CONSISTENCY RELATIONS OF RANK 2 CLUSTER SCATTERING DIAGRAMS OF AFFINE TYPE AND PENTAGON RELATION

KODAI MATSUSHITA

Abstract. In this paper, we prove the consistency relations of rank 2 cluster scattering diagrams of affine type by using the pentagon relation.

1. Introduction

The cluster algebras are commutative algebras introduced by Fomin and Zelevinsky [FZ02]. The positivity of coefficients of $F$-polynomials and the sign-coherence of $c$-vectors were important conjectures. These were proved in [GHKK18] by using scattering diagram methods. Scattering diagrams for cluster algebras are characterized by the consistency relations in their structure groups $G$.

The structure group $G$ of a given cluster scattering diagram is generated by the dilogarithm elements [GHKK18, Nak21]

\[
\Psi[n] := \exp \left( \sum_{j=1}^{\infty} \frac{(-1)^{j+1} j^2}{X_{jn}} \right).
\]

The precise definition of $G$ is given in §2. These elements satisfy pentagon relations:

\[
\Psi[n]^c \Psi[n']^c = \Psi[n]^c \Psi[n+n']^c \Psi[n]^c
\]

where \( \{n, n'\} = c^{-1} \).

In this paper, we prove the consistency relations of rank 2 cluster scattering diagrams of affine type, namely types $A_{1}^{(1)}$ and $A_{2}^{(2)}$. More precisely, we prove the following theorem. For simplicity, let \( \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \Psi[(n_1, n_2)] \).

Theorem 1. The following relations hold:

\[
\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^2 \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}^2 \cdots \begin{bmatrix} 2^j & 2^{j-1} \\ 2 & 2^{j-1} \end{bmatrix} \cdots \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix},
\]

\[
\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^4 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^4 \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}^4 \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix} \cdots
\]

\[
\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^4 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^4 \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}^4 \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}^4 \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}^4 \cdots
\]
Moreover, these formulas can be reduced to trivial relations by iteration of the pentagon relations.

The relations (1.3) and (1.4) are the (unique) consistency relations of type $A_1^{(1)}$ and type $A_2^{(2)}$, respectively.

We say a product of dilogarithm elements is ordered, (resp. anti-ordered) if, for any adjacent pair $\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \begin{bmatrix} n'_1 \\ n'_2 \end{bmatrix}$, the inequality $n_1/n_2 \leq n'_1/n'_2$ (resp. $n_1/n_2 \leq n'_1/n'_2$) holds. The consistency relations of scattering diagrams in $\mathbb{R}^2$ have the form of

\begin{equation}
\text{(1.5)} \quad \text{“anti-ordered product” = “ordered product”}
\end{equation}

It was shown that the consistency relations are generated by the pentagon relation [Nak21], and the above theorem provides an simplest examples involving the infinite product.

We remark that the relations (1.3) first proved by [Rei10] by using quiver representations. Also, the relations (1.3) and (1.4) were proved by cluster mutation technique by [Rea20].

In §2, we introduce dilogarithm elements and the pentagon relations. In §3, we prove a generalization of the formula (1.3). In §4, we prove a generalization of the formula (1.4) by using a results of §3.

2. Dilogarithm elements and pentagon relation

Let $N$ be a rank 2 lattice with a skew-symmetric bilinear form

$$\{\cdot, \cdot\}: N \times N \rightarrow \mathbb{Q}.$$ 

Let $e_1$, $e_2$ be a basis of $N$, and we define

$$N^+ := \{a_1 e_1 + a_2 e_2 \mid a_1, a_2 \in \mathbb{Z}_{\geq 0}, a_1 + a_2 > 0\}.$$ 

Let $k$ be a field of characteristic 0, and we define an $N^+$-graded Lie algebra $g$ over $k$ with generators $X_n$ such that

$$g = \bigoplus_{n \in N^+} g_n, \quad g_n = kX_n, \quad [X_n, X_{n'}] = \{n, n'\}X_{n+n'}.$$ 

