{i-2} \delta_{j}^{1}(\text{Id}_{X \otimes B^{j-1}} \otimes \mu_{1}^{B} \otimes \text{Id}_{B^{(i-j-1)}})$$

$$+ \sum_{j=1}^{i-2} (-1)^{j} \delta_{j-1}(\text{Id}_{X \otimes B^{j-1}} \otimes \mu_{2}^{B} \otimes \text{Id}_{B^{(i-j-2)}})$$

$$+ \sum_{j=1}^{i} (-1)^{j(i-j)} (\mu_{2}^{A} \otimes \text{Id}_{X}) \circ (\text{Id}_{B} \otimes \delta_{j}^{1}) \circ (\delta_{j}^{1} \otimes \text{Id}_{B^{(i-j)}}),$$

using the conventions of compositions from Equation (12.3). So, for example, if $\delta_{j}^{1}(x, b_{1}, \ldots, b_{j-1})$ contains $a_{1} \otimes y$, and $\delta_{i-j+1}(y, b_{j}, \ldots, b_{i-1})$ contains $a_{2} \otimes z$, the corresponding contribution (to the last term) in the $A_{\infty}$ relation with inputs $(x, b_{1}, \ldots, b_{j-1})$ is $(-1)^{(i+1)(i-j+1)}(a_{1} \cdot a_{2}) \otimes z$. With this understood, we can form the $A_{\infty}$ tensor products as in Section 12.6.

Iterating $\delta_{j}^{1}$, we obtain a map $^{X}\delta : X \otimes T^{\ast}(B) \to T^{\ast}(B) \otimes X$.

If $A$, $B$, and $C$ are DG algebras, and $^{C}Y_{B}$, $^{B}X_{A}$ are type $DA$ bimodules, we define their tensor product $^{C}Y \otimes X_{B}$ to be the $DA$ bimodule with operations given by $^{Y \otimes X}\delta^{1} = ^{Y}\delta^{1} \circ ^{X}\delta$, when suitable boundedness hypotheses are satisfied. For example, we have the following signed version of Proposition 2.7:

**Proposition 12.2.** Let $W_{1}$ be a disjoint union of finitely many intervals joining $Y_{1}$ to $Y_{2}$; and let $W_{2}$ be a disjoint union of finitely many intervals joining $Y_{2}$ to $Y_{3}$. Suppose moreover that $W_{1} \cup W_{2}$ has no closed components, i.e. it is a disjoint union of finitely many intervals joining $Y_{1}$ to $Y_{3}$. Given any two bimodules $^{A_{2}}X_{A_{1}}^{1}$ and $^{A_{3}}X_{A_{2}}^{2}$ adapted to $W_{1}$ and $W_{2}$ respectively, we can form their tensor product $^{A_{3}}X_{A_{2}}^{2} \otimes ^{A_{2}}X_{A_{1}}^{1}$ with actions $^{X_{3} \otimes X_{1}}\delta^{1}$ as defined above; and moreover, it is a bimodule that is adapted to $W_{1} \cup W_{2}$.

**Proof.** The proof from Proposition 2.7 shows that sums defining $^{X_{3} \otimes X_{1}}\delta^{1}$ are in fact finite. The $A_{\infty}$ relation holds by a straightforward application of the usual rules, applying sign conventions as in the proof of Proposition 12.1.

### 12.8. Homological perturbation lemma over $\mathbb{Z}$.

The homological perturbation lemma plays a central role in the construction of the bimodule associated to a minimum, Section 7; where it is used for type $DA$ bimodules. We review here how to introduce signs into this basic result, starting with the case of modules. The proof is standard; see [4] for a general discussion; see also [8, Lemma 9.6].

**Lemma 12.3.** Let $Y_{g}$ be a strictly unital module over $B$ with grading set $S$, and let $Z$ be a chain complex. Suppose that there are chain maps $f : Z \to Y$ (i.e. as the notation suggests, we are forgetting here about the right $B$-action) and $g : Y \to Z$ and a map $T : Y \to Y$ so that $f$ and $g$ preserve gradings, and $T$ preserves Alexander gradings and shifts $\Delta$ grading by $+1$. Suppose moreover that the following identities hold

$$g \circ f = \text{Id}_{Z}, \quad \text{Id}_{Y} + \partial T + T \partial = f \circ g, \quad T \circ T = 0, \quad f \circ T = 0, \quad T \circ g = 0.$$
Then $Z$ can be turned into a strictly unital type $A$ module, denoted $Z_B$; and there is an $A_\infty$ homotopy equivalence $\phi: Z_B \to Y_B$ with $\phi_1 = f$.

**Proof.** By hypothesis, $Z$ is already equipped with an action $m_1$. For $j > 1$, given $\bar{b} = b_1 \otimes \cdots \otimes b_{j-1}$, the operations $m_j$ are specified by the following pictures:

$$m_+(x \otimes \bar{b}) = \begin{array}{ccc} x & \bar{b} & + \\ f & \Delta & f \\ m_+ & m'_+ & m'_+ \\ & & \end{array} + \begin{array}{ccc} x & \bar{b} & + \\ f & \Delta & f \\ m_+ & m'_+ & m'_+ \\ & & \end{array} + \cdots$$

These graphs represent compositions of maps; and the primes indicate that maps are weighted with signs indicated as in Equation (12.4); i.e. $T': Y \otimes A \otimes t \to Y \otimes A \otimes t$ given by $T'(y \otimes a_1 \otimes \cdots \otimes a_t) = (-1)^t \cdot T(y) \otimes (a_1 \otimes \cdots \otimes a_t)$; while $m'$ is as in Equation (12.5); and the subscript $+$ indicates that we require there to be a non-zero number of algebra inputs into the corresponding node. Note that there are no algebra elements moving past the $m_+$-labeled nodes, so for those nodes, $m'_+ = m_+$.

