We study in this report the so-called Strictly Subgaussian (SSub) random variables (r.v.), which form a very interest subclass of Subgaussian (Sub) r.v., and obtain the exact exponential bounds for tail of distribution for sums of independent and disjoint such a variables, not necessary to be identical distributed, and give some new examples of SSub variables to show the exactness of our estimates.

We extend also these results on the case of sums of subgaussian martingale differences, and show that the mixture of (Strictly) Subgaussian r.v. forms also (Strictly) subgaussian variable.

Key words and phrases: Random variables (r.v.), centering, indicator, binary and Bernoulli’s r.v., variance, martingales, mixture, Grand Lebesgue Spaces (GLS), subgaussian norm, subgaussian (Sub) and strictly subgaussian (SSub) r.v., tail or concentrations inequalities, independence.

1 Introduction. Definitions. Notations. Examples.

Let \( \{\Omega, B, P\} \) be some non-trivial probability space with expectation \( E \).

Definition 1.1.

We say that the centered: \( E\xi = 0 \) numerical random variable (r.v.) \( \xi = \xi(\omega), \omega \in \Omega \) is subgaussian, or equally, belongs to the space \( \text{Sub}(\Omega) \), if there exists some non-negative constant \( \tau \geq 0 \) such that

\[
\forall \lambda \in \mathbb{R} \Rightarrow E \exp(\lambda \xi) \leq \exp[\lambda^2 \tau^2/2]. \tag{1.1}
\]

The minimal value \( \tau \) satisfying (1.1) is called a subgaussian norm of the variable \( \xi \), write
\[ \|\xi\|_{\text{Sub}} = \inf\{\tau, \ \tau > 0 : \forall \lambda \in R \Rightarrow E\exp(\lambda\xi) \leq \exp(\lambda^2\tau^2/2)\} . \]

Evidently,
\[ \|\xi\|_{\text{Sub}} = \sup_{\lambda \neq 0} \left[ \sqrt{2\ln E\exp(\lambda\xi)} / |\lambda| \right] . \] (1.2)

This important notion was introduced by J.P.Kahane [11]; V.V.Buldygin and Yu.V.Kozachenko in [6] proved that the set Sub(\Omega) relative the norm \|\cdot\| is complete Banach space which is isomorphic to subspace consisting only from the centered variables of Orlicz’s space over (\Omega, B, P) with \( N - \text{Orlicz-Young function} \) \( N(u) = \exp(u^2) - 1 \); see also [14].

For instance, let us consider the centered indicator (binary) random variable \( \xi_p \):
\[ P(\xi_p = 1 - p) = p; \quad P(\xi_p = -p) = 1 - p, \quad p \in (0, 1); \]
then (in our definitions and notations)
\[ \|\xi_p\|_{\text{Sub}} = \sqrt{\frac{1 - 2p}{2\ln((1 - p)/p)}} . \]

This important result was obtained independently in the articles [12], [8]; see also [5], [20], [25].

The detail investigation of this class or random variables with very interest applications into the theory of random fields reader may found in the book [7]; we reproduce here some main facts from this monograph.

If \( \|\xi\|_{\text{Sub}} = \tau \in (0, \infty) \), then
\[ \max\{P(\xi > x), P(\xi < -x)\} \leq \exp(-x^2/(2\tau^2)), \quad x \geq 0; \] (1.3)
and the last inequality is in general case non-improvable. It is sufficient for this to consider the case when the r.v. \( \xi \) has the centered Gaussian non-degenerate distribution.

Conversely, if \( E\xi = 0 \) and if for some positive finite constant \( K \)
\[ \max\{P(\xi > x), P(\xi < -x)\} \leq \exp(-x^2/K^2), \quad x \geq 0, \]
then \( \xi \in \text{Sub}(\Omega) \) and \( \|\xi\|_{\text{Sub}} < 4K \).

The subgaussian norm in the subspace of the centered r.v. is equivalent to the following Grand Lebesgue Space (GLS) norm:
\[ \|\xi\|_{\text{GLS}} := \sup_{s \geq 1} \left[ \frac{|\xi|_s}{\sqrt{s}} \right], \quad |\xi|_s = \left[E|\xi|^s\right]^{1/s} . \]

For the non-centered r.v. \( \xi \) the subgaussian norm may be defined as follows:
\[ \|\xi\|_{\text{Sub}} := \left\{ \left( \|\xi - E\xi\|_{\text{Sub}} \right)^2 + (E\xi)^2 \right\}^{1/2} . \]

More detail investigation of these spaces see in the monograph [16], chapter 1.
Denote in the sequel for brevity for any r.v. \( \eta \)
\[ \sigma^2(\eta) = \sigma^2 = \text{Var} \eta = E\eta^2 - (E\eta)^2. \]

**Definition 1.2.** (See [7], chapter 1.)
The subgaussian r.v. \( \xi \) is said to be *Strictly Subgaussian (SSub)*, iff
\[ \forall \lambda \in \mathbb{R} \Rightarrow Ee^{\lambda \xi} \leq e^{\lambda^2 \sigma^2(\xi)/2}, \quad (1.4) \]
or equally
\[ ||\xi||_{\text{Sub}} \leq \sigma(\xi) = ||\xi||_{L^2(\Omega)}. \quad (1.4\text{a}) \]

