A Calculation of the Full Neutrino Phase Space in Cold+Hot Dark Matter Models

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ABSTRACT

This paper presents a general-relativistic $N$-body technique for evolving the phase space distribution of massive neutrinos in linear perturbation theory. The method provides a much more accurate sampling of the neutrino phase space for the HDM initial conditions of $N$-body simulations in a cold+hot dark matter (CDM+HDM) universe than previous work. Instead of directly sampling the phase space at the end of the linear era, we first compute the evolution of the metric perturbations by numerically integrating the coupled, linearized Einstein, Boltzmann, and fluid equations for all particle species (CDM, baryons, photons, massless neutrinos, and massive neutrinos). (Details of this calculation are discussed in a separate paper.) We then sample the phase space shortly after neutrino decoupling at redshift $z = 10^9$ when the distribution is Fermi-Dirac. To follow the trajectory of each neutrino, we subsequently integrate the geodesic equations for each neutrino in the perturbed background spacetime from $z = 10^9$ to $z = 13.55$, using the linearized metric found in the previous calculation to eliminate discreteness noise. The positions and momenta resulting from this integration represent a fair sample of the full neutrino phase space and can be used as HDM initial conditions for $N$-body simulations of nonlinear structure evolution in CDM+HDM models. A total of $\sim 21$ million neutrino particles are used in a $100$ Mpc box, with $\Omega_{\text{cdm}} = 0.65$, $\Omega_{\text{hdm}} = 0.30$, $\Omega_{\text{baryon}} = 0.05$, and Hubble constant $H_0 = 50$ km s$^{-1}$ Mpc$^{-1}$. We find that correlations develop in the neutrino densities and momenta which are absent when only the zeroth-order Fermi-Dirac distribution is considered.

Subject headings: cosmology: theory — dark matter — gravitation

1. Introduction

As it has become increasingly difficult to explain cosmological observations in the context of the standard cold dark matter (CDM) model, the cold+hot dark matter (CDM+HDM) models have emerged as one of the promising alternatives that require only moderate modifications of the
CDM model. The excess small-scale power relative to the large-scale power in the standard CDM model (Davis et al. 1985, 1992a; Gelb, Gradwohl, & Frieman 1993; Gelb & Bertschinger 1993) motivates the addition of a hot component of massive neutrinos to the total mass density of the universe. The free streaming of the neutrinos suppresses the growth of small-scale perturbations while leaving the growth of large-scale perturbations unimpeded, and may therefore alleviate some of the problems of the standard CDM model. Several recent linear calculations (Schaefer, Shafi, & Stecker 1989; van Dalen & Schaefer 1992; Taylor & Rowan-Robinson 1992; Holtzman & Primack 1993) and N-body simulations (Davis, Summers, & Schlegel 1992b; Klypin et al. 1993) have found a better match of observations with the CDM+HDM models than with the CDM models, although a fair comparison between the models and the galactic scale data such as the epoch of galaxy formation and galaxy pairwise velocities awaits results from higher resolution N-body simulations in a large volume. Most workers (including ourselves) have assumed that the mass density fraction contributed by HDM is $\Omega_{\text{hdm}} \sim 0.3$ for an $\Omega_{\text{total}} = 1$ universe, corresponding to a neutrino mass of $m_{\nu} \sim 7$ eV, although Pogosyan & Starobinsky (1993) favor $0.17 \leq \Omega_{\text{hdm}} \leq 0.28$ for $H_0 = 50$ km s$^{-1}$ Mpc$^{-1}$.

The introduction of HDM into the theory brings about one complication due to the different behavior of the neutrinos and CDM. In the linear regime CDM behaves as a pressureless perfect fluid, but the neutrinos can be appropriately described only by their full phase space distribution obeying the Boltzmann equation. None of the earlier studies of HDM models of which we are aware has taken account of the full phase space information of the neutrinos.

