On the Post-Peak Structural Response due to Softening with Localization

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An analytical study is taken to investigate the relationship between material softening and structural softening through the use of a model problem in one dimension. With general nonlinear assumptions on the constitutive relations, it turns out that the governing equations can be viewed as a system of parametric equations, which couple the size effect and the nonlinear effect. Compared with the bilinear assumptions in previous literature, we find that the nonlinear assumptions herein capture more details in the post-peak structural response. After doing standard mathematical analysis to the nonlinear equations, we manage to derive necessary and sufficient conditions for the occurrence of four important post-peak cases, which are often observed in experiments. In particular, our analysis reveals that the mechanism of the snap-through phenomenon is due to the convexity change of the constitutive curve of the softening part. Mathematical examples are also given to illustrate the proposed procedures.

1 Introduction
Strain-softening, i.e., the decrease of stress with the increase of strain, is such a common phenomenon that has been recorded for a variety of materials, like concrete, rocks, ceramics, metals, etc. Bazant et al. gave a comprehensive review of this phenomenon and analyzed its mechanism from a continuum point of view. Moreover, it is well-known that strain softening is always accompanied by highly localized deformations of the specimen (2,3). Due to the importance of softening phenomenon in structural safety assessment, many efforts have been made in the past decades to investigate strain-softening with localization experimentally, numerically, and analytically, as reviewed by [4,5].

Snap-back may be one of the most interesting and perhaps most common structural instability phenomena observed in experiments. It shows that the load-displacement curve displays a positive slope after attaining the peak load. de Borst demonstrated the possibility of snap-back behavior on structural level by means of two concrete structures: a reinforced concrete and an unreinforced specimen. In order to simulate the highly localized failure mode in a strain-softening solid, a modified arc-length control method was used in that paper. Later, Rots and de Borst did a tensile test on concrete specimens and analyzed it by using the finite element method, with a particular attention on the snap-back behavior. He et al. studied the class II behavior (snap-back) of rock with a spring model, which was characterized by non-uniform failure. Unloading-reloading tests were also conducted in the post failure region in that paper. One of their results is that, if inelastic strain increases slower than the elastic strain decreases, rock shows class II behavior.

Jansen et al. did an experiment on concrete cylinders by using the feedback-control method. From two test series, the stress-displacement behavior for different height-diameter ratios with normal strength and high strength were
obtained. They found that the pre-peak segment of the stress-displacement curves agrees well with the pre-peak part of the stress-strain curves, while the post-peak segment shows a strong dependence on the geometric size, namely the radius-length ratio. More specifically, the longer the specimen is, the steeper the post-peak segment of the stress-displacement curves becomes. The feedback-control method was also used in Subramaniam et al. [10] to test concrete in torsion, and snap-back was also found in the experiment.

Some analytical studies were also taken to investigate softening with localization. With the use of a one-dimensional model, Schreyer and Chen [11] analyzed the snap-back phenomenon and found the important size effect on the instability. Due to the simplicity of bilinear assumptions on the constitutive relations, further features like snap-through were lost in the result, although in some experiments this feature was observed (see van Vilet and van Mier [12]). The same constitutive relations were also assumed in Chen et al. [13] to analyze the stability in some hierarchical structures. In a more complex setting with certain nonlinear assumptions on the constitutive relations, Sundara Raja Iyengar et al. [14] took an analytical study. By using the fictitious crack model (FCM) developed by Hillerborg, they found the effect of the softening exponent $n$ on the size effect and snap-back behavior of beams, while the stress-displacement relation was assumed as a general power law function. Dai et al. [15] constructed the analytical solutions for localizations in a hyperelastic slender cylinder. In order to consider a general nonlinear case, the constitutive relations for the two regions are set as: the loading and unloading segments of region A are two arbitrary functions $f_{11}$ and $f_{12}$ respectively, while the loading and softening segments of region B are two arbitrary functions $f_{21}$ and $f_{22}$ respectively. Moreover, we assume that the foregoing nonlinear functions are twice differentiable with $f'_{11} > 0$, $f'_{12} > 0$, $f'_{21} > 0$, $f'_{22} < 0$. The limit stress for region A, denoted by $\sigma_0$, is assumed to be slightly larger than that of region B, which is denoted by $\sigma_0$. The details are shown in Figure 1(b) (where $f_1$ is used to denote both the pre-peak and the post-peak segments, as region A only experiences loading or unloading). As the post-peak curve of the structure is our main concern, in the following derivation, for simplicity, we use $f_1$, $f_2$ to denote the post-peak curves of region A and region B respectively, unless otherwise specified.

