Decay and scattering amplitudes are not the same.

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Abstract
We study the lowest lying $\pi^+ \pi^-$ resonance $R$, bound by the long range Coulomb potential and destabilized by short range strong interactions mediating the dominant decay into two neutral pions. Parametrizing the corresponding partial decay width by

$$\Gamma \left( R \to 2\pi^0 \right) = \frac{2}{3} \alpha^3 \sqrt{m_{\pi^+}^2 - m_{\pi^0}^2} \left( \Delta a_D m_{\pi^+} \right)^2$$

the quantity $\Delta a_D$ depends within QCD & QED on four basic parameters: $\Lambda_{QCD}$, $m_u + m_d$, $\alpha$, $m_d - m_u$.

We are interested here only in the limit where the last two parameters vanish

$$\Delta a_D^0 = \Delta a_D \left( \Lambda_{QCD} , m_u + m_d , \alpha = 0 , m_d - m_u = 0 \right).$$

While isospin invariance implies $\Delta a_D^{I=0} = a_D^{I=0} - a_D^{I=2}$ it is shown that beyond the first order expansion in the strong interaction the identities $a_D^I = a_I$, where $a_I$ denote the strong interaction scattering lengths, do not hold.
1 Introduction, general and specific issues

The general physics situation addressed here deals with the interplay of a long range but weak binding force (specifically Coulomb attraction within $\pi^-\text{mesonic atoms or pionium}$) and strong but short range forces (repulsive and/or attractive) which however do not themselves give rise to binding (specifically destabilising Coulombic bound states, which become resonances due to the strong transition $\pi^-X \rightarrow \pi^0 Y$ in particular $\pi^-\pi^+ \rightarrow \pi^0 \pi^0$).

In this paper we will exclusively deal with the lowest lying $\pi^-\pi^+$ (pionium) resonance, denoted R, which in purely nonrelativistic Coulomb spectroscopic terms corresponds to the 1S state of the composing charged pions. Yet the above general situation not only applies also to $\pi^-p$ and $\pi^-d$ systems, i.e. pionic hydrogen and deuterium [1], [2], but applies as well to (almost) classical gravitationally bound systems, e. g. Halley’s comet, subject to the short distance solar radiation pressure, which eventually will dissolve the bulk of the comet upon successive near approaches.

For our system R, the (partial) width $\Gamma ( R \rightarrow 2\pi^0 )$ is given by the expression

$$
\Gamma ( R \rightarrow 2\pi^0 ) = f_B \frac{q_f}{8 \pi m_R^2} | T_{2 \leftarrow R} |^2
$$

$$
q_f = \frac{\sqrt{m_R^2 - 4 m_{\pi^0}^2}}{2}; \quad f_B = \frac{1}{2} : \text{Bose factor}
$$

$$
T_{2 \leftarrow R} = \langle 2\pi^0 ; k_1, k_2 | T | R ; p \rangle; \quad p = k_1 + k_2
$$

In eq. (1) state vectors are labelled by particle content in asymptotic outgoing plane waves, characterized by four momentum vectors $k_{1,2}$ and $p$, subject to energy momentum conservation. The relativistic normalization of one particle states is adopted throughout.

$\Gamma_R / m_R \sim 0.7 \times 10^{-9}$.

T denotes the momentum conservation reduced T-matrix or transition operator.
$S - 1 \atop i = T ; \langle b ; p_b | T | a ; p_a \rangle = (2\pi)^4 \delta^4(p_b - p_a)$

$$\langle b ; p_b | T | a ; p_a \rangle$$

(2)

It follows from the structure of the expression for the decay amplitude in eq. \( \text{(1)} \), that the strong i.e. the short range interactions dominate the transition matrix \( T \) on one hand, but also modify the structure of the resonance. The latter is described by a convenient wave function, best known from nonrelativistic interactions, represented by potentials in all relevant channels. But a relativistic Bethe Salpeter wave function or any variant thereof \([3]\) exhibits the same modification in principle.

The region, where the short range forces modify this wave function (in configuration space) is small compared to the main Coulomb dominated volume proportional to the cube of the pionium Bohr radius \( r_\pi = 2 / (\alpha m_\pi) \).

But this reduced volume is \textit{the dominant region} from where the decay of the resonance takes place.

