Boundary driven \(\text{XYZ}\) chain: Exact inhomogeneous triangular matrix product ansatz

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We construct an explicit matrix product ansatz for the steady state of a boundary driven \(\text{XYZ}\) spin-\(\frac{1}{2}\) chain for arbitrary local polarizing channels at the chain’s ends. The ansatz, where the Lax operators are written explicitly in terms of infinite-dimensional bidiagonal (triangular) site-dependent matrices, becomes exact either in the (Zeno) limit of infinite dissipation strength, or thermodynamic limit of infinite chain length. The solution is based on an extension of the newly discovered family of separable eigenstates of the model.

Introduction.— Exact and explicit solutions are indispensable for our advancement of understanding of statistical mechanics of interacting systems. While many such exact solutions in the realm of equilibrium statistical physics are known for over a half of century [1], say the Onsager’s and Baxter’s solutions to two-dimensional classical statistical models at thermal equilibrium or, sometimes equivalent, Bethe-ansatz solutions for quantum models, which, unlike the Hamiltonian and the complete eight vertex transfer matrix, break the spin reversal symmetry of the model and should have applications beyond the dissipative steady state paradigm.

Separable eigenstates of XYZ chain.— We consider a chain of \(N\) spins \(1/2\) described by XYZ hamiltonian \(H_N\) acting over \(2^N\) dimensional Hilbert space \(\mathcal{H}_N = \mathbb{C}^{2^N}\)

\[
H_N = \sum_{n=1}^{N-1} h_{n,n+1}, \quad h_{n,n+1} = \sum_{\alpha} \sigma_n^\alpha J_\alpha \sigma_{n+1}^\alpha,
\]

where \(\sigma_n^\alpha, n \in \{1, 2, \ldots N\}, \alpha \in \{x, y, z\}\) are Pauli operators embedded in \(\mathcal{H}\). It turns out that the natural parametrisation of the anisotropy coupling tensor \(J_\alpha\) is in terms of two complex parameters \(\eta, \tau\) and Jacobi \(\theta\)-functions, defining shorthand notation (following [19]):

\[
\theta_\alpha(u) \equiv \theta_\alpha(\pi u, e^{i2\pi\tau}), \quad \bar{\theta}_\alpha(u) \equiv \theta_\alpha(\pi u, e^{i\pi\tau})
\]

\[
\frac{J_x}{J_y} = e^{i\pi\eta} g(\frac{\eta}{\tau}), \quad \frac{J_y}{J_z} = e^{i\pi\eta} g(\frac{1+\tau}{2}), \quad \frac{J_z}{J_x} = g(\frac{1}{2}),
\]

where \(g(z) \equiv \bar{\theta}_1(\eta+z) / \bar{\theta}_1(z)\). Fixing the energy scale, say \(J = 1\), the remaining two independent coupling constants \(J_\alpha\) are uniquely parametrised — up to permutation of
the axes—by taking \( \eta, i \tau \in \mathbb{R} \). However, all the results of this Letter remain valid for arbitrary choice \( J, \eta, \tau \in \mathbb{C} \) parametrizing a general complex coupling tensor \( J_n \).

Our analysis starts by the following remarkable observation. Defining a one-parameter family of spinors

\[
|\psi_n\rangle \equiv |\psi(u + n\eta)\rangle = \begin{pmatrix} \theta_1(u + n\eta) \\ -\theta_2(u + n\eta) \end{pmatrix},
\]

where \( u \in \mathbb{C} \) is a free parameter, we find a family of spatially inhomogeneous separable eigenstates of XYZ model with boundary fields:

\[
(H_N - a_n\sigma^z \otimes I_2 - a_{n+1}I_2 \otimes \sigma^z + d_n I_4) |\psi_n\rangle \otimes |\psi_{n+1}\rangle = 0,
\]

where \( h = \sum_n J_n \sigma^x \otimes \sigma^x \) is a 4 \times 4 hamiltonian operator. Consistency of (6) requires that coefficients \( a_n, d_n \in \mathbb{C} \) satisfy a set of recurrence relations which can be explicitly solved [20]:

\[
a_n = a(u + n\eta), \quad d_n = f(\eta) + f(u + n\eta) - f(u + (n + 1)\eta),
\]

\[
a(u) = \frac{\bar{\theta}_1(u)\bar{\theta}_2(u)}{\theta_2(0)\theta_1(u)}, \quad f(u) = \frac{\bar{\theta}_1(u)\bar{\theta}_2(u)}{\theta_2(0)\theta_1(u)}.
\]

Note that this fixes the magnitude of the boundary fields \( a_1, a_N \), while their direction (chosen here along z-axis) is arbitrary so the result can be generalized to arbitrarily oriented boundary fields which not need be collinear.

Inhomogeneous bidiagonal Lax operators.— However, the choice (4) serves our purpose, which is to promote Eq. (6) to a divergence relation for local Lax operators [15, 16]

\[
[h_{n,n+1}, L_n L_m] = 2i[H_{n,m} - L_n I],
\]

where \( h_{n,n+1} = \sum_n L_n^\alpha \sigma^\alpha \) are the so-called Lax operators with components \( L_n^\alpha \in \text{End}(H_n) \) as well as \( I \in \text{End}(H_n) \) acting as linear operators over a suitable auxiliary space \( H_n \). Note that \( I \) acts trivially over the physical space \( H_N \).

We first show that the solution (6), together with an equivalent relation for a dual, bi-orthogonal spinor [20]

\[
\langle \psi_n^\dagger | = \langle \psi_1(u + n\eta), \theta_1(u + n\eta)\rangle, \quad \langle \psi_n^\dagger | \otimes \langle \psi_{n+1}^\dagger | h = \langle \psi_n^\dagger | \otimes \langle \psi_{n+1}^\dagger | (a_n\sigma^z \otimes I_2 + a_{n+1}I_2 \otimes \sigma^z + d_n I_4),
\]

provides a solution to (8) for 1-dimensional auxiliary space \( H_n = \mathbb{C} \),

\[
L_n = \frac{1}{\kappa(u + n\eta)} |\psi_n\rangle \langle \psi_n^\dagger |, \quad I = 1,
\]

where \( \kappa(u) = -i\theta_1(u)\theta_2(u)a(u) \). The proof follows from inserting (10) into Eq. (8), while facilitating divergence conditions Eqs. (6,9) and a trivially verifiable identity \( \sigma^z |\psi_n\rangle \langle \psi_n^\dagger | + |\psi_n\rangle \langle \psi_n^\dagger | \sigma^z = 2\theta_1(u + n\eta)\theta_2(u + n\eta)I_2 \).

Now, we are in position to state our main result:

**Theorem:** The operator divergence condition (8) is generally solved, for any auxiliary space \( H_n = \mathbb{C}^M \), with the following inhomogeneous bidiagonal ansatz

\[
L_n^\alpha = \sum_{j=1}^M s_{n-2(j-1)}^\alpha |j\rangle \langle j| + \sum_{j=1}^{M-1} s_{n-2(j-1)}^\alpha |j\rangle \langle j+1|,
\]

\[
I = \sum_{j=1}^M |j\rangle \langle j| - \sum_{j=1}^{M-1} |j+1\rangle \langle j| + 1
\]

where \( s_{n}^\alpha = s^\alpha(u + n\eta) \) and \( s^\alpha(u) = \frac{1}{2\alpha(u)} \left( \frac{\theta_1(u)}{\theta_2(u)} - \frac{\theta_2(u)}{\theta_1(u)} \right) \).

**Proof:** It is straightforward to check that (11) is equivalent to (10) for \( M = 1 \). For the general proof of (11) we make the following observation: diagonal elements of triangular matrices (bidiagonal ones being special cases thereof) form a commutative subalgebra \( \mathbb{C} \), hence the diagonal elements of Lax operators \( |j\rangle L_n^\alpha |j\rangle \) must all have the same functional form (independent of \( j \)) apart from a possible shift in the variable \( u \) (which may depend on \( j \)). Within the ansatz (11), the matrix elements \( |j\rangle L_n^\alpha |j'\rangle \) for \( j' \geq j + 2 \) all identically vanish. Hence we only need to check the case \( j' = j + 1 \) which is equivalent to study 2 \times 2 problem (in auxiliary space) with

\[
L_n = \begin{pmatrix} u_n & u_n \\ 0 & v_n \end{pmatrix}, \quad I = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}
\]

where \( u_n = \frac{1}{\kappa(u)} |\psi(u + n\eta)\rangle \langle \psi^\dagger(u + n\eta)|, \quad v_n = \frac{1}{\kappa(u + n\eta)} |\psi^\dagger(u + n\eta)| \langle \psi(u + n\eta)| \).

