Research Article

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On the value-distribution of iterated integrals of the logarithm of the Riemann zeta-function I: Denseness

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Abstract: We consider iterated integrals of $\log \zeta(s)$ on certain vertical and horizontal lines. Here, the function $\zeta(s)$ is the Riemann zeta-function. It is a well-known open problem whether or not the values of the Riemann zeta-function on the critical line are dense in the complex plane. In this paper, we give a result for the denseness of the values of the iterated integrals on the horizontal lines. By using this result, we obtain the denseness of the values of $\int_0^t \log \zeta(\frac{1}{2} + it') dt'$ under the Riemann Hypothesis. Moreover, we show that, for any $m \geq 2$, the denseness of the values of an $m$-times iterated integral on the critical line is equivalent to the Riemann Hypothesis.

Keywords: Riemann zeta-function, value-distribution, denseness, critical line

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1 Introduction and statement of results

In the present paper, we give some results for the value-distribution of iterated integrals of the logarithm of the Riemann zeta-function $\zeta(s)$. Many mathematicians have studied the value-distribution of the Riemann zeta-function and other $L$-functions. Here we should mention two remarkable results which are starting points of those studies.

Theorem (Bohr and Courant in 1914 [2]). For fixed $\frac{1}{2} < \sigma \leq 1$, the set $\{\zeta(\sigma + it) \mid t \in \mathbb{R}\}$ is dense in the complex plane.

Theorem (Bohr in 1916 [1]). For fixed $\frac{1}{2} < \sigma \leq 1$, the set $\{\log \zeta(\sigma + it) \mid t \in \mathbb{R}\}$ is dense in the complex plane.

Note that the latter theorem is an improvement of former one since the former is an immediate consequence from the latter. As developments of these theorems, the Bohr–Jessen limit theorem [3], Selberg’s limit theorem [17] (which is essentially included in [16]), and Voronin’s universality theorem [18] are well known. By these theorems, we can understand some properties of $\zeta(s)$ such as the exact value-distribution of $\zeta(s)$ and the complexity of the behavior of $\zeta(s)$ in the critical strip. As further developments of these results, there are many studies such as [4, 8, 10, 14, 15].

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Here, we mention some known facts for the denseness of the values $\zeta(\sigma + it)$ for $t \in \mathbb{R}$. In the case $\sigma > 1$ fixed, the values $\zeta(\sigma + it)$ is bounded. As for the case $\sigma < \frac{1}{2}$, it has been proved by Garunkštis and Steuding [7] that the values $\zeta(\sigma + it)$ for $t \in \mathbb{R}$ are not dense in the complex plane under the Riemann Hypothesis. Additionally, as we mentioned above, the denseness in the case $\frac{1}{2} < \sigma \leq 1$ has been proved. Hence, the remaining problem for the denseness is only the following.

**Problem 1.** Is the set $\{\log \zeta(\frac{1}{2} + it) \mid t \in \mathbb{R}\}$ dense in the complex plane?

For Problem 1, there is an interesting study by Kowalski and Nikeghbali [12]. They studied the Fourier transform of the probability measure which represents the probability of $\log \zeta(\frac{1}{2} + it) \in A$ with $A$ a Borel set. In particular, they gave a sufficient condition that the values $\zeta(\frac{1}{2} + it)$ for $t \in \mathbb{R}$ are dense in the complex plane (see [12, Corollary 9]). Hence, from their study, we might guess that the answer for Problem 1 could be yes. However, as they mentioned in their paper [12], their sufficient condition is rather strong. Therefore, it is also not strange that the answer for Problem 1 could be no. Moreover, Garunkštis and Steuding [7] showed that the set of $(\zeta(\frac{1}{2} + it), \zeta'(\frac{1}{2} + it))$ for $t \in \mathbb{R}$ is not dense in $\mathbb{C}^2$. As we can see from these works, it seems difficult to decide clearly the answer of Problem 1 at present. Hence, it is desirable to obtain some new information for this problem, and this is the objective of the paper.

In order to give new information of this theme, we consider the function $\eta_m(s)$ defined by

$$\eta_m(\sigma + it) = \int_0^t \eta_{m-1}(\sigma + it') \, dt' + c_m(\sigma),$$

where

$$\eta_0(\sigma + it) = \log \zeta(\sigma + it), \quad c_m(\sigma) = \frac{i^m}{(m-1)!} \int_\sigma^\infty (\alpha - \sigma)^{m-1} \log \zeta(\alpha) \, d\alpha.$$

Under this definition, the well-known function $S_m(t)$ is represented by $S_m(t) = \frac{1}{\pi} \Im \eta_m(\frac{1}{2} + it)$. The second author studied this function. In the present paper we discuss the topic related to [9, Section 2.4]. Since the function $\eta_m(s)$ is the $m$-times iterated integral of $\log \zeta(s)$ on the vertical line, we can expect that the function contains information related to the value-distribution of $\log \zeta(s)$. In particular, since $\eta_m(\frac{1}{2} + it)$ is the iterated integral on the critical line, the study of the value-distribution of this function might give new information on Problem 1. From this background, we study the value-distribution of the function $\eta_m(s)$ and prove the following theorem.