To lowest order in the strong interactions the resonance wave function remains unmodified, but if higher orders are included, this modification becomes important. Beyond this lowest order no obvious relation exists, even though higher order isospin asymmetries and electromagnetic corrections are neglected, which expresses the matrix element in eq. \( \text{(1)} \) in terms of exclusively the strong scattering amplitude of constituent pions on one hand and the purely Coulombic wave function for the resonance on the other hand.

Precisely such a relation has been derived by Deser, Goldberger, Baumann and Thirring \([4]\) for pionic atoms. Since its validity would encompass the much more general interplay of long range weak binding and short range strong but non binding forces, it can easily be falsified in potential models. The fact, that such models may not be applicable to pionium or more generally to pionic atoms is in this respect irrelevant.

In section 2 we will derive such a relation involving beyond the abovementioned quantities the dependence of the strong interaction scattering lengths on an appropriate coupling parameter \( \lambda \) for strong interactions

$$\Delta a^0_D = \lambda \frac{d}{d \lambda} \Delta a^0$$

$$\Delta a^0 = (a^{I=0} - a^{I=2}) (\Lambda_{QCD}, m_u + m_d, \alpha = 0, m_d - m_u = 0)$$

(3)
In eq. (3) the quantities $\Delta a^0, a_{l=0,2}^I$ refer to the scattering length in the limit specified.

Estimates of the lifetime of pionium are presented in section 3.

2 Resonance decay amplitude in the Lippmann-Schwinger framework

The amplitude $T_2 \leftarrow R$ in eq. (1) is proportional to the reduced amplitude in relative space coordinates as adapted to the center of mass frame

$$T_2 \leftarrow R = K \left< \varphi_{\vec{k}}^{+} \left| H_1 \left| R; \text{rel} \right> \right. \right. ; \left. \left. \vec{k} = \frac{1}{2} (\vec{k}_1 - \vec{k}_2) \right>$$

In eq. (4) $H_1$ denotes the interaction Hamiltonian in the Schrödinger representation. The state $\mid R; \text{rel} \rangle$ is normalized according to the scalar product in relative coordinate space. Because of the open decay channel the resonance is an eigenstate in the continuous spectrum of the total relative Hamiltonian, including all electromagnetic interactions

$$H_{\text{rel}} \mid R; \text{rel} \rangle = m_R \mid R; \text{rel} \rangle ; \quad H_{\text{rel}} = H_0 + H_1 \rightarrow H$$

Furthermore $\varphi_{\vec{k}}^{+}$ in eq. (4) denotes the outgoing scattering wave function obeying the Lippmann-Schwinger equation

$$\varphi_{\vec{k}}^{+} = \psi_{\vec{k}} + \frac{1}{E - i \varepsilon - H} H_1 \psi_{\vec{k}}$$

$$E = m_R = 2 \sqrt{k^2 + m_R^2} \quad \langle \vec{x} \mid \psi_{\vec{k}} \rangle = \exp \left( i \vec{k} \cdot \vec{x} \right)$$

In eq. (5) $\psi_{\vec{k}}$ denotes a plane wave.

A fully relativistic description of relative coordinate space involves the use of relative time and the corresponding Bethe Salpeter equation [3]. Finally the kinematic constant $K$ in eq. (4) is determined from the equivalent expression for the resonance width in eq. (1)

$$\Gamma \left( R \rightarrow 2\pi^0 \right) = f_B \int \left( 2\pi \right) \delta \left( E - m_R \right) \frac{d^3 k}{(2\pi)^3} \left< \varphi_{\vec{k}}^{+} \left| H_1 \left| R; \text{rel} \right> \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \r
From eq. (7) we determine the constant $K$ in eq. (4)

$$\Gamma \left( R \rightarrow 2\pi^0 \right) = f_B \frac{q_f m_R}{4\pi} \left| \langle \varphi^+_k | H_1 | R ; \text{rel} \rangle \right|^2$$

(8)

$$K = \sqrt{2} ( m_R )^{3/2}$$

Associated $2' \rightarrow 2$ scattering amplitudes

The resonance decay amplitude in eqs. (1) and (4) is associated with the scattering amplitude for the $2' \rightarrow 2$ pion reaction

$$\begin{array}{ccccccc}
\pi^0 & \pi^0 & \pi^+ & \pi^- \\
k_1 & k_2 & p_1 & p_2
\end{array}$$

$$\frac{d\sigma \left( 2' \rightarrow 2 \right)}{d\Omega} = f_B \left( \frac{q_f}{q_i} \right)_{cm} \frac{1}{64\pi^2 s} \left| T_{2 \leftarrow 2'} \right|^2$$

(9)

$$T_{2 \leftarrow 2'} = \langle 2 \pi^0 ; k_1 , k_2 | T | \pi^+ \pi^- ; p_1 , p_2 \rangle$$

All kinematic quantities in eq. (1) refer to the center of mass system.

