Dynamics of quasisolitons in degenerate fermionic gases

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We investigate the dynamics of the system of multiple bright and dark quasisolitons generated in a one-component ultracold Fermi gas via the phase imprinting technique in terms of atomic orbitals approach. In particular, we analyze the collision between two bright quasisolitons and find that quasisolitons are subject to the superposition principle.

Solitons are already well established phenomena in the area of dilute atomic quantum gases. They have been observed in several experiments with trapped bosonic atoms as well as described in many theoretical papers. The latter usually involves the mean-field approach and the resulting Gross-Pitaevskii equation for the condensate wave function. The intrinsic nonlinearity of this equation is originating from the interparticle interactions. As the effective interaction between atoms can be both repulsive or attractive, two types of solitonic excitations are possible in Bose-Einstein condensates.

Dark solitons were realized experimentally first\(^3\) whereas there have been problems with generating bright solitons implied by the occurrence of the collapse in the attractive condensates. Finally, two ways turned out to be successful. In experiments of Refs.\(^4\) a large condensate with positive scattering length is prepared and next the interatomic interactions are changed to attractive ones (with small negative value of the scattering length) by using Feshbach resonances. In this case the single bright soliton or a train of solitons were observed. Another way of creating bright solitons was demonstrated in the experiment of Ref.\(^5\) where the condensate still remains repulsive but appropriate engineering of atoms in a periodic potential allows for a change of the sign of the effective mass and results in the generation of bright gap solitons. Recently, another concept of bright matter waves in a repulsive condensate has been developed that involves degenerate Fermi gas as a “stabilizing medium”\(^5\).

In a series of papers\(^5\) we have shown that similar structures can also be created in a one-component ultracold fermionic gases even though to a very good approximation spin-polarized fermions at zero temperature do not interact. These structures exist both in the systems with uniform and nonuniform densities under a quasi-one-dimensional confinement. They can be generated by using the technique of phase imprinting and its appearance is related to the Fermi statistics. There exists the regime of parameters, defined by the condition that the width of the imprinted phase is bigger than the Fermi length, where only states with momentum close to the Fermi momentum are excited. Hence, the life time of generated structures can be long enough (in comparison with the Fermi time) allowing for observation and making them similar to solitons.

The method of phase imprinting has been already employed in the experiments where dark solitons have been generated\(^6\)\(^7\). The strong and short enough off-resonance laser pulse is passing first through the appropriately tailored absorption plate and next through the sample of atoms. Under such conditions the motion of atoms is frozen and the only result of the interaction of light with the atoms is imprinting the phase on the atomic wave functions. After the light is gone, the density and the phase change and desired structures (like solitons or vortices) appear in the system, according to the pattern written in the absorption plate.

In this paper we further investigate the motion of fermionic quasisolitons by analytical calculations based on the propagator technique. These calculations allow us for a detailed description of the collision between fermionic quasisolitons. We come about the conclusion that quasisolitons fulfill the superposition principle, i.e. the sum of two quasisolitons is a two-quasisoliton solution.

So, we consider a system of \(N\) fermionic atoms confined in a one-dimensional box with periodic boundary conditions at zero temperature. We assume that the many-body wave function of such a system is well described by the Slater determinant

\[
Ψ(x_1, ..., x_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \varphi_1(x_1) & \cdots & \varphi_1(x_N) \\ \vdots & \ddots & \vdots \\ \varphi_N(x_1) & \cdots & \varphi_N(x_N) \end{vmatrix}. \tag{1}
\]

