Classical and quantum dissipation
in non homogeneous environments

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Abstract

We generalize the oscillator model of a particle interacting with a thermal reservoir by introducing arbitrary nonlinear couplings in the particle coordinates. The equilibrium positions of the heat bath oscillators are promoted to space-time functions, which are shown to represent a modulation of the internal noise by the external forces. The model thus provides a description of classical and quantum dissipation in non homogeneous environments. In the classical case we derive a generalized Langevin equation with nonlinear multiplicative noise and a position-dependent fluctuation-dissipation theorem associated to non homogeneous dissipative forces. When time-modulation of the noise is present, a new force term is predicted besides the dissipative and random ones. The model is quantized to obtain the non homogenous influence functional and master equation for the reduced density matrix of the Brownian particle. The quantum evolution equations reproduce the correct Langevin dynamics in the semiclassical limit. The consequences for the issues of decoherence and localization are discussed.

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1. Introduction and summary.

In the Langevin approach to classical Brownian motion [1] the total system, globally isolated, is divided into two parts, the central particle and the environment. The effects of the environment on the otherwise free particle are taken into account by adding two force terms in Newton’s equation, a deterministic damping force $-M\gamma_0 \dot{x}$ and a random force $R(t)$ (for simplicity we consider only one-dimensional motions, since the generalization to higher dimensions is straightforward):

$$M\ddot{x} = -M\gamma_0 \dot{x} + R(t).$$  

(1)

The former expression is the Langevin equation, where $M$ stands for the mass of the Brownian particle, $\gamma_0$ is the damping constant, and $R(t)$ is a stationary Gaussian Markov process. The expectation value $\langle R(t) \rangle$ of the random force is assumed to be zero,

$$\langle R(t) \rangle = 0,$$  

(2)

so that the macroscopic equation for the expectation $\langle x(t) \rangle$ coincides with the classical equation of motion of a damped particle, $M\ddot{x}_{cl} = -M\gamma_0 \dot{x}_{cl}$. The additional requirement that, for $t \to \infty$, the particle reaches thermal equilibrium with the environment, so that $M\langle \dot{x}^2 \rangle/2 = k_B T/2$, requires that:

$$\langle R(t)R(s) \rangle = 2\gamma_0 M k_B T \delta(t-s),$$  

(3)

where $k_B$ is Boltzmann’s constant and $T$ is the absolute temperature. $R(t)$ is thus a white noise. Eq.(3) expresses the well known fluctuation-dissipation theorem for the particular case of white noise, in which the correlation function $K(t-s) \propto \delta(t-s)$.

However, there is no Lagrangian and therefore no action principle allowing to derive the phenomenological equation of classical Brownian motion, eq.(1). This shortcoming of the classical theory makes it very difficult to build a quantum theory of Brownian motion and to treat the quantum mechanics of open systems because canonical or path-integral quantization require the knowledge of the microscopic dynamics of the system, either Lagrangian or Hamiltonian.

A possible way to overcome this difficulty is to introduce a suitable mechanical model of dissipation. A particular one, the so-called oscillator model, has grown to a privileged position since its inception in the early sixties [2], [3], [4]). The oscillator model, in its classical version, provides an appealing alternative derivation of the Langevin equation of classical Brownian motion. Upon quantization, the model provides a theory of quantum Brownian motion which, in the semiclassical limit, reduces to the known phenomenological Langevin dynamics.

The first step towards a mechanical model of homogeneous classical and quantum dissipation is to consider the central particle and the dissipative environment as forming a large isolated dynamical system.

The second step is to consider the dissipative environment as composed by an infinite collection of independent harmonic oscillators $\{q_n\}$, with frequencies $\{\omega_n/2\pi\}$, linearly
coupled to the “central” particle $x$ of mass $M$. Then, if suitable assumptions are made on
the statistical distribution of the initial conditions of the oscillators, the central particle
exhibits Brownian motion with homogeneous noise in the limit of continuously distributed
frequencies.

The Lagrangian of this many-body mechanical model is
\[
L(x, \dot{x}, \{q_n, \dot{q}_n\}, t) = \frac{M}{2} \dot{x}^2 + \sum_n \left\{ \frac{m}{2} \dot{q}_n^2 - \frac{m}{2} \omega_n^2 (q_n - x)^2 \right\},
\]
(4)
where for simplicity all the oscillators are assumed to be of equal mass $m$. The first term
in the r.h.s. of eq.(4) is the free Lagrangian $L_0$ of the central particle, while the remaining
part will be referred to as the oscillator Lagrangian $L_{osc}$; it describes the whole environment
and its interaction with the central particle. The oscillator Lagrangian actually describes
a field-particle coupling, for any free field can be essentially viewed as an infinite collection
of harmonic oscillators.

The oscillators are assumed to be in thermal equilibrium at the initial time $t_0$ according
to a canonical distribution:
\[
\langle q_{i0} - x_0 \rangle = 0 ,
\]
\[
\langle \dot{q}_{i0} \rangle = 0 ,
\]
\[
m \omega_i \omega_j \langle (q_{i0} - x_0)(q_{j0} - x_0) \rangle = k_B T \delta_{ij} ,
\]
\[
m \langle \dot{q}_{i0} \dot{q}_{j0} \rangle = k_B T \delta_{ij} ,
\]
\[
\langle \dot{q}_{i0} (q_{j0} - x_0) \rangle = 0 ,
\]
(5)
where $\delta_{ij}$ is the Kronecker symbol and $q_{i0} = q_i(t_0)$, $\dot{q}_{i0} = \dot{q}_i(t_0)$, and $x_0 = x(t_0)$. The
classical homogeneous Brownian dynamics of the central particle is obtained by eliminating
the oscillators coordinates from the Euler-Lagrange equations. The resulting equation of
motion for the central particle $x$ is the generalized Langevin equation with homogeneous
noise, considered by Mori and Kubo in the context of nonequilibrium statistical mechanics
\[3\],
\[
M \ddot{x} = - \int_{t_0}^t ds K(t - s) \dot{x}(s) + R(t).
\]
(6)

