Combating errors in crosstalk channels

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Abstract

In order to harness quantum advantage in communication protocols, combating channel noises with resources available in the current age is an inevitable requirement. In a recent work [Combating quantum errors: an integrated approach, Rajni Bala, Sooryansh Asthana, V. Ravishankar, arXiv:2208.04555], a framework has been laid down for an error immune information transfer for those noisy channels in which the initial and the final states belong to the same Hilbert space. This information transfer scheme neither involves a costly resource like multi-party entanglement nor any sophisticated experimental technique, such as, dynamical decoupling. In the same spirit, in this work, we identify invariants for idealised crosstalk channels in which the initial and the final states may not have supports in same Hilbert spaces. We have also laid down a procedure that allows to retrieve full information in a state even after passing through a crosstalk channel. Finally, we determine constraints required to be imposed on the choice of basis modes to construct quantum error correction and rejection codes for crosstalk channels.

Keywords: Crosstalk, Information retrieval, Quantum error rejection, Quantum error correction, QuNits
1 Introduction

Recent times have witnessed an unprecedented surge of interest in quantum information theoretic protocols [1–4]. This interest owes to the promise of outperforming their classical counterparts [3] or to the promise of altogether novel applications, e.g., quantum key distribution [1, 2, 5]. However, since transmission of quantum states is an inevitable requirement in communication protocols, noise acts as a major impediment in implementation of all these protocols. This has led to various proposals, e.g., quantum error correcting codes [6], approximate quantum error correcting codes [7], error avoiding codes [8], ancilla-free error correcting codes [9], decoherence-free subspaces [10], dynamical decoupling [11], etc., for mitigating the effect of noise. However, these techniques either demand resources such as multiparty entanglement or implementation of interventions at appropriate time intervals, which requires sophisticated experimental techniques [12]. Generation of multiparty entanglement is still a challenge, especially in photonic systems [13]. One important example involves orbital angular momentum (OAM) of light, which is arguably at the forefront of realising higher dimensional states for quantum communication protocols [14, 15]. These considerations necessitate the need for a framework for combating errors that does not employ such costly resources and is implementable with the current state-of-art technology.

In a recent work [16], we have proposed such a new information encoding scheme. This scheme encodes information in invariants that allow transfer of information in an error-immune manner. The framework, with additional constraints on the choice of basis modes, also allows for constructions of quantum error correcting and rejecting codes.

The invariants that were identified in [16], correspond to noisy channels of a quNit in which the initial and the final states have support in the same Hilbert space. These quNits, in current state-of-art, are realised with modes which, in principle, belong to infinite-dimensional spaces such as OAM modes, Hermite-Gauss modes, radial modes [17]. The dominant error in the propagation of these modes is channel crosstalk on which we focus here [18–21]. Crosstalk channel causes a spillover to neighbouring modes, i.e., it results in an increase in number of modes whose superposition is involved in the final state as compared to the initial state. Since the framework, proposed in [16], does not take into account such errors, in this work, we identify invariants for crosstalk channels. In fact, we show that by imposing constraints in a hierarchical manner on the choice of basis modes, a complete hierarchy of schemes for combating crosstalk errors emerges. We briefly describe them below.

Suppose that the initial state involves superposition of $M$ modes that evolves stochastically to the final state involving superposition of $N$ modes. That is, they have supports in $M$ and $N$ dimensional spaces, viz., $\mathcal{H}^M$ and $\mathcal{H}^N$. Necessarily, $N \geq M$. Let there be $l$ errors which project a state to the subspaces represented by $\mathcal{H}^{E_1}, \mathcal{H}^{E_2}, \ldots, \mathcal{H}^{E_l}$ respectively. Clearly, $\mathcal{H}^N \equiv \mathcal{H}^M \cup \mathcal{H}^{E_1} \cup \mathcal{H}^{E_2} \cup \cdots \cup \mathcal{H}^{E_l}$. For crosstalk channels (described in sec. (2)),
which are of concern to us, the following possibilities may arise. If it so happens that $\mathcal{H}^M \cap \mathcal{H}^{E_k} \neq \emptyset$, it is possible to identify invariants for crosstalk channels (discussed in section (3)). Information encoded in these invariants get transferred in an error–immune manner. We show that identification of these invariants follows from the observation that the effect of the crosstalk channel, $\rho \in \mathcal{H}^M \xrightarrow{\xi_k} \rho' \in \mathcal{H}^N$, can effectively be described in terms of a generalised flip channel defined in $\mathcal{H}^N$. Furthermore, we show that complete information in a state can be retrieved by admitting a generalised flip channel in $\mathcal{H}^N$ where $N \geq N$ is to be suitably chosen (section (4)). At the other extreme, the condition $\mathcal{H}^M \cap \mathcal{H}^{E_k} = \emptyset$ for all possible $k$, makes the construction of quantum error rejection codes possible (section (5)). If further, the subspaces $\mathcal{H}^{E_k}$ are mutually orthogonal, i.e., the conditions

