AN ALGORITHM TO COMPUTE THE HILBERT DEPTH

ADRIAN POPESCU

Abstract. We give an algorithm which computes the Hilbert depth of a graded module based on a theorem of Uliczka. Partially answering a question of Herzog, we see that the Hilbert depth of a direct sum of modules can be strictly greater than the Hilbert depth of all the summands.

Key words: depth, Hilbert depth, Stanley depth.
2010 Mathematics Subject Classification: Primary 13C15, Secondary 13F20, 13F55, 13P10.

Introduction

Let $K$ be a field and $R = K[x_1, \ldots, x_n]$ be the polynomial algebra over $K$ in $n$ variables. On $R$ consider the following two grading structures: the $\mathbb{Z}$–grading in which each $x_i$ has degree 1 and the multigraded structure, i.e. the $\mathbb{Z}^n$–grading in which each $x_i$ has degree the $i$–th vector $e_i$ of the canonical basis.

After Bruns-Krattenthaler-Uliczka [4] (see also [11]), a Hilbert decomposition of a $\mathbb{Z}$–graded $R$–module $M$ is a finite family

$$\mathcal{H} = (R_i, s_i)_{i \in I}$$

in which $s_i \in \mathbb{Z}$ and $R_i$ is a $\mathbb{Z}$–graded $K$–algebra retract of $R$ for each $i \in I$ such that

$$M \cong \bigoplus_{i \in I} R_i(-s_i)$$

as a graded $K$–vector space.

The Hilbert depth of $\mathcal{H}$ denoted by $\text{hdepth}_1 \mathcal{H}$ is the depth of the $R$–module $\bigoplus_{i \in I} R_i(-s_i)$. The Hilbert depth of $M$ is defined as

$$\text{hdepth}_1(M) = \max\{\text{hdepth}_1 \mathcal{H} \mid \mathcal{H} \text{ is a Hilbert decomposition of } M\}.$$ 

We set $\text{hdepth}_1(0) = \infty$.

Theorem 0.1. (Uliczka [13]) $\text{hdepth}_1(M) = \max\{e \mid (1 - t)^e HP_M(t) \text{ is positive}\}$, where $HP_M(t)$ is the Hilbert–Poincaré series of $M$ and a Laurent series in $\mathbb{Z}[[t, t^{-1}]]$ is called positive if it has only nonnegative coefficients.

The support from the Department of Mathematics of the University of Kaiserslautern is gratefully acknowledged.
If $M$ is a multigraded $\mathbb{Z}^n$-module, then one can define $hdepth_n(M)$ as above by considering the $\mathbb{Z}^n$-grading instead of the standard one. There exists an algorithm for computing the $hdepth_n$ of a finitely generated multigraded module $M$ over the standard multigraded polynomial ring $K[x_1, \ldots, x_n]$ in Ichim and Moyano-Fernández’s paper [8] (see also [9]).

The main purpose of this paper is to provide an algorithm to compute $hdepth_1(M)$, where $M$ is a graded $R$-module (see Algorithm 1.3). This is part of the author’s Master Thesis [10].

A Stanley decomposition (see [12]) of a $\mathbb{Z}$-graded (resp. $\mathbb{Z}^n$-graded) $R$-module $M$ is a finite family

$$D = (R_i, u_i)_{i \in I}$$

in which $u_i$ are homogeneous elements of $M$ and $R_i$ is a graded (resp. $\mathbb{Z}^n$-graded) $K$-algebra retract of $R$ for each $i \in I$ such that $R_i \cap \text{Ann}(u_i) = 0$ and

$$M = \bigoplus_{i \in I} R_i u_i$$

as a graded $K$-vector space.

The Stanley depth of $D$ denoted by $\text{sdepth} D$ is the depth of the $R$-module $\bigoplus_{i \in I} R_i u_i$. The Stanley depth of $M$ is defined as

$$\text{sdepth}(M) = \max \{ \text{sdepth} D \mid D \text{ is a Stanley decomposition of } M \}.$$ 

We set $\text{sdepth}(0) = \infty$.

We talk about $\text{sdepth}_1(M)$ and $\text{sdepth}_n(M)$ if we consider the $\mathbb{Z}$-grading respectively the $\mathbb{Z}^n$-grading of $M$. The Hilbert depth of $M$ is greater than the Stanley depth of $M$ and can be strictly greater (an example can be found in [1]).

