We provide formulae for fidelity decay and parametric energy correlations for random matrix ensembles where time-reversal invariance of the original Hamiltonian is broken by the perturbation. Like in the case of a symmetry conserving perturbation a simple relation between both quantities exists. Fidelity freeze is observed for systems with even and odd spin.

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I. INTRODUCTION

Fidelity presently attracts considerable attention in diverse fields like quantum information, quantum chaotic systems and others [1, 2]. It measures the change of quantum dynamics of a state under a modification of the Hamiltonian. In quantum information, fidelity measures the deviation between a mathematical algorithm and its physical implementation.

Since fidelity requires knowledge of the entire wave function for the original and for the modified system a measurement of fidelity is a notoriously difficult task. However a number of experimental results have been obtained in microwave billiards, where the perturbation was achieved by varying some geometric parameter. There are two qualitatively different ways to do this, either by a global perturbation, e.g. by moving one wall [3], or a local perturbation, e.g. by varying the position of an impurity [4]. For the first case random matrix theory is applicable, and indeed a perfect agreement between experiment and theory was found [3].

On the other hand statistical properties of energy correlations between spectra of complex quantum systems which differ by a parameter-dependent variation have been studied experimentally and theoretically [5]. This quantity can be obtained with great accuracy from scattering experiments by analyzing the fluctuations of the resonances in the scattering cross-section [6, 7].

From an experimental point of view it is interesting to relate fidelity with spectral quantities. This allows an indirect measurement of fidelity via an analysis of the (parametric) scattering data and the problem of measuring the entire wave function is circumvented.

A simple differential relation between fidelity decay and parametric energy correlations was established in the case that the parameter dependent perturbation falls into the same symmetry class as the unperturbed system [8, 9]. This differential relation was derived earlier in energy space by Taniguchi, Simons and coworkers [10, 11] and it was identified with a continuity equation of the Calogero–Moser–Sutherland model [12]. In Ref. [13] similar expression were derived for parametric energy correlations in the case where the perturbation breaks the global symmetry of the original unperturbed system.

Recently billiard experiments could be performed in microwave resonators, where time reversal symmetry (TRS) was broken by a piece of ferrite [14] which plays the role of the perturbation. From the experimental results S–matrix elements could be determined and an estimate of the strength of the TRS–breaking was made. The experimental setup seems adequate for a measurement of parametric energy correlations and of fidelity decay by a TRS breaking perturbation.

In this paper we therefore analyze the expressions found in Refs [10, 13] for TRS–breaking perturbations under the aspect of fidelity and provide formulae for fidelity and parametric form factor as well as differential relations between them and discuss their consequences.

II. DEFINITIONS AND RESULTS

Fidelity amplitude is defined as a functional of the initial wave function. In an ergodic situation it seems reasonable to replace a specific initial state by a random one. In Ref. [12] the corresponding random matrix model for the fidelity amplitude was defined by ($\hbar = 1$)

$$f(\lambda_\parallel, \lambda_\perp, t) = \frac{1}{N} \langle \text{tr} \exp(itH) \exp(-itH_0) \rangle .$$  (1)
The Fourier transform of parametric energy correlations is defined by
\[ \tilde{K}(\lambda_{||}, \lambda_{\perp}, t) = \frac{1}{N} \langle \text{tr} \exp(\imath t H) \text{tr} \exp(-\imath t H_0) \rangle . \] (2)

It was named cross–form–factor in [3]. The brackets denote an ensemble average. The perturbed Hamiltonian \( H \) is given as
\[ H = H_0 + \lambda_{||} V_{||} + \lambda_{\perp} V_{\perp} . \] (3)

Let us first discuss the unperturbed Hamiltonian. We assume that for the unperturbed system \( H_0 \) TRS is conserved. The time reversal operator \( T \) acts differently on systems with integer spin and on systems with half–integer spin [10]. For even spin \( T \) is chosen from the Gaussian ensemble of matrices (GOE, \( \beta = 1 \)). For odd spin systems \( T \) acts via conjugation with the symplectic metric calculation and comparing results. In appendix we consider a much wider class of TRS breaking Hamiltonians as before. Observe that for \( \lambda_{||} = \lambda_{\perp} \) this corresponds to a perturbation by a Hermitian matrix, i.e. to a perturbation which is taken from the Gaussian unitary ensemble (GUE). This ensemble is called type A in [18]. Thus time reversal symmetry breaking can occur in different ways. Symbolically we may write the l. h. s. of equation (3) as \( \text{AI + } \lambda_{||} \text{AI + } \lambda_{\perp} \text{B} \) (case I) or as \( \text{AI + } \lambda_{||} \text{AI + } \lambda_{\perp} \text{C} \) (case II). Usually only the transition \( \text{AI + } \lambda \text{AI} \) is considered, when time–reversal invariance is discussed [20].

