Stability and curvature estimates for minimal graphs with flat normal bundles

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Abstract

It is well-known that a minimal graph of codimension one is stable, i.e. the second variation of the area functional is non-negative. This is no longer true for higher codimensional minimal graphs in view of an example of Lawson and Osserman. In this note, we prove that a minimal graph of any codimension is stable if its normal bundle is flat. We also prove minimal graphs of dimension no greater than six and any codimension is flat if the the normal bundle is flat and the density at infinity is finite. Such a Bernstein type theorem holds in any dimension if we assume additionally growth conditions on the volume element.

1 Introduction

The graph of a solution $f : D \subset \mathbb{R}^n \to \mathbb{R}$ of the minimal surface equation

$$\text{div}(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}) = 0 \quad (1.1)$$

is naturally a minimal hypersurface in $\mathbb{R}^{n+1}$. In general, we may consider the graph of a vector-valued function and ask when this is a minimal submanifold of the Euclidean space. The function then satisfies a nonlinear elliptic system. Indeed, a $C^2$ vector-valued function $f = (f^1, \cdots f^m) : D \to \mathbb{R}^m$ is said to be
a solution to the minimal surface system (see Osserman [OS1] or Law son-Osserman [LO]) if

\[ \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} (\sqrt{g} g^{ij} \frac{\partial f^\alpha}{\partial x^j}) = 0 \text{ for each } \alpha = 1 \cdots m \] (1.2)

where \( g^{ij} = (g_{ij})^{-1} \), \( g_{ij} = \delta_{ij} + \sum_{\beta=1}^{m} \frac{\partial f^\beta}{\partial x^i} \frac{\partial f^\beta}{\partial x^j} \), and \( g = \det g_{ij} \). The graph of \( f \), so called a minimal graph, is then a minimal submanifold of \( \mathbb{R}^{n+m} \) of dimension \( n \) and codimension \( m \).

The minimal surface system was first studied in Osserman [OS1], [OS2] and Lawson-Osserman [LO]. In contrast to the codimension one case, Lawson and Osserman [LO] discovered remarkable counterexamples to the existence, uniqueness and regularity of solutions to the minimal surface system in higher codimension. It is thus interesting to identify natural conditions under which theorems for the minimal surface equation can be generalized.

The difficulty of the higher codimensional problems is amplified by the complexity of the normal bundle. Given an \( n \) dimensional submanifold \( \Sigma \) of \( \mathbb{R}^{n+m} \), recall the normal bundle consists of the orthogonal complement of the tangent spaces of \( \Sigma \) in \( \mathbb{R}^{n+m} \). Near a point of \( \Sigma \), choose an orthonormal frame \( e_1, \cdots, e_n \) for the tangent bundle and \( e_{n+1}, \cdots, e_{n+m} \) for the normal bundle. The coefficients of the second fundamental form is denoted by \( h_{\alpha ij} = \langle \nabla e_i e_\alpha, e_j \rangle \). Recall from the Ricci equation, the curvature of the normal bundle is given by

\[ R_{\alpha \beta ij} = h_{\alpha ik} h_{\beta jk} - h_{\alpha jk} h_{\beta ik}. \] (1.3)

We say \( \Sigma \) has flat normal bundle if \( R_{\alpha \beta ij} \equiv 0 \), see for example [TE]. When \( \Sigma \) is a graph, we can in fact choose \( e_{n+1}, \cdots, e_{n+m} \) to be globally parallel sections. Equation (1.3) holds trivially when \( \Sigma \) is of codimension one, i.e. \( m = 1 \). In particular, any oriented hypersurface has flat normal bundle.

Recall a minimal submanifold is called stable if the second variation of the volume functional is non-negative with respect to any compact-supported variation fields. By a calibration argument, a minimal graph of codimension one is stable (in fact area-minimizing). This is no longer true in higher codimension by a counterexample of Lawson and Osserman [LO]. Nevertheless, we prove the following stability theorem.

**Theorem 1.1** If \( \Sigma \) is a minimal graph with flat normal bundle in the Euclidean space, then \( \Sigma \) is stable.
A different stability criterion for higher codimensional minimal graphs is studied in \[\text{[LW]}\]. We first generalize the curvature estimate of Schoen-Simon-Yau \[\text{[SSY]}\] and prove the following Bernstein type theorem.

**Theorem 1.2** Suppose \(\Sigma\) is the graph of an entire smooth function \(f : \mathbb{R}^n \rightarrow \mathbb{R}^m\) of the minimal surface system for \(n \leq 5\). If the normal bundle of \(\Sigma\) is flat and \(\text{Vol}(\Sigma \cap B_R) \leq cR^n\) for some constant \(c\), then \(f\) is a linear map.

