A new class of copula regression models for modelling multivariate heavy-tailed data

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Abstract

A new class of copulas, termed the MGL copula class, is introduced. The new copula originates from extracting the dependence function of the multivariate generalized log-Moyal-gamma distribution whose marginals follow the univariate generalized log-Moyal-gamma (GLMGA) distribution as introduced in Li et al. (2021). The MGL copula can capture nonelliptical, exchangeable, and asymmetric dependencies among marginal coordinates and provides a simple formulation for regression applications. We discuss the probabilistic characteristics of MGL copula and obtain the corresponding extreme-value copula, named the MGL-EV copula. While the survival MGL copula can be also regarded as a special case of the MGB2 copula from Yang et al. (2011), we show that the proposed model is effective in regression modelling of dependence structures. Next to a simulation study, we propose two applications illustrating the usefulness of the proposed model. This method is also implemented in a user-friendly R package: rMGLReg.

Keywords: MGL copula; exchangeable and asymmetric dependency; extreme-value copula; copula regression.

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1 Introduction

Modelling multivariate heavy-tailed data is an important challenge in actuarial statistics. While univariate risk models based on heavy tailed distributions are well developed (see e.g. Beirlant and Goegebeur (2003), Li et al. (2016), Leppisaari (2016), and Li et al. (2021)), predicting extreme loss through multivariate models has received much less attention, especially in the presence of additional covariate information.

Copulas have become highly popular in modelling flexible dependence structures of multi-dimensional data during the past several decades. The advantage of the copula concept is that it separates modelling of the marginals from the dependence structure (Joe, 2014). Multivariate loss often exhibits asymmetric dependence, with special emphasis on the joint upper tail dependence of the multivariate heavy tailed data. Elliptical copulas such as the Gaussian or Student-t copulas, and Archimedean copulas with exchangeable dependence structures, imply symmetric dependence. Extreme-value copulas are tailored for extremes and can capture tail dependence and asymmetry, while computation of joint densities can be excessively prohibitive in higher dimensions (Castruccio et al., 2016). Hence, copula families that are able to accommodate tail dependence and to capture dependence asymmetry, and are easy to implement, are highly desirable.

Some other methods have also been proposed to construct new classes of copulas in the literature, such as the Laplace transform method (Yang and Li, 2020), the geometric weighting method (Zhang et al., 2016), the vine copula approach (Aas et al., 2009, Shi and Yang, 2018), and the factor copula model (Oh and Patton, 2017), among others. In order to model multivariate heavy-tailed data, Yang et al. (2011) extracted a new copula family from the multivariate generalized beta distribution of the second kind, named the generalized beta copula or MGB2 copula, which features positive tail dependence in the joint upper tail and tail independence in the joint lower tail.

The main contribution of this article is to propose a new class of beta-type copulas, the MGL copula family, where the dependence function is extracted from a new multi-dimensional version of the univariate GLMGA distribution which was proposed in Li et al. (2021). We provide some important characteristics of this class and obtain the corresponding extreme-value copula (the MGL-EV copula). Inheriting heavy-tail features from the GLMGA distribution and its multivariate extension, the MGL and MGL-EV copulas are able to account for joint extreme events based on a positive tail dependence index. Furthermore, these new copulas can accommodate non-elliptical and asymmetric dependence,
and are easily extended for regression applications. We illustrate the usefulness of the proposed copula regression modelling using two insurance cases. The first example is about modelling the pairwise dependence between two continuous variables in the well-known Danish fire insurance data set, which was already considered for instance in Hashorva et al. (2017) and Lu and Ghosh (2021). In the second application we model the dependence between a continuous and a semi-continuous variable in a Chinese earthquake data set. This data set was already analyzed in Li et al. (2021) concerning the univariate losses. In both case studies we consider the evolution of the dependence as a function of time. We compare the performance of the proposed models with the MGB2 copula and other copula candidates in terms of goodness of fit and tail dependence measures.

The remainder of the paper is structured as follows. In Section 2, after recapitulating the univariate GLMGA distribution, we construct a multivariate extension and the corresponding copula. In Section 3 we report properties of the proposed copulas and obtain the corresponding extreme-value copulas. The copula regression modelling is discussed in Section 4. A simulation study is conducted in Section 5 and the two applications are discussed in Section 6. Finally we formulate some conclusions and future possible extensions. Proofs are deferred to the Appendix. The R package: rMGLReg can be found at https://github.com/lizhengxiao/rMGLReg.

2 From a multivariate GLMGA distribution to a MGL copula

The GLMGA three-parameter distribution model as proposed in Li et al. (2021), is obtained by mixing a generalized log-Moyal distribution (GlogM) (Bhati and Ravi, 2018) with the gamma distribution.

**Definition 2.1.** The random variable \( Y > 0 \) follows a GLMGA distribution \( (Y \sim \text{GLMGA}(\sigma, a, b)) \) if it admits the following stochastic representation:

\[
Y|\Theta \sim \text{GlogM}(\Theta, \sigma) \quad \text{and} \quad \Theta \sim \text{Gamma}(a, b),
\]

where GlogM(\( \Theta, \sigma \)) refers to the generalized log-Moyal distribution introduced in Bhati and Ravi (2018) with density and distribution function (cdf)

\[
f_{Y|\theta}(y|\theta) = \frac{\sqrt{\theta}}{\sqrt{2\pi\sigma}} \left( \frac{1}{y} \right)^{\frac{1}{2}+1} \exp \left[ -\frac{\theta}{2} \left( \frac{1}{y} \right)^{1/\sigma} \right],
\]

\[
F_{Y|\theta}(y|\theta) = \text{erfc} \left( \sqrt{\theta/2} y^{-\frac{1}{2\sigma}} \right). \tag{2.1}
\]
Here Gamma\((a, b)\) refers to the gamma distribution with density
\[
p(\theta) = \frac{b^a}{\Gamma(a)} \theta^{a-1} \exp(-b\theta),
\] (2.2)
for \(a, b > 0\), while \(\text{erfc}(\cdot)\) denotes the complementary error function given by \(\text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty \exp(-t^2)dt\).

The density function of the GLMGA distribution is then given by
\[
f(y) = \frac{(2b)^a}{\sigma B(a, \frac{1}{2})} \left(\frac{y^{-\frac{1}{\sigma}}}{y^{-\frac{1}{\sigma}} + 2b}\right)^{a+\frac{1}{2}},
\] (2.3)
for \(y > 0\), \(\sigma > 0\), \(a > 0\), \(b > 0\), with \(B(m, n) = \int_0^1 t^{m-1}(1-t)^{n-1}dt\) the beta function. The cdf and quantile function of the GLMGA distribution are given by
\[
F(y) = 1 - I_{\frac{1}{2}, a} \left(\frac{y^{-1/\sigma}}{y^{-1/\sigma} + 2b}\right), \quad y > 0,
\] (2.4)
\[
F^{-1}(p) = (2b)^{-\sigma} \left[\frac{I_{\frac{1}{2}, a}^{-1}(1-p)}{1 - I_{\frac{1}{2}, a}^{-1}(1-p)}\right]^{-\sigma}, \quad p \in (0, 1),
\] (2.5)
where \(I_{m,n}(\cdot)\) denotes the inverse of the beta cumulative distribution function \(I_{m,n}(\cdot)\) (or regularized incomplete beta function).

Moreover, the survival function \(\bar{F} = 1 - F\) of the GLMGA distribution allows for the expansion at infinity
\[
\bar{F}(y) = Cy^{-1/(2a)} \left\{1 + Dy^{-1/\sigma} (1 + o(1))\right\}, \quad y \to +\infty,
\] (2.6)
with \(C = \frac{2}{(2b)^{1/2}B(a, \frac{1}{2})} > 0\) and \(D = -\frac{2a+1}{12b}\). Hence, the GLMGA distribution is of Pareto-type with Pareto tail index \(1/(2\sigma)\).

Also, near 0 the distribution function \(F\) is regularly varying with index \(a/\sigma\):
\[
\lim_{t \to +0} \frac{F(ty)}{F(t)} = y^{a/\sigma}, \quad \text{for all } y > 0.
\] (2.7)

Finally, we mention that the GLMGA(\(\sigma, a, b\)) distribution is a special case of the four-parameter generalized beta distribution of the second kind GB2(\(\tau, \mu, \nu, p\)) by substituting \(\tau = a\), \(\mu = (2b)^{-\sigma}\), \(\nu = \frac{1}{2}\) and \(p = -\frac{1}{\sigma}\) with the density function given by \(f_{GB2}(y) = \frac{|p|}{B(\nu, \tau)\theta} \frac{\mu^{\nu} y^{\nu - 1}}{(y^{\nu + \mu} + \theta^{\nu + \mu})^{\nu + \tau}}\).
As noted in Li et al. (2021), the univariate GLMGA distribution can be used to accommodate the extreme risks and capture both tail and modal parts of heavy-tailed insurance data, and it occupies an interesting position in between the popular GB2 model and its subfamilies, such as the Lomax model.

2.1 The multivariate GLMGA distribution (MGL)

In this section we propose a multivariate extension of the univariate GLMGA distribution as a gamma mixture of the GlogM distribution from Bhati and Ravi (2018) using a common scale parameter \( a \) over all dimensions.

**Definition 2.2.** A \( d \)-dimensional random vector \( \mathbf{Y} = (Y_1, \ldots, Y_d)^T \) on \((0, \infty)^d\) follows a multivariate GLMGA distribution (denoted by \( \mathbf{Y} \sim \text{MGL}(\sigma, a, b) \)), with \( \sigma = (\sigma_1, \ldots, \sigma_d) \) and \( b = (b_1, \ldots, b_d) \) if

- each \( Y_j \) given \( \Theta = \theta \) follows a GlogM(\( \theta/b_j, \sigma_j \)) with density

\[
f_{Y_j|\Theta=\theta}(y_j|\theta) = \sqrt{\frac{\theta}{b_j}} \left( \frac{1}{y_j} \right)^{\frac{1}{2\sigma_j}+1} \exp \left[ -\frac{\theta/b_j}{2} \left( \frac{1}{y_j} \right)^{1/\sigma_j} \right], \quad y_j > 0, \quad (2.8)
\]

where \( b_j > 0 \) and \( \sigma_j > 0 \),

- \( Y_1, \ldots, Y_d \) are conditionally independent given \( \Theta \),

where the mixing variable \( \Theta \) follows a gamma distribution with shape parameter \( a \) and one unit rate, i.e. \( \Theta \sim \text{Gamma}(a, 1) \).

The above definition easily leads to the following multivariate GLMGA density by taking the expectation with respect to \( \Theta \):

\[
f(y_1, \ldots, y_d) = \frac{\Gamma(a + \frac{d}{2})}{\Gamma(a)\Gamma(\frac{1}{2})^d \prod_{j=1}^{d} \sigma_j y_j \left[ \sum_{j=1}^{d} ((2b_j)^{\alpha_j} y_j)^{-\frac{1}{\sigma_j}} + 1 \right]^{a + \frac{d}{2}}}, \quad (2.9)
\]

for \( y_j > 0 \), being \( \sigma_j > 0, a > 0, b_j > 0 \).

Since \( Y_j \) (\( j = 1, \ldots, d \)) are conditionally independent given \( \Theta \), the marginal distributions are obtained by setting \( d = 1 \) which leads to the densities in (2.3).

