Existence, Blow-Up and Exponential Decay Estimates for the Nonlinear Kirchhoff-Carrier Wave Equation in an Annular with Nonhomogeneous Dirichlet Conditions

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Abstract. This paper is devoted to the study of a nonlinear Kirchhoff-Carrier wave equation in an annular associated with nonhomogeneous Dirichlet conditions. At first, by applying the Faedo-Galerkin, we prove existence and uniqueness of the solution of the problem considered. Next, by constructing Lyapunov functional, we prove a blow-up result for solutions with a negative initial energy and establish a sufficient condition to obtain the exponential decay of weak solutions.

1. Introduction

In this paper, we are concerned with the following nonlinear Kirchhoff-Carrier wave equation in the annular

\[ u_{tt} - \mu \left( t, \|u(t)\|_0^2, \|u_x(t)\|_0^2 \right) \left( u_{xx} + \frac{1}{x} u_x \right) = f \left( x, t, u, u_x, u_t, \|u(t)\|_0^2, \|u_x(t)\|_0^2 \right), \quad \rho < x < 1, \quad 0 < t < T, \]

(1.1)

associated with nonhomogeneous Dirichlet conditions

\[ u(\rho, t) = g_0(t), \quad u(1, t) = g_1(t), \]

(1.2)

and initial conditions

\[ u(x, 0) = \tilde{u}_0(x), \quad u_t(x, 0) = \tilde{u}_1(x), \]

(1.3)

where \( \mu, f, g_0, g_1, \tilde{u}_0(x), \tilde{u}_1(x) \) are given functions; \( \rho \in (0, 1) \) is given constant. In Eq. (1.1), the nonlinear terms \( \mu \left( t, \|u(t)\|_0^2, \|u_x(t)\|_0^2 \right), \left( x, t, u, u_x, u_t, \|u(t)\|_0^2, \|u_x(t)\|_0^2 \right) \), depend on the integrals \( \|u(t)\|_0^2 = \int_\rho^1 x u^2(x, t) \, dx \)
and \( \|u_x(t)\|_0^2 = \int_\rho^1 x u_x^2(x, t) \, dx \).

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Eq. (1.1) herein is the bidimensional nonlinear wave equation describing nonlinear vibrations of the annular membrane \( \Omega = \{(x, y) : \rho^2 < x^2 + y^2 < 1\} \). In the vibration processing, the area of the annular membrane and the tension at various points change in time. The conditions \( u(\rho, t) = g_0(t) \) and \( u(1, t) = g_1(t) \) on the boundaries \( \Gamma_1 = \{(x, y) : x^2 + y^2 = \rho^2\} \) and \( \Gamma_2 = \{(x, y) : x^2 + y^2 = 1\} \) describe vibrations of a membrane which depends on \( g_0 \) and \( g_1 \) at the both boundaries of the annular membrane.

It is known that Kirchhoff [6] first investigated the following nonlinear vibration of an elastic string

\[
\rho u_{tt} = \left( P_0 + \frac{E h}{2L} \int_0^L \left| \frac{\partial u}{\partial y}(y, t) \right|^2 dy \right) u_{xx},
\]

(1.4)

where \( u = u(x, t) \) is the lateral displacement at the space coordinate \( x \) and the time \( t \), \( \rho \) is the mass density, \( h \) is the cross-section area, \( L \) is the length, \( E \) is the Young modulus, \( P_0 \) is the initial axial tension.

In [3], Carrier established the equation which models vibrations of an elastic string when changes in tension are not small

\[
\rho u_{tt} - \left( 1 + \frac{EA}{LT_0} \right) \int_0^L u^2(y, t) dy u_{xx} = 0,
\]

(1.5)

where \( u(x, t) \) is the \( x \)-derivative of the deformation, \( T_0 \) is the tension in the rest position, \( E \) is the Young modulus, \( A \) is the cross-section of a string, \( L \) is the length of a string and \( \rho \) is the density of a material. Clearly, if properties of a material depends on \( x \) and \( t \), there is a hyperbolic equation of the type (Larkin [7])

\[
u_{tt} - B \left( x, t, \int_0^1 u^2(y, t) dy \right) u_{xx} = 0.
\]

(1.6)

The Kirchhoff - Carrier equations of the form Eq. (1.1) received much attention. We refer the reader to, e.g., Cavalcanti et al. [1], [2], Ebihara, Medeiros and Miranda [4], Miranda et al. [15], Lasiecka and Ong [8], Hosoya, Yamada [5], Larkin [7], Long et al. [10]-[12], Medeiros [14], Menzala [16], Messaoudi [17], Ngoc et al. [18]-[22], Park et al. [23], [24], Rabello et al. [25], Santos et al. [26], Truong et al. [28], for many interesting results and further references. In these works, the results concerning local existence, global existence, asymptotic expansion, asymptotic behavior, decay and blow-up of solutions have been examined.

Recently, Gongwei Liu [13] studied the damped wave equation of Kirchhoff type with initial and Dirichlet boundary condition

\[
\left\{
\begin{align*}
u_{tt} - M \left( |\nabla u(t)|^2 \right) \Delta u + u_t &= g(u) \text{ in } \Omega \times (0, \infty), \\
u(x, 0) &= u_0(x), u_t(x, 0) &= u_1(x), \text{ } x \in \Omega, \\
u(x, t) &= 0 \text{ on } \partial \Omega \times (0, \infty),
\end{align*}
\right.
\]

(1.7)

where \( \Omega \) is a bounded domain with smooth boundary \( \partial \Omega \), \( g \) is a source term with exponential growth at the infinity to be specified later. Here \( M(s) \) is a positive \( C^1 \) function on \( [0, \infty) \) and \( M(s) \geq 1, |M'(s)| \leq s^\alpha \) for all \( s > 1, \alpha \geq 0 \) and for suitably chosen initial data, (1.7) possesses a global weak solution which decays exponentially. On the other hand, if conditions of \( M, g \) and initial data are suitable, the solution \( u \) of (1.7) blows-up at a finite \( T^* \).

In [15] Miranda and Jutucu dealt with the existence and uniqueness of solutions and exponential decay of solutions of an initial-homogeneous boundary value problem for the Kirchhoff equation.

In [1], [2], Cavalcanti also studied the existence and uniform decay of solutions of the Kirchhoff-Carrier equation.

In [28], the global existence and regularity of weak solutions for the linear wave equation

\[
u_{tt} - u_{xx} + Ku + \lambda u_t = f(x, t), \text{ } 0 < x < 1, t > 0
\]

(1.8)

with the initial conditions as in (1.3) and two-point boundary conditions. The exponential decay of solutions was given there by using Lyapunov method.
Motivated by the above work, we intend to study the existence and uniqueness, the blow-up and exponential decay of solutions for problem (1.1-1.3) under suitable conditions on \( f, \mu \) and initial data. Our paper is organized as follows.

First, we present preliminaries in Section 2, with the notations, definitions, list of appropriate spaces and required lemmas. We prove the existence and uniqueness a weak solution in Section 3 by using Faedo-Galerkin method, the linearization method and the weak compact method. Next, in Section 4, Prob. (1.1)-(1.3) is considered in case \( \mu = \mu \left( \|u(\cdot,t)\|_0^2 \right) \), \( f = -\lambda u_t + f(u) \), \( g_0(\cdot,t) = g_1(\cdot,t) \equiv 0 \), with \( \lambda > 0 \) is constant. If some auxiliary conditions are satisfied, we imply that the weak solution \( u \) of Prob. (1.1)-(1.3) blows-up at finite time.

Finally, in Section 5 with the case \( \mu = \mu \left( \|u(\cdot,t)\|_0^2 \right) \), \( f = -\lambda u_t + f(u) + F(x,t) \), \( g_0(\cdot,t) = g_1(\cdot,t) \equiv 0 \), if \( \int_0^{\|u(\cdot,t)\|_0^2} \mu(z)dz - \int_0^1 xdx \int_0^{\|u(\cdot,t)\|_0^2} f(z)dz > 0 \) and the initial energy and \( \|F(\cdot,0)\|_0 \) are small enough, we verify that the energy of the solution decays exponentially as \( t \to +\infty \). In the proofs, to obtain the blow-up and the exponential decay, we use the multiplier technique combined with a suitable Lyapunov functionals. Our results can be regarded as an extension and improvement of the corresponding results of [7], [10]-[12], [18]-[22], [28].

2. Preliminaries

First, we put \( \Omega = (\rho, 1) \), \( Q_T = \Omega \times (0, T) \), \( T > 0 \), and denote the usual function spaces used throughout the paper by the notations \( L^p = L^p(\Omega), H^m = H^m(\Omega). \) We denote the usual norm in \( L^2 \) by \( \| \cdot \| \) and we denote \( \| \cdot \|_X \) for the norm in the Banach space \( X \). We call \( X' \) the dual space of \( X \). We denote \( \mathcal{U}^p(0, T; X), 1 \leq p \leq \infty \) the Banach space of real functions \( u : (0, T) \to X \) measurable, such that \( \|u\|_{\mathcal{U}^p(0, T; X)} < +\infty \), with

\[
\|u\|_{\mathcal{U}^p(0, T; X)} = \left\{ \left( \int_0^T \|u(t)\|^p_X dt \right)^{\frac{1}{p}} \right\}, \text{ if } 1 \leq p < \infty,
\]

\[
\|u\|_{\mathcal{U}^\infty(0, T; X)} = \text{ess sup}_{0 < t < T} \|u(t)\|_X, \text{ if } p = \infty.
\]

With \( f \in C^1 \big( [0, 1] \times [0, T^*] \times R^3 \times R^2 \big) \), \( f = f(x, t, y_1, y_2, y_3, y_4, y_5) \), we put \( D_1f = \frac{\partial f}{\partial x}, D_2f = \frac{\partial f}{\partial t}, D_4f = \frac{\partial f}{\partial y_4} \) with \( i = 1, \ldots, 5 \), and \( D^\beta f = D_{\alpha_1} \ldots D_{\alpha_5} f, \alpha = (\alpha_1, \ldots, \alpha_5) \in Z^5, |\alpha| = \alpha_1 + \ldots + \alpha_5 = k, D^{(\alpha_1, \ldots, \alpha_5)} f = f \).

With \( \mu \in C^1 \big( [0, T^*] \times R^2 \big), \mu = \mu(t, y, z) \), we put \( D_1\mu = \frac{\partial \mu}{\partial x}, D_2\mu = \frac{\partial \mu}{\partial t}, D_3\mu = \frac{\partial \mu}{\partial y} \) and \( D^k \mu = D^k_1 \ldots D^k_5 \mu, \beta = (\beta_1, \ldots, \beta_5) \in Z^5, |\beta| = \beta_1 + \ldots + \beta_5 = k, D^{(\beta_1, \ldots, \beta_5)} \mu = \mu \).

On \( H^1, H^2 \), we use the following norms

\[
\|v\|_{H^1} = \left( \|v\|^2 + \|v_x\|^2 \right)^{\frac{1}{2}}, \tag{2.1}
\]

and

\[
\|v\|_{H^2} = \left( \|v\|^2 + \|v_x\|^2 + \|v_{xx}\|^2 \right)^{\frac{1}{2}}, \tag{2.2}
\]

respectively.

We remark that \( L^2, H^1, H^2 \) are the Hilbert spaces with respect to the corresponding scalar products

\[
\langle u, v \rangle = \int_\rho^1 xu(x)v(x)dx, \langle u, v \rangle = \langle u_x, v_x \rangle, \langle u, v \rangle + \langle u_x, v_x \rangle + \langle u_{xx}, v_{xx} \rangle. \tag{2.3}
\]

The norms in \( L^2, H^1 \) and \( H^2 \) induced by the corresponding scalar products (2.3) are denoted by \( \| \cdot \|_0, \| \cdot \|_1 \) and \( \| \cdot \|_2 \).

We then have the following lemmas.

Lemma 2.1. The following inequalities are fulfilled.
(i) \( \sqrt{\|v\|} \leq \|v\|_0 \leq \|v\| \) for all \( v \in L^2 \),

(ii) \( \sqrt{\|v\|_{H^1}} \leq \|v\|_1 \leq \|v\|_{H^1} \) for all \( v \in H^1 \).

**Lemma 2.2.** The embedding \( H^1_0 \hookrightarrow C^0(\overline{\Omega}) \) is compact and for all \( v \in H^1_0 \), we have

(i) \( \|v\|_{C^0(\overline{\Omega})} \leq \sqrt{1-\rho}\|v_x\| \),

(ii) \( \|v\| \leq \frac{1-\rho}{\sqrt{2}}\|v_x\| \),

(iii) \( \|v\|_0 \leq \frac{1-\rho}{\sqrt{2}}\|v_x\|_0 \).

Proofs of Lemma 2.1 and Lemma 2.2 are straightforward, so we omit the details.