To simulate post-peak experiments, we consider a structure with a serial arrangement of intact elastic and strain-softening zones. This model was used by several researchers, such as [19,20,21] in the early years. In [11], it was introduced to analyze strain-softening with bilinear assumptions on the constitutive relations. As shown in Figure 1(a), the structure is a bar of length $L = a + b$ with a unit cross-sectional area. That is to say, it is composed of two segments (segment A with length $a$ and segment B with length $b$). The two segments are usually described by similar constitutive equations, and the main difference is that the limit stress for B is slightly less than that of A. Therefore, if the stress on the structure is such that the strain in region B exceeds the value at the limit state, then softening will occur. It is assumed that softening occurs uniformly over a localized region B under quasi-static loading.

In order to consider a general nonlinear case, the constitutive relations for the two regions are set as: the loading and unloading segments of region A are two arbitrary functions $f_{11}$ and $f_{12}$ respectively, while the loading and softening segments of region B are two arbitrary functions $f_{21}$ and $f_{22}$ respectively. Moreover, we assume that the foregoing nonlinear functions are twice differentiable with $f'_{11} > 0$, $f'_{12} > 0$, $f'_{21} > 0$, $f'_{22} < 0$. The limit stress for region A, denoted by $\sigma_0$, is assumed to be slightly larger than that of region B, which is denoted by $\sigma_0$. The details are shown in Figure 1(b) (where $f_1$ is used to denote both the pre-peak and the post-peak segments, as region A only experiences loading or unloading). As the post-peak curve of the structure is our main concern, in the following derivation, for simplicity, we use $f_1$, $f_2$ to denote the post-peak curves of region A and region B respectively, unless otherwise specified.

As to the post-peak response, for a strain softening material with a serial setting (cf. Figure 1(a)), region A is in an unloading process and region B experiences strain softening. Given the values of strain in regions A and B, say $e_1$ and $e_2$, respectively, then the composite strain for the complete structure is given by

$$e = \frac{a e_1 + b e_2}{L} = (1-n)e_1 + ne_2,$$

where $n = b/L$. Since we consider it as a quasi-static problem, the composite stress is then given by

$$\sigma = f_1(e_1) = f_2(e_2).$$

In fact, one can easily see that (1) and (2) are also true if
Fig. 1. (a) One-dimensional model problem; (b) Stress-strain relations for A and B.

\[ f_2 \] is used to denote both the pre-peak and post-peak segments. Here, for the post-peak region, we consider only when \( \sigma \geq \sigma^* \) (\( \sigma^* \) represents the lowest stress value at which the bar breaks), and denote \( e_{11} \) and \( e_{21} \) the values such that \( f_1(e_{11}) = f_2(e_{21}) = \sigma^* \). Then, for the post-peak region, we have \( e_1 \in [e_{11}, e_{10}] \) and \( e_2 \in [e_{20}, e_{21}] \) (see Figure 1(b) for the definitions of \( e_{10} \) and \( e_{20} \)). From equation (2), we get \( e_2 = f_2^{-1}[f_1(e_1)] \) (or \( e_1 = f_1^{-1}[f_2(e_2)] \)). Thus (1) and (2) can be transformed into the system

\[
\begin{align*}
\sigma &= f_1(e_1) \\
e &= (1-n)e_1 + n f_2^{-1}[f_1(e_1)],
\end{align*}
\]

which can be viewed as the parametric equations for the engineering stress-strain curve. We note that \( n \) is actually the width (scaled by \( L \)) of the localization zone in the reference configuration, as material points in region B are in the localization zone in the post-peak region. Obviously, system (3) couples the size effect and nonlinear effect.

