Is the Polyakov path integral prescription too restrictive?

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In the first quantised description of strings, we integrate over target space co-ordinates $X^\mu$ and world sheet metrics $g_{\alpha\beta}$. Such path integrals give scattering amplitudes between the ‘in’ and ‘out’ vacuua for a time-dependent target space geometry. For a complete description of ‘particle creation’ and the corresponding backreaction, we need instead the causal amplitudes obtained from an ‘initial value formulation’. We argue, using the analogy of a scalar particle in curved space, that in the first quantised path integral one should integrate over $X^\mu$ and world sheet zweibiens. This extended formalism can be made to yield causal amplitudes; it also naturally allows incorporation of density matrices in a covariant manner. (This paper is an expanded version of [hep-th 9301044].)
1. Introduction.

When we quantise a field in curved space, many interesting phenomena arise. For example in an expanding Universe we have particle creation, which means that a spacetime which looks empty in terms of particle modes natural in the past, may look full of particles on using co-ordinates natural to the future. The Minkowski vacuum appears to have a particle flux for an accelerated observer. For black holes, imposing vacuum conditions at past null infinity gives Hawking radiation in the future, due to the time-dependent gravitational field of a collapsing object. ([1] and references therein.)

For a complete description of the physics, we need also the backreaction of the quantum field on the geometry. At this point we need a consistent theory of quantum gravity, to deal with the gravity loops which arise. For this reason backreaction questions have been traditionally restricted to semiclassical order (apart from recent studies on low dimensional gravity theories [2]).

Strings provide a consistent theory of quantised matter and gravity, so we would like to investigate the issue of particle creation and backreaction in string theory. Several examples of strings in time-dependent backgrounds were studied by de Vega et. al [3]. In their first quantised treatment of the string, they find that an existing string may get excited to higher mode levels in moving through the time-dependent geometry (e.g. spacetime with a gravitational wave), but it remains one string. Thus at their tree level calculation we do not see the creation of a ‘flux of strings’, which would be the analogue of the ‘creation of particles’ for field theory in curved spacetime.

In what way do we expect to see ‘created particles’ in a first quantised language? Suppose a Universe starts in the vacuum, and undergoes expansion which creates particles. Consider the two point propagator computed for spacetime points in the region where the ‘bath’ of created particles is nonvanishing. We may compare this situation to the computation of the propagator at finite temperature, where also a bath of ambient particles exists. In the latter case we know that the propagator depends on the temperature, and with this modified propagator Feynman diagrams can be computed to develop perturbation theory. With our ‘created particles’ there should be a similar modification of the propagator, except that the ‘bath’ would be neither exactly thermal nor time-independent in general.

In this paper we take the simple case of a scalar field, and show how a careful handling of the ‘sum over paths’ in the first quantised propagator allows us to obtain the modifications needed for incorporating the effects of created particles. If we compute the
analogue of the Polyakov path integral for a particle, we get a two point scattering amplitude between the ‘in’ and ‘out’ vacuum states. Such a naively computed first quantised propagator does not directly ‘see’ the created particles. What we need instead is a propagator that reproduces the causal amplitude obtained by taking the expectation value of field operators in a specified state. Our ‘first quantised’ calculation reproduces this propagator. A density matrix formalism develops naturally, and the particle fluxes produced in time dependent geometries become an essential part of the first quantised description. It appears that only such an extension of the Polyakov path integral can adequately describe the physics of quantised gravity plus matter.

For the string case we have the further issue of obtaining the background field equations by a $\beta$-function calculation. The tree level $\beta$-function gives the classical vacuum; but we need the higher string loop contributions to take into account the backreaction of created particles. We argue that our modified propagator, rather than the Polyakov prescription, should be used in computing the loop amplitudes.

The issue.

Consider the free massive scalar field. For the ‘propagator’ from one spacetime point $x_1$ to another spacetime point $x_2$, the analogue of the Polyakov path integral gives

$$D(x_2,x_1) = \int_0^\infty d\lambda D[X]e^{-i\int_0^1 d\tau(1/2)(X^2/\lambda + m^2\lambda) - \epsilon\lambda}$$

where $\tau$ parametrises the world line, and $X(0) = x_1$, $X(1) = x_2$. The action is multiplied by $i$ because we are in spacetime with Minkowski signature. The regulator term $-\epsilon\lambda$ ensures convergence of the path integral.

What precisely is the answer we obtain? In spacetimes with Euclidean signature there is a unique Green’s function, determined by the boundary condition that the amplitude to propagate to infinity falls to zero in all directions. Such is not the case for spacetimes with Minkowski signature, where various Green’s functions can be defined, differing from each other by solutions of the homogeneous field equation. For describing the choice of boundary conditions implicit in (1.1) let us assume that the spacetime does not expand ‘too rapidly’ in the far past ($t \to -\infty$) or in the far future ($t \to \infty$). Then we obtain naturally defined vacuum states for the Klein-Gordon field, $|0>_{in}$ and $|0>_{out}$ for $t \to -\infty$ and $t \to \infty$ respectively. For a time-dependent geometry, in general $|0>_{out} \neq |0>_{in}$. In this situation it is known that the RHS of (1.1) computes

$$D(x_2,x_1) = \frac{<0|T[\phi(z_2)\phi(z_1)]|0>_{in}}{<0_{out} |0>_{in}} \equiv G_F(x_2,x_1)$$

(1.2)
which is the analogue of the Feynman propagator in the time-dependent geometry.

Such ‘in-out’ amplitudes are relevant for computing scattering amplitudes where a finite number of ‘in’ particles scatter to a finite number of ‘out’ particles. By contrast, for an ‘initial value problem’, with the initial state the ‘in’ vacuum, we would compute amplitudes like \( \langle 0 | T[\phi(z_2)\phi(z_1)] | 0 \rangle_{in} \). In particular, we know that for a source term to Einstein’s equations we should use ‘true expectation values’ \( \langle \psi | T_{\mu\nu} | \psi \rangle \) for the stress tensor, not ‘in-out’ vacuum amplitudes \([1]\). We illustrate (in section 5 below) the difference between these two different kinds of amplitudes by a simple example in 1 + 1 spacetime. We compute the spatial average of \( \langle in \rangle \) and of \( \langle out \rangle \) in an Universe which is in the initial vacuum and undergoes sudden expansion at \( \eta = 0 \). (\( \eta \) is the conformal time.) For \( \eta < 0 \), \( \langle T_{00} \rangle_{in} \) in is just the vacuum energy. For \( \eta > 0 \) it becomes the vacuum energy plus the energy of created particles. But \( \langle T_{00} \rangle_{out} \) is just the vacuum energy in both ranges of \( \eta \). Worse, \( \langle T_{11} \rangle_{out} \) is complex, so it cannot be a source term for the gravitational field.

To compare ‘in-in’ and ‘in-out’ amplitudes we note that \( \langle in \rangle = C_{0} \langle out \rangle + C_{2} \langle out \rangle + C_{4} \langle out \rangle + \ldots \) where \( \langle out \rangle \) is a state with one pair of ‘out’ particles etc. The first quantised path integral described above computed only ‘in-out’ amplitudes. This may not appear to be a serious shortcoming of the first quantised approach in the particle case because one could compute successively \( \langle out \rangle \) and thus reconstruct the result of the initial value problem. But this is not an adequate treatment for the string case, which was the motivation for studying the first quantised approach in the first place. With strings we do not start with a Lagrangian governing the field; instead we discover backgrounds satisfying the field equations by demanding that string propagation in the background be ‘consistent’. But which propagator should we demand consistency for: the ‘in-in’ one or the ‘in-out’ one? These two choices would in general give different answers for the background field (considering \( \beta \)-functions to one loop or beyond). The discussion above indicates that the ‘in-in’ propagator is the one which correctly gives the stress tensor of the created flux of ‘out’ particles, and is therefore the one that should be involved in the multiloop contributions to the \( \beta \)-function.

Can we develop a natural extension of the Polyakov prescription so that we compute ‘in-in’ amplitudes? In the second quantised field language, ‘in-in’ perturbation theory is described in the ‘real time formalism’ of many-body theory. We outline this formalism
below, and then give our results on how this formalism arises naturally and covariantly in
the first-quantised ‘sum over paths’ language.

Note: In this paper we use the term ‘Polyakov path integral’ for any prescription where
one sums over spacetime co-ordinates on the world sheet /world line and the metrics on this
world sheet/world line. This approach was developed primarily for Euclidean target space
metrics. We are assuming here that the prescription of summing over co-ordinates and
metrics has been extended in some way to spacetime with Minkowski signature. Thus the
world sheet will also have Minkowski signature, and construction of higher genus surfaces
may require explicit use of interaction vertices.

The real time formalism.

To obtain a perturbation theory for the initial value problem, one uses the ‘real time
formalism’, developed in the context of time-dependent many body theory [6][7]. Suppose
we start in the vacuum \( |0>_{in} \) at \( t = -\infty \). We wish to compute the expectation value of a
product of observables \( O_i \), defined at times \( t_i \), in the state \( |0>_{in} \). This expectation value
can be expressed as \( \text{Tr}\{\rho_0 O_1 \ldots O_n\}/\text{Tr}\rho_0 \), with \( \rho_0 = |0>_{in} <0| \) the density matrix for
a pure state. Because both the bra and the ket states composing \( \rho_0 \) are given at \( t = -\infty \),
the interaction Hamiltonian acts along the ‘time path’ leading from \( t = -\infty \) to \( t = \infty \),
and then back to \( t = -\infty \). Correspondingly, we need propagators from any time on the
first or second time path segment to any other time on either path segment. We collect
these four propagators (appropriate to the four possible choices of time path segments)
into a 2x2 matrix propagator. Vertices arising from the interaction Hamiltonian acting
on the first time path segment have the usual coupling \(-ig\), while those on the second
path segment have coupling \( ig \). With this extension of rules, the computation of Feynman
diagrams gives correlation functions for the density matrix \( \rho_0 \), inserted at \( t = -\infty \).

Note that the above formalism computed ‘in-in’ correlation functions, regardless of
whether or not the ‘in’ vacuum differed from the ‘out’ vacuum. The density matrix \( \rho_0 \) can
be replaced by an arbitrary density matrix \( \rho \). A special class of \( \rho \) emerge in the development
of perturbation theory: density matrices expressible as ‘exponential of quadratic in the
field operator’. For such \( \rho \) the Wick decomposition holds: the expectation of a product of
several operators decomposes into sums of products of expectations for pairs of operators.
The propagator of the real time formalism encodes a choice of such a ‘special’ density
matrix. Deviations of \( \rho \) from this special class give ‘correlation kernels’, which are handled
perturbatively just like interaction vertices.
The relevance of the real time formalism for field theory in time dependent geometry is easy to see. Firstly, cosmology appears set up as an initial value problem, rather than a scattering problem. Secondly, ‘particle creation’ is generally obtained as a Bogoliubov transformation on a vacuum state. The particle flux produced this way is described by a density matrix of the form ‘exponential of quadratic in field’. This flux is therefore precisely of the form that can be incorporated into the propagator of the real time formalism, so that backreaction computations can be examined by loop calculations using the matrix propagator. Thirdly, the density matrix of the real time formalism can be taken to describe a mixed state, whereupon one can study correlations in an evolving distribution of particles. Note that the ‘periodic imaginary time’ formalism for finite temperature correlations applies only to time independent thermal distribution functions. But in a theory with gravity, a nonempty distribution of particles gives rise to a nonvanishing stress tensor, which leads to a time dependence of the spacetime geometry in general, and a corresponding evolution of the particle distribution. Thus the real time formalism must be used in developing kinetic theory for a theory with gravity; the imaginary time formalism can only provide quasi-static approximations.

