Evolution equations of p-Laplace type with absorption or source terms and measure data
Marie-Françoise Bidaut-Véron, Quoc-Hung Nguyen

To cite this version:
Marie-Françoise Bidaut-Véron, Quoc-Hung Nguyen. Evolution equations of p-Laplace type with absorption or source terms and measure data. 2014. hal-01060687

HAL Id: hal-01060687
https://hal.archives-ouvertes.fr/hal-01060687
Preprint submitted on 4 Sep 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Evolution equations of $p$-Laplace type with absorption or source terms and measure data

Marie-Françoise BIDAUT-VÉRON$^*$ Quoc-Hung NGUYEN$^†$

Abstract

Let $\Omega$ be a bounded domain of $\mathbb{R}^N$, and $Q = \Omega \times (0, T)$. We consider problems of the type

$$
\begin{cases}
  u_t - \Delta_p u \pm G(u) = \mu & \text{in } Q, \\
  u = 0 & \text{on } \partial \Omega \times (0, T), \\
  u(0) = u_0 & \text{in } \Omega,
\end{cases}
$$

where $\Delta_p$ is the $p$-Laplacian, $\mu$ is a bounded Radon measure, $u_0 \in L^1(\Omega)$, and $\pm G(u)$ is an absorption or a source term. In the model case $G(u) = \pm |u|^{q-1}u$ $(q > p-1)$, or $G$ has an exponential type. We prove the existence of renormalized solutions for any measure $\mu$ in the subcritical case, and give sufficient conditions for existence in the general case, when $\mu$ is good in time and satisfies suitable capacitary conditions.

1 Introduction

Let $\Omega$ be a bounded domain of $\mathbb{R}^N$, and $Q = \Omega \times (0, T)$, $T > 0$. We consider the quasilinear parabolic problem

$$
\begin{cases}
  u_t - A(u) \pm G(u) = \mu & \text{in } Q, \\
  u = 0 & \text{on } \partial \Omega \times (0, T), \\
  u(0) = u_0 & \text{in } \Omega,
\end{cases}
$$

where $\mu$ is a bounded Radon measure on $Q$, $u_0 \in L^1(\Omega)$. We assume that $A(u) = \text{div}(A(x, \nabla u))$ and $A$ is a Carathéodory function on $\Omega \times \mathbb{R}^N$, such that, for a.e. $x \in \Omega$, and any $\xi, \zeta \in \mathbb{R}^N$,

$$
A(x, \xi) \geq \Lambda_1 |\xi|^p, \quad |A(x, \xi)| \leq \Lambda_2 |\xi|^{p-1}, \quad \Lambda_1, \Lambda_2 > 0,
$$

for $p > 1$; and $G(u) = G(x, t, u)$, where $(x, t, r) \mapsto G(x, t, r)$ is a Carathéodory function on $Q \times \mathbb{R}$ with

$$
G(x, t, r) \geq 0, \quad \text{for a.e. } (x, t) \in Q \quad \text{and any } r \in \mathbb{R}.
$$

The model problem is relative to the $p$-Laplace operator: $A(u) = \Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$, and $G$ has a power-type $G(u) = \pm |u|^{q-1}u$ $(q > p-1)$, or an exponential type. Our aim is to give sufficient conditions on

---

$^*$Laboratoire de Mathématiques et Physique Théorique, CNRS UMR 7350, Faculté des Sciences, 37200 Tours France. E-mail: veronmf@univ-tours.fr

$^†$Laboratoire de Mathématiques et Physique Théorique, CNRS UMR 7350, Faculté des Sciences, 37200 Tours France. E-mail: Hung.Nguyen-Quoc@lmpt.univ-tours.fr
the measure μ in terms of capacity to obtain existence results. We denote by $\mathcal{M}_b(\Omega)$ and $\mathcal{M}_b(Q)$ the sets of bounded Radon measures on $\Omega$ and $Q$ respectively.

Next we make a brief survey of the main works on problem (1.1). First we consider the case of an absorption term:

$$
\begin{align*}
\left\{
\begin{array}{ll}
u_t - A(u) + G(u) = \mu & \text{in } Q, \\
u = 0 & \text{on } \partial\Omega \times (0, T), \\
u(0) = u_0 & \text{in } \Omega.
\end{array}
\right.
\end{align*}
$$

For $p = 2$, $A(u) = \Delta u$ and $G(u) = |u|^{q-1}u$ ($q > 1$), the pioneer results concern the case $\mu = 0$ and $u_0$ is a Dirac mass in $\Omega$, see [12]: existence holds if and only if $q < (N + 2)/N$. Then optimal results are given in [3], for any $\mu \in \mathcal{M}_b(Q)$ and $u_0 \in \mathcal{M}_b(\Omega)$. Here two capacities are involved: the elliptic Bessel capacity $\text{Cap}_{\mathcal{G}_\alpha,s}$ defined, for $\alpha > 0$, by

$$
\text{Cap}_{\mathcal{G}_\alpha,s}(E) = \inf\{||\varphi||_{L^s(\mathbb{R}^N)}^s : \varphi \in L^s(\mathbb{R}^N), \varphi \geq 0 \text{ on } E\},
$$

where $\mathcal{G}_\alpha$ is the Bessel kernel of order $\alpha$; and the capacity $\text{Cap}_{2,1,s}$ defined, for any compact set $K \subset \mathbb{R}^{N+1}$, by

$$
\text{Cap}_{2,1,s}(K) = \inf\{||\varphi||_{W^{2,1}(\mathbb{R}^{N+1})}^s : \varphi \in S(\mathbb{R}^{N+1}), \varphi \geq 1 \text{ in a neighborhood of } K\},
$$

and extended classically to Borel sets, where

$$
||\varphi||_{W^{2,1}(\mathbb{R}^{N+1})} = ||\varphi||_{L^s(\mathbb{R}^{N+1})} + ||\varphi_t||_{L^s(\mathbb{R}^{N+1})} + ||\nabla\varphi||_{L^s(\mathbb{R}^{N+1})} + \sum_{i,j=1,2,...,N} ||\varphi_{x,i,j}||_{L^s(\mathbb{R}^{N+1})}.
$$

In [3], Baras and Pierre proved that there exists a solution if and only if $\mu$ does not charge the sets of $\text{Cap}_{2,1,s}$-capacity zero and $u_0$ does not charge the sets of $\text{Cap}_{\mathcal{G}_\alpha,s}$-capacity zero.

The case where $\mathcal{G}$ has an exponential type was initiated by [17], and studied in the framework of Orlicz spaces in [29, 19], and very recently by [24] in the context of Wolff parabolic potentials.

For $p \neq 2$, most of the contributions are relative to the case $G(u) = |u|^{q-1}u$, $\mu = 0$, with $\Omega$ bounded, or $\Omega = \mathbb{R}^N$. The case where $u_0$ is a Dirac mass in $\Omega$ was studied in [18, 20] when $p > 2$, and [13] when $p < 2$. Existence and uniqueness hold in the subcritical case

$$
q < p_c := p - 1 + \frac{p}{N}.
$$

If $q \geq p$, and $q > 1$, there is no solution with an isolated singularity at $t = 0$. For $q < p_c$, and $u_0 \in \mathcal{M}_b(\Omega)$, the existence was obtained in the sense of distributions in [30], and for any $u_0 \in \mathcal{M}_b(\Omega)$ in [8]. The case $\mu \in L^1(Q)$, $u_0 = 0$ was treated in [14], and with $\mu \in L^1(\Omega)$, $u_0 \in L^1(\Omega)$ in [1], where $\mathcal{G}$ can be multivalued. A larger set of measures, introduced in [16], was studied in [26]. Let $\mathcal{M}_b(Q)$ be the set of Radon measures $\mu$ on $Q$ that do not charge the sets of zero $c_p^Q$-capacity, where for any Borel set $E \subset Q$,

$$
c_p^Q(E) = \inf_{E \subset U \text{ open} \subset Q} \inf_{\in \chi_U} ||u||_W : u \in W, u \geq \chi_U \text{ a.e. in } Q\},
$$

and $W$ is the space of functions $z \in L^p((0,T); W_0^{1,p}(\Omega) \cap L^2(\Omega))$ such that $z_t \in L^p((0,T); W^{-1,p}(\Omega)+L^2(\Omega))$ imbedded with the norm

$$
||z||_W = ||z||_{L^p((0,T); W_0^{1,p}(\Omega) \cap L^2(\Omega))} + ||z_t||_{L^p((0,T); W^{-1,p}(\Omega)+L^2(\Omega))}.
$$
It was shown that existence and uniqueness hold for any measure $\mu \in \mathcal{M}_b(Q) \cap \mathcal{M}_0(Q)$, called regular, or diffuse, and $p > 1$, and for any function $G \in C(\mathbb{R})$ such that $G(u)u \geq 0$. Up to our knowledge, up to now no existence results have been obtained for a measure $\mu \not\in \mathcal{M}_0(Q)$.

The case of a source term

\[
\begin{aligned}
  u_t - A(u) &= \mathcal{G}(u) + \mu & \text{in } Q, \\
  u &= 0 & \text{on } \partial \Omega \times (0, T), \\
  u(0) &= u_0 & \text{in } \Omega,
\end{aligned}
\]

(1.7)

with $\mathcal{G}(u) = u^\prime$ with nonnegative $u$ and $\mu, u_0$ was treated in [2] for $p = 2$, giving optimal conditions for existence. As in the absorption case the arguments of proofs cannot be extended to general $p$.

2 Main results

In Section 3, we introduce the notion of renormalized solutions, called R-solutions, of problem (1.1), and we recall at Theorem 3.4 the stability result that we proved in [7] for the problem without perturbation

\[
\begin{aligned}
  u_t - A(u) &= \mu & \text{in } Q, \\
  u &= 0 & \text{on } \partial \Omega \times (0, T), \\
  u(0) &= u_0 & \text{in } \Omega.
\end{aligned}
\]

(2.1)

under the assumption

\[ p > p_1 := (2N + 1)/(N + 1), \]

that we make in all the sequel. This condition ensures that the functions $u$ and $|\nabla u|$ are well defined in $L^1(Q)$. Combined with some approximation properties of the measures, Theorem 3.4 is the key point of our results.

In Section 4, we first give existence results of subcritical type, valid for any measure $\mu \in \mathcal{M}_b(Q)$. Let $G \in C(\mathbb{R}^+)$ be a nondecreasing function with values in $\mathbb{R}^+$, such that

\[ |\mathcal{G}(x, t, r)| \leq G(|r|) \text{ for a.e. } x \in \Omega \text{ and any } r \in \mathbb{R}, \]

(2.2)

\[ \int_1^\infty G(s)s^{-1-p_c}ds < \infty, \]

(2.3)

where $p_c$ is defined at (1.6).

**Theorem 2.1** Assume (1.4), (2.2), (2.3). Then, for any $\mu \in \mathcal{M}_b(Q)$ and $u_0 \in L^1(\Omega)$, there exists a R-solution $u$ of problem

\[
\begin{aligned}
  u_t - A(u) + \mathcal{G}(u) &= \mu & \text{in } Q, \\
  u &= 0 & \text{in } \partial \Omega \times (0, T), \\
  u(0) &= u_0 & \text{in } \Omega.
\end{aligned}
\]

(2.4)

**Theorem 2.2** Assume (1.4), (2.2), (2.3). There exists $\varepsilon > 0$ such that, for any $\lambda > 0$, any $\mu \in \mathcal{M}_b^+(Q)$ and any nonnegative $u_0 \in L^1(\Omega)$, if $\lambda + \mu(\Omega) + ||u_0||_{L^1(\Omega)} \leq \varepsilon$, then there exists a nonnegative R-solution $u$ of problem

\[
\begin{aligned}
  u_t - A(u) &= \lambda \mathcal{G}(u) + \mu & \text{in } Q, \\
  u &= 0 & \text{in } \partial \Omega \times (0, T), \\
  u(0) &= u_0 & \text{in } \Omega,
\end{aligned}
\]

(2.5)
In particular for any if $G(u) = |u|^{q-1} u$, condition (2.3) is equivalent to the fact that $q$ is subcritical: $0 < q < p_c$, where $p_c$ is defined at (1.6).