Let $\mathcal{L} := \{L \subset N^+ \mid N^+ + L \subset L, \#(N^+ \setminus L) < \infty\}$. For $L \in \mathcal{L}$, we define a Lie algebra ideal $g^L := \bigoplus_{n \in L} g_n$ and the quotient of $g$ by $g^L$

$$g_L := g/g^L = \bigoplus_{n \in N^+ \setminus L} g_n \quad \text{(as a vector space)}.$$ 

Let $G_L$ be a group with a set bijection

$$\exp_L: g_L \rightarrow G_L$$
and the product is defined by a Baker-Campbell-Hausdorff (BCH) formula:
\begin{equation}
\exp_L(X) \exp_L(Y) = \exp_L(X+Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \cdots).
\end{equation}
This product formula is well-defined because $g_L$ is nilpotent.

For $L, L' \in \mathcal{L}$ such that $L \subset L'$, there exists the canonical Lie algebra homomorphism $g_L \rightarrow g_{L'}$, which induces the group homomorphism $G_L \rightarrow G_{L'}$. Thus, by the inverse limit we obtain a Lie algebra $\hat{g}$ and a group $G$:

$$\hat{g} := \lim_{\leftarrow L \in \mathcal{L}} g_L, \quad G := \lim_{\leftarrow L \in \mathcal{L}} G_L.$$ 

There is a set bijection
\[ \exp : \hat{g} \rightarrow G, \quad (X_L)_{L \in \mathcal{L}} \mapsto (\exp_L(X_L))_{L \in \mathcal{L}}. \]

We use an infinite sum to express an element of $\hat{g}$.

We define important elements in $G$:

**Definition 1** (Dilogarithm element). For any $n \in \mathbb{N}^+$, define
\[ [n] := \exp \left( \sum_{j > 0} \frac{(-1)^{j+1}}{j^2} X_{jn} \right) \in G. \]
We call $[n]$ a **dilogarithm element** for $n$.

For $c \in \mathbb{Q}$ and $g = \exp(X) \in G$, we define $g^c := \exp(cX)$.

**Proposition 1** (Pentagon relation [GHKK18, Nak21]). Let $n, n' \in \mathbb{N}^+$. Then, the following relations hold in $G$:

1. If $\{n', n\} = 0$, then $[n'][n] = [n][n']$, 
2. If $\{n', n\} = c^{-1}$ ($c \in \mathbb{Q} \setminus \{0\}$), then
\[ [n'][c][n] = [n]c[n + n'][c][n']c \quad \text{(pentagon relation)}. \]

**3. Proof of formula (3.1)**

For a subset $I = \{i_1 < i_2 < i_3 < \cdots\}$ of $\mathbb{Z}$ and a sequence $(a_i)_{i \in I}$ of elements of $G$, we write
\[ \prod_{i \in I} a_i := a_{i_1}a_{i_2}a_{i_3} \cdots, \quad \prod_{i \in I} a_i := \cdots a_{i_1}a_{i_2}a_{i_1}. \]

For example, $\prod_{i \geq 0} a_i = a_0a_1a_2 \cdots$ and $\prod_{i \geq 0} a_i = \cdots a_2a_1a_0$.

The following is the main theorem of this section:

**Theorem 2.** If $\{n', n\} = c^{-1}$ ($c \in \mathbb{Q} \setminus \{0\}$), then
\[ [n']^{2c}[n]^{2c} = \prod_{p \geq 0} [n+p(n+n')]^{2c} \prod_{p \geq 0} [2^p(n+n')]^{4c/2p} \prod_{p \geq 0} [n'+p(n+n')]^{2c}. \]
The case of $c = 1$, $n = [(1, 0)]$, $n' = [(0, 1)]$ is nothing but the formula (1.3).

To prove this theorem, we introduce some notations and lemmas.

Let $L \in \mathcal{L}$. For two elements $g_1, g_2$ of $G$, let us denote $g_1 \equiv g_2 \mod L$ if their images in $G_L$ are identical. For example, if $n \in N^+$ is in $L$, then $[n] \equiv \exp(0) = 1_G \mod L$. By the definition of $G$, two elements $g_1, g_2$ of $G$ are identical if and only if $g_1 \equiv g_2 \mod L$ for all $L \in \mathcal{L}$.

