Reliability for Lindley Distribution with an Outlier

Hossein Jabbari Khamnei

Department of Statistics, Faculty of mathematical sciences, University of Tabriz, Tabriz, Iran

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Abstract. In this paper, we consider the problem of estimating \( R = P(Y < X) \), when \( Y \) has lindley distribution with parameter \( a \) and \( X \) has lindley distribution with presence of one outlier with parameters \( b \) and \( c \), such that \( X \) and \( Y \) are independent. The maximum likelihood estimator of \( R \) is derived and some results of simulation studies are presented.

1 Introduction

In reliability context inferences about \( R = P(Y < X) \), when \( X \) and \( Y \) are independently distributed, are a subject of interest. For example in mechanical reliability of a system if \( X \) is the strength of a component which is subject to stress \( Y \), then \( R \) is a measure of system performance. The system fails, if at any time the applied stress is greater than its strength. Stress-strength reliability has been discussed in Kapur and Lambersen (1977). Sathe and Dixit (2001) have done estimation of \( R \) in the negative binomial distribution. Baklizi and Dayyeh (2003) have done shrinkage estimation of \( R \) in exponential case, and recently Deiri (2011) has done estimation of \( R \) with presence of two outliers in the exponential and gamma cases, respectively. Jafari (2011) has obtained the moment, maximum likelihood and mixture estimators of \( R \) in Rayleigh distribution in the presence of one outlier and Jabbari, Abolhasani and Fathipour (2012) have discussed the estimation of \( R \) in the six parameter generalized Burr XII distribution with transformation method.

In this paper, we obtain the maximum likelihood estimator of \( R \) for lindley distribution with presence of one outlier generated from the same distribution.

The probability density function of the lindley distribution with parameter of \( a \) is given by:
\[
f(y; a) = \frac{a^2}{1+a} (1 + y) e^{-ay}, x > 0, a > 0.
\]

In this paper we assume that the random variables \((Y_1, Y_2, ..., Y_m)\) have lindley distribution with parameter \( a \) and the random variables \((X_1, X_2, ..., X_n)\) are such that one of them is from lindley distribution with parameter \( c \) and the remaining \((n-1)\) random variables are from lindley distribution with parameter \( b \).

The paper is organized as follows:

In section 2, we obtain the joint distribution of \((X_1, X_2, ..., X_n)\) in the presence of one outlier. Section 3 and section 4 discusses the method of maximum likelihood estimators of parameters and the MLE of \( R \) respectively. In section 5 simulation studies are presented and the results are summarized in section 6.

2 Joint distribution of \(X_1, X_2, ..., X_n\) in presence of an outlier

Assume \((X_1, X_2, ..., X_n)\) are such that one of them is distributed with p.d.f \( g(x; c) \) as lindley\((c)\) and remaining \((n-1)\) of them are distributed with p.d.f \( f(x; b) \) as lindley\((b)\). The joint distribution of \((X_1, X_2, ..., X_n)\) can be expressed as
\[
f(x_1, x_2, ..., x_n; b, c) = \frac{(n - 1)!}{n!} \prod_{i=1}^{n} f(x_i, b) \sum_{i=1}^{n} \frac{g(x_i; c)}{f(x_i; b)}
\]

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\[
\frac{(n - 1)!}{n!} \frac{b^{2n}}{(1 + b)^n} \prod_{i=1}^{n} (1 + x_i)e^{-b\sum_{i=1}^{n} x_i} \sum_{i=1}^{n} \frac{c^2}{b^2} \frac{(1 + x_i)e^{-bx_i}}{1 + b (1 + x_i)e^{-bx_i}} \\
= \frac{(n - 1)!}{n!} \frac{b^{2n-2}}{(1 + b)^{n-1}} \frac{c^2}{1 + c} \prod_{i=1}^{n} (1 + x_i)e^{-b\sum_{i=1}^{n} x_i} \sum_{i=1}^{n} (1 + x_i)e^{x_i(b-c)}
\]

(1)

See Dixit (1989), Dixit and Nasiri (2001), and Nasiri and Pazira (2009). From (1), the marginal distribution of \(X\) is

\[
f(x; b, c) = \frac{c^2}{n+c} (1 + x)e^{-cx} + \frac{n-1}{n} \frac{b^2}{1+b} (1 + x)e^{-bx}, x, b, c > 0
\]

We will use (2) to obtain \(R = P(Y < X)\).

