Quantum charging supremacy via Sachdev-Ye-Kitaev batteries

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The exactly-solvable Sachdev-Ye-Kitaev (SYK) model has recently received considerable attention in both condensed matter and high energy physics because it describes quantum matter without quasiparticles, while being at the same time the holographic dual of a quantum black hole. In this Letter, we examine SYK-based charging protocols of quantum batteries with N quantum cells. We demonstrate that the optimal charging power of our SYK quantum batteries displays a super-extensive scaling with N that stems from genuine quantum mechanical effects. To the best of our knowledge, this is the first quantum many-body battery model where fast charging occurs due to the maximally-entangling underlying quantum dynamics.

Introduction.—In the era of quantum supremacy for quantum computing [1], research on the potential usefulness of quantum mechanical resources (such as entanglement) in energy science has led a consistent number of authors to introduce and study “quantum batteries” (QB). A QB [2, 3] is a system composed of N identical quantum cells, where energy is stored and from which work can be extracted.

In 2013, Alicki and Fannes [2] suggested that “entangling unitary controls”, i.e. unitary operations acting globally on the state of the N quantum cells, lead to better work extraction capabilities from a QB, when compared to unitary operations acting on each quantum cell separately. Hovhannisyan et al. [4] were the first to demonstrate that entanglement generation leads to a speed-up in the process of work extraction, thereby leading to larger delivered power. Later on, the authors of Refs. [5, 6] focussed on the charging (rather than the discharging) phase and identified two types of charging schemes: i) the parallel charging scheme in which each of the N quantum cells is acted upon independently of the others; and ii) the collective charging scheme, where global unitary operations (i.e. the entangling unitary controls of Ref. [2]) acting on the full Hilbert space of the N quantum cells are allowed. In the collective charging case and for N ≥ 2, the power absorbed by a QB is larger than in the parallel scheme. This collective speed-up, stemming from entangling operations, in the charging phase of a QB has been dubbed “quantum advantage”.

In the quest for such quantum advantage and potential laboratory implementations of QBs—based, e.g., on circuit quantum electrodynamics and trapped-ion setups—the abstract concepts of “quantum cell” and “entangling operations” have been recently spelled out more explicitly [7–25]. Different concrete models of quantum cells and related QBs have been studied: i) Dicke models where arrays of N qubits (i.e. the proper battery) are coupled to a harmonic energy source [8–13]; and iii) disordered spin chains [14, 15]. These quantum cells can be charged by switching on either direct [7, 14, 15] or effective [8–13] interactions between them.

The problem, however, is that no genuine quantumness is rooted in the charging dynamics of all these microscopic QB models. Indeed, the authors of Ref. [24] proved that rigorous classical analogues of all the aforementioned models display an optimal charging power with the same scaling with N. At best, certain quantum models were found [24] to display a parametric advantage, independent of N, over their classical analogues. Finally, the authors of Ref. [25] derived a powerful bound for the charging power, which allows to distinguish between a genuine entanglement-induced speed-up and a “collective” speed-up with a fully classical counterpart. In agreement with Ref. [24], the conclusion of Ref. [25] is that all the many-body QB models proposed in the literature so far do not feature any genuine quantum advantage.

Motivated by this literature, here we propose a model of a QB which unequivocally presents a neat quantum...
advantage. Our implementation relies on the Sachdev-Ye-Kitaev (SYK) model [26–29], which has recently attracted a great deal of attention for its exact solvability and profound properties. The SYK model describes quantum matter with no quasiparticles. It displays fast scrambling [30, 31], has a nonzero entropy density at vanishing temperature [32, 33], all its eigenstates exhibit volume-law entanglement entropy [34], and is holographically connected to the dynamics of AdS2 horizons of quantum black holes [27, 28, 35, 36]. Proposals to realize the SYK Hamiltonian have been recently put forward and rely on ultra-cold atoms [37], graphene flakes with irregular boundaries [38], and topological superconductors [39, 40].

Many-body QBs and figures of merit.—Consider a QB made of N identical quantum cells (for a cartoon, see Fig. 1), which are governed by the following free and local Hamiltonian (ℏ = 1):

\[ \hat{H}_0 = \sum_{j=1}^N \hat{h}_j \, . \]  

(1)

At time \( t = 0 \), the system is prepared in its ground state \( |0\rangle \), physically representing the discharged battery. By suddenly switching on a suitable interaction Hamiltonian \( \hat{H}_1 \) for a finite amount of time \( \tau \) (and switching off \( \hat{H}_0 \)), one aims at injecting as much energy as possible into the quantum cells [5–7]. The time interval \( \tau \) is called the charging time of the protocol.

