A GENERALIZATION OF A Baire Theorem Concerning Barely Continuous Functions

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Abstract. We prove that if \( X \) is a paracompact space, \( Y \) is a metric space and \( f : X \to Y \) is a functionally fragmented map, then (i) \( f \) is \( \sigma \)-discrete and functionally \( F_\sigma \)-measurable; (ii) \( f \) is a Baire-one function, if \( Y \) is weak adhesive and weak locally adhesive for \( X \); (iii) \( f \) is countably functionally fragmented, if \( X \) is Lindelöf.

This result generalizes one theorem of Rene Baire on classification of barely continuous functions.

1. Introduction

A map \( f : X \to Y \) between topological spaces \( X \) and \( Y \) is said to be

- Baire-one, if it is a pointwise limit of a sequence of continuous maps \( f_n : X \to Y \);

- (functionally) \( F_\sigma \)-measurable or of the first (functional) Borel class, if the preimage \( f^{-1}(V) \) of any open set \( V \subseteq Y \) is a union of a sequence of (functionally) closed sets in \( X \);

- barely continuous, if the restriction \( f\upharpoonright F \) of \( f \) to any non-empty closet set \( F \subseteq X \) has a point of continuity.

Let us observe that the term ”barely continuous” belongs to Stephens [16]. However, barely continuous functions are also mentioned in literature as functions with the ”point of continuity property” (see, for instance, [13, 15]).

Among many other characterizations of Baire-one functions, the following classical Baire’s theorem is well-known [2].

Theorem A. For a complete metric space \( X \) and a function \( f : X \to \mathbb{R} \) the following conditions are equivalent:

(1) \( f \) is Baire-one;

(2) \( f \) is \( F_\sigma \)-measurable;

(3) \( f \) is barely continuous.

Recall that a map \( f : X \to Y \) between topological space \( X \) and a metric space \( Y \) is said to be fragmented, if for all \( \varepsilon > 0 \) and nonempty closed set \( F \subseteq X \) there exists a relatively open set \( U \subseteq F \) such that \( \text{diam} f(U) < \varepsilon \). The above notion was supposed by Jayne and Rogers [6] in connection with Borel selectors of certain set-valued maps.

Evidently, every barely continuous map between a topological and a metric spaces is fragmented. Moreover, if \( X \) is a hereditarily Baire space, then every fragmented function is barely continuous. The property of baireness of \( X \) is essential: let us consider a function \( f : \mathbb{Q} \to \mathbb{R} \), \( f(r_n) = 1/n \), where \( \mathbb{Q} = \{r_n : n \in \mathbb{N} \} \) is the set of all rational numbers. Notice that \( f \) is fragmented and everywhere discontinuous.

The next generalization of Baire’s theorem follows from [5, Corollary 7] and [1, Theorem 2.1].

Theorem B. Let \( X \) be a hereditarily Baire paracompact perfect space, \( Y \) is a metric space and \( f : X \to Y \). The following conditions are equivalent:

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(i) $f$ is $F_\sigma$-measurable and $\sigma$-discrete;

(ii) $f$ is fragmented.

Moreover, if $Y$ is a convex subset of a Banach space, they are equivalent to:

(iii) $f$ is Baire-one.

Let us observe that a similar result for $Y = \mathbb{R}$ was obtained by Mykhaylyuk [14]. The next theorem was recently proved in [10, Theorem 10].

**Theorem C.** If $X$ is a paracompact perfect space, $Y$ is a metric contractible locally path-connected space and $f : X \to Y$ is fragmented, then $f \in B_1(X,Y)$.

The aim of this note is to extend the above mentioned results on maps defined on paracompact spaces which are not necessarily perfect (recall that a topological space is perfect if every its open subset is $F_\sigma$).

The convenient tool of investigation of fragmented functions on non-perfect spaces is a concept of functional fragmentability introduced in [11]. We prove a technical auxiliary result (Lemma [2]) which connects regular families of functionally open sets in paracompact spaces with the notion of $\sigma$-discrete decomposability. As an application of this result we obtain (Theorem [3]) that for a paracompact space $X$, a metric space $Y$ and a functionally fragmented map $f : X \to Y$ the following propositions hold: (i) $f$ is $\sigma$-discrete and functionally $F_\sigma$-measurable; (ii) $f$ is a Baire-one function, if $Y$ is weak adhesive and weak locally adhesive for $X$; (iii) $f$ is countably functionally fragmented, if $X$ is Lindelöf.

