Loewy Filtration and Quantum de Rham Cohomology over Quantum Divided Power Algebra

Haixia Gu and Naihong Hu

Abstract. The paper explores the indecomposable submodule structures of quantum divided power algebra $A_q(n)$ defined in [22] and its truncated objects $A_q(n,m)$. An “intertwinedly-lifting” method is established to prove the indecomposability of a module when its socle is non-simple. The Loewy filtrations are described for all homogeneous subspaces $A_q(n)$ or $A_q(n,m)$, the Loewy layers and dimensions are determined. The rigidity of these indecomposable modules is proved. An interesting combinatorial identity is derived from our realization model for a class of indecomposable $u_q(sl_n)$-modules. Meanwhile, the quantum Grassmann algebra $\Omega_q(n)$ over $A_q(n)$ is constructed, together with the quantum de Rham complex $(\Omega_q(n), d^\bullet)$ via defining the appropriate $q$-differentials, and its subcomplex $(\Omega_q(n,m), d^\bullet)$. For the latter, the corresponding quantum de Rham cohomology modules are decomposed into the direct sum of some sign-trivial $u_q(sl_n)$-modules.

1. Introduction

1.1. For the generic parameter $q \in \mathbb{C}^*$, it is well-known that the finite dimensional representation theory of quantum groups $U_q(g)$ is essentially the same as that of the complex semisimple Lie algebras $g$ (see the independent work in 1988 by Lusztig [28] and Rosso [36]). The representation theory of quantum groups $U_q(g)$ at roots of unity was established in the early 90s by many authors (see Anderson-Polo-Wen [2], DeConcini-Kac-Procesi [13], [14], [15], [16], Lusztig [29], [30], [31], Andersen-Janzen-Soergel [1], etc.). In recent years, another exciting progress has been made towards geometric representation theory (eg. [4], [5], [8], [12], [10], [17], [20], etc.). The picture looks much close to the modular case (see [1], [13], [16], [29], [30], [31], [8], [6], [33] and references therein). Even for the restricted quantum group $u_q(sl_2)$, there has been drawing more attention to the category of finite-dimensional modules since the early 90s up to now, for instance, the work of DeConcini-Kac [13], Chari-Premet [11], Suter [38], Xiao [39], and recently, Kondo-Saito [27], etc. Their main problems focus on determining all simple modules of $u_q(sl_2)$; classifying and constructing the restricted indecomposable modules of $u_q(sl_2)$; decomposing

1991 Mathematics Subject Classification. Primary 17B10, 17B37, 20G05, 20G42, 81R50; Secondary 14F40, 81T70.

Key words and phrases. quantum divided power algebra, Loewy filtration, rigidity, $q$-differentials, quantum Grassmann algebra, quantum de Rham cohomology.

*N.H., Corresponding Author, supported in part by the NNSF (Grant 11271131) and the FUDP from the MOE of China.
u_q(\mathfrak{sl}_2) as principal indecomposable modules (PIMs) and decomposing the tensor product of a PIM and a module as a direct sum of PIMs; determining all finite dimensional indecomposable representations of \( u_q(\mathfrak{sl}_2) \); exploring the tensor product decomposition rules for all indecomposable modules of \( u_q(\mathfrak{sl}_2) \) with \( q \) being 2p-th root of unity \( (p \geq 2) \), respectively, etc.

1.2. In the representation theory of quantum groups at roots of unity, it is often assumed that the parameter \( q \) is a primitive \( \ell \)-th root of unity with \( \ell \) an odd prime. Recently, there has been increasing interest in the cases where \( \ell \) is an even integer. For example, in the study of knot invariants \((34)\), or in logarithmic conformal field theories where Feigin et al. \((18, 21)\) make a new correspondence between logarithmic conformal field theories based on the so-called triplet VOA \( W(p) \) and representation theory of the restricted quantum enveloping algebras. More precisely, they gave the following

**Conjecture 1.1.** \((21)\) Let \( p \geq 2 \), \( u_q(\mathfrak{sl}_2) \) be the restricted quantum enveloping algebra at 2p-th roots of unity. As a braided quasitensor category, \( W(p)\text{-mod} \) is equivalent to \( u_q(\mathfrak{sl}_2)\text{-mod} \). Here \( u_q(\mathfrak{sl}_2)\text{-mod} \) denotes the category of finite-dimensional \( u_q(\mathfrak{sl}_2)\text{-mod} \) modules.

They also proved the conjecture for \( p = 2 \). After the above conjecture, Tsuchiya and Nagatomo proved

**Theorem 1.2.** \((35)\) As abelian categories, these are equivalent for any \( p \geq 2 \).

These work motivated the investigations of the “quantum group-side” of the FGST’s correspondence, in particular, as tensor categories, see Kondo-Saito \((27)\) and Semikhatov \((37)\). Note that \( u_q(\mathfrak{sl}_2)\text{-mod} \) has a structure of a rigid tensor category, but it is not a braided tensor category if \( p \geq 3 \) (since \( u_q(\mathfrak{sl}_2) \) has no universal \( R \)-matrices for \( p \geq 3 \)). Kondo-Saito’s main result is to determine indecomposable decomposition of all tensor products of indecomposable \( u_q(\mathfrak{sl}_2)\text{-modules} \) in explicit formulas. These also suggest that Conjecture 1.1 needs to be modified; although \( W(p)\text{-mod} \) and \( u_q(\mathfrak{sl}_2)\text{-mod} \) are equivalent as abelian categories by Theorem 1.2, their natural tensor structures do not agree with each other.

On the other hand, Hu \((22)\) first defined the quantum divided power algebras \( A_q(n) \) and the restricted quantum divided power subalgebras \( A_q(n, 1) \) as \( u_q(\mathfrak{s}_n)\text{-module algebras} \) by defining the appropriate \( q \)-derivations, and thereby provided a realization model for some simple modules with highest weights \((\ell−1−s_i)\lambda_{i−1}+s_i\lambda_i \) \((0 \leq s_i < \ell)\). Recently, Semikhatov \((37)\) also exploited the divided-power quantum plane \( \mathbb{C}_q \) that is the rank 2 quantum divided power algebra \( A_q(2) \) and its \( u_q(\mathfrak{sl}_2)\text{-module algebra realization} \) to derive an explicit description of the indecomposable decompositions of \((\mathbb{C}_q)_{(np−1)} \) and of the space of 1-forms \((\Omega_n^1)_{(np−1)} \) for the Wess-Zumino de Rham complex on \( \mathbb{C}_q \) (at \( q \) a 2p-th root of 1).

Anyway, up to now, the study for the tensor category \( u_q(\mathfrak{sl}_2)\text{-mod} \) is sufficient enough and perfect. A natural question is to ask what about the tensor category \( u_q(\mathfrak{sl}_n)\text{-mod} \), for \( n > 2 \).

1.3. In contrast to the generic case, the category \( u_q(\mathfrak{sl}_n)\text{-mod} \) of finite dimensional \( u_q(\mathfrak{sl}_n)\text{-modules} \) is non-semisimple. So in this case it is necessary to pay more attention to studying indecomposable modules. While, category \( u_q(\mathfrak{sl}_n)\text{-mod} \) for \( n > 2 \) is more complicated than \( u_q(\mathfrak{sl}_2)\text{-mod} \), as witnessed by a Theorem of Feldvoss-Witherspoon \((19)\) stating that small quantum groups of rank at least
two are wild, which was a conjecture of Cibils (12), meanwhile, \( u_q(\mathfrak{sl}_2) \) is known to be tame (see 38, 39). In this paper, we will focus on the restricted quantum groups \( u_q(\mathfrak{sl}_n) \) for \( n > 2 \) and explore the indecomposable submodule structures for \( A_q(n) \) and its truncated objects \( A_q(n, m) \) by the method of filtrations analysis, among which Propositions 3.3—3.6 and Lemma 3.7 serve as the basic but essential observations for the whole story. Furthermore, we define the quantum Grassmann algebra \( \Omega_q(n) \) over \( A_q(n) \) and construct the quantum de Rham complex \( (\Omega_q(n), d^*) \) via defining the appropriate \( q \)-differentials \( d^* \) and its subcomplex \( (\Omega_q(n, m), d^*) \), describe the corresponding quantum de Rham cohomology modules \( H^*(\Omega_q) \) for \( \Omega_q = \Omega_q(n) \) or \( \Omega_q(n, m) \), as well as compute the dimensions of \( H^*(\Omega_q) \).

### 1.4. The paper is organized as follows. Section 2 collects some notation and the results on the quantum divided power algebra as \( u_q(\mathfrak{sl}_n) \)-module algebra from [22]. In Section 3, an important notion, named “energy degree” is introduced, which is crucial for the description of Loewy filtrations as well as Loewy layers of the \( s \)-th homogeneous subspaces \( A_q^{(s)}(n, m) \) (see Theorem 3.10). We develop a new “intertwinedly-lifting” method to prove the indecomposability of \( A_q^{(s)}(n, m) \) in the case when its socle is non-simple (see the proof of Theorem 3.8 (5) (ii)), and its rigidity (see Theorem 3.12) under the assumption that \( n \geq 3 \) and \( \text{char}(q) = l \geq 3 \). Thereby, we see that all \( A_q^{(s)}(n) \)'s are indecomposable and rigid (see Corollary 3.13), and the indecomposable decomposition of \( A_q(n) \) is \( A_q(n) = \bigoplus_{s=0}^{+\infty} A_q^{(s)}(n) \). As a by-product, since for different \( s \), \( A_q^{(s)}(n) \)'s are not isomorphic to each other, \( u_q(\mathfrak{sl}_n) \) \((n \geq 3)\) is of infinite representation type (cf. 3). Section 4 is devoted to defining the \( q \)-differentials by using the \( q \)-derivatives in [22], which are not the “differential calculus” in the sense of Woronowicz (40), as well as constructing the quantum de Rham complex \( \Omega_q(n) \) over \( A_q(n) \) (see Propositions 4.2 & 4.4), which is different from the Wess-Zumino de Rham complex used in [32], [37].

### 2. Some notation and earlier results

#### 2.1. Arithmetic properties of \( q \)-binomials. Let \( \mathbb{Z}[v, v^{-1}] \) be the Laurent polynomial ring in variable \( v \). For any integer \( n \geq 0 \), define

\[
[n]_v = \frac{v^n - v^{-n}}{v - v^{-1}}, \quad [n]_v! = [n]_v[n - 1]_v \cdots [1]_v.
\]

Obviously, \([n]_v, [n]_v! \in \mathbb{Z}[v, v^{-1}].\)

For integers \( m, r \geq 0 \), we have (31),

\[
\left[ \begin{array}{c}
  m \\
  r
\end{array} \right]_v = \prod_{i=1}^{r} \frac{v^{m-i+1} - v^{-m+i-1}}{v^i - v^{-i}} \in \mathbb{Z}[v, v^{-1}].
\]

Thus,

1. For \( 0 \leq r \leq m \), \( \left[ \begin{array}{c}
  m \\
  r
\end{array} \right]_v = [m]_v!/([r]_v!)[m-r]_v!);\)
2. For \( 0 \leq m < r \), \( \left[ \begin{array}{c}
  m \\
  r
\end{array} \right]_v = 0;\)
3. For \( m < 0 \), \( \left[ \begin{array}{c}
  m \\
  r
\end{array} \right]_v = (-1)^r [\frac{-m+r-1}{r}]_v);\)
4. Set \( [m]_v = 0 \), when \( r < 0 \).
Assume $k$ is an algebraically closed field of characteristic zero and $q \in k^*$. We briefly set

$$[n] := [n]_{v=q}, \quad [n]! := [n]_{v=q}!, \quad \binom{n}{r} := \binom{[n]}{[r]}_{v=q},$$

when $v$ is specialized to $q$, where $q$-binomials satisfy

$$\binom{n}{r} = q^{r-n} \binom{n-1}{r-1} + q^{r} \binom{n-1}{r}.$$

Define the characteristic of $q$ as in [22], \text{char}(q) := \min \{ \ell \mid \ell \in \mathbb{Z}_{\geq 0}, q^\ell \neq 1 \}.
\text{char}(q) = 0$ if and only if $q$ is generic. If $\text{char}(q) = \ell > 0$ and $q \neq \pm 1$, then either

1. $q$ is the $2\ell$-th primitive root of unity; or
2. $\ell$ is odd and $q$ is the $\ell$-th primitive root of unity.

Assume that the $\mathbb{Z}[v, v^{-1}]$-algebra $\mathcal{R}$ with $\phi : \mathbb{Z}[v, v^{-1}] \rightarrow \mathcal{R}$ and $v = \phi(v)$, is an integral domain satisfying $\mathbf{v}^{2\ell} = 1$ and $\mathbf{v}^{2t} \neq 1$ for all $0 < t < \ell$.

**Lemma 2.1.** ([31], Chapter 34) (1) If $t \geq 1$ is not divided by $\ell$, and $a \in \mathbb{Z}$ is divided by $\ell$, then $\phi\left(\binom{a}{t}\right) = 0$.

(2) If $a_1 \in \mathbb{Z}$ and $t_1 \in \mathbb{N}$, then $\phi\left(\binom{a_1}{t_1}\right) = \mathbf{v}^{2(a_1+1)t_1} \binom{a_1}{t_1}.$

(3) Let $a \in \mathbb{Z}$ and $t \in \mathbb{N}$, write $a = a_0 + \ell t$ with $a_0, a_1 \in \mathbb{Z}$ such that $0 \leq a_0 \leq \ell - 1$ and $t = t_0 + t_1$ with $t_0, t_1 \in \mathbb{N}$ such that $0 \leq t_0 \leq \ell - 1$.

\[ \phi\left(\binom{a}{t}\right) = \mathbf{v}^{(a_0-1-t_0)\ell} \left(\phi\left(\binom{a_0}{t_0}\right) \phi\left(\binom{a_1}{t_1}\right)\right). \]

(5) $\mathbf{v}^{2\ell + t} = (-1)^{\ell+1}$.

According to this proposition, it is easy to get the following.

**Lemma 2.2.** ([29], [31], [22], 1.5) Assume that $q \in k^*$, \text{char}(q) = \ell \geq 3$.

1. Let $m = m_0 + m_1 \ell, r = r_0 + r_1 \ell$ with $0 \leq m_0, r_0 < \ell, m_1, r_1 \geq 0$, and $m \geq r$. Then $\binom{m}{r} = \binom{m_0}{r_0} \binom{m}{m_1} \binom{m_1}{r_1}$ when $\ell$ is odd and $q$ is the $\ell$-th primitive root of unity; $\binom{m}{r} = (-1)^{(m_1+1)r_1 + m_0 r_1 - r_0 m_1} \binom{m_0}{r_0} \binom{m_1}{r_1}$ when $q$ is the $2\ell$-th primitive root of unity, where $\binom{m_1}{r_1}$ is an ordinary binomial coefficient.

(2) Let $m = m_0 + m_1 \ell, 0 \leq m_0 < \ell, m_1 \in \mathbb{Z}$, if $\ell$ is odd and $q$ is an $\ell$-th primitive root of unity, then $\binom{m}{r} = \binom{m_1}{r_1}$; if $\ell$ is the $2\ell$-th primitive root of unity, then $\binom{m}{r} = (-1)^{(m_1+1)\ell + m_0 m_1}$.

(3) If $m = m_0 + m_1 \ell, m' = m_0' + m_1' \ell \in \mathbb{Z}$ with $0 \leq m_0, m_0' < \ell$ satisfy
\[ q^m = q^{m'}, \quad \binom{m}{\ell} = \binom{m'}{\ell}, \text{ then } m = m'. \]

2.2. Quantum (restricted) divided power algebras. Following [22], 2.1, for any $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$, define the map $*: \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z}$ as $\alpha * \beta = \sum_{i>j} \alpha_i \beta_j$ and a bicharacter $\theta: \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow k$ of the additive group $\mathbb{Z}^n$ as $\theta(\alpha, \beta) = q^{\alpha \cdot \beta - \beta \cdot \alpha}$. Denote $\varepsilon_i = (0, \ldots, 1, 0, \ldots, 0)$.

The author introduced in [22] a quantum divided power algebra $\mathcal{A}_q(n)$ as follows. Define $\mathcal{A}_q(n) := \text{span}_k\{ x^{(\alpha)} \mid \alpha \in \mathbb{Z}_+^n \}$, with $x^{(0)} = 1, x^{(i)} = x_i$ and
\[ x^{(\alpha)} x^{(\beta)} = q^{\alpha \cdot \beta} \binom{\alpha + \beta}{\alpha} x^{(\alpha + \beta)} = \theta(\alpha, \beta) x^{(\beta)} x^{(\alpha)}, \]

where $\binom{\alpha + \beta}{\alpha} := \prod_{i=1}^n [\alpha_i + \beta_i]_{\alpha_i}^{-1}, [\alpha_i + \beta_i] = [\alpha_i, \beta_i].$
When $\text{char}(q) = \ell \geq 3$, denote $\tau = (\ell-1, \ldots, \ell-1) \in \mathbb{Z}_n^\ell$. Set

$$A_q(n, 1) := \text{span}_k \left\{ x^{(\alpha)} \in A_q(n) \mid \alpha \in \mathbb{Z}_n^\ell, \alpha \leq \tau \right\},$$

where $\alpha \leq \tau \iff \alpha_i \leq \tau_i$ for each $i$. Obviously, this is a subalgebra of $A_q(n)$ with dimension $s^n$, which is called the quantum restricted divided power algebra.