Inserting this ansatz into (1) [Eq.(8)2] and using the established identities, e.g. (6,9), the only nontrivial condition that remains connects \( u \) and \( v \), i.e. \( v = u - 2\eta \), which proves (11) for any \( u, M \).

Steady state of the boundary driven chain.— We wish to construct the nonequilibrium steady state (NESS) density matrix \( \rho \) of the Lindblad equation

\[
\frac{d}{dt} \rho = -i[H_{N+2}, \rho] + \Gamma D_\rho |\rho\rangle + \Gamma D_\rho |\rho\rangle = 0,
\]

at large dissipation strength \( \Gamma \), where \( D_\rho |\rho\rangle, \mu \in \{1, r\} \), denote the dissipators at the left and right ends of the chain of \( N+2 \) sites, which we label by 0 and \( N+1 \), respectively. They are of the form \( D_\rho |\rho\rangle = 2k_\mu k_\mu^\dagger - \{k_\mu^\dagger k_\mu, \rho \} \) with jump operators \( k_\mu, k_\mu^\dagger = (n^\mu_0 + n^\mu_0^\dagger) \cdot \sigma^\mu \) targeting polarizations \( \mu_\mu = \mu (\theta_1, \phi_1, \phi_2, \phi_3) \), while their direction (chosen here along \( \sigma^z \) of sign \( \sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta \)). Here, \( n_0^\mu = \mu \frac{\sigma^z}{2} - \mu_\mu = \mu \pm \frac{\sigma^z}{2} \), which together with \( n_0^\mu \) form an orthonormal basis of \( \mathbb{R}^3 \). The targeted states of the dissipators are single-site pure states \( \rho_\mu \) such that \( D_\rho |\rho_\mu\rangle = 0 \).
In our previous work [15, 16, 18] we have shown that in the regime of either large $\Gamma$ or large $N$, NESS can in the leading order be written as $\rho = \rho_l \otimes \rho_N \otimes \rho_r + O((N\Gamma)^{-1})$, where $\rho_N = \Omega^d / \text{Tr} (\Omega^d)$ is completely fixed with condition

$$
[H_N + \sum_{\alpha} (J_{\alpha} n_{\alpha}^0 \sigma_{\alpha}^1 + J_{\alpha} n_{\alpha}^1 \sigma_{\alpha}^N), \rho_N] = 0,
$$

(14)

and the matrix product ansatz

$$
\Omega = \langle w_l | L_1 L_2 \cdots L_N | w_r \rangle,
$$

(15)

where $L_n$ obey the divergence condition (8). The boundary vectors $|w_\mu\rangle \in \mathcal{H}_n$ are fixed by projecting the commutativity conditions to the boundary sites, yielding

$$
\langle w_l | V_1 = 0,
$$

(16)

$$
V_r |w_r\rangle = \varepsilon (1,-1,1,-1,\ldots)^T,
$$

(17)

$$
V_l = \sum_{\alpha} J_{\alpha} n_{\alpha}^0 L_{\alpha}^1 + il, \quad V_r = \sum_{\alpha} J_{\alpha} n_{\alpha}^0 L_{\alpha}^N - il
$$

while the commutativity (14) in the bulk follows from (8). Parameter $\varepsilon$ is arbitrary and in generic case we may fix it as $\varepsilon = 1$, while a special – homogeneous case $\varepsilon = 0$ should be treated separately.

We do not only have a fully explicit form of the Lax operators (11) but we can also solve the boundary equations explicitly (16,17) and determine the free complex variable $u$. Namely, we shall parametrize targeted boundary polarizations $n_i, n_r$ via two complex numbers,

$$
u_\mu = x_\mu + iy_\mu, \quad \mu \in \{1, r\},
$$

(18)

as (see [20])

$$
n_r^x = - \frac{\bar{\theta}_2 (iy_\mu)}{\theta_2 (iy_\mu)} \frac{\theta_1 (x_\mu)}{\bar{\theta}_2 (x_\mu)},
n_r^y = - \frac{\bar{\theta}_2 (iy_\mu)}{\theta_2 (iy_\mu)} \frac{\theta_1 (x_\mu)}{\bar{\theta}_2 (x_\mu)},
n_r^z = - \frac{\bar{\theta}_2 (iy_\mu)}{\theta_2 (iy_\mu)} \frac{\theta_1 (x_\mu)}{\bar{\theta}_2 (x_\mu)}
$$

(19)

We can now prove, see [20] for details, that (14) is satisfied with the choice

$$
u = u_1, \quad M = N + 1
$$

(20)

in (11), and Eq.(16) is solved by

$$
\langle w_l | = (1,1,0,\ldots,0).
$$

(21)

The solution of Eq (17) for $|w_r\rangle = (r_1, r_2, \ldots, r_{N+1})^T$ for $\varepsilon = 1$ is given by recurrence

$$
r_{k-1} = \frac{(-1)^k r_{k-1} (V_r)_{k-1,k}}{(V_r)_{k-1,k}}, \quad k = 1, \ldots, N + 1,
$$

(22)

$$
r_{N+1} = \frac{(-1)^N}{(V_r)_{N+1,N+1}}.
$$

(23)

which is valid if operator $V_r$ in (17) does not have zero eigenvalues, i.e. $\prod_{k} (V_r)_{kk} \neq 0$.

We stress that the vector on right hand side of (17) should be allowed since it is essentially in the joint kernel (null space) of all $L_{\alpha}^\beta$, disregarding the last component, namely $L_{\alpha}^\beta (1,-1,1,-1,\ldots)^T = (0,0,\ldots,0,0)^T$, where $*$ denotes any nonzero element. Next action of $L_{\alpha}^\beta$ creates another nonzero element $L_{\alpha}^\beta (0,0,\ldots,0,0,\ldots)^T = (0,0,\ldots,0,0,\ldots,\ldots)^T$, and so on. The property (14) is thus guaranteed by $\langle w_l | L_1^\alpha \cdots L_{N+1}^\beta | w_l \rangle = (1,1,0,\ldots,0,0,\ldots)^T = 0$. In the previous study [15, 16], on the other hand, Lax operators had a trivial joint kernel hence only $\varepsilon = 0$ applied there.

However, for a submanifold of fine tuned driving/coupling parameters the matrix $V_r$ can be singular $\det [V_r] = \prod_{k} (V_r)_{kk} = 0$ [17]. Such situation corresponds to a nonequilibrium steady state with large modulations of the local magnetization, see Fig. 1. In this case, the Eq.(17) for the right auxiliary vector $|w_r\rangle$ must be solved with $\varepsilon = 0$, while the recurrence (22) breaks down.

The most prominent NESS of this singular type is obtained if just the first diagonal term of $V_r$ vanish, $(V_r)_{11} = 0$, yielding unique solution of (17) with $\varepsilon = 0$: $|w_r\rangle = (1,0,0,\ldots,0)^T$. The respective right boundary polarization $n_r$ is given by Eq.(19) with $y_1 = y_T, x_r = x_1 + (N+1)\eta$, see [20]. Due to upper triagonal structure of all $L_{\alpha}^\beta$, every expression of the form $\langle w_l | L_1^\alpha L_2^\beta \cdots L_N^\beta | w_l \rangle$ will contain only one nonzero term rendering the steady state site-factorized,

$$\rho_N = (L_1 L_1^\dagger) \otimes (L_2 L_2^\dagger) \otimes \cdots \otimes (L_N L_N^\dagger)
$$

(23)

where $L_n$ is given by (10). It is easy to verify that the state is pure, and is fully characterized by the corresponding magnetization profile, given by Jacobi elliptic functions

$$
\langle \sigma_{\alpha}^x \rangle = A_x \sin (2K_k (mn + x_1)),
$$

(24)

$$
\langle \sigma_{\alpha}^y \rangle = A_y \cos (2K_k (mn + x_1)),
$$

$$
\langle \sigma_{\alpha}^z \rangle = A_z \cos (2K_k (mn + x_1))
$$

(23)

(explicit $A_\alpha$ given in [20]), where $k = \frac{(\theta_2 (0))^2}{\theta_3 (0)}$, $K_k = \frac{\pi}{2} \theta_3 (0)^2$, with periods 2/\eta (1/\eta) for $x, y$ ($z$) components, see upper panel of Fig. 1. The state (23) is an elliptic counterpart of the spin-helix state (see lower Panel of Fig. 1) appearing in models with uniaxial spin anisotropy (XXZ) [22, 23].

