Recognizing difference quotients of real functions.

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Abstract

For a real function \( f : [0, 1] \to \mathbb{R} \), the difference quotient of \( f \) is the function of two real variables \( \text{DQ}_f(a, b) = \frac{f(b) - f(a)}{b - a} \), which we view as defined on the triangle \( \mathcal{T} = \{(a, b) : 0 \leq a < b \leq 1\} \). In this paper we investigate how to determine whether a given function of two variables \( H(a, b) \) is the difference quotient of some real function \( f(x) \). We develop three independent methods for recognizing such a function \( H \) as a difference quotient, and corresponding methods for recovering the underlying function \( f \) from \( H \).

1 Introduction

Given a function \( f(x) \) defined at two distinct points \( a, b \in [0, 1] \), the difference quotient of \( f \) from \( a \) to \( b \) is the slope of the line segment connecting points \((a, f(a))\) and \((b, f(b))\). This we denote as

\[
\text{DQ}_f(a, b) = \frac{f(b) - f(a)}{b - a}.
\]

However, simplification and rearrangement of \( \text{DQ}_f \) may leave this function in a form that is unrecognizable as a difference quotient. Therefore, it is desirable to develop some means to determine whether a given function of two variables \( H(a, b) \) is the difference quotient of some function \( f(x) \) and, if so, a method for recovering \( f \) from \( H \). We will give three such tests for whether \( H \) is equal to \( \text{DQ}_f \) for some \( f \), and also corresponding methods to recover \( f \) from \( H \).

Throughout this paper, all variables are taken to be real, and all functions are assumed to take real values. We will restrict our attention to functions \( f(x) \) having domain \([0, 1]\). Therefore we will consider \( H \) to have as its domain either the square \([0, 1] \times [0, 1]\), or some subset thereof. Of course, if the difference quotient \( \text{DQ}_f \) extends continuously to the diagonal

\[
\mathcal{D} = \{(a, a) : a \in [0, 1]\},
\]

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then \( f \) must be differentiable and \( \text{DQ}_f = f' \) on \( D \).

The tests we provide are presented in an increasing order of specificity with regard to the sort of functions \( H \) to which they may be applied, and along the way, we will give concrete examples of functions \( H \) which we may test.

The first criteria, which we call the **Algebraic Criteria** and prove in Section 2, is meant to detect the difference quotient of a completely arbitrary real function \( f(x) \). Therefore the function \( H \) is not expected to be defined on \( D \). It must, however, be symmetric (ie \( H(a, b) = H(b,a) \)) in order to be a difference quotient, so, rather than include symmetry as a part of the criteria, we merely assume that \( H \) is defined only on the upper triangle

\[
\mathcal{T} = \{(a, b) : 0 \leq a < b \leq 1\}.
\]

If we wish to extend \( H \) to the lower triangle we may do so symmetrically by defining \( H(b, a) = H(a, b) \) for \( a < b \).

**Theorem 1** (Algebraic Criteria). Let \( H : \mathcal{T} \to \mathbb{R} \) be given. Then the following are equivalent:

1. There exists some \( f : [0, 1] \to \mathbb{R} \) such that \( \text{DQ}_f(a, b) = H(a, b) \).
2. For all \( 0 \leq a < b < c \leq 1 \),
   \[
   H(a, c) = \frac{H(a, b)(b - a) + H(b, c)(c - b)}{c - a}.
   \]
3. For all \( 0 < b < c \leq 1 \),
   \[
   H(0, c) = \frac{H(0, b)(b) + H(b, c)(c - b)}{c}.
   \]

In the proof of Theorem 1 which we give in Section 2 we construct the underlying function \( f \) assuming \( H \) satisfies the second and third items of Theorem 1. We enshrine that construction in the following corollary.

**Corollary 2.** For \( H(a, b) \) satisfying Items 2 and 3 of Theorem 1, the functions \( f(x) \) which make \( H = \text{DQ}_f \) are all functions of the form

\[
f(x) = xH(0, x) + C
\]

for a constant \( C \).