**Lemma 2.3.** (22, 2.4) Assume $\text{char}(q) = \ell (\geq 3)$, then the algebra $A_q(n)$ is generated by $x_i, x^{(\ell)}_i$ ($1 \leq i \leq n$). When $\ell$ is odd and $q$ is an $\ell$-th primitive root of 1, $x^{(\ell)}_i$ ($1 \leq i \leq n$) are central in $A_q(n)$, and $A_q(n) \cong A_q(n, 1) \otimes_k k[x^{(\ell)}_1, \ldots, x^{(\ell)}_n]$, as algebras.

Define an automorphism of $A_q(n)$ as $\sigma_i(x^{(\beta)}) = q^{\beta_i}x^{(\beta)}$. Obviously, $\sigma_i\sigma_j = \sigma_j\sigma_i$. In particular, $\sigma_i = \text{id}$ for $q = 1$. Define a $q$-derivative of $A_q(n)$ as $\frac{\partial}{\partial s_i}(x^{(\beta)}) = q^{-\varepsilon_i-s_i}x^{(\beta-s_i)}$. Briefly, denote it by $\partial_i$. Then one has $\partial_i\partial_j = \theta(\varepsilon_i, \varepsilon_j)\partial_i\partial_j$.

The $U_q(\mathfrak{sl}_n)$-module algebra structure of $A_q(n)$ can be realized by virtue of the generators $s^{\pm 1}_i, \Theta(\pm \varepsilon_i), x_i, \partial_i$ in the quantum Weyl algebra $W_q(2n)$ defined by (22).

**Proposition 2.4.** (22, 4.1) For any monomial $x^{(\beta)} \in A_q(n)$, set

\begin{align*}
e_i x^{(\beta)} &= (x_i\partial_{i+1}\sigma_i)(x^{(\beta)}) = [\beta_i+1] x^{(\beta+\varepsilon_i-\varepsilon_{i+1})}, \\
f_i x^{(\beta)} &= (\sigma_{i-1}x_{i+1}\partial_{i})(x^{(\beta)}) = [\beta_i+1] x^{(\beta-\varepsilon_i+\varepsilon_{i+1})}, \\
K_i x^{(\beta)} &= (\sigma_i\sigma_{i+1}^{-1})(x^{(\beta)}) = q^{\beta_i-\beta_{i+1}}x^{(\beta)}, \\
K_i^{-1} x^{(\beta)} &= (\sigma_i^{-1}\sigma_{i+1})(x^{(\beta)}) = q^{\beta_{i+1}-\beta_i}x^{(\beta)},
\end{align*}

where $e_i, f_i, K_i, K_i^{-1}$ ($1 \leq i \leq n-1$) are the generators of $U_q(\mathfrak{sl}_n)$.

This equips $A_q(n)$ with a $U$-module algebra, where $U = U_q(\mathfrak{sl}_n)$, or $u_q(\mathfrak{sl}_n) := U_q(\mathfrak{sl}_n)/(e_i^\ell, f_i^\ell, K_i^{2\ell}-1, \forall i < n)$ at roots of 1.

Denote by $|\alpha| := \sum \alpha_i$ the degree of $x^{(\alpha)} \in A_q(n)$. Set $A_q := A_q(n)$ or $A_q(n, 1)$, let $A_q^{(s)} := \text{span}_k \left\{ x^{(\alpha)} \in A_q \mid |\alpha| = s \right\}$ be the subspace of $A_q$ spanned by homogeneous elements of degree $s$.

**Theorem 2.5.** (22, 4.2) $A_q^{(s)}$ is a $U$-submodule of $A_q$.

1. If $\text{char}(q) = 0$, $A_q^{(s)}(n) \cong V(s\lambda_1)$ is a simple module generated by highest weight vector $s^{(\alpha)}$, where $s = (s, 0, \ldots, 0) = se_1 = s\lambda_1, \lambda_1$ is the first fundamental weight of $\mathfrak{sl}_n$.

2. If $\text{char}(q) = \ell \geq 3$, $A_q^{(s)}(n, 1) \cong V((l-1-s_1)\lambda_{l-1}+s_1\lambda_1)$ is a simple module generated by highest weight vector $s^{(\alpha)}$, where $s = (i-1)(\ell-1)+s_i, 0 \leq s_i < \ell$ for $1 \leq i \leq n$, and $s = (\ell-1, \ldots, \ell-1, s_i, 0, \ldots, 0) = (l-1-s_i)\lambda_{l-1}+s_1\lambda_1, \lambda_i = s_1+\cdots+s_i (i < n)$ is the $i$-th fundamental weight of $\mathfrak{sl}_n$.

Set $P_{a,b}(t) := (1 + t + t^2 + \cdots + t^b)^a$, for $a, b \in \mathbb{Z}_{\geq 0}$.

**Corollary 2.6.** $\dim A_q^{(s)}(n, 1) = the coefficient of t^s$ of polynomial $P_{n,\ell}(t) = \sum_{i=0}^{n} \binom{n+\ell-1}{n-i} \binom{n+i-\ell-1}{n-1}$.
Since $\dim \mathcal{A}_q^{(s)} < \infty$ for all $s \geq 0$, they are both noetherian and artinian modules. Thus they satisfy the conditions of the Krull-Schmidt theorem.

**Lemma 2.7. (Krull-Schmidt theorem)** Let $M$ be a module that is both noetherian and artinian, and let $M = M_1 \oplus \cdots \oplus M_n = N_1 \oplus \cdots \oplus N_m$, where $M_i$ and $N_j$ are indecomposable. Then $m = n$ and there exists a permutation $i \mapsto i'$ such that $M_i \cong N_{i'}, 1 \leq i \leq n$.

2.3. Quantum exterior algebra. Recall the Manin’s quantum exterior algebra $k[A_q^{(n)}] := k\{x_1, \cdots, x_n\}/(x_i^2, x_jx_i + q^{-1}x_ix_j, i < j)$, which is a $U$-module algebra with $e_i x_j = \delta_{i+1,j} x_i$, $f_i x_j = \delta_{ij} x_{i+1}$, $K_i x_j = q^{\varepsilon_i - \varepsilon_{i+1}} x_j$, for $U = U_q(\mathfrak{sl}_n)$ or $U_q(\mathfrak{gl}_n)$.

The known fact below is independent of $\text{char}(q)$.

**Lemma 2.8.** $k[A_q^{(n)}] = \bigoplus_{s=0}^n k[A_q^{(n)}]_{(s)}$ as $U$-modules, and

$$k[A_q^{(n)}]_{(s)} = \text{span}_k\{x_{i_1} \cdots x_{i_s} \mid 1 \leq i_1 \leq \cdots \leq i_s \leq n\} \cong V(\lambda_s)$$

is a simple module generated by highest weight vector $x_1 \cdots x_s$, where $\lambda_s$ is the $s$-th fundamental weight of $\mathfrak{sl}_n$.

2.4. Convention. In the rest of paper, we will focus our discussions on the case when $Q(q) \subseteq k$, $\text{char}(q) = \ell (\geq 3)$ and $U = U_q(\mathfrak{sl}_n)$ with $n > 2$ (since for the rank 1 case, there are sufficient discussions in the literature).

3. Loewy filtration of $\mathcal{A}_q^{(s)}(n, m)$ and its rigidity

3.1. Truncated objects $\mathcal{A}_q(n, m)$. Set $m = (m\ell - 1, \cdots, m\ell - 1) \in \mathbb{Z}_+^n$, $m \in \mathbb{N}$, and

$$\mathcal{A}_q(n, m) := \text{span}_k\{x^{(\alpha)} \in \mathcal{A}_q(n) \mid \alpha \leq m\},$$

$$\mathcal{A}_q^{(s)}(n, m) := \text{span}_k\{x^{(\alpha)} \in \mathcal{A}_q(n, m) \mid |\alpha| = s\},$$

then $\mathcal{A}_q(n, m) = \bigoplus_{s=0}^N \mathcal{A}_q^{(s)}(n, m)$, where $N = |m| = n(m\ell - 1)$.

**Proposition 3.1.** (1) $\mathcal{A}_q^{(s)}(n, m) \ (0 \leq s \leq N)$ are $U_q(\mathfrak{sl}_n)$-submodules.

(2) $\dim \mathcal{A}_q^{(s)}(n, m) = \text{the coefficient of } t^s \text{ of } t^{m\ell-1} \}$

$$\text{P}_n(m, m, m) = \sum_{i=0}^m (1 + t + t^2 + \cdots + t^{m-1})^n = \sum_{i=0}^m (1 + t + t^2 + \cdots + t^{m-1})^n = \sum_{i=0}^m (-1)^n \binom{n}{i} (n+s-imi-1).$$

**Proof.** (1) For any $x^{(\alpha)} \in \mathcal{A}_q^{(s)}(n, m)$:

(i) if $\alpha_i = m\ell - 1$, then Proposition 2.4, (2.1) yields $e_i x^{(\alpha)} = 0$;

(ii) if $\alpha_i < m\ell - 1$ and $\alpha_{i+1} = 0$, then Proposition 2.4, (2.1) yields $e_i x^{(\alpha)} = 0$.

Similarly, Proposition 2.4, (2.2)–(2.4) imply $f_i x^{(\alpha)}$, $K_i^{\pm 1} x^{(\alpha)} \in \mathcal{A}_q^{(s)}(n, m)$.

Hence, $\mathcal{A}_q^{(s)}(n, m)$ is a $U_q(\mathfrak{sl}_n)$-submodule.

(2) Note that $\{x^{(\alpha)} \in \mathcal{A}_q(n, m) \mid |\alpha| = s\}$ is a basis of $\mathcal{A}_q^{(s)}(n, m)$. The homomorphism $\phi: \mathcal{A}_q(n) \rightarrow k[t]$ with $\phi(x_i) = t$, $\phi(x^{(t)}_i) = t^i$ restricted to $\mathcal{A}_q^{(s)}(n, m)$ counts up the cardinal of the above basis set as the coefficients of $t^s$ of polynomial $P_n(m, m, m)$. The final identity is due to the expansion of generating function $(1 - t^{m\ell})^n (1 - t)^{-n}$. 

\[\square\]
3.2. Energy degrees and action rules. In this subsection, we introduce an important concept, the so-called “energy degree”. We will see that this captures the essential features of the submodule structures in the root of unity case.

For any rational number $x$, denote by $\lfloor x \rfloor$ the integer part of $x$.

**Definition 3.2.** For any $x^{(\alpha)} \in \mathcal{A}_q(n, m)$ or $\mathcal{A}_q(n)$, the energy degree of $x^{(\alpha)}$, denoted by $\text{Edeg } x^{(\alpha)}$, is defined as

$$\text{Edeg } x^{(\alpha)} := \sum_{i=1}^{n} \left\lfloor \frac{\alpha_i}{\ell} \right\rfloor = \sum_{i=1}^{n} \text{Edeg}_i x^{(\alpha)},$$

where $\text{Edeg}_i x^{(\alpha)}$ indicates the $i$-th energy degree of $x^{(\alpha)}$, i.e., $\text{Edeg}_i x^{(\alpha)} := [\frac{\alpha_i}{\ell}]$.

In general, for any $x \in \mathcal{A}_q(n, m)$ or $\mathcal{A}_q(n)$, define

$$\text{Edeg } x := \max \{ \text{Edeg } x^{(\alpha)} \mid x = \sum k_{\alpha} x^{(\alpha)}, \ k_{\alpha} \in k^\ast \}.$$

**Proposition 3.3.** $\text{Edeg } (u. x^{(\alpha)}) \leq \text{Edeg } x^{(\alpha)}$, for any $u \in u_q(\mathfrak{sl}_n)$ and $x^{(\alpha)} \in \mathcal{A}_q(n, m)$ or $\mathcal{A}_q(n)$. In particular, $\text{Edeg}_i (u. x^{(\alpha)}) \leq \text{Edeg}_i x^{(\alpha)}$ for each $i$.

**Proof.** It suffices to check the behavior of generators $e_i, f_i$, $K_i^{\pm 1}$ ($1 \leq i < n$) of $u_q(\mathfrak{sl}_n)$ acting on any basis element $x^{(\alpha)} \in \mathcal{A}_q(n, m)$ or $\mathcal{A}_q(n)$.

(1) Note that $e_i, x^{(\alpha)} = [\alpha_i+1]^{-1} x^{(\alpha+\epsilon_i+1)}$, by Proposition 2.4, (2.1).

If $e_i, x^{(\alpha)} \neq 0$, then $\ell \not| (\alpha_i + 1)$ and $\alpha_i > 0$. Observing

$$\text{Edeg } x^{(\alpha+\epsilon_i+1)} = \left[ \frac{\alpha_1}{\ell} \right] + \cdots + \left[ \frac{\alpha_{i-1}}{\ell} \right] + \left[ \frac{\alpha_i+1}{\ell} \right] + \left[ \frac{\alpha_{i+1}-1}{\ell} \right] + \cdots + \left[ \frac{\alpha_n}{\ell} \right],$$

we get

$$\text{Edeg } x^{(\alpha+\epsilon_i+1)} - \text{Edeg } x^{(\alpha)} = \left[ \frac{\alpha_i+1}{\ell} \right] + \left[ \frac{\alpha_{i+1}-1}{\ell} \right] - \left[ \frac{\alpha_i}{\ell} \right] - \left[ \frac{\alpha_{i+1}}{\ell} \right].$$

Obviously, $\left[ \frac{\alpha_i+1}{\ell} \right] - \left[ \frac{\alpha_{i+1}}{\ell} \right] \leq 0$. If $\left[ \frac{\alpha_i+1}{\ell} \right] - \left[ \frac{\alpha_{i+1}}{\ell} \right] > 0$, then $\alpha_i \equiv \ell-1(\text{mod } \ell)$.

Therefore, $\text{Edeg } (e_i, x^{(\alpha)}) \leq \text{Edeg } x^{(\alpha)}$.

Similarly, by Proposition 2.4, (2.2)–(2.4), we get $\text{Edeg } (f_i, x^{(\alpha)}) \leq \text{Edeg } x^{(\alpha)}$, $\text{Edeg } (K_i^{\pm 1}, x^{(\alpha)}) = \text{Edeg } x^{(\alpha)}$.

(2) In the proof of (1), we actually show that $\text{Edeg}_j (u. x^{(\alpha)}) \leq \text{Edeg}_j x^{(\alpha)}$ for each $j$ and for arbitrary $u \in u_q(\mathfrak{sl}_n)$, $x^{(\alpha)} \in \mathcal{A}_q(n, m)$ or $\mathcal{A}_q(n)$. \hfill $\square$

**Proposition 3.4.** Given $x^{(\alpha)}, x^{(\beta)} \in \mathcal{A}_q(n, m)$ or $\mathcal{A}_q(n)$ with $\text{Edeg } x^{(\alpha)} = \text{Edeg } x^{(\beta)}$. If $\text{Edeg}_i x^{(\alpha)} \neq \text{Edeg}_i x^{(\beta)}$ for some $i$ ($1 \leq i < n$), then for any $u, v \in u_q(\mathfrak{sl}_n)$, $u. x^{(\alpha)} \neq x^{(\beta)}$, $v. x^{(\alpha)} \neq x^{(\beta)}$. Namely, $x^{(\alpha)} \not\in u_q(\mathfrak{sl}_n). x^{(\beta)}$, $x^{(\beta)} \not\in u_q(\mathfrak{sl}_n). x^{(\alpha)}$.

**Proof.** Without loss of generality, we assume that $\text{Edeg}_i x^{(\alpha)} > \text{Edeg}_i x^{(\beta)}$. Since $\text{Edeg } x^{(\alpha)} = \text{Edeg } x^{(\beta)}$, there must exist a $j \neq i$ with $1 \leq j \leq n$ such that $\text{Edeg}_j x^{(\alpha)} < \text{Edeg}_j x^{(\beta)}$.

(1) If there exists $u \in u_q(\mathfrak{sl}_n)$ such that $u. x^{(\alpha)} = x^{(\beta)}$, by Proposition 3.3, $\text{Edeg}_r x^{(\beta)} = \text{Edeg}_r (u. x^{(\alpha)}) \leq \text{Edeg}_r x^{(\alpha)}$ for $1 \leq r \leq n$. It contradicts the fact $\text{Edeg}_j x^{(\alpha)} < \text{Edeg}_j x^{(\beta)}$ for some $j (\neq i)$. Hence, $u. x^{(\alpha)} \neq x^{(\beta)}$, for any $u \in u_q(\mathfrak{sl}_n)$. \hfill $\square$
(2) Using the assumption $\text{Edeg}_i x^{(\alpha)} > \text{Edeg}_i x^{(\beta)}$, by a similar argument of (1), we can derive $v, x^{(\beta)} \neq x^{(\alpha)}$, for any $v \in u_q(s_{l_n})$. □

The proof of Theorem 2.5 (2) (see [22], 4.2) motivates the following observation.

**Proposition 3.5.** Given $x^{(\alpha)}, x^{(\beta)} \in \mathcal{A}_q(\eta) \cup \mathcal{A}_q^{(s)}(\eta)$ with $x^{(\alpha)} = \text{Edeg}_i x^{(\beta)}$ for each $i (1 \leq i \leq n)$, then there exist $u, v \in u_q(s_{l_n})$ such that $u, x^{(\alpha)} = x^{(\beta)}, v, x^{(\beta)} = x^{(\alpha)}$. In this case, $u_q(s_{l_n}), x^{(\alpha)} = u_q(s_{l_n}), x^{(\beta)}$.

**Proof.** Put $m_j := \text{Edeg}_j x^{(\beta)}$ and $r := s - \sum_{j=1}^n m_j \ell$, for $x^{(\beta)} \in \mathcal{A}_q^{(s)}(\eta)$. Clearly, $0 \leq r \leq n(\ell - 1)$. Write $r = (i-1)(\ell - 1) + r_i$ with $1 \leq i \leq n$, $0 \leq r_i \leq \ell - 1$.

