On exact multiplicity for a second order equation with radiation boundary conditions

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Abstract

A second order ordinary differential equation with a superlinear term \( g(x, u) \) under radiation boundary conditions is studied. Using a shooting argument, all the results obtained in the previous work [2] for a Painlevé II equation are extended. It is proved that the uniqueness or multiplicity of solutions depend on the interaction between the mapping \( \frac{\partial g}{\partial u}(\cdot, 0) \) and the first eigenvalue of the associated linear operator. Furthermore, two open problems posed in [2] regarding, on the one hand, the existence of sign-changing solutions and, on the other hand, exact multiplicity are solved.

Keywords: Second order ODEs; Radiation boundary conditions; Multiple solutions; Electro-diffusion models.

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1 Introduction

In [2], the following problem arising on a two-ion electro-diffusion model (see [3], [5]) was studied:

\[
    u''(x) = Ku(x)^3 + L(x)u(x) + A
\]

with

\[
    u'(0) = a_0 u(0), \quad u'(1) = a_1 u(1).
\]

Here, \( K \) and \( A \) some given positive constants and \( L(x) := a_0^2 + (a_1^2 - a_0^2)x \).

Unlike the standard Robin condition, both coefficients \( a_0 \) and \( a_1 \) in the radiation boundary condition [2] are assumed to be positive.
It was proven that the problem has a negative solution; moreover, if $a_1 \leq a_0$ then there are no other solutions. When $a_1 > a_0$, the solution is still unique for $A \gg 0$ but, if $A$ is sufficiently small, then the problem has at least three solutions. Numerical evidence in [2] suggests that the number of solutions cannot be arbitrarily large and it was proven that, indeed, there exists exactly one negative solution, at most two positive solutions and that the set of solutions is bounded. It was conjectured that the maximum number of solutions is 3 (typically, one of them negative and the other two positive) but, however, none of the results in [2] prevents against the existence of many sign-changing solutions.

In this work, we study a generalization of the previous problem, namely the equation

$$u''(x) = g(x, u(x)) + p(x)$$

where $p \in C([0, 1])$ and $g : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, of class $C^1$ with respect to $u$ and superlinear, that is:

$$\lim_{|u| \rightarrow +\infty} \frac{g(x, u)}{u} = +\infty$$

uniformly in $x \in [0, 1]$. Without loss of generality, we shall assume that $g(x, 0) = 0$ for all $x \in [0, 1]$. As before, we look for those solutions satisfying the radiation boundary condition [2] with $a_0, a_1 > 0$. In the spirit of [2], we shall assume throughout the paper that

$$g \text{ is strictly increasing in } u$$

and

$$p(x) > 0 \text{ for all } x.$$  

For general multiplicity results avoiding conditions [5] and [6] see [1]. In the present setting, we shall demonstrate that all the results in [2] can be retrieved in a simple manner; furthermore, we shall give an answer to two questions that were left open. Specifically, it shall be seen that the set of solutions is bounded and contains always a negative solution, which tends uniformly to $-\infty$ as $p \rightarrow +\infty$ uniformly. Moreover, we shall extend the uniqueness statement in [2] by imposing the condition that $-\frac{\partial g}{\partial u}(x, u)$ is smaller than the first eigenvalue $\lambda_1$ of the associated linear operator for all $u$. Under a weaker condition, it shall be proved that uniqueness holds also if $p$ is large. As a complement of the uniqueness results, we shall also prove that if $-\frac{\partial g}{\partial u}(x, 0)$ lies below $\lambda_1$ then the problem has at least three solutions, provided that $\|p\|_{\infty}$ is small. Furthermore, under an extra condition, which is fulfilled in [1], the multiplicity result is sharp. This extends the corresponding result for the particular problem [1] and gives an answer to a question, sustained by numerical evidence but not proven in [2]:

**Theorem 1.1** Assume that (4), (5) and (6) hold. Then (3)-(2) has a negative solution. Moreover, if

$$\frac{\partial g}{\partial u}(\cdot, 0) \leq -\lambda_1$$

(7)
then there exists a constant $p_1 > 0$ such that problem (3)-(2) has at least three solutions, one of them negative, one of them positive and another one sign-changing, when $\|p\|_\infty < p_1$. If furthermore
\[
\frac{\partial g}{\partial u}(x,u) > \frac{g(x,u)}{u}
\]
for all $u \neq 0$ and all $x$, then (3)-(2) has no other solutions, provided that $p_1$ is small enough.

It follows that, under the previous assumptions, the number of solutions moves from 3 to 1 as $\|p\|_\infty$ gets large. Similarly, for each fixed $p$, if we take $\alpha_1$ as a parameter then uniqueness or multiplicity of solutions vary according to its different values. In general terms, multiplicity arises when $\alpha_1$ is sufficiently large and should not be expected if $\alpha_1$ is small. More precisely:

**Theorem 1.2** Assume that (4), (5) and (6) hold. Then there exist constants $a^* > a_*> 0$ such that:

1. If $a_1 > a^*$ then problem (3)-(2) has at least three solutions, one of them negative and another one sign-changing.
2. If $0 < a_1 < a^*$ then problem (3)-(2) has a unique (negative) solution.

