Zig-zagging in a Triangulation

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1 Introduction

We present an oblivious walk for point-location in 2-dimensional triangulations and a corresponding, strictly monotonically decreasing distance measure.

2 Problem Description

Let $T = (V, F, E)$ be a 2-dimensional triangular tiling of $\mathbb{R}^2$ in quad-edge representation [2], consisting of a set of vertices $V$, a set of faces $F$ and a set of half-edges $E$. We say $T$ is locally finite iff for any circle in $\mathbb{R}^2$ it holds that the set consisting of all edges (vertices, faces) in $T$ that intersect the circle (circumference or interior) is finite.

Let $e \in E$ be some half-edge of this mesh starting at vertex $e_1$ and ending at vertex $e_2$. With $\text{inv}(e)$ we denote the twin of $e$ such that $\text{inv}(e)_1 = e_2$ and $\text{inv}(e)_2 = e_1$. With $\text{face}(e)$ we denote the face $f \in F$ that is, by convention, to the left of the half-edge $e$. With $\text{next}(e)$ we denote the next half-edge (in the face winding order) that is also a side of $\text{face}(e)$, i.e.: $\text{face}(\text{next}(e)) = \text{face}(e)$. With $\text{prev}(e)$ we denote the previous half-edge (in the face winding order) that is also a side of $\text{face}(e)$, i.e.: $\text{face}(\text{prev}(e)) = \text{face}(e)$. Because we are dealing with a triangular mesh, in particular, it holds: $\text{next}(\text{next}(e)) = \text{prev}(e)$ and vice versa.

The problem of point-location [1] that we consider in this note can be formulated as follows: Given a point $p \in \mathbb{R}^2$ and an initial half-edge $e_{\text{init}} \in E$ find some half-edge $e_{\text{goal}} \in E$, using only $\text{next}(\cdot)$, $\text{prev}(\cdot)$ and $\text{inv}(\cdot)$ operations, such that $p \in \text{face}(e_{\text{goal}})$, i.e.: the point is on the face. We, additionally, require the algorithm to be oblivious, meaning that the next chosen half-edge only depends on the last-visited half-edge and the goal location.

3 Example of a Zig-Zagging Walk

A walk in a triangulation is a sequence of faces. We concern ourselves here only with directed walks, such walks attempt to close the distance between some starting edge (or its corresponding face) and a target point. Directed walks are
Figure 1: An example of a zig-zagging motion exhibited during a walk.

important in practice as they are frequently used to navigate meshes as efficient, flexible space-partitioning data-structures.

One possible application of a directed walk would be to translate mouse-clicks to selected faces in a CAD-application. Because such type of interaction exhibits a lot of locality —i.e.: clicks often happen in close proximity to each other— it is much more efficient to walk to the corresponding face from the last-visited one then to perform a, potentially exhaustive, sweep of the entire mesh, which, in representative cases, may measure in the order of millions or even billions of faces.

Before we turn to the exposition of the zig-zagging walk, its corresponding distance measure and termination proof, let us first investigate briefly why a zig-zagging motion is a useful and intuitive analogy in thinking about point-location in two dimensional triangulations.

Since an oblivious walk “forgets” its original point of departure as well as the route followed to get to the current face, it suffices to consider only the current half-edge and the corresponding current face (which, by our convention, is always to the left of the current half-edge) when thinking about the point-location problem.

As an example consider Figure 1. Assume that at some point during the walk we arrive at half-edge $e$ and corresponding face $f = \text{face}(e)$. The target point $p$ is “straight ahead” in the direction of the tip of $f$. However, going straight is not an option, because the only faces that are directly linked to $f$ (in the quad-edge representation) are $f_l = \text{face}(\text{inv(prev}(e)))$ and $f_r = \text{face}(\text{inv(next}(e)))$. Since “turning back” is not an option either, any route to the target point $p$ will have to go either over the left or over the right, hence, any route will make a “zig-zag” motion to reach the target.
4 Distance Measure