### 2. Preliminaries

Let $\mathcal{U} = (U_\xi : \xi \in [0, \alpha])$ be a transfinite sequence of subsets of a topological space $X$. Then $\mathcal{U}$ is regular in $X$, if

(a) each $U_\xi$ is open in $X$;

(b) $U_0 = \emptyset$, $U_\alpha = X$ and $U_\xi \subseteq U_\eta$ for all $0 \leq \xi \leq \eta < \alpha$;

(c) $U_\gamma = \bigcup_{\xi < \gamma} U_\xi$ for every limit ordinal $\gamma \in [0, \alpha]$.

**Proposition 1.** [12, Proposition 1] Let $X$ be a topological space, $(Y,d)$ be a metric space and $\varepsilon > 0$. For a map $f : X \to Y$ the following conditions are equivalent:

(1) $f$ is $\varepsilon$-fragmented;

(2) there exists a regular sequence $\mathcal{U} = (U_\xi : \xi \in [0, \alpha])$ in $X$ such that $\text{diam}(f(U_{\xi+1} \setminus U_\xi)) < \varepsilon$ for all $\xi \in [0, \alpha]$.

If a sequence $\mathcal{U}$ satisfies condition (2) of the previous proposition, then it is called $\varepsilon$-associated with $f$ and is denoted by $\mathcal{U}_\varepsilon(f)$.

We say that an $\varepsilon$-fragmented map $f : X \to Y$ is functionally $\varepsilon$-fragmented if $\mathcal{U}_\varepsilon(f)$ can be chosen such that every set $U_\xi$ is functionally open in $X$. Further, $f$ is functionally fragmented, if it is functionally $\varepsilon$-fragmented for each $\varepsilon > 0$.

A map $f$ is (functionally) countably fragmented, if $f$ is (functionally) fragmented and every sequence $\mathcal{U}_\varepsilon$ can be chosen to be countable.
3. A Lemma

Let $\mathcal{A}$ be a family of subsets of a topological space $X$. Then $\mathcal{A}$ is called

- discrete, if each point $x \in X$ has a neighborhood which intersects at most one set from $\mathcal{A}$;
- strongly functionally discrete or, briefly, sfd-family, if there exists a discrete family $(U_A : A \in \mathcal{A})$ of functionally open subsets of $X$ such that $\bigcup A \subseteq U_A$ for every $A \in \mathcal{A}$.

Let us observe that every discrete family is strongly functionally discrete in collectionwise normal space.

Lemma 2. (cf. [3, Theorem 2]) Let $\mathcal{U} = (U_\xi : \xi \in [0, \alpha])$ be a regular family of functionally open sets in a paracompact space $X$. Then there exists a sequence $(\mathcal{F}_n)_{n \in \omega}$ of families $\mathcal{F}_n = (F_{\xi,n} : \xi \in [0, \alpha])$ such that

1. $U_\xi \setminus \bigcup_{\eta < \xi} U_\eta = \bigcup_{n \in \omega} F_{\xi,n}$ for all $\xi \in [0, \alpha)$,
2. $\mathcal{F}_n$ is an sfd-family in $X$ for all $n \in \omega$,
3. $F_{\xi,n}$ is closed in $X$ for all $n \in \omega$ and $\xi \in [0, \alpha)$.

Proof. For every $\xi \in [1, \alpha]$ we denote $P_\xi = U_\xi \setminus \bigcup_{\eta < \xi} U_\eta$. Since every $P_\xi$ is functionally $G_\delta$ in $X$ as a difference of two functionally open sets, we can choose a sequence $(G_{\xi,n})_{n \in \omega}$ of functionally open sets such that

$$P_\xi = \bigcap_{n \in \omega} G_{\xi,n} \quad \text{for all } \xi \in [1, \alpha) \quad \text{and} \quad G_{\xi,n} \subseteq U_\xi \quad \text{for all } \xi \in [1, \alpha), n \in \omega.$$  

We put

$$I = \bigcup_{k \in \omega} \omega^k$$

and define by the induction on $k$ sequences $(\mathcal{U}_i : i \in I)$ and $(\mathcal{V}_i : i \in I)$ of open coverings of $X$ such that

(a) $\mathcal{U}_0 = \mathcal{U}$;
(b) $\mathcal{V}_i$ is a locally finite barycentric refinement of $\mathcal{U}_i$ for all $i \in \omega^k$;
(c) for all $i \in \omega^k$ and $n \in \omega$ we have $\mathcal{U}_{(i,n)} = (U_{\xi,(i,n)} : \xi \in [0, \alpha])$, where

$$C_{\xi,i} = \left\{ x \in X : \operatorname{St}(x, \mathcal{V}_i) \subseteq \bigcup_{\eta < \xi} U_\eta \right\} \quad \text{and} \quad U_{\xi,(i,n)} = G_{\xi,n} \setminus C_{\xi,i}$$

for all $k \in \omega$. Let us observe that the existence of families $\mathcal{V}_i$ follows from the paracompactness of $X$ (see [4, Theorem 5.1.12]).

