Structure formation in large-volume cosmological simulations of fuzzy dark matter: Impact of the non-linear dynamics

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Abstract: An ultra-light bosonic particle of mass around $10^{-22} \text{eV}/c^2$ is of special interest as a dark matter candidate, as it both has particle physics motivations, and may give rise to notable differences in the structures on highly non-linear scales due to the manifestation of quantum-physical wave effects on macroscopic scales, which could address a number of contentious small-scale tensions in the standard cosmological model, $\Lambda$CDM. Using a spectral technique, we here discuss simulations of such fuzzy dark matter (FDM), including the full non-linear wave dynamics, with a comparatively large dynamic range and for larger box sizes than considered previously. While the impact of suppressed small-scale power in the initial conditions associated with FDM has been studied before, the characteristic FDM dynamics are often neglected; in our simulations, we instead show the impact of the full non-linear dynamics on physical observables. We focus on the evolution of the matter power spectrum, give first results for the FDM halo mass function directly based on full FDM simulations, and discuss the computational challenges associated with the FDM equations. FDM shows a pronounced suppression of power on small scales relative to cold dark matter (CDM), which can be understood as a damping effect due to ‘quantum pressure’. In certain regimes, however, the FDM power can exceed that of CDM, which may be interpreted as a reflection of order-unity density fluctuations occurring in FDM. In the halo mass function, FDM shows a significant abundance reduction below a characteristic mass scale only. This could in principle alleviate the need to invoke very strong feedback processes in small galaxies to reconcile $\Lambda$CDM with the observed galaxy luminosity function, but detailed studies that also include baryons will be needed to ultimately judge the viability of FDM.

Key words: dark matter – large-scale structure of Universe – cosmology: theory – galaxies: haloes – methods: numerical – software: simulations

1 INTRODUCTION

The ‘standard cosmological model’ with cold dark matter and a cosmological constant ($\Lambda$CDM) has been extremely successful in describing a wide variety of cosmological observations across a broad range of physical scales (e.g. Frenk & White 2012; Bull et al. 2016). On small cosmological scales, however, challenges such as the ‘cusp–core problem’, the ‘missing satellite problem’, or the ‘too-big-to-fail problem’ have sometimes raised questions about the validity of $\Lambda$CDM (Weinberg et al. 2015; Del Popolo & Le Delliou 2017; Boylan-Kolchin et al. 2011). Since physical effects experienced by baryons can become relevant at these scales, it has proven difficult to disentangle the apparent discrepancies from baryonic physics. In addition, even though the cosmological properties of cold dark matter are constrained extremely well on large scales, its micro-physical nature is still completely unknown thus far.

In light of these small-scale questions and the enduring lack of any direct detection of the most well-studied dark matter candidates, in particular weakly interacting massive particles (WIMPs), models based on ultra-light (‘axion-like’) scalar particles have been gaining interest as alternative dark matter models. Due to their small masses, quantum effects are expected to cause interesting wave-like behaviour at small (i.e. galactic) scales compared to heavy particles like WIMPs or compact objects. These effects have also been proposed to elucidate some of the ‘small-scale problems’ of cold dark matter (CDM); for example, early numerical simulations showed that ultra-light scalars form cores in the centres of dark matter haloes (Schive et al. 2014a). Furthermore, light (pseudo-)scalar particles are a common feature of theories in particle physics, from the original axion in quantum chromodynamics (QCD) to a plethora of axion-like particles predicted by unified and early-universe theories such as string theories (Marsh 2016).

The amount of existing work studying structure formation with CDM utterly dwarfs that of such fuzzy dark matter (FDM), particularly concerning numerical simulations that reliably probe the non-linear regime. Computations have been performed using a variety of approaches and numerical methods, although many attempts were limited in scope (Zhang et al. 2019; Laguè et al. 2021, table 1). Correspondingly, our understanding of structure formation in FDM cosmologies is still comparatively spotty compared to CDM, where decades of experience have led to extremely detailed insights. Apart from the lower level of research attention, an important reason impeding insight into FDM has been that numerically solving the corresponding equations of motion incurs very large computational costs – much higher than those associated with corresponding calculations of $\Lambda$CDM. In particular, while it has been established that, in
the limit of large scales (or large particle mass), the FDM equations are equivalent to the Vlasov–Poisson equations of CDM (Mocz et al. 2018; Widrow & Kaiser 1993), the effects of FDM in (mildly) non-linear regimes of structure formation are still poorly understood when compared to CDM.

Due to the computational requirements, the cosmological volumes studied in simulations with full FDM dynamics have been especially limited (Woo & Chiu 2009; Schive et al. 2014a; Veltmaat et al. 2018; Mocz et al. 2020), which is an issue that this work seeks to improve upon. In particular, we would like to carry out simulations that smoothly connect the non-linear state reached in isolated FDM haloes to the still linear large-scale structure, thereby bridging, in particular, the regime of mildly non-linear evolution where differences in the temporal evolution compared to CDM can be expected. To this end, we carry out very large uni-grid FDM simulations with a spectral method, which fully retains the quantum-mechanical effects. Because there is still a dearth of precision studies of how FDM compares to CDM for traditional measures of large-scale structure, we focus our analysis on central measures of matter clustering, namely the power spectrum and the halo mass function, and compare to ordinary ΛCDM where appropriate.

While methods which forego a treatment of the full wave dynamics have been able to conduct simulations with volumes much closer to those attainable using traditional N-body and smoothed-particle hydrodynamics (SPH) approaches for CDM (Schive et al. 2016; Veltmaat & Niemeyer 2016; Zhang et al. 2018; Nori & Baldi 2018; Nori et al. 2019), these do not capture inherent wave phenomena such as interference effects, which can have a significant impact on the overall evolution at least on small scales (Li et al. 2019), leaving the validity of results obtained in this way unclear in the absence of similar computations solving the fundamental wave equations. In particular, while all simulations can easily incorporate the impact of the suppressed small-scale power spectrum present with FDM in the initial conditions, such methods either lack the wave nature of FDM entirely or only approximate it. Using our simulations presented in this work, we aim to clarify the reliability of such approximative results by explicitly omitting the suppression in the FDM initial conditions, starting instead from ‘standard’ ΛCDM initial conditions. In this way, we disentangle the two essential physical differences distinguishing FDM from CDM in cosmological numerical simulations: the initial conditions and the non-linear dynamics. This study is structured as follows. In section 2, we briefly review the theoretical basis for the equations of motion of FDM models. We then describe our numerical approaches for cosmological simulations of FDM in section 3. In section 4, we turn to an analysis of our results for the matter power spectrum, while in section 5 we report our findings for the halo mass function. Section 6 discusses halo profiles, and section 7 the differences expected in FDM due to modifications of the initial linear theory spectrum relative to ΛCDM if this is self-consistently computed. Finally, we give our conclusions in section 8.

2 THEORETICAL BACKGROUND

Fundamentally, fuzzy dark matter is identical to the well-studied scalar field dark matter model, which is perhaps also the simplest particle-based dark matter model at the theoretical level. Such a model is described by the simple scalar field action

\[ S = \frac{1}{2 \hbar^2} \int d^3 x \sqrt{-g} \left( \frac{1}{2} g^{\mu \nu} (\partial_\mu \phi)(\partial_\nu \phi) - \frac{1}{2} \frac{m^2 c^2}{\hbar^2} \phi^2 - \frac{\lambda}{\hbar^2 c^2} \phi^4 \right) \]  

(1) 

with the metric \( g^{\mu \nu} \) and its determinant \( g \), and the real scalar field \( \phi \), its mass \( m \) and a coupling strength \( \lambda \). This action is to be understood in the context of quantum field theory in a curved spacetime, i.e. \( \phi \) is a (‘second-quantised’) quantum field.

