GENERALIZING THE DE FINETTI–HEWITT–SAVAGE THEOREM

IRFAN ALAM

Abstract. The original formulation of de Finetti’s theorem says that an exchangeable sequence of Bernoulli random variables is a mixture of iid sequences of random variables. Following the work of Hewitt and Savage, this theorem is known for several classes of exchangeable random variables (for instance, for Baire measurable random variables taking values in a compact Hausdorff space, and for Borel measurable random variables taking values in a Polish space). Under an assumption of the underlying common distribution being Radon, we show that de Finetti’s theorem holds for a sequence of Borel measurable exchangeable random variables taking values in any Hausdorff space. This includes and generalizes the currently known versions of de Finetti’s theorem. We use nonstandard analysis to first study the empirical measures induced by hyperfinitely many identically distributed random variables, which leads to a proof of de Finetti’s theorem in great generality while retaining the combinatorial intuition of proofs of simpler versions of de Finetti’s theorem. The required tools from topological measure theory are developed with the aid of perspectives provided by nonstandard measure theory. One highlight of this development is a new generalization of Prokhorov’s theorem.

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1. INTRODUCTION

The goal of this paper is to establish a generalization of de Finetti’s theorem. The original formulation of this theorem states that a sequence of exchangeable random variables taking values in \{0, 1\} is uniquely representable as a mixture of independent and identically distributed (iid) random variables. We show that the same conclusion holds for any sequence of Radon distributed exchangeable random variables taking values in any Hausdorff space equipped with its Borel sigma algebra (see Theorem 4.7). This includes and extends the current generalizations of de Finetti’s theorem following the works of Hewitt and Savage [37] (who proved de Finetti’s theorem in the case when the state space is a compact Hausdorff space equipped with its Baire sigma algebra). An analysis of our proof reveals that a slightly weaker condition than Radonness of the underlying common distribution is sufficient—we only need the common distribution of the random variables to be tight and outer regular on compact sets (see the discussion following Theorem 4.7).

Dubins and Freedman [24] had constructed a counterexample that showed that de Finetti’s theorem does not hold for a particular exchangeable sequence of Borel measurable random variables taking values in some separable metric space. Thus,
one consequence of the current work is to show that the random variables in their counterexample did not have a tight distribution (as any tight probability measure on a metric space is also Radon). In general, there is a large class of Hausdorff spaces such that de Finetti’s theorem holds for any sequence of tightly distributed exchangeable random variables taking values in any such Hausdorff space equipped with its Borel sigma algebra (see the discussion following Theorem 4.7). Another consequence is that de Finetti’s theorem holds whenever the state space is a Radon space equipped with its Borel sigma algebra (see Corollary 4.10).

Our methods blend together topological measure theory and nonstandard analysis. We present some preparatory results from each of these areas through the perspective provided by looking at them jointly. An example of a classical technique benefitting from this joint perspective is the technique of pushing down Loeb measures, which we are able to interpret as the topological operation of finding a standard measure that an internal measure is nearstandard to (with respect to the A-topology on the space of all Borel probability measures on a given topological space). See Theorem 2.28, Remark 2.29, and Theorem 2.36 for more details.

The above formulation is useful in proving a generalization of Prokhorov’s theorem as an intermediate consequence (see Theorem 2.44 and Theorem 2.46). This version of Prokhorov’s theorem postulates the sufficiency of uniform tightness for relative compactness of a subset of the space of Borel probability measures on any topological space (such a result was previously known for the space of Radon probability measures on any Hausdorff space). Prokhorov’s theorem is used as a tool to allow pushing down certain internal measures on the space of all Radon probability measures on a Hausdorff space (see Theorem 3.11 and Theorem 3.12), a key step in preparation for our proof of the generalization to de Finetti–Hewitt–Savage theorem.

At the heart of our argument is a combinatorial result analogous to the approximate, finite version of de Finetti’s theorem obtained by Diaconis and Freedman [21]. The topological nonstandard measure theory developed herein establishes a hyperfinite version of such a result (see Theorem 4.1) as a sufficient condition for our proof. This hyperfinite version of the result of Diaconis and Freedman has a salient interpretation in terms of Bayes’ theorem, which ties in nicely with the relevance of de Finetti’s theorem in Bayesian statistics (see the discussion following the statement of Theorem 4.1; see also Appendix B for an alternative proof of Theorem 4.1 along these lines).

The rest of this section is divided into subsections that introduce the above concepts, provide historical context, and also give a more detailed overview of our methods.

1.1. Introducing de Finetti’s theorem and its history. We begin with the definition of exchangeable random variables.

Definition 1.1. A finite collection \(X_1, \ldots, X_n\) of random variables is said to be exchangeable if for any permutation \(\sigma \in S_n\), the random vectors \((X_1, \ldots, X_n)\) and \((X_{\sigma(1)}, \ldots, X_{\sigma(n)})\) have the same distribution. An infinite sequence \((X_n)_{n \in \mathbb{N}}\) of random variables is said to be exchangeable if any finite subcollection of the \(X_n\) is exchangeable.
See Feller [26, pp. 229-230] for some examples of exchangeable random variables. A well-known result of de Finetti says that an exchangeable sequence of Bernoulli random variables (that is, random variables taking values in \{0, 1\}) is conditionally independent given the value of a random parameter in [0, 1] (the parameter being sampled through a unique probability measure on the Borel sigma algebra of the closed interval [0, 1]). In a more technical language, we say that any exchangeable sequence of Bernoulli random variables is uniquely representable as a mixture of independent and identically distributed (iid) sequences of Bernoulli random variables. More precisely, we may write de Finetti’s theorem in the following form.

**Theorem 1.2** (de Finetti). Let \((X_n)_{n\in\mathbb{N}}\) be an exchangeable sequence of Bernoulli random variables. There exists a unique Borel probability measure \(\nu\) on the interval [0, 1] such that the following holds:

\[
P(X_1 = e_1, \ldots, X_k = e_k) = \int_{[0,1]} \prod_{j=1}^{k} \nu(p) \left(1 - p\right)^{\sum_{j=1}^{k} c_j} \cdot p^{\sum_{j=1}^{k} c_j} \nu(dp)
\]

(1.1)

for any \(k \in \mathbb{N}\) and \(e_1, \ldots, e_k \in \{0, 1\}\).

See de Finetti [17, 18] for the original works of de Finetti on this topic. The work of generalizing de Finetti’s theorem from \{0, 1\} to more general state spaces has been an enterprise spanning the better part of the twentieth century.

What counts as a generalization of Theorem 1.2? Notice that in equation (1.1), the variable of integration, \(p\), can be identified with the measure \(\mathbb{B}_p\) induced on \{0, 1\} by a coin toss where the chance of success (with success identified with the state 1) is \(p\). Clearly, all probability measures on the discrete set \{0, 1\} are of this form. Thus, \(\nu\) in (1.1) can be thought of as a measure on the set of all probability measures on \{0, 1\}. The integrand in (1.1) then represents the probability of getting \(\sum_{j=1}^{k} e_j\) successes in \(k\) independent coin tosses, while the integral represents the expected value of this probability with respect to \(\nu\).

With \(S = \{0, 1\}\), we can thus interpret (1.1) as saying that the probability that the random vector \((X_1, \ldots, X_k)\) is in the Cartesian product \(B_1 \times \ldots \times B_k\) of measurable sets \(B_1, \ldots, B_k \subseteq S\), is given by the expected value of \(\mu(B_1) \cdot \ldots \cdot \mu(B_k)\) as \(\mu\) is sampled (according to some distribution \(\nu\)) from the space of all Borel probability measures on \(S\). Thus, one possible direction in which to generalize Theorem 1.2 is to look for a statement of the following type (although we now know this to be incorrect in such generality following the work of Dubins and Freedman [24], it is still illustrative to explore the kind of statement that we are looking for).

**A first (incorrect) guess for a generalization of de Finetti.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and let \((X_n)_{n\in\mathbb{N}}\) be an exchangeable sequence of random variables taking values in some measurable space \((S, \mathcal{S})\) (called the state space). If \(\mathfrak{M}(S)\) denotes the set of all probability measures on \((S, \mathcal{S})\), then there is a unique probability measure \(\mathcal{P}\) on \(\mathfrak{M}(S)\) such that the following holds:

\[
P(X_1 \in A_1, \ldots, X_k \in A_k) = \int_{\mathfrak{M}(S)} \mu(A_1) \cdot \ldots \cdot \mu(A_k) d\mathcal{P}(\mu) \text{ for all } A_1, \ldots, A_k \in \mathcal{S}.
\]

(1.2)
The above statement is crude since we want a probability measure on the underlying set \( \mathcal{P}(S) \), yet we have not specified what sigma algebra on \( \mathcal{P}(S) \) we are working with. We shall soon see that there are multiple natural sigma algebras on \( \mathcal{P}(S) \). Since we want to integrate functions of the type \( \mu \mapsto \mu(A) \) on \( \mathcal{P}(S) \) for all \( A \in \mathcal{G} \), the smallest sigma algebra ensuring the measurability of all such functions is appropriate for this discussion. That minimal sigma algebra is \( C(\mathcal{P}(S)) \), the one generated by cylinder sets; that is, sets of the type

\[
\{ \mu \in \mathcal{P}(S) : \mu(A_1) \in B_1, \ldots, \mu(A_k) \in B_k \},
\]

where \( k \in \mathbb{N} \); \( A_1, \ldots, A_k \in \mathcal{G} \); and \( B_1, \ldots, B_k \in \mathcal{B}(\mathbb{R}) \), the Borel sigma algebra on \( \mathbb{R} \).

Hewitt and Savage \cite[Theorem 9.4, p. 489]{HewittSavage} observed that the methods used so far required some sense of separability of the state space \( S \) in an essential way. They were able to overcome this requirement by using new ideas from convexity theory—they looked at the set of exchangeable distributions on the product space \( S^\infty \) as a convex set, of which the (coordinate-wise) independent distributions (whose values at \( A_1 \times \cdots \times A_k \) are being integrated on the right side of (1.2)) are the extreme points. Using the Krein–Milman–Choquet theorems, they were thus able to extend de Finetti’s theorem to the case in which the state space \( S \) is a compact Hausdorff space with the sigma algebra \( \mathcal{G} \) being the collection of all Baire subsets of \( S \) (see \cite[Theorem 7.2, p. 483]{HewittSavage}). Thus in their terminology, Hewitt and Savage proved that all compact Hausdorff spaces equipped with their Baire sigma algebra are presentable:

**Theorem 1.4** (Hewitt–Savage). Let \( S \) be a compact Hausdorff space and let \( \mathcal{B}_a(S) \) denote the Baire sigma algebra on \( S \) (which is the smallest sigma algebra with respect to which any continuous function \( f : S \to \mathbb{R} \) is measurable). Then \( \mathcal{B}_a(S) \) is presentable.
What does the result of Hewitt and Savage say about the presentability of Borel sigma algebras, as opposed to Baire sigma algebras? As a consequence of their theorem, they were able to show that the Borel sigma algebra of an arbitrary Borel subset of the real numbers is presentable (see \cite{37}, p. 484), generalizing the earlier works of de Finetti \cite{18} and Dynkin \cite{25} (both of whom independently showed the presentability of the Borel sigma algebra on the space of real numbers).

For a topological space $T$, we will denote its Borel sigma algebra (that is, the smallest sigma algebra containing all open subsets) by $B(T)$. Recall that a Polish space is a separable topological space that is metrizable with a complete metric. A subset of a Polish space is called an analytic set if it is representable as a continuous image of a Borel subset of some (potentially different) Polish space. As pointed out by Varadarajan \cite{66, p. 219}, the result of Hewitt and Savage immediately implies that any state space $(S, \mathcal{S})$ that is analytic is also presentable. Here an analytic space refers to a measurable space that is isomorphic to $(T, B(T))$ where $T$ is an analytic subset of a Polish space, equipped with the subspace topology (see also, Mackey \cite[Theorem 4.1, p. 140]{51}). In particular, all Polish spaces equipped with their Borel sigma algebras are presentable.

Remark 1.5. Note that both Mackey and Varadarajan use the standard conventions in descriptive set theory of referring to a measurable space as a Borel space (thus, the original conclusion of Varadarajan was stated for “Borel analytic spaces”). We will not use descriptive set theoretic considerations in this work, and hence we decided to not use the adjective ‘Borel’ in quoting Varadarajan above, so as to avoid confusion with Borel subsets of topological spaces that we will generally consider in this paper.

The above observation of Varadarajan is the state of the art for modern treatments of de Finetti’s theorem for Borel sigma algebras on topological state spaces. For example, Diaconis and Freedman \cite[Theorem 14, p. 750]{21} reproved the result of Hewitt and Savage using their approximate de Finetti’s theorem for finite exchangeable sequences in any state space (wherein they needed a nice topological structure on the state space to be able to take the limit to go from their approximate de Finetti’s theorem on finite exchangeable sequences to the exact de Finetti’s theorem on infinite exchangeable sequences). They then concluded (see \cite[p. 751]{21}) that de Finetti’s theorem holds for state spaces that are isomorphic to Borel subsets of a Polish space. Since any Borel subset of a Polish space is also analytic, this observation is a special case of Varadarajan’s. In his monograph, Kallenberg \cite[Theorem 1.1]{41} has a proof of de Finetti’s theorem for any state space that is isomorphic to a Borel subset of the closed interval $[0,1]$, a formulation that is contained in the above.

As is justified from the above discussion, the generalization of de Finetti’s theorem to more general state spaces is sometimes referred to in the literature as the de Finetti–Hewitt–Savage theorem.

Due to a lack of counterexamples at the time, a natural question arising from the work of Hewitt and Savage \cite{37} was whether de Finetti’s theorem held without any topological assumptions on the state space $S$. This was answered in the negative by Dubins and Freedman \cite{24} who constructed a separable metric space $S$ on which de Finetti’s theorem does not hold for some exchangeable sequence of $S$-valued Borel
measurable random variables. In terms of the (pushforward) measure induced by the sequence on the countable product $S^\infty$ of the state space, Dubins [23] further showed that the counterexample in [24] is singular to the measure induced by any presentable sequence. This counterexample suggests that some topological conditions are typically needed in order to avoid such pathological cases, though it may be difficult to identify the most general set of conditions that work.

Let us define the following related concept for individual sequences of exchangeable random variables.

**Definition 1.6.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $(X_n)_{n \in \mathbb{N}}$ be an exchangeable sequence of random variables taking values in some state space $(S, \mathcal{S})$. Then the sequence $(X_n)_{n \in \mathbb{N}}$ is said to be presentable if it satisfies (1.2) for some unique probability measure $\mathbb{P}$ on $(\mathcal{P}(S), C(\mathcal{P}(S)))$.

Thus a state space $(S, \mathcal{S})$ is presentable if and only if all exchangeable sequences of $S$-valued random variables are presentable. It is interesting to note that any Borel probability measure on a Polish space (which is the setting for the modern treatments of de Finetti–Hewitt–Savage theorem) is automatically Radon (see Definition 2.3). Curiously enough, the counterexample of Dubins and Freedman was for a state space on which non-Radon measures are theoretically possible. The main result of this paper shows that the Radonness of the common distribution of the underlying exchangeable random variables is actually sufficient for de Finetti’s theorem to hold for any Hausdorff state space (equipped with its Borel sigma algebra). In particular, this implies that the exchangeable random variables constructed in the counterexample of Dubins and Freedman do not have a Radon distribution. Restricting to random variables with Radon distributions (which is actually not that restrictive as many areas of probability theory work under that assumption in any case) shows that there does not exist a non-presentable exchangeable sequence of this type. For brevity of expression, let us make the following definitions.

**Definition 1.7.** An identically distributed sequence $(X_n)_{n \in \mathbb{N}}$ of random variables taking values in a Hausdorff space $S$ equipped with its Borel sigma algebra $\mathcal{B}(S)$ is said to be Radon-distributed if the pushforward probability measure induced on $(S, \mathcal{B}(S))$ by $X_1$ is Radon. It is said to be tightly distributed if this pushforward measure is tight (see also Definition 2.2).

Focusing on Hausdorff state spaces, while the answer to the original question of whether de Finetti’s theorem holds without topological assumptions is indeed in the negative (as the counterexample of Dubins and Freedman shows), we are still able to show that the most commonly studied exchangeable sequences (that is, those that are Radon-distributed) taking values in any Hausdorff space are presentable, thus establishing an affirmative answer from a different perspective. Ignoring the various technicalities in the statement of our main result (Theorem 4.7), we can thus briefly summarize our contribution to the above question as follows.

**Theorem 1.8.** Any Radon-distributed exchangeable sequence of random variables taking values in a Hausdorff space (equipped with its Borel sigma algebra) is presentable.

A closer inspection of our proof shows that we will not use the full strength of the assumption of Radonness of the common distribution of exchangeable random
variables—the theorem is still true for sequences of exchangeable random variables whose common distribution is tight and outer regular on compact sets (see the discussion following Theorem 4.7).

Before we give an overview of our methods, let us first describe a common practice in statistics that is intimately connected to the reasoning behind a statement like equation (1.2) that we are trying to generalize for sequences of tightly distributed exchangeable random variables.

1.2. A heuristic strategy motivated by statistics. Let $\mathcal{S}$ be a sigma algebra on a state space $S$. Suppose we devise an experiment to sample values from an identically distributed sequence $X_1, \ldots, X_n$ (where $n \in \mathbb{N}$ can theoretically be as large as we please) of random variables from some underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $(S, \mathcal{S})$. Depending on the way the experiment is conducted, within each iteration of the experiment it might not be justified to assume that the sampled values are independent, but it might be reasonable to still believe that the distribution of $(X_1, \ldots, X_n)$ is invariant under permutations of indices. Depending on the application, one might be interested in the joint distribution of two (or more) of the $X_i$, which is difficult to establish without an assumption of independence. However, only under an assumption of exchangeability, it is not very difficult to show the following. (Theorem 4.1 is a nonstandard version of this statement, with the standard statement having a proof along the same lines, replacing the step where we use the hyperfiniteness of $N$ in that proof by an argument about taking limits.)

$$\mathbb{P}(X_1 \in A_1, \ldots, X_k \in A_k) = \lim_{n \to \infty} \mathbb{E}(\mu_{-n}(A_1) \cdot \ldots \cdot \mu_{-n}(A_k))$$ \hspace{1cm} (1.3)

for all $k \in \mathbb{N}$ and $A_1, \ldots, A_k \in \mathcal{S}$, where

$$\mu_{\omega,n}(A) = \frac{\#\{i \in [n]: X_i(\omega) \in A\}}{n} \text{ for all } \omega \in \Omega \text{ and } A \in \mathcal{S}.$$ \hspace{1cm} (1.4)

Here $[n]$ denotes the initial segment $\{1, \ldots, n\}$ of $n \in \mathbb{N}$. In (statistical) practice, for any $k \in \mathbb{N}$ and $A_1, \ldots, A_k \in \mathcal{S}$, we do multiple independent iterations of the experiment. For any $m \in \mathbb{N}$, we calculate the product $\mu_{\omega,m}^{(1)}(A_1) \cdot \ldots \cdot \mu_{\omega,m}^{(m)}(A_k)$ of the “empirical sample means” in the $m$th iteration of the experiment. The strong law of large numbers (which we can use because of the assumption that the experiments generating samples of $(X_1, \ldots, X_n)$ are independent) thus implies the following:

$$\lim_{m \to \infty} \frac{\sum_{j \in [m]} \mu_{\omega,m}^{(j)}(A_1) \cdot \ldots \cdot \mu_{\omega,m}^{(j)}(A_k)}{m} = \mathbb{E}(\mu_{-n}(A_1) \cdot \ldots \cdot \mu_{-n}(A_k)) \text{ almost surely.}$$ \hspace{1cm} (1.5)

By (1.5) and (1.3), we thus obtain the following for all $k \in \mathbb{N}$ and $A_1, \ldots, A_k \in \mathcal{S}$:

$$\mathbb{P}(X_1 \in A_1, \ldots, X_k \in A_k) = \lim_{n \to \infty} \lim_{m \to \infty} \frac{\sum_{j \in [m]} \mu_{\omega,m}^{(j)}(A_1) \cdot \ldots \cdot \mu_{\omega,m}^{(j)}(A_k)}{m}.$$ \hspace{1cm} (1.6)

Thus, only under an assumption of exchangeability of the values sampled in each experiment, as long as we have a method to repeat the experiment independently, we have the following heuristic idea to statistically approximate the joint probability $\mathbb{P}(X_1 \in A_1, \ldots, X_k \in A_k)$ for any $A_1, \ldots, A_k \in \mathcal{S}$:
(i) In each iteration of the experiment, sample a large number (this corresponds to \( n \) in (1.6)) of values.

(ii) Conclude a large number (this corresponds to \( m \) in (1.6)) of such independent experiments.

(iii) The average of the empirical sample means \( \mu_{\omega,n}(A_1) \cdot \ldots \cdot \mu_{\omega,n}(A_k) \) (as \( j \) varies in \([m]\)) is then an approximation to \( \mathbb{P}(X_1 \in A_1, \ldots, X_k \in A_k) \).

As hinted earlier, the above heuristic idea is at the heart of the intuition behind de Finetti’s theorem as well. How do we make this idea more precise to hopefully get a version of de Finetti theorem of the form (1.2)? Suppose for the moment that we have fixed some sigma algebra on \( \mathfrak{F} \) (we will come back to the issue of which sigma algebra to fix) such that the following natural conditions are met:

(i) For each \( n \in \mathbb{N} \), the map \( \omega \mapsto \mu_{\omega,n} \) is a \( \mathfrak{P}(S) \)-valued random variable on \( \Omega \).

(ii) For each \( A \in \mathcal{G} \), the map \( \mu \mapsto \mu(A) \) is a real-valued random variable on \( \mathfrak{P}(S) \).

For each \( n \in \mathbb{N} \), this would define a pushforward probability measure \( \nu_n \) on \( \mathfrak{P}(S) \) that is supported on \( \{ \mu_{\omega,n} : \omega \in \Omega \} \subseteq \mathfrak{P}(S) \), such that

\[
\int_{\mathfrak{P}(S)} \mu(A_1) \ldots \mu(A_k) d\nu_n(\mu) = \int_{\Omega} \mu_{\omega,n}(A_1) \ldots \mu_{\omega,n}(A_k) d\mathbb{P}(\omega)
\]

for all \( A_1, \ldots, A_k \in \mathcal{G} \). \hspace{1cm} (1.7)

Comparing (1.3) and (1.7), it is clear that we are looking for conditions that guarantee there to be a measure \( \nu \) on \( \mathfrak{P}(S) \) such that the following holds:

\[
\lim_{n \to \infty} \int_{\mathfrak{P}(S)} \mu(A_1) \ldots \mu(A_k) d\nu_n(\mu) = \int_{\mathfrak{P}(S)} \mu(A_1) \ldots \mu(A_k) d\nu(\mu)
\]

for all \( A_1, \ldots, A_k \in \mathcal{G} \). \hspace{1cm} (1.8)

Intuitively, equation (1.8) is a statement of convergence (in some sense) of \( \nu_n \) to \( \nu \). A naive candidate for \( \nu \) could come from (1.7) if the following are true:

(1) There exists an almost sure set \( \Omega' \subseteq \Omega \) such that for each \( A \in \mathcal{G} \), the limit \( \lim_{n \to \infty} \mu_{\omega,n}(A) \) exists for all \( \omega \in \Omega' \). Up to null sets in \( \Omega \), this would thus define a function \( \omega \mapsto \mu_{\omega} \) from \( \Omega \) to the space of all real-valued functions on \( \mathcal{G} \), where \( \mu_{\omega}(A) = \lim_{n \to \infty} \mu_{\omega,n}(A) \).

(2) The function \( \mu_{\omega} : \mathcal{G} \to [0,1] \) is actually a probability measure on \( (S, \mathcal{G}) \).

Indeed if these two conditions are true, then one may define \( \nu \) to be the pushforward on \( \mathfrak{P}(S) \) of the map \( \omega \mapsto \mu_{\omega} \). A weaker version of (1) is often interpreted as a generalization of the strong law of large numbers for exchangeable random variables—see, for instance, Kingman [44, Equation (2.2), p. 185], which can be easily modified to work for arbitrary \( (S, \mathcal{G}) \) to conclude that \( \lim_{n \to \infty} \mu_{\omega,n}(A) \) exists for all \( \omega \) in an almost sure set that depends on \( A \). Of course, an issue with this idea is that if we have too many (that is, uncountably many) different choices for \( A \in \mathcal{G} \), then there is no guarantee that an almost sure set would exist that works for all \( A \in \mathcal{G} \) simultaneously. The condition (2) is even more delicate, as showing countable additivity of \( \mu_{\omega} \) would require some control on the rates at which the sequences \( (\mu_{\omega,n}(A))_{n \in \mathbb{N}} \) converge for different \( A \in \mathcal{G} \).
Thus we seem to have reached a dead end in this heuristic strategy in the absence of having more information about the specific structure of our spaces and measures. We now describe a generalization of a slightly different type before explaining our method of proof.

1.3. Ressel’s Radon presentability and the ideas behind our proof. As we describe next, our strategy (motivated by the statistical heuristics from Section 1.2) for proving de Finetti’s theorem naturally leads to an investigation into a de Finetti style theorem first proved by Ressel in [55]. Ressel studied de Finetti-type theorems using techniques from abstract harmonic analysis. His insight was to look for indirect generalizations of de Finetti’s theorem; that is, those generalizations which do not prove (1.2) for a state space in a strict sense, but rather prove an analogous statement applicable to nicer classes of random variables, with the smaller space of Radon probability measures being considered (as opposed to the space of all Borel probability measures). Before we proceed, let us make some of these technicalities more precise.

Definition 1.10. Let a sequence of random variables \((X_n)_{n \in \mathbb{N}}\) taking values in a Hausdorff space \(S\) be called jointly Radon distributed if the pushforward measure induced by the sequence on \((S^\infty, \mathcal{B}(S^\infty))\) (the product of countably many copies of \(S\), equipped with its Borel sigma algebra) is Radon.

Definition 1.11. Let a jointly Radon distributed sequence of exchangeable random variables \((X_n)_{n \in \mathbb{N}}\) be called Radon presentable if there is a unique Radon measure \(P\) on the space \(\mathcal{P}_r(S)\) of all Radon measures on \(S\) (equipped with its weak topology) such that the following holds for all \(k \in \mathbb{N}:\)

\[
\mathbb{P}(X_1 \in B_1, \ldots, X_k \in B_k) = \int_{\mathcal{P}_r(S)} \mu(B_1) \cdot \ldots \cdot \mu(B_k) d\mathcal{P}(\mu)
\]

for all \(B_1, \ldots, B_k \in \mathcal{B}(S). \quad (1.9)

Note that (1.9) is an analog of (1.2). This terminology of Ressel is inspired from the similar terminology of presentable spaces introduced by Hewitt and Savage [37].

One of the results that Ressel proved (see [55, Theorem 3, p. 906]) says that all completely regular Hausdorff spaces are Radon presentable. Ressel’s theorem, in particular, shows that all Polish spaces and all locally compact Hausdorff spaces are Radon presentable (see [55, p. 907]). In fact, as we show in Appendix A (see Theorem A.6), there is a standard measure theoretic argument by which Ressel’s result on completely regular Hausdorff spaces implies the Hewitt–Savage generalization of de Finetti’s theorem (Theorem 1.4). Thus, although it appears to be in a slightly different form, Ressel’s result indeed is a generalization of the de Finetti–Hewitt–Savage theorem in a strict sense. Prior to the statement of his theorem, he remarked the following (see [55, p. 906]):
"It might be true that all Hausdorff spaces have this property."

This conjecture of Ressel was confirmed by Winkler [68] using ideas from convexity theory (similar in spirit to Hewitt–Savage [37]). Fremlin showed in his treatise [30] that a stronger statement is actually true. Replacing the requirement of being jointly Radon distributed with the weaker requirement of being jointly quasi-Radon distributed (this notion is defined in Fremlin [30, 411H, p. 5]) and marginally Radon distributed (that is, the individual common distribution of the random variables must be Radon), Fremlin [30, 459H, p. 166] showed that all such exchangeable sequences also satisfy (1.9). One of our main results generalizes this further to situations where no assumptions on the joint distribution of the sequence of exchangeable random variables are needed:

**Theorem 4.2.** Let $S$ be a Hausdorff topological space, with $\mathcal{B}(S)$ denoting its Borel sigma algebra. Let $\mathcal{P}_r(S)$ be the space of all Radon probability measures on $S$ and $\mathcal{B}(\mathcal{P}_r(S))$ be the Borel sigma algebra on $\mathcal{P}_r(S)$ with respect to the $A$-topology on $\mathcal{P}_r(S)$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $X_1, X_2, \ldots$ be a sequence of exchangeable $S$-valued random variables such that the common distribution of the $X_i$ is Radon on $S$. Then there exists a unique probability measure $\mathbb{P}$ on $(\mathcal{P}_r(S), \mathcal{B}(\mathcal{P}_r(S)))$ such that the following holds for all $k \in \mathbb{N}$:

$$\mathbb{P}(X_1 \in B_1, \ldots, X_k \in B_k) = \int_{\mathcal{P}_r(S)} \mu(B_1) \cdot \ldots \cdot \mu(B_k) d\mathbb{P}(\mu)$$

for all $B_1, \ldots, B_k \in \mathcal{B}(S)$. (4.14)

We have not yet described the concept of $A$-topology that appears in the above theorem. In general, if $S$ is a topological space and $\mathcal{S} = \mathcal{B}(S)$ is the Borel sigma algebra on $S$, then there are natural ways to topologize the space $\mathcal{P}(S)$ (respectively $\mathcal{P}_r(S)$) of Borel probability measures (respectively Radon probability measures) on $S$, which would thus lead to natural (Borel) sigma algebras on $\mathcal{P}(S)$ (respectively $\mathcal{P}_r(S)$). Although we had already established that any such sigma algebra on $\mathcal{P}(S)$ we work with under the aim of showing (1.2) should be at least as large as the cylinder sigma algebra $\mathcal{C}(\mathcal{P}(S))$, a potentially larger Borel sigma algebra on $\mathcal{P}(S)$ induced by some topology on $\mathcal{P}(S)$ would be desirable in order to be able to use tools from topological measure theory (an analogous statement applies for $\mathcal{P}_r(S)$ in the context of (1.9)).

For instance, perhaps the most common topology studied in probability theory is the topology of weak convergence (see Definition 1.9). The weak topology on $\mathcal{P}(S)$, however, is interesting only when there are many real-valued continuous functions on $S$ to work with. If $S$ is completely regular (which is true of all the settings in the previous generalizations of de Finetti’s theorem), for instance, then the weak topology on $\mathcal{P}(S)$ is a natural topology to work with. However, if the state space $S$ is not completely regular then the weak topology may actually be too coarse to be of any interest.

Indeed, as extreme cases, there are regular Hausdorff spaces that do not have any nonconstant continuous real-valued functions. Identifying the most general conditions on the topological space $S$ guaranteeing existence of at least one nonconstant continuous real-valued function was part of Urysohn’s research program...
Thus, we ideally want something finer than the weak topology when working with state spaces that are more general than completely regular spaces. A natural finer topology is the so-called $A$-topology (named after A.D. Alexandroff) defined through bounded upper (or lower) semicontinuous functions from $S$ to $\mathbb{R}$, as opposed to through bounded continuous functions. Thus, the $A$-topology on $\mathcal{P}(S)$ (respectively $\mathcal{P}_r(S)$) is the smallest topology such that the maps $\mu \mapsto \mathbb{E}_\mu(f)$ on either space are upper semicontinuous for each bounded upper semicontinuous function $f : S \to \mathbb{R}$. With respect to the Borel sigma algebra on $\mathcal{P}(S)$ or $\mathcal{P}_r(S)$ induced by this topology, the evaluation maps $\mu \mapsto \mu(A)$ are indeed measurable for any $A \in \mathcal{B}(S)$ (see Theorem 2.20 and Theorem 2.33), which is something we necessarily need in order to even write an equation such as (1.2) or (1.9) meaningfully. The next section is devoted to a thorough study of this topology.

How is a generalization of Ressel’s theorem in the form of Theorem 4.2 connected to our generalization of the classical de Finetti’s theorem as stated in Theorem 1.8 (see Theorem 4.7 for a more precise statement)? The idea is that any sequence of exchangeable random variables satisfying (1.9) must also satisfy the more classical equation (1.2) of de Finetti–Hewitt–Savage (see Theorem 4.6). This follows from elementary topological measure theory arguments that exploit the specific structure of the subspace topology induced by the $A$-topology. Thus, extending Ressel’s theorem to a wider class of exchangeable random variables also proves the classical de Finetti’s theorem for that class of exchangeable random variables. Let us now describe the intuition behind our proof idea, which will complete the story by showing that such an idea naturally leads to an investigation into a generalization of Ressel’s theorem in the form of Theorem 4.2.

The idea is to carry out the naive strategy from Section 1.2 using hyperfinite numbers from nonstandard analysis as tools to model large sample sizes. Fix a hyperfinite $N > \mathbb{N}$ and study the map $\omega \mapsto ^*\mu_{\omega,N}$ from $^*\Omega$ to $^*\mathcal{P}(S)$. This map induces an internal probability measure (through the pushforward) on the space $^*\mathcal{P}(S)$ of all internal probability measures on $^*S$. That is, this pushforward measure lives in the space $^*\mathcal{P}(^*\mathcal{P}(S))$. In view of (1.8) (and the nonstandard characterization of limits), we want to have a standard probability measure $\nu$ on $\mathcal{P}(^*\mathcal{P}(S))$ that is close to the above pushforward internal measure in the sense that the integral of a function of the type $\mu \mapsto \mu(^*A_1) \cdot \ldots \cdot \mu(^*A_k)$ with respect to this pushforward is infinitesimally close to its integral with respect to $^*\nu$ for any $k \in \mathbb{N}$ and $A_1, \ldots, A_k \in \mathcal{B}(S)$.

As the space $\mathcal{P}(S)$ (and hence the space $\mathcal{P}(^*\mathcal{P}(S))$) has a topology on it (namely, the $A$-topology), a natural way to look for a standard element in $\mathcal{P}(^*\mathcal{P}(S))$ close to a given element of $^*\mathcal{P}(^*\mathcal{P}(S))$ is to try to see if this given element has a unique
standard part (or if it is at least nearstandard). It turns out (see Section 2, more specifically Theorem 2.28 and Theorem 2.12) that there are certain natural sufficient conditions for this if the space $\mathcal{P}(S)$ is Hausdorff. This is, however, again too much to ask for in general (see Corollary 2.31)! It is shown in Topsoe [61] (and also proved in Theorem 2.35) that if the underlying space $S$ is Hausdorff then the space $\mathcal{P}_r(S)$ of all Radon probability measures on $S$ is also Hausdorff. Since the internal measures $\mu_{\omega,N}$, being supported on the hyperfinite sets $\{X_1(\omega), \ldots, X_N(\omega)\}$, are internally Radon for all $\omega \in *\Omega$, this move from $\mathcal{P}(S)$ to $\mathcal{P}_r(S)$ does not affect our strategy—the pushforward of the map $\omega \mapsto *\mu_{\omega,N}$ lives in $*\mathcal{P}(\mathcal{P}_r(S))$, in which we try to find its standard part in order to complete our proof.

