TWISTED HILBERT SCHEMES AND DIVISION ALGEBRAS

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Abstract. Let \( \mathcal{X}/S \) be any Severi–Brauer scheme of constant relative dimension \( n \) over a Noetherian base scheme \( S \). For each polynomial \( \phi(t) \in \mathbb{Q}[t] \), we construct a scheme \( \text{Hilb}_{\phi(t)}^{\text{tw}}(\mathcal{X}/S) \) that \( \text{étale} \) locally, on a cover \( S'/S \) splitting \( \mathcal{X}/S \), is the Hilbert scheme \( \text{Hilb}_{\phi(t)}(\mathcal{X}_{S'}/S') \) of the projective bundle \( \mathcal{X}_{S'}/S' \).

We then study curves of small degree on a Severi–Brauer variety in order to analyze examples. Our primary interest, in the case \( X \) is a Severi–Brauer variety with index \( n > 1 \) over a field \( k \), is the subscheme \( \text{Ell}_n(X) \) of \( \text{Hilb}_{\phi(t)}^{\text{tw}}(X/k) \) parametrizing curves that are smooth, geometrically connected, and of genus 1.

1. Introduction

This work originated from the idea that one could study deformations of curves on a Severi–Brauer variety to obtain algebraic information on the structure of the associated central simple algebra. Specifically, this work was an attempt to implement the following program:

(Step 1) for each polynomial \( \phi(t) \) of \( \mathbb{Q}[t] \), construct a variant of the Hilbert scheme parametrizing closed subschemes of a Severi–Brauer variety \( X \) with Hilbert polynomial \( \phi(t) \) geometrically;

(Step 2) for a fixed polynomial \( \phi(t) = rt + s \), classify possible curves \( C \subset X \) defined over the ground field with Hilbert polynomial \( \phi(t) \) over a splitting field for \( X \);

(Step 3) using a bend-and-break type argument, deform an irreducible curve \( C \subset X \) with this Hilbert polynomial to a curve \( C^* \subset X \) having the same Hilbert polynomial, and which, additionally, is geometrically a union of rational curves;

(Step 4) study the action of the absolute Galois group on the geometric irreducible components of \( C^* \) to obtain specific restrictions on the possible Galois splitting fields of \( X \).

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In this paper, we complete (Step 1) in much broader generality and we provide some initial analysis in (Step 2) for specific cases pertaining to curves of minimal degree in a Severi–Brauer variety.

To be precise, we prove in Section 2 (culminating in Theorem 2.5) that for any polynomial $\phi(t) \in \mathbb{Q}[t]$, and for any Severi–Brauer scheme $\mathcal{X}/S$ of relative dimension $n$ over a Noetherian base scheme $S$, there exists a scheme $\text{Hilb}^{\text{tw}}_{\phi(t)}(\mathcal{X}/S)$ parametrizing subschemes of $\mathcal{X}$ that are flat and proper over $S$ with Hilbert polynomial $\phi(t)$ over any étale splitting $S'/S$ for $\mathcal{X}/S$. It turns out that, with only minor changes, one can adapt the proof of representability for the usual Hilbert scheme of a projective bundle to this generalized setting of Severi–Brauer schemes.

We then turn, in Section 3, to the study of those Hilbert schemes $\text{Hilb}^{\text{tw}}_{\phi(t)}(X/k)$, for a Severi–Brauer variety $X$ over a fixed field $k$, that are associated to linear polynomials $\phi(t)$, i.e. to those Hilbert schemes parametrizing subschemes consisting of either curves or curves-and-points. In some cases of minimal degree subschemes, we get very precise information (e.g. in Example 3.11 we give a satisfying picture for the components of $\text{Hilb}^{\text{tw}}_{5}(X/k)$, for a Severi–Brauer variety $X$ associated to a degree 5 division algebra, that can possibly have a rational point).

Of particular interest, to the author, is the subscheme $\text{Ell}_n(X)$ of $\text{Hilb}^{\text{tw}}_{n}(X/k)$ parametrizing smooth and geometrically connected curves of genus one in a Severi–Brauer variety $X$ of index $n$. When $X$ contains no twisted linear subvarieties, i.e. when $X$ is associated to a division algebra, we observe (in Theorem 3.16) that $\text{Ell}_n(X)$ is geometrically irreducible of dimension $n^2$. For the program outlined above, we also observe that, when $X$ is associated to a division algebra of prime index $n = p > 2$, the only geometrically reducible curves $C \subset X$ appearing in the same component $\text{Ell}_n(X)$ and defined over the ground field are, geometrically, $p$-gons of lines (Lemma 3.12).

The relevance of this latter observation to the program outlined above was, in the author’s eyes, critical. Indeed, under the assumption that one could carry out (Step 3) of the above program to obtain such a curve $C^*$ defined over the ground field, it would follow that one could deform a smooth curve $C \subset X$ of degree $n$ and of genus 1 (if one exists) to a curve $C^*$ containing a point $x$ of minimal degree and splitting $X$. The Galois closure $E/k(x)$ of the residue field $k(x)$ of $x$ would then have Galois group admitting a quotient that is a transitive subgroup of the automorphism group of the $p$-gon $C^*_X$, i.e. either the cyclic group $\mathbb{Z}/p\mathbb{Z}$ or the full dihedral group $D_p$. With some minimal assumptions, this implies that the underlying division algebra is cyclic (i.e. if $p \leq 7$, etc.)
if the characteristic of $k$ is zero, and if $k$ contains a $p$th root of unity, see [RS96, Theorem 4.2]).

The largest obstacle to implementing the above program is (Step 3). It’s often possible to find a deformation of a given irreducible curve $C \subset X$, over a rational curve defined over the ground field $k$, where one of the fibers is a geometrically reducible curve having the same Hilbert polynomial over an algebraic closure of $k$. However, it’s been challenging, for the author, to show that the geometrically reducible curve one obtains is also defined over the ground field $k$. This depends on how precisely one can control the deformations of $C$.

This idea represents one step towards a converse to an avenue of current research in this area. That is to say, there has been a lot of research done towards determining when a Severi–Brauer variety either contains or, somewhat weaker, is split by a smooth and geometrically connected curve of genus 1: in some cases of small index [dJH12], if one fixes a Severi–Brauer surface and attempts to determine all such curves [Sal21], if one asks instead about splitting Brauer classes [CK12], or if one asks instead about splitting $\mu_n$-gerbes [AA21]. One theme present in a handful of this work is starting from structural assumptions on the underlying central simple algebra (e.g. assuming that it is cyclic) and proceeding from there. Here we’ve attempted (mostly unsuccessfully) to do the opposite.

**Notation.** We use the following notation throughout:

- if $k$ is a base field, then we write $\overline{k}$ to denote a fixed algebraic closure of $k$ and $k^s$ to denote the separable closure of $k$ inside $\overline{k}$

**Conventions.** We use the following conventions throughout:

- a variety is an integral scheme that is separated and of finite type over a base field
- a curve is a proper scheme of pure dimension one that is separated and of finite type over a base field.

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2. **Descent for Hilbert Schemes**

Let $\mathcal{X}/S$ be a Severi–Brauer scheme of relative dimension $n$ over a Noetherian scheme $S$. Concretely, this means there exists an étale cover


\[ S' = \{ S_i \}_{i \in I} \] of \( S \) and isomorphisms \( \mathcal{X}_{S_i} = \mathcal{X} \times_S S_i \cong \mathbb{P}^n_{S_i} \). We call the data consisting of an étale cover \( S' \) and isomorphisms \( \epsilon_i : \mathcal{X}_{S_i} \to \mathbb{P}^n_{S_i} \) a splitting of \( \mathcal{X}/S \). The splitting data \( (S_i, \epsilon_i)_{i \in I} \) of \( \mathcal{X}/S \) determines a Čech 1-cocycle \( \xi \) giving rise to a class in \( \check{H}^1(S, \text{PGL}_{n+1}/S) \).

Conversely, descent shows that every element \( \xi \) of \( \check{H}^1(S, \text{PGL}_{n+1}/S) \) is determined by the splitting data \( (S_i, \epsilon_i)_{i \in I} \), over some étale cover \( S' = \{ S_i \}_{i \in I} \) of \( S \), of some Severi–Brauer scheme \( \mathcal{X}/S \) that is uniquely determined by \( \xi \) up to isomorphism. For each Čech 1-cocycle \( \xi \) one can then choose splitting data \( (S_i, \epsilon_i)_{i \in I} \) and, for a polynomial \( \phi(t) \in \mathbb{Q}[t] \), descend the Hilbert schemes \( \text{Hilb}_{\phi(t)}(\mathbb{P}^n_{S_i}/S_i) \) defined over \( S_i \) to a scheme \( \text{Hilb}^{\text{tw}}_{\phi(t)}(\mathcal{X}/S) \) defined over \( S \).

The scheme \( \text{Hilb}^{\text{tw}}_{\phi(t)}(\mathcal{X}/S) \) represents the functor associating to any locally Noetherian \( S \)-scheme \( T \) the set of all subschemes of \( \mathcal{X}_T = \mathcal{X} \times_S T \) which are flat and proper over \( T \) and which, locally for the étale cover \( S'/S \), have Hilbert polynomial \( \phi(t) \). The goal for this section is to prove the representability of \( \text{Hilb}^{\text{tw}}_{\phi(t)}(\mathcal{X}/S) \), done in Theorem 2.5, by extending the construction of the Hilbert scheme of a projective bundle, e.g. as given in [Kol96], so that it also provides a construction of the scheme \( \text{Hilb}^{\text{tw}}_{\phi(t)}(\mathcal{X}/S) \) for any Severi–Brauer scheme \( \mathcal{X}/S \).

To start, recall from [Qui73, §8.4] that Quillen has constructed a universal vector bundle \( \mathcal{J} \) on the Severi–Brauer scheme \( \mathcal{X}/S \) having the following property: locally for an étale cover \( S'/S \) splitting \( \mathcal{X}/S \), \( \mathcal{J} \) admits isomorphisms

\[
\mathcal{J}|_{S_i} \cong \mathcal{O}_{\mathbb{P}^n_{S_i}}(-1)^{\oplus n+1}
\]

for each \( S_i \in S' \), compatible with the isomorphisms \( \mathcal{X}_{S_i} \cong \mathbb{P}^n_{S_i} \) of the splitting. We write \( \mathcal{Q} = \mathcal{J}^\vee = \text{Hom}(\mathcal{J}, \mathcal{O}_{\mathcal{X}}) \) to denote the dual of \( \mathcal{J} \) and we call \( \mathcal{Q} \) the Quillen bundle on the Severi–Brauer scheme \( \mathcal{X}/S \).