The homogeneous noise $R(t)$ and the homogeneous memory kernel $K(\tau)$ are, respectively,
\[
R(t) = \sum_n m \omega_n^2 \left\{ (q_{n0} - x_0) \cos[\omega_n (t - t_0)] + \frac{\dot{q}_{n0}}{\omega_n} \sin[\omega_n (t - t_0)] \right\} ,
\]
(7)
\[
K(\tau) = \sum_n m \omega_n^2 \cos (\omega_n \tau).
\]
(8)
Note that the initial oscillator coordinates and velocities are just the Fourier coefficients of $R(t)$. By using initial conditions (5), it is straightforward to show that $R(t)$ is connected to the memory kernel $K(\tau)$ by the fluctuation-dissipation theorem:

$$\langle R(t)R(s) \rangle = k_BT K(t-s). \quad (9)$$

The form of the correlation function depends on the oscillator distribution. For instance, white noise corresponds to an oscillator distribution function $G(\omega) = 2M\gamma_0/\pi m\omega^2$. In this case the correlation function is

$$K(\tau) = \int_0^\infty d\omega G(\omega)m\omega^2 \cos(\omega\tau) = 2\gamma_0 \delta(\tau) \quad (10)$$

and eq.(6) and (9) reduce to eq.(1) and (3) respectively.

It should be remarked that the dissipation-fluctuation theorem was originally obtained on grounds of internal consistency of the Langevin equation [5], while in the oscillator model it is a natural consequence of the form of the Lagrangian (4) and of the initial conditions (5).

The model Lagrangian (4) is the only quadratic one which is both reflection and translation invariant. These invariance properties must be shared by any model Lagrangian describing a globally isolated system, in order for eq.(1) to be invariant.

In fact, the homogeneous oscillator model can be effectively visualized as a set of masses $m$ bound to the mass $M$ of the central particle by linear forces [3]. In particular, the coordinate $x$ of the central particle represents the equilibrium position of the oscillators. This explains the presence of $x_0$ in the initial data of the oscillators, eq.(5). Lagrangian (4) has been first proposed in ref. [3], and it has been later considered in different contexts ([7], [8], [9], [10]).

Models with an oscillator Lagrangian which is not translation and reflection invariant produce non physical infinite force terms and ill-defined quantities which depend on frequency cutoffs, both in the classical and in the quantum regime ([4], [11], [12], [13]).

Consider now the case in which an external potential $V(x,t)$ is present. The usual generalization of eq.(6) is

$$M\ddot{x} + \partial_x V(x,t) = -\int_{t_0}^t dsK(t-s)\dot{x}(s) + R(t), \quad (11)$$

obtained by simply adding the external force $-\partial_x V(x,t) \equiv \partial V(x,t)/\partial x$ in the generalized Langevin equation. It can be derived from the oscillator model by simply adding the potential term $V(x,t)$ in Lagrangian (4). In many physical situations this equation or its white noise limit provide a satisfactory description of classical Brownian dynamics.

However, from a general point of view some inconsistencies arise in this approach.

At a phenomenological level one can see, by inspection of eq.(11), that the addition of an external potential $V(x,t)$ has affected the mechanical part of eq.(6), on the left hand side, leaving unchanged the environment-induced force terms, on the right hand side. Instead, a corresponding influence should be expected to appear also on these terms, since
the environment surrounding the Brownian particle is made up of particles as well. Thus, eq. (11) cannot be in general a correct description of Brownian dynamics in the presence of external fields.

There are many physical situations in which the non homogeneous nature of the environment has observable consequences on the dynamics of the central degree of freedom. Some examples are provided by systems escaping an oscillating barrier [4], stochastic resonance or stochastic parametric oscillators [15].

The purpose of the present paper is to provide a reliable model for the description of classical and quantum Brownian motion in non homogeneous environments. This is achieved by introducing a suitable generalization of the classical homogeneous oscillator model.

A true mathematical modelling of the bath and the central particle as forming a closed system is really sound as long as $V(x, t) = 0$. In this case $L_0$ and $L_{osc}$, as well as the total Lagrangian, share the same invariance properties. If we allow for external potentials $V(x, t)$ acting on the central particle, we are in fact considering situations in which the whole system (bath + particle) is open and it is then natural to consider it coupled to the rest of the universe through external forces which act both on the particle and on the thermal oscillators. In this way one treats the central particle and the oscillators on equal dynamical footing.

This approach enables one to study those physical situations where strong enough couplings with external fields do not leave the environment unaffected.

The generalized Lagrangian is introduced in Sec.2. Besides adding an external potential $V(x, t)$, we let external forces $F_n(x, t)$ act upon each oscillator $q_n$ with coupling strength $c_n$, in order to describe the action of the external potential on the environment. This is shown to be equivalent to shifting the equilibrium positions of the oscillators from $x$, the position of the central particle, to the positions $Q_n(x, t) = c_n F_n(x, t)/m\omega_n^2$. It is also shown that the functions $Q_n$ represent a space-time modulation of noise and dissipation.