Recall that always \( ||\xi||_{\text{Sub}} \geq \sigma(\xi) = ||\xi||_{L^2(\Omega)} \), so that
\[ \xi \in S\text{Sub}(\Omega) \iff E\xi = 0, \quad ||\xi||_{\text{Sub}} = \sigma(\xi) = ||\xi||_{L^2(\Omega)}. \quad (1.4\text{b}) \]

Many examples of strictly subgaussian distributions may be found in the book of V.V.Buldygin and Yu.V.Kozachenko [7], chapter 1. For instance, arbitrary mean zero Gaussian distributed r.v. is strictly subgaussian, including the case when this r.v. is equal to zero a.e.; the symmetric Rademacher’s r.v. \( \rho \) with distribution \( P(\rho = 1) = P(\rho = -1) = 1/2 \) belongs to the set \( S\text{Sub}(\Omega) \). The random variable \( \eta \) which has an uniform distribution on the symmetrical interval \((-b, b)\), \( b = \text{const} \in (0, \infty) \) is Strictly Subgaussian.

Consider also following the authors [7] the r.v. \( \zeta \) with the following density:
\[ f_\zeta(x) = \frac{\alpha + 1}{2\alpha} (1 - |x|^\alpha) I(|x| \leq 1), \quad \alpha = \text{const} \geq 0, \quad (1.5) \]
where \( I(A) = I(A, x) = 1, \ x \in A; \ I(A) = I(A, x) = 0, \ x \notin A \) is indicator function; then \( \zeta \in S\text{Sub}(\Omega) \).

This example is interesting because the kurtosis of the r.v. \( \zeta \) is zero if \( \alpha = \sqrt{10} - 3 \).

The convenience of these notions is following. Let \( \{\xi(i)\}, \ i = 1, 2, \ldots, n \) be (centered) independent subgaussian r.v. Denote
\[ S(n) = \sum_{i=1}^{n} \xi(i), \quad \Sigma^2(n) = \sum_{i=1}^{n} (||\xi(i)||_{\text{Sub}})^2. \quad (1.6) \]

Then \( ||S(n)||_{\text{Sub}} \leq \Sigma(n) \) and following
\[ \max(P(S(n)/\Sigma(n) > x), P(S(n)/\Sigma(n) < -x)) \leq e^{-x^2/2}, \ x \geq 0, \quad (1.7) \]
the tail or concentrations inequalities.

If in addition \( \xi(i) \) are identical distributed and \( \beta := ||\xi(1)||_{\text{Sub}} \in (0, \infty) \), then
\[ \sup_n ||S(n)/\sqrt{n}||_{\text{Sub}} = \beta \]
and
\[ \sup_n \max(P(S(n)/(\beta \sqrt{n}) > x), P(S(n)/(\beta \sqrt{n}) < -x)) \leq e^{-x^2/2}, \ x \geq 0, \quad (1.8) \]
If in addition the r.v. $\xi(i)$ are strictly subgaussian, the estimate (1.8) may be reinforced by lower estimate used the classical CLT:

$$\sup_n P(S(n)/(\beta\sqrt{n}) > x) \geq \lim_{n \to \infty} P(S(n)/(\beta\sqrt{n}) > x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-y^2/2} dy \geq C x^{-1} e^{-x^2/2}, \ x \geq 1.$$ (1.9)

This short report may be considered as a slight addition to the book of V.V.Buldygin and Yu.V.Kozachenko [7]; we give some new examples of Subgaussian and Strictly Subgaussian random variables, obtain the exponential exact bounds for tails of distribution for sums of independent Strictly Subgaussian random variables, extend this estimates on the sequence of martingale differences, investigate the mixture of ones distributions etc.

Applications of these notions in the non-parametrical statistics may be found in the articles [10], [13]. Another statistical applications is described in [9], [26]. The subgaussian r.v. appears also in the articles [23], [24] devoted to the non-linear Schrödinger’s equation. Some applications in the information and coding theory see in [18].

2 Mixture of subgaussian random variables

Let $Z = \{z\}$ be another set equipped some sigma-algebra and probability measure $\mu$. Let also $\xi_z, z \in Z$ be a family of random variables such that the function

$$z \to P(\xi_z \in A), \ A \in B$$

is $\mu$ – measurable. By definition, the random variable $\nu$, more precisely its distribution is called mixture of individual distributions $P(\xi_z \in A)$ relative the (weight) measure $\mu$, if

$$P(\nu \in A) = \int_Z P(\xi_z \in A) \mu(dz).$$ (2.1)

Another interpretation-conditional distribution.