The difference to most earlier studies of scalar field dark matter lies in the chosen parameters, specifically the mass \( m \), which is commonly assumed to be in the range of 100 GeV–TeV, in line with the concept of CDM. In contrast, an ultra-light scalar field with a mass around \( m^2 c^2 \approx 10^{-22} \text{ eV} \) is under consideration here. This vast difference in the considered mass range compared to heavy scalar particles has drastic phenomenological consequences. Additionally, the ultra-light particles require a non-thermal production mechanism so that the resulting dark matter is not ultra-hot, but still resembles CDM.

The term ‘fuzzy dark matter’, then, is typically used for an ultra-light scalar field without self-interactions, i.e. \( \lambda = 0 \). Thus, it corresponds to a limit (\( \lambda \to 0 \), or zero temperature, \( T \to 0 \)) of more general ultra-light scalar field models (see Ferreira 2020, for a review), such as superfluid dark matter. Such models are often called axion-like particle (ALP) models due to the phenomenological similarity to the axion of QCD, which yields the same action as eq. (1) originating from a periodic potential \( V(\phi) \propto \lambda^2 (1 - \cos(\phi/\mu)) \) for \( \phi \ll \mu_a \) (see e.g. Marsh 2016; Hui et al. 2017).

In order to perform numerical simulations of the non-linear structure formation in a universe with FDM, as they are traditionally done for CDM, it is necessary to use non-relativistic approximations. This is possible because virial velocities are small compared to \( c \) and the simulations will be restricted to structures forming on scales smaller than the Hubble horizon \( c/H_0 \). Even so, as will be detailed later, FDM simulations are only possible with massively increased computational cost compared to CDM.

As sketched, for example, in Marsh (2016); Hui et al. (2017), the non-relativistic equations of motions for the fuzzy dark matter field can be obtained as follows. First, the real scalar field is rewritten in terms of a complex field \( \psi \) as

\[ \psi = \frac{1}{2} \sqrt{\frac{\hbar^2 c}{2 \mu}} \text{Re} \left( \psi e^{-i m^2 c^2 t/\hbar^2} + \psi^* e^{i m^2 c^2 t/\hbar^2} \right). \]  

(2)

Next, the non-relativistic limit is taken, with the perturbed Friedmann–Robertson–Walker (FRW) metric

\[ ds^2 = \left( 1 + \frac{2 \Phi}{c^2} \right) c^2 dt^2 - a(t)^2 \left( 1 - \frac{2 \Phi}{c^2} \right) dx^2, \]  

(3)

where \( a \) is the scale factor and \( \Phi \) corresponds to the Newtonian potential. This results in the Schrödinger equation

\[ i\hbar \left( \frac{\partial}{\partial t} + \frac{\Phi}{2} \right) \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + m \Phi \psi, \]  

(4)

where \( \Phi \) obeys the usual Poisson equation.

Since the scalar particles are bosons with low velocities, most of the particles will be in the ground state. This enables the use of the mean field approximation, where \( \psi \) in eq. (4) is interpreted as the single macroscopic wave function of a Bose–Einstein condensate with mass density

\[ \rho = m |\psi|^2. \]  

(5)

Equation (4) and the Poisson equation can additionally be rewritten using the ‘comoving’ quantities

\[ \psi_c = a^{1/3} \psi, \quad \nabla_c = a \nabla, \quad \Phi_c = a \Phi, \]  

(6)

\(^1 \) \( c \) and \( \hbar \) have the usual meanings of the speed of light in vacuum and the reduced Planck constant, and are explicitly included in all equations.
to yield the non-linear Schrödinger–Poisson system of equations:  

$$i\hbar \partial_t \psi_c(t, x) = -\frac{\hbar^2}{2m a(t)^2} \nabla^2 \psi_c(t, x) + \frac{m}{a(t)} \Phi_c \psi_c(t, x) \quad (7)$$

$$\nabla^2 \Phi_c(t, x) = 4\pi G m \left( |\psi_c(t, x)|^2 - \langle |\psi_c(t)|^2 \rangle \right) \quad (8)$$

The field $\psi$ is now a function of time and space whose values are ordinary complex numbers, instead of the operator-valued field that was the starting point in eq. (1). The challenge for numerical simulations is then to solve the non-linear time evolution of these FDM equations.

The Schrödinger equation has the form of a diffusion equation and thus has a well-known conservation law described by the continuity equation

$$\partial_t \rho_c + \nabla \cdot \rho_c \mathbf{v}_c = 0, \quad (9)$$

with the density current

$$\rho_c \mathbf{v}_c = \frac{\hbar}{2i} \left( \psi_c^* \nabla \psi_c - \psi_c \nabla^* \psi_c \right) \quad (10)$$

$$(\rho_c = m \langle |\psi_c|^2 \rangle)$. The (comoving) velocity field $\mathbf{v}_c$ gives the peculiar velocity of matter at each point. The interpretation of the wave function becomes clearer when written in polar form,  

$$\psi_c = \frac{\rho_c}{\sqrt{m}} e^{i\alpha}, \quad (11)$$

that is, the velocity field is given by the gradient of the wave function’s phase. The wave function in polar components (eq. (11)) can be rewritten in hydrodynamical form using the Madelung (1927) transformation, which results in the continuity eq. (9) along with a modified Euler equation

$$\partial_t \mathbf{v}_c + \frac{1}{a^2} \nabla \mathbf{v}_c^2 = -\frac{1}{a^2} \nabla \mathbf{\Phi}_c + \frac{\hbar^2}{2 m a^2} \mathbf{v}_c - \frac{\nabla^2 \sqrt{\rho_c}}{\sqrt{\rho_c}} \quad (13)$$

(cf. e.g. Mocz et al. (2020)). This allows for a hydrodynamical interpretation of the density and velocity fields $\rho_c$ and $\mathbf{v}_c$.

It should be noted that eqs. (7) and (8) only have a single parameter given by the constant $\hbar/m$, which is related to the de Broglie wavelength as

$$\lambda_{db} = \frac{2\pi \hbar}{m v}. \quad (14)$$

Another relevant scale, also determined by $\hbar/m$, is the FDM Jeans length. Due to the existence of ‘quantum pressure’ resulting from the Heisenberg uncertainty principle, at least in linear theory, there exists a length scale where this pressure cancels out the gravitational attraction of matter, analogous to ordinary pressure e.g. for baryons. Perturbations at length scales larger than the Jeans length grow, while smaller perturbations oscillate. The comoving Jeans wave number $k_J$ at some redshift $z$ is given by (Hu et al. 2000)

$$k_J = \frac{6\Omega_m}{1+z} \left( \frac{m H_0}{\hbar} \right)^{1/2}. \quad (15)$$

### 3 NUMERICAL METHODOLOGY

Our simulations are performed within the same framework as ordinary cosmological CDM simulations. The simulation volume consists of a cubic box of side length $L$ with periodic boundary conditions, which is intended to sample the matter distribution in the universe. The volume is filled with matter whose average comoving density is the mean background density

$$\langle \rho_c \rangle = \Omega_m \rho_{crit} = \Omega_m \frac{3H_0^2}{8\pi G}. \quad (16)$$

In order to solve eqs. (7) and (8), a 2nd-order symmetrised split-step pseudo-spectral Fourier method (‘kick–drift–kick’) is employed. For a small time step $\Delta t$, the time evolution can be approximated as follows (Woo & Ciueh 2009; Edwards et al. 2018):

$$\psi_c(t + \Delta t, x) = e^{-i \Delta t \Lambda} \left[ e^{-\frac{i \Delta t}{2} \nabla^2} + \frac{m}{\pi \hbar} \frac{1}{a(t)^2} \frac{\mathbf{\Phi}_c(t-x)}{a(t)} \right] e^{-\frac{i \Delta t}{2} \nabla^2} \psi_c(t, x) \quad (17)$$

$$= \mathcal{T} e^{-i \Delta t \Lambda} \left[ e^{-\frac{i \Delta t}{2} \nabla^2} + \frac{m}{\pi \hbar} \frac{1}{a(t)^2} \frac{\mathbf{\Phi}_c(t+\Delta t, x)}{a(t)} \right] e^{-\frac{i \Delta t}{2} \nabla^2} \psi_c(t, x) \quad (18)$$

where $\mathcal{T}$ is the time ordering operator and, using the Baker–Campbell–Hausdorff formula, the time evolution operator has been split into three parts which do not mix functions of the position and derivative operators.

