On Bias and Rank

David Kazhdan  Tomer M. Schlank

August 20, 2018

Abstract

Given a hypersurface $X \subset \mathbb{P}^{N+1}_C$ Dimca gave a proof showing that the cohomologies of $X$ are the same as the projective space in a range determined by the dimension of the singular locus of $X$. We prove the analog of Dimca’s result case when $C$ is replaced with an algebraically closed field of finite characteristic and singular cohomology is replaced with $\ell$-adic étale cohomology. The Weil conjectures allow relating results about étale cohomology to counting problems over a finite field. Thus by applying this result, we are able to get a relationship between the algebraic properties of certain polynomials and the size of their zero set.

1 introduction

Let $X \subset \mathbb{P}^{N+1}_C$ be a hypersurface. If $X$ is smooth, Lefschetz hyperplane theorem implies that all the cohomologies of $X$ but the one in dimension $N$ are the same as for the projective spaces. This result has a qualitative generalisation due to Dimca [Dim12]:

Lemma 1.1. Let $X \subset \mathbb{P}^{N+1}_C$ be a projective hyper-surface of dimension $n$ such that the singular locus of $X$ is in at least in codimension $c$ then:

$$H^m(X(C)) = H^m(\mathbb{P}^N(C))$$

for $2N - (c + 2) \leq m \leq 2N$.

The first main result of this paper is Lemma 5.1 which is the analog of Dimca’s result case when $C$ is replaced with an algebraically closed field of finite characteristic and singular cohomology is replaced with $\ell$-adic étale cohomology. The Weil conjectures allows to relate results about étale cohomology to counting problems over finite field. Thus by applying Lemma 5.1 we able to get a relationship between
the algebraic properties of certain polynomials and the size of their zero set.

We thank Dennis Gaitsgory and Michael Finkelberg for useful discussions.

2 The results

More specifically Let $p$ be a prime and $q = p^m$, $k = \mathbb{F}_q$. We denote by $k_n$ the extension of $k$ of degree $n$ and by $\bar{k}$ the algebraic closure of $k$.

Let $V$ be a vector space over $k$ of dimension $N$, and let $F \in k[V]$ be a polynomial.

For every $n \in \mathbb{N}$, $F$ defines a map

$$F|_n : V(k_n) \to k_n$$

As a map of finite sets $F|_n$ induces a distribution on $k_n$ according to size of the fibres. In this paper we ask the following:

Assume that for $n$ large, this distribution is very far from uniform. Can we deduce a structural result on $F$? Can one say that it is “degenerate” in some sense? In [BL15], Bhowmick and Lovett prove a beautiful result along this lines using combinatorial methods. We give a similar result based on Lemma 5.1. To state our result in a precise way we need first to define a qualitative measurements of both the non-uniformity of the aforementioned distribution and the “degeneracy” of $F$. In next two subsections we deal with these two tasks one by one.

2.0.1 The bias of a polynomial

In the notation as above let $\nu_n$ be the uniform probability measure on the elements of $V(K_n)$. We denote

$$\mu^F_n := (F|_n)_*(\nu_n)$$

As the push-forward probability measure. Simply put we have for every $t \in k_n$

$$\mu^F_n(\{t\}) = \frac{\#(F|_n^{-1}(t))}{\#V(k_n)} = \frac{\#(F|_n^{-1}(t))}{q^{nN}}$$

We wish to measure how far is $\mu^F_n$ from the uniform distribution on $k_n$. As a measure we shall take

$$b_n = -\log_{q^n}(\max_{t,s \in k_n}(|\mu^F_n(t) - \mu^F_n(s)|)).$$
To get the asymptotic behaviour we put

\[ B(F) = \limsup(b_n^{-1}). \]

We call \( B(F) \) the bias of \( F \).

### 2.0.2 The rank of a polynomial

Now let \( G \in \bar{k}[V] \) be homogeneous of degree \( d > 1 \). We say that \( G \) admits an \( r \)-factorization if

\[ G = \sum_{i=1}^{r} Q_i P_i \]

for \( \deg Q_i, \deg P_i < d \). The minimal \( r \) such that \( G \) admits an \( r \)-factorization is called The rank of \( G \) and will be denoted by \( \mathcal{R}(G) \).