If the equality (2.1) there holds, then for all non-negative measurable function $h : R \to R$

$$E h(\nu) = \int_Z E h(\xi_z) \mu(dz).$$ (2.2)

**Theorem 2.1.** Let the random variables $\xi_z$ be Strictly Subgaussian with at the same subgaussian norm

$$||\xi_z||_{\text{Sub}} = \sigma = \text{const} \in (0, \infty),$$ (2.3)

Then the random variable $\nu$ is also Strictly Subgaussian with at the same norm.
Proof. We derive using the identity (2.2) for the functions \( h(x) = x, x^2, e^{\lambda x} \) correspondingly

\[
E\nu = \int_Z E\xi_z \mu(dz) = 0, \quad E\nu^2 = \int_Z E\xi_z^2 \mu(dz) = \int_Z \sigma^2 \mu(dz) = \sigma^2,
\]

\[
E e^{\lambda\nu} = \int_Z E e^{\lambda\xi_z} \mu(dz) \leq \int_Z e^{\lambda^2\sigma^2/2} \mu(dz) = e^{\lambda^2\sigma^2/2},
\]

Q.E.D.

Where the conditions of Theorem 2.1 are not satisfied, the variable \( \nu \) may be as a Strictly Subgaussian or no. The correspondent (very spectacular) example see in the aforementioned monograph [7], p. 14:

\[
P(\nu = 1) = P(\nu = -1) = \frac{1 - \gamma}{2}, \quad P(\nu = 0) = \gamma, \quad \gamma = \text{const} \in [0, 1];
\]

then \( \nu \in \text{SSub}(\Omega) \) iff \( 0 \leq \gamma \leq 2/3 \) or \( \gamma = 1 \); otherwise \( \nu \notin \text{SSub}(\Omega) \).

Let now \( \xi_1 \) be a centered Gaussian distributed r.v. with variance \( \sigma_1^2 \) and \( \xi_2 \) be also mean zero Gaussian variable with other variance \( \sigma_2^2 \), wherein \( 0 < \sigma_1^2 < \sigma_2^2 < \infty \) and \( \mu(\{1\}) = \mu(\{2\}) = 1/2 \). It is easy to verify that the r.v. \( \nu \) is not strictly subgaussian despite both the r.v. \( \xi_1, \xi_2 \) are strictly subgaussian.

Remark 2.1. If the random variables \( \xi(z) \) in the theorem 2.1 are only subgaussian with variable but uniformly bounded norm

\[
\sigma_+ := \sup_z ||\xi_z|| \text{ Sub} = \sup_z \sigma(z) \in (0, \infty), \quad (2.4)
\]

then the r.v. \( \nu \) is subgaussian with subgaussian norm less than \( \sigma_+ \).

3 Disjoint subgaussian random variables

Two r.v. \( \eta_1, \eta_2 \) are called by definition disjoint, if \( \eta_1 \cdot \eta_2 = 0 \) almost everywhere. The family \( \eta_j, j = 1, 2, \ldots, n; \ n < \infty \) of r.v. is named disjoint, if it is pairwise disjoint.

Denote \( \beta(j) = ||\eta_j|| \text{ Sub}, \ S(n) = \sum_{j=1}^n \eta_j \) and suppose \( E\eta_j = 0 \).

We intend to obtain in this section the subgaussian and other exponential estimations for the sums of disjoint random variables.

The first \( L_p(\Omega) \) estimates for \( S(n) \) was obtained in the famous work of H.P.Rosenthal [19]; the modern results with very interest generalization see in the articles of S.V.Astashkin and F.S.Sukochev [1], [2].

Let us introduce the following function:

\[
G_n(y_1, y_2, \ldots, y_n) \overset{def}{=} \inf_{\mu > 0} \left[ \mu^{-1} \ln \left( \sum_{j=1}^n e^{\mu y_j} - (n - 1) \right) \right]^{1/2} \quad (3.1)
\]

Theorem 3.1. Let \( \eta_j, \ j = 1, 2, \ldots, n \) be centered disjoint random variables. Then
\[ ||S(n)||_{\text{Sub}} \leq G_n(\beta_1^2, \beta_2^2, \ldots, \beta_n^2). \]  

**Proof.** Given: \( \eta_j = \eta_j \cdot I(A(j)), A(j) \cap A(i) = \emptyset, i \neq j, A(j) \in B; \)

\[ \int_{\Omega} e^{\lambda \eta_j} P(d\omega) \leq e^{\lambda^2 \beta_j^2/2}, \]

therefore

\[ \int_{A(j)} e^{\lambda \eta_j} P(d\omega) \leq e^{\lambda^2 \beta_j^2/2} - (1 - P(A_j)). \]

We deduce:

\[ E e^{\lambda S(n)} = E e^{\lambda \sum \eta_j} = E e^{\lambda \sum \eta_j I(A(j))} = E \prod_j e^{\lambda \eta_j I(A(j))} = \]

\[ \sum_j \int_{A(j)} e^{\lambda \eta_j} P(d\omega) + (1 - \sum_j P(A(j))) \leq \]

\[ \sum_j \left[ e^{\lambda^2 \beta_j^2/2} - (1 - P(A(j))) \right] + (1 - \sum_j P(A(j))) = \]

\[ \sum_j e^{\lambda^2 \beta_j^2/2} - (n - 1) \leq e^{\lambda^2 G_n^2(\beta_1^2, \beta_2^2, \ldots, \beta_n^2)/2} \]

(3.3)

on the basis of definition of the function \( G_n(\cdot) \). Theorem 3.1 is proved.

