Seifert manifolds and (1, 1)-knots

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Abstract

The aim of this paper is to investigate the relations between Seifert manifolds and (1, 1)-knots. In particular, we prove that every orientable Seifert manifold with invariants

\[ \{Oo, 0 \mid -1; (p, q), \ldots, (p, q), (l, l-1) \} \]

\( n \) times

has the fundamental group cyclically presented by \( G_n((x_1^q \cdots x_n^q)l x_n^{-p}) \) and, moreover, it is the \( n \)-fold strongly-cyclic covering of the lens space \( L(\left\lceil nlq-p \right\rceil, q) \), branched over the (1, 1)-knot \( K(q, q(ql-2), p-2q, p-q) \), if \( p \geq 2q \), and over the (1, 1)-knot \( K(p-q, 2q-p, q(ql-2), p-q) \), if \( p < 2q \).

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1 Introduction

Cyclic branched coverings of knots in \( S^3 \) with cyclically presented fundamental group have been deeply investigated in the recent years by many authors (see \cite{11, 10, 11, 12, 17, 20, 21, 22, 23, 33}). Their results have been included in an organic and more general context in \cite{24}, where it is proved that the fundamental group of every \( n \)-fold strongly-cyclic branched covering of a (1, 1)-knot admits a cyclic presentation encoded by a genus \( n \) Heegaard diagram. In \cite{5
this result has been improved, obtaining a constructive algorithm which explicit.
ly gives the cyclic presentation, starting from a representation of the
\((1,1)\)-knot via the mapping class group of the twice punctured torus (see [6]
for further details on this representation).

In [14], M. J. Dunwoody introduces a class of manifolds depending on
six integers, the so-called Dunwoody manifolds, with cyclically presented
fundamental groups. It is proved in [15] and [7] that the family of Dun-
woody manifolds coincides with the family of strongly-cyclic coverings of
lens spaces (possibly \(S^3\)), branched over \((1,1)\)-knots. As a consequence, any
\((1,1)\)-knot can be represented by four integers \(a, b, c, r\), and it will be denoted
by \(K(a, b, c, r)\).

In this paper we show that the orientable Seifert manifold with invaria-
tics \(\{Oo, 0 \mid -1; (p, q), \ldots, (p, q), (l, l - 1)\}\)
n times

has the fundamental group isomorphic to the cyclically presented group
\(G_n((x_1^q \cdots x_n^q)^l x_n^{-p})\), and it is the \(n\)-fold strongly-cyclic covering of the lens
space \(L(|nlq - p|, q)\), branched over the \((1,1)\)-knot \(K(q, q(nl - 2), p - 2q, p - q)\),
if \(p \geq 2q\), and over the \((1,1)\)-knot \(K(p - q, 2q - p, q(nl - 2), p - q)\), if \(p < 2q\).

2 Basic notions

A finite balanced presentation of a group \(<x_1, \ldots, x_n \mid r_1, \ldots, r_n>\) is
said to be a cyclic presentation if there exists a word \(w\) in the free group
\(F_n\) generated by \(x_1, \ldots, x_n\) such that \(r_k = \theta^{k-1}(w)\), \(k = 1, \ldots, n\), where
\(\theta : F_n \to F_n\) denotes the automorphism defined by \(\theta(x_i) = x_{i+1}\) (subscripts
mod \(n\)), \(i = 1, \ldots, n\). This presentation (and the related group) will be
denoted by \(G_n(w)\). For further details see [18].

A knot \(K\) in a closed, connected, orientable 3-manifold \(N^3\) is
called a \((1,1)\)-knot if there exists a Heegaard splitting of genus one
\((N^3, K) = (H, A) \cup_{\varphi} (H', A')\), where \(H\) and \(H'\) are solid tori, \(A \subset H\) and
\(A' \subset H'\) are properly embedded trivial arcs\(^1\), and \(\varphi : (\partial H', \partial A') \to (\partial H, \partial A)\)
is an attaching homeomorphism (see Figure 1). Obviously, \(N^3\) turns out to
be a lens space \(L(p, q)\), including \(S^3 = L(1, 0)\) and \(S^2 \times S^1 = L(0, 1)\).