**Theorem 1.** Let $\frac{1}{2} \leq \sigma < 1$. If the number of zeros $\rho = \beta + iy$ of $\zeta(s)$ with $\beta > \sigma$ is finite, then the set

$$\left\{ \int_0^t \log \zeta(\sigma + it') \, dt' \right\mid t \in [0, \infty) \right\}$$

is dense in the complex plane. Moreover, for each integer $m \geq 2$, the following statements are equivalent.

(I) The Riemann zeta-function does not have any zero whose real part is greater than $\sigma$.

(II) The set $\{\eta_m(\sigma + it) \mid t \in [0, \infty)\}$ is dense in the complex plane.

From this theorem, we see that the Riemann Hypothesis implies that the set

$$\left\{ \int_0^t \log \zeta\left(\frac{1}{2} + it'\right) \, dt' \right\mid t \in [0, \infty) \right\}$$

is dense in the complex plane. This implication seems to suggest that the answer of Problem 1 is yes. Moreover, the equivalence above would be a new type of statement which gives the relation between the denseness of values of the Riemann zeta-function and the Riemann Hypothesis.

Here, we mention the plan of the proof of Theorem 1 briefly. We introduce the function $\tilde{\eta}_m(\sigma + it)$ recursively by

$$\tilde{\eta}_m(\sigma + it) = \int_\sigma^\infty \tilde{\eta}_{m-1}(\alpha + it) \, d\alpha.$$
where \( \tilde{\eta}_m(\sigma + it) = \log \zeta(\sigma + it) \). This function is the \( m \)-times iterated integral of \( \log \zeta(\sigma + it) \) on the horizontal line. Our main focus in this paper is better understanding of Problem 1 and the value-distribution of \( \eta_m(\frac{1}{2} + it) \). However, the function \( \tilde{\eta}_m(s) \) is regular in the same region as in the case of \( \log \zeta(s) \), and also some properties of this function are similar to \( \log \zeta(s) \). From this observation, this function would be an interesting object itself, and we obtain the following theorem unconditionally.

**Theorem 2.** Let \( \frac{1}{2} \leq \sigma < 1 \), and let \( m \) be a positive integer. Let \( T_0 \) be any positive number. Then the set

\[
\{ \tilde{\eta}_m(\sigma + it) \mid t \in [T_0, \infty) \}
\]

is dense in the complex plane.

Theorem 1 can be obtained from Theorem 2 and the following lemma.

**Lemma 1.** Let \( m \) be a positive integer, and let \( t > 0 \). Then, for any \( \sigma \geq \frac{1}{2} \), we have

\[
\eta_m(\sigma + it) = i^m \tilde{\eta}_m(\sigma + it) + 2\pi \sum_{k=0}^{m-1} \frac{i^{m-1-k}}{(m-k)!k!} \sum_{\beta \in \mathbb{C}, \beta > \sigma} (\beta - \sigma)^{m-k}(t - y)^k.
\]

This lemma follows from [9, Lemma 1]. Actually the statement of the lemma is

\[
\eta_m(\sigma + it) = \frac{i^m}{(m-1)!} \int_{\sigma}^{\infty} (a - \sigma)^{m-1} \log \zeta(a + it) da + 2\pi \sum_{k=0}^{m-1} \frac{i^{m-1-k}}{(m-k)!k!} \sum_{\beta \in \mathbb{C}, \beta > \sigma} (\beta - \sigma)^{m-k}(t - y)^k
\]

for \( m \in \mathbb{Z}_{\geq 1}, t > 0, \) and \( \sigma \geq \frac{1}{2} \). Additionally, by integration by parts, it holds that

\[
\tilde{\eta}_m(\sigma + it) = \frac{1}{(m-1)!} \int_{\sigma}^{\infty} (a - \sigma)^{m-1} \log \zeta(a + it) da.
\]

Therefore, we obtain equation (1.1). Hence, our first purpose is to show Theorem 2. In the proof of Theorem 2, the following two propositions play an important role.

In the following, the symbol \( \text{meas} \cdot \) stands for the one-dimensional Lebesgue measure, and \( \text{Li}_m(z) \) means the polylogarithmic function defined as \( \sum_{n=1}^{\infty} \frac{z^n}{n^m} \) for \( |z| < 1 \).