Next we consider the general case, with no subcriticality assumptions, when $G$ is nondecreasing in $u$, and $G$ has a power type, or an exponential type. For $G(u) = |u|^{q-1} u$ for $q \geq p_c$, and $p \neq 2$, up to now the good capacities for solving the problem are not known. In the following, we search sufficient conditions on the measures $\mu$ and $u_0$ ensuring that there exists a solution. To our knowledge, the question of finding necessary conditions for existence is still an open problem.

In the sequel we give sufficient conditions for existence for measures that have a good behaviour in $t$, based on recent results of [9] relative to the elliptic case. We recall the notion of (truncated) Wolff potential: for any nonnegative measure $\omega \in \mathcal{M}^+(\mathbb{R}^N)$ any $R > 0$, $x_0 \in \mathbb{R}^N$,

$$W^R_{1,p}[\omega](x_0) = \int_0^R \frac{(r^{p-N}\omega(B(x_0,r)))^{\frac{1}{p-1}}}{r} dr.$$  \hspace{1cm} (2.6)

Any measure $\omega \in \mathcal{M}_b(\Omega)$ is identified with its extension by $0$ to $\mathbb{R}^N$. In case of absorption, we obtain the following:

**Theorem 2.3** Let $p < N$, $q > p - 1$, $\mu \in \mathcal{M}_b(Q)$, $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$. Assume that

$$|\mu| \leq \omega \otimes F,$$

where $\omega \in \mathcal{M}^+_b(\Omega)$, $F \in L^1((0,T))$, $F \geq 0$. \hspace{1cm} (2.7)

If $\omega$ does not charge the sets of $\text{Cap}_{G_{\lambda_1,\lambda_2}}$-capacity zero, then there exists a $R$-solution $u$ of problem

$$
\begin{cases}
  u_t - \mathcal{A}(u) + |u|^{q-1}u = f + \mu & \text{in } Q, \\
  u = 0 & \text{on } \partial \Omega \times (0,T), \\
  u(0) = u_0 & \text{in } \Omega. 
\end{cases}
$$  \hspace{1cm} (2.8)

From [3, Proposition 2.3], a measure $\omega \in \mathcal{M}_b(\Omega)$ does not charge the sets of $\text{Cap}_{G_{\lambda_1,\lambda_2}}$-capacity zero if and only if $\omega \otimes \chi_{(0,T)}$ does not charge the sets of $\text{Cap}_{G_{\lambda_1,\lambda_2}}$-capacity zero. Therefore, when $\mathcal{A}(u) = \Delta u$ and $\mu = \omega \otimes \chi_{(0,T)}$, $u_0 \in L^1(\Omega)$, we find again the existence result of [3]. Besides, in view of [16, Theorem 2.16], there exists data $\mu \in \mathcal{M}_b(Q)$ in Theorem 2.3 such that $\mu \notin \mathcal{M}_b(\Omega)$, see Remark 5.7, thus our result is the first one of existence for non diffuse measure. Otherwise our result can be extended to a more general function $G$, see Remark 5.9.

We also consider a source term. Denoting by $D = \sup_{x,y \in \Omega} |x - y|$ the diameter of $\Omega$, we obtain the following:

**Theorem 2.4** Let $p < N$, $q > p - 1$. Let $\mu \in \mathcal{M}^+_b(Q)$, and nonnegative $u_0 \in L^\infty(\Omega)$. Assume that

$$\mu \leq \omega \otimes \chi_{(0,T)},$$

with $\omega \in \mathcal{M}^+_b(\Omega)$. \hspace{1cm} (2.9)

Then there exist $\lambda_0$ and $b_0$, depending of $\mu, q, \Lambda_1, \Lambda_2, D$, such that, if

$$\omega(E) \leq \lambda_0 \text{Cap}_{G_{\lambda_1,\lambda_2}}(E), \quad \forall E \text{ compact } \subset \mathbb{R}^N,$$

there exists a nonnegative $R$-solution $u$ of problem

$$
\begin{cases}
  u_t - \mathcal{A}(u) = u^q + \mu & \text{in } Q, \\
  u = 0 & \text{on } \partial \Omega \times (0,T), \\
  u(0) = u_0 & \text{in } \Omega. 
\end{cases}
$$  \hspace{1cm} (2.10)
which satisfies, a.e. in $Q$,
\[ u(x, t) \leq CW_{1,p}^D[\omega](x) + 2||u_0||_{L^\infty(\Omega)}, \tag{2.11} \]
where $C = C(N, p, \Lambda_1, \Lambda_2)$.

In case where $\mathcal{G}$ is an exponential, we introduce the notion of maximal fractional operator, defined for any $\eta \geq 0, R > 0, x_0 \in \mathbb{R}^N$ by
\[ M^\eta_{p, R}[\omega](x_0) = \sup_{r \in (0, R)} \frac{\omega(B(x_0, r))}{r^{pN - p} h_\eta(r)} \quad \text{where} \quad h_\eta(r) = \inf((-\ln r)^{-\eta}, (\ln 2)^{-\eta}). \]

In the case of absorption, we obtain the following:

**Theorem 2.5** Let $p < N$ and $\tau > 0, \beta > 1, \mu \in M_0(\Omega), f \in L^1(\Omega)$ and $u_0 \in L^1(\Omega)$. Assume that
\[ |\mu| \leq \omega \otimes F, \quad \text{with} \quad \omega \in M^+_0(\Omega), \quad F \in L^1((0, T)), \quad F \geq 0, \]
and that one of the following assumptions is satisfied:
(i) $||F||_{L^\infty((0, T))} \leq 1$, and for some $M_0 = M_0(N, p, \beta, \tau, \Lambda_1, \Lambda_2, D)$,
\[ ||M^\tau_{p, 2D}[\omega]||_{L^\infty(\mathbb{R}^N)} < M_0; \tag{2.12} \]
(ii) there exists $\beta_0 > \beta$ such that $M^\frac{\beta}{\beta + 1}_{p, 2D}[\omega] \in L^\infty(\mathbb{R}^N)$.

Then there exists a $R$-solution to the problem
\[
\begin{cases}
  u_t - A(u) + (e^{|u|^p} - 1)\text{sign}u = f + \mu & \text{in } Q,
  \\
  u = 0 & \text{on } \partial\Omega \times (0, T),
  \\
  u(0) = u_0 & \text{in } \Omega.
\end{cases}
\]

In the case of a source term, we obtain:

**Theorem 2.6** Let $\tau > 0, l \in \mathbb{N}$ and $\beta \geq 1$ such that $l \beta > p - 1$. We set
\[ \mathcal{E}(s) = e^s - \sum_{j=0}^{l-1} \frac{s^j}{j!}, \quad \forall s \in \mathbb{R}. \tag{2.13} \]

Let $\mu \in M^+_b(\Omega)$, such that
\[ \mu \leq \omega \otimes \chi_{(0, T)}, \quad \text{with} \quad \omega \in M^+_0(\Omega). \]

Then, there exist $b_0$ and $M_0$ depending on $N, p, \beta, \tau, l, \Lambda_1, \Lambda_2, D$, such that if
\[ ||M^{(p-1)(\beta - 1)}_{p, 2D}[\omega]||_{L^\infty(\mathbb{R}^N)} \leq M_0, \quad \text{and} \quad ||u_0||_{L^\infty(\Omega)} \leq b_0, \]
the problem
\[
\begin{cases}
  u_t - A(u) = \mathcal{E}(\tau u^\beta) + \mu & \text{in } Q,
  \\
  u = 0 & \text{on } \partial\Omega \times (0, T),
  \\
  u(0) = u_0 & \text{in } \Omega, \tag{2.14}
\end{cases}
\]
admits a nonnegative $R$-solution $u$, which satisfies, a.e. in $Q$, for some $C = C(N, p, \Lambda_1, \Lambda_2)$,
\[ u(x, t) \leq CW_{1,p}^2[\omega](x) + 2b_0. \tag{2.15} \]
3 Renormalized solutions and stability theorem

Here we recall the definition of renormalized solutions of the problem without perturbation (2.1), given in [25] for $p > p_1$.

Let $\mathcal{M}_a(Q)$ be the set of measures $\mu \in \mathcal{M}_b(Q)$ with support on a set of zero $c_p^Q$-capacity, also called singular.

Let $\mathcal{M}^+_a(Q), \mathcal{M}^+_b(Q), \mathcal{M}^+_a(Q)$ be the positive cones of $\mathcal{M}_b(Q), \mathcal{M}_b(Q), \mathcal{M}_a(Q)$.

Recall that any measure $\mu \in \mathcal{M}_b(Q)$ can be written (in a unique way) under the form

$$\mu = \mu_0 + \mu_s, \text{ where } \mu_0 \in \mathcal{M}_b(Q), \mu_s = \mu^+_s - \mu^-_s, \text{ with } \mu^+_s, \mu^-_s \in \mathcal{M}^+_a(Q).$$

In turn $\mu_0 \in \mathcal{M}_0(Q)$ admits (at least) a decomposition under the form

$$\mu_0 = f - \text{div} g + h, \quad f \in L^1(Q), \quad g \in (L^p(Q))^N, \quad h \in L^p((0,T);W^{1,p}_b(\Omega)),$$

see [16]; and we write $\mu_0 = (f,g,h)$.

We set $T_k(r) = \max\{\min[r,k], -k\}$, for any $k > 0$ and $r \in \mathbb{R}$. If $u$ is a measurable function defined and finite a.e. in $Q$, such that $T_k(u) \in L^p((0,T);W^{1,p}_b(\Omega))$ for any $k > 0$, there exists a measurable function $w$ from $Q$ into $\mathbb{R}^N$ such that $\nabla T_k(u) = \chi_{|u| \leq k} w$, a.e. in $Q$, and for any $k > 0$. We define the gradient $\nabla u$ of $u$ by $w = \nabla u$.

**Definition 3.1** Let $u_0 \in L^1(\Omega), \mu = \mu_0 + \mu_s \in \mathcal{M}_b(Q)$. A measurable function $u$ is a renormalized solution, called **R-solution** of (2.1) if there exists a decomposition $(f,g,h)$ of $\mu_0$ such that

$$U = u - h \in L^\sigma(0,T;W^{1,\sigma}_b(\Omega)) \cap L^\infty(0,T;L^1(\Omega)), \quad \forall \sigma \in [1,m_c); \quad T_k(U) \in L^p((0,T);W^{1,p}_b(\Omega)), \quad \forall k > 0;$$

and:

(i) for any $S \in W^{2,\infty}(\mathbb{R})$ such that $S'$ has compact support on $\mathbb{R}$, and $S(0) = 0$,

$$-\int_Q S(u_0)\varphi(0)dx - \int_Q \varphi_t S(U) dx dt + \int_Q S'(U) A(x,t,\nabla u) \cdot \nabla \varphi dx dt + \int_Q S''(U) \varphi A(x,t,\nabla u) \nabla U dx dt = \int_Q f S'(U) \varphi dx dt + \int_Q g \nabla (S(U) \varphi) dx dt,$$

for any $\varphi \in L^p((0,T);W^{1,p}_b(\Omega)) \cap L^\infty(Q)$ such that $\varphi \in L^p((0,T);W^{1,p}_b(\Omega)) + L^1(\Omega)$ and $\varphi(., T) = 0$;

(ii) for any $\phi \in C(\overline{Q})$,

$$\lim_{m \to -\infty} \frac{1}{m} \int_{\{m \leq U < 2m\}} \phi A(x,t,\nabla u) \cdot \nabla U dx dt = \int_Q \phi d\mu^+_s,$$

$$\lim_{m \to +\infty} \frac{1}{m} \int_{\{-m > U > -2m\}} \phi A(x,t,\nabla u) \cdot \nabla U dx dt = \int_Q \phi d\mu^-_s.$$

In the sequel we consider the problem (1.1) where $\mu \in \mathcal{M}_b(Q), u_0 \in L^1(\Omega)$. We say that $u$ is a R-solution of problem (1.1) if $G(u) \in L^1(\Omega)$ and $u$ is a R-solution of (2.1) with data $(\mu + G(u), u_0)$.