Analysing equation (4) and the following ones of Ref. [13], we find expressions for fidelity amplitude and for cross–form factor. To present them concisely we define for case I the function
\[ Z^{(1)}(\lambda_{||}, \lambda_{\perp}, \tau) = \int_{\text{Max}(0,\tau,1)}^{\tau} \int_{0}^{u} \int_{0}^{v} \text{d}v \text{d}u \left( \frac{1 + 4\pi^2 \lambda_{||}^2 (\tau^2 - v^2)}{\sqrt{[u^2 - 2v(u + 1) - v^2]}} \right) \left( \frac{(1 - \tau + u)}{(1 - \tau - u)} \right) \left( \frac{\sqrt{v^2 - \tau^2}}{2\tau} \right) \right) \]
\[ e^{-2\pi^2 (\lambda_{||}^2 + \lambda_{\perp}^2)r(2u + 1 - \tau)} - 2\pi^2 (\lambda_{||}^2 - \lambda_{\perp}^2)(u^2), \] (8)

and for the case II the function
\[ Z^{(II)}(\lambda_{||}, \lambda_{\perp}, \tau) = \int_{-1}^{1} \int_{0}^{1 - |u|} \frac{u + 1 + t}{\sqrt{[(u - 1)^2 - v^2][u + 1)^2 - v^2]}} \text{d}v \text{d}u \left( 1 + \pi^2 \lambda_{||}^2 \right) \left( \frac{(1 - \tau + u)}{(1 - \tau - u)} \right) \left( \frac{\sqrt{v^2 - \tau^2}}{2\tau} \right) \left( \frac{\sqrt{v^2 - \tau^2}}{2\tau} \right) \]
\[ e^{-\pi^2 (\lambda_{||}^2 + \lambda_{\perp}^2)r(2u + 1 - \tau)} - \pi^2 (\lambda_{||}^2 - \lambda_{\perp}^2)(u^2), \] (9)

where \( \tau \) is time measured in units of Heisenberg time \( t_H = 2\pi/D \). Then in the large \( N \)–limit the fidelity as defined in Eq. (11) is given in both cases by
\[ f(\lambda_{||}, \lambda_{\perp}, \tau) = -\frac{1}{\pi^2} \frac{\partial}{\partial (\lambda_{||})} Z(\lambda_{||}, \lambda_{\perp}, \tau) . \] (10)

The cross form–factor is given by
\[ \tilde{K}(\lambda_{||}, \lambda_{\perp}, \tau) = \frac{4}{\beta} \tau^2 \int_{\text{Max}(0,\tau,1)}^{\tau} \int_{0}^{u} \int_{0}^{v} \text{d}v \text{d}u \left( \frac{1 + 4\pi^2 \lambda_{||}^2 (\tau^2 - v^2)}{\sqrt{[u^2 - 2v(u + 1) - v^2]}} \right) \left( \frac{(1 - \tau + u)}{(1 - \tau - u)} \right) \left( \frac{\sqrt{v^2 - \tau^2}}{2\tau} \right) \right) \]
\[ e^{-2\pi^2 (\lambda_{||}^2 + \lambda_{\perp}^2)r(2u + 1 - \tau)} - 2\pi^2 (\lambda_{||}^2 - \lambda_{\perp}^2)(u^2), \] (11)

From this follows the relation between fidelity and cross form–factor [10]
\[ f(\lambda_{||}, \lambda_{\perp}, \tau) = -\frac{\beta}{4\pi^2 \tau^2} \frac{\partial}{\partial (\lambda_{||})} \tilde{K}(\lambda_{||}, \lambda_{\perp}, \tau) . \] (12)

This relation can be derived through an universality argument without going through a lengthy supersymmetric calculation and comparing results. In appendix
A we present this derivation extending the method of Refs. [8, 9] to the case of TRS breaking.