3 Maximum likelihood estimators of parameters

Let \((Y_1, Y_2, ..., Y_m)\) be a random sample for \(Y\) with pdf,

\[
f(y; a) = \frac{a^2}{1 + a} (1 + y)e^{-ay}, x, a > 0
\]

the log likelihood function is given by

\[
L(a) = 2mln a - mln(1 + a) + \sum_{i=1}^{m} ln(1 + y_i) - a \sum_{i=1}^{m} y_i
\]

Taking the derivative with respect to \(a\) and equating to \(0\), we obtain the MLE of \(a\) as

\[
\hat{a} = \frac{m \sum_{i=1}^{m} y_i \pm \sqrt{(\sum_{i=1}^{m} y_i - m)^2 + 8m \sum_{i=1}^{m} y_i}}{2 \sum_{i=1}^{m} y_i}
\]

(3)

Now let \((X_1, X_2, ..., X_n)\) be a random sample for \(X\) with presence of one outlier with pdf,

\[
f(x; b, c) = \frac{1}{n + 1 + c} \frac{c^2}{1+ c} (1 + x)e^{-cx} + \frac{n-1}{n} \frac{b^2}{1+b} (1 + x)e^{-bx}; x, b, c > 0.
\]

From (1), the log likelihood function is given by

\[
L(b, c) = \ln \left(\frac{(n - 1)!}{n!}\right) + (2n - 2)lnb - (n - 1)ln(1 + b) + 2ln c - ln(1 + c) + \sum_{i=1}^{n} ln(1 + x_i) - b \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} e^{x_i(b-c)}
\]

Taking the derivatives with respect to \(b\) and \(c\) and equating the results to \(0\), we obtain the normal equations as

\[
\frac{\partial L(b, c)}{\partial b} = \frac{2n - 2}{b} - \frac{n - 1}{1 + b} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i e^{x_i(b-c)}
\]

(4)

\[
\frac{\partial L(b, c)}{\partial c} = \frac{2}{c} - \frac{1}{1 + c} - \sum_{i=1}^{n} x_i e^{x_i(b-c)}
\]

(5)

There is no closed-form solution to this system of equations, so we will solve for \(\hat{b}\) and \(\hat{c}\) iteratively, using the Newton-Raphson method. In our case we will estimate \(\hat{b} = (\hat{b}, \hat{c})\) iteratively:

\[
\hat{b}_{i+1} = \hat{b}_i - G^{-1}g
\]

(6)

where \(g\) is the vector of normal equations for which we want

\[
g = [g_1, g_2]
\]

With

\[
g_1 = \frac{2n-2}{b} - \frac{n-1}{1+b} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i e^{x_i(b-c)}
\]

\[
g_2 = \frac{2}{c} - \frac{1}{1+c} - \sum_{i=1}^{n} x_i e^{x_i(b-c)}
\]

\[
g_1 = \frac{2n-2}{b} - \frac{n-1}{1+b} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i e^{x_i(b-c)}
\]

\[
g_2 = \frac{2}{c} - \frac{1}{1+c} - \sum_{i=1}^{n} x_i e^{x_i(b-c)}
\]
and $G$ is the matrix of second derivatives

$$G = \begin{bmatrix} \frac{dg_1}{db} & \frac{dg_1}{dc} \\ \frac{dg_2}{db} & \frac{dg_2}{dc} \end{bmatrix}$$

