Hydrodynamic Limit of Multiple SLE

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Abstract

Recently del Monaco and Schleißinger addressed an interesting problem whether one can take the limit of multiple Schramm–Loewner evolution (SLE) as the number of slits $N$ goes to infinity. When the $N$ slits grow from points on the real line $\mathbb{R}$ in a simultaneous way and go to infinity within the upper half plane $\mathbb{H}$, an ordinary differential equation describing time evolution of the conformal map $g_t(z)$ was derived in the $N \to \infty$ limit, which is coupled with a complex Burgers equation in the inviscid limit. It is well known that the complex Burgers equation governs the hydrodynamic limit of the Dyson model defined on $\mathbb{R}$ studied in random matrix theory, and when all particles start from the origin, the solution of this Burgers equation is given by the Stieltjes transformation of the measure which follows a time-dependent version of Wigner’s semicircle law. In the present paper, first we study the hydrodynamic limit of the multiple SLE in the case that all slits start from the origin. We show that the time-dependent version of Wigner’s semicircle law determines the time evolution of the SLE hull, $K_t \subset \mathbb{H} \cup \mathbb{R}$, in this hydrodynamic limit. Next we consider the situation such that a half number of the slits start from $a > 0$ and another half of slits start from $-a < 0$, and determine the multiple SLE in the hydrodynamic limit. After reporting these exact solutions, we will discuss the universal long-term behavior of the multiple SLE and its hull $K_t$ in the hydrodynamic limit.

Key words: hydrodynamic limit; multiple Schramm–Loewner evolution (SLE); complex Burgers equation; Dyson model; Wigner’s semicircle law

1 Introduction

Construction and description of stochastic interacting systems consisting of an infinite number of particles have been important topics in probability theory and nonequilibrium statistical mechanics [27, 35, 28, 17, 22]. In the present paper we report a trial to characterize the infinite limit of stochastic interacting curves in a plane.

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Figure 1: Schematic picture of \( N \)-tuples of non-intersecting slits in \( \mathbb{H} \), \( (\gamma_1^N, \gamma_2^N, \ldots, \gamma_N^N) \), starting from \( \mathbf{x}^N = (x_1^N, x_2^N, \ldots, x_N^N) \in \mathbb{W}_N \).

Let \( i = \sqrt{-1} \) and denote the upper half of the complex plane \( \mathbb{C} \) as
\[
\mathbb{H} = \{ z : z = x + iy, x \in \mathbb{R}, y > 0 \}.
\]

For \( N \in \mathbb{N} \equiv \{1, 2, \ldots \} \), consider the Weyl chamber
\[
\mathbb{W}_N = \{ \mathbf{x} = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : x_1 < x_2 < \cdots < x_N \}.
\]

Given \( \mathbf{x}^N = (x_1^N, \ldots, x_N^N) \in \mathbb{W}_N \), we consider \( N \)-tuples of slits in \( \overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{R} \cup \{ \infty \} \) denoted as \( (\gamma_1^N, \gamma_2^N, \ldots, \gamma_N^N) \), such that \( \gamma_j^N \in \mathbb{H} \), \( j = 1, 2, \ldots, N \), are simple curves connecting \( x_j \) and \( \infty \), and they are non-intersecting, i.e.,
\[
\gamma_j^N \cap \gamma_k^N = \emptyset, \quad 1 \leq j < k \leq N.
\]

See Fig.1. Based on the theory of multiple Schramm–Loewner evolution (SLE) \[7, 8, 9, 24, 20\], del Monaco and Schleißinger \[13\] have considered a one-parameter \( (0 < \kappa \leq 4) \) family of probability laws \( \mathbb{P}^{x_N, \infty}_\kappa \) constructed from independent \( N \) copies of the one-dimensional Brownian motion as follows.

Let \( B_j(t), t \geq 0, j = 1, 2, \ldots, N \), be independent one-dimensional standard Brownian motions. The Dyson model with parameter \( \beta > 0 \) studied in random matrix theory \[14, 29, 17, 1, 22\] is a system of stochastic differential equations (SDEs) for the interacting particle system on \( \mathbb{R} \), \( \mathbf{X}^N(t) = (X_1^N(t), X_2^N(t), \ldots, X_N^N(t)) \),
\[
dX_j^N(t) = dB_j(t) + \frac{\beta}{2} \sum_{1 \leq k < N, \ k \neq j} \frac{dt}{X_j^N(t) - X_k^N(t)}, \quad t \in [0, T_x^N], \quad j = 1, 2, \ldots, N,
\]
with an initial configuration \( \mathbf{X}^N(0) = \mathbf{x}^N \in \mathbb{W}_N \), and \( T_x^N = \inf \{ t > 0 : \mathbf{X}^N(t) \notin \mathbb{W}_N \} \). Here we set \( \beta = 8/\kappa \geq 2 \) \[7, 8, 9\] and perform a time change,
\[
V_j^N(t) = X_j^N(kt/N), \quad j = 1, 2, \ldots, N.
\]
Then we have a system of SDEs for $V^N(t) = (V^N_1(t), \ldots, V^N_N(t))$,

$$dV^N_j(t) = \sqrt{\frac{\kappa}{N}} dB_j(t) + \frac{1}{N} \sum_{1 \leq k \leq N, k \neq j} \frac{4}{V^N_j(t) - V^N_k(t)} dt, \quad t \in [0, \infty), \quad j = 1, 2, \ldots, N, \quad (1.1)$$

with the initial configuration

$$V^N(0) = x^N \in \mathbb{W}_N.$$  

Here we have used the fact that $T^\kappa = \infty$, $\forall x^N \in \mathbb{W}_N$, with probability one for the Dyson model with $\beta \geq 1$ [22, 21]. With the solution $V^N(t), t \in [0, \infty)$ of (1.1), the multiple SLE is introduced as

$$\frac{\partial g^N_t(z)}{\partial t} = \frac{1}{N} \sum_{j=1}^{N} g^N_t(z) - \frac{2}{V^N_j(t)}, \quad t \geq 0, \quad g^N_0(z) = z \in \mathbb{H}. \quad (1.2)$$

Then, for $\kappa \in (0, 4]$, $\mathbb{L}^{x^N, \infty}$ is defined as the probability law of the $N$-tuples of slits such that they are parameterized by $t \in [0, \infty]$ as $(\gamma^N_1(t), \ldots, \gamma^N_N(t))$ with

$$\gamma^N_j(0) = x_j, \quad \gamma^N_j(\infty) = \infty, \quad j = 1, 2, \ldots, N,$$

and each realization of solution $g^N_t, t \in [0, \infty]$ for (1.2) determines a time evolution of the slits, $(\gamma^N_1(t), \ldots, \gamma^N_N(t)), t \in [0, \infty]$, in which $g^N_t$ is regarded as a time-dependent conformal map (a Loewner chain) onto $\mathbb{H}$, and the domain of definition of $g^N_t$ is identified with $\mathbb{H} \setminus \bigcup_{j=1}^{N} \gamma^N_j(0, t]$ for each $t \in [0, \infty]$;

$$g^N_t: \text{conformal map } \mathbb{H} \setminus \bigcup_{j=1}^{N} \gamma^N_j(0, t] \to \mathbb{H}, \quad t \in [0, \infty],$$

where $\gamma^N_j(0, t) \equiv \bigcup_{0 < s \leq t} \gamma^N_j(s), \ j = 1, 2, \ldots, N$.

Let $\mathfrak{M}$ be the space of probability measures on $\mathbb{R}$ equipped with its weak topology. For $T > 0$, $C([0, T] \to \mathfrak{M})$ denotes the space of continuous processes defined in the time period $[0, T]$ realized in $\mathfrak{M}$. Let $\delta_x(\cdot)$ be the Dirac measure centered at $x$; $\delta_x(\{y\}) = 1$ if $y = x$ and $\delta_x(\{y\}) = 0$ otherwise. We consider the empirical measure of the solution $V^N(t)$ of (1.1),

$$V^N_t(\cdot) = \frac{1}{N} \sum_{j=1}^{N} \delta_{V^N_j(t)}(\cdot), \quad t \in [0, T],$$

as an element of $C([0, T] \to \mathfrak{M})$, whose initial value is given by $V^N_0(\cdot) = \frac{1}{N} \sum_{j=1}^{N} \delta_{x^N_j}(\cdot)$.