More generally for \( F \in \bar{k}[V] \) of degree \( d > 1 \) (not necessarily homogeneous) we denote \( \mathcal{R}(F) \) := \( \mathcal{R}(\tilde{F}) \) Where \( \tilde{F} \) the homogeneous part of \( F \) of degree \( d \) considered as polynomial over \( \bar{k} \).

### 3 The relationship of bias and rank

We are now ready to state the main result

**Theorem 3.1.** Let \( b \in \mathbb{R}_{>0}, \ d \in \mathbb{N}_{\geq 2} \) then there exist a constant \( c(b,d) \) such that for every prime \( p > d, \ q = p^m \) a prime power, \( V \) a vector space over \( \mathbb{F}_q \) and \( F \in \mathbb{F}_q[V] \) a polynomial of degree \( d \) such that 

\[ \mathcal{R}(F) > c(b,d) \]

we have

\[ B(F) > b \]

The passage from the rank to the bias will be achieved by analysing the codimension of the singular locus of the fibers (or their respective projectivizations). We shall set up some notation.

First we denote the by \( X^F \) the hypersurface in \( \mathbb{P}^{N-1} \) defined by \( \tilde{F} \). For any \( t \in \mathbb{F}_q \) we denote by \( Y_t^F \) the projective variety in \( \mathbb{P}^N \) which is the closure of the affine hypersurface \( F = t \). When \( F \) is clear from the context we shall omit it from the notation.

We now have

\[ F|_n^{-1}(t) = Y_t^F(k_n) \setminus X^F(k_n). \]

**Definition 3.2.** We say that a projective \( X \) is \( c \)-regular for an integer \( c \) if the singular locus of \( X \) is of codimension \( c \) or more. We shall say that a polynomial \( F \) is \( c \)-good for an integer \( c \) if \( X^F \) and \( Y_t^F \) are \( c \)-regular for all \( t \in \bar{k} \).
To estimate $\# F^{-1}_n(t)$ we now use the Weil Conjectures proved by Deligne.

Now let $Z$ be a projective variety of dimension $e$ defined over a field $\mathbb{F}_q$ with $\overline{Z}$ the base change to the algebraic closure and $\phi_q$ the Frobenius map. By the Weil conjectures we have

$$\# Z(k_n) = \sum_{i=0}^{2e} (-1)^i \text{Tr} \left( \phi_q^n | H^i_{\text{et}}(\overline{Z}) \right)$$

By Weil II the eigenvalues of $\phi_q$ on $H^i_{\text{et}}(\overline{Z})$ have at most absolute values $q^{i/2}$ We thus get that if

$$M_Z := \sum_{i=0}^{2e} \dim (H^i_{\text{et}}(\overline{Z}))$$

Then for every integer $m$ we have

$$\left| \# Z(k_n) - \sum_{i=2e-m}^{2e} (-1)^i \text{Tr} \left( \phi_q^n | H^i_{\text{et}}(\overline{Z}) \right) \right| \leq M_Z q^{n(2e-m)}$$

Now we use the following lemma which is due to Dimca in [?] in the case of singular cohomology of complex varieties. We give in the last section a proof for $\ell$-adic cohomology over an algebraic closed field of any characteristic (Which is required for the use of the Weil conjectures)

**Lemma 3.3.** Let $Z \subset \mathbb{P}^{n+1}$ be a c-regular projective hyper-surface of dimension $n$

$$H^m(Z) = H^m(\mathbb{P}^n)$$

for $2n - (c + 2) \leq m \leq 2n$

Algebraic maps respects the Frobenius action. So from comparing with the projective space we get

**Lemma 3.4.** Let $Z$ be a c-regular projective variety of dimension $N$ over $\mathbb{F}_q$ then we have

$$\left| \# Z(k_n) - \sum_{i=N-\frac{N}{e}+1}^{N} q^{ni} \right| \leq M_Z q^{n(N - \frac{N}{e})}$$

