NEW MATRIX MODEL SOLUTIONS TO THE KAC-SCHWARZ PROBLEM

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ABSTRACT

We examine the Kac-Schwarz problem of specification of point in Grassmannian in the restricted case of gap-one first-order differential Kac-Schwarz operators. While the pair of constraints satisfying $[\kappa_1, W] = 1$ always leads to Kontsevich type models, in the case of $[\kappa_1, W] = W$ the corresponding KP $\tau$-functions are represented as more sophisticated matrix integrals.
1 Introduction

In the framework of string theory one is interested, among other things, in construction of the non-perturbative partition function of a string model. By definition this “non-perturbative partition function” \( \tau\{t\} \) is a generating functional for all the correlators in the given model and as such, it depends on some set of parameters \( \{t\} \), in which it can be expanded as a formal series. It also depends on particular model, i.e. on some other set of parameters \( \{M\} \), parametrizing the “module space of models”. Since “exponentiation of perturbations” - implicit in the concept of generating functional - effectively changes the action, i.e. parameters of the model, there are all reasons to believe that in fact the nature of both types of parameters is almost the same, and “universal partition function” \( \tau\{t|M\} \) essentially depends on some specific “combination” \( t \circ M \) of variables \( t \) and \( M \):

\[
\tau\{t|M\} \sim \tau\{t \circ M\}. \tag{1}
\]

So far little is known about \( \tau\{t|M\} \) in such a general situation \cite{1}. Particular example, studied already in some detail, is provided by the theory of Generalized Kontsevich Model (GKM), i.e. the family of matrix models, closely related to that of the simplest Landau-Ginzburg topological gravity. This set of models is parametrized by a function \( W(x) \) and universality classes are labeled by types of singularities of \( W(x) \). In the most popular case \( W(x) = W_p(x) \). Then the variables \( t \) form a sequence (i.e. a discrete one-parametric set, with discreteness reflecting the absense of particle-like degrees of freedom or “topological nature” of the models), which in the matrix model formulation are symmetric functions of eigenvalues of some matrix \( \Lambda \):

\[
t_k = \frac{1}{k} \text{tr} \Lambda^{-k}, \quad k = 1, 2, \ldots
\]

Partition function is essentially given by the matrix integral

\[
\tau(\Lambda|W) \sim \int dX \exp \left( -\text{tr} \int X W(x) dx + \text{tr} X W'(\Lambda) \right), \tag{2}
\]

and in this particular case \cite{3} can be stated explicitly \cite{3}:

\[
\tau(\Lambda|W) \sim \tau_p \left( t_k + \frac{p}{k(p-k)} \text{res} W^{1-\frac{1}{p}}(x) dx \right) \tag{3}
\]

The interesting property of \( \tau_p \) is that it is usually a KP \( \tau \)-function with \( t \)'s being just KP time-variables. This observation raises a lot of questions, which can be studied in the search for adequate generalizations of GKM and can help to make further steps towards exact formulation and proof of the fundamental property (1).

One of ideas, implied by the study of Kontsevich model, is to look for the independent characterization of the object \( \tau_p \), given that it is a KP \( \tau \)-function. Usually KP \( \tau \)-functions are considered as depending on two sets of variables:
the “times” \( t_k \) and the “point of the Sato Universal Grassmannian” \( g \) \(^1\). In particular our \( \tau_p(t) = \tau^{(KP)}(t|g_p) \), and the question is what is the way to characterize the point \( g_p \in \mathcal{GR} \) without explicit reference to the matrix integral \(^2\). Usually the point of \( \mathcal{GR} \), as of any homogeneous manifold, is characterized by its stability subgroup in the group \( GL(\infty) \) of symmetries of \( \mathcal{GR} \). This is the way, leading to the theory of “Virasoro and \( W \)-like constraints”. Alternative approach was explicitly formulated by V.Kac and A.Schwarz \(^3\). They suggested to associate the point \( g_p \) with the intersection of invariant spaces of some set of operators, acting on a linear bundle over \( \mathcal{GR} \), and proved that in particular case of GKM (at least for \( p = 2 \)) just two operators are enough to fix \( g_p \) unambiguously. This formulation is inspired by the theory of “reduced hierarchies” and “string equations”, as well as by older considerations of integrable hierarchies in terms of pseudodifferential operators. It is an appealing approach, because it allows to study the problem in much more generality, asking what happens for arbitrary choice of operators etc.

Unfortunately, this seems to remain an almost untouched field, at least we are not aware of exhaustive discussion even of the following basic problems:

(a) What is the way to find some set of operators, associated with any given point \( g \in \mathcal{GR} \) and what is the way to characterize the ambiguity of the set?

(b) What is the adequate basis in the space of all operators, acting on Grassmannian - adequate for this kind of problems?

(c) What characterizes the minimal set (at least the number of operators), needed to define the specific point \( g \in \mathcal{GR} \)?

(d) What - in full generality, i.e. for any \( g \in \mathcal{GR} \) - is the relation between the Kac-Schwarz problem of intersection of invariant subspaces and that, characterizing \( g \) as a stable point of some subalgebra of symmetries (\( GL(\infty) \)) of \( \mathcal{GR} \) (i.e. in terms of Virasoro or \( W \)-like constraints)? What (if any) is the group-theoretical interpretation of the relevant invariant subspaces (representations)?