Some details on the derivation of equations (8) to (12) from the pertinent formulae of Ref. [13] are given in appendix B.

III. DISCUSSION

The double integrals (8) and (9) can be evaluated numerically (see appendix B of Ref. [21] for a convenient parametrization). Figure 1 shows the fidelity decay in case I for different perturbation strengths $\lambda$. The results $f(\lambda, 0, \tau)$ (a pure GOE perturbation, dashed lines) and $f(0, \lambda, \tau)$ (a purely perpendicular perturbation, dashed-dotted lines) are shown as well for the same parameter.

With increasing perturbation the decays for the GOE and the GUE perturbation separate, and the freeze behavior get lost. For strong perturbations a recovery of fidelity at Heisenberg time is seen. This is already known from [23] where the cases $A + \lambda A$ and $AI + \lambda AI$ were discussed.

For small perturbations and for times much smaller than Heisenberg time fidelity decay is governed by Fermi’s golden rule. In this regime the crucial parameter is $\lambda^2 = \lambda_{||}^2 + \lambda_{\perp}^2$ which is related to the spreading width $\Gamma = 2\pi\lambda^2 D$ of an unperturbed state. This result holds independently of the universality class of the background. It is therefore interesting to look on the fidelity amplitude for fixed $\lambda$ but different ratios between orthogonal and parallel perturbation.

In figure 2 fidelity amplitude is plotted for small perturbation strength $\lambda = 0.1$ and for different ratios between $\lambda_{||}$ and $\lambda_{\perp}$ for case I and case II.

In case I we see that fidelity amplitude is a monotonous function of this ratio for all times. The slowest decay dominates for times larger than Heisenberg time. It was therefore predicted [22] that fidelity decay is much slower for perturbations which are purely off-diagonal in the eigenbasis of the original Hamiltonian.

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happens for $\lambda_\parallel = 0$, i.e. when the perturbation in the direction of $H_0$ is zero (freeze). In case II fidelity shows qualitatively the same behavior, i.e. a slower decay for perpendicular perturbations for times beyond Heisenberg time. This suggest to define for a general perturbation $V$

$$\text{tr} H_0 V = 0$$

as condition for a fidelity freeze, which is slightly more general than the one proposed in \[22\]. However in case II the freeze is much less pronounced than in case I, indicating that the diagonal elements of $V_\perp$, albeit $\text{tr} V_\perp = \text{tr} V_\perp H_0 = 0$, have some impact on the decay.

A careful look reveals that for times beyond Fermi’s golden rule but smaller than Heisenberg time in case II fidelity decay is slower for a parallel perturbation than for a perpendicular perturbation.

This becomes evident for strong perturbations. In figure 3 fidelity amplitude is plotted for the same ratios of $\lambda_\parallel$ and $\lambda_\perp$ as before but for strong overall perturbation $\lambda = 1.5$. Case I fidelity decay shows monotonous behavior as a function of $\lambda_\parallel / \lambda_\perp$ and fidelity decay is for all times for a perpendicular perturbation. However case II fidelity decay is more complicated. For times smaller than Heisenberg time decay is slowest for a purely parallel perturbation and fastest for a purely perpendicular one. At Heisenberg time a pronounced revival is seen for a purely parallel perturbation. The peak decreases as the share of the perpendicular perturbation increases. Finally for a purely perpendicular perturbation there is a minimum at Heisenberg time and no revival at all.

After Heisenberg time things change. Now decay becomes fastest for a purely parallel perturbation with only a tiny second revival at twice the Heisenberg time. For a purely perpendicular perturbation the freeze behavior comes in and at twice the Heisenberg time a sizable revival occurs, such that just as in case I for long times decay is slowest for a purely perpendicular perturbation. Somewhere between Heisenberg time and twice the Heisenberg time the two curves cross.

To understand this behavior qualitatively, we recall two peculiarities of the GSE: first the spectral rigidity is much higher than for the GUE or the GOE. It has been argued \[23\] that the revival at Heisenberg time is a signature of the high spectral rigidity. More generally high spectral rigidity favors a slow decay. Second the eigenvalues of the GSE are two–fold degenerate (Kramers degeneracy).

