Recognition and Complexity of Point Visibility Graphs

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Abstract A point visibility graph is a graph induced by a set of points in the plane, where every vertex corresponds to a point, and two vertices are adjacent whenever the two corresponding points are visible from each other, that is, the open segment between them does not contain any other point of the set. We study the recognition problem for point visibility graphs: Given a simple undirected graph, decide whether it is the visibility graph of some point set in the plane. We show that the problem is complete for the existential theory of the reals. Hence the problem is as hard as deciding the existence of a real solution to a system of polynomial inequalities. The proof involves simple substructures forcing collinearities in all realizations of some visibility graphs, which are applied to the algebraic universality constructions of Mnëv and Richter-Gebert. This solves a longstanding open question and paves the way for the analysis of other classes of visibility graphs. Furthermore, as a corollary of one of our construction, we show that there exist point visibility graphs that do not admit any geometric realization with points on a grid. We also prove that the problem of recognizing visibility graphs of points on a grid is decidable if and only if the existential theory of the rationals is decidable.

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1 Introduction

Visibility between geometric objects is a cornerstone notion in discrete and computational geometry, that appeared as soon as the late 1960s in pioneering experiments in robotics [18]. Visibility is involved in major themes that helped shape the field, such as art gallery and motion planning problems [6,8,22]. However, despite decades of research on those topics, the combinatorial structures induced by visibility relations in the plane are far from understood.

Among such structures, visibility graphs are arguably the most natural. In general, a visibility graph encodes the binary, symmetric visibility relation among sets of objects in the plane, where two objects are visible from each other whenever there exists a straight line of sight between them that does not meet any obstacle. More precisely, a point visibility graph associated with a set \( P \) of points in the plane is a simple undirected graph \( G = (P,E) \) such that two points of \( P \) are adjacent if and only if the open segment between them does not contain any other point of \( P \). Note that the points play both the roles of vertices of the graph and obstacles. In what follows, we will use the abbreviation PVG for point visibility graph.

We consider the recognition problem for point visibility graphs: Given a simple undirected graph \( G = (V,E) \), does there exists a point set \( P \) such that \( G \) is isomorphic to the visibility graph of \( P \)? More concisely, the problem consists of deciding the property of being a point visibility graph of some point set. As is often the case for geometric graphs, the recognition problem appears to be intractable under usual complexity-theoretic assumptions. In fact, recently, Roy gave an NP-hardness proof for this problem [26].

1.1 Our Results

We characterize the problem as complete for the existential theory of the reals; hence recognizing point visibility graphs is as hard as deciding the existence of a solution to an arbitrary system of polynomial inequalities over the reals. Equivalently, this amounts to deciding the emptiness of a semialgebraic set. This complexity class is intimately related to fundamental results on oriented matroids and pseudoline arrangements starting with the insights of Mnëv on the algebraic universality properties of these structures [21]. The notation \( \exists \mathbb{R} \) has been proposed recently by Schaefer [27] to refer to this class, motivated by the continuously expanding collection of problems in computational geometry that are identified as complete for it.

The only known inclusion relations for \( \exists \mathbb{R} \) are \( \text{NP} \subseteq \exists \mathbb{R} \subseteq \text{PSPACE} \). It is known from the Tarski–Seidenberg Theorem that the first-order theory of real closed fields is decidable, but polynomial space algorithms for problems in \( \exists \mathbb{R} \) have been proposed only much more recently by Canny [5].

Whenever a graph is known to be a point visibility graph, the description of the point set as a collection of pairs of integer coordinates constitutes a natural certificate.
Since it is not known whether $\exists R \subseteq NP$, we should not expect such a certificate to have polynomial size. In fact, we show that there exist point visibility graphs all realizations of which have an irrational coordinate, and point visibility graphs that require doubly exponential coordinates in any realization. We also prove that recognizing visibility graphs of points on a grid is decidable if and only if the existential theory of the rationals is decidable. This establishes an interesting connection between a natural graph drawing problem and Hilbert’s tenth problem over the rationals.

1.2 Related Work and Connections

The recognition problem for point visibility graphs has been explicitly stated as an important open problem by various authors [15], and is listed as the first open problem in a recent survey from Ghosh and Goswami [9].

A linear-time recognition algorithm has been proposed by Ghosh and Roy for planar point visibility graphs [10]. For general point visibility graphs they showed that recognition problem lies in $\exists R$. More recently, Roy [26] published an ingenious and rather involved NP-hardness proof for recognition of arbitrary point visibility graphs. Our result clearly implies NP-hardness as well, and, in our opinion, has a more concise proof.

