Existence of solutions for a system of mixed fractional differential equations

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ABSTRACT
The aim of this work is to investigate, by the help of Krasnoselskii's fixed point theorem, the existence of solutions for a system of fractional differential equations involving left and right Riemann–Liouville fractional derivatives. The presence of mixed type derivatives lead to great difficulties. We believe that the obtained results are new and will contribute to the development of fractional differential equations.

1. Introduction
The study of fractional differential equations is gaining more importance and attention, this is due to their various applications in different scientific disciplines such as in physics, chemistry, viscoelasticity, aerodynamics, electromagnetic . . . see [1–3] and the references therein.

Various techniques and methods are involved in the investigation of fractional differential equations. Some authors have used fixed point theory to show the existence of solutions [4–10]. In [11–13], authors have used lower and upper solutions method, to show the existence of positive solutions to fractional boundary value problems.

In [14], the author used cone theory to show the existence of solutions for a system of multi-order fractional differential equations with non-local boundary conditions, where each equation has an order that may be different from the order of the other equations, that is:

\[ D^{\alpha_i}u_i = f_i(t, u_1, u_2, \ldots, u_n), \]
\[ u_i(0) = 0, \quad 0 < \alpha_i < 1, \quad i \leq n, \]
\[ u_i(T) = 0, \]

where 0 ≤ t ≤ T, \( \bar{u} = (u_1, \ldots, u_n) \) and 0 < \( \alpha_i < 1, \forall i \).

\( D^{\alpha}u \) denote the standard Riemann–Liouville fractional derivatives and \( f: [0, T] \times R^n \to R \). In [15], the author showed the existence of positive solutions of the following fractional boundary value problem,

\[ D^{\alpha}u + a(t)f(u) = 0, \quad 0 < t < 1, \quad 1 < \alpha < 2 \]
\[ u(0) = 0, \quad u'(1) = 0, \]

where \( D^{\alpha} \) is the Riemann–Liouville fractional derivative of order, the functions \( a \) and \( f \) are given.

Recently, equations including both left and right fractional derivatives are attracting much attention as an interesting field in fractional differential equations theory and many results are obtained concerning the existence of solutions by the help of different methods, such as fixed point theory, upper and lower solutions method, variational method . . . [11,13,16–20]. In [20], the authors used Krasnoselskii's fixed point theorem to prove the existence of solutions for the following boundary value problem involving both left Riemann–Liouville and right Caputo-type fractional derivatives

\[ C^{\beta}D^{\alpha}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \]
\[ u(0) = u'(1) = 0, \]

where 0 < \( \alpha \leq 1, \quad 1 < \beta < 2, \quad f: [0, 1] \times R \to R \).

In this work, we consider the following system of fractional differential equations with boundary conditions that we denote by (5):

\[ D^{\beta}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \]
\[ D_0^\beta u(0) = D_0^\beta u(1) = 0, \]
\[ u(0) = u'(1) = 0, \]

where the function \( u = (u_1, u_2, \ldots, u_n) \) is an unknown function with, \( u_i: [0, 1] \to R \),

\[ D_{1-}^\beta (D_0^\beta u(t)) = (D_{1-}^\beta (D_0^\beta u_1(t)), D_{1-}^\beta (D_0^\beta u_2(t)), \ldots, D_{1-}^\beta (D_0^\beta u_n(t))), \]

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\(D^\alpha_{t-}\) and \(D^\beta_b\) denote, respectively, the left and the right Riemann–Liouville fractional derivative, \(1 < \alpha, \beta_i < 2,\) \(i \in \{1, \ldots, n\}, n \geq 2, f : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n,\)
\(f(t, u) = (f_1(t, u_1, u_2, \ldots, u_n), \ldots, f_n(t, u_1, u_2, \ldots, u_n)),\)
\(f_i \in C([0, 1] \times \mathbb{R}^n, \mathbb{R}).\)

This paper is organized as follows: in Section 2, we give some preliminary materials to be used later. In Section 3, we give sufficient conditions that guarantee the existence of solution and prove the main results. Finally we give an example illustrating the obtained results.

2. Preliminaries

We present the necessary definitions from fractional calculus theory. These definitions can be found in [1–3].