We treat the classical model first, and for it we derive the Euler-Lagrange equation of motion for the central particle. It turns out to be a modified generalized Langevin equation with a colored and multiplicative noise term $\tilde{R}(x, t)$ that can be expressed as the gradient of a stochastic potential $\tilde{V}(x, t)$. Further, a position-dependent memory kernel $\tilde{K}$ replaces the homogeneous kernel $K(\tau)$ in the dissipative force term. A new deterministic force is predicted besides the random and dissipative ones when time-modulation of the noise is present. A generalized position-dependent fluctuation-dissipation theorem is derived connecting the memory kernel $\tilde{K}$ and the noise $\tilde{R}$. The corresponding modification in the Fokker-Planck equation is also discussed.

Further theoretical and experimental effort is needed to verify the predictions of the model. To this end, we study in detail the different physically significant limits of the Brownian dynamics in the non homogeneous background. In particular it is shown that in the white noise limit the friction coefficient in eq.(1) acquires a space-time modulation: $\gamma_0 \rightarrow \gamma(x, t)$, and that a precise relation between $\gamma(x, t)$ and the stochastic potential $\tilde{V}(x, t)$ holds.

In Sec. 3 the model is quantized via Feynman path integration. The non homogeneous
influence functional and the master equation for the reduced density matrix of the central particle are obtained in closed form. We discuss the semiclassical limit and verify its consistency with the results obtained from the classical model. The expressions obtained are compared with those already derived in the literature in order to analyze the consequences for the issues of decoherence and localization.

2. The classical model.

2.a Langevin equation and fluctuation-dissipation theorem.

Our starting point is the total classical Lagrangian, eq.(4), of a central particle $x$ in interaction with an infinite set of harmonic oscillators $\{q_n\}$. We now consider that a non zero external potential $V(x, t)$ acting on the central particle is present. In this case the interpretation of $L_{osc}$ as the Lagrangian of the whole environment does not hold and there is no more reason to request its reflection and translation invariance. In order to take into account the influence of the external potential on the environment, we let the generic oscillator $q_n$ interact with the external field through a coupling function $F_n(x(t), t)$. The new total Lagrangian is then

$$L(x, \dot{x}, \{q_n, \dot{q}_n\}, t) = M\dot{x}^2 - V(x, t) + \sum_n \left( \frac{m}{2}(q_n^2 - \omega_n^2 q_n^2) + c_n q_n F_n(x, t) \right),$$

(12)

where the $c_n$'s are constants. This Lagrangian, already considered in [16], cannot lead to the correct generalized Langevin equation. The reason is that in the homogeneous limit, that is by choosing the simple linear particle-oscillator coupling $F_n = m\omega_n^2 x/c_n$, it does not reduce to Lagrangian (4), but to the original non invariant Feynman-Vernon Lagrangian, where the renormalization potential $\sum_n m\omega_n^2 x_n/2$ is missing.

In fact, it can be shown that the generalized Langevin equation obtained from the Lagrangian of eq.(12) contains a force term which is the gradient of the nonlinear renormalization potential $\Delta V = -\sum_n c_n F_n^2/2m\omega_n^2$.

In order to avoid the presence of this ill-defined quantity in the equations of motion we redefine the Lagrangian by adding the renormalization potential $-\Delta V$. It is convenient to introduce the quantities $Q_n(x(t), t) = c_n F_n(x(t), t)/m\omega_n^2$, with the physical dimensions of a coordinate. Then Lagrangian (12) becomes

$$L(x, \dot{x}, \{q_n, \dot{q}_n\}, t) = \frac{M}{2}\dot{x}^2 - V(x, t) + \sum_n \left( \frac{1}{2}mq_n^2 - \frac{1}{2}m\omega_n^2 [q_n - Q_n(x, t)]^2 \right).$$

(13)

It is to be noted that formally this Lagrangian and the homogeneous one, eq.(4), differ only in that the external potential $V(x, t)$ is present and the oscillator equilibrium positions are now represented by the functions $Q_n(x, t)$. 
Lagrangian (13) defines the most general oscillator model which is linear in the oscillator coordinates, thus preserving the exact solvability of the oscillator sector, and nonlinear in the central particle coordinates. This Lagrangian has been considered by Zwanzig \[7\] and by Lindenberg and Seshadri \[17\], in the classical regime for the case in which the $Q_n$’s do not depend on time.

The equation of motion for the central particle can be derived from Lagrangian (13) following the same procedure applied in the homogeneous case. Introducing the compact notations $x(t) = x$, $x(s) = y$, and $t - s = \tau$, one obtains the following generalized Langevin equation

$$M\ddot{x} + \partial_x V(x, t) = -\int_{t_0}^{t} ds\tilde{K}(x, y, \tau)\dot{y} - \int_{t_0}^{t} ds\Phi(x, y, \tau) + \tilde{R}(x, t). \quad (14)$$

The first and the third terms on the right hand side of the equation are the dissipation and fluctuation forces respectively. The second term in the r.h.s. is an additional force, neither random nor dissipative. Its physical origin is explained below.

The expressions for the fluctuating force $\tilde{R}(x, t)$, the memory kernel $\tilde{K}(x, y, \tau)$, and the deterministic kernel $\Phi(x, y, \tau)$ are

$$\tilde{R}(x, t) = \sum_n m\omega_n^2 \partial_x Q_n(x, t)\left\{(q_{n0} - Q_{n0}) \cos[\omega_n(t - t_0)] + \frac{\dot{q}_{n0}}{\omega_n} \sin[\omega_n(t - t_0)]\right\}, \quad (15)$$

$$\tilde{K}(x, y, \tau) = \sum_n \left\{\partial_x Q_n(x, t)\partial_y Q_n(y, s)\right\}m\omega_n^2 \cos(\omega_n\tau), \quad (16)$$

$$\Phi(x, y, \tau) = \sum_n \left\{\partial_x Q_n(x, t)\partial_s Q_n(y, s)\right\}m\omega_n^2 \cos(\omega_n\tau), \quad (17)$$

where $Q_{n0} = Q_n(x(t_0), t_0)$.