Moments of the MGL distribution are easy to calculate thanks to the gamma mixture structure.
Proposition 2.1. Suppose \( Y \sim MGL(\sigma, a, b) \). Then, when \( \max_j \sigma_j < 1/4 \)

\[
\mathbb{E}(Y) = \mathbb{E}[\mathbb{E}(Y|\theta)] = (2b_j)^{-\sum_j^d \sigma_j} \prod_{j=1}^d \Gamma\left(\frac{1}{2} - \sigma_j\right) \frac{B\left(\frac{1}{2} - \sum_{j=1}^d \sigma_j, \sum_{j=1}^d \sigma_j + a\right)}{B\left(\frac{1}{2}, a\right)},
\]

\[
\text{Cov}(Y) = \mathbb{E}[\text{Var}(Y|\theta)] + \text{VaR}[\mathbb{E}(Y|\theta)] = \Sigma,
\]

where the components \( \Sigma_{jj'} = \text{Cov}(Y_j, Y_{j'}) \) of the variance-covariance matrix \( \Sigma \) are given by

\[
\Sigma_{jj'} = \mathbb{E}(Y_j)\mathbb{E}(Y_{j'}) \left[ \frac{B(a + \sigma_j + \sigma_{j'}, a)}{B(a + \sigma_j, a + \sigma_{j'})} - 1 \right], \quad j \neq j'.
\]

The correlations are given by

\[
\text{Corr}(Y_j, Y_{j'}) = \rho_{jj'} = \sqrt{\frac{B(a+2\sigma_j, a)B(\frac{1}{2}-2\sigma_j, \frac{1}{2})}{B(a+\sigma_j, a+\sigma_{j'})B(\frac{1}{2}-\sigma_j, \frac{1}{2}-\sigma_{j'}) - 1}} \frac{B(a+2\sigma_{j'}, a)B(\frac{1}{2}-2\sigma_{j'}, \frac{1}{2})}{B(a+\sigma_{j'}, a+\sigma_j)B(\frac{1}{2}-\sigma_j, \frac{1}{2}-\sigma_{j'}) - 1}.
\]

Moreover

\[
\lim_{a \to \infty} \rho_{jj'} = 0, \quad \text{for} \quad j \neq j'.
\]

Next we show that the MGL distribution is closed under conditional distributions. This result can be used in order to simulate random samples from the MGL distribution. This property also allows insurers to derive the conditional mean, \( \mathbb{E}(Y_{r+1}|Y_1, ..., Y_r) \), incorporating past experience claim amount \( Y_1, ..., Y_r \), into future premium in a nonlinear fashion for experience ratemaking application in non-life actuarial science (Shi and Yang, 2018).

Proposition 2.2. Suppose \( Y \sim MGL(\sigma, a, b) \) and consider two complementary sub-vectors \( Y_1 = (Y_1, \cdots, Y_r)^T \) and \( Y_2 = (Y_{r+1}, \cdots, Y_d)^T \) of \( Y \). Then the conditional distribution of \( Y_1 \) given \( Y_2 \) equals \( MGL(\sigma^*, a^*, b^*) \) where \( a^* = a + \frac{d-r}{2} \) and \( b^*_j = b_j \left[ 1 + \sum_{j=r+1}^d y_j^{1/\sigma_j} / (2b_j) \right] \).

Figure 1 displays the scatter for the MGL distribution with simulated sample size 1,000 for different combinations of \( (\sigma_1, \sigma_2, a, b_1, b_2) \). Contour plots of the density function are also given in Figure 1. Positive dependences and the tail asymmetry features in the observed MGL data. The dependence parameters appear in the marginal distributions and the MGL features stronger lower tail dependence with smaller values of \( a \).
Figure 1: Contour plots of the MGL distribution with the simulated sample 1,000 (red points) when different values of parameters ($\sigma_1, \sigma_2, a, b_1, b_2$) are considered.
2.2 MGL copula and survival MGL copula

Although the MGL distribution may provide a useful tool for handling the multivariate heavy-tailed data, it suffers some limitations: the univariate marginal distributions belong to the same family, the dependence parameters appear in the marginal distributions (Frees and Valdez, 1998, Yang et al., 2011), and each margin contains the common parameter $a$. Considering the corresponding MGL copula and survival MGL copula allows to separate the modelling of the marginal and dependence structures. Based on Sklar’s theorem (Sklar, 1959), any joint cdf $H$ with continuous marginal cdfs $F_1, \ldots, F_d$ for a sequence of random variables $Y_1, \ldots, Y_d$ has a unique copula $C$ through

$$H(y_1, \ldots, y_d) = C(F_1(y_1), \ldots, F_d(y_d)),$$

where $C$ represents a multivariate joint distribution defined on a $d$-dimensional cube $(0,1)^d$ such that every marginal follows the uniform $(0,1)$ distribution. It is convenient to rewrite this as

$$H(y_1, \ldots, y_d) = C(u_1, \ldots, u_d),$$

with $u_j = F_j(y_j), \ j = 1, \ldots, d$.

**Definition 2.3.** The MGL copula is defined as

$$C_{MGL}(u_1, \ldots, u_d; a) = H(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)), \quad (u_1, \ldots, u_d) \in (0,1)^d$$

$$= E_{\Theta} \left[ \prod_{j=1}^{d} \text{erfc} \left( \sqrt{t(u_j; a)} \Theta \right) \right], \quad (2.13)$$

with $\Theta \sim \text{gamma}(a,1)$ and

$$t(u_j; a) = \frac{I_{\frac{1}{2}a}(1-u_j)}{1 - I_{\frac{1}{2}a}(1-u_j)}.$$

The corresponding copula density function is given by

$$c_{MGL}(u_1, \ldots, u_d; a) = \frac{h(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d))}{\prod_{j=1}^{d} f_j(F_j^{-1}(u_d))}$$

$$= \frac{\Gamma(a)^{d-1} \Gamma(a + \frac{d}{2})}{\Gamma(a + \frac{d}{2})^d} \prod_{j=1}^{d} \left( t(u_j; a) + 1 \right)^{a + \frac{1}{2}} \left( \sum_{j=1}^{d} t(u_j; a) + 1 \right)^{a + \frac{d}{2}}, \quad (u_1, \ldots, u_d) \in [0,1]^d, \quad (2.14)$$

8
where \( h \) denotes the joint density of the MGL(\( \sigma, a, b \)) distribution and \( f_j \) is the density of the univariate GLMGA distribution with parameters (\( \sigma_j, a_j, b_j \)).

Given that larger values of the common parameter \( a \) in the MGL distribution yield weaker dependence, in what follows we re-parameterize the MGL copula by substituting \( \delta = 1/a \), and the density and cdf are denoted with \( c^{MGL}(\cdot; \delta) \) and \( C^{MGL}(\cdot; \delta) \).

We next propose a simulation procedure for pseudo data from the MGL copula.

**Proposition 2.3.** Pseudo random vectors from \( C^{MGL}(\cdot; \delta) \) can be constructed using the following steps:

- Generate i.i.d. random samples \((U_1, \ldots, U_d)\) from the uniform \((0, 1)\) distribution, and let \( Z_j = \frac{1 - I_{1/2}^{-1}(1 - U_j)}{1 - I_{1/2}^{-1}(1 - U_j)} \), where \( k_j = \frac{1}{\delta} + \frac{j - 1}{2} \) for \( j = 1, \ldots, d \);
- Generate the random numbers \( M_1 = Z_1 \), and then \( M_j = (1 + \sum_{k=1}^{j-1} M_k) Z_j \) for \( j = 2, \ldots, d \);
- Compute \( U^*_j = 1 - I_{1/2}^{-1}(\frac{M_j}{1 + M_j}) \) for \( j = 1, \ldots, d \).

To illustrate the dependence structure, in Figure 2 simulated normalized scatter plots with \( d = 3 \) from the MGL copula are given with \( n = 5,000 \) for different values of copula parameter \( \delta \). The normalized random samples are defined as \( z_{ij} = \Phi^{-1}(u^*_{ij}) \) for \( j = 1, 2, 3 \) and \( i = 1, \ldots, 5,000 \), where \( \Phi(\cdot) \) is the standard normal distribution function and \( u^*_{ij} \) denote the copula realizations. Note the positive dependences and the tail asymmetry features among the three variables, while the MGL copula features stronger tail dependence in the lower tail with the larger value of the parameter \( \delta \).

To capture the upper tail dependence structure, we propose a survival MGL copula \( \bar{C}^{MGL}(\cdot; \delta) \).

**Definition 2.4.** The survival MGL copula is defined as

\[
\bar{C}^{MGL}(u_1, \ldots, u_d; \delta) = 1 - \sum_{j=1}^{d}(1 - u_j) + \sum_{J \subseteq \{1, \ldots, d\}} (-1)^{|J|} C^{MGL}((1 - u_1)^{1(1 \in J)}, \ldots, (1 - u_d)^{1(d \in J)}; \delta), \tag{2.15}
\]

where the sum extends over all \( 2^d \) subsets \( J \) of \( \{1, \ldots, d\} \) and \( |J| \) denotes the number of elements of \( J \) and \( 1(j \in J) \) the indicator of \( J \). For \( d = 2 \), \( \bar{C}^{MGL}(u_1, u_2; \delta) = u_1 + u_2 - 1 + C^{MGL}(1 - u_1, 1 - u_2; \delta) \).
The density function of the survival MGL copula is given by

\[ \tilde{c}^{MGL}(u_1, \ldots, u_d; \delta) = c^{MGL}(1 - u_1, \ldots, 1 - u_d; \delta). \] (2.16)

The one-parameter survival MGL with joint density function \( \tilde{c}^{MGL}(u_1, \ldots, u_d; \delta) \) can be regarded as a special case of the MGB2 copula (see Yang et al. (2011)) with \((d + 1)\)-parameters and joint density function \( c^{MGB2}(u_1, \ldots, u_d; p_1, \ldots p_d, q) \) given by

\[
c^{MGB2}(u_1, \ldots, u_d; p_1, \ldots p_d, q) = \frac{\Gamma(q)^{d-1}\Gamma\left(\sum_{i=1}^{d} p_i + q\right)}{\prod_{i=1}^{d} \Gamma(p_i + q)} \frac{\prod_{i=1}^{d} (1 + x(u_i; p_i, q))^{p_i+q}}{(1 + \sum_{i=1}^{d} x(u_i; p_i, q))^{\sum_{i=1}^{d} p_i + q}},
\]

in which \( x(u_i; p_i, q) = I_{p_i, q}^{-1}(u_i)/ (1 - I_{p_i, q}^{-1}(u_i)) \). The survival MGL density is obtained taking \( p_1 = p_2 = \ldots = p_d = \frac{1}{2} \) and \( q = \delta \).

![Figure 2: The normalized scatter plots with simulated sample size 5,000 for low (\( \delta = 0.1 \)), medium (\( \delta = 2 \)) and high dependence (\( \delta = 5 \)) of the 3-dimensional MGL copula (top row) and survival MGL copula (bottom row).](image-url)
3 Properties of the MGL copula

In this section we collect some basic properties of the MGL copula and specify these in the bivariate case. The $h$-function represents the conditional distribution function of a bivariate copula, which is defined as the partial derivative of the distribution function of the copula with respect to the first argument $h_{2|1}(u_2|u_1) := \Pr(U_2 \leq u_2|U_1 = u_1) = \partial C(u_1, u_2)/\partial u_1$, and with respect to second argument $h_{1|2}(u_1|u_2) := \Pr(U_1 \leq u_1|U_2 = u_2) = \partial C(u_1, u_2)/\partial u_2$ (Schepsmeier and Stöber, 2014).

**Proposition 3.1.** The $h$-function corresponding to the bivariate MGL copula is given by

$$h_{2|1}^{MGL}(u_2|u_1; \delta) = 1 - I_{\frac{1}{2} + \frac{1}{2}} \left[ \frac{t(u_2; \delta)}{t(u_1; \delta) + t(u_2; \delta) + 1} \right],$$

$$h_{1|2}^{MGL}(u_1|u_2; \delta) = 1 - I_{\frac{1}{2} + \frac{1}{2}} \left[ \frac{t(u_1; \delta)}{t(u_1; \delta) + t(u_2; \delta) + 1} \right],$$

for all $(u_1, u_2) \in [0, 1]^2$.

This result follows combining Proposition 2.2, 2.4 and 2.5.

