Local and Global Existence of Solution for Love Type Waves with Past History

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Abstract: In this paper, we consider an initial boundary value problem for nonlinear Love equation with infinite memory. By combining the linearization method, the Faedo–Galerkin method, and the weak compactness method, the local existence and uniqueness of weak solution is proved. Using the potential well method, it is shown that the solution for a class of Love-equation exists globally under some conditions on the initial datum and kernel function.

Keywords: nonlinear love-equation; local existence; global existence; infinite memory

1. Introduction

Love equation is a one-dimensional mathematical model that is used to determine a many physical phenomenon. This theory is a continuation of the Euler–Bernoulli beam theory and was developed in 1888 by Love. This kind of system appears in the models of nonlinear Love waves or Love type waves. It is a generalization of a model introduced by [1–3].

In order to completely study an evolutionary mathematical equation, a quantitative and qualitative study must be approached. An initial boundary value problem for a nonlinear Love equation with infinite memory has been considered by Zennir and et al. in [4] and the finite time blow up of weak solution has been shown under a relationship between the relaxation function $g$ and nonlinear sources, i.e., $\|u\| \to \infty$ when $t \to T^*$ ($T^*$ is a finite time). Next, a very general decay rate for solution of the same problem, by certain properties of convex functions combined with some estimates, has been obtained in [5]. These two last results are considered as a qualitative studies. Obviously, in order to complete the study, we have to address the problem in quantitative terms. This is the subject of our present article, from a different angle, where we proved in detail, with the use of the most modern methods, the local existence (on small temporal period $[0, T_{max}]$) and global existence of solution on $(0, \infty)$.

Investigations on the propagation of surface waves of Love-type are made by many authors in different models and many attempts to solve Love’s equation have been performed, in view of its wide applicability. To our knowledge, there are few results for damped equations of Love waves or Love type waves. However, the existence of solutions or blow up results, with different boundary conditions, have been extensively studied by many authors.
To begin with, one must goes to the origins of Love’s equation. It is derived in [6,7] by the energy method. Under the assumptions that the Kinetic energy per unit of length is

\[ e_1 = \frac{1}{2} F \rho \left[ \partial_t u^2 + \kappa^2 w^2 \partial_t u_x^2 \right], \]  

and the potential energy per unite of length is

\[ e_2 = \frac{1}{2} EF(u_x^2), \]  

where \( F \) is an area of cross-section, \( w \) is a cross-section radius of gyration about the central line.

Using in (2) the corrected form of tension, we have

\[ e_2 = \frac{1}{2} Fu_x (Eu_x + \rho \kappa^2 w^2 \partial_t u_x). \]  

Then, the variational equation of motion is given by

\[ \delta \int_{t_1}^{t_2} ds \int_0^L (e_1 - e_2) dx = 0, \]  

and we then obtain the equation of extensional vibrations of rods as

\[ \partial_{tt} u - \frac{E}{\rho} u_{xx} - 2\kappa^2 w^2 \partial_{tt} u_{xx} = 0. \]  

The parameters in (5) have the following meaning: \( u \) is the displacement, \( \kappa \) is a coefficient, \( E \) is the Young modulus of the material and \( \rho \) is the mass density.

This type of problem describes the vertical oscillations of a rod and was established from Euler’s variational equation of an energy functional associated with (5). A classical solution of problem (5), with null boundary conditions and asymptotic behavior, is obtained by using the Fourier method and method of small parameter.

In this article, Love-equation is considered as follows

\[ \partial_{tt} y - (\lambda_0 y_x + \lambda_1 \partial_t y_x + \lambda_2 \partial_{tt} y_x) + \lambda \int_{-\infty}^{t} \mu(t-s)y_{xx}(s)ds \]  

\[ = F[y] - \left( F[y] \right)_x + f(t,x), \quad 0 < t < T, \quad x \in \Omega = (0,L), \]  

subject to the homogeneous Dirichlet boundary conditions

\[ y(t,0) = y(t,L) = 0, \quad t \in (0,T), \]  

and the following initial conditions

\[ y(0,x) = \tilde{y}_0(x), \quad \partial_t y(0,x) = \tilde{y}_1(x), \quad x \in \Omega, \]  

where

\[ F[y] = F\left( t, x, y, y_x, \partial_t y, \partial_{tt} y \right) \in C^1 \left( \mathbb{R}^+ \times [0,L] \times \mathbb{R}^4 \right), \]  

\( y = y(t,x), y_x = \frac{\partial y}{\partial x}(t,x), y_{xx} = \frac{\partial^2 y}{\partial x^2}(t,x), \lambda, \lambda_0, \lambda_1, \lambda_2, L > 0 \) are constants. The past history in (6) is

\[ \int_{-\infty}^{t} \mu(t-s)y_{xx}(s)ds, \]  

which is considered as a damping term. It is well known that the damping terms play an important role in the studying the propagation mechanism of wave. It shows a behavior which is something between that of elastic solids and Newtonian fluids. Indeed, the stresses in these media
depend on the entire history of their deformation, not only on their current state of deformation or their current state of motion.

Equation (6) is a generalization of a class of symmetric regularized long wave equations, known in abbreviation as (SRLWEs), given by

\[ \partial_{tt} y - y_{xx} - \partial_{tt} y_{xxx} = -y \partial_t y_x - u_x \partial_t u. \]  

Equation (10) was proposed as a model for propagation of weakly nonlinear ion acoustic and space charge waves, it is explicitly symmetric in the \( x \) and \( t \) derivatives and is very similar to the regularized long wave equation which describes shallow water waves and plasma drift waves. The SRLWE and its symmetric version also arises in many other areas of mathematical physics.

The functions \( \mu \) and \( f \) satisfy

**Hypothesis 1.** \( \mu \in \mathcal{C}^1(\mathbb{R}^+, \mathbb{R}^+) \) is a nonincreasing function such that:

\[ \lambda_0 - \lambda \int_0^\infty \mu(s)ds = l > 0, \quad \mu(0) > 0, \quad \lim_{s \to \infty} \mu(s) = 0; \]

**Hypothesis 2.** \( \tilde{y}_0, \tilde{y}_1 \in \mathcal{H}^1_0(\Omega) \cap \mathcal{H}^2(\Omega); \)

**Hypothesis 3.** \( f \in \mathcal{H}^1((0, T) \times \Omega); \)

**Hypothesis 4.** \( F \in \mathcal{C}^1([0, T] \times [0, L] \times \mathbb{R}^4), \quad F(t, 0, 0, y_2, 0, y_4) = F(t, L, 0, y_2, 0, y_4) = 0 \) for all \( t \in [0, T], \quad y_2, y_4 \in \mathbb{R}. \)

Below, for \( F = F(t, x, y_1, \ldots, y_4) \), we denote \( D_1 F = \frac{\partial F}{\partial t}, D_2 F = \frac{\partial F}{\partial x}, D_{i+2} F = \frac{\partial^2 F}{\partial x^i \partial t}, \) with \( i = 1, \ldots, 4. \) Our interest in this paper arose in the first place in consequence of a query for existence of unique solution.

Equations of Love waves or Love-type waves have been studied by many authors, we refer to [8–13], and references therein. In [8] a higher order iterative scheme is established for a Dirichlet problem for a class nonlinear Love-type equations

\[ \partial_{tt} y - y_{xx} - \partial_{tt} y_{xxx} = f(x, t, y), \]

and the authors get a recurrence sequence that converges at a rate 1 to a local unique weak solution of the above mentioned equation. In [8] is considered the following nonlinear Love equation

\[ \partial_{tt} y - y_{xx} - \partial_{tt} y_{xxx} - \lambda_1 \partial_t y_{xx} + \lambda \partial_t y = F(x, t, y, y_x, \partial_t y, \partial_t y_x) \]

\[ - (G(x, t, y, y_x, \partial_t y, \partial_t y_x))_x + f(x, t), \]

with initial conditions and homogeneous Dirichlet boundary conditions and the authors established the existence of a unique local weak solution, a blow-up result for solutions with negative initial energy, the global existence and exponential decay of weak solution. In [9] is investigated the following Love equation

\[ \partial_{tt} y - (\mu(x, t)y_x)_x + f(y, \partial_t y) = F(x, t), \]
with initial conditions and boundary conditions of two-point type and the authors proved existence of a weak solution, uniqueness, regularity, and decay properties of solution. In [13] is investigated the following nonlinear Love equation

$$\partial_{tt} y - \left( B(x, t, y, \|y\|^2, \|y_x\|^2, \|\partial_t y\|^2, \|\partial_t y_x\|^2) (y_x + \lambda_1 \partial_t y_x + \partial_t y_x) \right) + \lambda \partial_t y = F(x, t, y, y_x, \partial_t y, \partial_t y_x, y(t), \|y_x(t)\|^2, \|\partial_t y(t)\|^2, \|\partial_t y_x(t)\|^2)$$

with initial conditions and homogeneous Dirichlet boundary conditions and the authors proved existence of solutions, exponential decay of solutions, and blow-up results for viscoelastic wave equations, have been extensively studied and many results have been obtained by many authors (see [14–16]).