where

$$\frac{dg_1}{db} = \frac{2 - 2n}{b^2} + \frac{n - 1}{(1 + b)^2} + \frac{\sum_{i=1}^{n} x_i^2 e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}} - \left(\frac{\sum_{i=1}^{n} x_i e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}}\right)^2$$

$$\frac{dg_1}{dc} = -\frac{\sum_{i=1}^{n} x_i^2 e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}} + \left(\frac{\sum_{i=1}^{n} x_i e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}}\right)^2$$

$$\frac{dg_2}{db} = -2 + \frac{1}{(1 + c)^2} + \frac{\sum_{i=1}^{n} x_i^2 e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}} - \left(\frac{\sum_{i=1}^{n} x_i e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}}\right)^2$$

The Newton-Raphson algorithm converges, as our estimate of $b$ and $c$ change by less than a tolerated amount with each successive iteration, to $\hat{b}$ and $\hat{c}$.

4 The maximum likelihood estimator of $R$

Let $Y \sim \text{lindley}(a)$ with pdf $h(y; a)$ and $X$ be distributed with pdf $f(x; b, c)$ given in (2). The parameter $R$ we want to estimate is

$$R = P(Y < X) = \int_{0}^{\infty} \int_{0}^{x} h(y; a) f(x; b, c) dy dx$$

$$= \frac{1}{b} \int_{0}^{\infty} \int_{0}^{x} \frac{a^2}{1 + a} (1 + y) e^{-ay} \frac{c^2}{1 + c} (1 + x) e^{-cx} dy dx$$

$$+ \frac{n - 1}{n} \int_{0}^{\infty} \int_{0}^{x} \frac{a^2}{1 + a} (1 + y) e^{-ay} \frac{b^2}{1 + b} (1 + x) e^{-bx} dy dx$$

$$= \frac{1}{n} \left[ c^2 (1 + c) + (1 + c)(3 + c)a + (3 + 2c)a^2 + a^3 \right]$$

$$+ \frac{n - 1}{n} \left[ b^2 (1 + b) + (1 + b)(3 + b)a + (3 + 2b)a^2 + a^3 \right]$$

Thus, by invariant property for MLEs, the MLE of $R$ is

$$\hat{R} = \frac{1}{2} \left[ \frac{\hat{c}^2 (\hat{c} + \hat{e}) + (1 + \hat{e})(3 + \hat{e})\hat{a} + (3 + 2\hat{e})\hat{a}^2 + \hat{a}^3)}{1 + \hat{e}(1 + \hat{a})(\hat{c} + \hat{a})} \right]$$

$$+ \frac{n - 1}{n} \left[ \frac{\hat{b}^2 (1 + \hat{b}) + (1 + \hat{b})(3 + \hat{b})\hat{a} + (3 + 2\hat{b})\hat{a}^2 + \hat{a}^3)}{1 + \hat{b}(1 + \hat{a})(\hat{b} + \hat{a})} \right]$$

where $\hat{a}, \hat{b}$, and $\hat{c}$ can be obtained from (3) and (6).

5 Simulation Study

In this section we generate random numbers from lindley distribution (with and without outlier) with accept-reject method by Maple software. Using these samples and the Newton-Raphson method we obtain the maximum likelihood estimators of parameters $a, b$ and $c$. Then we use them to calculate the MLE of $R$. The values of biases and MSEs of these estimates are presented in table 1, for $a=1$, $b=2$ and $c=1.6,1.7,1.8,1.9,2.1,2.2,2.3,2.4,2.5,3,4$ and in table 2, for $a=1$, $b=2$ , and the same values of $c$. All the results are based on 100 replications.