(c) When does a matrix-integral representation of \( \tau(t|g) \) exist (for which, if not any, \( g \)), and how is the number of matrix integrations (at least) related to the set of the corresponding Kac-Schwarz operators? \(^4\)

In these notes we will not achieve much progress in discussion of these problems. Our goal is modest: to show that the questions, including (e) are, perhaps, not senseless. The need for such demonstration does exist, since one could (and still can) simply assume, that existence of matrix integral representation for

\(^1\)This question is motivated by the belief that KP \( \tau \)-functions are usually associated with string models with no gauge fields (“\( c \leq 1 \)” set is the example). In the language of matrix models this implies the possibility to “decouple” in one or another way the integration over angular (unitary) matrices, thus leaving only that over eigenvalues. If inversed, the hypothesis would be that any KP \( \tau \)-function - which can be always formulated (after a kind of Fourier transform) as an “eigenvalue model” - can be actually lifted to some matrix integral where angular matrix integration can be performed exactly.
\( \tau(g) \) is an exclusive property of specific points \( g = g_p \). This suspicion could be partly supported by the failure (so far) to find such representation for the only generalization of \( g_p \)'s, which was ever discussed: for points \( g_{p,q} \) (associated - in some peculiar sense - with \( (p,q) \) rather than \( (p,1) \) minimal conformal models). Here it is known only that the analogue of the integral (2) defines the duality transformation \( \tau_{p,q} \to \tau_{q,p} \) \(^2\), but no matrix integral was so far discovered to represent \( \tau_{p,q} \) itself for \( q > 1 \). Perhaps, however, the failure is only due to the small number of solvable generalizations of GKM, which were studied so far, and further work in this direction can bring a solution. Parameters \( p,q \) of the \( (p,q) \)-models have an interpretation as "gap sizes" in terms of the Kac-Schwarz operator \(^3\) (see next Section). Before addressing the problems of \( q > 1 \) it can be reasonable to pay more attention to the deformations of Kac-Schwarz operators which preserve the unit gap size, \( q = 1 \).

In this paper we analyze generic gap-one Kac-Schwarz operator and show that the corresponding \( \tau(g) \) can indeed be lifted to some matrix integral, which is non-trivial multi-matrix generalization of GKM. We actually derive this integral representation only in the simplest case (for some special choice of parameters in the Kac-Schwarz problem), but it seems to exist at least in generic gap-one Situation. This successful experiment can probably encourage further investigation of the whole set of above-mentioned problems.

\section*{2 Kac-Schwarz operators}

\subsection*{2.1 KP \( \tau \)-function in Miwa coordinates}

KP \( \tau \)-function is most conveniently defined as a generating function of all the correlators in the system of free fermions \(^3\):

\begin{equation}
\tau^{(KP)}(t|g) = \langle 0 | e^{H} g | 0 \rangle ,
\end{equation}

where

\begin{align*}
H &= \sum_{k>0} t_k J_k, \quad g = \exp \sum_{m,n} \mathcal{G}_{mn} \psi_m \tilde{\psi}_n, \\
J(z) &= \sum_{k=-\infty}^{+\infty} J_k z^{-k-1} = \psi(z) \tilde{\psi}(z) : \\
\psi(z) &= \sum_{k=-\infty}^{+\infty} \psi_k z^k, \quad \tilde{\psi}(z) = \sum_{k=-\infty}^{+\infty} \tilde{\psi}_k z^{-k-1},
\end{align*}

\(^2\) \( \tau_{p,q} \neq \tau_{q,p} \)! The reference to minimal conformal models could erroneously suggest that they are equal, but the symmetry between \( p \) and \( q \) is broken by coupling to 2d gravity, for example \( \tau_{1,p} = 1 \) while \( \tau_{p,1} = \tau_p \), defined in \(^3\).

\(^3\) In the language of pseudodifferential operators the Kac-Schwarz operators are represented as polynomials in \( \frac{\partial}{\partial t_1} \). Then \( p \) and \( q \) are just the orders of these polynomials. See \(^3\) for more details and references.
\[
\{ \psi(z), \tilde{\psi}(z') \} = \delta(z - z'), \quad [J(z), J(z')] = \delta'(z - z'), \\
\langle 0 | \psi_k = 0 \text{ for } k \geq 0, \quad \langle 0 | \tilde{\psi}_k = 0 \text{ for } k < 0.
\]

The one-parameter discrete sequence of \( t \)-variables is in fact enough to generate any correlator of fermions, provided “big”, not only infinitesimal variations of \( t \) are allowed\(^4\). The basic formula is:

\[
\Psi(\lambda, \mu) \equiv \frac{\langle \psi(\lambda) \tilde{\psi}(\mu)e^{Hg} \rangle}{\langle e^{Hg} \rangle} = \frac{X(\lambda, \mu)\tau(t|g)}{\tau(t|g)},
\]

where “vertex operator” \( X(\lambda, \mu) \) performs the Backlund-Miwa \( GL(\infty) \) transformation of \( \tau \)-function:

\[
X(\lambda, \mu) = \prod_{n=1}^{\infty} e^{\eta(\lambda_n)} - \eta(\mu_n) = \prod_{n=1}^{\infty} \left( e^{V(\lambda_n)} - e^{-\eta(\lambda_n)} \right) \left( e^{-V(\mu_n)} - e^{-\eta(\mu_n)} \right),
\]

\[
V(\lambda) = \sum_{k>0} t_k \lambda^k, \quad \eta(\lambda) = \sum_{k>0} \frac{\lambda^k}{k} \frac{\partial}{\partial t_k}.
\]

Operator \( e^{n(\lambda_\alpha, \mu_\alpha)} \) shifts the time variables according to the rule

\[
t_k \rightarrow t_k \{ \lambda_\alpha, \mu_\alpha \} = t_k^{(0)} + \frac{1}{n} \sum_{\alpha=1}^{n} (\lambda_\alpha - \mu_\alpha)^{-k}.
\]

For \( e^{n(\lambda, \mu)} \), entering the definition of \( X(\lambda, \mu) \) \( n = 1 \). However, the same transformation with \( n > 1 \), implying insertion of \( n \) pairs of \( \psi \) and \( \tilde{\psi} \) operators under the average-sign in \[\text{(4)}, \] can be reduced to the \( n = 1 \) case with the help of Wick theorem for Gaussian functional integrals (also called Fay’s identity in the theory of \( \tau \)-functions):

\[
\Psi(\{ \lambda_\alpha \}, \{ \mu_\alpha \}) \equiv \Delta(\lambda)\Delta(\mu) \prod_{\alpha=1}^{n} X(\lambda_\alpha, \mu_\alpha)\tau(t|g) = \det_{1 \leq \alpha, \beta \leq n} \Psi(\lambda_\alpha, \mu_\beta),
\]

where

\[
\Delta(\lambda) = \prod_{\alpha > \beta} (\lambda_\alpha - \lambda_\beta) = \det_{1 \leq \alpha, \beta \leq n} \lambda_\alpha^{\beta-1}.
\]