In this paper, the attention is focused on the local and global existence of weak solution of the problem (6)–(8). In Section 2, combining the linearization method, the Faedo–Galerkin method, and the weak compactness method, the local existence and uniqueness of the weak solution of the problem (6)–(8) is proved. In Section 3, using the potential well method, it is shown that the solution for class of Love-equation exists globally under some conditions on the initial datum.

2. Existence and Uniqueness of Local Weak Solution

Definition 1. A function $y$ is said to be a weak solution of (6)–(8) if

$$y, \partial_t y, \partial_{tt} y \in L^\infty((0, T), H^1_0(\Omega) \cap H^2(\Omega))$$

and $y$ satisfies the variational equation

$$\int_0^L \partial_{tt} y w \, dx + \int_0^L (\lambda_0 y_x + \lambda_1 \partial_t y_x + \lambda_2 \partial_{tt} y_x) w_x \, dx$$

$$- \lambda \int_0^L \int_0^s \mu(s) y_x(t - s) \, dw_x \, ds \, dx$$

$$= \int_0^L f w \, dx + \int_0^L F[y] w \, dx + \int_0^L F'[y] w_x \, dx,$$  \hspace{1cm} (12)

for all test functions $w \in H^1_0(\Omega)$ and for almost all $t \in (0, T)$.

The following technical result will play an important role in the sequel. [4] For any $v \in C^1(0, T, H^1_0(\Omega))$ we have

$$\int_\Omega \int_0^\infty \mu(s) v_{xx}(t - s) \partial_t v(t) \, dx \, ds$$

$$= \frac{1}{2} \partial_t \int_0^\infty \mu(s) \int_\Omega \left| v_x(t - s) - v_x(t) \right|^2 \, dx \, ds + \frac{1}{2} \partial_t \int_0^\infty \mu(s) \int_\Omega \left| v_x(t) \right|^2 \, dx \, ds$$

$$- \frac{1}{2} \int_0^\infty \partial_t \mu(s) \int_\Omega \left| v_x(t - s) - v_x(t) \right|^2 \, dx \, ds + \frac{1}{2} \mu(t) \int_\Omega \left| v_x(t) \right|^2 \, dx. \hspace{1cm} (13)$$

Now, we will prove the existence of a unique local solution for (6)–(8). Our main result is as follows.
Theorem 1. Let \( y_0, y_1 \in H^1_0(\Omega) \cap H^2(\Omega) \) be given. Assume that (Hypothesis 1) – (Hypothesis 4) hold. Then there exists a \( T_0 \in (0, T] \) such that the problem (6)–(8) has a unique local solution \( y \) for which
\[
y, \partial_t y, \partial_t y \in L^\infty((0, T_0); H^1_0(\Omega) \cap H^2(\Omega)).
\] (14)

Proof of Theorem 1. Firstly, we will construct a sequence \( \{y_m\}_{m \in \mathbb{N}} \). Then, the Faedo–Galerkin method combined with the weak compactness method shows that \( \{y_m\}_{m \in \mathbb{N}} \) converges to \( y \) which is exactly a unique local solution of (6)–(8).

Step 1. Let \( T > 0 \) be fixed and \( M > 0 \) be arbitrarily chosen. We set
\[
K_M(f) = \sqrt{\|f\|^2_{L^2((0,1) \times (0,T))} + \|\partial_t f\|^2_{L^2((0,1) \times (0,T))} + \|f_x\|^2_{L^2((0,1) \times (0,T))}}.
\] (15)

Let
\[
\|F\|_{C^0([0,1] \times [0,T]) \cap [-M,M]^4} = \sup_{(x,t,y_1,\ldots,y_4) \in [0,1] \times [0,T] \times [-M,M]^4} |F(x,t,y_1,\ldots,y_4)|
\]

and
\[
F_M = \|F\|_{C^1([0,1] \times [0,T] \times [-M,M]^4)}
\]
\[
= \|F\|_{C^0([0,1] \times [0,T] \times [-M,M]^4)}
\]
\[
+ \sum_{i=1}^6 \|D_i F\|_{C^0([0,1] \times [0,T] \times [-M,M]^4)}.
\]

For some \( T_0 \in (0, T] \) and \( M > 0 \), we put
\[
W(M, T_0) = \left\{ v, \partial_t v \in L^\infty(0, T_0; H^1_0 \cap H^2) : \partial_{tt} v \in L^\infty(0, T_0; H^1_0) \right\},
\]
\[
W_1(M, T_0) = \left\{ v \in W(M, T_0) : \partial_{tt} v \in L^\infty(0, T_0; H^1_0 \cap H^2) \right\}.
\] (16)

Take \( y_0 \equiv 0 \) and define the sequence \( \{y_m\}_{m \in \mathbb{N}} \) as follows
\[
\int_0^L \partial_t y_m w dx + \int_0^L \left( \lambda_0 y_{xm} + \lambda_1 \partial_t y_{xm} + \lambda_2 \partial_{tt} y_{xm} \right) w_x dx
\]
\[
- \lambda \int_0^\infty \mu(s) ds \int_0^L y_{xm} w_x dx - \lambda \int_0^\infty \mu(s) \int_0^L y_{xm}(t-s) - y_{xm}) w_x dx ds
\]
\[
= \int_0^L f w dx + \int_0^L F_{m-1}[y] w dx + \int_0^L F_{m-1}[y] w_x dx, \quad t \in (0, T), \forall w \in H^1_0(\Omega),
\]
\[
y_m(0) = \tilde{y}_0, \quad \partial_t y_m(0) = \tilde{y}_1 \quad \text{on} \quad \Omega, \quad m \in \mathbb{N},
\]
where
\[
F_{m-1}[y] = F[y_{m-1}],
\]
\[
F \left( x, t, y_{m-1}, y_{xm-1}, \partial_t y_{m-1}, \partial_{tt} y_{xm-1} \right).
\] (18)
Let \( \{ w_j \}_{j=1}^{\infty} \) be an orthonormal basis of \( H_0^1(\Omega) \), formed by the eigenfunctions of the operator \( -\frac{\partial^2}{\partial x^2} \). Let also, \( V_k = \text{span}\{ w_1, w_2, \ldots, w_k \} \). We have

\[
\tilde{y}_{0k} = \sum_{j=1}^{k} a_j^{(k)} w_j, \\
\tilde{y}_{1k} = \sum_{j=1}^{k} \beta_j^{(k)} w_j
\]

for

\[
a_j^{(k)} = \int_0^L \tilde{y}_{0j} w_j dx, \\
\beta_j^{(k)} = \int_0^L \tilde{y}_{1j} w_j dx.
\]

Note that

\[
\tilde{y}_{0k} \to \tilde{y}_0 \quad \text{strongly in } H_0^1(\Omega) \cap H^2(\Omega), \\
\tilde{y}_{1k} = \to \tilde{y}_1 \quad \text{strongly in } H_0^1(\Omega) \cap H^2(\Omega).
\]

We seek \( k \) functions \( \varphi_{mj}^{(k)}(t) \in C^2[0, T], 1 \leq j \leq k \), such that the expression in the form

\[
y_m^{(k)} = \sum_{j=1}^{k} \varphi_{mj}^{(k)} w_j
\]

solves the problem

\[
\int_0^L \partial_t y_m^{(k)} w_j dx + \int_0^L \left( \lambda_0 y_m^{(k)} + \lambda_1 \partial_t y_m^{(k)} + \lambda_2 \partial_t y_m^{(k)} \right) w_j dx \\
- \lambda \int_0^\infty \mu(s) ds \int_0^L y_m^{(k)} w_j dx - \lambda \int_0^\infty \mu(s) \int_0^L (y_m^{(k)}(t-s) - y_m^{(k)}(s)) w_j dx ds \\
= \int_0^L f w_j dx + \int_0^L F_m w_j dx + \int_0^L F_m w_j dx, \quad 1 \leq j \leq k, \\
y_m^{(k)}(0) = \tilde{y}_{0k}(t), \quad \partial_t y_m^{(k)}(0) = \tilde{y}_{1k}.
\]

This leads to a system of ODE’s for unknown functions \( \varphi_{mj}^{(k)} \). Based on standard existence theory for ODE, the system (22) admits a unique solution \( \varphi_{mj}^{(k)}, 1 \leq j \leq k \) on the interval \( [0, T] \).

