STABILITY OF THE NON-ABELIAN X-RAY TRANSFORM
IN DIMENSION $\geq 3$

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Abstract. Non-abelian X-ray tomography seeks to recover a matrix potential $\Phi : M \to \mathbb{C}^{m \times m}$ in a domain $M$ from measurements of its so called scattering data $C_\Phi$ at $\partial M$. For $\dim M \geq 3$ (and under appropriate convexity and regularity conditions), injectivity of the forward map $\Phi \mapsto C_\Phi$ was established in [20]. In this article we extend [20] by proving a Hölder-type stability estimate. As an application we generalise a statistical consistency result for $\dim M = 2$ [14] to higher dimensions.

The injectivity proof in [20] relies on a novel method by Uhlmann-Vasy [27], which first establishes injectivity in a shallow layer below $\partial M$ and then globalises this by a layer stripping argument. The main technical contribution of this paper is a more quantitative version of these arguments, in particular proving uniform bounds on layer-depth and stability constants.

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1. Introduction

Let \((M, g)\) be a compact \(d\)-dimensional Riemannian manifold with strictly convex boundary \((d \geq 2)\) and \(\Phi : M \to \mathbb{C}^{m \times m} (m \geq 1)\) a continuous matrix-potential. Suppose \(\gamma : [0, \tau] \to M\) is a unit-speed geodesic with endpoints on \(\partial M\) and consider the linear matrix differential equation

\[
\dot{U}(t) + \Phi(\gamma(t))U(t) = 0, \quad U(\tau) = \text{id}.
\]  

(1.1)

This has a unique continuous solution \(U : [0, \tau] \to \text{Gl}(m, \mathbb{C})\) and we write \(C_\Phi(\gamma) = U(0) \in \text{Gl}(m, \mathbb{C})\) for its value at the boundary. The matrix \(C_\Phi(\gamma)\) is called scattering data or non-abelian X-ray transform of \(\Phi\) (along \(\gamma\)).

The partial data problem asks whether one can recover \(\Phi\) in an open set \(O \subset M\) from measuring \(C_\Phi(\gamma)\) for geodesics \(\gamma\) that do not leave \(O\). For \(d \geq 3\), in which case the problem is formally overdetermined, this is known to be possible for smooth matrix fields and \(O \subset M\) satisfying the so called foliation condition (see Definition 1.1 below). Precisely, [20] establishes injectivity of the map

\[
C^\infty(O, \mathbb{C}^{m \times m}) \ni \Phi \mapsto (C_\Phi(\gamma) : \gamma \in \Gamma_O),
\]  

(1.2)

where \(\Gamma_O\) is the set of unit-speed geodesics \(\gamma : [0, \tau] \to M\) with \(\gamma([0, \tau]) \subset O\) and both endpoints on \(\partial M\), so called \(O\)-local geodesics. In this article we refine the injectivity result by proving a Hölder-type stability estimate for the partial data problem (see Theorem 1.3 below).

Non-abelian X-ray tomography provides the mathematical basis for the novel imaging technology of polarimetric neutron tomography [21][10], which seeks to determine a magnetic field within a medium by probing it with neutron beams and measuring the spin-change that results from traversing the magnetic field. In this setting \(\Phi\) takes values in \(\mathfrak{so}(3)\) and encodes the magnetic field and \(C_\Phi(\gamma) \in SO(3)\) describes the resulting rotation of the spin-vector for a neutron travelling along \(\gamma\). For a survey on further applications of non-abelian X-ray tomography we refer to [15].

Even in the simplest example, when \(M\) is a Euclidean ball (thus geodesics are straight lines) and we have access to full data \((O = M)\), the inverse problem described above is very challenging. It is non-linear and (for \(m \geq 2\)) no explicit inversion formula is known or expected to exist. At the same time, real-life applications demand a computational approach to ‘solve’ the inverse problem, typically in the presence of statistical noise on the measurements. An attractive and widely used such approach is Stuart’s framework of Bayesian inverse problems [4], in which \(\Phi\) is estimated from draws of a ‘posterior probability measure’, which can be computed from a finite number of observations \(C_\Phi(\gamma_1), \ldots, C_\Phi(\gamma_n)\).
From a theoretical point of view this shifts the focus to a rigorous study of the performance of Bayesian algorithms. For the non-abelian X-ray transform this was initiated in [14], where the authors prove a statistical consistency result in dimension $d = 2$. This roughly asserts that potentials $\Phi$ can be recovered from $C\Phi$ by a Bayesian algorithm with arbitrary accuracy, as the number of measurements $n \to \infty$.

One of the key ingredients in the statistical analysis of non-linear inverse problems is a quantitative stability estimate with good control on the involved constants. This principle has emerged in a series of recent papers, including [14] for the two-dimension non-abelian X-ray transform, as well as [1] and [9], which analyse the Calderón-problem and an inverse problem for the Schrödinger equation respectively. In our case, to establish consistency in $L^2$-norm, we need a stability estimate of the form

$$\|\Phi - \Psi\|_{L^2} \leq C(\Phi, \Psi) \cdot d(C\Phi, C\Psi), \quad (1.3)$$

where $C(\Phi, \Psi) > 0$ is bounded over large classes of potentials $\Phi, \Psi$ and $d(\cdot, \cdot)$ is an appropriate (semi-)metric. In [14], the authors prove such an estimate in the two-dimensional case for $d(\cdot, \cdot)$ given by the distance in an $H^1$-Sobolev-space. Using an interpolation argument they derive further stability estimates with $d(\cdot, \cdot) = \|\cdot - \cdot\|_{L^2}^\mu$ and $\mu \in (0, 1)$. Our main theorem contains a version of this Hölder-type stability estimate for $d \geq 3$ and implies essentially the same consistency result as in the previous paragraph, however with the caveat of requiring priors of higher regularity and obtaining a slower rate of convergence.

The stability analysis in [14] is valid in the case that $M$ is a simpleootnote{$(M, g)$ is called simple, if it is compact, non-trapping, free of conjugate points and has a strictly convex boundary. Examples are the Euclidean disk and small perturbations of it.} surface ($d = 2$), one has access to full data ($O = M$) and $\Phi$ takes values in $u(m)$, the space of skew-hermitian matrices. In $d \geq 3$ it suffices to have access only to partial data ($O \subset M$ with foliation condition) and general potentials with values in $\mathbb{C}^{m \times m}$ are allowed.

In a Euclidean setting the two-dimensional result is satisfying also for higher dimensions, as one can reduce to $d = 2$ by recovering $\Phi$ slice by slice. Besides the applicability to a wider class of geometries, the main motivation for studying the case $d \geq 3$ intrinsically is the availability of partial data results ($O \neq M$). This is of direct relevance to real-life applications, where one might have access only to localised measurement data. In dimension $d = 2$ however, partial data result are less well understood and not available in general [2].

The restriction to $u(m)$-valued potentials in [14] is mainly due to the use of
Pestov-identities, which are not available for general matrix-potentials. However, using the techniques from a recent injectivity result [18], one might be able to extend [14] to general potentials.

The working horse behind many partial data results in \( d \geq 3 \), for non-abelian X-ray tomography as well as boundary rigidity and some other geometric inverse problems, is a ground-breaking technique of Uhlmann and Vasy [27]. While their method automatically provides a local stability estimate for the linearised problem, there are two less welcome features: The necessity of smooth data and the need to globalise. Let us elaborate on these points to explain the main technical contributions of this article.

With microlocal analysis at the core of the method, smoothness of the underlying data (in our case the potential \( \Phi \)) is not easily relaxed to lower regularity - in particular the constants in the local stability estimate depend continuously on \( \Phi \) only in the \( C^\infty \)-topology. However, statistical consistency demands better control and one of our main contributions is to show uniformity on _arbitrarily large_ \( C^k \)-balls (for \( k \geq 0 \) sufficiently large).

By ‘globalisation’ we mean the extension of injectivity from small neighbourhoods of boundary points to larger domains, or all of \( M \), via a layer stripping argument. As the initial domain of injectivity depends on the potentials \( \Phi \), the layer stripping argument becomes more delicate and another contribution of this paper is to carefully combine the arguments from [20] and [24] to globalise stability estimates for the non-abelian X-ray transform.

1.1. Notation and Background. We denote with \( SM = \{(x, v) \in TM : |v| = 1\} \) the unit-sphere bundle of \( M \) and write \( \pi : SM \to M \) for the projection onto the base variable. \( SM \) is itself a manifold with boundary and, writing \( \nu \) for the inward-pointing unit-normal to \( \partial M \), we can decompose \( \partial SM \) into

\[
\partial_\pm SM = \{(x, v) \in SM : x \in \partial M, \pm \langle \nu(x), v \rangle \geq 0\}.
\]

Let \( X \) be the geodesic vector field on \( SM \) and \( \varphi_t \) the geodesic flow. We then write \( \gamma_{x,v}(t) = \pi(\varphi_t(x, v)) \) for the geodesic adapted to \( (x, v) \in SM \) and \( \tau(x, v) \in [0, \infty] \) for the first time that \( \gamma_{x,v} \) exits \( M \). If \( \tau(x, v) < \infty \) for all \( (x,v) \in SM \), then \( M \) is called _non-trapping_. Further we say that \( \partial M \) is _strictly convex_ if its second fundamental form is positive definite everywhere.

If \( M \) is non-trapping and has strictly convex boundary, then \( \partial_+ SM \) parametrises all complete geodesics through \( M \) and the non-abelian X-ray transform may be viewed as map

\[
C(M, \mathbb{C}^{m \times m}) \to C(\partial_+ SM, GL(m, \mathbb{C})), \quad \Phi \mapsto C_\Phi.
\]
Precisely, if $U_\Phi : SM \to \mathbb{C}^{m \times m}$ denotes the unique continuous solution (differentiable along the geodesic flow) of the boundary value problem

$$(X + \Phi)U_\Phi = 0 \text{ on } SM \quad \text{and} \quad U_\Phi = \text{id} \text{ on } \partial_- SM,$$

then $C_\Phi = U_\Phi|_{\partial_+ SM}$. More generally, if $G \subset \text{GL}(m, \mathbb{C})$ is a matrix Lie-group with Lie-algebra $g$, one can show that the non-abelian X-ray transform maps $C(M, g) \to C(\partial_+ SM, G)$.

For an open set $O \subset M$ we write $M_O \subset \partial_+ SM$ for the (open) set of all $(x, v)$ for which $\gamma_{x,v}(t) \in O$ for $0 \leq t \leq \tau(x, v)$. It parametrises the set $\Gamma_O$ of $O$-local geodesics. The following condition, introduced in this form in [20], ensures that $O$ is scanned by sufficiently many geodesics emerging from $M_O$ and allows to prove an injectivity result as stated below.

**Definition 1.1.** An open subset $O \subset M$ satisfies the foliation condition, if there is a smooth, strictly convex function $\rho : O \to \mathbb{R}$ which is exhausting in the sense that the sets $O_{\geq c} = \{x \in O : \rho(x) \geq c\} \subset M$ are compact for all $c > \inf_O \rho$.

**Theorem 1.2.** [Paternain, Salo, Uhlmann, Zhou, 2019 [20]] Let $d \geq 3$ and suppose that $M$ has strictly convex boundary and $O$ satisfies the foliation condition. Suppose $\Phi, \Psi : M \to \mathbb{C}^{m \times m}$ are smooth. Then $C_\Phi = C_\Psi$ on $M_O$ implies that $\Phi = \Psi$ on $O$. \hfill \square

In fact, the authors of [20] consider a more general situation, where scattering data is defined with respect to attenuations $A(x, v) = \Phi(x) + A_x(v)$ that may depend on the direction $v$ up to first order. That is, $\Phi$ is a matrix potential as above and $A \in \Omega^1(O, \mathbb{C}^{m \times m})$ is a matrix-valued one-form. In that case a similar result holds true, but one only has injectivity up to the gauge $A \mapsto u^{-1}du + u^{-1}Au$ ($u : O \to \text{GL}(m, \mathbb{C})$ smooth).

For the full-data problem ($O = M$), the foliation condition reduces to the existence of a strictly convex function on $M$ and is set into relation with other geometric properties of $M$ in Section 2 of [20]: For example, if $M$ (with $\partial M$ strictly convex) supports a strictly convex function, it is automatically non-trapping and contractible. Conversely, if $M$ has non-negative sectional curvatures (or non-positive sectional curvatures and it is simply connected), then it admits a strictly convex function.

Let us conclude with a brief overview of the history of the problem. Assuming a flat background geometry and access to full data, the problem was first studied by Vertgeim [29], with further pioneering work by Novikov [16] and Eskin [5], who established injectivity in dimension $d \geq 3$ and $d = 2$ respectively (up to gauge in the general problem mentioned above).

In the geometric setting and for $d = 2$, the full data problem is typically
studied on simple surfaces \((M, g)\). There injectivity of the map \(C^\infty(M, g) \to C^\infty(\partial_+ SM, G), \Phi \mapsto C_\Phi\) was first obtained for \(G = U(m)\) (the unitary group) in [19]. In the case \(G = \text{Gl}(m, \mathbb{C})\) injectivity was established under a negative curvature assumption in [17] and, very recently, for general simple surfaces [18]. Partial data results on the other hand (even for \(m = 1\)) are less well understood in \(d = 2\) [2] and there is no analogue for (1.2) for smooth (non-analytic) potentials.

1.2. Main result. Let \((M, g)\) be compact, non-trapping and with strictly convex boundary.\(^2\) Our main analytical result is the following stability estimate for the non-abelian \(X\)-ray transform.

**Theorem 1.3.** Suppose \(d \geq 3\) and \(K \subset O \subset M\) are such that \(K\) is compact and \(O\) is open and satisfies the foliation condition. Then for smooth potentials \(\Phi, \Psi : M \to \mathbb{C}^{m \times m}\) we have

\[
\|\Phi - \Psi\|_{L^2(K)} \leq C(\Phi, \Psi) \cdot \|C_\Phi - C_\Psi\|_{L^2(M_O)},
\]

where \(C > 0\) and \(\mu \in (0, 1)\) obey an estimate of the form

\[
C(\Phi, \Psi) \vee \mu(\Phi, \Psi)^{-1} \leq \omega\left(\|\Phi\|_{C^k(M)} \vee \|\Psi\|_{C^k(M)}\right)
\]

for some non-decreasing function \(\omega : [0, \infty) \to [0, \infty)\) and an integer \(k \geq 0\).

Here the Lebesgue spaces \(L^2(K)\) and \(L^2(M_O)\) (with codomain \(\mathbb{C}^{m \times m}\) suppressed from the notation) are defined with respect to the natural Riemannian volume forms on the ambient manifolds \(M\) and \(\partial_+ SM\). Further, the space \(C^k(M)\) consist of functions which are \(k\)-times continuously differentiable up to \(\partial M\) and a choice of continuous norm \(\|\cdot\|_{C^k(M)}\) (defined with respect to some atlas) is assumed to be fixed throughout the discussion. Finally, the notation \(a \vee b\) is used for the maximum of two quantities \(a, b > 0\).

In view of the local stability estimates for the linearised problem in [20, Theorem 1.3] and the situation in \(d = 2\) [14, Corollary 1.4] one might expect a stronger result, with the right hand side of (1.6) being replaced by a Lipschitz-type bound \(\leq C(\Phi, \Psi) \cdot \|C_\Phi - C_\Psi\|_{F(M_O)}\) in terms of a suitable function-space \(F\), say of Sobolev-regularity \(H^1\) or even \(H^{1/2}\). However, our result is both in line with the available estimates for the related conformal boundary rigidity problem and it is sufficient to prove statistical consistency. Let us elaborate on these points:

The conformal boundary rigidity problem (determining a Riemannian metric \(g\) on \(M\) within a fixed conformal class from its boundary distance function) shares many features with the problem at hand: It is a gauge-free non-linear

\(^2\)For partial data results \((O \neq M)\) the geometry of \(M\) outside of \(O\) is irrelevant. In this case assuming the global conditions is a matter of convenience.
problem, which in dimension $d \geq 3$ is solved with Uhlmann-Vasy’s method, also requiring a layer stripping argument to propagate injectivity into the interior of $M$. It is thus natural to compare the available stability estimates [24, Theorem 1.4] and indeed, equation (3) there is of a similar form as (1.6) here. Both here and there, the passage to the weaker Hölder-type estimate is an artefact of the globalisation procedure, which employs interpolation at every step of the layer stripping argument.

To understand the statistical consequences of Theorem 1.3, we draw the comparison with the stability estimates in [14] on a simple surface $(M, g)$. When they eventually enter the consistency analysis, they take the following form, valid for smooth potentials $\Phi, \Psi : M \rightarrow \mathbb{R}^{m \times m}$:

$$
\|\Phi - \Psi\|_{L^2(M)} \leq C_k(\Phi, \Psi) \cdot \|\Phi - \Psi\|_{L^2(\partial_+ SM)}^{(k-1)/k}, \quad k \geq 2
$$

(1.8)

with

$$
C_k(\Phi, \Psi) = c_{1,k} \exp\left( \|\Phi\|_{C^k(M)} \vee \|\Psi\|_{C^k(M)} \right)
$$

(1.9)

for constants $c_{1,k}, c_{2,k} > 0$ only depending on $(M, g)$ and $m$. Note that this differs from [14, Corollary 2.3], which asserts a Lipschitz-type $L^2$-$H^1$ stability estimate and is proved using energy estimates (which are not available in the higher dimensional setting). The formulation above can be derived by means of their forward-estimates and the interpolation inequality $\| \cdot \|_{H^1} \lesssim_k \| \cdot \|_{L^2}^{(k-1)/k} \cdot \| \cdot \|_{H^k}^{1/k}$ ($k \geq 2$) (cf. Lemma 7.4) as described in the proof of Theorem 5.16 there.

This resembles the estimates given in Theorem 1.3 and indeed, in Section 6 we show that the statistical analysis of [14] carries over to the full data case ($O = M$) in $d \geq 3$: In a Bayesian framework, and under a suitable choice of priors $\Pi_n$ on $C(M, \mathbb{R}^{m \times m})$, the following consistency result holds true:

**Theorem 1.4** (Consistency). Let $\Phi_0 \in C^\infty(M, \mathbb{R}^{m \times m})$ and suppose we observe $(X_i, V_i)$ and $Y_i = \Phi_0(X_i, V_i) + \epsilon_i$ ($i = 1, \ldots, n$), where the directions $(X_i, V_i) \in \partial_+ SM$ are drawn uniformly at random and $\epsilon_i \in \mathbb{R}^{m \times m}$ is independent Gaussian noise. Then, as the sample size $n \rightarrow \infty$, the potential $\Phi_0$ can be recovered as $L^2$-limit (in probability) of the posterior means $\mathbb{E}_{\Pi_n} [\Phi| (X_i, V_i, Y_i)_{i=1}^n] \in C(M, \mathbb{R}^{m \times m})$.

