CONFORMAL TORI WITH ALMOST NON-NEGATIVE SCALAR CURVATURE

JIANCHUN CHU AND MAN-CHUN LEE

ABSTRACT. In this work, we consider sequence of metrics with almost non-negative scalar curvature on torus. We show that if the sequence is uniformly conformal to another sequence of metrics with uniformly controlled geometry, then it converges to a flat metric in the volume preserving intrinsic flat sense, $L^p$ sense and the measured Gromov-Hausdorff sense.

1. Introduction

In the study of Riemannian geometry, the notion of curvature plays a significant role. As the average of sectional curvature, the scalar curvature is one of the simplest curvature invariants on a Riemannian manifold. In general, the scalar curvature can be regarded as a kind of weak measure of the local geometry. It is tempting to ask which manifolds can admit metric of positive scalar curvature. When the underlying manifold is torus $\mathbb{T}^n$, the Geroch Conjecture predicted that metrics with non-negative scalar curvature must be flat. The problem was solved by Schoen-Yau [21, 22] for $n \leq 7$ using minimal surface method and Gromov-Lawson [14] for general $n$ using Atiyah-Singer index theorem for a twisted spinor bundle on a spin manifold.

In [13], Gromov conjectured a stability for the torus rigidity. Namely, a sequence of Riemannian manifolds with almost non-negative scalar curvature, which are diffeomorphic to tori, combined with appropriate compactness conditions should converge to a flat torus in some weak sense. In [20], Sormani had formulated the conjecture more concretely using the notion of intrinsic flat distance which is a distance between integral current spaces and was introduced by Sormani-Wenger [27]. This is believed to be the suitable notion for taking limits of manifolds with lower scalar curvature bounds, see also the recent work of the second named author, Naber and Neumayer in [19] which suggested that the geodesic distance should be replaced by the $L^p$ version of distance function $d_p$ at least when $n > 3$. For general dimension, it is unclear what conditions should serve as the non-collapsing assumption. While in $n = 3$, Sormani [26] gave a precise prediction of the non-collapsing conditions called the MinA condition in order to avoid bubbling occurring.
The first result in this direction is given by Gromov in [13] where if one assumes that a sequence of tori with almost non-negative scalar curvature converges in the $C^0$ sense to a $C^2$ metric, then one can show that the $C^2$ limit is a flat Riemannian metric. In [6], Bamler gave an alternative proof using the Ricci flow to perform regularization. It was recently generalized by Burkhardt-Guim [8] to the case when the limiting metrics $g_\infty$ are only $C^0$ metric and it was shown that $g_\infty$ is isometric to the flat torus as a metric space. Further progresses toward the conjecture formulated by Sormani [26] has been made in various cases. In [2], Allen, Hernandez-Vazquez, Parise, Payne, and Wang studied the warped product case. In [9], Cabrera Pacheco, Ketterer, and Perales studied the case of graphical tori. In [1], Allen studied the case when the sequence is conformal to the flat tori. In [19], the second named author with Naber and Neumayer considered the case when the entropy is small and proved the stability in the sense of $d_p$ convergence.

Motivated by the work of Allen in [1], we study the case when metrics are uniformly conformal to some metrics with well-controlled geometry on $\mathbb{T}^n$. We show that if the conformal factor is bounded in $L^{p_0}$ for some sufficiently large $p_0$, then the sequence will converge to a flat metric in the volume preserving intrinsic flat sense.

**Theorem 1.1.** Let $(M_i, g_i)$ be a sequence of $n$-dimensional Riemannian manifolds ($n > 2$). Suppose that $M_i$ is diffeomorphic to $\mathbb{T}^n$ and $g_i = u_i^{\frac{4}{n-2}} h_i$ for some metric $h_i$ such that

1. $|\text{Ric}(h_i)|_{h_i} + (\text{inj}(M_i, h_i))^{-1} + \text{diam}(M_i, h_i) \leq \Lambda$;
2. $\text{diam}(M_i, g_i) + \|u_i\|_{L^{p_0}(M_i, h_i)} \leq \Lambda$;
3. $\text{Vol}(M_i, g_i) \geq \Lambda^{-1}$;
4. $R(g_i) \geq -\delta_i$.

for some $\Lambda > 0$, $p_0 > \frac{n}{2} + \frac{2n(n+4)}{(n-2)(n+2)}$ and $\delta_i \to 0$. After passing to a subsequence, $(M_i, g_i)$ converges to a flat torus in the volume preserving intrinsic flat sense.

**Remark 1.1.** By the Hölder inequality, condition (ii) and (iii) will imply a lower bound of $\text{Vol}(M_i, h_i)$. In this case, the lower bound of injectivity radius of $h_i$ will follow from the work of Cheeger-Gromov-Taylor [10] if we strengthen the curvature bound of $h_i$ from $\text{Ric}(h_i)$ to $\text{Rm}(h_i)$.

In Theorem 1.1, $h_i$ serves as a sequence of well-behaved reference metrics within the conformal classes. In certain sense, if the metric within the conformal has almost non-negative scalar curvature, then it is almost the solution to the Yamabe problem. The volume condition (iii) is necessary in order to prevent collapsing. Condition (ii) is stronger than volume non-expanding which plays the role to rule out bubbling. If we strengthen condition (ii) from $L^{p_0}$ to $L^\infty$, better convergence can be obtained. Note that in this case, the upper bound of $\text{diam}(M_i, g_i)$ follows from condition (i).
Theorem 1.2. Let \((M_i, g_i)\) be a sequence of \(n\)-dimensional Riemannian manifolds \((n > 2)\). Suppose that \(M_i\) is diffeomorphic to \(\mathbb{T}^n\) and \(g_i = u_i^{-\frac{4}{n-2}}h_i\) for some metric \(h_i\) such that

(i) \(|\text{Ric}(h_i)|_{h_i} + (\text{inj}(M_i, h_i))^{-1} + \text{diam}(M_i, h_i) \leq \Lambda; \)
(ii) \(\sup_{M_i} u_i \leq \Lambda; \)
(iii) \(\text{Vol}(M_i, g_i) \geq \Lambda^{-1}; \)
(iv) \(R(g_i) \geq -\delta_i\)

for some \(\Lambda > 0\) and \(\delta_i \to 0\). After passing to a subsequence, \((M_i, g_i)\) converges to a flat torus in \(L^p\) for all \(p > 0\) modulo diffeomorphism. Moreover, \((M_i, g_i)\) converges to a flat torus in the measured Gromov-Hausdorff sense.

For detailed statement, we refer readers to Section 3. The main technique is motivated by that of [6, 8] which used the Ricci flow to regularize the metrics. In our case, we do not a-priori assume any convergence of \(g_i\) and thus Ricci flow’s estimates do not apply. Instead, we make use of the Yamabe flow which is a geometric heat flow evolving inside the conformal class to (partially) regularize the metric \(g_i\). Unlike the Ricci flow, its regularization ability is relatively limited and depends on the uniform geometry of \(h_i\). In our case, although the corresponding Yamabe flow \(g_i(t)\) is not uniformly regular in \(C^\infty\), we are still able to show that it converges to a fixed flat metric away from \(t = 0\) weakly modulo diffeomorphism. This reduces the problem to establishing uniform weak convergence of \(g_i(t)\) as \(t \to 0\). This will be done in Section 3.