It follows that $m' \circ (\text{Id} \otimes \Delta)$ is represented by the pictures:

$$m(x \otimes \bar{b}) = \begin{array}{ccc} x & \bar{b} & + \\ f & \Delta & f \\ m_+ & m'_+ & m'_+ \\ & & \end{array} + \begin{array}{ccc} x & \bar{b} & + \\ f & \Delta & f \\ m_+ & m'_+ & m'_+ \\ & & \end{array} + \cdots$$

These graphs represent compositions of maps; and the primes indicate that maps are weighted with signs indicated as in Equation (12.4); i.e. $T': Y \otimes A \otimes t \to Y \otimes A \otimes t$ given by $T'(y \otimes a_1 \otimes \cdots \otimes a_t) = (-1)^t \cdot T(y) \otimes (a_1 \otimes \cdots \otimes a_t)$; while $m'$ is as in Equation (12.5); and the subscript $+$ indicates that we require there to be a non-zero number of algebra inputs into the corresponding node. Note that there are no algebra elements moving past the $m_+$-labeled nodes, so for those nodes, $m'_+ = m_+$.
We must verify that the above specified maps satisfy the $A_\infty$ relation. To this end, consider $m \circ (\text{Id} \otimes \mu')(x \otimes \hat{b})$; and think of $\mu'$ as labeling some node just below the input labeled $\hat{b}$ in some tree above. The signs on $T'$, $m'$, and $\mu'$ ensure that if a $\mu'$-labeled node is commuted past an $m'$-marked node (provided this commutation makes sense; i.e. the output of the $\mu'$-marked node does not feed into the input of the $m'$-marked one), then the contributions of the two trees before and after the contribution add up to zero. Commute each $\mu'$-marked node as far down as possible, until it appears immediately above the $m'$-marked node into which its output is channeled (an even number of commutations), in which case the contribution of this tree coincides with the original tree in $m \circ (\text{Id} \otimes \mu')$.

With this observation in place, the proof from [8, Lemma 9.6] applies. Specifically, for trees of the second kind apply the $A_\infty$ relation to the $m'$-labeled node which takes its input from the $\mu'$-labeled node. The $A_\infty$ relation then guarantees that the $m'(x \otimes \mu'(b))$ is the count of trees as shown in the definition of $m'(x \otimes \hat{b})$, except that some $m'$-labeled node is replaced by two consecutive $m'$-labeled nodes, exactly one of which might have no algebra inputs.

Compare with terms in $(m \circ m') \circ (\text{Id}_Y \otimes \Delta)$. This latter contribution counts trees as in the definition of $m'$, except that some $m$-marked node is replaced by a $g$ marked node above an $f$-marked node. Applying the formula $\text{Id}_Y + \partial T + T \partial = f \circ g$, we find that the contribution of these trees can be thought of as the contributions of the following types of trees: trees as in the definition of $m'$, only with some $T'$-marked vertex erased (this is the contribution of the identity map); or trees as in the definition of $m'$, only with one $m'$-marked vertex consisting of $m'_1$ (i.e. with no incoming algebra element). This is the same as the contributions of $m'(x \otimes \mu'(b))$ from above, verifying the $A_\infty$ relation for the claimed $A_\infty$ module.

A morphism $\phi: Z \to Y$ is defined by letting $\phi_1 = f$ and $\phi_k$ with $k > 1$ be counts of trees similar to the ones used to define $m$, except that the final $g$-labeled node is removed. A morphism $\psi: Y \to Z$ is defined similarly, with $\psi_1 = g$ and $\psi_k$ with $k > 1$ be counts of trees as above, only with the initial $f$ node removed. Verification of the $A_\infty$ relations for these morphisms proceeds in a similar manner to the above proof.

For example, if $m_k = 0$ on $Y$ for $k > 2$, then the trees for computing $m_{j+1}$ on $Z$ are obtained by counting the trees appearing as above (without the primes on $T$ and $m$); but weighted by $(-1)^{\eta(j)}$, where

$$\eta(j) = \begin{cases} 0 & \text{if } j \equiv 0 \text{ or } 1 \pmod{4} \\ 1 & \text{if } j \equiv 2 \text{ or } 3 \pmod{4} \end{cases}$$

(12.8)

In the form we will need it, the lemma has the following form:

**Lemma 12.4.** Let $^A Y_B$ be a strictly unital type $DA$ bimodule over algebras $A$ and $B$ with grading set $S$, and let $^A Z$ be a type $D$ structure over $A$. Suppose that there are chain maps $f: ^A Z \to ^A Y$ (i.e. again forgetting here about the right $B$-action) and $g: ^A Y \to ^A Z$ and a type $D$ structure morphism $T: ^A Y \to ^A Y$ so that $f$ and $g$ preserve gradings, and $T$ preserves Alexander gradings and shifts $\Delta$ grading by $+1$. Suppose moreover that the following identities hold

$$g \circ f = \text{Id}_Z, \quad \text{Id}_Y + \partial T = f \circ g, \quad T \circ T = 0, \quad f \circ T = 0, \quad T \circ g = 0.$$
Then $Z$ can be turned into a strictly unital type $A$ module, denoted $Z_B$; and there is an $\mathcal{A}_\infty$ homotopy equivalence $\phi: Z_B \to Y_B$ with $\phi_1 = f$.

The proof is as in Lemma 12.4, with now a multiplication in $A$ appearing along the leftmost factor; see [19, Lemma 2.12).

12.9. Koszul duality and signs. The canonical type $DD$ bimodule is defined as in Section 2.3. As before, its the generators correspond to idempotent states $x = x_1 < \cdots < x_k$ over $\mathcal{A}$; or equivalently, the complementary idempotent $y = y_1 < \cdots < y_k$. We define the exterior grading of the generator associated to the idempotent state $x$ to be given by

$$\|y\| = y_1 + \cdots + y_k.$$  

(12.9)

The algebra element defining the differential is now defined by

$$A = \sum_{i=1}^{2n} (L_i \otimes R_i' + R_i \otimes L_i') - \sum_{i=1}^{2n} U_i \otimes E_i + \sum_{\{i,j\} \in M} C_{\{i,j\}} \otimes \llbracket E_i, E_j \rrbracket \in \mathcal{A} \otimes \mathcal{A}'$$

where

$$\llbracket E_i, E_j \rrbracket = E_i E_j + E_j E_i.$$  

With these choices, $K$ is still a type $DD$ bimodule.

Note that the pure algebra elements form a $Z$-basis for $\mathcal{A}$ and $\mathcal{A}'$. Our candidate inverse module has the form

$$\mathcal{Y}_{\mathcal{A}',\mathcal{A}} = \text{Mor}^A(\mathcal{A} \otimes \mathcal{A}', \mathcal{A} \otimes \mathcal{A}'; \mathcal{K}, \text{Id}_A).$$

It is a free $Z$-module with spanned by element $(a|b)$, where $a \in \mathcal{A}$ is a pure algebra element, $b \in \mathcal{A}$, and $a \in \text{Hom}(\mathcal{A}', Z)$ takes $a$ to 1 and all other pure algebra elements to 0.

