Optimal controlled teleportation

T. Gao¹,²(a), F. L. Yan²,³ and Y. C. Li³

¹ College of Mathematics and Information Science, Hebei Normal University - Shijiazhuang 050016, China
² Max-Planck-Institut für Quantenoptik - Hans-Kopfermann-Str. 1, D-85748 Garching, Germany, EU
³ College of Physics Science and Information Engineering, Hebei Normal University - Shijiazhuang 050016, China

received 18 March 2008; accepted in final form 22 October 2008
published online 26 November 2008

PACS 03.67.Hk - Quantum communication

Abstract – We study the general case of controlled quantum teleportation and give analytic expressions for the maximal probabilities of successful controlled teleportation of an unknown qubit via every kind of three-qubit state. We also give an explicit expression for the three-qubit states that can be used as quantum channel for controlled teleportation of an unknown state with unit probability and with unit fidelity. In addition, an exact value of localizable entanglement is determined. We obtain the necessary and sufficient condition that a three-qubit state can be collapsed to an EPR pair by a measurement on one qubit.

Introduction. – Quantum teleportation is one of the most profound results of quantum information theory. In the seminal work of Bennett et al. [1], they showed that an arbitrary unknown state of a qubit could be perfectly teleported from one place to another using a previously shared Einstein-Podolsky-Rosen (EPR) pair and by the transmission of two bits of classical information between the sender Alice and receiver Bob, where Alice knows neither the state to be teleported nor the location of the intended receiver. Due to its important applications in quantum communication [2–4] and quantum computation, quantum teleportation has been generalized by many authors to various cases [5–21]. On the experimental side, over the past several years quantum teleportation has also been experimentally demonstrated by several groups [22–24].

The controlled quantum teleportation scheme was first proposed by Karlsson and Bourennane [15], using ideas very similar to those used in the quantum-secret-sharing paper of Hillery et al. [25]. In [15,25], the entanglement property of the Greenberger-Horne-Zeilinger (GHZ) state is utilized for teleporting an unknown state instead of an EPR pair. According to this scheme, a third side is included, so that the quantum channel is supervised by this additional side. An unknown state can be perfectly transported from a sender to a spatially distant receiver via a previously shared quantum resource, the GHZ state, by means of local operations and classical communications (LOCC) under the permission of the third party. The signal state cannot be transmitted unless all three sides agree to cooperate. Controlled quantum teleportation is useful in various contexts in quantum information, such as quantum information processing, cryptographic conferencing [26–29], and controlled quantum secure direct communication [30]. Recently, a number of works on controlled quantum teleportation have been presented [16,17,20,21], where the authors restricted themselves to special quantum channels, such as the GHZ state or the W state. In this paper, we investigate the general case of controlled quantum teleportation, as well as the maximal successful probability of controlled teleportation. The maximal successful probability is a localizable entanglement.

The quantification of the entanglement contained in quantum states, is of central interest in the field of quantum information. For multipartite states, many more parameters are needed to describe the entanglement. As a result, many new entanglement measures have been designed, especially for pure states. In the context of spin chains, Verstraete, Popp, and Cirac [31] introduced a measure of entanglement which they called localizable entanglement (LE). The LE \( E_{ij} \) is defined as the maximal amount of entanglement that can be localized, on average, between the spins \( i \) and \( j \) by performing local measurements on the other spins. More specifically, every measurement basis specifies a pure state ensemble \( \varepsilon = \{ p_s, |\phi_s\rangle \} \) consisting of at least \( 2^{N-2} \) elements counted by the index \( s \). In this notation \( p_s \) denotes the probability of obtaining the two-spin state \( |\phi_s\rangle \) after performing the measurement \( |s\rangle \) on the rest of the system. The LE [31] is then given by

\[
E_{ij} = \max_\varepsilon \sum_s p_s E(|\phi_s\rangle),
\]

(1)

(a)E-mail: gaoting@hebtu.edu.cn
where $E(|\phi_i\rangle)$ is the chosen measure of entanglement of $|\phi_i\rangle$. Namely, one chooses two spins, and performs LOCC operations aiming at obtaining the largest bipartite entanglement between them. This quantity not only has a very well-defined physical meaning that treats entanglement as a truly physical resource, but also establishes a very close connection between entanglement and correlation functions. The determination of the LE is a formidable task since it involves optimization over all possible local measurement strategies, and thus cannot be determined in general. However, Verstraete, Popp, and Cirac [31] gave tight upper and lower bounds when the entanglement measure $E(|\phi_i\rangle)$ is chosen as the concurrence $C$ of $|\phi_i\rangle$.

