Maximal quantum Fisher information for general su(2) parametrization processes

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Quantum Fisher information is a key concept in the field of quantum metrology, which aims to enhance the parameter accuracy by using quantum resources. In this paper, utilizing a representation of quantum Fisher information for a general unitary parametrization process, we study unitary parametrization processes governed by su(2) dynamics. We obtain the analytical expression for the Hermitian operator of the parametrization and the maximal quantum Fisher information. We find that the maximal quantum Fisher information over the parameter space consists of two parts, one is quadratic in the time and the other oscillates with the time. We apply our result to the estimation of a magnetic field and obtain the maximal quantum Fisher information. We further discuss a driving field with a time-dependent Hamiltonian and find the maximal quantum Fisher information of the driving frequency attains optimum when it is in resonance with the atomic frequency.

I. INTRODUCTION

Last two decades have witnessed the dramatic development of quantum metrology [1][15], which is rooted in the theory of quantum parameter estimation. The estimation of unknown parameters plays an important role in physics and engineering. By taking advantage of quantum resources such as entanglement and squeezing, quantum metrology promises higher precision in parameter estimation than its classical counterpart. Therefore many practical applications have been raised to reap this benefit, including the detection of gravitational radiation [19][21], quantum frequency standards [9][11] and quantum imaging [22][24].

Fisher information characterizes the amount of information about the true value of a parameter that can be extracted from a probability distribution. Quantum Fisher information (QFI), which is a key concept in quantum metrology, is defined by maximizing the Fisher information over all possible measurements [1][25]. The quantum Cramer-Rao theorem asserts that the precision is bounded from below by the inverse of the QFI [26][27]. Due to its great importance in quantum metrology and parameter estimation, QFI has attracted a lot of attentions [28][39]. However, analytical expressions for QFI is difficult to obtain in most cases.

The QFI associated with a state ρ for a parameter θ is defined as [25][27]

\[ F = \text{Tr}(\rho L^2), \]

where Tr stands for trace and L is the symmetric logarithmic derivative operator, which is determined by \( \partial_\theta \rho = (L \rho + \rho L)/2 \). Consider a general unitary parametrization process \( U = \exp(-i t H) \) for an initial pure state \( |\psi\rangle \), with the time-independent Hamiltonian \( H \) depending on a parameter \( \theta \), the QFI is simply connected with the variance of a Hermitian operator \( \mathcal{H} \) in \( |\psi\rangle \), that is [39]

\[ F = 4 \left( \langle H^2 \rangle - \langle H \rangle^2 \right), \]

where

\[ \mathcal{H} \equiv i \left( \partial_\theta U^\dagger \right) U \]

is a Hermitian operator associated with the parametrization.

It has been shown that the variance of an Hermitian operator \( \mathcal{H} \) is maximized when the initial state \( |\psi\rangle \) is an equally weighted superposition of the eigenvectors \( |\lambda_M\rangle \) and \( |\lambda_m\rangle \), which correspond to the maximum and minimum eigenvalues \( \lambda_M \) and \( \lambda_m \) of the operator \( \mathcal{H} \), respectively [3][35], i.e.,

\[ |\psi\rangle = \frac{1}{\sqrt{2}} \left( |\lambda_M\rangle + e^{i \phi} |\lambda_m\rangle \right), \]

where \( e^{i \phi} \) is an arbitrary phase, and the maximal QFI (MQFI) is [38]

\[ F^{\text{max}} = (\lambda_M - \lambda_m)^2. \]