**Theorem 12.5.** The canonical type $DD$ bimodule is quasi-invertible, in the sense that the tensor product of $\mathcal{Y}_{\mathcal{A}',\mathcal{A}}$ with $\mathcal{A} \otimes \mathcal{A}'$ over either $\mathcal{A}$ or $\mathcal{A}'$ is homotopy equivalent to the identity type $DA$ bimodules over $\mathcal{A}'$ or $\mathcal{A}$ respectively.

**Proof.** To adapt the proof of Theorem 2.2, our goal is to show that $\mathcal{Y}$ is homotopy equivalent to its $Z$-submodule generated by idempotents. As in Section 2.4, this can be reduced to the case of algebras considered in [19]. Namely, we consider algebras $B_1 = B(2n, n, \emptyset)$ and $B_2 = B(2n, n, \{1, \ldots, 2n\})$. The proof of Lemma 2.3 still applies, showing that $\phi$ is a homotopy equivalence. In adapting the proof, note that Equation (2.5) remains true (with $\llbracket E_i, E_j \rrbracket = E_i \cdot E_j + E_j \cdot E_i$), bearing in mind that now $\alpha(E_i E_j) = -\alpha(E_j E_i) = C_i C_j$; so dually

$$\overline{\alpha}(C_i C_j) = E_i \cdot E_j - E_j \cdot E_i.$$  

Having reduced to the $Z$ analogue of the invertibility of the canonical bimodule over $B_1$ and $B_2$, the proof from [19, Theorem 3.16], showing that $H(\mathcal{Y})$ is a free $Z$-module generated by the idempotents in $\mathcal{A}$. Since $Z$ is free, it follows that $\mathcal{Y}$ is homotopy equivalent to this $Z$-submodule, as needed.  

$\square$
12.10. **The $A_\infty$ tensor product with signs.** Recall that if $A$ and $B$ are $DG$ algebras, $M_A$ is a module, and $A^B X$ is a type $DD$ bimodule, then we can form their tensor product $(M_A \boxtimes A^B X)=^B (M \boxtimes X)$. We explain here how signs work in this construction.

First, $M_A$ can be promoted to a type $DA$ bimodule $B M_A \otimes B$, with the same underlying vector space, and with actions given by

$$
\delta^1_\ell(x, (a_1 \otimes b_1), \ldots (a_{\ell-1} \otimes b_{\ell-1})) = (-1)^s(b_1 \cdots b_{\ell-1})m_\ell(x, a_1, \ldots, a_{\ell-1}),
$$

where

$$
s = \sum_j |b_j| + \sum_{i,j} |a_i||b_j|.
$$

We can think of $M_A \boxtimes A^B X$ as given by $B M_A \otimes B \boxtimes A^B X$, where the signs on the latter tensor product are as specified in Section 12.6.

For example, if $X_A$ is a module and $A^B, A^B P_i$ is the canonical type $DD$ bimodule, and $X$ is a generator with $X \cdot \textbf{I}_y = X$, then $\partial(X \otimes K_x)$ contains a term of the form $((-1)^{|x|}L'_\ell \otimes m_2(X, R_i)$ and another one of the form $-E_i \otimes m_2(X, U_i)$.

12.11. **Standard modules over $\mathbb{Z}$**. The notion of standard modules (as in Section 2.8) has a natural signed analogue. To state it, we modify Definition 2.10, replacing Property (DA-4), with the condition

$$
(12.10)
C^2 \otimes x + \sum_{\ell=0}^\infty (-1)^{\ell}|x| + \ell^2 \delta^1_\ell(x, (C^1, \ldots, C^\ell)) \in B_2 \otimes X.
$$

With this definition, it is now true that the tensor product of two standard type $DA$ bimodules is standard, generalizing Lemma 2.11 to the signed case. To see where the sign comes from, note that

$$
e = e(x, (C^1, \ldots, C^\ell)) = \ell \cdot |x| + \frac{\ell^2 - \ell}{2};
$$

and we multiply $(-1)^e$ by $(-1)^\ell$, to turn each $C^1$ to $-C^1$.

12.12. **Positive crossing with signs**. The exterior grading on the algebra is compatible with an exterior grading on all the bimodules considered earlier.

For example, consider the $DD$ bimodule of a positive crossing from Subsection 3.1, $A^1, A^1 P_i$. Generators of $P_i$ are determined by an idempotent state $y$ in $A^1_\ell$ and a label, which can be $N$, $S$, $W$, or $E$. Let

$$
(12.11)
|N| = |W| = |E| = 0 \quad |S| = -1.
$$

Let the exterior grading of each generator $X$ with $(1 \otimes \textbf{I}_y) \cdot X = X$ be given by $\|y\| + |X|$, where the terms are defined in Equations (12.9) and (12.11) respectively. To obtain a type $DD$ bimodule, the diagram from Equation (3.3) is modified as
follows:

\[
\begin{array}{ccc}
\text{W} & \text{N} & \text{E} \\
U_{i+1} \otimes R_i' + R_{i+1} \otimes L_i' & L_i \otimes L_i' & -L_i \otimes 1 \\
- R_i \otimes U_{i+1} - L_{i+1} \otimes R_i' & 1 \otimes L_i' & \end{array}
\]


together with outside arrows (connecting generators \(X\) of the same type) \((R_j \otimes L_j' + L_j \otimes R_j')(−1)^{|X|}\) for all \(j \in \{1, \ldots, 2n\} \setminus \{i, i+1\}\), \(-U_j \otimes E_{\tau(j)}\) for all \(j = 1, \ldots, 2n\), and \(C(x, y) \otimes [E_{\tau(x)}, E_{\tau(y)}]\) for all \(\{x, y\} \in M\).

For the type \(DA\) bimodule, the exterior grading is specified in Equation (12.11). The differential \(\delta^1\) is determined by

\[
\delta^1_b(E) = R_2 \otimes S, \quad \delta^1_b(W) = -L_1 \otimes S.
\]

For \(b \in \mathcal{B}\), the actions \(\delta^1_b(x, b)\) all give positive multiples (in \(\mathcal{B} \otimes \mathcal{P}^1\)). For \(\delta^1_b(S, a, b)\) if in \(b\) \(U_2\) dominates, then the contribution is positive; while if in \(b\) \(U_1\) dominates, the sign is negative. Recall that in all \(\delta^1\) actions, \(I(a, I(b, Y)) \in \{E, W\}\). When \(I(a, I(b, Y)) = E\) then the sign is \(-1\), when \(I(a, I(b, Y)) = W\) then the sign is \(+1\).