The gas is spin-polarized and therefore the s-wave scattering, dominant at low temperatures, is absent and effectively the gas becomes the system of noninteracting particles. The many-body Schrödinger equation is then equivalent to the set of one-particle equations, each of them describing the evolution of a particular single-particle orbital \(\varphi_1(x), ..., \varphi_N(x)\) under the same Hamiltonian. Thus these orbitals remain orthogonal during the evolution and the one-particle density matrix is at any time given by the following formula

\[
ρ_1(x', x'', t) = \frac{1}{N} \sum_{n=1}^{N} \varphi_n(x', t) \varphi_n^*(x'', t), \tag{2}
\]
and its diagonal part is the particle density

$$\rho(x, t) = \frac{1}{N} \sum_{n=1}^{N} |\phi_n(x, t)|^2. \tag{3}$$

Fermi momentum in one-dimensional space, defined as $hN/(2L)$, determines the characteristic length $\lambda_F$ via the relation $\lambda_F = h/p_F = 2L/N$. From now on we choose the Fermi length as a unit of length. This quantity does not change while realizing the thermodynamic limit procedure. Simultaneously, the expression $h/\varepsilon_F$ gives a unit of time. Hence, according to the formula \(6\), the single particle wave function evolves as

$$\varphi_n(x, t) = \frac{1}{\sqrt{2Lt}} \int_{-\infty}^{\infty} e^{i\frac{p_n}{\hbar}(x-x')} e^{ik_n x'} e^{i\phi(x')} dx', \tag{8}$$

where the wave number $k_n = p_n/\hbar$ was introduced.

At zero temperature particles occupy the lowest possible levels. The corresponding eigenstates are just the plane waves $\varphi(x) = \exp(i p_n x/\hbar)/\sqrt{L}$ with momenta $p_n$ quantized according to $p_n = 2\pi \hbar n/L$, where $n = 0, \pm 1, \pm 2, \ldots$. The evolution of the single-particle wave function can be calculated based on the propagator technique

$$\varphi_n(x, t) = \int K(x, x', t) \varphi_n(x', 0) dx', \tag{4}$$

where $K(x, x', t)$ is the propagator function for a one-dimensional box with periodic boundary conditions

$$K(x, x', t) = \frac{1}{L^3} \sum_n \exp \left[ i \left( \frac{p_n}{\hbar} (x - x') - \frac{p_n^2}{2m\hbar} t \right) \right]. \tag{5}$$

It is convenient now to increase simultaneously the number of particles and the length of the box in such a way that their ratio (i.e. the Fermi momentum) remains constant. Increasing both quantities to infinity defines the, so called, thermodynamic limit. In this limit the propagator takes the same form as for a free particle and is given by

$$K(x, x', t) = \sqrt{\frac{m}{2\hbar t}} \exp \left[ i \frac{m}{2\hbar t} (x - x')^2 \right]. \tag{6}$$

To simplify further calculations we assume the following form of the imprinted phase (see Fig. 1)

$$\phi(x) = \begin{cases} 0 & , x < -a \\ \frac{w}{\pi}(\pi + 1) & , -a \leq x \leq a \\ w & , x > a \end{cases}, \tag{7}$$

where the parameters $a$ and $w$ define the width and the jump of the imprinted phase, respectively. The graph is divided into three regions: a constant phase region $w$, a linearly increasing phase region $\pi/2$, and a constant phase region $w$. The phase pattern leading to the generation of a single bright-dark quasisolitons pair.

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FIG. 1: Phase pattern leading to the generation of a single bright-dark quasisolitons pair.

Successive frames show the fermionic density (normalized to $N/2$) of 501 atoms after writing a single phase step of height $\pi$. The corresponding times are 3, 5, 7, and 8 in units of $h/\varepsilon_F$.

Calculating the integral \(8\), one obtains an explicit formula for the evolution of the single particle wave function (in units of $1/\sqrt{\lambda_F}$)

$$\varphi_n(x, t) = \frac{1}{\sqrt{2Lt}} e^{-ik_n^2 \pi^2 + ik_n x} \left\{ (1 + i)(1 + e^{i\pi})/2 + C(u_{-a}) + iS(u_{-a}) - e^{iw} C(u_a) - e^{iw} iS(u_a) + e^{i[w + \frac{\pi}{2} x - \frac{\pi}{2} (k_n + \frac{\pi}{2})]} \left[ C(u_a') + iS(u_a') - C(u_{-a}') \right] \right\}, \tag{9}$$

where $C(z)$ and $S(z)$ are the Fresnel integrals \[11\] and their arguments are the following functions of position and time

