On the high frequency spectrum of a classical accretion disc

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ABSTRACT

We derive simple and explicit expressions for the high frequency spectrum of a classical accretion disc. Both stress-free and finite stress inner boundaries are considered. A classical accretion disc spectrum with a stress-free inner boundary departs from a Wien spectrum at large $\nu$, scaling as $\nu^{2.5}$ (as opposed to $\nu^3$) times the usual exponential cut-off. If there is finite stress at the inner disc boundary, the maximum disc temperature generally occurs at this edge, even at relatively modest values of the stress. In this case, the high frequency spectrum is proportional to $\nu^2$ times the exponential cut-off. If the temperature maximum is a local hot spot, instead of an axisymmetric ring, then an interior maximum produces a $\nu^2$ prefactor while an edge maximum yields $\nu^{1.5}$. Because of beaming effects, these latter findings should pertain to a classical relativistic disc. The asymptotics are in general robust and independent of the detailed temperature profile, provided only that the liberated free energy of differential rotation is dissipated locally, and may prove useful beyond the strict domain of classical disc theory. As observations continue to improve with time, our findings suggest the possibility of using the high energy spectral component of black hole candidates as a signature prediction of classical theory, as well as an diagnostic of the stress at the inner regions of an accretion disc.

1. Introduction

Well into its fifth decade (e.g. Lynden-Bell 1969, Shakura & Sunyaev 1973), classical thin disc accretion theory has withstood the test of time reasonably intact. While it is clear that accretion is considerably more complex than allowed for in classical disc theory (hereafter CDT), there are many examples in which its computed disc spectrum, a superposition of blackbody rings over the surface of the disc, is realised with some fidelity in observations. Even when there are temporal changes in the spectra, there is generally a rather well-defined
thin disc state to be found (e.g. Davis et al. 2005). A detailed theoretical examination of the disc spectrum is both useful and revealing.

In its classical and simplest formulation (Lynden-Bell 1969, Pringle 1981, Frank, King, & Raine 2002), the computation of the disc spectrum involves an integral that must in general be evaluated numerically. In this Letter, we show that the integral in question may, however, be performed analytically in the astrophysically relevant limit $E_\gamma \gg kT_{\text{max}}$, where $E_\gamma$ is the photon energy, $k$ the Boltzmann constant, and $T_{\text{max}}$ the maximum disc temperature. The resulting spectra are similar to, but depart from, a Wien spectrum. The computed frequency dependence is both robust and insensitive to the precise temperature profile. The form of the departure from a Wien spectrum changes (nearly discontinuously) beyond a moderate threshold value of the stress at the inner disc edge. It is this result that may prove useful to the analysis of the dynamical conditions at the inner disc edge. The calculational technique, however, should find application under any conditions in which the disc has a well-defined temperature maximum. In principle, the effect of several isolated maxima may be superposed, but in general the spectrum will be dominated at high $\nu$ by the highest temperature peak.

The calculations are presented in the next section, which is followed by a brief discussion of our results. Both relativistic and nonrelativistic discs are considered.

2. Disc spectrum

2.1. Preliminaries

In nonrelativistic CDT, the emitted flux per unit frequency $\nu$ from a thin disc is given by the expression (e.g. Frank, King, & Raine 2002):

$$F_\nu = \frac{4\pi h^3 c^2 \cos i}{r^2} \int_{R_*}^{\infty} \frac{RdR}{\exp[h\nu/kT(R)] - 1}$$  \hspace{1cm} (1)

Where $h$ is Planck’s constant, $c$ the speed of light, $r$ the distance to the source, $i$ the inclination angle (from a face-on orientation), $R_*$ the inner disc radius, and $T(R)$ the temperature profile. The integral is over all radii $R$ and formally extends to infinity.

