QUANTUM GROUPS, COHERENT STATES, SQUEEZING AND LATTICE QUANTUM MECHANICS

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Abstract. By resorting to the Fock–Bargmann representation, we incorporate the quantum Weyl–Heisenberg (q-WH) algebra into the theory of entire analytic functions. The main tool is the realization of the q–WH algebra in terms of finite difference operators. The physical relevance of our study relies on the fact that coherent states (CS) are indeed formulated in the space of entire analytic functions where they can be rigorously expressed in terms of theta functions on the von Neumann lattice. The rôle played by the finite difference operators and the relevance of the lattice structure in the completeness of the CS system suggest that the q–deformation of the WH algebra is an essential tool in the physics of discretized (periodic) systems. In this latter context we define a quantum mechanics formalism for lattice systems.

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1. Introduction

A great deal of attention and efforts have been devoted recently to the mathematical structures referred to as $q$-groups\textsuperscript{[1,2]}, newly discovered (or, better, rediscovered, in that they are but specific realizations of Hopf algebras) by theoretical physicists. These structures promise to be very rich of physical meaning, and, although their skeleton appears pretty well understood, there are however many properties which deserve more study to be fully under control.

The interest in $q$-groups arose almost simultaneously in statistical mechanics as well as in conformal theories, in solid state physics as in the study of topologically non-trivial solutions to nonlinear equations. As a matter of fact, the research in $q$-groups was indeed originated from physical problems. One of the main interesting features resides in the $C^*$-algebraic properties of the elements of their matricial representations, properties which can be expected lead to fruitful connections with noncommutative geometries. Studies in such directions are promising but at same time a comprehensive program is far from being formulated.

A different situation occurs with the dual structures, the $q$-algebras (care should be exerted here, as – by an abuse of language now almost universally adopted – the name $q$-groups has been adopted to designate $q$-algebras as well, even though they are in fact neither groups nor algebras in the customary sense).

$q$-algebras are deformations of enveloping algebras of Lie algebras and, like the latter, they have Hopf algebra features. The general properties of $q$-algebras are much better known than those of $q$-groups, in particular for the specific characteristics which relate them with concrete physical models. Among the $q$-algebras, the non-semisimple ones are the less easy to handle, as in fact it happens in general with non-semisimple structures. The $q$-deformation of the Weyl-Heisenberg algebra ($q$-WH), as well as the WH algebra, is not even a Hopf algebra; it has only the properties of a Hopf superalgebra\textsuperscript{[3]}.

In view of the fundamental rôle of WH algebra in quantization, we believe that its $q$-deformation deserves special attention. The WH algebra has in quantum physics such powerful applications as the harmonic oscillator, coherent states, Jordan-Wigner realizations, and so on. $q$-WH algebra appears to be equally useful in the study of the corresponding deformed structures.

In this paper we cast the study of the $q$-WH algebra in the frame of the Fock-Bargmann representation (FBR) of Quantum Mechanics (QM). The reason for this is that one of our objectives is to preserve, in our study of $q$-deformations, the analytic structure of the corresponding Lie algebras and therefore we need to operate in a scheme where analyticity is ensured.
As a first step we present a realization of the $q$-WH algebra in terms of finite difference operators. As a result we recognize that whenever a finite scale is involved in a self-contained physical theory, then a $q$-deformation of the algebra of dynamical observables occurs, with the $q$-parameter related with the finite spacing, namely carrying the information about discreteness. $q$-deformation is also expected in the presence of periodic conditions, since periodicity is a special form of invariance under finite difference operators.

As a first conjecture, the physical content of the $q$-deformation thus emerges in connection with discretized (periodic) systems.

The analytic properties of the FBR together with the von Neumann lattice topological structure in the complex plane, make then manifest the relation between the $q$-WH algebra and the coherent states (CS) in QM. Here, we obtain a formal relation between the coherent state generator and the commutator of $q$-WH creation and annihilation operators, which is thus recognized to be an operator in the CS space. Moreover, theta functions, in terms of which CS are expressed, also admit a representation in terms of $q$-deformed WH commutators on the von Neumann lattice, allowing us to get further insight in the basic unity of the various structures.

One additional successive step in the understanding of the physical meaning of $q$-deformations is then achieved by realizing that the commutator of $q$-WH creation and annihilation operators is, in the FBR, the squeezing generator for CS, a result which confirms a conjecture, previously\cite{4} formulated, whereby $q$-groups are the natural candidates to study squeezed coherent states.

The relation established between CS and the $q$-WH algebra is of course of great interest in view of the numerous interesting physical applications of the CS formalism, and it may also open rich perspectives in Quantum Field Theory (QFT) where the CS formalism is the key to study vacuum structure and boson condensation.

The relevance of $q$-deformation to discretized system physics naturally leads us to analyze the structure of Lattice Quantum Mechanics (LQM). We study it in configuration space as well as in momentum space and show that LQM is characterized in both cases by the algebra $E(2)$. The lattice CS, optimizing the lattice position-momentum uncertainty relation, show that just $q$-WH is the algebraic structure underlying the physics of lattice quantum systems. We find that the commutator between $q$-WH creation and annihilation operators acts as generator of the $U(1)$ subgroup of $E(2)$, giving rise to phase variations in the complex plane. In this context, in the presence of a periodic potential on the lattice, there emerges naturally a relation between $q$-WH algebra and the Bloch functions, which
further confirms the conjectured deeply rooted presence of $q$-deformation within the dynamical structure of periodic systems.

As a general remark, we should like to stress that it is only by fully exploiting the FBR that we succeed in incorporating $q$-deformation of the WH algebra into the theory of (entire) analytical functions. Such result may deserve by itself further attention: in this way, indeed, it appears possible to elucidate the deep rôle of $q$-WH algebra in the physics of lattice quantum systems, coherent states and squeezing.

Through this paper we shall use units such that all relevant physical quantities are dimensionless.

2. $q$-Weyl-Heisenberg algebra, Fock–Bargmann representation and finite difference operators

The quantum version of the Weyl-Heisenberg algebra

$$[a, a^\dagger] = I , \quad [N, a] = -a , \quad [N, a^\dagger] = a^\dagger ,$$

(2.1)

is realized in terms of the set of operators \{\(a_q, \bar{a}_q, N_q; q \in \mathbb{C}\)\}, with relations\(^{[1,2]}\):

$$[N_q, a_q] = -a_q , \quad [N_q, \bar{a}_q] = \bar{a}_q , \quad a_q \bar{a}_q - q^{-\frac{1}{2}} \bar{a}_q a_q = q^{\frac{1}{2}} N_q .$$

(2.2)

The structure lying behind (2.2) is a quantum superalgebra\(^{[3]}\). The notion of hermiticity associated with it has been studied in ref. \([4]\) in connection with the discussion of the squeezing of the generalized coherent states (GCS)\(_q\), defined in the usual Fock space \(\mathcal{K}\).