In the particle-particle/particle-mesh (P$^3$M) simulation performed by Davis et al. (1992b), the HDM particles were placed initially on a grid without perturbations at $1 + z = 20$, based on the argument that the neutrino Jeans length at this redshift is comparable to their simulation box size of 14 Mpc. The initial conditions for the CDM particles were generated with the Zel’dovich approximation from the pure CDM spectrum and scaled to the normalization $\sigma_8 \sim 0.45$. To simulate the thermal motion of the particles, a velocity drawn randomly from the Fermi-Dirac distribution was given to each HDM particle. The cosmological parameters $\Omega_{\text{cdm}} = 0.7$, $\Omega_{\text{hdm}} = 0.3$, and $H_0 = 50$ km s$^{-1}$ Mpc$^{-1}$ were used, with $32^3$ CDM and $32^3$ HDM particles.

In the particle-mesh (PM) simulations of Klypin et al. (1993), the initial conditions for CDM and HDM were generated with the Zel’dovich method from individual CDM and HDM power spectra, which differ significantly on scales smaller than the neutrino free-streaming distance since the growth of perturbations in HDM is suppressed. A thermal velocity drawn from the Fermi-Dirac distribution was added to each HDM particle. The simulations were performed in 14, 50, and 200 Mpc boxes with $128^3$ CDM and $6 \times 128^3$ HDM particles starting at redshift $1 + z = 15$. The parameters $\Omega_{\text{cdm}} = 0.6$, $\Omega_{\text{hdm}} = 0.3$, $\Omega_{\text{baryon}} = 0.1$, and $H_0 = 50$ km s$^{-1}$ Mpc$^{-1}$ were used.

In both groups’ simulations, the initial neutrino momenta were drawn from the Fermi-Dirac distribution independently of the neutrino positions. In general, however, the neutrino phase space distribution is a complicated function of positions, momenta (or velocities), and time, with
the Fermi-Dirac distribution being only the zeroth-order term. Velocity-position correlations in the neutrinos can arise from perturbations to the Fermi-Dirac distribution. Although the actual HDM correlations are initially small in the linear regime, they can play an important role in the nonlinear stage of evolution, and in the linear theory should be treated as being of the same order as all other perturbations.

In this paper, we obtain the neutrino initial conditions from the full phase space distribution in the linear theory of gravitational perturbations. Since the full phase space distribution depends on 6 canonical variables, it is numerically impractical to sample the full neutrino phase space at the start of $N$-body simulations. We resolve this difficulty by sampling the neutrino phase space with neutrino particles at a very early time of $z \sim 10^9$ when the spatial distribution of neutrinos is nearly uniform. At this time the phase space distribution is Fermi-Dirac to a very good approximation, and the neutrinos can be placed on a grid with momenta drawn from the Fermi-Dirac distribution. Since the neutrinos have already decoupled from other species ($z_{\nu, dec} \sim 10^{10}$), their trajectories simply follow geodesics in the perturbed Robertson-Walker spacetime.

Our strategy will be to first calculate the metric perturbations by integrating the coupled, linearized Einstein, Boltzmann, and fluid equations that govern the evolution of the metric and density perturbations of all particle species (CDM, photons, baryons, massless neutrinos, and massive neutrinos). Then we integrate the linearized geodesic equations for each neutrino from $z \sim 10^9$ until $z = 13.55$, after which we will switch to a fully nonlinear Newtonian integration. The high-redshift approach can be described as a “general-relativistic cosmological $N$-body integration” valid in the linear theory. It differs from the conventional $N$-body technique in that the gravitational forces are precomputed from the metric perturbations of the background spacetime rather than directly from the particles. The configuration of neutrinos at $z = 13.55$ will then represent a fair sample of the full phase space distribution at that redshift, and can be used directly as the HDM initial conditions for subsequent $N$-body simulations.

We leave the discussion of the first stage of our calculation on the Einstein, Boltzmann, and fluid equations to a separate paper (Ma and Bertschinger 1993). In the present paper we focus on the geodesic integration assuming the metric perturbations have been computed. We find the conformal Newtonian gauge a very convenient choice for this part of the calculation. We derive the linearized geodesic equations in this gauge in Section 2 and discuss the integration method in Section 3. We report the integration results in Section 4 where we show the effect of the perturbations in the neutrino phase space on the correlation between the HDM momenta and the density contrast. Section 5 includes a summary and a discussion of work in progress.