Consider the case where the inputs are not in \(\mathcal{B}\). We modify the rules from Equation (3.10) as follows:

\[
\begin{align*}
\delta^1(X, C_p \cdot a_1) &= (-1)^{|X|} C_{\tau(p)} \cdot \delta^1(X, a_1) \\
\delta^1_b(S, a_1 \cdot C_p, a_2) &= \delta^1_b(S, a_1, C_p \cdot a_2) \\
\delta^1_b(S, C_p \cdot a_1, a_2) &= C_{\tau(p)} \cdot \delta^1_b(S, a_1, a_2).
\end{align*}
\]

for all \(p \in M\). When \(\{1, 2\} \notin M\), the actions must be further modified by adding the following terms (compare Equation (3.11)):

\[
\begin{align*}
\delta^1_b(S, C_2) &\rightarrow -U_{\beta} R_1 \otimes W \\
\delta^1_b(S, C_1) &\rightarrow U_\alpha L_2 \otimes E \\
\delta^1_b(S, U_1 C_2) &\rightarrow -U_{\beta} U_1 L_2 \otimes E \\
\delta^1_b(S, U_1 C_2) &\rightarrow -U_{\beta} U_1 C_2 \rightarrow U_{\beta} U_1 C_2 \otimes \mathcal{E} \\
\delta^1_b(S, C_1 U_2) &\rightarrow U_\alpha R_1 U_2 \otimes W \\
\delta^1_b(S, C_1 U_2) &\rightarrow U_\alpha C_1 R_1 U_2 \otimes W \\
\delta^1_b(S, R_1 C_2) &\rightarrow -U_{\beta} R_1 \otimes N \\
\delta^1_b(S, R_1 C_2) &\rightarrow U_{\beta} C_1 R_1 \otimes N \\
\delta^1_b(S, L_2 C_1) &\rightarrow U_\alpha L_2 \otimes N \\
\delta^1_b(S, L_2 C_1) &\rightarrow U_\alpha L_2 \otimes N \\
\delta^1_b(S, R_1 C_1 U_2) &\rightarrow U_\alpha R_1 U_2 \otimes N \\
\delta^1_b(S, R_1 C_1 U_2) &\rightarrow U_\alpha R_1 U_2 \otimes N \\
\delta^1_b(S, U_1 L_2 C_2) &\rightarrow -U_{\beta} L_2 U_1 \otimes N \\
\delta^1_b(S, U_1 L_2 C_2) &\rightarrow U_{\beta} U_1 L_2 \otimes N
\end{align*}
\]

We now have the following:

**Proposition 12.6.** The operations defined above give \(\mathcal{P}^1\) the structure of a type \(DA\) bimodule (over \(\mathbb{Z}\)), \(\mathcal{A}^{(n,k,M)} \otimes \mathcal{P}^1\). Moreover, \(\mathcal{P}^1\) is standard (with signs specified in Equation (12.10)).
It is easy to verify that $g_1$ and $g_1'$ are homomorphisms of type $DD$ structures.

Along with self-arrows $(-1)X_{(L_k \otimes L_i)^2} - U_{(L_k \otimes L_i)^2}$ for $i \neq 1, 2$, and $C_{(L_k \otimes L_i)^2} \otimes U_{(L_k \otimes L_i)^2}$, consider maps $i_1, \ i_2, \ P_1 \otimes P_2$ and $g_1 : P_1 \otimes P_2 \rightarrow E$.

The proof of the above is obtained by modifying the proof of Proposition 3.3. We also have the following:

Proposition 12.7. Let $A = A(n, k, \gamma(M))$.

$a = A(n, 2n + 1 - k, M)$.

$P_i$ is dual to $P_i$, in the sense that

$A_{P_i} \otimes A_{-k} = A_{-k} \otimes A_{P_i}$.

Proof. This follows as in Proposition 3.4. For example, when $i = 1$ and 2 are not matched, then we find that $P_1 \otimes \mathbb{K}$ is given by:

$A_{P_1} = A(n, k, \gamma(M))$.
12.13. **The maximum with signs.** For the type $DD$ bimodule associated to a maximum $A_2\cdot A_2 \cdot \Omega_c$ from Section 5.1, the generators $P_x$ correspond to idempotent states $x$ for $A_2$. Let

$$(12.15) \quad \gamma(x) = \#(x \cap \{0, \ldots, c-1\}),$$

and define the exterior grading of $P_x$ to be the sum $\gamma(x) + \|y\|$, where the second term is as in Equation (12.9) and $y$ is specified by $(I_x \otimes I_y) \cdot P_x = P_x$ (i.e. $y = \psi'(x)$ as in Equation (5.5)). Define

$$\epsilon = \sum_x (-1)^{\gamma(x)} \cdot I_x.$$ 

Signs in the differential are specified by

$$A = (L_cL_{c+1} \otimes 1) + (R_{c+1}R_c \otimes 1) + \sum_{i=1}^{2n} \left( L_{\phi(i)} \otimes R_i' + R_{\phi(i)} \otimes L_i' \right) (\epsilon \otimes 1)$$

$$- C_{\{c,c+1\}} \otimes 1 \sum_{i=1}^{2n} U_{\phi(i)} \otimes E_i + \sum_{\{i,j\} \in M} C_{\{\phi(i),\phi(j)\}} \otimes [E_i, E_j].$$

**Lemma 12.8.** With the above definition, $A_2 \cdot A_2 \cdot \Omega_c$ is a type $DD$ bimodule over $\mathbb{Z}$, with the specified exterior grading.

**Proof.** The proof is mostly straightforward. To see that terms in the third sum anti-commute with $(L_cL_{c+1} \otimes 1)$, it helps to note that

$$(L_cL_{c+1}) \cdot (L_i \cdot \epsilon) + (L_i \cdot \epsilon) \cdot (L_cL_{c+1}) = 0$$

for $i \neq c, c+1$.

Note also that the algebra elements $U_{\phi(i)} \otimes E_i$ and $C_{\{\phi(i),\phi(j)\}} \otimes [E_i, E_j]$ anti-commute with each other. □

For the $DA$ bimodule, the exterior grading of the generator $Q_x$ associated to an allowed idempotent state $x$ (for $A_2$) is given by $\gamma(x)$ from Equation (12.15).

The actions by $\delta^1_1$ as specified in Equation (5.7) are given the following sign refinements:

$$\delta^1_1(X) = - C_{\{c,c+1\}} \otimes X + R_{c+1}R_c \otimes Y$$

$$\delta^1_1(Y) = - C_{\{c,c+1\}} \otimes Y + L_cL_{c+1} \otimes X$$

$$\delta^1_1(Z) = - C_{\{c,c+1\}} \otimes Z.$$ 

Similarly, the maps $\delta^2_1$ are defined now by

$$\delta^2_1(Q_x, a) = (-1)^{\gamma(x)}|a| \Phi(x)(a) \otimes Q_z,$$

where $z$ and $\Phi$ are as in Lemma 5.2, and $a$ is a pure algebra element. For example,

$$\delta^2_1(Q_y, C_{\{i,j\}}) = (-1)^{\gamma(y)} C_{\{\phi(i),\phi(j)\}} \otimes Q_y,$$

for $c \notin \{i, j\}$.