**Proposition 1.1.** Let \( m \) be a positive integer. Then for any \( \sigma \geq \frac{1}{2}, T \geq X^{1.35}, \varepsilon > 0 \), we have

\[
\lim_{X \to \infty} \frac{1}{T} \text{meas} \left\{ t \in [0, T] \left| \tilde{\eta}_m(\sigma + it) - \sum_{\beta \in \mathbb{C}} \frac{\text{Li}_{m+1}(p^{-\sigma - it})}{(\log p)^m} \right| < \varepsilon \right\} = 1.
\]

The important point of this proposition is that \( \tilde{\eta}_m(s) \) can be approximated by the Dirichlet polynomial even on the critical line. To prove this proposition, we must control exactly the contribution of nontrivial zeros of \( \zeta(s) \), and we therefore need a strong zero density estimate of the Riemann zeta-function like Selberg’s result [16, Theorem 1]. More precisely, we require that there exist numbers \( c > 0, A < 2m + 1 \) such that

\[
N(\sigma, T) \ll T^{1 - c(\sigma - \frac{3}{2})}(\log T)^A
\]

uniformly for \( \frac{1}{2} \leq \sigma \leq 1 \). Here, \( N(\sigma, T) \) is the number of zeros of \( \zeta(s) \) with multiplicity satisfying \( \beta > \sigma \) and \( 0 < \gamma \leq T \). Therefore, to prove Proposition 1.1, we need a strong zero density estimate comparable to the assumption by Bombieri and Hejhal [4]. On the other hand, when we discuss the denseness of \( \tilde{\eta}_m(s) \) for fixed \( \frac{1}{2} < \sigma < 1 \), it suffices to use the weaker estimate

\[
N(\sigma, T) \ll T^{1 - c(\sigma - \frac{3}{2})} + \varepsilon
\]

for every \( \varepsilon > 0 \). Hence, there is an essential difference of the depth between the discussion in the case \( \frac{1}{2} < \sigma < 1 \) and that in the case \( \sigma = \frac{1}{2} \) in Proposition 1.1. We will explain this discussion more closely later.

In contrast, we can prove the following proposition by almost the same method as in [1, 2].
Proposition 1.2. Let \( m \) be a positive integer, \( \frac{1}{2} \leq \sigma < 1 \). Let \( a \) be any complex number, and let \( \epsilon \) be any positive number. If we take a sufficiently large number \( N_0 = N_0(m, \sigma, a, \epsilon) \), then, for any integer \( N \geq N_0 \), there exists some Jordan measurable set \( \Theta_0 = \Theta_0(m, \sigma, a, \epsilon, N) \subset [0, 1)^{m(N)} \) with \( \text{meas}(\Theta_0) > 0 \) such that
\[
\left| \sum_{p \leq N} \frac{\text{Li}_{m+1}(p^\sigma \exp(-2\pi i \theta p))}{(\log p)^m} - a \right| < \epsilon
\]
for any \( \theta = (\theta_p)_{p=1}^{n(N)} \in \Theta_0 \).

Roughly speaking, Proposition 1.1 means that \( \tilde{\eta}_m(\sigma + it) \) “almost” equals the finite sum of polylogarithmic functions when the number of the terms of the sum is sufficiently large, and Proposition 1.2 that any complex number can be approximated by the finite sum of polylogarithmic functions when the number of the terms of the sum is sufficiently large.

Bohr developed his denseness results with Jessen from the viewpoint of probability theory in [3]. Following their method, the authors will continue our study in the paper with Mine [6] in a subsequent paper. They will give deeper results such as an analog of Lamzouri’s study [13] and the study of Lamzouri, Lester and Radziwill [14].

2 Proof of Proposition 1.1

In this section, we prove Proposition 1.1. In order to prove it, we prepare two lemmas.

Lemma 2. Let \( m \) be a positive integer, and \( \sigma \geq \frac{1}{2} \). Let \( T \) be large. Then, for \( 3 \leq X \leq T^{1+\epsilon} \), we have
\[
\frac{1}{T} \int_{1/4}^T \left| \tilde{\eta}_m(\sigma + it) - \sum_{2 \leq n \leq X} \frac{\Lambda(n)}{n^{\sigma + it}(\log n)^{m+1}} \right|^2 dt \ll \frac{X^{1-2\sigma}}{(\log X)^{2m}}.
\]
Here, we refer the following theorem to prove this lemma.

Lemma 3. Let \( m, k \) be positive integers. Let \( T \) be large, and \( X \geq 3 \) with \( X \leq T^{1+\epsilon} \). Then, for \( \sigma \geq \frac{1}{2} \), we have
\[
\int_{1/4}^T \left| \tilde{\eta}_m(\sigma + it) - i^m \sum_{2 \leq n \leq X} \frac{\Lambda(n)}{n^{\sigma + it}(\log n)^{m+1}} - Y_m(\sigma + it) \right|^{2k} dt \ll 2^k k \left( \frac{2m + 1}{2m} + \frac{C}{\log X} \right)^k X^{k(1-2\sigma)} + C^k k^{2k(m+1)} \frac{T^{1-2\sigma}}{(\log T)^{2k m}}.
\]
This lemma is [9, Theorem 5]. As we mentioned in the previous section, the proof of this lemma requires a strong zero density estimate like Selberg’s result. In fact, if we only knew the estimate
\[
N(\sigma, T) \ll T^{1-c(\sigma - \frac{1}{2})} (\log T)^A
\]
for some \( c > 0, A \geq 1 \), then the right hand side of (2.1) in the case \( k = 1 \) becomes
\[
O\left( \frac{X^{1-2\sigma}}{(\log X)^{2m}} + \frac{T^{1-2\sigma}}{(\log T)^{2m+1-A}} \right).
\]
Hence, the power of the logarithmic factor of the zero density estimate plays an important role in the case \( \sigma = \frac{1}{2} \).

