THE LEAST COMMON MULTIPLE OF SETS OF POSITIVE INTEGERS

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Abstract. We prove that \( \log \text{lcm}\{a \in A\} = n \log 2 + o(n) \) for almost every set \( A \subset \{1, \ldots, n\} \). We also study the typical behavior of the logarithm of the least common multiple of sets of integers in \( \{1, \ldots, n\} \) with prescribed size. For example, we prove that, for any \( 0 < \theta < 1 \), \( \log \text{lcm}\{a \in A\} = (1 - \theta)n^{\theta} \log n + o(n^{\theta}) \) for almost all sets \( A \subset \{1, \ldots, n\} \) of size \( \lfloor n^{\theta} \rfloor \).

Extremal values of \( \log \text{lcm}\{a \in A\} \) for sets \( A \) of prescribed size are also studied.

1. Introduction

The function \( \psi(n) = \log \text{lcm}\{k : 1 \leq k \leq n\} \) was introduced by Chebyshev in his study of the distribution of the prime numbers. It is a well known fact that the asymptotic relation \( \psi(n) \sim n \) is equivalent to the Prime Number Theorem, which was finally proved by Hadamard and de la Vallée Poussin. Chebyshev’s function can be generalized to \( \psi_f(n) = \log \text{lcm}\{f(k) : 1 \leq f(k) \leq n\} \) for a given polynomial \( f(x) \in \mathbb{Z}[x] \) and it is natural to try to obtain the asymptotic behavior for \( \psi_f(n) \). Some progress has been made in this direction. In [1], the Prime Number Theorem for arithmetic progressions is exploited to get the asymptotic estimate when \( f(x) = a_1x + a_0 \) is a linear polynomial:

\[
\psi_f(n) \sim \frac{n}{a_1 \phi(q)} \sum_{\substack{1 \leq l \leq q \\ (l, q) = 1}} \frac{1}{l},
\]

where \( q = a_1/(a_1, a_0) \). The first author [3] has extended this result to quadratic polynomials. For a given irreducible quadratic polynomial \( f(x) = a_2x^2 + a_1x + a_0 \in \mathbb{Z}[x] \) with \( a_2 > 0 \) the following asymptotic estimate holds:

\[
\psi_f(n) = \frac{1}{2} \left( \frac{n}{a_2} \right)^{1/2} \log \left( \frac{n}{a_2} \right) + B \left( \frac{n}{a_2} \right)^{1/2} + o(n^{1/2}),
\]

where the constant \( B = B(f) \) depends only on \( f \). In the particular case of \( f(x) = x^2 + 1 \), we have \( \psi_f(n) = n^{1/2} \log n^{1/2} + Bn^{1/2} + o(n^{1/2}) \) with \( B = \gamma - 1 - \frac{\log 2}{\log n} - \sum_{p \neq 2} \left( \frac{-1}{p} \right) \log p \), where \( \gamma \) is the Euler constant, \( \left( \frac{-1}{p} \right) \) is the Legendre’s symbol and the sum is considered over all odd prime numbers. It has recently been proved in [4] that the error term in the previous expression is \( O\left(n^{1/2}(\log n)^{-4/9 + \epsilon}\right) \) for each \( \epsilon > 0 \). When \( f(x) \) is a reducible polynomial the behavior is, however, different. We have (see Theorem 3 in [3]):

\[
\psi_f(n) \sim Cn^{1/2}
\]
where }C\text{ is an explicit constant depending only on }f. \text{ For example, when } f(x) = x^2 - 1 \text{ we have } 
abla f(n) \sim n^{1/2}.

The asymptotic behavior of } \psi_f(n) \text{ remains unknown for irreducible polynomials of degree } d \geq 3, \text{ but it is conjectured in [5] that this is given by

(3) \quad \psi_f(n) \sim \left(1 - \frac{1}{d}\right) \left(\frac{n}{a_d}\right)^{1/d} \log\left(\frac{n}{a_d}\right),

where }a_d > 0 \text{ is the coefficient of } x^d. \text{ For example, this conjecture would imply } \psi_f(n) \sim \frac{2}{3} n^{1/3} \log n \text{ for } f(x) = x^3 + 2.

The above discussed results are particular cases of a more general problem, which we study in this work. \text{ For any set } A \text{ of positive integers } A \text{ let us set

}\psi(A) = \log \text{lcm}\{a : a \in A\}.

Let us denote } A_f = \{f(k) : k \in \mathbb{Z}\} \text{ and } A_f(n) = A_f \cap \{1, \ldots, n\}. \text{ We observe that } \psi_f(n) = \psi(A_f(n)). \text{ Consider the two quadratic polynomials } f(x) = x^2 + 1 \text{ and } f(x) = x^2 - 1. \text{ In both cases we have } |A_f(n)| = n^{1/2} + O(1), \text{ but } \psi(A_f(n)) \sim \frac{1}{2} n^{1/2} \log n \text{ in the first case and } \psi(A_f(n)) \sim n^{1/2} \text{ in the second one.}

It is natural to ask if either of these compare well with the behavior of } \psi(A) \text{ for a random subset of } \{1, \ldots, n\} \text{ of cardinality } n^{1/2} \text{ or if both of these examples represent rare events. The main purpose of the present paper is to study these questions.