As the statistical analysis is conceptually independent of the remaining paper, a more detailed discussion of the underlying priors and the a comparison with [14] is postponed to Section 6. The theorem above is restated as Theorem 6.2 and the remarks thereafter.

Continuing our discussion of Theorem 1.3, we remark that estimate (1.7) is a way of saying that for $\Phi$ and $\Psi$ lying inside of a fixed ball $\{\| \cdot \|_{C^k(M)} \leq A\}$ ($A > 0$), one may choose the constants $C$ and $\mu$ uniformly. This is a stronger
result than the uniformity in [24, Theorem 1.4] (conformal boundary rigidity), which only holds over sufficiently small balls. However, similar to the just cited result, the required regularity $k$ for which (1.7) is true, is unknown. This is in stark contrast with the two-dimensional situation in (1.8), where one can freely choose $k \geq 2$. To the knowledge of the author, the available techniques to reduce the required regularity to some smaller $k' \ll k$ (cf. [6, Theorem 2b], where $k' = 2$), only yield uniformity for generic elements of $C^{k'}$, which is not sufficient for the statistical application mentioned above.

1.3. Key ideas and structure. Analysis of the non-abelian X-ray transform starts with a pseudo-linearisation identity that we will now describe. Given a potential $\Phi \in C^\infty(M, \mathbb{C}^{m \times m})$ we call any (smooth) solution $R : SM \to \text{Gl}(m, \mathbb{C})$ to $(X + \Phi)R = 0$ on $SM$ an integrating factor for $\Phi$. Smooth integrating factors always exist in our setting ($M$ compact, non-trapping & with strictly convex boundary) and can be used to express the non-abelian X-ray transform in terms of the linear, weighted X-ray transform

$$I_W f(x,v) = \int_0^{\tau(x,v)} Wf(\varphi_t(x,v))dt, \quad (x,v) \in \partial_+ SM,$$

(1.10)
defined for $W : SM \to \mathbb{C}^{m \times m}$ and $f : M \to \mathbb{C}^m$. Precisely, we have:

**Lemma 1.5.** Let $\Phi, \Psi \in C^\infty(M, \mathbb{C}^{m \times m})$ and suppose that $R_\Phi$ and $R_\Psi$ are smooth integrating factors for $\Phi$ and $\Psi$ respectively. Then we have

$$C_\Phi - C_\Psi = R_\Phi \cdot I_{W_{\Phi,\Psi}}(\Phi - \Psi) \cdot \alpha^* R_\Psi^{-1} \quad \text{on } \partial_+ SM,$$

(1.11)

where $\alpha(x,v) = \varphi_{\tau(x,v)}(x,v)$ is the scattering relation of $(M,g)$ and the weight $W_{\Phi,\Psi} : SM \to \text{End}(\mathbb{C}^{m \times m})$ is defined pointwise by $W_{\Phi,\Psi} A = R_\Phi^{-1} AR_\Psi$ for $A \in \mathbb{C}^{m \times m}$.

Note that the weighted X-ray transform in (1.11) is to be understood ‘one level higher’, identifying $\mathbb{C}^{m \times m} \cong \mathbb{C}^{m'}$ and $\text{End}(\mathbb{C}^{m \times m}) \cong \mathbb{C}^{m' \times m'}$ for $m' = m^2$.

**Proof.** Let $F_\Phi$ be a first integral for $R_\Phi^{-1}|_{\partial_- SM}$, that is $F_\Phi : SM \to \text{Gl}(m, \mathbb{C})$ solves $X F_\Phi = 0$ on $SM$ and $F_\Phi = R_\Phi^{-1}$ on $\partial_- SM$. Then $U_\Phi = R_\Phi F_\Phi$ satisfies (1.5) and $C_\Phi = U_\Phi|_{\partial_+ SM}$. Using the corresponding notation for $\Psi$ we have

$$U_\Phi - U_\Psi = R_\Phi \cdot (F_\Phi F_\Psi^{-1} - R_\Phi^{-1} R_\Psi) \cdot F_\Psi,$$

(1.12)

which, when restricted to $\partial_+ SM$, yields (1.11). To see this, note that $G = F_\Phi F_\Psi^{-1} - R_\Phi^{-1} R_\Psi$ satisfies $X G = -W_{\Phi,\Psi}(\Phi - \Psi)$ on $SM$ and $G = 0$ on $\partial_- SM$. The fundamental theorem of calculus now implies that $G|_{\partial_+ SM} = I_{W_{\Phi,\Psi}}(\Phi - \Psi)$ and since further $F_\Psi|_{\partial_+ SM} = \alpha^* R_\Psi^{-1}$, the proof is complete.

We can now summarise the content of the subsequent sections and lay out the general strategy to prove the main results of this article.
In section 2, we prove a forward estimate for the map $\Phi \mapsto R_\Phi$, which allows to translate stability estimates for the weighted X-ray transform into one for the non-abelian one. Further consequences are forward estimates for $\Phi \mapsto C_\Phi$, which are of interest in statistical applications. The techniques in this section are similar to the ones in [14], suitably adjusted to deal with dimension $d \geq 3$ and integrating factors taking values in the non-compact group $\text{Gl}(m, \mathbb{C})$.

Section 3 prepares the further analysis by proving a quantitative version of the microlocal technique (local inversion of scattering operators near elliptic points), introduced in the context of X-ray transforms by Uhlmann and Vasy [27]. We give a self-contained proof, emphasising quantitative bounds on the involved constants.

In section 4 we start the stability analysis by considering the weighted X-ray transform $f \mapsto I_W f$. By [20, Thm. 1.3], if $W : SM \to \text{Gl}(m, \mathbb{C})$ is a smooth invertible weight, then every convex boundary point $p \in \partial M$ has a neighbourhood $O$ such that for $K \subset O$ compact we have

$$\|f\|_{L^2(K)} \lesssim_K C \cdot \|I_W f\|_{H^1(M_O)}.$$  

(1.13)

Here both $C > 0$ and the maximal size of $O$ (say, measured by the largest radius $h > 0$ for which the ball $B(p, h) \subset O$) depend on $W$ and we will be concerned with understanding their behaviour as $W$ varies. Standard techniques imply that $C(W)$ and $h(W)$ depend continuously on $W$ in the $C^\infty$-topology, but this is not sufficient for our purposes. Using the quantitative analysis from Section 3, we can upgrade this to uniformity as long as $\|W\|_{C^k(SM)} \lesssim \|W^{-1}\|_{L^\infty(SM)}$ (for some $k \gg 1$) remains bounded.

In section 5 we use the local stability result from the previous section to successively derive further stability estimates. First, using a layer stripping argument similar to the one in [20], we extend stability to arbitrary sets satisfying the foliation condition. Next, we use the pseudo-linearisation formula to translate this into a stability estimate for the non-abelian X-ray transform and finish the proof of our main theorem.

Finally, in section 6 we illustrate the strength of Theorem 1.3 by proving a statistical consistency result similar to the one in [14]. This section is mostly expository, as the extension to higher dimensions and general $\mathbb{R}^{m \times m}$-valued potentials is fairly straightforward.

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2. Forward Estimates

In this section $(M, g)$ is a compact, non-trapping Riemannian manifold with strictly convex boundary $\partial M$ and dimension $d \geq 2$. Further, as it requires no additional effort, we work in a slightly more general setting and replace matrix potentials $\Phi : M \to \mathbb{C}^{m \times m}$ by attenuations $A : SM \to \mathbb{C}^{m \times m}$.

Recall that an integrating factor for $A$ is a solution to the transport equation $(X + A)R = 0$ on $SM$. The main result of this section then reads as follows:

**Theorem 2.1.** For every $A \in C^\infty(SM, \mathbb{C}^{m \times m})$ there exists an integrating factor $R_A \in C^\infty(SM, \text{Gl}(m, \mathbb{C}))$ with

$$
\|R_A^{-1}\|_{C^k(SM)} \leq c_{1,k} \exp(c_{2,k}\|A\|_{L^\infty(SM)}) \cdot (1 + \|A\|_{C^k(SM)})^k, \quad k \geq 0
$$

for constants $c_{1,k}, c_{2,k} > 0$ only depending on $M$ and $m$. If $A$ takes values in $u(m)$, the exponential factors can be dropped.

In order to define $R_A$, we use a standard trick to avoid differentiability issues at the glancing region $\partial SM$: We embed $M$ into the interior of a slightly larger manifold $M_1$ and extend $A$ smoothly to an attenuation $A_1 : SM_1 \to \mathbb{C}^{m \times m}$ with compact support in $SM_{\text{int}}$. Then

$$(X + A_1)U = 0 \text{ on } SM_1 \quad \text{and} \quad U = \text{id on } \partial SM_1$$

has a unique solution $U_{A_1} : SM_1 \to \text{Gl}(m, \mathbb{C})$, which is constant $\equiv \text{id}$ near $\partial SM_1$ and thus smooth on all of $SM_1$. Setting $R_A = U_{A_1}|_{SM}$ gives the desired integrating factor and the forward estimate above is a consequence of the following result, applied to the larger manifold $M_1$.

**Proposition 2.2.** Let $A \in C^\infty(SM, \mathbb{C}^{m \times m})$ and suppose $U_A \in C(SM, \text{Gl}(m, \mathbb{C}))$ solves $(X + A)U = 0$ on $SM$ and $U = \text{id}$ on $\partial SM$.

(i) Writing $\tau_\infty = \sup_{SM} \tau$, we have $\|U_A\|_{L^\infty(SM)} \leq m^{1/2} \exp(\tau_\infty \|A\|_{L^\infty(SM)})$.

(ii) If $\supp A \subset K$ for a compact set $K \subset SM_{\text{int}}$, then

$$
\|U_A\|_{C^k(SM)} \leq c_k e^{(2k+1)\tau_\infty \|A\|_{C^k(SM)}} (1 + \|A\|_{C^k(SM)})^k, \quad k \geq 1,
$$

for constants $c_k = c_k(m, K, M) > 0$.

(iii) The assertions remain true if $U_A$ is replaced by its inverse $U_A^{-1}$. Further, if $A$ takes values in $u(m)$, the exponential factors can be dropped.

**Proof of Theorem 2.1.** Following the construction outlined above, Proposition 2.2 yields an estimate of $R_A$ in terms of the norms $\|A_1\|_{C^k(SM_1)}$ and it remains to replace this by $\|A\|_{C^k(SM)}$. Formally, this can be achieved by using Seeley’s extension operator $E : C^\infty(SM, \mathbb{C}^{m \times m}) \to C^\infty(SM_1, \mathbb{C}^{m \times m})$ (Lemma 7.2). One can arrange (by multiplying with a fixed cut-off), that $\supp E A \subset K$ for all $A \in C^\infty(SM, \mathbb{C}^{m \times m})$ and a fixed $K \subset SM_{\text{int}}$. Then, as $E$ is continuous between the respective $C^k$-spaces, setting $A_1 = E A$ allows to estimate $\|A_1\|_{C^k(SM_1)} \lesssim \|A\|_{C^k(SM)}$ as desired. \(\square\)
2.1. Proof of Proposition 2.2. We start by constructing suitable commuting frames, adapting [14, Lemma 5.1] to arbitrary dimensions \( d \geq 2 \).

**Lemma 2.3.** Suppose \( \Sigma \subset \partial_+ SM \setminus \partial M \) is open and \( \{P_1, \ldots, P_{2d-2}\} \) is a commuting frame of \( T\Sigma \). Then these vector fields can be extended smoothly to the open set \( W_\Sigma = \{ \varphi_t(x,v) : (x,v) \in \Sigma, 0 \leq t \leq \tau(x,v) \} \subset SM \) to yield a commuting frame \( \{X,P_1, \ldots, P_{2d-2}\} \) of \( TW_\Sigma \).

**Remark 1.** The Lemma can be strengthened to allow \( \Sigma \subset \partial_+ SM \) with \( \Sigma \cap S\partial M \neq \emptyset \). In that case the extended vector fields are continuous on \( W_\Sigma \) and smooth on \( W_\Sigma \setminus \partial M \). (One can show that the map \( \Phi \) below is a homeomorphism on \( \Sigma \times \mathbb{R} \) and an immersion in \( \Sigma \setminus S\partial M \times \mathbb{R} \). Since we do not use the stronger result, we omit the details.)

**Proof.** Let \((N,g)\) be a no return extension of \( M \) (cf. Lemma 7.1) and denote the geodesic flow on \( N \) also by \( \varphi_t \). We claim that the map

\[
\Phi : \Sigma \times \mathbb{R} \to SN, \quad (x,v,t) \mapsto \varphi_t(x,v)
\]

is a diffeomorphism onto its image. Injectivity follows immediatley from the no-return property: If \( \Phi(x,v,t) = \Phi(y,w,s) \), then \( \gamma_{x,v} \) enters \( M \) both at times 0 and \( t-s \), which is impossible unless \( (x,v,t) = (y,w,s) \). It remains to prove that \( \Phi \) is an immersion, so let us compute its derivative at \( (x,v,t) \in \Sigma \times \mathbb{R} \): For a tangent vector \( \xi \oplus a \partial_t \in T_{(x,v)} \Sigma \oplus T_t \mathbb{R} \) we have

\[
\Phi_*(\xi \oplus a \partial_t) = d\varphi_t(x,v)(\xi) + aX(\varphi_t(x,v)) \in T_{\varphi_t(x,v)}SN.
\]

If \( \Phi_*(\xi \oplus a \partial_t) = 0 \), then the previous display implies \( \xi + aX(x,v) = 0 \) and as \( X \) is transversal to \( \Sigma \), we must have \( a = 0 \) and \( \xi = 0 \). Hence \( \Phi \) is an immersion and the claim follows from invariance of domain.

Now extend the vector fields \( P_1, \ldots, P_{2d-2} \) to \( t \)-independent smooth vector fields \( \bar{P}_1, \ldots, \bar{P}_{2d-2} \) on \( \Sigma \times \mathbb{R} \). Then \( \{\partial_t, P_1, \ldots, P_{2d-2}\} \) is a commuting frame on \( \Sigma \times \mathbb{R} \) which pushes forward along \( \Phi \) to a commuting frame \( \{X,P_1, \ldots, P_{2d-2}\} \) on \( \Phi(\Sigma \times \mathbb{R}) \subset SN \). Restricting to \( W_\Sigma = \Phi(\Sigma \times \mathbb{R}) \cap SM \) finishes the proof. \( \square \)

**Proof of Proposition 2.2.** Let us first remark why (iii) holds true. The inverse \( U_A^{-1} \) satisfies the equation \( XU_A^{-1} - U_A^{-1}A = 0 \) and forward estimates can be derived with the same arguments as for \( U_A \). Further, if \( A \) is \( u(m) \)-valued, then \( U_A \in U(m) \), which is compact. In particular \( \|U_A\|_{L_\infty(SM)} \) can be bounded by an absolute constant and no exponentials arise below.

To prove part (i), fix \( (x,v) \in SM \) and note that \( U(t) = U_A(\varphi_t(x,v)) \) solves

\[
\dot{U} + A(\varphi_t(x,v))U = 0 \quad \text{for} \quad 0 \leq t \leq \tau(x,v) \quad \text{and} \quad U(\tau(x,v)) = \text{id}.
\]

Let \( v(t) = |U(t)|_F^2 \) (with \( |.|_F \) the Frobenius norm), then \( \dot{v}(t) = 2\langle \dot{U}(t), U(t) \rangle_F = 2\langle -A(\varphi_t(x,v))U(t), U(t) \rangle_F \leq 2\|A(\varphi_t(x,v))\|_F \cdot \|U(t)\|_F^2 \), where we used the
Cauchy-Schwarz inequality and the sub-multiplicativity of the Frobenius-norm. Thus by Gronwall’s inequality (with reversed time) we have

\[ v(t) \leq v(\tau(x,v)) \exp \left( \int_t^{\tau(x,v)} 2\|A(\varphi_s(x,v))\|_F ds \right), \quad 0 \leq t \leq \tau(x,v). \]

Choose \( t = 0 \), such that the left hand side becomes \( |U_A(x,v)|^2 \). Note that \( v(\tau(x,v)) = |id|^2 = m \) and crudely bound the integral in the exponential by \( 2\tau_\infty\|A\|_{L^\infty(M)} \). This concludes the proof of (i).

In order to show (ii), we use the following inequality, which (in the unitary version) appears as part of Lemma 5.2 in [14]: If \( A, F : SM \to \mathbb{C}^{m \times m} \) are continuous and \( G \in C(SM, \mathbb{C}^{m \times m}) \) is the unique solution to \((X + A)G = -F\) on \( SM \) and \( G = 0 \) on \( \partial_- SM \), then

\[ \|G\|_{L^\infty(SM)} \leq m\tau_\infty \exp(2\tau_\infty \|A\|_{L^\infty(SM)}) \cdot \|F\|_{L^\infty(SM)}. \quad (2.1) \]

We repeat its proof: One readily checks that

\[ G(x,v) = -U_A(x,v) \int_0^{\tau(x,v)} U_A^{-1}F(\varphi(t)(x,v))dt, \quad (x,v) \in SM \]

and thus \( \|G\|_{L^\infty(SM)} \leq \tau_\infty \|U_A\|_{L^\infty(SM)} \|U_A^{-1}\|_{L^\infty(SM)} \|F\|_{L^\infty(SM)} \). The norms of \( U_A^{-1} \) can be bounded with (i) and thus (2.1) follows.

To proceed, take \( \Sigma \subset \partial_- SM \backslash S\partial M \) a small open subset (such that it admits a commuting frame). Let \( P_1, \ldots, P_{2d-2} \) be the vector fields on \( W_\Sigma \), as provided by Lemma 2.3 and write \( P^\alpha = P_1^{\alpha_1} \cdots P_{2d-2}^{\alpha_{2d-2}} \) for a multi-index \( \alpha \in \mathbb{Z}^{2d-2} \). We claim that

\[ \|U_A\|_{k,\Sigma} \overset{\text{def}}{=} \sup_{j+|\alpha| = k} \|X_j P^\alpha U_A\|_{L^\infty(W_\Sigma)} \lesssim_{k,\Sigma} e^{(2k+1)\tau_\infty} \|A\|_{L^\infty(SM)} \|A\|_{C^k(SM)} \]

for all \( k \in \mathbb{Z}_{\geq 0} \). Since finitely many such sets \( \Sigma_1, \ldots, \Sigma_n \) suffice to ensure \( K \subset \bigcup_i W_\Sigma_i \), we have \( \|U_A\|_{C^k(SM)} \leq \sum_i \sum_{l \leq k} \|U_A\|_{l,\Sigma_i} \lesssim_k e^{2(k+1)\tau_\infty} \|A\|_{L^\infty(SM)} (1 + \|A\|_{C^k(SM)})^k \) and (ii) follows.

