LOWER BOUND ON THE DENSITY OF STATES FOR PERIODIC SCHRODINGER OPERATORS

SERGEY MOROZOV, LEONID PARNOVSKI, AND IRINA PCHELINTSEVA

ABSTRACT. We consider Schrödinger operators $-\Delta + V$ in $\mathbb{R}^d$ ($d \geq 2$) with smooth periodic potentials $V$ and prove a uniform lower bound on the density of states for large values of the spectral parameter.

1. Introduction

Let $H = -\Delta + V$ be a Schrödinger operator in $L_2(\mathbb{R}^d)$ with a smooth periodic potential $V$. We will assume throughout that $d \geq 2$. The integrated density of states (IDS) for $H$ is defined as

$$N(\lambda) := \lim_{L \to \infty} L^{-d} N(\lambda; H^{(L)}_D), \quad \lambda \in \mathbb{R}. \quad (1.1)$$

Here $H^{(L)}_D$ is the restriction of $H$ to the cube $[0, L]^d$ with the Dirichlet boundary conditions, and $N(\lambda; \cdot)$ is the counting function of the discrete spectrum below $\lambda$. For $H_0 := -\Delta$ the IDS can be easily computed explicitly (e.g. using the representation (2.6) below):

$$N_0(\lambda) = (2\pi)^{-d} d^{-1} \omega_d \lambda^{d/2}. \quad (1.2)$$

Here $\omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of the unit sphere $S^{d-1}$ in $\mathbb{R}^d$.

The asymptotic behaviour of the function (1.1) for large values of the spectral parameter was recently studied in a number of publications, see [1], [3], and references therein.

Our article concerns the high–energy behaviour of the Radon–Nikodým derivative of IDS

$$g := dN/d\lambda,$$

which is called the density of states (DOS) (see [4]). Our main result is that for big values of $\lambda$

$$g(\lambda) \geq g_0(\lambda) (1 - o(1)), \quad (1.3)$$

where

$$g_0(\lambda) = dN_0(\lambda)/d\lambda = (2\pi)^{-d} \omega_d \lambda^{(d-2)/2}/2.$$
We remark that (1.3) should be understood in the sense of measures; in particular, we do not claim that \( g(\lambda) \) is everywhere differentiable.

It has been proved in [2] that the spectrum of \( H \) contains a semi-axis \([\lambda_0, +\infty)\); this statement is known as the Bethe-Sommerfeld conjecture (see the references in [2] for the history of this problem). This result has an obvious reformulation in terms of IDS: each point \( \lambda \geq \lambda_0 \) is a point of growth of \( N \). It was also proved in [2] that for each \( n \in \mathbb{N} \) and \( \varepsilon = \lambda^{-n} \) we have

\[
N(\lambda + \varepsilon) - N(\lambda) \ll \varepsilon^{(d-2)/2}. \tag{1.4}
\]

Later, when the second author discussed the results and methods of [2] with Yu. Karpeshina, she suggested that using the technique from that paper, one should be able to prove the opposite bound

\[
N(\lambda + \varepsilon) - N(\lambda) \gg \varepsilon^{(d-2)/2} \tag{1.5}
\]

when \( \lambda \) is sufficiently large, not just with \( \varepsilon = \lambda^{-n} \) (when the proof is relatively straightforward given [2]), but also uniformly over all \( \varepsilon \in (0, 1] \). In this paper we prove that for big \( \lambda \)

\[
N(\lambda + \varepsilon) - N(\lambda) \geq \frac{\omega_d}{2(2\pi)^d}\varepsilon^{(d-2)/2} (1 - o(1)). \tag{1.6}
\]

Note that (1.6) implies the claimed bound (1.3).

The proof of (1.6) is heavily based on the technique of [2] and uses various statements proved therein. In order to minimise the size of our paper, we will try to quote as many results as we can from [2], possibly with some minor modifications when necessary.

Acknowledgement. As we have already mentioned, this paper is a result of observations and suggestions made by Yu. Karpeshina; we are very grateful to her for sharing them with us and allowing us to use them. The authors were supported by the EPSRC grant EP/F029721/1.

