On Quantum Channels.

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The existence of non-local correlations or entanglement in multipartite quantum systems [1, 2] is one of the cornerstones on which the newly established field of quantum information theory is build. The main gain of quantum over classical information processing stems from the fact that we are allowed to perform operations on entangled states: through the quantum correlations, an operation on a part of the system affects the whole system. One of the most challenging open problems is to clarify and quantify how entanglement behaves when part of an entangled state is sent through a quantum channel.

Of central importance in the description of a quantum channel or completely positive map (CP-map) is the dual state associated to it. This state is defined over the tensor product of the Hilbert space itself (the input of the channel) with another one of the same dimension (the output of the channel). It is clear that there appears a natural tensor product structure, and indeed the notion of entanglement will be crucial in the description of quantum channels.

In a typical quantum information setting, Alice wants to send one qubit (eventually entangled with other qubits) to Bob through a quantum channel. The channel acts linearly on the input state, and the consistency of quantum mechanics dictates that this map be completely positive (CP) [3]. This implies that the map is of the form [4]

$$\Phi(\rho) = \sum_i A_i \rho A_i^\dagger.$$  

Moreover the map is trace-preserving if no loss of the particle can occur. A natural way of describing the class of CP-maps is by using the duality between maps and states, first observed by Jamiolkowski [5] and since then rediscovered by many. We review some nice properties of CP-maps based on this dual description, and show how to obtain the extreme points of the convex set of trace-preserving CP-maps.

The dual state is defined on a Hilbert space that is the tensor product of two times the original Hilbert space on which the map acts, and is therefore naturally endowed with a notion of entanglement. Unitary evolution for example corresponds to maximal correlations between the input- and output state, and this kind of evolution leads to a dual state that is maximally entangled. We will show how normal forms derived for entangled states lead to interesting parameterizations of CP-maps, and will discuss some issues concerning the use of quantum channels to distribute entanglement.

It thus turns out that the techniques developed for describing entanglement can directly be applied for describing the evolution of a quantum system. Concepts as quantum steering and teleportation have a direct counterpart. A quantum channel for example will be useful for distributing entanglement if and only if the dual state associated to it is entangled, and optimal decompositions of states as derived in the case of entanglement of formation will yield very appealing parameterizations of quantum channels.
product of two Hilbert spaces of dimension $n$

$$|A| = \sum_{ij}^{n} a_{ij} |i\rangle\langle j|.$$ 

Define

$$|I| = \sum_{i}^{n} |i\rangle\langle i|$$

an unnormalized maximally entangled state and $A$ the operator with elements $\langle i|A|j\rangle = a_{ij}$, then

$$|A| = A \otimes I_n |I|.$$ 

Moreover it holds that

$$X \otimes Y |A| = XA \otimes Y |I| = XAY^T \otimes I_n |I| = I_n \otimes YA^T X^T |I|.$$ 

The symbol $|I|$ will solely be used to denote the unnormalized maximally entangled state $|I| = \sum_i |ii\rangle$. We are now ready for the following fundamental Theorem of de Pillis [6]:

**Theorem 1** A linear map $\Phi$ acting on a matrix $X$ is Hermitian-preserving if and only if there exist operators $\{A_i\}$ and real numbers $\lambda_i$ such that

$$\Phi(X) = \sum_{i} \lambda_i A_i X A_i^\dagger$$

**Proof:** Suppose the map $\Phi$ acts on a $n \times n$ matrix. Then due to linearity, $\Phi$ is completely characterized if we know how it acts on a complete basis of $n \times n$ matrices, for example on all matrices $|e_i\rangle\langle e_j|$, $1 \leq i, j \leq n$ with $|e_i\rangle$ a complete orthonormal base in Hilbert space. Let us define the $n^2 \times n^2$ positive matrix

$$|I\rangle\langle I| = \left(\begin{array}{cccc} |e_1\rangle\langle e_1| & \cdots & |e_1\rangle\langle e_n| \\ \vdots & \ddots & \vdots \\ |e_n\rangle\langle e_1| & \cdots & |e_n\rangle\langle e_n| \end{array}\right),$$

being the matrix notation of a maximally entangled state in a $n \otimes n$ Hilbert space. It follows that all the information of a map $\Phi$ is encoded in the state

$$\rho_\Phi = I_n \otimes \Phi(|I\rangle\langle I|),$$

as the $n^2$ $n \times n$ blocks represent exactly the action of the map on the complete basis $|e_i\rangle\langle e_j|$. If $\Phi$ is Hermitian-preserving, then $\Phi(|e_i\rangle\langle e_j|)$ has to be equal to the Hermitian conjugate of $\Phi(|e_j\rangle\langle e_i|)$, and this implies that $\rho_\Phi$ is Hermitian. Let us therefore consider the eigenvalue decomposition of $\rho_\Phi = \sum_i \lambda_i |\chi_i\rangle\langle \chi_i|$. Using the trick $|A| = (A \otimes I)|I|$, we easily arrive at the conclusion that $\Phi(X) = \sum_i \lambda_i A_i X A_i^T$, where $\{\lambda_i\}$ are the eigenvalues and where the operators $\{A_i^T\}$ are the reshaped versions of the eigenvectors of $\rho_\Phi$. 

A central ingredient in the proof was the introduction of the matrix

$$\rho_\Phi = I_n \otimes \Phi(|I\rangle\langle I|)$$

with $|I\rangle = \sum_i |i\rangle|i\rangle$ a maximally entangled state. We define this Hermitian matrix $\rho_\Phi$ as being the dual state corresponding to the map $\Phi$. It was already explained that it encodes all the information about the map, and its eigenvectors give rise to the operators $A_i$. The above lemma characterizes all possible Hermitian preserving maps, and therefore surely all positive and completely positive maps. For example, let us consider the positive map that corresponds to taking the transpose of the density operators of a qubit:

$$\lambda_1 = 1 \quad A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (3)$$

$$\lambda_2 = 1 \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (4)$$

$$\lambda_3 = 1 \quad A_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} / \sqrt{2} \quad (5)$$

$$\lambda_4 = -1 \quad A_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} / \sqrt{2} \quad (6)$$
Not all Hermitian-preserving maps are physical in quantum mechanics however: if a map acts on a subsystem, then it should conserve positivity of the complete density operator. This extra assumption leads to the condition of complete positivity, meaning that $I_m \otimes \Phi$ is positive for all $m$. Of course, this implies that the dual state $\rho_\Phi$ is not only Hermitian but also positive (i.e. all its eigenvalues are positive), as it is defined as the action of the map $I_n \otimes \Phi$ on a maximally entangled state. The positive eigenvalues can then be absorbed into the (Kraus) operators $\{A_i\}$, and we have therefore proven the Kraus representation Theorem (Choi[1]):