In this letter, we investigate the general case of controlled quantum teleportation — i.e., controlled teleportation of an unknown state with unit fidelity from a sender to a remote receiver under the control of a third agent by using a general three-qubit state as the quantum channel — and its maximal probability of success $p_{\text{max}}$. The maximal probability of success $p_{\text{max}}$ is a kind of LE, but the measure of entanglement is not concurrence. This LE has a very important physical meaning, but is more difficult to determine. However, we give the analytic expression for the maximal probability of success $p_{\text{max}}$ and the exact value of LE in [31] for three-qubit states. Moreover, we give the necessary and sufficient condition that a general three-qubit state can collapse to an EPR pair with a certain (nonzero) probability by means of measurement on one qubit. In addition, for any given three-qubit state, we show in detail how to choose the measurement basis to achieve maximal probability of success for controlled teleportation. More surprisingly, we find that there do exist many states that cannot be converted to GHZ states under LOCC but can nevertheless be used for perfect controlled teleportation, i.e., controlled teleportation of a qubit with unit fidelity and unit probability. To our knowledge, we are the first to show this fact. Indeed, we prove that a tripartite entangled state $|\Psi\rangle$ can be used for perfect controlled teleportation if and only if $|\Psi\rangle$ is LOCC equivalent to one of the following states:

$$a_0|000\rangle + a_1|100\rangle + \frac{1}{\sqrt{2}}|111\rangle,$$  

$$a_0 > 0, \quad a_1 \geq 0, \quad a_0^2 + a_1^2 = \frac{1}{2}. \quad (2)$$

Here the first qubit belongs to the controller. The GHZ class (classified under LOCC) is only a very small part of the states that can be used for perfect controlled teleportation.

**Controlled teleportation via the three-qubit state of general form.** Acín et al. [32] proved that for every pure state of a composite system, 123, there exist orthonormal bases $\{|01\rangle, |11\rangle\}, |02\rangle, |12\rangle$, and $|03\rangle, |13\rangle$ for systems 1, 2 and 3, respectively, such that

$$|\Psi\rangle_{123} = a_0|000\rangle_{123} + a_1 e^{i\theta}|100\rangle_{123} + a_2|101\rangle_{123} + a_3|110\rangle_{123} + a_4|111\rangle_{123},$$

$$a_i > 0, \quad 0 \leq \mu \leq \pi, \quad \Sigma_{i=0}^4 a_i^2 = 1. \quad (3)$$

Suppose that Alice is to deliver an unknown state $|\psi\rangle_4 = \alpha|0\rangle_4 + \beta|1\rangle_4$, $|\alpha|^2 + |\beta|^2 = 1$ (4) to a distant receiver Bob supervised by the third trusted party Charlie via the general pure three-qubit state (3), where particle 1 belongs to Charlie, particle 2 is on Alice’s side, while Bob has particle 3. Throughout the paper, we let $a_0 \neq 0$, since if $a_0 = 0$, then $|\Psi\rangle_{123}$ is a tensor product state of a pure state of particle 1 and a pure state of particles 2 and 3, but not a true tripartite entangled state.

The controller Charlie measures his particle in the basis

$$|x\rangle = \cos \frac{\theta}{2}|0\rangle + e^{i\varphi} \sin \frac{\theta}{2}|1\rangle,$$

$$|x\rangle^\perp = \sin \frac{\theta}{2}|0\rangle - e^{i\varphi} \cos \frac{\theta}{2}|1\rangle,$$  

(5)

and broadcasts his measurement result to Alice and Bob. Here $\theta \in [0, \pi], \varphi \in [0, 2\pi]$.

The tripartite state $|\Psi\rangle_{123}$ can be reexpressed as

$$|\Psi\rangle_{123} = \sqrt{p_1}|x\rangle_1|\Phi_1\rangle_{23} + \sqrt{p_2}|x\rangle_1|\Phi_2\rangle_{23}. \quad (6)$$

Here

$$p_1 = \sin^2 \frac{\theta}{2} + a_0^2 \cos \theta + a_0 a_1 \cos (\mu - \varphi) \sin \theta,$$

$$p_2 = \cos^2 \frac{\theta}{2} - a_0^2 \cos \theta - a_0 a_1 \cos (\mu - \varphi) \sin \theta,$$

$$|\Phi_1\rangle_{23} = \frac{1}{\sqrt{p_1}} \left\{ \left[ a_0 \cos \frac{\theta}{2} + a_1 e^{i(\mu - \varphi)} \sin \frac{\theta}{2} \right]|00\rangle_{23} + e^{-i\varphi} \sin \frac{\theta}{2} \left[ a_2 |01\rangle_{23} + a_3 |10\rangle_{23} + a_4 |11\rangle_{23} \right] \right\},$$

$$|\Phi_2\rangle_{23} = \frac{1}{\sqrt{p_2}} \left\{ \left[ a_0 \sin \frac{\theta}{2} - a_1 e^{i(\mu - \varphi)} \cos \frac{\theta}{2} \right]|00\rangle_{23} - e^{-i\varphi} \cos \frac{\theta}{2} \left[ a_2 |01\rangle_{23} + a_3 |10\rangle_{23} + a_4 |11\rangle_{23} \right] \right\}. \quad (7)$$

