ON POINTWISE DECAY OF LINEAR WAVES ON A SCHWARZSCHILD BLACK HOLE BACKGROUND

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Abstract. We prove sharp pointwise $t^{-3}$ decay for scalar linear perturbations of a Schwarzschild black hole without symmetry assumptions on the data. We also consider electromagnetic and gravitational perturbations for which we obtain decay rates $t^{-4}$, and $t^{-6}$, respectively. We proceed by decomposition into angular momentum $\ell$ and summation of the decay estimates on the Regge-Wheeler equation for fixed $\ell$. We encounter a dichotomy: the decay law in time is entirely determined by the asymptotic behavior of the Regge-Wheeler potential in the far field, whereas the growth of the constants in $\ell$ is dictated by the behavior of the Regge-Wheeler potential in a small neighborhood around its maximum. In other words, the tails are controlled by small energies, whereas the number of angular derivatives needed on the data is determined by energies close to the top of the Regge-Wheeler potential. This dichotomy corresponds to the well-known principle that for initial times the decay reflects the presence of complex resonances generated by the potential maximum, whereas for later times the tails are determined by the far field. However, we do not invoke complex resonances at all, but rely instead on semiclassical Sigal-Soffer type propagation estimates based on a Mourre bound near the top energy.

1. Introduction

The study of linear waves on fixed black hole backgrounds has a long history in mathematical relativity and very recently, major progress has been made on various aspects of the problem, see, e.g., [46], [17], [31], [32], [13], [13], [14], [4], [30], [29], [18], [18], [18] to name just a few of the more recent contributions. We refer the reader to the excellent lecture notes by Dafermos and Rodnianski [11] for the necessary background and a more detailed list of references. Understanding the behavior of linear waves on fixed backgrounds is supposed to be a necessary prerequisite for the study of the stability of black hole spacetimes in full general relativity, one of the major open problems in the field. The goal of this paper is to prove point-wise in time decay estimates for linear waves on the background of a Schwarzschild black hole. To be precise, let

$$g = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$

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be the Schwarzschild metric on \((t,r,\theta,\varphi) \in \mathbb{R} \times (2M, \infty) \times (0, \pi) \times (0, 2\pi)\). Introdu-
cing the tortoise coordinate

\[
x = r + 2M \log \left( \frac{r}{2M} - 1 \right)
\]

reduces the wave equation \(\Box_g \psi = 0\) to the form

\[
-\partial_t^2 \psi + \partial_r^2 \psi - \frac{F}{r} \frac{dF}{dr} \psi + \frac{F}{r^2} \Delta_g \psi = 0
\]

where \(F = \frac{dr}{dt}\). Our main result is as follows:

**Theorem 1.1.** The following decay estimates hold for solutions \(\psi\) of \((1.1)\) with data \(\psi[0] = (\psi_0, \psi_1)\):

\[
\|\langle x \rangle^{-\frac{3}{2}} \psi(t)\|_{L^2} \lesssim (t)^{-3} \|\langle x \rangle^{\frac{3}{2}} (\nabla^5 \partial_x \psi_0, \nabla^5 \psi_0, \nabla^4 \psi_1)\|_{L^2}\]

\[
\|\langle x \rangle^{-1} \psi(t)\|_{L^\infty} \lesssim (t)^{-3} \|\langle x \rangle^{4} (\nabla^{10} \partial_x \psi_0, \nabla^{10} \psi_0, \nabla^9 \psi_1)\|_{L^1}
\]

where \(\nabla\) stands for the angular derivative. Here \(L^p := L^p_x(\mathbb{R}; L^p(S^2))\) and \(\langle x \rangle := (1 + |x|^2)^{1/2}\).

The rate \(t^{-3}\) is well-known to be optimal for radial data, i.e., vanishing angular momentum, see for example \(16\). The same applies to the weight \(\langle x \rangle^{-1}\). We remark that Tataru \(46\) has recently obtained a striking result of this flavor but for essentially smooth data (he apparently needs a large number of angular derivatives). On the other hand, he derives his result in the greater generality of a Kerr background (for small parameter \(a\)) and also obtains a Huygens principle. We expect that our methods can be generalized to cover these as well as other scenarios, but we do not pursue this here. Another result in this direction, albeit for Schwarzschild de-Sitter, is due to Bony and Häfner \(7\). By means of a resonance expansion they prove local exponential decay in that setting for compactly supported data.

Let us mention two (related) extensions of Theorem \(1.1\). The first extension concerns the type of black hole perturbation we can cover. As stated above, Theorem \(1.1\) applies to scalar perturbations. However, one has similar statements (but with better decay, see below) for gravitational and electromagnetic perturbations of the Schwarzschild black hole which appear as \(\sigma = -3\) and \(\sigma = 0\), respectively, in the Regge-Wheeler potential, see \(14\) below. In the case of \(\sigma = -3\) one needs the data to be perpendicular to the spherical harmonics \(Y_0\) and \(\{Y_{r,1}\}_{r=1}^\infty\), and for \(\sigma = 0\) one needs to require orthogonality of the data to \(Y_0\). These conditions eliminate a gauge freedom inherent in the problem (such as changing the mass or the charge). We can cover these other values of \(\sigma\) for two reasons: (i) the decay bounds in \(17\) apply to them, and (ii) the WKB analysis in Section \(2\) which is the only place where \(\sigma\) plays a role in this paper, is insensitive to this modification.

The second extension concerns faster rates of decay. In fact, Theorem \(1.1\) actually gives an arbitrary rate of decay, i.e., \(t^{-N}\) for any \(N\), provided the data are perpendicular to the first few spherical harmonics (the exact number depending on \(N\)). This follows immediately by inspection of our proof, since \(17\) establishes

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1. The notation \(a \pm \varepsilon\) stands for \(a \pm \varepsilon\) where \(\varepsilon > 0\) is arbitrary (the choice determines the constants involved). Also, instead of \((\nabla^{10}, \nabla^9)\) in \(153\) one needs less, namely \((\nabla^{9+1}, \nabla^9)\) where \(\sigma > 8\) is arbitrary, see the proof in Section \(6\) for details.

2. From the point of view of the decay estimates in \(17\), these values need to be excluded as they are precisely the ones that give rise to a zero energy resonance.
accelerated rates as in Price’s law \cite{37, 38} for a fixed spherical harmonic. One formulation of this result reads as follows:

**Theorem 1.2.** Suppose that $\psi[0] \perp Y_j$ where $Y_j$ are the spherical harmonics on $S^2$ with eigenvalues less than $\ell(\ell + 1)$ with $\ell > 0$. Then one has the following faster rates of decay for solutions $\psi$ of (1.1) with data $\psi[0] = (\psi_0, \psi_1)$:

\begin{equation}
\|⟨x⟩^{-m}\psi(t)\|_\infty \lesssim (⟨t⟩)^{(2\ell+2)}\|⟨x⟩^{m}(\nabla^{n+1}\partial_x\psi_0, \nabla^{n+1}\psi_0, \nabla^n\psi_1)\|_1
\end{equation}

The implicit constant depends on $\ell$ and $n,m$ are sufficiently large integers which grow linearly in $\ell$.

The decay predicted by Price’s law is $t^{-2\ell-3}$ but at the moment we only obtain $t^{-2\ell-2}$, see \cite{17}. In particular, for gravitational perturbations we take $\ell = 2$ and for electromagnetic ones $\ell = 1$ leading to the decay rates $t^{-6}$ and $t^{-4}$, respectively, as stated in the abstract. Note that according to Price’s law one should have $t^{-7}$ and $t^{-5}$, respectively.

1.1. Extension to more general data. As stated, Theorems 1.1 and 1.2 require the initial data to vanish at the bifurcation sphere $x \to -\infty$. This is clearly a disadvantage of the result from the physical point of view since one would like to cover more general perturbations. However, there exists a classical construction by Kay and Wald \cite{28} which enables one to overcome this restriction. In order to explain this clever geometric argument, we have to briefly digress into some more advanced aspects of the Schwarzschild geometry. As is well-known, the Schwarzschild coordinates $(t, r, \theta, \phi)$ cover only a small portion of a bigger manifold which is referred to as maximally extended Schwarzschild or the Kruskal extension, see, e.g., \cite{22, 49}.

This is shown by introducing a new coordinate system $(T, R, \theta, \phi)$ which is related to the Schwarzschild coordinates by

\[ R^2 - T^2 = \left( \frac{r}{2M} - 1 \right) e^{r/(2M)}, \quad t = 2M \log \left( \frac{R + T}{R - T} \right). \]

In Kruskal coordinates the Schwarzschild metric reads

\[ g = \frac{32M^3}{r} \left(-dT^2 + dR^2\right) + r^2(d\theta^2 + \sin^2 \theta \, d\phi^2) \]

for $R > |T|$ and $r$ is now interpreted as a function of $T$ and $R$. However, the singularity at $r = 2M$ (which corresponds to $R = |T|$) has disappeared and nothing prevents us from allowing all values of $T$ and $R$ provided that $R^2 - T^2 > -1$. This yields the celebrated Kruskal extension. A spacetime diagram of the Kruskal extension is depicted in Fig. 1 and the wedge $S$ (which consists of the two shaded regions in Fig. 1) represents the original Schwarzschild manifold. The Kruskal spacetime is globally hyperbolic and in order to describe the Kay-Wald argument, we consider two Cauchy surfaces $\Sigma_0$ and $\Sigma_1$ at $T = 0$ and some small $T > 0$, respectively. The intersection $\Sigma_0 \cap S$ corresponds to the initial surface $t = 0$ in Theorems 1.1 and 1.2. Suppose now we prescribe initial data on $\Sigma_0$ (sufficiently regular, with sufficient decay at spatial infinity but not necessarily vanishing at the bifurcation sphere $T = R = 0$) and consider the wave equation $\Box_g \psi = 0$ with these data. We are interested in the future development in the original Schwarzschild wedge $S$. According to the domain of dependence property of the wave equation (see \cite{22}) the development to the future of $\Sigma_1$ in $S$ (the domain $D$ in Fig. 1) is
entirely determined by the values of $\psi$ and $\psi_T$ on $\Sigma_1 \cap \mathcal{S}$. Now we prescribe initial data $(f, g)$ on $\Sigma_1$ such that

1. $f(R, \theta, \varphi) = -f(-R, \theta, \varphi)$, $g(R, \theta, \varphi) = -g(-R, \theta, \varphi)$,
2. $(f, g)$ coincide with $(\psi, \psi_T)$ on $\Sigma_1 \cap \mathcal{S}$,
3. $(f, g)$ are as regular as $(\psi, \psi_T)$ on $\Sigma_1$.

It is obvious from the spacetime diagram Fig. 1 that this can be done. Then we consider the solution $\tilde{\psi}$ of $\Box_g \tilde{\psi} = 0$ with data $(f, g)$ on $\Sigma_1$. By the aforementioned domain of dependence property we have $\tilde{\psi} = \psi$ in $\mathcal{D}$. The key observation now is the existence of the discrete isometry $(T, R, \theta, \varphi) \mapsto (T, -R, \theta, \varphi)$ which leaves the line $R = 0$ invariant and guarantees that property (1) of the data $(f, g)$ is propagated by the wave flow, i.e., we have $\tilde{\psi}(T, R, \theta, \varphi) = -\tilde{\psi}(T, -R, \theta, \varphi)$ which in particular implies $\psi(T, 0, \theta, \varphi) = 0$ for all $T$. As a consequence, by evaluating $\psi$ and $\psi_T$ on $\Sigma_0$, we obtain new initial data on $\Sigma_0$ which vanish at the bifurcation sphere and lead to the same solution in $\mathcal{D}$ as the original data $(\psi, \psi_T)|_{\Sigma_0}$. If the data are sufficiently regular, they have to vanish at least linearly in $R$ at the bifurcation sphere which yields exponential decay with respect to the tortoise coordinate $x$ as $x \to -\infty$ and our Theorems 1.1 and 1.2 apply.

We remark in passing that the discrete isometry which lies at the heart of the Kay-Wald argument is a very fragile property which cannot be expected to hold in more general spacetimes. Recently, Dafermos and Rodnianski [12] devised a more robust method based on vector field multipliers which is capable of extending decay estimates up to the horizon.

1.2. Strategy of proof of Theorems 1.1 and 1.2. The strategy for the proof of Theorem 1.1 is to decompose the solution into spherical harmonics and then to sum the resulting decay estimates. The wave equation (1.1) at fixed angular momentum turns into a wave equation in $1+1$ dimensions, namely in the time
variable $t$ and the “radial” variable $x$. The angular derivatives in the estimates (1.2) and (1.3) then arise as weights given by powers of the angular momentum. This procedure is not expected to yield the optimal bounds as far as the number of angular derivatives is concerned. The specific numbers appearing in (1.2) above are a result of the Mourre estimate approach to the “top of the barrier scattering” which we develop in this paper. This Mourre estimate is non-classical in the sense that it needs to take into account that the top energy is trapping. We deal with the issue by means of the Heisenberg uncertainty principle (or the ground state of the semiclassical harmonic oscillator). The transition from our Mourre estimate to the decay in time is accomplished by means of Sigal-Soffer propagation theory going back to [43], but the implementation we follow is [26]. The further losses in terms of angular derivatives in (1.3) as compared to (1.2) are due to the Bernstein inequality and the $L^\infty$ bound on the spherical harmonics.

It is likely that Theorem 1.1 can be improved with regard to the number of angular derivatives required by a more detailed analysis of the spectral measure (at fixed angular momentum) for energies near the potential maximum. This would involve a reduction to Weber’s equation and an explicit perturbative analysis of the Jost solutions instead of the more indirect Mourre-Sigal-Soffer method. However, since it would complicate this paper we have chosen not to follow that route. We emphasize that the number of derivatives $\nabla$ appearing in our theorems is determined exclusively by the analysis near the maximum energy. There is a sizable literature on the topic of scattering near a potential maximum and, more generally, on scattering in the presence of trapping energies, see for example [2], [6], [8], [19], [23], [34], [39], and [44] and the references in these papers. However, we find that the available results in that direction are either not sharp enough for our purposes, or actually can be sidestepped completely with the Mourre approach we follow here.

For energies near zero, the Mourre estimates become degenerate. Therefore we need to rely on a WKB-type analysis of a semiclassical problem which we describe in detail in Section 1.3 (the semiclassical parameter being $\hbar = \ell^{-1}$ where $\ell$ is the angular momentum). The main issue for low lying energies is that the errors of the perturbative analysis of the spectral measure (and the Jost solutions) have to be controlled simultaneously for all energies near zero and all small $\hbar$. This was accomplished in [10] and [9].

We remark that the technical part of this paper is entirely devoted to large angular momenta - in other words, to the analysis of the semiclassical equation. In fact, for angular momenta $0 \leq \ell \leq \ell_0$ where $\ell_0$ is large we invoke the bounds from [17] and [16]. The constants appearing in the decay bounds in these papers grow rapidly in the angular momentum (in some super-exponential fashion). This precludes us from summing them in $\ell$ and necessitates the separate WKB/Mourre analysis of this paper. However, since the latter only applies to large $\ell$ the methods developed in [17] and [16] are of crucial importance for Theorem 1.1.