The main tool in finding a standard part of this pushforward is Theorem 2.28, which is used in conjunction with Theorem 2.12 (originally from Albeverio et al. [4, Proposition 3.4.6, p. 89]). This technique is called “pushing down Loeb measures” and is well-known in the nonstandard literature (see, for example, Albeverio et al. [4, Chapter 3.4] or Ross [56, Section 3]). It is often used to construct a standard measure that is close in some sense to an internal (nonstandard) measure. The way we develop the theory of $A$-topology allows us to interpret this classical technique of pushing down Loeb measures as actually taking a standard part in a legitimate nonstandard space (of internal measures). See, for example, Theorem 2.28, Remark 2.29, and Theorem 2.36.

Using Theorem 2.12 as described above requires us to first show the existence of large compact sets in $\mathcal{P}_r(S)$ in some sense, which is shown to be the case in Theorem 3.11 using a version of Prokhorov’s theorem in this setting (see Theorem 2.46). It is in this proof that we need the Radonnes of the underlying distribution of $X_1$, thus explaining how our statistical heuristic naturally leads to an investigation of a generalization Ressel’s theorem to sequences of Radon-distributed exchangeable random variables, rather than the classical presentability of Hewitt and Savage.

After setting up this abstract machinery for pushing down Loeb measures, the main computational result that is sufficient for Theorem 4.2 is Theorem 4.1, which, as mentioned earlier, is the nonstandard version of (1.3) from our statistical heuristic in Section 1.2. The fact that this is a sufficient condition follows naturally from the general topological measure theory of hyperfinitely many identically distributed random variables that is developed in Section 3. It should be pointed out that the proof of Theorem 4.1 uses a similar combinatorial construction as Diaconis–Freedman’s proof of the finite, approximate version of de Finetti’s theorem in [21]. In fact, the proof shows that the two results are different ways to express the same idea (see also the discussion following the statement of Theorem 4.1). The form of the result presented here can be given an intuitive underpinning based on Bayes’ theorem (this is made more precise in Appendix B, where an alternative proof of Theorem 4.1 is provided). This is noteworthy from the point of view that Theorem 4.1 is the key ingredient in our proof of the generalization of a result (namely de Finetti’s theorem) usually considered foundational for Bayesian statistics (see Savage [57, Section 3.7], and Orbanz–Roy [52]).

In some sense, we prove a highly general de Finetti’s theorem using the same underlying basic idea that works for the simplest versions of de Finetti’s theorem
(that being the idea of approximating using empirical sample means), the technical machinery from topological measure theory and nonstandard analysis notwithstanding. The bulk of this paper (Sections 2 and 3) is devoted to setting up this technical machinery.

For a more thorough introduction to exchangeability, see Aldous [6], Kingman [44], and Kallenberg [41]. Besides a recent note of the author on a nonstandard proof of de Finetti’s theorem for Bernoulli random variables (see Alam [2]), there is some precedence in the use of nonstandard analysis in this field, as Hoover [38, 39] studied the notions of exchangeability for multi-dimensional arrays using nonstandard methods in the guise of ultraproducts. In view of this work, Aldous [6, p. 179] had also expressed the hope of nonstandard analysis being useful in other topics in exchangeability. Another example is Dacunha-Castelle [16] who also used ultraproducts to study exchangeability in Banach spaces. Our general reference for the nonstandard analysis used in this paper is Albeverio et al. [Chapters 1-3][4], while Ross [56] is also recommended for background on the concept of $S$-integrability. While we assume familiarity with the basics of nonstandard extensions (a very quick overview can be found in Alam [3]; see also Loeb [49] for a more thorough introduction), we provide some background on Loeb measures as well as on nonstandard extensions of topological spaces in Section 2.2. The quick overview in [3] and Section 2.2 are sufficient to cover all the non-probability theoretic pre-requisites of this paper.

1.4. Outline of the paper. In Section 2, the main object of study is the space of probability measures on a topological space $S$, under the $A$-topology. Section 2.2 outlines some standard techniques in nonstandard measure theory that we will be using throughout. The rest of Section 2 develops basic results on the so-called $A$-topology on the space of probability measures. While some of this material can be thought of as a review of known results, the theory is developed from scratch, with frequent interplay with the perspectives provided from nonstandard analysis. This is an attempt to unify the classical technique of pushing down Loeb measures with statements about the topology on the space of probability measures. Thus, while not all results on $A$-topology are new, many proofs are new and the material is presented in a way that makes the exposition as self-contained as possible. A highlight of this section is a quick nonstandard proof of a generalization of Prokhorov’s theorem (see Theorem 2.44; see also Section 2.6 for a historical discussion on Prokhorov’s theorem).

In Section 3, we only assume that the sequence $(X_n)_{n \in \mathbb{N}}$ is identically distributed and derive several useful foundational results as applications of the theory built in Section 2. In particular, we study the structure of the hyperfinite empirical distributions derived from (the nonstandard extension of) an identically distributed sequence of random variables, and the measures they induce on the space of all Radon probability measures on the state space.

In Section 4, we exploit the added structure provided by exchangeability that allows us to use the foundational results from Section 3 to prove our generalizations of de Finetti’s theorem. Section 4.3 briefly mentions some other possible versions and generalizations of de Finetti’s theorem that we did not consider in this paper, along with a discussion on potential future work.
2. Background from nonstandard and topological measure theory

2.1. General topology and measure theory notations. All measures considered in this paper are countably additive, and unless otherwise specified, probability measures. We will usually work with probability measures on the Borel sigma algebra $\mathcal{B}(T)$ of a topological space $T$ (thus $\mathcal{B}(T)$ is the smallest sigma algebra that contains all open subsets of $T$).

**Definition 2.1.** A subset of a topological space is called a $G_\delta$ set if it is a countable intersection of open sets. A topological space is called a $G_\delta$ space if all of its closed subsets are $G_\delta$ sets.

Let us recall the various notions of separation in topological spaces (for further topological background, we refer the interested reader to Kelley [43]):

$(T_1)$ A space $T$ is called Fréchet if any singleton subset of $T$ is closed.
$(T_2)$ A space $T$ is called Hausdorff if any two points in it can be separated via open sets. That is, given any two distinct points $x$ and $y$ in $T$, there exist disjoint open sets $G_1$ and $G_2$ such that $x \in G_1$ and $y \in G_2$.
$(T_3)$ A space $T$ is called regular if any closed set and a point outside that closed set can be separated via open sets. That is, given a closed set $F \subseteq T$ and given $x \in T \setminus F$, there exist disjoint open sets $G_1$ and $G_2$ such that $x \in G_1$ and $y \in G_2$.
$(T_{3\frac{1}{2}})$ A space $T$ is called completely regular if any closed set and a point outside that closed set can be separated via some bounded real-valued function. That is, given a closed set $F \subseteq T$ and $x \in T \setminus F$, there is a continuous function $f : T \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$ for all $y \in F$.
$(T_4)$ A space $T$ is called normal if any two disjoint subsets of $T$ can be separated by open sets. That is, given closed sets $F_1, F_2 \subseteq T$ such that $F_1 \cap F_2 = \emptyset$, there exist disjoint open sets $G_1$ and $G_2$ such that $F_1 \subseteq G_1$ and $F_2 \subseteq G_2$.
$(T_5)$ A space $T$ is called hereditarily normal if all subsets of $T$ (under the subspace topology) are normal.
$(T_6)$ A space $T$ is called perfectly normal if it is a normal $G_\delta$ space.

We now recall the definitions of some important classes of probability measures.

**Definition 2.2.** For a Hausdorff space $T$, a Borel probability measure $\mu$ is called tight if given any $\epsilon \in \mathbb{R}_{>0}$, there is a compact subset $K_\epsilon$ such that the following holds:

$$\mu(K_\epsilon) > 1 - \epsilon.$$  

(2.1)

An alternative way to write the above condition for tightness is the following:

$$\mu(T) = \sup \{ \mu(K) : K \text{ is a compact subset of } T \}.$$  

(2.2)

If a measure $\mu$ satisfies (2.2) with the occurrence of $T$ replaced by any Borel subset of $T$, then we call it a Radon measure. More formally we make the following definition (the second line in the equality following from the fact that we are only considering probability, and in particular finite, measures).
Definition 2.3. For a Hausdorff space $T$, a Borel probability measure $\mu$ is called Radon if for each Borel set $B \in \mathcal{B}(T)$, the following holds:

$$
\mu(B) = \sup\{\mu(K) : K \subseteq B \text{ and } K \text{ is compact}\} = \inf\{\mu(G) : B \subseteq G \text{ and } G \text{ is open}\}.
$$

Note that the Hausdorffness of the topological space $T$ was assumed in the previous definitions so as to ensure that the compact sets appearing in them were Borel measurable (as a compact subset of any Hausdorff space is automatically closed). While not typically done (as many results do not generalize to those settings), these definitions can be made for arbitrary topological spaces if we replace the word “compact” by “closed and compact”. See Schwarz [58, pp. 82-88] for more details on this generalization (Schwarz uses the phrase ‘quasi-compact’ instead of ‘compact’ in this discussion). In this paper, we will always have an underlying assumption of Hausdorffness of $T$ during any discussions involving tight or Radon measures.

Remark 2.4. It is clear that all Radon measures are tight. Note that any Borel probability measure on a $\sigma$-compact Hausdorff space (that is, a Hausdorff space that can be written as a countable union of compact spaces) is tight. Vakhania–Tarladze–Chobanyan [64, Proposition 3.5, p. 32] constructs a non-Radon Borel probability measure on a particular compact Hausdorff space (the construction being attributed to Dieudonné). Thus, not all tight measures are Radon.

Definition 2.5. Let $T$ be a topological space and let $K \subseteq \mathcal{B}(T)$. We say that a Borel probability measure $\mu$ is outer regular on $K$ if we have the following:

$$
\mu(B) = \inf\{\mu(G) : B \subseteq G \text{ and } G \text{ is open}\} \text{ for all } B \in K.
$$

In our generalization of the de Finetti–Hewitt–Savage theorem, we will work under the assumption that the underlying common distribution of the given exchangeable random variables is tight and outer regular on compacts.

2.2. Review of nonstandard measure theory and topology. Assuming familiarity with basic nonstandard methods, we outline here a construction of Loeb measures, both to establish the notation we will use and to make the rest of the exposition as self-contained as possible. The goal of this discussion is to describe the method of pushing down Loeb measures, which is one of the main tools in our work as it allows us to precisely talk about when a nonstandard measure on the nonstandard extension of a topological space is, in a reasonable sense, infinitesimally close to a standard measure (this idea will be made more precise at the end of our discussion on Alexandroff topology in the next subsection; see, for example, Theorem 2.28 and Remark 2.29).

Following are some general notations that we will follow. For two nonstandard numbers $x, y \in {^*\mathbb{R}}$, we will write $x \approx y$ to denote that $x - y$ is an infinitesimal. The set of finite nonstandard real numbers will be denoted by $^*\mathbb{R}_{\text{fin}}$ and the standard part map $\text{st} : {^*\mathbb{R}}_{\text{fin}} \rightarrow \mathbb{R}$ takes a finite nonstandard real to its closest real number. We follow the superstructure approach to nonstandard extensions, as in Albeverio et al. [4]. In particular, we fix a sufficiently saturated nonstandard extension of a superstructure containing all standard mathematical objects under study. The nonstandard extension of a set $A$ (respectively a function $f$) is denoted by $^*A$.
This page contains a detailed explanation of the Loeb measure and its properties. The text delves into the concept of Loeb algebra and its relation to topological spaces, particularly focusing on the Loeb measure and its definition for nonstandard points. The text references the Definetti–Hewitt–Savage theorem and discusses the completion of a probability space. The notation and definitions are formal, using symbols and terms specific to measure theory and nonstandard analysis.
pathological situation is remedied in Hausdorff spaces. Indeed, given two standard points \( x_1 \) and \( x_2 \) in a Hausdorff space \( T \), one may separate them by open sets (say) \( G_1 \) and \( G_2 \) respectively, so that \(^*G_1\) and \(^*G_2\) are disjoint, thus making \( \text{st}^{-1}(x_1) \) and \( \text{st}^{-1}(x_2) \) also disjoint.

Conversely, thinking along the same lines, if the standard inverses of any two distinct points are disjoint, then those points can be separated by disjoint open sets. Thus, we have the following nonstandard characterization of Hausdorffness (see also [4, Proposition 2.1.6 (i), p. 48]):

**Lemma 2.6.** A topological space \( T \) is Hausdorff if and only if for any distinct elements \( x, y \in T \) we have \( \text{st}^{-1}(x) \cap \text{st}^{-1}(y) = \emptyset \).

Regardless of whether \( T \) is Hausdorff or not, (2.6) allows us to naturally talk about \( \text{st}^{-1}(A) \) for subsets \( A \subseteq T \). That is, we define:

\[
\text{st}^{-1}(A) := \{ y \in ^*T : y \in \text{st}^{-1}(x) \text{ for some } x \in A \}.
\]

(2.7)

We define the set of nearstandard points of \(^*T\) as follows:

\[
\text{Ns}(^*T) := \text{st}^{-1}(T).
\]

Thus, by Lemma 2.6, if \( T \) is Hausdorff then \( \text{st} : \text{Ns}(^*T) \to T \) is a well-defined map.

Using the notation in (2.7), there are succinct nonstandard characterizations of open, closed, and compact sets, which we note next (see [4, Proposition 2.1.6, p. 48], with the understanding that Albeverio et al. only use the set function \( \text{st}^{-1} \) when the underlying space is Hausdorff, but that is not needed for these characterizations).

**Theorem 2.7.** Let \( T \) be a topological space.

(i) A set \( G \subseteq T \) is open if and only if \( \text{st}^{-1}(G) \subseteq ^*G \).

(ii) A set \( F \subseteq T \) is closed if and only if for all \( x \in ^*F \cap \text{Ns}(^*T) \), the condition \( x \in \text{st}^{-1}(y) \) implies that \( y \in F \).

(iii) A set \( K \subseteq T \) is compact if and only if \( ^*K \subseteq \text{st}^{-1}(K) \).

The following technical consequence of Theorem 2.7 will be useful in Section 3.

**Lemma 2.8.** Suppose \( (K_i)_{i \in I} \) is a collection of closed subsets of a Hausdorff space \( T \) (where \( I \) is an index set). Suppose that \( K := \cap_{i \in I} K_i \) is compact. Then for any open set \( G \) with \( K \subseteq G \), we have:

\[
^*K \subseteq \left( \bigcap_{i \in I} ^*K_i \right) \cap \text{Ns}(^*T) \subseteq ^*G.
\]

(2.8)

**Proof.** The first inclusion in (2.8) is true since \( ^*K \subseteq ^*K_i \) for all \( i \in I \) (which follows because \( K \subseteq K_i \) for all \( i \in I \)), and since \( K \) is compact (so that all elements of \( ^*K \) are nearstandard by Theorem 2.7(iii)). To see the second inclusion in (2.8), suppose we take \( x \in \cap_{i \in I} (^*K_i \cap \text{Ns}(^*T)) \). Since \( T \) is Hausdorff, \( x \in \text{Ns}(^*T) \) has a unique standard part, say \( \text{st}(x) = y \in T \). Since \( K_i \) is closed for each \( i \in I \), it follows from the nonstandard characterization of closed sets (Theorem 2.7(ii)) that \( y \in K_i \) for all \( i \in I \). As a consequence, \( y \in K \subseteq G \). Thus by the nonstandard characterization of open sets (see Theorem 2.7(ii)), it follows that \( x \in ^*G \), thus completing the proof. \( \square \)
If $T$ is a topological space and $T' \subseteq T$ is viewed as a topological space under the subspace topology (thus a subset $G' \subseteq T'$ is open in $T'$ if and only if $G' = T' \cap G$ for some open subset $G$ of $T$), then there are multiple ways to interpret (2.7). There is a similar issue in general when we have two topological spaces in which we could be taking standard inverses. We will generally use ‘$\text{st}$’ and ‘$\text{st}^{-1}$’ for all such usages when the underlying topological space is clear from context. If it is not clear from context, then we mention the space in a subscript. Thus in the above situation where $T' \subseteq T$, we denote by $\text{st}^{-1}_{T'}$ and $\text{st}^{-1}_{T}$ the corresponding set functions on subsets of $T$ and $T'$ respectively. Thus, for subsets $A \subseteq T$ and $A' \subseteq T'$, we have:

$$\text{st}^{-1}_{T}(A) = \{ x \in {}^*T : \exists y \in A \text{ such that } x \in {}^*G \text{ for all open neighborhoods } G \text{ of } y \text{ in } T \},$$

and

$$\text{st}^{-1}_{T'}(A') = \{ x \in {}^*T' : \exists y \in A' \text{ such that } x \in {}^*G' \text{ for all open neighborhoods } G' \text{ of } y \text{ in } T' \}.$$

The following useful relation is immediate from the fact that the nonstandard extension of a finite intersection of sets is the same as the intersection of the nonstandard extensions.

**Lemma 2.9.** Let $T$ be a topological space and let $T' \subseteq T$ be viewed as a topological space under the subspace topology. For a subset $A \subseteq T' \subseteq T$, we have:

$$^*T' \cap \text{st}^{-1}_{T}(A) \subseteq \text{st}^{-1}_{T'}(A).$$

Using the notation in (2.7), Lemma 2.6 can be immediately modified to obtain the following nonstandard characterization of Hausdorffness, which will be useful in the sequel.

**Lemma 2.10.** A topological space $T$ is Hausdorff if and only if for any disjoint collection $(A_i)_{i \in I}$ of subsets of $T$ (indexed by some set $I$), we have

$$\text{st}^{-1} \left( \bigcup_{i \in I} A_i \right) = \bigcup_{i \in I} \text{st}^{-1}(A_i),$$

(2.9)

where $\sqcup$ denotes a disjoint union.

Given an internal probability space $({}^*T, {}^*\mathcal{B}(T), \nu)$, if we know that $\text{st}^{-1}(B)$ is Loeb measurable with respect to the corresponding Loeb space $({}^*T, L({}^*\mathcal{B}(T)), L\nu)$ for all Borel sets $B \in \mathcal{B}(T)$, then one can define a Borel measure on $(T, \mathcal{B}(T))$ by defining the measure of a Borel set $B$ as $L\nu(\text{st}^{-1}(B))$. The fact that this defines a Borel measure in this case is easily checked. This measure is not a probability measure, however, except in the case that the set of nearstandard points $\text{Ns}({}^*T) := \text{st}^{-1}_{T}(T)$ is Loeb measurable with Loeb measure equaling one.

Thus, in the setting of an internal probability space $({}^*T, {}^*\mathcal{B}(T), \nu)$, where $T$ is a topological space, there are two things to ensure in order to obtain a natural standard probability measure on $(T, \mathcal{B}(T))$ corresponding to the internal measure $\nu$:

(i) The set $\text{st}^{-1}(B)$ must be Loeb measurable for any Borel set $B \in \mathcal{B}(T)$.

(ii) It must be the case that $L\nu(\text{Ns}({}^*T)) = 1$. 

Verifying when $\text{st}^{-1}(B)$ is Loeb measurable for all Borel sets $B \in \mathcal{B}(T)$ is a tricky endeavor in general, and has been studied extensively. It is interesting to note that if the underlying space $T$ is regular, then this condition is equivalent to the Loeb measurability of $\text{Ns}(^\ast T)$ (this was investigated by Landers and Rogge as part of a larger project on universal Loeb measurability; see [45, Corollary 3, p. 233]). Prior to Landers and Rogge, the same result was proved for locally compact Hausdorff spaces by Loeb [48]. Also, Henson [34] gave characterizations for measurability of $\text{st}^{-1}(B)$ when the underlying space is either completely regular or compact. See also the discussion after Theorem 3.2 in Ross [56] for other relevant results in this context. We will, however, not assume any additional hypotheses on our spaces, and hence we must study sufficient conditions for (i) and (ii) that work for any Hausdorff space.

The results in Albeverio et al. [4, Section 3.4] are appropriate in the general setting of Hausdorff spaces. Their discussion is motivated by the works of Loeb [17, 48] and Anderson [9, 8]. We now outline the key ideas to motivate the main result in this theme (see Theorem 2.12, originally from [4, Theorem 3.4.6, p. 89]), which we will heavily use in the sequel.

If the underlying space $T$ is Hausdorff, then an application of Lemma 2.10 shows that the collection $\{B \in \mathcal{B}(T) : \text{st}^{-1}(B) \in L(B(T))\}$ is a sigma algebra if and only if $\text{Ns}(^\ast T)$ is Loeb measurable. Thus in that case (that is, when $T$ is Hausdorff), one would need to show that $\text{st}^{-1}(F)$ is Loeb measurable for all closed subsets $F \subseteq T$ (or the corresponding statement for all open subsets of $T$).

Thus, under the assumptions that $\text{st}^{-1}(F)$ is Loeb measurable for all closed subsets $F \subseteq T$, and that $L_{\nu}(\text{Ns}(^\ast T)) = 1$, the map $L_{\nu} \circ \text{st}^{-1} : B(T) \to [0, 1]$ does define a probability measure on $(T, B(T))$ whenever $T$ is Hausdorff. This is the content of [4, Proposition 3.4.2, p. 87], which further uses the completeness of the Loeb measures and some nonstandard topology to show that $L_{\nu} \circ \text{st}^{-1}$ is actually a regular, complete measure on $(T, B(T))$ in this case. Under what conditions can one guarantee that $\text{st}^{-1}(F)$ is Loeb measurable for all closed subsets $F \subseteq T$? Note that if we replace $F$ by a compact set, then this is always true (for all sufficiently saturated nonstandard extensions):

**Lemma 2.11.** Let $T$ be a topological space and let $\tau$ be the topology on $T$. Then we have, for any compact subset $K \subseteq T$:

$$\text{st}^{-1}(K) = \bigcap\{^\ast O : K \subseteq O \text{ and } O \in \tau\}.$$ 

As a consequence, for any compact set $K \subseteq T$, the set $\text{st}^{-1}(K)$ is universally Loeb measurable with respect to $(^\ast T, ^\ast B(T))$. That is, for any internal probability measure $\nu$ on $(^\ast T, ^\ast B(T))$ and any compact $K \subseteq T$, we have $\text{st}^{-1}(K) \in L_{\nu}(^\ast B(T))$.

Furthermore, we have:

$$L_{\nu}(\text{st}^{-1}(K)) = \inf\{LP(^\ast O) : K \subseteq O \text{ and } O \in \tau\} \text{ for all compact subsets } K \subseteq T.$$ 

See [4, Lemma 3.4.4 and Proposition 3.4.5, pp. 88-89] for a proof of Lemma 2.11 (note that $T$ is assumed to be Hausdorff in [4] but is not needed for this proof). Thus, if we require that there are arbitrarily large compact sets with respect to $(^\ast T, ^\ast B(T), \nu)$ in the sense that

$$\sup\{L_{\nu}(\text{st}^{-1}(K)) : K \text{ is a compact subset of } T\} = 1,$$ 

(2.10)
then the completeness of the Loeb space \( (*T, L(*B(T)), L\nu) \) allows us to conclude that \( L\nu(\text{Ns}(T)) = 1 \) and that \( st^{-1}(F) \) is Loeb measurable for all closed sets \( F \subseteq T \). In this case, if \( T \) is also assumed to be Hausdorff, then \( L\nu \circ st^{-1} \) is thus shown to be a Radon measure on \((T, B(T))\) (see [4, Corollary 3.4.3, p. 88] for a formal proof). In view of Lemma 2.11, we thus immediately obtain the following result; see also [4, Theorem 3.4.6, p. 89] for a detailed proof of a slightly more general form.

**Theorem 2.12.** Let \( T \) be a Hausdorff space with \( B(T) \) denoting the Borel sigma algebra on \( T \). Let \( (*T, *B(T), \nu) \) be an internal, finitely additive probability space and let \( (*T, L(*B(T)), L\nu) \) denote the corresponding Loeb space. Let \( \tau \) denote the topology on \( T \).

Then \( st^{-1}(K) \in L(*B(T)) \) for all compact \( K \subseteq T \). Assume further that for each \( \epsilon \in \mathbb{R}_{>0} \), there is a compact set \( K_\epsilon \) with

\[
\inf \{ L\nu(*O) : K_\epsilon \subseteq O \text{ and } O \in \tau \} \geq 1 - \epsilon. \tag{2.11}
\]

Then \( L\nu \circ st^{-1} \) is a Radon probability measure on \( T \).

Note that Theorem 2.12 is a special case of [4, Theorem 3.4.6, p. 89], which we have chosen to present here in this simplified form because we do not need the full power of the latter result in our current work. In the next section, we will study a natural topology on the space of all Borel probability measures on a topological space \( T \). It will turn out that under the conditions of Theorem 2.12, the measure \( \nu \) on \((*T, *B(T))\) is nearstandard to \( L\nu \circ st^{-1} \) in the nonstandard topological sense (see Theorem 2.28). Also, the subspace of Radon probability measures is always Hausdorff (see Theorem 2.35), so that Theorem 2.12 will allow us to push down a natural nonstandard measure in a unique way on the space of all (Radon) probability measures in our proof of de Finetti’s theorem. We finish this subsection with a corollary that follows from the definition of tightness.

**Corollary 2.13.** Let \( T \) be a Hausdorff space and let \( \mu \) be a tight probability measure on it. Then \( L^*\mu \circ st^{-1} \) is a Radon probability measure on \( T \).

### 2.3. The Alexandroff topology on the space of probability measures on a topological space.

For a topological space \( T \) and a function \( f : T \to \mathbb{R} \), we say:

(i) \( f \) is upper semicontinuous at \( x_0 \in T \) if for every \( \alpha \in \mathbb{R} \) with \( \alpha > f(x_0) \), there is an open neighborhood \( U \) of \( x_0 \) such that \( \alpha > f(x) \) for all \( x \in U \).

(ii) \( f \) is lower semicontinuous at \( x_0 \in T \) if for every \( \alpha \in \mathbb{R} \) with \( \alpha < f(x_0) \), there is an open neighborhood \( U \) of \( x_0 \) such that \( \alpha < f(x) \) for all \( x \in U \).

A function \( f : T \to \mathbb{R} \) is called upper (respectively lower) semicontinuous if \( f \) is upper (respectively lower) semicontinuous at every point in \( T \). The following characterization of upper/lower semicontinuity is immediate from the definition.

**Lemma 2.14.** A function \( f : T \to \mathbb{R} \) is upper semicontinuous if and only if the set \( \{ x \in T : f(x) < \alpha \} \) is open for every \( \alpha \in \mathbb{R} \).

A function \( f : T \to \mathbb{R} \) is lower semicontinuous if and only if the set \( \{ x \in T : f(x) > \alpha \} \) is open for every \( \alpha \in \mathbb{R} \).

As a consequence, a function \( f : T \to \mathbb{R} \) is upper semicontinuous if and only if \( -f \) is lower semicontinuous.
For a topological space $T$, we will denote the set of all bounded upper semicontinuous functions on $T$ by $USC_b(T)$. Similarly, $LSC_b(T)$ will denote the set of all bounded lower semicontinuous functions on $T$.

**Remark 2.15.** It is immediate from the definition that the indicator function of an open set is lower semicontinuous, and that the indicator function of a closed set is upper semicontinuous.

For a topological space $T$, let $\mathcal{B}(T)$ denote the Borel sigma algebra of $T$—that is, $\mathcal{B}(T)$ is the smallest sigma algebra containing all open sets. Consider the set $\mathfrak{P}(T)$ of all Borel probability measures on $T$. For each bounded measurable $f: T \to \mathbb{R}$, define the map $E_f: \mathfrak{P}(T) \to \mathbb{R}$ by

$$E_f(\mu) := \mathbb{E}_\mu(f) = \int_T f d\mu. \quad (2.12)$$

**Definition 2.16.** Let $T$ be a topological space. The $A$-topology on the space of Borel probability measures $\mathfrak{P}(T)$ is the weakest topology for which the maps $E_f$ are upper semicontinuous for all $f \in USC_b(T)$.

The “$A$” in $A$-topology refers to A.D. Alexandroff [7], who pioneered the study of weak convergence of measures and gave many of the results that we will use. In the literature, the term ‘weak topology’ is sometimes used in place of ‘$A$-topology’; see, for instance, Topsøe [61, p. 40]. However, following Kallianpur [42], Blau [11], and Bogachev [13], we will reserve the term weak topology for the smallest topology on $\mathfrak{P}(T)$ that makes the maps $E_f$ continuous for every bounded continuous function $f: T \to \mathbb{R}$. For a bounded Borel measurable function $f: T \to \mathbb{R}$ and $\alpha \in \mathbb{R}$, define the following sets:

$$\mathcal{U}_{f,\alpha} := \{ \mu \in \mathfrak{P}(T) : E_\mu(f) < \alpha \}, \quad (2.13)$$

and

$$\mathcal{L}_{f,\alpha} := \{ \mu \in \mathfrak{P}(T) : E_\mu(f) > \alpha \}. \quad (2.14)$$

By Definition 2.16 and Lemma 2.14, the $A$-topology on $\mathfrak{P}(T)$ is the smallest topology under which $\mathcal{U}_{f,\alpha}$ is open for all $f \in USC_b(T)$ and $\alpha \in \mathbb{R}$. More formally, the $A$-topology on $\mathfrak{P}(T)$ is induced by the subbasis $\{ \mathcal{U}_{f,\alpha} : f \in USC_b(T), \alpha \in \mathbb{R} \}$. Also, by the last part of Lemma 2.14, this collection is actually equal to the collection $\{ \mathcal{L}_{f,\alpha} : f \in LSC_b(T), \alpha \in \mathbb{R} \}$. These observations are summarized in the following useful description of the $A$-topology.

**Lemma 2.17.** Let $T$ be a topological space, and $\mathfrak{P}(T)$ be the set of all Borel probability measures on $T$. The $A$-topology on $\mathfrak{P}(T)$ is generated by the subbasis

$$\{ \mathcal{U}_{f,\alpha} : f \in USC_b(T), \alpha \in \mathbb{R} \} = \{ \mathcal{L}_{f,\alpha} : f \in LSC_b(T), \alpha \in \mathbb{R} \}. \quad (2.15)$$

**Remark 2.18.** Note that, by Lemma 2.14, a function is continuous if and only if it is both upper and lower semicontinuous. Thus, by Lemma 2.17, the $A$-topology also makes the maps $E_f$ continuous for every bounded continuous function $f: T \to \mathbb{R}$, thus implying that the $A$-topology is, in general, finer than the weak topology on $\mathfrak{P}(T)$. The two topologies coincide if $T$ has a rich topological structure. For example, in Kallianpur [42, Theorem 2.1, p. 948], it is proved that the the $A$-topology and the weak topology on $\mathfrak{P}(T)$ are the same if $T$ is a completely regular Hausdorff space such that it can be embedded as a Borel subset of a compact Hausdorff space. This, in particular, means that the two topologies are the same
if the underlying space $T$ is a Polish space (that is, a complete separable metric space) or is a locally compact Hausdorff space.

Remark 2.19. While we are focusing on Borel probability measures on topological spaces, we could have analogously defined the $A$-topology on the space of all finite Borel measures on a topological space as well. Although we will not work with non-probability measures, we are not losing too much generality in doing so. In fact, Blau [11, Theorem 1, p. 24] shows that the space of finite Borel measures on a topological space $T$ is naturally homeomorphic to the product of $\mathcal{P}(T)$ and the space of positive reals. Thus, from a practical point of view, most results that we will obtain for $\mathcal{P}(T)$ will also hold for the $A$-topology on the space of all finite measures (some results such as Prokhorov’s theorem that talk about subsets of finite measures will hold in that setting with an added assumption of uniform boundedness that is inherently satisfied by all sets of probability measures).

By Remark 2.15, we know that $\{\mu \in \mathcal{P}(T) : \mu(G) > \alpha\}$ is open for any open subset $G \subseteq T$ and $\alpha \in \mathbb{R}$; and similarly, $\{\mu \in \mathcal{P}(T) : \mu(F) < \alpha\}$ is open for any closed subset $F \subseteq T$ and $\alpha \in \mathbb{R}$. Lemma 2.22 will show that the $A$-topology is generated by either of these types of subbasic open sets as well. We first use the above facts to show that the evaluation maps are Borel measurable with respect to the $A$-topology.

Theorem 2.20. Let $B$ be a Borel subset of a topological space $T$. Let $\mathcal{P}(T)$ be the space of all Borel probability measures on $T$ equipped with the $A$-topology. Then the evaluation map $e_B : \mathcal{P}(T) \to [0, 1]$ defined by $e_B(\mu) := \mu(B)$ is Borel measurable.

Proof. Consider the collection

$$\mathcal{B} = \{B \in \mathcal{B}(T) : e_B \text{ is Borel measurable}\}.$$

This collection contains $T$, since $f_T$ is the constant function 1, which is continuous. It is also closed under taking relative complements. That is, if $A \subseteq B$ and $A, B \in \mathcal{B}$ then $B \setminus A \in \mathcal{B}$ as well, since $f_{B \setminus A} = f_B - f_A$ in that case. Finally, $\mathcal{B}$ is closed under countable increasing unions. That is, if $(B_n)_{n \in \mathbb{N}} \subseteq \mathcal{B}$ is a sequence of sets such that $B_n \subseteq B_{n+1}$ for all $n \in \mathbb{N}$, then $B := \bigcup_{n \in \mathbb{N}} B_n \in \mathcal{B}$ as well (this is because $f_B = \lim_{n \to \infty} f_{B_n}$ is a limit of Borel measurable functions in that case). Thus, $\mathcal{B}$ is a Dynkin system.

Furthermore, $\mathcal{B}$ contains all open sets since for any open set $G \subseteq T$, the set $\{\mu \in \mathcal{P}(T) : \mu(G) > \alpha\}$ is Borel measurable (in fact, open) for all $\alpha \in \mathbb{R}$. Thus, by Dynkin’s $\pi$-$\lambda$ theorem, it contains, and hence is equal to, $\mathcal{B}(T)$, completing the proof. \qed

Lemma 2.22 finds other useful subbases for the $A$-topology. We first need the following intuitive fact from probability theory as a tool in its proof.

Lemma 2.21. Suppose $P_1$ and $P_2$ are probability measures on the same space and $X$ is a bounded random variable such that

$$P_1(X > x) \geq P_2(X > x) \quad \text{for all } x \in \mathbb{R}. \quad (2.16)$$

Then, we have $E_{P_1}(X) \geq E_{P_2}(X)$. 