These are the modifications needed for the constructions from Section 5 (notably, Theorem 5.3 and Proposition 5.4) to hold over $\mathbb{Z}$.
12.14. The minimum with signs. We consider now changes needed to adapt Section 7 to work over $\mathbb{Z}$.

First, recall that in Section 7.1 we defined a type bimodule $\mathcal{A}_2^\dagger \mathcal{U}_c$ associated to a minimum, whose generators $P_y$ correspond to allowed idempotent states $y$ for $\mathcal{A}_1'$. The exterior grading is given by $\|y\| + \gamma(y)$ from Equations (12.9) and (12.15) respectively. To specify the differential, we refine Equation (7.3) as follows:

$$A = (1 \otimes L'_c L'_{c+1}) + (1 \otimes R'_{c+1} R'_c) + \sum_{j=1}^{2n} \left( R_j \otimes L'_{\phi(j)} + L_j \otimes R'_{\phi(j)} \right) (1 \otimes c)$$

$$- 1 \otimes E_c U_{c+1} - \sum_{j=1}^{2n} U_j \otimes E_{\phi(j)} + U_\alpha \otimes [E_{\phi(\alpha)}, E_c] E_{c+1}$$

$$- C_{\{\alpha, \beta\}} \otimes [E_{\phi(\alpha)}, E_c] [E_{c+1}, E_{\phi(\beta)}] + \sum_{\{i,j\} \in \mathcal{M}_2 \setminus \{\alpha, \beta\}} C_{\{i,j\}} \otimes [E_{\phi(i)}, E_{\phi(j)}].$$

With these choices, it is clear that $\mathcal{A}_2^\dagger \mathcal{U}_c$ is a type $\mathcal{D}\mathcal{D}$ bimodule over $\mathbb{Z}$.

We can alternatively replace the roles of $c$ with $c + 1$ and $\alpha$ with $\beta$ to obtain a homotopy equivalent bimodule.

Next, we turn to the signed version of the $\mathcal{D}\mathcal{A}$ bimodule of a minimum.

Consider first the larger bimodule given in Section 7.3, denoted $M$. That module is generated by two generators $X$ and $Y$. We specify the exterior grading by defining $|X| = 1$ and $|Y| = 0$. (This coincides with $\gamma$.) Signs are put in the model from Equation (7.6), as follows:

$$C \otimes (C_1, C_2)$$

$$X$$

$$U_2 - U_\alpha \otimes C_1 - C \cdot U_2 \otimes C_1 \cdot C_2$$

$$C \otimes (C_2, C_1)$$

$$Y$$

$$U_1 + U_\beta \otimes C_2 + C \cdot U_1 \otimes C_1 \cdot C_2$$

With these signs, the bimodule relations hold; indeed, homological perturbation theory gives the following analogue of Lemma 7.7:

**Lemma 12.9.** The operations described above make $\mathcal{A}_2^\dagger \mathcal{A}_1$ into a type $\mathcal{D}\mathcal{A}$ bimodule, with the generating set described in Lemma 7.6
Proof. We verify the $A_\infty$ relations with these sign choices. Half of the non-trivial cases are shown below; the remaining half follow the same way:
The above relations hold since $|U_i| = 0 = |Y|$, $|X| = 1 = |C|$. Another consistency check is offered by considering the $A_\infty$ relation with a single algebra input, that is $C_1 C_2$. □

With signs, the homotopy operator $h^1: \Theta \to \Theta$ considered in Section 7.3. The operator is characterized by the property that for pure algebra element $a$,

\[
h^1(X \otimes a) = \begin{cases} 
- Y \otimes a' & \text{if there is an } a' \in \Gamma \text{ with } a = U_1 a' \\
0 & \text{otherwise}
\end{cases}
\]

\[
h^1(Y \otimes a) = \begin{cases} 
- X \otimes a' & \text{if there is an } a' \in \Gamma \text{ with } a = U_2 a' \\
0 & \text{otherwise},
\end{cases}
\]

where $\Gamma$ is as in Equation (7.5). This operator satisfies the equation

\[i \circ \pi = \text{Id} + \partial \circ h^1 + h^1 \circ \partial.\]

For the purpose of the next lemma, we adapt the associated element from Definition 7.3:

**Definition 12.10.** For a preferred sequence, where each pure algebra generator appears with coefficient $+1$, there is at most one pure algebra generator (again with appearing with coefficient $+1$) $b \in \mathcal{B}(2n,k) \subset A_2$, characterized by the following properties:

(PS-1) $b = L_{\psi(x_1)} \cdot b$

(PS-2) For all $i$ with $1 \leq i \leq 2n$, $w_i(b) = \sum_{j=1}^{m} w_{i+2}(a_j)$.

**Lemma 12.11.** The $A_\infty$ action of standard sequences on $U_1$ are given by

\[
\delta^1_3(XL_1, C_1, C_2) = C \otimes XL_1
\]

\[
\delta^1_3(YR_2, C_2, C_1) = C \otimes YR_2
\]
with further actions governed by the diagram

The contribution of a path of $k$ algebra elements in this diagram is multiplied by $(-1)^{\eta(k)+k-1}$, where $\eta$ is as in Equation (12.8), to give the $A_\infty$ operations on the type $DA$ bimodule; i.e. if $a_1, \ldots, a_k$ is a preferred sequence of pure algebra elements with $a_1 = 1_x \cdot a_1$ and $a_k = a_k \cdot 1_y$ and $b$ is a pure algebra element as in Definition (12.10), then $\delta^1_{k+1}(T_x, a_1, \ldots, a_k)$ is given by $(-1)^{\#(C_1(a_1, \ldots, a_k))+(k)+k-1} b \otimes T_y$.

Proof. The homological perturbation lemma (applied to type $DA$ bimodules) endows $XL_1 \oplus YR_2$ with the structure of a type $DA$ bimodule. To find the sign, observe in the descriptions of the new $\delta^1_{k+1}$ actions for $k > 1$, only the $\delta^1_2$ actions on $M$ appear, and the $h^1$ nodes contribute $(-1)^{\sum_{i=1}^{k-1} i} = \eta(k)$; moreover each $\delta^1_{k+1}$ action uses the homotopy operator $h^1$ a total of $k - 1$ times.

