On the Divisibility of Character Values of the Symmetric Group

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Abstract. Fix a partition \( \mu = (\mu_1, \ldots, \mu_m) \) of an integer \( k \) and positive integer \( d \). For each \( n > k \), let \( \chi_\mu^\lambda \) denote the value of the irreducible character of \( S_n \) at a permutation with cycle type \((\mu_1, \ldots, \mu_m, 1^{n-k})\). We show that the proportion of partitions \( \lambda \) of \( n \) such that \( \chi_\mu^\lambda \) is divisible by \( d \) approaches 1 as \( n \) approaches infinity.

Let \( k \) be a positive integer, and \( \mu = (\mu_1, \ldots, \mu_m) \) a partition of \( k \). For a partition \( \lambda \) of an integer \( n \geq k \), let \( \chi_\mu^\lambda \) denote the value of the irreducible character of \( S_n \) corresponding to \( \lambda \) at an element with cycle type \((\mu_1, \ldots, \mu_m, 1^{n-k})\). The purpose of this article is to prove:

Main Theorem. For any positive integers \( k \) and \( d \), and any partition \( \mu \) of \( k \),

\[
\lim_{n \to \infty} \frac{\#\{\lambda \vdash n \mid \chi_\mu^\lambda \text{ is divisible by } d\}}{p(n)} = 1.
\]

Here \( p(n) \) denotes the number of partitions of \( n \).

In particular, for any integer \( d \), the probability that an irreducible character of \( S_n \) has degree divisible by \( d \) converges to 1 as \( n \to \infty \).

Recall the theorem of Lassalle [4, Theorem 6], which implies that there exists an integer \( A_\mu^\lambda \) such that

\[
\chi_\mu^\lambda = \frac{f_\lambda}{(n)_k} A_\mu^\lambda.
\]

Here \( (n)_k = n(n-1) \cdots (n-k+1) \), and \( f_\lambda \) is the degree of the irreducible character of \( S_n \) corresponding to \( \lambda \). Therefore, in order to prove the
main theorem, we focus on the divisibility properties of $f_\lambda$. For each prime number $q$, let $v_q(m)$ denote the $q$-adic valuation of an integer $m$, in other words, $q^{v_q(m)}$ is the largest power of $q$ that divides $m$. Also write $\log n = \log_q n$. The main theorem will follow from the following result:

**Theorem A.** For every prime number $q$ and non-negative integer $m$,  
$$\lim_{n \to \infty} \frac{\# \{ \lambda \vdash n \mid v_q(f_\lambda) \leq m + (q-1)\log n \}}{p(n)} = 0.$$  

In the rest of this article, we first prove Theorem A, and then show that it implies the main theorem.

1. **Proof of Theorem A**

The proof of Theorem A is based on the theory of $q$-core towers. This construction originated in the seminal paper [5] of Macdonald, and was developed further by Olsson in [6]. We now recall the relevant aspects.

Let $[q]$ denote the set $\{0, \ldots, q-1\}$, and consider the disjoint union  
$$T_q = \coprod_{i=0}^{\infty} [q]^i.$$  
The set $T_q$ can be regarded as a rooted $q$-ary tree with root $\emptyset$. The children of a vertex $(a_1, \ldots, a_i) \in [q]^i$ are the vertices $(a_1, \ldots, a_i, a_{i+1})$, where $a_{i+1} \in [q]$. A partition $\lambda$ is said to be a $q$-core if no cell in its Young diagram has hook length divisible by $q$. Denote the set of all $q$-core partitions by $C_q$. The $q$-core tower construction associates to each partition $\lambda$ of $n$ a function $T^\lambda_q : T_q \to C_q$ known as the $q$-core tower of $\lambda$. For a partition $\lambda$, define:

$$w_i(\lambda) = \sum_{x \in [q]^i} \card{T^\lambda_q(x)}.$$  

Then the $q$-core tower satisfies the following constraint:

$$\sum_{i=0}^{\infty} w_i(\lambda)q^i = n. \quad (2)$$  

In particular, $T^\lambda_q(x) = \emptyset$ for all $i > \log_q n$. This function $\lambda \mapsto T^\lambda_q$ is a bijection from the set of partitions of $n$ onto the set of $q$-core towers satisfying the condition (2).

Let $n$ be a positive integer with $q$-ary expansion:  
\begin{equation*}  
n = a_0 + a_1q + \cdots + a_rq^r,  
\end{equation*}  
with $a_i \in [q]$ for $i = 1, \ldots, r$, and $a_r > 0$. 