The oscillators $\{q_n\}$ of the environment are assumed to be in thermal equilibrium at time $t_0$ around their effective equilibrium positions $Q_n(x, t)$ according to a canonical distribution, in complete analogy with the homogeneous case:

$$\langle q_{i0} - Q_{i0} \rangle = 0,$$

$$\langle \dot{q}_{i0} \rangle = 0,$$

$$m\omega_i\omega_j\langle (q_{i0} - Q_{i0})(q_{j0} - Q_{j0}) \rangle = k_B T \delta_{ij}, \quad (18)$$

$$m\langle \dot{q}_{i0}\dot{q}_{j0} \rangle = k_B T \delta_{ij},$$

$$\langle \dot{q}_{i0}(q_{j0} - Q_{j0}) \rangle = 0.$$
Comparing the definition (15) of $\tilde{R}$ with the above statistical conditions on the oscillators, one can prove that the fluctuating force is a Gaussian stochastic process with zero mean
\[ \langle \tilde{R}(x, t) \rangle = 0, \] (19)
and covariance
\[ \langle \tilde{R}(x, t) \cdot \tilde{R}(y, s) \rangle = k_B T \tilde{K}(x, y, \tau). \] (20)

It is remarkable that a fluctuation-dissipation theorem holds even in this non homogeneous generalization of the oscillator model. Furthermore, no restriction has been assumed on the different time scales of the noise and of the external potential. From the definition (15) and the statistical properties (19)-(20), it follows that the fluctuating force $\tilde{R}$ is a colored multiplicative process, in general not factorizable in the product of a time-dependent and of a space-dependent part.

If the functions $Q_n$ have no explicit time dependence, then from eq.(17) it follows that the last force term in the Langevin equation is absent. On the other hand, if the functions $Q_n$ do explicitly depend on time but not on $x(t)$, then from eq.(16) it follows that the dissipative force is absent, and we are left with a purely time-dependent force, a case already considered by Feynman and Vernon [2]. Finally, the case of homogeneous noise is regained as $V = 0$ and $Q_n(x, t) \equiv x, \forall n$, and eqs.(13)-(16) and (18)-(20) go over into the corresponding eqs.(4),(10),(7),(8) and (5),(2),(9).

The Langevin equation obtained above is very general because, as we earlier remarked, the noise $\tilde{R}$ and the correlation function $\tilde{K}$ do not factorize. However, when all the functions $Q_n$ are equal to the same function $Q(x, t)$, the equation simplifies into:
\[ M\ddot{x} + \partial_x V(x, t) = \partial_x Q(x, t) \left\{ R(t) - \int_{t_0}^t K(\tau) \left( \partial_s Q(y, s) + \dot{y} \partial_y Q(y, s) \right) ds \right\}, \] (21)
where $R(t)$ and $K(\tau)$ are those of the homogeneous case, given by eqs.(7)-(8), and, as before, $y$ stands for $x(s)$. As a consequence, the generalized fluctuation-dissipation theorem (20) reduces to the homogeneous form (9).

Inspection of eq.(21) shows that the effect of the new oscillator equilibrium position amounts to a space-time modulation of the fluctuating force, which can be rewritten as
\[ \tilde{R}(x, t) = \partial_x \left( R(t)Q(x, t) \right) = -\partial_x \tilde{V}(x, t). \] (22)
We are thus naturally led to the interpretation of $-Q(x, t)$ as a deterministic modulation of the fluctuating potential $\tilde{V}$. It is now clear that the last force term on the right hand side of eq.(21) derives from the time modulation alone of the stochastic force. Namely, the model predicts that if there is a time modulation of the noise, then, besides the usual dissipative and random terms, a new (non dissipative) force appears.

If the functions $Q_n(x, t)$ are different from each other, then the random force $\tilde{R}(x, t)$ in eq.(14) can still be written as the gradient of a stochastic potential $\tilde{V}(x, t)$, given by the expression
\[ \tilde{V}(x, t) = -\sum_n m\omega_n^2 Q_n(x, t) \left\{ (q_{n0} - Q_{n0}) \cos[\omega_n(t - t_0)] + \frac{\dot{q}_{n0}}{\omega_n} \sin[\omega_n(t - t_0)] \right\}. \] (23)
In this case we preserve the same interpretations of the functions $Q_n$. The above relation can be seen as a particular Fourier expansion of the random potential, in which every Fourier component is modulated in space and time by the functions $Q_n(x,t)$.

In the white noise limit $K(\tau) \to 2M\gamma_0\delta(\tau)$ and eq.(21) reduces to

$$M\ddot{x} + \partial_x V(x,t) = -M\gamma(x,t)\dot{x} + R(t)\partial_x Q(x,t) - M\gamma_0\partial_x Q(x,t)\partial_t Q(x,t),$$

where we have defined the effective friction coefficient as

$$\gamma(x,t) = \gamma_0 \left( \partial_x Q(x,t) \right)^2.$$ \hfill (25)

We see that the nonlinear coupling to the linear bath modifies the white-noise Langevin equation by introducing a nonlinear dissipative force, a multiplicative noise term, and a deterministic force due to the time-variations of the coupling. The friction coefficient $\gamma_0$ is modulated by the square of the function that modulates the white noise $R(t)$, as expected from the fluctuation-dissipation theorem (20). For $Q_n(x,t) = Q(x,t)$, the following relation holds:

$$\tilde{R}(x,t) = -\partial_x \tilde{V}(x,t) = R(t)\partial_x Q(x,t) = \sqrt{\frac{\gamma(x,t)}{\gamma_0}}R(t).$$ \hfill (26)

It is instructive to analyze the overdamped regime as well. In this case however, one must be aware of the fact that the very existence of the overdamped limit rests on the hypothesis that the scale of the time-variations of $Q(x,t)$ is slower than the relaxation time $1/\gamma_0$.

