RESOLUTION OF SINGULARITIES FOR $C^\infty$ FUNCTIONS
AND MEROMORPHY OF LOCAL ZETA FUNCTIONS

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To the memory of Professor Masatake Kuranishi.

Abstract. In this paper, we attempt to resolve the singularities of the zero variety of a $C^\infty$ function of two variables as much as possible by using ordinary blowings up. As a result, we formulate an algorithm to locally express the zero variety in the “almost” normal crossings form, which is close to the normal crossings form but may include flat functions. As an application, we investigate analytic continuation of local zeta functions associated with $C^\infty$ functions of two variables. As is well known, the desingularization theorem of Hironaka implies that the local zeta functions associated with real analytic functions admit the meromorphic continuation to the whole complex plane. On the other hand, it is recently observed that the local zeta function associated with a specific (non-real analytic) $C^\infty$ function has a singularity different from the pole. From this observation, the following questions are naturally raised in the $C^\infty$ case: how wide the meromorphically extendible region can be and what kinds of information essentially determine this region? This paper shows that this region can be described in terms of some kind of multiplicity of the zero variety of each $C^\infty$ function. By using our blowings up algorithm, it suffices to investigate local zeta functions in the almost normal crossings case. This case can be effectively analyzed by using real analysis methods; in particular, a van der Corput-type lemma plays a crucial role in the determination of the above region.

1. Introduction

In this paper, we study an integral of the form

$$Z(f, \varphi)(s) := \int_{\mathbb{R}^2} |f(x, y)|^s \varphi(x, y) dxdy, \quad s \in \mathbb{C},$$

where $f, \varphi$ are real-valued $C^\infty$ functions defined on a small open neighborhood $U$ of the origin in $\mathbb{R}^2$ and the support of $\varphi$ is contained in $U$. Since the integral in (1.1) locally converges on the region $\text{Re}(s) > 0$, $Z(f, \varphi)$ can be regarded as a holomorphic function there, which is called a local zeta function. We are interested in an issue: how local zeta functions can be analytically continued to a wider region. Since the analytic continuation issue has multiple connections with many mathematics, such
as partial differential equations, complex analysis, harmonic analysis, number theory, representation theory, and singularity theory among others, the theory of local zeta functions has been considerably evolved (c.f. [32], [1], [10], [27], etc.).

Observing the form of the integral in (1.1), we can see that the convergence of the integral induces the holomorphic extension of local zeta functions and, moreover, that their analytic continuation is deeply related to the geometry of the zero variety of \( f \). When \( f \) is real analytic, it was shown by Bernstein and Gel’fand [4] and M. Atiyah [3] that local zeta functions admit a meromorphic continuation to the whole complex plane. The most important idea of their works is to locally express the zero variety of \( f \) in the normal crossings form by using the desingularization theorem of Hironaka [22]. Their results were the first remarkable applications of Hironaka’s theorem to an important analytic issue.

More generally, understanding the geometric structure of the zero variety of a multivariate function is crucial in many important issues in analysis; harmonic analysis, partial differential equations, complex analysis, probability, etc. Since the works [4], [3], resolution of singularities has been recognized to be a powerful tool for these issues. It should be specially mentioned that reasonable resolutions of singularities recently give many strong results concerning harmonic analysis ([30], [37], [38], [16], [18], [21], [23], [25], [9], [26], etc.). For the application of desingularization theorems, some kind of analyticity of the corresponding function is usually required, but it is sometimes desirable to deal with a given issue in a more general setting. Therefore, it is meaningful to try to improve desingularization theorems for a wider class of functions. For example, in the study of oscillatory integrals and local zeta functions in [28], since desingularization theorems have been established in a certain class of \( C^\infty \) functions including the Denjoy-Carleman quasianalytic class (see also [5], [6]) and, as a result, many strong results obtained in the real analytic case can be generalized by using these theorems.

Unfortunately, since there exist \( C^\infty \) functions whose singularities cannot be completely resolved by using algebraic transforms only, the general \( C^\infty \) case is hard to deal with from a geometrical point of view. Furthermore, a distinctive phenomenon concerning these troublesome \( C^\infty \) functions has been recently observed in an analytic issue for local zeta functions. To be more specific, it was shown in [30] (see also [17]) that the local zeta function associated with a specific (non-real analytic) \( C^\infty \) function cannot be meromorphically extended to the whole complex plane (see Section 2.3). In other words, local zeta functions possibly have singularities different from poles. Then a new issue is naturally raised in the \( C^\infty \) case: how wide the meromorphically extendible region of local zeta functions can be and what kinds of information of a \( C^\infty \) function \( f \) essentially determine this region? (This issue will be more exactly formulated in Section 2.3.) For the investigation of this issue, it is necessary to understand geometrical properties of the zero varieties of general \( C^\infty \) functions. This geometrical issue itself seems interesting and important from
various motivations. The first half of this paper is devoted to the investigation of this issue.