**Theorem 2** A linear map $\Phi$ acting on a density operator $\rho$ is completely positive if and only if there exist operators $\{A_i\}$ such that

$$\Phi(\rho) = \sum_i A_i \rho A_i^\dagger.$$

Remarks:

- A CP-map is trace-preserving iff $\sum_i A_i^\dagger A_i = I_n$; this property is easily verified using the cyclicity of the trace. In terms of the (unique) dual state $\rho_\Phi$ associated to the map $\Phi$, this trace-preserving condition amounts to:

$$Tr_2(\rho_\Phi) = I_n.$$

Here the notation $Tr_2$ means the partial trace over the second subsystem. A CP-map is furthermore called bistochastic if also the condition

$$Tr_1(\rho_\Phi) = I_n$$

holds; this property is equivalent to the fact that the map is identity-preserving, i.e. $\Phi(I_n) = I_n$.

- The dual state $\rho_\Phi$ corresponding to a CP-map $\Phi$ is uniquely defined. The Kraus operators are obtained by considering the columns of a square root of $\rho_\Phi$ ($A_i$ is obtained by making a matrix out of the $i$th column of a square root of $X$, with $\rho_\Phi = XX^\dagger$). As the square root of a matrix is not uniquely defined, the Kraus operators are not unique. Each different “square root” $X$ of $\rho_\Phi$ ($\rho_\Phi = XX^\dagger$) gives rise to a different set of equivalent Kraus operators. This implies that all equivalent sets of Kraus operators are related by an isometry, and that the minimal number of Kraus operators is given by the rank of the density operator $\rho_\Phi$. Therefore we define the rank of a map to be the rank of the dual operator $\rho_\Phi$. This rank is bounded above by $n^2$ with $n$ the dimension of the Hilbert space. A unique Kraus representation can be obtained by for example enforcing the Kraus operators to be orthogonal, as these would correspond to the unique eigenvectors of $\rho_\Phi$. Note that a similar reasoning applies to all Hermitian preserving and all positive maps, although there an additional sign should be taken into account.

- By construction, we have proven that a map $\Phi$ acting on a $n$-dimensional Hilbert space is completely positive iff $I_n \otimes \Phi$ is positive: there is no need to consider auxiliary Hilbert spaces with dimension larger than the original one. The reasoning is as follows: if $I_n \otimes \Phi$ is positive, then $\rho_\Phi$ is positive, and therefore $\Phi$ has a Kraus representation, which implies complete positivity.

- Suppose $\Phi$ is positive but not completely positive. Then there exists a completely positive map $\tilde{\Phi}$ and a positive scalar $\epsilon$ such that

$$\Phi(\rho) = (1 + n\epsilon)\tilde{\Phi}(\rho) - \epsilon Tr(\rho)I_n.$$

The proof of this fact is elementary: take $\epsilon$ to be the opposite of the smallest eigenvalue of $\rho_\Phi$ (this eigenvalue is negative as otherwise $\Phi$ would be completely positive), and define the CP-map $\tilde{\Phi}(\rho) = (\Phi(\rho) + n\epsilon Tr(\rho)I/n)/(1 + n\epsilon)$ (this map is completely positive because the dual state $A_\tilde{\Phi}$ associated to it is positive and has therefore a Kraus representation). Note that the whole reasoning is also valid for general Hermitian-preserving maps. As an example, consider again the transpose map on a qubit. Then it can be checked that the minimal value of $\epsilon$ is 1 (this is true for the PT operation in arbitrary dimensions) and that the Kraus operators corresponding to $\tilde{\Phi}$ become

$$\{A_i\} = \{\sqrt{2/3}(1\ 0\ 0), \sqrt{2/3}(0\ 0\ 1), \sqrt{1/3}(0\ 1\ 0)/\sqrt{2}\}.$$
The map $\Phi$ is extremal if and only if there does not exist a matrix $\rho^T_i$, it is clear that $\rho^{T_1}_i$ will typically not longer be positive. This identity is very useful, and was used in the section on optimal teleportation with mixed states.

II. EXTREME POINTS OF CP-MAPS

The set of completely positive maps is a convex set: indeed, if $\Phi_1$ and $\Phi_2$ are CP-maps, then so is $x\Phi_1 + (1 - x)\Phi_2$. Due to the one to one correspondence between maps $\Phi$ and states $\rho$, it is trivial to obtain the extreme points of the set of completely positive maps: these are the maps with one Kraus operator, corresponding to $\rho_\delta$ having rank 1.

If however we consider the convex set of trace-preserving maps, the characterization of extreme points becomes more complicated. The knowledge of the set of extreme points of the trace-preserving CP-maps is very interesting from a physical perspective in the following way: suppose one has a multipartite state of qubits and one wants to maximize some convex functional of the state (e.g. the fidelity, ...) by performing local operations. Due to convexity, the optimal operation will correspond to an extreme point of the set of trace-preserving maps.

Let us now characterize all extremal trace-preserving maps:

**Theorem 3** Consider a TPCP-map $\Phi$ acting on a Hilbert space of dimension $n$ and of rank $m$. Consider the dual state $\rho_\delta = XX^\dagger$ with $X$ a $n^2 \times m$ matrix, and the $n^2$ matrices $X_i = X_i^\dagger(\sigma_i \otimes I_n)X$ (the matrices $\{\sigma_i\}$ form a complete basis for the Hermitian $n \times n$ matrices). Then $\Phi$ is extremal if and only if $m \leq n$ and if the set of linear equations $\{\text{Tr}(QX_i) = 0\}$ has only the trivial solution $Q = 0$.

This condition is equivalent to the following one given by Choi [4]: given $m^2$ Kraus operators $\{A_i\}$ of a map $\Phi$, then the map is extremal iff the $m^2$ matrices $\{A_i^\dagger A_j\}$, $1 \leq i, j \leq m$ are linearly independent.

**Proof:** The map $\Phi$ is extremal if and only if there does not exist a $R$ with the property that $RR^\dagger \neq I$ and such that $\text{Tr}_2(XRR^\dagger X^\dagger) = I$. This condition is equivalent to the fact that the set of equations

$$\text{Tr}_Q \left( X \left( RR^\dagger - I \right) X^\dagger \sigma_i \otimes I \right) = 0$$

does only have the trivial solution $Q = 0$. As there are $n^2$ independent generators $\sigma_i$ and due to the fact that $Q$ has $m^2$ degrees of freedom, it is immediately clear that there will always be a non-trivial solution if $m > n$, ending the proof.