Let \([a, b]\) be a finite interval \(\mathbb{R}\). The left and right Riemann–Liouville fractional integrals of order \(\alpha > 0\) of a function \(f\) are given respectively by
\[
I^\alpha_a f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) \, ds,
\]
\[
I^\alpha_b f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) \, ds.
\]
The left and right Riemann–Liouville fractional derivatives of order \(\alpha > 0\) of a function are given respectively by
\[
D^\alpha_{a+} f(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_a^t (t-s)^{n-\alpha-1} f(s) \, ds,
\]
\[
D^\alpha_{b-} f(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_t^b (s-t)^{n-\alpha-1} f(s) \, ds,
\]
where \(n = [\alpha] + 1\).

We have the following result.

**Lemma 2.1:** Let \(\alpha > 0\), the equality \(D^\alpha_{a+} u(t) = 0\), is valid if, and only if,
\[
u(t) = a_1^{\alpha-1} (t-a) + a_2^{\alpha-2} (t-a) + \cdots + a_n (t-a)^{\alpha-n}, \quad a_i \in \mathbb{R}.
\]
The equation \(D^\alpha_{b-} u(t) = 0\), has a unique solution
\[
u(t) = c_1 (b-t)^{\alpha-1} + c_2 (b-t)^{\alpha-2} + \cdots + c_n (b-t)^{\alpha-n}, \quad c_i \in \mathbb{R},
\]
where \(n = [\alpha] + 1\).

Let us recall the Krasnoselskii’s fixed point theorem.

**Theorem 2.2:** Let \(M\) be a closed bounded convex non-empty subset of a Banach space \(E\), Suppose that \(A\) and \(B\) map \(M\) into \(E\) such that
(i) \(x, y \in M\) implies \(Ax + By \in M\).
(ii) \(B\) is a contraction mapping.
(iii) \(A\) is completely continuous.

Then there exists \(z \in M\) with \(z = Az + Bz\).

3. Main result

We shall transform the system (S) to an equivalent system of integral equations. Consider the corresponding linear system:
\[
D^\alpha_{t-} (D^\beta_{0+} u_i(t)) = -y_i(t), \quad 0 < t < 1, \quad (1)
\]
\[
D^\beta_{0+} u_i(0) = D^\beta_{0+} u_i(1) = 0, \quad (2)
\]
\[
u_i(0) = u'_i(1) = 0, \quad (3)
\]
here \(i \in \{1, \ldots, n\}.

**Lemma 3.1:** Assume that \(y_i \in C(0, 1) \cap L_1(0, 1)\), for \(i \in \{1, \ldots, n\}\), then the boundary value problem (1–3), has a unique solution given by
\[
\nu_i(t) = \int_0^1 G_i(t, r)y_i(r) \, dr + g_i(t) \int_0^1 s^{\alpha-1}y_i(s) \, ds. \quad (4)
\]

Where
\[
G_i(t, r) = \frac{1}{\Gamma(\alpha) \Gamma(\beta_i)} \begin{cases}
\int_0^t (t^\beta_i - (1-s)^{\beta_i-1})(r-s)^{\alpha-1} \, ds, & 0 \leq r \leq t < 1, \\
\int_0^t (t^\beta_i - (1-s)^{\beta_i-1})(r-s)^{\alpha-1} \, ds, & 0 \leq t \leq r \leq 1,
\end{cases}
\]
\[
g_i(t) = \frac{1}{\Gamma(\alpha) \Gamma(\beta_i)} \left( \int_0^t (t^\beta_i - (1-s)^{\beta_i-1})(1-s)^{\alpha-1} \, ds - \frac{t^{\beta_i-1}}{\alpha + \beta_i - 2} \right).
\]

**Proof:** Applying the integral operator \(L^\alpha_a \) to Equation (1), we get
\[
D^\beta_{0+} u_i(t) = -r^\alpha_{t-} y_i(t) + c_1 (1-t)^{\alpha-1} + c_2 (1-t)^{\alpha-2}.
\]
According to conditions (2), it yields
\[
c_2 = 0, \quad c_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 s^{\alpha-1}y_i(s) \, ds. \quad (6)
\]
Substituting \(c_1\) and \(c_2\) by their values in (5), we obtain
\[
D^\beta_{0+} u_i(t) = -r^\alpha_{t-} y_i(t)
+ \frac{1}{\Gamma(\alpha)} (1-t)^{\alpha-1} \int_0^1 s^{\alpha-1}y_i(s) \, ds. \quad (7)
\]
Now, we apply the fractional integral \( f_i^{\beta_i} \) to Equation (7) to get