\(^1\)This means that there exists a disk \(D \subset H\) (resp. \(D' \subset H'\)) with \(A \cap D = A \cap \partial D = A\)
and \(\partial D - A \subset \partial H\) (resp. \(A' \cap D' = A' \cap \partial D' = A'\) and \(\partial D' - A' \subset \partial H'\)).
It is well known that the family of $(1, 1)$-knots contains all torus knots and all two-bridge knots in $S^3$. Several topological properties of $(1, 1)$-knots have recently been investigated in many papers (see references in \[6\]).

An integer 4-parametric representation of $(1, 1)$-knots have been developed in \[7\] (see also \[8\]). Every $(1, 1)$-knot, with the only exception of the “core” knot $\{P\} \times S^1 \subset S^2 \times S^1$, can be represented by four non-negative integers $a, b, c, r$, and the represented knot will be referred as $K(a, b, c, r)$.

A $(1, 1)$-knot $K(a, b, c, r)$, with $a + b + c > 0$, admits a natural $(1, 1)$-decomposition $(H, A) \cup \varphi (H', A')$ described by the genus one Heegaard diagram of Figure 2, where the labels $a, b$ and $c$ denote the correspondent number of parallel arcs, and the gluing between the circles $C'$ and $C''$ depends on the twist parameter $r$ in such a way that equally labelled vertices are identified together (observe that $r$ can be taken mod $2a + b + c$).

As a simple consequence of Seifert-Van Kampen Theorem, the fundamental group of the exterior of a $(1, 1)$-knot (as well as its first homology group) is generated by the two loops $\alpha, \gamma \subset \partial H$ depicted in Figure 3.

In the next section we need the following result.

**Lemma 1**

(i) If $a$ and $c$ are non-negative integers such that $\gcd(a, c) = 1$, then $K(a, 0, c, a)$ is a $(1, 1)$-knot in the lens space $L(c, a)$.

(ii) If $a, b, c$ are non-negative integers such that $\gcd(a - c, b + c) = 1$, then $K(a, b, c, a)$ is a $(1, 1)$-knot in the lens space $L(b + c, a + b)$.

(iii) If $a, b, c$ are non-negative integers such that $a > 0$ and $\gcd(a, b - c) = 1$, then $K(a, b, c, a + c)$ is a $(1, 1)$-knot in the lens space $L(|b - c|, a)$.

**Proof.** As proved in \[8\], $K(a, b, c, r)$ is equivalent to $K(a, c, b, 2a + b + c - r)$ and $K(a, 0, c, r)$ is equivalent to $K(a, c, 0, r)$. As a consequence, $K(a, 0, c, r)$ is equivalent to $K(a, 0, c, 2a + c - r)$. 

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Figure 2: Heegaard diagram for $K(a, b, c, r)$

Figure 3:
(i) If \( a = 0 \), then \( c = 1 \) and the result is straightforward. If \( 0 < a < c \), then applying the Singer move \([29]\) of type IB depicted in Figure 4, we obtain the canonical Heegaard diagram of \( L(c, a) \). If \( a \geq c \), the Singer move depicted in Figure 5 transforms the diagram of \( K(a, 0, a) \) in the diagram of \( K(a-c, 0, a-c) \), which is equivalent to \( K(a-c, 0, c, a-c) \). Since \( L(c, a-c) \) is homeomorphic to \( L(c, a) \), the result follows from the previous case \( a < c \).

(ii) If \( b = 0 \) the result follows from (i). If \( b > 0 \), by performing the Singer move of Figure 6, the diagram of \( K(a, b, c, a) \) becomes the diagram of \( K(a-c, b+1, c-1, a) \). If \( a \leq c \), after performing the move \( c \) times, we obtain the diagram of \( K(0, b, c, 0) \), which is the canonical Heegaard diagram of \( L(b+c, a) \), since \( \gcd(a+b, b+c) = \gcd(a-c, b+c) = 1 \). If \( a > c \), after performing the move \( c \) times, we obtain the diagram of \( K(a-c, b+c, 0, a-c) \), which is equivalent to \( K(a-c, 0, b+c, a-c) \). Now the result follows from (i), since \( L(b+c, a-c) \) is homeomorphic to \( L(b+c, a) \).