We recall some properties of R-solutions which we proved in [7, Propositions 2.8,2.10 and Remark 2.9]:

6
Theorem 3.4 Let \( \mu \in L^1(\Omega) \) and \( u_0 \in L^1(\Omega) \), and \( u \) be the (unique) \( R \)-solution of problem (1.1) with data \( \mu \) and \( u_0 \). Then

\[
\text{meas} \{ |u| > k \} \leq C(\|u_0\|_{L^1(\Omega)} + \|\mu\|(\Omega)) \frac{\mu_{\Omega \setminus N}}{\kappa} k^{-\rho_c}, \quad \forall k > 0, \tag{3.1}
\]

for some \( C = C(N, p, \Lambda_1, \Lambda_2) \).

Proposition 3.3 Let \( \{\mu_n\} \subset M_b(\Omega) \), and \( \{u_{0,n}\} \subset L^1(\Omega) \), with

\[
\sup_n |\mu_n|(Q) < \infty, \quad \text{and} \quad \sup_n \|u_{0,n}\|_{L^1(\Omega)} < \infty.
\]

Let \( \{u_n\} \) be a sequence of \( R \)-solutions of (1.1) with data \( \mu_n = \mu_{n,0} + \mu_{n,s} \) and \( u_{0,n} \), relative to a decomposition \( (f_n, g_n, h_n) \) of \( \mu_{n,0} \). Assume that \( \{f_n\} \) is bounded in \( L^1(\Omega) \), \( \{g_n\} \) bounded in \( (L^p(\Omega))^N \) and \( \{h_n\} \) converges in \( L^p(0,T; W_0^{1,p}(\Omega)) \).

Then, up to a subsequence, \( \{u_n\} \) converges to a function \( u \) a.e. in \( Q \) and in \( L^s(\Omega) \) for any \( s \in [1, m_c) \). Moreover, if \( \{\mu_n\} \) is bounded in \( L^1(\Omega) \), then \( \{u_n\} \) converges to \( u \) in \( L^s(0,T; W_0^{1,s}(\Omega)) \) in \( s \in [1, p - \frac{N}{N+1}] \).

Our results are based on the stability theorem that we obtained for problem (2.1) in [7], extending the elliptic result of [15, Theorem 3.4] to the parabolic case. Note that it is valid under more general assumptions on the operator \( A \), see [7]. Recall that a sequence \( \{\mu_n\} \subset M_b(\Omega) \) converges to \( \mu \in M_b(\Omega) \) in the narrow topology of measures if

\[
\lim_{n \to \infty} \int_Q \varphi d\mu_n = \int_Q \varphi d\mu, \quad \forall \varphi \in C(\Omega) \cap L^\infty(Q).
\]

Theorem 3.4 Let \( p > p_1, u_0 \in L^1(\Omega) \), and

\[
\mu = f - \text{div} g + h_t + \mu^+ + \mu^- \in M_b(\Omega),
\]

with \( f \in L^1(\Omega), g \in (L^p(\Omega))^N, h \in L^p((0,T); W_0^{1,p}(\Omega)) \) and \( \mu^+, \mu^- \in M_1^+(\Omega) \). Let \( u_{0,n} \in L^1(\Omega) \),

\[
\mu_n = f_n - \text{div} g_n + \text{div} h_n, \text{ and } \rho_n, \eta_n \in M_b(\Omega),
\]

with \( f_n \in L^1(\Omega), g_n \in (L^p(\Omega))^N, h_n \in L^p((0,T); W_0^{1,p}(\Omega)) \), and \( \rho_n, \eta_n \in M_b^+(\Omega) \), such that

\[
\rho_n = \rho_n^1 - \text{div} \rho_n^2 + \rho_{n,s}, \quad \eta_n = \eta_n^1 - \text{div} \eta_n^2 + \eta_{n,s},
\]

with \( \rho_n^1, \eta_n^1 \in L^1(\Omega), \rho_n^2, \eta_n^2 \in (L^p(\Omega))^N \) and \( \rho_{n,s}, \eta_{n,s} \in M_1^+(\Omega) \). Assume that

\[
\sup_n |\mu_n|(Q) < \infty,
\]

and \( \{u_{0,n}\} \) converges to \( u_0 \) strongly in \( L^1(\Omega) \), \( \{f_n\} \) converges to \( f \) weakly in \( L^1(\Omega) \), \( \{g_n\} \) converges to \( g \) strongly in \( (L^p(\Omega))^N \), \( \{h_n\} \) converges to \( h \) strongly in \( L^p((0,T); W_0^{1,p}(\Omega)) \), \( \{\rho_n\} \) converges to \( \mu^+ \) and \( \{\eta_n\} \) converges to \( \mu^- \) in the narrow topology of measures; and \( \{\rho_n^1\}, \{\eta_n^1\} \) are bounded in \( L^1(\Omega) \), and \( \{\rho_n^2\}, \{\eta_n^2\} \) bounded in \( (L^p(\Omega))^N \).

Let \( \{u_n\} \) be a sequence of \( R \)-solutions of

\[
\begin{cases}
  u_{n,t} - A(u_n) = \mu_n & \text{in } Q, \\
  u_n = 0 & \text{on } \partial\Omega \times (0,T), \\
  u_n(0) = u_{0,n} & \text{in } \Omega.
\end{cases}
\]
relative to the decomposition \((f_n + \rho_n^1 - \eta_n^1, g_n + \rho_n^2 - \eta_n^2, h_n)\) of \(\mu_{n,0}\). Let \(U_n = u_n - h_n\).

Then up to a subsequence, \(\{u_n\}\) converges a.e. in \(Q\) to a \(R\)-solution \(u\) of (2.1), and \(\{U_n\}\) converges a.e. in \(Q\) to \(U = u - h\). Moreover, \(\{\nabla u_n\}, \{\nabla U_n\}\) converge respectively to \(\nabla u, \nabla U\) a.e. in \(Q\), and \(\{T_k(U_n)\}\) converge to \(T_k(U)\) strongly in \(L^p((0,T);W_{0,1}^{1,p}(\Omega))\) for any \(k > 0\).

For applying Theorem 3.4, we require some approximation properties of measures, see [7]:

**Proposition 3.5** Let \(\mu = \mu_0 + \mu_s \in M_+^e(Q)\) with \(\mu_0 \in M_0^e(Q)\) and \(\mu_s \in M_+^a(Q)\).

(i) Then, we can find a decomposition \(\mu_0 = (f, g, h)\) with \(f \in L^1(Q), g \in (L^p(Q))^N, h \in L^p(0,T;W_{0,1}^{1,p}(\Omega))\) such that

\[
\|f\|_{L^1(Q)} + \|g\|_{(L^p(Q))^N} + \|h\|_{L^p(0,T;W_{0,1}^{1,p}(\Omega))} + \mu_s(\Omega) \leq 2\mu(Q).
\]  

(3.2)

(ii) Furthermore, there exist sequences of measures \(\mu_{0,n} = (f_n, g_n, h_n)\) and \(\mu_{s,n} \in C_c^\infty(Q)\) strongly converge to \(f, g, h\) in \(L^1(Q), (L^p(Q))^N\) and \(L^p(0,T;W_{0,1}^{1,p}(\Omega))\) respectively, and \(\mu_{s,n} \in (C_c^\infty(Q))^+\) converges to \(\mu_s\) and \(\mu_{0,n} + \mu_{s,n}\) converges to \(\mu\) in the narrow topology of measures, satisfying \(|\mu_{n}(Q)| \leq \mu(Q)\),

\[
\|f_n\|_{L^1(Q)} + \|g_n\|_{(L^p(Q))^N} + \|h_n\|_{L^p(0,T;W_{0,1}^{1,p}(\Omega))} + \mu_{s,n}(\Omega) \leq 2\mu(Q).
\]  

(3.3)

In particular we use in the sequel a property of approximation by nondecreasing sequences:

**Proposition 3.6** Let \(\mu \in M_+^a(Q)\). Let \(\{\mu_n\}\) be a nondecreasing sequence in \(M_+^a(Q)\) converging to \(\mu\) in \(M_0(Q)\). Then, there exist \(f_n, f \in L^1(Q), g_n, g \in (L^p(Q))^N\) and \(h_n, h \in L^p(0,T;W_{0,1}^{1,p}(\Omega))\), \(\mu_{n,s}, \mu_s \in M_+^a(Q)\) such that

\[
\mu = f - \text{div}\, g + h, \quad \mu_s = f_n - \text{div}\, g_n + (h_n) + \mu_{n,s},
\]

and \(\{f_n\}, \{g_n\}, \{h_n\}\) strongly converge to \(f, g, h\) in \(L^1(Q), (L^p(Q))^N\) and \(L^p(0,T;W_{0,1}^{1,p}(\Omega))\) respectively, and \(\{\mu_{n,s}\}\) converges to \(\mu_s\) (strongly) in \(M_0(Q)\) and

\[
\|f_n\|_{L^1(Q)} + \|g_n\|_{(L^p(Q))^N} + \|h_n\|_{L^p(0,T;W_{0,1}^{1,p}(\Omega))} + \mu_{s,n}(\Omega) \leq 2\mu(Q).
\]  

(3.4)

As a consequence of the above results, we get the following:

**Corollary 3.7** (i) Let \(u_0 \in L^1(\Omega)\) and \(\mu \in M_0(\Omega)\). Then there exists a \(R\)-solution \(u\) to the problem 2.1 with data \((\mu, u_0)\) such that \(u\) satisfies (3.1).

(ii) Furthermore, if \(v_0 \in L^1(\Omega)\) and \(\nu \in M_0(\Omega)\) such that \(u_0 \leq v_0\) and \(\mu \leq \nu\), then one can find \(R\)-solutions \(u\) and \(v\) to the problem 2.1 with respective data \((\mu, u_0)\) and \((\nu, v_0)\) such that \(u \leq v\), \(u\) satisfies (3.1) and

\[
\text{meas}\{|v| > k\} \leq C(\|v_0\|_{L^1(\Omega)} + \|\nu|_{(Q)})^\frac{k}{\alpha}\mu_p, \quad \forall k > 0.
\]  

(3.5)

**Proof.** (i) We approximate \(\mu\) by a smooth sequence \(\{\mu_n\}\) defined at Proposition 3.5-(ii) and apply Proposition 3.2 and Theorem 3.4.

(ii) We set \(w_0 = v_0 - u_0 \geq 0\) and \(\lambda = \omega - \mu \geq 0\). In the same way, we consider a nonnegative, smooth sequence \((\lambda_n, w_{0,n})\) of approximations of \((\lambda, w_{0,n})\) defined at Proposition 3.5-(ii). Let \(v_n\) be the solution of the problem with data \((\lambda_n + \mu_n, w_{0,n} + u_{0,n})\). Clearly, \(u_n \leq v_n\) and \((\lambda_n + \mu_n, w_{0,n} + u_{0,n})\) is an approximation of data \((\omega, v_0)\) in the sense of Theorem 3.4, then we reach the conclusion. ■
4 Subcritical case

We first consider the subcritical case with absorption. We obtain Theorem 2.1 as a direct consequence of Theorem 3.4 and Proposition 3.5. We follow the well-known technique introduced in [4] for the elliptic problem with absorption

\[-A(u) + G(u) = \omega \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,\]  

(4.1)

where \( \omega \in \mathcal{M}_b(\Omega) \), \( p > 1 \), and \( G \) is nondecreasing and odd, and \( \int_1^\infty G(s)s^{-(N-1)p/(N-p)}ds < \infty \).

**Proof of Theorem 2.1.** Let \( \mu = \mu_0 + \mu_s \in \mathcal{M}_b(\Omega) \), with \( \mu_0 \in \mathcal{M}_0(\Omega), \mu_s \in \mathcal{M}_s(\Omega) \), and \( u_0 \in L^1(\Omega) \). By Proposition 3.5, we can find \( f_n, g_{n,i}, h_{n,i} \in C_\infty(\Omega) \) which strongly converge to \( f_i, g_i, h_i \) in \( L^1(\Omega), (L^p(\Omega))' \) and \( L^p((0,T);W_{0,1}^{1,p}(\Omega)) \) respectively, for \( i = 1, 2 \), such that \( \mu_0^+ = (f_1, g_1, h_1) \), \( \mu_0^- = (f_2, g_2, h_2) \), and \( \mu_{n,0,i} = (f_{n,i}, g_{n,i}, h_{n,i}) \), converge respectively for \( i = 1, 2 \) to \( \mu_0^+, \mu_0^- \) in the narrow topology; and we can find nonnegative \( \mu_{n,s,i} \in C_\infty(\Omega), i = 1, 2 \), converging respectively to \( \mu_s^+, \mu_s^- \) in the narrow topology.