Herzog posed the following question (see also [11, Problem 1.67]): is $\text{sdepth}_n(R \oplus m) = \text{sdepth}_n(m)$, where $m$ is the maximal ideal in $R$? Since we implemented an algorithm to compute $hdepth_1$, we have tested whether $hdepth_1(R \oplus m) = hdepth_1(m)$ and as a consequence when $\text{sdepth}_n(R \oplus m) = \text{sdepth}_n(m)$. Proposition 2.6 says that Herzog’s question holds for $n \in \{1, \ldots, 5, 7, 9, 11\}$, but Remark 2.4 says that for $n = 6$ it holds $hdepth_1(R \oplus m) > hdepth_1 m$, which is a sign that in this case $\text{sdepth}_n(R \oplus m) > \text{sdepth}_n m$ and so Herzog’s question could have a negative answer for $n = 6$. This is indeed the case as it was shown later by Ichim and Zarojanu in [9]. Meanwhile Bruns et. al. [5] found another algorithm computing $hdepth_1$ and Chen [6] gave another one in the frame of ideals.

We owe thanks to Ichim who suggested us this problem and to Uliczka who found a mistake in a previous version of our algorithm.
1. hdepth Computation

In this section we introduce an algorithm which computes \(hdepth_1\) (Algorithm 1.3) and prove its correctness (Theorem 1.4). In the next section we provide some examples and some results related to [1, Problem 1.67].

Remark 1.1. The algorithm presented in this section is based on Theorem 0.1 and at a first glance it might look trivial. The difficulty lies in the fact that it is not clear how many coefficients of the infinite Laurent series have to be checked for positivity. This paper provides a bound up to which it suffices to check.

Recall first [3, Corollary 4.1.8] the definition of the Hilbert–Poincaré series of a module \(M\)

\[
HP_M(t) = \frac{Q(t)}{(1-t)^d} = \frac{G(t)}{(1-t)^d},
\]

where \(d = \dim M\) and \(Q(t), G(t) \in \mathbb{Z}[t],\ G(1) \neq 0\). In fact, note that \(G(1)\) is equal to the multiplicity of the module which is known to be positive.

The algorithm which we construct requires the module \(M\) as the input. Actually we only need the \(G(t)\) from (1) and the dimension of \(M\).

Definition 1.2. Let \(p(t) = \sum_{i=0}^{\infty} a_i \cdot t^i \in \mathbb{Z}[[t]]\) be a formal power series. By \(\text{jet}_j(p)\)

we understand the polynomial \(\text{jet}_j(p) = \sum_{i=0}^{j} a_i \cdot t^i\).

Algorithm 1.3. We now present the algorithm that computes the \(hdepth_1\) of a \(\mathbb{Z}\)-graded module \(M\). The algorithm uses the following procedures which can easily be constructed in any computer algebra system:

- \(\text{inverse}(\text{poly } p, \text{ int bound})\): computes the inverse of a power series \(p\) till the degree \(\text{bound}\),
- \(\text{hilbconstruct}(\text{module } M)\): computes the second Hilbert series of the module \(M\) - a way to do this in SINGULAR is to use the already built-in function \(\text{hilb}(\text{module } M, 2)\) which returns the list of coefficients of the second Hilbert series and construct the series,
- \(\text{positive}(\text{poly } f)\): returns 1 if \(f\) has all the coefficients nonnegative and 0 else,
- \(\text{sumcoef}(\text{poly } f)\): returns the sum of the coefficients of \(f\),
- \(\text{jet}(\text{poly } p, \text{ int } j)\): returns the \(\text{jet}_j p\). This procedure is already implemented in SINGULAR,
- \(\text{dim}(\text{module } M)\): returns the dimension of \(M\). This procedure is already implemented in SINGULAR.

Below we give the algorithm \(\text{hdepth}(\text{poly } g, \text{ int } \text{dim}_M)\). Hence in order to compute \(hdepth_1 M\), one considers \(g(t) = \text{hilbconstruct}( M )\) and \(\text{dim}_M = \text{dim}(M)\).
Algorithm hdepth\(_1\) (poly\(g\), int\(dim_M\))

Input:
- a polynomial \(g(t) \in \mathbb{Z}[t]\) (equal to \(\text{HP}_M(t)\))
- an integer \(dim_M = \text{dim } M\)

Output:
- \(hdepth_1 M\)