**Step 2.** Now we will prove that there exist constants \( M > 0 \) and \( T_\ast \in [0, T) \) such that \( y_m^{(k)} \in W(M, T_\ast) \), for all \( m, k \in \mathbb{N} \). We partially estimate the terms of the associated energy. We replace \( y_m \) and \( w \) with \( y_m^{(k)} \) in (17) and we get

\[
\int_0^L \partial_t y_m^{(k)} \partial_t y_m^{(k)} dx + \int_0^L \left( \lambda_0 y_m^{(k)} + \lambda_1 \partial_t y_m^{(k)} + \lambda_2 \partial_t y_m^{(k)} \right) \partial_t y_m^{(k)} dx \\
+ \lambda \int_0^L \int_0^\infty \mu(s) y_m^{(k)}(t-s) \partial_t y_m^{(k)} ds dx \\
= \int_0^L F_m y_m^{(k)} \partial_t y_m^{(k)} dx + \int_0^L F_m y_m^{(k)} \partial_t y_m^{(k)} dx + \int_0^L f \partial_t y_m^{(k)} dx.
\]
Using (13), we obtain

\[
\frac{1}{2} \partial t \left[ \int_{0}^{L} \left( \partial t y_{m}^{(k)} \right)^{2} + \left( \lambda_{0} + \lambda \int_{0}^{\infty} \mu(s) \right) \left| y_{xm}^{(k)} \right|^{2} + \lambda_{2} \left| \partial_{t} y_{xm}^{(k)} \right|^{2} dx \right] \right. \\
- \lambda \int_{0}^{\infty} \mu(s) \int_{0}^{L} \left| y_{xm}^{(k)}(t-s) - y_{xm}^{(k)}(t) \right|^{2} dx ds \\
+ \lambda_{1} \int_{0}^{L} \left| \partial_{t} y_{xm}^{(k)} \right|^{2} dx + \lambda \int_{0}^{\infty} \partial_{s} \mu(s) \int_{0}^{L} \left| y_{xm}^{(k)}(t-s) - y_{xm}^{(k)}(t) \right|^{2} dx ds \\
= \int_{0}^{L} F[y_{m}^{(k)}] \partial_{t} y_{m}^{(k)} dx + \int_{0}^{L} F[y_{m}^{(k)}] \partial_{t} y_{xm}^{(k)} dx + \int_{0}^{L} \partial_{t} y_{m}^{(k)} dx.
\]

(23)

Let us denote the left hand side of (23) as

\[
e^{(k)}(y_{m}) = \int_{0}^{L} \left[ \partial_{t} y_{m}^{(k)} \right]^{2} + \left( \lambda_{0} + \lambda \int_{0}^{\infty} \mu(s) ds \right) \left| y_{xm}^{(k)} \right|^{2} + \lambda_{2} \left| \partial_{t} y_{xm}^{(k)} \right|^{2} dx \\
- \lambda \int_{0}^{\infty} \mu(s) \int_{0}^{L} \left| y_{xm}^{(k)}(t-s) - y_{xm}^{(k)}(t) \right|^{2} dx ds \\
+ 2 \lambda_{1} \int_{0}^{L} \left| \partial_{t} y_{xm}^{(k)} \right|^{2} dx ds \\
+ 2 \lambda \int_{0}^{L} \int_{0}^{\infty} \partial_{s} \mu(s) \int_{0}^{L} \left| y_{xm}^{(k)}(t-s) - y_{xm}^{(k)}(t) \right|^{2} dx ds d\tau,
\]

and

\[
e^{(k)}(y_{xm}) = \int_{0}^{L} \left[ \partial_{t} y_{xm}^{(k)} \right]^{2} + \left( \lambda_{0} + \lambda \int_{0}^{\infty} \mu(s) ds \right) \left| y_{xm}^{(k)} \right|^{2} + \lambda_{2} \left| \partial_{t} y_{xm}^{(k)} \right|^{2} dx \\
- \lambda \int_{0}^{\infty} \mu(s) \int_{0}^{L} \left| y_{xm}^{(k)}(t-s) - y_{xm}^{(k)}(t) \right|^{2} dx ds \\
+ 2 \lambda_{1} \int_{0}^{L} \left| \partial_{t} y_{xm}^{(k)} \right|^{2} dx ds \\
+ 2 \lambda \int_{0}^{L} \int_{0}^{\infty} \partial_{s} \mu(s) \int_{0}^{L} \left| y_{xm}^{(k)}(t-s) - y_{xm}^{(k)}(t) \right|^{2} dx ds d\tau,
\]

and

\[
e^{(k)}(\partial_{t} y_{m}) = \int_{0}^{L} \left[ \partial_{t} y_{m}^{(k)} \right]^{2} + \left( \lambda_{0} + \lambda \int_{0}^{\infty} \mu(s) ds \right) \left| \partial_{t} y_{xm}^{(k)} \right|^{2} + \lambda_{2} \left| \partial_{t} y_{xm}^{(k)} \right|^{2} dx \\
- \lambda \int_{0}^{\infty} \mu(s) \int_{0}^{L} \left| \partial_{t} y_{xm}^{(k)}(t-s) - \partial_{t} y_{xm}^{(k)}(t) \right|^{2} dx ds \\
+ 2 \lambda_{1} \int_{0}^{L} \left| \partial_{t} y_{xm}^{(k)} \right|^{2} dx ds \\
+ 2 \lambda \int_{0}^{L} \int_{0}^{\infty} \partial_{s} \mu(s) \int_{0}^{L} \left| \partial_{t} y_{xm}^{(k)}(t-s) - \partial_{t} y_{xm}^{(k)}(t) \right|^{2} dx ds d\tau.
\]

Put

\[
e_{m}^{(k)}(t) = e^{(k)}(y_{m}) + e^{(k)}(y_{xm}) + e^{(k)}(\partial_{t} y_{m}).
\]

(24)
Then
\[ E_m^{(k)}(t) = E_m^{(k)}(0) + 2 \int_0^t \int_0^L f(s) \partial_t y_m^{(k)}(s) dx ds + 2 \int_0^t \int_0^L f_x(s) \partial_t y_m^{(k)}(s) dx ds + 2 \int_0^t \int_0^L f_{xx}(s) \partial_t y_m^{(k)}(s) dx ds \]
\[ + 2 \int_0^t \int_0^L f_m(s) \partial_t y_m^{(k)}(s) dx ds + 2 \int_0^t \int_0^L F_m(s) \partial_t y_m^{(k)}(s) dx ds \]
\[ + 2 \int_0^t \int_0^L \partial_t f_m(s) \partial_t y_m^{(k)}(s) dx ds + 2 \int_0^t \int_0^L \partial_t F_m(s) \partial_t y_m^{(k)}(s) dx ds \]
\[ + 2 \int_0^t \int_0^L \partial_t y_m^{(k)}(s) dx ds. \quad (25) \]

Now we will estimate
\[ A_m^{(k)} = \int_0^L |\partial_t y_m^{(k)}(0)|^2 dx + \lambda_2 \int_0^1 |\partial_t y_m^{(k)}(0)|^2 dx. \]

Let \( w_i = \partial_t y_m^{(k)} \) in (22). Then we take \( t \to 0_+ \) in the first term and we obtain
\[ \int_0^L |\partial_t y_m^{(k)}(0)|^2 dx + \lambda_2 \int_0^L |\partial_t y_m^{(k)}(0)|^2 dx \]
\[ + \int_0^L \left( \left[ \lambda_0 - \lambda \int_0^\infty \mu(s) ds \right] \tilde{y}_{0kx} + \lambda_1 \tilde{y}_{1kx} \right) \partial_t y_m^{(k)}(0) dx \]
\[ + \lambda \int_0^\infty \mu(s) \int_0^L (\tilde{y}_{0kx}(0) - \tilde{y}_{0kx}(-s)) \partial_t y_m^{(k)}(0) dx ds \]
\[ = \int_0^L f(0) \partial_t y_m^{(k)}(0) dx + \int_0^L F_m(0) \partial_t y_m^{(k)}(0) dx + \int_0^L F_m(0) \partial_t y_m^{(k)}(0) dx. \]