It remains to be proven that he condition obtained is equivalent to the one derived [4] by Choi [4]. This can be seen as follows: the condition $\text{Tr}_2(XRR^\dagger X^\dagger) = I$ is equivalent to the condition $\sum_{jk} A_i^\dagger A_j (\sum_{i} R_{ji} R_{ki}^\dagger - \delta_{jk}) = 0$ (this is readily obtained using the trick $|A\rangle = A \otimes I(|I\rangle$). Therefore a nontrivial solution of $Q$ is possible iff the set of matrices $\{A_i^\dagger A_j\}$, $1 \leq i, j \leq m$ are linearly dependent.

Note that the given proof is constructive and can therefore be used for decomposing a given TPCP-map into a convex combination of extremal maps: once a non-trivial $Q$ and therefore $R$ is obtained, one can scale it such that $RR^\dagger \leq I$, and define another $S = \sqrt{T - RR^\dagger}$. This $S$ is guaranteed to be another trace-preserving map up to a constant factor, and the original map is the sum of the maps parameterized by $XRR^\dagger X^\dagger$ and $XSS^\dagger X^\dagger$.

All TPCP maps $\Phi$ of rank 1 are of course extreme and correspond to unitary dynamics. One easily verifies that this implies that the dual $\rho_\delta$ is a maximally entangled state. The intuition behind this is as follows: by equation (7), $\rho_\delta$ characterizes the correlation between the output and the input of the channel. Maximal correlation happens iff the evolution occurs reversibly and thus unitarily, and therefore corresponds to maximal “entanglement” between in- and output. We will explore this connection between maps and entanglement more thoroughly in the following section.

One could go one step further, and try to characterize all extreme points of the convex set defined by all trace-preserving channels for which the extra condition holds that $\Phi(\rho_1) = \rho_2$ with $\rho_1$ and $\rho_2$ given density operators. (Note that $\rho_1$ and $\rho_2$ can be chosen completely arbitrary, as there will always exist at least one TPCP-map that transforms a given state into another given one: consider for example the map with its associated dual state $\rho_\delta = I \otimes \rho_2$.) Bistochastic channels are a special subset of this convex set of maps (in that case $\rho_1 = \rho_2 \simeq I$). An adaption of Theorem 3 leads to the following:
Theorem 4 Consider the convex set of trace-preserving CP-maps \( \Phi \) for which \( \Phi(\rho_1) = \rho_2 \) with \( \rho_1, \rho_2 \) given. Suppose \( \Phi \) is of rank \( m \), its dual state is \( \rho_\Phi = XX^\dagger \) with \( X \) a \( n \times n \) matrix, and that there are \( m \) Kraus operators \( \{A_i\} \). Then this map is extremal if and only if the set of \( 2m^2 \) linear equations

\[
\text{Tr}(QX^\dagger(\sigma_i \otimes I)X) = 0 \quad \text{Tr}(QX^\dagger(\rho_{ij}^T \otimes \sigma_i)X) = 0
\]

has only the trivial solution \( Q = 0 \), or equivalently if and only if the \( m^2 \) operators \( \{A_i^\dagger A_j \pm A_j A_i^\dagger\} \) \( (1 \leq i, j \leq m) \) are linearly independent.

Proof: The proof is completely analogous to the proof of Theorem 3 but here we have the extra condition

\[
\text{Tr} (X(RR^\dagger - I)X(\rho_{ij}^T \otimes \sigma_i)) = 0
\]

In terms of Kraus operators, this additional condition becomes

\[
\sum_{kj} A_j \rho_{ij} A_k^\dagger (\sum_i R_{ji} R_{ki}^* - \delta_{jk}) = 0
\]

which ends the proof.

A similar Theorem was stated by Landau and Streater in the special case of bistochastic maps. In analogy with the conclusions of Theorem 5, we conclude that the number of Kraus operators in an extremal TPCP-map of the kind considered in the above Theorem is bounded by \( \sqrt{2m^2} \).

Let us for example consider the case of qubits. Then the rank of an extremal \( \Phi \) is bounded by 2, and extremal rank 2 TPCP-maps obeying the condition \( \Phi(\rho_1) = \rho_2 \) typically exist. There is however a notable exception if \( \rho_1 = \rho_2 = I/2 \) (i.e. when \( \Phi \) is bistochastic): a bistochastic qubit map has a corresponding dual \( \rho_\Phi \) that is Bell-diagonal. A Bell-diagonal state is a convex sum of maximally entangled states, and therefore a rank 2 bistochastic map cannot be extremal. Note however that this is an accident, and for Hilbert space dimensions larger than 2 there exist extremal bistochastic channels that are not unitary. Sometimes the name “unital” is also used instead of “bistochastic”. The foregoing argument however shows that this terminology is not completely justified.

One could now add more constraints \( \Phi(\rho_{2i}) = \rho_{2i+1} \), and this would lead to similar conditions for extremality in terms of the Kraus operators. Note however that the \( \rho_i \) appearing in the constraints cannot be chosen completely arbitrary, as in general non-compatible constraints can arise due to the complete positivity condition on the physical maps. (Deciding whether a set of conditions \( \Phi(\rho_{2i}) = \rho_{2i+1} \) is physical can be solved using the techniques of semidefinite programming).

Let us now formulate another interesting Theorem:

Theorem 5 Given a Hilbert space of dimension \( n \) and a trace-preserving map \( \Phi \) of rank \( m \leq n \), then there exist pure states \( |\psi\rangle \) such that \( \Phi(|\psi\rangle\langle\psi|) \) are states of rank \( m - 1 \).