This paper is not self-contained, as it relies on the results of [10], [9], [17] and [16]. Needless to say, there is a long history concerning Price’s law, see [46] as well as [11]. We refer to those papers as well as to the introduction of our earlier paper [17] for a detailed list of references and more background. We would also like to mention that this paper as well as [10], [9], [17] and [16], are a result of those investigations into decay of wave equations on curved backgrounds with trapping metrics which began with the surface of revolution papers [40], [41].
1.3. Angular momentum decomposition. Restricting to spherical harmonics, the wave equation (1.1) takes the form

\[ \partial_t^2 \psi - \partial_x^2 \psi + V_{\ell,\sigma}(x)\psi = 0 \]

with the Regge-Wheeler potential

\[ V_{\ell,\sigma}(x) = \left( 1 - \frac{2M}{r(x)} \right) \left( \frac{\ell(\ell+1)}{r^2(x)} + \frac{2M\sigma}{r^4(x)} \right) \]

for \( \sigma = 1 \). However, as mentioned before we allow for other values of \( \sigma \) as well with the physically relevant ones being \( \sigma = -3, 0 \). We will take \( \ell \) large and study the semiclassical operator

\[ H = H(h) := -\hbar^2 \partial_x^2 + V(x; h) \]

with the normalization \( V(x; h) := h^2 V_{\ell,\sigma}(x) \), and \( V(x_{\max}; h) = 1 \) where \( x_{\max} \) is the location of the unique maximum of the potential. Thus \( h \sim \ell^{-1} \) as \( \ell \to \infty \). The maximum has the property that

\[ V'(x_{\max}; h) = 0, \quad V''(x_{\max}; h) \simeq -1 \]

uniformly in \( h \) and \( x_{\max} \simeq M \). For the cosine evolution one has for \( x' < x \)

\[
\begin{align*}
\cos \left( t\sqrt{\mathcal{H}_{\ell,\sigma}}(x, x') \right) &= \int_0^\infty \cos(t\lambda) \text{Im} \left[ \frac{f_+(x, \lambda; \ell) f_-(x', \lambda; \ell)}{W(f_+(\cdot, \lambda; \ell), f_-(\cdot, \lambda; \ell))} \right] d\lambda \\
&= \frac{2}{\pi\hbar^2} \int_0^\infty \cos(h^{-1}tE) \text{Im} \left[ \frac{f_+(x, E; h) f_-(x', E; h)}{W(f_+(\cdot, E; h), f_-(\cdot, E; h))} \right] E \, dE
\end{align*}
\]

with \( f_\pm \) being the outgoing Jost solutions for the original operator \( \mathcal{H}_{\ell,\sigma} \) and the semiclassical one, respectively. For the latter case this means that

\[ (-\hbar^2 \partial_x^2 + V(x; h)) f_\pm(x, E; h) = E^2 f_\pm(x, E; h) \]

\[ f_\pm(x, E; h) \sim e^{\pm \frac{\sqrt{E}x}{\hbar}} \quad x \to \pm \infty. \]

Furthermore, we write \( W(f, g) := fg' - f'g \) for the Wronskian of two functions \( f, g \).

The sine evolution is given by

\[ \frac{\sin \left( t\sqrt{\mathcal{H}_{\ell,\sigma}}(x, x') \right)}{\sqrt{\mathcal{H}_{\ell,\sigma}}} = \frac{2}{\pi\hbar} \int_0^\infty \sin(h^{-1}tE) e(x, x'; h) \, dE \]

with the semiclassical spectral measure

\[ e(x, x'; h) = \text{Im} \left[ \frac{f_+(x, E; h) f_-(x', E; h)}{W(f_+(\cdot, E; h), f_-(\cdot, E; h))} \right] \]

In order to control the semiclassical evolution we distinguish energies \( 0 < E < \varepsilon \), \( \varepsilon \leq E < 100 \) and \( 100 \leq E \). Here \( \varepsilon > 0 \) is some fixed small constant which does not depend on \( h \).

The regime of large energies is relatively easy, whereas the low lying energies as well as those near the maximum \( V = 1 \) represent the most difficult contributions to analyze. For small energies we follow the analysis of [10] and [9] which was specifically developed with this application in mind. In the former paper, the challenge was to carry out the WKB analysis for a smooth, positive, inverse square potential \textit{uniformly} for small \( h \) and small energies \( 0 < E < \varepsilon \). This was accomplished

\[ \text{Throughout, } a \simeq 1 \text{ means that } C^{-1} \leq a \leq C \text{ for some constant } C. \]
by means of Langer's uniformizing transformation which reduces the perturbative analysis to an Airy equation. We note that a novel feature was the modification of the potential, see Section 2. In this paper we have to go beyond [10] since the Regge-Wheeler potential exhibits inverse square decay only in the far field, whereas it decays exponentially towards the event horizon. This is where [9] applies, which develops a normal form reduction for the exponentially decaying region to the left of the maximum of the potential.

As mentioned above, we do not employ a uniformizing transformation for energies close to the maximum $V = 1$; this can indeed be done, and requires a perturbation theory around Weber's equation[4]. Instead, we prove a Mourre estimate near the maximum. This is somewhat unusual as the maximum energy is trapping and therefore needs to be excluded in the classical Mourre theory, see [20], [24]. However, a simple application of the uncertainty principle (or the ground state of the semiclassical harmonic oscillator) allows one to deal with this trapping case as well. Of crucial importance here is that the maximum of the potential is nondegenerate[5] (in fact, $V''(x_{\text{max}}) < 0$). Once the Mourre bound is established, we employ a semiclassical version of the propagation estimates of Hunziker, Sigal, Soffer [26], which in turn go back to the work of Sigal, Soffer [43], see Section 6 below.

1.4. Notations. In this paper we frequently employ the notation $a \lesssim b$ (for $a, b \in \mathbb{R}$) meaning that there exists an (absolute) constant $c > 0$ such that $a \leq cb$. We also use $a \gtrsim b$ and write $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$. Furthermore, $O(f(x))$ denotes a generic complex–valued function that satisfies $|O(f(x))| \lesssim |f(x)|$ in a domain of $x$ that is either stated explicitly or follows from the context. We write $O_{x}(f(x))$ to indicate that the respective function is real–valued. The symbol $\sim$ is reserved for asymptotic equality, i.e., $f(x) \sim g(x)$ as $x \to a$, where $f, g$ are two complex–valued functions, means that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 1.$$

2. Low lying energies and WKB

In this section we bound (1.9) for energies $0 < E < \varepsilon$. Our approach is based on the perturbative WKB analysis of the Jost solutions which was developed in [10] and [9]. More precisely, [10] applies to the case of $x \geq 0$ for which the potential decays like an inverse square. For $x \leq 0$ the potential exhibits exponential decay as $x \to -\infty$ and [9] develops the methods needed for that case. We present the main steps of the analysis developed in these papers but omit the most involved technical details so as not to disrupt the flow of the argument.

2.1. The far field. We begin with the former case, i.e., $x \geq 0$. In fact, we shall apply the analysis of this section to $x \geq x_{0}$ where $x_{0} < 0$ is a fixed constant. In

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4We will pursue this matter elsewhere. This approach seems needed in order to prove a Huyghen’s principle along the lines of this paper.

5Technically speaking, the methods of this paper apply to potentials with a unique maximum satisfying $V^{(j)}(\tilde{x}) = 0$ for $1 \leq j \leq k$, $V^{(k+1)}(\tilde{x}) \neq 0$. However, the number of derivatives $N$ in our decay estimates would then increase with $k$. 

fact, any $x_0 < 0$ is admissible, see below. Define a modified potential
\[ V_0(x; \hbar) := V(x; \hbar) + \frac{\hbar^2}{4} \langle x \rangle^{-2} \]
and denote by $x_1(E; \hbar) > 0$ the unique positive turning point for any\(^6\) $0 < E < \varepsilon$, i.e., the solution of $V_0(x; \hbar) = E^2$, $x > 0$.

2.1.1. Liouville-Green transform and reduction to a perturbed Airy equation. The analysis of \[^{10}\] was based on the following “Langer transform”, which in turn is a special case of the Liouville-Green transform, see \[^{35}, 33\]. See Lemma 3 in \[^{10}\] for essentially the same statement.

**Lemma 2.1.** With\(^7\) $Q_0 := V_0 - E^2$
\begin{equation}
\zeta = \zeta(x, E; \hbar) := \text{sign}(x - x_1(E; \hbar)) \left| \frac{3}{2} \int_{x_1(E; \hbar)}^x \sqrt{|Q_0(u, E; \hbar)|} \, du \right|^{\frac{1}{2}}
\end{equation}
defines a smooth change of variables $x \mapsto \zeta$ for all $x \geq x_0$. Let $q := -\frac{Q_0}{\zeta}$. Then $q > 0$, $\frac{dq}{d\zeta} = \zeta'$, and
\[-\hbar^2 f'' + (V - E^2) f = 0\]
transforms into
\begin{equation}
-\hbar^2 \ddot{w}(\zeta) = (\zeta + \hbar^2 \tilde{V}(\zeta, E; \hbar)) w(\zeta)
\end{equation}
under $w = \sqrt{\zeta} f = q^{\frac{1}{4}} f$. Here $\dot{\cdot} = \frac{d}{d\zeta}$ and
\[ \tilde{V} := \frac{1}{4} q^{-1} (x)^{-2} - q^{-1} \frac{d^2 q^{\frac{1}{4}}}{d\zeta^2} \]

**Proof.** It is clear that \[^{21}\] defines a smooth map away from the point $x = x_1(E; \hbar)$. Taylor expanding $Q_0(x, E; \hbar)$ in a neighborhood of that point and using that $V_0'(x_1(E; \hbar)) < 0$ implies that $\zeta(x, E; \hbar)$ is smooth around $x = x_1$ as well with $\zeta'(x_1, E; \hbar) > 0$. Next, one checks that
\[ \dot{w} = q^{-\frac{3}{4}} f' + \frac{dq^{\frac{1}{4}}}{d\zeta} f, \quad \ddot{w} = q^{-\frac{5}{4}} f'' + \frac{d^2 q^{\frac{1}{4}}}{d\zeta^2} f \]
and thus, using $-\hbar^2 f'' = (E^2 - V) f$,
\[ -\hbar^2 \ddot{w} = q^{-1} (E^2 - V) w - \hbar^2 q^{-\frac{3}{4}} \frac{d^2 q^{\frac{1}{4}}}{d\zeta^2} w \]
\[ = q^{-1} (-Q_0 + h^2 (x)^{-2} / 4) w - \hbar^2 q^{-\frac{1}{4}} \frac{d^2 q^{\frac{1}{4}}}{d\zeta^2} w \]
\[ = \zeta w(\zeta) + \hbar^2 (q^{-1} (x)^{-2} / 4 - q^{-\frac{1}{4}} \frac{d^2 q^{\frac{1}{4}}}{d\zeta^2}) w \]
as claimed. \(\square\)

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\(^6\)It was shown in \[^{10}\] that the WKB approximation can only be applied after this modification.

\(^7\)This condition will be tacitly in force throughout this section.

\(^8\)We warn the reader that what was called $E$ in \[^{10}\] is now called $E^2$. 
The Airy equation (2.2) provides a convenient way of solving the matching problem at the turning point $\zeta = 0$. We remark that it was assumed in [10] that the potential satisfies $V(x) = \mu x^{-2} + O(x^{-3})$ as $x \to \infty$. However, the methods of that paper equally well apply under the weaker assumption $V(x) = \mu x^{-2} + O(x^{-\sigma})$ as $x \to \infty$ where $\sigma > 2$; furthermore, one needs $\partial_x^k O(x^{-\sigma}) = O(x^{-\sigma-k})$ for all $k \geq 0$. This is relevant here since the Regge-Wheeler potential exhibits a $\frac{\log x}{x^3}$-correction to the leading $x^{-3}$-decay.

2.1.2. A basis for the perturbed Airy equation. For the Airy functions $\text{Ai}, \text{Bi}$ appearing below we refer the reader to Chapter 11 of [35]. To the left of the turning point a fundamental system of (2.2) is described by the following result, see Proposition 8 in [10].

**Proposition 2.2.** Let $\hbar_0 > 0$ be small. A fundamental system of solutions to (2.2) in the range $\zeta \leq 0$ is given by

$$
\begin{align*}
\phi_1(\zeta, E, \hbar) &= \text{Ai}(\tau)[1 + h a_1(\zeta, E, \hbar)] \\
\phi_2(\zeta, E, \hbar) &= \text{Bi}(\tau)[1 + h a_2(\zeta, E, \hbar)]
\end{align*}
$$

with $\tau := -h^{-\frac{2}{3}} \zeta$. Here $a_1, a_2$ are smooth, real-valued, and they satisfy the bounds, for all $k \geq 0$ and $j = 1, 2$, and with $\zeta_0 := \zeta(x_0, E)$,

$$
\sup_{\omega \leq \zeta \leq 0} |\partial^k E a_j(\zeta, E, \hbar)| \lesssim E^{-k}
$$

(2.3)

$$
|\partial^k E \partial_\tau a_j(\zeta, E, \hbar)| \lesssim E^{-k} \left[ h^{-\frac{4}{3}}(h^{-\frac{2}{3}} \zeta)^{-\frac{2}{3}} \chi_{[-1, \zeta \leq \zeta \leq 0]} + |\zeta|^{-\frac{2}{3}} \chi_{[\zeta_0 \leq \zeta \leq -1]} \right]
$$

uniformly in the parameters $0 < h < \hbar_0$, $0 < E < \varepsilon$.

*Proof.* This is essentially Proposition 8 in [10]. The two differences are (i) we work with $E^2$ instead of $E$ (ii) the potential has a $\frac{\log x}{x^3}$-correction to the leading inverse square decay rather than the $x^{-3}$-correction assumed in [10].

As far as (i) is concerned, we note the following. Let $E$ be as in [10] and assume $\tilde{E} := E$. If $|\partial_\tau a(E)| \lesssim E^{-1}$, then $b(\tilde{E}) := a(E)$ satisfies $|\partial_\tau b(\tilde{E})| \lesssim E^{-1}$ by the chain rule. So it makes no difference whether we work with $E$ or $\tilde{E}$. As for (ii), we note that the only change to the estimates in Section 3 of [10] is in Lemma 7, where one needs to replace $(x)^{-3} \beta_1(x, E)$ with $\langle x \rangle^{-3} \log x \beta_1(x, E)$, cf. (3.20) and (3.24) in that paper. However, inspection of the proof of Proposition 8 there reveals that this does not affect the resulting bound in any way, see the paragraph between (4.12) and (4.13) there.

We remark that Proposition 2.2 would fail if we had defined $\zeta$ in (2.1) with $Q = V - E^2$ instead of $Q_0$. In the region $\zeta \geq 0$ we have a basis of oscillatory solutions as described by the following result, see Proposition 9 in [10].

**Proposition 2.3.** Let $\hbar_0 > 0$ be small. In the range $\zeta \geq 0$ a basis of solutions to (2.2) is given by

$$
\begin{align*}
\psi_1(\zeta, E; \hbar) &= (\text{Ai}(\tau) + i\text{Bi}(\tau))[1 + h b_1(\zeta, E; \hbar)] \\
\psi_2(\zeta, E; \hbar) &= (\text{Ai}(\tau) - i\text{Bi}(\tau))[1 + h b_2(\zeta, E; \hbar)]
\end{align*}
$$

The usual WKB machinery requires solving two matching problems, namely between the Airy region and the oscillatory region on the one hand, and the Airy region and the exponential growth/decay region on the other hand; see for example [35].
with $\tau := -h^{-\frac{1}{2}}\xi$ and where $b_1, b_2$ are smooth, complex-valued, and satisfy the bounds for all $k \geq 0$, and $j = 1, 2$

$$|\partial^k_E b_j(\zeta, E; \hbar)| \leq C_k E^{-k}(\zeta)^{-\frac{3}{4}}$$

$$|\partial^k_E b_j(\zeta, E; \hbar)| \leq C_k E^{-k}h^{-\frac{1}{4}}(\zeta)^{-2}$$

uniformly in the parameters $0 < h < h_0$, $0 < E < \varepsilon$, $\zeta \geq 0$. 

2.1.3. The outgoing Jost solution for the far field. We can now draw the following conclusions from Propositions 2.2 and 2.3 about the outgoing Jost solution.