We observe furthermore that the expression

\[
H(a, c) = \frac{H(a, b)(b - a) + H(b, c)(c - b)}{c - a}
\]

from Item 2 of Theorem 1 may be rewritten as

\[
H(b, c)(b - c) - H(a, c)(a - c) + H(a, b)(a - b) = 0.
\]
The left hand side of this equation may easily be recognized as the determinant of the matrix

$$M = \begin{bmatrix} H(b, c) & H(a, c) & H(a, b) \\ a & b & c \\ 1 & 1 & 1 \end{bmatrix},$$

which will be referred to in Corollary 3 below. From this observation, we extract several secondary tests, which follow from Theorem 1. It is interesting to point out that, in the case $H = DQf$ for some function $f$, the dimension and null set of $M$ are in fact independent of $H$.

**Corollary 3.** Let $H : T \to \mathbb{R}$ be given. The following are equivalent:

1. There exists some $f : [0, 1] \to \mathbb{R}$ such that $DQf(a, b) = H(a, b)$.
2. For all $0 \leq a < b < c \leq 0$, $\det(M) = 0$.
3. For all $0 \leq a < b < c \leq 0$, $\dim(M) = 2$.
4. For all $0 \leq a < b < c \leq 0$, $\text{Null}(M) = \left\{ k \begin{bmatrix} b - c \\ c - a \\ a - b \\ c - a \end{bmatrix} : k \in \mathbb{R} \right\}$.

We now turn our attention to the difference quotients of differentiable functions $f(x)$. If $f$ is differentiable, then its difference quotient $DQf$ may now defined on $\mathcal{T} = T \cup D$ by

$$DQf(a, a) = f'(a).$$

With this definition in mind, we obtain the following test, which we call the Integrable Criteria and prove in Section 3.

**Theorem 4** (Integrable Criteria). Let $H : T \to \mathbb{R}$ be given, such that the function $h(x) = H(x, x)$ is (Lebesgue) integrable on $[0, 1]$. Then the following are equivalent:

1. There exists some $f : [0, 1] \to \mathbb{R}$ such that $DQf(a, b) = H(a, b)$.
2. For all $0 \leq a < b \leq 1$,

$$(b - a)H(a, b) = \int_a^b H(s, s)ds.$$

As with the Algebraic Criteria, the proof of the Integrable Criteria consists of an explicit construction of the functions $f$ for which $H = DQf$. This construction is recorded as the following corollary.

**Corollary 5.** For $H(a, b)$ satisfying Item 2 of Theorem 4 the functions $f(x)$ which make $H = DQf$ are all functions of the form

$$f(x) = \int_0^x H(s, s)ds + C$$

for a constant $C$. 

3
In Section 3 we will also observe that, for a differentiable function \( f \), the sum of the first partial derivatives of the difference quotient of \( f \) is the difference quotient of \( f' \).

**Theorem 6.** Let \( f \) be differentiable on \([0, 1]\). Then

\[
\left( \frac{\partial}{\partial a} + \frac{\partial}{\partial b} \right) \text{DQ}_f(a, b) = \text{DQ}_{f'}(a, b).
\]

Our final test, which we call the **Summation Criteria** and will prove in Section 4, is designed to detect the difference quotient of an analytic function \( f \). The key observation is that the difference quotient of the function \( f(x) = x^n \) is exactly

\[
\text{DQ}_f(a, b) = \sum_{i=0}^{n-1} a^i b^{n-1-i}.
\]

**Theorem 7** (Summation Criteria). Let \( H : \mathcal{T} \to \mathbb{R} \) be an analytic function \( H(a, b) = \sum_{i,j \geq 0} c_{ij} a^i b^j \) on \( \mathcal{T} \). The following are equivalent:

1. There exists some \( f : [0, 1] \to \mathbb{R} \) such that \( \text{DQ}_f(a, b) = H(a, b) \).
2. For each \( p \in \mathbb{N} \), there exists a constant \( c_p \) such that for all \( i, j \geq 0 \) with \( i + j = p \),

\[
c_{ij} = c_p.
\]

As with the earlier criteria, for functions \( H \) that satisfy the items of Theorem 7 we obtain an explicit construction of the functions \( f \) that will make \( H = \text{DQ}_f \).

**Corollary 8.** For \( H(a, b) \) satisfying Item 2 of Theorem 7 the functions \( f(x) \) which make \( H = \text{DQ}_f \) are all functions of the form

\[
f(x) = \left( \sum_{p=0}^{\infty} c_p x^{p+1} \right) + C
\]

for a constant \( C \).