The following can be proved (see Proposition 4.3.10 in [1]).

**Proposition 1.1** Assume that $(x^N)_{N \in \mathbb{N}}$ is a sequence of initial configurations such that $x^N \in \mathbb{W}_N$,

$$\sup_{N \geq 0} \frac{1}{N} \sum_{j=1}^{N} \log\{(x^N_j)^2 + 1\} < \infty,$$
and $V^N_0(\cdot)$ converges weakly to a measure $\mu_0(\cdot) \in \mathcal{M}$ as $N \to \infty$. Then for any fixed $T < \infty$,

$$(V^N_t(\cdot))_{t \in [0,T]} \rightharpoonup^{\mathcal{M}} (\mu_t(\cdot))_{t \in [0,T]} \quad \text{a.s. in } C([0,T] \to \mathcal{M}),$$

and the function defined by

$$M_t(z) = \int_{\mathbb{R}} \frac{2\mu_t(du)}{z-u}$$

solves the equation

$$\frac{\partial M_t(z)}{\partial t} = -2M_t(z)\frac{\partial M_t(z)}{\partial z}, \quad t \in [0,T], \quad z \in \mathbb{C} \setminus \mathbb{R},$$

under the initial condition

$$M_0(z) = \int_{\mathbb{R}} \frac{2\mu_0(du)}{z-u}.$$  \hspace{1cm} (1.3)

The function $(M_t(z))_{t \in [0,T]}$, $z \in \mathbb{C} \setminus \mathbb{R}$ defined by the Stieltjes transformation \{13\} of $\mu_t(\cdot)$ is called the Green’s function (or the resolvent) for the measure-valued process $(\mu_t(\cdot))_{t \in [0,T]}$. Equation \{14\} can be regarded as the complex Burgers equation in the inviscid limit (i.e., the (complex) one-dimensional Euler equation). Thus the $N \to \infty$ limit given by this theorem is called the hydrodynamic limit of the Dyson model \{10, 32, 35, 1, 6, 18, 2\}. Note that the dependence on the parameter $\kappa \in (0,4]$ disappears in the hydrodynamic limit.

Associated with this hydrodynamic limit of the Dyson model, the following limit theorem was proved by del Monaco and Schleißinger.

**Theorem 1.2 (del Monaco, Schleißinger \{13, Theorem 1.1\})** Under the same assumption as given in Proposition \{14\}, in $N \to \infty$, $(g^N_t)_{N \in \mathbb{N}}$ converges locally uniformly in distribution to the solution $g_t$ of the deterministic Loewner equation

$$\frac{\partial g_t(z)}{\partial t} = M_t(g_t(z)), \quad t \geq 0, \quad g_0(z) = z \in \mathbb{H}.$$ \hspace{1cm} (1.6)

Let $D_t, t \geq 0$ be the domain of definition of $g_t, t \geq 0$. By the Carathéodory kernel theorem (see, for instance, Theorem 1.8 on page 29 in \{31\}), the locally uniform convergence of $(g^N_t)_{N \in \mathbb{N}}$ to $g_t$ in $N \to \infty$ means that

$$\mathbb{H} \setminus \bigcup_{j=1}^N \gamma^N_j(0,t] \to D_t, \quad t \geq 0, \quad \text{in the sense of kernel convergence}.$$  

Moreover, del Monaco and Schleißinger proved the following. (See \{12\} for the tightness results of the limit.)

**Theorem 1.3 (del Monaco, Schleißinger \{13, Theorem 1.2\})** The set $K_t = \mathbb{H} \setminus D_t$ is bounded for every $t \geq 0$ and there exists $T > 0$ such that for every $t > T$, the boundary $\partial K_t \cap \mathbb{H}$ is an analytic curve in $\mathbb{H}$. 


In an earlier paper [2], the hydrodynamic limit of the Dyson model was studied by solving the complex Burgers equation (1.4) explicitly for some special initial configurations \( \mu_0 \). Here we regard the deterministic Loewner equation (1.6) for \( g_t, t \geq 0 \) as the **hydrodynamic limit** of the multiple SLE, and \( K_t \) as the **SLE hull in the hydrodynamic limit**. In the present paper first we characterize the hydrodynamic limit of the multiple SLE by solving (1.6) explicitly for two special initial configurations. After reporting these exact solutions, a universal shape of \( K_t \) in long-term limit is discussed, which can be regarded as the counterpart of the celebrated *Wigner’s semicircle law* realized in the hydrodynamic limit of the Dyson model [10, 32, 29, 17, 11, 22, 2]. The present study is an extension of the results reported as examples and remarks in Sections 3.4 and 4 in [13] and remarks in Section 2.5 in [12].

The paper is organized as follows. In Section 2 we explain a method to solve the coupled system of the complex Burgers equation (1.4) and the multiple SLE in the hydrodynamic limit (1.6). This method was briefly mentioned in Remark 3.11 in [13]. Section 3 is devoted to reporting the exact results for the system starting from a single source at the origin, \( \mu_0 = \delta_0 \). We can see the same statement as Proposition 3.1 and the same figure as Fig.2 in Section 4 of [13], but the explicit expression for the SLE hull (3.18) with (3.19) is first given in Proposition 3.2 in the present paper. New exact results are reported in Section 4 for the system starting from the two sources. The long-term asymptotics of the SLE and its hull \( K_t \) are generally discussed in Section 5. Concluding remarks are given in Section 6.

## 2 Preliminaries

### 2.1 Transformation from \( g_t \) to \( h_t \)

In Remark 3.11 in [13], the following transformation \( g_t \to h_t \) was introduced,

\[
g_t(z) = h_t(z) + 2tM_t(g_t(z)), \tag{2.1}
\]

where \( h_t \) is chosen as the following equalities hold,

\[
M_t(g_t(z)) = M_0(h_t(z)), \quad t \geq 0, \quad g_0(z) = h_0(z) = z \in \mathbb{H}. \tag{2.2}
\]

The compatibility of (2.1) and (2.2) is guaranteed by the fact that the solution \( M_t(z) \) of the complex Burgers equation (1.4) satisfies the functional equation

\[
M_t(z) = M_0(z - 2tM_t(z)).
\]

See, for instance, Theorem 1.2 (ii) in [2].

Then the following is verified.

**Lemma 2.1 (del Monaco, Schleißinger [13, Remark 3.11])** Assume that the function \( h_t(z) \) solves the following partial differential equation,

\[
\frac{\partial h_t(z)}{\partial t} = - \frac{M_0(h_t(z))}{1 + 2t \frac{\partial M_0}{\partial z}(h_t(z))}, \quad h_0(z) = z. \tag{2.3}
\]
Then \( g_t(z) \) is obtained from \( h_t(z) \) by the relation
\[
g_t(z) = h_t(z) + 2tM_0(h_t(z)). \tag{2.4}
\]

**Proof.** By the definition (2.1) of \( h_t \) and the condition (2.2) for \( h_t \), (2.4) is concluded. By differentiating (2.4) by \( t \), we obtain
\[
\frac{\partial g_t(z)}{\partial t} = \frac{\partial h_t(z)}{\partial t} + 2M_0(h_t(z)) + 2t \frac{\partial M_0}{\partial z}(h_t(z)) \frac{\partial h_t(z)}{\partial t}.
\]
On the other hand, by the deterministic Loewner equation (1.6) and the equality (2.2),
\[
\frac{\partial g_t(z)}{\partial t} = M_t(g_t(z)) = M_0(h_t(z)).
\]
Then we have the equation
\[
\frac{\partial h_t(z)}{\partial t} + 2M_0(h_t(z)) + 2t \frac{\partial M_0}{\partial z}(h_t(z)) \frac{\partial h_t(z)}{\partial t} = M_0(h_t(z)),
\]
which is equivalent with (2.3). Then the proof is completed. \( \blacksquare \)

### 2.2 SLE hull \( K_t \) in the hydrodynamic limit

For each \( t \in [0, \infty) \), the boundary of SLE hull in the hydrodynamic limit, \( \partial K_t \subset \mathbb{H} \cup \mathbb{R} \), is given by the inverse image \( g_t^{-1} \) of the closed support of \( \mu_t \);
\[
\partial K_t = g_t^{-1}(\text{supp } \mu_t), \quad t \in [0, \infty), \tag{2.5}
\]
and the SLE hull in the hydrodynamic limit is given by
\[
K_t = \bigcup_{0 \leq s \leq t} \partial K_s, \quad t \in [0, \infty). \tag{2.6}
\]