The above description of the real time method pertained to the second quantised field language. In order to apply the method to strings, we need to obtain it in a first quantised path integral language. At first one might think that this would not be possible, since a first quantised path integral seems to describe the propagation of just one particle, while the real time formalism is built to handle fluxes. Nevertheless, it turns out that one does find the entire structure of the formalism, when one carefully quantises the action of, say, a scalar particle. The phrase ‘first quantised’ is not really correct for the resulting path integral; what one gets is a ‘proper time representation’ for the real time propagator.\footnote{The author thanks L. Ford for pointing out this issue of terminology.} The fact that one reaches this object after starting with what looks like just one particle, reflects the ambiguity of the particle concept in curved space. The Polyakov path integral (and its particle analogue) chooses a prescription to define the propagation amplitude such that this ambiguity is suppressed. In restoring the ambiguity, one regains the freedom to have an ‘in-in’ formulation, and also the choice of an initial density matrix.

**Approach and results.**

We start with the geometric action $m \int ds$ for a scalar particle moving in a spacetime with Minkowski signature. $ds = (dX^\mu dX_\mu)^{1/2}$ has a sign ambiguity. Along a trajectory

\[1\]
contributing to the sum over paths, the path can change several times between being
timelike and being spacelike. At each change \(dX^\mu dX_\mu\) passes through zero, and we have
to make the choice of root afresh. We might try to evade the problem by passing to the
quadratic form of the action. One often thinks of this form of the path integral as being
a sum over functions \(X^\mu(\tau)\) and over metrics \(g_{\tau \tau}(\tau)\) on the world line of the particle. In
fact this would be the analogue for the scalar particle of the Polyakov prescription. But in
constraining the square root action to bring it to the quadratic form, what we really get
is a sum over the \(X^\mu(\tau)\) and over the \textit{einbein} \(\lambda(\tau)\) on the world line. Since the \textit{einbein}
is the square root of the metric, two signs of the \textit{einbein} correspond to the same
\(g_{\tau \tau}(\tau)\); this appears to be a reflection of the sign ambiguity in the original square root action.

To evaluate the path integral we gauge fix \(\lambda(\tau)\) to a constant by using the freedom
of diffeomorphisms along the world line. If the world line geometry were described by the
metric \(g_{\tau \tau}(\tau)\), it is evident that the only degree of freedom remaining of this geometry
would be the length \(\Lambda\) of the entire world line, and \(\Lambda\) should be integrated from 0 to \(\infty\).
But since we have instead a sum over the \(X^\mu(\tau)\) and over the \textit{einbein} \(\lambda(\tau)\) on the world line. Since the \textit{einbein}
is the square root of the metric, two signs of the \textit{einbein} correspond to the same
\(g_{\tau \tau}(\tau)\); this appears to be a reflection of the sign ambiguity in the original square root action.

Diffeomorphisms can fix \(\lambda\) to a positive constant in any interval where \(\lambda\) is positive, and
to a negative constant in any interval where it is negative. These two kinds of intervals
will alternate along the world line for a typical realisation of \(\lambda(\tau)\) in the path integral.
We can perform the integral over the value of the constants \(\Lambda_i\) obtained in these intervals,
with the integral running over positive or negative reals respectively for the two different
kinds of intervals. To complete the evaluation of the path integral, we need to know what
amplitude to assign to a crossover of \(\lambda\) from positive values to negative or the other way
round.

This amplitude is not supplied by the action, and so is supplemental information
needed to make sense of the propagation amplitude. It turns out that the freedom of choice
here is exactly that of a density matrix of the \textquoteleft special\textquoteright{} kind mentioned above (exponential
of quadratic in the field) \textit{and} the choice of a \textquoteleft time path\textquoteright{}. These are precisely the ingredients
which define the matrix propagator of the real time approach. The two different signs of
the \textit{einbein} \(\lambda\) on the world line correspond to the two segments of the time path of the real
time formalism, and so furnish the two dimensional vector space on which the 2x2 matrix
propagator acts.

More specifically, we can describe the first quantised propagator in the following terms.
We consider both the propagation \(e^{-i(\bar{\Phi} + m^2)}\) \textit{and} the propagation \(e^{i(\bar{\Phi} + m^2)}\) on the world
line. We may collect these two possibilities into a 2x2 matrix form, getting a world line
Hamiltonian \(\text{diag}[(\square + m^2), -(\square + m^2)]\). Near the mass shell \(\square + m^2 = 0\) the path integral needs regulation. If we add \(-i\epsilon I\) to the Hamiltonian (\(I\) is the 2x2 identity matrix) then we get a diagonal matrix propagator, with (1.2) and its complex conjugate as the first and second diagonal entries. But if we regulate instead by adding \(-i\epsilon M\), where \(M\) is a non-diagonal matrix, then we get the matrix propagator for perturbative calculation of Green’s functions in an ‘exponential of quadratic’ particle flux.

We compute \(M\) for a thermal distribution in Minkowski space; for the ‘closed time path’ mentioned above, and for the time path used by Niemi and Semenoff [8]. These two \(M\) matrices are not the same, which reflects the fact that the matrix propagator depends on both \(\rho\) and the choice of time path. For a curved space example, we consider a 1+1 spacetime with ‘sudden’ expansion. We compute \(M\) to obtain the matrix propagator appropriate to the initial value problem with state the ‘in’ vacuum; i.e. for \(\rho = \text{in}|0><0|\text{in}\) and time path beginning and ending at past infinity.

We conclude with a discussion of the significance of our results for strings, which were the motivation for this study of the first quantised formalism. The usual first quantised path integrals for strings would give the analogue of (1.2), which corresponds to specific boundary conditions at spacetime infinity. To handle phenomena involving particle fluxes, we propose extending the world sheet Hamiltonian to a 2x2 Hamiltonian as in the above particle case. Thus we would allow not only classical deformations of the background field (analogous to the ‘exponential of linear’ \(\rho\) in the particle case) but also particle flux backgrounds (corresponding to ‘exponential of quadratic’ density matrices). Studying \(\beta\)-function equations [4] for this extended theory should give not equations between classical fields but equations relating fields and fluxes. The latter kind of equation, we believe, would be natural to a description of quantised matter plus gravity.

The plan of this paper is as follows. Section 2 reviews finite temperature perturbation theory, and discusses the significance of ‘exponential of quadratic’ density matrices for the curved space theory. Section 3 translates the finite temperature results of flat space to first quantised language. Section 4 gives a curved space example. Section 5 discusses strings. Section 6 is a summary and discussion.
2. Propagators in the presence of a particle flux.

2.1. Review of the real time formalism.

The real time method has been extensively developed in recent years, in a variety of contexts. A good description of the closed time path and its meaning is given in [10], [11]. We give a brief summary below.

Consider a scalar field in Minkowski spacetime (metric signature +−−−):

\[ S = \int dx \left( \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \right) \] (2.1)

We have the following relation for Green’s functions [12]:

\[ <0|T\{\phi(x_1)\ldots\phi(x_n)}|0> = <0|U^{-1}(\infty)T\{\phi_0(x_1)\ldots\phi_0(x_n)}e^{-i\int_{t'=\infty}^{t} dt' H_{\text{int}}(t')}U(-\infty)|0> \] (2.2)

where

\[ U(t) = Te^{-i\int_{t'=\infty}^{t} dt' H_{\text{int}}(t')} \] (2.3)

\[ \phi(x) = U^{-1}(t)\phi_0(x)U(t) \] (2.4)

and \( \phi_0 \) is the free field.

From (2.3) we have

\[ U(-\infty)|0> = |0> \] (2.5)

If the vacuum is assumed stable:

\[ <0|U^{-1}(\infty) = e^{i\theta} <0| \] (2.6)

then we obtain the usual perturbation series

\[ <0|T\{\phi(x_1)\ldots\phi(x_n)}|0> = \frac{<0|T\{\phi_0(x_1)\ldots\phi_0(x_n)}e^{-i\int_{t'=\infty}^{t} dt' H_{\text{int}}(t')}|0>}{<0|e^{-i\int_{t'=\infty}^{t} dt' H_{\text{int}}(t')}|0>} \] (2.7)

If the vacuum is not stable (i.e., (2.6) does not hold), then we get interaction vertices from \( U^{-1}(\infty) \) in (2.2). The interaction Hamiltonian acts from \( t = -\infty \) to \( t = \infty \) because of the exponential in (2.2) and from \( t = \infty \) to \( t = -\infty \) because of \( U^{-1}(\infty) \). The Green’s function can be written as

\[ <0|T\{\phi(x_1)\ldots\phi(x_n)}|0> = <0|T_C\{\phi_0(x_1)\ldots\phi_0(x_n)}e^{-i\int_{C} dt' H_{\text{int}}(t')}|0> \] (2.8)
where $H_{\text{int}}$ is integrated along the time path $C$ running from $t = -\infty$ to $t = \infty$ and back to $t = -\infty$. $T_C$ represents path ordering along the path $C$.

We label field operators on the first part of $C$ with the subscript 1 on the second part of $C$ with the subscript 2. Thus for the two point function of the scalar field there are four possible combinations of subscripts, and the corresponding propagators are collected into a matrix:

$$D(x_2, x_1) = \begin{pmatrix} 
  in < 0|T(\phi(x_2)\phi(x_1))|0 >_{in} & in < 0|\phi(x_1)\phi(x_2)|0 >_{in} \\
  in < 0|\phi(x_2)\phi(x_1)|0 >_{in} & in < 0|\tilde{T}(\phi(x_2)\phi(x_1))|0 >_{in}
\end{pmatrix}$$  \hspace{1cm} (2.9)

where we have added the subscript ‘in’ to the vacuum to emphasize that it is the vacuum state at $t = -\infty$. ($\tilde{T}$ describes anti time ordering.)

Perturbation in the coupling $\lambda$ is described by the following diagram rules [13]. The external insertions are all of type 1. (This can be seen from the way we arrived at (2.8) from (2.2).) The vertices from the perturbation term must have either all legs of type 1 or all legs of type 2. Attach a factor $(−i\lambda)$ to the vertex in the former case, $(i\lambda)$ in the latter. The propagator is given by the matrix (2.9) and can connect vertices of type 1 to vertices of type 2. Summing all Feynman diagrams with these rules gives the correlation function for the initial value problem with initial state the vacuum at $t = -\infty$.

As a simple example where the second leg of the time path makes a difference, consider the free scalar field in $0 + 1$ spacetime dimensions. Let the only perturbation be the time dependent term $\mu \int dt \delta(t - t_0) : \phi^2(t) :$. We compute to first order in $\mu$ the two point correlator for field insertions at $t_1 < t_0 < t_2$, using time path $C$:

$$< \phi(t_2)\phi(t_1) > = 2\mu[< 0|\phi(t_2)\phi(t_0)|0 > < 0|\phi(t_0)\phi(t_1)|0 > + < 0|\phi(t_0)\phi(t_2)|0 > < 0|\phi(t_0)\phi(t_1)|0 >]$$  \hspace{1cm} (2.10)

Without the second leg of the time path, we just get the first term on the RHS. The latter answer gives the amplitude to start in the interaction picture vacuum at $t = -\infty$ and to also end in interaction picture vacuum at $t = \infty$. The missing term corresponds to correcting the state at $t = \infty$ away from the vacuum $|0>$, to be such that it evolves from $|0>$ after suffering the perturbation at $t_0$. Thus the complete answer (2.10) gives a Heisenberg picture expectation value, and is thus the only physically meaningful object to be computed in this situation.

We can rewrite (2.8) as

$$< 0|T\{\phi(x_1)\ldots\phi(x_n)\}|0 > = \text{Tr}\{\rho_0 T_C\{\phi_0(x_1)\ldots\phi_0(x_n)e^{-i\int_C dt' H_{\text{int}}(t')}\}} / \text{Tr}\rho_0$$  \hspace{1cm} (2.11)
with $\rho = |0><0|$ the density matrix of a pure state. But we can immediately extend
the above relation to the case where $\rho_0$ is replaced by an arbitrary density matrix $\rho$,
not necessarily representing a pure state, and not necessarily describing a distribution in
equilibrium. To develop diagrammatic perturbation theory, however, we need to have the
Wick decomposition for expectation of operator products. This is achieved for a special
class of density matrices, which have the form ‘exponential of quadratic in the field operator
$\phi$’. Departures of $\rho$ from this special form are handled perturbatively just like interaction
vertices. We discuss the significance of this special class of $\rho$ in the next subsection.