**Lemma 2.1.** Suppose that \( S \) is connected and write \( \pi : \mathcal{X} \to S \) for the structure map of \( \mathcal{X}/S \). Let \( \mathcal{F} \) be an \( S \)-flat coherent sheaf on \( \mathcal{X} \). Then there exists a numerical polynomial \( \phi(t) \in \mathbb{Q}[t] \) and an integer \( N \) so that the following equality holds

\[
\text{rk}(\pi_*(\mathcal{F} \otimes \mathcal{Q}^{\otimes t})) = \phi(t) \cdot \text{rk}(\mathcal{Q}^{\otimes t})
\]

for all integers \( t \geq N \).

**Proof.** Let \( S' = \{ S_i \}_{i \in I} \) be an étale cover splitting \( \mathcal{X}/S \) and write \( \pi_i : \mathcal{X}_{S_i} \to S_i \) for map coming from base change. Then, for all \( t \geq 1 \), there are isomorphisms

\[
\pi_*(\mathcal{F} \otimes \mathcal{Q}^{\otimes t})|_{S_i} \cong \pi_{i*}(\mathcal{F}|_{S_i} \otimes (\mathcal{O}_{\mathbb{P}^n_{S_i}}(1)^{\oplus n+1})^{\otimes t}) \cong \pi_{i*}(\mathcal{F}|_{S_i}(t)^{\oplus (n+1)^t}).
\]
Since $\pi_*(\mathcal{F}|_{S_i}(t))^{\oplus(n+1)^t} \cong \pi_*(\mathcal{F}|_{S_i}(t))^{\oplus(n+1)^t}$, the $\phi(t)$ of the lemma is necessarily the Hilbert polynomial of $\mathcal{F}|_{S_i}$ on $\mathcal{X}_i \cong \mathbb{P}^n_{S_i}$.

\textbf{Definition 2.2.} Let $\mathcal{X}/S$ be a Severi–Brauer scheme over a base $S$. Let $\mathcal{F}$ be an $S$-flat coherent sheaf on $\mathcal{X}$. For each connected component $S_\rho \subset S$ we define the \textit{reduced Hilbert polynomial} of $\mathcal{F}$ on $S_\rho$ to be the numerical polynomial $rh_{\mathcal{F}}(t) \in \mathbb{Q}[t]$ guaranteed to exist by Lemma 2.1. In other words, $rh_{\mathcal{F}}(t)$ is uniquely characterized by the existence of an integer $N \geq 0$ and equality

$$\text{rk}(\pi_*(\mathcal{F} \otimes \mathcal{O}_{\mathcal{X}}^{\otimes t})|_{S_\rho}) = rh_{\mathcal{F}}(t) \cdot \text{rk}(\mathcal{O}_{\mathcal{X}}^{\otimes t}) \quad \text{for all } t \geq N.$$ 

If the reduced Hilbert polynomial of $\mathcal{F}$ on $S_\rho$ is equal to $rh_{\mathcal{F}}(t)$ for all connected components $S_\rho \subset S$, then we call $rh_{\mathcal{F}}(t)$ the \textit{reduced Hilbert polynomial} of $\mathcal{F}$. When $\mathcal{F} = \mathcal{O}_V$ is the structure sheaf of a subscheme $V \subset \mathcal{X}$ we write $rh_V(t)$ instead of $rh_{\mathcal{O}_V}(t)$.

\textbf{Remark 2.3.} If $\mathcal{X}/S$ is a split Severi–Brauer scheme (i.e. if $\mathcal{X}/S$ is isomorphic over $S$ with a projective bundle $\mathbb{P}_S(\mathcal{E})$ for some vector bundle $\mathcal{E}$ on $S$) then, for any $S$-flat coherent sheaf $\mathcal{F}$ on $\mathcal{X}$, the reduced Hilbert polynomial $rh_{\mathcal{F}}(t)$ is just the usual Hilbert polynomial $h_{\mathcal{F}}(t)$ with respect to the line bundle $\mathcal{O}_{\mathcal{P}_S(\mathcal{E})}(1)$.

\textbf{Lemma 2.4.} Let $\mathcal{X}/S$ be a Severi–Brauer scheme over any scheme $S$. Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$. Then for every polynomial $\phi(t) \in \mathbb{Q}[t]$ there is a locally closed subscheme $S_{\phi(t)} \subset S$ with the property:

(f) given a morphism $T \to S$, the pullback $\mathcal{F}_T$ on $\mathcal{X}_T$ is flat over $T$ with reduced Hilbert polynomial $rh_{\mathcal{F}_T}(t) = \phi(t)$ if and only if $T \to S$ factors $T \to S_{\phi(t)} \subset S$.

\textbf{Proof.} The lemma holds étale locally over the base $S$. More precisely, let $S' = \{S_i\}_{i \in I}$ be any étale cover splitting $\mathcal{X}/S$ with $I$ a finite set and let $\epsilon_i : \mathcal{X}_i \to \mathbb{P}^n_{S_i}$ be isomorphisms realizing the splitting. Write $T_i = T \times_S S_i$ and $\mathcal{F}_i$ for the pullback of $\mathcal{F}$ to $\mathcal{X}_T$. Then for each of the indices $i \in I$, there is a locally closed subscheme $S_{i,\phi(t)} \subset S_i$ so that $\mathcal{F}_i$ is flat over $T_i$ with reduced Hilbert polynomial $rh_{\mathcal{F}_i}(t) = \phi(t)$ if and only if $T_i \to S_i$ factors $T_i \to S_{i,\phi(t)} \subset S_i$. Because of Remark 2.3, the reduced Hilbert polynomial $rh_{\mathcal{F}_i}(t)$ is just the Hilbert polynomial of $h_{\epsilon_i,\mathcal{F}_i}(t)$ and this follows from [Kol96, Theorem I.1.6] which ultimately refers to [Mum66, Lecture 8].

To see that the lemma also holds over $S$, we note that it’s possible to descend the $S_{i,\phi(t)}$ to a scheme $S_{\phi(t)} \subset S$ with $S_{\phi(t)} \times_S S_i = S_{i,\phi(t)}$. Indeed, both of the schemes $S_{i,\phi(t)} \times_S S_j$ and $S_{j,\phi(t)} \times_S S_i$ are uniquely characterized as subschemes of $S_i \times_S S_j$ by the given property with respect to the coherent sheaf $\mathcal{F}|_{S_i \times_S S_j} \cong \mathcal{F}|_{S_j \times_S S_i}$ on $\mathcal{X}_{S_i \times_S S_j}$. As it’s
clear that the cocycle condition on any triple product $S_i \times_S S_j \times_S S_k$ is satisfied, it follows that $S_{\phi(t)}$ exists as a scheme over $S$.

It remains to show that $S_{\phi(t)}$ has property (f). Both the flatness of $F_T$ and the computation for the reduced Hilbert polynomial $\text{rh}_F(t)$ can be checked étale locally for the cover $S'/S$. The claim follows then from the construction of $S_\phi$.

For any locally Noetherian $S$-scheme $T$, write $H_{x/S}(T)$ for the set

$$H_{x/S}(T) := \left\{ V \subset x_T \mid V \text{ is proper and flat over } T \right\}. $$

The association of $T$ to $H_{x/S}(T)$ defines a contravariant functor from the category of locally Noetherian $S$-schemes to the category of sets.

For a morphism $\rho: T' \to T$, the associated map $H_{x/S}(T) \to H_{x/S}(T')$ sends a subscheme $V \subset x_T$ to $V \times_T T' \subset x_{T'}$, where the fiber product is taken along the morphism $\rho$.

**Theorem 2.5.** Let $x/S$ be a Severi–Brauer scheme over a Noetherian base scheme $S$. Then, for every polynomial $\phi(t) \in \mathbb{Q}[t]$, there exists an $S$-scheme $\text{Hilb}^{\text{tw}}_{\phi(t)}(x/S)$ which represents the functor $H_{x/S}^{\phi(t)}$ from (1).

In particular, there is a subscheme

$$\text{Univ}^{\text{tw}}_{\phi(t)}(x/S) \subset x \times_S \text{Hilb}^{\text{tw}}_{\phi(t)}(x/S)$$

and, for any locally Noetherian $S$-scheme $T$, there is an equality

$$\text{Hom}_S(T, \text{Hilb}^{\text{tw}}_{\phi(t)}(x/S)) = H_{x/S}^{\phi(t)}(T)$$

where a map $f: T \to \text{Hilb}^{\text{tw}}_{\phi(t)}(x/S)$ corresponds to the subscheme

$$V \cong \text{Univ}^{\text{tw}}_{\phi(t)}(x/S) \times_{x \times_S \text{Hilb}^{\text{tw}}_{\phi(t)}(x/S)} T.$$ 

**Proof.** The proof we give here is, in essence, the same as [Kol96, Proof of Theorem I.1.4]. We’re going to break the proof into several steps. First, we construct an $S$-scheme $H$ together with a scheme $U \subset x \times_S H$ which end up being $\text{Hilb}^{\text{tw}}_{\phi(t)}(x/S)$ and $\text{Univ}^{\text{tw}}_{\phi(t)}(x/S)$ respectively. Once we’ve constructed $H$ there will be an obvious functorial map

$$\text{Hom}_S(T, H) \to H_{x/S}^{\phi(t)}(T)$$

defined as in the theorem statement. The next step will be to construct a map in the other direction

$$H_{x/S}^{\phi(t)}(T) \to \text{Hom}_S(T, H).$$

The proof will be complete once we show that these two maps are mutually inverse.
Throughout the proof, we’ll refer to the following diagram.

\[
\begin{array}{c}
\mathcal{X} \times_S T \xrightarrow{\tilde{\rho}} \mathcal{X} \times_S \mathcal{Y} \xrightarrow{p_1} \mathcal{X} \\
\downarrow \pi_T \quad \downarrow p_2 \\
T \xrightarrow{\tilde{\rho}} \mathcal{Y} \xrightarrow{\sigma} S
\end{array}
\]

For the first part of the proof, we use the following notation:

- \( \pi : \mathcal{X} \rightarrow S \) is the \( S \)-structure map of the Severi–Brauer scheme \( \mathcal{X}/S \) of relative dimension \( n \),
- \( \phi(t) \) is a fixed polynomial from \( \mathbb{Q}[t] \) and \( N > 0 \) is an integer (chosen to be divisible by \( n+1 \)) so that \( h^i(V, \mathcal{O}_V(N)) = 0 \) for any subscheme \( V \subset \mathbb{P}^n \) with Hilbert polynomial \( \phi(t) \) [Kol96, Theorem I.1.5],
- \( \mathcal{Y} = \text{Gr}_S(\phi(N), \pi_* \mathcal{L}) \) is the Grassmannian \( S \)-bundle of rank \( \phi(N) \) quotient bundles of the locally free \( \pi_* \mathcal{L} \), where \( \mathcal{L} = (\text{det} \mathcal{Q})^{\otimes N/(n+1)} \) is the given tensor power of the determinant of the Quillen bundle, with \( S \)-structure map \( \sigma : \mathcal{Y} \rightarrow S \),
- and \( p_1, p_2 : \mathcal{X} \times_S \mathcal{Y} \rightarrow \mathcal{X}, \mathcal{Y} \) are the first and second projections from the fiber product.