2. Geodesic Equations in Conformal Newtonian Gauge
Although many calculations of the general-relativistic linear perturbation theory have been carried out in the synchronous gauge, we find it most convenient to compute the trajectories of the neutrinos in the conformal Newtonian gauge (Mukhanov, Feldman, & Brandenberger 1992). The conformal Newtonian gauge has the advantage that spurious coordinate singularities do not arise, and the geodesic equations have simple forms which are easy to integrate. The metric in the conformal Newtonian gauge is given by

\[
\frac{ds^2}{a^2(\tau)} = -\left(1 + 2\psi\right)d\tau^2 + \left(1 - 2\phi\right)\gamma_{ij}dx^idx^j,
\]

where the scalar potentials \(\psi\) and \(\phi\) characterize the perturbations about a flat Robertson-Walker spacetime. We use Cartesian coordinates so that the 3-metric of \(\tau = \text{constant}\) hypersurfaces is \(\gamma_{ij} = \delta_{ij}\). It should be emphasized that \(\phi\) and \(\psi\) describe only the scalar mode of the metric perturbations. We do not consider the vector and the tensor modes in this paper.

The geodesic equations for a neutrino of mass \(m_\nu\) can be derived by minimizing the action

\[
S = \int d\tau L = -m_\nu \int \left(-\frac{ds^2}{a^2(\tau)}\right)^{1/2},
\]

where \(L\) is the Lagrangian and the metric in the conformal Newtonian gauge is given by Eq. (1). To linear order in the potentials, the Lagrangian is

\[
L = -m_\nu a\sqrt{1 - u^2} \left(1 + \frac{\psi + u^2\phi}{1 - u^2}\right),
\]

where \(\vec{u} = \frac{d\vec{x}}{d\tau}\) is the coordinate velocity and \(u^2 = \gamma_{ij}u^iu^j\). The conjugate momentum \(q_i\) is given by

\[
q_i \equiv \frac{\partial L}{\partial \dot{u}^i} = \frac{m_\nu a\gamma_{ij}u^j}{\sqrt{1 - u^2}} \left(1 - 2\phi - \frac{\psi + u^2\phi}{1 - u^2}\right),
\]

which can be inverted to give, to first order in \(\psi\) and \(\phi\),

\[
\dot{u}^i = \frac{dx^i}{d\tau} = \frac{\gamma^{ij}q_j}{\epsilon(q, \tau)} \left\{1 + \psi(\vec{x}, \tau) + \left[2 - \frac{q^2}{\epsilon^2(q, \tau)}\right] \phi(\vec{x}, \tau)\right\}
\]

with \(\epsilon(q, \tau) = \sqrt{q^2 + m_\nu^2a^2}\). The Euler-Lagrange equation of motion gives

\[
\frac{dq_i}{d\tau} = -\frac{m_\nu a}{\sqrt{1 - u^2}} \left(\partial_i\psi + u^2\partial_i\phi\right).
\]

Replacing \(\vec{u}\) on the right-hand side with Eq. (5), we obtain

\[
\frac{dq_i}{d\tau} = -\epsilon(q, \tau) \left[\partial_i\psi(\vec{x}, \tau) + \frac{q^2}{\epsilon^2(q, \tau)}\partial_i\phi(\vec{x}, \tau)\right].
\]

Eqs. (5) and (7) give the linearized geodesic equations for a particle moving in the perturbed spacetime characterized by scalar metric perturbations \(\psi\) and \(\phi\). In the weak-field, nonrelativistic \((q^2 \ll \epsilon^2)\) limit they reduce to the standard Newtonian equations.
3. Integration of Geodesic Equations