Set $\gamma := (\ell - 1, \cdots, \ell - 1, r_i, 0, \cdots, 0)$ and $\eta := (m_1 \ell, \cdots, m_n \ell) + \gamma$. Then $|\gamma| = r$, and $|\eta| = s$, that is, $x^{(\eta)} \in \mathcal{A}_q^{(s)}(\eta)$.

Write $\beta = \sum_{j=1}^n (m_j \ell + h_j) \varepsilon_j$ with $0 \leq h_j \leq \ell - 1$, $\sum_{j=1}^n h_j = r$ (since $|\beta| = s$). Denote by $k$ the last ordinal number with $h_k \neq 0$ for the $n$-tuple $(h_1, \cdots, h_n)$. So, $k \geq i$ if $r_i \neq 0$, and $k \geq i - 1$ if $r_i = 0$.

(1) Note that the pair $(x^{(\eta)}, x^{(\beta)})$ satisfies the hypothesis of our Proposition. Firstly, for the given pair $(x^{(\eta)}, x^{(\beta)})$, we can prove the following Claims (A), (B).

Claim (A): There exists $u_1 \in u_q(n^-)$, such that $u_1, x^{(\eta)} = x^{(\beta)}$.

Case (1). If $r_i \geq h_k$, then by Proposition 2.4, (2.2) & Lemma 2.2 (1), we get

$$f_{k-1}^{h_k} \cdots f_i^{h_k} x^{(\eta)} = \prod_{z=i+1}^k \prod_{j=1}^{h_k} [m_z \ell + j] x^{(\eta - h_k \varepsilon_i + h_k \varepsilon_k)}$$

$$= q^{h_k m_k \ell - \eta \varepsilon_i + h_k \varepsilon_k} \prod_{z=i+1}^k \prod_{j=1}^{h_k} [m_z \ell + j] x^{(\eta') x^{(h_k \varepsilon_k)}} \neq 0,$$

where $\eta' = \eta - h_k \varepsilon_i = \sum_{j=1}^i (m_j \ell + \ell - 1) \varepsilon_j + (m_i \ell + r_i - h_k) \varepsilon_i + \sum_{j=i+1}^n (m_j \ell) \varepsilon_j$.

Case (2). If $r_i < h_k$, then by Proposition 2.4, (2.2) & Lemma 2.2 (1), we get

$$f_{k-1}^{h_k - r_i} \cdots f_{i-1}^{h_k - r_i} \cdots f_i^{h_k} x^{(\eta)}$$

$$= \left(\prod_{z=i+1}^k \prod_{j=1}^{h_k - r_i} \prod_{j=1}^{m_z \ell + j} \right) \left(\prod_{z=i+1}^k \prod_{j=1}^{h_k + r_i} \prod_{j=1}^{m_z \ell + j} \right) \times [m_k \ell + r_i + 1] \cdots [m_k \ell + h_k] x^{(\eta - (h_k - r_i) \varepsilon_i - 1 - r_i \varepsilon_i + h_k \varepsilon_k)}$$

$$= \left(\prod_{z=i+1}^k \prod_{j=1}^{h_k - r_i} \prod_{j=1}^{m_z \ell + j} \right) \left(\prod_{z=i+1}^k \prod_{j=1}^{h_k + r_i} \prod_{j=1}^{m_z \ell + j} \right) \times [m_k \ell + r_i + 1] \cdots [m_k \ell + h_k] q^{h_k m_k \ell - \eta \varepsilon_i + h_k \varepsilon_k} x^{(\eta') x^{(h_k \varepsilon_k)}} \neq 0,$$

where $\eta' = \eta - (h_k - r_i) \varepsilon_i - 1 - r_i \varepsilon_i$.

Set $\beta' := \beta - h_k \varepsilon_k$. For the pair $(x^{(\eta')}, x^{(\beta')})$, using an induction on $\eta$ (at first, noting that the argument holds for $\eta = \varepsilon_1 = \lambda_1$), the same argument of the proof of Theorem 2.5 (2) (see [22], 4.2), there exists $u'_1 \in u_q(n^-)$ generated by $f_j (j < k - 1)$, such that $u'_1, x^{(\eta')} = x^{(\beta')}$.

Note that $f_j, x^{(h_k \varepsilon_k)} = 0$ $(j < k - 1)$ and $\Delta f_j = f_j \otimes 1 + K_j^{-1} \otimes f_j$, then

$$u'_1(x^{(\eta')}) x^{(h_k \varepsilon_k)} = (u'_1 x^{(\eta')}) x^{(h_k \varepsilon_k)} = x^{(\beta')} x^{(h_k \varepsilon_k)} = q^{\beta' + h_k \varepsilon_k - m_k h_k \ell} x^{(\beta)} \neq 0.$$

Combining with both cases (1) and (2), we get the claim as desired.
Conversely, for the given pair \((x^{(\beta)}, x^{(\eta)})\), we can prove the following Claim (B): There exists \(u_2 \in U_q(n^\perp)\), such that \(u_2 \cdot x^{(\beta)} = x^{(\eta)}\).

Case (i). If \(r_i \geq h_k\), then
\[
e_i^{h_k} \cdots e_{k-1}^{h_k} \cdot x^{(\eta+h_k \varepsilon_k)} = \left( \prod_{z=i+1}^{k-1} \prod_{j=1}^{h_k} [m_z \ell + j] \right) \left( m_i \ell + r_i + h_k \right) \cdots \left( m_i \ell + r_i + h_k + 1 \right) x^{(\eta)} \neq 0,
\]
where \(\eta' = \eta - h_k \varepsilon_i = \sum_{j=1}^{i-1} (m_j \ell + \ell - 1) \varepsilon_j + (m_i \ell + r_i + h_k) \varepsilon_i + \sum_{j=i+1}^{n} (m_j \ell) \varepsilon_j\).

Case (ii). If \(r_i < h_k\), then
\[
e_i^{r_i} \cdots e_{k-1}^{r_i} \cdot e_{k-1}^{h_k-r_i} \cdots e_{i+1}^{h_k-r_i} \cdot x^{(\eta+h_k \varepsilon_k)}
= \left( \prod_{z=i+1}^{k-1} \prod_{j=1}^{h_k-r_i} [m_z \ell + j] \right) \left( m_i \ell + \ell - 1 - (h_k - r_i) \right) \cdots \left( m_i \ell + \ell - 1 \right) \times \left( \prod_{z=i+1}^{k-1} \prod_{j=1}^{r_i} [m_z \ell + j] \right) x^{(\eta)} \neq 0,
\]
where \(\eta' = \eta - (h_k - r_i) \varepsilon_{i-1} - r_i \varepsilon_i\).

Inductively, for the pair \((\beta', \eta')\) \((\beta' := \beta - h_k \varepsilon_k)\), there exists \(u_2' \in U_q(n^\perp)\) generated by \(e_j (j < k-1)\) such that \(u_2' \cdot x^{(\beta')} = x^{(\eta')}\). Note that \(e_j \cdot x^{(h_k \varepsilon_k)} = 0\), \(K_j \cdot x^{(h_k \varepsilon_k)} = x^{(h_k \varepsilon_k)}\) for \(j < k-1\), and \(\Delta e_j = e_j \otimes K_j + 1 \otimes e_j\), then there are \(c, c' \in k^*\) such that
\[
u_2' \cdot x^{(\beta')} = c' (u_2' \cdot x^{(\beta')}) x^{(h_k \varepsilon_k)} = c x^{(\eta' + h_k \varepsilon_k)} \neq 0.
\]
Combining with both cases (i) and (ii), we get the claim as required.

(II) For the given pair \((x^{(\alpha)}, x^{(\beta)})\) satisfying the hypothesis of our Proposition, consider both pairs \((x^{(\eta)}, x^{(\beta)})\) and \((x^{(\alpha)}, x^{(\eta)}\)), by Claims (A) and (B), we see that there exists \(u_1, u_2 \in U_q(sl_n)\) such that \(u_1 \cdot x^{(\alpha)} = x^{(\beta)}, u_2 \cdot x^{(\alpha)} = x^{(\eta)}\). Set \(u = u_1 u_2\), then \(u \cdot x^{(\alpha)} = x^{(\beta)}\). Similarly, there exists \(v \in U_q(sl_n)\) such that \(v \cdot x^{(\beta)} = x^{(\alpha)}\). \(\square\)

The observation below is more crucial to understand the submodules structure of \(A_q(s^\alpha)(n, \mathfrak{m})\) and \(A_q(s^\beta)(n)\). Its proof is skillful.

**Proposition 3.6.** Given \(x^{(\alpha)}, x^{(\beta)} \in A_q(s^\alpha)(n, \mathfrak{m})\) or \(A_q(s^\beta)(n)\) with \(\text{Edeg} \cdot x^{(\alpha)} > \text{Edeg} \cdot x^{(\beta)}\). If \(\text{Edeg}_i \cdot x^{(\alpha)} \geq \text{Edeg}_i \cdot x^{(\beta)}\) for each \(i\), then there exists \(u \in U_q(sl_n)\) such that \(u \cdot x^{(\alpha)} = x^{(\beta)}\). That is, \(u \cdot (sl_n) \cdot x^{(\beta)} \subseteq U_q(sl_n) \cdot x^{(\alpha)}\).

**Proof.** (I) Assume \(\text{Edeg} \cdot x^{(\alpha)} = \text{Edeg} \cdot x^{(\beta)} + 1\). Then there exists \(j (1 \leq j \leq n)\) such that \(\text{Edeg}_j \cdot x^{(\alpha)} = \text{Edeg}_j \cdot x^{(\beta)} + 1\) and \(\text{Edeg}_i \cdot x^{(\alpha)} = \text{Edeg}_i \cdot x^{(\beta)}\) for \(i \neq j\).

Write \(\alpha = \sum_{i=1}^{n} (m_i \ell + r_i) \varepsilon_i\), where \(m_i = \text{Edeg}_i \cdot x^{(\alpha)}\) and \(0 \leq r_i \leq \ell - 1\), then \(0 \leq \sum_{i=1}^{n} r_i \leq n(\ell - 1)\). Note that \(|\alpha| = |\beta| = s = \ell \cdot \text{Edeg} \cdot x^{(\alpha)} + \sum_{i=1}^{n} r_i\). By the assumption above, we must have \(\sum_{i=1}^{n} r_i < (n-1)(\ell - 1)\). Otherwise, \(\sum_{i=1}^{n} r_i \geq (n-1)(\ell - 1)\). This implies that \(\text{Edeg} \cdot x^{(\alpha)}\) is the least among the \(Edeg^{(\theta)}\)’s, for any \(x^{(\theta)} \in A_q(s^\alpha)(n, \mathfrak{m})\). It contradicts the given condition \(\text{Edeg} \cdot x^{(\alpha)} > \text{Edeg} \cdot x^{(\beta)}\).

(1) When \(j < n\): as \(\sum_{i=1}^{n} r_i < (n-1)(\ell - 1)\), there exists \((h_1, \ldots, h_n) \in \mathbb{Z}_+^n\) with \(h_j = 0, h_{j+1} < \ell - 1, 0 \leq h_i \leq \ell - 1\) for \(i \neq j, j+1\) such that \(\sum_{i=1}^{h_j} h_i = \sum_{i=1}^{n} r_i\).

Set \(\gamma = \sum_{i=1}^{n} (m_i \ell + h_i \varepsilon_i)\), then \(|\gamma| = s\), i.e., \(x^{(\gamma)} \in A_q(s^\alpha)(n, \mathfrak{m})\). Obviously, \(\text{Edeg}_i \cdot x^{(\alpha)} = \text{Edeg}_i \cdot x^{(\gamma)}\) for each \(i\).
Again, we have $f_j, x^{(\gamma)} = [m_{j+1}\ell + h_{j+1} + 1], x^{(\gamma - \varepsilon_j, \varepsilon_{j+1})} (\neq 0) \in A_0^{(s)}(n, m)$, $E_{deg},_j(f_j, x^{(\gamma)}) = m_{j+1} - 1 = E_{deg},_j(x^{(\beta)}$ and $E_{deg}(f_j, x^{(\gamma)}) = E_{deg}(x^{(\beta)}$, for $i \neq j$.

Hence, for the pairs $(x^{(\alpha)}, x^{(\gamma)})$ and $(f_j, x^{(\gamma), x^{(\beta)}})$, by Proposition 3.5, there exist $u_1, u_2 \in u_q(\mathfrak{s}_l n)$ such that $u_1, x^{(\alpha)} = x^{(\gamma)}$ and $u_2, (f_j, x^{(\gamma)}) = x^{(\beta)}$. Set $u = u_2f_ju_1$, then $u, x^{(\alpha)} = x^{(\beta)}$.

(2) When $j = n$ as $\sum_{i=1}^n r_i < (n-1)(\ell-1)$, there exists $(h'_1, \ldots, h'_n) \in \mathbb{Z}_+^n$ with $h'_{n-1} < \ell-1, h'_n = 0, 0 \leq h'_i \leq \ell-1$ for $i \neq n-1, n$ such that $\sum_{i=1}^n h'_i = \sum_{i=1}^n r_i$.

Set $\gamma' = \sum_{i=1}^n (m_i\ell + h'_{i})e_i$, then $|\gamma'| = s$, i.e., $x^{(\gamma')} \in A_q^{(s)}(n, m)$. Obviously, $E_{deg}(x^{(\alpha)}) = E_{deg}(x^{(\beta)}) = E_{deg}(x^{(\gamma')})$ for each $i$.

Again, $e_{n-1}, x^{(\gamma')} = [m_{n-1}\ell + h'_{n-1} + 1], x^{(\gamma' + \varepsilon_{n-1} - \varepsilon_n)} (\neq 0) \in A_0^{(s)}(n, m)$, and $E_{deg}(e_{n-1}, x^{(\gamma')}) = E_{deg}(x^{(\beta)})$ for $i < n$, $E_{deg}(e_{n-1}, x^{(\gamma')}) = m_{n-1} = E_{deg}(x^{(\beta)})$.

Now for the pairs $(x^{(\alpha)}, x^{(\gamma')})$ and $(e_{n-1}, x^{(\gamma')}, x^{(\beta)})$, using Proposition 3.5, there exists $v_1, v_2 \in u_q(\mathfrak{s}_l n)$ such that $u_1, x^{(\alpha)} = x^{(\gamma')}$ and $v_2, (e_{n-1}, x^{(\gamma')}) = x^{(\beta)}$. Set $u = v_2e_{n-1}v_1$, then $u, x^{(\alpha)} = x^{(\beta)}$.

(II) Use an induction on $E_{deg}(x^{(\alpha)}) - E_{deg}(x^{(\beta)})$. As $E_{deg}(x^{(\alpha)}) > E_{deg}(x^{(\beta)})$, according to the proof of (I), it is clear that there are $x^{(\gamma_1)} \in A_0^{(s)}(n, m)$ with $E_{deg}(x^{(\alpha)}) = E_{deg}(x^{(\gamma_1)}) + 1$ and $E_{deg}(x^{(\alpha)}) \geq E_{deg}(x^{(\gamma_i)})$ for each $i$, and $u_1 \in u_q(\mathfrak{s}_l n)$ such that $u_1, x^{(\alpha)} = x^{(\gamma_1)}$. And for the pair $(x^{(\gamma_1)}, x^{(\beta)})$, by the inductive hypothesis, there is $u_2 \in u_q(\mathfrak{s}_l n)$ such that $u_2, x^{(\gamma_1)} = x^{(\beta)}$.

This completes the proof.

3.3. Equivalence and ordering on $n$-tuples. Note that the set of $n$-tuples of nonnegative integers indexes a basis of $A_q^{(s)}(n)$ via the mapping $\chi : \mathbb{Z}_+^n \to A_q^{(s)}(n)$ such that $\chi(\alpha) = x^{(\alpha)}$. Set $\mathbb{Z}_+^n(s) := \{\alpha \in \mathbb{Z}_+^n \mid |\alpha| = s\}$, $\mathbb{Z}_+^n(s, m) := \{\alpha \in \mathbb{Z}_+^n(s) \mid |\alpha| \leq m\}$. These index bases of $A_0^{(s)}(n)$ and $A_q^{(s)}(n, m)$, respectively.

Set $E_{\ell}(s) := E_{deg},_\ell(x^{(\alpha)})$ and $E_{\alpha}(\alpha) := (E_{\ell}(\alpha), \ldots, E_{\ell}(\alpha)).$ Define an equivalence ~ on $\mathbb{Z}_+^n(s, m)$ or $\mathbb{Z}_+^n(s)$ as follows: $\alpha \sim \beta \iff E_{\alpha}(\alpha) = E_{\beta}(\beta)$, for any $\alpha, \beta \in \mathbb{Z}_+^n(s, m)$ or $\mathbb{Z}_+^n(s)$. So, Proposition 3.4 shows that $\alpha \not\sim \beta \in \mathbb{Z}_+^n(s, m)$ or $\mathbb{Z}_+^n(s)$, then $x^{(\alpha)} \notin u_q(\mathfrak{s}_l n), x^{(\beta)}$ and $x^{(\beta)} \notin u_q(\mathfrak{s}_l n), x^{(\alpha)}$. While, Proposition 3.5 indicates that if $\alpha \sim \beta \in \mathbb{Z}_+^n(s, m)$ or $\mathbb{Z}_+^n(s)$, then $u_q(\mathfrak{s}_l n), x^{(\alpha)} = u_q(\mathfrak{s}_l n), x^{(\beta)}$.