As an example of mixed state density matrices, let $\rho$ at $t = -\infty$ be of thermal form
with temperature ($\beta$)$^{-1}$. Then the matrix propagator is

$$D(p) = \left( \frac{i}{2\pi[p^2 - m^2 + i\epsilon]} + 2\pi n(p)\delta(p^2 - m^2) \right) \frac{2\pi[(n(p) + \theta(-p_0)]\delta(p^2 - m^2)}{2\pi[n(p) + \theta(p_0)]\delta(p^2 - m^2)}$$

where

$$n(p) = (e^{\beta|p_0|} - 1)^{-1}$$

is the number density of particles for the free scalar field at temperature ($\beta$)$^{-1}$. With this
change of matrix propagator, perturbation theory has the same rules as for the vacuum
case.

Another approach leading to a similar diagrammatic structure was given by Niemi
and Semenoff [8]. In the periodic imaginary time description of temperature, we evolve
the field theory in imaginary time from $t = 0$ to $t = -i\beta$, and identify these two time
slices. In the ‘real time’ approach of Niemi and Semenoff we use instead the following path
$C_1$ in complex $t$ space to connect these two slices

$$C_1: \quad I: \quad -\infty \rightarrow \infty$$

$$II: \quad \infty \rightarrow \infty - i\beta/2$$

$$III: \quad \infty - i\beta/2 \rightarrow -\infty - i\beta/2$$

$$IV: \quad -\infty - i\beta/2 \rightarrow -\infty - i\beta$$

The concept of time ordered correlation functions is replaced by ‘path ordered’ correlation
functions, with $t$ running along the above path $C_1$. One argues that the parts $II, IV$ of
$C_1$ can be ignored. Field operators on part $I$ are labelled with the subscript 1 while on
part $III$ are labelled with the subscript 2. The matrix propagator becomes

$$D_{NS}(p) = \left( \frac{i}{2\pi[p^2 - m^2 + i\epsilon]} + 2\pi n(p)\delta(p^2 - m^2) \right) \frac{2\pi[(n(p)(n(p) + 1)]^{1/2}\delta(p^2 - m^2)}{2\pi[n(p)(n(p) + 1)]^{1/2}\delta(p^2 - m^2)}$$

$2\pi[n(p)\delta(p^2 - m^2)]$
Perturbation theory based on the contours $C$ and $C_1$ are not equivalent, even for zero temperature [11]. The propagators (2.12) and (2.13) are not the same, for $\beta \to \infty$. Thus the matrix propagator depends both on the particle distribution and on the time path.

Note that the temperature dependent terms in (2.12) or (2.15) are all on shell. This is a manifestation of the fact that these terms describe a flux of real particles. The expression $\delta(p^2 - m^2)$ does not have a Euclidean counterpart. (In Euclidean space one would consider $(p^2 + m^2)$ which is positive definite.) Thus temperature is a phenomenon of Minkowski signature spacetime, where Green’s functions are subject to the addition of solutions of the homogeneous field equation.

To understand the origin of the on shell terms, let us consider the element $D_{11}$ in (2.15). We can decompose the free scalar field in Minkowski space into Fourier modes. Each quantised mode is a harmonic oscillator with frequency $\omega(p) = (p^2 + m^2)^{1/2}$. The two point function $< T[\phi(x_2)\phi(x_1)] >$ reduces, for each Fourier mode, to $< T[\hat{q}(t_2)\hat{q}(t_1)] >$ for each harmonic oscillator. Thus we focus on a single oscillator (we suppress its momentum label). Let $t_2 > t_1$. Then

$$< \hat{q}(t_2)\hat{q}(t_1) > \equiv \sum_n e^{-\beta(n+\frac{1}{2})\omega} < n|T[\hat{q}(t_2)\hat{q}(t_1)]|n> / \sum_n e^{-\beta(n+\frac{1}{2})\omega}$$

(2.16)

The first term on the RHS corresponds to the stimulated emission of a quantum at $t_1$ with absorption at $t_2$. The second term corresponds to the annihilation at time $t_1$ of one of the existing quanta in the thermal bath, and the subsequent transport of a hole from $t_1$ to $t_2$, where another particle is emitted to replace the one absorbed from the bath. The Fourier transform in time of (2.16) is

$$\int_{-\infty}^{\infty} dt e^{-i\omega t} < T[\hat{q}(t_2)\hat{q}(t_1)] > = \frac{i}{\omega^2 - \omega'\omega + i\epsilon} + 2\pi < n > \delta(\omega'^2 - \omega^2)$$

(2.17)

which gives the matrix element $D_{11}$ in (2.15).

Thus the correction to $< T[\phi(x_2)\phi(x_1)] > = D_{11}(x_2, x_1)$ (and other elements of $D$) due to the particle flux is not an effect of interactions with the particles of the bath. This correction arises from the possible exchange of the propagating particle with identical real particles in the ambient flux. Using such a corrected propagator with the interaction vertices gives the interaction of the bath particles with the propagating particle. With this understanding of the propagator we see that the real-time formalism is really a covariant
concept. In a general spacetime, single particle wavefunctions are just solutions of the free field equation. If there is more than one particle populating the same wavefunction, then we must keep track of ‘exchange terms’ in computing correlation functions. The matrix propagator does this bookkeeping; the terms with \( \delta(p^2 - m^2) \) in the above would be replaced with the appropriate solutions of the wave equation in the spacetime.

2.2. ‘Exponential of quadratic’ density matrices.

When doing perturbation theory on flat space, we usually assume that the vacuum is stable: \( |0\rangle_{\text{out}} = |0\rangle_{\text{in}} \) (upto a phase). As we saw above, when we do not have this stability then we have to use a 2x2 matrix propagator. Stability may be lost either because the spacetime geometry is time-dependent, or because there exists a distribution of particles in the initial state. With gravity we would typically encounter both these causes: the time dependent geometry creates particle fluxes which evolve on (and modify) the geometry. Thus it becomes natural to construct the theory with arbitrary initial state and to not consider a ‘vacuum theory’ at all.

In perturbation theory we deal easily with a special class of states: the coherent states. These have the form \( e^{A[\phi]}|0\rangle \) where \( A \) is an operator linear in the field variable \( \phi \). The operator \( A \) generates a change in the classical value of the field; shifting the field variable by this classical value removes the effect of \( A \) from Feynman diagrams. In studying first quantised propagation about a background, this shift has already been made (to zeroth order in the coupling).

The discussion of the real time method revealed that there is another kind of field configuration that is also easy to deal with. This is the class of configurations expressed as density matrices \( \rho = e^{B[\phi]} \), where \( B \) is quadratic in the field operator. The special role of these ‘exponential of quadratic’ density matrices is due to the fact that Wick’s theorem extends to correlators computed with such \( \rho \). Thus for operators \( A_i \) linear in the field,

\[
<A_1 \ldots A_n> \equiv \frac{1}{\text{Tr} \rho} \text{Tr}\{\rho A_1 \ldots A_n\} = \sum_{\text{permutations}} <A_{i_1}A_{i_2} \ldots <A_{i_{n-1}}A_{i_n}> \quad (2.18)
\]

We sketch a proof of (2.18) in the appendix.

The physical importance of ‘exponential of quadratic’ density matrices is seen from the different instances where they occur:

a) The thermal distribution has such a density matrix:

\[
\rho_{\text{thermal}} = \prod_i e^{-\beta \omega_i a_i a_i^\dagger} \quad (2.19)
\]
b) In time dependent geometries one considers the Bogoliubov transformation relating different expansions of the field variable (we suppress the mode index):

\[ \phi = a_{in} f_{in} + a_{in}^\dagger f_{in}^* \]
\[ \phi = a_{out} f_{out} + a_{out}^\dagger f_{out}^* \]  

(2.20)

The ‘in’ and ‘out’ vacua are given by

\[ a_{in} |0 >_{in} = 0, \quad a_{out} |0 >_{out} = 0. \]  

(2.21)

The ‘in’ vacuum can be expressed as

\[ |0 >_{in} = e^{b_{out}^\dagger a_{out}^\dagger} |0 >_{out} \]  

(2.22)

so that it contains a flux of ‘out’ particles. If the initial state is \( |0 >_{in} \) then the density matrix for the initial value problem is \( \rho_0 = |0 >_{in} < 0 |. \) \( \rho_0 \) can be expressed as

\[ \rho_0 = e^{b_{out}^\dagger a_{out}^\dagger} |0 >_{out} < 0 | e^{b^*_a a_{out}} \]
\[ = \lim_{\beta \to \infty} [\rho / \text{Tr} \rho] \]  

(2.23)

where

\[ \rho = e^{b_{out}^\dagger a_{out}^\dagger} e^{-\beta a_{out}^\dagger a_{out}} e^{b^*_a a_{out}} \]  

(2.24)

and the limit is established by inserting complete set of ‘out’ number operator eigenstates between the exponentials in (2.23).

Thus the ‘exponential of quadratic’ density matrices

\[ \rho = \prod_i e^{\gamma a_i a_i^\dagger} e^{\alpha a_i a_i^\dagger} e^{-\beta a_i a_i^\dagger} \]  

(2.25)

unify the treatment of thermal fluxes and the fluxes created by gravitational fields. An ensemble like (2.19) might describe the distribution at the initial state of the Universe. In fact the notion of ‘thermal’ distributions has to be extended to the class (2.25) to be useful in curved space. Suppose we choose a time co-ordinate \( t \) and start with a distribution \( e^{-\beta H} \), thermal for the Hamiltonian giving evolution in \( t \). As the Universe expands, the distribution will not remain thermal, in general. Redshifting of wavelengths gives an obvious departure from thermal form if the field has a mass or is not conformally coupled.
But even a massless conformally coupled field departs from thermal form if the time coordinate $t$ is not appropriately chosen $[15]$. However, the density matrix remains within the class $\langle 2.25 \rangle$, if it starts in this class, even for a massive field.

The Bogoliubov transformation is important in obtaining the stress energy of particles created in the gravitational field. For example in the black hole geometry the state is the Kruskal vacuum. The Bogoliubov transform to the Schwarzschild modes allows us to see readily the $< T_{\mu\nu} >$ of emitted radiation at spatial infinity. The class $\langle 2.25 \rangle$ is closed under Bogoliubov transformations relating different field expansions like $\langle 2.20 \rangle$. It is the natural class to arise in a covariant formulation where the density matrix $\rho_0$ described above is one of the allowed initial state specifications.

How do we compute the backreaction on the gravitational field from particles described by $\langle 2.25 \rangle$? For a coherent state, the stress tensor is given, ignoring quantum corrections, by the classical value of $< T_{\mu\nu} >$ computed for the classical field value implied in the coherent state. But the density matrices $\langle 2.25 \rangle$ do not approximate a classical configuration; indeed the dispersion in $T_{\mu\nu}$ in each mode is always of the order of $T_{\mu\nu}$ itself $[16]$. Can we take into account the backreaction from such distributions? By incorporating the ‘exponential of quadratic’ density matrices in the propagator the real time formalism allows us to do precisely that. Thus even though such configurations are far from classical the Feynman diagram technique can handle them, provided we make the extension to the 2x2 matrix propagator.

What if $\rho$ was not well approximated by the form $e^{A[\phi]+B[\phi]}$? No easy route appears available for studying backreaction in that case. Small deviations from the above form can be handled perturbatively as ‘correlation kernels’, which behave as extra, nonlocal vertices for the perturbation series. For a description of the expansion in correlation kernels and interaction vertices, see $[14]$.

Thus in the real time method we are relying on the assumption that for the situations of interest the state is well approximated by coherent and ‘exponential of quadratic’ parts. From the discussion above about particle creation, we conclude that allowing coherent states alone will not be adequate for a theory of gravity, particularly in regions of strongly varying gravitational field. But if we set all couplings (other than gravity) to be small, we should be able to validate the above assumption about $\rho$, and carry out real time perturbation theory.

We should distinguish two different limits in which the physics of fluxes may be studied. One limit is where the collisions are so rapid that approximate thermal equilibrium
is maintained at all times, and we need only let $\beta$ be a function of time. The other limit is that of kinetic theory, where we assume that collisions are rare; particle wavefunctions evolve on the time-dependent background, and collisions between these particles are taken into account by perturbation theory. Our approach assumes the latter limit.