Let us denote by } \binom{n}{k} \text{ the set of all subsets } A \subset \{1, \ldots, n\} \text{ of size } k. \text{ The mean value of } \psi \text{ in } \binom{n}{k} \text{ is the quantity

\bar{\psi}(n, k) = \frac{1}{\binom{n}{k}} \sum_{A \in \binom{n}{k}} \psi(A) = \frac{1}{\binom{n}{k}} \sum_{|A|=k} \psi(A).

It is natural to wonder whether for a given polynomial } f \text{ and } k = |A_f(n)| \text{ the asymptotic } \bar{\psi}(n, k) \sim \psi_f(n) \text{ holds when } n \to \infty. \text{ If this is indeed the case, then we may also ask if } \psi(A) \sim \bar{\psi}(n, k) \text{ for almost all sets } A \in \binom{n}{k}. \text{ Our main theorem answers these questions.

Theorem 1.1. Let } c > 0, 0 < \theta \leq 1 \text{ and } k = cn^\theta + O(1). \text{ We then have that

\bar{\psi}(n, k) = \begin{cases} c(1 - \theta)n^\theta \log n - c(\log c)n^\theta + o(n^\theta), & 0 < \theta < 1, \\ c \log(1/c/\epsilon)n + o(n), & \theta = 1, \end{cases}

\text{ when } n \to \infty. \text{ Furthermore, for any } \epsilon > 0 \text{ we have that

\frac{|\{A \in \binom{n}{k} : |\psi(A) - \bar{\psi}(n, k)| < \epsilon \bar{\psi}(n, k)\}|}{\binom{n}{k}} \to 1

\text{ when } n \to \infty.

In particular, for } 0 < \theta < 1, \text{ our main theorem implies that } \psi(A) \sim c(1 - \theta)n^\theta \log n \text{ for almost every set } A \subset \{1, \ldots, n\} \text{ of size } \lfloor cn^\theta \rfloor.
We observe that for \( f(x) = a_0 + a_1 x + \cdots + a_d x^d \) we have \(|A_f(n)| = (n/a_d)^{1/d} + O(1)\). Thus, if \( f \) is irreducible of degree \( d \geq 2 \) and we assume conjecture \( \mathbf{3} \) (known to be true for \( d = 2 \)), then Theorem 1.1 implies that \( \psi_f(n) \sim \overline{\psi}(n,k) \) when \( k = |A_f(n)| = cn^\theta + O(1) \) with \( \theta = 1/d \) and \( c = a_d^{-1/d} \). However, although the asymptotic behavior of \( A_f(n) \) is like that of a typical set \( \mathcal{A} \) of the same size, there are some differences in the second term. For example, if \( f(x) = x^2 + 1 \), we have

\[
\psi_f(n) = \frac{1}{2}n^{1/2} \log n + Bn^{1/2} + o(n^{1/2}),
\]

for \( k = |A_f(n)| = \lfloor \sqrt{n-1} \rfloor \). The constant above is \( B = -0.06627563... \)

For \( d = 1 \), namely \( f(x) = a_1 x + a_0 \), \( (a_1, a_0) = 1 \), the situation is different. In this case, for \( k = |A_f(n)| \) it is easily seen that \( \overline{\psi}(n,k) \) and \( \psi_f(n) \) have different asymptotic behavior when \( a_1 > 1 \). More precisely, we have that

\[
\overline{\psi}(n,k) \sim \frac{\log a_1}{a_1 - 1} n, \quad \psi_f(n) \sim \frac{n}{\varphi(a_1)} \sum_{1 \leq l < \varphi(a_1)} \frac{1}{l}.
\]

While we have dealt above with sets of prescribed size, it is also a natural question to ask for the mean value of \( \psi(\mathcal{A}) \) over all subsets \( \mathcal{A} \subset \{1, \ldots, n\} \) (define \( \psi(\emptyset) = 0 \)). In other words, we may ask for the asymptotic behavior of

\[
\overline{\psi}_n = \frac{1}{2n} \sum_{\mathcal{A} \subset \{1, \ldots, n\}} \psi(\mathcal{A}).
\]