By introducing \(\hat{a}_q \equiv \bar{a}_q q^{N/2}\), and setting – without loss of generality – \(N_q \equiv N\), relations (2.2) can be rewritten as

$$[N, a_q] = -a_q , \quad [N, \hat{a}_q] = \hat{a}_q , \quad [a_q, \hat{a}_q] \equiv a_q \hat{a}_q - \hat{a}_q a_q = q^N .$$

(2.3)

In the following we shall discuss the functional realization of eqs. (2.3) by means of finite difference operators in the complex plane, in the Fock-Bargmann representation of Quantum Mechanics\(^{[5,6]}\).

Contrary to the usual coordinate or momentum representation of QM, where no condition of analyticity is imposed upon the wavefunction, in the FBR any state vector is described by an entire analytic function. The FBR operators, solution of the commutation relations (2.1), are\(^{[5,6]}\):

$$N \rightarrow z \frac{d}{dz} , \quad a^\dagger \rightarrow z , \quad a \rightarrow \frac{d}{dz} .$$

(2.4)
The Hilbert space of entire analytic functions $\mathcal{F}$ has a well defined inner product:
\[
< \psi_1 | \psi_2 > = \int \overline{\psi}_1(z) \psi_2(z) d\mu(z) ,
\]
with suitable measure $d\mu$, where, in terms of the complete orthonormal set of eigenkets \{\ket{n} | n \in \mathbb{N}, n \in \mathcal{K}\} of $N$, \(|\psi> = \sum_{n=0}^{\infty} c_n \ket{n} , \quad <\psi|\psi> = \sum_{n=0}^{\infty} |c_n|^2 = 1\). One associates with $|\psi>$ the function in $\mathcal{F}$
\[
\psi(z) = \sum_{n=0}^{\infty} c_n u_n(z) , \quad u_n(z) = \frac{z^n}{\sqrt{n!}} , \quad (n \in \mathbb{N}_+) , \quad u_0(z) = 1 .
\]

The set \{\{u_n(z)\}\} provides an orthonormal basis in $\mathcal{F}$. The rôle of the $\delta$-function is played in this representation, assuming $d\mu(z) = \frac{1}{\pi} e^{-|z|^2} dRez dImz$, by
\[
\delta(z,z') = \sum_{n=0}^{\infty} u_n(z) \overline{u}_n(z') = \exp(zz') ,
\]
in that
\[
f(z) = \int \delta(z,z') f(z') d\mu(z') .
\]

Eqs. (2.6) provide thus the most general representation of an entire analytic function in the $z$-plane. Note also that, from eqs. (2.4) and (2.6),
\[
a^\dagger u_n(z) = \sqrt{n+1} u_{n+1}(z) , \quad a u_n(z) = \sqrt{n} u_{n-1}(z) ,
\]
\[
N u_n(z) = a^\dagger a u_n(z) = z \frac{d}{dz} u_n(z) = n u_n(z) ,
\]
as expected in view of the correspondence $\mathcal{K} \rightarrow \mathcal{F}$ ($|n> \rightarrow u_n(z)$).

Let us now consider the finite difference operator $\mathcal{D}_q$ defined by:
\[
\mathcal{D}_q f(z) = \frac{f(qz) - f(z)}{(q-1)z} ,
\]
with $f(z) \in \mathcal{F} , \quad q = e^\zeta , \quad \zeta \in \mathbb{C}$ . $\mathcal{D}_q$ is the so called $q$-derivative operator\[7\], which, for $q \rightarrow 1$ (i.e. $\zeta \rightarrow 0$), reduces to the standard derivative. By using the representation (2.6) for $f$ and resorting to the last equality in eqs.(2.9), it may be written on $\mathcal{F}$ as
\[
\mathcal{D}_q = ((q-1)z)^{-1} \left(q^z \frac{d}{dz} - 1 \right) .
\]
Consistency between (2.10) and the above form of $D_q$ can be proven by first "normal ordering" the operator $(z \frac{d}{dz})^n$ in the form:

$$\left(z \frac{d}{dz}\right)^n = \sum_{m=1}^{n} S_n^{(m)} z^m \frac{d^m}{dz^m},$$

(2.12)

where $S_n^{(m)}$ denotes the Stirling numbers of the second kind defined by the recursion relations\[^{[8]}\]

$$S_{n+1}^{(m)} = m S_n^{(m)} + S_{n}^{(m-1)},$$

(2.13)

and then expanding in formal power series the exponential $\left(q^z \frac{dz}{dq} - 1\right)$, keeping in mind the identity:

$$\frac{1}{m!} (e^\xi - 1)^m = \sum_{n=m}^{\infty} S_n^{(m)} \xi^n n!.$$

(2.14)

$D_q$ generates, together with $z$ and $\frac{d}{dz}$, the algebra:

$$[D_q, z] = q^z \frac{d}{dz}, \quad [z \frac{d}{dz}, D_q] = -D_q, \quad [z \frac{d}{dz}, z] = z,$$

(2.15)

which can be recognized as a realization of relations (2.3) in the space $\mathcal{F}$. In the latter, operators $N, \hat{a}_q$ and $a_q$ can then be associated with

$$N \rightarrow z \frac{d}{dz}, \quad \hat{a}_q \rightarrow z, \quad a_q \rightarrow D_q,$$

(2.16)

with $\hat{a}_q = \hat{a}_{q=1} = a^\dagger$ and $\lim_{q \rightarrow 1} a_q = a$. The corresponding realization of (2.2) is

$$N \rightarrow z \frac{d}{dz}, \quad \tilde{a}_q \rightarrow zq^{-\frac{dz}{dq}}, \quad a_q \rightarrow D_q.$$

(2.17)

There follows that the commutator $[a_q, \hat{a}_q]$ acts in $\mathcal{F}$ as

$$[a_q, \hat{a}_q] f(z) = q^z \frac{dz}{dq} f(z) = f(qz).$$

(2.18)

All of this suggests\[^{[9]}\] that whenever one deals with some finite scale (e.g. with some discrete structure, lattice or periodic system) which cannot be reduced to the continuum by some limiting procedure, then a deformation of the operator algebra acting in $\mathcal{F}$ should arise. Deformation of the operator algebra is also expected whenever the system under study involves periodic (analytic) functions, since periodicity is but a peculiar invariance under finite difference operators.