**Special case of XXZ chain.** In the partially anisotropic case $J_x/J = J_y/J = 1, J_z/J = \Delta = \cos \gamma$ ($\gamma$ either real or imaginary) the divergence condition (6) is satisfied with $|\psi_n\rangle = (\cos \frac{\eta}{2} e^{-i\alpha_n/2}, \sin \frac{\eta}{2} e^{i\alpha_n/2})^T$, $u_n = u + n\gamma$ where $\alpha_{n+1} = \alpha_n - \gamma \sin \gamma, d_n = \Delta$, see [20]. Following the same line of argument as for XYZ case, we obtain explicit Lax operators in the bidiagonal...
form (11):

\[
s_n^\alpha (u) \pm i s_n^\nu (u) = \mp \frac{(\tan \frac{\theta}{2})^{\pm 1}}{\sin \gamma} e^{\pm i(n+\eta)}
\]

(25)

The Eq. (16) is satisfied with the same left boundary vector (21), provided that the parameters \(u, \theta\) in (25) relate to the spherical coordinates \(\phi_1, \theta_1\) of the left boundary polarization \(n_1\) via \(\theta = \theta_1, u = \phi_1\). The right boundary vector is calculated using Eq. (17) with either \(\epsilon = 1\) or \(\epsilon = 0\), as discussed above.

While in special cases the NESS can be obtained fully analytically, see (24) using our MPA, for generic parameters simplicity and sparse structure of our MPA allows efficient numerical calculus of arbitrary NESS observables for large chains, as exemplified in Fig. 2. A crucial advantage of our representation wrt to earlier results [15, 16] is a full control of the auxiliary vector \(|w_n\rangle\) via a recurrence (22) and explicit Lax operator expression (11) for fully anisotropic XYZ case.

Discussion. – We have proposed an analytic method of constructing inhomogeneous MPA on the basis of local divergence condition (6), by means of which we solve driven dissipative problem in the Zeno regime for a quantum spin chain, with boundary spins kept in fixed arbitrary quantum states. Based on our results, we identified parameters allowing to generate remarkably simple pure steady states with local magnetization described via Jacobi elliptic functions (24) depicted in Fig. 1. These states are elliptic counterparts of spin helix states [21–23] discussed recently in connection with cold atom experiments [24, 25], from one side, and in connection with remarkable underlying algebraic structure (phantom Bethe roots) [26–28], from the other side. In XYZ model context we can show that highly atypical quantum states of type (24) result from emergence of low-dimensional invariant subspaces in the spectrum of open XYZ spin chain, under special choice of boundary fields [29].

Our result enable efficient study of steady state properties of driven dissipative spin chains, reducing complexity from exponential degree \(2^{2N}\) to polynomial degree \(N^2\) thus allowing accessing hydrodynamic scales. From theoretical viewpoint, our construction is easily generalizable to other models satisfying property (6), see [22], e.g. for the Izergin-Korepin model.

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FIG. 1. Local magnetization profiles for factorized pure NESS in the elliptic XYZ case (upper panel) and for XXZ case (lower panel). \(x-, y-, z-\)spin projections are indicated with black, red and blue points respectively. Interpolating curves for \(x-, y-, z-\)spin projections are given by Jacobi elliptic functions (24) for XYZ model and trigonometric functions for XXZ model, and \(n\) is site number. For both cases, boundary polarizations are chosen so as to render \(\langle V_{11}\rangle = 0\) in Eq. (17) and \(N = 30\), so that the (17) is solved with \(\epsilon = 0\). Parameters: \(\eta = 0.12, \tau = i/2, u = 0.0477 + 0.12336i\) (upper panel), \(\gamma = 0.12, \theta_1 = \pi/3, u = \phi_1 = -0.2\pi\) (lower panel).

FIG. 2. Local magnetization profiles for elliptic XYZ case for a generic choice of parameters, when Eq. (17) is solved with \(\epsilon = 1\), i.e. by recurrence (22). \(x-, y-, z-\)spin projections are indicated, respectively, with black, red and blue points connected by lines for clarity. Boundary targeted magnetizations (site numbers \(n = 0, n = N + 1\)) are indicated by bullet symbols at the ends. Parameters: \(N = 100, \eta = 0.4511, \tau = i/2, u_1 = -0.89 + 0.4i, u_2 = 0.1 + 0.55i\). The corresponding anisotropy tensor eigenvalues are \(\{J_x, J_y, J_z\} = \{2.37994, 0.427449, 0.128303\}\).
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This Supplemental Material contains seven sections. In S-I we list and prove some useful identities for elliptic theta functions. They are used in S-II to prove the general divergence condition for XYZ Heisenberg Hamiltonian. In S-III we show how the divergence condition for partially anisotropic XXZ model follows from the general divergence condition via a limit procedure. Sections S-IV and S-V are central, as there we prove the bidiagonal Matrix Product Ansatz and obtain the left and the right auxiliary vectors. In S-VI we discuss how to obtain special family of NESS which generalize spin-helix states to fully anisotropic XYZ Hamiltonian. In S-VII we explicitly calculate observables for the simplest representative of the family, the fully factorized pure elliptic NESS.

S-I. SOME ELLIPTIC FUNCTION IDENTITIES

In this paper, we adopt the notations of elliptic theta functions \(\vartheta_\alpha(u, q)\) following Ref. [19]

\[
\vartheta_1(u, q) = 2\sum_{n=0}^{\infty} (-1)^n q^{n+\frac{1}{2}} \sin((2n + 1)u),
\]

\[
\vartheta_2(u, q) = 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2} \cos((2n + 1)u),
\]

\[
\vartheta_3(u, q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos(2nu),
\]

\[
\vartheta_4(u, q) = 1 + 2\sum_{n=1}^{\infty} (-1)^m q^{n^2} \cos(2nu).
\]

For convenience, we use the following shorthand notations \(\theta_\alpha, \bar{\theta}_\alpha\)

\[
\theta_\alpha(u) \equiv \vartheta_\alpha(\pi u, e^{i\pi \tau}), \quad \bar{\theta}_\alpha(u) \equiv \vartheta_\alpha(\pi u, e^{i\pi \tau}), \quad \text{Im}[\tau] > 0.
\]

The four types of theta functions satisfy the following identities [19]

\[
\bar{\theta}_2(u) = \bar{\theta}_1(u + \frac{1}{2}), \quad \bar{\theta}_3(u) = e^{i\pi(u + \frac{1}{2})} \bar{\theta}_1(u + \frac{1+\tau}{2}), \quad \bar{\theta}_4(u) = e^{i\pi(u + \frac{1+\tau}{2})} \bar{\theta}_1(u + \frac{1}{2}),
\]

\[
\theta_2(u) = \theta_1(u + \frac{1}{2}), \quad \theta_3(u) = e^{i\pi(u + \frac{1}{2})} \theta_1(u + 1+\tau), \quad \theta_4(u) = -e^{i\pi(u + \frac{1+\tau}{2})} \theta_1(u + \frac{1}{2}).
\]
We introduce some fundamental properties of elliptic functions [30]

\[
\begin{align*}
\bar{\theta}_1(-u) &= \bar{\theta}_1(u), & \bar{\theta}_1(u+1) &= -\bar{\theta}_1(u), & \bar{\theta}_1(u+\tau) &= -e^{-2i\pi(u+\bar{\tau})}\bar{\theta}_1(u), \\
\theta_1(-u) &= \theta_1(u), & \theta_1(u+1) &= -\theta_1(u), & \theta_1(u+2\tau) &= -e^{-2i\pi(u+\tau)}\theta_1(u), \quad (S4) \\
\theta(u+x)\theta(u-x)\theta(v+y)\theta(v-y) - \theta(u+y)\theta(u-y)\theta(v+x)\theta(v-x) &= \theta(u+v)\theta(u-v)\theta(x+y)\theta(x-y), & \theta & \equiv \bar{\theta}_1, \theta_1, \\
\frac{\theta_1(2u)}{\theta_1(u)} &= \frac{\bar{\theta}_1(u)\bar{\theta}_2(u)}{\bar{\theta}_1(2\tau)\bar{\theta}_2(\tau)} = \frac{\bar{\theta}_1(u)\bar{\theta}_1(u+\frac{1}{2})}{\bar{\theta}_1(\frac{3\tau}{2})\bar{\theta}_1(\frac{3}{2})}, \\
\frac{\bar{\theta}_1(u)}{\theta_1(\tau)} &= \frac{\theta_4(u)\theta_1(u)}{\theta_4(\frac{1}{2})\theta_1(\frac{1}{2})} = e^{i\pi(u+\bar{\tau})}\theta_1(u+\tau)\theta_1(u), \quad (S7)
\end{align*}
\]