Previous research has mainly focused on the estimation of an overall multiplicative factor of a Hamiltonian \( H \) [7][8][10][12][13][34][36]. In such cases, the Hamiltonian parameter estimation problem where the parameter does not appear as an overall multiplicative factor. Their study extends the quantum metrology to a more general case. Based on their work, we offered another approach to get the QFI for unitary parametrization processes in [39]. In this approach, the QFI is related to the spectral decomposition of the initial state and the Hamiltonian of the parametrization process

\[ \mathcal{H} = -t \partial_\theta H + i \sum_{k=1}^{\infty} \frac{(it)^{k+1}}{(k+1)!} H^k (\partial_\theta H), \]
where the superoperator $H^{\times k}$ denotes a $k$th-order nested commutator operation, $H^{\times k}() = [H, \ldots, [H, \cdot]]$. When $\theta$ is a multiplicative factor, as in the Mach-Zehnder interferometer setup, $H = \theta^k H^\prime$, ($k = 1, 2, \ldots$) [12][13], where $H^\prime$ is independent on the parameter to be estimated, the summation vanishes and only the first term in the right hand side of Eq. (10) contributes.

In this paper, we investigate a general unitary parametrization process governed by a su(2) Hamiltonian in the form of $r \cdot J$, where $r$ is a time-independent function of the parameter and $J$ is the generator of su(2) algebra. This Hamiltonian covers many interesting applications in physics. We derive the analytical expression of the MQFI and find that it can be divided into two parts, one is quadratic in the time $t$ due to the dependence of the norm $|r|$ on the parameter $\theta$, and the other oscillates with the time $t$ because of the dependence of the direction $e_r = r/|r|$ on the parameter $\theta$. Furthermore, we apply the theoretical expression for three practical examples. The first two examples undergo time-independent su(2) parametrization, and the third one subjects to time-dependent su(2) parametrization.

This paper is structured as follows. In Sec II, we derive the $\mathcal{H}$ operator and the MQFI for a general su(2) Hamiltonian. In Sec III, three applications of the theoretical result are listed. The first two are governed by time-independent su(2) Hamiltonians. We discuss the corresponding MQFI for the estimated parameters. In the third application, we further study the parameter estimation with a time-dependent su(2) Hamiltonian and give the expression of the $\mathcal{H}$ operator and the MQFI. Finally, the conclusion is given in Sec IV.

II. MAXIMAL QUANTUM FISHER INFORMATION FOR A GENERAL TIME-INDEPENDENT SU(2) HAMILTONIAN

In this section, we investigate the $\mathcal{H}$ operator and the MQFI for a general su(2) parametrization process. Assume the Hamiltonian takes the form of

$$H = r \cdot J,$$

(7)

where $J = (J_x, J_y, J_z)$ is the generator of su(2) algebra and $r = r(\theta)$ is a time-independent parametric curve in the parameter space. We denote the derivative of $H$ with respect to $\theta$ as

$$\partial_\theta H = v \cdot J,$$

(8)

where we define the velocity $v \equiv dr/d\theta$, which denotes the change of $r$ over the parameter space of $\theta$, including both of the change of $|r|$ over $\theta$ and the change of the unit vector $e_r = r/|r|$ over $\theta$. Thus the velocity $v$ can be decomposed as $v = v_r + v_n$, with the radial velocity $v_r$ and the normal velocity $v_n$ reading as

$$v_r = \frac{d|r|}{d\theta} e_r,$$

(9a)

and

$$v_n = |r| \frac{d}{d\theta} e_\perp,$$

(9b)

Utilizing the commutation relation for su(2) algebra $[a \cdot J, b \cdot J] = i (a \times b) \cdot J$, the first two commutators in Eq. (6) read

$$H^{\times k} \partial_\theta H = i(r \times v) \cdot J,$$

(10a)

$$H^{\times 2} \partial_\theta H = -[r \times (r \times v)] \cdot J = |r|^2 v - (r \cdot v) r \cdot J,$$

(10b)

where the relation $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$ is used in the last equality. It is shown that both $H^{\times k} \partial_\theta H$ and $H^{\times 2} \partial_\theta H$ are eigen operators of the superoperator $H^{\times 2k}$, i.e.,

$$H^{\times (2k+1)} \partial_\theta H = |r|^{2k} H^{\times k} \partial_\theta H,$$

(11a)

$$H^{\times (2k+2)} \partial_\theta H = |r|^{2k+2} H^{\times 2} \partial_\theta H,$$