3. The first quantised formalism.

3.1. The regulator matrix.

The Feynman propagator for a scalar field in Minkowski space can be written as

$$D_F(p) = \frac{i}{p^2 - m^2 + i\epsilon} = \int_{\tilde{\lambda}=0}^{\infty} d\tilde{\lambda} e^{i\tilde{\lambda}(p^2 - m^2 + i\epsilon)}$$

(3.1) can be used to express $G_F(p)$ in a first quantised language, with $p^2 = -\Box$. (See for example [17], [18].) The Hamiltonian on the world line is $\Box + m^2$, evolution takes place in a fictitious time for a duration $\tilde{\lambda}$, and this length $\tilde{\lambda}$ of the world line is summed over all values from 0 to $\infty$.

Let us write the matrix propagator (2.15) in a similar fashion

$$D_{NS}(p) = \int_{0}^{\infty} d\tilde{\lambda} e^{-i\tilde{\lambda}H - \epsilon\tilde{\lambda}M}$$

(3.2) where

$$H = \begin{pmatrix} -(p^2 - m^2) & 0 \\ 0 & (p^2 - m^2) \end{pmatrix}, \quad M = \begin{pmatrix} 1 + 2n(p) & -2\sqrt{n(p)(n(p) + 1)} \\ -2\sqrt{n(p)(n(p) + 1)} & 1 + 2n(p) \end{pmatrix}$$

(3.3)

($n(p)$ is given by (2.13).) This matrix world line Hamiltonian has the following structure. If we forget the term multiplying $\epsilon$, then the $a = 1$ component of the vector state on the world line evolves as $e^{i\tilde{\lambda}(p^2 - m^2)}$ while the $a = 2$ component evolves as $e^{-i\tilde{\lambda}(p^2 - m^2)}$. To define the path integral we need the regulation from $\epsilon$ at the mass shell $p^2 - m^2 = 0$. But the regulator matrix is not diagonal, for nonzero temperature. Thus transitions are allowed from state 1 to state 2. As we shall show below, in the limit $\epsilon \to 0^+$, which we must finally take, these transitions occur only on the mass shell. (The absolute values of the entries in $M$ are not significant, because $\epsilon$ goes to 0, but the relative values are.)

Similarly we can write the matrix propagator (2.12) in the form (3.2) with $H$ as in (3.3) but

$$M = \begin{pmatrix} 1 + 2n(p) & -2\sqrt{n(p)(n(p) + 1)e^{-\beta p_0/2}} \\ -2\sqrt{n(p)(n(p) + 1)e^{\beta p_0/2}} & 1 + 2n(p) \end{pmatrix}$$

(3.4)
Again we have the $a = 1, 2$ states propagating on the world line with Hamiltonians $\mp(p^2 - m^2)$. Transitions between $a = 1, 2$ are again of order $\epsilon$, but are different from those in (3.4).

Looking at these examples the following picture emerges. To obtain the matrix propagator in a many-body situation (with ‘exponential of quadratic’ density matrices) we need to consider both evolutions $e^{-i\tilde{\lambda}H}$ and $e^{i\tilde{\lambda}H}$ on the world line. A damping factor is needed to define the first quantised path integral, near the mass shell. But the regulator matrix $M$ need not be diagonal. The off-diagonal terms of $M$ encode the strength of an ‘exponential of quadratic flux’. These terms also depend on the choice of the time path, as we see from the difference in $M$ for the contours $C$ and $C_1$.

The limit $\epsilon \to 0$ implies that the effect of the regulator matrix $M$ is felt only on-shell (i.e. for $p^2 - m^2 = 0$). Equivalently, we may say that only world lines of infinite length ($\tilde{\lambda} = \infty$) see the regulator matrix. To see this, let $M$ and $M'$ be two different regulator matrices. We write

$$D_{M'} = \lim_{L \to \infty} \lim_{\epsilon \to 0^+} \{ [\int_0^L d\tilde{\lambda} e^{-iH\tilde{\lambda}-\epsilon M\tilde{\lambda}} + \int L^\infty d\tilde{\lambda} e^{-iH\tilde{\lambda}-\epsilon M\tilde{\lambda}}] + [\int L^\infty d\tilde{\lambda} e^{-iH\tilde{\lambda}-\epsilon M'\tilde{\lambda}} - \int L^\infty d\tilde{\lambda} e^{-iH\tilde{\lambda}-\epsilon M\tilde{\lambda}}] + [\int_0^L d\tilde{\lambda} e^{-iH\tilde{\lambda}-\epsilon M'\tilde{\lambda}} - \int_0^L d\tilde{\lambda} e^{-iH\tilde{\lambda}-\epsilon M\tilde{\lambda}}] \} \quad (3.5)$$

The first square bracket on the RHS is $D_M$, the last vanishes with the indicated limits, while the second has support only on world lines of infinite length. Let us take $M = I$, the identity matrix. Then $D_M = \text{diag}\{\frac{-i}{p^2 - m^2 + i\epsilon}, \frac{-i}{p^2 - m^2 - i\epsilon}\}$. Thus other propagators (obtained by using $M$ not the identity matrix) differ from this basic one only by the contribution of world lines of infinite length.

3.2. Quantising the relativistic particle.

What is the origin of the two components $a = 1, 2$ of the state on the world line? We would like to offer the following heuristic ‘derivation’ as a more physical description of the matrix structure in (3.2).

The geometric action for a scalar particle is

$$S = \int_{X_i}^{X_f} mds = \int_{X_i}^{X_f} m(X^\mu,\tau, X_{\mu,\tau})^{1/2} d\tau \equiv \int Ld\tau \quad (3.6)$$
where $\tau$ is an arbitrary parametrisation of the world line. The canonical momenta

\[ P_\mu = \frac{\partial L}{\partial X^{\mu,\tau}} = \frac{mX_{\mu,\tau}}{(X^{\mu,\tau}X_{\mu,\tau})^{1/2}} \] (3.7)

satisfy the constraints

\[ P_\mu P_\mu - m^2 = 0 \] (3.8)

We choose the range of the parameter $\tau$ as $[0, 1]$. Following the approach in [18], we impose the constraint at each $\tau$ through a $\delta$-function:

\[ \delta(p^2(\tau) - m^2) = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\lambda(\tau) e^{-i\lambda/2(p^2(\tau) - m^2)} \] (3.9)

The path integral amplitude to propagate from $X_i$ to $X_f$ becomes

\[ G(X_2, X_1) \equiv N \int \frac{D[X]D[P]D[\lambda]}{\text{Vol}[\text{Diff}]} e^{i \int_0^1 d\tau [P_{\mu}(\tau)X^{\mu,\tau}(\tau) - \lambda/2(p^2(\tau) - m^2)]} \] (3.10)

where $N$ is a normalisation constant, $P_\mu X^{\mu,\tau} = m(X^{\mu,\tau}X_{\mu,\tau})^{1/2}$ is the original action (3.6) and we have divided by the volume of the symmetry group, which is related to $\tau$-diffeomorphisms in the manner discussed below. (The $\delta$-function constraint on the momenta and dividing by $\text{Vol}[\text{Diff}]$ remove the two phase space co-ordinates redundant in the description of the particle path.)

There are two ways to consider the symmetry of the action (3.10). The action is invariant under

\[ S_1 : \]
\[ \delta X^{\mu}(\tau) = h(\tau)P^{\mu}(\tau) \]
\[ \delta P_\mu(\tau) = 0 \]
\[ \delta \lambda(\tau) = h(\tau)_{,\tau} \] (3.11)

and

\[ S_2 : \]
\[ \delta X^{\mu}(\tau) = \epsilon(\tau)\lambda(\tau)P^{\mu}(\tau) \]
\[ \delta P_\mu(\tau) = 0 \]
\[ \delta \lambda(\tau) = (\epsilon(\tau)\lambda(\tau))_{,\tau} \] (3.12)

The difference between $S_1$ and $S_2$ is best seen by considering the finite transformations on $\lambda$:

\[ S_1 : \] $\lambda'(\tau) = \lambda(\tau) + \frac{dh(\tau)}{d\tau}$ (3.13)

\[ S_2 : \] $\lambda'(\tau) = \frac{d\tau}{d\tau}(\tau)\lambda(\tau)$ (3.14)

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Using $S_1$ we can gauge fix any function $\lambda(\tau)$ to any other function $\lambda_1(\tau)$, provided $\lambda, \lambda_1$ have the same value of

$$\int_0^1 \lambda(\tau) d\tau \equiv \Lambda \quad (3.15)$$

With such gauge fixing we obtain not a Green’s function but a solution of the homogeneous Klein-Gordon equation. Thus the Fourier transform of $(3.10)$ gives

$$\tilde{G}(p) = \delta(p^2 - m^2) \quad (3.16)$$

for a suitable normalisation of the measure.

With $S_2$, $\lambda$ transforms as an einbein under the diffeomorphism $\tau \to \tau'(\tau)$. Note that for regular $\epsilon(\tau)$, $\lambda$ either changes sign for no $\tau$ or for all $\tau$. We take Diff as the group of regular diffeomorphisms connected to the identity; then $\lambda$ does not change sign under the allowed diffeomorphisms. These diffeomorphisms cannot gauge-fix $\lambda(\tau)$ to any preassigned function $\lambda_1(\tau)$. We have the obvious restriction coming from $(3.15)$, where $\Lambda$ may now be interpreted as the length of the world line. This restriction is usually assumed to mean that the length of the world line is the only remaining parameter after gauge-fixing. But what we find instead is that there is a further complication: there is a discrete infinity of classes, each with one or more continuous parameters. One class comes from configurations $\lambda(\tau)$ which are everywhere positive. This class can be gauge-fixed with the allowed diffeomorphisms to have

$$\dot{\lambda}(\tau) = 0, \quad \int_0^1 d\tau \lambda(\tau) = \Lambda \quad (3.17)$$

with $0 < \Lambda < \infty$. Similarly, the set of everywhere negative $\lambda(\tau)$ can be gauge-fixed as in $(3.17)$ but with $-\infty < \Lambda < 0$. Thus these classes give for the Fourier transform of $(3.10)$

$$G_{\lambda>0}(p) = \frac{1}{4\pi} \int_0^\infty d\lambda(\tau) e^{-i\lambda/2(p^2(\tau)-m^2)} = \frac{i}{p^2 - m^2 + i\epsilon},$$

$$G_{\lambda<0}(p) = \frac{1}{4\pi} \int_{-\infty}^0 d\lambda(\tau) e^{-i\lambda/2(p^2(\tau)-m^2)} = \frac{-i}{p^2 - m^2 - i\epsilon} \quad (3.18)$$

respectively. (We have added the $\epsilon$ term for convergence to each sector; this term was not in the action.) Keeping the first class alone gives the Feynman propagator for particles, while the second gives its complex conjugate.\footnote{A restriction to the range $(0, \infty)$ for $\Lambda$ can be naturally obtained using a Newton-Wigner formalism \cite{19}. Here the particle travels only forwards in the time co-ordinate $X^0$, thus it is not a co-variant approach.} Note that these two objects are the diagonal entries of $(2.13)$ for $\beta = \infty$.\footnote{A restriction to the range $(0, \infty)$ for $\Lambda$ can be naturally obtained using a Newton-Wigner formalism \cite{19}. Here the particle travels only forwards in the time co-ordinate $X^0$, thus it is not a co-variant approach.}
But in the path integral over $\lambda$ in (3.10) we also have the class of $\lambda(\tau)$ which are positive for $0 < \tau < \tau_1$, negative for $\tau_1 < \tau < 1$. The group of orientation preserving diffeomorphisms can gauge fix this to

$$\dot{\lambda}(\tau) = 0 \quad \text{for} \quad \tau \neq \tau_1, \quad \int_0^{\tau_1} d\tau \lambda(\tau) = \Lambda_1, \quad \int_{\tau_1}^1 d\tau \lambda(\tau) = \Lambda_2$$

with $0 < \Lambda_1 < \infty$, $-\infty < \Lambda_2 < 0$. We would like to identify this sector as the contribution to the amplitude to start with a state of type 1 and end with a state of type 2 (the off-diagonal element $D_{12}$ of the matrix propagator). A general sector has a given number of alternations in the sign of $\lambda$, and in each interval of constant sign we gauge fix $\lambda$ to a constant $\Lambda_i$. We can perform the integration over the variables $\Lambda_i$ appearing in each sector, but we should also specify a ‘transition amplitude’ for each point $\tau_i$ where $\lambda$ changes sign. We allow this amplitude to depend on the states on both sides of $\tau_i$, and also on whether $\lambda$ changes from positive to negative or vice versa. This amplitude is not supplied by the original action; it is supplementary information needed for determining the propagator.