We assume that there is a set \( I \subset \mathbb{R} \) and the elements of \( \text{supp } \mu_t \) are parameterized by \( \xi \in I \) as
\[
\text{supp } \mu_t = \{ \sigma_t(\xi) : \xi \in I \}, \quad t \in [0, \infty). \tag{2.7}
\]
Define
\[
\Gamma_t(\xi) = g_t^{-1}(\sigma_t(\xi)), \quad \xi \in I, \quad t \in [0, \infty).
\]
Then (2.5) gives
\[
\partial K_t = \{ \Gamma_t(\xi) : \xi \in I \}, \quad t \in [0, \infty).
\]
By (2.4), we have the equality
\[
\sigma_t(\xi) = h_t(\Gamma_t(\xi)) + 2tM_0(h_t(\Gamma_t(\xi))), \quad t \in [0, \infty). \tag{2.8}
\]
3 Exact Results for the System Starting from Single Source

3.1 SLE $g_t$ in the hydrodynamic limit

Consider the multiple SLE in the case that all slits start from the origin, which is obtained by taking the limit $x_j^N \to 0$, $1 \leq j \leq N$ for $x^N = (x_1^N, x_2^N, \ldots, x_N^N) \in \mathbb{W}$. In the hydrodynamic limit $N \to \infty$, this situation is realized by setting

$$\mu_0 = \delta_0, \quad \text{that is,} \quad \mu_0(du) = \delta(u)du,$$

(3.1)

where $\delta(u)$ denotes Dirac’s delta function. We say in this situation that the multiple SLE in the hydrodynamic limit starts from a single source located at the origin.

In this case, (1.5) is given by

$$M_0(z) = \int_{\mathbb{R}} \frac{2\delta_0(du)}{z - u} = \frac{2}{z}.$$

(3.2)

Under this initial condition the complex Burgers equation (1.4) is solved as

$$M_t(z) = \frac{1}{4t} \left( z - \sqrt{z^2 - 16t} \right) = \frac{4}{z + \sqrt{z^2 - 16t}}, \quad t \in [0, \infty).$$

(3.3)

Then the hydrodynamic limit of the multiple SLE (1.6) which we want to solve now is given by

$$\frac{\partial g_t(z)}{\partial t} = \frac{4}{g_t(z) + \sqrt{g_t(z)^2 - 16t}} \quad \text{for} \quad t \geq 0, \quad g_0(z) = z \in \mathbb{H}.$$  

(3.4)

Thanks to Lemma 2.1, however, we do not need to solve directly this equation (3.4) to determine $g_t$, $t \in [0, \infty)$. Only using the simple initial condition $M_0(z)$ given by (3.2), we can obtain Eq. (2.3) for $h_t(z)$, and through (2.4) $g_t(z)$ is determined. In the present case, Eq. (2.3) becomes

$$\frac{\partial h_t(z)}{\partial t} = -\frac{2}{h_t(z)} \left( 1 - \frac{4t}{h_t(z)^2} \right),$$

(3.5)

$$h_0(z) = z \in \mathbb{H}.$$  

(3.6)

It is easy to verify that Eq. (3.5) is equivalent with

$$\frac{\partial}{\partial t} \log h_t(z) = \frac{\partial}{\partial t} \left( -\frac{2t}{h_t(z)^2} \right).$$
Then we have \( h_t(z) = c_1 e^{-2t/h_t(z)^2} \), where \( c_1 \) is an integral constant. By the initial condition (3.6), the constant is determined as \( c_1 = z \). We rewrite the obtained equation as

\[
- \frac{4t}{h_t(z)^2} e^{-4t/h_t(z)^2} = \frac{4t}{z^2},
\]

(3.7)

Here we consider the Lambert \( W \) function (see, for instance, [11, 38]). This function is defined as the inverse function of the mapping \( x \mapsto x e^x \).

This mapping is not injective, and the Lambert \( W \) function has two real branches with a branching point at \((-e^{-1}, -1)\) in the real plane \((x, W) \in \mathbb{R}^2\). We take the upper branch \( W_0(x) \) defined for \( x \in (-e^{-1}, \infty) \). By this definition, we can show that

\[
W_0(x) e^{W_0(x)} = x,
\]

\[
W_0(0) = 0, \quad W_0(e) = 1,
\]

and

\[
W_0(x) \simeq x \quad \text{in} \quad x \to 0.
\]

(3.8)

Define the complex function \( W_0(z), z \in \mathbb{C} \) as an analytic continuation of \( W_0(x) \in (-e^{-1}, \infty) \subset \mathbb{R} \). \( W_0(z) \) is analytic at \( z = 0 \) having the expansion

\[
W_0(z) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} z^n,
\]

whose radius of convergence is \( e^{-1} \). The branch cut of \( W_0(z) \) on the complex plane \( \mathbb{C} \) is given by \((-\infty, -e^{-1}] \subset \mathbb{R}\). See [11] for more details.

Equation (3.7) is now rewritten as

\[
W_0 \left( \frac{-4t}{z^2} \right) = -\frac{4t}{h_t(z)^2},
\]

which gives

\[
h_t(z) = \pm i \sqrt{\frac{4t}{W_0(-4t/z^2)}} = \pm 2i \sqrt{\frac{1}{W_0(-4t/z^2)}}.
\]

(3.9)

If we take the limit \( t \to 0 \) in (3.9), we have \( h_0(z) = \pm i \sqrt{-z^2} \), where (3.8) was used. We choose the square root branch as \( \sqrt{-z^2} = -iz, z \in \mathbb{H} \), and hence \( h_0(z) = \pm z \). Due to the initial condition (3.6) of \( h_t \), we should choose the plus sign, and we obtain

\[
h_t(z) = 2i \sqrt{\frac{1}{W_0(-4t/z^2)}}.
\]

(3.10)

From this solution \( h_t(z) \), we obtain the solution \( g_t \) of (3.4) by (2.4) following Lemma 2.1. By (3.2), Eq. (2.4) becomes

\[
g_t(z) = h_t(z) + \frac{4t}{h_t(z)}.
\]

(3.11)

Insert (3.10) into (3.11), we obtain the exact solution of Eq. (3.4) as following.
Proposition 3.1  The exact solution of the multiple SLE in the hydrodynamic limit starting from the single source (3.1) is given by

\[ g_t(z) = 2i\sqrt{t} \left\{ \frac{1}{\sqrt{W_0(-4t/z^2)}} - \sqrt{W_0(-4t/z^2)} \right\}, \]

where \( W_0 \) is the Lambert \( W \) function satisfying (3.8).

3.2 SLE hull \( K_t \) in the hydrodynamic limit

Now we determine the SLE hull by (2.5) and (2.6). So far we did not use the explicit expression of the solution (3.3), but now we have to know the support of \( \mu_t \) in the expression (1.3) of the solution (3.3). By the same argument as given in Section 2 of [2], the following equality can be derived,

\[ \mu_t(du) = -3 \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi} M_t(u + i\varepsilon) \, du, \quad t \in [0, \infty), \quad u \in \mathbb{R}. \]

From (3.3), we have

\[ \mu_t(du) = \begin{cases} \frac{1}{8\pi t} \sqrt{16t - u^2} \, du, & \text{if } |u| < 4\sqrt{t}, \\ 0, & \text{if } |u| \geq 4\sqrt{t}, \end{cases} \]

(3.12)

for \( t \in [0, \infty) \). This is a time-dependent version of Wigner’s semicircle law in the present time change [10, 32] (see also Section 3.9 of [22]). Therefore, we can conclude that \( \text{supp} \mu_t = [-4\sqrt{t}, 4\sqrt{t}] \).