$$u_a = (a - x + k_n t)/\sqrt{\pi},$$

$$u_{-a} = (-a - x + k_n t)/\sqrt{\pi},$$

$$u_a' = u_a + \frac{w}{2\pi a} \sqrt{\pi},$$

$$u_{-a}' = u_{-a} + \frac{w}{2\pi a} \sqrt{\pi}. \tag{10}$$
The fermionic density (the diagonal part of one-particle density matrix) is just the sum of single-particle densities obtained from single-particle orbitals given by (9). Fig. 2 shows such a density for a system of 501 atoms at various times after writing a phase step of height $\pi$ and the width $4a_0$. As already reported in Ref. [3 11], the bright and the dark quasisolitons are generated. They propagate with different velocities and in opposite direction. The speed of bright quasisoliton is slightly higher than the speed of sound (equal to $p_F/m$) whereas opposite is true for the dark quasisoliton. We confirm all the details of dynamics of such a system described earlier in Ref. [3 11], especially its dependence on the ratio $a/\lambda_F$ of the width of the imprinted phase step and the Fermi length.

![Phase pattern used for generating two bright-dark quasisolitons pairs.](image)

**FIG. 3:** Phase pattern used for generating two bright-dark quasisolitons pairs. In this case bright quasisolitons move towards each other and collide.

The purpose of the present paper is, however, to investigate the collision between bright fermionic quasisolitons. To this end, we imprint the phase pattern as in Fig. 3 on the fermionic gas. It results in a creation of two pairs of bright-dark quasisolitons with bright quasisolitons moving towards each other. It turns out that the time evolution of individual orbitals can be expressed in terms of the evolution corresponding to the phase pattern already considered in Fig. 1 and given by the formula (9). First, let us notice that the phase shown in Fig. 3 can be split in the following way $\phi^{lr} = \phi^l + \phi^r - w$, where $\phi^l$ and $\phi^r$ are the left and the right cuts of the phase shown in Fig. 3 respectively. Next, using the formula (9) one gets

$$\phi_n^{lr}(x,t) = \phi_n^l(x,t) + \phi_n^r(x,t) - A,$$

where

$$A = \frac{1}{\sqrt{2}L} \int_{-\infty}^{\infty} e^{i\frac{x}{2L}(x-x')} e^{ik_nx'} e^{iw} dx' = \frac{1 + i}{\sqrt{2}L} e^{iw} e^{-ik_nx} \sin(k_nx)$$

and superscripts ‘l’ and ‘r’ indicate that the wave functions are propagated after the imprinting the single step patterns $\phi^l$ and $\phi^r$, respectively. The superscript ‘lr’ means the evolution according to the phase pattern plotted in Fig. 3.

![Graph showing the evolution of the density of fermionic quasisolitons](image)

**FIG. 4:** The relative (with respect to 1/L) oscillations of the expression $B + 2/L$ as a function of position for a particular ($n = 200$) single-particle orbital. The length of the box $L = 250a_0$ and the time is 12 in units of $\hbar/\varepsilon_F$. Other parameters: $c = 15, d_1 = 4, d_2 = 5$, in units of $\lambda_F$, and $w = \pi$ (the case of the last frame of Fig. 5).

Both orbitals $\phi_n^l(x,t)$ and $\phi_n^r(x,t)$ can be calculated based on the following rules that connect the time evolution due to the phase pattern given in Fig. 3 with the evolution resulting from patterns obtained by the shift ($\phi_n^{s}(x,t)$), the reflection ($\phi_n^{ref}(x,t)$), and a combination of the reflection and the shift ($\phi_n^{rs}(x,t)$) of that one shown in Fig. 3.

$$\phi_n^{s}(x,t) = e^{ik_nx} \phi_n(x-s,t),$$

$$\phi_n^{ref}(x,t) = \phi_n^{r}(-x,t),$$

$$\phi_n^{rs}(x,t) = e^{-ik_nx} \phi_n^{r}(-x-s,t).$$

(13)

