NONLINEAR DIFFERENTIAL EQUATION FOR KOROBOV NUMBERS

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Abstract. In this paper, we present nonlinear differential equations for the generating functions for the Korobov numbers and for the Frobenius-Euler numbers. As an application, we find an explicit expression for the $n$th derivative of $1/\log(1 + t)$.

Keywords: Korobov numbers; Frobenius-Euler numbers

1. Introduction

The Korobov polynomials $K_n(\lambda, x)$ of the first kind are given by
$$\frac{\lambda t}{(1 + t)^\lambda - 1} (1 + t)^x = \sum_{n \geq 0} K_n(\lambda, x) \frac{t^n}{n!}.$$ For example,
$$K_0(\lambda, x) = 1,$$
$$K_1(\lambda, x) = \frac{1}{2}(2x - \lambda + 1),$$
$$K_2(\lambda, x) = \frac{1}{12}(6x^2 - 1 + \lambda^2 - 6\lambda x),$$
$$K_3(\lambda, x) = \frac{1}{24}(2\lambda x - 2x^2 - \lambda + 2x + 1)(1 - 2x + \lambda).$$

When $x = 0$, $K_n(\lambda) = K_n(\lambda, 0)$ are called the Korobov numbers of the first kind or just Korobov numbers. Since 2002, Korobov polynomials and numbers have been received a lot of attention (see [13, 14]). In particular, these polynomials are used to derive some interpolation formulas of many variables and a discrete analog of the Euler summation formula (see [16]). The Frobenius-Euler numbers $H_n(\mu)$ are defined by the generating function (see [18, 12, 15, 17])
$$\left(\frac{1 - \mu}{1 - e^t - \mu}\right) = \sum_{n \geq 0} H_n(\mu) \frac{t^n}{n!}, \quad \mu \neq 1.$$

Recently, the degenerate Bernoulli and Euler polynomials related to Korobov polynomials are studied by several authors (2, 5, 12) and Kim and Kim-Kim derived some interesting identities of Frobenius-Euler polynomials and the Bernoulli polynomials of the second kind arising from nonlinear differential equations (see 4, 7, 8).

The main goal of this paper is to write a nonlinear differential equation satisfying the generating function $\frac{\lambda t}{(1 + t)^\lambda - 1}$ for Korobov numbers $K_n(\lambda)$, and a nonlinear differential equation satisfying the generating function $\left(\frac{1 - \mu}{e^t - \mu}\right)$ for Frobenius-Euler numbers $H_n(\mu)$, see next sections. Also, we
present in each case some applications for our nonlinear differential equations. For instance, we find an explicit expression for the $n$th derivative of $1/\log(1+t)$, see Corollary 3.

2. KOROBOV NUMBERS

Put $F = F(t) = F(t; \lambda) = \frac{1}{(1+t)^{\lambda} - 1} (\lambda \neq 0)$. By differentiating respect to $t$, we have

$$F^{(1)} = \frac{-1}{((1+t)^{\lambda} - 1)^2} \cdot \frac{\lambda(1+t)^{\lambda}}{1+t} = \frac{-\lambda}{1+t} \cdot \frac{(1+t)^{\lambda} - 1 + 1}{((1+t)^{\lambda} - 1)^2} = \frac{-\lambda}{1+t}(F + F^2).$$

Now, we let

$$F_{\lambda} = \frac{-1}{(1+t)^{\lambda} - 1} \cdot \frac{\lambda(1+t)^{\lambda}}{1+t} = \frac{-\lambda}{1+t} \cdot \frac{(1+t)^{\lambda} - 1 + 1}{((1+t)^{\lambda} - 1)^2} = \frac{-\lambda}{1+t}(F + F^2).$$

for all $N \geq 0$, and $a_i(N) = 0$ for all $i \geq N + 1$. Note that $F = F^{(0)} = \lambda a_0(0) F$, which implies that $a_0(0) = \frac{1}{\lambda}$. Also, by (1), we have $a_0(1) = a_1(1) = 1$. For $N + 1$, we have

$$F^{(N+1)} = \frac{d}{dt} \left( \frac{-1}{(1+t)^{\lambda} - 1} \cdot \frac{\lambda(1+t)^{\lambda}}{1+t} \sum_{i=1}^{N+1} a_{i-1}(N) F^i \right)$$