Now by lemma 5.4 since all the $Y^F_t$ are defined by polynomials of the same degree and number of variables we have some $M_F$ such that

$$M_{X^F}, M_{Y^F_t} \leq M_F$$
for all $t \in \overline{F}_q$. Now by lemma 3.4 and the formula:

$$\#F|_{n=1}^{-1}(t) = \#Y_t^F(k_n) - \#X_t^F(k_n)$$

we have:

**Lemma 3.5.** Let $F$ be a $c$-good homogenous polynomial of $N$ variables

over $\mathbb{F}_q$ then we have for every $t \in k_n$

$$\left| \#F|_{n=1}^{-1}(t) - q^{n(N-1)} \right| \leq M_F q^{n(N-\frac{c}{2})}$$

and

$$\left| \mu_n^F(\{t\}) - \frac{1}{q^n} \right| \leq \frac{M_F}{q^{n(\frac{c}{2}-1)}}$$

**Remark 3.6.** Not that the proof above really proves a stronger statement namely if $F$ and $G$ are degree $d$ polynomials with the same homogenous part and which are $c$-good, we have

$$\left| \mu_n^F(\{t\}) - \frac{1}{q^n} \right| \leq \frac{2M_F}{q^{n(\frac{c}{2}-1)}}$$

A conclusion of lemma 3.5 in that

$$\max_{t,s \in k_n} |\mu_n^F(\{t\}) - \mu_n^F(\{s\})| \leq \frac{2M_F}{q^{n(\frac{c}{2}-1)}}$$

and thus

$$b_n = -\log_q \left( \max_{t,s \in k_n} (|\mu_n^F(t) - \mu_n^F(s)|) \right) \geq \frac{c}{2} - 1 - \frac{\log_q(2M_F)}{n}$$

and

$$B(F) = \limsup_{n \to \infty} b_n \leq \frac{2}{c - 2}$$

**Remark 3.7.** We thus get a close relationship between the bias of $F$ and it’s singular locus codimension. A similar result in terms of Fourier analysis can be found in [CM10].

The main result now follows from the above bound together with Lemma 4.3 in the next section.
4 $c$-regularity and strength

In this section we shall connect $c$-regularity to notion of rank, one key result we shall use is Theorem A. [AH16]. It should be noted though that in [AH16] our notion of rank is called strength.

Now given $F$ a polynomial of degree $d$ over $\mathbb{F}_q$. For $t \in \overline{\mathbb{F}}_q$, denote by $\hat{F}_t(\bullet, Z)$ the homogenization of the polynomial $F - t$. Here $Z$ is the extra variable of the homogenization. Note that $\hat{F}_t = \hat{F}_t(\bullet, Z)$ is the defining polynomial of the variety $Y^F_t$ defined above.

Lemma 4.1. For every $t \in \overline{\mathbb{F}}_q$

$$\text{rank}(F) \leq \text{rank}(\hat{F}_t) \leq \text{rank}(F) + 1$$

Proof. Recall that by definition $\mathcal{R}(F) = \mathcal{R}(\tilde{F})$ where $\tilde{F}$ is the homogenous degree $d$ part of $F$. The first bound is now achieved by noticing that $\tilde{F} = \hat{F}_t(\bullet, 0)$. The second bound is achieved by the observation that

$$\hat{F}_t(\bullet, Z) = \tilde{F}(\bullet) + ZG(\bullet, Z)$$

for some $G$ homogenous of degree $d - 1$.

Remark 4.2. Since here we are only using that $F$ and $F - t$ have the same homogenous part taking into account remark 3.6 we get that for fixed $d$ and $c$. Then there exists $r, M > 0$ such that for any two polynomial $F, G$ of degree $d$ on $n + 1$ variables with the same homogenous part and rank $\mathcal{R}(F) = \mathcal{R}(G) \geq r$ we have

$$|\#F^{-1}(0) - \#G^{-1}(0)| \ll Mq^{n-c}$$

The following lemma now follows immediately from Theorem A. [AH16] and Lemma 4.1.

Lemma 4.3. Let $c, d > 0$ be integers there exists a number $A(c, d)$ such that every homogenous polynomial $F$ of degree $d$ with $\mathcal{R}(F) > A(c, d)$ we have that $F$ is $c$-good.

