Thinking Twice about Second-Price Ad Auctions

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Abstract

A number of recent papers have addressed the algorithmic problem of allocating advertisement space for keywords in sponsored search auctions so as to maximize revenue, most of which assume that pricing is done via a first-price auction. This does not realistically model the Generalized Second Price (GSP) auction used in practice, in which bidders pay the next-highest bid for keywords that they are allocated. Towards the goal of more realistically modelling these auctions, we introduce the Second-Price Ad Auctions problem, in which bidders’ payments are determined by the GSP mechanism.

We show that the complexity of the Second-Price Ad Auctions problem is quite different than that of the more studied First-Price Ad Auctions problem. First, unlike the first-price variant, for which small constant-factor approximations are known, we show that it is NP-hard to approximate the Second-Price Ad Auctions problem to any non-trivial factor, even when the bids are small compared to the budgets. Second, we show that this discrepancy extends even to the 0-1 special case that we call the Second-Price Matching problem (2PM). In particular, offline 2PM is APX-hard, and for online 2PM there is no deterministic algorithm achieving a non-trivial competitive ratio and no randomized algorithm achieving a competitive ratio better than 2. This stands in contrast to the results for the analogous special case in the first-price model, the standard bipartite matching problem, which is solvable in polynomial time and which has deterministic and randomized online algorithms achieving better competitive ratios. On the positive side, we provide a 2-approximation for offline 2PM and a 5.083-competitive randomized algorithm for online 2PM. The latter result makes use of a new generalization that we prove of a classic result on the performance of the “Ranking” algorithm for online bipartite matching.

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1 Introduction

The rising economic importance of online sponsored search advertising has led to a great deal of research focused on developing its theoretical underpinnings. (See, e.g., [18] for a survey). Since search engines such as Google, Yahoo! and MSN depend on sponsored search for a significant fraction of their revenue, a key problem is how to optimally allocate ads to keywords (user searches) so as to maximize search engine revenue [1, 2, 3, 5, 6, 14, 15, 20, 21, 24]. Most of the research on the dynamic version of this problem assumes that once the participants in each keyword auction are determined, the pricing is done via a first-price auction; in other words, bidders pay what they bid. This does not realistically model the standard mechanism used by search engines, called the Generalized Second Price mechanism (GSP) [10, 25].

In an attempt to model reality more closely, we study the Second-Price Ad Auctions problem, which is the analogue of the above allocation problem when bidders’ payments are determined by the GSP mechanism and there is only one slot for each keyword. The GSP mechanism for a given keyword auction reduces to a second-price auction when there is one slot per keyword – given the participants in the auction, it allocates the advertisement slot to the highest bidder, charging that bidder the bid of the second-highest bidder.

In the Second-Price Ad Auctions problem, we assume that there is a set of keywords $U$ and a set of bidders $V$, where each bidder $v \in V$ has a known daily budget $B_v$ and a non-negative bid $b_{u,v}$ for every keyword $u \in U$. The keywords are ordered by their arrival time, and as each keyword arrives, the algorithm (i.e., the search engine) chooses the bidders to participate in that particular auction and runs a second-price auction with respect to those participants. Thus, instead of selecting one bidder for each keyword, two bidders need to be selected by the algorithm. Of these two bidders, the bidder with the higher bid (where bids are always reduced to the minimum of the actual bid and the bidders’ remaining budget) is allocated that keyword’s advertisement slot at the price of the other bid.

This process results in an allocation and pricing of the advertisement slots associated with each of the keywords. The goal is to select the bidders participating in each auction to maximize the total profit extracted by the algorithm. For an example instance of this problem, see Figure 1.

We note that for simplicity of presentation we have chosen to present the model, the algorithms and the results as if the bidders are competing for a single slot. Obviously, our hardness results hold for the multi-slot problem as well.

1.1 Our Results

We begin by considering the offline version of the Second-Price Ad Auctions problem, in which the algorithm knows all of the original bids of the bidders (Section 3). Our main result here is that it is NP-hard to approximate the optimal solution to this problem to within a factor better than $m/R_{\text{min}}$, where $m$ is the number of keywords and $R_{\text{min}} \geq 1$ is a constant independent of $m$ such that no bidder bids more than $1/R_{\text{min}}$ of its initial budget on any keyword. Thus, even when bids are small compared to budgets, it is not possible in the worst case to get a good approximation to the optimal revenue. (We show that it is trivial to get a matching approximation algorithm.) This result stands in sharp contrast to the standard First-Price Ad Auctions problem, for which there is a 4/3-approximation to the offline problem [6, 24] (even for $R_{\text{min}} = 1$), and an $e/(e - 1)$-competitive algorithm to the online problem when bids are small compared to budgets [5, 21] (i.e., as $R_{\text{min}} \to \infty$).
We then turn our attention to a theoretically appealing special case that we call Second-Price Matching. In this version of the problem, all bids are either 0 or 1 and all budgets are 1. As before, the keywords are ordered by arrival time, and a keyword \( u \) can be matched to a bidder \( v \) with a profit of 1 only if \( b_{u,v} = 1 \), there is a distinct bidder \( v' \) with \( b_{u,v'} = 1 \), and neither \( v \) nor \( v' \) has been matched to a keyword that arrived before \( u \). For an example instance of this problem, see Figure 2.

Recall that for First-Price Matching or, as we know and love it, maximum bipartite matching, the offline problem can of course be solved optimally in polynomial time, whereas for the online problem we have the trivial 2-competitive deterministic greedy algorithm and the celebrated \( \epsilon/(\epsilon - 1) \)-competitive randomized algorithm due to Karp, Vazirani and Vazirani [16], both of which are best possible.

In contrast, we show that the Second-Price Matching problem is APX-hard (Section 4.1). We also give a 2-approximation algorithm for the offline problem (Section 4.2). We then turn to the online version of the problem. Here, we show that no deterministic online algorithm can get a competitive ratio better than \( m \), where \( m \) is the number of keywords in the instance and that no randomized online algorithm can get a competitive ratio better than 2 (Section 5.1). On the other hand, we present a randomized online algorithm that achieves a competitive ratio of \( 4\sqrt{\epsilon}/(\sqrt{\epsilon} - 1) \approx 5.08 \) (Section 5.2). To obtain this competitive ratio, we prove a generalization of the result due to Karp, Vazirani, and Vazirani [16] and Goel and Mehta [15] that the Ranking

![Figure 1](image-url)

Figure 1: An example of the Second-Price Ad Auctions problem: The nodes in \( U \) are keywords and the nodes in \( V \) are bidders. The number immediately to the right of each bidder represents its remaining budget, and the number next to each edge connecting a bidder to a keyword represents the bid of that bidder for that keyword. Figure (a) shows the situation when the first keyword arrives. For this keyword, the search engine selects the first bidder, whose bid is 4, and the third bidder, whose bid is 3. The keyword is allocated to the first bidder at a price of 3, thereby reducing that bidder’s budget by 3. Figure (b) shows the situation when the second keyword arrives. The first and fourth bidders are selected, and the keyword is allocated to the fourth bidder at a price of 2, thereby reducing its remaining budget to 2. As each keyword arrives, the bid of a bidder for that keyword is adjusted to the minimum of its original bid and its remaining budget. Thus, for example, when the third keyword arrives, as shown in Figure (c), the bid of the first bidder for that keyword is adjusted from its original value of 6 down to 3 since that is its remaining budget. The two bidders then selected are the first and the second, and the keyword is allocated to the second bidder at a price of 3.
algorithm for online bipartite matching achieves a competitive ratio of $e/(e-1)$.

1.2 Related Work

As discussed above, the related First-Price Ad Auctions problem\(^1\) has received a fair amount of attention. Mehta et al. [21] presented an algorithm for the online version that achieves an optimal competitive ratio of $e/(e-1)$ for the case when the bids are much smaller than the budgets (i.e., when $R_{min} \to \infty$), a result also proved by Buchbinder et al. [5]. When there is no restriction on the values of the bids relative to the budgets (i.e., when $R_{min}$ can be as low as 1), the best known competitive ratio is 2 [19]. For the offline version of the problem, a sequence of papers [19] [2] [11] [24] [6] culminating in a paper by Chakrabarty and Goel, and independently, a paper by Srinivasan, show that the offline problem can be approximated to within a factor of $4/(4-1/R_{min})$ and that there is no polynomial time approximation algorithm that achieves a ratio better than 16/15 unless $P = NP$ [6].