(11b)

where $k = 1, 2, \ldots$. By plugging Eq. (11) into Eq. (6), we have

$$\mathcal{H} = -t \partial_\theta H + i \sum_{k=0}^{\infty} \left(\frac{(it)^{2k+2}}{(2k+2)!} H^{\times (2k+1)}(\partial_\theta H) + \frac{i}{2k+3} H^{\times (2k+2)}(\partial_\theta H) \right)$$

$$= -t \partial_\theta H + i \left( \frac{H^{\times 2 \partial_\theta H}}{|r|} \frac{H^{\times \partial_\theta H}}{|r|^3} \right) - \sin(|r| t) \frac{H^{\times \partial_\theta H}}{|r|^3} \frac{\sin(|r| t)}{|r|^3} r \cdot J$$

(12)

and insert the first two commutators in Eq. (10) into Eq. (12), we find that

$$\mathcal{H} = \left( r \cdot v \right) (\sin(|r| t) - |r| t) r \cdot J - \frac{\sin(|r| t)}{|r|^3} v \cdot J$$

$$+ \frac{1 - \cos(|r| t)}{|r|^2} (r \times v) \cdot J.$$  

(13)

According to Eq. (9), we have that $(r \cdot v) r = |r|^2 v_r$, $r \times v = |r| e_r \times v_n = |r| (v_r \times v_n)/(dr)/d\theta$, therefore, $\mathcal{H}$ can be further written as

$$\mathcal{H} = A \cdot J$$

$$= \left( 1 - \cos(|r| t) \right) (v_r \times v_n) \cdot J - t v_r \cdot J - \frac{\sin(|r| t)}{|r|^3} v_n \cdot J$$

(14)

where

$$A = \left( 1 - \cos(|r| t) \right) (v_r \times v_n) - t v_r - \frac{\sin(|r| t)}{|r|^3} v_n.$$  

(15)

The three terms in Eq. (15) are perpendicular with each other and the norm of $A$ is $|A| = \cdots$
\[ \sqrt{v_r^2 t^2 + 4 \frac{v^2}{r^2} \sin^2 \left( \frac{|r|}{2} t \right)} \], therefore according to Eq. (5), the MQFI over the parameter \( \theta \) is
\[
F_{\theta}^{\text{max}} = |j| |A| - (-j)|A| \right|^2 \\
= 4j^2 \left[ v_r^2 t^2 + 4 \frac{v^2}{r^2} \sin^2 \left( \frac{|r|}{2} t \right) \right],
\]
where \( j \) is the maximal eigenvalue of \( J_z \). Equation (16) shows that the MQFI can be divided into two parts. The first part is quadratic in the time \( t \) and is proportional to the square of the radial velocity \( v_r \). The second part oscillates with the time \( t \) and is proportional to the ratio of the square of the normal velocity \( v_n \) to the square of \( r \). According to Eq. (16), this ratio is equal to the square of the derivative \( \delta e_r / \delta \theta \). Thus the quadratic term is due to the dependence of the norm \( |r| \) on the parameter \( \theta \), while the oscillation term is due to the dependence of the direction \( e_r \) on \( \theta \).

When \( t \) is small, we can expand Eq. (16) in \( t \) to the second order, and the MQFI is simplified to
\[
F_{\theta}^{\text{max}} = 4j^2 t^2 \left[ v_r^2 + v_n^2 \right] = 4j^2 t^2 v^2.
\]
That is, the MQFI grows quadratically with the time \( t \) and depends only on the norm of \( v \).