One can further consider taking \( n = \infty \) in this formula. Then one can think that all the information about time-variables is encoded in \( \lambda \)-variables, while

\[\text{(4)} \quad \text{This statement reflects nothing but the fact that universal enveloping of the Kac-Moody algebra} \hat{G}_k \text{ coincides with that of its Heisenberg-Cartan subalgebra, provided} G \text{ is simply laced and} k = 1 \text{. In the case of KP theory} G \text{ is just} U(1).\]

\[\text{(4)} \quad \text{The one-parameter discrete sequence of} \ t \text{-variables is in fact enough to generate any correlator of fermions, provided “big”, not only infinitesimal variations of} \ t \text{ are allowed.}\]

\[\text{(4)} \quad \text{The basic formula is:}\]

\[\text{(4)} \quad \text{where “vertex operator”} X(\lambda, \mu) \text{ performs the Backlund-Miwa} GL(\infty) \text{ transformation of} \ \tau \text{-function:}\]

\[\text{(4)} \quad \text{Operator} e^{n(\lambda_\alpha, \mu_\alpha)} = \prod_{\alpha=1}^{n} e^{\eta(\lambda_\alpha)} - \eta(\mu_\alpha) \text{ shifts the time variables according to the rule}\]

\[\text{(4)} \quad \text{For} e^{n(\lambda, \mu)} \text{, entering the definition of} X(\lambda, \mu) \text{ \( n = 1 \). However, the same transformation with} n > 1 \text{, implying insertion of} n \text{ pairs of} \psi \text{ and} \tilde{\psi} \text{ operators under the average-sign in} \ [\text{(4)}, \text{ can be reduced to the} n = 1 \text{ case with the help of Wick theorem for Gaussian functional integrals (also called Fay’s identity in the theory of} \ \tau \text{-functions):}\]

\[\text{(4)} \quad \text{One can further consider taking} n = \infty \text{ in this formula. Then one can think that all the information about time-variables is encoded in} \lambda \text{-variables, while}\]

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In (8) can be, if necessary, absorbed into $g$. In fact, once $\lambda$’s are allowed to be complex, the double set $\{\lambda, \mu\}$ is unnecessarily large for the purpose of fully parametrizing the $t$-space, and one can eliminate $\mu$’s by putting them all equal: $\mu_\alpha = \mu_*$. Usually one chooses $\mu_* = \infty$. Then eq.(10) can be regarded as a formula for $\tau$-function itself, expressed through Miwa coordinates:

$$\tau\{\Lambda|g\} = \frac{\det_{1\leq \alpha, \beta \leq N} \Psi_\alpha(\lambda_\beta)}{\Delta(\lambda)}$$

where

$$\Psi(\lambda, \mu) = \sum_{\alpha \geq 1} \mu^{-\alpha} \Psi_\alpha(\lambda).$$

Since $\Psi(\lambda, \mu) = \frac{1}{x^{\lambda}}(1 + o(\lambda - \mu))$, $\Psi_\alpha(\lambda) = \lambda^{\alpha-1}(1 + o(\lambda^{-1})).$

The set of functions in eq.(11) is not fixed unambiguously for given $g$: any linear triangular transformations

$$\Psi_\alpha(\lambda) \to \Psi_\alpha(\lambda) + \sum_{1 \leq \beta < \alpha} A_{\alpha\beta} \Psi_\beta(\lambda)$$

does not change the determinant in (11), and thus leaves $\tau$-function the same. This freedom can be used to bring all the functions to “canonical” form

$$\Psi_\alpha^{(can)}(\lambda) = \lambda^{\alpha-1} + \sum_{\beta \geq 1} S_{\alpha\beta} \lambda^{-\beta} = \lambda^{\alpha-1} + o(\lambda^{-1}),$$

but we are not going to use this “gauge-fixing” below. $\Psi_\alpha$’s can be considered as defining the coordinates in a bundle $\mathcal{B}$ over “infinite-dimensional Universal Grassmannian” $\mathcal{GR}$. The point of Grassmannian $g$ is associated with the class of equivalency

$$\mathcal{W}_g = \{\Psi_1(\lambda), \Psi_2(\lambda), \ldots\}$$

defined modulo transformations (14).

3 Formulation of the Kac-Schwarz problem

Idea of the Kac-Schwarz approach is to consider linear operators acting on $\mathcal{B}$ and look on their invariant subspaces. An example of such operator is the “$p$-reduction” constraint:

$$\lambda^p \mathcal{W}_g \subset \mathcal{W}_g.$$\footnote{For relation between Matrices $S_{\alpha\beta}$ in (15) and $G_{mn}$ in (1) see [4].}
which is known to specify the subset of $\mathcal{GR}$, on which the KP $\tau$-functions are (almost) independent on all the time variables $t_{pk}$. This constraint has a natural generalization when $\lambda^p$ is substituted by any function $W_p(\lambda)$ of power $p$, i.e. $W_p(\lambda) = \lambda^p (1 + o(\lambda^{-1}))$ (it actually does not need to be a polynomial). Then appropriate change of time-variables brings such constraint into the standard $p$-reduction one \[2\]. The variables on which $\tau$-function is (almost) independent are then given by $t_{pk}^{(W)} = \frac{1}{k} \text{tr}(W_p(\lambda))^{-k}$. The $W_p$-reduction constraint implies that

$$W_p(\lambda) \mathcal{W}_g \subset \mathcal{W}_g : \quad W_p(\lambda) \Psi_\alpha(\lambda) = \Psi_{\alpha+p} + \sum_{1 \leq \beta < \alpha+p} \Omega_{\alpha\beta} \Psi_\beta(\lambda)$$

and it provides an example of a “gap-$p$” operator, for which $O\mathcal{W}_g$ is a subspace of codimension $p$ in $\mathcal{W}_g$. Coefficients $\Omega$ are at this moment defined up to conjugation by transformations \[14\].