The most closely related papers to this one are the works of Abrams, Mendelevitch and Tomlin [1], and of Goel, Mahdian, Nazerzadeh and Saberi [14]. The latter looks at the online allocation problem when the search engine is committed to charging under the GSP scheme, with multiple slots per keyword. They study two models, both of which differ from the model studied in this paper, even for one slot. Their first model, called the strict model, allows a bidder’s bid to be above his remaining budget, as long as the remaining budget is strictly positive. However, as in our model, when a bidder is allocated a slot, that bidder is never charged more than his remaining budget. Thus, in the strict model, a bidder $v$ with a negligible amount of remaining budget can

\(^1\)This problem has also been called the Adwords problem [21] and the Maximum Budgeted Allocation problem [4] [4] [24]. It is an important special case of SMW [11] [11] [17] [19] [22] [26], the problem of maximizing utility in a combinatorial auction in which the utility functions are submodular, and is also related to the Generalized Assignment Problem (GAP) [7] [11] [12] [23].
Figure 3: In this example, \( R_{\text{min}} \) is equal to a constant \( c \), i.e., every bid is at most \( 1/c \) of the budget of the bidder. In the strict model, all keywords except the first must be allocated to the second bidder at a price of \( k \) (or the remaining budget if it’s smaller). Thus, the total profit on this input for the strict model is at most \( ck + 3 \). On the other hand, in our model, if we allocate the first keyword to the second bidder, and then the next \( c - 1 \) keywords to the second bidder, that bidder’s budget is reduced to \( k - 1 \). Thus, all of the remaining keywords can be allocated to the lower bidder at a price of \( k - 1 \), for a total revenue exceeding \( ck(k - 1) \). For \( k \) large, this ratio is \( \Omega(k) = \Omega(m) \).

keep his bids high indefinitely, and as long as bidder \( v \) is never allocated another slot, this high bid can determine the prices other bidders pay on many keywords. Their second model, called the non-strict model, differs from the strict model in that bidders can keep their bids positive even after their budget is depleted. Thus, even after a bidder’s budget is depleted, that bidder can determine the prices that other bidders pay on keywords indefinitely. However, a bidder is never charged more than his remaining budget for a slot. Therefore, if a bidder is allocated a slot after his budget is fully depleted, it gets the slot for free.

Under the assumption that bids are small compared to budgets, Goel et al. present a \( e/(e - 1) \)-competitive algorithm for the non-strict model and a 3-competitive algorithm for the strict model. They also show that \( OPT_{\text{strict}} \leq OPT_{\text{non-strict}} \leq 2OPT_{\text{strict}} \), where these quantities refer to the optimal offline revenue of the search engine in their models. Their algorithms build on the linear programming formulation of Abrams et al. [1] for the offline version of the strict model.

Interestingly, neither of their models or our model completely dominates the others in terms of the optimal offline revenue. However, it is fairly easy to show that neither \( OPT_{\text{strict}} \) nor \( OPT_{\text{non-strict}} \) are ever more than a constant factor larger than the optimal offline revenue from our model (see Appendix A for a proof of this). On the other hand, the optimal offline revenue in our model can be \( \Omega(m) \) times as big as \( OPT_{\text{non-strict}} \) and \( OPT_{\text{strict}} \), where \( m \) is the number of keywords, and this holds even when \( R_{\text{min}} \) is a large constant \( c \) (see Figure 3). This is not surprising, given the strong hardness of approximation in our model, versus the fact that constant factor approximations are available, even online, in theirs.

Notice also, that for the special case of Second-Price Matching the strict model and our model

\[ \text{In terms of gaming the system, this would be a great way for a bidder to potentially force his competitors to pay a lot for slots without backing those payments up with budget. This effect is even worse for the non-strict model.} \]
are identical, whereas the non-strict model reduces to standard maximum matching (on all those keywords which have at least two bidders bidding on them).

We feel that our model, in which bidders are not allowed to bid more than their remaining budget, is more natural then the strict and non-strict models. It seems inherently unfair that a bidder with negligible or no budget should be able to indefinitely set high prices for other bidders. We recognize of course that there may be technical issues involving the scale and distributed nature of search engines that could make it difficult to implement our model precisely.

2 Model and Notation

We define the Second-Price Ad Auctions (2PAA) problem formally as follows. The input is a set of ordered keywords $U$ and bidders $V$. Each bidder $v \in V$ has a budget $B_v$ and a nonnegative bid $b_{u,v}$ for every keyword $u \in U$. We assume that all of bidder $v$’s bids $b_{u,v}$ are less than or equal to $B_v$.

Let $B_v(t)$ be the remaining budget of bidder $v$ immediately after the $t$-th keyword is processed (so $B_v(0) = B_v$ for all $v$), and let $b_{u,v}(t) = \min(b_{u,v}, B_v(t))$. (Both quantities are defined inductively.) A solution (or second-price matching) to 2PAA chooses for the $t$-th keyword $u$ a pair of bidders $v_1$ and $v_2$ such that $b_{u,v_1}(t-1) \geq b_{u,v_2}(t-1)$, allocates the slot for keyword $u$ to bidder $v_1$ and charges bidder $v_1$ a price of $p(t) = b_{u,v_1}(t-1)$, the bid of $v_2$. (We say that $v_1$ acts as the first-price bidder for $u$ and $v_2$ acts as the second-price bidder for $u$.) The budget of $v_1$ is then reduced by $p(t)$, so $B_{v_1}(t) = B_{v_1}(t-1) - p(t)$. For all other bidders $v \neq v_1$, $B_v(t) = B_v(t-1)$. The final value of the solution is $\sum p(t)$, and the goal is to find a feasible solution of maximum value.

In the offline version of the problem, all of the bids are known to the algorithm beforehand, whereas in the online version of the problem, keyword $u$ and the bids $b_{u,v}$ for each $v \in V$ are revealed only when keyword $u$ arrives, at which point the algorithm must irrevocably map $u$ to a pair of bidders without knowing the bids for the keywords that will arrive later.

The special case referred to as Second-Price Matching (2PM) is the special case where $b_{u,v}$ is either 0 or 1 for all $(u,v)$ pairs and $B_v = 1$ for all $v$. Perhaps, however, it is more intuitive to think of it as a variant on maximum bipartite matching. Viewing it this way, the input is a bipartite graph $G = (U \cup V, E)$, and the vertices in $U$ must be matched, in order, to the vertices in $V$ such that the profit of matching $u \in U$ to $v \in V$ is 1 if and only if there is at least one additional vertex $v' \in V$ that is a neighbor of $u$ and is unmatched at that time.

Note that in 2PM, a keyword can only be allocated for profit if its degree is at least two. Therefore, we assume without loss of generality that for all inputs of 2PM, the degree of every keyword is is at least two.

For an input to 2PAA, let $R_{\min} = \min_{u,v} B_v/b_{u,v}$, and let $m = |U|$ be the number of keywords.

3 Hardness of Approximation of 2PAA

In this section, we show that it is NP-hard to approximate 2PAA to a factor better than $m/R_{\min}$ for any constant $R_{\min}$ independent of $m$ (Theorem 1) and provide a trivial algorithm with an approximation guarantee that meets this factor.

For a constant $c \geq 1$, let 2PAA$(c)$ be the version of 2PAA in which we are promised that $R_{\min} \geq c$. The proof of the following theorem (presented in full in Appendix C.1) uses a gadget similar to the instance presented in Figure 3. This gadget exploits the sensitivity of the problem to
small changes in budget. In particular, if the budget of a key bidder is reduced by a small amount,
then the optimal revenue is much higher than if it is not reduced. We prove our inapproximability
result by setting up the reduction so that the budget of this bidder is reduced if and only if the
answer to the problem we reduce from is “yes.”

**Theorem 1.** Let $c \geq 1$ be a constant integer. For any constant $c' > c$, it is NP-hard to approximate
$2PAA(c)$ to a factor of $m/c'$.

*Proof.* See Appendix C.1

**Theorem 2.** Let $c \geq 1$ be a constant integer. There is an $m/c$-approximation to $2PAA(c)$.

*Proof.* For each keyword $u \in U$, let $s_u$ be the second-highest bid for $u$. Consider the algorithm
that selects the $c$ keywords with the highest values of $s_u$ and then allocates these keywords to
get $s_u$ for each of them (i.e., chooses the two highest bidders for $u$). Since no bidder bids more
than $1/c$ of its budget for any keyword, no bids are reduced from their original values during this
allocation. Hence, the profit of this allocation is at least $(c/m) \sum_{u \in U} s_u$. Since the value of the
optimal solution cannot be larger than $\sum_{u \in U} s_u$, it follows that this is an $m/c$-approximation to
$2PAA(c)$. \hfill \Box

## 4 Offline Second-Price Matching

In this section, we turn our attention to the offline version of the special case of Second-Price
Matching (2PM). First, in Section 4.1, we show that 2PM is APX-hard. This stands in contrast
to the maximum matching problem, the corresponding special case of the First-Price Ad Auctions
problem, which is solvable in polynomial time. Second, in Section 4.2, we give a 2-approximation
for 2PM.

### 4.1 Hardness of Approximation

To prove our hardness result for 2PM, we use the following result due to Chlebík and Chelbková.

**Theorem 3** (Chlebík and Chelbková [8]). It is NP-hard to approximate Vertex Cover on 4-regular
graphs to within $53/52$.

The precise statement of our hardness result is the following theorem.

**Theorem 4.** It is NP-hard to approximate 2PM to within a factor of $364/363$.

*Proof.* See Appendix C.2 \hfill \Box

### 4.2 A 2-Approximation Algorithm

Consider an instance $G = (U \cup V, E)$ of the 2PM problem. We provide an algorithm that first
finds a maximum matching $f : U \rightarrow V$ and then uses $f$ to return a second-price matching that
contains at least half of the keywords matched by $f$.