Structural aspects of point visibility graphs have been studied by Kára, Pór, and Wood [15], Pór and Wood [25], and Payne et al. [24]. Many fascinating open questions revolve around the big-line-big-clique conjecture, stating that for all $k, \ell \geq 2$, there exists an $n$ such that every finite set of at least $n$ points in the plane contains either $k$ pairwise visible points or $\ell$ collinear points.

Visibility graphs of polygons are defined over the vertices of an arbitrary simple polygon in the plane, and connect pairs of vertices such that the open segment between them is completely contained in the interior of the polygon. This definition has also attracted a lot of interest in the past twenty years. Ghosh gave simple properties of visibility graphs of polygons and conjectured that they were sufficient to characterize visibility graphs [7]. These conjectures have been disproved by Streinu [31] via the notion of pseudo-visibility graphs, or visibility graphs of pseudo-polygons [23]. A similar definition is given by Abello and Kumar [1]. Roughly speaking, the relation between visibility and pseudo-visibility graphs is of the same nature as that between arrangements of straight lines and pseudolines. Although, as Abello and Kumar remark, these results somehow suggest that the difficulty in the recognition task is due to a stretchability problem, the complexity of recognizing visibility graphs of polygons remains open, and it is not clear whether the techniques described in this paper can help characterizing it. The influential surveys and contributions of Schaefer about $\exists R$-complete problems in computational geometry form an ideal point of entry in the field [27,28]. Among such problems, let us mention recognition of segment intersection graphs [16], recognition of unit distance graphs and realizability of linkages [14,28], recognition of disk and unit disk intersection graphs [20], computing the rectilinear crossing number of a graph [3], simultaneous geometric graph embedding [17], and recognition of $d$-dimensional Delaunay triangulations [2].
1.3 Outline of the Paper

In Sect. 2, we provide a simple visibility graph construction, a fan, all geometric realizations of which are guaranteed to preserve a specified collection of subsets of collinear points. The proofs are elementary and only require a series of basic observations.

The main result of the paper is given in Sect. 3. We first recall the main notions and tools used in the results from Mnëv [21] and Shor [29] for showing that realizability of abstract order types is complete for the existential theory of the reals. We then combine these tools with the fan construction to produce families of point visibility graphs that can simulate arbitrary arithmetic computations over the reals.

In Sect. 5, we give two applications of the fan construction. In the first, we show that there exists a point visibility graph that does not have any geometric realization on the integer grid. In other words, all geometric realizations of this point visibility graph are such that at least one of the points has an irrational coordinate. Another application of the fan construction follows, where we show that there are point visibility graphs every grid realization of which requires coordinates of values $2^{\sqrt{n}}$ where $n$ denotes the number of vertices of the PVG. Afterwards, we show that the recognition of visibility graphs of points on a grid (or, equivalently, with integer coordinates) might be undecidable by reducing from the solvability of a system of polynomial (in)equalities over the rationals.

1.4 Notation

For the sake of simplicity, we slightly abuse notation and do not distinguish between a vertex of a point visibility graph and its corresponding point in a geometric realization. We denote by $G[P']$ the induced subgraph of a graph $G = (P, E)$ with the vertex set $P' \subseteq P$. For a point visibility realization $R$ we denote by $R[P']$ the induced subrealization containing only the points $P'$. The PVG of this subrealization is in general not an induced subgraph of $G$. By $N(p)$ we denote the open neighborhood of a vertex $p$.

The line through two points $p$ and $q$ is denoted by $\ell(p, q)$ and the open segment between $p$ and $q$ by $\overline{pq}$. We will often call $\overline{pq}$ the sightline between $p$ and $q$, since $p$ and $q$ see each other iff $\overline{pq} \cap P = \emptyset$. We call two sightlines $\overline{p_1 q_1}$ and $\overline{p_2 q_2}$ non-crossing if $\overline{p_1 q_1} \cap \overline{p_2 q_2} = \emptyset$.

For each point $p$ all other points of $G$ lie on $\deg(p)$ many rays $R_1^p, \ldots, R_{\deg(p)}^p$ originating from $p$.

2 Point Visibility Graphs Preserving Collinearities

We first describe constructions of point visibility graphs, all the geometric realizations of which preserve some fixed subsets of collinear points.
2.1 Preliminary Observations

In the realization of a PVG, the point $p$ sees exactly $\text{deg}(p)$ many vertices, hence all other points lie on $\text{deg}(p)$ rays of origin $p$.