We prove (2.2) by induction over \( k \in \mathbb{Z}_{\geq 0} \). The case \( k = 0 \) follows from part (i), so let \( k \geq 1 \) and assume the result is true for \( k-1 \). Consider \( G = X_j P^\alpha U_A \) for an integer \( j \geq 0 \) and multi-index \( \alpha \) such that \( j + |\alpha| = k \). We have

\[ (X + A)G = [A, X_j P^\alpha]U_A \] on \( SM \) and \( G = 0 \) on \( \partial_- SM \),

where \([\cdot, \cdot]\) denotes the commutator and the zero boundary values follow from \( A \) having compact support and thus \( U_A \) being constant near \( \partial_- SM \). By (2.1) we conclude that

\[ \|X_j P^\alpha U_A\|_{L^\infty(W_\Sigma)} \lesssim e^{2\tau_\infty} \|A\|_{L^\infty(SM)} \cdot \|A\|_{C^k(SM)} \|U_A\|_{L^\infty(W_\Sigma)} \]
and since $[\mathcal{A}, X^j P^\alpha]$ is a differential operator on $SM$ of order $k - 1$ and with smooth coefficients $\lesssim_k ||\mathcal{A}||_{C^k(SM)}$, we have

$$\|X^j P^\alpha U_\mathcal{A}\|_{L^\infty(W_\Sigma)} \lesssim_k e^{2\tau_x} ||\mathcal{A}||_{L^\infty(SM)} \cdot ||\mathcal{A}||_{C^k(SM)} \cdot ||U_\mathcal{A}||_{k-1,\Sigma}. \quad (2.3)$$

Inequality (2.2) follows from the induction hypothesis and we are done. □

2.2. Consequences and further forward estimates. We first recall that the standard linear $X$-ray transform

$$\mathcal{I}: C^\infty(SM) \to C^\infty(\partial_+ SM), \quad \mathcal{I}F(x, v) = \int_0^{\tau(x,v)} F(\varphi_t(x, v))dt,$$

is continuous as map $H^k(SM) \to H^k(\partial_+ SM)$ for all $k \geq 0$ [22, Theorem 4.2.1]. Independently of Theorem 2.1, this yields the following:

**Corollary 2.4.** Let $f \in C^\infty(M, \mathbb{C}^m)$ and $W \in C^\infty(SM, \mathbb{C}^{m \times m})$. Then

$$\|I_W f\|_{H^k(\partial_+ SM)} \lesssim_k \|W\|_{C^k(SM)} \cdot \|f\|_{H^k(SM)} \quad k \geq 0. \quad (2.4)$$

**Proof.** As $I_W f = \mathcal{I}(W f)$, this follows immediately from the $H^k$-continuity of $\mathcal{I}$ (in its straightforward extension to vector-valued functions) and the fact that pull-back by $\pi: SM \to M$ yields a bounded linear map $\pi^*: H^k(M) \to H^k(SM)$. □

Further, using Lemma 1.5 (pseudo-linearisation) and Theorem 2.1 we obtain the following forward-estimates for the non-abelian $X$-ray transform:

**Corollary 2.5.** Let $\Phi, \Psi \in C^\infty(M, \mathbb{C}^{m \times m})$, then

$$\|C_\Phi - C_\Psi\|_{H^k(\partial_+ SM)} \leq c_k(\Phi, \Psi) \cdot \|\Phi - \Psi\|_{H^k(M)}, \quad k \geq 0,$$

where

$$c_k(\Phi, \Psi) = c_{1,k} \exp(c_{2,k} \|\Phi\|_{L^\infty(M)} + \|\Psi\|_{L^\infty(M)}) \cdot (1 + \|\Phi\|_{C^k(M)} + \|\Psi\|_{C^k(M)})^{2k}$$

for constants $c_{1,k}, c_{2,k}$ only depending on $(M, g)$ and $m$. Further, if $\Phi, \Psi$ take values in $u(m)$, the exponential factors can be dropped.

**Proof.** We use the pseudo-linearisation identity $C_\Phi - C_\Psi = R_\Psi \cdot I_{W_\Phi, \Psi}(\Phi - \Psi) \cdot \alpha^* R_\Psi^{-1}$ from Lemma 1.5 for the integrating factors provided by Theorem 2.1. The integrating factor $R_\Psi$, acting via multiplication on $H^k(\partial_+ SM)$, has operator norm $\leq \|R_\Psi\|_{C^k(\partial_+ SM)} \leq \|R_\Psi\|_{C^k(SM)}$. A similar bound holds for $\alpha^* R_\Psi^{-1}$, as $\alpha$ is a diffeomorphism and thus, by Corollary 2.4, we obtain

$$\|C_\Phi - C_\Psi\|_{H^k(\partial_+ SM)} \lesssim_k \|R_\Psi\|_{C^k(M)} \|I_{W_\Phi, \Psi}\|_{C^k(SM)} \|\Phi - \Psi\|_{H^k(M)} \|R_\Psi^{-1}\|_{C^k(SM)}.$$ As $\|I_{W_\Phi, \Psi}\|_{C^k(SM)} \leq \|R_\Psi\|_{C^k(SM)} \|R_\Psi\|_{C^k(SM)}$, the proof is finished by applying the estimates from Theorem 2.1. □

\(^3\)Alternatively one could start with a forward estimate for $\mathcal{I}$ with respect to different function spaces and obtain corresponding results for weighted and non-abelian $X$-ray transforms.
3. Robust Local Inversion of Scattering Operators

This section prepares the local stability estimate from Section 4 by proving a quantitative version of the microlocal argument that underlies Uhlmann and Vasy’s method from [27].

Their argument relies on the following phenomenon: In the context of Melrose’s scattering calculus, ellipticity of an operator near a boundary point yields local injectivity. More precisely, if $X$ is a manifold with boundary and $A : C_c^\infty(X^{\text{int}}) \to C^\infty(X^{\text{int}})$ is a (classical) scattering pseudodifferential operator ($\psi$do), then the leading order behaviour at $\partial X$ is captured by its scattering principal symbol, which is a smooth function $\sigma_{sc} : scT_{\partial X}^*X \to \mathbb{C}$, defined on the total space of the scattering cotangent bundle over $\partial X$. Ellipticity at $p \in \partial X$ then means that

\[
\inf_{\zeta \in scT_p^*X} |\sigma_{sc}(p, \zeta)| > 0
\]

(3.1)

and implies the existence of a neighbourhood $O \subset X$ of $p$ for which

\[
\ker A \cap \{u \in L^2(X) : \text{supp}(u) \subset O\} = 0.
\]

(3.2)

Together with the Fredholm property between appropriate function spaces this can be upgraded to a stability estimate for functions supported in $O$. The purpose of this section is to show that the size of $O$ as well as constants in a stability estimate depend on the operator $A$ in a robust way, as made precise in part (ii) of the theorem below.

Let $X$ be a compact manifold with boundary, fix a boundary defining function $\rho : X \to [0, \infty)$ and write $B(\partial X, h) = \{x \in X : \rho(x) < h\}$. The main theorem of this section is formulated in terms of the locally convex spaces

- $\Psi_{sc}^{m,\ell}(X) =$ Fréchet space of classical scattering $\psi$do’s of order $(m, \ell)$
- $H_{sc}^{s,r}(X) =$ Hilbert space of Sobolev-functions of regularity $(s, r)$,

discussed in Subsection 3.1 below, and reads as follows:

**Theorem 3.1** (Local inversion of scattering operators). Let $V \subset \partial X$ be open and $K \subset X$ compact with $K \cap \partial X \subset V$. Further, for $A \in \Psi_{sc}^{m,\ell}(X)$ put $\lambda(A) = \inf |\sigma_{sc}(A)(z, \zeta)|$, where the infimum is taken over $z \in V$ and $\zeta \in scT^*_zX$.

(i) If $\lambda(A) > 0$, then there are $h, C > 0$ such that for $u \in L^2(X)$ we have

\[
\|u\|_{L^2(X)} \leq C\|Au\|_{H_{sc}^{m,-(d+1+2\ell)/2}(X)} \quad \text{if supp}(u) \subset K \cap B(\partial X, h).
\]

(3.3)

(ii) As $A$ varies, the constants $h(A)$ and $C(A)$ satisfy

\[
C(A) \lor h(A)^{-1} \leq \omega(\|A\| \lor \lambda(A)^{-1})
\]

(3.4)

for a non-decreasing function $\omega : [0, \infty) \to [0, \infty)$ and a continuous $\Psi_{sc}^{m,\ell}$-semi-norm $\|\cdot\|$. 
Morally, the reason for (3.4) to hold, is the observation that microlocal constructions (such as a finite order parametrix) only ‘cost’ finitely many derivatives of the symbol and thus one should be able to work with symbols of finite regularity $C^{k+1}$ for some $k \gg 1$. Estimate (3.4) (with $\| \cdot \|$ locally bounding the symbol and its derivatives up to $k$th order) would follow from establishing continuity with respect to the $C^{k+1}$-topology and the compactness of the embedding $C^{k+1} \rightarrow C^k$.

While the reasoning described above seems to be part of the microlocal analysis folklore, the author is not aware of any reference for the claim, let alone in the setting of scattering pseudodifferential operators on manifolds. We thus chose to follow a slightly different path by keeping track of the ‘robustness’ of microlocal constructions as introduced in Definition 3.3 below. This allows for a rigorous treatment of uniformity on $C^k$-balls, without leaving the world of smooth microlocal analysis.

Remark 2. Given $\lambda_0 > 0$, a lower bound $\lambda(A) \geq \lambda_0$ guarantees a polynomial bound $C(A) \vee h(A)^{-1} \lesssim_\omega (1 + ||A||)^r$ (for some $r > 0$) as can be deduced from Theorem 3.8 below. In general our proof does not give any control on $\omega$ and $\| \cdot \|$. In order to obtain such control one would have to further open up the microlocal analysis machinery and keep track of the precise regularity requirements and growth rates at every step. This seems to be very tedious and would potentially obscure the simplicity of the argument. It is for that reason that we chose to only consider robustness below.

3.1. The scattering calculus. We summarise here some aspects of Melrose’s scattering calculus [13], mainly with the purpose of fixing notation and gathering the most relevant results at one place. For further reference we recommend [13][28][27, Section 2] and [25, Section 3.2].

First some general notation: We denote with $\bar{\mathbb{R}}^d$ the radial compactification of $\mathbb{R}^d$, obtained by glueing $\mathbb{R}^d$ and $[0, \infty) \times S^{d-1}$ along the identification $x \mapsto (|x|^{-1}, |x|^{-1}x)$. More generally, given a vector bundles $E \rightarrow X$, one can radially compactify the fibres to obtain a bundle $\bar{E} \rightarrow X$ [13, Section 1]. Further, we let $\dot{C}(X) = \bigcap_k \rho^k C^\infty (X)$ denote the space of functions which vanish to infinite order at $\partial X$. Note that the natural inclusion $\mathbb{R}^d \subset \bar{\mathbb{R}}^d$ induces an isomorphism $\mathcal{S}(\mathbb{R}^d) \cong \dot{C}(\bar{\mathbb{R}}^d)$.

We can now recall the definition of $\Psi_{sc}^{m,\ell}(X)$, the space of classical scattering pseudodifferential operators on $X$.

Definition 3.2. A linear operator $A : \dot{C}^\infty (X) \rightarrow \dot{C}^\infty (X)$ is in $\Psi_{sc}^{m,\ell}(X)$, if the following two conditions are satisfied:

\begin{align*}
\mathcal{S}(\mathbb{R}^d) \cong \dot{C}(\bar{\mathbb{R}}^d).
\end{align*}
(i) The Schwartz-kernel of $A$ is smooth away from the diagonal of $X \times X$ and vanishes to infinite order at the boundary.

(ii) In local coordinates $(x, y) = (x, y_1, \ldots, y_{d-1})$ with $x|_{\partial X} = 0$ we have

$$Au(x, y) = \int e^{i \xi x - i \eta y} \frac{u(x, y, \xi, \eta)}{x^2 + i \eta \cdot y - y'} d\xi d\eta \frac{dx' dy'}{(x')^{d+1}},$$  \hspace{1cm} (3.5)

for all $u \in C^\infty(X)$ with compact support within the chart domain, where $a : (0, \infty)_x \times \mathbb{R}^{d-1}_y \times \mathbb{R}_\xi \times \mathbb{R}^{d-1}_\eta \to \mathbb{C}$ is smooth and satisfies

$$x^\ell \langle (\xi, \eta) \rangle^{-m} a(x, y, \xi, \eta) \in C^\infty([0, \infty)_x \times \mathbb{R}^{d-1}_y \times \mathbb{R}^d_{(\xi, \eta)}).$$  \hspace{1cm} (3.6)

Note that we use the order convention from [27], that is, $\Psi_{sc}^{m, \ell}(X)$ increases as $m$ and $\ell$ increase. The definition above differs from the (equivalent) one given in [27] in that it describes $A$ in terms of the local model $[0, \infty)_x \times \mathbb{R}^{d-1}_y$ for $X$ rather than in terms of $\mathbb{R}^d$. The formulation here is for example used in [25, Proof of Prop.4.2] and has the advantage that $(\xi, \eta)$ provide natural coordinates for the scattering cotangent bundle introduced below.

For the sake of completeness we mention here that (3.6) could be replaced by the condition

$$|((x\partial_x)^k \partial_y^\alpha \partial_{(\xi, \eta)}^\beta a(x, y, \xi, \eta))| \lesssim_{k, \alpha, \beta} x^{-\ell} \langle (\xi, \eta) \rangle^{m-|\beta|}, \quad (k, \alpha, \beta) \in \mathbb{Z}^d_{\geq 0}$$  \hspace{1cm} (3.7)

to obtain the larger class $\Psi_{sc}^{m, \ell}(X)$ of (not necessarily classical) scattering $\psi$do’s.

The advantage of using classical operators is that their principal symbols can be realised as functions, rather than as elements in a quotient space. In particular there is a natural way to measure their magnitude (in the sense of size of semi-norms), which is crucial for the quantitative aspect of Theorem 3.1.

Finally, we remark that $\Psi_{sc}^{m, \ell}(X)$ has a natural Fréchet-space structure in which a sequence of operators $A_n$ converges to $0$ iff the (weighted) symbols $a_n$ in (3.6) converge to $0$ in the $C^\infty$-topology. In this topology $\Psi_{sc}^{m, \ell}(X) \subset \Psi_{sc}^{m', \ell'}(X)$ is a closed subspace whenever $m \leq m'$ and $\ell \leq \ell'$.

Let us briefly discuss some key aspects of the scattering calculus:

**Multiplication.** The product of two scattering $\psi$do’s is again a scattering $\psi$do, in fact multiplication of operators yields a bilinear continuous map

$$\Psi_{sc}^{m, \ell}(X) \times \Psi_{sc}^{m', \ell'}(X) \to \Psi_{sc}^{m+m', \ell+\ell'}(X).$$  \hspace{1cm} (3.8)

The continuity claim can be verified by keeping track of semi-norms, when proving that scattering operators are closed under multiplication and is a direct consequence of [28, Prop.3.5].

\footnote{The equivalent statement is false in $\Psi_{sc}^{m, \ell}(X)$, classicality is needed.}
Principal Symbol. The leading order behaviour of an operator $A \in \Psi_{sc}^{m,\ell}(X)$ at $\partial X$ can be described in coordinates, where $A$ takes form (3.5), by

$$\sigma_{sc}(A)(y, \xi, \eta) = x^\ell \langle (\xi, \eta) \rangle^{-m} a(x, y, \xi, \eta)|_{x=0},$$  \hspace{1em} (3.9)

which makes sense in view of the stated regularity in (3.6). In order to understand $\sigma_{sc}$ invariantly, one defines the so called scattering cotangent bundle $^{5}\text{sc}T^*X \to X$ with fibres having the following coordinate-description:

$$\text{sc}T^*_pX = \left\{ \left. \xi \frac{dx}{x^2} + \eta \cdot \frac{dy}{x^2} \right| _p \right\} \equiv \mathbb{R}^d_{(\xi, \eta)}.$$  \hspace{1em} (3.10)

Let $\text{sc}T^*X \to X$ be the ball-bundle obtained by radially compactifying the fibres of $\text{sc}T^*X$. Then under the identification indicated in (3.10), definition (3.9) yields a smooth map $\sigma_{sc}(A) : \text{sc}T^\ast_{\partial X}X \to \mathbb{C}$, defined on the total space of the pull-back of $\text{sc}T^*X$ to $\partial X$. We call $\sigma_{sc}(A)$ the scattering principal symbol of the operator $A$.

The principal symbol map $A \mapsto \sigma_{sc}(A)$ fits into a split exact sequence

$$0 \to \Psi_{sc}^{m,\ell-1}(X) \hookrightarrow \Psi_{sc}^{m,\ell}(X) \xrightarrow{\sigma_{sc}} C^\infty(\text{sc}T^*_{\partial X}X) \to 0$$  \hspace{1em} (3.11)

of Fréchet-spaces. By this we mean that there is a linear right-inverse $r_{m,\ell} : C^\infty(\text{sc}T^*_{\partial X}X) \to \Psi_{sc}^{m,\ell}(X)$ to $\sigma_{sc}$ and that all involved maps are continuous.

The scattering principal symbol is also called ‘principal symbol at finite points’ and can be complemented by $\sigma_p$, the ‘principal symbol at fibre infinity’. While the joint symbol $(\sigma_p, \sigma_{sc})$ is needed, e.g. for regularity questions, for our purposes it suffices to keep track of the boundary behaviour.

Exactness of (3.11) is stated in [13, Prop.20] (where the scattering principal symbol is called ‘normal operator’ and denoted $N_{sc}$), while a continuous linear right split (also called quantisation map) is discussed below equation (5.30) in the same notes.

Finally we remark here that our definition of $\sigma_{sc}$ differs from the one in [27], where the authors do not incorporate the pre-factor $\langle (\xi, \eta) \rangle^{-m}$ in (3.9), which implies that ellipticity is witnessed by a lower bound $|\sigma_{sc}| \gtrsim \langle (\xi, \eta) \rangle^m$ rather than $|\sigma_{sc}| \gtrsim 1$ as in (3.1).