First, recall that the Airy functions satisfy the following asymptotic expansions with $\xi = \frac{2}{3}x^\frac{3}{2}$:

$$\text{Ai}(x) = \frac{e^{-\xi}}{2\pi x^\frac{1}{4}} (1 + O(\xi^{-1})), \quad \text{Bi}(x) = \frac{e^{\xi}}{\pi x^\frac{1}{4}} (1 + O(\xi^{-1}))$$

$$\text{Ai}(-x) = \frac{1}{\pi^2 x^\frac{3}{4}} \left[ \cos(\xi - \frac{1}{4}\pi) (1 + O(\xi^{-2})) + \sin(\xi - \frac{1}{4}\pi) O(\xi^{-1}) \right],$$

$$\text{Bi}(-x) = \frac{1}{\pi^2 x^\frac{3}{4}} \left[ -\sin(\xi - \frac{1}{4}\pi) (1 + O(\xi^{-2})) + \cos(\xi - \frac{1}{4}\pi) O(\xi^{-1}) \right]$$

as $x \to \infty$, see [35]. Moreover,

$$W(\text{Ai}, \text{Bi}) = \text{Ai} \text{Bi}' - \text{Ai}' \text{Bi} = \frac{1}{\pi}$$

In what follows,

$$S_+(E; \hbar) := \int_{x_0}^{x(E; \hbar)} \sqrt{V_0(y; \hbar) - E^2} \, dy$$

$$T_+(E; \hbar) := E x(E; \hbar) - \int_{x_0}^{\infty} \left( \sqrt{V_0(y; \hbar) - E^2} - E \right) \, dy$$

One checks that $S_+(E; \hbar) \sim |\log E|$ as $E \to 0^+$, whereas $T_+(E; \hbar) \to T_+(0; \hbar)$, some finite number. Moreover,

$$|\partial^k_E S(E; \hbar)| + |\partial^k_E T_+(E; \hbar)| \leq C_k E^{-k}$$

for all $k \geq 1$. One has

$$f_+(x, E) = \sqrt{\pi} E^\frac{1}{2} h^{-\frac{1}{4}} e^{i(t_+(E) + \frac{3}{2})} q^{-\frac{1}{4}}(\zeta) \psi_2(\zeta, E).$$

This is obtained by matching the asymptotic behavior of $f_+$ with that of $\psi_2(\zeta)$ as $x \to \infty$. Moreover, by using the relation $w = q^\frac{1}{2}f$ from Lemma 2.1, we refer the reader to [10] for all the details. We now connect $\psi_2$ to the basis $\phi_j(\zeta, E)$ of Proposition 2.2:

$$\psi_2(\zeta, E) = c_1(E) \phi_1(\zeta, E) + c_2(E) \phi_2(\zeta, E)$$

where

$$c_1(E) = \frac{W(\psi_2(\cdot, E), \phi_2(\cdot, E))}{W(\phi_1(\cdot, E), \phi_2(\cdot, E))}, \quad c_2(E) = -\frac{W(\psi_2(\cdot, E), \phi_1(\cdot, E))}{W(\phi_1(\cdot, E), \phi_2(\cdot, E))}$$

By Proposition 2.2,

$$W(\phi_1(\cdot, E), \phi_2(\cdot, E)) = -h^{-\frac{1}{2}} W(\text{Ai}, \text{Bi}) + O(h^\frac{1}{2}) = -\pi^{-1} h^{-\frac{1}{4}} (1 + O(h))$$

\[10\] We suppress $h$ as argument in most functions, even though everything here does depend on $h$.
where we evaluated the Wronskian on the left-hand side at $\zeta = 0$. Next, by Propositions 2.2 and 2.3,

$$W(\psi_2(\cdot, E), \phi_2(\cdot, E)) = -h^{-\frac{3}{2}}[(Ai(0) - iBi(0))Bi'(0)$$

$$- (Ai'(0) - iBi'(0))Bi(0) + O(h)]$$

$$= -h^{-\frac{3}{2}}[W(Ai, Bi) + O(h)]$$

(2.7)

$$W(\bar{\psi}_2(\cdot, E), \phi_3(\cdot, E)) = -h^{-\frac{3}{2}}[(Ai(0) - iBi(0))Ai'(0)$$

$$- (Ai'(0) - iBi'(0))Ai(0) + O(h)]$$

$$= -h^{-\frac{3}{2}}[W(Ai, Bi) + O(h)]$$

so that

(2.8) \hspace{1cm} c_1(E) = 1 + O(h), \hspace{0.5cm} c_2(E) = -i + O(h)

where the $O(\cdot)$ terms satisfy $|\partial_x^k O(h)| \leq C_k h E^{-k}$. With $\zeta_0 = \zeta(x_0, E)$ we infer that

$$f_+(x_0, E) = \sqrt{\pi} e^{i\left(\frac{2}{\gamma} + \frac{\zeta}{2}\right)} E^{\frac{3}{2}} h^{-\frac{1}{2}} q^{-\frac{1}{4}} (\zeta_0) \psi_2(\zeta_0, E)$$

$$= \sqrt{\pi} e^{i\left(\frac{2}{\gamma} + \frac{\zeta}{2}\right)} E^{\frac{3}{2}} h^{-\frac{1}{2}} q^{-\frac{1}{4}} (\zeta_0) [c_1(E)\phi_1(\zeta_0, E) + c_2(E)\phi_2(\zeta_0, E)]$$

$$f'_+(x_0, E) = \sqrt{\pi} e^{i\left(\frac{2}{\gamma} + \frac{\zeta}{2}\right)} E^{\frac{3}{2}} h^{-\frac{1}{2}} q^{-\frac{1}{4}} (\zeta_0) \left[\psi_2(\zeta_0, E) - \frac{1}{4q} (\zeta_0) \psi_2(\zeta_0, E)\right]$$

$$= \sqrt{\pi} e^{i\left(\frac{2}{\gamma} + \frac{\zeta}{2}\right)} E^{\frac{3}{2}} h^{-\frac{1}{2}} q^{-\frac{1}{4}} (\zeta_0) [c_1(E)\phi_1(\zeta_0, E) + c_2(E)\phi_2(\zeta_0, E)]$$

where we have used that $\zeta' = q^{\frac{1}{2}}$, see Lemma 2.1. Furthermore,

$$\frac{q}{q}(\zeta_0) = -\zeta_0^{-1} + |\zeta_0|^2 \frac{V'_0(x_0)}{(V_0(x_0) - E^2)^{\frac{1}{2}}} = O(|\zeta_0|^{\frac{1}{2}})

with $O(\partial_x^k |\zeta_0|^{\frac{1}{2}}) = O(E^{-k})$ for $k \geq 1$. From Proposition 2.2 with $O_\mathbb{R}$ denoting a real-valued term,

$$\phi_1(\zeta_0, E) = Ai(-h^{-\frac{3}{2}}\zeta_0)(1 + O_\mathbb{R}(h))$$

$$\phi_2(\zeta_0, E) = Bi(-h^{-\frac{3}{2}}\zeta_0)(1 + O_\mathbb{R}(h))$$

$$\dot{\phi}_1(\zeta_0, E) = -h^{-\frac{3}{2}}Ai'(-h^{-\frac{3}{2}}\zeta_0)(1 + O_\mathbb{R}(h)) + O_\mathbb{R}(h)|\zeta_0|^\frac{1}{2}Ai(-h^{-\frac{3}{2}}\zeta_0)$$

$$\dot{\phi}_2(\zeta_0, E) = -h^{-\frac{3}{2}}Bi'(-h^{-\frac{3}{2}}\zeta_0)(1 + O_\mathbb{R}(h)) + O_\mathbb{R}(h)|\zeta_0|^\frac{1}{2}Bi(-h^{-\frac{3}{2}}\zeta_0)$$

which implies via 2.6 that

$$\phi_1(\zeta_0, E) = (4\pi)^{-\frac{3}{2}}(h^{-\frac{3}{2}}|\zeta_0|)^{\frac{3}{4}} e^{-\frac{3}{4}h^{-1}|\zeta_0|^{\frac{3}{2}}} (1 + O_\mathbb{R}(h))$$

$$\phi_2(\zeta_0, E) = \pi^{-\frac{1}{2}}(h^{-\frac{3}{2}}|\zeta_0|)^{-\frac{1}{2}} e^{\frac{3}{2}h^{-1}|\zeta_0|^2} (1 + O_\mathbb{R}(h))$$

$$\dot{\phi}_1(\zeta_0, E) = h^{-\frac{3}{2}}(4\pi)^{-\frac{3}{2}}(h^{-\frac{3}{2}}|\zeta_0|)^{\frac{3}{4}} e^{-\frac{3}{4}h^{-1}|\zeta_0|^{\frac{3}{2}}} (1 + O_\mathbb{R}(h))$$

$$\dot{\phi}_2(\zeta_0, E) = -h^{-\frac{3}{2}}(4\pi)^{-\frac{3}{2}}(h^{-\frac{3}{2}}|\zeta_0|)^{\frac{3}{4}} e^{\frac{3}{2}h^{-1}|\zeta_0|^2} (1 + O_\mathbb{R}(h))$$
In view of these properties and using that $e^{-\hbar^{-1}|\zeta_0|^2} = O_\hbar(h)$ where $\partial_k^k O_\hbar(h) = O(E^{-k} h)$, one obtains (with $c_2$ as above)
\begin{align}
  f_+(x_0, E) &= c_2 q^{-\frac{1}{2}}(\zeta_0)\dot{g}_2(\zeta_0, E) \left[ (1 + O_\hbar(h)) + i\left(\frac{1}{2} + O(h)\right)e^{-\hbar^{-1}S_+} \right] \\
  f'_+(x_0, E) &= c_2 q^{-\frac{1}{2}}(\zeta_0)\dot{g}_2(\zeta_0, E) \left[ (1 + O_\hbar(h)) - i\left(\frac{1}{2} + O(h)\right)e^{-\hbar^{-1}S_+} \right]
\end{align}
(2.9)
where $\gamma = \gamma(E, \hbar) := -\sqrt{\pi} e^{\frac{T_0(E)}{T_0(E)}} E^{\frac{2}{3}} h^{-\frac{1}{3}}$ and with
\[\frac{2}{3} |\zeta_0|^2 = S_+(E; \hbar) = S_+ = \int_{x_0}^{x_1(E; \hbar)} \sqrt{V_0(x; \hbar) - E^2} \, dx\]
being the action integral defined earlier. Furthermore, it follows from Propositions 2.2 and 2.3 that each differentiation in $E$ loses one power of $E$ (in particular, the $O(\hbar)$ terms have this property). For future reference, we remark that
\begin{align}
  \frac{f'_+(x_0, E)}{f_+(x_0, E)} &= -d_1 h^{-1}(1 + O_\hbar(h))[1 + O(e^{-\hbar^{-1}S_+})] \\
  \text{where } d_1 > 0 \text{ is a constant (depending on } x_0)\text{. In particular,}
\end{align}
(2.10)
\[\text{Im} \left[ \frac{f'_+(x_0, E)}{f_+(x_0, E)} \right] = h^{-1} O(e^{-\hbar^{-1}S_+(E; \hbar)})\]

2.2. Approaching the event horizon. We now deal with the potential for $x \leq x_0$. Here $x_0 < 0$ is chosen such that the Regge-Wheeler potential (setting $2M = 1$ for simplicity) can be written as
\[V(x; \hbar) = \sum_{n=1}^{\infty} c_{n-1}(h)e^{nx}\]
as a convergent series $x \leq x_0$ uniformly in $\hbar \in (0, \hbar_0)$. In fact, since the Lambert function $W(z)$ defined via $W(z)e^{W(z)} = z$ is analytic on $|z| < e^{-1}$, it follows that one can take any $x_0 < 0$. The coefficients have expansions in powers of $h$ and we normalize such that $c_0(h) = 1 + O(h^2)$. One can also check that $c_1(0) \neq 0$. The goal is to control the Jost solutions $f_-(x, E; \hbar)$ as $x \to -\infty$ uniformly for $(E, \hbar) \in (0, \varepsilon) \times (0, \hbar_0)$.

2.2.1. Transforming the problem to a compact interval. For notational convenience, we switch from $x$ to $-x$ and consider $x > |x_0|$. The problem is then to control $f_+$ for the problem
\begin{align}
  -\hbar^2 f''_+(x, E; \hbar) + V(x; \hbar) f_+(x, E; \hbar) = E^2 f_+(x, E; \hbar)
\end{align}
(2.11)
with
\[V(x; \hbar) = e^{-x}(1 + \sum_{n=1}^{\infty} c_n(h)e^{-nx})\]

We now transform this case into a semi-classical scattering problem on a bounded interval $(0, y_0)$ by introducing the new independent variable $y = 2e^{-\frac{x}{2}}$. Setting $f(x) = g(y)$ reduces finding the outgoing Jost solution to the equation for $g(y) = g(y, E; \hbar)$,
\[-\hbar^2 [g''(y) + y^{-1}g'(y)] + \left(\Omega(y; \hbar) - \frac{4E^2}{y^2}\right)g(y) = 0\]
with the normalization $g(y) \sim (y/2)^{-\frac{3}{4}}$ as $y \to 0^+$, and with

\[(2.12)\quad \Omega(y; h) = 1 + \sum_{n=1}^{\infty} \frac{c_n(h)}{4^n} y^{2n}\]

analytic in $|y| < y_0 := 2e^{-x_0/2}$. Finally, setting $\tilde{g}(y; h) := y^{\frac{3}{4}}g(y; h)$ yields the equation

\[(2.13)\quad -\hbar^2 \tilde{g}''(y; E; h) + \left(\Omega(y; h) - \left(\frac{\hbar^2}{4} + 4E^2\right)y^{-2}\right)\tilde{g}(y; E; h) = 0\]

with the normalization

\[(2.14)\quad \tilde{g}(y; E; h) \sim 2^{2i\frac{\hbar}{E}} y^{\frac{3}{2} - 2i\frac{\hbar}{E}}\]

as $y \to 0^+$. We remark that in the case $\Omega(y; h) \equiv 1$ the equation (2.13) is a modified Bessel equation with a basis given by the modified Bessel functions $I_{\nu}(\hbar^{-1}y)$ and $K_{\nu}(\hbar^{-1}y)$ with $\nu = \frac{2i\hbar}{E}$. It is shown in [9] by means of a suitable Liouville-Green transform that this basis leads to an actual basis of (2.13). We begin with the following normal form result from [9] which is based on a Liouville-Green transform (the variable $z$ below is a rescaling of $y$: $y = az$, with $\alpha := \sqrt{\hbar^2/4 + 4E^2}$). Recall that $f \approx 1$ means that $C^{-1} < f < C$ for some constant $C$.

**Lemma 2.4** ([9]). Let $\Omega$ be as above and $\alpha_0 > 0$ be sufficiently small. For all $0 < \alpha < \alpha_0$ there exists a $C^\infty$ diffeomorphism $w = w(z, \alpha) : I_0(\alpha) := (0, \alpha^{-1}y_0) \to J_0(\alpha) := (0, \alpha^{-1}w_0(\alpha))$ where $y_0$ is as above with the following properties, uniformly in $0 < \alpha < \alpha_0$:

- $w_0(\alpha) \approx 1$
- $w'(z, \alpha) \approx 1$ for all $z \in I_0(\alpha)$
- $|\partial_\alpha^k \partial_z^\ell w(z, \alpha)| \leq C_{k,\ell} z^{1+k} \alpha^\ell$ for all $k, \ell \geq 0$ and $z \in I_0(\alpha)$

Let $h := \alpha^{-1} h$. Then there exists a function $V_2(\cdot; \alpha; h)$ such that $\psi$ solves the rescaled form of (2.13), viz.

\[(2.15)\quad -h^2 \psi''(z) + \left(\Omega(\alpha z; h) - z^{-2}\right)\psi(z) = 0\]

on $I_0(\alpha)$ iff $\varphi(w) := (w'(z, \alpha))^2 \psi(z)$ (where $w = w(z, \alpha)$) solves

\[(2.16)\quad -h^2 \varphi''(w) + (1 - w^{-2}) \varphi(w) = h^2 V_2(w) \varphi(w)\]

on $J_0(\alpha)$. Furthermore, the potential $V_2(\cdot; \alpha; h)$ satisfies

\[|\partial_\alpha^k \partial_w^\ell V_2(w; \alpha; h)| \leq C_{k,\ell} w^{1+k} \alpha^{3+\ell}\]

for all $w \in J_0(\alpha)$ and $k, \ell \geq 0$.