Proof. By [9, Theorem 5], we have
\[
\frac{1}{T} \int_{1/4}^T \left| \eta_m(\sigma + it) - i^m \sum_{2 \leq n \leq X} \frac{\Lambda(n)}{n^{\sigma + it}(\log n)^{m+1}} - Y_m(\sigma + it) \right|^2 dt \ll \frac{X^{1-2\sigma}}{(\log X)^{2m}}.
\]
where
\[ Y_m(\sigma + it) = 2\pi \sum_{k=0}^{n-1} \frac{i^{m-k}}{(m-k)!k!} \sum_{\beta > \sigma, 0 < y < t} (\beta - \sigma)^{m-k}(t - y)^k. \] (2.2)

Further, by Lemma 1, we see that
\[ \eta_m(\sigma + it) - Y_m(\sigma + it) = i^m \tilde{\eta}_m(\sigma + it). \]

Hence we obtain this lemma.

\[ \square \]

**Lemma 4.** Let \( m \) be an integer, \( \sigma \geq \frac{1}{2} \). Let \( T \) be large. Then, for \( 3 \leq X \leq T^\frac{1}{2} \), we have
\[ \frac{1}{T} \int_0^T \left| \sum_{p \leq X} \frac{\text{Li}_{m+1}(p^{-\sigma-it})}{(\log p)^m} - \sum_{2 \leq n \leq X} \frac{\Lambda(n)}{n^{\sigma+it}(\log n)^{m+1}} \right|^2 \, dt \ll \frac{X^{1-2\sigma}}{(\log X)^{2m+1}}, \]
where the function \( \Lambda(n) \) is the von Mangoldt function.

**Proof.** By the definitions of the polylogarithmic function and the von Mangoldt function, we find that
\[ \sum_{p \leq X} \frac{\text{Li}_{m+1}(p^{-\sigma-it})}{(\log p)^m} = \sum_{2 \leq n \leq X} \frac{\Lambda(n)}{n^{\sigma+it}(\log n)^{m+1}} + O\left( \frac{X^{1-3\sigma}}{(\log X)^m} \right). \]

Here, we can write
\[ \left| \sum_{p \leq X} \sum_{x \leq k \leq 3} \frac{p^{-k(\sigma+it)}}{(k^{m+1}(\log p)^m) k^{2m+1}(\log p)^{2m}} \right|^2 = \sum_{p \leq X} \sum_{x \leq k \leq 3} \frac{p^{-2k\sigma}}{(k^{2m+1}(\log p)^{2m})} + \sum_{p_1 \leq X} \sum_{p_2 \leq X} \sum_{x \leq k_1 \leq 3} \sum_{x \leq k_2 \leq 3} \sum_{x \leq k \leq 3} \frac{1}{(k_1 k_2)^{m+1}(\log p_1 \log p_2)^{m}}. \]

Therefore, it holds that
\[ \int_0^T \left| \sum_{p \leq X} \sum_{x \leq k \leq 3} \frac{p^{-k(\sigma+it)}}{(k^{m+1}(\log p)^m)} \right|^2 \, dt = T \sum_{p \leq X} \sum_{x \leq k \leq 3} \frac{p^{-2k\sigma}}{(k^{2m+1}(\log p)^{2m})} + O\left( X^3 \left( \sum_{p \leq X} \sum_{x \leq k \leq 3} \sum_{x \leq k \leq 3} \frac{1}{p^{k\sigma} k^{m+1}(\log p)^m} \right)^2 \right) \ll T \frac{X^{1-2\sigma}}{(\log X)^{2m+1}} + \frac{X^5-2\sigma}{(\log X)^{2m+1}} \ll T \frac{X^{1-2\sigma}}{(\log X)^{2m+1}}. \]

Hence we have
\[ \int_0^T \left| \sum_{p \leq X} \frac{\text{Li}_{m+1}(p^{-\sigma-it})}{(\log p)^m} - \sum_{2 \leq n \leq X} \frac{\Lambda(n)}{n^{\sigma+it}(\log n)^{m+1}} \right|^2 \, dt \ll T \int_0^T \left| \sum_{p \leq X} \sum_{x \leq k \leq 3} \frac{p^{-k(\sigma+it)}}{(k^{m+1}(\log p)^m)} \right|^2 \, dt + T \frac{X^2-6\sigma}{(\log X)^{2m}} \ll T \frac{X^{1-2\sigma}}{(\log X)^{2m+1}}, \]
which completes the proof of this lemma.