To sample the neutrino phase space as accurately as possible, we integrate the geodesic equations (5) and (7) for $10 \times 128^3$ ($\sim 21$ million) massive neutrino particles. A cubic simulation box with sides $100$ Mpc is used. The cosmological parameters are taken to be $\Omega_{\text{cdm}} = 0.65$, $\Omega_{\text{hdm}} = 0.3$, $\Omega_{\text{baryon}} = 0.05$, and $H_0 = 50$ km $s^{-1}$ Mpc$^{-1}$. We start the integration shortly after neutrino decoupling at redshift $z \sim 10^9$ when perturbations in the neutrino density and momenta can be safely ignored. The neutrinos are placed initially on a $128^3$ grid, 10 per grid point, with the neutrino momenta drawn randomly from the Fermi-Dirac distribution

$$f(\vec{q}) \, d^3q \propto \frac{d^3q}{e^{q T_{\nu,0}/k} + 1},$$

where $T_{\nu,0} = (4/11)^{1/3} T_{\gamma,0}$ is the neutrino temperature today with $T_{\gamma,0} = 2.735$ K. We performed test runs with the same set of momenta but randomly generated initial positions and found no statistically significant difference at the end of the integration depending on whether the neutrinos were initially placed at random or on a grid.

We also tested the momentum pairing scheme used by Klypin et al. (1993) in their initial conditions. For every momentum drawn from the Fermi-Dirac distribution, they assigned an equal but opposite momentum to a second neutrino at the same grid point to preserve the local center of momentum. We performed two test runs, with the initial neutrino momenta drawn randomly in one run and paired up in opposite directions with the same magnitude in the other run. We found no statistically significant difference in the power spectrum at the end of the geodesic integration. We thus adopted the simpler scheme without pairing.

We integrated the geodesic equations from conformal time $\tau_i = 3 \times 10^{-4}$ Mpc ($z \sim 10^9$) to $\tau_f = 3 \times 10^3$ Mpc ($z = 13.55$), using 701 time steps with stepsize $\Delta (\log_{10} \tau) = 0.01$. The initial $\tau_i$ is chosen so that the largest $k$ in the simulation box is well outside the horizon ($k \tau \ll 1$) at the onset of the integration. The integration was stopped when the fluctuations were still in the linear regime. We used a leap-frog integration scheme in which the positions and momenta were advanced half a timestep out of phase to give a second-order accuracy in timestep size.

The evolution of the metric perturbations $\psi$ and $\phi$ in Eqs. (3) and (5) were precomputed from the coupled, linearized Einstein, Boltzmann, and fluid equations for all particle species including massive neutrinos (Ma and Bertschinger 1993). The resulting transfer functions were saved on a grid of 41 $k$- and 701 $\tau$-values. For the geodesic integration the initial $\psi$ and $\phi$ were generated as Gaussian random variables in $k$-space with the scale-invariant power spectrum $P_{\psi,\phi} \propto k^{-3}$ predicted by the simplest inflationary cosmology models. For later times, the Fourier components of $\psi$ and $\phi$ simply scale according to our linear theory computation. We normalized the amplitude to the COBE rms quadrupole fluctuation $Q_{\text{rms-PS}} = 14 \times 10^{-6}$ K (Seljak & Bertschinger 1993; Wright et al. 1992; Smoot et al. 1992) assuming the Sachs-Wolfe formula for a scale-invariant spectrum.
and $T_{0,\gamma} = 2.735$ K:

$$\frac{Q_{\text{rms-PS}}^2}{T_{0,\gamma}^2} = \frac{5}{108} \left[4\pi k^3 P_\psi(k, \tau_{\text{rec}})\right]_{k \to 0}. \quad (9)$$

The gradients of $\psi$ and $\phi$ in Eqs. (8) and (7) were first computed on a grid in $k$-space and then Fourier transformed to a grid in real space. The second-order Triangular-Shaped Cloud (TSC) interpolation scheme was then used to interpolate the gradients from the grid to the particle positions (see Ma 1993 for more details).

Test runs were performed with $N_{\text{grid}} = 32^3$ using $N_{\text{part}} = 10 \times 32^3$, $40 \times 32^3$, and $80 \times 32^3$, and $N_{\text{grid}} = 64^3$ using $N_{\text{part}} = 10 \times 64^3$ and $40 \times 64^3$. (The first factor in $N_{\text{part}}$ gives the number of samples of the momentum space at each initial position.) We tested the accuracy of the time integration using 351 and 701 timesteps respectively and found little difference in the final positions and velocities, indicating that 701 timesteps are sufficient. We also generated realizations of the potentials and the initial neutrino momenta using three different random number generators. No correlations in the random numbers were detected. Our large production run had $N_{\text{grid}} = 128^3$ and $N_{\text{part}} = 10 \times 128^3$ ($\sim 21$ million). The geodesic integration required a total of $\sim 1.5$ Gbytes of memory and $\sim 140$ CPU hours on the Convex C3880 supercomputer at the National Center for Supercomputing Applications.