We answer this question by considering a more general problem. For a given \( \delta \), \( 0 < \delta < 1 \), select the elements of \( \mathcal{A} \) as outputs of the independent events \( a \in \mathcal{A}, 1 \leq a \leq n \), with the same probability \( P(a \in \mathcal{A}) = \delta \). We denote in the sequel this probability space by \( \mathcal{S}(n;\delta) \). That is, \( \mathcal{S}(n;\delta) \) is the set of subsets of \( \{1, \ldots, n\} \) equipped with the probability measure given by \( P(X) = \delta^{|X|}(1-\delta)^{n-|X|} \) for any subset \( X \) of \( \{1, \ldots, n\} \). Observe that \( \overline{\psi}_n \) is just the expected value of \( \psi(\mathcal{A}) \) in the probability space \( \mathcal{S}(n;1/2) \). In Proposition 2.1 we obtain an explicit expression for \( \mathbb{E}(\psi(\mathcal{A})) \) in the probability space \( \mathcal{S}(n;\delta) \). This proposition immediately implies the following theorem which proves, in particular, that

\[
\overline{\psi}_n = n \log 2 + o(n).
\]

**Theorem 1.2.** Let \( c > 0 \) and \( 0 < \theta \leq 1 \). In the probability space \( \mathcal{S}(n;cn^{\theta-1}) \) the expected value of \( \psi(\mathcal{A}) = \log \text{lcm}\{a : a \in \mathcal{A}\} \) satisfies

\[
\mathbb{E}(\psi(\mathcal{A})) = \begin{cases} 
  c(1-\theta)n^\theta \log n - c(\log c)n^\theta + o(n^\theta), & 0 < \theta < 1, \\
  c\log (1/c) n + o(n), & \theta = 1.
\end{cases}
\]

Furthermore, the variance satisfies \( V(\psi(\mathcal{A})) \ll n^\theta \log^2 n \).
Clearly, $E(|A|) = cn^\theta$ and even more, $|A| \sim cn^\theta$ for almost every set $A$, so $S(n;cn^{\theta-1})$ is the appropriate probability space to simulate sets $A$ of this size. We observe that the estimate for $E(\psi(A))$ in Theorem 1.2 is similar to that obtained in Theorem 1.1 for $\psi(n,k)$ when $k = cn^\theta + O(1)$.

Note that the upper bound for the variance implies concentration. Indeed, for any $\epsilon > 0$ we have that $P(|\psi(A)| - E(\psi(A))| < cE(\psi(A)) \rightarrow 1$ as $n \rightarrow \infty$. In other words, for almost all sets $A$ in the probability space $S(n;cn^{\theta-1})$ and $0 < \theta < 1$ (respectively $\theta = 1$) the asymptotic estimates $|A| \sim cn^\theta$ and $\psi(A) \sim c(1-\theta)n^\theta \log n$ (respectively $\psi(A) \sim \frac{\epsilon \log(1/\epsilon)}{1-\epsilon} n$) hold.

Our reason for considering $S(n;\delta)$ is not only because it is an interesting and natural probability space but also because it is close to the probability space considered in Theorem 1.1, where all the sets of size $\delta n$ are chosen with the same probability. It appears to be difficult to prove Theorem 1.1 directly. For this reason, our strategy will be to prove Theorem 1.2 first and then deduce Theorem 1.1 from it.

The above discussion tells about what to expect from $\psi(A)$ in this setting, at least in most cases. As we have seen, sets $A_f$ are exceptional cases and their difference from the expected value depends on the irreducibility of $f$. What we still do not know is how far from $E(\psi(A))$ the exceptional cases are. We finish our work studying the extremal behavior of $\psi(A)$ for sets of prescribed size.

**Theorem 1.3.** Let $c > 0$ and $0 < \theta < 1$. We then have that

1) $\max_{A \subset \{1, \ldots, n\}}\psi(A) \geq cn^\theta(\log n)(1 + o(1))$.

2) $\min_{A \subset \{1, \ldots, n\}}\psi(A) \leq (\log n)^{2+\frac{1}{1-\theta}} + o(1)$.

2. Chebyshev’s function for random sets in $S(n;\delta)$ and Proof of Theorem 1.2

When $A$ is a set of positive integers, we consider the quantity $\psi(A) = \log \lcm\{a : a \in A\}$. The following lemma provides us with an explicit expression for $\psi(A)$.

**Lemma 2.1.** For any set of positive integers $A$ we have $\psi(A) = \sum_m \Lambda(m) I_A(m)$, where $\Lambda$ denotes the classical Von Mangoldt function and

$$I_A(m) = \begin{cases} 1 & \text{if } A \cap \{m, 2m, 3m, \ldots\} \neq \emptyset, \\ 0 & \text{otherwise}. \end{cases}$$

**Proof.** Observe that $p^k \mid \lcm\{a : a \in A\} \iff A \cap \{p^s, 2p^s, 3p^s, \ldots\} \neq \emptyset$ for $s = 1, \ldots, k$. \qed

Note that if $A = \{1, \ldots, n\}$ then $\psi(A) = \sum_{m \leq n} \Lambda(m)$ is the classical Chebychev’s function $\psi(n)$. 


