Derivative corrections to Dirac-Born-Infeld Lagrangian
and non-commutative gauge theory

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Abstract

We consider the constraints on the effective Lagrangian of the rank-one gauge field on D-branes imposed by the equivalence between the description by ordinary gauge theory and that by non-commutative gauge theory in the presence of a constant $B$ field. It is shown that we can consistently construct the two-derivative corrections to the Dirac-Born-Infeld Lagrangian up to the quartic order of field strength and the most general form which satisfies the constraints up to this order is derived.
1. Introduction

There are two different descriptions of the effective Lagrangian of the gauge fields on D-branes in flat space, with metric \( g_{ij} \), in the presence of a constant Neveu-Schwarz–Neveu-Schwarz two-form gauge field (\( B \) field) \( B_{ij} \). The first one is the conventional one in terms of ordinary gauge fields with ordinary gauge invariance. The gauge transformation and field strength are familiar, which are for rank-one gauge theory,

\[
\delta_\lambda A_i = \partial_i \lambda, \\
F_{ij} = \partial_i A_j - \partial_j A_i, \\
\delta_\lambda F_{ij} = 0.
\]

(1.1)

In this formulation, the \( B \)-dependence of the effective Lagrangian \( \mathcal{L} \) is only in the combination \( B + F \).

The other one is in terms of non-commutative gauge fields \[1\] where the algebra of functions is deformed to a non-commutative, associative one defined by

\[
f(x) \ast g(x) = \exp \left( \frac{i}{2} \theta^{ij} \frac{\partial}{\partial \xi^i} \frac{\partial}{\partial \zeta^j} \right) f(x + \xi) g(x + \zeta) \bigg|_{\xi = \zeta = 0} \\
= fg + \frac{i}{2} \theta^{ij} \partial_i f \partial_j g + O(\theta^2),
\]

(1.2)

with

\[
\theta^{ij} = -(2\pi \alpha')^2 \left( \frac{1}{g + 2\pi \alpha' B} \frac{1}{g - 2\pi \alpha' B} \right)^{ij},
\]

(1.3)

and the gauge transformation and field strength for rank-one gauge theory are correspondingly deformed to

\[
\hat{\delta}_\lambda \hat{A}_i = \partial_i \hat{\lambda} + i \hat{\lambda} \ast \hat{A}_i - i \hat{A}_i \ast \hat{\lambda}, \\
\hat{F}_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i - i \hat{A}_i \ast \hat{A}_j + i \hat{A}_j \ast \hat{A}_i, \\
\hat{\delta}_\lambda \hat{F}_{ij} = i \hat{\lambda} \ast \hat{F}_{ij} - i \hat{F}_{ij} \ast \hat{\lambda}.
\]

(1.4)

The effective Lagrangian \( \hat{\mathcal{L}}(\hat{F}) \) in the latter formulation takes the same form as the one \( \mathcal{L}(F) \) in the former \[2\] except that the product of functions is replaced with the \( \ast \) product
and that Lorentz indices are contracted by the metric $G_{ij}$ which is different from the metric $g_{ij}$ used in the description in terms of ordinary gauge theory:

$$G_{ij} = g_{ij} - (2\pi\alpha')^2 (Bg^{-1}B)_{ij}, \quad (1.5)$$

$$\left(G^{-1}\right)^{ij} = \left(\frac{1}{g + 2\pi\alpha'B} \frac{1}{g - 2\pi\alpha'B}\right)^{ij}. \quad (1.6)$$

The $B$-dependence in the latter formulation is encoded in $\theta$, $G$ and the coupling constant.

The equivalence of the two descriptions is recently discussed in detail \cite{2} which is realized by the transformation between the non-commutative gauge field $\hat{A}$ and the ordinary one $A$,

$$\hat{A}(A) + \delta\lambda \hat{A}(A) = \hat{A}(A + \delta\lambda A), \quad (1.7)$$

with infinitesimal $\lambda$ and $\hat{\lambda}(\lambda, A)$. The two Lagrangians in terms of $A$ and $\hat{A}$ should be related as

$$\mathcal{L}(B + F) = \hat{\mathcal{L}}(\hat{F}) + \text{total derivative}, \quad (1.8)$$

under the transformation (1.7). This was verified in \cite{2} for the Dirac-Born-Infeld (DBI) Lagrangian\footnote{For a recent review of the Dirac-Born-Infeld theory see \cite{3} and references therein.} in the approximation that field strength is slowly varying. We should note here that the relation between $\hat{A}$ and $A$ (1.7) is determined \textit{independently} of the form of the effective Lagrangian and thus the condition (1.8) imposes constraints on the form of the Lagrangian as is argued in \cite{2} that the DBI Lagrangian is the only Lagrangian which satisfies (1.8) in the approximation that field strength is slowly varying.

The applicability of the argument that the two formulations are equivalent and the condition (1.8) should be satisfied is, however, more general and not restricted to such cases. It would be a non-trivial question whether the condition (1.8) can indeed be satisfied when we include derivative corrections to the DBI Lagrangian. If we assume that it is possible, the next question to be raised is to what extent the Lagrangian is constrained by the condition (1.8). In the present paper, we consider the questions for two-derivative corrections to the DBI Lagrangian up to the quartic order of field strength in rank-one gauge theory. We derive the most general structure which satisfies the condition (1.8) up to this order.
The organization of this paper is as follows. In Section 2, we first consider how $F^4$ terms are determined by the requirement (1.8) to illustrate our method for this simplest case. We then extend our discussions to two-derivative corrections in Section 3. Section 4 is devoted to conclusions and discussions.

2. Determination of $F^4$ terms

It is already argued in [2] that the DBI Lagrangian satisfies the condition (1.8) in the approximation that field strength is slowly varying as we mentioned in Section 1 and most of the calculations in the present section are nothing but the reorganization of those in [2]. However, in addition to the purpose of the illustration of our method which will be applied to the determination of derivative corrections in the next section, it would be instructive to see how $F^4$ terms are determined from the information on the $F^2$ term alone without the information on the whole form of the Lagrangian in order to extend our consideration to derivative corrections of which we do not know the whole structure.