For some half-edge \( e \) and point \( p \) such that \( e \) does not intersect \( p \) we define the oriented distance of \( e \) to \( p \) denoted \( e \downarrow p \) as a pair \([d, \alpha]\) where \( d \) is defined to be the Euclidian distance of point \( p \) to the closest point \( e_p \) on the (finite!) line segment \( e_1; e_2 \) corresponding to half-edge \( e \) and \( \alpha \) is defined to be the (smallest) angle between \( e_1; e_2 \) and the ray \( p; e_p \) or zero in case \( p = e_p \). Since the shortest ray from \( p \) to \( e \) ends at either \( e_1 \), \( e_2 \) or the orthogonal projection of \( p \) on \( e_1; e_2 \), it follows \( \alpha \geq \pi/2 \). For some examples of how this oriented distance measure is defined under various relative configurations of the current edge and the target point, see Figure 2.

We now define a strict partial ordering \( < \) on oriented distances such that

\[ [d, \alpha] < [d', \alpha'] \text{ iff } d < d' \lor (d = d' \land \alpha < \alpha') \]

In order to see how this distance measure allows us to compare various candidate successor edges consider the example in Figure 3. Here we show a classification of the points in the space that is to the left of the line supporting the current half-edge \( e \) (but not on the current face). So let \( p \in \mathbb{R}^2 \) be the target point and let \( l = \text{inv}(\text{next}(e)) \) and \( r = \text{inv}(\text{prev}(e)) \) be the two possible successor half-edges.
edges, now let $[d_l, α_l] = ll ⊖ p$ and $[d_r, α_r] = rr ⊖ p$, we classify the space where the target point $p$ may reside as follows:

- $d_l < d_r$ and the point of $l$ closest to $p$ is $l_1$ (I)
- $d_l < d_r$ and the point of $l$ closest to $p$ is strictly in-between $l_1$ and $l_2$ (II)
- $d_l = d_r$ and $α_l < α_r$ (III)
- $d_l = d_r$ and $α_l = α_r$ (IV)
- $d_l > d_r$ and $α_l > α_r$ (III)
- $d_l > d_r$ and the point of $r$ closest to $p$ is strictly in-between $r_1$ and $r_2$ (II)
- $d_l > d_r$ and the point of $r$ closest to $p$ is $r_2$ (I)

Note that area $I_l$ or $I_r$ becomes a line in case the triangle is right in its left or right base angle respectively and empty in case the corresponding angle becomes obtuse. All the other areas remain non-empty no matter how obtuse or how thin the triangle becomes.

5 Oriented Distance Walk

We are now in a position to describe the algorithm properly. Let $e ∈ E$ be some initial half-edge, and let $p ∈ R^2$ be the goal location. The algorithm then goes as follows:
Figure 4: Lemma 5.1 step 1: $p$ lies somewhere in the gray cone

1. if $p$ is to the right–of $e$ then $e \leftarrow \text{inv}(e)$
2. while $p \notin \text{face}(e)$
3. \hspace{1em} $l, r \leftarrow \text{inv}(\text{next}(e)), \text{inv}(\text{prev}(e))$
4. \hspace{1em} if $(l \ll p) < (r \ll p)$ then $e \leftarrow l$ else $e \leftarrow r$

In line 1 we bootstrap the algorithm by ensuring the invariant that $p$ is always to the left of $e$. In line 2 we check the termination condition that $p$ is on the current face (i.e.: the face to the left of the current half-edge). In line 3 we compute the two possible successor edges. In line 4 we pick the successor edge that has the smallest oriented distance to the target point.

Note that, in this algorithm, ties are broken by defaulting to the right-successor edge. Clearly, it is possible to have a leftmost version by defaulting to the left instead, or a non-deterministic version by allowing either the left or the right-successor edge whenever $l \ll p$ is not strictly less or greater-than $r \ll p$. For the remainder we will assume the non-deterministic version of the algorithm, all the results that follow will then hold also for (both) deterministic versions, as a corollary.

We shall first prove a number of properties of the walk which consequently will allow us to prove the main result, i.e.: guaranteed termination of the zig-zagging walk. First we show that the point will, as an invariant, always be to the left of the current half-edge.