Proof: Let us first consider the case \( m = n \), and define \( m \) Kraus operators \( \{A_i\} \) corresponding to \( \Phi \). Given a pure state \( |\psi\rangle \), then \( \Phi \) maps this state to one that is not full rank iff there exists a pure state \( |\chi\rangle \) such that

\[
\langle\chi|\Phi(|\psi\rangle\langle\psi|)|\chi\rangle = 0 = \sum_i |\langle\chi|A_i|\psi\rangle|^2.
\]

Writing \( |\chi\rangle = \sum_i y_i|i\rangle \), \( |\psi\rangle = \sum_i x_i|i\rangle \) and \( \langle j|A_i|k\rangle = A_{ik}^j \), then the previous equation amounts to solving the following set of bilinear equations:

\[
\forall i = 1 : n, \sum_{k=1}^n (\sum_{j=1}^{m-n} x_j A_{ik}^j) y_k = 0.
\]

This set of equations always has a non-trivial solution. Indeed, the parameters \( x_j \) can always be chosen such that the matrix \( \hat{A} = \sum_j x_j A_{jk}^j \) is singular (if all \( A_i \) are full rank then this can be done by fixing all but one of them, and then choosing the remaining parameter such that the determinant vanishes; if one of the \( A_i \) is rank deficient then the solution is of course direct). Then the parameters \( y_k \) can be chosen such that the vector \( y \) is in the right kernel of \( \hat{A} \) (the right kernel is not zero-dimensional as the dimension of the matrix \( \hat{A} \) is \( n \times n \), and therefore \( \Phi(|\psi\rangle\langle\psi|) \) is not full rank. If \( m < n \), then the right kernel of \( \hat{A} \) is at least \( n - m + 1 \) dimensional, such that \( n - m + 1 \) linearly independent \( |\chi\rangle \) can be found such that \( \langle\chi|\Phi(|\psi\rangle\langle\psi|)|\chi\rangle = 0 \), which ends the proof.

In general, it is thus proven that one can always find states \( |\psi\rangle \) such that the rank of \( \Phi(|\psi\rangle\langle\psi|) \) is smaller than the rank of the map, which is surprising. Note that the bound in the Theorem is generically tight, i.e. the minimal rank
of the output state will typically be \( m - 1 \); this follows from the fact that decreasing the rank of the matrix \( \tilde{A} \) with two units would need \( n(n - 1)/2 \) independent degrees of freedom, while there are only \( n - 1 \) available.

Note that extremal TPCP-maps always fulfill the conditions of the Theorem. In particular, extremal qubit channels are generically of rank 2, and the previous Theorem implies that there always exist pure states that remain pure after the action of a rank 2 extremal map (This was also observed by Ruskai et al.\[10\]).

The above Theorem has also some consequences for the study of entanglement. Applying the foregoing proof to the dual state \( \rho_B \), we can easily prove the following: if the rank of a mixed state \( \rho \) defined in an \( n \times n \) dimensional Hilbert space is given by \( m \leq n \), then there always exist at least \( (n - m + 1) \) linearly independent product states orthogonal to it.

Let us now consider an example of the use of extremal maps. Suppose we want to characterize the optimal local trace-preserving operations that one has to apply locally to each of the qubits of a 2-qubit entangled mixed state, such as to maximize the fidelity (i.e. the overlap with a maximally entangled state). This problem is of interest in the context of teleportation \[11, 12\] as the fidelity of the state used to teleport is the standard measure of the quality of teleportation. Badziag and the Horodecki’s \[13\] discovered the intriguing property that the fidelity of a mixed state can be enhanced by applying an amplitude damping channel to one of the qubits. This is due to the fact that the fidelity is both dependent on the quantum correlations and on the classical correlations, and enhancing the classical correlations by mixing (and hence losing quantum correlations) can sometimes lead to a higher fidelity.

With the help of the previous analysis of extremal maps, we are in the right position to find the optimal trace-preserving map that maximizes the fidelity. Indeed, the optimization problem is to find the trace-preserving CP-maps \( \Phi_A, \Phi_B \) such as to maximize the fidelity \( F \) defined as

\[
F(\rho, \Phi_A, \Phi_B) = \langle \psi | \Phi_A \otimes \Phi_B (\rho) | \psi \rangle = \text{Tr} \left\{ \rho \left( \Phi_A^\dagger \otimes \Phi_B^\dagger (|\psi\rangle \langle \psi|) \right) \right\}
\]

with \( |\psi\rangle \) the maximally entangled state. This problem is readily seen to be jointly convex in \( \Phi_A \) and \( \Phi_B \), and therefore the optimal strategy will certainly consist of applying extremal (rank 2) maps \( \Phi_A, \Phi_B \). As we just have derived an easy parametrization of these maps, it is easy to devise a numerical algorithm that will yield the optimal solution.

Note that the problem, although convex in \( \Phi_A \) and \( \Phi_B \), is bilinear and therefore can have multiple (local) maxima. This problem disappears when only one party (Alice or Bob) applies a map (i.e. \( \Phi_B = I \)). This problem was studied in more detail by Rehacek et al.\[14\], where a heuristic algorithm was proposed to find the optimal local trace-preserving map to be applied by Bob. As the optimization problem is however convex, the powerful techniques of semidefinite programming \[14\] should be applied, for which an efficient algorithm exists that is assured to converge to the global optimum. Indeed, due to linearity the problem now consists of finding the 2-qubit state \( \rho_B = I \) such that the fidelity is maximized. As we already know, the algorithm will converge to a \( \rho_B \) of maximal rank 2 in the case of qubits. Exactly the same reasoning holds for systems in higher dimensional Hilbert spaces: if only one party is to apply a trace-preserving operation to enhance the fidelity, the above semidefinite program will produce the optimal local map that maximally enhances the fidelity.

Other situations in which extremal maps will be encountered are for example the problem of optimal cloning\[15\]: given an unknown input state \( \rho \), one wants to construct the optimal trace-preserving CP-map such as to yield an output for which the fidelity with \( \rho \otimes \rho \) is maximal. This can again be rephrased as a semidefinite program whose unique solution will be given by an extremal trace-preserving CP-map.

### III. QUANTUM CHANNELS AND ENTANGLEMENT

The physical interpretation of the dual state corresponding to a CP-map or quantum channel is straightforward. It is the density operator that corresponds to the state that can be made as follows: Alice prepares a maximally entangled state \( |I\rangle \), and sends one half of it to Bob through the channel \( \Phi \). This results into \( \rho_B \).

A perfect quantum channel is unitary and the corresponding state \( \rho_B \) is a maximally entangled state. This corresponds to the case of perfect transmission of qudits, and indeed a maximally entangled state is the state with perfect quantum correlations. Consider now a completely depolarizing channel. In that case it is possible to transmit a classical bit perfectly, and indeed \( \rho_B \) corresponds to a separable state with maximal classical correlations. As a third example, consider the complete amplitude damping channel. Then \( \rho_B \) is a separable pure state with no correlations whatever between Alice and Bob. It is therefore clear that the study of the character of correlation present in the quantum state \( \rho_B \) tells us a lot about the character of the quantum channel.