The paper is organized as follows. Section 2 is devoted to present several general aspects of the problem and state uniqueness and related results. In section 3, we define a shooting-type operator that will be used to derive the proofs of Theorems 1.1 and 1.2. Some open questions are briefly exposed in a last section.

## 2 Uniqueness and related results

This section is devoted to introduce general results concerning problem (3)-(2) that shall be used in the proofs of the main results. In the first place, we observe that solutions are bounded:

**Theorem 2.1** Assume that (4) holds. Then there exists a constant $C$ such that every solution $u$ of (3)-(2) satisfies $\|u\|_{C^2} \leq C$.

**Proof:** Let $u$ be a solution. Multiply the equation by $u$ and integrate to obtain
\[
a_1u(1)^2 - a_0u(0)^2 = \int_0^1 [u'(x)^2 + g(x,u(x))u(x) + p(x)u(x)] \, dx.
\]
Setting $\varphi(t) := [(a_1 - a_0)t + a_0]u(t)^2$, it is seen that
\[
a_1u(1)^2 - a_0u(0)^2 = \int_0^1 \varphi'(x) \, dx \leq \frac{C}{\varepsilon} \|u\|_{L^2}^2 + \varepsilon \|u'\|_{L^2}^2.
\]
for arbitrary \( \varepsilon > 0 \) and \( C \) depending on \( \varepsilon, a_1 \) and \( b_1 \). Choose for example \( \varepsilon = \frac{1}{2} \) and set \( M > 2C + \frac{1}{2} \), then by superlinearity there exists a constant \( K \) (depending only on \( M \) and \( \|A\|_{L^2} \)) such that
\[
\frac{1}{2} \|u'\|_{L^2}^2 + 2C\|u\|_{L^2}^2 \geq \|u'\|_{L^2}^2 + M\|u\|_{L^2}^2 - K.
\]
This implies \( \|u\|_{\infty} \leq \|u\|_{H^1} \leq \sqrt{2K} \) and the proof follows using (3).

Next, we may state an uniqueness result in terms of the first eigenvalue \( \lambda_1 \) of the (self-adjoint) linear operator \(-u''\) under the boundary conditions (2). To this end, let us simply recall that, by the standard Sturm-Liouville theory, \( \lambda_1 \) can be computed as the minimum of \(-\int_0^1 u'' u \, dx\) over all the smooth functions satisfying (2) such that \( \|u\|_{L^2} = 1 \).

Theorem 2.2 Assume there exists an interval \( I \subset \mathbb{R} \) such that, for all \( u \in I \),
\[
\frac{\partial g}{\partial u}(x, u) \geq -\lambda_1 \quad \text{for all } x \in [0, 1]
\]
and the inequality is strict for some \( x \) independent of \( u \). Then (3)-(2) has at most one solution \( u \) such that \( u(x) \in I \) for all \( x \).

Proof: Let \( u_1, u_2 : [0, 1] \to I \) be solutions of (3)-(2) and define \( w := u_1 - u_2 \), then \( w \) satisfies the boundary condition and
\[
w''(x) = g(x, u_1(x)) - g(x, u_2(x)) = \frac{\partial g}{\partial u}(x, \xi(x))w(x)
\]
for some \( \xi(x) \) between \( u_1(x) \) and \( u_2(x) \). Fix an open interval \( J \neq \emptyset \) such that \( \frac{\partial g}{\partial u}(x, \xi(x)) > -\lambda_1 \) for \( x \in J \) and suppose \( w \neq 0 \) in \( J \), then
\[
0 = \int_0^1 \left(w''w - \frac{\partial g}{\partial u}(x, \xi(x))w^2 \right) \, dx < \int_0^1 (w''w + \lambda_1 w^2) \, dx \leq 0,
\]
because \( \lambda_1 \) is the first eigenvalue. This contradiction proves that \( w \equiv 0 \) over \( J \) and consequently \( w \equiv 0 \).

Remark 2.3 As shown in [7], \( \lambda_1 \) is a strictly decreasing continuous function of \( a_1 \) and, moreover, \( \lambda_1 \geq 0 \) if and only if \( a_1 \leq -\frac{a_0}{a_1 + 1} \). In particular, when (7) holds, the latter inequality is a sufficient condition for uniqueness. However, it was proved in [2], the (sharp) condition for uniqueness in the particular case (1) is weaker, namely: \( a_1 \leq a_0 \). This is due to fact that, in this specific case, it is verified that \( \lambda_1 \geq -a_1^2 \) and hence
\[
-\lambda_1 \leq a_1^2 \leq L(x) + 3Ku^2 = \frac{\partial g}{\partial u}(x, u).
\]
The next result shows that the failure of \(8\) does not necessarily imply multiplicity: this fact was already observed in \(2\) where, as mentioned, it was proven the solution of \(1\)-\(2\) is unique also when \(A\) is large. The latter property can be easily deduced in the general case with \(p \equiv A\) from the next two results. The first of them establishes that, for \(p\) large, solutions are negative; the second one proves that, under suitable assumptions, there cannot be two solutions with the same sign.