One has

$$\phi_n^l(x,t) = e^{-ik_n(x+c+d_1/2)} \phi_n(x+c+d_1/2, t; a = d_1/2),$$

$$\phi_n^r(x,t) = e^{ik_n(x+c+d_2/2)} \phi_n(-x+c+d_2/2, t; a = d_2/2).$$

(14)

The relation (14) can be rewritten now in terms of the single-particle densities

$$\rho_n^l(x,t) = \rho_n^l(x,t) + \rho_n^r(x,t) + \frac{1}{L} + B,$$

$$B = 2Re(\phi_n^l(x,t)\phi_n^{ref}(x,t) - \phi_n^l(x,t)A - \phi_n^r(x,t)A^*)$$

(15)

and it turns out that approximately

$$\rho_n^{lr}(x,t) = \rho_n^l(x,t) + \rho_n^r(x,t) - \frac{1}{L}. $$

(16)
In fact, what is left, i.e. the expression \( B + 2/L \), oscillates as a function of position around zero with a relative (in comparison with 1/L) amplitude usually of the order of 10^{-4} right after the quasisolitons are created. This amplitude increases in time but remains at the low level (10^{-2}) even after the collision of quasisolitons. An example of that is given in Fig. 4 where we consider the box of length \( L = 250 \rho_F \) (it corresponds to the last frame of Fig. 5). Only when the imprinted phase step gets very sharp the relative amplitude becomes as large as 10^{-1}. The total density is the sum of single-particle densities and equals

\[
\rho_l(x,t) = \rho_0(x,t) + \rho^{lr}(x,t) = \frac{N}{2L}.
\]  

(17)

Here, the density is normalized to the number of particles divided by 2. Therefore, in the thermodynamic limit, i.e. when the size of the box and the number of atoms increase to infinity while keeping their ratio constant, one gets the following rule

\[
\rho^{lr}(x,t) = \rho_0(x,t) + \rho^{lr}(x,t) - 1.
\]  

(18)

First, this formula generalizes the results already obtained in Ref. [10] (and expressed by solution (24)) to any perturbation of the initial density. Secondly, it happens that going from \([10]\) to \([17]\) the “noise” present for each orbital (see Fig. 4), even if it is large, adds destructively causing that the relation \([18]\) is satisfied almost perfectly. It has to be noticed that the same property (18) is shared by the solutions of the wave equation. Here, however, single-particle densities satisfy the nonlinear hydrodynamic equations according to the Madelung representation of the Schrödinger equation \([12]\)

\[
\frac{\partial \rho_n}{\partial t} + \frac{\partial}{\partial x} (\rho_n v_n) = 0
\]

\[
\frac{\partial v_n}{\partial t} + \frac{\partial}{\partial x} \left( \frac{v_n^2}{2} - \frac{\hbar^2}{2m} \frac{1}{\sqrt{\rho_n}} \frac{\partial^2}{\partial x^2} \sqrt{\rho_n} \right) = 0 \tag{19}
\]

where \( \rho_n \) and \( v_n \) are the density and the velocity fields, respectively, associated with the orbital \( \varphi_n^F(x,t) \).

In Fig. 5 we illustrate the collision of two bright fermionic quasisolitons. Successive frames show the total density calculated based on the formulas (11), (14), and (18). No interference, as a consequence of (18), is observed when both peaks meet at the center of the box which distinguishes this case from collision of usual wave packets. It is not surprising, since building the quasisoliton we add probabilities (see (4)) rather than the amplitudes. After the collision both quasisolitons reappear, however no phase shift, usually expected when two solitons collide, is observed. This is another consequence of formula (18). So, what we have is a solution (in a sense of (4)) of a set of nonlinear hydrodynamic equations (19) that is a sum of quasisolitons not only well before and after the collision but all the time.

In conclusion, we have investigated the dynamics of fermionic quasisolitons generated in ultracold fermionic gas by using the technique of phase imprinting. We obtained an analytic formula that describes the propagation of a bright-dark quasisoliton pair. Based on this formula we were able to study the collision of two bright fermionic quasisolitons and found that the quasisolitons are subject to the superposition principle — the sum of two quasisolitons is a two-quasisoliton solution.

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