Sobolev-spaces. A natural scale of Hilbert-spaces that scattering operators act on, is provided by $H^{s,r}_{sc}(X)$. On $\mathbb{R}^d$ (the radial compactification of $\mathbb{R}^d$) these spaces can be defined in terms of the standard Sobolev-space on $\mathbb{R}^d$ as

$$H^{s,r}(\mathbb{R}^d) = \langle z \rangle^{-r}H^s(\mathbb{R}^d).$$  \hspace{1em} (3.12)

Formally, one checks that the one-forms $\frac{dx}{x^2}, \frac{dy}{x^2}, \ldots, \frac{dy_{d-1}}{x}$ (for local coordinates $(x, y)$ as in (3.5)) span a locally free sheaf $\mathcal{E}$ over $C^\infty(X)$. The vector bundle $\text{sc}T^*X \to X$ is then defined fibre-wise by $\text{sc}T^*_pX = \mathcal{E}(X)/I_p\mathcal{E}(X)$, where $I_p \subset C^\infty(X)$ is the ideal of functions vanishing at $p$, and the smooth structure is chosen such that the natural map $\mathcal{E}(X) \to C^\infty(X, \text{sc}T^*X)$ is an isomorphism.
In general, $H_{sc}^{s,r}(X)$ is defined by locally identifying $X$ with open subsets of $\overline{\mathbb{R}}^d$. For $s \geq 0$ they are related to the standard Sobolev-spaces $H^s(X)$ as follows:

$$
\begin{cases}
H^s(X) \subset H_{sc}^{s,r}(X) & \text{for } r \leq -\frac{d+1}{2} \\
H_{sc}^{s,r}(X) \subset H^s(X) & \text{for } r \geq -\frac{d+1}{2} + 2s
\end{cases}
$$

(3.13)

An operator $A \in \Psi_{sc}^{m,\ell}(X)$ then is continuous as map $A : H_{sc}^{s,r}(X) \to H_{sc}^{s-m,r-\ell}(X)$ and indeed the inclusion

$$
\Psi_{sc}^{m,\ell}(X) \hookrightarrow \mathcal{B}(H_{sc}^{s,r}(X), H_{sc}^{s-m,r-\ell}(X))
$$

(3.14)

into the space of bounded linear operators is continuous. The statements above are proved in [27, Section 2] and [28, Section 3.8], modulo continuity of (3.14), which follows from the open mapping theorem.

### 3.2. Robust maps

We propose the notion of robustness\(^6\) to keep track of parameter dependence in a smooth setting. This subsection is devoted to establish its basic properties (used both in the remainder of Section 3 and in Section 4) and give some key examples.

**Definition 3.3.** Let $f : U \to F$ be a (possibly non-linear) continuous map between an open set $U \subset E$ of a locally convex space into another locally convex space $F$. We say that $f$ is robust, if for any continuous semi-norm $\| \cdot \|$ on $F$ there exists a non-decreasing function $\omega : [0, \infty) \to [0, \infty)$ and continuous semi-norm $\| \cdot \|'$ on $E$ such that

$$
\|f(x)\| \leq \omega(\|x\|').
$$

(3.15)

The concatenation of robust maps, when defined, is robust again. Further, it is clear that for linear maps robustness follows from continuity. This can be extended to non-linear maps with a certain scaling behaviour (e.g. homogeneity) and even holds true uniformly for continuous families of such maps. Precisely:

**Lemma 3.4.** Let $0 \in T \subset \mathbb{R}$ be an open interval, $E$ and $F$ locally convex spaces. Suppose $f : T \times E \to F$ is a continuous map that satisfies

$$
f(t, \lambda x) = \omega(\lambda)f(t, x) \quad \text{for all } (t, x) \in T \times E, \lambda > 0
$$

(3.16)

and some non-decreasing function $\omega : [0, \infty) \to [0, \infty)$. Then there exists $\epsilon > 0$ such that the family $(f(t, \cdot) : |t| < \epsilon)$ is uniformly robust in the following sense: For every continuous semi-norm $\| \cdot \|$ on $F$ there exists a continuous semi-norm $\| \cdot \|'$ on $E$ such that

$$
\|f(t, x)\| \leq \omega(\|x\|') \quad \text{for all } |t| < \epsilon, x \in E.
$$

(3.17)

\(^6\)While the idea of ‘robustness’ is certainly familiar to many analysts, it does not seem to have an established name. We chose to give it one, as it is quite useful in the sequel.
Proof. Note that the collection of sets \( \{(t, x) : |t| < \epsilon, \|x\|' < 1\} \), where \( \epsilon > 0 \) and \( \| \cdot \|' \) runs through continuous semi-norms on \( E \), constitutes a basis for the \((0, 0)\)-neighbourhoods of \( T \times E \). Since \( \{(t, x) : \|f(t, x)\| < 1\} \subset T \times E \) is an open neighbourhood of \((0, 0)\), we can thus find \( \epsilon > 0 \) and \( \| \cdot \|' \) on \( E \) such that

\[
\{(t, x) : |t| < \epsilon, \|x\|' < 1\} \subset \{(t, x) : \|f(t, x)\| < 1\}. \tag{3.18}
\]

Now take \( |t| < \epsilon \) and \( x \in E \), then \((t, x/\|x\|')\) lies in the left set and thus

\[
\|f(t, x)\| = \omega(\|x\|') \cdot \|f(t, x/\|x\|')\| < \omega(\|x\|'), \tag{3.19}
\]
as desired. \( \square \)

A genuinely non-linear example of a robust map is given in the Lemma below. Apart from serving as an example, it will be useful in the parametrix construction below, where all the non-linearity can be reduced to a map of this form.

**Lemma 3.5.** Let \( M \) be a compact manifold (with or without boundary) and \( \chi : \mathbb{C} \to \mathbb{C} \) a smooth map with all derivatives bounded. Then \( \|\chi \circ u\|_{C^k(M)} \lesssim_{k, \chi} \|u\|_{C^k(M)}^{k, \chi} \) for all \( u \in C^\infty(M) \). In particular, \( u \mapsto \chi \circ u \) defines a robust map \( C^\infty(M, \mathbb{C}) \to C^\infty(M, \mathbb{C}) \).

**Proof.** First note that the case \( \partial M \neq 0 \) follows from the boundaryless case, as one can always embed \( M \) into a closed manifold \( N \) and decompose the map as \( C^\infty(M) \to C^\infty(N) \to C^\infty(N) \to C^\infty(M) \), i.e. into a Seeley extension (cf. Lemma 7.2), the concatenation map over \( N \) and restriction respectively. We may thus assume that \( \partial M = \emptyset \) and, to ease notation, only discuss the case of real-valued functions and \( \chi : \mathbb{R} \to \mathbb{R} \). Choose local coordinates \( x^1, \ldots, x^d \) and note that \( \partial^\alpha(\chi \circ u) \) (for \( \alpha \in \mathbb{Z}^d \) with \( |\alpha| = k \)) may be written as finite linear combination of terms of the form

\[
P_{m, \beta} u \overset{\text{def}}{=} (\chi^{(m)} \circ u) \cdot \prod_{i=1}^m \partial^{\beta_i} u, \quad m \leq k, \beta_i \in \mathbb{Z}^d_{\geq 0} \text{ with } \sum_{i=1}^m |\beta_i| = k. \tag{3.20}
\]

Let \( K \subset M \) be a compact set inside the chart that supports \( x_1, \ldots, x_d \). Then, since \( \chi \) has bounded derivatives,

\[
\|P_{m, \beta} u\|_{L^\infty(K)} \lesssim_{m, \beta} \|u\|_{C^k(M)}^{m, \beta} \quad \text{for all } u \in C^\infty(M) \tag{3.21}
\]
and

\[
\|P_{m, \beta} u_n - P_{m, \beta} u\|_{L^\infty(K)} \to 0 \quad \text{when } u_n \to u \text{ in } C^\infty(M). \tag{3.22}
\]

Since \( \|\partial^\alpha \|_{L^\infty(K)} \) provide a defining family of semi-norms for \( C^\infty(M) \), where \( \alpha \) runs through multi-indices in \( \mathbb{Z}^d_{\geq 0} \) and \( K \) through compacts inside of chart domains, the previous two displays establish the desired robustness. \( \square \)
Example 3.6. Given \( t > 0 \), take a smooth function \( \chi : \mathbb{C} \to \mathbb{C} \) with \( \chi(z) = z^{-1} \) for \( |z| > t \). Then outside the compact set \( \{|z| \leq t\} \) we have \( \partial_z^m \chi = k!(-1)^m z^{-m} \) and \( \partial_z^m \chi = 0 \) for all \( m \in \mathbb{Z}_{\geq 0} \). Thus the derivatives of \( \chi \) are all bounded and the previous Lemma applies.

Finally, we record an elementary idea to patch together robustness estimates:

Lemma 3.7. Suppose \( U \subset E \) is an open subset of a locally convex space and \( \lambda : U \to (0, \infty) \) is a continuous function. Let \( U_t = \{ x \in U : \lambda(x) > t \} \) and suppose we are given a family \( (C_t : U_t \to (0, \infty) : t > 0) \) of continuous functions that satisfy a robustness estimate of the form

\[
C_t(x) \leq \omega_t(\|x\|), \quad x \in U_t, t > 0
\]

for non-decreasing functions \( \omega_t : [0, \infty) \to [0, \infty) \) \( (t > 0) \) and a semi norm \( \| \cdot \| \) that is independent of \( t \). Then \( C(x) = \inf \{ C_t(x) : \lambda(x) \geq t/2 \} \) satisfies

\[
C(x) \leq \omega(\|x\| \lor \lambda(x)^{-1}), \quad x \in U.
\]

for a non-decreasing function \( \omega : [0, \infty) \to [0, \infty) \).

Proof. Put \( \omega(s) = \sup \{ \omega_t(s) : st > 1 \} \), then for \( x \in U \) we have \( C(x) \leq C_{\lambda(x)}(x) \leq \omega_{\lambda(x)}(\|x\|) \leq \omega_{\lambda(x)}(\|x\| \lor \lambda(x)^{-1}) \leq \omega(\|x\| \lor \lambda(x)^{-1}) \).

3.3. Proof of Theorem 3.1. The theorem follows from the following more general result, which does not require ellipticity in all fibre directions.

Proposition 3.8 (Microlocal version). Let \( t > 0 \) and \( \Gamma \subset U \subset \sigma^c T_{\partial X} \) subsets such that \( \Gamma \) is compact and \( U \) is open.

(i) For every \( A \in \Psi_{sc}^{m,\ell}(X) \) with \( |\sigma_{sc}(A)| > \lambda \) on \( U \) there exist constants \( h, C > 0 \) as well as a continuous semi-norm \( \| \cdot \|_0 \) on \( \Psi^{0,0}_{sc}(X) \) with the following property: If \( Q = q_1 Q_2 \in \Psi^{0,0}_{sc}(X) \) is the product of a function \( q_1 \in C^\infty(X) \) and an operator \( Q_2 \in \Psi^{0,0}_{sc}(X) \) such that

\[
\|\rho q_1\|_{L^\infty(X)} \|Q_2\|_0 < h \quad \text{and} \quad \text{supp} \sigma_{sc}(Q_2) \subset \Gamma,
\]

then we have

\[
\|u\|_{L^2(X)} \leq C(\|Q\|_0 \cdot \|Au\|_{H^{-m,-(\ell+1+2\ell)/2}(X)} + 2\|Q\|_0 u\|_{L^2(X)}).
\]

(ii) As \( A \) varies in the open set of operators with \( |\sigma_{sc}(A)| > t \) on \( U \), both \( C(A) \) and \( h(A) \) depend continuously on \( A \) and obey an estimate

\[
C(A) \lor h(A)^{-1} \leq \omega_t(\|A\|)
\]

for \( \omega_t : [0, \infty) \to [0, \infty) \) non-decreasing and of polynomial growth and a semi-norm \( \| \cdot \| \) on \( \Psi^{m,\ell}_{sc}(X) \). Both \( \| \cdot \|_0 \) and \( \| \cdot \| \) can be taken independently of \( \lambda \) and \( A \).

Let us first demonstrate how Theorem 3.1 follows from this result:
Proof of Theorem 3.1. By continuity we have $|σ_{sc}(A)| ≥ λ(A)$ on $U = π^{-1}(V) ⊂ T^*_ΩX$, so that (for any choice of $0 < t < λ(A)$), we can apply Theorem 3.8 and denote with $h'$ and $C'$ the constants from (i). Let $Q = q_1q_2 ∈ Ψ^{0,0}(X)$ be the product of two functions $q_1, q_2 ∈ C^∞(X)$ with $1_{B(ΩX,h)} ≤ q_1 ≤ 1_{B(ΩX,2h)}$ (for $h > 0$ to be chosen) and $1_K ≤ q_2 ≤ 1_{V'}$ (for $V'$ a neighbourhood of $K$ with $V' ∩ ∂X = V$). Now choose $h > 0$ such that

$$∥ρq_1∥_{L^∞(X)}∥q_2∥ ≤ 2h∥q_2∥ = h',$$  

then (3.25) is satisfied and we obtain (3.26). Since $\{u : supp(u) ⊂ K ∩ B(ΩX,h)\} ⊂ ker(id-Q)$, this concludes the proof of part (i) (with $∥Q∥₀$ being absorbed into the constant, i.e. we set $C = C''∥Q∥₀$). Part (ii) is a consequence of Lemma 3.7 and the fact that $∥·∥$ is independent of $t$. □

Proof of Proposition 3.8. Fix $t > 0$ and suppose $A$ satisfies $|σ_{sc}(A)| > t$ on $U$. We start by constructing a first order parametrix $A^+$ for $A$. Fix a smooth function $χ : C → C$ with $χ(z) = z^{-1}$ for $|z| > t$, then $vit(f) = χ ◦ f$ defines a robust map on $C^∞(scT^*_ΩX)$ (Lemma 3.5). Then

$$A^+ = r ◦ inv_t ◦ σ_{sc}(A) ∈ Ψ^{−m,−ℓ}_sc(X),$$  

where $r : C^∞(scT^*_ΩX) → Ψ^{−m,−ℓ}_sc(X)$ is a continuous right split for the sequence in (3.11). Note that $σ_{sc}(A^+A − id) = inv_λ(σ_{sc}A)σ_{sc}A − 1 ≡ 0$ on $]|σ_{sc}A| > λ|$, so putting $Λ = scT^*_ΩX \setminus U$, the residual $R = id − A^+A$ satisfies

$$R ∈ Ψ^{0,0}_sc(X) and supp σ_{sc}(R) ⊂ Λ.$$  

Now $A → A^+$ and $A → R$ are robust maps on $U = \{A ∈ Ψ^{m,ℓ}_sc(X) : |σ_{sc}(A)| > t on U\}$, taking values in $Ψ^{−m,−ℓ}_sc(X)$ and $Ψ^{0,0}_sc(X)$ respectively. For $A^+$ this follows from the robustness of $inv_λ$ and for $R$ from continuity of multiplication.

We now estimate the operator norm of $QR$, acting on $L^2(X) = H^{0,0}_{sc,d+1/2}(X)$. To this end, we write $QR = (ρq_1) ◦ (ρq_2R)$ and treat the two factors separately. To estimate the second factor, consider the bilinear continuous map

$$Ψ^{0,0}_sc(Ω) × Ψ^{0,0}_sc(Λ) → Ψ^{0,−1}_sc(Ω) → Ψ^{0,0}_sc(Ω) ⊂ B(L^2(X)),$$  

where the involved spaces and maps are defined as follows: For $K ⊂ scT^*_ΩX$ closed we write $Ψ^{0,0}_sc(K)$ for the closed subspace of operators $P ∈ Ψ^{0,0}_sc(X)$ with supp $σ_{sc}(P) ⊂ K$. Then the first map in (3.31) is multiplication, which takes values in $Ψ^{0,−1}_sc(Ω)$ as $Λ ∩ Γ = 0$. Now $ρq_2R ∈ B(L^2(X))$ is the image of $(Q_2,R)$ under the map (3.31) and hence its operator norm is bounded by $∥Q_2∥₀ ∥R∥₀$ for a continuous seminorm $∥·∥₀$ on $Ψ^{0,0}_sc(X)$. Further, multiplication by $ρq_1$ has operator norm $≤ ∥ρq_1∥_{L^∞(X)}$ and overall we can estimate

$$∥QR∥_{L^2(X) → L^2(X)} ≤ ∥ρq_1∥_{L^∞(X)} ∥Q_2∥₀ ∥R∥₀.$$  

(3.32)
Put \( h = \| R \|_0^{-1}/2 \). Then \( h(A)^{-1} \) is a robust function on \( U_t \) and if \( Q \) obeys (3.25), the operator norm of \( QR \) is bounded by \( 1/2 \), which means that \( \text{id} - QR \) is invertible in \( \mathcal{B}(L^2(X)) \).

Denote \( N = (\text{id} - QR)^{-1} \in \mathcal{B}(L^2(X)) \), then
\[
u = N QA^+ Au + N(\text{id} - Q)u \quad \text{for all } u \in L^2(X).
\]
and the result follows from the following considerations: \( \| N \|_{L^2(X) \to L^2(X)} \leq 2 \), further the \( L^2(X) \)-operator norm is continuous on \( \Psi_{\text{sc}}^{0,0}(X) \) and can be added to \( \| \cdot \|_0 \). We then obtain (3.26) with \( C(A) \) being twice the \( H_{\text{sc}}^{0,-\frac{d+1}{2}}(X) \to H_{\text{sc}}^{-m,-\ell-\frac{d+1}{2}}(X) \) operator norm of \( A^+ \).

So far we have established that \( C, h^{-1} : U_t \to \mathbb{R} \) are robust, which entails part (ii) with \( \| \cdot \| \) potentially depending on \( t \). However, our arguments only depend on \( t \) through the choice of \( \text{inv}_t \) and as the semi-norms in Lemma 3.5 are independent of \( \chi = \text{inv}_t \), also \( \| \cdot \| \) is independent of \( t \). Finally, the polynomial growth assertion follows as \( C(A) \) and \( h(A) \) are defined through linear maps and \( \text{inv}_t \), which is of ‘polynomial robustness’ in the sense of Lemma 3.5.

3.4. Vector valued case. Theorem 3.1 works equally well for operators that act between sections of vector bundles. In this section we discuss the necessary changes in the case of trivial bundles (which is all we need in the sequel).

Let us write \( A \in \Psi^{m,\ell}(X; \mathbb{C}^k) \) for \( k \times k \)-matrices of operators in \( \Psi^{m,\ell}(X) \), understood to act between vector-valued functions in the obvious way. The scattering principal symbol is then a matrix-valued map \( \sigma_{\text{sc}}(A) : \mathcal{S}\mathcal{C}^{\partial X} X \to \mathbb{C}^{k \times k} \) and, using the notation \( |M| = (M^*M)^{1/2} \in \mathbb{C}^{k \times k} \) for matrices \( M \in \mathbb{C}^{k \times k} \), ellipticity of \( A \) is witnessed by an inequality of the form
\[
|\sigma_{\text{sc}}(A)| > \lambda \iff \forall t \in \mathbb{C}^k : \langle \sigma_{\text{sc}}(A)t, t \rangle > \lambda |t|^2.
\]
Using this notation, Theorem 3.1 holds true for \( u \in L^2(X, \mathbb{C}^k) \) and is proved in the same way as the scalar case.
4. Local Stability of the Weighted X-ray transform

This section is devoted to the proof of the following theorem, which is a robust version of Theorem 1.3 in [20].

