Hydrodynamic signatures and spectral properties of the quantum vortex

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We characterize the low-lying excitations of a quantum vortex in a quasi-two-dimensional Bose-Einstein condensate (BEC) using the standard definition of the density of states (DOS) and a modified version that is sensitive to complementary aspects of the excitation’s spectrum. The latter proves to be particularly relevant to studying the polaronic state realized when an impurity is embedded in a quantum vortex. We establish that the impurity becomes sensitive to the transversal fluctuations of the vortex, via its remnant kelvon mode, and to the phase fluctuations of the BEC Nambu-Goldstone mode. The presence of the vortex yields an anomalous excitation spectrum with a finite energy gap and non-linear DOS at low energies. We find that the high sensitivity of the kelvon mode to external potentials provides a channel of quantum-level control over impurities trapped in a vortex. This extra control channel may be of practical use for the proposal of using vortex-trapped impurities as qubit units for quantum information processing.

I. INTRODUCTION

Bose-Einstein condensates (BEC) host a variety of topological defects emerging collectively from the interactions among its constituent atoms. Notable examples are vortex configurations occurring in three-dimensional and in quasi-two-dimensional (quasi-2D) BECs. Their vorticity is quantized and the singularity in the phase of the BEC’s order parameter causes the fluid to deplete with a long-range density profile [1]. Although a single-charge vortex is dynamically stable and topologically protected from dissipation [2], the system is bound up with multiple dynamics. For one, the vortex is intrinsically subject to the transversal excitations of its filament, known as kelvon modes; these can precipitate the vortex off its rotational axis, triggering an outwards precessional motion that eventually drives it to the boundary [3–11]. Once the vortex is removed then so is the phase singularity, and the BEC is left to evolve according to its intrinsic dynamics alone: the fluid undergoes quantum diffusion [12], driven by the Nambu-Goldstone (NG) mode inherent to the BEC state. One of our main results is to show how these two modes—a kelvon and the NG mode—have also an exceptional capacity to drive the dynamics of impurities embedded in the quantum fluid.

The physics of a BEC in the presence of impurities (e.g., heterogeneous atomic species) has aroused interest in both theory and experiment over the past decade [13–20]; of particular interest are cases when the BEC hosts one or multiple vortices, motivated by the possibility of impurities becoming bound in their core or hopping across the Abrikosov lattice [21–24]. These systems naturally give rise to problems of quantum many-body physics, specifically, along the branch of polaron physics [25, 26]; spectral properties of such many-body systems are ubiquitously extracted via a variety of measures broadly referred to as densities of states—energy-distributions of excited states. We introduce and analyze a class of densities of states, which we call hydrodynamic, that naturally arise in this problem and which charac-
\[ \hat{H} = \int d^2r \hat{\Phi}^\dagger(r) \left[ -\frac{\hbar^2}{2M} \nabla^2 + \mu + V(r) \right] \hat{\Phi}(r) + \frac{1}{2}g \int d^2r \hat{\Phi}^\dagger(r) \hat{\Phi}^\dagger(r) \hat{\Phi}(r) \hat{\Phi}(r), \]  

(1)

describing the field of a bosonic species of mass \(M\) interacting by an effective contact potential of strength \(g > 0\). Atomic annihilation and creation operators \(\hat{\Phi}\) and \(\hat{\Phi}^\dagger\) satisfy the commutation relation \([\hat{\Phi}(r), \hat{\Phi}^\dagger(r')] = \delta(r-r')\) and \(\hat{N} = \int d^2r \hat{\Phi}^\dagger(r) \hat{\Phi}(r)\) is the number operator, whose expectation value is fixed by the chemical potential \(\mu\). The external potential \(V\) is comprised of two contributions: a highly anisotropic trapping potential \(V_t\), rendering the system quasi-2D; a small pinning potential \(V_p\) acting by an effective contact potential of strength \(\xi\). The quasi-2D BEC cloud. The mean-field \(\Phi_0\) satisfies the time-independent Gross-Pitaevskii (GP) equation,

\[ (\hat{H} - \mu) \Phi_0 = 0, \]

(3)

with \(h = -\hbar^2/(2M)\nabla^2 + V\), subject to the boundary condition consistent with the external potential \(V\) and to the normalization condition

\[ \langle \hat{N} \rangle_0 = n_0 \int d^2r |\Phi_0(r)|^2 = N_0, \]

(4)

where \(\langle \ldots \rangle_0\) represents the expected value in the many-body state of the BEC, which can be used to determine \(\mu\).

We choose the natural microscopic units to be the coupling energy \(n_0g\) and the coherence length \(\xi = \hbar/\sqrt{2Mn_0g}\), so that quantities in the Hamiltonian (1) become rescaled as \(r \rightarrow \xi r\), \(\Phi \rightarrow \xi^{-1}\Phi\), \(\mu \rightarrow (n_0g)\mu\), \(h \rightarrow (n_0g)^2\hbar\) and \(\hat{H} \rightarrow (n_0g)^2(\xi^2)n_0\hat{H}\); substituting for (2) and by virtue of (3), we have

\[ \hat{H} = F_0 + \frac{1}{n_0\xi^2} \hat{H}_B + \ldots \]

(5)

where \(F_0 = \int d^2r \Phi_0^\dagger \left( -\frac{\hbar^2}{2M} \nabla^2 + \mu + \frac{1}{2} |\Phi_0|^2 \right) \Phi_0\) is the (classical) free energy of the BEC and

\[ \hat{H}_B = \frac{1}{2} \int d^2r \begin{pmatrix} \hat{\phi} \\ \hat{\phi}^\dagger \end{pmatrix}^\dagger \begin{pmatrix} \sigma_3 \hat{H}_B \left( \begin{pmatrix} \hat{\phi} \\ \hat{\phi}^\dagger \end{pmatrix} \right) \end{pmatrix}, \]

(6)

with the Bogoliubov operator

\[ \hat{H}_B = \sigma_3 \left( h - \mu + 2 |\Phi_0|^2 \right) \begin{pmatrix} \Phi_0^\dagger \\ \Phi_0 \end{pmatrix}, \]

(7)

where \(\sigma_i (i = 1, 2, 3)\) denote Pauli matrices. Higher-order contributions of Eq. (5) are neglected within the Bogoliubov approximation. This requires \(\langle \hat{N} \rangle_0 = \langle \hat{\phi} \hat{\phi}^\dagger \rangle \approx 1\), i.e., in the limit of both weak thermal and quantum depletions. We then expand the excitation operators as

\[ \begin{pmatrix} \hat{\phi}(r) \\ \hat{\phi}^\dagger(r) \end{pmatrix} = \sum_{\lambda=0}^\infty \left( X_\lambda(r) \hat{b}_\lambda + \sigma_1 \bar{X}_\lambda(r) \hat{b}_\lambda^\dagger \right), \]

