Supersymmetry Breakings and
Fermat’s Last Theorem

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Abstract. A mechanism of supersymmetry breaking in two or four-dimensions is
given, in which the breaking is related to the Fermat’s last theorem. It is shown
that supersymmetry is exact at some irrational number points in parameter space,
while it is broken at all rational number points except for the origin. Accordingly,
supersymmetry is exact almost everywhere, as well as broken almost everywhere on
the real axis in the parameter space at the same time. This is the first explicit
mechanism of supersymmetry breaking with an arbitrarily small change of parameters
around any exact supersymmetric model, which is possibly useful for realistically
small non-perturbative supersymmetry breakings in superstring model building. As
a byproduct, we also give a convenient superpotential for supersymmetry breaking
only for irrational number parameters. Our superpotential can be added as a “hidden”
sector to other useful supersymmetric models.

MSC Numbers: 11D41, 81Q60, 81T30, 81T60

1This work is supported in part by NSF grant # PHY-91-19746.
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1. Introduction

Supersymmetric theory in two-dimensions \((D = 2)\) has interesting features related to superconformal field theory and superstring theory, because their phase structures can be described well by Landau-Ginzburg and Calabi-Yau hypersurface models \([1]\), which are easy to analyze. It is known that the change of physical aspects of the theory including the topological changes of the target space-time occurs, when the parameters in these models are varied. If we also need to understand the target space-time supersymmetry breaking in superstring theory for realistic phenomenology, it is imperative to comprehend the associated \(N = 2\) supersymmetry breaking on the world-sheet \([2]\). In this connection, the study of \(N = 2\) world-sheet supersymmetry breaking in these models must be the first step for our ultimate goal of realistic model building.

Entirely independent of this development related to superstring theory in physics, there has been recently some excitement in mathematics about the possible complete proof \([3]\) for what is called “Fermat’s last theorem” (FLT) \([4]\). This theorem dictates that there exist no integral solutions \(l, m, n \in \mathbb{Z} - \{0\}\) for the algebraic equation

\[
l^p + m^p = n^p, \quad p \in \{3, 4, 5, \cdots\}.
\]

Even though there seem to be small gaps in the recent proof \([3]\), its validity has been widely accepted nowadays. (We do not address ourselves to the question of the FLT itself, but we take its validity for granted. This principle about mathematical strictness is common to other formulations in physics such as path-integrals, or renormalizations, \textit{etc.}.)

In this paper, we give the first application of this FLT \([4]\) to physical models, in particular to an \(N = (2, 2)\) supersymmetric models in \(D = 2\) or \(N = 1\) supersymmetric ones in \(D = 4\). We use the model in ref. \([1]\) for \(D = 2, N = (2, 2)\) supersymmetric chiral supermultiplets, and show that the supersymmetry is broken spontaneously for some values of the parameters involved in the model. In particular, we confirm the interesting and peculiar fact that the breaking occurs at points found in any arbitrarily small neighbourhood of each exactly supersymmetric point in the parameter space. We also give as a by-product a superpotential that gives supersymmetry breakings for all irrational values of parameters, while it is exact for all rational values of parameters.
2. Example of Supersymmetry Breaking by FLT

Our mechanism will be useful not only for models in $D = 2$, but also for realistic superunifications in $D = 4$. However, for a later purpose of generalizing the superpotential into a non-polynomial one, we temporarily stick to $D = 2$, following the notation in ref. [1]. We also note that because of the parallel structure between the $D = 2$, $N = (2, 2)$ and $D = 4$, $N = 1$ supersymmetries, it will be straightforward to switch to the latter notation.

Consider a $D = 2$, $N = (2, 2)$ supersymmetric system [1] with seven chiral superfields $\Phi_i$ ($i = 1, 2, \cdots, 7$), and specify the total action as

$$I \equiv I_K + I_W \ ,$$

$$I_K \equiv \sum_i \int d^2x d^4\theta \overline{\Phi}_i \Phi_i$$

$$= \sum_i \int d^2x \left[ -(\partial_\mu \overline{A}_i)(\partial^\mu A_i) + i\overline{\chi}_{-i} \partial_+ \chi_{-i} + i\overline{\chi}_{+i} \partial_- \chi_{+i} + |F_i|^2 \right] \ ,$$

$$I_W \equiv -\int d^2x d\theta^+ d\theta^- W(\Phi_i)\bigg|_{\theta^+=0} = \text{h.c.}$$

$$= -\int d^2x \left[ \sum_i F_i \frac{\partial W}{\partial \Phi_i} + \sum_{i,j} \frac{\partial^2 W}{\partial \Phi_i \Phi_j} \chi_{-i} \chi_{+j} \right] \bigg|_{\theta^+=0} = \text{h.c.} \ .$$