Furthermore, if we set

\[ \mu_n = \mu_{n,0,1} - \mu_{n,0,2} + \mu_{n,s,1} - \mu_{n,s,2}, \]

then \( |\mu_n|(Q) \leq |\mu|(Q) \). Consider a sequence \( \{u_{0,n}\} \subset C_\infty(\Omega) \) which strongly converges to \( u_0 \) in \( L^1(\Omega) \) and satisfies \( ||u_{0,n}||_{1,\Omega} \leq ||u_0||_{L^1(\Omega)} \).

Let \( u_n \) be a solution of

\[ \begin{cases} \{ u_n \} - A(u_n) + G(u_n) = \mu_n & \text{in } Q, \\ u_n = 0 & \text{on } \partial \Omega \times (0,T), \\ u_n(0) = u_{0,n} & \text{in } \Omega. \end{cases} \]

We can choose \( \varphi = \varepsilon^{-1}T_\varepsilon(u_n) \) as test function of above problem. Since

\[ \int_Q (\varepsilon^{-1}T_\varepsilon(u_n))_t dxdt = \int \Omega \varepsilon^{-1}T_\varepsilon(u_n(T))dx - \int \Omega \varepsilon^{-1}T_\varepsilon(u_n,0)dx \geq -||u_{0,n}||_{L^1(\Omega)}, \]

there holds from (1.2)

\[ \int_Q G(x,t,u_n)\varepsilon^{-1}T_\varepsilon(u_n)dxdt \leq |\mu_n|(Q) + ||u_{0,n}||_{L^1(\Omega)} \leq |\mu|(Q) + ||u_0||_{L^1(\Omega)}. \]

Letting \( \varepsilon \to 0 \), we obtain

\[ \int_Q |G(x,t,u_n)|dxdt \leq |\mu|(Q) + ||u_0||_{L^1(\Omega)}. \]

Next we apply the estimate (3.1) of Proposition 3.2 to \( u_n \), with initial data \( u_{0,n} \) and measure data \( \mu_n - G(u_n) \in L^1(\Omega) \). We get for any \( s > 0 \) and any \( n \in \mathbb{N} \),

\[ \text{meas} \{ |u_n| \geq s \} \leq M s^{-p_c}, \quad M = C(|\mu|(Q) + ||u_0||_{L^1(\Omega)}), \quad C = C(N, p, A_1, A_2). \]

For any \( L > 1 \), we set \( G_L(s) = \chi_{[L,\infty)}(s)G(s) \), and \( |u_n|^+(s) = \inf \{ a > 0 : \text{meas} \{ |u_n| > a \} \leq s \} \). For any \( s \geq 0 \), we obtain

\[ \int_{|u_n| \geq L} G(|u_n|)dxdt = \int_Q G_L(|u_n|)dxdt \leq \int_0^\infty G_L(|u_n|^+(s))ds \]

(4.2)

Since \( |G(x,t,u_n)| \leq G(|u_n|) \), we deduce that \( \{ G(u_n) \} \) is equi-integrable. Then, from Proposition 3.3, up to a subsequence, \( \{ u_n \} \) converges to some function \( u \), a.e. in \( Q \), and \( \{ G(u_n) \} \) converges to \( G(u) \) in \( L^1(\Omega) \). Therefore, applying Theorem 3.4, \( u \) is a R-solution of (2.4). \( \blacksquare \)
Next we study the subcritical case with a source term. We proceed by induction by constructing a nondecreasing sequence of solutions. Here we meet a difficulty, due to the possible nonuniqueness of the solutions, that we solve by using Corollary 3.7.

**Proof of Theorem 2.2.** Let \( \{u_n\}_{n \geq 1} \) be defined by induction as nonnegative R-solutions of

\[
\begin{cases}
(u_1)_t - A(u_1) = \mu & \text{in } Q, \\
u_1 = 0 & \text{on } \partial \Omega \times (0, T), \\
u_1(0) = u_0 & \text{in } \Omega,
\end{cases}
\]

\[
\begin{cases}
(u_{n+1})_t - A(u_{n+1}) = \mu + \lambda G(u_n) & \text{in } Q, \\
u_{n+1} = 0 & \text{on } \partial \Omega \times (0, T), \\
u_{n+1}(0) = u_0 & \text{in } \Omega,
\end{cases}
\]

From Corollary 3.7 we can assume that \( \{u_n\} \) is nondecreasing and satisfies, for any \( s > 0 \) and \( n \in \mathbb{N} \)

\[
\text{meas } \{|u_n| \geq s\} \leq C_1 K_n s^{-p_c},
\]

where \( C_1 \) does not depend on \( s, n, \) and

\[
K_1 = (|u_0|)_{\text{L}^1(\Omega)} + |\mu|(Q) \frac{1}{p_c} + |\mu|(|\mu| + \lambda |G(u_0)|) \frac{1}{p_c},
\]

\[
K_{n+1} = (|u_0|)_{\text{L}^1(\Omega)} + |\mu|(|\mu| + \lambda |G(u_n)|) \frac{1}{p_c},
\]

for any \( n \geq 1 \). Take \( \varepsilon = \lambda + |\mu|(Q) + |u_0|_{\text{L}^1(\Omega)} \leq 1 \). Denoting by \( C_i \) some constants independent on \( n, \varepsilon, \) there holds \( K_1 \leq C_2 \varepsilon, \) and for \( n \geq 1 \),

\[
K_{n+1} \leq C_3 \varepsilon (|G(u_n)|)_{\text{L}^1(\Omega)} + 1).
\]

From (4.2) and (4.3), we find

\[
|G(u_n)|_{\text{L}^1(Q)} \leq |Q| G(2) + \int_{\{u_n \geq 2\}} G(u_n) dx dt \leq |Q| G(2) + C_4 K_n \int_2^\infty G(s) s^{-1+p_c} ds.
\]

Thus, \( K_{n+1} \leq C_5 \varepsilon (K_n + 1) \). Therefore, if \( \varepsilon \) is small enough, \( \{K_n\} \) is bounded. Since \( \{u_n\} \) is nondecreasing, from (4.2) and the relation \( G(x, t, u_n) \leq G(u_n) \), we deduce that \( \{G(u_n)\} \) converges. Then by Theorem 3.4, up to a subsequence, \( \{u_n\} \) converges to a R-solution \( u \) of (2.5).

**Remark 4.1** Theorems 2.1 and 2.2 are still valid for operators \( A \) also depending on \( t \), satisfying conditions analogous to (1.2), (1.3).

## 5 General case with absorption terms

In the sequel we combine the results of Theorem 3.4 with delicate techniques introduced in [9] for the elliptic problem (4.1), for proving Theorems 2.3 and 2.5. In these proofs the use of the elliptic Wolff potential is an essential tool.

We recall a first result obtained in [9, Corollary 3.4 and Theorem 3.8] for the elliptic problem without perturbation term, inspired from [27, Theorem 2.1]:

**Theorem 5.1** Let \( 1 < p < N, \Omega \) be a bounded domain of \( \mathbb{R}^N \) and \( \omega \in \mathcal{M}^b(\Omega) \) with compact support in \( \Omega. \) Suppose that \( u_n \) is a solution of problem

\[
\begin{cases}
-A(u_n) = \varphi_n * \omega & \text{in } \Omega, \\
u_n = 0 & \text{on } \partial \Omega,
\end{cases}
\]

\[
\begin{cases}
-A(u_n) = \varphi_n * \omega & \text{in } \Omega, \\
u_n = 0 & \text{on } \partial \Omega,
\end{cases}
\]

Thus, \( K_{n+1} \leq C_5 \varepsilon (K_n + 1) \). Therefore, if \( \varepsilon \) is small enough, \( \{K_n\} \) is bounded. Since \( \{u_n\} \) is nondecreasing, from (4.2) and the relation \( G(x, t, u_n) \leq G(u_n) \), we deduce that \( \{G(u_n)\} \) converges. Then by Theorem 3.4, up to a subsequence, \( \{u_n\} \) converges to a R-solution \( u \) of (2.5).

**Remark 4.1** Theorems 2.1 and 2.2 are still valid for operators \( A \) also depending on \( t \), satisfying conditions analogous to (1.2), (1.3).

## 5 General case with absorption terms

In the sequel we combine the results of Theorem 3.4 with delicate techniques introduced in [9] for the elliptic problem (4.1), for proving Theorems 2.3 and 2.5. In these proofs the use of the elliptic Wolff potential is an essential tool.

We recall a first result obtained in [9, Corollary 3.4 and Theorem 3.8] for the elliptic problem without perturbation term, inspired from [27, Theorem 2.1]:

**Theorem 5.1** Let \( 1 < p < N, \Omega \) be a bounded domain of \( \mathbb{R}^N \) and \( \omega \in \mathcal{M}^b(\Omega) \) with compact support in \( \Omega. \) Suppose that \( u_n \) is a solution of problem

\[
\begin{cases}
-A(u_n) = \varphi_n * \omega & \text{in } \Omega, \\
u_n = 0 & \text{on } \partial \Omega,
\end{cases}
\]

Thus, \( K_{n+1} \leq C_5 \varepsilon (K_n + 1) \). Therefore, if \( \varepsilon \) is small enough, \( \{K_n\} \) is bounded. Since \( \{u_n\} \) is nondecreasing, from (4.2) and the relation \( G(x, t, u_n) \leq G(u_n) \), we deduce that \( \{G(u_n)\} \) converges. Then by Theorem 3.4, up to a subsequence, \( \{u_n\} \) converges to a R-solution \( u \) of (2.5).

**Remark 4.1** Theorems 2.1 and 2.2 are still valid for operators \( A \) also depending on \( t \), satisfying conditions analogous to (1.2), (1.3).
where \( \varphi_n \) is a sequence of mollifiers in \( \mathbb{R}^N \). Then, up to subsequence, \( u_n \) converges a.e in \( \Omega \) to a renormalized solution \( u \) of

\[
\begin{align*}
-\mathcal{A}(u) &= \omega \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

in the elliptic sense of [15], satisfying

\[
-\kappa W_{1,p}^{2D}[\omega^-] \leq u \leq \kappa W_{1,p}^{2D}[\omega^+] \tag{5.1}
\]

where \( \kappa \) is a constant which only depends of \( N, p, \Lambda_1, \Lambda_2 \).

Next we give a general result for the parabolic problem (1.5) with absorption:

**Theorem 5.2** Let \( p < N \), and assume that \( \gamma \mapsto \mathcal{G}(x,t,\gamma) \) is nondecreasing and odd, for a.e. \((x,t)\) in \( Q \).

Let \( \mu_1, \mu_2 \in M_b^1(Q) \) such that there exist \( \{\omega_n\} \subset M_b^\infty(\Omega) \) and nondecreasing sequences \( \{\mu_{1,n}\}, \{\mu_{2,n}\} \) in \( M_b^1(Q) \) with compact support in \( Q \), converging to \( \mu_1, \mu_2 \), respectively in the narrow topology, and satisfying

\[
\mu_{1,n}, \mu_{2,n} \leq \omega_n \otimes \chi(0,T), \quad \text{and} \quad \mathcal{G}(n + \kappa W_{1,p}^{2D}[\omega_n]) \in L^1(Q),
\]

where the constant \( \kappa \) is given at Theorem 5.1. Let \( u_0 \in L^1(\Omega) \), and \( \mu = \mu_1 - \mu_2 \).