After Charlie’s measurement, the quantum channel is collapsed to $|\Phi_1\rangle_{23}$, and $|\Phi_2\rangle_{23}$ with probabilities $p_1$, and $p_2$, respectively. Here it is necessary for $p_1$ and $p_2$ to be greater than 0, otherwise, particle 1 is not entangled with the other two particles 2 and 3, and as a result Charlie has no control over Alice and Bob. In this paper, we suppose that not only $a_0 > 0$, but also $a_2$, $a_3$, and $a_4$ are not equal to 0 at the same time, which is equivalent to $p_1 > 0$ and $p_2 > 0$. Note that $p_1 = a_0^2$ and $p_2 = 1 - a_0^2$ in the case of $\theta = 0$; $p_1 = 1 - a_0^2$ and $p_2 = a_0^2$ in the case of $\theta = \pi$. Suppose $\theta \neq 0, \pi$, and let $t = \cot \frac{\theta}{2}$. Then there are

$$p_1 = \frac{1}{t^2 + 1} a_0^2 (\cos^2 (\mu - \varphi) + |\mu| + t a_0 \cos (\mu - \varphi))^2 \geq \frac{a_0^2}{t^2 + 1} + \frac{a_0^2}{t^2 + 1}$$

and

$$p_2 = \frac{1}{t^2 + 1} a_0^2 (\cos^2 (\mu - \varphi) + |\mu| + t a_0 \cos (\mu - \varphi))^2 \geq \frac{a_0^2}{t^2 + 1} + \frac{a_0^2}{t^2 + 1}.$$

Therefore, $p_1 > 0$ and $p_2 > 0$ hold if and only if $a_0 > 0$, and $a_2$, $a_3$, and $a_4$ are not equal to 0 at the same time.

By Schmidt decomposition, there is

$$|\Phi_1\rangle_{23} = \sqrt{\lambda_0}|00\rangle_{23} + \sqrt{\lambda_1}|11\rangle_{23}, \quad (7)$$

$$|\Phi_2\rangle_{23} = \sqrt{\lambda_2}|00\rangle_{23} + \sqrt{\lambda_3}|11\rangle_{23}. \quad (8)$$
where \( \{|0\rangle_2, |1\rangle_2\} \) and \( \{|0\rangle_4, |1\rangle_4\} \) are orthonormal bases of system 2 (system 3), and

\[
\lambda_{10} = \frac{1 - \sqrt{1 - C^2}}{2}, \quad \lambda_{11} = \frac{1 + \sqrt{1 - C^2}}{2},
\]

\[
\lambda_{20} = \frac{1 - \sqrt{1 - C^2}}{2}, \quad \lambda_{21} = \frac{1 + \sqrt{1 - C^2}}{2},
\]

\[
C_1 = \frac{|a_0a_4e^{-i\theta}\sin \theta + 2(a_1a_4e^{i\theta} - a_2a_3)e^{-2i\theta}\sin^2 \frac{\theta}{2}|}{p_1},
\]

\[
C_2 = \frac{|a_0a_4e^{-i\theta}\sin \theta - 2(a_1a_4e^{i\theta} - a_2a_3)e^{-2i\theta}\cos^2 \frac{\theta}{2}|}{p_2}.
\]

If the outcome of Charlie’s measurement is \( |x\rangle_1 \), then the state of particles 2, 3 and 4 is

\[
|\psi\rangle_{4}|\Phi_{123}\rangle = \sqrt{\lambda_{10}^2 + \lambda_{11}^2} |\phi^+\rangle_{24} + \sqrt{\lambda_{10}^2 + \lambda_{11}^2} |\phi^+\rangle_{24} + \sqrt{\lambda_{10}^2 + \lambda_{11}^2} |\phi^+\rangle_{24} + \sqrt{\lambda_{10}^2 + \lambda_{11}^2} |\phi^+\rangle_{24},
\]

(9)

where \( |\phi^\pm\rangle_{24} = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) \) and \( |\psi\rangle_{24} = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) \) Alice makes a Bell measurement on particles 2 and 4, and conveys her measurement outcome to Bob over a classical communication channel.

In order to achieve teleportation, Bob needs to introduce an auxiliary particle \( b \) with initial state \( |0\rangle_b \) [33], and performs a collective unitary transformation

\[
U_{3b} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{\lambda_{10}}{\lambda_{11}}} & \sqrt{\frac{\lambda_{11}}{\lambda_{10}}} \\ 0 & 0 & -\sqrt{1 - \frac{\lambda_{10}}{\lambda_{11}}} & \sqrt{\lambda_{10}} \end{pmatrix}
\]

on the state of particles 3 and \( b \). Then the measurement on his auxiliary particle \( b \) follows. If his measurement result is \( |0\rangle_b \), Bob can fix up the state of his particle 3, recovering \( |\psi\rangle \), by applying an appropriate local unitary operation. The achievable successful probability of teleporting the unknown state in (4) via \( |\Phi_{123}\rangle \) is

\[
2 \left( \frac{\sqrt{\lambda_{10}}^2 + \lambda_{11}^2}{\sqrt{\lambda_{10}}} \right)^2 \left( \frac{\sqrt{\lambda_{11}}^2 + \lambda_{10}^2}{\sqrt{\lambda_{11}}} \right)^2 = 2\lambda_{10}.
\]

Similarly, if the result of Charlie’s measurement is \( |x\rangle_1 \), the achievable successful probability of teleporting the state in (4) via \( |\Phi_{23}\rangle \) is \( 2\lambda_{20} \).