**Remark 2.3.** On \( L^2 \), two norms \( v \mapsto \|v\| \) and \( v \mapsto \|v\|_0 \) are equivalent. So are two norms \( v \mapsto \|v\|_{H^1} \) and \( v \mapsto \|v\|_1 \) on \( H^1 \), and four norms \( v \mapsto \|v\|_{H^1}, v \mapsto \|v\|_1, v \mapsto \|v_x\| \) and \( v \mapsto \|v_x\|_0 \) on \( H^1_0 \).

**Lemma 2.4.** We have

\[
\|v\|_{C^0(\overline{\Omega})} \leq a_0\|v\|_{H^1}, \text{ for all } v \in H^1
\]

where \( a_0 = \sqrt{\frac{1}{2(1-\rho)}} \sqrt{1 + \sqrt{1 + 16(1-\rho)^2}}. \)

**Proof of Lemma 2.4.** It is well known that the embedding \( H^1 \hookrightarrow C^0(\overline{\Omega}) \) is compact. Since \( C^1(\overline{\Omega}) \) is dense in \( H^1 \), we only show that (2.4) holds for all \( v \in C^1(\overline{\Omega}) \). For all \( v \in C^1(\overline{\Omega}) \), and \( x, y \in \overline{\Omega} \), we have

\[
v^2(x) = v^2(y) + 2 \int_y^x v(z)v_x(z)dz.
\]

Integrating over \( y \) from \( \rho \) to 1 to get

\[
(1-\rho)v^2(x) = \|v\|^2 + 2 \int_\rho^1 dy \int_y^x v(z)v_x(z)dz
\]

\[
= \|v\|^2 + 2 \int_\rho^1 dy \int_y^x v(z)v_x(z)dz - 2 \int_\rho^1 dy \int_y^x v(z)v_x(z)dz
\]

\[
\leq \|v\|^2 + 2(1-\rho) \int_\rho^1 v(z)v_x(z)dz + 2 \int_\rho^1 (1-z)v(z)v_x(z)dz
\]

\[
\leq \|v\|^2 + 4(1-\rho) \int_\rho^1 (1-z)v(z)v_x(z)dz.
\]

Note that \( a_0^2 = \frac{1+\sqrt{1+16(1-\rho)^2}}{2(1-\rho)} \) satisfies \( 1 + 4(1-\rho) \frac{1}{a_0^2} = (1-\rho)a_0^2 \) applying the inequality \( 2ab \leq \frac{a^2}{\alpha_0} + \frac{b^2}{\beta} \), for all \( a, b \in \mathbb{R} \), we deduce from (2.5), that

\[
(1-\rho)v^2(x) \leq \|v\|^2 + 2(1-\rho) \left( \frac{2}{a_0^2}\|v\|^2 + \frac{a_0^2}{2} \right)
\]

\[
= \left( 1 + 4(1-\rho) \frac{1}{a_0^2} \right)\|v\|^2 + (1-\rho)a_0^2\|v_x\|^2
\]

\[
= (1-\rho)a_0^2\|v\|_{H^1}^2.
\]

Hence (2.4) holds. Lemma 2.4 is complete. \(\Box\)

Now, we define the following bilinear form

\[
a(u,v) = \int_\rho^1 xu_x(x)v_x(\alpha)dx, \text{ for all } u, v \in H^1_0.
\]
Lemma 2.5. The symmetric bilinear form $a(\cdot, \cdot)$ defined by (2.6) is continuous on $H^1_0 \times H^1_0$ and coercive on $H^1_0$, i.e.,

(i) $|a(u, v)| \leq \|u\|_0\|v\|_0$,

(ii) $a(v, v) \geq \|v\|_0^2$,

for all $u, v \in H^1_0$.

Lemma 2.6. There exists the Hilbert orthonormal base $\{w_j\}$ of the space $L^2$ consisting of eigenfunctions $w_j$ corresponding to eigenvalues $\lambda_j$ such that

(i) $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots$, $\lim_{j \to +\infty} \lambda_j = +\infty$,

(ii) $a(w_j, v) = \lambda_j \langle w_j, v \rangle$ for all $v \in H^1_0$, $j = 1, 2, \ldots$.

Furthermore, the sequence $\{w_j/\sqrt{\lambda_j}\}$ is also the Hilbert orthonormal base of $H^1_0$ with respect to the scalar product $a(\cdot, \cdot)$.

On the other hand, we also have $w_j$ satisfying the following boundary value problem

\[
\left\{
\begin{array}{l}
Aw_j \equiv -\left(w_{jxx} + \frac{1}{\rho}w_{jx}\right) = -\frac{1}{\rho} \frac{d}{dx} (\sqrt{\lambda_j} w_{jx}) = \lambda_j w_{jx}, \quad \text{in } \Omega, \\
\end{array}
\right.
\quad (2.7)
\]

The proof of Lemma 2.6 can be found in [27, p.87, Theorem 7.7], with $H = L^2$, and $a(\cdot, \cdot)$ as defined by (2.6). \(\Box\)

We also note that the operator $A : H^1_0 \rightarrow (H^1_0)'$ in (2.7) is uniquely defined by the Lax-Milgram’s lemma, i.e.

\[a(u, v) = \langle Au, v \rangle \quad \text{for all } u, v \in H^1_0.\]  

\[\text{(2.8)}\]

Lemma 2.7. On $H^1_0 \cap H^2$, two norms $v \mapsto \|v\|_{H^2 \cap H^1}$ and $v \mapsto \|v\|_2$ are equivalent and

\[C_{1\rho} \|v\|_{H^2 \cap H^1} \leq \|v\|_2 \leq C_{2\rho} \|v\|_{H^2 \cap H^1}, \quad \text{for all } v \in H^1_0 \cap H^2,\]

where

\[C_{1\rho} = \sqrt{(1 - \rho)\rho^3}, \quad C_{2\rho} = \sqrt{1 + \frac{2}{\rho^2}}.\]  

\[\text{Proof of Lemma 2.7.} \quad \text{For all } v \in H^1_0 \cap H^2, \text{ we have}\]

\[\|Av\|_0 \leq \|v_{xx}\|_0 + \|v_{jx}\|_0 \leq \|v_{xx}\|_0 + \frac{1}{\rho}\|v_{jx}\|_0.\]

\[\text{Hence}\]

\[\|v\|_2^2 = \|v\|_0^2 + \|Av\|_0^2 \leq \|v\|_0^2 + \frac{2}{\rho^2}\|v_{xx}\|_0^2 \leq C_{2\rho}^2 \|v\|_{H^2 \cap H^1}^2.\]

On the other hand

\[\|Av\|_0^2 = \|v_{xx}\|_0^2 + \|v_{jx}\|_0^2 + 2 \int \rho v_x(x)v_{xx}(x)dx \geq \|v_{xx}\|_0^2 + \|v_{jx}\|_0^2 + \int \rho v_x(x)v_{xx}(x)dx.\]
Therefore
\[ \|\varphi_x\|_0^2 + \|\varphi_v\|_0^2 \leq \|\mathcal{A}\varphi\|_0^2 - \frac{2}{\rho} \int_0^1 \varphi_x(x)\varphi_x(x)dx \]
\[ \leq \|\mathcal{A}\varphi\|_0^2 + \frac{2}{\rho} \|\varphi_x\|_0 \|\varphi_v\|_0 \]
\[ \leq \|\mathcal{A}\varphi\|_0^2 + \frac{1}{\rho} \left( \frac{1}{\rho} \|\varphi_x\|_0^2 + \rho^2 \|\varphi_v\|_0^2 \right), \]
this implies that
\[ \|\varphi_x\|_0^2 + \|\varphi_v\|_0^2 \leq \frac{1}{1-\rho} \left( \|\mathcal{A}\varphi\|_0^2 + \frac{1-\rho^2}{\rho^3} \|\varphi_v\|_0^2 \right) \]
\[ \leq \frac{1}{C_{1,0}} \|\varphi\|_2^2. \]
Lemma 2.7 is complete. □

3. The existence and uniqueness theorem

First, we make the following assumptions:

(H1) : \( g_0, g_1 \in C^3 ([0, T^*]) \);

(H2) : \( \bar{u}_0 \in H^2, \bar{u}_1 \in H^1, \bar{u}_0(\rho) - \bar{g}_0(0) = \bar{u}_0(1) - \bar{g}_1(0) = 0; \)

(H3) : \( \mu \in C^1 \left([0, T^*] \times \mathbb{R}_+^2 \right), \mu (t, y, z) \geq \mu_* > 0, \forall (t, y, z) \in [0, T^*] \times \mathbb{R}_+^2; \)

(H4) : \( f \in C^1 \left([\rho, 1] \times [0, T^*] \times \mathbb{R}^3 \times \mathbb{R}_+^2 \right), \) satisfying the following conditions
\[ f (\rho, t, y_1, y_2, y_3, y, z) = f (1, t, y_1, y_2, y_3, y, z) = 0, \]
\[ \forall (t, y_1, y_2, y_3, y, z) \in [0, T^*] \times \mathbb{R}^3 \times \mathbb{R}_+^2. \]

Put
\[ \varphi (x, t) = \frac{1}{1-\rho} \left( [g_1 (t) - g_0 (t)] x + g_0 (t) - \rho g_1 (t) \right). \]

By the transformation \( \varphi (x, t) = u (x, t) - \varphi (x, t), \) Prob. (1.1)-(1.3) reduces to the following problem with the homogeneous boundary conditions
\[
\begin{align*}
\begin{cases}
\varphi_t + \mu [\varphi](t)\mathcal{A} \varphi = f(t)\varphi(x, t), & \rho < x < 1, \, 0 < t < T, \\
\varphi (\rho, t) = \varphi (1, t) = 0, & \\
\varphi (x, 0) = \bar{\varphi}_0 (x), & \varphi_1 (x, 0) = \bar{\varphi}_1 (x), \\
\end{cases}
\end{align*}
\]
in which
\[
\begin{align*}
\mu [\varphi](t) &= \mu \left( t, \|\varphi\|_0^2, \|\varphi_x\|_0^2 \right), \\
f(t) &= f \left( x, t, \varphi + \varphi_x + \varphi_{xx}, \varphi_t + \varphi_{xt}, \|\varphi\|_0^2, \|\varphi_x\|_0^2 \right), \\
\phi (t) &= \frac{1}{2} \phi (t) \mu [\varphi](t), \\
\bar{\varphi}_0 (x) &= \bar{\varphi}_0 (x) - \varphi (x, 0), & \bar{\varphi}_1 (x) &= \bar{\varphi}_1 (x) - \varphi (x, 0), \\
(\bar{\varphi}_0, \bar{\varphi}_1) &\in (H_0^2 \cap H^2) \times H_0^2.
\end{align*}
\]
Consider $T^* > 0$ fixed, let $M > 0$, we put
\[ K_M(f) = \| f \|_{C^1(A_M)} = \| f \|_{C^1(A_M)} + \sum_{i=1}^{2} \| D_i f \|_{C^1(A_M)}, \]
\[ R_M(\mu) = \| \mu \|_{C^1(A_M)} = \sum_{i=0}^{3} \| D_i \mu \|_{C^1(A_M)}, \]
\[ \| f \|_{C^1(A_M)} = \sup \{ | f (x, t, y_1, y_2, y_3) | : (x, t, y_1, y_2, y_3) \in A_M \}, \]
\[ \| \mu \|_{C^1(A_M)} = \sup \{ | \mu (t, t, z) |, \} \]

where
\[ A_M = \{ (x, t, y_1, y_2, y_3, z) \in [0, 1] \times [0, T^*] \times \mathbb{R}^2 : | y_1 | \leq \frac{1}{\rho} \mu M + M^*, \]
\[ | y_2 | \leq \frac{M}{| C_1 \rho |}, \]
\[ | y_3 | \leq \frac{1}{\rho} M + M^*, \]
\[ 0 \leq y \leq \left( \frac{(1 - \rho) M}{2 \rho \mu C_1 \rho} + \frac{1}{2} M^* \right)^2, \]
\[ 0 \leq z \leq \left( \frac{M}{\sqrt{\mu} C_1 \rho} + \frac{1}{2} M^* \right)^2, \] (3.4)
\[ \bar{A}_M = \{ (t, y, z) \in [0, T^*] \times \mathbb{R}^2 : 0 \leq y \leq \left( \frac{(1 - \rho) M}{2 \rho \mu C_1 \rho} + \frac{1}{2} M^* \right)^2, \]
\[ 0 \leq z \leq \left( \frac{M}{\sqrt{\mu} C_1 \rho} + \frac{1}{2} M^* \right)^2, \]
\[ M^* = \| \varphi \|_{C^1([0, T^*])}. \]

Now, for each $M > 0$ and $T \in (0, T^*)$, we consider the sets
\[ W(M, T) = \{ v \in L^\infty (0, T; H_0^2) : v' \in L^\infty (0, T; H_0^1), \quad v'' \in L^2 (Q_T), \}
\[ \| v \|_{L^\infty (0, T; H_0^2)} \leq \frac{M}{\sqrt{\mu} C_1 \rho}, \quad \| v' \|_{L^\infty (0, T; H_0^1)} \leq M, \quad \| v'' \|_{L^2 (Q_T)} \leq M, \]
\[ W_1 (M, T) = \{ u \in W(M, T) : u_{tt} \in L^\infty (0, T; L^2) \}, \] (3.5)

and we establish the linear recurrent sequence $[v_m]$ as follows. We shall choose as first initial term $v_0 = 0$, suppose that
\[ v_{m-1} \in W_1 (M, T), \] (3.6)

and associate with the problem (3.1) the following variational problem: Find $v_m \in W_1 (M, T)$ ($m \geq 1$) so that
\[ \{ \langle v_m' (t), w \rangle + \mu_m (t) a(v_m (t), w) = \langle F_m (t), w \rangle , \quad \forall w \in H_0^1, \}
\[ v_m (0) = \varphi_0, \quad v_m' (0) = \varphi_1, \]

where
\[ \mu_m (t) = \mu (0, \| v_{m-1} (t) \|_{0}^2, \| \nabla v_{m-1} + \varphi_3 \|_0^2), \]
\[ F_m (t) = f (v_{m-1}) (x, t). \] (3.8)
The existence of a sequence \(\{v_m\}\) defined by (3.6) - (3.8) is established by our following theorem.