**Remark 1.** Bivariate random samples can be generated using the inverse conditional distribution function of a given parametric bivariate copula. This function can be easily inverted using the quantile function of the beta distribution, yielding the corresponding inverse $h$-function of MGL copula.

$$h_{2|1}^{-1}(u_2|u_1; \delta) = 1 - I_{\frac{1}{2} + \frac{1}{2}} \left[ \frac{(t(u_1; \delta) + 1)t(u_2; 2\delta/(2 + \delta))}{(t(u_1; \delta) + 1)t(u_2; 2\delta/(2 + \delta)) + 1} \right],$$

$$h_{1|2}^{-1}(u_1|u_2, \delta) = 1 - I_{\frac{1}{2} + \frac{1}{2}} \left[ \frac{(t(u_2; \delta) + 1)t(u_1; 2\delta/(2 + \delta))}{(t(u_1; \delta) + 1)t(u_2; 2\delta/(2 + \delta)) + 1} \right].$$

**Rank based measures of association** Kendall’s $\tau$ and Spearman’s $\rho$ are two well-known rank based measures of association in copula modelling. Unlike Pearson correlation coefficient, Kendall’s $\tau$ and Spearman’s $\rho$ solely depend on the copula function $C$ through

$$\tau = 4 \int_0^1 \int_0^1 C(u, v) dC(u, v) - 1,$$

$$\rho = 12 \int_0^1 \int_0^1 C(u, v) dudv - 3.$$

(Fredricks and Nelsen, 2007, Nelsen, 2007)
For a bivariate copula these correlations can be evaluated using
\[
\tau = 1 - 4 \int_{[0,1]^2} h_{2|1}(v|u)h_{1|2}(u|v)dudv,
\]
\[
\rho = 3 - 12 \int_{[0,1]^2} h_{2|1}(v|u)dudv,
\]
where \(h_{2|1}(\cdot|\cdot)\) and \(h_{1|2}(\cdot|\cdot)\) denote the \(h\)-function of the bivariate MGL copula given in (3.1) and (3.2) respectively. Although there is no closed form solution, the integrations on the support of unit hypercubes can well be approximated using numerical methods.

Tail dependence behaviour It is important to understand how the parameter \(\delta \in (0, \infty)\) influences the level of dependence in the the bivariate MGL copula. One can show that the copula approaches the independence copula as \(\delta\) approaches 0, that is
\[
\lim_{\delta \to +0} C_{MGL}(u_1, u_2; \delta) = u_1u_2.
\]
Moreover, it is easy to check that the copula becomes singular as \(\delta \to +\infty\).

Tail dependence, also known as extremal dependence or asymptotic dependence, quantifies the probability of concurrence of extreme events in the upper tail or lower tail of a bivariate distribution. The indices of lower and upper tail dependence of a copula \(C\) are defined by
\[
\lambda_l = \lim_{u \to 0^+} \mathbb{P}(U_2 \leq u|U_1 \leq u) = \lim_{u \to 0^+} \frac{C(u, u)}{u},
\]
\[
\lambda_u = \lim_{u \to 1^-} \mathbb{P}(U_2 > u|U_1 > u) = \lim_{u \to 1^-} \frac{1 - 2u + C(u, u)}{u},
\]
provided the limits \(\lambda_l\) and \(\lambda_u\) exist in \([0, 1]\) (Joe, 1997). The copula \(C\) is said to be lower, respectively upper, asymptotically tail dependent if \(\lambda_l \neq 0\), respectively \(\lambda_u \neq 0\). Moreover, copulas of elliptically symmetric distributions have \(\lambda_l = \lambda_u\).

The next proposition shows that the MGL copula is able to accommodate joint extreme events on the lower tail, but not on the upper tail. The lower tail dependence index reduces to 0 as \(\delta\) approaches 0.

**Proposition 3.2.** The copula \(C_{MGL}\) is asymptotic lower tail dependent with the indices of lower and
upper tail dependence given respectively by

\[ \lambda_{\text{MGL}}^M = 2 - 2I_{\frac{3}{2} + \frac{1}{2}} \left( \frac{1}{2} \right), \]
\[ \lambda_{\text{MGL}}^U = 0.\] (3.5) (3.6)

Similarly the survival copula \( \tilde{C}^{\text{MGL}} \) allows asymptotic upper tail dependence. In Figure 3 with provide contour plots with low, medium and high dependence as measured by Kendall’s \( \tau \) for the MGL and survival MGL copula. The value of parameter \( \delta \) describes the strength of the relationship with higher values of \( \delta \) implying stronger dependence. The lower tail dependence and non upper tail dependence are observed in MGL copula, and upper tail dependence is observed with the survival MGL copula.

**Extreme-value copula** Here we study the corresponding domain of attraction of the survival MGL copula which is able to capture the upper tail dependence. Following the definition of convergence of dependence structures, as for instance described in Chapter 8.3.2 in Beirlant et al. (2004), a copula \( C \) is attracted to an extreme value copula \( C_0 \) if the limit

\[ \lim_{s \to 0} \frac{1 - C(1 - su_1, ..., 1 - su_d)}{s} = \ell(u_1, ..., u_d) \]

exists for all \((u_1, ..., u_d) \in [0, \infty]^d\). Then \( \ell \) is named the stable tail dependence function. The corresponding copula \( C_0 \) is then obtained from \( \ell \) through

\[ - \log C_0(\exp(-u_1), ..., \exp(-u_d)) = \ell(u_1, ..., u_d). \] (3.7)

In the bivariate case \( d = 2 \) the stable tail dependence function \( \ell \) can be represented in terms of the Pickands dependence function \( A \):

\[ \ell(u_1, u_2) = (u_1 + u_2)A \left( \frac{u_2}{u_1 + u_2} \right), \]

which is necessarily convex and satisfies the boundary condition \( \max(1 - w, w) \leq A(w) \leq 1 \). The
Figure 3: Contour plots of the joint distribution and density function of the MGL and survival MGL copula with low, medium and high dependence. First column: low dependence with Kendall’s tau = 0.1, second column: medium dependence with Kendall’s tau = 0.5, and third column: high dependence with Kendall’s tau = 0.75.
extreme value copula is then represented as
\[ C_0(u_1, u_2) = \exp \left[ \log(u_1 u_2) A \left( \frac{\log u_2}{\log(u_1 u_2)} \right) \right], \]
and the upper tail dependence coefficient is given by \( \lambda_u = 2 - 2A(1/2) \).

The density function and the \( h \)-function of the bivariate extreme-value copula are given by
\[
\begin{align*}
c_0(u_1, u_2) &= \frac{C_0(u_1, u_2)}{u_1 u_2} \left[ \left( \frac{\partial \ell(z_1, z_2)}{\partial z_1} \right) - \frac{\partial^{2} \ell(z_1, z_2)}{\partial z_1 \partial z_2} \right] \bigg|_{z_1 = -\log u_1, z_2 = -\log u_2}, \\
h_{2|1}(u_2|u_1) &= \frac{C_0(u_1, u_2)}{u_1} \frac{\partial \ell(z_1, z_2)}{\partial z_1} \bigg|_{z_1 = -\log u_1, z_2 = -\log u_2}, \\
h_{1|2}(u_1|u_2) &= \frac{C_0(u_1, u_2)}{u_2} \frac{\partial \ell(z_1, z_2)}{\partial z_2} \bigg|_{z_1 = -\log u_1, z_2 = -\log u_2}.
\end{align*}
\]

**Proposition 3.3.** The extreme value copula \( \bar{C}^{MGL-EV} \) of the survival MGL copula is given by
\[
\bar{C}^{MGL-EV}(u_1, u_2; \delta) = \exp \left[ \log \left( \frac{u_1 u_2}{1 - A_\delta \left( \frac{u_1}{u_1 + u_2} \right)} \right) \right],
\]
where the Pickands dependence function \( A_\delta \) is given by
\[
A_\delta(w) = w I_{\frac{3}{2} + \frac{1}{\delta}} \left[ \frac{(1 - w)^{-\delta}}{(1 - w)^{-\delta} + w^{-\delta}} \right] + (1 - w) I_{\frac{3}{2} + \frac{1}{\delta}} \left[ \frac{w^{-\delta}}{(1 - w)^{-\delta} + w^{-\delta}} \right].
\]

**Remark 2.** The MGL copula also has a limiting lower tail copula given by
\[
C^{MGL-EV}(u_1, u_2; \delta) = \frac{(u_1 + u_2) \left( 1 - A_\delta \left( \frac{u_1}{u_1 + u_2} \right) \right)}{2 \left( 1 - A_\delta \left( \frac{1}{2} \right) \right)},
\]
where \( A_\delta \) is the Pickands dependence function of \( \bar{C}^{MGL-EV} \).

Figure 4 displays the contour plots for the joint distribution and density function with \( \delta = 1 \), next to the Pickands dependence function \( A_\delta \) for different values of the dependence parameter \( \delta \).

### 4 Copula regressions

The proposed models can be used to model response variables of any dimension in order to improve the model fitting by introducing regression analysis that accounts for the dynamic dependence patterns
conditioning on different values of covariates. For regression on a copula parameter, Hua and Xia (2014) propose a copula model that incorporates both regression on each marginal of bivariate response variables and regression on the dependence parameter for the response variables. Acar et al. (2011) apply a nonparametric approach for calibrating the dependence parameters according to the covariates, where the dependence parameter is allowed to change along the covariates. Here, we assume that the $d$-dimensional pseudo-copula data $(u_1, ..., u_d)$ follow the survival MGL/survival MGL-EV copula and propose the copula parameter $\delta$ to be modelled as a function of the explanatory variables $1$. In order to avoid boundary problems in optimization, we consider a log link function obtaining real values for the copula parameter $\delta_i$:

$$u_{i1}, ..., u_{id} | x_i \sim \text{survival MGL}(\delta_i),$$

$$\log(\delta_i) = x_i^T \beta,$$

where $x_i = (1, x_{i1}, ..., x_{ik})$ denotes the vector of covariates and $\beta = (\beta_0, \beta_1, ..., \beta_k)$ the vector of coefficients.

The pseudo log-likelihood function $\log L(\cdot; \beta)$ based on pseudo-copula data $(u_1, ..., u_d)^T$ for $i = \ldots$
1, \ldots, n is given by
\[
\log L(u_1, \ldots, u_d; \beta) = \sum_{i=1}^{n} \log \tilde{c}_{\text{MGL}}(u_{i1}, \ldots, u_{id}; \delta_i)
\]
\[
= (d - 1) \sum_{i=1}^{n} \log \Gamma\left(\frac{1}{\delta_i}\right) + \sum_{i=1}^{n} \log \Gamma\left(\frac{1}{\delta_i} + \frac{d}{2}\right) - d \sum_{i=1}^{n} \log \Gamma\left(\frac{1}{\delta_i} + \frac{1}{2}\right)
\]
\[
+ \sum_{i=1}^{n} \left(\frac{1}{\delta_i} + \frac{1}{2}\right) \sum_{j=1}^{d} \log \left(\frac{I_{\frac{1}{2},\frac{1}{\delta_i}}^{-1}(u_{ij})}{1 - I_{\frac{1}{2},\frac{1}{\delta_i}}^{-1}(u_{ij})}\right)
\]
\[
- \sum_{i=1}^{n} \left(\frac{1}{\delta_i} + \frac{d}{2}\right) \log \left(\sum_{j=1}^{d} \frac{I_{\frac{1}{2},\frac{1}{\delta_i}}^{-1}(u_{ij})}{1 - I_{\frac{1}{2},\frac{1}{\delta_i}}^{-1}(u_{ij})} + 1\right) .
\] (4.1)

Similarly, the covariates can be also introduced into the parameter $\delta$ in the survival MGL-EV copula:
\[
u_{i1}, \ldots, u_{id} | x_i \sim \text{survival MGL-EV}(\delta_i),
\]
\[
\log(\delta_i) = x_i^T \beta,
\]

with the pseudo log-likelihood function given by
\[
\log L(u_1, \ldots, u_d; \beta) = \sum_{i=1}^{n} \log \tilde{c}_{\text{MGL-EV}}(u_{i1}, \ldots, u_{id}; \delta_i)
\]
\[
= \sum_{i=1}^{n} \log \tilde{c}_{\text{MGL-EV}}(u_{i1}, \ldots, u_{id}; \delta_i) - \sum_{i=1}^{n} \sum_{j=1}^{d} \log u_j
\]
\[
+ \sum_{i=1}^{n} \log \sum_{m=1}^{d} (-1)^{d-m} \sum_{\pi: |\pi| = m} \prod_{B \in \pi} D_B \ell(z_{i1}, \ldots, z_{id}) | z_1 = - \log u_{i1}, \ldots, z_d = - \log u_{id},
\] (4.2)

where $D_B := \frac{\partial^{|B|}}{\prod_{j \in B} \partial z_j}$ is defined as the high order partial differentiation operation, $\pi$ runs through the set of all partitions of the set $\{1, \ldots, d\}$ and $B \in \pi$ denotes that $B$ runs through the list of all elements of the partition $\pi$, $|\pi|$ denotes the number of sets in $\pi$.