We recall that \( S(n; \delta) \) is the probability space with sets \( A \subset \{1, \ldots, n\} \) such that all its events \( \{a \in A\} \) are independent and \( P(a \in A) = \delta \) for any \( 1 \leq a \leq n \).

### 2.1. Expectation

First of all we give an explicit expression for the expected value of \( \psi(A) \) in \( S(n; \delta) \).

**Proposition 2.1.** Let \( \delta \in (0, 1) \) be fixed. Then in \( S(n; \delta) \) we have

\[
E(\psi(A)) = n \frac{\delta \log(\delta^{-1})}{1 - \delta} \left(1 + \frac{1}{\log(\delta^{-1})} \sum_{r \geq 1} \varepsilon \left(\frac{n}{r}\right) \left(1 - \frac{\delta}{r}\right)^r\right),
\]

where \( \varepsilon(x) = \frac{\psi(x)}{x} - 1 \), denotes the error term in the Prime Number Theorem.

**Proof.** By linearity, Lemma 2.1 clearly implies \( E(\psi(A)) = \sum_{m \leq n} \Lambda(m) E(I_{\mathcal{A}}(m)) \). On the other hand,

\[
E(I_{\mathcal{A}}(m)) = P(A \cap \{m, 2m, \ldots\} \neq \emptyset) = 1 - \prod_{r \leq n/m} P(rm \notin \mathcal{A}) = 1 - (1 - \delta)^{\lfloor n/m \rfloor}.
\]

Thus,

\[
E(\psi(A)) = \sum_{m \leq n} \Lambda(m) \left(1 - (1 - \delta)^{\lfloor n/m \rfloor}\right).
\]

We observe that \( \lfloor n/m \rfloor = r \) whenever \( \frac{n}{r+1} < m \leq \frac{n}{r} \), so we split the sum into intervals \( J_r = \left(\frac{n}{r+1}, \frac{n}{r}\right] \), obtaining

\[
E(\psi(A)) = \sum_{r \geq 1} \left(1 - (1 - \delta)^r\right) \sum_{m \in J_r} \Lambda(m)
\]

\[
= \sum_{r \geq 1} \left(1 - (1 - \delta)^r\right) \left(\psi\left(\frac{n}{r}\right) - \psi\left(\frac{n}{r+1}\right)\right) 
\]

\[
= \delta \sum_{r \geq 1} \psi\left(\frac{n}{r}\right) (1 - \delta)^{r-1}
\]

\[
= \delta n \sum_{r \geq 1} \frac{(1 - \delta)^{r-1}}{r} + \delta n \sum_{r \geq 1} \varepsilon \left(\frac{n}{r}\right) \frac{(1 - \delta)^{r-1}}{r}
\]

\[
= \frac{\delta}{1 - \delta} \left(n \log(\delta^{-1}) + n \sum_{r \geq 1} \varepsilon \left(\frac{n}{r}\right) \frac{(1 - \delta)^r}{r}\right).
\]

\[ \square \]

**Corollary 2.1.** If \( \delta n \to \infty \) then

\[
E(\psi(A)) = n \frac{\delta \log(\delta^{-1})}{1 - \delta} \left(1 + O\left(e^{-C\sqrt{\log(n}\delta}}\right)\right),
\]

in \( S(n; \delta) \) when \( n \to \infty \).
Proof. We get bounds for the error term, taking into account the error term in the Prime Number Theorem. It is known that 
\[ \varepsilon(x) = O(e^{-C_1 \sqrt{\log x}}) \text{ for some } C_1 > 0. \]
For small values of \( x \) we use \[ \varepsilon(x) = O(1). \]
For \( T = \sqrt{\log(\delta n)} \) we have
\[
\sum_r \varepsilon \left( \frac{n}{r} \right) \frac{(1 - \delta)^r}{r} \leq \sum_{r \leq T} \varepsilon \left( \frac{n}{r} \right) \frac{(1 - \delta)^r}{r} + \sum_{r > T} \varepsilon \left( \frac{n}{r} \right) \frac{(1 - \delta)^r}{r}
\]
\[
\ll \left( \max \frac{\varepsilon(x)}{n/T} \right) \sum_{r \leq T} \frac{(1 - \delta)^r}{r} + \sum_{r > T} \frac{(1 - \delta)^r}{r}
\]
\[
\ll e^{-C_1 \sqrt{\log(\delta)}} \log(\delta^{-1}) + \frac{1}{T} (1 - \delta)^T \delta
\]
\[
\ll \log(\delta^{-1}) \left( e^{-C_1 \sqrt{\log(\delta)}} + \frac{1 - \delta}{\log(\delta^{-1})} \left( (1 - \delta)^{1/\delta} \delta^{(T-1)} \right) \right)
\]
Taking into account inequalities \( (1 - \delta)^{1/\delta} < e^{-1} \) and \( (1 - \delta)/\log(\delta^{-1}) \leq 1 \) for \( 0 < \delta < 1 \), and the fact that \( \frac{\delta n}{\log(\delta n)} > \sqrt{\delta n} \) if \( \delta n \) is large enough, we get
\[
\frac{1}{\log(\delta^{-1})} \sum_k \varepsilon \left( \frac{n}{k} \right) \frac{(1 - \delta)^k}{k} \ll e^{-C_1 \sqrt{\log(\delta n)}} + e^{-\sqrt{\log(\delta n)}} \ll e^{-C_2 \sqrt{\log(\delta n)}}
\]
for some \( C_2 > 0. \)