Now, we differentiate system (3) with respect to \( e_1 \) to obtain

\[
\begin{align*}
\frac{d\sigma}{de_1} &= f_1'(e_1) \\
\frac{de}{de_1} &= (1-n) + nf_2^{-1}[f_1(e_1)].
\end{align*}
\]

If \( (1-n)f_2'(e_2) + nf_1'(e_1) \neq 0 \), we have

\[
\frac{d\sigma}{de} = \frac{f_1'(e_1)f_2'(e_2)}{(1-n)f_2'(e_2) + nf_1'(e_1)},
\]

\[
\frac{d^2\sigma}{de^2} = \frac{n[f_1'(e_1)]^2f_2''(e_2) + (1-n)f_1''(e_1)[f_2'(e_2)]^3}{[n f_1'(e_1) + (1-n)f_2'(e_2)]^3}.
\]

In order to analyze the sign of (5), we define

\[
g(e_1, e_2;n) = nf_1'(e_1) + (1-n)f_2'(e_2),
\]

\[m(e_1, e_2) = \frac{f_2'(e_2)}{f_2'(e_2) - f_1'(e_1)},
\]

\[G(e_1, e_2) = [f_1'(e_1)]^2f_2''(e_2) - [f_2'(e_2)]^2f_1''(e_1).
\]

The above three functions can be viewed as functions of either \( e_1 \) or \( e_2 \) by the relations between them as shown above. We note that \( m(e_1, e_2) \) depends on the slopes (the first-order derivatives) of the constitutive curves, \( G(e_1, e_2) \) depends on the convexities (the second-order derivatives) of the constitutive curves and \( g(e_1, e_2;n) \) depends on the size parameter \( n \). We also point out that \( m(e_1, e_2) = n \) is equivalent to \( g(e_1, e_2;n) = 0 \). We shall see that \( n \) has an important influence on the structural response.

3 Post-peak Curves and Conditions

Assuming that \( f_2'(e_{20}) = 0 \) and \( f_2''(e_{20}) < 0 \), that is to say \( e_{20} \) is a local maximum of \( f_2 \). Then, for different \( f_1, f_2 \) and \( n \) the four cases shown in Figure 2 can arise. Next, we shall establish the conditions for each case.

3.1 Case A: Stable Softening

For the structure to be in stable softening (i.e., \( d\sigma/de < 0 \)), from (5), it is easy to see the necessary and sufficient condition is

\[
g(e_1, e_2;n) = nf_1'(e_1) + (1-n)f_2'(e_2) > 0, \text{ for } e_1 \in [e_{11}, e_{10}].
\]

From which, we get

\[
b = \frac{n}{L} > n_0 := \max_{e_1 \in [e_{11}, e_{10}]} m(e_1, e_2) = \max_{e_2 \in [e_{20}, e_{21}]} m(e_1, e_2).\]


3.2 Cases B and C: Snap-Through

Now, we focus on the interval \( n \in (0, n_0) \). There are several possibilities, as shown in Figure 2. Before analyzing the remaining cases, we point out that the initial part (i.e., the part close to the peak) of the post-peak curve is in a state of stable softening for the conditions imposed on \( f_1 \) and \( f_2 \). In fact, \( g(e_{10}, e_{20}) = n f'_1(e_{10}) \), and at the peak point, we have \( d\sigma/de < 0 \). By continuity, there must be a part of the post-peak curve for \( e \) close to \( \delta_0 \) (\( \delta_0 = (1 - n)e_{10} + ne_{20} \)) in which \( d\sigma/de < 0 \). Also, at \( e_1 = e_{10}, e_2 = e_{20} \), we have \( d^2\sigma/de^2 = f'_2(e_{20})/n^2 < 0 \). This would be useful for our later derivation.