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2. A-priori estimates along the Yamabe flow

In the following, we will consider a Riemannian manifold \((M, h)\) which satisfies

\[
(A) : \quad \left\{ \begin{array}{l}
|\text{Ric}(h)|_h \leq \Lambda; \\
\text{diam}(M, h) \leq \Lambda; \\
\text{inj}(M, h) \geq \Lambda^{-1}
\end{array} \right.
\]

for some \(\Lambda > 0\). For later use, here we recall the Sobolev inequality and the Poincaré inequality: for \(f \in W^{1,2}(M), \)

\[
(2.1) \quad \left( \int_M |f|^{\frac{2n}{n-2}} d\mu_h \right)^{\frac{n-2}{2}} \leq C_S \int_M |\nabla^h f|^2 d\mu_h + C_S \int_M f^2 d\mu_h,
\]

and

\[
(2.2) \quad \int_M |f - f|_{\mu_h}^2 d\mu_h \leq C_P \int_M |\nabla^h f|^2_{\mu_h}, \quad f = \frac{1}{\text{Vol}(M, h)} \int_M f d\mu_h.
\]
Note that the lower bound of injective radius implies the lower bound of volume (see [7]), and then constants $C_S$ (see [17]) and $C_P$ (see [20]) depend only on $n$ and $\Lambda$.

Let $g_0 = u_0^{\frac{4}{n-2}} h$ be a metric inside the conformal class of $h$. Suppose that

$$
\begin{align*}
(B) : \quad & \|u_0\|_{L^{p_0}(M,h)} \leq \Lambda; \\
& \text{Vol}(M,g_0) \geq \Lambda^{-1}; \\
& R(g_0) \geq -\delta \geq -1
\end{align*}
$$

for some $p_0, \Lambda, \delta > 0$. In this section, we will use the Yamabe flow to regularize $g_0$ slightly. This is the family of metric $g(t)$ which solves

$$
\begin{align*}
\frac{\partial g}{\partial t} = -R(g)g, \\
g(0) = g_0.
\end{align*}
$$

Equivalently, if we write $g(t) = u(t)^{\frac{4}{n-2}} h$, then the function $u(t)$ solves

$$
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{n+2}{4} \left( \frac{4(n-1)}{n-2} \Delta_h u - R_h u \right), \\
u(0) &= u_0,
\end{align*}
$$

where $N = \frac{n+2}{n-2}$ and $R_h$ denotes the scalar curvature of $h$. We will establish a-priori estimates of $g(t)$ or equivalently $u(t)$ along the Yamabe flow.

### 2.1. Lower bound of the Yamabe flow

We first show that if $p_0$ is sufficiently large, then assumption $(B)$ implies the positive lower bound of $u_0$. We begin with the following lemma, which is a global version of [16, (4.8)].

**Lemma 2.1.** If $p_0 > \frac{2n}{n-2}$, then there is a constant $\varepsilon_0(n, \Lambda, p_0) > 0$ such that

$$
\left( \int_M u_0^{\varepsilon_0} d\mu_h \right) \left( \int_M u_0^{-\varepsilon_0} d\mu_h \right) \leq 4e^2.
$$

**Proof.** The proof is similar to that of the local version [16, (4.8)]. For the sake of completeness, we include the proof here. Since $g_0 = u_0^{\frac{4}{n-2}} h$, then the scalar curvature of $g_0$ is

$$
R_{g_0} = u_0^{-\frac{n+2}{n-2}} \left( R_h u_0 - \frac{4(n-1)}{n-2} \Delta_h u_0 \right).
$$

Define

$$
w = \log u_0 - \frac{1}{\text{Vol}(M,h)} \int_M (\log u_0) d\mu_h.
$$

Direct calculation shows

$$
R_{g_0} = u_0^{\frac{4}{n-2}} \left( R_h - \frac{4(n-1)}{n-2} \frac{\Delta_h u_0}{u_0} \right)
$$

$$
= u_0^{\frac{4}{n-2}} \left( R_h - \frac{4(n-1)}{n-2} \left( \Delta_h w + |\nabla_h w|^2_h \right) \right).
$$
Recalling $R_{g_0} \geq -1$,

\[(2.9) \quad |\nabla^h w|_h^2 = -\Delta_h w + \frac{n-2}{4(n-1)} \left( R_h - R_{g_0} u_0^{\frac{4}{n-2}} \right) \leq -\Delta_h w + C u_0^{\frac{4}{n-2}} + C\]

for some $C(n, \Lambda)$. Since $p_0 > \frac{2n}{n-2} > \frac{4}{n-2}$, we obtain

\[(2.10) \quad \int_M |\nabla^h w|_h^2 d\mu_h \leq \int_M (C u_0^{\frac{4}{n-2}} + C) d\mu_h \leq C \|u_0\|_{p_0}^{p_0} + C \leq C(n, \Lambda_0, p_0).\]

Combining this with the Sobolev inequality and the Poincaré inequality,

\[(2.11) \quad \left( \int_M |w|^{\frac{2n}{n-2}} d\mu_h \right)^\frac{n-2}{n} \leq C_S \int_M |\nabla^h w|_h^2 d\mu_h + C_S \int_M |w|^2 d\mu_h \leq C_S(1 + C_P) \int_M |\nabla^h w|_h^2 d\mu_h \leq C(n, \Lambda, p_0).\]

Define two constants $\beta$ and $\beta^*$ by

\[(2.12) \quad \frac{4\beta}{n-2} = p_0, \quad \frac{1}{\beta} + \frac{1}{\beta^*} = 1.\]

Then $p_0 > \frac{2n}{n-2}$ implies $\beta > \frac{n}{2}$ and $\beta^* < \frac{n}{n-2}$. Using (2.9), (2.10) and the Hölder inequality, for $p > 0$, we compute

\[(2.13) \quad \int_M |w|^p |\nabla^h w|_h^2 d\mu_h \leq p \int_M |w|^{p-1} |\nabla^h w|_h^2 d\mu_h + C \int_M |w|^p |\nabla^h u_0^{\frac{4}{n-2}}| d\mu_h + C \int_M |w|^p d\mu_h \leq \frac{1}{2} \int_M |w|^p |\nabla^h w|_h^2 d\mu_h + (Cp)^p \int_M |\nabla^h w|_h^2 d\mu_h + C \left( \int_M |w|^{p\beta^*} d\mu_h \right)^\frac{1}{p^*} + C \left( \int_M |w|^{p\beta} d\mu_h \right)^\frac{1}{p},\]

which implies

\[(2.14) \quad \int_M |\nabla (|w|^{\frac{p+2}{2}})|_h^2 d\mu_h \leq C(p+2)^2 \left( \int_M |w|^{p\beta^*} d\mu_h \right)^\frac{1}{p^*} + (C(p+2))^{p+2},\]
Combining this with the Sobolev inequality and the Hölder inequality,
\[
\left(\int_M |w|^{\frac{n(p+2)}{n-2}} d\mu_h\right)^{\frac{n-2}{n}} \leq C(p+2)^2 \left(\int_M |w|^{p\beta^*} d\mu_h\right)^{\frac{1}{p\beta^*}} + C\int_M |w|^{p+2} d\mu_h + (C(p+2))^{p+2}
\]
\[
\leq C(p+2)^2 \left(\int_M |w|^{(p+2)\beta^*} d\mu_h\right)^{\frac{1}{p\beta^*}} + (C(p+2))^{p+2}.
\]

It then follows that
\[
\|w\|_{L^n(p+2)} \leq \frac{C(n,p+2)}{n} \|w\|_{L^{(p+2)\beta^*}} + C(p+2), \quad \text{for } p > 0.
\]