For example,

\begin{align*}
\delta^1_3(YR_2, U_2, C_1) &= -U_2 YR_2 \\
\delta^1_4(YR_2, U_2^2, C_1, C_1) &= -U_2^2 YR_2 \\
\delta^1_4(YR_2, L_2, U_1, R_2) &= -YR_2 \\
\delta^i_b(YR_2, L_2, U_1, U_2, U_1, R_2) &= YR_2
\end{align*}

These are the modifications needed for the constructions from Section 7 (notably, Theorem 7.4 and Lemma 7.5) to hold over $\mathbb{Z}$.

12.15. Invariance over $\mathbb{Z}$. We can now define $Q_{\mathbb{Z}}(D)$ to be the chain complex over $\mathbb{Z}$ defined by forming the tensor products of the above bimodules. If $D$ denotes the diagram for $\tilde{K}$ with the minimum removed, then $Q_{\mathbb{Z}}(D)$ can be used to define a complex denoted $C_{\mathbb{Z}}(\tilde{K})$ over $\mathbb{Z}[U, V]/UV = 0$ following methods from Section 8.2.

By its construction, $Q_{\mathbb{Z}}(D)$ and hence also $C_{\mathbb{Z}}(\tilde{K})$ inherits the exterior $\mathbb{Z}/2\mathbb{Z}$ grading, which was constructed in order to impose a sign convention. The homology of the complex, thought of as equipped with an absolute $\mathbb{Z}/2\mathbb{Z}$ grading, cannot be a knot invariant: it evidently changes under certain Reidemeister 1 moves. However, as a relative $\mathbb{Z}/2\mathbb{Z}$ grading, it agrees with the Maslov grading. This can be shown by identifying the generators of the complex with Kauffman states, and proving that both the exterior and the Maslov gradings change parity under clock transformations. Since any two Kauffman states can be connected by a sequence of clock transformations (see [3]) it follows that the two relative $\mathbb{Z}/2\mathbb{Z}$-gradings agree.
Let $J_Z(\vec{K})$ denote the homology of $C_Z(\vec{K})$, thought of as a module over $\mathbb{Z}[U, V]/UV = 0$, equipped with its Alexander and Maslov gradings. To see that $J_Z(\vec{K})$ is a knot invariant, we follow the logic from Section 8, with a few adaptations.

Theorem 4.1 can be proved by direct computation, in the spirit of the proof of the analogous result for the earlier algebras [19, Theorem 6.1]. We leave the details to the interested reader.

The proof of the trident relations with a maximum, Proposition 5.5, works with the following adjustments. Consider $A_3 \mathcal{P}_2^2 \boxtimes A_4 \Omega_1$, and let $X$ be a generator of this tensor product. $X$ can be decomposed as a tensor product of a generator $P_2$, which has type $((N, S, W, E))$ and a generator of $\Omega_1$, which is determined by its idempotent $x$ in $A_4$. Let $y$ denote the corresponding idempotent in $A'_1$. The type of $X$ agrees with the type of its $P_2$ factor. Let

$$\sigma(X) = \begin{cases} 
-1 & \text{if } X \text{ is of type } S \\
+1 & \text{otherwise}
\end{cases}, \quad \begin{cases} 
-1 & \text{if } 0 \in x \\
1 & \text{otherwise}
\end{cases}$$

Clearly, $\sigma(X)(-1)^{\|y\|}$ is $(-1)$ to the mod two grading of $X$.

The module $A_3 \mathcal{P}_2^2 \boxtimes A_4 \Omega_1$ has outside arrows (connecting generators $X$ of the same type) of the form $-C_{\{1,3\}} \otimes 1; \sigma \cdot (L_{j+2} \otimes R_j' + R_{j+2} \otimes L_j')$ for all $j = 1, \ldots, 2n$; $-U_{j+2} \otimes E_j$ for all $j = 1, \ldots, 2n$; and $C_{\{m+2,\alpha+2\}} \otimes [E_m, E_\alpha]$ for all $\{m, \alpha\} \in M_1$ with $1 \notin \{m, \alpha\}$, and $C_{\{2,\alpha+2\}} \otimes [E_1, E_\alpha]$ with $\{1, \alpha\} \in M_1$. The arrows connecting different types appear in the following sign-refined version of Equation (5.9):
Consider the more symmetric bimodule $A^3.A'_1 \mathcal{T}$ with the same generators, and differentials

\begin{equation}
N \mapsto -R_2 \otimes E_1 + L_1 \otimes L_3 - 1 \label{12.18}
\end{equation}

equipped with the above self-arrows; compare Equation (5.10). The map

$h^1: A_3^2 \otimes A^4_1 \Omega_1 \to A^3_3.A'_1 \mathcal{T}$

defined by

\begin{equation*}
h^1(X) = \begin{cases}
S + (R_2 \otimes E_1) \cdot W & \text{if } X = \text{some generator} \\
X & \text{otherwise}
\end{cases}
\end{equation*}

gives a homotopy equivalence between the two $DD$ bimodules.

The relation from Proposition 5.5 now works.

For the proof of the trident relation involving minima, Lemma 7.9, we worked in the dual algebra $A'$. That algebra can be given signs as explained earlier. Moreover, as needed in the proof, we can construct the $DA$ bimodule of crossings over $A'$. This is constructed by following the rules from Section 6.1, and using the signs outlined in Section 12.12. So, for example, we have relations as in Equation (12.12) and further extensions as in Equation (12.14), except now with $E$-variables instead of $C$-variables. With these straightforward changes, we find that, as in the proof of Lemma 7.9, $A'_3 \otimes A'_4 \otimes A'_1 \Omega_1$ is the bimodule whose arrows are obtained from Equation (12.18), switching the prime markings from the second to the first tensor factor, equipped with further self-arrows from each generator $X$ to itself of the form $-U_1E_3 \otimes 1$, $-E_2 \otimes U_1$, $\sigma \cdot L_{j+2} \otimes R_j$ for $j = 1, \ldots, 2n$ (with $\sigma$ defined as before), $\sigma R_{j+2} \otimes L_j$ for $j = 1, \ldots, 2n$, $-E_{j+2} \otimes U_j$ for $j = 1, \ldots, 2n$, $[E_{m+2}, E_{\ell+2}] \otimes C_{(m, \ell)}$ for all $(m, \ell) \in M_1$; and additional self-arrows of the form $[E_{\beta+2}, E_3] \cdot E_1 \otimes U_3$ and $-[E_1, E_{\alpha+2}] \cdot [E_3, E_{\beta+2}] \otimes C_{\alpha, \beta}$. By a natural symmetry of this bimodule, the trident relation involving a minimum holds.