**Lemma 1** Let $q \in N(p)$ be a degree-one vertex in $G[N(p)]$. Then all points lie on one side of the line $\ell(p,q)$.

**Proof** If the angle between two consecutive rays is smaller than $\pi$, then every vertex on one ray sees every vertex on the other ray. Hence one of the angles between two rays of origin $p$ must be at least $\pi$ (see Fig. 1). $\square$

**Corollary 1** If $G[N(p)]$ is an induced path, then the order of the path and the order of the rays coincide.

**Proof** By Lemma 1 the two endpoints of the path lie on rays on the boundary of empty halfspaces. Thus all other rays form angles which are smaller than $\pi$, and thus they see their two neighbors of the path on their neighboring rays. $\square$

**Observation 1** Let $q, q \neq p$, be a point that sees all points of $N(p)$. Then $q$ is the second point (not including $p$) on one of the rays emerging from $p$.

**Proof** Assume $q$ is not the second point on one of the rays. Then $q$ cannot see the first point on its ray which is a neighbor of $p$. $\square$

This also yields the following observation.

**Observation 2** Let $q, q \neq p$, be a point that is not the second point on one of the rays from $p$ and sees all but one of the neighbors of $p$, say $r$. Then $q$ lies on the ray of $r$.

2.2 Fans

We have enough tools by now to show the uniqueness of a PVG obtained from the following construction, which is depicted in Fig. 2. Consider a set $S$ of segments between two lines $\ell$ and $\ell'$ intersecting in a point $p$, such that each endpoint of a segment lies on $\ell, \ell'$ or another segment. For each intersection of a pair of segments, construct a ray of origin $p$ and going through this intersection point. Add two segments $s_1$ and $s_2$ between $\ell$ and $\ell'$, such that $s_1$ is the closest and $s_2$ is the second closest segments to $p$. 

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We now put a point on each intersection of the segments and rays and construct the PVG of this set of points. We call this graph the fan of S and denote it by fan(S). Since we have the choice of the position of the segments $s_1$ and $s_2$ we can avoid any collinearity between a point on $s_1$ or $s_2$ and points on other segments, except for the obvious collinearities on one ray. Thus every point sees all points on $s_1$ except for the one of the ray it lies on.

Lemma 2 All realizations of a fan preserve collinearities between points that lie on one segment and between points that lie on one ray.

Proof We first show that the distribution of the points onto the rays of $p$ is unique. By construction, the points on $s_2$ see all the points on $s_1$, which are exactly the neighbors of $p$. Thus by Observation 1 the points from $s_2$ are the second points of a ray. Since there is exactly one point for each ray on $s_2$, all the other points are not second points on a ray. By construction, each of the remaining points see all but one point of $s_1$. Observation 2 gives a unique ray a point lies on. The order of the rays is unique by Corollary 1. On each ray the order of the points is as constructed, since the PVG of points on one ray is an induced path.

Now we have to show that the points originating from one segment are still collinear. Consider three consecutive rays $R_1$, $R_2$, $R_3$. We consider a visibility between a point $p_1$ on $R_1$ and one point $p_3$ on $R_3$ that has to be blocked by a point on $R_2$. Let $p_2$ be the original blocker from the construction.

For each point $q$ on $R_2$ that lies closer to $p$ than $p_2$ there is a sightline blocked by $q$, and for each point $q$ that lies further away from $p$ than $p_2$ there is a sightline blocked by $q$. For each point $q$ on $R_2$ we pick one sightline between a point on $R_1$ and another point on $R_3$ that is blocked by $q$. This set of sightlines is non-crossing, since the segments only intersect on rays, hence on $R_2$. So we have a set of non-crossing sightlines and the same number of blockers available. Since the order on each ray is fixed, and the sightlines intersect $R_2$ in a certain order, the blocker for each sightline is uniquely determined and has to be the original blocker. By transitivity of collinearity all points from the segments remain collinear over all the rays. \(\square\)

3 $\exists \mathbb{R}$-Completeness Reductions and Universality

The existential theory of the reals ($\exists \mathbb{R}$) is a complexity class defined by the following complete problem. We are given a well-formed quantifier-free formula $F(x_1, \ldots, x_k)$
using the numbers 0 and 1, addition and multiplication operations, strict and non-strict comparison operators, Boolean operators, and the variables \( x_1, \ldots, x_k \), and we are asked whether there exists an assignment of real values to \( x_1, \ldots, x_k \), such that \( F \) is satisfied. This amounts to deciding whether a system of polynomial inequalities admits a solution over the reals. The first main result connecting this complexity class to a geometric problem is the celebrated result of Mnëv, who showed that realizability of order types, or—in the dual—stretchability of pseudoline arrangements, is complete in this complexity class [21]. In what follows, we use the simplified reduction due to Shor [29].