Given a matching $f$, call an edge $(u, v) \in E$

4Note that $f$ is a partial function.
otherwise. Recall that we have assumed without loss of generality that the degree of every keyword in $U$ is at least two. Therefore, every keyword $u \in U$ that is matched by $f$ must have at least one up-edge or down-edge. Theorem 5 shows that the following algorithm, called ReverseMatch, is a 2-approximation for 2PM.

**ReverseMatch Algorithm:**

| Initialization: | Find an arbitrary maximum matching $f: U \to V$ on $G$. |
|-----------------|-------------------------------------------------------|
| Constructing a 2nd-price matching: |
| Consider the matched keywords in reverse order of their arrival. |
| For each keyword $u$: |
| If keyword $u$ is adjacent to a down-edge $(u, v)$: |
| Assign keyword $u$ to bidder $f(u)$ (with $v$ acting as the second-price bidder). |
| Else: |
| Choose an arbitrary bidder $v$ that is adjacent to keyword $u$. |
| Remove the edge $(f^{-1}(v), v)$ from $f$. |
| Assign keyword $u$ to bidder $f(u)$ (with $v$ acting as the second-price bidder). |

**Theorem 5.** The ReverseMatch algorithm is a 2-approximation.

**Proof.** See Appendix C.3.

5 Online Second-Price Matching

In this section, we consider the online 2PM problem, in which the keywords arrive one-by-one and must be matched by the algorithm as they arrive. We start, in Section 5.1, by giving a simple lower bound showing that no deterministic algorithm can achieve a competitive ratio better than $m$, the number of keywords. Then we move to randomized online algorithms and show that no randomized algorithm can achieve a competitive ratio better than 2. In Section 5.2, we provide a randomized online algorithm that achieves a competitive ratio of $4\sqrt{e}/(\sqrt{e} - 1) \approx 5.083$.

5.1 Lower Bounds

The following theorem establishes our lower bound on deterministic algorithms, which matches the trivial algorithm of arbitrarily allocating the first keyword to arrive, and refusing to allocate any of the remaining keywords.

**Theorem 6.** For any $m$, there is an adversary that creates a graph with $m$ keywords that forces any deterministic algorithm to get a competitive ratio no better than $1/m$.

**Proof.** The adversary shows the algorithm a single keyword (keyword 1) that has two adjacent bidders, $a_1$ and $b_2$. If the algorithm does not match keyword 1 at all, a new keyword 2 arrives that is adjacent to two new bidders $a_2$ and $b_2$. The adversary continues in this way until either $m$ keywords arrive or the algorithm matches a keyword $k < m$. In the first case, the algorithm’s performance is at most 1 (because it might match keyword $m$), whereas the adversary can match all $m$ keywords. Hence, the ratio is at most $1/m$.  

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In the second case, the adversary continues as follows. Suppose without loss of generality that
the algorithm matches keyword \( k \) to \( a_k \). Then each keyword \( i \), for \( k + 1 \leq i \leq m \), has one edge
to \( a_k \) and one edge to a new bidder \( c_k \). Since the algorithm cannot match any of these keywords
for a profit, its performance is 1. The adversary can clearly match each keyword \( i \) for profit, for
\( 1 \leq i \leq k - 1 \), and if it matches keyword \( k \) to \( b_k \), then it can use \( a_k \) as a second-price bidder for
the remaining keywords to match them all to the \( c_i \)’s for profit. Hence, the adversary can construct
a second-price matching of size at least \( m \).

The following theorem establishes our lower bound of 2 for any (randomized) online algorithm
for 2PM.

**Theorem 7.** The competitive ratio of any randomized algorithm for 2PM must be at least 2.

**Proof.** See Appendix C.4.

### 5.2 A Randomized Competitive Algorithm

In this section, we provide an algorithm that achieves a competitive ratio of \( 2\sqrt{e}/(\sqrt{e} - 1) \approx 5.083 \). The result builds on a new generalization of the result that the Ranking algorithm for online
bipartite matching achieves a competitive ratio of \( e/(e - 1) \approx 1.582 \). This was originally shown by
Karp, Vazirani, and Vazirani [16], though a mistake was recently found in their proof by Krohn
and Varadarajan and corrected by Goel and Mehta [15].

The online bipartite matching problem is merely the first-price version of 2PM, i.e., the problem
in which there is no requirement for there to exist a second-price bidder to get a profit of 1 for a
match. The Ranking algorithm chooses a random permutation on the bidders \( V \) and uses that to
choose matches for the keywords \( U \) as they arrive. This is described more precisely below.

| Ranking Algorithm: |
|--------------------|
| **Initialization:** |
| Choose a random permutation (ranking) \( \sigma \) of the bidders \( V \). |
| **Online Matching:** |
| Upon arrival of keyword \( u \in U \):
  | Let \( N(u) \) be the set of neighbors of \( u \) that have not been matched yet. |
  | If \( N(u) \neq \emptyset \), match \( u \) to the bidder \( v \in N(u) \) that minimizes \( \sigma(v) \). |

Karp, Vazirani, and Vazirani, and Goel and Mehta prove the following result.

**Theorem 8** (Karp, Vazirani, and Vazirani [16] and Goel and Mehta [15]). The Ranking algorithm
for online bipartite matching achieves a competitive ratio of \( e/(e - 1) + o(1) \).

In order to state our generalization of this result, we define the notion of a **left \( k \)-copy** of a
bipartite graph \( G = (U \cup V, E) \). Intuitively, a left \( k \)-copy of \( G \) makes \( k \) copies of each keyword
\( u \in U \) such that the neighborhood of a copy of \( u \) is the same as the neighborhood of \( u \). More
precisely, we have the following definition.

**Definition 9.** Given a bipartite graph \( G = (U_G \cup V, E_G) \), define a **left \( k \)-copy** of \( G \) to be a graph
\( H = (U_H \cup V, E_H) \) for which \( |U_H| = k|U_G| \) and for which there exists a map \( \zeta : U_H \to U_G \) such that

- for each \( u_G \in U_G \) there are exactly \( k \) vertices \( u_H \in U_H \) such that \( \zeta(u_H) = u_G \), and
for all \( u_H \in U_H \) and \( v \in V \), \((u_H, v) \in E_H \) if and only if \((\zeta(u_H), v) \in E_G \).

Our generalization of Theorem 8 describes the competitive ratio of Ranking on a graph \( H \) that is a \( k \)-copy of \( G \). It’s proof, presented in Appendix B, is a modification to the proof of Theorem 8 presented by Birnbaum and Mathieu [4].

**Theorem 10.** Let \( G = (U_G \cup V, E_G) \) be a bipartite graph that has a maximum matching of size \( \text{OPT}_1 \), and let \( H = (U_H \cup V, E_H) \) be a left \( k \)-copy of \( G \). Then the expected size of the matching returned by Ranking on \( H \) is at least

\[
   k \text{OPT}_1 \left( 1 - \frac{1}{e^{1/k}} + o(1) \right).
\]

**Proof.** See Appendix B.

Using this result, we are able to prove that the following algorithm, called RankingSimulate, achieves a competitive ratio of \( 4\sqrt{e}/(\sqrt{e} - 1) \).

**RankingSimulate Algorithm:**

*Initialization:*
- Set \( M \), the set of matched bidders, to \( \emptyset \).
- Set \( R \), the set of reserved bidders, to \( \emptyset \).
- Choose a random permutation (ranking) \( \sigma \) of the bidders \( V \).

*Online Matching:*
- Upon arrival of keyword \( u \in U \):
  - Let \( N(u) \) be the set of neighbors of \( u \) that are not in \( M \) or \( R \).
  - If \( N(u) = \emptyset \), do nothing.
  - If \( |N(u)| = 1 \), let \( v \) be the single bidder in \( N(u) \).
    - With probability \( 1/2 \), match \( u \) to \( v \) and add \( v \) to \( M \), and
    - With probability \( 1/2 \), add \( v \) to \( R \).
  - If \( |N(u)| \geq 2 \), let \( v_1 \) and \( v_2 \) be the two distinct bidders in \( N(u) \) that minimize \( \sigma(v) \).
    - With probability \( 1/2 \), match \( u \) to \( v_1 \), add \( v_1 \) to \( M \), and add \( v_2 \) to \( R \), and
    - With probability \( 1/2 \), match \( u \) to \( v_2 \), add \( v_2 \) to \( R \), and add \( v_1 \) to \( M \).

Let \( G = (U_G \cup V, E_G) \) be the bipartite input graph to 2PM, and let \( H = (U_H \cup V, E_H) \) be a left 2-copy of \( H \). In the arrival order for \( H \), the two copies of each keyword \( u_G \in U \) arrive in sequential order.

**Lemma 11.** Fix a ranking \( \sigma \) on \( V \). For each bidder \( v \in V \), let \( X_v \) be the indicator variable for the event that \( v \) is matched by Ranking on \( H \), when the ranking is \( \sigma \). Let \( X'_v \) be the indicator variable for the event that \( v \) is matched by RankingSimulate on \( G \), when the ranking is \( \sigma \). Then \( \mathbb{E}(X'_v) = X_v/2 \).

