Rational Formulas for Traces in zero-dimensional Algebras

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Abstract

We present a rational expression for the trace of the multiplication map $\times_r : A \to A$ in a finite-dimensional algebra $A := K[x_1, \ldots, x_n]/I$ in terms of the generalized Chow form of $I$. Here, $I \subset K[x_1, \ldots, x_n]$ is a zero-dimensional ideal, $K$ is a field of characteristic zero, and $r(x_1, \ldots, x_n)$ a rational function whose denominator is not a zero divisor in $A$. If $I$ is a complete intersection in the torus, we get numerator and denominator formulas for traces in terms of sparse resultants.

1 Introduction

Traces in finite dimensional algebras play a fundamental role in Commutative Algebra and Algebraic Geometry. Recent applications of traces include the evaluation of symmetric functions, the effective Nullstellensatz, the computation of radicals of ideals and algorithms for solving polynomial systems [ABRW AS BW DG FGS JRS KP SS Kun]. For developments in the analytical counterpart of this algebraic tool (residues) see [AGV AY BGVY BY1 BY2 BY3 CDS1 CDS2 CM Elk Tsi].

The main results of this paper are explicit rational formulas for the computation of traces of the multiplication map $\times_r : A \to A$ in terms of the generalized Chow form of $I$ in the case $I \subset K[x_1, \ldots, x_n]$ is a zero-dimensional ideal and $r(x_1, \ldots, x_n)$ a rational function whose denominator is not a zero divisor in $A := K[x_1, \ldots, x_n]/I$. If $I$ is a complete intersection in the torus, we get numerator and denominator formulas for traces in terms of sparse resultants.

The importance of having such formulas is that they allow to develop computational techniques for solving polynomial equations as in [JRS], to bound arithmetic aspects of the membership problem as in [Elk] or to compute invariants in “general” (as opposed to “generic”) situations like the computations given in [Ped EY] to mention some examples. In all these cases, from generic formulae like the results presented here, one can perform a suitable specialization and get results in a general case.

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To state properly our results, let $\mathbb{K}$ be a field of characteristic zero, $S := \mathbb{K}[x_1, \ldots, x_n]$, $\mathcal{I} \subset S$ a zero-dimensional ideal, and $A := S/\mathcal{I}$. Let $p, q \in S$ such that $q$ is not a zero divisor in $A$. Then, $r := \frac{p}{q}$ induces a $\mathbb{K}$-linear map

$$\text{Times}_r : A \to A \quad a \mapsto r \cdot a.$$ (1)

We are interested in the computation of the trace of this map. First, we relate $\text{Trace}(\text{Times}_r)$ with an algebraic object depending on $\mathcal{I}$ and $r$: the generalized Chow form of the ideal $\mathcal{I}$ (see [Phi] for the definition of this eliminating polynomial).

Then, we focus on the case where $\mathcal{I}$ is given by a complete intersection in the torus. We show that, in this case, the generalized Chow form may be replaced with a sparse resultant (in the sense of [CLO, GKZ]), and we exhibit numerator and denominator formulas for traces in terms of sparse resultants, similar to the denominator formulas obtained for residues in [CDS2]. In particular, our trace formulas can be used for computing global residues.

Finally, we compare our formulas for the denominator of the trace with those obtained from the formulas proposed in [CDS2] for denominators of residues in the torus.

Effective procedures for the computation of traces can be derived from our formulas. In order to do this, one has to deal with Chow forms and resultants. In the case $\mathcal{I}$ is a radical ideal, the generalized Chow form of $\mathcal{I}$ coincides with the generalized Chow form of the variety $V(\mathcal{I})$ and there are effective algorithms for its computation (see, for instance, [JKSS]). Algorithms for the computation of sparse resultants can be found in [CE, CLO, EP, JS] and the references given therein.

On the other hand, our factorization formulas for denominators of traces and residues could lead to a better understanding of non-generic situations, namely, those coefficient vectors for which the denominators vanish.

The paper is organized as follows: In Section 2 we show the general formula for the computation of the trace based on the generalized Chow form, and then we focus on the case where we have a generic complete intersection in the torus. We give rational expressions for both the numerator and the denominator of the trace. We compare our results with those obtained in [CDS2] for the computation of residues in Section 3. Section 4 is concerned with multidimensional residues on affine space. By using results of Jouanolou we recover known denominator formulas for residues given in [Ts, El, CDS1, CDS2] and also give an algebraic proof of the Euler-Jacobi formula.

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2 Rational expressions for traces

2.1 A Chow form–based formula

As in the introduction, let $\mathbb{K}$ be a field of characteristic zero, and $S := \mathbb{K}[x_1, \ldots, x_n]$. Let $\mathcal{I}$ be a zero-dimensional ideal of $S$, and set $A := S/\mathcal{I}$.