Let us begin with some preparations. The equation (1.7) is solved in the expansion with respect to $\theta$ [2]. For rank-one gauge theory, it is given by\

\[ \hat{A}_i = A_i - \frac{1}{2} \theta^{kl} A_k (\partial_i A_l + F_{il}) + O(\theta^2), \] (2.1)\

\[ \hat{\lambda} = \lambda + \frac{1}{2} \theta^{kl} \partial_k \lambda A_l + O(\theta^2). \] (2.2)

For the field strength, it follows from the solution for $\hat{A}$ that\

\[ \hat{F}_{ij} = F_{ij} - \theta^{kl} (F_{ik} F_{lj} + A_k \partial_l F_{ij}) + O(\theta^2). \] (2.3)

We need to expand $G^{-1}$, $\theta$, the * product (1.2) and $\hat{F}_{ij}$ with respect to $\alpha'$:

\[ (G^{-1})^{ij} = (g^{-1})^{ij} + (2\pi \alpha')^2 (g^{-1} B g^{-1})^{ij} + O(\alpha'^4), \] (2.4)\

\[ \theta^{ij} = -(2\pi \alpha')^2 (g^{-1} B g^{-1})^{ij} + O(\alpha'^4), \] (2.5)\

\[ f * g = f g - \frac{i}{2} (2\pi \alpha')^2 (g^{-1} B g^{-1})^{kl} \partial_k f \partial_l g + O(\alpha'^4), \] (2.6)\

\[ \hat{F}_{ij} = F_{ij} + (2\pi \alpha')^2 (g^{-1} B g^{-1})^{kl} (F_{ik} F_{lj} + A_k \partial_l F_{ij}) + O(\alpha'^4). \] (2.7)

\[ \text{\footnote{It was pointed out in [4] that there are ambiguities in perturbative solutions to the equation (1.7) which are related to gauge transformation and to field redefinition. However, it is easily verified that our results will not be modified essentially by the presence of such ambiguities up to the order we are discussing.}} \]
The discussions presented in this paper do not depend on the dimension of space-time on which the gauge theory is defined, namely, the dimension of world-volume of the D-brane. We only need to multiply an appropriate power of \( \alpha' \) to the Lagrangian to make the action dimensionless.

Let us first verify that the \( F^2 \) term

\[
\mathcal{L}(F) = \text{Tr}(g^{-1} F g^{-1} F) + O(\alpha'^2) \equiv (g^{-1})^{ij} F_{jk} (g^{-1})^{kl} F_{li} + O(\alpha'^2),
\]

satisfies the condition (1.8). The left-hand side of (1.8) is

\[
\mathcal{L}(B + F) = \text{Tr}(g^{-1} (B + F) g^{-1} (B + F)) + O(\alpha'^2)
\]

\[
= \text{Tr}(g^{-1} B)^2 + 2 \text{Tr}(g^{-1} B g^{-1} F) + \text{Tr}(g^{-1} F)^2 + O(\alpha'^2)
\]

\[
= \text{Tr}(g^{-1} F)^2 + \text{constant} + \text{total derivative} + O(\alpha'^2).
\]

The non-commutative counterpart, the right-hand side of (1.8), is

\[
\hat{\mathcal{L}}(\hat{F}) = \text{Tr}(G^{-1} \hat{F} \ast G^{-1} \hat{F}) + O(\alpha'^2) = \text{Tr}(g^{-1} F g^{-1} F) + O(\alpha'^2).
\]

Thus (1.8) is satisfied for (2.8). This may seem trivial but is important: The \( F^2 \) term (2.8) is qualified as an initial term of a consistent Lagrangian in the \( \alpha' \) expansion. Let us define the initial term condition as follows: If \( f(F) \) satisfies

\[
f(B + F) = f(F) + \text{total derivative},
\]

we say that \( f(F) \) satisfies the initial term condition. In this terminology, \( \text{Tr}(g^{-1} F g^{-1} F) \) satisfies the initial term condition.

Now we go on to the order \( O(\alpha'^2) \). The non-commutative side \( \text{Tr}(G^{-1} \hat{F} \ast G^{-1} \hat{F}) \) is evaluated as follows:

\[
\text{Tr}(G^{-1} \hat{F} \ast G^{-1} \hat{F}) = \text{Tr}(G^{-1} \hat{F} G^{-1} \hat{F}) + O(\alpha'^4)
\]

\[
= \text{Tr}(g^{-1} \hat{F} )^2 + 2 (2\pi \alpha')^2 \text{Tr}(g^{-1} B)^2 (g^{-1} F)^2 + O(\alpha'^4)
\]

\[
= \text{Tr}(g^{-1} F)^2 + 2 (2\pi \alpha')^2 \text{Tr} g^{-1} B (g^{-1} F)^3 - \frac{1}{2} (2\pi \alpha')^2 \text{Tr} g^{-1} B g^{-1} F \text{Tr} (g^{-1} F)^2
\]

\[
+ 2 (2\pi \alpha')^2 \text{Tr} (g^{-1} B)^2 (g^{-1} F)^2 + \text{total derivative} + O(\alpha'^4). \quad (2.12)
\]
The existence of $O(BF^3)$ terms and $O(B^2F^2)$ terms in (2.12) implies that the corresponding terms must exist on the commutative side as well. The sources for such terms are $\text{Tr}(g^{-1}(B + F))^4$ and $(\text{Tr}(g^{-1}(B + F)))^2$ which are expanded as follows:

$$\text{Tr}(g^{-1}(B + F))^4 = \text{Tr}(g^{-1}F)^4 + 4\text{Tr}g^{-1}B(g^{-1}F)^3$$
$$+ 4\text{Tr}(g^{-1}B)^2(g^{-1}F)^2 + 2\text{Tr}(g^{-1}Bg^{-1}F)^2$$
$$+ \text{constant} + \text{total derivative}, \quad (2.13)$$

$$(\text{Tr}(g^{-1}(B + F)))^2 = (\text{Tr}(g^{-1}F))^2 + 4\text{Tr}(g^{-1}Bg^{-1}F)\text{Tr}(g^{-1}F)^2$$
$$+ 2\text{Tr}(g^{-1}B)^2\text{Tr}(g^{-1}F)^2 + 4(\text{Tr}(g^{-1}Bg^{-1}F))^2$$
$$+ \text{constant} + \text{total derivative}. \quad (2.14)$$

By comparing the $O(BF^3)$ terms in (2.12) with those in (2.13) and (2.14), we can uniquely determine the structure at $O(\alpha'^2)$ as

$$\frac{1}{2}(2\pi\alpha')^2\text{Tr}(g^{-1}(B + F))^4 - \frac{1}{8}(2\pi\alpha')^2(\text{Tr}(g^{-1}(B + F)))^2. \quad (2.15)$$

Now the comparison of the $O(B^2F^2)$ terms provides a consistency condition. There are three missing terms in (2.12) in comparison with (2.15). The two of them are combined into total derivative:

$$(2\pi\alpha')^2 \left[ \text{Tr}(g^{-1}Bg^{-1}F)^2 - \frac{1}{2}(\text{Tr}(g^{-1}Bg^{-1}F))^2 \right] = \text{total derivative}. \quad (2.16)$$

The last one

$$- \frac{1}{4}(2\pi\alpha')^2\text{Tr}(g^{-1}B)^2\text{Tr}(g^{-1}F)^2 \quad (2.17)$$

can be taken care of by the $B$-dependence of the coupling constant. The Lagrangian $\mathcal{L}(F)$ in terms of the ordinary gauge theory should be multiplied by $\sqrt{\text{det}g/g_s}$ where $g_s$ is the string coupling constant and we write the corresponding factor on the non-commutative side as $\sqrt{\text{det}G/G_s}$. The new coupling constant $G_s$ can depend on $B$. The $B$-dependence of $G_s$ was determined in [2] by using the DBI Lagrangian. In our point of view, we cannot use the DBI Lagrangian: We are now determining its form. It is determined perturbatively by the presence of the term (2.17) as follows:

$$\frac{\sqrt{\text{det}G}}{G_s} = \frac{\sqrt{\text{det}g}}{g_s} \left[ 1 - \frac{1}{4}(2\pi\alpha')^2\text{Tr}(g^{-1}B)^2 + O(\alpha'^4) \right]. \quad (2.18)$$
This completes the consistency check of the $O(B^2F^2)$ terms.