We see that each of Case B and Case C represents a snap-through case. Here, snap-through is defined to be the point at which the slope of the force-displacement curve becomes infinite. As a result, when displacement (elongation) crosses this point, the force may experience a sudden drop. Firstly, let us consider the similarities between Case B and Case C. There are two turning points (the points at which \( d\sigma/de = \infty \)) in both curves. From (5), it can be seen that this is equivalent to that the equation

\[
g(e_1, e_2; n) = n f'_1(e_1) + (1 - n) f'_2(e_2) = 0 \tag{12}
\]

has two roots, say \( e^*_1 \) and \( e^*_2 \) (\( e^*_1 > e^*_2 \)). The following theorem provides a necessary and sufficient condition for the occurrence of the two turning points.

**Theorem 3.1.** If two turning points occur, then the function \( G(e_1, e_2) \) must change sign at least once for \( e_1 \in [e_{11}, e_{10}] \). On the other hand, if the sign of \( G(e_1, e_2) \) changes only once for \( e_1 \in [e_{11}, e_{10}] \), then for any \( n \in [n_1, n_0] \), two turning points occur, where \( n_1 = m(e_{11}, e_{21}) \).

**Proof.** If two turnings occur, then the sign of the function \( g(e_1, e_2) \) changes twice (cf. Case B or Case C in Figure 2).

So, we get

\[
g(e_{11}, e_{21}; n) = n f'_1(e_{11}) + (1 - n) f'_2(e_{21}) > 0 \tag{13}
\]

Thus,

\[
\frac{f'_2(e_{21})}{f'_1(e_{11})} > -\frac{n}{1 - n} \tag{14}
\]

Since

\[
-\frac{n}{1 - n} > -\frac{n_0}{1 - n_0} \tag{15}
\]

we have

\[
\frac{f'_2(e_{21})}{f'_1(e_{11})} = \min_{e_1 \in [e_{11}, e_{10}]} \frac{f'_2(e_{22})}{f'_1(e_{11})} \tag{16}
\]

Suppose that the minimum is attained at \( e_{12} \) (the corresponding \( e_2 \) is given by \( f_2^{-1}[f_1(e_{12})] = e_{22} \)). That is,

\[
f'_2(e_{22})/f'_1(e_{11}) = \min_{e_1 \in [e_{11}, e_{10}]} \{f'_2(e_{22})/f'_1(e_1)\} \tag{17}
\]

If we view \( f'_2(e_{22})/f'_1(e_1) \) as a function of \( e_2 \), then according to the Lagrange Mean Value theorem, there exists an \( \xi \in (e_{22}, e_{21}) \) such that

\[
\frac{d}{de_2} \left[ \frac{f'_2(e_{22})}{f'_1(e_1)} \right]_{e_2=\xi} = \frac{f'_2(e_{22})}{f'_1(e_1)} \frac{f'_1(e_{12}) - f'_1(e_{11})}{e_{21} - e_{22}} > 0. \tag{18}
\]

As

\[
\frac{d}{de_2} \left[ \frac{f'_2(e_{22})}{f'_1(e_1)} \right] = \left[ \frac{f'_1(e_1)}{f'_1(e_1)} \right]^2 \frac{f'^2_2(e_{22}) - [f'_2(e_{22})]^2 f'^2_1(e_1)}{f'_1(e_1)^3}, \tag{19}
\]

we have

\[
[f'_1(\xi)]^2 f'^2_2(\xi) - [f'_2(\xi)]^2 f'^2_1(\xi) > 0, \quad \text{where } \xi = f_1^{-1}[f_2(\xi)]. \tag{20}
\]

which implies that \( G(\xi, \xi) > 0 \). Since \( G(e_{10}, e_{20}) = \left| f'_1(e_{10}) \right|^2 f'_2(e_{20}) < 0 \), the sign of \( G(e_1, e_2) \) changes for \( e_1 \in [e_{11}, e_{10}] \).