**Theorem 2.4** Let \(4\) hold. Then there exists \(p_0\) such that, if \(p(x) \geq p_0\) for all \(x \in [0,1]\), then all the solutions of \(4\)-\(3\) are negative.

**Proof:** Due to the superlinearity of \(g\), for each \(M \geq 0\) we may define the quantity

\[ N_M := \inf_{x \in [0,1], u \geq 0} \{ g(x, u) - Mu \} > -\infty. \]

Then

\[ g(x, u) \geq Mu + N_M \tag{9} \]

for all \(u \geq 0\). Let \(M > 0\) to be determined, fix \(p_0 > -N_M\) and let \(u\) be a solution of \(3\)-\(2\) such that \(u(x) \geq 0\) for some \(x \in [0,1]\). In view of \(6\), the inequality \(u''(x) \geq g(x, u(x)) + p_0\) implies that

\[ u''(x) > Mu(x) \tag{10} \]

whenever \(u(x) \geq 0\). We deduce that, if \(x_0 \in [0,1]\) is such that \(u(x_0)\) and \(u'(x_0)\) are nonnegative, then \(u(x)\) and \(u'(x)\) are strictly positive for \(x > x_0\). Multiply \(10\) by \(u'\) and integrate to obtain, for \(x > x_0\):

\[ u'(x)^2 > u'(x_0)^2 + M(u(x)^2 - u(x_0)^2). \tag{11} \]

If \(u(0) > 0\), then \(u'(0) > 0\) and

\[ u(1)^2 - u(0)^2 = \int_0^1 2u(x)u'(x)dx > 2a_0u(0)^2. \]

Thus,

\[ u(1)^2 - u(0)^2 > \frac{2a_0}{1 + 2a_0}u(1)^2 \tag{12} \]

and fixing \(M = a_1^2 \frac{1 + 2a_0}{2a_0}\) we obtain, from \(11\) and \(12\):

\[ a_1^2u(1)^2 > M\frac{2a_0}{1 + 2a_0}u(1)^2 = a_1^2u(1)^2. \]

This contradiction proves that there are no positive solutions when \(p_0 > -N_M\).

On the other hand, if \(u(0) \leq 0\) then \(u\) vanishes at a (unique) value \(x_0\), with \(u'(x_0) \geq 0\). Fix \(M = a_1^2\), then \(11\) yields

\[ a_1^2u(1)^2 = u'(1)^2 > u'(x_0)^2 + a_1^2u(1)^2 \geq a_1^2u(1)^2, \]

a contradiction. \(\square\)
Theorem 2.5 Assume there exists an interval \( I \subset \mathbb{R} \neq 0 \) such that
\[
\frac{\partial g}{\partial u}(x, u) > \frac{g(x,u)+p(x)}{u}
\] holds for all \( x \in [0,1] \) and \( u \in I \). Then there exists at most one solution \( u \) of (2)-(3) such that \( u(x) \in I \) for all \( x \).

Proof: Let \( u_1, u_2 : [0,1] \to I \) be two different solutions, then \( u_1(0) \neq u_2(0) \).

Suppose for example that \( u_1 < u_2 \) over \( [0,x_0) \), then
\[
u''_1(x) = \frac{g(x,u_1(x)) + p(x)}{u_1(x)} u_1(x) < \frac{g(x,u_2(x)) + p(x)}{u_2(x)} u_1(x)
\]
and hence
\[
u''_1(x) u_2(x) > u_1(x) u''_2(x) \quad x < x_0.
\]
We conclude that
\[
u'_1(x_0) u_2(x_0) > u_1(x_0) u'_2(x_0),
\]
along with contradiction yields if \( x_0 = 1 \). Thus, we may suppose that \( u_1 \) and \( u_2 \) meet for the first time at \( x_0 \), then \( u_1(x_0) = u_2(x_0) \) and \( u'_1(x_0) \geq u'_2(x_0) \). Again, this contradicts (14).