(8)

in a complete basis of complex vector-valued functions \(X_\lambda(r) = (u_\lambda(r), v_\lambda(r))^T\) (\(X_\lambda\) its complex-conjugate), whose components \(u_\lambda\) and \(v_\lambda\) are the particle and hole components, respectively. By requiring that each \(b_\lambda\) and \(\hat{b}_\lambda^\dagger\) inherit bosonic commutation relations, the diagonalization of (6) becomes an eigenproblem for the \(X_\lambda\) as

\[ \hat{H}_B X_\lambda = \omega_\lambda X_\lambda. \]

(9)

Bogoliubov operators \(\hat{H}_B\) are non-hermitian and, in general, admit complex eigenvalues [2]; the operator we study presently, however, possesses a fully real spectrum [5] and the bi-orthonormality relations

\[ \langle X_\lambda, X_{\lambda'} \rangle = \int d^2r X_\lambda(r)^\dagger \sigma_3 X_{\lambda'} = \delta_{\lambda,\lambda'}, \]

(10)

\[ \langle \sigma_1 X_\lambda, X_{\lambda'} \rangle = \int d^2r \left( \sigma_1 X_\lambda \right)^\dagger \sigma_3 X_{\lambda'} = 0, \]

(11)

hold for every normalized eigenstate, where we introduce the bilinear product \(\langle \cdot, \cdot \rangle\); the diagonalized form of (6) thus reads

\[ \hat{H}_B = \frac{1}{2} \alpha \hat{b}^2 + \sum_{\lambda \neq 0} \omega_\lambda \hat{b}_\lambda^\dagger \hat{b}_\lambda, \]

(12)

where the first term accounts for the dynamics of the phase degree-of-freedom of the BEC [12]. We explain the structure and importance of this term by observing that the variation of Eq. (4), i.e., \(\delta \langle \hat{N} \rangle_0 = \delta N_0\), yields the condition

\[ \int \frac{d^2r}{A} \left( \Phi_0 \Theta_0 + \Phi_0 \Theta_0^\dagger \right) = 1, \]

(13)

where we introduce the adjoint BEC wave function

\[ \Theta_0(r) = \left( N_0 \frac{\partial}{\partial N_0} \right) \frac{\partial \Phi_0}{\partial \mu}(r). \]

(14)
It is because Eq. (3) holds that $\Phi_0$ and $\Theta_0$ comprise, respectively, the zero-mode of $H_B$ (the Nambu-Goldstone mode of the BEC state [30]) and its adjoint, as
\[
H_B \left( \frac{\Phi_0}{\Phi_0} \right) = 0, \quad H_B \left( \frac{\Theta_0}{\Theta_0} \right) = \alpha \left( \frac{\Phi_0}{-\Phi_0} \right). \tag{15}
\]

The second of these equations is obtained by taking the derivative $d/d\rho_0$ of the first and then multiplying it by $\alpha$, where we identify $\alpha \equiv N_0(\partial \mu/\partial N_0)$ [12]. These solutions make up the $\lambda = 0$ mode in (8) as $X_0 = (\Theta_0 + \Phi_0, \Theta_0 - \Phi_0)^T / \sqrt{2A}$, which is normalized, per Eq. (13), as $\langle X_0, X_0 \rangle = 1$, thus determining $\alpha$. The associated operators $\hat{b}_0$ and $\hat{b}_0^\dagger$ comprise the Hermitian operator $\hat{P} = \frac{1}{\sqrt{2}} (\hat{b}_0 + \hat{b}_0^\dagger)$ [31]. For conciseness, we refer to the mode $X_0$ as the Nambu-Goldstone (NG) mode henceforth. The basis (8) thus becomes complete. (The zero-mode pertaining to the position of the vortex core, though related to the present discussion, is present only when the vortex is off-axis [32].)

Making the potential $V$ be isotropic in the plane and assuming a vortex to be on the axis of a disk-shaped quasi-2D BEC, a vortex solution of the GP Eq. (3) can be written in polar coordinates as
\[
\Phi_\nu(r, \varphi) = e^{i\nu \varphi} \phi_\nu(r), \tag{16}
\]
where $\nu$ is an integer, the quantized vorticity, and $\phi_\nu$ satisfies a reduced form of Eq. (3) in the radial coordinate [33]. Inserting (16) in Eq. (7), it becomes apparent that solutions of Eq. (9) can be separated, in polar coordinates, as
\[
X_\lambda(r, \varphi) = e^{i\varphi(m+\nu \sigma_3)} X_{m,n}(r), \tag{17}
\]
where $m$ is an integer, the angular momentum of the excitation, and $n$, the number of nodes in the radial coordinate, is the quantum number associated to the non-zero eigenstates of the reduced Bogoliubov operator
\[
\sigma_3 H_m = \left( \frac{\hbar - \mu + 2\phi_\nu^2}{2 \phi_\nu^2} \frac{\phi_\nu^2}{\hbar - \mu + 2\phi_\nu^2} \right) + \frac{1}{r^2} (m + \nu \sigma_3)^2, \tag{18}
\]
acting on $X_{m,n}(r) = (\alpha_{m,n}(r), \beta_{m,n}(r))^T$. The label $\lambda \neq 0$ is, thus, explicitly identified with the pair of quantum numbers $(m, n)$ of solutions of the eigenproblem
\[
H_m X_{m,n} = \omega_{m,n} X_{m,n}. \tag{19}
\]
Note that, by Eqs. (14) and (15), the NG mode has angular momentum $m = 0$.

B. Densities of states

The local density of states (LDOS) of Bogoliubov excitations is typically defined in terms of the imaginary part of the associated Green’s function [34]. In particular, we can consider an angular momentum-resolved LDOS (AM-LDOS), given by
\[
D^{(B)}_m(r; \omega) = \sum_n \delta^{(B)}_{m,n}(r) \delta(\omega - \omega_{m,n}), \tag{20}
\]
with the spatial density $\delta^{(B)}_{m,n}$ given by
\[
\delta^{(B)}_{m,n}(r) = |u_{m,n}(r)|^2 - |v_{m,n}(r)|^2. \tag{21}
\]
However, a test particle of a different species (i.e., an atomic impurity) that interacts with the species comprising the BEC can be found to be sensitive to AM-LDOS of a distinct form:
\[
D^{(H)}_m(r; \omega) = \sum_n \delta^{(H)}_{m,n}(r) \delta(\omega - \omega_{m,n}), \tag{22}
\]
with the spatial density $\delta^{(H)}_{m,n}$ given by
\[
\delta^{(H)}_{m,n}(r) = |u_{m,n}(r)|^2 + \Phi_0(r)|v_{m,n}(r)|^2, \tag{23}
\]
which is due, physically, to the interactions with the bosonic system being exclusively density interactions. The derivation of this AM-LDOS is illustrated in Appendix (A). We shall refer to the quantities in, and derived from, Eqs. (20) and (22) as the Bogoliubov and the hydrodynamic densities of states, respectively. We coin the latter hydrodynamic densities due to the $\delta^{(H)}_{m,n}$ being identical to the density degree of freedom used in the hydrodynamic formalism for the excitations of a BEC [29].