The time-steady surface temperature profile of a thin Keplerian disc is given by (Pringle 1981, Balbus & Hawley 1998):

$$T^4(R) = \frac{3GM\dot{M}}{8\pi R^3\sigma} \left[ 1 - \left( \frac{R_*}{R} \right)^{1/2} \right]$$  \hspace{1cm} (2)
where \( G \) is gravitational constant, \( \sigma \) the Stefan-Boltzmann constant, \( M \) the central mass, \( \dot{M} \) the steady mass accretion rate. This result is calculated by assuming that the stress vanishes at the inner edge \( R = R_* \). But \( R_* \) might equally well be regarded as an integration constant (arising from angular momentum flux conservation) whose value is determined by specifying the stress at some radius. To avoid confusion, we will write

\[
T^4(R) = \frac{3GM\dot{M}}{8\pi R^3\sigma} \left[ 1 - \left( \frac{R_0}{R} \right)^{1/2} \right]
\]

where \( R_0 \) is an arbitrary constant with dimensions of a length whose value we leave unspecified. The choice \( R_0 = R_* \) corresponds to what we shall view as the special case of the stress vanishing at the inner edge of the disc. Other notational conventions we will use are

\[
\Omega^2 = \frac{GM}{R^3},
\]

for the Keplerian angular velocity \((s^{-1})\), \( \Sigma \) for the height-integrated disc surface density, and the characteristic temperatures

\[
T_0^4 = \frac{3GM\dot{M}}{8\pi R_0^3\sigma}, \quad T_*^4 = \frac{3GM\dot{M}}{8\pi R_*^3\sigma}.
\]

Care should be taken to distinguish \( T_* \) and \( T(R_*) \), which are quite distinct. The dominant \( R\phi \) component of the density-weighted velocity stress tensor (dimensions of velocity\(^2\)) is denoted \( W_{R\phi} \).

The temperature reaches a maximum, \( T_{\text{max}} \), when \( R_{\text{max}} = (49/36)R_0 \) (Kubota et al. 1998). This is near the inner edge if \( R_0 \) also is, but there may be no temperature maximum if \( R_0 \ll R_* \). This corresponds to the case of a significant stress at the inner edge. “Significant” means a value close to the local \( \dot{M}\Omega/(2\pi\Sigma) \) (Balbus & Hawley 1998). We will show that the two cases of an interior temperature maximum and a boundary maximum lead to distinct observational signatures.

### 2.2. Finite stress modification

The condition of angular momentum conservation may quite generally be written (Balbus & Hawley 1998)

\[
-\frac{\dot{M}R^2\Omega}{2\pi} + \Sigma R^2 W_{R\phi} = \text{constant} = -\frac{\dot{M}R_*^2\Omega_*}{2\pi} + \Sigma_* R_*^2 W
\]
where $W$ is the selected value of $W_{R\phi}$ at $R = R_*$. (Both $\Sigma_*$ and $\Omega_*$ are evaluated at $R = R_*$.)

This value of $W_{R\phi}$ is to be distinguished from a fiducial characteristic value

$$W_{\text{char}} = \frac{\dot{M}_\Omega}{2\pi \Sigma_*}, \quad (7)$$

which will appear (as a normalization for $W$) in the equations.

If we now solve equation (6) for $W_{R\phi}$, we obtain

$$W_{R\phi} = \frac{\dot{M}_\Omega}{2\pi \Sigma} \left[ 1 - \left( \frac{R_0}{R} \right)^{1/2} \right], \quad (8)$$

where

$$R_0 = R_* \left( 1 - \frac{W}{W_{\text{char}}} \right)^2, \quad (9)$$

Equating the energy radiated by (each side of) the disc to the energy extracted from the differential rotation, we have (Balbus & Hawley 1998):

$$2\sigma T^4 = -\Sigma W_{R\phi} \frac{d\Omega}{d\ln R} = -\frac{\dot{M}}{4\pi} \frac{d\Omega^2}{d\ln R} \left[ 1 - \left( \frac{R_0}{R} \right)^{1/2} \right], \quad (10)$$

leading immediately to (3). Equation (9) tells us precisely how the $R_0$ constant is related to the imposed stress $W$ and inner boundary $R_*$, and is for that reason very useful. As noted, the formal location of the temperature maximum from (3) is $R_{\text{max}} = (49/36)R_0$. The question is, at what value of $W$ does this radius move from within the disc proper to the inner edge? Setting $R_{\text{max}} = R_*$ and using (9) leads to

$$W = \frac{W_{\text{char}}}{7}. \quad (11)$$

Thus, when $W$ exceeds $0.1429W_{\text{char}}$, the temperature maximum lies on the inner disc boundary, and when the stress drops below this, the temperature maximum moves off the boundary to within the disc interior. As we shall now see, the location of the temperature maximum makes a significant difference to the emitted spectrum.

