GEOMETRIC CRITERIA FOR TAME RAMIFICATION

JOHANNES NICASIE

Abstract. We prove an A’Campo type formula for the tame monodromy zeta function of a smooth and proper variety over a discretely valued field $K$. As a first application, we relate the orders of the tame monodromy eigenvalues on the $\ell$-adic cohomology of a $K$-curve to the geometry of a relatively minimal $snec$-model, and we show that the semi-stable reduction theorem and Saito’s criterion for cohomological tameness are immediate consequences of this result. As a second application, we compute the error term in the trace formula for smooth and proper $K$-varieties. We see that the validity of the trace formula would imply a partial generalization of Saito’s criterion to arbitrary dimension.

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1. Introduction

Let $R$ be a henselian discrete valuation ring with quotient field $K$ and algebraically closed residue field $k$, and let $\ell$ be a prime number different from the characteristic of $k$. We denote by $K^t$ a tame closure of $K$. Let $X$ be a smooth and proper $K$-variety. In Section 2, we compute the zeta function $\zeta_X(t)$ of the tame monodromy action on the graded tame $\ell$-adic cohomology $H(X \times_K K^t, \mathbb{Q}_\ell)$ in terms of an $snec$-model $\mathcal{X}$ of $X$ over $R$ (Theorem 2.6.2). This zeta function completely determines the class of $H(X \times_K K^t, \mathbb{Q}_\ell)$ in the Grothendieck ring of $\ell$-adic representations of the tame inertia group $G(K^t/K)$. Our formula for the monodromy zeta function is an arithmetic analog of a formula obtained by A’Campo [AC75] for the zeta function of the monodromy action on the cohomology of the Milnor fiber at a complex hypersurface singularity. The main additional complication in the arithmetic setting is that we need to prove a tameness property of the complex of $\ell$-adic tame nearby cycles associated to $\mathcal{X}$ (Proposition 2.5.2). This property allows us to compute $\zeta_X(t)$ pointwise on the special fiber $\mathcal{C}_s$ of $\mathcal{X}$, using Grothendieck’s description of the stalks of the tame nearby cycles on a divisor with normal crossings [SGA7a, I.3.3].

We present two applications of our arithmetic A’Campo formula. In Section 3 we consider the case where $X$ is a $K$-curve $C$. In Theorem 3.2.3 and Corollary 3.2.4 we relate the orders of the tame monodromy eigenvalues on $H^1(C \times_K K^t, \mathbb{Q}_\ell)$ to the geometry of the special fiber $\mathcal{C}_s$ of a relatively minimal $snec$-model $\mathcal{C}$ of $C$. We show how Saito’s criterion for cohomological tameness (Theorem 3.3.2)
and the semi-stable reduction theorem (Theorem 3.4.2 and Corollary 3.4.3) are immediate consequences of this result. Our methods also allow to determine the degree of the minimal extension of $K$ where $C$ acquires semi-stable reduction, if $C$ is cohomologically tame (Corollary 3.4.4). Our approach is similar in spirit to the one in Saito’s paper [Sa87], but our proof substantially simplifies the combinatorial analysis of $C_s$. For different proofs of Saito’s criterion, see [Sa04, St05] (using logarithmic geometry) and [Ha10] (using a geometric analysis of the behaviour of sncd-models under base change). For a survey on the semi-stable reduction theorem for curves, see [Ab00].

As a second application, in Section 4, we compute the error term in the trace formula for $X$ on an sncd-model $\mathcal{X}$ of $X$ over $R$. The trace formula was introduced in [NS07b] and further studied in [Ni09a, Ni09b, Ni11]. It expresses a certain measure for the set of rational points on $X$ in terms of the Galois action on the tame $\ell$-adic cohomology of $X$. We expect that the trace formula is valid if $X$ is geometrically connected and cohomologically tame, and $X(K_t)$ non-empty. We’ve proven this if $k$ has characteristic zero [Ni11, 6.5], if $X$ is a curve [Ni11, §7], and if $X$ is an abelian variety [Ni09b, 2.9]. Our formula for the error term shows that (assuming the existence of an sncd-model), our conjecture is equivalent to a partial generalization of Saito’s criterion to arbitrary dimension (Question 4.2.4).

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Notations. Let $R$ be a henselian discrete valuation ring, with quotient field $K$ and algebraically closed residue field $k$. We fix a uniformizer $\pi$ in $R$. We denote by $p = 0$ the characteristic of $k$, and we fix a prime $\ell$ different from $p$. We denote by $P$ the wild inertia subgroup of $G(K_s/K)$, and we choose a topological generator $\varphi$ of the tame inertia group $G(K_t/K)$. We denote by $N'$ the set of strictly positive integers that are not divisible by $p$. We fix an algebraic closure $\overline{Q}_\ell$ of $Q_\ell$, and we denote by $Q_\ell^a$ the algebraic closure of $Q$ in $\overline{Q}_\ell$.

If $X$ is a separated scheme of finite type over $K$, then we have a canonical $G(K_t/K)$-equivariant isomorphism

$$H^m(X \times_K K^t, \mathbb{Q}_\ell) \cong H^m(X \times_K K^t, \mathbb{Q}_\ell)^P$$

for every integer $m \geq 0$. We say that $X$ is cohomologically tame if $P$ acts trivially on $H^m(X \times_K K^a, \mathbb{Q}_\ell)$ for all $m \geq 0$.

If $Y$ is a separated scheme of finite type over a field $F$ and $p'$ is a prime different from the characteristic of $F$, then we denote by $\chi(Y)$ the $p'$-adic Euler characteristic (with proper supports) of $Y$:

$$\chi(Y) = \sum_{m \geq 0} (-1)^m \dim H^m_{c}(Y \times_F F^s, \mathbb{Q}_{p'})$$

with $F^s$ a separable closure of $F$. It is well-known that the Euler characteristic $\chi(Y)$ is independent of $p'$; if $F$ is a finite field, then by the Grothendieck-Lefschetz trace formula, $\chi(Y)$ equals minus the degree of the Hasse-Weil zeta function of $Y$ [De73 1.5.4]. The general case follows by a spreading out argument and proper base change (if $F$ has characteristic zero, it can also be deduced from the comparison
with singular cohomology). The Euler characteristic $\chi(Y)$ is equal to the Euler characteristic without supports, i.e.,

$$\chi(Y) = \sum_{m \geq 0} (-1)^m \dim H^m(Y \times_F F^s, \mathbb{Q}_p').$$

If $F$ has characteristic zero this result is due to Grothendieck; the general case was proven by Laumon [La81]. We denote by

$$(\cdot)_s : (\text{Sch}/R) \to (\text{Sch}/k) : X \mapsto X \times_R k$$

the special fiber functor from the category of $R$-schemes to the category of $k$-schemes. We denote by $(\cdot)_{\text{red}}$ the endofunctor on the category of schemes that maps a scheme $S$ to its maximal reduced closed subscheme $S_{\text{red}}$. All regular schemes are assumed to be locally Noetherian. When we speak of a local ring $(A, m_A, k_A)$, we mean that $A$ is a local ring with maximal ideal $m_A$ and residue field $k_A$.

For every integer $d > 0$, we denote by $\Phi_d(t) \in \mathbb{Z}[t]$ the cyclotomic polynomial whose roots are the primitive $d$-th roots of unity.

2. The tame monodromy zeta function

2.1. Models. We recall some standard definitions and fix some terminology. All of the results in this section are well-known, but we include them here for lack of suitable reference. All definitions, results and proofs in Section 2.1 are formulated in such a way that they are valid over an arbitrary discrete valuation ring $R$.

Let $\mathcal{X}$ be a regular flat $R$-scheme, and let $x$ be a point of the special fiber $\mathcal{X}_s$. We say that $\mathcal{X}_s$ has strict normal crossings at $x$ if there exist a regular system of parameters $(x_1, \ldots, x_m)$ and a unit $u$ in $\mathcal{O}_{\mathcal{X},x}$ and elements $N_1, \ldots, N_m$ in $\mathbb{N}$ such that

$$(2.1) \quad \pi = u \prod_{i=1}^m (x_i)^{N_i}.$$ 

Since every regular local ring is a UPD, this is equivalent to saying that every tuple of non-associated prime factors of $\pi$ in $\mathcal{O}_{\mathcal{X},x}$ is part of a regular system of parameters. We say that $\mathcal{X}$ is strictly semi-stable at $x$ if $\mathcal{X}_s$ has strict normal crossings at $x$ and the local ring $\mathcal{O}_{\mathcal{X}_s,x}$ is reduced. This is equivalent to the property that each exponent $N_i$ in (2.1) is either zero or one.

An $\text{sncd}$-model over $R$ is a regular flat separated $R$-scheme of finite type $\mathcal{X}$ such that $\mathcal{X}_s$ has strict normal crossings at every point of $\mathcal{X}_s$. An $\text{sncd}$-model $\mathcal{X}$ is called semi-stable if $\mathcal{X}_s$ is reduced. This is equivalent to the property that $\mathcal{X}$ is strictly semi-stable at every point of $\mathcal{X}_s$.

Lemma 2.1.1. Let $\mathcal{X}$ be a regular flat $R$-scheme, and let $x$ be a point of $\mathcal{X}_s$. We denote by $d$ the dimension of $\mathcal{X}$ at $x$. Let $E_1, \ldots, E_n$ be the irreducible components of $\mathcal{X}_s$ that pass through $x$, endowed with their induced reduced structure. For every non-empty subset $J$ of $\{1, \ldots, n\}$, we denote by $E_J$ the schematic intersection of the closed subschemes $E_j$ of $\mathcal{X}$ with $j$ in $J$. Then the following properties are equivalent.

1. The special fiber $\mathcal{X}_s$ has strict normal crossings at $x$.
2. For every non-empty subset $J$ of $\{1, \ldots, n\}$, the scheme $E_J$ is regular and of dimension $d - |J|$ at $x$. 

Proof. Let \((x_1, \ldots, x_n)\) be a maximal tuple of non-associated prime factors of \(\pi\) in \(\mathcal{O}_{\mathcal{X}, x}\). Then the irreducible components of \(\text{Spec} \mathcal{O}_{\mathcal{X}, x}\) are precisely the integral closed subschemes \(\text{Spec} (\mathcal{O}_{\mathcal{X}, x}/(x_i))\) of \(\text{Spec} \mathcal{O}_{\mathcal{X}, x}\), with \(i \in \{1, \ldots, n\}\). It follows from \([\text{EGA}1, 7.2.8.1]\), \([\text{EGA}4a, 16.3.7\) and \(17.1.7]\) and \([\text{EGA}4b, 5.1.9]\) that \((x_1, \ldots, x_n)\) is part of a regular system of parameters in \(\mathcal{O}_{\mathcal{X}, x}\) if and only if condition (2) is satisfied.

\[\square\]

Corollary 2.1.2. Let \(\mathcal{X}\) be a regular flat \(R\)-scheme locally of finite type. The set of points of \(\mathcal{X}_s\) where \(\mathcal{X}_s\) has strict normal crossings is open in \(\mathcal{X}_s\).

Proof. Let \(x\) be a point of \(\mathcal{X}_s\) such that \(\mathcal{X}_s\) has strict normal crossings at \(x\), and denote by \(d\) the dimension of \(\mathcal{X}\) at \(x\). Let \(E_1, \ldots, E_n\) be as in Lemma 2.1.1. Replacing \(\mathcal{X}\) by a suitable open neighbourhood of \(x\), we may assume that \(E_1, \ldots, E_n\) are the only irreducible components of \(\mathcal{X}_s\).

Choose a non-empty subset \(J\) of \(\{1, \ldots, n\}\). Then \(E_J\) is regular and of dimension \(d - |J|\) at \(x\), by Lemma 2.1.1. The regular locus of \(E_J\) is open in \(E_J\) because \(E_J\) is locally of finite type over the field \(k\) \([\text{EGA}4d, 6.12.5]\). Thus, shrinking \(\mathcal{X}\), we may assume that \(E_J\) is regular for every non-empty subset \(J\) of \(\{1, \ldots, n\}\). We may also assume that \(E_J\) has pure dimension \(d - |J|\). Then \(\mathcal{X}_s\) has strict normal crossings at every point of \(\mathcal{X}_s\), by Lemma 2.1.1.

\[\square\]

Lemma 2.1.3. Let \(\varphi : (A, m_A, k_A) \to (B, m_B, k_B)\) be a local homomorphism of regular local rings. Then the following are equivalent:

(1) the morphism \(\varphi\) is flat, and \(m_A B = m_B\),
(2) there exists a regular system of parameters \((a_1, \ldots, a_m)\) in \(A\) such that \((\varphi(a_1), \ldots, \varphi(a_m))\) is a regular system of parameters in \(B\),
(3) a tuple \((a_1, \ldots, a_m)\) of elements in \(A\) is a regular system of parameters if and only if \((\varphi(a_1), \ldots, \varphi(a_m))\) is a regular system of parameters in \(B\),
(4) the morphism of \(k_B\)-vector spaces

\[\psi : \frac{m_A}{m_A^2} \otimes_{k_A} k_B \to \frac{m_B}{m_B^2}\]

induced by \(\varphi\) is an isomorphism.

