Large N limit of O(N) vector models

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Abstract

Using a simple identity between various partial derivatives of the energy of the vector model in 0+0 dimensions, we derive explicit results for the coefficients of the large N expansion of the model. These coefficients are functions in a variable $\rho^2$, which is the expectation value of the two point function in the limit $N = \infty$. These functions are analytic and have only one (multiple) pole in $\rho^2$. We show to all orders that these expressions obey a given general formula. Using this formula it is possible to derive the double scaling limit in an alternative way. All the results obtained for the double scaling limit agree with earlier calculations.

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1 Introduction

The $O(N)$ vector model has for some time been an interesting testing ground for ideas and techniques that are widely used in the more complicated matrix-models. The latter models recently gained importance as theories of two-dimensional gravity coupled to matter with central charge $c \leq 1$ \cite{3,4,5}. In this article we investigate the large-$N$ expansion of the vectormodel, obtaining analytic expressions for the various terms in this expansion. This process can be easily continued to any desired order, leading to exact closed expressions in every order. We will show that the results obtained here are in agreement with the well-known perturbative expansions of vectormodels, as well as with the double-scaling limit, considered earlier in \cite{1,2,6}. As an extra bonus the method used to obtain these expressions is useful to find non-perturbative approximations for the energy of the vector model for any value of $N$. The $O(N)$ vector model is defined by the following partition function:

$$e^{-E} = \int d^N x e^{-\beta x^2 - g(x^2)^2}$$  \hspace{1cm} (1)

A crucial observation is now that for this model all correlation functions (or, in this simple case, the moments of the integral) can be obtained by successive derivations with respect to the parameter $\beta$. We find for the first two correlators:

$$\frac{\partial E}{\partial \beta} = <x^2>$$  \hspace{1cm} (2)

$$\frac{\partial^2 E}{\partial \beta^2} = <x^2>^2 - <(x^2)^2> = \left(\frac{\partial E}{\partial \beta}\right)^2 - \frac{\partial E}{\partial g}$$

We rewrite the last formula in a form that will be of greater use:

$$\frac{\partial E}{\partial g} = \left(\frac{\partial E}{\partial \beta}\right)^2 - \frac{\partial^2 E}{\partial \beta^2}$$  \hspace{1cm} (3)

It is indeed this formula that can be used to find a series expansion in the coupling constant for any value of $N$, and will also form the basis for the large-$N$ expansion of the model. Let us show how this comes about. The only required input is the value of the energy for the free model (i.e. without the interaction term). Since in this case the integral is simply gaussian, this value is easily found to be: $E = \frac{N}{2} \log(\beta) + c^e$. Using this expression as the
lowest order approximation to the energy one can calculate with it the RHS of (3). We then find in the LHS the derivative of the energy with respect to \( g \), given up to zeroth order. Upon integration we find the energy up to first order. Explicitly:

\[
E = \frac{N}{2} \log(\beta) + \frac{N(N + 2)}{4\beta^2} g + O(g^2)
\]

This first order approximation to the energy can again be used in the RHS of (3), giving us, after integration, the energy to order \( g^2 \). It is clear that by just repeating this simple algorithm the energy can be found to arbitrary large order in the coupling constant for any value of \( N \). The first few orders are given by:

\[
E = \frac{N}{2} \log(\beta) + \frac{N(N + 2)}{4\beta^2} g - \frac{N(N + 2)(N + 3)}{4\beta^2} g^2
\]

\[
+ \frac{N(N + 2)(5N^2 + 34N + 60)}{12\beta^{12}} g^3 - \frac{N(N + 2)(7N^3 + 79N^2 + 310N + 420)}{8\beta^8} g^4
\]

Since the zero-dimensional integral simply counts the number of diagrams in every order of perturbation, we explicitly find the number of diagrams with one, two, three, ... index-loops in every order.