2.3. Large $\nu$ limit of $F_\nu$.

2.3.1. Small stress: interior $T_{\text{max}}$

Consider the integral

$$I = \int_{R_*}^{\infty} \frac{R \, dR}{\exp[h\nu/kT(R)] - 1}, \quad (12)$$
with $T(R)$ given by (3). When $\hbar \nu \gg k T_{\text{max}}$ we may safely ignore the $-1$ in the denominator across the entire domain of integration, since it leads only to exponentially small corrections. Thus,

$$I = \int_{R_*}^{\infty} R \exp[-\hbar \nu / k T(R)] \, dR$$

(13)

The function $\beta \equiv 1/k T(R)$ has a sharp minimum at $R = R_{\text{max}}$, which renders the integral an ideal candidate for an asymptotic expansion based on Laplace’s method (Bender & Orszag 1978). Under these conditions, the entire contribution to the integral comes from a small region near $R = R_{\text{max}}$. Expanding $\beta$ and remembering the first derivative vanishes at $R = R_{\text{max}}$:

$$\beta = \frac{1}{k T_{\text{max}}} + \frac{\beta''_{\text{min}} (R - R_{\text{max}})^2}{2} + ...$$

(14)

where

$$\beta''_{\text{min}} = \frac{d^2}{dR^2} \left[ \frac{1}{k T(R)} \right]_{R=R_{\text{max}}}$$

(15)

The integral (13) transforms to

$$I \simeq R_{\text{max}} \exp(-\hbar \nu / k T_{\text{max}}) \int_{-\infty}^{\infty} \exp(-\hbar \nu \beta''_{\text{min}} x^2 / 2) \, dx$$

(16)

where $x = R - R_{\text{max}}$, and we set $R = R_{\text{max}}$ since only this neighborhood contributes. Extending the limits of integration introduces only exponentially small corrections. Hence,

$$I \simeq R_{\text{max}} \left( \frac{2\pi}{\beta''_{\text{min}} \hbar \nu} \right)^{1/2} \exp(-\hbar \nu / k T_{\text{max}})$$

(17)

For the distribution (3), $\beta''_{\text{min}}$ works out to

$$R_0^2 \beta''_{\text{min}} = 3^{7/2} / (7^{5/4} \times 2^{1/2} \times k T_0) = 2.90426 / (k T_0)$$

(18)

This leads to an emission spectrum

$$F_{\nu} = 2.002 \frac{4\pi \cos i}{c^2} (\hbar k T_0)^{1/2} \frac{R_0^2}{\tau^2} \nu^{5/2} \exp(-\hbar \nu / k T_{\text{max}})$$

(19)

where $T_{\text{max}} = 0.487871 T_0$. The solution for a classical zero-stress inner boundary is obtained by setting by using the inner edge of the disc $R_*$ for $R_0$ and $T_*$ for $T_0$. The key point is that the frequency dependence differs from a Wien spectrum by a factor of $\nu^{-1/2}$. 
Fig. 1.— Plot comparing a $F_{\nu} e^{h\nu/kT_{\text{max}}}$, renormalised for display, with the large $\nu$ asymptotic result for the case of vanishing stress. Solid line is from numerical evaluation; dotted line is large $\nu$ asymptotic form (eq. [17]).