Let $f$ be a $C^\infty$ function defined near the origin in $\mathbb{R}^2$. When the set defined by $f(x, y) = 0$ is restricted to the real space $\mathbb{R}^2$, this restricted set sometimes has very few information and is not always useful for precise analysis. In the case where $f$ is real analytic, the defining region of $f$ can be naturally extended to the complex region in $\mathbb{C}^2$. The zero variety in $\mathbb{C}^2$ of the extended $f$ is so-called a holomorphic plane curve, which has been very widely studied. Actually, many fruitful results about these curves improve the investigation of local zeta functions associated with real analytic functions. For example, the theory of toric varieties based on the geometry of Newton polyhedra gives quantitative results about poles of local zeta functions ([43], [12], [13], [35], [7], [28], [29], etc.). On the other hand, when a $C^\infty$ function $f$ is extended to the complex space, the conjugate variables must be considered in general, which makes it difficult to understand geometric properties of the zero variety of $f$ in $\mathbb{C}^2$. Therefore, we give up handling this variety itself and instead look for an essentially important subset in it, which is easier to deal with.

With the aid of the factorization formula for $C^\infty$ functions of V. S. Rychkov [39], an important curve in the zero locus of $f$ in $\mathbb{C}^2$ is defined, which will be called the decisive curve defined by $f$, and this curve has sufficient information for our analysis. The decisive curve defined by $f$ consist of branches in $\mathbb{C}^2$ parametrized by using the Puiseux series of one real variable. Although the singularity of this curve might not be completely resolved by using algebraic transforms only, this curve can be locally expressed as in almost normal crossings form via finite compositions of ordinary blowings up. To be more exact, there exist a two-dimensional $C^\infty$ real manifold $Y$ and a proper map $\pi : Y \to \mathbb{R}^2$ such that $f \circ \pi$ can be locally expressed at any point on the zero locus of the map $\pi$ as

$$(1.2) \quad (f \circ \pi)(x, y) = u(x, y)x^a \left(y^m + \varepsilon_1(x)y^{m-1} + \cdots + \varepsilon_m(x)\right),$$

where $a, m$ are nonnegative integers and $u, \varepsilon_k$ are real-valued $C^\infty$ functions satisfying that $u(0, 0) \neq 0$ and $\varepsilon_k$ are flat at the origin. Note that in the real analytic case, since $\varepsilon_k$ must be zero functions, $f \circ \pi$ can be locally expressed in ordinary normal crossings form, which implies that each local zeta function can be meromorphically extended to the whole complex plane by using an elementary method ([1], see also Section 11).

After a desingularization theorem was shown by Hironaka [22], more elementary constructive proofs for the theorem have been given ([12], [5], etc.), which reveal the situation of resolution of singularities more clearly and make the theorems more applicable to the analysis. Our method is in this direction and the proof in this paper is self-contained if the result of Rychkov in [39] is agreed. It is expected that our almost desingularization theorem will be established in the general dimensional case and will be usefully applied to various analytic issues.
Throughout the above geometric process, it is sufficient to deal with local zeta functions associated with functions of the form \((1.2)\), which is considered as a model in the \(C^\infty\) case. The latter half of this paper is devoted to the investigation of the analytic continuation of local zeta functions in this model case. In this case, these analytic continuation can be effectively investigated by using real analysis methods; the most important tool is a van der Corput-type lemma. The original van der Corput’s lemma gives an estimate for one-dimensional oscillatory integrals, which is explained in [11]. This lemma has been rewritten in various forms according to the purposes. Our analysis needs one of the versions used in [8] (see also [17]). Note that these van der Corput type lemmas play key roles in recent studies in harmonic analysis, which investigate estimations of oscillatory integral operators, oscillatory integrals, Fourier restrictions, maximal operators, critical integrability indices, the sizes of sublevel sets and so on (([36], [37], [38], [21], [23], [25], [9], [20], etc.). As a result, we show that the meromorphically extendible region of local zeta functions associated with \((1.2)\) contains the region \(\text{Re}(s) > -1/m\). The above mentioned analysis has been essentially performed in the recent paper [31].

After the above explained investigation, we give an answer to the meromorphic extension issue for local zeta functions in the \(C^\infty\) case. For this purpose, we introduce a quantity \(\mu_0(f)\) for a given \(C^\infty\) function \(f\). In general, the double formal power series has the factorization formula by using the Puiseux series. Through the above explained resolution process, the multiplicities of real roots in this factorization formula essentially appear in the index \(m\) in the expression \((1.2)\). The maximum of the multiplicities of real roots in the factorization formula is denoted by \(\mu_0(f)\). Then we can see that the meromorphically extendible region always contains the region \(\text{Re}(s) > -1/\mu_0(f)\). This result is optimal in the uniform sense. Note that the quantity \(\mu_0(f)\) is an invariant of \(f\), i.e., it is independent of the choice of coordinates (see Section 8.3).