(iii) Since \( K(a, b, c, a) \) is equivalent to \( K(a, c, b, a) \), we can always suppose \( c \leq b \). If \( c > 0 \), then, by performing the Singer move of Figure 7, the diagram of \( K(a, b, c, a+c) \) becomes the diagram of \( K(a, b-1, c-1, a+c-1) \). After performing the move \( c \) times, we obtain the diagram of \( K(a-c, b+c, 0, a) \), which is equivalent to \( K(a, 0, b-c, a) \). The result now follows from (i).

Observe that, since \( K(a, b, c, a+b+c) \) is equivalent to \( K(a, c, b, a) \), \( K(a, b, c, a+b+c) \) is a \((1, 1)\)-knot in the lens space \( L(b+c, a+c) \), if \( \gcd(a-b, b+c) = 1 \).

An \( n \)-fold cyclic covering \( M^3 \) of a 3-manifold \( N^3 \), branched over a knot \( K \subset N^3 \), is called strongly-cyclic if the branching index of \( K \) is \( n \). This means that the fiber in \( M^3 \) of each point of \( K \) consists of a single point. In this case, the homology class \( m \) of a meridian loop around \( K \) is mapped by the monodromy \( \omega : H_1(N^3-k) \to \mathbb{Z}_n \) of the covering in a generator of \( \mathbb{Z}_n \) (up to equivalence, we can always suppose \( \omega(m) = 1 \)). Observe that a cyclic branched covering of a knot \( K \) in \( S^3 \) is always strongly-cyclic and is uniquely determined, up to equivalence, since \( H_1(S^3-K) \cong \mathbb{Z} \). Obviously, this property is no longer true for a knot in a more general 3-manifold. Necessary and sufficient conditions for the existence and uniqueness of strongly-cyclic branched coverings of \((1, 1)\)-knots have been obtained in [5].
3 Main results

Let \( n, p, q, l \) be positive integers such that \( q < p \) and \( \gcd(p, q) = 1 \). In the following we denote by \( \Sigma(n, p, q, l) \) the orientable Seifert manifold \([28]\) with invariants

\[
\{Oo, 0 \mid -1;(p, q), \ldots, (p, q), (l, l - 1)\}
\]

having \( S^2 \) as orbit space, \( n \) exceptional fibres of type \((p, q)\) and, for \( l > 1 \), an exceptional fibre of type \((l, l - 1)\).

Observe that, in particular, the manifolds \( \Sigma(n, 2, 1, 1) \) are precisely the Neuwirth manifolds \( M_n \) introduced in [25], studied in [9], and generalized in [16], [31] and [27].

Proposition 2 The fundamental group of \( \Sigma(n, p, q, l) \) is cyclically presented by \( G_n(w) \), where \( w = (x_1^q \cdots x_n^q) x_n^{l-p}. \)

Proof. Following [26], a standard presentation of \( G = \pi_1(\Sigma(n, p, q, l)) \) is

\[
\langle y_1, \ldots, y_n, y, h \mid [y_i, h], [y, h], y^p h^q, y^l h^{l-1}, y_1 \cdots y_n y h; i = 1, \ldots, n \rangle.
\]

Since \( \gcd(p, q) = 1 \), there exist \( \alpha, \beta \in \mathbb{Z} \) such that \( q\beta - p\alpha = 1 \).

From the relations we have \( y^l h^{l-1} = (yh)^{l-1} \). By introducing the new generator \( x = yh \), the presentation becomes

\[
\langle y_1, \ldots, y_n, x, h \mid [y_i, h], [x, h], y^p h^q, x^l h^{-1}, y_1 \cdots y_n x; i = 1, \ldots, n \rangle.
\]

Now we define \( x_i = y_i^\beta h^\alpha \), for \( i = 1, \ldots, n \). Then we have the following new presentation for \( G \):

\[
\langle y_1, \ldots, y_n, x, h, x_1, \ldots, x_n \mid [y_i, h], [x, h], y^p h^q, x^l h^{-1}, y_1 \cdots y_n x, x_i^{-1} y_i^\beta h^\alpha; i = 1, \ldots, n \rangle.
\]

From the relations we obtain \( x_i^q = y_i^q h^{g\alpha} = y_i^{1+p\alpha} h^{g\alpha} = y_i (y_i^{p\alpha} h^{q})^{\alpha} = y_i \) and \( x_i^p = y_i^{p\beta} h^{p\alpha} = y_i^{p\beta} h^{q\beta - 1} = (y_i^{p\alpha} h^{q})^{\beta} h^{-1} = h^{-1} \).