The probability \( p \) of successful controlled teleportation of an unknown qubit (4) using the state in (3) is

\[
p = 2p_1\lambda_{10} + 2p_2\lambda_{20} = 1 - R(\theta, \varphi).
\]

(10)

Here

\[
R = R(\theta, \varphi) = \sqrt{P(\theta, \varphi) + 2Q(\theta, \varphi)},
\]

\[
P = P(\theta, \varphi) = p_1^2(1 - C_1^2),
\]

\[
Q = Q(\theta, \varphi) = p_2^2(1 - C_2^2).
\]

The three-qubit states used for perfect controlled teleportation. – If there exists \( (\theta_0, \varphi_0) \) such that \( P(\theta_0, \varphi_0) = 0 \) or \( Q(\theta_0, \varphi_0) = 0 \), then Alice and Bob share an EPR pair with some finite (nonzero) probability \( p_1 \) or \( p_2 \) by Charlie’s measurement in the basis (5) with \( (\theta, \varphi) = (\theta_0, \varphi_0) \). Furthermore, if there exists \( (\theta_0, \varphi_0) \) such that \( P(\theta_0, \varphi_0) = Q(\theta_0, \varphi_0) = 0 \), then an EPR pair occurs with certainty after Charlie’s measurement in the basis (5) with \( (\theta, \varphi) = (\theta_0, \varphi_0) \). The tripartite state (3) with the property that both \( P(\theta, \varphi) \) and \( Q(\theta, \varphi) \) are equal to zero at the same point \( (\theta, \varphi) \) can be used for perfect teleportation.

We first investigate the condition \( P(\theta, \varphi) = 0 \).

\[
P(\theta, \varphi) = 0 \iff | \Phi_{123} \rangle \text{ is a Bell state} \iff \text{the concurrence } C_1 \text{ of } | \Phi_{123} \rangle \text{ is 1.}
\]

Since \( P(0, \varphi) = a_2^2 \neq 0 \), we suppose \( \theta \in (0, \pi] \). Let

\[
|\phi_1\rangle_{23} = \frac{ae^{i\alpha}|00\rangle_{23} + a_2|01\rangle_{23} + a_3|10\rangle_{23} + a_4|11\rangle_{23}}{\sqrt{a^2 + a_2^2 + a_3^2 + a_4^2}}
\]

\[
= \frac{\sqrt{p_1}}{\sqrt{a^2 + a_2^2 + a_3^2 + a_4^2}} |\Phi_1\rangle_{23}.
\]

(12)

Here

\[
ae^{i\alpha} = t\alpha_0 e^{i\varphi_0} + a_1 e^{i\mu}, \quad t = \cot \frac{\theta}{2},
\]

while \( a = |\alpha_0 e^{i\varphi_0} + a_1 e^{i\mu}| \) is the absolute value of the complex number \( ae^{i\alpha} \), and \( \alpha \) is the argument of \( ae^{i\alpha} \).

Note that the concurrence \( C_1 \) of \( | \Phi_{123} \rangle \) is equal to the concurrence \( C(| \phi_1 \rangle_{23}) \) of \( | \phi_1 \rangle_{23} \), and \( C(| \phi_1 \rangle_{23}) \) is 1, if and only if \( |C(| \phi_1 \rangle_{23})|^2 = 1 \). \( C(| \phi_1 \rangle_{23}) \) is 1 means that

\[
(a^2 - a_2^2)^2 + 2(aa_2 - a_3a_4)^2 + 2(aa_3 - a_2a_4)^2 + 8a_2a_3a_4(1 + \cos \alpha) + (a_2^2 - a_3^2)^2 = 0.
\]

(14)

From eq. (14), we see that \( C_1 = 1 \) if and only if either

\[
a_2 = a_3 = 0, \quad a = a_4 \neq 0;
\]

or

\[
a_2 = a_3 \neq 0, \quad a = a_4 = 0;
\]

or

\[
a_2 = a_3 \neq 0, \quad a = a_4 \neq 0, \quad \alpha = \pi.
\]

(17)

From the expressions in eq. (15), we derive that \( P(\theta, \varphi) = 0 \), if and only if, either \( \cos \theta = 1 - 2a_2^2 \) in the
case $a_1 = a_2 = a_3 = 0$, or $\cos(\varphi - \mu) = \frac{\sin^2 \theta}{\cos \varphi - \mu} + \frac{\sin^2 \varphi}{\cos \mu - \varphi}$ in the case $a_1 \neq 0$ and $a_2 = a_3 = 0$.

By the expressions in eq. (16), we get $P(\theta, \varphi) = 0$, if and only if, either $\theta = \pi$ in the case $a_1 = 0$, or $\varphi = \mu + \pi$ and $\cot \frac{\mu}{2} = \frac{a_1}{a_2}$ in the case $a_1 \neq 0$.