\[
\begin{aligned}
u_i(t) &= -t_i^{\beta_i} f_i^{\beta_i} y_i(t) + \frac{1}{\Gamma(\alpha)} \bigg( t_i^{\beta_i - 1} (1 - t)^{\alpha - 1} \bigg) \int_0^1 s^{\alpha - 1} y_i(s) \, ds \\ &+ C_3 t^{\beta_i - 1} + C_4 t^{\beta_i - 2}.
\end{aligned}
\] (8)

From conditions (3) we deduce

\[
c_4 = 0,
\]

\[
c_3 = \frac{1}{\Gamma(\beta_i) \Gamma(\alpha)} \bigg( t_i^{\beta_i - 2} \bigg) \int_0^1 (1 - s)^{\beta_i - 2} \bigg( \int_0^1 (r - s)^{\alpha - 1} y_i(r) \, dr \bigg) \, ds \\ - \bigg( \int_0^1 (1 - s)^{\beta_i - 2} (1 - s)^{\alpha - 1} \, ds \bigg) \\ &\times \bigg( \int_0^1 s^{\alpha - 1} y_i(s) \, ds \bigg).
\]

Substituting \( c_3 \) and \( c_4 \) in Equation (8), yields

\[
u_i(t) = -\frac{1}{\Gamma(\beta_i) \Gamma(\alpha)} \bigg( \int_0^t (t - s)^{\beta_i - 1} \\ \times \bigg( \int_0^1 (r - s)^{\alpha - 1} y_i(r) \, dr \bigg) \, ds \\ + \frac{1}{\Gamma(\alpha)} \bigg( t_i^{\beta_i - 1} (1 - t)^{\alpha - 1} \bigg) \int_0^1 s^{\alpha - 1} y_i(s) \, ds \\ + \frac{t^{\beta_i - 1}}{\Gamma(\beta_i) \Gamma(\alpha)} \bigg( \int_0^1 (1 - s)^{\beta_i - 2} \bigg) \int_0^1 (r - s)^{\alpha - 1} y_i(r) \, dr \\ - \bigg( \frac{1}{\alpha + \beta_i - 2} \bigg) \bigg( \int_0^1 s^{\alpha - 1} y_i(s) \, ds \bigg) \bigg) \bigg) \\ + \frac{1}{\Gamma(\alpha) \Gamma(\beta_i)} \bigg( \int_0^t (t - s)^{\beta_i - 1} (1 - s)^{\alpha - 1} \, ds \\ \times \bigg( \int_0^1 s^{\alpha - 1} y_i(s) \, ds \bigg) \\ - \frac{t^{\beta_i - 1}}{\alpha + \beta_i - 2} \Gamma(\beta_i) \Gamma(\alpha) \bigg( \int_0^1 s^{\alpha - 1} y_i(s) \, ds \bigg)
\]

that gives (4). The proof is achieved. \( \blacksquare \)

**Lemma 3.2:** The functions \( g_i \) and \( G_i \), \( i \in \{1, \ldots, n\} \) are continuous and satisfy the following properties:

\[
0 \leq G_i(t, r) \leq \frac{1}{(\alpha + \beta_i - 2) \Gamma(\beta_i) \Gamma(\alpha)} \quad 0 \leq t, r \leq 1
\] (9)

\[
g_i(t) \leq 0, |g_i(t)| \leq \frac{1}{(\alpha + \beta_i - 2) \Gamma(\beta_i) \Gamma(\alpha)}, \quad 0 \leq t \leq 1.
\] (10)