Assuming that this constraint is satisfied, in the overdamped regime one has that the function modulating the fluctuation goes into its inverse:

$$\dot{x} = -\frac{1}{\partial_x Q} \left( \frac{\partial_x V}{M\gamma_0\partial_x Q} - \frac{R}{M\gamma_0} + \partial_t Q \right).$$ \hfill (27)

When $\partial_t Q = 0$, according to the above expression the particle moves under the influence of an effective force $-\partial_x V/\partial_x Q^2$. Compared to the homogeneous case, the modulation of the noise can strongly modify the intensity of the external force in the overdamped limit but cannot change its direction. However, when $\partial_t Q \neq 0$, also the sign of the effective force can be reversed if the scale of time-variations of $Q(x,t)$ is small enough.

Finally, if $Q$ does not explicitly depend on time, eq.(27) suggests a simple method to determine $\partial_x Q$ and so the nonlinear coupling $Q$ as well. From a measurement of the velocity $\dot{x}$ and the knowledge of the external potential $V(x,t)$, it is possible to infer the non homogenous nature of the environment and to determine $\partial_x Q$ given that, on average, $\partial_x Q^2 = -\partial_x V/M\gamma_0\dot{x}$.

2b. Fokker-Planck equation.

In the case of white noise the Langevin equation

$$M\ddot{x}(t) + \partial_x V(x,t) = -M\gamma_0\dot{x}(t) + R(t),$$ \hfill (28)
is equivalent to the Fokker-Planck equation
\[ \partial_t W = -pM^{-1}\partial_x W + \partial_p \left( W\partial_x V + \gamma_0 pW \right) + M\gamma_0\beta^{-1}\partial_p^2 W \] (29)
for the phase-space probability density \( W(x, p, t) \), where \( p \) is the momentum of the particle and \( \beta^{-1} = k_B T \).
In the overdamped regime the above equation reduces to the diffusion equation for the configurational probability density \( P(x, t) = \int dp W(x, p, t) \):
\[ \partial_t P = \left( M\gamma_0 \right)^{-1} \left[ \partial_x \left( P\partial_x V \right) + \beta^{-1}\partial_x^2 P \right]. \] (30)
Considering now the case of non homogeneous noise, the Fokker-Planck equation equivalent to the white noise Langevin equation (24) is
\[ \partial_t W = -pM^{-1}\partial_x W + \partial_p \left[ \left( \gamma_0 + M\gamma_0\partial_x Q\partial_t Q + \partial_x V \right) W \right] + M\gamma\beta^{-1}\partial_p^2 W. \] (31)
It is to be noted that since the same function \( \gamma(x, t) \) appears both in the drift and in the diffusion term, the equilibrium (stationary) solution of eq.(31), if \( Q \) and \( V \) do not depend on time and if such a solution exists, is just the Maxwell-Boltzmann distribution \( W_{eq}(x, p) \propto \exp \left[ -\beta \left( p^2/2 + V(x) \right) \right] \).
In the overdamped regime, one obtains the diffusion equation for the configurational probability density \( P(x, t) \):
\[ \partial_t P = \partial_x \left[ \frac{\partial_x V}{M\gamma} + \frac{\beta^{-1}}{M\gamma_0}\frac{\partial_t Q}{\partial_x Q} - \frac{\partial_x^2 Q}{(\partial_x Q)^3} \right] P + \frac{\beta^{-1}}{M\gamma_0} \partial_x \left[ \frac{P}{(\partial_x Q)^2} \right] \partial_x P. \] (32)
This equation can be used to study specific examples of nonlinear couplings with given external potentials. It is easy to see that the modifications induced by the non homogeneous terms may lead to interesting non perturbative effects. We shall present in a forthcoming paper a detailed treatment of the non homogeneous Brownian dynamics in bistable and in periodic potentials.

3. The quantum model

3.a Influence functional.

We shall now discuss the quantum version of the generalized oscillator model, by quantizing the Lagrangian (13) via Feynman path integration, and derive (see Sec.3.b) the relevant evolution equations for the reduced density matrix \( \rho \) of the central Brownian particle.

We begin by considering the action \( S[x, q] \) of a system formed by a central particle \( x \) and just one oscillator \( q \) of natural frequency \( \omega/2\pi \):
\[ S[x, q] = S_0[x] + S_{osc}[q, x] = \int_{t_a}^{t_b} dt \left\{ \frac{M}{2} \dot{x}^2 - V(x, t) \right\} + \int_{t_a}^{t_b} dt \frac{m}{2} \left\{ \dot{q}^2 - \omega^2(q - Q(x, t))^2 \right\}, \] (33)
where \( t_a \) and \( t_b \) are two arbitrary initial and final times, respectively. The functions \( S_0 \) and \( S_{osc} \) are respectively the actions of the isolated particle and of the oscillator, while \( Q(x, t) \) is an arbitrary space-time function.