\[ \square \]
Proof of Proposition 1.1. By Lemma 2 and Lemma 4, for $X \leq T^{1/2}$, we find that
\[
\frac{1}{T} \int_{1/4}^{T} \left| \eta_m(\sigma + it) - \sum_{p \leq X} \frac{\text{Li}_{m+1}(p^{-\sigma - it})}{\log p} \right|^2 dt \ll \frac{1}{T} \int_{1/4}^{T} \left| \eta_m(\sigma + it) - \sum_{2 \leq n \leq X} \frac{\Lambda(n)(n^{-\sigma})}{\log n^{m+1}} \right|^2 dt
\]
\[+ \frac{1}{T} \int_{1/4}^{T} \left| \sum_{p \leq X} \frac{\text{Li}_{m+1}(p^{-\sigma - it})}{\log p} - \sum_{2 \leq n \leq X} \frac{\Lambda(n)(n^{-\sigma})}{\log n^{m+1}} \right|^2 dt \ll \frac{X^{1-2\alpha}}{(\log X)^{2m}}.
\]
By using this estimate, for any fixed $\epsilon > 0$, we have
\[
\frac{1}{T} \text{meas} \left\{ t \in [0, T] \left| \left| \eta_m(\sigma + it) - \sum_{p \leq X} \frac{\text{Li}_{m+1}(p^{-\sigma - it})}{\log p} \right| \geq \epsilon \right\} \leq \frac{X^{1-2\alpha}}{\epsilon^2(\log X)^{2m}} + \frac{1}{T}.
\]
Hence, for any $T \geq X^{135}$, it holds that
\[
\frac{1}{T} \text{meas} \left\{ t \in [0, T] \left| \left| \eta_m(\sigma + it) - \sum_{p \leq X} \frac{\text{Li}_{m+1}(p^{-\sigma - it})}{\log p} \right| \geq \epsilon \right\} \to 0
\]
as $X \to +\infty$. Thus, we obtain Proposition 1.1.

3 Proof of Proposition 1.2

In this section, we prove Proposition 1.2 by the method of [11, Chapter VIII.3] and [19]. First of all, we will show the following elementary geometric lemma.

Lemma 5. Let $N$ be a positive integer larger than two. Suppose that the positive numbers $r_1, r_2, \ldots, r_N$ satisfy the condition
\[
r_{n_0} \leq \sum_{n=1}^{N} r_n,
\]
where $r_{n_0} = \max\{r_n \mid n = 1, 2, \ldots, N\}$. Then we have
\[
\left\{ \sum_{n=1}^{N} r_n \exp(-2\pi i \theta_n) \in \mathbb{C} \mid \theta_n \in [0, 1) \right\} = \left\{ z \in \mathbb{C} \mid |z| \leq \sum_{n=1}^{N} r_n \right\}.
\]
Proof. By [5, Proposition 3.3], it immediately follows that
\[
\left\{ \sum_{n=1}^{N} r_n \exp(-2\pi i \theta_n) \in \mathbb{C} \mid \theta_n \in [0, 1) \right\}
\]
is the closed circle with center origin and radius $\sum_{n=1}^{N} r_n$. Note that their $T_n$ becomes zero under assumption (3.1).

Next, we introduce the following definitions.

Definition 3.1. Let $m$ be a positive integer and $\mathcal{M}$ a finite subset of the set of prime numbers. For $\sigma \geq \frac{1}{2}$ and $\theta = (\theta_p)_{p \in \mathcal{M}} \in [0, 1)^{\mathcal{M}}$, we define the functions
\[
\varphi_{m, \mathcal{M}}(\sigma, \theta) := \sum_{p \in \mathcal{M}} \frac{\exp(-2\pi i \theta_p)}{p^{\sigma}(\log p)^m},
\]
\[
\tilde{\eta}_{m, \mathcal{M}}(\sigma, \theta) := \sum_{p \in \mathcal{M}} \frac{\text{Li}_{m+1}(p^{-\sigma})\exp(-2\pi i \theta_p)}{(\log p)^m} = \sum_{p \in \mathcal{M}} \sum_{k=1}^{\infty} \frac{\exp(-2\pi i k \theta_p)}{k^{m+1} p^{\sigma}(\log p)^m},
\]
respectively.
Definition 3.2. Let \( p_n \) be the \( n \)-th prime number. Put
\[
\varphi^{(0)} = (\varphi^{(0)}_{p_n})_{n \in \mathbb{N}} = \left( 0, \frac{1}{2}, 0, \frac{1}{2}, \ldots \right) \in [0, 1]^\mathbb{N},
\]
and
\[
Y_{m, \sigma} = \sum_{p} \sum_{k=1}^{\infty} \frac{\exp(-2\pi ik\varphi^{(0)}_p)}{k^{m+1} p^{\sigma}(\log p)^m}.
\]
Note that the series for \( Y_{m, \sigma} \) is convergent for \( \sigma > \frac{1}{2} \).