Replacing \((p+2)\beta^*\) by \(p\), and writing \(\gamma = n\beta^*\frac{n-2}{2}\), we obtain
\[
\|w\|_{L^\gamma} \leq (Cp^\gamma) \|w\|_{L^p} + C, \quad \text{for } p > 2\beta^*.
\]

Recalling \(\beta^* < \frac{n}{n-2}\), we have \(\gamma > 1\). Combining (2.11) and (2.17), we obtain
\[
\|w\|_{L^\gamma^k} \leq (C\gamma^{k-1}) \frac{C}{\gamma^{k-1}} \|w\|_{L^\gamma^{k-1}} + C\gamma^{k-1}, \quad \text{for } k \in \mathbb{N},
\]
and so
\[
\|w\|_{L^\gamma^k} \leq C(n,\Lambda,p_0)\gamma^k, \quad \text{for } k \in \mathbb{N}.
\]

This shows
\[
\|w\|_{L^p} \leq C_0(n,\Lambda,p_0)p, \quad \text{for } p \geq 1.
\]

Choosing \(\varepsilon_0(n,\Lambda,p_0)\) such that \(C_0\varepsilon_0 \leq \frac{1}{2}\), then
\[
\int_M e^{\varepsilon_0|w|} d\mu_h = \sum_{k=0}^{\infty} \int_M \frac{\varepsilon_0^k|w|^k}{k!} d\mu_h \leq \sum_{k=0}^{\infty} \frac{(C_0\varepsilon_0)^k}{k!} \leq e \sum_{k=0}^{\infty} (C_0\varepsilon_0)^k \leq 2e.
\]

Recalling the definition of \(w\) (2.7), we obtain
\[
\left(\int_M u_0^{\varepsilon_0} d\mu_h\right) \left(\int_M \frac{1}{u_0^{\varepsilon_0}} d\mu_h\right) = \left(\int_M e^{\varepsilon_0w} d\mu_h\right) \left(\int_M e^{-\varepsilon_0w} d\mu_h\right) \leq 4e^2.
\]

\[\square\]

**Lemma 2.2.** If \(p_0 > \frac{2n}{n-2}\), then there is a constant \(c(n,\Lambda,p_0) > 0\) such that
\[
\inf_M u_0(x) \geq c(n,\Lambda,p_0).
\]

**Proof.** By (2.6), we have
\[
\frac{4(n-1)}{n-2} \Delta_h u_0 = R_h u_0 - R_{g_0} u_0 \frac{n+2}{n-2}.
\]
Using $R_{g_0} \geq -1$, the function $v = u_0^{-1}$ satisfies
\[
\Delta_h v = -v^2 \Delta_h u_0 + 2v^3 |\nabla^h u_0|^2_h
\]
(2.25)
\[
\geq -\frac{(n-2)v}{4(n-1)} \left( R_h - R_{g_0} u_0^{\frac{4}{n-2}} \right)
\]
\[
\geq -C(n)\Lambda v - C(n) u_0^{\frac{1}{n-2}}.
\]
Define two constants $\beta$ and $\beta^*$ by
(2.26)
\[
\frac{4\beta}{n-2} = p_0, \quad \frac{1}{\beta} + \frac{1}{\beta^*} = 1.
\]
Then $p_0 > \frac{2n}{n-2}$ implies $\beta > \frac{n}{2}$ and $\beta^* < \frac{n}{n-2}$. For $p > 1$, multiplying $-v^{p-1}$ on both sides of (2.25) and integrating on $(M, h)$, we have
(2.27)
\[
\frac{4(p-1)}{p^2} \int_M |\nabla^h v|^2_h d\mu_h = -\int_M (v^{p-1} \Delta_h v) d\mu_h
\]
\[
\leq C \int_M v^p d\mu_h + C \int_M (u_0^{\frac{4}{n-2}} v^p) d\mu_h
\]
\[
\leq C \int_M v^p d\mu_h + C \left( \int_M u_0^{\frac{1}{n-2}} d\mu_h \right)^{\frac{1}{\beta}} \left( \int_M v^{p\beta^*} d\mu_h \right)^{\frac{1}{\beta^*}}
\]
for some $C(n, \Lambda)$. Combining this with $\|u_0\|_{L^{p_0}} \leq \Lambda$, the Sobolev inequality and the Hölder inequality, for $p > 1$,
(2.28)
\[
\frac{p-1}{p^2} \left( \int_M v^{\frac{p}{n-2}} d\mu_h \right)^{\frac{n-2}{n}} \leq C(n, \Lambda, p_0) \left( \int_M v^{p\beta^*} d\mu_h \right)^{1/\beta^*}.
\]
Recalling $\beta^* < \frac{n}{n-2}$, we may apply the iteration method to conclude that
(2.29)
\[
\|v\|_{L^\infty} \leq C(n, \Lambda, p_0) \|v\|_{L^{p_0}},
\]
where $\varepsilon_0$ is the constant in Lemma 2.1.

It remains to control $L^{\varepsilon_0}$ bound of $v = u_0^{-1}$. Thanks to Lemma 2.1, it suffices to establish the positive lower bound of $\|u_0\|_{L^{p_0}}$. We may assume $\varepsilon_0 = \frac{2n\sigma}{n-2}$ for some $\sigma < 1$. Otherwise, the required estimate follows from $\operatorname{Vol}(M, g_0) \geq \Lambda^{-1}$ and the Hölder inequality. For any $\gamma \in (0, \sigma)$,
(2.30)
\[
\Lambda^{-1} \leq \operatorname{Vol}(M, g_0) = \int_M u_0^{\frac{2n}{n-2}} d\mu_h = \int_M u_0^\frac{\varepsilon_0\gamma + 2n(1-\gamma)}{n-2} d\mu_h
\]
\[
\leq \left( \int_M u_0^{\frac{\varepsilon_0\gamma}{\sigma}} d\mu_h \right)^{\frac{\sigma}{\gamma}} \left( \int_M u_0^{\frac{2n(1-\gamma)}{(n-2)(\sigma-\gamma)}} d\mu_h \right)^{\frac{\sigma-\gamma}{\sigma}}.
\]
Since $p_0 > \frac{2n}{n-2}$, we may choose $\gamma < \sigma$ such that
(2.31)
\[
\frac{2n\sigma(1-\gamma)}{(n-2)(\sigma-\gamma)} = p_0.
\]
Using \( \|u_0\|_{L^{p_0}} \leq \Lambda \), we obtain

\[
\int_M u_0^{p_0} d\mu_g \geq c(n, \Lambda, p_0) > 0,
\]

which completes the proof. \(\square\)

Using Lemma 2.2, we establish the lower bound of the Yamabe flow:

**Proposition 2.1.** Let \( g(t) = u(t)h \) be a solution of the Yamabe flow \((2.3)\). Suppose that \( g(t) \) exists on \( M \times [0, T] \). If \( p_0 > \frac{2n}{n-2} \), then there are constants \( \hat{T} \) and \( c \) depending only on \((n, \Lambda, p_0)\) such that

\[
\inf_{M \times [0, \hat{T} \wedge T]} u \geq c > 0.
\]

**Proof.** Applying the minimum principle to \((2.4)\) and using Lemma 2.2, for \((x, t) \in M \times [0, T]\),

\[
(2.33) \quad u(x, t) \geq \left(-C(n) \Lambda t + \inf_M u_0^{\frac{n-2}{4}}\right)^{\frac{n-2}{4}} \geq \left(-C(n) \Lambda t + c_0(n, \Lambda, p_0)\right)^{\frac{n-2}{4}}.
\]

Choosing \( \hat{T} = \frac{c_0}{2C(n)\Lambda} \), we complete the proof. \(\square\)

2.2. **Upper bound of the Yamabe flow.** Next, we will show that if \( p_0 \) is sufficiently large, then along the Yamabe flow, \( u(t) \) will be bounded from above instantaneously.