This way of looking at quantum channels gives a nice way of unifying statics and dynamics in one framework: the future is entangled (or at least correlated) with the past. Just as a measurement in the future gives us information about the prepared system (through the use of the quantum Bayes rule), a measurement on Bob’s side enables Alice to refine her knowledge of her local system (through the use of the quantum steering Theorem)\[14\]. It is therefore
clear that the description of entanglement will shed new light on the question of describing correlations between the states of the same system at two different instants of time, and vice-versa. Therefore we expect that many useful results concerning entanglement can directly be applied to quantum channels. On the other hand, a lot of work has been done concerning the quantification of the classical capacity of a quantum channel. These results offer a nice starting point for the study of classical correlations present in a quantum state.

A. Quantum capacity

The quantum capacity of a quantum channel is related to the asymptotic number of uses of the channel needed for obtaining states whose fidelity tends to one. To transmit quantum information with high fidelity, one indeed needs almost perfect singlets. It is immediately clear that ideas of entanglement distillation will be crucial: sending one part of an EPR through the channel will result in a mixed state, and these mixed states will have to be purified.

Let us first establish a result that was already intrinsically used by many [12, 18, 19, 20]:

**Theorem 6** A quantum channel $\Phi$ can be used to distribute entanglement if and only if $\rho_\Phi$ is entangled. If $\rho_\Phi$ is separable, then the Kraus operators of the map $\Phi$ can be chosen to be projectors, and the map $\Phi$ is entanglement breaking.

**Proof:** The if part is obvious, as $\rho_\Phi$ is the state obtained by sending one part of a maximally entangled state through the channel. To prove the only if part, assume that $\rho_\Phi$ is separable. Then all Kraus-operators can be chosen to be projectors (corresponding to the decomposition with separable pure states), destroying all entanglement. □

It is also possible to make a quantitative statement:

**Theorem 7** Suppose we want to use the channel $\Phi$ to distribute entanglement by sending one part of an entangled state through the channel. The maximal attainable fidelity (i.e. overlap with a maximally entangled state) corresponds to the largest eigenvalue of $\rho_\Phi$. This maximal fidelity is obtained if Alice sends one half of the state described by the eigenvector of $\rho_\Phi$ corresponding to its largest eigenvalue.

**Proof:** Suppose Alice prepares the entangled state $|\chi\rangle$ and sends the second part to Bob through the channel $\Phi$ with Kraus-operators $\{A_i\}$. We want to find the state $|\chi\rangle$ such that

$$\langle I | \sum_i I \otimes A_i |\chi\rangle \langle \chi | \sum_i I \otimes A_i^\dagger |I\rangle = \langle \chi | \rho_\Phi |\chi\rangle$$

is maximized, which immediately gives the stated result. □

The above result is amazing: it tells us that it is not always the best strategy to send one part of a maximally entangled state through the channel. It would be tempting to conjecture that the entanglement of distillation of the obtained state represents the quantum capacity of the given channel.

Note that the eigenvalues and eigenvectors of $\rho_\Phi$ got an appealing interpretation: these represent the fidelities that are obtained by sending one half of the eigenvectors through the channel. Note also that the reduction criterion [21, 22],

$$I \otimes \text{Tr}_2(\rho_\Phi) - \rho_\Phi = I - \rho_\Phi$$

implies that $\rho_\Phi$ is entangled if its largest eigenvalue exceeds $1/n$. This is of course in complete accordance with the previous Theorem, as the maximal fidelity for a separable state is also given by $1/n$.

A more sophisticated treatment of the quantum capacity of a quantum channel would involve ideas of coding and of quantum error correction, although only partial results have been obtained yet; the following is an incomplete list of papers where interesting results have been obtained [18, 23, 24, 25, 26, 27, 28].

B. Classical Capacity

Let us now move towards the well-studied problem of classical capacity of a quantum channel. The central result is the Holevo- Schumacher- Westmoreland Theorem [24, 30], which tells us that the classical product state capacity of a quantum channel $\Phi$ is given by

$$\chi(\Phi) = \max_{\rho_j} \left\{ S(\Phi(\sum_j p_j \rho_j)) - \sum_j p_j S(\rho_j) \right\}.$$ (11)
Let us now ask the following question: what would be the analogy and the interpretation of this formula in the dual picture of states $\rho_{\Phi}$? Using formula (7), it holds that

$$\Phi(\rho_j) = \text{Tr}_1(\rho_{\Phi}(\rho_j^T \otimes I)).$$

Suppose Alice and Bob share the state $\rho_{\Phi}$. Then the above formula describes how Bob has to update his local density operator when Alice did a measurement with corresponding POVM-element $\rho_j^T$. Reasoning along the lines of the HSW-Theorem, the natural interpretation would now be that formula (11) will give us a measure of how much (secret) classical randomness Alice and Bob can create using the state $\rho_{\Phi}$: if Alice implements a POVM measurement with elements $\{p_j, \rho_j^T\}$, this drives the system at Bob’s side into a particular direction, and a measurement of Bob will reveal some information about the (random) outcome of Alice. Note that we interpret the presence of a bipartite state as being a particular kind of quantum channel. Note that the question of creating shared randomness has also been discussed in [31, 32].

The foregoing discussion suggests the following definition for the classical random correlations $C_{cl}$ present in a quantum state $\rho$:

$$C_{cl}^{AB}(\rho_{AB}) = \max_{\{E_j\}} \left( S(\rho_B) - \sum_j p_j S(\rho_j^B) \right)$$

(12)

$$p_j = \text{Tr}\rho(E_j \otimes I)$$

(13)

$$\rho_j^B = \frac{1}{p_j} \text{Tr}_1(\rho(E_j \otimes I))$$

(14)

Here $\{E_j\}$ presents the elements of the POVM implemented by Alice. Observe that there is an asymmetry in the definition, in that $C_{cl}^{A}$ is not necessarily equal to $C_{cl}^{B}$. This definition coincides with the one given by Henderson and Vedral [33], where they introduced this measure because it fulfilled the condition of monotonicity under local operations.

In general, the classical mutual information obtained by the actions of Alice and Bob to obtain classical randomness will be smaller than the derived quantity (12), as coding is needed to achieve the Shannon capacity. This coding could be implemented by doing joint measurements, but we do not expect that the upper bound is tight; a better rate could be obtained if also public classical communication is allowed (A. Winter, unpublished).

IV. ONE-QUBIT CHANNELS

In the case of qubit channels, much more explicit results can be obtained, due to the fact that we have a fairly good insight into the properties of mixed states of two qubits. In this section we highlight some questions about qubit channels that can be solved analytically.