We would like to identify $\lambda > 0$ with particles of the type 1 field and $\lambda < 0$ with particles of the type two field in the language of section 2. To obtain the matrix propagator element $D_{ji}$ we need to add together all sectors for $\lambda(\tau)$ beginning as type $i$ and ending as type $j$. Take the example (3.2), (3.3) and write

$$D(p) = \int_0^{\infty} d\lambda e^{-i\lambda H - \epsilon \lambda M'} = \int_0^{\infty} d\lambda e^{-i\lambda H' - \epsilon \lambda M'}$$

$$H' = \begin{pmatrix} -[p^2 - m^2 + i\epsilon(1 + 2n(p))] & 0 \\ 0 & [p^2 - m^2 - i\epsilon(1 + 2n(p))] \end{pmatrix},$$

$$M' = \begin{pmatrix} 0 & -2\sqrt{n(p)(n(p) + 1)} \\ -2\sqrt{n(p)(n(p) + 1)} & 0 \end{pmatrix}$$

Expanding (3.20) in a perturbation series in $M'$,

$$D(p) = \int_0^{\infty} d\lambda_1 e^{-iH'\lambda_1} + \int_0^{\infty} d\lambda_1 d\lambda_2 e^{-iH'\lambda_1}(-\epsilon M')e^{-iH'\lambda_2} + \ldots$$

$$= D^0 + D^0(-\epsilon M')D^0 + \ldots$$

where

$$D^0 = \begin{pmatrix} \frac{i}{p^2 - m^2 + i\epsilon} & 0 \\ 0 & \frac{i}{p^2 - m^2 - i\epsilon} \end{pmatrix}$$
$D_{12}$ for example gets a contribution from every term with an odd number of $M'$ insertions:

$$D_{12}(p) = -\epsilon D_{11}^0 M'_{12} D_{22}^0 - \epsilon^3 D_{11}^0 M'_{12} D_{22}^0 M'_{21} D_{11}^0 M'_{12} D_{22}^0 + \ldots$$

$$= \sum_{m=0}^{\infty} \left( \frac{1}{(p^2 - m^2 + \tilde{\epsilon}^2)} \right)^m (2\tilde{\epsilon} \sqrt{\frac{n(p)(n(p)+1)}{1+2n(p)}})^{2m-1}$$

$$= 2\pi \delta(p^2 - m^2) \sqrt{n(p)(n(p)+1)}$$

where $\tilde{\epsilon} = \epsilon(1 + 2n(p)).$

Thus suppose the propagators for $\lambda > 0, \lambda < 0$ are $\pm i(p^2 - m^2)$ respectively. Let us choose the amplitude for $\lambda$ to change sign by requiring that momentum $p$ goes to $p$, and the associated amplitude is $2\epsilon \sqrt{\frac{n(p)(n(p)+1)}{1+2n(p)}} = \tilde{\epsilon} \text{sech}(\beta|p_0|/2)$. Then summing all the sectors that contribute to each $D_{ji}$ we get the matrix propagator (2.15).

To obtain (2.12), we must take the amplitude for $\lambda$ to go from positive to negative as $\tilde{\epsilon} \text{sech}(\beta|p_0|/2)e^{\beta p_0/2}$, and for negative to positive as $\tilde{\epsilon} \text{sech}(\beta|p_0|/2)e^{-\beta p_0/2}$.

Note: To achieve the interpretation of a matrix of propagators as arising from the above ‘sum over sectors’, the matrix itself must be very special. From the symmetry of (2.15) we find that $M$ must have the form $$\begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$ Because of the limit $\epsilon \to \infty$ only the ratio $b/a$ affects the obtained propagator. As we saw above, this ratio (which directly gave the amplitude for reversing world line orientation) got determined by the value of $D_{12}$. But now there are no free parameters to adjust for obtaining $D_{11}$. A direct calculation gives

$$D_{11}(p) = D_{11}^0 + \epsilon^2 D_{11}^0 M'_{12} D_{22}^0 M'_{21} D_{11}^0 + \ldots$$

$$= \sum_{m=0}^{\infty} \frac{i(2\tilde{\epsilon} \sqrt{\frac{n(p)(n(p)+1)}{1+2n(p)}})^{2m-1} 2m}{(p^2 - m^2 + i\tilde{\epsilon})(p^2 - m^2 + \tilde{\epsilon}^2)^m}$$

$$= \frac{i}{p^2 - m^2 + i\tilde{\epsilon}} + 2\pi \delta(p^2 - m^2)(1 + 2n(p))$$

which agrees with $D_{11}$ in (2.15). Thus the matrix of propagators obtained from the real time formalism has, at least for this example, the special form required for the first quantised interpretation to work. (2.12) also has the required form. $M$ has unequal off diagonal terms but their product remains the same as in the previous example; thus we get the same $D_{11}$, in agreement with (2.12).

Thus we have verified in these examples that the amplitude for einbein sign change reflects the presence of particle flux in the first quantised formalism. In section 4 we will present an example where the flux created by spacetime expansion is handled in a similar way.
3.3. BRST formalism.

One might wonder if the more formal BRST quantisation of the relativistic particle would resolve the above issues about different possible quantisations. We follow the notation in [4]. We introduce the canonical conjugate $\pi$ for $\lambda$ ($[\lambda, \pi] = i$) and ghosts $(\eta^1, P_1)$, $(\eta^2, P_2)$ ($[\eta^i, P_j] = -i\delta^i_j$) for the two constraints $\pi = 0$ and $\frac{1}{2}(P^2 - m^2) = 0$ respectively. The BRST charge

$$Q = \eta^1 \pi + \frac{\eta^2}{2}(P^2 - m^2)$$ \hspace{1cm} (3.26)

is nilpotent ($Q^2 = 0$), and gives the variations:

\begin{align*}
\delta X^\mu &= -i[X^\mu, Q] = \eta^2 P^\mu \quad \delta P_\mu = 0 \\
\delta \lambda &= \eta^1 \\ 
\delta \pi &= 0 \\
\delta \eta^1 &= 0 \quad \delta P_1 = -\pi \sim 0 \\
\delta \eta^2 &= 0 \quad \delta P_2 = -\frac{1}{2}(P^2 - m^2) \sim 0
\end{align*} \hspace{1cm} (3.27)

The equation of motion (obtained after gauge fixing) gives $\dot{\eta}^2 = \eta^1$, which agrees with (3.11).

But we can define another nilpotent BRST charge

$$Q' = \eta^1' \pi' \lambda' + \frac{1}{2}\eta^2'(P^2 - m^2)$$ \hspace{1cm} (3.28)

which generates the symmetry

\begin{align*}
\delta X'^{\mu'} &= \eta^2' \lambda' P^{\mu'} \quad \delta P_{\mu'} = 0 \\
\delta \lambda' &= \lambda' \eta^1' \\
\delta \pi' &= -\eta^1' \pi' \sim 0 \\
\delta \eta^1' &= 0 \quad \delta P'_1 = -\lambda' \pi' \sim 0 \\
\delta \eta^2' &= 0 \quad \delta P'_2 = -\frac{1}{2}(P^{2'} - m^2) \sim 0
\end{align*} \hspace{1cm} (3.29)

The equation of motion gives $\dot{\eta}^{2'} = \eta^1' \lambda'$, which suggests that we should identify the above symmetry with $S_2$.

The symmetries $Q$ and $Q'$ are related through the identifications

$$\pi = \pi' \lambda', \quad \lambda = log\lambda'$$ \hspace{1cm} (3.30)

all other primed variables equalling the unprimed ones. From (3.30) we find that $-\infty < \lambda < \infty$ corresponds to $0 < \lambda' < \infty$. If we perform a path integral with the primed variables
and sum over both positive and negative $\lambda'$ then we are summing over more than is being summed in the unprimed variable path integral.

We thus see sources of ambiguity on the quantisation of the relativistic particle working with a Fadeev-Popov approach in sec 3.2 and a BRST approach in sec. 3.3. In fact the action we start with, (3.6), is itself ambiguous because of the two possible signs of the square root. The particle trajectory would keep switching in general between timelike and spacelike, and at each switch we have to choose afresh the sign of the real or imaginary quantity obtained in these two cases respectively. This suggests that the world line configuration should be described by the pair $\{X^\mu(\tau), \sigma(\tau)\}$ with $\sigma = \pm 1$ giving the choice of root. Evaluating the quadratic form of the action (given in (3.10)) classically we find the sign of $\lambda$ to be related to the sign of the square root chosen for (3.6).

The above discussion suggests a close connection between the ambiguities found in three different approaches to the quantum relativistic particle, and it would be good to determine if they indeed are the same. For the rest of this paper we simply adopt as basic the picture of two complex conjugate propagations on the world line, with switching between them possible through the regulator matrix.

3.4. Summary.

We may describe our quantisation of the relativistic scalar particle in the following colloquial terms. To obtain the usual Feynman propagator the particle was allowed to move both backwards and forwards in target space time. But it moved only forwards in its proper time. Now we are allowing the particle to move both backwards and forwards in proper time also; this corresponds to flipping between the two signs of the world line einbein. Giving the amplitude to reverse orientation of proper time amounts to specifying an ‘exponential of quadratic’ density matrix,. Putting this amplitude to zero lets us recover the Feynman propagator, by restricting to the positive sign of the einbein.

Thus we find a covariant description of the notion of density matrix. We see that in the first quantised description, there is very little difference between causal perturbation theory for pure states and for mixed states.

Note: $\text{tr}\rho$ is finite only for $4\alpha\gamma < 1 - e^{-\beta}$^2 (see appendix). Consider the initial value problem for charged scalar field in a small box with a (suddenly switched on) strong electric field. We may wish to expand the field in modes which are eigenfunctions of the

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^3 I am indebted to A. Vilenkin for this way of expressing the result.
Hamiltonian. The above inequality is violated at the point where the electric field becomes strong enough to destabilise the vacuum [20]. For stronger fields the mode coefficients are \( b, d^\dagger, [b,d^\dagger] = -i \) instead of \( a, a^\dagger, [a,a^\dagger] = 1 \). We expect that black holes would also be characterised by a similar violation. The density matrix is still ‘exponential of quadratic in the field’ but (2.25) is not a useful representation. We will ignore this limitation of the form (2.25) because for our examples it will be adequate.

4. A curved space example: spacetime with expansion.

Consider the free scalar field ((2.1) with \( \lambda = 0 \)) propagating in 1 + 1 spacetime with metric

\[
 ds^2 = C(\eta)[d\eta^2 - dx^2], \quad -\infty < \eta < \infty, \quad 0 \leq x < 2\pi 
\]

\[
 C(\eta) = A + B \tanh \kappa \eta, \quad A > B \geq 0 
\]

The conformal factor \( C(\eta) \) tends to \( A \pm B \) at \( \eta \to \pm\infty \). The limit \( \kappa \to \infty \) gives a step function for \( C(\eta) \); the Universe jumps from scale factor \( A - B \) to \( A + B \) at \( \eta = 0 \). We will work in this limit to ensure simpler expressions.