The presence of the structure (2.13) in turn requires the existence of the corresponding structure on the non-commutative side, which is

$$\frac{1}{2}(2\pi\alpha')^2\text{Tr}(G^{-1}\hat{F})^4_{\text{arbitrary}} - \frac{1}{8}(2\pi\alpha')^2(\text{Tr}(G^{-1}\hat{F})^2)^2_{\text{arbitrary}},$$

where the subscripts “arbitrary” imply that the ordering of the four field strengths in each term is arbitrary. Since the product in the non-commutative gauge theory is the non-commutative $\ast$ product, we have to specify the ordering of field strengths as in the case of $F^4$ terms in the Yang-Mills theory. However, the non-commutativity becomes relevant only at higher orders in the $\alpha'$ expansion:

$$\frac{1}{2}(2\pi\alpha')^2\text{Tr}(G^{-1}\hat{F})^4_{\text{arbitrary}} - \frac{1}{8}(2\pi\alpha')^2(\text{Tr}(G^{-1}\hat{F})^2)^2_{\text{arbitrary}}$$

$$= \frac{1}{2}(2\pi\alpha')^2\text{Tr}(G^{-1}\hat{F})^4 - \frac{1}{8}(2\pi\alpha')^2(\text{Tr}(G^{-1}\hat{F})^2)^2 + O(\alpha'^4),$$

where the product in the second line is ordinary one, so the ordering problem does not matter at the order we are discussing. Inversely, the consideration at the present order alone cannot constrain the ordering. It would be interesting to see if the discussion at higher orders can determine or constrain the ordering.

To summarize, we have seen that the Lagrangian

$$\mathcal{L}(B + F) = \sqrt{\text{det} g} \left[ \text{Tr}(g^{-1}(B + F))^2 + \frac{1}{2}(2\pi\alpha')^2\text{Tr}(g^{-1}(B + F))^4 \right.$$

$$- \frac{1}{8}(2\pi\alpha')^2(\text{Tr}(g^{-1}(B + F))^2)^2 + O(\alpha'^4) \left. \right],$$

and its non-commutative counterpart

$$\hat{\mathcal{L}}(\hat{F}) = \sqrt{\text{det} G} \left[ \text{Tr}(G^{-1}\hat{F} \ast G^{-1}\hat{F}) + \frac{1}{2}(2\pi\alpha')^2\text{Tr}(G^{-1}\hat{F})^4_{\text{arbitrary}} \right.$$

$$- \frac{1}{8}(2\pi\alpha')^2(\text{Tr}(G^{-1}\hat{F})^2)^2_{\text{arbitrary}} + O(\alpha'^4) \left. \right],$$

coincide up to total derivative under the definition of $G_s$ (2.18), namely, the condition (1.8) is satisfied. Thus we uniquely determined the $F^4$ terms as

$$\mathcal{L}(F) = \frac{\sqrt{\text{det} g}}{g_s} \left[ \text{Tr}(g^{-1}F)^2 + \frac{1}{2}(2\pi\alpha')^2\text{Tr}(g^{-1}F)^4 \right.$$

$$- \frac{1}{8}(2\pi\alpha')^2(\text{Tr}(g^{-1}F)^2)^2 + O(\alpha'^4) \left. \right],$$

(2.23)
from the requirement (1.8) alone. The resulting Lagrangian coincides with the $\alpha'$ expansion of the DBI Lagrangian if it is multiplied by

$$-\frac{(2\pi\alpha')^2}{4(2\pi)^p(\alpha')^{2-p}}$$

when the dimension of the space-time is $p+1$.

We would like to make a comment here. From the fact that we have determined the $F^4$ terms uniquely from the $F^2$ term it follows that it is impossible to organize the $F^4$ terms so as to satisfy (1.8) without the $F^2$ term. In other words, no $F^4$ structure can satisfy the initial term condition defined in (2.11). We can show this explicitly by writing the most general $F^4$ terms and seeing if they satisfy (2.11). It is not difficult to see that the $O(B^2F^2)$ terms on the left-hand side cannot be arranged to total derivative.

3. Determination of two-derivative corrections

Since we have explained our strategy in detail in Section 2, it would not be difficult to apply it to the two-derivative corrections to the DBI Lagrangian. In the first part of this section, we construct one of consistent Lagrangians with two derivatives up to the quartic order of field strength. We then derive the most general form of the Lagrangian up to this order in the second part.

3.1 Construction of a consistent Lagrangian

The two-derivative corrections to the DBI Lagrangian can first appear at order $O(\alpha')$ compared with the $F^2$ term. Using the integration by parts and the Bianchi identity, any term of order $O(\alpha')$ can be transformed to the following form up to an overall constant:

$$\mathcal{L}(F) = (g^{-1})^{nm}(g^{-1})^{ij}\partial_n F_{jk}(g^{-1})^{kl}\partial_m F_{li}.$$  \hspace{1cm} (3.1)

It is possible to absorb this term into the $F^2$ term (2.8) by field redefinition. However, we do not know in which definition of the gauge field the transformation (2.11) is valid in general so we should not make such redefinition in determining the possible form of

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§ We absorbed an appropriate power of $\alpha'$ into the overall constant as well to simplify the following expressions.
the Lagrangian. It is easily seen that (3.1) satisfies the initial term condition (2.11) since $\partial_i(B+F)_{jk} = \partial_iF_{jk}$ for a constant $B$.