This kind of constraints, however, is not enough to restrict the point $g$ strongly enough: there remains still a large freedom (there are plenty of KdV ($p = 2$) $\tau$-functions, for example). The idea is then to impose more constraints, associated with some other operators, and fix $g$ more strictly as the point associated with intersection of all the invariant subspaces in $\mathcal{B}$. Therefore it is a natural problem to consider after \[18\] the system of two constraints:

$$W_p(\lambda) \mathcal{W}_g \subset \mathcal{W}_g, \quad K\mathcal{W}_g \subset \mathcal{W}_g$$

Let us assume further that $K$ is a gap-one operator. This is a very restrictive condition. It implies that the coordinates of the point $g$ in Grassmannian can be represented by

$$\mathcal{W}_g = \{ \Psi_1(\lambda), K_1(\lambda) \Psi_1(\lambda), \ldots \}, \quad \text{i.e.} \quad \Psi_\alpha(\lambda) = K_1^{\alpha-1} \Psi_1(\lambda).$$

This condition already fixes the “gauge freedom” of linear triangular transformations \[14\]. Let us agree that the other constraint is written in the form \[18\] in exactly this basis, thus eliminating the ambiguity in the choice of coefficients $\Omega$. Then we get the basic linear equation on $\Psi_1(\lambda)$, arising from \[18\] for $\alpha = 1$:

$$\left( P_p(K_1) - W_p(\lambda) \right) \Psi_1(\lambda) =$$

$$= \left( K_1^p + \sum_{1 \leq \beta < \alpha+p} \Omega_{1\beta} K_1^{\beta} - W_p(\lambda) \right) \Psi_1(\lambda) = 0,$$

while the rest of the equations \[18\] is essentially a consistency condition for the system \[19\], determining the coefficients $\Omega_{\alpha\beta}$ for all $\alpha > 1$. $\tau$-function in this
case is given by
\[
\tau\{\Lambda|g\} = \frac{\det_{\alpha\beta} K_{1}^{\alpha-1} \Psi_{1}(\lambda_{\beta})}{\Delta(\lambda)},
\]
(22)

So far we did not ask anything from \(K_{1}\) except for being a gap-one operator. Under this condition it is senseless to take \(K_{1}\) to be a function: then (21) will be either true identically (for the adequate rigid choice of \(\Omega_{1,\beta}\)), giving a \(\tau\)-function of the factorized form,
\[
\tau\{\Lambda|g\} = \prod_{\alpha=1}^{\infty} \Psi_{1}(\lambda_{\alpha})
\]
(23)
for any
\[
\Psi_{1}(\lambda) = 1 + o(\lambda^{-1})
\]
(24)
or instead have only a delta-function like solution for \(\Psi_{1}\), which does not satisfy the normalization condition (24).

Generic linear operator \(O\) acting on \(B\) is non-local. One can, however consider it as belonging to the universal enveloping of an algebra, generated by functions and first-order differential operators. This can serve as excuse for selecting \(K_{1}\) to be the first-order differential operator:
\[
K_{1}(\lambda) = A(\lambda) \frac{\partial}{\partial \lambda} + B(\lambda) = h(\lambda) \left( A(\lambda) \frac{\partial}{\partial \lambda} \right) h(\lambda)^{-1},
\]
\[
h(\lambda) = \exp \left( -\int \frac{B(x)}{A(x)} dx \right).
\]
(25)

Rescaling Baker-Ahiezer function,
\[
\Psi_{1}(\lambda) = \exp \left( -\int \frac{B(x)}{A(x)} dx \right) \Phi(\lambda) = h(\lambda)\Phi(\lambda),
\]
(26)
we obtain equation (21) in the form:
\[
\left( \mathcal{P}_{p} \left( A(\lambda) \frac{\partial}{\partial \lambda} \right) - W_{p}(\lambda) \right) \Phi(\lambda) = 0
\]
(27)
with arbitrary polinomial \(\mathcal{P}_{p}\) of power \(p\).

Moreover, the consistency condition
\[
[K_{1}, W] = F(W)
\]
(28)
requires that
\[
A(\lambda) = \frac{F(W)}{W^{p}},
\]
(29)
while the unity-gap requirement allows $F$ to be either of power less than two in $W$, $F(W) = W^\sigma(1 + o(W^{-1}))$, $\sigma \leq 1$. The same unity-gap condition implies that $B(\lambda)$ is of power one. It is of course non-vanishing, this is important for (21) to be true. Finally one should pick up solutions to (27) satisfying normalization requirement (24).

Thus, if restricted to a system of two constraints, one defined by a function, another - by a gap-one first-order differential operator, the Kac-Schwarz problem actually depends only on the choice of the power-$p$ function $W_p$, polynomial $P_p$ and the power-one function $F(W)$. We shall now study this problem for two particular choices of $F(W)$: $F(W) = \text{const}$ and $F(W) = \text{const} \cdot W$. In the first case for any $W_p$ and $P_p$ the answer is represented by Generalized Kontsevich model, while in the second case it is a slightly more sophisticated model, which can be considered as peculiar average of GKM over external matrix field (compare with [10]).

4 The case of $F(W) = c^{-1}$ (the GKM)

Perform an integral transformation
\[ \Phi(\lambda) = \int dx e^{cW(\lambda)} f(x) \] (30)
and substitute it into (27):
\[ \left( P \left( c^{-1} \frac{\partial}{\partial W(\lambda)} \right) - W(\lambda) \right) \Phi(\lambda) = \int dx e^{cW(\lambda)} f(x) (P(x) - W(\lambda)) = 0. \] (31)
This equation implies that the integrand at the r.h.s. is total derivative w.r.to $x$, so that the integral vanishes for appropriate choice of integration contour. This implies in turn that
\[ f(x) = \exp \left( -c \int x P(x) dx \right). \] (32)