The fields $\psi$ and $\mathbf{\Phi}$ are discretised on a uniform Cartesian mesh with $N^3$ grid points to allow for numerical computations using the Fast Fourier Transform (FFT). Accordingly, the numerical algorithm of the pseudo-spectral method performs the following operations:

- $\psi_c \leftarrow e^{-i \frac{\Delta t}{2} \nabla^2} \psi_c$ (kick) \hspace{1cm} (20a)
- $\psi_c \leftarrow FFT^{-1} \left( e^{-i \frac{\Delta t}{2} \nabla^2} FFT(\psi_c) \right)$ (drift) \hspace{1cm} (20b)
- $\mathbf{\Phi}_c \leftarrow FFT^{-1} \left( \frac{1}{k^2} FFT \left(4\pi G m \left( |\psi_c|^2 - \langle |\psi_c|^2 \rangle \right)\right) \right)$ (update potential) \hspace{1cm} (20c)
- $\psi_c \leftarrow e^{-i \frac{\Delta t}{2} \nabla^2} \psi_c$ (kick) \hspace{1cm} (20d)
- Go to step (20a) \hspace{1cm} (20e)

Consecutive executions of steps (20a) and (20d) (i.e. except for the initial and final time steps) can be combined into a single operation $\psi_c \leftarrow e^{-i \frac{\Delta t}{2} \nabla^2} \mathbf{\Phi}_c \psi_c$ to improve performance.

The choice of the time step $\Delta t$ in steps (20a) to (20e) is determined by the requirement that the phase difference in the exponentials must not exceed $2\pi$, at which point the time step would be incorrectly ‘aliased’ to a smaller time step corresponding to the phase difference subtracted by a multiple of $2\pi$ due to the periodicity of the exponential function. Both the kicks (steps (20a) and (20d)) and the drift (step (20b)) yield separate constraints for $\Delta t$, which must be simultaneously fulfilled.
The resulting time step criterion is
\[ \Delta t < \min \left( \frac{4 m}{3 \pi \hbar} a^2 \Delta x^2, \frac{2 \pi \hbar}{m a} \phi_c \right), \tag{21} \]
where \( \Delta x = L/N \) is the spatial resolution and \( \phi_c \) is the maximum (absolute) value of the potential. The constraint involving the resolution \( \Delta x \) stems from the drift operation, while the constraint with the potential \( \phi_c \) results from the kick operation. The dependence \( \Delta t \propto \Delta x^2 \) seems to be typical for all numerical approaches to the Schrödinger-Poisson system of eqs. (7) and (8), and can be viewed as a reflection of the relation of the Schrödinger equation to diffusion problems.

Another constraint on the validity of the discretisation becomes apparent when considering the velocity field (eq. (12)). Since this is given by the gradient of the wave function’s phase, whose difference between two points can be at most \( 2\pi \), it follows that the discretised velocity field cannot exceed a maximum value (depending on the concrete form of the discretised gradient operator) of about
\[ v_{\text{max}} = \frac{\hbar \pi}{m \Delta x}. \tag{23} \]
Velocities \( v \geq v_{\text{max}} \) cannot be represented in a simulation with resolution \( \Delta x \). Comparing to eq. (14), it becomes apparent that this statement is equivalent to a constraint on the resolution, which should be good enough to resolve the de Broglie wavelength of the largest velocities:
\[ \Delta x < \frac{\hbar}{m v_{\text{max}}} = \frac{1}{2} \lambda_B (v_{\text{max}}). \tag{23} \]
The requirement to resolve structures on the scale of the de Broglie wavelength, given by eq. (23), in combination with the time step requirement \( \Delta t \propto \Delta x^2 \) (eq. (21)), is one of the main reasons why FDM simulations require so many more computational resources than traditional particle-based CDM simulations.

To enable our simulation work, the pseudo-spectral algorithm (steps (20a) to (20c)) has been implemented as a module in the AREPO code (Springel 2010; Weinberger et al. 2020) in a similar approach to Mocz et al. (2017, 2020). Step (20c) is performed using AREPO’s existing Poisson solver algorithm, while the split-step solution to the Schrödinger equation is solved using newly developed, highly parallel code inspired by the same algorithm. It, too, makes use of the ‘Fastest Fourier Transform in the West (FFTW)’ library (Frigo & Johnson 2005) to perform the FFT.

This new implementation enables simulations of meshes which are several orders of magnitude larger than any previous work. The split-step scheme and the integration with AREPO also make it suitable for future simulations including baryons. All FDM simulations reported in this work were performed using this pseudo-spectral FDM AREPO module, whereas the CDM simulations were done using AREPO’s standard TreePM method.

### 3.1 Initial conditions

As mentioned previously, there is in fact a correspondence between the Vlasov–Poisson equations which describe the evolution of a CDM fluid and the Schrödinger–Poisson equations, with the latter reducing to the former for \( h/m \to 0 \) (Widrow & Kaiser 1993; Mocz et al. 2018; Garny et al. 2020). A wave function \( \psi(x) \) can be constructed from a phase space distribution function \( f(x,p) \) by means of a Wigner or Husimi transform. This relationship illustrates why FDM behaves like CDM on scales larger than \( \lambda_B \). At first, this was in fact considered as an alternate method to simulate CDM (Widrow & Kaiser 1993). However, one can also take the equations seriously and actually consider the Schrödinger–Poisson equations as the ‘true’ description of dark matter. In this case, the correspondence allows one to ‘translate’ a phase space distribution function to the wave function formalism. This is very useful because it enables direct comparisons between structure formation with CDM and FDM from the same initial conditions (ICs).

In general, a wave function on a discretised lattice can be constructed using the prescription
\[ \psi(x_n) \propto \sum_{n'=0}^{N-1} \sum_{n''=0}^{N-1} \sqrt{f(x_{n'},v_n)} e^{i \frac{\pi}{N} n' v_n + R_n}, \tag{24} \]
where \( n, n' = (n_1', n_2', n_3') \) are discrete grid indices, \( x_n = n \Delta x \) and \( v_n = 2\pi \frac{\hbar}{m N_{\Delta x}^3} \) are discrete phase space grid points, and \( R_n \) is a random phase which is required to ensure that different velocity components are uncorrelated (cf. Widrow & Kaiser 1993; Mocz et al. 2018).