Theorem 1.2 follows from Corollary 2.1 on observing that
\[
\frac{\log(\delta^{-1})}{1 - \delta} = ((1 - \theta) \log n - \log c)(1 + O(n^{\theta - 1}))
\]
when \( 0 < \theta < 1. \)

2.2. Variance. The second part of Theorem 1.2 is an immediate consequence of the following proposition.

Proposition 2.2. In \( S(n; \delta) \) we have \( V(\psi(A)) \ll \delta n \log^2 n. \)

Proof. By linearity of expectation we have that
\[
V(\psi(A)) = \mathbb{E} \left( \psi^2(A) \right) - (\mathbb{E} \left( \psi(A) \right))^2
\]
\[
= \sum_{m, l \leq n} \Lambda(m) \Lambda(l) \left( \mathbb{E} (I_A(m) I_A(l)) - \mathbb{E} (I_A(m)) \mathbb{E} (I_A(l)) \right).
\]
We observe that if \( \Lambda(m) \Lambda(l) \neq 0 \) then \( l \mid m, \ m \mid l \) or \( (m, l) = 1 \). Let us now study the term \( \mathbb{E}(I_A(m) I_A(l)) \) in these cases.
(i) If \( l \mid m \) then
\[
\mathbb{E}(I_A(m)I_A(l)) = 1 - (1 - \delta)^{[n/m]}.
\]
(ii) If \( (l, m) = 1 \) then
\[
\mathbb{E}(I_A(m)I_A(l)) = 1 - (1 - \delta)^{[n/m]} - (1 - \delta)^{[n/l]} + (1 - \delta)^{[n/m] + [n/l] - [n/(ml)]}.
\]
Both of these relations are subsumed in
\[
\mathbb{E}(I_A(m)I_A(l)) = 1 - (1 - \delta)^{[n/m]} - (1 - \delta)^{[n/l]} + (1 - \delta)^{[n/m] + [n/l] - [n/(ml)]}.
\]
Thus, on using (4), we obtain
\[
\Lambda(m)\Lambda(l)\left(\mathbb{E}(I_A(m)I_A(l)) - \mathbb{E}(I_A(m))\mathbb{E}(I_A(l))\right)
= \Lambda(m)\Lambda(l)(1 - \delta)^{[n/m] + [n/l] - [n/(ml)]} \left(1 - (1 - \delta)^{[n/(ml)]}\right).
\]
Finally on using the inequality \( 1 - (1 - x)r \leq rx \) we have
\[
V(\psi(A)) \leq 2\delta n \sum_{1 \leq l \leq m \leq n} \frac{\Lambda(l)\Lambda(m)}{m} (1 - \delta)^{[n/m] + [n/l] - [n/(ml)]}
\]
\[
\leq 2\delta n \sum_{p \leq n} \sum_{1 \leq j \leq k} \frac{\log p \log p}{p^j} (1 - \delta)^{[n/p^j] - [n/p^k]}
\]
\[
\leq 2\delta n \sum_{p \leq n} \sum_{1 \leq j \leq k} \frac{k \log^2 p}{p^k} (1 - \delta)^{[n/p^j]}
\]
\[
\leq 2\delta n \sum_{p \leq n} \frac{k \log^2 p}{p^k}
\]
\[
\ll \delta n \log^2 n,
\]
as we wanted to prove. \( \square \)

3. Proof of Theorem 1.1

Let us set \( \overline{\psi}(n, k) = \frac{1}{\binom{n}{k}} \sum_{|A|=k} \psi(A) \) and \( \overline{\psi^2}(n, k) = \frac{1}{\binom{n}{k}} \sum_{|A|=k} \psi^2(A) \).