**Proof.** It is easy to establish the invariant that for all \( v \in V \), \( X_v = 1 \) if and only if RankingSimulate puts \( v \) in either \( M \) or \( R \). Furthermore, each bidder \( v \in V \) is put in \( M \) or \( R \) at most once by RankingSimulate. The lemma follows because each time RankingSimulate adds a bidder \( v \) to \( M \) or \( R \), it matches it with probability \( 1/2 \).

\(^4\)Note that once \( \sigma \) is fixed, \( X_v \) is deterministic.
Theorem 12. The competitive ratio of RankingSimulate is $2\sqrt{e}/(\sqrt{e} - 1) \approx 5.083$.

Proof. For a permutation $\sigma$ on $V$, let RankingSimulate($\sigma$) be the matching of $G$ returned by RankingSimulate, and let Ranking($\sigma$) be the matching of $H$ returned by Ranking. Lemma 11 implies that, conditioned on $\sigma$, $\mathbb{E}(|\text{RankingSimulate}(\sigma)|) = |\text{Ranking}(\sigma)|/2$. By Theorem 10

$$
\mathbb{E}(|\text{RankingSimulate}(\sigma)|) = \frac{1}{2} \mathbb{E}(|\text{Ranking}(\sigma)|) \geq OPT_{1P} \left(1 - 1/e^{1/2} + o(1)\right).
$$

Fix a bidder $v \in V$. Let $P_v$ be the profit from $v$ obtained by RankingSimulate. Suppose that $v$ is matched by RankingSimulate to keyword $u \in U_G$. Recall that we have assumed without loss of generality that the degree of $u$ is at least 2. Let $v' \neq v$ be another bidder adjacent to $u$. Then, given that $v$ is matched to $u$, the probability that $v'$ is matched to any keyword is no greater than $1/2$. Therefore, $\mathbb{E}(P_v|v \text{ matched}) \geq 1/2$. Hence, the expected value of the second-price matching returned by RankingSimulate is

$$
\sum_{v \in V} \mathbb{E}(P_v) = \sum_{v \in V} \mathbb{E}(P_v|v \text{ matched}) \mathbb{P}(v \text{ matched}) \\
\geq \frac{1}{2} \sum_{v \in V} \mathbb{P}(v \text{ matched}) \\
= \frac{1}{2} \mathbb{E}(|\text{RankingSimulate}(\sigma)|) \\
\geq \frac{1}{2} OPT_{1P} \left(1 - 1/e^{1/2} + o(1)\right) \\
\geq \frac{1}{2} OPT_{2P} \left(1 - 1/e^{1/2} + o(1)\right),
$$

where $OPT_{2P}$ is the size of the optimal second-price matching on $G$. \hfill \Box

6 Conclusion

In this paper, we have shown that the complexity of the Second-Price Ad Auctions problem is quite different from that of the more studied First-Price Ad Auctions problem, and that this discrepancy extends to the special case of 2PM, whose first-price analogue is bipartite matching. On the positive side, we have given a 2-approximation for offline 2PM and a 5.083-competitive algorithm for online 2PM.

Some open questions remain. Closing the gap between 2 and $364/363$ in the approximability of offline 2PM is one clear direction for future research, as is closing the gap between 2 and 5.083 in the competitive ratio for online 2PM. Another question we leave open is whether the analysis for RankingSimulate is tight, though we expect that it is not.
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A Discussion of Related Models

In this section, we prove the statements claimed in the introduction regarding the relationships between the optimal solutions of our model and the strict and non-strict models of Goel et al. [14]. Given an instance $A$, let $OPT_{2P}$ be the optimal solution value in our model; let $OPT_{\text{strict}}$ be the optimal solution value under the strict model; and let $OPT_{\text{non-strict}}$ be the optimal solution value under the non-strict model. We prove the following theorem.

**Theorem 13.** For any instance $A$, $OPT_{\text{non-strict}} \leq (2+1/R_{\text{max}})OPT_{\text{strict}} \leq 8(2+1/R_{\text{max}})OPT_{2P}$.

The first inequality follows from the work of Goel et al. [14], so we need only prove that $OPT_{\text{strict}} \leq 8OPT_{2P}$.

The core of our argument is a reduction from 2PAA to the First-Price Ad Auctions problem (1PAA), in which only one bidder is chosen for each keyword and that bidder pays the minimum of its bid and its remaining budget. Given an instance $A$ of 2PAA, we construct an instance $A'$ of 1PAA problem by replacing each bid $b_{u,v}$ by $b'_{u,v} \triangleq \max_{v' \neq v : b_{u,v} \leq b_{u,v'}} b_{u,v'}$. Denote by $OPT_{1P}(A')$ the optimal value of the first-price model on $A'$. The following two lemmas prove Theorem 13 by relating both $OPT_{\text{non-strict}}(A)$ and $OPT_{2P}(A)$ to $OPT_{1P}(A')$.

**Lemma 14.** $OPT_{\text{non-strict}}(A) \leq OPT_{1P}(A')$.

**Proof.** For an instance $A$, we can view a non-strict second-price allocation of $A$ as a pair of (partial) functions $f_1$ and $f_2$ from the keywords $U$ to the bidders $V$, where $f_1$ maps each keyword to the bidder to which it is allocated and $f_2$ maps each keyword to the bidder acting as its second-price bidder. Thus, if $f_1(u) = v$ and $f_2(u) = v'$ then $u$ is allocated to $v$ and the price $v$ pays is $b_{u,v'}$. We have, for all such $u, v,$ and $v'$, that $b_{u,v'} \leq b'_{u,v}$.

We construct the first-price allocation on $A'$ defined by $f_1$ and claim that the value of this first-price allocation is at least the value of the non-strict allocation defined by $f_1$ and $f_2$. It suffices to show that for any bidder $v$, the profit that the non-strict allocation gets from $v$ is at most the profit that the first-price allocation gets from $v$, or in other words,

$$\min \left( B_v, \sum_{u : f_1(u) = v} b_{u,f_2(u)} \right) \leq \min \left( B_v, \sum_{u : f_1(u) = v} b'_{u,v} \right).$$

This inequality follows trivially from the fact that $b_{u,f_2(u)} \leq b'_{u,f_1(u)}$ for all allocated keywords $u$, and hence the lemma follows.

**Lemma 15.** $OPT_{1P}(A') \leq 8OPT_{2P}(A)$.

**Proof.** Given an optimal first-price allocation of $A'$, we can assume without loss of generality that each bidder’s budget can only be exhausted by the last keyword allocated to it, or, more formally, if $u_1, u_2, \ldots, u_k$ are the keywords that are allocated to a bidder $v$ and they come in that order, then

---

Footnote 5: Recall that this problem has also been called the Adwords problem [21] and the Maximum Budgeted Allocation problem [26] [27].
we can assume that $\sum_{i=1}^{k-1} b'_{ui,v} < B_v$. The reason for this is that if for some $j < k$, $\sum_{i=1}^{j-1} b'_{ui,v} < B_v$ and $\sum_{i=1}^{j} b'_{ui,v} \geq B_v$, then we can ignore the allocation of $u_{j+1}, \ldots u_k$ to $v$ without losing any profit.

With this assumption, we design a randomized algorithm that constructs a second-price allocation on $A$ whose expected value in our model is at least $1/8$ of the first-price allocation’s value. Viewing the first-price allocation of $A'$ as a (partial) function $f$ from the keywords $U$ to the bidders $V$ and denoting by $s(u,v)$ the bidder $v'$ for which $b_{vu'} = b'_{vu}$, the algorithm is as follows.

**Random Construction:**

Randomly mark each bidder with probability $1/2$.

For each unmarked bidder $v$:

Let $S_v = \emptyset$.

For each keyword $u$ such that $f(u) = v$:

If $s(u,v)$ is marked: $S_v = S_v \cup \{u\}$.

Assume that $S_v = \{u_1, u_2, \ldots u_k\}$, where $u_1, u_2, \ldots u_k$ come in that order.

If $\sum_{i=1}^{k} b'_{ui,v} \leq B_v$:

Let $f_1(u_i) = v$ and $f_2(u_i) = s(u_i,v)$ for all $i \leq k$.

Else:

If $\sum_{i=1}^{k} b'_{ui,v} \geq b'_{uk,v}$: let $f_1(u_i) = v$ and $f_2(u_i) = s(u_i,v)$ for all $i \leq k - 1$.

Else: let $f_1(u_k) = v$ and $f_2(u_k) = s(u_k,v)$.

We claim that for the $f_1$ and $f_2$ defined by this construction, whenever $f(u_i)$ is set to $v$, the profit from that allocation is $b'_{ui,v}$. This is not trivial because in our model, if a bidder’s remaining budget is smaller than its bid for a keyword, it changes its bid for that keyword to its remaining budget. However, one can easily verify that in all cases, if we set $f_1(u_i) = v$ and $f_2(u_i) = s(u_i,v)$, the remaining budget of $v$ is at least $b'_{ui,v} = b_{ui,s(u_i,v)}$. Thus, the (modified) bid of $f_1(u_i)$ for $u_i$ is still at least the original bid of $f_2(u_i)$ for $u_i$.