Associated to the LDOS is the density of states of states (DOS), which provides a measure for counting excitations of a many-body system [29]. The angular momentum-resolved DOS (AM-DOS) are obtained by integrating in space each of the AM-LDOS, yielding
\[
D^{(O)}_m(\omega) = \sum_n c^{(O)}_{m,n} \delta(\omega - \omega_{m,n}), \tag{24}
\]
where $O = B, H$, with weights $c^{(B)}_{m,n} = 1$ for the Bogoliubov densities of states, by virtue of Eq. (10), and
\[
c^{(H)}_{m,n} = \int d^2 r \delta^{(H)}_{m,n}(r) \tag{25}
\]
for the hydrodynamic density of states. The (total) LDOS and DOS are recovered upon summing over all $m$, i.e.,
\[
D^{(O)}(\omega) = \sum_m D^{(O)}_m(\omega) \tag{26}
\]
for the DOS; as in Eq. (24), this is for $O = B, H$.

For comparison with numerical results, we consider a large, homogeneous (i.e., vorticity zero and no boundary effects) quasi-2D BEC, in which case excitations have the Bogoliubov spectrum $\omega_B(k) = \sqrt{k^2 + 2\mu}$, with $k$ the mode’s wave-vector, or momentum, and $k = ||k||$. The Bogoliubov and the hydrodynamic DOS of excitations of
the homogeneous quasi-2D BEC can then be explicitly computed in the continuum approximation of momentum space, i.e., approximating the sum over modes by an integral over $k$, yielding

$$D_B^{(3)}(\omega) = \frac{A}{4\pi} \frac{\omega}{\sqrt{\mu^2 + \omega^2}} = \frac{A}{4\pi} \frac{\omega}{\mu} - O(\omega^3), \quad (27)$$

$$D_B^{(1)}(\omega) = \frac{\mu A}{4\pi} \left( 1 - \frac{\mu}{\sqrt{\mu^2 + \omega^2}} \right) = \frac{A}{8\pi} \frac{\omega^2}{\mu} - O(\omega^4), \quad (28)$$

with $A = A/\xi^2$ the area covered by the BEC in natural units.

C. Vortex nucleation and external potentials

We envision the on-axis vortex to be nucleated by phase imprinting via Laguerre-Gauss beams [35], that is, without imposing any laboratory-frame rotation on the fluid.

We consider the anisotropy of the trapping potential $V_{tr}$ to be produced by a tight harmonic potential in the $z$-direction of energy $\hbar \omega_z \gg \mu$, yielding the effective interaction strength $g = 2\sqrt{2\pi\hbar^2a/(Ml_z)}$, where $a > 0$ is the $s$-wave scattering length and $l_z = \sqrt{\hbar/(M\omega_z)}$ is the characteristic length of the harmonic potential.

The in-plane radial trap is a box potential of radius $R \gg \xi$, making Eq. (3) subject to the boundary condition $\Phi_0(R, \phi) = 0$. In natural units, the radius $R = R\xi$ is given by $R = \sqrt{N/(\pi n_0 \xi^2)}$, where $n_0 \xi^2 = l_z/(4\sqrt{2\pi}a)$ is independent of the in-plane radial size ($R$ is related to a dimensionless coupling strength introduced in Refs. [4] and [36]). For the pinning potential, we consider $V_p(r) = \epsilon \exp(-r^2/w^2)$, i.e., a Gaussian beam with maximum optical potential $\epsilon$ and waist length $w$ [8, 37]; we shall refer to pinning configurations in terms of the ordered pair $(\epsilon, w)$.

For the numerical calculations, we consider experiments with $^7$Li BECs [38] wherein positive values of $a$ of tens of nanometers are accessible via Feshbach resonances, while typical trapping frequencies of hundreds of kiloHertz yield $l_z$ of hundreds of nanometers [39], resulting $n_0 \xi^2 \gg 1$ and, thus, guaranteeing a regime of negligible quantum depletion consistent with the Bogoliubov approximation. Moreover, in these conditions, a BEC of up to $N \approx 10^5$ atoms may yield up to $R \approx 200$, while experiments with box potentials up to $R \approx 70\mu m$ [40] yield, in laboratory units, $n_0\hbar/h \approx 30$kHz, or $n_0/k_B \approx 200$nK, and $\xi \approx 300$nm.

III. RESULTS

We give an account of a numerical analysis of the quantities presented in Sec. II B. We begin by briefly explaining the numerical methods used and the motivation for the inclusion of a pinning potential. Then, we present and discuss results for the Bogoliubov and the hydrodynamic DOS, Eq. (26) for $O = B, H$, where we will encounter details that motivate an investigation of the low-lying states of definite angular momentum. Using the AM-DOS and the AM-LDOS, we make the physical origin of the hydrodynamic signatures clear. We follow up with a detailed account of the dependence of low-energy modes on the pinning potential showing, in particular, the sensitivity of the LCLS and of its hydrodynamic weight to this perturbation. Finally, we provide a scaling analysis by which we identify anomalies in the low-lying excitation spectrum.

A. Numerical vortex solutions

Combining Eq. (3) and (16) yields the reduced radial equation

$$\left( \hbar + \frac{\mu^2}{r^2} - \mu + \phi_0^2(r) \right) \phi_\nu(r) = 0. \quad (29)$$

We generate numerical solutions of (29) for $\nu = 1$ using a combination of imaginary-time evolution and an $R$-asymptotic approximation, as outlined in Appendix B; results are presented in Fig. 1(a). We then use these to obtain numerical solutions of Eq. (19) using a discretization-based solver.

Single-charge (i.e., $|\nu| = 1$) vortices are dynamically stable, meaning that the associated Bogoliubov operator (7) possesses only real eigenvalues. However, due to a negative energy of the LCLS, they are energetically unstable, meaning that the spectrum of $H_B$, Eq. (12), has a negative eigenvalue and, therefore, that the mean-field (29) is energetically unstable. The existence of the LCLS has long been recognized and known to trigger the vortex’s spiraling-out motion [3–11]. In Fig. 1 we show features of the BEC wave function and its adjoint near the vortex core, as well as the Bogoliubov and hydrodynamic spatial densities of the LCLS.