For the Newton-Raphson algorithm, the first- and second-order derivatives of $\log L(u_1, \ldots, u_d; \beta)$ with respected to $\beta$ are required at each iteration. In our model, there is a closed form for the derivatives for the survival MGL copula, but no simple form for the survival MGL-EV copula is obtained. However they can be obtained numerically, requiring multiple calculations of the log-likelihood. We
show how to calculate the gradient of the log-likelihood (4.1) in the Appendix E. The maximum likelihood (ML) estimates are consistent and asymptotically normal if the copula family is correctly specified. The asymptotic variance of the estimates depends on the estimation method; see sections 5.4, 5.5, and 5.9 in Joe (2014) for a detailed review of asymptotic theory when estimates are obtained using the joint likelihood, the two-step approach with parametric margins, and the two-step approach with nonparametric ranks, respectively.

5 Simulation Study

In this section, we check first the accuracy of the ML estimators based on the proposed $d$-dimensional MGL copula regression model discussed in Section 4 with respect to sample size $n$. We generate $N = 1,000$ data sets from $n = 100$ to $n = 2,000$ from the $d$-dimensional MGL copula regression model with $k = 2$, $\mathbf{x}_i^T = (1, x_{i1}, x_{i2})$, $\beta = (-0.6, 0.5, 0.2)^T$ with the covariates $x_{i1}$ and $x_{i2}$ being generated from the standard normal distribution. We consider $d = 2$ and $d = 10$ to yield bivariate and high dimensional copula regressions for simulated pseudo-copula data.

Figures 5 and 6 show how the bias, asymptotic variance and mean squared error (MSE) vary with respect to sample size $n$ in case of $d = 2$ and $d = 10$ respectively. It can be observed that estimation of the model parameter $\beta_0$ is less accurate in smaller samples, while more stability is observed with the estimators of $\beta_1$ and $\beta_2$. As the sample size increases the estimators close up to the true values, with smaller bias, asymptotic variance and MSE.

Figure 5: Bias (left), asymptotic variance (middle) and MSE (right) of the parameter estimates $(\beta_0, \beta_1, \beta_2)$ for MGL copula regression in case of $d = 2$. The sample size runs from $n = 100$ to $n = 1,000$. The plots are obtained by averaging over 1,000 samples.

Dynamic dependence modelling has been a popular research topic in actuarial science (Hua and
Figure 6: Bias (left), asymptotic variance (middle) and MSE (right) of the parameter estimates $(\beta_0, \beta_1, \beta_2)$ for MGL copula regression in case of $d = 10$. The sample size runs from $n = 100$ to $n = 1,000$. The plots are obtained by averaging over 1,000 samples.

Xia, 2014), with special emphasis to upper tail dependence. In order to study dynamic upper tail dependence structures, we apply the copula regression model to simulated bivariate data using the survival MGL copula with a time covariate. We generated random samples of size $n = 1,200$ from the bivariate survival MGL copula, and the sample size for each time point. The dependence parameter is assumed to be a function of the claim duration in months ($t \in \{1, \ldots, 24\}$) according to $\delta_t = \exp(-1 + 0.1t)$. Figure 7 presents the boxplots of the parameter estimates from 2,000 Monte Carlo simulations. The median estimates of $\beta_0$ and $\beta_1$ are very close to the true values. To demonstrate the approximate normality of the estimators the normal QQ plots of the estimated parameters are given in Figure 8, which show acceptable results. Figure 7 shows the predictive curve between the copula parameters $\delta_t$ and duration $t$. The red line shows the dependence pattern from the true model. The gray lines are generated based on the the estimates of the copula parameter $\hat{\delta}_t$ from 2,000 Monte Carlo simulations. It can be observed that the dependence in the upper tail is close to independence at smaller $t$ and increases with $t$. 

19
Figure 7: Boxplots of the parameter estimates from 2,000 survival MGL copula simulated samples of size \( n = 1,200 \) (left panel). The predictive value of copula parameter \( \delta_i \) for different values of the covariate \( t \) (right panel).

Figure 8: Normal QQ plots of the ML parameter estimates from the survival MGL copula regression simulations with sample size \( n = 1,200 \).
6 Real-data illustrations

We now illustrate the proposed methods with two practical examples which are investigated without and with covariates.

6.1 Danish fire insurance data

As the first example, we fit the bivariate copula and regression models to the Danish fire insurance data set which was collected from the Copenhagen Reinsurance Company and comprises 2167 fire losses over the period 1980-1990. The claims have been adjusted for inflation to reflect 1985 values and are expressed in millions of Danish Krone and can be found in the R package: fitdistrplus. The total claims in the multivariate data set is divided into building loss, contents loss and profit loss. This data set was already analyzed in Hashorva et al. (2017) and Lu and Ghosh (2021) among others. Here we model the dependence between building loss and contents loss, and we consider the observations where both components are non-zero. There is a total of \( n = 1502 \) observations that are positive in both variables.

Figure 9 displays the scatter plot of the log transformed data \((Y_{i1}, Y_{i2})\) and of the pseudo-copula data \((u_{i1}, u_{i2})\) \((i = 1, \ldots, n)\) based on the kernel smoothing method

\[
u_{ij} = \hat{F}_n(y_{ij}) = \int_{-\infty}^{y_{ij}} \frac{1}{n} \sum_{j=1}^{n} K_h(Y_{ij} - u)du, \quad \text{for} \quad j = 1, 2,
\]

where \(K(\cdot)\) is a kernel function and \(K_h = K(\cdot/h)/h\). Here we have chosen the standard Gaussian kernel and \(h = 0.2\). From these positive right upper tail dependence appears. The empirical value of Kendall’s \( \tau \) equals 0.085.

Table 1 reports the estimation results, AIC and BIC values of the survival MGL and the survival MGL-EV copula, along with four other families of copulas with positive upper tail indices, the MGB2 copula discussed in Yang et al. (2011), the Gumbel copula, the Student \( t \) copula, and the Gaussian copula. The Gumbel copula is an extreme-value copula and also belongs to the Archimedean family, whereas the Student \( t \) copula and Gaussian copula belongs to the elliptical copulas. We estimate the copula parameters \( \gamma \).

- For the survival MGL, \( \gamma = \delta \),
- For the survival MGL-EV, \( \gamma = \delta \),
• for the MGB2 copula, $\gamma$ is a 3-vector of $(p_1, p_2, q)^T$,
• for the Gumbel copula, $\gamma = \delta$,
• for the Gaussian copula, $\gamma = \rho$,
• for the Student $t$ copula, $\gamma = (\rho, v)^T$ and the degree of freedom $v$ is determined by ML.

In terms of the AIC and BIC values, the MGB2 and survival MGL are preferred over the other four families of copulas. In addition, in order to analyze the model fitting in upper and lower regions, we consider the squared fit error over a region $A \subseteq [0, 1]$, defined as (see Li et al. (2014))

$$e_A(C) = \left( \frac{1}{m(A)} \int \int_A |C(u, v) - C^{emp}(u, v)|^2 dudv \right)^{\frac{1}{2}},$$

where $m(A)$ is the Lebesgue measure of a set $A$, $C$ the fitted copula and $C^{emp}$ is the empirical copula which is defined as $C^{emp}(t_1, t_2) = \sum_{i=1}^{n} 1_{y_{i1} < t_1} 1_{y_{i2} < t_2} / n$. It can be observed that the survival MGL copula possesses the better performance over the regions $[0, 0.05]^2$ and $[0.95, 1]^2$.

To further investigate the tail behavior of the proposed models, we also consider the tail-weighted measures of dependence proposed by Krupskii and Joe (2015) and applied in Krupskii et al. (2018). The measures provide useful tools for summarizing the strength of dependence in different joint tails for each pair of variables with value close to 0 or 1 corresponding to very weak or strong dependence in the tails respectively. Unlike the goodness-of-fit procedures such as AIC and BIC statistics, the tail-weighted measures of dependence can be used as additional scalar measures to distinguish bivariate copulas with roughly the same overall monotone dependence, as well as for assessing the adequacy of fit of multivariate copulas in the tails.

The empirical and model-based tail-weighted measures of dependence in the upper tail are given respectively by:

$$\varrho_U(a, p) = \text{Cor} \left[ a \left( 1 - \frac{1 - R_{i1}}{p} \right), a \left( 1 - \frac{1 - R_{i2}}{p} \right) \bigg| 1 - R_{i1} < p, 1 - R_{i2} < p \right],$$

$$\rho_U(a, p; C) = \frac{C(p, p)m_{12} - m_1m_2}{\left\{ [C(p, p)m_{11} - m_1^2] [C(p, p)m_{22} - m_2^2] \right\}^{\frac{1}{2}}},$$
where the notation $\hat{\text{Cor}}[y_{i1}, y_{i2} | (y_{i1}, y_{i2}) \in B]$ is shorthand for

$$
\frac{\sum_{i \in J_B} y_{i1} y_{i2} - n_B^{-1} \sum_{i \in J_B} y_{i1} \sum_{i \in J_B} y_{i2}}{\left[ \sum_{i \in J_B} y_{i1}^2 - n_B^{-1} \left( \sum_{i \in J_B} y_{i1} \right)^2 \right]^{1/2}} \frac{\left[ \sum_{i \in J_B} y_{i2}^2 - n_B^{-1} \left( \sum_{i \in J_B} y_{i2} \right)^2 \right]^{1/2}}{2},
$$

with $J_B = \{i : (y_{i1}, y_{i2}) \in B\}$, $n_B$ is the cardinality of $J_B$, and

$$
m_{12} = \frac{1}{p^2} \int_0^p \int_0^p a'(1 - \frac{u_1}{p}) a' \left(1 - \frac{u_2}{p}\right) C(u_1, u_2) du_1 du_2,
$$

$$
m_{1} = \frac{1}{p} \int_0^p a'(1 - \frac{u_1}{p}) C(u_1, p) du_1,
$$

$$
m_{2} = \frac{1}{p} \int_0^p a'(1 - \frac{u_2}{p}) C(p, u_2) du_2,
$$

$$
m_{11} = \frac{1}{p} \int_0^p 2a \left(1 - \frac{u_1}{p}\right) a' \left(1 - \frac{u_1}{p}\right) C(u_1, p) du_1,
$$

$$
m_{22} = \frac{1}{p} \int_0^p 2a \left(1 - \frac{u_2}{p}\right) a' \left(1 - \frac{u_2}{p}\right) C(p, u_2) du_2,
$$

where $a(\cdot)$ is a weighting function, $a'(\cdot)$ is the first-order derivative, $p$ a truncation level, and $C$ a fitted copula. In this case, we follow the method in Krupskii and Joe (2015) by using the power function for $a(u) = u^k$ with $k = 6$ and the truncation level $p = 0.5$. The power function with large $k$ puts more weight in the tail. The results of the power function for $k = 5$ and $k = 7$ are also investigated for comparison. Table 2 reports the estimation results of tail-weighted measures of dependence for the six families of copulas. To account for variability in parameter estimates, we use the bootstrap to construct 95% confidence intervals. One can see that the model-based estimates of tail-weighted measures of dependence are quite close to the empirical estimates for survival MGL copula. The MGB2 copula model slightly overestimates the empirical upper tail dependence, while the Gumbel model slightly underestimates the empirical $\hat{\varrho}_U(a; p = 0.5)$ for different values of $k$. Moreover, the narrow confidence intervals for the tail-weighted measures of dependence indicates that the survival MGL model is appropriate for modelling dependence in the upper tails for this case.