Proof. If (1) holds, then \(A\) and \(B\) have the same dimension by \([\text{Ma80}, 13.B]\), so that (2) follows from (1). Conversely, (2) implies immediately that \(m_A B = m_B\), and flatness of \(\varphi\) follows from the local criterion for flatness (in the formulation of \([\text{EGA}4d, 6.9]\)) by induction on the dimension of \(A\). Thus (1) and (2) are equivalent. The implication (3) \(\Rightarrow\) (2) is trivial. By \([\text{EGA}4a, 17.1.7]\), a tuple \((c_1, \ldots, c_m)\) of elements in the maximal ideal of a regular local ring \((C, m_C, k_C)\) is a regular system of parameters in \(C\) if and only if the residue classes of the elements \(c_i\) in the \(k_C\)-vector space \(m_C/m_C^2\) form a basis. This shows that (2) \(\Rightarrow\) (4) \(\Rightarrow\) (3).

\[\square\]

Lemma 2.1.4. Let \(f : Y \to X\) be a morphism of schemes that is locally of finite presentation. Let \(y\) be a point of \(Y\) and set \(x = f(y)\). Assume that \(X\) is regular at \(x\) and that \(Y\) is regular at \(y\). Then \(f\) is étale at \(y\) if and only if the residue field at \(y\) is a finite separable extension of the residue field at \(x\) and the morphism \(\mathcal{O}_{X, x} \to \mathcal{O}_{Y, y}\) satisfies the equivalent properties of Lemma 2.1.3.

Proof. This follows from the characterization of étale morphisms in \([\text{EGA}4d, 17.6.1(c')]\).

\[\square\]
The following proposition describes the local structure of semi-stable \textit{sncd}-models.

**Proposition 2.1.5.** Assume that \(k\) is perfect. Let \(X\) be a regular flat \(R\)-scheme locally of finite type, and let \(x\) be a closed point of \(X_s\). Then the following are equivalent:

1. the \(R\)-scheme \(X\) is strictly semi-stable at \(x\),
2. the point \(x\) admits an open neighbourhood \(U\) in \(X\) such that there exist integers \(m \geq n > 0\) and an étale \(R\)-morphism

\[
g : U \to Y = \text{Spec } R[y_1, \ldots, y_m]/(\pi - \prod_{j=1}^n y_j)
\]

such that \(g(x) = 0\). Here \(O\) denotes the origin in \(Y_s \subset \mathbb{A}_k^n\).

**Proof.** Assume that \(X\) is strictly semi-stable at \(x\). Then we have an expression of the form (2.1) in \(O_{X,x}\), with each \(N_i\) either zero or one. Permuting the local parameters \(x_i\) if necessary, we may assume that there exists an element \(n\) of \(\{1, \ldots, m\}\) such that \(N_i = 1\) for \(i = 1, \ldots, n\) and \(N_i = 0\) for \(i > n\). We can also arrange that \(u = 1\) by replacing \(x_1\) by \(ux_1\).

The local parameters \(x_i\) are germs of regular functions on \(X\), and we choose a connected open neighbourhood \(U\) of \(x\) in \(X\) such that \(x_i\) is defined on \(U\) for every \(i\). Then, by equation (2.1) and our assumptions, there exists a unique morphism of \(R\)-schemes

\[
g : U \to Y = \text{Spec } R[y_1, \ldots, y_m]/(\pi - \prod_{j=1}^n y_j)
\]

such that \(x_i = y_i \circ g\) for every \(i\).

Note that \(g(x)\) is the origin \(O\) in \(Y_s\), and that the pullback by \(g\) of the regular system of parameters \((y_1, \ldots, y_m)\) in \(O_{Y,O}\) is the regular system of parameters \((x_1, \ldots, x_n)\) in \(O_{X,x}\). Since \(k\) is perfect, we also know that the residue field at \(x\) is separable over the residue field \(k\) at \(O\). Thus, by Lemma 2.1.4, the morphism \(g\) is étale at \(x\). Shrinking \(U\), we may assume that \(g\) is étale everywhere. This shows that (1) implies (2).

Conversely, assume that \(X\) and \(x\) satisfy (2). We put \(x_i = y_i \circ g\) for \(i = 1, \ldots, m\). Since \(h\) is étale, we know by Lemma 2.1.4 that \((x_1, \ldots, x_m)\) is a regular system of parameters in \(O_{X,x}\), and this system satisfies the equation

\[
\pi = \prod_{j=1}^n x_j.
\]

Thus \(X\) is strictly semi-stable at \(x\).

\[\square\]

Let \(X\) be a regular flat \(R\)-scheme, and let \(x\) be a point of \(X_s\). We say that \(X_s\) has normal crossings at \(x\) if there exists an étale morphism of \(R\)-schemes \(h : Z \to X\) such that \(Z_s\) has strict normal crossings at some point \(z\) of \(h^{-1}(x)\). Note that \(Z\) is regular and \(R\)-flat since \(h\) is étale \([\text{EGA4d}] 17.5.8\) and 17.6.1]. If \(X\) is locally of finite type over \(R\), then it follows from Corollary 2.1.2 that the locus of points of \(X_s\) where \(Z_s\) has normal crossings is open in \(X_s\), since the image of an étale morphism of \(R\)-schemes \(Z \to X\) is open in \(X\) \([\text{EGA4b}] 2.4.6\).

We say that \(X\) is semi-stable at \(x\) if \(X_s\) has normal crossings at \(x\) and, moreover, \(O_{X_s,x}\) is reduced. We call \(X\) an \textit{ncld-model} if \(X\) is separated and of finite type.
over $R$ and $\mathcal{X}_s$ has normal crossings at every point of $\mathcal{X}_s$. An ncd-model $\mathcal{X}$ is called semi-stable if $\mathcal{X}_s$ is reduced. This is equivalent to the property that $\mathcal{X}$ is semi-stable at every point of $\mathcal{X}_s$.

It follows from Proposition 2.1.5 that, if $k$ is perfect, the generic fiber $\mathcal{X} \times_k K$ of a proper semi-stable ncd-model $\mathcal{X}$ is smooth over $K$. This implies that, if $k$ is perfect, the generic fiber of a proper semi-stable ncd-model is also smooth over $K$, since it has a $K$-smooth étale cover. The properness assumption is needed to ensure that every collection of open subsets of $\mathcal{X}$ that covers $\mathcal{X}_s$ also covers $\mathcal{X}$.

**Proposition 2.1.6.** Let $\mathcal{X}$ be a regular flat $R$-scheme, and let $x$ be a point of $\mathcal{X}_s$. Let $y$ be a geometric point centered at $x$, and denote by $\mathcal{Y}$ the strict henselization of $\mathcal{X}$ at $y$. Then $\mathcal{Y}$ is a regular flat $R$-scheme, and $\mathcal{X}_s$ has normal crossings at $x$ if and only if $\mathcal{Y}_s$ has strict normal crossings at $y$.

**Proof.** The scheme $\mathcal{Y}$ is regular [EGA4d, 18.8.13] and the local homomorphism $O_{\mathcal{X},x} \to O(\mathcal{Y})$ satisfies the equivalent properties of Lemma 2.1.3 by [EGA4d, 18.8.8(iii)]. Assume that $\mathcal{X}_s$ has normal crossings at $x$, and choose an étale morphism of $R$-schemes $h : \mathcal{Z} \to \mathcal{X}$ such that $\mathcal{Z}_s$ has strict normal crossings at some point $z$ of $h^{-1}(x)$. Then by [EGA4d, 18.8.4], there exists a morphism of $\mathcal{X}$-schemes $\mathcal{Y} \to \mathcal{Z}$ that maps $y$ to $z$. Applying Lemma 2.1.4 to $h$, we see that the local homomorphism $O_{\mathcal{X},z} \to O(\mathcal{Y})$ also satisfies the equivalent properties of Lemma 2.1.3, which immediately implies that $\mathcal{Y}_s$ has strict normal crossings at $y$.

Suppose, conversely, that $\mathcal{Y}_s$ has strict normal crossings at $y$, and choose an equation of type (2.1) in $O(\mathcal{Y})$. By construction [EGA4d, 18.8.7], the ring $O(\mathcal{Y})$ is a direct limit of local rings that are essentially étale over $O_{\mathcal{X},x}$, thus we can find such a local ring $A$ such that $u, x_1, \ldots, x_m$ lift to $A$, $u$ is a unit in $A$ and the equality (2.1) holds in $A$. The ring $O(\mathcal{Y})$ is also the strict henselization of $A$ at $y$, so that the tuple $(x_1, \ldots, x_m)$ is a regular system of parameters in $A$ by Lemma 2.1.3. Since $A$ is essentially étale over $O_{\mathcal{X},x}$, it follows from [EGA4c, 8.8.2] that we can find an étale $\mathcal{X}$-scheme $\mathcal{Z}$ and a point $z$ of $\mathcal{Z}$ lying over $x$ such that $A$ and $O_{\mathcal{X},z}$ are isomorphic as $O_{\mathcal{X},x}$-algebras. Then $\mathcal{Z}_s$ has strict normal crossings at $z$. It follows that $\mathcal{X}_s$ has normal crossings at $x$. □

Let $\mathcal{X}$ be a regular flat $R$-scheme, and let $\mathcal{Y} \rightarrow \mathcal{X}$ be an étale morphism. Then, as was already mentioned above, $\mathcal{Y}$ is regular and $R$-flat. Let $y$ be a point of $\mathcal{Y}_s$, and denote by $x$ its image in $\mathcal{X}_s$. The following properties follow easily from Lemma 2.1.4 and Proposition 2.1.6:

- if $\mathcal{X}_s$ has strict normal crossings at $x$ then $\mathcal{Y}_s$ has strict normal crossings at $y$;
- if $\mathcal{X}_s$ is strictly semi-stable at $x$ then $\mathcal{Y}_s$ is strictly semi-stable at $y$;
- $\mathcal{Y}_s$ has normal crossings at $y$ if and only if $\mathcal{X}_s$ has normal crossings at $x$;
- $\mathcal{Y}$ is semi-stable at $y$ if and only if $\mathcal{X}$ is semi-stable at $x$.

(Note that $\mathcal{Y}_s$ is reduced at $y$ if and only if $\mathcal{X}_s$ is reduced at $x$, by [EGA4d, 17.5.7].)

**Proposition 2.1.7.** Let $\mathcal{X}$ be a regular flat $R$-scheme, and let $x$ be a point of $\mathcal{X}_s$ such that $\mathcal{X}_s$ has normal crossings at $x$. Then $\mathcal{X}_s$ has strict normal crossings at $x$ if and only if every irreducible component of $\mathcal{X}_s$ that passes through $x$ (endowed with its reduced induced structure) is regular at $x$.

**Proof.** If $\mathcal{X}_s$ has strict normal crossings at $x$, then every irreducible component of $\mathcal{X}_s$ that passes through $x$ is regular at $x$, by Lemma 2.1.1. Conversely, assume
that every irreducible component of $\mathcal{X}_s$ that passes through $x$ is regular at $x$. Let $(x_1, \ldots, x_n)$ be a tuple of non-associated prime factors of $\pi$ in $\mathcal{O}_{\mathcal{X}, x}$. It’s enough to show that this tuple is part of a regular system of parameters in $\mathcal{O}_{\mathcal{X}, x}$. By [EGA4d 18.8.8(iii)] and Lemma 2.1.3, we can verify this in a strict henselization $A$ of $\mathcal{O}_{\mathcal{X}, x}$.

Locally at $x$, each of the equations $x_i = 0$ defines an irreducible component of $\mathcal{X}_s$, so that $x_i$ is part of a regular system of parameters by [EGA4a 17.1.7]. It follows from [EGA4d 18.8.8(iii)] and Lemma 2.1.3 that the image of $x_i$ in $A$ is still part of a regular system of parameters. In particular, this element is prime. Moreover, [Ma80 4.C(ii)] implies that $x_i$ is not associated to $x_j$ in $A$ if $i$ and $j$ are distinct elements of $\{1, \ldots, n\}$, because $x_i$ and $x_j$ are not associated in $\mathcal{O}_{\mathcal{X}, x}$ and $A$ is faithfully flat over $\mathcal{O}_{\mathcal{X}, x}$. Thus $x_1, \ldots, x_n$ are non-associated prime factors of $\pi$ in $A$. It follows that $(x_1, \ldots, x_n)$ is part of a regular system of parameters in $A$, because $(\text{Spec } A)_s$ has strict normal crossings at its unique closed point, by Proposition 2.1.6.

\[\square\]

Definition 2.1.8. If $X$ is a proper $K$-scheme, then a model of $X$ is a flat proper $R$-scheme $\mathcal{X}$ endowed with an isomorphism of $K$-schemes

$$\mathcal{X} \times_R K \to X.$$ We say that $\mathcal{X}$ is an ncd-model (resp. sncd-model) of $X$ if, moreover, $\mathcal{X}$ is regular and $\mathcal{X}_s$ has normal crossings (resp. strict normal crossings) at every point of $\mathcal{X}_s$. This implies that $X$ is regular.

