Fast Track Communication

Algebraic Bethe ansatz for the totally asymmetric simple exclusion process with boundaries

N Crampé

Laboratoire Charles Coulomb (L2C), UMR 5221 CNRS-University Montpellier 2, Montpellier, France

E-mail: nicolas.crampe@univ-montp2.fr

Received 3 December 2014, revised 13 January 2015
Accepted for publication 14 January 2015
Published 28 January 2015

Abstract

We study the one-dimensional totally asymmetric simple exclusion process in contact with two reservoirs including also a fugacity at one boundary. The eigenvectors and the eigenvalues of the corresponding Markov matrix are computed using the modified algebraic Bethe ansatz, a method introduced recently to study the spin chain with non-diagonal boundaries. We provide in this case a proof of this method.

Keywords: algebraic Bethe ansatz, out-of-equilibrium system, integrable models, exclusion process
Mathematics Subject Classification: 82B23, 81R12

Introduction

The one-dimensional totally asymmetric simple exclusion process (TASEP), describing the diffusion of particles with hard-core interactions, is one of the most studied models of non-equilibrium statistical mechanics (see [1, 2] for reviews). The stationary state of this model has been computed in [3] using the method so-called matrix ansatz (see [4] for a review). Then, a link with the integrable spin chains has been found in [5] allowing one to use the results obtained in the context of the integrable systems to study the TASEP. For example, the algebraic Bethe ansatz has been used in [6] to compute the spectral gap of diffusion models using the previous works on the integrable quantum spin chain [7, 8].

For the computation of the current fluctuations, the situation is more complicated. The matrix ansatz has only been developed recently in [9] for the TASEP then generalized in [10] for the ASEP. The use of the algebraic Bethe ansatz was only possible for a discrete set of the parameters of the ASEP [11]. The main result of this paper is to provide the algebraic Bethe ansatz for any parameters of the TASEP using the recent results introduced in the context of
the spin chains [12, 13]. Let us mention that the results of the papers [12, 13] have been conjectured by investigating models of small size. Then, the result of [12] has been proven in [14]. The results of this paper are proven and therefore support the conjectures of [13]. The proofs proposed here could be also useful to study other models.

The plan of this paper is as follows. In section 1, we present the model and give the eigenvalues of the generalized Markov matrix as well as the associated Bethe equations. Then, we introduce the transfer matrix associated to the TASEP in section 2.1 and we give the outline of the modified algebraic Bethe ansatz in section 2.2. More technical proofs are given in section 3.

1. TASEP and its eigenvalues

We consider the TASEP on a finite segment of size $L$ in contact with two reservoirs. The dynamics of the model is defined by the following rules: each site can be occupied by at most one particle, a particle attempts to hop on its right neighboring site with rate 1 unless this site is occupied, a particle may appear at the site 1 with rate $\alpha$ if this site is empty and a particle may disappear at the site $L$ with rate $\beta$ (see figure 1).

A configuration of the system is given by an $L$-tuple $(\tau_1, \tau_2, ..., \tau_L)$ where $\tau_i = 1$ if a particle is present at the site $i$ and $\tau_i = 0$ otherwise. The probabilities of each configuration at time $t$, $P_t(\tau_1, \tau_2, ..., \tau_L)$, can be encompassed in the following vector

$$P_t = \begin{pmatrix} P_t(0,0,0) \\ P_t(0,0,1) \\ \vdots \\ P_t(1,1,1) \end{pmatrix} = \sum_{\tau_1, \tau_2, ..., \tau_L = 0,1} P_t(\tau_1, \tau_2, ..., \tau_L) e^{\tau_1} \otimes e^{\tau_2} \otimes ... \otimes e^{\tau_L}.$$  

(1)

where $e^0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then, the time evolution of the probability is given by the following master equation

$$\frac{dP_t}{dt} = MP_t.$$  

(2)

The Markov matrix, $M$, for the TASEP is given by

$$M = B_1 + \sum_{k=1}^{L-1} w_{k,k+1} + \tilde{B}_L,$$  

(3)
where the subscripts indicate which sites the matrices $w$, $B$ and $\widetilde{B}$ act on non-trivially and

$$B = \begin{pmatrix} -\alpha & 0 \\ \alpha & 0 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} 0 & \beta \\ 0 & -\beta \end{pmatrix}. \tag{4}$$

For the boundary rates, we will use also the convenient notations $a = \frac{1}{\alpha} - 1$ and $b = \frac{1}{\beta} - 1$.