## 2 Large N expansion

The simple formula (3) can however be used to give far more powerfull results than just a series expansion in the coupling constant for general \( N \). In fact, with a little extra input it is all we need to find explicit expressions for the coefficients of the large \( N \) expansion of the integral. The large \( N \) expansion of the vector model consists in taking the number of field components \( N \) to \( \infty \), at the same time rescaling the coupling constant \( g \) to \( g/N \). The \( 1/N \) expansion is then the series expansion around \( N = \infty \) of the model thus defined. It can equivalently be viewed as an expansion in the number of indexloops of a given diagram. For large \( N \) the expansion can then be used as an approximation to the original integral. It differs significantly from the perturbation series in that every order provides us with a nonperturbative approximation to the original function. The general formula (3) needs a little
modification in this case because of the rescaling of g and is easily found to be:
\[
\frac{\partial E}{\partial g} = \frac{1}{N} \left( \frac{\partial E}{\partial \beta} \right)^2 - \frac{\partial^2 E}{\partial \beta^2} \tag{6}
\]
The following expansion of the energy is proposed:
\[
E = NE_0 + E_1 + \frac{1}{N} E_2 + \ldots = \sum_{i=0}^{\infty} N^{1-i} E_i
\]
Putting this into (6) and identifying the terms with equal powers of \( \frac{1}{N} \), we find an infinite series of identities:
\[
\frac{\partial E_0}{\partial g} = \left( \frac{\partial E_0}{\partial \beta} \right)^2 \\
\frac{\partial E_1}{\partial g} = 2 \left( \frac{\partial E_0}{\partial \beta} \right) \left( \frac{\partial E_1}{\partial \beta} \right) - \frac{\partial^2 E_0}{\partial \beta^2} \\
\frac{\partial E_2}{\partial g} = 2 \left( \frac{\partial E_0}{\partial \beta} \right) \left( \frac{\partial E_2}{\partial \beta} \right) + \left( \frac{\partial E_1}{\partial \beta} \right)^2 - \frac{\partial^2 E_1}{\partial \beta^2} \tag{7}
\]
and so on, with the \( p \) th formula given by:
\[
\frac{\partial E_p}{\partial g} = \sum_{i=0}^{p} \left( \frac{\partial E_i}{\partial \beta} \right) \left( \frac{\partial E_{p-i}}{\partial \beta} \right) - \frac{\partial^2 E_{p-1}}{\partial \beta^2} 	ag{8}
\]
We can rewrite this in a more convenient way as:
\[
\frac{\partial E_p}{\partial g} = 2 \left( \frac{\partial E_0}{\partial \beta} \right) \left( \frac{\partial E_p}{\partial \beta} \right) + \left\{ \sum_{i=1}^{p-1} \left( \frac{\partial E_i}{\partial \beta} \right) \left( \frac{\partial E_{p-i}}{\partial \beta} \right) - \frac{\partial^2 E_{p-1}}{\partial \beta^2} \right\} \tag{9}
\]
In this form, the second part of the RHS of (9) consists entirely of lower order coefficients in the large \( N \) expansion, while only the first term depends on \( E_p \). It is this property that will allow a recursive determination of the coefficients. We know that the energy is a dimensionless quantity, and that therefore it can only depend on the dimensionless combination \( g/\beta^2 \). We thus find the following relation:
\[
\frac{\partial E_p}{\partial \beta} = -2 \frac{g}{\beta} \frac{\partial E_p}{\partial g} \tag{10}
\]
Using (10) in (9) we find:
\[
E_p = \int_0^g dg' \frac{F_{p-1}}{1 + \frac{4g'^2}{\beta^2} \rho^2}, \tag{11}
\]
where $F_{p-1}$ is defined as being equal to the second term in the RHS of (9) and $\rho^2$ is the expectation value of $\frac{x^2}{N}$ in the $N = \infty$ limit. (i.e. the derivative of $E_0$ with respect to $\beta$). Since $\rho^2$ and $E_0$ can be obtained from the saddle point approximation of the integral, this formula will allow us to find all the coefficients in the large N expansion using recursion.