On \( \mathcal{Y} \) there is a short exact sequence

\[
0 \rightarrow \mathcal{U} \rightarrow \sigma^* \pi_* \mathcal{L} \rightarrow \mathcal{V} \rightarrow 0
\]

with \( \mathcal{V} \) the universal quotient bundle of rank \( \phi(N) \) and \( \mathcal{U} \) the universal subbundle. Pulling back to \( \mathcal{X} \times_S \mathcal{Y} \) we get a map

\[
p_2^* \mathcal{U} \rightarrow p_2^* \sigma^* \pi_* \mathcal{L} = p_1^* \pi^* \pi_* \mathcal{L} \rightarrow p_1^* \mathcal{L}
\]

by composing with the \((\pi^*, \pi_*^*)\)-adjunction map. Let \( \mathcal{C} \) be the cokernel of this composition. The projection \( p_2 : \mathcal{X} \times_S \mathcal{Y} \rightarrow \mathcal{Y} \) realizes \( \mathcal{X} \times_S \mathcal{Y} \) as a Severi–Brauer scheme over \( \mathcal{Y} \) so that we can apply Lemma 2.4 to the sheaf \( \mathcal{C} \otimes (p_1^* \mathcal{L}') \). In this way, we get a subscheme \( H \subset \mathcal{Y} \) fitting into a Cartesian diagram

\[
\begin{array}{c}
\mathcal{X} \times_S H \xrightarrow{\mathcal{X} \times_S \mathcal{Y}} \mathcal{X} \times_S \mathcal{Y} \xrightarrow{\mathcal{X} \times_S H} \mathcal{X} \times_S \mathcal{Y} \\
\downarrow p_2 \quad \downarrow i \\
H \xrightarrow{i} \mathcal{Y}
\end{array}
\]
so that $C' = i'^*(C \otimes p_1^* L')$ is flat over $H$ with reduced Hilbert polynomial $rh_{C'}(t) = \phi(t)$. Further, since $C'$ is a quotient of $O_{\mathcal{X} \times S \overline{H}}$, the sheaf $C'$ defines a closed subscheme $U \subset \mathcal{X} \times_S H$. From the construction of $H$, any morphism $f : T \to H$ of $S$-schemes produces an element of $H^0(\mathcal{X}/S)(T)$ by pulling back $U \subset \mathcal{X} \times_S H$ along the induced $f_{\mathcal{X}} : \mathcal{X} \times_S T \to \mathcal{X} \times_S H$. This is the definition of (2).

Conversely, from any subscheme $V \subset \mathcal{X} \times_S T$ flat over $T$ with reduced Hilbert polynomial $rh_V(t) = \phi(t)$ we can identify a morphism $f : T \to H$ as follows. For this part of the proof, we use additionally:

- $\rho_{\mathcal{X}} : \mathcal{X} \times_S T \to \mathcal{X}$ is the first projection from the fiber product $\mathcal{X} \times_S T$ taken with respect to $\rho$,
- and $\pi_T : \mathcal{X} \times_S T \to T$ is the second projection.

Tensoring the ideal sheaf sequence for $V$ with $\rho_{\mathcal{X}}^* \mathcal{L}$ and pushing forward along $\pi_T$ gives the exact sequence

\begin{equation}
0 \to \pi_T^*(I_V \otimes \rho_{\mathcal{X}}^* \mathcal{L}) \to \pi_T^* \rho_{\mathcal{X}}^* \mathcal{L} \to \pi_T^*(O_V \otimes \rho_{\mathcal{X}}^* \mathcal{L}).
\end{equation}

The rightmost arrow of this sequence is surjective since the coherent sheaf $R^1 \pi_T^*(I_V \otimes \rho_{\mathcal{X}}^* \mathcal{L}) = 0$ vanishes; this can be checked after splitting $\mathcal{X}/S$ and using our choice of $N$, cf. [Kol96, Theorem I.1.5]. Moreover, each of the terms in (5) is locally free by [Har77, Theorem III.12.11] and $\text{rk}(\pi_T^*(O_V \otimes \rho_{\mathcal{X}}^* \mathcal{L})) = \phi(N)$.

The base change map

$$\rho^* \pi_* \mathcal{L} \to \pi_T^* \rho_{\mathcal{X}}^* \mathcal{L}$$

is an isomorphism since it is after an étale (fppf) extension of $S$ by [Nit05, Lemma 5.4]. Hence the surjection $\psi : \rho^* \pi_* \mathcal{L} \to \pi_T^*(O_V \otimes \rho_{\mathcal{X}}^* \mathcal{L})$ defines a map

$$\tilde{\rho} : T \to \mathcal{Y} \quad \text{with} \quad \tilde{\rho}^*(\sigma^* \pi_* \mathcal{L} \to \mathcal{V}) = \psi$$

by the functorial description of $\mathcal{Y}$. If we let $\tilde{\rho}_{\mathcal{X}} : \mathcal{X} \times_S T \to \mathcal{X} \times_S \mathcal{Y}$ denote the map obtained by base change, then we find that

$$\tilde{\rho}_{\mathcal{X}}^* \mathcal{L} = \tilde{\rho}_{\mathcal{X}} \text{coker} \left( p_2^* \mathcal{U} \to p_2^* \sigma^* \pi_* \mathcal{L} = p_1^* \pi^* \pi_* \mathcal{L} \to p_1^* \mathcal{L} \right)$$

$$= \text{coker} \left( \pi_T^* \pi_T^*(I_V \otimes \rho_{\mathcal{X}}^* \mathcal{L}) \to \pi_T^* \rho_{\mathcal{X}}^* \rho_{\mathcal{X}}^* \pi_* \mathcal{L} = \rho_{\mathcal{X}}^* \rho_{\mathcal{X}}^* \pi_* \mathcal{L} \to \rho_{\mathcal{X}}^* \mathcal{L} \right).$$

The composition factors through the adjunction

$$\pi_T^* \pi_T^*(I_V \otimes \rho_{\mathcal{X}}^* \mathcal{L}) \to I_V \otimes \rho_{\mathcal{X}}^* \mathcal{L},$$

which induces the isomorphism $\rho_{\mathcal{X}}^* (C \otimes p_1^* \mathcal{L}') \cong \rho_{\mathcal{X}}^* C \otimes \rho_{\mathcal{X}}^* \mathcal{L} \cong O_V$. Since $V$ is flat over $T$ with reduced Hilbert polynomial $rh_V(t) = \phi(t)$, this implies that $\rho = i \circ f$ factors via a morphism $f : T \to H$ since $H$ satisfies property (f). The association sending $V$ to $f$ defines the map...
in (3). With a moment’s thought (and also noting that the factorization above is unique by [Sta19, Tag 01L7]), it’s clear the maps (2) and (3) are mutually inverse. This completes the proof.

**Definition 2.6.** We’ll call \(\text{Hilb}_{\phi(t)}^{\text{tw}}(\mathcal{X}/S)\) the Hilbert scheme of \(\mathcal{X}/S\) that parameterizes subschemes with reduced Hilbert polynomial \(\phi(t)\). The superscript \(\text{tw}\) is a reminder that this is a twist of one of the usual Hilbert schemes of a projective bundle as the next remark notes.

**Remark 2.7.** If \(\mathcal{X}/S\) is split, i.e. if \(\mathcal{X}/S\) is a projective bundle \(\mathbb{P}^n_S(\mathcal{E})\) for some vector bundle \(\mathcal{E}\) on \(S\), then the above theorem recovers the usual Hilbert scheme \(\text{Hilb}_{\phi(t)}(\mathcal{X}/S)\). This also shows the following statement: if \(\mathcal{X}/S\) is any Severi–Brauer scheme over a Noetherian base scheme \(S\), and if \(S'/S\) is an étale cover splitting \(\mathcal{X}/S\), then there are splitting isomorphisms

\[
\text{Hilb}_{\phi(t)}^{\text{tw}}(\mathcal{X}/S) \times_S S' \cong \text{Hilb}_{\phi(t)}(\mathcal{X}'/S')
\]

as claimed in the beginning of this section. Consequently, the scheme \(\text{Hilb}_{\phi(t)}^{\text{tw}}(\mathcal{X}/S)\) inherits any property of \(\text{Hilb}_{\phi(t)}(\mathcal{X}'/S')\) that is étale local on the base, e.g. properness, finite-typeness over \(S\), smoothness if it holds over \(S'\).

The infinitesimal theory of \(\text{Hilb}_{\phi(t)}^{\text{tw}}(\mathcal{X}/S)\) can also be checked on an étale cover of the base, so we get the following corollary using the fact that the scheme \(\text{Hilb}_{\phi(t)}^{\text{tw}}(\mathcal{X}/S)\) is étale locally, e.g. on a cover \(S'/S\) splitting \(\mathcal{X}/S\), isomorphic to \(\text{Hilb}_{\phi(t)}(\mathbb{P}^n_{S'}/S')\).