A morphism of models of $X$ is an $R$-morphism $h$ such that the induced morphism $h_K$ between the generic fibers commutes with the respective isomorphisms to $X$. In particular, $h_K$ is an isomorphism. We say that a regular model $\mathcal{X}$ of $X$ is relatively minimal if every morphism to another regular model is an isomorphism. We say that $\mathcal{X}$ is minimal if, up to isomorphism, it is the unique relatively minimal regular model. The analogous terminology applies to ncd-models and sncd-models.

2.2. The case of curves. From now on, we’ll assume that $R$ is henselian and that $k$ is algebraically closed. To describe the geometry of models of curves, we gather some results from [Li02]. Beware that the author of [Li02] uses the term “normal crossings” where we use “strict normal crossings”, see [Li02 9.1.6 and 9.1.7]. The key lemma for minimality issues of sncd-models is [Li02 9.3.35]. Unfortunately, this statement is not entirely correct. In our situation, it can be corrected and generalized as follows.

Lemma 2.2.1. Let $\mathcal{X}$ be a regular flat $R$-scheme of pure dimension two. Let $x$ be a point of $\mathcal{X}_s$ where $\mathcal{X}_s$ has normal crossings. Then $\mathcal{X}_s$ has strict normal crossings at $x$ unless $x$ lies on precisely one irreducible component $\Gamma$ of $\mathcal{X}_s$ and $x$ is a singular point of $\Gamma$ (with its reduced induced structure).

Proof. Since $\mathcal{O}_{\mathcal{X}, x}$ has dimension two and $\mathcal{X}_s$ has normal crossings at $x$, the point $x$ can lie on at most two irreducible components of $\mathcal{X}_s$. If $x$ lies on only one irreducible component $\Gamma$ of $\mathcal{X}_s$, then $\mathcal{X}_s$ has strict normal crossings at $x$ if and only if $\Gamma$ is regular at $x$, by Proposition 2.1.7. So we may assume that $x$ lies on two distinct irreducible components of $\mathcal{X}_s$. Then $\pi$ has precisely two non-associated prime factors $x_1$ and $x_2$ in $\mathcal{O}_{\mathcal{X}, x}$, and it is enough to show that $(x_1, x_2)$ is a regular system of parameters in $\mathcal{O}_{\mathcal{X}, x}$. 

2.
Let $A$ be the henselization of $\mathcal{O}_{\mathcal{X},x}$. It is a regular [EGA4d 18.6.10] and the local homomorphism $\mathcal{O}_{\mathcal{X},x} \to A$ satisfies the equivalent properties in Lemma 2.1.3 by [EGA4d 18.6.6(iii)]. Thus it suffices to prove that $(x_1, x_2)$ is a regular system of parameters in $A$. We know by Proposition 2.1.6 that $(\text{Spec } A)$ has strict normal crossings at $x$, so that we only have to show that $x_1$ and $x_2$ are non-associated prime factors of $\pi$ in $A$. For $i = 1, 2$, the ring $A/(x_i)$ is reduced because it is the henselization of the reduced local ring $\mathcal{O}_{\mathcal{X},x}/(x_i)$ [EGA4d 18.6.8 and 18.6.9]. Moreover, $x_1$ and $x_2$ have no common prime factor in $A$ because $x_1$ is not a zero-divisor in $B = \mathcal{O}_{\mathcal{X},x}/(x_2)$ so that it cannot be a zero divisor in the faithfully flat $B$-algebra $A/(x_2)$. It follows that $x_1$ and $x_2$ are the two non-associated prime factors of $\pi$ in $A$. 

Proposition 2.2.2. Let $\mathcal{X}$ be a regular flat $R$-scheme of pure dimension two. Let $x$ be a closed point of $\mathcal{X}_s$. We denote by $f : \mathcal{X}' \to \mathcal{X}$ the blow-up of $\mathcal{X}$ at $x$, and by $E = f^{-1}(x)$ the exceptional divisor of $f$.

1. If $\mathcal{X}_s$ has normal crossings at $x$, then $\mathcal{X}'_s$ has strict normal crossings at every point of $E$, and $E$ meets the other irreducible components of $\mathcal{X}'_s$ in at most two points.

2. Assume that $\mathcal{X}'_s$ has normal crossings at every point of $E$. Then $\mathcal{X}_s$ has normal crossings at $x$ if and only if $E$ meets the other irreducible components of $\mathcal{X}'_s$ in at most two points.

3. Assume that $\mathcal{X}'_s$ has normal crossings at every point of $E$. Then $\mathcal{X}_s$ has strict normal crossings at $x$ if and only if $E$ meets the other components of $\mathcal{X}'_s$ in at most two points and $E$ does not intersect any other component twice.

Proof. Even though we are not dealing with proper $R$-schemes, we can still borrow most of the arguments from [Li02], using the intersection theory on regular two-dimensional schemes developed in [Li68].

First, we prove (1). We choose an étale morphism of $R$-schemes $\mathcal{Y} \to \mathcal{X}$ such that $\mathcal{Y}_s$ has strict normal crossings at some point $y$ lying over $x$. Then we can apply [Li02 9.2.31] to the blow-up of $\mathcal{Y}$ at $y$. Since blowing up commutes with flat base change [Li02 8.1.12], it follows that $\mathcal{X}'_s$ has normal crossings at every point of $E$ and that $E$ intersects the other components of $\mathcal{X}'_s$ in at most two points. The exceptional curve $E$ is regular, so that Lemma 2.2.1 implies that $\mathcal{X}'_s$ has strict normal crossings at every point of $E$.

Now we prove (3). We only indicate where the proof of [Li02 9.3.35] must be modified. On line 5 of the proof, it is tacitly assumed that $\bar{\Gamma}$ intersects $E$ in at most one point. This is not always the case under the hypotheses of [Li02 9.3.35]; we added it as an assumption in (3). The argument in [Li02 9.3.35] can be copied verbatim to prove point (3) of our proposition.

Finally, we prove (2). Assume that $E$ verifies the conditions in the statement. We will prove that $\mathcal{X}_s$ has normal crossings at $x$. The converse implication follows from (1). We may assume that $\mathcal{X}$ is a strictly henselian local scheme, by Proposition 2.1.6 and the fact that blowing up commutes with flat base change [Li02 8.1.12]. Then it follows from [EGA4d 18.6.8] that $E$ cannot meet any other irreducible component of $\mathcal{X}'_s$ twice, so that we can deduce from (3) that $\mathcal{X}_s$ has strict normal crossings at $x$. \qed
Let $C$ be a smooth, proper, geometrically connected $K$-curve of genus $g$. The curve $C$ admits a relatively minimal regular model $\mathcal{C}$ [Li02, 10.1.8], and every regular model of $C$ admits a morphism to some relatively minimal regular model [Li02, 9.3.19]. This morphism is a composition of contractions of irreducible components in the special fiber.

Assume that $g \geq 1$. Then $C$ is minimal [Li02, 9.3.21]. Repeatedly blowing up $C$ at points where $C_s$ does not have normal crossings, we obtain a minimal $ncd$-model $C'$ of $C$. Blowing up $C'$ at the self-intersection points of the irreducible components of its special fiber, we obtain a minimal $sncd$-model. This can be proved as in [Li02, 9.3.36], invoking Proposition 2.2.2 instead of [Li02, 9.3.35].

Now suppose that $g = 0$. This case is treated in [Sh66, pp. 155–157] and [Li02, Exercise 9.3.1]. Under our assumptions ($R$ henselian and $k$ algebraically closed), the Brauer group of $K$ is trivial [Gr68, 6.2], so that the conic $C$ is isomorphic to $\mathbb{P}^1_K$. The $R$-scheme $\mathbb{P}^1_R$ is a relatively minimal regular model of $\mathbb{P}^1_K$ which is not minimal. It is also a relatively minimal $ncd$-model and $sncd$-model of $\mathbb{P}^1_K$. Moreover, every relatively minimal regular model of $\mathbb{P}^1_K$ is smooth over $R$, and its special fiber is isomorphic to $\mathbb{P}^1_k$.

2.3. Constructions on $sncd$-models. Let $\mathcal{X}$ be an $sncd$-model over $R$. We put $X = \mathcal{X} \times_R K$. We write

$$\mathcal{X}_s = \sum_{i \in I} N_i E_i$$

where $E_i$, $i \in I$ are the irreducible components of $\mathcal{X}_s$, and $N_i$ is the multiplicity of $E_i$ in the Cartier divisor $\mathcal{X}_s$ on $\mathcal{X}$. For each $i \in I$, we denote by $N_i'$ the largest divisor of $N_i$ that is not divisible by $p$. Note that $N_i' = N_i$ if $p = 0$.

Let $J$ be a non-empty subset of $I$. We set

$$N'_J = \gcd\{N'_j \mid j \in J\}.$$ 

Moreover, we put

$$E_J = \bigcap_{j \in J} E_j$$

$$E^o_J = E_J \setminus \bigcup_{i \in I \setminus J} E_i.$$ 

The set

$$\{E^o_J \mid \emptyset \neq J \subset I\}$$

is a partition of $\mathcal{X}_s$ into locally closed subsets. We endow all $E_J$ and $E^o_J$ with their reduced induced structures. The schemes $E_J$ and $E^o_J$ are regular, by Lemma 2.1.1. For every $i \in I$, we write $E^o_i$ instead of $E^o_{\{i\}}$.

**Lemma 2.3.1.** For every non-empty subset $J$ of $I$, there exist integral affine open subschemes $\mathcal{U}_1, \ldots, \mathcal{U}_r$ of $\mathcal{X}$ such that

- $E_J$ is contained in $\bigcup_{i=1}^r \mathcal{U}_i$,
- on each open subscheme $\mathcal{U}_i$, we can write $\pi = u_i(v_i)^{N'_J}$ with $u_i$, $v_i$ regular functions on $\mathcal{U}_i$ such that $u_i$ is a unit.
Proof. Let \( x \) be a closed point of \( E^o_J \). Since \( \mathcal{X} \) is an sncd-model, we can find a regular system of parameters \( (x_1, \ldots, x_m) \) and a unit \( u \) in \( \mathcal{O}_{\mathcal{X}, x} \) such that

\[
\pi = u \prod_{j=1}^m (x_j)^{M_j}
\]

for some \( M_1, \ldots, M_m \) in \( \mathbb{N} \). Permuting the parameters \( x_j \), we may assume that there exists an \( n \in \{1, \ldots, m\} \) such that \( M_j > 0 \) for \( j = 1, \ldots, n \) and \( M_j = 0 \) for \( j > n \). The irreducible components of \( \mathcal{X}_a \) that pass through \( x \) are the components \( E_i \) with \( i \in J \), and they are locally defined by the equations \( x_j = 0 \), for \( j = 1, \ldots, n \). This correspondence yields a bijection between the set \( J \) and the set \( \{1, \ldots, n\} \). Modulo this identification, we have \( N_j = M_j \) for every \( j \in J \).

The elements \( u \) and \( x_1, \ldots, x_m \) are germs of regular functions on \( \mathcal{X} \), and we choose an affine integral open neighbourhood \( V \) of \( x \) in \( \mathcal{X} \) such that \( u \) and \( x_1, \ldots, x_m \) are all defined on \( V \) and \( u \) is a unit in \( \mathcal{O}(V) \). Then we have the equation

\[
\pi = u \prod_{j \in J} (x_j)^{N_j}
\]

in \( \mathcal{O}(V) \). Writing

\[
v = \prod_{j \in J} (x_j)^{N_j/N_j'},
\]

we obtain the equation \( \pi = uv^{N_j'} \). Therefore, \( E^o_J \) can be covered by finitely many open subschemes \( \mathcal{U}_i \) of \( \mathcal{X} \) as in the statement of the lemma. \( \square \)

We keep the notations of Lemma 2.3.1. We write

\[
\mathcal{U}_i = \text{Spec} \ A_i
\]

for \( i = 1, \ldots, r \), and we define a finite étale covering of \( \text{Spec} \ A_i \) by

(2.2) \[
\mathcal{V}_i = \text{Spec} \ A_i[t_i]/((t_i)^{N_j'} - u_i) \to \mathcal{U}_i.
\]

These coverings glue to a finite étale covering

(2.3) \[
\widetilde{\mathcal{U}} \to \mathcal{U}
\]

of degree \( N_j' \), the gluing data being given by \( t_i = v_j t_j/v_i \) over \( \mathcal{U}_{ij} := \mathcal{U}_i \cap \mathcal{U}_j \) (note that \( v_j/v_i \) is regular on \( \mathcal{V}_j \times \mathcal{U}_j \mathcal{U}_{ij} \), because this scheme is normal, and \( (v_j/v_i)^{N_j'} = u_i/u_j \in \mathcal{O}(\mathcal{U}_{ij}) \)). We put

\[
\widetilde{E}^o_j = \widetilde{\mathcal{U}} \times_{\mathcal{U}} E^o_j.
\]

This is a finite étale covering of \( E^o_j \) of degree \( N_j' \). Up to \( E^o_J \)-isomorphism, it is independent of the choices of \( \mathcal{U}_i, u_i \) and \( v_i \). In fact, we have the following alternative construction.