On the other hand, suppose that for \( e_1 \in [e_{11}, e_{10}] \), \( G(e_1, e_2) \) changes sign once. Now, we consider the function \( m(e_1, e_2) \) (cf., (8); we regard it as a function of \( e_2 \)). We now
show that for any $n \in [n_1, n_0]$, equation (12) has two roots. In fact, it is easy to get

$$\frac{dm}{de_2} = \frac{-G(e_1, e_2)}{[f'_2(e_2) - f'_1(e_1)]^2 f'_1(e_1)}. \quad (21)$$

Thus, $\frac{dm}{de_2}$ also changes sign once. We also note that $n_0$ is a maximum of $m(e_1, e_2)$, say, attained at $e_{2n}$. Then, $\frac{dm}{de_2} = 0$ at $e_2 = e_{2n}$. On the other hand,

$$\left. \frac{dm}{de_2} \right|_{e_2=e_0} = \frac{-G(e_{10}, e_{20})}{[f'_2(e_{20}) - f'_1(e_{10})]^2 f'_1(e_{10})} > 0. \quad (22)$$

So, the curve $m(e_1, e_2)$ should have the characteristics shown in Figure 3. Thus, for any $n \in [n_1, n_0]$, $n = m(e_1, e_2)$ has two roots, which then implies that $g(e_1, e_2; n)$ has two zeros. This completes the proof of the second part of the theorem.

If $f_1$ is linear in the post-peak region, then $G(e_1, e_2) = [f'_1(e_1)]^2 f'_2(e_2)$. Consequently, the sign of $G(e_1, e_2)$ depends on the convexity of $f_2$. We have the following corollary:

**Corollary 3.1.** For $f_1$ being linear in the post-peak region, if two turning points occur, then the convexity of $f_2$ must change at least once for $e_1 \in [e_{11}, e_{10}]$. On the other hand, if the convexity of $f_2$ changes once, then for any $n \in [n_1, n_0]$, two turning points occur in the post-peak curve.

**Remark 3.1.** Usually, $f''_1(e_1)$ should be small ($f''_1(e_1) = 0$ for $f_1$ being linear). Thus, the sign of $G(e_1, e_2)$ is primarily determined by the sign of $f''_2(e_2)$. So, one may say that a necessary condition for the snap-through (i.e., there are two turning points in the post-peak curve) is the change of the convexity of the constitutive curve of the softening part.

Now, let us consider the differences between Case B and Case C. Recall that $e'_{11}$ and $e'_{12}(e'_{11} > e'_{12})$ are the two roots of equation (13). For Case B, we have

$$\delta^*_2 = (1 - n)e'_{12} + nf^*_2 - |f'_1(e'_{12})| > \delta_0. \quad (23)$$

While for Case C, we have

$$\delta^*_2 = (1 - n)e'_{12} + nf^*_2 - |f'_1(e'_{12})| \leq \delta_0. \quad (24)$$

In other words, in Case C the post-peak curve has entered the pre-peak region, while in Case B it has not. We find that, for given $f_1$ and $f_2$, there are some conditions for the occurrence of Case C. For simplicity, we assume that the sign of $G(e_1, e_2)$ changes once, say,

$$\begin{cases}
G(e_1, e_2) < 0, & e_2 \in [e_{20}, e_{21}], \\
G(e_{12}, e_{22}) = 0, \\
G(e_1, e_2) > 0, & e_2 \in (e_{22}, e_{21}].
\end{cases} \quad (25)$$

The following theorem provides a necessary and sufficient condition for Case C.

**Theorem 3.2.** Under assumption (25) and $n \in [n_1, n_0]$, a necessary and sufficient condition for the occurrence of Case C is

$$f''_2(e'_{22})(e'_{22} - e_{20}) \geq f'_1(e'_{12})(e'_{12} - e_{10}). \quad (26)$$

**Proof.** From (24), we have

$$(1 - n)e'_{12} + ne'_{22} \leq (1 - n)e_{10} + ne_{20}, \text{ where } e'_{22} = f^{-1}_2(f'_1(e'_{12})).$$

On the other hand,

$$g(e'_{12}, e'_{22}; n) = nf'_1(e'_{12}) + (1 - n)f'_2(e'_{22}) = 0. \quad (27)$$

From the above two equations, we can get (26) immediately. Also, from (26) and (28) one can immediately deduce (27). This completes the proof.