**Lemma 3.1.** If \( j \leq k \), then for \( s = 1, 2 \) we have that
\[
\overline{\psi^s}(n, j) \leq \overline{\psi^s}(n, k) \leq \overline{\psi^s}(n, j) + (k^s - j^s) \log^s n.
\]

**Proof.** Suppose \( j < k \). There are \( \binom{n-j}{k-j} \) ways to add \( k - j \) new elements to a set \( A \in \binom{[n]}{k} \) in order to obtain a set of \( \binom{[n]}{k} \). Thus, for \( s = 1, 2 \) we have
\[
\psi^s(A) \leq \binom{n-j}{k-j}^{-1} \sum_{|A\cap A'|=\emptyset, |A'|=k-j} \psi^s(A \cup A'),
\]
and then

\[
\sum_{|A| = j} \psi^s(A) \leq \left( \frac{n - j}{k - j} \right)^{-1} \sum_{|A'| = k, |A'| = k \setminus j} \psi^s(A \cup A')
\]

\[
= \left( \frac{n - j}{k - j} \right)^{-1} \sum_{|A'| = k} \sum_{|A \cup A'| = A''} \psi^s(A'')
\]

\[
= \left( \frac{n - j}{k - j} \right)^{-1} \sum_{|A'| = k} \psi^s(A'') \sum_{|A \cup A'| = A''} 1
\]

\[
= \left( \frac{n}{k} \right) \sum_{|A'| = k} \psi^s(A'),
\]

which proves the first inequality.

For the second inequality we observe that for any set \( A \in \binom{[n]}{k} \) and any partition in two sets \( A = A' \cup A'' \) with \(|A'| = j, |A''| = k - j|\) we have that \( \psi(A) \leq \psi(A') + \psi(A'') \leq \psi(A') + (k - j) \log n \). Similarly,

\[
\psi^2(A) \leq \psi^2(A') + (k - j) \log n)^2
\]

\[
= \psi^2(A') + 2\psi(A')(k - j) \log n + (k - j)^2 \log^2 n
\]

\[
\leq \psi^2(A') + 2j(k - j) \log^2 n + (k - j)^2 \log^2 n
\]

\[
= \psi^2(A') + (k^2 - j^2) \log^2 n.
\]

Thus, for \( s = 1, 2 \) we have

\[
\psi^s(A) \leq \left( \frac{k}{j} \right)^{-1} \sum_{A' \subseteq A, |A'| = j} \left( \psi^s(A') + (k^s - j^s) \log^s n \right) \leq \left( \frac{k}{j} \right)^{-1} \sum_{A' \subseteq A, |A'| = j} \psi^s(A') + (k^s - j^s) \log^s n.
\]

Then,

\[
\sum_{|A| = k} \psi^s(A) \leq \left( \frac{k}{j} \right)^{-1} \sum_{|A'| = j} \sum_{A' \subseteq A, |A'| = k} \psi^s(A') + \left( \frac{n}{k} \right)(k^s - j^s) \log^s n
\]

\[
= \left( \frac{k}{j} \right)^{-1} \sum_{|A'| = j} \psi^s(A') \sum_{A' \subseteq A, |A'| = k} 1 + \left( \frac{n}{k} \right)(k^s - j^s) \log^s n
\]

\[
= \left( \frac{k}{j} \right)^{-1} \left( n - j \right) \left( \frac{n - j}{k - j} \right) \sum_{|A'| = j} \psi^s(A') + \left( \frac{n}{k} \right)(k^s - j^s) \log^s n
\]

\[
= \left( \frac{n}{k} \right) \sum_{|A'| = j} \psi^s(A') + \left( \frac{n}{k} \right)(k^s - j^s) \log^s n,
\]

and the second inequality holds. \( \square \)
Since $|A|$ is the sum of $n$ Boolean independent variables with expectation $E(|A|) = k$ we can use Chernoff’s inequality to get that, for each positive real value $r$,

\[ P(|A| - k \geq r) \leq 2e^{-\frac{r^2}{2}}. \]

We use this inequality in the proof of the next proposition.

**Proposition 3.1.** For $s = 1, 2$ we have that 

\[ \psi^s(n, k) = E(\psi^s(A)) + O(k^{s-1/2} \log^{s+1/2} n) \]

where $E(\psi(A))$ is the expectation of $\psi(A)$ in $S(n; k/n)$.

**Proof.** Observe that

\[
E(\psi^s(A)) - \psi^s(n, k) = -\psi^s(n, k) + \sum_{j=0}^{n} \left( \frac{k}{n} \right)^j \left( 1 - \frac{k}{n} \right)^{n-j} \sum_{|A|=j} \psi^s(A)
\]

\[
= -\psi^s(n, k) + \sum_{j=0}^{n} \left( \frac{k}{n} \right)^j \left( 1 - \frac{k}{n} \right)^{n-j} \left( \frac{n}{j} \psi^s(n, j) \right)
\]

\[
= \sum_{j=0}^{n} \left( \frac{k}{n} \right)^j \left( 1 - \frac{k}{n} \right)^{n-j} \left( \frac{n}{j} \right) \left( \psi^s(n, j) - \psi^s(n, k) \right),
\]

for $s = 1, 2$. Whenever $j$ and $k$ are close to each other, say $|j - k| < r$ for a given $1 \leq r \leq k$,

then from the previous lemma we have

\[
|\psi^s(n, j) - \psi^s(n, k)| \leq \begin{cases} 
  r \log n, & s = 1, \\
  3kr \log^2 n, & s = 2,
\end{cases}
\]

which implies

\[ \sum_{|j-k|<r} \left( \frac{k}{n} \right)^j \left( 1 - \frac{k}{n} \right)^{n-j} \left( \frac{n}{j} \right) (\psi^s(n, j) - \psi^s(n, k)) \ll k^{s-1} r \log^s n. \]