The invariant amplitude $T_{2 \leftarrow 2'}$ depends on the standard Lorentz invariants

$$s = ( p_1 + p_2 )^2 ; \quad t = ( p_1 - k_1 )^2 ; \quad u = ( p_1 - k_2 )^2$$

$$k_1 + k_2 = p_1 + p_2 = ( E , \vec{0} )$$

$$q_f = \frac{\sqrt{s - 4 m^2_{\pi^0}}}{2} ; \quad q_i = \frac{\sqrt{s - 4 m^2_{\pi^+}}}{2}$$

(10)

Analogous to the relative coordinate space decay amplitude

$$\langle \varphi^+_k | H_1 | R ; \text{rel} \rangle$$

in eq. (1), the corresponding $2' \rightarrow 2$ scattering amplitude is

$$T_{2 \leftarrow 2'} = K' \langle \varphi^+_k | H_1 | \psi_{\vec{p}} \rangle ; \quad \vec{k} = \frac{1}{2} ( \vec{k}_1 - \vec{k}_2 )$$

$$\left| \varphi^+_k \right> = \left| 2 \pi^0 , \text{rel} ; \vec{k} + \right>$$

$$\vec{p} = \frac{1}{2} ( \vec{p}_1 - \vec{p}_2 ) ; \quad \left| \psi_{\vec{p}} \right> = \left| \pi^+ \pi^- , \text{rel} ; \vec{p} \right>$$

(11)
$| \psi_{\vec{p}} \rangle$ in eq. (11) refers to the plane wave asymptotic $\pi^+ \pi^-$ state.

The expression for the cross section in eq. (9) becomes

$$d \sigma ( 2' \rightarrow 2 ) = f_B \int \delta ( E - E' ) \left( \frac{d^3 k}{(2\pi)^3} \right)$$

$$\left| \left< \varphi_{\vec{k}}^+ | H_1 | \psi_{\vec{p}} \right> \right|^2 \frac{1}{v_{lab}}$$

(12)

$$E = 2 \sqrt{k^2 + m_{\pi}^2} \quad ; \quad E' = 2 \sqrt{p^2 + m_{\pi}^2}$$

$$v_{lab} = \frac{2 \rho E}{E^2 - 2 m_{\pi}^2}$$

Thus we obtain the expression for the differential cross section equivalent to eq. (11)

$$d \sigma ( 2' \rightarrow 2 ) \quad \frac{1}{d \Omega} = f_B \left( \frac{q_f}{q_i} \right)_{cm} \frac{1}{16 \pi^2} \frac{E^2 - 2 m_{\pi}^2}{2} \left| \left< \varphi_{\vec{k}}^+ | H_1 | \psi_{\vec{p}} \right> \right|^2$$

(13)

Parametrizing the differential cross section in the form

$$d \sigma ( 2' \rightarrow 2 ) \quad \frac{1}{d \Omega} = f_B \left( \frac{q_f}{q_i} \right)_{cm} | F |^2$$

(14)

and comparing eqs. (11) and (14) it follows

$$F = \frac{1}{8 \pi E} \ T_{2 \leftarrow 2'}$$

$$= \frac{m_{\pi}^+}{4 \pi} \sqrt{1 + \frac{2 \rho^2}{m_{\pi}^2}} \left< \varphi_{\vec{k}}^+ | H_1 | \psi_{\vec{p}} \right>$$

(15)
From eq. (15) we deduce the kinematical constant $K'$

$$K' = 2 \, m_{\pi^+} \, E \sqrt{1 + \frac{2 \, p^2}{m_{\pi^+}^2}} \quad ; \quad E = \sqrt{s}$$

(16)

The nonrelativistic limit involves the shift of the energy $E \to E - 2 \, m_{\pi^+}$ to zero value at $\pi^+\pi^-$ threshold.