**Corollary 2.8.** Let \(\mathcal{X}/S\) be a Severi–Brauer scheme over \(S\). Let \(s \in S\) be a point, let \(F\) be a field, and let \(p : \text{Spec}(F) \to s\) be a morphism. Let \(V \subset \mathcal{X}_F\) be a subscheme with ideal sheaf \(\mathcal{I}_V\) and reduced Hilbert polynomial \(\text{rh}_V(t) = \phi(t)\). Then the following are true:

- (1) The Zariski tangent space of \(\text{Hilb}_{\phi(t)}^{\text{tw}}(\mathcal{X}_F/F)\) at the \(F\)-point given by \(V\) via Theorem 2.5 is naturally isomorphic to

\[
\text{Hom}_{\mathcal{O}_{\mathcal{X}_F}}(\mathcal{I}_V, \mathcal{O}_V) = \text{Hom}_{\mathcal{O}_S}(\mathcal{I}_V/\mathcal{I}_V^2, \mathcal{O}_V).
\]

- (2) The dimension of every irreducible component of \(\text{Hilb}_{\phi(t)}^{\text{tw}}(\mathcal{X}_F/F)\) at the \(F\)-point defined by \(V\) is at least

\[
\dim_F \text{Hom}_{\mathcal{O}_{\mathcal{X}_F}}(\mathcal{I}_V, \mathcal{O}_V) - \dim_F \text{Ext}^1_{\mathcal{O}_{\mathcal{X}_F}}(\mathcal{I}_V, \mathcal{O}_V) + \dim_s S.
\]

- (3) If \(V \subset \mathcal{X}_F\) is (étale) locally unobstructed, then the dimension of every irreducible component of \(\text{Hilb}_{\phi(t)}^{\text{tw}}(\mathcal{X}/S)\) at any point in the image of the point defined by \(V\) is at least

\[
\dim_F \text{Hom}_{\mathcal{O}_V}(\mathcal{I}_V/\mathcal{I}_V^2, \mathcal{O}_V) - \dim_F H^1(V, \mathcal{Hom}(\mathcal{I}_V/\mathcal{I}_V^2, \mathcal{O}_V)) + \dim_s S.
\]
Moreover, in either of the cases (2) or (3) above, if the lower bound given for the dimension is equal to the dimension of every irreducible component of $\text{Hilb}_{\phi(t)}^{tw}(\mathcal{X}/S)$ at the point defined by $V$, then the map

$$\text{Hilb}_{\phi(t)}^{tw}(\mathcal{X}/S) \to S$$

is a local complete intersection morphism at that point.

**Proof.** This is a combination of [Kol96, Theorems I.2.10 and I.2.15]. See [Kol96, Definition I.2.11] for the definition of locally unobstructed subschemes. □

### 3. Classifying subschemes

From now on, we work in the following setting: we fix a base field $k$, a $k$-central simple $k$-algebra $A$, and we let $X = \text{SB}(A)$ be the associated Severi–Brauer variety of $A$. We use the triple $(d, n, m)$ to refer to the degree, index, and exponent of $A$ respectively, i.e.

$$d = \deg(A), \quad n = \text{ind}(A), \quad m = \exp(A).$$

In this section, we analyze the subschemes of $X$ corresponding to points in the Hilbert scheme $\text{Hilb}_m^{tw}(X/k)$. We assume throughout this section that $d > 2$ so that $X$ is not a curve itself.

**Lemma 3.1.** Let $C \subset X$ be a curve. Let $p$ be a prime number. Then the degree $\deg(C)$ satisfies the following

$$v_p(\deg(C)) \geq \begin{cases} v_p(n) & \text{if } p \text{ is odd} \\ v_p(n) - 1 & \text{if } p = 2. \end{cases}$$

In other words, the integer $n$ divides $\deg(C)$ if $n$ is odd and the integer $n/2$ divides $\deg(C)$ if $n$ is even.

**Proof.** Let $F$ be any splitting field for $X$. The degree $\deg(C)$ is defined as the unique integer so that there is an equality $[C_F] = \deg(C)[L]$ inside the Chow group $\text{CH}_1(X_F)$ where $L \subset X_F \cong \mathbb{P}_F^{d-1}$ is any line. The degree $\deg(C)$ is independent of the choice of splitting field $F$.

To check the given divisibility relations, it suffices to work $p$-locally in the group $\text{CH}_1(X_F) \otimes \mathbb{Z}_{(p)}$. By a corestriction-restriction argument, it therefore suffices to assume $n = p^r$ is a prime power of $p$. The case that $p$ is odd is [Mac21, Theorem 4.7]. So we can assume that $p = 2$.

By first replacing $k$ with some (possibly large) field extension that doesn’t change the index of $A$, we can assume that $m = \exp(A) = 2$. Let $e \in \text{CH}^1(X)$ be the class of a divisor that has degree $m = \exp(A)$ over any splitting field for $X$. Let $p$ be any closed point of $X$ with $[k(p) : k] = n = 2^r$ and set $F = k(p)$. Let $q \in X_F$ be any $F$-rational
point inside $p_F$. Then since $\text{CH}_0(X) = \mathbb{Z}$ is generated by the class of the point $p$, we find (by restricting to $F$ the relation $e \cdot [C] = a[p]$ that holds over $k$ for some integer $a \geq 1$) that

$$e_F \cdot [C_F] = 2\text{deg}(C)[q] = r[p_F] = a2^r[q].$$

In other words, we find $2^{r-1}$ divides the degree $\text{deg}(C)$. \hfill \Box

**Remark 3.2.** In fact, when the index $n$ is a power of 2, it follows that for any field $F$ splitting $X$, the image of $\text{CH}_1(X)$ inside $\text{CH}_1(X_F) = \mathbb{Z}$ by restriction is exactly $(n/2)\mathbb{Z}$. Generators of the image are exactly the restrictions of the classes $c_{n-2}(\zeta_X(1))c_n(\zeta_X(1))^{(d/n)-1}$ and $c_1(\zeta_X(m))^{d-2}$ constructed in [KM19, Appendix A].

**Example 3.3.** In this example, we construct some curves in Severi–Brauer varieties associated to central simple $k$-algebras of index $n = 2^r$ with 2-adic valuation of the degree strictly smaller than $r$.

For an example of a curve $C$ with minimal possible degree in $X$, let $A = Q_1 \otimes Q_2$ be a biquaternion algebra of index 4 split by a biquadratic extension $F = k(\sqrt{a}, \sqrt{b})$ with Galois group $\text{Gal}(F/k) = (\mathbb{Z}/2\mathbb{Z})^\oplus 2$. Let $p$ be a point on $X$ with residue field $k(p) = F$ and identify the points in $p_F$ with elements of $\text{Gal}(F/k)$ so that the action of $G$ on $p_F$ is the canonical one. In $X_F \cong \mathbb{P}^3_F$, let $L_1$ be the line passing through the points $(0,0)$ and $(0,1)$ and let $L_2$ be the line passing through the points $(1,0)$ and $(1,1)$. Then $L_1 \cup L_2$ forms a Galois orbit, hence it descends to the ground field $k$ to give a curve $C \subset X$ with $\text{deg}(C) = 2$.

If the division algebra $A = Q_1 \otimes \cdots \otimes Q_r$ is an $r$-fold ($r > 2$) product of distinct quaternion algebras split by a multi-quadratic extension $F/k$ with $\text{Gal}(F/k) \cong (\mathbb{Z}/2\mathbb{Z})^\oplus r$, then there is a point $p$ on $X$ with residue field $F$. One can pass a Galois orbit of lines through $p_F$ which is essentially an $r$-dimensional cube $I_r = [0,1]^r$ with lines replacing the edges of the cube. The curve $C \subset X$ that one gets from this cube has degree $\text{deg}(C) = r2^{r-1}$ (equal to the number of edges of the $r$-cube). If $r = 3$ then this curve has arithmetic genus $h^1(C, \mathcal{O}_C) = 2^{3-2}(3) - 1$ (equal to the number of faces of the 3-cube minus 1).

In general, any division algebra $A$ of index $n = 2^r$ is split by a separable field extension $F/k$ of degree $[F : k] = 2^r$. For this field $F$, there is a point $p$ on $X$ with residue field $k(p) = F$ and $p_{k^s}$ contains $2^r$ points over a separable closure $k^s$ of $k$. Passing a line between every pair of points in $p_{k^s}$ gives a $\text{Gal}(k^s/k)$ orbit that descends to a curve $C \subset X$. The degree of $C$ is equal to the number of edges in the complete graph $K_n$, i.e. $\text{deg}(C) = \binom{n}{2} = 2^{r-1}(2^r - 1)$. 


Lemma 3.4. Assume that $A$ is a division $k$-algebra, i.e. assume $n = d$. Let $V \subset X$ be any subscheme of $X$ containing an irreducible component $C$ of dimension $\dim(C) \geq 1$. Then $V$ is geometrically nondegenerate.

Furthermore, if $C \subset X$ is any geometrically integral curve in $X$ with degree $\deg(C) = p$ for some integer $p \geq 1$, then the geometric genus of $C$ is bounded above by

$$g_{geom}(C) \leq (d - 2)\frac{q(q - 1)}{2} + qr$$

where $q,r$ are the quotient and remainder of dividing $p - 1$ by $d - 2$, i.e. where $p - 1 = q(d - 2) + r$ and $0 \leq r < d - 2$.

Proof. Let $\overline{k}$ be an algebraic closure of $k$ and $k^s$ a separable closure of $k$ inside $\overline{k}$. To see that $V_{\overline{k}}$ is nondegenerate in $X_{\overline{k}} = \mathbb{P}_k^{d-1}$, it suffices to show that $C_{\overline{k}}$ is contained in no hyperplane.

Assume to the contrary that there is a hyperplane $H'$ containing $C_{\overline{k}}$, and let $H$ be the unique hyperplane inside $X_{k^s}$ with $H \times_k k^s = H'$. Let $\alpha$ be a 1-cocycle of $G = \text{Gal}(k^s/k)$ representing $X$ in $H^1(G, \text{PGL}_n(k^s))$. Then $C_{k^s}$ is contained in each of the (finitely many) Galois orbits of $H$ coming from $\alpha$. In particular, we get an inclusion

$$C_{k^s} \subset \bigcap_{g \in G} gH$$

with the right hand side a Galois invariant linear subspace of $X_{k^s}$. Since this subspace would necessarily descend to the field $k$, this contradicts the fact that $A$ was assumed to be a division $k$-algebra (which implies that $X$ has no twisted linear subvarieties).