Here \(B_R\) is the ball of radius \(R\) in \(\mathbb{R}^{n+m}\) centered at the origin.

Ecker and Huisken \[\text{[EH]}\] prove the following Bernstein type result in the codimension one case.

**Theorem** Suppose \(\Sigma\) is the graph of an entire smooth solution \(f : \mathbb{R}^n \rightarrow \mathbb{R}\) of the minimal surface equation. \(f\) is a linear map if
\[
\sqrt{1 + |Df|^2(x)} = o(\sqrt{|x|^2 + |f(x)|^2}).
\]

We generalize Ecker and Huisken’s theorem \([\text{EH}]\) to the higher codimensional case.

**Theorem 1.3** Suppose \(\Sigma\) is the graph of an entire smooth solution \(f : \mathbb{R}^n \rightarrow \mathbb{R}^m\) of the minimal surface system. \(f\) is a linear map if the following three conditions hold:

1) the normal bundle of \(\Sigma\) is flat.
2) \[
\sqrt{\det(I + (df)^T df)}(x) = o(\sqrt{|x|^2 + |f(x)|^2}).
\]
3) \(\text{Vol}(\Sigma \cap B_R) \leq c(n)R^n\) for some constant \(c(n)\).

Higher codimensional Bernstein type theorems have been studied by many authors \[\text{[FC]}, \text{[HJW]}, \text{[JX]}, \text{[WA3]}\] assuming various conditions.

The author would like to thank Professor C.-L. Terng for suggesting him to look at mean curvature flows of submanifolds of flat normal bundles in the spring of 2002. This note was written up while the author was teaching a graduate course at Columbia in which he went over curvature estimates for minimal hypersurfaces. During when he realized the key formulae (see equations \[\text{[2.2]}\] and \[\text{[2.3]}\]) to generalize Schoen-Simon-Yau \[\text{[SSY]}\] and Ecker-Huisken \[\text{[EH]}\] to the flat normal bundle case were contained in his earlier work \[\text{[WA1]}\] and \[\text{[WA2]}\]. With these formulae, the derivations follow straightforward from \[\text{[SSY]}\] and \[\text{[EH]}\].

We remark that mean curvature flows of submanifolds with flat normal bundles are studied in a recent paper by Smoczyk, Wang and Xin \[\text{[SWX]}\].
2 Preliminary

Let $\Sigma$ be the graph of $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $\{e_1, \ldots, e_n\}$ and $\{e_{n+1}, \ldots e_{n+m}\}$ be local orthonormal bases for $T\Sigma$ and $N\Sigma$, respectively.

Let $\Omega$ denote the n-form $dx^1 \wedge \cdots \wedge dx^n$ where $x^1, \ldots, x^n$ are coordinates on $\mathbb{R}^n$. We extend $\Omega$ to $\mathbb{R}^{n+m}$ and consider the function $*\Omega = \Omega(e_1, \cdots, e_n)$ on $\Sigma$. Notice $*\Omega$ is the Jacobian of the natural projection from $\Sigma$ to $D$ and $*\Omega > 0$ on $\Sigma$. In terms of $f$, we have $*\Omega = \frac{1}{\sqrt{\det(I + (df)^Tdf)}}$.

We recall the following formula derived in [WA2] (Proposition 3.1) and [WA3]

$$\Delta *\Omega + *\Omega|A|^2 + 2\sum_{k=1}^{n} \sum_{\alpha,\beta,i<j} \Omega_{\alpha \beta ij} h_{\alpha ik} h_{\beta jk} = 0 \quad (2.1)$$

where $\Omega_{\alpha \beta ij} = \Omega(e_1, \ldots, e_\alpha, \cdots, e_\beta, \cdots, e_n)$ with $e_\alpha$ occupying the $i$-th place and $e_\beta$ occupying the $j$-the place. Anti-symmetrizing the $\alpha$ and $\beta$ indexes, we obtain

$$\Delta *\Omega + *\Omega|A|^2 + 2\sum_{\alpha<\beta,i<j} \Omega_{\alpha \beta ij} R_{\alpha \beta ij} = 0 \quad (2.2)$$

This formula essentially appeared in [FC]. The parabolic analogue was re-discovered by the author in the study of mean curvature flows in higher codimension.

Another basic equation is equation (7.2) in [WA1].

$$\Delta |A|^2 = 2|\nabla A|^2 - 2\sum_{i,j,m,k} \left( \sum_\alpha h_{\alpha ij} h_{\alpha mk} \right)^2 - 2\sum_{\alpha,\beta,i,j} (R_{\alpha \beta ij})^2 \quad (2.3)$$

In codimension one case, this is the so called Simon’s identity which has been enormously useful in the study of minimal hypersurfaces.