4. Numerical Results

An image of an intermediate output (timestep 351) from one of the $N_{\text{part}} = 10 \times 32^3$ test runs is shown in Fig. 1. The corresponding redshift is $z \sim 4.9 \times 10^5$. Each side in the figure is 100 Mpc comoving, and the particles in the simulation box have been projected onto the $x - y$ plane. At the starting $z \sim 10^9$, the particles were placed on a $32^3$ grid, 10 per grid point, and were given momenta drawn randomly from the Fermi-Dirac distribution. In this figure, one sees that the neutrinos have begun to spread out from the grid points. In fact, the size of each “ball” is approximately the comoving horizon distance $c\tau \sim 0.95$ Mpc at this moment since the neutrinos are still relativistic.

Fig. 2 shows the same projection in a 100 Mpc box of the last output (timestep 701) from the $N_{\text{part}} = 10 \times 128^3$ run. The corresponding redshift is $z = 13.55$. As one can see, small perturbations are developing in the otherwise uniform distribution of the neutrinos. We present quantitative analyses of this output below.

To check the integration results, we computed the HDM power spectrum from the final output of the $N_{\text{part}} = 10 \times 128^3$ run and made comparison with the prediction from the linear theory. The density field $\delta$ was first computed on a $128^3$ spatial grid from the positions of the neutrinos by the TSC interpolation scheme, and then Fourier transformed into $k$-space. We calculated the power per ln $k$ at a given $k$, $4\pi k^3 P(k)$, by taking a spherical shell of radius $k$ and thickness $\Delta k$ centered
at the origin of the $k$-grid and averaging the contribution to $4\pi k^3 P(k)$ from the grid points that lie within the shell. The shot noise due to the finite number of particles was subtracted and the TSC window was deconvolved. The result is shown in Fig. 3 and is compared with the linear theory predictions for the CDM and HDM power spectra. The agreement provides an important check of the accuracy of our geodesic integration code. From smaller simulations we conclude that the deviations from the ensemble-average power HDM spectrum are due to sampling fluctuations.

The output shown in Fig. 3 is used as the initial conditions for the HDM particles in our $N$-body simulations of structure formation in this CDM+HDM model. To compare our initial conditions to those of others, we recall that the initial positions and velocities of the particles in $N$-body simulations are conventionally generated from the power spectrum using the Zel’dovich (1970) approximation. In this procedure, the positions of the particles are displaced from a regular grid:

$$\vec{x}(\tau) = \vec{x}_0 + \vec{\epsilon}(\vec{x}_0, \tau),$$

where $\vec{x}_0$ gives the position of the grid and $\vec{\epsilon}(\vec{x}_0, \tau)$ is the displacement field. The displacements are computed from the density perturbation field by solving $\vec{\nabla} \cdot \vec{\epsilon} = -\delta$. For small displacements, $\vec{\epsilon}(\vec{x}_0, \tau)$ is approximated by $D_+^\prime(\tau) \vec{\epsilon}(\vec{x}_0)$, where $D_+^\prime(\tau)$ denotes the growth factor of the perturbations, and the velocities of particles are given by

$$\vec{v} \equiv \frac{d\vec{x}}{d\ln a} = \frac{d\ln D_+^\prime}{d\ln a} \vec{\epsilon}(\vec{x}_0, \tau).$$