The parameterization of \( \text{supp} \mu_t \) assumed as (2.7) is now realized as

\[ \sigma_t(\xi) = 4\sqrt{t} \xi, \quad \xi \in I \equiv [-1, 1]. \]

(3.13)

By (3.2), Eq. (2.8) is given as

\[ \sigma_t(\xi) = h_t(\Gamma_t(\xi)) + \frac{4t}{h_t(\Gamma_t(\xi))}, \quad t \in [0, \infty). \]

(3.14)

Put

\[ h_t(\Gamma_t(\xi)) = v + iw, \quad v = v(\sigma_t(\xi), t) \in \mathbb{R}, \quad w = w(\sigma_t(\xi), t) \in \mathbb{R}. \]

Since \( \sigma_t(\xi) \in \mathbb{R} \), Eq. (3.14) gives

\[ v^2 - w^2 - \sigma_t(\xi)v + 4t = 0, \quad 2v = \sigma_t(\xi). \]

They are solved as

\[ v = \frac{\sigma_t(\xi)}{2}, \quad w = \frac{1}{2} \sqrt{16t - \sigma_t(\xi)^2}, \]
and we obtain
\[ h_t(\Gamma_t(\xi)) = \frac{1}{2} \left\{ \sigma_t(\xi) + i\sqrt{16t - \sigma_t(\xi)^2} \right\}. \] (3.15)

On the other hand, the solution (3.10) gives the equality,
\[ W_0 \left( -\frac{4t}{\Gamma_t(\xi)^2} \right) = -\frac{4t}{h_t(\Gamma_t(\xi))^2}, \quad t \in [0, \infty). \]

By (3.15), it gives
\[ W_0 \left( -\frac{4t}{\Gamma_t(\xi)^2} \right) = 1 - \frac{\sigma_t(\xi)^2}{8t} + \frac{i \sigma_t(\xi)}{8t} \sqrt{16t - \sigma_t(\xi)^2}. \]

Since the Lambert function \( W_0 \) is defined as the inverse function of the mapping \( x \mapsto xe^x \), the above equation is equivalent with
\[ -\frac{4t}{\Gamma_t(\xi)^2} = \left( 1 - \frac{\sigma_t(\xi)^2}{8t} + \frac{i \sigma_t(\xi)}{8t} \sqrt{16t - \sigma_t(\xi)^2} \right) \times \exp \left( 1 - \frac{\sigma_t(\xi)^2}{8t} + \frac{i \sigma_t(\xi)}{8t} \sqrt{16t - \sigma_t(\xi)^2} \right), \]
which gives
\[ \Gamma_t(\xi)^2 = -4t \left( 1 - \frac{\sigma_t(\xi)^2}{8t} - \frac{i \sigma_t(\xi)}{8t} \sqrt{16t - \sigma_t(\xi)^2} \right) \times \exp \left( -1 + \frac{\sigma_t(\xi)^2}{8t} - \frac{i \sigma_t(\xi)}{8t} \sqrt{16t - \sigma_t(\xi)^2} \right). \] (3.16)

Here we use the parameterization (3.13). Moreover, we put
\[ \xi = \sin \varphi, \quad \varphi \in [-\pi/2, \pi/2]. \] (3.17)

Then we see that (3.16) is simplified and we obtain
\[ \tilde{\Gamma}_t(\varphi) \equiv \Gamma_t(\sin \varphi) = 2i\sqrt{t} \exp \left( -i\varphi - \frac{e^{2i\varphi}}{2} \right). \]

The above results are summarized as follows.

**Proposition 3.2** Assume that the multiple SLE in the hydrodynamic limit starts from a single source at the origin, \( \mu_0 = \delta_0 \). Then the SLE hull is given by
\[ K_t = \sqrt{t} K, \quad t \in [0, \infty), \] (3.18)

with
\[ K = K_1 = \left\{ \tilde{\Gamma}_s(\varphi) : \tilde{\Gamma}_s(\varphi) = 2ir \exp \left( -i\varphi - \frac{e^{2i\varphi}}{2} \right), \quad -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}, 0 \leq r \leq 1 \right\}. \] (3.19)
Figure 2: The curve shows the boundary $\partial K$ of (3.19). Here $K \cap \mathbb{R} = [-2\sqrt{e}, 2\sqrt{e}] = [-3.297\cdots, 3.297\cdots]$ and $\max_{z \in K} \Im z = 2/\sqrt{e} = 1.213\cdots$.

It is easy to verify that

$$\Re \tilde{\Gamma}_t(-\varphi) = - \Re \tilde{\Gamma}_t(\varphi), \quad \Im \tilde{\Gamma}_t(-\varphi) = \Im \tilde{\Gamma}_t(\varphi), \quad -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2},$$

$$\max_{\varphi \in [-\pi/2, \pi/2]} \Im \tilde{\Gamma}_t(\varphi) = \Im \tilde{\Gamma}_t(0) = 2\sqrt{\frac{t}{e}},$$

$$\tilde{\Gamma}_t(\pm (\pi/2 - \varepsilon)) = \pm 2\sqrt{te}(1 - \varepsilon^2) + i\frac{4\sqrt{te}}{3}\varepsilon^3 + O(\varepsilon^4), \quad \varepsilon > 0. \quad (3.20)$$

The boundary $\partial K = \partial K_1$ is shown by a curve in Fig.2. Remark that the interval $K_t \cap \mathbb{R} = \sqrt{t}K \cap \mathbb{R} = [-2\sqrt{te}, 2\sqrt{te}]$ corresponds to the branch cut of $W_0(-4t/z^2)$, which is considered as a complex function of $z$ for each time $t > 0$ in (3.10); $-4t/z^2 \in (-\infty, -e^{-1}] \iff |z| \leq 2\sqrt{te}, z \in \mathbb{R}$.

The expansion (3.20) implies that in the vicinity of the edges $\pm 2\sqrt{te} \in \mathbb{R}$, the boundary $\{z = x + iy : z \in \partial K_t = \sqrt{t}\partial K\}$ behaves as

$$y \approx \begin{cases} \frac{\sqrt{2}}{3}(te)^{-1/4}(2\sqrt{te} - x)^{3/2}, & \text{if } x \leq 2\sqrt{te}, \\ \frac{\sqrt{2}}{3}(te)^{-1/4}(x + 2\sqrt{te})^{3/2}, & \text{if } x \geq -2\sqrt{te}. \end{cases} \quad (3.21)$$

4 Exact Results for the System Starting from Two Sources

4.1 SLE $g_t$ in the hydrodynamic limit

Next we consider the multiple SLE in the case that a half number of slits start from $x = a > 0$ and another half of slits start from $x = -a < 0$, which is obtained for an even $N$ by taking the limit $x_j^N \to -a, 1 \leq j \leq N/2$, and $x_j^N \to a, N/2 + 1 \leq j \leq N$ for
\( x^N = (x_1^N, x_2^N, \ldots, x_N^N) \in \mathbb{W} \). In the hydrodynamic limit \( N \to \infty \), this situation is realized by setting

\[
\mu_0 = \frac{1}{2} (\delta_a + \delta_{-a}), \quad \text{that is,} \quad \mu_0(du) = \frac{1}{2} \{\delta(u - a) + \delta(u + a)\} du. \tag{4.1}
\]

We say in this situation that the multiple SLE in the hydrodynamic limit starts from the two sources.

In this case, the initial condition of the complex Burgers equation is given by

\[
M_0(z) = \int_R \frac{2\delta_0(du)}{z - u} = \frac{1}{z - a} + \frac{1}{z + a} = \frac{2z}{z^2 - a^2}. \tag{4.2}
\]

Equation (2.3) for \( h_t(z) \) given in Lemma 2.1 is then

\[
\frac{\partial h_t(z)}{\partial t} = -\frac{1}{1 - 2t} \left( \frac{1}{h_t(z) - a} + \frac{1}{h_t(z) + a} \right) \frac{2h_t(z)}{h_t(z)^2 - a^2}, \tag{4.3}
\]

\[
h_0(z) = z \in \mathbb{H}. \tag{4.4}
\]

Equation (2.4) in Lemma 2.1 becomes

\[
g_t(z) = h_t(z) + 2t \left( \frac{1}{h_t(z) - a} + \frac{1}{h_t(z) + a} \right) \tag{4.5}
\]

We have found that (4.3) is equivalent with

\[
\frac{\partial}{\partial t} \log \sqrt{h_t(z)^2 - a^2} = \frac{\partial}{\partial t} \left( -\frac{2th_t(z)^2}{(h_t(z)^2 - a^2)^2} \right). \tag{4.3}
\]