Remark 2.6 Condition (13) implies that the function \( \frac{g(x,u)+p(x)}{u} \) increases in \( u \) when \( I \subset \mathbb{R}^+ \) and decreases when \( I \subset \mathbb{R}^- \). Moreover, if \( 0 \in \partial I \) then \( sp \leq 0 \), where \( s \) denotes the sign of the elements of \( I \). In particular, if the condition holds for all \( u \neq 0 \), then \( p = 0 \). This case is well known in the literature (see e.g. [4]) and implies that if \( u_0 \neq 0 \) is a critical point of the associated functional \( J \), then \( u_0 \) is transversal to the Nehari manifold introduced after the pioneering work [6], namely:
\[
N := \{ u \in H^1(0,1) \setminus \{0\} : DJ(u)(u) = 0 \}.
\]

Indeed, setting \( I(u) := DJ(u)(u) \) it is readily seen that \( T_u N = ker(DI(u_0)) \) and \( DI(u_0)(u_0) > 0 \). For the particular case of problem [4], condition (13) simply reads \( \frac{A}{u^3} < 2K \), so the previous result applies with \( I = (-\infty,0) \) and \( I = \left( \sqrt{\frac{A}{2K}} , +\infty \right) \).

The next theorem generalizes another result from [2], concerning the behaviour of the solutions as \( p \) increases. We know that all solutions are negative if \( p \geq p_0 \gg 0 \) and it is readily verified (e.g. by the method of upper and lower solutions) that a solution always exists; however, if the assumptions of Theorem 2.2 or Theorem 2.5 are not satisfied, then there might be more than one negative solution. As we shall see, all possible solutions tend uniformly to \( -\infty \) as \( p \) tends uniformly to \( +\infty \). In order to emphasize the dependence on \( p \), any solution shall be denoted \( u_p \), despite the fact that it might not be unique.
Theorem 2.7 Assume that (4) holds and let $u_p$ be a solution of (3)-(2). Then $u_p \to -\infty$ uniformly when $p \to +\infty$ uniformly.

Proof: Let $p \geq p_0$ for some large constant $p_0$. From Theorem 2.4, we may assume $u_p < 0$. Fix $x_p$ such that $\max_{x \in [0,1]} u_p(x) = u_p(x_p)$, then $x_p < 1$. Suppose $u_p(x_p) > -M$ and fix $p_0$ large enough, such that,

$$g(x,u) + p_0 > Ma_0, \quad \text{for all } u \geq -(1 + a_0)M.$$  \hfill (15)

It follows that $x_p = 0$. Consider the maximum value $\delta \leq 1$ such that $u_p''(x) \geq 0$ for all $x \in [0,\delta]$, then $u_p'(0) = a_0u_p(0) > -Ma_0$, for $x \leq \delta$. Hence, $u_p(\delta) > u_p(0) - \delta Ma_0 \geq -M(1 + a_0)$ and by (15) we conclude that $u_p''(\delta) > 0$. Thus, $\delta = 1$ and, in particular, $u_p(x) > -M(1 + a_0)$. Using (15) again, it follows that $u_p''(1) > Ma_0$ for all $x$. Then $u_p'(1) > u_p'(0) + Ma_0 > 0$, a contradiction. 

Combining the previous result with Theorems 2.2 and 2.5 we deduce that, in fact, the solution is typically unique when $p$ is large. Indeed, due to superlinearity we observe that, on the one hand, $\frac{\partial g}{\partial u}(x,u)$ cannot remain bounded from above as $u \to -\infty$ and, on the other hand, the function $\frac{g(x,u)}{u}$ cannot be increasing in $u$ over any interval $(-\infty, C)$. In other words, it is reasonable to expect that either condition (8) holds or $\frac{\partial g}{\partial u}(x,u) - \frac{g(x,u)}{u} \geq k$ when $u \ll 0$. Any of these conditions, which are fulfilled in the particular case (1), ensures the applicability of Theorems 2.2 or 2.5 when $p$ is large. Thus, the following corollary is obtained:

Corollary 2.8 Assume that (4) holds. Moreover, assume there exists $C \leq 0$ such that one of the following conditions holds:

1. Condition (8) holds for all $u \leq C$,

\[ \sup_{x \in [0,1], u \leq C} u \frac{\partial g}{\partial u}(x,u) - g(x,u) < +\infty. \]

Then there exists $p_0$ such that problem (3)-(2) has a unique solution, which is negative, for all $p \geq p_0$.

Proof: From Theorem 2.7 there exists $\tilde{p}$ such that if $u$ is a solution for $p \geq \tilde{p}$ then $u(x) \leq C$ for all $x$. If the first condition holds, then the proof follows directly from Theorem 2.2. Otherwise, there exists a constant $M$ such that

\[ \frac{\partial g}{\partial u}(x,u) - \frac{g(x,u)}{u} > \frac{M}{u} \]

for all $u \leq C$ and, by Theorem 2.5, the result follows taking $p_0$ as the maximum value between $\tilde{p}$ and $M$. □
3 A shooting operator for problem (3)-(2)

This section is devoted to proof Theorems 1.1 and 1.2 by means of a shooting-type operator. To this end, let us firstly state the following lemma, which ensures that, if (5) holds, then the graphs of two different solutions of (3) with initial condition \( u'(0) = a_0 u(0) \) do not intersect. More generally,

**Lemma 3.1** Let \( u_1 \) and \( u_2 \) be solutions of (3) defined over an interval \([0, b]\) such that \( u_1(0) > u_2(0) \) and \( u'_1(0) > u'_2(0) \) and assume that (5) holds. Then \( u_1 > u_2 \) and \( u'_1 > u'_2 \) on \([0, b]\).