If the change of \( |r| \) over the parameter space is zero, i.e., \( v_r = \frac{dr}{d\theta} e_r = 0 \), then the MQFI is reduced to
\[
F_{\theta}^{\text{max}} = 16j^2 \frac{v^2}{r^2} \sin^2 \left( \frac{|r|}{2} t \right).
\]
In this case, the MQFI always oscillates with the time \( t \) so that its value is bounded. At the time points \( t = (2k + 1)\pi/|r|, k = 0, 1, 2, \ldots \), the MQFI reaches the optimal value,
\[
F_{\theta}^{\text{op}} = 16j^2 \frac{v^2}{r^2}.
\]

### III. APPLICATIONS

In this section, we apply the theoretical expression for three practical examples. The dynamics of the first two examples are governed by time-independent su(2) Hamiltonians, and the third one is governed by a time-dependent su(2) Hamiltonian.

**A. time-independent Hamiltonians**

**Case 1.** Let us consider the Hamiltonian (7) with \( r \) in the explicit form in the spherical coordinates,
\[
r = r (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta),
\]
where \( r \) represents the amplitude of the external magnetic field and \( \theta, \varphi \) denotes the direction of the field. Suppose \( \theta \) is the parameter we want to estimate. The velocity vector is readily obtained as \( v = r (\cos \theta \cos \varphi, \cos \theta \sin \varphi, -\sin \theta) \). It is obvious that \( v_r = 0 \). According to Eq. (18), the MQFI over \( \theta \) is
\[
F_{\theta}^{\text{max}} = 16j^2 \sin^2 \left( \frac{rt}{2} \right),
\]
and when \( t = (2k + 1)\pi/r, k = 0, 1, 2, \ldots \), \( F_{\theta}^{\text{max}} \) reaches its optimal value as \( 16j^2 \).

Similarly, if \( \varphi \) is the parameter to be estimated, the MQFI can be derived as
\[
F_{\varphi}^{\text{max}} = 16j^2 \sin^2 \theta \sin^2 \left( \frac{rt}{2} \right),
\]
and when \( t = (2k + 1)\pi/r, k = 0, 1, 2, \ldots \), and \( \theta = \frac{\pi}{2} \), \( F_{\varphi}^{\text{max}} \) reaches its optimal value as \( 16j^2 \). In both cases, the MQFI is bounded.

Finally, if \( r \) is the parameter to be estimated, then \( v^2 = v_r^2 = 1 \) and the second term in Eq. (16) vanishes. Thus the MQFI reads
\[
F_{r}^{\text{max}} = 4j^2 t^2.
\]
In this case, the MQFI is unbounded and grows quadratically with the time \( t \).

**Case 2.** Let us consider an ensemble of \( N \) two-level atoms interacting with a static magnetic field. The Hamiltonian of this system can be written as
\[
H = \omega_0 J_z + \lambda J_x,
\]
where \( \omega_0 \) is the atomic transition frequency and \( \lambda \) is the Rabi frequency, which is proportional to the amplitude of the static magnetic field.

Here we shall find the MQFI of \( \omega_0 \) first. Compare with the previous section, the coefficient vector is \( r = (\lambda, 0, \omega_0) \). Denote the norm of \( |r| \) as \( K = \sqrt{\lambda^2 + \omega_0^2} \), then the velocity is \( v = dr/d\omega_0 = (0, 0, 1)^T \), and the radial velocity \( v_r = d|r|/d\omega_0 = \omega_0/K \), the normal velocity \( v_n = v^2 - v_r^2 = \lambda^2/K^2 \), thus the corresponding MQFI is readily obtained as
\[
F_{\omega_0}^{\text{max}} = 4j^2 \left[ \frac{\omega_0^2}{K^2} t^2 + 4\frac{\lambda^2}{K^2} \sin^2 \left( \frac{Kt}{2} \right) \right].
\]

When \( t \) is very large, the oscillating term can be neglected. However, this term may also be important in some cases. We plot the MQFI \( F_{\omega_0}^{\text{max}} \) with and without the oscillating term in Fig. 1. In Fig. 1(a), when we take \( \lambda = 10\omega_0 = 1 \), we see that the effect of the oscillating term is very strong. In Fig. 1(b), we take \( \lambda = \omega_0 = 1 \), the amplitude of the oscillating term is extremely small in comparison with the quadratic term. And in Fig. 1(c), where we take \( \lambda = 0.1\omega_0 = 1 \), the oscillating term could be neglected even when \( t \) is very small.
Here place the static magnetic field in the preceding subsection, we go beyond the previous situations and investigate dependent, generally the unitary operator will involve a per are time-independent. If the Hamiltonian is time-dependent, this result is similar to Eq. (25) and we omit the discussion here. 