### 3 Coefficients of the large N expansion

The saddle point equation of the integral is found by rescaling $x$ to $\sqrt{N}x$ and finding the extrema of the effective action. This gives rise to the following equation for $\rho^2$:

$$1 - 2\beta \rho^2 - 4g \rho^4 = 0 \quad (12)$$

for which we choose the regular solution (finite when $g = 0$):

$$\rho^2 = -\frac{\beta}{4g}(1 - \sqrt{(1 + \frac{4g}{\beta^2})}) \quad (13)$$

Taking the derivative of (12) with respect to $\beta$ and $g$ we easily find the following identities:

$$\frac{\partial \rho^2}{\partial \beta} = -\frac{\rho^2}{\beta + 4g \rho^2} = -\frac{\rho^4}{1 - \beta \rho^2} \quad (14)$$

$$\frac{\partial \rho^2}{\partial g} = \frac{-2\rho^4}{\beta + 4g \rho^2} = \frac{-2\rho^6}{1 - \beta \rho^2} \quad (15)$$

Note that in the last expressions all explicit $g$-dependence is eliminated. We now find an expression for $E_0$ by integrating the relation $\rho^2 = \frac{\partial E_0}{\partial \beta}$, leading to:

$$E_0 = \frac{\beta \rho^2 - \log(\beta \rho^2)}{2} \quad (16)$$

This is all one needs to obtain the coefficients of the large N expansion to arbitrarily high order in $\frac{1}{N}$. Using formula (15), we can change integration variables from $g$ to $\rho^2$. In doing so we completely eliminate any explicit dependence on $g$. The expressions will be given as functions of $\rho^2$ only. (or, more correctly, as functions of the dimensionless quantity $\beta \rho^2$). After the change of variables formula (11) becomes:
The integrals encountered in this problem are relatively simple and can all be carried out exactly. Note that this would have been quite more complicated without the change of variables. Since all lower order coefficients are needed to find a coefficient of higher order, we have to determine the expressions one by one in rising order. Beginning with $E_1$ we find without too much difficulty:

$$F_0 = \frac{\rho^4}{1 - \beta \rho^2}$$

$$E_1 = -\frac{\beta}{2} \int_{\frac{1}{2\beta}}^{\rho^2} \frac{1}{1 - \beta \rho}$$

$$= \frac{1}{2} \ln(2 - 2\beta \rho^2)$$

(18)

In higher orders the expressions become more elaborate, but no serious complications arise. The integrals are always of the same type (simple integrals of rational functions) and can easily be carried out exactly. For $E_2$ we find in an analogous way:

$$F_1 = \left( \frac{\partial E_1}{\partial \beta} \right)^2 - \frac{\partial^2 E_1}{\partial \beta^2}$$

$$= -\frac{\rho^4}{4} \frac{1 + 2\beta \rho^2 - 8\beta^2 \rho^4}{(1 - \beta \rho^2)^2}$$

$$E_2 = -\frac{1}{24} \frac{(2\beta \rho^2 - 1)^2(\beta \rho^2 + 4)}{(1 - \beta \rho^2)^3}$$

(19)

Explicit results for higher order are collected in an appendix. A closer look at those results reveals that from $E_2$ on the coefficients appear to obey some general form:

$$E_p = \frac{(1 - 2\beta \rho^2)^p}{(1 - \beta \rho^2)^{3(p-1)}} P_p(\beta \rho^2)$$

(20)

Where $P_p$ is a polynomial of degree at most $2p - 3$. This form holds for all the expressions that are given in the appendix. We will now prove by induction that it is true for all orders.
We already know that the energy is a dimensionless quantity. It thus depends only on the dimensionless variable $\beta \rho^2$, which we will from now on denote as $\alpha$. The derivative of a function of $\alpha$ with respect to $\beta$ is given by

$$\frac{\partial}{\partial \beta} f(\alpha) = \frac{\partial f}{\partial \alpha} \frac{\partial \alpha}{\partial \beta} = \frac{\alpha 1 - 2\alpha \frac{\partial f}{\partial \alpha}}{\beta 1 - \alpha \frac{\partial f}{\partial \alpha}}$$

(21)

If the function $f$ happens to be of the form (20), the derivative with respect to $\beta$ is given by:

$$\frac{\partial f}{\partial \beta} = \frac{\alpha^2 (2\alpha - 1)^p}{\beta^2 (1 - \alpha)^{3p-1}} Q_p(\alpha)$$