**Proof:** By calculus, we get, if \( 0 \leq r \leq t \leq 1 \):

\[
G_i(t, r) = \frac{1}{\Gamma(\beta_i) \Gamma(\alpha)} \int_0^r \bigg( t_i^{\beta_i - 1} (1 - s)^{\beta_i - 2} \\ - (t - s)^{\beta_i - 1} (r - s)^{\alpha - 1} \, ds \\ \geq \frac{1}{\Gamma(\beta_i) \Gamma(\alpha)} \int_0^r \bigg( t_i^{\beta_i - 1} \\ - (t - s)^{\beta_i - 1} (r - s)^{\alpha - 1} \, ds \bigg) \geq 0,
\]

and

\[
G_i(t, r) \leq \frac{1}{\Gamma(\beta_i) \Gamma(\alpha)} \int_0^r \bigg( t_i^{\beta_i - 1} (1 - s)^{\beta_i - 2} (r - s)^{\alpha - 1} \, ds \\ \leq \frac{1}{\Gamma(\beta_i) \Gamma(\alpha)} \int_0^r \bigg( t_i^{\beta_i + \alpha - 3} \, ds \\ \leq \frac{1}{(\beta_i + \alpha - 2) \Gamma(\beta_i) \Gamma(\alpha)}.
\]

Now, if \( 0 \leq t \leq r \leq 1 \), then

\[
G_i(t, r) = \frac{1}{\Gamma(\beta_i) \Gamma(\alpha)} \bigg( t_i^{\beta_i - 1} \int_0^t (1 - s)^{\beta_i - 2} (r - s)^{\alpha - 1} \, ds \\ - \int_0^t (t - s)^{\beta_i - 1} (r - s)^{\alpha - 1} \, ds \\ \geq \frac{t_i^{\beta_i - 1}}{\Gamma(\beta_i) \Gamma(\alpha)} \bigg( \int_0^t (1 - s)^{\beta_i - 2} (r - s)^{\alpha - 1} \, ds \\ - \int_0^t (r - s)^{\alpha - 1} \, ds \\ \geq \frac{t_i^{\beta_i - 1}}{\alpha + \beta_i - 2} \Gamma(\beta_i) \Gamma(\alpha) \bigg( \int_0^t (r - s)^{\alpha - 1} \, ds \\ - \int_0^t (r - s)^{\alpha - 1} \, ds \bigg) \geq 0,
\]
and
\[ G_i(t, r) \leq \frac{1}{\Gamma(\beta_i)\Gamma(\alpha)} \int_0^r (1 - s)^{\beta_i - 1} (1 - s)^{\alpha - 1} \, ds \]
\[ \leq \frac{1}{\Gamma(\beta_i)\Gamma(\alpha)} \int_0^r (r - s)^{\alpha + \beta_i - 3} \, ds \]
\[ \leq \frac{1}{(\alpha + \beta_i - 2)\Gamma(\beta_i)\Gamma(\alpha)}. \]

Similarly, we obtain
\[ g_i(t) = \frac{1}{\Gamma(\alpha)\Gamma(\beta_i)} \left( \int_0^t (t - s)^{\beta_i - 1} (1 - s)^{\alpha - 1} \, ds \right) \]
\[ \leq \frac{1}{(\beta_i + \alpha - 2)\Gamma(\beta_i)\Gamma(\alpha)}. \]

\[ |g_i(t)| = -g_i(t) = \frac{1}{\Gamma(\alpha)\Gamma(\beta_i)} \left( \int_0^t (t - s)^{\beta_i - 1} (1 - s)^{\alpha - 1} \, ds \right) \]
\[ \leq \frac{1}{(\beta_i + \alpha - 2)\Gamma(\beta_i)\Gamma(\alpha)}. \]

The proof is complete.

Define \( E \) the Banach space of all functions
\[ u \in C^0[0, 1] = C[0, 1] \times \cdots \times C[0, 1] \]
equipped with the norm
\[ ||u|| = \sum_{i=1}^n \max_{t \in [0, 1]} |u_i(t)|. \]

Define the integral operators \( A \) and \( B \) on \( E \) by
\[ Au(t) = (A_1 u_1(t), A_2 u_2(t), \ldots, A_n u_n(t)), \]
\[ Bu(t) = (B_1 u_1(t), B_2 u_2(t), \ldots, B_n u_n(t)), \]
where
\[ A_i u_i(t) = \int_0^t G_i(t, r) f_i(r, u(r)) \, dr, \]
\[ B_i u_i(t) = g_i(t) \int_0^1 s^{\alpha - 1} f_i(s, u(s)) \, ds. \]

Lemma 3.3: The function \( u \in E \) is a solution of the system (S) if, and only if, \( A_i u_i(t) + B_i u_i(t) = u_i(t) \) for all \( t \in [0, 1] \) and \( i \in \{1, \ldots, n\} \).