Recall formula (7)

$$\Phi(\rho) = \text{Tr}_1(\rho_{\Phi}(\rho^T \otimes I_n))$$

(15)

which is almost exactly the same expression as if Alice were measuring the POVM-element $\rho$ on the joint state $\rho_{\Phi}$; the difference it that the partial transpose of this state has to be taken. It is now natural to look at the R-picture of the dual state $\rho_{\Phi}$ associated to the map [34], where $\rho$ is parameterized by a real $4 \times 4$ matrix

$$R_{ij} = \text{Tr}(\rho \sigma_i \otimes \sigma_j),$$

$0 \leq \sigma_i \leq 3$. In the R-representation, a partial transpose corresponds to a multiplication of the third column or row with a minus sign. Let us therefore define $R_{\Phi}$ to be the parameterization of $\rho_{\Phi}^T$ in the R-picture, i.e. the R-picture of $\rho_{\Phi}$ in which the third row is multiplied by $-1$. Note that the first row of $R_{\Phi}$ is given by $[1; 0; 0; 0]$, as this corresponds to the trace-preserving condition.

If $x$ is the Bloch vector corresponding, then the action of the map with corresponding $\rho_{\Phi}^T$ or $R_{\Phi}$ is the following:

$$\begin{pmatrix} 1 \\ x' \end{pmatrix} = R_{\Phi} \begin{pmatrix} 1 \\ x \end{pmatrix}$$

(16)

. One can easily prove that the image of the Bloch sphere yields an ellipsoid, where the local density operator of Alice is represented by the center of the ellipsoid. This implies that the knowledge of the ellipsoid corresponds to the
complete knowledge of the quantum channel up to local unitaries at the input. (Note that not all ellipsoids correspond to physical maps, but that there is some restriction on the ratio of the axis).

Let us now consider the analogue of local unitary (LU) and local filtering (SLOCC) equivalence classes as known for mixed states of two qubits \([34]\). What we are looking for are normal forms \(\Omega\) (where \(\Omega\) is a map) such that

\[
\Phi(\rho) = \Omega(\mathbf{A}\rho\mathbf{A}^\dagger)B^\dagger \quad \text{with} \quad A, B \in SU(2) \quad \text{or} \quad \in SL(2, \mathbb{C}).
\]

The proof is immediate given the Lorentz singular value decomposition. The first case corresponds to a

\[
\text{decomposition of the lower } 3 \times 3 \text{ block of } R, \text{ taking into account that the orthogonal matrices have determinant } +1
\]

(see also Fujiwara and Algoet \([35]\) and King and Ruskai \([36]\) for a different approach but with the same result).

Let us next move to SLOCC equivalence classes; it is clear that the Lorentz singular value decomposition \([34]\) is all we need:

**Theorem 8** Given a 1-qubit trace-preserving CP-map \(\Phi\) and its dual \(R_\Phi\). Then the SLOCC normal form \(\Omega\) of \(R_\Phi\) is proportional to one of the following unique normal forms:

\[
R_\Phi = \begin{pmatrix}
1 & 0 & 0 & 0 \\
x & \lambda_1 & 0 & 0 \\
y & 0 & \lambda_2 & 0 \\
z & 0 & 0 & \pm \lambda_3 \\
\end{pmatrix}
\]

by local unitary transformations, where \(\lambda_1 \geq \lambda_2 \geq |\lambda_3|\) and \(x, y \geq 0\); one just has to take the singular value decomposition of the lower \(3 \times 3\) block of \(R\), taking into account that the orthogonal matrices have determinant \(+1\).

In the case of a qubit channel \(\Phi\), the dual state \(\rho_\Phi\) is a mixed state of two qubits. It is possible to obtain an explicit parameterization of all extremal qubit maps (see also Ruskai et al. \([10]\) for a different approach):
Theorem 9 The set of dual states $\rho_\Phi$ corresponding to extreme points of the set of completely positive trace preserving maps $\Phi$ on 1 qubit is given by the union of all maximally entangled pure states, and all rank 2 states $\rho$ for which $Tr_2(\rho_\Phi)$ is equal and $Tr_1(\rho_\Phi)$ is not equal to the identity. The Kraus operators corresponding to the rank 1 extreme points are unitary, while the ones corresponding to the rank 2 extreme points have a representation of the form:

$$A_1 = U \begin{pmatrix} s_0 & 0 \\ 0 & s_1 \end{pmatrix} V^\dagger \quad A_2 = U \begin{pmatrix} 0 \\ \sqrt{1 - s_0} \sqrt{1 - s_1} \end{pmatrix} V^\dagger$$

with $U, V$ unitary.

Proof: We have already proven that extremal TPCP-maps have maximal rank 2. Due to the duality between maps and states, it is sufficient to consider rank 2 density operators of two qubits $\rho_\Phi$ for which $Tr_2(\rho_\Phi) = I_2$. A real parameterization of all 2-qubit density operators $\rho$ is given by the real $4 \times 4$ matrix $R$ with coefficients

$$R_{ij} = Tr (\rho \sigma_i \otimes \sigma_j)$$

where $0 \leq i, j \leq 3$. An appropriate choice of local unitary bases can always make the $R_{1:3,1:3}$ block diagonal, and the trace-preserving condition translates into $R_{0,1:3} = 0$. Therefore $R$ is given by:

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1 & \lambda_1 & 0 & 0 \\ t_2 & 0 & \lambda_2 & 0 \\ t_3 & 0 & 0 & \lambda_3 \end{pmatrix}.$$  

The corresponding $\rho$ is given by

$$\rho = \frac{1}{4} \begin{pmatrix} 1 + t_3 + \lambda_3 & 0 & t_1 - it_2 & \lambda_1 - \lambda_2 \\ 0 & 1 + t_3 - \lambda_3 & \lambda_1 + \lambda_2 & t_1 - it_2 \\ t_1 + it_2 & \lambda_1 + \lambda_2 & 1 - t_3 - \lambda_3 & 0 \\ \lambda_1 - \lambda_2 & t_1 + it_2 & 0 & 1 - t_3 + \lambda_3 \end{pmatrix},$$

and the positivity of $\rho$ constrains the allowed range of the 6 parameters. Let us now impose that the rank of the corresponding $\rho$ is 2. This implies that linear combinations of $3 \times 3$ minors of $\rho$ be zero, and after some algebra one obtains the following conditions:

$$t_3(\lambda_3 + \lambda_1 \lambda_2) = 0$$
$$t_2(\lambda_2 + \lambda_1 \lambda_3) = 0$$
$$t_1(\lambda_1 + \lambda_2 \lambda_3) = 0$$