This is solved as \( \sqrt{h_t(z)^2 - a^2} = c_2 e^{-2th_t(z)^2/(h_t(z)^2 - a^2)^2} \) with an integral constant \( c_2 \). By the initial condition (4.4), it is determined as \( c_2 = \sqrt{z^2 - a^2} \), and we obtain the equation

\[
h_t(z)^2 - a^2 = (z^2 - a^2)e^{-4th_t(z)^2/(h_t(z)^2 - a^2)^2}. \]

We solve this equation for \( z \) as

\[
z = \sqrt{a^2 + (h_t(z)^2 - a^2)e^{4th_t(z)^2/(h_t(z)^2 - a^2)^2}},
\]
which satisfies the initial condition (4.4); $h_0(z) = z$. Define a function

$$V_t(z) = \sqrt{1 + (z^2 - 1)e^{4tz^2/(z^2 - 1)^2}}, \quad t \in [0, \infty), \quad z \in \mathbb{H}. \quad (4.6)$$

Then the solution $h_t(z)$ of (4.3) under (4.4) is expressed as

$$h_t(z) = aV_t^{-1} \left( \frac{z}{a} \right), \quad (4.7)$$

where $V_t^{-1}$ denotes the inverse function of $V_t$ for each $t \in [0, \infty)$.

Insert (4.7) into (4.5), we obtain the exact solution $g_t$ as following.

**Proposition 4.1** The exact solution of the multiple SLE in the hydrodynamic limit starting from the two sources (4.1) is given by

$$g_t(z) = a \left\{ V_t^{-1} \left( \frac{z}{a} \right) + \frac{4(t/a^2)V_t^{-1}(z/a)}{(V_t^{-1}(z/a))^2 - 1} \right\}, \quad t \in [0, \infty), \quad (4.8)$$

where $V_t^{-1}$ is the inverse function of $V_t$ defined by (4.6).

### 4.2 SLE hull $K_t$ in the hydrodynamic limit

The complex Burgers equation (1.4) was solved under the two-source initial condition (4.1) in Section 6.5 of [30], in Section 2.3 of [39], and in Section 4.2 of [2]. See also [4]. The critical time is given by

$$t_c = t_c(a) = \frac{a^2}{4},$$

and the support of $\mu_t$, $t \geq 0$ is determined as

$$\text{supp} \mu_t = \begin{cases} 
\{x \in \mathbb{R} : ab_-(t/a^2) \leq |x| \leq ab_+(t/a^2)\}, & \text{if } 0 \leq t < t_c, \\
\{x \in \mathbb{R} : 0 \leq |x| \leq ab_+(t/a^2)\}, & \text{if } t \geq t_c,
\end{cases} \quad (4.9)$$

where

$$b_{\pm}(t) = \sqrt{(1 + 2t) \pm 2\sqrt{t(t + 2)}} \left\{ (1 - t) \pm \sqrt{t(t + 2)} \right\}. \quad (4.10)$$

It is easy to verify that

$$b_{\pm}(t)^2 = 1 + 10t - 2t^2 \pm 2(2 + t)\sqrt{t(2 + t)}$$

and

$$b_+(t)b_-(t) = (1 - 4t)^{3/2}.$$
Then (4.10) is written as
\[ b_-(t) = \sqrt{1 + 10t - 2t^2 - 2(2 + t)\sqrt{t(2 + t)}} \]
\[ = \frac{(1 - 4t)^{3/2}}{\sqrt{1 + 10t - 2t^2 + 2(2 + t)\sqrt{t(2 + t)}}}, \]
\[ b_+(t) = \sqrt{1 + 10t - 2t^2 + 2(2 + t)\sqrt{t(2 + t)}}. \] (4.11)

Since the system is symmetric with respect to the imaginary axis in \( \mathbb{C} \), we need to consider only the non-negative part of the support
\[ \text{supp} \mu_t^+ = \{ u \in \text{supp} \mu_t : u \geq 0 \}, \quad t \geq 0. \]

Moreover, the above formulas (4.8) and (4.9) implies that we can set \( a = 1 \) in calculation, since the system obeys the diffusion scaling and hence the general solution will be obtained by just setting \( t \rightarrow t/a^2, z \rightarrow z/a \) for \( a > 0 \). The parameterization for \( \text{supp} \mu_t^+ \) is given by
\[ \sigma_t^+(\xi) = \begin{cases} (1 - \xi)b_-(t) + \xi b_+(t), & \text{if } 0 \leq t < 1/4, \\ \xi b_+(t), & \text{if } t \geq 1/4, \end{cases} \quad \xi \in I^+ \equiv [0, 1]. \] (4.12)

By (4.2), Eq. (2.8) is given as
\[ \sigma_t^+(\xi) = h_t(\Gamma_t^+(\xi)) + \frac{4t h_t(\Gamma_t^+(\xi))}{h_t(\Gamma_t^+(\xi))^2 - 1}, \quad t \in [0, \infty). \] (4.13)

Put
\[ h_t(\Gamma_t^+(\xi)) = v + iw, \quad v = v(\sigma_t^+(\xi), t) \in \mathbb{R}, \quad w = w(\sigma_t^+(\xi), t) \in \mathbb{R}. \]

Since \( \sigma_t^+(\xi) \geq 0 \), Eq. (4.13) gives
\[ v^3 - \sigma_t^+(\xi)v^2 - \{ 3w^2 - (4t - 1) \} v + \sigma_t^+(\xi)(w^2 + 1) = 0, \]
\[ w \{ w^2 + 2\sigma_t^+(\xi)v - 3v^2 - (4t - 1) \} = 0. \] (4.14)

We need the solution such that \( w \) is not identically zero. Then we have chosen the solution of (4.14) as
\[ v = v(\sigma_t^+(\xi), t) \equiv -\frac{(S(\sigma_t^+(\xi), t) - \sigma_t^+(\xi))^2 - 3(4t - 1)}{6S(\sigma_t^+(\xi), t)}, \]
\[ w = w(\sigma_t^+(\xi), t) \equiv -\sqrt{3v(\sigma_t^+(\xi), t)^2 - 2\sigma_t^+(\xi)v(\sigma_t^+(\xi), t) + 4t - 1}, \quad t \in [0, \infty), \] (4.15)

where
\[ S(\sigma, t) = \sqrt{\sigma^3 + 9\sigma^2\sqrt{-\sigma^2 - 2(2t^2 - 10t - 1) + 3(4t - 1)^3} - 9\sigma(2t + 1)}. \] (4.16)

On the other hand, the solution (4.7) gives
\[ \Gamma_t^+(\xi) = V_t(h_t(\Gamma_t^+(\xi))), \quad t \in [0, \infty). \]

The results are summarized as follows.
Figure 3: The boundaries of the SLE hull in the hydrodynamic limit, $\partial K_t$, are shown for $t = 0.05$ (the thickest curve), 0.25, 0.5, 1, 2, and 4 (the thinnest curve), when the multiple SLE in the hydrodynamic limit starts from the two sources, $\mu_0 = (\delta_1 + \delta_{-1})/2$. As $t \to \infty$, $\partial K_t$ seems to approach to $\sqrt{t}\partial K$.

**Proposition 4.2** Assume that the multiple SLE in the hydrodynamic limit starts from the two sources $\mu_0 = (\delta_a + \delta_{-a})/2$, $a > 0$. Then the SLE hull is given by

$$K_t = K_t^+ \cup K_t^-, \quad t \in [0, \infty),$$

with

$$K_t^+ = \left\{ \Gamma_s^+ (\xi) : \Gamma_s^+ (\xi) = a V_{s/2} \left( a \left\{ v(\sigma_{s/2}^+ (\xi), s/a^2) + iw(\sigma_{s/2}^+ (\xi), s/a^2) \right\} \right) : \xi \in [0, 1], 0 \leq s \leq t \right\},$$

$$K_t^- = - \overline{K_t^+} = \left\{ -z = -x + iy : z = x + iy \in K_t^+ \right\},$$

where $v(\sigma_t^+ (\xi), t)$ and $w(\sigma_t^+ (\xi), t)$ are given by (4.15) with (4.10), (4.12) and (4.16).