B. Total densities of states

Figure 2 shows plots of the Bogoliubov and the hydrodynamic DOS for multiple values of $R$ and fixed $(\epsilon, w)$. We have chosen to represent the Dirac delta function in the continuum approximation of momentum space, i.e., approximating the sum over modes by an integral over $k$, yielding

$$\delta(\omega) \sim \frac{1}{\pi} \frac{\delta \omega}{\omega^2 + \delta \omega^2}, \quad (30)$$

with the width $\delta \omega = O(R^{-2})$; this width is of the energy scale of a single particle in a rigid wall potential of size $R$ and yields the Dirac delta in the limit $R \to \infty$. We see that, apart from noise, the numerical results are in good agreement with the analytical results in Eqs. (27) and (28) for the homogenous BEC.
Both $D^{(B)}_m$ and $D^{(H)}_m$ deviate from the numerical result only in the low-energy region of the spectrum, around $\omega \sim 0$, as shown in the insets of Figs. 2. This deviation is twofold: i) the low-lying hydrodynamic DOS is dominated by a peak at energy $\omega_{-1,0}$ of the LCLS (or remnant kelvon mode) [inset of Fig. 2(b)]; ii) the Bogoliubov DOS shows that the energy $\omega_{0,0} = \omega_{0,0}(R)$, the energy of the first non-kevonic excitation, appears to become larger than the typical inter-level spacing with increasing $R$ [highlighted in the inset of Fig. 2(a)]. Additionally, the presence of the NG mode (especially non-exclusive) is not inconsistent with the Bogoliubov spectrum $\omega_{B}(k)$, which accounts only for density excitations. (Although excitations of the Bogoliubov spectrum entail phase excitations, the NG mode is an excitation of the phase exclusively.)

### C. Angular momentum-resolved densities of states

#### 1. AM-DOS

Figure 3 shows plots of the AM-DOS for selected values of $(\epsilon, w)$ and $R$ and angular momenta $m = 0, \pm 1, \pm 2$. Notable features of the hydrodynamic DOS are reproduced here: the NG mode, at $m = 0$, and the LCLS, at $m = -1$, tower over all other low-lying states.

There is a visible growth of the hydrodynamic weight, Eq. (25) (the height of the peaks in each $D^{(H)}_m$), with energy that ties in with the known breakdown of the hydrodynamic approximation beyond low energies [29]: it signals a departure from a collective sound-wave (phononic) picture of excitations, wherein the spectrum is $\sim k$, to a single-particle (atomic) one, with the spectrum $\sim k^2$. The transition from the phononic to the atomic picture is accompanied by a decrease in the magnitude of the hole-component $v_\lambda$ relative to the particle-component $u_\lambda$ of the excitation [29]. Thus, in this sense, the hydrodynamic weight $c^{(H)}$, Eq. (25), is a measure of the particle-hole imbalance of a bosonic state. We explain this observation by noting that the hydrodynamic spatial density $\varrho^{(H)}_\lambda$, Eq. (23), amounts to an interference pattern between the amplitudes of the particle and hole components $u_\lambda$ and $v_\lambda$; this interference is essentially destructive, since $\Phi_0 v_\lambda$ has only a phase $e^{i\pi}$ relative to $\Phi_0 u_\lambda$. Thus, the discrepancy between the magnitudes of $u_\lambda$ and $v_\lambda$ determines the intensity of the interference pattern $\varrho^{(H)}_\lambda$ and then $c^{(H)}_\lambda$, being its integral, functions as a global measure of the particle-hole imbalance of the mode $X_\lambda$.

We note, moreover, that the presence of the vortex is known to lift the angular momentum-degenerate excitations of an otherwise homogeneous BEC [41], a feature that we highlight in Fig. 3.

Contrarily to the DOS, the AM-DOS of a finite system are sparse (compare Figs. 2 and 3), that is, states are separable within each angular momentum sector. It follows that the spatial dimension of the AM-LDOS can be represented faithfully in terms of individual spatial densities $\varrho^{(O)}_{m,n}$, Eqs. (21) and (23), alone. These are displayed in Fig. 4. In particular, the Bogoliubov density of the LCLS is shown to be orders-of-magnitude larger at
FIG. 2. Plots of the (a) Bogoliubov and (b) hydrodynamic DOS (solid), compared with the plots of Eqs. (27) and (28) (dashed), for system sizes $R = 50$ (red), $R = 125$ (orange) and $R = 200$ (blue) (appearing in ascending order in the plots), and pinning $(\epsilon, w) = (0.1, 1)$. Distinct scales in the vertical axes show that $D^{(B)}$ bounds $D^{(H)}$. Insets zoom into the low-energy features in the boxed regions of the respective plots: (a) the highlighted energies are the energy of the first density (non-kelvonic) excitation $\omega_{0 \alpha}$, for the two system sizes $R = 125$ and $R = 200$; (b) peaks belonging to the NG mode and LCLS are labelled; the energy $\omega_{-1,0}$ is the energy of the LCLS at the selected configuration of the pinning potential. The notation of the energy levels follows Eq. (19).

This happens because $\varrho^{(B)}$ in Eq. (21) is sign-indefinite, so that a bosonic state can be locally particle-like (hole-like) in regions of positive (negative) sign; accordingly, the state is locally characterized by an accumulation (depletion) of atoms proportional to its magnitude. Complementarily to the hydrodynamic weight $c^{(H)}_{m,n}$, which acts as a global measure, the Bogoliubov spatial density $\varrho^{(B)}$ acts as a local measure of particle-hole character of a state. The LCLS is, therefore, largely more particle-like in magnitude than most other low-lying states [4], whence its hydrodynamic weight derives exceptional magnitude. The one exception is the NG mode, which is likewise particle-like but delocalized: its Bogoliubov AM-LDOS is similar in magnitude to that of the first non-kelvonic excitation, for instance, but the hydrodynamic AM-DOS of the latter is vanishing—in fact, the hydrodynamic weight of the first is negligible while that of the NG surpasses the LCLS (see Fig. 3). As energy increases, the magnitudes of the Bogoliubov and hydrodynamic densities become generically comparable, as modes become progressively more particle-like. These features are patent in Fig. 4.