Proof of Proposition 1.2. Fix a complex number \( a \) and \( \frac{1}{2} \leq \sigma < 1 \). Let \( U \) be a positive real parameter. We take a sufficiently large number \( N = N(U, m, \sigma, a) \) for which the estimates
\[
|a - Y_{m, \sigma}| \leq \sum_{p \in \mathcal{M}} \frac{1}{p^\sigma(\log p)^m},
\]
\[
\frac{1}{p_{\min}^\sigma(\log p_{\min})^m} \leq \sum_{p \in \mathcal{M} \setminus \{p_{\min}\}} \frac{1}{p^\sigma(\log p)^m}
\]
are satisfied, where \( \mathcal{M} = \mathcal{M}(U, N) \) is defined as \( \{ p | p \ \text{is prime}, \ U < p \leq N \} \), and \( p_{\min} \) is the minimum of \( \mathcal{M} \). Note that the existence of such \( N \) is guaranteed by \( \sum_{p} \frac{1}{p^\sigma(\log p)^m} = \infty \). Then, by Lemma 5, the function
\[
\varphi_{m, \mathcal{M}}(\sigma, \cdot) : \{ 0, 1 \}^{\mathcal{M}} \ni \theta \mapsto \varphi_{m, \mathcal{M}}(\sigma, \theta) \in \left\{ z \in \mathbb{C} \mid |z| \leq \sum_{p \in \mathcal{M}} \frac{1}{p^\sigma(\log p)^m} \right\}
\]
is surjective. Hence, there exists some \( \varphi^{(1)}_p = \varphi(p, \sigma, U)^{(1)} = (\varphi^{(1)}_{p})_{p \in \mathcal{M}} \in [0, 1]^{\mathcal{M}} \) such that
\[
\varphi_{m, \mathcal{M}}(\sigma, \varphi^{(1)}_p) = a - Y_{m, \sigma}.
\]
By using the prime number theorem, we find that
\[
\tilde{h}_{m, \mathcal{M}}(\sigma, \varphi^{(1)}_p) = \varphi_{m, \mathcal{M}}(\sigma, \varphi^{(1)}_p) + \sum_{p \in \mathcal{M}} \sum_{k=2}^{\infty} \frac{\exp(-2\pi ik\varphi^{(1)}_p)}{k^{m+1} p^{\sigma}(\log p)^m}
\]
\[
= a - Y_{m, \sigma} + O \left( \frac{1}{(\log U)^m} \right).
\]
For any prime number \( p \), we put
\[
\varphi^{(2)}_p = \begin{cases} 
\varphi^{(0)}_p & \text{if } p \notin \mathcal{M}, \\
\varphi^{(1)}_p & \text{if } p \in \mathcal{M}.
\end{cases}
\]
Then it holds that
\[
\sum_{p \leq N} \text{Li}_{m+1}(p^{-\sigma} \exp(-2\pi i\varphi^{(2)}_p)) \big/(\log p)^m = \sum_{p \in \mathcal{M}} \text{Li}_{m+1}(p^{-\sigma} \exp(-2\pi i\varphi^{(1)}_p)) + \sum_{p > U} \text{Li}_{m+1}(p^{-\sigma} \exp(-2\pi i\varphi^{(0)}_p)) \big/(\log p)^m
\]
\[
= \tilde{h}_{m, \mathcal{M}}(\sigma, \varphi^{(1)}_p) + Y_{m, \sigma} + \sum_{p > U} \text{Li}_{m+1}(p^{-\sigma} \exp(-2\pi i\varphi^{(0)}_p)) \big/(\log p)^m,
\]\nand additionally, by using the prime number theorem and simple calculations of alternating series,
\[
\sum_{p > U} \text{Li}_{m+1}(p^{-\sigma} \exp(-2\pi i\varphi^{(0)}_p)) \big/(\log p)^m = \sum_{p > U} \frac{\exp(-2\pi i\varphi^{(0)}_p)}{p^\sigma(\log p)^m} + O \left( \sum_{p > U} \frac{1}{p^{2\sigma(\log p)^m}} \right) \ll \frac{1}{(\log U)^m}.
\]
Hence, by taking a sufficiently large \( U = U(\varepsilon) \) and noting the continuity of the function
\[
\sum_{p \leq N} \text{Li}_{m+1}(p^{-\sigma} \exp(-2\pi i\varphi_p)) \big/(\log p)^m
\]
with respect to \( \varphi_p \in [0, 1]^{\mathcal{M}(N)} \), we obtain this proposition. \( \square \)
4 Proof of Theorem 2

In this section, we prove Theorem 2. Here, we use the following lemma related with Kronecker’s approximation theorem.

**Lemma 6.** Let $A$ be a Jordan measurable subregion of $[0, 1]^N$, and let $a_1, \ldots, a_N$ be real numbers linearly independent over $\mathbb{Q}$. Set, for any $T > 0$,

$$I(T, A) = \{ t \in [0, T] \mid ((a_1 t), \ldots, (a_N t)) \in A \}.$$

Then we have

$$\lim_{T \to +\infty} \frac{\text{meas}(I(T, A))}{T} = \text{meas}(A).$$

**Proof.** This lemma is [11, Appendix 8, Theorem 1].

Let us start the proof of Theorem 2.