**Remark 3.1.** It is interest to note that our estimation does not depend on the partition

\[ R = \{ A(j), j = 1, 2, \ldots, n; \Omega \setminus \bigcup_j A(j) \}. \]

**Example 3.1.** Let \( \nu : \Omega \to R \) be a centered stepwise (simple) r.v. (measurable function): 

\[ \nu = \sum_{j=1}^{m} c(j)[I(A(p(j))) - p(j)],\ m = \text{const} \leq \infty,\ c(j) = \text{const}, \]

and \( \{ A(p(j)) \} \) are pairwise disjoint events. We conclude using triangle inequality for the subgaussian norm and the completeness of the space Sub(\( \Omega \)) in the case when \( m = \infty \):

\[ ||\nu||_{\text{Sub}} \leq \sum_{j=1}^{m} |c(j)| Q(p(j)). \]

The application of theorem 3.1 gives more exact estimation.

**Another approach to the problem of exponential tail estimates for sums of disjoint random variables.**

We recall briefly first of all here for reader conventions some definitions and facts from the theory of GLS spaces.
Recently, see [27], [28], [29], [30], [31], [32], [14], [33], [16], [25] etc. appear the so-called Grand Lebesgue Spaces (GLS)

\[ \mathcal{G}(\psi) = \mathcal{G} = \mathcal{G}(\psi; B); \quad B = \text{const} \in (1, \infty) \]

spaces consisting on all the random variables (measurable functions) \( f : \Omega \to \mathbb{R} \) with finite norms

\[ \| f \|_{\mathcal{G}(\psi)} \overset{\text{def}}{=} \sup_{p \in (A; B)} \left[ \frac{|f|^p}{\psi(p)} \right], \quad (3.4) \]

\[ |f|^p \overset{\text{def}}{=} [\mathbb{E}|f|^p]^{1/p}, \quad 1 \leq p \leq \infty. \]

Here \( \psi = \psi(p), \quad p \in [1, B) \) is some continuous positive on the open interval \((1; B)\) function such that

\[ \inf_{p \in (A; B)} \psi(p) > 0. \quad (3.5) \]

We will denote

\[ \text{supp}(\psi) \overset{\text{def}}{=} [1; B) \]

or by abuse of notations \( \text{supp}(\psi) = B \).

The set of all such a functions with the support \( \text{supp}(\psi) = (1; B) \) will be denoted by \( \Psi(1; B) = \Psi(B) \).

This spaces are rearrangement invariant; and are used, for example, in the theory of Probability, theory of Partial Differential Equations, Functional Analysis, theory of Fourier series, Martingales, Mathematical Statistics, theory of Approximation etc.

Notice that the classical Lebesgue – Riesz spaces \( L_p \) are extremal case of Grand Lebesgue Spaces, see [25].

Let a function \( \xi : \Omega \to \mathbb{R} \) be such that

\[ \exists B > 1 \Rightarrow \forall p \in [1, B) \quad |\xi|^p < \infty. \]

Then the function \( \psi = \psi_\xi(p) \) may be naturally defined by the following way:

\[ \psi_\xi(p) := |\xi|^p; \quad p \in [1, B). \quad (3.6) \]

The finiteness of the \( G_\psi \) – norm for some r.v. \( \xi \) allows to obtain the exact exponential tail inequalities for the distribution \( \xi \); for instance,

\[ \sup_{p \geq 1} \left[ \frac{|\xi|^p}{p^m} \right] < \infty \Leftrightarrow \exists C > 0, \forall x \geq 0 \Rightarrow P(|\xi| > x) \leq e^{-C x^{1/m}}, \quad m = \text{const} > 0, \quad (3.7) \]

see [14], [16], chapter 1, section 3.

Let us return to the the problem of exponential estimations for sums of disjoint variables.

**Proposition 3.1.** Let \( \xi, \eta \) be two disjoint r.v. belonging to some space \( G_\psi, \psi \in G_\Psi \) with \( \text{supp} \psi = B \in (1, \infty) \). Then

\[ \text{Proposition 3.1.} \]
\[ \| \xi + \eta \| G_\psi \leq \left[ (\| \xi \| G_\psi)^B + (\| \eta \| G_\psi)^B \right]^{1/B}, \]  
where at \( B = \infty \)

\[ \| \xi + \eta \| G_\psi \leq \max[\| \xi \| G_\psi, \| \eta \| G_\psi]. \]  

(3.8a)

**Proof.** Denote for brevity \( a = \| \xi \| G_\psi, b = \| \eta \| G_\psi. \) Then

\[ |\xi|_p \leq a \psi(p), \ |\eta|_p \leq b \psi(p); \ |\xi|^p_p \leq a^p \psi^p(p), \ |\eta|^p_p \leq b^p \psi^p(p). \]

Since the r.v. \( \xi, \eta \) are disjoint,

\[ |\xi + \eta|^p_p = |\xi|^p_p + |\eta|^p_p = (a^p + b^p) \cdot \psi^p(p). \]

Therefore,

\[ |\xi + \eta|_p \leq \psi(p) \cdot (a^p + b^p)^{1/p} \leq \psi(p) \cdot (a^B + b^B)^{1/B}. \]

It remains to divide on the \( \psi(p) \) and take maximum over \( p; 1 \leq p < B. \)

The generalization on the sum of \( n \) disjoint variables is clear.