Thus a perpendicular perturbation has two effects: first it breaks time reversal invariance and drives the GSE into a GUE. Since the latter has lower spectral rigidity, this has as consequence that the peak at Heisenberg time becomes less and lesser pronounced and for times smaller than Heisenberg time decay is enhanced by the perpendicular perturbation. Second it breaks Kramers degeneracy, thus the number of independent levels and therefore level density and Heisenberg time double. This leads to the pronounced peak at twice the (original) Heisenberg time, A comparison with the plot of fidelity amplitude $f(\sqrt{2}\lambda, \tau/2)$ of a GUE, with a GUE perturbation $\text{A} + \lambda \text{A}$ shows indeed good agreement.

In figure 4 the cross form factor is plotted in both cases for the same five ratios between $\lambda_\parallel$ and $\lambda_\perp$ as before. Qualitatively the behavior is similar to fidelity amplitude. In case I the form factor is smallest for a purely parallel perturbation for all times. In case II before Heisenberg time the form factor is smallest for a purely perpendicular perturbation and largest for a purely parallel one. After Heisenberg time the order is inverted. At Heisenberg time a logarithmic singularity occurs, which is typical for the GSE. For strong perturbations the cross form–factor develops peaks at Heisenberg time and for case II at twice the Heisenberg time (not shown here). It has its cause in the algebraic decay of the cross form–factor at these specific times \[8\]. At all other times it decays exponentially.
In conclusion we presented the analytic formulae for fidelity amplitude and cross–form factor for parametric RMT ensembles, where the time reversal invariance of the unperturbed system is broken by the perturbation. The general perturbation is split into a parallel component, sharing the symmetries of the original Hamiltonian and a perpendicular component which maximally breaks this symmetry.

Both possibilities of TRS breaking, even spin GOE→GUE and odd spin GSE→GUE, were discussed on equal footing. In the first case a strong freeze effect occurs for a purely perpendicular perturbation. It can be explained by the absence of diagonal elements of the perturbation in the eigenbasis of the unperturbed Hamiltonian. In case II long time decay is slowest for a purely perpendicular perturbation. It can be explained as a more general condition for a reduced fidelity decay. This leads us to propose the behavior "freeze". We propose to call it "weak fidelity freeze".

The full Hilbert space is involved in the condition with respect to a random initial state as considered here, to which all states of the Hilbert space contribute. For an arbitrary initial state this condition will in general not suffice to attenuate fidelity decay.

In the differential relation between fidelity and cross–form factor only the parallel perturbation strength enters. The relation might be verified experimentally for instance in a billiard experiment as described in [14]. It might be used to measure fidelity indirectly via spectral correlations.

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Appendix A: Derivation of Eq. (12) based on a universality argument

In this section we demonstrate on the example of the transition GOE → GUE (case I) how the method of Refs. [8, 9] can be extended to the case of symmetry breaking. We introduce new variables

$$\tilde{\lambda}_\parallel = \frac{\lambda_\parallel}{2}, \quad \tilde{H}_0 = H_0 + \frac{\lambda_\parallel}{2} V_\parallel.$$  \hspace{1cm} (A1)

For $\tilde{H}_0$ we allow for a general probability measure in the GOE universality class and denote it by $d\nu(\tilde{H}_0)$, while those of $V_\parallel$ and $V_\perp$ are Gaussian measures as before ($dV_\parallel$ and $dV_\perp$ include the normalization constants). Since the probability measure of $\tilde{H}_0$ is assumed to be general, it should be typical and free from any special constraint besides the matrix symmetry.

Now we define

$$\tilde{\mathcal{F}}_{\alpha\beta,\gamma\delta} = \left( \frac{1}{z_1 - \tilde{H}_0} \right)_{\alpha\beta} \left( \frac{1}{z_2 - \tilde{H}} \right)_{\gamma\delta}$$

$$= \left( \frac{1}{z_1 - \tilde{H}_0 + \lambda_\parallel V_\parallel} \right)_{\alpha\beta}$$

$$\times \left( \frac{1}{z_2 - \tilde{H}_0 - \lambda_\perp V_\perp} \right)_{\gamma\delta}.$$  \hspace{1cm} (A2)