**Proposition 2.2.** Let \( g(t) = u(t)h \) be a solution of the Yamabe flow \((2.3)\). Suppose that \( g(t) \) exists on \( M \times [0, T] \) for some \( T \leq 1 \). If \( \frac{n}{2} + \frac{2n(n+4)}{(n-2)(n+2)} < p_0 < \infty \), then for any \( \left(\frac{2p_0}{n} - \frac{4(n+4)}{(n-2)(n+2)}\right)^{-1} \alpha < 1 \), there is a constant \( C(\alpha, n, \Lambda, p_0) \) such that

\[
\sup_M u(\cdot, t) \leq C t^{-\alpha}, \quad \text{for } t \in [0, T].
\]

If \( p_0 = \infty \), then there is a constant \( C(n, \Lambda) \) such that

\[
\sup_{M \times [0, T]} u \leq C.
\]

**Proof.** When \( \frac{n}{2} + \frac{2n(n+4)}{(n-2)(n+2)} < p_0 < \infty \), we split the proof into two steps:

**Step 1.** \( \int_0^T \int_M u^{\frac{(n+2)p_0}{n}} d\mu_g dt \leq C(n, \Lambda, p_0) \).

It is clear that \( p_0 > N = \frac{n+2}{n-2} \). Multiplying both sides of \((2.4)\) by \( u^{p_0-N} \) and integrating by parts,

\[
(2.34) \quad \frac{N}{p_0} \frac{\partial}{\partial t} \left( \int_M u^{p_0} d\mu_g \right) + \frac{4N(n-1)(p_0-N)}{(p_0-N+1)^2} \int_M |\nabla h u^{\frac{p_0-N+1}{2}}|^2 d\mu_g = -\frac{n+2}{4} \int_M R_h u^{p_0-N+1} d\mu_g \leq C(n) \Lambda \int_M u^{p_0-N+1} d\mu_g.
\]
Integrating this on \([0, T]\) and using \(\|u_0\|_{L^{p_0}} \leq \Lambda\), we obtain

\[
\sup_{t \in [0, T]} \left( \int_M u(t)^{p_0} \, d\mu_h \right) + \int_0^T \int_M \left| \nabla^{\frac{p_0}{n} + 1} u \right|^2 \, d\mu_h \, dt \\
\leq C(n) \Lambda \int_0^T \int_M u^{p_0 - N + 1} \, d\mu_h \, dt + C(n, \Lambda, p_0).
\] (2.35)

Combining this with the Sobolev inequality,

\[
\sup_{t \in [0, T]} \left( \int_M u(t)^{p_0} \, d\mu_h \right) + \int_0^T \left( \int_M u \, d\mu_h \right)^{\frac{n-2}{n}} \, dt \\
\leq C(n) \Lambda \int_0^T \int_M u^{p_0 - N + 1} \, d\mu_h \, dt + C(n, \Lambda, p_0).
\] (2.36)

Using \(N > 1\) and the Young inequality,

\[
C(n) \Lambda \int_0^T \int_M u^{p_0 - N + 1} \, d\mu_h \, dt \leq \frac{1}{2} \sup_{t \in [0, T]} \left( \int_M u(t)^{p_0} \, d\mu_h \right) + C(n, \Lambda, p_0).
\] (2.37)

Then

\[
\sup_{t \in [0, T]} \left( \int_M u(t)^{p_0} \, d\mu_h \right) + \int_0^T \left( \int_M u \, d\mu_h \right)^{\frac{n-2}{n}} \, dt \leq C(n, \Lambda, p_0).
\] (2.38)

By the Hölder inequality,

\[
\int_0^T \int_M u \left( \frac{n+2}{n} N + 1 \right) \, d\mu_h \, dt = \int_0^T \int_M u^{p_0 - N + 1} + \frac{2m}{n} \, d\mu_h \, dt \\
\leq \left( \int_0^T \left( \int_M u \left( \frac{n+2}{n} N + 1 \right) \, d\mu_h \right)^{\frac{n-2}{n}} \, dt \right) \sup_{t \in [0, T]} \left( \int_M u^{p_0}(t) \, d\mu_h \right)^{\frac{2}{n}} \leq C(n, \Lambda, p_0).
\] (2.39)

**Step 2.** \(\sup_M u(\cdot, t) \leq C(\alpha, n, \Lambda, p_0) t^{-\alpha}\).

Define \(v = t^\alpha u\). For \(p > \max(\alpha^{-1}, 2)\), multiplying both sides of (2.4) by \(t^{\alpha - 1}\),

\[
t^\alpha v^{p-1} \frac{\partial u^N}{\partial t} - N(n-1) v^{p-1} \Delta h v = -\frac{n+2}{4} R_h v^p.
\] (2.40)

We compute the first term of the left hand side:

\[
t^\alpha v^{p-1} \frac{\partial u^N}{\partial t} = t^{\alpha p} v^{p-1} \frac{\partial u^N}{\partial t} = \frac{N t^{\alpha p}}{N + p - 1} \frac{\partial u^{N+p-1}}{\partial t} \\
= \frac{N}{N + p - 1} \frac{\partial}{\partial t} (t^{\alpha p} u^{N+p-1}) - \frac{\alpha N p}{N + p - 1} u^{N+p-1} \\
= \frac{N}{N + p - 1} \frac{\partial}{\partial t} (v^{p-1} u^{N-1}) - \frac{\alpha N p}{N + p - 1} v^{p-1} u^{N+p-1}.
\] (2.41)
Then

$$\frac{N}{N + p - 1} \frac{\partial}{\partial t} (v^pu^{N-1}) - N(n-1)v^{p-1}\Delta hv$$

$$= -\frac{n+2}{4} R_h v^p + \frac{\alpha Np}{N + p - 1} v^{p-\frac{1}{\alpha}} u^{N+\frac{1}{\alpha}-1}$$

$$\leq C(n)\Lambda v^p + C(\alpha, n) v^{p-\frac{1}{\alpha}} u^{N+\frac{1}{\alpha}-1}.$$

Integrating both sides on $(M, h)$,

$$\frac{N}{N + p - 1} \frac{\partial}{\partial t} \left( \int_M v^pu^{N-1}d\mu_h \right) + \frac{4N(n-1)(p-1)}{p^2} \int_M |\nabla^h v|^2 \, d\mu_h$$

$$\leq C(\alpha, n, \Lambda) \int_M \left( v^p + v^{p-\frac{1}{\alpha}} u^{N+\frac{1}{\alpha}-1} \right) \, d\mu_h.$$

Recalling that $v(0) = 0$, the above inequality shows

$$\sup_{t \in [0, T]} \left( \int_M v^p(t)u^{N-1}(t)d\mu_h \right) + \int_0^T \int_M |\nabla^h v|^2 \, d\mu_h \, dt$$

$$\leq C(\alpha, n, \Lambda) p \int_0^T \int_M \left( v^p + v^{p-\frac{1}{\alpha}} u^{N+\frac{1}{\alpha}-1} \right) \, d\mu_h \, dt.$$

By the Sobolev inequality,

$$\sup_{t \in [0, T]} \left( \int_M v^p(t)u^{N-1}(t)d\mu_h \right) + \int_0^T \left( \int_M \frac{v^{\frac{n-2}{n}}}{n} \, d\mu_h \right)^{\frac{n}{n-2}} \, dt$$

$$\leq C(\alpha, n, \Lambda) p \int_0^T \int_M \left( v^p + v^{p-\frac{1}{\alpha}} u^{N+\frac{1}{\alpha}-1} \right) \, d\mu_h \, dt.$$