Then
\[ A_m^{(k)} \leq \int_0^L \left( \left[ \lambda_0 - \lambda \int_0^\infty \mu(s) ds \right] \tilde{y}_{0kx} + \lambda_1 \tilde{y}_{1kx} + F_m(0) \right) \partial_t y_m^{(k)}(0) dx \]
\[ + \lambda \int_0^\infty \mu(s) \int_0^L (\tilde{y}_{0kx}(0) - \tilde{y}_{0kx}(-s)) \partial_t y_m^{(k)}(0) dx ds \]
\[ + \int_0^L f(0) \partial_t y_m^{(k)}(0) dx + \int_0^L F_m(0) \partial_t y_m^{(k)}(0) dx, \]
\[ \leq \int_0^L \left[ \left[ \lambda_0 - \lambda \int_0^\infty \mu(s) ds \right] \tilde{y}_{0kx} + \lambda_1 \tilde{y}_{1kx} + 2F_m(0) + f(0) \right. \]
\[ + \lambda \int_0^\infty \mu(s) (y_{0kx}(0) - y_{0kx}(-s)) \right] dx \]
\[ \leq \zeta, \quad \text{for all } m, k \in \mathbb{N}. \quad (26) \]
In the last inequality we have used that \( f^L_0 |F_m(0)| dx \) is a constant independent of \( m \). Note that \( \xi \) is a constant depending only on \( f, \tilde{y}_0, \tilde{y}_1, F, \lambda_0, \lambda, \lambda_1, \lambda_2 \) and \( \int_0^\infty \mu(s) ds \). Equations (20), (24) and (26) imply that

\[
E_m^{(k)} (0) = \int_0^L \left[ |\tilde{y}_{1k}|^2 + \left( \lambda_0 - \lambda \int_0^\infty \mu(s) ds \right) |\tilde{y}_{0kx}|^2 + |\tilde{y}_{1kx}|^2 dx \right] \nonumber
\]

\[
+ \lambda \int_0^\infty \mu(s) \int_0^L \left| \tilde{y}_{0kx}(-s) - \tilde{y}_{0kx}(0) \right|^2 dx ds \nonumber
\]

\[
+ \int_0^L \left[ |\tilde{y}_{1kx}|^2 + \left( \lambda_0 - \lambda \int_0^\infty \mu(s) ds \right) |\tilde{y}_{0kx}|^2 + |\tilde{y}_{1kx}|^2 dx \right] \nonumber
\]

\[
+ \lambda \int_0^\infty \mu(s) \int_0^L \left| \tilde{y}_{0kxx}(-s) - \tilde{y}_{0kxx}(0) \right|^2 dx ds \nonumber
\]

\[
+ A_m^{(k)} + \int_0^L \left( \lambda_0 - \lambda \int_0^\infty \mu(s) ds \right) |\tilde{y}_{1kx}|^2 dx \nonumber
\]

\[
+ \lambda \int_0^\infty \mu(s) \int_0^L |\tilde{y}_{1kx} - \tilde{y}_{1kx}|^2 dx ds \nonumber
\]

\[
\leq \xi_0 \quad \text{for all } m, k \in \mathbb{N},
\]

(27)

where \( \xi_0 \) is a constant depending only on \( f, \tilde{y}_0, \tilde{y}_1, F, \lambda_0, \lambda, \lambda_1, \lambda_2 \) and \( \int_0^\infty \mu(s) ds \).

Now we estimate the other terms of (25). By the Cauchy—Schwartz's inequality, we obtain

\[
E_m^{(k)} (t) \leq \xi_0 \nonumber
\]

\[
+ ||f||_{L^2(\Omega \times (0,T))}^2 + \int_0^t \int_0^L |\partial_t y_m^{(k)}|^2 dx ds; \nonumber
\]

\[
+ ||f_x||_{L^2(\Omega \times (0,T))}^2 + \int_0^t \int_0^L |\partial_{tt} y_m^{(k)}|^2 dx ds; \nonumber
\]

\[
+ ||\partial_y f||_{L^2(\Omega \times (0,T))}^2 + \int_0^t \int_0^L |\partial_t y_m^{(k)}|^2 dx ds; \nonumber
\]

\[
+ 2 \int_0^t \int_0^L F_m(s) \partial_t y_m^{(k)}(s) dx ds \nonumber
\]

\[
+ 2 \int_0^t \int_0^L F_m(s) \partial_t y_m^{(k)}(s) dx ds + 2 \int_0^t \int_0^L F_m(s) \partial_t y_m^{(k)}(s) dx ds \nonumber
\]

\[
+ 2 \int_0^t \int_0^L \partial_t F_m(s) \partial_t y_m^{(k)}(s) dx ds + 2 \int_0^t \int_0^L \partial_t F_m(s) \partial_t y_m^{(k)}(s) dx ds \nonumber
\]

\[
+ 2 \int_0^t \int_0^L F_m(s) \partial_t y_m^{(k)}(s) dx ds. \nonumber
\]

(28)

By (Hypothesis 1) and (24), we have

\[
E_m^{(k)} (t) \leq \xi_0 + ||f||_{H^1(\Omega \times (0,T))}^2 + c \int_0^t E_m^{(k)} (s) ds \nonumber
\]

\[
+ 2 \int_0^t \int_0^L F_m(s) \partial_t y_m^{(k)}(s) dx ds \nonumber
\]

\[
+ 2 \int_0^t \int_0^L F_m(s) \partial_t y_m^{(k)}(s) dx ds + 2 \int_0^t \int_0^L F_m(s) \partial_t y_m^{(k)}(s) dx ds \nonumber
\]

\[
+ 2 \int_0^t \int_0^L \partial_t F_m(s) \partial_t y_m^{(k)}(s) dx ds + 2 \int_0^t \int_0^L \partial_t F_m(s) \partial_t y_m^{(k)}(s) dx ds \nonumber
\]

\[
+ 2 \int_0^t \int_0^L F_m(s) \partial_t y_m^{(k)}(s) dx ds. \nonumber
\]

(29)
We have
\[
E_m^{(k)}(t) \leq \xi_0 + \|f\|^2_{L^1(\Omega \times (0,T))} + c \int_0^t E_m^{(k)}(s) ds \\
+ T_s F_M^2 + \int_0^t \int_0^t |\partial_t y_{xm}^{(k)}(s)|^2 dxds + T_s F_M^2 + \int_0^t \int_0^t |\partial_t y_m^{(k)}(s)|^2 dxds \\
+ 2 \int_0^t \int_0^t F_{xm}(s) \partial_t y_{xm}^{(k)}(s) dx ds + T_s F_M^2 + \int_0^t \int_0^t |\partial_t y_m^{(k)}(s)|^2 dxds \\
+ 2 \int_0^t \int_0^t \partial_t F_{xm}(s) \partial_t y_{xm}^{(k)}(s) dx ds + 2 \int_0^t \int_0^t \partial_t F_m(s) \partial_t y_m^{(k)}(s) dx ds.
\]

By (Hypothesis 1), (9) and (24), we arrive to
\[
\partial_t F_{xm}(t) = D_1 F[y_{m-1}] + D_3 F'[y_{m-1}] \partial_t y_{xm-1} + D_5 F[y_{m-1}] \partial_t y_{xm-1} \\
+ D_5 F[y_{m-1}] \partial_t y_{xm-1} + D_6 F[y_{m-1}] \partial_t y_{xm-1}.
\]

Then
\[
E_m^{(k)}(t) \leq \xi_0 + \|f\|^2_{L^1(\Omega \times (0,T))} \\
+ 2 T_s \left[1 + 2(1 + 4M)^2 \right] F_M^2 + c \int_0^t E_m^{(k)}(s) ds.
\]

We choose \(M > 0\) sufficiently large such that
\[
\xi_0 + \|f\|^2_{L^1(\Omega \times (0,T))} \leq \frac{1}{2} M^2,
\]
and then choose \(T_s \in (0, T]\) small enough such that
\[
\left(\frac{1}{2} M^2 + T_s \left[1 + 2(1 + 4M)^2 \right] F_M^2\right) \exp[2T_s] \leq M^2,
\]
and
\[
k_{T_s} = 2 \sqrt{F_M^2} \exp[T_s] < 1.
\]

Then
\[
y_m^{(k)} \in W(M, T_s), \text{ for all } m \text{ and } k.
\]