**Theorem 4.1.** Let $(M, g)$ be a compact Riemannian manifold of dimension $d \geq 3$ and suppose $p \in \partial M$ is a point of strict convexity. Then there exists a smooth function $\tilde{x} : M \to \mathbb{R}$, strictly convex near $p$ and satisfying

$$\tilde{x} \leq 0 = \tilde{x}(p) \quad \text{and} \quad |d \tilde{x}|_g \leq 1,$$

such that for all invertible matrix weights $W : SM \to \text{Gl}(m, \mathbb{C})$ we have:

1. There exist $C, h > 0$ with the following property: For $0 < c < h$ let $B = B(p, c/2)$ and $O = \{\tilde{x} > -c\}$, then

$$\|f\|_{L^2(B)} \leq C\|I_W f\|_{H^1(M_O)} \quad \text{for } f \in L^2(M). \tag{4.1}$$

2. As $W$ varies, the maps $W \mapsto C(W)$ and $W \mapsto h(W)$ are continuous in the $C^\infty$-topology and obey

$$h(W)^{-1} \vee C(W) \leq \omega(\|W\|_{C^k(SM)} \vee \|W^{-1}\|_{L^\infty(SM)}) \tag{4.2}$$

for some non-decreasing $\omega : [0, \infty) \to [0, \infty)$ and an integer $k \geq 1$.

3. Under small perturbations of $M$ and $p$, in a sense made precise below, one can choose $\omega$ and $k$ to be constant.

Let us remark on a few aspects of the theorem: The bound $|d \tilde{x}|_g \leq 1$ can always be achieved by scaling $\tilde{x}$ and is included as it ensures that the set $O = \{\tilde{x} > -c\}$ ($c > 0$) contains the geodesic ball $B(p, c/2)$.

In order to make the perturbation result from part (iii) precise, assume that $M = \{\vartheta \leq 0\}$ for a smooth function $\vartheta : M \to \mathbb{R}$ that is strictly convex in a neighbourhood $U$ of $p$. Then for $t > 0$ small, also the boundary of the manifolds $M_t = \{\vartheta \leq -t\} \subset M$ is strictly convex in $U$ and the Theorem applies to the weighted X-ray transform of $M_t$ (defined via integrals over the shorter geodesics with endpoints on $\partial M_t$). Then (iii) means that estimate (4.2) can be made uniform for $t > 0$ sufficiently small and $q \in \partial M_t \cap U$ close to $p$.

**Remark 3.** The compactness condition is non-essential and has only been included to simplify bound (4.2). For non-compact $M$ and $h(W)$ replaced by $h(W) \vee h^*$ for a fixed upper bound $h^* > 0$, the relevant sets from part (i) lie within a compact subset $L \subset M$ and (4.2) remains true after replacing the right hand side by $\omega(\|W\|_{C^k(L)} \vee \|W^{-1}\|_{L^\infty(L)})$. Here the semi-norm $\|\cdot\|_{C^k_{L}(SM)}$ is defined in local coordinates by taking the supremum over $L$ of derivatives up to order $k$. In particular, if $M$ can be embedded into a compact manifold $M'$, then $\|\cdot\|_{C^k_{L}(SM)} \lesssim \|\cdot\|_{C^k(SM')}$. 
Proof of Theorem 4.1. The proof essentially consists of a careful inspection of the Uhlmann-Vasy method, which is comprised of the following steps:

1. In a neighbourhood of $p$, the normal operator $I_W^* I_W$ is modified to a ‘localised normal operator’ $A_W^\chi$, defined over an auxiliary manifold $X$ with $p \in O = M \cap X$.
2. The operator $A_W^\chi$ is shown to lie in the class $\Psi^{-1,0}_{sc}(X)$ (Definition 3.2), elliptic near the ‘artificial boundary’ $\partial X$. By Theorem 3.1, it is thus locally invertible in a neighbourhood of $\partial X$.
3. A posteriori, the auxiliary manifold $X$ is chosen such that the domain of injectivity includes $O = M \cap X$. Stability estimates for $A_W^\chi$ can then be translated into ones for $I_W$.

Using Theorem 3.1(i), the constants in the resulting stability estimate are then uniform under some control on $\sigma_{sc}(A_W^\chi)$ and $\|A_W^\chi\|$ (for a semi-norm $\|\cdot\|$ on $\Psi^{-1,0}_{sc}(X)$). As $A_W^\chi$ depends homogeneously and (in the $C^\infty$-topology) continuously on $W$, this easily translates to uniformity in terms of $W$ and eventually yields (ii).

We will now discuss the three steps above in more detail. However, as the method has been used in several previous articles (e.g. [27][24][25][20]), the exposition below will be brief and focus on the application of our quantitative result from the previous section.

**Step (1).** We embed $M$ into a a closed manifold $(N, g)$ of the same dimension and extend the weight smoothly to $W : SN \to \mathbb{C}^{m \times m}$. As $p \in \partial M$ is a point of strict convexity, it admits a neighbourhood $U \subset N$ and coordinates $(\tilde{x}, y) : U \to \mathbb{R} \times \mathbb{R}^{d-1}$ for which

$$\{\tilde{x} \geq 0\} \cap M = \{p\} \quad \text{and} \quad \tilde{x} \text{ is strictly convex near } p$$  (4.3)

(cf. [20, Section 3] for a construction). The following constructions are carried out with respect to a small parameter $0 < c < c_0$ (and $c_0$ chosen later), noting dependencies when necessary. Change coordinates to $(x, y) = (\tilde{x} + c, y)$, such that $\{x \geq 0\}$ is the intersection of $U$ with a compact manifold $X \subset N$ with strictly concave boundary near $p$. Consider the parametrisation

$$\mathbb{R}_x \times \mathbb{R}_y^{d-1} \times \mathbb{R}_\lambda \times S^{d-2}_\omega \to SU, \quad (x, y, \lambda, \omega) \mapsto \frac{\lambda \partial_x + \omega \partial_y}{|\lambda \partial_x + \omega \partial_y|_g},$$  (4.4)

with vectors parallel to $\partial_x$ missing in the image (they are negligible, as eventually we are interested in geodesics that are ‘nearly tangent’ to $\partial X$). Pulling back the geodesic flow via (4.4) yields integral curves

$$\gamma_{x,y,\lambda,\omega}(t) = \left(\gamma_{x,y,\lambda,\omega}^{(1)}(t), \gamma_{x,y,\lambda,\omega}^{(2)}(t)\right) \in \mathbb{R} \times \mathbb{R}^{d-1}$$  (4.5)
and one may consider the following ‘localised normal operators’, acting on smooth functions $f : [0, \infty)_x \times \mathbb{R}^{d-1}_y \to \mathbb{C}^m$ with suitable decay at $x = 0$:

$$A_W^x f(x, y) = x^{-2}e^{-1/x} \int \int \int W^*(x, y, \lambda, \omega)(Wf) (\gamma_{x, y, \lambda, \omega}(t), \hat{\gamma}_{x, y, \lambda, \omega}(t)) e^{1/\gamma_{x, y, \lambda, \omega}(t)} \chi(x, y, \lambda, \omega/x) \ dt d\lambda d\omega. \quad (4.6)$$

This corresponds to equation (4.1) in [20]. Let us discuss the ingredients of (4.6) in detail: Without loss of generality we may assume that the interior of the box $B = [0, 2c]_x \times [-1, 1]^{d-1}_y$ contains the portion of $M$ within $U \cap X$. Further, the ‘localising function’ $\chi$ is assumed to satisfy

$$\text{supp} \chi \subset B \times S^{d-2} \times [-C_0, C_0] \quad (4.7)$$

for some $C_0 > 0$ and will later be chosen such that $A_W^x$ is elliptic in an appropriate sense. The domain of integration in (4.6) is $[-\delta_0, \delta_0]_t \times \mathbb{R}_\lambda \times S^{d-2}_\omega$, where $\delta_0 > 0$ is chosen small enough to satisfy the following criteria: First we ask that the curves (4.5), starting from $B$, do not leave the coordinate chart for $|t| \leq \delta_0$. Second, and after decreasing $c_0$ if necessary, we ask that

$$\gamma_{x, y, \lambda, \omega}^{(1)}(t) \geq \frac{C_1}{2} \left( t + \frac{\lambda}{C_1} \right)^2 + \left( x - \frac{\lambda^2}{2C_1} \right), \quad (x, y) \in B, |t|, |\lambda| < \delta_0, \quad (4.8)$$

for some $C_1 > 0$. (See equation (3.2) in [27], where the this inequality is derived for $C_1$ essentially being a lower bound of the Hessian of $\bar{x}$ near $p$).

**Step (2).** Note that $A_W^x$ may be viewed as operator $C_c^\infty(X^{\text{int}}, \mathbb{C}^m) \to C_c^\infty(X^{\text{int}}, \mathbb{C}^m)$ with Schwartz-kernel compactly contained in $(U \cap X)^2$. The crux is now that $A_W^x$ fits into Melrose’ scattering calculus in the sense that $A_W^x \in \Psi^{-1.0}_X(X)$ and, upon a judicious choice of localiser $\chi$, is elliptic near $\partial X \cap M$. In particular, Theorem 3.1 (local inversion of scattering operators) can be used.

In order to give a precise statement, we recall that that the constructions above depend on a parameter $c > 0$ and there is a whole family of operators $A_W^x(c)$, defined over sub-manifolds $X_c \subset N$. We may assume that there is a flow $\psi_c$ on $N$, defined for small $c > 0$, for which $X_c = \psi_c(X_0)$.

**Theorem 4.2.** Upon choosing $c_0, \lambda_0 > 0$ sufficiently small, we have:

(i) For all smooth localisers $\chi$ with (4.7), the operator $A_W^x(c) \in \Psi^{-1.0}_X(X_c)$.

Further, allowing $\chi$ to depend continuously on $c$, the map

$$[0, c_0) \times C^\infty(SN, \mathbb{C}^{m \times m}) \to \Psi^{-1.0}_X(X_0), \quad (c, W) \mapsto \psi_c^* A_W^x(c) \quad (4.9)$$

is continuous with respect to the natural Fréchet-topologies.

---

7 Note that in [20], the authors write $\chi = \chi(\lambda/x)$, suppressing the dependency on $(x, y, \omega)$.
(ii) There exists a localiser \( \chi \), smooth, satisfying (4.7) and depending continuously on \( c \), such that for all \((c,W)\) we have

\[
|\sigma_{sc}(A^X_W(c)(z,\zeta))| \geq \lambda_0 W_\ast(p)^2, \quad z \in \partial X_c \cap M, \zeta \in \mathcal{W}^*_c X_c. \tag{4.10}
\]

Sketch of Proof. The proof is essentially carried out in Section 4 of [20]. We sketch the main aspects, highlighting dependencies on the weight.

Either by first computing the Schwartz-kernel ([20, Lem.4.1.]) or directly (akin to [25]), one verifies that \( A^X_W \) has an oscillatory integral expression of the form (3.5) and the pseudodifferential-property as well as the continuous dependence can be checked directly. We note here that continuous dependence on \( c \) is already implicitly used in [14] and continuous dependence on \( W \) is akin to continuous dependence on the metric as stated in e.g. [25, Prop.4.2].

To prove (ii), we first fix \( c \). Then the symbol in said oscillatory integral expression, restricted to \( x = 0 \), takes the form

\[
a(0,y,\xi,\eta) = \iint e^{i\xi(\hat{\lambda}t + \alpha(0,y,0,\omega))i^2 + i\eta \omega t} \cdot e^{-\hat{\lambda}t - \alpha(0,y,0,\omega)i^2} \times W^*W(0,y,0,\omega)\chi(0,y,\hat{\lambda},\omega)d\lambda d\theta d\omega,
\]

where \( \alpha(x,y,\lambda,\omega) = (d/dt)^2 \gamma^{(1)}_{x,y,\lambda,\omega}(t) > 0 \) (say, for \((x,y) \in B\)) and the integral domain is \( \mathbb{R}_\hat{\lambda} \times \mathbb{R}_t \times S^{d-2}_\omega \). For the particular choice \( \chi(x,y,\hat{\lambda},\omega) = \exp(-\hat{\lambda}^2/(2\alpha(x,y,\lambda,\omega))) \) (multiplied with a cut-off in \((x,y)\) to ensure that it is supported in \( B\)), the integral in the last display can further be evaluated to obtain a non-zero multiple of

\[
\langle \xi \rangle^{-1} \int_{S^{d-2}} (W^*W)(0,y,0,\omega)e^{-|\eta_\omega/\langle \xi \rangle|^2/2\alpha(0,y,0,\omega)}d\omega,
\]

which corresponds to the second display below equation (4.10) in [20]. Following the reasoning of [20, proof of Prop.4.3] below said expression yields

\[
\langle (\xi,\eta) \rangle a(0,y,\xi,\eta) \geq 2\lambda_0 \cdot \|W^{-1}\|^{-2}_{L^\infty(SN)} \tag{4.13}
\]

for a constant \( \lambda_0 \) only depending on the local geometry near \( p \). Here it was used that \( W^*W(0,y,0,\omega) \) is bounded from below by the square of the smallest singular value of \( W \), which is in turn lower-bounded by \( \|W^{-1}\|^{-2}_{L^\infty(SN)} \).

The localiser \( \chi \) above has full support in \( \hat{\lambda} \) and thus fails to satisfy (4.7). In the proof of [20, Prop.4.3] \( \chi \) is thus approximated by localisers with compact \( \hat{\lambda} \)-support, thus obeying (4.7) for some \( C_0 > 0 \). From (4.11) it follows that the approximation is uniform in \( W \), at least under an a priori bound \( \|W\|_{L^\infty(SN)} \leq 1 \). This proves part (ii) for all \( W \) with \( \|W\|_{L^\infty(SN)} \leq 1 \) and the general case follows from a scaling argument, noting that both sides of (4.10) are homogeneous in \( W \) of degree 2.

Finally we comment on the \( c \)-dependency: Note that \( \alpha(x,y,\lambda,\omega) \) (and thus \( \chi \))
implicitly depends on $c$ through the choice of $x = \bar{x} + c$. However, the dependence is clearly continuous and $\alpha$ can be bounded in terms of the geometry near $p$. In particular, the bound (4.10) is uniform in $c$.

Let us now assume that $W$ is invertible at every point, such that $\|W^{-1}\|_{L^\infty(SN)}^{-1} > 0$. Then by Theorem 4.2 the operator $A^x_W(c)$ is locally elliptic for suitably chosen $\chi$ and sufficiently small $c > 0$. In particular Theorem 3.1 can be applied and, for constants $C, \hbar > 0$ (depending on $W$ and $c$), we obtain

$$\|f\|_{L^2(X)} \leq C\|A^x_W(c)f\|_{H^{1/2+(d+1)/2}(X,c)}, \quad \text{if supp } f \subset M \cap B(\partial X_c, h).$$

(4.14)

As $A^x_W$ is homogeneous in $W$, we can apply Lemma 3.4 to upgrade continuity of (4.9) to robustness. Further using (4.10), the uniformity statement of Theorem 3.1(ii) translates to a bound

$$\omega(W, c) + h(W, c)^{-1} \leq \omega(\|W\|_{C^k(SN)} \vee \|W^{-1}\|_{L^\infty(SN)}),$$

valid for sufficiently small $c > 0$ and all smooth weights $W : SN \to \text{Gl}(m, \mathbb{C})$. Here $\omega : [0, \infty) \to [0, \infty)$ is a non-decreasing function and $k \geq 1$.

**Step (3).** From now on we argue with a fixed weight $W : SN \to \text{Gl}(m, \mathbb{C})$ and keep track of how the arising constants depend on

$$A = \|W\|_{C^k(SN)} \vee \|W^{-1}\|_{L^\infty(SN)} > 0.$$  

(4.16)

Fix $0 < c < h$, then $O_c = M \cap X_c$ lies in $M \cap B(\partial X_c, h)$ and consequently (dropping the $c$-subscripts from now on)

$$\|f\|_{L^2(O)} \lesssim_A \|A_W^x f\|_{H^{1/2+(d+1)/2}(X)}, \quad f \in L^2(M),$$

(4.17)

where it is understood that $f$ is extended by zero outside of $M$. In order to translate this into a stability estimate for $I_W$, we factor the operator $A_W^x$ as

$$A_W^x f = x^{-2}e^{-1/x}L^\mu_W \tilde{I}_W(e^{1/x} f), \quad f \in L^2(M)$$

(4.18)

where the operators $L^\mu_W$ and $\tilde{I}_W$ are defined as follows: We may assume that $\tilde{U}$ (viewed as a manifold with boundary) is simple and denote with $\tau_U : \tilde{U} \to [0, \infty)$ its exit time. Note that we can write $\chi(x, y, \lambda/\omega)d\lambda d\omega = \mu(z, v)dv$ on $SU$ for a smooth function $\mu : SU \to \mathbb{R}$ with compact support. Then

$$L^\mu_W : \text{C}(\partial_+ \tilde{U}) \to \text{C}(U), \quad L^\mu_W u(z) = \int_{S_{zN}} W^*(z, v)u^z(z, v)\mu(z, v)dv,$$

(4.19)

where $u^z$ extends $u$ constant along the geodesic flow. Further $\tilde{I}_W$ is the weighted X-ray transform, defined with respect to the manifold $\tilde{U}$ and (4.18) is evident, as $f|_U$ is supported in $M \cap U$ and thus no additional mass is collected by integrating along complete geodesics in $\tilde{U}$.
Define $\bar{\mathcal{M}} \subset \partial_+ S\bar{U}$ to consist of initial conditions $(z, v)$ for which $z \in X$, the geodesic $\gamma_{z,v}(t)$ enters $B$ for some $0 \leq t \leq \tau_U(z, v)$, but does not hit $\partial X \cap M$. After decreasing $h$ if necessary, we can assume that

$$\{ \varphi_t(z, v) : 0 \leq t \leq \tau_U(z, v) \} \cap \text{supp } \mu = \emptyset \text{ for } (z, v) \in \partial_+ S\bar{U} \setminus \bar{\mathcal{M}}. \quad (4.20)$$

Indeed, assume that $h < C_1/C_0^2 \land \delta_0/(2C_0)$, where $\delta_0, C_0, C_1$ are the constants from (4.7) and (4.8). Then if the integral curve starting at $(z, v) \in \partial_+ S\bar{U}$ enters the support of $\mu$ at, say $(x, y, \lambda, \omega)$, we must have $0 < x < 2c < 2h$ and $|\lambda/x| < C_0$, which implies that $|\lambda| < \delta_0$ and $x - \lambda^2/(2C_1) > x(1-hC_0^2/(2C_1)) > x(1-hC_0^2/C_1) > 0$. In particular the right hand side of (4.8) is non-negative and the curve cannot intersect $M \cap \partial X$.