Since $T \leq 1$, then $v = t^\alpha u \leq u$, and so

$$\sup_{t \in [0, T]} \left( \int_M v^{p+N-1}(t)d\mu_h \right) + \int_0^T \left( \int_M \frac{v^{\frac{n}{n-2}}}{n} \, d\mu_h \right)^{\frac{n}{n-2}} \, dt$$

$$\leq C(\alpha, n, \Lambda) p \int_0^T \int_M \left( v^p + v^{p-\frac{1}{\alpha}} u^{N+\frac{1}{\alpha}-1} \right) \, d\mu_h \, dt.$$

Using $N > 1$, we obtain

$$\sup_{t \in [0, T]} \left( \int_M v^p(t)d\mu_h \right) + \int_0^T \left( \int_M \frac{v^{\frac{n}{n-2}}}{n} \, d\mu_h \right)^{\frac{n}{n-2}} \, dt$$

$$\leq C p \int_0^T \int_M \left( v^p + v^{p-\frac{1}{\alpha}} u^{N+\frac{1}{\alpha}-1} \right) \, d\mu_h \, dt + C.$$

(2.47)
for some $C(\alpha, n, \Lambda)$. Combining this with the Hölder inequality,

$$\int_0^T \int_M v^{(n+2)p_n} d\mu_h dt = \int_0^T \int_M v^{p_0} \frac{2p_0}{n} d\mu_h dt$$

(2.48)

$$\leq \left( \int_0^T \left( \int_M v^{\frac{np}{n}} d\mu_h \right)^{\frac{n-2}{n}} dt \right) \sup_{t \in [0,T]} \left( \int_M v^p(t) d\mu_h \right)^{\frac{2}{n}}$$

$$\leq \left( Cp \int_0^T \int_M \left( v^p + v^{p-\frac{1}{\alpha}} u^{N+\frac{1}{\alpha}-1} \right) d\mu_h dt + C \right)^{\frac{n+2}{n}}.$$

Then

(2.49)

$$\left( \int_0^T \int_M v^{\frac{(n+2)p}{n}} d\mu_h dt \right)^{\frac{n+p}{n+2}} \leq Cp \int_0^T \int_M \left( v^p + 1 \right) \left( u^{N+\frac{1}{\alpha}-1} + 1 \right) d\mu_h dt + C$$

$$\leq C p \left( \left( \int_0^T \int_M v^{\beta p} d\mu_h dt \right)^{\frac{1}{\beta}} + 1 \right) \left( \left( \int_0^T \int_M u^{\frac{(n+2)p_0}{n}} - N+1 d\mu_g dt \right)^{\frac{1}{\beta}} + 1 \right),$$

for some $C(\alpha, n, \Lambda, p_0)$. Recalling $\alpha > \left( \frac{2p_0}{n} - \frac{4(n+4)}{(n-2)(n+2)} \right)^{-1}$ and $N = \frac{n+2}{n-2}$, we obtain

(2.50)

$$\frac{1}{\beta} + \frac{1}{\beta^*} = 1, \quad \beta^* = \frac{(n+2)p_0}{n} - N + 1$$

Combining (2.49) with Step 1,

(2.51)

$$\left( \int_0^T \int_M v^{\frac{(n+2)p}{n}} d\mu_h dt \right)^{\frac{n+p}{n+2}} \leq C p \left( \int_0^T \int_M v^{\beta p} d\mu_h dt \right)^{\frac{1}{\beta}} + C p$$

for some $C(\alpha, n, \Lambda, p_0)$. Recalling $\alpha > \left( \frac{2p_0}{n} - \frac{4(n+4)}{(n-2)(n+2)} \right)^{-1}$ and $N = \frac{n+2}{n-2}$, we obtain

(2.52)

$$\beta^* = \frac{(n+2)p_0}{n} - N + 1 = \frac{(n+2)p_0}{n} - \frac{4}{n-2} > \frac{n+2}{2},$$

which implies $\beta < \frac{n+2}{n}$. Applying the iteration method and Step 1, we obtain

(2.53)

$$\sup_{M \times [0,T]} (t^\alpha u) = \sup_{M \times [0,T]} v \leq C(\alpha, n, \Lambda, p_0) \left( \int_0^T \int_M v^{p_0} d\mu_h dt \right)^{\frac{1}{p_0}}.$$

Combining this with (2.38), $v \leq u$ and $T \leq 1$,

(2.54)

$$\sup_{M \times [0,T]} (t^\alpha u) \leq C(\alpha, n, \Lambda, p_0) \left( \int_0^T \int_M u^{p_0} d\mu_h dt \right)^{\frac{1}{p_0}} \leq C(\alpha, n, \Lambda, p_0).$$
When $p_0 = \infty$, applying the maximum principle to (2.4) and using $T \leq 1$, for any $(x, t) \in M \times [0, T]$,

\[
(2.55) \quad u(x, t) \leq \left( C(n) \Lambda t + \sup_{M} u^\frac{4}{n-2} \right)^\frac{n-2}{4} \leq C(n, \Lambda).
\]

\[\square\]

2.3. More estimates along the Yamabe flow.

**Proposition 2.3.** Let $g(t) = u(t)^{\frac{4}{n-2}} h$ be a solution of the Yamabe flow (2.3). If $p_0 > \frac{n}{2} + \frac{2n(n+4)}{(n-2)(n+2)}$, then there is a constant $T_0(n, \Lambda, p_0)$ such that $g(t)$ exists on $[0, T_0]$. Moreover, for all $[a, T_0] \subset (0, T_0]$, there is a constant $\lambda(a, n, \Lambda, p_0) > 1$ such that

\[
\begin{align*}
\lambda^{-1} h &\leq g(t) \leq \lambda h; \\
|\nabla h g(t)|_h &\leq \lambda; \\
-\delta &\leq R g(t) \leq \lambda
\end{align*}
\]

on $M \times [a, T_0]$.

**Proof.** Denote the maximal existence time of the Yamabe flow (2.3) by $T_{\text{max}}$. First, we establish the lower bound of $T_{\text{max}}$. By [5, Lemma 2.2], for any $\beta \in (0, 1)$, the $C^{1,\beta}$ harmonic radius of $h$ is bounded from below by $r(\beta, n, \Lambda)$. Within harmonic radius, the Laplacian operator of $h$ can be expressed as $\Delta_h = h_{ij} \partial_i \partial_j$, and then (2.4) can be written as

\[
(2.57) \quad \frac{\partial u}{\partial t} - (n - 1) u^{-\frac{4}{n-2}} h_{ij} \partial_i \partial_j u = -\frac{n-2}{4} R_h u^{\frac{n-6}{n-2}}.
\]

Write $T_0 = \min(\hat{T}, 1)$, where $\hat{T}$ is the constant in Proposition 2.1. We will show $T_{\text{max}} > T_0$. If $T_{\text{max}} \leq T_0$, then by Proposition 2.1 and 2.2 for any $a \in (0, T_{\text{max}})$,

\[
(2.58) \quad C^{-1}(a, n, \Lambda, p_0) \leq \inf_{M \times [a, T_{\text{max}}]} u \leq \sup_{M \times [a, T_{\text{max}}]} u \leq C(a, n, \Lambda, p_0).
\]

This shows the equation (2.57) is uniformly parabolic on $[a, T_{\text{max}}]$ with $L^\infty$ inhomogeneous term. Since $h$ is uniformly equivalent to the Euclidean metric $g_{\text{euc}}$ in the harmonic coordinate system, we obtain uniform $C^{\delta, \frac{\delta}{2}}$ estimate on a slightly smaller ball by [18] for some $\delta(a, n, \Lambda, p_0)$. The covering argument shows

\[
(2.59) \quad \|u\|_{C^{\delta, \frac{\delta}{2}}(M \times [a, T_{\text{max}}])} \leq C(a, n, \Lambda, p_0).
\]

Combining this with the parabolic Schauder estimates [12], we obtain the higher order estimates of $u$, which contradicts with the definition of $T_{\text{max}}$. Then we obtain $T_{\text{max}} > T_0$.