**Lemma 1.** If $\{n', n\} = c^{-1}$ ($c \in \mathbb{Q} \setminus \{0\}$), we obtain

\begin{equation}
[n']^{2c} [n]^{2c} = [n]^{2c} [2n + n']^{2c} [n + n']^{2c} [n']^{c},
\end{equation}

\begin{equation}
[n']^{2c} [n]^{c} = [n]^{c} [n + n']^{2c} [n + 2n']^{c} [n']^{2c}.
\end{equation}

**Proof.** The equality (3.2) can be proved by repeatedly applying the pentagon relation:

\[
[n']^{2c} [n]^{c} = [n']^{c} [n']^{c} [n]^{c} = [n']^{c} [n]^{c} [n + n']^{c} [n']^{c} = [n]^{c} [n + n']^{c} [n + 2n']^{c} [n']^{2c}.
\]

Note that $\{n', n + n'\} = c^{-1}$, thus we can use the pentagon identity in the last equality.

If $\{n', n\} = c^{-1}$, then $\{n, n'\} = (-c)^{-1}$. By the equality (3.4),

\begin{equation}
[n]^{-c} [n']^{2(-c)} = [n]^{2(-c)} [n + 2n']^{-c} [n + n']^{2(-c)} [n']^{-c}.
\end{equation}

Taking the inverse of both sides of (3.4), we obtain the equality (3.3). □

The following is a key lemma:

**Lemma 2.** Let $l$ be a non-negative integer, and let $n, n' \in N^+$. If $\{n', n\} = c^{-1}$,

\begin{equation}
[n']^{2c} \left( \prod_{0 \leq p \leq l} [n + 2pn']^{c} \right) = [n]^{c} \left( \prod_{1 \leq p \leq 2l+1} [n + pn']^{2c} \right) [n + (2l + 2)n']^{c} [n']^{2c}
\end{equation}

**Proof.** We will prove it by induction on $l$.

If $l = 0$, the equality (3.5) is nothing but (3.3).

Let $l > 0$. Suppose that the claim is true in the case of $l - 1$, then by the induction hypothesis,

\[
[n]^{2c} \left( \prod_{0 \leq p \leq l} [n + 2pn']^{c} \right) = [n]^{2c} \left( \prod_{0 \leq p \leq l-1} [n + 2pn']^{c} \right) [n + 2ln']^{c}
\]

\[
= [n]^{c} \left( \prod_{1 \leq p \leq 2l-1} [n + pn']^{2c} \right) [n + 2ln']^{c} [n']^{2c} [n + 2ln']^{c}
\]
\[ [n]^c \left( \prod_{1 \leq p \leq 2l+1} [n + pn']^{2c} \right) [n + (2l + 2)n']^{c}[n']^{2c}. \]

In the last equality, we use 
\[ [n']^{2c}[n + 2ln']^c = [n + 2ln']^c[n + (2l + 1)n']^{2c}[n + (2l + 2)n']^{c}[n']^{2c}, \]
which is a specialization of (3.3). \( \square \)

Now we consider the limit of Lemma 2

Lemma 3. If \( \{n', n\} = c^{-1} \), then

\[
(3.6) \quad [n']^{2c} \left( \prod_{p \geq 0} [n + 2pn']^c \right) = [n]^c \left( \prod_{p \geq 1} [n + pn']^{2c} \right) [n']^{2c},
\]

\[
(3.7) \quad \left( \prod_{p \geq 0} [n' + 2pn]^{c} \right) [n]^{2c} = [n]^{2c} \left( \prod_{p \geq 1} [n' + pn]^{2c} \right) [n']^{c}.\]

Proof. Let \( L \in \mathcal{L} \). Then, by the cofiniteness of \( L \), there exists some positive integer \( l \) such that \( n + 2ln' \in L \). Then, by Lemma 2, we obtain

\[
[n']^{2c} \left( \prod_{p \geq 0} [n + 2pn']^c \right) \equiv [n']^{2c} \left( \prod_{0 \leq p \leq l} [n + 2pn']^c \right) \mod L
\]

\[
= [n]^c \left( \prod_{1 \leq p \leq 2l+1} [n + pn']^{2c} \right) [n + (2l + 2)n']^{c}[n']^{2c}
\]

\[
\equiv [n]^c \left( \prod_{p \geq 1} [n + pn']^{2c} \right) [n']^{2c} \mod L.
\]

Thus, the equality (3.6) holds.