**Proposition 2.3.2.** Let \( J \) be a non-empty subset of \( I \), and denote by \( \mathcal{Y} \) the normalization of

\[
\mathcal{X} \times_R (R[s]/(s^{N_j'} - \pi)).
\]

Then \( \widetilde{E}^o_j \) and \( \mathcal{Y} \times_{\mathcal{X}} E^o_j \) are isomorphic as \( E^o_j \)-schemes.
Proof. We set $R' = R[s]/(s^{N_i} - \pi)$ and $\mathcal{X}' = \mathcal{X} \times_R R'$. We denote by $K'$ the quotient field of $R'$. It is a finite separable extension of $K$. The morphism

$$\mathcal{Y} \times_R K' \to \mathcal{X}' \times_R R' \cong \mathcal{X} \times_R K'$$

is an isomorphism, because $\mathcal{X} \times_R K$ is regular so that $\mathcal{X} \times_R K'$ is regular [EGA4b 6.7.4], and thus normal.

It is enough to show that, in the notation of (2.3), $\mathcal{W}$ is isomorphic to $\mathcal{Y} \times_\mathcal{X} \mathcal{W}$ as a $\mathcal{W}$-scheme. Since normalization commutes with open immersions, we may assume that $\mathcal{W} \cong \mathcal{X}$.

The scheme $\mathcal{W}$ is regular and $R$-flat, because $\mathcal{W} \to \mathcal{X}$ is étale and $\mathcal{W}$ is regular and $R$-flat. In particular, $\mathcal{W}$ is normal. The elements $t_i v_i \in \mathcal{O}(\mathcal{V}_i)$ glue to a regular function $w$ on $\mathcal{W}$. We have $w^{N_i} = \pi$ on $\mathcal{W}$ because this holds on every open $\mathcal{V}_i$.

There is a unique morphism of $\mathcal{W}$-schemes

$$g : \mathcal{W} \to \mathcal{X}$$

such that $s \circ g = w$, and it factors uniquely through a morphism

$$h : \mathcal{W} \to \mathcal{Y}$$

because $\mathcal{W}$ is normal. One sees from the local description in (2.2) that the induced morphism

$$h_{K'} : \mathcal{W} \times_R K' \to \mathcal{Y} \times_R K' \cong \mathcal{X} \times_R K'$$

is an isomorphism, since $\pi = u_i v_i^{N_i}$ and $v_i$ is a unit on $\mathcal{V}_i \times_R K$ for every $i$ in $\{1, \ldots, r\}$. Thus $h$ is birational, because $\mathcal{W}$ and $\mathcal{Y}$ are $R$-flat. Moreover, $\mathcal{W} \to \mathcal{Y}$ is finite, so that $h$ is finite. Since $\mathcal{Y}$ is normal, we can conclude by [EGA3a 4.4.9] that $h$ is an isomorphism. \hfill $\square$

2.4. Tame nearby cycles. Let $\mathcal{Y}$ be a separated $R$-scheme of finite type. Let $\Lambda$ be either $\mathbb{Q}_\ell$, or $\mathbb{Z}_\ell$, or a Noetherian torsion ring that is killed by an element of $\mathbb{N}$. We denote by $D^b_c(\mathcal{Y}, \Lambda)$ the bounded derived category of constructible sheaves of $\Lambda$-modules on $\mathcal{Y}$. If $\Lambda$ is a torsion ring this is simply the full subcategory of the derived category of étale sheaves of $\Lambda$-modules on $\mathcal{Y}$ consisting of complexes with bounded and constructible cohomology. If $\Lambda = \mathbb{Q}_\ell$ or $\mathbb{Z}_\ell$ the definition is more delicate; see [De80 1.1.2] (note that the finiteness conditions in c) and d) of [De80 1.1.2] are fulfilled, since $\mathcal{Y}$ is of finite type over the algebraically closed field $k$).

We denote by $R\psi_{\mathcal{Y}}(\Lambda)$ and $R\psi'^{\flat}_{\mathcal{Y}}(\Lambda) \in D^b_c(\mathcal{Y}, \Lambda)$ the complex of nearby cycles, resp. tame nearby cycles, with coefficients in $\Lambda$ associated to $\mathcal{Y}$. If $\Lambda$ is torsion, these objects were defined in [SGA7a Exp.I] and [SGA7b Exp.XIII], and the fact that $R\psi_{\mathcal{Y}}(\Lambda)$ is constructible was proven in [SGA4 4.2 Th. finitude(3.2)]. It follows that $R\psi'^{\flat}_{\mathcal{Y}}(\Lambda)$ is constructible, because

$$R^i \psi'^{\flat}_{\mathcal{Y}}(\Lambda) \cong (R^i \psi_{\mathcal{Y}}(\Lambda))^P$$

for every $i$ in $\mathbb{N}$ [SGA7a, 1.2.7.2].

For every integer $n > 0$, the object $R\psi_{\mathcal{Y}}(\mathbb{Z}/\ell^n)$ has finite Tor-dimension, and it is compatible with reduction of the coefficients modulo powers of $\ell$ [KW01 D.8]. Thus we can define the object $R\psi_{\mathcal{Y}}(\Lambda)$ in $D^b_c(\mathcal{Y}, \Lambda)$ when $\Lambda = \mathbb{Z}_\ell$ or $\Lambda = \mathbb{Q}_\ell$ by passing to the limit; see [De80 1.1.2(c)] and [KW01 p. 354].
Let $M$ be a $(\mathbb{Z}/\ell^n)$-module with continuous $P$-action. Since $P$ is a pro-$p$-group and $p$ is different from $\ell$, the module $M^P$ is a direct summand of $M$. It is split off by the averaging map

$$M \to M^P : m \mapsto \frac{1}{|P/P_m|} \sum_{g \in P/P_m} g \cdot m$$

where $P_m$ denotes the stabilizer of $m$, which is an open subgroup of $P$ and thus of finite index. It follows that

$$(M \otimes_{\mathbb{Z}/\ell^n} \mathbb{Z}/\ell^m)^P \cong M^P \otimes_{\mathbb{Z}/\ell^n} \mathbb{Z}/\ell^m$$

for all integers $n \geq m > 0$ and that the functor $(\cdot)^P$ is exact on the category of $(\mathbb{Z}/\ell^n)$-modules with continuous $P$-action.

Using these properties, we deduce from (2.4) that $R\psi_t^Y(\mathbb{Z}/\ell^n)$ has finite Tor-dimension for every integer $n > 0$ and that $R\psi_t^Y$ is compatible with reduction of the coefficients modulo powers of $\ell$. Thus, we can define $R\psi_t^Y(\Lambda)$ in $D^b_c(Y_{/\Lambda})$ when $\Lambda = \mathbb{Z}_\ell$ or $\Lambda = \mathbb{Q}_\ell$ by passing to the limit, and we still have an isomorphism (2.4) in those cases.

2.5. Tame nearby cycles on divisors with strict normal crossings. We keep the notations of Section 2.4. The following lemma and proposition constitute the key technical result of this section. The proofs were suggested to me by L. Illusie and T. Saito.

**Lemma 2.5.1.** Let $Y$ be a regular flat separated $R$-scheme of finite type, of pure dimension $n$. Consider an integer $q$ in $\{1, \ldots, n\}$ and a tuple $(M_1, \ldots, M_q)$ in $(\mathbb{Z}_{>0})^q$. For each $i \in \{1, \ldots, q\}$, we denote by $M'_i$ the largest divisor of $M_i$ that is not divisible by $p$. We put

$$\mu = \gcd\{M'_i \mid i \in \{1, \ldots, q\}\}.$$

Let $y$ be a closed point of $Y$. Assume that there exist a regular system of parameters $(y_1, \ldots, y_n)$ and a unit $v$ in $O_{Y,y}$ such that

$$\pi = v^\mu \prod_{i=1}^q (y_i)^{M_i}.$$  

Then there exists an integral affine open neighbourhood

$$U = \Spec B$$

of $y$ in $Y$ such that $y_1, \ldots, y_q$ are regular functions on $U$ and such that for each $m \in \mathbb{N}$, the sheaf

$$R^m \psi^Y_t(L)$$

is constant on the subscheme

$$U = \Spec (B/(y_1, \ldots, y_q))$$

of $Y$. 

**Proof.** It is enough to consider the case where $\Lambda$ is torsion. For each $i \in \{1, \ldots, q\}$, we write

$$M_i = e_i M'_i$$

with $e_i \in \mathbb{N}$. If $p = 0$ then all $e_i$ are equal to one; if $p > 1$ then all $e_i$ are powers of $p$. 

Shrinking $\mathcal{Y}$, we may assume that $\mathcal{Y}$ is integral and affine, say, $\mathcal{Y} = \text{Spec } B$, and that $v$ and $y_1, \ldots, y_q$ are regular functions on $\mathcal{Y}$, with $v$ a unit in $B$.  Then the equation (2.5) holds in $B$.  We may also assume that $\mathcal{Y}$ is an sncd-model, by Corollary (2.1.2) and that $y_i$ is a prime element in $B$ for $i = 1, \ldots, q$.  We put $U = \text{Spec } (B/(y_1, \ldots, y_q))$.

By Bézout’s theorem, there exist integers $\alpha_1, \ldots, \alpha_q$ such that

$$\mu = \sum_{i=1}^q \alpha_i M_i'.$$

We put

$$\mathcal{Z} = \text{Spec } R[z_1, \ldots, z_n]/(\pi - \prod_{i=1}^q (z_i)^{M_i}')$$

and we consider the morphism $f : \mathcal{Y} \to \mathcal{Z}$ defined by

$$z_i \mapsto v^\alpha_i(y_i)^\epsilon_i$$

for $i = 1, \ldots, q$,

$$z_i \mapsto y_i$$

for $i = q + 1, \ldots, n$.

Then $f$ maps $y \in \mathcal{Y}$ to the origin in $\mathcal{Z}$, and $f(U)$ is contained in the closed subscheme $V = \text{Spec } k[z_{q+1}, \ldots, z_n]$ of $\mathcal{Z}$.

We denote by $\theta$ the base change morphism (2.6)

$$\theta : f^* R^m \psi^t_{\mathcal{Z}}(\Lambda) \to R^m \psi^t_{\mathcal{Y}}(\Lambda)$$

of $\Lambda$-sheaves on $\mathcal{Y}$ [SGA7b, XIII.2.1.7.2].  We claim that $\theta$ is an isomorphism.  Assuming this for now, it suffices to prove that $R^m \psi^t_{\mathcal{Z}}(\Lambda)$ is constant on $V$.  Consider the morphism

$$g : \mathcal{Z} \to \mathcal{Z}' = \text{Spec } R[z'_1, \ldots, z'_q]/(\pi - \prod_{i=1}^q (z'_i)^{M_i'})$$

defined by

$$z'_i \mapsto z_i$$

for $i = 1, \ldots, q$.

The morphism $g$ is smooth.  By smooth base change, we have

$$R^m \psi^t_{\mathcal{Z}}(\Lambda) \cong g^* R^m \psi^t_{\mathcal{Z}'}(\Lambda)$$

for each $m \in \mathbb{N}$ [SGA7b, XIII.2.1.7.2].  Thus the restriction of $R^m \psi^t_{\mathcal{Z}}(\Lambda)$ to $V = g^{-1}(0)$ is constant.