**Proof.** This is done by setting

\[dw := \sqrt{z^{-2} - \Omega(\alpha z; h)} dz \]

More precisely, with $z_t$ being the turning point defined by $z_t^{-2} - \Omega(\alpha z_t; h) = 0$, this means that

\[\int_1^w \sqrt{1 - v^{-2}} dv = \int_{z_t}^z \sqrt{\Omega(\alpha u; h) - u^{-2}} du\]

provided $z > z_t$ and

\[\int_w^1 \sqrt{1 + v^{-2}} dv = \int_z^{z_t} \sqrt{-\Omega(\alpha u; h) + u^{-2}} du\]
provided $0 < z < z_t$. Note that $w \to 0$ as $z \to 0$. The properties of $w$ stated above are now shown by calculus. The potential $V_2$ is given by

$$V_2(w) = (w'(z))^{-\frac{3}{2}} \partial_x^2(w'(z))^{-\frac{1}{2}} = \frac{3}{4} \frac{(w''(z))^2}{(w'(z))^4} - \frac{1}{2} \frac{w'''(z)}{(w'(z))^3}.$$ 

We refer the reader to [9] for further details. □

We remark that the proof also shows that $w(z) = z + O(z^2)$ and $w'(z) = 1 + O(z)$ as $z \to 0$. One now concludes the following concerning a basis of (2.13). Let $\nu := \sqrt{\frac{\hbar^2}{4} + 4E^2}$. Since (2.15) is a rescaled form of (2.13), one can now obtain a system of fundamental solutions to the latter equation from a perturbative analysis of (2.16). The modified Bessel functions $I_{\nu}(z)$ and $K_{\nu}(z)$, which are both analytic on $\mathbb{C} \setminus (-\infty, 0]$, give rise to a fundamental system of the homogeneous equation on the left-hand side of (2.16). In our case $\nu = 2\frac{E}{\hbar}$. Recall the asymptotics

$$I_{\nu}(z) = \left(\frac{z}{2}\right)^{\nu} \frac{1}{\Gamma(\nu + 1)}(1 + O(z^2)) \quad z \to 0.$$ 

Note that for our purposes it suffices to consider real $z$. Moreover, $I_{\nu}(x)$ grows exponentially as $x \to -\infty$, whereas $K_{\nu}(x)$ decays exponentially as $x \to \infty$.

**Corollary 2.5.** Let $\alpha := \sqrt{\frac{\hbar^2}{4} + 4E^2}$ with $\hbar$ and $E > 0$ small. There exists a fundamental system of (2.13), denoted by $(\tilde{g}_0, \tilde{g}_1)$, of the form

$$\tilde{g}_0(y, E; \hbar) = (w(z)/w'(z))^{\frac{\nu}{4}} I_{\nu} \left( \frac{w(z)}{\hbar_1} \right) (1 + \alpha_1(y, E; \hbar))$$

$$\tilde{g}_1(y, E; \hbar) = (w(z)/w'(z))^{\frac{\nu}{4}} I_{-\nu} \left( \frac{w(z)}{\hbar_1} \right) (1 + \alpha_2(y, E; \hbar))$$

where $w(z) = w(z, E; \hbar)$ is as in Lemma 2.4 and with $z = \frac{y}{\alpha}$, $\hbar_1 = \frac{\hbar}{\alpha}$. The $c_j$ satisfy for all $k, \ell \geq 0$,

$$|\partial_E^k \partial_y^\ell c_j(y, E; \hbar)| \leq C_{k, \ell} \alpha^{-k}$$

and all $0 < y < y_0$.

**Proof.** This follows from two facts: (i) a basis of the homogeneous equation (2.16) is given by

$$\phi_0(w, E; \hbar) := \sqrt{w}I_{\nu} \left( \frac{w}{\hbar_1} \right), \quad \phi_1(w, E; \hbar) := \sqrt{w}I_{-\nu} \left( \frac{w}{\hbar_1} \right)$$

and (ii): the equations for $c_{1,2}$ are contractive; in fact, they are given by the usual Volterra equation involving the homogeneous basis and the potential $V_2$. For $c_1$ one has (suppressing $E$ and $\hbar$ as arguments)

$$c_1(w) = -\hbar_1^{-1} \int_0^w \int_u w \phi_0^{-2}(v) dv V_2(u) \phi_0(u) (1 + \hbar_1 c_1(u)) du$$

which implies the desired bounds on $c_1$ via Lemma 2.4 and the well-known asymptotic behavior of the modified Bessel functions. For this see [9]. □
2.2.2. The outgoing Jost solution towards the event horizon. From (2.17), Lemma 2.4 and Corollary 2.5 we conclude that
\[
\tilde{g}_1(y, E; \hbar) = \left(\frac{y}{\alpha}\right)^{\frac{1}{2}} \left(\frac{i y / 2 \hbar}{1 - 2i \frac{E}{\hbar}}\right)^{-2i \frac{\hbar}{E}} + o(1)
\]
as \(y \to 0\). In view of (2.14) this implies that the outgoing Jost solution is represented as
\[
f_-(x, E; \hbar) = \sqrt{\alpha \Gamma(1 - iv)} \frac{\Gamma(1 - iv)}{(-i \hbar)^{iv}} \tilde{g}_1(y, E; \hbar)
\]
for all \(x \leq x_0\) with \(y = \alpha z = 2e^\frac{2}{E} \hbar\) and \(\nu = 2 \frac{E}{\hbar^2}\). In particular, using the standard asymptotic behavior of \(I_{-iv}\), see [9], one obtains for \(x = x_0\) that
\[
\begin{align*}
f_-(x_0, E; \hbar) &= \gamma_\alpha e^{\frac{1}{2} S_-(E; h)} (1 + O(\hbar)) \\
f'_-(x_0, E; \hbar) &= \gamma'_\alpha e^{\frac{1}{2} S_-(E; h)} (1 + O(\hbar))
\end{align*}
\]
with constants \(\gamma_\alpha, \gamma'_\alpha \approx 1\) depending on \(E, \hbar\), as well as a suitable action \(S_-(E; \hbar)\) which is analytic for \(|E| \leq \hbar\) with \(S_-(E; \hbar) > 0\) for small real-valued \(E\), and \(T_-(E; \hbar)\) some real-valued function of real \(E\) analytic on \(|E| \leq \hbar\). We remark that \(\frac{\tilde{\gamma}}{\tilde{\gamma}} > 0\), which is most important in Section 2.3. Furthermore, each derivative in \(E\) costs at most a power of \(\hbar^{-1}\). It is important that one does not lose \(E^{-1}\) as in the \(x \geq 0\) case, but only \(O(\hbar^{-1})\) as such a loss is negligible compared to the size of \(e^{\frac{1}{2} S_-(E; h)}\).

2.3. The Wronskian of the outgoing Jost solutions. From Sections 2.1.3 and 2.2.2 it is now a simple matter to determine the Wronskian between the outgoing Jost solutions.

Lemma 2.6. Define
\[S(E; h) := S_+(E; h) + S_-(E; h), \quad T(E; h) := T_+(E; h) + T_-(E; h)\]
One has
\[
W(f_+(\cdot, E; \hbar), f_-(\cdot, E; \hbar)) = \gamma_0 E \hbar^{-1} e^{\frac{1}{2} S(E; h)} (1 + O(\hbar))
\]
where \(\gamma_0 \neq 0\) is an absolute constant, and \(|\partial E O(\hbar)| \leq C_k \hbar E^{-k}\) for all \(0 < E \ll 1, 0 < \hbar \ll 1\) and \(k \geq 1\).

Proof. This follows from (2.9) and (2.18). \(\Box\)

Due to the growth of the action \(S(E; h)\) one can now conclude the following important size estimate on the Wronskian:
\[
|W(f_+(\cdot, E; h), f_-(\cdot, E; h))| \ll \hbar^{-1} E e^{\frac{1}{2} S(E; h)} \gtrsim N(\mu E)^{1-N}, \quad N := h^{-1}
\]
for all \(0 < E < \epsilon\). More precisely, one uses that \(S_+(E; h) = -\log E + \alpha_0 + o(1)\) as \(E \to 0^+\) uniformly in small \(\hbar\), whereas \(S_-(E; h) > 0\) for small \(E\). Then \(\mu := e^{-\alpha_0}\). In other words, the Wronskian blows up as \(E \to 0^+\) as a power law with large power since \(\hbar\) is very small.
2.4. The spectral measure \( e(E, x, x'; \hbar) \) near the maximum of the potential.

We now derive the contribution of energies \( 0 < E < \varepsilon \) to the desired pointwise decay of \((\ref{1.9})\) in time. We shall fix \( x = x' = x_0 \) since this case can be treated most easily from the previous sections; moreover, the region near the maximum of the potential is in some sense the most important one. The case of general \( x, x' \) is considered in Section \(2.5\). First, one has

\[
e(E; x_0, x_0; \hbar) = \text{Im} \left[ \frac{f_-(x_0, E; \hbar) f_+(x_0, E; \hbar)}{f'_-(x_0, E; \hbar) f'_+(x_0, E; \hbar) - f'_+(x_0, E; \hbar) f'_-(x_0, E; \hbar)} \right]
\]

\[
= \text{Im} \left[ \frac{f'_-(x_0, E; \hbar) - f'_+(x_0, E; \hbar)}{f_-(x_0, E; \hbar) - f_+(x_0, E; \hbar)} \right]^{-1}
\]

\[
= \hbar \text{Im} \left[ \frac{\alpha(E; \hbar)}{1 - \alpha(E; \hbar) \beta(E; \hbar)} \right]
\]

where

\[
\alpha(E; \hbar) := \hbar^{-1} \frac{f_-(x_0, E; \hbar)}{f'_+(x_0, E; \hbar)}, \quad \beta(E; \hbar) := \frac{f'_+(x_0, E; \hbar)}{f_+(x_0, E; \hbar)}
\]

From \((\ref{2.18})\) one has\(^{11}\)

\[
\alpha(E; \hbar) = d_0 \left( 1 + O(e^{-2\hbar^{-1}S_-}) \right) \left( 1 + O_R(\hbar) \right)
\]

where \( d_0 > 0 \) is a constant that depends on \( x_0, E, \hbar \) with \( d_0 \simeq 1 \). The \( O(\hbar) \)-terms in the numerator and denominator are not necessarily the same. Similarly, from \((\ref{2.19})\), with a constant \( d_1 > 0 \),

\[
\beta(E; \hbar) = -d_1 \left( 1 + O(e^{-2\hbar^{-1}S_+}) \right) \left( 1 + O_R(\hbar) \right) = -d_1(\hbar) \left( 1 + O(e^{-2\hbar^{-1}S_+}) \right)
\]

Due to the exponential decay of \( V \) as \( x \to -\infty \), the functions \( f_-(x, E; \hbar) \) and \( f'_+(x, E; \hbar) \) are analytic in \( E \) in a disk \(|E| \lesssim \hbar\). In particular, \( \alpha(E; \hbar) \) is analytic around \( E = 0 \) in the same neighborhood. Moreover, due to \( f_-(x, E; \hbar) = f'_-(x, E; \hbar) \), one checks that \( \text{Re} f_-(x_0, E; \hbar) \) and \( \text{Im} f_-(x_0, E; \hbar) \) are even and odd in \( E \), respectively. Thus, it follows that \( \text{Im} \alpha(E; \hbar) \) is odd in \( E \), whereas \( \text{Re} \alpha(E; \hbar) \) is even. Moreover, for any \( k, n \geq 0 \),

\[
(\ref{2.20}) \quad |\partial_{k,n}^E \text{Im} \alpha(E; \hbar)| \leq C_{k,n}\hbar^n
\]

which follows from the fact that \( S_-(E; \hbar) > 0 \) uniformly in \(-\varepsilon < E < \varepsilon\) as well as the differentiability properties of \( S_-(E; \hbar) \) in \( E \), see \((\ref{3})\). In view of these properties,

\[
e(E; x_0, x_0; \hbar) = \hbar \text{Im} \left[ \frac{\alpha(E; \hbar)}{1 + d_1 \alpha(E; \hbar) - \alpha(E; \hbar)(d_1 + \beta(E; \hbar))} \right]
\]

\[
= \hbar \text{Im} \left[ \frac{\alpha(E; \hbar)}{1 + d_1 \alpha(E; \hbar)} + O_R(e^{-2\hbar^{-1}S_+(E; \hbar)}) \right]
\]

Since \( S_+(E; \hbar) \sim -\log E \) as \( E \to 0^+ \),

\[
O(e^{-2\hbar^{-1}S_+(E; \hbar)}) = O(E^N), \quad N = \hbar^{-1}
\]

Moreover, the imaginary part in \((\ref{2.21})\), i.e.,

\[
\eta(E; \hbar) := \text{Im} \left[ \frac{\alpha(E; \hbar)}{1 + d_1 \alpha(E; \hbar)} \right]
\]

\(^{11}\)This \( \alpha \) is not related to \( \alpha \) appearing in Section \(2.2\).
is an odd function in $E$ (and analytic near $E = 0$) and it satisfies the bounds
\[
|\partial^k_E \eta(E; \hbar)| \leq C_{k,n} \hbar^n
\]
cf. (2.20), and $\partial^k_E \eta(0; \hbar) = 0$ for even $k$. It is now easy to bound (1.9): for any $n \geq 0$, and all $t \geq 0$, and any $0 \leq k \ll \hbar^{-1}$,
\[
\lim_{\hbar \to 0} \frac{1}{\hbar^2} \int_0^\infty \sin(\hbar^{-1} t E) \text{Im} \left[ \frac{f_+(x_0, E; \hbar) f_-(x_0, E; \hbar)}{W(f_+(x_0, E; \hbar), f_-(x_0, E; \hbar))} \right] \chi_\varepsilon(E) dE \\
\lesssim C_{k,n} \hbar^n (t)^{-k}
\]
by integrating by parts (here $\chi_\varepsilon$ is a smooth localizer to energies $E < \varepsilon$). In other words, by taking $\hbar$ sufficiently small one can achieve any rate of decay. Moreover, we note the important property that small energies do not present any kind of obstruction to the problem of summing in the angular momentum $\ell$; in fact, the contributions of low lying energies to the decay estimates decay rapidly in $\ell$.

2.5. The weighted $L^2$ bound on the spectral measure. Here we generalize the analysis of Section 2.4 to allow for general $x, x'$. More precisely, we claim the following result which is a routine application of the basis representations which we have obtained above, cf. Section 8 in [17].

**Lemma 2.7.** Let $0 \leq M \ll \hbar^{-1}$. The spectral measure as defined in (1.11) satisfies the bounds,
\[
\sup_{0 < E < \varepsilon} \|(x)^{-k - \frac{1}{2}} \partial^k_E e(E, x, x'; \hbar)(x')^{-k - \frac{1}{2}}\|_{L^2_{x,x'}} \leq C_{k,n} \hbar^n
\]
for any $n \geq 0$ and any $0 \leq k \leq M$. Moreover, for any choice of $x, x' \in \mathbb{R}$, one has the property
\[
\lim_{E \to 0^+} \partial^j_E e(E, x, x'; \hbar) = 0
\]
for any $0 \leq j \leq \frac{M}{2}$.

2.6. The decay estimate for small energies. It is now a simple matter to establish the desired decay estimate for (1.9) for small energies.

**Proposition 2.8.** Let $0 \leq M \ll \hbar^{-1}$ be given. Then for any $n \geq 0$, $0 \leq k \leq M$, and $t \geq 0$ one has the bounds
\[
\left\| (x)^{-k - \frac{1}{2}} \int_0^\infty \sin(\hbar^{-1} t E) e(E, x, x'; \hbar) \chi_\varepsilon(E) dE(x')^{-k - \frac{1}{2}} \right\|_{L^2_{x,x'}} \leq C_{k,n} \hbar^n (t)^{-k}
\]
\[
\left\| (x)^{-k - \frac{1}{2}} \int_0^\infty \cos(\hbar^{-1} t E) e(E, x, x'; \hbar) \chi_\varepsilon(E) E dE(x')^{-k - \frac{1}{2}} \right\|_{L^2_{x,x'}} \leq C_{k,n} \hbar^n (t)^{-k}
\]
uniformly in $0 \leq \hbar \leq \hbar_0$.