For the standard CDM model with $\Omega = 1$, the growth factor in the matter-dominated era is equal to the expansion factor, and $f(\Omega) \equiv d\ln D_+^\prime/d\ln a = 1$. The growth rate, however, does not behave so simply in models such as the CDM+HDM models where more than one particle species contributes to $\Omega$. This is illustrated by the power spectra shown in Fig. 3 for the standard CDM model and our CDM+HDM model. The growth rate of CDM in the CDM+HDM model matches the growth rate in the standard CDM model only at small $k$. At large $k$ where neutrino free-streaming is important, we have $\delta_{\text{hdm}} \ll \delta_{\text{cdm}}$, and $f(\Omega, k) = d\ln D_+^\prime/d\ln a < 1$ and is $k$-dependent. We calculated $f(\Omega, k)$ for CDM at $z = 13.55$ in our CDM+HDM model from the output of the linear theory integration; the result is shown in Fig. 3. In the limit $\delta_{\text{cdm}} = \delta_{\text{baryon}}$ and $\delta_{\text{hdm}} = 0$, the growth rate can be computed analytically to be $f = (\sqrt{1 + 24\Omega_c} - 1)/4$ where $\Omega_c = \Omega_{\text{cdm+baryon}}$ (Bond, Efstathiou, & Silk 1980). For our parameters, $f = 0.805$. As one can see in Fig. 3, $f$ is indeed approaching this value at high $k$.

If one did not take into consideration the $k$-dependence of the growth rate and instead used $f(\Omega, k) = 1$ to obtain the CDM initial velocities in CDM+HDM models, one would give the CDM particles excessive initial velocities on small scales, leading to earlier gravitational collapses in the simulations. We estimated this effect in the linear theory on scales below the free-streaming distance. We find that using $f = 1$ instead of $f = 0.805$ gives an initial amplitude in the linear growing mode that is too large by a factor 1.093. Since galaxies form later in CDM+HDM models than in the standard CDM model due to neutrino free-streaming, the epoch of structure formation
is one of the crucial factors that will determine the fate of the model. By overestimating \( f(\Omega, k) \), one underestimates the severity of the problem with late galaxy formation in CDM+HDM models. Klypin et al. (1993) set \( f = 1 \) for the CDM. However, this error was cancelled by an opposite effect (Primack, private communication): they used the baryon transfer function for the CDM. The baryonic perturbations are smaller than the CDM by up to 15%. As a result, the CDM particles were given less power, which counteracted their excessive velocities so that the two errors essentially cancelled.

In the simulations by Davis et al. (1992b), the initial HDM momenta were drawn randomly from the Fermi-Dirac distribution. In Klypin et al. (1993), this thermal velocity was added to the velocity arising from the Zel’dovich approximation for each HDM particle. At their starting \( z \sim 15 \), the thermal component was about a factor of 4 larger. Neither group included the actual correlations between neutrino positions and momenta that develop through the Boltzmann equation in the linear regime. To incorporate these first-order effects, we retained the full phase space information by sampling the phase space at \( z \sim 10^9 \) with 21 million neutrino particles and following their trajectories in the perturbed background spacetime until low redshifts.

To estimate the importance of these neutrino phase space perturbations, we mimicked the approach of Klypin et al. to generate an “equivalent” set of initial positions and momenta at \( z = 13.55 \) for \( 10 \times 128^3 \) neutrinos, using the same realization for \( \delta \) as for \( \phi \) and \( \psi \) in our geodesic integrations. A randomly-drawn thermal velocity was also added to the Zel’dovich velocity for each neutrino particle.

We calculated \( \delta_{\text{hdm}} \) from the particle positions and examined the correlation between the rms neutrino velocities and \( \delta_{\text{hdm}} \) in the two cases. If the neutrinos obeyed the zeroth-order Fermi-Dirac distribution, the neutrino velocities should be uncorrelated with \( \delta_{\text{hdm}} \); any correlation would indicate deviations from the Fermi-Dirac distribution. Our results are plotted in Fig. 6. We see that correlations in the velocities and the density perturbations have developed in the Boltzmann integration case between \( z \sim 10^9 \) and \( z = 13.55 \). The more clustered neutrinos appear to have higher rms velocities and therefore higher temperature, possibly resulting from the increase in the kinetic energy when the neutrinos fall into the CDM potential wells. This correlation is absent when the initial conditions are generated with the conventional Zel’dovich approach because the dominant thermal contribution to the neutrino velocities is drawn from the zeroth-order Fermi-Dirac distribution with constant temperature.