(22)

With $Q_p$ a polynomial. The second derivative is given by:

$$\frac{\partial^2 f}{\partial \beta^2} = \frac{\alpha^2 (2\alpha - 1)^p}{\beta^2 (1 - \alpha)^{3p+1}} Q_p^*(\alpha)$$

(23)

Now we assume that all $E_i$ obey the form (20), $i=1,...,p-1$. In that case $F_{p-1}$ is given by:

$$F_{p-1} = \frac{\alpha^2 (2\alpha - 1)^{p-1}}{\beta^2 (1 - \alpha)^{3p-2}} Q_p^*(\alpha)$$

$$= \rho^4 \frac{(2\alpha - 1)^{p-1}}{(1 - \alpha)^{3p-2}} Q_p^*(\alpha)$$

(24)

With this form for $F_{p-1}$, the coefficient $E_p$ is given by:

$$E_p = -\frac{1}{2} \int_\frac{1}{2}^\alpha \frac{(2\alpha' - 1)^{p-1}}{(1 - \alpha')^{3p-2}} Q_p^*(\alpha')d\alpha'$$

(25)

It is clear that also this integral is of the form proposed in (20). The integral can be performed by expanding the numerator in powers of $1 - \alpha'$. The integral is then reduced to a sum of simple integrals of inverse powers of $(1 - \alpha')$, the highest power being $3p - 2$. After integration the result is therefore still a sum of inverse powers of $1 - \alpha$, this time with highest power $3(p - 1)$. The integral is therefore given by a polynomial divided by $(1 - \alpha)^{3(p-1)}$. Furthermore, since the integral vanishes for $\alpha = \frac{1}{2}$, the RHS must at least
be proportional to \((2\alpha - 1)\). But, since the integrand is proportional to \((2\alpha' - 1)^{p-1}\), the integral itself is proportional to \((2\alpha - 1)^p\). We have thus proved by induction that the form proposed for the \(p\)th coefficient in the large \(N\) expansion of the energy of the vector model is indeed correct. This general form can be used to obtain the double scaling limit of the vector model. Two remarks are at order here: in the proof we assume that \(E_1\) has the form proposed in (20). This is clearly not the case. However, the derivative of \(E_1\) has the correct form, and therefore the proof remains valid. Secondly, one sees that the general form bears a striking resemblance to the form conjectured for the coefficients of the large \(N\) expansion of matrix models. In [7] the following ansatz was proposed for these coefficients:

\[
E_h = \frac{(1 - a^2)^{2h-1}P_h(a^2)}{(2 - a^2)^{5(h-1)}} \quad h \geq 2
\]  

(26)

with \(P_h\) a polynomial and \(a^2\) the solution of a quadratic equation similar to (12).

4 Double scaling limit

From (20) we find that all the coefficients exhibit a pole at \(\alpha = 1\). This pole corresponds to the branch point singularity of \(\rho^2\), namely: \(g_c = -\frac{\beta^2}{4}\). If we define \(\epsilon\) by \(\epsilon\beta^2 = g - g_c\), \(\alpha\) can be developed around its critical value and we find: \(\alpha = 1 - 2\sqrt{\epsilon} + O(\epsilon)\). Substituting this in the general form we have found for the coefficients of the large \(N\) expansion we find that near the critical point they behave as:

\[
E_p \approx \left(\frac{1}{8}\right)^{p-1} P_p(1) \epsilon^{-\frac{3}{2}(p-1)}
\]  

(27)

This is of course not true for \(E_0\) and \(E_1\). It is however straightforward to obtain their expansion around the critical point with the expressions we have found:

\[
E_0(g) - E_0(0) = \frac{1}{4} - \frac{\log(2)}{2} + \epsilon - \frac{8}{3} \epsilon^{\frac{3}{2}} + O(\epsilon^2)
\]

\[
E_1 = \frac{1}{2} \log(4\sqrt{\epsilon}) + O(\epsilon)
\]  