We claim that the expected value of the second-price allocation defined by $f_1$ and $f_2$ is at least $1/8OPT_{1P}(A')$. For each bidder $v$, let $X_v$ be the random variable denoting the profit that $f_1$ and $f_2$ get from $v$, and let $Y_v$ be the profit that $f$ gets from $v$. We have $OPT_{1P}(A') = \sum_v Y_v$, so it suffices to show that $E[X_v] \geq 1/8Y_v$ for all $v \in V$.

Consider any $v \in V$ that is unmarked. Let $T_v = \{u : f(u) = v\}$. If $\sum_{u \in S_v} b'_{u,v} \leq B_v$ then $X_v = \sum_{u \in S_v} b'_{u,v}$. If $\sum_{u \in S_v} b'_{u,v} > B_v$ then $X_v \geq \sum_{u \in S_v} b'_{u,v}/2$. Thus, in both case, we have

$$E[X_v | v \text{ is unmarked}] \geq E[\sum_{u \in S_v} b'_{u,v}/2 | v \text{ is unmarked}] = \sum_{u \in T_v} b'_{u,v}/4 = Y_v/4,$$

which implies

$$E[X_v] \geq E[X_v | v \text{ is unmarked}] Pr[v \text{ is unmarked}] = 1/2 \cdot Y_v/4 = Y_v/8.$$

\[\square\]

**B Proof of Theorem 10**

In this appendix, we provide a full proof of Theorem 10. The proof presented here is quite similar to the simplified proof of Theorem 8 presented by Birnbaum and Mathieu [4]. For intuition into
the proof presented here, the interested reader is referred to that work.

Let $G = (U_G \cup V, E_G)$ be a bipartite graph and let $H = (U_H \cup V, E_H)$ be a left $k$-copy of $G$. Let $\zeta : U_H \to U_G$ be a map that satisfies the conditions of Definition \[.\] Let $M_G \subseteq E_G$ be a maximum matching of $G$.

Let $\text{Ranking}(H, \pi, \sigma)$ denote the matching constructed on $H$ for arrival order $\pi$, when the ranking is $\sigma$. Consider another process in which the vertices in $V$ arrive in the order given by $\sigma$ and are matched to the available vertex $u \in U_H$ that minimizes $\pi(u)$. Call the matching constructed by this process $\text{Ranking}'(H, \pi, \sigma)$. It is not hard to see that these matchings are identical, a fact that is proved in \[.\]

**Lemma 16** (Karp, Vazirani, and Vazirani \[.\]). For any permutations $\pi$ and $\sigma$, $\text{Ranking}(H, \pi, \sigma) = \text{Ranking}'(H, \pi, \sigma)$.

The following monotonicity lemma shows that removing vertices in $H$ can only decrease the size of the matching returned by Ranking.

**Lemma 17.** Let $\pi_H$ be an arrival order for the vertices in $U_H$, and let $\sigma_H$ be a ranking on the vertices in $V$. Suppose that $x$ is a vertex in $U_H \cup V$, and let $H' = (U_H', V_H', E_{H'}) = H \setminus \{x\}$. Let $\pi_{H'}$ and $\sigma_{H'}$ be the orderings of $U_H'$ and $V_H'$ induced by $\pi_H$ and $\sigma_H$, respectively. Then $\text{Ranking}(H', \pi_{H'}, \sigma_{H'}) \subseteq \text{Ranking}(H, \pi_H, \sigma_H)$.

**Proof.** Suppose first that $x \in U_H$. In this case, $V = V_{H'}$ and $\sigma_H = \sigma_{H'}$. Let $Q_t(H) \subseteq V$ be the set of vertices matched to vertices in $U_H$ that arrive at or before time $t$ (under arrival order $\pi_H$ and ranking $\sigma_H$), and let $Q_t(H') \subseteq V$ be the set of vertices matched to vertices in $U_{H'}$ that arrive at or before time $t$ (under arrival order $\pi_{H'}$ and ranking $\sigma_{H'}$). We prove by induction on $t$ that $Q_{t-1}(H') \subseteq Q_t(H)$, which by substituting $t = n$ is sufficient to prove the claim. The statement holds when $t = 1$, since $Q_0(H') = \emptyset$. Now suppose we have $Q_{t-2}(H') \subseteq Q_{t-1}(H)$, we prove $Q_{t-1}(H') \subseteq Q_t(H)$. Suppose that $t$ is at or before the time that $x$ arrives in $\pi_H$. Then clearly $Q_{t-1}(H') = Q_{t-1}(H) \subseteq Q_t(H)$. Now suppose that $t$ is after the time that $x$ arrives in $\pi_H$. Let $u$ be the vertex that arrives at time $t - 1$ in $\pi_H$. If $u$ is not matched by $\text{Ranking}(H', \pi_{H'}, \sigma_{H'})$, then $Q_{t-1}(H') = Q_{t-2}(H') \subseteq Q_{t-1}(H) \subseteq Q_t(H)$. Now suppose that $u$ is matched by $\text{Ranking}(H', \pi_{H'}, \sigma_{H'})$, say to vertex $v'$. We show that $v' \in Q_t(H)$, which by the induction hypothesis, is enough to prove that $Q_{t-1}(H') \subseteq Q_t(H)$. Note that $u$ arrives at time $t$ in $\pi_H$. Let $v$ be the vertex to which $u$ is matched by $\text{Ranking}(H, \pi_H, \sigma_H)$. If $v = v'$, we are done, so suppose that $v \neq v'$. Since $v \notin Q_{t-1}(H)$, it follows by the induction hypothesis that $v \notin Q_{t-2}(H')$. Therefore, vertex $v$ is available to be matched to $u$ when it arrives in $\pi_{H'}$. Since $\text{Ranking}(H', \pi_{H'}, \sigma_{H'})$ matched $u$ to $v'$ instead, $v'$ must have a lower rank than $v$ in $\sigma_{H'}$. Since $\text{Ranking}(H, \pi_H, \sigma_H)$ chose $v$, vertex $v'$ must have already been matched when vertex $u$ arrived at time $t$ in $\pi_H$, or, in other words, $v' \in Q_{t-1}(H) \subseteq Q_t(H)$.

Now suppose that $x \in V$. In this case, $U_H = U_{H'}$, and $\pi_H = \pi_{H'}$. Let $R_t(H) \subseteq U_H$ be the set of vertices matched to vertices in $V$ that are ranked less than or equal to $t$ (under arrival order $\pi_H$ and ranking $\sigma_H$), and let $R_t(H') \subseteq U_H$ be the set of vertices matched to vertices in $V$ that are ranked less than or equal to $t$ (under arrival order $\pi_{H'}$ and ranking $\sigma_{H'}$).

\[footnote{For those familiar with the proof in \[.\], the main difference between the proof of Theorem \[.\] presented here and the proof of Theorem \[.\] presented in \[.\] appears in Lemma \[.\]. Instead of letting $u$ be the single vertex that is matched to $v$ by the perfect matching, as is done in \[.\], we choose $u$ uniformly at random from one of the $k$ vertices that correspond to the vertex that is matched to $v$ by the perfect matching. The rest of the proof is essentially the same, but we present its entirety here for completeness.}
are ranked less than or equal to $t$ (under order $\pi_H$ and ranking $\sigma_H$). Then by Lemma 16, we can apply the same argument as before to show that $R_{t-1}(H') \subseteq R_t(H)$ for all $t$, which by substituting $t = n$, is sufficient to prove the claim.

We define the following notation. For all $u_G \in U_G$, let $\zeta^{-1}(u_G)$ be the set of all $u_H \in U_H$ such that $\zeta(u_H) = u_G$, and for any subset $U'_G \subseteq U_G$, let $\zeta^{-1}(U'_G)$ be the set of all $u_H \in U_H$ such that $\zeta(u_H) \in U'_G$. The following lemma shows that we can assume without loss of generality that $M_G$ is a perfect matching.

**Lemma 18.** Let $U' \subseteq U_G$ and $V' \subseteq V$ be the subset of vertices that are in $M_G$. Let $G'$ be the subgraph of $G$ induced by $U' \cup V'$, and let $H'$ be the subgraph of $H$ induced by $\zeta^{-1}(U') \cup V'$. Then the expected size of the matching produced by Ranking on $H'$ is no greater than the expected size of the matching produced by Ranking on $H$.

**Proof.** The proof follows by repeated application of Lemma 17 for all $x$ that are not in $\zeta^{-1}(U') \cup V'$.

In light of Lemma 18, to prove Theorem 10 it is sufficient to show that the expected size of the matching produced by Ranking on $H'$ is at least $(1 - 1/e^{1/k} - \omega(1))|M_G|$. To simplify notation, we instead assume without loss of generality that $G = G'$, and hence $G$ has a perfect matching. Let $n = OPT_1P = |M_G| = |V|$. Henceforth, fix an arrival order $\pi$. To simplify notation, we write $\text{Ranking}(\sigma)$ to mean $\text{Ranking}(H, \pi, \sigma)$.