### 2 Algebraic Criteria

The Algebraic Criteria is the most general of the tests we will develop in this paper. It is designed to detect the difference quotient of an arbitrary function \( f : [0, 1] \to \mathbb{R} \), and thus, the function \( H \) being tested need only be defined on the upper triangle \( \mathcal{T} \).

For such a function \( H \), we will show that the following items are equivalent.

**Algebraic Criteria**

1. There exists some \( f : [0, 1] \to \mathbb{R} \) such that \( \text{DQ}_f(a, b) = H(a, b) \).
2. For all \( 0 \leq a < b < c \leq 1 \),

\[
H(a, c) = \frac{H(a, b)(b - a) + H(b, c)(c - b)}{c - a}.
\]
3. For all $0 < b < c \leq 1$,

$$H(0, c) = \frac{H(0, b)(b) + H(b, c)(c - b)}{c}.$$ 

The fact that Item 1 implies Item 2 follows directly from the definition of the difference quotient, and of course Item 2 implies Item 3 after setting $a = 0$. Therefore, we proceed to a more interesting fact that Item 3 implies Item 1.

For a fixed constant $C \in \mathbb{R}$, we define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 
C & x = 0 \\
 xH(0, x) + C & x \neq 0.
\end{cases}$$

With this choice of $f$, we will show that Item 3 implies $H = \text{DQ}_f$. Note that it is easy to show that two real valued functions have the same difference quotient if and only if they differ by a constant. Therefore showing that $H = \text{DQ}_f$ will immediately establish Corollary 2 as well.

**Proof.** Assume that Item 3 of the Algebraic Criteria holds, and let $0 \leq i < j \leq 1$ be given. We wish to show that, for the function $f$ defined above, $\text{DQ}_f(i, j) = H(i, j)$.

We have two values of $i$ to check:

Case $i = 0$:

$$\text{DQ}_f(0, j) = \frac{f(j) - f(0)}{j - 0} = \frac{jH(0, j) + C - C}{j} = H(0, j).$$

Case $0 < i$: Note that the condition present in Item 3 of the Algebraic Criteria may be rearranged to give that, for all $0 < b < c \leq 1$,

$$H(b, c) = \frac{cH(0, c) - bH(0, b)}{c - b}.$$ 

We therefore have that

$$\text{DQ}_f(i, j) = \frac{f(j) - f(i)}{j - i} = \frac{(jH(0, j) + C) - (iH(0, i) + C)}{j - i} = \frac{jH(0, j) - iH(0, i)}{j - i} = H(i, j).$$

This establishes the desired result.

**2.1 Application to Linear Algebra**

Continuing with our analysis of an arbitrary function $H$ defined on $T$, for any $0 \leq a < b < c \leq 1$, we define the matrix

$$M = \begin{bmatrix} 
H(b, c) & H(a, c) & H(a, b) \\
 a & b & c \\
 1 & 1 & 1
\end{bmatrix}.$$ 

We will now show the equivalence of the following items from Corollary 3.
1. There exists some $f : [0, 1] \to \mathbb{R}$ such that $DQ_f(a, b) = H(a, b)$.
2. For all $0 \leq a < b < c \leq 0$, $\det(M) = 0$.
3. For all $0 \leq a < b < c \leq 0$, $\dim(M) = 2$.
4. For all $0 \leq a < b < c \leq 0$, $\text{Null}(M) = \left\{ k \begin{bmatrix} \frac{b-c}{c-a} \\ \frac{a-b}{c-a} \\ 1 \end{bmatrix} : k \in \mathbb{R} \right\}$.

It may easily be shown that the condition in Item 2 of the Algebraic Criteria may be rewritten as the condition $\det(M) = 0$. Therefore, the Algebraic Criteria gives us the equivalence of Items 1 and 2 from Corollary 3. Moreover, basic linear algebra shows that, among the items of Corollary 3, Item 4 implies Item 3, which in turn implies Item 2. Thus, we have the following implications.

$$1 \iff 2 \iff 3 \iff 4$$

We will establish the remaining implications by showing that Item 1 of Corollary 3 implies Item 4 of Corollary 3.