2. Preliminaries

We study the Schrödinger operator

\[
H = -\Delta + V(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d \tag{2.1}
\]

with the potential \( V \) being infinitely smooth and periodic with the lattice of periods \( \Lambda \). We denote the lattice dual to \( \Lambda \) by \( \Lambda^\dagger \), fundamental cells of these lattices are denoted by \( \Omega \) and \( \Omega^\dagger \), respectively. We choose \( \Omega^\dagger \) to be the first Brillouin zone and introduce

\[
Q := \sup \{ \|\xi\| : \xi \in \Omega^\dagger \}. \tag{2.2}
\]
Let
\[ D := -i \nabla, \quad D(k) := D + k. \] (2.3)

The Floquet-Bloch decomposition allows to represent our operator (2.1) as a direct integral (see e.g. [4]):
\[ H = \int_{\Omega^\dagger} \oplus H(k) \, dk, \] (2.4)
where
\[ H(k) = D(k)^2 + V(x) \] (2.5)
is the family of ‘fibre’ operators acting in \( L^2(\Omega) \). The domain of each \( H(k) \) is the set of periodic functions from \( H^2(\Omega) \). The spectrum of \( H \) is the union over \( k \in \Omega^\dagger \) of the spectra of the operators (2.5).

We denote by \( | \cdot |_o \) the surface area Lebesgue measure on the unit sphere \( S^{d-1} \) in \( \mathbb{R}^d \) and put \( \omega_d := |S^{d-1}|_o = 2\pi^{d/2}/\Gamma(d/2) \). Finally, \( N(\lambda) := (2\pi)^{-d} \int_{\Omega^\dagger} \# \{ j : \lambda_j(k) < \lambda \} \, dk \) (2.6) is the integrated density of states of the operator (2.1). It is known (see e.g. [4]) that the definitions (1.1) and (2.6) are equivalent.

The main result of the paper is

**Theorem 2.1.** For sufficiently big \( \lambda \) and any \( \varepsilon > 0 \) the integrated density of states of \( H \) satisfies (1.6).

By \( B(R) \) we denote the ball of radius \( R \) centered at the origin. Given two positive functions \( f \) and \( g \), we say that \( f \gg g \), or \( g \ll f \), or \( g = O(f) \) if the ratio \( g/f \) is bounded. We say \( f \asymp g \) if \( f \gg g \) and \( f \ll g \). Whenever we use \( O, o, \gg, \ll, \asymp \) notation, the constants involved can depend on \( d \) and norms of the potential in various Sobolev spaces \( H^s \); the same is also the case when we use the expression ‘sufficiently large’.

By \( \lambda = \rho^2 \) we denote a point on the spectral axis. We also denote by \( v \) the \( L_\infty \)-norm of the potential \( V \), and put \( J := [\lambda - 20v, \lambda + 20v] \). Let
\[ \mathcal{A} := \{ \xi \in \mathbb{R}^d, \, ||\xi||^2 - \lambda \leq 40v \}. \] (2.7)
Notice that the definition of \( \mathcal{A} \) obviously implies that if \( \xi \in \mathcal{A} \), then \( ||\xi| - \rho| \ll \rho^{-1} \).

Any vector \( \xi \in \mathbb{R}^d \) can be uniquely decomposed as \( \xi = n + k \) with \( n \in \Lambda^\dagger \) and \( k \in \Omega^\dagger \). We call \( n = [\xi] \) the ‘integer part’ of \( \xi \) and \( k = \{ \xi \} \) the ‘fractional part’ of \( \xi \).

By \( \text{vol}(\cdot) \) we denote the Lebesgue measure in \( \mathbb{R}^d \). The identity matrix is denoted by \( I \). For any \( h \in L_2(\Omega) \) we introduce its Fourier coefficients
\[ h_n := (\text{vol} \Omega)^{-1/2} \int_\Omega h(x) \exp \left( -i \langle n, x \rangle \right) \, dx, \quad n \in \Lambda^\dagger. \] (2.8)
For $\xi \in \mathbb{R}^d \setminus \{0\}$ we define $r = r(\xi) := |\xi|$ and $\xi' := \xi/|\xi|$. We put
\[ R = R(\rho) := \rho^{1/(36d^2(d+2))} \] (2.9)
so that the condition stated after equation (5.15) in [2] is satisfied. For $j \in \mathbb{N}$ let
\[ \Theta'_j := \Lambda^\dagger \cap B(jR) \setminus \{0\}. \]
Let $M := 5d^2 + 7d$. We introduce the set
\[ B := \{ \xi \in A \, | \, \langle \xi, \eta' \rangle > \rho^{1/2}, \text{ for all } \eta \in \Theta'_6M \}. \] (2.10)
In other words, $B$ consists of all points $\xi \in A$ the projections of which to the directions of all vectors $\eta \in \Theta'_6M$ have lengths larger than $\rho^{1/2}$. We also denote $D := A \setminus B$.