To compute the fluctuations of the current, an off-diagonal element of the Markov matrix is multiplied by a fugacity $e^{\mu}$ to keep track of the number of particles jumping through a bond [15]. We choose here to count the number of particles entering in the system and the corresponding generalized Markov matrix is

$$M(\mu) = B_1(\mu) + \sum_{k=1}^{L-1} w_{k,k+1} + \widetilde{B}_L$$

where $B(\mu) = \begin{pmatrix} -\alpha & 0 \\ 0 & 0 \end{pmatrix}$. \tag{5}

Its largest eigenvalue is the generating function of the cumulants of the current in the long time limit.

The main result of this paper is to compute the eigenvalues and the eigenvectors of $M(\mu)$ with the algebraic Bethe ansatz. Before giving details concerning the computations of the eigenvectors, we summarize the results about the eigenvalues. The eigenvalues of the generalized Markov matrix are

$$\lambda = -\beta - \sum_{p=1}^{L} \frac{u_p}{u_p - 1}, \tag{6}$$

where $u_1, \ldots, u_L$ are solutions of the Bethe equations

$$\left( au_j + e^{\mu} \right) \left( \frac{u_j - 1}{u_j} \right)^{L-1} = \left( au_j + 1 \right) \left( u_j + b \right) \times \prod_{k=1, k \neq j}^{L} \left( u_j - \frac{1}{u_k} \right), \quad \text{for } j = 1, 2, \ldots, L. \tag{7}$$

These results are a direct consequence of the general results of section 2.2. The eigenvalues (6) are given by

$$\lambda = -\frac{1}{2} \frac{dA(x)}{dx} \bigg|_{x=1, z=1}, \tag{8}$$

where $\Lambda(x)$ is the eigenvalues of the transfer matrix (18). The Bethe equations (7) are obtained by setting $z_i = 1$ in (17).

As usual, we consider only the solutions of the Bethe equations such that the Bethe roots are two-by-two different. We solved numerically Bethe equations (7) for systems of small size ($L = 1, 2, 3$) and we compared with a direct diagonalisation of the generalized Markov matrix. We showed that, in these cases, the spectrum obtained by the Bethe equations is complete.

The comparison of our result with previous results may be fruitful: Bethe equations (7) must be a limit of the Bethe equations obtained for the XXZ spin chain [16], Bethe vectors (16) are conjectured in [13] for the XXZ spin chain and the eigenvalue (6) with the largest real part must be compared to the one obtained with matrix ansatz [9, 10].
Markovian model. The Markovian model is recovered for $e^\mu = 1$. In this case, Bethe equations (7) split into two cases:

- For $u_j \neq -1/\alpha$ ($j = 1,..., L$), the factors $(au_j + 1)$ can be simplified on both sides of the Bethe equations to transform them into

$$\left(\frac{u_j - 1}{u_j}\right)^L = (u_j + b) \prod_{k=1 \atop k \neq j}^L (u_j - \frac{1}{u_k}) \quad \text{for } j = 1, 2, ..., L. \quad (9)$$

By solving these Bethe equations (9) for small size systems, we show that they seem to have only one solution corresponding to the stationary state of the TASEP (i.e. with vanishing eigenvalue $\lambda = 0$). Although the result for the eigenvalue is very simple, it seems that there are no simple expressions for the Bethe roots (see figure 2 for an example). It would be very interesting to compare the results obtained here and the matrix ansatz [3].

- Since all the Bethe roots must be distinct, we may choose without loss of generality $u_L = -1/\alpha$ which is a solution of the $L$th Bethe equation\(^1\). The $L - 1$ remaining Bethe equations become

$$\left(\frac{u_j - 1}{u_j}\right)^L = (u_j + b) (u_j + a) \prod_{k=1 \atop k \neq j}^{L-1} (u_j - \frac{1}{u_k}) \quad \text{for } j = 1, 2, ..., L - 1. \quad (10)$$

The associated eigenvalues can be written as $\lambda = -\alpha - \beta - \sum_{p=1}^{L-1} \frac{u_p}{u_p - 1}$. The Bethe equations (10) have been used previously in [6] to compute the spectral gap. They show that all the eigenvalues except the stationary state are obtained.