D. Features of low-lying states

1. Effect of the pinning potential

The following results show how the LCLS is strongly dependent on the pinning potential, while other modes
FIG. 4. Plots of the Bogoliubov (blue, $O = B$) and hydrodynamic (red, $O = H$) spatial densities for the first few states of angular momentum $m = 0, \pm 1, \pm 2$ at system size $R = 50$ and pinning $(\epsilon, w) = (0.1, 0.1)$, representing the AM-LDOS at each value $\omega = \omega_{m,n}$ along the right-vertical axes. Magnitudes of spatial densities can be inferred from the scale on the left-vertical axis of each panel; each Bogoliubov spatial density has locally a larger magnitude than its hydrodynamic counterpart. $m = \pm 2$: densities are nearly identical except at $r \approx 0$ (more visible with increasing energies). $m = +1$: the Bogoliubov density is hole-like (negative) at $r \approx 0$. $m = 0$: the lowest state (NG mode) is plotted in distinct colors—purple for $O = B$, orange for $O = H$; unlike most other states, the hydrodynamic spatial density is comparable with its Bogoliubov counterpart. $m = -1$: the lowest state (LCLS) is plotted in likewise distinct colors and in dotted lines, shown here for comparison—a clearer picture displayed in Fig. 1(b.2); here, the Bogoliubov density is purposefully shown to exceed the plot range.

have only a negligible (and indirect) dependence. We begin by noting that the pinning energy competes against the dominant centrifugal barrier $(m + \nu \sigma_3)^2/r^2$ at the vortex core $r \lesssim 1$ in Eq. (18). A mode $(m, n)$ is, thus, insensitive to the pinning as long as it cannot penetrate the centrifugal barrier: there may be a non-negligible dependence on the pinning potential only in case $\omega_{m,n} \gtrsim (m \pm \nu)^2$. This observation ensures that we will find a negligible effect for all low-energy modes except at angular momenta $m = \pm 1$; we have found the numerical evidence to support this observation, and so we focus our discussion on the case $m = -1$, for definiteness. In Fig. 5, we depict the eigenvalues and hydrodynamic weights $c^{(H)}$, Eq. (25) (shown as the size of plot markers), at low energies, as functions of the pinning parameters. The strong dependence of the LCLS on the pinning potential is visible and, furthermore, we see that the $c^{(H)}$ cross over at avoided level crossings, suggesting that the LCLS enters the energy region of (non-kelvonic) density excitations. In order to clarify these features, we derived the minimal model described next.

We consider the reduced Bogoliubov Eq. (19) with the Hamiltonian rewritten as $\mathcal{H} = \mathcal{H}^{(0)} + \Delta \mathcal{H}$, where

$$
\Delta \mathcal{H} = \sigma_3 V_p + \sigma_3 \delta_{\text{BEC}},
$$

with

$$
\delta_{\text{BEC}} = \begin{pmatrix}
-\delta \mu + 2\delta \dot{\phi}^2 & \delta \phi^2 \\
\delta \dot{\phi}^2 & -\delta \mu + 2\delta \phi^2
\end{pmatrix},
$$

where $\delta \mu = \delta \mu(p, w) + \delta \mu(\epsilon, w)$ and $\delta \phi^2 = \delta \phi^2(p, w) + \delta \phi^2(\epsilon, w)$. We have only a negligible (and indirect) dependence. We begin by noting that the pinning energy competes against the dominant centrifugal barrier $(m + \nu \sigma_3)^2/r^2$ at the vortex core $r \lesssim 1$ in Eq. (18). A mode $(m, n)$ is, thus, insensitive to the pinning as long as it cannot penetrate the centrifugal barrier: there may be a non-negligible dependence on the pinning potential only in case $\omega_{m,n} \gtrsim (m \pm \nu)^2$. This observation ensures that we will find a negligible effect for all low-energy modes except at angular momenta $m = \pm 1$; we have found the numerical evidence to support this observation, and so we focus our discussion on the case $m = -1$, for definiteness. In Fig. 5, we depict the eigenvalues and hydrodynamic weights $c^{(H)}$, Eq. (25) (shown as the size of plot markers), at low energies, as functions of the pinning parameters. The strong dependence of the LCLS on the pinning potential is visible and, furthermore, we see that the $c^{(H)}$ cross over at avoided level crossings, suggesting that the LCLS enters the energy region of (non-kelvonic) density excitations. In order to clarify these features, we derived the minimal model described next.

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$$
\Delta \mathcal{H} = \sigma_3 V_p + \sigma_3 \delta_{\text{BEC}},
$$

with

$$
\delta_{\text{BEC}} = \begin{pmatrix}
-\delta \mu + 2\delta \dot{\phi}^2 & \delta \phi^2 \\
\delta \dot{\phi}^2 & -\delta \mu + 2\delta \phi^2
\end{pmatrix},
$$

where $\delta \mu = \delta \mu(p, w) + \delta \mu(\epsilon, w)$ and $\delta \phi^2 = \delta \phi^2(p, w) + \delta \phi^2(\epsilon, w)$. We have only a negligible (and indirect) dependence. We begin by noting that the pinning energy competes against the dominant centrifugal barrier $(m + \nu \sigma_3)^2/r^2$ at the vortex core $r \lesssim 1$ in Eq. (18). A mode $(m, n)$ is, thus, insensitive to the pinning as long as it cannot penetrate the centrifugal barrier: there may be a non-negligible dependence on the pinning potential only in case $\omega_{m,n} \gtrsim (m \pm \nu)^2$. This observation ensures that we will find a negligible effect for all low-energy modes except at angular momenta $m = \pm 1$; we have found the numerical evidence to support this observation, and so we focus our discussion on the case $m = -1$, for definiteness. In Fig. 5, we depict the eigenvalues and hydrodynamic weights $c^{(H)}$, Eq. (25) (shown as the size of plot markers), at low energies, as functions of the pinning parameters. The strong dependence of the LCLS on the pinning potential is visible and, furthermore, we see that the $c^{(H)}$ cross over at avoided level crossings, suggesting that the LCLS enters the energy region of (non-kelvonic) density excitations. In order to clarify these features, we derived the minimal model described next.