We are following the notation in ref. [1] except for the names of component fields in the $D = 4$ style, and $\partial_\pm \equiv \partial_0 + \partial_1$, $\partial_\pm \equiv \partial_0 - \partial_1$. We specify superpotential $W$ as

$$M^{-1}W(\Phi_i) = \Phi_4 \left[ \Phi_1^p + \Phi_2^p - (t\Phi_3)^p \right] + \Phi_5 \frac{\sin(\pi\Phi_1)}{\pi\Phi_1} + \Phi_6 \frac{\sin(\pi\Phi_2)}{\pi\Phi_2} + \Phi_7 \frac{\sin(\pi\Phi_3)}{\pi\Phi_3} \ ,$$

where $M \neq 0$ is a real number supplying a mass dimension, and

$$t \in \mathbb{R} - \{0\} \ .$$

is a parameter. (There are many other ways of putting such parameters. For example, we can have $r$, $s$ such that the first term in (2.4) is $\Phi_4 [(r\Phi_1)^p + (s\Phi_2)^p - (t\Phi_3)^p]$.) Note that our superpotential $W(\Phi_i)$ is regular even at $\Phi_i = 0$, due to the property of the function $(\sin x)/x$. The corresponding bosonic potential $V$ in component is obtained as usual by eliminating the $F$-auxiliary field:
\[ M^{-2}V = \sum_i \left| \frac{\partial W}{\partial \Phi_i} \right|^2 \bigg|_{\theta^\pm = \bar{\theta}^\pm = 0} \]

\[ = \left| A_1^p + A_2^p - (tA_3)^p \right|^2 + \left| \frac{\sin(\pi A_1)}{\pi A_1} \right|^2 + \left| \frac{\sin(\pi A_2)}{\pi A_2} \right|^2 + \left| \frac{\sin(\pi A_3)}{\pi A_3} \right|^2 \]

\[ + \left| pA_4A_1^{p-1} + A_5 \frac{\cos(\pi A_1)}{A_1} - A_5 \frac{\sin(\pi A_1)}{\pi A_1^2} \right|^2 \]

\[ + \left| pA_4A_2^{p-1} + A_6 \frac{\cos(\pi A_2)}{A_2} - A_6 \frac{\sin(\pi A_2)}{\pi A_2^2} \right|^2 \]

\[ + \left| ptA_4(tA_3)^{p-1} + A_7 \frac{\cos(\pi A_3)}{A_3} - A_7 \frac{\sin(\pi A_3)}{\pi A_3^2} \right|^2 , \quad (2.6) \]

We now try to minimize the potential (2.6) to see the supersymmetry for the vacuum. First of all we can easily see that the last six terms can vanish by the vacuum expectation values (v.e.v.s)

\[ A_1 = l \, , \quad A_2 = m \, , \quad A_3 = n \, ; \quad l, m, n \in \mathbb{Z} - \{0\} \, , \quad (2.7) \]

\[ A_4 = 0 \, , \quad A_5 = 0 \, , \quad A_6 = 0 \, , \quad A_7 = 0 . \quad (2.8) \]

Now the only remaining term is

\[ \left| A_1^p + A_2^p - (tA_3)^p \right|^2 \quad (2.9) \]

If we also use the v.e.v.s (2.7) here, our question is whether or not we can satisfy the following equation:

\[ \left| l^p + m^p - (tn)^p \right|^2 = 0 . \quad (2.10) \]

It is not difficult at all to solve this for \( t \) as

\[ t = \frac{1}{n} \left( (p + m^p)^{1/p} \right) = \frac{l}{n} \left[ 1 + \left( \frac{m}{l} \right)^p \right]^{1/p} \equiv t(l, m, n) . \quad (2.11) \]

As long as the real number \( t \) is chosen such that (2.11) is satisfied, the model has a supersymmetric vacuum. However, a simple consideration of the FLT with (2.10) reveals that there is some more meaning in this equation, which turns out to be exciting.