Then there exists a \( R \)-solution \( u \) of problem (1.5). Moreover if \( u_0 \in L^\infty(\Omega) \), and \( \omega_n \leq \gamma \) for any \( n \in \mathbb{N} \), for some \( \gamma \in M_b^\infty(\Omega) \), then a.e. in \( Q \),

\[
|u(x,t)| \leq \kappa W_{1,p}^{2D}[\gamma](x) + ||u_0||_{L^\infty(\Omega)}, \tag{5.2}
\]

For proving this result, we need two Lemmas:

**Lemma 5.3** Let \( \mathcal{G} \) satisfy the assumptions of Theorem 5.2 and \( \mathcal{G} \in L^\infty(Q \times \mathbb{R}) \). For \( i = 1, 2 \), let \( u_{0,i} \in L^\infty(\Omega) \) be nonnegative, and \( \lambda_i = \lambda_{i,0} + \lambda_{i,n} \in M_b^\infty(Q) \) with compact support in \( Q \), \( \gamma \in M_b^\infty(\Omega) \) with compact support in \( \Omega \) such that \( \lambda_i \leq \gamma \otimes \chi(0,T) \). Let \( \lambda_{i,0} = (f_i, g_i, h_i) \) be a decomposition of \( \lambda_{i,0} \) into functions with compact support in \( Q \).

Then, there exist R-solutions \( u, u_1, u_2, \) to problems

\[
\begin{align*}
\begin{cases}
u_i - \mathcal{A}(u_i) + \mathcal{G}(u_i) &= \lambda_1 - \lambda_2 & \text{in } Q, \\
u &= 0 & \text{on } \partial \Omega \times (0,T), \\
u(0) &= u_{0,i} - u_{0,2} & \text{in } \Omega,
\end{cases}
\end{align*}
\]

relative to decompositions \( (f_{1,n} - f_{2,n} - \mathcal{G}(u_n), g_{1,n} - g_{2,n}, h_{1,n} - h_{2,n}), (f_{i,n} - \mathcal{G}(u_{i,n}), g_{i,n}, h_{i,n}) \), such that a.e. in \( Q \),

\[
-||u_{0,2}||_{L^\infty(\Omega)} - \kappa W_{1,p}^{2D}[\gamma](x) \leq -u_2(x,t) \leq u(x,t) \leq u_1(x,t) \leq \kappa W_{1,p}^{2D}[\gamma](x) + ||u_{0,1}||_{L^\infty(\Omega)}, \tag{5.5}
\]

and

\[
\int_Q |\mathcal{G}(u)| \, dx \, dt \leq \sum_{i=1,2} |\lambda_i(Q) + ||u_{0,i}||_{L^1(\Omega)}| \quad \text{and} \quad \int_Q \mathcal{G}(u_i) \, dx \, dt \leq \lambda_i(Q) + ||u_{0,i}||_{L^1(\Omega)}, \quad i = 1, 2. \tag{5.6}
\]
Furthermore, assume that $\mathcal{H}, \mathcal{K}$ have the same properties as $\mathcal{G}$, and $\mathcal{H}(x, t, s) \leq \mathcal{G}(x, t, s) \leq \mathcal{K}(x, t, s)$ for any $s \in (0, +\infty)$ and a.e. in $Q$. Then, one can find solutions $u_i(\mathcal{H}), u_i(\mathcal{K})$, corresponding to $\mathcal{H}, \mathcal{K}$ with data $\lambda_i$, such that $u_i(\mathcal{H}) \geq u_i \geq u_i(\mathcal{K})$, $i = 1, 2$.

Assume that $\omega_i, \theta_i$ have the same properties as $\lambda_i$ and $\omega, \theta \leq \lambda_i$, $u_{0,i,1}, u_{0,i,2} \in L^{\infty}(\Omega)$, $u_{0,i,2} \leq u_{0,i,1}$. Then one can find solutions $u_i(\omega_i), u_i(\theta_i)$, corresponding to $(\omega, u_{0,i,2}, \theta_i, u_{0,i,1})$, such that $u_i(\omega_i, u_{0,i,2}) \leq u_i \leq u_i(\theta_i, u_{0,i,1})$.

**Proof.** Let $\{\varphi_{1,n}\}, \{\varphi_{2,n}\}$ be sequences of mollifiers in $\mathbb{R}$ and $\mathbb{R}^N$, and $\varphi_n = \varphi_{1,n} \varphi_{2,n}$. Set $\gamma_n = \varphi_{2,n} \ast \gamma$, and for $i = 1, 2$, $u_{0,i,n} = \varphi_{2,n} \ast u_{0,i}$,

\[
\lambda_{i,n} = \varphi_n \ast \lambda_i = f_{i,n} - \text{div}(g_{i,n}) + (h_{i,n})_t + \lambda_{i,s,n},
\]

where $f_{i,n} = \varphi_n \ast f_i, g_{i,n} = \varphi_n \ast g_i, h_{i,n} = \varphi_n \ast h_i, \lambda_{i,s,n} = \varphi_n \ast \lambda_{i,s}$, and

\[
\lambda_n = \lambda_{1,n} - \lambda_{2,n} = f_n - \text{div}(g_n) + (h_n)_t + \lambda_{s,n},
\]

where $f_n = f_{1,n} - f_{2,n}, g_n = g_{1,n} - g_{2,n}, h_n = h_{1,n} - h_{2,n}, \lambda_{s,n} = \lambda_{1,s,n} - \lambda_{2,s,n}$. Then for $n$ large enough, $\lambda_{1,n}, \lambda_{2,n}, \lambda_n \in C_c^\infty(Q), \gamma_n \in C_c^\infty(\Omega)$. Thus there exist unique solutions $u_n, u_{i,n}, v_{i,n}, i = 1, 2$, of problems

\[
\begin{cases}
(u_n)_t - A(u_n) + G(u_n) = \lambda_{1,n} - \lambda_{2,n} & \text{in } Q, \\
u_n = 0 & \text{on } \partial\Omega \times (0, T), \\
u_n(0) = u_{0,i,n} - u_{0,2,n} & \text{in } \Omega,
\end{cases}
\]

\[
\begin{cases}
(u_{i,n})_t - A(u_{i,n}) + G(u_{i,n}) = \lambda_{i,n} & \text{in } Q, \\
u_{i,n} = 0 & \text{on } \partial\Omega \times (0, T), \\
u_{i,n}(0) = u_{i,n} & \text{in } \Omega,
\end{cases}
\]

\[-A(w_n) = \gamma_n \text{ in } \Omega, \quad w_n = 0 \text{ on } \partial\Omega,
\]

such that

\[-\|u_{0,2}\|_{L^{\infty}(\Omega)} - w_n(x) \leq -u_{2,n}(x, t) \leq u_n(x, t) \leq u_{1,n}(x, t) \leq w_n(x) + \|u_{0,1}\|_{L^{\infty}(\Omega)} \quad \text{a.e. in } Q.
\]

Otherwise, as in the Proof of Theorem 2.1, (i), there holds

\[
\int_Q |G(u_n)| dxdt \leq \sum_{i=1,2} (\lambda_i(\Omega) + \|u_{0,i,n}\|_{L^1(\Omega)}), \quad \text{and} \quad \int_Q |G(u_{i,n})| dxdt \leq \lambda_i(\Omega) + \|u_{0,i,n}\|_{L^1(\Omega)}, \quad i = 1, 2.
\]

From Proposition 3.3, up to a common subsequence, $\{u_n, u_{1,n}, u_{2,n}\}$ converge to some $(u, u_1, u_2)$, a.e. in $Q$. Since $G$ is bounded, in particular, $\{G(u_n)\}$ converges to $G(u)$ and $\{G(u_{i,n})\}$ converges to $G(u_i)$ in $L^1(Q)$. Thus, (5.6) is satisfied. Moreover $\{\lambda_{i,n} - G(u_{i,n}), f_{i,n} - G(u_{i,n}), g_{i,n}, h_{i,n}, \lambda_{i,s,n}, u_{0,i,n}\}$ is an approximation of $\{\lambda - G(u_i), f_i - G(u_i), g_i, h_i, \lambda_{i,s}, u_{0,i}\}$, and $\{\lambda_n - G(u_n), f_n - G(u_n), g_n, h_n, \lambda_{s,n}, u_{0,1,n} - u_{0,2,n}\}$ is an approximation of $\{\lambda_1 - \lambda_2 - G(u), f - G(u), g, h, \lambda, u_{0,1} - u_{0,2}\}$, in the sense of Theorem 3.4. Thus, we can find (different) subsequences converging a.e. to $u, u_1, u_2$. R-solutions of (5.3) and (5.4). Furthermore, from Theorem 5.1, up to a subsequence, $\{w_n\}$ converges a.e. in $Q$ to a renormalized solution of

\[-A(w) = \gamma \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial\Omega,
\]

such that $w \leq \kappa W_{1,p}^{2p} [\gamma]$, a.e. in $\Omega$. Hence, we get the inequality (5.5). The other conclusions follow in the same way. \hfill \blacksquare

12
Lemma 5.4 Let $\mathcal{G}$ satisfy the assumptions of Theorem 5.2. For $i = 1, 2$, let $u_{0,i} \in L^\infty(\Omega)$ be nonnegative, $\lambda_i \in \mathcal{M}^+_\gamma(\Omega)$ with compact support in $\Omega$, and $\gamma \in \mathcal{M}^+_\gamma(\Omega)$ with compact support in $\Omega$, such that
\[
\lambda_i \leq \gamma \otimes \chi_{(0,T)}, \quad \text{and} \quad \mathcal{G}([u_{0,i}||L^\infty(\Omega) + \kappa \mathbb{W}^{2D}_{1,p}([\gamma])]) \in L^1(\Omega).
\] (5.7)
Let $\lambda_{i,0} = (f_i, g_i, h_i)$ be a decomposition of $\lambda_{i,0}$ into functions with compact support in $Q$.

Then, there exist R-solutions $u, u_1, u_2$ of the problems (5.3) and (5.4), respectively relative to the decompositions $(f_1 - f_2 - G(u), g_1 - g_2, h_1 - h_2)$, $(f_i - \mathcal{G}(u_i), g_i, h_i)$, satisfying (5.5) and (5.6).

Moreover, assume that $\omega_1, \theta_1$ have the same properties as $\lambda_i$ and $\omega_1 \leq \lambda_1 \leq \theta_1$, $u_{0,1,1}, u_{0,1,2} \in L^\infty(\Omega)$, $0 \leq u_{0,1,2} \leq u_{0,1} \leq u_{0,1,1}$. Then, one can find solutions $u_i(\omega_1, u_{0,1,2}), u_i(\theta_1, u_{0,1,1})$, corresponding with $(\omega_1, u_{0,1,2})$, $(\theta_1, u_{0,1,1})$, such that $u_i(\omega_1, u_{0,1,2}) \leq u_i \leq u_i(\theta_1, u_{0,1,1})$.