The full model Hamiltonian can be thus written as

\[ \hat{H}(t) = \hat{H}_0 + \lambda(t)(\hat{H}_1 - \hat{H}_0) \, , \]  

(2)

where \( \lambda(t) \) is a classical parameter that represents the external control exerted on the system, and which is assumed to be given by a step function equal to 1 for \( t \in [0, \tau] \) and zero elsewhere. Accordingly, denoting by \( |\psi(t)\rangle \) the state of the system at time \( t \), its total energy \( E_N^{\psi}(t) = \langle \psi(t)|\hat{H}(t)|\psi(t)\rangle \) is constant for all values of \( t \) but \( t = 0 \) and \( t = \tau \) (the switching points).

The energy injected into the \( N \) quantum cells can be expressed in terms of the mean local energy at the end of the protocol, \( E_N(\tau) = \langle \psi(\tau)|\hat{H}_0|\psi(\tau)\rangle \). In writing the previous equation, we have set to zero the ground-state energy \( \langle 0|\hat{H}_0|0\rangle \). Other crucial figures of merit are the average charging power \( P_N(\tau) \) and its optimal value

\[ P_N(\tau^*) = \max_{\tau > 0} P_N(\tau) \, , \]  

(3)

obtained at time \( \tau^* \). In the following, we will be mainly interested in the scaling of the optimal charging power \( P_N(\tau^*) \) with the number \( N \) of quantum cells.

SYK-based charging protocols.—We consider a QB where each quantum cell is a spin-1/2 system. In the absence of charging operations, the system is described by the non-interacting Hamiltonian (1), with \( \hat{h}_j = \omega_0 \hat{\sigma}_j^y/2 \). Here, \( \omega_0 > 0 \) represents a magnetic field strength (with units of energy) and \( \hat{\sigma}_j^\alpha = (x, y, z) \) are the usual Pauli matrices. The battery energy \( E_N(\tau) \) will be measured in units of the energy scale \( \omega \). At time \( t = 0 \), the quantum cells are initialized in the ground state of \( \hat{H}_0 \), \( |0\rangle = \bigotimes_{j=1}^N |\downarrow(0)\rangle_j \), where \( \hat{\sigma}_j^y |\downarrow(0)\rangle_j = -|\uparrow(0)\rangle_j \).

For the charging Hamiltonian \( \hat{H}_1 \), we use the complex SYK (c-SYK) [28, 41, 42] model Hamiltonian:

\[ \hat{H}_1^{c-SYK} = \sum_{i,j,k,l=1}^N J_{i,j,k,l} \hat{c}_i^\dagger \hat{c}_j^\dagger \hat{c}_k \hat{c}_l \, , \]  

(4)

where \( \hat{c}_j^\dagger (\hat{c}_j) \) is a spinless fermionic creation (annihilation) operator [43]. This has to be understood in its spin-1/2 representation, which is obtained by the usual Jordan-Wigner (JW) transformation \( \hat{c}_j^\dagger = \hat{\sigma}_j^z \hat{\sigma}_j^x (\Pi_{m=1}^{j-1} \hat{\sigma}_m^z)^{1/2} \), where \( \hat{\sigma}_j^\pm = (\hat{\sigma}_j^x \pm i \hat{\sigma}_j^y)/2 \). For more details on the JW transformation and numerical calculations see Ref. [44].

The couplings \( J_{i,j,k,l} \) are zero-mean Gaussian-distributed complex random variables, with variance \( \langle J_{i,j,k,l}^2 \rangle = J^2/N^2 \), satisfying \( J_{i,j,k,l} = J^*_{k,l,i,j} \) and \( J_{i,j,k,l} = -J_{j,i,k,l} = -J_{j,i,l,k} \). In the following, we average any quantity of interest \( \mathcal{O} \) over the distribution of \( \{J_{i,j,k,l}\} \), and denote by \( \langle \mathcal{O} \rangle \) the averaged quantity, i.e., \( \langle \mathcal{O} \rangle = \int P(\{J_{i,j,k,l}\}) \mathcal{O}(\{J_{i,j,k,l}\}) \, d\{J_{i,j,k,l}\} \).