With our choice of integral transformation the powers of operator $K_1 = h(\lambda) \frac{\partial}{\partial W(\lambda)} h(\lambda)^{-1}$ act on the Baker-Akhiezer function $\Psi_1(\lambda) = h(\lambda) \Phi(\lambda)$ just by insertion of powers of $x$ under the integral sign in (30):
\[ K_1^\alpha \Psi_1(\lambda) \sim h(\lambda) \int dx e^{cW(\lambda)} f(x)x^\alpha. \] (33)

In some special cases, e.g. for $p = 2$, $F(W) = W^2$ is still allowed, but we do not consider such exceptional cases here. Note also that our definition of “power-$p$” function implies that the coefficient in front of the $p$-th power is unity, thus “function of power $p$” is not identically zero.
where “∼” sign means equivalence up to linear triangular transformation which leave \( \tau \)-function intact. Because of this we obtain from (22):

\[
\tau \{ \Lambda | g \} = \frac{1}{\Delta(\lambda)} \prod_{a=1}^{n} h(\lambda_a) \int dx_\alpha f(x_\alpha) e^{e^{\alpha} W(\lambda_\alpha)} \Delta(x). \tag{34}
\]

With the help of the Harish-Chandra-Itzykson-Zuber formula for unitary matrix integration,

\[
\frac{1}{\text{Vol}_U(n)} \int_{n \times n} [dU] e^{trU X U^{-1} Y} = \det_{\alpha \beta} \frac{\Delta(x) \Delta(y)}{\Delta(x_\alpha) \Delta(y_\beta)},
\]

\[
\text{Vol}_U(n) = \frac{(2\pi)^{n(n+1)/2}}{\prod_{k=1}^{n} k!}, \tag{35}
\]

and explicit expression (32) this eigenvalue integral can be rewritten as a matrix integral:

\[
\tau \{ \Lambda | g \} \sim \int_{n \times n} dX \exp \left( c \, \text{tr} \int_{\Lambda} P(x) dx + c \, \text{tr} X W(\Lambda) \right). \tag{36}
\]

Following [2] we can now change the variable \( \lambda \rightarrow \tilde{\lambda}(\lambda) \) so that

\[
W(\lambda) = P(\tilde{\lambda}). \tag{37}
\]

Since both \( P \) and \( W \) are functions of the same power \( p \) this is allowed change of \( \lambda \)-variables \( \tilde{\lambda} = \lambda(1 + o(\lambda^{-1})) \), and the normalization condition

\[
\Phi(\lambda) = 1 + o(\lambda^{-1}) \tag{38}
\]

is actually not affected.

In these new variables (36) acquires the standard form of Generalized Kontsevich model [2]:

\[
\tau \{ \Lambda | g \} \rightarrow \tau \{ \tilde{\Lambda} | g P \} \sim \int_{n \times n} dX \exp \left( c \, \text{tr} \int_{\Lambda} P(x) dx + c \, \text{tr} X P(\tilde{\Lambda}) \right). \tag{39}
\]

### 5 The case of \( F(W) = W \)

#### 5.1 Solution for \( \Phi(W) \)

In this case it is convenient to perform the change of variables \( \tilde{\lambda} = W^{1/p}(\lambda) \) at intermediate stages of calculation. After this change the main equation (27)
turns into:

\[
\left( P_p \left( c^{-1} \frac{\partial}{\partial \lambda} \right) - \hat{\lambda}^p \right) \Phi(\lambda) = 0 \tag{40}
\]

Let us perform integral transformation

\[
\Phi(\hat{\lambda}) = \hat{\lambda}^b \int_C df(u) e^{c\hat{\lambda}u}. \tag{41}
\]

Note that it is different from the one used in Kontsevich case in the previous section, since \( \hat{\lambda} \) appears in the exponent instead. This is adequate to the new form of the differential operator involved. Substitution of (41) into (40) gives

\[
\int df(u) e^{c\hat{\lambda}u} \left( \tilde{P}_p(\hat{\lambda}u) - \hat{\lambda}^p \right) = 0, \tag{42}
\]

where \( \tilde{P}_p(y) \) is some power-\( p \) polynomial, built from \( P_p(x) \). It is actually simpler to say what is \( P(x) \) for a given \( \tilde{P}(y) \):

\[
\text{if } \tilde{P}(y) = y^k \text{ then }
\]

\[
P^{(k)}(x) = c^{-k}(cx-b)(cx-b-1)\ldots(cx-b+1-k) = c^{-k} \frac{\Gamma(cx+1-b)}{\Gamma(cx+1-b-k)} \tag{43}
\]

It remains to adjust function \( f(u) \) to the polynomial \( \tilde{P}_p(x) \) so that the integrand in (42) becomes full derivative.

We discuss the way to solve equation (40) in full generality in subsection 5.3 below, while now we instead consider a simple example. In this example, instead

8Direct relation is as follows. Let \( \tilde{P}^{(k)}(y) \) be associated with monomial \( P(x) = x^k \). These satisfy an obvious recurrent formula:

\[
\tilde{P}^{(k+1)}(y) = \frac{1}{c} \left( (cy+b)\tilde{P}^{(k)}(y) + y \frac{\partial}{\partial y} \tilde{P}^{(k)}(y) \right)
\]

and as a corollary the generating functional

\[
\tilde{P}(cv|y) = \sum_{k \geq 0} \frac{(cv)^k}{k!} \tilde{P}^{(k)}(y) = \exp \left( bv + c(e^v - 1) y \right).
\]

9In generic theory of matrix models one often does not need to care about exact choice of integration contour for which the integral of total derivative is actually vanishing. To make the model physically sensible and even to respect the normalization conditions like (24) - necessary to have the standard interpretation of eq. (11) for \( \tau \)-functions, a rather sophisticated choice can be required, especially when exponential factors are present in the integrand. The choice of a contour is usually a separate problem to be addressed independently, see for example the next footnote.
of adjusting $f(u)$ to a given $\hat{\mathcal{P}}_p$ we do the opposite: choose some specifically simple $f(u)$ and consider only $\hat{\mathcal{P}}_p(x)$ - and thus $\mathcal{P}_p(x)$ - associated with it. Namely, let us take

$$f(u) = \frac{1}{(u^p - 1)^{r+1}}. \quad (44)$$

Then

$$\hat{\mathcal{P}}_p(uz) = (uz)^p - \frac{rp}{c}(uz)^{p-1} \quad (45)$$

and according to (43)

$$\mathcal{P}_p(x) = c^{-p}(cx - b)(cx - b - 1)\ldots(cx + 1 - b - p - rp) \quad (46)$$

We see that this is not the most general polynomial of degree $p$: this is because we restricted ourselves to a very special choice of function $f(u)$ in (41). This choice simplifies considerably the matrix integral representation of the $\tau$-function, to be derived in the next subsection. After this description we return to consideration of generic polynomials $\mathcal{P}_p(x)$ in (40).