To construct the non-commutative counterpart $\hat{L}(\hat{F})$ of this Lagrangian, we have to replace the derivatives in (3.1) with covariant derivatives defined by

$$D_i \hat{F}_{jk} = \partial_i \hat{F}_{jk} - i \hat{A}_i \ast \hat{F}_{jk} + i \hat{F}_{jk} \ast \hat{A}_i,$$  \hspace{1cm} (3.2)

since non-commutative gauge fields are non-commutative even when the rank is one. Now the non-commutative Lagrangian becomes

$$\hat{L}(\hat{F}) = (G^{-1})^{nm}(G^{-1})^{ij} \hat{D}_n \hat{F}_{jk} \ast (G^{-1})^{kl} \hat{D}_m \hat{F}_{li},$$  \hspace{1cm} (3.3)

Since the gauge transformation of $\hat{D}_i \hat{F}_{jk}$ is

$$\hat{\delta}_\lambda (\hat{D}_i \hat{F}_{jk}) = i \lambda \ast \hat{D}_i \hat{F}_{jk} - i \hat{D}_i \hat{F}_{jk} \ast \lambda,$$  \hspace{1cm} (3.4)

the action made from (3.3) is gauge invariant.

Now let us evaluate (3.3) in the $\alpha'$ expansion. The covariant derivative of field strength $\hat{D}_n \hat{F}_{ij}$ is expanded as

$$\hat{D}_n \hat{F}_{ij} = \partial_n F_{ij}$$

$$+(2\pi \alpha')^2 (g^{-1} B g^{-1})^{kl} (\partial_n (F_{ik} F_{lj}) + F_{nk} \partial_l F_{ij} + A_k \partial_n \partial_l F_{ij}) + O(\alpha'^4).$$  \hspace{1cm} (3.5)

Using this result, (3.3) is evaluated as follows:

$$\left((G^{-1})^{nm}(G^{-1})^{ij} \hat{D}_n \hat{F}_{jk} \ast (G^{-1})^{kl} \hat{D}_m \hat{F}_{li}\right)$$

$$= (G^{-1})^{nm}(G^{-1})^{ij} \hat{D}_n \hat{F}_{jk}(G^{-1})^{kl} \hat{D}_m \hat{F}_{li} + O(\alpha'^4)$$

$$= (g^{-1})^{nm}(g^{-1})^{ij} \hat{D}_n \hat{F}_{jk}(g^{-1})^{kl} \hat{D}_m \hat{F}_{li} + O(\alpha'^4)$$

$$+(2\pi \alpha')^2 \left[(g^{-1} B g^{-1})^{nm}(g^{-1})^{ij} \partial_n F_{jk}(g^{-1})^{kl} \partial_m F_{li}\right]$$

$$+2(g^{-1})^{nm}(g^{-1} B g^{-1})^{ij} \partial_n F_{jk}(g^{-1})^{kl} \partial_m F_{li} + O(\alpha'^4)$$

$$= (g^{-1})^{nm}(g^{-1})^{ij} \partial_n F_{jk}(g^{-1})^{kl} \partial_m F_{li}$$

$$+(2\pi \alpha')^2 \left[2(g^{-1})^{nm}(g^{-1})^{ij} \partial_n F_{jk}(g^{-1})^{kl}(g^{-1} B g^{-1})^{pq} \partial_p (F_{iq} F_{qi}) + F_{mp} \partial_q F_{li}\right]$$

$$-\frac{1}{2} \text{Tr}(g^{-1} B g^{-1} F)(g^{-1})^{nm}(g^{-1})^{ij}(g^{-1})^{kl} \partial_n F_{jk} \partial_m F_{li}$$

$$+(g^{-1} B g^{-1})^{nm}(g^{-1})^{ij} \partial_n F_{jk}(g^{-1})^{kl} \partial_m F_{li}$$

$$+2(g^{-1})^{nm}(g^{-1} B g^{-1})^{ij} \partial_n F_{jk}(g^{-1})^{kl} \partial_m F_{li}$$

$$+ \text{total derivative} + O(\alpha'^4).$$  \hspace{1cm} (3.6)
Lorentz indices in most of the expressions in what follows are contracted with respect to
the metric \( g_{ij} \) so that we simplify the expressions by making \( g^{-1} \) implicit as
\[
A_i B_i \equiv (g^{-1})^{ij} A_j B_j, \quad \partial^2 \equiv (g^{-1})^{ij} \partial_i \partial_j, \quad (3.7)
\]
unless the other metric \( G_{ij} \) is explicitly used. With this convention, (3.6) is expressed as
\[
(G^{-1})^{nm}(G^{-1})^{ij} \hat{D}_n \hat{F}_{jk} \ast (G^{-1})^{kl} \hat{D}_m \hat{F}_{li}
= \partial_n F_{ij} \partial_n F_{ji}
+ (2\pi \alpha')^2 \left[ 2 \partial_n F_{ij} \partial_n F_{jk} (B_{kl} F_{li} + F_{kl} B_{li}) + 2 \partial_n F_{ij} F_{nk} B_{kl} \partial_l F_{ji} 
- \frac{1}{2} B_{kl} F_{li} \partial_n F_{ij} \partial_n F_{ji} + B_{nk} B_{kl} \partial_n F_{ij} \partial_l F_{ji} + 2 \partial_n F_{ij} \partial_n F_{jk} B_{kl} B_{li} \right]
+ O(\alpha'^4) + \text{total derivative.} \quad (3.8)
\]
We can easily guess the \( O(\alpha'^2) \) terms which we have to add to (3.1) to satisfy the condition
(1.8) from the \( O(B \partial^2 F^3) \) part of (3.8). By replacing \( B \) in the \( O(B \partial^2 F^3) \) part of (3.8)
with \( F \) and taking into account the symmetry factors, we have the following Lagrangian:
\[
\mathcal{L}(F) = \frac{\sqrt{\text{det} g}}{g_s} \left[ \partial_n F_{ij} \partial_n F_{ji} + 2(2\pi \alpha')^2 \partial_n F_{ij} \partial_n F_{jk} F_{kl} F_{li} 
+ (2\pi \alpha')^2 F_{nk} F_{kl} \partial_n F_{ij} \partial_l F_{ji} - \frac{1}{4} (2\pi \alpha')^2 F_{kl} F_{li} \partial_n F_{ij} \partial_n F_{ji} + O(\alpha'^4) \right]. \quad (3.9)
\]
By expanding \( \mathcal{L}(B + F) \), we can see that it generates the \( O(B \partial^2 F^3) \) terms in (3.8). The
consistency check of \( O(B^2 \partial^2 F^2) \) part can be done just as in the case of Section 2. There
is one missing term in (3.8) compared with the \( O(B^2 \partial^2 F^2) \) part of \( \mathcal{L}(B + F) \). Precisely
the same definition of the coupling constant \( G_s \) in the non-commutative gauge theory as
(2.18) produces the missing term. There is again the ordering ambiguity in the \( O(\partial^2 F^4) \)
terms on the non-commutative side \( \hat{\mathcal{L}}(\hat{F}) \) but it does not matter at the order we are
considering just as in the preceding section.