We emphasize that our choice of battery and charging Hamiltonians is such that \( [\hat{H}_0, \hat{H}_1] \neq 0 \), a condition which ensures energy injection into the QB by the charging protocol (2). Note, finally, that the Hamiltonian in Eq. (4) is invariant under particle-hole symmetry (PHS) in the thermodynamic limit \( N \to \infty \). Extra terms, however, need to be added to it in order to enforce PHS at any finite \( N \) [41]:

\[ \hat{H}_1^{c-SYK \text{(PHS)}} = \hat{H}_1^{c-SYK} + \frac{1}{2} \sum_{i,j,k,l=1}^N J_{i,j,k,l} \times (\delta_{i,k} \hat{c}_i^\dagger \hat{c}_l - \delta_{i,l} \hat{c}_i^\dagger \hat{c}_k - \delta_{j,k} \hat{c}_j^\dagger \hat{c}_l + \delta_{j,l} \hat{c}_j^\dagger \hat{c}_k) \, . \]  

(5)

Hereafter, we will always use this version of the c-SYK model. We have however checked that our main findings do not qualitatively change if PHS is not enforced and (4), rather than (5), is used as charging Hamiltonian.

In the following, we will also consider charging Hamiltonians based on a bosonic version of the SYK model (b-SYK) [41]:

\[ \hat{H}_1^{b-SYK} = \sum_{i,j,k,l=1}^N J_{i,j,k,l} \hat{b}_i^\dagger \hat{b}_j^\dagger \hat{b}_k \hat{b}_l \, . \]  

(6)

Here, \( \hat{b}_j^\dagger (\hat{b}_j) \) creates (annihilates) an hard-core boson. The following relations are obeyed: \( \{\hat{b}_j, \hat{b}_j^\dagger\} = 1 \) and \( [\hat{b}_i, \hat{b}_j] = 0 \) for \( i \neq j \). Hence, \( \hat{b}_j^\dagger \) can be directly written in
its spin representation as $\hat{b}_j^\dagger = \sigma_j^+$. Similarly to $J_{i,j,k,l}$, the quantities $\tilde{J}_{i,j,k,l}$ in Eq. (6) are random, Gaussian-distributed variables, with variance $\langle\langle \tilde{J}_{i,j,k,l}^2 \rangle\rangle = J^2/N^3$, satisfying $J_{i,j,k,l} = \tilde{J}_{k,l,i,j}$ and $J_{i,j,k,l} = J_{j,i,k,l} = \tilde{J}_{i,j,l,k}$ (in order to comply with the bosonic commutation rules of the model). For PHS to hold, we enforce the site indices $i,j,k,l$ in Eq. (6) to be all different [41].

Finally, we will also examine charging protocols based on the Ising Hamiltonian,

$$\hat{H}_1^{\text{Ising}} = -K \sum_{j=1}^N \hat{\sigma}_j^x \hat{\sigma}_{j+1}^x.$$  

This charging Hamiltonian is known to display no quantum advantage [25] and will therefore work for us as reference model, to be compared against the c- and b-SYK charging models.

Microscopy of the charging dynamics in energy space.—As an indicator of the speed at which the dynamics occurs, we start by looking at the evolution in time of the occupations of the energy levels. Consider the spectral decomposition of the Hamiltonian (1):

$$\hat{H}_0 = \sum_{k=0}^N \epsilon_k \sum_{i,j} |k,i \rangle \langle k,j|, \quad \text{where } \epsilon_k = k\omega_0 \text{ denote its eigenvalues and the index } i \text{ accounts for the degenerate eigenvectors.}$$

We are interested in the dynamics of the populations:

$$p_k(\tau) = \sum_i |\langle k,i|\psi(\tau)\rangle|^2.$$  

Figure 2 displays $p_k(\tau)$ for three cases, corresponding to the three charging Hamiltonians introduced above: c-SYK (a), b-SYK (b), and Ising (c). While in the latter two cases the charging protocol generates a dynamics that is clearly local in energy space, this is not the case for the c-SYK model. This charging model generates a non-local population dynamics in energy space, which manifests as a sudden macroscopic population of excited levels. Indeed, after an ultrashort “thermalization” time, a central band of excited energy levels appears uniformly populated. This non-locality in energy space is a direct realization of the global charging dynamics envisioned by the authors of Ref. [5]. We finally note that panel (c) is the only one to display recurrences, since it corresponds to the charging dynamics dictated by the only integrable model we have studied here, i.e. the Ising model (7).