Let $f : U_H \to V$ be a map such that for all $v \in V$, there are exactly $k$ vertices $u \in U_H$ such that $f(u) = v$. The existence of such a map $f$ follows from the assumption that $G$ has a perfect matching. For any vertex $v \in V$ let $f^{-1}(v)$ be the set of $u \in U_H$ such that $f(u) = v$. We proceed with the following two lemmas.

**Lemma 19.** Let $u \in U_H$, and let $v = f(u)$. For any ranking $\sigma$, if $v$ is not matched by $\text{Ranking}(\sigma)$, then $u$ is matched to a vertex whose rank is less than the rank of $v$ in $\sigma$.

**Proof.** If $v$ is not matched by $\text{Ranking}(\sigma)$, then since there is an edge between $u$ and $v$, it was available to be matched to $u$ when it arrived. Therefore, by the behavior of Ranking, $u$ must have been matched to a vertex of lower rank.

**Lemma 20.** Let $u \in U_H$, and let $v = f(u)$. Fix an integer $t$ such that $1 \leq t \leq n$. Let $\sigma$ be a permutation, and let $\sigma'$ be the permutation obtained from $\sigma$ by removing vertex $v$ and putting it back in so its rank is $t$. If $v$ is not matched by $\text{Ranking}(\sigma')$, then $u$ must be matched by $\text{Ranking}(\sigma)$ to a vertex whose rank in $\sigma$ is less than or equal to $t$.

**Proof.** For the proof, it is convenient to invoke Lemma 16 and consider $\text{Ranking}'(\sigma)$ and $\text{Ranking}'(\sigma')$ instead of $\text{Ranking}(\sigma)$ and $\text{Ranking}(\sigma')$. In the process by which $\text{Ranking}'$ constructs its matching, call the moment that the $t$th vertex in $V$ arrives time $t$. For any $1 \leq s \leq n$, let $R_s(\sigma)$ (resp., $R_s(\sigma')$) be the set of vertices in $U_H$ matched by time $s$ in $\sigma$ (resp., $\sigma'$). By Lemma 19, if $v$ is not matched by $\text{Ranking}(\sigma')$, then $u$ must be matched to a vertex $v'$ in $\text{Ranking}(\sigma')$ such that $\sigma'(v') < \sigma'(v)$. Hence $u \in R_{t-1}(\sigma')$. We proceed the lemma by showing that $R_{t-1}(\sigma') \subseteq R_t(\sigma)$. Let $\tilde{t}$ be the time that $v$ arrives in $\sigma$. Then if $\tilde{t} \geq t$, the two orders $\sigma$ and $\sigma$ are identical through time $t$, which implies that $R_{t-1}(\sigma') = R_{t-1}(\sigma) \subseteq R_t(\sigma)$.

Now, in the case that $\tilde{t} < t$, we prove that for $1 \leq s \leq t$, $R_{s-1}(\sigma') \subseteq R_s(\sigma)$. The proof, which is similar to the proof of Lemma 17, proceeds by induction on $s$. When $s = 0$, the claim clearly holds,
since \( R_0(\sigma') = \emptyset \). Now, supposing that \( R_{s-2}(\sigma') \subseteq R_{s-1}(\sigma) \), we prove that \( R_{s-1}(\sigma') \subseteq R_s(\sigma) \). If \( s \leq \hat{t} \), then the two orders \( \sigma \) and \( \sigma' \) are identical through time \( s \), so \( R_{s-1}(\sigma') = R_{s-1}(\sigma) \subseteq R_s(\sigma) \). Now suppose that \( s > \hat{t} \). Then the vertex that arrives at time \( s - 1 \) in \( \sigma' \) is the same as the vertex that arrives at time \( s \) in \( \sigma \). Call this vertex \( w \). If \( w \) is not matched by Ranking\('(\sigma')\), then \( R_{s-1}(\sigma') = R_{s-2}(\sigma') \), and we are done by the induction hypothesis. Now suppose that \( w \) is matched to vertex \( x' \) by Ranking\('(\sigma')\) and to vertex \( x \) by Ranking\'(\(\sigma\)). If \( x = x' \), then again we are done by the induction hypothesis, so suppose that \( x \neq x' \). Since \( x \) was available at time \( s - 1 \) in \( \sigma \), we have \( x \notin R_{s-1}(\sigma) \), and by the induction hypothesis \( x \notin R_{s-2}(\sigma') \). Hence, \( x \) was available at time \( s - 1 \) in \( \sigma' \). Since Ranking\'(\(\sigma')\) matched \( w \) to \( x' \), it must be that \( \pi(x') < \pi(x) \). This implies that \( x' \) must be matched when \( w \) arrives at time \( s \) in \( \sigma \), or in other words, \( x' \in R_{s-1}(\sigma) \subseteq R_s(\sigma) \). By the induction hypothesis, we are done.

\[ \text{Lemma 21. For } 1 \leq t \leq n, \text{ let } x_t \text{ denote the probability over } \sigma \text{ that the vertex ranked } t \text{ in } V \text{ is matched by Ranking}(\sigma). \text{ Then } \]

\[ 1 - x_t \leq \frac{1}{kn} \sum_{s=1}^{t} x_s. \]

**Proof.** Let \( \sigma \) be permutation chosen uniformly at random, and let \( \sigma' \) be a permutation obtained from \( \sigma \) by choosing a vertex \( v \in V \) uniformly at random, taking it out of \( \sigma \), and putting it back so that its rank is \( t \). Note that both \( \sigma \) and \( \sigma' \) are distributed uniformly at random among all permutations. Let \( u \) be a vertex chosen uniformly at random from \( f^{-1}(v) \). Note that conditioned on \( \sigma \), \( u \) is equally likely to be any of the \( kn \) vertices in \( U_H \). Let \( R_t \) be the set of vertices in \( U_H \) that are matched by Ranking\(\sigma\) to a vertex of rank \( t \) or lower in \( \sigma \). Lemma 20 states that if \( v \) is not matched by Ranking\'(\(\sigma')\), then \( u \in R_t \). The expected size of \( R_t \) is \( \sum_{1 \leq s \leq t} x_s \). Hence, the probability that \( u \in R_t \), conditioned on \( \sigma \), is \((1/(kn))\sum_{1 \leq s \leq t} x_s \). The lemma follows because the probability that \( v \) is not matched by Ranking\'(\(\sigma')\) is \( 1 - x_t \).

We are now ready to prove Theorem 10.

**Proof of Theorem 10.** For \( 0 \leq t \leq n \), let \( S_t = \sum_{1 \leq s \leq t} x_s \). Then the expected size of the matching returned by Ranking on \( H \) is \( S_n \). Rearranging (1) yields, for \( 1 \leq t \leq n \),

\[ S_t \geq \left( \frac{kn}{kn+1} \right) (1 + S_{t-1}), \]

which by induction implies that \( S_t \geq \sum_{1 \leq s \leq t} (kn/(kn + 1))^s \), and hence

\[ S_n \geq \sum_{s=1}^{n} \left( \frac{kn}{kn+1} \right)^s = kn \left( 1 - \left( 1 - \frac{1}{kn+1} \right)^n \right) = kn \left( 1 - \frac{1}{e^{1/k}} + o(1) \right). \]

**C Other Missing Proofs**

In this appendix, we provide the missing proofs from the body of the paper.
C.1 Proof of Theorem 1

Fix a constant $c' > c$, and let $n_0$ be the smallest integer such that for all $n \geq n_0$,

$$c' \cdot \frac{c(n^5 + n + 2)}{cn^2 + n + 2} \geq c(n^3 + cn^2 + n + 2) \quad (2)$$

and

$$\frac{n/2 + 1}{2} \geq c \quad . \quad (3)$$

Note that since $n_0$ depends only on $c'$, it is a constant.

We reduce from PARTITION, in which the input is a set of $n \geq n_0$ items, and the weight of the $i$-th item is given by $w_i$. If $W = \sum_{i=1}^{n} w_i$, then the question is whether there is a partition of the items into two subsets of size $n/2$ such that the sum of the $w_i$’s in each subset is $W/2$. It is known that this problem (even when the subsets must both have size $n/2$) is NP-hard [13].

Suppose that there is an $m/c'$-approximation algorithm to 2PAA($c$); we will show that constructing the following instance of 2PAA($c$) (illustrated in Figure 4) allows us to use the $m/c'$-approximation to solve the PARTITION instance:

- First, create $n + 2$ keywords $c_1, \ldots, c_n, e_1, e_2$. Second, create an additional set $G = \{ g_{i,k} : 1 \leq i \leq n^2 \text{ and } 1 \leq k \leq c \}$ of $cn^2$ keywords. The keywords arrive in the order $c_1, \ldots, c_n, e_1, e_2, g_{1,1}, \ldots, g_{1,c}, \ldots, g_{n^2,1}, \ldots, g_{n^2,c}$.

- Create $n^2 + 4$ bidders $a, d_1, d_2, f, h_1, \ldots, h_n$. Set the budgets of $a$, $d_1$, and $d_2$ to $cW(1 + n/2)$. Set the budget of $f$ to $cW(n^3 + 1)$. For $1 \leq i \leq n^2$, set the budget of $h_i$ to $cWn^3$.