Let $p, q \in S$ such that $q$ is not a zero divisor in $A$. Set $d := \max\{\deg(p), \deg(q)\}$, where $\deg$ denotes total degree in the variables $x_1, \ldots, x_n$. We introduce new variables $U_\alpha$ for $\alpha \in (\mathbb{Z}_{\geq 0})^n$, $|\alpha| \leq d$. Let $U := (U_{\alpha}, |\alpha| \leq d)$ and $Ch_{d, \mathcal{I}}(U)$ be the generalized Chow form of $\mathcal{I}$ (see [Phi]): if $\mathcal{U}(x) := \sum_{|\alpha| \leq d} U_{\alpha} x^\alpha$, then $Ch_{d, \mathcal{I}}(U) = \prod_{\xi \in \mathcal{V}(\mathcal{I})} \mathcal{U}(\xi)^{m(\xi)}$, where $m(\xi)$ is the multiplicity of $\xi$ with respect to $\mathcal{I}$, i.e. the dimension of the local ring $A_\xi := S_\xi/\mathcal{I}S_\xi$.

**Theorem 2.1.** Let $\text{Times}_r$ be the map defined in (1). Then,

$$\text{Trace}(\text{Times}_r) = \frac{\sum_{|\alpha| \leq d} P_{\alpha} \frac{\partial Ch_{d, \mathcal{I}}(q)}{\partial U_{\alpha}}(q)}{Ch_{d, \mathcal{I}}(q)}$$

(2)

Observe that as $q$ is not a zero divisor in $A$, then the denominator of (2) is not zero.

**Proof.** Let $T$ be a new variable and set

$$\mathcal{X}(T) := Ch_{d, \mathcal{I}}(q + T p) = C_0 + C_1 T + \text{higher order terms in } T.$$

Using the identity $Ch_{d, \mathcal{I}}(q + T p) = \prod_{\xi \in \mathcal{V}(\mathcal{I})} (q(\xi) + T p(\xi))^{m(\xi)}$, we get

$$\mathcal{X}(T) = \prod_{\xi \in \mathcal{V}(\mathcal{I})} q(\xi)^{m(\xi)} + \left( \sum_{\xi \in \mathcal{V}(\mathcal{I})} m(\xi)p(\xi)q(\xi)^{m(\xi)-1} \prod_{\xi' \in \mathcal{V}(\mathcal{I}), \xi' \neq \xi} q(\xi')^{m(\xi')} \right) T + \text{higher order terms in } T.$$

From these identities, recalling that the trace of $\text{Times}_r$ is equal to $\sum_{\xi \in \mathcal{V}(\mathcal{I})} m(\xi)r(\xi)$, it is easy to see that

$$\text{Trace}(\text{Times}_r) = \frac{C_1}{C_0} = \frac{\mathcal{X}'(0)}{\mathcal{X}(0)}.$$

By applying the chain rule to compute the numerator, we get Identity (2). ⊓⊔

2.2 Traces in the torus

Now we turn our attention to generic complete intersections in the torus. We will give rational formulas for traces of linear maps in terms of sparse resultants.

Consider a system of $k$ generic Laurent polynomials $f_i := \sum_{a \in A_i} c_{ia} t^a$, where $i = 1, \ldots, k$, $A_i$ is a finite subset of $\mathbb{Z}^k$, the coefficients $c_{ia}$ are indeterminates over $\mathbb{Q}$, $a = (a_1, \ldots, a_k) \in \mathbb{Z}^k$ and $t^a = t_1^{a_1} t_2^{a_2} \ldots t_k^{a_k}$. For $i = 1, \ldots, k$, the support of $f_i$ is the set of exponent vectors $A_i \subset \mathbb{Z}^k$, and its Newton polytope is the convex hull $P_i = \text{conv}(A_i) \subset \mathbb{R}^k$.

We assume that the lattice affinely generated by $A_1, \ldots, A_k$ (that is, the $\mathbb{Z}$-submodule of
\[ \mathbb{Z}^k \] generated by the vectors that are differences of two points in a set \( A_i \) is a \( k \)-dimensional affine sublattice of \( \mathbb{Z}^k \).

Let \( K \) be an algebraic closure of \( \mathbb{Q}(c_{a_i})_{1 \leq i \leq k}, a \in A_i \) and \( S := K[t_1, t_1^{-1}, \ldots, t_k, t_k^{-1}] \). Then \( A := S/(f_1, \ldots, f_k) \) is a finite-dimensional \( K \)-vector space (see [PS]). Let \( A, A' \) be finite subsets of \( \mathbb{Z}^k \) and consider a generic rational function of the form \( r(t) := \frac{p(t)}{q(t)} \) where \( p(t) = \sum_{a \in A} p_a t^a \), \( q(t) = \sum_{a \in A'} q_a t^a \), and \( p_a, q_a \) are new indeterminates. As \( q(t) \) is a generic denominator, it is invertible in \( A \). We can then consider the \( K \)-linear map \( \text{Times}_r : A \to A \) as defined in [1].

For any family of finite subsets \( B_1, \ldots, B_s \subset \mathbb{Z}^k \) we denote \( L(B_1, \ldots, B_s) \) the affine lattice generated by \( B_1, \ldots, B_s \):

\[
L(B_1, \ldots, B_s) = \left\{ \sum_{1 \leq i \leq s} \lambda_i b^{(i)} \mid b^{(i)} \in B_i, \lambda_i \in \mathbb{Z} \text{ for all } 1 \leq i \leq s \text{ and } \sum_{1 \leq i \leq s} \lambda_i = 1 \right\}.
\]

Let \( A_0 := A \cup A' \) and \( f_0 = \sum_{a \in A_0} c_{0a} t^a \), where \( c_{0a} \) are new indeterminates.