Notice that

$$C_{\xi,i} \subseteq \bigcup_{\eta < \xi} U_\eta,$$

because $\mathcal{V}_i$ is an open covering of $X$. Therefore, since $(P_\xi : \xi \in [0, \alpha])$ is a partition of $X$, $\mathcal{U}_{(i,n)}$ defined in (c) covers $X$ for all $n \in \omega$.

For every $x \in X$ we put

$$\mu(x) = \min \{ \xi \in [0, \alpha) : x \in U_\xi \}$$
and show that

\[(3.1) \quad \forall x \in X \exists i \in I : \text{St}(x, \mathcal{Y}_i) \subseteq U_\mu(x).\]

Assume to the contrary that there exists \(x \in X\) such that (3.1) is not true. Since each family \(\mathcal{Y}_i\) is locally finite refinement of \(\mathcal{U}\), for every \(i \in I\) there is \(\xi_i\) such that \(\text{St}(x, \mathcal{Y}_i) \subseteq U_{\xi_i}\). Let \(\xi(x) = \min\{\xi_i : i \in I\}\). Then \(\xi(x) > \mu(x)\). Therefore, \(x \not\in P_{\xi(x)}\) and we can take \(j \in \omega\) such that \(x \not\in G_{\xi(x),j}\).

From the definition of the sequence \(\mathcal{U}(i,j)\) it follows that \(x \not\in U_{\xi(i,j)}\). Since \(\text{St}(x, \mathcal{Y}_i) \subseteq U_{\xi(x)}\), we have \(x \not\in U_{\xi(i,j)}\) for all \(\xi > \xi(x)\). Therefore,

\[(3.2) \quad x \not\in \bigcup_{\xi \geq \xi(x)} U_{\xi(i,j)}\]

By (b) there exists \(\beta \in [0, \alpha)\) such that \(\text{St}(x, \mathcal{Y}(i,j)) \subseteq U_{\beta(i,j)}\). It follows from (3.2) that \(\beta < \xi(x)\).

The inclusion \(U_{\beta(i,j)} \subseteq U_\beta\) contradicts to the choice of \(\xi(x)\).

Let \((\mathcal{Y}_i : i \in I) = (\mathcal{H}_n : n \in \omega)\). Now for all \(\xi \in [0, \alpha)\) and \(n \in \omega\) we put

\[D_{\xi,n} = \{x \in P_\xi : \text{St}(x, \mathcal{H}_n) \subseteq U_\xi\}\]

for all \(\xi \in [0, \alpha)\). Property (3.1) implies that \(P_\xi \subseteq \bigcup_{n \in \omega} F_{\xi,n}\). Now assume that \(x \in F_{\xi,n}\) for some \(\xi\) and \(n\). Put \(O = \text{St}(x, \mathcal{H}_n) \cap U_\mu(x)\). Then \(O \cap D_{\xi,n} \neq \emptyset\). Take any \(y \in O\). Since \(y \in U_\mu(x)\) and \(y \in P_\xi\), \(\mu(x) \geq \xi\). The inclusions \(\text{St}(y, \mathcal{H}_n) \subseteq U_\xi\) and \(y \in \text{St}(x, \mathcal{H}_n)\) imply that \(x \in U_\xi\). Hence, \(\mu(x) \leq \xi\).

Therefore, \(\mu(x) = \xi\). Then \(x \in P_\xi\). Moreover, it follows that the family \(\mathcal{F}_n = (F_{\xi,n} : \xi \in [0, \alpha))\) is discrete in \(X\).

Since \(X\) is paracompact, \(X\) is collectionwise normal, which implies that \(\mathcal{F}_n\) is strongly functionally discrete family for all \(n \in \omega\).

\[\square\]

4. AN APPLICATION OF LEMMA TO CLASSIFICATION OF FRAGMENTED FUNCTIONS

Let \(X\) be a topological space. Recall that a topological space \(Y\) is

- **an adhesive for \(X\)**, if for any disjoint functionally closed sets \(A\) and \(B\) in \(X\) and for any two continuous maps \(f, g : X \to Y\) there exists a continuous map \(h : X \to Y\) such that \(h|_A = f|_A\) and \(h|_B = g|_B\);

- **a weak adhesive for \(X\)**, if for any two points \(y, z \in Y\) and disjoint functionally closed sets \(A\) and \(B\) in \(X\) there exists a continuous map \(h : X \to Y\) such that \(h|_A = y\) and \(h|_B = z\);

- **a locally weak adhesive for \(X\)**, if for every \(y \in Y\) and every neighborhood \(V \subseteq Y\) of \(y\) there exists a neighborhood \(U\) of \(y\) such that \(U \subseteq V\) and for every \(z \in U\) there exists a continuous map \(h : X \to V\) with \(h|_A = y\) and \(h|_B = z\).