3 Relations between decay and scattering amplitudes

We extrapolate the rate formula from above threshold to the resonance position with an arbitrary incident intensity $I$

$I = v_{\text{lab}} \, \varrho \quad ; \quad \varrho = |\chi|^2 \, (E)$

$$\Gamma (E \; ; \varrho \to 2\pi^0) = \int d\Omega \, \frac{d\sigma \, (2' \to 2)}{d\Omega} \, I$$

$$= f_B \frac{q_f \, E}{4 \, \pi} \left| \left\langle \varphi_{\frac{1}{2}}^+ | H_1 | \psi_p \right\rangle \chi \right|^2$$

(17)

to $E \to m_R$, i.e. below $\pi^+\pi^-$ threshold. It follows, that the resonance is described equivalently by the appropriate choice of the amplitude $\chi \to \chi_R$ from eq. (8)

$$\left\langle \varphi_{\frac{1}{2}}^+ | H_1 | R \right\rangle_{\text{rel}} = \chi_R \lim_{E \to m_R} \left\langle \varphi_{\frac{1}{2}}^+ | H_1 | \psi_{\bar{p}} \right\rangle$$

$$p^2 \to \frac{m_R^2 - 4 \, m_{\pi^+}^2}{4} < 0$$

(18)

We define the extrapolated scattering length $a_R$ below $\pi^+\pi^-$ threshold in accordance with eq. (13)

$$\lim_{E \to m_R} \left\langle \varphi_{\frac{1}{2}}^+ | H_1 | \psi_{\bar{p}} \right\rangle = \frac{4 \, \pi}{m_{\pi^+}} \, a_R \sqrt{2}$$

(19)

$$a_R = a_R \left( ^{\pi^+ \pi^-} \right)$$
The combinatorial factor $\sqrt{2}$ included in the definition of $a_R$ in eq. (19) accounts for the two identical pions in an algebraic way. The quotes in " $2 \pi^0$ " refer to the bose symmetrized and normalized state.

Substituting $a_R$ in eq. (17) we obtain upon extrapolating to the mass of the resonance

$$\Gamma ( R \rightarrow 2\pi^0 ) = \frac{4 \pi q_f m_R}{m_{\pi^+}^2} \left| a_R \chi_R \right|^2 \tag{20}$$

It is rewarding in view of sequential approximations to use as unit of density the Coulomb density $\left| \chi_C \right|^2$ for the $\pi^+ \pi^-$ system

$$\chi_C = \left( \alpha^3 \frac{m_{3\pi^+}}{8\pi} \right)^{1/2} ; \quad \xi_R = \frac{\chi_R}{\chi_C} \tag{21}$$

The expression for the resonance width in eq. (20) becomes

$$\Gamma ( R \rightarrow 2\pi^0 ) = \alpha^3 q_f \frac{m_R}{2m_{\pi^+}} \left| m_{\pi^+} a_R \xi_R \right|^2 \tag{22}$$

$$a_D = a_R \xi_R$$

In eq. (22) the quantity $a_D$ denotes the decay equivalent scattering length, to be distinguished from $a_R$ which is indeed the scattering length, extrapolated to the resonance mass and for the reaction " $2 \pi^0 \rightarrow \pi^+ \pi^-$.

Up to this point all equations were exact, to be evaluated in QCD, QED theory, restricted to two light quark flavors with masses $m_u$, $m_d$.

Hence $a_D$ is a function of 4 basic parameters

$$a_D = a_D \left( \Lambda_{QCD}, m_u + m_d, \alpha, m_d - m_u \right) \tag{23}$$

We are only interested here in the limit, where the last two of the four parameters in eq. (23) $\alpha$ and $m_d - m_u$ tend to zero, whereby only the lowest order contributions to the resonance width $\Gamma ( R \rightarrow 2\pi^0 )$ are retained:

$$\alpha, m_d - m_u \rightarrow 0 ; \quad m_{\pi^0} \rightarrow m_{\pi^+}$$

$$\Gamma ( R \rightarrow 2\pi^0 ) \rightarrow \Gamma^0 ( R \rightarrow 2\pi^0 ) ; \quad m_R \rightarrow 2m_{\pi^+}$$

$$a_D \rightarrow a_D^0 = a_D \left( \Lambda_{QCD}, m_u + m_d, \alpha = 0, m_d - m_u = 0 \right) \tag{24}$$
The momentum variables \( \vec{p}, \vec{k} \) also tend to zero in the above limit according to eq. (18).