$$= \frac{-1}{(1+t)^{\lambda + 1}} \sum_{i=1}^{N+1} a_{i-1}(N) F^i + \frac{-\lambda}{1+t} \sum_{i=1}^{N+1} \lambda a_{i-1}(N) F^i \sum_{i=1}^{N+2} \lambda(i-1)a_{i-2}(N) F^i,$$

which, by (1), gives

$$F^{(N+1)} = \frac{-1}{(1+t)^{\lambda + 1}} \sum_{i=1}^{N+1} a_{i-1}(N) F^i + \frac{-\lambda}{1+t} \sum_{i=1}^{N+1} \lambda a_{i-1}(N) F^i \sum_{i=1}^{N+2} \lambda(i-1)a_{i-2}(N) F^i.$$ 

By assumption, $F^{(N+1)} = \frac{-1}{(1+t)^{\lambda + 1}} \sum_{i=1}^{N+1} a_{i-1}(N + 1) F^i$, and, by comparing the coefficients of $F^i$ on both sides, we obtain the following recurrence relation

$$a_{i-1}(N + 1) = (N + i\lambda) a_{i-1}(N) + \lambda(i-1)a_{i-2}(N), \quad i = 2, \ldots, N + 1$$

with $a_j(N + 1) = 0$ whenever $j \geq N + 2$,

$$a_0(N + 1) = (N + \lambda)a_0(N), \quad a_{N+1}(N + 1) = (N+1)a_N(N).$$

Recalling that $a_0(0) = \frac{1}{\lambda}$ and $a_0(1) = a_1(1) = 1$, by induction on $N$, we have

$$a_0(N + 1) = (N + \lambda)(N - 1 + \lambda) \cdots (1 + \lambda) = (N+\lambda)_N,$$

$$a_{N+1}(N + 1) = a_1(1) \prod_{j=2}^{N+1} (j\lambda) = \lambda^N(N + 1)!. $$

In next lemma, we treat the general case.
**Lemma 1.** The coefficients \( a_j(N), \ j = 1, 2, \ldots, N, \) satisfy

\[
a_j(N) = j\lambda \sum_{i=0}^{N-j} (N + (j + 1)\lambda - 1) a_{j-1}(N - i - 1).
\]

**Proof.** By (2), we have

\[
a_j(N + 1) = j\lambda a_{j-1}(N) + (N + (j + 1)\lambda) a_j(N)
\]

\[
= j\lambda a_{j-1}(N) + j\lambda(N + (j + 1)\lambda) a_{j-1}(N - 1)
\]

\[
+ (N + (j + 1)\lambda)(N + (j + 1)\lambda - 1)a_j(N - 1).
\]

By induction and the initial condition \( a_j(j) = \lambda^{j-1}j! \), we obtain

\[
a_j(N + 1) = j\lambda \sum_{i=0}^{N+1-j} (N + (j + 1)\lambda) a_{j-1}(N - i),
\]

as required. \( \square \)

By Lemma 1, we can state the following result.

**Theorem 2.** The function \( F = F(t) = \frac{1}{(1+t)\lambda^{j-1}j!} \) with \( \lambda \neq 0 \) satisfies the nonlinear differential equation

\[
F(N) = \frac{(-1)^N}{(1+t)^N} \sum_{i=1}^{N+1} a_{i-1}(N) F^i,
\]

where \( a_0(N) = (N + \lambda - 1)_{N-1} \) with \( a_0(0) = \frac{1}{\lambda} \), and

\[
a_j(N) = j\lambda \sum_{i=0}^{N-j} (N + (j + 1)\lambda - 1) a_{j-1}(N - i - 1),
\]

for \( j = 1, 2, \ldots, N \).