- For $1 \leq i \leq n$, bidders $a$, $d_1$, and $d_2$ bid $c(w_i + W)$ on keyword $c_i$.

- For $j \in \{1, 2\}$, bidder $d_j$ bids $cW$ on keyword $e_j$. Bidder $f$ bids $cW/2$ on both $e_1$ and $e_2$.

- For $1 \leq i \leq n^2$ and $1 \leq k \leq c$, keyword $g_{i,k}$ receives a bid of $W(n^3 + 1)$ from bidder $f$ and a bid of $Wn^3$ from bidder $h_i$.

This reduction can clearly be performed in polynomial time. Furthermore, it can easily be checked that [3] implies that no bidder bids more than $1/c$ of its budget on any keyword.

We first show that if the PARTITION instance is a “yes” instance, then there exists a feasible solution to the 2PAA($c$) instance of value at least $cW(n^5 + n + 2)$. Let $U \subseteq [n]$ be such that $|U| = n/2$ and $\sum_{i \in U} w_i = \sum_{i \in \overline{U}} w_i = W/2$.

We construct a solution to the 2PAA($c$) instance as follows. For every $i \in U$, allocate $c_i$ to $d_1$, and for every $i \in \overline{U}$, allocate $c_i$ to $d_2$. For each of these allocations, choose $a$ as the second-price bidder. This will reduce the budget of $d_1$ and $d_2$ to exactly $cW/2$, and hence the bids from $d_1$ to $e_1$ and from $d_2$ to $e_2$ will be both be reduced to $cW/2$. Allocate $e_1$ to $f$ choosing $d_1$ as the second-price bidder, and allocate $e_2$ to $f$ choosing $d_2$ as the second-price bidder. This will reduce the budget of $f$ to $cWn^3$. The profit from the solution constructed so far is $cW(n + 2)$. Now allocate $g_{1,1}, g_{1,2}, \ldots, g_{1,c-1}$ to $f$, choosing $h_1$ as the second-price bidder. This will reduce the budget of $f$
Figure 4: The 2PAA(c) instance of the reduction. Each bidder’s budget is shown above its node, and the bids of bidders for keywords is shown near the corresponding edge.

to $Wn^3$. Hence, it can act as the second-price bidder for each of the remaining keywords in $G$. Allocate $g_{1,c}$ to $h_1$, choosing $f$ as the second-price bidder, and then, for $2 \leq i \leq n^2$ and $1 \leq k \leq c$, allocate $g_{i,k}$ to $h_i$, choosing $f$ as the second-price bidder. The profit obtained for each keyword in $G$ in this assignment is $Wn^3$. Since $|G| = cn^2$, the total profit of the solution constructed is $cW(n + 2) + cWn^5 = cW(n^5 + n + 2)$.

We now show that if there is a second-price matching in the 2PAA(c) instance of value at least $cW(n^3 + cn^2 + n + 2)$, then there must be a partition of $w_1, \ldots, w_n$. In such a matching, at most $cW(n + 2)$ units of profit can be obtained from keywords $c_1, \ldots, c_n, e_1, e_2$, since the initial second-highest bids on those keywords sum to $cW(n + 2)$. Hence, at least $cW(n^3 + cn^2)$ profit must come from the keywords in $G$.

Suppose that the budget of $f$ is greater than $cWn^3$ after keywords $e_1$ and $e_2$ are allocated. Note that at least $c$ of the keywords in $G$ must be allocated to reach a profit of $cW(n^3 + cn^2)$ on these keywords. Consider what happens after the first $c$ of the keywords in $G$ are assigned. For each of these keywords, $f$ must have been the first-price bidder, so its budget is reduced to an amount greater than 0 and less than or equal to $cW$. Hence, for each keyword in $G$ allocated henceforth, $f$ is the second-price bidder, and the profit is at most $cW$. Since there are at most $c(n^2 - 1)$ more keywords in $G$, the total profit from the keywords in $G$ is at most $cWn^3 + c^2W(n^2 - 1)$, which contradicts the fact that at least $cW(n^3 + cn^2)$ units of profit must come from $G$. Hence, we conclude that the budget of $f$ is less than or equal to $cWn^3$ after keywords $e_1$ and $e_2$ are allocated.

The budget of $f$ can only be smaller than $cWn^3$ if $f$ acts as the first-price bidder for both $e_1$ and $e_2$. But this can happen only if the budgets of both $d_1$ and $d_2$ are reduced to an amount less than or equal to $cW/2$. For $j \in \{1, 2\}$, let $U_j \subseteq [n]$ be the set of indices $i$ such that $d_j$ acts as the
first-price bidder for \( i \). For both \( j \), we have that

\[
\sum_{i \in U_j} c(w_i + W) \geq \frac{cW}{2} + \frac{cWn}{2} .
\]  

(4)

Rearranging (4) yields \( \sum_{i \in U_j} W \geq W/2 + Wn/2 - \sum_{i \in U_j} w_i \), which implies \( W|U_j| \geq W/2 + Wn/2 - W \), and hence \( |U_j| \geq n/2 - 1/2 \). By integrality, then, \( |U_j| \geq n/2 \) for both \( j \). Hence \( |U_j| = n/2 \) for both \( j \), and using (4) again, we have \( \sum_{i \in U_j} cw_i + cW|U_j| \geq cW/2 + cWn/2 \) which implies that \( \sum_{i \in U_j} w_i \geq W/2 \) for both \( j \). Therefore, the partition defined by \( U_1 \) and \( U_2 \) is a solution to the PARTITION instance.

To conclude the proof, note that the number of keywords in the 2PAA\((c)\) instance is \( cn^2 + n + 2 \). Hence, if the PARTITION instance is a “yes” instance, then by (2), we can run the \( m/c'\)-approximation algorithm to find a second-price matching of value at least \( cW(n^3 + cn^2 + n + 2) \), and if the PARTITION instance is a “no” instance, then the value of the solution returned by the algorithm must be strictly less than \( cW(n^3 + cn^2 + n + 2) \). Hence, an \( m/c'\)-approximation algorithm for 2PAA\((c)\) can be used to solve PARTITION.

### C.2 Proof of Theorem 4

The proof is by a reduction from vertex cover. Given a graph \( G \), we construct an instance \( f(G) \) of 2PM as follows. First, for each edge \( e \in E(G) \), we create a keyword with the same label (called an edge keyword), and for each vertex \( v \in V(G) \), we create a bidder with the same label (called a vertex bidder). Bidder \( v \) bids for keyword \( e \) if vertex \( v \) is one of the two end points of edge \( e \). (Recall that in 2PM, if a bidder makes a non-zero bid for a keyword, that bid is 1.) In addition, for each edge \( e \), we create a unique bidder \( x_e \) who also bids for \( e \). Furthermore, for each vertex \( v \), we create a gadget containing two keywords \( h_v \) and \( l_v \) and two bidders \( y_v \) and \( z_v \). We let \( v \) and \( y_v \) bid for \( h_v \); and \( y_v \) and \( z_v \) bid for \( l_v \). The keywords arrive in an order such that for each \( v \in V(G) \), keyword \( h_v \) comes before \( l_v \), and the edge keywords arrive after all of the \( h_v \)'s and \( l_v \)'s have arrived. An example of this reduction is shown in Figure 5.

The following lemma provides the basis of the proof.

**Lemma 22.** Let \( \text{OPT}_{VC} \) and \( \text{OPT}_{2P} \) be the size of the minimum vertex cover of \( G \) and the maximum second-price matching on \( f(G) \), respectively. Then

\[
\text{OPT}_{2P} = 2|V(G)| + |E(G)| - \text{OPT}_{VC}
\]

Proof. We first show that given a vertex cover \( S \) of size \( \text{OPT}_{VC} \) of \( G \), we can construct a solution to the 2PM instance whose value is \( 2|V(G)| + |E(G)| - \text{OPT}_{VC} \). For each vertex \( v \notin S \), we allocate \( h_v \) to \( v \) (with \( y_v \) acting as the second-price bidder) and \( l_v \) to \( y_v \) (with \( z_v \) acting as the second-price bidder), getting a profit of 2 from the gadget for \( v \). For each vertex \( v \in S \), we allocate \( h_v \) to \( y_v \) (with \( v \) acting as the second-price bidder) and ignore \( l_v \), getting a profit of 1 from the gadget for \( v \). We then allocate each edge keyword \( e \) to \( x_e \). During each of these edge keyword allocations, at least one of the two vertex bidders that bid for \( e \) is still available, since \( S \) is a vertex cover. Hence, for each of these allocations, there is a bidder that can act as a second-price bidder, and the profit from the allocation is 1. This allocation yields a second-price matching of size \( 2|V(G)| + |E(G)| - \text{OPT}_{VC} \). Therefore, \( \text{OPT}_{2P} \geq 2|V(G)| + |E(G)| - \text{OPT}_{VC} \).
To show that $OPT_{2P} \leq 2|V(G)| + |E(G)| - OPT_{VC}$, we start with an optimal solution to $f(G)$ of value $OPT_{2P}$ and construct a vertex cover of $G$ of size $2|V(G)| + |E(G)| - OPT_{2P}$. To do this, we first claim that there exists an optimal solution of $f(G)$ in which every edge-keyword is allocated for a profit of 1. Consider any optimal second-price matching of the instance. Let $e$ be an edge-keyword $e$ that is not allocated for a profit of 1. If it is adjacent to a vertex bidder that is unassigned when $e$ arrives, then $e$ can be allocated to $x_e$ for a profit of 1, which can only increase the value of the solution. Suppose, on the other hand, that both of its vertex bidders are not available when $e$ arrives. Let $v$ be a vertex bidder that bids for $e$. Since it is not available, $h_v$ must have been allocated to $v$. We can transform this second-price matching to another one in which $h_v$ is assigned to $y_v$, $l_v$ is ignored and $e$ is assigned to $x_e$, with $v$ acting the second-price bidder in both cases. This does not decrease the total profit of the solution. Hence, we can perform these transformations for each edge keyword $e$ that is not allocated for a profit of 1 until we obtain a new optimal solution in which each edge keyword is allocated for a profit of 1.