Next, we establish the required estimates. The estimate $\lambda^{-1} h \leq g(t) \leq \lambda h$ follows from (2.58). Within $C^{1,\beta}$ harmonic radius, $h$ and $g_{\text{euc}}$ are uniformly
comparable in $C^{1,\beta}$. Using (2.59), $|R_h| \leq n\Lambda$, the parabolic $L^p$ estimate [12] and Sobolev embedding theorem, for $p > 1$, 
(2.60) $\|u\|_{C^1} \leq C(a, n, \Lambda, \delta, p_0)$, $\|u\|_{W^{2,p}} \leq C(p, a, n, \Lambda, \delta, p_0)$.
onumber
on $[a, T_0]$. This shows $|\nabla^h g(t)|_h \leq \lambda$.

The scalar curvature $R_g(t)$ satisfies (see [11] Lemma 2.2)),
(2.61) $\left( \frac{\partial}{\partial t} - (n-1)\Delta_g(t) \right) R_g(t) = R^2_g(t)$.

The lower bound $R_g(t) \geq -\delta$ follows from the minimum principle. For the upper bound of $R_g(t)$ on $[a, T_0]$, (2.60) shows for $p > 1$,
(2.62) $\|R_g(t)\|_{L^p} \leq C(p, a, n, \Lambda, \delta, p_0)$.

Combining this with (2.59), the equation (2.61) has $C^\delta$ parabolic coefficient and $L^p$ inhomogeneous term for any $p > 1$ in the local coordinate system. Then $R_g(t) \leq \lambda$ follows from the parabolic $L^p$ estimate [12] and Sobolev embedding theorem.

Lastly, we will establish the weak convergence of $g(t)$ as $t \to 0$.

Proposition 2.4. Let $g(t) = u(t)^{\frac{4}{n-2}} h$ be a solution of the Yamabe flow (2.3) on $[0, T_0]$, where $T_0$ is the constant in Proposition [2.3]. If $\frac{n}{2} + \frac{2n(n+4)}{(n-2)(n+2)} < p_0 < \infty$, then for any
(2.63) $\left| \text{Vol}(M, g(t)) - \text{Vol}(M, g_0) \right| \leq C t^{1-\alpha}$.

If $p_0 = \infty$, then for any $p > 0$, there is a constant $C(p, n, \Lambda)$ such that
(2.64) $\int_M |g(t) - g(0)|^p h d\mu_h \leq C(t + t^p)$.

Proof. When $\frac{n}{2} + \frac{2n(n+4)}{(n-2)(n+2)} < p_0 < \infty$, along the Yamabe flow (2.3), using $R_g(t) \geq -\delta \geq -1$ (see (2.56)), we obtain
(2.65) $\frac{d}{dt} \text{Vol}(M, g(t)) = -\frac{n}{2} \int_M R_g(t) d\mu_g(t) \leq \frac{n}{2} \text{Vol}(M, g(t))$.

It then follows that
(2.66) $\text{Vol}(M, g(t)) \leq e^\frac{n}{2} \text{Vol}(M, g_0) \leq \text{Vol}(M, g_0) + C(n) t \text{Vol}(M, g_0)$.

By volume comparison theorem, we have $\text{Vol}(M, h) \leq C(n, \Lambda)$. Combining this with $\|u_0\|_{L^{p_0}} \leq \Lambda$ and $p_0 > \frac{2n}{n-2}$, we obtain
(2.67) $\text{Vol}(M, g_0) = \int_M u_0^{\frac{4}{n-2}} d\mu_h \leq C(n, \Lambda, p_0)$

and so
(2.68) $\text{Vol}(M, g(t)) \leq \text{Vol}(M, g_0) + C(n, \Lambda, p_0) t$. 

It suffices to estimate the lower bound. By Proposition 2.1, 2.2 and $R_g(t) \geq -\delta \geq -1$ (see (2.56)),

$$\frac{d}{dt} \text{Vol}(M, g(t)) = -\frac{n}{2} \int_M u^{N+1} R_g d\mu_h$$

$$\geq -\frac{C}{t^\alpha} \int_M u^N(R_g + 1) d\mu_h$$

$$= -\frac{C}{t^\alpha} \int_M u^N R_g d\mu_h - \frac{C}{t^\alpha} \int_M u^{-1} d\mu_g(t)$$

$$\geq -\frac{C}{t^\alpha} \int_M (R_h u - C(n) \Delta_h u) d\mu_h - \frac{C}{t^\alpha} \text{Vol}(M, g(t))$$

$$\geq -\frac{C}{t^\alpha} (\text{Vol}(M, g(t)))^{\frac{\alpha}{2N}} - \frac{C}{t^\alpha} \text{Vol}(M, g(t)),$$

where $C(\alpha, n, \Lambda, p_0) > 0$. Using (2.67) and (2.68), we obtain $\text{Vol}(M, g(t)) \leq C(n, \Lambda, p_0)$, and then

$$\frac{d}{dt} \text{Vol}(M, g(t)) \geq -C(\alpha, n, \Lambda, p_0) \frac{1}{t^\alpha}.$$

This implies

$$\frac{d}{dt} \text{Vol}(M, g(t)) \geq \text{Vol}(M, g(t)) - C(\alpha, n, \Lambda, p_0) t^{1-\alpha}$$

Combining (2.68) and (2.71), we obtain

$$|\text{Vol}(M, g(t)) - \text{Vol}(M, g_0)| \leq (\alpha, n, \Lambda, p_0) t^{1-\alpha}.$$

When $p_0 = \infty$, for $t \in [0, T_0]$, Proposition 2.1 and 2.2 show

$$C(n, \Lambda)^{-1} \leq u(t) \leq C(n, \Lambda), \quad C(n, \Lambda)^{-1} h \leq g(t) \leq C(n, \Lambda) h.$$

Using $R_g(t) \geq -\delta \geq -1$ (see (2.56)), we compute

$$\int_M |R_g(t)| d\mu_h \leq \int_M (R_g(t) + 1) d\mu_h + C$$

$$\leq C \int_M u^N(R_g + 1) d\mu_h + C$$

$$\leq C \int_M (R_h u - C(n) \Delta_h u) d\mu_h + C$$

$$\leq C(n, \Lambda).$$

Therefore,

$$\int_M |g(t) - g(0)| h d\mu_h = \int_M \left| \int_0^t \frac{\partial g}{\partial s} ds \right| h d\mu_h$$

$$\leq C \int_0^t \int_M |R_g(s)| d\mu_h ds$$

$$\leq C(n, \Lambda) t.$$
Theorem 3.1. Let \( (M, g_{i,0}) \) be a sequence of Riemannian manifolds such that

\( (a) \) \( M_i \) is diffeomorphic to \( \mathbb{T}^n \);

\( (b) \) \( g_{i,0} = u_i^{-2} h_i \) for some metric \( h_i \) on \( M_i \) satisfying assumption \( (A) \);

\( (c) \) \( g_{i,0} \) satisfies assumption \( (B) \) for \( \delta = i^{-1} \to 0 \) and \( p_0 > \frac{n}{2} + \frac{2n(n+4)}{(n-2)(n+2)} \).