To study the existence of solution for the system (S), we make the following hypothesis:

\[ (H) \] The functions \( f_i(t, 0) \) are continuous and not identically null on \([0, 1]\) and there exists non-negative functions \( K_i \in L_1(0, 1) \), such that:
\[ |f_i(t, x) - f_i(t, y)| \leq K_i(t) \sum_{j=1}^n |x_j - y_j|, \]
\[ t \in [0, 1], x, y \in R, i \in \{1, \ldots, n\}. \]

with
\[ \sum_{i=1}^n \frac{\|K_i\|_{L_1}}{(\alpha + \beta_i - 2)\Gamma(\beta_i)\Gamma(\alpha)} < \frac{1}{4}. \]

Let \( M = \{u \in P, ||u|| \leq R\} \), where \( R \) is chosen such
\[ R \geq 4 \sum_{i=1}^n \left( \frac{L_i}{(\alpha + \beta_i - 2)\Gamma(\beta_i)\Gamma(\alpha)} \right), \]
where \( L_i = \max_{t \in [0, 1]} |f_i(t, 0)| \).

Clearly, \( M \) is a non-empty, bounded and convex subset of the Banach space \( E \).

Theorem 3.4: Under the hypothesis \((H)\), the system (S) has at least one non-trivial solution in \( M \).

Proof: We shall use Krasnoselskii’s fixed point theorem, for this the proof will be done in three steps.

Step 1: \( Au + Bv \in M \) for all \( u, v \in M \). In fact by hypothesis \((H)\) and the properties of the functions \( G_i \) given in Lemma 3.3, we get for all \( i \in \{1, \ldots, n\} \),
\[ ||A_i u_i(t)|| \leq \int_0^1 G_i(t, r) |f_i(r, u(r))| \, dr \]
\[ \leq \frac{1}{(\alpha + \beta_i - 2)\Gamma(\beta_i)\Gamma(\alpha)} \int_0^1 |f_i(r, u(r)) - f_i(r, 0)| \, dr \]
\[ \leq \frac{1}{(\alpha + \beta_i - 2)\Gamma(\beta_i)\Gamma(\alpha)} \int_0^1 \left( |K_i(r)| \sum_{j=1}^n |u_j(r)| + L_i \right) \, dr \]
\[ \leq \frac{R||K_i||_{L_1} + L_i}{(\alpha + \beta_i - 2)\Gamma(\beta_i)\Gamma(\alpha)}. \]

By taking the maximum over \( t \in [0, 1] \), it follows
\[ ||A_i u_i|| \leq \frac{R||K_i||_{L_1} + L_i}{(\alpha + \beta_i - 2)\Gamma(\beta_i)\Gamma(\alpha)}. \]

Summing the \( n \) inequalities in (13), it yields
\[ ||Au|| \leq \sum_{i=1}^n \left( \frac{R||K_i||_{L_1} + L_i}{(\alpha + \beta_i - 2)\Gamma(\beta_i)\Gamma(\alpha)} \right) \leq R \]
\[ \leq \frac{R}{2}. \]
By the properties of the functions $g_i$, we have
\[
|B_iu(t)| \leq |g_i(t)| \int_0^1 s^{\alpha - 1}|f_i(s, v(s))| \, ds
\leq \frac{1}{(\alpha + \beta_1 - 2)\Gamma(\beta_1)\Gamma(\alpha)} \int_0^1 |f_i(r, v(r)) - f_i(r, 0)| \, dr
\leq \frac{R||K||_{L_1} + L_i}{(\alpha + \beta_1 - 2)\Gamma(\beta_1)\Gamma(\alpha)}.
\]

Taking the supremum over $[0, 1]$, then summing the obtained inequalities according to $i$ from 1 to $n$, we get $||Bv|| \leq R/2$. Hence $||Au + Bv|| \leq ||Au|| + ||Bv|| \leq R$ so, $Au + Bv \in M$ for all $u, v \in M$.