We consider the reduced Bogoliubov Eq. (19) with the Hamiltonian rewritten as $\mathcal{H} = \mathcal{H}^{(0)} + \Delta \mathcal{H}$, where

$$
\Delta \mathcal{H} = \sigma_3 V_p + \sigma_3 \delta_{\text{BEC}},
$$

with

$$
\delta_{\text{BEC}} = \begin{pmatrix}
-\delta \mu + 2\delta \dot{\phi}^2 & \delta \phi^2 \\
\delta \dot{\phi}^2 & -\delta \mu + 2\delta \phi^2
\end{pmatrix},
$$

where $\delta \mu = \delta \mu(p, w) + \delta \mu(\epsilon, w)$ and $\delta \phi^2 = \delta \phi^2(p, w) + \delta \phi^2(\epsilon, w)$.
Taking the bilinear product
\[ \langle X_0^{(0)}, H X \rangle \], for \( i = 0, 1 \), yields the algebraic equation
\[
\begin{pmatrix}
\omega_0^{(0)} + \Delta_{00} & \Delta_{01} \\
\Delta_{10} & \omega_1^{(0)} + \Delta_{11}
\end{pmatrix}
\begin{pmatrix}
a_0 \\
a_1
\end{pmatrix}
= \omega
\begin{pmatrix}
a_0 \\
a_1
\end{pmatrix},
\]
where \( \Delta_{ij} = \langle X_i^{(0)}, \Delta H X_j^{(0)} \rangle \), with the eigenvalues
\[
\omega_{n=0,1} = \tilde{\omega} + \Delta_+ \mp \sqrt{\Omega - \Delta_-^2 + |\Delta_{01}|^2},
\]
where \( \tilde{\omega} = (\omega_1^{(0)} + \omega_0^{(0)})/2 \), the half-gap \( \Omega = (\omega_1^{(0)} - \omega_0^{(0)})/2 \) and \( \Delta_\pm = (\Delta_{00} \pm \Delta_{11})/2 \); the minus- (plus-) signed branch in Eq. (33) is the \( n = 0 \) (\( n = 1 \)) solution. The comparisons of Eqs. (33) with the numerical results in Fig. 5 reveal the qualitative agreement of the minimal model; the quantitative inaccuracy results simply from the truncated subspace in Eq. (32) and is of no bearing to the following analysis.

Further comparing Eq. (33) with numerics in Fig. 5, we observe \( \Delta_{00} \) to be much larger than \( \Delta_{01} \) and \( \Delta_{11} \); indeed, we found \( \Delta_{01} \approx 10^{-2} \Delta_{00} \) while \( 10^{-2} < \Delta_{11}/\Delta_{01} \lesssim 1 \) across the sampled values of \((\epsilon, \omega)\). To clarify these disparities, we consider the quantity
\[
\eta_{ij} = \frac{\langle X_i^{(0)}, \sigma_3 V_p X_j^{(0)} \rangle}{\langle X_i^{(0)}, \sigma_3 V_p X_j^{(0)} \rangle},
\]
that is, the ratio of the contributions to the \( \Delta_{ij} \), Eq. (30): the term \( V_p \) is the potential energy due to the pinning potential; the term \( \delta_{\text{BEC}} \), Eq. (31), is the potential energy due the deformation of the BEC caused by the pinning potential. (Note that both terms are effects of the application of the pinning potential on the system, but that \( V_p \) is the direct effect while \( \delta_{\text{BEC}} \) is an indirect effect on its modes, in particular.) We found that \( \eta_{11} > 0 \) with \( 0.1 < \eta_{11} < 20 \) and, for \( i = 0, 1, \eta_{0i} < 0 \) (negative due to the deformation term) with \( 10^{-2} < |\eta_{0i}| < 0.5 \), increasing with \( \omega \) in all cases—that is, the term \( V_p \) dominates over \( \delta_{\text{BEC}} \) for both \( \Delta_{00} \) and \( \Delta_{01} \) and vice-versa for \( \Delta_{11} \).
This shows, since $\Delta_{00} \gg \Delta_{01} \gg \Delta_{11}$, that the LCLS is strongly dependent on the pinning potential directly, due to its exceptionally large amplitude at the vortex core; other modes are, at most, negligibly dependent on the pinning potential and indirectly so via the deformation of the BEC, as shown by $\Delta_{11}$.

We analyze the crossover of the $c^{(H)}$ by considering the probability of observing the pinning potential-free LCLS (i.e., the state $X_0^{(0)}$) given a state $X_n$, that is, the overlap $|a_{0,n}|^2 = |\langle X_0^{(0)}, X_n \rangle|^2$, plotted in Fig. 6. This figure of merit differentiates the states that possess a large, particle-like density at the vortex core for each configuration of the pinning potential. In particular, we can conclude that the LCLS (i.e., the mode with quantum numbers $m = -1$ and $n = 0$) eventually loses the characteristic hydrodynamic weight resulting from the large, core-localized density, as this becomes, due to the pinning potential, a feature of higher energy states. Hence the negligible magnitude of the $\theta_{\text{LCLS}}^{(0)}$ seen in Fig. 1(c.2) relative to Fig. 1(b.2). Moreover, we notice that the crossover $a_{0,0}$ with $a_{0,1}$ is relatively steep while $a_{0,1}$ with $a_{0,2}$ is markedly smoother and can be seen to take place at a value $|a_{0,2}|^2 < 0.5$. This suggests that, as the intensity of the pinning potential increases (and with it the energy of a density at the vortex core), the large hydrodynamic weight characteristic of the LCLS (for a weak pinning potential) becomes spread out across a number of modes instead of concentrated in a single one, and, so, there will be a number of modes with increased hydrodynamic weights instead of a single dominant one. Thus, we can think of the large hydrodynamic weight,
initially concentrated in the LCLS, as becoming diluted under a sufficiently intense pinning potential. This effect is shown in terms of the hydrodynamic AM-DOS in Fig. 7.

2. Scaling and $R$-dependence

We carried out scaling analyses concerning the discrepancy between the gap (that is, the energy of the lowest non-kelvonic excitation $\omega_{0,0}$) and the mean inter-level spacing $\Delta \omega$, as apparent in the inset of Fig. 2(a). Thus, we obtained information on the appropriate low-energy, continuum description of density excitations; results are shown in Fig. 8. Indeed, we find distinct scalings between the energy of the first density excitation and the mean inter-level spacing, indicative of a gap that scales anomalously as $O(R^{-1/2})$. For comparison, we show the prediction for a homogeneous BEC (given by the Bogoliubov spectrum $\omega_{B}$) to scale as $R^{-1}$; this gap is a trivial finite-size effect, as it scales equally to the mean inter-level spacing and approaches a massless spectrum. Notice that the deviation in the exponent of the mean inter-level spacing from that of the homogeneous spectrum, though small, is not accounted for by the error margin. Considering that the group velocity must vanish at the gap, this deviation is suggestive of a non-analytic, momentum-dependent correction to the linear dispersion.

Finally, it shows in Fig. (6) that the energy level of the LCLS does not cross zero at the presented system size, that is, that it does not represent an energetic instability. Indeed, we have found that this mode stabilizes spontaneously for a system size $R \gtrsim 73$, in qualitative agreement with Ref. [4]. (Eventual quantitative discrepancies are attributed to the fact that a quasi-2D BEC produced by a box potential is considered here.)