**Theorem 3.1.** Let the assumptions \( (H_1) \) - \((H_4)\) hold. Then there exist positive constants \( M, T\) such that the problem (3.7), (3.8) has a solution \(v_m \in W_{1}(M,T)\).

**Proof of Theorem 3.1.** The proof consists of three steps.

**Step 1. The Faedo - Galerkin approximation** (introduced by Lions [9]). Consider the basis \(\{w_j\}\) for \(H_0^1\) as in Lemma 2.6. Put

\[
v_m^{(j)}(t) = \sum_{j=1}^{k} c_{m_j}^{(j)}(t)w_j,
\]

where the coefficients \(c_{m_j}^{(j)}\) satisfy the system of linear differential equations

\[
\begin{aligned}
\left\{ 
&\left\langle \ddot{v}_m^{(j)}(t), w_j \right\rangle + \mu(t) \dot{a}(v_m^{(j)}(t), w_j) = \left\langle F_m(t), w_j \right\rangle, \\
&v_m^{(j)}(0) = \dot{v}_0, \quad \ddot{v}_m^{(j)}(0) = \ddot{v}_0, \quad j = 1, ..., k,
\end{aligned}
\]

with

\[
\begin{aligned}
&\ddot{v}_0 = \sum_{j=1}^{k} a_{m_j}^{(j)} w_j \rightarrow \ddot{v}_0 \text{ strongly in } H_0^1 \cap H^2, \\
&\dddot{v}_0 = \sum_{j=1}^{k} \dddot{v}_m^{(j)} w_j \rightarrow \dddot{v}_0 \text{ strongly in } H_0^1.
\end{aligned}
\]

The system (3.10) can be rewritten in form

\[
\begin{aligned}
&\left\{ 
&\dddot{c}_{m_j}^{(j)}(t) + \lambda_j \mu(t) c_{m_j}^{(j)}(t) = F_m(t), \\
&c_{m_j}^{(j)}(0) = a_{m_j}^{(j)}, \quad \dddot{c}_{m_j}^{(j)}(0) = \dddot{v}_m^{(j)}, \quad 1 \leq j \leq k,
\end{aligned}
\]

in which

\[F_m(t) = \left\langle F_m(t), w_j \right\rangle, \quad 1 \leq j \leq k.\]

Note that by (3.6), it is not difficult to prove that the system (3.12) has a unique solution \(c_{m_j}^{(j)}(t), 1 \leq j \leq k\) on interval \([0, T]\), so let us omit the details.

**Step 2. A priori estimates.**

We put

\[
S_m^{(j)}(t) = \left\| v_m^{(j)}(t) \right\|_0^2 + \left\| \dot{v}_m^{(j)}(t) \right\|_0^2 + \mu(t) \left( \left\| v_m^{(j)}(t) \right\|_0^2 + \left\| \ddot{v}_m^{(j)}(t) \right\|_0^2 \right) + \int_0^t \left\| \dddot{v}_m^{(j)}(s) \right\|_0^2 ds,
\]

and

\[
\sigma_m(t) = g_1(t) - \dot{\varphi}(t) \mu(t), \quad \sigma_m(t) = -\rho g_1(t) + \dot{\varphi}(t) \mu(t).
\]

Then, it follows from (3.10), (3.14), (3.15), that

\[
S_m^{(j)}(t) = S_m^{(j)}(0) + \int_0^t \mu(t) \left( \left\| v_m^{(j)}(s) \right\|_0^2 + \left\| \ddot{v}_m^{(j)}(s) \right\|_0^2 \right) ds
\]

\[
+ 2 \int_0^t \left\langle F_m(s), v_m^{(j)}(s) \right\rangle ds + 2 \int_0^t \left\langle F_m(s), \dot{v}_m^{(j)}(s) \right\rangle ds
\]

\[
+ 2 \int_0^t \sigma_m(s) v_m^{(j)}(s) ds + 2 \int_0^t \sigma_m(s) v_m^{(j)}(s) ds
\]

\[
+ \int_0^t \left\| \dddot{v}_m^{(j)}(s) \right\|_0^2 ds
\]

\[= S_m^{(j)}(0) + \sum_{j=1}^{6} I_j.
\]
Lemma 3.2. We have the following estimates

(i) \( |\mu'_m(t)| \leq \beta_M, \) a.e. \( t \in (0, T), \)
(ii) \( |F_m(x, t)| \leq F_1(M), \) a.e. \( (x, t) \in Q_T = (\rho, 1) \times (0, T), \)
(iii) \( |F_m(t)| \leq \tilde{F}_2(M), \) a.e. \( t \in (0, T), \)
(iv) \( |\phi'_m(t)| \leq \bar{\delta}_M, \) a.e. \( t \in (0, T), \)
(v) \( |\phi''_m(t)| \leq \sigma_{\rho M}, \) a.e. \( t \in (0, T), \)

where

\[
\beta_M = \tilde{K}_M(\mu) \left[ 1 + 4 \left( \frac{M}{\sqrt{3}C_{1p}} + \sqrt{\frac{1-\rho^2}{2}M^*} \right) \left( M + \sqrt{\frac{1-\rho^2}{2}M^*} \right) \right],
\]

\[
F_1(M) = K_M(f) + \left( \|g_0\|_{C^0([0, T])} + \|g_1\|_{C^0([0, T])} \right) \left( 1 + \frac{\tilde{K}_M(\mu)}{\rho(1-\rho)} \right),
\]

\[
F_2(M) = 2K_M(f) \left[ \frac{1}{2} + \frac{M}{\sqrt{3}C_{1p}} + \sqrt{\frac{1-\rho^2}{2}M^*} \right]
+ \|\phi\|_{C^0([0, T])} \left( 1 + \frac{1}{\rho^2} \tilde{K}_M(\mu) \right) \sqrt{\frac{1-\rho^2}{2}},
\]

\[
\bar{\delta}_M = \|g_0''\|_{C^0([0, T])} + \|\phi''\|_{C^0([0, T])} \tilde{K}_M(\mu) + \|\phi\|_{C^0([0, T])} \beta_M.
\]

Proof of Lemma 3.2.
Proof (i). Note that

\[
\mu'_m(t) = D_1\mu[v_{m-1}](t) + 2D_2\mu[v_{m-1}](t)(v_{m-1}(t) + \varphi(t), v'_{m-1}(t) + \varphi'(t))
+ 2D_3\mu[v_{m-1}](t)(\nabla v_{m-1}(t) + \varphi_x(t), v'_{m-1}(t) + \varphi'_x(t)),
\]

with \( D_1\mu[v_{m-1}](t) = D_1\mu(t, \|v_{m-1}(t) + \varphi(t)\|_0^2, \|\nabla v_{m-1}(t) + \varphi_x(t)\|_0^2), i = 1, 2, 3, \) it implies from (3.6) that \( |\mu'_m(t)| \leq \beta_M. \)

Proof (ii). By (3.2) and (3.8), we have

\[
|F_m(x, t)| \leq K_M(f) + |\varphi_x| + \frac{1}{x} |\bar{\varphi}(t)| \tilde{K}_M(\mu)
\leq K_M(f) + |g_0''(t)| + |g_1''(t)| + \frac{1}{\rho} \frac{|g_0(t)| + |g_1(t)|}{1-\rho} \tilde{K}_M(\mu)
\leq \tilde{F}_1(M).
\]

Proof (iii). Note that

\[
F_{m3}(t) = D_1f[v_{m-1}](t) + D_3f[v_{m-1}](t) (\nabla v_{m-1}(t) + \varphi_x)
+ D_4f[v_{m-1}](t) (\Delta v_{m-1}(t) + \varphi_{xx}) + D_5f[v_{m-1}](t) (\nabla v'_{m-1}(t) + \varphi'_x)
- \bar{\varphi}''(t) - \frac{1}{x^2} \bar{\varphi}(t) \mu_m(t),
\]

where \( D_3f[v_{m-1}] = D_3f(x, t, v_{m-1} + \varphi, \nabla v_{m-1} + \varphi_x, v'_{m-1} + \varphi'_x, \|v_{m-1} + \varphi\|_0^2, \|\nabla v_{m-1} + \varphi_x\|_0^2) ), i = 1, \cdots, 7. \)
Lemma 3.2 (i)-(iii) and the following inequality we obtain

By Lemma 3.2 (iv) and the following inequality (iv), we have

Proof (i). The proof of Lemma 3.2 is complete. □

Proof (iv). We have

\[ \sigma_1'(t) = g_1''(t) - \dot{\varphi}(t) \mu_1(t) - \dot{\varphi}(t) \mu_1'(t), \]

so

\[ |\sigma_1'(t)| \leq |g_1''(t)| + |\dot{\varphi}(t)| \mu_1(t) + |\dot{\varphi}(t)| \mu_1'(t) \leq \delta_1M. \]

Proof (v). It is similarly to the proof of (iv).

The proof of Lemma 3.2 is complete. □

Applying Lemma 3.2, we now estimate the terms \( I_j \) on the right-hand side of (3.16) as follows. By Lemma 3.2 (i)-(iii) and the following inequality

\[ S_m(t) \geq \|v_0^{(k)}(t)\|_0^2 + \|v_1^{(k)}(t)\|_0^2 + \mu_1 \left[ \|v_2^{(k)}(t)\|_0^2 + \|A\upsilon_1^{(k)}(t)\|_0^2 \right], \]

we obtain

\[ I_1 = \int_0^t \mu_1'(s) \left[ \|v_2^{(k)}(s)\|_0^2 + \|A\upsilon_1^{(k)}(s)\|_0^2 \right] \, ds \leq \frac{\bar{\mu}_M}{\mu_1} \int_0^t S_m(s) \, ds, \]

\[ I_2 = 2 \int_0^t \left\langle F_m(s), \upsilon_1^{(k)}(s) \right\rangle \, ds \leq \frac{1}{2} \frac{\rho^2}{2} TF_1^2(M) + \int_0^t S_m(s) \, ds, \]

\[ I_3 = 2 \int_0^t \left\langle F_m(s), \upsilon_1^{(k)}(s) \right\rangle \, ds \leq TF_2^2(M) + \int_0^t S_m(s) \, ds. \]

Using integration by parts leads to

\[ I_4 = 2 \int_0^t \sigma_1'(s) v_1^{(k)}(1, s) \, ds \]

\[ = -2\sigma_1(0)v_{0,0}(1) + 2\sigma_1(0)v_1^{(k)}(1, t) + 2v_1^{(k)}(1, t) \int_0^t \sigma_1'(s) \, ds \]

\[ -2 \int_0^t \sigma_1'(s)v_1^{(k)}(1, s) \, ds \]

\[ = -2\sigma_1(0)v_{0,0}(1) + f^{(1)} + f^{(2)} + f^{(3)}. \]

By Lemma 3.2 (iv) and the following inequality
it is not difficult to estimate the following terms $I^{(1)}_4$, $I^{(2)}_4$, $I^{(3)}_4$

\[
|I^{(1)}_4| = \left|2\sigma_{1m}(0)v_m^{(1)}(1, t)\right| \leq 2\alpha_0 \sigma_{1m}(0) \sqrt{s_m^{(1)}(t)} \leq \frac{8\alpha_0^2 \sigma_{1m}(0)}{\rho \mu C_{1p}} \sqrt{s_m^{(1)}(t)},
\]

\[
|I^{(2)}_4| = 2\alpha_0^2 \sigma_{1m}(0) \int_0^t s_{mm}^{(2)}(s) ds \leq \frac{8\alpha_0^2 \sigma_{1m}(0)}{\rho \mu C_{1p}} \sqrt{s_m^{(1)}(t)} + T \frac{2\sigma_{1m}^2 \alpha_0^2}{\rho \mu C_{1p}} + \int_0^t s_m^{(1)}(s) ds.
\]