1: if \(\text{positive}(g) = 1\) then
2: \(\text{return } dim_M;\)
3: end if
4: poly \(f = g;\)
5: int \(c, d, \beta;\)
6: \(\beta = \text{deg}(g);\)
7: for \(d = dim_M\) to \(d = 0\) do
8: \(d = d - 1;\)
9: \(f = \text{jet}( g \cdot \text{inverse}((1 - t)^{dim_M - d}, \beta ));\)
10: if \(\text{positive}(f) = 1\) then
11: \(\text{return } d;\)
12: end if
13: \(c = \text{sumcoef}(f);\)
14: if \(c < 0\) then
15: while \(c < 0\) do
16: \(\beta = \beta + 1;\)
17: \(f = \text{jet}( g \cdot \text{inverse}((1 - t)^{dim_M - d}, \beta ));\)
18: \(c = \text{sumcoef}(f);\)
19: end while
20: end if
21: end for

Theorem 1.4. Given a \(\mathbb{Z}\)-graded module \(M\), Algorithm \([3]\) correctly computes

\[
\max \{ n \mid (1 - t)^n \cdot \text{HP}_M(t) \text{ is positive } \}
\]

where \(\text{HP}_M(t) = \frac{G(t)}{(1 - t)^{\text{dim } M}}\) is the Hilbert-Poincaré series of \(M\). Hence, by Theorem 0.1, the algorithm computes the Hilbert depth of a module \(M\) for \(g = G(t)\) and \(dim_M = \text{dim } M\).

Proof. Note that \(G(1)\) is the multiplicity of the module \(M\) and hence \(G(1) > 0\).

Assume that \(M \neq 0\). Denote the bound \(\beta\) at the end of the loop where \(d = i\) by \(\beta_i\). In order to prove this theorem one has to show the following two claims:

- the maximum from (2) does not exceed \(\text{dim } M\),
- after the bound \(\beta_i\) degree, the coefficients are nonnegative.
For the first part consider \( G(t) = \sum_{\mu=0}^{g} a_{\mu} \cdot t^{\mu} \). Note that

\[
(1-t)^{\dim M + 1} \cdot HP_M(t) = (1-t) \cdot G(t) = a_0 + (a_1 - a_0) \cdot t + \ldots + (a_g - a_{g-1}) \cdot t^g - a_g \cdot t^{g+1}.
\]

If all coefficients would be nonnegative, we would obtain

\[
0 \geq a_g \geq a_{g-1} \geq \ldots \geq a_2 \geq a_1 \geq a_0 \geq 0
\]

which implies that \( G(t) = 0 \). This will lead to a contradiction with \( M \neq 0 \). The same holds for \( (1-t)^{\dim M + \alpha} \cdot HP_M(t) \) by considering \( (1-t)^{\dim M + \alpha - 1} \cdot HP_M(t) \) instead of \( G(t) \), where \( \alpha \geq 0 \). Thus the maximum from (2) is smaller or equal than \( \dim M \).

Note that if \( G(t) \) already has all the coefficients nonnegative, then the algorithm stops by returning \( \dim M \), and the result is correct since in this case \( hdepth_1 M = \dim M \).

For the second part we need to show that at each step \( i \) the coefficient of the term of order \( \beta_i \) in \( \frac{G(t)}{(1-t)^{\dim M - i}} \) is nonnegative and the coefficients of the terms of higher order are increasing (and hence nonnegative). Apply induction on \( i \). For the first step, \( d = \dim M - 1 \), \( f = \frac{G(t)}{(1-t)} \) and all the coefficients of the terms of order \( \geq \beta_{\dim M - 1} = \deg G(t) \) are equal to the sum of the coefficients \( G(1) > 0 \).

For the general step \( i \), assume that at the beginning of loop \( d = i \), we started with

\[
\frac{G(t)}{(1-t)^{\dim M - i}} = \sum_{\mu=0}^{\infty} b_{\mu} \cdot t^{\mu}
\]

which satisfied all the desired properties by induction: the bound \( \beta_i \) was increased (if required), such that the coefficient sum \( c_i := \sum_{\mu=0}^{\beta_i} b_{\mu} > 0 \) and all coefficients of higher order terms are nonnegative, i.e. \( b_{\mu} \geq 0 \) for \( \mu \geq \beta_{i-1} \).

We now consider the next step, \( d = i - 1 \), and compute the new \( f \) as in line 9 of the algorithm. In order to check that the coefficients of the terms of order higher than the bound \( \beta_i \) are nonnegative. We have:

\[
\frac{G(t)}{(1-t)^{\dim M - (i-1)}} = b_0 + (b_0 + b_1) \cdot t + \ldots + \left( \sum_{\mu=0}^{\beta_i} b_{\mu} \right) \cdot t^{\beta_i} + (c_i + b_{\beta_i+1}) \cdot t^{\beta_i+1} + \ldots
\]

By induction, \( 0 < b_{\beta_i} \leq b_{\beta_i+1} \leq b_{\beta_i+2} \leq \ldots \) and since \( c_i > 0 \) we obtain \( c_i + b_{\beta_i+\nu} > 0 \) for \( \nu \geq 0 \).