For the case of a ‘cold’ or ‘single-stream’ distribution function as in the case of CDM, where each point in space has a single well-defined value for the velocity, this value can be directly related to the wave function’s phase as in eq. (12). The construction of the wave function then simply reduces to the polar decomposition (eq. (11)), with the absolute value and phase determined by the density and velocity at each point, respectively:
\[ |\psi(x)| = \sqrt{\rho(x) \frac{m}{\hbar}}. \tag{25} \]
\[ \nabla \arg(\psi(x)) = \nabla \alpha(x) = \frac{m}{\hbar} v(x). \tag{26} \]
Equation (26) can be easily solved numerically by applying the spectral method to the equation
\[ \nabla^2 \alpha(x) = \frac{m}{\hbar} \nabla \cdot v(x). \tag{27} \]
The ICs were generated using the N-GenIC code (Springel 2015), which employs the Zel’Dovich approximation to generate a perturbed particle distribution, for an ordinary ΛCDM cosmological simulation at the starting redshift \( z = 1/a - 1 = 127 \) with an input power spectrum following Efstathiou et al. (1990, 1992), i.e. of the form
\[ P(k) \propto k^2 \left( 1 + \left( \frac{a k + (bk)^2}{(c k)^2} \right)^3 \right)^{-5/2}, \tag{28} \]
with \( a = 6.4/G h^{-1} \text{Mpc}, b = 3.0/G h^{-1} \text{Mpc}, c = 1.7/G h^{-1} \text{Mpc}, \Gamma = 0.2 a h = 0.21, \) and \( v = 1.13. \) A wave function was constructed from the same ICs using the procedure in eqs. (25) and (26).

### 3.2 Resolution and convergence tests

Before using the newly-developed FDM AREPO module for simulations of larger cosmological boxes, extensive tests were performed on boxes of comoving size \( L = 1 h^{-1} \text{Mpc}. \) This not only allowed for verification of the code’s correctness by comparing to CDM simulations with the same ICs and to other implementations of FDM, but also to study the behaviour of the pseudo-spectral method. Of particular interest are the resolution requirements, e.g. to what extent the numerical results remain valid and the convergence of the matter power spectrum is compromised when the velocity constraint (eq. (22)) is violated. Moreover, the differences to the behaviour of CDM simulations are of interest even in this small test volume.

5 See section 3.4.

6 The cosmological parameters are the same as in section 3.4.
Although the simulation volume is too small to be cosmologically representative at \( z = 0 \) since even the largest scales become non-linear by then, there are opportunities to observe what differences (or similarities) become apparent in both the linear (early times) and non-linear (late times) regimes of both dark matter models. The legend indicates the grid size for each included simulation. The power spectrum evolved using linear perturbation theory is shown for comparison. The bottom panel shows the ratio of the power spectra to the result from linear theory.

Fig. 2. Dark matter power spectra for cosmological FDM simulations with different resolutions, box size \( L = 1 \, h^{-1} \) Mpc, and FDM mass \( mc^2 = 1.75 \times 10^{-23} \) eV, at \( z = 0 \) evolved from CDM ICs. The legend indicates the grid size for each included simulation. The power spectrum evolved using linear perturbation theory is shown for comparison. The bottom panel shows the ratio of the power spectra to the result from linear theory.

A more quantitative indication of numerical convergence is given in Fig. 2 by the power spectra of the density field. This demonstrates that, for the given setup, grid sizes of \( 128^3 \) and smaller lead to significant deviations from higher-resolution results even on large scales.
which is determined by the time step size given by eq. (21). The drift criterion, which results in the following limit given by the drift criterion, which is inversely proportional to the resolution requirements (e.g. eq. (22)) and for higher redshifts. Thus, the number of time steps is roughly inversely proportional to the total number of time steps is roughly inversely proportional to the drift criterion, which is inversely proportional to the resolution requirements. The concrete values of CPU time correspond to the Cobra cluster at MPCDF for reference.

Figure 3. The expected computational cost of cosmological simulations when running to $z = 0$ as a function of box size $L$ for a mass size of $N^3 = 8192^3$ and different FDM masses. Shaded regions indicate areas where the velocity constraint eq. (22) is violated for the corresponding mass at $z = 0$ due to lack of resolution. The CPU times correspond to the Cobra cluster at MPCDF for reference.

Figure 4. The expected computational cost of cosmological simulations with different FDM masses as a function of target redshift $z_{\text{end}}$ for a mass size of $N^3 = 8192^3$ at the maximum allowed box size $L = L_{\text{max}}$, as given by the velocity constraint eq. (22). The points where the curves intersect their associated shaded regions in Fig. 3 would correspond to $z_{\text{end}} = 0$ in this figure.

3.3 Computational cost analysis

As mentioned before, the required computational resources – both runtime cost and memory – are a significant obstacle when attempting to perform large-scale simulations of FDM. The run-time cost depends on three simulation parameters: the FDM mass $m$, the (comoving) box size $L$, and the number of mesh points $N^3$ (where $N$ is the number of points per dimension).

For each individual time step, the operations in steps (20a) to (20e) are performed. The cost of a single time step only depends on the mesh size $N^3$, while $m$ and $L$ only enter the computation through multiplication as constant factors. It is dominated by the cost of the FFT operations, which scales as $O(N^3 \ln(N^3))$. The total cost then depends on the number of time steps which have to be performed, which is determined by the time step size given by eq. (21). The drift criterion $\Delta x^2$ is typically more stringent, especially in light of the resolution requirements (e.g. eq. (22)) and for higher redshifts. Thus, the number of time steps is roughly inversely proportional to the limit given by the drift criterion, which results in the following approximate behaviour of the computation time $C$:

$$C \propto \frac{N^5 \ln(N^3)}{mL^2}. \quad (29)$$

This relation was also verified in practice using measurements of small test computations.

Particularly striking is the strong dependence of eq. (29) on $N^5$, which rapidly leads to exploding computational costs for larger $N$. At fixed $N$, the cost can be decreased by choosing a larger box size $L$ because this reduces the resolution $\Delta x$. However, as mentioned before, the resolution requirements are quite strict due to the inherent scale in the Schrödinger–Poisson system, which must be resolved in order to yield reliable results.

To give a tangible reference for the absolute costs involved on the one hand, but also the code’s performance on the other hand, Fig. 3 shows the expected cost of simulations with mesh size $N^3 = 8192^3$ to $z = 0$ in CPU hours on the state-of-the-art Cobra supercomputer at the Max Planck Computing and Data Facility (MPCDF). Fig. 3 displays the CPU time for a range of masses $m$ with variable box size $L$. Even when allowing for the velocity criterion eq. (22) to be violated, it becomes clear that such a simulation would take at least $10^7$ CPUh. Even worse, the resolution requirements severely limit the possible simulation volume to $L < 10 \, h^{-1} \, \text{Mpc}$ for the mass range of interest. Since the de Broglie wavelength increases with smaller masses, the resolution requirements can be loosened by choosing much smaller values for $m$, but this does not nearly go far enough as the physically viable range is bounded around $\gtrsim 10^{-22} \, \text{eV}$. The computational cost for the largest possible box size $L_{\text{max}}$ at a given mass $m$ (intersections of the graphs with the vertical lines) decreases only weakly with decreasing $m$.

Fig. 4 shows the same computation time at the largest possible box size due to eq. (22) for each mass, but now for the case where the simulation is stopped earlier than $z = 0$. Unfortunately, the computational cost does not depend very strongly on the final redshift.

\[\text{https://www.mpcdf.mpg.de/services/supercomputing/cobra}\]

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Table 1. List of performed simulations with important characteristics. The lengths given for the box sizes and resolutions are comoving.

| Type | Res. el. | L / h⁻¹ Mpc | mc² / eV | Resolution |
|------|----------|-------------|----------|------------|
| FDM  | 8640³    | 10          | 7 × 10⁻²³ | 1.16 h⁻¹ kpc |
| FDM  | 4320³    | 10          | (3.5, 7) × 10⁻²³ | 2.31 h⁻¹ kpc |
| FDM  | 3072³    | 10          | (3.5, 7) × 10⁻²³ | 3.26 h⁻¹ kpc |
| FDM  | 2048³    | 10          | (3.5, 7) × 10⁻²³ | 4.88 h⁻¹ kpc |
| FDM  | 4320³    | 5           | 7 × 10⁻²³ | 1.16 h⁻¹ kpc |
| FDM  | 3072³    | 5           | (3.5, 7) × 10⁻²³ | 1.63 h⁻¹ kpc |
| FDM  | 2048³    | 5           | (3.5, 7) × 10⁻²³ | 2.44 h⁻¹ kpc |
| FDM  | 1024³    | 5           | (3.5, 7) × 10⁻²³ | 4.88 h⁻¹ kpc |
| CDM  | 2048³    | 10          | — | 9.69 × 10³ h⁻¹ M☉ |
| CDM  | 1024³    | 10          | — | 7.75 × 10⁴ h⁻¹ M☉ |
| CDM  | 512³     | 10          | — | 6.20 × 10⁵ h⁻¹ M☉ |
| CDM  | 1024³    | 5           | — | 9.69 × 10³ h⁻¹ M☉ |
| CDM  | 512³     | 5           | — | 7.75 × 10⁴ h⁻¹ M☉ |

(approximately ∝ z⁻³/₅ in the region shown); it is merely lowered by a factor of ≤ 5 even when stopping at z = 6, and still remains at several million CPU hours for a N³ = 8192³ simulation.