$$\mathcal{H}^M \cap \mathcal{H}^{E_l} = \emptyset, \text{ and } \mathcal{H}^{E_k} \cap \mathcal{H}^{E_l} = \emptyset, \quad \forall \ k, l, k \neq l,$$

make the construction of quantum error correction codes possible (section (6)). These codes, like the conventional ones (please see [22] and references therein), require specific choice of basis modes, stabiliser measurements and appropriate transformations for detection and correction of errors. However, they do not require any ancillary system, thanks to the availability of arbitrarily large-dimensional systems. Thus, these codes harness the potential provided by higher dimensional states as well as idealised crosstalk channels. The sets of basis modes and stabilisers that are employed in error rejecting and correcting codes for crosstalk channel have been identified. This work shows a nice interplay between the consumption of resources (by resources we mean the choice of a specific set of basis modes) and consequent retrieval of information/state for a crosstalk channel. The three scenarios, viz., identification of invariants, quantum error rejection code and quantum error correction code have been shown schematically in figure (1). Finally, in section (7), we discuss several practical situations in which crosstalk noise is observed and possibility of employing our results in those situations. Section (8) concludes the paper with closing remarks.

## 2 Noisy channel: Idealised Crosstalk

In this section, we describe the channel of interest in this paper, viz., an idealised crosstalk channel with the following Kraus operators:

$$E_{\pm k} = \sqrt{p_k} \sum_{n=-\infty}^{\infty} |n \pm k\rangle \langle n|, \quad (1)$$

where $p_k$ is the probability with which the errors (spillovers) $E_{+k}$ and $E_{-k}$ corrupt a state. The channel is idealised in the sense that probabilities are functions of only $k$, however, in a general case, probabilities would have been
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Fig. 1 An initial state $\rho$ undergoes a noisy evolution, $\mathcal{E}$, which is given as, $\rho \to \rho' \equiv p_0 \rho + p_1 (\rho_{+1} + \rho_{-1})$. Case (a) All the three states, viz., $\rho_{\pm 1}$ and $\rho$ have overlapping supports, i.e., they involve superpositions of common modes. This allows for identification of invariants. (b) The states $\rho_{\pm 1}$ involve superpositions of common modes. However, they are orthogonal to the initial state $\rho$. This allows for construction of a quantum error rejection code. (c) All the three states $\rho, \rho_{+1}, \rho_{-1}$ involve superposition of mutually orthogonal modes, i.e., they have supports in orthogonal subspaces. This allows for construction of a quantum error correction code.

functions of both $n$ and $k$. The values of $k$ describe the errors which are dominant in a given crosstalk channel, depending upon the physical scenarios.

The error operators, defined in equation (1) act on infinite-dimensional spaces because the modes, in principle, may belong to infinite-dimensional spaces. However, in practical scenarios, finite number of modes are employed to transfer information. In such cases, the noisy channel in equation (1) can also be mimicked by finite-dimensional error operators, i.e,

$$E_{\pm k} = \sqrt{p_k} \sum_{n=l_{\text{min}}}^{l_{\text{max}}} |n \pm k \rangle \langle n|,$$

where the set

$$\mathcal{B} \equiv \{ |l_{\text{min}} \rangle, \ldots, |l_{\text{max}} \rangle \},$$

consists of the modes which are employed for information transfer.

In the next section, we identify invariants that allow for error-immune information transfer even after the transmission of a state through a crosstalk channel.
3 Identification of invariants for idealised crosstalk channel

In [16], a framework has been laid down for identifying invariant quantities under noisy evolutions of the kind, $\mathcal{H}^M \overset{\mathcal{E}_1}{\rightarrow} \mathcal{H}^M$. However, as discussed in section (2), in a crosstalk channel—the initial and the final states may involve superpositions of $M$ and $N$ modes respectively [20, 23, 24], i.e., $\mathcal{H}^M \overset{\mathcal{E}_2}{\rightarrow} \mathcal{H}^N$. In this section, we undertake the task of identifying invariants for this channel. As precursors, we (i) highlight how invariants are legitimate carriers of information, and, (ii) determine invariants for a generalised flip channel.

**Invariants as carriers of information:** For the state passing through a noisy channel, the invariants can be expressed as functions of expectation values of observables. For our consideration, these invariants are either simply the expectation values or their ratios. Expectation values can be legitimately treated as random variables (since their values depend on the state chosen) and so can be these invariants. Being invariants, they naturally yield us error-immune information when a state passes through a noisy channel.