To analyze the algebraic meaning of (2.10), we develop a useful corollary of the FLT. We can easily prove that the FLT also implies that the algebraic equation

\[ x^p + y^p = (tz)^p \, , \quad p \in \{3, 4, 5, \cdots\} \, , \quad t \in \mathbb{Q} - \{0\} \quad (2.12) \]
has no integral number solutions $x, y, z \in \mathbb{Z} - \{0\}$. This can be easily proved by inserting hypothetical rational number $t = a/b$ ($a, b \in \mathbb{Z} - \{0\}$) into (2.12), and multiply both sides by $b^p$. The result is obviously incompatible with the FLT.

By the use of this corollary, it is now obvious that if $t \in \mathbb{Q} - \{0\}$, there are no solutions for $A_1, A_2, A_3$, that can put (2.10) to zero. In other words, the vacuum of the system breaks supersymmetry for any choice of $t \in \mathbb{Q} - \{0\}$.

On the other hand, we know that supersymmetric vacuum is realized for $t = t(l, m, n)$ in (2.11). (From now on, $t(l, m, n)$ always denotes (2.11), avoiding the messy expression.) Notice the important point here that since $l, m, n$ are all arbitrary non-zero integers, $t = t(l, m, n)$ can be made arbitrarily near to any rational number. As a matter of fact, we can prove this rather easily, as follows. Let $u \equiv L/N > 0$, $(L, N \in \mathbb{Z} - \{0\})$ be a positive arbitrary rational number. (The case of $u < 0$ can be also proven in a similar way.) Then for any arbitrarily small positive real number $\epsilon > 0$, we can show the existence of $t(l, m, n)$ such that

$$u < t(l, m, n) = \frac{l}{n} \left[1 + \left(\frac{m}{l}\right)^p\right]^{1/p} < u + \epsilon$$

for an appropriate choice of $(l, m, n) \in (\mathbb{Z} - \{0\})^3$. We can first choose $l = KL$, $n = KN$ for some large positive integer $K$. We next choose $m$ and the appropriately large enough integer $K$ satisfying

$$0 < \left(\frac{m}{KL}\right)^p \frac{L}{pN} < \epsilon$$

for any small $\epsilon > 0$. Because this implies that

$$1 < 1 + \left(\frac{m}{KL}\right)^p < 1 + \frac{p}{u} \epsilon < \left(1 + \frac{\epsilon}{u}\right)^p = \frac{1}{u^p} (u + \epsilon)^p,$$

yielding (2.13).

Let us now introduce two sets for the parameter $t$, depending on the supersymmetry of the vacuum: The set of “supersymmetric parameters”

$$S \equiv \left\{t \mid t = n^{-1}(l^p + m^p)^{1/p} \in \mathbb{R} - \{0\}, \forall(l, m, n) \in (\mathbb{Z} - \{0\})^3\right\},$$

and the set of “non-supersymmetric parameters” $B \equiv \mathbb{R} - \{0\} - S$. Obviously $S \cap B = \emptyset$, and $S \cup B = \mathbb{R} - \{0\}$. Additionally, $B \supset \mathbb{Q} - \{0\}$, according to the FLT.

Remarkably, all the points in the set $S$ are countable and isolated, but yet they are covering almost everywhere on the real number field, as we have seen above. Moreover, for any neighbourhood of an arbitrary supersymmetric model for $t \in S$, there exists infinitely many broken supersymmetric models for $t' \in B$, and vice versa! This means that we can break any supersymmetric model by arbitrarily small magnitude, by choosing appropriate
The peculiar feature of the dependence of the “rationality” of the parameter $t$ is rather unexpected by the general wisdom of supersymmetry breaking based on the Witten’s index $\text{Tr}(-1)^F$ [5]. Because usually any small “continuous” change of parameters in the system, such as from irrational numbers to rational numbers, does not trigger any supersymmetry breaking [5]. This has been also the general principle for the renormalization effects, where the quantum corrections will preserve the classical supersymmetry. For our peculiar models, supersymmetry is broken rather “frequently” in the parameter space, each time the parameters deviate from a point in $S$ to a point in $B$. This apparent discrepancy from the usual wisdom seems to be attributed to the following aspects in our models. First of all, the Witten’s index is ill-defined for our models due to the presence of a massless superfield, as well as the infinite degeneracy [5], as will be seen in the next section. Second, we expect that the index might have some implicit dependence on the “rationality” of the parameters. These aspects enabled the models to escape from the topological constraints of supersymmetry breaking, which usually forbids such a peculiar fashion as the dependence on “rationality” of parameters. Additionally, the degeneracy of the vacua prevents us from performing analysis for renormalization group flows similar to that in ref. [6].