**Proof:** Set \( u(x) = u_1(x) - u_2(x) \), then \( u''(x) = \theta(x)u(x) \) on \([0, b]\), where \( \theta(x) := \frac{\partial g}{\partial u}(x, \xi(x)) > 0 \). Thus, the result follows since \( u(0), u'(0) > 0 \). \( \square \)

Next, we define our shooting operator as follows. For each fixed \( \lambda \in \mathbb{R} \), let \( u_\lambda \) be the unique solution of problem

\[
\begin{aligned}
\left\{ \begin{array}{l}
  u''(x) = g(x, u(x)) + p(x) \\
  u(0) = \lambda, \quad u'(0) = a_0 \lambda
\end{array} \right.
\]

and define the function \( T : D \rightarrow \mathbb{R} \), by

\[
T(\lambda) = \frac{u'_\lambda(1)}{u_\lambda(1)},
\]

where \( D \subset \mathbb{R} \) is the set of values of \( \lambda \) such that the corresponding solution \( u_\lambda \) of (16) is defined on \([0, 1]\), with \( u_\lambda(1) \neq 0 \). Thus, solutions of (3)-(2) that do not vanish on \( x = 1 \) can be characterized as the functions \( u_\lambda \), where \( \lambda \in D \) is such that \( T(\lambda) = a_1 \). By continuity arguments, it is easy to verify that, for each \( s \in \mathbb{R} \), there exists \( \lambda \) such that \( u_\lambda(1) = s \). By Lemma 3.1, this value of \( \lambda \) is unique; in particular, there exists a unique \( \lambda_0 \) such that \( u_{\lambda_0} = 0 \). Thus, we conclude that

\[
D = (\lambda_*, \lambda_0) \cup (\lambda_0, \lambda^*)
\]

for some \( \lambda_* \geq -\infty \) and \( \lambda^* \leq +\infty \).

From (6), it follows that \( \lambda_0 < 0 \) and, furthermore: if \( \lambda > 0 \) then \( u_\lambda \) is positive and if \( \lambda_0 \leq \lambda \leq 0 \) then \( u_\lambda \) vanishes exactly once in \([0, 1]\). In particular, \( u_{\lambda_0} < 0 \) in \([0, 1]\) and, since \( u''_{\lambda_0}(x) > 0 \) when \( x \) is close to 1, we conclude that \( u'_{\lambda_0}(1) > 0 \). Hence,

\[
\lim_{\lambda \rightarrow \lambda_0^-} T(\lambda) = -\infty, \quad \lim_{\lambda \rightarrow \lambda_0^+} T(\lambda) = +\infty.
\]

We claim that also

\[
\lim_{\lambda \rightarrow (\lambda_*^*)^-} T(\lambda) = +\infty, \quad \lim_{\lambda \rightarrow (\lambda_*)^+} T(\lambda) = +\infty.
\]

Indeed, observe firstly that, because solutions of (16) do not cross each other, \( \lim_{\lambda \rightarrow (\lambda_*^*)^-} u_\lambda(1) = +\infty \). On the other hand, multiplying (3) by \( u' \) it is easy to see, given \( M > 0 \) that

\[
|u_\lambda'(1)| \geq \sqrt{M} \mathcal{O}(|u_\lambda(1)|)
\]

8
for $|\lambda|$ sufficiently large. This implies that
\[ |T(\lambda)| = \left| \frac{u_\lambda'(1)}{u_\lambda(1)} \right| > \sqrt{M} \]
and the claim follows.

The previous considerations show the existence of $\lambda_{\text{min}} \in (\lambda_0, \lambda^*)$ such that $T(\lambda_{\text{min}}) \leq T(\lambda)$ for all $\lambda \in (\lambda_0, \lambda^*)$. The value $a_{\text{min}} := T(\lambda_{\text{min}}) > 0$ depends on $p$ and, in this context, Theorem 2.4 simply states that if $p \geq p_0$ for some large enough constant $p_0$ then $a_{\text{min}} > a_1$. Also, we easily deduce some of the conclusions of Theorems 1.1 and 1.2, as shown in the following figure.

By continuity, there exists $\lambda < \lambda_0$ such that $T(\lambda) = a_1$; the corresponding $u_\lambda$ is a negative solution. Uniqueness of negative solutions does not follow directly from this setting, unless an extra assumption like (13) is assumed for $u < 0$ (see Proposition 3.2 below). However, recall that if $a_1 < \frac{a_0}{a_0 + 1}$, then $\lambda_1 > 0$ so (8) is satisfied; thus uniqueness holds if $a_1$ is small.