Proof. With \( \lambda \) denoting the Lebesgue measure on \( \mathbb{R} \), we have the following representation of the expected value of any bounded random variable \( X \) (see, for example, Lo [46, Proposition 2.1]):

\[
\mathbb{E}_P(X) = \int_{(0,\infty)} \mathbb{P}(X > x) d\lambda(x) - \int_{(-\infty,0)} \mathbb{P}(X < x) d\lambda(x).
\] (2.17)

Let \( \mathbb{P}_1, \mathbb{P}_2 \) and \( X \) be as in the statement of the lemma. Then, using (2.16), we obtain the following for each \( x \in \mathbb{R} \):

\[
\mathbb{P}_1(X < x) = 1 - \mathbb{P}_1(X \geq x) = 1 - \mathbb{P}_1 \left( \bigcap_{n \in \mathbb{N}} \left\{ X > x - \frac{1}{n} \right\} \right) = 1 - \lim_{n \to \infty} \mathbb{P}_1 \left( X > x - \frac{1}{n} \right) \leq 1 - \lim_{n \to \infty} \mathbb{P}_2 \left( X > x - \frac{1}{n} \right) = \mathbb{P}_2(X < x). \] (2.18)

Using (2.17), (2.16) and (2.18), we thus obtain:

\[
\mathbb{E}_{\mathbb{P}_1}(X) = \int_{(0,\infty)} \mathbb{P}_1(X > x) d\lambda(x) - \int_{(-\infty,0)} \mathbb{P}_1(X < x) d\lambda(x) \geq \int_{(0,\infty)} \mathbb{P}_2(X > x) d\lambda(x) - \int_{(-\infty,0)} \mathbb{P}_2(X < x) d\lambda(x) = \mathbb{E}_{\mathbb{P}_2}(X),
\]

completing the proof. \( \square \)

Lemma 2.22. For each Borel set \( B \in \mathcal{B}(T) \), let

\[
\mathcal{U}_{B,\alpha} := \{ \mu \in \mathcal{P}(T) : \mu(B) < \alpha \},
\] (2.19)

and \( \mathcal{L}_{B,\alpha} := \{ \mu \in \mathcal{P}(T) : \mu(B) > \alpha \}. \) (2.20)

Then the topology on \( \mathcal{P}(T) \) generated by \( \{ \mathcal{U}_{F,\alpha} : \alpha \in \mathbb{R} \text{ and } F \text{ is closed} \} \) as a subbasis is the same as the topology on \( \mathcal{P}(T) \) generated by \( \{ \mathcal{L}_{G,\alpha} : \alpha \in \mathbb{R} \text{ and } G \text{ is open} \} \) as a subbasis. Both of these topologies equal the A-topology on \( \mathcal{P}(T) \).

Proof. If \( G \) is an open subset of \( T \) and \( \alpha \in \mathbb{R} \), then we have

\[
\mathcal{L}_{G,\alpha} = \bigcup_{\epsilon \in \mathbb{R}, \epsilon > \alpha} \mathcal{U}_{T \setminus G, 1 - \alpha + \epsilon}.
\] (2.21)

Since the complement of an open set is closed, this shows that a basic open set in the topology on \( \mathcal{P}(T) \) generated by \( \{ \mathcal{L}_{G,\alpha} : \alpha \in \mathbb{R} \text{ and } G \text{ is open} \} \) as a subbasis, is a finite intersection of sets that are unions of elements in the collection \( \{ \mathcal{U}_{F,\alpha} : \alpha \in \mathbb{R} \text{ and } F \text{ is closed} \} \). That is, a basic open set in the topology on \( \mathcal{P}(T) \) generated by \( \{ \mathcal{L}_{G,\alpha} : \alpha \in \mathbb{R} \text{ and } G \text{ is open} \} \) as a subbasis, is also open in the topology on \( \mathcal{P}(T) \) generated by \( \{ \mathcal{U}_{F,\alpha} : \alpha \in \mathbb{R} \text{ and } F \text{ is closed} \} \) as a subbasis. A similar argument shows that a basic open set in the latter topology is also open in the former topology, thus proving that the two topologies are equal.
Let \( \tau_1 \) be the \( A \)-topology and \( \tau_2 \) be the topology induced by \( \{ E_{G,\alpha} : G \text{ open, } \alpha \in \mathbb{R} \} \) as a subbasis. From the discussion preceding this lemma, it is clear that \( \tau_2 \subseteq \tau_1 \). Conversely, let \( U \in \tau_1 \) and \( \nu \in U \). By Lemma 2.17, there exist finitely many \( f_1, \ldots, f_k \in \text{LSC}_b(T) \) and \( \beta_1, \ldots, \beta_k \in \mathbb{R} \) such that the following holds:

\[
\nu \in \cap_{i=1}^k E_{G,\beta_i} \subseteq U. \tag{2.22}
\]

Let \( E_\nu(f_i) = \delta_i > \beta_i \) for all \( i \in \{1, \ldots, k\} \). For each \( i \in \{1, \ldots, k\} \) and \( \alpha \in \mathbb{R} \), let \( G_{i,\alpha} = \{ x \in T : f_i(x) > \alpha \} \), which is an open set by Lemma 2.14. Define

\[
E_{\alpha,\epsilon} := \cap_{i=1}^k E_{G_{i,\alpha}, \nu(G_{i,\alpha}) - \epsilon} \text{ for all } \alpha \in \mathbb{R} \text{ and } \epsilon \in \mathbb{R}_{>0}. \tag{2.23}
\]

Note that \( \nu \in E_{\alpha,\epsilon} \) for all \( \alpha \in \mathbb{R} \) and \( \epsilon \in \mathbb{R}_{>0} \), where \( E_{\alpha,\epsilon} \) is a subbasic set for the topology \( \tau_2 \). Thus it is sufficient to prove the following claim.

**Claim 2.23.** There exists \( n \in \mathbb{N} \) and \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \), \( \epsilon_1, \ldots, \epsilon_n \in \mathbb{R}_{>0} \) such that

\[
\cap_{j=1}^n E_{\alpha_j,\epsilon_j} \subseteq \cap_{i=1}^k E_{f_i,\beta_i} \subseteq U. \tag{2.24}
\]

**Proof of Claim 2.23.** Suppose, if possible, that the claim is not true. Then for each \( n \in \mathbb{N} \) and any \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \) and \( \epsilon_1, \ldots, \epsilon_n \in \mathbb{R}_{>0} \), there must exist some \( \mu \in \mathfrak{P}(T) \) such that \( \mu \notin \cap_{i=1}^k E_{G_{i,\alpha_j}, \nu(G_{i,\alpha_j}) - \epsilon_j} \) for all \( j \in \{1, \ldots, n\} \), but \( \mu \notin \cap_{i=1}^k E_{f_i,\beta_i} \). By transfer, the following internal set is non-empty for each \( n \in \mathbb{N} \), \( \bar{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \) and \( \bar{\epsilon} := (\epsilon_1, \ldots, \epsilon_n) \in (\mathbb{R}_{>0})^n \).

\[
B_{\bar{\alpha}, \bar{\epsilon}} := \{ \mu \in * \mathfrak{P}(T) : \mu(G_{i,\alpha_j}) > \nu(G_{i,\alpha_j}) - \epsilon_j \text{ for all } i \in \{1, \ldots, k\}, j \in \{1, \ldots, n\} \}
\]

but \( *E_{\mu}(f_i) \leq \beta_i \) for some \( i \in \{1, \ldots, k\} \).

By the same argument (after concatenating different finite sequences of \( \bar{\alpha} \)'s and \( \bar{\epsilon} \)'s), we note that the collection \( \cup_{n \in \mathbb{N}} \{ B_{\bar{\alpha}, \bar{\epsilon}} : \bar{\alpha} \in \mathbb{R}^n, \bar{\epsilon} \in (\mathbb{R}_{>0})^n \} \) has the finite intersection property. By saturation, there exists \( \mu \in * \mathfrak{P}(T) \) such that the following holds:

\[
\exists i_0 \in \{1, \ldots, k\} \text{ such that } *E_{\mu}(f_{i_0}) \leq \beta_{i_0} < E_\nu(f_{i_0}) \text{ but } \\
\mu(G_{i_0,\alpha}) > \nu(G_{i_0,\alpha}) - \epsilon \text{ for all } \alpha \in \mathbb{R}, \epsilon \in \mathbb{R}_{>0}. \tag{2.25}
\]

But this implies that \( L\mu(G_{i_0,\alpha}) \geq L^*\nu(G_{i_0,\alpha}) \) for all \( \alpha \in \mathbb{R}_{>0} \), which yields:

\[
L\mu(\text{st}(f_{i_0})) > \alpha \geq \lim_{\epsilon \to 0} L\mu(\text{st}(f_{i_0}) > \alpha + \epsilon) \geq \lim_{\epsilon \to 0} L^*\nu(\text{st}(f_{i_0}) > \alpha + \epsilon) = L^*\nu(\text{st}(f_{i_0}) > \alpha). \tag{2.26}
\]

By Lemma 2.21 and (2.26), we thus obtain:

\[
E_{L\mu}(\text{st}(f_{i_0})) \geq E_{L^*\nu}(\text{st}(f_{i_0})). \tag{2.27}
\]

However, using the fact that finitely bounded internally measurable functions are \( S \)-integrable and that \( \beta_{i_0} \) and \( E_\nu(f_{i_0}) \) are real numbers, taking standard parts in the first inequality of (2.25) yields

\[
E_{L\mu}(\text{st}(f_{i_0})) < E_{L^*\nu}(\text{st}(f_{i_0})),
\]

which directly contradicts (2.27), completing the proof. \( \square \)
In the rest of the paper, we will interchangeably use either of the collections in Lemma 2.17 and Lemma 2.22 as a subbasis, depending on convenience. Lemma 2.22 also allows us to show that for any Borel set \( B \in \mathcal{B}(T) \), the evaluation map \( \mu \mapsto \mu(B) \) is Borel measurable on \( \mathfrak{P}(T) \).

If \( T \) is a topological space, then for any subset \( T' \subseteq T \), we can view \( T' \) as a topological space under the subspace topology. By routine measure theoretic arguments, it is clear that the Borel sigma algebra on \( T' \) with respect to the subspace topology contains precisely those sets that are intersections of \( T' \) with Borel subsets of \( T \). That is,

\[
\mathcal{B}(T') = \{ B \cap T' : B \in \mathcal{B}(T) \} \quad \text{for all } T' \subseteq T. \tag{2.28}
\]

Indeed, the collection on the right side of (2.28) is a sigma algebra that contains all open subsets of \( T' \) under the subspace topology (as any open subset of \( T' \) is of the type \( G \cap T' \) for some open, and hence Borel, subset of \( T \)). Using a similar argument, we can show the following functional version of (2.28):

**Lemma 2.24.** Let \( T' \) be a subspace of a topological space \( T \). For any bounded \( \mathcal{B}(T) \)-measurable function \( f : T \to \mathbb{R} \), its restriction \( f|_{T'} : T' \to \mathbb{R} \) is \( \mathcal{B}(T') \)-measurable.

*Proof.* Consider the collection

\[
\mathcal{C} := \{ f : T \to \mathbb{R} : f|_{T'} \text{ is } \mathcal{B}(T')\text{-measurable} \}. \tag{2.29}
\]

By (2.28), the collection \( \mathcal{C} \) contains the indicator function \( 1_B \) of each \( B \in \mathcal{B}(T) \). The collection \( \mathcal{C} \) is clearly an \( \mathbb{R} \)-vector space closed under increasing limits. Thus \( \mathcal{C} \) contains all bounded \( \mathcal{B}(T) \)-measurable functions by the monotone class theorem. \( \square \)

Thus if \( T' \) is a subspace of a topological space \( T \) and \( \mu \in \mathfrak{P}(T') \), then one can naturally define an “extension” \( \mu' \in \mathfrak{P}(T) \) of \( \mu \) as follows:

\[
\mu'(B) := \mu(B \cap T') \quad \text{for all } B \in \mathcal{B}(T). \tag{2.30}
\]

That \( \mu' \) is well-defined follows from (2.28), and the fact that \( \mu' \) is a Borel probability measure on \( T \) follows from the fact that \( \mu \) is a Borel probability measure on \( T' \). We had put scare quotes around the word ‘extension’ to emphasize that \( \mu \) is not necessarily a restriction of its extension \( \mu' \) in this sense. Indeed, \( T' \) could be a non-Borel subset of \( T \) or it might not be known whether it is a Borel subset of \( T \), in which cases \( \mu' \) might not even be defined on a typical Borel subset of \( T' \). This will be the situation in Section 4, when we will have to extend a probability measure defined on the space \( \mathfrak{P}_r(S) \) of all Radon probability measures on a topological space \( S \) to a Borel probability measure on \( \mathfrak{P}(S) \), the space of all Borel probability measures on \( S \) (thus \( \mathfrak{P}(S) \) will play the role of \( T \) and \( \mathfrak{P}_r(S) \) will play the role of \( T' \)). We will study the subspace topology on the space of Radon probability measures in the next subsection. Let us now summarize our discussion on the extension of a Borel measure on a subspace so far and prove a natural correspondence of expected values in the following lemma.

**Lemma 2.25.** Let \( T \) be a topological space and let \( T' \subseteq T \) be a subspace. Let \( \mu \in \mathfrak{P}(T') \) be a Borel probability measure on \( T' \) and let \( \mu' \) be its extension,
defined in (2.30). Then \( \mu' \in \mathcal{P}(T) \). Furthermore, we have:

\[
\mathbb{E}_{\mu'}(f) = \mathbb{E}_{\mu}(f|_{T'}) \quad \text{for all bounded } \mathcal{B}(T)\text{-measurable functions } f : T \to \mathbb{R}.
\] (2.31)

**Proof.** Only (2.31) remains to be proven. This follows from (2.30) and the monotone class theorem. □

Before we proceed, let us recall the concept of nets which often play the same role in abstract topological spaces that sequences play in metric spaces. This discussion is mostly borrowed from a combination of Kelley [43, Chapter 2] and Bogachev [13, Chapter 2].

A directed set \( D \) is a set with a partial order \( \succeq \) on it such that for any pair of elements \( i, j \in D \), there exists an element \( k \in D \) having the property \( k \succeq i \) and \( k \succeq j \). For a topological space \( T \), a net in \( T \) is a function \( f \) from a directed set \( D \) into \( T \), with \( f(i) \) usually written as \( x_i \) for each \( i \in D \). Mimicking the notation for sequences, we denote a generic net by \((x_i)_{i \in D}\).

For a net \((c_i)_{i \in D}\) of real numbers, we define the superior and inferior limits as follows:

\[
\limsup_{i \in D} (c_i) \coloneqq \operatorname{lub}\{c \in \mathbb{R} : \forall k \in D \exists j \succeq k \text{ such that } c_j \geq c\},
\] (2.32)

\[
\liminf_{i \in D} (c_i) = - \limsup_{i \in D} (-c_i),
\] (2.33)

where \( \operatorname{lub}(A) \) (for a set \( A \subseteq \mathbb{R} \)) denotes the least upper bound of \( A \).

An net \((x_i)_{i \in D}\) in a topological space \( T \) is said to converge to a point \( x \in T \) (written \((x_i)_{i \in D} \to x\)) if for each open neighborhood \( U \) of \( x \), there exists \( k \in D \) such that \( x_i \in U \) for all \( i \succeq k \). This definition clearly coincides with the usual definition of convergence of a sequence (thinking of \( \mathbb{N} \) as a directed set with the usual order on it). The following generalizes the characterization of closure in metric spaces using sequences to abstract topological spaces using nets (see Kelley [43, Theorem 2.2] for a proof):

**Theorem 2.26.** Let \( T \) be a topological space and let \( A \subseteq T \). A point \( x \) belongs to the closure of \( A \) if and only if there is a net in \( A \) converging to \( x \).

With the language of nets, we can prove the following useful characterizations of convergence in the \( A \)-topology, originally due to Alexandroff (see Topsoe [61, Theorem 8.1, p. 40] for a similar result).

**Theorem 2.27.** Let \( T \) be a topological space and \( \mathcal{P}(T) \) be the space of Borel probability measures on \( T \), equipped with the \( A \)-topology. For a net \((\mu_i)_{i \in D}\) in \( \mathcal{P}(T) \), the following are equivalent:

(i) \((\mu_i)_{i \in D} \to \mu\).

(ii) \(
\limsup_{i \in D} (\mathbb{E}_{\mu_i}(f)) \leq \mathbb{E}_{\mu}(f) \quad \text{for all } f \in \text{USC}_b(T).
\)

(iii) \(
\liminf_{i \in D} (\mathbb{E}_{\mu_i}(f)) \geq \mathbb{E}_{\mu}(f) \quad \text{for all } f \in \text{LSC}_b(T).
\)

(iv) \(
\limsup_{i \in D} (\mu_i(F)) \leq \mu(F) \quad \text{for all closed sets } F \subseteq T.
\)

(v) \(
\liminf_{i \in D} (\mu_i(G)) \geq \mu(G) \quad \text{for all open sets } G \subseteq T.
\)
Proof: The equivalences (ii) $\iff$ (iii) and (iv) $\iff$ (v) are clear from (2.33) and the last part of Lemma 2.14 (along with the fact that a set is open if and only if its complement is closed). We will prove (i) $\iff$ (ii) and omit the very similar proof of (i) $\iff$ (iv).

Throughout this proof, for any function $f \in USC_b(T)$, define
\[
S_f := \{c \in \mathbb{R} : \forall k \in D \exists j \succ k \text{ such that } \mathbb{E}_{\mu_j}(f) \geq c\}.
\] (2.34)

**Proof of (i) $\implies$ (ii)** Assume (i)—that is, $(\mu_i)_{i \in D} \to \mu$. Let $f \in USC_b(T)$ and $\beta := \mathbb{E}_\mu(f)$. We want to show that $\beta$ is at least as large as the least upper bound of $S_f$ (see (2.32)). In other words, we want the show that $\beta$ is an upper bound of $S_f$. To that end, let $c \in S_f$. Suppose, if possible, that $c > \beta = \mathbb{E}_\mu(f)$. Then $\mu$ would be in the subbasic open set $\mathcal{U}_{f,c} = \{\gamma \in \mathcal{P}(T) : \mathbb{E}_\gamma(f) < c\}$. Since $(\mu_i)_{i \in D} \to \mu$, there would exist a $k \in D$ such that $\mu_i \in \mathcal{U}_{f,c}$ for all $i \succeq k$. That is,
\[
\mathbb{E}_{\mu_i}(f) < c \text{ for all } i \succeq k.
\] (2.35)

Since $c \in S_f$, there would also exist $j \succ k$ such that $\mathbb{E}_{\mu_j}(f) \geq c > \beta$. But this contradicts (2.35), so we know that it is not possible for $c > \beta$ to be true. Since $c$ was an arbitrary element of $S_f$, it is now clear that $\beta = \mathbb{E}_\mu(f)$ is an upper bound of $S_f$, completing the proof of (i) $\implies$ (ii).

**Proof of (ii) $\implies$ (i)** Assume (ii)—that is, $\limsup_{i \in D}(\mathbb{E}_{\mu_i}(f)) \leq \mathbb{E}_\mu(f)$ for all $f \in USC_b(T)$. Suppose, if possible, that $(\mu_i)_{i \in D} \not\to \mu$. Then there would exist finitely many maps $f_1, \ldots, f_n \in USC_b(T)$ and real numbers $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, such that the set
\[
U := \bigcap_{t=1}^n \{\gamma \in \mathcal{P}(T) : \mathbb{E}_\gamma(f_t) < \alpha_t\}
\]
is a basic open neighborhood of $\mu$, and such that for any $k \in D$, one may find $j \succ k$ such that $\mu_j \not\in U$. Thus:

For all $k \in D$, there exists $j \succ k$ such that $\mathbb{E}_{\mu_j}(f_t) \geq \alpha_t$ for some $t \in \{1, \ldots, n\}$.
\] (2.36)

Since $\limsup_{i \in D}(\mathbb{E}_{\mu_i}(f_t)) \leq \mathbb{E}_\mu(f_t)$, we also know that $\mathbb{E}_\mu(f_t)$ is an upper bound of $S_{f_t}$ for all $t \in \{1, \ldots, n\}$. Since $\mu \in U$, we conclude that $\alpha_t$ is strictly larger than the least upper bound of $S_{f_t}$ for all $t \in \{1, \ldots, n\}$. In particular, $\alpha_t \not\in S_{f_t}$ for any $t \in \{1, \ldots, n\}$. By the definition of $S_{f_t}$, this means that for each $t \in \{1, \ldots, n\}$, there exists a $k_t \in D$ such that for all $j \succ k_t$, we have $\mathbb{E}_{\mu_j}(f_t) < \alpha_t$. Since $D$ is a directed set, there exists $\tilde{k}$ such that $\tilde{k} \succ k_t$ for all $t \in \{1, \ldots, n\}$. We thus conclude:
\[
\mathbb{E}_{\mu_j}(f_t) < \alpha_t \text{ for all } j \succ \tilde{k} \text{ and } t \in \{1, \ldots, n\}.
\] (2.37)

But (2.36) and (2.37) contradict each other, thus showing that the net $(\mu_i)_{i \in D}$ must in fact converge to $\mu$. This completes the proof of (i) $\implies$ (ii).

Returning to the theme of Loeb measures, we are now in a position to show that for any internal probability $\nu$ on $(^*T, ^*B(T))$, if $L\nu \circ st^{-1}$ is a legitimate Borel
probability measure on \((T, \mathcal{B}(T))\), then \(\nu\) is infinitesimally close to \(L\nu \circ \text{st}^{-1}\) in the sense that the former is nearstandard to the latter in \(\ast \mathfrak{P}(T)\). Combined with Theorem 2.12, we also have sufficient conditions for when this happens.

**Theorem 2.28.** Let \(T\) be a Hausdorff space. Suppose \((\ast T, \mathcal{B}(T), \nu)\) is an internal probability space, and let \((\ast T, L(\ast \mathcal{B}(T)), L\nu)\) be the associated Loeb space. If \(L\nu \circ \text{st}^{-1} : \mathcal{B}(T) \to [0, 1]\) is a Borel probability measure on \(T\), then \(\nu\) is nearstandard in \(\ast \mathfrak{P}(T)\) to \(L\nu \circ \text{st}^{-1}\). That is,

\[
\nu \in \text{st}^{-1}(L\nu \circ \text{st}^{-1}).
\]  

(2.38)

**Proof.** Let \(\nu\) be as in the statement of the theorem. Thus, \(L\nu \circ \text{st}^{-1} \in \mathfrak{P}(T)\), which implicitly also requires that \(\text{st}^{-1}(B) \in L(\ast \mathcal{B}(T))\) for all \(B \in \mathcal{B}(T)\). For brevity, denote \(L\nu \circ \text{st}^{-1}\) by \(\mu\). Suppose \(G_1, \ldots, G_n\) are finitely many open sets and \(\alpha_1, \ldots, \alpha_n \in \mathbb{R}\) are such that the set

\[
\mathcal{U} \coloneqq \bigcap_{i=1}^{n} \{ \gamma \in \mathfrak{P}(T) : \gamma(G_i) > \alpha_i \} 
\]

(2.39)

is a basic open neighborhood of \(\mu\) in \(\mathfrak{P}(T)\).

Note that in a Hausdorff space, a subset \(G\) is open if and only if \(\text{st}^{-1}(G) \subseteq \ast G\) (see Theorem 2.7(i)). Since \(\mu \in \mathcal{U}\), we thus obtain:

\[L\nu(\ast G_i) \geq L\nu(\text{st}^{-1} G_i) = \mu(G_i) > \alpha_i\text{ for all }i \in \{1, \ldots, n\}.
\]

Since the \(\alpha_i\) are real, it thus follows that

\[\nu(\ast G_i) > \alpha_i\text{ for all }i \in \{1, \ldots, n\}.
\]

By the definition (2.39) of \(\mathcal{U}\), it is thus clear that \(\nu \in \ast \mathcal{U}\). Since \(\mathcal{U}\) was an arbitrary neighborhood of \(\mu\), it thus follows that \(\nu \in \text{st}^{-1}(\mu)\), completing the proof. \(\square\)

**Remark 2.29.** For an internal probability measure \(\nu\) on \(\ast T\), whenever \(L\nu \circ \text{st}^{-1}\) is a probability measure on the underlying topological space \(T\), we typically call the measure \(L\nu \circ \text{st}^{-1}\) as being obtained by “pushing down” the Loeb measure \(L\nu\). In fact, Albeverio et al. [1, Section 3.4] denotes \(L\nu \circ \text{st}^{-1}\) by \(\text{st}(L\nu)\), calling it the standard part of \(\nu\). Theorem 2.28 makes this precise by showing that \(L\nu \circ \text{st}^{-1}\) is indeed nearstandard to \(\nu \in \ast \mathfrak{P}(T)\) when we equip the space of probability measures \(\mathfrak{P}(T)\) with a natural topology. In Section 2.4, we show that the subset \(\mathfrak{P}_r(T)\) of Radon probability measures on \(T\) is Hausdorff, which will allow us to show that \(L\nu \circ \text{st}^{-1}\) is actually the standard part of \(\nu\) as an element of \(\ast \mathfrak{P}_r(T)\) (see Theorem 2.36).

Theorem 2.28 applied together with Corollary 2.13 implies that the nonstandard extension of a tight measure is nearstandard to a Radon measure. Thus, while not all tight measures are Radon, each tight measure is close to a Radon measure from a topological point of view. More precisely, for each tight measure, there is a Radon measure such that the former belongs to each open neighborhood of the latter. We record this as a corollary.

**Corollary 2.30.** Let \(T\) be a Hausdorff space and \(\mu\) be a tight probability measure on it. Then there exists a Radon measure \(\mu'\) on \(T\) such that \(\mu \in \mathcal{U}\) for all open neighborhoods \(\mathcal{U}\) of \(\mu'\) in \(\mathfrak{P}(T)\).
Proof. By Corollary 2.13 and Theorem 2.28, we have that $\mu' := L^* \mu \circ \sigma^{-1}$ is a Radon probability measure such that $\mu \in \mathfrak{U}(\mu')$. Also, by definition of $\sigma^{-1}$, we have that $\mu \in \mathfrak{U}$ for any open neighborhood $\mathcal{U}$ of $\mu'$ in $\mathfrak{P}(\mathcal{T})$. By transfer, we have that $\mu \in \mathcal{U}$ for any open neighborhood $\mathcal{U}$ of $\mu'$ in $\mathfrak{P}(\mathcal{T})$. □

This, in particular, shows that the $A$-topology is not always Hausdorff. We end this subsection with this corollary.

**Corollary 2.31.** There exists a topological space $\mathcal{T}$ such that the $A$-topology on its space of Borel probability measures $\mathfrak{P}(\mathcal{T})$ is not Hausdorff.

**Proof.** There is a Hausdorff space $\mathcal{T}$ and a Borel probability measure $\mu$ on it such that $\mu$ is tight but not Radon (in fact, $\mathcal{T}$ may be taken to be a compact Hausdorff space; see Vakhania–Tarindaze–Chobanyan [64, Proposition 3.5, p.32] for an example/construction). By Corollary 2.30, there is a Radon probability measure $\mu'$ (thus $\mu \neq \mu'$ necessarily) such that $\mu$ and $\mu'$ cannot be separated by disjoint open sets in $\mathfrak{P}(\mathcal{T})$. As a consequence, $\mathfrak{P}(\mathcal{T})$ is not Hausdorff. □

2.4. **Space of Radon probability measures under the Alexandroff topology.** In de Finetti’s theorem, one wants to construct a second-order probability—a probability measure with certain properties on a space of probability measures. Our strategy will be to first create a nonstandard internal probability measure on the nonstandard extension of our space of probability measures and then “push it down” to get a standard Borel probability measure with the properties we desire of it. However, as is clear from the discussion in Section 2.3 (see, for example, Theorem 2.12), this general procedure usually requires the underlying space of probability measures that we are constructing our measure on to be Hausdorff. As Corollary 2.31 shows, the space $\mathfrak{P}(\mathcal{T})$ of all Borel probability measures that we have studied so far may be too wild! We want to identify a large collection of Borel measures that is Hausdorff under the subspace topology. The subspace of Radon probability measures on a Hausdorff space $\mathcal{T}$ that we will focus on in this subsection serves our purposes adequately (see Theorem 2.35).

Recall the concept of Radon probability measures on an arbitrary Hausdorff space $\mathcal{T}$ from Definition 2.3. The space of all Radon probability measures on $\mathcal{T}$ is denoted by $\mathfrak{P}_r(\mathcal{T})$, and we equip it with the subspace topology induced by the $A$-topology on $\mathfrak{P}(\mathcal{T})$. We require the Hausdorffness of $\mathcal{T}$ to ensure that compact subsets are Borel measurable (as a compact subset of a Hausdorff space is closed).

Being a subspace of $\mathfrak{P}(\mathcal{T})$, a subbasis of $\mathfrak{P}_r(\mathcal{T})$ can be obtained by intersecting all sets of a given subbasis of $\mathfrak{P}(\mathcal{T})$ with $\mathfrak{P}_r(\mathcal{T})$. Hence, by Lemma 2.17 and Lemma 2.22, we have the following result on various subbases of $\mathfrak{P}_r(\mathcal{T})$.

**Lemma 2.32.** Let $\mathcal{T}$ be a Hausdorff space. Then the topology on $\mathfrak{P}_r(\mathcal{T})$ as a subspace of $\mathfrak{P}(\mathcal{T})$ under the $A$-topology is generated by either of the following collections as a subbasis:

1. $\{\{\mu \in \mathfrak{P}_r(\mathcal{T}) : \mu(G) > \alpha\} : G$ an open subset of $\mathcal{T}$ and $\alpha \in \mathbb{R}\}$.
2. $\{\{\mu \in \mathfrak{P}_r(\mathcal{T}) : \mu(F) < \alpha\} : F$ a closed subset of $\mathcal{T}$ and $\alpha \in \mathbb{R}\}$.
3. $\{\{\mu \in \mathfrak{P}_r(\mathcal{T}) : E_\mu(f) > \alpha\} : f \in LSC_b(\mathcal{T})$ and $\alpha \in \mathbb{R}\}$.
4. $\{\{\mu \in \mathfrak{P}_r(\mathcal{T}) : E_\mu(f) < \alpha\} : f \in USC_b(\mathcal{T})$ and $\alpha \in \mathbb{R}\}$. 
Henceforth, we will call the subspace topology on \( P_r(T) \) as the \( A \)-topology on \( P_r(T) \), and we will use either of the subbases from Lemma 2.32 for this topology on \( P_r(T) \), depending on convenience. Using these subbases, the proofs of most of the results on \( P(T) \) from Section 2.3 carry over to \( P_r(T) \) almost immediately. We state below the analogs of Theorem 2.20 and Theorem 2.27 respectively (with the similar proofs omitted).

**Theorem 2.33.** Let \( B \) be a Borel subset of a Hausdorff space \( T \). Let \( P_r(T) \) be the space of all Radon probability measures on \( T \). Then the evaluation map \( e_B : P_r(T) \to [0,1] \) defined by \( e_B(\mu) := \mu(B) \) is \( B(P_r(T)) \)-measurable.

**Theorem 2.34.** Let \( T \) be a Hausdorff space and \( P_r(T) \) be the space of Radon probability measures on \( T \), equipped with the \( A \)-topology. For a net \( (\mu_i)_{i \in D} \in P_r(T) \), the following are equivalent:

1. \( (\mu_i)_{i \in D} \to \mu \).
2. \( \limsup_{i \in D} \mathbb{E}_{\mu_i}(f) \leq \mathbb{E}_\mu(f) \) for all \( f \in USC_b(T) \).
3. \( \liminf_{i \in D} \mathbb{E}_{\mu_i}(f) \geq \mathbb{E}_\mu(f) \) for all \( f \in LSC_b(T) \).
4. \( \limsup_{i \in D} \mu_i(F) \leq \mu(F) \) for all closed sets \( F \subseteq T \).
5. \( \liminf_{i \in D} \mu_i(G) \geq \mu(G) \) for all open sets \( G \subseteq T \).

With these results motivated from the results in Section 2.3 out of the way, we now show why \( P_r(T) \) is inherently a better space to work with than \( P(T) \)—we show that \( P_r(T) \) is Hausdorff (see also Topsøe [61, Theorem 11.2, p. 49]).

**Theorem 2.35.** If \( T \) is a Hausdorff space, then \( P_r(T) \) is also Hausdorff.

**Proof.** Let \( T \) be a Hausdorff space. Suppose \( \mu, \nu \) are two distinct elements of \( P_r(T) \). Since they are distinct Borel measures, there exists an open set \( G \subseteq T \) such that \( \alpha := \nu(G) \) and \( \beta := \mu(G) \) are distinct. Without loss of generality, assume \( \alpha < \beta \). Since \( \mu \) and \( \nu \) are Radon measures, we can find a compact set \( K \) such that \( K \subseteq G \) and the following holds:

\[
\nu(K) \leq \nu(G) = \alpha < \alpha + \frac{3(\beta - \alpha)}{4} < \mu(K) \leq \beta = \mu(G). \tag{2.40}
\]

Since \( T \) is Hausdorff, all compact subsets of \( T \) are closed. In particular \( K \) is closed. Consider the subbasic open set \( \mathfrak{U} \) defined by:

\[
\mathfrak{U} := \left\{ \gamma \in P_r(T) : \gamma(K) < \alpha + \frac{\beta - \alpha}{4} \right\}.
\]

By (2.40), it is clear that \( \nu \in \mathfrak{U} \) and \( \mu \notin \mathfrak{U} \). For each \( \gamma \in \mathfrak{U} \), by Radonness, there exists and open set \( G_\gamma \) such that \( K \subseteq G_\gamma \subseteq G \) and we have:

\[
\gamma(G_\gamma) < \alpha + \frac{\beta - \alpha}{2} \quad \text{for all } \gamma \in \mathfrak{U}. \tag{2.41}
\]
Thus the following set, being the complement of a closed set (owing to the fact that an arbitrary intersection of closed sets is closed), is open:

\[ U := Pr(T) \setminus \left( \bigcap_{\gamma \in \Psi} \left\{ \theta \in Pr(T) : \theta(G_\gamma) \leq \alpha + \frac{\beta - \alpha}{2} \right\} \right) . \]

By (2.40), it is clear that

\[ \mu(G_\gamma) \geq \mu(K) > \alpha + \frac{3(\beta - \alpha)}{4} > \alpha + \frac{\beta - \alpha}{2} \]

for all \( \gamma \in \Psi \).

As a consequence, we have \( \nu \in U \). Furthermore, by (2.41), it is clear that \( \Psi \cap U = \emptyset \), thus completing the proof. \( \square \)

Since nonstandard extensions of Hausdorff spaces admit unique standard parts (of nearstandard elements), we have the following form of Theorem 2.28 for \( Pr(T) \):

**Theorem 2.36.** Let \( T \) be a Hausdorff space. Suppose \((T, B(T), \nu)\) is an internal probability space, and let \((T, L(B(T)), L\nu)\) be the associated Loeb space. If \( L\nu \circ st^{-1} : B(T) \to [0,1] \) is a Radon probability measure on \( T \), then \( \nu \) is nearstandard in \( *Pr(T) \) to \( L\nu \circ st^{-1} \). That is,

\[ st(\nu) = L\nu \circ st^{-1} \in Pr(T) . \] (2.42)

**Proof.** We use \( st_{*Pr(T)}^{-1} \) and \( st_{Pr(T)}^{-1} \) to denote standard inverses on subsets of \( *Pr(T) \) and \( Pr(T) \) respectively. By Theorem 2.28 and the given information, we have that

\[ \nu \in st_{*Pr(T)}^{-1}(L\nu \circ st^{-1}) \cap *Pr(T) . \]

By Lemma 2.9, we have

\[ \nu \in st_{Pr(T)}^{-1}(L\nu \circ st^{-1}) . \]

Since \( Pr(T) \) is Hausdorff, this completes the proof. \( \square \)

Knowing that \( Pr(T) \) is Hausdorff for any Hausdorff space \( T \) allows us to apply results such as Theorem 2.12 to uniquely push down internal measures on \((*Pr(T), *B(Pr(T)))\). Thus in the next section, we will take \( T = Pr(S) \) for a Hausdorff topological space \( S \), and construct a nonstandard measure living in \( *Pr(Pr(S)) \) that we will be able to push down to a Radon measure on \( Pr(S) \). We begin this theme here with a uniqueness result about Radon presentability in Theorem 2.38.