For the sake of clarity, we shall assume here and in the following that the initial density matrix of the total system factorizes in the product of the density matrices of the two subsystems. In the coordinate representation, denoting by \( f_i \) and \( f_i' \) two different values of an observable quantity \( f(x, t) \) at time \( t_i \), the former statement reads

\[
\rho(x_a, x_a', q_a, q_a', t_a) = \rho(x_a, x_a', t_a) \cdot \rho_q(q_a - \bar{Q}_a, q_a' - \bar{Q}_a). \tag{34}
\]

Some remarks should be made about the form of this density matrix. Equation (34) describes the result of a measurement in which the state of the central particle has not been affected by the environment (\[6\], \[10\], \[18\]). The density matrix \( \rho(x_a, x_a', t_a) \) represents the initial state of the central particle, and \( \rho_q(q_a - \bar{Q}_a, q_a' - \bar{Q}_a) \) is the equilibrium density matrix of a harmonic oscillator at an inverse temperature \( \beta = 1/k_B T \) with its equilibrium position at \( q_a = Q_a \). It has the expression

\[
\rho_q(q_a - \bar{Q}_a, q_a' - \bar{Q}_a) = F(\beta) \exp \left\{ - \frac{m\omega}{2\hbar} \left[ (q_a - \bar{Q}_a)^2 + (q_a' - \bar{Q}_a)^2 \right] \coth(\beta\hbar\omega) - \frac{2(q_a - \bar{Q}_a)(q_a' - \bar{Q}_a)}{\sinh(\beta\hbar\omega)} \right\}, \tag{35}
\]

where \( F(\beta) = \sqrt{m\omega/\pi\hbar \coth(\beta\hbar\omega/2)} \) is a normalization factor such that \( \int dq \rho_q(q, q) = 1 \).

The oscillator density matrix only depends on \( q_a \) and \( q_a' \) through the differences \( q_a - Q_a \) and \( q_a' - \bar{Q}_a \), where \( \bar{Q}_a \), in analogy with the case of homogeneous dissipation \[18\], can be shown to be given by

\[
\bar{Q}_a = \frac{Q(x_a, t_a) + Q(x_a', t_a)}{2}. \tag{36}
\]

This expression accounts for the fact that, compared to the homogeneous case, in which \( \bar{Q}_a = (x_a + x_a')/2 \), the equilibrium position of the oscillator at time \( t \) has been shifted from \( x(t) \) to \( Q(x(t), t) \).

The density matrix evolves in time from \( t = t_a \) to \( t = t_b \) through the action of the total density matrix propagator \( J(b|a) \equiv J(x_b, x_b', q_b, q_b', t_b|x_a, x_a', q_a, q_a', t_a) \):

\[
\rho(x_b, x_b', q_b, q_b', t_b) = \int dx_a dx_a' dq_a dq_a' J(b|a) \rho(x_a, x_a', q_a, q_a', t_a). \tag{37}
\]

The total propagator \( J(b|a) \) is simply given by the product of the two propagators of the total wave function:

\[
J(b|a) = \int_{x_a}^{x_b} Dx \int_{x_a'}^{x_b'} Dx' \int_{q_a}^{q_b} Dq \int_{q_a'}^{q_b'} Dq' \exp \left\{ \frac{i}{\hbar} \left( S[x, q] - S[x', q'] \right) \right\}, \tag{38}
\]

where \( S[x, q] \) is given by eq. \((33)\). The reduced density matrix of the central particle at \( t = t_b \) is obtained by integrating the total density matrix in the oscillator coordinate,

\[
\rho(x_b, x_b', t_b) = \int dq_b \rho(x_b, x_b', q_b, q_b, t_b). \tag{39}
\]
Following the same procedure as in the homogeneous case (2, 11), the time evolution law of the reduced density matrix is

$$\rho(x_b, x'_b, t_b) = \int dx_a dx'_a J_{eff}(x_b, x'_b, t_b|x_a, x'_a, t_a) \rho(x_a, x'_a, t_a),$$

where the effective propagator $J_{eff}$ can be written as

$$J_{eff}(x_b, x'_b, t_b|x_a, x'_a, t_a) = \int_{x_a}^{x_b} Dx \int_{x'_a}^{x'_b} Dx' \exp \left( \frac{i}{\hbar} S[x, x'] \right).$$

Here we have introduced the effective action

$$S[x, x'] = S_0[x] - S_0[x'] + \Phi[x, x'],$$

where $\Phi[x, x']$ denotes the influence phase [2], and must not be confused with the deterministic kernel $\Phi(x, y, \tau)$ defined by eq. (17). Introducing, as before, the compact notation $x' = x'(t), x = x(t), y' = x'(s), y = x(s)$, and $t - s = \tau$, the expression for the influence phase reads

$$\Phi[x, x'] = \frac{m\omega^2}{2} \int_{t_a}^{t_b} dt \int_{t_a}^{t} ds \left[ Q(x', t) - Q(x, t) \right] \left\{ \partial_y Q(y, s) + \partial_y Q(y', s) \right\} \cos(\omega \tau)$$

$$+ \frac{m\omega^2}{2} \int_{t_a}^{t_b} dt \int_{t_a}^{t} ds \left[ Q(x', t) - Q(x, t) \right] \left\{ \partial_y Q(y, s) + \partial_y Q(y', s) \right\} \cos(\omega \tau)$$

$$+ i \frac{m\omega^3}{2} \coth(\beta \hbar \omega/2) \int_{t_a}^{t_b} dt \int_{t_a}^{t} ds \left[ Q(x', t) - Q(x, t) \right] \left\{ Q(y', s) - Q(y, s) \right\} \cos(\omega \tau).$$

Because of the dynamical and statistical independence of the oscillators, in the case of an environment with infinite degrees of freedom the influence phase can be calculated by simply summing up the influence phases of all the oscillators.