**Invariants for a generalised flip channel:** We briefly recapitulate the method of determination of invariants for a generalised flip channel obtained in [16]. A state $\rho$, after passing through a generalised flip channel, changes to a state $\rho'$ as follows:

$$\rho \rightarrow \rho' = \sum_{r=0}^{N-1} F_r \rho F_r^\dagger \equiv \sum_{r=0}^{N-1} p_r X^r \rho (X^r)^\dagger,$$

(4)

where $p_r$ is the probability with which an error $X^r = \sum_{k=0}^{N-1} |k + r \rangle \langle k|$ has corrupted a state (here, the summation is modulo $N$). Employing the cyclicity property of trace, we see that the expectation values of operators,

$$I_1^{(m)} = \langle X^m \rangle,$$

(5)

are invariant and thus the encoded information remains error–free. We designate these invariants as the *invariants of the first family.*

Additional invariants are identified by employing the relation $Z X = \omega X Z$, where $Z = \sum_{k=0}^{N-1} \omega^k |k \rangle \langle k|$, and $\omega$ is the $N$th root of unity. As shown in [16], the quantities,

$$I_2^{(m,l)} = \frac{\langle Z^l \rangle}{\langle X^m Z^l \rangle},$$

(6)

also remain invariants. Hence, information encoded in $I_2^{(m,l)}$ also remains immune to error. We designate the invariants $I_2^{(m,l)}$ as the *invariants of the second family.* Equipped with these results, we move on to identify invariants for an idealised crosstalk channel.

It might appear that the channel, $\mathcal{E}_2$, is distinctly different from the channel $\mathcal{E}_1$, rendering construction of invariants for $\mathcal{E}_2$ more difficult and invariants...
derived for a generalised flip channel (given in equations (5) and (6)) will not be useful. However, the channel $\mathcal{H}^M \xrightarrow{\mathcal{E}_2} \mathcal{H}^N$ can be looked upon as a noisy channel in the larger Hilbert space $\mathcal{H}^N$ with the proviso that the suitable conditions are imposed on the supports of the initial states. That is to say, this scenario can be looked upon as a propagation of an $N$-dimensional quantum system\(^1\) in a noisy channel $\mathcal{H}^N \xrightarrow{\mathcal{E}_1} \mathcal{H}^N$.

Indeed, let $\rho$ be the initial state having a support over an $M$-dimensional space, viz., $\mathcal{H}^M$. Consider the propagation of this state in an idealised crosstalk channel, which causes a spillover to at most $l$ modes. Then, equation (2) assumes the form,

$$E_{\pm k} = \sqrt{p_k} \sum_{n=l_{\text{min}}}^{l_{\text{max}}} | n \pm k \rangle \langle n |, \quad 1 \leq k \leq l. \quad (7)$$

After passing through the crosstalk channel, the final state $\rho'$ is given by,

$$\rho' \equiv \sum_{k=0}^{l} E_{\pm k} \rho E_{\pm k}^\dagger \equiv \sum_{k=0}^{l} p_k \rho^{(k)}_{\pm}. \quad (8)$$

The value $k = 0$ corresponds to no error which leaves the state unchanged. Note that $\rho'$ can be looked upon as an incoherent sum of $\rho^{(k)}_{\pm}$ having supports in corresponding subspaces $\mathcal{H}^{E_{\pm k}}$. Clearly,

$$\mathcal{H}^N \equiv \mathcal{H}^M \cup \mathcal{H}^{E_{+1}} \cup \cdots \cup \mathcal{H}^{E_{+l}} \cup \mathcal{H}^{E_{-1}} \cup \cdots \cup \mathcal{H}^{E_{-l}}. \quad (9)$$

We now show that the effect of the channel $\mathcal{E}_2$ is identical to that of an appropriately chosen generalised flip channel in $\mathcal{H}^N$.

### 3.1 Mapping an idealised crosstalk channel to a generalised flip channel

A crosstalk channel acts on the modes which, in principle, belong to infinite-dimensional spaces. It follows from equation (7) that the effect of crosstalk channel can be understood as causing a translation by $k$ units on the real number line $\mathbb{R}^1$. On the other hand, the effect of a generalised flip channel can be understood as an error causing a shift of points by $k$ units on a one-dimensional circle $\mathbb{S}^1$. However, for the special choice of basis modes employed in equation (3), the actions of the two channels are effectively the same.

As an illustration, consider, a crosstalk channel that causes a spillover by one unit. The corresponding error operators are, $E_{\pm 1} \equiv \sqrt{p_1} \sum_{k=1}^{M} | k \pm 1 \rangle \langle k |$.