Hence, we deduce from (3.19) and (3.20) that

\[
I_4 \leq 2\alpha_0 \sigma_{1m}(0) \sqrt{s_{mm}(1)} + \frac{8\alpha_0^2 \sigma_{1m}(0)}{\rho \mu C_{1p}} \sqrt{s_m^{(1)}(t)} + T \frac{2\sigma_{1m}^2 \alpha_0^2}{\rho \mu C_{1p}} + \int_0^t s_m^{(1)}(s) ds.
\]

Similarly, by using integration by parts

\[
I_5 = 2\int_0^t \sigma_{pm}(s) s_{mm}^{(1)}(\rho, s) ds
\]

\[
= -2\sigma_{pm}(0) \sigma_{0m}(\rho) + 2\sigma_{pm}(0) v_m^{(1)}(\rho, t)
\]

\[
+ 2v_m^{(1)}(\rho, t) \int_0^s \sigma_{pm}(s) ds - 2\int_0^t \sigma_{pm}(s) v_m^{(1)}(\rho, s) ds,
\]

from Lemma 3.2 (v) and the following inequality

\[
|v_m^{(1)}(\rho, t)| \leq \left\|v_m^{(1)}(t)\right\|_{C^0(\Omega)} \leq \alpha_0 \sqrt{s_m^{(1)}(t)},
\]

we also have

\[
I_5 \leq 2\left|\sigma_{pm}(0) \sigma_{0m}(\rho)\right| + \frac{8\alpha_0^2 \sigma_{pm}(0)}{\rho \mu C_{1p}} \sqrt{s_m^{(1)}(t)} + T \frac{2\sigma_{pm}^2 \alpha_0^2}{\rho \mu C_{1p}} + \int_0^t s_m^{(1)}(s) ds.
\]
Eq. (3.10) can be rewritten as follows
\[
\left\langle \varepsilon_m^{(i)}(t), w_j \right\rangle + \mu_m(t) \left\langle Av_m^{(i)}(t), w_j \right\rangle = \left\langle F_m(t), w_j \right\rangle, \quad j = 1, \ldots, k. \tag{3.23}
\]

Hence, it follows after replacing \(w_j\) with \(v_m^{(i)}(t)\), that
\[
\|v_m^{(i)}(t)\|^2 = -\mu_m(t) \left\langle Av_m^{(i)}(t), v_m^{(i)}(t) \right\rangle + \left\langle F_m(t), v_m^{(i)}(t) \right\rangle \leq \left( \mu_m(t) \left\| Av_m^{(i)}(t) \right\|_0 + \|F_m(t)\|_0 \right) \|v_m^{(i)}(t)\|_0 \leq \left( \mu_m(t) \left\| Av_m^{(i)}(t) \right\|_0 + \|F_m(t)\|_0 \right)^2 \leq 2\mu_m^2(t) \left\| Av_m^{(i)}(t) \right\|_0^2 + 2\|F_m(t)\|_0^2 \leq 2K_M(\mu) S_m^{(i)}(t) + (1 - \rho^2) F_1^2(M). \tag{3.24}
\]

Integrating in \(t\) to get
\[
I_6 = \int_0^t \|v_m^{(i)}(s)\|^2 ds \leq T \left( 1 - \rho^2 \right) F_1^2(M) + 2K_M(\mu) \int_0^t S_m^{(i)}(s) ds. \tag{3.25}
\]

It follows from (3.16), (3.18), (3.21), (3.22) and (3.25), that
\[
S_m^{(i)}(t) \leq S_m^{(i)}(0) + TD_1(M) + D_2(M) \int_0^t S_m^{(i)}(s) ds, \tag{3.26}
\]

where
\[
S_m^{(i)} = 2S_m^{(i)}(0) + 4 \left( \left| \sigma_{1m}(0)\hat{\theta}_{0x}(1) \right| + \left| \sigma_{jm}(0)\hat{\theta}_{0x}(\rho) \right| \right) \\
+ \frac{16\alpha_0^2 \left( \sigma_{1m}^2(0) + \sigma_{jm}^2(0) \right)}{\rho^2 C_1 \rho}, \tag{3.27}
\]

\[
D_1(M) = 3 \left( 1 - \rho^2 \right) F_1^2(M) + 2F_2^2(M) + 2\left( 1 + 8T^* \right) \frac{\alpha_0^2}{\rho \mu C_1^2 \rho} \left( \sigma_{1m}^2 + \sigma_{jm}^2 \right),
\]

\[
D_2(M) = 2 \left( 4 + \frac{\tilde{R}_M}{\mu^*} + 2K_M(\mu) \right).
\]

It remains to estimate the term \(S_m^{(i)}(0)\).

Notice that \(\mu_m(0) = \mu(0) \|\varphi(0)\|_0^2, \|\tilde{\theta}_{0x}(1)\|_0^2, \|\tilde{\theta}_{0x}(\rho)\|_0^2\) is independent of \(m\), so \(S_m^{(i)}(0), \sigma_{1m}(0), \sigma_{jm}(0)\) are also independent of \(m\), because of
\[
S_m^{(i)}(0) = \|\tilde{\theta}_{1x}(1)\|_0^2 + \|\tilde{\theta}_{1x}(\rho)\|_0^2 + \mu_m(0) \left( \|\varphi(0)\|_0^2 + \|\hat{\theta}_{0x}\|_0^2 \right),
\]

\[
\sigma_{1m}(0) = g_x^*(0) - \hat{\varphi}(0) \mu_m(0),
\]

\[
\sigma_{jm}(0) = -\rho g_r^*(0) + \hat{\varphi}(0) \mu_m(0).
\]

By means of the convergences in (3.11), we can deduce the existence of a constant \(M > 0\), it is independent of \(k\) and \(m\) such that
\[
S_m^{(i)} = 2S_m^{(i)}(0) + 4 \left( \left| \sigma_{1m}(0)\hat{\theta}_{0x}(1) \right| + \left| \sigma_{jm}(0)\hat{\theta}_{0x}(\rho) \right| \right) \\
+ \frac{16\alpha_0^2 \left( \sigma_{1m}^2(0) + \sigma_{jm}^2(0) \right)}{\rho^2 C_1 \rho} \leq \frac{1}{2}M^2, \quad \text{for all } m \text{ and } k. \tag{3.28}
\]
Therefore, we can choose $T \in (0, T^*)$, such that
\[
\left(\frac{1}{2}M^2 + TD_1(M)\right)\exp (TD_2(M)) \leq M^2,
\] (3.29)
and
\[
k_T = \left(1 + \frac{1}{\sqrt{t_k}}\right)\sqrt{T \left(\bar{\mu}_M + F_M^2\right) e^{T(\bar{\mu}_M^2/2)}} < 1,
\] (3.30)
where
\[
\bar{\mu}_M = \frac{8}{\mu_c} \left(\frac{M}{\sqrt{\mu_c} C_{\rho}} + \sqrt{\frac{1 - \rho^2}{2} M^2}\right)^2 M^2 K_M^2(\mu),
\] (3.31)
\[
F_M = \left(1 + \sqrt{2}\right) K_M(f)
\]
\[
+ 2 \sqrt{2}\left(\frac{M}{\sqrt{\mu_c} C_{\rho}} + \sqrt{\frac{1 - \rho^2}{2} M^2}\right) \left(K_M(f) + \frac{\|\Phi\|_{L^1(0,T)}}{\rho} K_M(\mu)\right) \sqrt{1 - \rho^2/2}.
\]
Combining (3.26), (3.28) and (3.29), we get
\[
S_{m^k}(t) \leq M^2 e^{-TD_2(M)} + D_2(M) \int_0^t S_{m^k}(s)ds.
\] (3.32)
By using Gronwall’s Lemma, (3.32) yields
\[
S_{m^k}(t) \leq M^2 e^{-TD_2(M)dD_2(M)} \leq M^2,
\] (3.33)
for all $t \in [0, T]$, for all $m$ and $k$. It implies that
\[
\bar{v}_{m^k} \in W(M, T), \text{ for all } m \text{ and } k.
\] (3.34)

**Step 3. Limiting process.** From (3.34), there exists a subsequence of $\{u_{m^k}\}$, still so denoted, such that
\[
\begin{cases}
\bar{v}_{m^k} \to v_m & \text{in } L^\infty(0, T; H_0^1 \cap H^2) \text{ weakly*}, \\
v_{m^k} \to v_m' & \text{in } L^\infty(0, T; H_0^1) \text{ weakly*}, \\
\bar{v}_{m^k}' \to v_m'' & \text{in } L^2(\Omega_T) \text{ weakly}, \\
v_{m^k} \in W(M, T).
\end{cases}
\] (3.35)
Passing to limit in (3.10), we have $v_m$ satisfying (3.7), (3.8) in $L^2(0, T)$. On the other hand, it follows from (3.7) and (3.35) that $v_m'' = -\mu_m(t) Av_m + F_m \in L^\infty(0, T; L^2)$, hence $v_m \in W_1(M, T)$ and the proof of Theorem 3.1 is complete. □

In order to get the existence and uniqueness, we shall use the following Banach space (see [9])
\[
W_1(T) = \{v \in L^\infty(0, T; H_0^1): v' \in L^\infty(0, T; L^2)\},
\]
with respect to the norm $\|v\|_{W_1(T)} = \|v\|_{L^\infty(0, T; H_0^1)} + \|v'\|_{L^\infty(0, T; L^2)}$.

By the result given in Theorem 3.1 and by the compact imbedding theorems, we now prove the main results in this section as follows.

**Theorem 3.3.** Let (H1) – (H4) hold. Then, there exist positive constants $M, T$ satisfying (3.28)-(3.30) such that Prob. (3.1) - (3.2) has a unique weak solution $v \in W_1(M, T)$. Furthermore, the linear recurrent sequence $\{v_m\}$ defined by (3.7), (3.8) converges to the solution $v$ strongly in the space $W_1(T)$ with the estimate
\[
\|v_m - v\|_{W_1(T)} \leq \frac{2M}{1 - k_T} k_T^m, \text{ for all } m \in \mathbb{N}.
\] (3.36)
Proof of Theorem 3.3.
(a) The existence. First, we shall prove that \{v_m\} is a Cauchy sequence in \(W_1(T)\). Let \(w_m = v_{m+1} - v_m\). Then \(w_m\) satisfies the variational problem

\[
\begin{align*}
\langle w_m'(t), w \rangle + \mu_{m+1}(t)a(w_m(t), w) &= -[\mu_{m+1}(t) - \mu_m(t)] (Av_m(t), w) \\
&+ \langle F_{m+1}(t) - F_m(t), w \rangle, \forall w \in H^1_0,
\end{align*}
\]

where

\[
\mu_m(t) = \mu[v_{m-1}](t) = \mu \left(t, \|v_{m-1}(t) + \varphi(t)\|_0^2, \|\nabla v_{m-1} + \varphi_x\|_0^2 \right),
\]

\[
F_m(t) = \int[v_{m-1}](x, t).
\]

Taking \(w = w_m'\) in (3.37), after integrating in \(t\), we get

\[
Z_m(t) = \int_0^t \mu_{m+1}(s) \|w_{m-1}(s)\|_0^2 ds - 2 \int_0^t \mu_{m+1}(s) (Av_m(s), w_m'(s)) ds
\]

\[
+ 2 \int_0^t \langle F_{m+1}(s) - F_m(s), w_m'(s) \rangle ds
\]

\[
= J_1 + J_2 + J_3,
\]

where

\[
Z_m(t) = \|w_m'(t)\|_0^2 + \mu_{m+1}(t) \|w_{m-1}(t)\|_0^2 \geq \|w_m'(t)\|_0^2 + \mu_m \|w_{m-1}(t)\|_0^2.
\]

All integrals on the right-hand side of (3.39) will be estimated as below.

The integral \(J_1\). By (3.40), we have

\[
|J_1| \leq \int_0^t |\mu_{m+1}(s)| \|w_{m-1}(s)\|_0^2 ds \leq \frac{\bar{\mu}_M}{\mu_m} \int_0^t Z_m(s) ds,
\]

where \(\bar{\mu}_M\) as in Lemma 3.2 (i).