Introducing delta–distributions of matrix arguments we can express $\tilde{\mathcal{F}}_{\alpha\beta,\gamma\delta}$ as

$$\tilde{\mathcal{F}}_{\alpha\beta,\gamma\delta} = \int dH_1dH_2 \delta(H_1 - \tilde{H}_0 + \lambda_\parallel V_\parallel) \mathcal{F}_{\alpha\beta,\gamma\delta}$$

$$\times \delta(H_2^{(R)} - \tilde{H}_0 - \lambda_\perp V_\perp) \delta(H_2^{(I)} - \lambda_\perp V_\perp).$$  \hspace{1cm} (A3)
where $H_1$ is an $N \times N$ real symmetric matrix, $H_2$ is an $N \times N$ hermitian matrix and $H_2^{(R)} = \text{Re} H_2$, $H_2^{(I)} = \text{Im} H_2$. Moreover

$$F_{\alpha\beta,\gamma\delta} = \left(\frac{1}{z_1 - H_1}\right)_{\alpha\beta} \left(\frac{1}{z_2 - H_2}\right)_{\gamma\delta}.$$  \hspace{1cm} (A4)

All three delta–distributions can be Fourier transformed. We find

$$\tilde{F}_{\alpha\beta,\gamma\delta} = \int d\lambda_1 d\lambda_2 d\lambda_3 dH_2 dH_2 e^{2\pi i \text{tr} \lambda_1 (H_1 - H_0 + \lambda_1 V_0)} \times e^{2\pi i \text{tr} \lambda_2 (H_2^{(R)} - H_0 - \lambda_2 V_0)} \times e^{2\pi i \text{tr} \lambda_3 (H_2^{(I)} - \lambda_2 V_0)} F_{\alpha\beta,\gamma\delta}. \hspace{1cm} (A5)$$

Here $\lambda_{1,2,3}$ are matrices which have the same symmetry as their real space counterparts, namely $H_1$, $H_2^{(R)}$ and $H_2^{(I)}$. This means $\lambda_1$ and $\lambda_2$ are $N \times N$ real symmetric matrices and $\lambda_3$ is an $N \times N$ real antisymmetric matrix. The integration domain is the real axis for all independent entries of $\lambda_n$, $n = 1, 2, 3$. The expectation value of $\tilde{F}_{\alpha\beta,\gamma\delta}$ can be written as

$$\langle F_{\alpha\beta,\gamma\delta} \rangle = \int d\nu_1 d\nu_2 d\nu_3 \tilde{F}_{\alpha\beta,\gamma\delta} \times e^{-(1/4)\text{tr} \nu_1^2 + (1/4)\text{tr} \nu_2^2} = \int d\nu_1 d\nu_2 d\nu_3 dH_1 dH_2 F_{\alpha\beta,\gamma\delta} \times e^{-(2\pi \lambda_1)^2} \text{tr} (\lambda_1 - \lambda_2)^2 + (2\pi \lambda_1)^2 \text{tr} (\lambda_3)^2 \times e^{2\pi i \text{tr} \left( \lambda_1 (H_1 - H_0) + \lambda_2 (H_2^{(R)} - H_0) + \lambda_3 H_2^{(I)} \right)}.$$  \hspace{1cm} (A6)

Here the brackets $\langle \ldots \rangle$ do not simply mean the expectation value. Rather $\langle F_{\alpha\beta,\gamma\delta} \rangle$ is defined to be the expectation value of $\tilde{F}_{\alpha\beta,\gamma\delta}$.

Now we introduce the notation

$$\text{tr} \frac{\partial^2}{\partial H_1 \partial H_2^{(R)}} = \sum_{j=1}^N \frac{\partial^2}{\partial (H_1^{(R)})_{jj}} \int \frac{\partial^2}{\partial (H_2^{(R)})_{jj}} + \frac{1}{2} \sum_{j<l}^N \frac{\partial^2}{\partial (H_1^{(R)})_{jl}} \int \frac{\partial^2}{\partial (H_2^{(R)})_{jl}}. \hspace{1cm} (A7)$$

Then it follows from partial integrations that

$$\langle \frac{\partial^2}{\partial H_1 \partial H_2^{(R)}} F_{\alpha\beta,\gamma\delta} \rangle = -(2\pi)^2 \langle \text{tr} (\lambda_1 - \lambda_2)^2 F_{\alpha\beta,\gamma\delta} \rangle$$  \hspace{1cm} (A8)

and

$$\frac{\partial}{\partial (\lambda_2^{(R)})_{\alpha\beta}} \langle F_{\alpha\beta,\gamma\delta} \rangle = -(2\pi)^2 \langle \text{tr} (\lambda_1 - \lambda_2)^2 F_{\alpha\beta,\gamma\delta} \rangle. \hspace{1cm} (A9)$$

Here repeated indices stand for summations from 1 to $N$.