Since \( \{n, n'\} = (-c)^{-1} \), by (3.6), we obtain

\[
(3.8) \quad [n]^{-2c} \left( \prod_{p \geq 0} [n' + 2pn]^{-c} \right) = [n']^{-c} \left( \prod_{p \geq 1} [n' + pn]^{-c} \right) [n]^{-2c}.
\]

The the equality (3.7) is obtained by taking the inverse of the both sides of (3.8). \( \square \)

Proof of Theorem 2 For \( L \in \mathcal{L} \) and \( k \in \mathbb{Z}_{\geq 0} \), let \( P_L(k) \) be the following assertion: for \( n, n' \in \mathbb{N}^+ \) and \( c \in \mathbb{Q} \setminus \{0\} \), if \( \{n', n\} = c^{-1} \) and \( k(n + n') \in L \), then

\[ [n']^{2c}[n]^{2c} \]
Since hypothesis, for any \( L \) we obtain (3.9) 
\[
\prod_{p \geq 0} [n + p(n + n')]^{2c} \prod_{p \geq 0} [2^p(n + n')]^{4c/2p} \prod_{p \geq 0} [n' + p(n + n')]^{2c} \mod L.
\]

For any \( L \in \mathcal{L} \), there exists some positive integer \( k \) such that \( k(\frac{n + n'}{L}) \in L \). Thus, if \( P'_L(k) \) is true for any \( k \in \mathbb{Z}_{>0} \) and \( L \in \mathcal{L} \), then a relation (3.9) holds for any \( L \in \mathcal{L} \), and Theorem 2 is proved. Fix \( L \in \mathcal{L} \), and we prove \( P'_L(k) \) by induction on \( k \).

If \( k = 1 \), the right hand side of (3.9) is equivalent to \( [n]^{2c}[n']^{2c} \) because \( l_n + l'_n \in L \) for any \( l, l' \in \mathbb{Z}_{\geq 1} \). Since 
\[
[n']^{c}[n]^{c} = [n]^{c}[n + n']^{c} = [n]^{c}[n']^{c} \mod L,
\]
we obtain \( [n']^{2c}[n']^{2c} = [n]^{2c}[n']^{2c} \mod L \).

Let \( k \geq 2 \), and we suppose a proposition \( P'_L(k - 1) \) is true. By the equality (3.3), 
\[
[n']^{2c}[n]^{2c} = ([n']^{2c}[n]^{c})^{c}
\]
\[
= [n]^{c}[n + n']^{2c}/[n + 2n']^{c} \cdot (n')^{c}/[n']^{2c}
\]
\[
= [n]^{c}[n + n']^{2c}/[n + 2n']^{c} \cdot [n + n']^{2c}/[n + n']^{2c}.
\]

Since \( (k - 1)(n + (n + n')) \in L \) and \( \{n + 2n', n\} = (c/2)^{-1} \), by the induction hypothesis, 
\[
[n + 2n']^{c}[n]^{c}
\]
\[
= [n + 2n']^{(c/2) - 2}[n]^{(c/2) - 2}
\]
\[
= \prod_{p \geq 0} [n + p(2n + 2n')]^{2-(c/2)} \prod_{p \geq 0} [2^p(2n + 2n')]^{4-(c/2)/2p}
\]
\[
\times \prod_{p \geq 0} [(n + 2n') + p(2n + 2n')]^{2-(c/2)} \mod L
\]
\[
= \prod_{p \geq 0} [n + 2p(n + n')]^{c} \prod_{p \geq 1} [2^p(n + n')]^{4c/2p} \prod_{p \geq 0} [(n + 2n') + 2p(n + n')]^{c}.
\]