3.4 Simulations

All our simulations were performed using the cosmological parameters Ω_m = 0.3, Ω_λ = 0, Ω_b = 0.7, H₀ = 70 km s⁻¹ Mpc⁻¹ (h = 0.7), and σ₉ = 0.51 with ICs as described in section 3.1. The same random seed was shared for the generation of all ICs at z = 127 in order to allow for a direct comparison between different dark matter models and resolutions. In order to avoid the onset of resolution effects that would affect even our highest-resolution simulations, the simulations were run until z = 3. For comoving box sizes of 5 h⁻¹ Mpc and 10 h⁻¹ Mpc, and masses mc² of 3.5 × 10⁻²³ eV and 7 × 10⁻²³ eV, simulations with different resolutions – up to mesh sizes of N³ = 8640³ – were performed. A detailed list of the different simulations is given in Table 1.

To date, these are the largest three-dimensional cosmological simulations of structure formation including the full FDM dynamics to low redshifts, with a simulation volume which is at least 64 times larger and a mesh which has at least 600 times more resolution elements than any comparable existing work. The largest simulation required more than 7 × 10⁹ CPU h on the Coba cluster to complete. Previous efforts (using various methods) have only reached box sizes of 2.5 h⁻¹ Mpc (Mina et al. 2020, using an adaptive mesh refinement (AMR) method, until z = 2.5) or 1.7 h⁻¹ Mpc (Mocz et al. 2020, also with a pseudo-spectral method, until z = 5.5) for full FDM simulations, and 2.5 h⁻¹ Mpc with a hybrid method (Velmaat et al. 2018, i.e. not including the FDM dynamics everywhere in the simulation volume) for similar cosmological simulations.

3.5 Grid-based halo finding

The halo mass function (HMF) is an important measure of large-scale structure. For typical CDM simulations, it is determined using an algorithm like friends-of-friends (FoF) in AREPO, which identifies haloes by connecting simulation particles (point masses) whose distance to other particles is below a certain threshold. When using the pseudo-spectral method, however, there are no particles, and the density field is instead represented by a Cartesian mesh. This means that widely-used algorithms, which operate on the particle distribution, cannot be used to analyse the FDM simulations presented here.

Because of this, a modified version of the FoF algorithm was developed in this work to enable the determination of the HMF for a discretised density mesh. Instead of a linking length, the grid-based halo finder uses a density threshold as a parameter. Adjacent cells in the mesh are linked if their density exceeds the density threshold. The grid-based halo finder is part of the pseudo-spectral FDM AREPO module.

Fig. 5 demonstrates that this new halo finder performs well in comparison to the standard particle-based FoF algorithm. Using a CDM simulation with 512³ particles, the density was represented both using the original particle data and a Cartesian mesh with 1024³ grid points whose density values were determined using cloud-in-cell (CiC) mass assignment. The HMF was determined with the particle and grid data as input for FoF and the new grid-based halo finder, respectively. As evident in Fig. 5, both procedures show excellent agreement. Differences arise in the lightest haloes, which are limited by resolution since density is always spread out over at least one cell volume, and slight variations are present for the most massive haloes, where statistical effects play a role since there are only a few haloes per mass bin.

4 DARK MATTER POWER SPECTRUM

The matter power spectrum gives important insight into how matter clusters at different length scales, and its evolution in time is by now quite accurately known for ΛCDM (e.g. Jenkins et al. 1998), even when baryonic effects are included (Springel et al. 2018). The power
panels show the ratio of the power spectra to the result from linear theory. For Figure 6.

$$P(k) / P_{\text{linear}}$$

**Figure 6.** Dark matter power spectra at different redshifts for a high-resolution cosmological FDM simulation with box size $L = 10\, h^{-1}\, \text{Mpc}$, FDM mass $m_{\text{c}} = 7 \times 10^{-23}\, \text{eV}$, mesh size $N^3 = 8640^3$, and CDM ICs. The power spectrum evolved using linear perturbation theory is shown for comparison. The lower panels show the ratio of the power spectra to the result from linear theory. For $z = 63$, the dashed line additionally indicates the FDM Jeans scale (eq. (15)).
spectrum is also a particularly important diagnostic for a comparison of FDM with CDM, because the former is expected to suppress structure formation at scales smaller than $\lambda_{\text{BH}}$ due to the Heisenberg uncertainty principle. Due to the use of CDM ICs, our results demonstrate to what extent the dynamics of the Schrödinger–Poisson eqs. (7) and (8) affect the clustering of matter compared to CDM, even without the initial suppression present in a ‘realistic’ FDM cosmology (cf. section 7).

In Fig. 6, we show the time evolution of the matter power spectra of FDM at our highest resolution of $8640^3$ cells in a $10\,h^{-1}\text{Mpc}$ box. From the top left to the bottom right panel, we show redshifts from $z = 63$ to $z = 3$, and compare to expectations from linear theory. In particular, the bottom of each panel gives the ratio of the measured FDM power to the linear theory power spectrum used for initializing the simulation. At high redshifts and at small $k$, FDM accurately follows linear theory, with modes growing independently with the same linear growth factor, just like CDM does, so that the pattern of random fluctuations of our particular realization around the smooth initial input spectrum is preserved in time. Beginning at scales around $k \approx 10\,h\,\text{Mpc}^{-1}$ and a redshift of $z \approx 15$, signs of mildly non-linear evolution are apparent, which manifest themselves in a stronger than linear growth of power. This non-linear evolution becomes quickly more pronounced in a way reminiscent of CDM, except that on the smallest resolved scales, for $k \geq 1000\,h\,\text{Mpc}^{-1}$, the non-linear growth appears sluggish and lagging behind that seen on larger scales.

It is now important to examine on which scales and at which times these results are quantitatively reliable. As we have discussed earlier, the stringent numerical requirements of FDM make this more involved than for CDM, because here even the large-scale linear growth requires a fairly high resolution to get right, and it is not readily clear how numerical limitations will manifest themselves in the results. In Fig. 7 we first compare results for FDM simulations in a $10\,h^{-1}\text{Mpc}$ box using different resolutions, with grid sizes from $2048^3$ to $8640^3$. Focusing on the three redshifts of $z = 31$, $z = 10$, and $z = 3$, we see that the results still appear well converged at the high redshifts of $z = 31$ and $z = 10$ (although the lowest-resolution simulation with $N^3 = 2048^3$ slightly starts to lag behind at $z = 10$ on smaller scales). However, this ceases to be true at the lower redshift of $z = 3$, where the growth of the lower-resolution simulations now clearly trails behind, and this effect occurs on an extended range of scales, including fairly large ones, quite unlike in CDM, where the impact of spatial resolution limits is typically constrained to fairly
expectation, for fixed box size and grid resolution, it is possible to
test this by examining how they vary with box size and axion particle mass
at a fixed epoch of \( z = 63 \). This is shown in Figs. 8 and 9. Here, the
impact of eq. (23) is clearly visible: The resolution requirements become more stringent with larger axion masses and at fixed grid size, a larger box size directly implies worse resolution. Thus, at fixed grid size, if axion mass and box size are scaled by the same factor, the resulting power spectra are identical except at small scales, which are impacted by the change in Jeans length with different axion masses. Accordingly, due to resolution effects and contrary to the physical expectation, for fixed box size and grid resolution, it is possible to see more small-scale power when the particle mass is lowered, since lower masses have less stringent resolution requirements.