**Step 2:** The mapping $B$ is contraction on $M$. Indeed, let $u, v \in M$, then by using hypothesis (H) it yields
\[
|B_iu(t) - B_iu(t)| \leq |g_i(t)| \int_0^1 s^{\alpha - 1}|f_i(s, u(s)) - f_i(s, v(s))| \, ds
\leq \frac{1}{(\alpha + \beta_1 - 2)\Gamma(\beta_1)\Gamma(\alpha)} \int_0^1 |K_i(r)| \sum_{j=1}^n |u_j - v_j| \, dr
\leq \frac{||K||_{L_1} ||u - v||}{(\alpha + \beta_1 - 2)\Gamma(\beta_1)\Gamma(\alpha)}.
\]

By taking the maximum over $t \in [0, 1]$, we get
\[
||B_iu - B_iu|| \leq \frac{||K||_{L_1} ||u - v||}{(\alpha + \beta_1 - 2)\Gamma(\beta_1)\Gamma(\alpha)}.
\]

Summing the $n$ inequalities in (15), then taking (11) into account, it yields:
\[
||Bu - Bv|| \leq \left( \sum_{i=1}^n \frac{||K||_{L_1}}{(\alpha + \beta_1 - 2)\Gamma(\beta_1)\Gamma(\alpha)} \right) ||u - v||
\leq \frac{||u - v||}{4}.
\]

**Step 3:** The operator $A$ is completely continuous on $M$. In fact, using the properties of the function $g_i$ and the hypothesis (H), we show the continuity of $A$. From (14), we deduce that $AM \subset M$. Now we prove that $(Au)$ is equicontinuous on $M$. Denote $L = \sup\{|f_i(t, u(t))|, 0 \leq t \leq 1, u \in M, i = 1, \ldots, n\}$. Let $u \in M$, $0 \leq t_1 \leq t_2 \leq 1$, we have
\[
|A_iu(t_1) - A_iu(t_2)| = \int_{t_1}^{t_2} |G_i(t_1, r) - G_i(t_2, r)||f_i(r, u(r))| \, dr
\]

Consequently, $|A_iu(t_1) - A_iu(t_2)| \to 0$, when $t_1 \to t_2, \forall i \in \{1, \ldots, n\}$. Hence $(Au)$ is equicontinuous on $M$. Finally, by Arzela–Ascoli’s theorem, it follows that $A$ is completely continuous mapping on $M$. Then By Krasnoselskii fixed point theorem, we conclude that there exists at least one fixed point for the operator $A + B$ in $M$. The proof is complete.

Now, we give an example to illustrate the usefulness of our main results.

**4. Example**

Consider the following two-dimensional fractional order system
\[
\begin{align*}
D^{1.2}_0 u_1(t) &= \left( \frac{1 - 2t}{10} \right) \left( u_2 - \frac{1}{2(1 + u_2^2)} \right), \\
D^{1.2}_0 u_2(t) &= \left( \frac{e^{-t}}{60} \right) \left( t^2 + 2 \frac{1}{2} \left( 3u_1 - \frac{1}{1 + u_2^2} \right) \right).
\end{align*}
\]

Let us check hypothesis (H). We have
\[
\begin{align*}
f_1(t, u) &= \left( \frac{1 - 2t}{10} \right) \left( u_2 - \frac{1}{2(1 + u_2^2)} \right), \\
f_2(t, u) &= \left( \frac{e^{-t}}{60} \right) \left( t^2 + 2 \frac{1}{2} \left( 3u_1 - \frac{1}{1 + u_2^2} \right) \right).
\end{align*}
\]
By computations, we obtain
\[
\|K_1\|_{L^1} = \int_0^1 \frac{3}{20} (1 - t) \, dt = 0.075,
\]
\[
\|K_2\|_{L^1} = \int_0^1 \frac{e^{-t}}{40} \, dt = 1.580 \times 10^{-2},
\]
\[
\sum_{i=1}^n \|K_i\|_{L^1} = \frac{\|K_i\|_{L^1}}{2} \Gamma(\beta_i) \Gamma'(\alpha) = 0.10495 < \frac{1}{4}.
\]

Let \( M = \{u \in E, \|u\| \leq R\} \), where \( R \) is chosen such
\[
R = 0.5 > 4 \sum_{i=1}^n \left( \frac{L_i}{(\alpha + \beta_i - 2) \Gamma(\beta_i) \Gamma'(\alpha)} \right) = 0.47032.
\]
Hence, we deduce by Theorem 3.4 that the system \((S)\) has at least one non-trivial solution \( u^* \in M \) such that \( \|u^*\| \leq 0.5 \).

Disclosure statement
No potential conflict of interest was reported by the authors.

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