Using Eq. (S5) and Landen’s type of transformations (S6)-(S7), we get useful identities

\[
\begin{align*}
\frac{1}{\theta_1(u)\theta_1(u+\tau)} & \left[ \theta_1(u-v)\theta_1(u+v+\tau) + e^{-2i\pi\nu}\theta_1(u+v)\theta_1(u-v+\tau) \right] = -\frac{\theta_1(\tau)\theta_1(2v)}{\theta_1(v-\tau)\theta_1(v)}, \\
\bar{\theta}_1(2u) &= 2e^{2i\pi u}\frac{\bar{\theta}_1(u)\bar{\theta}_1(u+\frac{1}{2})\bar{\theta}_1(u+\frac{3}{2})\bar{\theta}_1(u+\frac{1}{2}+\frac{\tau}{2})}{\bar{\theta}_1(\frac{\tau}{2})\bar{\theta}_1(\frac{1}{2})\bar{\theta}_1(\frac{1}{2})}, \\
\theta_1(2u) &= 2e^{2i\pi u}\frac{\theta_1(u)\theta_1(u+\frac{1}{2})\theta_1(u+\tau)\theta_1(u+\tau+\frac{1}{2})}{\theta_1(\tau+\frac{1}{2})\theta_1(\frac{1}{2})\theta_1(\tau)}, \quad (S10)
\end{align*}
\]

Introduce the derivative of functions \(\ln \theta_1(u)\) and \(\ln \bar{\theta}_1(u)\)

\[
\zeta(u) = \frac{\theta'_1(u)}{\theta_1(u)}, \quad \bar{\zeta}(u) = \frac{\bar{\theta}'_1(u)}{\bar{\theta}_1(u)}. \quad (S11)
\]

The functions \(\zeta(u)\) and \(\bar{\zeta}(u)\) possess the following properties

\[
\begin{align*}
\bar{\zeta}(u) &= -\zeta(-u), & \bar{\zeta}(u+1) &= \zeta(u), & \zeta(u+\tau) &= \bar{\zeta}(u) - 2i\pi, \\
\zeta(u) &= -\zeta(-u), & \zeta(u+1) &= \zeta(u), & \zeta(u+2\tau) &= \zeta(u) - 2i\pi, \\
2\zeta(2u) &= \zeta(u) + \bar{\zeta}(u + \frac{1}{2}), & \bar{\zeta}(u) &= i\pi + \zeta(u) + \zeta(u+\tau), \\
2\zeta(u) &= 2\pi + \zeta(u) + \zeta(u+\frac{1}{2}) + \zeta(u+\frac{3}{2}) + \zeta(u+\frac{5}{2}) + \zeta(u+\frac{7}{2}). \quad (S15)
\end{align*}
\]

The functions \(\bar{\theta}_1(u), \theta_1(u), \zeta(u), \bar{\zeta}(u)\) satisfy the identities

\[
\begin{align*}
\frac{\bar{\theta}_1(u+\frac{3}{2})}{\bar{\theta}_1(u)} &= \frac{\bar{\theta}_1(\frac{1}{2})}{\theta'_1(0)}[\bar{\zeta}(\frac{1}{2}) + \bar{\zeta}(\frac{u+3}{2}) - \bar{\zeta}(u)], \\
\frac{\bar{\theta}_1(\tau)}{\theta_1(\tau)} &= \frac{1}{\theta'_1(0)}[\zeta(u) + \zeta(b) + \zeta(c) - \zeta(a + b + c)], \quad (S17) \\
\frac{\theta_1(\tau)}{\theta'_1(0)} &= \frac{e^{i\pi\tau}\theta_1(\tau)\theta'_1(0)}{\theta_1(\tau)\theta'_1(0)}[\zeta(u) + \zeta(b) + \zeta(c) - \zeta(a + b + c)] \quad (S18)
\end{align*}
\]

for some arbitrary constants \(a, b, c, d\). Some other useful relations are:

\[
\begin{align*}
2\theta_1(x+y)\theta_1(x-y) &= \theta_4(x)\theta_3(y) - \theta_4(y)\theta_3(x), \\
2\theta_4(x+y)\theta_4(x-y) &= \theta_4(x)\theta_3(y) + \theta_4(y)\theta_3(x), \\
2\theta_4(x+y)\theta_1(x-y) &= \theta_1(x)\theta_2(y) - \theta_1(y)\theta_2(x). \quad (S21)
\end{align*}
\]

**Proof of Eq. (S16):** Define auxiliary functions

\[
c_1(u) = \frac{\bar{\theta}_1(u+\frac{1}{2})}{\bar{\theta}_1(u)}, \quad c_2(u) = \frac{\bar{\theta}_1(\frac{1}{2})}{\theta'_1(0)}[\bar{\zeta}(\frac{1}{2}) + \bar{\zeta}(\frac{u+1}{2}) - \bar{\zeta}(u)]. \quad (S22)
\]
These two functions possess identical properties, as follows

- double periodicity: \( c_1(u + 1) = c_1(u) \), \( c_1(u + \tau) = -c_1(\tau) \),
- \( c_2(u + 1) = c_2(u) \), \( c_2(u + \tau) \overset{\text{(S15)}}{=} -c_2(\tau) \),

\[
\begin{align*}
\text{zeros: } & \quad c_1(\frac{1}{2}) = 0, \quad c_2(\frac{1}{2}) \overset{\text{(S12)}}{=} 0, \\
\text{poles: } & \quad c_1(0) \rightarrow \infty, \quad c_2(0) \rightarrow \infty,
\end{align*}
\]

 derivative at certain point: \( \lim_{u \to 0} \frac{\partial}{\partial u} c_1(u) = \lim_{u \to 0} \frac{\partial}{\partial u} c_2(u) = \bar{c}_1(\frac{1}{2}) \).

These properties imply \( c_1(u) \equiv c_2(u) \). Following this method, one can prove all the other identities.

## S-II. PROOF OF DIVERGENCE CONDITION FOR XYZ MODEL (6)

The anisotropy exchange constants \( J_x, J_y, J_z \) in XYZ model are parameterized as

\[
J_x = e^{i\pi\eta} \frac{\tilde{\theta}_1(\eta + \frac{\tau}{2})}{\tilde{\theta}_1(\frac{\tau}{2})}, \quad
J_y = e^{i\pi\eta} \frac{\tilde{\theta}_1(\eta + \frac{1+\tau}{2})}{\tilde{\theta}_1(\frac{1+\tau}{2})}, \quad
J_z = \frac{\tilde{\theta}_1(\eta + \frac{1}{2})}{\tilde{\theta}_1(\frac{1}{2})},
\]

where \( \eta \) is generic complex number. Introduce two parameters \( J_{\pm} \) as

\[
J_{\pm} = J_x \pm J_y.
\]

Existence of factorized eigenstates \( |\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \ldots \otimes |\psi_N\rangle \) in the spectrum of the XYZ transfer matrix \( T \) (8-vertex model) under special conditions (on special manifolds) has been established long ago \([31, 32]\). Since the transfer matrix \( T \) and the XYZ quantum spin-\( \frac{1}{2} \) Hamiltonian commute, \( [T, H] = 0 \), \( T \Psi = \Lambda \Psi \) necessarily entails \( H \Psi = E_0 \Psi \) (if the eigenvalue \( E_0 \) is nondegenerate). Further, since the Hamiltonian \( H \) is a sum of local terms \( h_{n,n+1} \), the eigenvalue condition for a factorized state necessarily implies a local divergence condition, which can be written as

\[
h |\psi_n\rangle \otimes |\psi_{n+1}\rangle = (a_n \sigma^z \otimes I_2 - a_{n+1} I_2 \otimes \sigma^z + d_n I_4) |\psi_n\rangle \otimes |\psi_{n+1}\rangle,
\]

where \( h = \sum_\alpha J_\alpha \sigma^\alpha \otimes \sigma^\alpha \) is a \( 4 \times 4 \) hamiltonian density operator. Indeed, under the condition \( a_1 \equiv a_{N+1} \), in a periodic system \( H = \sum_{n=1}^N h_{n,n+1} \) the above entails

\[
H \Psi = E_0 \Psi, \quad E_0 = \sum_{n=1}^N d_n.
\]

Here our aim is to prove (S26) and establish compact form for the coefficients \( a_n, a_{n+1}, d_n \). The Eq. (S26) contains 4 scalar equations and only 3 unknowns so there must be a consistency condition. The consistency condition has the form

\[
(\langle \psi_n^+ | \otimes \langle \psi_{n+1}^+ | ) h (|\psi_n\rangle \otimes |\psi_{n+1}\rangle) = 0 \quad \text{(S28)}
\]

\[
\langle \psi_n^+ | \psi_n \rangle = 0.
\]