The orientation of an ordered triple of points \((p, q, r)\) indicates whether the three points form a clockwise or a counterclockwise cycle, or whether the three points are collinear. Let \( P = \{p_1, \ldots, p_n\} \) and an orientation \( O \) of each triple of points in \( P \) be given. The pair \((P, O)\) is called an (abstract) order type. We say that the order type \((P, O)\) is realizable if there are coordinates in the plane for the points of \( P \), such that the orientations of the triples of points match those prescribed by \( O \).

In order to reduce the order type realizability problem to solvability of a system of strict polynomial inequalities, we have to be able to simulate arithmetic operations with order types. This uses standard constructions introduced by von Staudt in his “algebra of throws” [30].

### 3.1 Arithmetics with Order Types

To carry out arithmetic operations using orientation predicates, we associate numbers with points on a line, and use the cross-ratio to encode their values.

The cross ratio \((a, b; c, d)\) of four points \(a, b, c, d \in \mathbb{R}^2\) is defined as

\[
(a, b; c, d) := \frac{|a, c| \cdot |b, d|}{|a, d| \cdot |b, c|},
\]

where \(|x, y|\) is the determinant of the matrix obtained by writing the two vectors as columns. The two properties that are useful for our purpose is that the cross-ratio is invariant under projective transformations, and that for four points on one line, the cross-ratio is given by \(\frac{\overrightarrow{ac} \cdot \overrightarrow{bd}}{\overrightarrow{ad} \cdot \overrightarrow{bc}}\), where \(\overrightarrow{xy}\) denotes the oriented distance between \(x\) and \(y\) on the line.

We will use the cross-ratio the following way: We fix two points on a line and call them 0 and 1. On the line through those points we call the point at infinity \(\infty\). For a point \(a\) on this line the cross-ratio \(x := (a, 1; 0, \infty)\) results in the distance between 0 and \(a\) scaled by the distance between 0 and 1. Because the cross-ratio is a projective invariant we can fix one line and use the point \(a\) for representing the value \(x\). In this way, we have established the coordinates on one line.

For computing on this line, the gadgets for addition and multiplication depicted in Fig. 3 can be used. Let us detail the case of multiplication. We are given the points \(0 < 1 < x < y < \infty\) on the line \(\ell\), and wish to construct a point on \(\ell\) that represents the value \(x \cdot y\). Take a second line \(\ell_\infty\) that intersects \(\ell\) in \(\infty\), and two points \(a, b\) on this line. Construct the segments \(\overrightarrow{by}, \overrightarrow{b1}\) and \(\overrightarrow{ax}\). Denote the intersection point of \(\overrightarrow{ax}\) and \(\overrightarrow{b1}\) by \(c\).
Call $d$ the intersection point of $\overline{by}$ and $\ell(0, c)$. The intersection point of $\ell$ and $\ell(d, a)$ represents the point $x \cdot y =: z$ on $\ell$, i.e., $(z, 1; 0, \infty) = (x, 1; 0, \infty) \cdot (y, 1; 0, \infty)$. In a projective realization of the gadget in which the line $\ell_{\infty}$ is indeed the line at infinity, the result can be obtained by applying twice the intercept theorem, in the triangles with vertices $0, d, y$ and $0, d, z$, respectively. To add the cross ratios of two points on a line, a similar construction is given in Fig. 3.

### 3.2 The Reduction for Order Types

Using the constructions above we can already model a system of strict polynomial inequalities. However, it is not clear how we can determine the complete order type of the points without knowing the solution of the system. Circumventing this obstacle was the main achievement of Mnëv [21]. We cite one of the main theorems in a simplified version.

**Theorem 1** [29] Every primary semialgebraic set $V \subseteq \mathbb{R}^d$ is stably equivalent to a semialgebraic set $V' \subseteq \mathbb{R}^n$, with $n = \text{poly}(d)$, for which all defining equations have the form $x_i + x_j = x_k$ or $x_i \cdot x_j = x_k$ for certain $1 \leq i \leq j < k \leq n$, where the variables $1 = x_1 < x_2 < \cdots < x_n$ are totally ordered.