Power, bounds, and quantum charging supremacy.— Quantitative conclusions on the charging performances of SYK QBs—as compared to those of other reference many-body QBs—can be drawn from the analysis of the optimal power $P_N(\tau^*)$ in Eq. (3) and its scaling with $N$. Such comparisons need to be made with great care. We note that the time-evolution operator is $U(t) = \exp(-i\hat{H}_1 t)$. The charging Hamiltonian $\hat{H}_1$ contains an energy scale, i.e. $J/J\bar{J}$ for the c-SYK (b-SYK) model and $K$ for the Ising model. We here want to: 1) rule out trivial power enhancements determined by an increase in the energy scale, i.e. obtained by multiplying the energy couplings by a factor $\alpha > 1$, and 2) compare the three models in a fair manner—“fair” in the sense that, trivially, an Ising charging protocol with $K \geq J,J$, for example, may outperform c- and b-SYK charging protocols, and we want to avoid that.

To rule out these spurious effects, we first consider the rescaled charging Hamiltonians [6],

$$\hat{\mathcal{H}}_1 = \frac{\hat{H}_1}{||\hat{H}_1||},$$  

where $||\hat{O}|| = \mu_{\hat{O}}^{(\text{max})} - \mu_{\hat{O}}^{(\text{min})}$ defines the norm of the Hermitian operator $\hat{O}$, $\mu_{\hat{O}}^{(\text{max})}$ ($\mu_{\hat{O}}^{(\text{min})}$) being its maximum (minimum) eigenvalue. The charging Hamiltonian (9) allows a fair comparison between different QB models. In Fig. 3, we report the optimal charging power $\langle\langle P_N(\tau^*)\rangle\rangle$ as a function of $N$, calculated for the c-SYK,
In contrast, $\Delta_r \hat{H}_0^2$ is connected with the distance traveled in the Hilbert space [25]. An enhancement of it can be linked to shortcuts in the Hilbert space: going through highly entangled states, it is possible to reduce the length of the trajectory in such space, consequently enhancing the charging power [25]. This is an exquisite quantum effect, which has no classical analogue. Any increase of the average optimal power that can be linked to $\Delta_r \hat{H}_0^2$, can be considered as a smoking gun of the quantum supremacy of a QB model, unreproducible by classical dynamics. For the sake of completeness, a detailed derivation of the bound (10) can be found in Ref. [44].

If the battery Hamiltonian $\hat{H}_0$ is made of a sum of local terms, as in the case of Eq. (1), it is possible write $\Delta_r \hat{H}_0^2$ as the sum of two terms: $
abla^\perp \hat{H}_0^2 = \Delta_r^\perp \hat{H}_0^2 + \Delta_r^\parallel \hat{H}_0^2$, with [25]

\begin{align}
\Delta_r^\perp \hat{H}_0^2 &\equiv \frac{1}{\tau} \int_0^\tau dt \sum_i \left[ \langle \hat{h}_i^2 \rangle_t - \langle \hat{h}_i \rangle_t^2 \right], \\
\Delta_r^\parallel \hat{H}_0^2 &\equiv \frac{1}{\tau} \int_0^\tau dt \sum_{i \neq j} \left[ \langle \hat{h}_i \hat{h}_j \rangle_t - \langle \hat{h}_i \rangle_t \langle \hat{h}_j \rangle_t \right].
\end{align}

The quantity (11), being a sum of local terms, scales linearly with $N$ (i.e. is extensive) by construction. On the other hand, $\Delta_r^\parallel \hat{H}_0^2$, whose explicit form can be immediately linked to correlations between sites $i$ and $j$, may display a super-linear scaling with $N$. Due to the non-linearity of the bound (10), which applies to a single disorder realization, averaging over disorder is not straightforward.

Through the Cauchy–Schwarz inequality, though, it is however possible to rewrite it as $\langle P_N(\tau) \rangle = 2 \langle \sqrt{\Delta_r^\parallel \hat{H}_0^2} \right\rangle^2 \langle \Delta_r^\perp \hat{H}_0^2 \rangle \leq 2 \sqrt{\langle \Delta_r^\parallel \hat{H}_0^2 \rangle} \langle \Delta_r^\perp \hat{H}_0^2 \rangle$. This means that one can separately study the averaged quantities $\langle \Delta_r^\parallel \hat{H}_0^2 \rangle$ and $\langle \Delta_r^\perp \hat{H}_0^2 \rangle$. In the following we are interested in the scaling at the optimal time $\tau^*$, thus we focus on

\begin{equation}
\langle P_N(\tau^*) \rangle = 2 \sqrt{\langle \Delta_r^\parallel \hat{H}_0^2 \rangle} \langle \Delta_r^\perp \hat{H}_0^2 \rangle.
\end{equation}