**Proof.** This follows directly from Lemma 2.7 by repeated integrations by parts. □

We remark that these estimates immediately transfer to $L^1 \to L^\infty$ bounds by means of Bernstein’s inequality. In fact, one can also establish the following result with the optimal $(x)^{-k}$ weights by means of a more careful treatment of the oscillatory integrals as in [17].
Proposition 2.9. Let \( 0 \leq M \ll h^{-1} \) be given. Then for any \( n \geq 0, 0 \leq k \leq M, \) and \( t \geq 0 \) one has the bounds
\[
\| \langle x \rangle^{-k} \int_0^\infty \sin (\hbar^{-1} tE) e(E, x, x'; h) \chi_x(E) dE \langle x' \rangle^{-k} \|_{1 \to \infty} \leq C_{k,n} h^n(t)^{-k}
\]
\[
\| \langle x \rangle^{-k} \int_0^\infty \cos (\hbar^{-1} tE) e(E, x, x'; h) \chi_x(E) E dE \langle x' \rangle^{-k} \|_{1 \to \infty} \leq C_{k,n} h^n(t)^{-k}
\]
uniformly in \( 0 \leq h \leq h_0. \)

We call the reader’s attention to the fact that these bounds decay rapidly with the angular momentum \( \ell \simeq h^{-1} \). In other words, energies (in the original formulation of the Regge-Wheeler equation) of size \( \ll \ell^2 \) do not present any obstruction to the summation in \( \ell \). This is a reflection of the expectation that any such obstruction should result from the local behavior of the potential around the maximum due to complex resonances. In the related context of the surfaces of revolutions, this corresponds to the principle that the growth of the constants \( C(\ell) \) in the decay estimates of (11) is determined by the local geometry of the manifold rather than its asymptotic behavior at the ends. In particular, if the surface contains a large trapping set (such as an equatorial section of a sphere) then the constants grow exponentially in \( \ell \), rendering summation impossible.

3. Energies close to the top, Mourre and Sigal-Soffer estimates
For energies in the range \( \varepsilon < E < 100 \) we establish a Mourre estimate which then allows us to invoke the semiclassical Sigal-Soffer type decay bounds of Section 6.
Thus, let \( p := -ih \partial_x, H := p^2 + V \) as above, and \( A := \frac{1}{2}(px + xp) \). Note that the Mourre estimate is shown here to hold in a neighborhood of a trapping energy (namely, \( E = 1 \)). For notational convenience, we shift the location of the maximum to \( x_{\text{max}} = 0 \) in this section.

Lemma 3.1. For \( \varepsilon > 0 \) and \( h \) small, there exists a fixed constant \( c_0 > 0 \) so that
\[
\chi_I(H) \frac{i}{\hbar} [H, A] \chi_I(H) \geq c_0 h \chi_I(H)
\]
where \( \chi_I \) is the indicator of \( I := [\varepsilon/2, 100] \).

Proof. We split \( I = I_0 \cup I_1 \) where \( I_0 := [\varepsilon/2, 1 - \varepsilon/2] \) and \( I_1 := [1 - 2\varepsilon, 100] \). We start with the latter, and write \( I \) instead of \( I_1 \) for simplicity. First,
\[
\frac{i}{\hbar} [H, A] = 2p^2 - xV'(x; h) \geq p^2 - xV'
\]
Hence, with \( g_I \) being a smooth cutoff function adapted to \( I \),
\[
g_I(H) \frac{i}{\hbar} [H, A] g_I(H)
\]
\[
\geq g_I(H)(p^2 - xV')g_I(H)
\]
\[
\geq g_I(H)(p^2 - xV')F^2 + F^2(p^2 - xV') + (p^2 - xV')F' + F'(p^2 - xV'))g_I(H)
\]
\[
\geq g_I(H)[2F(p^2 - xV')F + [F, [F, p^2]] + F^2Hg_I(H) + Hg_I(H)F^2
\]
\[
+ 2F\langle -xV' - V \rangle]g_I(H)
\]
Here \( 1 = F + \bar{F} \) is a smooth partition of unity with \( F(x) = 1 \) on \([-x_1, x_1]\) where \( x_1 > 0 \) will be a large number depending only on \( V \). Moreover, \( \bar{g}_I \) is another
function adapted to $I$ with $\tilde{g}_I g_I = g_I$. By $F(-xV') F \geq cx^2 F^2$ for some $c > 0$ depending on $x_1$ and the Heisenberg uncertainty principle implies that

$$F(p^2 - xV') F \geq c F(p^2 + x^2) F \geq 2c_0 h F^2$$

The uncertainty principle here is being used in the form

$$\|p\psi\|_2^2 + \|x\psi\|_2^2 \geq 2\|p\psi\|_2 \|x\psi\|_2 \geq h\|\psi\|_2^2$$

which immediately follows from the fact that $[p, x] = -i\hbar$, see for example [21]. Furthermore,

$$[F, [F, p^2]] = -2\hbar^2 (F')^2$$

and

$$g_I(\tilde{F}^2 H \tilde{g}_I + H \tilde{g}_I \tilde{F}^2) g_I$$

$$= g_I(\tilde{F}^2 (H - 1) \tilde{g}_I + (H - 1) \tilde{g}_I \tilde{F}^2 + (\tilde{F}^2 \tilde{g}_I + \tilde{g}_I \tilde{F}^2)) g_I$$

$$= g_I(2\tilde{F}(H - 1) \tilde{g}_I \tilde{F} + 2\tilde{F} \tilde{g}_I \tilde{F} + [\tilde{F}, [\tilde{F}, (H - 1) \tilde{g}_I]] + [\tilde{F}, [\tilde{F}, \tilde{g}_I]]) g_I$$

$$\geq g_I(2(1 - \varepsilon) \tilde{F}^2 - O(\hbar^2)) g_I \geq g_I \tilde{F}^2 g_I - C \hbar^2 g_I F^2 g_I$$

where we used that $\|[\tilde{F}, [\tilde{F}, \tilde{g}_I]]\| \lesssim \hbar^2$, see Lemma 3.2 below. Finally, from the shape of our potential $V(x; \hbar)$ one verifies easily that $x_1$ can be chosen such that $-xV' - V \geq 0$ for all $|x| \geq x_1$ whence $\tilde{F}^2(-xV' - V) \geq 0$. In view of the preceding,

$$g_I(\hbar^{-1} [H, A] g_I(H)) \geq c_0 \hbar^2 g_I(H)$$

as desired. Finally, on the interval $I_0$ one can use [32] directly since one has a classical nontrapping condition on energies in that range. This then gives the desired Mourre estimate in that range of energies, see Theorem 1 in [20].

The following commutator bound was used in the previous proof.

**Lemma 3.2.** Let $F$ and $g$ be smooth and compactly supported. Then

$$\|[F(x), [F(x), g(H)]]\| \leq C \hbar^2$$

where $C = C(F, g)$.

**Proof.** For simplicity we show that $\|[F(x), g(H)]\| \leq C \hbar$, the double commutator being an obvious variation thereof. By the commutator expansion formula (6.3) one has

$$\|[F(x), g(H)]\| \lesssim C(F\|[x, g(H)]\|$$

Now $g(H) = \tilde{g}(\tilde{H})$ where $\tilde{H} := H(H + 1)^{-1}$ and $\tilde{g}$ is again smooth and compactly supported. Hence one can expand with the bounded $\tilde{H}$ to conclude that

$$\|[F(x), g(H)]\| \lesssim C(F, g\|[x, \tilde{H}]\| = C(F, g\|(H + 1)^{-1} x, x\|(H + 1)^{-1}\| \leq C(F, g)\hbar$$

Here we used that $\|[x, H]\| = -2i\hbar p$ and $\|p(H + 1)^{-1}\| \lesssim \|p(1 + p^2)^{-1}\| \lesssim 1$. Since we are dealing with wave rather than the Schrödinger equation, we need to derive a Mourre estimate for $\sqrt{\tilde{H}}$ rather than $H$. However, this is an easy consequence of the Kato square root formula.

**Corollary 3.3.** For $\varepsilon > 0$ and $\hbar$ small, there exists a fixed constant $\tilde{c}_0 > 0$ so that

$$\chi_I(H) \frac{1}{\hbar} [\sqrt{\tilde{H}}, A] \chi_I(H) \geq \tilde{c}_0 b \chi_I(H)$$

where $\chi_I$ is the indicator of $I := [\varepsilon/2, 100]$. 
Proof. One uses that
\[ H^{-\frac{1}{2}}\chi_1(H) = \frac{1}{\pi} \int_0^\infty (H + \lambda)^{-\frac{1}{2}} \lambda^{-\frac{1}{2}} \, d\lambda \chi_1(H) \]
whence by Lemma 3.1
\[
\chi_1(H) \frac{i}{\hbar}[\sqrt{H}, A] \chi_1(H)
= \chi_1(H) \sqrt{H} \frac{i}{\hbar}[A, H^{-\frac{1}{2}}] \sqrt{H} \chi_1(H)
= \frac{1}{\pi} \chi_1(H) \sqrt{H} \int_0^\infty \frac{i}{\hbar}[A, (H + \lambda)^{-1}] \lambda^{-\frac{1}{2}} \, d\lambda \sqrt{H} \chi_1(H)
= \frac{1}{\pi} \sqrt{H} \int_0^\infty (H + \lambda)^{-1} \chi_1(H) \frac{i}{\hbar}[A] \chi_1(H)(H + \lambda)^{-1} \lambda^{-\frac{1}{2}} \, d\lambda \sqrt{H}
\geq \frac{1}{\pi} \sqrt{H} \int_0^\infty (H + \lambda)^{-1} c_0 h \chi_1(H)(H + \lambda)^{-1} \lambda^{-\frac{1}{2}} \, d\lambda \sqrt{H}
\geq \tilde{c}_0 h \chi_1(H)
\]
and we are done. \(\Box\)

In order to apply the time-decay result from Section 6, we need to verify the basic commutator assumption (6.1). For the definition of \(\text{ad}_A^k\) we refer the reader to that section.

Lemma 3.4. For any smooth function \(g\) on the line with support in \((0, \infty)\) one has
\[ \|\text{ad}_A^k(g(\sqrt{H}))\| \leq C(k, g) h^k \]
for all \(k \geq 1\).

Proof. For the purposes of this proof, we call any smooth function \(g\) on the line with support in \((0, \infty)\) admissible. First, there exists another admissible function \(\tilde{g}\) with \(g(\sqrt{H}) = \tilde{g}(H)\). Second, with \(\tilde{H} = H(H + 1)^{-1}\) for any admissible \(g\) there exists \(\tilde{g}\) admissible such that \(g(H) = \tilde{g}(\tilde{H})\). So it suffices to consider \(\text{ad}_A^k(g(\tilde{H}))\) with admissible \(g\).

As a preliminary calculation, note that
\[
i[H, A] = h(2H - (2V + xV')) =: h(2H + V_1)
i[\tilde{H}, A] = (H + 1)^{-1}i[H, A](H + 1)^{-1} = h(H + 1)^{-1}(2H + V_1)(H + 1)^{-1}
\]
whence \(\|\tilde{H}, A\| \leq C h\). At the next level,
\[
i[i[H, A], A] = h(2i[H, A] + i[V_1, A]) = h^2(4H + 2V_1 - xV'_1)
\]
For \(\tilde{H}\) we use the general identity
\[ [SBS, A] = SB[S, A] + S[B, A]S + [S, A]BS \]
to conclude that
(3.3)
\[
i[i[\tilde{H}, A], A] = i h \{ (H + 1)^{-1}(2H + V_1)[(H + 1)^{-1}, A] + (H + 1)^{-1}[2H + V_1, A](H + 1)^{-1} + [(H + 1)^{-1}, A](2H + V_1)(H + 1)^{-1} \}
\]
Inserting
\[
i[(H + 1)^{-1}, A] = -(H + 1)^{-1}i[H, A](H + 1)^{-1}
\]
into (3.3) implies that \(|i[H, A]| \| \leq C^2\hbar^2\). Continuing in this fashion implies
\[
\| \text{ad}_A^k(\hbar H) \| \leq C(k) \hbar^k \quad \forall \, k \geq 1
\]
Next, we transfer this estimate to \(\text{ad}_A^k(g(\hbar H))\) via an almost analytic extension of an admissible function \(g\). This refers to a smooth function \(G_N(z)\) in the complex plane of compact support such that \(g = G_N\) on the real axis and with
\[
|\langle \partial_z G_N(z) \rangle| \leq C_N |\text{Im} \, z|^N
\]
for an arbitrary but fixed positive integer \(N\). One then has the Helffer-Sjöstrand formula
\[
g(\hbar H) = \frac{1}{\pi} \int_{\mathbb{C}} \langle \partial_z G_N(z) (\hbar H - z)^{-1} \rangle m(dz)
\]
where \(m\) is the Lebesgue measure on \(\mathbb{C}\), see [15, Chapter 2]. The desired estimate now follows from
\[
\text{ad}_A^k(g(\hbar H)) = \frac{1}{\pi} \int_{\mathbb{C}} \langle \partial_z G_N(z) \text{ad}_A^k((\hbar H - z)^{-1}) \rangle m(dz)
\]
For example, for \(k = 1\)
\[
\text{ad}_A^1((\hbar H - z)^{-1}) = -(\hbar H - z)^{-1} [\hbar H, A] (\hbar H - z)^{-1}
\]
and therefore
\[
\| \text{ad}_A^1((\hbar H - z)^{-1}) \| \leq C |\text{Im} \, z|^2 \hbar
\]
Inserting this into (3.5) and using (3.4) yields
\[
\| [g(\hbar H), A] \| \leq C\hbar
\]
The cases of higher \(k\) are analogous. The larger \(k\) is, the larger \(N\) needs to be. \(\Box\)

We are now ready to state the main decay estimate for intermediate energies.