(28)
This is exactly the same behaviour found in [1] (because of a different definition of $\epsilon$, one has to replace $\epsilon$ in the above formulae with $4\epsilon$). The behaviour of $E_0$ and $E_1$ was dubbed "non-universal" and therefore these terms were omitted in the double scaling limit. We will follow this procedure and define the energy as $E = \sum_{p=2}^{\infty} \frac{1}{N^{p-1}} E_p$. Near the critical point the energy is approximated by:

$$E \simeq \sum_{p=2}^{\infty} N^{-(p-1)} (\epsilon^{-\frac{4}{2}})^{p-1} P_p(1) \left(\frac{1}{8}\right)^{p-1}$$

(29)

One remarks that in every term we now have a competition between a suppression factor proportional with an inverse power of $N$ and a divergence proportional to a negative power of $\epsilon$. The double scaling limit consists in taking $N$ to $\infty$ and $\epsilon$ to 0, while keeping the combination $N\epsilon^\frac{3}{2}$ finite (this is the same combination as found in [1]). In this way we sum the contributions from all loops. If we define the double scaling variable as: $z = (N\epsilon^\frac{3}{2})^{-1}$ the energy is given by the series expansion: $E = \sum z^{p-1} (\frac{1}{8})^{p-1} P_p(1)$. We will now prove that the complete series expansion is indeed the one proposed in [1]. We find that in the double scaling limit ($\alpha \approx 1 - 2\sqrt{\epsilon}$) the following relation holds:

$$\frac{\partial f(\alpha)}{\partial \beta} \simeq -\alpha \frac{\partial f(\alpha)}{\partial \alpha} \simeq \frac{1}{2\beta} \frac{\partial f(\alpha)}{\partial \epsilon}$$

(30)

Therefore the first and second derivatives with respect to $\beta$ are given by:

$$\frac{\partial E_p}{\partial \beta} \simeq -\frac{3(p-1)\alpha}{4\beta} b_{p-1} \epsilon^{-\frac{4p-1}{2}}$$

$$\frac{\partial^2 E_p}{\partial \beta^2} \simeq \frac{3(p-1)(3p-1)\alpha^2}{16\beta^2} b_{p-1} \epsilon^{-\frac{4p-1}{2}}$$

(31)

(and similarly for $E_1$, where $E_p$ is assumed to have the form $b_{p-1} \epsilon^{-\frac{4}{2}(p-1)}$)

Another useful relation is given by:

$$F_{p-1} = 2\sqrt{\epsilon} \frac{\partial E_p}{\partial \epsilon}$$

(32)

which we find by changing variables in (25) from $\alpha$ to $\epsilon$ and taking the derivative of both sides with respect to $\epsilon$. Using these relations in the definition
of $F_{p-1}$ and identifying equal powers of $\epsilon$ we readily find a recursion relation for the coefficients $b_p$:

$$-4pb_p = \frac{3}{4} \sum_{q=2}^{p-1} b_{q-1}b_{p-q}(p-q)(q-1) - \frac{3}{4}p(p-1)b_{p-1}$$  \hspace{1cm} (33)

Now, from the definition of the energy in the double scaling limit we find:

$$E = \sum_{p=2}^{\infty} b_{p-1}z^{p-1} = \sum_{p=1}^{\infty} b_pz^p$$  \hspace{1cm} (34)

Therefore the recursion relation (33) can be cast into a differential equation for the energy. This can be done by multiplying the relation with $z^{p-1}$ and summing $p$ from 2 to $\infty$.

$$-4p b_p z^{p-1} = \frac{3}{4} \sum_{q=2}^{p-1} b_{q-1}b_{p-q}(p-q)(q-1)z^{p-1} - \frac{3}{4}p(p-1)b_{p-1}z^{p-1}$$  \hspace{1cm} (35)

The LHS is immediately found to be equal to:

$$-4 \frac{\partial E}{\partial z} + 4b_1$$  \hspace{1cm} (36)

While the first term in the RHS is equal to:

$$\frac{3}{4} \sum_{q=2}^{\infty} b_{q-1}(q-1)z^{q-1})^2 = \frac{3}{4} (z \frac{\partial E}{\partial z})^2$$  \hspace{1cm} (37)