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\(^1\)This is possible because $\mathcal{H}^M \subset \mathcal{H}^N$, and hence any state belonging to $\mathcal{H}^M$ also belongs to $\mathcal{H}^N$. 
We, now, consider the state, 
\[ |\psi\rangle = \sum_{i=1}^{M} \alpha_i |i\rangle \equiv \sum_{i=0}^{M+1} \alpha_i |i\rangle; \quad \alpha_0, \alpha_{M+1} \equiv 0, \] 
(10)

that is employed to transfer information. Then, upto normalisation factors,
\[ E_{+1} |\psi\rangle \equiv \sum_{i=1}^{M} \alpha_i |i+1\rangle, \quad E_{-1} |\psi\rangle \equiv \sum_{i=1}^{M} \alpha_i |i-1\rangle. \]
(11)

Consider, now, the generalised flip channel acting on \( \mathcal{H}^{M+2} \) via the error operators, \( F_{+1} = \sqrt{p_1}X \) and \( F_{-1} = F_{+1}^\dagger = \sqrt{p_1}X^\dagger \), where the flip operator is \( X = \sum_{k=0}^{M+1} |k+1\rangle \langle k| \) (with addition modulo \( M + 2 \)). Again, upto normalisation factors, the effect of error operators \( F_{\pm 1} \) on the state \( |\psi\rangle \) is given by,
\[ F_{+1} |\psi\rangle \equiv \sum_{i=1}^{M} \alpha_i |i+1\rangle, \quad F_{-1} |\psi\rangle \equiv \sum_{i=1}^{M} \alpha_i |i-1\rangle, \]
(12)

which is no different from equation (11), thereby establishing that they have the same effect on the family of states chosen in equation (10). This argument extends to crosstalk channels that shift a mode by \( l \) units provided we employ a generalised flip channel defined in \((M + 2l)\)-dimensional space.

Consider a crosstalk channel that shifts a mode by \( l \) units. The error operators given in equation (2) assume the following form,
\[ E_{\pm r} = \sqrt{p_r} \sum_{l}^{l+M-1} |k \pm r\rangle \langle k|, \quad 1 \leq r \leq l. \]
(13)

The initial state would have the form,
\[ |\phi\rangle = \sum_{j=l}^{l+M-1} \beta_j |j\rangle. \]
(14)

Upto normalisation factors, the effect of these operators on the state \( |\phi\rangle \) is,
\[ E_{+r} |\phi\rangle \equiv \sum_{j=l}^{l+M-1} \beta_j |j + r\rangle, \quad E_{-r} |\phi\rangle \equiv \sum_{j=l}^{l+M-1} \beta_j |j - r\rangle, \]
(15)

which can be reproduced by the generalised flip channel defined in \((M + 2l)\)-dimensional space with operators,
\[ F_{+r} = \sqrt{p_r}X^r, \quad F_{-r} = F_{+r}^\dagger = \sqrt{p_r}(X^r)^\dagger; \quad 1 \leq r \leq l, \]
(16)
where $X^r = \sum_{k=0}^{M+2l-1}|k+r\rangle\langle k|$ (with addition modulo $M+2l$) as may be verified easily.

In summary, the effect of a crosstalk error causing a spillover by $l$ units is equal to that of a generalised flip errors in $\mathcal{H}^N$ provided the initial state is chosen appropriately and $N \geq M+2l$.

These results allow us to import invariants of a generalised flip channel obtained in [16] for crosstalk channels which can be used for error–immune information transfer.

4 Complete information retrieval from a state after passing through an idealised crosstalk channel

In this section, we lay down a scheme that allows for complete information retrieval from a state by employing the invariants identified in the last section. To start with, we consider an example of a qutrit passing through a crosstalk channel causing a spillover by one unit.

4.1 Example: qutrit passing through a crosstalk channel causing crosstalk by one unit

The error operators of a crosstalk channel causing a spillover by one unit are given by,

$$E_0 = \sqrt{p_0} \sum_{k=1}^{3} |k\rangle\langle k|, \quad E_{\pm1} \equiv \sqrt{p_1} \sum_{k=1}^{3} |k \pm 1\rangle\langle k|.$$  

Suppose that a state,

$$\rho = \sum_{i,j=1}^{3} \rho_{i,j} |i\rangle\langle j|,$$  

is employed to transfer information. The resulting state, $\rho'$, after passing through this channel, in accordance with equation (8), is given by,

$$\rho' \equiv E_0 \rho E_0^\dagger + E_{+1} \rho E_{+1}^\dagger + E_{-1} \rho E_{-1}^\dagger \equiv p_0 \rho + p_1 \rho_{+1}^{(1)} + p_1 \rho_{-1}^{(1)}.$$  

The states $\rho_{\pm1}^{(1)}$ would have the forms,

$$\rho_{\pm1}^{(1)} = \sum_{i,j=1}^{3} \rho_{i,j} |i \pm 1\rangle\langle j \pm 1|.$$  

Since the corresponding final state $\rho'$ involves superpositions of five modes, we define the corresponding flip and phase operators as $X = \sum_{j=0}^{4} |j + 1\rangle \langle j|$ and $Z = \sum_{j=0}^{4} \omega^j |j\rangle \langle j|$ (with addition modulo 5 and $\omega$ is fifth root of unity).