We further investigate the dynamic dependence introducing the covariate $Year$ into the dependence parameter in the survival MGL and survival MGL-EV regression model respectively. The natural cubic splines are used to allow flexible relationships between $Year$ and the dependence parameter. The log link function is considered for the dependence parameter $\delta_i$:

$$
\log \delta_i = \text{ns}_i(Year) = \beta_1 b_1(Year) + ... + \beta_k b_k(Year),
$$

23
Figure 9: Scatter plots of the log transformed data (*left panel*) and the pseudo copula data based on the nonparametric kernel smoothing method (*right panel*).

| Copula          | Estimates and standard errors | Loglike | AIC    | BIC    | $e_{A[0.05]}^2$ | $e_{A[0.95]}^2$ |
|-----------------|------------------------------|---------|--------|--------|----------------|----------------|
| Gaussian        | $\hat{\rho} = 0.252$        | -       | -      | -      | 35.60          | -69.20         |
|                 | (0.027)                      | -       | -      | -      | -63.89         | 2.23           |
| Student $t$     | $\hat{\rho} = 0.193$        | $\hat{v} = 3.400$ | - | - | 64.08          | -124.17        |
|                 | (0.032)                      | (0.457) | -      | -      | -113.54        | 9.34           |
| Gumbel          | $\hat{\delta} = 1.211$      | -       | -      | -      | 79.13          | -156.25        |
|                 | (0.022)                      | -       | -      | -      | -150.94        | 0.79           |
| Survival MGL   | $\hat{\delta} = 0.892$      | -       | -      | -      | 115.97         | -229.93        |
|                 | (0.067)                      | -       | -      | -      | -224.62        | **0.27**       |
| Survival MGL-EV| $\hat{\delta} = 0.655$      | -       | -      | -      | 81.76          | -161.52        |
|                 | (0.040)                      | -       | -      | -      | -156.20        | 0.79           |
| MGB2            | $\hat{p}_1 = 0.233$         | $\hat{p}_2 = 1.123$ | $\hat{q} = 0.939$ | - | - | **127.82** | **249.64** |
|                 | (0.046)                      | (0.561) | (0.209) | - | - | **233.69** | **0.28** |

Notes: The square fit error $e_{A}(C)$ in regions $A = [0, 0.05]^2$ and $A = [0.95, 1]^2$ are rescaled by $\times 10^8$. The standard error is reported in…… brackets.
Table 2: Estimates of empirical and fitted tail-weighted measures of dependence in the upper tail $(\varrho_U(a;p)$ and $\rho_U(a;p))$ and the model-based 95% confidence intervals for the Danish fire insurance data set.

| Models          | Estimates $a(u) = u^5, p = 0.5$ | Estimates $a(u) = u^6, p = 0.5$ | Estimates $a(u) = u^7, p = 0.5$ |
|-----------------|---------------------------------|---------------------------------|---------------------------------|
| Empirical       | 0.434                           | 0.427                           | 0.419                           |
|                 | (0.357,0.503)                   | (0.348,0.499)                   | (0.339,0.492)                   |
| Gaussian        | 0.101                           | 0.099                           | 0.097                           |
|                 | (0.078,0.123)                   | (0.076,0.121)                   | (0.074,0.118)                   |
| Student t       | 0.352                           | 0.360                           | 0.366                           |
|                 | (0.276,0.364)                   | (0.282,0.372)                   | (0.286,0.378)                   |
| Gumbel          | 0.317                           | 0.324                           | 0.330                           |
|                 | (0.290,0.344)                   | (0.297,0.352)                   | (0.303,0.358)                   |
| Survival MGL    | (0.392,0.448)                   | (0.400,0.458)                   | (0.406,0.465)                   |
| Survival MGL-EV | 0.306                           | 0.314                           | 0.319                           |
|                 | (0.280,0.333)                   | (0.287,0.340)                   | (0.292,0.346)                   |
| MGB2            | 0.434                           | 0.444                           | 0.452                           |
|                 | (0.378,0.487)                   | (0.387,0.498)                   | (0.393,0.506)                   |

Notes: The 95% confidence intervals are reported in brackets and the confidence intervals that contain the empirical value are shown in bold font. The results are based on 200 bootstrap samples.

where $n_s(x)$ denote the natural cubic splines, with $b_1(x),...,b_p(x)$ denoting the spline basis and $\beta_1,..,\beta_k$ the regression coefficients. We have chosen the 50% percentile of $Year$ as one knot for the natural cubic spline and there are three coefficients $\beta_k, k = 1, ..., 3$ to be estimated in the copula regression.

Table 3 reports the estimates and standard errors of the regression coefficients, together with the log-likelihood and information statistics of the survival MGL and survival MGL-EV copula regression model. The estimation results are also reported for the Gumbel regression model. In terms of the AIC and BIC values it can be observed that the survival MGL copula provide a better overall fitting than the Gumbel regression. Figure 10 presents the relationship between $Year$ and the dependence parameter. A non-linear relationship appears with an initial increasing dependence followed by a final decrease.

6.2 Chinese earthquake loss data

As the second example, we consider an earthquake loss data set concerning the Chinese mainland, which contains risk information on 291 earthquake events with magnitude greater than 4.0 from 1990
Table 3: Estimates and goodness fit for the survival MGL and Gumbel regression models for the Danish fire insurance data set.

| Parameters | Survival MGL Estimates | S.E. | Survival MGL-EV Estimates | S.E. | Gumbel Estimates | S.E. |
|------------|------------------------|------|---------------------------|------|------------------|------|
| $\beta_1$  | 0.205                  | 0.243 | -0.022                    | 0.194 | -0.422           | 0.341 |
| $\beta_2$  | -0.319                 | 0.186 | -0.968                    | 0.148 | -3.483           | 0.266 |
| $\beta_3$  | -0.191                 | 0.225 | -0.192                    | 0.174 | -0.446           | 0.316 |
| Loglike    | 116.69                 |      | 82.34                     |      | 79.69            |      |
| AIC        | -227.37                |      | -158.69                   |      | -153.39          |      |
| BIC        | -211.43                |      | -142.75                   |      | -137.44          |      |

Notes: In order to avoid boundary problem in MLE procedures, we consider a log link function obtaining real values for Gumbel copula regression: $\log(\delta_i - 1) = n_s(Y)$ for all $\delta_i > 1$.

Figure 10: The predicted value of copula parameter with different value of the covariate Year for the Danish fire insurance data set. The gray lines are generated based on the simulated coefficients for the natural cubic spline, and their values are generated using a multivariate normal distribution with mean equal to the MLE’s $\hat{\beta}_k$ for $k = 1, 2, 3$, and the covariance matrix being the inverse of the Hessian matrix.
to 2015. The data set is collected from the “China earthquake yearbook” and also analyzed in Li et al. (2021). The data set contains the occurrence time, location, the number of casualties and the total economic loss of each earthquake event. Among them, casualties are defined as fatalities and injured people, which are due to damage to occupied buildings. Table 4 reports the major earthquake disasters in China since 1990. The total economic damage is expressed in millions of Chinese Yuan (CNY) and are adjusted for inflation to reflect values in 2015. In particular, the 2008 earthquake in Sichuan is the most damaging earthquake. It did cost about 69,227 lives and caused 845.11 billion CNY direct total economic damage.

Here we focus on the dependence structure between the total economic losses and the number of casualties. The pairwise Kendall’s tau is 0.548 and the Spearman’s rho 0.704. The total economic loss variable is of continuous nature and is defined as the positive economic loss associated with an earthquake impact as determined in the weeks and sometimes months after the event. The number of casualties are semi-continuous data with heavy tails which are used to measure the earthquake risks associated with the fatalities and injured people. The kurtosis and skewness of these two variables indicate a heavy tailed nature of these variables. 36.08% of the earthquake events showed no casualties and so the casualties variable exhibits over-dispersion with a significant fraction of zero observations.

For the continuous total economic loss outcome, we consider the univariate heavy-tailed GLMGA distribution as proposed in Li et al. (2021). Three other competitive heavy-tailed distributions, namely, log-gamma, Fréchet, GlogM (Bhati and Ravi, 2018), and double-Pareto-Lognormal (DPLN) distribution (Reed and Jorgensen, 2004) are discussed in details for comparison. The estimation results and model selections are reported in Appendix F.

For the semi-continuous number of casualties variable $Y_2$ we fit a composite model. The composite model assumes a threshold $u$ below which $Y_2$ is modelled using a count distribution, whereas above $u$ a heavy-tailed distribution such as the generalized Pareto (GP) distribution with location parameter $\mu \in \mathbb{R}$, shape parameter $\sigma > 0$ and scale parameter $\xi \in \mathbb{R}$ (Pickands, 1975) can be used. It is a usual procedure in actuarial science to combine two distributions in a so-called splicing or composite model, see e.g. Bakar et al. (2015), Leppisaari (2016), and Grün and Miljkovic (2019).
Table 4: Major Chinese earthquakes since 1990.

| Time of Occurrence | Location  | Magnitude | Deaths | Injuries | The number of casualties | The total economic losses |
|--------------------|-----------|-----------|--------|----------|--------------------------|--------------------------|
| 2013/08/31         | Yunnan    | 5.9       | 3      | 63       | 66                       | 2834                     |
| 2012/06/30         | Xinjiang  | 6.6       | 0      | 52       | 52                       | 1967                     |
| 2013/11/23         | Jilin     | 5.5       | 0      | 25       | 25                       | 1990                     |
| 2005/11/26         | Jiangxi   | 5.7       | 13     | 775      | 788                      | 2023                     |
| 2009/07/09         | Yunnan    | 6.0       | 1      | 372      | 373                      | 2154                     |
| 2014/12/06         | Yunnan    | 5.9       | 1      | 22       | 23                       | 2377                     |
| 2011/03/10         | Yunnan    | 5.8       | 25     | 314      | 339                      | 2385                     |
| 1996/02/03         | Yunnan    | 7.0       | 309    | 17057    | 17366                    | 2500                     |
| 2013/08/12         | Tibet     | 6.1       | 0      | 87       | 87                       | 2707                     |
| 2014/11/22         | Sichuan   | 6.3       | 5      | 78       | 83                       | 4232                     |
| 2008/08/30         | Sichuan   | 6.1       | 41     | 1010     | 1051                     | 4462                     |
| 2012/09/07         | Yunnan    | 5.7       | 81     | 834      | 915                      | 4771                     |
| 2014/10/07         | Yunnan    | 6.6       | 1      | 331      | 332                      | 5110                     |
| 2015/07/03         | Xinjiang  | 6.5       | 3      | 260      | 263                      | 5430                     |
| 2015/04/25         | Tibet     | 8.1       | 27     | 860      | 887                      | 10302                    |
| 2014/08/03         | Yunnan    | 6.5       | 617    | 3143     | 3760                     | 19849                    |
| 2010/04/14         | Qinghai   | 7.1       | 2698   | 11000    | 13698                    | 22847                    |
| 2013/07/22         | Gansu     | 6.6       | 95     | 2414     | 2509                     | 24416                    |
| 2013/04/20         | Sichuan   | 7.0       | 196    | 13019    | 13215                    | 66514                    |
| 2008/05/12         | Sichuan   | 8.0       | 69227  | 375783   | 445010                   | 845110                   |
The density and distribution function of a two-component model for $Y_2$ is expressed as

$$f_{Y_2}(y) = \begin{cases} \omega f_{d,Y_2}(y), & y \leq u, \\ (1 - \omega) f_{c,Y_2}(y), & y > u, \end{cases} \quad (6.1)$$

$$F_{Y_2}(y) = \begin{cases} \omega F_{d,Y_2}(y), & y \leq u, \\ \omega + (1 - \omega) F_{c,Y_2}(y), & y > u, \end{cases} \quad (6.2)$$

where $f_{d,Y_2}$ and $F_{d,Y_2}$ denote the density and cdf of the corresponding right-truncated count distribution, and $f_{c,Y_2}$ and $F_{c,Y_2}$ of the heavy-tailed continuous component. In the truncated count component, we consider the right-truncated negative binomial Type II distribution (NBII) with mean parameter $\lambda$ and dispersion parameter $\phi$ with the variance being $1 + \phi\lambda^2$. For more details on truncated negative binomial regression and likelihood functions, see e.g. Shi et al. (2015). In practical applications one usually tries to set the threshold as low as possible, subject to the GP distribution providing an acceptable fit. Here a threshold value of $u = 20$ is chosen. For estimation purpose, we fix the mixing weight $w$ which can be expressed as $\hat{w} = (n - n_c)/n$ by assuming the exceedance times of the threshold $u$ to follow a homogeneous Poisson process, where $n$ is the total number of earthquake events and $n_c$ is the number of observed exceedances over the threshold $u$ (Leppisaari, 2016). There are a total of 194 observations in the count data set exceeding the threshold $u = 20$. Therefore, the estimated for weight $\hat{w} = 194/291 = 66.67\%$ for the casualties data. Moreover, the parameters in right-truncated negative binomial distribution are estimated by using MLE with $\hat{\lambda} = 37.42$ and $\hat{\phi} = 5.45$. The ML estimates for GP distribution is with the location parameter $\mu = 20$, the shape $\hat{\sigma} = 1.87$, and the scale $\hat{\xi} = 56.58$. The results for the goodness fit of the count distribution can be found in Appendix G.