It deserves saying that in order to obtain correct asymptotics of (41) as $\hat{\lambda} \to -\infty$, the contour $C$ in (41) should be chosen to encircle all the singularities at the negative part of the real line.\footnote{Actually for integer $r$ the whole integral is just equal to residue at $u = 1$, while for half-integer $r$ it is twice the integral along the real line between $u = 1$ and $u = \infty$.} Asymptotic behaviour is actually dictated by the vicinity of $u = 1$ and $\Phi(\hat{\lambda}) \sim e^{c\hat{\lambda}^b r}$. In order that $\Psi_1(\lambda)$ has correct asymptotics (i.e. tends to one as $\hat{\lambda} \to \infty$) we need this expression to be completely compensated by the coefficient $h(\lambda) = \exp\left(-c\int \frac{\hat{\lambda} B(x)dx}{x}\right)$, distinguishing $\Psi_1(\lambda)$ from $\Phi(\lambda)$. This is an extra restriction on parameter $b + r$ in (41): it is expressed through parameters of Kac-Schwarz operator $K_1$. If $B(\lambda) = \hat{\lambda} + a + o(\lambda^{-1}) = W^{1/p}(\lambda) + a + o(\lambda^{-1})$, then the requirement is: $b + r = ac$.

### 5.2 Matrix integral representation

In this section we show how solution to the equation (41) with $f(u)$ given by (44) can be lifted to a matrix integral.

The Kac-Schwarz operator $K_1$ acts on $\Phi(\lambda)$ represented as (41) by insertions of powers of $u\hat{\lambda}$:

$$K_1^\alpha \Psi_1(\lambda) \sim h(\lambda) \int du f(u) e^{c\hat{\lambda}u} (u\hat{\lambda})^\alpha, \quad (47)$$

where the “$\sim$” sign means equality modulo linear triangular transformations which do not change the point of $GR$ and the value of $\tau$-function.
change of integration variable \( v = u\hat{\lambda} \) the latter one is represented as

\[
\tau\{\Lambda|g\} = \frac{1}{\Delta(\lambda)} \left( \prod_{\beta=1}^{n} \frac{h(\lambda_{\beta})}{\lambda_{\beta}} \right) \det_{1 \leq \alpha,\beta \leq n} \left( \int dv v^{\alpha-1} f\left(\frac{v}{\lambda_{\beta}}\right) \right) = \\
= \frac{1}{\Delta(\lambda)} \frac{\lambda^{p(r+1)}}{(vp - \lambda p)^{r+1}} \int_{0}^{\infty} x^r dx e^{-x(v^p - \hat{\lambda}^p)}
\]

and obtain:

\[
\tau\{\Lambda|g\} = \frac{1}{\Delta(\lambda)} \left( \prod_{\beta} \frac{\lambda^{p(r+1) - 1} h(\lambda_{\beta})}{\Gamma(r + 1)} \int x_{\beta}^{r} dx_{\beta} e^{v_{\beta} e^{\lambda_{\beta}} (\hat{\lambda}^p - v^p)} \right) \Delta(v) = \\
= \frac{\Delta(W_p(\lambda))}{\Delta(\lambda)} \left( \prod_{\beta} \frac{\lambda^{p(r+1) - 1} h(\lambda_{\beta})}{\Gamma(r + 1)} \int x_{\beta}^{r} dx_{\beta} \int_{0}^{\infty} x_{\beta}^{r} dx_{\beta} \right) \times \\
\times \Delta^2(x) \Delta^2(v) \frac{e^{x_{\alpha} W_p(\lambda)}}{\Delta(x) \Delta(W_p(\lambda))} \frac{\det_{(\alpha,\beta)} e^{x_{\alpha} W_p(\lambda)}}{\Delta(x) \Delta(v)} \Delta(v)
\]

(we substituted \( \hat{\lambda}^p = W_p(\lambda) \)). In this formula it is already easy to recognize the integrals over angular variables, and we finally get:

\[
\tau\{\Lambda|g\} = \frac{(\det W_p(\Lambda))^{r+1}}{(\operatorname{Vol}_{\Lambda(n)})^2} \frac{\det h(W_p^{1/p}(\Lambda))}{W_p^{1/p}(\Lambda)} S_{R(p,n)}(\Lambda) \times \\
\times \int_{n \times n} dV e^{trV} S_{R(p,n)}(V) \int_{n \times n} dX (\det X)^r e^{trX(W_p(\Lambda) - V^p)}
\]

The integral over \( X \) is a peculiar version of Generalized Kontsevich model (with "zero-time" \( r \), no "potential" term in the action and integration over positive definite matrices \( X \) only). Instead the \( W_p(\lambda) \) acquires a matrix-valued but \( \lambda \)-independent shift by \(-V^p\) and GKM partition function is further averaged over \( V \) with a complicated weight. This weight includes

\[
S_{R(p,n)}(V) \equiv \frac{\Delta(v^p)}{\Delta(v)} = \\
= \prod_{\alpha > \beta} \frac{v^p_{\alpha} - v^p_{\beta}}{v_{\alpha} - v_{\beta}} = \prod_{\alpha > \beta} (v^p_{\alpha} - 1)^{v_{\alpha} - 1} v_{\beta} + \ldots + v^p_{\beta} - 1)
\]
which is already a symmetric function of $v$’s, espressible as a polynomial in variables $\text{tr} V^k = \sum_{1 \leq \alpha \leq n} v^k_\alpha$. The presence of this function makes integrand explicitly $n$-dependent. Similarly in the preintegral factor

$$S_{RW(p,n)}(V) \equiv \prod_{\alpha > \beta} W_p(\lambda_\alpha) - W_p(\lambda_\beta)$$

(53)

$S_{R(p,n)}(V)$ is actually a character of the irreducible representation of $GL(\infty)$, associated with the Young table with rows of length $l_\beta = (p-1)(n-\beta)$, $1 \leq \beta \leq n$ (a “regular ladder” table) $^{11} S_{RW(p,n)}$ for non-monomial $W_p$ is a character of reducible representation, parametrized by the coefficients of $W$.