**Proof.** From Lemma 5.3 there exist R-solutions $u_n, u_{i,n}$ to problems
\[
\begin{aligned}
&\begin{cases}
(u_n)_t - \mathcal{A}(u_n) + T_n(\mathcal{G}(u_n)) = \lambda_1 - \lambda_2 & \text{in } Q, \\
u_n = 0 & \text{on } \partial \Omega \times (0,T), \\
u_n(0) = u_{0,1} - u_{0,2} & \text{in } \Omega,
\end{cases}
&\begin{cases}
(u_{i,n})_t - \mathcal{A}(u_{i,n}) + T_n(\mathcal{G}(u_{i,n})) = \lambda_i & \text{in } Q, \\
u_{i,n} = 0 & \text{on } \partial \Omega \times (0,T), \\
u_{i,n}(0) = u_{0,i} & \text{in } \Omega,
\end{cases}
\end{aligned}
\]
relative to the decompositions $(f_1 - f_2 - T_n(\mathcal{G}(u_n)), g_1 - g_2, h_1 - h_2) - (f_i - T_n(\mathcal{G}(u_i)), g_i, h_i)$; and they satisfy, a.e. in $Q$,
\[
-||u_{0,2}||L^\infty(\Omega) - \kappa \mathbb{W}^{2D}_{1,p}([\gamma]) \leq -u_{2,n}(x,t) \leq u_{n}(x,t) \leq u_{1,n}(x,t) \leq \kappa \mathbb{W}^{2D}_{1,p}([\gamma]) + ||u_{0,1}||L^\infty(\Omega),
\] (5.8)
\[
\int_Q |T_n(\mathcal{G}(u_n))| dx dt \leq \sum_{i=1,2} (\lambda_i(\Omega) + ||u_{0,i}||L^1(\Omega)), \quad \text{and} \quad \int_Q T_n(\mathcal{G}(u_{i,n})) dx dt \leq \lambda_i(\Omega) + ||u_{0,i}||L^1(\Omega).
\]
As in Lemma 5.3, up to a common subsequence, $\{u_n, u_{1,n,2}, u_{2,n}\}$ converges a.e. in $Q$ to $\{u_1, u_2\}$ for which (5.5) is satisfied a.e. in $Q$. From (5.7), (5.8) and the dominated convergence Theorem, we deduce that $\{T_n(\mathcal{G}(u_n))\}$ converges to $\mathcal{G}(u)$ and $\{T_n(\mathcal{G}(u_{i,n}))\}$ converges to $\mathcal{G}(u_i)$ in $L^1(Q)$. Thus, from Theorem 3.4, $u$ and $u_i$ are respective R-solutions of (5.3) and (5.4) relative to the decompositions $(f_1 - f_2 - G(u), g_1 - g_2, h_1 - h_2) - (f_i - \mathcal{G}(u_i), g_i, h_i)$, and (5.5) and (5.6) hold. The last statement follows from the same assertion in Lemma 5.3.

**Proof of Theorem 5.2.** By Proposition 3.6, for $i = 1, 2$, there exist $f_{i,n}, f_i \in L^1(Q); g_{i,n}, g_i \in (L^p(Q))^N$ and $h_{i,n}, h_i \in L^p((0,T); W_0^{1,p}(\Omega)), \{\mu_{i,n,s}, \mu_{i,s} \in \mathcal{M}^+_\gamma(\Omega) \text{ such that} \\
\mu_i = f_i - \text{div } g_i + (h_i)_t + \mu_{i,s}, \quad \mu_{i,n} = f_{i,n} - \text{div } g_{i,n} + (h_{i,n})_t + \mu_{i,n,s},
\]
and $\{f_{i,n}, g_{i,n}, h_{i,n}\}$ strongly converge to $f_i, g_i, h_i$ in $L^1(Q), (L^p(Q))^N$ and $L^p((0,T); W_0^{1,p}(\Omega))$ respectively, and $\{\mu_i\}, \{\mu_{i,s}\}$ converge to $\mu_i, \mu_{i,s}$ (strongly) in $\mathcal{M}_0(\Omega)$, and
\[
||f_{i,n}||L^1(\Omega) + ||g_{i,n}||L^p(\Omega) + ||h_{i,n}||L^p((0,T); W_0^{1,p}(\Omega)) + ||\mu_{i,n,s}||(\Omega) \leq 2\mu(\Omega).
\]
By Lemma 5.4, there exist R-solutions $u_n, u_{i,n}$ to problems
\[
\begin{aligned}
&\begin{cases}
(u_n)_t - \mathcal{A}(u_n) + \mathcal{G}(u_n) = \mu_{1,n} - \mu_{2,n} & \text{in } Q, \\
u_n = 0 & \text{on } \partial \Omega \times (0,T), \\
u_n(0) = T_n(u_0) & \text{in } \Omega,
\end{cases}
&\begin{cases}
(u_{i,n})_t - \mathcal{A}(u_{i,n}) + \mathcal{G}(u_{i,n}) = \mu_{i,n} - \mu_{2,n} & \text{in } Q, \\
u_{i,n} = 0 & \text{on } \partial \Omega \times (0,T), \\
u_{i,n}(0) = T_n(u_0) & \text{in } \Omega,
\end{cases}
\end{aligned}
\]
Since As in the proof of Lemma 5.4, up to a common subsequence \( \{\) obtain that \( |G| \) for \( i = 1, 2, \) relative to the decompositions \((f_i, g_i, h_i, n)\), such that \( \{u_i,n\} \) is nonnegative and nondecreasing, and \(-u_{2,n} \leq u_n \leq u_{1,n}; \) and

\[
\int_Q |G(u_n)|dxdt, \int_Q G(u_n)dxdt \leq \mu_1(Q) + \mu_2(Q) + ||u_0||_{L^1(\Omega)}. \tag{5.9}
\]

As in the proof of Lemma 5.4, up to a common subsequence \( \{u_{n}, u_{1,n}, u_{2,n}\} \) converge a.e. in \( Q \) to \( \{u_{1}, u_{2}\} \). Since \( \{G(u_{i,n})\} \) is nondecreasing, and nonnegative, from the monotone convergence Theorem and (5.9), we obtain that \( \{G(u_{i,n})\} \) converges to \( G(u_i) \) in \( L^1(Q) \), \( i = 1, 2 \). Finally, \( \{G(u_n)\} \) converges to \( G(u) \) in \( L^1(Q) \), since \( |G(u_n)| \leq G(u_{1,n}) + G(u_{2,n}) \). Thus, we can see that

\[
\{\mu_{1,n} - \mu_{2,n} - G(u_n), f_{1,n} - f_{2,n} - G(u_n), g_{1,n} - g_{2,n}, h_{1,n} - h_{2,n}, \mu_{1,n} - \mu_{2,n}, T_n(u_0)\}
\]

is an approximation of \( \{\mu_2 - G(u), f_2 - G(u), g_2, h_2, \mu_2 - \mu_2, 0\} \), in the sense of Theorem 3.4. Therefore, \( u \) is a R-solution of (1.1), and (5.2) holds if \( u_0 \in L^\infty(\Omega) \) and \( \omega_n \leq \gamma \) for any \( n \in \mathbb{N} \) and some \( \gamma \in M^+(\Omega) \).

As a consequence of Theorem 5.2, we get a result for problem (2.1), used in Section 6:

**Corollary 5.5** Let \( u_0 \in L^\infty(\Omega) \), and \( \mu \in M_b(\Omega) \) such that \( |\mu| \leq \omega \otimes \chi_{(0,T)} \) for some \( \omega \in M^+(\Omega) \). Then there exist a R-solution \( u \) of (2.1), such that

\[
|u(x,t)| \leq \kappa W^D_{1,p}[\omega](x) + ||u_0||_{L^\infty(\Omega)}, \quad \text{for a.e.} \ (x,t) \in Q, \tag{5.10}
\]

where \( \kappa \) is defined at Theorem 5.1.

**Proof.** Let \( \{\phi_n\} \) be a nonnegative, nondecreasing sequence in \( C^\infty_c(Q) \) which converges to 1, a.e. in \( Q \). Since \( \{\phi_n\mu^+\}, \{\phi_n\mu^-\} \) are nondecreasing sequences, the result follows from Theorem 5.2.

### 5.1 The power case

First recall some results relative to the elliptic case for the model problem

\[
-\Delta_p u + |u|^{p-1} u = \omega \quad \text{in} \ \Omega, \quad u = 0 \quad \text{on} \ \partial \Omega, \tag{5.11}
\]

with \( \omega \in M_b(\Omega), q > p - 1 > 0 \).

For \( p = 2 \), it is shown in [2] that (5.11) admits a solution if and only if \( \omega \) does not charge the sets of Bessel Cap\(G_{p,\frac{N}{N-q}}\)-capacity zero. For \( p \neq 2 \), existence holds for any measure \( \omega \in M_b(\Omega) \) in the subcritical case

\[
q < p_c := N(p-1)/(N-p) \tag{5.12}
\]

from [4]. Some necessary conditions for existence have been given in [5, 6]. From [9, Theorem 1.1], a sufficient condition for existence is that \( \omega \) does not charge the sets of Cap\(G_{p,\frac{N}{N-q}}\)-capacity zero, and it can be conjectured that this condition is also necessary.

Next we prove Theorem 2.3. We use the following result of [9]:
Proposition 5.6 Let $q > p - 1$ and $\nu \in M_0^+(\Omega)$.

If $\nu$ does not charge the sets of $\text{Cap}_{p, \frac{1}{p}+}$-capacity zero, there exists a nondecreasing sequence $\{\nu_n\} \subset M_0^+(\Omega)$ with compact support in $\Omega$ which converges to $\nu$ strongly in $M_0(\Omega)$ and such that $W_{1,p}^1[\nu_n] \subset L^q(\mathbb{R}^N)$, for any $n \in \mathbb{N}$ and $R > 0$.

**Proof of Theorem 2.3.** Let $f \in L^1(Q)$, $u_0 \in L^1(\Omega)$, and $\mu \in M_b(\Omega)$ such that $|\mu| \leq \omega \otimes F$, where $F \in L^1((0,T))$ and $\omega$ does not charge the sets of $\text{Cap}_{p, \frac{1}{p}+}$-capacity zero. From Proposition 5.6, there exists a nondecreasing sequence $\{\omega_n\} \subset M_0^+(\Omega)$ with compact support in $\Omega$ which converges to $\omega$, strongly in $M_0(\Omega)$, such that $W_{2,p}^1[\omega_n] \subset L^q(\mathbb{R}^N)$. We can write

$$f + \mu = \mu_1 - \mu_2, \quad \mu_1 = f^+ + \mu^+, \quad \mu_2 = f^- + \mu^-,$$

and $\mu^+, \mu^- \leq \omega \otimes F$. We set

$$Q_n = \{(x,t) \in \Omega \times \left(\frac{1}{n},T - \frac{1}{n}\right) : d(x,\partial \Omega) > \frac{1}{n}\}, \quad F_n = T_n(\chi_{\frac{1}{n}T - \frac{1}{n}}F),$$

$$\mu_{1,n} = T_n(\chi_{Q_n} f^+) + \inf\{\mu^+, \omega_n \otimes F_n\}, \quad \mu_{2,n} = T_n(\chi_{Q_n} f^-) + \inf\{\mu^-, \omega_n \otimes F_n\}.$$  

Then $\{\mu_{1,n}\}, \{\mu_{2,n}\}$ are nondecreasing sequences with compact support in $Q$, and

$$\mu_{1,n}, \mu_{2,n} \leq \tilde{\omega}_n \otimes \chi_{(0,T)}, \quad \text{with} \quad \tilde{\omega}_n = n(\chi_{\Omega} + \omega_n),$$

and $(n + \kappa W_{2,p}^1[\tilde{\omega}_n])^q \subset L^1(Q)$. Besides, $\omega_n \otimes F_n$ converges to $\omega \otimes F$ strongly in $M_b(Q)$. Indeed we easily check that

$$||\omega_n \otimes F_n - \omega \otimes F||_{M_b(Q)} \leq ||F_n||_{L^1((0,T))}||\omega_n - \omega||_{M_b(\Omega)} + ||\omega||_{M_b(\Omega)}||F_n - F||_{L^1((0,T))}.$$  

Observe that for any measures $\nu, \theta, \eta \in M_b(Q)$, there holds

$$|\inf\{\nu, \theta\} - \inf\{\nu, \eta\}| \leq |\theta - \eta|,$$

hence $\{\mu_{1,n}\}, \{\mu_{2,n}\}$ converge to $\mu_1, \mu_2$ respectively in $M_0(\Omega)$. Therefore, the result follows from Theorem 5.2.  

**Remark 5.7** From Theorem 2.3, we deduce the existence for any measure $\omega \in M_b(\Omega)$ for $p < p_c$, where $p_c$ is defined at (5.12), since $p_c$ is the critical exponent of the elliptic problem (5.11). Note that $p_c > p_1$. Let $M_{0,c}(\Omega)$ be the set of Radon measures $\omega$ on that do not charge the sets of zero $c^1_p$-capacity, where, for any compact set $K \subset \Omega$,

$$c_1^p(K) = \inf\{\int_\Omega |\nabla \varphi|^p dx : \varphi \geq \chi_K, \varphi \in C_c^\infty(\Omega)\}.$$  

From [16, Theorem 2.16], for any $F \in L^1((0,T))$ with $\int_0^T F(t) dt \neq 0$, and $\omega \in M_b(\Omega)$,

$$\omega \in M_{0,c}(\Omega) \iff \omega \otimes F \in M_b(Q).$$

If $q \geq p_c$, there exist measures $\omega \in M_0^+(\Omega)$ which do not charge the sets of $\text{Cap}_{p, \frac{1}{p}+}$-capacity zero, such that $\omega \notin M_{0,c}(\Omega)$. As a consequence, Theorem 2.3 shows the existence for some measures $\mu \notin M_0(\Omega)$.
Remark 5.8 Let $\mathcal{G} : Q \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that the map $s \mapsto \mathcal{G}(x,t,s)$ is nondecreasing and odd, for a.e. $(x,t)$ in $Q$. Let $\mu \in \mathcal{M}(Q)$, $f \in L^1(Q)$, $u_0 \in L^1(\Omega)$ and $\omega \in \mathcal{M}_0^{\infty}((\Omega)$) such that (2.7) holds.