This paper is organized as follows. In Section 2, after exactly describing our analytic issues of local zeta functions and introducing the quantity \(\mu_0(f)\), we state our main theorem. Sections 3–8 are the geometrical part of this paper. In Section 3, we state the most important theorem from a geometrical point of view, which gives an almost resolution of singularities for \(C^\infty\) functions. In Section 4, we recall process of blowings up, which is well-known in the study of algebraic geometry, etc. In Section 5, we explain an important factorization formula for \(C^\infty\) functions given by Rychkov [39]. By using many tools in Sections 4-5, we actually attempt to resolve the singularities of the zero variety of \(C^\infty\) functions as much as possible and, as a result, we obtain a desired desingularization theorem. In Section 7, we give a proof of the theorem stated in Section 3. Since the quantity \(\mu_0(f)\) used in the theorem plays important roles in the analytic continuation of local zeta functions, we precisely investigate its properties in Section 8. Sections 9–13 are the analytic part of this paper. In Section 9, after using almost resolution of singularities and decomposing
the integral in (1.1), we can see that it suffices to consider the model case as in (1.2). We state the most important result from an analytical point of view, which describes the meromorphically extendible region in the model case. In Section 10, we give a proof of the main theorem stated in Section 2 by using results in Section 9. In Section 11, we prepare many useful analytic lemmas for the subsequent analysis. In Section 12, geometric and analytic properties of the above model function are investigated. In Section 13, we give a proof for the theorem stated in Section 9 by using many tools prepared in Sections 11-12. At present, there have been very few results about the meromorphic extension issue of local zeta functions and there are many open issues which should be investigated in the future. We discuss these matters in Section 14.

Notation and symbols.

- We denote by \( \mathbb{Z}_+, \mathbb{R}_+ \) the subsets consisting of all nonnegative numbers in \( \mathbb{Z}, \mathbb{R} \), respectively. For \( s \in \mathbb{C} \), \( \text{Re}(s) \) expresses the real part of \( s \).
- For \( n \in \mathbb{N} \), we denote by \( \mathbb{P}^n(\mathbb{C}) \) (or \( \mathbb{P}^n(\mathbb{R}) \)) the \( n \)-dimensional complex projective space (or real projective space).
- For \( R = \mathbb{R} \) or \( \mathbb{C} \), \( R[t], R[[t]], R\{t\} \) are the rings of polynomials, formal power series, convergent power series in \( t \) with coefficients from \( R \), respectively. Moreover, \( R[[x,y]], R\{x,y\} \) are the rings of double formal power series and double convergent power series, respectively.
- For an open set \( U \) in \( \mathbb{R}^2 \), \( C^\infty(U) \) denotes the set of real analytic functions on \( U \).
- By (1.1), \( Z(f, \varphi)(s) \) is defined as an integral. When \( Z(f, \varphi) \) can be regarded as a function on some region, this function is also denoted by the same symbol.

2. Description of the problems and the main result

Let \( U \) be a small open neighborhood of the origin in \( \mathbb{R}^2 \) and let \( f, \varphi \in C^\infty(U) \) satisfy the conditions in the Introduction. Moreover, we usually assume that \( f \in C^\infty(U) \) is non-flat and satisfies

\[
(2.1) \quad f(0,0) = 0 \quad \text{and} \quad \nabla f(0,0) = (0,0).
\]

Unless (2.1) is satisfied, every problem addressed in this paper is easy. As for \( \varphi \in C_0^\infty(U) \), we sometimes give the following conditions

\[
(2.2) \quad \varphi(0,0) > 0 \quad \text{and} \quad \varphi \geq 0 \text{ on } U.
\]

In order to investigate the analytic continuation of local zeta functions, we only use the half-plane of the form \( \text{Re}(s) > -\rho \) with \( \rho > 0 \). This is the reason why we observe the situation of analytic continuation through the integrability of integrals of the form (1.1). Of course, it is desirable to deal with various kinds of regions in
the study of analytic continuation and this advanced issue should be investigated in the future.