We make the following ansatz:

\[
|\psi_n\rangle = \begin{pmatrix} \theta_1(u) \\ -\theta_4(u) \end{pmatrix} \equiv |\psi(u)\rangle, \quad |\psi_{n+1}\rangle \equiv |\psi(u + \eta)\rangle \quad \text{(S29)}
\]

\[
|\psi_n^+\rangle \equiv \langle \psi^+ | (u) \equiv (\theta_4(u), \theta_1(u)) \quad \text{(S30)}
\]

Inserting the ansatz (S29), (S30) into the divergence condition (S26), the coefficients \( a_n, a_{n+1}, d_n \) are given by

\[
2(a_n - a_{n+1}) = J_- \begin{bmatrix} \theta_4(u) \theta_1(u + \eta) & \theta_1(u) \theta_4(u + \eta) \\ \theta_1(u) \theta_4(u + \eta) & \theta_4(u) \theta_1(u + \eta) \end{bmatrix},
\]

\[
2(a_n + a_{n+1}) = J_+ \begin{bmatrix} \theta_4(u) \theta_1(u + \eta) & \theta_1(u) \theta_4(u + \eta) \\ \theta_1(u) \theta_4(u + \eta) & \theta_4(u) \theta_1(u + \eta) \end{bmatrix},
\]

\[
2d_n = J_- \begin{bmatrix} \theta_4(u) \theta_4(u + \eta) & \theta_1(u) \theta_4(u + \eta) \\ \theta_1(u) \theta_4(u + \eta) & \theta_4(u) \theta_4(u + \eta) \end{bmatrix},
\]

\[
2d_n = J_+ \begin{bmatrix} \theta_4(u) \theta_4(u + \eta) & \theta_1(u) \theta_4(u + \eta) \\ \theta_1(u) \theta_4(u + \eta) & \theta_4(u) \theta_4(u + \eta) \end{bmatrix},
\]
while the consistency condition (S28) becomes

\[ 4J_z - J_+ \left[ \frac{\theta_4(u)\theta_1(u + \eta)}{\theta_1(u)\theta_4(u + \eta)} + \frac{\theta_1(u)\theta_4(u + \eta)}{\theta_4(u)\theta_1(u + \eta)} \right] + J_- \left[ \frac{\theta_4(u)\theta_4(u + \eta)}{\theta_1(u)\theta_1(u + \eta)} + \frac{\theta_1(u)\theta_1(u + \eta)}{\theta_4(u)\theta_4(u + \eta)} \right] = 0. \]  

(S32)

The above consistency condition can be proved, while Eq. (S31) can be simplified using elliptic theta function identities.

For the convenience of calculation, we only use the notations \( \theta_1 \) and \( \bar{\theta}_1 \) in the proof. Using Landen’s type of transformation (S7), we rewrite the anisotropy constants

\[ J_x = \frac{\theta_1(\frac{\tau}{2} - \eta)\theta_4(\frac{\tau}{2} + \eta)}{\theta_1(\frac{\tau}{2})}, \quad J_y = \frac{\theta_1(\frac{3\tau}{2} + \eta)\theta_4(\frac{3\tau}{2} + \eta)}{\theta_1(\frac{3\tau}{2} + \eta)}, \quad J_z = e^{i\pi \eta} \frac{\theta_1(\eta + \frac{1}{2} + \tau)\theta_1(\eta + \frac{1}{2})}{\theta_1(\frac{1}{2} + \tau)\theta_1(\frac{1}{2})}. \]  

(S33)

Using Eqs. (S5) and (S10), we get

\[ J_- = -2e^{-i\pi \eta} \frac{\theta_1^2(\eta)}{\theta_1^2(\tau)}, \quad J_+ = -2 \frac{(\eta + \tau)\theta_1(\eta - \tau)}{\theta_1^2(\tau)}. \]  

(S34)

Then, we have

\[ -J_+ \left[ \frac{\theta_4(u)\theta_1(u + \eta)}{\theta_1(u)\theta_4(u + \eta)} + \frac{\theta_1(u)\theta_4(u + \eta)}{\theta_4(u)\theta_1(u + \eta)} \right] + J_- \left[ \frac{\theta_4(u)\theta_4(u + \eta)}{\theta_1(u)\theta_1(u + \eta)} + \frac{\theta_1(u)\theta_1(u + \eta)}{\theta_4(u)\theta_4(u + \eta)} \right] 
= 2\theta_1(\eta + \tau)\theta_1(\eta - \tau) \frac{\theta_1^2(\eta)}{\theta_1(\tau)} \left[ e^{-i\pi \eta} \frac{\theta_1(u + \tau\theta_1(u + \eta) + \eta + \tau)}{\theta_1(u + \eta)\theta_1(u + \eta)} + e^{i\pi \eta} \frac{\theta_1(u + 2\tau)\theta_1(u + \eta)}{\theta_1(u + \eta)\theta_1(u + \eta)} \right] 
= -2 \frac{2e^{i\pi \eta}}{\theta_1(u)\theta_1(u + \tau)} \left[ \theta_1(u - \eta)\theta_1(u + \eta + \tau) + e^{-2i\pi \eta} \theta_1(u + \eta)\theta_1(u - \eta + \tau) \right] 
= 2e^{i\pi \eta} \frac{\theta_1(\tau)\theta_1(2\tau)}{\theta_1(\eta - \tau)} \left[ e^{-i\pi \eta} \frac{\theta_1(\eta + \frac{1}{2} + \tau)\theta_1(\eta + \frac{1}{2})}{\theta_1(\frac{1}{2} + \tau)\theta_1(\frac{1}{2})} \right] \]  

(S35)

So, the consistency condition (S32) is proved. From Eqs. (S31)-(S32), we get the expression of \( a_n \)

\[ a_n \equiv a(u) = -J_z + \frac{J_+}{2} \frac{\theta_4(u)\theta_1(u + \eta)}{\theta_1(u)\theta_4(u + \eta)} + \frac{J_-}{2} \frac{\theta_1(u)\theta_4(u + \eta)}{\theta_4(u)\theta_1(u + \eta)} \]  

(S36)

Analogously, \( a_{n+1} \) can be obtained

\[ a_{n+1} = a(u + \eta). \]  

(S37)

We can simplify the expression of \( d_n \) as follows

\[ d_n \equiv d(u, \eta) = -J_z + \frac{J_+}{2} \frac{\theta_4(u)\theta_1(u + \eta)}{\theta_1(u)\theta_4(u + \eta)} + \frac{\theta_1(u)\theta_4(u + \eta)}{\theta_4(u)\theta_1(u + \eta)} \]  

(S38)
To sum up, the divergence condition (S26) has the form
\[ h |\psi(u)\rangle \otimes |\psi(u + \eta)\rangle = [a(u) \sigma^z \otimes I_2 - a(u + \eta)I_2 \otimes \sigma^z + d(u, \eta)I_4] |\psi(u)\rangle \otimes |\psi(u + \eta)\rangle \] (S39)
where \(a(u), d(u, \eta)\) are given by (S36), (S38).
Note now that the oddness property of the \(\bar{\theta}_1(\eta) = -\bar{\theta}_1(\eta)\) implies an equivalent form of the divergence condition for the “downhill” gradient,
\[ h |\psi(u)\rangle \otimes |\psi(u - \eta)\rangle = [-a(u) \sigma^z \otimes I_2 + a(u - \eta)I_2 \otimes \sigma^z + d(u, -\eta)I_4] |\psi(u)\rangle \otimes |\psi(u - \eta)\rangle \] (S40)
From the definition (S30) and properties of the elliptic functions in Eqs. (S3), (S4), we obtain
\[ \langle \psi^+ \rangle = (\theta_4(u), \theta_1(u)) = -ie^{i\pi(u+\hat{\tau})}\eta(u + \tau), \theta_4(u + \tau)) \]
\[ = ie^{i\pi(u+\hat{\tau})}(\theta_4(u + \tau + 1), -\theta_4(u + \tau + 1)). \] (S41)
With the help of the properties of \(\bar{\theta}_a(u)\) and \(\bar{\zeta}(u)\), we note that
\[ a(u + \tau + 1) = -a(u), \]
\[ a(u + \eta + \tau + 1) = -a(u + \eta), \]
\[ d(u + \tau + 1, \eta) = d(u, \eta). \] (S42, S43, S44)
Consequently, due to \(h^T = h\), the transposed form of (S39) implies yet another divergence relation for bra vectors
\[ \langle \psi^+_n \rangle \otimes \langle \psi^+_n \rangle |h = \langle \psi^+_n \rangle \otimes \langle \psi^+_n \rangle [-a(u) \sigma^z \otimes I_2 + a(u + \eta)I_2 \otimes \sigma^z + d(u, \eta)I_4]. \] (S45)

**S-III. DIVERGENCE CONDITION FOR XXZ CHAIN**

Divergence condition for XXZ model, analogous to (6) for XYZ model, was already given elsewhere [21, 22]. Here we show how to derive it directly from the (6) in the limit XYZ \(\rightarrow\) XXZ. Note that only when \(\tau\) is purely imaginary and \(\eta\) is real or purely imaginary, the XYZ bulk Hamiltonian is Hermitian, specifically as follows
- when \(\text{Im}[\eta] = \text{Re}[\tau] = 0\), \(J_x \geq J_y \geq J_z\).
- when \(\text{Re}[\eta] = \text{Re}[\tau] = 0\), \(J_x \leq J_y \leq J_z\).