Introduce an ordering \geq on $\mathbb{Z}_+^n$ as follows: $\alpha \geq \beta \iff E_{\alpha}(\alpha) \geq E_{\beta}(\beta) \iff E_{\ell}(\alpha) \geq E_{\ell}(\beta)$ for each $i$. So, Proposition 3.6 means that if $\alpha \geq \beta \in \mathbb{Z}_+^n(s, m)$ or $\mathbb{Z}_+^n(s)$, then $u_q(\mathfrak{s}_l n), x^{(\beta)} \subseteq u_q(\mathfrak{s}_l n), x^{(\alpha)}$. Actually, Proposition 3.6 captures an essential feature between the ordering relations \geq on the set of $n$-tuples of energy-degrees $\{E_{\alpha}(\alpha)\}$ and the including relations of submodules of $A_0^{(s)}(n, m)$ or $A_q^{(s)}(n)$.

This will be useful to analyze their indecomposability.

3.4. Socle of $A_0^{(s)}(n, m)$. Given $0 \leq s \leq N \{N = |m|\}$, denote by $E(s)_0$ (resp. $E(s)$) the lowest (resp. highest) energy degree of elements of $A_0^{(s)}(n, m)$.

The following observation will be essential to describing the whole picture of the submodules structure of $A_0^{(s)}(n, m)$ in a more explicit manner.

**Lemma 3.7.** Suppose $n \geq 3$ and $\text{char}(q) = \ell \geq 3$. Given $s$ with $0 \leq s \leq N$, where $N = |m| = n(m\ell - 1)$.

1. When $0 \leq s \leq \ell - 1$: $E(s)_0 = 0 = E(s)$.

2. When $(\ell-1)+1 \leq s \leq n(\ell-1)$: $E(s)_0 = 0$, and $1 \leq E(s) \leq E(n(\ell-1))$, where $n = n'\ell + r (0 \leq r < \ell), E(n(\ell-1)) = n - n' - 1 + \delta_{n, n'}.$ More precisely,
where $s \in \mathbb{N}$ and $r = (r_1, \ldots, r_n)$ if $r > n$. Taking $\alpha$ and $\gamma$ for some $j_1$. Based on the above observation, the conclusion (1) is clear. As for (4), we note that for any $\alpha \in \mathbb{Z}_+^n(s, m)$, denote by $\gamma(\alpha) := \alpha - \ell \cdot \mathcal{E}(\alpha) = (r_1, \ldots, r_n)$ the rest $n$-tuple of $\alpha$ with respect to its energy-degree $n$-tuple. Clearly, $\gamma(\alpha) \leq \tau$.

Now from the definitions of $E(s)_0$ and $E(s)$, there are at least $\alpha, \beta \in \mathbb{Z}_+^n(s, m)$ such that $E(s)_0 = \gamma(\alpha) + \mathcal{E}(\beta)$, as well as $s = |\alpha| = \ell \cdot E(s)_0 + \gamma(\alpha)$ with $|\gamma(\alpha)| = \sum_{i=1}^n r_i$ as large as possible, and $s = |\beta| = \ell \cdot E(s) + |\gamma(\beta)|$ with $|\gamma(\beta)|$ as smallest as possible.

Based on the above observation, the conclusion (1) is clear. As for (4), we note that for any $x^{(\alpha)} \in \mathcal{A}^n_0(n, m)$, $\alpha$ is of the form $(m-1)\ell + a_1, \ldots, (m-1)\ell + a_n$ with $\gamma(\alpha) = (a_1, \ldots, a_n) \leq \tau$ such that $|\gamma(\alpha)| = (n-1)(\ell-1) + h$ with $0 \leq h \leq \ell-1$, and $\mathcal{E}(\alpha) = (m-1, \ldots, m-1)$. So, $E(s)_0 = E(s) = n(m-1)$.

(2) When $\ell \leq s \leq n(\ell-1):$ it is clear that $E(s)_0 = 0$, as for the extreme case $s = n(\ell-1)$, taking $\alpha = \tau$, we get that $s = |\tau|$, $\gamma(\tau) = \tau$ and $\mathcal{E}(\tau) = 0$, i.e., $E(s)_0 = 0$.

In order to estimate $E(s)$, now we can assume that $j(\ell-1)+1 \leq s \leq (j+1)(\ell-1)$ for $1 \leq j \leq n-1$. Let us consider the general case $s = j(\ell-1) + h$ with $0 \leq h < \ell$ and $1 \leq j \leq n$. Write $j = j' + r_j$ with $0 \leq r_j < \ell$. Then rewrite $s = j(\ell-1) + h = j'(\ell-1) + r_j \ell - r_j - h$. Clearly, when $h \geq r_j$, $E(s) = j'(\ell-1) + r_j = j - j'$; and when $h < r_j$, $E(s) = j'(\ell-1) + r_j - 1 = j - j' + 1$. Particularly, when $j = j(\ell-1)$ with $h > 0$, we get $E(s) = j-j'-1 = \delta_{j,j'}.\delta_{j,j'}$. So we obtain $j-j' \leq E(s) \leq j-j'$, for $j(\ell-1)+1 \leq s \leq (j+1)(\ell-1)$ with $1 \leq j \leq n-1$.

(3) When $n(\ell-1)+1 \leq s \leq N-\ell$: Firstly, we rewrite $N - \ell = n(m\ell-1) - \ell = (n(m-1)-1)\ell + n(\ell-1)$. So now for the $s$ given above, we can put it into a certain strictly smaller interval: $k\ell + n(\ell-1) \leq (k-1)\ell + n(\ell-1) \leq s \leq k\ell + n(\ell-1)$, for some $k$ with $1 \leq k \leq n(m-1)-1$. Namely, $s = k\ell + h + n(\ell-1)$ with $0 \leq h \leq \ell-1$.

Secondly, write $k = k'n + r$ ($0 \leq r < n$). Note $n(m-1)-1 = (m-2)n + (n-1)$. Taking $\alpha = (k'+1, \ldots, k'+1, k', \ldots, k')\ell + (h, \ell-1, \ldots, \ell)$, we obtain $|\alpha| = s$, i.e., $\alpha \in \mathbb{Z}_+^n(s, m)$, $\mathcal{E}(\alpha) = (k'+1, \ldots, k'+1, k', \ldots, k')\ell + (h, \ell-1, \ldots, \ell)$, we obtain $|\gamma(\alpha)| = (n-1)(\ell-1) + h$ large enough. So, $E(s)_0 = |\mathcal{E}(\alpha)| = k$.

Finally, as for the estimate of $E(s)$, for $k\ell + (n-1)(\ell-1) \leq s \leq k\ell + n(\ell-1)$, in view of (2), from $n = n'\ell+r$, we get that $(n-1)' = n'-1$ if $r = 0$, and $(n-1)' = n'$ if $r > 0$. Therefore, $E((n-1)(\ell-1)) = (n-1)-(n-1)'-1 + \delta_{n-1,n'\ell} = n-n'-1$ if $r = 0, 1$; and $E((n-1)(\ell-1)) = n-n'-2$ if $r > 1$. So, for the above $s$, we get

$$k + \left(n-n'-1 + \sum_{i=2}^{\ell-1} \delta_{i,n-n'\ell}\right) \leq E(s) \leq k + \left(n-n'-1 + \delta_{n,n'\ell}\right).$$
only if $k + (n - n' - 1 + \delta_{n,n'}) \leq n(m - 1)$. Otherwise, $E(s) = n(m - 1)$.

This completes the proof. \hfill \Box

**Theorem 3.8.** Assume that $n \geq 3$ and $\text{char}(q) = \ell \geq 3$. Then for the $u_q(\mathfrak{sl}_n)$-modules $A^{(s)}_q(n, m)$ with $0 \leq s \leq N$, one has

1. For any nonzero $y \in A^{(s)}_q(n, m)$ with energy degree $\text{Edeg}(y)$, assume that the submodule $\mathfrak{V}_y = u_q(\mathfrak{sl}_n).y$ is simple, then $\text{Edeg}(y) = E(s)$. 

2. $\text{Soc}_q A^{(s)}_q(n, m) = \text{span}_k \{ x^{(\alpha)} \in A^{(s)}_q(n, m) | |E(\alpha)| = E(s) \}$. 

3. $A^{(s)}_q(n, m) = \sum_{\alpha \in \mathbb{Z}^\nu_+(n, m)} \text{soc}(\mathfrak{V}_\alpha)$, where $\mathfrak{V}_\alpha = u_q(\mathfrak{sl}_n).x^{(\alpha)}$.

4. When $0 \leq s \leq \ell - 1$, or $N - (\ell - 1) \leq s \leq N : A^{(s)}_q(n, m)$ is simple, where $\eta = (s, 0, \ldots, 0)$ for $0 \leq s < \ell$, or $\eta = (m\ell - 1, \ldots, m\ell - 1, (m - 1)\ell + h)$ with $s = |\eta| = n(m - 1)\ell + (n - 1)(\ell - 1) + h$, $1 \leq h \leq \ell - 1$, and $x^{(\alpha)}$ is the respective highest weight vector.

5. When $\ell \leq s \leq N - \ell : A^{(s)}_q(n, m)$ is indecomposable. Moreover,

(i) for $(\ell - 1) + 1 \leq s \leq n(\ell - 1) : \text{Soc}_q A^{(s)}_q(n, m) = \mathfrak{V}_\eta$ is simple, where $\eta = (\ell - 1, \ldots, \ell - 1, h, 0, \ldots, 0)$ with $s = |\eta| = j(\ell - 1) + h$, $(1 \leq h \leq \ell - 1, 1 \leq j < n)$, and $x^{(\alpha)}$ is the highest weight vector;

(ii) for $n(\ell - 1) + 1 \leq s \leq N - \ell : \text{Soc}_q A^{(s)}_q(n, m) = \bigoplus_{(\ell(\alpha), k_\alpha) \in E} \mathfrak{V}_\eta(\alpha)$ is non-simple, where $\eta(\alpha) = \{(\kappa_\alpha, 0, \ell(\alpha), \kappa_\alpha - \ell(\alpha) + h) \mid \sum \kappa_i = \kappa, 0 \leq \kappa_i \leq m - 1\}$ with $s = |\eta(\alpha)| = n\ell + h(n - 1)(\ell - 1), (1 \leq \kappa \leq n(m - 1) - 1, 0 \leq h \leq \ell - 1)$, and $x^{(\alpha)}$'s are the respective highest weight vectors.

**Proof.** (1) If $\text{Edeg}(y) > E(s)_{0}$, then by Definition 3.2, in the expression of $y = \sum_{\alpha} k_\alpha x^{(\alpha)}$, there exists some $\beta \in \mathbb{Z}^\nu_+(s, m)$, $k_\beta \neq 0$ such that $|E(\beta)| = \text{Edeg}(y)$. By Proposition 3.6, we can find $u \in u_q(\mathfrak{sl}_n)$ such that $u.x^{(\beta)} \neq 0$ (then $u.y \neq 0$) but $\text{Edeg}(u.y) = \text{Edeg}(u.x^{(\beta)} < \text{Edeg}(y)$, so we get a proper submodule $(0 \neq) \mathfrak{U}_u.y \not\subseteq \mathfrak{V}_y$. It is a contradiction. So the assertion is true.

(2) follows from the conclusion (1), together with Propositions 3.4-3.6. Since for those $\alpha, \beta \in \mathbb{Z}^\nu_+(s, m)$ with $|E(\alpha)| = |E(\beta)| = E(s)_0$, if $\alpha \sim \beta$, then $\mathfrak{V}_{x^{(\alpha)}} = \mathfrak{V}_{x^{(\beta)}}$; by Proposition 3.5; and if $\alpha \sim \beta$, then by Propositions 3.4 & 3.6, $\mathfrak{V}_{x^{(\alpha)}} \cap \mathfrak{V}_{x^{(\beta)}} = 0$.

(3) For any $\alpha \in \mathbb{Z}^\nu_+(s, m)$ with $|\alpha| = s$, according to the pre-ordering $\preceq$ defined in subsection 3.3, we assert that there exists a $\omega \in \mathbb{Z}^\nu_+(s, m)$ with $|E(\omega)| = E(s)$, such that $\omega \succeq \alpha$. Actually, this fact follows from the proof of Lemma 3.7. Since $s = \ell : |E(\alpha)| + |\gamma(\alpha)| = \ell E(s) + h$ (0 $\leq h \leq n(\ell - 1)$), if $|E(\alpha)| < E(s)$, writing $|\gamma(\alpha)| = k\ell + r$ (0 $\leq r < \ell$), then $h = h\ell + r$ and $k \geq k - h = E(s) - |E(\alpha)|$. Construct $\omega = (E_1(\alpha) + j_1, \ldots, E_n(\alpha) + j_n)$ and $\gamma(\omega) = (h_1, \ldots, h_n)$, such that each $E_j(\alpha) + j_j \leq m - 1$, and $\sum h_j = k - h$. Taking $\omega = \ell E(\omega) + \gamma(\omega)$, we get $\omega \succeq m$, $|\omega| = s$, i.e., $\omega \in \mathbb{Z}^\nu_+(s, m)$ and $|E(\omega)| = E(s)$, such that $\omega \succeq \alpha$.

Again, from Proposition 3.6, together with its proof, there is $u \in u_q(\mathfrak{sl}_n)$ such that $u.x^{(\omega)} = x^{(\alpha)}$. Hence, we arrive at the result as stated.

(4) In these two extreme cases, by Lemma 3.7, we have $E(s)_0 = E(s)$. Note that the generating sets of (3) in both cases only contain one equivalent class with respect to the equivalent relation $\sim$ defined in subsection 3.3. Thus, the above conclusions (2) & (3) give us the desired result below:

$A^{(s)}_q(n, m) = \text{Soc}_q A^{(s)}_q(n, m) = \mathfrak{V}_\eta$
is simple, here \( \eta = (s, 0, \ldots, 0) \) for \( 0 \leq s < \ell \), or \( \eta = (m\ell - 1, \ldots, m\ell - 1, (m - 1)\ell + h) \) with \( s = |\eta| = n(m - 1)\ell + (n - 1)(\ell - 1) + h \) \((0 \leq h < \ell)\), \( x^{(n)} \) is the respective highest weight vector (by Theorem 2.5 (2), or Proposition 3.5).

(5) By Lemma 3.7, (2) & (3), we have \( E(s) = E(s) \). Consequently, (2) & (3) give rise to the fact that \( \text{Soc} \mathcal{A}_q^{(s)}(n, m) \subseteq \mathcal{A}_q^{(s)}(n, m) \).

(i) When \( \ell \leq s \leq n(\ell - 1) \): Due to Lemma 3.7, \( E(s) = 0 \). Then those \( n \)-tuples \( \alpha \in \mathbb{Z}_+^n(s, m) \) with \( |E(\alpha)| = E(s) = 0 \) (namely, \( \alpha \leq \tau \)) are equivalent to each other with respect to \( \sim \), and \( \eta = (\ell - 1, \ldots, \ell - 1, h, 0, \ldots, 0) \) is one of their representatives, here \( \eta = j(\ell - 1) + h = s \) \((1 \leq h \leq \ell - 1, 1 \leq j < n)\), i.e., \( \eta \in \mathbb{Z}_+^n(s, m) \).

Hence, \( \text{Soc} \mathcal{A}_q^{(s)}(n, m) = \mathbb{M}_q \) is simple, where \( x^{(n)} \) is the highest weight vector, by Theorem 2.5 (2). Consequently, \( \mathcal{A}_q^{(s)}(n, m) \) is indecomposable.

(ii) When \( n(\ell - 1) + 1 \leq s \leq n(m\ell - 1) - \ell \): Thanks to Lemma 3.7, we can set \( s = k\ell + h + (n - 1)(\ell - 1) \) \((1 \leq k \leq n(m - 1) - 1 \text{ and } 0 \leq h \leq \ell - 1)\), then \( E(s) = 0 \), and \( \kappa + 1 \leq E(s) \leq n(m - 1) \) under the assumption \( n > 2 \) (see Lemma 3.7).

Consider the set of equivalent classes of \( n \)-tuples \( \eta \in \mathbb{Z}_+^n(s, m) \) with \( s = |\eta| \) and \( |E(\eta)| = E(s) = \kappa \). Denote it by \( \wp \). Clearly, those \( \eta \in \wp \) can be constructed as follows: For the given \( \kappa \), set \( \kappa = (\kappa_1, \ldots, \kappa_n) \) \((0 \leq \kappa_i \leq m - 1)\) with \( \sum \kappa_i = \kappa \), \( \gamma = (\ell - 1, \ldots, \ell - 1, h) \) with \( \gamma = (n - 1)(\ell - 1) + h \). Take \( \eta := (\eta(s)) = \ell \cdot \kappa + \gamma \), then \( E(\eta(s)) = \mathbb{M}_q \cdot \gamma \) \((\eta(s)) = \gamma, \text{ as well as } |\eta| = k\ell + h + (n - 1)(\ell - 1) = s \), i.e., \( \eta \in \mathbb{Z}_+^n(s, m) \). So, \( \wp = \{ \eta(s) \in (k\ell + (\ell - 1), \ldots, \kappa_n - 1, \ell - 1, k\ell - h + (\ell - 1), k\ell + h) \mid \sum \kappa_i = \kappa \} \), \( 0 \leq \kappa_i \leq m - 1 \).

According to Proposition 3.5 and the above conclusion (1), we see that \( x^{(\eta(s))} \) is the highest weight vector of the simple module \( \mathcal{A}_q(\eta(s)) \). As the \( n \)-tuples in \( \wp \) are not equivalent with each other with respect to \( \sim \), from the proof of the above conclusion (2), we obtain that \( \text{Soc} \mathcal{A}_q^{(s)}(n, m) = \bigoplus_{\eta(s) \in \wp} \mathbb{M}_q(\eta(s)) \) is non-simple.

Now we claim that \( \mathcal{A}_q^{(s)}(n, m) \) is indecomposable.