Before going over to the systematic approximation defined in eq. (24) we represent the resonance width \( \Gamma ( R \rightarrow 2\pi^0 ) \) in the form

\[
\Gamma ( R \rightarrow 2\pi^0 ) = \Gamma^0 ( R \rightarrow 2\pi^0 ) ( 1 + \delta )
\]

(25)

\[
\delta = \delta ( \alpha, m_d - m_u ) = O ( \alpha, ( m_d - m_u )^2 )
\]

In eq. (25) only the relevant expansion parameters of the correction factor \( \delta \) are exhibited explicitly.

The dependence of \( \delta \) dominated by the first order \( \alpha \) correction has been discussed in this workshop by A. Rusetsky and H. Sazdjan [6], [7] and amounts numerically to

\[
\delta \sim 0.06
\]

enhancing in lowest nontrivial order the limiting width \( \Gamma^0 ( R \rightarrow 2\pi^0 ) \), to which exclusive attention is directed in the following.

**The limiting situation** : \( \alpha, m_d - m_u \rightarrow 0 \)

From eqs. (22) and (24) we deduce the following expression for the limiting width \( \Gamma^0 ( R \rightarrow 2\pi^0 ) \)

\[
\Gamma^0 ( R \rightarrow 2\pi^0 ) = \alpha^3 \sqrt{m^2_{\pi^+} - m^2_{\pi^0}} \left| m_{\pi^+} a^0_D \right|^2
\]

(27)

\[
a^0_D = a^0_R \xi^0_R
\]

In the limiting situation u-d isospin is an exact symmetry, which implies the general decomposition into \( I = 0, 2 \) channels remembering the channel definition in eq. (19)

\[
a^0_D, R = a^0_D, R \left( \pi^+ \pi^- \right)
\]

\[
\left| \pi^+ \pi^- \right> = \sqrt{\frac{2}{3}} \left| I = 0 \right> - \sqrt{\frac{1}{3}} \left| I = 2 \right>
\]

(28)

\[
\left| \pi^+ \pi^- \right> = \sqrt{\frac{1}{3}} \left| I = 0 \right> + \sqrt{\frac{2}{3}} \left| I = 2 \right>
\]

It follows from eq. (28) relinquishing the subscript R on \( a^0_R \), the scattering length

\[
a^0_R \rightarrow a^0 : a^0 = \frac{\sqrt{2}}{3} \Delta a^0_D ; a^0_D = \frac{\sqrt{2}}{3} \Delta a^0_D
\]

(29)

\[
\Delta a^0 = a^{I=0} - a^{I=2} ; \Delta a^0_D = a^{I=0} - a^{I=2}
\]

8
In the following we drop also the superscript 0 on the quantities \( \Delta a^0 \rightarrow \Delta a \) and \( \Delta a^0_D \rightarrow \Delta a_D \).

Substituting eq. (29) in eq. (27) we obtain

\[
\Gamma^0 \left( R \rightarrow 2\pi^0 \right) = \frac{2}{9} \alpha^3 \sqrt{m_{\pi^+}^2 - m_{\pi^0}^2} \left| m_{\pi^+} \Delta a_D \right|^2
\]

For the purpose of getting orders of magnitude into perspective we define the reference width, substituting the quantity \( \Delta a \) for \( \Delta a_D \) in eq. (30)

\[
\Gamma_a \left( R \rightarrow 2\pi^0 \right) = \frac{2}{9} \alpha^3 \sqrt{m_{\pi^+}^2 - m_{\pi^0}^2} \left| m_{\pi^+} \Delta a \right|^2
\]

\[
\Gamma_{ref} = \frac{1}{72} \alpha^3 \sqrt{m_{\pi^+}^2 - m_{\pi^0}^2}
\]

Substituting the reference width eq. (30) becomes

\[
\Gamma^0 \left( R \rightarrow 2\pi^0 \right) = \Gamma_{ref} \left( \frac{m_{\pi^+} \Delta a}{0.25} \right)^2
\]

\[
\Gamma_{ref} = \frac{5.397}{10^{-9}} \sqrt{m_{\pi^+}^2 - m_{\pi^0}^2} = 0.1917 \text{ eV}
\]

\[
\tau_{ref} = \frac{1}{\Gamma_{ref}} = \left( 3.434 \right) 10^{-15} \text{ sec}
\]