If all the $Q_n$'s are taken to be equal to a certain function $Q$, the influence phase assumes the form

$$\Phi[x, x'] = \frac{1}{2} \int_{t_a}^{t_b} dt \int_{t_a}^{t} ds \left[ Q(x', t) - Q(x, t) \right] \left\{ \partial_y Q(y, s) + \partial_y Q(y', s) \right\} K(\tau)$$

$$+ \frac{1}{2} \int_{t_a}^{t_b} dt \int_{t_a}^{t} ds \left[ Q(x', t) - Q(x, t) \right] \left\{ \partial_y Q(y, s) + \partial_y Q(y', s) \right\} K(\tau)$$

$$+ i \int_{t_a}^{t_b} dt \int_{t_a}^{t} ds \left[ Q(x', t) - Q(x, t) \right] \left\{ Q(y', s) - Q(y, s) \right\} \alpha(\tau),$$

where

$$\alpha(\tau) = \frac{1}{2} \sum_{n} m\omega_n^3 \coth(\beta \hbar \omega_n/2) \cos(\omega_n \tau),$$

and the function $K(\tau)$ is the same memory kernel defined in the classical homogeneous problem, eq. (8).
Given the expressions (41) and (44), the corresponding evolution law (37) for the reduced density matrix is completely determined and provides the general description of the dynamics of a quantum Brownian particle acted upon by a multiplicative stochastic nonhomogeneous potential $\tilde{V}(x, t) = -Q(x, t)R(t)$.

### 3.b Master equation and decoherence.

Starting from the functional integral formulation, we can also write the master equation for the reduced density matrix, which provides an alternative description of quantum Brownian motion.

A differential evolution equation for the density matrix $\rho(x, x', t)$ exists if the effective Lagrangian corresponding to the effective action (42) is local in time. This happens in the white noise limit, in which the influence phase reads

$$\Phi[x, x'] = M\gamma_0 \int_{t_a}^{t_b} dt \left( Q(x', t) - Q(x, t) \right) \left( \dot{Q}(x, t) + \dot{Q}(x', t) \right)$$

$$+ M\gamma_0 \int_{t_a}^{t_b} dt \left( Q(x', t) - Q(x, t) \right) \left( \dot{x}\partial_x Q(x, t) + \dot{x}'\partial_{x'} Q(x', t) \right)$$

$$+ i\frac{M\gamma_0}{\beta\hbar} \int_{t_a}^{t_b} dt \left( Q(x', t) - Q(x, t) \right)^2.$$  \hspace{1cm} (46)

The white noise master equation can now be derived following the same procedure applied by Caldeira and Leggett in the linear homogeneous case [11]. Starting from the effective action (42) with the nonlinear influence phase (46) we obtain

$$\frac{\partial \rho}{\partial t} = \frac{i\hbar}{2M} \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x'^2} \right) \rho - \frac{i}{\hbar} \left( V(x', t) - V(x, t) \right) \rho$$

$$- i\frac{M\gamma_0}{\beta\hbar} \left( Q(x', t) - Q(x, t) \right) \left( \frac{\partial Q(x, t)}{\partial t} + \frac{\partial Q(x', t)}{\partial t} \right) \rho$$

$$- \frac{\gamma_0}{2} \left( Q(x, t) - Q(x', t) \right) \left( \frac{\partial Q(x, t)}{\partial x} \frac{\partial}{\partial x} - \frac{\partial Q(x', t)}{\partial x'} \frac{\partial}{\partial x'} \right) \rho$$

$$- \frac{2M\gamma_0}{\beta\hbar^2} \left( Q(x, t) - Q(x', t) \right)^2 \rho.$$  \hspace{1cm} (47)

The first two terms on the right hand side of eq.(47) represent the mechanical evolution without bath-particle interaction. The fourth one describes dissipation and the last one the random fluctuations which destroy the quantum coherence. The third term is present only when the bath-particle coupling is time-dependent, and it can completely modify the usual decoherence patterns as we will clarify below.

The Caldeira-Leggett master equation in a homogeneous environment is recovered for $Q \equiv x$, that is for $\partial_x Q \equiv 1$. 

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In the classical limit eq.(47) reduces to the Fokker-Planck dynamics discussed in Sec.2b. This can be best understood by transforming the master equation (47) for the density matrix in the evolution equation for the associated Wigner function. The task is achieved by introducing the “relative coordinate” \( Y = x' - x \) and the “center of mass coordinate” \( X = (x + x')/2 \) and by Fourier transforming eq.(47) respect to the \( Y \) variable. In this way, one obtains the evolution equation in phase space for the Wigner function \( \hat{W}(x, p, t) \) defined as

\[
\hat{W}(x, p, t) = \int dY \rho \left( X - \frac{Y}{2}, X + \frac{Y}{2} \right) e^{ipY/\hbar}.
\] (48)

It is then easy to prove that, in the limit of small \( \hbar \), the evolution equation for the Wigner function \( \hat{W}(x, p, t) \) reduces to the Fokker-Planck equation (31) for the classical probability density \( W(x, p, t) \), where \( p \) is the momentum of the particle.

The quantum model provides very interesting information concerning the dynamics of decoherence and the transition from quantum to classical. We consider in particular the coefficient \( g(x, x', t) \) of \( \rho(x, x', t) \) in the last term of the master equation (47),

\[
g(x, x', t) = \frac{2M\gamma_0}{\beta\hbar^2} \left( Q(x, t) - Q(x', t) \right)^2,
\] (49)

which is always positive and represents the space-time dependent decay rate of the off-diagonal elements in the density matrix; it becomes very large in the semiclassical limit, since it is proportional to \( \hbar^{-2} \) \cite{19}, and it reduces to a quadratic function of the difference \( |x - x'| \) in the homogeneous case. In the non homogeneous case this term, as well as the dissipative one, loses its invariance properties under reflection and translation, because of its dependence on the noise modulating function \( Q(x, t) \).