### 4 Martingale case

Let \( F \) be non-trivial sigma subalgebra of the source sigma-algebra \( B. \) The r.v. \( \eta \) is said to be **conditional subgaussian**, if there is a **non-random** non-negative constant \( \tau \) for which

\[ \forall \lambda \in R \Rightarrow E e^{\lambda \eta}/F \leq e^{\lambda^2 \tau^2/2}. \]

(4.1)

The minimal value of the constant \( \tau \) from the inequality (4.1) is called **conditional subgaussian norm** of the r.v. \( \eta \) relative the sigma-algebra \( F, \) write

\[ \| \eta \| \text{Sub}(F) := \inf\{ \tau, \ \tau > 0 : \forall \lambda \in R \Rightarrow E e^{\lambda \eta}/F \leq e^{\lambda^2 \tau^2/2} \}. \]

(4.2)

The set of all r.v. with \( \| \eta \| \text{Sub}(F) < \infty \) relative this norm and ordinary algebraic operations forms by definition the complete Banach space \( \text{Sub}(\Omega, F). \)

For example, arbitrary centered bounded r.v. \( \eta = \eta(\omega), \ \omega \in \Omega \) belongs to any space \( \text{Sub}(\Omega, F). \)

Obviously, \( \| \eta \| \text{Sub}(\Omega) \leq \| \eta \| \text{Sub}(\Omega, F). \)

If in the equality the value \( \tau \) may be selected such that

\[ \tau = \| \eta \| \text{Sub}(F) = \sqrt{\text{Var}(\eta)/F} = \sqrt{E \eta^2}/F, \]

then as before the r.v. \( \eta \) may be named **Strictly conditional subgaussian**: \( \eta \in \text{SSub}(\Omega, F). \)

Recall that sequence \( (X(i), F(i)), \ i = 0, 1, 2, \ldots, n \) where the \( X(i) \) are random variables and the \( F(i) \) are sigma-algebras, is a martingale if the following conditions
are satisfied:

1. The sequence of sigma-algebras \( \{F(i)\} \) forms a filtration, i.e. \( F(0) \subset F(1) \subset F(2) \ldots \subset F(n) \); usually, \( F(0) \) is the trivial sigma-algebra \((\emptyset, \Omega)\) and \( F(n) \) is sigma-subalgebra of source sigma-algebra \( B \).

2. \( X(i) \in L_1(\Omega, P) \) and

\[
X_{i-1} \overset{a.e.} = E X(i)/F(i-1), \; i = 1, 2, \ldots, n - 1.
\]

We suppose in the sequel \( E X(i) = 0 \) and introduce the correspondent sequence of martingale-differences \( \xi(i) \) as follows:

\[
\xi(0) = 0, \; \xi(i) = X(i+1) - X(i), \; i = 1, 2, \ldots, n - 1.
\]

Denote

\[
\theta(j) = ||\xi(j)||_{\text{Sub}(\Omega,F(j-1))}, \; j = 1, 2, \ldots, n, \; \Delta(n) = \sqrt{\sum_{j=1}^{n} \theta^2(j)}, \; (4.3)
\]

if there exists.

**Theorem 4.1.**

\[
||X(n)||_{\text{Sub}(\Omega)} \leq \Delta(n); \quad (4.4)
\]

\[
\max(P(X(n)/\Delta(n) > x), P(X(n)/\Delta(n) < -x) \leq e^{-x^2/2}, \; x \geq 0. \quad (4.4a)
\]

**Proof** is alike to the one in the article of K.Azuma [3]; see also [18]. Namely, let \( \lambda = \text{const} \in R \); then

\[
E e^{\lambda X(n)} = E e^{\lambda \sum_{j=1}^{n-1} \xi(j)} = E \left[ E \left[ e^{\lambda \sum_{j=1}^{n-1} \xi(j)} \right] / F(n-2) \right] =
\]

\[
E \left[ e^{\lambda \sum_{j=1}^{n-2} \xi(j)} \cdot E e^{\lambda \xi(n-1) / F(n-2)} \right] \leq
\]

\[
E \left[ e^{\lambda \sum_{j=1}^{n-2} \xi(j)} \cdot e^{\lambda^2 \theta^2(n-1)/2} \right] \leq \ldots \leq e^{\lambda^2 \sum_{j=1}^{n} \theta^2(j)/2} = e^{\lambda^2 \Delta^2(n)/2}.
\]

Thus,

\[
||X(n)||_{\text{Sub}(\Omega)} \leq \Delta(n). \quad (4.6)
\]

This completes the proof of theorem 4.1.

Note that this estimate is exponential exact if for instance \( \{\xi(j)\} \) are independent and strictly subgaussian.
5 Another examples of subgaussian random variables

A. Symmetrized beta distribution.

Let us consider a symmetrical r.v. \( \xi = \xi_{\alpha, \beta} \) with the density

\[
f(x) = f_{\alpha, \beta}(x) = 0.5 \frac{|x|^\alpha - 1 (1 - |x|) ^{\beta - 1}}{B(\alpha, \beta)} I(|x| < 1), \quad \alpha, \beta = \text{const} > 0, \tag{5.1}
\]

where as usually \( B(\alpha, \beta) \) denotes the beta-function.