**Proof of Theorem 2.** Let $\varepsilon > 0$ be any small number, $a$ any fixed complex number, $\frac{1}{2} \leq \sigma < 1$, and let $T_0$ be any positive number. Define $S_M(\theta_1, \ldots, \theta_M; \sigma, m)$ and $S_{M,N}(\theta_{M+1}, \ldots, \theta_N; \sigma, m)$ by

$$S_M(\theta_1, \ldots, \theta_M; \sigma, m) = \sum_{n \leq M} \frac{\log m!}{m^2} (P_n^m e^{-2ni\theta_n})^{-\sigma},$$

$$S_{M,N}(\theta_{M+1}, \ldots, \theta_N; \sigma, m) = \sum_{n \leq M} \frac{\log m!}{m^2} (P_n^m e^{-2ni\theta_n})^{-\sigma}.$$

Then, by Proposition 1.2, we can take a sufficiently large $M_0 = M_0(m, \sigma, a, \varepsilon)$ so that for any $M > M_0$, there exists some Jordan measurable subset $\Theta_M^{(M)} = \Theta_M^{(M)}(m, \sigma, a, \varepsilon, M)$ of $[0, 1]^M$ such that $\delta_M := \text{meas}(\Theta_1^{(M)}) > 0$ and

$$|S_M(\theta_1, \ldots, \theta_M; \sigma, m) - a| < \varepsilon$$

for any $(\theta_1, \ldots, \theta_M) \in \Theta_1^{(M)}$. We also find that

$$\int_0^1 \cdots \int_0^1 |S_{M,N}(\theta_{M+1}, \ldots, \theta_N; \sigma, m)|^2 d\theta_{M+1} \cdots d\theta_N$$

$$= \int_0^1 \cdots \int_0^1 \left| \sum_{M < n \leq N} \sum_{k_1, k_2} \left( \frac{P_n}{k_1^2} \cdot (\log p_n)^2 \right) \right|^2 d\theta_{M+1} \cdots d\theta_N$$

$$= \sum_{M < n_1 \leq N} \sum_{M < n_2 \leq N} \sum_{k_1, k_2} \left( \frac{P_{n_1} P_{n_2}}{k_1^2 (\log p_{n_1})^2} \right) \int_0^1 \int_0^1 e^{-2\pi i (k_1 \theta_{n_1} + k_2 \theta_{n_2})} d\theta_{M+1} \cdots d\theta_N$$

$$= \sum_{M < n_1 \leq N} \sum_{M < n_2 \leq N} \sum_{k_1, k_2} \frac{1}{k^2 (\log p_n)^2} \leq \sum_{M < n_1 \leq N} \frac{1}{(\log p_{n_1})^2}.$$

Note that the last sum tends to zero as $M \to +\infty$. Therefore, there exists some large number $M_1 = M_1(m, \varepsilon)$ such that, for any $N > M > M_1$, it holds that

$$\text{meas}(\{ (\theta_{M+1}, \ldots, \theta_N) \in [0, 1)^{N-M} \mid |S_{M,N}(\theta_{M+1}, \ldots, \theta_N; \sigma, m)| < \varepsilon \}) > \frac{1}{2}.$$

Here we denote the set of content of meas(·) in the above inequality by $\Theta_2^{(M,N)} = \Theta_2^{(M,N)}(M, N, \varepsilon)$.

We put $M_2 = \max\{M_0, M_1\}$ and $\Theta_3 = \Theta_1^{(M_2)} \times \Theta_2^{(M_1, N)}$ for any $N > M_2$. Then $\Theta_3$ is a subset of $[0, 1)^N$ satisfying $\text{meas}(\Theta_3) > \frac{M_2}{2}$. Hence, putting

$$\mathcal{I}(T) = \left\{ t \in [T_0, T] \mid \left( \left\{ \frac{t}{2\pi} \log p_1 \right\}, \ldots, \left\{ \frac{t}{2\pi} \log p_N \right\} \right) \in \Theta_3 \right\}$$
and applying Lemma 6, for any positive integer \( N > M_2 \), there exists some large number \( T_N > T_0 \) such that
\[
\text{meas}\left\{ t \in [T_0, T] \left| \left| \tilde{\eta}_m(\sigma + it) - \sum_{n \leq N} \frac{\text{Li}_{m+1}(p_n \sigma - it)}{(\log p_n)^m} \right| < \varepsilon \right. \right\} > \left( 1 - \frac{\delta_{M_1}}{4} \right) T
\]
for any \( N \geq N_0, T \geq p_N^{135} \).