Define \( a(n) = \sum_{i=0}^{r} a_i \).

Recall the following Theorem:

**Theorem 1 ([5 Equation (3.3)])**. For a partition \( \lambda \), let \( w(\lambda) = \sum_{i=0}^{r} w_i(\lambda) \). For any partition \( \lambda \) of \( n \) and any prime \( q \),

\[
v_q(f_\lambda) = \frac{w(\lambda) - a(n)}{q - 1}.
\]

Theorem [1] can be used to find constraints on partitions with small values of \( v_q(f_\lambda) \). Suppose that \( v_q(f_\lambda) \leq b \). By Theorem [1] this is equivalent to

\[
w(\lambda) \leq a(n) + b(q - 1).
\]

The expansion (*) implies that \( r \leq \log n < r + 1 \), so that \( a(n) \leq (q - 1)(r + 1) \leq (q - 1)(\log n + 1) \). So if \( v_q(f_\lambda) \leq b \), then

\[
w(\lambda) \leq (q - 1)(\log n + 1 + b).
\]

Thus an upper bound for the number \( p_b(n) \) of partitions \( \lambda \) of \( n \) such that \( v_q(f_\lambda) \leq b \) can be obtained by counting the number of \( q \)-core towers with \( (q - 1)(\log n + 1 + b) \) or fewer cells. The total number of vertices in the first \( r + 1 \) rows of \( T_q \), i.e., in \( \coprod_{i=0}^{r}[q]^i \), is:

\[
1 + q + \cdots + q^r = \frac{q^{r+1} - 1}{q - 1} < qn,
\]

since \( q^r \leq n \). Let \( c_q(n) \) denote the number of \( q \)-core partitions of \( n \). Set \( N_b = (q - 1)(\log n + b + 1) \). Let \( \tilde{c}_q(n) \) denote \( \max\{c_q(i) \mid 1 \leq i \leq n\} \). There are \( (w + N - 1) \) ways to distribute \( w \) cells into \( N \) nodes. Thus

\[
p_b(n) \leq \tilde{c}_q(N_b)^N_b \left( \frac{qn + N_b}{N_b} \right)
\]

\[
\leq \tilde{c}_q(N_b)^N_b (qn + N_b)^N_b
\]

It is known that, for every integer \( q \), there exists a polynomial \( f_q(n) \) such that \( \tilde{c}_q(n) \leq f_q(n) \) for all \( n \geq 0 \). Indeed, for \( q = 2 \), it is well-known that \( c_2(n) \leq 1 \), and for \( q = 3 \), using a formula of Granville and Ono [2 Section 3, p. 340], \( c_3(n) \leq 3n + 1 \). For \( q \geq 4 \), the existence of \( f_q(n) \) follows from Anderson [1 Corollary 7].

We get:

\[
p_b(n) \leq f_q(N_b)^N_b (qn + N_b)^N_b,
\]

whence

\[
\log p_b(n) \leq N_b[\log f_q(N_b) + \log(qn + N_b)].
\]
Taking $b = m + (q - 1) \log n$ gives $N_b = (q - 1)(q \log n + m + 1)$. Thus $\log p_b(n) = o(n^\epsilon)$ for every $\epsilon > 0$. On the other hand, the Hardy-Ramanujan asymptote [3] for $p(n)$ implies that $\log p(n)$ grows faster than $n^{2-\epsilon}$ for any $\epsilon > 0$. Thus Theorem A follows.

2. Proof of the Main Theorem

The identity (1) implies that

$$v_q(\chi^\lambda_{\mu}) \geq v_q(f_\lambda) - v_q((n)_k).$$

Using Legendre’s formula on the valuation of a factorial, that $v_q(n!) = \frac{n-a(n)}{q-1}$, we have:

$$v_q((n)_k) = v_q\left(\frac{n!}{(n-k)!}\right) = k + \frac{a(n-k) - a(n)}{q-1} \leq k + (q - 1) \log n.$$

Hence if $v_q(f_\lambda) \geq m + (q - 1) \log n$, then $v_q(\chi^\lambda_{\mu}) \geq (m - k)$. Thus taking $m = k + b$ in Theorem A tells us that

$$\lim_{n \to \infty} \frac{\#\{\lambda \vdash n \mid v_p(\chi^\lambda_{\mu}) \leq b\}}{p(n)} = 0.$$

From this the main theorem follows.

References

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