When \(\tau \rightarrow +i\infty\), the system degenerates into XXZ chain as follows:
\[ J_x \rightarrow 1, \quad J_y \rightarrow 1, \quad J_z \rightarrow \cos(\pi \eta) \leq 1, \quad \text{Im}[\eta] = 0, \] (S46)
\[ J_x \rightarrow 1, \quad J_y \rightarrow 1, \quad J_z \rightarrow \cos(\pi \eta) \geq 1, \quad \text{Re}[\eta] = 0. \] (S47)
Assume that \(u = \frac{\tau}{2} + \hat{u}\). In the limit \(\tau \rightarrow +i\infty, u \rightarrow \hat{u} + i\infty\), we have
\[ \lim_{\tau \rightarrow +i\infty} e^{\frac{i\tau}{2}} \bar{\theta}_1(u) = \lim_{\tau \rightarrow +i\infty} \theta_1(u) = ie^{-i\pi \hat{u}}, \]
\[ \lim_{\tau \rightarrow +i\infty} e^{\frac{i\tau}{2}} \bar{\theta}_2(u) = e^{-i\pi \hat{u}}, \lim_{\tau \rightarrow +i\infty} \theta_4(u) = 1, \quad \hat{u} \text{ is finite.} \] (S48)
Under such assumption, we get
\[ \lim_{\tau \rightarrow +i\infty} a(u) = -i \sin(\pi \eta), \] (S49)
\[ \lim_{\tau \rightarrow +i\infty} d(u, \eta) = \cos(\pi \eta), \] (S50)
\[ \lim_{\tau \rightarrow +i\infty} -\frac{\theta_4(u)}{\theta_1(u)} = e^{i\pi (\hat{u} + \frac{1}{2})}. \] (S51)
Now we see the spin-helix structure and the divergence condition (S39) degenerates into the XXZ type [26–28]
\[ h^{XXZ} |\tilde{\psi}(\hat{u})\rangle \otimes |\tilde{\psi}(\hat{u} + \eta)\rangle = [-i \sin(\pi \eta) \sigma^x \otimes I_2 + i \sin(\pi \eta)I_2 \otimes \sigma^z + \cos(\pi \eta)I_4] |\tilde{\psi}(\hat{u})\rangle \otimes |\tilde{\psi}(\hat{u} + \eta)\rangle, \] (S52)
where \(h^{XXZ} = \sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \cos(\pi \eta) \sigma^z \otimes \sigma^z\) and \(|\tilde{\psi}(u)\rangle = (1, e^{i\pi (u + \frac{1}{2})})^T\).
S-IV. OBTAINING THE LEFT AUXILIARY VECTOR $\langle w_l |$. PROVING EQ. (21).

Our Ansatz for Zeno-limit MPA of a dissipative spin chain of length $N + 2$, with sites numbering $0, 1, \ldots, N + 1$ has the usual form

\[ \rho_{NESS} = \rho_l \otimes R_{NESS} \otimes \rho_r \]  
\[ R_{NESS} = \frac{\Omega_N \Omega_N^\dagger}{\text{Tr}(\Omega_N \Omega_N^\dagger)} \]  
\[ \Omega_N = \langle w_l | L_1 L_2 \ldots L_N | w_r \rangle \]

where $\rho_l = \sum_a n^a l \sigma^a$, $\rho_r = \sum_a n^a r \sigma^a$ are targeted left and right boundary magnetizations, $|n|_l = 1$, $L_n = \sum_a L^a n \sigma^a$ are site-dependent Lax matrices satisfying the divergence relation

\[ [h_{n,n+1}, L_n L_{n+1}] = 2i(J L_{n+1} - L_n I). \]

and $\langle w_l |$, $| w_r \rangle$ are suitable vectors in auxiliary space guaranteeing the commutation of the $R_{NESS}$ with the dissipation-projected XYZ Heisenberg Hamiltonian

\[ [R_{NESS}, h_D] = 0, \]
\[ h_D = \sum_{n=1}^{N-1} h_{n,n+1} + \sum_{\alpha=x,y,z} J_\alpha n^\alpha_{l} \sigma^\alpha_{l} + \sum_{\alpha=x,y,z} J_\alpha n^\alpha_{r} \sigma^\alpha_{r}. \]

Nonzero matrix elements of the Lax matrix $L_n$ is given by

\[ (L_n)_{j,j} = (L_n)_{j,j+1} = \frac{1}{\kappa(u_{n-2(j-1)})} \langle \psi(u_{n-2(j-1)}) \rangle \langle \psi^+(u_{n-2(j-1)}) \rangle, \]
\[ u_n = u + n, \eta \]
\[ \kappa(u) = -ia(u) \theta_1(u) \theta_4(u), \]

where $\langle \psi^+(u)|$, $|\psi(u)\rangle$ are given by (S30), (S29), and $u$ is some initial phase to be determined later, see (S73), (S74).

Explicitly, the components $L^\alpha_n$ and the matrix $I$ are given by

\[ L^\alpha_n = \sum_{j=1}^{N+1} s^\alpha_{n-2(j-1)} |j\rangle \langle j| + \sum_{j=1}^{N} s^\alpha_{n-2(j-1)} |j\rangle \langle j+1|, \]
\[ s^z(u) = \frac{i}{a(u)}, \]
\[ s^x(u) - is^y(u) = \frac{i}{a(u)} \theta_1(u), \]
\[ s^x(u) + is^y(u) = \frac{i}{a(u)} \theta_4(u), \]
\[ I = \sum_{j=1}^{N+1} |j\rangle \langle j| - \sum_{j=1}^{N} |j\rangle \langle j+1|. \]

Denoting $J_\alpha n^\alpha_{l/r} = h^\alpha_{l/r}$, and following the procedure outlined in [15, 16], from (S57) we get two conditions to be satisfied,

\[ \langle w_l | V_1 L_2 \ldots L_N | w_r \rangle = 0 \]
\[ \langle w_l | L_1 \ldots L_{N-1} V_r | w_r \rangle = 0, \]

where

\[ V_i = h_i \cdot L_i + i I, \]
\[ V_r = h_r \cdot L_N - i I. \]
The condition (S67) can be satisfied in its local form
\[ \langle w_1 | V_i = 0 \] (S71)
with an appropriate choice of \( u_1 \) in (S60) and \( \langle w_1 | \). Namely, let constants \( \alpha_k^- \) parametrize the left boundary field of the integrable XYZ Hamiltonian in the standard way. It turns out that boundary field of the form \((h_i)_\alpha = J_\alpha (n_i)_\alpha\) where \((n_i)_\alpha\) are components of a unit vector \(|n_i| = 1\) correspond to a choice
\[ \{ \alpha_1^-, \alpha_2^-, \alpha_3^- \} = \left\{ n_1, \frac{1}{2} + iv_1, \frac{\tau}{2} + v_2 \right\}, \] (S72)
where \(v_1, v_2\) are real numbers. Let us choose \( u_1 = u + \eta \) as
\[ u_1 = \frac{1}{2} + \alpha_1^- + \alpha_2^- + \alpha_3^- \] (S73)
or
\[ u_1 = \frac{1}{2} + \alpha_1^- - \alpha_2^- - \alpha_3^- . \] (S74)
Then, one can show that
\[ (V_1)_{11} = (V_1)_{23} = 0, \] (S75)
\[ (V_1)_{12} = -(V_1)_{22} \neq 0, \] (S76)
and consequently \( \langle w_1 | V_i = 0 \) is satisfied with the choice
\[ \langle w_1 | = (1, 1, 0, 0, \ldots, 0), \] (S77)
so, remarkably, the “left boundary” condition \( \langle w_1 | V_i = 0 \) is satisfied with the universal (anisotropy independent) choice of \( w_1 \).