Therefore, for any \( N \geq \max\{N_0, M_2 + 1\}, T \geq \max\{T_N, p_N^{135}\} \), there exists some \( t_0 \in [T_0, T] \) such that
\[
\left\{ \frac{t_0}{2\pi} \log p_1, \ldots, \frac{t_0}{2\pi} \log p_N \right\} \in \Theta_3
\]
and
\[
\left| \tilde{\eta}_m(\sigma + it_0) - \sum_{n \leq N} \frac{\text{Li}_{m+1}(p_n \sigma - it_0)}{(\log p_n)^m} \right| < \varepsilon.
\]
Then we have
\[
\left| \tilde{\eta}_m(\sigma + it_0) - a \right| \leq \left| \tilde{\eta}_m(\sigma + it_0) - \sum_{n \leq N} \frac{\text{Li}_{m+1}(p_n \sigma - it_0)}{(\log p_n)^m} \right| + \left| \sum_{n > N} \frac{\text{Li}_{m+1}(p_n \sigma - it_0 \log p_n)}{(\log p_n)^m} - a \right|
\]
\[
+ \left| \sum_{M_1 < n \leq N} \frac{\text{Li}_{m+1}(p_n \sigma - it_0 \log p_n)}{(\log p_n)^m} \right| < 3\varepsilon.
\]
This completes the proof of Theorem 2.

\section{5 Proof of Theorem 1}

In this section, we prove Theorem 1. Here, we prepare the following lemma.

**Lemma 7.** Let \( \sigma \geq \frac{1}{2} \) and let \( m \) be a positive integer. Then we have
\[
\eta_m(s) = Y_m(s) + O_m(\log t),
\]
where \( Y_m \) is defined by (2.2).

**Proof.** This lemma is [9, equation (2.2)].

**Proof of Theorem 1.** First, we show Theorem 1 in the case \( m = 1 \). If the number of zeros \( \rho = \beta + iy \) of \( \zeta(s) \) with \( \beta > \sigma \) is finite, then there exists a sufficiently large \( T_0 \) such that \( Y_1(\sigma + it) \equiv b \) for \( t \geq T_0 \), where \( b \) is a positive real number. Therefore, by Lemma 1, we have
\[
\int_0^t \log \zeta(\sigma + it') dt' = i \tilde{\eta}_1(\sigma + it) + b
\]
for any \( t \geq T_0 \). By this formula, we obtain
\[
\int_0^t \log \zeta(\sigma + it') dt' \bigg| \ t \in [0, \infty) \bigg\} \supset \int_0^t \log \zeta(\sigma + it') dt' \bigg| \ t \in [T_0, \infty) \bigg\} \cap \{ i \tilde{\eta}_1(\sigma + it) + b \ | \ t \in [T_0, \infty) \}.
\]
If a set \( A \subset \mathbb{C} \) is dense in \( \mathbb{C} \), then for any \( c_1 \in \mathbb{C} \setminus \{0\} \) and \( c_2 \in \mathbb{C} \), the set \( \{ c_1 a + c_2 \ | \ a \in A \} \) is also dense in \( \mathbb{C} \). By this fact and Theorem 2, the set \( \{ i \tilde{\eta}_1(\sigma + it) + b \ | \ t \in [T_0, \infty) \} \) is dense in \( \mathbb{C} \). Thus, it follows that the set \( \{ \int_0^t \log \zeta(\sigma + it') dt' \ | \ t \in [0, \infty) \} \subset \{ i \tilde{\eta}_1(\sigma + it) + b \ | \ t \in [T_0, \infty) \} \) is dense in \( \mathbb{C} \) under this assumption.

Next, for \( m \in \mathbb{Z}_{\geq 2} \), we show the equivalence of (I) and (II). The implication (I) \( \Rightarrow \) (II) is clear since the equation \( \eta_m(\sigma + it) = i^m \tilde{\eta}_m(\sigma + it) \) holds by assuming (I).
In the following, we show the inverse implication $(\text{II}) \Rightarrow (\text{I})$. By Lemma 7, if $(\text{I})$ is false, then the estimate $|\eta_m(\sigma + it)| \gg m t^{-1}$ holds. Therefore, for some $T_2 > 0$, we have

$$|\eta_m(\sigma + it) \mid t \in [T_2, \infty)) \subset \mathbb{C} \setminus \{ z \mid |z| \leq 1 \}. $$

Here, $\overline{A}$ means the closure of the set $A$. Since $|\eta_m(\sigma + it) \mid t \in [0, T_2])$ is a piecewise smooth curve of finite length, we have $\mu(\{ \eta_m(\sigma + it) \mid t \in [0, T_2]) = 0$. Here $\mu$ is the Lebesgue measure in $\mathbb{C}$. Therefore, we obtain

$$\{ z \mid |z| \leq 1 \} \not\subset \{ \eta_m(\sigma + it) \mid t \in [0, T_2]) \}. $$

Hence, if $(\text{I})$ is false, then the set $\{ \eta_m(\sigma + it) \mid t \in [0, \infty))$ is not dense in $\mathbb{C}$. Thus, we obtain the implication $(\text{II}) \Rightarrow (\text{I}).$

\[\Box\]

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