The integral \(J_2\). By (H3), it is clear to see that

\[
|\mu_{m+1}(t) - \mu_m(t)| \leq 2 \bar{K}_M(\mu) \left( \frac{M}{\sqrt{\mu} C_1p} + \sqrt{\frac{1 - \rho^2}{2} M^*} \right) \left( \|w_{m-1}(t)\|_0 + \|\nabla w_{m-1}(t)\|_0 \right)
\]

\[
\leq 2 \sqrt{2} \bar{K}_M(\mu) \left( \frac{M}{\sqrt{\mu} C_1p} + \sqrt{\frac{1 - \rho^2}{2} M^*} \right) \|w_{m-1}\|_{W_1(T)}.
\]

Hence

\[
|J_2| = 2 \int_0^t \left| \mu_{m+1}(s) - \mu_m(s) \right| \langle Av_m(s), w_m'(s) \rangle ds
\]

\[
\leq 2 \int_0^t \left| \mu_{m+1}(s) - \mu_m(s) \right| \|Av_m(s)\|_0 \|w_m'(s)\|_0 ds
\]

\[
\leq 4 \sqrt{2} \bar{K}_M(\mu) \left( \frac{M}{\sqrt{\mu} C_1p} + \sqrt{\frac{1 - \rho^2}{2} M^*} \right) \frac{M}{\sqrt{\mu}} \|w_{m-1}\|_{W_1(T)} \int_0^t \|w_m'(s)\|_0 ds
\]

\[
\leq T \bar{\mu}_M \|w_{m-1}\|_{W_1(T)}^2 + \int_0^t Z_m(s) ds,
\]
where \( \tilde{\mu}_M = \frac{8}{\mu_*} \left( \frac{M}{\sqrt{\mu_* C_{1p}}} + \sqrt{\frac{1 - \rho^2}{2} M^*} \right)^2 M^2 \tilde{p}^2 (\mu). \)

The integral \( J_2 \). By \( (H_4) \) it yields

\[
\|F_{m+1}(t) - F_m(t)\|_0 \\
\leq K_M(f) \left( \|w_{m-1}(t)\|_0 + \|\nabla w_{m-1}(t)\|_0 + \|w_{m-1}(t)\|_0 \right) \\
+ 2 K_M(f) \left( \frac{M}{\sqrt{\mu_* C_{1p}}} + \sqrt{\frac{1 - \rho^2}{2} M^*} \right) \sqrt{\frac{1 - \rho^2}{2} (\|w_{m-1}(t)\|_0 + \|\nabla w_{m-1}(t)\|_0)} \\
+ \frac{1}{\rho} \left\| \phi \right\|_{C([-1,1])} \left( \frac{1 - \rho^2}{2} \right) \left| \mu_{m+1}(t) - \mu_m(t) \right| \\
\leq K_M(f) \left( \sqrt{2} \|w_{m-1}(t)\|_1 + \|w_{m-1}(t)\|_0 \right) \\
+ 2 \sqrt{2} K_M(f) \left( \frac{1 - \rho^2}{2} \right) \sqrt{\frac{1 - \rho^2}{2} \tilde{K}_M(\mu) \left( \frac{M}{\sqrt{\mu_* C_{1p}}} + \sqrt{\frac{1 - \rho^2}{2} M^*} \right) \|w_{m-1}\|_{W_1(T)} \right) \\
+ \frac{1}{\rho} \left\| \phi \right\|_{C([-1,1])} \left( \frac{1 - \rho^2}{2} \right) \left| \mu_{m+1}(t) - \mu_m(t) \right| \\
\leq F_M \|w_{m-1}\|_{W_1(T)}.
\]

where \( F_M \) as in (3.31). Hence

\[
J_2 = 2 \int_0^t \langle F_{m+1}(s) - F_m(s), w'_m(s) \rangle ds \\
\leq TF_M^2 \|w_{m-1}\|^2_{W_1(T)} + \int_0^t Z_m(s)ds.
\]

Combining (3.39), (3.41), (3.43) and (3.45), we obtain

\[
Z_m(t) \leq T \left( \tilde{\mu}_M + F_M^2 \right) \|w_{m-1}\|^2_{W_1(T)} + \left( 2 + \frac{\mu_M}{\mu_*} \right) \left( \|w_{m-1}\|_{W_1(T)} \right) \\
\int_0^t Z_m(s)ds.
\]

Using Gronwall’s Lemma, we deduce from (3.46) that

\[
\|w_{m}\|_{W_1(T)} \leq k_T \|w_{m-1}\|_{W_1(T)} \forall m \in \mathbb{N},
\]

where \( k_T \) as in (3.30).

It implies that

\[
\|v_m - v_{m+p}\|_{W_1(T)} \leq \|v_0 - v_{1}\|_{W_1(T)} (1 - k_T)^{-1} k_T^p \leq \frac{2M}{1 - k_T} k_T^p \forall m, p \in \mathbb{N}.
\]

It follows that \( \{v_m\} \) is a Cauchy sequence in \( W_1(T) \). Then there exists \( v \in W_1(T) \) such that

\[
v_m \rightharpoonup v \text{ strongly in } W_1(T).
\]

Note that \( v_m \in W_1(M, T) \), then there exists a subsequence \( \{v_{m_j}\} \) of \( \{v_m\} \) such that

\[
\begin{cases}
  v_{m_j} \rightharpoonup v^* & \text{in } L^{\infty}(0, T; H^1_0(t) \cap H^2) \text{ weakly*}, \\
  v'_{m_j} \rightharpoonup v' & \text{in } L^{\infty}(0, T; H^1_0(t)) \text{ weakly*}, \\
  v''_{m_j} \rightharpoonup v'' & \text{in } L^2(Q_T) \text{ weakly*}, \\
  v & \text{in } W(M, T).
\end{cases}
\]
We also note that
\[
\left| \mu_m(t) - \mu[v](t) \right| \leq 2 \sqrt{2} \hat{K}_M \left( \frac{M}{\sqrt{\mu_C} C_{1p}} + \sqrt{\frac{1 - \rho^2}{2}} M^* \right) \| v_m - v \|_{W_1(T)}. \tag{3.51}
\]
Hence, it follows from (3.49) and (3.51) that
\[
\mu_m \to \mu[v] \text{ strongly in } L^\infty(0, T).	ag{3.52}
\]
On the other hand, we have
\[
\left\| F_m - \bar{f} \right\|_{L^\infty(0, T; L^2)} \leq F_M \| v_m - v \|_{W_1(T)}. \tag{3.53}
\]
Hence, from (3.49) and (3.51), we obtain
\[
F_m \to \bar{f} \text{ strongly in } L^\infty(0, T; L^2). \tag{3.54}
\]
Finally, passing to limit in (3.7), (3.8) as \( m = m_j \to \infty \), it implies from (3.49), (3.50), (3.52) and (3.54) that there exists \( u \in W(M, T) \) satisfying the equation
\[
\langle v''(t), w \rangle + \mu[v](t) a(v(t), w) = \left\{ \bar{f} \right\}(t), w \rangle,
\]
for all \( w \in H^1_0 \) and the initial conditions
\[
v(0) = \tilde{v}_0, \quad v'(0) = \tilde{v}_1. \tag{3.56}
\]
Furthermore, from the assumptions \((H_3), (H_4)\) we obtain from (3.50), (3.52), (3.54) and (3.55), that
\[
v'' = -\mu[v](t) Av(t) + \bar{f} \text{ strongly in } L^\infty(0, T; L^2),
\]
thus we have \( v \in W_1(M, T) \). The existence of a weak solution of Prob. (3.1) - (3.2) is proved.

(b) The uniqueness. Let \( v_1, v_2 \in W_1(M, T) \) be two weak solutions of Prob. (3.1) - (3.2). Then \( v = v_1 - v_2 \) satisfies the variational problem
\[
\begin{cases}
\langle v''(t), w \rangle + \mu_1(t)a(v(t), w) = -[\mu_1(t) - \mu_2(t)] \langle Av_2(t), w \rangle & + \langle \bar{f}_1(t) - \bar{f}_2(t), w \rangle, \\
v(0) = v'(0) = 0,
\end{cases}
\tag{3.58}
\]
where \( \bar{f}_i(x, t) = \bar{f}[v_i](t), \mu_i(t) = \mu[v_i](t), i = 1, 2. \)

We take \( w = v' \) in (3.58), and integrate in \( t \) to get
\[
Z(t) = \int_0^t \mu_1(s) \| v_2(s) \|_0^2 ds - 2 \int_0^t [\mu_1(s) - \mu_2(s)] \langle Av_2(s), v'(s) \rangle ds
+ 2 \int_0^t \langle \bar{f}_1(s) - \bar{f}_2(s), v'(s) \rangle ds,
\tag{3.59}
\]
with \( Z(t) = \| v'(t) \|_0^2 + \mu_1(t) \| v_2(t) \|_0^2 \). Put \( K'_M = \frac{\tilde{\rho}_M}{\mu_0} + \frac{2M}{\sqrt{\mu_0}} R_M \hat{K}_M (\mu) + 2K_M(f, \mu) \), where
\[
R_M = \frac{2}{\sqrt{\mu_0}} \left( \frac{M}{\sqrt{\mu_C} C_{1p}} + \sqrt{\frac{1 - \rho^2}{2}} M^* \right) \left( \frac{1 - \rho}{\sqrt{2 \rho}} + 1 \right),
\]
\[
K_M(f, \mu) = K_M \left( f \right) \left( \frac{1 - \rho}{\sqrt{2 \rho}} + 1 \right) \frac{\mu}{\sqrt{\mu_0}} + R_M + 1
\]
\[
+ \frac{1}{\rho} \| \overline{\varphi} \|_{C^0(0, T')} \sqrt{\frac{1 - \rho^2}{2} \hat{K}_M (\mu) R_M},
\]
it follows from (3.59) that
\[
Z(t) \leq K_M \int_0^t Z(s)ds, \text{ for all } t \in [0, T].
\]

Using Gronwall’s Lemma, it follows that \(Z(t) \equiv 0\), i.e., \(v_1 \equiv v_2\).
Therefore, Theorem 3.3 is proved. □

4. Blow-up result

In this section, Prob. (1.1) – (1.3) is considered with \(f = -\lambda u_t + f(u), \mu = \mu(\|u_x(t)\|_0^2), g_0 (t) = g_1 (t) \equiv 0\), as follows
\[
\begin{align*}
\frac{\partial u}{\partial t} - \mu(\|u_x(\cdot)\|_0^2)(u_{xx} + \frac{1}{2}u_x) + \lambda u_t &= f(u), \quad \rho < x < 1, \quad 0 < t < T, \\
u(x,0) &= \bar{u}_0(x), \quad u_t(x,0) = \bar{u}_1(x),
\end{align*}
\]
where \(\lambda > 0, 0 < \rho < 1\) are given constants and \(\bar{u}_0, \bar{u}_1, \mu, f\) are given functions satisfying conditions specified later. First, we assume that
\(\mu \in C^1(\mathbb{R}_+), \mu(y) \geq \mu_\ast > 0, \forall y \in \mathbb{R}_+; \mu(y) \geq \mu \ast > 0, \forall y \in \mathbb{R}_+; \mu(y) \leq \mu \ast > 0, \forall y \in \mathbb{R}_+;\)
\(f \in C^1(\mathbb{R}), f(0) = 0.\)

Next, in order to obtain a blow-up result in Theorem 4.2 below, we make more the following assumptions.

\[\begin{align*}
(H_2) & \quad \mu \in C^1(\mathbb{R}_+), \text{ and there exist the constants } \mu \ast > 0, \mu_\ast > 0 \text{ such that } \\
(i) & \quad \mu(y) \geq \mu \ast > 0, \forall y \geq 0, \\
(ii) & \quad y \mu(y) \leq \mu \ast \int_0^y \mu(z)dz, \forall y \geq 0;
\end{align*}\]

\[\begin{align*}
(P_2) & \quad f \in C^1(\mathbb{R}), f(0) = 0 \text{ and there exist the constants } p > 2, d_1 > 2, \\
(i) & \quad y f(y) \geq d_1 \int_0^y f(z)dz, \forall y \in \mathbb{R}, \\
(ii) & \quad \int_0^y f(z)dz \geq d_1 |y|^p, \forall y \in \mathbb{R};
\end{align*}\]

\[\begin{align*}
(P_3) & \quad d_1 > 2\mu \ast, \text{ with } d_1, \mu \ast \text{ as in } (H_2)(ii), (P_2)(ii).
\end{align*}\]

Note that we can give an example of two following functions \(f\) and \(\mu\) which satisfy \((H_2)\) and \((P_2)\).

**Example.** Consider the function \(\mu \in C^1(\mathbb{R}_+)\) with
\[
\mu(z) = \mu_\ast + z^0, \quad \forall z \geq 0,
\]
and
\[
f(z) = \begin{cases} \lambda \cdot z^2 & \text{for } z \geq 1, \\ z^2 & \text{for } z < 1. \end{cases}
\]
where \( \mu, > 0 \) and \( q > 1 \) are constants. It is obvious that \((\hat{H}_2^i)(i)\) holds. On the other hand, we have

\[
\int_0^y \mu(z)dz = \int_0^y (\mu_z + z^q)dz = y\left(\mu_z + \frac{y^q}{q + 1}\right)
\geq \frac{1}{q + 1}y\mu(y), \ \forall y \geq 0.
\]

Hence, \((\hat{H}_2^i)(i)\) holds with \( \mu_1 = q + 1 \).