Our proof will use the following generalization of the monotone class theorem (see Dellacherie and Meyer [19, Theorem 21, p. 13-I] for a proof of this result).

**Theorem 2.37.** Let \( \mathbb{H} \) be an \( \mathbb{R} \)-vector space of bounded real-valued functions on some set \( S \) such that the following hold:

(i) \( \mathbb{H} \) contains the constant functions.

(ii) \( \mathbb{H} \) is closed under uniform convergence.

(iii) For every uniformly bounded increasing sequence of nonnegative functions \( f_n \in \mathbb{H} \), the function \( \lim_{n \to \infty} f_n \) belongs to \( \mathbb{H} \).

If \( C \) is a subset of \( \mathbb{H} \) which is closed under multiplication, then the space \( \mathbb{H} \) contains all bounded functions measurable with respect to \( \sigma(C) \) - the smallest sigma algebra with respect to which all functions in \( C \) are measurable.
Theorem 2.38. Let \( S \) be a Hausdorff space and let \( \mathcal{P}_r(S) \) be the space of all Radon probability measures on \( S \) under the A-topology. Suppose \( \mathcal{P}, \mathcal{D} \in \mathcal{P}_r(\mathcal{P}_r(S)) \) are such that the following holds:

\[
\int_{\mathcal{P}_r(S)} \mu(B_1) \cdot \ldots \cdot \mu(B_n) d\mathcal{P}(\mu) = \int_{\mathcal{P}_r(S)} \mu(B_1) \cdot \ldots \cdot \mu(B_n) d\mathcal{D}(\mu)
\]

for all \( n \in \mathbb{N} \) and \( B_1, \ldots, B_n \in \mathcal{B}(S) \). (2.43)

Then it must be the case that \( \mathcal{P} = \mathcal{D} \).

Proof. For \( m \in \mathbb{N} \), let \( \mathcal{M}([0, 1]^m) \) denote the space of all bounded Borel measurable functions \( f \colon [0, 1]^k \to \mathbb{R} \). For each \( m \in \mathbb{N} \), consider the following collection of functions:

\[
\mathcal{G}_m := \{ f \in \mathcal{M}([0, 1]^m) : \mathbb{E}_{\mathcal{P}} [f(\mu(B_1), \ldots, \mu(B_m))] = \mathbb{E}_{\mathcal{D}} [f(\mu(B_1), \ldots, \mu(B_m))] \text{ for all } B_1, \ldots, B_m \in \mathcal{B}(S) \}.
\]

Note that the expected values in the definition of \( \mathcal{G}_m \) are well-defined because of Theorem 2.33. It is clear that for each \( m \in \mathbb{N} \), the collection \( \mathcal{G}_m \) contains all polynomials over \( m \) variables. Indeed, the collection \( \mathcal{G}_m \) is an \( \mathbb{R} \)-vector space (that is, closed under finite linear combinations), and for a monomial \( f \colon [0, 1]^m \to \mathbb{R} \) of the type \( f(x_1, \ldots, x_m) = x_1^{a_1} \cdots x_m^{a_m} \) (where \( a_1, \ldots, a_m \in \mathbb{Z}_{\geq 0} \)), the expectation \( \mathbb{E}_{\mathcal{P}} [f(\mu(B_1), \ldots, \mu(B_m))] \) is equal to \( \mathbb{E}_{\mathcal{D}} [f(\mu(B_1), \ldots, \mu(B_m))] \) be (2.43). That \( \mathcal{G}_m \) satisfies the conditions in Theorem 2.37 is also clear by dominated convergence theorem. It is straightforward to verify that the smallest sigma algebra on \([0, 1]^m\) with respect to which all polynomials are measurable is the Borel sigma algebra on \([0, 1]^m\). Since the set of polynomials over \( m \) variables is closed under multiplication, it thus follows from Theorem 2.37 that for each \( m \in \mathbb{N} \), the collection \( \mathcal{G}_m \) contains all bounded Borel measurable functions \( f \colon [0, 1]^m \to \mathbb{R} \).

Let \( \mathcal{G} \) be the collection of those Borel subsets of \( \mathcal{P}_r(S) \) that are assigned the same measure by \( \mathcal{P} \) and \( \mathcal{D} \). More formally, we define:

\[
\mathcal{G} := \{ \mathcal{B} \in \mathcal{B}(\mathcal{P}_r(S)) : \mathcal{P}(\mathcal{B}) = \mathcal{D}(\mathcal{B}) \}.
\]

(2.44)

Taking \( f \) to be the indicator function of a measurable rectangle in \([0, 1]^m\), we have thus shown that \( \mathcal{G} \) contains the following collection of cylinder sets:

\[
\mathcal{C} := \{ C(B_1, \ldots, B_m, A_1, \ldots, A_m) : m \in \mathbb{N}; B_1, \ldots, B_m \in \mathcal{B}(S); A_1, \ldots, A_m \in \mathcal{B}(\mathbb{R}) \},
\]

(2.45)

where

\[
C(B_1, \ldots, B_m, A_1, \ldots, A_m) := \{ \mu \in \mathcal{P}_r(S) : \mu(B_1) \in A_1, \ldots, \mu(B_m) \in A_m \}
\]

for all \( m \in \mathbb{N}; B_1, \ldots, B_m \in \mathcal{B}(S); A_1, \ldots, A_m \in \mathcal{B}(\mathbb{R}) \).

It is clear that the collection \( \mathcal{C} \) contains the basic open subsets with respect to the subbasis (i) in Lemma 2.32. Thus all basic open subsets of \( \mathcal{P}_r(S) \) are elements of \( \mathcal{G} \). Since \( \mathcal{G} \) is a sigma algebra, all finite unions of basic open sets are in \( \mathcal{G} \). (In fact, all countable unions are in \( \mathcal{G} \), but we do not need this fact here.) Let \( \mathcal{E} \) be a compact subset of \( \mathcal{P}_r(S) \) and let \( \epsilon \in \mathbb{R}_{>0} \) be given. Since \( \mathcal{P} \) and \( \mathcal{D} \) are Radon measures, we find an open subset \( \mathcal{U} \) of \( \mathcal{P}_r(S) \) such that we have \( \mathcal{U} \subseteq \mathcal{U} \) and

\[
\mathcal{P}(\mathcal{U} \setminus \mathcal{E}) < \epsilon \text{ and } \mathcal{D}(\mathcal{U} \setminus \mathcal{E}) < \epsilon.
\]

(2.46)
Cover $\mathcal{C}$ by finitely many basic open subsets contained in $\Omega$ and let $\mathcal{V}$ be the union of these basic open subsets. Then, we have (using (2.46)):
\[ P(\mathcal{V}\setminus\mathcal{C}) < \epsilon \text{ and } Q(\mathcal{V}\setminus\mathcal{C}) < \epsilon. \] (2.47)

Being, a finite union of basic open sets, we have $\mathcal{V} \in \mathcal{G}$, or in other words:
\[ P(\mathcal{V}) = Q(\mathcal{V}). \] (2.48)

Using (2.47) and (2.48) (and the triangle inequality), we thus obtain:
\[ |P(\mathcal{C}) - Q(\mathcal{C})| < 2\epsilon. \] (2.49)

Since $\mathcal{C}$ was an arbitrary compact subset of $\mathcal{P}(S)$ and $\epsilon \in \mathbb{R}_{>0}$ was arbitrary, this shows that the measures $P$ and $Q$ agree on all compact subsets of $\mathcal{P}(S)$. Since they are Radon measures, it is thus clear now that they agree on all Borel subsets of $\mathcal{P}(S)$, completing the proof. □

Remark 2.39. Instead of using Theorem 2.37 (after showing that all polynomials in $m$ variables are in $\mathcal{G}_m$ for all $m \in \mathbb{N}$), we could have used the Stone-Weierstrass theorem to first show that all continuous functions on $[0,1]^m$ are in $\mathcal{G}_m$ for all $m \in \mathbb{N}$ and then approximate indicator functions of open subsets of $[0,1]^m$ by increasing sequences of continuous functions to complete the proof using the monotone class theorem. Theorem 2.37 achieved the same in a quicker manner.

In the above proof, the only place where Radonness was used was in extending the uniqueness result from the cylinder sigma algebra on $\mathcal{P}(S)$ to the Borel sigma algebra on $\mathcal{P}(S)$. In particular, the same argument shows that without working with Radon measures, one still has uniqueness if we focus on measures over the smallest sigma algebra generated by cylinder sets. We formally record this as a theorem in the next subsection that is devoted to other sigma algebras on $\mathcal{P}(S)$.

2.5. Useful sigma algebras on spaces of probability measures. Let $S$ be a topological space and $\mathcal{P}(S)$ be the space of all Borel probability measures on $S$. So far, we have studied the $A$-topology and the Borel sigma algebra $\mathcal{B}(\mathcal{P}(S))$ on $\mathcal{P}(S)$ arising out of it. As Remark 2.18 shows, the $A$-topology coincides with the more commonly studied weak topology (which is the smallest topology that makes the map $\mu \mapsto \mathbb{E}_\mu(f)$ continuous for each bounded continuous $f : S \to \mathbb{R}$) in the cases when $S$ is a Polish space or when $S$ is a locally compact Hausdorff space. Let $\mathcal{B}(\mathcal{P}(S))$ denote the Borel sigma algebra on $\mathcal{P}(S)$ with respect to the weak topology.

For general spaces, the $A$-topology is typically richer than the weak topology, and the corresponding Borel sigma algebra on the space of all probability measures is a very natural sigma algebra to work with from a topological measure theoretic standpoint. However, the Borel sigma algebra arising from the $A$-topology might be too large in some cases—it might contain more events than we might hope to have a grip on in some applications. There are other sigma algebras on spaces of probability measures on $S$ that are also used in practice, some that make sense even if $S$ is not a topological space. In fact, constructing a measurable space out of the space of all probability measures (on some space) is the first foundational step needed to talk about prior distributions in a Bayesian nonparametric setting. In Bayesian nonparametrics, it is generally agreed that any reasonable sigma algebra
on the space of all probability measures on some measurable space \((S, \mathcal{S})\) must make the evaluation functions (that is, the functions \(\mu \mapsto \mu(A)\) for each \(A \in \mathcal{S}\)) measurable. Let us give a name for the smallest sigma algebra with this property.

**Definition 2.40.** Let \((S, \mathcal{S})\) be a measurable space and let \(C(S)\) be the smallest sigma algebra on \(\mathfrak{P}(S)\), the space of all probability measures on \(S\), such that for each \(A \in \mathcal{S}\), the evaluation function \(\mu \mapsto \mu(A)\) is measurable.

As explained above, the sigma algebra \(C(S)\) is ubiquitous in the nonparametric Bayesian analysis literature. To mention just one classic example, this was the sigma algebra used by Ferguson [27] in his pioneering work on the Dirichlet processes.

When the underlying space \(S\) has a topological structure, then it is useful to see how this sigma algebra relates to the Borel sigma algebras arising out of the natural topologies on \(\mathfrak{P}(S)\) (namely the \(A\)-topology and the weak topology). Theorem 2.20 and Remark 2.18 show that \(\mathcal{B}(\mathfrak{P}(S))\) contains both \(C(S)\) and \(\mathcal{B}_w(\mathfrak{P}(S))\). In a metric space, the indicator function of an open set is a pointwise limit of uniformly bounded continuous functions, so that by routine measure theory we obtain the following whenever \(S\) is a metric space:

\[\{\{\mu \in \mathfrak{P}(S) : \mu(G) > \alpha\} : G \text{ open in } S \text{ and } \alpha \in \mathbb{R}\} \subseteq \mathcal{B}_w(\mathfrak{P}(S)).\]

In particular, the proof of Theorem 2.20 also shows that if \(S\) is a metric space, then \(C(S) \subseteq \mathcal{B}_w(\mathfrak{P}(S)) \subseteq \mathcal{B}(\mathfrak{P}(S))\). Finally, it is not very difficult to observe (for example, see Gaudard and Hadwin [33, Theorem 2.3, p. 171]) that these two sigma algebras actually coincide if \(S\) is a separable metric space. We summarize this discussion in the next theorem.

**Theorem 2.41.** Let \(S\) be a topological space and let \(\mathfrak{P}(S)\) denote the space of all Borel probability measures on \(S\). Let \(\mathcal{B}(\mathfrak{P}(S))\) and \(\mathcal{B}_w(\mathfrak{P}(S))\) be the Borel sigma algebras on \(\mathfrak{P}(S)\) with respect to the \(A\)-topology and the weak topology respectively. Let \(C(S)\) be the smallest sigma algebra on \(\mathfrak{P}(S)\) that makes the evaluation functions measurable. Then we have:

1. \(C(S) \subseteq \mathcal{B}(\mathfrak{P}(S))\) and \(\mathcal{B}_w(\mathfrak{P}(S)) \subseteq \mathcal{B}(\mathfrak{P}(S))\).
2. If \(S\) is metrizable, then \(C(S) \subseteq \mathcal{B}_w(\mathfrak{P}(S)) \subseteq \mathcal{B}(\mathfrak{P}(S))\).
3. If \(S\) is a separable metric space, then \(C(S) = \mathcal{B}_w(\mathfrak{P}(S)) \subseteq \mathcal{B}(\mathfrak{P}(S))\).
4. If \(S\) is a complete separable metric space, then \(C(S) = \mathcal{B}_w(\mathfrak{P}(S)) = \mathcal{B}(\mathfrak{P}(S))\).

With the requisite terminology now established, we finish this section by formally writing our observations at the end of Section 2.4 as a version of Theorem 2.38 for the space of all probability measures (not necessarily Radon). Theorem 2.41(iii) allows us to say something more in the case when \(S\) is a separable metric space.

**Theorem 2.42.** Let \(S\) be a topological space and let \(\mathfrak{P}(S)\) be the space of all Radon probability measures on \(S\) under the \(A\)-topology. Let \(C(\mathfrak{P}(S))\) be the smallest sigma algebra such that for any \(B \in \mathcal{B}(S)\), the evaluation function \(e_B : \mathfrak{P}(S) \to \mathbb{R}\), defined by \(e_B(\nu) = \nu(B)\), is measurable. Then \(C(\mathfrak{P}(S)) \subseteq \mathcal{B}(\mathfrak{P}(S))\).
Suppose \( P, Q \) are two probability measures on \((\mathcal{P}(S), C(\mathcal{P}(S)))\) such that the following holds:

\[
\int_{\mathcal{P}(S)} \mu(B_1) \cdots \mu(B_n) d\mathcal{P}(\mu) = \int_{\mathcal{P}(S)} \mu(B_1) \cdots \mu(B_n) d\mathcal{Q}(\mu)
\]

for all \( n \in \mathbb{N} \) and \( B_1, \ldots, B_n \in \mathcal{B}(S) \).

Then it must be the case that \( P = Q \).

Furthermore, if \( S \) is a separable metric space, then \( C(\mathcal{P}(S)) \) in the above result may be replaced by the Borel sigma algebra \( B_w(\mathcal{P}(S)) \) induced by the weak topology on \( \mathcal{P}(S) \).

2.6. Generalizing Prokhorov’s theorem—tightness implies relative compactness for probability measures on any Hausdorff space. Prokhorov [54, Theorem 1.12] famously proved that a collection \( \mathcal{A} \) of Borel probability measures on a Polish space \( T \) (that is, a complete and separable metric space) is relatively compact (that is, the closure \( \overline{\mathcal{A}} \) of \( \mathcal{A} \) is compact) if and only if \( \mathcal{A} \) satisfies the following property that is now known as tightness (being a property that is uniformly satisfied by all measures in \( \mathcal{A} \), it is sometimes called “uniform tightness” to avoid confusion with tightness of a particular measure as defined in Definition 2.2).

(Tightness of \( \mathcal{A} \)): For each \( \epsilon \in \mathbb{R}_{>0} \), there exists a compact set \( K_\epsilon \subseteq T \) such that 

\[ \mu(K_\epsilon) \geq 1 - \epsilon \]

for all \( \mu \in \mathcal{A} \).

Those topological spaces \( T \) for which a collection \( \mathcal{A} \subseteq \mathcal{P}(T) \) is relatively compact if and only if \( \mathcal{A} \) is tight are called Prokhorov spaces. Thus, Prokhorov [54] proved that all Polish spaces are Prokhorov spaces. Anachronistically, Alexandroff [7, Theorem V.4] had earlier shown that all locally compact Hausdorff spaces are also Prokhorov spaces. What is the topology on \( \mathcal{P}(T) \) that is under consideration in the above results? As is clear from Remark 2.18, there is not a lot of choice in the results described so far, as the \( A \)-topology and the weak topology on \( \mathcal{P}(T) \) are the same when \( T \) is a Polish space or a locally compact Hausdorff space.

With respect to the \( A \)-topology, tightness of a set \( \mathcal{A} \subseteq \mathcal{P}(T) \) is known to not be a necessary condition for the relative compactness of \( \mathcal{A} \). Nice counterexamples were independently constructed by Varadarajan [65], Fernique [28], and Preiss [53]. See Topsøe [62, p. 191] for a description of these counterexamples, and also for more history of Prokhorov’s theorem. The situation is slightly better when we restrict to the space of Radon probability measures (and look for relative compactness in that space). For example, Topsøe (see the comments following Theorem 3.1 in [62]) proves Prokhorov’s theorem for the space \( \mathcal{P}_r(T) \) of all Radon probability measures on a regular topological space \( T \). (Thus for a regular space \( T \), the set of probability measures \( \mathcal{A} \) is relatively compact in \( \mathcal{P}_r(T) \) equipped with the \( A \)-topology if and only if it is tight.)

With the knowledge that tightness is not a necessary condition for relative compactness in \( \mathcal{P}(T) \) in general, our focus here is on a result in the other direction—to see if tightness is still sufficient for relative compactness without too many additional assumptions. It is in this sense that we are looking for a generalization of Prokhorov’s theorem. The sufficiency of tightness seems to be known, in many cases, for the relative compactness on spaces of Radon measures equipped with
either the weak topology or the $A$-topology. For example, Bogachev [12, Theorem 8.6.7, p. 206, vol. 2] shows that tightness is sufficient for relative compactness in the space of Radon probability measures, equipped with the weak topology, on any completely regular Hausdorff space. Under the $A$-topology, Topsøe [61, Theorem 9.1(iii), p. 43] (see also [60]) has proved that uniform tightness is sufficient for relative compactness in the space of Radon probability measures over any Hausdorff space.

Remark 2.43. The above discussion seems to allude to the fact that relative compactness under the weak topology is a more restrictive notion than under the $A$-topology. This is technically correct, even though compactness in the weak topology is less restrictive than in the $A$-topology. Indeed, by Remark 2.18, it is clear that the weak topology on $P(T)$ (and hence on $P_r(T)$) is coarser than the $A$-topology. Hence any set that is compact in $P(T)$ (respectively $P_r(T)$) with the $A$-topology is also compact in $P(T)$ (respectively to $P_r(T)$) with the weak topology. On the other hand, the closure of a set with respect to the $A$-topology on $P(T)$ (respectively $P_r(T)$) is contained in the closure of that set with respect to the weak topology on $P(T)$ (respectively $P_r(T)$). This last fact, which can be seen by Theorem 2.26 and Remark 2.18, shows that a set that is relatively compact under the $A$-topology might fail to be so under the weak topology.

Our next result (Theorem 2.44) proves the sufficiency of tightness for relative compactness in the $A$-topology on the space of all probability measures on a Hausdorff space $T$. This is a slight variation of the same result that is known for the space of all Radon probability measures, and its proof can be readily adapted to show the latter result as well (see Theorem 2.46). The proof of Theorem 2.44 is short as most of the work has already been done in setting up the convenient framework of topological and nonstandard measure theory in the previous subsections. To the best of the author’s knowledge, this generalization of Prokhorov’s theorem is new.

Theorem 2.44 (Prokhorov’s theorem). Let $T$ be a Hausdorff space, and let $P(T)$ be the space of all Borel probability measures on $T$, equipped with the $A$-topology. Let $\mathfrak{A} \subseteq P(T)$ be such that for any $\epsilon \in \mathbb{R}_{>0}$, there exists a compact set $K_\epsilon \subseteq T$ for which

$$\mu(K_\epsilon) \geq 1 - \epsilon \text{ for all } \mu \in \mathfrak{A}.$$  \hfill (2.50)

Then the closure of $\mathfrak{A}$ in $P(T)$ is compact.

Proof. Let $\mathfrak{A}$ be as in the statement of the theorem. Let $\bar{\mathfrak{A}}$ be its closure in $P(T)$ with respect to the $A$-topology. By the nonstandard characterization of compactness (see Theorem 2.7(iii)), it suffices to show that $^*\mathfrak{A} \subseteq \text{st}^{-1}(\bar{\mathfrak{A}})$. Since $\mathfrak{A}$ is closed, any nearstandard element in $^*\mathfrak{A}$ must be nearstandard to an element of $\mathfrak{A}$ (this follows from the nonstandard characterization of closed sets; see Theorem 2.7(ii)). Thus, it suffices to show that all elements in $^*\mathfrak{A}$ are nearstandard. Toward that end, let $\nu \in ^*\mathfrak{A}$. For each $\epsilon \in \mathbb{R}_{>0}$, let $K_\epsilon$ be as in the statement of the theorem. We now prove the following claim.

Claim 2.45. $L\nu(^*K_\epsilon) \geq 1 - \epsilon$ for all $\epsilon \in \mathbb{R}_{>0}$. 
Proof of Claim 2.45. Suppose, if possible, that there is some \( \epsilon \in \mathbb{R}_{>0} \) such that 
\[ L \nu (\ast K_{\epsilon}) < 1 - \epsilon. \]
Since \( \epsilon \in \mathbb{R}_{>0} \), this implies that \( \nu (\ast K_{\epsilon}) < 1 - \epsilon \) as well. By transfer, we conclude that \( \nu \) belongs to \( \ast \mathcal{U} \), where \( \mathcal{U} \) is the following subbasic open subset of \( \mathcal{P}(T) \):
\[ \mathcal{U} := \{ \gamma \in \mathcal{P}(T) : \gamma(K_{\epsilon}) < 1 - \epsilon \}. \] (2.51)

Note that \( \mathcal{U} \) is indeed a subbasic open subset of \( \mathcal{P}(T) \), since \( K_{\epsilon} \), being a compact subset of the Hausdorff space \( T \), is closed in \( T \). By the definition of closure, we know that any open neighborhood of an element in the closure of \( \mathfrak{A} \) must have a nonempty intersection with \( \mathcal{U} \). By transfer, we thus find an element \( \mu \in \mathcal{U} \cap \mathfrak{A} \). But this is a contradiction (in view of (2.50) and (2.51)), thus completing the proof of the claim.

Claim 2.45 now completes the proof using Theorems 2.12 and 2.28 (in view of the fact that \( \ast K \subseteq \text{st}^{-1}(K) \) for all compact \( K \subseteq T \)). \( \square \)

Using Lemma 2.32, the proof of Theorem 2.44 carries over immediately to give Prokhorov’s theorem for the space of Radon probability measures.

Theorem 2.46 (Prokhorov’s theorem for spaces of Radon probability measures). Let \( T \) be a Hausdorff space and let \( \mathcal{P}_r(T) \) be the space of all Radon probability measures on \( T \), equipped with the A-topology. Let \( \mathfrak{A} \subseteq \mathcal{P}_r(T) \) be such that for any \( \epsilon \in \mathbb{R}_{>0} \), there exists a compact set \( K_{\epsilon} \subseteq T \) for which
\[ \mu(K_{\epsilon}) \geq 1 - \epsilon \text{ for all } \mu \in \mathfrak{A}. \] (2.52)
Then the closure of \( \mathfrak{A} \) in \( \mathcal{P}_r(T) \) is compact.

Proof. As in the proof of Theorem 2.44, it suffices to show that all elements in \( \ast \mathfrak{A} \) are nearstandard (in the current setting, \( \mathfrak{A} \) is the closure of \( \mathfrak{A} \) in the space \( \mathcal{P}_r(T) \), and the nearstandardness in question is with respect to the A-topology on \( \mathcal{P}_r(T) \)).

Toward that end, let \( \nu \in \ast \mathfrak{A} \). Then we see that \( \nu \) is nearstandard by Theorem 2.12 and Theorem 2.36, in view of the following analog of Claim 2.45 (which has the same proof as that of Claim 2.45, with the subbasic open set \( \{ \gamma \in \mathcal{P}_r(T) : \gamma(K_{\epsilon}) < 1 - \epsilon \} \) used as the analog of (2.51) from the earlier proof):
\[ L \nu (\ast K_{\epsilon}) \geq 1 - \epsilon \text{ for all } \epsilon \in \mathbb{R}_{>0}. \]
\( \square \)

3. Hyperfinite empirical measures induced by identically Radon distributed random variables

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space. Let \( S \) be a Hausdorff space equipped with its Borel sigma algebra \( \mathcal{B}(S) \). Suppose \( X_1, X_2, \ldots \) is a sequence of identically distributed \( S \)-valued random variables on \( \Omega \)—that is, the pushforward measure \( \mathbb{P} \circ X_i^{-1} \) on \( (S, \mathcal{B}(S)) \) is the same for all \( i \in \mathbb{N} \). Note that the de Finetti–Hewitt–Savage theorem requires the stronger condition of exchangeability, which we will assume in the next section when we prove that theorem. However, the results in this section are more abstract and preparatory in nature, and they are applicable to all identically distributed sequences of random variables.
Throughout this section, we will further assume that the common distribution of the $X_i$ is Radon. This is for ease of presentation as we will, however, not use the full strength of this hypothesis—we will only have occasion to use the fact that this distribution is tight and outer regular on compact subsets of $S$. By tightness, there exists an increasing sequence of compact subsets $(C_n)_{n \in \mathbb{N}}$ of $S$ such that:

$$\mathbb{P}(X_1 \in C_n) > 1 - \frac{1}{n} \text{ for all } n \in \mathbb{N}. \quad (3.1)$$

The results up to Lemma 3.14 only require tightness of the underlying distribution. We will also need outer regularity on compact subsets from Lemma 3.15 onwards.

For each $\omega \in \Omega$ and $n \in \mathbb{N}$, define the empirical measure $\mu_{\omega,n}$ on $\mathcal{B}(S)$ as follows:

$$\mu_{\omega,n}(A) := \frac{\#\{i \in [n] : X_i(\omega) \in A\}}{n} \text{ for all } A \in \mathcal{B}(S). \quad (3.2)$$

Nonstandardly, we also have for each $\omega \in {}^*\Omega$ and each $N \in {}^*\mathbb{N}$, the hyperfinite empirical measure $\mu_{\omega,N}$ defined by the following:

$$\mu_{\omega,N}(A) := \frac{\#\{i \in [N] : X_i(\omega) \in A\}}{N} \text{ for all } A \in {}^*\mathcal{B}(S). \quad (3.3)$$

Although we are calling $\mu_{\omega,N}$ a hyperfinite empirical measure because $N \in {}^*\mathbb{N}$, we do not need to assume $N > \mathbb{N}$ (that is, $N \in {}^*\mathbb{N}\setminus\mathbb{N}$) in this section. Also, we are abusing notation by using $(X_i)$ to denote both the standard sequence $(X_i)_{i \in \mathbb{N}}$ of random variables and the nonstandard extension of this sequence. More precisely, if $X : \Omega \times \mathbb{N} \to S$ is defined by $X(\omega,i) := X_i(\omega)$ for all $\omega \in \Omega$ and $n \in \mathbb{N}$, then for any $i \in {}^*\mathbb{N}$, the internal random variable $X_i : {}^*\Omega \to {}^*S$ is defined as follows:

$$X_i(\omega) = {}^*X(\omega,i) \text{ for all } \omega \in {}^*\Omega \text{ and } i \in {}^*\mathbb{N}.\]$$

The notation fixed above will be valid for the rest of this section which studies the structure of these empirical measures within the space of all Radon probability measures on $S$. We divide the exposition into four subsections. Section 3.1 deals with some basic properties that are satisfied by almost all hyperfinite empirical measures. Section 3.2 deals with the study of the pushforward measure induced on the space $^*\mathcal{P}_1(S)$ of internal Radon measures on $^*S$ by the map $\omega \mapsto \mu_{\omega,N}$. The goal of Section 3.3 is to show in a precise sense that the standard part of a hyperfinite empirical measure evaluated at a Borel set is almost surely given by the standard part of the measure of the nonstandard extension of that Borel set (see Theorem 3.19). Section 3.4 synthesizes the theory built so far in order to express some Loeb integrals on the space of all internal Radon probability measures in terms of the corresponding integrals on the standard space of Radon probability measures on $S$.

3.1. Hyperfinite empirical measures as random elements in the space of all internal Radon measures. Being supported on a finite set, it is clear that $\mu_{\omega,n}$ is, in fact, a Radon probability measure on $S$ for all $\omega \in \Omega$ and $n \in \mathbb{N}$.
Furthermore, for each $n \in \mathbb{N}$, the map $\omega \mapsto \mu_{\omega,n}$ is a measurable function from $(\Omega, \mathcal{F})$ to $(\mathbb{Q}_i(S), \mathcal{B}(\mathbb{Q}_i(S)))$. We record this as a lemma.

**Lemma 3.1.** For each $n \in \mathbb{N}$, the map $\mu_{\cdot,n} : \Omega \to \mathbb{Q}_i(S)$ defined by (3.2) is Borel measurable. Furthermore, for any $B \in \mathcal{B}(S)$, the map $\mu_{\cdot,n}(B) : \Omega \to [0,1]$ (that is, $\omega \mapsto \mu_{\omega,n}(B)$) is Borel measurable for each $n \in \mathbb{N}$.

**Proof.** The proof is immediate from the measurability of the $X_i$, in view of the observation that for each $n \in \mathbb{N}, \omega \in \Omega$, and $B \in \mathcal{B}(S)$, we have:

$$
\mu_{\omega,n}(B) = \frac{1}{n} \left( \sum_{i=1}^{n} 1_B(X_i(\omega)) \right).
$$

By transfer, we obtain the following immediate consequence.

**Corollary 3.2.** For each $N \in \mathbb{N}$, the map $\mu_{\cdot,N} : \Omega \to \mathbb{Q}_i(S)$ is an internally Borel measurable function from $\Omega$ to $\mathbb{Q}_i(S)$. That is, $\mu_{\cdot,N} : \Omega \to \mathbb{Q}_i(S)$ is internal and the set $\{ \omega \in \Omega : \mu_{\omega,N} \in \mathcal{B} \}$ belongs to $\mathcal{F}$ whenever $\mathcal{B} \in \mathcal{B}(\mathbb{Q}_i(S))$. Furthermore, for each $B \in \mathcal{B}(S)$, the map $\mu_{\cdot,N}(B) : \Omega \to [0,1]$ is internally Borel measurable.

By the usual Loeb measure construction, we have a collection of complete probability spaces indexed by $\Omega$, namely $(\mathbb{S}, L, \mathcal{N}_N, L\mu_{\cdot,N})_{\omega \in \Omega}$.

We now prove that with respect to the Loeb measure $L^*\mathbb{P}$, almost all $L\mu_{\cdot,N}$ assign full mass to the set $\mathbb{N}_N(S)$ of nearstandard elements of $\mathbb{S}$. This implicitly requires us to first show that for all $\omega$ in an $L^*\mathbb{P}$ almost sure subset of $\Omega$, the set $\mathbb{N}_N(S)$ is in the Loeb sigma algebra $L_{\omega,N}(\mathcal{B}(S))$ corresponding to the internal probability space $(\mathbb{S}, L^*\mathbb{P}, L\mu_{\cdot,N})$.

**Lemma 3.3.** Let $S$ be a Hausdorff space and $N \in \mathbb{N}$. There is a set $E_N \in L(\mathcal{F})$ with $L^*\mathbb{P}(E_N) = 1$ such that for any $\omega \in E_N$, we have $L\mu_{\cdot,N}(\mathbb{N}_N(S)) = 1$.

**Proof.** Let $(C_n)_{n \in \mathbb{N}}$ be as in (3.1). By the transfer of the second part of Lemma 3.1, the function $\omega \mapsto \mu_{\cdot,N}(\cdot^{C_n})$ is an internal random variable for each $n \in \mathbb{N}$. Since it is finitely bounded, it is $\mathcal{S}$-integrable with respect to the Loeb measure $L^*\mathbb{P}$. Thus, for each $n \in \mathbb{N}$, the $[0,1]$-valued function $L_{\cdot,N}(\cdot^{C_n})$ defined by $\omega \mapsto L\mu_{\cdot,N}(\cdot^{C_n})$, is Loeb measurable, and furthermore we have:

$$
\begin{align*}
\mathbb{E}_{L^*\mathbb{P}}(L\mu_{\cdot,N}(\cdot^{C_n})) &\approx \mathbb{E}_{\mathbb{P}}(\mu_{\cdot,N}(\cdot^{C_n})) \\
&= \mathbb{E}_{\mathbb{P}} \left[ \sum_{i=1}^{N} \frac{1}{N} \mathbb{1}_{\cdot^{C_n}}(X_i) \right] \\
&= \frac{1}{N} \left( \sum_{i=1}^{N} \mathbb{P}(X_i \in \cdot^{C_n}) \right) \\
&\geq \frac{1}{N} \left( N \left( 1 - \frac{1}{n} \right) \right) \\
&= 1 - \frac{1}{n},
\end{align*}
$$
where the last line follows from (3.1) and the fact that each $X_i$ has the same distribution.

Being a limit of measurable functions, $\lim_{n \to \infty} L\mu_{\omega,N}(^*C_n) = L\mu_{\omega,N}(\bigcup_{n \in \mathbb{N}} ^*C_n)$, is also measurable. Thus, by the monotone convergence theorem, we obtain:

$$
E_{L^*P} [L\mu_{\omega,N}(\bigcup_{n \in \mathbb{N}} ^*C_n)] = \lim_{n \to \infty} E_{L^*P}(L\mu_{\omega,N}(^*C_n)) \\
\geq \lim_{n \to \infty} \left(1 - \frac{1}{n}\right) = 1.
$$

(3.5)

But $L\mu_{\omega,N}[\bigcup_{n \in \mathbb{N}} ^*C_n] \leq 1$ for all $\omega \in ^*\Omega$. Therefore, by (3.5), we get:

$$
L^*P(E_N) = 1,
$$

(3.6)

where

$$
E_N = \{\omega : L\mu_{\omega,N}[\bigcup_{n \in \mathbb{N}} ^*C_n] = 1\} \in L(^*\mathcal{F}).
$$

(3.7)

Since each $C_n$ is compact, we have $^*C_n \subseteq \mathbf{Ns}(^*S)$ for all $n \in \mathbb{N}$. Thus for each $\omega \in E_N$, we have the following inequality for the inner measure with respect to $\mu_{\omega,N}$ (see (2.3)):

$$
\mu_{\omega,N}[\mathbf{Ns}(^*S)] \geq L\mu_{\omega,N}(^*C_n) \text{ for all } n \in \mathbb{N},
$$

By taking the limit as $n \to \infty$ on the right side and using the definition (3.7) of $E_N$, we obtain:

$$
\mu_{\omega,N}[\mathbf{Ns}(^*S)] \geq \lim_{n \to \infty} L\mu_{\omega,N}(^*C_n) = L\mu_{\omega,N}[\bigcup_{n \in \mathbb{N}} ^*C_n] = 1 \text{ for all } \omega \in E_N.
$$

Since

$$
1 = \mu_{\omega,N}[\mathbf{Ns}(^*S)] \leq \mu_{\omega,N}[\mathbf{Ns}(^*S)] \leq 1,
$$

it follows that $\mathbf{Ns}(^*S)$ is Loeb measurable, and that $L\mu_{\omega,N}[\mathbf{Ns}(^*S)] = 1$ for all $\omega \in E_N$. 