To proceed, take $K \subset O$ compact (such as the geodesic ball $B(p, c/2)$, when $\tilde{x}$ is scaled to satisfy $|\nabla \tilde{x}| \leq 1$). We then have for all $f \in L^2(M)$

$$\|f\|_{L^2(K)} \lesssim_K \| e^{-1/x} f\|_{L^2(O)} \lesssim_K \| x^{-d-1/2} e^{-1/2} L_W^\mu \tilde{I}_W f\|_{H^1(U)}, \quad (4.21)$$

where the first estimate follows from the fact that $e^{1/x}$ and all its derivatives are bounded on $K$ and the second estimate follows from equation (4.17) and inclusion $H^{1,\ell}(X) \subset x^{-\ell} H^{1,0} \subset x^{-\ell} H^1(X)$ for $\ell = -(d+1)/2$. Note that the function on the right hand side in (4.21) is compactly supported in $U$ (due to the support condition on $\mu$) and that $x^{-(d-1)/2} e^{-1/2}$ and all of its derivatives extend by zero to a bounded function on $U$. Thus

$$\|f\|_{L^2(K)} \lesssim_{K,A} \| L_W^\mu \tilde{I}_W f\|_{H^1(U)}, \quad \text{for all } f \in L^2(M) \quad (4.22)$$

and it remains to bound the operator norm of $L_W^\mu$ and relate $\tilde{I}_W$ to the transform $I_W$ we are actually interested in.

**Lemma 4.3.** For all $k \geq 0$ the operator $L_W^\mu : H^k(\bar{\mathcal{M}}) \to H^k(U)$ is bounded with operator norm $\lesssim \|W\|_{C^k(S\bar{U})}$.

**Proof of Lemma 4.3.** We prove the lemma in a slightly more general setting, when $\bar{\mathcal{M}} \subset \partial_+ S\bar{U}$ is any open subset with closure not intersecting $S\bar{U}$ and $\mu : S\bar{U} \to \mathbb{R}$ is a smooth function with compact support satisfying (4.20). The lemma then follows from factorising $L_W^\mu$ as

$$H^k(\bar{\mathcal{M}}) \xrightarrow{E} H^k_c(\partial_+ S\bar{U}^\text{int}) \xrightarrow{\sharp} H^k(S\bar{U}) \xrightarrow{\mu W^*} H^k(S\bar{U}) \xrightarrow{\pi} H^k(U) \quad (4.23)$$

with the following factors: $E$ is an extension operator (cf. Lemma 7.2), which may be chosen to map to compactly supported functions, as $\bar{\mathcal{M}}$ is assumed to have compact closure in $\partial_+ S\bar{U}^\text{int} = \partial_+ S\bar{U} \setminus S\partial \bar{U}$. Due to condition (4.20), the precise choice of $E$ is irrelevant. Next, the map $\sharp$, defined below (4.19), is continuous, as under the isomorphism

$$\{(z, v, t) \in \partial_+ S\bar{U}^\text{int} \times \mathbb{R} : 0 < t < \tau_U(z, v) \} \cong S\bar{U}, \quad (z, v, t) \mapsto \varphi_t(z, v) \quad (4.24)$$
it corresponds to pull-back by the projection \( \text{pr}_1 : \partial_+ S \bar{U}^{\text{int}} \times \mathbb{R} \to \partial_+ S \bar{U}^{\text{int}} \). Multiplication by \( \mu W^* \) is clearly bounded with operator norm \( \lesssim_\mu \| W \|_{C^k(S \bar{U})} \).

Finally, \( \pi_* \) is the push-forward along the base-projection, which is well known (and easily checked in coordinates) to be \( H^k \)-continuous. \( \square \)

Lemma 4.4. \( \| \tilde{I}_W f \|_{H^k(M)} \lesssim_k \| I_W f \|_{H^k(M_O)} \) (\( k \geq 0 \)) for all \( f \in L^2(M) \).

Proof of Lemma 4.4. Define \( \beta : M_O \to \tilde{M} \) by \( \beta(z,v) = \varphi_{-\tau_u(z,-v)}(z,v) \), then \( \beta^*(I_W f) = I_W f \) on \( M_O \). Now pull-back \( \beta^* : H^k(\beta(M_O)) \to H^k(M_O) \) (\( k \geq 0 \)) is an isomorphism, as \( \beta \) extends across the closure of \( M_O \) to a diffeomorphism onto its image. Thus \( \| \tilde{I}_W \|_{H^k(M)} \lesssim \| \tilde{I}_W f \|_{H^k(\beta(M_O))} \lesssim \| I_W f \|_{H^k(M_O)} \), where the first inequality follows from the fact that \( f \) is supported in \( M \) and thus \( \text{supp} \tilde{I}_W f \subset \beta(M_O) \). \( \square \)

We can now finish the proof of Theorem 4.1. Using (4.22) together with the previous two lemmas yields \( \| f \|_{L^2(K)} \lesssim_{K,A} \| I_W f \|_{H^1(M_O)} \) for all \( f \in L^2(M) \) and, taking \( K \) to be the geodesic ball \( B = B(p,c/2) \) and making the \( W \)-dependency explicit again,

\[
\| f \|_{L^2(B)} \leq C'(W) \| I_W f \|_{H^1(M_O)}, \quad f \in L^2(M),
\]

(4.25)

where \( C'(W) \leq \omega'(\| W \|_{C^k(SN)} \lor \| W^{-1} \|_{L^{-1}_{\infty}(SN)}) \) for \( \omega' : [0, \infty) \to [0, \infty) \) non-decreasing. One can further replace the norms on \( SN \) by their counterparts on \( SM \), as \( \| I_W f \|_{H^1(M_O)} \) only depends on \( W|_{SM} \). Thus (i) and (ii) are proved.

Finally, part (iii) is clear from the above: When \( p \) is slightly perturbed to some \( p' \in \partial M \), the ball \( B(p', c/2) \) remains within \( O \) and \( K \) may be chosen accordingly. Small perturbations of \( M \) correspond to an affine change of variables in \( \tilde{x} \) and are thus inconsequential. This concludes the proof. \( \square \)
5. Proof of the Stability Estimate

Let \((M, g)\) be compact, non-trapping and with strictly convex boundary \(\partial M\). We complete the proof of Theorem 1.3.

5.1. Layer Stripping Argument. We first derive a (global) stability estimate for the linearised problem.

**Theorem 5.1.** Let \(d \geq 3\) and suppose that \(K \subset O \subset M\), such that \(K\) is compact and \(O\) is open and satisfies the foliation condition. Then for \(f \in C^\infty(M, \mathbb{C}^m)\) and \(W \in C^\infty(SM, \text{GL}(m, \mathbb{C}))\), we have

\[
\|f\|_{L^2(K)} \leq C(f, W) \cdot \|I_W f\|_{\mu(W)}^{\mu(W)} L^2(O),
\]

where \(C > 0\) and \(\mu \in (0, 1)\) obey an estimate

\[
C(f, W) \vee \mu(W)^{-1} \leq \omega(\|f\|_{C^2(M)} \vee \|W\|_{C^k(SM)} \vee \|W^{-1}\|_{L^\infty(SM)})
\]

for some non-decreasing \(\omega : [0, \infty) \to [0, \infty)\) and an integer \(k \geq 2\).

Let us outline the strategy of proof for Theorem 5.1. Using the strictly convex exhaustion function on \(O\), we can stratify \(K\) into finitely many layers, where the number of layers depends on the weight \(W\). As each layer has a strictly convex boundary, one can use the local stability result in Theorem 4.1 and propagate the stability estimate into the interior of \(O\) layer by layer via an induction argument. More concretely, Theorem 4.1 allows to bound the norm of \(f\) within a certain layer in terms of the weighted X-ray transform, defined with respect to geodesics confined to that layer. As we are actually interested in the transform along complete geodesics in \(M\), an error occurs. By virtue of our forward estimates, this error can be bounded in terms of the magnitude of \(f\) in the previous layers, which is controlled by the induction hypothesis.

**Remark 4.** The Hölder-exponent \(\mu\) in the theorem is of order \(2^{-N}\), where \(N\) is the number of layers needed to stratify \(K\). This in turn is of order \(N = O(h^{-1})\), where \(h\) is the 'depth' from Theorem 4.1. The integer \(k\) that appears in the theorem is essentially the same as in the local stability estimates (Theorem 4.1), in particular a hypothetical universal bound \(k \leq c_d\) in Theorem 4.1 would remain true in Theorem 5.1.

**Remark 5.** For a fixed weight \(W\), the result can be improved to allow control on the Hölder-exponent \(\mu\) at the cost of needing bounds on higher derivatives of \(f\). Precisely, for any \(\mu \in (0, 1)\) we have \(\|f\|_{L^2(K)} \leq \omega(\|f\|_{C^\ell(M)}) \cdot \|I_W f\|_{L^2(O)}^{\mu}\) for \(\omega : (0, \infty) \to (0, \infty)\) non-decreasing (and dependent on the fixed weight \(W\)) and \(\ell \gg 1\) sufficiently large. To see this, one needs to amend Lemma 5.2 below by using different interpolation spaces.
We first discuss some notation and auxiliary results that are used in the proof of Theorem 5.1. In the following we fix a strictly convex function $\rho : O \to \mathbb{R}$ with compact super-level sets $O_{\geq c} = \{ x \in O : \rho(x) \geq c \}$ for $c > \inf_O \rho$. Then
\[
M_c = \{ x \in M^{\text{int}} : \rho(x) \leq c \}
\] (5.3)
is a (possibly non-compact) manifold with strictly convex boundary and geodesics in $M_c$ with endpoints on the level set $\{ \rho = c \}$ can be parametrised by the set
\[
\mathcal{M}_c = \{ (x, v) \in SM^{\text{int}} : \rho(x) = c, d\rho(v) \leq 0, \gamma_{x,v}(\tau(x, v)) \in O \}.
\] (5.4)
We denote with $I_W^c : \mathcal{M}_c \to \mathbb{C}^m$ the weighted X-ray transform on $M_c$, defined via integrals along the portion of geodesics within $M_c$. The following Lemma compares this with the full X-ray transform on $M$ and provides the key estimate that drives the layer stripping argument.

**Lemma 5.2** (Error-bound). Let $f \in C^\infty(M, \mathbb{C}^m)$ and $W \in C^\infty(SM, \mathbb{C}^{m \times m})$, then for all $0 < \mu \leq 1$ we have
\[
\| I_W^c f \|^2_{H^1(M_c)} \lesssim_{\mu} C(f, W) \left( 1 + \frac{\| f \|^2_{L^2(O_{\geq c})}}{\| I_W f \|^2_{L^2(M_\mu)}} \right) \cdot \| I_W f \|^\mu_{L^2(M_\mu)},
\] (5.5)
where $C(f, W) > 0$ is bounded when $\| f \|_{H^2(M)} \vee \| W \|_{C^2(SM)}$ is bounded.

**Proof.** Each geodesic in $M_c$ with endpoints on the level set $\rho = c$ can be extended to a complete $O$-local geodesic in $M$ and we denote the corresponding map between initial conditions by
\[
\beta_c : \mathcal{M}_c \to \mathcal{M}_O \subset \partial_+ SM, \quad (x, v) \mapsto \varphi_{-\tau(x, -v)}(x, v).
\] (5.6)
The weighted X-ray transform on $M_c$ can then be written as
\[
I_W^c(x, v) = I_W(1_{M_c} f)(\beta_c(x, v)),
\] (5.7)
where $1_{M_c}$ is the indicator function of $M_c$. As $\beta$ extends smoothly to the closure of $\mathcal{M}_c$, pull-back by $\beta^{-1}$ defines a bounded map $H^s(\beta(\mathcal{M}_c)) \to H^s(\mathcal{M}_c)$ and for all $s \in \mathbb{R}$ we have
\[
\| I_W^c f \|_{H^s(M_c)} \lesssim_{c, s} \| I_W(1_{M_c} f) \|_{H^s(M_\mu)} \leq \| I_W f \|_{H^s(M_\mu)} + \| I_W(1_{O_{\geq c}} f) \|_{H^s(M_\mu)}.
\] (5.8)
The last term accounts for the error that is made by integrating along complete geodesic in $M$ rather than the portion within $M_c$. We can bound this error by a forward-estimate (Cor. 2.4), as long as the truncated function $1_{O_{\geq c}} f$ is of regularity $H^s$. This restricts the choice of $s$ to $|s| < 1/2$, for which we obtain
\[
\| I_W^c f \|_{H^s(M_c)} \lesssim_{c, s} \| I_W f \|_{H^s(M_\mu)} + \| W \|_{C^1(SM)} \| f \|_{H^s(O_{\geq c})}
\] (5.9)
In order to estimate the $H^1$-norm of $I_W f$, we employ the interpolation inequality $\| \cdot \|_{H^1}^2 \leq \| \cdot \|_{L^2} \cdot \| \cdot \|_{H^2}$ on $\mathcal{M}_c$ and estimate the $H^2$-term via the forward-estimate\(^8\)

$$\| I_W f \|_{H^2(\mathcal{M}_c)} \lesssim \| W \|_{C^2(SM)} \| f \|_{H^2(O)}. \tag{5.10}$$

Combining the estimates in the preceding displays (for $s = 0$) and bounding the first factor in $\| I_W f \|_{L^2(\mathcal{M}_O)} = \| I_W f \|_{L^2(\mathcal{M}_O)}^{1-\mu} \| I_W f \|_{L^2(\mathcal{M}_O)}^{\mu}$ via another forward estimate, the result follows. \(\square\)

The next result is a technical Lemma that provides a convenient stratification of $K$ into layers. The parameter $h > 0$ below will later be the ‘intial penetration depth’ from Theorem 4.1.

**Lemma 5.3.** Suppose $K \subset O$ and $|\nabla \rho| \geq 1$ on $K$.

(i) For every $h > 0$ there exists a radius $0 < r(h) \leq h$ (non-decreasing in $h$) such that for $p \in K \cap \partial \mathcal{M}_c$ with $\text{dist}(p, \partial \mathcal{M}) > h/2$ we have

$$B(p, r(h)) \cap \mathcal{M}_c \subset \bigcup_{(x, v) \in \beta(M_c)} \gamma_{x, v}([0, \tau(x, v)]). \tag{5.11}$$

(ii) For all $h > 0$ there are finitely many numbers

$$\sup_K \rho = c_0 > c_1 \geq \cdots \geq c_N > c_{N+1} = \inf_K \rho \quad (N = O(h^{-1}))$$

as well as points $p_{ij} \in K$ ($i = 0, \ldots, N, j = 1, \ldots, J_i$) with the following properties: We have $p_{0j} \in \partial \mathcal{M}, p_{ij} \in \{\rho = c_i\}$ ($i = 1, \ldots, N$) and

$$\{x \in K : c_i \geq \rho(x) \geq c_{i+1}\} \subset \bigcup_{j=1}^{J_0} B(p_{0j}, h) \cup \bigcup_{j=1}^{J_i} B(p_{ij}, r) \tag{5.12}$$

for $i = 0, \ldots, N$ (where the second union is redundant for $i = 0$).

**Proof.** Let us denote the set on the right hand side of Lemma 5.11 by $V_c$. It is straightforward to see that $\partial \mathcal{M}_c \subset V_c$ and that $V_c$ is open. In particular, $V_c$ contains an open ball around each point of $\partial \mathcal{M}_c$. As the set of points on $\partial \mathcal{M}_c$ with $\text{dist}(\cdot, \partial \mathcal{M}) \geq h/2$ is compact, the radius of the balls can be chosen uniformly (depending on $h$), which is equivalent to the statement of Lemma 5.3(i).

For part (ii) we let $N(h) = 2[(\sup_K \rho - \inf_K \rho)/r(h)]$ and put $c_i = c_{i-1} - r/2$ for $i = 1, \ldots, N$, where $c_0 = \sup_K \rho$. The boundary points $p_{0j}, \ldots, p_{0\theta}$ are then chosen such that the $h$-balls around them cover the compact set $\partial \mathcal{M} \cap K$. Now let $x \in K$ be such that $c_i \leq \rho(x) \leq c_{i+1}$ for some $i = 0, \ldots, N$. If $i = 0$ or $\text{dist}(x, \partial \mathcal{M}) < h/2$, then $x \in B(p_{0j}, h)$ for some $j = 1, \ldots, J_0$. If $i \geq 1$ and $\text{dist}(x, \partial \mathcal{M}) \geq h/2$ we claim that $d(x, p) < r$ for some $p \in \partial \mathcal{M}_c$. Due

\(^8\)This follows from Corollary 2.4, applied to a suitable compact extension of $\mathcal{M}_c$. 
STABILITY OF THE NON-ABELIAN X-RAY TRANSFORM IN DIMENSION \( \geq 3 \)

\[ \rho = c_i \]

\[ p_{i1}, p_{i2}, p_{i3} \]

\[ K \]

\[ p_{01}, p_{02}, p_{03} \]

Figure 1. The layers from Lemma 5.3

to the compactness of \( \partial M_{c_i} \cap \{ \text{dist}(\cdot, \partial M) \geq h/2 \} \), finitely many such points \( p_{i1}, \ldots, p_{iJ} \) suffice to establish (5.12), so it remains to verify the claim. Indeed, if we let \( t \mapsto c_x(t) \) be the unit-length curve with \( c_x(0) = x \) and \( d\rho(\dot{c}_x(t)) = |\nabla \rho(c_x(t))| \), then \( \rho \) increases along \( c_x \) and by [20, Lem.2.5] the curve stays in \( O \) until it hits the boundary of \( M \). Let \( \ell \geq 0 \) be the first time for which \( p = c_x(\ell) \in \partial M \cup \partial M_{c_i} \). Then

\[ d(x, p) \leq \ell \leq \int_0^\ell d\rho(\dot{c}_x(t))dt = \rho(p) - \rho(x) \leq c_i - c_{i+1} \leq r/2 \]  

and we must have \( p \in \partial M_{c_i} \) and \( x \in B(p, r) \), as desired. \( \square \)

The next Lemma is of importance for the full data problem \( (O = M) \) and allows to perturb convex foliations in a way that shifts the point of degeneracy.

**Lemma 5.4.** Suppose \( \rho : M \to \mathbb{R} \) is a smooth and strictly convex. Then there exists another \( \tilde{\rho} : M \to \mathbb{R} \), smooth and strictly convex, such that \( \rho \) and \( \tilde{\rho} \) achieve their global minima at different points.