3. Mass Spectrum around Supersymmetric Vacuum

To understand our model better, we next study the mass spectrum of the system, when there exist supersymmetric vacuum solutions. To this end, we require $t \in S$ in this section, satisfying (2.11) or equivalently

$$l^p + m^p = (tn)^p \quad p \in \{3, 4, 5, \cdots \} .$$  (3.1)

As is easily seen, there can be infinitely many other solutions for the v.e.v.s of $A_1$, $A_2$, $A_3$, once there exists one set of solutions $(l, m, n)$, because if we re-scale it as $l' = ql$, $m' = qm$, $n' =qn$, $\forall q \in \mathbb{Z} - \{0\}$, the new set $(l', m', n')$ also satisfies (2.10). To put it differently, we can first choose one arbitrary set $l, m, n \in \mathbb{Z} - \{0\}$, while keeping the parameters of the model to be the same value as the original value: $t = t(l, m, n) = t(l', m', n')$. Thus this model has infinitely many supersymmetric vacua at $A_1 = ql$, $A_2 = qm$, $A_3 = qn$, $\forall q \in \mathbb{Z} - \{0\}$. (There may be even other solutions than these, which we do not care about so much here.)

The mass spectrum of the superpotential (2.4) around the supersymmetric v.e.v.s $\Phi_1 = l$, $\Phi_2 = m$, $\Phi_3 = n$ under the condition (3.1) can be easily analyzed by appropriate field
redefinitions. We first expand each superfield around their v.e.v.s, as

\[ \Phi_1 = l + \varphi_1 \, , \, \Phi_2 = m + \varphi_2 \, , \, \Phi_3 = n + \varphi_3 \, , \]
\[ \Phi_4 = \varphi_4 \, , \, \Phi_5 = \varphi_5 \, , \, \Phi_6 = \varphi_6 \, , \, \Phi_7 = \varphi_7 \ . \]  

(3.2)

We can directly use the superfields, because we are considering here a supersymmetric case with no v.e.v.s for any $F$-components of them. The superfields $\varphi_i$ denote the fluctuations around their v.e.v.s. Here we rely on (2.8) with no v.e.v.s for $A_4$, · · ·, $A_7$, and the stability of these solutions will be confirmed later as the absence of tachyons. Using also the expansion of the function $(\sin x)/x$, we easily get the quadratic part of $W$:

\[ W^{(2)} = 2\varphi_1 (a\varphi_4 + d\varphi_5) + 2\varphi_2 (b\varphi_4 + e\varphi_6) + 2\varphi_1 (c\varphi_4 + f\varphi_7) \ , \]

(3.3)

where

\[ a \equiv \frac{1}{2}l^{p-1} \, , \, b \equiv \frac{1}{2}m^{p-1} \, , \, c \equiv \frac{1}{2}p(tn)^{p-1} \, , \]
\[ d \equiv \frac{(-1)^l}{2l} \, , \, e \equiv \frac{(-1)^m}{2m} \, , \, f \equiv \frac{(-1)^n}{2n} \ . \]  

(3.4)

All the tadpole terms linear in $\varphi_i$ have disappeared in $W$ because of (3.1). After the superfield redefinitions

\[ \varphi_1 \equiv \frac{1}{2} (\bar{\varphi}_1 + \bar{\varphi}_5) \, , \, \varphi_5 \equiv \frac{1}{2} d^{-1} (\bar{\varphi}_1 - \bar{\varphi}_5 - 2a\varphi_4) \ , \]
\[ \varphi_2 \equiv \frac{1}{2} (\bar{\varphi}_2 + \bar{\varphi}_6) \, , \, \varphi_6 \equiv \frac{1}{2} e^{-1} (\bar{\varphi}_2 - \bar{\varphi}_6 - 2b\varphi_4) \ , \]
\[ \varphi_3 \equiv \frac{1}{2} (\bar{\varphi}_3 + \bar{\varphi}_7) \, , \, \varphi_7 \equiv \frac{1}{2} f^{-1} (\bar{\varphi}_3 - \bar{\varphi}_7 - 2c\varphi_4) \ , \]