Figure 3 shows time dependence of the boundary of the SLE hull $\partial K_t$ in the hydrodynamic limit, when $a = 1$ and $\mu_0 = (\delta_1 + \delta_{-1})/2$. When $t < t_c(1) = 1/4$, $\partial K_t$ consists of two separated curves, which are symmetric with respect to the imaginary axis. These two curves coalesce at the critical time $t = t_c(1) = 1/4$, and then $\partial K_t$ grows as a single curve in $\mathbb{H}$.

### 4.3 $a^2/t$-expansion

The numerical plots of $\partial K_t$ for various $t$ in Fig 3 show that the double-peak structure in $\partial K_t$ disappears gradually as $t \to \infty$ and the SLE hull $K_t$ seems to approach to a dilatation by factor $\sqrt{t}$ of the shape $K$ shown in Fig 2. Here we clarify such a long-term asymptotic in $t/a^2 \to \infty$ of $K_t$ starting from the two sources, $\mu_0 = (\delta_a + \delta_{-a})/2$, $a > 0$.

We consider the case $t > a^2/4$, in which $\text{supp } \mu_t$ is a single interval and is parametrized as

$$\sigma_t (\xi) = \xi ab_+(t/a^2), \quad \xi \in I \equiv [-1, 1].$$

(4.17)
When we include $a > 0$, Eq. (4.13) should be read as
\[
\sigma_t(\xi) = h_t(\Gamma_t(\xi)) + \frac{4 t h_t(\Gamma_t(\xi))}{h_t(\Gamma_t(\xi))^2 - a^2},
\]
which is rewritten as
\[
\frac{1}{t} h_t(\Gamma_t(\xi))^2 - \frac{1}{t} \sigma_t h_t(\Gamma_t(\xi)) + 4 = \left(1 - \frac{\sigma_t(\xi)}{h_t(\Gamma_t(\xi))}\right) \frac{a^2}{t}.
\] (4.18)

By (4.10), (4.17) has the following expansion,
\[
\sigma_t(\xi) = \sigma^{(0)} + \sigma^{(1)} \frac{a^2}{t} + O\left(\frac{a^2}{t^2}\right)
\] (4.19)

with
\[
\sigma^{(0)} = \sigma^{(0)}(t, \xi) = 4 \xi \sqrt{t}, \quad \sigma^{(1)} = \frac{1}{8}.
\] (4.20)

Then (4.18) implies that $h_t(\Gamma_t(\xi))$ can be also expanded with respect to $a^2/t$ as
\[
h_t(\Gamma_t(\xi)) = h^{(0)} + h^{(1)} + O\left(\frac{a^2}{t^2}\right).
\] (4.21)

When we put (4.19) and (4.21) into (4.18), the 0-th order terms and the first order terms in the $a^2/t$-expansion give the following equations, respectively,
\[
(h^{(0)})^2 - \sigma^{(0)} h^{(0)} + 4 t = 0,
\] (4.22)
\[
2(h^{(0)})^2 h^{(1)} - \sigma^{(0)} h^{(1)} (\sigma^{(1)} + h^{(1)}) = t \left(1 - \frac{\sigma^{(0)}}{h^{(0)}}\right).
\] (4.23)

Equation (4.22) is the same as (3.14), and hence we have
\[
h^{(0)} = h^{(0)}(t, \xi) = 2 \sqrt{t}(\xi + i \sqrt{1 - \xi^2}).
\]

Then (4.23) is solved as
\[
h^{(1)} = h^{(1)}(\xi) = \frac{(h^{(0)})^2 \sigma^{(0)} \sigma^{(1)} + t(h^{(0)} - \sigma^{(0)})}{(h^{(0)})^2 (2h^{(0)} - \sigma^{(0)})} = \frac{3 \xi - i \sqrt{1 - \xi^2}}{8(\xi + i \sqrt{1 - \xi^2})}.
\]

If we use the parameterization (3.17), we obtain the following expressions,
\[
h^{(0)} = \tilde{h}^{(0)}(t, \varphi) = 2 i \sqrt{t} e^{-i \varphi},
\]
\[
h^{(1)} = \tilde{h}^{(1)}(\varphi) = -\frac{i}{8} e^{i \varphi} (3 \sin \varphi - i \cos \varphi), \quad \varphi \in [-\pi/2, \pi/2].
\] (4.24)
Now we put (4.21) into
\[\Gamma_t(\xi) = a V_t(\Gamma_t(\xi))/a = \sqrt{a^2 + (h_t(\Gamma_t(\xi))^2 - a^2) \exp \left( \frac{4th_t(\Gamma_t(\xi))^2}{(h_t(\Gamma_t(\xi))^2 - a^2)^2} \right)} .\]

We obtain the expansion
\[\Gamma_t(\xi) = h_t(0) e^{2t/(h_t(0))^2} \times \left[ 1 + \left\{ \frac{2(h_t(0))^2 h_t(1) - t}{2(h_t(0))^2} + \frac{4t - (h_t(0))^2 h_t(1)}{(h_t(0))^4} + \frac{t}{2(h_t(0))^2} e^{-4t/(h_t(0))^2} \right\} \frac{a^2}{t} + O((a^2/t)^2) \right] .\]

Using (4.24), we will arrive at the result,
\[\tilde{\Gamma}_t(\varphi) \equiv \Gamma_t(\sin \varphi) = 2t \sqrt{t} \exp \left( -i\varphi - \frac{e^{2i\varphi}}{2} \right) \left[ 1 + \frac{1}{8} \left\{ 1 - \exp \left( 2i\varphi + e^{2i\varphi} \right) \right\} \frac{a^2}{t} + O((a^2/t)^2) \right] , \quad \varphi \in [-\pi/2, \pi/2] .\]

Therefore, the SLE hull behaves as
\[K_t = \sqrt{t} K(1 + O(a^2/t)) \equiv \{ \sqrt{t} z(1 + O(a^2/t)) : z \in \mathcal{K} \} , \quad \text{in } a^2/t \to 0, \]
where \(\mathcal{K}\) is given by (3.19). It implies that for any \(a > 0\),
\[\lim_{t \to \infty} \frac{K_t}{\sqrt{t}} = \mathcal{K} \quad (4.25)\]
for the SLE hull in the hydrodynamic limit starting from the two sources \(\mu_0 = (\delta_a + \delta_{-a})/2\).

In Section 5 we will discuss that the asymptotic behavior (4.25) is universal for any \(\mu_0\) such that \(\text{supp} \mu_0 \) on \(\mathbb{R}\) is bounded from either side by constants.

### 4.4 Critical curve \(\partial K_t\)

Here we assume \(a = 1\) for simplicity of expressions. At the critical time \(t_c = t_c(1) = 1/4\), (4.11) gives
\[b_-(t_c) = 0, \quad b_+(t_c) = \frac{3\sqrt{3}}{2} .\]

Let
\[\sigma_{t_c}(\xi) = \xi b_+(t_c) = \frac{3\sqrt{3}}{2} \xi , \quad \xi \in [-1, 1] ,\]

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Figure 4: The boundary of the SLE hull in the hydrodynamic limit at $t = t_c = 1/4$, $\partial K_{t_c}$, when the system starts from the two sources, $\mu_0 = (\delta_1 + \delta_{-1})/2$. The curve is symmetric with respect to the imaginary axis. The three osculation points on $\mathbb{R}$ are located at 0 and $\pm x_c = \pm \sqrt{1 + 2e^{3/4}} = \pm 2.287\cdots$. At the edges $x = \pm x_c$, the curve $\partial K_{t_c}$ shows the power law $\sim |x \mp x_c|^{3/2}$. In the vicinity of the origin, the curve is $V$-shaped with an inner angle $2\pi/3$.

which parameterizes the whole interval of $\text{supp} \mu_{t_c} = [-3\sqrt{3}/2, 3\sqrt{3}/2]$. The equations to determine the boundary of the SLE hull

$$\partial K_{t_c} = \{ \Gamma_{t_c}(\xi) = V_{t_c}(v + iw) : v = v(\xi) \in \mathbb{R}, w = w(\xi) \in \mathbb{R}, \xi \in [-1, 1] \}$$

are given by

$$8v^3 - 12\sqrt{3}v^2 + \frac{27}{2} \xi^2 v - \frac{3\sqrt{3}}{2} \xi = 0,$$

$$w^2 = 3v(v - \sqrt{3}\xi),$$

(4.26)

where

$$V_{t_c}(z) = V_{1/4}(z) = \sqrt{1 + (z^2 - 1)e^{3z^2/(z^2-1)^2}}.$$  

(4.27)

Figure 4 shows the critical curve $\partial K_{t_c}$, which is symmetric with respect to the imaginary axis.