**Proof.** Suppose \( \rho \) achieves its minimum at the point \( x^* \in M \) and let \( V \) be a smooth vector field on \( M \) which is tangent to \( \partial M \) and non-vanishing at \( x^* \). Denote the flow of \( V \) by \( (\psi_t : t \geq 0) \), then \( \psi_t^* \rho \in C^\infty(M, \mathbb{R}) \) is strictly convex for \( t \) sufficiently small and achieves its (unique) minimum at \( x_t^* = \psi_{-t}(x^*) \). Since \( V(x^*) \neq 0 \), we have \( x_t^* \neq x^* \) for \( t > 0 \) sufficiently small and thus \( \tilde{\rho} = \psi_t^* \rho \) and \( \rho \) achieve their minimum at different points. \( \square \)

**Proof of Proposition 5.1.** Let \( \rho : O \to \mathbb{R} \) be a strictly convex exhausting function and denote \( \rho_* = \inf_O \rho \). We first reduce to the situation that

\[ |\nabla \rho(x)| \geq 1 \quad \text{and} \quad \rho(x) > \rho_* \quad \text{for} \ x \in K. \]  

(5.14)
Indeed, after scaling $\rho$ if necessary, (5.14) can only fail, when $O = M$ [20, Lemma 2.5] and in this case we argue as follows: Take $\tilde{\rho}$ as in Lemma 5.4, then we may choose $\epsilon > 0$ such that $M = \{ \rho \geq \rho_x + \epsilon \} \cup \{ \tilde{\rho} \geq \inf_M \tilde{\rho} + \epsilon \} = K \cup \tilde{K}$. Then $K$ and $\tilde{K}$ satisfy (5.14) for $\rho$ and $\tilde{\rho}$ respectively and the corresponding stability estimates (5.1) can be combined to bound $\|If\|_{L^2(M)}$.

In the remaining proof we argue with fixed $f \in C^\infty(M, \mathbb{C}^m)$ and $W \in C^\infty(SM, \text{Gl}(m, \mathbb{C}))$ and keep track of the dependency of our constructions on

$$A = \|f\|_{C^2(M)} \vee \|W\|_{C^k(SM)} \vee \|W^{-1}\|_{C^\infty(SM)} \quad (5.15)$$

for an integer $k \geq 2$ to be specified. Let us first summarise the consequences of Theorem 4.1: Each $p \in K$ is a strictly convex boundary point of either $M$ itself or of the manifold $M_c$, defined in (5.3). We can thus apply Theorem 4.1, either with respect to the local X-ray transform $I_W$ on $M$ or the one on $M_c$, which we denote by $I_{W}^c$. Thus, for all $f \in L^2(M, \mathbb{C}^m)$ we have

$$\|f\|_{L^2(B(p,h))} \leq C \|I_W f\|_{H^1(M_O)}, \quad p \in K \cap \partial M \quad (5.16)$$

$$\|f\|_{L^2(B(p,r) \cap M_c)} \leq C \|I_{W}^c f\|_{H^1(M_c)}, \quad p \in K \cap \partial M \setminus B(\partial M, h/2) \quad (5.17)$$

where $C, h > 0$ depend on $W$ and $r = r(h)$ is as in Lemma 5.3(i). By part (iii) of the theorem, the choice of regularity $k$ that appears in (4.2) can be made uniform over the compactum $K$, and will be fixed from now on (assuming $k \geq 2$ without loss of generality). We then have $C \vee h^{-1} \lesssim_A 1$.

We proceed by stratifying $K$ into layers $\{x \in O : c_i \geq \rho(x) > c_{i+1}\} \ (i = 0, \ldots, N)$ for $c_0, \ldots, c_{N+1}$ as in Lemma 5.3(ii) with $N \lesssim h^{-1} \lesssim_A 1$. We will prove inductively that

$$\|f\|_{L^2(O_{c_i})} \lesssim_A \|I_W f\|^2_{L^2(M_O)}, \quad i = 1, \ldots, N + 1 \quad (5.18)$$

which implies (5.1). For $i = 0$ this is a straightforward consequence of (5.16). Indeed, for every $p \in \partial M$, we can use the interpolation inequality $\| \cdot \|^2_{H^1} \leq \| \cdot \|^2_{L^2} \cdot \| \cdot \|_{H^2}$ on $M_O$ and a forward estimate (Thm. 2.4) to obtain

$$\|f\|_{L^2(B(p_{0,j}, h))} \lesssim_A \|I_W f\|^{1/2}_{L^2(M_O)}, \quad j = 1, \ldots, J_0, \quad (5.19)$$

where the points $p_{01}, \ldots, p_{0J_0}$ are as in Lemma 5.3(ii). As the corresponding $h$-balls cover $O_{c_1}$, this implies (5.18) for $i = 1$. Next assume the estimate has been established for some $1 \leq i < N$ and consider the points $p_{i1}, \ldots, p_{iJ_i}$ from the Lemma. Then (5.17) and Lemma 5.2, combined with the induction hypothesis which allows to bound the bracketed term in (5.5), yield

$$\|f\|_{L^2(B(p_{ij}, r) \cap M_{c_i})} \lesssim_A \|I_W f\|^2_{L^2(M_O)}^{-(i+1)} \quad (5.20)$$
A similar bound can be achieved on the balls $B(p_{ij}, h)$ (decreasing the Hölder-exponent as in the proof of Lemma 5.2) and together with the induction hypothesis we conclude (5.18) for $i + 1$. This finishes the proof. \(\square\)

5.2. **Proof of Theorem 1.3.** We conclude the main stability theorem by combining the linear estimates from the previous section with pseudo-linearisation formula and the bounds on integrating factors from Theorem 2.1.

**Proof of Theorem 1.3.** Let $\Phi, \Psi \in C^\infty(M, C^{m \times m})$ and recall from Lemma 1.5, that $C_\Phi - C_\Psi = R_\Phi I_{W_\Phi, \Psi} (\Phi - \Psi) \alpha^* R_\Psi^{-1}$, where $W_\Phi, \Psi A = R_\Phi^{-1} AR_\Psi$ and we may choose smooth integrating factors $R_\Phi$ and $R_\Psi$ as in Theorem 2.1.

Now for $K \subset O \subset M$ as in the theorem, we can apply Theorem 5.1 to obtain

$$\|\Phi - \Psi\|_{L^2(K)} \leq C(\Phi - \Psi, W_\Phi, \Psi) \cdot \|C_\Phi - C_\Psi\|_{L^2(M_O)} (5.21)$$

with $C(\Phi - \Psi, W_\Phi, \Psi) \vee \mu(W_\Phi, \Psi)^{-1}$ bounded above by

$$\omega(\|\Phi - \Psi\|_{C^2(M)} \vee \|W_\Phi, \Psi\|_{C^k(SM)} \vee \|W^{-1}_\Phi, \Psi\|_{L^\infty(SM)}) (5.22)$$

for a non-decreasing function $\omega : [0, \infty) \to [0, \infty)$. It remains to bound the norms in the previous display in terms of $\|\Phi\|_{C^k(M)} \vee \|\Psi\|_{C^k(M)}$. Note that $\|W_{\Phi, \Psi}\|_{C^k(SM)} \lesssim \|R_\Phi\|_{C^k(M)} \cdot \|R_\Psi\|_{C^k(M)}$, hence the proof is finished by using the bounds from Theorem 2.1. \(\square\)
6. Statistical application

In this section we demonstrate the scope of our stability estimate (Theorem 1.3) by showing how it can be used to establish a statistical consistency result. We will focus on the full data problem \((O = M)\) and discuss the two dimensional results from [14] alongside with the case \(d \geq 3\). Let us therefore assume \((M, g)\) is a compact Riemannian manifold with strictly convex boundary and that we are in either of the following cases:

(A) \(d = 2\) and \(M\) is simple
(B) \(d \geq 3\) and \(M\) admits a strictly convex function

In both cases we assume for simplicity\(^9\) that (as a smooth manifold) \(M\) is the closed unit-ball in \(\mathbb{R}^d\). We further assume that the potentials \(\Phi\) take values in either \(\mathfrak{so}(m) = \{A \in \mathbb{R}^{m \times m} : A^T = -A\}\) or \(\mathfrak{gl}_m(\mathbb{R}) = \mathbb{R}^{m \times m}\) and write \(g\) to denote either choice.

The plan for this section is as follows: We first record all necessary estimates at one place, then give a brief overview of the Bayesian approach of inverse problems and recall the main statistical theorem from [14], including a sketch of its proof. Finally, in the last subsection, we explain how the proof can be amended to obtain a consistency result in case (B).

In order to keep the overlap with [14] at a minimum, the discussion below is brief and heavily relies on [14]. For more background on the statistical framework we refer to the books [7] and [8].

6.1. Available Estimates. In both cases the available forward- and stability-estimates take the following form:

\[
\|C\Phi - C\Psi\|_{L^2(M)} \leq c_1(\Phi, \Psi) \cdot \|\Phi - \Psi\|_{L^2(M)} \quad (6.1)
\]

\[
\|C\Phi\|_{L^\infty(M)} \leq c_2(\Phi) \quad (6.2)
\]

\[
\|\Phi - \Psi\|_{L^2(M)} \leq C(\Phi, \Psi) \cdot \|C\Phi - C\Psi\|_{L^2(\partial_+ SM)} \mu(\Phi, \Psi) \quad (6.3)
\]

Here \(c_1(\Phi, \Psi), c_2(\Phi), C(\Phi, \Psi) > 0\) and \(\mu(\Phi, \Psi) \in (0,1)\) may depend on the potentials. The validity of the estimates and the uniformity properties of the constants can be summarised as follows:

- The forward estimates (6.1) and (6.2) are the same in case (A) and (B) and hold true for smooth potentials \(\Phi, \Psi : M \to g\). If \(g = \mathfrak{so}(m)\), then \(c_1\) and \(c_2\) are constant, due to the compactness of \(SO(m)\). If \(g = \mathfrak{gl}_m(\mathbb{R})\), then \(c_1(\Phi, \Psi)\) and \(c_2(\Phi)\) are uniform on \(L^\infty\)-balls.

---

\(^9\)For \(d = 2\) and \(d = 3\) no generality is lost, as any manifold satisfying (A) or (B) is automatically diffeomorphic to a Euclidean ball. In higher dimensions this might fail, but the author is not aware of any counterexamples.
In case (A) and for $g = \mathfrak{so}(m)$ we can choose any integer $k \geq 2$. Then (6.3) holds true for smooth $\Phi, \Psi : M \to \mathfrak{so}(m)$ with $\mu(\Phi, \Psi) = (k-1)/k$ and $C(\Phi, \Psi)$ uniform on $C^k$-balls.

In case (B) and for $g = \mathfrak{gl}_m(\mathbb{R})$ there exists an integer $k \gg 1$ such that (6.3) holds true for $\Phi, \Psi : M \to \mathfrak{gl}_m(\mathbb{R})$ with both $C(\Phi, \Psi)$ and $\mu(\Phi, \Psi)$ uniform on $C^k$-balls.

Here we say that a quantity is ‘uniform on $F$-balls’ (for $F = L^\infty(M, g)$ or $F = C^k(M, g)$) if its supremum (resp. infimum) over $\{\Phi, \Psi : M \to g$ smooth : $\|\Phi\|_F \vee \|\Psi\|_F \leq A\}$ is finite (resp. $> 0$) for all $A > 0$.

The forward estimates are proved in Corollary 2.5 for a general non-trapping manifold (with strictly convex boundary). The stability estimate for case (B) is the content of our main theorem (Thm. 1.3) and the version for case (A) is discussed below the main theorem.

**Remark 6.** An important difference between case (A) and (B) lies in the role of ‘regularity parameter’ $k$ and Hölder-exponent $\mu$, which -in the statistical analysis below- determine the choice of prior and the rate of contraction respectively. In case (A) one can effectively choose the Hölder exponent arbitrarily close to 1 (by sending $k \to \infty$), while in case (B), our method of proof yields an unknown $k$ and there is no control over the Hölder-exponent. See also remark 9.

### 6.2. Statistical Background

The statistical question we are concerned with arises in following experimental setup: Suppose for $\Phi \in C(M, g)$ we observe the data $(X_i, V_i, Y_i)_{i=1}^n$, where

$$Y_i = C\Phi(X_i, V_i) + \epsilon_i, \quad i = 1, \ldots, n,$$

with directions $(X_i, V_i)$ $(i = 1, \ldots, n)$ drawn independently and uniformly\(^{10}\) from $\partial_+SM$ and independent additive noise given by

$$\epsilon_i = (\epsilon_{ij} : 1 \leq j \leq \dim g) \in \mathbb{R}^{\dim g} \equiv g \quad \text{for } \epsilon_{ij} \sim N(0,1) \text{ i.i.d.} \quad (6.5)$$

We write $P_\Phi^n = \mathcal{L}(D_n|\Phi)$ for the law of $D_n = (X_i, V_i, Y_i : 1 \leq i \leq n)$, arising from (6.4) with potential $\Phi$. The statistical experiment just described is then encoded in the collection of probability measures $(P_\Phi^n : \Phi \in C(M,g))$ on the sample space $\mathcal{D}^n = (\partial_+SM \times g)^n$.

The Bayesian approach to estimate $\Phi$ from a sample $D^n = ((X_i, V_i, Y_i) : 1 \leq i \leq n) \in \mathcal{D}^n$ is to choose a prior $\Pi_n$ on $C(M, g)$ and compute the posterior probability under the sample $D^n$ of a (Borel-measurable) set $B \subset C(M, g)$.

---

\(^{10}\)Uniform here means that the law of $(X_i, Y_i)$ is the standard Riemannian volume-form on $\partial_+SM$, normalised to have mass 1.
according to the formula

\[
\Pi_n(\Phi \in B|D^n) = \frac{\int_B p^n_\Phi(D^n)\Pi(\mathrm{d}\Phi)}{\int p^n_\Phi(D^n)\Pi(\mathrm{d}\Phi)},
\]

(6.6)

where \(p^n_\Phi(D^n)\) is the likelihood of \(D^n\) being generated from \(\Phi\). Precisely, \(p^n_\Phi = p_\Phi^{1} \otimes \cdots \otimes p_\Phi^{1}\) (\(n\)-times), where \(\log p_\Phi^{1}(x, v, y) = \frac{-1}{2}|C_\Phi(x, v) - y|^2_F - \frac{\dim g}{2} \log(2\pi)\)

for \((x, v, y) \in D^1\) and \(|\cdot|_F\) is the Frobenius-norm.

Given the posterior one can estimate \(\Phi\), for example by the posterior mean which in our setting exists as Bochner-integral in \(C(M, g)\). From a frequentist perspective one then asks how well \(\Phi\) is estimated, when the data is generated from (6.4) with a ‘true’ potential \(\Phi_0\) and a first such quality assessment is given by the posterior consistency results below.

### 6.3. Posterior consistency in case (A).

In order to state the posterior consistency result of [14], we first review the construction of priors (in arbitrary dimension \(d \geq 2\)), focusing on their key example based on Matérn-Whittle-processes.

For a given choice of regularity parameter \(\alpha > d/2\), define a base prior \(\Pi = \Pi(\alpha)\) on \(C(M, \mathbb{R})\) as law of a centred Gaussian process \((f(x) : x \in M)\) with covariance \(\mathbb{E}f(x)f(y) = \int_{\mathbb{R}^d} e^{i(x-y)\xi}\langle\xi\rangle^{-2\alpha} \mathrm{d}\xi\), where it is understood that \(M \subset \mathbb{R}^d\). This so called Matérn-Whittle-process of regularity \(\alpha\) is a standard prior choice in non-parametric Bayesian statistics (Example 11.8 in [7]) and satisfies

\[
\text{RKHS}(\Pi) = H^\alpha(M, \mathbb{R}), \quad \Pi(C^k(M, \mathbb{R})) = 1 \text{ for } k \in \mathbb{Z} \cap [0, \alpha - d/2)
\]

(6.7)

where RKHS(\(\cdot\)) stands for the ‘reproducing kernel Hilbert-space’.

The prior \(\Pi_1\) on \(C(M, g)\) is then obtained by drawing each component (in an identification \(g \equiv \mathbb{R}^{\dim g}\)) independently from \(\Pi\). For \(n \geq 2\) the prior \(\Pi_n\) is defined by scaling \(\Pi_1\), precisely

\[
\Pi_n = \mathcal{L}\left(n^{-\frac{d}{4\alpha+2\alpha}} \Phi\right), \quad \text{for } \Phi \sim \Pi_1,
\]

(6.8)

where \(\mathcal{L}(\cdot)\) denotes the law of a random variable.

Then in case (A) (\(M\) is a simple surface), the following result holds true:

**Theorem 6.1** (Thm. 3.2 in [14]). *Suppose we are in case (A) above, \(g = \mathfrak{so}(m)\) and \(\alpha > 3\). Then for every \(\Phi_0 \in C^\infty(M, \mathfrak{so}(m))\), there is a \(\gamma > 0\) such that

\[
\Pi_n(\Phi : \|\Phi - \Phi_0\|_{L^2(M)} \geq n^{-\gamma}|D^n|) \to 0 \quad \text{as } n \to \infty
\]

(6.9)

in \(P^n_{\Phi_0}\)-probability. Here \(\Pi_n(\cdot|D^n)\) are the posteriors, defined in (6.6), with respect to the scaled Matérn-Whittle-priors in (6.8) of regularity \(\alpha\).
Remark 7 (Generalisations). The theorem remains true for a larger class of base-priors (specified in [14, Condition 3.1]). Further, the regularity of $\Phi_0$ can be relaxed and, by varying $\alpha$, one has control over the rate of contraction $\gamma$ (Remark 3.3 in [14]).

Remark 8. The scaling rate $\nu = d/(4\alpha + 2d)$ in (6.8) is chosen such that, writing $t_* = 2t/(2 + t)$ for $t > 0$, we have

$$
(4\nu/(1 - 4\nu))_* = d/\alpha,
$$

which arises as exponent in a classical $L^2$-entropy bound for the unit-ball $B^n \subset H^1(M, g) = \text{RKHS}(\Pi_1)$ (cf. Lemma 7.5).

Sketch of proof. Let $\delta_n = n^{-\alpha/(2\alpha + d)}(= n^{\nu - 1/2})$. Using (6.10) and a theorem of Li-Linde [12, Thm.1.2], one computes the small ball probability

$$
- \log \Pi_n(\|\Phi\|_{L^2(M)} \leq \delta_n) \lesssim n\delta_n^2.
$$

The event in the last probability can be changed to $\|\Phi - \Phi_0\|_{L^2(M)} \leq \delta_n$ by a standard argument (Anderson’s Lemma, cf. [8, Cor.2.6.18]) and expressed in terms of the likelihoods $p^n_\Phi, p^n_{\Phi_0}$ by using the forward estimates. A general contraction theorem ([14, Thm.5.13]) then implies that, for some sufficiently large $m' > 0$, we have

$$
\Pi_n(\Phi : h(p^n_\Phi, p^n_{\Phi_0}) \leq m'\delta_n|D^n) \xrightarrow{P_{\Phi_0}} 1, \quad \text{as } n \to \infty.
$$

Here $h(p^n_\Phi, p^n_{\Phi_0})$ denotes the Hellinger-distance, which is $\approx \|C_\Phi - C_{\Phi_0}\|_{L^2(\partial_1, M)}$, as the scattering data is $SO(m)$-valued ([14, Lem.5.14]).