Let \( \beta > 0, p > 2 \) and \( k > 1 \), we define the function \( f \in C^1(\mathbb{R}) \) with

\[
f(z) = \beta|z|^{p-2}z\ln^k(z^2 + e), \ \forall z \in \mathbb{R}.
\]

It is clearly that \( f(0) = 0 \). By integration by parts, we obtain

\[
\int_0^y f(z)dz = \frac{1}{p}yf(y) - \frac{2k\beta}{p}\int_0^y \frac{|z|^{p-2}z}{e + z^2}\ln^{k-1}((z^2 + e))dz
\geq \frac{1}{p}yf(y), \ \forall y \in \mathbb{R},
\]

since \( \int_0^y \frac{|z|^{p-2}z}{e + z^2}\ln^{k-1}((z^2 + e))dz \geq 0 \) for all \( y \in \mathbb{R} \).

Thus, \((\hat{H}_3^1)(i)\) and \((\hat{H}_4^1)\) hold when \( d_1 = p > 2(q + 1) > 2 \).

For \( y \geq 0 \): \( \int_0^y f(z)dz = \beta\int_0^y |z|^{p-2}z\ln^k((z^2 + e))dz \geq \beta\int_0^y |z|^{p-2}zdz = \frac{\beta}{p}|y|^p. \)

For \( y < 0 \): \( \int_0^y f(z)dz = \beta\int_0^y |z|^{p-2}z\ln^k((z^2 + e))dz = \beta\int_0^{-y} |z|^{p-2}z\ln^k((z^2 + e))dz
\geq \beta\int_0^{-y} |z|^{p-2}zdz = \frac{\beta}{p}|-y|^p = \frac{\beta}{p}|y|^p. \)

Therefore

\[
\int_0^y f(z)dz \geq d_1|y|^p \quad \text{for all} \quad y \in \mathbb{R},
\]

where \( d_1 = \frac{\beta}{p} > 0 \). Thus, \((\hat{H}_3^1)(ii)\) is true.

Put

\[
H(0) = -\frac{1}{2}||\hat{u}_1||^2_0 - \frac{1}{2}\int_0^{\hat{u}_1(0)} \mu(z)dz + \int_0^1 \int_0^{\hat{u}(x)} f(z)dz.
\]

**Theorem 4.2.** Let \((\hat{H}_3^1) - (\hat{H}_4^1)\) hold. Then, for any \((\hat{u}_0, \hat{u}_1) \in (H_0^1 \cap H^2) \times H_0^1\) such that \( H(0) > 0 \), the weak solution \( u = u(x,t) \) of Prob. \((4.1)\) blows-up in finite time.

**Proof of Theorem 4.2.** It consists of two steps, in which the Lyapunov functional \( L(t) \) is constructed in step 1 and then the blow-up is proved in step 2.

**Step 1.** We define the energy associated with \((4.1)\) by

\[
E(t) = \frac{1}{2}||u'(t)||^2_0 + \frac{1}{2}\int_0^{\hat{u}_0(t)} \mu(z)dz - \int_0^1 dx \int_0^{\hat{u}(x,t)} f(z)dz.
\]

and we put \( H(t) = -E(t), \ \forall t \in [0, T_\star) \). Multiplying \((4.1)_1\) by \( xu'(x,t) \) and integrating the resulting equation over \((p, 1)\), we have

\[
H'(t) = \lambda ||u'(t)||^2_0 \geq 0.
\]
Hence, we can deduce from (4.5) and $H(0) > 0$ that
\[
H(t) \geq H(0) > 0, \; \forall t \in [0, T_*),
\]
so
\[
\begin{align*}
0 < H(0) &\leq H(t) \leq \int_{\rho}^{1} xdx \int_{0}^{\eta(x,t)} f(z)dz; \\
\|u'(t)\|_{0}^{2} + \int_{0}^{\eta(x,t)} \mu(z)dz &\leq 2 \int_{\rho}^{1} xdx \int_{0}^{\eta(x,t)} f(z)dz, \; \forall t \in [0, T_*).
\end{align*}
\]

We define the functional
\[
L(t) = H^{1-\eta}(t) + \epsilon \Psi(t),
\]
where
\[
\Psi(t) = \langle u'(t), u(t) \rangle + \frac{\lambda}{2} \|u(t)\|_{0}^{2},
\]
for $\epsilon$ small enough and
\[
0 < 2\eta < 1, \; 2/(1 - 2\eta) \leq p.
\]

In what follows, we show that there exists a constant $\lambda_1 > 0$ such that
\[
L'(t) \geq \lambda_1 \left[H(t) + \|u'(t)\|_{0}^{2} + \|u(t)\|_{L^{p}} + \|u_x(t)\|_{0}^{2} \right].
\]

Multiplying (4.1) by $xu(x,t)$ and integrating over $[\rho, 1]$, it leads to
\[
\Psi'(t) = \|u'(t)\|_{0}^{2} - \|u_x(t)\|_{0}^{2} \mu \left(\|u(t)\|_{0}^{2}\right) + \langle f(u(t)), u(t) \rangle.
\]

Therefore
\[
L'(t) = (1 - \eta) H^{-\eta}(t) H'(t) + \epsilon \Psi'(t) \geq \epsilon \Psi'(t).
\]

By $(\tilde{H}_{2}^{2})$, $(\tilde{H}_{2}^{3})$, we obtain
\[
\begin{align*}
\|u_x(t)\|_{0}^{2} \mu \left(\|u_x(t)\|_{0}^{2}\right) &\leq \mu_1 \int_{0}^{\eta(x,t)} \mu(z)dz, \\
\langle f(u(t)), u(t) \rangle &\geq d_1 \int_{\rho}^{1} xdx \int_{0}^{\eta(x,t)} f(z)dz, \\
\int_{\rho}^{1} xdx \int_{0}^{\eta(x,t)} f(z)dz &\geq \bar{d}_1 \rho \|u(t)\|_{L^{p}}^{p}.
\end{align*}
\]
Hence, combining (4.4), (4.12) and (4.14) give

\[ \Psi'(t) = \|u'(t)\|^2 - \|u(t)\|^2 + \mu \left( \|u(t)\|^2 + \langle f(u(t)), u(t) \rangle \right) \]

\[ \geq \|u'(t)\|^2 - \|u(t)\|^2 + \mu \int_0^{\|u(t)\|^2} \mu(z)dz + d_1 \int_0^{\|u(t)\|^2} \frac{df(z)}{dz}dz \]

\[ = \|u'(t)\|^2 - \|u(t)\|^2 + d_1 \delta_1 \int_0^{\|u(t)\|^2} \mu(z)dz + \frac{1}{2} \int_0^{\|u(t)\|^2} \mu(z)dz \]

\[ + d_1 (1 - \delta_1) \left[ H(t) + \left( 1 + \frac{d_1}{2} (1 - \delta_1) \right) \|u'(t)\|^2 \right] \]

\[ + d_1 \delta_1 \int_0^{\|u(t)\|^2} f(z)dz + \frac{1}{2} \left[ d_1 - 2 \mu_1 - \delta_1 d_1 \right] \int_0^{\|u(t)\|^2} \mu(z)dz \]

\[ \geq d_1 (1 - \delta_1) H(t) + \left( 1 + \frac{d_1}{2} (1 - \delta_1) \right) \|u'(t)\|^2 \]

\[ + d_1 \delta_1 \int_0^{\|u(t)\|^2} f(z)dz + \frac{1}{2} \left[ d_1 - 2 \mu_1 - \delta_1 d_1 \right] \int_0^{\|u(t)\|^2} \mu(z)dz. \]

By \( d_1 > 2 \mu_1 \), we can choose \( \delta_1 \in (0, 1) \) such that

\[ d_1 - 2 \mu_1 - \delta_1 d_1 > 0. \]  

(4.16)

By using the inequalities (4.13), (4.15), (4.16), we obtain (4.11) with choosing \( \lambda_1 > 0 \) small enough. From the formula of \( L(t) \) and (4.6), we can choose \( \varepsilon \) small enough such that

\[ L(t) \geq L(0) > 0, \quad \forall t \in [0, T_\varepsilon). \]

(4.17)

Using the inequality \( \left( \sum_{i=1}^{3} x_i \right) \geq 3^{-1} \sum_{i=1}^{3} x_i \) for all \( r > 1 \) and \( x_1, ..., x_3 \geq 0 \), we deduce from (4.8) - (4.10) that

\[ L^{1/(1-\eta)}(t) \leq \text{Const} \left( H(t) + \|u(t), u'(t)\|^{1/(1-\eta)} + \|u(t)\|^{2/(1-\eta)} \right). \]

(4.18)

Using Young’s inequality, we have

\[ \|u(t), u'(t)\|^{1/(1-\eta)} \leq \text{Const} \left( \|u(t)\|_{s}^{2} + \|u'(t)\|_{s}^{2} \right), \]

(4.19)

where \( s = 2/(1 - 2\eta) \leq p \) by (4.10).

Now, we shall need the following lemma

**Lemma 4.3.** Let \( s = 2/(1 - 2\eta) \leq p \), we obtain

\[ \|v\|_{s}^{2} + \|v\|_{0}^{2/(1-\eta)} \leq \frac{2}{\rho} \left( \|v\|_{s}^{2} + \|v\|_{s}^{2} \right), \quad \text{for any } v \in H^{s}_{0}. \]

(4.20)

Proof of Lemma 4.3 is straightforward, so we omit the details.
Step 2. Blow-up.
It follows from (4.18) - (4.20) that
\[ L^{1/(1-\eta)}(t) \leq \text{Const} \left( H(t) + \|u'(t)\|_0^2 + \|u_x(t)\|_0^2 + \|u(t)\|_T^p \right), \forall t \in [0, T_*). \] 
(4.21)

Using (4.11), (4.21) it yields
\[ L'(t) \geq \bar{\lambda}_2 L^{1/(1-\eta)}(t), \forall t \in [0, T_*), \]
(4.22)
where \( \bar{\lambda}_2 \) is a positive constant. By integrating (4.22) over \((0, t)\), it gives
\[ L^{\eta[1/(1-\eta)]}(t) \geq \frac{1}{L^{-\eta[1/(1-\eta)]}(0)} - \frac{\bar{\lambda}_2 \eta}{1-\eta} t, \quad 0 \leq t < \frac{1}{\bar{\lambda}_2 \eta} (1-\eta) L^{-\eta[1/(1-\eta)]}(0). \] 
(4.23)

Therefore, \( L(t) \) blows-up in a finite time given by \( T_* = \frac{1}{\bar{\lambda}_2 \eta} (1-\eta) L^{-\eta[1/(1-\eta)]}(0). \)
Theorem 4.2 is proved completely. \( \square \)

5. Exponential decay of solutions
This section investigates the decay of the solution of Prob. (1.1) - (1.3) corresponding to \( f = -\lambda u_t + f(u) + F(x, t), \mu = \mu(\|u_t(t)\|_0^2) \) and \( g_0(t) = g_1(t) \equiv 0 \), as follows

\[ \begin{cases} u_{tt} - \mu(\|u_x(t)\|_0^2)u_{txx} + \frac{1}{\bar{\lambda}} u_t = f(u) + F(x, t), & \rho < x < 1, \ t > 0, \\ u(\rho, t) = u(1, t) = 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = \bar{u}_1(x), \end{cases} \]
(5.1)
where \( \mu, f, F, \bar{u}_0, \bar{u}_1 \) are given functions and \( \lambda > 0, 0 < \rho < 1 \) are the given constants.