Since the battery energy is measured in units of $\omega_0$ and time in units of $1/J$, the averaged charging power $\langle P_N(\tau^*) \rangle$ is measured in units of $\omega_0 J$. Given these choices, we need to specify only the energy scales of the b-SYK and Ising charging protocols: below we set $J = J$ and $K = 0.4 J$ [46]. In Fig. 4 we display results on the bound (13) for a c-SYK QB. Panel (a) shows the relevant quantities for the bound (13), as functions of $N$. While $\langle \Delta_r^\perp \hat{H}_0^2 \rangle$ is extensive in $N$, we clearly see that both $\langle \Delta_r^\parallel \hat{H}_0^2 \rangle$ and $\langle \Delta_r^\parallel \hat{H}_0^2 \rangle$ display a super-linear scaling with $N$, which is compatible with a quadratic $\sim N^2$ growth. This, together with Eq. (13), suggests a super-linear scaling with $N$ of the optimal charging power,

\begin{equation}
\langle P_N(\tau^*) \rangle \sim N^{1+k}, \quad \text{with } k > 0,
\end{equation}

where $k \approx 0.5$. For the first time in the literature on QB models [7–25], we are thus in a situation where the

![FIG. 3. (Color online) The dependence of the averaged optimal charging power $\langle P_N(\tau^*) \rangle$ on the number $N$ of quantum cells. This is the only plot of our Letter where data have been calculated by using the rescaled Hamiltonian (9). This implies that the averaged optimal charging power shown in this plot is measured in units of $\omega_0$. In red, we show the optimal power calculated for the c-SYK model with PHS. In blue (black) we show the same quantity for the b-SYK (Ising) model. Here and in all the rest of the figures, data for both types of SYK models have been obtained after averaging over $N_{\text{dis}} = 10^3$ (for $N = 4, \ldots, 10$), $5 \times 10^2$ (for $N = 11, 12$), and $10^2$ (for $N = 13, \ldots, 16$) instances of disorder in the couplings $\{J_{i,j,k,l}\}$ and $\{\tilde{J}_{i,j,k,l}\}$.

b-SYK, and Ising rescaled charging Hamiltonians. For the case of the c- and b-SYK models, data have been obtained after averaging over many disorder realizations. Results in this figure are independent of the microscopic energy scale appearing in $\hat{H}_1$.

We see that the c-SYK is the only model for which $\langle P_N(\tau^*) \rangle$ clearly increases with $N$, thereby presenting a qualitative advantage over the b-SYK and Ising QBs. Concerning the b-SYK QB, its poor performance with respect to its fermionic cousin, the c-SYK QB, indicates that random pair hopping, which both models (5) and (6) share, is not enough to guarantee a quantum advantage. The non-local JW strings for fermions are crucial, as they maximize entanglement production during the time evolution and therefore correlations between the $N$ quantum cells.

In order to certify whether the highlighted charging advantage of the c-SYK model has an exquisite quantum origin, we now consider the following bound [25]:

\begin{equation}
P_N(\tau) \leq 2 \sqrt{\Delta_r \hat{H}_0^2} \Delta_r \hat{H}_1^2 ,
\end{equation}

where $\Delta_r \hat{H}_1^2 \equiv (1/\tau) \int_0^\tau dt [\langle \hat{H}_1^2 \rangle_t - \langle \hat{H}_1 \rangle_t^2]$ and $\langle \hat{O} \rangle_t \equiv \langle \psi(t) | \hat{O} | \psi(t) \rangle$. Here, $\Delta_r \hat{H}_1^2$ represents the charging speed in the Hilbert space: larger values of such quantity correspond to trivial increases of the charging speed [25].