Using the expressions in eq. (17), we obtain the following results: If $a_1 = 0$, then $P(\theta, \varphi) = 0$, and only if, $a_2 = a_3$, $\varphi = \pi$ and $\cot \frac{\mu}{2} = \frac{a_2}{a_3}$. If $\mu = 0$, then $P(\theta, \varphi) = 0$, if and only if, $a_2 = a_3$, $\varphi = \pi$ and $\cot \frac{\mu}{2} = \frac{a_2}{a_3}$ in the case $a_1 > a_3$, or $a_2 = a_3$, $\varphi = \pi$ and $\cot \frac{\mu}{2} = \frac{a_2}{a_3}$ in the case $a_1 < a_3$.

Similarly, we derive after some lengthy but straightforward computation that $Q(\theta, \varphi) = 0$, if and only if, the coefficients of the tripartite state in (3) satisfy $a_2 = a_3$.

Thus, we find that a three-qubit state in (3) can be collapsed to an EPR pair with a certain (non zero) probability by a measurement on the first qubit, if and only if, $a_2 = a_3$.

Next we characterize the states such that both $P(\theta, \varphi)$ and $Q(\theta, \varphi)$ are equal to 0 at the same point $(\theta, \varphi)$.

By the above discussion, we need only to find the condition such that $Q(\theta_0, \varphi_0) = 0$ for each case with $P(\theta_0, \varphi_0) = 0$. Note that $a_0 \neq 0$. If the expressions in eq. (15) hold, then $Q(\theta_0, \varphi_0) = Q(\theta, \varphi)(a_2 = a_3 = 0, a_0 = a_4 = 1 - 2a_3^2 - 2a_0^2)^2 = 0$, and only if, $a_2 = a_3 = 0$ and $a_4 = \frac{\mu}{\sqrt{3}}$. If the expressions in eq. (16) hold, then $Q(\theta_0, \varphi_0) = Q(\theta, \varphi)(a_2 = a_3, a_0 = a_4 = 1) + a_3^2(2 - a_0^2) > 0$.

Therefore, the three-qubit state in the generalized Schmidt decomposition (3) can be used for perfect teleportation if and only if, it is the state in eq. (2).

**Optimal controlled teleportation.** Now we investigate how to achieve the maximum of probability of successful controlled teleportation of an unknown qubit state (4) via the quantum channel (3).

Obviously, the maximum of (10) is

$$p_{\text{max}} = \max\{p\} = 1 - \min\{R(\theta, \varphi)\} = 1 - R_{\text{min}}. \quad (18)$$

The maximal probability of success $p_{\text{max}}$ is a kind of LE with a very important physical meaning, but the entanglement measure is $1 - \sqrt{1 - C^2}$, not the concurrence $C$.

In order to reach the maximal probability $p_{\text{max}}$ of controlled teleportation, the supervisor Charlie only needs to choose an optimal measurement basis, i.e. he selects $\theta_0$ and $\varphi_0$ such that $R_{\text{min}} = R(\theta_0, \varphi_0)$.

Note that the minimum of $R(\theta, \varphi)$ should occur at the boundary points, or the points such that $P(\theta, \varphi) = 0$, or $Q(\theta, \varphi) = 0$, or

$$\frac{\partial R}{\partial \theta} = 0, \quad \frac{\partial R}{\partial \varphi} = 0. \quad (19)$$

Combining the last two equations gives

$$\frac{\partial P}{\partial \theta} \frac{\partial Q}{\partial \varphi} - \frac{\partial Q}{\partial \theta} \frac{\partial P}{\partial \varphi} = 0, \quad (20)$$

$$P(\theta, \varphi) = Q(\theta, \varphi) = 0. \quad (21)$$

$$P(\theta, \varphi) = Q(\theta, \varphi) = 0. \quad (22)$$

Let us now look at the general case —the quantum channel with parameters satisfying $a_0 a_1 a_2 a_3 a_4 \sin \mu \neq 0$. Suppose $\sin \theta \neq 0$, and $P(\theta, \varphi) = Q(\theta, \varphi) = 0$.

By (20), there is

$$2b_1 x^2 + (b_2 \cos \varphi + b_3 \sin \varphi)x
-a_0^2 b_1 + b_1 \cos 2\varphi + b_3 \sin 2\varphi = 0, \quad (23)$$

where

$$x = a_0 \cot \theta, \quad g_1 = a_2 a_3 a_4,$$
$$g_2 = a_0^2 (a_0^2 - (2a_3^2 + 3a_0^2) a_3^2, \quad g_3 = 2a_3^2 + 2a_0^2,$$
$$b_1 = a_1 g_1 \sin \mu, \quad b_2 = a_2 a_1 \sin \mu (3a_1 g_1 \sin \mu - g_2),$$
$$b_3 = 2a_1 a_2 g_1 \sin \mu + g_1 (1 - 2a_3^2 - 3a_0^2 \cos 2\mu - 2a_2^2),$$
$$b_4 = a_1 \sin \mu [g_1 (1 - 6a_3^2) \cos \varphi + 2a_2^2] - 2a_1 g_2^2 \sin \mu],$$
$$b_5 = g_1 (1 - 6a_3^2) \cos \mu - (2a_3^2 + 2a_2^2) \cos \mu + a_1 g_2^2 \cos 2\mu.$$