Let us note that a simultaneous shift of $H_1$ and $H_2^{(R)}$ in $\langle \ldots \rangle$ induces a shift of $H_0$. Although such a shift modifies the measure $d\nu(H_0)$, the universality of the spectral correlation function implies that $\langle F_{\alpha\beta,\gamma\delta} \rangle$ is asymptotically invariant in the limit $N \rightarrow \infty$. Therefore we obtain the following estimate

$$\langle \text{tr} \left( \frac{\partial}{\partial H_1} + \frac{\partial}{\partial H_2^{(R)}} \right)^2 F_{\alpha\beta,\gamma\delta} \rangle = -(2\pi)^2 \langle \text{tr} (\lambda_1 + \lambda_2)^2 F_{\alpha\beta,\gamma\delta} \rangle \approx 0.$$  \hspace{1cm} (A10)

From this it follows that

$$\frac{\partial}{\partial (\lambda_2^{(R)})_{\alpha\beta}} \langle F_{\alpha\beta,\gamma\delta} \rangle = \frac{\text{tr}}{\partial H_1 \partial H_2^{(R)}} F_{\alpha\beta,\gamma\delta} \approx -\pi^2 \langle \text{tr} (\lambda_1 + \lambda_2)^2 F_{\alpha\beta,\gamma\delta} \rangle \approx 0.$$  \hspace{1cm} (A11)

In order to show that the estimate $(A10)$ is indeed correct, let us pay attention to Eq. $(A6)$. Proper unfolding of the energy level correlations requires an $O(1)$ scaling of the eigenvalue density of $H_0$. Each element of the perturbation $V_0$ is set to be $O(1)$, because it should equally scale as the mean level spacing. When the eigenvalue density is scaled as $O(1)$, since there are $N$ eigenvalues, each eigenvalue $E_{0j}$ of $H_0$ should typically be $O(N)$. Then the RHS of the identity

$$\text{tr}(\tilde{H}_0)^2 = \sum_{j=1}^N (\tilde{E}_{0j})^2$$  \hspace{1cm} (A12)

becomes $O(N^3)$. In the LHS, on the other hand, we have $O(N^2)$ terms, each of which is the square of an element of $\tilde{H}_0$. Therefore each element of $\tilde{H}_0$ is estimated as $O(N^{1/2})$. Then the main contribution to the integral over the matrix $\tilde{H}_0$ with respect to the measure $d\nu(\tilde{H}_0)$ in equation $(A6)$ comes from a region where the elements of $\lambda_1$ and $\lambda_2$ are of order $O(N^{-1/2})$. Only in that region a rapid oscillation of the exponential factor is avoided.

It can be seen from the Gaussian factor in Eq. $(A6)$ that the elements of $\lambda_1 - \lambda_2$ are scaled as $O(1)$. Because of the identity

$$(\lambda_1 - \lambda_2)^2 = -2(\lambda_1 \lambda_2 + \lambda_2 \lambda_1) + (\lambda_1 + \lambda_2)^2,$$  \hspace{1cm} (A13)

the elements of $(\lambda_1 - \lambda_2)^2$ are approximated by the elements of $-2(\lambda_1 \lambda_2 + \lambda_2 \lambda_1)$. Hence we find an estimate

$$\text{tr} (\lambda_1 - \lambda_2)^2 \approx -4\text{tr}(\lambda_1 \lambda_2),$$  \hspace{1cm} (A14)

which implies Eq. $(A11)$. We notice that this estimate can only be fulfilled when $\text{tr}(\lambda_1 \lambda_2)$ is negative.

On the other hand, we can readily find

$$\frac{\partial}{\partial H_1 \partial H_2^{(R)}} F_{\alpha\beta,\gamma\delta} = \text{tr} \left( \frac{1}{z_1 - H_1} \right)^2 \left( \frac{1}{z_2 - H_2} \right)^2 \hspace{1cm} (A15)$$

and

$$\frac{\partial}{\partial z_1 \partial z_2} F_{\alpha\beta,\gamma\delta} = \frac{\partial^2}{\partial z_1 \partial z_2} \left( \frac{1}{z_1 - H_1} \right)_{\alpha\beta} \left( \frac{1}{z_2 - H_2} \right)_{\beta\alpha} \hspace{1cm} (A16)$$