To understand the association, we create uniform transformed data $(u_1, u_2)$ by applying the probability integral transformation to the economic losses and the number of casualties. The empirical Spearman’s rho and Kendall’s tau are 0.634 and 0.479 respectively. Figure 11 displays the pairwise dependence structures evaluated at $(u_1, u_2)$ as a function of the Year, which indeed indicates changes in dependence over the years.

In order to accommodate the asymmetric feature exhibited, we apply a mixed copula to model the
dependence between the continuous and semi-continuous outcomes. The modelling of mixed copula
is also discuss in Shi and Yang (2018) when the outcome follows a semi-continuous distribution. The
idea is easily extended to the general mixed case. Here we also refer to the most recent work of Chang
and Joe (2019), which proposes a copula regression to handle mixed continuous and discrete response
variables.

For a portfolio of $n$ observations $(y_{i1}, y_{i2}; i = 1, \ldots, n)$, the joint density function of $(Y_1, Y_2)$ can
be written as

$$f_{Y_1, Y_2}(y_{i1}, y_{i2}) = \begin{cases} f_{Y_1}(y_{i1}) \left[ h_{2|1}(F_{Y_1}(y_{i1}), F_{Y_2}(y_{i2})) - h_{0|1}(F_{Y_1}(y_{i1}), F_{Y_2}(y_{i2} - 1)) \right], & y_{i2} \leq u, \\
           f_{Y_1}(y_{i1}) f_{Y_2}(y_{i2}) c(F_{Y_1}(y_{i1}), F_{Y_2}(y_{i2})), & y_{i2} > u, \end{cases} \tag{6.3}$$

where the density $f_{Y_i}(\cdot)$ and cdf $F_{Y_i}(\cdot)$ of the marginal distributions ($i = 1, 2$) are specified by (2.3),
(2.4) and (6.1), (6.2) respectively. Here $h_{2|1}(u_1, u_2) = \partial C(u_1, u_2)/\partial u_1$ is the $h$-function of bivariate
copula. The log-likelihood function with copula parameters $\gamma$ and marginal parameters $\xi$ is given by

$$\ell(\gamma, \xi) = \sum_{i=1}^{n} \log f_{Y_1, Y_2}(y_{i1}, y_{i2}). \tag{6.4}$$

Here we use the inference functions of margins (IFM) method proposed discussed in Joe (1997) and
Nelsen (2007) to reduce the computational burden. Compared with the standard ML method which
is a full likelihood approach estimating all parameters simultaneously, the IFM method is a two-
step approach. In the first step the marginals are fitted independently to obtain the estimates of the parameters $\xi$. In the second step the joint log-likelihood of the mixed copula given in (6.4) is maximized over the copula-related parameters fixed as estimated in the first step of the method.

Table 5 summarizes the copula parameter estimates, together with the log-likelihood and information statistics of the survival MGL and the survival MGL-EV copula, along with four other copula candidates.

- For the survival MGL, $\gamma = \delta$,
- For the survival MGL-EV, $\gamma = \delta$,
- for the MGB2 copula, $\gamma$ is a 3-vector of $(p_1, p_2, q)^T$,
- for the Gumbel copula, $\gamma = \delta$,
- for the Gaussian copula, $\gamma = \rho$,
- for the Student $t$ copula, $\gamma = (\rho, v)^T$ and the degree of freedom $v$ is determined by ML.

In terms of the BIC, the survival MGL copula is preferred. In order to assess the quality of the copula fit in the tails, we focus on the model fit in the upper tails. The model fit errors $e_A(C)$ in upper regions, e.g., $A = [0.95, 1]$ and $A = [0.99, 1]$, are reported in Table 5. One can see that although survival MGL-EV and Gumbel copula might be more appropriate for modelling the data in $[0.95, 1]$, the survival MGL copula has a best performance in the region $[0.99, 1]$.

The introduction of covariates into the dependence parameter can improve the model fitting. We introduce here the covariate $Year$ into the dependence parameter in the survival MGL and survival MGL-EV regression model. The natural cubic splines are used to allow flexible relationships between $Year$ and the dependence parameter. We use the log link function to obtain real value number $\log \delta$, leading to the model

$$ \log \delta_i = \text{ns}_i(Year) = \beta_1 b_1(Year) + \ldots + \beta_k b_k(Year), $$

where $\text{ns}_i(x)$ denote natural cubic splines with spline basis $b_1(x), \ldots, b_p(x)$. The 33.3% and 66.7% percentiles of $Year$ are used as two knots for the natural cubic spline, and there are four coefficients $\beta_k, k = 1, \ldots, 4$ to be estimated in the copula regression. The IFM method is used to estimate the regression coefficients.
Table 5: Estimates and goodness fit of candidate of copulas for the Chinese earthquake loss data set.

| Copula       | Estimates       | Loglike | AIC       | BIC       | $e_{[0.95,1]^2}$ | $e_{[0.99,1]^2}$ |
|--------------|-----------------|---------|-----------|-----------|------------------|------------------|
| Gaussian     | $\hat{\rho} = 0.698$ | -       | -         | -3022.35  | 6046.69          | 6050.36          | 66.85            | 0.29             |
|              | (0.028)         |         |           |           |                  |                  |                  |                  |
| Student t    | $\hat{\rho} = 0.691$ | $\hat{v} = 4.450$ | -       | -3018.91  | 6041.82          | 6049.17          | 57.44            | 0.21             |
|              | (0.036)         | (2.096) |           |           |                  |                  |                  |                  |
| Gumbel       | $\hat{\alpha} = 1.922$ | -       | -         | -3009.92  | 6021.84          | 6025.51          | 1.71             | 0.08             |
|              | (0.096)         |         |           |           |                  |                  |                  |                  |
| Survival MGL | $\hat{\delta} = 2.763$ | -       | -         | -3009.46  | 6020.92          | **6024.59**      | 2.66             | **0.06**         |
|              | (0.247)         |         |           |           |                  |                  |                  |                  |
| Survival MGL-EV | $\hat{\delta} = 1.861$ | -       | -         | -3009.88  | 6021.75          | 6025.42          | **1.71**         | 0.08             |
|              | (0.160)         |         |           |           |                  |                  |                  |                  |
| MGB2         | $\hat{p}_1 = 1.485$ | $\hat{p}_2 = 1.322$ | $\hat{q} = 0.816$ | -3007.30  | **6020.60**      | 6031.62          | 1.97             | 0.07             |
|              | (2.347)         | (0.785) | (0.530)   |           |                  |                  |                  |                  |

Notes: The square fit error $e_A(C)$ in regions $A = [0.95,1]^2$ and $A = [0.99,1]^2$ are resealed by $\times 10^8$. The standard error is reported in brackets.

Table 6 reports the estimates and standard errors of regression coefficients, together with the log-likelihood and information statistics for the survival MGL and survival MGL-EV copula regression. The estimation results are also reported for a corresponding Gumbel regression model. In terms of the AIC and BIC value the survival MGL copula regression provides a better overall fit.

Figure 12 presents the relationship between Year and the dependence parameter. One can see that the non-linear relationship appear in three cases, which displays the tendency of dependence rising up at the beginning and declining later on. The magnitude 8.0 Sichuan earthquake from 2008 was the strongest earthquake in China in over 50 years. After 2008 the Chinese government has conducted many disaster management-related projects, responding to the need for disaster prevention and mitigation by integrating livelihood assistance, disaster risk reduction, sharing knowledge and practice, technical support, capacity building, and policy advocacy after 2008, resulting in a declining trend of the dependence.
Table 6: Estimates and goodness fit for survival MGL and Gumbel regression models for the Chinese earthquake loss data set.

| Parameters | Survival MGL Estimates | S.E. | Survival MGL-EV Estimates | S.E. | Gumbel Estimates | S.E. |
|------------|------------------------|------|---------------------------|------|-----------------|------|
| $\beta_1$  | 0.634 0.333            | 0.333| 0.333 0.389              | 0.311| -0.250 0.495    | 0.495|
| $\beta_2$  | 1.722 0.292            | 1.327| 1.327 0.311              | 1.087| 1.087 0.366     | 0.366|
| $\beta_3$  | 1.846 0.386            | 1.051| 1.051 0.355              | 0.357| -0.608 0.470    | 0.470|
| $\beta_4$  | -0.250 0.341           | -0.211| -0.211 0.357            | -0.422| -0.422 0.480    | 0.480|
| Loglike    | -3002.22               | -3004.76| -3004.89               |      |                 |      |
| AIC        | 6012.43                | 6017.52| 6017.78                |      |                 |      |
| BIC        | 6027.12                | 6032.22| 6032.48                |      |                 |      |

Notes: In order to avoid boundary problem in MLE procedures, we consider a log link function obtaining real values for Gumbel copula regression: $\log(\delta_i - 1) = n_s(Y)\text{ for all } \delta_i > 1.$

Figure 12: The predicted value of copula parameter with different value of covariate Year for the Chinese earthquake loss data set. The gray lines are generated based on the simulated coefficients for the natural cubic spline, and their values are generated by a multivariate normal distribution with mean equal to the ML estimates of $\beta_k$ for $k = 1, ..., 4,$ and the covariance matrix being the inverse of the Hessian matrix.
7 Summary and concluding remarks

The main proposals in this paper are the MGL copula model for accommodating non-elliptical and asymmetric dependence structures, and dynamic dependence modelling using corresponding copula regression models. Based on the heavy-tailedness from the univariate GLMGA distribution, this new copula class and its survival version can capture positive lower/upper tail dependence. The proposed copula features asymmetric relationships using only one dependence parameter and demonstrates flexibility for modelling multi-dimensional asymmetry. The probabilistic characteristics of the proposed copula are discussed and the corresponding extreme-value copula is obtained. The proposed copula model is effective in regression modelling of the dependence structure using covariate information. ML estimation can be quite easily performed as the joint pdf is given in closed form even in high dimensions. We also implement the proposed method in a user-friendly R package: rMGLReg that can provide a nice visualization tool for interpreting the proposed copula and serve as a convenient tool for actuarial practitioners to investigate the nonlinear dynamic dependence pattern.

Considering bivariate copulas as building blocks for many multivariate dependence models using for instance vine copulas, the potential of the proposed copula for building up more complex multivariate dependence models should be the subject of future research. There are ongoing studies on its potential in modelling real datasets that have more dimensions and more complex dependence structure.

Supplementary Material

R-package for github routine: R package: rMGLReg containing code to display the properties of the proposed models and perform the estimation methods described in the paper. The package also contains all datasets used as examples in the paper. The package can be found at https://github.com/lizhengxiao/rMGLReg for more details.