In the rest of the section we quote some results from [2] which we will use in this paper. Our approach is slightly different from that of [2]. In particular, we consider arbitrary lattice of periods $\Lambda$, not equal to $(2\pi \mathbb{Z})^d$. We also use a different form of the Floquet-Bloch decomposition (so that the operators on fibers (2.5) are defined on the same domain). This leads to several straightforward changes in the formulation of the results from [2]. These changes are:

1. The lattices $(2\pi \mathbb{Z})^d$ and $\mathbb{Z}^d$ are replaced by $\Lambda$ and $\Lambda^\dagger$, respectively. The ‘integer’ and ‘fractional’ parts are now defined with respect to $\Lambda^\dagger$ (see above);
2. The matrices $F$ and $G$ are replaced by the unit matrix $I$ throughout;
3. The Fourier transform is now defined by (2.8), and the exponentials $e_m$ introduced at the beginning of Section 5 in [2] are redefined as
\[ e_m(x) := (\text{vol } \Omega)^{-1/2} e^{i\langle m, x \rangle}, \quad m \in \Lambda^\dagger; \]
4. The operators $H(k)$ are now given by (2.5) on the common domain $\mathcal{D}$.

The main result we will need follows from Corollary 7.15 of [2]:

**Proposition 2.2.** There exist mappings $f, g : A \to \mathbb{R}$ which satisfy the following properties:

(i) $f(\xi)$ is an eigenvalue of $H(k)$ with $\{\xi\} = k$; $|f(\xi) - |\xi|^2| \leq 2v$. $f$ is an injection (if we count all eigenvalues with multiplicities) and all eigenvalues of $H(k)$ inside $J$ are in the image of $f$.
(ii) If $\xi \in A$, then $|f(\xi) - g(\xi)| \leq \rho^{-d-3}$. 
(iii) For any $\xi \in \mathcal{B}$
\begin{equation}
\begin{split}
g(\xi) &= |\xi|^2 \\
&+ \sum_{j=1}^{2M} \sum_{n_1, \ldots, n_j \in \Theta'_M} \sum_{2 \leq n_1, \ldots, n_j \leq 2M} C_{n_1, \ldots, n_j} (\xi, n_1)^{-n_1} \ldots (\xi, n_j)^{-n_j}.
\end{split}
\end{equation}

(iv) Let $I = [a, b] \subset \mathcal{A}$ be a straight interval of length $L := |b - a| \ll \rho^{-1}$. Then there exists an integer vector $n$ such that $|g(b + n) - g(a)| \ll L\rho + \rho^{-d-3}$. Moreover, suppose $m \neq 0$ is an integer vector such that the interval $I + m$ is also entirely inside $\mathcal{A}$. Then there exist two different integer vectors $n_1$ and $n_2$ such that $|g(b + n_1) - g(a)| \ll L\rho + \rho^{-d-3}$ and $|g(b + n_2) - g(a + m)| \ll L\rho + \rho^{-d-3}$.

Remark 2.3. Formula (2.11) implies that
\begin{equation}
\frac{\partial g}{\partial r}(\xi) \asymp \rho, \quad \text{for any } \xi \in \mathcal{B}.
\end{equation}

For each positive $\delta < v$ we denote by $\mathcal{A}(\delta)$, $\mathcal{B}(\delta)$, and $\mathcal{D}(\delta)$ the intersections of $g^{-1}\left(\left[\rho^2 - \delta, \rho^2 + \delta\right]\right)$ with $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{D}$, respectively.

It is proved in Lemma 8.1 of [2] that
\begin{equation}
\text{vol}(\mathcal{D}(\delta)) \ll \rho^{d-7/3}\delta.
\end{equation}

The following statement (Corollary 8.5 of [2]) gives a sufficient condition for the continuity of $f$:

Lemma 2.4. There is a constant $C_1$ with the following properties. Let
\begin{equation}
I := \{\xi(t) : t \in [t_{\min}, t_{\max}]\} \subset \mathcal{B}(v).
\end{equation}
be a straight interval of length $L < \rho^{-1}\delta$. Suppose that there is a point $t_0 \in [t_{\min}, t_{\max}]$ with the property that for each non-zero $n \in \Lambda^\dagger$
\begin{equation*}
g(\xi(t_0) + n)
\end{equation*}
is either outside the interval
\begin{equation*}
\left[ g(\xi(t_0)) - C_1\rho^{-d-3} - C_1\rho L, g(\xi(t_0)) + C_1\rho^{-d-3} + C_1\rho L \right]
\end{equation*}
or not defined. Then $f(\xi(t))$ is a continuous function of $t$.