The termination of the algorithm is trivial since we know that in the last loop we would consider \( \frac{G(t)}{(1-t)^{\dim M}} = HP_M(t) \) which is positive by the definition, and hence it will return \( hdepth_1 M = 0 \). \( \square \)
Remark 1.5. The maximum from the statement of [13, Theorem 3.2] (see here Theorem 0.1) is always smaller than \( \dim M \). This was not shown in Uliczka’s proof and it has to be proved in Theorem 1.4.

2. Computational Experiments

The following examples illustrate the usage of the implementation of the algorithm in SINGULAR, which can be found in the Appendix. Note that in the outputs we print exactly the jet we considered in our computations followed by “+...”.

Example 2.1. Consider the ring \( \mathbb{Q}[x, y_1, \ldots, y_5] \) and consider the ideal \( I = (x) \cap (y_1, \ldots, y_5) \).

```plaintext
ring R=0,(x,y(1..5)),ds;
ideal i=intersect(x,ideal(y(1..5)));
module m=i;
"dim M = ",dim(m);
// dim M = 5
hdepth( hilbconstruct( m ), dim(m) );
// G(t)= 1+t-4t^2+6t^3-4t^4+t^5
// G(t)/(1-t)^ 1 = 1+2t-2t^2+4t^3+t^5 +...
// G(t)/(1-t)^ 2 = 1+3t+t^2+5t^3+5t^4+6t^5 +...
// hdepth= 3
```

Example 2.2. Consider a module \( M \) for which \( \text{HP}_M(t) = \frac{2 - 3t - 2t^2 + 2t^3 + 4t^4}{(1-t)^{\dim M}} \).

Denote by \( \dim M \) the dimension of \( M \).

```plaintext
ring R = 0, t, ds;
poly g = 2-3*t-2*t^2+2*t^3+4*t^4;
hdepth( g, dim(M) );
// G(t)= 2-3t-2t^2+2t^3+4t^4
// G(t)/(1-t)^ 1 = 2-t-3t^2+t^3+3t^4+3t^5 +...
// G(t)/(1-t)^ 2 = 2+t-2t^2+3t^3+3t^4+3t^5 +...
// G(t)/(1-t)^ 3 = 2+3t+t^2-2t^3-2t^4+t^5 +...
// G(t)/(1-t)^ 4 = 2+5t+6t^2+4t^3+2t^4+3t^5 +...
Hence, it results hdepth_1 M = dim M - 4.
```

As seen in the proof, we had to increase our bound if the coefficient sum was \( \leq 0 \).

Note that in this example, the coefficient sum of \( \text{jet}_4 \left( \frac{G(t)}{(1-t)} \right) \) is zero and thus we increase the bound to 5 (the coefficient sum of the jet_5 will be equal to 3 > 0).

Example 2.3. Consider \( R = K[x_1, \ldots, x_n] \) for \( n \in \{4, 5, \ldots, 19\} \) and \( m \) the maximal ideal. We computed \( \text{hdepth}_1 m \), \( \text{hdepth}_1 (R \oplus m) \), \ldots, \( \text{hdepth}_1 (R^6 \oplus m) \) and \( \text{hdepth}_1 (R^{100} \oplus m) \). We obtain the following results:
Remark 2.4. Note that for $n = 6$ we have $\text{hdepth}_1(R \oplus m) = 4 > 3 = \text{hdepth}_1 m$. This is a sign that in this case $\text{sdepth}_n(R \oplus m) > \text{sdepth}_n(m)$ and so Herzog’s question could have a negative answer for $n = 6$. The difference $\text{hdepth}_1(R \oplus m) - \text{hdepth}_1 m$ can be $> 1$ as one can see for $n = 18$.

Note that $\text{hdepth}_1(R^s \oplus m) - \text{hdepth}_1 m$ increases when $s$ and $n$ increase. For example $\text{hdepth}_1(R^{100} \oplus m) - \text{hdepth}_1 m = 5$ for $s = 100$ and $n = 19$.

Lemma 2.5. Let $n \in \mathbb{N}$ be such that $\text{hdepth}_1 m = \text{hdepth}_1(R \oplus m)$. Then $\text{sdepth}_n m = \text{sdepth}_n(R \oplus m)$.