3. $q$-Weyl-Heisenberg algebra and coherent states
In the following discussion of the connection of the $q$-WH algebra with the CS formalism we shall confine ourselves to the case of a single complex variable (one dimensional case). Extension to many complex variables would proceed along the same lines as the extension of customary CS to several degrees of freedom and it is straightforward. We begin by observing that it is just by exploiting the theory of entire analytic functions that the Fock–Bargmann representation provides a simple and transparent frame to study the usual CS. Actually, writing the latter in the form

$$|\alpha> = \mathcal{D}(\alpha)|0> \quad ; \quad a|\alpha> = \alpha|\alpha> \quad , \quad a|0> = 0 \quad , \quad \alpha \in \mathbb{C} \quad , \quad (3.1)$$

$$|\alpha> = \exp\left(\frac{-|\alpha|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} n > = \exp\left(\frac{-|\alpha|^2}{2}\right) \sum_{n=0}^{\infty} u_n(\alpha)|n> \quad , \quad (3.2)$$

the relation between the CS and the entire analytic function basis $\{u_n(z)\}$ is easily made explicit: $u_n(z) = e^{\frac{1}{2}|z|^2} <n|z>$. The unitary displacement operator $\mathcal{D}(\alpha)$ in (3.1) is given by:

$$\mathcal{D}(\alpha) = \exp(\alpha a^{\dagger} - \bar{\alpha} a) = \exp\left(\frac{-|\alpha|^2}{2}\right) \exp(\alpha a^{\dagger}) \exp(-\bar{\alpha} a) \quad , \quad (3.3a)$$

$$= \exp\left(\frac{|\alpha|^2}{2}\right) \exp(-\bar{\alpha} a) \exp(\alpha a^{\dagger}) \quad , \quad (3.3b)$$

and the following relations hold

$$\mathcal{D}(\alpha)\mathcal{D}(\beta) = \exp(iIm(\alpha\bar{\beta}))\mathcal{D}(\alpha + \beta) \quad , \quad (3.4)$$

$$\mathcal{D}(\alpha)\mathcal{D}(\beta) = \exp(2iIm(\alpha\bar{\beta}))\mathcal{D}(\beta)\mathcal{D}(\alpha) \quad . \quad (3.5)$$

Eq. (3.5) is the well known integral representation of the Heisenberg commutation relations (2.1), also called the Weyl integral representation (see e.g. ref. [6]).

In order to relate the $q$-deformed Heisenberg algebra (2.3) (or (2.2)) to the CS generator $\mathcal{D}(\alpha)$, we observe that, upon rewriting eq. (3.4) with $\alpha = (q - 1)z$ and $\beta = z$, one gets

$$<n|\mathcal{D}((q - 1)z)|z> = <n|\exp\left(iIm(q - 1)|z|^2\right)|qz> = \exp\left(\frac{q - \bar{q}}{2}|z|^2\right) \exp\left(-\frac{|qz|^2}{2}\right) u_n(qz) \quad . \quad (3.6a)$$

$$= \exp\left(\frac{q - \bar{q}}{2}|z|^2\right) \exp\left(-\frac{|qz|^2}{2}\right) u_n(qz) \quad . \quad (3.6b)$$
On the other hand we have, still from (3.2),
\[
< n|q^n|z > = \exp\left(-\frac{|z|^2}{2}\right)u_n(qz) .
\] (3.7)

Thus, from (2.3), (3.6b) and (3.7),
\[
< n|[a_q, \hat{a}_q]|z > = \exp\left(-(1 - \bar{q})(1 + q)\frac{|z|^2}{2}\right) < n|\mathcal{D}((q - 1)z)|z > .
\] (3.8)

Use of (3.6a) gives
\[
< n|[a_q, \hat{a}_q]|z > = \exp\left((|q|^2 - 1)\frac{|z|^2}{2}\right) < n|qz > ,
\] (3.9)

for any \(n > (\text{and for any } |z| >)\). Thus we can write
\[
\exp\left((1 - |q|^2)\frac{|z|^2}{2}\right)[a_q, \hat{a}_q]|z > = |qz > .
\] (3.10)

Finally, use of eq. (3.3a) leads to
\[
\mathcal{D}(\alpha)f(z) = \exp\left(-\frac{\alpha|^2}{2}\right)\exp(\alpha z)f(z - \bar{\alpha}) , \quad f \in \mathcal{F} .
\] (3.11)

By setting \(\alpha = (1 - \bar{q})\bar{z}\) in eq. (3.11), we have then, in view of eq. (2.18),
\[
[a_q, \hat{a}_q]f(z) = \exp\left(-(1 - \bar{q})(1 + q)\frac{|z|^2}{2}\right)\mathcal{D}((1 - \bar{q})\bar{z})f(z) .
\] (3.12)

These results establish the relation of the quantum algebra (2.3) (or (2.2)) with the theory of CS (eqs. (3.8)÷(3.10)) and with the theory of entire analytic functions (eqs. (2.18), (3.7) and (3.12)).

It is interesting to observe that the commutator \([a_q, \hat{a}_q]\) acts as shift operator from \(|z >\) to \(|qz >\) (notice that the exponential factor in eq. (3.10) simply generates the correct normalization); it acts as the \(z\)-dilatation operator \((z \to qz)\) in the space of entire analytic functions and it acts as the \(U(1)\) generator of phase variations in the \(z\)-plane when \(q = e^\zeta\), with \(\zeta\) pure imaginary, \(\zeta = i\theta : z \to e^{i\theta}z\).

We also observe that eqs. (3.8), (3.10) and (3.12) provide a nonlinear realization of the quantum algebra (2.3) in terms of \(a\) and \(a^\dagger\). Conversely, the nonlinear operator \(\mathcal{D}(\alpha)\) is represented by the linear form \([a_q, \hat{a}_q]\) in the algebra.
We finally recall that in order to extract a complete set of CS \( \{|z_n>\} \) (\( \{z_n\} \) a discrete set of points in the \( z \)-plane minus the origin) from the overcomplete set \( \{|z>\} \) it is necessary to introduce a regular lattice \( L \) \([5,6]\). By closely following the procedure of ref. [6] we recall that the set \( \{|z_n>\} \) (with exclusion of the vacuum state \( |0>\) ) is complete if the lattice elementary cell has area \( S = \pi \) \([6,10]\) \( (L \) is called, in this case, the von Neumann lattice). The points (or lattice vectors) \( z_n \) are given by \( z_n = \sum_i \mu_i \omega_i \), \( i = 1, 2 \) with \( \mu_i \in \mathbb{Z} \). The lattice periods \( \omega_i \) are assumed to be linearly independent, \( i.e. \) \( \text{Im} \omega_1 \omega_2 \neq 0 \). The proof of completeness is established by invoking square integrability along with analyticity\([11]\).

The possibility of extracting the complete set \( \{|z_n>\} \) makes more significant the above presented relation between \( q \)-algebras and CS, and once more we stress the central rôle of the underlying discrete lattice structure.

4. \( q \)-Weyl-Heisenberg and theta functions

The lattice structure is of crucial relevance in the relation between the theta functions and the complete system of CS.

A deep interrelation, essentially amounting to equivalence, is known to exist among three different ways of viewing theta functions:

a) as classical holomorphic functions in (several) complex variables \( z \) and the period (matrix) \( \tau \);

b) as matrix coefficients of a representation of the Heisenberg and metaplectic group;

c) as sections of line bundles on abelian varieties or the moduli space of abelian varieties.