By inspection of the proof of Lemma 8.3 of [2] we obtain

Lemma 2.5. For large enough $\rho$ and $\delta < \rho^{-1}$ the following estimates hold uniformly over $a \in \Lambda^\dagger \setminus \{0\}$: if $d \geq 3$,
\begin{equation}
\text{vol}\left(\mathcal{B}(\delta) \cap (\mathcal{B}(\delta) + a)\right) \ll (\delta^2\rho^{d-3} + \delta\rho^{-d});
\end{equation}

\begin{equation}
\text{vol}\left(\mathcal{B}(\delta) \cap (\mathcal{B}(\delta) + a)\right) \ll (\delta\rho^{d-3} + \delta\rho^{-d}).
\end{equation}
if \( d = 2 \),

\[
\text{vol}(\mathcal{B}(\delta) \cap (\mathcal{B}(\delta) + \mathbf{a})) \begin{cases} \ll \delta^{3/2}, & |\mathbf{a}| \leq 2\rho - 1, \\ \ll \delta^{3/2} + \delta \rho^{-2}, & |\mathbf{a}| - 2\rho < 1, \\ = 0, & |\mathbf{a}| \geq 2\rho + 1. \end{cases} \tag{2.15}
\]

3. Prevalence of regular directions

**Lemma 3.1.** For \( \rho \) big enough and

\[
0 < \delta \leq \rho^{-d-3}
\]

there exists a set \( \mathcal{F} = \mathcal{F}(\rho) \) on the unit sphere \( S^{d-1} \) in \( \mathbb{R}^d \) with

\[
|\mathcal{F}| \geq \omega_d(1 - o(1)) \tag{3.1}
\]

such that \( f(\mathbf{\xi}) \) is a simple eigenvalue of \( H(\{\mathbf{\xi}\}) \) continuously depending on \( r := |\mathbf{\xi}| \) for every \( \mathbf{\xi} = (r, \mathbf{\xi}') \in f^{-1}(\{\rho^2 - \delta, \rho^2 + \delta\}) \) with \( \mathbf{\xi}' := \mathbf{\xi}/|\mathbf{\xi}| \in \mathcal{F} \).

**Proof.** It is enough to consider \( \delta := \rho^{-d-3} \). For each \( \mathbf{\xi}' \in S^{d-1} \) let

\[
I_{\mathbf{\xi}}(\delta) := \{r\mathbf{\xi}', r > 0\} \cap \mathcal{B}(\delta). \tag{3.2}
\]

Let \( \mathcal{F}_1 := \{\mathbf{\xi}' \in S^{d-1} | I_{\mathbf{\xi}}(\delta) \neq \emptyset, \overline{I_{\mathbf{\xi}}(\delta)} \cap \mathcal{D}(\delta) = \emptyset\} \).

For any \( \mathbf{\eta} \in \Theta_{6M} \) the area of the set of points \( \mathbf{\xi}' \in S^{d-1} \) satisfying

\[
\langle r\mathbf{\xi}', \mathbf{\eta}' \rangle \leq \rho^{1/2}
\]

is evidently \( O(\rho^{-1/2}) \) if \( r \geq \rho/2 \) (the latter is true for all \( r\mathbf{\xi}' \in \mathcal{A} \)). Since the number of elements in \( \Theta_{6M} \) is \( O(R^d) \), by (2.9) and (2.10) we have

\[
|S^{d-1} \setminus \mathcal{F}_1| = o(1). \tag{3.3}
\]

By definition \( \mathcal{B}(\delta) = \mathcal{B} \cap g^{-1}(\{\rho^2 - \delta, \rho^2 + \delta\}) \), hence (2.12) implies that for big \( \rho \) the length \( l_{\mathbf{\xi}}(\delta) \) of \( I_{\mathbf{\xi}}(\delta) \) satisfies

\[
l_{\mathbf{\xi}}(\delta) \asymp \delta \rho^{-1}, \quad \mathbf{\xi}' \in \mathcal{F}_1. \tag{3.4}
\]

Let

\[
\mathcal{F} := \{\mathbf{\xi}' \in \mathcal{F}_1 \mid f \text{ is continuous on } I_{\mathbf{\xi}}(\delta)\},
\]

and

\[
\mathcal{E}(\delta) := \{\mathbf{\xi} \in \mathcal{B}(\delta) \mid \mathbf{\xi}' \in \mathcal{F}_1 \setminus \mathcal{F}\}.
\]

Lemma 2.4 tells us that for each point \( \mathbf{\xi} \in \mathcal{E}(\delta) \) there is a non-zero vector \( \mathbf{n} \in \Lambda^1 \) such that

\[
g(\mathbf{\xi} + \mathbf{n}) - g(\mathbf{\xi}) \leq C_1(\rho^{-d-3} + \rho l_{\mathbf{\xi}}(\delta)) \ll (\rho^{-d-3} + \delta). \tag{3.5}
\]