Equation (21) subtracted from eq. (22) gives

$$8d_1 x^2 + 4x^3 (d_2 \sin \varphi + d_3 \cos \varphi)$$
$$+ 2x (d_4 + d_5 \sin \varphi) + d_6 \cos \varphi + d_7 \sin \varphi + d_8 \cos \varphi = 0, \quad (24)$$

where

$$d_1 = a_1 g_1 (g_3 + a_3^2) \cos \mu - a_0^2 g_2 - g_4^2,$$
$$d_2 = 2a_2 a_1 \sin \mu [a_1 g_1 (2a_3^2 + 2a_0^2) \cos \mu - 2a_1^2 (a_0^2 - 2a_3^2) g_2],$$
$$d_3 = g_1 (3a_0^2 - 2a_1^2 + 4a_3^2 a_2^2 + 2a_0^2 a_2^2 - 2a_0 a_2^2$$
$$+ 4a_1 g_2 \cos \mu - 2a_0^2 a_2^2 \sin^2 \mu)$$
$$- 2a_1 \cos \mu (6g_1^2 - a_0^2 - 2a_1^2) g_2,$$
$$d_4 = 2a_0^2 a_2^2 (a_0^2 - 2a_3^2 a_2^2 - 3a_0^2 a_2^2) + (1 - 2a_3^2 - 2a_1^2$$
$$+ 4a_0^2 a_2^2 - 4a_1^2 g_2 - 16a_3^2 g_1^2 \cos \mu$$
$$+ 2a_1 g_1 \cos \mu (5a_0^2 - 2a_0^2 - a_1^2 - 3a_0^2 a_2^2 + 2a_1^2$$
$$+ 8a_3^2 a_0^2 - 2a_0^2 a_1^2 + 6a_3^2 a_1^2),$$
$$d_5 = 4(a_0^2 - 2a_3^2) g_2^2 + a_3 g_1 (g_3 - a_3^2) \cos 3\mu$$
$$+ a_1 g_1 \cos \mu (2a_0^2 - (1 - 2a_0^2)^2 + a_0^2 (1 + 2a_1^2 - 5a_0^2))$$
$$+ 4a_0^2 a_2^2 - 8a_0^2 a_2^2 + 2a_0^2 \cos 2\mu (2a_0^2 - a_1^2 - 2g_2^2),$$
$$d_6 = a_1 \sin \mu [g_1 (2a_0^2 (1 + 2a_0^2 + 2a_3^2 - 2a_0^2 - 4a_1^2)$$
$$+ 4a_0^2 a_2^2 + 4a_1 g_2 (g_3 - a_3^2) \cos \mu + 4a_1 (6a_3^2 - a_0^2) g_2 - 3g_1^2 \cos \mu),$$
$$d_7 = g_1 (1 - 4a_0^2 a_2^2 - 4a_0^2 a_2^2 - 2a_0^2 a_1^2 (4a_0^2$$
$$- 2a_0^2 [2 + 4a_1^2 - 2a_0^2 (3 - 6a_0^2)] - 2a_0^2 \cos^2 \mu [2a_0^2 a_2^2$$
$$+ 3 - 12a_3^2 a_0^2 - 4a_0^2 (1 + a_0^2) + a_3 (5a_0^2 - 7 + 2a_0^2)].$$
Using some algebra, we can prove that any \( x \) such that \( \sin x = 0 \) is not minimum point of \( R(\theta, \phi) \). Thus, we can take \( x \neq 0 \).

We first discuss the case \( k_1(\phi) \neq 0 \). From (25), we have

\[
x = \frac{b_2}{k_1(\phi)}.
\]

Substituting this value for \( x \) into eq. (23), we derive

\[
V^6 \{ 2b_1(c_1 + c_4)^2 + b_3(c_5 + c_7)(c_1 + c_4) \\
+ (c_5 + c_7)^2(b_1 a_2^2 - b_4 - 4b_1(c_6^2 + c_7^2)c_6^2) \\
+ [((c_2 - c_3)(2b_1 c_1 - b_2 c_4 - b_3 c_5 - b_2 c_7) \\
+ (c_5 - c_7)(3b_2 c_1 - b_3 c_4 + 2b_5 c_5 + 4b_4 c_6 - 2b_5 c_7 \\
+ 4b_1 c_6 c_7)] = 0,
\]

where \( V = \cot \phi \). Determining the solution(s) \( (\theta_1, \phi_1) \) of (26) and (25) satisfying the two equations in (19) if such solutions exist, and then determining the minimum point \( (\theta_1, \phi_1) \) such that \( \min \{ R(\theta_1, \phi_1) \} = R(\theta_1, \phi_1) \), we obtain

\[
R_{\min} = \begin{cases} 
\min \{ R(\theta_0, \phi_0), R(0, \phi) \}, & \text{if } a_2 = a_3, \\
\min \{ R(\theta_1, \phi_1), R(0, \phi) \}, & \text{if } a_2 \neq a_3. 
\end{cases}
\]