\(^1\) We may choose any other Bethe root equal to $-1/\alpha$ but, by invariance of the Bethe equations by permutations, we recover the same solutions.
In conclusion, for $e^\mu = 1$, the complete spectrum is obtained by solving Bethe equations (9) and (10).

2. Transfer matrix and algebraic Bethe ansatz

2.1. Transfer matrix

As usual in the context of the algebraic Bethe ansatz, one diagonalizes, instead of the Markov matrix, the transfer matrix. The central objects to construct the transfer matrix are the $R$-matrix, solution of the Yang–Baxter equation, and the $K$-matrix, solution of the reflection equation (see [17] for a review about the transfer matrix for the exclusion processes). For the TASEP, they are given explicitly by

$$R(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 1 & 1 - x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad K(x) = \begin{pmatrix} (a + x)x & 0 \\ xa + 1 & 0 \\ e^{\mu} [1 - x^2] & xa + 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \widetilde{K}(x) = \frac{1}{xb + 1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (11)$$

Then, one defines the monodromy matrix by

$$B_0(x) = R_{0L}(x/z_L) \cdots R_{01}(x/z_1) K_0(x) R_{10}(xz_1) \cdots R_{L0}(xz_L) = \begin{pmatrix} \mathcal{A}(x) & \mathcal{B}(x) \\ \mathcal{C}(x) & \mathcal{D}(x) \end{pmatrix}, \quad (12)$$

where $z_1, \ldots, z_L$ are called inhomogeneity parameters and the transfer matrix by [18]

$$t(x) = \text{Tr}_0(B_0(x)) = \frac{1}{xb + 1} (\mathcal{A}(x) + xb \mathcal{D}(x) + \mathcal{C}(x)). \quad (13)$$

The important features of the transfer matrix are that they commute for different spectral parameters (i.e. $[t(x), t(y)] = 0$) and that the generalized Markov matrix is obtained by

$$-\frac{1}{2} \frac{dt(x)}{dx} \bigg|_{x=1,z=1} = M(\mu). \quad (14)$$

Then, the eigenvectors of the generalized Markov matrix $M(\mu)$ can be computed by putting $z_i = 1$ in the eigenvectors of the transfer matrix.

The monodromy matrix satisfies also the reflection equation and one deduces that

$$[\mathcal{A}(x), \mathcal{C}(y)] = 0,$$

$$\mathcal{D}(x)\mathcal{C}(y) = \frac{x(xy - 1)}{y - x} \mathcal{D}(y)\mathcal{C}(x) - \frac{y(xy - 1)}{y - x} \mathcal{C}(x)\mathcal{D}(y) - \mathcal{C}(x)\mathcal{A}(y). \quad (15)$$

Unfortunately, there exists no relation allowing us to move $\mathcal{A}$ from the left to the right of $\mathcal{C}$ which complicates our tasks: we will come back to this point in section 3.2. This feature is particular to the TASEP and is due to the $0$ on the diagonal of the $R$-matrix (11). We can also show that $\mathcal{D}(x) = 0$.

To conclude this section, we would like to mention that the problem of finding exact methods to solve the problem with non-diagonal boundaries (i.e. $K$ and $\widetilde{K}$ are not diagonal) has attracted a lot of attention. The problem lies in the fact that the usual methods are based on the existence of one simple particular eigenvector which does not exist in this case. Therefore,

$^2$ In the case of the TASEP model, the proof given in [18] cannot be used directly because the crossing symmetry is not satisfied by the $R$-matrix (11). However, it is valid for the partially asymmetric simple exclusion process and then we can take the limit to get the result for the TASEP [17].
numerous approaches have been modified and generalized to deal with this problem: the algebraic Bethe ansatz \cite{7, 19–22}, the functional Bethe ansatz \cite{8, 23–25}, the coordinate Bethe ansatz \cite{26}, the separation of variables \cite{27, 28}, the $q$-Onsager approach \cite{29} and the matrix ansatz \cite{9, 10, 30}. Recently, inhomogeneous $T-Q$ relations have been studied in \cite{16, 31, 32} where they obtained eigenvalues and Bethe equations for generic boundaries. These results have permitted one to conjecture a modified algebraic Bethe ansatz to get the eigenvectors \cite{12, 13} (proved in the case of the open XXX chain in \cite{14}). It is this last method we used in this paper to find the eigenvalues and the eigenvectors of the generalized Markov matrix.