If $\omega\{x : W_{1,p}^{2D}[\omega](x) = \infty\} = 0$, then, (1.5) has a $R$-solution with data $(f + \mu, u_0)$. The proof is similar to the one of Theorem 2.3, after replacing $\omega$ by $\chi_{W_{1,p}^{2D}[\omega] \leq n} \omega$. Note that $\omega\{x : W_{1,p}^{2D}[\omega](x) = \infty\} = 0$ if and only if $\omega \in \mathcal{M}_{0,\infty}(\Omega)$, see [21].

Remark 5.9 As in [9], from Theorem 5.2, we can extend Theorem 2.3 given for $G(u) = |u|^{q-1} u$, to the case of a function $G(x,t,\cdot)$, odd for a.e. $(x,t) \in Q$, such that

$$ |G(x,t,u)| \leq G(|u|), \quad \int_1^\infty G(s)s^{-q-1}ds < \infty,$$

where $G$ is a nondecreasing continuous, under the condition that $\omega$ does not charge the sets of zero capacity, where for any Borel set $E \subset \mathbb{R}^N$,

$$ \text{Cap}_p \frac{\mathcal{G}}{p, \infty}(E) = \inf \{||\varphi||_{L_{\frac{2}{1+p}}(\mathbb{R}^N)} \geq \mathcal{G}_p \varphi \geq 1 \}$$

where $L_{\frac{2}{1+p}}(\mathbb{R}^N)$ is the Lorentz space of order $(q/(q - p + 1), 1)$.

5.2 The exponential case

Theorem 2.5 extends the elliptic result of [9, Theorem 1.2] to the parabolic case. For the proof, we use the following property of [9, Theorem 2.4]:

Proposition 5.10 Suppose $1 < p < N$. Let $\nu \in \mathcal{M}^+_\mathbb{N}(\Omega)$, $\beta > 1$, and $\delta_0 = ((12\beta)^{-1})p \ln 2$. There exists $C = C(N,p,\beta,D)$ such that, for any $\delta \in (0,\delta_0)$,

$$ \int_\Omega \exp(\delta \frac{(W_{1,p}^{2D}[\nu])^\beta}{||M_{p,2D}[\nu]||_{L_\infty}(\mathbb{R}^N)}) dx \leq \frac{C}{\delta_0 - \delta}. $$

Proof of Theorem 2.5. Let $Q_n$ be defined at (5.14), and $\omega_n = \omega \chi_{\Omega_n}$, where $\Omega_n = \{x \in \Omega : d(x,\partial \Omega) > 1/n\}$. We still consider $\mu_1, \mu_2, F_n, \mu_{1,n}, \mu_{2,n}$ as in (5.13), (5.15).

Case (i): Assume that $||F||_{L_\infty((0,T))} \leq 1$ and (2.12) holds. We have $\mu_{1,n}, \mu_{2,n} \leq n \chi_{\Omega} + \omega$. For any $\varepsilon > 0$, there exists $c_{\varepsilon} = c_{\varepsilon}(\varepsilon, N, p, \beta, \kappa, D) > 0$ such that

$$ (n + \kappa W_{1,p}^{2D}[n \chi_{\Omega} + \omega])^\beta \leq c_{\varepsilon} n \frac{\lambda^\beta}{n^\alpha} + (1 + \varepsilon) \kappa^\beta (W_{1,p}^{2D}[\omega])^\beta $$

a.e. in $\Omega$. Thus,

$$ \exp(\tau(n + \kappa W_{1,p}^{2D}[n \chi_{\Omega} + \omega])^\beta) \leq \exp(\tau c_{\varepsilon} n \frac{\lambda^\beta}{n^\alpha}) \exp(\tau(1 + \varepsilon) \kappa^\beta (W_{1,p}^{2D}[\omega])^\beta).$$

If (2.12) holds with $M_0 = (\delta_0/\tau \kappa^\beta)^{(n-1)/\beta}$ then we can chose $\varepsilon$ such that

$$ \tau(1 + \varepsilon) \kappa^\beta ||M_{p,2D}[\nu]||_{L_\infty(\mathbb{R}^N)} < \delta_0.$$
From Proposition 5.10, we get \( \exp(\tau(1 + \varepsilon)\kappa^2 W_{1,p}^{2D}(\omega)) \in L^1(\Omega) \), which implies \( \exp(\tau(n + \kappa^2 W_{1,p}^{2D}[n\chi_\Omega + \omega])) \in L^1(\Omega) \) for all \( n \). We conclude from Theorem 5.2.

Case (ii): Assume that there exists \( \varepsilon > 0 \) such that \( M_{\beta,2D}^{(p-1)/(\beta+e)'}[\omega] \in L^\infty(\mathbb{R}^N) \). Now we use the inequality \( \mu_1, \mu_2 \leq n(\chi_\Omega + \omega) \). For any \( \varepsilon > 0 \) and any \( n \in \mathbb{N} \) there exists \( \varepsilon_{n, \omega} > 0 \) such that

\[
(n + \kappa W_{1,p}^{2D}[n(\chi_\Omega + \omega)])^\beta \leq \varepsilon_{n, \omega} + \varepsilon(W_{1,p}^{2D}[\omega])^{\beta_n}.
\]

Thus, from Proposition 5.10, we obtain that \( \exp(\tau(n + \kappa^2 W_{1,p}^{2D}[n(\chi_\Omega + \omega)])) \in L^1(\Omega) \) for any \( n \in \mathbb{N} \). We conclude from Theorem 5.2.

\section{General case with source term}

The results of this Section are based on Corollary 5.5 and elliptic techniques of Wolff potential used in [27], [28] and [22, Theorem 2.5].

\subsection{The power case}

Recall some results of [27], [28] for the nonnegative solutions of equation

\[
-\Delta_p u = u^q + \omega \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega.
\]  

(6.1)

It was proved that if \( \omega(E) \leq C \text{Cap}_{\mathbb{R}^N}(E) \) for any compact of \( \mathbb{R}^N \), with \( C \) small enough, problem (6.1) has at least a solution, and conversely if there exists a solution, and \( \omega \) has a compact support, then there exists a constant \( C' \) such that

\[
\omega(E) \leq C' \text{Cap}_{\mathbb{R}^N}(E) \quad \text{for any compact set } E \text{ of } \mathbb{R}^N.
\]

For proving Theorem 2.4 we use the following property of Wolff potentials, shown in [27]:

\begin{theorem}
Let \( q > p - 1, 0 < p < N, \omega \in M_+(\Omega) \). If for some \( \lambda > 0 \),

\[
\omega(E) \leq \lambda \text{Cap}_{\mathbb{R}^N}(E) \quad \text{for any compact set } E \subset \mathbb{R}^N,
\]

(6.2)

then \( (W_{1,p}^{2D}[\omega])^q \in L^1(\Omega) \), and there exists \( M = M(N, p, q, \text{diam}(\Omega)) \) such that, a.e. in \( \Omega \),

\[
W_{1,p}^{2D} ((W_{1,p}^{2D}[\omega])^q) \leq M \lambda^{\frac{q-1}{q-1-p}} W_{1,p}^{2D}[\omega] < \infty.
\]

(6.3)

We deduce the following:

\begin{lemma}
Let \( \omega \in M_+(\Omega) \), and \( b \geq 0 \) and \( K > 0 \). Suppose that \( \{u_m\}_{m \geq 1} \) is a sequence of nonnegative functions in \( \Omega \) that satisfies

\[
u_1 \leq K W_{1,p}^{2D}[\omega] + b,
\]

\[
u_{m+1} \leq K W_{1,p}^{2D}[u_m + \omega] + b \quad \forall m \geq 1.
\]

Assume that \( \omega \) satisfies (6.2) for some \( \lambda > 0 \). Then there exist \( \lambda_0 \) and \( b_0 \), depending on \( N, p, q, K, D \), such that if \( \lambda \leq \lambda_0 \) and \( b \leq b_0 \), then \( W_{1,p}^{2D}[\omega] \in L^q(\Omega) \) and for any \( m \geq 1 \),

\[
u_m \leq 2\beta_p K W_{1,p}^{2D}[\omega] + 2b, \quad \beta_p = \max(1, 3^{\frac{q-1}{2}}).
\]

(6.4)
\end{lemma}
Proof. Clearly, (6.4) holds for \( m = 1 \). Now, assume that it holds at the order \( m \). Then

\[
\dot{u}_m^p \leq 2^{q-1}(2\beta_p)^q K^q(\mathbf{W}_{1,p}^{2D}[\omega])^q + 2^{q-1}(2b)^q.
\]

Using (6.3) we get

\[
u_{m+1} \leq K \mathbf{W}_{1,p}^{2D} \left[ 2^{q-1}(2\beta_p)^q K^q(\mathbf{W}_{1,p}^{2D}[\omega])^q + 2^{q-1}(2b)^q + \omega \right] + b
\]

\[
\leq \beta_p K \left( 1 + \mathbf{W}_{1,p}^{2D}(2b)^q + 2^{2D}[\omega] \right) + b
\]

\[
\leq \beta_p K (A_1 M \lambda_m^{\mu - \frac{1}{2}} + 1) \mathbf{W}_{1,p}^{2D}[\omega] + \beta_p K \mathbf{W}_{1,p}^{2D}(2b)^q + b
\]

\[
= \beta_p K (A_1 M \lambda_m^{\mu - \frac{1}{2}} + 1) \mathbf{W}_{1,p}^{2D}[\omega] + A_2 b \mathbf{\frac{W}{1}}^p + b,
\]

where \( M \) is as in (6.3) and

\[
A_1 = \left( 2^{q-1}(2\beta_p)^q K^q \right)^{1/(p-1)}, \quad A_2 = \beta_p K 2^{q/(p-1)} |B_1|^{1/(p-1)} (p')^{-1}(2D)^{p'}.
\]

Thus, (6.4) holds for \( m = n + 1 \) if we prove that

\[
A_1 M \lambda_m^{\mu - \frac{1}{2}} \leq 1 \quad \text{and} \quad A_2 b \mathbf{\frac{W}{1}}^p \leq b,
\]

which is equivalent to

\[
\lambda \leq \left( A_1 M \right)^{-\frac{1}{p-1}} \quad \text{and} \quad b \leq A_2^{-\frac{p-1}{p-1}}.
\]

Therefore, we obtain the result with \( \lambda_0 = (A_1 M)^{-(p-1)/2} \) and \( b_0 = A_2^{-1} \). }

Proof of Theorem 2.4. From Corollary 3.7 and 5.5, we can construct a sequence of nonnegative nondecreasing R-solutions \( \{u_m\}_{m \geq 1} \), defined in the following way: \( u_1 \) is a R-solution of (2.1), and \( u_{m+1} \) is a nonnegative R-solution of

\[
\begin{align*}
(u_{m+1})_t - A(u_{m+1}) &= \dot{u}_m^p + \mu \quad \text{in } Q, \\
u_{m+1} &= 0 \quad \text{on } \partial Q \times (0, T), \\
u_{m+1}(0) &= u_0 \quad \text{in } \Omega.
\end{align*}
\]