**Corollary 3.5.** One has for small \(\hbar\) and all \(t \geq 0\), as well as any \(\alpha \geq 0\),
\[
\| \langle x \rangle^{-\alpha} e^{i L_{\mathbb{C}} T} \chi_1(\hbar H) \langle x \rangle^{-\alpha} \|_{2 \rightarrow 2} \leq C(\alpha) (\hbar t)^{-\alpha}
\]
Furthermore,
\[
\| \langle x \rangle^{-\alpha} e^{i L_{\mathbb{C}} T} \chi_1(\hbar H) f \|_{\infty} \leq C(\alpha) \hbar^{-1} (\hbar t)^{-\alpha} \| \langle x \rangle^\alpha f \|_1
\]
**Proof.** By Corollary 3.3 and the previous lemma, we conclude from Proposition 6.1 that for any admissible function \(g\) (as defined in the previous proof) and any \(\alpha \geq 0\)
\[
\| \langle A \rangle^{-\alpha} e^{-i L_{\mathbb{C}} T} g(\hbar H) \langle A \rangle^{-\alpha} f \|_2 \leq C \langle \hbar t \rangle^{-\alpha} \| f \|_2
\]
To derive (3.6) from this estimate, we pick another admissible \(\tilde{g}\) so that \(\tilde{g}(\hbar H) g(\hbar H) = g(\hbar H)\) where \(H = H(H + 1)^{-1}\) as before. Moreover, the support of \(\tilde{g}\) is taken to lie strictly within \((0, 1)\). Then
\[
\langle x \rangle^{-\alpha} e^{i L_{\mathbb{C}} T} \chi_1(\hbar H) \langle x \rangle^{-\alpha} = \langle x \rangle^{-\alpha} \tilde{g}(\hbar H) \langle A \rangle^\alpha \langle A \rangle^{-\alpha} e^{i L_{\mathbb{C}} T} \chi_1(\hbar H) \langle A \rangle^{-\alpha} \langle A \rangle^\alpha \tilde{g}(\hbar H) \langle x \rangle^{-\alpha}
\]
It therefore suffices to prove that
\[
\| \langle x \rangle^{-\alpha} \tilde{g}(\hbar H) \langle A \rangle^\alpha \| \leq C(\alpha)
\]
The logic here is that the cutoff \(\tilde{g}(\hbar H)\) guarantees that \(H = p^2 + V\) is bounded, whence also \(p^2\) is bounded. But then \(p\) is bounded, so \(A\) should be at most as large as \(x\) which justifies (3.9). By complex interpolation, it suffices to prove that (3.9)
holds for positive integers $\alpha$. Moreover, composing with the adjoints shows that this is the same as

$$\| \langle x \rangle^{-\alpha} \tilde{g}(\tilde{H}) \langle A \rangle^{2\alpha} \tilde{g}(\tilde{H}) \langle x \rangle^{-\alpha} \| \leq C(\alpha)^2$$

For example, set $\alpha = 1$. Then one checks that

$$\langle A \rangle^2 = 1 + \frac{1}{4}(px + xp)^2 = 1 - \frac{\hbar^2}{4} + x^2 = 1 - \frac{\hbar^2}{4} - hx^2 + xHx$$

Since $V = O(|x|^{-2})$, it suffices to bound $xHx$. Let $G$ denote the almost analytic extension of $\tilde{g}$ as in the proof of the previous lemma. Then

$$\langle x \rangle^{-1} \tilde{g}(\tilde{H})xHx\tilde{g}(\tilde{H})\langle x \rangle^{-1} = \langle x \rangle^{-1} \langle x\tilde{g}(\tilde{H}) + [\tilde{g}(\tilde{H}), x]H(\tilde{g}(\tilde{H})x - [\tilde{g}(\tilde{H}), x]\rangle \langle x \rangle^{-1}$$

It is clear that the terms involving no commutators are bounded. For the commutators in the second line we use the Helffer-Sjöstrand formula as before, viz.

$$[\tilde{g}(\tilde{H}), x] = \frac{1}{\pi} \int_{\mathcal{C}} \partial_z G(z)[(\tilde{H} - z)^{-1}, x] m(dz)$$

$$= \frac{1}{\pi} \int_{\mathcal{C}} \partial_z G(z)[(\tilde{H} - z)^{-1}, x, \tilde{H}][H - z]^{-1} m(dz)$$

$$= \frac{1}{\pi} \int_{\mathcal{C}} \partial_z G(z)[(\tilde{H} - z)^{-1}(\tilde{H} + 1)^{-1}(-2i\hbar p)(\tilde{H} + 1)^{-1}(\tilde{H} - z)^{-1} m(dz)$$

In particular, $[\tilde{g}(\tilde{H}), x]$ is a bounded operator. Inserting this into (3.10) concludes the argument for $\alpha = 1$. For $\alpha > 1$ the argument is similar. We begin by expanding for $\ell \geq 1$ an integer

$$\langle A \rangle^{2\ell} = (1 + (xp + px)^2/4)^\ell$$

$$= \sum \text{const} \cdot x^{m_1}p^{n_1}x^{m_2}p^{n_2} \cdots x^{m_s}p^{n_s}$$

where the sum extends over integer $m_i, n_i$ with

$$\sum_i n_i \leq 2\ell, \quad \sum_i m_i \leq 2\ell$$

Moreover, using the commutator $[p, x] = -i\hbar$ to move powers of $p$ through powers of $x$, the general term in (3.12) may be written as $x^k p^{2k} x^k$ where $k \leq \ell$. Hence, we need to show that

$$\langle x \rangle^{-\ell} \tilde{g}(\tilde{H})x^k p^{2k} x^k \tilde{g}(\tilde{H}) \langle x \rangle^{-\ell}$$

with $0 \leq k \leq \ell$ is a bounded operator. First, the operator in (3.13) is nonnegative, and moreover bounded above by

$$\langle x \rangle^{-\ell} \tilde{g}(\tilde{H})x^k H x^k \tilde{g}(\tilde{H}) \langle x \rangle^{-\ell}$$

since $p^2 \leq p^2 + V = H$. Note that if we can move $x^k$ across the spectral cut-offs, then we are done since $0 \leq k \leq \ell$. To accomplish this, we start from the following identity, which is proved by induction: for every $k \geq 2$

$$[x^k, H] = -2i\hbar \sum_{j=1}^{k-1} x^{k-j-1} p x^j$$

and $[x, H] = -2i\hbar p$. Several comments are in order: first, domain considerations are irrelevant due to the cutoff $\tilde{g}(\tilde{H})$ which is always applied. In fact, we may
use this formally and in the end justify the procedure a posteriori by obtaining a bound on the $L^2$-operator norm. Second, the total weight in $x$ on the right-hand side of (3.15) is $k - 1$. And third, in any given term $x^{k-j-1}px^j$ we can shift the position of $p$ arbitrarily using the commutator $[p,x] = -2i\hbar$. To proceed, one has

$$[x^k, \hat{H}] = (H + 1)^{-1}[x^k, H](H + 1)^{-1}$$

so that

$$(3.16) \quad [g(\hat{H}), x^k] = \frac{1}{\pi} \int_C \partial_z G(z)(\hat{H} - z)^{-1}(H + 1)^{-1}[x^k, H](H + 1)^{-1}(\hat{H} - z)^{-1} m(dz)$$

Inserting (3.15) into the right-hand side of (3.16) and in view of the preceding comments we arrive at an expression of the form

$$\frac{1}{\pi} \int_C \partial_z G(z)(\hat{H} - z)^{-1}(H + 1)^{-1}x^{k-1}p(H + 1)^{-1}(\hat{H} - z)^{-1} m(dz)$$

If $k - 1 = 0$ we are done since $p(H + 1)^{-1}$ is bounded. Otherwise, commuting $x^{k-1}$ through $(H + 1)^{-1}$ to the left reduces the weight by another power. In other words, one obtains $x^{k-2}$. Because of this reduction of the degree, the process must terminate after at most $k$ commutations, and we are done with the proof of the first estimate (3.9).

Heuristically speaking, the second bound (3.17) is derived from the first by means of the following principle, known as Bernstein’s inequality: if $\varphi \in L^2(\mathbb{R})$ satisfies $\text{supp}(\hat{\varphi}) \subset [-R, R]$ (with $\hat{\varphi}$ being the Fourier transform), then $\varphi \in L^\infty(\mathbb{R})$ with the bound

$$\|\varphi\|_\infty \leq \|\hat{\varphi}\|_1 \leq (2R)^\frac{1}{2} \|\varphi\|_2$$

where the second inequality is obtained by Cauchy-Schwartz followed by Plancherel’s theorem.

To see the relevance of this, let $g_t(H)$ with $g_t$ smooth be as above. Since $p^2 + V \leq 100$ on the support of $g_t(H)$, one sees again at least heuristically that also $\partial_z^2 \leq 100\hbar^{-2}$ which restricts the Fourier support to size $\leq C\hbar^{-1}$. These operator inequalities can be interpreted in the sense of positive operators, or via quadratic forms, say. Ignoring the distinction between $H$ and the “free” case in which $H = H_0 := p^2$, we obtain via Bernstein that

$$\|\langle x \rangle^{-\alpha} g_t(H)f\|_\infty \lesssim \hbar^{-\frac{1}{2}} \|\langle x \rangle^{-\alpha} f\|_2$$

Replacing $L^2$ on the right-hand side costs another $\hbar^{-\frac{1}{2}}$ by duality, so that one loses $\hbar^{-1}$ in total over the $L^2$-bound, which is what (3.7) claims. Note that we passed the weight in $x$ through $g_t(H)$ onto $f$ which is another technical issue, next to the distinction between $H$ and $H_0$.

In order to rigorously implement these ideas it is advantageous to work with resolvents rather than the (distorted) Fourier transform. To be specific, we write

$$(3.18) \quad \langle x \rangle^{-\alpha} e^{i\frac{p^2}{100}} \chi_I(H) \langle x \rangle^{-\alpha}$$

$$= \langle x \rangle^{-\alpha} (1 + H)^{-1} \langle x \rangle^{-\alpha} e^{i\frac{p^2}{100}} (1 + H)^2 \chi_I(H) \langle x \rangle^{-\alpha} \langle x \rangle^{-\alpha} (1 + H)^{-1} \langle x \rangle^{-\alpha}$$

Note that $(1 + H)^2 \chi_I(H)$ is just another cut-off. Therefore, the $L^2$-decay bound applies to

$$\langle x \rangle^{-\alpha} e^{i\frac{p^2}{100}} (1 + H)^2 \chi_I(H) \langle x \rangle^{-\alpha}$$
and it suffices to prove that
\[(3.19) \quad \| \langle x \rangle^{-\alpha}(1 + H)^{-1}(x)\alpha f \|_\infty \leq C(\alpha)\hbar^{-\frac{3}{2}}\| f \|_2 \]
which by duality then implies the corresponding $L^1 \to L^2$ estimate and thus implies (3.7). To prove (3.19) we represent the Green function, i.e., the kernel of $(1 + H)^{-1}$, in the form
\[(3.20) \quad (1 + H)^{-1}(x, x') = \hbar^{-2} \psi_+(x)\psi_-(x') W(\psi_+, \psi_-) \chi_{[x > x']} + \hbar^{-2} \psi_-(x)\psi_+(x') W(\psi_+, \psi_-) \chi_{[x < x']} \]
with $W$ denoting the Wronskian, and where $\psi_{\pm}$ are the Jost solutions to $1 + H$ which are defined uniquely by
\[-\hbar^2 \psi''_\pm + V \psi = \psi_\pm \]
\[\psi_\pm(x) \sim e^{\mp x} \text{ as } x \to \pm \infty \]
These solutions are given in terms of Volterra integral equations in the form
\[(3.21) \quad \psi_+(x) = \psi_{+,0}(x) - \hbar^{-1} \int_x^\infty e^{\frac{x-y}{\hbar}} V(y)\psi(y) dy \]
where $\psi_{+,0} := e^{+x}$ and symmetrically for $\psi_-$. By the maximum principle (or elementary convexity arguments - recall that $V > 0$) one sees that $\psi_{\pm} > 0$ on the line. In view of (3.21) therefore $0 < \psi_\pm < \psi_{\pm,0}$ and the Green function in (3.20) satisfies
\[0 < (1 + H)^{-1}(x, x') \leq C\hbar^{-1} e^{-\frac{|x-x'|}{\hbar}} \]
We used here that $W(\psi_+, \psi_-) \geq c\hbar^{-1}$ which follows by differentiating and/or evaluating (3.21) at $x = 0$. In conclusion, in order to prove (3.19) we need to show that the kernel
\[\hbar^{-1} \langle x \rangle^{-\alpha} e^{-\frac{|x-y|}{\hbar}} \langle x' \rangle^{\alpha} \]
is bounded as an operator from $L^2 \to L^\infty$ with norm $\leq C\hbar^{-\frac{3}{2}}$. But this follows from Cauchy-Schwarz and we are done.

4. LARGE ENERGIES

This is comparatively easier than the other two regimes of energies. Indeed, the energy $E$ is so much larger than the potential that the free case becomes dominant. Technically speaking, we use the classical WKB ansatz without turning points.

4.1. The WKB ansatz for large energies. We shall use the outgoing Jost solutions $f_+(x, E; \hbar)$ which are defined uniquely as solutions to the equations
\[-\hbar^2 f''_+(x, E; \hbar) + V f_+(x, E; \hbar) = E^2 f_+(x, E; \hbar) \]
\[f_+(x, E; \hbar) \sim e^{\pm i\frac{x^2}{\hbar}} \text{ as } x \to \pm \infty \]
A global (at least on $x \geq 0$) representation of $f_+(x, E; \hbar)$ is given by the WKB ansatz
\[(4.1) \quad f_+(x, E; \hbar) = E^{\frac{3}{2}} e^{i\frac{x^2}{\hbar}} T_+(E; \hbar) Q^{-\frac{1}{2}}(x, E; \hbar) e^{\frac{1}{2} \int_0^x \sqrt{Q(y, E; \hbar)} dy} (1 + h\alpha_+(x, E; \hbar)) \]
where $Q(x, E; \hbar) := E^2 - V(x; \hbar)$ and
\[T_+(E; \hbar) := \int_0^\infty (E - \sqrt{Q(y, E; \hbar)}) dy \]
To see that (4.4) holds for $a_j$ involving the oscillatory kernel needs to be expanded further depending on the asymptotics at $x = +\infty$. This representation is valid for $x \geq 0$, which is justified by the bounds

$$|a_+(x, E; h)| \lesssim (x)^{-3} E^{-2} \quad \forall E \geq 100, \ x \geq 0$$

To obtain these estimates we start from the following equation for $a(x)$, which is obtained by inserting the ansatz (4.1) into the defining equation for $f_+$:

$$\dot{h}(\psi^2 \dot{a}) = -\psi^2 V_2(1 + ha), \quad a(\infty, E; h) = \dot{a}(\infty, E; h) = 0$$

where

$$\psi(x) := Q^{-\frac{1}{2}}(x, E; h)e^{\frac{i}{\hbar} \int_0^x \sqrt{Q(y, E; h)} \, dy}$$

and

$$V_2(x) = \frac{5}{16} \left( \frac{\dot{Q}(x)}{Q(x)} \right)^2 - \frac{1}{4} \frac{\dot{Q}(x)}{Q(x)}$$

$$= \frac{5}{16} \left( \frac{\dot{V}(x)}{E^2 - V(x)} \right)^2 + \frac{1}{4} \frac{\dot{V}(x)}{E^2 - V(x)} = O(E^{-2}(x)^{-4})$$

using that $V$ decays at least as fast as an inverse square. The solution of (4.3) is uniquely given in terms of the Volterra integral equation

$$a(x, E; h) = \int_x^\infty (1 - e^{\frac{i}{\hbar} \int_x^y \sqrt{Q(u, E; h)} \, du})V_2(y, E; h)(1 + ha(y, E; h)) \, dy$$

In addition to (4.2), this integral equation implies the derivative bounds

$$|\partial_x^k \partial_y^j a_+(x, E; h)| \lesssim (x)^{-3-j} E^{-2-k} \quad \forall E \geq 100, \ x \geq 0$$

and all $k \geq 0, j \geq 0$. While these statements are routine, we now give some indication on how they are obtained. Write

$$a(x, E; h) = a_0(x, E; h) + h \int_x^\infty k(x, y; h, E)V_2(y, E; h)a(y, E; h) \, dy$$

$$a_0(x, E; h) := \int_x^\infty k(x, y; h, E)V_2(y, E; h) \, dy$$

$$k(x, y; h, E) := 1 - e^{\frac{i}{\hbar} \int_x^y \sqrt{Q(u, E; h)} \, du}$$

To see that (4.4) holds for $a_0$ we expand the defining integral of $a_0$ as follows:

$$a_0(x, E; h) = \int_x^\infty V_2(y, E; h) \, dy + \int_x^\infty \partial_y \left[ e^{\frac{i}{\hbar} \int_x^y \sqrt{Q(u, E; h)} \, du} \frac{V_2(y, E; h)}{\sqrt{Q(y, E; h)}} \right] \, dy$$

$$= \int_x^\infty V_2(y, E; h) \, dy - i\hbar \int_x^\infty \frac{V_2(y, E; h)}{\sqrt{Q(x, E; h)}} \, dy - i\hbar \int_x^\infty e^{\frac{i}{\hbar} \int_x^y \sqrt{Q(u, E; h)} \, du} \partial_y \left[ \frac{V_2(y, E; h)}{\sqrt{Q(y, E; h)}} \right] \, dy$$

The first two terms here satisfy the bounds (4.4) by inspection, whereas the integral involving the oscillatory kernel needs to be expanded further depending on the number of derivatives, i.e., the size of $j + k$. Note that each further expansion improves the decay of the integrand by one power of $E$ and $y$, respectively.
4.2. Decay estimates in the regime of large energies. The WKB considerations of Section 4.1 imply the following decay estimate. For the definition of the spectral measure \( e(E, x, x'; \hbar) \) see the low energies regime.