2.2. Modified algebraic Bethe ansatz

The modified algebraic Bethe ansatz states that eigenvectors of the transfer matrix are given by a product of $L$ matrices $\mathcal{C}$ where $L$ is the number of sites of the model. Therefore, eigenvectors of $t(u_0)$ are given by the Bethe vector

$$\Phi(u_1, u_2, ..., u_L) = \mathcal{C}(u_1)\mathcal{C}(u_2)\ldots\mathcal{C}(u_L)|\Omega\rangle,$$

(16)

where $|\Omega\rangle = e^1 \otimes e^1 \otimes \ldots \otimes e^1$ and $\{u_1, u_2, ..., u_L\}$ are solutions of Bethe equations:

$$\left(au_j + e^\theta\right) \prod_{\ell=1}^{L} \frac{(u_j - z_{\ell})(u_j z_{\ell} - 1)}{u_j z_{\ell}} = \left(u_j + \hbar\right) \left(au_j + 1\right) \prod_{k=1 \atop k \neq j}^{L} \left(u_j - \frac{1}{u_k}\right), \quad \text{for } j = 1, 2, ..., L.\quad (17)$$

The associated eigenvalues are

$$\Lambda(u_0) = u_0^{L+1} \frac{b + u_0}{bu_0 + 1} \prod_{k=1}^{L} \frac{u_0 u_k - 1}{u_k - u_0} - \frac{(au_0 + e^\theta)(u_0^2 - 1)}{(au_0 + 1)(bu_0 + 1)}$$

$$\times \prod_{j=1}^{L} \left[(u_0 - z_j) \left(u_0 - \frac{1}{z_j}\right) \prod_{k=1}^{L} \frac{u_k}{u_k - u_0}\right].\quad (18)$$

Let us emphasize that the main difference with the usual algebraic Bethe ansatz is that the number of matrices $\mathcal{C}$ must be equal to the number of sites of the model. A vector constructed with less than $L$ matrices $\mathcal{C}$ (in particular $|\Omega\rangle$) is not an eigenvector of the transfer matrix. Another difference is the presence of the matrix $\mathcal{C}$ in the transfer matrix and the necessity to compute its action on $\Phi$: it is this computation which forces us to take $L$ matrices $\mathcal{C}$ in $\Phi$ (see section 3.1). The rest of the paper is devoted to proving these results. As usual in the context of the algebraic Bethe ansatz, the completeness of such a solution is not proven but it is conjectured to be supported by numerical evidence (as explained previously in section 1).

As for the usual algebraic Bethe ansatz, we need the actions of the matrices $\mathcal{F}$ and $\mathcal{D}$ on a product of $\mathcal{C}$. In section 3.2, we show the following relations
Due to the presence of the operator $\mathcal{C}$ in the transfer matrix, we need also the following relation proven in section 3.1

$$\mathcal{C}(u_0)\Phi(u_1,\ldots,u_L) = \left(\sum_{p=0}^{L} \frac{u_0 u_p \left(u_p^2 - 1\right)}{u_p u_0 - 1} \prod_{k=0}^{L} \left(\frac{u_p (u_p u_k - 1)}{u_k - u_p}\right)\right) \omega \right) |\Omega\rangle$$

$$- \sum_{p=0}^{L} \frac{a u_p (u_p^2 - 1)}{a u_p + 1} \prod_{j=1}^{L} \left(\left(\frac{u_p - z_j}{u_p - 1}\right)\left(\frac{u_p - 1}{z_j}\right)\right)$$

$$\times \prod_{k=0}^{L} \left(\frac{u_k}{u_k - u_p}\right) \Phi(u_k) \right) |\Omega\rangle, \quad (19)$$

$$\mathcal{D}(u_0)\Phi(u_1,\ldots,u_L) = \sum_{p=0}^{L} \frac{u_p^2 - 1}{u_p u_0 - 1} \prod_{k=0}^{L} \left(\frac{u_p (u_p u_k - 1)}{u_k - u_p}\right) \Phi(u_k) \right) |\Omega\rangle. \quad (20)$$