5 $c$-Regularity and Cohomology

This section is dedicated to the proof of the following lemma:

Lemma 5.1. Let $X \subset \mathbb{P}^{N+1}$ be a $c$-regular projective hyper-surface of dimension $n$

$$H^m(X) = H^m(\mathbb{P}^N)$$

for $2N - (c + 2) \leq m \leq 2N$
Proof. Consider the constant \( \ell \)-adic sheaf \( Q_\ell \) on \( \mathbb{A}^{N+2} \). The sheaf \( Q_\ell \) is perverse on \( \mathbb{A}^{N+2} \). Let \( F : \mathbb{A}^{N+2} \to \mathbb{A}^1 \) be a homogenous polynomial. Let \( \phi_F \) be the functor of vanishing cycles attached to \( F \). We get that \( \phi_F(Q_\ell[N+1]) \) is perverse on \( \mathbb{A}^{N+2} \) and concentrated on the critical locus of \( F \), that is the singular locus of \( F^{-1}(0) \) by assumption the singular locus is of dimension \( s := N+1-c \) so by perversity the stalk of \( \phi_F(Q_\ell[N+1]) \) at any point has no non-zero homology only in degrees \([0,s]\). Consider now the short exact sequence of perverse sheaves on \( F^{-1}(0) \)\[
0 \to Q_\ell[N+1] \to \psi_F(Q_\ell[N+1]) \to \phi_F(Q_\ell[N+1]) \to 0.
\]
Here \( \psi_F \) is the nearby cycle functor. We conclude that the stalks of \( \psi_F(Q_\ell) \) are concentrated in degrees 0 and \([-N-1,-N+s-1]\).

Now let \( i : F^{-1}(0) \to \mathbb{A}^{N+2} \) be the close embedding and \( j \) the embedding of the open complement. We have the Wang cofiber sequence

\[
i^* j_* j^* Q_\ell \to \psi_F(Q_\ell) \xrightarrow{T-id} \psi_F(Q_\ell)
\]

So the stalks of \( i^* j_* j^* Q_\ell \) are concentrated at degrees \([-1,0]\) and \([-N-2,-N+s-1]\) in particular let \( p : \mathbb{A}^0 \to \mathbb{A}^{N+2} \) be the inclusion of the origin. We get that \( p^* j_* Q_\ell \) is concentrated at degrees \([-1,0]\) and \([-N-2,-N+s-1]\). The following lemma is a special case of Lemma 6.1 in [?].

**Lemma 5.2.** Let \( J : \mathbb{A}^{N+2} \setminus \{0\} \to \mathbb{A}^{N+2} \) be the embedding and let \( A \) be a \( \mathbb{G}_m \) invariant sheaf on \( \mathbb{A}^{N+2} \setminus \{0\} \). Let \( \pi : \mathbb{A}^{N+2} \to \mathbb{A}^0 \) be the projection then

\[
\pi_*(J_!(A)) = 0
\]

**Corollary 5.3.** The map \( \pi_* j_* Q_\ell \to \pi_* p_* p^* j_* Q_\ell \equiv p^* J_* Q_\ell \) is an equivalence.

**Proof.** We have a cofiber sequence

\[
J_* J' j_* Q_\ell \to j_* Q_\ell \to p_* p^* j_* Q_\ell
\]

and therefore it is enough to show that \( \pi_* J_* J' j_* Q_\ell = 0 \) by 5.2 we reduced to show that \( J' j_* Q_\ell = \kappa_* Q_\ell \) is \( \mathbb{G}_m \) invariant where

\[
\kappa : F^{-1}(\mathbb{A}^1 \setminus \{0\}) \to \mathbb{A}^{N+2} \setminus \{0\}
\]

is the embedding. This is immediate from the homogeneity of \( F \). \( \square \)
Now let
\[ \hat{\pi}: F^{-1}(A^1 \setminus \{0\}) \to A^0 \]
be unique map. We have that \( \hat{\pi}_* \hat{\pi}^* Q_\ell \) is concentrated in degrees \([-1, 0] \]
and \([-N - 2, -N + s - 1]\). So \( H^m(F^{-1}(A^1 \setminus \{0\})) = 0 \) for \( 2 \leq m \leq N - s \).