The idea, used in the above proof, of showing that the expected value of a probability is one in order to conclude that the concerned probability is equal to one almost surely, can be turned around and used to show that a certain probability is zero almost surely, by showing that the expected value of that probability is zero. We use this idea to prove next that almost surely, $L\omega,N$ treats the nonstandard extension of a countable disjoint union as if it were the disjoint union of the nonstandard extensions, the leftover portion being assigned zero mass.

**Lemma 3.4.** Let $S$ be a Hausdorff space and $N \in ^*\mathbb{N}$. Let $(B_n)_{n \in \mathbb{N}}$ be a sequence of disjoint Borel sets. There is a set $E_{(B_n)_{n \in \mathbb{N}}} \in L(^*\mathcal{F})$ with $L^*P(E_{(B_n)_{n \in \mathbb{N}}}) = 1$ such that

$$
L\mu_{\omega,N}[\bigsqcup_{n \in \mathbb{N}} ^*B_n] = \sum_{n \in \mathbb{N}} L\mu_{\omega,N}(^*B_n) \text{ for all } \omega \in E_{(B_n)_{n \in \mathbb{N}}},
$$

(3.8)

where $\sqcup$ denotes a disjoint union.
Remark 3.5. Note that the above lemma does not follow from the disjoint additivity of the measure $L\mu_{\omega,N}$, because $\bigcup_{n\in\mathbb{N}}^* B_n \subseteq^* (\bigcup_{n\in\mathbb{N}}^* B_n)$ with equality if and only if the $B_n$ are empty for all but finitely many $n$. Also, the almost sure set $E_{(B_n)_{n\in\mathbb{N}}}$ depends on the sequence $(B_n)_{n\in\mathbb{N}}$. Since there are potentially uncountably many such sequences, therefore we cannot expect to find a single $L^*\mathbb{P}$-almost sure set on which equation (3.8) is always valid for all disjoint sequences $(B_n)_{n\in\mathbb{N}}$ of Borel sets.

Proof of Lemma 3.4. Let $(B_n)_{n\in\mathbb{N}}$ be a disjoint sequence of Borel sets and let

$$B := \bigcup_{n\in\mathbb{N}}^* B_n.$$ 

For each $m \in \mathbb{N}$, let $B_{(m)} := \bigcup_{n\in[m]}^* B_n$. Consider the map $\omega \mapsto \mu_{\omega,N} \left[^* (B \setminus B_{(m)}) \right]$, which is internally Borel measurable by Corollary 3.2. Since this map is finitely bounded, it is $S$-integrable with respect to the Loeb measure $L^*\mathbb{P}$. In particular, for each $m \in \mathbb{N}$, the $[0,1]$-valued function $L\mu_{\omega,N} \left[^* (B \setminus B_{(m)}) \right]$, defined by $\omega \mapsto L\mu_{\omega,N} \left[^* (B \setminus B_{(m)}) \right]$, is Loeb measurable. Taking expected values and using $S$-integrability, we obtain:

$$E_{L^*\mathbb{P}} \left[ L\mu_{\omega,N} \left[^* (B \setminus B_{(m)}) \right] \right] \approx E_{S^*\mathbb{P}} \left[ \mu_{\omega,N} \left[^* (B \setminus B_{(m)}) \right] \right]$$

$$= E_{S^*\mathbb{P}} \left[ \sum_{i=1}^N \frac{1}{N} \mathbb{I}_{^* (B \setminus B_{(m)})}(X_i) \right]$$

$$= \frac{1}{N} \sum_{i=1}^N \mathbb{P}(X_i \in ^* (B \setminus B_{(m)}))$$

$$= \frac{1}{N} \left[ N^* \mathbb{P}(X_1 \in ^* (B \setminus B_{(m)})) \right]$$

$$= \mathbb{P}(X_1 \in ^* (B \setminus B_{(m)}))$$

$$= \mathbb{P}(X_1 \in B) - \mathbb{P}(X_1 \in B_{(m)}). \quad (3.9)$$

Since the expression in (3.9) is a real number, we have the following equality:

$$E_{L^*\mathbb{P}} \left[ L\mu_{\omega,N} \left[^* (B \setminus B_{(m)}) \right] \right] = \mathbb{P}(X_1 \in B) - \mathbb{P}(X_1 \in B_{(m)}) \quad \text{for all } m \in \mathbb{N}. \quad (3.10)$$

Note that for each $\omega \in ^* \Omega$, the limit

$$\lim_{m \to \infty} L\mu_{\omega,N} \left[^* (B \setminus B_{(m)}) \right]$$

exists and is equal to $L\mu_{\omega,N} \left[ \cap_{m\in\mathbb{N}}^* (B \setminus B_{(m)}) \right]$, because $(^* (B \setminus B_{(m)}))_{m\in\mathbb{N}}$ is a decreasing sequence of measurable sets. Also, by the upper monotonicity of the measure induced by $X_1$ on $S$, we know that

$$\lim_{m \to \infty} \mathbb{P}(X_1 \in B_{(m)}) = \mathbb{P}(X_1 \in \cup_{m\in\mathbb{N}} B_{(m)}) = \mathbb{P}(X_1 \in B).$$

Using this in (3.10), followed by an application of the dominated convergence theorem, we thus obtain the following:

$$0 = \lim_{m \to \infty} E_{L^*\mathbb{P}} \left[ L\mu_{\omega,N} \left[^* (B \setminus B_{(m)}) \right] \right]$$

$$= E_{L^*\mathbb{P}} \left[ \lim_{m \to \infty} L\mu_{\omega,N} \left[^* (B \setminus B_{(m)}) \right] \right]. \quad (3.11)$$
Also, since \( \lim_{m \to \infty} L\mu_{\omega,N} \left[ \ast \left( B \setminus B_{(m)} \right) \right] \geq 0 \), it follows from (3.11) that there is an \( L^*\mathbb{P} \)-almost sure set \( E_{(B_n)}{n \in \mathbb{N}} \) such that
\[
\lim_{m \to \infty} L\mu_{\omega,N} \left[ \ast \left( B \setminus B_{(m)} \right) \right] = 0 \quad \text{for all } \omega \in E_{(B_n)}{n \in \mathbb{N}}.
\] (3.12)

But for each \( \omega \in E_{(B_n)}{n \in \mathbb{N}} \), we have the following:
\[
L\mu_{\omega,N} \left[ \ast \left( B \setminus B_{(m)} \right) \right] = L\mu_{\omega,N} \left( \ast B \right) - L\mu_{\omega,N} \left( B_{(m)} \right) \\
= L\mu_{\omega,N} \left( \ast B \right) - L\mu_{\omega,N} \left( \bigcup_{n \in [m]} B_n \right) \\
= L\mu_{\omega,N} \left( \ast B \right) - \sum_{n \in [m]} L\mu_{\omega,N} \left( \ast B_m \right) \quad \text{for all } m \in \mathbb{N}.
\] (3.13)

The proof is completed by letting \( m \to \infty \) in (3.13), followed by an application of (3.12).
\[\square\]

The specific form of the set \( E_N \) allows us to use Theorem 2.12 to show that for each \( N \in \ast \mathbb{N} \), the measure \( L\mu_{\omega,N} \circ \text{st}^{-1} \) is Radon for all \( \omega \in E_N \), and that \( \mu_{\omega,N} \) is nearstandard in \( \ast \mathcal{P}_r(S) \) to this measure. This is proved in the next lemma.

**Lemma 3.6.** Let \( S \) be a Hausdorff space. Let \( N \in \ast \mathbb{N} \) and \( E_N \) be as in (3.7). For all \( \omega \in E_N \), we have:

(i) \( L\mu_{\omega,N} \circ \text{st}^{-1} \in \mathcal{P}_r(S) \).

(ii) \( \mu_{\omega,N} \in \text{Ns}(\ast \mathcal{P}_r(S)) \), with \( \text{st}(\mu_{\omega,N}) = L\mu_{\omega,N} \circ \text{st}^{-1} \).

**Proof.** By the definition (3.7), we know that
\[
L\mu_{\omega,N} \left( \bigcup_{n \in \mathbb{N}} \ast C_n \right) = 1 \quad \text{for all } \omega \in E_N,
\]
where the \( C_n \) are compact subsets of \( S \).

By the upper monotonicity of the probability measure \( L\mu_{\omega,N} \) and the fact that \( \left( \ast C_n \right)_{n \in \mathbb{N}} \) is an increasing sequence, we obtain:
\[
\lim_{n \to \infty} L\mu_{\omega,N} \left( \ast C_n \right) = 1 \quad \text{for all } \omega \in E_N.
\] (3.14)

Therefore, given \( \epsilon \in \mathbb{R}_{>0} \), there exists an \( n_\epsilon \) such that \( L\mu_{\omega,N} \left( \ast C_n \right) > 1 - \epsilon \) for all \( \omega \in E_N \) and \( n \in \mathbb{N}_{>n_\epsilon} \). Thus the tightness condition (2.11) holds for \( \mu_{\omega,N} \) whenever \( \omega \in E_N \). Theorem 2.12 now completes the proof.
\[\square\]

Let \( \tau_{\mathcal{P}_r(S)} \) denote the \( A \)-topology on \( \mathcal{P}_r(S) \). For \( \mu \in \mathcal{P}_r(S) \), let \( \tau_\mu \) denote the set of all open neighborhoods of \( \mu \) in \( \mathcal{P}_r(S) \). That is,
\[
\tau_\mu := \{ \mathcal{U} \in \tau_{\mathcal{P}_r(S)} : \mu \in \mathcal{U} \}.
\]

Also, for any open set \( \mathcal{U} \in \tau_{\mathcal{P}_r(S)} \), let \( \tau_{\mathcal{U}} \) be the subspace topology on \( \mathcal{U} \). In other words, we define
\[
\tau_{\mathcal{U}} := \{ \mathcal{V} \in \tau_{\mathcal{P}_r(S)} : \mathcal{V} = \mathcal{W} \cap \mathcal{U} \text{ for some } \mathcal{W} \in \tau_{\mathcal{P}_r(S)} \} = \{ \mathcal{V} \in \tau_{\mathcal{P}_r(S)} : \mathcal{V} \subseteq \mathcal{U} \}.
\]

For internal sets \( A, B \), we use \( \mathfrak{I}(A, B) \) to denote the internal set of all internal functions from \( A \) to \( B \).
Lemma 3.7. Let $S$ be Hausdorff and $N \in ^*\mathbb{N}$. Let $E_N$ be as defined in (3.7). For each internal subset $E \subseteq E_N$, there exists an internal function $U : E \to ^* \tau_{\mathcal{P}_r(S)}$ such that

$$
\mu_{\omega,N} \in U_{\omega} \text{ and } U_{\omega} \subseteq \text{st}^{-1}(L_{\mu_{\omega,N} \circ \text{st}^{-1}}) \text{ for all } \omega \in E.
$$

Proof. Fix an internal set $E \subseteq E_N$. For each open set $U \in \tau_{\mathcal{P}_r(S)}$, define the following set of internal functions:

$$
\mathcal{G}_U := \left\{ f \in \mathfrak{F}(E, ^* \tau_{\mathcal{P}_r(S)} ) : f(\omega) \in ^* U \text{ and } \mu_{\omega,N} \in f(\omega) \text{ for all } \omega \in E \cap \mu_{\omega,N}^{-1}(^* U) \right\}.
$$

Since $E$ is internal and $\mu_{\omega,N}^{-1}(^* U)$ is internal by Lemma 3.1, therefore the set $\mathcal{G}_U$ is internal for all $U \in \tau_{\mathcal{P}_r(S)}$ by the internal definition principle (see, for example, Loeb [49, Theorem 2.8.4, p. 54]). Also, $\mathcal{G}_U$ is nonempty for each $U \in \tau_{\mathcal{P}_r(S)}$. Indeed, if $E \cap \mu_{\omega,N}^{-1}(^* U) = \emptyset$, then $\mathcal{G}_U = \mathfrak{F}(E, ^* \tau_{\mathcal{P}_r(S)} )$. Otherwise, if $\omega \in E \cap \mu_{\omega,N}^{-1}(^* U)$, then define $f(\omega) := ^* U$, and define $f$ (internally) arbitrarily on the remainder of $E$. It is clear that this function $f$ is an element of $\mathcal{G}_U$.

Now let $U_1, U_2$ be two distinct open subsets of $\mathcal{P}_r(S)$. Define a function $f$ on $E$ as follows:

$$
f(\omega) := \begin{cases} ^* U_1 \cap ^* U_2 & \text{if } \omega \in E \cap \mu_{\omega,N}^{-1}(^* U_1) \cap \mu_{\omega,N}^{-1}(^* U_2) \\ ^* U_1 & \text{if } \omega \in [E \cap \mu_{\omega,N}^{-1}(^* U_1)] \setminus \mu_{\omega,N}^{-1}(^* U_2) \\ ^* U_2 & \text{if } \omega \in [E \cap \mu_{\omega,N}^{-1}(^* U_2)] \setminus \mu_{\omega,N}^{-1}(^* U_1) \\ ^* \mathcal{P}_r(S) & \text{if } \omega \in E \setminus [\mu_{\omega,N}^{-1}(^* U_1) \cup \mu_{\omega,N}^{-1}(^* U_2)] \end{cases}.
$$

The above function is clearly in $\mathcal{G}_{U_1} \cap \mathcal{G}_{U_2}$. In general, to show the finite intersection property of the collection $\{ \mathcal{G}_U : U \in \tau_{\mathcal{P}_r(S)} \}$, the same recipe of “disjointifying” the union of finitely many open sets $U_1, \ldots, U_k$ works. More precisely, for a subset $A \subseteq \mathcal{P}_r(S)$, let $A^{(0)}$ denote $A$ and $A^{(1)}$ denote the complement $\mathcal{P}_r(S) \setminus A$. If $U_1, \ldots, U_k$ are finitely many open subsets of $\mathcal{P}_r(S)$, then for each $\omega \in E$, define $(i_1(\omega), \ldots, i_k(\omega)) \in \{0, 1\}^k$ to be the unique tuple such that $\omega \in E \cap \left( \bigcap_{j \in \{k\}} U_j^{i_j(\omega)} \right)$. Then the function $f$ on $E$ defined as follows is immediately seen to be a member of $\bigcap_{j \in \{k\}} \mathcal{G}_{U_j}$:

$$
f(\omega) := \bigcap_{j \in \{k\} : i_j(\omega) = 1} ^* U_j \text{ for all } \omega \in E.
$$

Thus the collection $\{ \mathcal{G}_U : U \in \tau_{\mathcal{P}_r(S)} \}$ has the finite intersection property. Let $U_0$ be in the intersection of the $\mathcal{G}_U$ (which is nonempty by saturation). It is clear from the definition of the sets $\mathcal{G}_U$ that $\mu_{\omega,N} \in U_{\omega}$ for all $\omega \in E$. We now show that $U_{\omega} \subseteq \text{st}^{-1}(L_{\mu_{\omega,N} \circ \text{st}^{-1}})$ for all $\omega \in E$.

By Lemma 3.6, we know that $\mu_{\omega,N} \in \text{st}^{-1}(L_{\mu_{\omega,N} \circ \text{st}^{-1}})$ for all $\omega \in E$. Thus for each $\omega \in E$, we have $\mu_{\omega,N} \in ^* U$ for all $U \in \tau_{L_{\mu_{\omega,N} \circ \text{st}^{-1}}}$, hence, for each $\omega \in E$, we have $\omega \in E \cap \mu_{\omega,N}^{-1}(^* U)$ for all $U \in \tau_{L_{\mu_{\omega,N} \circ \text{st}^{-1}}}$. Therefore, by the definition of the collections $\mathcal{G}_U$, we deduce that $U_{\omega} \in ^* \mathcal{P}_r(S)$ for all $U \in \tau_{L_{\mu_{\omega,N} \circ \text{st}^{-1}}}$. As a consequence, $U_{\omega} \subseteq ^* U$ for all $U \in \tau_{L_{\mu_{\omega,N} \circ \text{st}^{-1}}}$ and $\omega \in E$. Hence, $U_{\omega} \subseteq \bigcap_{U \in \tau_{L_{\mu_{\omega,N} \circ \text{st}^{-1}}}} ^* U = \text{st}^{-1}(L_{\mu_{\omega,N} \circ \text{st}^{-1}})$ for all $\omega \in E$, as desired. \qed
For each $N \in \mathbb{N}$, since $E_N$ is a Loeb measurable set of (inner) measure equaling one, there exists an increasing sequence $(F_{N,n})_{n \in \mathbb{N}}$ of internal subsets of $E_N$ such that the following holds:

$$\ast \mathbb{P}(F_{N,n}) > 1 - \frac{1}{n} \text{ for all } n \in \mathbb{N}. \tag{3.15}$$

Lemma 3.7 applied to the internal sets $F_{N,n}$ will imply that the pushforward (internal) measure on $\ast \mathcal{P}_r(S)$ induced by the random variable $\mu_{\ast,N}$ is such that its Loeb measure assigns full measure to $\mathcal{N}(\ast \mathcal{P}_r(S))$. This is the content of our next result.

More precisely, for each $N \in \ast \mathbb{N}$, define an internal finitely additive probability $P_N$ on $(\ast \mathcal{P}_r(S), \ast \mathcal{B}(\mathcal{P}_r(S)))$ as follows:

$$P_N(\mathfrak{A}) := \ast \mathbb{P}(\{ \omega : \mu_{\ast,N} \in \mathfrak{A} \}) = \ast \mathbb{P}(\mu_{\ast,N}^{-1}(\mathfrak{A})) \text{ for all } \mathfrak{A} \in \ast \mathcal{B}(\mathcal{P}_r(S)). \tag{3.16}$$

That this is indeed an internal probability follows from Corollary 3.2. As promised, we now show that the corresponding Loeb measure $L_P$ is concentrated on near-standard elements of $\ast \mathcal{P}_r(S)$.

**Theorem 3.8.** Let $S$ be a Hausdorff space. Let $N \in \ast \mathbb{N}$ and let $P_N$ be as in (3.16). Let

$$(\ast \mathcal{P}_r(S), L_P(\ast \mathcal{B}(\mathcal{P}_r(S))), L_P)$$

be the associated Loeb space. Then the set $\mathcal{N}(\ast \mathcal{P}_r(S))$ is Loeb measurable, with

$$L_P(\mathcal{N}(\ast \mathcal{P}_r(S))) = 1.$$

**Proof.** Let $E_N$ be as in (3.7) and let $(F_{N,n})_{n \in \mathbb{N}} \subseteq E_N$ be as in (3.15). Fix $n \in \mathbb{N}$. With $E := F_{N,n}$, apply Lemma 3.7 to obtain an internal function $U : F_{N,n} \rightarrow \ast \tau_{\mathcal{P}_r(S)}$ such that

$$\mu_{\ast,N} \in U_\omega \text{ and } U_\omega \subseteq \mathcal{N}(L \mu_{\ast,N} \circ st^{-1}) \text{ for all } \omega \in F_n.$$

In particular, $U_\omega \subseteq \mathcal{N}(\ast \mathcal{P}_r(S))$ for all $\omega \in F_{N,n}$, so that $\cup_{\omega \in F_{N,n}} U_\omega \subseteq \mathcal{N}(\ast \mathcal{P}_r(S))$.

By transfer (of the fact that if $f : I \rightarrow \tau_{\mathcal{P}_r(S)}$ is a function, then the set $U := \cup_{i \in I} f(i)$, with the membership relation given by $x \in U$ if and only if there exists $i \in I$ with $x \in f(i)$, is open), we have the following conclusions:

$$U := \cup_{\omega \in F_{N,n}} U_\omega \subseteq \mathcal{N}(S) \text{ and } U \subseteq \ast \tau_{\mathcal{P}_r(S)} \subseteq \ast \mathcal{B}(\mathcal{P}_r(S)).$$

Since $\mu_{\ast,N} \in U_\omega$ for all $\omega \in F_{N,n}$, we have $F_{N,n} \subseteq \mu_{\ast,N}^{-1}(U)$. Hence it follows from (3.16) that

$$P_N(\mathcal{N}(\ast \mathcal{P}_r(S))) \geq L_P(U) = L^* \mathbb{P}(\mu_{\ast,N}^{-1}(U)) \geq L^* \mathbb{P}(F_{N,n}).$$

Using (3.15) and observing that $n \in \mathbb{N}$ was arbitrary, we thus obtain the following:

$$P_N(\mathcal{N}(\ast \mathcal{P}_r(S))) \geq 1 - \frac{1}{n} \text{ for all } n \in \mathbb{N}.$$

This clearly implies that

$$1 = P_N(\mathcal{N}(\ast \mathcal{P}_r(S))) \leq P_N(\mathcal{N}(\ast \mathcal{P}_r(S))) \leq 1,$$

so that $P_N(\mathcal{N}(\ast \mathcal{P}_r(S))) = P_N(\mathcal{N}(\ast \mathcal{P}_r(S))) = 1$. As a consequence, $\mathcal{N}(\ast \mathcal{P}_r(S))$ is Loeb measurable with $L_P(\mathcal{N}(\ast \mathcal{P}_r(S))) = 1$, completing the proof. \( \square \)
The next lemma provides a useful dictionary between Loeb integrals with respect to \( LP_N \) and those with respect to \( L^*P \).

**Lemma 3.9.** Let \( S \) be a Hausdorff space and \( N \in \mathbb{N} \). Let \( P_N \) be as in (3.16). For any bounded \( LP_N \)-measurable function \( f : *\mathcal{P}_r(S) \to \mathbb{R} \), we have:

\[
\int_{*\mathcal{P}_r(S)} f(\mu) dLP_N(\mu) = \int_{*\Omega} f(\mu_\omega) dL^*P(\omega).
\]  
(3.17)

**Proof.** First fix an internally Borel set \( \mathfrak{B} \in *\mathcal{B}(\mathcal{P}_r(S)) \) and let \( f = 1_\mathfrak{B} \). Then the left side of (3.17) is equal to \( LP_N(\mathfrak{B}) = st(P_N(\mathfrak{B})) \), which also equals the following by (3.16):

\[
st[*P(\mu,N^{-1}(\mathfrak{B}))] = L^*P(\{\omega \in *\Omega : \mu_\omega,N \in \mathfrak{B}\}) = \int_{*\Omega} 1_\mathfrak{B}(\mu_\omega) dL^*P(\omega).
\]

Thus (3.17) is true when \( f \) is the indicator function of an internally Borel subset of \( *\mathcal{P}_r(S) \). That is:

\[
LP_N(\mathfrak{B}) = L^*P(\mu,N^{-1}(\mathfrak{B})) \text{ for all } \mathfrak{B} \in *\mathcal{B}(\mathcal{P}_r(S)).
\]  
(3.18)

Now, let \( \mathfrak{A} \) be a Loeb measurable set—that is, \( \mathfrak{A} \in LP_N(*\mathcal{B}(\mathcal{P}_r(S))) \) and \( f = 1_\mathfrak{A} \). By the fact that the Loeb measure of a Loeb measurable set equals its inner and outer measure with respect to the internal algebra \( *\mathcal{B}(\mathcal{P}_r(S)) \), we obtain sets \( \mathfrak{A}_\epsilon, \mathfrak{A}' \in *\mathcal{B}(\mathcal{P}_r(S)) \) for each \( \epsilon \in \mathbb{R}_{>0} \), such that \( \mathfrak{A}_\epsilon \subseteq \mathfrak{A} \subseteq \mathfrak{A}' \) and such that the following holds:

\[
LP_N(\mathfrak{A}) - \epsilon < LP_N(\mathfrak{A}_\epsilon) \leq LP_N(\mathfrak{A}) \leq LP_N(\mathfrak{A}') < LP_N(\mathfrak{A}) + \epsilon.
\]  
(3.19)

Using (3.18) in (3.19) yields the following for each \( \epsilon \in \mathbb{R}_{>0} \):

\[
LP_N(\mathfrak{A}) - \epsilon < L^*P(\mu,N^{-1}(\mathfrak{A}_\epsilon)) \leq LP_N(\mathfrak{A}) \leq L^*P(\mu,N^{-1}(\mathfrak{A}')) < LP_N(\mathfrak{A}) + \epsilon.
\]  
(3.20)

Since \( \mu,N^{-1}(\mathfrak{A}_\epsilon), \mu,N^{-1}(\mathfrak{A}') \) are members of \( *\mathcal{F} \) by Lemma 3.1, it follows from (3.20) that for any \( \epsilon \in \mathbb{R}_{>0} \) we have:

\[
LP_N(\mathfrak{A}) - \epsilon \leq \sup\{L^*P(E) : E \in *\mathcal{F} \text{ and } E \subseteq \mu,N^{-1}(\mathfrak{A}_\epsilon)\}
\]

\[
\leq \sup\{L^*P(E) : E \in *\mathcal{F} \text{ and } E \subseteq \mu,N^{-1}(\mathfrak{A})\}
\]

\[
= *P(\mu,N^{-1}(\mathfrak{A}))
\]

and

\[
LP_N(\mathfrak{A}) + \epsilon \geq \inf\{L^*P(E) : E \in *\mathcal{F} \text{ and } \mu,N^{-1}(\mathfrak{A}') \subseteq E\}
\]

\[
\geq \inf\{L^*P(E) : E \in *\mathcal{F} \text{ and } \mu,N^{-1}(\mathfrak{A}) \subseteq E\}
\]

\[
= \overline{P}(\mu,N^{-1}(\mathfrak{A})).
\]

Since \( \epsilon \in \mathbb{R}_{>0} \) is arbitrary, it thus follows that \( *P(\mu,N^{-1}(\mathfrak{A})) = \overline{P}(\mu,N^{-1}(\mathfrak{A})) \), both being equal to \( LP_N(\mathfrak{A}) \). This shows that \( \mu,N^{-1}(\mathfrak{A}) \) is Loeb measurable and that the following holds:

\[
LP_N(\mathfrak{A}) = L^*P(\mu,N^{-1}(\mathfrak{A})) \text{ for all } \mathfrak{A} \in LP_N(*\mathcal{B}(\mathcal{P}_r(S))).
\]  
(3.21)

This proves (3.17) for indicator functions of Loeb measurable sets. Since the functions \( f \) satisfying (3.17) are clearly closed under taking \( \mathbb{R} \)-linear combinations,
the result is true for simple functions (that is, those Loeb measurable functions that take finitely many values). The result for general bounded Loeb measurable functions follows from this (and the dominated convergence theorem) since any bounded measurable function can be uniformly approximated by a sequence of simple functions.

The result in (3.21) is interesting and useful in its own right. We record this observation as a corollary of the above proof.

**Corollary 3.10.** Let $S$ be a Hausdorff space and let $N \in \ast \mathbb{N}$. Let $P_N$ be as in (3.16). For any $A \in L_{P_N}(\mathcal{B}(\mathcal{P}_r(S)))$, the set $\mu \cdot N^{-1}(A)$ is $L^\ast P$-measurable. Furthermore, we have:

$$LP_N(A) = L^\ast P \left[ \mu \cdot N^{-1}(A) \right] \text{ for all } A \in L_{P_N}(\ast \mathcal{B}(\mathcal{P}_r(S))).$$

### 3.2. An internal measure induced on the space of all internal Radon probability measures.

Armed with a way to compute the $LP_N$ measure of a large collection of sets, we are in a position to use Prokhorov’s theorem (Theorem 2.46) to verify that $P_N$ satisfies the tightness condition (2.50) from Theorem 2.12.

**Theorem 3.11.** Let $S$ be a Hausdorff space and let $N \in \ast \mathbb{N}$. Let $P_N$ be as in (3.16). Given $\epsilon \in \mathbb{R}_{>0}$, there exists a compact set $\mathcal{K}(\epsilon) \subseteq \mathcal{P}_r(S)$ such that

$$LP_N(\ast \Omega) \geq 1 - \epsilon \text{ for all open sets } \Omega \text{ such that } \mathcal{K}(\epsilon) \subseteq \Omega.$$

**Proof.** Let $(C_n)_{n \in \mathbb{N}}$ be the increasing sequence of compact subsets of $S$ fixed in (3.1). Recall the $L^\ast P$ almost sure set $E_N$ from (3.7):

$$E_N = \{ \omega \in \ast \Omega : L\mu_{\omega,N} [\cup_{n \in \mathbb{N}} \ast C_n] = 1 \}
= \{ \omega \in \ast \Omega : \lim_{n \to \infty} L\mu_{\omega,N} (\ast C_n) = 1 \}
= \bigcap_{\ell \in \mathbb{N}} \left( \bigcup_{m \in \mathbb{N}} \bigcap_{n \in \mathbb{N} \geq m} \left\{ \omega \in \ast \Omega : \mu_{\omega,N} (\ast C_n) \geq 1 - \frac{1}{\ell} \right\} \right).$$

Note that \( \left( \bigcup_{m \in \mathbb{N}} \bigcap_{n \in \mathbb{N} \geq m} \left\{ \omega \in \ast \Omega : \mu_{\omega,N} (\ast C_n) \geq 1 - \frac{1}{\ell} \right\} \right)_{\ell \in \mathbb{N}} \) is a decreasing sequence of Loeb measurable sets. Hence the fact that $L^\ast P(E_N) = 1$ implies the following:

$$1 = \lim_{\ell \to \infty} L^\ast P \left( \bigcup_{m \in \mathbb{N}} \bigcap_{n \in \mathbb{N} \geq m} \left\{ \omega \in \ast \Omega : \mu_{\omega,N} (\ast C_n) \geq 1 - \frac{1}{\ell} \right\} \right). \quad (3.22)$$

Let $\epsilon \in \mathbb{R}_{>0}$ be given. By (3.22), there exists an $\ell_{\epsilon} \in \mathbb{N}$ such that we have

$$L^\ast P \left( \bigcup_{m \in \mathbb{N}} \bigcap_{n \in \mathbb{N} \geq m} \left\{ \omega \in \ast \Omega : \mu_{\omega,N} (\ast C_n) \geq 1 - \frac{1}{\ell_{\epsilon}} \right\} \right) > 1 - \frac{\epsilon}{4} \text{ for all } \ell \in \mathbb{N} \geq \ell_{\epsilon}. \quad (3.23)$$
Now \( \left( \bigcap_{n \in \mathbb{N} \geq m} \{ \omega \in \{ \ast \omega : \mu_{\omega,N} (\ast C_n) \geq 1 - \frac{1}{\ell} \} \right) \) is an increasing sequence of Loeb measurable sets. By (3.23), we thus find an \( m_\varepsilon \in \mathbb{N} \) for which the following holds:

\[
L^*P \left( \bigcap_{n \in \mathbb{N} \geq m} \{ \omega \in \{ \ast \omega : \mu_{\omega,N} (\ast C_n) \geq 1 - \frac{1}{\ell} \} \right) > 1 - \frac{\varepsilon}{2}
\]

for all \( \ell \in \mathbb{N} \geq \ell_\varepsilon \) and \( m \in \mathbb{N} \geq m_\varepsilon \). (3.24)

Let \( n_\varepsilon = \max\{\ell_\varepsilon, m_\varepsilon\} \in \mathbb{N} \). By (3.24), the following internal set contains \( N \geq n_\varepsilon \):

\[
\mathcal{G}_\varepsilon := \left\{ n_0 \in \{ \ast N \geq n_\varepsilon \} : \left[ \bigcap_{n_\varepsilon \leq n \leq n_0} \{ \omega \in \{ \ast \omega : \mu_{\omega,N} (\ast C_n) \geq 1 - \frac{1}{n_0} \} \right] > 1 - \varepsilon \right\}.
\]

(3.25)

By overflow, we obtain an \( N_\varepsilon > \mathbb{N} \) in \( \mathcal{G}_\varepsilon \). As a consequence, we conclude that for any \( n_0 \in \{ \ast N \geq n_\varepsilon \} \) we have the following:

\[
L^*P \left[ \bigcap_{n_\varepsilon \leq n \leq n_0} \{ \omega \in \{ \ast \omega : \mu_{\omega,N} (\ast C_n) \geq 1 - \frac{1}{n} \} \right] \\
\geq L^*P \left[ \bigcap_{n_\varepsilon \leq n \leq N_\varepsilon} \{ \omega \in \{ \ast \omega : \mu_{\omega,N} (\ast C_n) \geq 1 - \frac{1}{N_0} \} \right] \\
\geq 1 - \varepsilon.
\]

(3.26)

For each \( n \in \mathbb{N} \), consider the set \( \mathcal{K}_n \) defined as follows:

\[
\mathcal{K}_n := \left\{ \gamma \in \mathcal{P}_r(S) : \gamma(C_n) \geq 1 - \frac{1}{n} \right\}.
\]

Since compact subsets of a Hausdorff space are closed, the set \( \mathcal{K}_n \) is the complement of a subbasic open subset of \( \mathcal{P}_r(S) \), and is hence closed for each \( n \in \mathbb{N} \). Since the nonstandard extension of a finite intersection is the intersection of the
nonstandard extensions, Corollary 3.10 implies that for each $n_0 \in \mathbb{N}_{\geq n_\epsilon}$, we have:

$$LP_N \left( \bigcap_{n \in \mathbb{N}_{n_\epsilon \leq n \leq n_0}} \mathcal{R}_n \right) = LP_N \left( \bigcap_{n \in \mathbb{N}_{n_\epsilon \leq n \leq n_0}} *\mathcal{R}_n \right)$$

$$= \mathcal{L}^* \mathcal{P} \left( \left\{ \omega \in *\Omega : \mu_{\omega, N} \in \bigcap_{n \in \mathbb{N}_{n_\epsilon \leq n \leq n_0}} *\mathcal{R}_n \right\} \right)$$

$$= \mathcal{L}^* \mathcal{P} \left( \left\{ \omega \in *\Omega : \mu_{\omega, N} \in \bigcap_{n \in \mathbb{N}_{n_\epsilon \leq n \leq n_0}} *\mathcal{R}_n \right\} \right). \quad (3.27)$$

Using (3.27) and (3.26), we thus conclude the following:

$$LP_N \left( \bigcap_{n \in \mathbb{N}_{n_\epsilon \leq n \leq n_0}} *\mathcal{R}_n \right) \geq 1 - \epsilon \quad \text{for all } n_0 \in \mathbb{N}_{\geq n_\epsilon}. \quad (3.28)$$

Since $LP_N$ is a finite measure and $\left( \bigcap_{n \in \mathbb{N}_{n_\epsilon \leq n \leq n_0}} *\mathcal{R}_n \right)$ is a decreasing sequence of $LP_N$-measurable sets, we may take the limit as $n_0 \to \infty$ in (3.28) to obtain the following:

$$LP_N \left( \bigcap_{n \in \mathbb{N}_{n_\epsilon \leq n \leq n_0}} *\mathcal{R}_n \right) \geq 1 - \epsilon. \quad (3.29)$$

Define $\mathcal{R}_{(\epsilon)}$ as follows:

$$\mathcal{R}_{(\epsilon)} := \bigcap_{n \in \mathbb{N}_{n_\epsilon \leq n \leq n_0}} \mathcal{R}_n. \quad (3.30)$$

Since arbitrary intersections of closed sets are closed, it follows that $\mathcal{R}_{(\epsilon)}$ is a closed subset of $\mathcal{P}_r(S)$. It is also relatively compact by Theorem 2.46. Being a relatively compact closed set, it follows that $\mathcal{R}_{(\epsilon)}$ is a compact subset of $\mathcal{P}_r(S)$. Let $\Omega$ be any open subset of $\mathcal{P}_r(S)$ containing $\mathcal{R}_{(\epsilon)}$. We make the following immediate observation using Lemma 2.8:

$$*\mathcal{R}_{(\epsilon)} \subseteq \left( \bigcap_{n \in \mathbb{N}_{n_\epsilon \leq n \leq n_0}} *\mathcal{R}_n \right) \cap \text{Ns}(\mathcal{P}_r(S)) \subseteq *\Omega. \quad (3.31)$$

By (3.31) and Theorem 3.8, we thus obtain:

$$LP_N(*\Omega) \geq LP_N \left( \bigcap_{n \in \mathbb{N}_{n_\epsilon \leq n \leq n_0}} *\mathcal{R}_n \right) \cap \text{Ns}(\mathcal{P}_r(S)) = LP_N \left( \bigcap_{n \in \mathbb{N}_{n_\epsilon \leq n \leq n_0}} *\mathcal{R}_n \right).$$

Using (3.29) now shows that $LP_N(*\Omega) \geq 1 - \epsilon$, thus completing the proof. \qed
Theorem 3.11, Theorem 2.11, and Theorem 2.28 now immediately lead to the following result.