Appendices

A Proof of Proposition 2.1

From the model specification we obtain the following properties:

\[ E(Y_j|\Theta) = \left( \frac{\Theta}{2\sigma_j} \right)^{\sigma_j} \frac{\Gamma(\frac{1}{2}) - \sigma_j}{\Gamma(\frac{1}{2})} \]
Note that if \( j = j' \),
\[
\text{Cov}(Y_j, Y_{j'}|\Theta) = \begin{cases} 
\left( \frac{\Theta}{2b_j} \right)^{2\sigma_j} \frac{\Gamma\left(\frac{1}{2}-\sigma_j\right) \Gamma\left(\frac{1}{2}-2\sigma_j\right) - \Gamma^2\left(\frac{1}{2}\right)}{\Gamma^2\left(\frac{1}{2}\right)} & \text{if } j = j' \\
0 & \text{if } j \neq j',
\end{cases}
\]
and
\[
E(\Theta^r) = \frac{\Gamma(r + a)}{\Gamma(a)} \quad \text{and} \quad E(\Theta) = a \quad \text{and} \quad \text{Var}(\Theta) = a.
\]

These properties lead us to the unconditional mean of the univariate MGL distribution (also known as GLMGA distribution):
\[
\begin{align*}
E(Y_j) &= E[E(Y_j|\Theta)] = (2b_j)^{-\sigma_j} \frac{B\left(\frac{1}{2} - \sigma_j, a + \sigma_j\right)}{B\left(\frac{1}{2}, a\right)}, \\
\text{Var}(Y_j) &= E[\text{Var}(Y_j|\Theta)] + \text{Var}[E(Y_j|\Theta)] = \left[E(Y_j)\right]^2 \left[ \frac{B(a + 2\sigma_j, a)B\left(\frac{1}{2} - 2\sigma_j, \frac{1}{2}\right)}{B(a + \sigma_j, a + \sigma_j)B\left(\frac{1}{2} - \sigma_j, \frac{1}{2} - \sigma_j\right) - 1} \right].
\end{align*}
\]

It is straightforward to see that in the case where \( j \neq j' \) we have,
\[
\begin{align*}
\text{Cov}(Y_j, Y_{j'}) &= E[\text{Cov}(Y_j, Y_{j'}|\Theta)] + \text{Cov}[E(Y_j|\Theta), E(Y_{j'}|\Theta)], \\
&= \text{Cov}\left[ \left( \frac{\Theta}{2b_j} \right)^{-\sigma_j} \frac{\Gamma\left(\frac{1}{2} - \sigma_j\right)}{\Gamma\left(\frac{1}{2}\right)}, \left( \frac{\Theta}{2b_{j'}} \right)^{-\sigma_{j'}} \frac{\Gamma\left(\frac{1}{2} - \sigma_{j'}\right)}{\Gamma\left(\frac{1}{2}\right)} \right] \\
&= (2b_j)^{-\sigma_j} (2b_{j'})^{-\sigma_{j'}} \frac{\Gamma\left(\frac{1}{2} - \sigma_j\right) \Gamma\left(\frac{1}{2} - \sigma_{j'}\right)}{\Gamma^2\left(\frac{1}{2}\right)} \text{Cov}[\Theta^{\sigma_j}, \Theta^{\sigma_{j'}}] \\
&= \frac{E(Y_j)E(Y_{j'})}{B(a + \sigma_j, a + \sigma_{j'})} \left[ \frac{B(a + \sigma_j + \sigma_{j'}, a)}{B(a + \sigma_j, a + \sigma_{j'})} - 1 \right].
\end{align*}
\]

For \( j \neq j' \) we then obtain
\[
\begin{align*}
\text{Corr}(Y_j, Y_{j'}) &= \frac{\text{Cov}(Y_j, Y_{j'})}{\sqrt{\text{Var}(Y_j)\text{Var}(Y_{j'})}} \\
&= \frac{B(a + \sigma_j + \sigma_{j'}, a)}{B(a + \sigma_j, a + \sigma_{j'})} - 1 \\
&= \sqrt{\left[ \frac{B(a + 2\sigma_j, a)B\left(\frac{1}{2} - 2\sigma_j, \frac{1}{2}\right)}{B(a + \sigma_j, a + \sigma_j)B\left(\frac{1}{2} - \sigma_j, \frac{1}{2} - \sigma_j\right)} - 1 \right] \left[ \frac{B(a + 2\sigma_{j'}, a)B\left(\frac{1}{2} - 2\sigma_{j'}, \frac{1}{2}\right)}{B(a + \sigma_{j'}, a + \sigma_{j'})B\left(\frac{1}{2} - \sigma_{j'}, \frac{1}{2} - \sigma_{j'}\right)} - 1 \right]}.
\end{align*}
\]

Note that if \( \sigma_j = \sigma_{j'} := \sigma \), (2.12) simplifies to
\[
\begin{align*}
\text{Corr}(Y_j, Y_{j'}) &= \frac{B(a + 2\sigma, a)}{B(a + \sigma, a + \sigma)} - 1 \\
&= \frac{B(a + 2\sigma, a)B\left(\frac{1}{2} - 2\sigma, \frac{1}{2}\right)}{B(a + \sigma, a + \sigma)B\left(\frac{1}{2} - \sigma, \frac{1}{2} - \sigma\right)} - 1.
\end{align*}
\]
Since for any \( m \), \( \lim_{a \to +\infty} \frac{\Gamma(a+m)}{\Gamma(a)^m} = 1 \), we have that for any fixed \( \sigma \),
\[
\lim_{a \to +\infty} \frac{B(a + \sigma_j + \sigma_{j'}, a)}{B(a + \sigma_j, a + \sigma_{j'})} - 1 = \lim_{a \to +\infty} \frac{\Gamma(a + \sigma_j + \sigma_{j'})\Gamma(a)}{\Gamma(a + \sigma_j)\Gamma(a + \sigma_{j'})} - 1 = 0.
\]
This implies \( \lim_{a \to +\infty} \text{Corr}(Y_j, Y_{j'}) = 0 \) if \( j \neq j' \).

**B Proof of Proposition 2.3**

Random samples \( (Y_1, ..., Y_d) \) from the MGL(\( \sigma, a, b \)) distribution can be simulated using the conditional distribution given in Proposition 2.2 through the following steps:

- \( Y_1 \) is generated using the quantile function of the GLMGA(\( \sigma_1, a, b_1 \)) distribution:
  \[
  Y_1 = (2b_1)^{-\sigma_1} \left[ \frac{I_{1/2,a}(1 - U_1)}{1 - I_{1/2,a}(1 - U_1)} \right]^{-\sigma_1} = (2b_1 M_1)^{-\sigma_1}.
  \]

- \( Y_2 \) is generated using the quantile function of the GLMGA(\( \sigma_2, a_2, b_2^* \)) distribution with \( a_2 = a + \frac{1}{2} \) and \( b_2^* = b_2 \left[ 1 + \frac{1}{\sigma_1} \right] / (2b_1) \):
  \[
  Y_2 = (2b_2^*)^{-\sigma_2} \left[ \frac{I_{1/2,a_2}(1 - U_2)}{1 - I_{1/2,a_2}(1 - U_2)} \right]^{-\sigma_2} = (2b_2^* Z_2)^{-\sigma_2} = [2b_2 M_2]^{-\sigma_2}.
  \]

- \( \ldots \)

- \( Y_d \) is generated using the quantile function of the GLMGA(\( \sigma_d, a_d, b_d^* \)) distribution with \( a_d = a + \frac{1}{2} \) and \( b_d^* = b_d \left[ 1 + \sum_{j=1}^{d-1} \frac{1}{\sigma_j} \right] / (2b_j) \):
  \[
  Y_d = (2b_d^*)^{-\sigma_d} \left[ \frac{I_{1/2,a_d}(1 - U_d)}{1 - I_{1/2,a_d}(1 - U_d)} \right]^{-\sigma_d} = (2b_d^* Z_d)^{-\sigma_d} = [2b_d M_d]^{-\sigma_d}.
  \]

Finally, the random samples \( (U_1^*, ..., U_d^*) \) from \( C^{MGL}(\cdot; a) \) can be obtained by
\[
U_j^* = F(Y_j; \sigma_j, a, b_j) = 1 - I_{1/2,a} \left( \frac{M_j}{1 + M_j} \right), \quad \text{for} \quad j = 1, ..., d,
\]
where \( F(\cdot; \sigma_j, a, b_j) \) is the cdf of the univariate GLMGA distribution given in (2.4). The random
samples of $C^{MGL}(:, \delta)$ are generated substituting $\delta = 1/a$.

C Proof of Proposition 3.2

We first define $Y_1 = F_1^{-1}(U_1)$ and $Y_2 = F_2^{-1}(U_2)$, where $U_1, U_2$ are independent uniformly $(0,1)$ distributed and $F_j$ ($j = 1, 2$) represent the cdf of the GLMGA distribution with parameters $(\sigma_j, a, b_j)$ respectively. Using Proposition 2.2 the vector $(Y_1, Y_2)$ satisfies

$$Y_1 | Y_2 = y_2 \sim \text{GLMGA}(\sigma_1, a + 1/2, b_1(1 + y_2^{-1/\sigma_2}b_2))$$

$$Y_2 | Y_1 = y_1 \sim \text{GLMGA}(\sigma_2, a + 1/2, b_2(1 + y_1^{-1/\sigma_1}b_1))$$

For the upper tail dependence index, we use

$$\lambda_u = \lim_{u \to 1^-} \Pr [Y_1 > F_1^{-1}(u) | Y_2 = F_2^{-1}(u)] + \lim_{u \to 1^-} \Pr [Y_2 > F_2^{-1}(u) | Y_1 = F_1^{-1}(u)]$$

$$= \lim_{u \to 1^-} I_{1/2, a+1/2} \left( \frac{I_{1/2, a+1/2}^{-1}(1-u)}{1 + I_{1/2, a+1/2}^{-1}(1-u)} \right) + \lim_{u \to 1^-} I_{1/2, a+1/2} \left( \frac{I_{1/2, a+1/2}^{-1}(1-u)}{1 + I_{1/2, a+1/2}^{-1}(1-u)} \right)$$

$$= 0.$$

For the lower tail dependence index, we use

$$\lambda_l = \lim_{u \to 0^+} \Pr [Y_1 \leq F_1^{-1}(u) | Y_2 = F_2^{-1}(u)] + \lim_{u \to 0^+} \Pr [Y_2 \leq F_2^{-1}(u) | Y_1 = F_1^{-1}(u)]$$

$$= 2 - 2 \lim_{u \to 0^+} I_{1/2, a+1/2} \left( \frac{I_{1/2, a+1/2}^{-1}(1-u)}{1 + I_{1/2, a+1/2}^{-1}(1-u)} \right)$$

$$= 2 - 2 I_{1/2, a+1/2} \left( \frac{1}{2} \right)$$

$$= 2 - 2 I_{1/2, a+1/2} \left( \frac{1}{2} \right).$$

D Domain of attraction and extreme-value copula

For the proof of Proposition 3.3 we need the following intermediate result concerning the regular variation of the function $t(:, a)$. 

37
Lemma .1. The function \( t(u; a) = \frac{I^{-1}{\frac{a}{2}}(1-u)}{I^{-1}{\frac{a}{2}}(1-u)} \) is regularly varying at the origin with index \(-1/a\):

\[
\lim_{s \to 0} \frac{t(su; a)}{t(s; a)} = u^{-1/a}, \quad u > 0.
\]

Proof. Since the cdf \( F \) of a GLMGA distribution is regularly varying near 0 with index \( a/\sigma \), its inverse, the quantile function given in (2.5), is also regularly varying at 0 with index \( \sigma/a \). Since \( t \) is proportional to the \(-1/\sigma\) power of the quantile function the result follows. □

Proof of Proposition 3.3 Clearly, the boundary values of \( \ell \) are given by \( \ell(u_1, 0) = u_1, \ell(0, u_2) = u_2 \) and \( \ell(0, 0) = 0 \). Moreover

\[
\lim_{s \to 0^+} \frac{1 - \bar{C}_{MGL}(1 - su_1, 1 - su_2; \delta)}{s} = \lim_{s \to 0^+} \frac{su_1 + su_2 - C_{MGL}(su_1, su_2; \delta)}{s} = u_1 + u_2 - \lim_{s \to 0^+} u_1 \Pr (U_2 \leq su_2 | U_1 = su_1) - \lim_{s \to 0^+} u_2 \Pr (U_1 \leq su_1 | U_2 = su_2).
\]

Let \( Y_1 = F_1^{-1}(U_1) \) and \( Y_2 = F_2^{-1}(U_2) \) where \( F_j \) is the distribution function of the univariate GLMGA distribution with parameters \((\sigma_j, a, b_j), j = 1, 2\). The conditional probability function can be evaluated using Proposition 2.2 and is given by

\[
\Pr (U_2 \leq su_2 | U_1 = su_1) = \Pr (Y_2 \leq F_2^{-1}(su_2) | Y_1 = F_1^{-1}(su_1))
= 1 - I_{\frac{1}{2}, a + \frac{1}{2}} \left[ \frac{t(su_2; a)}{t(su_1; a) + t(su_2; a) + 1} \right]
= 1 - I_{\frac{1}{2}, a + \frac{1}{2}} \left[ \frac{t(su_2; a)}{t(s; a)} + \frac{t(su_1; a)}{t(s; a)} + \frac{1}{t(s; a)} \right].
\]

A similar expression holds for \( \Pr (U_1 \leq su_1 | U_2 = su_2) \). Since \( 1/t(s; a) \to 0^+ \) as \( s \to 0^+ \) and the only
The density of the MGL copula is given by

\[
\ell(u_1, u_2) = \lim_{s \to 0^+} \frac{1 - \tilde{C}_{\text{MGL}}(1 - su_1, 1 - su_2; \delta)}{s}
\]

where the stable tail dependence function \( \delta \) corresponds to complete dependence, whereas the upper bound, \( A_\delta(w) = \max(1 - w, w) \), corresponds to complete dependence, whereas the upper bound, \( A_\delta(w) = 1 \), corresponds to independence. □