Now consider an optimal second-price matching in which all edge keywords are allocated for a profit of 1. Let $T \subseteq V(G)$ be the set of vertices represented by vertex bidders that are not allocated any keywords in this second-price matching. Then $|T| = 2|V(G)| + |E(G)| - |OPT_{2P}|$, and $T$ is a vertex cover, which implies $OPT_{VC} \leq 2|V(G)| + |E(G)| - OPT_{2P}$. The lemma follows.

Now, suppose that we have an $\alpha$-approximation for 2PM. We will show how to use this approximation algorithm and our reduction to obtain an $((8\alpha - 7)/\alpha)$-approximation for Vertex Cover on 4-regular graphs. By Theorem 3, this means that $(8\alpha - 7)/\alpha \geq 53/52$, and hence $\alpha \geq 364/363$, unless $P = NP$.

To construct this $((8\alpha - 7)/\alpha)$-approximation algorithm, given a 4-regular graph $G$, run the above reduction to obtain a 2PM instance $f(G)$. Then use the $\alpha$-approximation to obtain a second-price matching $M$ whose value is at least $OPT_{2P}/\alpha$. Now, just as in the proof of Lemma 22, we can assume that in $M$, every edge keyword $e$ is allocated to $x_e$. Hence, the set of vertices $T$ associated
with the vertex bidders that are not allocated a keyword form a vertex cover, and

\[ |T| \leq 2|V(G)| + |E(G)| - OPT_{2P}/\alpha \]
\[ = 2|V(G)| + |E(G)| - (2|V(G)| + |E(G)| - OPT_{VC})/\alpha \]
\[ = (1 - 1/\alpha)(2|V(G)| + |E(G)|) + OPT_{VC}/\alpha \]

(5)

Since \( G \) is 4-regular, we have \( OPT_{VC} \geq m/4 = (2|V(G)| + |E(G)|)/8 \), and hence by (5), we conclude that \( |T| \leq ((8\alpha - 7)/\alpha)OPT_{VC} \), which finishes the proof of the theorem.

C.3 Proof of Theorem 5

Since the number of vertices matched by \( f \) is an upper bound on the profit of the maximum second-price matching on \( G \), we need only to prove that the second-price matching contains at least half of the keywords matched by \( f \). By the behavior of the algorithm, it is clear that whenever a vertex \( u \) is matched to \( f(u) \) in the second-price matching, the profit obtained is 1. Furthermore, every time an an edge is removed from \( f \), a new keyword is added to the second-price matching. Thus, the theorem follows.

C.4 Proof of Theorem 7

We invoke Yao’s Principle [27] and construct a distribution of inputs for which the best deterministic algorithm achieves an expected performance of (asymptotically) \( 1/2 \) the value of the optimal solution.

Our distribution is constructed as follows. The first keyword arrives, and it is adjacent to two bidders. Then the second keyword arrives, and it is adjacent to one of the two bidders adjacent to the first keyword, chosen uniformly at random, as well as a new bidder; then the third keyword arrives, and it is adjacent to one of the bidders adjacent to the second keyword, chosen uniformly at random, as well as a new bidder; and so on, until the \( m \)-th keyword arrives. We call this a normal instance. To analyze the performance of the online algorithms, we also define a restricted instance to be one that is exactly the same as a normal instance except that one of the two bidders of the first keyword is marked unavailable, i.e., he can not participate in any auction.

Clearly, an offline algorithm that knows the random choices beforehand can allocate each keyword to the bidder that will not be adjacent to the keyword that arrives next. In this way, it can ensure that for each keyword, there is a bidder that can act as a second-price bidder. Hence for a normal instance, the optimal second-price matching obtains a profit of \( m \).

Consider the algorithm Greedy, which allocates a keyword to an arbitrary adjacent bidder if and only if there is another available bidder to act as a second-price bidder. Our proof consists of two steps: first, we will show that the expected performance of Greedy on the normal instance is \( (m + 1)/2 \), and second we will prove that Greedy is the best algorithm in expectation for both types of instances.

Let \( X_k^* \) and \( Y_k^* \) be the expected profit of Greedy on a normal and a restricted instance of \( k \) keywords, respectively (where \( X_0^* \) and \( Y_0^* \) are both defined to be 0). Given the first keyword of a normal instance, Greedy allocates it to an arbitrary bidder. Then, with probability \( 1/2 \), it is faced with a normal instance of \( k - 1 \) keywords, and with probability \( 1/2 \), it is faced with a restricted instance of \( k - 1 \) keywords. Therefore, for all integers \( k \geq 1 \),

\[ X_k^* = 1/2(X_{k-1}^* + Y_{k-1}^*) + 1 \]  

(6)
On the other hand, given the first keyword of a restricted instance, Greedy just waits for the second keyword. Then, with probability $1/2$, the second keyword chooses the marked bidder, giving Greedy a restricted instance of $k-1$ keywords, and with probability $1/2$, the second keyword chooses the unmarked bidder, giving Greedy a normal instance of $k-1$ keywords. Therefore, for all $k$,

$$Y_k^* = 1/2(X_{k-1}^* + Y_{k-1}^*) .$$  \hfill (7)

From (6) and (7) we have, for all $k$,

$$Y_k^* = X_k^* - 1 .$$  \hfill (8)

Plugging (8) for $k = m - 1$ into (6) for $k = m$ yields

$$X_m^* = X_{m-1}^* + 1/2 ,$$  \hfill (9)

and hence, by induction $X_m^* = (m + 1)/2$.

Now, we prove that Greedy is the best among all algorithms on these two types of instances. In fact, we make it easier for the algorithms by telling them beforehand how many keywords in the instance they will need to solve. Let $X_m$ and $Y_m$ be the expected number of keywords in the second-price matching produced by the best algorithms that “know” that they are solving a normal instance of size $m$ and a restricted instance of size $m$, respectively. Let $A_m$ and $B_m$ denote these optimal algorithms.

We prove that $X_m \leq X_m^*$ and $Y_m \leq Y_m^*$ for all $m$ by induction. The base case in which $m = 1$ is easy, since no algorithm can obtain a profit of more than one on a normal instance of one keyword or more than zero on a restricted instance of one keyword. We now prove the induction step.

First, consider $A_m$. When the first keyword arrives, $A_m$ has two choices: either ignore it or allocate it to one of the bidders. If $A_m$ ignores the first keyword, its performance is at most the performance of $A_{m-1}$ on the remaining keywords, which constitute a normal instance of $m - 1$ keywords. On the other hand, if $A_m$ allocates the first keyword to one of the bidders, then with probability $1/2$, it is faced with a normal instance of $m - 1$ keywords, and with probability $1/2$ it is faced with a restricted instance of $m - 1$ keywords. The performance of $A_m$ on these instances is at most those of $A_{m-1}$ and $B_{m-1}$, respectively. Thus, by the induction hypothesis, (8), and (9), we have

$$X_m \leq \max\{X_{m-1}^* + 1/2(X_{m-1} + Y_{m-1}) + 1\}$$

$$\leq \max\{X_{m-1}^* + 1/2(X_{m-1}^* + Y_{m-1}^*) + 1\}$$

$$= \max\{X_{m-1}^* + 1/2(X_{m-1}^* + X_{m-1}^* - 1) + 1\}$$

$$= X_{m-1}^* + 1/2$$

$$= X_m^* .$$

Next, consider $B_m$. When the first keyword arrives, $B_m$ cannot allocate it for a profit. If it allocates it for a profit of 0, then it is faced with a restricted instance of $m - 1$ keywords. If it does not allocate the keyword, then with probability $1/2$, $B_m$ is faced with a normal instance of $m - 1$ keywords, and with probability $1/2$, it is faced with a restricted instance of $m - 1$ keywords. Its performance on these instances is at most those of $A_{m-1}$ and $B_{m-1}$, respectively. Thus, by the
induction hypothesis and (7), we have

\[ Y_m \leq \max\{Y_{m-1}, 1/2(X_{m-1} + Y_{m-1})\} \]
\[ \leq \max\{Y^*_m, 1/2(X^*_m + Y^*_m)\} \]
\[ = Y^*_m. \]

This completes our proof.