As in [Stu Section 1], for \( I \subset \{0,1,\ldots,k\} \), the collection of supports \( \{A_i\}_{i \in I} \) is said to be essential if \( \text{rank}(L(A_i, i \in I)) = \#I - 1 \) and \( \text{rank}(L(A_j, j \in J)) \geq \#J \) for each proper subset \( J \) of \( I \).

Let \( M \) be the mixed volume of the sequence of polytopes \( P_1, \ldots, P_k \) (see [CLO] for a definition). From now on, we will assume that \( M > 0 \); otherwise, our problem has no interest. Then, \( A_0, A_1, \ldots, A_k \) has a unique essential subset containing \( A_0 \). Moreover, the sparse resultant operator \( \text{Res}_{A_0, \ldots, A_k} \) as defined in [Stu] is not constantly one and, if \( \{A_i\}_{i \in I} \) is the essential subset, it coincides with the resultant \( \text{Res}_{A_i, i \in I} \) considered with respect to the lattice \( L(A_i, i \in I) \) (see [Stu Corollary 1.1]).

Throughout this section, we are going to use some results and notation from [Min] that we now recall.

For a family of finite sets \( B_0, \ldots, B_s \subset \mathbb{Z}^k \), if \( \{B_i\}_{i \in I} \) is the unique essential subfamily, consider the orthogonal decomposition \( L(B_0, \ldots, B_s) = \text{sat}(L(B_i, i \in I)) \oplus L^1 \), where \( \text{sat}() \) denotes saturation with respect to the ambient lattice and \( L^1 \) is the orthogonal complement (recall that, if \( L \) is a sublattice of a lattice \( \mathcal{L} \), the saturation of \( L \) with respect to \( \mathcal{L} \) is defined as \( \text{sat}(L) = (\mathbb{Q} \otimes \mathbb{Z} \mathcal{L}) \cap \mathcal{L} \)). If \( L^1 = 0 \), we define \( e_{B_0, \ldots, B_s} := 1 \). Otherwise, denote with \( p \) the projection onto the second factor and define \( e_{B_0, \ldots, B_s} \) as the normalized mixed volume of the family \( \{p(\text{conv}(B_i))\}_{i \notin I} \) in \( L^1 \) (see [Min]).

**Remark 2.2.** The reader should be cautious when comparing our statements with Minimair’s results in [Min]. Indeed, in Remark 3 of [Min] the resultant is defined as a power of an irreducible polynomial. In this paper, a resultant is always an irreducible polynomial.

Assuming that \( A_0, A_1, \ldots, A_k \) has a unique essential subset containing \( A_0 \), the following Poisson-type product formula holds ([Min Lemma 13]):

\[
\text{Res}_{A_0, \ldots, A_k}(f_0, f_1, \ldots, f_k)^{e_{A_0, \ldots, A_k}} = C \prod_{\beta \in V^*(f_1, \ldots, f_k)} f_0(\beta),
\]

where \( C \in K^* \), and \( V^* \) is the set of zeros over \( K^* \) with respect to the lattice \( L(A_0, \ldots, A_k) \).
For every \( \omega \in \mathbb{Z}^k \), every set \( A \subset \mathbb{Z}^k \) and any polynomial \( f \) in \( k \) variables with support \( A \), we denote \( a_A(\omega) = -\min\{\langle \omega, v \rangle : v \in A \} \), \( A^\omega \) the set of points in \( A \) that lie in the face with inward normal vector \( \omega \), \( f^\omega \) the polynomial formed by the monomials of \( f \) lying in \( A^\omega \), and \( H^\omega \) the lattice of integer points contained in the hyperplane orthogonal to \( \omega \) in \( \mathbb{Z}^k \).

Under the previous assumptions and notation, if \( A'_0 \subset A_0 \) and \( f'_0 \) is a generic polynomial with support \( A'_0 \), the main result in [Min] establishes a relation between the specialized resultant \( \text{Res}_{A_0, A_1, \ldots, A_k}(f'_0, f_1, \ldots, f_k) \) and \( \text{Res}_{A'_0, A_1, \ldots, A_k}(f'_0, f_1, \ldots, f_k) \) ([Min, Theorem 1]):

\[
\text{Res}_{A_0, A_1, \ldots, A_k}(f'_0, f_1, \ldots, f_k)^{e_{A_0, A_1, \ldots, A_k}} = \\
= \text{Res}_{A'_0, A_1, \ldots, A_k}(f'_0, f_1, \ldots, f_k)^{e_{A'_0, A_1, \ldots, A_k}} \cdot \prod_{\omega \in A_0} \text{Res}_{A_1^\omega, \ldots, A_k^\omega}(f_1^\omega, \ldots, f_k^\omega)^{e_{A_1^\omega, \ldots, A_k^\omega}(a_{A_0}(\omega) - a_{A'_0}(\omega)) \div \text{dim}(L(A_0, A_1, \ldots, A_k))},
\]

where the product ranges over all the primitive inward normal vectors to the facets of the convex hull of \( A_1 + \cdots + A_k \).