A **primary semialgebraic set** is a set defined by polynomial equations and strict polynomial inequalities with coefficients in $\mathbb{Z}$. Although we cannot give a complete definition of **stable equivalence** within the context of this paper, let us just say that two semialgebraic sets $V$ and $V'$ are stably equivalent if one can be obtained from the other by rational transformations and so-called **stable projections**, and that stable equivalence implies **homotopy equivalence**. From the computational point of view, the important property is that $V$ is the empty set if and only $V'$ is, and that the size of the description of $V'$ in the theorem above is polynomial in the size of the description of $V$. In the proof of universality for PVGs we will only use that we construct PVGs that contain a subset of points whose order type is the one constructed by Shor, which has a certain wanted realization space. We call the description of a semialgebraic set $V'$ given in the theorem above the **Shor normal form**.

We can now encode the defining relations of a semialgebraic set given in Shor normal form using abstract order types by simply putting the points $0, 1, x_1, \ldots, x_n, \infty$.
in this order on $\ell$. To give a complete order type, the orientations of triples including the points of the gadgets and the positions of the gadget on $\ell_\infty$ have to be specified. This can be done exploiting the fact that the distances between the points $a_i$ and $b_i$ of each gadget and their position on $\ell_\infty$ can be chosen freely. We refer to the references mentioned above for further details. We next show how to implement these ideas to construct a graph $G_V$ associated with a primary semialgebraic set $V$, such that $G_V$ has a PVG realization if and only if $V \neq \emptyset$.

4 $\exists \mathbb{R}$-Completeness of PVG Recognition

The idea to show that PVG recognition is complete in $\exists \mathbb{R}$ is to encode the gadgets described in the previous section in a fan.

We therefore consider the gadgets not as a collection of points with given order types, but as an arrangement of segments. This arrangement can be fixed in a fan if the radial ordering around $p$, the origin of the fan, is known.

We will consider the addition and multiplication gadgets given in Fig. 3, and for a copy $g_i$ of the addition gadget, denote by $a_i, b_i, c_i, \text{ and } d_i$ the points corresponding to $g_i$, and similarly for the multiplication gadget. We describe how to place these points, such that we are able to describe the complete order type. In addition, we are able to fix the order of the $x$-coordinates of the intersection points of the arrangement, such that it does not restrict realizability. This allows us to place $p$ at $(0, -C)$ for a large number $C$, such that the order of the $x$-coordinates of intersection points agrees with the radial ordering around $p$.

Theorem 2 The recognition of point visibility graphs is $\exists \mathbb{R}$-complete.

Proof Given a primal semialgebraic set $V$, we construct a graph whose realization space as point visibility graph is stably equivalent to $V$. For a primary semialgebraic set $V$ we compute the Shor normal form and denote the corresponding primary semialgebra set by $V'$. For $V'$, we construct the order type that implements the calculations on the lines $\ell$ and $\ell_\infty$ using the construction of Shor [29]: We iteratively place the points $a_i, b_i$ of the gadgets on $\ell_\infty$ and the points $c_i$ and $d_i$ on a vertical segment $B$ that starts on $\ell$. The points of $g_i$ are placed closer to the intersection point of $B$ and $\ell_\infty$ at each step. This can be done in a way that allows us to determine a possible order of the $x$-coordinates of all intersection points of the segments constructed. First, using a projective transformation we can assume that $\ell_\infty$ is indeed the line at infinity. Then we place a new gadget “close” to the intersection point of $B$ and $\ell_\infty$. This corresponds to choosing segments with a higher absolute values for the slopes. We choose the slope of a new segment $s$ large enough, so that all intersection points with the segments constructed before lie in an interval just to the left of the point on $\ell$ that $s$ is originating from. This interval to the left is indicated by the gray box in Fig. 4 (left).

So it only remains to determine the relative $x$-position of the intersection points of segments within one gadget among all intersection points. The intersection points of the segments in a multiplication gadget are the closest points to the vertical segment $B$. This is achieved by constructing $c_i$ and $d_i$, the points on $B$, close enough together.
Moving $c_i$ towards $d_i$ leads to an intersection point lying on $B$ in the limit. By continuity, the intersection point can be placed close to $B$, see Fig. 4 (right).