It remains to prove our claim.  We’ll use the local computations in [SGA7a, I.3.3] of the tame nearby cycles on a divisor with strict normal crossings (these computations assume a purity property that was later proven by Gabber [Fu00]).  Let $\pi$ be a geometric point of $\mathcal{Y}$ and denote by $\overline{\pi}$ its image $f \circ \pi$ in $\mathcal{Z}$.  It is enough to show that, for all integers $m \geq 0$, the morphism

$$R^m \psi^t_{\mathcal{Z}}(\Lambda)_{\overline{\pi}} \to R^m \psi^t_{\mathcal{Y}}(\Lambda)_{\overline{\pi}}$$

obtained from (2.6) by passing to the stalks at $\overline{\pi}$, is an isomorphism.  We denote by $\mathcal{B}_{\overline{\pi}}$ and $\mathcal{Z}_{\overline{\pi}}$ the strict localization of $\mathcal{Y}$ at $\overline{\pi}$, resp. $\mathcal{Z}$ at $\overline{\pi}$, and we set $Y = \mathcal{B}_{\overline{\pi}} \times_R K$.
and $Z = \mathcal{Z}_\pi \times_K K$. Then $f$ induces a morphism of $K$-schemes $Y \to Z$ and we can identify (2.7) with the morphism

$$(2.8) \quad H^m(Z \times_K K^t, \Lambda) \to H^m(Y \times_K K^t, \Lambda)$$

(see [SGA7a I.2.3]).

Denote by $I$ the subset of $\{1, \ldots, q\}$ consisting of indices $i$ such that $y_i$ vanishes at $\pi$, and set $\nu = \gcd\{M'_i \mid i \in I\}$. We denote by $K'$ the unique degree $\nu$ extension of $K$ in $K^t$ (obtained by adding a $\nu$-th root of a uniformizer) and by $R'$ the normalization of $R$ in $K'$.

Since every unit in $\mathcal{O}(\mathcal{Z}_\pi)$ is a $\nu$-th power, the proof of Proposition 2.3.2 shows that the normalization of $\mathcal{Z}_\pi \times_R R'$ is the disjoint union of $\nu$ copies of $\mathcal{Z}_\pi$, which are transitively permuted by the Galois action of $G(K'/K) \cong \mu_\nu(k)$. The $R'$-structure of such a copy $\mathcal{C}$ of $\mathcal{Z}_\pi$ is determined by the choice of a $\nu$-th root of $\pi$ in $\mathcal{O}(\mathcal{Z}_\pi)$. The special fiber of the $R'$-scheme $\mathcal{C}$ is a divisor with strict normal crossings with multiplicities $M_i/\nu$, $i \in I$. The generic fiber of the normalization of $\mathcal{Z}_\pi \times_R R'$ is canonically isomorphic to $Y \times_K K'$. A similar description holds for the normalization of $\mathcal{Z}_\pi \times_R R'$. Therefore, by base change to $K'$, we may assume that $\nu = 1$.

Let us recall how, in the case $\nu = 1$, the cohomology of $Y \times_K K^t$ was computed in [SGA7a, I.3.3]. For every element $d$ of $\mathbb{N}'$, we set

$$(2.9) \quad Y_d = \Spec \left( \mathcal{O}(Y)[s_{d,i} \mid i \in I]/(s_{d,i}^d - y_i)_{i \in I} \right), \quad Z_d = \Spec \left( \mathcal{O}(Z)[t_{d,i} \mid i \in I]/(t_{d,i}^d - z_i)_{i \in I} \right).$$

For $d = 1$, we simply get $Y$ and $Z$. For every element $c$ of $\mathbb{N}'$, we define a morphism of $Y$-schemes $Y_{cd} \to Y_d$ and a morphism of $Z$-schemes $Z_{cd} \to Z_d$ by mapping $s_{cd,i}$ to $(s_{d,i})^c$ and $t_{cd,i}$ to $(t_{d,i})^c$ for every $i$. All these morphisms are finite and étale, because $c$ is not divisible by $p$ and $s_{cd,i}$, $t_{cd,i}$ are units on $Y$, resp. $Z$. In this way, we obtain projective systems of Galois coverings of $Y$ and $Z$, and by passing to the limit, we get procoverings $\tilde{Y} \to Y$ and $\tilde{Z} \to Z$.

Now the crucial point is the following. We choose a sequence of elements $v_d$ in $\mathcal{O}(Y)$, for $d \in \mathbb{N}'$, such that $v_1 = v$ and such that $v_d = (v_{cd})^c$ for all $c, d \in \mathbb{N}'$. This is possible because $v$ is a unit on the strictly henselian local scheme $\mathcal{Z}_\pi$ and the elements in $\mathbb{N}'$ are not divisible by $p$. For every $d$ in $\mathbb{N}'$, the morphism of $Y$-schemes

$$f_d : Y_d \to Z_d \times_Z Y : t_{d,i} \mapsto (v_d)^{\alpha_i}(s_{d,i})^{c_i}$$

is an isomorphism. We can construct its inverse as follows. Fix an element $d$ in $\mathbb{N}'$. Since the exponents $c_i$ are either one (for $p = 0$) or powers of $p$ (for $p > 0$), we know that $d$ is prime to $c_i$ for every $i$ in $I$. We choose for every $i$ an integer $\beta_i$ such that $d$ divides $e_i\beta_i - 1$, and we denote the quotient $(e_i\beta_i - 1)/d$ by $\gamma_i$. Then the $Y$-morphism

$$Z_d \times_Z Y \to Y_d : s_{d,i} \mapsto (v_d)^{-\alpha_i\beta_i}(y_i)^{-\gamma_i}(t_{d,i})^{\beta_i}$$

is inverse to $f_d$.

The isomorphisms $f_d$ are compatible with the transition morphisms in the projective systems $(Y_d)_{d \in \mathbb{N}'}$ and $(Z_d)_{d \in \mathbb{N}'}$, so that we obtain by passing to the limit an isomorphism of procoverings

$$(2.9) \quad \tilde{f} : \tilde{Y} \to \tilde{Z} \times_Z Y.$$
The procovering $\tilde{Y} \to Y$ factors through a morphism $\tilde{Y} \to Z \times K K^i$ because we can repeatedly take $d$-th roots of $\pi$ in $O(\tilde{Z})$ for all $d$ in $N'$. Indeed, in $O(\tilde{Z})$, $\pi$ equals a unit times a monomial in the elements $z_i$, $i \in I$, and we can always take the $d$-th root of a unit in $O(\tilde{Z})$ since $O(\tilde{Z})$ is strictly henselian and $d$ is not divisible by $p$. Via the morphism $\tilde{f}$ in \((2.9)\), the $K'$-structure on $\tilde{Z}$ induces a $K'$-structure on $\tilde{Y}$ so that the procovering $\tilde{Y} \to Y$ factors through $\tilde{Y} \to Y \times K K^i$.

By \cite[1.3.3.1]{SGA7}, the schemes $\tilde{Y}$ and $\tilde{Z}$ have trivial cohomology, so that the $E_2$-terms of the Hochschild-Serre spectral sequences associated to the procoverings $\tilde{Y} \to Y \times K K^i$ and $\tilde{Z} \to Z \times K K^i$ are concentrated in degrees $(*, 0)$ and we can use them to compute the cohomology of $Y \times K K^i$ and $Z \times K K^i$ \cite[III.2.21(b)]{SGA7}. The isomorphism \((2.8)\) induces an isomorphism between these spectral sequences. It follows that \((2.8)\) is an isomorphism for every integer $m \geq 0$.

Proposition 2.5.2. We follow the notations introduced in Section 2.3. Let $J$ be a non-empty subset of $I$, and fix an integer $m \geq 0$. The restriction of

$$R^m \psi_{\mathcal{X}}^i(\Lambda)$$

to $E^0_J$ is lisse, and tamely ramified along the irreducible components of $E_J \setminus E^0_J$.

More precisely, the sheaf

$$R^m \psi_{\mathcal{X}}^i(\Lambda)$$

becomes constant on the finite étale covering $\tilde{E}^0_J$ of $E^0_J$ of degree $N^i_J \in N'$.

Proof. We may assume that $\mathcal{X}$ is connected, and we denote its dimension by $n$. We put $q = |J|$. We may suppose that $E_J$ is non-empty. This implies that $q \leq n$. We choose a bijection between $J$ and $\{1, \ldots, q\}$.

Let $x$ be a closed point of $E^0_J$. There exist a regular system of parameters $(x_1, \ldots, x_n)$ and a unit $u$ in $O_{\mathcal{X}, x}$ such that

$$\pi = u \prod_{i=1}^q (x_i)^{N^i_i}.$$ 

Let $\mathcal{U}$ be a connected affine open neighbourhood of $x$ in $\mathcal{X}$ such that $x_1, \ldots, x_n$ and $u$ are regular functions on $\mathcal{U}$, and $u$ is a unit in $O(\mathcal{U})$. Consider the finite étale covering

$$f : \mathcal{U} = \text{Spec} (O(\mathcal{U})[v]/(v^{N^i_J} - u)) \to \mathcal{U}$$

and let $y$ be a point on $\mathcal{U}$ that is mapped to $x$ by $f$. Since $f$ is étale, we have an isomorphism

$$f^* R^m \psi_{\mathcal{X}}^i(\Lambda) \cong R^m \psi_{\mathcal{Y}}^i(\Lambda)$$

of constructible $\Lambda$-sheaves on $\mathcal{Y}$.

The locally closed subset $E^0_J$ of $\mathcal{X}$ might not be connected, but by the local computations in \cite[1.3.3]{SGA7}, it is enough to show that

$$R^m \psi_{\mathcal{X}}^i(\Lambda)$$

becomes constant on every connected component of $\tilde{E}^0_J$. By construction of the covering $\tilde{E}^0_J$, there is an isomorphism of $E^0_J$-schemes

$$\tilde{E}^0_J \times \mathcal{X} \mathcal{U} \cong \mathcal{U} \times \mathcal{X} E^0_J.$$ 

Thus it suffices to show that

$$R^m \psi_{\mathcal{U}}^i(\Lambda)$$
is constant on a Zariski-open neighbourhood of \( y \) in \( \mathcal{Y} \times \mathcal{X} \). This follows from Lemma 2.5.1 because
\[
(f^*x_1, \ldots, f^*x_n)
\]
is a regular system of parameters in \( \mathcal{O}_{\mathcal{Y}, y} \) by Lemma 2.1.4 and
\[
\pi = v^{N_i} \prod_{i=1}^d (f^*x_i)^N_i
\]
in \( \mathcal{O}_{\mathcal{Y}, y} \).
\[\square\]

2.6. The tame monodromy zeta function.

**Definition 2.6.1.** Let \( Y \) be a separated \( K \)-scheme of finite type. The tame monodromy zeta function \( \zeta_Y(t) \) of \( Y \) is defined by
\[
\zeta_Y(t) = \prod_{m \geq 0} \det(t \cdot \text{Id} - \varphi | H^m_c(Y \times_K K_t, \mathbb{Q}_\ell))^{(-1)^{m+1}} \in \mathbb{Q}_\ell(t).
\]

**Theorem 2.6.2.** We follow the notations introduced in Section 2.3. Let \( Z \) be a subscheme of \( \mathcal{X} \). We have
\[
\prod_{m \geq 0} \det(t \cdot \text{Id} - \varphi | H^m_c(Z, \mathbb{Q}_\ell))^{(-1)^{m+1}} = \prod_{i \in I} (t^{N'_i} - 1)^{-\chi(E^\circ_i)}
\]
and, for every element \( d \in \mathbb{Z}_{>0} \),
\[
\sum_{m \geq 0} (-1)^m \text{Trace}(\varphi^d | H^m_c(Z, \mathbb{Q}_\ell)) = \sum_{N'_i | d} N'_i \chi(E^\circ_i) \cap Z).
\]

In particular, if \( \mathcal{X} \) is proper, then the tame monodromy zeta function of \( X = \mathcal{X} \times_K K \) is given by
\[
\zeta_X(t) = \prod_{i \in I} (t^{N'_i} - 1)^{-\chi(E^\circ_i)}
\]
and for every element \( d \in \mathbb{Z}_{>0} \), we have
\[
\sum_{m \geq 0} (-1)^m \text{Trace}(\varphi^d | H^m(X \times_K K_t, \mathbb{Q}_\ell)) = \sum_{N'_i | d} N'_i \chi(E^\circ_i).
\]

**Proof.** Equations (2.12) and (2.13) follow from (2.10) and (2.11), by taking \( Z = \mathcal{X} \) and applying the spectral sequence for tame nearby cycles \([SGA7a, I.2.7.3]\). For every endomorphism \( M \) on a finite dimensional vector space \( V \) over a field \( F \) of characteristic zero, we have the identity \([De73, 1.5.3]\)
\[
\det(\text{Id} - t \cdot M | V)^{-1} = \exp(\sum_{d > 0} \text{Trace}(M^d | V) \frac{t^d}{d})
\]
in \( F[[t]] \). Using this identity, (2.10) can easily be deduced from (2.11) (for a similar argument, see \([AC75, \S 1]\)). So it suffices to prove (2.11). Both sides of (2.11) are additive w.r.t. partitions of \( Z \) into subvarieties, so that we may assume that \( Z \) is contained in \( E^\circ_j \) for some non-empty subset \( J \) of \( I \), and that \( Z \) is normal. We choose a normal compactification \( \overline{Z} \) of \( Z \), and a closed point \( z \) on \( Z \).
By the spectral sequence for hypercohomology, we have
\[
\sum_{m \geq 0} (-1)^m \text{Trace}(\varphi^d | H^m_c(Z, R\psi^t_X(Q_\ell)|_Z)) = \sum_{a, b \geq 0} (-1)^{a+b} \text{Trace}(\varphi^d | H^a_c(Z, R^b\psi^t_X(Q_\ell)|_Z))
\]

By Proposition 2.5.2, the sheaf \( R^b\psi^t_X(Q_\ell)|_Z \) is lisse, and tamely ramified along the irreducible components of \( Z \setminus Z \). By the local computation in [SGA7a, I.3.3], the action of \( \varphi \) on \( R^b\psi^t_X(Q_\ell)|_Z \) has finite order. By [NS07b, 5.1] and [SGA7a, I.3.3], we have
\[
\sum_{a, b \geq 0} (-1)^{a+b} \text{Trace}(\varphi^d | R^b\psi^t_X(Q_\ell)|_Z) = \chi(Z) \cdot \sum_{b \geq 0} (-1)^b \text{Trace}(\varphi^d | R^b\psi^t_X(Q_\ell)|_Z)
\]
\[
= \begin{cases} 
0 & \text{if } |J| > 1 \text{ or } J = \{i\} \text{ with } N'_i \nmid d, \\
N'_i \chi(Z) & \text{if } J = \{i\} \text{ with } N'_i | d.
\end{cases}
\]

(in [NS07b, 5.1], the condition that \( Y \) is normal should be added to the statement of the lemma). This concludes the proof. \( \square \)

**Corollary 2.6.3.** Assume that \( \mathcal{X} \) is proper. If \( X = \mathcal{X} \times_R K \) is cohomologically tame, then
\[
\chi(X) = \sum_{i \in I} N'_i \chi(E_i^o).
\]
The converse implication holds if \( X \) is geometrically irreducible and of dimension one.