Thus we have succeeded in constructing a Lagrangian (3.9) with two derivatives up to
the quartic order of field strength which satisfies the condition (1.8) under the definition
of \( G_s \) (2.18). However, the Lagrangian (3.9) may not be the unique one which satisfies
the requirement (1.8). Let us reconsider the procedure by which we obtained (3.9),
namely, replacing \( B \) in the \( O(B \partial^2 F^3) \) part of (3.8) with \( F \). The resulting Lagrangian can
surely produce $O(B \partial^2 F^3)$ part of (3.8) as we have seen. However, it may not the unique possibility. Take the second term

$$2 \partial_n F_{ij} \partial_n F_{jk} (B_{kl} F_{li} + F_{kl} B_{li})$$

(3.10)
on the right-hand side of (3.8) as an example. The term $\partial_n F_{ij} \partial_n F_{jk} F_{kl} F_{li}$ generates (3.10) when we replace $F$ with $B + F$ but $\partial_n F_{ij} \partial_n (F_{jk} F_{kl} F_{li})$ also generates (3.10) with extra unwanted terms. There are possibilities that such extra terms can be arranged to total derivative so as to satisfy (1.8). We will consider such possibilities in the next subsection.

At any rate, the fact that we found a consistent form of two-derivative corrections (3.9) at least at the current order in the $\alpha'$ expansion is non-trivial and interesting itself. It remains to be investigated whether it persists to higher orders.

### 3.2 Solutions to the initial term condition

Let us go back to the problem whether or not the Lagrangian (3.9) is the unique one which satisfies (1.8). Assume that there is another Lagrangian $\tilde{\mathcal{L}}(F)$ satisfying (1.8) which coincides with $\mathcal{L}(F)$ (3.9) at $O(\partial^2 F^2)$ but differs at $O(\partial^2 F^4)$. Then the difference $\tilde{\mathcal{L}}(F) - \mathcal{L}(F)$ also satisfies the condition (1.8) but it does not have $O(\partial^2 F^2)$ part. Therefore the $O(\partial^2 F^4)$ part of it must satisfy the initial term condition (2.11). Now the problem of uniqueness reduced to the question whether there are solutions of the form $O(\partial^2 F^4)$ to the initial term condition.

What we should do is now clear. First write the most general terms of order $O(\partial^2 F^4)$ and replace $F$ with $B + F$. The resulting terms are quadratic with respect to $B$ so that we should look for combinations of terms such that the $O(B)$ and $O(B^2)$ parts are arranged to total derivatives respectively.

Any term of order $O(\partial^2 F^4)$ can be transformed to the following form using the integration by parts and the Bianchi identity [5]:

$$\mathcal{L} = \sum_{i=1}^{7} b_i J_i,$$

(3.11)

where

$$J_1 = \partial_n F_{ij} \partial_n F_{jk} F_{kl} F_{li}, \quad J_2 = \partial_n F_{ij} \partial_n F_{jk} F_{kl} F_{li},$$

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We will call this basis \{J_i\} as the Andreev-Tseytlin basis. This basis is useful when we consider field redefinition because the first three coefficients \(b_1, b_2\) and \(b_3\) in this basis do not change under field redefinition and unambiguous \[\text{[6]}. However, the following basis will turn out to be more convenient for the problem at hand:

\[
\mathcal{L} = \sum_{i=1}^{5} a_i J_i + a_6 J'_6 + a_7 J'_7, \tag{3.13}
\]

where

\[
J'_6 = F_{ij} \partial_n F_{ji} F_{kl} \partial_n F_{lk}, \quad J'_7 = F_{ij} \partial_n F_{jk} F_{kl} \partial_n F_{li}. \tag{3.14}
\]

The two bases are related as follows:

\[
J'_6 = -\frac{1}{2} J_1 - \frac{1}{2} J_6, \quad J'_7 = -2 J_2 - J_7, \\
J_6 = -J_1 - 2 J'_6, \quad J_7 = -2 J_2 - J'_7. \tag{3.15}
\]

Let us denote the \(O(B^n)\) part of \(J_i\) with its \(F\) replaced by \(B + F\) as \(J_i(B^n)\) (and similarly for \(J'_i\)).

First consider the \(O(B^2)\) part. Explicit expressions for \(J_i(B^2)\) and \(J'_i(B^2)\), and their variations with respect to \(A_i\) are

\[
J_1(B^2) = \partial_n F_{ij} \partial_n F_{ji} B_{kl} B_{lk}, \quad \delta J_1(B^2) = 4 B_{ki} B_{lk} \delta A_i \partial^2 \partial_j F_{ij}, \\
J_2(B^2) = \partial_n F_{ij} \partial_n F_{jk} B_{kl}, \quad \delta J_2(B^2) = 2 B_{km} B_{mj} \delta A_i \partial^2 \partial_k F_{mj}, \\
J_3(B^2) = B_{ni} B_{im} \partial_n F_{kl} \partial_m F_{lk}, \quad \delta J_3(B^2) = 4 B_{km} B_{mj} \delta A_i \partial^2 \partial_k F_{mj}, \\
J_4(B^2) = \partial_n F_{ni} \partial_m F_{im} B_{kl} B_{lk}, \quad \delta J_4(B^2) = 2 B_{ki} B_{lk} \delta A_i \partial^2 \partial_j F_{ij}, \\
J_5(B^2) = -\partial_n F_{ni} \partial_m F_{ij} B_{jm} B_{km}, \quad \delta J_5(B^2) = 2 B_{km} B_{mj} \delta A_i \partial^2 \partial_k F_{mj}, \\
J'_6(B^2) = B_{ij} \partial_n F_{ji} B_{kl} \partial_n F_{lk}, \quad \delta J'_6(B^2) = 4 B_{ij} B_{kl} \delta A_i \partial^2 \partial_j F_{lk}, \\
J'_7(B^2) = B_{ij} \partial_n F_{jk} B_{kl} \partial_n F_{li}, \quad \delta J'_7(B^2) = 4 B_{ij} B_{kl} \delta A_i \partial^2 \partial_l F_{jk}. \tag{3.16}
\]
where total derivatives are neglected. We can see the four different structures $B_{kl}B_{lk}δA_i$, $B_{im}B_{mj}δA_i$, $B_{km}B_{ml}δA_i$ and $B_{ij}B_{kl}δA_i$ in $δJ_i(B^2)$ and $δJ'_i(B^2)$. They must vanish respectively for a combination of $J_i(B^2)$ and $J'_i(B^2)$ to be total derivative. Solving this condition, we found that the most general combinations of $J_i(B^2)$ and $J'_i(B^2)$ which are total derivatives are

$$\mathcal{L} = a_1(J_1 - 2J_4) + a_3(J_3 + 2J_5) + a_6(J'_6 - 2J'_7).$$

(3.17)

This is a necessary condition for the initial term condition [2.11]. It follows that the number of independent solutions to the initial term condition is three at most.