Otherwise, if no solutions following the above conditions exist, there is no minimum point for the case \( P(\theta, \phi)Q(\theta, \phi) \sin \theta \neq 0 \) (i.e. the above \( (\theta_1, \phi_1) \) does not exist), and the minimum

\[
R_{\min} = \begin{cases} 
\min \{ R(\theta_2, \phi_2), R(\theta_0, \phi_0), R(0, \phi) \}, & \text{if } a_2 = a_3, \\
\min \{ R(\theta_2, \phi_2), R(0, \phi) \}, & \text{if } a_2 \neq a_3. 
\end{cases}
\]

Now let us look at the case \( k_1(\phi) = 0 \). If it has common solution(s) with \( k_2(\phi) = 0 \), and there are one or more solutions \( (\theta_1, \phi_1) \) to eqs. (23) and (24) satisfying the two equations in (19), then we have

\[
R_{\min} = \begin{cases} 
\min \{ R(\theta_0, \phi_0), R(0, \phi) \}, & \text{if } a_2 = a_3, \\
\min \{ R(\theta_2, \phi_2), R(0, \phi) \}, & \text{if } a_2 \neq a_3. 
\end{cases}
\]

where \( R(\theta_2, \phi_2) = \min \{ R(\theta_1, \phi_1) \} \). Otherwise, the minimum \( R_{\min} \) is the same as that in (27).

Note that in the above three expressions for \( R_{\min} \), we use the properties \( R(0, \phi) = R(\pi, \phi) \) and \( R(\theta_0, \phi_0) = \sqrt{\theta_0(\theta_0, \phi_0)} = \sqrt{P(\theta_0, \phi_0)} \), where \( R(\theta_0, \phi_0) = Q(\theta_0, \phi_0) \).

For the quantum channel (3) with \( a_1 a_2 a_3 \sin \mu = 0 \), we also obtain the exact values of the maximal probabilities of success for controlled teleportation [34].

The exact value of the localizable entanglement in ref. [31]. – According to the definition \( E_{ij} = \max \sum \rho_j E(|\phi_j\rangle) \) for the LE in [31], the maximal probability \( p_{\max} \) in (18) is a kind of LE, but the measure of entanglement in (18) is not the concurrence. If \( E(|\phi_j\rangle) \) is chosen to be the concurrence of \( |\phi_2\rangle \), the exact value of the LE in [31] for a three-qubit state is

\[
E_{23} = \max \{ p_1 c_1 + p_2 c_2 = 2 \sqrt{a_1^2 a_3^2 - 2a_1 a_2 a_3 \cos \mu + (a_1^2 + a_3^2)a_4^2} \}
\]

The proof is as follows: Let

\[
f(\theta, \phi) = p_1 c_1 + p_2 c_2 = 2 \left( \frac{1}{4} a_1^2 a_3^2 \sin^2 \theta \right. \\
+ a_1 a_3 \sin \theta \sin^2 \theta \left[ a_1 a_4 \cos(\phi - \mu) - a_2 a_3 \cos \phi \right]
\]

\[50001-p5\]
Suppose initially that \( \sin \theta \neq 0 \), \( C_1 \neq 0 \), and \( C_2 \neq 0 \). From \( \frac{\partial f(\theta, \varphi)}{\partial \varphi} = 0 \), we have

\[
\begin{align*}
& a_0 a_4 a_2 a_3 \sin \varphi + a_1 a_4 \sin(\mu - \varphi) \\{ [a_1 a_4 \cos(\mu - \varphi) - a_2 a_3 \cos \varphi] \sin \theta + a_0 a_4 \cos \theta \} = 0.
\end{align*}
\]

Note that there is a \((\bar{\theta}, \varphi)\) such that \([a_1 a_4 \cos(\mu - \varphi) - a_2 a_3 \cos \varphi] \sin \theta + a_0 a_4 \cos \theta = 0\). By substituting \(a_1 a_4 \cos(\mu - \varphi) - a_2 a_3 \cos \varphi\) with \(-a_0 a_4 \cot \bar{\theta}\) in (29), we obtain \(f(\bar{\theta}, \varphi) = E_{23}\).

When \(\sin \theta = 0\),

\[
f(\theta, \varphi) = 2 \sqrt{a_0^2 a_4^2 - 2 a_0 a_4 a_2 a_3 a_4 \cos \mu + a_2^2 a_3^2} \leq E_{23}.
\]

When \(C_1 = 0\), but \(\sin \theta \neq 0\), then \(f(\theta, \varphi) = | \cos \frac{\theta}{2} | E_{23} \leq E_{23}\). When \(C_2 = 0\), but \(\sin \theta \neq 0\), then \(f(\theta, \varphi) = | \sin \frac{\theta}{2} | E_{23} \leq E_{23}\). Therefore, \(E_{23}\) in eq. (28) is the maximum of \(f(\theta, \varphi) = p_1 E(\Phi_1) + p_2 E(\Phi_2)\). Thus, we have the exact value \(E_{23}\) of the measure of localized entanglement in [31] for three-qubit states.