**Proof.** We begin by assuming that Item 1 holds. We therefore replace $H$ with $DQ_f$ in the matrix $M$ and seek the solution set for the following matrix equation.

$$\begin{bmatrix} \frac{f(c)-f(b)}{c-b} & \frac{f(c)-f(a)}{c-a} & \frac{f(b)-f(a)}{b-a} \\ a & b & c \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{0}$$

Pulling this matrix equation apart componentwise, we obtain the following system of scalar equations.

$$\begin{cases} x \left( \frac{f(x)-f(b)}{c-b} \right) + y \left( \frac{f(x)-f(a)}{c-a} \right) + z \left( \frac{f(b)-f(a)}{b-a} \right) = 0 \\ xa + yb + zc = 0 \\ x + y + z = 0 \end{cases}$$

Applying standard algebraic techniques to the latter two equations, we may solve for each of $x$ and $z$ in terms of $y$, obtaining the following.

$$z = y \left( \frac{b-a}{a-c} \right)$$
$$x = y \left( \frac{a-b}{a-c} \right)$$

Thus, $\text{Null}(M)$ is contained in the set

$$\left\{ k \begin{bmatrix} \frac{b-c}{c-a} \\ \frac{a-b}{c-a} \\ 1 \end{bmatrix} : k \in \mathbb{R} \right\}.$$

However, it is easy to check that any vector in this set is also in $\text{Null}(M)$. This completes our proof.  

$\square$
2.2 Example

For $0 \leq a < b \leq 1$, define

$$H(a, b) = \begin{cases} 0 & a, b \in \mathbb{Q} \\ 0 & a, b \notin \mathbb{Q} \\ \frac{1}{b-a} & a \notin \mathbb{Q}, b \in \mathbb{Q} \\ -\frac{1}{b-a} & a \in \mathbb{Q}, b \notin \mathbb{Q} \\ \end{cases}$$

We will verify that this function $H$ satisfies the third item of the Algebraic Criteria, and is therefore the difference quotient of some function $f$. We will then use Corollary 2 to find the function $f$.

Let $0 < b < c \leq 1$ be given. We have four cases to check.

Case $b, c \in \mathbb{Q}$:

$$\frac{H(0, b)(b) + H(b, c)(c-b)}{c} = \frac{0 \cdot b + 0 \cdot (c-b)}{c} = 0 = H(0, c).$$

Case $b, c \notin \mathbb{Q}$:

$$\frac{H(0, b)(b) + H(b, c)(c-b)}{c} = \frac{1}{b-0} \cdot b + 0 \cdot (c-b) \cdot c = \frac{1}{c-0} = H(0, c).$$

Case $b \notin \mathbb{Q}, c \in \mathbb{Q}$:

$$\frac{H(0, b)(b) + H(b, c)(c-b)}{c} = \frac{1}{b-0} \cdot b + \frac{-1}{c-b} \cdot (c-b) \cdot c = 0 = H(0, c).$$

Case $b \in \mathbb{Q}, c \notin \mathbb{Q}$:

$$\frac{H(0, b)(b) + H(b, c)(c-b)}{c} = \frac{0 \cdot b + \frac{-1}{c-b} \cdot (c-b) \cdot c}{c} = \frac{-1}{c-0} = H(0, c).$$

Since $H$ satisfies the Algebraic Criteria, we know that $H$ is the difference quotient of a function $f : [0, 1] \rightarrow \mathbb{R}$. We may apply Corollary 2, choosing $C = 1$, to find one such $f$. That is, $H = \text{DQ}_f$, where

$$f(x) = \begin{cases} 1 & x = 0 \\ xH(0, x) + 1 & x \neq 0 \end{cases}$$

Thus, we have $f(0) = 1$, and if $x \neq 0$ then, by the definition of $H$, we have

$$f(x) = \begin{cases} x \cdot 0 + 1 & x \in \mathbb{Q} \\ x \cdot \frac{-1}{x} + 1 & x \notin \mathbb{Q} \end{cases}.$$
Combining cases and simplifying, we see that $H$ is the difference quotient of the Dirichlet function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}.$$ 

### 3 Integrable Criteria

In this section we extend the definition of the difference quotient of $f$ to the diagonal $D = \{(a,a) : 0 \leq a \leq 1\}$ by $\text{DQ}_f(a,a) = f'(a)$, provided that the derivative exists. With this definition, the fundamental theorem of calculus now suggests the Integrable Criteria, which may be used to detect whether a function $H$ defined on the closed triangle

$$T = \{(a,b) : 0 \leq a \leq b \leq 1\}$$

is the difference quotient of a differentiable function. The Integrable Criteria states that for such a function $H$, the following are equivalent.