Identity (3) enables us to use \( \text{Res}_{A_0, \ldots, A_k} \) instead of the generalized Chow form used in the previous section for trace computations. The analogue of Theorem 2.1 is the following:

**Theorem 2.3.** Under the previous assumptions and notations,

\[
\text{Trace}(\text{Times}_r) = d_{A_0, \ldots, A_k} \sum_{a \in A} p_a \frac{\partial \text{Res}_{A_0, \ldots, A_k}}{\partial c_{0a}}(q, f_1, \ldots, f_k) / \text{Res}_{A_0, \ldots, A_k}(q, f_1, \ldots, f_k),
\]

where \( d_{A_0, \ldots, A_k} := [\mathbb{Z}^k : L(A_0, \ldots, A_k)] e_{A_0, \ldots, A_k} \).

**Proof.** Suppose first that \( [\mathbb{Z}^k : L(A_0, \ldots, A_k)] = 1 \). In this case, Identity (3) reads as follows:

\[
\text{Res}_{A_0, \ldots, A_k}(f_0, f_1, \ldots, f_k)^{e_{A_0, \ldots, A_k}} = C \prod_{\gamma \in V(f_1, \ldots, f_k)} f_0(\gamma),
\]

with \( C \in K^* \). Now, as in the proof of Theorem 2.1 we make the substitution \( f_0 \mapsto q +Tp \), and get

\[
\text{Res}_{A_0, \ldots, A_k}(q + Tp, f_1, \ldots, f_k)^{e_{A_0, \ldots, A_k}} = \text{Res}_{A_0, \ldots, A_k}(q, f_1, \ldots, f_k)^{e_{A_0, \ldots, A_k}} + \\
+ e_{A_0, \ldots, A_k} \text{Res}_{A_0, \ldots, A_k}(q, f_1, \ldots, f_k)^{e_{A_0, \ldots, A_k} - 1} \frac{\partial \text{Res}_{A_0, \ldots, A_k}}{\partial T}(q + Tp, f_1, \ldots, f_k) |_{T=0} + \text{higher order terms in } T.
\]

By using the chain rule, we have that

\[
\frac{\partial \text{Res}_{A_0, \ldots, A_k}}{\partial T}(q + Tp, f_1, \ldots, f_k) = \sum_{a \in A} \frac{\partial \text{Res}_{A_0, \ldots, A_k}}{\partial c_{0a}}(q + Tp, f_1, \ldots, f_k) p_a.
\]
The claim follows by substituting this identity in the previous one and noticing that \( \text{Trace}(\text{Times}_r) \) is the quotient of the coefficient of the degree 1 term in \( T \) by the coefficient of degree 0 in \( T \) in the polynomial obtained.

In the general case, we also use Identity 8. By raising both sides of this equality to the power \([\mathbb{Z}^k : L(A_0, \ldots, A_k)]\), the claim is proved as in the previous case if we show that

\[
\prod_{\beta \in \mathbb{V}(\mathbb{F})} f_0(\beta)^{[\mathbb{Z}^k : L(A_0, \ldots, A_k)]} = \prod_{\gamma \in \mathbb{V}(\mathbb{F})} f_0(\gamma).
\]

By using normal Smith form reduction (this change preserves resultants, see [Min]), we can suppose that \( L(A_0, \ldots, A_k) = d_1 \mathbb{Z} \oplus \ldots \oplus d_k \mathbb{Z} \), with \([\mathbb{Z}^k : L(A_0, \ldots, A_k)] = d_1 d_2 \ldots d_k\). Then, 8 follows straightforwardly from [Min Corollary 5].

**Example 2.4.** Consider the following trivariate system:

\[
\begin{align*}
  f_1 &= c_{11} + c_{12} t_1^2 + c_{13} t_2^2 \\
  f_2 &= c_{21} + c_{22} t_1^2 + c_{23} t_2^2 \\
  f_3 &= c_{31} + c_{32} t_1^2 + c_{33} t_2^2 + c_{34} t_3^2,
\end{align*}
\]

and let \( \mathcal{A} = \{(0, 2, 0)\}, \mathcal{A}' = \{(0, 0, 0)\} \). In this case, the family \( \{A_0, A_1, A_2, A_3\} \) is essential, so \( e_{A_0, \ldots, A_3} = 1 \), but \([\mathbb{Z}^3 : L(A_0, \ldots, A_3)] = 8\). A straightforward computation shows that

\[
\text{Trace}(\text{Times}_{t_2}) = 8 \cdot c_{11} c_{32} c_{23} + c_{21} c_{12} c_{34} - c_{31} c_{12} c_{23} - c_{12} c_{23} c_{34} + c_{12} c_{23} c_{34} - c_{32} c_{13} c_{23}.
\]

This is a generalization of the example which appears at the end of [Ped], where the intersection of two perpendicular cylinders with a sphere is considered:

\[
\begin{align*}
  f_1 &= -1 + t_1^2 + t_2^2 \\
  f_2 &= -1 + t_2^2 + t_3^2 \\
  f_3 &= -1 + t_1^2 + t_2^2 + t_3^2,
\end{align*}
\]

and the trace of the multiplication by \( t_2^2 \) is shown to be 8.