Now, we are in position to compute the action of the transfer matrix on $\Phi(u_1, u_2,\ldots,u_L)$:

$$t(u_0)\Phi(u_1,\ldots,u_L) = \frac{1}{b u_0 + 1} \left(\mathcal{A}(u_0) + b u_0 \mathcal{D}(u_0) + \mathcal{C}(u_0)\right) \Phi(u_1,\ldots,u_L) \quad (22)$$

$$= \Lambda(u_0)\Phi(u_1, u_2,\ldots,u_L) + \sum_{p=0}^{L} F(u_0, u_p) U_p \prod_{k=0}^{L} \Phi(u_k) |\Omega\rangle, \quad (23)$$

where $F(u, x) = \frac{u(x^2 - 1)}{(u - x)(u + x)}$ and

$$U_p = \left(b + u_p\right) u_p \prod_{k=0}^{L} \left(\frac{u_p u_k - 1}{u_k - u_p}\right) - \frac{a u_p + e^u}{a u_p + 1}$$

$$\times \prod_{j=1}^{L} \left(\left(\frac{u_p - z_j}{u_p - 1}\right)\left(\frac{u_p - 1}{z_j}\right)\right) \prod_{k=0}^{L} \left(\frac{u_k}{u_k - u_p}\right). \quad (24)$$

Relation (23), called the off-shell equation, is obtained using relations (19), (20) and (21) and particularizing the elements $p = 0$ in the sum. Bethe equations (17) imply the vanishing of $U_p$ (for $p = 1, 2,\ldots,L$) and we obtain that $\Phi(u_1, u_2,\ldots,u_L)$ is an eigenvector of $t(u_0)$ with the eigenvalue $\Lambda(u_0)$. 

J. Phys. A: Math. Theor. 48 (2015) 08FT01
3. Actions of $A$, $C$ and $D$ on the Bethe vector $\Phi$

In the previous section 2.2, we gave the outline of the modified algebraic Bethe ansatz but the central relations (19)–(21) are only proven in this section. These proofs are more technical and we prefer, for clarity, to write them separately.

3.1. Proof of relation (21)

Relation (21) is a new type of relation to prove in comparison to the usual algebraic Bethe ansatz. To demonstrate it, let us introduce the following vector

$$\Psi(x) = e^{\mu} \frac{1 - x^2}{x(ax + 1)} \prod_{j=1}^{L} \left( x - z_j \right) \prod_{\ell=0}^{L} \left( \frac{u_\ell}{u_\ell - x} \mathcal{C}(u_\ell) \right) \mathcal{G}(x)^{-1} |\Omega\rangle. \tag{25}$$

We are going to show that the entries of this vector have only poles at $x = 0$ and $x = u_p$ ($p = 0, 1, ..., L$).

Firstly, we perform a change of basis using the factorizing twist introduced in [33] to obtain a simple explicit formula for $\mathcal{C}(x)$. The factorizing twist is

$$F_{12...L} = F_{L-1,L} F_{L-2,L-1,L} ... F_{1,23...L}, \tag{26}$$

where

$$F_{j,j+1...L} = F_{j,j+1...L}(z_j, ..., z_L) = 1 - \tilde{R}_j + \tilde{R}_j R_{j+1}(z_j, ..., z_L) ... R_{j+1}(z_{j+1}). \tag{27}$$

We have introduced the matrix $\tilde{R} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Using the results of [33], one gets

$$\mathcal{C}^F(x) = F_{12...L} \mathcal{C}(x) F_{12...L}^{-1} = 1 - \frac{x^2}{ax + 1} \sum_{i=1}^{L} \xi(z_i + a) \sigma_i^+$$

$$\times \prod_{i=1}^{L} \left( 1 - ax \right) \tilde{R}_j + \frac{x - z_j}{z_j - z_j} \left( 1 - \tilde{R}_j \right)$$

$$+ e^{\mu} \prod_{j=1}^{L} \left( 1 - ax \right) \tilde{R}_j + \frac{z_j - x}{z_j} \left( 1 - \tilde{R}_j \right) \right). \tag{28}$$

Secondly, by noting that $\mathcal{C}^F(x)$ is an upper triangular matrix, we can compute the determinant of $\mathcal{C}(x)$:

$$\det(\mathcal{C}(x)) = \det(\mathcal{C}(x)) = \left( e^{\mu} \frac{1 - x^2}{ax + 1} \right)^{2L} \prod_{j=1}^{L} \left( 1 - ax \right)^{2L-1}. \tag{29}$$