Further insights into these numerical effects can also be obtained by examining how they vary with box size and axion particle mass at a fixed epoch of \( z = 3 \). This is shown in Figs. 8 and 9. Here, the impact of eq. (23) is clearly visible: The resolution requirements become more stringent with larger axion masses and at fixed grid size, a larger box size directly implies worse resolution. Thus, at fixed grid size, if axion mass and box size are scaled by the same factor, the resulting power spectra are identical except at small scales, which are impacted by the change in Jeans length with different axion masses. Accordingly, due to resolution effects and contrary to the physical expectation, for fixed box size and grid resolution, it is possible to...
growth and then shows a quite similar overall shape of the non-linear power spectrum. While FDM power is still somewhat suppressed at \( z \leq 7 \) for non-linear scales, the difference to CDM is much smaller than at earlier redshifts. This is true until a characteristic scale of around \( k \approx 1000 \text{ h Mpc}^{-1} \), where the FDM non-linear power drops significantly below the CDM power. This scale appears to be related to the ‘quantum pressure’ effects in FDM, and is thus ultimately related to the particle mass.

Curiously, just before the non-linear FDM power decays away from CDM towards small scales, it manages to slightly exceed it. This effect could be a reflection of the interference patterns in the dark matter density field resulting from the wave-like nature of FDM, which generates transient structures of size \( \lambda_{dB} \) that are clearly visible in high-resolution images that zoom in on large haloes.\(^\text{12}\) These types of order-unity density fluctuations are absent in CDM. If the associated enhancement of power localized around a characteristic scale \( k \) could be measured by some tracer sufficiently accurately, it could be a tell-tale sign of FDM.

5 HALO MASS FUNCTION

One of the most important and fundamental outcomes of structure formation are gravitationally collapsed structures of dark matter, so-called haloes, which are in turn the sites of baryonic galaxy formation processes. The abundance of haloes as a function of mass and epoch is thus fundamental to any cosmological model.

For the CDM scenario, the (extended) Press–Schechter formalism (Press & Schechter 1974; Sheth & Tormen 1999) has proven to be a simple and quite reliable approach to estimate the HMF and its evolution based on the linear theory power spectrum and the linear growth factor alone. Comparisons to full \( N \)-body simulations have both verified the basic approach and led to the calibration of more

\(^{12}\) A similar effect (although in a different cosmological context) has been found in Mocz et al. (2020).

Figure 11. Projected dark matter density at \( z = 3 \) of a 3 \text{ h}^{-1} \text{ Mpc} (comoving) slab in a high-resolution cosmological box simulation of FDM with box size \( L = 10 \text{ h}^{-1} \text{ Mpc} \), FDM mass \( mc^2 = 7 \times 10^{-23} \text{ eV} \), mesh size \( N^3 = 8640^3 \), and CDM ICs. The largest haloes identified with the halo finder are marked with circles whose radii indicate the haloes’ virial radii \( R_{200} \).

Figure 12. HMF of cosmological FDM and CDM simulations at \( z = 3 \) with a box size of \( L = 10 \text{ h}^{-1} \text{ Mpc} \), CDM ICs, and FDM masses of \( mc^2 = 7 \times 10^{-23} \text{ eV} \) and \( 3.5 \times 10^{-23} \text{ eV} \) (top and bottom panels, respectively). The HMF derived for CDM by Sheth & Tormen (1999) is shown for comparison. The dotted lines show the fitting function determined by Schive et al. (2016) for the given FDM mass, while the dashed lines show a fit of the data to a similar function (eq. (30)) with two free parameters. The lower sub-panels show the ratios of the mass functions to the result of the CDM simulation.
in a collisionless N-body simulation with truncated initial fluctuation spectrum, similar to how warm dark matter models are often treated (Wang & White 2007). Ultimately, full non-linear simulations of the Schrödinger–Poisson system are required to obtain quantitatively reliable results. However, due to the numerical challenges involved in large-volume simulations of FDM, such determinations have not been obtained thus far.

Our new simulations for the first time allow corresponding measurements, although the results are not representative of a fully consistent FDM cosmology due to the use of CDM ICs. However, our simulations allow us to demonstrate how strongly the FDM dynamics alone impact the HMF, and accordingly, to what extent their omission might affect the result. To this end, we study the HMF using the grid-based halo finder introduced in section 3.5, whose operation on our largest simulation is demonstrated in Fig. 11. In Fig. 12, we show the obtained mass functions for the FDM and CDM cases for a set of example simulations. As expected from the power spectra, the HMF of FDM exhibits a lack of low-mass haloes, but agrees for massive haloes. To obtain essentially perfect agreement with CDM for the most massive haloes, however, requires sufficiently high resolution in the FDM calculation for the target redshift. As we have seen earlier, the numerical resolution requirements become ever more stringent towards lower redshift, and once they start to be compromised, the growth on large (even still linear) scales becomes damped, which then manifests itself in halo masses which are biased low. This is clearly seen in Fig. 12 in the comparison of the low- and high-resolution FDM results for the HMF.

More interesting, however, is the abundance reduction of low-mass haloes in FDM. Indeed, the delayed onset of structure formation becomes apparent even more clearly in the HMF than in the power spectrum, with almost no objects present before \( z = 7 \). At late times, there is a dearth of low-mass haloes in our results, but the deficit appears not as strong as the cut-off predicted by Kulkarni & Ostriker (2020). Rather, it appears that power transfer to smaller scales is sufficiently strong in FDM that the break in the HMF is weaker than predicted by simplified models based on ‘quantum pressure’-filtered versions of the Press–Schechter formalism. This reinforces the notion that simply applying a Jeans filtering at the linear level and assuming that all smaller scales never grow and thus can be ignored does not necessarily yield sufficient quantitative accuracy to predict the non-linear outcome of FDM. On the other hand, the simulations shown in Fig. 12 do not yet include the effect of ICs appropriate for the FDM model, which feature a strong cut-off even in the initial power spectrum (see section 7) and are thus expected to result in additional suppression of low-mass haloes.

Also shown in Fig. 12 are two fits, given by dotted and dashed lines, in a fashion similar to Schive et al. (2016). While the dotted line uses exactly the same function as given in Schive et al. (2016) for the corresponding particle mass \( m \), the dashed line is a modified fit of the form

\[
\frac{dn}{dM}_{\text{FDM}}(M, z) = \left( 1 + \left( \frac{M}{M_0} \right) \left( \frac{mc^2}{10^{-22}\text{eV}} \right)^{\frac{\alpha}{2}} \right)^{-2.2} \frac{dn}{dM}_{\text{CDM}}(M, z),
\]

where the Sheth & Tormen (1999) mass function has been used for the CDM mass function \( \frac{dn}{dM}_{\text{CDM}}(M, z) \). In other words, the mass parameter \( M_0 \) and the inner exponent \( \alpha \) are allowed to vary compared to the values found in Schive et al. (2016) (where they are set to \( M_0 = 1.6 \times 10^{10}M_\odot \) and \( \alpha = -1.1 \)). The result shows that while an appropriate fit to our mass function can be obtained using a very similar functional form, the shape changes slightly, and the peak mass \( M_0 \) shifts to much lower halo masses. This indicates that either the method used by Schive et al. (2016) underestimates the number of low-mass haloes, or that the inclusion of FDM-appropriate initial conditions (as discussed in section 7) causes a strong impact on the formation of low-mass haloes. This question can only be resolved by performing another set of full FDM simulations with self-consistent initial conditions.