Therefore \( G \) admits the presentation

\[
\langle x, h, x_1, \ldots, x_n \mid [x_i^q, x_i^{-q}], [x, x_1^l], x_i^{q} x_i^{-q}, x_i^l h^{-1}, x_1^q \cdots x_n^q x_1^{-1} x_n^{-q}, x_i^{p} x_i^{-p}; i = 1, \ldots, n \rangle,
\]

which obviously becomes

\[
\langle x, h, x_1, \ldots, x_n \mid x^l h^{-1}, x_1^{q} \cdots x_n^{q}, x_i^{p} h; i = 1, \ldots, n \rangle.
\]
The Seifert manifold $\Sigma(n, p, q, l)$ is the $n$-fold strongly-cyclic covering of the lens space $L(|nlq - p|, q)$, branched over the $(1, 1)$-knot $K(q, q(nl - 2), p - 2q, p - q)$, if $p \geq 2q$, and over the $(1, 1)$-knot $K(p - q, 2q - p, q(nl - 2), p - q)$, if $p < 2q$.

**Proof.** (i) Suppose $p \geq 2q$. By (iii) of Lemma $\text{[14]}$, $K = K(q, q(nl - 2), p - 2q, p - q)$ is a $(1, 1)$-knot in $L(|nlq - p|, q)$. Now suppose that $K$ admits the $n$-fold strongly-cyclic branched covering $C_n(K)$ defined by $\omega(\alpha) = 0$ and $\omega(\gamma) = 1$. In this case $C_n(K)$ is the Dunwoody manifold $D(q, q(nl - 2), p - 2q, n, p - q, 0)$ (see $\text{[13]}$ and $\text{[7]}$). Its defining genus $n$ Heegaard diagram, having cyclic symmetry of order $n$, is depicted in Figure $\text{[8]}$, where $a = q, b = q(nl - 2), c = p - 2q, r = p - q$, and the circle $C_i''$, for $i = 1, \ldots, n$, according to the twist parameter $r$. In order to check if the above diagram really represents a manifold, it is convenient to consider the cellular decomposition dual to the one associated to the diagram. In this way, $C_n(K)$ is obtained by pairwise identification of the regions of the 2-cell tessellation of the boundary of a 3-ball, as depicted in Figure $\text{[9]}$. The tessellation consists of $2n$ regions, $R_1', \ldots, R_n'$ around the North pole $N$ and
Heegaard genus is greater than two, and \( \pi \) the diagram in the following way. The poles \( M \) \( n \) \( C \) and \( \pi \) prime, with we note that \( A \) In order to obtain \( A \) \( \pi \) meridians and each meridian \( i \) is composed by \( p - 2q \) edges \( e_{i,j} \) \( (j = 1, \ldots, p) \). Observe that, when \( p = 2q \) (i.e., \( p = 2 \) and \( q = 1 \)), the points \( B_{i-1} \) and \( A_i \) coincide.

In order to obtain \( C_n(K) \), the region \( R_i \) is glued to \( R_i'' \), for \( i = 1, \ldots, n \), by an orientation reversing homeomorphism in such a way that point \( S \) of \( R_i'' \) matches point\(^2 \) \( B_{i-1} \) of \( R_i'. \) By this gluing we have: \( e_{i,j} \equiv e_{i-1,j+q} \) for \( j = 1, \ldots, q(nl - 1) \), \( e_{i-1,j} \equiv e_{i,j} \) for \( j = 1, \ldots, q \), \( e_{i,j} \equiv e_{i,j+q} \) for \( j = 1, \ldots, p - 3q \), and \( e_{i,p-3q+k} \equiv e_{i,q(nl-1)+k} \) for \( k = 1, \ldots, q \). As a consequence of all identifications, we have \( e_{i,j} \equiv e_{i,j+nq} \equiv \cdots \equiv e_{i,j+n(l-1)q} \) for \( j = 1, \ldots, nq \), and moreover \( e_{i-1,j} \equiv e_{i,q(nl-1)+j} \) for \( j = 1, \ldots, q \). Since \( \gcd(p, q) = 1 \), all edges of the arcs \( SB_{i-1}, B_{i-1}A_i \), and \( A_iN \) match, and we will call them \( x_i \). Therefore, the \( q(nl - 2) \) edges of the arc \( B_iA_i \) become: \( q \) times \( x_{i+1} \), \( q \) times \( x_{i+2} \), \ldots, \( q \) times \( x_i \) \( (l - 1) \) times) and then \( q \) times \( x_{i+1} \), \( q \) times \( x_{i+2} \), \ldots, \( q \) times \( x_{i-2} \) (see Figure 10). Since each edge \( x_i \) appears \( p \) times consecutively, its endpoints coincide. Moreover, since every edge \( x_i \) has \( S \) as endpoint, the cellular decomposition of \( C_n(K) \) is composed by one vertex, \( n \) edges, \( n \) regions and one 3-ball. By Seifert criterion, \( C_n(K) \) is actually a closed, orientable 3-manifold.