**Lemma 4.1.** Let \( \chi_{>100}(E) \) be a smooth cutoff function supported in \((100, \infty)\) and equal to 1 on \((200, \infty)\). Then for all \( t > 0 \),

\[
\sup_{x \in \mathbb{R}} \left| (x)^{-\frac{k}{2}} \hbar^{-1} \int_{\mathbb{R}} \cos(\hbar^{-1} t E) e(E, x, x'; \hbar) E \chi_{>100}(E) dE \langle x' \rangle^{-k} f(x') dx' \right|
\]

\[\leq C \hbar^{-2} (t)^{-k} \int \left| f'(y) \right| |f(y)| dy\]

\[\sup_{x \in \mathbb{R}} \left| (x)^{-\frac{k}{2}} \hbar^{-1} \int_{\mathbb{R}} \sin(\hbar^{-1} t E) e(E, x, x'; \hbar) \chi_{>100}(E) dE \langle x' \rangle^{-k} f(x') dx' \right|
\]

\[\leq C \hbar^{-1} (t)^{-k} \int |f(y)| dy\]

Moreover, the same bounds hold as weighted \( L^2 \) \( \to L^2 \) estimates, but with \( \langle \cdot \rangle^{-k - \frac{1}{2}} \) instead of \( \langle \cdot \rangle^{-k} \).

**Proof.** This is essentially the same as in Section 9 of [17]. The only difference being the factor \( \hbar \). However, we leave it to the reader to check that the proofs in [17] easily carry over to this case as well. As for the \( L^2 \) \( \to L^2 \) bounds, they follow from the \( L^1 \) \( \to L^\infty \) ones by means of Hölder’s inequality.

5. The proof of Theorems 1.1 and 1.2

We begin by reducing general data to those of fixed angular momentum. Thus

\[
(5.1) \quad \psi_0(x, \omega) = \sum_{\ell=0}^{\infty} \sum_{\ell \leq j \leq \ell} \langle \psi_0(x, \cdot), Y_{\ell, j} \rangle \bar{Y}_{\ell, j}(\omega)
\]

where \( \{ Y_{\ell, j} \}_{j \geq \ell} \) is the usual orthonormal basis of spherical harmonics in the space of \( Y \in C^\infty(S^2) \) with \( -\Delta_{S^2} Y = \ell(\ell + 1)Y \). One has \( \| Y_{\ell, j} \|_2 \leq C \langle \ell \rangle^{\frac{1}{2}} \) where \( C \) is an absolute constant. Now let \( Y \) be a normalized spherical harmonic with \( -\Delta_{S^2} Y = \ell(\ell + 1)Y \), and set \( \hbar = \ell^{-1} \). Consider data \( \psi[0] = (f, g)Y = (\psi_0, \psi_1) \).

Let \( \psi(t) \) denote the evolution of \( \psi[0] \) under the wave equation (1.1), as given by (1.9) and (1.10). Then by Lemma 4.1 Corollary 3.3 and Proposition 2.8 provided \( \ell \) is large, one obtains

\[
\| \langle x \rangle^{-k - \frac{1}{2}} \psi(t) \|_{L^2(R; L^2(S^2))} \leq (t)^{-k} \hbar^{-k-1} \| \langle x \rangle^{k + \frac{1}{2}} (\hbar^{-1} \partial_x \psi_0, \hbar^{-1} \psi_0, \psi_1) \|_{L^2(R; L^2(S^2))}
\]

for any \( 0 \leq k \ll \ell \) and \( t > 0 \). Starting from general data \( \psi[0] = (\psi_0, \psi_1) \), performing a decomposition as in (5.1) we may sum up the \( L^2 \)-bound over \( \ell \gg k \), whereas for the finitely many remaining \( \ell \) we invoke the decay estimates from (10) (for \( \ell = 0 \)) and (17) (for \( \ell > 0 \)). In this way one obtains (1.2). The reason why \( \langle x \rangle^{-\frac{3}{2}} \) weights are required stems from the fact the corresponding \( L^1 \) \( \to L^\infty \) bounds in (10) and (17) need \( \langle x \rangle^{-3} \) for \( t^{-3} \) decay, and then we lose another \( \langle x \rangle^{-\frac{1}{2}} \) due to Hölder’s inequality. On another technical note, the weights \( \langle x \rangle^{-k - \frac{1}{2}} \) for \( k = 3 \) (as in our case) essentially retain the orthogonality properties of the spherical harmonics which allows one to sum up the fixed \( \ell \) bounds without any losses in \( \ell \).
For the pointwise bounds we write (5.1) in the form
\[ \psi[0](x,\omega) = (\psi_0, \psi_1)(x,\omega) = \sum_{\ell=0}^{\infty} \sum_{-\ell \leq j \leq \ell} (f_{\ell,j}(x), g_{\ell,j}(x)) Y_{\ell,j}(\omega) \]

The evolution of these data is given by
\[ \psi(t,x,\omega) = \sum_{\ell=0}^{\infty} \sum_{-\ell \leq j \leq \ell} \psi_{\ell,j}(t,x) Y_{\ell,j}(\omega) \]

where \( \psi_{\ell,j} \) is the evolution of \((f_{\ell,j}, g_{\ell,j})\) under (1.5). Therefore, setting \( f_{\ell,j} = 0 \) for ease of notation, and using the bound \( \|Y_{\ell,j}\|_\infty \lesssim \ell^{\frac{1}{2}} \) yields
\[ \|\langle x \rangle^{-4} \psi(t)\|_{L^\infty(R; L^\infty(S^2))} \lesssim \sum_{\ell=0}^{\infty} \langle \ell \rangle^{\frac{1}{2}} \sum_{-\ell \leq j \leq \ell} \|\langle x \rangle^{-4} \psi_{\ell,j}(t)\|_{L^\infty_x} \]
\[ \lesssim \langle t \rangle^{-3} \sum_{\ell=0}^{\infty} \langle \ell \rangle^{\frac{1}{2}} \sum_{-\ell \leq j \leq \ell} \|\langle x \rangle^{4} g_{\ell,j}(x)\|_{L^1_x} \]
\[ \lesssim \langle t \rangle^{-3} \sum_{\ell=0}^{\infty} \langle \ell \rangle^{-\frac{7}{2}} \sum_{-\ell \leq j \leq \ell} \|\langle x \rangle^{4} ((-\Delta_S)^{\frac{3}{2}} \psi_0(x,\omega), Y_{\ell,j}(\omega))_{S^2}\|_{L^1_x} \]

where we invoked the pointwise bounds of Lemma 4.1, Corollary 3.5, and Proposition 2.8 for large \( \ell \), and \[16\] and \[17\] for the remaining \( \ell \). This bound can now be summed since
\[ \sum_{\ell=0}^{\infty} \langle \ell \rangle^{-\frac{7}{2}} \sum_{-\ell \leq j \leq \ell} \|\langle x \rangle^{4} ((-\Delta_S)^{\frac{3}{2}} \psi_0(x,\omega), Y_{\ell,j}(\omega))_{S^2}\|_{L^1_x} \lesssim \|\langle x \rangle^{4} ((-\Delta_S)^{\frac{3}{2}} \psi_0(x,\omega))_{S^2}\|_{L^1_{x,\omega}} \]

This implies the estimate \[13\] and Theorem 1.1 is proved. The proof of Theorem 1.2 is analogous.

6. Semiclassical Sigal-Soffer propagation estimates

In this section we present a semiclassical version of the abstract theory from \[26\]. Our arguments are very close to \[26\], but some care is required in keeping track of powers of \( \hbar \). The main result is as follows. In this section \( H \) and \( A \) are self-adjoint operators on a Hilbert space. \( H = H(\hbar) \) and \( A = A(\hbar) \) depend on a small parameter \( \hbar \in (0, \hbar_0] \) but with domains independent of \( \hbar \). We assume the bounds
\[ \|\text{ad}_A^k(g(H))\| \leq C(k, g) \hbar^k \quad (6.1) \]
for all \( k \geq 0, \hbar \in (0, \hbar_0] \) and smooth, compactly supported functions \( g \) on the line. As usual, \( \text{ad}_A^k(g(H)) \) are the \( k \)-fold iterated commutators defined inductively as \( \text{ad}_A^1(g(H)) = [g(H), A] \) and
\[ \text{ad}_A^k(g(H)) = [\text{ad}_A^{k-1}(g(H)), A] \quad \forall k \geq 2. \]
Lemma 6.2. Suppose \( I \subset \mathbb{R} \) is a compact interval so that\(^{12}\)

\[
\chi_I(H) \frac{i}{\hbar}[H, A] \chi_I(H) \geq \theta \hbar \chi_I(H)
\]

for some \( \theta > 0 \). Both \( I \) and \( \theta \) are independent of \( \hbar \). Then for any smooth \( g_I \) with support in \( I \) one has for all \( t \in \mathbb{R} \)

\[
\| \langle A \rangle^{-\alpha} e^{-i \frac{t}{\hbar} g_I(H)} \langle A \rangle^{-\alpha} f \| \leq C \langle \hbar t \rangle^{-\alpha} \| f \|
\]

for any \( \alpha \geq 0 \) where \( C \) depends on \( \alpha, \theta, g_I \) and \( I \), but not on \( \hbar \). Moreover, \( h_0 \) needs to be taken sufficiently small depending on these parameters.

The proof requires some preparatory work. First, recall the commutator expansion formula going back to [22, 33], and subsequently refined in [26, 45, 27, 8]:

\[
[g(H), f(A)] = \sum_{k=1}^{n-1} \frac{f^{(k)}(A)}{k!} \text{ad}^k_A(g(H)) + R_n
\]

where \( f, g \) are smooth, compactly supported functions on the line and the error \( R_n \) satisfies the bound

\[
\| R_n \| \leq C_n \| \text{ad}^n_A(g(H)) \| \sum_{k=0}^{n+2} \int \langle x \rangle^{k-n-1} |f^{(k)}(x)| \, dx
\]

with a constant \( C_n \) depending only on \( n \geq 1 \). This error bound is obtained by means of the Helffer-Sjöstrand formula involving almost analytic extensions of \( f \), see [15, Chapter 2]. For the expansion (6.4) and the error bound (6.5) see Appendix B in [23], in particular (B.8) and (B.14).

In particular, if \( f \) is of order at most \( p \) meaning that \( f \) is a smooth function on the line obeying the bound

\[
|f^{(k)}(x)| \leq C_k \langle x \rangle^{p-k}
\]

for each \( k \geq 0 \), then (6.4) can be applied with \( n > p \).

We now proceed as in [26]. Throughout this section, the assumptions of Proposition 6.1 will be in force.

Lemma 6.2. Let \( f \geq 0 \) be or order \( p < 4 \), nonincreasing and with \( f(x) = 0 \) for \( x \geq 0 \). Furthermore, assume that \( f = f_2^3 + f_2^3 + f_3^3 \) where \( f_2, f_3 \) are smooth. Let \( 1 \leq s < \infty, a \in \mathbb{R}, A_s := (hs)^{-1}(A - a) \), and fix \( \varepsilon \in (0, 1] \) as well as \( n \geq 2 \). Then with \( g_I \) as above

\[
g_I(H)[H, f(A_s)]g_I(H) \leq s^{-1} h \theta g_I(H) f'(A_s) g_I(H) + s^{-1-\varepsilon} g_I(H) f_1(A_s) g_I(H)
\]

\[+ s^{-2n-1-\varepsilon} g_I^2(H)\]

uniformly in \( a \in \mathbb{R} \) and \( h \in (0, h_0] \). Here, \( f_1 \) is of the same order \( p < 4 \) as \( f \), and vanishes on \( x \geq 0 \), and it depends only on \( f, g_I \) and \( n \).

\(^{12}\)It would be perhaps more natural to expect \( \chi_I(H) \frac{i}{\hbar}[H, A] \chi_I(H) \geq \theta \chi_I(H) \), see [20, 24]. The loss of an \( \hbar \) in the lower bound is due to the fact that we establish the Mourre estimate at an energy which is trooping, namely the top of the potential barrier.
Proof. We replace $H$ with $H_b := Hb(H)$ where $b$ is a smooth cutoff function with $bg = g$ (for simplicity, we write $g$ instead of $g_1$). Then
\[ B_k := i\hbar^{-k}a(\hbar B_b), \quad k \geq 1 \]
satisfy the bounds $\|B_k\| \leq C_k$ by assumption for all $k \geq 1$. We begin by showing that
\[ i[H_b, f(A_s)] \simeq -s^{-1}(f'(A_s))^{1/2}B_1(-f'(A_s))^{1/2} \tag{6.7} \]
where $\simeq$ throughout this proof will mean equality up to addition of a quadratic form remainder $\text{rem} = \text{rem}(s)$ satisfying the bound
\[ \pm \text{rem}(s) \leq s^{-(1+\varepsilon)}f_1(A_s) + s^{-(2n-1-\varepsilon)}\text{Id} \]
uniformly in $\hbar, a$ and with $f_1$ as above. Clearly, any term of the form $\text{rem}$ is admissible for the lemma and can be ignored. Write $f = F^2$ and expand by means of (6.4)
\[ i[H_b, f(A_s)] = i[H_b, F(A_s)]F(A_s) + F(A_s)i[H_b, F(A_s)] \]
\[ = \sum_{k=1}^{n-1} \frac{1}{k!}s^{-k}(F^{(k)}(A_s)B_kF(A_s) + F(A_s)B_k^*F^{(k)}(A_s)) \tag{6.8} \]
\[ + s^{-n}(R_n F(A_s) + F(A_s)R_n^*) \tag{6.9} \]
From (6.5) and since $n \geq 2$ and $F$ is of order $<1$, one concludes that $R_n$ is bounded uniformly in $s, a, \hbar$. We now claim that only the term $k = 1$ is significant, i.e.,
\[ i[H_b, f(A_s)] \simeq s^{-1}(F'(A_s)B_1F(A_s) + F(A_s)B_1F'(A_s)) \tag{6.10} \]
Indeed, we first check that the terms in (6.8) for $k \geq 2$ are subsumed in the $f_1$ expression of $\text{rem}$. To see this, we note that
\[ |F^{(k)}(A_s)B_kF(A_s)\psi, \psi| \leq \|B_k\|\|F(A_s)\psi\|_2\|F^{(k)}(A_s)\psi\|_2 \]
\[ \leq C\sqrt{\langle F(A_s)\psi, \psi\rangle \langle F^{(k)}(A_s)\psi, \psi\rangle} \leq \langle f_1\psi, \psi \rangle \]
provided $f_1$ is an upper envelope for both $F^2$ and $(F^{(k)})^2$ with some multiplicative constant. Second, for (6.10) one uses that
\[ \pm (P^*Q + Q^*P) \leq P^*P + Q^*Q \]
with $Q := \|R_n\|s^{-1/2}F(A_s)$, $P^* := \|R_n\|^{-1}s^{-n+1/2}R_n^*$. The $Q^*Q$ expression is again subsumed into $f_1$, whereas for $P^*P$ we obtain
\[ \|R_n\|^{-2}s^{-2n+1+\varepsilon}R_n^* \leq s^{-2n+1+\varepsilon} \]
This establishes our claim (6.10). By assumption we can write $F = u^2$, $-F' = v^2$ with $u, v$ of order $<1$ whence $\|B_1, u(A_s)\| \leq Cs^{-1}$ and $\|B_1, v(A_s)\| \leq C s^{-1}$. Therefore, the right-hand side of (6.10) is of the form
\[ s^{-1}(F'(A_s)B_1F(A_s) + F(A_s)B_1F'(A_s)) \]
\[ = -s^{-1}(v^2(A_s)B_1u^2(A_s) + u^2(A_s)B_1v^2(A_s)) \]
\[ \simeq -2s^{-1}uv(A_s)B_1uv(A_s) \]
whence (6.7) since $f' = -2(uv)^2$. The remainder that arises here is of the $f_1$-form as can be seen by arguing as in (6.11).
To invoke the Mourre estimate \((6.2)\), we choose \(G\) smooth and compactly supported in \(I\) and with \(bgG = gG = g\). Then
\[
G(H)B_1G(H) = G(H)\frac{i}{\hbar}[H, A]G(H) \geq \theta hG^2(H)
\]
We now claim that
\[
(6.12) \quad s^{-1}G(H)\eta B_1\eta G(H) \simeq s^{-1}\eta G(H)B_1G(H)\eta
\]
where \(\eta(A) := (-f'(A_s))^{\frac{1}{2}}\). It is clear that this claim will finish the proof. One has
\[
s^{-1}G(H)\eta B_1\eta G(H) - s^{-1}\eta G(H)B_1G(H)\eta
\]
\[
= s^{-1}(\eta G B_1[\eta, G] + [G, \eta]B_1G\eta + [G, \eta]B_1[\eta, G])
\]
and there is the expansion
\[
[G, \eta] = \sum_{k=1}^{n-1} \frac{s^{-k}}{k!} \eta^{(k)}(A_s) h^{-k}\text{ad}_A^k(G(H)) + s^{-n}R
\]
The expansion of \([\eta, G]\) is the adjoint of this one. To prove \((6.12)\), we observe that \(h^{-k}\text{ad}_A^k(G(H))\) is uniformly bounded in \(k\), and we also gain \(s^{-k-1} \leq s^{-2}\) with \(k \geq 1\) since \((6.12)\) is of order \(s^{-1}\), and each step in the expansion gains another \(s^{-1}\). The other issues, such as the domination by \(f_1\) etc. are very similar to what we have done before, and we skip them. These details are identical to those in [26, Lemma 2.1], see in particular the paragraph leading up to (2.10) in that reference. \(\Box\)

The following is the semiclassical analogue in this context of the key propagation estimate of Theorem 1.1 in [26].