We focus on the limiting behaviour of the quantities \( \langle \varphi^+_k | H_1 | R; \text{rel} \rangle \) in eq. (8) in comparison with the scattering amplitude \( \langle \varphi^+_k | H_1 | \psi_{\vec{p}} \rangle \) in eq. (13) and their limiting relation in eq. (18)

\[
\langle \varphi^+_k | H_1 | R; \text{rel} \rangle \rightarrow -\frac{2}{3} \frac{4 \pi}{m_{\pi^+}} \chi_C \Delta a_D
\]

\[
\langle \varphi^+_k | H_1 | \psi_{\vec{p}} \rangle \chi_C \rightarrow -\frac{2}{3} \frac{4 \pi}{m_{\pi^+}} \chi_C \Delta a
\]

Following S. Deser et al. [4] we consider the purely Coulombic bound state of \( \pi^+ \pi^- \), which we shall denote \( | \psi_C \rangle \) as unperturbed state together
with the limiting perturbing Hamiltonian
\[ H_1 \to H_1^{str.} \]  
and the eigenstate(s) \( | \Psi \rangle \) of the full Hamiltonian
\[ H = H_C + H_1^{str.} ; \quad H_C = H_0 + V_C \]  
From the two equations
\[ H | \Psi \rangle = E | \Psi \rangle ; \quad H_C | \psi_C \rangle = E_C | \psi_C \rangle \]
\[ \Delta E = E - E_C ; \quad E_C = 2 m_{\pi^+} - \frac{1}{4} \alpha^2 m_{\pi^+} \]
the limiting relation for the energy shift \( \Delta E \) of the resonance follows
\[ \Delta E = \frac{\langle \psi_C | H_1^{str.} | \Psi \rangle}{\langle \psi_C | \Psi \rangle} \]
In the limit we are considering the energy shift becomes
\[ \Delta E = \langle \psi_C | H_1^{str.} + P \left( \frac{1}{E_C - H} \right) H_1^{str.} | \psi_C \rangle \]
\[ = -\chi_C^2 \frac{1}{4 m_{\pi^+}^2} \text{Re} T(\pi^+ \pi^-; \pi^+ \pi^-) \]
\[ = -\alpha^3 \frac{m_{\pi^+}}{6} T(2 a^I = 0 + a^I = 2) \]
It is instructive to follow S. Deser et al. \[4\] and extend the relations in eq. (38) to the (complex) elastic \( \pi^+ \pi^- \) scattering amplitude
\[ P \left( \frac{1}{E_C - H} \right) \to \frac{1}{E_C - H - i \varepsilon} \]
\[ \Delta E - i \gamma / 2 = -\chi_C^2 \frac{1}{4 m_{\pi^+}^2} T(\pi^+ \pi^-; \pi^+ \pi^-) \to \]
\[ \gamma = \Gamma_a (R \to 2\pi^0) \]
\[ \text{(39)} \]
The quantity $\gamma = \Gamma_a \left( R \to 2\pi^0 \right)$ in eqs. (31) and (39) is the limiting width of the state $| \psi_C \rangle$ as it decays, to all orders in the strong interaction, to $2\pi^0$.

This corresponds to the substitution in eq. (33)

$$| R; \text{rel} \rangle \rightarrow | \psi_C \rangle$$

$$\Delta a_D \rightarrow \Delta a$$

However the state $| R; \text{rel} \rangle$ as it evolves from without external perturbation to a time $t = 0$ say, is not $| \psi_C \rangle$, but rather the full incoming Schrödinger state $| \varphi^-; C \rangle$, adiabatically evolving from $| \psi_C \rangle$

$$| \varphi^-; C \rangle = | \psi_C \rangle + \frac{1}{E_R - H - i \varepsilon} \ H_1^{\text{str.}} \ | \psi_C \rangle$$

with the energy $E_R$ including the energy shift $\Delta E$ in eq. (38).

In the limit we are considering it follows

$$| \varphi^-; C \rangle \rightarrow \chi_C | \varphi^-; \pi^+ \pi^- \rangle$$

$$\langle \varphi^+ \mid H_1 \mid R; \text{rel} \rangle \rightarrow \chi_C$$

$$\langle \varphi^+; 2\pi^0 \mid H_1 \mid \varphi^-; \pi^+ \pi^- \rangle$$

$$\vec{k} = \vec{p} = 0 \ ; \ H_1^{\text{str.}} \rightarrow H_1$$

From the structure of limiting amplitudes in eq. (42) we infer, dropping the superscript $\text{str.}$ in the following

$$\langle \varphi^+; 2\pi^0 \mid H_1 \mid \varphi^-; \pi^+ \pi^- \rangle =$$

$$\langle \psi; 2\pi^0 \mid H_1 \left( \frac{1}{1 - G_0 H_1} \right)^2 \mid \psi; \pi^+ \pi^- \rangle$$

$$G_0 = ( E_{\text{thr}} - H_0 + i \varepsilon )^{-1}$$

We recall that $\psi$ in eq. (43) refers to plane wave states with vanishing momentum at threshold.