As an application for Theorem 2 let us consider the limit \( \lim_{\lambda \to 0} \lambda F(t) \). By the fact that \( \lim_{\lambda \to 0} \lambda F(t) = \frac{1}{\log(1+t)} \), we obtain

\[
\frac{d^N}{dt^N} \frac{1}{\log(1+t)} = \frac{(-1)^N}{(1+t)^N} \sum_{i=2}^{N+1} \lim_{\lambda \to 0} \lambda^{2-i} a_{i-1}(N; \lambda) \frac{1}{\log^i(1+t)}.
\]

Combining with the previous result in [4], we obtain

\[
\lim_{\lambda \to 0} \lambda^{2-i} a_{i-1}(N; \lambda) = (i - 1)!(N - 1)! H_{N-1,i-2},
\]

where \( 2 \leq i \leq N + 1 \) and \( H_{N,j} \) is given by

\[
H_{N,j} = \begin{cases} 
1, & j = 0, \\
\frac{1}{\sum_{i=1}^{N} H_{i,j-1}}, & j = 1, \\
\sum_{i=1}^{N} \frac{1}{H_{i,j-1}}, & 2 \leq j \leq N,
\end{cases}
\]

with \( H_{0,j-1} = 0 \) when \( j \geq 2 \). Hence, by Theorem 2 we can state the following corollary.
Corollary 3. For all $N \geq 1$,

$$\frac{d^N}{dt^N} \frac{1}{\log(1+t)} = \frac{(-1)^N(N - 1)!}{(1 + t)^N} \sum_{i=2}^{N+1} \frac{(i - 1)!H_{N-1,i-2}}{\log^i(1+t)}.$$

For example,

$$\frac{d}{dt} \frac{1}{\log(1+t)} = -\frac{1}{1 + t} \frac{1}{\log^2(1+t)},$$

$$\frac{d^2}{dt^2} \frac{1}{\log(1+t)} = \frac{1}{(1 + t)^2} \left( \frac{1}{\log^2(1+t)} + \frac{2}{\log^3(1+t)} \right),$$

$$\frac{d^3}{dt^3} \frac{1}{\log(1+t)} = \frac{-1}{(1 + t)^3} \left( \frac{2}{\log^2(1+t)} + \frac{6}{\log^3(1+t)} + \frac{6}{\log^4(1+t)} \right).$$

As another application, let us consider the generating function for the Korobov numbers

$$\frac{\lambda t}{(1 + t)^\lambda - 1} = \sum_{n \geq 0} K_n(\lambda) \frac{t^n}{n!}.$$ More generally, the Korobov numbers of order $m$ are defined via the following generating function

$$\left( \frac{\lambda t}{(1 + t)^\lambda - 1} \right)^m = \sum_{n \geq 0} K_n^{(m)}(\lambda) \frac{t^n}{n!}.$$ 

Theorem 4. For all $N \geq 1$,

$$\min_{(n,N)} \sum_{i=0}^{\min(n,N)} \lambda^{-N+1}(n) a_{N-i}(N) K_{n-i}^{(N+1-i)}(\lambda)$$

$$= \begin{cases} 
N!(N)_n, & 0 \leq n \leq N; \\
(-1)^N \sum_{\ell=0}^{n-N-1} (N)_\ell \frac{K_{n-\ell}^{(\lambda)}(\lambda) (\ell)_n}{n!} (N+1)_n, & n \geq N + 1.
\end{cases}$$

Proof. Note that

$$F = \frac{1}{(1 + t)^\lambda - 1} = \frac{1}{\lambda} \left( \frac{1}{t} + \sum_{n \geq 0} K_{n+1}(\lambda) \frac{t^n}{(n+1)!} \right).$$

Thus,

$$F^{(N)} = \frac{1}{\lambda} \left( (-1)^N N! t^{-N-1} + \sum_{n \geq N} K_{n+1}(\lambda)(n) \frac{t^{n-N}}{(n+1)!} \right),$$

which implies

$$t^{N+1} F^{(N)} = \frac{1}{\lambda} \left( (-1)^N N! + \sum_{n \geq N} K_{n+1}(\lambda)(n) \frac{t^{n+1}}{(n+1)!} \right).$$
Multiplying both sides by \((1 + t)^N\), we obtain
\[
(1 + t)^N t^{N+1} F^{(N)} = \frac{1}{\lambda} \left( (-1)^N N! \sum_{n=0}^{N} (N)_n \frac{t^n}{n!} + \sum_{\ell=0}^{N} (N)_\ell \sum_{n \geq N} K_{n+1}(\lambda)(n) \frac{t^{n+1}}{(n+1)!} \right)
\]
(4)