**Theorem 5.1.** If

\[
B(\alpha, \beta) \leq 1, \tag{5.2}
\]

then the r.v. \( \xi = \xi_{\alpha, \beta} \) is strictly subgaussian.

Note that the condition (5.2) of theorem 5.1 is satisfied if for instance \( \alpha \geq 1, \beta \geq 1 \).

**Proof.** We have:

\[
\mathbb{E}_\xi x^{2k+1} = 0, \quad \mathbb{E}_\xi x^{2k} = \frac{B(2k + \alpha, \beta)}{B(\alpha, \beta)}, \quad k = 0, 1, \ldots,
\]

so that

\[
\sigma^2 := \text{Var}(\xi) = \frac{B(2 + \alpha, \beta)}{B(\alpha, \beta)} = \frac{\alpha(\alpha + 1)}{(\alpha + \beta)(\alpha + 1)};
\]

\[
\mathbb{E} e^{\lambda \xi} = \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!} \cdot \mathbb{E}_\xi x^{2k} = \sum_{k=0}^{\infty} \frac{B(2k + \alpha, \beta)}{B(\alpha, \beta)} \cdot \frac{\lambda^{2k}}{(2k)!};
\]

\[
e^{\lambda^2 \sigma^2 / 2} = \sum_{k=0}^{\infty} \frac{\lambda^{2k} \sigma^{2k}}{2^k k!}.
\]

It is sufficient to prove that

\[
\frac{B(2k + \alpha, \beta)}{B(\alpha, \beta)} \cdot \frac{1}{(2k)!} \leq \frac{\sigma^{2k}}{2^k k!}, \quad k = 1, 2, \ldots; \tag{5.3}
\]

see [7], p.8; the case \( k = 0 \) is trivial.

The inequality (5.3) is equivalent to the following:

\[
\frac{B(2k + \alpha, \beta)}{B(\alpha, \beta)} \cdot \frac{1}{(2k)!} \leq \frac{B^{2k}(\alpha + 2, \beta)}{B^{2k}(\alpha, \beta) \cdot 2^k \cdot k!}; \tag{5.4}
\]

We denote

\[
\theta(k) = \left[ \frac{B(2k + \alpha, \beta)}{B(\alpha, \beta)} \cdot \frac{1}{(2k)!} \right] \cdot \left[ \frac{B^{2k}(\alpha + 2, \beta)}{B^{2k}(\alpha, \beta) \cdot 2^k \cdot k!} \right], \quad k = 0, 1, \ldots; \tag{5.5}
\]
then \( \theta(0) = 1 \). Further, the expression for the fraction \( \theta(k+1)/\theta(k) \), \( k = 0, 1, 2, \ldots \) has a form:

\[
\frac{\theta(k+1)}{\theta(k)} = \frac{\Gamma^2(\alpha)\Gamma^2(\beta)}{\Gamma^2(\alpha + \beta)} \cdot \frac{k+1}{k(2k+1)} \times \frac{(2k+\alpha)(2k+\alpha+1)}{(2k+\alpha+\beta+1)(2k+\alpha+\beta)} < B^2(\alpha, \beta) < 1;
\]

therefore \( \theta(k) \leq 1, k = 0, 1, 2, \ldots \), Q.E.D.

**B. Symmetrized gamma distribution.**

We consider here the r.v. \( \gamma = \gamma_{\alpha,\beta} \) with the (symmetrical) density of distribution

\[
g_{\alpha,\beta}(x) = 0.5 \beta |x|^\alpha e^{-|x|^\beta} \big/ \Gamma((\alpha+1)/\beta), \ x \in \mathbb{R}, \ \alpha = \text{const} > -1, \ \beta = \text{const} > 0. \quad (5.6)
\]

We have: \( \mathbb{E}\gamma^{2k+1} = 0, \ k = 0, 1, 2, \ldots \);

\[
\mathbb{E}\gamma^{2k} = \frac{\Gamma((\alpha+2k+1)/\beta)}{\Gamma((\alpha+1)/\beta)}. \quad (5.7)
\]

In particular,

\[
\text{Var}(\gamma) = \frac{\Gamma((\alpha+3)/\beta)}{\Gamma((\alpha+1)/\beta)}. \quad (5.7a)
\]

We suppose in the sequel \( \beta > 2 \); obviously, when \( \beta < 2 \), the r.v. \( \gamma \) is even not subgaussian.

The r.v. \( \gamma \) is subgaussian iff \( \beta \geq 2 \).

In order to formulate the next result, we need to introduce some preliminary notations.

\[
\theta = \max(\alpha - \beta + 1, 1), \quad k_0 = \max(1, (\beta - \alpha - 1)/2),
\]

\[
G_1 = \sqrt{\pi} \cdot \Gamma \left( \frac{\alpha+1}{\beta} \right) \cdot e^{-3/8} \cdot e^{-\theta/(2\beta)},
\]

\[
G_2 = 0.25 e \cdot \frac{\Gamma((\alpha+3)/\beta)}{\Gamma((\alpha+1)/\beta)} \cdot \left[ 2\beta^{-1}(1 + \theta/(2k_0) \right]^{2/\beta},
\]

\[
G = [(\max(1, G_1)) \cdot G_2]^{\beta/(\beta-2)},
\]

\[
\zeta(k) = \zeta_{\alpha,\beta}(k) := \Gamma((\alpha+2k+1)/\beta) \cdot \frac{2^k k!}{(2k)!} \cdot \frac{\Gamma^{k-1}(\alpha+1)/\beta}{\Gamma((\alpha+3)/\beta)}.
\]

Recall that we consider the values \( \beta \) greatest than 2.