(1) Denote \( \mathcal{K}(\kappa) := \{ 0 \leq \kappa = (\kappa_1, \ldots, \kappa_n) \leq (m - 1, \ldots, m - 1) \mid \sum \kappa_i = \kappa \} \). Now let us lexicographically order the \( n \)-tuples in \( \mathcal{K}(\kappa) \) as follows.

\[
\begin{align*}
\kappa &> \kappa - \varepsilon_{n-1} + \varepsilon_n \\
&> \cdots \\
&> \kappa - \varepsilon_j + \varepsilon_{j+1} > \kappa - \varepsilon_j + \varepsilon_{j+2} > \cdots > \kappa - \varepsilon_j + \varepsilon_n \\
&> \cdots \\
&> \kappa - \varepsilon_i + \varepsilon_{i+1} > \kappa - \varepsilon_i + \varepsilon_{i+2} > \cdots > \kappa - \varepsilon_i + \varepsilon_n \quad (i-th \ line \ appears \ if \ \kappa_i > 0) \\
&> \cdots \\
&> \kappa - \varepsilon_1 + \varepsilon_2 > \kappa - \varepsilon_1 + \varepsilon_3 > \cdots > \kappa - \varepsilon_1 + \varepsilon_n > \cdots.
\end{align*}
\]

So \( \mathcal{K}(\kappa), > \) is a totally ordered set. Actually, the lexicographic order \( > \) on each line exactly coincides with the pre-order \( \succeq \) given by the type-A weight system (relative to its prime root system \( \{ \varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i < n \} \)), i.e., \( \kappa + \varepsilon_i - \varepsilon_{i+1} \kappa \Rightarrow \kappa + \varepsilon_i - \varepsilon_{i+1} \Rightarrow \kappa \). The latter pre-order will be used in dealing with the \( u_q(\mathfrak{sl}_n) \)-action below. Now we suppose that \( (\mathcal{K}(\kappa + i), > ) \) is totally ordered for each \( 0 \leq i \leq E(s) - \kappa \).

(II) For any two successive \( n \)-tuples \( (\kappa, \kappa') \) in \( \mathcal{K}(\kappa) \), \( \kappa \succeq \kappa' \), either (i): \( \kappa \succeq \kappa' \) lies in the same line of some \( \kappa'' \) in \( \mathcal{K}(\kappa) \), as shown in the Figure above, then there exist \( i < j < n \), such that \( \kappa'' > 0 \), \( \kappa = \kappa'' - \varepsilon_i + \varepsilon_j \) and \( \kappa' = \kappa'' - \varepsilon_i + \varepsilon_{j+1} \), that is, \( \kappa = \kappa'' + \varepsilon_j - \varepsilon_{j+1} > \kappa' \); or (ii): \( \kappa \) lies in the end of the \( j \)-th line of some \( \kappa'' \) in \( \mathcal{K}(\kappa) \).
i.e., $k''_i > 0$, $k = k''_j - e_j + e_n$, and $k'$ lies in the ahead of $i$ in the $i$-th line with $k''_i = 0$ for $i < j$ and $k''_i > 0$, and $k'' = k''_1 - e_1 + e_{i+1}$. Even for the latter, $k \neq k''$, but we have $k'' = k + e_j - e_n \equiv k$ and $k'' = k' + e_j - e_{i+1} \equiv k'$. 

Now both cases reduce to treat the general case: $(k+e_i-e_j, k)$, with the pre-order $k+e_i-e_j \triangleright k$, where $k+e_i-e_j = (k_1, \cdots, k_i+1, \cdots, k_j-1, \cdots, k_n)$, and $j (> i)$ is the first index such that $k_j \neq 0$. So, there exists a $\Delta+e_i \in K(k+1)$, such that $k+e_i \triangleright k+e_i-e_j$ and $k+e_i \triangleright k$ (Note the pre-order $\triangleright$ here defined as before in subsection 3.3).

To $k+e_i$, we can associate two equivalent $n$-tuples: $\theta_i \sim \bar{\theta}_i \in \mathbb{Z}^n_+ (s, m)$, where

$\theta_i = (k_1 \ell + (k_1+1) \ell, \cdots, k_n \ell + h, \ell), \quad \bar{\theta}_i = (k_1 \ell + (k_1+1) \ell, \cdots, k_n \ell + h, \ell),$

with $\mathcal{E}(\theta_i) = \mathcal{E} (\bar{\theta}_i) = k+e_i$. According to the formulae (1) & (2) in [22, 4.5] and Proposition 4.6 of [22], there are quantum root vectors $f_{\alpha_i}, e_{\alpha_i} \in u_q(\mathfrak{s}_n)$ associated to positive root $\alpha_i = e_i - e_j$, such that $f_{\alpha_i} \cdot \eta(\theta_i) = c_1 \vec{x}(\eta(\theta_i))$, and $e_{\alpha_i} \cdot \eta(\theta_i) = c_2 \vec{x}(\eta(\theta_i))$, $(c_1, c_2 \in k^*)$. However, $u_q(\mathfrak{s}_n)$, $\eta(\theta_i)$, $\eta(\bar{\theta}_i)$, is, that is, $u_q(\mathfrak{s}_n) \oplus u_q(\mathfrak{s}_n+e_i-e_j) \subset u_q(\theta_0) = u_q(\bar{\theta}_0)$.

In summary, for any two successive $n$-tuples $(k, k')$ in $K(k)$ with $k \triangleright k'$, either $u_q(\mathfrak{s}_n) \oplus u_q(\mathfrak{s}_n')$ for $k \triangleright k'$, or $u_q(\mathfrak{s}_n) \oplus u_q(\mathfrak{s}_n') \oplus u_q(\mathfrak{s}_n'')$ for $k'' \triangleright k'$ and $k'' \triangleright k$, can be embedded into a larger indecomposable highest weight submodule generated by highest weight vector $x^\eta(\mathfrak{a}+e_i)$, or the sum of two larger indecomposable highest weight submodules by highest weight vectors $x^\eta(\mathfrak{a}+e_i)$ and $x^\eta(\mathfrak{a}+e_{i+1})$, all lying in a higher energy degree $\kappa + 1$. Because $(K(k), \triangleright)$ is totally ordered, taking over all the two successive $n$-tuples pairs $(k_i, k_{i+1})$, for $i = 1, 2, \cdots, \#K(k)$, we prove that $\text{Soc} A^{(s)}_q(n, m) = \bigoplus_{\gamma \in \mathcal{P}(k)} \mathfrak{u}_q(\mathfrak{s}_n)$ can be pairwise intertwinedly embedded into the sum of larger indecomposable highest weight submodules with generators lying in a higher energy degree $\kappa + 1$.

(III) Finally, note that each $(K(k), \triangleright)$ is totally ordered, for every $i = 0, 1, \cdots, E(s)-\kappa$. Repeating the proof for $K(k)$ in (II), we can lift pairwise intertwinedly the sum of highest weight submodules at each energy level into the sum of highest weight submodules with highest weight vectors lying in a higher one level, up to the top energy level $E(s)$, so that $\text{Soc} A^{(s)}_q(n, m)$ is indecomposable.

We complete the proof. 

\[ \square \]

Remark 3.9. We develop a new “intertwined-lifting” method to prove the indecomposability of $A^{(s)}_q(n, m)$ in the case when its socle submodule is non-simple. Note that the indecomposability of $A^{(s)}_q(n, m)$ when its socle is non-simple depends on our assumption $n > 2$. Its argument is subtle and more interesting. An intrinsic reason for resulting in the indecomposability in this case is revealed by the existing difference between $E(s)$ and $E(s)$ as depicted in our result, see Lemma 3.7 (3), occurred only under the above assumption. Although our module model $A^{(s)}_q(n, m)$ is still valid to the analysis of the submodule structures in the rank 1 case, namely, for $u_q(\mathfrak{s}_2)$, there exists an essential difference between our case here $u_q(\mathfrak{s}_n)$ with $n > 2$ and $u_q(\mathfrak{s}_2)$. While, the indecomposable modules for the latter has been completely solved in different perspectives by many authors, like Chari-Premet [11], Suter [38], Xiao [39], etc. Recently, for the even order of root of unity case, Semikhatov [37] distinctly analyzed the submodules structure of the divided-power quantum plane for the Lusztig small quantum group $\mathfrak{u}_q(\mathfrak{s}_2)$ using a different way.
3.5. Loewy filtration of $A_s^q(n, m)$ and Loewy layers. As shown in Theorem 3.8 (5), for the given $s$ with $\ell \leq s \leq N-\ell$, $A_s^q(n, m)$ is indecomposable. We will adopt a method of the filtration analysis to explore the submodule structures for the indecomposable module $A_s^q(n, m)$.

Set $V_0 = \text{Soc} A_s^q(n, m)$, and for $i > 0$,

$$V_i = \text{span}_k \left\{ x^{(\alpha)} \in A_s^q(n, m) \mid E(s)_{0} \leq \text{Edeg} x^{(\alpha)} \leq E(s)_{0} + i \right\}.$$ 

Obviously, $V_{i-1} \subseteq V_i$, for any $i$.

Denote $K_i(s) := K(E(s)_{0} + i) = \{ k = (\kappa_1, \cdots, \kappa_n) \mid |k| = E(s)_{0} + i, \kappa_i \leq m-1 \},$ for $0 \leq i \leq E(s)_{0} - E(s)_{0}.$

Set $\eta_i = (\ell-1, \cdots, \ell-1, h_i, 0, \cdots, 0)$ and $s_i = |\eta_i| = (t_i-1)(\ell-1) + h_i$, for $1 \leq t_i \leq n$ and $0 \leq h_i \leq \ell - 1.$ Write $\eta(\kappa, i) := \ell \cdot \kappa + \eta_i, s.$ Set $s_i^{(s)} := \{ \eta(\kappa, i) \in \mathbb{Z}_+^n(s, m) \mid s = (E(s)_{0} + i) \ell + s_i \}.$ Particularly, for $n(\ell - 1) + 1 \leq s \leq N - \ell$, $s_{0}^{(s)} = \emptyset$, as defined in Theorem 3.8. Note that for any $\ell \leq s \leq N - \ell$, one has $t_i < n$, for $i > 0$.

**Theorem 3.10.** Suppose $n \geq 3$ and $\text{char}(q) = \ell \geq 3$. For the indecomposable $u_q(sl_n)$-modules $A_s^q(n, m)$ with $\ell \leq s \leq N - \ell$, one has

1. $V_i$’s are $u_q(sl_n)$-submodules of $A_s^q(n, m)$, and the filtration

$$(*) \quad 0 \subset V_0 \subset V_1 \subset \cdots \subset V_{E(s)_{0} - E(s)_{0}} = A_s^q(n, m)$$

is a Loewy filtration of $A_s^q(n, m)$.

2. $x^{(\eta(\kappa, i))} \in A_s^q(n, m)$ are primitive vectors of $V_i$ (relative to $V_{i-1}$), for all $\kappa \in K_i(s)$, and $u_q(sl_n)(x^{(\eta(\kappa, i))} + V_{i-1}) \cong u_q(sl_n), x^{(\eta(n))} = \mathfrak{V}_n.$ Its $i$-th Loewy layer

$$V_i / V_{i-1} = \text{span}_k \left\{ x^{(\alpha)} + V_{i-1} \mid \text{Edeg} x^{(\alpha)} = E(s)_{0} + i \right\} = \bigoplus_{\eta(\kappa, i) \in K_i(s)^{\ell}} u_q(sl_n), (x^{(\eta(\kappa, i))} + V_{i-1})$$

$$\cong \left( \# K_i(s)^{\ell} \right) \mathfrak{V}_n$$

is the direct sum of $\# K_i(s)$ isomorphic copies of simple module $\mathfrak{V}_n = A_s^q(n, 1)$.

**Proof.** By definition of $E(s)_{0}$, $\text{Edeg} (u, x^{(\alpha)}) \geq E(s)_{0},$ only if $u(x^{(\alpha)}) \neq 0$, for any $u \neq u_q(sl_n)$, $x^{(\alpha)} \in V_i$. Meanwhile, Proposition 3.3 gives rise to $\text{Edeg} (u, x^{(\alpha)}) \leq E(s)_{0} + i$. Thus, Definition 3.2 implies that $V_i$ is a $u_q(sl_n)$-submodule of $A_s^q(n, m)$. So we get a filtration $(*)$ of submodules of $A_s^q(n, m)$.

On the other hand, if $\text{Edeg} x^{(\alpha)} = E(s)_{0} + i$, then $x^{(\alpha)} \notin V_{i-1}$, by definition, $V_i / V_{i-1}$ is spanned by $\left\{ x^{(\alpha)} + V_{i-1} \mid \text{Edeg} x^{(\alpha)} = E(s)_{0} + i \right\}$. Assert that $x^{(\eta(\kappa, i))}$ is a primitive vector of $V_i$ relative to $V_{i-1}$ ($i \geq 1$). In fact,

$$e_j, x^{(\eta(\kappa, i)))} = \begin{cases} \left[ \kappa_j \ell + \ell \right] x^{(\eta(\kappa, i)) + \epsilon_j - \epsilon_j+1} = 0, & j < t_i, \\ \left[ \kappa_j \ell + \delta_j + h_i + 1 \right] x^{(\eta(\kappa, i)) + \epsilon_j - \epsilon_j+1}, & j \geq t_i. \end{cases}$$

(i) When $t_i = n$: since $e_j, x^{(\eta(\kappa, i))} = 0$ for $1 \leq j < n$, $x^{(\eta(\kappa, i))}$ is a maximal weight vector.
(ii) When $t_i < n$: either $e_j, x^{(\eta_{(s, i)})} = 0 \in \mathcal{V}_{i-1}$ for $j < t_i$, or $e_j, x^{(\eta_{(s, i)})} = c \cdot x^{(\eta_{(s, i)}) + \varepsilon_j - \varepsilon_{j+1}} \in \mathcal{V}_{i-1}$ for $j \geq t_i$ and $c \in k^*$. So, $x^{(\eta_{(s, i)})}$ is a primitive vector of $\mathcal{V}_i$ relative to $\mathcal{V}_{i-1}$ ($i \geq 1$).

Set $\nabla_{\eta_{(s, i)}} := u_q(\mathfrak{sl}_n), (x^{(\eta_{(s, i)})}) + \mathcal{V}_{i-1})$. By Proposition 3.5 & Theorem 2.5 (2), we get that

$$\nabla_{\eta_{(s, i)}} \cong u_q(\mathfrak{sl}_n), x^{(\eta_{(s, i)})}/(u_q(\mathfrak{sl}_n), x^{(\eta_{(s, i)})} \cap \mathcal{V}_{i-1}) \cong u_q(\mathfrak{sl}_n), x^{(\eta_{(s, i)})} = A_q(s_i)(n, 1).$$

So, $\nabla_{\eta_{(s, i)}}$ is a simple submodule of $\mathcal{V}_i/\mathcal{V}_{i-1}$.

For any $s, s' \in \mathcal{K}_i^{(s)}$ with $s \neq s'$, i.e., $\eta_{(s, i)} \not\succcurlyeq \eta_{(s', i)}$, by Proposition 3.4, $\nabla_{\eta_{(s, i)}}, \nabla_{\eta_{(s', i)}}$ are simple submodules of $\mathcal{V}_i/\mathcal{V}_{i-1}$ with $\nabla_{\eta_{(s, i)}} \cap \nabla_{\eta_{(s', i)}} = 0$, but $\nabla_{\eta_{(s, i)}} \cong \nabla_{\eta_{(s', i)}} \cong \mathfrak{g}_{s_{n, i}} = A_q(s_i)(n, 1)$. As $\varphi_i(s)$ parameterizes the generator set of $\mathcal{V}_i/\mathcal{V}_{i-1}$, $\mathcal{V}_i/\mathcal{V}_{i-1} = \bigoplus_{s \in \mathcal{K}_i^{(s)}} \nabla_{\eta_{(s, i)}} \cong (\#\mathcal{K}_i^{(s)}) \mathfrak{g}_{s_{n, i}}$.

As shown in the proof of Theorem 3.8 (5), $\mathcal{V}_i/\mathcal{V}_{i-2}$ is indecomposable for any $i$ ($1 < i \leq E(s) - E(s_0)$. Hence, the filtration ($\ast$) is not contractible and has the shortest length such that $\mathcal{V}_i/\mathcal{V}_{i-1}$ are semisimple, then it is a Loewy filtration (for definition, see [23]).

As a consequence of Theorem 3.10, we obtain an interesting combinatorial identity below.

**Corollary 3.11.**

(i) $\#\mathcal{K}_i^{(s)} = \sum_{j=0}^{E(s) - E(s_0)} \binom{n}{j} (-1)^j \binom{n+(E(s_0)+i)-jm-1}{n-1}.$

(ii) $\sum_{s: E(s) = E(s_0)} \binom{n+s-im-1}{n-1} = \sum_{i=0}^{E(s)-E(s_0)} \left( \sum_{j=0}^{E(s)-E(s_0)} \binom{n+j}{j} \binom{n+(E(s_0)+i)-jm-1}{n-1} \right) \times \left( \sum_{i=0}^{E(s)-E(s_0)} \binom{n+(E(s_0)+i)-jm-1}{n-1} \right)$, where $s = (E(s_0)+i)\ell + s_i$.

**Proof.** (i) From the definition of $\mathcal{K}_i^{(s)}$, $\#\mathcal{K}_i^{(s)}$ is equal to the coefficient of $t^{E(s_0)+i}$ of polynomial $P_{n,m}(t) = (1 + t + t^2 + \ldots + t^{m-1})^n$. So, it is true, similar to Corollary 2.6.

(ii) follows from (i), Proposition 3.1 & Corollary 2.6, as well as

$$(\otimes) \quad A_q(s)(n, m) \cong \bigoplus_{i=0}^{E(s)-E(s_0)} \mathcal{V}_i/\mathcal{V}_{i-1} \cong \bigoplus_{i=0}^{E(s)-E(s_0)} (\#\mathcal{K}_i^{(s)}) A_q(s_i)(n, 1),$$

as vector spaces.