On the other hand, by Theorem 2, we have
\[
(1 + t)^N t^{N+1} F^{(N)} = (-1)^N \lambda \sum_{i=1}^{N+1} a_{i-1}(N) \left( \frac{1}{(1 + \lambda t^{1/\lambda} - 1)^i} \right) t^{iN+1}
\]
\[
= (-1)^N \lambda \sum_{i=1}^{N+1} \lambda^{-i} a_{i-1}(N) \left( \frac{\lambda t}{1 + \lambda t^{1/\lambda} - 1} \right)^i t^{N+1-i}
\]
\[
= (-1)^N \lambda \sum_{i=0}^{N} \lambda^{-i-1} a_{N-i}(N) \left( \frac{\lambda t}{1 + \lambda t^{1/\lambda} - 1} \right)^{N+1-i} t^i.
\]

Thus by the generating function for the Korobov numbers, we obtain
\[
(1 + t)^N t^{N+1} F^{(N)} = (-1)^N \lambda \sum_{i=0}^{N} \lambda^{-i-N} a_{N-i}(N) \left( \frac{\lambda t}{1 + \lambda t^{1/\lambda} - 1} \right)^{N+i} \sum_{m \geq 0} K^{(N+1-i)}(\lambda) \frac{t^m}{m!}
\]
\[
= (-1)^N \sum_{n \geq 0} \left( \sum_{i=0}^{\min(n,N)} \lambda^{-n-i} a_{N-i}(N) K^{(N+1-i)}(\lambda) \right) \frac{t^n}{n!}
\]

By combining this equation with (4), we complete the proof. \(\square\)

3. FROBNIUS-EULER NUMBERS

Set \(F = F(t) = F(t; \lambda, \mu) = \frac{1}{(1+\lambda t)^{1/\mu} - \mu} \) \((\lambda \neq 0)\). By differentiating respect to \(t\), we have
\[
F^{(1)} = \frac{-1}{((1+\lambda t)^{1/\mu} - \mu)^2} \cdot \frac{(1 + \lambda t)^{1/\mu} - \mu + \mu}{1 + \lambda t} = \frac{-1}{1 + \lambda t} \cdot \frac{(1 + \lambda t)^{1/\mu} - \mu + \mu}{((1+\lambda t)^{1/\mu} - \mu)^2} = -\frac{1}{1 + \lambda t}(F + \mu F^2).
\]

Now, we let
\[
F^{(N)} = \frac{(-1)^N}{(1+\lambda t)^N} \sum_{i=1}^{N+1} b_{i-1}(N; \lambda, \mu) F^i = \frac{(-1)^N}{(1+\lambda t)^N} \sum_{i=1}^{N+1} b_{i-1}(N) F^i,
\]
for all $N \geq 0$, and $a_i(N) = 0$ for all $i \geq N + 1$. Note that $F = F^{(0)} = b_0(0)F$, which implies that $b_0(0) = 1$. Also, by (5), we have $b_0(1) = 1$ and $b_1(1) = \mu$. For $N + 1$, we have

$$F^{(N+1)} = \frac{d}{dt} \left( \frac{(-1)^N}{(1 + \lambda t)^N} \sum_{i=1}^{N+1} b_{i-1}(N)F^i \right)$$

which, by (5), gives

$$F^{(N+1)} = \frac{(-1)^{N+1}\lambda N}{(1 + \lambda t)^{N+1}} \sum_{i=1}^{N+1} b_{i-1}(N)F^i + \frac{(-1)^N}{(1 + \lambda t)^N} \sum_{i=1}^{N+1} i b_{i-1}(N)F^{i-1}F^{(1)},$$

which, by (5), gives

$$F^{(N+1)} = \frac{(-1)^{N+1}\lambda N}{(1 + \lambda t)^{N+1}} \sum_{i=1}^{N+1} b_{i-1}(N)F^i + \frac{(-1)^N}{(1 + \lambda t)^N} \sum_{i=1}^{N+1} i b_{i-1}(N)F^{i-1}F^{(1)},$$