2.3.2. Moderate-to-large stress: boundary $T_{\text{max}}$

If the stress $W$ exceeds $W_{\text{char}}/7$, the temperature maximum moves to the boundary. In that case, equations (3) and (9) may be combined to yield the temperature at the inner disc edge, $T(R_{*})$:

$$T(R_{*}) = T_{*} w^{1/4}, \quad \text{with } w \equiv W/W_{\text{char}}.$$  \hfill (20)

Another quantity of interest we shall require is the temperature gradient at $R = R_{*}$. This is most conveniently expressed in the form

$$\left( \frac{d \ln R}{d \ln \beta} \right)_{R=R_{*}} = - \left( \frac{d \ln R}{d \ln T} \right)_{R=R_{*}} = \frac{w}{2(7w - 1)}. \hfill (21)$$

The high frequency behaviour of (13) is obtained by a simple integration by parts. Now the first derivative $\beta'$ term dominates and one finds

$$I \approx \left( \frac{R_{*}}{h\nu} \right) \left( \frac{\exp[-h\nu/kT(R_{*})]}{\beta'(R_{*})} \right) = R_{*}^{2} \left( \frac{kT_{*}}{h\nu} \right) \frac{w^{5/4}}{2(7w - 1)} \exp(-h\nu/[kT_{*}w^{1/4}]), \hfill (22)$$

where

$$\beta'_{*} = - \frac{1}{kT_{*}^{2}} \left( \frac{d T}{d R} \right)_{R=R_{*}}. \hfill (23)$$
This gives a spectral flux of
\[ F_\nu = \left( \frac{R_*}{r} \right)^2 \frac{2\pi \cos i}{c^2} (kT_*) \frac{w^{5/4}}{7w - 1} \nu^2 \exp\left(-\frac{\nu}{[kT_* w^{1/4}]}\right) \] (24)

This is a less steep frequency dependence than is present in (19). The case of small stress and an interior maximum corresponds to a flatter region of high temperature, and a correspondingly larger disc area is able to contribute. A boundary maximum is more steeply cut-off as one moves outward, and less of the disc is able to contribute, reducing the high \( \nu \) emission relative to the interior maximum case.

Figures (1) and (2) show two representative examples illustrating the region of validity of our approximation. Deviations from the leading asymptotic behavior are expected to be \( O(kT_{\text{max}}/\hbar\nu) \) in both cases (Bender & Orszag 1978). Quantitatively, the agreement between asymptotic and exact integrals is excellent when \( \hbar\nu/kT_{\text{max}} \gtrsim 5 \). When the ratio is 10, the results are indistinguishable, at which point the flux is about \( 5 \times 10^{-3} \) of its peak value.

\[ \begin{array}{c}
\text{Frequency/}(kT_{\text{max}}) \\
0.001 & 0.01 & 0.1 & 1 & 10 & 100 \\
F_\nu e^{\hbar\nu/}\left(kT_{\text{max}}\right) & 10^{-6} & 10^{-4} & 10^{-2} & 10^0 & 10^2 & 10^4 & 10^6 \\
W/W_{\text{char}}=2/3 & \nu^2 \\
\end{array} \]

Fig. 2.— As in figure (1) for the finite stress case \( W = (2/3)W_{\text{char}} \).

2.4. A localised hot spot

The axisymmetric form of the emission integral (12) of classical disc theory is not preserved when relativistic physics is included. The most important deviation is due to beaming
from the portion of the disk approaching the observer. To extract the asymptotic behaviour of the spectrum, however, we need not explicitly invoke the full machinery of general relativity. It will suffice to note that the emission integral will be of the form

$$I = \nu^3 \int_{\mathcal{S}} \frac{F(s_1, s_2) \, dS}{\exp[\hbar \nu / kT(s_1, s_2)] - 1}$$  \hspace{1cm} (25)$$

where $\mathcal{S}$ is the effective “working surface,” $s_1$ and $s_2$ surface coordinates, $F$ an unspecified function of coordinates (but, importantly, not $\nu$) and $dS$ an area element. If $T$ now has a maximum within the disc interior localised at some particular location $(s_1, s_2)$, the high $\nu$ contribution is dominated by a small two-dimensional neighbourhood of this point. Even if the disc shape is globally complicated, locally it can always be represented as a flat plane with Cartesian $(x, y)$ coordinates.