Setting \( \overline{u}_m = \sup_{t \in (0, T)} u_m(t) \) for all \( m \geq 1 \), there holds

\[
\overline{u}_1 \leq \kappa \mathbf{W}_{1,p}^{2D}[\omega] + ||u_0||_{L^\infty(\Omega)},
\]

\[
\overline{u}_{m+1} \leq \kappa \mathbf{W}_{1,p}^{2D}[\overline{u}_m] + \omega + ||u_0||_{L^\infty(\Omega)} \quad \forall m \geq 1.
\]

From Lemma 6.2, we can find \( \lambda_0 = \lambda_0(N, p, q, D) \) and \( b_0 = b_0(N, p, q, D) \) such that if (2.9) is satisfied with \( \lambda_0 \) and \( b_0 \); then

\[
u_m \leq \overline{u}_m \leq 2\beta_p \kappa \mathbf{W}_{1,p}^{2D}[\omega] + 2||u_0||_{L^\infty(\Omega)} \quad \forall m \geq 1.
\]

(6.5)

Thus \( \{u_m\} \) converges a.e. in \( Q \) and in \( L^q(Q) \) to some function \( u \), for which (2.11) is satisfied in \( \Omega \) with \( c = 2\beta_p\kappa \). Finally, one can apply Theorem 3.4 to the sequence of measures \( \{u_m^p + \mu\} \), and obtain that \( u \) is a R-solution of (2.10).
6.2 The exponential case

We end this Section by proving Theorem 2.6. We first recall an approximation property, which is a consequence of [22, Theorem 2.5]:

**Theorem 6.3** Let \( \tau > 0, b \geq 0, K > 0, l \in \mathbb{N} \) and \( \beta \geq 1 \) such that \( l \beta > p - 1 \). Let \( \mathcal{E} \) be defined by (2.13). Let \( \{ v_m \} \) be a sequence of nonnegative functions in \( \Omega \) such that, for some \( K > 0 \),

\[
\begin{align*}
    v_1 &\leq K W^{2D}_{1,p}[\mu] + b, \\
v_{m+1} &\leq K W^{2D}_{1,p}[\mathcal{E}(\tau v_m^\beta) + \mu] + b, \quad \forall m \geq 1.
\end{align*}
\]

Then, there exist \( b_0 \) and \( M_0 \), depending on \( N, p, \beta, \tau, l, K, D \), such that if \( b \leq b_0 \) and

\[
|| M^{(p-1)(\beta-1)}_{p,2D} ||_{L^\infty(\mathbb{R}^N)} \leq M_0,
\]

then, setting \( c_p = 2 \max(1, \frac{2}{p-1}) \),

\[
\exp(\tau(K c_p W^{2D}_{1,p}[\mu] + 2b_0)^\beta) \in L^1(\Omega),
\]

\[
v_m \leq K c_p W^{2D}_{1,p}[\mu] + 2b_0, \quad \forall m \geq 1.
\]

**Proof of Theorem 2.6.** From Corollary 3.7 and 5.5 we can construct a sequence of nonnegative nondecreasing R-solutions \( \{ u_m \}_{m \geq 1} \) defined in the following way: \( u_1 \) is a R-solution of problem (2.1), and by induction, \( u_{m+1} \) is a R-solution of

\[
\begin{align*}
    \left\{ \begin{array}{ll}
        (u_{m+1})_t - A(u_{m+1}) = \mathcal{E}(\tau u_m^\beta) + \mu & \text{in } Q, \\
        u_{m+1} = 0 & \text{on } \partial \Omega \times (0, T), \\
        u_{m+1}(0) = u_0 & \text{in } \Omega.
    \end{array} \right.
\]
\]

And, setting \( \overline{u}_m = \sup_{t \in (0,T)} u_m(t) \), there holds

\[
\begin{align*}
    \overline{u}_1 &\leq K W^{2D}_{1,p}[\omega] + ||u_0||_{L^\infty(\Omega)}, \\
    \overline{u}_{m+1} &\leq K W^{2D}_{1,p}[\mathcal{E}(\tau \overline{u}_m^\beta) + \omega] + ||u_0||_{L^\infty(\Omega)}, \quad \forall m \geq 1.
\end{align*}
\]

Thus, from Theorem 6.3, there exist \( b_0 \in (0,1) \) and \( M_0 > 0 \), depending on \( N, p, \beta, \tau, l, D \), such that, if (6.6) holds, then (6.7) is satisfied with \( v_m = \overline{u}_m \). As a consequence, \( u_m \) is well defined. Thus, \( \{ u_m \} \) converges a.e. in \( Q \) to some function \( u \), for which (2.15) is satisfied in \( \Omega \). Furthermore, \( \{ \mathcal{E}(\tau u_m^\beta) \} \) converges to \( \mathcal{E}(\tau u^\beta) \) in \( L^1(Q) \). Finally, one can apply Theorem 3.4 to the sequence of measures \( \{ \mathcal{E}(\tau u_m^\beta) + \mu \} \), and obtain that \( u \) is a R-solution of (2.14). \( \blacksquare \)

**Remark 6.4** In [22, Theorem 1.1], when \( A = \Delta_p \), we showed that there exist \( M = M(N, p, \beta, \tau, l, D) \) such that if

\[
|| M^{(p-1)(\beta-1)}_{p,2D} ||_{L^\infty(\mathbb{R}^N)} \leq M,
\]

then the problem

\[
\begin{align*}
    \left\{ \begin{array}{ll}
        -\Delta_p v = \mathcal{E}(\tau v^\beta) + \omega & \text{in } \Omega, \\
        v = 0 & \text{on } \partial \Omega.
    \end{array} \right.
\]
\]

has a renormalized solution in the sense of [15]. We claim the following:
Let \( A = \Delta p \) and \( u_0 \equiv 0 \). If (6.9) has a renormalized solution \( v \) and \( \omega \in \mathcal{M}_{0,c}(\Omega) \), then the problem (2.14) in Theorem 2.6 admits a \( R \)-solution \( u \), satisfying \( u(x,t) \leq v(x) \) a.e in \( Q \).

Indeed, since \( \omega \in \mathcal{M}_{0,c}(\Omega) \), there holds \( \mu \in \mathcal{M}_0(Q) \). Otherwise, for any measure \( \eta \in \mathcal{M}_0(Q) \) the problem

\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta p u &= \eta & \text{in } Q, \\
u &= 0 & \text{on } \partial \Omega \times (0, T), \\
u &= 0 & \text{in } \Omega,
\end{align*}
\]

has a (unique) \( R \)-solution, and the comparison principle is valid, see [26]. Thus, as in the proof of Theorem 2.6, we can construct a unique sequence of nonnegative nondecreasing \( R \)-solutions \( \{u_m\}_{m \geq 1} \), defined in the following way: \( u_1 \) is a \( R \)-solution of problem (2.1) and satisfies \( u_1 \leq v \) a.e in \( Q \); and by induction, \( u_{m+1} \) is a \( R \)-solution of (6.8) and satisfies \( u_{m+1} \leq v \) a.e in \( Q \). Then \( \{\mathcal{E}(ru_m^p)\} \) converges to \( \mathcal{E}(ru^p) \) in \( L^1(Q) \).

Finally, \( u := \lim_{n \to \infty} u_n \) is a solution of (2.14). Clearly, this claim is also valid for power source term.

References

[1] Andreianov B., Sbihi K., and Wittbold P., On uniqueness and existence of entropy solutions for a nonlinear parabolic problem with absorption, J. Evol. Equ. 8 (2008), 449-490.
[2] Baras P. and Pierre M., Critère d’existence de solutions positives pour des équations semi-linéaires non monotones, Ann. I.H.P. 2 (1985), 185-212.
[3] Baras P. and Pierre M., Problèmes paraboliques semi-linéaires avec données mesures, Applicable Anal. 18 (1984), 111-149.
[4] Benilan P., Boccardo L., Gallouet T., Gariepy R., Pierre M. and Vázquez J., An \( L^1 \)-theory of existence and uniqueness of solutions of nonlinear elliptic equations, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 22 (1995), no. 2, 241–273.
[5] Bidaut-Véron M.F., Removable singularities and existence for a quasilinear equation, Adv. Nonlinear Studies 3 (2003), 25-63.
[6] Bidaut-Véron M.F., Necessary conditions of existence for an elliptic equation with source term and measure data involving the \( p \)-Laplacian, E.J.D.E., Conference 08 (2002), 23–34.
[7] Bidaut-Véron M.F. and Nguyen-Quoc H., Stability properties for quasilinear parabolic equations with measure data, Arxiv...
[8] Bidaut-Véron M.F., Chasseigne E. and Véron L. Initial trace of solutions odf some quasilinear parabolic equations with absorption, J. Funct. Anal. 193 (2002), 140-205.
[9] Bidaut-Véron M.F., Nguyen Quoc H. and Véron L., Quasilinear Emden-Fowler equations with absorption terms and measure data, J. Math. Pures Appl. 102 (2014), 315-337.
[10] Boccardo L. and Gallouet T., Nonlinear elliptic and parabolic equations involving measure data, J Funct. Anal. 87 (1989), 149-169.
[11] Boccardo L., Gallouet T., and Orsina L., Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data, Ann. Inst. H. Poincaré Anal. non Lin. 13 (1996), 539-555.
[12] Brezis H. and Friedman A., Nonlinear parabolic equations involving measures as initial conditions, J.Math.Pures Appl. 62 (1983), 73-97.
[13] Chen X., Qi Y. and Wang M., *Singular solutions of parabolic p-Laplacian with absorption*, T.A.M.S. 3589 (2007), 5653-5668.

[14] Dall’Aglio A. and Orsina L., *Existence results for some nonlinear parabolic equations with nonregular data*, Diff. Int. Equ. 5 (1992), 1335-1354.

[15] Dal Maso G., Murat F., Orsina L., and Prignet A., *Renormalized solutions of elliptic equations with general measure data*, Ann. Scuola Norm. Sup. Pisa, 28 (1999), 741-808.

[16] Droniou J., Porretta A. and Prignet A., *Parabolic capacity and soft measures for nonlinear equations*, Potential Anal. 19 (2003), 99-161.

[17] Fujita H., On the nonlinear equations $\Delta u + e^u = 0$ and $\partial v/\partial t = \Delta v + e^v$, Bull. Amer. Math. Soc. 75 (1969), 132-135.

[18] Gmira A., *On quasilinear parabolic equations involving measure data*, Asymptotic Anal. 3 (1990), 43-56.

[19] Ioku N., *The Cauchy problem for heat equations with exponential nonlinearities*, J. Diff. Equ. 251 (2011), 1172-1194.

[20] Kamin S. and Vazquez J. L., *Singular solutions of some nonlinear parabolic equations*, J. Analyse Math. 59 (1992), 51-74.

[21] Mikkonen P., *On the Wolff potential and quasilinear elliptic equations involving measures*, Ann. Acad. Sci. Fenn. Ser AI, Math. Dissert. 104 (1996).

[22] Nguyen Quoc H. and Verón L., *Quasilinear and Hessian equations with exponential reaction and measure data*, Ark. for Rat. Mech. Anal. 214 (2014), 235-267.

[23] Nguyen Quoc H., *Potential estimates and quasilinear parabolic equations with measure data*, Arkiv 1405-2587.

[24] Nguyen Phuoc T., *Parabolic equations with exponential nonlinearity and measure data*, preprint (2014).

[25] Petitta F., *Renormalized solutions of nonlinear parabolic equations with general measure data*, Ann. Math. Pura Appl. 187 (2008), 563-604.

[26] Petitta F., Ponce A. and Porretta A., *Diffuse measures and nonlinear parabolic equations*, J. Evol. Equ. 11 (2011), 861-905.

[27] Phuc N. and Verbitsky I., *Quasilinear and Hessian equations of Lane-Emden type*. Ann. of Math. 168 (2008), 859–914.

[28] Phuc N. and Verbitsky I., *Singular quasilinear equations and Hessian equations and inequalities*, J. Funct. Anal. 256 (2009), 1875-1906.

[29] Ruf B. and Terraneo E., *The Cauchy problem for a semilinear equation with singular initial data*, Prog. Nonlinear Diff. Equ. Appl. 50 (2002) 295-309.

[30] Zhao J., *Source-type solutions of a quasilinear degenerate parabolic equation with absorption*, Chin. Ann. of MathB,1 (1994), 89-104.