Another peculiarity of the matrix integral is specific choice of integration contours: $X$ is required to be positive-definite matrix, while integration contours for eigenvalues of $V$ are going back and forth the negative real line (encircling all the singularities which can appear on it).

5.3 Generic polynomial

In order to discuss generic solution of (40) let us make a Laplace transform in the variable $\log(h \lambda^c)$. After the substitution of

$$\Phi(\lambda) = \int dk \phi(k) e^{h \lambda k}$$

(54)

eq (55)

Further, let $s_i$, $i = 1, \ldots, p$ be the roots of $P_p(k) = \prod_{i=1}^p (k - s_i)$. Obviously, solution to the equation (55) is

$$\phi(k) = \frac{(p/c)^c k}{\prod_{i=1}^p \Gamma\left(\frac{c}{p}(k - s_i) + 1\right)} = \frac{(p/c)^c k}{\prod_{i=1}^p \left(\sin\frac{\pi c}{p}(k - s_i)\right) \Gamma\left(\frac{c}{p}(s_i - k)\right)}.$$

(56)

11 Strange as it is, this property is not unfamiliar in the theory of matrix model: similar phenomenon occurs in the case of the very important Brezin-Gross-Witten model: see 12.

12 This is because the character of representation is given by the Weyl formula:

$$\chi_{R(t)}(V) = \frac{\det v_{\alpha+\beta+n-\beta}}{\Delta(v)} = \frac{\det v_{\alpha+\beta+1+n-\beta-1}}{\Delta(v)}.$$ 

Taking $l_{n-\beta+1} = p(\beta - 1)$, we obtain $l_\beta = (p-1)(n-\beta)$. 

13
In the case when $P_p(k)$ has the peculiar form (46) this expression can be significantly simplified to include only three Γ-functions:

$$\phi(k) = \frac{e^{ck}}{\Gamma(ck - b + 1)} = e^{ck} \frac{\sin \pi(b - ck)}{\pi} \Gamma(b - ck),$$

for $s_i = \frac{b + i - 1}{c}, \ 1 \leq i \leq p$

while for $s_i = \frac{b + i - 1}{c}, \ 1 \leq i \leq p - 1, \ s_p = \frac{b + p - 1 + pr}{c}$

$$\phi(k) = \frac{e^{ck}}{\Gamma(ck - b + 1) \Gamma\left(\frac{ck - b - p + 1}{p} + 1\right)} =$$

$$= e^{ck} \frac{\sin \pi(b - ck) \sin \pi \frac{ck - b - p + 1}{p}}{\pi^2 \pi} (-r) \times$$

$$\times \Gamma(b - ck) \Gamma\left(\frac{b + p - 1 - ck}{p} + r\right) \Gamma\left(\frac{ck - b - p + 1}{p} + 1\right).$$ (57)

Substituting further integral representations for Γ-functions, we obtain in this peculiar case (57):

$$\Phi(\lambda) = \int dk (ck)^c k^{\frac{1}{p}} \int_0^\infty \frac{dv}{v} \int_0^\infty dy \int_0^\infty \frac{dy}{y} e^{-cv - y - \sum y_i} \times$$

$$\times (\sin factors).$$ (58)

while in the general situation (56)

$$\Phi(\hat{\lambda}) = \int dk \frac{p}{c} \lambda^c \prod_{i=1}^p \int_0^\infty \frac{dy_i}{y_i} e^{-y_i - \sum y_i} (c/p)(k - s_i) \times (\sin factors).$$ (59)

Integration over $k$ now gives rise to δ-functions, implying that

$$\log \left( e^{\pm i\pi} e^{\frac{c}{p} \hat{\lambda}^{1/p}} \right) \rightarrow \hat{y} = -y \frac{(\pm v)^p}{\lambda^p}$$ (60)

in the case of (58) and

$$\prod_i y_i = \frac{p}{c} \hat{\lambda}^p$$ (61)

In terms of Γ-functions transition from (56) to (57) involves the use of identity (which is actually a corollary of our reasoning with the functional equation)

$$\prod_{i=1}^p \Gamma(x + \frac{i}{p}) = (2\pi)^{-\frac{1}{2p}} x^{p-\frac{p+1}{2}} \Gamma(px).$$
Let us now concentrate on the case of (58). The role of \( \sin \)-factors is reduced to "\( \pm \)" in (60), which can be accounted for by changing the region of integration over \( v \) from the half-line \( (0, \infty) \) to a contour \( C \), going back and forth from infinity to 0. One can further change variable of integration: \( y = x^{\hat{\lambda} p} \), so that (58) turns into:

\[
\Phi(\lambda) \sim \hat{\lambda}^{p(r+1)+b-1} \int_C dv \int_0^\infty dxx^p e^{cv-x(v^p-\hat{\lambda}^p)},
\]

what is just the expression we used in (10) - and thus can be of use in the further work with matrix models, aimed at going beyond the most simple GKM class.

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APPENDIX. SYMMETRY OPERATORS ON \( \mathcal{GR} \)

This Appendix describes explicitly how the \( GL(\infty) \) transformations act on Grassmannian.