Now suppose that $C$ is a geometrically integral curve in $X$. To bound the geometric genus of $C$, it suffices to work over the algebraic closure. In particular, we can assume that $C$ is an integral curve nondegenerate inside of $\mathbb{P}^{d-1}$. Then the claim is just Castelnuovo’s bound [Har81]. □

Remark 3.5. The above proof even shows the more general statement that, if $A$ is any central simple $k$-algebra and if $V \subset X$ is any subscheme of $X$ containing an irreducible component $C$ of dimension $\dim(C) \geq 1$, then $V_{\overline{k}}$ is set-theoretically contained in a hyperplane of $X_{\overline{k}}$ if and only if $V$ is set-theoretically contained in a twisted linear subvariety $Y \subset X$.

In particular, if $A$ is a division $k$-algebra, then both $V_{\overline{k}}$ and $(V_{\overline{k}})_\text{red}$ are geometrically nondegenerate.

Remark 3.6. Assume that $A$ is a $k$-division algebra so that $d = n$. Then for any geometrically integral curve $C \subset X$ with $\deg(C) \leq n$, we find $g_{geom}(C) \leq 1$ by applying Lemma 3.4 (and if $\deg(C) < n$ then also $g_{geom}(C) < 1$).
Suppose $C \subset X$ as above has both $\deg(C) \leq n$ and $g_{\text{geom}}(C) = 0$. Then, in this case, since $C$ is geometrically integral, the normalization $C''$ of $C$ is smooth, geometrically connected, and geometrically rational. So there is a point $p$ on $C''$ with $[k(p) : k] = 2$. As $C''$ maps to $X$, this can only happen if $d = 2$.

It follows that any smooth and geometrically connected curve $C \subset X$ with $\deg(C) \leq n$ has genus $g_{\text{geom}}(C) = h^1(C, \mathcal{O}_C) = 1$ and $\deg(C) = n$.

Typically the arithmetic genus $h^1(C, \mathcal{O}_C)$ gives more information about a curve $C$ and its embedding $C \subset X$. The next two lemmas give technical tools that can allow one to determine the possible values for the arithmetic genera of curves in $X$ in some cases.

**Lemma 3.7.** Let $p$ be a prime number. Suppose that $V \subset X$ is any subscheme whose irreducible components $P_i \subset V$ have $\dim(P_i) \leq 1$. Then we have

$$v_p(h^0(V, \mathcal{O}_V) - h^1(V, \mathcal{O}_V)) \geq \begin{cases} v_p(n) & \text{if } p \text{ is odd} \\ v_p(n) - 1 & \text{if } p = 2. \end{cases}$$

In other words, the integer $n$ divides $\chi(V, \mathcal{O}_V)$ if $n$ is odd and the integer $n/2$ divides $\chi(V, \mathcal{O}_V)$ if $n$ is even.

**Proof.** It follows from flat base change [Sta19, Tag 02KH] that this can be checked geometrically. This fits into a commutative diagram

$$
\begin{array}{ccc}
K(X_{\overline{k}}) & \longrightarrow & \mathbb{Z} \\
\uparrow & & \uparrow \\
K(X) & \longrightarrow & \mathbb{Z}
\end{array}
$$

where the horizontal arrows are pushforwards along the structure map (with the identification $K(\text{Spec}(k)) = \mathbb{Z} = K(\text{Spec}(\overline{k}))$ of Grothendieck rings) and the vertical arrows are induced by extension of scalars. The class $[\mathcal{O}_V]$ in $K(X)$ sits in the topological filtration $\tau_1(X) \subset K(X)$ generated by coherent sheaves supported in dimension 1 or less.

The image of $\tau_1(X)$ under the left vertical map is given in [Mac21, Theorem 4.7] under the assumptions that $p$ is odd and $n = p^r$. In that particular case, $\tau_1(X) = p^r \tau_1(X_{\overline{k}})$ with generators $p^r[\mathcal{O}_{P_1}]$ and $p^r[\mathcal{O}_q]$ for the class of a $\overline{k}$-rational point $q \in X_{\overline{k}}$. The horizontal arrows in the diagram above take the class of a coherent sheaf $[\mathcal{F}]$ to $\chi(X, \mathcal{F})$. Writing $[\mathcal{O}_V] = ap^r[\mathcal{O}_{P_1}] + bp^r[\mathcal{O}_q]$ it follows that

$$p^r(a + b) = \chi(X, \mathcal{O}_V) = \chi(V, \mathcal{O}_V) = h^0(V, \mathcal{O}_V) - h^1(V, \mathcal{O}_V).$$
Since it suffices to check the claim with coefficients in $\mathbb{Z}_{(p)}$, the case that $p$ is odd follows from the particular case above by a restriction-corestriction argument. The case $p = 2$ follows by a similar argument using Lemma 3.1 to show $\tau_1(X) \subset 2^{r-1}\tau_1(X_{\tau})$ when $n = 2^r$. □

**Lemma 3.8.** Suppose that $V \subset X$ is any subscheme whose irreducible components $P_i \subset V$ have $\dim(P_i) \leq 1$. Assume that $\text{rh}_V(t) = rt + s$ for some integers $r, s$ with $r \geq 1$. Then $h^1(V, \mathcal{O}_V) \leq \frac{1}{2}(r^2 - 3r) + h^0(V, \mathcal{O}_V)$.

**Proof.** This is a bit overkill but, since we have

$$\text{Hilb}^{tw}_{rt+s}(X/k) \times_k \overline{k} \cong \text{Hilb}_{rt+s}(\mathbb{P}^{d-1}_k/\overline{k})$$

it’s enough to show that the right hand side is empty whenever there is an inequality $h^1(V, \mathcal{O}_V) > \frac{1}{2}(r^2 - 3r) + h^0(V, \mathcal{O}_V)$. This is proved in [Har66, Corollary 5.7]. More specifically, Hartshorne shows there that $\text{Hilb}_{rt+s}(\mathbb{P}^{d-1}_k/\overline{k})$ is nonempty if and only if one has $m_0 \geq m_1 \geq 0$ when $rt + s$ is written as

$$rt + s = \binom{t}{1} - \binom{t - m_0}{1} + \binom{t + 1}{2} - \binom{t + 1 - m_1}{2}$$

$$= m_0 + m_1 t + \frac{1}{2}(m_1 - m_1^2).$$

Comparing coefficients in the above gives

$$r = m_1 \quad \text{and} \quad s = \chi(V_{\overline{k}}, \mathcal{O}_{V_{\overline{k}}}) = \chi(V, \mathcal{O}_V) = m_0 + \frac{1}{2}(m_1 - m_1^2).$$

Equivalently, since $\chi(V, \mathcal{O}_V) = h^0(V, \mathcal{O}_V) - h^1(V, \mathcal{O}_V)$ this implies

$$h^0(V, \mathcal{O}_V) - s = h^1(V, \mathcal{O}_V) = \frac{1}{2}(m_1^2 - m_1) - m_0 + h^0(V, \mathcal{O}_V).$$

Now $\text{Hilb}_{rt+s}(\mathbb{P}^{d-1}_k/\overline{k})$ is nonempty if and only if $m_0 \geq m_1 \geq 0$ if and only if $r \geq 0$ and $0 \leq h^1(V, \mathcal{O}_V) \leq \frac{1}{2}(r^2 - 3r) + h^0(V, \mathcal{O}_V)$. □

**Proposition 3.9.** Suppose that $A$ is a division $k$-algebra with index $n$.

Let $V \subset X$ be any subscheme with $\text{rh}_V(t) = f(n)t + s$ where $f(n) = n$ if $n$ is odd and $f(n) = n/2$ if $n$ is even. Then the following are true.

1. There is a unique irreducible component $C \subset V$ with $\dim(C) = 1$. Moreover, $C$ is generically reduced and $\deg(C) = f(n)$.
2. If the index $n = p$ is prime, then the curve $C \subset V$ is geometrically generically reduced.
3. If $s = 0$ and if the index $n = p$ is prime, then the curve $C \subset V$ is also geometrically connected.
Proof. As $\text{rh}_V(t) = f(n)t + s$ has degree $\deg(\text{rh}_V(t)) = 1$, and since $\text{rh}_V(t)$ is geometrically the Hilbert polynomial of $V_{\overline{k}}$, the dimension of any irreducible component $P_i$ of $V$ satisfies $\dim(P_i) \leq 1$. If there were multiple components $P_i$ of $\dim(P_i) = 1$, then it would follow that $\deg(P_i) < f(n)$ which is impossible by Lemma 3.1.

Similarly, if the unique irreducible component $C \subset V$ of dimension $\dim(C) = 1$ had length $\text{length}_{\overline{k}(C)}\mathcal{O}_{C, \eta} > 1$ at the generic point $\eta$ of $C$, then we would find $\deg(\text{red}(C)) < \deg(C)$ which also contradicts Lemma 3.1. Hence the curve $C \subset V$ is also generically reduced, proving (1).

To prove (2), i.e. to show that $C$ is geometrically generically reduced when $n = p$ is prime, we can assume $n = p > 2$. If $C$ is geometrically irreducible, then geometrically we have

$$\deg(C) = m_{\eta} \deg(C_{\overline{k}}) \quad \text{with} \quad m_{\eta} = \text{length}_{\overline{k}(C)} \mathcal{O}_{C, \eta},$$

where $\eta$ is the generic point of $C_{\overline{k}}$. If $m_{\eta} = p$, then $(C_{\overline{k}})_{\text{red}}$ is a line in $X_{\overline{k}} \cong \mathbb{P}_{\overline{k}}^{m-1}$, hence topologically degenerate, contradicting Remark 3.5. So we must have $m_{\eta} = 1$ in which case $C_{\overline{k}}$ is generically reduced.

On the other hand, if $C$ is geometrically reducible, then the Galois group $G = \text{Gal}(k^s/k)$ acts transitively on the irreducible components of $C_{\overline{k}}$ and all of these irreducible components have the same degree. Since there are at least two irreducible components of $C_{\overline{k}}$, there must be exactly $p$, say $C_1, \ldots, C_p$, each with degree $\deg(C_i) = 1$. Hence $C$ is also geometrically generically reduced in this case.