The invariants of first family are found to be,

$$I^{(1)}_1 = \langle X \rangle = \rho_{1,2} + \rho_{2,3}; \quad I^{(2)}_1 = \langle X^2 \rangle = \rho_{1,3}. \quad (21)$$

Similarly, the invariants of second family are found to be,

$$I^{(1,1)}_2 = \frac{\langle Z \rangle}{\langle XZ \rangle} = \frac{\omega \rho_{1,1} + \omega^2 \rho_{2,2} + \omega^3 \rho_{3,3}}{\omega^2 \rho_{1,2} + \omega^3 \rho_{2,3}},$$

$$I^{(1,2)}_2 = \frac{\langle Z^2 \rangle}{\langle X^2 Z \rangle} = \frac{\omega^2 \rho_{1,1} + \omega^4 \rho_{2,2} + \omega^6 \rho_{3,3}}{\omega^2 \rho_{1,2} + \omega^3 \rho_{2,3}}, \quad (22)$$

$$I^{(2,1)}_2 = \frac{\langle Z \rangle}{\langle X^2 Z \rangle} = \frac{\omega \rho_{1,1} + \omega^2 \rho_{2,2} + \omega^3 \rho_{3,3}}{\omega^2 \rho_{1,3}},$$

$$I^{(2,2)}_2 = \frac{\langle Z^2 \rangle}{\langle X^2 Z^2 \rangle} = \frac{\omega^2 \rho_{1,1} + \omega^4 \rho_{2,2} + \omega^6 \rho_{3,3}}{\omega^2 \rho_{1,3}}. \quad (23)$$

It is important to note that the numerators and denominators in equations (22) and (23) have same dependence on the channel parameters $p_0$ and $p_1$, that is why their ratios, i.e., $I^{(1,1)}_2, I^{(1,2)}_2, I^{(2,1)}_2$ and $I^{(2,2)}_2$ are independent of them. Given these invariants, we next, enumerate the steps to retrieve full information in the state $\rho$ as follows:

- **Step 1:** The invariant $I^{(2)}_1$ determines the off–diagonal density matrix element $\rho_{1,3}$.

- **Step 2:** Having knowledge of $\rho_{1,3}$, we next employ the invariants $I^{(2,1)}_2$ and $I^{(2,2)}_2$, together with the trace condition $\rho_{1,1} + \rho_{2,2} + \rho_{3,3} = 1$, to find the values of the diagonal elements of the density matrix $\rho$, i.e., $\rho_{1,1}, \rho_{2,2},$ and $\rho_{3,3}$.

- **Step 3:** Equipped with the diagonal elements of the density matrix $\rho$, we employ the invariants $I^{(1,1)}_2$ and $I^{(1,2)}_2$ to retrieve the off–diagonal density matrix elements $\rho_{1,2}$ and $\rho_{2,3}$.

In this manner, we have determined diagonal as well as off–diagonal elements of the density matrix $\rho$. Since the equations get undetermined if $\rho_{1,3} = 0$, it is necessary that the initial state should be prepared accordingly.

In the following section, we describe the scheme that allows for full information retrieval from a state passing through a crosstalk channel that causes a spillover by at most $l$ units.
4.2 Complete information retrieval from a state passing through a crosstalk channel causing a spillover by $l$ units

Consider a state involving superposition of $M$ modes,

$$\rho = \sum_{i,j=l}^{t+M-1} \rho_{i,j} |i\rangle \langle j|,$$

that is employed to transfer information through a crosstalk channel. Suppose that the crosstalk channel is such that it can cause spillover to at most $l$ modes. In that case, the final state, after passing through this channel, takes the form,

$$\rho' \equiv \sum_{k=0}^{l} E_{\pm k} \rho E_{\pm k}^\dagger \equiv \sum_{k=0}^{l} p_k \rho_{\pm}^{(k)},$$

where $\rho_{\pm}^{(0)} \equiv \rho$ and $\rho_{\pm}^{(k)}$ are given by,

$$\rho_{\pm}^{(k)} = \sum_{i,j=l}^{t+M-1} \rho_{i,j} |i \pm k\rangle \langle j \pm k|, \quad \forall k \in \{1, \ldots, l\}. \quad (26)$$

From equations (25) and (26), it is clear that the final state $\rho'$ has a support in an $(M + 2l)$–dimensional Hilbert space.