The intersection point of segments in an addition gadget, that does not lie on $\ell$ or $\ell_{\infty}$, lies just to the left of the interval with the old segments that lie left of $y$. By construction of the gadget the intersection points of the vertical segment starting in $y$ has the same $x$-coordinate as $y$, see Fig. 5 (left). After constructing all gadgets we use a projective transformation to perturb the representation, such that the order of the different $x$-coordinates of intersection points is preserved. For points with the same $x$-coordinates that appear above $y$ in an addition gadget, this perturbation will place the points in an interval to the left of $y$ as shown in Fig. 5 (right).

Now we want to use the fan construction to construct a graph. Therefore, we place the point $p$, the origin of the fan, on the coordinates $(0, -C)$ for some large negative
C. If we choose $C$ large enough, then the order of all intersection points of segments around $p$ agrees with the $x$-coordinates.

Here we have the problem that collinearities between points that lie on different segments might occur.

What we can show this way is a *sandwich version* of the $\exists \mathbb{R}$-completeness: Let $H$ be the graph of the fan we obtain from this construction by assuming no collinearities appear between points that do not lie on a common segment or ray. Furthermore, we consider all possible combinations of collinearities that can appear in the construction and let $F$ be the intersection of all the graph obtained from those fans. Then there is a graph $G$ with $F \subseteq G \subseteq H$ that is a PVG if and only if $V$ is nonempty. The possibilities to choose the graph $G$ can be exponentially many. Hence for a polynomial reduction we have to determine one of these graphs which we denote by $G_V$, that is realizable as PVG in the case that $V$ is nonempty. We do this by showing that all unwanted collinearities can be avoided.

Therefore, we choose the positions of the point $a_i$ (and also $b_i$ in a multiplication gadget), such that all slopes of segments to $a_i$ and $b_i$ are algebraically independent of the coordinates of the points used so far. Therefore, we assume the points of the gadgets $g_1, \ldots, g_{i-1}$ are already placed. After placing the points of the gadget $g_i$, all newly created intersection points contain a part of these algebraically independent numbers. Thus the lines through two old points cannot go through a new point and vice versa.

Thus we can avoid all unwanted collinearities. We can construct the PVG realization of $G_V$ if and only if $V$ is nonempty, and if we obtain a PVG realization we know that the Now there exists a PVG of $G_V$ realization if $V$ and $V'$ are nonempty. The graph $G_V$ can clearly be constructed in polynomial time in the size of $V$, this PVG recognition is complete in the existential theory of the reals. □

**Remark 1** Note that since the construction contains a copy of the order type from Shor’s construction, a stronger Mnev-type universality result should hold, namely that the realization space of the visibility graph is stably equivalent to the primary semialgebraic set it is constructed from. We refer the reader to [4] for more details on the definitions involved.

### 5 Visibility Graphs of Points on a Grid

From the result of Canny [5], we know it is possible to recognize point visibility graphs in polynomial space. However, even when the answer to the decision problem is positive, it is not clear that the realization of the graph can be provided as a set of points with integer coordinates. In fact, we show in this section that there exist point visibility graphs that cannot be realized by points on a grid. We also show that there exist visibility graphs of points on a grid that require a doubly exponential grid size.

We then turn to the problem of recognizing visibility graphs of points on a grid, and show that the problem is decidable if and only if the existential theory of the rationals is decidable, a well-known, major open problem.
Theorem 3  There exists a point visibility graph every geometric realization of which has at least one point with one irrational coordinate.

Proof  We use the so-called Perles configuration of 9 points on 9 lines illustrated in Fig. 6. It is known that for every geometric realization of this configuration in the Euclidean plane, one of the points has an irrational number as one of its coordinate [13]. We combine this construction with the fan construction described in the previous section. Hence we pick two lines $\ell$ and $\ell'$ intersecting in a point $p$, such that all lines of the configuration intersect both $\ell$ and $\ell'$ in the same wedge. Note that up to a projective transformation, the point $p$ may be considered to be on the line at infinity and $\ell$ and $\ell'$ taken as parallel. We add two non-intersecting segments $s_1$ and $s_2$ close to $p$, that do not intersect any line of the configuration. We then shoot a ray from $p$ through each of the points, and construct the visibility graph of the original points together with all the intersections of the rays with the lines and the two segments $s_1$, $s_2$. From Lemma 2, all the collinearities of the original configuration are preserved, and every realization of the graph contains a copy of the Perles configuration.  