**Theorem 5.2.** If the following conditions
\( \forall k < G \Rightarrow \zeta_{\alpha,\beta}(k) \leq 1 \) \hspace{1cm} (5.9)

there holds, then the r.v. \( \gamma = \gamma_{\alpha,\beta} \) is strictly subgaussian.

**Proof.**

It is sufficient to prove as before the following inequality for all the values \( k = 0, 1, 2, \ldots \)

\[
\zeta(k) = \zeta_{\alpha,\beta}(k) := \Gamma((\alpha + 2k + 1)/\beta) \cdot \frac{2^k k! \cdot \Gamma^{k-1}((\alpha + 1)/\beta)}{(2k)! \cdot \Gamma((\alpha + 3)/\beta)} \leq 1. \hspace{1cm} (5.10)
\]

Since \( \beta > 2 \), \( \lim_{k \to \infty} \zeta(k) = 0 \). We conclude omitting complicated computation more exact estimate using Stirling’s formula:

\[ k \geq G \Rightarrow \zeta_{\alpha,\beta}(k) \leq 1. \]

We derive taking into account the condition (5.9) that (5.10) is true for all the integer values \( k = 1, 2, \ldots, \), Q.E.D.

**Remark 5.1.** Example. Let \( \alpha = \beta \to \infty \); then condition (5.9) is satisfied; moreover,

\[ \lim_{\beta \to \infty} \sup_{k \geq 1} \zeta_{\beta,\beta}(k) = 0. \]

Therefore, there exists a value \( Z \) such that for all the values \( \alpha = \beta > Z \) the r.v. \( \gamma_{\beta,\beta} \) is strictly subgaussian.

**Remark 5.2.** Recall that the condition \( \gamma(1) \leq 3 \) or in detail the inequality

\[ \Gamma((\alpha + 5)/\beta)\Gamma((\alpha + 1)/\beta) \leq 3\Gamma^2((\alpha + 3)/\beta) \hspace{1cm} (5.11) \]

is necessary for strictly subgaussianess. Denote \( \epsilon = 1/\beta \) and assume \( \alpha = \text{const}, \epsilon \to 0+ \Leftrightarrow \beta \to \infty \). Taking into account the behavior of the Gamma function near to the value \( 0+ \):

\[ \Gamma(\epsilon) \sim 1/\epsilon, \epsilon \to 0+, \]

we deduce from (5.11) as \( \beta \to \infty \)

\[ \frac{1}{\alpha + 5} \cdot \frac{1}{\alpha + 1} \leq \frac{3}{(\alpha + 3)^2}, \]

which is not true as \( \alpha \to -1 + 0 \).

Moreover, the solution of the last inequality subject to the limitation \( \alpha > -1 \) has a form \( \alpha \geq 3(\sqrt{3} - 1) \approx 2.19 \).

Thus, for all the values \( \alpha \) from the interval \( -1 < \alpha < 3(\sqrt{3} - 1) \) there exists a positive the value \( \beta_0 \) such that for all the values \( \beta > \beta_0 \) the random value \( \gamma = \gamma_{\alpha,\beta} \) is subgaussian but not strictly subgaussian.

Another (but more simple) example of strictly subgaussian r.v. with unbounded support may be constructed by means of section 2. Namely, let the r.v. \( \xi_1 \) has an
uniform distribution on the symmetric interval \([-b, b]\), \(b = \text{const} > 0\) and let \(\xi_2\) has a normal (Gaussian) mean zero distribution with at the same variation \(\text{Var} \xi_2 = b^2/3\). The arbitrary non-trivial mixture of these distributions has an unbounded support, is not Gaussian and is strictly subgaussian by theorem 2.1.

6 Acknowledgement.

Authors are very grateful to prof. S.V.Astashkin and L.Maligranda for sending Your remarkable articles and comments.

References

[1] Astashkin S.V. and Sukochev F.A. Comparison of sums of independent and disjoint functions in symmetric spaces, Math. Notes 76:34 (2004), 449 - 454.

[2] Astashkin S.V. and Sukochev F.A. Independent functions and the geometry of Banach spaces. Russian Math. Surveys, 65:6, 1003 - 1081, 2010.

[3] Azuma K. Weighted sums of certain dependent random variables. Tohoku Mathematical Journal, 19, 357 - 367, 1967.

[4] Bentkus V. On Hoeffdings inequalities. The Annals of Probability 32(2), 1650 – 1673, (2004).

[5] Berend D. and Kontorovich A. On the concentration of the missing mass. Electron. Commun. Probab., 18(3):17, 2013.

[6] Buldygin V.V., Kozachenko Yu.V. About subgaussian random variables. Ukrainian Math. Journal, 1980, 32, No 6, 723 - 730.

[7] Buldygin V.V., Kozachenko Yu.V. Metric Characterization of Random Variables and Random Processes. 1998, Translations of Mathematics Monograph, AMS, v.188.