By (33), there exists a subsequence of \(\{y_m^{(k)}\}\), such that
\[
y_m^{(k)} \rightarrow y_m \text{ in } L^\infty(0, T_s; H^1_0 \cap H^2) \text{ weakly*},
\]
\[
\partial_t y_m^{(k)} \rightarrow \partial_t y_m \text{ in } L^\infty(0, T_s; H^1_0 \cap H^2) \text{ weakly*},
\]
\[
\partial_{tt} y_m^{(k)} \rightarrow \partial_{tt} y_m \text{ in } L^\infty(0, T_s; H^1_0 \cap H^2) \text{ weakly*},
\]
\[
y_m \in W(M, T_s).
\]

Passing to limit in (22) we see that \(y_m\) satisfies (17), (18) in \(L^2(0, T_s)\). Furthermore, (17)_1 and (36)_4 give
\[
\lambda_0 y_m + \lambda_1 \partial_t y_m + \lambda_2 \partial_{tt} y_m + \lambda \int_0^\infty \mu(s) y_m(t-s) ds
\]
\[
= \partial_{tt} y_m - F[y_m] - \left( F[y_m]\right)_x - f
\]
\[
\equiv \psi_m \in L^\infty((0, T_s); H^1_0(\Omega) \cap H^2(\Omega)).
\]

We deduce that, if
\[
y_m \in L^\infty((0, T_s); H^1_0(\Omega) \cap H^2(\Omega)),
\]
Then
\[ \partial_t y_m, \partial_t y_m \in L^\infty((0, T_\ast); H^1_0(\Omega) \cap H^2(\Omega)). \]

So, we obtain \( y_m \in W_1(M, T_\ast). \)

**Step 3.** Consider the Banach space
\[ W_1(T_\ast) = \{ v \in L^\infty(0, T_\ast; H^1_0) : \partial_t v \in L^\infty(0, T_\ast; H^0_0) \}, \]
endowed with the norm
\[ \| v \|_{W_1(T_\ast)} = \| v \|_{L^\infty(0, T_\ast; H^1_0)} + \| \partial_t v \|_{L^\infty(0, T_\ast; H^0_0)}. \]

We will show the convergence of \( \{ y_m \}_{m \in \mathbb{N}} \) to the solution of our problem. Let \( w_m = y_{m+1} - y_m. \) Then \( w_m \) satisfies
\[
\begin{align*}
\int_0^L \partial_t w_m \omega \, dx + \int_0^L (\lambda_0 \omega_{xm} + \lambda_1 \partial_t \omega_{xm} + \lambda_2 \partial_t \omega_{xm}) \omega \, dx \\
- \lambda \int_0^L \int_0^{\infty} \mu(s) \omega_{xm}(t-s) \, ds \, \omega \, dx \\
= \int_0^L \left( F[w_{m+1} - F[w_m] \right) \omega \, dx + \int_0^L \left( F[w_{m+1}] - F[w_m] \right) \omega \, dx,
\end{align*}
\]
\[ w_m(0) = \partial_t w_m(0) = 0. \]

Consider (39) with \( w = \partial_t w_m. \) Then, integrating in \( t, \) we obtain
\[
\begin{align*}
\int_0^L \left[ \partial_t w_m \right]^2 + \left( \lambda_0 - \lambda \int_0^{\infty} \mu(s) \, ds \right) |\omega_{xm}|^2 + \lambda_2 |\partial_t \omega_{xm}|^2 \right] \, dx \\
+ \lambda \int_0^{\infty} \mu(s) \int_0^L |\omega_{xm}(t-s) - \omega_{xm}(t)|^2 \, dx \, ds \\
+ 2\lambda_1 \int_0^t \int_0^L |\partial_t \omega_{xm}|^2 \, dxds - \lambda \int_0^t \int_0^{\infty} \partial_t \mu(s) \int_0^L |\omega_{xm}(t-s) - \omega_{xm}(t)|^2 \, dxd\tau \\
= 2 \int_0^t \int_0^L \left( F_{m+1}(s) - F_m(s) \right) \partial_t w_m(s) \, dxds \\
+ 2 \int_0^t \int_0^L \left( F_{m+1}(s) - F_m(s) \right) \partial_t \omega_{xm}(s) \, dxds.
\end{align*}
\]

By (Hypothesis 2)–(Hypothesis 4), (15) and (36), we have
\[
\int_0^L |F_{m+1}(s) - F_m(s)|^2 \, dx \leq 2 F_M \int_0^L |w_{m-1}|^2 \, dx.
\]
Then
\[
E_m(t) \leq 2 F_M \int_0^L |w_{m-1}|^2 \, dx + \int_0^t \int_0^L |\partial_t w_m|^2 \, dxds \\
+ 2 F_M \int_0^L |w_{m-1}|^2 \, dx + \int_0^t \int_0^L |\partial_t \omega_{xm}|^2 \, dxds \\
\leq 4 F_M \int_0^L |w_{m-1}|^2 \, dx T_\ast + \int_0^t E_m(s) \, ds,
\]
where
\[ E_m(t) = 2 F_M \int_0^L |w_{m-1}|^2 \, dx + \int_0^t \int_0^L |\partial_t w_m|^2 \, dxds. \]
where

$$E_{m}(t) = \int_{0}^{L} \left[ |\partial_{t} w_{m}|^{2} + \left( \lambda_{0} - \lambda \int_{0}^{\infty} \mu(s) ds \right) |w_{xm}|^{2} + \lambda_{2} |\partial_{t} w_{xm}|^{2} \right] dx$$

$$+ \lambda \int_{0}^{\infty} \mu(s) \int_{0}^{L} |w_{xm}(t-s) - w_{xm}(t)|^{2} dx ds$$

$$+ 2\lambda_{1} \int_{0}^{L} |\partial_{t} w_{xm}|^{2} dx ds$$

$$- \lambda \int_{0}^{L} \int_{0}^{\infty} \partial_{t} \mu(s) \int_{0}^{L} |w_{xm}(t-s) - w_{xm}(t)|^{2} dx ds d\tau.$$ 

By (41), using Gronwall’s Lemma, we get

$$\int_{0}^{L} |w_{m}| dx \leq k_{T_{*}} \int_{0}^{L} |w_{m-1}| dx \quad \forall m \in \mathbb{N}. \quad (42)$$

So,

$$\int_{0}^{L} |y_{m} - y_{m+p}| dx \leq M(1 - k_{T_{*}})^{-1} k_{T_{*}}^{m}, \quad \forall m, p \in \mathbb{N}. \quad (43)$$

From here, it follows that \( \{y_{m}\}_{m \in \mathbb{N}} \) is a Cauchy sequence in \( W_{1}(T_{*}) \). Therefore there exists \( y \in W_{1}(T_{*}) \) such that

$$y_{m} \to y \text{ strongly in } W_{1}(T_{*}). \quad (44)$$

Note that \( y_{m} \in W_{1}(M, T_{*}) \). Hence, there exists a subsequence \( \{y_{m_{j}}\}_{j \in \mathbb{N}} \) of \( \{y_{m}\}_{m \in \mathbb{N}} \) such that

$$y_{m_{j}} \to y \quad \text{in } L^{\infty}((0, T_{*}); H_{0}^{1}(\Omega) \cap H^{2}(\Omega)) \text{ weakly*},$$

$$\partial_{t} y_{m_{j}} \to \partial_{t} y \quad \text{in } L^{\infty}((0, T_{*}); H_{0}^{1}(\Omega) \cap H^{2}(\Omega)) \text{ weakly*},$$

$$\partial_{tt} y_{m_{j}} \to \partial_{tt} y \quad \text{in } L^{\infty}((0, T_{*}); H_{0}^{1}(\Omega)) \text{ weakly*}, \quad y \in W(M, T_{*}). \quad (45)$$

By (15), (18) and (45), we obtain

$$\int_{0}^{L} |F_{m}(t) - F[y](t)| dx \leq 2(1 + 2M) F_{M} \int_{0}^{L} |y_{m-1} - y| dx. \quad (46)$$

Then (44) and (46) imply

$$F_{m} \to F[y] \quad \text{strongly in } L^{\infty}((0, T_{*}); L^{2}(\Omega)), \quad (47)$$