\begin{align}
\Delta_r^\perp \hat{H}_0^2 &\equiv \frac{1}{\tau} \int_0^\tau dt \sum_i \left[ \langle \hat{h}_i^2 \rangle_t - \langle \hat{h}_i \rangle_t^2 \right], \\
\Delta_r^\parallel \hat{H}_0^2 &\equiv \frac{1}{\tau} \int_0^\tau dt \sum_{i \neq j} \left[ \langle \hat{h}_i \hat{h}_j \rangle_t - \langle \hat{h}_i \rangle_t \langle \hat{h}_j \rangle_t \right].
\end{align}
FIG. 4. (Color online) Panel (a) The relevant quantities for the bound (13), evaluated at the optimal time \( \tau^\ast \), and averaged over disorder: time-averaged variances \( \langle \Delta_{\tau^\ast} \hat{H}_0^2 \rangle \) (blue triangles, in units of \( \omega_0^2 \)), \( \langle \Delta_{\tau^\ast} \hat{H}_1^2 \rangle \) (green squares, in units of \( J^2 \)), \( \langle \Delta_{\tau^\ast}^{\text{Ent}} \hat{H}_0^2 \rangle \) (black circles, in units of \( \omega_0^2 \)), as functions of \( N \). Dashed curves denote linear (green) and quadratic (blue, black) fits to the numerical results. The four data points corresponding to the smallest \( N \) have been always eliminated from the fits. Panel (b) The optimal power (red) \( \langle P_N(\tau^\ast) \rangle \) and the quantity in the right-hand-side of Eq. (13) (blue) are plotted as functions of \( N \), in a log-log scale and in units of \( \omega_0 J \). Dashed lines correspond to power laws \( \sim N^{1+k} \) (\( k = 0.5 \): red; \( k = 0 \): orange) and are plotted as guides to the eye. Data in this figure refer to the c-SYK QB model.

Power enhancement is genuinely linked to \( \Delta_{\tau^\ast} \hat{H}_0^2 \), a fact that hints at a quantum supremacy (i.e. supremacy over any classical battery) displayed by the c-SYK model with respect to the charging task.

The left- and right-hand-side members of the inequality (13) are displayed in panel (b), in red and blue, respectively. We clearly see a super-linear scaling with \( N \) (\( k = 0.5 \) corresponds to the red dashed straight line). We have also considered the b-SYK and Ising models, showing that, in both cases, all the quantities \( \langle \Delta_{\tau^\ast} \hat{H}_0^2 \rangle \), \( \langle \Delta_{\tau^\ast} \hat{H}_1^2 \rangle \), and \( \langle \Delta_{\tau^\ast}^{\text{Ent}} \hat{H}_0^2 \rangle \) are extensive in \( N \) [44]. In agreement with the results shown in Figs. 2–4, we thus conclude that these two QB models do not display any quantum advantage. Note that there is no contradiction between the scaling of the optimal charging power shown in Fig. 3 for the c-SYK charging protocol and the \( \sim N^{3/2} \) scaling seen in Fig. 4(b). The point is that, in the former, the rescaled Hamiltonian (9) was used. We have checked that the ratio between the two optimal charging powers yields the correct bandwidth of the c-SYK model, which scales linearly with \( N \).

We finally note that optimal charging powers scaling super-linearly with \( N \) have been found in Refs. [7, 8]. As shown in Ref. [25], such super-linear scalings do not stem from \( \Delta_{\tau^\ast} \hat{H}_0^2 \) but, rather, from \( \Delta_{\tau^\ast} \hat{H}_1^2 \). They have therefore no quantum origin. Furthermore, the models used in Refs. [7, 8] display a pathology in the thermodynamic limit [25], in the sense that they do not scale linearly with \( N \), but super-linearly. This is ultimately at the origin of the spurious super-extensive scaling of the optimal charging power found in Refs. [7, 8]. In this Letter, we have eliminated these problems by choosing the appropriate scaling [26–29, 41, 42] with \( N \) of the variance \( \langle J_{i,j,k,l}^2 \rangle = J^2/N^3 \) of the c-SYK coupling parameters.

Summary.—In summary, we have presented a numerical study of QBs where the charging Hamiltonian is based on the c- and b-SYK models [26, 27, 29]. We have used two independent strategies to show that fermionic SYK QBs display a truly genuine quantum advantage, i.e. a speed-up in the charging dynamics that stems from entanglement and is therefore unreproducible by any classical battery. This is in stark contrast with all known previous QB models [24, 25]. The first strategy, consists in comparing different QB models on equal footing, by using the rescaled Hamiltonian approach, Eq. (9). In this approach, a truly quantum advantage manifests as a linear scaling of the optimal charging power on the number \( N \) of quantum cells—see Fig. 3. The second strategy uses recently proposed bounds on the charging power, first demonstrated in Ref. [25]. In this case, the quantum advantage emerges as a non-zero super-linear scaling with \( N \) of the correlation-induced time-averaged variance (12) of the local quantum battery Hamiltonian (1). We hope that this work will stimulate further studies on QBs and experimental realizations of them on the basis of scalable solid-state technology.