Since \( |g(\mathbf{\xi}) - \rho^2| \leq \delta \), this implies

\[
g(\mathbf{\xi} + \mathbf{n}) - \rho^2 \leq C_2(\rho^{-d-3} + \delta) =: \delta_1 \ll \rho^{-d-3} = \delta,
\]
and thus $\xi + n \in A(\delta_1)$; notice that $C_2 > 1$ and so $\delta_1 > \delta$. Therefore, each point $\xi \in E(\delta)$ also belongs to the set $(A(\delta_1) - n)$ for a non-zero $n \in \Lambda^\dagger$; obviously, $|n| \ll \rho$. In other words,

$$E(\delta) \subset \bigcup_{n \in \Lambda\cap B(\delta)} (A(\delta_1) - n) = \bigcup_{n \neq 0} (B(\delta_1) - n) \cup \bigcup_{n \neq 0} (D(\delta_1) - n).$$  \hfill (3.6)

To proceed further, we need more notation. Denote $D_0(\delta_1)$ to be the set of all points $\nu$ from $D(\delta_1)$ for which there is no non-zero $n \in \Lambda^\dagger$ satisfying $\nu - n \in B(\delta)$; $D_1(\delta_1)$ to be the set of all points $\nu$ from $D(\delta_1)$ for which there is a unique non-zero $n \in \Lambda^\dagger$ satisfying $\nu - n \in B(\delta)$; and $D_2(\delta_1)$ to be the rest of the points from $D(\delta_1)$ (i.e. $D_2(\delta_1)$ consists of all points $\nu$ from $D(\delta_1)$ for which there exist at least two different non-zero vectors $n_1, n_2 \in \Lambda^\dagger$ satisfying $\nu - n_j \in B(\delta)$). Then Lemma 8.7 of [2] implies that we can rewrite (3.6) as

$$E(\delta) \subset \bigcup_{n \neq 0} (B(\delta_1) - n) \cup \bigcup_{n \neq 0} (D_1(\delta_1) - n).$$  \hfill (3.7)

This, obviously, implies

$$E(\delta) \subset \bigcup_{n \neq 0} \left( (B(\delta_1) - n) \cap B(\delta) \right) \cup \bigcup_{n \neq 0} \left( (D_1(\delta_1) - n) \cap B(\delta) \right),$$  \hfill (3.8)

since $E(\delta) \subset B(\delta)$.

The definition of the set $D_1(\delta_1)$ and (2.13) imply that

$$\text{vol} \left( \bigcup_{n \neq 0} \left( (D_1(\delta_1) - n) \cap B(\delta) \right) \right) \leq \text{vol} \left( D_1(\delta_1) \right) \leq C_1 \rho^d \ll \delta_1 \rho^{d-7/3} \ll \rho^{d-7/3}. \hfill (3.9)$$

For $d \geq 3$ Lemma [2.5], inequality $\delta < \delta_1$, and the fact that the union in (3.8) consists of no more than $C \rho^d$ terms imply

$$\text{vol} \left( \bigcup_{n \neq 0} \left( (B(\delta_1) - n) \cap B(\delta) \right) \right) \ll \rho^d (\delta_1^2 \rho^{-d} + \delta_1 \rho^{-d}) \ll \delta (\rho^{d-6} + 1). \hfill (3.10)$$
For $d = 2$ we obtain by Lemma 2.5

\[
\text{vol} \left( \bigcup_{n \in \Lambda^\dagger \setminus \{0\}} \left( B(\delta) \cap (B(\delta_1) + n) \right) \right) \\
\leq \sum_{n \in \Lambda^\dagger \setminus \{0\}} \text{vol} \left( \left( B(\delta) \cap (B(\delta_1) + n) \right) \right) \\
+ \sum_{n \in \Lambda^\dagger \setminus \{0\}} \text{vol} \left( B(\delta) \cap (B(\delta_1) + n) \right) \\
\ll \delta^{3/2} \rho^2 + \rho(\delta^{3/2} + \delta_1 \rho^{-2}) \ll \delta \rho^{-1/2},
\]

where we have used that

\[
\# \left\{ n \in \Lambda^\dagger \left| \left| n \right| - 2\rho \right| < 1 \right\} \ll \rho.
\]

Applying (3.9), (3.10), and (3.11) to (3.8) we obtain for all $d \geq 2$

\[
\text{vol} \mathcal{E}(\delta) \ll \delta \rho^{d-7/3}. \tag{3.12}
\]

By definition,

\[
\mathcal{E}(\delta) = \bigcup_{\xi' \in \bar{F}_1 \setminus F} I_{\xi'}(\delta).
\]

Hence by (3.4)

\[
|F_1 \setminus F|_o \ll \delta^{-1} \rho^{2-d} \text{vol} \mathcal{E}(\delta). \tag{3.13}
\]

Combining (3.12) and (3.13) we conclude that for big $\rho$

\[
|F_1 \setminus F|_o = o(1). \tag{3.14}
\]

We have

\[
|S^{d-1} \setminus F|_o = |S^{d-1} \setminus F|_o + |F_1 \setminus F|_o. \tag{3.15}
\]

Substituting (3.3) and (3.14) into (3.15) we obtain (3.1).