**Integrable Criteria**

1. There exists some $f : [0,1] \to \mathbb{R}$ such that $\text{DQ}_f(a,b) = H(a,b)$ on $T$.
2. The function $h(s) = H(s,s)$ is integrable on $[0,1]$, and for all $0 \leq a < b \leq 1$,

$$ (b-a)H(a,b) = \int_a^b H(s,s)ds. $$

The fact that Item 1 implies Item 2 follows directly from the fundamental theorem of calculus. It remains to prove the reverse implication.

**Proof.** Assume that the function $H : T \to \mathbb{R}$ satisfies the assumptions of the second item of the Integrable Criteria. Define the function $f : [0,1] \to \mathbb{R}$ by

$$ f(x) = \int_0^x H(s,s)ds. $$

Then, by the additivity of the integral, we have

$$ \text{DQ}_f(a,b) = \int_0^b H(s,s)ds - \int_0^a H(s,s)ds = \frac{1}{b-a} \int_a^b H(s,s)ds. $$

By rearranging the equation in Item 2 of the Integrable Criteria, we immediately obtain $\text{DQ}_f(a,b) = H(a,b)$. 

Since all functions having the same difference quotient vary only by a constant, we immediately obtain the conclusion found in Corollary 5.
3.1 Example

For $0 \leq a < b \leq 1$, define $H(a, b)$ to be the average value of $g(x) = e^{x^2}$ on the interval $[a, b]$. It therefore makes sense to extend this definition to the diagonal by $H(a, a) = e^{a^2}$, for $0 \leq a \leq 1$.

We check that Item 2 of the Integrable Criteria is satisfied.

Let $0 \leq a < b \leq 1$ be given. The basic calculus definition of average value of $g(x)$ on the interval $[a, b]$ is

$$\frac{1}{b - a} \int_a^b g(s)ds.$$ 

Thus we have

$$(b - a)H(a, b) = \frac{b - a}{b - a} \int_a^b e^{s^2}ds = \int_a^b H(s, s)ds.$$ 

Finally, according to Corollary 1, we have that one of the functions $f$ which makes $H = DQ_f$ is

$$f(x) = \int_0^x e^{s^2}ds.$$ 

3.2 Sum of Partial Derivatives

In this subsection we make note that the sum of the partial derivatives of $DQ_f$ is equal to $DQ_{f'}$,

$$\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial b}\right)DQ_f(a, b) = DQ_{f'}(a, b).$$

That is, if we define $D^1([0, 1])$ to be the set of all differentiable functions from $[0, 1]$ to $\mathbb{R}$, $D^1(\mathcal{T})$ to be the set of functions mapping $\mathcal{T}$ to $\mathbb{R}$ both of whose partial derivatives exist, $F([0, 1])$ to be the set of functions from $[0, 1]$ to $\mathbb{R}$, and $F(\mathcal{T})$ to be the set of functions from $\mathcal{T}$ to $\mathbb{R}$, then the following diagram commutes.

\[
\begin{array}{ccc}
D^1([0, 1]) & \xrightarrow{\frac{d}{dx}} & F([0, 1]) \\
\downarrow DQ & & \downarrow DQ \\
D^1(\mathcal{T}) & \xrightarrow{\frac{\partial}{\partial a} + \frac{\partial}{\partial b}} & F(\mathcal{T})
\end{array}
\]

Proof. Using the quotient rule, we observe that

$$\frac{\partial}{\partial a} \frac{f(b) - f(a)}{b - a} = -f'(a)(b - a) - (-1)(f(b) - f(a))$$

and

$$\frac{1}{(b - a)^2}$$
Thus a bit of arithmetic immediately gives
\[
\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial b}\right) \text{DQ}_f(a, b) = \frac{f'(b) - f'(a)}{b - a}.
\]

\[\square\]

4 Summation Criteria

Our final test is meant to detect the difference quotient of an analytic function. It comes from the basic observation that the difference quotient of the function \( f(x) = x^n \) is

\[
\text{DQ}_f(a, b) = \frac{b^n - a^n}{b - a} = \sum_{i=0}^{n-1} a^i b^{n-1-i}.
\]

We see that this is the sum of all terms of the form \( a^i b^j \) such that \( i, j \geq 0 \) and \( i + j = n - 1 \). This observation gives rise to the Summation Criteria, which states for power series

\[
H(a, b) = \sum_{i,j \geq 0} c_{ij} a^i b^j
\]

converging absolutely on the closed triangle \( \overline{T} \), that the following are equivalent.