[8] Buldygin V.V., Moskvichova K.K. The sub-Gaussian norm of a binary random variable. Theor. Probability and Math. Statist., Kiev, KSU, 2012, 86, p. 33-49.

[9] S. X. Chen and J. S. Liu. Statistical applications of the Poisson-binomial and conditional Bernoulli distributions. Statist. Sinica, 7(4):875892, 1997.

[10] Gaivoronsky E.I., Ostrovsky E.I. Non-asymptotical estimate of deviation of multidimensional function of distribution. Theory Probab. Applications, 1991, 36, Issue 3, 111 - 115.
[11] **Kahane J.P.** *Properties locales des fonctions a series de Fourier aleatoires.* Studia Math. (1960), **19**, No 1, 1-25.

[12] **Kearns M. and Saul L.** *Large deviation methods for approximate probabilistic inference.* In Proceedings of the Fourteenth conference on Uncertainty in artificial intelligence, pages 311 – 319. Morgan Kaufmann Publishers Inc., 1998.

[13] **Kiefer J.** *On large Deviations of the Empiric D.F. of vector chance variables and a Law of Iterated Logarithm.* Pacific J.Math., 1961, **11**, N° 2, 649 - 660.

[14] **Kozachenko Yu.V., Ostrovsky E.I.** *Banach spaces of random variables of subgaussian type.* Theory Probab. And Math. Stat., Kiev, (1985), p. 42 – 56 (in Russian).

[15] **Maligranda L.** *The K-functional for symmetric spaces.* Lect. Notes in Math. 1984. V. 1070, 169182. Proc. Conf. ”Interpolation spaces and Allied Topics in Analysis”, Lund, Aug. 29Sept. 1, 1983.

[16] **Ostrovsky E.I.** *Exponential Estimations for Random Fields.* Moscow - Obninsk, OINPE, (1999), (in Russian).

[17] **Pinelis Iosif.** *Exact inequalities for sums of asymmetric random variables, with applications.* arXiv:math/0602556v2 [math.PR] 24 May 2006

[18] **Raginsky M. and Sason I.** *Concentration of Measure Inequalities in Information Theory, Communications, and Coding.* Foundations and Trends in Communications and Information Theory, vol. 10, no. 1 – 2, pp. 1246, 2013.

[19] **Rosenthal H.P.** *On the subspaces of $L_p$, $(p > 2)$ spanned by sequences of independent random variables.* Israel J. Math. 8:3 (1970), 273 - 303.

[20] **Schlemm E.** *The Kearns-Saul inequality for Bernoulli and Poisson - binomial distributions.* arXiv:1405.4496v1 [math.PR] 18 May 2014

[21] **Serov A.A., Zubkov A.M.** *A full proof of universal inequalities for the distribution function of the binomial law.* arXiv:1207.3838v1 [math.PR] 16 Jul 2012

[22] **Zubkov A.M., Serov A.A.** *Bounds for the number of Boolean functions admitting affine approximations of a given accuracy.* Discrete Math. Appl., 2010, **20**, N° 5 - 6, p. 467 - 486.

[23] **B’enyi R., Oh T., Pocovnicu O.** *On the probabilistic Cauchy theory of the cubic nonlinear Schrödinger equation on $R^d$, $d \geq 3$.* arXiv:1405.7327v1 [math.AP] 28 May 2014

[24] **B’enyi R., Oh T., Pocovnicu O.** *Wiener randomization on domain and an application to almost sure well-posedness of NLS.* arXiv:1405.7326v1 [math.AP] 28 May 2014
[25] Ostrovsky E., Sirota L. Exact value for subgaussian norm of centered indicator random variable. arXiv:1405.6749v1 [math.PR] 26 May 2014

[26] Ryiabinin A.A. Probabilities of large deviations in some scheme of summing dependent random variables. Proceedings of scientific seminars of LOMI, Leningrad, Nauka, 1989, V. 11 p. 138-144.

[27] Fiorenza A. Duality and reflexivity in grand Lebesgue spaces. Collect. Math. 51, (2000), 131-148.

[28] Fiorenza A. and Karadzhov G.E. Grand and small Lebesgue spaces and their analogs. Consiglio Nationale Delle Ricerche, Instituto per le Applicazioni del Calcolo Mauro Picone”, Sezione di Napoli, Rapporto tecnico 272/03, (2005).

[29] Iwaniec T. and Sbordone C. On the integrability of the Jacobian under minimal hypotheses. Arch. Rat.Mech. Anal., 119(1992), 129-143.

[30] Iwaniec T, Koskela P. and Onninen J. Mapping of Finite Distortion: Monotonicity and Continuity. Invent. Math. 144(2001), 507-531.

[31] Jawerth B. and Milman M. Extrapolation theory with applications. Mem. Amer. Math. Soc. 440(1991).

[32] Karadzhov G.E. and Milman M. Extrapolation theory: new results and applications. J. Approx. Theory, 113(2005), 38-99.

[33] Liflyand E., Ostrovsky E., Sirota L. Structural Properties of Bilateral Grand Lebesgue Spaces. Turk. J. Math.; 34 (2010), 207-219.