Now we notice that for every $\xi' \in F$ the interval $I_{\xi'}(\delta)$ has the following property: for each point $\xi \in I_{\xi'}(\delta)$ and each non-zero vector $n \in \Lambda^\dagger$ such that $\xi + n \in A$ we have $\left| g(\xi + n) - g(\xi) \right| > 2\rho^{-d-3}$. This implies $f(\xi + n) - f(\xi) \neq 0$. Therefore, $f(\xi)$ is a simple eigenvalue of $H(\{\xi\})$ for each $\xi \in I_{\xi'}(\delta)$. The lemma is proved. \[ \square \]

4. **Some properties of operators on the fibers**

For $m \in \mathbb{R}$ let

\[
V^{(m)} := \left( \sum_{n \in \Lambda^\dagger} \left| n \right|^{2m} \left| V_n \right|^2 \right)^{1/2}.
\]
Since \( V \) is smooth, \( V^{(m)} \) is finite for any \( m \geq 0 \). Recall that \( Q \) is defined by (2.2).

**Lemma 4.1.** Fix \( m \in \mathbb{N} \) and \( \varkappa \in (0, 1) \). For \( k \in \Omega^\dagger \) let \( \psi \) be a normalized eigenfunction of \( H(k) \):

\[
H(k)\psi = \zeta \psi
\]

with the eigenvalue

\[
\zeta \geq \max \left\{ 36Q^2 \varkappa^{-2}, (1 + m \varkappa)^{2/(d-1)} \varkappa^{-2d/(d-1)} \right\}. \tag{4.2}
\]

Then there exists \( M_m = M_m(d, \Lambda, V) \in \mathbb{R}_+ \) such that for all \( n \in \Lambda^\dagger \) with

\[
|n| \geq (1 + m \varkappa) \sqrt{\zeta} \tag{4.3}
\]

the Fourier coefficients of \( \psi \) satisfy

\[
|\psi_n| < M_m \varkappa^{-m} |n|^{-(3m+1)/2}. \tag{4.4}
\]

**Proof.** We proceed by induction. Suppose that either \( m = 1 \), or \( m > 1 \) and the statement is proved for \( m - 1 \). Substituting the Fourier series

\[
\psi(x) = (\text{vol } \Omega)^{-1/2} \sum_{n \in \Lambda^\dagger} \psi_n \exp\left( i \langle n, x \rangle \right), \quad x \in \Omega
\]

into (4.1) and equating the coefficients at \( \exp\left( i \langle n, x \rangle \right) \) on both sides, we obtain by (2.5):

\[
|n + k|^2 \psi_n + \sum_{l \in \Lambda^\dagger} V_{n-l} \psi_l = \zeta \psi_n. \tag{4.5}
\]

Since \(|k| \leq Q\), by (4.2) and (4.3) we have

\[
2 |n||k| \leq \varkappa |n|^2 / 6 + 6 \varkappa^{-1} Q^2 \leq \varkappa |n|^2 / 3. \tag{4.6}
\]

For \( \varkappa \in (0, 1) \), it follows from (4.3) that

\[
|n|^2 - \zeta \geq (1 - (1 + \varkappa)^{-2}) |n|^2 = \varkappa (2 + \varkappa) (1 + \varkappa)^{-2} |n|^2 \geq \varkappa |n|^2 / 2. \tag{4.7}
\]

Combining (4.6) and (4.7) we obtain

\[
|n + k|^2 - \zeta \geq |n|^2 - 2 |n||k| - \zeta \geq \varkappa |n|^2 / 6,
\]

and thus by (4.5)

\[
|\psi_n| < 6 \varkappa^{-1} |n|^{-2} \sum_{l \in \Lambda^\dagger} |V_{n-l}|. \tag{4.8}
\]