Now we give an example to show the structural variations of $A_q(s)(n, m)$ by increasing the degree $s$. For $n = 3, m = 2$ and $\ell = 3$, in the following picture, each point represents one simple submodule of a Loewy layer, and each arrow represents the linked relationships existed among the simple subquotients. For example, $a \rightarrow b$ means that there exists $u \in u_q(\mathfrak{sl}_3)$ such that $u.a = b$. 

\[\]
3.6. Rigidity of $A_{q}^{(s)}(n, m)$. As we known, both the radical filtration and the socle filtration of a module $M$ are the Loewy filtrations, and $\text{Rad}^{-k} M \subseteq \text{Soc}^{k} M$, where $r = \ell M$ is the Loewy length of $M$. In this subsection, we will prove the coincidence of both filtrations for $A_{q}^{(s)}(n, m)$, that is the following result.

**Theorem 3.12.** Suppose $n \geq 3$ and $\text{char}(q) = \ell \geq 3$. Then $A_{q}^{(s)}(n, m)$ is a rigid $u_{q}(\mathfrak{sl}_{n})$-module, and $\ell \ell A_{q}^{(s)}(n, m) = E(s) - E(s)_{0} + 1$.

**Proof.** By the definition of rigid module, it suffices to prove that the filtration (*) in Theorem 3.10 is both socle and radical.

(1) Note $\text{Soc}^{0} A_{q}^{(s)}(n, m) = 0$, $\text{Soc}^{1} A_{q}^{(s)}(n, m) = \mathcal{V}_{0}$, by Theorem 3.10. Assume that we have proved $\text{Soc}^{i} A_{q}^{(s)}(n, m) = \mathcal{V}_{i-1}$, for $i \geq 1$. We are going to show $\mathcal{V}_{i}/\mathcal{V}_{i-1} = \text{Soc}(A_{q}^{(s)}(n, m)/\mathcal{V}_{i-1})$, i.e., $\text{Soc}^{i+1} A_{q}^{(s)}(n, m) = \mathcal{V}_{i}$.

As $\mathcal{V}_{i}/\mathcal{V}_{i-1} = \bigoplus_{(\underline{\alpha}, \underline{\beta}) \in \mathcal{V}_{i}} \mathfrak{m}_{E_{\underline{\alpha}, \underline{\beta}}}(\subset A_{q}^{(s)}(n, m)/\mathcal{V}_{i-1})$ is semisimple, $\mathcal{V}_{i}/\mathcal{V}_{i-1} \subseteq \text{Soc}(A_{q}^{(s)}(n, m)/\mathcal{V}_{i-1})$. Note that $\mathcal{V}_{i}$ is spanned by $\{ x^{(\alpha)} \in A_{q}^{(s)}(n, m) \mid \text{Edeg}(x^{(\alpha)}) = E(s)_{0} + i \}$, for each $i \geq 1$. Similarly to Theorem 3.8 (1), we assert that for any nonzero $y \in \mathcal{V}_{i+1} \subseteq A_{q}^{(s)}(n, m)/\mathcal{V}_{i}$ with energy degree $\text{Edeg}(y) \geq E(s)_{0} + i$, assume that the submodule $\mathfrak{m}_{y} = u_{q}(\mathfrak{sl}_{n}).(y + \mathcal{V}_{i-1})$ is simple, then $\text{Edeg}(y) = E(s)_{0} + i$, that is, $y \in \mathcal{V}_{i}$. This gives the desired result.

In fact, if $\text{Edeg}(y) > E(s)_{0} + i$, that is, $\text{Edeg}(y) = E(s)_{0} + j$ with $j > i$, then by Definition 3.2, in the expression of $y = \sum_{\alpha} k_{\alpha} x^{(\alpha)}$, there exists some $\beta \in \mathbb{Z}_{+}^{n} (s, m)$, $k_{\beta} \neq 0$ such that $|E(\beta)| = \text{Edeg}(y)$. Write $\underline{\alpha} = E(\beta)$. Then there exists $\eta(\underline{\alpha}, j) = \ell \cdot \underline{\alpha} + \eta_{j} \in \mathbb{Z}_{+}^{n} (s, m)$ (where $\eta_{j} = (\ell - 1, \ldots, \ell - 1, h_{j}, 0, \ldots, 0)$), with $|\underline{\alpha}| = E(s)_{0} + j \geq j$, (so $\exists k_{\beta} \neq 0$), such that $\eta(\underline{\alpha}, j) \sim \beta$, by the remark in subsection 3.3. Since $j > i \geq 1$, by the note previous to Theorem 3.10, $t_{j} < n - 1$. So, there is $\bar{n}_{j} = (\bar{h}_{1}, \ldots, \bar{h}_{n})$ with $\bar{h}_{0} = 0$, $\bar{h}_{i+1} < \ell - 1$ and $\bar{h}_{k} \leq \ell - 1$ and $|\bar{n}_{j}| = s_{j} = |\eta_{j}|$, such that $\bar{n}_{j} \sim \eta_{j}$, and $\bar{n}(\underline{\alpha}, j) = \ell \cdot \underline{\alpha} + \bar{n}_{j} \sim \eta(\underline{\alpha}, j) \sim \beta$. By Proposition 3.5, we can find $u \in u_{q}(\mathfrak{sl}_{n})$ such that $u \cdot x^{(\beta)} = x^{(\beta)}$. Clearly, for $x^{(\beta)}$, there exists an $f_{\alpha} \in u_{q}(\mathfrak{sl}_{n})$, such that $f_{\alpha} \cdot x^{(\beta)} \neq 0$ (then $f_{\alpha}u(y) \neq 0$) but $\text{Edeg}(f_{\alpha}u, x^{(\beta)}) = \text{Edeg}(x^{(\beta)}) - 1$, so $\text{Edeg}(f_{\alpha}u, y) = \text{Edeg}(f_{\alpha}u, x^{(\beta)}) = \text{Edeg}(u, x^{(\beta)}) - 1 = \text{Edeg}(x^{(\beta)}) - 1 < \text{Edeg}(y)$. Thereby, we get a proper submodule $\mathfrak{m}_{f_{\alpha}u, y} \subset \mathfrak{m}_{u, y} = \mathfrak{m}_{y}$, by Proposition 3.6. It is a contradiction. So the above assertion is true.

The definitions of rigid module, socle filtration, radical filtration can be found in [23, 8, 14], or some relevant elegant investigations on rigidity of a module and Loewy filtration in [23, 25].
(2) By Theorem 3.10, \((\ast)\) is a Loewy filtration of \(A_q^{(s)}(n, m)\), so its Loewy length \(r = \ell \ell A_q^{(s)}(n, m) = E(s) - E(s)_0 + 1\). Then for \(0 \leq i \leq E(s) - E(s)_0\), we have

\[
\text{Rad}^i(A_q^{(s)}(n, m)) \subseteq \text{Soc}^{r-i}(A_q^{(s)}(n, m)) = V_{E(s) - E(s)_0 - i}.
\]

For \(i = 1\): if there exists a \((0 \neq y) \in V_{E(s) - E(s)_0 - 1}\), and \(y \notin \text{Rad}^1(A_q^{(s)}(n, m))\), then by definition, there is a maximal proper submodule \(V \subset A_q^{(s)}(n, m)\) such that \(y \notin V\). Since \(V\) is maximal, \(u_q(\mathfrak{sl}_n)y + V = A_q^{(s)}(n, m) = \sum_{|\alpha| = E(s)} u_q(\mathfrak{sl}_n)x^{(\alpha)}\), by Theorem 3.10 (3). However, \(E_{\text{deg}} u_q y \leq E_{\text{deg}} y = E(s) - 1\), so we derive that \(\{ x^{(\alpha)} \in A_q^{(s)}(n, m) \mid E_{\text{deg}} x^{(\alpha)} = E(s)\} \subseteq V\). Therefore, \(V = A_q^{(s)}(n, m)\), it is contrary to the above assumption. This means \(\text{Rad}^1(A_q^{(s)}(n, m)) = V_{E(s) - E(s)_0 - 1}\).

Assume we have proved that \(\text{Rad}^i(A_q^{(s)}(n, m)) = V_{E(s) - E(s)_0 - i}\) for \(i \geq 1\). Note that \(\text{Rad}^{i+1}(A_q^{(s)}(n, m)) \subseteq V_{E(s) - E(s)_0 - i - 1} \subset V_{E(s) - E(s)_0 - i} = \text{Rad}^i(A_q^{(s)}(n, m))\). By definition, \(\text{Rad}^{i+1}(A_q^{(s)}(n, m))\) is the intersection of all maximal submodule of \(\text{Rad}^i(A_q^{(s)}(n, m))\). According to Theorem 3.10 (2), we have that \(V_{E(s) - E(s)_0 - i - 1}\) is spanned by \(\{ x^{(\alpha)} \in A_q^{(s)}(n, m) \mid E_{\text{deg}} x^{(\alpha)} = E(s) - i\} \). Using the similar argument for \(i = 1\), we can derive \(\text{Rad}^{i+1}(A_q^{(s)}(n, m)) = V_{E(s) - E(s)_0 - i - 1}\).

Consequently, the filtration \((\ast)\) is a radical filtration. \(\square\)

Denote by \(A_q^{(s)}(n)\) the \(s\)-th homogenous space of \(A_q(n)\).

**Corollary 3.13.** Suppose that \(n \geq 3\) and \(\text{char}(q) = \ell \geq 3\). Then \(u_q(\mathfrak{sl}_n)\)-submodules \(A_q^{(s)}(n)\) of \(A_q(n)\) are indecomposable and rigid.

**Proof.** Since for any \(s \in \mathbb{N}\), there is \(m \in \mathbb{N}\) such that \((m-1)\ell \leq s \leq m\ell - 1\), then \(A_q^{(s)}(n, m) = A_q^{(s)}(n)\). By Theorems 3.8 and 3.12, \(A_q^{(s)}(n)\) is indecomposable and rigid \(u_q(\mathfrak{sl}_n)\)-module. \(\square\)

4. Quantum Grassmann algebra and quantum de Rham cohomology

4.1. \(q\)-differential over \(A_q(n)\). Denote by \(\wedge_q(n) = k\{dx_1, \ldots, dx_n\}/((dx_i)^2, dx_j dx_i + q^{-1} dx_i dx_j, i < j\) the quantum exterior algebra over \(k\). Let \(\wedge_q(n)_{(s)}\) be the \(s\)-th homogeneous subspace of \(\wedge_q(n)\), as we know

\[
\wedge_q(n)_{(s)} = \text{span}_k \{ dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_s} \mid 1 \leq i_1 < i_2 < \cdots < i_s \leq n \}.
\]

Identifying \(\wedge_q(n)_{(1)}^{(s)}\) with the \(u_q(\mathfrak{sl}_n)\)-module \(V(\lambda_1)\) with highest weight vector \(dx_1\), then \(\wedge_q(n)_{(1)}^{(s)} \cong k[A_q^{(s)}]\), as \(u_q(\mathfrak{sl}_n)\)-modules.

**Definition 4.1.** Define a linear mapping \(d : A_q(n) \longrightarrow A_q(n) \otimes_k \wedge_q(n)_{(1)}\) as

\[
d x^{(\alpha)} = \sum_{i=1}^n \partial_i x^{(\alpha)} \otimes dx_i = \sum_{i=1}^n q^{-\varepsilon_i * \alpha} x^{(\alpha - \varepsilon_i)} \otimes dx_i, \quad \forall x^{(\alpha)} \in A_q(n).
\]

Then \(d\) is called the \(q\)-differential on \(A_q(n)\).

**Proposition 4.2.** The \(q\)-differential \(d\) is a \(u_q(\mathfrak{sl}_n)\)-module homomorphism, that is, \(d(u x) = u d x, \) for \(u \in u_q(\mathfrak{sl}_n), \) \(x \in A_q(n)\), provided that \(\wedge_q(n)_{(1)} \cong V(\lambda_1)\) as \(u_q(\mathfrak{sl}_n)\)-module with highest weight vector \(dx_1\).
Proof. It suffices to consider the actions of generators of \( u_q(\mathfrak{sl}_n) \) on the basis elements \( x^{(\beta)} \) of \( A_q(n) \).

(1) For \( e_i \) (\( i = 1, \ldots, n-1 \)): On the one hand, noting that
\[
q^{-\epsilon_j x^{(\beta+\epsilon_i-\epsilon_{i+1})}} = q^{-\epsilon_j x^{\beta}}, \quad \text{for } j \leq i, \text{ or } j > i+1,
\]
\[
q^{-\epsilon_{i+1} x^{(\beta+\epsilon_i-\epsilon_{i+1})}} = q^{-\epsilon_{i+1} x^{\beta}} q^{-1}, \quad \text{for } j = i+1,
\]
we have
\[
d(e_i, x^{(\beta)}) = d([\beta_i+1] x^{(\beta+\epsilon_i-\epsilon_{i+1})})
\]
\[
= [\beta_i+1] \sum_{j=1}^{n} q^{-\epsilon_j x^{(\beta+\epsilon_i-\epsilon_{i+1})}} x^{(\beta+\epsilon_i-\epsilon_{i+1}-\epsilon_j)} \otimes dx_j
\]
\[
= [\beta_i+1] \left( \sum_{j<i \text{ or } j>i+1} q^{-\epsilon_j x^{(\beta-\epsilon_i+\epsilon_{i+1})}} \otimes dx_j
\]
\[
+ q^{-\epsilon_i x^{(\beta-\epsilon_{i+1})}} \otimes dx_i + q^{-\epsilon_{i+1} x^{(\beta-2\epsilon_{i+1})}} \otimes dx_{i+1} \right).
\]

On the other hand, as \( \Delta(e_i) = e_i \otimes K_i + 1 \otimes e_i \), we have
\[
e_i, (x^{(\beta-\epsilon_j)} \otimes dx_j) = \begin{cases} 
[\beta_i+1] x^{(\beta-\epsilon_i+\epsilon_{i+1})} \otimes dx_j, & j < i, \text{ or } j > i+1 \\
[\beta_i] q x^{(\beta-\epsilon_i+\epsilon_{i+1})} \otimes dx_i, & j = i \\
x^{(\beta-\epsilon_i)} \otimes dx_i + q^{-1} [\beta_i+1] x^{(\beta+\epsilon_i-2\epsilon_{i+1})} \otimes dx_{i+1} & j = i+1
\end{cases}
\]
Observing that \( q^{-\epsilon_j x^{\beta}} [\beta_i+1] = q^{-\epsilon_j x^{\beta}} [\beta_i] q + q^{-\epsilon_{i+1} x^{\beta}} \), we finally obtain
\[
e_i, d(x^{(\beta)}) = e_i, \sum_{j=1}^{n} q^{-\epsilon_j x^{(\beta-\epsilon_i)} \otimes dx_j} = \sum_{j=1}^{n} q^{-\epsilon_j x^{(\beta-\epsilon_i)}} e_i, (x^{(\beta-\epsilon_j)} \otimes dx_j)
\]
\[
= d(e_i, x^{(\beta)}).
\]

(2) Similarly, we can check that \( d(f_i, x^{(\beta)}) = f_i, (dx^{(\beta)}) \), for \( 1 \leq i < n \).

(3) For \( K_i \) (\( i = 1, \ldots, n-1 \)):
\[
K_i, dx^{(\beta)} = \sum_{j=1}^{n} q^{-\epsilon_j x^{\beta}} K_i, x^{(\beta-\epsilon_i)} \otimes K_i, dx_j
\]
\[
= \sum_{j=1}^{n} q^{-\epsilon_j x^{\beta}} q^{\delta_{i-j}-\delta_{i+1-j}+\delta_{i+1,j}} x^{(\beta-\epsilon_j)} \otimes q^{\delta_{i-j}-\delta_{i+1,j}} dx_j
\]
\[
= d(K_i, x^{(\beta)}).
\]
This completes the proof. \( \square \)

4.2. Quantum Grassmann algebra and quantum de Rham Complex. It is a well-known fact that there exists a braiding \( \triangleq : \wedge_q(n)_{(1)} \otimes A_q(n) \to A_q(n) \otimes \wedge_q(n)_{(1)} \), which is a \( u_q(\mathfrak{sl}_n) \)-module homomorphism. This \( \triangleq \) also induces braiding \( \triangleq_q : \wedge_q(n)_{(2)} \otimes A_q(n) \to A_q(n) \otimes \wedge_q(n)_{(2)} \). Now let us define the quantum Grassmann algebra as follows.
Definition 4.3. Let $\Omega_q(n) := \mathcal{A}_q(n) \otimes \Lambda_q(n)$ with product
\[
(x^{(\alpha)} \otimes \omega_s) \cdot (x^{(\beta)} \otimes \omega_r) = x^{(\alpha)} \otimes (\omega_s \otimes x^{(\beta)}) \otimes \omega_r,
\]
where $\omega_s \in \Lambda_q(n(s))$, $\omega_r \in \Lambda_q(n(r))$.

$\Omega_q(n)$ is said the quantum Grassmann algebra over $\mathcal{A}_q(n)$. $\Omega_q(n) = \bigoplus_{s=0}^{n} \Omega_q(n)^{(s)}$, where $\Omega_q(n)^{(s)} := \mathcal{A}_q(n) \otimes \Lambda_q(n(s))$.

Define the linear mappings as follows.
\[
d^s : \Omega_q(n)^{(s)} \rightarrow \Omega_q(n)^{(s+1)},
\]
\[
d^s (x^{(\alpha)} \otimes dx_{i_1} \wedge \cdots \wedge dx_{i_s}) = \sum_{j=1}^{n} q^{-\varepsilon_j s} x^{(\alpha - \varepsilon_j)} \otimes dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_s}.
\]

Specially, $d^0 = d$ for $s = 0$; $d^n = 0$ for $s = n$.