Next, we generalize a Lemma of [SSY] to higher codimension.

**Lemma 2.1**

$$|\nabla A|^2 - |\nabla |A||^2 \geq \frac{2}{n}|\nabla |A||^2$$

**Proof.**

$$|\nabla A|^2 - |\nabla |A||^2 = \sum_{\alpha ij k} h_{\alpha ij,k}^2 - |A|^2 \sum_{\alpha ij} (\sum_k h_{\alpha ij} h_{\alpha ij,k})^2$$
By expanding the right hand side, it is not hard to see

\[ |\nabla A|^2 - |\nabla A|\|^2 = \frac{1}{2|A|^2} \sum_{\alpha \beta i j k r s} (h_{\alpha ij} h_{\beta rs,k} - h_{\beta rs} h_{\alpha ij,k})^2 \]

Recall the \( h_{\alpha ij} \) are simultaneously diagonalizable. So the expression is equal to

\[ \sum_{\alpha i j k} h_{\alpha ij,k}^2 - |A|^{-2} \sum_{\alpha i} (\sum_{\alpha i} h_{\alpha ii} h_{\alpha ii,k})^2 \]

\[ \geq \sum_{\alpha i j k} h_{\alpha ij,k}^2 - |A|^{-2} \sum_{\alpha i} h_{\alpha ii}^2 \sum_{\alpha i k} h_{\alpha ii,k}^2 \]

\[ \geq \sum_{\alpha, i \neq j, k} h_{\alpha ij,k}^2 \]

\[ \geq \sum_{\alpha, i \neq j} h_{\alpha ii,i}^2 + \sum_{\alpha, i \neq j} h_{ij,j}^2 \]

\[ = 2 \sum_{\alpha, i \neq j} h_{\alpha ii,i}^2 \]

On the other hand, by diagonalization, we have

\[ |\nabla A|^2 \leq \sum_{\alpha, i k} h_{\alpha ii,k}^2 = \sum_{\alpha, i \neq k} h_{\alpha ii,k}^2 + \sum_i h_{\alpha ii,i}^2 \]

By the minimal surface equation \( \sum_i h_{\alpha ii} = 0 \), thus

\[ |\nabla A|^2 \leq \sum_{\alpha, i \neq k} h_{\alpha ii,k}^2 + \sum_i (\sum_{\alpha, j \neq i} h_{\alpha jj,i})^2 \leq \sum_{i \neq j} h_{ii,j}^2 + (n-1) \sum_{i \neq j} h_{\alpha ii,j}^2 = n \sum_{\alpha, i \neq j} h_{\alpha ii,j}^2 \]

\[ \square \]

### 3 Proofs of Theorem 1.1. and 1.2.

*Proof of Theorem 1.1.* Since the normal bundle of \( \Sigma \) is flat, by (2.2), we have

\[ \Delta * \Omega + *\Omega |A|^2 = 0. \]
As \( \ast \Omega > 0 \), this equation implies the first nonzero eigenvalue of the operator \(-\Delta - |A|^2\) is non-negative or that

\[
\int_\Sigma u^2 |A|^2 \leq \int_\Sigma |\nabla u|^2 \tag{3.1}
\]

for any \( C^1 \) function \( u \). This follows from a well known argument, see for example Lemma 1.24 (page 21) of [CM]. Indeed, take the log of \( \ast \Omega \) and compute

\[
\Delta \log \ast \Omega = -|A|^2 - |\nabla \log \ast \Omega|^2.
\]

Multiply both sides by \( u^2 \), integrate by parts, apply the Cauchy-Schwarz inequality, and the result is obtained.

We recall that for a minimal submanifold of \( \mathbb{R}^{n+m} \), the stability condition is equivalent to

\[
\int_\Sigma \sum_{i,j} \langle \nabla e_i e_j, V \rangle^2 \leq \int_\Sigma \sum_{i} |(\nabla e_i V)^\perp|^2
\]

for any compact-supported section \( V \) of the normal bundle.

Since the normal bundle is flat, we can find parallel sections \( e_\alpha \) of the normal bundle and write \( V = V^\alpha e_\alpha \). Then the stability condition is equivalent to

\[
\int_\Sigma \sum_{i,j} \left( \sum_{\alpha} V^\alpha h_{aij} \right)^2 \leq \int_\Sigma \sum_{\alpha} |\nabla V^\alpha|^2 \tag{3.2}
\]

Apply (3.1) to \( u = V^\alpha \) and sum over \( \alpha \), we derive

\[
\int_\Sigma \sum_{\alpha} (V^\alpha)^2 |A|^2 \leq \int_\Sigma \sum_{\alpha} |\nabla V^\alpha|^2 \tag{3.3}
\]

This clearly implies (3.2).