It was proved in [9] Theorem 2.7] that any topological space \(Y\) is an adhesive for every strongly zero dimensional space \(X\); a path-connected space \(Y\) is an adhesive for any compact space \(X\) each point of which has a base of neighborhoods with discrete boundaries; \(Y\) is an adhesive for any space \(X\) if and only if \(Y\) is contractible. Moreover, it is easy to see that every (locally) path-connected space is a (locally) weak adhesive for any \(X\).

A family \(\mathcal{B}\) of subsets of a topological space \(X\) is said to be a base for a map \(f : X \to Y\), if for every open set \(V \subseteq Y\) there exists a subfamily \(\mathcal{B}_V\) of \(\mathcal{B}\) such that \(f^{-1}(V) = \bigcup_{B \in \mathcal{B}_V} B\).

A map \(f : X \to Y\) is \(\sigma\)-discrete, if there is a sequence \((\mathcal{H}_n)_{n \in \omega}\) of discrete families of sets in \(X\) such that the family \(\bigcup_{n \in \omega} \mathcal{H}_n\) is a base for \(f\).
Theorem 3. Let $X$ be a paracompact space, $Y$ be a metric space and $f: X \to Y$ be a functionally fragmented map. Then

(1) $f$ is $\sigma$-discrete and functionally $F_\sigma$-measurable;

(2) $f$ is a Baire-one function, if $Y$ is weak adhesive and weak locally adhesive for $X$;

(3) $f$ is countably functionally fragmented, if $X$ is Lindelöf.

Proof. 1) For every $n \in \mathbb{N}$ we choose a family $\mathcal{U}_{1/n}(f) = (U_{\xi,n} : \xi \in [0, \alpha_n])$ consisting of functionally open sets $U_{\xi,n}$. We claim that the family $\mathcal{P} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n$ is a base for $f$, where $\mathcal{P}_n = (U_{\xi,n} \setminus \bigcup_{\eta<\xi} U_{\eta,n} : \xi \in [0, \alpha_n])$, $n \in \mathbb{N}$. Indeed, fix an open set $V$ in $Y$ and take any $x \in f^{-1}(V)$. Find $n \in \mathbb{N}$ such that an open ball $B$ with the center at $f(x)$ and radius $1/n$ contains in $V$. Since $\mathcal{P}_n$ is a partition of $X$, there exists $\xi \in [0, \alpha_n]$ such that $x \in P_{\xi,n} = U_{\xi,n} \setminus \bigcup_{\eta<\xi} U_{\eta,n}$. Evidently, $f(P_{\xi,n}) \subseteq B \subseteq V$.

By Lemma 2 for every $n \in \mathbb{N}$ there exists a sequence $(\mathcal{F}_{n,k})_{k \in \omega}$ of families $\mathcal{F}_{n,k} = (F_{\xi,n,k} : \xi \in [0, \alpha_n])$ which satisfies conditions (1)–(3) of Lemma 2. Properties (1) and (2) imply that the family $\mathcal{B} = \bigcup_{k,n} \mathcal{F}_{n,k}$ is a $\sigma$-discrete base for $f$ consisting of closed sets. It follows that $f$ is $F_\sigma$-measurable and a $\sigma$-discrete map. Finally, [7, Proposition 2.6 (iv)] implies that $f$ is functionally $F_\sigma$-measurable.

Property 2) follows from 1) and [8, Theorem 3.2].

3) It is enough to show that every regular sequence consisting of functionally open sets in a Lindelöf space $X$ is countable.

Let $\mathcal{U} = (U_{\xi} : \xi \in [0, \alpha])$ be a regular covering of $X$ by functionally open sets $U_{\xi}$. There exists a sequence $(\mathcal{F}_n)_{n \in \omega}$ of families in $X$ such that conditions (1)–(3) of Lemma 2 are valid. Notice that every family $\mathcal{F}_n$ is at most countable, since it is discrete and $X$ is Lindelöf. We consider an enumeration $\{F_k : k \in \omega\}$ of the family $\bigcup_{n \in \omega} \mathcal{F}_n$. Let $\varphi : [0, \alpha) \to 2^\omega$ be a map,

$$\varphi(\xi) = \{k \in \omega : F_k \subseteq P_\xi\}.$$  

Since $(\varphi(\xi) : \xi \in [0, \omega_1))$ is a family of mutually disjoint subsets of $\omega$, it is at most countable. \hfill \Box

We do not know the answer to the following question.

**Question 1.** Is it true that every fragmented Baire-one real-valued function defined on a paracompact Hausdorff space is functionally fragmented?

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