B. time-dependent Hamiltonian

Hitherto all the Hamiltonians considered in this paper are time-independent. If the Hamiltonian is time-dependent, generally the unitary operator will involve a time-ordering procedure and Eq. (24) fails. In this subsection, we go beyond the previous situations and investigate a time-dependent Hamiltonian. In the following we replace the static magnetic field in the preceding subsection with a time-dependent driving field, and the Hamiltonian is

$$H = \omega_0 J_z + \lambda [J_x \cos(\omega t) + J_y \sin(\omega t)].$$

(27)

Here \(\omega\) is the frequency of the driving field. This Hamiltonian could reflect many realistic physical systems, such as in the Ramsey spectroscopy and NMR techniques [10-14]. The \(\mathcal{H}\) operator of this Hamiltonian cannot be written in the form of Eq. (6) and the previous method must be changed. We can circumvent this problem by utilizing a rotating transform. In a rotating frame, the original state vector \(|\psi\rangle\) is transformed as

$$|\tilde{\psi}\rangle = R|\psi\rangle = e^{i\omega t J_z} |\psi\rangle,$$

(28)

and the effective Hamiltonian in this new frame is

$$H_{\text{eff}} = RH R^\dagger - iR \frac{\partial R^\dagger}{\partial t} = \Delta J_z + \lambda J_x,$$

(29)

which is time-independent. Here \(\Delta = \omega_0 - \omega\) denotes the detuning between the atom transition and the driving magnetic field.

Therefore, the unitary evolution in the original frame is readily obtained as

$$U = U_1 U_2 = e^{-i\omega t J_z} e^{-i H_{\text{eff}} t},$$

(30)

with \(U_1 = R^\dagger = e^{-i\omega t J_z}\) and \(U_2 = e^{-i H_{\text{eff}} t}\). In such a way, we turn the time-dependent Hamiltonian of Eq. (27) into two time-independent ones, \(H_1 = \omega J_x\), and \(H_{\text{eff}}\). The \(\mathcal{H}\) operator for two consecutive unitary operation \(U = U_1 U_2\) can be derived as

$$\mathcal{H} = i \partial_\theta \left( U_2^\dagger U_1^\dagger \right) U_1 U_2 = \mathcal{H}_1 + U_2^\dagger \mathcal{H}_2 U_2,$$

(31)

with \(\mathcal{H}_1 = i \partial_\theta U_1^\dagger U_1\) and \(\mathcal{H}_2 = i \partial_\theta U_2^\dagger U_2\). Here \(\theta\) can be \(\lambda, \omega_0\), or \(\omega\). For the case in Eq. (30), if \(\lambda\) or \(\omega_0\) is the parameter to be estimated, it is obvious that \(\mathcal{H}_1 = 0\) and one can simply get the maximal QFI by replacing \(\omega_0\) in Eq. (25) and Eq. (26) with \(\Delta\). When \(\Delta = 0\), \(F^\text{max}_\lambda\) attains optimal value as \(4J^2 t^2\).

Next, let us find the \(\mathcal{H}\) operator with respect to \(\omega\). The Hamiltonian associated with \(\mathcal{H}_2 = H_{\text{eff}}\), and the corresponding coefficient vector reads \(r = (\lambda, 0, \Delta)^T\). Then by utilizing Eq. (9), we can obtain that

$$v_x = \frac{\Delta}{K^2} (0, 1, 0),$$

$$v_y = \frac{\lambda}{K^2} (\Delta, 0, -\lambda),$$

$$v_x \times v_y = -\frac{\Delta}{K^2} (0, 1, 0).$$

(32)