Since \( \{n + n', n\} = c^{-1} \) and \( \{n + 2n', n + n'\} = c^{-1} \), by Lemma 3 
\[
[n']^{2c}[n]^{2c} \equiv [n]^{c}[n + n']^{2c} \prod_{p \geq 0} [n + 2p(n + n')]^{c} \prod_{p \geq 1} [2^p(n + n')]^{4c/2p}
\]
\[
\times \prod_{p \geq 0} [(n + 2n') + 2p(n + n')]^{c}
\]
\[
\times [n + n']^{2c}[n + 2n']^{c} \mod L
\]
\[
= [n]^{c}[n]^{c} \left( \prod_{p \geq 1} [n + p(n + n')]^{c} \right) [n + n']^{2c} \prod_{p \geq 1} [2^p(n + n')]^{4c/2p}
\]
\[
\times [n + n']^{2c} \prod_{p \geq 1} [(n + 2n') + p(n + n')]^c \\
\times [n + 2n']^c [n + 2n']^c [n']^{2c} \\
= \prod_{p \geq 0} [n + p(n + n')]^{2c} \prod_{p \geq 0} [2^p(n + n')]^{4c/2p} \prod_{p \geq 0} [n' + p(n + n')]^{2c}
\]

This completes the proof of Theorem 2. □

4. Proof of formula (1.4)

The formula (1.4) is the case of \(c = -1\), \(n = [(0, 1)]\), \(n' = [(1, 0)]\) of the following theorem:

**Theorem 3.** If \(\{n', n\} = c^{-1} (c \in \mathbb{Q})\), then we obtain

\[
[n']^c [n]^{4c} = \prod_{p \geq 0} ([2p + 1)n + pn']^{4c} [(4p + 4)n + (2p + 1)n']^c \\
\times [2n + n']^{2c} \prod_{p \geq 0} [2^p(2n + n')]^{4c/2p} \\
\times \prod_{p \geq 0} ([(2p + 1)n + (p + 1)n']^{4c}[4pn + (2p + 1)n']^c).
\]

To prove this theorem, we consider some lemmas.

**Lemma 4.** If \(\{n', n\} = c^{-1} (c \in \mathbb{Q} \setminus \{0\})\), then we obtain

\[
[n']^c \left( \prod_{0 \leq p \leq l} [n + pn']^{2c} \right) \\
= [n]^{2c} \left( \prod_{1 \leq p \leq l} [(2n + (2p - 1)n']^c [n + pn']^{4c} \right) \\
\times [2n + (2l + 1)n']^c [n + (l + 1)n']^{2c} [n']^c
\]

**Proof.** We prove it by induction on \(l\). The case of \(l = 0\) is nothing less than the equality (3.2).

Let \(l > 0\). Suppose that the claim is true in the case of \(l - 1\), then

\[
[n']^c \left( \prod_{0 \leq p \leq l-1} [n + pn']^{2c} \right) \\
= [n']^c \left( \prod_{0 \leq p \leq l-1} [n + pn']^{2c} \right) [n + ln']^{2c}
\]
Thus, the equality (4.1) holds. Let
\[ L = \prod_{p \leq l} ([2n + (2p - 1)n']^c[n + pn']^{4c}) \]
In the last equality, we use
\[ [n']^c [n + ln']^{2c} = [n + ln']^{2c} [2n + (2l + 1)n']^c [n + (l + 1)n']^{2c} [n']^2, \]
which is a specialization of (3.2).

Now we consider the limit of Lemma 4.

**Lemma 5.** If \( \{n', n\} = c^{-1} \), then we obtain

(4.1)
\[ [n']^c \left( \prod_{p \geq 0} [n + pn']^{2c} \right) = [n]^{2c} \left( \prod_{p \geq 1} [2n + (2p - 1)n']^c[n + pn']^{4c} \right) [n']^c, \]
(4.2)
\[ \left( \prod_{p \geq 0} [n' + pn]^{2c} \right) [n]^c = [n]^{c} \left( \prod_{p \geq 1} [n' + pn]^{4c} [2n' + (2p - 1)n]^c \right) [n']^{2c}. \]