Proof. By [4] and [2] we have $\text{hdepth}_1 m = \left \lceil \frac{n}{2} \right \rceil = \text{sdepth}_n m$. It is enough to see that the following inequalities hold:

$\text{hdepth}_1 m = \text{sdepth}_n m \leq \text{sdepth}_n(R \oplus m) \leq \text{hdepth}_n(R \oplus m) \leq \text{hdepth}_1(R \oplus m).$

\[\Box\]

Proposition 2.6. If $n \in \{1, \ldots, 5, 7, 9, 11\}$ then $\text{sdepth}_n m = \text{sdepth}_n(R \oplus m)$, that is Herzog’s question has a positive answer.

Proof. Note that $\text{hdepth}_1 m = \text{hdepth}_1(R \oplus m)$ for $n$ as above and apply Lemma 2.5.

\[\Box\]
As stated before, Algorithm 1.3 was implemented as a procedure for the computer algebra system SINGULAR [7]. This procedure was used in order to obtain the results from Figure 1. The additional procedures which have been used were defined in Algorithm 1.3. In addition, we printed some information which we find useful for understanding the algorithm.

```c
proc hdepth ( poly g, int dim_M )
{
    int d;
    ring T = 0,t,d;
    "G(t)=",g;
    if (positiv(g)==1)
        {return("hdepth=",dim_M);}
    poly f=g;
    number ag;
    int c1;
    int bound;
    bound = deg(g);
    for (d = dim_M; d>=0; d--)
    {
        f = jet( g*inverse( (1-t)^((dim_M-d),bound) , bound ) , bound );
        if (positiv(f) == 1)
            {
                "G(t)/(1-t)^" ,dim_M-d,"=" ,f,"+...";
                "hdepth=",d;
                return();
            }
        c1=sumcoef(f);
        if (c1<=0)
            {
                while ( c1<0 )
                    {
                        bound = bound + 1;
                        f = jet( g*inverse( (1-t)^((dim_M-d),bound) , bound ) , bound );
                        c1 = sumcoef(f);
                    }
                "G(t)/(1-t)^" ,dim_M-d,"=" ,g,"+...";
            }
    }
}
```
References

[1] A.M. Bigatti, P. Gimenez, E. Sáenz-de-Cabezón: Monomial Ideals, Computations and Applications, Springer, 2013
[2] C. Biro, D.M. Howard, M.T. Keller, W.T. Trotter, S.J. Young, Interval partitions and Stanley depth, J. Combin. Theory Ser. A 117 (2010), 475-482.
[3] W. Bruns, J. Herzog: Cohen-Macaulay rings, Revised edition, Cambridge University Press (1998).
[4] W. Bruns, C. Krattenthaler, J. Uliczka: Stanley decompositions and Hilbert depth in the Koszul complex, J. Commut. Algebra 2 (2010), 327-357
[5] W. Bruns, J. Moyano-Fernández, J. Uliczka: Hilbert regularity of ZZ-graded modules over polynomial rings, (2013), arXiv:AC/1308.2917
[6] R.-X. Chen: How to compute the Hilbert depth of a graded ideal, (2013), arXiv:AC/1308.3205
[7] W. Decker, G.-M. Greuel, G. Pfister, H. Schönemann: SINGULAR 3-1-6 — A computer algebra system for polynomial computations. http://www.singular.uni-kl.de (2013).
[8] B. Ichim, J. J. Moyano-Fernández, How to compute the multigraded Hilbert depth of a module, to appear in Mathematische Nachrichten, arXiv:AC/1209.0084.
[9] B. Ichim, A. Zarojanu: An algorithm for computing the multigraded Hilbert depth of a module, (2013), to appear in Experimental Mathematics, arXiv:AC/1304.7215
[10] A. Popescu: Standard Bases over Principal Ideal Rings, Master Thesis at Technische Universität Kaiserslautern (2013).
[11] Y.H. Shen: Lexsegment ideals of Hilbert depth 1, (2012), arXiv:AC/1208.1822v1.
[12] R.P. Stanley: Linear Diophantine equations and local cohomology, Invent. Math. 68 (1982) 175-193.
[13] J. Uliczka: Remarks on Hilbert series of graded modules over polynomial rings, Manuscripta Math. 132 (2010), 159-168.

ADRIAN POPECȘCU, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KAIERSLAUTERN, ERWIN-SCHRÖDINGER-STR., 67663 KAIERSLAUTERN, GERMANY
E-mail address: popescu@mathematik.uni-kl.de