By assumption, $F^{(N+1)} = \frac{(-1)^{N+1}\lambda N}{(1 + \lambda t)^{N+1}} \sum_{i=1}^{N+1} b_{i-1}(N + 1)F^i$, and, by comparing the coefficients of $F^i$ on both sides, we obtain the following recurrence relation

$$b_{i-1}(N + 1) = (N\lambda + i)b_{i-1}(N) + \mu (i - 1)b_{i-2}(N), \quad i = 2, \ldots, N + 1$$

with $b_j(N + 1) = 0$ whenever $j \geq N + 2$,

$$b_{0}(N + 1) = (N\lambda + 1)b_{0}(N), \quad b_{N+1}(N + 1) = \mu (N + 1)b_{N}(N).$$

Recalling that $b_0(1) = 1$, $b_0(1) = 1$ and $b_1(1) = \mu$, by induction on $N$, we have

$$b_{0}(N + 1) = (N\lambda + 1)((N - 1)\lambda + 1)\cdots(\lambda + 1) = (N\lambda + 1)\lambda_N,$$

$$b_{N+1}(N + 1) = \mu^{N+1}(N + 1)!,$$

where $(x \mid \lambda)_n = x(x - \lambda)\cdots(x - (n-1)\lambda)$.

In next lemma, we treat the general case.

**Lemma 5.** The coefficients $b_j(N)$, $j = 1, 2, \ldots, N$, satisfy

$$b_j(N) = j\mu \sum_{i=0}^{N-j} ((N - 1)\lambda + j + 1 | \lambda)b_{j-1}(N - i - 1).$$

**Proof.** By (6), we have

$$b_j(N + 1) = j\mu b_{j-1}(N) + (N\lambda + j + 1)b_j(N)$$

$$= j\mu b_{j-1}(N) + j\mu (N\lambda + j + 1)b_{j-1}(N - 1)$$

$$+ (N\lambda + j + 1)((N - 1)\lambda + j + 1)b_j(N - 1).$$

By induction and the initial condition $b_j(j) = \mu^j j!$, we obtain

$$b_j(N + 1) = j\mu \sum_{i=0}^{N+1-j} (N\lambda + j + 1|\lambda)b_{j-1}(N - i),$$

as required. \(\square\)
By Lemma 5 we can state the following result.

**Theorem 6.** The function \( F = F(t) = \frac{1}{(1+\lambda t)^{1-\mu}} \) with \( \lambda \neq 0 \) satisfies the nonlinear differential equation

\[
F^{(N)} = \frac{(-1)^N}{(1+\lambda t)^N} \sum_{i=1}^{N+1} b_{i-1}(N)F^i,
\]

where \( b_0(N) = ((N-1)\lambda + 1)\lambda_{N-1}(N \geq 1) \) with \( b_0(0) = 1 \), and

\[
b_j(N) = j\mu \sum_{i=0}^{N-j} ((N-1)\lambda + j + 1)\lambda b_{j-1}(N - i - 1),
\]

for \( j = 1, 2, \ldots, N \).

By considering the proof of Theorem 6 and taking \( \lambda \to 0 \), we obtain the following result.

**Theorem 7.** Let \( F = F(t) = \frac{1}{e^{t} - \mu} \). Then

\[
F^{(N)} = (-1)^N \sum_{i=1}^{N+1} b_{i-1}(N; \mu)F^i,
\]

where \( b_0(N; \mu) = b_0(N-1; \mu) \), \( b_N(N; \mu) = \mu Nb_{N-1}(N-1; \mu) \), and \( b_{i-1}(N; \mu) = ib_{i-1}(N-1; \mu) + \mu(i-1)b_{i-2}(N-1; \mu) \), for \( 2 \leq i \leq N \) with \( b_0(0; \mu) = b_0(1; \mu) = 1 \) and \( b_1(1; \mu) = \mu \).

By the recurrence relation of \( b_j(N; \mu) \), see Theorem 7 and by induction on \( N \), we obtain \( b_0(N; \mu) = 1 \) and \( b_{N}(N; \mu) = \mu^N N! \), for all \( N \geq 0 \).

Along the lines of the proof of Lemma 5 and taking \( \lambda \to 0 \), we obtain that the coefficients \( b_j(N; \mu) \), \( j = 1, 2, \ldots, N \), satisfy

\[
b_j(N; \mu) = j\mu \sum_{i=0}^{N-j} (j + 1) b_{j-1}(N - i - 1; \mu) \text{ with } b_0(N; \mu) = 1.
\]

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