In the remaining part of this section we will give an explicit factorization of the denominator of the trace. The following result is straightforward due to the irreducibility of the resultant of a system of generic polynomials ([GKZ]), and the fact that the degree of the numerator of the right hand side of 7 with respect to the variables \( c_{ia}, q_a \) is strictly less than the degree of the denominator.

**Lemma 2.5.** Suppose that \( \mathcal{A}_0 = \mathcal{A}' \) (i.e. \( \mathcal{A} \subset \mathcal{A}' \)). Then either the trace is identically zero, or the right hand side of 7 is the irreducible representation of the trace as a rational function in \( \mathbb{Q}(c_{ia}, p_a, q_a) \).

Suppose now that \( \mathcal{A} \) is not contained in \( \mathcal{A}' \). As the trace is linear in the monomial expansion of \( p \), it is enough to consider the following situation: \( p(t) = t^a \), with \( a \not\in \mathcal{A}' \).

Without loss of generality, let \( \{A_0, \ldots, A_j\} \) (\( j \leq k \)) be the unique essential subfamily of \( \mathcal{A}_0, \ldots, \mathcal{A}_k \). Let \( \delta_{\mathcal{A}'} := e_{\mathcal{A}' \cdot A_1, \ldots, A_j} e_{A_0, A_1, \ldots, A_j}^{-1} [L(A_0, A_1, \ldots, A_j) : L(\mathcal{A}', A_1, \ldots, A_j)] \).
For each facet of the Minkowski sum $P_1 + \ldots + P_j$, we consider its primitive inward normal vector $\omega$ and define $\mu_{\omega} := \min\{ (b, \omega), \ b \in A' \} - \min\{ (b, \omega), \ b \in A_0 \}$. Observe that $\mu_{\omega} \geq 0$ and equality may hold. Set $\delta_{\omega} := \mu_{\omega} e_{A_1} \ldots e_{A_j} e_{A_0}^{1} \omega_{\perp} : L(A_1^\perp, \ldots, A_j^\perp)$, where $\omega_{\perp}$ is the lattice of integer points contained in the hyperplane orthogonal to $\omega$ in $L(A_0, A_1, \ldots, A_j)$. With this notation, Identity (4) gives us the following:

**Proposition 2.6.** In the situation described above, we have that the denominator of (5) has the following irreducible factorization:

$$\text{Res}_{A_1, A_2, \ldots, A_k}(q, f_1, \ldots, f_k)^{A'} \prod_{\omega} \text{Res}_{A_1^\perp, A_2^\perp, \ldots, A_j^\perp}(f_1^\omega, \ldots, f_k^\omega)^{\delta_{\omega}}$$

where $\omega$ ranges over the primitive inward normal vectors of the facets of $P_1 + \ldots + P_j$.

**Example 2.7.** Consider the following system

$$\begin{align*}
  f_1 &= c_1 t_1 + c_1 t_1^2 + c_3 t_2^2 \\
  f_2 &= c_2 t_1 + c_2 t_1^2 + c_3 t_2^2.
\end{align*}$$

We set $q := q_1 + q_2 t_1 + q_3 t_2$, $a := (2, 0)$. The Newton polygon $P_1 + P_2$ is a pentagon whose vertices are $(0, 3), (1, 3), (3, 1), (3, 0), (1, 1)$. The inward normal vectors $\omega$ of this polygon satisfying $\mu_{\omega} > 0$ are $(-1, -1)$ and $(-1, 0)$. The facet resultants associated with these edges are $c_{22} c_{12} - c_{13} c_{23}$ and $c_{23}$ respectively and

$$\text{Res}_{A_0, A_1, A_2}(q, f_1, f_2) = \text{Res}_{A_1, A_2}(q, f_1, f_2) c_{23} \left( c_{22} c_{12} - c_{13} c_{23} \right).$$

This is the irreducible decomposition of the denominator of $\text{Trace}(\text{Times}_{t_{a/q}})$. Indeed, computing explicitly, we get that its numerator is

$$2c_{13} c_{2}^2 c_{23}^2 q_2 c_{21} q_1 - 3c_{13} c_{2} c_{23} q_2 c_{21} q_1 - 2c_{13} c_{21} c_{2}^2 q_2^2 c_{23} c_{12} + 2c_{13} c_{21}^2 q_2^2 c_{23} c_{22} c_{12} - c_{13} c_{21}^2 c_{2}^2 q_1^2$$

which is an irreducible polynomial and does not divide (5).

### 3 Traces and Residues in the torus

In this section, we will compare our denominator formulas for traces in the torus with those that can be obtained by applying the results given in CDS2. As in Section 2.2, we will be dealing with a system of $k$ generic Laurent polynomials $f_i := \sum_{a \in A_i} c_{ia} t^a$, where $i = 1, \ldots, k$, $A_i$ is a finite subset of $\mathbb{Z}^k$, and the $c_{ia}$’s are indeterminates over $\mathbb{Q}$.