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\[ \sim N^2 \]

\[ \sim N \]

\[ \sim N^{3/2} \]

\[ \sim N^2 \]

\[ \sim N \]

\[ \sim N^{3/2} \]
Supplemental Material for:
“Quantum charging supremacy via SYK batteries”

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In this Supplemental Material we provide additional information on the way in which the c-SYK model is mapped onto a spin-1/2 model, and include also a few details on the numerical calculations. We then present a formal derivation of Eq. (10) in the main text. We finally provide explicit numerical evidence for the lack of a quantum advantage in b-SYK and Ising QBs.

On the JW transformation and other details on the numerical calculations

The c-SYK model [S1] for finite N is best handled numerically after mapping it onto a spin model. This is accomplished through the JW transformation. For the sake of clarity, we here report the c-SYK model Hamiltonian [cf. Eq. (4) in the main text]:

\[ \hat{H}_{c-SYK}^{c} = \sum_{i,j,k,l=1}^{N} J_{i,j,k,l} \hat{c}_{i}^{\dagger} \hat{c}_{j} \hat{c}_{k} \hat{c}_{l}. \] (S1)

Here, \( \hat{c}_{j}^{\dagger} (\hat{c}_{j}) \) creates (annihilates) a complex spinless fermion and the usual fermionic anticommutation relations, \{\( \hat{c}_{1}, \hat{c}_{2} \)\} = \( \delta_{i,j} \), \( \{\hat{c}_{i}, \hat{c}_{j}\} = 0 \), hold true. The JW transformation, which maps spinless fermions into spin-1/2 degrees of freedom, reads as follows:

\[ \hat{c}_{j}^{\dagger} = \hat{\sigma}_{z}^{+} \prod_{m=1}^{j-1} \hat{\sigma}_{m}^{\dagger} \hat{\sigma}_{j}^{\dagger}, \]

\[ \hat{c}_{j} = \hat{\sigma}_{z}^{-} \prod_{m=1}^{j-1} \hat{\sigma}_{m} \hat{\sigma}_{j}, \] (S2)

where \( \hat{\sigma}_{j}^{\pm} = (\hat{\sigma}_{j}^{x} \pm i \hat{\sigma}_{j}^{y})/2 \).

Applying such transformation to the model in Eq. (S1), one has to distinguish three cases [S2]:

- All indices are different (\( i \neq j \neq k \neq l \)). In this case

\[ \hat{c}_{i}^{\dagger} \hat{c}_{j}^{\dagger} \hat{c}_{k} \hat{c}_{l} = \beta \prod_{\xi = \zeta_{1}+1}^{\zeta_{2}-1} \hat{\sigma}_{\xi}^{\dagger} \prod_{\chi = \zeta_{3}+1}^{\zeta_{4}-1} \hat{\sigma}_{\chi}^{-} \hat{\sigma}_{j}^{\dagger} \hat{\sigma}_{j} \hat{\sigma}_{k} \hat{\sigma}_{l}, \] (S3)

where \( \{\zeta_{1}, \zeta_{2}, \zeta_{3}, \zeta_{4}\} = \{i, j, k, l\} \) are the four reordered indices, such that \( \zeta_{1} < \zeta_{2} < \zeta_{3} < \zeta_{4} \), and \( \beta = \text{sign}(i-j) \text{sign}(k-l) \).

- Two indices are equal (e.g. \( j = l \) and \( i \neq j \neq k \)). In this case:

\[ \hat{c}_{i}^{\dagger} \hat{c}_{j}^{\dagger} \hat{c}_{j} \hat{c}_{k} = \hat{\sigma}_{i}^{\dagger} \prod_{\xi = \zeta_{1}}^{\zeta_{2}-1} \hat{\sigma}_{\xi}^{\dagger} \hat{\sigma}_{j}^{\dagger} \hat{\sigma}_{j} \hat{\sigma}_{k}, \] (S4)

where \( \{\zeta_{1}, \zeta_{2}\} = \{i, k\} \) are reordered such that \( \zeta_{1} < \zeta_{2} \).

- Indices are equal in pairs (e.g. \( j = k \) and \( i = l \)). In this case

\[ \hat{c}_{i}^{\dagger} \hat{c}_{j}^{\dagger} \hat{c}_{j} \hat{c}_{i} = \hat{\sigma}_{i}^{\dagger} \hat{\sigma}_{j}^{\dagger} \hat{\sigma}_{j} \hat{\sigma}_{i}. \] (S5)

As mentioned in the main text, in order to enforce PHS, one needs to add extra terms of the form \( \hat{c}_{i}^{\dagger} \hat{c}_{k} \) to Eq. (S1) [cf. (5) in the main text]. We can again use the JW transformation in order to write each of these one-body contributions in terms of spin-1/2 operators:

\[ \hat{c}_{i}^{\dagger} \hat{c}_{k} = \hat{\sigma}_{i}^{\dagger} \prod_{\xi = \zeta_{1}}^{\zeta_{2}-1} \hat{\sigma}_{\xi}^{\dagger} \hat{\sigma}_{k}, \] (S6)

where \( \{\zeta_{1}, \zeta_{2}\} = \{i, k\} \) are reordered such that \( \zeta_{1} < \zeta_{2} \).