When the function \( Q \) depends explicitly on time, eq.(47) describes a decoherence process with multiple nonequilibrium scales due to time-modulation of the noise. Explicitly, the different scales are represented by the decoherence rate \( g \), the damping rate \( \gamma \), the external potential \( V(x, t) \), and the modulating function \( Q(x, t) \).

Overall, these nonequilibrium effects are expressed by the third term on the r.h.s. of the master equation; formally, it is similar to the terms derived from a potential, that do not involve space-derivatives, with the crucial difference that it cannot be written as a difference of the type \( V(x', t) - V(x, t) \), and so its effect is not equivalent, as in the classical case, to adding a suitable external potential. When the physical situation is such that the bath-particle coupling is strongly time-dependent, possibly with very fast variations, this term yields a fastly-oscillating contribution in the solution and so can reduce or even inhibit the decoherence process.

Generalizations of eq.(47) may be considered in order to take into account other effects such as bath-particle interactions nonlinear in the oscillator coordinates or the finite size of the bath.

In the first case the decoherence rate \( g(x, x', t) \) does not depend quadratically on the difference \( Q(x, t) - Q(x', t) \) (or on the difference \( x - x' \) in the homogeneous case), because such a dependence is a direct consequence of the oscillator Lagrangian being quadratic. In
the second case quantum interference is suppressed only as long as the system is confined inside the thermal bath, which is assumed to possess a finite scale $\xi$, of the order of its linear dimensions.

The variable $Y = x - x'$ is conjugate to the variable $p$ (see the definition of the Wigner function) which in the semiclassical limit reduces to the momentum variable, and actually represents the quantum uncertainty on the position of the Brownian particle [20]. We then expect expression (48) to be replaced by a function $g(x, x', t)$ which becomes negligible for $|x - x'| \gg \xi$.

A finite value of (48) for $|x - x'| \gg \xi$, both in the homogeneous and in the nonhomogeneous models, is possible only if the bath is assumed to be infinitely extended. This point clarifies some questions recently raised [21] concerning the decoherence process.

In this paper we generalized the homogeneous model of classical and quantum dissipation in order to include general bath-particle interactions nonlinear in the particle coordinates. The linear-dissipation theory emerges as a particular case of the model. Moreover, we have shown that the nonlinear bath-particle coupling leads to the appearance of nonhomogeneous random and dissipative forces and of a further deterministic time-dependent force, in the Langevin equation at the classical level, and in the master equation for the quantum case. We illustrated how this yields new interesting consequences both in the classical and in the quantum regime.
References

[1] N.G. Van Kampen, *Stochastic Processes in Physics and Chemistry* (Elsevier, Amsterdam, 1981).

[2] R.P. Feynman and F.L. Vernon, Ann. Phys. (N.Y.) 24 (1963) 118; R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).

[3] G.W. Ford, M. Kac, and P. Mazur, J. Math. Phys. 6 (1965) 504.

[4] P. Ullersma, Physica 32 (1966) 27.

[5] R. Kubo, Progr. Theor. Phys. 29 (1965) 255; H. Mori, Progr. Theor. Phys. 33 (1965) 423; M. Toda, R. Kubo, and V. Saito, *Statistical Physics, Vol. II* (Springer Verlag, Berlin, 1983).

[6] H. Grabert *et al.*, Phys. Rep. 168 (1988) 115.

[7] R. Zwanzig, J. Stat. Phys. 9 (1973) 215.

[8] V. Hakim and V. Ambegaokar, Phys. Rev. A 32 (1985) 423.

[9] G.W. Ford and M. Kac, J. Stat. Phys. 46 (1987) 803.

[10] P. Schramm and H. Grabert, J. Stat. Phys. 49 (1987) 767.

[11] A.O. Caldeira and A.J. Leggett, Physica A 121 (1983) 115.

[12] W. Eckhardt, Physica A 141 (1987) 81.

[13] B.L. Hu, J.P. Paz, and Y. Zhang, Phys. Rev. D 45 (1992) 2843.

[14] C.R. Doering and J.C. Gadoua, Phys. Rev. Lett. 69 (1992) 318.

[15] R.L. Stratonovich, *Topics in the Theory of Random Noise, Vol. II* (Gordon and Breach, New York, 1981).

[16] K. Möhring and U. Smilansky, Nucl. Phys. A 338 (1980) 227; H. Metiu and G. Schön, Phys. Rev. Lett. 53 (1984) 13; A.O. Caldeira and A. J. Legget, Ann. Phys. (N.Y.) 149 (1983) 374.

[17] K. Lindenberg and V. Seshadri, Physica A 109 (1981) 483; K. Lindenberg and B.J. West, *The Nonequilibrium Statistical Mechanics of Open and Closed Systems* (VCH, New York, 1990).

[18] M. Patriarca, Ph.D. Thesis, University of Perugia, 1993.

[19] W.H. Zurek, Progr. Theor. Phys. 89 (1993) 281.
[20] A. Schmidt, J. Low Temp. 49 (1982) 609.

[21] M.R. Gallis and G.N. Fleming, Phys. Rev. A 42 (1990) 38; M.R. Gallis, Phys. Rev. A 45 (1992) 47; Phys. Rev. A 48 (1993) 1028.