Assumption (25) can be made even more complicated, in that case we may draw the fairly complicated post-peak curves in [7], which were obtained by numerical methods. It should be pointed out that inequality (26) is another requirement among $f_1, f_2$ and $n$. For given $f_1$ and $f_2$, it provides another bound (say $n_2$) for $n$, since $e'_{22}$ and $e'_{12}$ are related to $n$ as equation (28) shows.

For $f_1$ being linear in the post-peak region, it is easy to show that (26) becomes

$$f''_2(e'_{22}) \geq \frac{f''_2(e_{20}) - f'_2(e_{20})}{e'_{22} - e_{20}}. \quad (29)$$

This implies that for the constitutive relation $\sigma = f_2(e_2)$, the secant line joining the point $e'_{22}$ and the peak $e_{20}$ should be steeper than the tangent line at $e'_{22}$ (see Figure 4). Combined with Corollary 3.1, we have the following corollary:

**Corollary 3.2.** For $f_1$ being a linear function, if the convexity of $f_2$ changes once, then a necessary and sufficient condition for the occurrence of Case C is $n \in [n_1, n_0]$, and inequality (29) holds.
3.3 Case D: Snap-back

In Case D, there is a snap-back in the structural response. Here, we say that snap-back occurs when the slope of the force-displacement curve becomes positive and remains positive in the post-peak response. Obviously, in this case there is only one turning point (see Figure 2). The following theorem provides a critical $n$ for the occurrence of Case D.

**Theorem 3.3.** If $G(e_1, e_2)$ changes sign once (cf. (25)) or does not change sign for $e_1 \in [e_{11}, e_{10}]$, then for Case D to occur, a necessary and sufficient condition is $n < n_1$.

**Proof.** First, suppose that $G(e_1, e_2)$ changes sign once. Then $m(e_1, e_2)$ has the characteristics shown in Figure 3. It is obvious that a necessary and sufficient condition for the occurrence of Case D is that there exists only one root for equation (12). While from Figure 3, it is easy to see that a necessary and sufficient condition is $n < n_1$. Second, suppose that $G(e_1, e_2)$ does not change sign. Since $dm/de_2 > 0$ (as $dm/de_2 > 0$ at $e_2 = e_{20}$), we have

$$n_0 = \max_{e_2 \in [e_{20}, e_{21}]} m(e_1, e_2) = m(e_{11}, e_{21}) = n_1. \quad (30)$$

Obviously, a necessary and sufficient condition for $n = m(e_1, e_2)$ to have one and only one root (i.e., $g(e_1, e_2; n)$ has one and only one zero) is $n < n_1$. Thus we complete the proof.

**Remark 3.2.** In this section, we derive some requirements on the constitutive functions, together with three critical values $(n_0, n_1, n_2)$ of the size parameter. Providing the constitutive requirements are met, the structural response may have different behaviors for $n$ in different intervals according to the above critical values. Thus, the results also show the important size effects.

4 Illustrative Examples

In this section, we give two examples to illustrate the theoretical results obtained in Section 3. The following two examples can be referred to as two different physical processes. One can easily check that the functions in the following examples satisfy the conditions we have proposed, in particular, (25).

Example 1: Consider the following constitutive relations:

$$f_{11}(e_1) = \csc \frac{9\pi}{20} \sin \left(50\pi e_1\right),$$

$$f_{12}(e_1) = \left(50\pi \csc \frac{9\pi}{20}\right) e_1 - \frac{9\pi}{20} \csc \frac{9\pi}{20} + 1,$$

$$f_{21}(e_2) = \sin \left(50\pi e_2\right),$$

$$f_{22}(e_2) = 65625 \left[\frac{1}{3}(e_2 - 0.03)^3 - 0.0004(e_2 - 0.03)\right] + 0.65.$$  

Here $\sigma_0 = 1$, and we take $\sigma^* = 0.301$. The details are shown in Figure 5(a).