Next consider the $O(B)$ part. $J_i(B)$ and $J'_i(B)$ are

$$J_1(B) = 2∂_nF_{ij}∂_kF_{jkl}, \quad J_4(B) = 2∂_nF_{ni}∂_mF_{im}B_{kl}F_{lk},$$

$$J_3(B) = 2B_{ni}F_{im}∂_nF_{kl}∂_mF_{lk}, \quad J_5(B) = -∂_nF_{ni}∂_mF_{ij}B_{jk}F_{km} - ∂_nF_{ni}∂_mF_{ij}F_{jkl}B_{km},$$

$$J'_6(B) = 2B_{ij}∂_nF_{ji}F_{kli}F_{lk}, \quad J'_7(B) = 2B_{ij}∂_nF_{jkl}F_{kli},$$

(3.18)

where we omitted $J_2(B)$ since its appearance in solutions is forbidden by the necessary condition (3.17). In this case, we can divide $δJ_i(B)$ and $δJ'_i(B)$ into two parts: terms with $B_{ij}δA_i$ and terms with $B_{kl}δA_i$. Both parts must vanish respectively for a combination of $J_i(B)$ and $J'_i(B)$ to be total derivative.

Consider the terms with $B_{ij}δA_i$. The relevant combinations in (3.17) are

$$δ[J_1(B) - 2J_4(B)] = 8B_{ij}δA_i∂_j∂_n(F_{ni}∂_mF_{im} - ∂_nF_{ni}F_{im}),$$

$$δ[J_3(B) + 2J_5(B)] = 2B_{ij}δA_i∂_n(∂_jF_{kl}∂_mF_{lk} + 2∂_lF_{lk}∂_nF_{kj}),$$

$$δ[J'_6(B) - 2J'_7(B)] = 4B_{ij}δA_i∂_n(∂_jF_{kl}∂_mF_{lk} - 2∂_lF_{lk}∂_nF_{kj}),$$

(3.19)

where total derivatives are neglected. We found that the only combination where the terms with $B_{ij}δA_i$ vanish is

$$δ \left[ (J_3(B) + 2J_5(B)) + \frac{1}{2}(J'_6(B) - 2J'_7(B)) \right] = \text{terms with } B_{kl}δA_i.$$  

(3.20)

\footnote{We can show that the $B_{ij}δA_i$ part of $δ[J_3(B) - 2J_4(B)]$ is not proportional to that of $δ[J_3(B) + 2J_5(B)]$ (or equivalently to that of $δ[J'_6(B) - 2J'_7(B)]$) as follows. The latter contains the structure $B_{ij}δA_iA_j$ while it is absent from the former. Moreover, both of them have the structure $BδA∂^3Aδ^jA$ but the index $j$ in each term of the former belongs to one of the derivatives in $∂^jA$ while it is not the case for the latter.}

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Thus we have obtained an additional necessary condition for the initial term condition \( (2.11) \), which is
\[
L = a_3 \left( (J_3 + 2J_5) + \frac{1}{2} (J_6' - 2J_7') \right).
\]
(3.21)

Now the number of independent solutions reduced to one at most. In order to determine if the particular combination \( (3.21) \) satisfies the initial term condition \( (2.11) \), it remains to evaluate the \( B_{kl} \delta A_i \) part of \( (3.20) \) to see if it vanishes. It would be straightforward to do that but it requires rather tedious calculations. We will take a different approach in the following instead of performing such explicit evaluation.

Let us repeat the calculations in the previous subsection with \( (3.1) \) replaced by
\[
L(F) = \partial_i F_{ik} \partial_j F_{jk},
\]
(3.22)

although this Lagrangian is proportional to \( (3.1) \) up to total derivative as we mentioned before. The non-commutative counterpart of this Lagrangian \((G^{-1})^{ij}(G^{-1})^{kl}(G^{-1})^{nm}\hat{D}_i \hat{F}_{jn} * \hat{D}_k \hat{F}_{ln}\) is expanded with respect to \( \alpha' \) as follows:
\[
(G^{-1})^{ij}(G^{-1})^{kl}(G^{-1})^{nm}\hat{D}_i \hat{F}_{jn} * \hat{D}_k \hat{F}_{ln} = \partial_i F_{ik} \partial_j F_{jk} + (2\pi\alpha')^2 [2B_{kl}\partial_i F_{lk} \partial_n F_{nj} + 2B_{kl}\partial_i F_{lk} \partial_n F_{nj} - \frac{1}{2}B_{kl}\partial_i F_{lk} \partial_n F_{nj} + 2B_{im}B_{ml}\partial_i F_{lj} \partial_k F_{kj} + B_{jm}B_{mn}\partial_i F_{lj} \partial_k F_{kn}] + O(\alpha'^4) + \text{total derivative}.
\]
(3.23)

As we have done in the previous case, it is not difficult to find the following Lagrangian
\[
\mathcal{L}(F) = \frac{\sqrt{\text{det} g}}{g_s} \left[ \partial_i F_{ik} \partial_j F_{jk} + (2\pi\alpha')^2 F_{im} \partial_i F_{lj} \partial_k F_{kj} + (2\pi\alpha')^2 F_{jm} F_{mn} \partial_i F_{lj} \partial_k F_{kn} - \frac{1}{4}(2\pi\alpha')^2 F_{kl} \partial_i F_{lj} \partial_k F_{kn} + O(\alpha'^4) \right],
\]
(3.24)

which generates the \( O(B \partial^2 F^3) \) and \( O(B^2 \partial^2 F^2) \) parts of \( (3.23) \) under the definition of \( G_s \) \( (2.18) \). This is another Lagrangian which satisfies the condition \( (1.8) \).