Then the Yamabe flow \( g_i(t) \) exists on \( \mathbb{T}^n \times [0,T_0] \), where \( T_0(n, \Lambda, p_0) \) is the constant in Proposition 2.3. There is a sequence of diffeomorphisms \( \Phi_i \) of \( \mathbb{T}^n \) and a flat metric \( g_{\infty} \) on \( \mathbb{T}^n \) such that after passing to subsequence, \( \Phi_i^* g_i(t) \) converges to \( g_{\infty} \) on \( \mathbb{T}^n \) in \( C^0_{\text{loc}}(\mathbb{T}^n \times (0,T_0]) \).

Proof. By [5], we can find a sequence of diffeomorphism \( \Phi_i \) such that \( \Phi_i^* h_i \) converges to some \( C^{1,\beta} \) metric \( h_{\infty} \) on \( \mathbb{T}^n \) in \( C^{1,\gamma} \) for all \( \gamma < \beta < 1 \). From now on, we will fix the \( \Phi_i \) and pull back all \( g_{i,0} \), so we may assume without loss of generality that all metrics \( h_i \) and \( g_{i,0} \) are defined on a fixed \( \mathbb{T}^n \). For notational convenience, we will omit \( \Phi_i^* \).

Applying Proposition 2.3 to each \( g_i(t) \), we obtain a sequence of Yamabe flows \( g_i(t) \) on \( \mathbb{T}^n \times [0,T_0] \) which is uniform bounded in \( C^1 \) on any \( [a,T_0] \subset (0,T_0] \). Our goal is to show that \( g_i(t) \) converges to a flat torus in a weak sense.

Claim 3.1. There is \( i_k \to +\infty \), \( g_{i_k}(T_0) \to g_{\text{flat}} \) in \( C^0 \) as \( k \to \infty \).

Proof of Claim [3.4]. Since \( g_1(T_0) \) is uniformly bounded in \( C^1 \) with respect to \( h_i \), uniformly equivalent to \( h_i \) and \( h_i \to h_{\infty} \) in \( C^{1,\gamma} \). By passing to subsequence, we may assume \( g_i(T_0) \to g_{\infty}(T_0) \) in \( C^\gamma \) for all \( \gamma \in (0,1) \). In particular, for \( \varepsilon > 0 \) sufficiently small, there is \( N \in \mathbb{N} \) such that for all \( i > N \),

\[
(1 - \varepsilon)^{\tilde{h}} \leq g_i(T_0) \leq (1 + \varepsilon)^{\tilde{h}}, \quad |\nabla^{\tilde{h}} g_i(T_0)|_{\tilde{h}} \leq C(\tilde{h}, n, \Lambda, p_0),
\]

where \( \tilde{h} = g_{\infty}(T_0) \). Note that the constant \( C \) depends on \( \tilde{h} \), but is independent of \( i \). In the following, all constants may possibly depend on \( \tilde{h} \), but are independent of \( i \).
Let \( \bar{g}_i(s) \) be the \( \bar{h} \)-flow (the Ricci-Deturck flow with reference metric \( \bar{h} \)) starting from \( g_i(T_0) \). By \cite{24} Lemma 4.3 with \( \delta = 0 \) (see also \cite{25}), there is a constant \( S_0(\bar{h}, n, \Lambda, p_0) > 0 \) such that \( \bar{g}_i(s) \) exists on \( \mathbb{T}^n \times [0, S_0] \) and satisfies
\[
\begin{cases}
|\nabla^h \bar{g}_i(s)|_\bar{h} + s^\frac{1}{2} |\nabla^h \mathcal{L} \bar{g}_i(s)|_\bar{h} \leq C(\bar{h}, n, \Lambda, p_0); \\
(1 - 2\varepsilon)\bar{h} \leq \bar{g}_i(s) \leq (1 + 2\varepsilon)\bar{h}.
\end{cases}
\tag{3.2}
\]
provided that \( \varepsilon \) is sufficiently small depending only on \( n \). The Ricci-Deturck flow is diffeomorphic to the Ricci flow and with the same initial data. We here make use of the strictly parabolicity of the Ricci-Deturck flow so that the compactness holds on the given chart in the classical sense. (Alternatively, one may also apply Hamilton’s compactness \cite{15} and obtain \( \Phi_t \) from the compactness of Ricci flow.)

By Shi’s estimates \cite{23} or \cite{25} Theorem 4.3, we can pass \( \bar{g}_i(s) \to \bar{g}_\infty(s) \) in \( C^\infty_{\text{loc}} \) on \( (0, S_0] \). Since the Ricci-Deturck flow preserved the lower bound of the scalar curvature,
\[
\tag{3.3}
R_{\bar{g}_i(s)} \geq R_{\bar{g}_i(0)} = R_{g_i(T_0)} \geq -i^{-1},
\]
where we used (2.56) in the last inequality. Letting \( i \to \infty \), we obtain \( R_{\bar{g}_\infty(s)} \geq 0 \) for \( s \in (0, S_0] \). By the torus rigidity \cite{14} \cite{21} \cite{22} and uniqueness of the Ricci-Deturck flow, \( \bar{g}_\infty(s) \equiv g_{\text{flat}} \) on \( \mathbb{T}^n \) for \( s \in (0, S_0] \). Combining (3.2) and the equation of \( \bar{h} \)-flow \cite{25}, (1.5)
\[
\frac{\partial \bar{g}_{kl}}{\partial s} = \bar{g}^{ij} \nabla_i \bar{h} \nabla_j \bar{g}_{kl} + \bar{g}^{-1} * \bar{g} * \bar{h}^{-1} * \text{Rm}(\bar{h}) + \bar{g}^{-1} * \bar{g}^{-1} * \nabla^h \bar{g} * \nabla^h \bar{g},
\tag{3.4}
\]
we obtain
\[
\left| \frac{\partial \bar{g}_{kl}}{\partial s} \right|_{\bar{h}} \leq C(\bar{h}, n, \Lambda, p_0) s^{-\frac{1}{2}},
\tag{3.5}
\]
and then
\[
\left| \bar{g}_i(s) - g_i(T_0) \right|_{\bar{h}} = \left| \bar{g}_i(s) - \bar{g}_i(0) \right|_{\bar{h}} \leq C(\bar{h}, n, \Lambda, p_0) s^{\frac{1}{2}}.
\tag{3.6}
\]
Combining this with \( \bar{g}_{ik}(s) \to g_{\text{flat}} \) in \( C^\infty_{\text{loc}} \) on \( (0, S_0] \), we \( g_{ik}(T_0) \to g_{\text{flat}} \) in \( C^0 \) as \( k \to \infty \).

Next, we claim that \( g_i(t) \) converges to the same flat metric \( g_{\text{flat}} \).

**Claim 3.2.** For all \( t \in (0, T_0] \), \( g_{ik}(t) \) converges to \( g_{\text{flat}} \) in \( C^0 \) as \( k \to \infty \), where \( g_{\text{flat}} \) is the flat metric obtained from Claim 3.1.