These equations, supplemented with the fact that diagonal elements of a positive semidefinite matrix are always bigger than the elements in the same column, lead to the conclusion that all $t_i$ but one have to be equal to zero if $\rho$ is rank 2. Without loss of generality, we can choose $t_1 = t_2 = 0$ and parameterize $\lambda_1 = \cos(\alpha)$, $\lambda_2 = \cos(\beta)$. We thus arrive at the canonical form

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha) & 0 & 0 \\ 0 & 0 & \cos(\beta) & 0 \\ \sin(\alpha) \sin(\beta) & 0 & 0 & -\cos(\alpha) \cos(\beta) \end{pmatrix}. \quad (22)$$
Suppose that \( \sin(\alpha) \sin(\beta) = 0 \) (this condition is equivalent to \( \text{Tr}_1(\rho_B) = I/2 \)). Then the state corresponding to this \( R \) is Bell-diagonal and thus a convex sum of two maximally entangled states, and therefore the map corresponding to this state cannot be extremal. In the other case, an extremal rank 2 TPCP-map is obtained, which can easily be shown to yield the given Kraus representation, where \( s_0 = \sqrt{1 - \cos(\alpha + \beta)/2} \) and \( s_1 = \sqrt{1 - \cos(\alpha - \beta)/2} \).

Note that the corresponding Theorem for bistochastic qubit channels is not very useful, as extremal TPCP qubit channels are always unitary. Theorem \( 6 \) however is very interesting, and indicates that there always exist pure states that remain pure after the action of the extremal qubit channel: indeed, if the basis vectors \( \{|i\rangle\} \) are chosen according to the unitary \( V \) in (20), then it is easily checked that the states \( |\psi\rangle \approx s_2 \sqrt{1 - s_2^2} |0\rangle \pm s_1 \sqrt{1 - s_1^2} |1\rangle \) remain pure by the action of the extremal map. Note that these two states are the only ones with this property, and note also that they are not orthogonal to each other.

B. Quantum capacity

Let us now move on to the relation between 1-qubit quantum channels and entanglement. We can now make use of the plethora of results derived for mixed states of two qubits. Let us first consider Theorem \( 9 \) about entanglement breaking channels. In the case of mixed states of two qubits, a state is entangled iff it violates the reduction criterion \( I \otimes \rho_B - \rho \geq 0 \). But in the case of the dual state \( \rho_B \), it holds that \( \rho_B = I/2 \), and therefore it holds that a quantum channel \( \Phi \) can be used to distribute entanglement iff the maximal eigenvalue of \( \rho_B \) exceeds \( 1/2 \). In the light of Theorem \( 10 \) it follows that such a non-entanglement breaking channel can always be used to distribute an entangled state with fidelity larger than \( 1/2 \), which implies on its turn that it can be used to distill entanglement.

Consider now an entanglement breaking channel, i.e. a channel for which \( \rho_B \) is separable. In this case all the Kraus operators can be chosen to be projectors. An explicit way of calculating this Kraus representation exists. Indeed, in the section about entanglement of formation of two qubits, a constructive way of decomposing a separable mixed state of two qubits as a convex combination of separable pure states was given. It was furthermore proven that a separable state of rank 2 or 4 can always be written as a convex combination of 2 respectively 4 separable pure states, thus giving rise to 2 respectively 4 rank one Kraus operators. Surprisingly, most separable rank 3 mixed states of two qubits can only be written as a convex combination of 4 separable pure states. This implies that a generic entanglement breaking channel of rank 3 needs 4 Kraus operators if these are to be chosen rank 1. Let us also mention that the set of separable states is not of measure zero, implying that the set of entanglement breaking channels is also not of measure zero.

The results of Wootters \( 37 \) can of course also be applied to non-entanglement-breaking channels. A direct application of the formalism of Wootters yields the following Theorem:

**Theorem 10** Given a 1-qubit channel \( \Phi \) and the state \( \rho_B \) associated to it. If \( C \) is the concurrence of \( \rho_B \), then the channel has a Kraus representation of the form:

\[
\Phi(\rho) = \sum_i p_i (U_i \hat{C} V_i^\dagger) \rho (U_i \hat{C} V_i)^\dagger
\]

\[
\hat{C} = \frac{1}{2} \begin{pmatrix} \sqrt{1+C} & \sqrt{1-C} \\ \sqrt{1+C} & \sqrt{1-C} \end{pmatrix}
\]

where \( U_i, V_i \) are unitary matrices.

Proof: The Theorem is a direct consequence of the fact that a mixed state with concurrence \( C \) can be written as a convex sum of pure states all with concurrence equal to \( C \).

The geometrical meaning in the context of channels is the following: each trace-preserving CP-map is a convex combination of contractive maps in unique different directions, where each contraction has the same magnitude.

Let us next address the question of calculating the quantum capacity of the one-qubit channel. Clearly, Theorem \( 8 \) tells us what states to send through the channel such as to maximize the fidelity of the shared entangled states. In general, the quantum capacity cannot be calculated as we even don’t have a way of calculating the entanglement of distillation of mixed states of two qubits (which is a simpler problem).

In the case of unital channels of rank 2 however, the eigenvectors of \( \rho_B \) are maximally entangled and the quantum capacity can be calculated explicitly:

**Theorem 11** Consider a bistochastic qubit channel \( \Phi \) of rank 2. Then its quantum capacity is given by \( C_Q = 1 - H(p) \), where \( p \) is the maximal eigenvalue of \( \rho_B \) and \( H(p) = -p \log_2(p) - (1-p) \log_2(1-p) \).
Proof: A unital qubit channel exhibits the nice property that no loss whatever occurs by sending a maximally entangled state through the channel: it can easily be shown (see Bennett et al.\textsuperscript{[18]}) that sending a quantum system through the channel is equivalent to using the standard teleportation channel induced by the (non-maximally entangled state) $\rho_\Phi$. Because we can use the state $\rho_\Phi$, obtained by sending a Bell state through the channel, to perfectly simulate the channel, this is clearly the optimal thing to do, and the quantum capacity of the channel is therefore equal to the distillable entanglement of $\rho_\Phi$. Now Rains\textsuperscript{[38]} has proven that the distillable entanglement of a Bell diagonal state of rank 2 is given by $E_{\text{dist}}(\rho) = 1 - S(\rho)$, which ends the proof of the Theorem.

More general, the quantum capacity of bistochastic qubit channel is always equal to the entanglement of distillation of the corresponding dual states (due to the arguments in the previous proof).

As a last remark, we observe that the channels of the non-generic kind that touch the Bloch sphere at exactly one point are never entanglement-breaking: this follows from the fact that the concurrence of $\rho_\Phi$ always exceeds 0 in that case.