**Lemma 5.1** (Target in Half-space). *At every iteration of the walk it holds that $p$ lies in the half-space to the left of $e$.*

**Proof.** By induction on the number of iterations. The basis is ensured as a postcondition of line 1. For the inductive step assume, w.l.o.g., that the left successor edge $l$ is chosen in line 4, which implies, in the non-deterministic version, $l \ll p \leq r \ll p$. Now, for contradiction, assume that $p$ lies strictly to the right of $l$. By inductive hypothesis we have that $p$ is to the left of $e$ which places it in the cone formed by $l$ and $e$ (cf. Figure 4). In addition we have the
Figure 5: Lemma 5.1 step 2: $p$ lies somewhere in the gray, truncated cone

Figure 6: Lemma 5.1 step 3: $p$ lies somewhere in the gray cone

Figure 7: Lemma 5.2 step 1: $p$ lies somewhere in the gray zone
Figure 8: Lemma 5.2, step 2a: $p$ lies somewhere in the gray zone

Figure 9: Lemma 5.2, step 2b: $p$ lies somewhere in the gray cone
loop invariant $p \notin \text{face}(e)$ which places $p$ further inside the now truncated cone formed by $l$, $e$, and $r$ (cf. Figure 6). Now let $[d_l, \alpha_l] = l \parallel p$ and $[d_r, \alpha_r] = r \parallel p$, and note that a ray cast from any point in the truncated cone to $l$ must pass through $r$ hence we obtain: $d_l \geq d_r$. Now, by assumption, $l \parallel p \leq r \parallel p$ which implies $d_l \leq d_r$. From the latter two facts it follows that $d_l = d_r$ and $\alpha_l \leq \alpha_r$, which places $p$ still further inside the cone formed by $l$ and the line orthogonal to $r$ starting at the tip vertex (cf. Figure 4). However, we see from the diagram that this implies $\alpha_l = \alpha_r + \angle ltr$ which directly contradicts our assumption that $\alpha_l \leq \alpha_r$. □

Lemma 5.2 (Monotonicity). With every step of the zig-zagging walk the oriented distance of the current edge to the target strictly decreases.

Proof. Assume for contradiction that the oriented distance does not strictly decrease, i.e.: $(e \parallel p) \leq (l \parallel p)$ and $(e \parallel p) \leq (r \parallel p)$. First note that by Lemma 5.1 we have that $p$ is to the left of $e$, and by the loop invariant we have that $p$ is outside of the current face (cf. Figure 7). Now let $e_p$ be the point on $e$ closest to $p$. We make a case distinction. As a first case consider $e_p \neq e_1$ and $e_p \neq e_2$, i.e.: $e_p$ is properly contained between $e_1$ and $e_2$. In this case the projection of $p$ on $e$ must have been orthogonal (cf. Figure 5). However this means that the ray from $p$ to $e$ must have passed through one of the two sides $r$ or $l$ and through the interior of the face which contradicts our assumption that $(e \parallel p) \leq (l \parallel p)$ and $(e \parallel p) \leq (r \parallel p)$. For the second case consider $e_p = e_1$ or $e_p = e_2$. So, w.l.o.g, assume $e_p = e_1$. Now let $[d_l, \alpha_l] = l \parallel p$ and $[d_e, \alpha_e] = e \parallel p$. Since $e_1 = l_1$ is a shared point with $l$ it must hold $d_l \leq d_e$, and hence, by assumption that $(e \parallel p) \leq (l \parallel p)$, it must follow that $d_l = d_e$ and $\alpha_e \leq \alpha_l$, this places $p$ in the cone between $e$ and the line orthogonal to $l$ emanating from $e_1$ (cf. Figure 6). However, we see from the diagram that this implies $\alpha_e = \alpha_l + \angle le$, which directly contradicts our assumption that $\alpha_e \leq \alpha_l$. □

We are now in position to formulate and prove finite time termination theorem. For this we need local finiteness of $T$ to rule out the possibility of asymptotic walks in dense tessellations.