For a generic value of $x$, the determinant does not vanish which allows us to take the inverse of $\mathcal{C}^F(x)$ (and also of $\mathcal{C}(x)$ which justifies the definition of $\Psi(x)$).
Thirdly, we can determine the entries of $C^F(x)\Omega\rangle\rangle^{-1}\Omega\rangle$.

\[ C^F(x)\Omega\rangle\rangle^{-1}\Omega\rangle = \frac{(ax + 1)e^{-\mu}}{(1 - x^2)\prod_{j=1}^{L}(x - z_j)(x - \frac{1}{z_j})} \times \sum_{\epsilon_1, \ldots, \epsilon_L=0,1} \prod_{j=1}^{L} f_{\epsilon_j}(x, z_j) \epsilon^{\epsilon_1} \otimes \cdots \otimes \epsilon^{\epsilon_L}, \tag{30} \]

where $f_{\epsilon}(x, z) = (e - 1)e^{-\mu}(x + z)x + e(z - x)/z$. We demonstrate relation (30) by showing that, with this expression and expression (28) of $C^F$, we get $C^F(x)\Omega\rangle\rangle^{-1}\Omega\rangle = \Omega\rangle$.

Finally, we remark that $V(x)$ is equal to the rhs of (25) replacing all the $C$ by $C^F$ (since $\Omega\rangle\rangle^{-1}\Omega\rangle = \Omega\rangle$) and we deduce that $V(x)$ has only poles at $x = 0$ and $x = u_p$ ($p = 0, 1, \ldots, L$). Let us remark that, if we take less than $L + 1$ matrices $C$ in the definition of $V(x)$, there is a pole at infinity and the corresponding residue does not take a nice form.

Therefore, the only non-trivial residues of $\mathcal{V}(x)$ are

\[ \text{Res} \mathcal{V}(x) \big|_{x=0} = e^{\mu} \prod_{l=0}^{L} \mathcal{C}(u_l)\mathcal{C}(0)^{-1}\Omega\rangle, \tag{31} \]

\[ \text{Res} \mathcal{V}(x) \big|_{x=u_p} = -e^{\mu} \frac{1 - u_p^2}{(u_p - z_j)(u_p - \frac{1}{z_j})} \prod_{l=0}^{L} \left( \frac{u_l}{u_l - u_p} \right) \prod_{k\neq p}^{L} \mathcal{C}(u_k)^{-1}\Omega\rangle. \tag{32} \]

By using that $\mathcal{C}(0)^{-1}\Omega\rangle = \frac{1}{\mathcal{F}_{12\ldots L}\mathcal{C}(0)^{-1}\Omega\rangle} = e^{-\mu}\Omega\rangle$ and that the sum over all the residues of a rational function vanishes, we prove relation (21).

3.2. Proof of relations (19) and (20)

As mentioned in section 2.1, the relations of type (19) and (20) are usually proven using the commutation relation between $A$, $D$, and $C$. Unfortunately in the case of the TASEP model, no relation permuting $A$ and $C$ exists. To overcome this problem, we use the transfer matrix associated to the partially asymmetric simple exclusion process (PASEP) depending on the parameter $q$ such that we recover the TASEP in the limit $q \to 0$.

The $R$-matrix associated to the PASEP is given by

\[ R^{(q)}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (x - 1)q & (q - 1)x & 0 \\ 0 & qx - 1 & qx - 1 & 0 \\ 0 & q - 1 & x - 1 & qx - 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{33} \]

One gets $R^{(q)}(x) = R(x)$ where $R(x)$ is the $R$-matrix of the TASEP (11). We indicate by the superscript $(q)$ the objects defined in section 2.1 but for the $R$-matrix $R^{(q)}$. From now, the computations are similar to those one performs usually in the context of the algebraic Bethe ansatz.
The reflection equation satisfied by the monodromy matrices $B^q(x)$ allows us to get the following commutation relations

\[
\overline{\mathcal{F}}^{(q)}(x)\overline{\mathcal{F}}^{(q)}(y) = \frac{q^2 xy - 1}{q(x - y)(qxy - 1)}\overline{\mathcal{F}}^{(q)}(y)\overline{\mathcal{F}}^{(q)}(x) - \frac{(q - 1)(q^2 xy - 1)x}{q(x - y)(qxy - 1)}\overline{\mathcal{F}}^{(q)}(x)\overline{\mathcal{F}}^{(q)}(y) + \frac{xy(q - 1)(y^2 - 1)(q^2 x^2 - 1)}{(qxy - 1)(qx^2 - 1)(qy^2 - 1)}\overline{\mathcal{F}}^{(q)}(x)\overline{\mathcal{F}}^{(q)}(y),
\]