For a given Laurent polynomial $p$ in $S := K[t_{1}^{-1}, \ldots, t_{k}^{-1}]$, where $K$ was defined in Section 2.2 the global residue of the differential form

$$\phi_p := \frac{p}{f_1 \ldots f_k} \frac{dt_1}{t_1} \wedge \ldots \wedge \frac{dt_k}{t_k}$$
equals, with our notation,

\[ \text{Residue}_f^T(p) := \text{Trace}(\text{Times}_p), \tag{9} \]

where \( J_f^T \) denotes the affine toric Jacobian \( J_f^T := \det \left( t_j \frac{\partial f_i}{\partial t_j} \right)_{1 \leq i,j \leq k} \).

In [CDS2], formulas for the denominator of the rational expression (9) were proposed. In particular, for any Laurent monomial \( t^a \in S \), by replacing \( p \) with \( t^a J_f^T \) in (9), we get that

\[ \text{Trace}(\text{Times}_t^a) = \text{Residue}_f^T(t^a J_f^T), \]

and so, the formulas in [CDS2] can be used for computing the denominator of the trace of \( \text{Times}_t^a \). Now, we will compare the denominator formula obtained in this way with our results in Section 2.2.

Assume that, for \( i = 1, \ldots, k \), \( A_i = \Delta_i \cap \mathbb{Z}^k \) for an integral polytope \( \Delta_i \) in \( \mathbb{R}^k \) and, without loss of generality, that \( A_i \subset (\mathbb{Z}_{>0})^k \).

Under these assumptions, the support of each of the polynomials \( t_j \frac{\partial f_i}{\partial t_j} \) (\( j = 1, \ldots, k \)) is \( A_i \), and therefore, the support of \( J_f^T \) is contained in \( \Delta \cap \mathbb{Z}^k \), where \( \Delta := \Delta_1 + \cdots + \Delta_k \).

Thus, in order to obtain a denominator for \( \text{Trace}(\text{Times}_t^a) \) it suffices to get denominator formulas for the residues in the torus of those monomials \( t^m \) with \( m \in (\Delta \cap \mathbb{Z}^k) + a \).

First, we introduce some notation. For each facet of \( \Delta \) with primitive inward normal vector \( \omega \), let \( a_\omega \in \mathbb{Z} \) be defined as

\[ a_\omega := -\min\{ \langle b, \omega \rangle : b \in \Delta \}, \]

and for every \( m \in \mathbb{Z}^k \), let

\[ \mu_\omega^-(m) := -\min\{0, \langle m, \omega \rangle + a_\omega - 1\}, \]
\[ \delta_\omega^-(m) := \mu_\omega^-(m)[\omega^\perp : L(A_1^\omega, \ldots, A_k^\omega)]. \]

Then, [CDS2] Theorem 3.2] states that \( \prod_\omega \text{Res}_{A_1^\omega, \ldots, A_k^\omega}(f_1^\omega, \ldots, f_k^\omega) \delta_\omega^-(m) \), where the product runs over all the primitive inward normal vectors of facets of \( \Delta \), is a denominator for \( \text{Residue}_f^T(t^m) \). Therefore, if

\[ \delta_\omega := \max\{ \delta_\omega^-(m) : m \in (\Delta \cap \mathbb{Z}^k) + a \}, \]

the following polynomial is a denominator for \( \text{Trace}(\text{Times}_t^a) = \text{Residue}_f^T(t^a J_f^T) \):

\[ \prod_\omega \text{Res}_{A_1^\omega, \ldots, A_k^\omega}(f_1^\omega, \ldots, f_k^\omega) \delta_\omega. \]

Let us estimate the exponents \( \delta_\omega \). For \( m \in (\Delta \cap \mathbb{Z}^k) + a \), write \( m = m_\Delta + a \) with \( m_\Delta \in \Delta \cap \mathbb{Z}^k \). We have that \( \langle m, \omega \rangle + a_\omega - 1 = \langle m_\Delta, \omega \rangle + a_\omega + \langle a, \omega \rangle - 1 \geq \langle a, \omega \rangle - 1 \).

We will consider two cases separately:
\[\mu_{\omega}(m) = 0. \text{ Therefore, } \delta_{\omega}' = 0.\]

\[\langle a, \omega \rangle \leq 0: \text{ taking } m_{\Delta} \in \Delta \cap \mathbb{Z}^k \text{ so that } \langle m_{\Delta}, \omega \rangle + a_{\omega} = 0, \text{ for } m := m_{\Delta} + a, \text{ we get } \langle m, \omega \rangle + a_{\omega} - 1 = \langle a, \omega \rangle - 1 \leq -1. \text{ Then, } \mu_{\omega}(m) = 1 - \langle a, \omega \rangle \geq 1 \text{ and therefore, } \delta_{\omega}' = (1 - \langle a, \omega \rangle)L(A_1^\omega, \ldots, A_k^\omega).\]

We deduce the following formula for a denominator of Trace(\(\times a\)):

\[
\prod_{\omega: \langle a, \omega \rangle \leq 0} \text{Res}_{A_1^\omega, \ldots, A_k^\omega}(f_1^\omega, \ldots, f_k^\omega)^{(1 - \langle a, \omega \rangle)L(A_1^\omega, \ldots, A_k^\omega)}. \tag{10}
\]