**Theorem 3.12.** Suppose that $S$ is a Hausdorff space. Let $N \in {^*}\mathbb{N}$ and let $P_N$ be as in (3.16). Let

\[(^*\mathfrak{F}_r(S), L_{P_N}(^*\mathcal{B}(\mathfrak{F}_r(S))), L_{P_N})\]

be the associated Loeb space. Then $L_{P_N} \circ \text{st}^{-1}$ is a Radon measure on the Hausdorff space $\mathfrak{F}_r(S)$. Furthermore, $P_N$ is nearstandard to $L_{P_N} \circ \text{st}^{-1}$ in $^*\mathfrak{F}(\mathfrak{F}_r(S))$—that is, we have:

\[P_N \in \text{st}^{-1}(L_{P_N} \circ \text{st}^{-1}) \subseteq ^*\mathfrak{F}(\mathfrak{F}_r(S)).\]

It is worthwhile to point out two useful observations arising from the statement of Theorem 3.12. Firstly, we were able to say that $P_N$ is nearstandard to $L_{P_N} \circ \text{st}^{-1}$ in $^*\mathfrak{F}(\mathfrak{F}_r(S))$, but we can still not say that the standard part of $P_N$ is $L_{P_N} \circ \text{st}^{-1}$. This is because $^*\mathfrak{F}(\mathfrak{F}_r(S))$ is not necessarily Hausdorff and even though $L_{P_N} \circ \text{st}^{-1} \in \mathfrak{F}_r(\mathfrak{F}_r(S))$, we do not know whether $P_N$ belongs to $^*\mathfrak{F}_r(\mathfrak{F}_r(S))$ or not (so we are not able to use the standard part map $\text{st} : \text{Ns}(^*\mathfrak{F}_r(\mathfrak{F}_r(S))) \to \mathfrak{F}_r(\mathfrak{F}_r(S))$ in this context).

Secondly, since $L_{P_N} \circ \text{st}^{-1}$ is a measure on $\mathcal{B}(\mathfrak{F}_r(S))$, it is in particular the case that $\text{st}^{-1}(\mathfrak{B})$ is $L_{P_N}$-measurable for all $\mathfrak{B} \in \mathcal{B}(\mathfrak{F}_r(S))$. This observation is useful enough that we record it as a corollary.

**Corollary 3.13.** Let $S$ be a Hausdorff space and let $P_N$ be as in (3.16). For each $\mathfrak{B} \in \mathcal{B}(\mathfrak{F}_r(S))$, the set $\text{st}^{-1}(\mathfrak{B}) \subseteq ^*\mathfrak{F}_r(S)$ is $L_{P_N}$-measurable.

### 3.3 Almost sure standard parts of hyperfinite empirical measures

We now return to studying properties of the measures $L_{\mu_\omega,N}$ for $N \in {^*}\mathbb{N}$. Corollary 3.13 immediately leads us to the following.

**Lemma 3.14.** Let $S$ be a Hausdorff space. Let $N \in {^*}\mathbb{N}$ and let $E_N$ be the $L^*\mathbb{P}$-almost sure set fixed in (3.7). Then for each $B \in \mathcal{B}(S)$, the set $\text{st}^{-1}(B)$ is $L_{\mu_\omega,N}$-measurable for all $\omega \in E_N$. Furthermore, for each $B \in \mathcal{B}(S)$, the function $\omega \mapsto L_{\mu_\omega,N}(\text{st}^{-1}(B))$ thus defines a $[0,1]$-valued random variable almost everywhere on $({^*}\Omega, L(^*\mathcal{F}), L^*\mathbb{P})$.

**Proof.** It was proved as part of Lemma 3.6 that for each $B \in \mathcal{B}(S)$, the set $\text{st}^{-1}(B)$ is $L_{\mu_\omega,N}$-measurable for all $\omega \in E_N$. Thus, the function $\omega \mapsto L_{\mu_\omega,N}(\text{st}^{-1}(B))$ is defined $L^*\mathbb{P}$-almost surely on $^*\Omega$ for all $B \in \mathcal{B}(S)$.

Now fix $B \in \mathcal{B}(S)$. Since $L^*\mathbb{P}(E_N) = 1$ and $(^*\Omega, L(^*\mathcal{F}), L^*\mathbb{P})$ is a complete probability space, showing that the map $\omega \mapsto L_{\mu_\omega,N}(\text{st}^{-1}(B))$ is Loeb measurable is equivalent to showing that for any $\alpha \in \mathbb{R}$, the set $\{\omega \in E_N : L_{\mu_\omega,N}(\text{st}^{-1}(B)) > \alpha\}$ is Loeb measurable. Toward that end, fix $\alpha \in \mathbb{R}$. Note that by Lemma 3.6, we obtain the following:

\[\{\omega \in E_N : L_{\mu_\omega,N}(\text{st}^{-1}(B)) > \alpha\} = \{\omega \in E_N : [\text{st}(\mu_\omega,N)](B) > \alpha\} = E_N \cap [\mu_\omega,N^{-1}(\text{st}^{-1}(\{\nu \in \mathfrak{F}_r(S) : \nu(B) > \alpha\}))].\]

By Theorem 2.33 and Corollary 3.13, we also have the following:

\[\text{st}^{-1}(\{\nu \in \mathfrak{F}_r(S) : \nu(B) > \alpha\}) \in L_{P_N}(^*\mathcal{B}(\mathfrak{F}_r(S))).\]

The proof is now completed by Corollary 3.10. \qed
The next two lemmas are preparatory for Theorem 3.18 that shows that for each Borel set $B \in \mathcal{B}(S)$, the $L_{\mu_{\omega}, N}$ measures of $\text{st}^{-1}(B)$ and $^*B$ are almost surely equal to each other.

**Lemma 3.15.** Let $S$ be a Hausdorff space and let $N \in {}^*\mathbb{N}$. Let $C$ be a compact subset of $S$. Then

$$L_{\mu_{\omega}, N}(\text{st}^{-1}(C)) = L_{\mu_{\omega}, N}(^*C)$$

for $L^p$-almost all $\omega \in {}^*\Omega$.

**Proof.** Let $C \subseteq S$ be a compact set. Let $E_N \subseteq {}^*\Omega$ be as in (3.7). By Lemma 3.6, we know that $\text{st}^{-1}(C)$ is $L_{\mu_{\omega}, N}$-measurable for all $\omega \in E_N$. Since $C$ is compact, we also have $^*C \subseteq \text{st}^{-1}(C)$. It is thus clear from the definition of standard parts that the following holds:

$$\text{st}^{-1}(C) \setminus C \subseteq {}^*O \setminus C = (^*O \setminus C)$$

for all open sets $O$ such that $C \subseteq O$. (3.32)

Using Lemma 3.14 and Corollary 3.2 respectively, we know that the maps $\omega \mapsto L_{\mu_{\omega}, N}[\text{st}^{-1}(C) \setminus C]$ and $\omega \mapsto L_{\mu_{\omega}, N}(^*O \setminus C)$ are $L^p$ measurable for all open sets $O$ containing $C$. Taking expected values and using (3.32), we obtain the following for any open set $O$ containing $C$:

$$E_{L^p}[L_{\mu_{\omega}, N}(\text{st}^{-1}(C) \setminus C)] \leq E_{L^p}[L_{\mu_{\omega}, N}(^*O \setminus C)].$$

But, by $S$-integrability of the map $\omega \mapsto \mu_{\omega, N}(^*O \setminus C)$, we also obtain the following:

$$E_{L^p}[L_{\mu_{\omega}, N}(^*O \setminus C)] = \frac{1}{N} \sum_{i \in [N]} \mathbb{P}[X_i \in O \setminus C]$$

$$= \mathbb{P}[X_1 \in O \setminus C].$$

Using this in (3.33), taking infimum as $O$ varies over open sets containing $C$, and using the fact that the distribution of $X_1$ is outer regular on compact subsets of $S$, we obtain the following:

$$E_{L^p}[L_{\mu_{\omega}, N}(\text{st}^{-1}(C) \setminus C)] = 0.$$ (3.34)

As a result, there exists a Loeb measurable set $E_{C, N} \in L(\mathcal{F})$ such that

$$[L_{\mu_{\omega}, N}(\text{st}^{-1}(C) \setminus C)] = 0$$

for all $\omega \in E_{C, N}$, completing the proof. \qed

**Remark 3.16.** We have only used the facts that the common distribution of the random variables $X_1, X_2, \ldots$ is tight and that it is outer regular on compact subsets of $S$. Tightness was used in (3.1) and all subsequent results that depended on it, while outer regularity on compact subsets was used to obtain (3.34). The results that follow are consequences of the results obtained so far, and, as such, they also only require the common distribution to be tight and outer regular on compact subsets. For simplicity, however, we will continue working under the assumption that the common distribution of the random variables $X_1, X_2, \ldots$ is Radon.

We can strengthen Lemma 3.15 to work for all closed sets, as we show next.
Lemma 3.17. Let $S$ be a Hausdorff space and let $N \in \mathbb{N}$. Let $F$ be a closed subset of $S$. Then we have the following:

$$L_{\mu,\omega,N}(\mathbf{st}^{-1}(F)) = L_{\mu,\omega,N}(\mathbf{st}^{-1}(F))$$

for all $\omega \in \mathbb{N}$. \hfill (3.35)

Proof. Let $E_N$ be as in (3.7). Thus there is an increasing sequence of compact sets \{C_n\}$_{n \in \mathbb{N}} \subseteq S$ such that

$$L_{\mu,\omega,N}(\bigcup_{n \in \mathbb{N}} C_n) = 1 \text{ for all } \omega \in E_N.$$

Using the upper monotonicity of $L_{\mu,\omega,N}$, we rewrite the above as follows:

$$\lim_{n \to \infty} L_{\mu,\omega,N}(C_n) = 1 \text{ for all } \omega \in E_N. \tag{3.36}$$

Let $F \subseteq S$ be closed. Since $F \cap C_n$ is compact for all $n \in \mathbb{N}$, by Lemma 3.15, there exist $L^*P$-almost sure sets $(E^{(n)})_{n \in \mathbb{N}}$ such that the following holds:

$$L_{\mu,\omega,N}(\mathbf{st}^{-1}(F \cap C_n)) = L_{\mu,\omega,N}(\mathbf{st}^{-1}(F \cap C_n))$$

for all $\omega \in E^{(n)}$, where $n \in \mathbb{N}$. \hfill (3.37)

Let $E_F := E_N \cap \bigcap_{n \in \mathbb{N}} E^{(n)} \big).$ Being a countable intersection of almost sure sets, $E_F$ is also $L^*P$-almost sure. Letting $\omega \in E_F$ and taking limits as $n \to \infty$ on both sides of (3.37), we obtain the following in view of (3.36):

$$\lim_{n \to \infty} L_{\mu,\omega,N}(\mathbf{st}^{-1}(F \cap C_n)) = L_{\mu,\omega,N}(\mathbf{st}^{-1}(F)) \text{ for all } \omega \in E_F. \tag{3.38}$$

Using the upper monotonicity of the measure $L_{\mu,\omega,N}$ on the left side of (3.38), we obtain the following:

$$L_{\mu,\omega,N}(\bigcup_{n \in \mathbb{N}} \mathbf{st}^{-1}(F \cap C_n)) = L_{\mu,\omega,N}(\mathbf{st}^{-1}(F)) \text{ for all } \omega \in E_F. \tag{3.39}$$

But, we also have the following:

$$\bigcup_{n \in \mathbb{N}} \mathbf{st}^{-1}(F \cap C_n) = \mathbf{st}^{-1}(\bigcup_{n \in \mathbb{N}} (F \cap C_n)) = \mathbf{st}^{-1}(F \cap (\bigcup_{n \in \mathbb{N}} C_n)),$$

so that

$$\mathbf{st}^{-1}(F) \setminus \bigcup_{n \in \mathbb{N}} \mathbf{st}^{-1}(F \cap C_n) = \mathbf{st}^{-1}(F) \setminus \mathbf{st}^{-1}(F \cap (\bigcup_{n \in \mathbb{N}} C_n))$$

$$= \mathbf{st}^{-1}(F \cap (\bigcap_{n \in \mathbb{N}} S \setminus C_n))$$

$$\subseteq \bigcap_{n \in \mathbb{N}} \mathbf{st}^{-1}(S \setminus C_n)$$

$$= \bigcap_{n \in \mathbb{N}} [\mathbf{st}^{-1}(S) \setminus \mathbf{st}^{-1}(C_n)].$$

Thus, for any $\omega \in E_F$, the following holds:

$$L_{\mu,\omega,N} \left[ \mathbf{st}^{-1}(F) \setminus \bigcup_{n \in \mathbb{N}} \mathbf{st}^{-1}(F \cap C_n) \right] \leq \lim_{n \to \infty} L_{\mu,\omega,N} \left[ \mathbf{st}^{-1}(S) \setminus \mathbf{st}^{-1}(C_n) \right]$$

$$= \lim_{n \to \infty} \left[ L_{\mu,\omega,N}(\mathbf{Ns}(S)) - L_{\mu,\omega,N}(\mathbf{st}^{-1}(C_n)) \right]$$

$$= \lim_{n \to \infty} \left[ 1 - L_{\mu,\omega,N}(\mathbf{st}^{-1}(C_n)) \right], \tag{3.40}$$

where the last line follows from Lemma 3.15 and the fact that $L_{\mu,\omega,N}(\mathbf{Ns}(S)) = 1$ for all $\omega \in E_F \subseteq E_N$. Using (3.36) and (3.40), we thus obtain the following:

$$L_{\mu,\omega,N} \left[ \mathbf{st}^{-1}(F) \setminus \bigcup_{n \in \mathbb{N}} \mathbf{st}^{-1}(F \cap C_n) \right] \leq 1 - \lim_{n \to \infty} L_{\mu,\omega,N}(\mathbf{st}^{-1}(C_n)) = 1 - 1 = 0.$$
Since $\bigcup_{n \in \mathbb{N}} \text{st}^{-1}(F \cap C_n) \subseteq \text{st}^{-1}(F)$, we thus conclude that

$$L_{\mu,\omega,N} \left[ \bigcup_{n \in \mathbb{N}} \text{st}^{-1}(F \cap C_n) \right] = L_{\mu,\omega,N}(\text{st}^{-1}(F)). \quad (3.41)$$

Using (3.41) in (3.39) completes the proof. \(\square\)

Having proved (3.35) for closed sets, it is easy to generalize it for all Borel sets using the standard measure theory trick of showing that the collection of sets satisfying (3.35) form a sigma algebra. This is the next result.

**Theorem 3.18.** Let $S$ be a Hausdorff space and let $N \in \ast \mathbb{N}$. Let $B$ be a Borel subset of $S$. Then we have the following:

$$L_{\mu,\omega,N}(\text{st}^{-1}(B)) = L_{\mu,\omega,N}(\ast B) \text{ for } L^\ast \mathbb{P}-\text{almost all } \omega \in \ast \Omega. \quad (3.42)$$

**Proof.** Let $E_N$ be as in (3.7). By Lemma 3.6, we know that $\text{st}^{-1}(B)$ is $L_{\mu,\omega,N}$-measurable for all $\omega \in E_N$ and $B \in \mathcal{B}(S)$. Consider the following collection:

$$G := \{B \in \mathcal{B}(S) : \exists E_B \in L(\ast \mathcal{F}) \left[ (L^\ast \mathbb{P}(E_B) = 1) \land (\forall \omega \in E_B \cap E_N \left( L_{\mu,\omega,N}(\text{st}^{-1}(B)) = L_{\mu,\omega,N}(\ast B)) \right) \right] \} \quad (3.43)$$

By Lemma 3.17, we know that $G$ contains all closed sets. In order to show that $G$ contains all Borel sets, by Dynkin’s theorem, it thus suffices to show that $G$ is a Dynkin system, that is, to show the following:

(i) $S \in G$.

(ii) If $B \in G$, then $S \setminus B \in G$ as well.

(iii) If $(B_n)_{n \in \mathbb{N}}$ is a sequence of mutually disjoint elements of $G$, then $\bigcup_{n \in \mathbb{N}} B_n \in G$.

(i) is immediate from Lemma 3.17, with $E_S := E_N$. To see (ii), take $B \in G$ and let $E_B$ be as (3.43). Note that for any $\omega \in E_B \cap E_N$, we have:

$$L_{\mu,\omega,N}(\ast (S \setminus B)) = L_{\mu,\omega,N}(\ast (S \setminus \ast B)) = L_{\mu,\omega,N}(\ast S) - L_{\mu,\omega,N}(\ast B) = L_{\mu,\omega,N}(\ast \text{st}^{-1}(S)) - L_{\mu,\omega,N}(\text{st}^{-1}(B)) = L_{\mu,\omega,N}(\text{st}^{-1}(S) \setminus \text{st}^{-1}(B)) = L_{\mu,\omega,N}(\text{st}^{-1}(S \setminus B)).$$

In the above argument, the third line used the fact that $S$ and $B$ are in $G$, the fourth line used the fact that $\text{st}^{-1}(B) \subseteq \text{st}^{-1}(S)$, and the fifth line used the fact that $\text{st}^{-1}(S) \setminus \text{st}^{-1}(B) = \text{st}^{-1}(S \setminus B)$ (which can be seen to follow from Lemma 2.10 since $S$ is Hausdorff).

We now prove (iii). Let $(B_n)_{n \in \mathbb{N}}$ be a sequence of mutually disjoint elements of $G$ and let $B := \bigcup_{n \in \mathbb{N}} B_n$. By Lemma 2.10 and the fact that $B_n \in G$ for all $n \in \mathbb{N},$
we have the following for all $\omega \in {}^*\Omega$:

$$L\mu_{\omega,N}(st^{-1}(B)) = L\mu_{\omega,N}(st^{-1}(\bigcup_{n\in\mathbb{N}} B_n))$$

$$= L\mu_{\omega,N}(\bigcup_{n\in\mathbb{N}} st^{-1}(B_n))$$

$$= \sum_{n\in\mathbb{N}} L\mu_{\omega,N}(st^{-1}(B_n))$$

$$= \sum_{n\in\mathbb{N}} L\mu_{\omega,N}(^*B_n). \tag{3.44}$$

Let $E_{(B_n)_{n\in\mathbb{N}}}$ be as in Lemma 3.4 and define $E_B := E_{(B_n)_{n\in\mathbb{N}}}$. Using (3.44) and (3.8), we thus obtain the following:

$$L\mu_{\omega,N}(st^{-1}(B)) = L\mu_{\omega,N}(\bigcup_{n\in\mathbb{N}} st^{-1}(B_n)) = L\mu_{\omega,N}(^*B)$$

for any $\omega \in E_B \cap E_N$, completing the proof. □

Recall that by Lemma 3.6, if $S$ is Hausdorff then $\mu_{\omega,N} \in Ns(\mathcal{P}_r(S))$, with $st(\mu_{\omega,N}) = L\mu_{\omega,N} \circ st^{-1}$ for all $\omega \in E_N$. Thus Theorem 3.18 shows the following:

**Theorem 3.19.** Let $S$ be a Hausdorff space. For any Borel set $B \in \mathcal{B}(S)$, we have

$$st(\mu_{\omega,N}(^*B)) = (st(\mu_{\omega,N}))(B)$$

for almost all $\omega \in {}^*\Omega$. \hfill (3.45)

We point out an interesting interpretation of Theorem 3.19. For each Borel set $B \in \mathcal{B}(S)$, the Loeb measure $L\mu_{\omega,N}(^*B)$ can almost surely be computed by either of the following two-step procedures:

(i) First find $\mu_{\omega,N}(^*B) \in {}^*[0,1]$ and then take the standard part of this finite nonstandard real number, which is the direct way.

(ii) First take the standard part of the internal measure $\mu_{\omega,N} \in \mathcal{P}_r(S)$, and then compute the measure $st(\mu_{\omega,N})(B)$ of $B$ with respect to this standard part.

Since the intersection of countably many almost sure sets is almost sure, we have thus shown the almost sure commutativity of the following diagram for any countable subset $C \subseteq \mathcal{B}(S)$:

![Diagram](image)

3.4. **Pushing down certain Loeb integrals on the space of all Radon probability measures.** We finish this section by relating certain nonstandard integrals over the space $({}^*\mathcal{P}_r(S), {}^*\mathcal{B}(\mathcal{P}_r(S)), P_N)$ to those over $(\mathcal{P}_r(S), \mathcal{B}(\mathcal{P}_r(S)), LP_N \circ st^{-1})$.

**Theorem 3.20.** Suppose $S$ is a Hausdorff space. Let $N \in {}^*\mathbb{N}$ and let $P_N$ be as in (3.16). Let $({}^*\mathcal{P}(S), LP_N({}^*\mathcal{B}(\mathcal{P}_r(S))))$ be the associated Loeb space. Then for
any Borel subset $B$ of $\mathcal{P}_r(S)$, we have:

\[ \ast \int_{\mathcal{P}_r(S)} \mu(\ast B)dP_N(\mu) \approx \int_{\mathcal{P}_r(S)} \mu(B)d\mathcal{P}_N(\mu), \tag{3.46} \]

where $\mathcal{P}_N = L_{N} \circ \text{st}^{-1} \in \mathcal{P}_r(S)$.

**Proof.** Fix $B \in \mathcal{B}(S)$. By Corollary 3.2 and (3.16), the function $\mu \mapsto \mu(\ast B)$ is internally Borel measurable on $\ast \mathcal{P}_r(S)$. Since it is finitely bounded (by one), it is $S$-integrable. Using this and Lemma 3.9, we thus obtain the following:

\[
\ast \mathbb{E}_{P_N}(\mu(\ast B)) \approx \int_{\mathcal{P}_r(S)} \text{st}(\mu(\ast B))dL_{N}\mu(\mu(\ast B))
= \int_{\Omega} \text{st}(\mu_{\omega,N}(\ast B))dL_{\ast \mathbb{P}(\omega)}
= \int_{\Omega} \text{st}(\mu_{\omega,N})(\ast B)dL_{\ast \mathbb{P}(\omega)},
\]

where we used Theorem 3.19 in the last line. Writing the last integral as a Lebesgue integral of tail probabilities, we make the following conclusion:

\[
\ast \mathbb{E}_{P_N}(\mu(\ast B)) \approx \int_{[0,1]} L_{\ast \mathbb{P}}((\text{st}(\mu_{\omega,N}))(\ast B) > y)d\lambda(y)
= \int_{[0,1]} L_{\ast \mathbb{P}}[\mu_{\omega,N}^{-1}(\{\nu \in \mathcal{P}_r(S) : \nu(\ast B) > y\})]d\lambda(y)
= \int_{[0,1]} L_{N}(\text{st}^{-1}(\{\nu \in \mathcal{P}_r(S) : \nu(\ast B) > y\}))d\lambda(y),
\]

where the last line follows from Corollary 3.10. (This also uses the fact that the set $\{\nu \in \mathcal{P}_r(S) : \nu(\ast B) > y\}$ is Borel measurable, in view of Theorem 2.33.)

Defining $\mathcal{P}_N := L_{N} \circ \text{st}^{-1}$ and noting that $\mathcal{P}_N$ is a Radon probability measure on $\mathcal{P}_r(S)$ (by Theorem 3.12), we obtain the following:

\[
\ast \mathbb{E}_{P_N}(\mu(\ast B)) \approx \mathcal{P}_N(\{\nu \in \mathcal{P}_r(S) : \nu(\ast B) > y\})d\lambda(y)
= \int_{\mathcal{P}_r(S)} \mu(B)d\mathcal{P}_N(\mu),
\]

thus completing the proof. \(\square\)

Note that the same proof idea can be used to prove the version of (3.46) for multiple closed sets. Indeed, we have the following theorem.

**Theorem 3.21.** Suppose $S$ is a Hausdorff space. Let $N \in \ast \mathbb{N}$ and let $P_N$ be as in (3.16). Let $(\ast \mathcal{P}(S), L_{P_N}(\ast \mathcal{B}(\mathcal{P}_r(S))), L_{P_N})$ be the associated Loeb space. Then for finitely many Borel subsets $B_1, \ldots, B_k$ of $\mathcal{P}_r(S)$, we have:

\[
\ast \int_{\mathcal{P}_r(S)} \mu(\ast B_1) \cdot \cdots \cdot \mu(\ast B_k)dP_N(\mu) \approx \int_{\mathcal{P}_r(S)} \mu(B_1) \cdot \cdots \cdot \mu(B_k)d\mathcal{P}_N(\mu), \tag{3.47}
\]

where $\mathcal{P}_N = L_{P_N} \circ \text{st}^{-1}$. 

The proof goes exactly the same way as that of Theorem 3.20, once we know that the set \{\nu \in \mathcal{P}_r(S) : \nu(B_1) \cdots \nu(B_k) > y\} is Borel measurable in \mathcal{P}_r(S) for all \(y \in [0,1]\). But this follows from the fact that a product of measurable functions is measurable (and that for each \(i \in [k]\), the function \(\nu \mapsto \nu(B_i)\) is measurable by Theorem 2.20).

Combining with Lemma 3.9, we can interject a \(+\)-integral in the approximate equation (3.47), which will be useful in our proof of de Finetti’s theorem in the next section. We state that as a corollary,

**Corollary 3.22.** Suppose \(S\) is a Hausdorff space. Let \(N \in \ast\mathbb{N}\) and let \(P_N\) be as in (3.16). Let \((\ast\mathcal{P}(S), L_{P_N}(\ast\mathcal{B}(\mathcal{P}_r(S))))\) be the associated Loeb space. Let \(\mathcal{P}_N = L_{P_N} \circ \ast^{-1}\), which is a Radon measure on \(\mathcal{P}_r(S)\). Then for finitely many Borel subsets \(B_1, \ldots, B_k\) of \(S\), we have:

\[
\ast \int_{\ast\Omega} \mu_{\omega,N}(B_1) \cdots \mu_{\omega,N}(B_k) d\ast\mu(\omega) \approx \int_{\mathcal{P}_r(S)} \mu(B_1) \cdots \mu(B_k) d\mathcal{P}_N(\mu). \tag{3.48}
\]

### 4. de Finetti–Hewitt–Savage theorem

#### 4.1. Uses of exchangeability and a generalization of Ressel’s Radon presentability.

The previous section built a theory of hyperfinite empirical measures arising out of any sequence of identically Radon distributed random variables taking values in a Hausdorff space. If we further require the random variables to be exchangeable, then the theory from Section 3 gives new tools to attack de Finetti style theorems in great generality. Let us first consider an exchangeable sequence of random variables taking values in any measurable space \(S\). We define hyperfinite empirical measures \(\mu_{\omega,N}\) in the same manner as in the previous section. If \(N > N\), then the joint distribution of any finite subcollection of the random variables is given by the expected values of hyperfinite empirical measures. This is proved in the next theorem, which is the main technical result that yields general forms of de Finetti’s theorem in view of Corollary 3.22.

**Theorem 4.1.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of \(S\)-valued exchangeable random variables, where \((S, \mathcal{S})\) is some measurable space. For each \(N > \mathbb{N}\) and \(\omega \in \ast\Omega\), define the internal probability measure \(\mu_{\omega,N}\) as follows:

\[
\mu_{\omega,N}(A) := \frac{\#\{i \in [N] : X_i(\omega) \in A\}}{n} \text{ for all } A \in \ast\mathcal{S}. \tag{4.1}
\]

Then we have:

\[
\ast\mathbb{P}(X_1 \in A_1, \ldots, X_k \in A_k) \approx \int_{\ast\Omega} \mu_{\omega,N}(A_1) \cdots \mu_{\omega,N}(A_k) d\ast\mu(\omega)
\text{ for all } k \in \mathbb{N} \text{ and } A_1, \ldots, A_k \in \ast\mathcal{S}. \tag{4.2}
\]

It should be pointed out that Theorem 4.1 may be viewed as a consequence of transferring Diaconis–Freedman’s finite, approximate version of de Finetti’s theorem [20, Theorem (13)] into the hyperfinite setting. We will provide two alternate proofs that underscore other ways of thinking about this result. The proof of
Theorem 4.1 in the main body of the paper uses a similar combinatorial construction as Diaconis–Freedman’s proof, with a key difference being that we can use inclusion-exclusion to give softer combinatorial arguments while still obtaining the same bounds. This proof does not use the hyperfiniteness of \( N \) in an essential way, and, as such, it can actually be thought of as a proof of the aforementioned result in Diaconis–Freedman (see (4.7), (4.10), (4.11), (4.12), and compare with [21, Theorem (13), p. 749]).

Our second proof of Theorem 4.1 is carried out in Appendix B. This proof illustrates an important explanatory advantage of stating Theorem 4.1 as a less quantitative version of Diaconis–Freedman’s result in the hyperfinite setting—such a statement is still strong enough to be sufficient in the proof of the infinitary de Finetti’s theorem, while the particular form of the statement ensures that it can be both predicted and understood by a reasoning based on Bayes’ theorem. This nicely ties in with the fact that de Finetti’s theorem is often interpreted as a foundational result for Bayesian statistics (see, for example, Savage [57, Section 3.7]; see also Orbanz and Roy [52] for a recent discussion in connection with the foundations of statistical modeling).

To better understand this idea, let us analyze (4.2) from the perspective of Bayes’ theorem. Suppose any two of the \( A_1, \ldots, A_k \) are either disjoint or equal, and let \( C_1, \ldots, C_n \) be the distinct sets appearing in the finite sequence \( A_1, \ldots, A_k \). In that case, writing the Cartesian product \( A_1 \times \ldots \times A_k \) as \( \vec{A} \) and the random vector \( (X_1, \ldots, X_k) \) as \( \vec{X} \), the internal Bayes’ theorem expansion (conditioning on the various possible values of the empirical sample means of the distinct sets \( C_1, \ldots, C_n \)) of the left side of (4.2) is the following:

\[
*P((X_1, \ldots, X_k) \in \vec{A}) = \sum_{(t_1, \ldots, t_n) \in [N]^n} *P(\vec{X} \in \vec{A}|\vec{Y} = (t_1, \ldots, t_n)) \cdot *P\left(\mu_{\cdot, N}(C_1) = \frac{t_1}{N}, \ldots, \mu_{\cdot, N}(C_n) = \frac{t_n}{N}\right).
\]  

(4.3)

In this case, assuming that the set \( C_i \) appears in the finite sequence \( A_1, \ldots, A_k \) with a frequency \( k_i \) (where \( i \in [n] \)), the right side of (4.2) can be written as the following hyperfinite sum by the (transfer of the) definition of expected values:

\[
\int_{\Omega} \mu_{\cdot, N}(A_1) \cdots \mu_{\cdot, N}(A_k) d*P(\omega) = \sum_{(t_1, \ldots, t_n) \in [N]^n} \left(\frac{t_1}{N}\right)^{k_1} \cdots \left(\frac{t_n}{N}\right)^{k_n} \cdot *P\left(\mu_{\cdot, N}(C_1) = \frac{t_1}{N}, \ldots, \mu_{\cdot, N}(C_n) = \frac{t_n}{N}\right).
\]

(4.4)

If \( t_1, \ldots, t_n > \mathbb{N} \) are such that the corresponding term in the internal sum (4.3) is nonzero, then the ratio of that term with the corresponding term on the right side of (4.4) can be shown to be infinitesimally close to one. By an application of underflow and the fact that the partial sums in (4.3) and (4.4) are both infinitesimals when \( t_1, \ldots, t_n \) are all bounded by a standard natural number, it can be shown that the two expansions (4.3) and (4.4) are infinitesimally close, proving (4.2) for any \( A_1, \ldots, A_k \) for which any two of them are either disjoint or equal. This was the
idea in the nonstandard proof of de Finetti’s theorem for exchangeable Bernoulli random variables in Alam [2]. Such an argument can then be modified to a proof of Theorem 4.1 by writing the event \( \{ X_1 \in A_1, \ldots, X_k \in A_k \} \) represented by arbitrary sets \( A_1, \ldots, A_k \in \mathcal{S} \) as a finite disjoint union of events represented by sets of the above type.

A conceptual benefit of this approach is that the idea of the proof is in some sense immediate after expressing the expansions (4.3) and (4.4). Indeed, the two expansions should be expected to be close to each other since the “majority” of the terms are very close to each other, while the rest add up to infinitesimals! While this is a quick way of understanding why Theorem 4.1 holds, the details of the term-by-term comparison between (4.3) and (4.4) may get computationally involved. We therefore present a shorter proof below that replaces the exact combinatorial formulas by simpler estimates using inclusion-exclusion. A complete proof based on the above Bayes’ theorem idea is included in Appendix B as an alternative.