Figure 13. ‘Stacked’ halo profiles at \( z = 3 \) for several bins of virial mass \( M_{200} \) (given as intervals \([\cdot, \cdot]\) in \( L = 10h^{-1}\text{ Mpc} \) cosmological box simulations of FDM \( m_{\text{had}}^2 = 7 \times 10^{-22}\text{eV} \), \( h = 0.72 \times 10^{-30} \text{m}^3 \text{s}^{-1} \text{kg}^{-1} \) and CDM \( 512^3 \) particles), using CDM ICs. The number of haloes contained in each mass bin is stated in parentheses. The top panel shows the results for FDM, while the bottom panel shows those for CDM. Thin dashed lines show NFW fits (eq. (32)) to the region within the virial radius \( R_{200} \), which is indicated with thick lines. For FDM, the inner region of the halo has been excluded from the NFW fit and instead been fit to the soliton density profile (eq. (31)), as indicated by dot-dashed lines. The FDM grid resolution is shown as a vertical dotted line.
6 HALO PROFILES

The internal density structure of cosmological dark matter haloes is a further key outcome of non-linear structure formation whose importance can hardly be overstated. This is because of its crucial influence on the size, kinematics and morphology of galaxies forming in the dark matter haloes, as well as on the gravitational lensing strength of these objects. These properties are of course decisive for the viability of a cosmological model in the first place. For cold dark matter models, the spherically averaged density profiles have density cusps at their centres (Navarro et al. 1996), giving rise to a particular rotation curve shape of galaxies.

One of the often cited motivations for considering FDM is the so-called cusp–core problem in ACDM, combined with the expectation that FDM naturally produces cored density profiles that may be potentially easier to reconcile with rotation curve data of certain galaxies. By considering a spherically-symmetric ground state of FDM, one arrives at solutions called solitons, whose density profiles are flat towards their centres. The production of such solitonic cores has been predicted on theoretical grounds and has been verified with simulations of small cosmological volumes (Schive et al. 2014a).

Here, we are interested in testing whether we also see them in our comparatively large-volume simulations, which have however quite limited ability to resolve the internal structure of individual haloes.

In Fig. 13 we therefore consider spherically averaged density profiles, stacked for several different mass ranges, in our highest-resolution simulation at $z = 3$. The stacking greatly reduces halo-to-halo scatter and allows us to clearly identify the mean density profile. For comparison, we also show in the bottom panel equivalent density profiles for haloes in a corresponding CDM N-body simulation. As before, the centres of haloes have been identified as the densest cell in the FDM case, or the particle with the minimum gravitational potential in CDM, while the total halo mass has been determined with the spherical overdensity definition in both cases. The measurements of the density profiles include all mass around the halo centres, i.e. beyond $R_{200}$, the mean profiles eventually slow their decline and asymptote to the cosmic mean density.

In the FDM case, we clearly see the formation of solitonic cores in low-mass haloes, as illustrated by the dot-dashed lines, which are fits to the analytic description of the soliton density profile by Schive et al. (2014b):

$$\rho_{c,\text{soliton}}(r_c) = \frac{1.9 \times 10^9 a^{-1\langle mc^2\rangle}^{-2}}{\left(1 + 0.091 \left(\frac{r_c}{r_s}\right)^3\right)^8} M_\odot/kpc^3.$$

(31)

Note that this profile depends on the core radius $r_s$ only, without further free parameters. Our simulations thus accurately reproduce earlier work on the shape of the innermost mean profile, where the ‘quantum pressure’ dominates. In the outer parts of the haloes, the run of the mean density follows the Navarro–Frenk–White (NFW) form (Navarro et al. 1996)

$$\rho_{c,\text{NFW}}(r_c) = \frac{\rho_0}{r_c \left(1 + \frac{r_c}{R_s}\right)^2}.$$

(32)

rather well. It is thus tempting to use a combination of a solitonic core and a NFW profile as comprehensive description of the non-linear density structure of FDM haloes, and to use this to forecast the rotation curve of galaxies and the core size of galaxies as a function of their virial mass (Burbert 2020). We caution, however, that this neglects the potentially significant impact of order-unity density fluctuations of the FDM wave function outside the solitonic core, which are completely lost in the spherical averaging and the halo stacking. Likewise, it is unclear how severely baryons can impact the innermost density profile. These baryonic effects are quite uncertain in CDM, but even less is known about them in FDM, where they could potentially be stronger due to the lower dark matter densities in the centres of haloes. It will thus likely require cosmological hydrodynamic simulations of galaxy formation in FDM to arrive at firm predictions about rotation curve shapes. Our results here confirm that such calculations are in principle feasible down to intermediate redshifts in sufficiently large volumes to allow population studies of galaxies, especially if the resolution can be made spatially adaptive (also see e.g. Mocz et al. 2020).

7 FUZZY DARK MATTER INITIAL CONDITIONS

Finally, we would like to briefly consider the additional source of differences in FDM relative to CDM introduced due to modifications of the initial conditions. Thus far, we had focused for this study on studying differences between CDM and FDM resulting just from the different dynamics induced by the Schrödinger–Poisson system of equations as opposed to the Vlasov–Poisson system governing collisionless dynamics of classical particles. To this end, we had used identical initial perturbations, retaining the CDM power spectrum for definiteness. However, a physically consistent model of a FDM cosmology will witness modifications of its initial conditions relative to CDM, particularly in terms of a significant reduction in small-scale power below a scale set by the ‘quantum pressure’, yielding something resembling the truncated power spectra of warm dark matter models. In linear theory, this is determined by the Jeans scale $k_s$ (eq. (15)), above which perturbations evolve in an oscillatory rather than growing fashion. It is however not clear how scales $k > k_s$ react to non-linear power transfer from larger scales once scales around $k_s$ become mildly or fully non-linear. This can only be accurately treated with explicit (e.g. spectral) FDM simulations like the ones we carry out here.

In Fig. 14 we show a comparison of two FDM simulations carried out either with CDM-like initial conditions or instead with ICs predicted by the Boltzmann code axionCAMB (Hložek et al. 2017) for an axion cosmology with our adopted particle mass of $2.5 \times 10^{-22} \text{ eV}$—in this comparison. For definiteness, we chose a 2 h$^{-1}$ Mpc box size with 2400$^3$ mesh points to represent the wave function; the remaining cosmological parameters are the same as before (section 3.4). We corresponding to the HMF-weighted mean soliton mass of the binned haloes. Although it is not necessarily to be expected that the drop in density beyond the flat central plateau of the stacked profile will be accurately described by the soliton profile due to the varying soliton radii contained in each mass bin, the general agreement with the soliton profile demonstrates the presence of the cores described in previous work in our simulations.
The lower panels show the ratio of the power spectra to the result from CDM linear theory. For small excess power relative to the linear theory CDM power spectrum, still shows a stalled growth at the smallest scales, but scales around negligible small-scale power and only agrees with the CDM linear theory. In comparison, the FDM model with self-consistent ICs shows behaviour based on linear theory, where scales with transition region oscillatory behaviour is seen. This is the expected much smaller scales, the growth has completely stalled, while in the transition region power transfer to smaller scales, as evidenced by the change in shape of its small-scale power spectrum.

Finally, at redshift \( z = 3 \), both simulations have become even closer in power transfer to smaller scales, as evidenced by the change in shape of its small-scale power spectrum.