Passing to limit in (17) and (18), as \( m = m_{j} \to \infty \). By (44), (45) and (47), it follows that there exists \( y \in W(M, T_{*}) \) satisfying

$$\int_{0}^{L} \partial_{tt} w dx + \int_{0}^{L} (\lambda_{0} y_{x} + \lambda_{2} \partial_{t} y_{x} + \lambda_{2} \partial_{tt} y_{x}) w_{x} dx$$

$$- \lambda \int_{0}^{L} \int_{0}^{\infty} \mu(s) y_{x}(t-s) ds w_{x} dx$$

$$= \int_{0}^{L} f w dx + \int_{0}^{L} F[y] w dx + \int_{0}^{L} F[y] w_{x} dx, \quad (48)$$
for all test function \( w \in H_0^1(\Omega) \), for almost all \( t \in (0, T) \), and satisfying the initial conditions. Let \( y_1, y_2 \) be two weak solutions of (6)–(8), such that

\[
y_1, y_2 \in W_1(M, T_\ast).
\]

Then \( v = y_1 - y_2 \) satisfies

\[
j \int_0^L \partial_t vw \, dx + \int_0^L \left( \lambda_0 v_x + \lambda_1 \partial_t v_x + \lambda_2 \partial_t y_x \right) w_x \, dx \\
- \lambda \int_0^L \int_0^\infty \mu(s) v_x(t-s) ds \, w_x \, dx \\
= \int_0^L \left( F[y_1] - F[y_2] \right) w \, dx + \int_0^L \left( F[y_1] - F[y_2] \right) w_x \, dx,
\]

for all test function \( w \in H_0^1(\Omega) \) and for almost all \( t \in [0, T] \). Taking \( \partial_t v = w \) in (50) and integrating with respect to \( t \), for

\[
e(t) = \int_0^L \left[ \partial_t w^2 + \left( \lambda_0 - \lambda \int_0^\infty \mu(s) \, ds \right) |w_x|^2 + \lambda_2 |\partial_t w_x|^2 \right] \, dx \\
+ \lambda \int_0^\infty \mu(s) \int_0^L |w_x(t-s) - w_x(t)|^2 \, dx \, ds \\
+ 2\lambda_1 \int_0^L \int_0^\infty |\partial_t w_x|^2 \, dx \, ds \\
- \lambda \int_0^L \int_0^\infty |\partial_t w_x|^2 \, dx \, ds \\
- \lambda \int_0^L \left( F[u_1] - F[u_2] \right) \partial_t v \, dx + \int_0^L \left( F[y_1] - F[y_2] \right) \partial_t y_x \, dx, \tag{51}
\]

we obtain

\[
e(t) = \int_0^L \left( F[u_1] - F[u_2] \right) \partial_t v \, dx + \int_0^L \left( F[y_1] - F[y_2] \right) \partial_t y_x \, dx.
\]

On the other hand, by (Hypothesis 2)–(Hypothesis 4), we deduce from (15), that

\[
\int_0^1 |F[u - 1] - F[u_2]| \, dx \leq 2c(1 + 2M) F_M \, e^{1/2}(s). \tag{52}
\]

Then

\[
e(t) \leq 4c(1 + 2M) F_M \int_0^t \, e(s) \, ds.
\]

Thanks again to Gronwall’s Lemma, we have \( e \equiv 0 \), i.e., \( y_1 \equiv y_2 \). This completes the proof. \( \square \)

3. Global Solution

In this section, we consider the equation

\[
\begin{aligned}
\partial_t y - \left( \lambda_0 y_x + \lambda_1 \partial_t y_x + \lambda_2 \partial_t y_x \right)_x + \lambda \int_{-\infty}^t \mu(t-s) y_x(s) \, ds \\
= |y|^{p-2} y - (|y_x|^{p-2} y_x)_x + f(t, x) \quad x \in \Omega, \ 0 < t < T_\ast,
\end{aligned}
\]

subject to the boundary conditions (7) and to the initial conditions (8). Here \( p > 2 \). We use methods introduced in \([17–24]\). Assume that \( f \in L^2(\Omega \times \mathbb{R}^+) \). We introduce the energy functional \( E(t) \) associated with Equation (53)

\[
E(t) = \frac{1}{2} \int_0^L |\partial_t y|^2 \, dx + \frac{1}{2} \lambda_2 \int_0^L |\partial_t y_x|^2 \, dx + f(t), \tag{54}
\]
where
\[
J(t) = \frac{1}{2} \int_0^L \left( \lambda_0 - \lambda \int_0^\infty \mu(s) ds \right) |y_x|^2 dx + \frac{1}{2} \lambda \int_0^L \int_0^\infty \mu(s) |y_x(t) - y_x(t-s)|^2 ds dx
+ \frac{1}{p} \int_0^L |y_x|^p dx - \frac{1}{p} \int_0^L |y|^p dx.
\] (55)

Now, we introduce the stable set as follows
\[
W = \{ y \in H^1_0(\Omega) \cap H^2(\Omega) : I(t) > 0, J(t) < d \} \cup \{ 0 \},
\] (56)

where
\[
I(t) = \int_0^L \left( \lambda_0 - \lambda \int_0^\infty \mu(s) ds \right) |y_x|^2 dx + \lambda \int_0^L \int_0^\infty \mu(s) |y_x(t) - y_x(t-s)|^2 ds dx
+ \int_0^L |y_x|^p dx - \int_0^L |y|^p dx,
\] (57)

and
\[
d = \inf \{ \sup_{y \in H^1_0(\Omega) \cap H^2(\Omega)} I(y) \}. \] (58)

In addition, we introduce the “Nehari manifold”
\[
\mathcal{N} = \left\{ y \in H^1_0(\Omega) \cap H^2(\Omega) \setminus \{ 0 \} : I(t) = 0 \right\}.
\]

It is readily seen that the potential depth \( d \) is also characterized by
\[
d = \inf_{y \in \mathcal{N}} I(t).
\] (59)

This characterization of \( d \) shows that
\[
\text{dist} \left( 0, \mathcal{N} \right) = \min_{y \in \mathcal{N}} \| y \|_{H^1_0(\Omega) \cap H^2(\Omega)}.
\] (60)

Suppose that (Hypothesis 1) holds. Let \( y \) be a solution of Equation (53). Then the energy functional (54) is a nonincreasing function and for all \( t \geq 0, \epsilon > 0 \), we have
\[
\partial_t E(t) \leq -\left( \lambda_1 - \frac{\epsilon}{2} \right) \int_0^L |\partial_1 y_x|^2 dx + \frac{1}{2} \lambda \int_0^1 \int_0^\infty \partial_1 \mu(s) |y_x(t) - y_x(t-s)|^2 ds dx
+ \frac{1}{2\epsilon} \int_0^L |f|^2 dx.
\] (61)

**Proof.** Multiplying (53) by \( \partial_1 y(x,t) \) and integrating over \([0,L]\), we obtain
\[
\partial_t E(t) = -\lambda_1 \int_0^L |\partial_1 y_x|^2 dx + \int_0^L f \partial_1 y dx
\leq -\lambda_1 \int_0^L |\partial_1 y_x|^2 dx + \frac{1}{2\epsilon} \int_0^L |f|^2 dx + \frac{\epsilon}{2} \int_0^L |\partial_1 y_x|^2 dx,
\]
which completes the proof. \( \square \)
We will prove the invariance of the set $W$. That is, if for some $t_0 > 0$ and $y(t_0) \in W$, then $y(t) \in W$, $\forall t \geq t_0$. $d$ is a positive constant.

**Proof.** We have

$$J(\nu t) = \frac{\nu^2}{2} \int_0^L \left( \lambda_0 - \lambda \int_0^\infty \mu(s)ds \right) |y_x|^2 dx + \frac{\nu^2}{2} \lambda \int_0^L \int_0^\infty \mu(s) |y_x(t) - y_x(t - s)|^2 ds dx + \frac{\nu^p}{p} \int_0^L |y_x|^p dx - \frac{\nu^p}{p} \int_0^L |y|^p dx.$$

Using (Hypothesis 1), we get

$$J(\nu y) \geq K(\nu),$$

where

$$K(\nu) = \frac{\nu^2}{2} \int_0^L |y_x|^2 dx - \frac{\nu^p}{p} \int_0^L |y|^p dx.$$