IV. DISCUSSION AND CONCLUSIONS

We have acquired new insights into the spectral properties of a quantum vortex in a quasi-2D BEC by analyzing the symplectic and hydrodynamic densities of states (DOS). Our discussion was particularly focused on two modes of the system. On the one hand, the lowest core-localized state (LCLS), which is the remnant kelvonic mode surviving the dimensionality reduction imposed by the trap. On the other, the Nambu-Goldstone (NG) mode inherent to the BEC state. Most strikingly, we have shown that these modes have an exceptional capacity to interact with impurities embedded in the BEC. Therefore, they can have a sizeable influence on the dynamics of these impurities, or, more generally, heterogeneous atomic species. Regarding the NG mode, we propose that this effect may be observable in the dynamics of degenerate fermion-BEC mixtures as well as heterogeneous BEC mixtures [42–45]. A proper account of the NG mode also becomes important to the polaron physics of the impurity-BEC system, as impurities can become exceptionally sensitive to the phase fluctuations of the BEC. Physically, this effect is rooted in the particular way excitations interact with an impurity, of which strength results from an interference pattern between their particle and hole components.

The LCLS will greatly affect impurities trapped in the core of the vortex, due to its localized character. It can stimulate transitions between quantum states of the impurity, be them orbital states or internal degrees of freedom, by virtue of the LCLS being an eigenstate of angular momentum. In turn, the LCLS is highly sensitive to the action of a pinning potential (this is the physical fundament of that mechanism of vortex stabilization [8, 28]). It follows that the LCLS affords a novel mechanism of control over a vortex-trapped impurity at the quantum level: its energy can be tuned to a transition between two states of distinct angular momentum of the impurity, by means of the pinning potential, and stimulate the transition. As a matter of fact, there is no fundamental reason for this mechanism to be limited to the single-vortex configuration considered presently—we can expect local properties of the excitations to hold in different physical setups. For instance, in the Abrikosov lattice of a quasi-2D BEC, each vortex in the array is bound to possess a qualitatively similar remnant kelvonic mode [46]. By allowing for such a degree of local, quantum-level control, this channel may be of practical use for a proposal based on vortex-trapped impurities as qubit units for quantum information processing [22, 47, 48].

Based on the symplectic DOS, we have inferred anomalies in the spectrum of excitations, namely, that the excitations of a large (but finite) BEC hosting a vortex are slightly (but non-vanishingly) gapped. This is anomalous with respect to the typical Bogoliubov spectrum, which is gapless and linear. One implication of a gapped dispersion is that the group velocity must vanish at the gap. Our scaling analysis of the mean inter-level spacing is consistent with this condition: it presents a non-negligible deviation from linearity, suggesting a non-analytic dependence in momentum. Physically, we attribute these results to the non-local, long-range profile of the vortex: its decays as $\sim 1/r^2$ results in the log($R$)-divergent energy of the BEC and, notably, in a logarithmically-modified dispersion of its kelvons (in a three-dimensional BEC) [1]. To our knowledge, however, the theory to support the long-wavelength dispersion of the in-plane, non-kelvonic density excitations is yet to be established. This may be important to the quantum treatment of vortex dynamics in a quasi-2D BEC [49].

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Appendix A: Decay width of an impurity bounded in a quantum vortex

We motivate the introduction of the hydrodynamic densities of states by obtaining the decay width between states of an impurity bounded in a quantum vortex.

We consider the BEC described by Hamiltonian (1) to be in the presence of atoms of a distinct species, described by a Hamiltonian

$$\hat{H}_{\text{imp}} = \int d^2r \hat{\Psi}^\dagger(\mathbf{r}) \left[ \frac{-\hbar^2}{2M_2} \nabla^2 + g_{12} \hat{\Phi}^\dagger(\mathbf{r}) \hat{\Phi}(\mathbf{r}) \right] \hat{\Psi}(\mathbf{r}), \tag{A1}$$

that is, a field of dilute (i.e., non-interacting) atoms of mass $M_2$ interacting with BEC atoms by a contact potential of strength $g_{12} > 0$; we identify the atoms of mass $M_2$ as impurities with respect to the BEC, since they are assumed to be dilute and of a distinct species; the total Hamiltonian will be $\hat{H} = \hat{H}_{\text{BEC}} + \hat{H}_{\text{imp}}$.

Substituting the BEC fields for (2) in (A1) yields, to leading and sub-leading orders,

$$\hat{H}_{\text{imp}} = \frac{\chi}{n_0 \xi^2} \int d^2r \hat{\Psi}^\dagger \hat{\Psi}^\dagger + \frac{\chi \gamma^2}{(n_0 \xi^2)^2} \int d^2r \hat{\Psi}^\dagger \left( \hat{\Phi}_\nu \hat{\phi} + \hat{\Phi}_\nu^\dagger \hat{\phi}^\dagger \right) \hat{\Psi} + \ldots, \tag{A2}$$

scaled to the natural units $n_0 g$ and $\xi$, according to the prescriptions in the main text, having introduced a Schrödinger operator $\hat{h}_{\text{imp}} = -\nabla^2 + \gamma^2 |\phi|_r^2$ and parameters $\chi = M/M_2$ and $\gamma^2 = M_2 g_{12}/(M g)$; performing a further scaling $\hat{H}_{\text{imp}} \rightarrow \frac{\chi}{n_0 \xi^2} \hat{H}_{\text{imp}}$, the total Hamiltonian can be written as

$$\hat{H} = F_0 + \frac{1}{n_0 \xi^2} \left( \hat{H}_B + \chi \hat{H}_{\text{imp}} \right).$$

Bosonic field operators $\hat{\phi}$ and $\hat{\phi}^\dagger$ are expanded as in Eq (8), a basis of eigenfunctions of $\hat{H}_B$, resulting

$$\hat{\Phi}_\nu(\mathbf{r}) \hat{\phi}(\mathbf{r}) + \text{h.c.} = \sum_{m,n} \left( \zeta_{m,n}(\mathbf{r}) \hat{b}_{m,n} + \zeta_{m,n}^*(\mathbf{r}) \hat{b}^\dagger_{m,n} \right),$$

where the identification $\lambda = (m, n)$ is made. Impurity field operators $\hat{\Psi}$ and $\hat{\Psi}^\dagger$ can be expanded in a basis of eigenfunctions of $\hat{h}_{\text{imp}}$: the vortex profile of the BEC density $|\Phi(\mathbf{r})|^2$ – along with the condition that $g_{12} > 0$ – essentially guarantees the existence of bound states of the impurities localized at the vortex core [22]; these are also eigenstates of angular momentum in the plane, since the density $|\Phi(\mathbf{r})|^2$ is cylindrically symmetric.