The argument for the limiting substitutions in eq. (42) is like this: in the matrix element $\langle \varphi^+ \mid H_1 \mid R; \text{rel} \rangle$ the action of $H_1 \rightarrow H_1^{\text{str.}}$ restricts
the configurations composing both states \(|\varphi^+\rangle\) and \(|R; rel\rangle\) to normal strong interaction relative distances \(d_{rel} \leq \sim 1 - 4\) fm.

Those configurations are in the sense of the limit considered insensitive to the two key parameters governing the resonance at its determining distance, i.e. its Bohr radius

\[ a_B = \frac{2}{(m_{\pi^+} \alpha)} \sim 400\text{ fm} \]

As a consequence also the key mass square difference

\[ m_{\pi^+}^2 - m_{\pi^0}^2 \sim (35.51\text{ MeV})^2 \]

plays no significant role. Hence the dominant configurations relevant for the transition amplitude within the state \(|R; rel\rangle\) are the same as those in which \(R\) is (almost) bound, i.e. for \(\alpha \neq 0\) but \(m_{\pi^+} = m_{\pi^0}\). Of course the resonance \(R\) is always decaying into two (and more) photons.

In the above situation the (almost) bound state \(R\) is by no means described by the Coulomb wave function, in particular at the distances within \(d_{rel}\).

It follows combining eqs. (43) and (33)

\[
- \Delta a_D \to
\frac{3 m_{\pi^+}}{8 \pi} \left\langle \psi ; 2 \pi^0 | H_1 \left( \frac{1}{1 - G_0 H_1} \right)^2 | \psi ; \pi^+ \pi^- \right\rangle
\]

\[- \Delta a \to
\frac{3 m_{\pi^+}}{8 \pi} \left\langle \psi ; 2 \pi^0 | H_1 \left( \frac{1}{1 - G_0 H_1} \right) | \psi ; \pi^+ \pi^- \right\rangle
\]

(44)

Eq. (44) shows that decay and scattering amplitudes are not the same.

\(\Delta a_D\) and \(\Delta a\) according to eq. (44) can not be related to each other without detailed knowledge of \(H_1\).

Let us introduce the coupling strength \(\lambda\) - always remaining in the 2 flavor SU2 symmetric QCD limit, with fixed \(m_{\pi^+} = m_{\pi^0}\) - through the substitution

\[
H_0, G_0 \quad \to \quad H_0, G_0
\]

\[
H_1 \quad \to \quad \lambda H_1
\]

(45)

\[
\Delta a_D, \Delta a \quad \to \quad \Delta a_D (\lambda), \Delta a (\lambda)
\]
Then it follows from eq. (44)

\[ \Delta a_D (\lambda) = \lambda \frac{d}{d\lambda} \Delta a (\lambda) \]  

In the limit considered the quantities \( \Delta a_D, \Delta a \) depend within QCD on the two basic parameters \( \Lambda QCD^2 \) and \( m_u = m_d \).

An equivalent set is \( f_\pi \), the pion decay constant, and \( m_{\pi^+} = m_{\pi^0} \). It follows from the relation in eq. (46) that the pion mass is to be held constant, whereas at least in lowest two orders of chiral perturbation theory the variation of \( \lambda \) is equivalent to a variation of \( f_\pi^{-2} \)

\[ \lambda \frac{d}{d\lambda} \sim \varrho \frac{d}{d\varrho} \]  

\[ \varrho = k \frac{m_\pi^2}{f_\pi^2} ; \ k \ \text{arbitrary fixed constant} \]  

Estimates of pionium lifetime

We use eqs. (46) and (47) to estimate \( \Delta a_D \) and the lifetime of pionium, with the shorthand notation \( m_{\pi^+} \rightarrow m_\pi \). The lowest order (tree level) values \(10\) are

\[ m_\pi a_1 (1) = 0 = \frac{7 m_\pi^2}{32 \pi f_\pi^2} = 0.1562 \]  

\[ m_\pi a_1 (1) = 2 = -\frac{2 m_\pi^2}{32 \pi f_\pi^2} = -0.0446 \]  