**Proof of Theorem 1.2.** From the previous considerations, the problem has a negative solution, which is unique if $a_1$ is sufficiently small. Moreover, the equation $T(\lambda) = a_1$ has, over the interval $(\lambda_0, \lambda^*)$ at least two solutions when $a_1 > a_{\text{min}}$ is large and no solutions when $a_1$ is small. Finally, observe that, as the value of $a_1$ increases, at least one of those solutions is located in $(\lambda_0, 0)$.

**Remark 3.2** If $\lambda^* > 0$ or, equivalently, if $u_0$ is defined on $[0, 1]$, then we deduce that the problem has also a positive solution when $a_1 \gg 0$.

Under appropriate conditions, a lower bound for $a_{\text{min}}$ is easily obtained as follows:

**Proposition 3.1** Assume that (5) and (6) hold. If there exists $r \leq a_0$ such that $g(x, u) + p(x) > r^2 u$ for all $u \geq 0$ and all $x$, then $a_{\text{min}} > r$.  

9
Proof: Fix $\lambda \in (\lambda_0, \lambda^*)$ and let $v(x) := e^{rx}$. Define $x_0$ as the minimum value such that $u_{\lambda}$ is positive after $x_0$ and observe that

$$v(x)u''_{\lambda}(x) > v(x)r^2u_{\lambda}(x) > v(x)r^2u_{\lambda}(x) = v''(x)u_{\lambda}(x)$$

for $x > x_0$. Thus,$$
v(1)[u'_{\lambda}(1) - ru_{\lambda}(1)] > v(x_0)[u'_{\lambda}(x_0) - ru_{\lambda}(x_0)] \geq v(x_0)[u'_{\lambda}(x_0) - ru_{\lambda}(x_0)] \geq 0
$$

and we conclude that $T(\lambda) = \frac{u'_1(1)}{u_1(1)} > r$. □

Remark 3.3 For problem (1), the previous proposition implies that $a_{\min} > \min\{a_0, a_1\}$, which provides an alternative proof of the fact that the problem has no positive nor sign-changing solutions when $a_1 \leq a_0$.

In order to complete the proof of Theorem 1.1, let us make a more careful description of the graph of $T$. With this aim, compute

$$T'(\lambda) = \frac{\partial}{\partial \lambda} \left( \frac{u'_1(1)}{u_1(1)} \right) = \frac{u_1(1) \frac{\partial u'_1}{\partial \lambda}(1) - u'_1(1) \frac{\partial u_1}{\partial \lambda}(1)}{u_1(1)^2}$$

and set $w_{\lambda} := \frac{\partial u_{\lambda}}{\partial \lambda}$, then

$$T'(\lambda) = \frac{u_1(1)w'_1(1) - u'_1(1)w_1(1)}{u_1(1)^2}.$$ 

Moreover, observe that $w_{\lambda}$ solves the linear problem

$$\begin{cases}
  w''_{\lambda}(x) = \frac{\partial g}{\partial u}(x, u_{\lambda}(x))w_{\lambda}(x) \\
  w_{\lambda}(0) = 1, \quad w'_1(0) = a_0,
\end{cases} \quad (17)$$

and hence

$$u_1(1)w'_1(1) - u'_1(1)w_1(1) = \int_0^1 (u_{\lambda}(x)w''_{\lambda}(x) - u''_{\lambda}(x)w_{\lambda}(x)) \, dx =$$

$$= \int_0^1 \left( u_{\lambda}(x) \frac{\partial g}{\partial u}(x, u_{\lambda}(x)) - g(x, u_{\lambda}(x)) - p(x) \right) w_{\lambda}(x) \, dx. \quad (18)$$

Taking into account that $w_{\lambda}(x) > 0$ for all $x$ and that $u_{\lambda}$ is negative for $\lambda < \lambda_0$ and positive for $\lambda > 0$, the following proposition is obtained:

Proposition 3.2 Assume that (4), (5) and (6) hold. Then:

1. $T$ is strictly decreasing for $\lambda < \lambda_0$, provided that (13) holds for $u < 0$.

2. $T$ is strictly increasing for $\lambda > C$, provided that (13) holds for $u > C \geq 0$.

Remark 3.4 In particular, the previous proposition shows that, when (5) and (6) are assumed, the conclusions of Theorem 2.5 are retrieved in a simple manner.
Assume firstly that $p = 0$. Although (6) obviously fails, the operator $T$ is well defined, with $\lambda_0 = 0$. Moreover, using the L’Hôpital rule we deduce that

$$\lim_{\lambda \to 0} T(\lambda) = \lim_{\lambda \to 0} \frac{u'_\lambda(1)}{u_\lambda(1)} = \lim_{\lambda \to 0} \frac{w'_\lambda(1)}{w_\lambda(1)} = \frac{\Phi'(1)}{\Phi(1)},$$

where $\Phi := w_0$, that is, the unique solution of the linear initial value problem

$$\Phi''(x) = \frac{\partial g}{\partial u}(x,0)\Phi(x), \quad \Phi'(0) = a_0, \Phi(0) = a_0. \tag{19}$$

Thus, $T$ can be extended continuously to a positive function defined over $(\lambda^*, \lambda^*)$, which tends to $+\infty$ as $\lambda \to (\lambda^*)^+$ or $\lambda \to (\lambda^*)^-$. Furthermore, if $\lambda_0$ holds for $u \neq 0$ then it decreases strictly on $(\lambda_0, 0)$ and increases strictly on $[0, \lambda^*)$.