Theorem 5.3. If $T$ is locally finite, the zig-zagging walk always terminates.

Proof. Let $p \in \mathbb{R}^2$ be the target point and $e \in E'$ the initial half-edge, let $[d, \alpha] = e \parallel p$, and let $E' = \{ e' \in E \mid (e' \parallel p) \leq (e \parallel p) \}$. We say $E'$ is the neighborhood of $e$ and $p$. Note that all edges in the neighborhood $E'$ intersect a disc centered at $p$ with radius $d$, hence, by local finiteness of $T$, it follows $E'$ is a finite set. Now by Lemma 5.2 we have that the zig-zagging walk visits a sequence of half-edges in $E'$ that exhibits a strictly monotonically decreasing chain of oriented distances, and, since $E'$ is finite, this implies termination. □

If we define the local size of the mesh as the size of the neighborhood as defined in the proof, we actually get the stronger result that the zig-zagging walk always terminates in a number of steps that is bounded by the local size of the mesh.
5.1 Spiraling in a Tetrahedralization

One of the obvious next questions to consider is whether this result can be generalized from two to three dimensions. Although we defer formal treatment to future work, we already give some intuition and a proof sketch as to how this might be done.

First observe that, in two dimensions, we can define the “current neighborhood circle” to be the smallest circle surrounding the target point that intersects the current half-edge. A zig-zagging walk then consists of steps that change the angle of the current half-edge to make it more tangent to the current neighborhood circle (cf. Figure 10) eventually followed by steps that approach the target and thereby shrink the radius of the current neighborhood circle (cf. Figure 11).

In three dimensions the analogon to the neighborhood circle would be the smallest sphere surrounding the target point that intersects the face. A spiraling
walk through a tetrahedralization would then consist, analogously, of steps that change the orientation of the current face to make it more tangent to the current neighborhood sphere (cf. Figure 12) eventually followed by steps that approach the target and thereby shrink the radius of the current neighborhood sphere (cf. Figure 13).

In the two dimensional case a single angle was sufficient to characterize the orientation of the current half-edge w.r.t. the target point. In the three dimensional case we need at least 2 angles to characterize the orientation of the current face w.r.t. the target point.

In particular we define the ray as the shortest line segment from the target point to the closest point on the face, we must then fix a roll roll axis as a line-segment that lies in the face and intersects the ray.

Given such a roll axis we then define the pitch axis as the line that is orthogonal to both the ray as well as the roll axis. We further define the pitch
angle of the face as the angle between the ray and the roll axis and we define the roll angle as the angle between the face and the plane defined by the pitch and roll axis.

We then define a pitch minimizing roll axis to be any roll axis that minimizes the pitch angle, and we define a minimizing roll axis to be any pitch minimizing roll axis that minimizes the roll angle, i.e.: it minimizes firstly the pitch angle and secondly the roll angle. We refer to the latter as the minimal pitch angle and the minimal roll angle.

For some face \( f \) and some point \( p \) where \( f \) does not contain \( p \), we can now define the oriented distance of \( f \) to \( p \), notation: \( f \bowtie p \), as the triple \([d, \alpha, \beta]\) where \( d \) is the Euclidean distance of the target point to the closest point on the face, \( \alpha \) is the minimal pitch angle and \( \beta \) is the minimal roll angle. We then define a strict partial order as follows:

\[
[d, \alpha, \beta] < [d', \alpha', \beta'] \text{ iff } d < d' \lor (d = d' \land \alpha < \alpha') \lor (d = d' \land \alpha = \alpha' \land \beta < \beta')
\]

6 Discussion and Future Work

We defer to future work the formal treatment of the three dimensional case (for which we already gave some intuitions in the previous section), and, possibly, a full generalization to \( n \)-dimensional simplexes.

Another possible direction that we are interested in is to look for a robust version of the zig-zagging walk that does not require infinite precision in the evaluation of the predicates. The hope there is that the clear termination proof in the infinite precision case can function as a starting point in proving termination in the finite precision case.

References

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