Proof of Theorem 4.1. Let \( N > \mathbb{N} \) and \( (A_1, \ldots, A_k) \in *\mathcal{S}^k \) be a finite sequence of internal events. Consider the following equation obtained by rewriting the internal product of internal sums on the left as an internal sum of internal products by (the transfer of) distributivity:

\[
\prod_{i \in [k]} \left( \sum_{j \in [N]} \mathbb{1}_{A_i}(X_j) \right) = \sum_{(\ell_1, \ldots, \ell_k) \in [N]^k} \left( \prod_{i \in [k]} \mathbb{1}_{A_i}(X_{\ell_i}) \right).
\] (4.5)

We separate the terms in the sum on the right of (4.5) according to whether there is any repetition in \( (\ell_1, \ldots, \ell_k) \) or not. Let

\[
\mathcal{R} := \{ (\ell_1, \ldots, \ell_k) \in [N]^k : \ell_\alpha = \ell_\beta \text{ for some } \alpha \neq \beta \}.
\]

An exact value of \( \#(\mathcal{R}) \) can be found using the (internal) inclusion-exclusion principle. However, the following immediate combinatorial estimate will be sufficient for our needs (for each of the \( N \) numbers in \([N]\), there are at most \( \binom{k}{2} N^{k-2} \) elements of \([N]^k\) in which that number is repeated at least twice):

\[
\#(\mathcal{R}) \leq N \binom{k}{2} N^{k-2} = \left( \binom{k}{2} \right) N^{k-1}.
\] (4.6)

Dividing both sides of (4.5) by \( N^k \) and noting that \( \frac{1}{N} \sum_{j \in [N]} \mathbb{1}_{A_i}(X_j) \) is the same as \( \mu_{\cdot , N}(A_i) \) for each \( i \in [k] \), we obtain the following:

\[
\prod_{i \in [k]} \mu_{\cdot , N}(A_i) = \frac{1}{N^k} \sum_{\ell_1, \ldots, \ell_k \in \mathcal{R}} \left( \prod_{i \in [k]} \mathbb{1}_{A_i}(X_{\ell_i}) \right) + \frac{1}{N^k} \sum_{\ell_1, \ldots, \ell_k \in [N]^k \setminus \mathcal{R}} \left( \prod_{i \in [k]} \mathbb{1}_{A_i}(X_{\ell_i}) \right).
\]
Taking expected values and using (4.6) thus yields:

\[
0 \leq \mathbb{E} \left( \prod_{i \in [k]} \mu_{\cdot, N}(A_i) \right) - \mathbb{E} \left[ \frac{1}{N^k} \sum_{(\ell_1, \ldots, \ell_k) \in [N]^k \setminus \mathcal{R}} \left( \prod_{i \in [k]} 1_{A_i}(X_{\ell_i}) \right) \right]
\]

\[
= -\mathbb{E} \left[ \frac{1}{N^k} \sum_{(\ell_1, \ldots, \ell_k) \in \mathcal{R}} \left( \prod_{i \in [k]} 1_{A_i}(X_{\ell_i}) \right) \right]
\]

\[
\leq \frac{\#(\mathcal{R})}{N^k}
\]

\[
\leq \frac{\binom{k}{2} N^{k-1}}{N^k}
\]

\[
= \frac{\binom{k}{2}}{N}
\approx 0.
\]

As a consequence of (4.8), and using the linearity of expectation, we thus obtain the following:

\[
\mathbb{E} \left( \prod_{i \in [k]} \mu_{\cdot, N}(A_i) \right) \approx \frac{1}{N^k} \sum_{(\ell_1, \ldots, \ell_k) \in [N]^k \setminus \mathcal{R}} \mathbb{E} \left( \prod_{i \in [k]} 1_{A_i}(X_{\ell_i}) \right).
\]

(4.9)

By exchangeability, we also have the following:

\[
\mathbb{E} \left( \prod_{i \in [k]} 1_{A_i}(X_{\ell_i}) \right) = \mathbb{P}(X_{\ell_1} \in A_1, \ldots, X_{\ell_k} \in A_k)
\]

\[
= \mathbb{P}(X_1 \in A_1, \ldots, X_k \in A_k) \text{ for all } (\ell_1, \ldots, \ell_k) \in [N]^k \setminus \mathcal{R},
\]

(4.10)

which allows us to conclude the following from (4.9):

\[
\mathbb{E} \left( \prod_{i \in [k]} \mu_{\cdot, N}(A_i) \right) \approx \frac{\#([N]^{k \setminus \mathcal{R}}) \mathbb{P}(X_1 \in A_1, \ldots, X_k \in A_k)}{N^k}.
\]

(4.11)

From (4.6), it is clear that

\[
1 > \frac{\#([N]^{k \setminus \mathcal{R}})}{N^k} \geq \frac{N^k - \binom{k}{2} N^{k-1}}{N^k} = 1 - \frac{\binom{k}{2}}{N} \approx 1,
\]

so that

\[
\frac{\#([N]^{k \setminus \mathcal{R}})}{N^k} \approx 1.
\]

(4.12)

(4.13)

Using (4.13) in (4.11) yields the following:

\[
\mathbb{E} \left( \prod_{i \in [k]} \mu_{\cdot, N}(A_i) \right) \approx \mathbb{P}(X_1 \in A_1, \ldots, X_k \in A_k),
\]

thus completing the proof. □
We are in a position to prove the following generalization of Ressel [55, Theorem 3, p. 906].

**Theorem 4.2.** Let $S$ be a Hausdorff topological space, with $\mathcal{B}(S)$ denoting its Borel sigma algebra. Let $\mathfrak{P}_r(S)$ be the space of all Radon probability measures on $S$ and $\mathcal{B}(\mathfrak{P}_r(S))$ be the Borel sigma algebra on $\mathfrak{P}_r(S)$ with respect to the $A$-topology on $\mathfrak{P}_r(S)$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $X_1, X_2, \ldots$ be a sequence of exchangeable $S$-valued random variables such that the common distribution of the $X_i$ is Radon on $S$. Then there exists a unique probability measure $\mathcal{P}$ on $(\mathfrak{P}_r(S), \mathcal{B}(\mathfrak{P}_r(S)))$ such that the following holds for all $k \in \mathbb{N}$:

$$\mathbb{P}(X_1 \in B_1, \ldots, X_k \in B_k) = \int_{\mathfrak{P}_r(S)} \mu(B_1) \cdot \ldots \cdot \mu(B_k) d\mathcal{P}(\mu)$$

for all $B_1, \ldots, B_k \in \mathcal{B}(S)$. \hspace{1cm} (4.14)

**Proof.** Let $N > \mathbb{N}$ and let $P_N$ be as in (3.16). Let $\mathcal{P}$ be $L P_N \circ \text{st}^{-1}$, which is a Radon probability measure on $\mathfrak{P}_r(S)$ by Theorem 3.12. The right side of (4.14) is the same as the right side of (3.48), while the left sides of the two equations are infinitesimally close in view of Theorem 4.1. This shows the existence of a measure $\mathcal{P} \in \mathfrak{P}_r(\mathfrak{P}_r(S))$ satisfying (4.14). The uniqueness follows from Theorem 2.38. \hspace{1cm} $\square$

We end this subsection with some immediate remarks on the proof of Theorem 4.2.

**Remark 4.3.** Note that the proof of Theorem 4.2 showed that $\mathcal{P}$ could be taken as $L P_N \circ \text{st}^{-1}$ for any $N > \mathbb{N}$, and all of these would have given the same (Radon) measure on $\mathfrak{P}_r(S)$. Following Theorem 3.12 this shows that, in the nonstandard extension $\mathfrak{P}(\mathfrak{P}_r(S))$ of $\mathfrak{P}(\mathfrak{P}_r(S))$, the internal measures $P_N$ are nearstandard to $\mathcal{P}$ for all $N > \mathbb{N}$. From the nonstandard characterization of limits in topological spaces, it thus follows that $\mathcal{P}$ is a limit of the sequence $(P_n)_{n \in \mathbb{N}}$ in the $A$-topology on $\mathfrak{P}(\mathfrak{P}_r(S))$ (and hence in the weak topology as well, since the $A$-topology is finer than the weak topology), where for each $n \in \mathbb{N}$, the probability measure $P_n$ on $(\mathfrak{P}_r(S), \mathcal{B}(\mathfrak{P}_r(S)))$ is defined as follows (this definition of $(P_n)_{n \in \mathbb{N}}$ ensures, by (3.16) and transfer, that $P_N$ is the $N^\text{th}$ term in the nonstandard extension of the sequence $(P_n)_{n \in \mathbb{N}}$ for each $N > \mathbb{N}$):

$$P_n(B) := \mathbb{P} (\{\omega \in \Omega : \mu_{\omega,n} \in B\}) = \mathbb{P} (\hat{\mu}_{n,-1}(B)) \text{ for all } B \in \mathcal{B}(\mathfrak{P}_r(S)). \hspace{1cm} (4.15)$$

Thus our proof shows that the canonical (pushforward) measure on $\mathcal{B}(\mathfrak{P}_r(S))$ induced by the empirical distribution of the first $n$ random variables does converge (as $n \to \infty$) to a (Radon) measure on $\mathcal{B}(\mathfrak{P}_r(S))$ which witnesses the truth of Radon presentability. This gives a different (standard) way to understand the measure $\mathcal{P}$ in Theorem 4.2, and also connects the proof to the heuristics from statistics described in Section 1.2.

**Remark 4.4.** While Remark 4.3 shows that the measure $\mathcal{P}$ in Theorem 4.2 can be thought of as a limit of the sequence $(P_n)_{n \in \mathbb{N}}$, we cannot say that it is the limit of this sequence (as the space $\mathfrak{P}(\mathfrak{P}_r(S))$, where this sequence lives, may not be Hausdorff). While this was not intended, the use of nonstandard analysis allowed
us to canonically find a useful limit point of this sequence using the machinery built in Theorem 2.12 and Theorem 2.46. The usefulness of nonstandard analysis in this context is thus highlighted by the observation that without invoking this machinery, it is not clear why there should be a Radon limit of this sequence at all.

**Remark 4.5.** Following Lemma 2.25 (thinking of $T'$ as $\mathcal{P}_r(S)$ and $T$ as $\mathcal{P}(S)$), we can canonically get a sequence $(P'_n)_{n\in\mathbb{N}}$ in $\mathcal{P}(\mathcal{P}(S))$ that can be seen to have $\mathcal{P}' \in \mathcal{P}(\mathcal{P}(S))$ as a limit point. We make this way of thinking precise when we next prove a generalization of the classical version of de Finetti’s theorem (as opposed to Ressel’s “Radon presentable” version).

### 4.2. Generalizing classical de Finetti’s theorem

While Theorem 4.2 is already a generalization of de Finetti’s theorem, its conclusion is slightly different from classical statements of de Finetti’s theorem that postulate the existence of a probability measure on the space of all probability measures (as opposed to a Radon measure on the space of all Radon measures). This can be easily remedied using ideas from Lemma 2.24 and Lemma 2.25, but at the cost of uniqueness. By Theorem 2.42, we still have uniqueness if we focus on probability measures on the smallest sigma algebra on $\mathcal{P}(S)$ that makes all evaluation functions measurable. As pointed out in Theorem 2.42, this is the same as uniqueness for Borel measures on $\mathcal{P}(S)$ if $S$ is a separable metric space. We prove this generalization next. In fact, we prove a slightly stronger result that has the above conclusion for any sequence $(X_n)_{n\in\mathbb{N}}$ of random variables satisfying (4.14).

**Theorem 4.6.** Let $S$ be a Hausdorff topological space, with $\mathcal{B}(S)$ denoting its Borel sigma algebra. Let $\mathcal{P}(S)$ (respectively $\mathcal{P}_r(S)$) be the space of all Borel probability measures (respectively Radon probability measures) on $S$, and let $\mathcal{B}(\mathcal{P}(S))$ (respectively $\mathcal{B}(\mathcal{P}_r(S))$) be the Borel sigma algebra on $\mathcal{P}(S)$ (respectively $\mathcal{P}_r(S)$) with respect to the $A$-topology on $\mathcal{P}(S)$ (respectively $\mathcal{P}_r(S)$). Let $\mathcal{C}(\mathcal{P}(S))$ be the smallest sigma algebra on $\mathcal{P}(S)$ such that for any $B \in \mathcal{B}(S)$, the evaluation function $e_B: \mathcal{P}(S) \to \mathbb{R}$, defined by $e_B(\nu) = \nu(B)$, is measurable. Also let $\mathcal{B}_w(\mathcal{P}(S))$ be the Borel sigma algebra induced by the weak topology on $\mathcal{P}(S)$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $X_1, X_2, \ldots$ be a sequence of $S$-valued random variables. Suppose that there exists a unique probability measure $\mathcal{P}$ on $(\mathcal{P}_r(S), \mathcal{B}(\mathcal{P}_r(S)))$ such that the following holds for all $k \in \mathbb{N}$:

$$
\mathbb{P}(X_1 \in B_1, \ldots, X_k \in B_k) = \int_{\mathcal{P}_r(S)} \mu(B_1) \cdots \mu(B_k) d\mathcal{P}(\mu)
$$

for all $B_1, \ldots, B_k \in \mathcal{B}(S)$. (4.16) Then there exists a probability measure $\mathcal{P}$ on $(\mathcal{P}(S), \mathcal{B}(\mathcal{P}(S)))$ such that the following holds for all $k \in \mathbb{N}$:

$$
\mathbb{P}(X_1 \in B_1, \ldots, X_k \in B_k) = \int_{\mathcal{P}(S)} \mu(B_1) \cdots \mu(B_k) d\mathcal{P}(\mu)
$$

for all $B_1, \ldots, B_k \in \mathcal{B}(S)$. (4.17)
Also, there is a unique probability measure $\mathcal{Q}_c$ on $(\mathfrak{P}(S), \mathcal{C}(\mathfrak{P}(S)))$ satisfying the following for all $k \in \mathbb{N}$:

$$\mathbb{P}(X_1 \in B_1, \ldots, X_k \in B_k) = \int_{\mathfrak{P}(S)} \mu(B_1) \cdot \cdots \cdot \mu(B_k) d\mathcal{Q}_c(\mu)$$  \hspace{1cm} \text{for all } B_1, \ldots, B_k \in \mathcal{B}(S).  \hspace{1cm} (4.18)$$

Furthermore, if $S$ is a separable metric space, then $\mathcal{C}(\mathfrak{P}(S)) = \mathcal{B}_w(\mathfrak{P}(S))$, so that there is a unique probability measure $\mathcal{Q}_c$ on $(\mathfrak{P}(S), \mathcal{B}_w(\mathfrak{P}(S)))$ satisfying (4.18).

Proof. Let $\mathcal{P} \in \mathfrak{P}(\mathfrak{P}_1(S))$ be the (Radon) measure obtained in (4.16). Define $\mathcal{Q}: \mathcal{B}(\mathfrak{P}(S)) \to [0,1]$ as follows:

$$\mathcal{Q}(B) := \mathcal{P}(B \cap \mathfrak{P}_1(S)) \text{ for all } B \in \mathcal{B}(\mathfrak{P}(S)).$$  \hspace{1cm} (4.19)$$

By Lemma 2.25, this defines a probability measure on $(\mathfrak{P}(S), \mathcal{B}(\mathfrak{P}(S)))$ (in fact, $\mathcal{P}$ is the same as $\mathcal{P}^*$ in the terminology of Lemma 2.25). Equation (4.17) now follows from (4.16) and (2.31) (within Lemma 2.25).

Call $\mathcal{Q}_c$ the restriction of $\mathcal{P}$ to $\mathcal{C}(\mathfrak{P}(S)) \subseteq \mathcal{B}(\mathfrak{P}(S))$. Note that for each $k \in \mathbb{N}$ and $B_1, \ldots, B_k \in \mathcal{B}(S)$, the map $\mu \mapsto \mu(B_1) \cdot \cdots \cdot \mu(B_k)$ is $\mathcal{C}(\mathfrak{P}(S))$ measurable as well, so that we have the following:

$$\int_{\mathfrak{P}(S)} \mu(B_1) \cdot \cdots \cdot \mu(B_k) d\mathcal{Q}_c(\mu) = \int_{[0,1]} \mathcal{Q}_c[\mu(B_1) \cdot \cdots \cdot \mu(B_k) > y] d\lambda(y)$$  \hspace{1cm} (4.18)$$

$$= \int_{[0,1]} \mathcal{Q}[\mu(B_1) \cdot \cdots \cdot \mu(B_k) > y] d\lambda(y)$$

$$= \int_{\mathfrak{P}(S)} \mu(B_1) \cdot \cdots \cdot \mu(B_k) d\mathcal{Q}_c(\mu).$$

Together with Theorem 2.42, this shows that there is a unique probability measure $\mathcal{Q}_c$ on $(\mathfrak{P}(S), \mathcal{C}(\mathfrak{P}(S)))$ satisfying (4.21). Theorem 2.41(iii) now completes the proof. \hspace{1cm} \square

In view of Theorem 4.2, the above result immediately yields our main theorem.

**Theorem 4.7.** Let $S$ be a Hausdorff topological space, with $\mathcal{B}(S)$ denoting its Borel sigma algebra. Let $\mathfrak{P}(S)$ be the space of all Borel probability measures on $S$ and $\mathcal{B}(\mathfrak{P}(S))$ be the Borel sigma algebra on $\mathfrak{P}(S)$ with respect to the $A$-topology on $\mathfrak{P}(S)$. Let $\mathcal{C}(\mathfrak{P}(S))$ be the smallest sigma algebra on $\mathfrak{P}(S)$ such that for any $B \in \mathcal{B}(S)$, the evaluation function $e_B: \mathfrak{P}(S) \to \mathbb{R}$, defined by $e_B(\nu) = \nu(B)$, is measurable. Also let $\mathcal{B}_w(\mathfrak{P}(S))$ be the Borel sigma algebra induced by the weak topology on $\mathfrak{P}(S)$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $X_1, X_2, \ldots$ be a sequence of exchangeable $S$-valued random variables such that the common distribution of the $X_i$ is Radon on $S$. Then there exists a probability measure $\mathcal{Q}$ on $(\mathfrak{P}(S), \mathcal{B}(\mathfrak{P}(S)))$ such that the
following holds for all $k \in \mathbb{N}$:

$$\mathbb{P}(X_1 \in B_1, \ldots, X_k \in B_k) = \int_{\Psi(S)} \mu(B_1) \cdot \ldots \cdot \mu(B_k) d\mathcal{P}(\mu)$$

for all $B_1, \ldots, B_k \in \mathcal{B}(S)$.  \hspace{1cm} (4.20)

There is a unique probability measure $\mathcal{P}_c$ on $(\Psi(S), \mathcal{C}(\Psi(S)))$ satisfying the following for all $k \in \mathbb{N}$:

$$\mathbb{P}(X_1 \in B_1, \ldots, X_k \in B_k) = \int_{\Psi(S)} \mu(B_1) \cdot \ldots \cdot \mu(B_k) d\mathcal{P}_c(\mu)$$

for all $B_1, \ldots, B_k \in \mathcal{B}(S)$.  \hspace{1cm} (4.21)

Furthermore, if $S$ is a separable metric space, then $\mathcal{C}(\Psi(S)) = \mathcal{B}_w(\Psi(S))$, so that there is a unique probability measure $\mathcal{P}_c$ on $(\Psi(S), \mathcal{B}_w(\Psi(S)))$ satisfying (4.21).

As explained in Remark 3.16, our proof of Theorem 4.7 did not use the full strength of the assumption that the common distribution of the exchangeable random variables $X_1, X_2, \ldots$ is Radon. The same proof would work if we assumed this common distribution to be tight and outer regular on compact subsets of $S$ (indeed, the proof of Theorem 4.2 would go through under these assumptions, while the rest of the steps in our proof of Theorem 4.7 are consequences of the conclusion of Theorem 4.2).

In practice, a natural situation in which the latter condition always holds is when $S$ is a Hausdorff $G_\delta$ space—that is, when all closed subsets of $S$ are $G_\delta$ sets (as any finite Borel measure on such a space is actually outer regular on all closed subsets, and in particular on all compact subsets).

In the point-set topology literature, $G_\delta$ spaces typically arise in discussions on perfectly normal spaces. Following are some commonly studied examples of spaces that are perfectly normal (as described in Gartside [32, p. 274], these are actually examples of stratifiable spaces, which are automatically perfectly normal):

(i) All CW complexes are perfectly normal. See Lundell and Weingram [50, Proposition 4.3, p. 55].

(ii) All Lašnev spaces (that is, all continuous closed images of metric spaces, where a continuous map $g: T \to T'$ is called closed if $g(F)$ is closed in $T'$ whenever $F$ is closed in $T$) are perfectly normal. This, in particular, includes all metric spaces. See Slaughter [59] for more details.

(iii) If $T$ is a compact-covering image of a Polish space (here, a continuous map $f: T \to T'$ is called a compact-covering if every compact subset of $T'$ is the image of a compact subset of $T$; see Michael–Nagami 1973 and the references therein for more details on compact-covering images of metric spaces), then the space $C_k(T)$ of continuous real-valued functions on $T$ (equipped with the compact-open topology) is perfectly normal. In particular, this implies that $C_k(T)$ is completely normal whenever $T$ is a Polish space. See Gartside and Reznichenko [Theorem 34, p. 111][31].

The above discussion shows that we could have stated Theorem 4.7 for any exchangeable sequence of tightly distributed random variables taking values in a
Hausdorff state space that is either a CW complex, a Lašnev space, or a space of continuous real-valued functions on a Polish space (with the compact-open topology). This, however, would not be a more general statement than that of Theorem 4.7, as it is easy to see that any tight finite measure on a Hausdorff $G_δ$ space is automatically Radon. It is still instructive to keep in mind these settings where one only needs to verify tightness of the common distribution in order for de Finetti–Hewitt–Savage theorem to hold.

Remark 4.8. Dubins and Freedman [24] had constructed an exchangeable sequence of random variables taking values in a separable metric space for which the conclusion of de Finetti’s theorem does not hold. An indirect consequence of the above discussion is that any random variable in such an example must not have a tight distribution.

Remark 4.9. We emphasize again that besides tightness of the underlying common distribution, one only needs outer regularity on compact subsets in order for de Finetti-Hewitt-Savage theorem to hold. Though we have not been able to find any natural examples of Hausdorff spaces in which all compact subsets (but not all closed subsets) are $G_δ$ sets, such spaces (if they exist) might yield more classes of examples where de Finetti–Hewitt–Savage theorem holds for any exchangeable sequence of tightly distributed random variables.

Note that all finite Borel measures on any $\sigma$-compact space are tight. Combined with the above examples of perfectly normal spaces, this gives us classes of state spaces for which de Finetti–Hewitt–Savage theorem holds unconditionally (namely, any $\sigma$-compact perfectly normal space would be an example). While instructive from the point of view of examples, this is not surprising as such spaces are also examples of Radon spaces (that is, spaces on which every finite Borel measure is Radon), so that Theorem 4.7 automatically holds for any exchangeable sequence of random variables on such state spaces. Other examples of Radon spaces are Polish spaces, which is the setting for modern treatments of de Finetti’s theorem. In this sense, Theorem 4.7 includes and generalizes the currently known versions of de Finetti’s theorem for sequences of Borel measurable exchangeable random variables taking values in a Hausdorff state space. We finish this subsection by recording the observation that Theorem 4.7 theorem holds unconditionally for any Radon state space.

Corollary 4.10. Let $S$ be a Radon space. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $X_1, X_2, \ldots$ be a sequence of exchangeable $S$-valued random variables. Then there exists a probability measure $\mathcal{Q}$ on the space $(\mathcal{P}(S), \mathcal{B}(\mathcal{P}(S)))$ such that (4.20) holds. Also, there is a unique probability measure $\mathcal{Q}_c$ on $(\mathcal{P}(S), \mathcal{C}(\mathcal{P}(S)))$ such that (4.21) holds.

4.3. Comments and possible future work. Starting from a result on an exchangeable sequence of $\{0, 1\}$-valued random variables, de Finetti’s theorem has had generalizations in several directions. While the classical form of de Finetti’s theorem was known to be true for Polish spaces, Dubins and Freedman [24] had shown that some form of topological condition on the state space is necessary. Theorem 4.7 shows that we actually do not need any topological conditions on the state space besides Hausdorffness as long as we focus on exchangeable sequences of Radon distributed random variables (by the discussion following Theorem 4.7, we
actually only need to assume that the common distribution of the random variables is tight and outer regular on compact subsets).

Since properties of the common distribution were crucially used in our proof, the question of the most general state space under which de Finetti’s theorem holds (without any assumptions on the common distribution) is quite natural. Corollary 4.10 provides some answers (in the form of Radon spaces), but the leap from Theorem 4.7 to Corollary 4.10 is rather trivial. It would be instructive to investigate if there are other classes of state spaces for which de Finetti’s theorem holds unconditionally. Along these lines, it would also be instructive to find examples of state spaces for which tightness of the underlying common distribution is sufficient for an exchangeable sequence of random variables to be presentable. Radon spaces are again trivial examples, while Hausdorff $G_δ$ spaces (see examples in (i), (ii), and (iii)) provide some non-trivial examples. Remark 4.9 provides a potential strategy for finding more examples, though carrying out this project seems to be beyond the scope of the current paper.

There are other formulations of de Finetti’s theorem that we have completely ignored in the present treatment. For example, a useful formulation says that an infinite sequence of exchangeable random variables is conditionally independent with respect to certain sigma algebras. See Kingman [44] for a description of such a version of de Finetti’s theorem along with some applications.

Another setting in which de Finetti’s theorem is traditionally generalized is the setting of exchangeable arrays, with the main result in that setting sometimes called the Aldous–Hoover–Kallenberg representation theorem (See Aldous [5, 6], Hoover [39, 38], and Kallenberg [40, 41]). This is a highly fruitful setting from the point of view of both theoretical and practical applications. Indeed, it has been recently used in graph limits, random graphs, and ergodic theory (see Diaconis and Janson [22], and also Austin [10]) on one hand, and statistical network modeling (see Caron and Fox [15], as well as Veitch and Roy [67]) on the other. While we did not cover exchangeable arrays, an obvious future direction is to try to see if similar techniques allow us to treat that setting as well. In view of Hoover’s existing work based on ultraproducts in this setting, it seems likely that there are areas that would benefit from a more concerted nonstandard analytic treatment.

Finally, there are existing generalizations of de Finetti’s theorem for random variables indexed by continuous time as well (see Bühlmann [14], Freedman [29], as well as Accardi and Lu [1]), which is yet another area where a nonstandard analytic treatment using hyperfinite time intervals could be useful.

Appendix A. Concluding the theorem of Hewitt and Savage from the theorem of Ressel

In this appendix, we prove that the theorem of Ressel showing Radon presentability of completely regular Hausdorff spaces ([55, Theorem 3, p. 906]) implies the theorem of Hewitt and Savage on the presentability of the Baire sigma algebra of compact Hausdorff spaces ([37, Theorem 7.2, p. 483]). Since we will have occasion to talk about the presentability of Baire sigma algebras and Radon presentability in the same context, it is desirable to reduce the risk of confusion by introducing more precise notation for the relevant sigma algebras.
Notation A.1. For a Hausdorff space $S$, let $\mathcal{B}_a(S)$ denote its Baire sigma algebra, the smallest sigma algebra with respect to which all continuous functions $f : S \to \mathbb{R}$ are measurable. Let $\mathcal{B}(S)$ denote its Borel sigma algebra, the smallest sigma algebra containing all open subsets of $S$ (it is clear that $\mathcal{B}_a(S) \subseteq \mathcal{B}(S)$). Let $\mathcal{P}_r(S)$ denote the set of all Radon probability measures on $S$, and let $\mathcal{P}_{Ba}(S)$ denote the set of all Baire probability measures on $S$. Let $C(\mathcal{P}_r(S))$ be the smallest sigma algebra on $\mathcal{P}_r(S)$ that makes all maps of the form $\mu \mapsto \mu(B)$ measurable, where $B \in \mathcal{B}(S)$. Let $C(\mathcal{P}_{Ba}(S))$ be the smallest sigma algebra on $\mathcal{P}_{Ba}(S)$ that makes all maps of the form $\mu \mapsto \mu(A)$ measurable, where $A \in \mathcal{B}_a(S)$.

Note that any compact Hausdorff space is normal (see, for example, Kelley [43, Theorem 9, chapter 5]), and in particular completely regular. The key idea in going from Ressel’s result to that of Hewitt–Savage is that on any completely regular Hausdorff space, a tight Baire measure has a unique extension to a Radon measure (see Bogachev [12, Theorem 7.3.3, p. 81, vol. 2]). In particular, since every Baire measure on a $\sigma$-compact space is tight, it follows that every Baire measure on a completely regular $\sigma$-compact Hausdorff space admits a unique extension to a Radon measure on that space. See Bogachev [12, Corollary 7.3.4, p. 81, vol. 2] for this result. Bogachev also has a formula for this unique extension on [12, p. 78, vol. 2]. We record these facts as a lemma.

Lemma A.2. Let $S$ be a completely regular $\sigma$-compact Hausdorff space. For a subset $A \subseteq S$, let $\tau_A(S)$ denote the collection of those open subsets of $S$ that contain $A$. For every $\mu \in \mathcal{P}_{Ba}(S)$, there is a unique element $\hat{\mu} \in \mathcal{P}_r(S)$ such that $\hat{\mu}(A) = \mu(A)$ for all $A \in \mathcal{B}_a(S)$. Furthermore, $\hat{\mu}$ is precisely given by the following formula:

$$\hat{\mu}(B) = \inf_{U \in \tau_B(S)} \sup_{A \subseteq U} \mu(A).$$  \hfill (A.1)

As a consequence, we obtain the following lemma.

Lemma A.3. Let $S$ be a completely regular $\sigma$-compact Hausdorff space. Consider the map $\hat{} : \mathcal{P}_{Ba}(S) \to \mathcal{P}_r(S)$ defined by $\hat{\mu} = \hat{\mu}$ for all $\mu \in \mathcal{P}_{Ba}(S)$ (where $\hat{\mu}$ is as in (A.1)). Then $\hat{}$ is a bijection.

Furthermore, for a set $A \in C(\mathcal{P}_{Ba}(S))$, define $\hat{A}$ to be its image under $\hat{}$ (thus $\hat{A} := \{\hat{\mu} : \mu \in A\}$). Then $\hat{A} \in C(\mathcal{P}_r(S))$ for all $A \in C(\mathcal{P}_{Ba}(S))$.

Proof. If $\mu$ and $\nu$ are distinct elements of $\mathcal{P}_{Ba}(S)$, then there exists an $A \in \mathcal{B}_a(S)$ such that $\mu(A) \neq \nu(A)$, which implies $\hat{\mu}(A) \neq \hat{\nu}(B)$, so that $\hat{\mu} \neq \hat{\nu}$. Thus $\hat{}$ is an injection. That it is also a surjection follows from the fact that for any $\mu \in \mathcal{P}_r(S)$, its restriction $\hat{\mu}_{|\mathcal{B}_a(S)}$ to the Baire sigma algebra is a Baire measure that has a unique Radon extension by Lemma A.2, so that it must be the case that $\mu = \hat{\mu}_{|\mathcal{B}_a(S)}$ for all $\mu \in \mathcal{P}_r(S)$.

Consider the collection $\mathcal{G}$ of sets $A \in C(\mathcal{P}_{Ba}(S))$ for which $\hat{A}$ is an element of $C(\mathcal{P}_r(S))$, that is,

$$\mathcal{G} := \{A \in C(\mathcal{P}_{Ba}(S)) : \hat{A} \in C(\mathcal{P}_r(S))\}. \hfill (A.3)$$
We want to show that $G$ equals $C(\mathcal{P}_B(S))$. It is not very difficult to see that for any collection $(A_n)_{n \in \mathbb{N}} \subseteq C(\mathcal{P}_B(S))$, we have the following:

$$\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} \hat{A}_n.$$ 

Hence, by the fact that $C(\mathcal{P}_r(S))$ is a sigma algebra, it follows that $G$ is closed under countable unions. Furthermore, if $A \in C(\mathcal{P}_B(S))$, then we have the following (the inclusion from left to right follows from the injectivity of $\hat{\cdot}$, while the inclusion from right to left follows from the fact that $\hat{\cdot}$ is a bijection):

$$\overline{\mathcal{P}_B(S)} \setminus A = \mathcal{P}_r(S) \setminus \hat{A}. \quad (A.4)$$

This shows that $G$ is closed under complements as well. Since $\emptyset \in G$, it thus follows that $G$ is a sigma algebra. Thus by Dynkin’s $\pi$-$\lambda$ theorem, it suffices to show that $G$ contains a $\pi$-system (that is, a collection of sets that is closed under finite intersections) that generates $C(\mathcal{P}_B(S))$. A convenient $\pi$-system of that type is the following (that this is a $\pi$-system is trivial, and the fact that the smallest sigma algebra containing it coincides with $C(\mathcal{P}_B(S))$ follows from the fact that any map on $\mathcal{P}_B(S)$ of the type $\mu \mapsto \mu(A)$ for some $A \in \mathcal{P}_B(S)$ is measurable on the former sigma algebra):

$$\mathcal{A} := \{ \mathcal{A}^{A_1,\ldots,A_n}_{B_1,\ldots,B_n} : n \in \mathbb{N}, A_1,\ldots,A_n \in \mathcal{B}_a(S) \text{ and } B_1,\ldots,B_n \in \mathcal{B}(\mathbb{R}) \}, \quad (A.5)$$

where for any $n \in \mathbb{N}$, $A_1,\ldots,A_n \in \mathcal{B}_a(S)$ and $B_1,\ldots,B_n \in \mathcal{B}(\mathbb{R})$, the set $\mathcal{A}^{A_1,\ldots,A_n}_{B_1,\ldots,B_n}$ is defined as follows:

$$\mathcal{A}^{A_1,\ldots,A_n}_{B_1,\ldots,B_n} := \{ \mu \in \mathcal{P}_B(S) : \mu(A_1) \in B_1,\ldots,\mu(A_n) \in B_n \}. \quad (A.6)$$

For $n \in \mathbb{N}$, consider the sets $A_1,\ldots,A_n \in \mathcal{B}(S)$ and $B_1,\ldots,B_n \in \mathcal{B}(\mathbb{R})$. Define the collection $\mathcal{B}^{A_1,\ldots,A_n}_{B_1,\ldots,B_n}$ as follows:

$$\mathcal{B}^{A_1,\ldots,A_n}_{B_1,\ldots,B_n} := \{ \mu \in \mathcal{P}_r(S) : \mu(A_1) \in B_1,\ldots,\mu(A_n) \in B_n \} \in C(\mathcal{P}_r(S)). \quad (A.7)$$

It thus suffices to show the following claim.

**Claim A.4.** We have $\mathcal{A}^{A_1,\ldots,A_n}_{B_1,\ldots,B_n} = \mathcal{B}^{A_1,\ldots,A_n}_{B_1,\ldots,B_n}$ for all $A_1,\ldots,A_n \in \mathcal{B}_a(S)$ and $B_1,\ldots,B_n \in \mathcal{B}(\mathbb{R})$.

**Proof of Claim A.4.** Note that for any $A,B \in C(\mathcal{P}_B(S))$, we have the following (the inclusion from left to right is trivial, while the inclusion from right to left follows from the injectivity of the map $\hat{\cdot}$):

$$\overline{A \cap B} = \overline{A} \cap \overline{B}.$$ 

Since $\mathcal{A}^{A_1,\ldots,A_n}_{B_1,\ldots,B_n} = \bigcap_{i \in [n]} \mathcal{A}^{A_i}_{B_i}$ and $\mathcal{B}^{A_1,\ldots,A_n}_{B_1,\ldots,B_n} = \bigcap_{i \in [n]} \mathcal{B}^{A_i}_{B_i}$, it suffices to show the following set equality:

$$\overline{\mathcal{A}_B^A} = \overline{\mathcal{B}_B^A} \text{ for any } B \in \mathcal{B}(\mathbb{R}) \text{ and } A \in \mathcal{B}_a(S). \quad (A.8)$$

Toward that end, let $B \in \mathcal{B}(\mathbb{R})$ and $A \in \mathcal{B}_a(S)$. If $\mu \in \mathcal{A}_B^A$, then we have $\hat{\mu}(A) = \mu(A) \in B$, so that $\hat{\mu} \in \mathcal{B}_B^A$. Thus the left side of (A.8) is contained in the right side of (A.8). Conversely, if $\mu \in \mathcal{B}_B^A$, then $\mu = \mu|_{\mathcal{B}_a(S)}$, where $\mu|_{\mathcal{B}_a(S)} \in \mathcal{A}_B^A$, completing the proof.