And the second term finally:

$$-\frac{3}{4} z \frac{\partial^2 (zE)}{\partial z^2}$$  \hspace{1cm} (38)

Defining $\xi$ as $\frac{\partial E}{\partial z}$ this leads to the following differential equation:

$$3z^2 \frac{\partial \xi}{\partial z} - 3z^2 \xi^2 + 6z\xi - 16\xi = 5/12$$  \hspace{1cm} (39)

One can also derive an equation for the partition function $\zeta = e^{-E}$:

$$z^2 \frac{\partial^2 \zeta}{\partial z^2} + 2z \frac{\partial \zeta}{\partial z} - \frac{16}{3} \frac{\partial \zeta}{\partial z} + \frac{5}{36} \zeta = 0$$  \hspace{1cm} (40)

This is exactly the same equation as found in [1], apart from a rescaling from $z$ to $z/8$. We have therefore proved that the double scaling limit defined in this way confirms the earlier derivations.
5 Conclusions

In this paper we have explicitly obtained the large $N$ expansion of the vector model. The coefficients are found to be analytic functions of a variable $\rho^2$, which is the expectation value of the two point function in the limit $N = \infty$. Remarkably enough all these functions (apart from the first two) obey some general form, which clearly illustrates the divergence of the coefficients. It is seen that all coefficients diverge at the same critical value of the coupling, and in such a way that a double scaling limit may be taken. This alternative derivation is in a sense closer to the philosophy of the double scaling limit because it clearly illustrates the competition between the large $N$ suppression and the divergence at critical coupling. The results derived in this paper are however far more general than their application to the double scaling limit. They also allow us to obtain information of the model away from the critical region and to find approximations for the energy for finite values of $N$.

A Appendix

We collect some important results in this appendix. The first six coefficients in the large $N$ expansion are given by:

\begin{align*}
E_0 &= \frac{\beta \rho^2 - \log(\beta \rho^2)}{2} \\
E_1 &= \frac{1}{2} \ln(2 - 2\beta \rho^2) \\
E_2 &= -\frac{1}{24} \frac{(2\beta \rho^2 - 1)^2(\beta \rho^2 + 4)}{(1 - \beta \rho^2)^3} \\
E_3 &= -\frac{5}{16} \frac{(2\beta \rho^2 - 1)^3 \beta \rho^2}{(1 - \beta \rho^2)^6} \\
E_4 &= \frac{(2\beta \rho^2 - 1)^4(8\beta^5 \rho^{10} - 56\beta^4 \rho^8 + 164\beta^3 \rho^6 - 4696\beta^2 \rho^4 - 1841\beta \rho^2 + 896)}{5760(1 - \beta \rho^2)^9} \\
E_5 &= -\frac{(2\beta \rho^2 - 1)^5 \beta \rho^2(706\beta^2 \rho^4 + 823\beta \rho^2 - 399)}{256(1 - \beta \rho^2)^{12}}
\end{align*}
The first terms of the series expansions of these expressions are given by:

\[ E_0 = g - 4g^2 + \frac{80}{3}g^3 - 224g^4 + \frac{1072}{5}g^5 + \ldots \]
\[ E_1 = 2g - 20g^2 + \frac{704}{3}g^3 - 2976g^4 + \frac{197632}{5}g^5 + \ldots \]
\[ E_2 = -24g^2 + \frac{2048}{3}g^3 - 14976g^4 + 296960g^5 + \ldots \]
\[ E_3 = 640g^3 - 33280g^4 + 1126400g^5 + \ldots \]
\[ E_4 = -26880g^4 + \frac{10604544}{5}g^5 + \ldots \]
\[ E_5 = 1548288g^5 + \ldots \]

Notice that the sum \( \sum_i E_i \) gives the correct expansion of the ”\( N = 1 \) vector-model”, which is just a one-dimensional integral:

\[ E = 3g - 48g^2 + 1584g^3 - 78336g^4 + \frac{25671168}{5}g^5 + \ldots \]

(where we have put \( \beta = \frac{1}{2} \))

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