Summation Criteria

1. There exists some \( f : [0, 1] \to \mathbb{R} \) such that \( \text{DQ}_f(a, b) = H(a, b) \) on \( \overline{T} \).

2. For each \( p \in \mathbb{N} \), there exists a constant \( c_p \) such that for all \( i, j \geq 0 \) with \( i + j = p \),

\[ c_{ij} = c_p. \]

The fact that Item 1 implies Item 2 follows immediately from the observation above regarding the difference quotient of a power of \( x \). It therefore remains to show that Item 2 implies Item 1.

**Proof.** Assume that Item 2 of the Summation Criteria holds and define \( f : [0, 1] \to \mathbb{R} \) by

\[
f(x) = \left(\sum_{p=0}^{\infty} c_p x^{p+1}\right).
\]

Then

\[
\text{DQ}_f(a, b) = \sum_{p=0}^{\infty} c_p b^{p+1} - \sum_{p=0}^{\infty} c_p a^{p+1} = \sum_{p=0}^{\infty} c_p \frac{b^{p+1} - a^{p+1}}{b - a} = \sum_{p=0}^{\infty} \sum_{i=0}^{p} c_p a^i b^{p-i}.
\]

Observe that for each \( p \geq 1 \), and for each \( 0 \leq i \leq p \), setting \( j = p - i \) we have \( c_{ij} = c_p \). Therefore, the latter sum may be rewritten
$$DQ_f(a, b) = \sum_{p=0}^{\infty} \left( \sum_{i,j \geq 0} c_{ij} a^i b^j \right).$$

This is indeed a reordering of the power series

$$H(a, b) = \sum_{i,j \geq 0} c_{ij} a^i b^j.$$

Since $H$ was assumed to be absolutely convergent on $T$, reorderings of $H$ are equal to $H$, so we have

$$DQ_f(a, b) = H(a, b).$$

\[ \Box \]

### 4.1 Example

For $0 \leq a \leq b \leq 1$, define

$$H(a, b) = \sum_{i,j \geq 0} \frac{1}{(i+j)!} a^i b^j.$$

Before applying the Summation Criteria to $H$, we must verify that $H$ converges absolutely on the closed triangle $T$. Since each $c_{ij}$ is positive and $0 \leq a \leq b \leq 1$, we have

$$\sum_{i,j \geq 0} \left| \frac{1}{(i+j)!} a^i b^j \right| < \sum_{i,j \geq 0} \frac{1}{(i+j)!}.$$

We regroup the summation as follows.

$$\sum_{i,j \geq 0} \frac{1}{(i+j)!} = \sum_{p=0}^{\infty} \sum_{i+j=p} \frac{1}{(i+j)!}.$$

Looking at the inner summation, we observe that

$$\sum_{i+j=p} \frac{1}{(i+j)!} = \sum_{i=0}^{p} \frac{1}{p!} = (p+1) \frac{1}{p!}.$$

Thus, we rewrite

$$\sum_{p=0}^{\infty} \sum_{i,j \geq 0} \frac{1}{(i+j)!} = \sum_{p=0}^{\infty} \frac{p+1}{p!}.$$
Finally, the ratio test guarantees the convergence of \( \sum_{p=0}^{\infty} \frac{p+1}{p!} \), so we conclude that \( H \) does converge absolutely on \( T \).

We observe that, by setting \( c_p = \frac{1}{p!} \) for all \( p \geq 0 \), if \( i, j \geq 0 \) with \( i + j = p \), then \( c_{ij} = c_p \), so that the second item of the Summation Criteria is satisfied. Using Corollary 8, we find one of the functions \( f(x) \) for which \( H(a, b) = \text{DQ}_f(a, b) \) is

\[
f(x) = \sum_{p=0}^{\infty} \frac{1}{p!} x^{p+1} = x \sum_{p=0}^{\infty} \frac{1}{p!} x^p.
\]

We recognize the series above as the expansion for \( f(x) = xe^x \).