6 Conclusion

According to the results of simulation, when the value of parameters $b$ and $c$ are close to each other, the biases and MSEs are often around zero and when the difference between $b$ and $c$ is greater than 1, the biases and MSEs increase.
Table 1: Biases and (MSE)s of the MLEs of $R$, for $a=1$, $b=2$, and different values of $c$

| $(m,n)$ | $(10,10)$ | $(20,10)$ | $(30,10)$ | $(40,10)$ | $(50,10)$ | $(60,10)$ | $(10,20)$ | $(20,20)$ | $(30,20)$ | $(40,20)$ | $(50,20)$ | $(60,20)$ | $(10,100)$ | $(20,100)$ | $(30,100)$ | $(40,100)$ | $(50,100)$ | $(60,100)$ |
|--------|-----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| 1.6    | 0.00160   | 0.00188  | -0.00469 | -0.00301 | 0.00715  | -0.00418 | -0.00499 | -0.03139 | 0.00776  | 0.00425  | 0.08441  | 0.01131  | 0.000015  | 0.000036  | 0.000059  | 0.000085  | 0.000121  | 0.000147  |
| 1.7    | 0.00195   | 0.00293  | 0.00646  | 0.00757  | 0.00113  | -0.00419 | 0.00658  | 0.01312  | 0.01178  | 0.00095  | 0.00579  | 0.00888  | 0.01131  | 0.000015  | 0.000036  | 0.000059  | 0.000085  | 0.000121  | 0.000147  |
| 1.8    | -0.00252  | 0.00266  | 0.00809  | 0.00716  | 0.00101  | 0.00799  | 0.01411  | 0.00685  | 0.00853  | -0.00135 | 0.00652  | 0.00888  | 0.00911  | 0.000015  | 0.000036  | 0.000059  | 0.000085  | 0.000121  | 0.000147  |
| 1.9    | 0.00359   | 0.00354  | -0.00386 | 0.00863  | 0.00188  | -0.00129 | 0.01297  | 0.00374  | 0.01019  | 0.000015  | 0.000036  | 0.000059  | 0.000085  | 0.000121  | 0.000147  |
| 2.1    | -0.00002  | -0.00219  | -0.00800 | 0.00374  | 0.00254  | 0.00558  | 0.00854  | 0.01070  | 0.01078  | 0.000015  | 0.000036  | 0.000059  | 0.000085  | 0.000121  | 0.000147  |
| 2.2    | 0.00107   | 0.00135   | 0.00646  | 0.00394  | 0.00038  | 0.00446  | 0.00629  | 0.02297  | 0.00832  | 0.000015  | 0.000036  | 0.000059  | 0.000085  | 0.000121  | 0.000147  |
| 2.3    | 0.00096   | 0.00128   | 0.00646  | 0.00621  | 0.00616  | 0.00776  | 0.00888  | 0.00400  | 0.00726  | 0.000015  | 0.000036  | 0.000059  | 0.000085  | 0.000121  | 0.000147  |
| 2.4    | 0.00104   | -0.00135 | -0.00184 | 0.00477  | 0.00311  | 0.00795  | 0.01112  | 0.00688  | 0.00505  | 0.000015  | 0.000036  | 0.000059  | 0.000085  | 0.000121  | 0.000147  |
| 2.5    | -0.00424  | 0.00229   | 0.11499  | 0.00711  | 0.04781  | 0.00599  | 0.00477  | 0.00516  | 0.00283  | 0.000015  | 0.000036  | 0.000059  | 0.000085  | 0.000121  | 0.000147  |
| 3      | -0.00431  | 0.00290   | 0.00573  | 0.00960  | 0.12725  | 0.00251  | 0.00252  | 0.00608  | 0.00297  | 0.000015  | 0.000036  | 0.000059  | 0.000085  | 0.000121  | 0.000147  |
| 4      | -0.00150  | 0.00272   | 0.01788  | 0.00916  | 0.01984  | 0.00507  | 0.01143  | 0.00703  | 0.00401  | 0.000015  | 0.000036  | 0.000059  | 0.000085  | 0.000121  | 0.000147  |

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