Now let
\[ \iota: F^{-1}(0) \setminus \{0\} \to A^{N+2} \setminus \{0\} \]
be the embedding. \( A^{N+2} \setminus \{0\} \) behaves cohomologically like a sphere of dimension \( 2N + 3 \) so we get by Alexander duality (for \( \iota \) and \( \kappa \)) that
\[ H^m(F^{-1}(0) \setminus \{O\}) = 0 \]
for
\[ 2N + 3 - c = N + 2 + s \leq m \leq 2N. \]

Let us now denote \( S := A^{N+2} \setminus \{O\} \) and \( K = F^{-1}(0) \setminus \{O\} \) and consider the map of \( \mathbb{G}_m \) bundles:

\[
\begin{array}{ccc}
K & \xrightarrow{\psi} & S \\
\downarrow & & \downarrow \\
X & \xrightarrow{\psi} & \mathbb{P}^N \\
\end{array}
\]

By taking the associated Gysin sequences we get:

\[
\begin{array}{cccccccccccccccc}
\longrightarrow & H^{m+1}(S) & \longrightarrow & H^m(\mathbb{P}^{N+1}) & \xrightarrow{\psi} & H^{m+2}(\mathbb{P}^{N+1}) & \longrightarrow & H^{m+2}(S) & \longrightarrow \\
\downarrow & & & & & & & & & \\
\longrightarrow & H^{m+1}(K) & \longrightarrow & H^m(X) & \xrightarrow{\psi \psi} & H^{m+2}(X) & \longrightarrow & H^{m+2}(K) & \longrightarrow \\
\end{array}
\]

Now by the above computation
\[ H^{2N}(K) = 0 \]
and
\[ H^{2N+1}(K) = 0 \]
since \( X \) is of dimension \( N \) so \( H^{2N-1}(X) = 0 \). Now since \( H^{2N-2}(K) = 0 \) by the above computation we get that \( H^{2N-3}(X) = 0 \) as well. We proceed by decreasing induction and get the result for odd \( m \). Similarly the result for \( H^{2N}(X) \) follows from \( V \) being irreducible (Note that of \( c < 2 \) the statement of the lemma is vacuous) \( \square \)
This additional lemma is useful for achieving the bounds in the proof.

**Lemma 5.4.** Let $k$ be a field and $N$ and $d$ be integers. There exists a universal bound $M(N,d)$ such that

For every non-zero homogenous polynomial $F \in k[x_0, ..., x_N]$ the zero locus of $F$, the variety $V_F \subset \mathbb{P}^N$ satisfies

$$M_{V_F} := \sum_{i=0}^{2e} \dim \left( H^i_{\text{et}}(\overline{V}_F) \right) \leq M(N,d)$$

**Proof.** Let $W_{N,d} := k[x_0, ..., x_N]$ be the vector space of degree $d$ homogenous polynomials (note that $\dim W_{N,d} = \binom{N+d}{d}$). Denote by

$$X_{N,F} \subset \mathbb{P}^N \times \mathbb{P}(W_{N,d})$$

The variety of points $(v, F)$ such that $F(v) = 0$. We have a proper projection map

$$\pi: X_{N,F} \to \mathbb{P}(W_{N,d})$$

The sheaf $C = \pi_* \mathbb{Q}_\ell$ is constructible on $\mathbb{P}(W_{N,d})$. By the proper base change theorem the stalk of $C$ over a polynomial $F \in \mathbb{P}(W_{N,d})(k)$ is the $\ell$-adic cohomology complex of $V_F$. By the boundedness properties of constructible sheaves we are done. \qed

**References**

[AH16] Tigran Ananyan and Melvin Hochster, *Small subalgebras of polynomial rings and stillman’s conjecture*, arXiv preprint arXiv:1610.09268 (2016).

[BL15] Abhishek Bhowmick and Shachar Lovett, *Bias vs structure of polynomials in large fields, and applications in effective algebraic geometry and coding theory*, arXiv preprint arXiv:1506.02047 (2015).

[CM10] Brian Cook and Akos Magyar, *On restricted arithmetic progressions over finite fields*, arXiv preprint arXiv:1011.5302 (2010).

[Dim12] Alexandru Dimca, *Singularities and topology of hypersurfaces*, Springer Science & Business Media, 2012.