A note on holomorphic extensions

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Abstract. We give a criterion of holomorphy for some type formal power series. This gives a stronger form of a Rothstein’s type extension theorem for a particular ring of holomorphic functions.

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We consider the set $R \subset \mathbb{C}[[z_1, z_2]], z_1, z_2 \in \mathbb{C}^k \times \mathbb{C}^m$ of formal power series of the form

$$f(z) = f(z_1, z_2) = \sum_{n} P_n(z_2)z_1^n$$

where $P_n$ is a polynomial in $m$ variables of total degree $d^0 P_n \leq C_0 + C_1 ||n||$, for some constants $C_0, C_1 > 0$. One easily checks that $R$ is a local sub-ring of $\mathbb{C}[[z_1, z_2]]$. For the notion of $\Gamma$-capacity, that generalizes the notion of capacity in one complex variable, we refer to [Ro].

Theorem. Let

$$f(z_1, z_2) = \sum_{n} P_n(z_2)z_1^n$$

be a formal power series of the two complex variables $(z_1, z_2) \in \mathbb{C}^k \times \mathbb{C}^m$. We assume that $(P_n)$ is a sequence of polynomials in $m$ variables of total degree

$$\deg P_n \leq C_0 + C_1 ||n||.$$

We assume that for a set $K \in \mathbb{C}^m$ of positive $\Gamma$-capacity, $z_2 \in K$ being fixed, the formal power series $f(z_1, z_2)$ converges.

Then for some $C_2 > 0$, the formal power series $f$ defines a holomorphic function in a neighborhood of the axes $\{z_1 = 0\}$ of the form,

$$U = \{(z_1, z_2) \in \mathbb{C}^k \times \mathbb{C}^m; ||z_1|| \leq \frac{C_2}{1 + ||z_2||}\}.$$

Compare with Rothstein’s theorem (see [Siu] p.25). Our theorem is motivated and has applications in problems of holomorphic dynamics and small divisors ([PM]) where power series in the ring $A$ appear naturally.

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\textbf{Γ-capacity.}

We refer to [Ro]. Let $K \subset \mathbb{C}^m$. The Γ-projection of $K$ on $\mathbb{C}^{m-1}$ is the set $\Gamma_{m-1}(K)$ of $z = (z_1, \ldots, z_{m-1}) \in \mathbb{C}^{m-1}$ such that

$$K \cap \{(z, w) \in \mathbb{C}^m\}$$

has positive capacity in the complex plane $C_z = \{(z, w) \in \mathbb{C}^m\}$. We define

$$\Gamma_1^1(K) = \Gamma_2^1 \circ \Gamma_3^2 \circ \ldots \Gamma_{m-1}^m(K).$$

Finally, the Γ-capacity is defined as

$$\Gamma\text{-Cap}(K) = \sup_{A \in U(m, C)} \text{Cap}_{\Gamma}^1(A(K)).$$

where $A$ runs over all unitary transformations of $\mathbb{C}^m$.

We have the following lemma ([Ro] Lemma 2.2.8 p.92)

\textbf{Lemma.} Let $K \subset \mathbb{C}^m$, $K \neq \mathbb{C}^m$ and assume that the intersection of $K$ with any complex line which is not a subset of $K$ has inner capacity zero. Then the Γ-capacity of $K$ is zero.

Thus we are reduced to prove the theorem for $m = 1$

\textbf{Bernstein lemma.}

We recall (see [Ra] p.156):

\textbf{Lemma (Bernstein).} Let $K \subset \mathbb{C}$ be a non-polar set, and $\Omega$ be the component of $\mathbb{C} - K$ containing $\infty$.

If $P$ is a polynomial of degree $n$, then for $z \in \mathbb{C}$

$$|P(z)| \leq ||P||_{C^0(K)} e^{ng_\Omega(z, \infty)}$$

where $g_\Omega$ is the Green function of $\Omega$.

\textbf{Proof of the theorem.}

We are reduced to prove the theorem for $m = 1$. For $z_2 \in K$, let $R(z_2)$ be the radius of convergence in $z_1$ of $f(z_1, z_2)$. Let $K_i = \{z_2 \in K; R(z_2) \geq 1/i\}$. Since a countable union of polar sets is polar, there is $K_i$ non-polar. We can take a non-polar sub-compact $L \subset K_i$ so that there exists $\rho_0 > 0$ such that for all $z_2 \in L$

$$\lim sup_{||n|| \to +\infty} \frac{|P_n(z_2)|}{\rho_0^{||n||}} < +\infty.$$

Define

$$\varphi(z_2) = \lim sup_{||n|| \to +\infty} \frac{|P_n(z_2)|}{\rho_0^{||n||}}.$$
The function $\varphi$ is lower semi-continuous, and

$$L = \bigcup_{p \geq 1} L_p$$

where $L_p = \{ z \in L; \varphi(z_2) \leq p \}$ is closed. By Baire theorem for some $p$, $L_p$ has non-empty interior (with respect to $L$), thus some $L_p$ has positive capacity. Finally we found a compact set $C = L_p$ of positive capacity such that there exists $\rho_1 > 0$ such that for any $z_2 \in C$ and $n$,

$$|P_n(z_2)| \leq \rho_1^{||n||}.$$  

Now using Bernstein lemma we conclude that for any $z_2 \in C$, for all $n$,

$$|P_n(z_2)| \leq \rho_1^{||n||}e^{(C_0 + C_1||n||)g_\Omega(z_2, \infty)}.$$  

Finally using the asymptotic

$$g_\Omega(z_2, \infty) = \log |z_2| + O(1)$$

we obtain the extension to the desired domain.

**Remark.**

1. We can improve on the domain of extension if we control the growth of the degrees of the polynomials $(P_n)$. For instance, the same proof shows that if

$$\limsup_{n} \frac{1}{||n||} \deg P_n = 0$$

we have a holomorphic extension to a domain

$$U = \{(z_1, z_2) \in \mathbb{C}^k \times \mathbb{C}^m; ||z_1|| \leq C\}$$

for some $C > 0$.

2. As N. Sibony has pointed out recently to me, the condition positive $\Gamma$-capacity in the theorem can be replaced by non-pluri-polar set (which is stronger and more natural) using the definition of capacity in higher dimension and the techniques of [Al] and [Si]. Essentially one writes down a general Bernstein lemma in higher dimension (similar to lemma 6.5 in [Al]) and use it as we do in dimension 1. The main dynamical applications, where we seek for generic conditions, work as well with the version with $\Gamma$-capacity. For this first version we content ourselves with the above statement.

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