**Proof.** The opposite of the degree of \( \zeta_X(t) \) is equal to the tame Euler characteristic
\[
\chi_{\text{tame}}(X) = \sum_{m \geq 0} (-1)^m \dim H^m(X \times_K K^*, Q_\ell).
\]
By Theorem 2.6.2, we find
\[
\chi_{\text{tame}}(X) = \sum_{i \in I} N'_i \chi(E_i^o).
\]
If \( X \) is cohomologically tame, then \( \chi(X) = \chi_{\text{tame}}(X) \). If \( X \) is geometrically irreducible and of dimension one, then the converse implication holds as well, because the wild inertia acts trivially on \( H^m(X \times_K K^*, Q_\ell) \) for \( m \in \{0, 2\} \). \( \square \)

**Remark 2.6.4.** If \( X \) is a curve, then one can use [Ab00, 3.3] instead of Proposition 2.5.2 to prove Theorem 2.6.2. This case suffices for the applications in Section 3. The case of dimension \( > 1 \) is needed in Section 4.
3. Saito’s criterion for cohomological tameness, and the semi-stable reduction theorem

3.1. Numerical criteria for cohomological tameness. Let $C$ be a smooth, projective, geometrically connected curve of genus $g$, and let $C^{\varphi}$ be an $sncd$-model for $C$, with

$$C^{\varphi} = \sum_{i \in I} N_i E_i.$$

For every non-empty subset $J$ of $I$, we define $E_J$ and $E_o^J$ as in Section 2.3. For every integer $d \geq 0$, we denote by $I_d$ the subset of $I$ consisting of the indices $i$ such that $d|N_i$. For every $i \in I$, we put

$$\kappa_i = -(E_i \cdot E_i)$$
$$\nu_i = (E_i \cdot K_{C/R})$$

where $K_{C/R}$ is a relative canonical divisor. We denote by $g_i$ the genus of $E_i$. The component $E_i$ is called principal if $g_i > 0$ or $E_i \backslash E_o^i$ contains at least three points.

Recall the following well-known identities, for each $j \in I$:

$$\sum_{i \in I} N_i (E_i \cdot E_j) = 0,$$  \hspace{1cm} (3.1)
$$2g_j - 2 = \nu_j - \kappa_j,$$  \hspace{1cm} (3.2)
$$2g - 2 = \sum_{i \in I} N_i \nu_i.$$  \hspace{1cm} (3.3)

The first formula is obtained by intersecting $C^{\varphi}$ with $E_j$ [Li02, 9.1.21], the second and third follow from the adjunction formula [Li02, 9.1.37].

Since $\varphi$ acts trivially on $H^m(C \times K K^t, \mathbb{Q}_\ell)$ for $m \in \{0, 2\}$, it follows from (2.10) that the characteristic polynomial

$$P_C(t) = \det(t \cdot \operatorname{Id} - \varphi | H^1(C \times K K^t, \mathbb{Q}_\ell))$$

is given by

$$P_C(t) = (t - 1)^2 \prod_{i \in I} (t^{n_i} - 1)^{-\chi(E_o^i)}.$$  \hspace{1cm} (3.4)

Lemma 3.1.1. We have

$$\chi(C) = \sum_{i \in I} N_i \chi(E_o^i).$$

Proof. By (3.3), we have

$$\chi(C) = 2 - 2g = -\sum_{i \in I} N_i \nu_i.$$

Solving $\nu_i$ from equation (3.2), we find

$$\chi(C) = \sum_{i \in I} N_i (\chi(E_i) - \kappa_i).$$
denote by \( R \) elements. Assume that \( \text{properness} \), the point \( a \) and /a similar fashion. We may assume that \( B´ezout’s \) theorem, we can find integers \( m \).

Note that \( N \in \mathbb{N} \). We take an integral affine étale neighbourhood \( K \) for each \( d \). For each \( \ell \) \( \mathbb{N} \), \( \mathbb{N}_d \). \( \mathbb{N}_j \) such that \( \alpha_i \mathbb{N}_j + \alpha_j \mathbb{N}_i \mathbb{N}_j = d \). We set \( N = \gcd(N_i, N_j) \). By Bézout’s theorem, we can find integers \( m_i \) and \( m_j \) such that \( N = m_i N_i + m_j N_j \). Note that \( N \) divides \( d \), so that \( N \) must belong to \( \mathbb{N}^t \).

Let \( B \) be a point of \( E^o_{(i,j)} \). There exist a regular system of parameters \( (x, y) \) and a unit \( u \) in \( \mathcal{O}_{E, B} \) such that
\[
\pi = u x^{N_i} y^{N_j}.
\]

We take an integral affine étale neighbourhood \( U = \text{Spec} B \) of \( B \) in \( \mathcal{C} \) such that \( u, x, y \) are regular functions on \( U \) and \( u = v^N \) for some unit \( v \) in \( B \). Then we can define a morphism
\[
f : U \to V = \text{Spec} R[x', y']/(\pi - (x')^{N_i} (y')^{N_j}).
\]

Lemma 3.1.2. For each \( d \in \mathbb{N} \), we denote by \( K(d) \) the unique extension of \( K \) in \( \mathbb{N}^t \) of degree \( d \). For each couple \((i, j)\) in \( I \times I \), we consider the set
\[
D_{i,j} = \{ \alpha_i N_i + \alpha_j N_j \mid \alpha_i, \alpha_j \in \mathbb{N} \}.
\]

We denote by \( S \) the subset of \( I \times I \) consisting of the couples \((i, j)\) such that \( E^o_{(i,j)} \neq \emptyset \), and we put
\[
D_\varphi = \mathbb{N} \cap ( \cup_{(i,j) \in S} D_{i,j} )
\]

Then for each \( d \in \mathbb{N} \), the set \( C(K(d)) \) is non-empty if and only if \( d \) belongs to \( D_\varphi \).

Proof. Assume that \( C(K(d)) \) is non-empty. Let \( a \) be an element of \( C(K(d)) \), and denote by \( R(d) \) the normalization of \( R \) in \( K(d) \). By the valuative criterion for properness, the point \( a \) extends uniquely to a section \( \psi_a \) in \( \mathcal{C}(R(d)) \). We denote by \( a_0 \) the image in \( \mathcal{C}_s \) of the closed point of \( \text{Spec} R(d) \). The point \( a_0 \) belongs to \( E^o_{(i,j)} \), for some couple \((i, j)\) in \( S \), and this couple is unique up to transposition.

There exist elements \( x, y, u \) in \( \mathcal{O}_{\mathcal{C}, a_0} \) such that \( u \) is a unit and \( \pi = u x^{N_i} y^{N_j} \). If we denote by \( v_{K(d)} \) the normalized discrete valuation on \( K(d) \), then the equality
\[
\pi = \psi_a^*(u) \psi_a^*(x)^{N_i} \psi_a^*(y)^{N_j}
\]
in \( R(d) \) implies that
\[
d = N_i \cdot v_{K(d)}(\psi_a^*(x)) + N_j \cdot v_{K(d)}(\psi_a^*(y)).
\]

It follows that \( d \in D_{i,j} \).

So let us show the converse implication. Let \( d \) be an element of \( \mathbb{N} \cap D_{i,j} \), for some \((i, j)\) in \( S \). We'll treat the case where \( i \neq j \), the other case can be known in a similar fashion. We may assume that \( d \notin D_{i,i} \) and \( d \notin D_{i,j} \). Then there exist elements \( \alpha_i, \alpha_j \) in \( \mathbb{N}_0 \) such that \( \alpha_i N_i + \alpha_j N_j = d \). We set \( N = \gcd(N_i, N_j) \). By Bézout’s theorem, we can find integers \( m_i \) and \( m_j \) such that \( N = m_i N_i + m_j N_j \). Note that \( N \) divides \( d \), so that \( N \) must belong to \( \mathbb{N}^t \).

Let \( B \) be a point of \( E^o_{(i,j)} \). There exist a regular system of parameters \((x, y)\) and a unit \( u \) in \( \mathcal{O}_{E, B} \) such that
\[
\pi = u x^{N_i} y^{N_j}.
\]
by \( x' \mapsto v^{m_i}x \) and \( y' \mapsto v^{m_i}y \). This morphism is étale at every point of \( U \times \mathscr{C} b \), by Lemma 2.1.4. Since \( R(d) \) is strictly henselian, it suffices to show that \( V(R(d)) \) contains a section that maps the closed point of Spec \( R(d) \) to the origin in \( V_s \); this section will then lift to \( U \). We can construct such a section by sending \( x' \) to \( \pi(d)^{\alpha_i} \) and \( y' \) to \( \pi(d)^{\alpha_j} \), where \( \pi(d) \) is an element of \( R(d) \) such that \( \pi(d)^{\alpha_i} = \pi(d)^{\alpha_j} \). This concludes the proof. □

**Corollary 3.1.3.** The set \( C(K) \) is non-empty if and only if there exists an element \( \alpha \) in \( I \) such that \( N\alpha = 1 \). The set \( C(K^1) \) is non-empty if and only if there exists an element \( \beta \) in \( I \) with \( p \nmid N\beta \).

We will repeatedly use the following elementary lemma.

**Lemma 3.1.4.** Let \( I' \) be a non-empty subset of \( I \) such that \( \bigcup_{i \in I'} E_i \) is connected. Then \( \sum_{i \in I'} \chi(E_i) \leq 0 \), unless \( \bigcup_{i \in I'} E_i \) is a tree of rational curves. In the latter case, \( \sum_{i \in I'} \chi(E_i) = 2 \).

**Proof.** This is easily proven by induction on \( |J| \); see [Ro04, 2.2]. □

**Lemma 3.1.5.** Fix an integer \( d > 1 \), and let \( I' \) be a subset of \( I_d \) such that \( \bigcup_{i \in I'} E_i \) is a connected component of \( \bigcup_{i \in I_d} E_i \). Then we have

\[
\sum_{i \in I'} \chi(E_i) \leq 0.
\]

**Proof.** First, suppose that \( I' = I \), and that

\[
\sum_{i \in I'} \chi(E_i) > 0.
\]

Then \( C_s \) is a tree of rational curves, by Lemma 3.1.4. By Lemma 3.1.1 we find that \( \chi(C) \) is at least \( 2d > 2 \), which is impossible.

Hence, we may assume that there exists an index \( a \in I' \) such that \( E_a \) meets a component of \( C_s \) whose multiplicity is not divisible by \( d \). By (3.1), the intersection of \( E_a \) with

\[
\bigcup_{i \in I \setminus I_d} E_i
\]

contains at least two points. For each \( i \in I' \), we put

\[
E_i' = E_i \setminus (\bigcup_{j \in I' \setminus \{i\}} E_j).
\]

Then

\[
\sum_{i \in I'} \chi(E_i') \leq 2
\]

by Lemma 3.1.4, and since \( \bigcup_{i \in I'} E_i' \) is an open subset of \( \bigcup_{i \in I'} E_i' \) whose complement contains at least two points, we find

\[
\sum_{i \in I'} \chi(E_i') \leq 0.
\]

□
Proposition 3.1.6. The rational function
\[ Q_C(t) = (t-1)^2 \prod_{i \in I} (t^{N_i} - 1)^{-\chi(E_i^0)} \]
is a polynomial in \( \mathbb{Z}[t] \). It is divisible by the characteristic polynomial \( P_C(t) \) of the tame monodromy operator \( \varphi \) on \( H^1(C \times_K K^1, Q_\ell) \).