Suppose now that $s = 0$ and $n = p > 2$ is a prime number. For (3), we suppose that the unique curve $C \subset V$ is geometrically disconnected and aim for a contradiction. Since the degree of $C$ is $\deg(C) = p$, there are exactly $p$ connected components $C_1, \ldots, C_p$ of $C_{\overline{k}}$ with $\deg(C_i) = 1$. Hence $(C_i)_{\text{red}} \cong \mathbb{P}_{\overline{k}}^{1}$. Considering the ideal sheaf sequence

$$0 \to \mathcal{N} \to \mathcal{O}_{C_{\overline{k}}} \to \mathcal{O}_{(C_{\overline{k}})_{\text{red}}} \to 0,$$

we find that $\text{Supp}(\mathcal{N})$ is a finite set of closed points. Therefore

$$h^1(C, \mathcal{O}_C) = h^1(C_{\overline{k}}, \mathcal{O}_{C_{\overline{k}}}) = h^1((C_{\overline{k}})_{\text{red}}, \mathcal{O}_{(C_{\overline{k}})_{\text{red}}}) = 0.$$  

But $h^1(C, \mathcal{O}_C) = h^1(V, \mathcal{O}_V) = h^0(V, \mathcal{O}_V) \neq 0$ by the assumption that the reduced Hilbert polynomial of $V$ is $\text{rh}_V(t) = pt$. Hence if $n = p > 2$ is prime, then $C$ is geometrically connected. \hfill $\square$

**Remark 3.10.** Proposition 3.9 implies that, if $A$ is a division $k$-algebra of prime index $n = p$, then any reduced curve $C \subset X$ with $\deg(C) = p$ is geometrically reduced [Sta19, Tag 04KS].

**Example 3.11.** If $A$ is a division $k$-algebra with index $n = 5$, then we can say something about possible subschemes $V \subset X$ with $\text{rh}_V(t) = 5t$. 

Let $C \subset V$ be the unique curve sitting in $V$ with degree $\deg(C) = 5$. Then $C$ is geometrically connected (by Proposition 3.9) and $C$ is either reduced or nonreduced. If $C$ is reduced, then $C$ is geometrically reduced (by Remark 3.10). In this case $h^0(C, \mathcal{O}_C) = 1$ so that Lemma 3.8 implies $h^1(C, \mathcal{O}_C) \leq \frac{1}{5}(25 - 15) + 1 = 6$. With Lemma 3.7 we get that 5 divides $1 - h^1(C, \mathcal{O}_C)$. So either $h^1(C, \mathcal{O}_C) = 1$ or $h^1(C, \mathcal{O}_C) = 6$.

If $C$ is reduced and geometrically reducible, then $C_k$ is the union of 5 irreducible components $C_1, \ldots, C_5$ each with $\deg(C_i) = 1$. The singular points of $C_k$ form a Galois orbit, so that they span $X_k \cong \mathbb{P}^4_k$ linearly. In particular, $C_k$ is a union of 5 lines passing through at least 5 points.

We show in Lemma 3.12 below that $C_k$ is essentially a 5-gon of lines with $h^1(C, \mathcal{O}_C) = 1$. Since $\chi(C, \mathcal{O}_C) = 0$, this implies $C = V$.

Otherwise $C$ is geometrically integral and the normalization of $C$ is a smooth genus 1 curve by Remark 3.6. Then either $C$ is smooth and $h^1(C, \mathcal{O}_C) = 1$, or $h^1(C, \mathcal{O}_C) = 6$ and $C$ is singular [Sta19, Tag 0CE4]. In the former case we again find $V = C$ and, in the latter case, we find $V = C \cup p$ for some Artinian subscheme $p \subset X$ with $h^0(p, \mathcal{O}_p) = 5$. But, since 5 divides the degree of any closed point of $X$, we find that $p$ is a closed point with $[k(p) : k] = 5$.

When $C$ is nonreduced, we consider instead the reduced subscheme $C_{\text{red}} \subset C$ which still has $\deg(C_{\text{red}}) = 5$ since $C$ is generically reduced. The scheme $C_{\text{red}}$ is both geometrically connected (since this is true for $C$ by Proposition 3.9) and geometrically reduced (from Remark 3.10). Hence $h^0(C_{\text{red}}, \mathcal{O}_{C_{\text{red}}}) = 1$ and, similar to the reduced case, we find that either $h^1(C_{\text{red}}, \mathcal{O}_{C_{\text{red}}}) = 1$ or $h^1(C_{\text{red}}, \mathcal{O}_{C_{\text{red}}}) = 6$ from both Lemma 3.8 and Lemma 3.7.

Now from the ideal sheaf sequence for $C_{\text{red}} \subset C$ we find (in)equalities

$$h^0(C, \mathcal{O}_C) > h^0(C_{\text{red}}, \mathcal{O}_{C_{\text{red}}}) \quad \text{and} \quad h^1(C, \mathcal{O}_C) = h^1(C_{\text{red}}, \mathcal{O}_{C_{\text{red}}}).$$

It follows that if we write $V = C \cup p$ as the disjoint union of $C$ and a possibly empty Artinian scheme $p$, then

$$0 = \chi(V, \mathcal{O}_V) = \chi(C, \mathcal{O}_C) + r > \chi(C_{\text{red}}, \mathcal{O}_{C_{\text{red}}}) + r$$

with $r = h^0(p, \mathcal{O}_p) \geq 0$. So $\chi(C_{\text{red}}, \mathcal{O}_{C_{\text{red}}}) < 0$ and $h^1(C_{\text{red}}, \mathcal{O}_{C_{\text{red}}}) = 6$. But then $h^0(V, \mathcal{O}_V) = 6$ as well since we have $h^1(C, \mathcal{O}_C) = h^1(V, \mathcal{O}_V)$. Since there are no closed points on $X$ of degree less than 5, it follows that $V = C$ and $h^0(C, \mathcal{O}_C) = 6$.

**Lemma 3.12.** Let $A$ be a division $k$-algebra of prime index $n = p > 2$. Let $V \subset X$ be any subscheme with $\text{fri}_V(t) = pt$ and let $C \subset V$ be the unique curve from Proposition 3.9. If $C$ is geometrically reducible, then $C_k$ is a $p$-gon of lines through $p$ points spanning $X_k \cong \mathbb{P}^{p-1}_k$. 
Proof. Since $C$ is geometrically reducible and of degree $\deg(C) = p$, we know that $(C_{\text{red}})_{\overline{k}}$ is the union of $p$-lines $L_1, ..., L_p$. The Galois group $G = \text{Gal}(k^s/k)$ acts transitively on the set of these lines $\{L_1, ..., L_p\}$, giving a map $G \to S_p$ whose image contains a $p$-cycle. Proposition 3.9 shows that the curve $C$ is geometrically connected, so we can find an element $g \in G$ so that $L_1 \cap gL_1 \neq \emptyset$. After possibly relabeling the lines $L_1, ..., L_p$ we can assume

1. $L_k = g^{k-1}L_1$ for all $k \leq p$,
2. $L_i \cap L_{i+1} \neq \emptyset$ for all $i = 1, ..., p-1$
3. and $L_p \cap L_1 \neq \emptyset$ also.

Assume that there is a hyperplane $H \subset X_{\overline{k}} \cong \mathbb{P}^{p-1}_{\overline{k}}$ containing the lines $L_1, ..., L_{p-1}$. Then, in particular, this $H$ contains the set of all singular points of $C_{\overline{k}}$ which are the union of Galois orbits under $G$. Since $H$ doesn’t contain $L_p$, we have that $gH$ doesn’t contain $gL_p = L_1$.

So the intersection of these translates is empty,

$$\bigcap_{g \in G} gH = \emptyset.$$ 

But this would imply $C$ is smooth, a contradiction since $C$ is assumed singular. Hence no $H$ can contain $L_1, ..., L_{p-1}$.

Now we go inductively. Starting with $L_1$, we add $L_2$ with $L_1 \cap L_2 \neq \emptyset$ by our choice of labeling. We get that $L_1 \cup L_2$ is contained in a linear subspace $H_1$ with $\dim(H_1) = 2$. Now we consider $L_3$. Adding $L_3$ to $L_1 \cup L_2$, we know that $L_2 \cap L_3 \neq \emptyset$ so that $L_1 \cup L_2 \cup L_3$ is contained in a linear subspace $H_2$ with $\dim(H_2) = 3$. Further, we know $L_1 \cup L_2 \cup L_3$ is not contained in $H_1$ since, if it were, then $L_1 \cup \cdots \cup L_{p-1}$ would then be contained in a hyperplane. So $\#(L_3 \cap H_1) = 1$.

Repeating this process, we find for all $i < p-1$ linear subspaces $H_i$ of dimension $\dim(H_i) = i+1$ so that each $H_k$ contains $L_1 \cup \cdots \cup L_{k+1}$, and $\#(L_{k+1} \cap H_{k-1}) = 1$. Finally, we have that $\#(L_p \cap (L_1 \cup \cdots \cup L_{p-1})) \geq 2$ and we claim that actually equality holds. If this inequality were strict, then $L_p \cap L_i \neq \emptyset$ for some $i \neq 1, p-1$. But then

$$gL_p \cap gL_i = L_1 \cap L_{i+1} \neq \emptyset$$

and $1 < i + 1 \leq p - 1$, contradicting that $\#(H_{i-1} \cap L_{i+1}) = 1$.

We think of $(C_{\text{red}})_{\overline{k}}$ as a graph with singular points as vertices and lines as edges; the above shows that the graph associated to $(C_{\text{red}})_{\overline{k}}$ is a $p$-gon on exactly $p$-vertices. These $p$-vertices form a Galois orbit under $G$, so they must span $X_{\overline{k}}$. Corollary 3.13 below shows that $C = C_{\text{red}}$, which completes the proof. □
**Corollary 3.13.** Let $A$ be a division $k$-algebra of prime index $n = p$. Let $V \subset X$ be any subscheme with $rh_V(t) = pt$ and let $C \subset V$ be the unique curve found in Proposition 3.9. If $C$ is geometrically reducible, then $C$ is reduced, $h^1(C, \mathcal{O}_C) = 1$, and $V = C$.

**Proof.** The proof of Lemma 3.12 describes how to construct $(C_{\text{red}})_{\mathbb{T}}$ as a union of lines. One can use this construction to compute $h^1(C, \mathcal{O}_C) = 1$ from the exact sequence (7) below and the observation that $h^1(C, \mathcal{O}_C) = h^1(C_{\text{red}}, \mathcal{O}_{C_{\text{red}}}) = h^1((C_{\text{red}})_{\mathbb{T}}, \mathcal{O}_{(C_{\text{red}})_{\mathbb{T}}})$ where the first equality comes from $C$ being generically reduced and the second from flat base change.