The \( O(\partial^2 F^4) \) terms of both Lagrangians \( (3.9) \) and \( (3.24) \) can be expressed in the Andreev-Tseytlin basis \( (3.12) \), as follows:
\[
\mathcal{L}_1 = \frac{\sqrt{\text{det} g}}{g_s} \left[ \partial_n F_{ij} \partial_n F_{ji} + (2\pi\alpha')^2 \left( 2J_2 + J_3 - \frac{1}{4} J_1 \right) + O(\alpha'^4) \right],
\]
(3.25)
$$\mathcal{L}_2 = \frac{\sqrt{\det g}}{g_s} \left[ \partial_i F_{ik} \partial_j F_{jk} + (2\pi\alpha')^2 \left( J_5 - \frac{1}{8} J_6 + \frac{1}{2} J_7 \right) \right.
+ \text{total derivative} + O(\alpha'^4) \right], \quad (3.26)$$

where we have used

$$F_{jm} F_{mn} \partial_i F_{ij} \partial_k F_{kn} = -\frac{1}{4} J_4 - J_5 - \frac{1}{8} J_6 + \frac{1}{2} J_7 + \text{total derivative}. \quad (3.27)$$

Since

$$\partial_n F_{ij} \partial_n F_{ji} = -2\partial_i F_{ik} \partial_j F_{jk} + \text{total derivative}, \quad (3.28)$$

the $O(\partial^2 F^4)$ part of $(-2\mathcal{L}_2) - \mathcal{L}_1$ satisfies the initial term condition (2.11), which is

$$(-2\mathcal{L}_2) - \mathcal{L}_1 = (2\pi\alpha')^2 \sqrt{\det g} \left[ \frac{1}{4} J_1 - 2 J_2 - J_3 - 2 J_5 + \frac{1}{4} J_6 - J_7 + \text{total derivative} + O(\alpha'^4) \right]$$

$$= \frac{(2\pi\alpha')^2 \sqrt{\det g}}{g_s} \left[ -J_3 - 2 J_5 - \frac{1}{2} J_6' + J_7' + \text{total derivative} + O(\alpha'^4) \right]. \quad (3.29)$$

This precisely coincides with the combination appeared in the necessary condition (3.21).

Thus we have shown that the combination of $O(\partial^2 F^4)$ terms (3.21) satisfies the initial term condition (2.11) and that it is the only solution since we have shown that the number of independent solutions is one at most.

To summarize, we derived the most general form of the Lagrangian with two derivatives which satisfies (1.8) up to the quartic order of field strength. Our result is

$$\mathcal{L}(F) = \frac{\sqrt{\det g}}{g_s} \left[ a\partial_n F_{ij} \partial_n F_{ji} + a(2\pi\alpha')^2 \left( -\frac{1}{4} J_1 + 2 J_2 + J_3 \right) \right.
+ b(2\pi\alpha')^2 \left( -J_3 - 2 J_5 - \frac{1}{2} J_6' + J_7' \right) + O(\alpha'^4) \right]$$

$$= \frac{\sqrt{\det g}}{g_s} \left[ a\partial_n F_{ij} \partial_n F_{ji} + (a - b)(2\pi\alpha')^2 \left( -\frac{1}{4} J_1 + 2 J_2 + J_3 \right) \right.
+ b(2\pi\alpha')^2 \left( -2 J_5 + \frac{1}{4} J_6 - J_7 \right) + O(\alpha'^4) \right], \quad (3.30)$$

where $a$ and $b$ are arbitrary constants. Furthermore, the definition of $G_s$ coincides with that for the Lagrangian without derivatives in the previous section so that we can superpose the two Lagrangians without violating the condition (1.8). The resulting expression
with the appropriate overall factor of $\alpha'$ for $p + 1$ space-time dimensions is

\[
\mathcal{L}(F) = \frac{(2\pi\alpha')^2 \sqrt{\text{det} g}}{g_s(2\pi)^p(\alpha')^{p+1}} [c F_{ij} F_{ji} + a(2\pi\alpha') \partial_n F_{ij} \partial_n F_{ji} \\
+ c(2\pi\alpha')^2 \left( \frac{1}{2} F_{ij} F_{jk} F_{kl} F_{li} - \frac{1}{8} F_{ij} F_{kl} F_{ik} \right) \\
+ (a - b)(2\pi\alpha')^3 \left( -\frac{1}{4} \partial_n F_{ij} \partial_n F_{ji} F_{kl} F_{lk} + 2 \partial_n F_{ij} \partial_n F_{jk} F_{kl} F_{li} + F_{ni} F_{im} \partial_n F_{kl} \partial_m F_{lk} \right) \\
+ b(2\pi\alpha')^3 \left( 2\partial_n F_{ni} \partial_m F_{ij} F_{jk} F_{km} + \frac{1}{4} \partial^2 F_{ij} F_{ji} F_{kl} F_{lk} - \partial^2 F_{ij} F_{jk} F_{kl} F_{li} \right) + O(\alpha'^4)],
\]

(3.31)

where $a$, $b$ and $c$ are arbitrary constants. This is the most general form of the Lagrangian which satisfies the condition (1.8) up to two derivatives and up to the quartic order of field strength.

4. Conclusions and discussions

We considered the constraints on the form of the effective Lagrangian of the rank-one gauge field on D-branes imposed by the condition that the two descriptions in terms of the ordinary and non-commutative gauge theories in the presence of a constant $B$ field are equivalent and are related by (1.8). We first explained how the form of $F^4$ terms is uniquely determined from the information on the $F^2$ term alone by the condition (1.8). We then applied our method to two-derivative terms and derived the most general form of them up to the quartic order of field strength. The result is summarized in (3.31).

Our result shows that the equivalence of the two descriptions can persist beyond the approximation that the field strength is slowly varying at least to the first non-trivial order in the $\alpha'$ expansion. Moreover, we found that the requirement of the equivalence highly constrains the form of the effective Lagrangian. Not only the equivalence of the two descriptions is important conceptually but also it may be useful practically. We hope that our approach provides a new perspective on the analysis of the dynamics of the gauge field on D-branes.

Finally, let us compare our result with ones obtained from other methods and discuss possible future direction of our approach. It would be helpful to discuss how our final

\[\text{We can show that there are no non-vanishing } O(F^3) \text{ and } O(\partial^2 F^3) \text{ terms in rank-one gauge theory.}\]
result (3.31) behaves under field redefinition in comparing with results in the literature. As we mentioned before, the coefficients in front of the \(O(\partial^2 F^2)\) term, \(J_4, J_5, J_6\) and \(J_7\) in the Andreev-Tseytlin basis change under field redefinition. We can make them vanish if we redefine the gauge field \(A_i\) as follows:

\[
\tilde{A}_i = A_i - \frac{a}{2c} (2\pi \alpha') \partial_j F_{ji} - \frac{a^2}{8c^2} (2\pi \alpha')^2 \partial^2 \partial_j F_{ji} - \frac{a^3}{16c^3} (2\pi \alpha')^3 \partial^4 \partial_j F_{ji}
\]

\[+(2\pi \alpha')^3 \left(-\frac{a-b}{8c} \partial_n F_{in} F_{kl} F_{kl} - \frac{a-2b}{2c} \partial_n F_{ik} F_{li} F_{lk} - \frac{a-b}{2c} \partial_n F_{kn} F_{ij} F_{lk} \right). \tag{4.1}
\]