For the purpose of information retrieval, we first fix the the dimension of the space in which the flip operators are to be defined. This requires us to take into account the following point. The dimensionality of the space should be such that the relation,

$$\langle X^{M-1} Z^r \rangle_\rho = f(\omega, r) \langle X^{M-1} \rangle_\rho,$$

holds where $f(\omega, r)$ is a function of only $\omega$ and $r$. Since the initial state $\rho$ involves superposition of $M$ modes, condition (27) is satisfied iff the operator $X^{M-1}$ is not its own inverse. It holds in the minimum dimension $2M - 1$. This is because in $(2M - 1)$–dimensional space, the operator $X$ satisfies the relation, $X^{2M-1} = 1$ leading to $(X^{M-1})^{-1} = X^M$. On the other hand, since the final state has support in $(M + 2l)$ dimensional space, the minimum dimension for information retrieval is given by,

$$N \equiv \max[2M - 1, M + 2l]. \quad (28)$$

Thus, in the $N$–dimensional space, operators $X^r$ and $Z^r$ have the forms:

$$X^r = \sum_{k=0}^{N-1} |k + r\rangle \langle k|, \quad Z^r = \sum_{k=0}^{N-1} \omega^{nk} |k\rangle \langle k|, \quad (29)$$
where addition is modulo $N$ and $\omega$ is $N$th root of identity. Following the reasoning in section (3.1), the invariants of the first family provide the following information on the initial state $\rho$,

$$I_1^{(r)} = \langle X^r \rangle = \sum_{k=l}^{l+M-1-r} \rho_{k,k+r}; \quad 1 \leq r \leq M - 1.$$  \hspace{1cm} (30)

In a similar manner, the second family of invariants yields the relation,

$$I_2^{(r,s)} = \frac{\langle Z^s \rangle}{\langle X^r Z^s \rangle} = \frac{\sum_{k=l}^{l+M-1} \omega^{ks} \rho_{k,k}}{\sum_{k=l}^{l+M-1-r} \omega^{ks} \rho_{k,k+r}}; \quad 1 \leq r, s \leq M - 1.$$  \hspace{1cm} (31)

Similar to the section (4.1), the steps of the scheme that allows full information retrieval of a state are as follows:

- **Step 1:** The first family invariant, $I_1^{(M-1)}$ determines the off–diagonal element $\rho_{l,l+M-1}$.
- **Step 2:** The second family of invariants,

$$I_2^{(M-1,s)} = \frac{\sum_{k=l}^{l+M-1} \omega^{ks} \rho_{k,k}}{\omega^{ls} \rho_{l,l+M-1}}; \quad 1 \leq s \leq M - 1,$$

yield the simultaneous set of $(M-1)$ equations. These equations, together with the trace condition $\sum_{i=l}^{l+M-1} \rho_{i,i} = 1$, allow to retrieve each of the diagonal element $\rho_{i,i}$.

It remains to retrieve the other off–diagonal elements for which the following invariants are employed,

$$I_1^{(p)} = \sum_{k=l}^{l+M-1-p} \rho_{k,k+p} ; \quad I_2^{(p,s)} = \frac{\sum_{k=l}^{l+M-1} \omega^{ks} \rho_{k,k}}{\sum_{k=l}^{l+M-1-p} \omega^{ks} \rho_{k,k+p}}; \quad 1 \leq s \leq M - 1 - p,$$

where $p$ can take values from the set $\{1, 2, \cdots, M-2\}$. The index $p$ corresponds to the retrieval of off–diagonal elements lying on the line $S_p$ as shown in the figure (2). This is explicitly detailed in the table (1).

### 5 Quantum error rejection code (QERC)

In the preceding sections, we have identified invariants for crosstalk channels that allows for error–immune information transfer. Employing these invariants, we have laid down a scheme which allows for full information retrieval from a state. In this section, we determine the additional constraints to be imposed on the choice of basis modes to construct QERC.
Fig. 2 $\rho_{i,j}$ represents the $ij$th element of the initial density matrix $\rho$, $D$ represents its diagonal, and $S_i$ represents its $i$th super–diagonal (which we use for convenience).

| Invariants | Elements of density matrix |
|------------|---------------------------|
| $I_{1}^{(M-1)}$ & $\rho_{i,j}$; $j - i = M - 1$ | $S_{M-1}$ |
| $I_{2}^{(M-1,s)}$, $1 \leq s \leq M - 1$ & $\rho_{i,i}$, $\forall i \in \{l, \ldots, l + M - 1\}$ | $D$ |
| $I_{1}^{(M-2,1)}$, $I_{2}^{(M-2,1)}$ & $\rho_{i,j}$; $j - i = M - 2$ | $S_{M-2}$ |
| $I_{1}^{(M-3,1)}$, $I_{2}^{(M-3,1)}$, $I_{2}^{(M-3,2)}$ & $\rho_{i,j}$; $j - i = M - 3$ | $S_{M-3}$ |
| $I_{1}^{(1,1)}$, $I_{2}^{(1,1)}$, $I_{2}^{(1,2)}$, $\ldots$, $I_{2}^{(1,M-2)}$ & $\rho_{i,j}$; $j - i = 1$ | $S_1$ |

Table 1 Sets of invariants employed and correspondingly retrieved density matrix elements.