Consider the time evolution down to redshift \( z = 3 \), and show in each redshift panel of the plot also the linear theory power spectrum of the corresponding CDM cosmology, for comparison. At \( z = 63 \), the FDM model with CDM-like ICs follows linear growth accurately only to a scale of order the Jeans scale set by the ‘quantum pressure’. On much smaller scales, the growth has completely stalled, while in the transition region oscillatory behaviour is seen. This is the expected behaviour based on linear theory, where scales with \( k \ll k_j \) should not grow, while those with \( k \gg k_j \) follow the same linear growth as CDM. In comparison, the FDM model with self-consistent ICs shows negligible small-scale power and only agrees with the CDM linear power on the largest scales that can be followed with this box size.

At redshift \( z = 15 \), the FDM simulation with self-consistent ICs still shows a stalled growth at the smallest scales, but scales around \( k_j \) have already entered mild non-linear evolution, as reflected in small excess power relative to the linear theory CDM power spectrum. Apparently, this already couples significantly to smaller scales, so that scales beyond \( k_j \) begin to catch up with the growth, such that in this instance, the power spectrum happens to agree coincidentally with the linear CDM power at around \( k \approx 200 \, h \, \text{Mpc}^{-1} \), while scales \( k \gtrsim 1000 \, h \, \text{Mpc}^{-1} \) are not yet affected and still prevented from growth. Similarly, the self-consistent FDM model begins to witness power transfer to smaller scales, as evidenced by the change in shape of its small-scale power spectrum.

By redshift \( z = 7 \), these trends have greatly accelerated. Now the non-linear evolution of the FDM model started from CDM-like initial conditions has produced a large power excess relative to linear theory for nearly all \( k \) resolved in the simulation, except for the largest scales (which are still linear), and the smallest scales (which are still suppressed by ‘quantum pressure’). Interestingly, however, the model with self-consistent initial conditions is clearly in the process of catching up to this evolution, with its small-scale power being progressively filled in by the non-linear evolution on larger scales.
their total matter power spectrum. We expect these trends to continue towards lower redshifts (which are unfortunately inaccessible with this resolution), so that the discrepancy introduced by starting from CDM-like initial conditions instead of self-consistent FDM ones becomes largely forgotten on small scales, simply because it is overwhelmed by non-linear evolution. However, there may still be an important impact on other quantities, such as the HMF (which, however, requires a larger box size, as in section 5, to measure). How large this is remains to be seen. In principle, excess oscillatory perturbations on small scales that cannot grow should not give rise to additional collapsed haloes, but due to the non-linear coupling to larger scales this may only be approximately true.

8 SUMMARY AND CONCLUSIONS

In this study, we have carried out simulations of fuzzy dark matter models with the pseudo-spectral method. This can be considered the most accurate approach to solve the Schrödinger–Poisson system numerically, without neglecting certain aspects of the temporal evolution of the wave function. In particular, this can account for oscillatory, order-unity fluctuations of the local density due to the quantum-mechanical effects in the axion-like dynamics.

Unfortunately, the numerical resolution requirements to faithfully follow the FDM dynamics are much harder to fulfill than for the familiar $N$-body techniques applicable in the CDM case. Even large-scale modes require a very fine mesh, otherwise the velocities appearing in the dynamics cannot properly be represented because then the spatial variation of the quantum-mechanical phase factor is not resolved. Yet, making the mesh fine enough to resolve the de Broglie wavelength $\lambda_{dB}$ of the largest velocities drives down the time step, which depends quadratically and not linearly on the spatial resolution. In practice, this means that one cannot trivially simulate large cosmological volumes at low resolution in a way similar to the standard practice in CDM. Rather, one is relegated to very small volumes and to the high-redshift regime, whereas a push to large box sizes and low redshift quickly becomes extremely expensive, as we highlighted in this paper.

In this work, we could nevertheless compute the largest-volume simulations with full FDM dynamics thus far by using rather large grid sizes of up to $8640^3$ cells. This allowed us to gain new insights into the evolution of the non-linear power spectrum in these cosmologies, and to make the first direct measurements of the halo mass function in such models, especially with respect to how the FDM dynamics affect these observables without starting from an already-suppressed power spectrum in the ICs. Our main findings can be summarized as follows:

- Once sufficient resolution is available, the FDM power spectrum in our spectral simulations follows the CDM evolution very closely on large scales, even once highly non-linear evolution has set in on smaller scales.
- Lacking resolution in pseudo-spectral simulations of FDM manifests itself in a spurious ‘freezing’ of further structure formation once the resolution requirements become violated. The evolution of the power spectrum then stops, and it remains roughly constant with decreasing redshift. This can be detected by comparing the power on the largest scales to linear perturbation theory or a CDM simulation, and observing that the FDM model falls behind.
- The maximum simulation volume which can be achieved in the pseudo-spectral method without violating resolution requirements increases for lower particle masses $m$. The computational time required for the corresponding maximum box size varies only slowly for different values of $m$. Stopping a simulation before $\tau_{end} = 0$ relaxes the resolution requirements, thus allowing for larger box sizes, and reduces computational time, but not very strongly (the cost approximately scales as $\propto z_{end}^{-1/2}$ at fixed box size, which parallels the scaling of the maximum particle velocity in linear perturbation theory).
- On small scales, FDM structure formation is suppressed compared to CDM due to the effects of the ‘quantum pressure’, which also induces a small-scale cut-off in the initial power spectrum of self-consistent cosmological FDM models. The onset of non-linear structure formation is thus generally delayed in FDM compared to CDM. However, on scales comparable to $\lambda_{dB}$, the difference between FDM and CDM is reduced, and interference patterns on these scales in FDM can even lead to a small temporary excess of power compared to CDM. Mildly trans-linear and fully non-linear evolution in FDM in any case leads to significant power transfer to scales smaller than the initial FDM Jeans scale, even to the extent that oscillatory perturbations on these scales, if present, can become completely overwhelmed and buried by the power transfer from larger scales.
- We could, for the first time, measure the halo mass function directly from spectral FDM simulations. For massive haloes, we find the same abundance of haloes as in CDM, consistent with expectations based on the agreement of their large-scale linear power spectra. The formation of the first haloes is however delayed for FDM with respect to CDM, and there are considerably fewer low-mass haloes forming compared to CDM. The low-mass cut-off of the HMF we find is less pronounced than found by methods which take into account the suppressed FDM ICs, even for those which neglect the non-linear FDM dynamics. This demonstrates that the FDM ICs have a larger impact on the HMF than the dynamics at later times, although the suppression caused by the latter is still sizeable. However, since the cut-off appears at much lower halo masses compared to the FDM ICs, the impact of this effect is much more difficult to observe.
- Despite our large grid sizes, our simulations have only limited resolving power for the internal structure of dark matter haloes. However, we could still clearly detect characteristic solitonic cores in spherically averaged density profiles of haloes, with a profile shape that matches results of previous work well. These cores are particularly large for low-mass haloes, where they also substantially reduce the central dark matter densities relative to CDM.

Our results thus highlight that it is, in principle, possible to compute accurate cosmological simulation results for FDM in representative volumes, albeit at considerable computational cost. We will apply our implementation to FDM initial conditions (similar to section 7) in larger volumes in the future, providing physical observables in a fully consistent FDM scenario. However, to conclusively understand the combined dynamics of FDM and baryons in the centres of galaxies, and thus to arrive at reliable predictions about whether these scenarios can resolve, e.g., the cusp–core tension, will ultimately require to add baryonic physics to these simulations, and to make them spatially more adaptive inside individual haloes such that the resolution can be increased there in a targeted fashion. First attempts in this direction have been carried out (e.g. Mocz et al. 2019; Veltmaat et al. 2020), but the associated numerical challenges are still formidable.

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DATA AVAILABILITY

The data underlying this article will be shared on request to the corresponding author.

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