On the other hand, if $|j - k| \geq r$, we use the trivial estimate

\[
|\psi^s(n, j) - \psi^s(n, k)| \leq (k^s + j^s) \log^s n,
\]

which gives

\[ \sum_{|j-k|\geq r} \left( \frac{k}{n} \right)^j \left( 1 - \frac{k}{n} \right)^{n-j} \left( \frac{n}{j} \right) (\psi^s(n, j) - \psi^s(n, k)) \]

\[ \leq \log^s n \sum_{|j-k|\geq r} \left( \frac{k}{n} \right)^j \left( 1 - \frac{k}{n} \right)^{n-j} \left( \frac{n}{j} \right) (k^s + j^s). \]

We bound the summands separately:

\[ k^s \log^s n \sum_{|j-k|\geq r} \left( \frac{k}{n} \right)^j \left( 1 - \frac{k}{n} \right)^{n-j} \left( \frac{n}{j} \right) \leq k^s \log^s n P(|A| - k \geq r), \]
and
\[
\log^s n \sum_{j \geq k+r} (k/n)^j (1 - k/n)^{n-j} \binom{n}{j} j^s
\]
\[= \log^s n \sum_{j \leq k-r} (k/n)^j (1 - k/n)^{n-j} \binom{n}{j} j^s + \log^s n \sum_{j \geq k+r} (k/n)^j (1 - k/n)^{n-j} \binom{n}{j} j^s\]
\[\leq k^s \log^s n \mathbb{P}(|A| - k \geq r) + \log^s n \sum_{j \geq k+r} (k/n)^j (1 - k/n)^{n-j} \binom{n}{j} j^s.\]

We rewrite the last sum in (9) as (recall that \( j \leq n \))
\[
\sum_{j \geq k+r} (k/n)^j (1 - k/n)^{n-j} \binom{n}{j} j^s
\]
\[= k^s \sum_{j \geq k+r} \frac{(k/n)^j (1 - k/n)^{n-j-s} \binom{n}{j}}{n-k} \frac{j^s}{n-k} \frac{(n-j)}{(n-j-s)},\]

and observe that if \( r \geq 2 \) then \( j \geq k + 2 \), so
\[\frac{j^s}{(n-k)^s} \binom{n-j}{j} \leq \begin{cases} \frac{n-j+1}{n-j+1} \frac{(n-j+1)}{(n-k)^s} \frac{j}{j-1} & \text{if } s = 1 \\ \frac{n-j+1}{n-j+1} \frac{(n-j+1)}{(n-k)^s} \frac{j}{j-1} & \text{if } s = 2 \end{cases} \leq 2,
\]

thus
\[\sum_{j \geq k+r} \frac{(k/n)^j (1 - k/n)^{n-j} \binom{n}{j}}{n-k} \frac{j^s}{n-k} \frac{(n-j)}{(n-j-s)} \leq 2k^s \mathbb{P}(|A| - k \geq r - s).\]

Finally, we combine the estimates (5), (6), (7), (8) and (10) to get
\[|\mathbb{E}(\psi^s(A)) - \overline{\psi}^s(n, k)| \leq k^s \log^s n \left( \frac{r}{k} + \mathbb{P}(|A| - k \geq r - s) \right),\]
\[\leq k^s \log^s n \left( \frac{r}{k} + e^{-\frac{(r-k)^2}{4k^2}} \right).\]

By taking \( r \sim 10\sqrt{k \log k} \) the result holds.