To test whether gravitational infall is responsible for the density-velocity correlation, we also computed for each neutrino particle the velocity components parallel and perpendicular to the gravitational acceleration \( \mathbf{g} = -\nabla \phi \) at the location of the neutrino: \( v_\parallel = \mathbf{v} \cdot \mathbf{g} \) and \( v_\perp = \sqrt{v^2 - v_\parallel^2} \). Then we calculated the conditional rms \( v_\parallel \) and \( v_\perp \) for a given \( \delta_{\text{hdm}} \). The results are shown in Fig. 7, where the solid curve represents \( \langle v_\parallel^2 \rangle^{1/2} \) and the dashed curve represents \( \langle v_\perp^2 \rangle^{1/2}/\sqrt{2} \). The component along the gravitational acceleration is larger than the orthogonal components by \( \sim 2\% \). Thus, the velocity-density correlation is not simply due to a uniform gravitational infall. This
is because the velocity dispersion (temperature) of the neutrinos and not just the bulk (fluid) velocity is higher in the denser regions.

For comparison, Fig. 8 shows the rms velocity versus the density for our CDM particles at redshift \( z = 13.55 \). The positions and the velocities were generated using the method described earlier. As one can see from Figs. 6 and 8, CDM is more clustered than HDM (the range of \( \delta \) is larger), and the rms velocities of the CDM particles are \( \sim 30 \) km s\(^{-1}\) compared to \( \sim 95 - 105 \) km s\(^{-1}\) for the HDM particles. The CDM velocities show no significant correlation with \( \delta_{\text{cdm}} \). No correlation is expected in linear theory.

Fig. 9 is a contour plot of neutrino density in the velocity component-\( \delta_{\text{hdm}} \) plane from the last output of the \( N_{\text{part}} = 10 \times 128^3 \) geodesic integrations. The rapid decline with \( v \) is due to the (approximately) Fermi-Dirac distribution (with rms \( v \sim 55 \) km s\(^{-1}\)) and the decline with \( |\delta| \) is due to the Gaussian distribution of the potential. However, the contours are asymmetric about \( \delta_{\text{hdm}} = 0 \), showing a positive correlation in the velocities and the density perturbations in the HDM component. One sees that the overdense regions contain hotter neutrinos than the underdense regions.

5. Conclusion and Work in Progress

Motivated by the CDM+HDM models, we have presented a general-relativistic \( N \)-body technique that provides an accurate sampling of the full neutrino phase space at all times when the linear perturbation theory is valid. Although the evolution of the neutrino phase space distribution can be solved from the Boltzmann equation, we know of no practical scheme for computing and sampling the final distribution except for the Monte Carlo method we have employed. In this method we first compute the metric perturbations about a Robertson-Walker spacetime by integrating the coupled, linearized Boltzmann, Einstein, and fluid equations for all particle species, including the massive neutrinos. Then we sample the massive neutrino phase space right after neutrino decoupling at \( z \sim 10^9 \) when the distribution is Fermi-Dirac to a very good approximation. We subsequently integrate the linearized geodesic equations for individual neutrinos to obtain their trajectories in the perturbed background spacetime described by the metric perturbations found in the previous calculation. This technique is valid only in the linear regime. It differs from the conventional \( N \)-body simulation method in that the gravitational forces are precomputed from the metric perturbations of the background spacetime using continuum linear theory rather than from the particles directly. The resulting neutrino positions and velocities can be used as the HDM initial conditions for subsequent \( N \)-body simulations of the nonlinear evolution of structures in the CDM+HDM models.

The same method could be used to generate initial conditions for the pure HDM model. Although these would differ from what previous workers have assumed, because of the very large
damping of small-scale fluctuations we are doubtful that there would be significant differences in one’s conclusions about the model.