Moreover, \( \pi_1(C_n(K)) \) admits a balanced presentation with \( x_1, \ldots, x_n \) as generators and with relators obtained by walking around the boundaries of the regions \( R_i' \). As a consequence, \( \pi_1(C_n(K)) = G_n((x_1^n \cdots x_n^n)^r) \), and therefore it is isomorphic to \( \pi_1(\Sigma(n,p,q,l)) \).

In order to prove that \( C_n(K) \) and \( \Sigma(n,p,q,l) \) are actually homeomorphic, we note that \( \pi_1(C_n(K)) \) has non-trivial centre. So, either \( C_n(K) \) is prime or is the connected sum \( C_n(K) = M \# M' \), where \( \pi_1(M') \) is trivial and \( M \) is prime, with \( \pi_1(M) = \pi_1(C_n(K)) \).

If \( \Sigma(n,p,q,l) \) is large (in the sense of \(^2 \)), then also \( M \) is a large Seifert manifold (see \(^2 \)). So \( M \) and \( \Sigma(n,p,q,l) \) are homeomorphic, with Heegaard genus at least \( n - 2 \) (see \(^2 \)). If \( M' \) is not homeomorphic to \( S^3 \), then its Heegaard genus is greater than two, and \( C_n(K) \) has Heegaard genus greater than \( n \). But this is impossible since \( C_n(K) \) admits a genus \( n \) Heegaard

\(^2 \)The subscript \( i \) is considered mod \( n \).
splitting by [24]. As a consequence, $M' = S^3$ and $C_n(K) = \Sigma(n, p, q, l)$.

If $\Sigma(n, p, q, l)$ is not large, then one of the following possibilities holds: (a) $n = 3$, $p = 2$, $q = 1 = l$; (b) $n = 2$, $p = 2$, $q = 1$, $l > 1$; (c) $n = 2$, $p = 3$, $q = 1$, $l = 2$. Since for $q = 1$ the given decomposition of $C_n(K)$ coincides with the decomposition $P(p, \ldots, p; l)$ given in [27], the result follows from Proposition 4.1.

(ii) Let $p < 2q$. By (ii) of Lemma 1, $K(p - q, 2q - p, q(nl - 2), p - q)$ is a $(1, 1)$-knot in $L([nlq - p], q)$. Now suppose that $K$ admits the $n$-fold strongly-cyclic branched covering $C_n(K)$ defined by $\omega(\alpha) = 1$ and $\omega(\gamma) = 1$. In this case $C_n(K)$ is the Dunwoody manifold $D(p - q, 2q - p, q(nl - 2), n, p - q, 1)$. Its defining genus $n$ Heegaard diagram, having cyclic symmetry of order $n$, is depicted in Figure 5 where $a = p - q, b = 2q - p, c = q(nl - 2), r = p - q$, and the circle $C_i'$ must be glued to the circle $C''_{i+1}$, for $i = 1, \ldots, n$, according to the twist parameter $r$. Again it is better to refer to the dual decomposition (see Figure 3). This time each meridian is composed by $p$ edges, and precisely $p - q$ from $S$ to $B_i$, $2q - p$ from $B_i$ to $A_i$ and $p - q$ from $A_i$ to $N$. Moreover, the arc connecting $B_{i-1}$ with $A_{i}$ is composed by $q(nl - 2)$ edges. In order to obtain $C_n(K)$, the region $R_{i-1}'$ is glued to $R_i''$ in such a way that point $N$ of $R_{i-1}'$ matches point $B_{i-1}$ of $R_i''$.