Before proving Theorem 1.2, we first generalize an integral curvature estimate of [SSY].

**Theorem 3.1** Let \( \Sigma^n \) be a minimal graph with flat normal bundle, for \( p \in [2, 2 + \sqrt{2/n}) \) and \( \phi > 0 \), \( \phi \in C^1_c(\Sigma) \), we have
\[
\int |A|^{2p} \phi^{2p} \leq C(n, p) \int |\nabla \phi|^{2p}.
\]

Proof. Set \( u = |A|^{1+q} f \) in the stability inequality (3.1), we obtain

\[
\int |A|^{4+2q} f^2 \leq \int |f \nabla |A|^{1+q} + |A|^{1+q} \nabla f|^2
\]

\[
= (1 + q)^2 \int f^2 |\nabla |A||^2 |A|^{2q} + \int |A|^{2+2q} |\nabla f|^2 + 2(1 + q) \int f |A|^{1+2q} \nabla f \cdot \nabla |A|
\]

(3.4)

We shall estimate the first term using the inequality

\[
|A| \Delta |A| + |A|^4 \geq \frac{2}{n} |\nabla |A||^2
\]

which follow from Lemma 2.1 and equation (2.3).

Multiply both sides by \( |A|^{2q} f^2 \) and integrate by parts, we derive

\[
\frac{2}{n} \int |\nabla |A||^2 |A|^{2q} f^2
\]

\[
\leq \int |A|^{4+2q} f^2 - 2 \int f |A|^{1+2q} \nabla f \cdot \nabla |A|^2 - (1 + 2q) \int f^2 |A|^{2q} |\nabla |A||^2
\]

(3.5)

Substitute the inequality (3.4) for \( \int |A|^{4+2q} f^2 \) into (3.5), we obtain

\[
(\frac{2}{n} - q^2) \int |A|^{2q} |\nabla |A||^2 f^2 \leq \int |A|^{2+2q} |\nabla f|^2 + 2q \int f |A|^{1+2q} \nabla f \cdot \nabla |A|^2
\]

Using the inequality \( 2xy \leq \epsilon x^2 + \frac{1}{\epsilon} y^2 \), we arrive at

\[
(\frac{2}{n} - q^2 - \epsilon) \int f^2 |A|^{2q} |\nabla |A||^2 \leq (1 + \frac{q}{\epsilon}) \int |\nabla f|^2 |A|^{2+2q}
\]

(3.6)

We go back to (3.4) and apply \((x + y)^2 \leq 2x^2 + 2y^2 \) and obtain

\[
\int |A|^{4+2q} f^2 \leq 2(1 + q)^2 \int f^2 |\nabla |A||^2 |A|^{2q} + 2 \int |A|^{2+2q} |\nabla f|^2.
\]

(3.7)
In view of (3.6), if \(2n - q^2 - \epsilon q > 0\), we have
\[
\int |A|^{4+2q}f^2 \leq C \int |A|^{2+2q}|
abla f|^2
\]
for some constant \(C\).

Take \(p = 2 + q\) and \(f = \phi^p\) and use \(xy \leq \frac{x^a}{a} + \frac{y^b}{b}\) for \(\frac{1}{a} + \frac{1}{b} = 1\), we achieve
\[
\int |A|^{2p}\phi^{2p} \leq C(n, p) \int |
abla \phi|^{2p}
\]

The condition \(\frac{2}{n} - q^2 > 0\) translates to \(2 \leq p < 2 + \sqrt{2/n}\) in view of \(p = 2 + q\).

\[\square\]

Take \(\phi\) to be the standard cut-off function supported in \(B_R\) and \(\equiv 1\) in \(B_{R/2}\), we have
\[
\int_{\Sigma \cap B_{R/2}} |A|^{2p} \leq C(n, p)R^{-2p}Vol(\Sigma \cap B_R) \tag{3.8}
\]
for \(p \in [2, 2 + \sqrt{2/n}]\).

We recall the following sub-mean-value inequality for minimal submanifolds ([BDM], [MS]).