For \( \eta \to -\infty \) it is natural to expand \( \phi \) as

\[
 \phi(\eta,x) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega_n^-}}(a_n e^{-i\omega_n^- \eta + ix} + a_n^\dagger e^{i\omega_n^- \eta - ix}) 
\]

with

\[
 \omega_n^- = (n^2 + (A - B)m^2)^{1/2} > 0 
\]

We define the ‘in’ vacuum by

\[
 a_n|0 >_{in} = 0, \quad \text{for all} \quad n 
\]

Similarly, for \( \eta \to \infty \) we write

\[
 \phi(\eta,x) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega_n^+}}(\tilde{a}_n e^{-i\omega_n^+ \eta + ix} + \tilde{a}_n^\dagger e^{i\omega_n^+ \eta - ix}) 
\]

with

\[
 \omega_n^+ = (n^2 + (A + B)m^2)^{1/2} > 0 
\]
The ‘out’ vacuum is defined through
\[ \tilde{a}_n|0>_{out} = 0, \quad \text{for all } n \] (4.8)

The ‘out’ vacuum does not equal the ‘in’ vacuum, even in the free theory:
\[ |0>_{out} = C_0 e^{\frac{2\pi}{\hbar} a_0^\dagger a_0} \prod_{n>0} C_n e^{b_n a_n a_n^\dagger} |0>_{in} \] (4.9)
\[ b_n = -\frac{\omega_n^+ - \omega_n^-}{\omega_n^+ + \omega_n^-}, \quad C_n = (1 - b_n^2)^{1/2}, \quad n \geq 0. \] (4.10)

As mentioned in the introduction, evaluating the analogue of the Polyakov path integral gives \[ \mathcal{Z} \] (we denote the pair (\eta, x) by z)
\[ <z_2| \int_0^\infty d\lambda D[X] e^{-i\int_0^1 d\tau (1/2)(\dot{X}^2/\lambda + m^2\lambda)} |z_1> = \frac{\text{out} <0|T[\phi(z_2)\phi(z_1)]|0>_{in}}{\text{out} <0|0>_{in}} \] (4.11)
(We have integrated out p in the phase space path integral.) It is not surprising that both vacua appear in this quantity; after all the action and measure are covariantly given and should not distinguish the past or future as special. Using \(-i\) in place of \(i\) in the exponential gives \[ \text{in} <0|T[\phi(z_2)\phi(z_1)]|0>_{out} / \text{in} <0|0>_{out}. \] (\(\tilde{T}\) denotes anti-time-ordering.)

We wish to compute the propagator appropriate to the initial value problem based on the ‘in’ vacuum at \(t = -\infty\). This implies a density matrix \(\rho \equiv \rho_0 = |0>_{in} \text{ in } <0|\). To develop a perturbation theory using \(\rho_0\) we need a real time contour running from \(\eta = -\infty\) to \(\eta = \infty\), and then back to \(\eta = -\infty\) where we insert \(\rho_0\) and take a trace to close the path. This perturbation theory requires a matrix propagator \(D\):
\[ D(z_2, z_1) = \begin{pmatrix} \text{in} <0|T[\phi(z_2)\phi(z_1)]|0>_{in} & \text{in} <0|\phi(z_1)\phi(z_2)|0>_{in} \\ \text{in} <0|\phi(z_2)\phi(z_1)|0>_{in} & \text{in} <0|T[\phi(z_2)\phi(z_1)]|0>_{in} \end{pmatrix} \] (4.12)

Our goal is to see if there exists a choice of regulator matrix \(M\) such that evaluating
\[ D = \int_0^\infty \lambda e^{-i\lambda H - \epsilon \lambda M}, \quad H = \begin{pmatrix} (\Box + m^2) & 0 \\ 0 & -(\Box + m^2) \end{pmatrix} \] (4.13)
gives the matrix propagator (4.12).

Let us discuss the general structure of the world line Hamiltonian and regulator matrix in (4.13). \(H\) and \(M\) have a ‘2x2 matrix of operators’ structure, on account of the two signs of the einbein \(\lambda\) possible on the world line. The operator \(\Box + m^2\) acts on the space of all functions of (\(\eta, x\)). Formally, each operator in the 2x2 matrix \(M\) also acts on this
space, but as noted at the end of section 3.1 the only space affected by these operators
is the space of ‘on shell’ wavefunctions, i.e., solutions of $(\Box + m^2)f(\eta, x) = 0$. In the flat
space examples (3.3), (3.4) we had translational invariance in space and time, so each
operator could connect one Fourier mode $p$ only to itself. Thus for each $p$, $M$ was just
a 2x2 matrix of numbers. In the present example, we retain $x$-translational invariance,
so different $x$-Fourier modes (labelled by $n$) are not mixed by $M$. But time translational
invariance is broken by the expansion, and so the two solutions of the wave equation for
fixed $n$ can be transformed into each other by the elements of $M$. Thus for each $n$ we
need to consider a 4x4 matrix, which acts on a column vector $(\{f^1_n, f^2_n\}^+, \{f^1_n, f^2_n\}^-)$. Here
the first pair of functions propagate on the world line as $e^{i(\Box + m^2)}$ while the second pair propagates as $e^{-i(\Box + m^2)}$. Within each type (+ or −) we have two linearly independent
solutions of $(\Box + m^2)f = 0$. Thus for example the element $M_{41}$ of the regulator matrix
gives the amplitude for the einbein to change sign from positive to negative, and for the
wavefunction on the world line to change from $f^1_n$ to $f^2_n$.

Let us set up the calculation of Green’s functions in the first quantised formalism. We
need eigenfunctions of the world line Hamiltonian:

$$(\Box + m^2)\psi_s(\eta, x) = -H\psi_s(\eta, x) = s\psi_s(\eta, x)$$ (4.14)

The following is a complete set: $(-\infty < n < \infty)$

$$m^2 + \frac{n^2}{A+B} < s < m^2 + \frac{n^2}{A-B} :$$

$$\psi_{\nu_+,n}(\eta, x) = \frac{e^{inx}}{\sqrt{2\pi}} e^{-i\nu_+\eta}, \quad \eta > 0$$

$$\frac{e^{inx}}{\sqrt{2\pi}} \left[ \frac{1}{2} (1 + \frac{\nu_+}{\nu_-}) e^{-i\nu_-\eta} + \frac{1}{2} (1 - \frac{\nu_+}{\nu_-}) e^{i\nu_-\eta} \right], \quad \eta < 0$$

$$\tilde{\nu}_+ = -(A + B)(m^2 - s) - n^2]^{1/2} > 0, \quad \nu_- = [(A - B)(m^2 - s) + n^2]^{1/2} > 0$$ (4.15)

$$-\infty < s < m^2 + \frac{n^2}{A + B} :$$

$$\psi_{\nu_-,n}(\eta, x) = \frac{e^{inx}}{\sqrt{2\pi}} e^{-i\nu_-\eta}, \quad \eta > 0$$

$$\frac{e^{inx}}{\sqrt{2\pi}} \left[ \frac{1}{2} (1 + \frac{\nu_-}{\nu_+}) e^{-i\nu_+\eta} + \frac{1}{2} (1 - \frac{\nu_-}{\nu_+}) e^{i\nu_+\eta} \right], \quad \eta < 0$$
\[ \nu_{\pm}^2 = (A \pm B)(m^2 - s) + n^2, \quad \text{sign}(\nu_+) = \text{sign}(\nu_-) \]  
\[ (4.16) \]

These functions are normalised as
\[ (\psi_{\nu+}, n', \psi_{\nu+,n}) = \int d\eta dx C(\eta) \psi_{\nu+}^*(\eta, x) \psi_{\nu+,n}(\eta, x) \]
\[ = \delta_{n', n} \pi (A + B) \frac{\nu_+^2 + \tilde{\nu}_+^2}{2\nu_- \tilde{\nu}_+} \delta(\nu' - \nu_+) \]
\[ (4.17) \]

\[ (\psi_{\nu+}, n', \psi_{\nu+,n}) = \delta_{n', n} \pi (A + B) \left[ \frac{(\nu_- + \nu_+)^2}{2\nu_- \nu_+} \delta(\nu' - \nu_+) + \frac{\nu_-^2 - \nu_+^2}{2\nu_- \nu_+} \delta(\nu' + \nu_+) \right] \]
\[ (4.18) \]

Let us first recover the propagator (4.11) in this formalism. Let \( \eta', \eta > 0 \). The range \( m^2 + \frac{n^2}{A+B} < s < m^2 + \frac{n^2}{A-B} \) gives the contribution
\[ \sum_n \int_{\tilde{\nu}_+ = 0}^{\sqrt{2B/(A-B)|n|}} d\tilde{\eta}_+ d\tilde{\eta}_+ < \eta', x' | \psi_{\nu+}, n > < \psi_{\nu+}, n | \int_0^\infty d\lambda e^{i(-s+i\lambda)\lambda} |\psi_{\nu+}, n > \]
\[ = \sum_n \frac{e^{i(x'-x)}}{2\pi} \int_0^{\sqrt{2B/(A-B)|n|}} d\nu_+ \frac{(-i)}{\pi} e^{-\nu_+ (\eta' + \eta)} \frac{2\nu_- \tilde{\nu}_+}{\nu_-^2 + \tilde{\nu}_+^2} \frac{1}{(\nu_-^2 - n^2 - m^2 (A + B))} \]
\[ (4.19) \]

Similarly the range \(-\infty < s < m^2 + \frac{n^2}{A+B} \) provides the contribution
\[ \sum_n \frac{e^{i(x'-x)}}{2\pi} \int_{-\infty}^{\infty} d\nu_+ \frac{i}{2\pi} \left[ e^{i\nu_+ (\eta' - \eta)} + \frac{\nu_- - \nu_+}{\nu_+ + \nu_-} e^{i\nu_+ (\eta' + \eta)} \right] \frac{1}{(\nu_-^2 - n^2 - m^2 (A + B) + i\epsilon)} \]
\[ (4.20) \]

There is a branch cut in the complex \( \nu_+ \) plane joining \( \nu_+ = \pm i \sqrt{\frac{2B}{A-B}} |n| \). The \( \nu_+ \) integral in (4.20) has a discontinuous jump across this cut for the part multiplying \( e^{i\nu_+ (\eta' + \eta)} \). The contribution from (4.19) can be added to (4.20), however, with the identification \( \tilde{\nu}_+ = i\nu_+ \). This results in a contour passing over the cut. Evaluating the resulting contour integrals one obtains the result
\[ D_{11}^{01}(\eta', x', \eta, x) = \sum_n \frac{e^{i(x'-x)}}{2\pi} \frac{1}{2\omega_n} e^{-i\omega_n^+ |\eta' - \eta|} + \frac{1}{2\omega_n^+ \omega_n^- + \omega_n^+} e^{-i\omega_n^+ (\eta' + \eta)}, \quad \text{for } \eta, \eta' > 0 \]
\[ (4.21) \]

which may be readily verified in the operator language using (4.6) and (4.9).

We now proceed to the ‘in-in’ matrix propagator. From the expansion of the field operator \( \phi \) in creation and annihilation modes, we can readily compute each element of
the matrix propagator. We try to find a regulator matrix $M$ which will reproduce this propagator by the first quantised calculation. To display the 4x4 matrix $M$, we need to choose a basis in the space of solutions of the wave equation for each $x$-Fourier mode $n$. We choose the basis $(\omega_n^- > 0)$

\[
\begin{align*}
  f_n^1 &= \frac{e^{inx}}{\sqrt{2\pi}} e^{-i\omega_n^- \eta}, & \quad \eta < 0 \\
  &= \frac{e^{inx}}{\sqrt{2\pi}} \left[ \frac{1}{2} (1 + \frac{\omega_n^-}{\omega_n^+}) e^{-i\omega_n^+ \eta} + \frac{1}{2} (1 - \frac{\omega_n^-}{\omega_n^+}) e^{i\omega_n^+ \eta} \right], & \quad \eta > 0
\end{align*}
\]

(4.22)

\[
f^2_n(\eta, x) = f^*_{n}(\eta, x)
\]

The result we get is that the required propagator is obtained for

\[
M = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & \frac{4B_n}{1+4B_n^2} & \frac{-2}{1+4B_n^2} \\
\frac{-2}{1+4B_n^2} & \frac{4B_n}{1+4B_n^2} & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

(4.23)

where

\[
B_n = -\frac{1}{2} \frac{\omega_n^+ - \omega_n^-}{\omega_n^+ + \omega_n^-}
\]

(4.24)

(In this computation we need to note that there are many equivalent descriptions of the delta function. For example $\lim_{\delta \to 0} \delta x^2 + \delta^2 = \pi \delta(x)$, but $\lim_{\delta \to 0} \delta (x+i\delta)^2 = 0$; thus any amount of the latter can be added to the former to get a different representation of the $\delta$-function.)