The \( d \)-dimensional extreme value copula The extreme value copula \( \tilde{C}_{\text{MGL}-\text{EV}} \) of the survival MGL copula if given by

\[
\tilde{C}_{\text{MGL}-\text{EV}}(u_1, \ldots, u_d; \delta) = \exp \left[ -\ell(\text{log } u_1, \ldots, \text{log } u_d) \right],
\]

where the stable tail dependence function \( \ell : [0, \infty)^d \to [0, \infty) \) is given by

\[
\ell(u_1, \ldots, u_d) = \sum_{j=1}^{d} u_j I_{\frac{1}{2} \delta + \frac{1}{2}} \left( 1 - \frac{u_j^{-1/a}}{\sum_{j=1}^{d} u_j^{-1/a}} \right).
\]

The density of the \( d \)-dimensional extreme-value copula \( \tilde{C}_{\text{MGL}-\text{EV}} \) is of the form

\[
\tilde{c}_{\text{MGL}-\text{EV}}(u_1, \ldots, u_d; \delta) = \frac{\tilde{C}_{\text{MGL}-\text{EV}}(u_1, \ldots, u_d; \delta)}{\prod_{j=1}^{d} u_j} \frac{(-1)^{d-m} \sum_{\pi : |\pi| = m} \prod_{B \in \pi} D_B \ell(z_1, \ldots, z_d) | z_1 = -\log u_1, \ldots, z_d = -\log u_d}{\prod_{j=1}^{d} u_j}.
\]

E The gradient of the log-likelihood for survival MGL copula

We obtain derivatives of the log-likelihood (4.1) in Section 4 with respect to model parameters. With \( \delta_i = \exp(x_i^T \beta) \), \( \phi'(x) := \frac{\partial \log \Gamma(x)}{\partial x} |_{x=x} \), \( \phi''(x) := \frac{\partial^2 \log \Gamma(x)}{\partial x^2} |_{x=x} \), \( m'(x, \frac{1}{\delta_i}) = \frac{\partial h^{-1}(x)}{\partial z} |_{z = \frac{1}{\delta_i}} \) and
\[
m''(x; \frac{1}{\sigma_i}) = -\frac{\partial^2 I^{-1}(z)}{\partial z^2} \bigg|_{z = \frac{1}{\sigma_i}}, \text{the first-order derivatives are given by}
\]

\[
\frac{\partial \ell(u_1, \ldots, u_d; \beta)}{\partial \beta_h} = -\frac{x_{ij}}{\delta_i} \left\{ (d - 1) \sum_{i=1}^{n} \phi'\left(\frac{1}{\delta_i}\right) + \sum_{i=1}^{n} \phi'\left(\frac{1}{\delta_i} + \frac{d}{2}\right) - d \sum_{i=1}^{n} \phi'\left(\frac{1}{\delta_i} + \frac{1}{2}\right) \right. \\
+ \sum_{i=1}^{n} \sum_{j=1}^{d} \log \frac{I_{\frac{1}{2}, \frac{1}{\delta_i}}(u_{ij})}{1 - I_{\frac{1}{2}, \frac{1}{\delta_i}}(u_{ij})} \\
+ \sum_{i=1}^{n} \left( \frac{1}{\delta_i} + \frac{1}{2} \right) \sum_{j=1}^{d} m'(u_{ij}, \frac{1}{\delta_i}) + \sum_{i=1}^{n} \left( \frac{1}{\delta_i} + \frac{1}{2} \right) \sum_{j=1}^{d} m'(u_{ij}, \frac{1}{\delta_i}) \\
+ \sum_{i=1}^{n} \log \left( \sum_{j=1}^{d} \frac{I_{\frac{1}{2}, \frac{1}{\delta_i}}(u_{ij})}{1 - I_{\frac{1}{2}, \frac{1}{\delta_i}}(u_{ij})} + 1 \right) \\
+ \sum_{i=1}^{n} \left( \frac{1}{\delta_i} + \frac{d}{2} \right) \left[ \sum_{j=1}^{d} \frac{I_{\frac{1}{2}, \frac{1}{\delta_i}}(u_{ij})}{1 - I_{\frac{1}{2}, \frac{1}{\delta_i}}(u_{ij})} + 1 \right]^{-1} \sum_{j=1}^{d} \frac{m'(u_{ij}, \frac{1}{\delta_i})}{\left[1 - I_{\frac{1}{2}, \frac{1}{\delta_i}}(u_{ij})\right]^2} \right\},
\] (2)

for \( h = 0, \ldots k \).

Equating (2) to zero, the maximum likelihood (ML) estimator \( \hat{\beta}_h \) of \( \beta_h \) is obtained by using the function \texttt{MGL.reg} in R package: \texttt{rMGLReg} to minimize the negative log-likelihood with a given gradient.

**F Economic loss: marginal modelling**

In Table 7 we provide the estimates, log-likelihood values (LL), as well as the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC) values, defined respectively as AIC = \(-2\ell + 2p\) and BIC = \(-2\ell + p \log n\) where \( \ell \) denotes the log-likelihood value, \( p \) the number of model parameters and \( n \) the number of observations. We use the \texttt{optim()} function in R which uses the Nelder-Mead method. Parameters are estimated by the MLE and standard errors are calculated using the observed information matrix. It is clear from Table 7 that the GLMGA provide a better fit than the other four models, as it has the highest log-likelihood and smallest AIC and BIC value.

We also provide goodness-of-fit measures and the bootstrap P-values for the corresponding goodness-of-fit tests. In Table 8 we consider the Kolmogorov-Smirnov (KS), Cramér-von Mises (CvM) and Anderson-Darling (AD) test statistics and corresponding P-values, choosing for the models with small
Table 7: Earthquake economic losses: model selection measures.

| Distribution | Estimates | #Par. | LL     | AIC    | BIC    |
|--------------|-----------|-------|--------|--------|--------|
| GlogM        | $\hat{\sigma}$ | 1.426 (0.061) | 2      | -1899.76 | 3803.52 | 3810.86 |
|              | $\hat{\mu}$   | 15.475 (2.042) |        |         |        |
| GLMGA        | $\hat{\sigma}$ | 0.820 (0.074) | 3      | -1871.01 | 3748.02 | 3759.04 |
|              | $\hat{b}$      | 0.005 (0.003) |        |         |        |
|              | $\hat{a}$      | 0.697 (0.153) |        |         |        |
| Log-gamma    | $\hat{\alpha}$ | 3.547 (0.280) | 2      | -1878.35 | 3760.70 | 3768.05 |
|              | $\hat{\beta}$ | 1.215 (0.103) |        |         |        |
| Fréchet      | $\hat{\alpha}$ | 0.385 (0.014) | 2      | -1927.01 | 3858.02 | 3865.37 |
|              | $\hat{\beta}$ | 209.940 (33.806) |      |         |        |
| DPLN         | $\lambda_1$    | 3.803 (2.759) | 4      | -1874.11 | 3756.23 | 3770.92 |
|              | $\lambda_2$    | 2.144 (0.021) |        |         |        |
|              | $\hat{\tau}$   | 2.142 (0.021) |        |         |        |
|              | $\hat{\nu}$    | 4.409 (0.022) |        |         |        |

*The standard errors of estimates are reported in parentheses.

values of the KS, CvM and AD test statistics, or large values of the corresponding P-values. The P-values are obtained using the bootstrap method as developed in Calderín-Ojeda and Kwok (2016). Here again the GLMGA model is prevailing with a P-value above 0.7, which provides a strong evidence for the best fit.

In Figure 13 the QQ-plots of the log-transformed empirical quantiles against the log-transformed estimated quantiles of the 5 competing models are given. The correlation coefficients $R$ of these QQ-plots are also given in Table 8: $R$ measures the degree of linearity in the QQ-plot and hence also the goodness-of-fit with respect to the corresponding model. These QQ-plots also provide interesting information concerning the estimates of the VaR at extreme quantile levels. We can judge the appropriateness of the VaR estimates using the different competing models by comparing the model estimates of an extreme quantile $F^{-1}(p)$ with the quantile level $p$ close to $1 - \frac{1}{n}$ with the empirical VaR, which is then close to the maximum value of the data set. In Table 9 we compare the empirical 95%, 99%, 99.8% Var with the estimates of the model VaR obtained from the different models. We report the relative deviations from the empirical VaR. Note that the Fréchet and DPLN models are lower than the empirical estimate, while the GLMGA model gives much more conservative estimates than GlogM and Log-gamma models.
Figure 13: Earthquake economic losses: QQ-plots of the log-transformed empirical quantiles against the log-transformed estimated model quantiles.
Table 8: Earthquake economic losses: goodness-of-fit measures.

| Distribution | R  | Kolmogorov-Smirnov | Anderson-Darling | Cramer-von Mises |
|--------------|----|-------------------|-----------------|-----------------|
|              |    | Statistic | P-value | Statistic | P-value | Statistic | P-value |
| GlogM        | 0.975 | 0.089 | 0.000 | 5.765 | 0.000 | 0.907 | 0.000 |
| GLMGA        | 0.997 | 0.028 | 0.810 | 0.282 | 0.589 | 0.035 | 0.710 |
| Log-gamma    | 0.990 | 0.077 | 0.000 | 1.954 | 0.000 | 0.346 | 0.000 |
| Fréchet      | 0.950 | 0.115 | 0.000 | 9.200 | 0.000 | 1.398 | 0.000 |
| DPLN         | 0.991 | 0.040 | 0.302 | 0.525 | 0.130 | 0.074 | 0.200 |

*The bootstrap P-values are computed using parametric bootstrap with 1000 simulation runs.

Table 9: Earthquake economic losses: estimates of VaR_{0.95}, VaR_{0.99} and VaR_{0.998}, relative difference (in percentage) with respect to the empirical VaR.

| Model       | 95%  | Diff. % | 99%  | Diff. % | 99.8% | Diff. % |
|-------------|------|---------|------|---------|-------|---------|
| Empirical   | 2574.97 | 25349.91 | 49635.53 |
| GlogM       | 41844.12 | 15.25 | 4140052.88 | 162.32 | 40891187.73 | 892.48 |
| GLMGA       | 3577.63 | 0.39 | 50153.12 | 0.98 | 701400.43 | 0.53 |
| Log-gamma   | 5582.17 | 1.17 | 82065.15 | 2.24 | 1013583.58 | 1.22 |
| Fréchet     | 3621.54 | 0.41 | 11055.97 | -0.56 | 24068.97 | -0.95 |
| DPLN        | 2509.55 | -0.03 | 11198.31 | -0.56 | 37585.05 | -0.92 |

G The numbers of casualties: marginal modelling

The ML estimates for truncated count distribution is performed via the `gamlss` function of the `gamlss` and `gamlss.tr` package in R, and the estimates for GP distribution is performed via the `fevd` function of the `extRemes` package in R (see Gilleland and Katz (2016) for details).

To demonstrate the goodness fit of the truncated count distribution below the threshold and the tail behavior above the threshold of casualties data, we use randomized (normal) quantile residuals defined by $r_i^c = \Phi^{-1}\left[F_{Y_i}(y_i)\right]$ for $i = 1, \ldots, n_c$ with $j = c$ and $i = 1, \ldots, n - n_c$ with $j = d$, where $\Phi^{-1}(\cdot)$ is the inverse function of the cdf of the standard normal distribution and $F_{Y_i}(\cdot)$ denotes the cdf of the right-truncated count distribution and GP distribution as given in (6.2) respectively. The distribution of $r_i^c$ and $r_i^d$ converge to standard normal if parameters are consistently estimated, see Dunn and Smyth (1996), and hence a normal QQ-plot of randomized quantile residuals should follow the 45 degree line. Figure 14 displays the normal QQ-plot for the number of casualties supporting the condition that residuals of right truncated negative binomial distribution and GP distribution are normally distributed.
Figure 14: Normal QQ-plots of quantile residuals for the number of casualties based on the truncated negative binomial distribution ($r_i^d$ in the left panel), and GP distribution ($r_i^c$ in the right panel) with the sample size $n - n_c$ and $n_c$ respectively.

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