By differentiating the second term in the last equality with respect to $\nu$, we obtain

$$\frac{d}{d\nu} K(\nu) = v l \int_0^L |y_x|^2 dx - v^{p-1} \int_0^L |y|^p dx.$$

For $\nu_1 = 0$ and $\nu_2 = \left( \frac{1}{L} \int_0^L |y_x|^2 dx \right)^{-(p-2)}$, we have

$$\frac{d}{d\nu} K(\nu_2) = 0, \quad K(\nu_1) = 0.$$

Since

$$\frac{d^2}{d\nu^2} K(\nu)|_{\nu=\nu_2} < 0,$$

we arrive to

$$\sup_{\nu \geq 0} J(\nu) \geq \sup_{\nu \geq 0} K(\nu) = K(\nu_2)$$

$$= \frac{1}{2} \left( \frac{L}{L} \int_0^L |y_x|^2 dx \right)^{\frac{p}{2}} \times \left( \frac{1}{L} \int_0^L |y|^p dx \right) + \frac{1}{p} \left( \frac{1}{L} \int_0^L |y_x|^2 dx \right)^{\frac{p}{2}} \times \left( \frac{1}{L} \int_0^L |y|^p dx \right) - \frac{1}{p} L^{\frac{p}{2}} \left( \frac{1}{L} \int_0^L |y|^p dx \right)^{\frac{p}{2}} \left( \frac{1}{L} \int_0^L |y_x|^2 dx \right)^{\frac{p}{2}} - \frac{1}{p} L^{\frac{p}{2}} \left( \frac{1}{L} \int_0^L |y|^p dx \right)^{\frac{p}{2}} \left( \frac{1}{L} \int_0^L |y_x|^2 dx \right)^{\frac{p}{2}} = L^{\frac{p}{2}} \left( \frac{1}{2} - \frac{1}{p} \right) \left( \frac{1}{L} \int_0^L |y|^p dx \right)^{\frac{p}{2}} \left( \frac{1}{L} \int_0^L |y_x|^2 dx \right)^{\frac{p}{2}}.$$
By Sobolev-Poincare’s inequality, we deduce that \( K(\nu^2) > 0 \). Then

\[
\sup_{\nu \geq 0} j(\nu) \geq d > 0.
\]  

(62)

Then, by the definition of \( d \), we conclude that \( d > 0 \). This completes the proof. \( \square \)

\( W \) is a bounded neighborhood of 0 in \( H^1_0(\Omega) \cap H^2(\Omega) \).

**Proof.** For \( y \in W \) and \( y \neq 0 \), we have

\[
J(t) = \frac{1}{2} \int_0^L \left( \lambda_0 - \lambda \int_0^\infty \mu(s)ds \right) |y_x|^2 dx
\]

\[
+ \frac{1}{2} \lambda \int_0^L \int_0^\infty \mu(s)|y_x(t) - y_x(t - s)|^2 ds dx + \frac{1}{p} \int_0^L |y_x|^p dx - \frac{1}{p} \int_0^L |y|^p dx
\]

\[
= \left( \frac{p-2}{2p} \right) \left[ \int_0^L \left( \lambda_0 - \lambda \int_0^\infty \mu(s)ds \right) |y_x|^2 dx
\]

\[
+ \lambda \int_0^L \int_0^\infty \mu(s)|y_x(t) - y_x(t - s)|^2 ds dx \right]
\]

\[
+ I(t)
\]

\[
\geq \left( \frac{p-2}{2p} \right) \left[ \int_0^L \left( \lambda_0 - \lambda \int_0^\infty \mu(s)ds \right) |y_x|^2 dx
\]

\[
+ \lambda \int_0^L \int_0^\infty \mu(s)|y_x(t) - y_x(t - s)|^2 ds dx \right].
\]

By (Hypothesis 1), we get

\[
J(t) \geq \left( \frac{p-2}{2p} \right) \left[ \int_0^L \int y_x^2 dx + \lambda \int_0^L \int_0^\infty \mu(s)|y_x(t) - y_x(t - s)|^2 ds dx \right]
\]

\[
\geq \left( \frac{p-2}{2p} \right) \int_0^L |y_x|^2 dx,
\]

and

\[
\int_0^L |y_x|^2 dx \leq \frac{1}{t} \left( \frac{2p}{p-2} \right) J(t)
\]

\[
< \frac{1}{t} \left( \frac{2p}{p-2} \right) d = R.
\]  

(63)

Consequently, for any \( y \in W \), we have \( y \in B \), where

\[
B = \left\{ y \in H^1_0(\Omega) \cap H^2(\Omega) : \int_0^L |y_x|^2 dx < R \right\}.
\]  

(64)

This completes the proof. \( \square \)

Now, we will show that our local solution \( y \) is global in time. For this purpose it suffices to prove that the norm of the solution is bounded, independently of \( t \). This is equivalent to prove the following theorem.

**Theorem 2.** Suppose that (Hypothesis 1) and

\[
C^p t^{(1-p)} \left( \frac{2p}{p-2} E(0) \right)^{(p-2)} < 1,
\]

(65)
Proof. Since \( y_0(0) \in W \), then

\[
I(0) = \int_0^L \left( \lambda_0 - \lambda \int_0^L \mu(s) ds \right) |y_{xx}(0)|^2 dx \\
+ \lambda \int_0^L \int_0^L \mu(s) |y_{xx}(0) - y_{xx}(-s)|^2 ds dx \\
+ \int_0^L |y_{xx}(0)|^p dx - \int_0^L |y(0)|^p dx > 0.
\]

Consequently, by continuity, there exists \( T_m \leq T \) such that

\[
I(t) = \int_0^L \left( \lambda_0 - \lambda \int_0^L \mu(s) ds \right) |y_s|^2 dx \\
+ \lambda \int_0^L \int_0^L \mu(s) |y_s(t) - y_s(t-s)|^2 ds dx \\
+ \int_0^L |y_s|^p dx - \int_0^L |y|^p dx \leq 0, \forall t \in [0, T_m].
\]

This gives

\[
J(t) = \frac{1}{2} \int_0^L \left( \lambda_0 - \lambda \int_0^L \mu(s) ds \right) |y_s|^2 dx \\
+ \frac{1}{2} \lambda \int_0^L \int_0^L \mu(s) |y_s(t) - y_s(t-s)|^2 ds dx \\
+ \frac{1}{p} \int_0^L |y_s|^p dx - \frac{1}{p} \int_0^L |y|^p dx \\
\geq \frac{p-2}{2p} \left[ \int_0^L \lambda_0 - \lambda \int_0^L \mu(s) ds \right] |y_s|^2 dx \\
+ \lambda \int_0^L \int_0^L \mu(s) |y_s(t) - y_s(t-s)|^2 ds dx \\
+ \frac{1}{p} J(t)
\]

By (Hypothesis 1), we easily see that

\[
\int_0^L |y_s|^2 dx \leq \frac{1}{t} \left( \frac{2p}{p-2} \right) J(t) \\
\leq \frac{1}{t} \left( \frac{2p}{p-2} \right) E(t) \\
\leq \frac{1}{t} \left( \frac{2p}{p-2} \right) E(0), \quad \forall t \in [0, T_m].
\]

We then exploit (Hypothesis 1), \( p > 2 \) and the embedding \( H^1_0 \to L^p \)

\[
\left( \int_0^L |y|^p dx \right)^{1/p} \leq C \left( \int_0^L |y_s|^2 dx \right)^{1/2},
\]

for some \( C = C(p, \Omega) > 0 \), we get
\[
\int_0^L |y|^p dx \leq C^p \left( \int_0^L |y_x|^2 dx \right)^{p/2} \\
\leq C^p \left( \int_0^L |y_x|^2 dx \right)^{(p-2)/2} \int_0^L |y_x|^2 dx \\
\leq C^p l^{-1} \left( \frac{2p}{p-2} \right)^{(p-2)} E(0)^{(p-2)} \left( l \int_0^L |y_x|^2 dx \right) \\
\leq \beta l \left( \int_0^L |y_x|^2 dx \right),
\]

where \( \beta = C^p l^{(1-p)} \left( \frac{2p}{p-2} E(0) \right)^{(p-2)} \). This means, by the definition of \( l \),

\[
\int_0^L |y|^p dx \leq \beta \int_0^L |y_x|^2 dx \\
\leq \left( \lambda_0 - \lambda \int_0^\infty \mu(s) ds \right) \int_0^L |y_x|^2 dx \\
\leq \left( \lambda_0 - \lambda \int_0^\infty \mu(s) ds \right) \int_0^L |y_x|^2 dx \\
+ \lambda \int_0^L \int_0^\infty \mu(s) |y_x(t) - y_x(t-s)|^2 ds dx.
\]

Therefore \( I(t) > 0 \) for all \( t \in [0, T_m] \), in view of the following relation

\[
\lim_{t \to T_m} C^p l^{1-p} \left( \frac{2p}{p-2} E(0) \right)^{(p-2)} \leq \beta < 1. \tag{66}
\]

This shows that the solution \( y \in W \), for all \( t \in [0, T_m] \). By repeating this procedure we extend \( T_m \) to \( T \). This completes the proof. \( \square \)

The next Theorem shows that the local solution is global in time.