**Theorem 12.12.** The bigraded module $J_Z(\vec{K})$ over $\mathbb{Z}[u, v]/uv = 0$, which is the homology of the complex $C_Z(D)$, is an invariant of the oriented knot $\vec{K}$. \hfill \square

Obviously, $J_Z(\vec{K}) = H(C_Z(\vec{K}))$ refines $J(\vec{K}) = H(C(\vec{K}))$ considered earlier: since $C_Z(\vec{K}) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong C(\vec{K})$. 

so their homologies are related by a universal coefficient theorem.

12.16. **Crossing change morphisms over** \( \mathbb{Z} \). Proposition 11.1 has the following obvious generalization:

**Proposition 12.13.** Let \( K_+ \) and \( K_- \) be two knots represented by knot projections \( D_+ \) and \( D_- \), that differ in a single crossing, which is positive in \( D_+ \) and negative in \( D_- \). Then, there are maps

\[
c_+ : J_2(K_-) \to J_2(K_+) \quad \text{and} \quad c_- : J_2(K_+) \to J_2(K_-),
\]

where \( c_- \) preserves bidegree and \( c_+ \) is of degree \((-1, -1)\), so that \( c_+ \circ c_- = U \) and \( c_- \circ c_+ = U \).

**Proof.** We insert signs into the homomorphisms from Lemma 11.2. The map \( \phi_- : \mathcal{A}_1, \mathcal{A}_2 \mathcal{P}_1 \to A_1, A_2 \mathcal{N}_1 \) is represented by

\[
\begin{array}{c}
W_+ = U_2 \otimes R_1' + R_2 R_1 \otimes L_2' \quad U_1 \otimes L_2' + L_1 L_2 \otimes R_1' \quad L_2 \otimes U_1 + R_1 \otimes L_1' L_2' \quad -R_1 \otimes U_2 - L_2 \otimes R_1' \\
\end{array}
\]

\[
= W_- = U_2 \otimes L_1' + L_1 L_2 \otimes R_2' \quad U_1 \otimes R_2' + R_2 R_1 \otimes L_1' \quad R_2 \otimes U_1 + L_1 \otimes R_1' R_1' \quad -L_1 \otimes U_2 - R_2 \otimes L_1' L_2'
\]

(where once again we have suppressed self-arrows of Type (P-1), (P-2), and (P-3), which now occur with signs; e.g. including self-arrows at each vertex labeled by \(- U_2 \otimes E_1 - U_1 \otimes E_2\) and \( \phi_+ : \mathcal{N}_1 \to \mathcal{P}_1 \) is represented (with self-arrows suppressed) by

\[
\begin{array}{c}
W_- = U_2 \otimes L_1' + L_1 L_2 \otimes R_2' \quad U_1 \otimes R_2' + R_2 R_1 \otimes L_1' \quad R_2 \otimes U_1 + L_1 \otimes R_1' R_1' \quad -L_1 \otimes U_2 - R_2 \otimes L_1' L_2'
\end{array}
\]

It is straightforward to check that \( \phi_+ \) and \( \phi_- \) are \( DD \) bimodule homomorphisms.

Letting

\[
h_+(N_+) = h_+(W_+) = h_+(E_+) = 0
\]

\[
h_+(S_+) = (L_2 \otimes 1) \otimes E_+
\]
and
\[
\begin{align*}
  h_-(N_-) &= h_-(W_-) = h_-(S_-) = 0 \\
  h_-(E_-) &= (R_2 \otimes 1) \otimes S_-, \\
\end{align*}
\]
it is straightforward to verify that
\[
\begin{align*}
  dh_+ &= (U_2 \otimes 1) \text{Id}_{P_1} - \phi_+ \circ \phi_- \\
  dh_- &= (U_2 \otimes 1) \text{Id}_{N_1} - \phi_- \circ \phi_+, \\
\end{align*}
\]

For any prime \( p \), we can define \( J(K, s; \mathbb{Z}/p\mathbb{Z}) \) to be the portion of \( H(C_\mathbb{Z}(\bar{K}) \otimes \mathbb{Z}/p\mathbb{Z}) \) in Alexander grading \( s \). We can define invariants \( \nu_p(\bar{K}) \) for any prime \( p \) by
\[
\nu_p(\bar{K}) = -\max\{s | U^d \cdot H(\bar{K}, s; \mathbb{Z}/p\mathbb{Z}) \neq 0 \forall d \geq 0\}.
\]
It would be interesting to find examples of knots \( \bar{K} \) for which \( \nu_p(\bar{K}) \neq \nu_q(\bar{K}) \) for \( p \neq q \).
13. Fast Computations

13.1. Working with standard $D$ modules. We introduce a shorthand. Given a sequence of algebra elements $(\beta_1, \ldots, \beta_k)$ in $B$, we say that a standard sequence $(a_1, \ldots, a_\ell)$ augments the sequence, if it is obtained from $(\beta_1, \ldots, \beta_k)$ by inserting copies of $-C$.

Let $X$ be a standard type $DA$ module, and let $\gamma_k(x, \beta_1, \ldots, \beta_k)$ be a suitable signed count of the sum of $\delta_{k+m}^1$ over all the ways of extending the sequence $(\beta_1, \ldots, \beta_k)$ by sequences of $-C$. Specifically,

\[ \gamma_k(x, \beta_1, \ldots, \beta_k) = \sum_{\ell=k}^{\infty} (-1)^{\ell(\ell+1)/2} \prod_{i=1}^{\ell} |a_i| \delta_{\ell+1}^1(x, a_1, \ldots, a_\ell), \]

where the sum is taken over all standard sequences $(a_1, \ldots, a_\ell)$ augmenting $(\beta_1, \ldots, \beta_k)$.

To justify our use of $\gamma$, note that if $X$ is also adapted to a one-manifold with $W$ with $\partial W = Y_1 \cup Y_2$ with non-empty $Y_2$, then the sum defining $\gamma_k(x, \beta_1, \ldots, \beta_k)$ is finite. If the sum were not finite, then there would be output algebra elements with arbitrarily large weight, since $X$ respects the Alexander multigrading, and $\sum_p C_p$ has positive weight everywhere. But this contradicts the universal weight bound supplied by the $\Delta$-grading, which is guaranteed since each term in the sum contributes the same $\Delta$ grading.