**Lemma 6.3.** Let \(0 < \theta' < \theta\) and \(g_t\) be as in Proposition 6.1. Let \(\chi^\pm\) be the indicator functions of \(\mathbb{R}^\pm\), respectively. Then for any \(t \geq 0\) and any \(m \geq 0\),
\[
\|\chi^- (A - a - \hbar \theta' t)e^{-i\frac{4\pi}{\hbar}g_1(H)}\chi^+ (A - a)\| \leq C_m \langle \hbar t \rangle^{-m}
\]
uniformly in \(a, \theta\) where \(C_m\) only depends on \(m, \theta', \theta\), and \(g_1\).

**Proof.** Define for any \(s \geq 1\)
\[
A_{s,t} := (hs)^{-1}(A - a - h\theta t)
\]
Choose \(F \geq 0\) smooth, nonincreasing of order 0 and \(F(x) = 0\) for \(x \geq 0\). We shall prove the estimate
\[
(6.13) \quad \|F(A_{s,t})e^{-i\frac{4\pi}{\hbar}g_1(H)}\chi^+ (A - a)\| \leq C \langle \hbar t \rangle^{-m}
\]
To see that this implies the lemma, set \(s = t\) in \((6.13)\) and note that if \(F = 1\) on \((-\infty, -\delta]\) with \(\delta > 0\) small, then
\[
F(A_{s,t})\chi^- (A - a - \hbar \theta' t) = \chi^- (A - a - \hbar \theta' t), \quad \theta' := \theta - \delta
\]
Define
\[
\phi_s(t) := g_1(H)f(A_{s,t})g_1(H), \quad f = F^2
\]
\[
\psi_t := e^{-i\frac{4\pi}{\hbar}g_1(H)}\chi^+ (A - a)\phi
\]
where \(\phi\) is an arbitrary unit vector. Then \((6.13)\) will follow from the claim: for every positive integer \(m\),
\[
(6.14) \quad \langle \phi_s(t) \rangle_t := \langle \psi_t, \phi_s(t)\psi_t \rangle \leq C \hbar^{-m}s^{-m}
\]
uniformly in $\hbar, a$ and $0 \leq t \leq s, 1 \leq s$. Note that $\langle \phi_s(t) \rangle_t \geq 0$ by construction. Fix some $m$. Differentiating yields

$$
\partial_t \langle \phi_s(t) \rangle_t = \langle \psi_t, D_t \phi_s(t) \psi_t \rangle = \langle D_t \phi_s \rangle_t
$$

$$
D_t \phi_s(t) = \frac{i}{\hbar} [H, \phi_s(t)] + \partial_t \phi_s(t)
$$

$$
= g_t(H) \frac{i}{\hbar} [H, f(A_s, t)] g_t(H) - s^{-1} g_t(H) f(A_s, t) g_t(H)
$$

By (6.14), for any $n \geq 1$,

$$
0 \leq \langle \phi_s(0) \rangle_0 \leq C_n s^{-2n}
$$

The point here is that $f^{(k)}(A_s, 0) \chi^+(A - a) = 0$ for all $k \geq 0$ so that only the remainder in the commutator expansion contributes. Next, apply Lemma 6.2 with adjoints. □

The second term satisfies

$$
\langle \phi_s(t) \rangle_t \leq C(s^{-2n} + \hbar^{-1} s^{-1})
$$

which implies (6.14) with $m = 1$. The idea is now to bootstrap using (6.15). Indeed, we can apply (6.16) to $f_1$ to conclude that (6.14) holds with $m = 2$. Iterating this procedure concludes the proof. □

Proof of Proposition 6.7. This follows from Lemma 6.3 as follows. Let $t \geq 0$. First, write

$$
\langle A \rangle^{-\alpha} = \langle A \rangle^{-\alpha} \chi^+(A + \frac{1}{2} \hbar \dot{\theta} t) + \langle A \rangle^{-\alpha} \chi^-(A + \frac{1}{2} \hbar \dot{\theta} t)
$$

The second term satisfies

$$
\| \langle A \rangle^{-\alpha} \chi^-(A + \frac{1}{2} \hbar \dot{\theta} t) \| \leq C \hbar^{-\alpha} t^{-\alpha}
$$

in the sense of operator norms. The first term we subject to the evolution: with $a = -\frac{1}{2} \hbar \dot{\theta} t$,

$$
e^{-i \frac{\hbar \dot{\theta} t}{2}} g_t(H) \chi^+(A - a) = \chi^-(A - a - \frac{3}{4} \frac{\hbar \dot{\theta} t}{2}) e^{-i \frac{\hbar \dot{\theta} t}{2}} g_t(H) \chi^+(A - a)
$$

$$
+ \chi^+(A - a - \frac{3}{4} \frac{\hbar \dot{\theta} t}{2}) e^{-i \frac{\hbar \dot{\theta} t}{2}} g_t(H) \chi^+(A - a)
$$

The second term here satisfies

$$
\| \langle A \rangle^{-\alpha} \chi^+(A - a - \frac{3}{4} \frac{\hbar \dot{\theta} t}{2}) e^{-i \frac{\hbar \dot{\theta} t}{2}} g_t(H) \chi^+(A - a) \| \leq C \hbar^{-\alpha} t^{-\alpha}
$$

while the first satisfies the same bound without the weights $\langle A \rangle^{-\alpha}$ by Lemma 6.3 which concludes the proof for positive times. For negative times one passes to the adjoints. □
References

[1] Abramowitz, M., Stegun, I. Handbook of mathematical functions with formulas, graphs, and mathematical tables. Reprint of the 1972 edition. Dover Publications, Inc., New York, 1992.

[2] Alexandrova, I., Bony, J., Ramond, T. Resolvent and scattering matrix at the maximum of the potential. Serdica Math. J. 34 (2008), no. 1, 267–310.

[3] Amrein, W., Boutet de Monvel, A., Georgescu, V. C0-groups, commutator methods and spectral theory of N-body Hamiltonians. Progress in Mathematics, 135. Birkhäuser Verlag, Basel, 1996.

[4] Andersson, L., Blue, P. Hidden symmetries and decay for the wave equation on the Kerr spacetime. Preprint arXiv:0908.2265.

[5] Balogh, Charles B. Asymptotic expansions of the modified Bessel function of the third kind of imaginary order. SIAM J. Appl. Math. 15 (1967), 1315–1323.

[6] Bony, J.-F., Fujiié, S., Ramond, T., Zerzeri, M. Microlocal solutions of Schrödinger equations at a maximum point of the potential, preprint 2009.

[7] Bony, J.-F., Häfner, D. Decay and non-decay of the local energy for the wave equation on the de Sitter-Schwarzschild metric. Comm. Math. Phys. 282 (2008), no. 3, 697–719.

[8] Briet, P.; Combes, J.-M., Duclos, P. On the location of resonances for Schrödinger operators in the semiclassical limit. II. Barrier top resonances. Comm. Partial Differential Equations 12 (1987), no. 2, 201–222.

[9] Costin, O., Donninger, R., Schlag, W., Tanveer, S. Semiclassical low energy scattering for one-dimensional Schrödinger operators with exponentially decaying potentials. Preprint arXiv:1105.4221, 2011.

[10] Costin, O., Schlag, W., Staubach, W., Tanveer, S. Semiclassical analysis of low and zero energy scattering for one-dimensional Schrödinger operators with inverse square potentials. J. Funct. Anal. 255 (2008), no. 9, 2321–2362.

[11] Dafermos, M., Rodnianski, I. Lectures on black holes and linear waves. preprint 2008.

[12] Dafermos, M., Rodnianski, I. The red-shift effect and radiation decay on black hole spacetimes. Comm. Pure Appl. Math. 62 (2009), no. 7, 859-919.

[13] Dafermos, M., Rodnianski, I. A proof of the uniform boundedness of solutions to the wave equation on slowly rotating Kerr backgrounds. Preprint arXiv:0805.4309.

[14] Dafermos, M., Rodnianski, I. Decay for solutions of the wave equation on Kerr exterior spacetimes I-II: The cases |a| << M or axial symmetry. Preprint arXiv:1010.5132.

[15] Davies, E. B. Spectral Theory and Differential Operators. Cambridge, 1995.

[16] Donninger, R., Schlag, W. Decay estimates for the one-dimensional wave equation with an inverse power potential. Int. Math. Res. Not. 2010, no. 22, 4276–4300.

[17] Donninger, R., Schlag, W., Soffer, A. A proof of Price’s law on Schwarzschild black hole manifolds for all angular momenta. Adv. Math. 226 (2011), no. 1, 484-540.

[18] Finster, F., Kamran, N., Smoller, J., Yau, S.-T. Decay of solutions of the wave equation in the Kerr geometry. Comm. Math. Phys. 264 (2006), no. 2, 465-503.

[19] Gérard, C., Grisiz, A. Precise estimates of tunneling and eigenvalues near a potential barrier. J. Differential Equations 72 (1988), no. 1, 149–177.

[20] Graf, G. The Mourre estimate in the semiclassical limit. Lett. Math. Phys. 20 (1990), no. 1, 47–54.

[21] Gustafson, S., Sigal, I. M. Mathematical concepts of quantum mechanics. Universitext. Springer-Verlag, Berlin, 2003.

[22] Hawking, S., Ellis, G. The large scale structure of space-time. Cambridge Monographs on Mathematical Physics, No. 1. Cambridge University Press, London-New York, 1973.

[23] Helffer, B., Sjöstrand, J. Semiclassical analysis of Harper’s equation III. Bull. Soc. Math. France, Mémoire 39, 1990.

[24] Hislop, P., Nakamura, S. Semiclassical resolvent estimates. Ann. Inst. H. Poincaré Phys. Théor. 51 (1989), no. 2, 187–198.

[25] Hunziker, W., Sigal, I. M. Time-dependent scattering theory of N-body quantum systems. Rev. Math. Phys. 12 (2000), no. 8, 1033–1084.

[26] Hunziker, W., Sigal, I. M., Soffer, A. Minimal escape velocities. Comm. Partial Differential Equations 24 (1999), no. 11-12, 2279–2295.

[27] Ivrii, V., Ja., Sigal, I. M. Asymptotics of the ground state energies of large Coulomb systems. Ann. of Math. (2) 138 (1993), no. 2, 243–335.
[28] Kay, B., Wald, R. Linear stability of Schwarzschild under perturbations which are nonvanishing on the bifurcation 2-sphere. Classical Quantum Gravity 4 (1987), no. 4, 893-898.

[29] Marzuola, J., Metcalfe, J., Tataru, D., Tohaneanu, M. Strichartz estimates on Schwarzschild black hole backgrounds. Comm. Math. Phys. 293 (2010), no. 1, 37-83

[30] Metcalfe, J., Tataru, D., Tohaneanu, M. Price’s Law on Nonstationary Spacetimes Preprint arXiv:1104.5437

[31] Luk, J. Improved decay for solutions to the linear wave equation on a Schwarzschild black hole. Ann. Henri Poincaré 11 (2010), no. 5, 805-880.

[32] Luk, J. A Vector Field Method Approach to Improved Decay for Solutions to the Wave Equation on a Slowly Rotating Kerr Black Hole. Preprint arXiv:1009.0671

[33] Miller, Peter D. Applied asymptotic analysis. Graduate Studies in Mathematics, 75. American Mathematical Society, Providence, RI, 2006.

[34] Nakamura, S. Semiclassical resolvent estimates for the barrier top energy. Commun. Partial Differ. Eq. 16(4/5) (1991), 873883.

[35] Olver, F. W. J. Asymptotics and Special Functions, A K Peters, Ltd., Wellesley, MA, 1997.

[36] Mourre, E. Absence of singular continuous spectrum for certain selfadjoint operators. Comm. Math. Phys. 78 (1980/81), no. 3, 391–408.

[37] Price, R. Nonspherical perturbations of relativistic gravitational collapse. I. Scalar and gravitational perturbations. Phys. Rev. D (3), 5:2419–2438, 1972.

[38] Price, R. Nonspherical perturbations of relativistic gravitational collapse. II. Integer-spin, zero-rest-mass fields. Phys. Rev. D (3), 5:2439–2454, 1972.

[39] Ramond, T. Semiclassical study of quantum scattering on the line. Comm. Math. Phys. 177 (1996), no. 1, 221–254.

[40] Schlag, W., Soffer, A., Staubach, W. Decay for the wave and Schrödinger evolutions on manifolds with conical ends, Part I. Trans. Amer. Math. Soc. 362 (2010), no. 1, 19-52

[41] Schlag, W., Soffer, A., Staubach, W. Decay for the wave and Schrödinger evolutions on manifolds with conical ends, Part II. Trans. Amer. Math. Soc. 362 (2010), no. 1, 289-318

[42] Sigal, I. M., Soffer, A. Long-range many-body scattering. Invent. Math 99 (1990), 115-143.

[43] Sigal, I. M., Soffer, A. Local decay and velocity bounds. Preprint, Princeton University 1988.

[44] Sjöstrand, J. Semiclassical Resonances Generated by Nondegenerate Critical Points Pseudodifferential Operators (Oberwolfach, 1986), Lecture Notes in Math., Vol. 1256, Springer-Verlag, Berlin, 1987, pp. 402429.

[45] Skibsted, E. Propagation estimates for N-body Schroedinger operators. Comm. Math. Phys. 142 (1991), no. 1, 67–98.

[46] Tataru, D. Local decay of waves on asymptotically flat stationary space-times, preprint 2009.

[47] Tataru, D., Tohaneanu, M. A local energy estimate on Kerr black hole backgrounds. Int. Math. Res. Not. IMRN 2011, no. 2, 248-292.

[48] Tohaneanu, M. Strichartz estimates on Kerr black hole backgrounds. Preprint arXiv:0910.1545

[49] Wald, R. General relativity. University of Chicago Press, Chicago, IL, 1984.

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