(34)

\[
\mathcal{D}^{(q)}(x)\mathcal{D}^{(q)}(y) = \frac{(x - qy)(xy - 1)}{(qxy - 1)(x - y)}\mathcal{D}^{(q)}(y)\mathcal{D}^{(q)}(x) + \frac{(q - 1)(y^2 - 1)y}{(x - y)(qy^2 - 1)}\mathcal{D}^{(q)}(x)\mathcal{D}^{(q)}(y)
- \frac{q - 1}{qxy - 1}\mathcal{D}^{(q)}(x)\overline{\mathcal{F}}^{(q)}(y),
\]

(35)

where $\overline{\mathcal{F}}^{(q)}(x) = \mathcal{F}^{(q)}(x) + \frac{(1 - q)^2}{q^2 - 1}\mathcal{D}(x)$. We see that relation (35) gives back relation (15) in the limit $q \to 0$ whereas relation (34) is not defined in this limit.

We can also determine the values of $\overline{\mathcal{F}}^{(q)}(x)$ and $\mathcal{D}^{(q)}(x)$ on the vector $|\Omega\rangle$ and we get

\[
\overline{\mathcal{F}}^{(q)}(x) |\Omega\rangle = q^L \frac{x(x^2 - 1)(qx + a)}{(xa + 1)(q^2 x^2 - 1)} \prod_{j=1}^{L} \frac{(z_j - x)(1 - xz_j)}{(z_j - qx)(1 - qxz_j)} |\Omega\rangle,
\]

(36)

\[
\mathcal{D}^{(q)}(x) |\Omega\rangle = |\Omega\rangle.
\]

(37)

By using these previous relations, we are able to compute $\mathcal{F}^{(q)}(u_0)\Phi^{(q)}(u_1, ..., u_L)$ and $\mathcal{D}^{(q)}(u_0)\Phi^{(q)}(u_1, ..., u_L)$. The results are not singular in the limit $q \to 0$ and we get relations (19) and (20).

**Acknowledgments**

I thank warmly S Belliard, V Caudrelier, E Ragoucy and M Vanicat for their interests and their suggestions.

**References**

[1] Derrida B 1998 An exactly soluble non-equilibrium system: the asymmetric simple exclusion process Phys. Rep. 301 65

[2] Schütz G M 2001 Exactly solvable models for many-body systems far from equilibrium Phase Transit. Crit. Phenom. 19 1

[3] Derrida B, Evans M R, Hakim V and Pasquier V 1993 Exact solution of a 1d asymmetric exclusion model using a matrix formulation J. Phys. A: Math. Gen. 26 1493

[4] Blythe R A and Evans M R 2007 Nonequilibrium steady states of matrix product form: a solver’s guide J. Phys. A: Math. Theor. 40 333–441