We prove that if \( \int_{0}^{\bar{u}_0(x)} \mu(z)dz - p \int_{0}^{1} \int_{0}^{\bar{u}_0(x)} f(z)dz > 0 \) and if the initial energy, \( \|F(t)\|_0 \) are small enough, then the energy of the solution decays exponentially as \( t \to +\infty \). For this purpose, we make the following assumptions

\( (\bar{\mathcal{H}}_3) \) \( f \in C^1(\mathbb{R}), f(0) = 0 \) and there exist the constants \( \alpha, \beta > 2, \ d_2 \), \( d_2 > 0 \), such that

(i) \( yf(y) \leq d_2 \int_{y}^{\alpha} f(z)dz, \) for all \( y \in \mathbb{R}, \)

(ii) \( \int_{y}^{\beta} f(z)dz \leq d_2 \left( |y|^{\alpha} + |y|^d \right), \) for all \( y \in \mathbb{R}, \)

\( (\bar{\mathcal{H}}_4) \) \( F \in L^{\infty}(\mathbb{R}^+; L^2) \cap L^1(\mathbb{R}^+; L^2), \) \( F' \in L^2(\mathbb{R}^+; L^2) \) and there exist two constants \( C_0 > 0, \gamma_0 > 0 \) such that \( \|F(t)\|_0 \leq C_0 e^{-\gamma_0 t}, \) for all \( t \geq 0. \)

We will show that the example of \( f \) in Section 4 also satisfies \( (\bar{\mathcal{H}}_3) \). Let \( \beta > 0, p > 2k \) and \( k > 1 \), we set
\[ f(z) = \beta |z|^{p-2} z \ln^k \left( z^2 + \epsilon \right), \quad \forall z \in \mathbb{R}. \]

We know that \( f \in C^1(\mathbb{R}) \) and \( f(0) = 0 \). For \( y \geq 0, \)
\[
\int_{0}^{y} f(z)dz = \frac{1}{p} yf(y) - \frac{2k \beta}{p} \int_{0}^{y} \frac{|z|^p}{e + z^2} \ln^{k-1} \left( z^2 + \epsilon \right)dz \\
\geq \frac{1}{p} yf(y) - \frac{2k \beta}{p} \ln^{k-1} \left( y^2 + \epsilon \right) \int_{0}^{y} \frac{|z|^p}{e + z^2}dz \\
\geq \frac{1}{p} yf(y) - \frac{2k \beta}{p^2} |y|^p \ln^{k-1} \left( y^2 + \epsilon \right) \\
\geq \frac{1}{p} yf(y) - \frac{2k \beta}{p^2} |y|^p \ln^k \left( y^2 + \epsilon \right) = \frac{p - 2k}{p^2} yf(y). 
\]
Note that
\[ \int_0^y \frac{\|y\|^p}{e + z^2} \ln^{k-1}(z^2 + e) \, dz = \int_0^y \frac{\|y\|^p}{e + z^2} \ln^{k-1}(z^2 + e) \, dz \geq 0 \text{ for all } y \in \mathbb{R}. \]

By the inequality \( \ln (1 + x) \leq x \) for all \( x \geq 0 \), we also have
\[ \int_0^y f(z) \, dz \geq \frac{p - 2k}{p^2} y f(y) \text{ for } y < 0. \]

Hence, \( (H_3) (i) \) holds when \( d_2 = \frac{p^2}{p - 2k} > 0 \).

By the inequality \( \ln (1 + x) \leq x \) for all \( x \geq 0 \), we also have
\[ \int_0^y f(z) \, dz \leq \frac{1}{p} y f(y) = \frac{1}{p} \|y\|^p \ln^{k} \left( y^2 + e \right) = \frac{1}{p} \|y\|^p \left[ 1 + \ln \left( 1 + \frac{y^2}{e} \right) \right] \leq \frac{1}{p} \|y\|^p \left( 1 + \frac{y^2}{e} \right)^k. \]

Using the inequality \( (a + b)^r \leq 2^{r-1} (a^r + b^r) \), for all \( r > 1 \) and \( a, b \geq 0 \), we deduce that
\[ \int_0^y f(z) \, dz \leq \frac{1}{p} \|y\|^p \left( 1 + \frac{y^2}{e} \right)^k \leq \frac{2^{k-1}}{p} \left( |y|^p + |y|^{p+2k} \right) \text{ for all } y \in \mathbb{R}. \]

Thus, \( (H_3) \) holds when \( \alpha = p > 2, \beta = p + 2k > 2 \) and \( d_2 = \frac{2^{k-1}}{p} > 0 \).

First, we construct the following Lyapunov functional
\[ L(t) = E(t) + \delta \Psi(t), \tag{5.2} \]

where \( \delta > 0 \) is chosen later and
\[ E(t) = \frac{1}{2} \|u'(t)\|_0^2 + \frac{1}{2} \int_0^{\|u_{(t)}\|_0^2} \mu(z) \, dz - \int_0^1 x \, dx \int_0^{\omega(x,t)} f(z) \, dz \]
\[ = \frac{1}{2} \|u'(t)\|_0^2 + \left( \frac{1}{2} - \frac{1}{p} \right) \int_0^{\|u_{(t)}\|_0^2} \mu(z) \, dz + \frac{1}{p} I(t), \tag{5.3} \]

\[ \Psi(t) = \langle u'(t), u(t) \rangle + \frac{1}{2} \|u(t)\|_0^2, \tag{5.4} \]

where
\[ I(t) = I(u(t)) = \int_0^{\|u_{(t)}\|_0^2} \mu(z) \, dz - \int_0^1 x \, dx \int_0^{\omega(x,t)} f(z) \, dz. \tag{5.5} \]

Then we have the following theorem.

**Theorem 5.1.** Assume that \((H'_2), (H_3), (H_4)\) hold. Let \((\bar{u}_0, \bar{u}_1) \in (H^1_0 \cap H^2) \times H^1_0\) such that \( I(0) > 0 \) and the initial energy \( E(0) \) satisfy
\[ \eta^* = \mu_* - p d_2 (1 - \rho) \left[ \sqrt{\left( \frac{1 - \rho}{\rho} \right)^2 R^{\alpha-2} + \sqrt{\left( \frac{1 - \rho}{\rho} \right)^2 R^{\beta-2}}} \right] > 0, \tag{5.6} \]
where
\[ R_\varepsilon = \left( \frac{2pE_\varepsilon}{(p-2)\mu_\varepsilon} \right)^{1/2}, \quad E_\varepsilon = \left( E(0) + \frac{1}{2} \rho_\varepsilon \right) \exp (\rho_\varepsilon), \quad \rho_\varepsilon = ||F||_{L^1([\varepsilon])}. \]

Let \( \mu_{\text{max}} = \max_{0 \leq z \leq R_\varepsilon^2} \mu (z) < \frac{p \mu_\varepsilon}{2 \varepsilon^2} + \eta' \), with \( \mu_\varepsilon, d_2 \) as in \((H_2'), (H_3')(i)\).

Then, there exist positive constants \( \bar{C}, \bar{\gamma} \) such that
\[ \|u'(t)\|^2_0 + \|u_s(t)\|^2_0 \leq C \exp(-\bar{\gamma}t), \text{ for all } t \geq 0. \]  

(5.7)

**Proof of Theorem 5.1.**

First, we need the following lemmas.

**Lemma 5.2.** The energy functional \( E(t) \) defined by (5.3) satisfies

(i) \[ E'(t) \leq \frac{1}{2} ||F(t)||_0 + \frac{1}{2} \| F(t) \|_0 \| u'(t) \|^2_0, \]

(ii) \[ E'(t) \leq - \left( \lambda - \frac{\varepsilon_1}{2} \right) \| u'(t) \|^2_0 + \frac{1}{2} \| F(t) \|^2_0, \]

for all \( \varepsilon_1 > 0 \).

**Proof of Lemma 5.2.** Multiplying (5.1) by \( xu'(x, t) \) and integrating over \([\rho, 1]\), we get
\[ E'(t) = -\lambda \| u'(t) \|^2_0 + \langle F(t), u'(t) \rangle. \]  

(5.9)

On the other hand
\[ \langle F(t), u'(t) \rangle \leq \frac{1}{2} || F(t) ||_0 + \frac{1}{2} \| F(t) \|_0 \| u'(t) \|^2_0. \]  

(5.10)

It follows from (5.9) and (5.10) that (5.8) holds.

Similarly,
\[ \langle F(t), u'(t) \rangle \leq \frac{1}{2} \varepsilon_1 \| u'(t) \|^2_0, \text{ for all } \varepsilon_1 > 0. \]  

(5.11)

It follows from (5.9) and (5.11) that (5.8) holds.

Lemma 5.2 is proved completely. \( \square \)

**Lemma 5.3.** Assume that \((H_2'), (H_3'), (H_4)\) hold. Let \( I(0) > 0 \) and (5.6) hold. Then \( I(t) > 0, \forall t \geq 0 \).

**Proof of Lemma 5.3.** By the continuity of \( I(t) \) and \( I(0) > 0 \), there exists \( T_1 > 0 \) such that
\[ I(t) = I(u(t)) \geq 0, \forall t \in [0, T_1], \]

(5.12)

this implies
\[ E(t) \geq \frac{1}{2} \| u'(t) \|^2_0 + \frac{1}{2} \left( \frac{1}{p} - \frac{1}{p} \right) \mu(z) \int_0^{\|u(t)\|_0^2} \mu(z) dz \]
\[ \geq \frac{1}{2} \| u'(t) \|^2_0 + \frac{(p-2)\mu_\varepsilon}{2p} \| u_s(t) \|^2_0, \forall t \in [0, T_1]. \]  

(5.13)

Combining (5.8), (5.13) and using Gronwall’s inequality to obtain
\[ \| u_s(t) \|^2_0 \leq \frac{2p}{(p-2)\mu_\varepsilon} E(t) \leq \frac{2pE_\varepsilon}{(p-2)\mu_\varepsilon} \equiv R_\varepsilon^2, \forall t \in [0, T_1]. \]

(5.14)
where $E$, as in (5.6).

Hence, it follows from $(\overline{H}_3)(ii)$ and (5.14) that

$$
p \int_\rho^1 x dx \int_0^{\varphi(x,t)} f(z) dz \leq pd_2 \left( \|u(t)\|_{L^2}^\beta + \|u(t)\|_{L^p}^\beta \right) \leq pd_2 (1 - \rho) \left( \sqrt{\frac{1 - \rho}{\rho}} \|u_\varepsilon(t)\|_{L^2}^\beta + \sqrt{\frac{1 - \rho}{\rho}} \|u_\varepsilon(t)\|_{L^p}^\beta \right) \leq pd_2 (1 - \rho) \left( \sqrt{\frac{1 - \rho}{\rho}} R_{t}^{\beta - 2} + \sqrt{\frac{1 - \rho}{\rho}} R_{t}^{\beta - 2} \right) \|u_\varepsilon(t)\|_{L^2}^\beta.
$$

Therefore, $I(t) \geq \eta^* \|u_\varepsilon(t)\|_{L^2}^\beta > 0$, $\forall t \in [0, \tilde{T}_1]$, where $\eta^*$ as in (5.6).

Now, we put $T_\infty = \sup \{ T > 0 : I(t) > 0, \forall t \in [0, T] \}$. If $T_\infty < +\infty$, then by the continuity of $I(t)$, we have $I(T_\infty) \geq 0$. By the same arguments as above, we can deduce that there exists $\tilde{T}_\infty > T_\infty$ such that $I(t) > 0$, $\forall t \in [0, T_\infty]$. Hence, we conclude that $I(t) > 0$, $\forall t \geq 0$.

Lemma 5.3 is proved completely. □

**Lemma 5.4.** Assume that $(H_3^*)$, $(\overline{H}_3)$, $(\overline{H}_4)$ hold. Let $I(0) > 0$ and (5.6) hold. Put

$$
E_1(t) = \|u'(t)\|_{L^2}^\beta + \int_0^{\varphi_{u(t)}(t)} \mu(z) dz + I(t).
$$

Then there exist the positive constants $\tilde{\beta}_1, \tilde{\beta}_2$ such that

$$
\tilde{\beta}_1 E_1(t) \leq \mathcal{L}(t) \leq \tilde{\beta}_2 E_1(t), \ \forall t \geq 0,
$$

for $\delta$ is small enough.

**Proof of Lemma 5.4.** It is easy to see that

$$
\mathcal{L}(t) = \frac{1}{2} \|u'(t)\|_{L^2}^2 + \left( \frac{1}{2} - \frac{1}{p} \right) \int_0^{\varphi_{u(t)}(t)} \mu(z) dz + \frac{1}{p} I(t) + \delta(u'(t), u(t)) + \frac{\delta \lambda}{2} \|u(t)\|_{L^2}^2.
$$

From the following inequalities

$$
\delta(u'(t), u(t)) \leq \frac{1}{2} \delta \|u'(t)\|_{L^2}^2 + \frac{1}{2} \delta \|u(t)\|_{L^2}^2
$$

$$
\int_0^{\varphi_{u(t)}(t)} \mu(z) dz \geq \mu_*\|u_\varepsilon(t)\|_{L^2}^2,
$$

it follows that

$$
\mathcal{L}(t) \leq \frac{1}{2} \|u'(t)\|_{L^2}^2 + \left( \frac{1}{2} - \frac{1}{p} \right) \int_0^{\varphi_{u(t)}(t)} \mu(z) dz + \frac{1}{p} I(t) + \delta(u'(t), u(t)) + \frac{\delta \lambda}{2} \|u(t)\|_{L^2}^2 + \left( \frac{1}{2} - \frac{1}{p} \right) \int_0^{\varphi_{u(t)}(t)} \mu(z) dz + \frac{1}{p} I(t) + \frac{\delta \lambda}{2} \|u(t)\|_{L^2}^2 + \frac{1}{2} \delta \|u'(t)\|_{L^2}^2 + \frac{1}{2} \delta \|u(t)\|_{L^2}^2 + \frac{1}{2} \delta \|u'(t)\|_{L^2}^2 + \frac{1}{2} \delta \|u(t)\|_{L^2}^2.
$$

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we deduce that
\[
\mathcal{L}(t) \geq \frac{1}{2} \|u'(t)\|_{t_0}^2 + \left(\frac{1}{2} - \frac{1}{p}\right) \int_{t_0}^{\|u(t)\|_{t_0}} \mu(z)dz + \frac{1}{p} I(t) + \delta(u'(t), u(t))
\]  
(5.20)

\[
\geq \frac{1}{2} \|u'(t)\|_{t_0}^2 + \left(\frac{1}{2} - \frac{1}{p}\right) \int_{t_0}^{\|u(t)\|_{t_0}} \mu(z)dz + \frac{1}{p} I(t) - \frac{1}{2} \delta \|u'(t)\|_{t_0}^2 - \frac{\delta(1 - \rho)^2}{4\rho} \|u(t)\|_{t_0}^2
\]

\[
\geq \frac{1}{2} \|u'(t)\|_{t_0}^2 + \left(\frac{1}{2} - \frac{1}{p}\right) \int_{t_0}^{\|u(t)\|_{t_0}} \mu(z)dz + \frac{1}{p} I(t) - \frac{1}{2} \delta \|u'(t)\|_{t_0}^2 - \frac{\delta(1 - \rho)^2}{4\rho u_x} \int_{t_0}^{\|u(t)\|_{t_0}} \mu(z)dz
\]

\[
= \frac{1 - \delta}{2} \|u'(t)\|_{t_0}^2 + \left(\frac{1}{2} - \frac{1}{p} - \frac{\delta(1 - \rho)^2}{4\rho u_x}\right) \int_{t_0}^{\|u(t)\|_{t_0}} \mu(z)dz + \frac{1}{p} I(t)
\]

\[
\geq \tilde{\beta}_1 E_1(t),
\]

where we choose \(\tilde{\beta}_1 = \min\left\{\frac{1}{2}, \frac{1}{2} - \frac{1}{p} - \frac{\delta(1 - \rho)^2}{4\rho u_x}\right\}\), with \(\delta\) is small enough, \(0 < \delta < \min\{1; \frac{4\rho u_x}{(1 - \rho)^2}\left(\frac{1}{2} - \frac{1}{p}\right)\}\).