Then $1 = h^1(C, \mathcal{O}_C) = h^1(V, \mathcal{O}_V) = h^0(V, \mathcal{O}_V)$ as $rh_V(t) = pt$. Since $0 \neq H^0(C, \mathcal{O}_C) \subset H^0(V, \mathcal{O}_V)$, we find that $h^0(C, \mathcal{O}_C) = 1$. Hence $C$ is reduced and $V = C$. □

It can be difficult to say anything complete regarding the schemes $\text{Hilb}^{tw}_{n_t}(X/k)$, as done in Example 3.11, for general $X$. However, we can still analyze specific irreducible components of $\text{Hilb}^{tw}_{n_t}(X/k)$, for some particular cases of Severi–Brauer variety $X$, to some benefit.

From now on we write

$$(6) \quad \psi_X : \text{Univ}^{tw}_{\phi(t)}(X/k) \to \text{Hilb}^{tw}_{\phi(t)}(X/k)$$

for the canonical map coming from the projection. (By slight abuse of notation, we use the same $\psi_X$ regardless of the function $\phi(t)$ under consideration). For each irreducible component $V \subset \text{Hilb}^{tw}_{\phi(t)}(X/k)$ we let $\eta_V$ denote the generic point of $V$. If $\phi(t) = rt + s$ is linear then, for each such $V$, the generic fiber $\psi_X^{-1}(\eta_V)$ is the union of a curve and a finite number points.

There may be more than one irreducible curve in the fiber $\psi_X^{-1}(\eta_V)$. Proposition 3.9 can sometimes show the curve in $\psi_X^{-1}(\eta_V)$ is irreducible.

**Corollary 3.14.** Suppose that $A$ is a division $k$-algebra of index $n$. Define $f(n)$ to be the following function.

$$f(n) = \begin{cases} n & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even} \end{cases}$$

Assume that $V \subset \text{Hilb}^{tw}_{f(n)t+s}(X/k)$ is an irreducible component and let $V_{\text{sm}} \subset V$ denote the locus of points smooth in $V$.

If $V$ has a smooth $k$-rational point, i.e. $V_{\text{sm}}(k) \neq \emptyset$, then $\psi_X^{-1}(\eta_V)$ contains a unique irreducible and geometrically connected curve.
Proof. Because of the isomorphisms
\[
\text{Hilb}_{f(n)t+s}^\text{tw}(X/k) \times_k k(\eta_V) \cong \text{Hilb}_{f(n)t+s}^\text{tw}(X_{k(\eta_V)}/k(\eta_V)),
\]
the generic point \(k(\eta_V)\) corresponds to a subscheme \(\psi_{X}^{-1}(\eta_V) \subset X_{k(\eta_V)}\) with reduced Hilbert polynomial \(r_{\text{H}}(\eta_V)(t) = f(n)t + s\).

The Severi–Brauer variety \(X_{k(\eta_V)}\) is associated to the central simple algebra \(A_{k(\eta_V)}\) which, because of the assumption that \(V_{\text{sm}}(k) \neq \emptyset\), has index \(n\) as well (apply Lemma A.1 to the Azumaya algebra \(A \otimes_k \mathcal{O}_{V,x}\) where \(x \in V_{\text{sm}}(k)\)). Now the claim follows from Proposition 3.9. \(\square\)

Of particular interest is the following component of \(\text{Hilb}_{nt}^\text{tw}(X/k)\).

Definition 3.15. Let \(\text{Ell}_n(X) \subset \text{Hilb}_{nt}^\text{tw}(X/k)\) denote the union of the irreducible components \(V\) of \(\text{Hilb}_{nt}^\text{tw}(X/k)\) whose generic fiber \(\psi_{\text{X}}^{-1}(\eta_V)\) is a smooth and geometrically connected curve of genus 1.

The scheme \(\text{Ell}_n(X)\) is never empty. To see this, note that over an algebraic closure \(\overline{k}\) one can always find a smooth genus 1 curve \(C\). If \(D\) is any closed point on \(C\), then the complete linear system associated to the divisor \(nD\) gives a closed immersion to \(\mathbb{P}^{n-1}_{\overline{k}} \subset X_{\overline{k}}\) whenever \(n \geq 3\).

Let \(x\) be any closed point of \(H = \text{Hilb}_{nt}^\text{tw}(X/k)\) defined by a complete linear system as above. By base change we get a morphism
\[
\psi_X|_C : C = \psi_X^{-1}(\text{Spec}(\mathcal{O}_{H,x})) \to \text{Spec}(\mathcal{O}_{H,x})
\]
with special fiber \(C_{k(x)}\) geometrically isomorphic with the curve \(C\). By [Sta19, Tag 01V9] there is an open \(U' \subset \text{Spec}(\mathcal{O}_{H,x})\) and an open \(U \subset C\) containing \(C_{k(x)} \subset U\) such that the restriction
\[
\psi_X|_U : U \to U'
\]
is smooth. The complement \(C \setminus U\) is closed in \(C\) and, since the map \(\psi_X|_C\) is proper, the image \(\psi_X|_C(C \setminus U)\) is closed in \(\text{Spec}(\mathcal{O}_{H,x})\).

Now every nonempty closed subset of \(\text{Spec}(\mathcal{O}_{H,x})\) contains \(x\) and, by construction, the set \(\psi_X|_C(C \setminus U)\) is closed and does not contain \(x\). Thus \(\psi_X|_C(C \setminus U) = \emptyset\) and necessarily \(C = U\) so that \(\psi_X|_C\) is smooth. Since \(\psi_X|_C\) is a smooth, proper, flat, and finitely presented morphism it follows from [Sta19, Tag 0E0N] that \(\psi_X|_C\) has geometrically connected fibers. The generic fiber \(\psi_X^{-1}(\eta_V)\) for any irreducible component \(V \subset H\) containing \(x\) is therefore a smooth and geometrically connected curve of genus 1 implying that \(V \subset \text{Ell}_n(X)\).

Theorem 3.16. Assume that \(A\) is a division \(k\)-algebra of index \(n\). Then the following are true:

1. \(\text{Ell}_n(X)\) is geometrically irreducible with \(\dim(\text{Ell}_n(X)) = n^2\);
(2) If either $A$ is cyclic or, if $A$ contains a maximal subfield $F \subset A$ whose Galois closure $E/k$ is a Galois extension of degree $2n$ with dihedral Galois group, then $\Ell_n(X)(k) \neq \emptyset$.

Proof. We first prove (2). In either case, let $x$ be a point of $X$ with $k(x)$ either a cyclic Galois extension $E/k$ of $k$ of degree $n$ (in the first case) or a maximal subfield $k(x) \subset A$ with Galois closure $E/k$ a dihedral Galois extension of degree $2n$ (in the second case). The field $E$ splits $X$ and $k(x) \otimes_k E \cong E^{\oplus n}$ either way. Let $H \subset \Gal(E/k)$ be a cyclic subgroup of order $n$. Pick an $E$-rational point $p$ in $x_E$ and let $L$ be the line through $p$ and $g p$ for any generator $g$ of $H$.

The union of the $H$ translates of $L$ forms a $\Gal(E/k)$-orbit which descends to a scheme $V \subset X$ defined over $k$. Geometrically, the scheme $V_k$ is an $n$-gon of lines through the points $x_{\overline{k}}$. Hence $V$ has $\mathrm{rh}_V(t) = nt$. We claim the point defined by $V$ in $\Hilb_{nt}^{\mathrm{tw}}(X/k)$ is contained in $\Ell_n(X)$.

Actually, as $V_k$ is the scheme-theoretic union of lines we can use the exact sequence [Sta19, Tag 0C4J]

$$0 \to \mathcal{O}_{C \cup D} \to \mathcal{O}_C \oplus \mathcal{O}_D \to \mathcal{O}_{C \cap D} \to 0$$

where $V_k = C \cup D$, with $C$ a chain of $n-1$ lines and $D$ a line closing the $n$-gon, to compute that $h^1(V, \mathcal{O}_V) = 1$ and that $h^1(V_k, \mathcal{O}_{V_k}(1)) = 0$ by tensoring the exact sequence with $\mathcal{O}_{X_k}(1)$. Since $V_k$ has lci singularities, one can apply [Har10, Proposition 29.9] to find that $V_k$ is smoothable.

More precisely, we find that $\Hilb_{nt}^{\mathrm{tw}}(X/k)$ is smooth at the $k$-rational point defined by $V \subset X$ and, over an algebraic closure, there is an integral curve passing through both the point corresponding to $V_k \subset X_k$ and the subset of $\Ell_n(X_k)$ parametrizing smooth and connected curves. In particular, the embedding $V \subset X$ defines a point of $\Ell_n(X)(k)$ completing the proof of (2).

Now we prove part (1). First, if $n = 3$, note that $\Hilb_{3n}^{\mathrm{tw}}(X/k)$ is a form of $\mathbb{P}^9$. In this case, the central division $k$-algebra $A$ associated to $X$ is cyclic [KMRT98, Theorem 19.2]. By part (2) above, the set $\Hilb_{3n}^{\mathrm{tw}}(X/k)(k) \neq \emptyset$ is nonempty. Hence $\Ell_3(X) = \Hilb_{3n}^{\mathrm{tw}}(X/k) \cong \mathbb{P}^9$.

(Alternatively, one can avoid the use of (2) by using Bertini’s theorem).

So we can assume $n > 3$. Let $U' \subset \Hilb_{nt}^{\mathrm{tw}}(X/k) \times_k \overline{k}$ be the open set consisting of all smooth and connected curves of degree $n$ and genus 1. Write $U$ for the image of $U'$ inside $\Hilb_{nt}^{\mathrm{tw}}(X/k)$. Then $U \subset \Ell_n(X)$ is open and irreducible since $U'$ is [Ein86, Theorem 8]. We’ll show that $U$ is dense in $\Ell_n(X)$; this will prove the first claim since $U' = U \times_k \overline{k}$.

Let $V$ be an irreducible component of $\Ell_n(X)$. Since the generic fiber $\psi_X^{-1}(\eta_V)$ is smooth and geometrically connected, there is an open subset $W \subset V$ such that for any point $x$ in $W$, the curve $\psi_X^{-1}(x)$ is smooth (see
and geometrically irreducible (by [Sta19, Tag 0559]) of degree $n$ by assumption and, by Remark 3.6, of genus 1. In particular, we have $W \subset U$ showing that $U$ is (topologically, but possibly not scheme-theoretically) dense in $\text{Ell}_n(X)$.