We are currently performing a high resolution particle-particle particle-mesh (P\(^3\)M) \(N\)-body simulation of the nonlinear evolution of the density perturbations, using the positions and velocities from the last output of our large geodesic integration as the HDM initial conditions. The initial conditions for the CDM particles were generated from the CDM power spectrum with a modified form of the Zel’dovich approximation taking into account the wavenumber-dependence of the growth rate \(f = \frac{d \ln D_+}{d \ln a}\). A total of \(10 \times 128^3\) HDM and \(128^3\) CDM particles are used in a 100 Mpc comoving box starting at \(z = 13.55\). If computer time permits, we will also perform an “equivalent” simulation with the same parameters but with the Zel’dovich initial conditions adopted by other groups. Our simulation box will be large enough to include most of the important long-wavelength power absent in smaller boxes, and the \(P^3M\) force calculation will give us much higher resolution than particle-mesh (PM) simulations. In addition, we will be able to make a fair comparison of the two different treatments of the initial conditions and examine the importance of correlations in the neutrino phase space. Until that time it would be premature for us judge the merits of the approximate methods used by previous workers.

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Fig. 1.— Neutrinos in a 100 Mpc box at $z \sim 4.9 \times 10^5$ (timestep 351, $c\tau = 0.95$ Mpc) from one of the $N_{\text{part}} = 10 \times 32^3$ test runs. Projection of the box onto the $x - y$ plane is shown. Each ball is a horizon-radius shell of relativistic neutrinos expanding away from their original grid positions.
Fig. 2.— Neutrinos in a 100 Mpc box at $z = 13.55$ (timestep 701) from the $N_{\text{part}} = 10 \times 128^3$ run. Projection of the box onto the $x-y$ plane is shown. The grayscale represents the projected density.
Fig. 3.— The power per $\ln k$ (fill circles) computed from $10 \times 128^3$ neutrino particles at the end of the geodesic integration ($z = 13.55$). The CDM and HDM power spectra computed from the linear theory are shown (dotted curves) for comparison.
Fig. 4.— The dashed curves are the power per \(\ln k\) of CDM and HDM in a hybrid model with \(\Omega_{\text{cdm}} = 0.65\), \(\Omega_{\text{hdm}} = 0.3\) and \(\Omega_{\text{baryon}} = 0.05\) at redshift \(z = 13.55\). For comparison, the solid curve represents the CDM power in a CDM model with \(\Omega_{\text{cdm}} = 0.95\) and \(\Omega_{\text{baryon}} = 0.05\) at the same redshift.
Fig. 5.— The growth rate $f(\Omega, k) = d\ln D_+/d\ln a$ of CDM at $z = 13.55$ in the CDM+HDM model with $\Omega_{\text{cdm}} = 0.65$, $\Omega_{\text{hdm}} = 0.3$ and $\Omega_{\text{baryon}} = 0.05$. 
Fig. 6.— The rms neutrino velocities $\langle (d|\vec{x}|/d\ln a)^2 \rangle^{1/2}$ versus the density perturbation $\delta$ at $z = 13.55$. The solid line is computed from $10 \times 128^3$ neutrinos at the end of the geodesic equation integration. The dashed line represents an “equivalent” set of initial conditions generated from the Zel’dovich method. The density bins have size $\Delta \delta_{\text{hdm}} = 0.02$ and the rms velocities of the particles in each bin are shown.
Fig. 7.— The rms neutrino velocity components versus $\delta_{\text{bhm}}$. The solid curve represents $\left\langle v_\parallel^2 \right\rangle^{1/2}$, where $v_\parallel$ is the velocity component parallel to the gravitational acceleration; the dashed curve represents the perpendicular velocity component $\left\langle v_\perp^2 \right\rangle^{1/2}/\sqrt{2}$. 
Fig. 8.— The rms CDM velocities versus $\delta_{\text{cdm}}$. The velocities are entirely parallel to the direction of gravity.
Fig. 9.— Contour plot of constant particle number in the neutrino velocity component-$\delta_{\text{bhm}}$ plane. The absolute values of all three velocity components are shown. The five contours from bottom up correspond to $10^5$, $10^4$, 1000, 100 and 10 neutrinos per pixel. Each pixel has a width $\Delta \delta = 0.02$ and a height $\Delta (dx/d \ln a) = 10 \text{ km s}^{-1}$. 
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