As was the case in the examples of the last section, $M$ does not have enough independent entries to reproduce any arbitrary 2x2 matrix of propagators. In the present example any propagator has the freedom of 4 constants for each $x$-Fourier mode $n$: we can add $\sum_{i,j=1}^2 C_{ij} f^{(i)}(z) f^{(j)}(z')$ where $f^{(i)}$ are given in (4.22). A 2x2 matrix of propagators thus has 16 free parameters for each $n$. But the entries in the diagonal 2x2 blocks of $M$ are not all independent, in the sense that some changes to $M$ do not affect the final propagator obtained. (For example, scaling $M \to cM$ leaves the propagator unchanged.) Thus at first it seems a coincidence that we could find $M$ to obtain (4.12).

To understand this coincidence we recall that (4.12) came from a ‘closed time path’ formulation. This formulation is really covariant (and not dependent on any choice of ‘time’). Each spacetime point is covered twice in the description (alternatively we can say that there is a doubling of fields [21]). Over this double cover of spacetime we are looking for a first quantised description of the propagator. Clearly such a propagator can be represented by the usual path integral in each cover, together with an amplitude to switch between covers. Thus all matrix propagators arising in the closed time path formalism will have the special form that arises from allowing orientation changes on the world line in the first quantised language.

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5. Strings.

Let us now come to strings. Requiring consistent propagation of the string determines the field equations for the background on which the string moves. But use of ‘in-out’ and ‘in-in’ propagators might give different field equations, as we argue in this section.

To obtain the necessary ingredients for the argument, we first consider a simple case: strings in a constant temperature bath, in a time independent background. Since we are at constant temperature, we can use for the moment the periodic imaginary time trick to take into account the temperature \cite{22}.

We expect that the vanishing $\beta$-function conditions will reproduce, in some approximation, the Einstein equation

$$G_{\mu\nu} = <T_{\mu\nu}> \quad (5.1)$$

where $T_{\mu\nu}$ is the stress-tensor of the thermal distribution. The tree level $\beta$-function calculation is a local one: local on the world sheet and local in target space. At tree level we thus obtain $G_{\mu\nu} = 0$; we are unable to ‘see’ the thermal bath. The $\beta$-function also has a contribution at one loop in the string coupling expansion: we must consider tadpoles attached to the world sheet by necks thinner than the world sheet cut-off scale \cite{23}. The loop of the tadpole can wind around the compact imaginary time direction. This takes into account the value of temperature and produces the contribution $<T_{\mu\nu}>$ required for the field equation (5.1) \cite{24}.

Let us now ask how the $\beta$-function equations would reproduce (5.1) if we are not permitted use of the periodic imaginary time trick to take into account the particle distribution. The question is pertinent, because as mentioned in the introduction time independent distributions are unnatural in a theory of gravity, and any time dependence would invalidate the periodic imaginary time trick. There is no ‘analytic continuation of time’ for a general geometry. \footnote{It is possible to study some time dependent situations by making a canonical transformation on the target space co-ordinates $X^\mu$ and taking the new $X^0$ to be imaginary and compact; this does not however have the physical appeal of the real-time method.}

In the absence of the imaginary time winding mode, neither the tree nor the loop contributions to the $\beta$-function see the thermal distribution. But considering the extended propagator-vertex structure of the real time formalism we should be able to recover the RHS of (5.1). The one loop tadpole is composed of two propagators and two vertices (including the vertex joining the tadpole to the rest of the world sheet). These vertices
can be type 1 or type 2, in the notation of section 2, and the propagators will be the corresponding elements of the 2x2 matrix of propagators. These propagators depend on the ensemble distribution function, and so ‘see’ the temperature. It would be interesting to explicitly carry out this calculation, but here we will just assume that the real time method will work for strings the same way it does with particles. Note that such a calculation does not need the restriction to time-independent distribution functions.

We now argue, again with the scalar field analogy, that there is a direct connection between the choice of propagator and the value of $< T_{\mu \nu} >$ seen by the $\beta$-function calculation for the string background. Consider the scalar field in flat space and ignore the mass term for the moment (since the string case has massless fields on the world sheet). Then Einstein’s equations read

$$R_{\mu \nu}(x) = < \partial_\mu \phi(x) \partial_\nu \phi(x) > = \lim_{x \to x'} \partial_\mu \partial_\nu' < \phi(x) \phi(x') > \quad (5.2)$$

(We ignore any subtraction in the definition of $T_{\mu \nu}$ since the string case is ultraviolet finite and has no subtraction.) Using the analogue of the Polyakov path integral

$$< \phi(x) \phi(x') > = \int_0^\infty d\lambda D[X]_{X(1)=x', X(0)=x} e^{-i \int_0^1 d\tau (1/2)X, ^2 / \lambda} = \int_0^\infty d\lambda D[X]_{X(\lambda)=x', X(0)=x} e^{-i \int_0^1 ds (1/2)X, ^2} \quad (5.3)$$

where $ds = \lambda d\tau$ is the length element along the world line. Then

$$\partial_\mu \partial_\nu' < \phi(x) \phi(x') > = \int_0^\infty d\lambda D[X]_{X(\lambda)=x', X(0)=x} \frac{dX_\mu}{ds}(0) \frac{dX_\nu}{ds}(\lambda) e^{-i \int_0^\lambda ds (1/2)X, ^2} \quad (5.4)$$

Let us compare the above with the $\beta$-function equation for string theory in a finite temperature bath described in the imaginary-time formalism [24]:

$$\beta_{\mu \nu} = R_{\mu \nu} + 2 \nabla_\mu \nabla_\nu = \frac{e^{2\phi}}{2\pi \beta V d} \sum_{g \geq 1} < \partial X_\mu \partial X_\nu >_g \quad (5.5)$$

($\phi$ is the dilaton, $V$ is the volume of spacetime, $d$ is the number of space dimensions.) The RHS of (5.5) is the analogue of (5.4). (The factor $(\beta V)^{-1}$ in (5.3) compensates for the $X^\mu$ zero mode; without this compensation we would compute the stress tensor not at a point $x$ but instead integrated over all the range of coordinates $X^\mu$.) Examining the derivation of (5.4), we expect that in the string case also the stress tensor contribution to the $\beta$-function comes in a direct way from the propagator of the first quantised language. (Thus in (5.5) it comes from the propagator in spacetime with a compact Euclidean time.
direction.) If we have a general time dependent spacetime (with Minkowski signature) and we compute amplitudes by the Polyakov prescription, then the propagator is
\[ \langle 0 | T \{ \phi(x)\phi(x') \} | 0 \rangle_{\text{out}} < 0 | 0 \rangle_{\text{in}} \] and so we expect to incorporate
\[ \langle 0 | T_{\mu\nu} | 0 \rangle_{\text{out}} < 0 | 0 \rangle_{\text{in}} \] in the background field equations obtained from the \( \beta \)-function constraint. This is the wrong kind of object to have in the gravity equation, as we illustrate with the following simple example.

We consider the free scalar field on the 1 + 1 spacetime \((4.1), (4.2)\). We compute ‘in-out’ vacuum expectation values and ‘in-in’ vacuum expectation values, for the stress tensor. To achieve a parallel with the string case, we consider no subtraction in defining \( T_{\mu\nu} \), but compute the contribution from different field modes separately, to get finite answers. (We can imagine an ultraviolet cutoff regulates the sum over different modes, to make sense of sum over modes.) More specifically, we compute the contribution to
\[
< T_{\mu\nu}(\eta) >_{\text{in out}} \equiv \int_{x=0}^{2\pi} dx_{\text{in}} < 0 | T_{\mu\nu}(\eta, x) | 0 >_{\text{in}}
\]
\[
< T_{\mu\nu}(\eta) >_{\text{in in}} \equiv \int_{x=0}^{2\pi} dx_{\text{out}} < 0 | T_{\mu\nu}(\eta, x) | 0 >_{\text{in}} / < 0 | 0 >_{\text{in}}
\]

from the field modes with \( x \)-Fourier component \( \pm n \):
\[
\phi_{\pm n} = a_{\pm n} f_{\pm n} + a_{\pm n}^\dagger f_{\pm n}^\dagger
\]
\[
f_{\pm n} = \frac{e^{\pm inx}}{\sqrt{2\pi\sqrt{2\omega_-}}} e^{-i\omega_- \eta} , \eta < 0
\]
\[
= \frac{e^{\pm inx}}{\sqrt{2\pi\sqrt{2\omega_-}}} \left[ \frac{\omega_+ + \omega_-}{2\omega_+} e^{-i\omega_+ \eta} + \frac{\omega_+ - \omega_-}{2\omega_+} e^{i\omega_+ \eta} \right] , \eta > 0
\]

The stress tensor for the scalar field is
\[
T_{\mu\nu} = \frac{1}{2} (\partial_\mu \phi \partial_\nu \phi + \partial_\nu \phi \partial_\mu \phi) - \frac{1}{2} g_{\mu\nu} \partial_\lambda \phi \partial^\lambda \phi + \frac{1}{2} m^2 \phi^2
\]
Define
\[
T_{\mu\nu}[f, g] = \frac{1}{2} (\partial_\mu f \partial_\nu g + \partial_\nu f \partial_\mu g) - \frac{1}{2} g_{\mu\nu} \partial_\lambda f \partial^\lambda g + \frac{1}{2} m^2 fg
\]
The expansion \((5.7)\) is in terms of the ‘in’ vacuum creation and annihilation operators. We recall that
\[
|0 >_{\text{out}} = C e^{ba_n^\dagger a_{-n}^\dagger} |0 >_{\text{in}}
\]
where \( b = - (\omega_+ - \omega_-)/(\omega_+ + \omega_-) \). We have

\[
<T_{00}>_{\text{in in}} = T_{00}[f_n, f_n^*] + T_{00}[f_{-n}, f_{-n}^*]
\]

\[
<T_{11}>_{\text{in in}} = T_{11}[f_n, f_n^*] + T_{11}[f_{-n}, f_{-n}^*]
\]

\[
<T_{00}>_{\text{in out}} = T_{00}[f_n, f_n^*] + T_{00}[f_{-n}, f_{-n}^*] + b\{T_{00}[f_n^*, f_n^*] + T_{00}[f_{-n}^*, f_{-n}^*]\}
\]

\[
<T_{11}>_{\text{in out}} = T_{11}[f_n, f_n^*] + T_{11}[f_{-n}, f_{-n}^*] + b\{T_{11}[f_n^*, f_n^*] + T_{11}[f_{-n}^*, f_{-n}^*]\}
\]

(5.12)

\(T_{01}, T_{10}\) vanish.

From (5.8) we get

\[
<T_{00}>_{\text{in in}} = \omega_-, \eta < 0 = \omega_+ + 2\omega_+ < N_{\text{out}}>, \eta > 0
\]

\[
<T_{11}>_{\text{in in}} = \frac{n^2}{\omega_-}, \eta < 0 = \frac{n^2}{\omega_+} (\omega_+^2 + \omega_-^2) - \frac{m^2}{2\omega_+^2} (\omega_+^2 - \omega_-^2) \cos(2\omega_+ t), \eta > 0
\]

\[
<T_{00}>_{\text{in out}} = \omega_-, \eta < 0 = \omega_+ + 2\omega_+ < N_{\text{out}}>, \eta > 0
\]

\[
<T_{11}>_{\text{in out}} = \frac{n^2}{\omega_-} - \frac{(\omega_-^2 - n^2)}{\omega_-} \frac{\omega_- - \omega_+}{\omega_+ + \omega_-} e^{2i\omega_- \eta}, \eta < 0
\]

\[
= \frac{n^2}{\omega_+} - \frac{(\omega_+^2 - n^2)}{\omega_+} \frac{\omega_+ - \omega_-}{\omega_+ + \omega_-} e^{-2i\omega_+ \eta}, \eta > 0
\]

(5.13)

Here \( < N_{\text{out}} > = (\omega_+ - \omega_-)^2/4\omega_+ \omega_- \) is the number of ‘out’ particles in the ‘in’ vacuum.