In practice, if $X$ and $Y$ are standard type $DA$ bimodules, then their $\gamma$-operations of $X \otimes Y$ are determined by the $\gamma$-operations of $X$ and $Y$. Moreover, to determine the chain complex $C$, we need only understand these $\gamma$-operations. (Retaining only the $\gamma$-operations, we can think of $X$ as a curved module over $B$, and the tensor product then corresponds to a suitable curved analogue of the tensor product. This is the point of view taken in [12]; compare also [7].)

We describe these data explicitly in the case of $\mathcal{O}_1$. Since the idempotent of $XL_1$ does not appear in the output algebras, we will describe the part of the module with input $YR_2$. The function $\gamma_{2k+1}(YR_2, \beta_1, \ldots, \beta_{2k+1})$ is determined by the graph:

![Graph](image)

\[ \gamma_{2k+1}(YR_2, \beta_1, \ldots, \beta_{2k+1}) \]

as given in the following lemma (compare Equation (7.4)):
Proposition 13.1. Suppose that $\beta_1, \ldots, \beta_{2k+1}$ is a sequence of algebra elements labeling the edges in the graph of Equation (13.2), to give a path from $Y R_2$ to itself, so that the intermediate vertices alternate between $X$ and $Y$, then letting

$$v_1 = \left( \sum_{i=1}^{2k+1} w_1(\beta_i) \right) - k; \quad v_2 = \left( \sum_{i=1}^{2k+1} w_2(\beta_i) \right) - k.$$

$$\gamma_{2k+1}(x, \beta_1, \ldots, \beta_{2k+1}) = (-1)^k U^{x_1}_{\alpha_1} U^{x_2}_{\beta_1} \otimes x.$$

Proof. Clearly, $v_1$ and $v_2$ are the number of algebra elements $C_2$ and $C_1$ to be added to the sequence to make it a sequence as in Equation (7.4). Thus, we see that the output algebra element contains $\pm U^{x_1}_{\alpha_1} U^{x_2}_{\beta_1}$, as in Definition 7.3.

We wish to compute the sign in $\gamma_{2k+1}(x, \beta_1, \ldots, \beta_{2k+1})$. By Lemma (12.11), suppressing the module generator $x = Y R_2$ from the notation,

$$\delta_{j+1}^1(\beta_1, (-C_1)^{j+1}, \beta_2, (-C_2)^{j+1}, \ldots, \beta_{2k+1}, (-C_1)^{j_1}) = (-1)^{\#C_1 + \#C_2}\delta_{j+1}^1(\beta_1, C_1^{j+1}, \beta_2, C_2^{j+1}, \ldots, \beta_{2k+1}, C_1^{j_1})$$

$$= (-1)^{\#C_2 + \#(j+1)} U^{x_2}_{\alpha_2} U^{x_{1}}_{\beta_1} = (-1)^{\#C_1} U^{x_2}_{\alpha_2} U^{x_{1}}_{\beta_1},$$

since $j-1 \equiv \#C_1 + \#C_2 \pmod{2}$. Moreover, since $|Y R_2| = 0 = |\beta_i|$, and $|C_i| = 1$,

$$\epsilon(Y R_2, \beta_1, (-C_1)^{j+1}, \beta_2, (-C_2)^{j+1}, \ldots, \beta_{2k+1}, (-C_1)^{j_1})$$

$$= \left( \sum_{i=1}^{j-1} i \right) - j_1 - (j_1 + j_2 + 1) - (j_1 + j_2 + j_3 + 2) + \cdots - (j_1 + \cdots + j_{2k+1} + 2k).$$

Note that $\eta(m) \equiv \sum_{i=1}^{m-1} i \pmod{2}$. It follows that

$$\epsilon(Y R_2, \beta_1, (-C_1)^{j+1}, \beta_2, (-C_2)^{j+1}, \ldots, \beta_{2k+1}, (-C_1)^{j_1})$$

$$\equiv \eta(j) + \eta(2k+1) + \sum_{i=0}^{k} j_{2i+1} = \eta(j) + \eta(2k+1) + \#C_1.$$

Combining Equations (13.3), (13.4), and the observation that $\eta(2k+1) \equiv k \pmod{2}$, the result follows. □

13.2. Contracting arrows. In practice, we inductively construct the complexes, say, over a field as follows. The top gives a standard type $D$ module, tensor that with the next type $DA$ bimodule, to obtain a new standard type $D$ module. Next, cancel differentials until algebra elements appearing the coefficients of $\delta^1(x)$ all have non-zero weight, and then proceed. More formally:

Definition 13.2. A $\Delta$-graded standard type $D$ structure $X$ over $A$ is said to be small, if for any two generators $x, y \in X$ with $\Delta(y) = \Delta(x) - 1$, the $A \otimes y$ coefficient of $\delta^1(x)$ is $0$.

Since all algebra elements $a \in A$ with $\Delta(a) = 0$ are in the idempotent ring, the above condition is equivalent to the condition that for all $x \in X$ with $I_x \cdot x = x$, $I_x y$ does not appear with non-zero coefficient in $\delta^1(x)$.

The cancellation is done via the following:
Lemma 13.3. Any finitely generated, $\Delta$-graded standard type $D$ module over $A \otimes F$ (where $F$ is a field; consider for example $F = \mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Q}$), is homotopy equivalent to a small, finitely generated, $\Delta$-graded standard type $D$ module over $A \otimes F$.

Proof. Given a type $D$ module $Y$ with a generating set $y_1, \ldots, y_m$, with the property that $y_i = I_{x_i} \cdot y_i$ for some idempotent state $x_i$. We can write

$$\delta^1(y_i) = \sum_{i,j} a_{i,j} \otimes y_j,$$

where $I_{x_i}, a_{i,j}, I_{x_j} = a_{i,j}$. If $Y$ is not small, we can assume without loss of generality (after renumbering generators $y_i$ and possibly rescaling some by an element of $F$, if needed) that $I_{x_1} = I_{x_2} = a_{1,2}$. Then, we can find a new type $D$ structure $Y'$ with two fewer generators, obtained by canceling $y_1$ and $y_2$. Explicitly, $Y'$ has generators $y'_3, \ldots, y'_m$ with $a'_{i,j} = a_{i,j} - a_{i,2} \cdot a_{1,j}$. Here, $Y'$ is the submodule of $Y$ with $y'_j = y_j - a_{j,2} \cdot y_1$. Clearly, if $Y$ is standard, then so is $Y'$.

By applying this cancellation after each step, we gain some control over the complexity of the calculations. Software for implementing this algorithm will be included with this preprint submission.
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