Finally, we will restate the result in Proposition 2.4 in this context: here \(A = \{a\}, A' = \{0\}\) and \(A_0 = \{0, a\}\). The first resultant appearing in the factorization equals 1. On the other hand, it follows from the definition that \(\mu_{\omega} = -\min\{0, \langle a, \omega \rangle\}\), and so, \(\delta_{\omega} = -\langle a, \omega \rangle L(A_1^\omega, \ldots, A_k^\omega)\) if \(\langle a, \omega \rangle \leq 0\) and \(\delta_{\omega} = 0\) otherwise. We conclude that

\[
\prod_{\omega: \langle a, \omega \rangle \leq 0} \text{Res}_{A_1^\omega, \ldots, A_k^\omega}(f_1^\omega, \ldots, f_k^\omega)^{-\langle a, \omega \rangle L(A_1^\omega, \ldots, A_k^\omega)} \tag{11}
\]

is a denominator for Trace(\(\times a\)).

By comparing (11) with (10) we get a slightly improvement on the exponent of each of the factors in the denominator.

### 4 Multidimensional Residues in \(\mathbb{C}^n\)

In the case where the underlying variety is the zero locus of a regular sequence of \(n\) polynomials in \(\mathbb{C}[x_1, \ldots, x_n]\) without zeroes in the infinity, some results of Jouanolou on generalized discriminants given in [Jou] (see also [GKZ]) will allow us to use our formulas in order to recover known results about residues ([Elk] [CDS1] [CDS2]).

#### 4.1 The denominator of the residue

Let \(f := (f_1, \ldots, f_n)\) with \(f_i := \sum_{|a| \leq d_i} c_{ia}x^a, i = 1, \ldots, n\), be a generic system of polynomials in \(\mathbb{K}[x_1, \ldots, x_n]\) of respective degrees \(d_1, \ldots, d_n\). Set \(J_f := \det(\partial f_i/\partial x_j)\) for the Jacobian determinant of the system. Let us observe that \(\deg(J_f) = \rho := \sum_{i=1}^n d_i - n\).

For every \(\beta \in (\mathbb{Z}_{\geq 0})^n\), the global residue associated with the data \((x^\beta, f)\) can be obtained as

\[
\text{Residue}_f(x^\beta) = \text{Trace}(\text{Times}_{x^\beta/J_f}),
\]

where the linear maps in the right hand side of the equation are defined in the quotient ring \(\mathbb{K}[x_1, \ldots, x_n]/(f_1, \ldots, f_n)\). Applying (5), we obtain the following expression for the residue as a rational function in the coefficients of \(f\):

\[
\text{Residue}_f(x^\beta) = \frac{\partial \text{Res}_{D,d_1,\ldots,d_n}(f_0,f_1,\ldots,f_n)}{\partial x_0}\bigg|_{x_0 \to J_f},
\]

\[
\text{Res}_{D,d_1,\ldots,d_n}(J_f,f_1,\ldots,f_n)
\]
where $D := \max \{|\beta|, \rho\}$, $f_0$ is a generic polynomial of degree $D$, $\text{Res}_{D,d_1,\ldots,d_n}$ is the classic resultant of a generic system of $n+1$ polynomials with respective degrees $D, d_1, \ldots, d_n$, and $\frac{\partial}{\partial \alpha_i}$ stands for the derivative with respect to the coefficient of the monomial $x^\beta$ in the first polynomial.

Due to Proposition 2.6 if $f_i^0 := \sum_{|a|=d_i} c_a x^a$ is the homogeneous part of degree $d_i$ of the polynomial $f_i$, $i = 1, \ldots, n$, the denominator vanishes. As $\frac{\partial}{\partial \alpha_i}$ vanishes. Hence, the expression on the left hand side of (14) still has sense and is a finite determinant.

In the case when the toric variety associated to the sparse system is smooth.

Let us show now that $\text{Res}_{D,d_1,\ldots,d_n}(J_f, f_1, \ldots, f_n) = \text{Res}_{d_1,\ldots,d_n}(f_0^1, \ldots, f_n^1) D-\rho \text{Res}_{d_1,\ldots,d_n}(J_f, f_1, \ldots, f_n)$. (12)

Now, according to [Jou], the following identity holds:

$$\text{Res}_{d_1,\ldots,d_n}(J_f, f_1, \ldots, f_n) = \text{Res}_{d_1,\ldots,d_n}(f_0^1, \ldots, f_n^1) \text{Disc}(f).$$

Here, $\text{Disc}(f)$ denotes the discriminant of the polynomial system $f = (f_1, \ldots, f_n)$. Then, we can factor the denominator of the residue further. This result

Remark 4.1. Identity (13) can be also proved by applying the results about principal A determinants given in [GKZ] Chapter 10 to the polynomial $f_1 + \sum_{i=2}^n T_i f_i$, with $T_2, \ldots, T_n$ new variables. Theorem 1.2 in [GKZ] Chapter 10 essentially implies (13).