(3.5)

we get

\[ W^{(2)} = \frac{1}{2} M \left( \bar{\varphi}_1^2 + \bar{\varphi}_2^2 + \bar{\varphi}_3^2 - \bar{\varphi}_5^2 - \bar{\varphi}_6^2 - \bar{\varphi}_7^2 \right) \ , \]

(3.6)

up to some appropriate but non-essential normalization for each field. Since (3.6) is for superpotential mass terms, the signature of each term does not matter, and the absence of tachyons or tadpoles is also guaranteed. The point here is that the $\varphi_4$-superfield has a zero mass, and all the other superfields are massive. This is also consistent with our initial assumption about the absence of other v.e.v.s for $\varphi_4, \cdots, \varphi_7$. The presence of massless chiral field $\varphi_4$ causes practical difficulty when computing the Witten’s index $\text{Tr}(-1)^F$ [5]. The trouble is also caused by the enormous degeneracy related to the rescalings of $(l, m, n)$.

The degeneracy with respect to supersymmetric vacua can be also seen by searching for the valleys of our potential (2.6), including also higher order terms in addition to the mass terms above. For example, as long as the conditions

\[ A_5 = (-1)^{l-1} p^l A_4 \, , \, A_6 = (-1)^{m-1} p^m A_4 \, , \, A_7 = (-1)^{n-1} p(tn)^p A_4 \ , \]

(3.7)
are satisfied, the $A_4$-field can be arbitrarily large, keeping the potential (2.6) to be zero. In other words, there exists such a valley in the potential, and the v.e.v.s for the $A_4$-field is indefinite at the classical level.

Notice that nothing is too particular about (canonical or path-integral) quantization around the supersymmetric vacuum. This is because our action (2.1) has ordinary kinetic and mass terms, but all the peculiar effect came from higher order terms in the function $(\sin x)/x$ rather “non-perturbatively”.

It is usually believed that the essential features of $D=2$ supersymmetric systems, such as for relevant or marginal operators [6], are determined by the lowest-order terms like the mass terms or the cubic interactions, but our superpotential $W$ does not obey this tendency. In this sense, we regard the effects by the higher-order terms as “non-perturbative” ones, because those infinitely higher-order terms can not be reached by summing up any finite number of terms.

4. Other Examples of Superpotentials

The mechanism proposed in this paper provides us with other interesting by-products than the FLT itself, such as superpotentials that have supersymmetric vacua only for rational v.e.v.s. of some bosonic field.

Take for example, a superpotential $W_Q$ defined by

$$W_Q(\Phi, \tilde{\Phi}_1, \tilde{\Phi}_2, \tilde{\Phi}_3) \equiv \tilde{\Phi}_2 \frac{\sin(\pi \tilde{\Phi}_1)}{\pi \tilde{\Phi}_1} + \tilde{\Phi}_3 \frac{\sin(\pi \tilde{\Phi}_1)}{\pi \tilde{\Phi}_1} .$$

The corresponding bosonic potential is

$$V_Q = + \left| \frac{\sin(\pi A\tilde{A}_1)}{\pi A\tilde{A}_1} \right|^2 + \left| \frac{\sin(\pi \tilde{A}_1)}{\pi \tilde{A}_1} \right|^2 + \left| A_2 \frac{\sin(\pi A\tilde{A}_1) - \pi A\tilde{A}_1 \cos(\pi A\tilde{A}_1)}{\pi A^2 \tilde{A}_1} \right|^2$$

$$+ \left| A_2 \frac{\sin(\pi A\tilde{A}_1) - \pi A\tilde{A}_1 \cos(\pi A\tilde{A}_1)}{\pi A^2 \tilde{A}_1} + A_3 \frac{\sin(\pi \tilde{A}_1) - \pi \tilde{A}_1 \cos(\pi \tilde{A}_1)}{\pi \tilde{A}_1^2} \right|^2 .$$

The last two terms in (4.2) can be made zero by the v.e.v.s

$$\tilde{A}_2 = \tilde{A}_3 = 0 ,$$

while the first two terms will vanish only at

$$\tilde{A}_1 \equiv n , \quad A\tilde{A}_1 \equiv l , \quad (n, l \in \mathbb{Z} - \{0\}) .$$
This implies that

\[ A = \frac{l}{n} \in \mathbb{Q} - \{0\} , \quad A_1 \in \mathbb{Z} - \{0\} , \quad (4.5) \]

in particular, the v.e.v.s of \( A \) must always be a non-zero rational numbers.