Such an equivalence is thoroughly analyzed by Mumford in ref. [12], where its mathematical structure is explored in full detail. Here it is our aim to show how these profound mathematical results are related to some physically interesting features. For the sake of completeness, we simply summarize hereafter the basic mathematical concepts on which our analysis will be based, reporting them in the concise "flow chart" in Tab.I. In such chart \( \mathcal{H} \) denotes the Hilbert space (\( \mathcal{H}^* \) its dual) of representations of the Heisenberg group \( H \). \( H \) acts unitarily on the elements of \( \mathcal{H} \), which are holomorphic functions, by

\[
U_{(A,B)} f(Z) = \exp \{i\pi \text{Tr} [A^t (TAQ + B)]\} f(Z + TAQ + B) \ . \quad (4.1)
\]

\( A, B, Z, T, \) and \( Q \) are complex matrices and in Tab.I \( \iota \) means isomorphism, \( \mathcal{H}_{\pm \infty} \) are the completions of \( \mathcal{H} \) to the space of holomorphic functions such that \( f(X) = \mathcal{O}(\|X\|^n e^{\frac{1}{2} \Xi (X,X)}) \), for some integer \( n \), with quadratic (negative if \( \text{Im} T \)
is negative) weight $\Xi(X, X) = X^t(\text{Im}T)^{-1}X$, $e_z = \sum_{N \in \mathbb{Z}} e_\ast(N/2)\delta_N$, (where $e_\ast(N/2) \in \{\pm1\}$ is a quadratic form on $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$), $\sigma$ denotes the cocycle of the group action, $W_T \equiv \{\text{span of } f \mapsto X \cdot f\}$, and $f_T$ is the (highest weight) element in the Heisenberg representation, *unique* up to a scalar, which is annihilated by $W_T$.

\[ H^* \otimes H \]
\[ \| \]
\[ \left[ \text{All functions } f \text{ on } \mathbb{R}^{\otimes 2} \right] \]
\[ \left[ \text{on which } H \otimes H \text{ acts} \right] \]
\[ \| \]
\[ \left[ \text{Space spanned by functions} \right] \]
\[ \left[ < U_{(A,B)}f, g >, f \in H_\infty, g \in H_{-\infty} \right] \]
\[ \bigcup \bigcup \]
\[ W_T \text{ right } - \text{ annihilator} \quad \sigma (\mathbb{Z}^{\otimes 2}) - \text{left } - \text{ invariants} \]
\[ \| \]
\[ \left[ \text{Space spanned by functions} \right] \]
\[ \left[ < U_{(A,B)}f_T, f >, f \in H_{-\infty} \right] \]
\[ \| \]
\[ e^{-\frac{\tau \Xi}{2}} \quad \left[ \text{Fock space} \right] \quad \left[ \text{Quasi } - \text{ periodic space} \right] \]
\[ F(\mathbb{C}, T)_{-\infty} \quad l^2(\mathbb{R}^{\otimes 2}/\mathbb{Z}^{\otimes 2}) \]
\[ \| \]
\[ \bigcup \bigcup \]
\[ H^* \quad H \]
\[ \left[ \text{The unique function} \right] \]
\[ \left[ \vartheta_Q \begin{bmatrix} A \\ B \end{bmatrix}(T) = < U_{(A,B)}f_T, e_z > \right] \]

*Tab.I.*
In general,
\[
\vartheta^{Q} \left[ \begin{array}{c} A \\ B \end{array} \right] (T) = \sum_{N \in \mathbb{Z}} \exp \left\{ i\pi \text{Tr} \left[ (N + A)^{t}(N + A)Q + 2(N + A)^{t}(Z + B) \right] \right\} .
\] (4.2)

We shall relate, in what follows, the general scheme of the "flow chart" with the properties of coherent states, in a quite elementary way, which, indeed, is nothing but a rephrasing of the flow chart depicted above.

In order to do so, we look for the common eigenvectors $|\theta>$ of the CS operators $\mathcal{D}(z_n)$ associated to the regular lattice $L$. A common set of eigenvectors exists only if all the $\mathcal{D}(z_n)$ commute, which happens, of course, when the $\mathcal{D}(\omega_i)$ commute. From eq. (3.4) we see that $\mathcal{D}(\omega_i)$, $i = 1, 2$, commute when $\text{Im} \omega_1 \bar{\omega}_2 = k\pi$ with $k$ integer, namely just when $L$ is a Von Neumann lattice. Without loss of generality, we set $k = 1$, which is sufficient to guarantee that operators $\mathcal{D}(z_n)$ generate a complete set of CS.

The states $|\theta>$ are characterized by two real numbers $\epsilon_1$ and $\epsilon_2$, $|\theta>$ $\equiv$ $|\theta_{\epsilon}>$, since, as required, they are eigenvectors of $\mathcal{D}(\omega_i)$:
\[
\mathcal{D}(\omega_i)|\theta_{\epsilon}> = e^{i\pi \epsilon_i}|\theta_{\epsilon}> , \quad i = 1, 2 \quad 0 \leq \epsilon_i \leq 2 .
\] (4.3)

The vector $|\theta_{\epsilon}>$ corresponds to a point on the two-dimensional torus. $|\theta_{\epsilon}>$ belongs to a space which is the extension\footnote{see eq. (4.2)} of the Hilbert space where the operators $\mathcal{D}(z)$ act. The action of $\mathcal{D}(z)$ on $|\theta_{\epsilon}>$ generates a set of generalized coherent states $|\theta_{z}>$. The system of CS $|\theta_{\epsilon}>$ in the FBR may be associated with an entire analytic function, say $\theta(z)$, $|\theta_{\epsilon}> \rightarrow \theta(z)$. Use of eqs. (4.3) and (3.4) gives
\[
\mathcal{D}(z_m)|\theta_{\epsilon}> = e^{i\pi F_{\epsilon}(m)}|\theta_{\epsilon}> ,
\] (4.4)
with $z_m = m_1 \omega_1 + m_2 \omega_2$ an arbitrary lattice vector and
\[
F_{\epsilon}(m) = m_1 m_2 + m_1 \epsilon_1 + m_2 \epsilon_2 .
\] (4.5)

Moreover, eq. (3.11) with $\bar{\alpha} = -z_m$ shows that eq. (4.4) may be written as
\[
\theta_{\epsilon}(z + z_m) = \exp(i\pi F_{\epsilon}(-m)) \exp\left( \frac{|z|^2}{2} \right) \exp(\bar{z}_m z) \theta_{\epsilon}(z) ,
\] (4.6)
which is the functional equation for the theta functions\footnote{see eq. (4.2)} (see eq. (4.2)). The strict relation between the theta functions and the CS system is thus established.
We emphasize that such relation is obtained by considering the CS system corresponding to the admissible lattice \( L \).