Recall that, a QERC merely detects whether error has corrupted a state or not [25]. In order to construct a QERC, it is necessary that each error projects the initial state to a space which has no overlap with it. That is,

$$\mathcal{H}^M \perp \mathcal{H}^E,$$

where $\mathcal{H}^E \equiv \mathcal{H}^{E+1} \cup \cdots \cup \mathcal{H}^{E+l} \cup \mathcal{H}^{E-1} \cup \cdots \cup \mathcal{H}^{E-1}$. (33)

By virtue of this orthogonality, one may employ an observable,

$$O = \Pi^M,$$

that provides an outcome only if the state is uncorrupted. A more general observable, $O = c_1\Pi^M + c_2\Pi^E$, $c_1 \neq c_2$ can be employed for detection of occurrence of errors. Here, $\Pi^M$ and $\Pi^E$ represent the projections having supports
in $\mathcal{H}^M$ and $\mathcal{H}^E$. If the measurement outcome is $c_1$, then, there is no corruption. On the other hand, the outcome $c_2$ corresponds to the case when the post-measurement state is corrupted and hence can be rejected.

5.1 QERC for a crosstalk channel that causes a spillover by at most $l$ units

For a crosstalk channel that shifts a mode by $l$ units, the basis modes should be chosen at a distance, $\Delta_r = (l + 1)$ to preserve the condition of orthogonality of the initial state with all the corrupted states. Therefore, if an $M$–dimensional state is to be transferred through a crosstalk channel causing a spillover by $l$ units, the following set of basis modes,

$$B_r \equiv \{|l\rangle, |l + \Delta_r\rangle, |l + 2\Delta_r\rangle, \cdots, |l + (M - 1)\Delta_r\rangle\},$$

(35)
can be employed. Thus, any $M$–dimensional state,

$$|\psi\rangle = \sum_{i=0}^{M-1} \alpha_i |l + i\Delta_r\rangle,$$

(36)
can be employed as it remains orthogonal to any of the corrupted states of a crosstalk channel, thus fulfilling the requirement of QERC. For this state, measurement of the observable,

$$O = \Pi^M = \sum_{j=0}^{M-1} |l + j\Delta_r\rangle \langle l + j\Delta_r|,$$

(37)
gives the outcome only when the state is uncorrupted.

6 Quantum error correction code (QECC)

In the preceding section, we have constructed a quantum error rejecting code for an $M$–dimensional state passing through a crosstalk channel that may cause spillover by at most $l$ units. The demand of error correction codes imposes the most stringent condition that the errors projects the initial state onto mutually orthogonal spaces $[22, 26]$. That is,

$$\mathcal{H}^M \perp \mathcal{H}^{E_k}, \quad \mathcal{H}^{E_j} \perp \mathcal{H}^{E_k}; \quad j, k \in \{\pm 1, \pm 2, \cdots, \pm l\}. $$

(38)

By virtue of this orthogonality, an observable can be constructed which can distinguish each of these states and hence, each error can be detected. Such an observable is given as,

$$O = c_M \Pi^M + \sum_{k=1}^{l} c_{\pm k} \Pi^{E_{\pm k}},$$

(39)
with distinct eigenvalues $c_M$, $c_{\pm k}$ belonging to the respective projections $\Pi^M$, $\Pi^{E_{\pm k}}$.

Each outcome of this observable provides information about the subspace to which the post-measurement state belongs. If an error has occurred, its effect can be undone by performing an appropriate transformation on the state.

**Example:** To start with, we consider an idealised crosstalk channel that causes a spillover by one unit. In this case, the error operators are given as $E_{\pm 1}$. For a state involving superposition of three modes, the basis modes $\{|1\rangle, |4\rangle, |7\rangle\}$ satisfy the requirement of a QECC. The effect of this crosstalk channel on the basis modes is shown in figure (3).

![Choice of modes for a state passing through a crosstalk channel of one unit.](image)

The states $|\psi_\pm\rangle$ have the forms,

$$|\psi_+\rangle = \alpha |2\rangle + \beta |5\rangle + \gamma |8\rangle, \quad |\psi_-\rangle = \alpha |0\rangle + \beta |3\rangle + \gamma |6\rangle.$$  

(42)

It is to be noted that for this choice of modes, $\langle \psi_\pm | \psi \rangle = 0$ and $\langle \psi_+ | \psi_- \rangle = 0$. Detection of errors can be performed by measuring the stabiliser given in equation (39) for which the projections are:

$$\Pi^M = |1\rangle \langle 1| + |4\rangle \langle 4| + |7\rangle \langle 7|, \quad \Pi^{E_{+1}} = |2\rangle \langle 2| + |5\rangle \langle 5| + |8\rangle \langle 8|,$$

(43)

$$\Pi^{E_{-1}} = |0\rangle \langle 0| + |3\rangle \langle 3| + |6\rangle \langle 6|.$$  

(44)

Since each outcome of the stabiliser provides information about occurrence of a particular error, effect of which can be undone by performing an appropriate transformation. The measurement outcomes, corresponding errors and required transformations for error correction have been given in table (2).
We, next, move on to propose a QECC that allows to retrieve state after passing through an idealised crosstalk channel that causes a spillover by \( l \) units.