We are now in condition of completing the proof of Theorem 1.1. To this end, we shall need the following lemma:

**Lemma 3.5** Assume that (5) and (7) holds. Then $\Phi'(1) < a_1 \Phi(1)$.

**Proof:** Let $\varphi_1$ be the (unique) eigenfunction corresponding to $\lambda_1$ such that $\varphi_1(0) = 1$, then it is readily verified that $\varphi_1(x) > 0$ for all $x$. Moreover, it is seen from (19) that also $\Phi(x) > 0$ for all $x$. Then

$$\varphi_1(x)\Phi''(x) = \frac{\partial g}{\partial u}(x,0)\Phi(x)\varphi_1(x) \leq -\lambda_1 \Phi(x)\varphi_1(x) = \Phi(x)\varphi''_1(x)$$

and the inequality is strict for some $x$. Integration yields

$$\varphi(1)\Phi'(1) < \Phi(1)\varphi'(1) = a_1 \Phi(1)\varphi(1)$$

and the proof follows.

**Proof of Theorem 1.1** In view of the previous Lemma, the proof is an immediate corollary of the following proposition, slightly more general:

**Proposition 3.3** Assume that (4), (5) and (6) hold and that $\Phi'(1) < a_1 \Phi(1)$. Then there exists a constant $p_1 > 0$ such that problem (3)-(2) has at least three solutions when $\|p\|_{\infty} < p_1$. Moreover, one of the solutions is negative, one of them positive and another one sign-changing. If furthermore $\lambda_0$ holds with $p = 0$ for all $u \neq 0$, then there exists $p_1$ such that the problem has exactly three solutions, provided that $\|p\|_{\infty} < p_1$. Moreover, exactly one of the solutions is negative and another one changes sign.

**Proof:** From the previous considerations we know that, if $p$ is small, then $\lambda^* > 0$ and $T(0) < a_1$; thus, the existence of at least three solutions follows. Clearly, one of the solutions is negative, another one is positive and another one changes sign.

From now on, assume that $\lambda_0$ with $p = 0$ holds for all $u \neq 0$. Consider, for arbitrary $p$, the mapping

$$R_p(\lambda) := u'_\lambda(1) - a_1 u_\lambda(1).$$
Let us firstly take \( p = 0 \). From the previous computations, we know that 
\[ \text{sgn}(T'(\lambda)) = \text{sgn}(u_\lambda) = \text{sgn}(\lambda) \text{ for } \lambda \neq 0 \] 
and \( T(0) = \frac{\Phi'(1)}{\Phi(1)} < a_1 \), whence \( R_0 \) has exactly three roots \( \{0, \lambda_\pm\} \) with \( \lambda_- < 0 < \lambda_+ \). Moreover, write as before 
\[ T'(\lambda) = \frac{u_\lambda(1)w_\lambda'(1) - w_\lambda'(1)w_\lambda(1)}{u_\lambda(1)^2} = \frac{w_\lambda(1)}{u_\lambda(1)} \left( \frac{w_\lambda'(1)}{w_\lambda(1)} - T(\lambda) \right) \]
to deduce that 
\[ \frac{w_\lambda'(1)}{w_\lambda(1)} > T(\lambda_\pm) = a_1. \]

Next, observe that 
\[ R'_0(\lambda) = w_\lambda'(1) - a_1w_\lambda(1) = w_\lambda(1) \left( \frac{w_\lambda'(1)}{w_\lambda(1)} - a_1 \right), \]
so \( R'_0(\lambda_\pm) > 0 \). On the other hand, \( R'_0(0) = \Phi'(1) - a_1\Phi(1) < 0 \) and, by continuity, we conclude that if \( p \) is close to 0 then \( R_p \) has exactly three roots. Furthermore, \( T(0) \) is close to \( \frac{\Phi'(1)}{\Phi(1)} < a_1 \), so the equation \( T(\lambda) = a_1 \) has at least one solution in \( (\lambda_0, 0) \). Finally, observe that if \( p \) is small then \( u_0 \) is defined in \( [0,1] \); thus, \( \lambda^* > 0 \) and letting \( p \) be smaller if necessary we conclude that the equation \( T(\lambda) = a_1 \) has also a solution in \( (0, \lambda^*) \) \( \square \)