To conclude the proof of Theorem 1.1 we combine Proposition 2.2 and Proposition 3.1 with \( k \sim cn^\theta \) to get
\[
\frac{1}{k} \sum_{A \in \binom{[n]}{k}} (\psi(A) - \overline{\psi}(n, k))^2 = \overline{\psi}^2(n, k) - \overline{\psi}^2(n, k)
\]
\[= V(\psi(A)) + \left( \overline{\psi}^2(n, k) - \mathbb{E}(\psi^2(A)) \right) + \left( \mathbb{E}(\psi(A)) - \overline{\psi}(n, k) \right) (\mathbb{E}(\psi(A)) + \overline{\psi}(n, k))\]
\[\ll n^\theta \log^2 n + n^{3\theta/2} \log^{5/2} n + \left( \binom{n}{k}^2 \log^{3/2} n \right) \left( n^\theta \log n \right)\]
\[\ll n^{3\theta/2} \log^{5/2} n.
\]
Then Chebyshev’s inequality implies that
\[
\left| \{ A \in \binom{[n]}{k} : |\psi(A) - \psi(n,k)| \geq \epsilon \psi(n,k) \} \right| \leq \frac{\sqrt{\psi^2(n,k) - \psi^2(n,k)}}{\epsilon \psi(n,k)} \leq \frac{n^{3/4} \log^{5/4} n}{\epsilon n^{\theta/4}} = \frac{\log^{1/4} n}{\epsilon n^{\theta/4}} \to 0,
\]
which concludes the proof of Theorem 1.1.

4. Extremal values

The first part of Theorem 1.3 follows from the next proposition on taking \( \delta = cn^{\theta-1} \).

**Proposition 4.1.** Let \( \delta = \delta(n) \) be a function such that \( 0 < \delta < 1 \) and \( \lim_{n \to \infty} \delta n = \infty \). Then
\[
\max_{A \subseteq \{ 1, \ldots, n \}} \psi(A) \sim n \min(1, \delta \log n).
\]

**Proof.** Let \( A \subseteq \{ 1, \ldots, n \} \) and \( |A| = \lfloor \delta n \rfloor \). It is clear that
\[
\psi(A) \leq \sum_{a \in A} \log a \leq |A| \log n \leq \delta n \log n.
\]
On the other hand, it is clear that we always have \( \psi(A) \leq \psi(n) \sim n \). Thus
\[
\psi(A) \lesssim n \min(1, \delta \log n).
\]
For the lower bound we distinguish two separate cases:

- If \( \lfloor \delta n \rfloor \geq (1 - \frac{1}{\log n}) \pi(n) \), then we consider any set \( A \) of \( \lfloor \delta n \rfloor \) elements containing the largest \( (1 - \frac{1}{\log n}) \pi(n) \) primes in \( \{ 1, \ldots, n \} \) and get
\[
\psi(A) \geq \sum_{p \in A} \log p \geq \left( 1 - \frac{1}{\log n} \right) \sum_{p \leq n} \log p \geq n(1 + o(1)).
\]

- If \( \lfloor \delta n \rfloor < (1 - \frac{1}{\log n}) \pi(n) \), then we consider a set \( A \) of \( \lfloor \delta n \rfloor \) largest primes in \( \{ 1, \ldots, n \} \) and get (denoting the \( i \)-th prime by \( p_i \))
\[
\psi(A) = \sum_{p \in A} \log p \geq \lfloor \delta n \rfloor \log p_{\pi(n) - \lfloor \delta n \rfloor} \geq \lfloor \delta n \rfloor \log p_{\pi(n)/\log n} \geq \delta n \log n(1 + o(1)).
\]
\[ \square \]

To prove part ii) of Theorem 1.3 we need some notation and results concerning smooth numbers. A number \( n \) is a \( y \)-**smooth number** if all the prime factors of \( n \) are \( \leq y \). It is usual to denote by
\[
\Psi(x; y) = | \{ n \leq x : \text{ p | } n \implies p \leq y \} |
the number of \( y \)-smooth numbers not greater than \( x \). Canfield, Erdős and Pomerance proved that for any \( \epsilon > 0 \) we have

\[
\Psi(x; y) = \frac{x}{u^{u+o(u)}},
\]

where \( u = \log x / \log y \) and \( y \geq (\log x)^{1+\epsilon} \). As a consequence of this result we have

\( \Psi(x; \log^t x) = x^{1-1/t+o(1)} \)

for any \( t > 1 \).

We prove that there exists a set \( A \) such that \( |A| = \lfloor cn^\theta \rfloor \) and \( \psi(A) \leq (\log n)^{2+\theta/(1-\theta)+o(1)} \).

Let \( t \) be a real number satisfying \( \Psi(n; \log^t n) = \lfloor cn^\theta \rfloor \). By (11) we know that \( t = \frac{1}{1-\theta} + o(1) \). For this \( t \) we consider the set \( A \) of \( \log^t n \)-smooth integers \( \leq n \), namely

\[
A = \{ m \leq n : p | m \implies p \leq \log^t n \}.
\]

By construction we have that \( |A| = \lfloor cn^\theta \rfloor \). It is clear that

\[
\lcm\{a \in A\} = \prod_{p \leq \log^t n} p^{\lfloor \log n / \log p \rfloor}.
\]

Thus

\[
\psi(A) = \sum_{p \leq \log^t n} \log p \lfloor \log n / \log p \rfloor \leq \sum_{p \leq \log^t n} \log n \leq (\log n)^{2+\frac{\theta}{1-\theta}+o(1)},
\]

as was to be shown.

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