Therefore, according to Eq. (14), we have that

$$\mathcal{H}_2 = \frac{1}{K^2} \left[ \lambda \Delta [K' t - \sin(K' t)] J_x 
+ \lambda K' [1 - \cos(K' t)] J_y 
+ [\Delta^2 K' t + \lambda^2 \sin(K' t)] J_z \right].$$

(33)

On the other hand, the Hamiltonian associated with \(\mathcal{H}_1\) reads \(H_1 = \omega J_z\), where the parameter enters in the Hamiltonian as a multiplicative factor. According to Eq. (6), we have \(\mathcal{H}_1 = -t J_z\). Based on the formula...
Figure 2. The MQFI $F_{\omega}^{\text{max}}$ as a function of the detuning $\Delta$ in (a) and $\lambda$ in (b). In (a) we take $\lambda = 1, t = 1$. $F_{\omega}^{\text{max}}$ attains optimum when $\Delta = 0$. In (b), we take $\Delta = 0$ and $\lambda = 1$, $F_{\omega}^{\text{max}}$ grows quadratically with time $t$. Here we set $j = 1$ in both figures.

$\exp(A)B\exp(-A) = \exp(A^\dagger)B = \sum_{k=0}^{\infty} \frac{A^\dagger AB}{n!}$, we obtain

$$U_2^\dagger H_1 U_2 = -\frac{t}{K^2} \left[ \Delta \left[ 1 - \cos(K t) \right] J_x + \lambda K' \sin(K t) J_y + \left[ \frac{\Delta^2 + \lambda^2 \cos(K t)}{2} \right] J_z \right].$$

where we denote $K' = \sqrt{\lambda^2 + \Delta^2}$.

Combining Eq. (31), Eq. (33) and Eq. (34), we have

$$\mathcal{H} = A' \cdot J = \frac{\sin(K t) - K' t \cos(K t)}{K^3} \left( -\lambda \Delta J_x + \lambda^2 J_z \right) + \frac{\left[ 1 - \cos(K t') - K' \sin(K t') \right]}{K^2} \lambda J_y.$$  

(35)

Based on the norm of the vector $A'$, the MQFI is readily obtained as

$$F_{\omega}^{\text{max}} = 4j^2 \frac{\lambda^2}{K^4} \left[ 2 + K^2 t^2 - 2K' t \sin(K t') - 2 \cos(K t') \right].$$  

(36)

Figure 2.(a) shows that when $\Delta = 0$ (resonance), $F_{\omega}^{\text{max}}$ attains its optimal value. This can be proved theoretically by taking the derivative of Eq. (36),

$$\frac{d}{d\Delta} F_{\omega}^{\text{max}}|_{\Delta = 0} = 0.$$  

(37)

When $\Delta = 0$, we plot $F_{\omega}^{\text{max}}$ in Fig. 2(b) with $\lambda = 1$. As $t$ grows large, the MQFI can be approximated as $F_{\omega}^{\text{max}} \approx 4j^2 t^2$ and the oscillation can also be neglected.

IV. CONCLUSION

In summary, we investigate a unitary parametrization process governed by a $\text{su}(2)$ Hamiltonian $r \cdot J$ and find the MQFI when the Hamiltonian is independent on time. We find that the MQFI can be divided into two parts. One part is quadratic in the time $t$ due to the dependence of the norm $|r|$ on the parameter $\theta$, and the other part oscillates with the time $t$ because of the dependence of the direction $e_r = r/|r|$ on the parameter $\theta$.

We apply this result to a typical scenario and find the MQFI corresponding to the amplitude and the direction of the magnetic field. We also investigate the MQFI for an ensemble of atoms interacting with a static field. We find that the in some cases the oscillating terms can contribute significantly and thus can not be neglected. We further investigate a time-dependent driving field and find that the MQFI of the field frequency reaches optimum when the driving field is in resonance with the atomic transition frequency. Moreover, the MQFI of the amplitude of the external field also reaches optimum when the field is in resonance with the atoms.

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