A solution of (4.6) can be expressed as follows:\[6\]:

\[
\theta_\epsilon(z) = \sum_m e^{-i\pi F_\epsilon(m)} D(-\bar{z}_m) f(z) \tag{4.7a}
\]

\[
= \sum_m e^{-i\pi F_\epsilon(m)} \exp\left(-\frac{|z_m|^2}{2}\right) \exp\left(-\bar{z}_m z\right) f(z) , \tag{4.7b}
\]

where \( f(z) \) is an arbitrary entire function such that the series (4.7b) is convergent.

We now have all the ingredients necessary to establish the relation between \( q \)-WH algebra and theta functions. Let us make first explicit the dependence of the deformation parameter \( q \) on the lattice periods \( \omega_i \) by writing \( q = q_m = e^{\zeta_m} \), with \( \zeta_m \) a vector on the von Neumann lattice \( L \).

By setting \( z_m = (q_m - 1)z \), from eqs. (4.7b) and (3.11), (3.12) we obtain

\[
\theta_\epsilon(q_m z) = [a_{q_m}, \hat{a}_{q_m}] \theta_\epsilon(z) \tag{4.8}
\]

which, by use of (4.6), gives

\[
[a_{q_m}, \hat{a}_{q_m}] \theta_\epsilon(z) = \exp\left(i\pi F_\epsilon(-m)\right) \exp\left(-(1 - \bar{q}_m)(1 + q_m) \frac{|z|^2}{2}\right) \theta_\epsilon(z) . \tag{4.9}
\]

Moreover, use of (3.12) in eq. (4.7a) gives

\[
\theta_\epsilon(z) = \sum_m \exp\left(-i\pi F_\epsilon(m)\right) \exp\left((1 - \bar{q}_m)(1 + q_m) \frac{|z|^2}{2}\right) [a_{q_m}, \hat{a}_{q_m}] f(z) . \tag{4.10}
\]

Eqs. (4.8) ÷ (4.10) exhibit the relation between the \( q \)-algebra (2.3) and the theta functions: eq. (4.8) is the functional equation for the theta function corresponding to (4.2), (4.6) and, together with (4.9), shows the explicit action of \([a_q, \hat{a}_q]\) on \( \theta_\epsilon(z) \). Eq. (4.10) expresses the solution of (4.6) and of (4.8) (i.e. the theta function) in terms of the application of the operator \([a_{q_m}, \hat{a}_{q_m}]\) to the arbitrary entire function \( f(z) \).

The natural way to exhibit the topological as well as the dynamical meaning of this representation is to recall that theta functions are actually the quantum propagator for systems whose space of dynamical variables \( \mathcal{M} \), transitive under some compact Lie group \( \mathcal{G} \), is multiply connected\[14\]. In particular, in the simple case in which \( \mathcal{G} \sim SO(2) \), \( \mathcal{M} \) is just the 1-torus (circle) \( T^1 \). Assuming \( \mathcal{G} \sim SO(2) \otimes SO(2) \), due to the composition properties of theta functions,
or, equivalently, of the CS displacement operators (eq. (3.4)) would lead just to $\mathcal{M} \sim T^{(2)} \sim T^{(1)} \times T^{(1)}$, the 2-torus CS-state manifold of the $\theta$’s$^{[12]}$. In the former case (extension to the latter is immediate) the free dynamics, for which the orbits are geodesics and hence the lagrangian $\mathcal{L}$ is the manifold metrics, leads – by path integration in the universal covering space of $\mathcal{M}$ – to the propagator

$$ K(\varphi_1, t_1 | \varphi_2, t_2) = \int \mathcal{D}[\varphi(t)] e^{iS[\varphi(t)]/\hbar} \Rightarrow $$

$$ \Rightarrow K_\mu(\phi, \tau) = \frac{1}{2\pi} \exp \left( i \left[ \frac{\mu \phi}{2\pi} - \frac{\mu^2}{8\pi^2 \Theta} \right] \right) \vartheta_3 \left( \frac{1}{2} \phi - \frac{\mu}{8\pi \Theta}; -\frac{1}{2\pi \Theta} \right), \quad (4.11) $$

where $S[\varphi]$ denotes the action integral $S[\varphi(t)] = \int_{t_{in}}^{t_{fin}} \mathcal{L}[\varphi(t)] dt$, $\Theta$ is a dimensionless parameter depending on the system (e.g. the moment of inertia for a rigid rotator) and on the process duration $\tau = t_{fin} - t_{in}$; whereas $\phi = \varphi_{fin} - \varphi_{in}$ is the corresponding variation of the variable parametrizing the system state (e.g. the polar angle defining the position on $T^{(1)}$). $\mu$ is the phase coherence parameter whereby the global rotation $\mathcal{R}: \varphi_{fin} \rightarrow \varphi_{fin} + 2\pi$ implies $\mathcal{R}(K_\mu) = e^{i\mu} K_\mu$. $\vartheta_3(z; s)$ is the standard Jacobi theta function$^{[13]}$

$$ \vartheta_3(z; s) = \sum_{n=-\infty}^{+\infty} e^{i\pi sn^2} e^{2inz} . \quad (4.12) $$

Eq. (4.11) shows how the representative functions $\theta_e$ carry indeed dynamical as well as topological information, in that their structure is non-trivial only in view of the homotopical non-triviality of the configuration space.

It is finally interesting to realize the action of $[a_{q_m}, \hat{a}_{q_m}]$ on a conformal image $\tilde{\mathcal{F}}$ of $\mathcal{F}$ in terms of the variables $u$ and $\frac{d}{du}$ defined by

$$ u = \log z , \quad \frac{d}{du} = z \frac{d}{dz} . \quad (4.13) $$

These have canonical commutator $[u, \frac{d}{du}] = \mathbb{I}$ in $\tilde{\mathcal{F}}$. We have (cfr. (2.18)), by setting $q_m = e^{\zeta_m}$ and $f(z) = f(\exp(\log z)) = \tilde{f}(u)$,

$$ [a_{q_m}, \hat{a}_{q_m}] f(z) = q_m \frac{d}{du} f(z) = \exp(\zeta_m \frac{d}{du}) \tilde{f}(u) = \tilde{f}(u + \zeta_m) = f(q_m z) , \quad (4.14) $$

i.e. the commutator $[a_{q_m}, \hat{a}_{q_m}]$ generates a translation of $u$ by the lattice vector $\zeta_m$. Thus, in the $u, \frac{d}{du}$ variables, the relation of the $q$-algebra (2.3) with the CS generator $\mathcal{D}(\alpha_m)$, which acts indeed as a “translation” operator on the lattice $L$
is even more transparent. Also, the functional equation (4.8) for theta functions looks more familiar when written in terms of the $u$ variable in $\tilde{F}$:

$$\tilde{\theta}_\epsilon(u + \zeta_m) = [a_{q_m}, \hat{a}_{q_m}]\tilde{\theta}_\epsilon(u) \quad .$$ \hfill (4.15)

These results lead us once more to conclude that the existence of a quantum deformed dynamical algebra signals the presence of a lattice length in the theory and provides the natural framework for the physics of discretized systems.