We see that \( < T_{00}>_{\text{in in}} \) has a direct physical interpretation. For \( \eta < 0 \) (before expansion) we are in the ‘in’ vacuum state, and we just get the vacuum energy \( \omega_-/2 \) for each of the two modes \( n, -n \) considered here. For \( \eta > 0 \) (after expansion) we get the vacuum energy \( \omega_+/2 \) for each mode, plus the energy of created ‘out’ particles.

\( < T_{00}>_{\text{in out}} \) on the other hand gives only the vacuum energies; there is no contribution from created particles. Worse, \( < T_{11}>_{\text{in out}} \) is complex, and so cannot be on the RHS of Einstein’s equation for the gravitational field.

This result is not surprising; we know that in the ‘in-out’ formalism the effective field equation for a real field can become complex [11]. The ‘in-in’ formalism, by contrast, is causal and yields real expectation values for self-adjoint operators. Thus it appears essential to extend the first quantised path integral prescription for strings so that we can get the 2x2 matrix propagators of the real time formalism. Extending the discussion of
section 3 to the string case we would obtain a sum over both signs of the zweibein on the world sheet (instead of just a sum over metrics.) The amplitude to flip sign of the zweibein would encode a density matrix which describes ‘exponential of quadratic’ string distributions (initially present or ‘created’.) Because the string is two dimensional, there is an interesting variety of ways that the zweibein can change sign, but for the purposes of the present paper we can just assume that the string is expanded into particle modes, in which case the above approach for particles applies.

Note: Leblanc [25] studied the real time formalism for open and closed strings, for the case of constant temperature in flat space. The propagator was computed in the ‘thermo-field dynamics’ language, which used the Niemi-Semenoff time path, and so was given by (3.2), (3.3). For this time-independent situation amplitudes were computed and the Hagedorn temperature recovered. To extend the string calculations to arbitrary spacetime geometry, as required in our approach, needs further progress in defining and computing amplitudes in Minkowski signature spacetimes. We will discuss this issue in more detail elsewhere.

6. Discussion.

Let us summarise the arguments and results of this paper. Requiring consistent propagation for the first quantised string should determine the background geometry. Do the background field equations incorporate the stress tensor of ‘created strings’ in time dependent geometries? To investigate this question we studied a simpler theory in first quantised language: the relativistic scalar particle. From the example of strings at constant temperature, we expect that backreaction from particle fluxes will show up at one string loop level in the $\beta$-function calculation. But this one loop contribution computes the stress tensor as two derivatives of the propagator, and so the backreaction seen by the $\beta$-function calculation depends on the choice of propagator. The Polyakov prescription of summing over target space co-ordinates $X^\mu$ and world sheet metrics gives an ‘in-out’ vacuum propagator. Using the scalar field analogy, we then argue that the gravity field equations obtained at one loop level will have an object like $\text{out} < 0 | T_{\mu\nu} | 0 > \text{in} / \text{out} < 0 | 0 > \text{in}$ for the backreaction. We demonstrated by a simple example that such quantities are in general not real, and not the correct ones to have in the field equation. We need instead a ‘true expectation value’ of the stress tensor, which will be achieved if we have a causal ‘in-in’ propagator, instead of the ‘in-out’ propagator.
In achieving a causal formulation of perturbation theory we need to extend the propagator to a $2\times2$ matrix of propagators \([11]\). This closed time path formulation (called the real time formulation in non-equilibrium many body theory) also naturally handles perturbation theory in the presence of particle fluxes. In fact the perturbation scheme is built around precisely the same kind of fluxes (exponential of quadratic) which are obtained in particle creation effects.

Accepting the necessity of a real time formalism for strings, we are faced with the question: can this formalism be obtained in a natural way in a first quantised language? We start with the square root action of the scalar particle, and try to reach the quadratic form that is the analogue of the Polyakov action. We find that the proper time in the latter description does not necessarily go forward. We need to supply the amplitudes for any state $|\psi>$ on the world line to connect to any other state $|\psi'>$ when the proper time reverses direction. Setting all these amplitudes to zero is a special choice that forbids reversal of proper time; choosing the forward direction of this time reproduces the ‘in-out’ vacuum propagator.

But it is also possible to set these amplitudes to be nonzero. With a particular choice of the amplitudes we found the matrix propagator for perturbation theory in a thermal bath. With another choice we found the causal ‘in-in’ propagator for a Universe which is in the ‘in’ vacuum but where expansion creates a bath of ‘out’ particles. In both these cases the real time description involved a density matrix of the kind (2.25). For the thermal bath we have $\alpha = \gamma = 0$, $\beta$ finite. For the ‘created particles’ case we had a limit for the coefficients in which a pure state density matrix was obtained.

For strings, the passage from the Nambu-Goto action to a quadratic action would bring in two signs of world sheet zweibien; restricting to one sign would give the Polyakov amplitude. As remarked at the end of section 3.1, only infinitely long world lines contribute to the orientation flip amplitude of the world line. The analogue of this for the string case would be that the effect of $\rho$ is felt only through the boundary of the moduli space of Riemann surfaces, where a homologically trivial or non-trivial cycle is pinched. A $\beta$-function calculation for the string world sheet theory would have to take into account such pinches while considering the small handle contribution studied by Fishler and Susskind. Such a calculation would yield a relation between the classical fields and the particle fluxes, rather than just among the classical fields giving the background. (The constant temperature case quoted in section 5 is an example of this.) The classical field strength describes a coherent state; the flux describes an ‘exponential of quadratic’ distribution of...
particles. The natural way in which field and fluxes have been mixed in our formalism leads us to expect that in a theory of gravity ‘effective equations’ of the quantum theory should have both these components, instead of having field values alone.

It would be interesting to obtain a non-perturbative treatment (like a matrix model approach) for a situation like an evaporating black hole where the backreaction from a flux is important in determining the evolution of the geometry. Based on the discussion of this paper, it appears that for this problem we should be looking not for a conformal theory but an ‘extended conformal theory’ (with both orientations of the world sheet allowed). The conformal invariance condition would set up a relation between the coherent field (contributing to the $\beta$-function from tree level onwards) and the orientation reversing amplitude representing ‘string flux’ (which contributes to the $\beta$-function from one string loop onwards). The conformal theory describes just one string, and so should be an inadequate description of this situation.

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Appendix A. Wick theorem for ‘exponential of quadratic’ density matrices.

We wish to establish Wick’s theorem for density matrices of the form

\[ \rho = e^{\gamma a^a} e^{\alpha a^\dagger} e^{-\beta a^\dagger a} \tag{A.1} \]

A string of creation and annihilation operators can be brought to normal ordered form in the same way as for the usual Wick theorem in the vacuum. What we need to show in addition is that

\[ \frac{1}{\text{Tr}\rho} \text{Tr}\{ \rho a^\dagger \ldots a^\dagger a \ldots a \} = \sum \frac{\text{Tr}\{ \rho a^\dagger a \}}{\text{Tr}\rho} \ldots \frac{\text{Tr}\{ \rho a a \}}{\text{Tr}\rho} \tag{A.2} \]

where the RHS has a summation over all possible pairings of the \(a^\dagger, a\) operators on the LHS. We sketch below some of the steps involved in the derivation.

Note

\[ e^{\alpha a^a} e^{\beta a^\dagger a} = e^{-\beta a^\dagger a} e^{\alpha' a^\dagger a^\dagger} \tag{A.3} \]

with \(\alpha' = \alpha e^{2\beta}\). A straightforward calculation gives

\[ \text{Tr}\rho = (1 - e^{-\beta})^{-1}[1 - \frac{4\alpha\gamma}{(1 - e^{-\beta})^2}]^{-1/2} \tag{A.4} \]

which we may rewrite as

\[ \text{Tr}\rho = \text{Tr}\{e^{\gamma a^a} e^{-\beta a^\dagger a} e^{\alpha' a^\dagger a^\dagger} \} = (1 - e^{-\beta})^{-1}[1 - \frac{4\alpha'\gamma}{(1 - e^{-\beta})^2}]^{-1/2} \tag{A.5} \]

From (A.1) we see that in (A.2) there must be either an even number \((2p)\) of ‘\(a\)’ oscillators and an even number \((2q)\) of ‘\(a^\dagger\)’ oscillators, or an odd number \((2p + 1)\) of ‘\(a\)’ and an odd number \((2q + 1)\) of ‘\(a^\dagger\)’ oscillators. Assume first that we have the former case. Then the LHS of (A.2) is obtained as

\[ \text{Tr}\{\rho(a^\dagger)^{(2q)}(a)^{2p} \} = \frac{1}{\text{Tr}\rho} (\partial_{\alpha'})^q (\partial_{\gamma})^p \text{Tr}\rho \tag{A.6} \]

(Partial derivatives are taken with \(\alpha', \beta, \gamma\) as independent variables, unless otherwise mentioned.) In particular,

\[ <a^\dagger a^\dagger> = \frac{1}{\text{Tr}\rho} \partial_{\alpha'} \text{Tr}\rho = 2\gamma e^{-2\beta}/K \equiv A \]

\[ <a a> = \frac{1}{\text{Tr}\rho} \partial_{\gamma} \text{Tr}\rho = 2\alpha' e^{-2\beta}/K \equiv B \tag{A.7} \]

\[ <a^\dagger a> = -\frac{1}{\text{Tr}\rho} \partial_{\beta}[\text{Tr}\rho]_{\alpha,\gamma} = e^{-\beta}(1 - e^{-\beta})/K \equiv C \]
where in computing $C$, $\partial_\beta$ is a partial derivative with $\alpha, \gamma$ held fixed. Here

$$K = (1 - e^{-\beta})^2 - 4\alpha'\gamma e^{-2\beta} \quad \text{(A.8)}$$

We find

$$\text{Tr}\rho = e^\beta[C^2 - AB]^{1/2} = K^{-1/2} \quad \text{(A.9)}$$

For fixed $\beta$

$$\partial_\alpha' A = 2A^2 \quad \partial_\gamma A = 2C^2$$

$$\partial_\alpha' B = 2C^2 \quad \partial_\gamma B = 2B^2 \quad \text{(A.10)}$$

Using the above formulae, we can establish (A.2) by induction. Suppose (A.2) holds with $2p$ operators ‘$a$’ and $2q$ operators ‘$a^\dagger$’. A typical term on the RHS would have the form $FA^{n_1}B^{n_2}C^{n_3}$, where $F$ is a constant and $n_1, n_2, n_3 \geq 0$. To establish the result for $2p$ operators ‘$a$’ and $2q + 2$ operators ‘$a^\dagger$’ we get for the LHS of (A.2):

$$e^\beta[C^2 - AB]^{1/2} \partial'_\alpha[e^\beta(C^2 - AB)^{1/2}FA^{n_1}B^{n_2}C^{n_3}] = F[A^{n_1+1}B^{n_2}C^{n_3} + 2n_1A^{n_1+1}B^{n_2}C^{n_3} + 2n_2A^{n_1}B^{n_2-1}C^{n_3} + 2n_3A^{n_1+1}B^{n_2}C^{n_3}] \quad \text{(A.11)}$$

The first term on the RHS of (A.11) gives the pairing of the two new operators $a^\dagger$ with each other. The second term gives the $n_1$ ways to choose an existing pair $(a^\dagger a^\dagger)$ and to contract the new $a^\dagger$ operators with members of this pair instead. The third term corresponds to choosing an $(aa)$ pair in the original expression and contracting the ‘$a$’ operators with the new ‘$a^\dagger$’ operators instead. The last term corresponds to exchanging the $a^\dagger$ in an existing $a^\dagger a$ pair with one of the new $a^\dagger$ operators. It is easily seen that this generates all the new terms required on the RHS of (A.2) for the induction to hold.

To work with the case of an odd number of $a$ and $a^\dagger$ operators we start with the expression $C\text{Tr}\rho = \text{Tr}\{e^{\gamma aa}e^{-\beta a^\dagger a}e^{a'a^\dagger} a^\dagger a\}$ in place of $\text{Tr}\rho$, and proceed as above to introduce extra $a^\dagger a^\dagger$ and $aa$ pairs in the induction.
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