**Theorem 3.2** If \(\Delta u \geq -Qu\) in \(\Sigma \cap B_R\), \(u \geq 0\), \(Q \geq 0\) and \(Q \in L^q\) for some \(q > n/2\) then
\[
\sup_{\Sigma \cap B_{R/2}} u \leq C(R^{-n} \int_{\Sigma \cap B_R} u^2)^{1/2}
\]
where \(C\) is a constant depending on \(n, p, R^{2q-n} \int_{B_R} Q^q\) and the isoparametric constant.

**Proof of Theorem 1.2.** We recall from (2.3) \(|A|^2\) satisfies \(\Delta |A|^2 + 2|A|^4 \geq 0\) take \(u = |A|^2\) and \(Q = 2|A|^2\), we have
\[
\sup_{B_{R/2}} |A|^2 \leq C(R^{-n} \int_{B_R} |A|^4)^{1/2}
\]
for some \(C\) that depends on \(R^{2q-n} \int_{B_R} |A|^{2q}\). This quantity is bounded by (3.8) and finite density assumption.
Suppose there exists a $q$ satisfying $2 \leq q < 2 + \sqrt{2/n}$ and $q > \frac{4}{2}$. Take $p - 2$ in Theorem 3.1, we have

$$
\int_{\Sigma \cap B_{R/2}} |A|^4 \leq C R^{-4} Vol(\Sigma \cap B_R).
$$

We have

$$
\sup_{B_{R/2}} |A|^2 \leq C(R^{-(n+4)}Vol(\Sigma \cap B_R))^{1/2}
$$

Applying the assumption $Vol(\Sigma \cap B_R) \leq c(n) R^n$ and let $R \to \infty$, we obtain $|A|^2 = 0$. The relation satisfied by $q$ requires that

$$
n < 4 + \sqrt{8/n}
$$

or $n = 1 \cdots 5$

4 Proof of Theorem 1.3

Proof of Theorem 1.3. We follow the proof of Ecker-Huisken [EH] closely. Since $R_{\alpha \beta ij} = 0$, by (2.2) and (2.3), we obtain

$$
\Delta \ast \Omega + |A|^2 \ast \Omega = 0 \quad (4.1)
$$

and

$$
\Delta |A|^2 = 2|\nabla A|^2 - 2 \sum_{i,j,m,k} (\sum_{\alpha} h_{\alpha ij} h_{\alpha mk})^2 \geq 2|\nabla A|^2 - 2|A|^4 \quad (4.2)
$$

These two equations correspond to equation(2) and equation (3) in [EH].

$R_{\alpha \beta ij} = 0$ implies the matrices $A_{\alpha} = [h_{\alpha ij}]$ are pairwise commutative and thus simultaneously diagonalizable. As in [SSY], we can show

$$
|\nabla A|^2 \geq (1 + \frac{2}{n})|\nabla|A||^2.
$$

This is then identical to equation (4) in [EH].

As in [EH], for $p \geq 2$ and $q(1 - 2/n) \leq p - 1 + 2/n$, we derive

$$
\Delta(|A|^p \ast \Omega^{-q}) \geq (q - p)|A|^{p+2} \ast \Omega^{-q}.
$$

Choose $q = p \geq (n - 1)/2$, we obtain
\[ \Delta(|A|^p v^p) \geq 0. \]

The sub-mean value inequality for subharmonic functions together with the volume growth assumption yield

\[ |A|^p v^p(0) \leq c(n) R^{-n/2} \left( \int_{\Sigma \cap B_R} |A|^{2p} v^{2p} \right)^{1/2}. \quad (4.3) \]

On the other hand, for \( p \geq \max(3, n-1) \) fixed,

\[ \Delta(|A|^{p-1} v^p) \geq |A|^{p+1} v^p \]

Multiply this equation by \( |A|^{-1} v^p \eta^{2p} \) where \( \eta \) is a test function and integrate by parts, we arrive at

\[ \int_{\Sigma} |A|^{2p} v^{2p} \eta^{2p} \leq c(p) \int_{\Sigma} v^{2p} |\nabla \eta|^{2p}. \]

Take \( \eta \) to be the standard cut-off function such that \( \eta \equiv 1 \) on \( B_R \), \( \eta \equiv 0 \) outside \( B_{2R} \) and \( |\nabla \eta| \leq \frac{2}{R} \). Combine this with \( 4.3 \) apply the growth on the volume and \( v \), and let \( R \to \infty \), we see that \( |A| \equiv 0. \]

Comparing with Ecker-Huisken’s proof, we notice the arguments only differ in that we need to make the assumption of the growth of the volume. A minimal graph of codimension one is area-minimizing and a comparison argument gives this area bound. However, in the higher codimensional case, we only prove the stability. It is interesting to investigate whether a minimal graph with flat normal bundle is area-minimizing.

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