$$= \text{tr} \left( \frac{1}{z_1 - H_1} \right)^2 \left( \frac{1}{z_2 - H_2} \right)^2.$$
so that
\[
\left\langle \frac{\partial^2}{\partial H_1 \partial H_2^2} \mathcal{F}_{\alpha \alpha, \beta \beta} \right\rangle = \frac{\partial^2}{\partial z_1 \partial z_2} \left( \mathcal{F}_{\alpha \beta, \beta \alpha} \right). \tag{A17}
\]
Comparing (A11) and (A17), we arrive at
\[
\frac{\partial}{\partial (\lambda^2)} \left( \mathcal{F}_{\alpha \alpha, \beta \beta} \right) \approx - \frac{\partial^2}{\partial z_1 \partial z_2} \left( \mathcal{F}_{\alpha \beta, \beta \alpha} \right), \tag{A18}
\]
which gives the required relation (12) between the fidelity and parametric spectral correlation, respectively cross form-factor.

**Appendix B: Derivation of equations (8) and (12)**

from Ref. [13]

In Ref. [13], called THSA in the following, the Fourier-transform of the cross form-factor was derived as a three-fold integral
\[
K(\bar{x}, x_o, x_u, \omega) = \text{Re} \int d\lambda d\lambda_1 d\lambda_2 W e^{F_{\pm}}, \tag{B1}
\]
where the integration domains are in case I defined by \(\lambda \in [-1, 1], \lambda_1 \in [0, \infty], \lambda_2 \in [0, \infty]\) and in case II by \(\lambda \in [0, \infty], \lambda_1 \in [-1, 1], \lambda_2 \in [0, 1]\). Setting the parameter \(\bar{x} = x_u/2\) the expressions for \(F\) and \(W\) (equations (5) and (6) of THSA) are given by
\[
F_{\pm} = \pm \kappa \pi \omega (\lambda_1 \lambda_2 - \lambda) \pm \frac{\pi^2}{2} \left( \lambda^2 + \lambda_2^2 - \lambda^2 - 1 \right)
- \frac{\pi^2}{4} \left( 2\lambda_1^2 \lambda_2^2 - 2\lambda^2 - \lambda_1^2 - \lambda_2^2 + 1 \right) \tag{B2}
\]
\[
W = \frac{(\lambda_1 \lambda_2 - \lambda^2)(1 - \lambda^2)}{(\lambda_1^2 + \lambda_2^2 + \lambda^2 - 2\lambda_1 \lambda_2 - 1)^2} \times \left( 1 + \frac{\pi^2}{4} \frac{\omega^2}{\kappa} (\lambda_1^2 + \lambda_2^2 + \lambda^2 - 2\lambda_1 \lambda_2 - 1) \right) \tag{B3}
\]
Here the plus sign applies to case I and the minus sign to case II. The parameter \(\kappa\) has the value \(\kappa = 1\) (case I) and \(\kappa = 2\) (case II). This factor does not appear in THSA, however it does appear in Ref. [24]. We introduced it, such that \(\tilde{K}(t)\) is related to \(K(\omega)\) in both cases via
\[
\tilde{K}(\tau) = \int d\omega e^{-2\pi i \tau \omega} K(\omega). \tag{B4}
\]
In THSA the function \(W\) differs in case I and case II by a relative minus sign between two summands in the last line of equation (B3). This seems to be wrong. Moreover in the same line the factor \(1/\kappa\) in the second summand is missing in THSA.

Fourier transformation yields \(\delta(\tau - \lambda_1 \lambda_2/2 + \lambda)\) in case I and \(\delta(\tau - \lambda + \lambda_1 \lambda_2)\) in case II, which allows to integrate over \(\lambda\). Equations (8) to (12) are obtained through the transformations
\[
u = \frac{1}{2} \left( \sqrt{\lambda_1^2 \lambda_2^2 - \lambda_1 - \lambda_2 + 1} \right) \quad \text{case I} \quad \text{(B5)}
\nu = \sqrt{\lambda_1^2 \lambda_2^2 - \lambda_1 - \lambda_2 + 1} \quad \text{case II.} \quad \text{(B6)}
\]
The parameters are identified as \(\lambda_\parallel = x_o/2\) and \(\lambda_\perp = x_u/\sqrt{2}\).

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