By Proposition 4.1, $d$ is a homomorphism of $u_q(\mathfrak{sl}_n)$-modules, then by definition, it follows readily that $d^s (s = 1, \ldots, n)$ are homomorphisms of $u_q(\mathfrak{sl}_n)$-modules.

Proposition 4.4. $(\Omega_q(n), d^s)$ is a complex, i.e., $d^{s+1}d^s = 0$, for $s = 0, 1, \ldots, n$.

Proof. Observing the relationships between $d^s$ and $d^0$ for $s = 1, \ldots, n$, it is enough to check the case $s = 0$.

Consider the actions of $d^1 d^0$ over the basis elements of $\mathcal{A}_q(n)$.

For any $x^{(\beta)} \in \mathcal{A}_q(n)$,
\[
d^1 d^0 (x^{(\beta)}) = d^1 \left( \sum_{j=1}^{n} q^{-\varepsilon_j s} x^{(\beta - \varepsilon_j)} \otimes dx_j \right)
\]
\[
= \sum_{j=1}^{n} \sum_{i=1}^{n} q^{-\varepsilon_j s} q^{-\varepsilon_i s} (\beta - \varepsilon_j) x^{(\beta - \varepsilon_i - \varepsilon_j)} \otimes dx_i \wedge dx_j
\]
\[
= \sum_{i < j} (q^{-\varepsilon_j s} - q^{-\varepsilon_i s} \beta - \varepsilon_j) x^{(\beta - \varepsilon_i - \varepsilon_j)} \otimes dx_i \wedge dx_j
\]
\[
= \sum_{i < j} q^{-\varepsilon_i s - \varepsilon_j s} x^{(\beta - \varepsilon_i - \varepsilon_j)} \otimes dx_i \wedge dx_j = 0.
\]

Thus, $d^1 d^0 = 0$. By definition, it is easy to see that $d^{s+1}d^s = 0$, $s = 1, \ldots, n$.

This completes the proof. \hfill \Box

For the complex $(\Omega_q(n), d^s)$ given in Proposition 4.4, that is,
\[
0 \rightarrow \Omega_q(n)^{(0)} \xrightarrow{d^0} \cdots \xrightarrow{d^{i-1}} \Omega_q(n)^{(s)} \xrightarrow{d^s} \Omega_q(n)^{(s+1)} \xrightarrow{d^{s+1}} \cdots \xrightarrow{d^{n-1}} \Omega_q(n)^{(n)} \xrightarrow{d^n} 0,
\]
when $q = 1$, this is the standard de Rham complex of polynomial algebra with $n$ variables. Thus, we call it the quantum de Rham complex.

4.3 Quantum de Rham subcomplex $(\Omega_q(n, m), d^s)$ and its cohomologies. Now define $\Omega_q(n, m) := \bigoplus_{s=0}^{n} \Omega_q(n, m)^{(s)}$, where $\Omega_q(n, m)^{(s)} = \mathcal{A}_q(n, m) \otimes \Lambda_q(n(s))$.

Note that $d^s(\Omega_q(n, m)^{(s)}) \subseteq \Omega_q(n, m)^{(s+1)}$, for $s = 0, 1, \ldots, n$. So, we get a quantum de Rham subcomplex $(\Omega_q(n, m), d^s)$. 
For $\gamma \in \mathbb{Z}_+^n$, denote briefly by $\Omega^{(s)}_\gamma$ the weight space corresponding to the weight $\gamma = \sum_{i=1}^n \gamma_i \epsilon_i$ of $\Omega_q(n, m)^{(s)}$, then
\[
\Omega^{(s)}_\gamma = \text{span}_k \left\{ x^{(h_\gamma - \sum_{j=1}^s \epsilon_j)} \otimes dx_{i_1} \land \cdots \land dx_{i_s} \in \Omega_q(n, m)^{(s)} \mid 0 \leq h_\gamma - \sum_{j=1}^s \epsilon_{i_j} \leq m \right\}.
\]

**Lemma 4.5.** Given $\gamma \in \mathbb{Z}_+^n$ with $k_\gamma$ coordinates equal to $m_\ell$ and $h_\gamma$ coordinates equal to $0$. Then $\Omega^{(s)}_\gamma \neq 0$ if and only if $k_\gamma \leq s$, and dim $\Omega^{(s)}_\gamma = \binom{n - k_\gamma - h_\gamma}{s - k_\gamma}$.

**Proof.** For the given $\gamma \in \mathbb{Z}_+^n$ with $\gamma_1 = \cdots = \gamma_{k_\gamma} = m_\ell$ and $\gamma_{k_\gamma + 1} = \cdots = \gamma_{n_\gamma} = 0$, if $\Omega^{(s)}_\gamma \neq 0$, there exists a pairwise distinct sequence $(j_1, \cdots, j_s)$, such that $\{j_1, \cdots, j_s\} \cap \{1, \cdots, n_\gamma\} = \emptyset$ and $0 \neq x^{(h_\gamma - \sum_{i=1}^s (m_\ell - 1) \epsilon_{j_i})} \otimes dx_{j_1} \land \cdots \land dx_{j_s} \in \Omega^{(s)}_\gamma$, then the pairwise distinct sequence $(1, \cdots, i_{k_\gamma})$ is a subsequence of $(j_1, \cdots, j_s)$, so, $k_\gamma \leq s$. And vice versa. Hence, dim $\Omega^{(s)}_\gamma = \binom{n - k_\gamma - h_\gamma}{s - k_\gamma}$.

**Theorem 4.6.** For the quantum de Rham subcomplex $(\Omega_q(n, m), d^s)$ below,
\[
0 \longrightarrow \Omega_q(n, m)^{(0)} \longrightarrow \cdots \longrightarrow \Omega_q(n, m)^{(s)} \longrightarrow \Omega_q(n, m)^{(s+1)} \longrightarrow \cdots \longrightarrow \Omega_q(n, m)^{(n)} \longrightarrow 0,
\]
one has
\[
H^s(\Omega_q(n, m)) = \text{Ker } d^s / \text{Im } d^{s-1} \cong \bigoplus_{1 \leq i_1 < \cdots < i_s \leq n} k \left[ x^{(\sum_{j=1}^s (m_\ell - 1) \epsilon_{i_j})} \otimes dx_{i_1} \land \cdots \land dx_{i_s} \right],
\]
as $k$-vector spaces, and dim $H^s(\Omega_q(n, m)) = \binom{n}{s}$, for $s = 0, 1, \cdots, n$.

**Proof.** Note the facts that $\Omega_q(n, m)^{(s)} = \bigoplus_{\gamma \in \mathbb{Z}_+^n} \Omega^{(s)}_\gamma$ and each differential $d^s$ preserves the weight-grading. It suffices to consider the restriction of the complex to weight $\gamma$, for any given $\gamma$.
\[
0 \longrightarrow \Omega^{(0)}_\gamma \longrightarrow \cdots \longrightarrow \Omega^{(s)}_\gamma \longrightarrow \Omega^{(s+1)}_\gamma \longrightarrow \cdots \longrightarrow \Omega^{(n)}_\gamma \longrightarrow 0.
\]
If $\gamma$ has $k_\gamma$ coordinates equal to $m_\ell$ and $h_\gamma$ coordinates equal to 0, then by Lemma 4.5, dim $\Omega^{(s)}_\gamma = \binom{n - k_\gamma - h_\gamma}{s - k_\gamma}$.

(1) Consider the action of $d^0$ on $A_q(n, m)$.
For $\gamma = 0$, it is clear that $d^0 x^{(\gamma)} = dx^{(\gamma)} = 0$.
For $\gamma \neq 0$, there exists $\gamma_j \neq 0$ for some $j$ ($1 \leq j \leq n$), then
\[
d^0 x^{(\gamma)} = dx^{(\gamma)} = \sum_{i=1}^n q^{-\epsilon_i} \gamma^{.} \otimes dx_i \neq 0.
\]
So, $\text{Ker } d^0 \cong k$, $H^0(\Omega_q(n, m)) = \text{Ker } d^0 / \text{Im } d^{-1} \cong k$, dim $H^0(\Omega_q(n, m)) = 1$.

(2) Consider the behavior of $d^{s-1}$, $d^s$ at $\Omega^{(s)}$.
For $\gamma \neq 0 \in \mathbb{Z}^n_\geq 0$, by Lemma 4.5, $k_\gamma < s$ if and only if $\Omega^{(s-1)}_\gamma \neq 0$; $k_\gamma = s$ if and only if $\Omega^{(s-1)}_\gamma = 0$ and $\Omega^{(s)}_\gamma \neq 0$; and $k_\gamma > s$ if and only if $\Omega^{(s)}_\gamma = 0 (= \Omega^{(s-1)}_\gamma)$.

Obviously, when $k_\gamma > s$, Im $d^{s-1}|_{\Omega^{(s-1)}_\gamma} = \text{Ker } d^s|_{\Omega^{(s)}_\gamma} = 0$. Namely, this case is no contribution to $H^s(\Omega_q(n, m))$. So, it suffices to consider the cases $k_\gamma \leq s$.

We are now in a position to show the following assertions by induction on $s \geq 1$:
Case (i): When $\Omega^{(s-1)}_\gamma \neq 0$, i.e., $k_\gamma < s$, we must have

$$\text{Im } d^{s-1}|_{\Omega^{(s-1)}_\gamma} = \text{Ker } d^s|_{\Omega^{(s)}_\gamma}, \quad \text{dim } d^s|_{\Omega^{(s)}_\gamma} = \left(\begin{array}{c} n-k_\gamma-h_\gamma-1 \\ s-k_\gamma \end{array}\right).$$

So, this case is also no contribution to $H^s(\Omega_q(n, m))$.

Case (ii): When $\Omega^{(s-1)}_\gamma = 0$ but $\Omega^{(s)}_\gamma \neq 0$, i.e., $k_\gamma = s$, we must have

$$\Omega^{(s)}_\gamma = \text{span}_k \left\{ x^{(\sum_{j=1}^m (m-1)\varepsilon_j)} \otimes dx_i \wedge \cdots \wedge dx_s \right\} = \text{Ker } d^s|_{\Omega^{(s)}_\gamma}.$$

In summary, the above analysis leads to

$$H^s(\Omega_q(n, m)) = \text{Ker } d^s / \text{Im } d^{s-1} = \bigoplus_{\gamma \in \mathbb{Z}^n_r} \text{Ker } d^s|_{\Omega^{(s)}_\gamma} / \text{Im } d^{s-1}|_{\Omega^{(s-1)}_\gamma}$$

and $\text{dim } H^s(\Omega_q(n, m)) = \binom{n}{s}$.

Proofs of cases (i) & (ii):

For $s = 1$: Assume that $0 \neq \gamma \in \mathbb{Z}^n_r$, without loss of generality.

When $\Omega^{(0)}_\gamma \neq 0$, i.e., $k_\gamma = 0$: $0 < \gamma \leq m$, dim $\Omega^{(0)}_\gamma = 1$. This means $\Omega^{(1)}_\gamma \neq 0$.

Now assume $0 \neq \sum_{j=1}^n a_j x^{(\gamma-\varepsilon_j)} \otimes dx_j \in \text{Ker } d^1$ with $a_j \in k$, i.e.,

$$d^1 \left( \sum_{j=1}^n a_j x^{(\gamma-\varepsilon_j)} \otimes dx_j \right) = \sum_{i<j} (a_i q^{-\varepsilon_i \gamma} - a_j q^{-\varepsilon_j \gamma}) x^{(\gamma-\varepsilon_i - \varepsilon_j)} \otimes dx_i \wedge dx_j = 0,$$

we obtain a system of equations with indeterminates $a_i \ (i = 1, \ldots, n)$:

$$a_i q^{-\varepsilon_i \gamma} - a_j q^{-\varepsilon_j \gamma} = 0, \quad \forall \ 1 \leq i < j \leq n,$$

and its solution is $a_j = a_i q^{-\varepsilon_j \gamma + \varepsilon_i \gamma}$, for $1 \leq i < j \leq n$, that is,

$$a_j = a_i q^{-\varepsilon_j \gamma}, \quad \forall \ 1 \leq j \leq n.$$

So, $\dim \text{Ker } d^1|_{\Omega^{(1)}_\gamma} = 1$, $\text{Ker } d^1|_{\Omega^{(1)}_\gamma} = \text{Im } d^0|_{\Omega^{(0)}_\gamma}$, $\text{Im } d^0|_{\Omega^{(0)}_\gamma} = \text{Ker } d^1|_{\Omega^{(1)}_\gamma}$, moreover, $\dim d^1(\Omega^{(1)}_\gamma) = \dim \Omega^{(1)}_\gamma - \dim \text{Ker } d^1|_{\Omega^{(1)}_\gamma} = \binom{n-k_\gamma-h_\gamma-1}{1}$.

When $\Omega^{(0)}_\gamma = 0$ but $\Omega^{(1)}_\gamma \neq 0$, i.e., $k_\gamma = 1$: by Lemma 4.5, $\exists$! $i$, such that $\gamma_i = m\ell$
and $\dim \Omega^{(1)}_\gamma = 1$. This implies $\gamma = m\ell \varepsilon_i$, and $\Omega^{(1)}_\gamma = \text{span}_k \left\{ x^{(m\ell-1)\varepsilon_i} \otimes dx_i \right\} = \text{Ker } d^1|_{\Omega^{(1)}_\gamma}$.

Now for $s > 1$, suppose for any $s' \leq s$, the assertions are true. We consider the case $s+1$:

Assume that

$$d^{s+1} \left( \sum_{i_1 < \cdots < i_{s+1}} a_{i_1 \cdots i_{s+1}} x^{(\gamma-\sum_{j=1}^{s+1} \varepsilon_j)} \otimes dx_{i_1} \wedge \cdots \wedge dx_{i_{s+1}} \right) = 0,$$

we can obtain a system of linear equations with indeterminates $a_{i_1 \cdots i_{s+1}} \ (1 \leq i_1 < \cdots < i_{s+1} \leq n)$,

$$\sum_{j=1}^{s+2} a_{i_1 \cdots \hat{i_j} \cdots i_{s+2}} (-1)^{j-1} q^{-\varepsilon_j \gamma} = 0, \quad \forall \ 1 \leq i_1 < \cdots < i_{s+2} \leq n.$$
Set \( P = \{ i_1 \cdots i_{s+1} \mid i_1 < \cdots < i_{s+1}, x^{(\gamma - \sum_{j=1}^{s+1} \epsilon_{i_j})} \otimes dx_{i_1} \wedge \cdots \wedge dx_{i_{s+1}} \neq 0 \} \),
\( Q = \{ i_1 \cdots i_{s+2} \mid i_1 < \cdots < i_{s+2}, x^{(\gamma - \sum_{j=1}^{s+2} \epsilon_{i_j})} \otimes dx_{i_1} \wedge \cdots \wedge dx_{i_{s+2}} \neq 0 \} \). Denote \( p = \# P \), \( q = \# Q \). Order lexicographically the words in \( P \) and \( Q \) respectively in column to get two column vectors \( P, Q \). Write \( X = (a_{i_1 \cdots i_{s+1}})_{i_1 \cdots i_{s+1} \in P} \). Thereby, we express the system \((\circ)\) of \( q \) linear equations with \( p \) indeterminates \( a_{i_1 \cdots i_{s+1}} \) as a matrix equation \( AX = 0 \), where the coefficients matrix \( A \) is of size \( q \times p \).

When \( \Omega^{(s)}_\gamma \neq 0 \), i.e., \( k_\gamma \leq s \): there exists a unique longest word \( j_1 \cdots j_k \), such that each \( \gamma_{j_k} = m \ell \). By definition, \( j_1 \cdots j_k \) must be a subword of any word \( i_1 \cdots i_{s+1} \) in \( P \), and each \( \gamma_{i_j} \neq 0 \). Now set \( b = \min \{ i \mid \gamma_i \neq 0, 1 \leq i \leq n \} \). Owing to the lexicographic order adopted in \( X \), it is easy to see that there is a diagonal submatrix \( \text{diag}\{ q^{\epsilon_{j_k}}, \cdots, q^{\epsilon_{j_s}} \} \) with order \((n-k_\gamma-h_{\gamma}^{-1})\) in the top right corner of \( A \), which is provided by the front \((n-k_\gamma-h_{\gamma}^{-1})\) equations corresponding to those words \( i_1 \cdots i_{s+2} \) with the beginning letter \( i_1 = b \). Thus, \( \text{rank } A \geq (n-k_\gamma-h_{\gamma}^{-1}) \), and \( \dim \text{Ker } d^{s+1}|_{\Omega^{(s+1)}_\gamma} = \dim \Omega^{(s+1)}_\gamma - \text{rank } A \leq (n-k_\gamma-h_{\gamma}^{-1}) = (n-k_\gamma-h_{\gamma}^{-1}) \).

Note that \( \text{Im } d^{s} \subseteq \text{Ker } d^{s+1} \) and \( d^{s} \Omega^{(s)}_\gamma \subseteq \Omega^{(s+1)}_\gamma \). By the inductive hypothesis, \( \dim d^{s}|_{\Omega^{(s)}_\gamma} = (n-k_\gamma-h_{\gamma}^{-1}) \), so \( \dim \text{Ker } d^{s+1}|_{\Omega^{(s+1)}_\gamma} \geq (n-k_\gamma-h_{\gamma}^{-1}) \).