**Proof of (S75):** The left boundary field of the integrable XYZ Hamiltonian is standardly parameterized as \([30, 33, 34]\)
\[
(h_i)_x = e^{-i \pi \sum_{k=1}^{3} \alpha_k^- - \frac{\tau}{2}} \frac{\bar{\theta}_1(\eta)}{\theta_1(\frac{\tau}{2})} \prod_{k=1}^{3} \frac{\bar{\theta}_1(\alpha_k^- - \frac{\tau}{2})}{\theta_1(\alpha_k^-)} ,
\]
\[
(h_i)_y = e^{-i \pi \sum_{k=1}^{3} \alpha_k^- - \frac{\tau}{2}} \frac{\bar{\theta}_1(\eta)}{\theta_1(\frac{1+\tau}{2})} \prod_{k=1}^{3} \frac{\bar{\theta}_1(\alpha_k^- - \frac{1+\tau}{2})}{\theta_1(\alpha_k^-)} ,
\]
\[
(h_i)_z = \frac{\bar{\theta}_1(\eta)}{\theta_1(\frac{1}{2})} \prod_{k=1}^{3} \frac{\bar{\theta}_1(\alpha_k^- - \frac{1}{2})}{\theta_1(\alpha_k^-)} , \] (S78)
and Eqs. (S75) can then be written as
\[ F(u_1) = -a(u_1), \] (S79)
\[ F(u_1 - 2\eta) = a(u_1 - 2\eta), \] (S80)
where
\[ F(u) = (h_i)_z + \frac{(h_i)_x}{2} \left[ - \frac{\theta_2(u)}{\theta_1(u)} + \frac{\theta_1(u)}{\theta_4(u)} \right] + \frac{i(h_i)_y}{2} \left[ \frac{\theta_4(u)}{\theta_1(u)} + \frac{\theta_1(u)}{\theta_2(u)} \right]. \] (S81)
Using Eqs. (S5), (S6) and (S9), we get
\[ \frac{- \theta_4(u)}{\theta_1(u)} + \frac{\theta_1(u)}{\theta_4(u)} = 2e^{i \pi (u + \frac{1+\tau}{2})} \frac{\bar{\theta}_1(u + \frac{1+\tau}{2})}{\theta_1(u)\theta_1(\frac{1}{2})} , \]
\[ \frac{- \theta_4(u)}{\theta_1(u)} - \frac{\theta_1(u)}{\theta_4(u)} = 2e^{i \pi (u + \frac{1+\tau}{2})} \frac{\bar{\theta}_1(u + \frac{1+\tau}{2})}{\theta_1(u)\theta_1(\frac{1}{2})} . \] (S82)
With the help of Eqs. (S5) and (S82), we finally simplify the expression of \( F(u) \) after tedious calculations

\[
F(u) = a(u) + 2\frac{\theta_1(\eta)}{\theta_3(\eta)} \prod_{l=0}^{3} \frac{\theta_1(2u + 2l - 1)}{\theta_1(2u + 2l - 1)}
\]

\[
= -a(u) + 2\frac{\theta_1(\eta)}{\theta_3(\eta)} \prod_{k=1}^{3} \theta_1(\alpha_k^l).
\]

(S83)

where

\[
\chi_0 = -\alpha_1^\alpha - \alpha_2^\alpha - \alpha_3^\alpha, \quad \chi_1 = \alpha_2^\alpha + \alpha_3^\alpha - \alpha_1^\alpha,
\]

\[
\chi_2 = \alpha_1^\alpha - \alpha_2^\alpha + \alpha_3^\alpha, \quad \chi_3 = \alpha_1^\alpha + \alpha_2^\alpha - \alpha_3^\alpha.
\]

The value of \( u_1 \) is

\[
u_1 = \frac{1}{2} - \chi_0 = \frac{1}{2} + \chi_1 + 2\eta,
\]

or

\[
u_1 = \frac{1}{2} - \chi_1 = \frac{1}{2} + \chi_0 + 2\eta.
\]

(S85)

From the explicit expression of \( F(u) \) in (S83), it is straightforward that

\[
F(u_1) = -a(u_1), \quad F(u_1 - 2\eta) = a(u_1 - 2\eta).
\]

Eqs. (S79)–(S80) are thus proven.

Finally, parameter \( u \) from the Eqs.(3),(11) in the main text can be found as \( u = u_1 - \eta \). Accounting for (S72), we get

\[
u = \frac{\tau}{2} + iv_1 + 1 + v_2 = x_1 + iy_1,
\]

\[
y_1 = v_1 + \frac{\tau}{2i}, \quad x_1 = 1 + v_2.
\]

(S86)

(S87)

The components of the targeted left boundary magnetization are related to components of the boundary fields via \( n_\alpha^1 = \chi_\alpha^1/J_\alpha \). Substituting this into (S78), using (S24), (S87) and using basis relations for elliptic theta-functions, one obtains

\[
n_\mu^x = -\frac{\bar{\theta}_2(\mu) \bar{\theta}_1(x_\mu)}{\bar{\theta}_3(\mu) \bar{\theta}_4(x_\mu)},
\]

\[
n_\mu^y = -\frac{\bar{\theta}_1(\mu) \bar{\theta}_2(x_\mu)}{\bar{\theta}_3(\mu) \bar{\theta}_4(x_\mu)},
\]

\[
n_\mu^z = -\frac{\bar{\theta}_4(\mu) \bar{\theta}_3(x_\mu)}{\bar{\theta}_3(\mu) \bar{\theta}_4(x_\mu)},
\]

i.e. the Eq. (19) for \( n_1 \) and Eq. (20) in the main text.

We can check that \( n_\mu^x, n_\mu^y, n_\mu^z \) are indeed components of a unit vector. We shall use an identity ([19], p. 487):

\[
\bar{\theta}_3(x + iy)\bar{\theta}_3(x - iy)\bar{\theta}_3^2(0) = \bar{\theta}_2(x)\bar{\theta}_4^2(y) - \bar{\theta}_2^2(x)\bar{\theta}_4(x)\bar{\theta}_4^2(y).
\]

(S88)

For purely imaginary \( \tau \) and real \( x, \bar{\theta}_\alpha(x), \alpha = 1, 2, 3, 4 \) and \( \bar{\theta}_\beta(ix), \beta = 2, 3, 4 \) are real, while \( \bar{\theta}_1(ix) \) is purely imaginary. Using Eq. (S88), we get

\[
(n_\mu^x)^2 + (n_\mu^y)^2 + (n_\mu^z)^2 - 1
\]

\[
= \frac{1}{\bar{\theta}_4^2(\mu)\bar{\theta}_4^2(x_\mu)} \left[ \bar{\theta}_2^2(\mu)\bar{\theta}_3^2(x_\mu) - \bar{\theta}_2^2(\mu)\bar{\theta}_3^2(x_\mu) + \bar{\theta}_2^2(\mu)\bar{\theta}_3^2(x_\mu) - \bar{\theta}_3^2(\mu)\bar{\theta}_3^2(x_\mu) \right]
\]

\[
= \frac{\bar{\theta}_3^2(0)\bar{\theta}_3^2(x_\mu + iy_\mu)}{\bar{\theta}_4^2(\mu)\bar{\theta}_4^2(x_\mu)} \left[ \bar{\theta}_3(x - iy) - \bar{\theta}_3(x + iy_\mu - x_\mu) \right]
\]

\[
= 0.
\]

(S89)
To obtain the part of the Eq.(19) in the main text concerning $\mathbf{n}$, we choose
\[
\{\alpha_1^+, \alpha_2^+, \alpha_3^+\} = \left\{ \eta, \frac{1 + \tau}{2} - iy_r, \frac{\tau}{2} + x_r \right\},
\]
and proceed analogously.