To this end we list from Ecker et al. \(9\) the contributions through two loop
order to both scattering lengths \( a^I = 0 \) and \( a^I = 2 \)

| order | \( m_\pi a^I = 0 \) | \( m_\pi a^I = 2 \) | \( m_\pi \Delta a \) |
|-------|-----------------|-----------------|-----------------|
| 1     | 0.16            | -0.045          | 0.205           |
| 2     | 0.04            | 0.003           | 0.037           |
| 3 \( (I) \) | 0.017          | 0.0007          | 0.0163          |
| 3 \( (II) \) | 0.006          | -0.0023         | 0.0083          |
| total \( (I) \) | 0.217          | -0.0413         | 0.258           |
| total \( (II) \) | 0.206          | -0.0443         | 0.250           |

We base our estimate on the one loop contribution to \( a^I = 0 \) which is of the form

\[
a^I = 0^{(2)} = k_1 \rho^2 L ; \quad L = - \log \rho \sim 4
\]  

(50)

The numerical value of the logarithm in the one loop contribution to \( a^I = 0 \) is quite accurately 4 as in eq. (50) , when all nonlogarithmic terms of order \( \rho^2 \) are absorbed into the argument of the logarithm. The constant \( k \) in eq. (47) can then be chosen such that \( \rho = \exp -4 \) according to eq. (50) .

It then follows

\[
\lambda \frac{d}{d \lambda} a^I = 0^{(2)} = (1 - L^{-1}) a^I = 0^{(2)} \sim \frac{3}{4} a^I = 0^{(2)} = 0.03
\]  

(51)

Neglecting all other corrections from higher orders and from the \( I = 2 \) channel we obtain as our estimate

\[
\Delta a_D \sim 0.28 ; \quad \Delta a \sim 0.25 \quad \rightarrow \quad \frac{\Delta a_D}{\Delta a} \sim 1.12
\]  

(52)

With the ratio \( \Delta a_D / \Delta a \) given in eq. (52) we obtain from eq. (32)

\[
\Gamma^0 (R \rightarrow 2\pi^0) = 1.25 \Gamma_{ref} \left( \frac{m_\pi + \Delta a}{0.25} \right)^2
\]  

\[
= 0.240 \text{ eV} \left( \frac{m_\pi + \Delta a}{0.25} \right)^2
\]  

(53)
or equivalently for the lifetime, using the abbreviation $\Gamma^0 ( R \to 2\pi^0 ) \to \Gamma^0$

$$\tau_0 = 1 / \Gamma^0 = (2.74) \times 10^{-15} \text{ sec} \left( \frac{0.25}{m_{\pi^+} \Delta a} \right)^2$$

(54)

Finally we apply the radiative corrections as estimated by A. Rusetsky and H. Sazdjian \[6\] \cite{6} according to eqs. (25) and (26) and find for the resonance width the estimate

$$\Gamma ( R \to 2\pi^0 ) \sim 1.33 \Gamma_{ref} \left( \frac{m_{\pi^+} \Delta a}{0.25} \right)^2 \frac{1 + \delta}{1.06}$$

$$= 0.255 \text{ eV} \left( \frac{m_{\pi^+} \Delta a}{0.25} \right)^2 \frac{1 + \delta}{1.06}$$

(55)

or equivalently for the lifetime

$$\tau ( R \to 2\pi^0 ) = (2.58) \times 10^{-15} \text{ sec} \left( \frac{0.25}{m_{\pi^+} \Delta a} \right)^2 \frac{1.06}{1 + \delta}$$

(56)

If we omit the correction proportional to $(\Delta a_D / \Delta a)^2$ the corresponding estimate for the lifetime would be

$$\tau ( R \to 2\pi^0 ) = (3.24) \times 10^{-15} \text{ sec} \left( \frac{0.25}{m_{\pi^+} \Delta a} \right)^2 \frac{1.06}{1 + \delta}$$

(57)

The results on the above lifetime to date are from L. Nemenov et al. \[11\]

$$\tau ( R \to 2\pi^0 ) = (2.9^{+\infty}_{-2.1}) \times 10^{-15} \text{ sec}$$

(58)

In conclusion we are looking forward in suspense to the measurement or better the analytic deduction of the lifetime of pionium from the study of the breakup reaction in targets with appropriately chosen thickness by the DIRAC collaboration \[12\].
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