5. $q$-Weyl-Heisenberg algebra and the squeezing generator

Let us consider the harmonic oscillator Hamiltonian

$$H = \frac{1}{2}(\hat{p}_z^2 + \hat{z}^2) \quad ,$$ \hfill (5.1)

where $\hat{p}_z = -i\frac{d}{dz}$ and $[\hat{z}, \hat{p}_z] = i$, over a Hilbert space of states identified with the space of entire analytic functions $\mathcal{F}$. Introduce, as customary, the operators

$$\alpha = \frac{1}{\sqrt{2}}(\hat{z} + i\hat{p}_z) \quad , \quad \alpha^\dagger = \frac{1}{\sqrt{2}}(\hat{z} - i\hat{p}_z) \quad , \quad [\alpha, \alpha^\dagger] = \mathbb{I} \quad ,$$ \hfill (5.2)

in terms of which

$$H = \alpha^\dagger \alpha + \frac{1}{2} \quad .$$ \hfill (5.3)

The ground state is described by the Gaussian wavefunction $\psi^0(z) = (\pi)^{-\frac{1}{4}}\exp(-\frac{z^2}{2})$. It is immediate to observe that, via the definition (5.2), we have

$$2z\frac{d}{dz}f(z) = [(\alpha^2 - \alpha^\dagger^2) - 1]f(z) \quad , \quad f \in \mathcal{F} \quad ,$$ \hfill (5.4)

and the operator $[a_q, \hat{a}_q]$ acts as the squeezing operator $\hat{S}(\zeta)$ \cite{15} on any state $\psi(z)$ (see also eq. (2.18)):

$$[a_q, \hat{a}_q]\psi(z) = \exp\left(\zeta z\frac{d}{dz}\right)\psi(z) = \frac{1}{\sqrt{q}}\exp\left(\frac{\zeta}{2}(\alpha^2 - \alpha^\dagger^2)\right)\psi(z)$$

$$= \frac{1}{\sqrt{q}}\hat{S}(\zeta)\psi(z) = \frac{1}{\sqrt{q}}\psi_s(z) \quad ,$$ \hfill (5.5)

with $q = e^\zeta$ and $\psi_s(z)$ denoting the squeezed state. In the case of the ground state $\psi^0_s(z) = \psi^0(qz) = \left(\frac{q}{\pi}\right)^{\frac{1}{4}}\exp(-\frac{q^2z^2}{2})$. The minimum Heisenberg uncertainty relation holds for $\psi^0(z)$ as well as for $\psi^0_s(z)$ and $\psi_s(z)$:

$$\Delta z\Delta p_z = \frac{1}{2} \quad ,$$ \hfill (5.6)
with \( \Delta z = \sqrt{\frac{1}{2}} = \Delta p_z \) for \( \psi^0(z) \) and \( \Delta z = \frac{1}{q} \sqrt{\frac{1}{2}}, \Delta p_z = q \sqrt{\frac{1}{2}} \) for \( \psi_s(z) \) (and of course \( \psi^0_s \)).

We thus conclude that the \( q \)-deformation parameter plays the rôle of squeezing parameter and the commutator \([a_q, \hat{a}_q]\) is, up to a numerical factor, the squeezing generator with respect to the operators \( \alpha \) and \( \alpha^\dagger \):

\[
[a_q, \hat{a}_q] = \frac{1}{\sqrt{q}} \exp \left( \frac{\zeta}{2} (\alpha^2 - \alpha^\dagger^2) \right).
\] (5.7)

On the other hand, it is well known\(^{[15]} \) that the right hand side of (5.7) is an \( SU(1,1) \) group element. In fact by defining \( K_+ = \frac{1}{2} \alpha^2, K_+ = \frac{1}{2} \alpha^\dagger \), \( K_z = \frac{1}{2} (\alpha^\dagger \alpha + \frac{1}{2}) = \frac{1}{2} \alpha \), one easily checks they close the algebra \( su(1,1) \).

Let us finally observe that in view of the holomorphy conditions on \( f(z) \)

\[
(z = x + iy)
\]

\[
\frac{d}{dz} f(z) = \frac{d}{dx} f(z) = -i \frac{d}{dy} f(z), \quad f \in \mathcal{F},
\] (5.8)

one finds

\[
z \frac{d}{dz} = x \frac{d}{dx} + iy \frac{d}{dy} = -ix \frac{d}{dy} + iy \frac{d}{dx} = xp_y - yp_x \equiv L,
\] (5.9)

with \( L \) an angular momentum operator. For \( \zeta = i\theta \), with \( \theta \) real, the commutator \([a_q, \hat{a}_q]\) acts then in \( \mathcal{F} \) as the \( U(1) \) group element

\[
[a_q, \hat{a}_q] = e^{i\theta L},
\] (5.10)

as already observed above.

6. Lattice Quantum Mechanics.

In this section we recall first the structure of Lattice Quantum Mechanics (LQM) in order to relate it to \( q \)-WH and construct lattice CS minimizing the lattice position-momentum uncertainty relation.

We limit ourselves to 1-dimensional lattice (extension to more dimensions is straightforward). 1-dimensional lattice quantum system is defined on the configurational Hilbert space \( \mathcal{G} = l^2(\epsilon \mathbb{Z}) \) where \( \mathbb{Z} \) denotes the set of integers \( n \) and \( \epsilon \) is the lattice spacing.

The hermitian position operator, \( \hat{x}_\epsilon \), is defined as

\[
\hat{x}_\epsilon f(x_n) = x_n f(x_n) = \epsilon n f(x_n), \quad f \in \mathcal{G},
\] (6.1)
whereas the hermitian lattice momentum operator $\hat{p}_\epsilon$ is defined by

$$\hat{p}_\epsilon f(x_n) = -i D_\epsilon f(x_n) \quad ,$$  \hspace{1cm} (6.2)

where $D_\epsilon$ is the symmetrized, finite difference gradient

$$D_\epsilon f(x_n) = (2\epsilon)^{-1} [f(x_{n+1}) - f(x_{n-1})] \quad .$$  \hspace{1cm} (6.3)

The dual momentum space representation of the above operators is therefore

$$\hat{x}_\epsilon f(k) = i \frac{d}{dk} f(k) \quad ,$$  \hspace{1cm} (6.4a)

$$\hat{p}_\epsilon f(k) = \epsilon^{-2} \sin(k\epsilon) f(k) \quad ,$$  \hspace{1cm} (6.4b)

respectively, where $f(k)$ is the Fourier conjugate of $f(x_n)$, $k$ belonging to the first Brillouin zone (BZ), $|k| \leq \pi/\epsilon$.