**Remark 3.6** In particular, all the assumptions of the previous proposition are fulfilled for problem \( \square \) if (and only if) \( a_1 > a_0 \). Indeed, in this case it is readily seen that \( \lambda_1 < -a_1^2 \) and hence 
\[ \frac{\partial g}{\partial u}(x,0) = L(x) \leq a_1^2 < -\lambda_1. \]

## 4 Open questions

1. Numerical experiments for the particular case \( \square \) suggest that \( T'' > 0 \) for \( \lambda > \lambda_0 \). If this is true, then an exact multiplicity result yields for arbitrary \( p \), depending on whether \( a_{\text{min}} \) is smaller, equal or larger than \( a_1 \). It would be interesting to investigate if this fact could be verified for the general case, under appropriate conditions, using the differential equation for \( z_\lambda := \frac{\partial w_\lambda}{\partial \lambda} = \frac{\partial^2 u_\lambda}{\partial \lambda^2} \), namely 
\[ z_\lambda''(x) = \frac{\partial}{\partial u}(x, u_\lambda(x))z_\lambda(x) + \frac{\partial^2 g}{\partial u^2}(x, u_\lambda(x))w_\lambda(x)^2, \quad z_\lambda(0) = z_\lambda'(0) = 0. \]

2. Is it possible to obtain an exact multiplicity result also for \( a_1 \) large? Observe that, in such a case, the behaviour of \( T \) can be controlled near \( \lambda_0 \), but it is not easy to see what happens as \( \lambda \) gets closer to \( \lambda_* \) or \( \lambda^* \). In more precise terms, we may set \( \varepsilon := \frac{1}{a_1} \) and 
\[ R_\varepsilon(\lambda) := \varepsilon u_\lambda'(1) - u_\lambda(1). \]
Then \( R_0(\lambda) = -u_\lambda(1) \) decreases from \( +\infty \) to \( -\infty \) over \( (\lambda_*, \lambda^*) \). Furthermore, \( R'_0(\lambda) = -w_\lambda(1) < 0 \) for all \( \lambda \); thus, if \( \varepsilon \) is small, then \( R_\varepsilon \) has, near
\( \lambda_0 \), a unique root. However, for \( \varepsilon \neq 0 \) the graph of \( R_\varepsilon \) bends in such a way that it tends to \( \pm \infty \) as \( \lambda \) gets closer to \( \lambda^* \) and \( \lambda_* \) respectively. This ensures the existence of at least three solutions for \( \varepsilon \) small, although there might be more. Clearly, there exists \( \lambda_1 \) such that \( R_\varepsilon \) increases with \( \varepsilon \) for \( \lambda > \lambda_1 \) and decreases when \( \lambda < \lambda_1 \); moreover, if \( K \subset (\lambda_*, \lambda^*) \) is a compact neighborhood of \( \lambda_0 \), then \( R_\varepsilon \) vanishes exactly once in \( K \) when \( \varepsilon = \varepsilon(K) \) is small. This is due to the fact that \( R_\varepsilon \) tends to \( R_0 \) over \( K \) for the \( C^1 \) norm. However, it is not clear which condition would be appropriate in order to prevent against a possible ‘strange’ behaviour of \( R_\varepsilon \) outside compact sets. For example, taking into account the superlinearity, we might impose the assumption that \( \frac{\partial g}{\partial u}(x, u) \) tends uniformly to \( +\infty \) as \( |u| \to +\infty \). This would ensure that \( R_\varepsilon \) has positive derivative near the endpoints of its domain but, still, it might change sign many times.

3. How does the graph of \( T \) vary with respect to \( p' \)? Suppose for simplicity that \( p \) is a constant and let \( y_p := \frac{\partial u_\lambda}{\partial p} \). Then

\[
y_p'' = \frac{\partial g}{\partial u}(x, u_\lambda(x)) y_p + 1 \quad y_p(0) = y'_p(0) = 0
\]

and the sign of \( \frac{\partial T}{\partial p} \) coincides with the sign of the integral

\[
\int_0^1 \left( u_\lambda(x) \frac{\partial g}{\partial u}(x, u_\lambda(x)) - g(x, u_\lambda(x)) - p \right) y_p(x) + u_\lambda(x) \, dx.
\]

If (13) holds for \( u < 0 \), then \( \frac{\partial T}{\partial p} < 0 \) for \( \lambda < \lambda_0 \). In particular, the (unique) value \( \lambda < \lambda_0 \) for which \( T(\lambda) = a_1 \) moves to the left as \( p \) increases. This is consistent with the fact that the negative solution tends uniformly to \( -\infty \) as \( p \to +\infty \). It seems difficult to obtain similar conclusions for \( \lambda \in (\lambda_0, 0) \) since \( u_\lambda \) changes sign but, in general, if (13) is satisfied for \( u > C \geq 0 \), then \( \frac{\partial T}{\partial p} > 0 \) for \( \lambda \geq C \). For example, this is the case in problem (1), with

\[
C = \sqrt[3]{\frac{\lambda}{2\pi}}.
\]

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