Also note that point visibility graphs that can be realized with rational coordinates do not necessarily admit a realization that can stored in polynomial space in the number of vertices of the graph. To support this, consider a line arrangement $\mathcal{A}$, and add a point $p$ in an unbounded face of the arrangement, such that all intersections of lines are visible in an angle around $p$ that is smaller than $\pi$. Construct rays $\ell$ and $\ell'$ through the extremal intersection points and $p$. From Lemma 2, the fan of this construction gives a PVG that fixes $\mathcal{A}$. Since there are line arrangements that require integer coordinates of values $2^{2\Theta(|\mathcal{A}|)}$ [11] and the fan has $\Theta(|\mathcal{A}|^3)$ points we get the following worst-case lower bound on the coordinates of points in a realization of a PVG.

Corollary 2  There exists a point visibility graph with $n$ vertices every realization of which requires coordinates of values $2^{2\Theta(\sqrt[3]{n})}$.

We now prove that the recognition problem for visibility graphs on a grid is decidable if and only if the existential theory of the rationals is decidable. The definition of the latter is analogous to that of the existential theory of the reals, except we now seek a solution in $\mathbb{Q}^k$.

The computational complexity of answering the question “Does this object have a realization on a grid?” is unknown for various types of objects. Among those objects are, most prominently, polytopes and oriented matroids and (non-simple) order types. Matiyasevich [19] showed that the existential theory of the integers is undecidable by
giving a negative solution to Hilbert’s tenth problem: Deciding whether a diophantine equation has a solution is undecidable. This cannot be directly applied to a grid realization of a PVG, since a realization of a PVG with rational coordinates, which can be obtained by a rational solution of the inequality system, leads to a grid realization by scaling. Hence for those geometric realizations on the grid the decidability of Hilbert’s tenth problem over the rationals is of interest. Grünbaum [12] conjectured in 1972 that there is no algorithm that enumerates all arrangements in the rational projective plane, which is equivalent to the recognition problem of order types that can be represented on a grid. This conjecture is still open.

Similar to work of Sturmfels [32] for oriented matroids and polytopes we show the following theorem.

**Theorem 4** The realization problem for visibility graphs of points on a grid is decidable if and only if the existential theory of the rationals is decidable.

Before proving this theorem we point out a connection to finding an upper bound on the grid size of a PVG that is realizable on a grid.

**Corollary 3** Suppose the recognition problem for PVG on a grid is undecidable. Then there is no computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that each PVG with $n$ points that is realizable on a grid can be drawn on a grid of size $f(n) \times f(n)$.

**Proof** We suppose that a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ exists, such that every PVG that is realizable on a grid with $n$ vertices can be represented on a grid of size $f(n) \times f(n)$. Using this function we can give an algorithm that decides whether a graph $G$ with $|V(G)| = n$ has a realization as a PVG on a grid.

We first compute $f(n)$, then for each $x \in [f(n)]^{2n}$ we check whether $G$ is the PVG of the point set $(x(v_1), y(v_1), \ldots, x(v_n), y(v_n)) = x$. If there is such an $x$ the algorithm returns the realization, otherwise no realization exists.

This algorithm is clearly an effective decision procedure, and thus the recognition problem for PVG on a grid is decidable—a contradiction to the assumption.

**Proof of Theorem 4** To prove this theorem we construct a set of graphs from a semi-algebraic set $V$, such that one of the graphs has a rational representation as PVG if and only if $V$ has. As a shortcut we use the result of Sturmfels [32], that the problem of realizing a line arrangement with rational coordinates is complete in the existential theory of the rationals. The plan is to encode a given (pseudo)line arrangement in a fan by placing the origin of the fan $p$, such that all intersection points lie in one halfspace. By Lemma 2, a realization of the resulting PVG leads to a realization of the line arrangement. However, we know neither the radial order of all intersection points around $p$ nor which points of the fan, that do not lie on different lines are collinear. Thus, we apply the fan construction for all possible radial orderings and all possible additional blocked visibilities. This gives a finite set of graphs, one of which has a rational PVG realization if and only if the arrangement has a rational realization: From a rational realization of the line arrangement we obtain a rational PVG by applying the fan construction as described with a rational origin $p$. This graph is contained in the set of graphs we constructed. On the other hand, from a rational PVG representation of one of the graphs we obtain a rational PVG representation by Lemma 2.
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References