By replacing (13) in (12), we get

$$\text{Res}_{d_1,\ldots,d_n}(J_f, f_1, \ldots, f_n) = \text{Res}_{d_1,\ldots,d_n}(f_0^1, \ldots, f_n^1) \text{Disc}(f).$$

with

$$\text{Res}_{d_1,\ldots,d_n}(J_f, f_1, \ldots, f_n) = \frac{A_\beta(f)}{\text{Res}_{d_1,\ldots,d_n}(f_0^1, \ldots, f_n^1) D+1-\rho \text{Disc}(f)},$$

with $A_\beta(f) = \partial \text{Res}_{d_1,\ldots,d_n}(f_0, f_1, \ldots, f_n) / \partial f_0 | f_0 \rightarrow J_f$.

Let us show now that $A_\beta(f)$ is also divisible by the discriminant $\text{Disc}(f)$. Recall that $\text{Disc}(f)$ is an irreducible polynomial in the coefficients of $f$ which vanishes if and only if the system $f$ has a multiple root provided that $f$ does not have roots at infinity (see [GKZ]).

Suppose that we specialize $f$ in such a way that $\text{Res}_{d_1,\ldots,d_n}(f_0^1, \ldots, f_n^1) \neq 0$ but $\text{Disc}(f)$ vanishes. Hence, the expression on the left hand side of (14) still has sense and is a finite complex number. On the right hand side of this identity, the denominator vanishes. As $\text{Res}_{d_1,\ldots,d_n}(f_0^1, \ldots, f_n^1) = 0$ (see for instance the “Principle of Continuity” in [GH] p. 657), then $A_\beta(f)$ must be zero also. This implies that $\text{Disc}(f)$ divides $A_\beta(f)$, and hence

$$\text{Res}_{d_1,\ldots,d_n}(J_f, f_1, \ldots, f_n) = \frac{A_\beta'(f)}{\text{Res}_{d_1,\ldots,d_n}(f_0^1, \ldots, f_n^1) D-\rho+1},$$

with $A_\beta'(f) \in \mathbb{K}[c_{ia}]$ (c.f. [Elk] Proposition 2.1), [BGVY] Proposition 5.15).

Remark 4.2. In [CDS2], a formula like (15) is given for systems of generic sparse polynomials, where the denominator is now a product of sparse facet resultants. Our technique is limited only to the homogenous (dense) case, due to the fact that there is still not known formula like (15) for sparse systems. In [GKZ] Chapter 10 some general formulae is given, but only in the case when the toric variety associated to the sparse system is smooth.
4.2 An algebraic proof of the Euler-Jacobi vanishing theorem

The following result is very well known in the literature, see for instance [Mac] or [AGV, Chapter 5, Corollary 4]. We will recover it by using Identity (15) and algebraic methods.

**Theorem 4.3** (Euler-Jacobi Formula). For a system \( f = (f_1, \ldots, f_n) \) as before, and any polynomial \( h \) of degree less than the degree of the Jacobian of the system, \( \text{Res} \ f(h) = 0 \).

**Proof.** It suffices to show that \( \text{Res} \ f(x^\beta) = 0 \) for every monomial of degree at most \( \rho := \deg(J_f) \).

For \( i = 1, \ldots, n \), let \( f_{t,i} := \sum_{|a| \leq d_i} c_{ia} t^{d_i-|a|} x^a \in \mathbb{K}[t][x_1, \ldots, x_n] \) and let \( f_t \) be the generic system \( f_t := (f_{t,1}, \ldots, f_{t,n}) \).

Fix \( \beta \in (\mathbb{Z}_{\geq 0})^n \) with \( |\beta| < \rho \). In what follows, we will relate the global residue of \( x^\beta \) with respect to the original system \( f \) with its global residue with respect to the system \( f_t \).

First, let us observe that \( \{\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{A}^n : f_t(\xi) = 0\} = \{t\eta = (t\eta_1, \ldots, t\eta_n) \in \mathbb{A}^n : f(\eta) = 0\} \)

and that \( J_{f_t}(t\eta) = t^\rho J_f(\eta) \), which implies that

\[
\text{Res} f_t(x^\beta) = \sum_{f_t(\xi) = 0} \frac{\xi^\beta}{J_{f_t}(\xi)} = \sum_{f(\eta) = 0} \frac{(t\eta)^\beta}{t^\rho J_f(\eta)} = t^{||\beta|-\rho|} \sum_{f(\eta) = 0} \eta^\beta J_f(\eta) = t^{||\beta|-\rho|} \text{Res} f(x^\beta).
\]

On the other hand, taking into account that \( f^0_t = f^0_{t,i} \), Identity (15) applied to system \( f_t \) states that

\[
\text{Res} f_t(x^\beta) = \frac{A'_\beta(f_t)}{\text{Res}_{d_1,\ldots,d_n}(f^0_1,\ldots,f^0_n)^{D-\rho+1}}.
\]

Finally, combining the previous identities we deduce:

\[
A'_\beta(f_t) = t^{||\beta|-\rho|} A'_\beta(f).
\]

Now, \( A'_\beta(f_t) \) is a polynomial in \( \mathbb{K}[t,c_{ia}] \), while \( A'_\beta(f) \in \mathbb{K}[c_{ia}] \) does not depend on \( t \) and \( ||\beta|-\rho| < 0 \). Then, the above equality is only possible if \( A'_\beta(f) = 0 \) and so, \( \text{Res} f(x^\beta) = 0 \).

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