Similarly, we can prove that
\[
\mathcal{L}(t) \leq \frac{1}{2} \|u'(t)\|_{t_0}^2 + \left(\frac{1}{2} - \frac{1}{p}\right) \int_{t_0}^{\|u(t)\|_{t_0}} \mu(z)dz + \frac{1}{p} I(t)
\]  
(5.21)

\[
+ \frac{1}{2} \delta \|u'(t)\|_{t_0}^2 + \frac{\delta(1 - \rho)^2}{4\rho} \|u(t)\|_{t_0}^2 + \frac{\delta}{4\rho} \int_{t_0}^{\|u(t)\|_{t_0}} \mu(z)dz
\]

\[
\leq \frac{1 + \delta}{2} \|u'(t)\|_{t_0}^2 + \left(\frac{1}{2} - \frac{1}{p} + \frac{\delta(1 + \lambda)(1 - \rho)^2}{4\rho u_x}\right) \int_{t_0}^{\|u(t)\|_{t_0}} \mu(z)dz + \frac{1}{p} I(t)
\]

\[
\leq \tilde{\beta}_2 E_1(t),
\]

where \(\tilde{\beta}_2 = \max\left\{\frac{1 + \delta}{2}, \frac{1}{2} - \frac{1}{p} + \frac{\delta(1 + \lambda)(1 - \rho)^2}{4\rho u_x}\right\}\).

Lemma 5.4 is proved completely. \(\square\)

**Lemma 5.5.** Assume that (H2), (H3), (H4) hold. Let \(I(0) > 0\) and (5.6) hold. The functional \(\Psi(t)\) defined by (5.4) satisfies
\[
\Psi'(t) \leq \|u'(t)\|_{t_0}^2 - \frac{\delta t d_z}{p} I(t) + \frac{1}{2} \varepsilon_2 \|F(t)\|_{t_0}^2
\]  
(5.22)

\[
- \frac{\delta}{2\etr} \left[\frac{d_z}{p} \left(P u_x + \eta u_x - \mu u\right) - \frac{\delta t d_z \eta' - \varepsilon_2 (1 - \rho)^2}{4\rho}\right] \int_{t_0}^{\|u(t)\|_{t_0}} \mu(z)dz,
\]

for all \(\varepsilon_2 > 0\), \(\delta_1 \in (0, 1)\).
Proof of Lemma 5.5. By multiplying (5.1) by \( xu(x, t) \) and integrating over \([\rho, 1]\), we obtain
\[
\Psi'(t) = ||u'(t)||^2_0 - ||u_x(t)||^2_0 \mu \left( ||u_x(t)||^2_0 \right) + (f(u(t)), u(t)) + (F(t), u(t)).
\] (5.23)

By the following inequalities
\[
-d ||u_x(t)||^2_0 \mu \left( ||u_x(t)||^2_0 \right) \leq -\mu_\epsilon ||u_x(t)||^2_0,
\] (5.24)
\[
\langle f(u(t)), u(t) \rangle \leq d_2 \int_0^1 x dx \int_0^{\mu(x,t)} f(z) dz
\]
\[
= \frac{d_2}{p} \int_0^{||u_x(t)||^2_0} \mu(z) dz - I(t)
\]
\[
I(t) \geq \eta ||u_x(t)||^2_0,
\]
\[
\langle F(t), u(t) \rangle \leq \frac{\varepsilon_2}{2} ||u(t)||^2_0 + \frac{1}{2\varepsilon_2} ||F(t)||^2_0
\]
\[
\leq \varepsilon_2 \left( \frac{(1 - \rho)^2}{4p} ||u_x(t)||^2_0 + \frac{1}{2\varepsilon_2} ||F(t)||^2_0 \right), \forall \varepsilon_2 > 0,
\]
we deduce that
\[
\Psi'(t) = ||u'(t)||^2_0 - ||u_x(t)||^2_0 \mu \left( ||u_x(t)||^2_0 \right) + (f(u(t)), u(t)) + (F(t), u(t))
\] (5.25)
\[
\leq ||u'(t)||^2_0 - \mu_\epsilon ||u_x(t)||^2_0 + \frac{d_2}{p} \int_0^{||u_x(t)||^2_0} \mu(z) dz - I(t)
\]
\[
+ \varepsilon_2 \left( \frac{(1 - \rho)^2}{4p} ||u_x(t)||^2_0 + \frac{1}{2\varepsilon_2} ||F(t)||^2_0 \right)
\]
\[
= ||u'(t)||^2_0 - \left( \mu_\epsilon - \varepsilon_2 \left( \frac{(1 - \rho)^2}{4p} \right) \right) ||u_x(t)||^2_0 + \frac{d_2}{p} \int_0^{||u_x(t)||^2_0} \mu(z) dz
\]
\[
- \frac{\delta_1d_2}{p} I(t) - \left( \frac{(1 - \delta_1)d_2}{p} \right) ||u_x(t)||^2_0 + \frac{1}{2\varepsilon_2} ||F(t)||^2_0
\]
\[
\leq ||u'(t)||^2_0 - \left( \mu_\epsilon - \varepsilon_2 \left( \frac{(1 - \rho)^2}{4p} \right) \right) ||u_x(t)||^2_0 + \frac{d_2}{p} \int_0^{||u_x(t)||^2_0} \mu(z) dz
\]
\[
- \frac{\delta_1d_2}{p} I(t) - \left( \frac{(1 - \delta_1)d_2}{p} \right) \eta^* ||u_x(t)||^2_0 + \frac{1}{2\varepsilon_2} ||F(t)||^2_0
\]
\[
= ||u'(t)||^2_0 - \left( \mu_\epsilon + \frac{(1 - \delta_1)d_2}{p} \eta^* - \varepsilon_2 \left( \frac{(1 - \rho)^2}{4p} \right) \right) ||u_x(t)||^2_0
\]
\[
+ \frac{d_2}{p} \int_0^{||u_x(t)||^2_0} \mu(z) dz - \frac{\delta_1d_2}{p} I(t) + \frac{1}{2\varepsilon_2} ||F(t)||^2_0
\]
\[
\leq ||u'(t)||^2_0 - \left( \mu_\epsilon + \frac{(1 - \delta_1)d_2}{p} \eta^* - \varepsilon_2 \left( \frac{(1 - \rho)^2}{4p} \right) \right) \left( \frac{1}{\mu_{\max}} \right) \int_0^{||u_x(t)||^2_0} \mu(z) dz
\]
\[
+ \frac{d_2}{p} \int_0^{||u_x(t)||^2_0} \mu(z) dz - \frac{\delta_1d_2}{p} I(t) + \frac{1}{2\varepsilon_2} ||F(t)||^2_0
\]
\[
= ||u'(t)||^2_0 - \frac{1}{\mu_{\max}} \left( \frac{d_2}{p} \left( \frac{p \mu_\epsilon}{d_2} + \eta^* - \mu_{\max} \right) - \frac{\delta_1d_2 \eta^*}{p} - \varepsilon_2 \left( \frac{(1 - \rho)^2}{4p} \right) \right) \int_0^{||u_x(t)||^2_0} \mu(z) dz
\]
\[
- \frac{\delta_1d_2}{p} I(t) + \frac{1}{2\varepsilon_2} ||F(t)||^2_0.
\]
Hence, the lemma 5.5 is proved by using some simple estimates. □

Now we continue to prove Theorem 5.1.

Then, we deduce from (5.2), (5.8)(ii) and (5.22) that

\[
\mathcal{L}'(t) \leq - \left( \lambda - \frac{\varepsilon_1}{2} - \delta \right) \|u'(t)\|_0^2 - \frac{\delta_1 d_2}{p} I(t) + \left( \frac{1}{2} + \delta \right) \|F(t)\|_0^2 - \delta \frac{d_2}{p} \left( \frac{\mu_0}{d_2} + \eta' - \mu'_{\max} \right) - \frac{\delta_1 d_2 \eta'}{p} - \varepsilon_2 \left( 1 - \rho \right)^2 / 4
\]

(5.26)

for all \( \delta, \varepsilon_1, \varepsilon_2 > 0, \delta_1 \in (0, 1). \)

Because of \( \mu'_{\max} < \frac{\mu_0}{d_2} + \eta' \) and

\[
\lim_{\delta_1 \to 0^+, \varepsilon_1 \to 0} \left[ \frac{d_2}{p} \left( \frac{\mu_0}{d_2} + \eta' - \mu'_{\max} \right) - \frac{\delta_1 d_2 \eta'}{p} - \varepsilon_2 \left( 1 - \rho \right)^2 / 4 \right] = \frac{d_2}{p} \left( \frac{\mu_0}{d_2} + \eta' - \mu'_{\max} \right) > 0,
\]

we can choose \( \delta_1 \in (0, 1) \) and \( \varepsilon_2 > 0 \) such that

\[
\theta_1 = \frac{d_2}{p} \left( \frac{\mu_0}{d_2} + \eta' - \mu'_{\max} \right) - \frac{\delta_1 d_2 \eta'}{p} - \varepsilon_2 \left( 1 - \rho \right)^2 / 4 > 0.
\]

(5.27)

Then, for \( \varepsilon_1 \) small enough such that \( 0 < \varepsilon_1 / 2 < \lambda \) and if \( \delta > 0 \) such that

\[
\theta_2 = \lambda - \frac{\varepsilon_1}{2} - \delta > 0, \quad 0 < \delta < \min \left\{ 1, \frac{4 \mu_0}{(1 - \rho)^2} \left( \frac{1}{2} - \frac{1}{p} \right) \right\},
\]

(5.28)

it follows from (5.17), (5.26)-(5.28) that

\[
\mathcal{L}'(t) \leq - \tilde{\beta}_3 E_1(t) + \tilde{C}_0 e^{-2\gamma_0 t}
\]

(5.29)

\[
\leq - \frac{\tilde{\beta}_3}{\tilde{\beta}_2} \mathcal{L}(t) + \tilde{C}_0 e^{-2\gamma_0 t} \leq - \gamma \mathcal{L}(t) + \tilde{C}_0 e^{-2\gamma_0 t},
\]

where \( \tilde{\beta}_3 = \min \left\{ \frac{\delta_1}{\mu'_{\max}}, \frac{\delta_1 d_2}{p}, \frac{\delta_1}{\varepsilon_1} \right\}, \quad 0 < \gamma < \min \left\{ \frac{\tilde{\beta}_3}{\tilde{\beta}_0}, \frac{2 \gamma_0}{\rho_0} \right\}, \quad \tilde{C}_0 = \frac{1}{2} \left( \frac{1}{\varepsilon_1} + \frac{\delta_1}{\varepsilon_1} \right) C_0^2.
\]

On the other hand, we have

\[
\mathcal{L}(t) \geq \tilde{\beta}_1 \min \left\{ 1, \mu_0 \right\} (\|u'(t)\|_0^2 + \|u(t)\|_0^2) - \gamma \mathcal{L}(t) + \tilde{C}_0 e^{-2\gamma_0 t},
\]

(5.30)

Combining (5.29) and (5.30) we get (5.7). Theorem 5.1 is proved completely. □

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