**Theorem 3.** Suppose that (Hypothesis 1), \( p > 2 \) and (65) hold. If \( \tilde{y}_0 \in W, \tilde{y}_1 \in H_0^1(\Omega) \). Then the local solution \( y \) is global in time such that \( y \in G_T \), where

\[
G_T = \left\{ y : \begin{array}{l}
y \in L^\infty (\mathbb{R}^+; H_0^1(\Omega) \cap H^2(\Omega)), \\
\partial_t y \in L^\infty (\mathbb{R}^+; H_0^1(\Omega))
\end{array} \right\}. \tag{67}
\]

**Proof.** Now, it is enough to show that the following norm

\[
\int_0^L |\partial_t y|^2 dx + \int_0^L |y_x|^2 dx \tag{68}
\]

is bounded independently of \( t \). To achieve this, we use (54), (55) and (61). We get
\begin{align*}
E(0) & \geq E(t) = f(t) + \frac{1}{2} \int_0^L |\partial_t y|^2 \, dx \\
& \geq \left( \frac{p-2}{2p} \right) \left[ t \int_0^L |y|^{2p} \, dx + \lambda \int_0^L \int_0^\infty \mu(s)|y_x(t) - y_x(t-s)|^2 \, ds \, dx \right] \\
& + \frac{1}{2} \int_0^L |\partial_t y|^2 \, dx + \frac{1}{p} I(t) \\
& \geq \left( \frac{p-2}{2p} \right) \left[ t \int_0^L |y|^{2p} \, dx + \lambda \int_0^L \int_0^\infty \mu(s)|y_x(t) - y_x(t-s)|^2 \, ds \, dx \right] \\
& + \frac{1}{2} \int_0^L |\partial_t y|^2 \, dx + \frac{1}{p} I(t) \\
& \geq \left( \frac{1(p-2)}{2p} \right) \int_0^L |y|^{2p} \, dx + \frac{1}{2} \int_0^L |\partial_t y|^2 \, dx.
\end{align*}

Since \( I(t) \) and \( \lambda \int_0^L \int_0^\infty \mu(s)|y_x(t) - y_x(t-s)|^2 \, ds \, dx \) are positive, we conclude that
\[
\left( \frac{1(p-2)}{2p} \right) \int_0^L |y|^{2p} \, dx + \frac{1}{2} \int_0^L |\partial_t y|^2 \, dx \leq CE(0),
\]
where \( C \) is a positive constant depending only on \( p \) and \( l \). This completes the proof. \( \square \)

4. Conclusions

By imposing less conditions with the help of some special results, we obtained local and global existence results extending some earlier results known in the existing literature. The main results in this manuscript are the following. Theorem 1 for local existence of solution and Theorem 3 for the global existence in time based on the potential depth.

This article is considered as an essential link in a series of articles by the same authors in the same type of equations. Our research falls within the scope of interests of many researchers in the modern era, according to the general objectives and broad scope of its application areas.

The importance of this research, although it is theoretical, lies in the following:

1. We proved our results without need of how to decrease for the kernel function.
2. In our previous work, we restricted on the case where \( f \equiv 0 \), which is not the case in these studies.
3. There are several generalizations and contributions that are very important in terms of the system itself. We proposed a damped system related to a large number (infinite) of sources, each one has functionality and physical properties, and we look at the overlapping of these terms. Which makes the problem have very wide applications and important in terms of applications in modern science, especially when it comes in bounded domain.
4. Quantitatively, we used and developed the stable set method named potential well method, with details to study the global existence of solution commensurate with the bounded domain after proving the local existence of solution using a usual method, Faedo–Galerkin method.

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References

1. Dutta, S. On the propagation of Love type waves in an infinite cylinder with rigidity and density varying linearly with the radial distance. Pure Appl. Geophys. 1972, 98, 35–39. [CrossRef]

2. Paul, M.K. On propagation of Love-type waves on a spherical model with rigidity and density both varying exponentially with the radial distance. Pure Appl. Geophys. 1964, 59, 33–37. [CrossRef]

3. Radchova, V. Remark to the comparison of solution properties of Love’s equation with those of wave equation. Appl. Math. 1978, 23, 199–207. [CrossRef]

4. Zennir, K.; Miyasita, T.; Papadopoulos, P. Local and Global nonexistence of solution for Love-equation with infinite memory. J. Int. Equ. Appl. 2020, in press.

5. Zennir, K.; Biomy, M. General decay rate of solution for Love-equation with past history and absorption. Mathematics 2020, 8, 1632. [CrossRef]

6. Brepta, R.; Prokopec, M. Stress Waves and Shocks in Solids; Academia: Prague, Czech, 1972.

7. Love, A.E.H. A Treatise on the Mathematical Theory of Elasticity; Cambridge University Press: Cambridge, UK, 1952.

8. Duy, N.T.; Long, N.T. On a high-order iterative scheme for a nonlinear Love equation. Appl. Math. 2015, 60, 285–298.

9. Long, N.; Ngoc, L. On a nonlinear wave equation with boundary conditions of two-point type. J. Math. Anal. Appl. 2012, 385, 1070–1093. [CrossRef]

10. Ngoc, L.T.P.; Duy, N.T.; Long, N.T. A linear recursive scheme associated with the Love’s equation. Acta Math. Vietnam. 2013, 38, 551–562. [CrossRef]

11. Ngoc, L.T.P.; Long, N.T. Existence, blow-up and exponential decay for a nonlinear Love equation associated with Dirichlet conditions. Appl. Math. 2016, 61, 165–196. [CrossRef]

12. Triet, N.A.; Mai, V.T.T.; Ngoc, L.T.P.; Long, N.T. A Dirichlet problem for a nonlinear wave equation of Kirchhoff—Love type. Nonl. Funct. Anal. Appl. 2017, 22, 595–626.

13. Triet, N.A.; Mai, V.T.T.; Ngoc, L.T.P.; Long, N.T. Existence, blow-up and exponential decay for Kirchhoff-love equations with Dirichlet conditions. Electr. J. Diff. Equ. 2018, 167, 1–26.

14. Cavalcanti, M.M.; Domingos Cavalcanti, V.N.; Prates Filho, J.S. Existence and exponential decay for a Kirchhoff Carrier model with viscosity. J. Math. Anal. Appl. 1998, 228, 181–205. [CrossRef]

15. Dafermos, C.M. On the existence and the asymptotic stability of solution to the equations of linear thermo-elasticity. Arch. Ration. Mech. Anal. 1968, 29, 241–271. [CrossRef]

16. Giorgi, C.; Muñoz Rivera, J.; Pata, V. Global Attractors for a Semilinear Hyperbolic Equation in visco-elasticity. J. Math. Anal. Appl. 2001, 260, 83–99. [CrossRef]

17. Atangana. A.; Doungmo Goufo, E.F. Solution of Diffusion Equation with Local Derivative with New Parameter. Therm. Sci. 2015, 19, 231–238. [CrossRef]

18. Cavalcanti, M.M.; Domingos Cavalcanti, V.N.; Lasiecka, I. Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping-source interaction. J. Diff. Equ. 2007, 236, 407–459. [CrossRef]

19. Cavalcanti, M.M.; Domingos Cavalcanti, V.N.; Soriano, J.A. Global solvability and asymptotic stability for the wave equation with nonlinear boundary damping and source term. Prog. Nonl. Diff. Equ. Appl. 2005, 66, 161–184.

20. Doungmo Goufo, E.F. Evolution equations with a parameter and application to transport-convection differential equations. Turk. J. Math. 2017, 41, 636–654. [CrossRef]

21. Doungmo Goufo, E.F.; Khan, Y.; Mugisha, S. Control parameter & solutions to generalized evolution equations of stationarity, relaxation and diffusion. Results Phy. 2018, 9, 1502–1507.

22. Levine, H.A.; Park, S.R.; Serrin, J. Global existence and global nonexistence of solutions of the Cauchy problem for a nonlinearly damped wave equation. J. Math. Anal. Appl. 1998, 228, 181–205. [CrossRef]

23. Oukouomi Noutchie, S.C.; Doungmo Goufo, E.F. Exact Solutions of Fragmentation Equations with General Fragmentation Rates and Separable Particles Distribution Kernels. Math. Prob. Eng. 2014. [CrossRef]
24. Zennir, K.; Guesmia, A. Existence of solutions to nonlinear $\kappa$–th-order coupled Klein-Gordon equations with nonlinear sources and memory terms. *Appl. Math. E-Notes* **2015**, *15*, 121–136.

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