Then the Lagrangian (3.31) is rewritten in terms of \(\tilde{F}_{ij} = \partial_i \tilde{A}_j - \partial_j \tilde{A}_i\) as

\[
\mathcal{L} = \frac{(2\pi \alpha')^2 \sqrt{\det g}}{g_s (2\pi)^p (\alpha')^{4\frac{p-1}{2}}} \left[ c \tilde{F}_{ij} \tilde{F}_{ji} + c (2\pi \alpha')^2 \left( \frac{1}{2} \tilde{F}_{ij} \tilde{F}_{jk} \tilde{F}_{kl} \tilde{F}_{li} - \frac{1}{8} \tilde{F}_{ij} \tilde{F}_{ji} \tilde{F}_{kl} \tilde{F}_{lk} \right) \right.
\]
\[+(a-b) (2\pi \alpha')^3 \left(-\frac{1}{4} \partial_n \tilde{F}_{ij} \partial_n \tilde{F}_{ji} \tilde{F}_{kl} \tilde{F}_{lk} + 2 \partial_n \tilde{F}_{ij} \partial_n \tilde{F}_{jk} \tilde{F}_{li} \tilde{F}_{lk} + \tilde{F}_{ni} \tilde{F}_{im} \partial_n \tilde{F}_{kl} \partial_m \tilde{F}_{lk} \right) \right]
\[+O(\alpha'^4) \right]. \tag{4.2}
\]

where total derivatives are neglected. It can be seen from this expression that the condition (1.8) determines the coefficients which do not change under field redefinition almost uniquely except some overall constants.

Now let us compare (4.2) with results obtained from other methods. The derivative corrections to the DBI Lagrangian were derived from the string four-point amplitude \[5\] or from the two-loop \(\beta\)-function in the open string \(\sigma\) model \[6\]. The \(O(\partial^2 F^4)\) terms in the bosonic string case are proportional to\[7\]

\[-\frac{1}{4} J_1 - 2 J_2 + J_3 \tag{4.3}\]

while our result is proportional to

\[-\frac{1}{4} J_1 + 2 J_2 + J_3. \tag{4.4}\]

These are very close but differ in a sign. We do not understand the origin of such discrepancy. For the superstring case, it was found that \(O(\partial^2 F^4)\) terms vanish \[5, 8\]. This **Very recently, the derivative corrections to the D-brane action were derived from the method of generalized boundary state for bosonic string theory \[7\] and for superstring theory \[8\].**

†† This expression is slightly different from (4) in \[3\] but the author was informed of a misprint in (4) of \[3\]: the last coefficient \(b_3\) should have sign +.
is consistent with our result because our method did not determine the overall factor and allows it to vanish. It should be clarified whether or not the discrepancy is characteristic of bosonic strings.

In this paper, we have concentrated on rank-one gauge theory. One of possible extensions of our approach is to consider higher-rank gauge theory. In particular, it would be an interesting question whether our approach can constrain the ordering of non-Abelian field strengths. The perturbative solution to (1.7) for the higher-rank case is already presented in [2] and in fact it is not difficult to see that the calculations presented in Section 2 can be extended to the higher-rank case as well at the order we have considered. However, the discussion at this order could not determine the ordering of field strengths. It would deserve to extend our consideration to higher orders to discuss the problem.

Another motivation for extension to higher orders is to investigate how terms with different numbers of derivatives are related by the condition (1.8). The constraints on terms without derivatives presented in Section 2 and those on two-derivative corrections in Section 3 are almost independent at the order which we have considered except that the definitions of $G_s$ on both sides must be the same to superpose the two Lagrangians. However, the independence may not persist to higher orders. This problem would be more important for the higher-rank case where the separation between field strengths and covariant derivatives becomes ambiguous.

*Note added*

In proving that the combination (3.21) satisfies the initial term condition, it was assumed that the non-commutative counterpart of the relation (3.28) holds as well. However, it turned out that this is not the case since from

\[
\hat{D}_n \hat{F}_{ij} \ast \hat{D}_n \hat{F}_{ji} = 2 \hat{F}_{ik} \ast \hat{D}_j \hat{D}_i \hat{F}_{jk} + \text{total derivative},
\]

\[
-2 \hat{D}_i \hat{F}_{ik} \ast \hat{D}_j \hat{F}_{jk} = 2 \hat{F}_{ik} \ast \hat{D}_i \hat{D}_j \hat{F}_{jk} + \text{total derivative},
\]

it follows that

\[
\hat{D}_n \hat{F}_{ij} \ast \hat{D}_n \hat{F}_{ji} + 2 \hat{D}_i \hat{F}_{ik} \ast \hat{D}_j \hat{F}_{jk} = 2 \hat{F}_{ik} \ast [\hat{D}_j, \hat{D}_i] \hat{F}_{jk} + \text{total derivative}
\]
\begin{align*}
&= -4i \hat{F}_{ij} \hat{F}_{jk} \hat{F}_{ki} + \text{total derivative} \\
&= -2(2\pi\alpha')^2 B_{nm} F_{ij} \partial_n F_{jk} \partial_m F_{ki} + O(\alpha'^4) + \text{total derivative}.
\end{align*}

Thus the conclusion which we can derive from the fact that the Lagrangians $\mathcal{L}_1$ and $\mathcal{L}_2$ satisfy the condition (1.8) is not that the combination

$$
\mathcal{F}(F) \equiv \frac{1}{4} J_1 + 2J_2 + J_3 + 2J_5 - \frac{1}{4} J_6 + J_7
$$

satisfies the initial term condition but that

$$
\mathcal{F}(B + F) = \mathcal{F}(F) - 2B_{nm} F_{ij} \partial_n F_{jk} \partial_m F_{ki} + \text{total derivative},
$$

so that there is no solution of the form $O(\partial^2 F^4)$ to the initial term condition.

This does not change our final result (3.31), however the reason why we can add the part proportional to $b$ in (3.31) is not that it satisfies the initial term condition but that we can add the term $2 \hat{F}_{ik} * [\hat{D}_j, \hat{D}_i] \hat{F}_{jk} = -4i \hat{F}_{ij} \hat{F}_{jk} \hat{F}_{ki}$ which vanishes in the commutative limit when we construct the Lagrangian on the non-commutative side. This ambiguity in constructing $\hat{\mathcal{L}}(\hat{F})$ from $\mathcal{L}(F)$ is characteristic of the rank-one gauge theory because the $F^3$ term no longer vanishes for higher-rank cases and if we could succeed in generalizing the part proportional to $b$ in (3.31) to the higher-rank cases, its existence would be naturally understood by the fact that the $F^3$ term satisfies the initial term condition.

Furthermore, several important developments in our understanding have been made recently [9]. It has turned out that it is in general possible to constrain the effective Lagrangian without assuming the form of the field redefinition (2.1) and its form is rather regarded as a consequence of the compatibility of the description by non-commutative gauge theory with that by ordinary gauge theory. Moreover, it has turned out that gauge-invariant but $B$-dependent corrections to (2.1) are generally possible and necessary for some cases including the case of bosonic string theory, which resolves the discrepancy between (4.3) and (4.4).

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