If \( m = 1 \) we estimate the sum on the r.h.s. by \( V^{(0)} \) using Cauchy–Schwarz inequality (since \( \psi \) is normalized) and obtain (4.4) with \( M_1 := 6V^{(0)} \).
If \( m > 1 \), we estimate
\[
\sum_{l \in \Lambda^1 : \|l-n\| \leq |n|^{1/d}} |V_{n-l}^1| \leq \sup_{m : |m| > |n|^{1/d}} |\psi_m| \sum_{l \in \Lambda^1 : \|l\| \leq |n|^{1/d}} |V_l|. \tag{4.9}
\]
By (4.3), (4.2), and monotonicity of the function \( q(t) = t - t^{1/d} \) for \( t > 1 \) we have
\[
|n| - |n|^{1/d} \geq (1 + m \varepsilon) \sqrt{\zeta} - ((1 + m \varepsilon) \sqrt{\zeta})^{1/d} \geq (1 + (m-1) \varepsilon) \sqrt{\zeta}.
\]
We can thus apply the induction hypothesis obtaining
\[
\sup_{m : |m| > |n|^{1/d}} |\psi_m| \leq \varepsilon^{1-m} M_{m-1} (1 - |n|^{(1-d)/d})^{1-3m/2} |n|^{1-3m/2}. \tag{4.10}
\]
Since \( \varepsilon \in (0, 1) \), by (4.3) and (4.2) we have
\[
|n| \geq (1 + m \varepsilon) \sqrt{\zeta} \geq (\varepsilon^{-1} + m)^{d/(d-1)} > 2^{d/(d-1)},
\]
thus
\[
(1 - |n|^{(1-d)/d})^{1-3m/2} < 2^{3m/2 - 1}. \tag{4.11}
\]
Let
\[
W := \sup_{r > 1} r^{-d} \# \{ l \in \Lambda^1 : |l| \leq r \}.
\]
Clearly, \( W < \infty \). Then by Cauchy–Schwarz inequality
\[
\sum_{l \in \Lambda^1 : \|l\| \leq |n|^{1/d}} |V_l| \leq W^{1/2} V(0) |n|^{1/2}. \tag{4.12}
\]
Substituting (4.10), (4.11), and (4.12) into (4.9) we get
\[
\sum_{l \in \Lambda^1 : \|l-n\| \leq |n|^{1/d}} |V_{n-l}| < 2^{3m/2 - 1} \varepsilon^{1-m} W^{1/2} V(0) M_{m-1} |n|^{3(1-m)/2}. \tag{4.13}
\]
On the other hand, since \( \|\psi\| = 1 \), by Cauchy–Schwarz we have
\[
\sum_{l \in \Lambda^1 : \|l-n\| > |n|^{1/d}} |V_{n-l}| < |n|^{3(1-m)/2} \sum_{l \in \Lambda^1 : \|l\| > |n|^{1/d}} |l|^{3(m-1)d/2} |V_l||\psi_n|.
\]
\[
\leq |n|^{3(1-m)/2} \left( \sum_{l \in \Lambda^1 : \|l\| > |n|^{1/d}} |l|^{3(m-1)d/2} |V_l|^2 \right)^{1/2} \leq V(3(m-1)d/2) |n|^{3(1-m)/2}. \tag{4.14}
\]
Inserting (4.13) and (4.14) into (4.8) we obtain (4.1) with
\[
M_m := 6(2^{3m/2 - 1} W^{1/2} V(0) M_{m-1} + V(3(m-1)d/2)).
\]
\[\square\]
Lemma 4.2. For any $\eta \in (0, 1)$ there exists $\zeta_0 > 0$ such that if $\zeta(k) \geq \zeta_0$ is a simple eigenvalue of $H(k)$ for some $k \in \Omega^!$ then

$$|\nabla_k \zeta| \leq 2(1 + \eta) \sqrt{\zeta}. \quad (4.15)$$

Proof. Let $\psi(k)$ be the eigenfunction corresponding to $\zeta(k)$ with

$$\|\psi(k)\| = 1. \quad (4.16)$$

Then

$$\nabla_k \zeta(k) = \nabla_k (\psi(k), H(k)\psi(k)) = \left(\psi(k), (\nabla_k H(k))\psi(k)\right). \quad (4.17)$$

By (2.5) and (2.3),

$$\nabla_k H(k) = 2D(k).$$

Substituting this into (4.17) we obtain:

$$|\nabla_k \zeta(k)| \leq 2\|D(k)\psi(k)\| = 2\left(\sum_{n \in \Lambda^!} |n + k|^2 |\psi_n(k)|^2\right)^{1/2}. \quad (4.18)$$

Let

$$m := \left[(d + 1)/3\right] + 1 \quad (4.19)$$

and

$$\varkappa := \eta/(2m + 1). \quad (4.20)$$

We put

$$\zeta_0 := \max \left\{36Q^2 \varkappa^{-2}, (1 + m\varkappa)^{2/(d-1)} \varkappa^{-2d/(d-1)}\right\} \quad (4.21)$$

and assume that $\zeta := \zeta(k) \geq \zeta_0$. Since by (2.2) $|k| \leq Q$, by (4.16), (4.21), and (4.20) we have

$$\sum_{|n| < (1 + m\varkappa)\sqrt{\zeta}} |n + k|^2 |\psi_n(k)|^2 < (1 + (m + 1/6)\varkappa)^2 \zeta < (1 + \eta/2)^2 \zeta. \quad (4.22)$$