[5] Sandow S 1994 Partially asymmetric exclusion process with open boundaries Phys. Rev. E 50 2660
[6] de Gier J and Essler F 2005 Bethe ansatz solution of the asymmetric exclusion process with open boundaries Phys. Rev. Lett. 95 240601
[7] Cao J, Lin H, Shi K and Wang Y 2003 Exact solutions and elementary excitations in the XXZ spin chain with unparallel boundary fields Nucl. Phys. B 663 487
[8] Nepomechie R I 2004 Bethe ansatz solution of the open XXZ chain with non-diagonal boundary terms J. Phys. A: Math. Gen. 37 433
[9] Lazarescu A and Mallick K 2011 Exact formula for the statistics of the current in the TASEP with open boundaries J. Phys. A: Math. Theor. 44 315001
[10] Gorissen M, Lazarescu A, Mallick K and Vanderzande C 2012 Exact current statistics of the ASEP with open boundaries Phys. Rev. Lett. 109 170601
[11] de Gier J and Essler F 2011 Current large deviation function for the open asymmetric simple exclusion process Phys. Rev. Lett. 107 010602
[12] Belliard S and Crampe N 2013 Heisenberg XXX model with general boundaries: eigenvectors from algebraic Bethe ansatz SIGMA 9 072
[13] Belliard S 2015 Modified algebraic Bethe ansatz for XXZ chain on the segment-I-triangular cases Nucl. Phys. B 892 1–20
[14] Zhang X, Li Y-Y, Cao J, Yang W-L, Shi K and Wang Y 2014 Retrieve the Bethe states of quantum integrable models solved via off-diagonal Bethe ansatz arXiv:1407.5294
[15] Lebowitz J L and Spohn H 1999 A Gallavotti-cohen-type symmetry in the large deviation functional for stochastic dynamics J. Stat. Phys. 95 333
[16] Cao J, Yang W, Shi K and Wang Y 2013 Off-diagonal Bethe ansatz solutions of the anisotropic spin-1/2 chains with arbitrary boundary fields Nucl. Phys. B 877 152
[17] Crampe N, Ragoucy E and Vanicat M 2014 Integrable approach to simple exclusion processes with boundaries Review and progress J. Stat. Mech. P11032
[18] Sklyanin E K 1988 Boundary conditions for integrable quantum systems J. Phys. A: Math. Gen. 21 2375
[19] Melo C S, Martins M J and Ribeiro G A P 2005 Bethe ansatz for the XXX-S chain with non-diagonal open boundaries Nucl. Phys. B 711 565
[20] Yang W-L and Zhang Y-Z 2007 On the second reference state and complete eigenstates of the open XXZ chain J. High Energy Phys. JHEP04(2007)044
[21] Belliard S, Crampe N and Ragoucy E 2013 Algebraic Bethe ansatz for open XXX model with triangular boundary matrices Lett. Math. Phys. 103 493
[22] Pimenta R A and Lima-Santos A 2013 Algebraic Bethe ansatz for the six vertex model with upper triangular K-matrices J. Phys. A: Math. Theor. A 46 455002
[23] Murgan R and Nepomechie R I 2005 Bethe ansatz derived from the functional relations of the open XXZ chain for new special cases J. Stat. Mech. P08002
[24] Galleas W 2008 Functional relations from the Yang–Baxter algebra: eigenvalues of the XXZ model with non-diagonal twisted and open boundary conditions Nucl. Phys. B 790 524
[25] Frahm H, Grelik J H, Seel A and Wirth T 2011 Functional Bethe ansatz methods for the open XXX chain J. Phys. A: Math. Theor. 44 015001
[26] Crampe N, Ragoucy E and Simon D 2010 Eigenvectors of open XXZ and ASEP models for a class of non-diagonal boundary conditions J. Stat. Mech. P11038
[27] Crampe N and Ragoucy E 2012 Generalized coordinate Bethe ansatz for non-diagonal boundaries Nucl. Phys. B 858 502
[28] Frahm H, Seel A and Wirth T 2008 Separation of variables in the open XXX chain Nucl. Phys. B 802 351
[29] Baseilhac P and Koizumi K 2007 Exact spectrum of the XXZ open spin chain from the q-Onsager algebra representation theory J. Stat. Mech. P09006
Baseilhac P and Belliard S 2013 The half-infinite XXZ chain in Onsager’s approach Nucl. Phys. B 873 550

[30] Crampe N, Ragoucy E and Simon D 2011 Matrix coordinate Bethe ansatz: applications to XXZ and ASEP models J. Phys. A: Math. Theor. A 44 405003

[31] Cao J, Yang W, Shi K and Wang Y 2013 Off-diagonal Bethe ansatz and exact solution a topological spin ring Phys. Rev. Lett. 111 137201

Cao J, Yang W, Shi K and Wang Y 2013 Off-diagonal Bethe ansatz solution of the XXX spin-chain with arbitrary boundary conditions Nucl. Phys. B 875 152

[32] Nepomechie R I 2013 Inhomogeneous T-Q equation for the open XXX chain with general boundary terms: completeness and arbitrary spin J. Phys. A: Math. Theor. 46 442002

[33] Maillet J M and Sanchez de Santos J 2000 Drinfel’d twists and algebraic Bethe ansatz Am. Math. Soc. Transl. 201 137