Over $G$ we have the following commutation relation between position and momentum

$$[\hat{x}_\epsilon, \hat{p}_\epsilon] f(x_n) = \frac{i}{2} (f(x_{n+1}) + f(x_{n-1})) \quad .$$  \hspace{1cm} (6.5)

The operators $\hat{P}_\epsilon = \cos(\epsilon \hat{p})$, $\hat{p} = -i \frac{d}{dx}$, and $\hat{x}_\epsilon$ generate the algebra $E(2)$:

$$[\hat{x}_\epsilon, \hat{p}_\epsilon] = i \hat{P} \quad , \quad [\hat{x}_\epsilon, \hat{P}] = -i \epsilon^2 \hat{p}_\epsilon \quad , \quad [\hat{P}, \hat{p}_\epsilon] = 0 \quad .$$  \hspace{1cm} (6.6)

Relations (6.6) follow of course from the simple observation that

$$f(x_{n\pm 1}) \equiv f(x_n \pm \epsilon) = \exp(\pm i \epsilon \hat{p}) f(x_n) \quad ,$$  \hspace{1cm} (6.7)

provided the derivatives of $f(x_n)$ exist at any order.

Let us notice that the above $E(2)$ algebra contracts to the Weyl–Heisenberg algebra in the limit $\epsilon \to 0$: the discrete lattice spacing $\epsilon$ plays thus the rôle of deformation parameter.

Upon introducing, in momentum space, the angle $\phi = k\epsilon$, $-\pi \leq \phi \leq \pi$, on the unit circle, we set

$$L_3 = -i \frac{d}{d\phi} = -i \frac{d}{\epsilon dk} \quad , \quad L_1 = \cos \phi \quad , \quad L_2 = \sin \phi \quad .$$  \hspace{1cm} (6.8)

The algebra (6.6) takes then the usual form of the (angular momentum) $E(2)$ algebra:

$$[L_3, L_1] = i L_2 \quad , \quad [L_3, L_2] = -i L_1 \quad , \quad [L_1, L_2] = 0 \quad .$$  \hspace{1cm} (6.9)
Following standard procedures\textsuperscript{[17,18]}, one gets the uncertainty inequalities

\[
\Delta^2(\hat{x}_\epsilon) \Delta^2(\hat{p}_\epsilon) \geq \frac{1}{4}(\cos(k\epsilon))^2 , \\
\Delta^2(\hat{x}_\epsilon) \Delta^2(\cos(k\epsilon)) \geq \frac{1}{4}\epsilon^2(\sin(k\epsilon))^2 ,
\]

where \( \langle \hat{A} \rangle = \int dk \Psi^*(k) \hat{A} \Psi(k) \) denotes quantum expectation on the lattice and \( \Delta^2(\hat{A}) = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2 \) the (square) variance. The continuum limit \( \epsilon \to 0 \) corresponds to opening the circle into a line. In \( d \) dimensions the limit \( \epsilon \to 0 \), is an isometric and conformal mapping of the torus on the plane (decompactification)\textsuperscript{[16]}. The states minimizing the uncertainty products of eqs.(6.10) must satisfy\textsuperscript{[17,18]}

\[
(\hat{x}_\epsilon + i\gamma \hat{p}_\epsilon) \Psi = \lambda \Psi ,
\]

where \( \lambda = \langle \hat{x}_\epsilon \rangle + i\gamma \langle \hat{p}_\epsilon \rangle \), and \( \gamma \) is connected with the variances \( \Delta (\hat{x}_\epsilon) \), \( \Delta (\hat{p}_\epsilon) \) of position and momentum. Relation (6.11) becomes, in momentum space,

\[
\left[ \frac{d}{d(\epsilon k)} + \bar{\gamma} \sin(\epsilon k) \right] \Psi(k) = -i\bar{\lambda} \Psi(k) ,
\]

where \( \bar{\lambda} = \lambda \epsilon^{-1} \), \( \bar{\gamma} = \gamma \epsilon^{-2} \). Its solution is

\[
\Psi(k) = G \exp\left[ \bar{\gamma} \cos(\epsilon k) - i\bar{\lambda} \epsilon k \right] ,
\]

where the normalization constant \( G \) is given by \( G = \frac{2\pi}{\epsilon} I_0(2\bar{\gamma}) \), \( I_0 \) denoting the modified Bessel function of the first kind of order 0. Notice that, in the continuum limit \( \epsilon \to 0 \), the Fourier transform \( \tilde{\Psi}(x) \) of (6.13) becomes

\[
\tilde{\Psi}(x) = (\gamma \pi)^{-\frac{3}{2}} \exp\left\{ - \left[ (2\gamma)^{-1}(x - \langle \hat{x}\rangle)^2 + i(\bar{\rho})(x - \langle \hat{x}\rangle) \right] \right\} ,
\]

which is just the minimum uncertainty wavefunction given by Schrödinger\textsuperscript{[19]}. By setting \( z = \langle x \rangle + i\gamma \langle p \rangle \), eq. (6.14) defines the usual coherent states\textsuperscript{[6]}. The wave-function (6.13) is the coherent state for a system with discretized position and momentum or, equivalently, endowed with some periodic constraint.

7. \textit{q}-\textit{Weyl-Heisenberg algebra and lattice quantum mechanics}

We aim now to show that also the structure of LQM is underlined by \( q \)-WH. In order to do so, we introduce a conformal image \( \tilde{\mathcal{F}} \) of the configurational Hilbert space \( \mathcal{G} \) by defining \( z = e^{ix} \), in terms of which

\[
i\hat{p} f(x_n) = \frac{d}{dx} f(x_n) = iz \frac{d}{dz} \tilde{f}(z_n) , \quad z_n = e^{inx} , \quad \tilde{f} \in \tilde{\mathcal{F}} .
\]

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The assumption that the derivatives of \( f(x_n) \) of any order exist (cf. eq (6.7)) implies of course that \( \tilde{f}(z_n) \) is an analytic function, which we also assume to be entire, on the unit circle \( C \). \( z_n = e^{ix_n} \) is of course a discrete set of points in the \( z \)-plane belonging to \( C \). The function \( \tilde{f}(z_n) \) may thus be represented by means of the expansion in the basis \( u_m(z_n) \) as in eq. (2.6).