For $|n| \geq (1 + m\varkappa)\sqrt{\zeta}$ we apply Lemma 4.1 obtaining

$$\sum_{|n| \geq (1 + m\varkappa)\sqrt{\zeta}} |n + k|^2 |\psi_n(k)|^2 \leq M_m^{2m} \varkappa^{-2m} \sum_{|n| \geq (1 + m\varkappa)\sqrt{\zeta}} |n + k|^2 |n|^{-3m-1}. \quad (4.23)$$

By (1.19), the r. h. s. of (4.23) is finite and is $O(\zeta^{-1/2})$. Thus, increasing $\zeta_0$ if necessary, by (4.18), (4.22), and (4.23) we obtain (4.15). $\square$
5. Proof of Theorem 2.1

It is enough to prove

**Theorem 5.1.** For any $\alpha \in (0, 1)$ there exists $\rho_0 > 0$ big enough such that for all $\rho \geq \rho_0$

$$N(\rho^2 + \delta) - N(\rho^2 - \delta) \geq (1 - \alpha)(2\pi)^{-d}\omega_d \rho^{d-2}$$  (5.1)

for any

$$0 < \delta \leq \rho^{-d-3}.$$  (5.2)

Indeed, the original statement of Theorem 2.1 can be obtained by partitioning of the interval $[\lambda, \lambda + \varepsilon]$ into subintervals with lengths not exceeding $2\lambda^{(-d-3)/2}$ and adding up estimates (5.1) on this subintervals (with $\rho^2$ being respective middle points).

**Proof.** We first express the growth of IDS in terms of the function $f$ of Proposition 2.2:

$$N(\rho^2 + \delta) - N(\rho^2 - \delta) = (2\pi)^{-d} \text{vol}\left(f^{-1}[\rho^2 - \delta, \rho^2 + \delta]\right).$$  (5.3)

We can write

$$\text{vol}\left(f^{-1}[\rho^2 - \delta, \rho^2 + \delta]\right) = \int_{S^{d-1}} \int_{0}^{\infty} \chi(r, \xi') r^{d-1} dr d\xi',$$  (5.4)

where $\chi$ is the indicator function of $f^{-1}([\rho^2 - \delta, \rho^2 + \delta])$. To obtain a lower bound we can restrict the integration in (5.4) to $\xi' \in \mathcal{F}$ defined in Lemma 3.1. Then for any $\eta \in (0, 1)$ there exists $\rho_0 > 0$ such that for any $\rho \geq \rho_0$ we have

$$|\mathcal{F}|_o \geq (1 - \eta)\omega_d,$$  (5.5)

and for any $\xi' \in \mathcal{F}$ the support of $\chi(\cdot, \xi')$ contains an interval $[r_1, r_2]$ with

$$(1 - \eta)\rho \leq r_1 < r_1 + (1 - \eta)\rho^{-1}\delta \leq r_2.$$  (5.6)

Indeed, the first inequality in (5.6) follows from Proposition 2.2(ii),(iii). The last inequality in (5.6) follows from Lemmata 3.1 and 4.2.

Thus for all $\rho \geq \rho_0$ by (5.4) and (5.6) we obtain

$$\int_{S^{d-1}} \int_{0}^{\infty} \chi(r, \xi') r^{d-1} dr d\xi' \geq \int_{\mathcal{F}} (1 - \eta)^d \rho^{d-2} \delta d\xi' \geq (1 - \eta)^{d+1}\omega_d \rho^{d-2}\delta,$$  (5.7)

Combining (5.3), (5.4), and (5.7), and choosing $\eta$ small enough we arrive at (5.1). The theorem is proved. □
References

1. Yulia E. Karpeshina, *On the density of states for the periodic Schrödinger operator*, Ark. Mat. 38 (2000), no. 1, 111–137. MR MR1749362 (2001g:47088)

2. Leonid Parnovski, *Bethe-Sommerfeld conjecture*, Ann. Henri Poincaré 9 (2008), no. 3, 457–508. MR MR2419769

3. Leonid Parnovski and Roman Shterenberg, *Asymptotic expansion of the integrated density of states of a two-dimensional periodic Schrödinger operator*, Invent. Math. 176 (2009), no. 2, 275–323.

4. Michael Reed and Barry Simon, *Methods of modern mathematical physics. IV. Analysis of operators*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978. MR MR0493421 (58 #12429c)

Department of Mathematics, University College London, Gower Street, London, WC1E 6BT, UK