If we set $\xi = \pm 1$ and $w = 0$, the equations (4.26) become

$$(v \mp \sqrt{3}) \left(8v^2 \mp 4\sqrt{3}v + \frac{3}{2} \right) = 0,$$

$$3v(v \mp \sqrt{3}) = 0.$$  

Then we find the solution

$$h_{t_c}(\Gamma_{t_c}(\pm 1)) = \pm \sqrt{3} \equiv \pm v_c.$$  

On the other hand, if we set $\xi = 0$ and $w = 0$, the equations (4.26) give

$$h_{t_c}(\Gamma_{t_c}(0)) = 0.$$  

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They determine the three real values in the curve $\partial K_{t_\varepsilon} \in \mathbb{R}$,

$$\Gamma_{t_\varepsilon}(\pm 1) = \pm V_{t_\varepsilon}(\pm v_c) = \pm \sqrt{1 + 2e^{3/4}}$$

$$= \pm 2.287 \cdots \equiv x_c,$$

$$\Gamma_{t_\varepsilon}(0) = V_{t_\varepsilon}(0) = 0,$$

which are the three osculation points of $\partial K_{t_\varepsilon}$ on $\mathbb{R}$; $\partial K_{t_\varepsilon} \cap \mathbb{R} = \{-x_c, 0, x_c\}$.

If we put $z = v + \delta$ in (4.27) with $v \in \mathbb{R}$, $\delta \in \mathbb{C}$, $v \neq 0$, $|\delta| \ll 1$, we have the following expansion,

$$V_{t_\varepsilon}(v + \delta) = \sqrt{1 + (v^2 - 1)e^{v^2/(v^2-1)^2}}$$

$$\times \left[ 1 + \frac{e^{v^2/(v^2-1)^2}}{1 + (v^2 - 1)e^{v^2/(v^2-1)^2}} v^3(v^2 - 3)\right]$$

$$+ \frac{e^{v^2/(v^2-1)^2}}{1 + (v^2 - 1)e^{v^2/(v^2-1)^2}} v^2(v^8 - 6v^6 + 18v^4 - 14v^2 + 9)\delta^2$$

$$- \frac{e^{2v^2/(v^2-1)^2}}{1 + (v^2 - 1)e^{v^2/(v^2-1)^2}} v^6(v^2 - 3)^2\delta^2 + O(\delta^3) \right].$$

In the vicinity of the right edge $x = x_c$ of $\partial K_{t_\varepsilon}$, the above gives

$$V_{t_\varepsilon}(v_c + \delta) = x_c + \frac{9e^{3/4}}{4x_c} \delta^2 + O(\delta^3). \quad (4.28)$$

If we set $\xi = 1 - \varepsilon$, $v = v_c + \varepsilon + O(\varepsilon^2)$ in (4.26) assuming $0 < \varepsilon \ll 1$, the coefficient $c$ is determined as $c = -7\sqrt{3}/9$ and we obtain $w = \pm \sqrt{2}c^{1/2} + O(\varepsilon^{3/2})$. We put

$$\delta = -\frac{7\sqrt{3}}{9} \varepsilon - \sqrt{2}c^{1/2} + O(\varepsilon^{3/2})$$

in (4.28). Then we have

$$V_{t_\varepsilon}(v_c + \delta) = x_c - \frac{9e^{3/4}}{2x_c} \varepsilon + \frac{7\sqrt{6}e^{3/4}}{2x_c} i\varepsilon^{3/2} + O(\varepsilon^2).$$

The above result implies that in the vicinity of the right and left edges $\pm x_c$, the boundary of the SLE hull, $\{z = x + iy : z \in \partial K_{t_\varepsilon}\}$, behaves as

$$y \simeq \begin{cases} 
\frac{14\sqrt{6}}{27} \sqrt{x_c \over x_c^2 - 1} (x_c - x)^{3/2}, & \text{if } x \leq x_c, \\
\frac{14\sqrt{6}}{27} \sqrt{x_c \over x_c^2 - 1} (x + x_c)^{3/2}, & \text{if } x \geq -x_c.
\end{cases}$$

The singularities of the curve $\partial K_{t_\varepsilon}$ at the edges $x = \pm x_c$ are governed by the power law with exponent $3/2$, which is common to the single-source solution as shown by (3.21).
Next we consider the vicinity of the origin. Equation (4.27) gives
\[ V_t(c) = \sqrt{\frac{3}{2} \delta^2 + O(\delta^3)}, \quad |\delta| \ll 1. \] (4.29)

If we assume \( 0 < \xi \ll 1, |v| \ll 1 \) and \( |w| \ll 1 \), then the function (4.26) behaves as
\[ v = v(\xi) \simeq \frac{\sqrt{3}}{2^{4/3}} \xi^{1/3}, \quad w = w(\xi) \simeq \sqrt{3} v(\xi). \]

Then if we put
\[ \delta = \delta(\xi) = v(\xi) + iw(\xi) \simeq \frac{\sqrt{3}}{2^{4/3}} (1 + i\sqrt{3}) \xi^{1/3} = \frac{\sqrt{3}}{2^{4/3}} e^{i\pi/3} \xi^{1/3} \]
in (4.29), we obtain the estimate
\[ \Gamma_t(\xi) = V_t(\delta(\xi)) \simeq \sqrt{\frac{3^3}{2^{7/3}} e^{i\pi/3} \xi^{4/3}} = \frac{3^{3/2}}{2^{7/6}} e^{i\pi/6} \xi^{2/3}, \quad 0 < \xi \ll 1. \]

This result implies that in the vicinity of the origin, the boundary of the SLE hull, \( \{ z = x + iy : z \in \partial K_t \} \), behaves as
\[ y \simeq \pm \frac{x}{\sqrt{3}}. \]
That is, \( \partial K_t \) behaves linearly in the vicinity of the origin, and the right and left asymptotic lines make angles with respect to the positive real axis given by \( \pi/6 \) and \( \pi - \pi/6 \), respectively.

Note that after the critical time, \( t > t_c \), the singularity at the origin disappears in the curve \( \partial K_t \), since for \( t > t_c \) there is no gap in \( \text{supp} \mu_t \) as given by (4.9) and the map between \( \xi \in \text{supp} \mu_t \) and \( \Gamma_t(\xi) \in \partial K_t \) is analytic as shown by (4.13).

5 Long-Term Asymptotics

5.1 Wigner’s semicircle law as a long-term asymptotics

For a function of \( t \) and \( z \), \( f(t, z) \), here we use the following abbreviations for partial differentials,
\[ \dot{f} = \frac{\partial f}{\partial t}, \quad f' = \frac{\partial f}{\partial z}. \]
Consider the complex Burgers equation (1.4) for \( M_t(z) \), \( t \geq 0, z \in \mathbb{H} \), which is now written as
\[ \dot{M}_t(z) = -2M_t(z)M'_t(z). \] (5.1)
Let \( c > 0 \) and define
\[ M_t(z, c) \equiv cM_{ct}(cz). \] (5.2)
We see that
\[
\frac{\partial M_t(z, c)}{\partial t} = c^3 \dot{M}_{ct}(cz), \quad \frac{\partial M_t(z, c)}{\partial z} = c^2 M'_{ct}(cz).
\]
If we set \( t \to c^2 t, z \to cz \) in (5.1), we have the equation
\[
\dot{M}_{ct}(cz) = -2M_{ct}(cz)M'_{ct}(cz).
\]
Multiply the both sides by \( c^3 \). Then we obtain the equation
\[
\frac{\partial M_t(z, c)}{\partial t} = -2M_t(z, c) \frac{\partial M_t(z, c)}{\partial z}.
\]
Hence, if \( M_t(z) \) solves the complex Burgers equation (5.1), then (5.2) with any \( c > 0 \) also solves it.