The dimension of $\text{Ell}_n(X)$ can be determined geometrically, i.e. over an algebraic closure, and this is done in [Ein86, Theorem 8]. Essentially, if $C \subset X$ is smooth of degree $n$ and genus 1 then one can compute

$$h^0(C, \mathcal{N}_{C/X}) = n^2 \quad \text{and} \quad h^1(C, \mathcal{N}_{C/X}) = 0$$

using the normal bundle sequence (and the Euler sequence for $X$).

This shows both that $\dim(\text{Ell}_n(X)) \leq n^2$, from Corollary 2.8 (1), and that $\dim(\text{Ell}_n(X)) \geq n^2$, from Corollary 2.8 (3); moreover this shows that $\text{Ell}_n(X)$ is smooth along $U$.

\[\square\]

**Remark 3.17.** The proof of (2) in Theorem 3.16 is an extension of an argument due to Jason Starr, cf. [Sta17]. Starr’s original goal is to use the fact that $V$ defines a smooth $k$-rational point on $\text{Hilb}^w_{nt}(X/k)$ to construct a smooth genus 1 curve on a Severi–Brauer variety $X$ over a large (also called ample) field $k$ (e.g. a $p$-special field or the fraction field of a Henselian DVR).

We can elaborate on Starr’s result in the setting of Theorem 3.16, i.e. when $A$ is a division $k$-algebra satisfying the assumptions of (2). Indeed, the scheme $\text{Ell}_n(X)$ is projective so we can construct a smooth curve $C$ with a $k$-rational point mapping to the $k$-point $x$ associated to the $n$-gon $V$ constructed in the proof of Theorem 3.16 (2) as follows.

Let $y$ be any point of $\text{Ell}_n(X)$ whose associated subscheme $C \subset X$ is a smooth geometrically connected curve of genus 1. Let $I = \{x, y\}$. Consider the blowup $\text{Bl}_I(\text{Ell}_n(X))$ with center the points $I \subset \text{Ell}_n(X)$.

For any embedding $\text{Ell}_n(X) \subset \mathbb{P}^N$, we get an embedding of the blowup

$$\text{Bl}_I(\text{Ell}_n(X)) \subset \mathbb{P}^N \times \mathbb{P}^{n^2-1} \subset \mathbb{P}^M$$

by composing with the Segre embedding. Now a general linear section of the correct codimension intersects $\text{Bl}_I(\text{Ell}_n(X))$ in a curve (smooth over $x$) by Bertini’s theorem [Jou83, Théorème 6.10 et Corollaire 6.11]. A general linear section of the same codimension also intersects the exceptional divisor $\mathbb{P}^{n^2-1} \subset \text{Bl}_I(\text{Ell}_n(X))$ over $x$ in a $k$-rational point and the exceptional divisor over $y$ in some number of points. So we can choose a linear section $E \subset \text{Bl}_I(\text{Ell}_n(X))$ doing all three things at once.

The normalization $E''$ of $E$ is a curve with all the stated properties.

Over a large (also called ample) field $k$, any irreducible curve having a smooth $k$-rational point has infinitely many $k$-rational points. Thus the curve $E''$ has infinitely many $k$-rational points and the image along
the composition of the normalization and blowdown

\[ E' \to E \to \text{Bl}_I(\text{Ell}_n(X)) \to \text{Ell}_n(X) \]

has nontrivial intersection with the open subset of \( \text{Ell}_n(X) \) consisting of smooth and geometrically connected genus 1 curves.

**Example 3.18.** If \( A \) is a cyclic division \( k \)-algebra of index \( n \), there are lots of field extensions \( F/k \) where \( X_F \) contains a smooth geometrically connected curve of genus 1 and where the algebra \( A_F \) has index \( n \). When \( n = p^r \) is a power of a prime \( p \), Remark 3.17 shows this holds for a minimal \( p \)-special field \( F/k \) contained in an algebraic closure \( \bar{k}/k \). When the index \( n \) is arbitrary one can instead use the field \( k((t)) \), which is the fraction field of a Henselian DVR, and apply Remark 3.17. The index remains \( n \) here since \( A_{k((t))} \) specializes to \( A \) (Lemma A.1).

One can also construct “generic” examples for an arbitrary division algebra \( A \) of index \( n \) as follows. If \( n = p \) is prime, then one can first replace the base field \( k \) by an extension \( F/k \) with \( A_F \) a cyclic division algebra of index \( p \) if necessary. Next, one again extends the base field but now to the function field \( L = F(\text{Ell}_p(X_F)) \) of the scheme \( \text{Ell}_p(X_F) \). Since \( \text{Ell}_p(X_F) \) has a smooth \( F \)-rational point by Theorem 3.16, the algebra \( A_L \) is nonsplit (hence of index \( p \)) by [GS17, Lemma 5.4.7]. The generic fiber \( \psi^{-1}_X(\eta_{\text{Ell}_p(X_F)}) \) is then a smooth and geometrically connected genus 1 curve on \( X_L \).

If \( n \) is not prime, one can use [RTY08] to get a field extension \( F/k \) with \( A_F \) cyclic of index \( n \) and with the restriction \( \text{Br}(k) \to \text{Br}(F) \) an injection. Setting \( L = F(\text{Ell}_n(X_F)) \) then, as above, \( X_L \) contains a smooth and geometrically connected curve of genus 1. In this situation [GS17, Lemma 5.4.7] shows that the restriction \( \text{Br}(F) \to \text{Br}(L) \) is an injection and Lemma A.1 below shows that \( A_L \) remains index \( n \) (actually, both statements can be obtained from Lemma A.1).

The period of any curve \( C \) constructed in this way is equal to the index \( n \) of the division \( k \)-algebra \( A \). The period is divisible by \( n \), by Lemma 3.1, since \( C \subset X_L \) and \( X_L \) has index \( n \). But the map \( C \subset X_L \) defines a rational point in \( \text{Pic}_C^k(L) \) so that \( C \) has period dividing \( n \). The index of \( C \) is then a multiple of \( n \) in the interval \( n \leq \text{ind}(C) \leq n^2 \).

For an example when one can explicitly determine the index, note that there exist smooth and geometrically connected curves \( E \) of genus 1 with period \( n \) and index \( n^2 \) over some number fields \( k \) by [CS10]. For any Severi–Brauer variety \( X \) containing one of these curves \( E \subset X \) as a smooth closed subscheme, the generic curve \( C \) constructed above has index \( n^2 \) since \( C \) specializes to \( E \) through a sequence of DVRs (cf. the proof of Lemma A.1) so that one can apply [GLL13, Theorem 8.2].
However, if the exponent of $A$ is strictly less than $n$, the index of any generic curve $C$, constructed as above, is then strictly less than $n^2$. Indeed, if $H$ is a divisor of $X_L$ of degree $\exp(A)$ then

$$[C \cap H] = [C][H] = m[p]$$

holds in $\text{CH}_0(X_L)$ for some point $p$ of degree $\text{ind}(A)$ and $m = \exp(A)$. The left hand side of this equation has degree a multiple of the index of $C$ whereas the right hand side has degree $\text{ind}(A)\exp(A) < \text{ind}(A)^2$.

### Appendix A. On Azumaya algebras

**Lemma A.1.** Let $R$ be a Noetherian regular local ring with maximal ideal $m$, residue field $k = R/m$, and fraction field $F$. Suppose that $A$ is an Azumaya $R$-algebra. Then there is an inequality $\text{ind}(A_k) \leq \text{ind}(A_F)$.

**Proof.** We consider the $R$-schemes $X_m = \text{SB}_m(A)$ which are étale forms of the Grassmannian $R$-schemes $\text{Gr}_R(m, n)$, where $n$ is the square root of the rank of $A$, and for varying $m$. The $F$ and $k$ fibers of the structure map over $R$ are canonically

$$\text{SB}_m(A_F) \cong \text{SB}_m(A) \times_R F \quad \text{and} \quad \text{SB}_m(A_k) \cong \text{SB}_m(A) \times_R k,$$

which have an $F$-rational point, or a $k$-rational point respectively, if and only if the index $\text{ind}(A_F)$, or $\text{ind}(A_k)$ respectively, divides $m$ [Bla91, Proposition 3]. We’ll show that the assumption $R$ is regular guarantees that $\text{SB}_m(A_k)(k) \neq \emptyset$ whenever $\text{SB}_m(A_F)(F) \neq \emptyset$.

For this, we first note that $R$ admits a sequence of discrete valuation rings $R_0, ..., R_t$ with maximal ideals $m_0, ..., m_t$ for some $t \geq 0$ with the following properties:

1. $\text{Frac}(R_0) = F$,
2. $R_i/m_i \cong \text{Frac}(R_{i+1})$
3. $R_t/m_t \cong k$.

One can take a regular sequence $(a_0, ..., a_{t-1})$ of generators for $m$ and define $R_i = (R/(a_0, ..., a_{i-1}))(a_i)$ (cf. [Sta19, Tag 00NQ, Tag 0AFS]). Now the valuative criterion for properness [Har66, Theorem 4.7] shows

$$(X_m)_{R_i}(\text{Frac}(R_i)) \neq \emptyset \implies (X_m)_{R_{i+1}}(\text{Frac}(R_{i+1})) \neq \emptyset.$$

One can conclude by induction. \qed

**Example A.2.** The assumption that $R$ is regular cannot be dropped from the statement of Lemma A.1. Here’s an example from Qixiao Ma. Fix a field $k$. Let $X/k$ be any Severi–Brauer variety having $X(k) = \emptyset$. 

...
Let $x \in X$ be a closed point. Consider the pushout $\tilde{X}$ in the cocartesian diagram below.

$$
\begin{array}{ccc}
x & \longrightarrow & \text{Spec}(k) \\
\downarrow & & \downarrow \\
X & \longrightarrow & \tilde{X}
\end{array}
$$

Let $\tilde{x} \in \tilde{X}$ denote the canonical (singular) $k$-rational point of $\tilde{X}$ and $\mathcal{O}_{\tilde{X},\tilde{x}}$ the local ring. If $A$ is the central simple algebra associated to $X$, then the Azumaya algebra $A \otimes_k \mathcal{O}_{\tilde{X},\tilde{x}}$ is split over the generic point and nontrivial over the closed point by construction.

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