**S-V. Calculating the Right Auxiliary Vector $|w_t\rangle$. Showing Consistency of the Recurrence (22).**

While the universal form of $\langle w_1 \rangle$ satisfying $\langle w_1 \rangle V_1 = 0$ can be found, the analogical local right boundary condition $V_t |w_t\rangle = 0$ cannot be satisfied, since, in general, the operator $V_t$ does not have zero eigenvalues. Instead, a milder condition
\[
\langle w_1 \rangle (L_1)_{\alpha_1} (L_2)_{\alpha_2} \ldots (L_{N-1})_{\alpha_{N-1}} V_t |w_t\rangle = 0,
\]
valid for any $\alpha_1, \ldots, \alpha_{N-1} \in \{x, y, z\}$, can be satisfied by a nontrivial choice
\[
V_t |w_t\rangle = |1, -1, 1, -1, \ldots\rangle.
\]
Indeed, due to a special band form of all $L_n^\alpha$ with equal elements within each row (S62) we have
\[
\langle w_1 \rangle (L_1)_{\alpha_1} (L_2)_{\alpha_2} \ldots (L_{N-1})_{\alpha_{N-1}} V_t |w_t\rangle =
\langle w_1 \rangle (L_1)_{\alpha_1} (L_2)_{\alpha_2} \ldots (L_{N-1})_{\alpha_{N-1}} |1, -1, 1, -1, \ldots\rangle
= \langle w_1 \rangle (L_1)_{\alpha_1} (L_2)_{\alpha_2} \ldots (L_{N-2})_{\alpha_{N-2}} |0, \ldots, 0, \ast\rangle
= (1, 1, 0, \ldots |0, 0, \ast, \ldots, \ast\rangle = 0
\]
where we mark by the asterisk sign $\ast$ potentially nonzero entries of the right ket vector. If $\det V_t \neq 0$ (the generic case), a unique nontrivial solution for $|w_t\rangle$ exists, given by the explicit recurrence (22) in the main text.

On the other hand, under special conditions rendering $\det[V_t] = 0$, which entails $(V_t)_{nn} = 0$ for some $n$, the recurrence (22) in the main text breaks down. Instead, the vector $|w_t\rangle$ can be chosen such as to satisfy the relation (S68) locally, as shown in the next section.

**S-VI. Spin-Helix States Analouges for XYZ Model. The Case of $\det V_t = 0$.**

Elements of $V_t$ in the $k$-th row have the form
\[
(V_t)_{kk} = \text{const} \times \left[ \frac{G(v)}{a(v)} + 1 \right],
\]
\[
(V_t)_{k-1,k} = \text{const} \times \left[ -\frac{G(v + 2\eta)}{a(v + 2\eta)} + 1 \right],
\]
\[
v = u_1 + (N + 1 - 2k)\eta,
\]
\[
k = 1, 2, \ldots N + 1
\]
where $G(x)$ is obtained from $F(x)$ in (881) by substitutions $\alpha_k^- \rightarrow \alpha_k^+$. An inspection of the above shows that both elements $(V_t)_{kk} = (V_t)_{k-1,k} = 0$ vanish if one of the two conditions is satisfied,
\[
v = \frac{1}{2} - \alpha_1^+ - \alpha_2^+ + \alpha_3^+,
\]
\[
v = \frac{1}{2} - \alpha_1^+ + \alpha_2^+ - \alpha_3^+,
\]
where \( \{\alpha_1^+, \alpha_2^+, \alpha_3^+\} \) \( \{\eta, \frac{1 + \tau}{2} - iy_r, \frac{\tau}{2} + x_r\} \).
By doing some simple substitutions on \( F(u) \) and using the expression in (S83), the above property can be easily proved. The condition (S95) turns out to be equivalent to

\[
x_t = x_1 + (N + 1 - 2M_0) \eta, \quad y_t = y_h, \quad M_0 = 0, 1, \ldots N, \tag{S100}
\]

used to get the factorized periodically modulated state (24) in the main text, for \( M_0 = 0 \). Then the equation (S92) cannot be solved because of singularity in (22), while the equation

\[
V_t |w_t\rangle = V_t(r_1, r_2, \ldots r_{N+1})^T = 0 \tag{S101}
\]

is readily solved with

\[
r_k = \delta_{k, M_0+1}. \tag{S102}
\]

Now, the periods of the \( G(u) \) are \( G(u + 2n_0 + 2\tau m_0) = G(u) \) where \( n_0, m_0 \) are integers. It follows that if periodicity conditions are satisfied

\[
\eta M_1 = 2n_0 + 2\tau m_0, \tag{S103}
\]

\[
M_1 < N, \tag{S104}
\]

there could be several rows of \( V_t \) which become zero simultaneously. Then, the zero eigenvalue of \( V_t \) has degeneracy, and a unique solution of (S101) for \( |w_t\rangle \) does not exist. Proper consideration of this case needs considering subleading corrections to the Zeno limit of NESS and is out of our present scope.

**S-VII. DERIVATION OF EQ. (24). OBTAINING ONE-POINT OBSERVABLES (MAGNETIZATION PROFILE) IN FIG. 1.**

Pure NESS Eq. (23) in the main text has the form of a factorized product of the qubits Eq.(3). Let us calculate the expectations \( \langle \sigma_n^+ \rangle, \langle \sigma_n^- \rangle \). Using (3) and denoting \( u_n = u + n\eta \), we obtain

\[
\langle \sigma_n^+ \rangle = -\frac{\theta_1^*(u_n)\theta_3(u_n)}{\theta_1^*(u_n)\theta_1(u_n) + \theta_3^*(u_n)\theta_3(u_n)}, \tag{S105}
\]

\[
\langle \sigma_n^- \rangle = -\frac{\theta_1(u_n)\theta_3^*(u_n) - \theta_4(u_n)\theta_4^*(u_n)}{\theta_1(u_n)\theta_1^*(u_n) + \theta_4(u_n)\theta_4^*(u_n)}, \tag{S106}
\]

where \( * \) denotes complex conjugation. Using \( \theta_1^*(u) = \theta_1(u^*) \), \( \theta_4^*(u) = \theta_4(u^*) \) and the formulae for elliptic functions (S19-S21), we readily obtain

\[
\langle \sigma_n^+ \rangle = 2\text{Re}[\langle \sigma_n^+ \rangle] = -\frac{\theta_1(x)\bar{\theta}_2(iy)}{\theta_4(x)\bar{\theta}_3(iy)}, \tag{S107}
\]

\[
\langle \sigma_n^- \rangle = 2\text{Im}[\langle \sigma_n^- \rangle] = -\frac{\bar{\theta}_1(x)\theta_2(iy)}{\bar{\theta}_4(x)\theta_3(iy)}, \tag{S108}
\]

\[
\langle \sigma_n^z \rangle = -\frac{\bar{\theta}_3(x)\bar{\theta}_4(iy)}{\theta_4(x)\theta_3(iy)}, \tag{S109}
\]

where \( x = \text{Re}[u + n\eta] = \text{Re}[u] + n\eta, \ y = \text{Im}[u + n\eta] = \text{Im}[u] \), assuming \( \eta \) being real. Finally, using the relation to well known Jacobi elliptic functions sn, cn, dn and denoting

\[
k = \left( \frac{\theta_2(0)}{\theta_3(0)} \right)^2, \quad k' = \sqrt{1 - k^2} = \left( \frac{\theta_4(0)}{\theta_3(0)} \right)^2, \quad k_2 = \frac{1}{2}\pi (\bar{\theta}_3(0))^2, \tag{S110}
\]

\[
A_x = -\sqrt{k} \frac{\theta_2(i \text{Im}[u])}{\theta_3(i \text{Im}[u])}, \quad A_y = -i \left( \frac{k}{k'} \frac{\theta_1(i \text{Im}[u])}{\theta_3(i \text{Im}[u])} \right), \quad A_z = -\frac{1}{\sqrt{k'}} \frac{\bar{\theta}_3(i \text{Im}[u])}{\theta_3(i \text{Im}[u])}, \tag{S111}
\]

Note that in our physical case \( 0 \leq k \leq 1 \) is real. For general case (e.g. for complex \( \tau \) parameter) one can use

\[
A_x = -\frac{\theta_3(0)}{\theta_2(0)} \frac{\theta_2(i \text{Im}[u])}{\theta_3(i \text{Im}[u])}, \quad A_y = -i \frac{\theta_4(0)}{\theta_2(0)} \frac{\theta_1(i \text{Im}[u])}{\theta_3(i \text{Im}[u])}, \quad A_z = -\frac{\theta_4(0)}{\theta_3(0)} \frac{\theta_4(i \text{Im}[u])}{\theta_3(i \text{Im}[u])}, \tag{S112}
\]
valid for arbitrary choice of $\tau$ and $\eta$. We finally obtain

$$\langle \sigma^x_n \rangle = A_x \text{ sn}(2K_k (\text{Re}[u] + n\eta), k), \quad \text{(S113)}$$

$$\langle \sigma^y_n \rangle = A_y \text{ cn}(2K_k (\text{Re}[u] + n\eta), k), \quad \text{(S114)}$$

$$\langle \sigma^z_n \rangle = A_z \text{ dn}(2K_k (\text{Re}[u] + n\eta), k), \quad \text{(S115)}$$

leading to Eq. (24) in the main text.