For the purpose of this derivation, we consider a two-level truncated basis comprised of the lowest-energy states of angular momenta $\ell \neq 0$ and $\ell' = 0$, i.e., we let $\hat{\Psi}(\mathbf{r}) = \Psi_0(\mathbf{r}) \hat{a}_0 + \Psi_\ell(\mathbf{r}) \hat{a}_\ell$, yielding the effective Hamiltonian

$$\hat{H}_{\text{eff}} = \hat{H}_B + \Delta \left( \hat{a}_\ell^\dagger \hat{a}_\ell - \hat{a}_0^\dagger \hat{a}_0 \right) + D \chi \gamma^2 \int d^2r \Psi_\ell \zeta_{\ell,n} \Psi_0; \tag{A3}$$

with $2\Delta$ the energy gap, where $g_{\ell,0}^{(n)} = \delta_{\ell,m} g_{\ell}^{(n)}$ and $g_{0,\ell}^{(n)} = \delta_{-\ell,m} g_{-\ell}^{(n)}$.

The selection rules above are made apparent from the fact that $L \zeta_{m,n} = m \zeta_{m,n}$ and $L \Psi_\ell = \ell \Psi_\ell$, with $L = -i \partial/\partial \varphi$ the two-dimensional angular momentum operator. An effective model for the dynamics of the impurity coupled to the bosonic bath follows from (A3) by projecting onto a single-particle subspace of the impurity in the rotating-wave approximation and considering only the bosonic modes of angular momentum $m = \ell$:

$$\hat{H}_{\text{eff}} = \sum_n \omega_\ell n \hat{b}_\ell^\dagger \hat{b}_\ell + D \sigma_3 + \ldots$$

$$+ \sum_n \left( e^{-i(\omega_\ell n - \Delta)} g_{\ell,0}^{(n)} \hat{b}_\ell n \sigma_+ + \text{h.c.} \right),$$

with $\sigma_+$ the raising operator of impurity levels. A standard approach to this problem is to employ the Wigner-Weisskopf approximation [50], which predicts a decay width

$$\Gamma_{\ell \rightarrow 0} = \frac{\pi}{n_0 \xi^2} \sum_n |g_{\ell}^{(n)}|^2 \delta(\Delta - \omega_\ell, n).$$

Then, considering that the impurity is localized in the vortex core of a large BEC, the $|g_{\ell}^{(n)}|^2$ can be approximated to rewrite the width as

$$\Gamma_{\ell \rightarrow 0} \approx \frac{\pi^2 \gamma^4}{n_0 \xi^2} \int d^2r D_{\ell}^{(H)}(\mathbf{r}; \Delta) |\Psi_\ell(\mathbf{r})|^2 |\Psi_0(\mathbf{r})|^2,$$

where $D_{\ell}^{(H)}(\mathbf{r}; \Delta) = \sum_n |\zeta_{\ell,n}(\mathbf{r})|^2 \delta(\Delta - \omega_\ell, n)$ is a hydrodynamic AM-LDOS, Eq. (22).

Appendix B: Computation of the radial BEC profile

A box potential, such as considered in this work, endows the BEC wave function with a nearly-homogeneous
FIG. 9. Asymptotic solutions \( f_i \), for orders \( i = 0, 1, 2 \), of Eq. (B1) for \( \nu = 1 \) and \( \mu > 1 \).

profile, with the exception of an exponentially-fast depletion of the wave function at the border \([51]\). On the one hand, as the size of the BEC increases, it becomes numerically non-trivial to compute its full profile, seeing as the depletion becomes steeper and more abrupt at the scale of the system, as illustrated in Fig. 1(a); on the other hand, provided that any other non-homogeneous feature is sufficiently localized within the bulk, that is, away from the border of the BEC, the computation of its wave function and the computation of the wave function near the border can be asymptotically separated. We employ this observation to compute the solution of Eq. (29) by introducing an ansatz of the form

\[
\phi_{\nu}(r) = \phi_{\Sigma}(r)\phi_{\Omega}(r),
\]

where \( \phi_{\Sigma} \) is the wave function in the bulk surface of the quasi-2D BEC and \( \phi_{\Omega} \) the wave function near the border.

The bulk wave function \( \phi_{\Sigma} \) is required to satisfy Eq. (29) in absence of \( V_{tr} \)—that is, it is not required to satisfy the boundary condition \( \phi_{\nu}(R) = 0 \), but rather \( \phi_{\Sigma}(r \to \infty) = \sqrt{\mu} \) by the asymptotic separation argument. The solution can be obtained using imaginary-time evolution on a transformed radial coordinate \( \theta = 2\arctan(r) \), with \( \theta \in [0, \pi] \); we did not update the chemical potential at each step of the evolution \([52]\) but rather sampled the parametric \( \mu \)-dependence of the numerical solutions. Further, because \( V_{tr} \) is exponentially localized at the origin, the asymptotic expansion \( \phi_{\Sigma}(r) = \sqrt{\mu} \left(1 - \frac{\nu^2}{2\mu r^2}\right) + \ldots \) holds for large \( r \) \([33]\).

We now solve for \( \phi_{\Omega} \) using an asymptotic approximation. We begin by writing \( \phi_{\Sigma}(r) = f(x) \), for \( i = 0, 1, 2 \), with the coordinate \( x = \sqrt{\mu/2}(R - r) \); the resulting equation reads

\[
-\frac{1}{2} f'' + \left(\frac{1}{\sqrt{2\mu R}} + \frac{x}{\mu R^2}\right) f' + \frac{\nu^2}{\mu R^2} f - \left[1 - \left(1 - \frac{\nu^2}{\mu R^2}\right)f\right] f = O(R^{-3}), \tag{B1}
\]

where we keep terms up to second order in \( R^{-1} \); likewise, we expand \( f \) asymptotically to order \( R^{-2} \):

\[
f(x) = f_0(x) + R^{-1} f_1(x) + R^{-2} f_2(x) + \ldots \tag{B2}
\]

Considering \( x \in [0, \infty) \) by the asymptotic separation argument, we require the boundary condition \( f(0) = 0 \) and that \( f(x) \) be bounded as \( x \to \infty \). To 0th order, the equation is \(-\frac{1}{2} f_0'' - (1 - f_0^2) f_0 = 0\), with solution \( f_0(x) = \tanh(x) \), in agreement with Ref. \([51]\). The 1st and 2nd-order equations are not worthwhile displaying here; we mention only that we solved for the \( f_i \) analytically with the aid of symbolic computation software; results are plotted in Fig. 9.

This method was validated against fully imaginary time-evolved solutions for small system sizes \((R < 70)\). Because \( R^2 \) amounts to a dimensionless coupling strength \([4]\), this approximation constitutes a strong-coupling approximation. Moreover, further terms in the \( R \gg 1 \) asymptotic expansion of \( \phi_{\Sigma} \) can be obtained to compute Eq. (B2) to arbitrary order in \( R^{-1} \).

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