$\square$
As a corollary, we now have a way to define a natural measure on $C(P_{\text{Ba}}(S))$ corresponding to any measure on $C(P_r(S))$ in the case when $S$ is completely regular, Hausdorff, and $\sigma$-compact.

**Corollary A.5.** Let $S$ be a completely regular $\sigma$-compact Hausdorff space. Let $^\sim : P_{\text{Ba}}(S) \to P_r(S)$ be as in Lemma A.3. Suppose $\mathcal{P}$ is a probability measure on $C(P_r(S))$. Define a map $\hat{\mathcal{P}} : C(P_{\text{Ba}}(S)) \to [0, 1]$ as follows:

$$\hat{\mathcal{P}}(A) := \mathcal{P}(^\sim A) \quad \text{for all } A \in C(P_{\text{Ba}}(S)).$$  \hspace{1cm} (A.9)

Then $\hat{\mathcal{P}}$ is a probability measure on $C(P_{\text{Ba}}(S))$.

**Proof.** The fact that $\hat{\mathcal{P}}$ is well-defined follows from Lemma A.3. Its countable additivity follows from that of $\mathcal{P}$ and the fact that the map $^\sim$ is injective. Finally, the fact that $\hat{\mathcal{P}}(P_{\text{Ba}}(S)) = 1$ follows from the surjectivity of the map $^\sim$ (as we have $P_{\text{Ba}}(S) = P_r(S)$, whose measure with respect to $\mathcal{P}$ is one). \hfill $\Box$

We are now able to show that the main result in Hewitt–Savage [37] is a direct consequence of the theorem of Ressel on the Radon presentability of completely regular Hausdorff spaces.

**Theorem A.6** (Hewitt–Savage [37, Theorem 7.2, p. 483]). Suppose all completely regular spaces are Radon presentable as in Definition 1.11. Let $S$ be a compact Hausdorff space equipped with its Baire sigma algebra $B_a(S)$. Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and let $(X_n)_{n \in \mathbb{N}}$ be a sequence of exchangeable random variables (with respect to the Baire sigma algebra $B_a(S)$). In other words, suppose the following holds:

$$\mathbb{P}(X_1 \in A_1, \ldots, X_k \in A_k) = \mathbb{P}(X_{\sigma(1)} \in A_1, \ldots, X_{\sigma(k)} \in A_k)$$

for all $k \in \mathbb{N}$, $\sigma \in S_k$, and $A_1, \ldots, A_k \in B_a(S)$. \hspace{1cm} (A.10)

Then there is a unique probability measure $\mathcal{Q}$ on $C(P_{\text{Ba}}(S))$ such that

$$\mathbb{P}(X_1 \in A_1, \ldots, X_k \in A_k) = \int \mu(A_1) \cdot \cdots \cdot \mu(A_k) d\mathcal{Q}(\mu)$$

for all $A_1, \ldots, A_k \in B_a(S)$. \hspace{1cm} (A.11)

**Proof.** We will only prove the existence of a probability measure $\mathcal{Q}$ on $C(B_a(S))$ satisfying (A.11), with uniqueness following more elementarily from Hewitt–Savage [37, Theorem 9.4, p. 489].

Since $S$ is compact Hausdorff, so is the countable product $S^\infty$ under the product topology (this follows from Tychonoff’s theorem). Furthermore, Bogachev [12, Lemma 6.4.2 (iii), p. 14, vol. 2] implies the following:

$$B_a(S^\infty) = \bigotimes B_a(S),$$  \hspace{1cm} (A.12)

where $\bigotimes B_a(S)$ denotes the product sigma algebra on $S^\infty$ induced by the Baire sigma algebra $S$ (thus $\bigotimes B_a(S)$ is the smallest sigma algebra on $S^\infty$ that makes the projection $\pi_i : S^\infty \to S$ Baire measurable for each $i \in \mathbb{N}$). Let $\nu \in P_{\text{Ba}}(S^\infty)$ be the distribution of the $S^\infty$-valued Baire measurable random variable $(X_n)_{n \in \mathbb{N}}$ (the
Baire measurability of this random variable follows from the Baire measurability of the \( X_i \) together with (A.12).

Let \( \hat{\nu} : \mathcal{P}_{B}(S^\infty) \rightarrow \mathcal{P}_{r}(S^\infty) \) be as in Lemma A.3. Consider \( \hat{\nu} \in \mathcal{P}_{r}(S^\infty) \). We show in the next claim that the Baire exchangeability of the sequence \((X_n)_{n \in \mathbb{N}}\) implies the exchangeability of the measure \( \hat{\nu} \). In particular, let \( \Omega' := S^\infty, F' := B(S^\infty) \), and \( \hat{\nu}' := \hat{\nu} \). Consider the sequence of Borel measurable \( S \)-valued random variables \((Y_n)_{n \in \mathbb{N}}\) where, for each \( n \in \mathbb{N} \), the map \( Y_n : \Omega' \rightarrow S \) is the projection onto the \( n^{th} \) coordinate. Then we have the following claim:

**Claim A.7.** The sequence \((Y_n)_{n \in \mathbb{N}}\) is a jointly Radon distributed sequence of exchangeable random variables taking values in a completely regular Hausdorff space.

**Proof of Claim A.7.** The fact that \((Y_n)_{n \in \mathbb{N}}\) is a jointly Radon distributed sequence is immediate from the construction. Thus we only need to check the exchangeability of the \((Y_n)_{n \in \mathbb{N}}\) as Borel measurable random variables.

To that end, suppose \( k \in \mathbb{N} \) and \( B \in B(\mathbb{R}^k) \). Let \( \psi \in \mathcal{P}_{r}(S^k) \) be the Borel distribution of \((Y_1, \ldots, Y_k)\). That is, \( \psi \) is the measure on \((\mathbb{R}^k, B(\mathbb{R}^k))\) given by the pushforward \( P' \circ (Y_1, \ldots, Y_k)^{-1} \) (which is Radon, being the marginal of a Radon distribution on \( S^\infty \)). Let \( \psi' \) be its restriction to the Baire sigma algebra on \( S^k \)—that is, \( \psi' := \psi |_{\mathcal{B}_{a}(S^k)} \). Let \( \sigma \in S_k \), and let \( \psi_{\sigma} \) be the pushforward \( P' \circ (Y_{\sigma(1)}, \ldots, Y_{\sigma(k)}) \in \mathcal{P}_{r}(S^k) \) induced by the permuted random vector \((Y_{\sigma(1)}, \ldots, Y_{\sigma(k)})\), with \( \psi'_{\sigma} := \psi_{\sigma} |_{\mathcal{B}_{a}(S^k)} \) being its restriction to the Baire sigma algebra on \( S^k \). It suffices to show that \( \psi = \psi_{\sigma} \).

Note that for any \( A \in \mathcal{B}_{a}(S^k) \), we have the following chain of equalities:

\[
\psi'(A) = P'((Y_1, \ldots, Y_k) \in A) \\
= \hat{\nu}(A) \\
= \nu(A) \\
= P((X_1, \ldots, X_k) \in A) \\
= P((X_{\sigma(1)}, \ldots, X_{\sigma(k)}) \in A) \\
= P'((Y_{\sigma(1)}, \ldots, Y_{\sigma(k)}) \in A), \\
= \psi_{\sigma}(A) \\
= \psi'_{\sigma}(A). \quad (A.13)
\]

In the above, equation (A.13) follows from the Baire-exchangeability of \((X_1, \ldots, X_k)\), while the other lines follow from the fact that \( A \in \mathcal{B}_{a}(S^k) \).
Note that $\psi = \hat{\psi}'$ and $\psi_\sigma = \hat{\psi}'_\sigma$ by Lemma A.2. By (A.1), we thus have the following for any $B \in \mathcal{B}(S^k)$ (where we use (A.14) in the third line):

$$
\psi(B) = \hat{\psi}'(B)
= \inf_{U \in \tau_{n}(S^k)} \sup_{A \subseteq U} \psi'(A)
= \inf_{U \in \tau_{n}(S^k)} \sup_{A \subseteq U} \psi'(A)
= \hat{\psi}'_\sigma(A)
= \psi_\sigma(B)
$$

for all $B \in \mathbb{R}^k$, which completes the proof of the claim.

Since completely regular Hausdorff spaces are Radon presentable, we obtain a unique Radon measure $\mathcal{P}$ on $(\mathfrak{P}_r(S), C(\mathfrak{P}_r(S)))$ such that the following holds:

$$
\mathcal{P}'(Y_1 \in B_1, \ldots, Y_k \in B_k) = \int_{\mathfrak{P}_r(S)} \mu(B_1) \cdot \ldots \cdot \mu(B_k) d\mathcal{P}(\mu)
$$

for all $B_1, \ldots, B_k \in \mathcal{B}(S)$. \hspace{1cm} (A.15)

Define $\mathcal{Q} := \mathcal{P} : C(\mathfrak{P}_{Ba}(S^{\infty})) \to [0, 1]$ as in Lemma A.5. We claim that $\mathcal{Q}$ satisfies (A.11). Indeed, if $k \in \mathbb{N}$ and $A_1, \ldots, A_k \in \mathcal{B}_a(S)$, then we have:

$$
\mathbb{P}(X_1 \in A_1, \ldots, X_k \in A_k) = \nu(A_1 \times \ldots \times A_k)
= \hat{\nu}(A_1 \times \ldots \times A_k)
= \mathcal{P}'(Y_1 \in A_1, \ldots, Y_k \in A_k)
= \int_{\mathfrak{P}_r(S)} \mu(A_1) \cdot \ldots \cdot \mu(A_k) d\mathcal{P}(\mu)
= \mathcal{P}'(\{\mu \in \mathfrak{P}_r(S) : \mu(A_1) \cdot \ldots \cdot \mu(A_k) > y\}) d\lambda(y)
= \mathcal{P}'(\mathfrak{A}_y) d\lambda(y),
$$

where

$$
\mathfrak{A}_y := \{\mu \in \mathfrak{P}_{Ba}(S) : \mu(A_1) \cdot \ldots \cdot \mu(A_k) > y\}.
$$

As a consequence, we have the following:

$$
\mathbb{P}(X_1 \in A_1, \ldots, X_k \in A_k) = \int_{[0,1]} \mathcal{P}'(\mathfrak{A}_y) d\lambda(y)
= \int_{[0,1]} \mathcal{Q}(\mathfrak{A}_y) d\lambda(y)
= \int_{[0,1]} \mathcal{Q}(\{\mu \in \mathfrak{P}_{Ba}(S) : \mu(A_1) \cdot \ldots \cdot \mu(A_k) > y\}) d\lambda(y)
= \int_{[0,1]} \mu(A_1) \cdot \ldots \cdot \mu(A_k) d\mathcal{P}(\mu),
$$

which completes the proof. \hspace{1cm} $\square$
Appendix B. A proof of Theorem 4.1 using internal Bayes’ theorem

In this appendix, we will carry out an alternative proof of Theorem 4.1, which was the key ingredient in our proof of the generalization of de Finetti–Hewitt–Savage theorem. The proof that we will present here is a refinement of the Bayes’ theorem-based idea from [2]. We restate Theorem 4.1 for convenience.

**Theorem 4.1.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of \(S\)-valued exchangeable random variables, where \((S, \mathcal{G})\) is some measurable space. For each \(N > \mathbb{N}\) and \(\omega \in *\Omega\), define the internal probability measure \(\mu_{\omega,N}\) as follows:

\[
\mu_{\omega,N}(A) := \frac{\# \{i \in [N] : X_i(\omega) \in A \}}{n} \quad \text{for all } A \in *\mathcal{G}.
\]

Then we have:

\[
*\mathbb{P}(X_1 \in A_1, \ldots, X_k \in A_k) \approx *\int \mu_{\omega,N}(A_1) \cdots \mu_{\omega,N}(A_k) d*\mathbb{P}(\omega)
\]

for all \(k \in \mathbb{N}\) and \(A_1, \ldots, A_k \in *\mathcal{G}\). (B.2)

It turns out that one difficulty in a direct generalization of the method in [2] is that the sets \(A_i\) were all either \(\{0\}\) or \(\{1\}\) in [2], while they may have intersections in (B.2). We get around this difficulty by observing that it suffices to prove (B.2) for tuples \((A_1, \ldots, A_k)\) such that \(A_i\) and \(A_j\) are either disjoint or equal for all \(i, j \in [k]\).

**Definition B.1.** Call a finite tuple \((A_1, \ldots, A_k)\) of sets **disjointified** if for all \(i, j \in [k]\), we have \(A_i \cap A_j = \emptyset\) or \(A_i \cap A_j = A_i = A_j\). In the setting of Theorem 4.1, call an event **disjointified** if it is of the type \(\{X_1 \in A_1, \ldots, X_k \in A_k\}\) for some disjointified tuple \((A_1, \ldots, A_k)\).

**Lemma B.2.** Let \(N > \mathbb{N}\). In the setting of Theorem 4.1, suppose that

\[
*\mathbb{P}(X_1 \in B_1, \ldots, X_k \in B_k) \approx *\int *\mathbb{P}(\omega) \mu_{\omega,N}(B_1) \cdots \mu_{\omega,N}(B_k) \]

for all \(k \in \mathbb{N}\) and \(B_1, \ldots, B_k \in *\mathcal{G}\) such that \((B_1, \ldots, B_k)\) is disjointified.

Then (B.2) holds.

**Proof.** Suppose (B.3) holds. Let \(A_1, \ldots, A_k \in *\mathcal{G}\) be fixed. We can write the event \(\{X_1 \in A_1, \ldots, X_k \in A_k\}\) as a disjoint union of disjointified events. Indeed, for \(d \in \{0, 1\}\) and a set \(A \subseteq S\), let \(A^d\) be equal to \(A\) if \(d = 1\), and let it be equal to the complement \(S \setminus A\) if \(d = 0\). For a tuple \(a = (a_1, \ldots, a_k) \in \{0, 1\}^k\) of zeros and ones, define the following set:

\[
[A_1, \ldots, A_k]^a := \bigcap_{i \in [k]} A_i^{a_i}.
\]

(B.4)

Being a finite intersection of \(*\)-measurable sets, \([A_1, \ldots, A_k]^a\) is a \(*\)-measurable set for all \(a \in \{0, 1\}^k\). For \(i \in [k]\), define \(D_i := \{(a_1, \ldots, a_k) \in \{0, 1\}^k : a_i = 1\}\). For a tuple \(\tilde{a} = (\tilde{a}_1, \ldots, \tilde{a}_k) \in D_1 \times \ldots \times D_k\) of \(k\)-tuples, we define

\[
[A_1, \ldots, A_k]^{\tilde{a}} := \{X_1 \in [A_1, \ldots, A_k]^{\tilde{a}_1}, \ldots, X_k \in [A_1, \ldots, A_k]^{\tilde{a}_k}\}.
\]

(B.5)
It is clear that the event \([A_1, \ldots, A_k]^\tilde{a}\) is disjointified for each \(\tilde{a} \in \mathcal{D}_1 \times \cdots \times \mathcal{D}_k\), and that
\[ [A_1, \ldots, A_k]^\tilde{a} \cap [A_1, \ldots, A_k]^\tilde{b} = \emptyset \] if \(\tilde{a}, \tilde{b}\) are distinct elements of \(\mathcal{D}_1 \times \cdots \times \mathcal{D}_k\).

We thus have the following representation as a disjoint union of disjointified events:
\[ \{X_1 \in A_1, \ldots, X_k \in A_k\} = \bigcup_{\tilde{a} \in \mathcal{D}_1 \times \cdots \times \mathcal{D}_k} [A_1, \ldots, A_k]^\tilde{a}. \quad \text{(B.6)} \]

For any internal probability measure \(\mu\) on \((^*S, ^*\mathcal{S})\), its finite additivity yields the following for all \(k \in \mathbb{N}\):
\[ \mu(A_i) = \mu \left( \bigcup_{\tilde{a}_i \in \mathcal{D}_i} [A_1, \ldots, A_k]^\tilde{a}_i \right) = \sum_{\tilde{a}_i \in \mathcal{D}_i} \mu ([A_1, \ldots, A_k]^\tilde{a}_i) \] for all \(i \in [k]\). \quad \text{(B.7)}

Taking the product of the terms in (B.7) as \(i\) varies over \([k]\), and switching the order of \(\sum\) and \(\prod\) using distributivity of multiplication over addition (which is a legal move since these are finite sums and products), we have the following observation for any internal probability measure \(\mu\) on \((^*S, ^*\mathcal{S})\):
\[ \prod_{i \in [k]} \mu(A_i) = \sum_{\tilde{a} = (\tilde{a}_1, \ldots, \tilde{a}_k) \in \mathcal{D}_1 \times \cdots \times \mathcal{D}_k} \left[ \prod_{i \in [k]} \mu ([A_1, \ldots, A_k]^\tilde{a}_i) \right] \] for all \(k \in \mathbb{N}\). \quad \text{(B.8)}

Applying (B.8) to the internal measure \(\mu_{\omega, N}\) for each \(\omega \in \mathbb{N}\) and then integrating with respect to \(^*\mathbb{P}\), we obtain the following by the linearity of the expectation:
\[ \ast \int_{^*\Omega} \left( \prod_{i \in [k]} \mu(A_i) \right) d^*\mathbb{P}(\omega) \]
\[ = \sum_{\tilde{a} = (\tilde{a}_1, \ldots, \tilde{a}_k) \in \mathcal{D}_1 \times \cdots \times \mathcal{D}_k} \ast \int_{^*\Omega} \left( \prod_{i \in [k]} \mu_{\omega, N} ([A_1, \ldots, A_k]^\tilde{a}_i) \right) d^*\mathbb{P}(\omega) \]
\[ = \sum_{\tilde{a} = (\tilde{a}_1, \ldots, \tilde{a}_k) \in \mathcal{D}_1 \times \cdots \times \mathcal{D}_k} \ast \mathbb{P} (X_1 \in [A_1, \ldots, A_k]^\tilde{a}_1, \ldots, X_k \in [A_1, \ldots, A_k]^\tilde{a}_k), \]
where the last line follows from the hypothesis of the theorem. The proof is now completed by (B.5) and (B.6).

For the rest of the paper, we fix the following set-up. Let \(N > \mathbb{N}\). We have established in Lemma B.2 that it suffices to show (B.3). Toward that end, let \(A_1, \ldots, A_k \in ^*\mathcal{S}\) be such that the tuple \((A_1, \ldots, A_k)\) is disjointified. For some \(n \in \mathbb{N}\), let \(C_1, \ldots, C_n\) be the distinct (disjoint) sets appearing in the tuple \((A_1, \ldots, A_k)\). For each \(i \in [n]\), let \(C_i\) appear in \((A_1, \ldots, A_k)\) with a frequency \(k_i\) (this necessarily implies that \(k_1 + \cdots + k_n = k\)).

For each \(i \in [n]\), let \(Y_i : ^*\Omega \to [N]\) be defined as follows:
\[ Y_i(\omega) := \# \{ j \in [N] : X_j(\omega) \in C_i \} = \sum_{j \in [N]} 1_{C_i}(X_j(\omega)) \text{ for all } \omega \in ^*\Omega. \quad \text{(B.9)} \]
Thus \( \mu_{\omega,N}(C_i) = \frac{Y_i(\omega)}{N} \) for all \( \omega \in \Omega \).

Let \( \tilde{A}, \tilde{X} \), and \( \tilde{Y} \) denote the tuples \((A_1, \ldots, A_k), (X_1, \ldots, X_k), \) and \((Y_1, \ldots, Y_n)\) respectively. The following lemma follows from elementary combinatorial arguments.

**Lemma B.3.** Suppose that \( t_i \in \mathbb{N} \) are such that \( t_i \geq k_i \) for all \( i \in [n] \), and such that \( *P(\tilde{Y} = (t_1, \ldots, t_n)) > 0 \). Then we have:

\[
*P(\tilde{X} \in \tilde{A} | \tilde{Y} = (t_1, \ldots, t_n)) = \frac{1}{N(N-1) \cdots (N - (k-1)) \cdot (t_1 - 1)! \cdots (t_n - k_n)!}.
\]

\[ \text{(B.10)} \]

**Proof:** Let \( t_1, \ldots, t_n \) be as in the statement of the lemma. Define the following event:

\[
E_{t_1, \ldots, t_n} := \{X_1, \ldots, X_{t_1} \in C_1; \]
\[
X_{t_1+1}, \ldots, X_{t_1+t_2} \in C_2; \]
\[
\vdots \]
\[
X_{t_1+\cdots+t_{n-1}+1}, \ldots, X_{t_1+\cdots+t_n} \in C_n; \]
\[
X_i \in S \setminus C_1 \sqcup \cdots \sqcup C_n \text{ for all other } i \in [N].\]

By exchangeability and the fact that the \( C_i \) are disjoint, we have the following:

\[
*P(\tilde{Y} = (t_1, \ldots, t_n)) = N_1 *P(E_{t_1, \ldots, t_n}), \quad \text{(B.11)}
\]

and

\[
*P(\tilde{X} \in \tilde{A} \text{ and } \tilde{Y} = (t_1, \ldots, t_n)) = N_2 *P(E_{t_1, \ldots, t_n}), \quad \text{(B.12)}
\]

where

\[
N_1 = \text{Number of ways to choose } t_i \text{ spots of the } i^{\text{th}} \text{ kind in } [N] \text{ as } i \text{ varies over } [n]
\]

\[
= \binom{N}{t_1} \binom{N-t_1}{t_2} \cdots \binom{N-t_1-\cdots-t_{n-1}}{t_n}. \quad \text{(B.13)}
\]

and

\[
N_2 = \text{Number of ways to choose } (t_i - k_i) \text{ spots of the } i^{\text{th}} \text{ kind in } [N] \text{ as } i \text{ varies over } [n]
\]

\[
= \binom{N-k}{t_1-k_1} \binom{N-k-(t_1-k_1)}{t_2-k_2} \cdots \binom{N-k-(t_1+\cdots+t_{n-1}-k_1 \cdots -k_{n-1})}{t_n-k_n}. \quad \text{(B.14)}
\]

Since it is given that \( *P(\tilde{Y} = (t_1, \ldots, t_n)) > 0 \), we thus have \( *P(E_{t_1, \ldots, t_n}) > 0 \) by (B.11). By (B.11), (B.12), (B.13), and (B.14), we therefore obtain (B.10) after simplification. \( \square \)

**Corollary B.4.** Suppose that \( t_i \in \mathbb{N} \) such that \( *P(\tilde{Y} = (t_1, \ldots, t_n)) > 0 \). Then we have:

\[
*P(\tilde{X} \in \tilde{A} | \tilde{Y} = (t_1, \ldots, t_n)) \approx \left( \frac{t_1}{N} \right)^{k_1} \cdots \left( \frac{t_n}{N} \right)^{k_n} \text{ for all } (t_1, \ldots, t_n) \in [N]^n.
\]

\[ \text{(B.15)} \]
Proof. Suppose that the $t_i \in [N]$ are such that $\mathbb{P}(\overline{Y} = (t_1, \ldots, t_n)) > 0$. If $t_i \geq k_i$ for all $i \in [n]$. Then by Lemma B.3, we obtain the following:

$$\frac{\mathbb{P}(\overline{X} \in \overline{A} | \overline{Y} = (t_1, \ldots, t_n))}{(\frac{N}{k_1})^{k_1} \cdots (\frac{N}{k_n})^{k_n}} = \frac{1}{1 - \frac{N}{k_1}} \cdots \frac{1}{1 - \frac{N}{k_n}} \cdot \prod_{i \in [n]} \left( \prod_{j \in [k_i - 1]} \left( 1 - \frac{j}{t_i} \right) \right)
$$

(B.16)

$$< \frac{1}{1 - \frac{N}{k_1}} \cdots \frac{1}{1 - \frac{N}{k_n}} \approx 1. \quad \text{(B.17)}$$

Note that if $t_i > N$ for all $i \in [n]$, then both $\frac{1}{1 - \frac{N}{k_1}} \cdots \frac{1}{1 - \frac{N}{k_n}} \approx 1$ and

$$\prod_{i \in [n]} \left( \prod_{j \in [k_i - 1]} \left( 1 - \frac{j}{t_i} \right) \right) \approx 1,$$

so that (B.16) implies that

$$\frac{\mathbb{P}(\overline{X} \in \overline{A} | \overline{Y} = (t_1, \ldots, t_n))}{(\frac{N}{k_1})^{k_1} \cdots (\frac{N}{k_n})^{k_n}} \approx 1 \text{ if } t_1, \ldots, t_n > N, \quad \text{(B.18)}$$

which, in particular, implies (B.15) in this case.

Now, if $t_j$ is in $[N]$ for some $j \in [n]$ but such that $t_i \geq k$ for all $i \in [n]$ and $\mathbb{P}(\overline{Y} = (t_1, \ldots, t_n)) > 0$, then the inequality in (B.17) implies that

$$\mathbb{P}(\overline{X} \in \overline{A} | \overline{Y} = (t_1, \ldots, t_n)) < 2 \left( \frac{t_1}{N} \right)^{k_1} \cdots \left( \frac{t_n}{N} \right)^{k_n} \approx 2 \left( \frac{t_j}{N} \right)^{k_j} < 0,$$

so that

$$\mathbb{P}(\overline{X} \in \overline{A} | \overline{Y} = (t_1, \ldots, t_n)) \approx 0 \approx \left( \frac{t_1}{N} \right)^{k_1} \cdots \left( \frac{t_n}{N} \right)^{k_n},$$

proving (B.15) in that case as well.

Finally, if $t_i < k_i$ for any $i \in [n]$, then $\mathbb{P}(\overline{X} \in \overline{A} | \overline{Y} = (t_1, \ldots, t_n)) = 0$, while

$$\left( \frac{t_1}{N} \right)^{k_1} \cdots \left( \frac{t_n}{N} \right)^{k_n} = 0 \text{ in that case as well. This completes the proof.} \quad \square$$

We record (B.18) in the proof of Corollary B.4 as its own result.

**Corollary B.5.** Suppose that $t_i > N$ such that $\mathbb{P}(\overline{Y} = (t_1, \ldots, t_n)) > 0$. Then we have the following approximate equality:

$$\frac{\mathbb{P}(\overline{X} \in \overline{A} | \overline{Y} = (t_1, \ldots, t_n))}{(\frac{N}{k_1})^{k_1} \cdots (\frac{N}{k_n})^{k_n}} \approx 1 \text{ if } t_1, \ldots, t_n > N. \quad \text{(B.19)}$$

By (B.17) and underflow applied to Corollary B.5, we obtain the following.

**Corollary B.6.** Given $\epsilon \in \mathbb{R}_{>0}$, there is an $m_\epsilon$ satisfying the following.

$$1 - \epsilon < \frac{\mathbb{P}(\overline{X} \in \overline{A} | \overline{Y} = (t_1, \ldots, t_n))}{(\frac{N}{k_1})^{k_1} \cdots (\frac{N}{k_n})^{k_n}} < 1 + \epsilon$$

if $t_1, \ldots, t_n > m_\epsilon$ are such that $\mathbb{P}(\overline{Y} = (t_1, \ldots, t_n)) > 0.$
The proof of Corollary B.4 also leads to the following observation.

**Corollary B.7.** For each \( m \in \mathbb{N} \), define the set

\[
L_m := \{(t_1, \ldots, t_n) \in [N]^n : \text{there is } j \in [n] \text{ such that } t_j \leq m \in [n]\}. \tag{B.19}
\]

Then, we have the following for all \( m \in \mathbb{N} \):

\[
0 \approx \sum_{(t_1, \ldots, t_n) \in L_m} \ast \mathbb{P}(\vec{X} \in \vec{A}|\vec{Y} = (t_1, \ldots, t_n)) \ast \mathbb{P} \left( \mu_{.,N}(C_1) = \frac{t_1}{N}, \ldots, \mu_{.,N}(C_n) = \frac{t_n}{N} \right)
\]

\[
\approx \sum_{(t_1, \ldots, t_n) \in L_m} \left( \frac{t_1}{N} \right)^{k_1} \cdots \left( \frac{t_n}{N} \right)^{k_n} \ast \mathbb{P} \left( \mu_{.,N}(C_1) = \frac{t_1}{N}, \ldots, \mu_{.,N}(C_n) = \frac{t_n}{N} \right).
\]

**Proof.** Let \( m \in \mathbb{N} \) and \( L_m \) be as in the statement of the corollary. Noting that the event \( \{ \mu_{.,N}(C_1) = \frac{t_1}{N}, \ldots, \mu_{.,N}(C_n) = \frac{t_n}{N} \} \) is the same as the event \( \{ \vec{Y} = (t_1, \ldots, t_n) \} \), we obtain the following from (B.17) (we also use the fact that if \( t_i < k \) for any \( i \in [n] \), then \( \ast \mathbb{P}(\vec{X} \in \vec{A}|\vec{Y} = (t_1, \ldots, t_n)) = 0 \):

\[
\sum_{(t_1, \ldots, t_n) \in L_m} \ast \mathbb{P}(\vec{X} \in \vec{A}|\vec{Y} = (t_1, \ldots, t_n)) \ast \mathbb{P} \left( \mu_{.,N}(^*C_1) = \frac{t_1}{N}, \ldots, \mu_{.,N}(^*C_n) = \frac{t_n}{N} \right)
\]

\[
\leq 2 \sum_{(t_1, \ldots, t_n) \in L_m} \left( \frac{t_1}{N} \right)^{k_1} \cdots \left( \frac{t_n}{N} \right)^{k_n} \ast \mathbb{P} \left( \mu_{.,N}(^*C_1) = \frac{t_1}{N}, \ldots, \mu_{.,N}(^*C_n) = \frac{t_n}{N} \right)
\]

\[
\leq 2 \sum_{j \in [n]} \left( \sum_{r \in [m]} \left( \sum_{(t_1, \ldots, t_n) \in [N]^n} \left( \frac{t_j}{N} \right)^{k_j} \ast \mathbb{P} \left( \mu_{.,N}(^*C_1) = \frac{t_1}{N}, \ldots, \mu_{.,N}(^*C_n) = \frac{t_n}{N} \right) \right) \right)
\]

\[
\leq 2 \sum_{j \in [n]} \left( \sum_{(t_1, \ldots, t_n) \in [N]^n} \frac{m}{N} \mathbb{P} \left( \mu_{.,N}(^*C_j) \leq \frac{m}{N} \right) \right)
\]

\[
= \frac{2m}{N} \sum_{j \in [n]} \ast \mathbb{P} \left( \mu_{.,N}(^*C_j) \leq \frac{m}{N} \right)
\]

\[
\leq \frac{2mn}{N} \approx 0,
\]

completing the proof. \( \square \)

We now have all the ingredients for our proof of Theorem 4.1.

**Proof of Theorem 4.1.** Conditioning on the various possible values of \( Y_i \) as \( i \) varies in \( [n] \), and noting that the event \( \{ \mu_{.,N}(C_1) = \frac{t_1}{N}, \ldots, \mu_{.,N}(C_n) = \frac{t_n}{N} \} \) is the same
as the event \( \{ \vec{Y} = (t_1, \ldots, t_n) \} \), we obtain:

\[
\*P( (X_1, \ldots, X_k) \in \vec{A} )
= \sum_{(t_1, \ldots, t_n) \in \mathbb{N}^n} \*P( \vec{X} \in \vec{A} | \vec{Y} = (t_1, \ldots, t_n) ) \cdot \*P \left( \mu_1 \cdot \mu_n(C_n) = \frac{t_n}{N} \right)
\]  
(B.20)

Now, by the definition of expected values, we have the following equality:

\[
\int_{\*\omega} \mu_1 \cdot \mu_n(A_1) \cdot \mu_1 \cdot \mu_n(A_k) d\*P(\omega)
= \sum_{(t_1, \ldots, t_n) \in \mathbb{N}^n} \left( \frac{t_1}{N} \right)^{k_1} \cdots \left( \frac{t_n}{N} \right)^{k_n} \cdot \*P \left( \mu_1 \cdot \mu_n(C_n) = \frac{t_n}{N} \right).
\]  
(B.21)

Let \( \epsilon \in \mathbb{R}_{>0} \) and let \( m_\epsilon \in \mathbb{N} \) be as in Corollary B.6. By that corollary, we obtain:

\[
\sum_{(t_1, \ldots, t_n) \in \mathbb{N}^n} \*P( \vec{X} \in \vec{A} | \vec{Y} = (t_1, \ldots, t_n) ) \cdot \*P \left( \mu_1 \cdot \mu_n(C_n) = \frac{t_n}{N} \right)
> (1 - \epsilon) \sum_{(t_1, \ldots, t_n) \in \mathbb{N}^n} \left( \frac{t_1}{N} \right)^{k_1} \cdots \left( \frac{t_n}{N} \right)^{k_n} \cdot \*P \left( \mu_1 \cdot \mu_n(C_n) = \frac{t_n}{N} \right).
\]

By taking standard parts and using Corollary B.7, the above yields the following inequality:

\[
\text{st} \left[ \sum_{(t_1, \ldots, t_n) \in \mathbb{N}^n} \*P( \vec{X} \in \vec{A} | \vec{Y} = (t_1, \ldots, t_n) ) \cdot \*P \left( \mu_1 \cdot \mu_n(C_n) = \frac{t_n}{N} \right) \right] \geq (1 - \epsilon) \text{st} \left[ \sum_{(t_1, \ldots, t_n) \in \mathbb{N}^n} \left( \frac{t_1}{N} \right)^{k_1} \cdots \left( \frac{t_n}{N} \right)^{k_n} \cdot \*P \left( \mu_1 \cdot \mu_n(C_n) = \frac{t_n}{N} \right) \right].
\]

Since \( \epsilon \in \mathbb{R}_{>0} \) is arbitrary, we thus obtain:

\[
\text{st} \left[ \sum_{(t_1, \ldots, t_n) \in \mathbb{N}^n} \*P( \vec{X} \in \vec{A} | \vec{Y} = (t_1, \ldots, t_n) ) \cdot \*P \left( \mu_1 \cdot \mu_n(C_n) = \frac{t_n}{N} \right) \right] \geq \text{st} \left[ \sum_{(t_1, \ldots, t_n) \in \mathbb{N}^n} \left( \frac{t_1}{N} \right)^{k_1} \cdots \left( \frac{t_n}{N} \right)^{k_n} \cdot \*P \left( \mu_1 \cdot \mu_n(C_n) = \frac{t_n}{N} \right) \right].
\]  
(B.22)

But the reverse inequality to (B.22) is also true because of (B.17) and the fact that \( \*P( \vec{X} \in \vec{A} | \vec{Y} = (t_1, \ldots, t_n) ) = 0 \) if \( t_j < k \) for any \( j \in [n] \). This completes the proof by (B.20) and (B.21). \( \Box \)
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Irfan Alam: Department of Mathematics, Louisiana State University, Baton Rouge, LA 70802, USA

E-mail address: irfanalamisi@gmail.com

URL: http://www.math.lsu.edu/~ialam1