Grassmannian, which is of interest in the theory of Cartanian \( \tau \)-functions is the set of hyperplanes of one-half dimension in a linear space. In order to parametrize it one can first pick up a "reference" hyperplane, by separating the basis in the full vector space into two equal parts: \( \{e_\alpha, e_{-\alpha}\} \). Then the basis on every other hyperplane can be decomposed as:

\[
E_\alpha = e_\alpha + \sum_\beta S_{\alpha\beta} e_{-\beta},
\]

or just

\[
E = e_+ + Se_-.
\]

Infinitesimal action of \( GL(\infty) \) (rotations) on the original vector space,

\[
e_+ \rightarrow e_+ + ae_+ + be_-,
\]

\[
e_- \rightarrow e_- + ce_+ + de_-
\]
moves $E$ into $(I + a + Sc)e_+ + (S + b + Sd)e_-$. Now rotations of the “positive”

vectors only, $e_+ \rightarrow (I + a + Sc)^{-1}e_+$, should be used to bring the result back

into original form, so that

$$S \rightarrow (I + a + Sc)^{-1}(S + b + Sd),$$

i.e. $GL(\infty)$ act on $S$ by rational (non-linear) transformation. Infinitesimally,

$$\delta S = b + aS - Sd - ScS,$$

(65)

or

$$\delta S_{\alpha\beta} = b_{\alpha\beta} + \sum_{\gamma} a_{\alpha\gamma}S_{\gamma\beta} - \sum_{\gamma} S_{\alpha\gamma}d_{\gamma\beta} - \sum_{\gamma,\delta} S_{\alpha\gamma}C_{\gamma\delta}S_{\delta\beta}.$$  

(66)

After this small exercise from the theory of homogeneous spaces let us turn to
description of Grassmannian in terms of Baker-Akhiezer functions. Infinitesimal
action of $GL(\infty)$ act on $\tau$-function is in fact described by the operator $X(\lambda, \mu)$ in

(7):

$$\delta \tau = X(\lambda, \mu)\tau = \Psi(\lambda, \mu)\tau.$$  

The question is now what is the action of the same transformation on $\Psi(\lambda', \mu')$.

The answer can be straightforwardly derived as follows:

$$\delta \Psi(\lambda', \mu') = \delta \left( \frac{X(\lambda', \mu')\tau}{\tau} \right) = -\frac{\delta \tau X(\lambda', \mu')}{\tau^2} + \frac{X(\lambda', \mu')\delta \tau}{\tau} =$$

$$= -\Psi(\lambda', \mu')\Psi(\lambda, \mu) + \frac{1}{\tau}X(\lambda', \mu')(\Psi(\lambda, \mu)\tau).$$  

(67)

The second item at the r.h.s. is just

$$\Psi\left(\{\lambda', \lambda\}, \{\mu', \mu\}\right) = \Psi(\lambda', \mu')\Psi(\lambda, \mu) - \Psi(\lambda', \mu)\Psi(\lambda, \mu'),$$  

(68)

where Wick theorem was applied for $n = 2$, and we finally obtain

$$\delta \Psi(\lambda', \mu') = -\Psi(\lambda', \mu')\Psi(\lambda, \mu).$$  

(69)

---

One could use the already known result from ref. [12], saying that

$$\delta \Psi(\lambda', \mu)\Psi(\lambda, \mu) = \left( e^{\eta(\lambda', \mu)} - 1 \right) \Psi(\lambda, \mu).$$

Then, using the Wick theorem in the form

$$e^{\eta(\lambda', \mu')\Psi(\lambda, \mu)} = \frac{(\lambda - \lambda')(\mu - \mu')}{(\lambda - \mu')(\lambda' - \lambda')} e^{V(\lambda) - V(\mu)} e^{\eta(\{\lambda', \lambda\}, \{\mu', \mu\})\tau(t|g)} =$$

$$= \frac{\psi(\{\lambda', \lambda\}, \{\mu', \mu\})}{\Psi(\lambda', \mu')} = \frac{\Psi(\lambda', \mu')\Psi(\lambda, \mu) - \Psi(\lambda', \mu)\Psi(\lambda, \mu')}{\Psi(\lambda', \mu')}$$

one obtains:

$$\left( e^{\eta(\lambda', \mu')} - 1 \right) \Psi(\lambda, \mu) = -\frac{\Psi(\lambda', \mu')\Psi(\lambda, \mu')}{\Psi(\lambda', \mu')}.$$
Expanding this relation in inverse powers of $\mu'$, we get:

$$\delta \Psi_\alpha(\lambda') = -\Psi(\lambda', \mu)\Psi_\alpha(\lambda).$$

(70)

This can be now compared with eq.(66), where

$$e_\alpha \sim \lambda^\alpha, \quad e_{-\alpha} \sim \lambda^{-\alpha}, \quad E_\alpha \sim \Psi_\alpha(\lambda).$$

Substitution of

$$\Psi(\lambda, \mu) = \sum_{\alpha, \beta} \lambda^\beta \frac{1}{\mu^\alpha} (\delta_{\beta, \alpha} - 1 + S_{\alpha \beta})$$

into (69) gives:

$$\delta S_{\alpha \beta} = (\lambda^{\alpha - 1} + \sum_\gamma S_{\alpha \gamma} \lambda^\gamma) (\mu^{-\beta - 1} + \sum_\gamma \mu^{-\gamma} S_{\gamma \beta}) =$$

$$= \frac{\lambda^{\alpha - 1}}{\mu^{\beta + 1}} + \sum_\gamma \frac{\lambda^{\alpha - 1}}{\mu^\gamma} S_{\gamma \beta} + \sum_\gamma S_{\alpha \gamma} \frac{\lambda^\gamma}{\mu^{\beta + 1}} + \sum_{\gamma, \delta} S_{\alpha \gamma} \frac{\lambda^\gamma}{\mu^\delta} S_{\delta \beta},$$

(71)

in accordance with the general rule (66).

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the pole is annihilated by operator in the brackets. This operator vanishes as $\lambda' = \mu'$, in
accordance with appearance of the pole of $\Psi(\lambda', \mu') = \frac{1}{\lambda' - \mu'} (1 + o(\lambda' - \mu'))$ in denominator.
As usual, the action of such shift operator is singular when $\lambda' = \mu$ or $\mu' = \lambda$, while the zeroes
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