We have thus seen that the superpotential \( W_Q \) is generating rational number v.e.v.s for the \( A \)-component of the chiral superfield \( \Phi \). The other three \textit{tilded} superfields are playing the role of auxiliary superfields, and \( V_Q \) can be minimized at \( \tilde{A}_1 = \tilde{A}_2 = \tilde{A}_3 = 0 \).

As for the Witten’s index of this model, it is ill-defined due to the “valley” structure of the potential \( V_Q \) along the direction of an arbitrary large value of \( A_3 \) as seen from the last term in (4.2). Thus we can not see the conservation of topology depending on the value of the v.e.v.s.

An interesting application of \( W_Q \) is the following superpotential:

\[ W_1 = \Phi_3(\Phi_1 - r) + \Phi_4(\Phi_2 - s) + W_Q(\Phi_1, \tilde{\Phi}_5, \tilde{\Phi}_6, \tilde{\Phi}_7) + W_Q(\Phi_2, \tilde{\Phi}_8, \tilde{\Phi}_9, \tilde{\Phi}_{10}) , \quad (4.6) \]

yielding the bosonic potential

\[ V_1 \equiv + |A_1 - r|^2 + |A_2 - s|^2 + |f(\pi \tilde{A}_5)|^2 + |f(\pi \tilde{A}_8)|^2 + |\tilde{A}_6 g(\pi A_1 \tilde{A}_5) + \tilde{A}_7 g(\pi \tilde{A}_5)|^2 + |\tilde{A}_9 g(\pi A_2 \tilde{A}_8) + \tilde{A}_7 g(\pi \tilde{A}_8)|^2 \]
\[ + |\tilde{A}_6 g(\pi A_1 \tilde{A}_5) + A_3|^2 + |\tilde{A}_9 g(\pi A_2 \tilde{A}_9) + A_4|^2 , \quad (4.7) \]

where

\[ f(x) \equiv \frac{\sin x}{x} , \quad g(x) \equiv f'(x) \equiv \frac{x \cos x - \sin x}{x^2} . \quad (4.8) \]

As before, the 2nd. and 3rd. lines in (4.7) vanish at the v.e.v.s

\[ A_3 = A_4 = \tilde{A}_6 = \tilde{A}_7 = \tilde{A}_9 = \tilde{A}_{10} = 0 , \quad (4.9) \]

while the 3rd. and 4th. term vanish, when

\[ A_1 \tilde{A}_5 \equiv m , \quad A_2 \tilde{A}_8 \equiv n , \quad \tilde{A}_5 \equiv k , \quad \tilde{A}_8 \equiv l , \quad (k,l,m,n \in \mathbb{Z} - \{0\}) , \quad (4.10) \]

or equivalently,

\[ A_1 , A_2 \in \mathbb{Q} - \{0\} , \quad (4.11) \]

\[ \tilde{A}_5 , \tilde{A}_8 \in \mathbb{Z} - \{0\} . \]

Considering these with the remaining first two terms in (4.7) easily reveals that

(i) \( \forall (r,s) \in (\mathbb{Q} - \{0\})^2 \equiv S \implies V_{\min} = 0 \): supersymmetric vacuum
\( \forall (r, s) \in \mathbb{R}^2 - S \equiv B \implies V_{\min} > 0: \) non-supersymmetric vacuum

The interesting feature here is that depending on the “rationality” of the parameters \((r, s)\), the system has either supersymmetric or non-supersymmetric vacua. Needless to say, we could have chosen only \(r\) as a one-dimensional parameter, or as many as we wish like \((r_1, r_2, \cdots, r_n)\) by adding \(n\) copies of \(W_Q\).

5. Concluding Remarks

In this paper we have presented an explicit mechanism characterized by the superpotential (2.4), in which supersymmetry breakings occur with arbitrarily small changes of parameters around isolated exact supersymmetric models, depending on the “rationality” of the parameter \(t\). On the real axis in the parameter space of \(t\), supersymmetry is found to be exact almost everywhere, as well as is spontaneously broken almost everywhere, at the same time. We believe the validity of our result, as long as the FLT [3,4] is acceptable.

At first sight, there appeared to be an incompatibility of this result with the general wisdom about the non-zero Witten’s index [5] of a supersymmetric model. We understand that this is attributed to the ill-defined Witten’s index of our model due to the massless superfield, and also to its possible dependence on the “rationality” of the parameter \(t\) in some implicit way.