We have taken $f_{12}$ being a linear function, which represents the physical situation that at the peak region A of the structure has entered the plastic state. We find the critical values of $n$ based on the theoretical analysis in Section 3: $n_0 = 0.142$, $n_1 = 0.0158$, $n_2 = 0.110$ (a bound for $n$ found from inequality (29)). Specifically, Case A occurs if $n > 0.142$; Case B occurs if $0.110 < n \leq 0.142$; Case C occurs if $0.0158 < n \leq 0.110$; Case D occurs if $n < 0.0158$. Accordingly, by taking $n$ to be in different intervals, we get the four cases as we have predicted in Section 3. They are shown in Figure 5(b).

To reflect the size effect on the localization zone, curves of the width of the localization zone in the current configuration versus the total elongation are shown in Figure 6. This width is denoted by $d$, whose expression is given by $d = n(1 + e_2)$. Here, for the purpose of clearness, we have used different scales for different curves. It can be seen that this width increases slowly in the pre-peak region and increases rapidly in the post-peak region. For the stable softening case ($n = 0.167$), there is only one value of $d$ for a given $e$. For the snap-back case ($n = 0.0156$), there are two values of $d$ for a given $e$. Also, $d$ increases very fast, as $e$ decreases in the post-peak region. For the two snap-through cases ($n = 0.1$ and $n = 0.125$), there are three values of $d$ for $e$ in some intervals. Thus, $d$ may jump from a small value to a large value for $e$ in these intervals, i.e., the localization zone may suddenly widen. Thus, the size parameter $n$ has an important influence on the localization zone.
Example 2: In this example, region A is assumed to be in nonlinear elasticity (loading or unloading), so the constitutive functions $f_{11}$ and $f_{12}$ are the same. The constitutive relations are listed below.

$$f_{11}(e_1) = f_{12}(e_1) = -11000(e_1 - 0.01)^2 + 1.1,$$

and

$$f_{21}(e_2) = -\frac{10^6}{81}(e_2 - 0.009)^2 + 1,$$

$$f_{22}(e_2) = 65625\left[\frac{1}{3}(e_2 - 0.029)^3 - 0.0004(e_2 - 0.029)\right] + 0.65.$$

Here $\sigma_0 = 1$, and we take $\sigma^* = 0.333$. The details are shown in Figure 7(a). Critical values of $n$ are: $n_0 = 0.174$, $n_1 = 0.0614$, $n_2 = 0.136$ (a bound for $n$ found from inequality (26)). The intervals for different cases are: Case A occurs if $n > 0.174$; Case B occurs if $0.136 < n \leq 0.174$; Case C occurs if $0.0614 \leq n \leq 0.136$; Case D occurs if $n < 0.0614$. The curves for $n$ taking four values in these four different intervals are shown in Figure 7(b), which agree with our theoretical predictions in Section 3. Curves of the width of the localization zone in the current configuration versus the total elongation are also shown (see Figure 8). Once again, from these curves, one can see the important influence of the size parameter $n$ on the localization zone.

5 Concluding Remarks and Future Tasks

An analytical study is performed on the post-peak structural response of strain-softening with localization. In a general nonlinear setting, after taking standard mathematical analysis to the parametric equations, we manage to handle the nonlinear and size effects. Qualitative requirements on the constitutive functions and quantitative requirements on the size effect are derived, especially for the snap-through
phenomenon. The results are consistent with earlier experimental and computational results. It seems that the four cases studied analytically here are quite representative. The theoretical and computational results may be of value for the verification of computational algorithms and can shed some light on the mechanisms of instabilities associated with strain-softening.

Especially, we have shown that the convexity change is a necessary condition for the snap-through phenomenon. As softening with localization is important for understanding the failure evolution in structures, future work will focus on considering structures with different configurations.

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