**Proof.** Let \( L \in \mathcal{L} \). Then, there exist some positive integer \( l \) such that \( n + ln' \in L \). Then, by Lemma 4 we obtain

\[ [n']^c \prod_{p \geq 0} [n + pn']^{2c} \mod L \]
\[ = [n]^{2c} \left( \prod_{1 \leq p \leq l} ([2n + (2p - 1)n']^c[n + pn']^{4c}) \right) \times [2n + (2l + 1)n']^c [n + (l + 1)n']^{2c} [n']^c \]
\[ \equiv [n]^{2c} \left( \prod_{p \geq 1} [2n + (2p - 1)n']^c[n + pn']^{4c} \right) [n']^c \mod L \]

Thus, the equality (4.1) holds.
Since \( \{n, n'\} = (-c)^{-1} \), by (4.1), we obtain
\[
[n]^{-c} \left( \prod_{p \geq 0} [n' + pn]^{-2c} \right) = [n']^{-2c} \left( \prod_{p \geq 1} [2n' + (2p - 1)n]^{-c}[n' + pn]^{-4c} \right) [n]^{-c}.
\]

By taking the inverse of both sides of this equality, we have the equality (4.2). \( \square \)

**Proof of Theorem 3.** Using the pentagon relations, Theorem 2 and Lemma 4, we can calculate as follows:

\[
[n']^c[n]^{4c} = [n']^c[n]^2[n]^{2c} = [n']^2c[2n + n']^c[n + n']^2c[n']^c[n]^{2c} = [n']^2c[2n + n']^c[n + n']^2c[2n + n']^c[n + n']^2c[n]^{c}
\]

\[
\begin{align*}
&= [n']^2c[2n + n']^c \\
&\quad \times \prod_{p \geq 0} [n + p(2n + n')]^{2c} \\
&\quad \times \prod_{p \geq 0} [2^p(2n + n')]^{4c/2p} \\
&\quad \times \prod_{p \geq 0} [n + n' + p(2n + n')]^{2c} \\
&\quad \times [2n + n']^c[n + n']^2c[n']^c \\
&= [n]^{2c} \times [n]^{2c} \left( \prod_{p \geq 1} ([2n + (2p - 1)(2n + n')]^c[n + p(2n + n')]^{4c}) \right) [2n + n']^c \\
&\quad \times \prod_{p \geq 0} [2^p(2n + n')]^{4c/2p} \times [2n + n']^c \\
&\quad \times \left( \prod_{p \geq 1} ([n + n'] + p(2n + n')]^{4c}[2(n + n') + (2p - 1)(2n + n')]^c \right) \\
&\quad \times [n + n']^2c[n + n']^2c[n']^c \\
&= \prod_{p \geq 0} ([n + p(2n + n')]^{4c}[2n + (2p + 1)(2n + n')]^c) \\
&\quad \times [2n + n']^2c \prod_{p \geq 0} [2^p(2n + n')]^{4c/2p} \\
&\quad \times \prod_{p \geq 0} ([n + n'] + p(2n + n')]^{4c}[2(n + n') + (2p - 1)(2n + n')]^c
\end{align*}
\]
\[ \prod_{p \geq 0} \left( \frac{[2(p + 1)n + pn']^{4c}}{[4pn + (2p + 1)n']^c} \right) \times \prod_{p \geq 0} \left( \frac{[2n + n']^{2c}}{[2p(2n + n')]^{4c/2p}} \right) \times \prod_{p \geq 0} \left( \frac{[(2p + 1)n + (p + 1)n']^{4c}}{[4pn + (2p + 1)n']^c} \right) \]

This completes the proof. \(\square\)

**References**

[FZ02] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529, 2002.

[GHKK18] Mark Gross, Paul Hacking, Sean Keel, and Maxim Kontsevich. Canonical bases for cluster algebras. *Journal of the American Mathematical Society*, 31(2):497–608, 2018.

[Nak21] Tomoki Nakanishi. Cluster algebras and scattering diagrams, part III. cluster scattering diagrams, 2021.

[Rea20] Nathan Reading. A combinatorial approach to scattering diagrams. *Algebr. Comb.*, 3(3):603–636, 2020.

[Rei10] Markus Reineke. Poisson automorphisms and quiver moduli. *Journal of the Institute of Mathematics of Jussieu*, 9(3):653–667, 2010.