Proof. By (3.4), we have \( P_C(t) = Q_C(t) \) if \( p = 0 \), so that we may assume that \( p > 0 \). The prime factorization of \( Q_C(t) \) is given by
\[ Q_C(t) = (t-1)^2 - \sum_{i \in I} \chi(E_i^0) \prod_{d > 1} \Phi_d(t)^{-\sum_{i \in I_d} \chi(E_i^0)}. \]

It follows from Lemma 3.1.4 that
\[ 2 - \sum_{i \in I} \chi(E_i^0) \geq 0. \]

By Lemma 3.1.5 we know that that
\[ -\sum_{i \in I_d} \chi(E_i^0) \geq 0 \]
for all \( d > 1 \). It follows that \( Q_C(t) \) is a polynomial. By formula (3.4), we have
\[ \frac{Q_C(t)}{P_C(t)} = \prod_{d > 0} \Phi_{d_p}(t)^{-\sum_{i \in I_{d_p}} \chi(E_i^0)} \]
which is a polynomial by Lemma 3.1.5. \( \square \)

Corollary 3.1.7. The following are equivalent:
(1) the curve \( C \) is cohomologically tame,
(2) we have
\[ \sum_{i \in I} N_i \chi(E_i^0) = \sum_{i \in I} N_i' \chi(E_i^0), \]
(3) we have
\[ P_C(t) = (t-1)^2 \prod_{i \in I} (t^{N_i} - 1)^{-\chi(E_i^0)} = (t-1)^2 \prod_{i \in I} (t^{N_i} - 1)^{-\chi(E_i^0)}. \]

Proof. The equivalence of (1) and (2) follows from Corollary 2.6.3 and Lemma 3.1.1. Point (3) implies (2) by comparing degrees. If (2) holds, then \( P_C(t) \) and \( Q_C(t) \) are monic polynomials of the same degree, so they coincide because \( P_C(t) \) divides \( Q_C(t) \) by Proposition 3.1.6. Hence, (2) implies (3). \( \square \)

3.2. Tame models.

Definition 3.2.1. Let \( d \) be an element of \( \mathbb{N} \). We say that \( C \) is \( d \)-tame if \( I \neq I_d \) and, for each \( i \in I_d \), we have \( \chi(E_i^0) = 0 \).

In particular, \( C \) is always 0-tame.

Lemma 3.2.2. Let \( d \) be an element of \( \mathbb{Z}_{>1} \). Then \( C \) is \( d \)-tame iff for each \( i \in I_d \), the following properties hold:
- \( E_i \cong \mathbb{P}_k^1 \),
- \( E_i \setminus E_i^0 \) consists of precisely two points,
- if \( E_i \) intersects \( E_j \) with \( j \in I \setminus \{i\} \), then \( N_j \) is not divisible by \( d \).
Proof. The “if” part is trivial, so let us prove the converse implication. Assume that \( \mathcal{C} \) is \( d \)-tame, and let \( \alpha \) be an element of \( I \) such that \( d \| N_\alpha \). We know that \( \chi(E^\alpha_\alpha) = 0 \). So either \( E_\alpha = E^\alpha_\alpha \) and \( E_\alpha \) is an elliptic curve, or \( E_\alpha \) is a rational curve and \( E_\alpha \setminus E^\alpha_\alpha \) consists of precisely two points. The first possibility cannot occur, since it would imply that \( I = I_d = \{ \alpha \} \).

Assume that \( E_\alpha \) meets precisely one other component \( E_\beta \) of \( \mathcal{C}_s \). Then \( E_\alpha \cap E_\beta \) consists of exactly two points. Assume that \( d \| N_\beta \). Since \( \mathcal{C} \) is \( d \)-tame, we see that \( E_\beta \) is rational, and that \( E_\beta \) meets no other components of \( \mathcal{C}_s \). Hence, \( I = \{ \alpha, \beta \} \). This contradicts the fact that \( I \neq I_d \).

So we may assume that \( E_\alpha \) meets precisely two other components \( E_\beta \) and \( E_\gamma \) of \( \mathcal{C}_s \), each of them in exactly one point. It suffices to show that \( d \nmid N_\beta \) and \( d \nmid N_\gamma \). If \( d \| N_\beta \), then \( d \| N_\gamma \), by equation \((3.6)\) (applied to \( j = \alpha \)). Repeating the arguments (with \( \alpha \) replaced by \( \beta \), resp. \( \gamma \)), we find that \( \mathcal{C}_s \) is a loop of rational curves, and that \( d \| N_i \) for each \( i \in I \). This contradicts the \( d \)-tameness of \( \mathcal{C} \). \( \square \)

**Theorem 3.2.3.** We fix an integer \( d > 1 \). Assume that \( \mathcal{C} \) is a relatively minimal sncl-model of \( C \). If \( I \neq I_d \), then the following are equivalent:

1. the polynomial
   \[
   Q_C(t) = (t - 1)^2 \prod_{i \in I}(t^{N_i} - 1)^{-\chi(E^\alpha_i)} \in \mathbb{Z}[t]
   \]
   has no root whose order in \( \mathbb{G}_m(\mathbb{Q}^a) \) is divisible by \( d \),
2. \( \mathcal{C} \) is \( d \)-tame.

Moreover, if \( I = I_d \) and (1) holds, then \( g = 1 \).

**Proof.** It is obvious that (2) implies (1). Assume, conversely, that (1) holds. For each integer \( m > 0 \), we denote by \( I_{=m} \) the subset of \( I \) consisting of the indices \( i \) with \( N_i = m \). By our assumption, we have

\[
(3.5) \sum_{i \in I_{\geq n}} \chi(E^\alpha_i) = 0
\]

for each \( n \in \mathbb{Z}_{\geq 0} \), since this is the exponent of the cyclotomic polynomial \( \Phi_{d_n}(t) \) in the prime factorization of \( Q_C(t) \). Taking linear combinations of these equations, we see that

\[
(3.6) \sum_{i \in I_{\geq n}} \chi(E^\alpha_i) = 0
\]

for each \( n \in \mathbb{Z}_{\geq 0} \). In particular, if \( I = I_d \), Lemma \((3.1)\) implies that \( g = 1 \). Hence, we may assume that \( I \neq I_d \).

Suppose that \( \mathcal{C} \) is not \( d \)-tame. Then there exists an index \( \alpha \in I \) such that \( d \| N_\alpha \) and \( \chi(E^\alpha_\alpha) \neq 0 \). We choose such an \( \alpha \) with maximal \( N_\alpha \). By \((3.6)\), we may assume that \( \chi(E^\alpha_\alpha) > 0 \). Then \( E_\alpha \) is rational, and \( E_\alpha \) meets exactly one other component \( E_\beta \) of \( \mathcal{C}_s \), in precisely one point. By \((3.1)\), we know that \( N_\beta = \kappa_\alpha N_\alpha \). Since \( \mathcal{C} \) is relatively minimal, \( \kappa_\alpha \) must be at least 2, by Castelnuovo’s criterion and Proposition \((2.2.2)\). It follows that \( \beta \) belongs to \( I_d \), and that \( N_\beta \geq 2N_\alpha \). By maximality of \( N_\alpha \), we must have \( \chi(E^\alpha_\beta) = 0 \). Hence, \( E_\beta \) is rational and meets precisely one component \( E_\gamma \) of \( \mathcal{C}_s \) distinct from \( E_\alpha \), in exactly one point. Again applying \((3.1)\), Castelnuovo’s criterion and Proposition \((2.2.2)\) we find that

\[
\kappa_\beta N_\beta = N_\alpha + N_\gamma
\]
and $\kappa_3 \geq 2$, so that $d|N_3$ and $N_3 > N_3$. Repeating the arguments, we produce an infinite chain of rational curves in $\mathcal{C}_3$, which is impossible. 

\textbf{Corollary 3.2.4.} Assume that $\mathcal{C}$ is a relatively minimal sncd-model of $C$, and that $C$ is cohomologically tame. Suppose either that $g \neq 1$ or that $C$ is an elliptic curve. Then for each integer $d > 1$, the following are equivalent:

1. the model $\mathcal{C}$ is $d$-tame,
2. $\varphi$ has no eigenvalue on $H^1(C \times K^t, Q_\ell)$ whose order in $\mathbb{G}_m(Q_\ell)$ is divisible by $d$.

In particular, $\mathcal{C}$ is $p$-tame.

\textit{Proof.} By Corollary \textbf{3.1.7} $Q_C(t) = P_C(t)$. Clearly, property (1) cannot hold if $I = I_d$. Neither can (2): by Theorem \textbf{3.2.3} the conjunction of (2) and $I = I_d$ would imply that $g = 1$, so that $C$ would have a rational point, by our assumptions. This contradicts $I = I_d$, by Corollary \textbf{3.1.3}. Hence, we may assume that $I \neq I_d$. From Theorem \textbf{3.2.3} we get the equivalence of (1) and (2). Then $p$-tameness follows from the fact that the pro-$p$-part of $G(K^t/K)$ is trivial, so that the order of an eigenvalue of $\varphi$ in $\mathbb{G}_m(Q_\ell)$ cannot be divisible by $p$. \hfill \Box

3.3. Saito’s criterion for cohomological tameness.

\textbf{Definition 3.3.1.} We denote by $\text{Jac}(C)$ the Jacobian of $C$. We say that $C$ is pseudo-wild if $g = 1$, $C(K^t)$ is empty, and one of the following holds:

- $p > 3$,
- $p = 2$, and $\text{Jac}(C)$ has reduction type $I_n$ ($n \geq 0$), IV or IV$^*$,
- $p = 3$ and $\text{Jac}(C)$ has reduction type $I_n$, $I_n^*$ ($n \geq 0$), III, or III$^*$.

\textbf{Theorem 3.3.2} (Saito’s criterion for cohomological tameness). Assume that $\mathcal{C}$ is a relatively minimal sncd-model of $C$. The following are equivalent:

1. the curve $C$ is cohomologically tame,
2. one of the following two conditions is satisfied:
   - $\mathcal{C}$ is $p$-tame,
   - $C$ is pseudo-wild.

In (2), the two conditions are disjoint, i.e., at most one of them can hold.

\textit{Proof.} Let us first explain why the conditions in (2) are disjoint. Assume that $C$ is pseudo-wild. By Corollary \textbf{3.1.3} emptiness of $C(K^t)$ implies that $p|N_i$ for all $i \in I$. Hence, $\mathcal{C}$ cannot be $p$-tame.

It follows immediately from Corollary \textbf{3.1.7} that (2) implies (1). It remains to show that (1) implies (2). By Corollary \textbf{3.2.3} it suffices to consider the case where $g = 1$ and $C(K)$ is empty. We may also assume that $I = I_p$, since otherwise, Theorem \textbf{3.2.3} implies that $\mathcal{C}$ is $p$-tame.

The equality $I = I_p$ implies that $C(K^t)$ is empty, by Corollary \textbf{3.1.3}. Since $C$ is cohomologically tame, the same holds for its Jacobian $\text{Jac}(C)$. Looking at the Kodaira-Néron reduction table, and applying Corollary \textbf{3.2.3} we see that an elliptic $K$-curve $E$ is cohomologically tame iff one of the following conditions holds:

- $p > 3$,
- $p = 2$ and $E$ has reduction type $I_n$ ($n \geq 0$), IV or IV$^*$,
- $p = 3$ and $E$ has reduction type $I_n$, $I_n^*$ ($n \geq 0$), III, or III$^*$.

Applying this criterion to the case $E = \text{Jac}(C)$, we find that $C$ is pseudo-wild. \hfill \Box
Corollary 3.3.3. The following are equivalent:

(1) $C$ is pseudo-wild,
(2) $C$ is cohomologically tame, and $C(K^t)$ is empty.

Proof. The implication (1) ⇒ (2) follows from Theorem 3.3.2. Conversely, assume that (2) holds, and suppose that $C$ is a relatively minimal sncd-model of $C$. Emptiness of $C(K^t)$ implies $I = I_p$, by Corollary 3.1.3 so that $C$ is not $p$-tame. Hence, by Theorem 3.3.2 $C$ is pseudo-wild. □

3.4. The semi-stable reduction theorem. Recall the following well-known lemma.

Lemma 3.4.1. If the wild inertia $P$ acts continuously on a finite dimensional $\mathbb{Q}_\ell$-vector space $V$, then the action factors through a finite quotient of $P$.