Near the temperature maximum $T_{\text{max}}$ we proceed as in subsection 2.3.1, expanding the $\beta$ function around the coordinates $x_{\text{max}}, y_{\text{max}}$:

$$\beta = \frac{1}{kT_{\text{max}}} + \beta_{xx} \frac{(x - x_{\text{max}})^2}{2} + \beta_{yy} \frac{(y - y_{\text{max}})^2}{2} + \ldots$$  \hspace{1cm} (26)$$

where the subscript $x$ or $y$ denotes partial differentiation with the other variable held fixed. The first order partial derivatives vanish at the maximum, as does the mixed derivative $\beta_{xy}$ for the proper choice of coordinates. The second order (nonmixed) derivatives are understood to be evaluated at $x_{\text{max}}, y_{\text{max}}$. Exactly the same reasoning as before leads to the emission (double) integral

$$I = \nu^3 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-\hbar \nu \beta_{xx} x^2 / 2) \exp(-\hbar \nu \beta_{yy} y^2 / 2) \, dx \, dy$$  \hspace{1cm} (27)$$

where $F_{\text{max}}$ is $F$ evaluated at $x_{\text{max}}, y_{\text{max}}$. This yields

$$I = \frac{2\pi F_{\text{max}}}{k(\beta_{xx} \beta_{yy})^{1/2}} \nu^2 \exp[-\hbar \nu / kT_{\text{max}}]$$  \hspace{1cm} (28)$$

The frequency dependence is exactly that of a classical axisymmetric disc with a temperature maximum at the inner boundary. On the other hand, for a disc with a localised hot spot at the inner boundary, exactly the same techniques we have been using show that the large $\nu$ asymptotic form is

$$I \sim \nu^{3/2} \exp(-\hbar \nu / kT_{\text{max}})$$  \hspace{1cm} (Hot spot on inner boundary.)  \hspace{1cm} (29)$$

3. Discussion

We have shown that at large photon energies, CDT yields a mathematically simple—and possibly observationally interesting—difference in the frequency dependence evinced by
a thin disc and a true Wien spectrum. For a Novikov-Thorne relativistic disk, the difference is yet more pronounced. Of greater astrophysical significance, perhaps, is our finding that a spectral changes of comparable magnitude and simplicity occur in going from zero to moderate stress at the disc’s inner boundary. This arises because, unless the stress is very small (cf eq.[11]), the maximum disc temperature is reached on this inner boundary[1]. By contrast, a zero stress constraint always results in a temperature peak within the disc interior. These different locations of the maxima cause measurably different high energy spectra: a $\nu^{5/2}$ ($\nu^2$) power law multiplying an exponential for the case of an interior (edge) maximum. (Recall that a Wien spectrum has a $\nu^3$ power law prefactor.) For a relativistic disc, or any other disc in which the maximum is a localised hot spot, the scalings are $\nu^2$ ($\nu^{3/2}$). These findings stand on their own, but since the question of the presence or absence of an inner stress, which is likely to be magnetic in origin (e.g. Agol & Krolik 2000), is a lively and contested issue (Beckwith, Hawley, & Krolik 2008), the current results may not be devoid of practical significance.

At the very least, our findings are useful benchmarks for numerical calculations of disc spectra. Real discs, on the other hand, live in messy accretion environments, often with many spectral components. Principal sources of confusion at high frequencies include emission from a hot corona and Comptonisation of soft photons. Even here, there is some utility in knowing the precise frequency dependence of a thermal disc, if only as a baseline from which to mark differences. Moreover, there are discs with minimal coronal components, and the models studied by Shimura & Takahara (1995; see also the discussion of Davis et al. 2005) indicate that the effects of Comptonisation can be well-modelled by replacing the Planck function in equation [1] by a “dilute blackbody” form. This modification would not change the frequency dependence of our formulae. Asymptotic expansions in the high frequency limit of spectral integrals, together with data that promises to be ever more accurate, will both strengthen and deepen our understanding of compact X-ray sources. Development and observational applications of these findings, as well as detailed comparisons with relativistic disc models (Novikov & Thorne 1973), are currently being pursued.

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1This is true provided the stress follows the precepts of CDT and causes local dissipational heating as described by equation [10]. Whether magnetic stresses behave in this manner is currently being investigated. (I thank J. Krolik for drawing my attention to this important point.)
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