We find that
\[
M_0(z, c) = c M_0(cz)
\]
\[
= c \int_{\mathbb{R}} \frac{2 \mu_0(du)}{cz - u} = \int_{\mathbb{R}} \frac{2 \mu_0(du)}{z - u/c},
\]
and hence, if the support of \( \mu_0 \) is bounded from either side by constants,
\[
\text{supp} \mu_0 \subset [-L, L] \quad \text{for some } L > 0,
\]
then
\[
\lim_{c \to \infty} M_0(z, c) = \frac{2}{z} \int_{\mathbb{R}} \mu_0(du) = \frac{2}{z}.
\]
Therefore, for any fixed \( T < \infty \), if (5.3) is satisfied,
\[
\lim_{c \to \infty} M_T(z, c) = \lim_{c \to \infty} c M_{ct}(cz) = \frac{4}{z + \sqrt{z^2 - 16T}}
\]
is concluded by the result (3.3).

Let \( 1(\omega) \) be an indicator function; \( 1(\omega) = 1 \) if the condition \( \omega \) is satisfied, and \( 1(\omega) = 0 \) otherwise. We can prove the following for the hydrodynamic limit of the Dyson model.

**Proposition 5.1** For any initial distribution \( \mu_0 \) satisfying (5.3),
\[
\lim_{t \to \infty} \sqrt{t} M_t(\sqrt{t}z) = \frac{4}{z + \sqrt{z^2 - 16}},
\]
\[
\lim_{t \to \infty} \sqrt{t} \mu_t(\sqrt{t}du) = 1(|u| \leq 4) \frac{1}{8\pi} \sqrt{16 - u^2} du.
\]

**Proof.** In (5.4), if we set \( c = \sqrt{t/T} \), we have
\[
\lim_{t \to \infty} \sqrt{\frac{t}{T}} M_t \left( \sqrt{\frac{t}{T}}z \right) = \frac{4}{z + \sqrt{z^2 - 16T}}.
\]
Since $T$ is an arbitrary positive number, we can replace $z/\sqrt{T}$ by $z$ and obtain (5.5). On the other hand, as given by (3.12), for $\mu_0 = \delta_0$,

$$M_t(z) = \frac{4}{z + \sqrt{z^2 - 16t}} = \int_{\mathbb{R}} \frac{2}{z - u} 1(|u| \leq 4t) \frac{1}{8\pi t} \sqrt{16t - u^2} du.$$

If we set $t = 1$, the above implies that

$$(\text{RHS}) \text{ of Eq.} (5.5) = \int_{\mathbb{R}} du \frac{2}{z - u} \frac{1}{8\pi} \sqrt{16 - u^2}.$$

By the definition of Stieltjes transformation (1.3), we see that

$$(\text{LHS}) \text{ of Eq.} (5.5) = \lim_{t \to \infty} \sqrt{t} \int_{\mathbb{R}} \frac{2\mu_t(du)}{\sqrt{t}z - u} = \lim_{t \to \infty} \int_{\mathbb{R}} du \frac{2\mu_t(du)}{z - u/\sqrt{t}}.$$

If we change the integral variable as $u \to w \equiv u/\sqrt{t}$, the above is written as

$$\lim_{t \to \infty} \int_{\mathbb{R}} \frac{2\sqrt{t}\mu_t(\sqrt{t}dw)}{z - w}.$$

Then (5.6) is concluded. The proof is hence completed. \hfill \blacksquare

5.2 Long-term asymptotics of hydrodynamic limit of the multiple SLE

The statement (5.5) in Proposition 5.1 means the asymptotics

$$M_t(z) \simeq \frac{1}{\sqrt{t}} \frac{4}{z/\sqrt{t} + \sqrt{(z/\sqrt{t})^2 - 16}} = \frac{4}{z + \sqrt{z^2 - 16t}}, \quad \text{in } t \to \infty. \quad (5.7)$$

This property was stated as Remark 2.13 in [12]. Here we claim that, by the consideration given in Section 3.1 (5.7) implies that

$$g_t(z) \simeq 2i\sqrt{t} \left\{ \frac{1}{\sqrt{W_0(-4t/z^2)}} - \sqrt{W_0(-4t/z^2)} \right\}, \quad \text{in } t \to \infty,$$

and hence

$$K_t \simeq \sqrt{t} \mathcal{K}, \quad \text{in } t \to \infty,$$

where $\mathcal{K}$ is given by (3.19). Therefore, the following theorem will be established.

Theorem 5.2 For any $\mu_0$ satisfying (5.3), the hydrodynamic limit of the multiple SLE shows the following long-term asymptotics,

$$\lim_{t \to \infty} \frac{1}{\sqrt{t}} g_t(\sqrt{t}z) = 2i \left\{ \frac{1}{\sqrt{W_0(-4t/z^2)}} - \sqrt{W_0(-4t/z^2)} \right\}, \quad z \in \mathbb{H} \setminus \mathcal{K},$$

where $\mathcal{K}$ is given by (3.19).
6 Concluding Remarks

For \( N \in \mathbb{N} \), the \( N \)-tuple SLE is a coupled system of (a modification of) the Dyson model \( V^N(t) \) driven by a set of independent Brownian motions \( B(t) = (B_1(t), \ldots, B_N(t)) \) on \( \mathbb{R} \) and the differential equation of conformal map \( g_t^N \) onto \( \mathbb{H} \) driven by \( V^N(t) \) \([3, 24, 13, 12]\). The probability law of solution \( g_t^N \) governs the statistical ensemble of \( N \) random slits in \( \mathbb{H} \). Recently del Monaco and Schleißinger \([13]\) discussed the system in the limit \( N \to \infty \). In the case of simultaneous growth of slits in \( \mathbb{H} \) without any intersection, they showed that the limit is a deterministic system consisting of a \((1 + 1)\)-dimensional partial differential equation called the complex Burgers equation \((1.4)\) for \( M_t(z), t \geq 0, z \in \mathbb{C} \setminus \mathbb{R} \), and the ordinary equation for \( g_t \) \((1.6)\) driven by \( M_t(\cdot) \). Since the complex Burgers equation has been studied in discussing the hydrodynamic limit of the Dyson model \([10, 32, 1, 6, 18, 2]\), here we have regarded the limit system as the hydrodynamic limit of the multiple SLE.

The Dyson model is a dynamical extension of the eigenvalue statistics studied in random matrix theory \([14, 29, 17, 1, 22]\). The most fundamental probability-law for eigenvalues of random matrices is Wigner’s semicircle law, since it can be considered as the law of large numbers for the eigenvalue statistics of random matrices. The universality of Wigner’s semicircle law has been extensively studied. See \([15, 16, 36, 37]\) and references therein. It is also well-known that if we consider the complex Burgers equation under the initial condition \( \mu_0 = \delta_0 \), it has a unique solution, which we denote as \( M_t^{Wigner}(z) \) here, since it is identified with the Stieltjes transformation of the measure following the time-dependent version of Wigner’s semicircle law (see, for instance, Exercise 4.3.18 in \([1]\), and \([2]\)).

Following \([13]\), we gave the conformal map \( g_t(z) \) driven by \( M_t^{Wigner}(\cdot) \) using the Lambert function \( W_0 \) in Proposition 3.1. We would like to emphasize the importance of its hull \( K_t \) given by Proposition 3.2 since it describes the time evolution of the hydrodynamic limit of an infinite number of slits growing in \( \mathbb{H} \). As demonstrated in Sections 4.3 and 5, we expect that the hull \( K = K_1 \) given by \((3.19)\) and Fig. 2 will provide a universal shape describing the long-term behavior of systems.

One of the advantage of SLE \([34, 26, 25]\) is that it enables us to clarify the probability laws of random fractal-geometry, e.g., the SLE slits and its hull, by analyzing the stochastic differential equation for a conformal map. As Wigner’s semicircle distribution plays a role of the law of large numbers in random matrix theory, the shape \( K \) will be considered to represent the law of large numbers for the infinite system of interacting random curves in \( \mathbb{H} \) generated by the multiple SLE.

As well as the random matrix theory provides a universal viewpoint for statistical and stochastic systems, it has the great variety of ensembles depending on symmetry and geometrical restrictions. In suitable setting, we can also observe crossover phenomena \([23]\) and phase transitions with critical phenomena \([5, 6, 11, 33, 20]\). It will be an interesting future problem to clarify how to lift these rich structures in the random matrix theory up to the level of multiple SLE.

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