C. Classical capacity

Far more progress has been made concerning the classical capacity of quantum channels: it is known that the classical capacity using product inputs is given by the Holevo-\chi quantity. Here the geometrical picture derived in section 6.4 can sharpen our intuition. Consider for example the case of a unital channel. It is immediately clear that Holevo-\chi will be maximized by choosing a mixture of two states that lie on the opposite side of the major axis of the ellipsoid. This implies that the optimal input states are orthogonal. King and Ruskai\textsuperscript{[39\textsuperscript{1}]\textsuperscript{,}38} even proved that entangled inputs cannot help in the case of unital channels, and we conclude that the classical capacity of the unital channels is completely understood.

Consider however a non-unital channel of the generic kind. As proven before, this channel can be interpreted as the succession of a filter, a unital channel, and another filter. The critical source of noise or decoherence and irreversibility in a channel is the mixing, and the previous analysis tells us that this mixing can always be interpreted to happen in a unital way, whereas the in- and output of the unital channel is reversibly but non-orthogonally filtered. It follows that orthogonal inputs will not appear orthogonally in the unital channel, and typically orthogonal inputs will not achieve capacity. This strange fact was indeed discovered by Fuchs\textsuperscript{[40]}, and it appears to be generic for non-unital channels.

Let us now have a look at the non-generic family of channels, whose ellipsoids touch the Bloch sphere at exactly one point. It happens that the so-called stretched channel belongs to this family, and this channel has the property that its (product) capacity is only achieved for an input ensemble with three states\textsuperscript{[41]}. This is surprising but not too surprising given the geometrical picture, as one of the input states corresponds to the pure output state, while the other two ones are chosen to lie symmetric around the axis connecting the maximally entangled state with the pure output state. Note however that most of the non-generic states achieve capacity with 2 input states.

Let us now move to calculate the classical capacity of the extremal qubit channels. In the case of extremal qubit channels, it is possible to reduce the problem of calculating the classical (Holevo) capacity to an optimization problem over the ensemble average. The problem to be solved is as follows: find the optimal ensemble $\{\rho_i, p_i\}$ such that

$$S(\sum_i p_i \Phi(\rho_i)) - \sum_i p_i S(\Phi(\rho_i))$$

is maximized. We assume that $\Phi$ is rank 2 and therefore has a Kraus representation of the form\textsuperscript{[20]}. It is clear that only pure states $\{\rho_i\}$ have to be considered. It is easily seen that in the case of qubits, the entropy of a state is a convex monotonously increasing function of the determinant of the density operator: $S(\rho) = H(1/2(1 - \sqrt{1 - 4 \det(\rho^2)})$ with $H(p) = p \log(p) + (1 - p) \log(1 - p)$ the Shannon entropy function. Inspired by the analysis of 2-qubit channels by Uhlmann in terms of anti-linear operators\textsuperscript{[42]}, we make the following observation:

$$\det(A_1^T \psi \langle \psi | A_1^T + A_2^T \psi \langle \psi | A_2^T) = |\psi^T (A_1^T \sigma_y A_2 - A_2^T \sigma_y A_1) \psi|.$$

(25)

Here $\psi$ is the vector notation (in the computational basis) of $|\psi\rangle$, and $\sigma_y$ is a Pauli matrix. Suppose now that we add an additional constraint to the problem, namely that the ensemble average $\rho$ is given. Taking a square root $X$ of $\rho = XX^T$, all possible pure state decompositions can be written as $X' = UX$ with $U$ an arbitrary isometry (note that the columns of $UX$ represent all unnormalized pure states in the decomposition). With this additional constraint, the problem can be solved exactly as we solved the entanglement of formation problem. A constructive way of obtaining the optimal decomposition of $\rho$ is as follows: take a square root $X$ of $\rho$, and calculate the singular value decomposition of the symmetric matrix $X^T (A_1^T \sigma_y A_2 - A_2^T \sigma_y A_1) X = V \Sigma V^T$. Call $C = \sigma_1 - \sigma_2$ the concurrence.
V. CONCLUSION

We have shown that the natural description of quantum channels or positive linear maps is given by a dual quantum state associated to the map. This dual state is defined over a Hilbert space that is naturally endowed with a tensor product structure of the in- and output of the channel. We showed that the techniques developed in the context of entanglement are of direct use in describing positive maps. We derived a characterization of the extreme points of the convex set of trace-preserving completely positive maps, and gave some generalizations. We discussed some new results about the classical and quantum capacity of a quantum channel, and in the case of one-qubit channels we showed how to exploit the duality between qubit channels and mixed states of two qubits to obtain useful parameterizations.

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with \( \{\sigma_i\} \) the singular values of the above symmetric matrix. Then the optimal decomposition is obtained by choosing \( U = V^*O \) with \( O \) the real orthogonal matrix that is chosen such that the diagonal entries of the matrix \( R = O^T(\text{Diag}[\sigma_1, -\sigma_2] - C \rho)O \) vanish. For given ensemble average \( \rho \), the classical capacity is therefore given by the following formula: \( S(\Phi(\rho)) - f(C) \) (see also Uhlmann [12]).

To derive an explicit formula for the classical capacity of the extremal channels, we still have to do an optimization over all possible ensemble averages \( \rho \). Note that the previous analysis already learned us that the capacity will always be reached with an ensemble of two input states. Both the terms \( \Phi(\rho) \) and \( C \) can easily be extremized separately, but unfortunately even if the eigenvalues of \( \rho \) are fixed, the optimal eigenvectors for maximizing \( S(\rho) \) and minimizing \( C \) are not compatible. However, the capacity can easily be calculated numerically, as it just an optimization problem over three real parameters.

On the other hand, we have seen that the definition of the classical capacity had a direct counterpart in giving an appealing definition for the number of classical correlations present in a (mixed) bipartite state \( C_{cl} \) (see [12]). The techniques used in the foregoing paragraph are perfectly adequate to give an exact expression of this quantity if the shared quantum state is a rank 2 bipartite state \( \rho \) of qubits. Indeed, a mixed bipartite state of two qubits can just be seen as a more general kind of quantum channel.
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[43] Actually, Choi derived the different problem of characterizing the extremal points of the (not necessarily trace-preserving) CP-maps that leave the identity unaffected, but his arguments are readily translated to the present situation. Note also that his proof was much more involved.

[44] In some sense one could argue that this was expected due to the fact that space and time play analogous roles in the theory of relativity. It is very nice however that in the non-relativistic case considered here, the duality is already present. This gives hope that it should be possible to generalize the current findings to the relativistic case.

[45] This was first observed by Michael Horodecki