Recalling that

\[
[p, e^{ix}] = e^{ix}, \quad [p, e^{-ix}p] = -e^{-ix}p, \quad [e^{-ix}p, e^{ix}] = 1, \quad (7.2)
\]

provides a realization of the WH algebra (2.1) on the unit circle, suggests that the realization (2.4) of the FBR may be adopted in \( \tilde{F} \) for the LQM as well. Eqs. (6.3), (6.5) and (6.7) are now written, respectively, as

\[
i\hat{p}\tilde{f}(x_n) = \mathcal{D}_\epsilon f(x_n) = \frac{i}{2\log q} \left(q^N - q^{-N}\right) \tilde{f}(z_n) = i\epsilon^{-1}\sin(\epsilon N)\tilde{f}(z_n) \quad , \quad (7.3)
\]

\[
[\hat{x}_\epsilon, \hat{p}_\epsilon]f(x_n) = i\cos(\epsilon N)\tilde{f}(z_n) \quad , \quad (7.4)
\]

and

\[
f(x_{n+1}) = e^{i\epsilon \hat{p}}f(x_n) = q^N \tilde{f}(z_n) = \tilde{f}(qz_n) = \tilde{f}(z_{n+1}) \quad , \quad (7.5)
\]

where we have set \( q = e^{i\epsilon} \), and \( q^N \) was introduced through the \( q \)-WH algebra (2.3).

This makes clear that the algebraic structure of LQM is intimately related with the \( q \)-WH algebra, the \( q \)-deformation parameter being determined by the discrete lattice length \( \epsilon = -i\log q \).

The same conclusion can be reached constructing LQM in momentum space. In such a case one may consider a conformal image \( \tilde{\mathcal{H}} \) of the Hilbert space in the momentum representation by setting \( z = e^{i\phi} \), where we introduced once more the angular variable over \( C \), \( \phi = k\epsilon \), \( -\pi \leq \phi \leq \pi \), so that (cf. eq.(6.8))

\[
-i\frac{d}{d\phi} = -\frac{i}{\epsilon}\frac{d}{dk} = z\frac{d}{dz} \quad . \quad (7.6)
\]

The functions belonging to \( \tilde{\mathcal{H}} \) are assumed to be entire analytic functions for which expansions of the form (2.6) hold. On the other hand,

\[
L_3f(\phi) = -i\frac{d}{d\phi}f(\phi) = z\frac{d}{dz}\tilde{f}(z) = N\tilde{f}(z) \quad , \quad \tilde{f} \in \tilde{\mathcal{H}} \quad , \quad (7.7)
\]

and

\[
f(\phi + \epsilon) = e^{i\epsilon L_3}f(\phi) = q^N \tilde{f}(z) = \tilde{f}(qz) \quad . \quad (7.8)
\]
The $E(2)$ algebra (6.9) is now realized in the momentum FBR as

$$[L_1, L_3] \tilde{f}(z) = -i L_2 \tilde{f}(z), \ [L_2, L_3] \tilde{f}(z) = i L_1 \tilde{f}(z), \ [L_1, L_2] \tilde{f}(z) = 0, \quad (7.9)$$

with $\tilde{f} \in \mathcal{H}$, and the identifications (see eq.(6.8))

$$L_1 = \frac{z + \bar{z}}{2}, \ L_2 = \frac{z - \bar{z}}{2i}, \ L_3 = z \frac{d}{dz}, \ L_+ = z, \ L_- = \bar{z}. \quad (7.10)$$

One can check that in this representation $[a_q, \hat{a}_q]$ is nothing but the $e^{i\epsilon L_3}$ group element of $E(2)$, which recovers eq. (5.12). We also note that $z^n = e^{in\phi}$, $n$ integer, is the eigenfunction of $L_3$ associated with the integer eigenvalue $n$ of the number operator in the FBR.

$$L_3 z^n = N z^n = n z^n. \quad (7.11)$$

The functions $z = e^{ik\phi}$ and $z^n$ play also a crucial rôle in the Bloch functions theory$^{[20]}$. Suppose we have a periodic potential $V(x_n) = V(x_n + \epsilon)$ on the lattice. Bloch theorem then ensures the existence of solutions of the related Schrödinger equation of the form

$$\psi(x_n) = e^{\pm ikx_n} v_k(x_n) \quad , \quad (7.12)$$

with $v_k(x_n) = v_k(x_n + \epsilon)$. $\psi(x_n)$ is the Bloch function.

Let us limit ourself for simplicity to considering the plus sign in (7.12). $\psi(x_n)$ has the property

$$\psi(x_n + \epsilon) = e^{ik\epsilon} \psi(x_n) = z \psi(x_n) \quad . \quad (7.13)$$

We thus see that the choice of the variable $z = e^{ik\epsilon}$ turns out to be natural in the case of periodic potentials:

$$\psi(x_n) = z^n v_k(x_n) \quad , \quad \psi(x_n + \epsilon) = z^{n+1} v_k(x_n) \quad . \quad (7.14)$$

Since $z^n = (z_n)^k$, from eq. (7.5)

$$q_N (z_n)^k = (qz_n)^k = (e^{i\epsilon} e^{ix_n})^k = e^{i\epsilon(n+1)} = z^{n+1}, \quad (7.15)$$

where $q_N$ is understood as defined on $\tilde{F}$. We thus have in $\tilde{F}$

$$\psi(x_n + \epsilon) = [a_q, \hat{a}_q] (z_n)^k u_k(x_n) = [a_q, \hat{a}_q] \psi(x_n) \quad , \quad (7.16)$$

namely the condition implementing the Bloch function periodicity features is realized by the same operator in $q$-WH algebra acting, in the FBR, as dilatation.
(see eq.(2.18) and compare eq. (7.13) with (7.14)-(7.16)). This shows that Bloch functions provide in fact a representation of the $q$-WH algebra.

8. Conclusions

In this paper we have considered the $q$-deformation of the WH algebra in connection with typical problems in QM: discretized systems, coherent states, squeezing, lattice quantum mechanics, finite difference operators, periodic systems. These have been proposed as physically relevant examples where $q$-deformation play a rôle. Such a collection of applications is not only interesting on its own, but also as a laboratory where problems characterized by the common feature of a discrete structure are treated in the framework of (entire) analytic functions theory.

The underlying philosophy is that whenever one deals with a quantum system defined on a countable set of degrees of freedom, then one has to work in the space of the related analytical functions and the structure of the operator algebra is the $q$-deformed one. In this sense, the word deformation does not sound as the most appropriate, since $q$-algebras provide an algorithm of wide physical application, and not an exception for deformed situations. Such a general philosophy appears to find very concrete support in the results discussed in this paper.

From the point of view of group theory, we used the well known mapping of a $q$-algebra into the universal enveloping algebra of the corresponding Lie structure; to be specific, the mapping of finite difference operators into functions of differential operators, which can be indeed achieved only by operating on $C^\infty$ functions. This was the main reason to work with FBR and to introduce the conformal image of both configuration and momentum space in the study of LQM.

An interesting and natural development, namely the extension from QM to QFT by considering the infinite volume limit of the lattice system, is at present under study. We conjecture that the unitarily inequivalent representations of the canonical commutation relations are parametrized by the $q$-deformation parameter in such a limit. Different values of the lattice spacing are thus described by inequivalent representations. In this framework finite temperature and dissipative systems\[^{21}\] may find an appropriate unified description.

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