**Proof.** By Proposition \cite{23}, for \( a \in (0, T_0] \), \( g_i(t) \) is uniformly equivalent to \( h_i \) on \( [a, T_0] \) and \( -i^{-1} \leq R_{g_i(t)} \leq \lambda \). Using (2.61), we compute
\[
\frac{\partial}{\partial t} \left( \int_{\mathbb{T}^n} R_{g_i(t)} d\mu_{g_i(t)} \right) = \left( 1 - \frac{n}{2} \right) \int_{\mathbb{T}^n} R^2_{g_i(t)} d\mu_{g_i(t)} \geq -C(a, n, \Lambda, p_0) \left( \int_{\mathbb{T}^n} R_{g_i(t)} d\mu_{g_i(t)} + i^{-1} \right).
\tag{3.7}
\]
Then along the Ricci flow, we have

\[ (3.10) \quad \int_{\mathbb{T}^n} R_{g_i(t)} d\mu_{g_i(t)} \leq C(a, n, \Lambda, p_0) \left( \int_{\mathbb{T}^n} R_{g_i(T_0)} d\mu_{g_i(T_0)} + i^{-1} \right). \]

We now estimate the integral of scalar curvature on the right hand side. We will make use of the smooth convergence of the \( \bar{h} \)-flow. But for convenience, we will work on the Ricci flow instead which is equivalent to \( \bar{h} \)-flow via a diffeomorphism. Denote the corresponding Ricci flow by \( \hat{g}_i(s) \). The scalar curvature \( R_{\hat{g}_i(s)} \) satisfies (see e.g., [28, (2.5.5)])

\[ (3.11) \quad \left( \frac{\partial}{\partial s} - \Delta_{\hat{g}_i(s)} \right) R_{\hat{g}_i(s)} = 2 |\text{Ric}(\hat{g}_i(s))|^2_{\hat{g}_i(s)}. \]

By pulling back, (3.2) and (3.3) imply

\[ (3.12) \quad -i^{-1} \leq R_{\hat{g}_i(s)} \leq C(\bar{h}, n, \Lambda, p_0) s^{-\frac{1}{2}}. \]

Then along the Ricci flow, we have

\[ (3.13) \quad \frac{\partial}{\partial s} \left( \int_{\mathbb{T}^n} R_{\hat{g}_i(s)} d\mu_{\hat{g}_i(s)} \right) = \int_{\mathbb{T}^n} \left( 2 |\text{Ric}(\hat{g}_i(s))|^2_{\hat{g}_i(s)} - |R_{\hat{g}_i(s)}|^2 \right) d\mu_{\hat{g}_i(s)} \]

\[ \geq - C(a, \bar{h}, n, \Lambda, p_0) s^{-\frac{1}{2}} \left( \int_{\mathbb{T}^n} R_{\hat{g}_i(s)} d\mu_{\hat{g}_i(s)} + i^{-1} \right). \]

This shows

\[ (3.14) \quad \int_{\mathbb{T}^n} R_{g_i(T_0)} d\mu_{g_i(T_0)} = \int_{\mathbb{T}^n} R_{\hat{g}_i(0)} d\mu_{\hat{g}_i(0)} \]

\[ \leq C(a, \bar{h}, n, \Lambda, p_0) \left( \int_{\mathbb{T}^n} R_{\hat{g}_i(S_0)} d\mu_{\hat{g}_i(S_0)} + i^{-1} \right). \]

Pulling the above back to the \( \bar{h} \)-flow and using (3.8), for all \( t \in [a, T_0] \),

\[ (3.15) \quad \int_{\mathbb{T}^n} |R_{g_i(t)}| d\mu_{g_i(t)} \leq C(a, \bar{h}, n, \Lambda, p_0) \left( \int_{\mathbb{T}^n} R_{\hat{g}_i(S_0)} d\mu_{\hat{g}_i(S_0)} + i^{-1} \right). \]

By the similar computation of (2.75),

\[ (3.16) \quad \int_{\mathbb{T}^n} |g_i(t) - g_i(T_0)| d\mu_{g_i} \leq C \int_{a}^{T_0} \int_{M} |R_{g_i(s)}| d\mu_{g_i} ds \]

\[ \leq C(a, \bar{h}, n, \Lambda, p_0) \left( \int_{\mathbb{T}^n} R_{\hat{g}_i(S_0)} d\mu_{\hat{g}_i(S_0)} + i^{-1} \right). \]
Using Claim 3.1 and the smooth convergence of \( \tilde{g}_i(S_0) \) to \( g_{\text{flat}} \), for all \( t \in [a, T_0] \), we see that \( g_i(t) \to g_{\text{flat}} \) in \( L^1 \) as \( i \to \infty \). By the uniform \( C^1 \) estimates from Proposition 2.3, we can improve the convergence to \( C^0 \). This completes the proof since \( a \in (0, T_0] \) is arbitrary.

We now prove Theorem 1.1 and 1.2.

**Corollary 3.1 (Theorem 1.1).** Under the assumption in Theorem 3.1, if \( \text{diam}(M_i, g_{i,0}) \) are uniformly bounded from above, then after passing to subsequence, \( g_{i,0} \) converges to a flat metric \( g_{\text{flat}} \) in the volume preserving intrinsic flat sense.

**Proof.** By (2.56), the Yamabe flow \( g_i(t) \) in Theorem 3.1 satisfies

\[
\frac{\partial g_i}{\partial t} = -R_{g_i}g_i \leq i^{-1}g_i,
\]

which implies

\[
g_i(t) \leq e^{-i^{-1}t}g_i(0) = e^{-i^{-1}t}g_{i,0}.
\]

In particular,

\[
g_i(T_0) \leq e^{-i^{-1}T_0}g_{i,0}.
\]

By Theorem 3.1, \( \Phi_i^*g_i(T_0) \) converges to \( g_{\text{flat}} \) in \( C^0 \). By relabelling the index, we may assume without loss of generality that

\[
\left(1 - \frac{1}{i}\right)g_{\text{flat}} \leq \Phi_i^*g_{i,0}.
\]

The intrinsic flat convergence follows from (3.19), volume convergence in Proposition 2.4 and [3, Theorem 2.1].

**Corollary 3.2 (Theorem 1.2).** Under the assumption in Theorem 3.1, if \( u_i \) are uniformly bounded from above, then there is a sequence of diffeomorphisms \( \Phi_i \) of \( T^n \) such that after passing to subsequence, \( \Phi_i^*g_{i,0} \) converges to \( g_{\text{flat}} \) in \( L^p(T^n, g_{\text{flat}}) \) for all \( p > 0 \). Moreover, \( g_{i,0} \) converges to a flat metric in the measured Gromov-Hausdorff sense.

**Proof.** The \( L^p \) convergence follows from Proposition 2.4 and Theorem 3.1. As shown in proof of Theorem 3.1 \( \Phi_i^*h_i \) converges to \( h_\infty \) in \( C^{1,\gamma} \) for any \( \gamma \in (0, 1) \), then \( h_i \leq C(n, \Lambda)g_{\text{flat}} \). Combining this with the uniform bound of \( u_i \) and (3.19),

\[
\left(1 - \frac{1}{i}\right)g_{\text{flat}} \leq \Phi_i^*g_{i,0} \leq C(n, \Lambda)g_{\text{flat}}.
\]

The measured Gromov-Hausdorff convergence follows from (3.20), the \( L^p \) convergence and [4, Theorem 1.2].
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(Jianchun Chu) Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, IL 60208
Email address: jianchun@math.northwestern.edu

(Man-Chun Lee) Mathematics Institute, Zeeman Building, University of Warwick, Coventry CV4 7AL; Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, IL 60208
Email address: Man.C.Lee@warwick.ac.uk, mclee@math.northwestern.edu