Once the Hamiltonian is written in the spin-1/2 representation (spin operators do commute on different sites), one can safely write its matrix representation in the usual computational basis where the operator \( \hat{\sigma}_{j}^{z} \) is diagonal. Notice that, for the b-SYK Hamiltonian [Eq. (6)], the JW string is not required.

In order to evaluate the properties of the time-evolved state during our charging protocol (\( h = 1 \)),

\[ |\psi(\tau)\rangle = e^{-i\hat{H}_{c}^{c} \tau} \left( \bigotimes_{j=1}^{N} |\psi(y)_{j}\rangle \right), \] (S7)

we numerically integrated the equation of motion for \( |\psi(\tau)\rangle \) using a fixed-step-size fourth-order Runge-Kutta method. To ensure convergence, typical integration time steps of order \( \Delta t \approx 10^{-3} \) (in units of \( 1/J \)) were used. We checked that our choice of \( \Delta t \) is always conservative (i.e., it guarantees convergence in time of all our results, within an error bar that is negligible on the scale of the figures).

Derivation of Eq. (10) in the main text

From the Heisenberg equation of motion for \( 0 \leq t \leq \tau \) we get:

\[ \left( \frac{dE_{N}(t)}{dt} \right)^{2} = \left| \langle [\hat{H}_{0}, \hat{H}_{1}] \rangle_{t} \right|^{2}. \] (S8)
The Schrödinger-Robertson (SR) inequality \cite{S3} yields: 
\[ |\langle [\hat{H}_0, \hat{H}_1] \rangle|^2 \leq 4 \langle \delta_i \hat{H}_0^2 \rangle \langle \delta_i \hat{H}_1^2 \rangle, \]
where \( \delta_i \hat{H}^2 \equiv \langle \hat{H}^2 \rangle_t - \langle \hat{H} \rangle_t^2 \). Taking the square root of Eq. (S8), using the SR inequality, applying the integral \( \int_0^\tau dt/\tau \) to both members of Eq. (S8), and using \( E_N(0) = 0 \), we finally get the inequality:
\[ P_N(\tau) \equiv \frac{E_N(\tau)}{\tau} \leq 2 \int_0^\tau \frac{dt}{\tau} \sqrt{\langle \delta_i \hat{H}_0^2 \rangle \langle \delta_i \hat{H}_1^2 \rangle}. \] (S9)

Using the Cauchy-Schwarz inequality with respect to the scalar product induced by \( \int_0^\tau dt/\tau \), we finally get Eq. (10) in the main text, i.e.
\[ P_N(\tau) \leq 2 \sqrt{\Delta_0 \Delta_1}. \] (S10)

**Power and bounds for the b-SYK and the Ising model**

In the main text it has been shown that a QB charged through the c-SYK model is able to outperform any classical battery, since both \( \langle \Delta_0 \hat{H}_0^2 \rangle \) and \( \langle \Delta_1 \hat{H}_1^2 \rangle \) grow quadratically with \( N \) (see Fig. 4 in the main text). Time fluctuations of \( \hat{H}_0 \) are thus super-extensive. On the other hand, as expected, \( \langle \Delta_0 \hat{H}_0^2 \rangle \) is extensive in \( N \). This suggests that the bound (13), as well as the optimal power, scale as \( N^{3/2} \):
\[ \langle P_N(\tau^*) \rangle \sim N^{1+\frac{1}{2}} \quad \text{(for the c-SYK model)}, \] (S11)
a fact that is fully confirmed by our numerical calculations.
In Fig. S1, we show the same quantities for the b-SYK model [panels (a)-(c)] and for the Ising model [panels (b)-(d)]. It is evident that, in both cases, all of the above mentioned time-averaged variances, as well as the optimal charging power, grow linearly in $N$, 

$$\langle P_N(\tau^*) \rangle \sim N \quad \text{for the b-SYK & Ising models}. \quad (S12)$$

This rules out the possibility to have a genuine quantum speed-up in the charging process, by using the b-SYK and Ising charging Hamiltonians.

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