### 6.1 QECC for crosstalk channel causing spillover by \( l \) units

In the last example, we have constructed a QECC for a state involving superposition of three modes passing through a crosstalk channel causing a spillover by one unit. In that example, the consecutive modes differ from each other by 3 units. Employing the same observation for crosstalk channel causing a spillover by \( l \) units, the consecutive basis modes should be chosen at a distance of \( \Delta = (2l + 1) \). So, for an \( M \)-dimensional system, the following set of basis modes can be employed,

\[
B_c \equiv \{ |l\rangle, |l + \Delta\rangle, \ldots, |l + (M - 1)\Delta\rangle \},
\]  

(45)

The most general \( M \)-dimensional pure state employing these basis modes is given as,

\[
|\psi\rangle \equiv \sum_{j=0}^{M-1} \alpha_j |l + j\Delta\rangle.
\]  

(46)

The projections, for basis modes \( B_c \), that determine the stabiliser given in equation (39) are,

\[
\Pi^M \equiv \sum_{j=0}^{M-1} |l + j\Delta\rangle \langle l + j\Delta|; \quad \Delta = 2l + 1,
\]  

(47)

\[
\Pi^{E \pm k} \equiv \sum_{j=0}^{M-1} |l \pm k + j\Delta\rangle \langle l \pm k + j\Delta|, \quad 1 \leq k \leq l,
\]  

(48)

Depending on the outcome of stabiliser, one may apply an appropriate transformation on the state as given in the table (3).
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| Measurement outcomes | Error | Transformation |
|----------------------|-------|---------------|
| $c_0$                | $E_0$ | No error      |
| $c_{\pm 1}$         | $E_{\pm 1}$ | $\sum_{j=0}^{M-1} |l \pm 1 + j\Delta|$ |
| $c_{\pm 2}$         | $E_{\pm 2}$ | $\sum_{j=0}^{M-1} |l + j\Delta|$ |
|                      |       |               |
| $c_{\pm l}$         | $E_{\pm l}$ | $\sum_{j=0}^{M-1} |l \pm l + j\Delta|$ |

Table 3 Measurement outcomes of the stabiliser, detected error and the corresponding transformation.

7 Relation with propagation of orbital angular momentum (OAM) modes in turbulent atmosphere

We, now, discuss the possibility of practical realisations of the schemes discussed in the paper. It has been observed that propagation of OAM modes through atmospheric turbulence can be modelled via crosstalk channels [20, 21, 23, 24, 27]. There are a number of studies which conform to idealised crosstalk channels considered in this paper. The idealised crosstalk channel is observed experimentally in [27] for OAM modes in weak turbulence regime. In addition, there are also analytical and simulation studies. In [24], the influence of atmospheric turbulence on the propagation of beams having uniform amplitude carrying OAM has been studied. In this work, it is observed that the crossover probabilities are functions of only $k$ and independent of the initial OAM mode index, i.e., $n$ which matches with the idealised crosstalk channel. The respective symbols $k$ and $n$ represent the units by which spillover takes place and initial mode index as also given in equation (2). Similarly, in [23], for weak turbulence regime, it is found that the crossover probabilities are independent of initial OAM mode and depends only upon the difference in initial and final OAM mode indices, which is $k$. Additionally, it has been numerically shown in [20] that the crossover probabilities, for Laguerre–Gauss (LG) modes in weak turbulence regime and for lower OAM mode indices, depend only upon the difference in initial and final OAM mode indices, i.e., they depend only upon $k$. In fact, the above result hold for OAM modes $2 \leq n \leq 7$ in weak turbulence regime.

However, it has been numerically observed that in strong turbulence, for LG modes, crossover probabilities, in general, are functions of initial and final OAM mode indices [20, 21]. That is, the crossover probabilities, depend on both $n$ and $k$ which do not obey our assumptions. Should these modellings be confirmed by experiments, our assumptions would need further refinement.
8 Conclusion

In summary, we have identified invariants for idealised crosstalk channels that allow for error-immune information transfer. In fact, we have laid down a scheme employing which full information in a state can be retrieved. We have also constructed quantum error rejection and quantum error correction codes for an idealised crosstalk channel.

This study opens up many avenues from information transfer perspective. For example, invariants for multipartite states for crosstalk channel may be employed for broadcasting error–free information. In fact, collaborative information transfer can be implemented by encoding information in a set of invariants whose measurements require collaboration among distinct participants. Full information retrieval from multipartite states passing through crosstalk channel also constitutes an interesting study.

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Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Disclosures

The authors declare no conflicts of interest.

Author contribution statement

R.B. and V.R. conceived the idea and contributed in the entire work. S. A. has contributed significantly in construction of quantum error correction code and in showing the equivalence of
generalised flip channels and crosstalk channels. All the authors have written the manuscript collectively.