By (6.7) it follows for $0 \leq k < \alpha - d/2$ that the events $F'(A) = \{\|\Phi\|_{C^k(M)} \leq A\} (A > 0)$ have $\Pi_n$-mass approaching 1 as $n \to \infty$ (Fernique’s theorem, cf. [8, Thm.2.1.20]), which suggests that one can intersect the event in (6.12) with $F'(A)$ without destroying the limit. To make this precise one shows, using Borell’s isoperimetric inequality [3], that the slightly smaller events $F_n(A) = \{\Phi_1 + \Phi_2 : \|\Phi_1\|_{L^2(M)} \leq \delta_n, \|\Phi_2\|_{H^k(M)} \leq A\} \cap F(A)$ obey

$$
- \log \Pi_n(F_n(A)^c) \geq \omega(A)n\delta_n^2 \quad \text{and} \quad \log \mathcal{N}(F_n, h, \delta_n) \lesssim A n\delta_n^2
$$

with $\omega(A)$ unbounded and non-decreasing in $A$ ([14, Lem.5.17]). Then, for $A > 0$ sufficiently large, [14, Thm.5.13] indeed implies that

$$
\Pi_n(\Phi : \|C_\Phi - C_{\Phi_0}\|_{L^2(\partial_1, SM)} \leq A\delta_n, \|\Phi\|_{C^k(M)} \leq A|D^n) \xrightarrow{P_{\Phi_0}} 1,
$$

as $n \to \infty$ [14, Thm.5.19]. If $\alpha > 3$, we may choose $k \in \mathbb{Z} \cap [2, \alpha - d/2)$ and apply stability estimate (6.3) with Hölder-exponent $(k - 1)/k$. Thus on the event in the previous display we have

$$
\|\Phi - \Phi_0\|_{L^2(M)} \leq (A'\delta_n)^{(k-1)/k}
$$

(6.15)
for some $A' > 0$ which incorporates the constant from the stability estimate. Choosing a slightly slower rate $0 < \eta < (k - 1)/k$, the constant $A'$ can be absorbed in the limit $n \to \infty$ and thus
\[ \Pi_n(\Phi : \|\Phi - \Phi_0\|_{L^2(M)} \leq \delta_n^n, \|\Phi\|_{C^k(M)} \leq A'|D_n) \to 1 \] (6.16)
in $P_{\Phi_0}$-probability. Dropping the constraint $\|\Phi\|_{C^k(M)} \leq A'$ yields (6.9) and finishes the proof. \hfill \Box

6.4. Posterior consistency in case (B). The proof above can be adapted to case (B) ($M$ of dimension $d \geq 3$, supporting a strictly convex function) and $g = gl_m(\mathbb{R})$ to obtain the following result:

**Theorem 6.2.** Suppose we are in case (B) above and $g = gl_m(\mathbb{R})$. Then there exist $\alpha > 0$ and $\gamma > 0$, such that for all $\Phi_0 \in C^\infty(M, g)$ we have
\[ \Pi_n(\Phi : \|\Phi - \Phi_0\|_{L^2(M)} \geq n^{-\gamma}|D_n) \to 0 \quad \text{as } n \to \infty \] (6.17)
in $P_{\Phi_0}$-probability. Here $\Pi_n(\cdot|D_n)$ is again the posterior defined in (6.6) with respect to the scaled Matérn-Whittle-priors in (6.8) of regularity $\alpha$. \hfill \Box

Under the hypotheses of the theorem and essentially with the same arguments as in [14] one can use the theorem above to derive a consistency result for the posterior mean. This is defined as $\bar{\Phi}_n(D_n) = E_{\Pi_n}[\Phi|D_n]$ and exists as Bochner-integral in $C(M, g)$. Using the precise exponential convergence rate in (6.17) (above withhold for simplicity), one then shows that
\[ P_{\Phi_0}(\|\bar{\Phi}_n(D_n) - \Phi_0\|_{L^2(M)} > n^{-\gamma}) \to 0, \quad \text{as } n \to \infty, \] (6.18)
which gives precisely Theorem 1.4 as stated in the introduction.

**Remark 9.** In comparison with Theorem 6.1, the theorem has two shortcomings: First, the rate of contraction, while being polynomial, is unknown. Second, and more importantly, the required regularity of the prior (the choice of $\alpha$) is unknown as well and thus the theorem does not provide a precise guideline for the choice of prior in applications.

Possibly the latter issue can be alleviated by choosing a prior with $C^\infty$-smooth sample paths, such as a squared exponential prior. However, as our ignorance of $\alpha$ rather seems to be an artefact of the proof of the underlying stability estimate than an intrinsic feature of the inverse problem, it is questionable whether such a prior choice is advisable.

**Sketch of proof of Theorem 6.2.** Let us first discuss the case $g = so(m)$. Then, as we have identical forward estimates as in case (A) and the general contraction theory is independent of the dimension, the proof of Theorem 6.1 extends
verbatim to case (B) up to equation (6.14). That is, for \( A > 0 \) large enough (and \( 0 \leq k < \alpha - d/2 \)) we have, as \( n \to \infty \)

\[
\Pi_n(\Phi : \| C_\Phi - C_{\Phi_0} \|_{L^2(\partial_1 SM)} \leq A\delta_n, \| C_\Phi \|_{C^k(M)} \leq A|D_n|) \xrightarrow{P_{\phi_0}} 1. \tag{6.19}
\]

To proceed, one chooses \( \alpha > 0 \) so large, that \( \alpha - d/2 \) exceeds the regularity parameter \( k \) from Theorem 1.3. Then stability estimate (6.3) implies that on the event in (6.19) we have

\[
\| \Phi - \Phi_0 \|_{L^2(M)} \leq (A'\delta_n)\mu, \tag{6.20}
\]

where \( A' \) incorporates the constant from the stability estimate and (in the notation of (6.3)) \( \mu = \inf \mu(\Phi, \Phi_0) > 0 \), where the infimum is taken over \( \{ \Phi : \| C_\Phi \|_{C^k(M)} \leq A \} \). The proof is then finished as in case (A).

For \( g = gl_m(\mathbb{R}) \), (6.19) remains true, but one has to take some care in its derivation, as the scattering data now assumes values in the non-compact group \( Gl(m, \mathbb{C}) \) and the forward estimates are only uniform on \( L^\infty \)-balls. We will explain the necessary changes in the following:

As for the small ball probabilities, (6.11) has to be replaced by

\[
- \log \Pi_n(\| \Phi \|_{L^2(M)} \leq \delta_n, \| \Phi \|_{L^\infty} \leq A) \lesssim_A n\delta_n^2, \tag{6.21}
\]

which follows from (6.11) and the Gaussian correlation inequality [11] 

\[
\Pi_n(\| \Phi \|_{L^2(M)} \leq \delta_n, \| \Phi \|_{L^\infty} \leq A) \geq \Pi_n(\| \Phi \|_{L^2(M)} \leq \delta_n)\Pi_n(\| \Phi \|_{L^\infty(M)} \leq A),
\]

noting that \( - \log \Pi_n(\| \Phi \|_{L^\infty(M)} \leq A) = o(1) \) as \( n \to \infty \) due to Fernique’s theorem. Mutatis mutandis, the same arguments as in case (A) imply (6.12). Next, the comparison between Hellinger- and \( L^2 \)-distance in the general case (and with essentially the same proof) takes the form

\[
\omega(\| \Phi \|_{L^\infty(M)})^{-1}\| C_\Phi - C_{\Phi_0} \|_{L^2(\partial_1 M)} \lesssim h(p^1_n, p^0_n) \lesssim \| C_\Phi - C_{\Phi_0} \|_{L^2(\partial_1, M)} \tag{6.22}
\]

for a non-decreasing function \( \omega : [0, \infty) \to [0, \infty) \) coming from (6.2). As we use the lower bound only on the event \( \mathcal{F}(A) = \{ \| \Phi \|_{C^k(M)} \leq A \} \), this adjustment is unproblematic, as \( \omega \) can be controlled.

Finally we note that the proof of (6.13) is completely independent of the forward-estimates and only uses the upper bound in (6.22). In particular [14, Thm.5.13] can again be used to conclude (6.19), as desired. \( \square \)
7. Appendix

7.1. Extensions. Let $M$ be a compact manifold with boundary. By an ‘extension’ of $M$ we mean a larger manifold $N$ (of the same dimension) with interior containing $M$ as embedded sub-manifold. For example by gluing two copies of $M$ along the common boundary, one can always extend $M$ to a closed manifold.

If $N$ is an extension of $M$, then smooth functions and tensors on $M$ can themselves be extended to $N$ and one can ask them obey certain geometric or functional analytic properties. We record here two useful constructions:

Lemma 7.1 (No return extension). Suppose $(M, g)$ is a compact Riemannian manifold with strictly convex boundary. Then there exists a complete extension $(N, g)$ with the property that geodesics that leave $M$ never re-enter and do not get trapped. Precisely, if $(x, v) \in \partial SM$ and $K \subset N$ is compact, then $\gamma_{x,v}(t) \in N \setminus M$ for all $t > 0$ and $\gamma_{x,v}(t) \in N \setminus K$ for $t \gg 1$ sufficiently large.

Proof. As a smooth manifold, $N$ is obtained by gluing $M$ and the cylinder $[0, \infty) \times \partial M$ along $\partial M$. The metric on $M$ can then be extended smoothly to all of $N$ such that on $[0, \infty) \times \partial M$ it takes the form $\tilde{g} = ds^2 + \tilde{h}_s$, where $\{\tilde{h}_s\}$ is a family of Riemannian metrics on $\partial M$, depending smoothly on $s \geq 0$.

We now construct $(\tilde{h}_s)$, agreeing with $\tilde{h}_s$ for $s$ near zero, such that $g = ds^2 + h_s$ satisfies the desired properties. First note that $2\partial_s \tilde{h}_s|_{s=0}$ is positive definite, as it coincides with the second fundamental form of $\partial M$. Thus by continuity there is an $\epsilon > 0$ such that $\partial_s \tilde{h}_s$ is positive definite for all $0 \leq s < 2\epsilon$. Let $\xi_i : [0, \infty) \to [0, 1]$ ($i = 1, 2$) be smooth and monotonic with $\xi_1 + \xi_2 = 1$ and $1_{[0, \epsilon)} \leq \xi_1 \leq 1_{[0, 2\epsilon]}$ and set $h_s = b_1(s)\tilde{h}_s + s\xi_2(s)k$, where $k$ is a Riemannian metric on $\partial M$ that will be chosen later. We want to arrange that

$$h_s > 0 \quad \text{and} \quad \mathbb{I}_s \equiv 2\partial_s h_s > 0 \quad \text{for all } s \geq 0, \tag{7.1}$$

where $\mathbb{I}_s$ is the second fundamental form of $\{s\} \times \partial M \subset (N, g)$ and ‘$>$’ is to be understood in the sense of positive-definiteness of symmetric bilinear forms on $T\partial M$. First note that, since $\xi_2(s) = 0$ for $s < \epsilon$, we have $h_s \geq \xi_1(s)\tilde{h}_s + \epsilon\xi_2(s)k > 0$ for all $s \geq 0$. Next,

$$\partial_s h_s = \xi'_2(s) \left( sk - \tilde{h}_s \right) + \left[ \xi_1(s)\partial_s \tilde{h}_s + \xi_2(s)k \right], \tag{7.2}$$

and we can argue as follows: As $(\tilde{h}_s/s : \epsilon \leq s \leq 2\epsilon)$ is a compact family of Riemannian metrics, it can be majorised by some $k$ in the sense that $sk - \tilde{h}_s \geq 0$ on $[\epsilon, 2\epsilon]$. Hence, since $\xi'_2$ is non-negative with support contained in $[\epsilon, 2\epsilon]$, the first term in (7.2) is non-negative. The second term is easily seen to be positive and thus (7.1) follows.

Let us verify that $(N, g)$ is indeed complete and has the no-return/non-trapping
property. Take $p : N \to \mathbb{R}$ a smooth function, non-positive on $M$ and agreeing with projection onto the first factor on $[0, \infty) \times \partial M \subset N$. Then $p$ is proper and $|dp|_g$ is bounded, which implies that $(N, g)$ must be complete. Further, the Hessian of $p$ on $[0, \infty) \times \partial M$ is given by the second fundamental form in (7.1) and thus $p$ is strictly convex. Then for $(x, v) \in \partial_SM$ the function $q(t) = p(\gamma_{x,v}(t)) (t \geq 0)$ satisfies $q(0) = 0$, $q'(0) > 0$ and further, as long as $q(t) \geq 0$, we must have $q''(t) = \|q(s)|\dot{\gamma}(t), \dot{\gamma}(t)\| \geq c > 0$. This shows that $q(t) \geq 0$ for all $t \geq 0$ and that $q$ is unbounded. This immediately implies the no-return property ($\gamma(t) \in \{p \geq 0\}$ for $t \geq 0$) and shows that $\gamma$ is not trapped.

\[ \square \]

**Lemma 7.2** (Seeley, 1963). Suppose $M$ is a compact manifold with boundary and $N$ is an extension. Then there exists a linear operator $E : C^\infty(M) \to C^\infty(N)$ which is continuous and has closed range in the all of the following functional settings:

\[ E : H^s(M) \to H^s(N) \ (s \in \mathbb{R}), \quad E : C^k(M) \to C^k(N) \ (k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}) \]  

**7.2. Sobolev spaces.** Let $M$ be a compact manifold (with or without boundary) of dimension $d \geq 1$ and $O \subset M$ an open set. We collect here some well-known results (interpolation inequality, metric entropy bound) concerning the Sobolev-spaces $H^s(O)$ ($s \in \mathbb{R}$), briefly discussing their proofs in the manifold case, which is avoided in many available references.

To avoid any notational ambiguity we first discuss our definition of $H^s(O)$, assuming the notion of $H^s(N)$ for a closed manifold $N$ to be known (cf. [26, Ch. 4.3]). For $M$ a compact manifold with boundary we then let $H^s(M) = \{u = U|_{M^{int}} : U \in H^s(N)\}$, where $N$ is any closed extension of $M$. Similarly, elements in $H^s(O)$ are defined as restrictions (to $O^{int}$) of functions in $H^s(M)$.

**Lemma 7.3.** Suppose that $\partial M = \emptyset$. Then there are smooth functions $\varphi_k : M \to \mathbb{R}, \ (k = 1, 2, \ldots)$ such that for all $s \in \mathbb{R}$ an equivalent norm on the Sobolev-space $H^s(M)$ is given by

\[ \|u\|^2_s = \sum_{k \geq 1} k^{2s/d} |\langle u, \varphi_k \rangle|^2, \quad u \in H^s(M). \]  

**Proof.** Let $g$ be a Riemannian metric on $M$, then the differential operator $1 + \Delta_g$ has positive principal symbol and its spectrum consists of eigenvalues $0 < \lambda_1^2 \leq \lambda_2^2 \leq \cdots \to \infty$. Let $\varphi_k \ (k = 1, 2, \ldots)$ be the corresponding eigenfunctions (normalised to $\|\varphi_k\|_{L^2(M)} = 1$), then the Lemma follows from standard spectral theory.

Let us nevertheless sketch the main ideas leading to the result: For $u \in \mathcal{D}'(M)$ one writes $\hat{u}_k = \langle u, \varphi_k \rangle$ for its Fourier-coefficients, and formally defines

\[ P^s u \overset{\text{def}}{=} \sum_{k \geq 1} \lambda_k^s \hat{u}_k \varphi_k, \quad s \in \mathbb{R}. \]
A priori it is not clear that the operator $P^s$ is well defined, but the theory of complex powers of elliptic operators (cf. Theorem 10.1, Theorem 10.2, Proposition 10.3 and Theorem 11.2 in [23]) yields that $P^s$ is a classical, elliptic operator of order $s$ with

$$P^{2j} = (1 + \Delta_g)^j \quad \text{for } j \in \mathbb{Z}_{\geq 0}, \quad P^s P^t = P^{s+t} \text{ for } s, t \in \mathbb{R} \quad (7.6)$$

and the series in (7.5) converges in $\mathcal{D}'(M)$. In particular $\|u\|'_s = \|P^s u\|_{L^2} = \left(\sum_{k \geq 1} \lambda_k^{2s} |\hat{u}_k|^2\right)^{1/2}$ defines a compatible norm on $H^s(M)$ and the result follows from the the asymptotic equivalence $\lambda_k^2 \sim k^{2/d}$ (Proposition 13.1 in [23]). 

**Lemma 7.4** (Interpolation inequality). Suppose $s_0 < s_1$ and let $s_0 = (1 - \theta)s_0 + \theta s_1$ ($\theta \in [0,1]$). Then for all $u \in H^{s_0}(O)$ we have

$$\|u\|_{H^{s_0}(O)} \leq C \|u\|_{H^{s_0}(O)}^{-\theta} \|u\|_{H^{s_1}(O)}^\theta \quad (7.7)$$

for a constant $C > 0$ only depending on $O, s_0, s_1$.

**Proof.** Extend $M$ to a closed manifold $N$, such that $O \subset M \subset N$. Extend $u$ to a function $U \in H^{s_0}(N)$ and consider the following inequality (for the norms $\| \cdot \|_s$ on $H^s(N)$ defined in the previous Lemma):

$$\|U\|_{s_0}^2 = \sum_{k \geq 1} \left( k^{2s_0/d} |\hat{U}_k|^2 \right)^{1-\theta} \left( k^{2s_1/d} |\hat{U}_k|^2 \right)^\theta \leq \left( \sum_{k \geq 1} k^{2s_0/d} |\hat{U}_k|^2 \right)^{1-\theta} \left( \sum_{k \geq 1} k^{2s_1/d} |\hat{U}_k|^2 \right)^\theta = \|U\|_{s_0}^{2(1-\theta)} \|U\|_{s_1}^{2\theta}$$

Here $\hat{U}_k = \langle U, \varphi_k \rangle$ and we have used the Hölder-inequality for the exponents $1/(1-\theta)$ and $1/\theta$. This implies that $\|u\|_{H^{s_0}(O)} \leq \|U\|_{s_0}^{(1-\theta)} \|U\|_{s_1}^\theta$ for all extensions $U$ and the Lemma follows by choosing $U = Eu$ as in Lemma 7.2.

Next, recall the notation $N(X, d, \epsilon)$ for the smallest number of $\epsilon$-balls needed to cover a (totally bounded) metric space $(X, d)$. Then:

**Lemma 7.5** (Metric entropy bound). Let $B^s \subset H^s(M)$ ($s > 0$) be the unit-ball. Then, as $\epsilon \to 0$, we have $\log N(B^s, \| \cdot \|_{L^2(M)}, \epsilon) = O(\epsilon^{-s/d})$.

**Proof.** Using the representation as sequence space from Lemma 7.3, the Lemma is easily proved in the case $\partial M = \emptyset$ by the same arguments as in [8, Theorem 4.3.36]. The case $\partial M \neq \emptyset$ follows immediately by extending $M$ to a closed manifold $N$ and realising $H^s(M)$ as closed subspace of $H^s(N)$ via an extension operator $E$ as in Lemma 7.2. \[\square\]
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