Therefore, we get \( \dim \text{Ker } d^{s+1}|_{\Omega^{(s+1)}_\gamma} = (n-k_\gamma-h_{\gamma}^{-1}) = \dim d^{s}|_{\Omega^{(s)}_\gamma} \), and

\[
\text{Ker } d^{s+1}|_{\Omega^{(s+1)}_\gamma} = \text{Im } d^{s}|_{\Omega^{(s)}_\gamma},
\]

\[
\text{Im } d^{s+1}|_{\Omega^{(s+1)}_\gamma} = \left( n-k_\gamma-h_{\gamma}^{-1} \right) \left( n-k_\gamma-h_{\gamma}^{-1} \right) = \left( n-k_\gamma-h_{\gamma}^{-1} \right) = \text{rank } A.
\]

When \( \Omega^{(s)}_\gamma = 0 \) but \( \Omega^{(s+1)}_\gamma \neq 0 \), i.e., \( k_\gamma = s+1 \): there are \( s+1 \) \( \gamma_i \)’s equal to \( m \ell \) and \( \dim \Omega^{(s+1)}_\gamma = 1 \). In this case, set \( \gamma_{i_1} = \gamma_{i_2} = \cdots = \gamma_{i_{s+1}} = m \ell \). Then \( \gamma = \sum_{j=1}^{s+1} m \ell \epsilon_{i_j} \) with \( 1 \leq i_1 < \cdots < i_{s+1} \leq n \), and

\[
\Omega^{(s+1)}_\gamma = \text{span } k \{ x^{(\sum_{j=1}^{s+1} (m \ell - 1) \epsilon_{i_j})} \otimes dx_{i_1} \wedge \cdots \wedge dx_{i_{s+1}} \} = \text{Ker } d^{s+1}|_{\Omega^{(s+1)}_\gamma}.
\]

This completes the proof.  \( \square \)

4.4. Cohomology modules. We will concern the module structure on \( H^*(\Omega_{q}(n, m)) \).

DEFINITION 4.7. Let \( V(\epsilon_1, \cdots, \epsilon_{n-1}) \) be a one-dimensional \( u_q(\mathfrak{s} \mathfrak{l}_n) \)-module. It is called a sign-trivial module if for \( 0 \neq v \in V(\epsilon_1, \cdots, \epsilon_{n-1}) \), \( e_i.v = f_i.v = 0 \) and \( K_i.v = \epsilon_i.v \), where \( \epsilon_i = \pm 1, \) for \( i = 1, \cdots, n-1 \).

THEOREM 4.8. For any \( s \) \((0 \leq s \leq n)\), each cohomology group \( H^*(\Omega_{q}(n, m)) \) is isomorphic to the direct sum of \( \binom{n}{s} \) (sign)-trivial \( u_q(\mathfrak{s} \mathfrak{l}_n) \)-modules when \( q \) is an \( \ell \)-th (resp. \( 2\ell \)-th but \( m \) is odd) root of unity or \( m \) is even.

PROOF. When \( s = 0 \), the statement is clear.

It suffices to consider the cases when \( 1 \leq s \leq n \). By Theorem 4.6, we have

\[
H^s(\Omega_{q}(n, m)) \cong \text{span } k \{ x^{(\sum_{j=1}^{s} (m \ell - 1) \epsilon_{i_j})} \otimes dx_{i_1} \wedge \cdots \wedge dx_{i_s} \mid 1 \leq i_1 < \cdots < i_s \leq n \}.
\]

Denote by \( \{ \sum_{j=1}^{s} (m \ell - 1) \epsilon_{i_j} \} \otimes dx_{i_1} \wedge \cdots \wedge dx_{i_s} \subseteq H^s(\Omega_{q}(n, m)) \), where \( \gamma = \sum_{j=1}^{s} (m \ell) \epsilon_{i_j} \).
Consider the actions of the generators $e_i$, $f_i$, $K_i$, $K_i^{-1}$ ($1 \leq i \leq n-1$) of $u_q(\mathfrak{sl}_n)$ on $\left[ x^\left(\sum_{j=1}^{(m-1)\epsilon_i} \right) \otimes dx_{i_1} \wedge \cdots \wedge dx_{i_s} \right]$.

(1) For any $e_h$: if $e_h(x^\left(\sum_{j=1}^{(m-1)\epsilon_i} \right) \otimes dx_{i_1} \wedge \cdots \wedge dx_{i_s}) \neq 0$, then $h+1 \in \{i_1, \ldots, i_s\}$ and $h \notin \{i_1, \ldots, i_s\}$. Write $\gamma = \sum_{j=1}^s (m\ell)\epsilon_i + \varepsilon_h - \varepsilon_{h+1}$, then $e_h(x^\left(\sum_{j=1}^{(m-1)\epsilon_i} \right) \otimes dx_{i_1} \wedge \cdots \wedge dx_{i_s}) \in \Omega_{\gamma}^{(s)} \cap \text{Ker} d^s = \text{Ker} d^s\vert_{\Omega_{\gamma}^{(s)}}$.

Since $k_{\gamma'} = s-1$, $h_{\gamma'} = h_{\gamma'-1} = n-s-1$, by Lemma 4.5, $\dim \Omega_{\gamma'}^{(s-1)} = 1$. So now the problem reduces to Case (i) in the proof of Theorem 4.6. We then obtain $\text{Ker} d^s\vert_{\Omega_{\gamma'}^{(s)}} = \text{Im} d^{s-1}\vert_{\Omega_{\gamma'}^{(s-1)}} \neq 0$. Thus $e_h(x^\left(\sum_{j=1}^{(m-1)\epsilon_i} \right) \otimes dx_{i_1} \wedge \cdots \wedge dx_{i_s}) \in \text{Im} d^{s-1}$, namely, $e_h(x^\left(\sum_{j=1}^{(m-1)\epsilon_i} \right) \otimes dx_{i_1} \wedge \cdots \wedge dx_{i_s}) = 0$.

Therefore, $e_h$ acts trivially on $H^s(\Omega_\gamma(n, m))$.

(2) Dually, we can show that $f_h$ trivially acts on $H^s(\Omega_\gamma(n, m))$.

(3) For $K_i^{\pm 1}$: we have

$$K_i^{\pm 1}. [ x^\left(\sum_{j=1}^{(m-1)\epsilon_i} \right) \otimes dx_{i_1} \wedge \cdots \wedge dx_{i_s} ] = q^{\pm (\gamma_i - \gamma_{i+1})} x^\left(\sum_{j=1}^{(m-1)\epsilon_i} \right) \otimes dx_{i_1} \wedge \cdots \wedge dx_{i_s}.$$  

Notice that $\gamma_i - \gamma_{i+1} = \pm m\ell$ or $0$, for $\gamma = \sum_{j=1}^s (m\ell)\epsilon_i$, where $r_i \in \mathbb{Z}$.

(i) When $q$ is the $\ell$-th primitive root of unity or $m$ is even, $q^{\pm (\gamma_i - \gamma_{i+1})} = 1$, the submodule generated by $[ x^\left(\sum_{j=1}^{(m-1)\epsilon_i} \right) \otimes dx_{i_1} \wedge \cdots \wedge dx_{i_s} ]$ is a trivial module.

(ii) When $q$ is the $2\ell$-th primitive root of unity but $m$ odd, $q^{\pm (\gamma_i - \gamma_{i+1})} = \pm 1$, then the submodule generated by $[ x^\left(\sum_{j=1}^{(m-1)\epsilon_i} \right) \otimes dx_{i_1} \wedge \cdots \wedge dx_{i_s} ]$ is a sign-trivial module.

Hence, $H^s(\Omega_\gamma(n, m))$ is isomorphic to the direct sum of $\binom{n}{s}$ sign-trivial $u_q(\mathfrak{sl}_n)$-modules. \hfill \Box

4.5. Quantum de Rham cohomologies $H^s(\Omega_\gamma(n))$. In this final subsection, we turn to give a description of the cohomologies for the quantum de Rham complex $(\Omega_\gamma(n), d^\bullet)$. Actually, Lemma 4.5 and the result of Case (i) in the proof of Theorem 4.6 are still available to the $(\Omega_\gamma(n), d^\bullet)$.

**Proposition 4.9.** For the quantum de Rham complex $(\Omega_\gamma(n), d^\bullet)$ over $\mathcal{A}_\gamma(n)$:

$$0 \longrightarrow \Omega_\gamma(n)^{(0)} \xrightarrow{d^s} \cdots \xrightarrow{d^{s-1}} \Omega_\gamma(n)^{(s)} \xrightarrow{d^s} \Omega_\gamma(n)^{(s+1)} \xrightarrow{d^{s+1}} \cdots \xrightarrow{d^{n-1}} \Omega_\gamma(n)^{(n)} \xrightarrow{d^n} 0,$$

one has $H^s(\Omega_\gamma(n)) = \delta_{0, s} k$, for any $s = 0, 1, \ldots, n$.

**Proof.** Clearly, we have $H^0(\Omega_\gamma(n)) = k$.

For any given $\gamma \in \mathbb{Z}_n^+$, since each $d^\bullet$ preserves the weight-grading, we have

$$0 \longrightarrow \Omega_\gamma(n)^{(0)} \xrightarrow{d^s} \cdots \xrightarrow{d^{s-1}} \Omega_\gamma(n)^{(s)} \xrightarrow{d^s} \Omega_\gamma(n)^{(s+1)} \xrightarrow{d^{s+1}} \cdots \xrightarrow{d^{n-1}} \Omega_\gamma(n)^{(n)} \xrightarrow{d^n} 0.$$

By definition, $H^s(\Omega_\gamma(n)) = \bigoplus_{\gamma \in \mathbb{Z}_n^+} \text{Ker} d^s\vert_{\Omega_\gamma(n)^{(s)}} / \text{Im} d^{s-1}\vert_{\Omega_\gamma(n)^{(s-1)}}$. So, for the given $0 < \gamma < \mathbb{Z}_n^+$, there exists an $m \in \mathbb{N}$, such that $m\ell \geq |\gamma|$. This means that $\Omega_\gamma(n)^{(s)} = \Omega_\gamma(n, m)^{(s)}$, for any $s \geq 1$, and $k_\gamma = 0$. So, Lemma 4.5 is adapted to our case, namely, $\Omega_\gamma(n, m)^{(s-1)} = \Omega_\gamma(n, m)^{(s-1)} \neq 0$, for $1 \leq s \leq n$. According to the proof of Theorem 4.6, the result of Case (i) works, that is, $\text{Ker} d^s\vert_{\Omega_\gamma(n, m)^{(s)}} = \text{Im} d^{s-1}\vert_{\Omega_\gamma(n, m)^{(s-1)}}$, for any given $\gamma$. This implies $H^s(\Omega_\gamma(n)) = 0$ for $s \geq 1$. \hfill \Box
References

[1] H.H. Andersen; J.C. Jantzen and W. Soergel, *Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p: independence of p*, Astérisque 220 (1994), 321 pp.

[2] H.H. Andersen; P. Polo and K. Wen, *Representations of quantum algebras*, Invent. Math. 104 (1991), 1–59.

[3] I. Assem; D. Simson and A. Skowronski, *Elements of the Representation Theory of Associative Algebras, 1: Techniques of Representation Theory*, London Math. Soc. Students Texts 65, Cambridge University Press, 2006.

[4] E. Backelin and K. Kremnizer, *Quantum flag varieties, equivariant quantum D-modules, and localization of quantum groups*, Adv. in Math. 203 (2006), 408–429.

[5] —, *Localization for quantum groups at a root of unity*, J. Amer. Math. Soc. 21 (4) (2008), 1001–1018.

[6] R. Bezrukavnikov; I. Mirkovic and D. Rumynin, *Localization of modules for a semisimple Lie algebra in prime characteristic*, (with an appendix by Bezrukavnikov and Simon Riche), Ann. of Math. 167 (3) (2008), 945–991.

[7] C. Bowman; S. Doty and S. Martin, *Decomposition of tensor products of modular irreducible representations for SL_3* (with an appendix by C. Ringel), Intern. Elec. J. Algebra, 9 (2011), 177–219.

[8] K. A. Brown and I. Gordon, *The ramification of centres: Lie algebras in positive characteristic and quantised enveloping algebras*, Math. Z. 238 (4) (2001), 733–779.

[9] N. Cantarini, *The quantized enveloping algebra U_q(sl_n) at the roots of unity*, Comm. Math. Phys. 211 (1) (2000), 207–230.

[10] N. Cantarini; G. Carnovale and M. Costantini, *Spherical orbits and representations of U_q(g)*, Transform. Groups 10 (1) (2005), 29–62.

[11] V. Chari and A. Premet, *Indecomposable restricted representations of quantum sl_2*, Intern. Math. Res. Notices 12 (1997), 541–553.

[12] C. De Concini and V. G. Kac, *Representations of quantum groups at roots of 1*, “Operator algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory” (Paris 1989), pp. 471–506, Progr. Math. 92, Birkhäuser Boston, Boston, MA, 1990.

[13] C. De Concini; V. G. Kac and C. Procesi, *Quantum coadjoint action*, J. Amer. Math. Soc. 5 (1) (1992), 151–189.

[14] —, *Some remarkable degenerations of quantum groups*, Comm. Math. Phys. 157 (1993), 404–427.

[15] C. De Concini and C. Procesi, *Quantum Groups, “D-modules, Representation Theory, and Quantum Groups*” (Venice, 1992), pp. 31–140, Lecture Notes in Math., 1565, Springer, Berlin, 1993.

[16] C. De Concini; C. Procesi; N. Reshetikhin and M. Rosso, *Hopf algebras with trace and representations*, Invent. Math. 161 (1) (2005), 1–44.

[17] B. Feigin; A. Gainutdinov; A. Semikhatov and I. Tipunin, *Modular group representations and fusion in logarithmic conformal field theories and in the quantum group center*, Comm. Math. Phys. 265 (2006), 47–93.

[18] J. Feldvoss and S. Witherspoon, *Support varieties and representation type of small quantum groups*, Intern. Math. Res. Notices 7 (2010), 1346–1362.

[19] P. Fiebig, *Sheaves on affine Schubert varieties, modular representations, and Lusztig’s conjecture*, J. Amer. Math. Soc. 24 (1) (2011), 133–181.

[20] A. Gainutdinov; A. Semikhatov I. Tipunin and B. Feigin, *The Kazhdan-Lusztig correspondence for the representation category of the triplet W-algebra in logarithmic conformal field theories*, Theoret. Math. Phys. 148 (3) (2006), 1210–1235.

[21] Naihong Hu, *Quantum divided power algebra, q-derivatives, some new quantum groups*, J. Algebra 232 (2000), 507–540.

[22] J. E. Humphreys, *Representations of Semisimple Lie Algebras in the BGG Category O*, Graduate Studies in Math., vol. 94, Amer. Math. Soc., Providence R. I., 2008.

[23] R. S. Irving, *Projective modules in the category O_S: Loewy series*, Trans. Amer. Math. Soc. 291 (2) (1985), 733–754.
[25] —, The socle filtration of a Verma module, Ann. Sci. École Norm. Sup. (4) 21 (1) (1988), 47–65.
[26] J. C. Jantzen, Lectures on Quantum Groups, Graduate Studies in Mathematics, vol. 6, Amer. Math. Soc., Providence R.I, 1996.
[27] Hiroki Kondo and Yoshihisa Saito, Indecomposable decomposition of tensor products of modules over the restricted quantum universal enveloping algebra associated to $sl_2$, J. Algebra 330 (1) (2011), 103–129.
[28] G. Lusztig, Quantum deformations of certain simple modules over enveloping algebras, Adv. Math. 70 (1988), 237–249.
[29] —, Modular representations and quantum groups, Contemp. Math. 82, 59–77, 1989.
[30] —, Finite dimensional Hopf algebras arising from quantized universal enveloping algebras, J. Amer. Math. Soc. 3 (1990), 257–296.
[31] —, Introduction to Quantum Groups, Birkhauser, Boston, Progress in Math. 110, 1993.
[32] Yu. I. Manin, Notes on quantum groups and de Rham complexes, Teor. Mat. Fiz. 92 (1992), 425–450.
[33] I. Mirkovic and D. Rumynin, Geometric representation theory of restricted Lie algebras of classical type, Transformation Groups 2 (2001), 175–191.
[34] J. Murakami and K. Nagatomo, Logarithmic knot invariants arising from restricted quantum groups, Intern. J. Math. 19 (2008), 1203–1213.
[35] K. Nagatomo and A. Tsuchiya, The triplet vertex operator algebra $W(p)$ and the restricted quantum group $u_q(sl_2)$ at $q = e^{2\pi i/p}$, Exploring new structures and natural constructions in mathematical physics, 1–49, Adv. Stud. Pure Math., 61, Math. Soc. Japan, Tokyo, 2011. arXiv: 0902.4607.
[36] M. Rosso, Finite dimensional representations of the quantum analog of the enveloping algebra of a complex simple Lie algebra, Comm. Math. Phys. 117 (1988), 581–593.
[37] A. M. Semikhatov, Quantum $sl_2$ action on a divided-power quantum plane at even roots of unity, Theor. and Math. Phys. 164 (1) (2010), 853–868.
[38] R. Suter, Modules over $u_q(sl_2)$, Comm. Math. Phys. 163 (1994), 359–393.
[39] J. Xiao, Finite dimensional representations of $U_q(sl_2)$ at root of unity, Can. J. Math. 49 (1997), 772–787.
[40] S. L. Woronowicz, Differential calculus on compact matrix pseudogroups (quantum groups), Comm. Math. Phys. 122 (1989), 125–170.

Department of Mathematics, East China Normal University, Minhang Campus, Dong Chuan Road 500, Shanghai 200241, PR China
E-mail address: alla0824@126.com

Department of Mathematics, East China Normal University, Min Hang Campus, Dong Chuan Road 500, Shanghai 200241, PR China
E-mail address: nhhu@math.ecnu.edu.cn