Proof. Continuity of the action implies that $V$ admits a $\mathbb{Z}_\ell$-lattice $M$ that is stable under the action of $P$ [Se68, §1.1]. The image $P_0$ of $P$ in the automorphism group $\text{Aut}(M)$ is a pro-$p$-group, so that it has trivial intersection with the pro-$\ell$-group $\ker(\text{Aut}(M) \to \text{Aut}(M/\ell M))$. It follows that $P_0$ is isomorphic to a subgroup of the finite group $\text{Aut}(M/\ell M)$. □

Theorem 3.4.2 (Cohomological criterion for semi-stable reduction; Deligne - Mumford, Saito). Assume that $\mathcal{D}$ is a relatively minimal sncd-model of $C$. We assume that either $g \neq 1$, or $C$ is an elliptic curve. Then $\mathcal{D}$ is semi-stable iff the monodromy action on

$$H^1(C \times_K K^s, \mathbb{Q}_\ell)$$

is unipotent.

Proof. Let $\mathcal{C}$ be the relatively minimal sncd-model of $C$ obtained by blowing up the self-intersection points of the irreducible components of $\mathcal{D}_s$. We write

$$\mathcal{C}_s = \sum_{i \in I} N_i E_i$$

as before.

If $\mathcal{D}$ is semi-stable, then $C$ is cohomologically tame, by Corollary 3.1.7. The action of $\phi$ on

$$H^1(C \times_K K^t, \mathbb{Q}_\ell)$$

is unipotent, by formula (3.4). It follows that the monodromy action on

$$H^1(C \times_K K^s, \mathbb{Q}_\ell)$$

is unipotent.

Conversely, suppose that the monodromy action on

$$H^1(C \times_K K^s, \mathbb{Q}_\ell)$$

is unipotent. Then $C$ is cohomologically tame, by Lemma 3.4.1. We denote by $I_{>1}$ the subset of $I$ consisting of indices $i$ with $N_i > 1$. By Corollary 3.2.3, we know that $\mathcal{C}$ is $d$-tame for all $d > 1$. In particular, $\chi(E_i^p) = 0$ for each $i \in I_{>1}$. Hence, by Lemma 3.1.3, we must have $I \neq I_{>1}$ if $g \neq 1$. If $g = 1$, then we also have $I \neq I_{>1}$ by Corollary 3.1.3 since $C$ has a rational point.
Let \( j \) be an element of \( I_{>1} \). Then by Lemma 3.2.2, \( E_j \) is rational, and \( E_j \setminus E_j^o \) consists of exactly two points. Since \( I \neq I_{>1} \), the component \( E_j \) fits into a sequence of components

\[
E_{j_0}, \ldots, E_{j_{a+1}}
\]
satisfying the conditions of [Ha10, 5.1], so that the component \( E_j \) is contracted by \( h \). Hence, \( \mathcal{D} \) is semi-stable. \( \square \)

**Corollary 3.4.3** (Semi-stable reduction theorem; Deligne-Mumford). There exists a finite separable extension \( K' \) of \( K \) such that every relatively minimal \( ncd \)-model of \( C \times_K K' \) is semi-stable.

**Proof.** The monodromy action on

\[
H^1(C \times_K K^*, \mathbb{Q}_\ell)
\]
is quasi-unipotent, by the monodromy theorem [SGA7a, I.1.3]. This also follows immediately from (3.4) and Lemma 3.4.1. \( \square \)

**Corollary 3.4.4.** Assume that \( \mathcal{C} \) is a relatively minimal \( ncd \)-model of \( C \), and that \( C \) is cohomologically tame. Suppose either that \( g \neq 1 \) or that \( C \) is an elliptic curve. The degree \( e \) of the minimal extension of \( K \) where \( C \) acquires semi-stable reduction is equal to

\[
\text{lcm} \{ N_i | i \in I, E_i \text{ is principal} \}.
\]

**Proof.** Note that \( E_i \) is principal iff \( \chi(E_i^o) < 0 \), or \( E_i = E_i^o \) and \( E_i \) is an elliptic curve. The latter possibility only occurs if \( C \) is an elliptic curve with good reduction, in which case the statement is obvious.

It follows from Corollary 3.2.4 that

\[
(3.7) \quad e = \text{lcm} \{ d \in \mathbb{Z}_{>1} | \mathcal{C} \text{ is not} \ d\text{-tame} \}
\]

\[
(3.8) \quad = \text{lcm} \{ N_i | \chi(E_i^o) \neq 0 \text{ or } I = I_{N_i} \}.
\]

On the other hand, by the implication (1) \( \Rightarrow \) (3) in Corollary 3.1.7, we know that \( e \) divides

\[
\text{lcm} \{ N_i | \chi(E_i^o) < 0 \}.
\]

This value divides the right hand side of (3.8). It follows that

\[
e = \text{lcm} \{ N_i | \chi(E_i^o) < 0 \}.
\]

\( \square \)

If \( g > 1 \), Corollary 3.4.3 was proven by a different method in [Ha10, 7.5].

**Remark 3.4.5.** If \( C \) has genus at least one, then it is not hard to show that the minimal \( ncd \)-model \( \mathcal{D} \) of \( C \) is semi-stable if and only if the minimal regular model \( \mathcal{E} \) of \( C \) is a semi-stable \( ncd \)-model. The “if” part is obvious. The model \( \mathcal{D} \) is obtained by blowing up \( \mathcal{E} \) at points of \( \mathcal{E}_s \) where \( \mathcal{E}_s \) does not have normal crossings. Such a point is never a regular point of \( \mathcal{E}_s \), so that the exceptional divisor of the blow-up has multiplicity at least two in \( \mathcal{D}_s \). This means that \( \mathcal{D} \to \mathcal{E} \) must be an isomorphism if \( \mathcal{D} \) is semi-stable.
3.5. Some counterexamples. To conclude this section, we discuss some examples that show that certain conditions in the statements of the above results cannot be omitted.

1. Theorem 3.4.2 is false if we take for $D$ a relatively minimal $snecd$-model of $C$. If $C$ is an elliptic curve of type $I_1$, then its minimal $ncd$-model $C$ is semi-stable, but the special fiber of its minimal $snecd$-model contains a component of multiplicity 2 (the exceptional curve of the blow-up of $C$ at the self-intersection point of $C$).

2. Theorem 3.4.2 can fail for curves of genus one without rational point, namely, for non-trivial $E$-torsors over $K$, with $E$ an elliptic $K$-curve with semi-stable reduction.

3. Theorem 3.3.2 is false if we don’t assume that $C$ is relatively minimal. If $N_i$ is divisible by $p$, then blowing up a point of $E_i$ destroys $p$-tameness of the model.

4. Corollaries 3.2.4 and 3.4.4 can fail for genus one curves without rational point, for instance, for non-trivial $E$-torsors over $K$, with $E$ an elliptic $K$-curve with good reduction.

5. Corollary 3.4.4 is false if we do not assume that $C$ is cohomologically tame. For instance, if $k$ has characteristic 2 and $R$ is the ring of Witt vectors over $k$, then the elliptic $K$-curve with Weierstrass equation $y^2 = x^3 + 2$ has reduction type $II$, while it acquires good reduction over $K(\sqrt{2})$.

4. The trace formula

4.1. The rational volume and the trace formula. Let $X$ be a smooth and proper $K$-variety. Recall that a weak Néron model for $X$ is a separated smooth $R$-scheme of finite type $\mathfrak{X}$, endowed with an isomorphism $\mathfrak{X} \times_R K \cong X$, such that the natural map

$$\mathfrak{X}(R) \to \mathfrak{X}(K) = X(K)$$

is a bijection. Such a weak Néron model always exists, and it can be constructed by taking a Néron smoothening of a proper $R$-model of $X$ [BLR90, 3.1.3].

It follows from [LS03 4.5.3] and [Ni11 5.2] (see also [NS09 5.4] for an erratum) that the value

$$s(X) = \chi(\mathfrak{X}_s) \in \mathbb{Z}$$

only depends on $X$, and not on the choice of weak Néron model.

Definition 4.1.1. We call $s(X)$ the rational volume of $X$.

Remark 4.1.2. It is quite non-trivial that $s(X)$ is independent of the choice of weak Néron model. If $k$ has positive characteristic, we do not know any proof of this result that does not use the change of variables formula for motivic integrals. If $k$ has characteristic zero, it can be deduced from the trace formula (Corollary 4.2.3).

The value $s(X)$ is a measure for the set of rational points on $X$. In particular, $s(X) = 0$ if $X(K) = \emptyset$, since in this case, $X$ is a weak Néron model of itself. In [Ni11], we’ve shown that under a certain tameness condition on $X$, the value $s(X)$ admits a cohomological interpretation in terms of a trace formula. To study this formula, we introduce the following definition.
Definition 4.1.3. We define the error term $\varepsilon(X)$ by
\[
\varepsilon(X) = \sum_{m \geq 0} (-1)^m \text{Trace}(\varphi | H^m(X \times_K \mathbb{K}^t, \mathbb{Q}_\ell)) - s(X).
\]
We say that the trace formula holds for $X$ if $\varepsilon(X) = 0$, i.e., if
\[
s(X) = \sum_{m \geq 0} (-1)^m \text{Trace}(\varphi | H^m(X \times_K \mathbb{K}^t, \mathbb{Q}_\ell)).
\]

In [Ni09b, §1], we raised the following question.

Question 4.1.4. Let $X$ be a smooth, proper, geometrically connected $K$-variety. Assume that $X$ is cohomologically tame and that $X(K^t)$ is non-empty. Is it true that the trace formula holds for $X$?

We’ve proven that this question has a positive answer if $k$ has characteristic zero [Ni11, 6.5], if $X$ is a curve [Ni11, §7] and also if $X$ is an abelian variety [Ni09b, 2.9]. The condition that $X(K^t)$ is non-empty can not be omitted, by [Ni11, 7.7]. If $X$ is not cohomologically tame, it would be quite interesting to relate $\varepsilon(X)$ to other measures of wild ramification. However, by the example in [Ni11, 7.7], the value $\varepsilon(X)$ can not always be computed from the Chow motive of $X$, if we don’t impose the condition $X(K^t) \neq \emptyset$.

4.2. Computation of the error term. In [Ni11, 7.3], we gave an explicit formula for the error term $\varepsilon(X)$ in terms of an sncd-model for $X$, if $X$ is a curve. Thanks to Theorem 2.6.2 we can generalize this result to arbitrary dimension.

Theorem 4.2.1. Let $X$ be a smooth and proper $K$-variety, and assume that $X$ admits an sncd-model $\mathcal{X}$. We denote by $\{E_i | i \in I\}$ the set of irreducible components of $\mathcal{X}$, and we write
\[
\mathcal{X}^s = \sum_{i \in I} N_i E_i.
\]
We define the subset $I^w$ of $I$ by
\[
I^w = \{i \in I \mid N_i = p^a \text{ for some } a \in \mathbb{Z}_{>0}\}.
\]
Then the error term $\varepsilon(X)$ is given by
\[
\varepsilon(X) = \sum_{i \in I^w} \chi(E_i^p).
\]
Note that the set $I^w$ is empty if $p = 0$.

Proof. Since $\mathcal{X}$ is a regular proper $R$-model of $X$, its $R$-smooth locus $Sm(\mathcal{X})$ is a weak Néron model for $X$ (see the remark following [BLR90, 3.1.2]). Therefore,
\[
s(X) = \sum_{N_i = 1} \chi(E_i^p).
\]
On the other hand, by Theorem 2.6.2 we have
\[
\sum_{m \geq 0} (-1)^m \text{Trace}(\varphi | H^m(X \times_K \mathbb{K}^t, \mathbb{Q}_\ell)) = \sum_{N' = 1} \chi(E_i^p).
\]
It follows that
\[
\varepsilon(X) = \sum_{i \in I^w} \chi(E_i^p).
\]
Corollary 4.2.2. The value
\[ \sum_{i \in I^w} \chi(E_i^w) \]
only depends on $X$, and not on the sncd-model $X$. 

Corollary 4.2.3. If $I^w = \emptyset$, then the trace formula holds for $X$. In particular, if $p = 0$, then the trace formula holds for every smooth and proper $K$-variety.

By Theorem 4.2.1 Question 4.1.4 implies the following one.

Question 4.2.4. Let $X$ be a smooth, proper, geometrically connected $K$-variety. Assume that $X$ is cohomologically tame, that $X(K^t)$ is non-empty, and that $X$ admits an sncd-model $X$, with $X_s = \sum_{i \in I} N_i E_i$. Is it true that
\[ \sum_{i \in I^w} \chi(E_i^w) = 0? \]

A positive answer to this question would form a partial generalization of Saito’s criterion for cohomological tameness (Theorem 3.3.2) to arbitrary dimension.

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KULeuven, Department of Mathematics, Celestijnenlaan 200B, 3001 Heverlee, Belgium

E-mail address: johannes.nicaise@wis.kuleuven.be