In complete analogy with point (i), it is easy to see that the gluing gives rise to a cellular decomposition of $C_n(K)$, composed by one vertex, $n$ edges, $n$ regions and one 3-ball (see the 2-skeleton in Figure 11). So $\pi_1(S(n, p, q, l))$ is isomorphic to $\pi_1(C_n(K))$ and $\Sigma(n, p, q, l)$ is homeomorphic to $C_n(K)$ when it is large. The only small Seifert manifold occurs when $n = q = l = 2, p = 3$. To achieve the result, it suffices to prove that $D = D(1, 1, 4, 2, 1, 1)$ is the 2-fold covering of $S^3$, branched over the Montesinos knot $m(-1; 1/2, 2/3, 2/3)$ (see [3, Chapter 12]). A genus two Heegaard diagram of $D$ is depicted in Figure 12 where the circle $C_1'$ (resp. $C_2'$) must be glued to the circle $C_2''$ (resp. $C_1''$) in such a way that equally labelled vertices match. The application of the Takahashi algorithm [32] to this diagram (as depicted in Figure 13) shows that the manifold $D$ is the 2-fold branched covering of the knot $K$ represented by the 3-bridge diagram of Figure 14. A Wirtinger presentation of the fundamental group of the exterior of $K$ can be easily computed from this diagram, obtaining $\pi_1(S^3 - K) = \langle x, y, z \mid yz^{-1}xy^{-1}x^{-1}zy^{-1}xz^{-1}x^{-1}zy^{-1}z^{-1}x^{-1}zy^{-1}x^{-1}\rangle$. The Alexander polynomial of $K$ can be obtained by a standard application of Fox differential calculus (see [3, Chapter 9]), and we have $\Delta_K(t) = 1 - 4t + 5t^2 - 4t^3 + t^4$. By the sequence of Reidemeister moves depicted in Figures 15-17, the knot $K$
admits the 9 crossing diagram of Figure 8. Since the knot 8_{21} of Rolfsen’s table is the only knot with crossing number \leq 9 with Alexander polynomial $1 - 4t + 5t^2 - 4t^3 + t^4$ (see Table 3), then $K$ is precisely the knot 8_{21}. Since 8_{21} is the Montesinos knot $m(-1; 1/2, 2/3, 2/3)$ (see Table 2), which uses a slightly different notation), the statement is proved. 

Observe that Theorem 3 refines and extends the results of [13, Theorem 4.2] and [30, Theorem 5.1]. The proof of Theorem 3 also gives the following:

**Corollary 4** The Seifert manifold $\Sigma(n, p, q, l)$ is the Dunwoody manifold $D(q, q(nl - 2), p - 2q, n, p - q, 0)$ if $p \geq 2q$ and the Dunwoody manifold $D(p - q, 2q - p, q(nl - 2), n, p - q, 1)$ if $p < 2q$.

**Remark** Since $\Sigma(n, p, p - 1, 1) = \Sigma(n - 1, p, p - 1, p)$, this manifold is at the same time the $n$-fold strongly-cyclic covering of the lens space $L(pn - p - n, p - 1)$, branched over the $(1,1)$-knot $K(1, p - 2, (p - 1)(n - 2), 1)$ and the $(n - 1)$-fold strongly-cyclic covering of the lens space $L(p(pn - p - n), p - 1)$, branched over the $(1,1)$-knot $K(1, p - 2, (p - 1)(np - p - 2), 1)$, when $p > 2$. Moreover, the Neuwirth manifold $M_n = \Sigma(n, 2, 1, 1) = \Sigma(n - 1, 2, 1, 2)$ is at the same time the $n$-fold strongly-cyclic covering of the lens space $L(n - 2, 1)$, branched over the $(1,1)$-knot $K(1, n - 2, 0, 1)$ and the $(n - 1)$-fold strongly-cyclic covering of the lens space $L(2n - 4, 1)$, branched over the $(1,1)$-knot $K(1, 2n - 4, 0, 1)$.
Figure 9:

Figure 10: Case $p \geq 2q$
Figure 11: Case $p < 2q$

Figure 12: $D(1, 1, 4, 2, 1, 1)$
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