1. Abello, J., Kumar, K.: Visibility graphs and oriented matroids. Discrete Comput. Geom. 28, 449–465 (2002)
2. Adiprasito, K.A., Padrol, A., Theran, L.: Universality theorems for inscribed polytopes and Delaunay triangulations. Discrete Comput. Geom. 54, 412–431 (2015)
3. Bienstock, D.: Some provably hard crossing number problems. Discrete Comput. Geom. 6, 443–459 (1991)
4. Björner, A., Las Vergnas, M., Sturmfels, B., White, N., Ziegler, G.M.: Oriented Matroids, 2nd edn. Cambridge University Press, Cambridge (1999)
5. Canny, J.: Some algebraic and geometric computations in PSPACE. In: STOC, pp. 460–467. ACM, New York (1988)
6. de Berg, M., Cheong, O., van Kreveld, M., Overmars, M.: Computational Geometry: Algorithms and Applications, 3rd edn. Springer, Berlin (2008)
7. Ghosh, S.K.: On recognizing and characterizing visibility graphs of simple polygons. Discrete Comput. Geom. 17, 143–162 (1997)
8. Ghosh, S.K.: Visibility Algorithms in the Plane. Cambridge University Press, Cambridge (2007)
9. Ghosh, S.K., Goswami, P.P.: Unsolved problems in visibility graphs of points, segments, and polygons. ACM Comput. Surv. 46, 22 (2013)
10. Ghosh, S.K., Roy, B.: Some results on point visibility graphs. In: WALCOM. LNCS, vol. 8344, pp. 163–175. Springer, Berlin (2014)
11. Goodman, J.E., Pollack, R., Sturmfels, B.: The intrinsic spread of a configuration in R^d. J. Am. Math. Soc. 3, 639–651 (1990)
12. Grünbaum, B.: Arrangements and Spreads. Regional Conference Series in Mathematics, vol. 10. AMS, Providence, RI (1972)
13. Grünbaum, B.: Convex Polytopes. Graduate Texts in Mathematics, vol. 221, 2nd edn. Springer, New York (2003)
14. Kapovich, M., Millson, J.J.: Universality theorems for configuration spaces of planar linkages. Topology 41, 1051–1107 (2002)
15. Kára, J., Pór, A., Wood, D.R.: On the chromatic number of the visibility graph of a set of points in the plane. Discrete Comput. Geom. 34, 497–506 (2005)
16. Kratochvíl, J., Matoušek, J.: Intersection graphs of segments. J. Comb. Theory, Ser. B 62, 289–315 (1994)
17. Kynčl, J.: Simple realizability of complete abstract topological graphs in P. Discrete Comput. Geom. 45, 383–399 (2011)
18. Lozano-Pérez, T., Wesley, M.A.: An algorithm for planning collision-free paths among polyhedral obstacles. Commun. ACM 22, 560–570 (1979)
19. Matiyasevich, Y.V.: Enumerable sets are diophantine. Dokl. Akad. Nauk SSSR 191, 279–282 (1970)
20. McDiarmid, C., Müller, T.: Integer realizations of disk and segment graphs. J. Comb. Theory, Ser. B 103, 114–143 (2013)
21. Mnëv, N.E.: The universality theorems on the classification problem of configuration varieties and convex polytopes varieties. In: Topology and Geometry—Rohlin Seminar. LNM, pp. 527–543. Springer, Berlin (1988)
22. O’Rourke, J.: Art Gallery Theorems and Algorithms. Oxford University Press, New York (1987)
23. O’Rourke, J., Streinu, I.: Vertex-edge pseudo-visibility graphs: characterization and recognition. In: SoCG, pp. 119–128. ACM, New York (1997)
24. Payne, M.S., Pór, A., Valtr, P., Wood, D.R.: On the connectivity of visibility graphs. Discrete Comput. Geom. 48, 669–681 (2012)
25. Pór, A., Wood, D.R.: On visibility and blockers. J. Comput. Geom. 1, 29–40 (2010)
26. Roy, B.: Point visibility graph recognition is NP-hard. arXiv:1406.2428 (2014)
27. Schaefer, M.: Complexity of some geometric and topological problems. In: GD, LNCS, vol. 5849, pp. 334–344. Springer, Berlin (2009)
28. Schaefer, M.: Realizability of graphs and linkages. In: Pach, J. (ed.) Thirty Essays on Geometric Graph Theory. Springer, Berlin (2012)
29. Shor, P.W.: Stretchability of pseudolines is NP-hard. Appl. Geom. Discrete Math. 4, 531–554 (1991)
30. Staudt, K.G.C.: Geometrie der Lage. F. Korn, Nuremberg (1847)
31. Streinu, I.: Non-stretchable pseudo-visibility graphs. Comput. Geom. 31, 195–206 (2005)
32. Sturmfels, B.: On the decidability of diophantine problems in combinatorial geometry. Bull. Am. Math. Soc. 17, 121–124 (1987)