**Conclusion.** In summary, we have shown the necessary and sufficient condition that a three-qubit state can be collapsed to an EPR pair by an appropriate measurement on one qubit. In addition, all of the tripartite states that can be used for perfect controlled teleportation have been obtained. Finally, we have determined one kind of LE —the maximal probability of success for controlled teleportation for a general three-qubit state, and the exact value of another LE studied in [31].

***

We thank Prof. J. I. Cirac for his fruitful discussions and for his hospitality during our stay at Max-Planck-Institut für Quantenoptik. We also thank the anonymous referees for their helpful comments and suggestions. This work was supported by the National Natural Science Foundation (NSF) of China under Grant No. 10671054, Hebei NSF of China under Grant No. 07M006, and the Key Project of Science and Technology Research of Education Ministry of China under Grant No. 207011.

**REFERENCES**

[1] Bennett C. H., Brassard G., Crepeau C., Joyza R., Peres A. and Wootters W. K., *Phys. Rev. Lett.*, 70 (1993) 1895.

[2] Gisin N., Ribordy G., Tittel W. and Zbinden H., *Rev. Mod. Phys.*, 74 (2002) 145.

[3] Wang X. B., Hiroshima T., Tomita A. and Hayashi M., *Phys. Rep.*, 448 (2007) 1.

[4] Long C. L., Deng F. G., Wang C., Li X. H., Wen K. and Wang W. Y., *Front. Phys. China*, 2 (2007) 251.

[5] Fuji M., *Phys. Rev. A*, 68 (2003) 050302.

[6] An N. B., *Phys. Rev. A*, 68 (2003) 022321.

[7] Bowen W. P., Treps N., Buchler B. C., Schnabel R., Tapol C. H., Bachor H.-A., Simul T. and Lam P. K., *Phys. Rev. A*, 67 (2003) 032302.

[8] Johnson T. J., Bartlett S. D. and Sanders B. C., *Phys. Rev. A*, 66 (2002) 042326.

[9] Vaidman L., *Phys. Rev. A*, 49 (1994) 1473.

[10] Braunstein S. L. and Kimble H. J., *Phys. Rev. Lett.*, 80 (1998) 869.

[11] Son W., Lee J., Kim M. S. and Park Y.-J., *Phys. Rev. A*, 64 (2001) 064304.

[12] Bruss D., DiVincenzo D. P., Ekert A., Fuchs C. A., Macchiavello C. and Smolin J. A., *Phys. Rev. A*, 57 (1998) 2368.

[13] Gordon G. and Rigolin G., *Phys. Rev. A*, 73 (2006) 042309.

[14] Gao T., Yan F. L. and Wang Z. X., *Quantum Inf. Comput.*, 4 (2004) 186.

[15] Karlsson A. and Bourennane M., *Phys. Rev. A*, 58 (1998) 4394.

[16] Yang C. P., Chu S. I. and Han S., *Phys. Rev. A*, 70 (2004) 022329.

[17] Pati A. K., *Phys. Rev. A*, 61 (2000) 022308.

[18] Agrawal P. and Pati A. K., *Phys. Lett. A*, 305 (2002) 12.

[19] Pati A. K. and Agrawal P., *J. Opt. B: Quantum Semiclass. Opt.*, 6 (2004) S844.

[20] Yan F. L. and Wang D., *Phys. Lett. A*, 316 (2003) 297.

[21] Deng F. G., Li C. Y., Li Y. S., Zhou H. Y. and Wang Y., *Phys. Rev. A*, 72 (2005) 022338.

[22] Bouwmeester D., Pan J. W., Matthe K., Eibl M., Weinfurter H. and Zeilinger A., Nature, 390 (1997) 575.

[23] Furusawa A., Sorensen J. L., Braunstein S. L., Fuchs C. A., Kimble H. J. and Polzik E. S., Science, 282 (1998) 706.

[24] Nielsen M. A., Knill E. and Laflamme R., Nature, 396 (1998) 52.

[25] Hillery M., Bužek V. and Berthiaume A., *Phys. Rev. A*, 59 (1999) 1829.

[26] Aoun B. and Tarifi M., e-print quant-ph/0401076.

[27] Biham E., Huttner B. and Mor T., *Phys. Rev. A*, 54 (1996) 2651.

[28] Townsend P. D., Nature, 385 (1997) 47.

[29] Bose S., Vedral V. and Knight P. L., *Phys. Rev. A*, 57 (1998) 822.

[30] Gao T., Z. Naturforsch., 59a (2004) 597.

[31] Verstraete F., Pop M. and Cirac J. I., *Phys. Rev. Lett.*, 92 (2004) 027901.

[32] Acín A., Andriano V., Costa L., Jané E., Latorre J. I. and Tarrach R., *Phys. Rev. Lett.*, 85 (2000) 1560.

[33] Li W. L., Li C. F. and Guo G. C., *Phys. Rev. A*, 61 (2000) 034301.

[34] Gao T., Yan F. L. and Li Y. C., *Sci. China Ser. G*, 51 (2008) 1529.