One of the interesting aspects of our model is the possibility of arbitrarily small supersymmetry breaking. This is because the breaking scale can be made as small as we wish, due to the “arbitrarily” small breaking effect on the bosonic potential by shifting the parameters from the exact supersymmetric values \(t = t(l, m, n), \forall (l, m, n) \in (\mathbb{Z} - \{0\})^3\) to an arbitrarily close rational numbers \(t' \in \mathbb{Q} - \{0\}\) for \(W\).

As is well-known, there is non-renormalization theorem applied to the \(F\)-type superpotential terms. Considering this theorem, we can conclude that the peculiar structure of our superpotential will be maintained even at the quantum level, if we have started with an exactly supersymmetric classical vacuum. This is the case only when the classical vacuum preserves supersymmetry, because the non-renormalization theorem applies only to the system with tree-level supersymmetry. It is therefore interesting to see what kind of quantum corrections will be generated, when the starting classical vacuum is non-supersymmetric such as the case \(t \in \mathbb{Q} - \{0\}\). It is also amusing that the supersymmetry is protecting a supersymmetric model against any quantum perturbations, that might shift the parameters away from the original set \(S\) to “next” infinitesimally close irrational numbers in \(B\), with such “infinite” accuracy. Furthermore, it is especially in supersymmetric models in which the positivity of the potential plays an important role, because of supersymmetry breakings related
to the non-zero vacuum energy. In ordinary models in physics, the “rationality” of constants and/or fields does not matter unlike our model, in which supersymmetry distinguishes them. From these viewpoints, together with the renormalizability for the $D = 2$ case, we believe that our models are not just of “accidental” interest, but they signal more fundamentally significant connection between supersymmetric field theories in $D = 2$ and the FLT in number theory.

We also mention the most important practical application of our model. Our superpotential (2.4) can be treated as a “hidden” sector added to other useful $D = 2$, $N = (2, 2)$ supersymmetric models, in order to break supersymmetry with small magnitude. This is because the presence of our superfields will not interfere with the fundamental structure of other sectors, such as the mass spectrum or manifold structures, except for the supersymmetry breaking at a global minimum. We are sure that there can be more to be done for interesting applications of our models combined with other useful models. Another interesting application is the $D = 4$ locally supersymmetric unifications [7], in which the renormalizability of the superpotential is no longer crucial, once supergravity is included. The usage of our superpotential as a “hidden” sector may well have some advantage over the conventional Polony-type superpotential [7], due to the possibility of small supersymmetry breaking of $\mathcal{O}(10^{-15})$ needed for realistic model building which keeps the zero-ness of the cosmological constant.

To our knowledge, our models have provided the mechanism which presents a peculiar link between the FLT in number theory [3,4] and the vacuum structure of supersymmetry in such an explicit way for the first time. The only well-known connection between number theory and supersymmetry has been via topological effects, such as instantons and monopoles in supersymmetric models. (However, see ref. [8] in which string coupling constant is parametrized by rational numbers.) Traditionally, supersymmetry has been always supposed to act on general real (or complex) fields rather continuously without distinguishing rational number parameters from irrational ones. We believe that our models have opened a new direction to the studies of such an important issue as supersymmetry breaking for the purpose of realistic model building as well as for purely mathematical or theoretical interest.

We are indebted to W.W. Adams, S.J. Gates, Jr., T. Hübsch, and J. Swank for valuable discussions.
References

1. E. Witten, Nucl. Phys. **B403** (1993) 159.

2. See e.g., M. Green, J.H. Schwarz and E. Witten, *Superstring Theory*, Vols. I and II, Cambridge University Press (1987).

3. Andrew Wiles, to be published.

4. P. de Fermat, ‘*Observatio*’ in “Arithmetica of Diophantus” (1621), unpublished (1637);
   K.A. Ribet, Notices of Amer. Math. Soc. **40** (1993) 575;
   D.A. Cox, Amer. Math. Monthly **101** (1994) 3.

5. E. Witten, Nucl. Phys. **B202** (1982) 253.

6. C. Vafa and N.P. Warner, Phys. Lett. **218B** (1989) 51.

7. See, e.g., H.P. Nilles, Phys. Rep. **110C** (1984) 1.

8. H. Nishino, Mod. Phys. Lett. **A7** (1992) 1805;
   H. Nishino and S.J. Gates, Int. Jour. Mod. Phys. **8** (1993) 3371.