2.1. Newton data. Let \( \overline{f}(x, y) \in \mathbb{R}[[x, y]] \) be the Taylor series of \( f(x, y) \) at the origin, i.e.,

\[
\overline{f}(x, y) = \sum_{(j,k) \in \mathbb{Z}^2_+} c_{jk} x^j y^k \quad \text{with} \quad c_{jk} = \frac{1}{j!k!} \partial_x^j \partial_y^k f(0, 0).
\]

The Newton polygon of \( f \) is the integral polygon

\[
\Gamma_+(f) = \text{the convex hull of the set } \bigcup \{(j, k) + \mathbb{R}_+^2 : c_{jk} \neq 0\} \text{ in } \mathbb{R}^2_+(i.e., \text{the intersection of all convex sets which contain } \bigcup \{(j, k) + \mathbb{R}_+^2 : c_{jk} \neq 0\}).
\]

The flatness of \( f \) at the origin is equivalent to the condition \( \Gamma_+(f) = \emptyset \).

The Newton distance \( d(f) \) of \( f \) is defined by

\[
d(f) = \inf\{\alpha > 0 : (\alpha, \alpha) \in \Gamma_+(f)\}.
\]

We set \( d(f) = \infty \) when \( f \) is flat at the origin. Since the Newton distance depends on the coordinates system \((x, y)\) on which \( f \) is defined, it is sometimes denoted by \( d_{(x,y)}(f) \). The height of \( f \) is defined by

\[
\delta_0(f) = \sup_{(x,y)} \{d_{(x,y)}(f)\},
\]

where the supremum is taken over all local smooth coordinate systems \((x, y)\) at the origin. A given coordinate system \((x, y)\) is said to be adapted to \( f \), if the equality \( \delta_0(f) = d_{(x,y)}(f) \) holds. Note that the height \( \delta_0(f) \) can be determined by the Taylor series \( \overline{f} \in \mathbb{R}[[x, y]] \) only. From their definitions, \( d(f) \) and \( \delta_0(f) \) roughly indicate some kind of flatness of \( f \) at the origin (when they are larger, the flatness of \( f \) becomes stronger).

Remark 2.1. (1) We can determine \( \delta_0(f) \) for \( f \) not satisfying the conditions (2.1) from its definition. When \( f(0,0) \neq 0 \), we have \( \delta_0(f) = 0 \). When \( f(0,0) = 0 \) and \( \nabla f(0,0) \neq (0,0) \), we have \( \delta_0(f) = 1 \) by using the implicit function theorem.

(2) The existence of adapted coordinates (in the two-dimensional case) is shown in [43], [37], [24], etc. Furthermore, useful necessary and sufficient conditions for their adaptedness have been obtained in [43], [1], [24] (they will be explained in Section 8.4). We remark that the existence of adapted coordinates is not obvious. The definition of the adapted coordinate can be directly generalized in higher dimensional case. In the three-dimensional case, it is known in [43] that there exists a real analytic function admitting no adapted coordinate.
2.2. **Holomorphic extension problem.** First, let us consider the following quantities:

\[ h_0(f, \varphi) := \sup \left\{ \rho > 0 : \text{the domain to which } Z(f, \varphi) \text{ can be holomorphically continued contains the half-plane } \Re(s) > -\rho \right\}, \]

\[ h_0(f) := \inf \{ h_0(f, \varphi) : \varphi \in C_0^\infty(U) \}. \]

It is obvious that \( h_0(f) \) is invariant under the change of coordinates. We remark that if \( \varphi \) satisfies (2.2), then \( h_0(f, \varphi) = h_0(f) \) holds; but otherwise, this equality does not always hold. Indeed, there exists \( \varphi \in C_0^\infty(U) \) with \( \varphi(0,0) = 0 \) such that \( h_0(f, \varphi) > h_0(f) \) (see e.g. [7], [29]).

From the form of the integral in (1.1), the relationship between the holomorphy and the convergence of the integral implies that the quantity \( h_0(f) \) is deeply related to the following famous index:

\[ c_0(f) := \sup \left\{ \mu > 0 : \text{there exists an open neighborhood } V \text{ of the origin in } U \text{ such that } |f|^{-\mu} \in L^1(V) \right\}, \]

which is called the *log canonical threshold* or the *critical integrability index*. The index \( c_0(f) = c_0(f) \) always holds. In fact, the inequality \( h_0(f) \geq c_0(f) \) is obvious; while the opposite inequality can be easily seen by Theorem 5.1 in [30].

In the real analytic case, since all the singularities of the extended \( Z(f, \varphi) \) are poles on the real axis, the leading pole exists at \( s = -h_0(f, \varphi) \). In the seminal work of Varchenko [43], when \( f \) is real analytic and satisfies some nondegeneracy conditions (see Section 8.4, below), \( h_0(f) \) can be expressed as \( h_0(f) = 1/d(f) \), where \( d(f) \) is the Newton distance of \( f \). An interesting work [9] treating the equality \( c_0(f) = 1/d(f) \) is from another approach. We remark that these results deal with the general dimensional case. In the same paper [43], Varchenko more deeply investigates the two-dimensional case. Indeed, without any assumption, he shows that the equality

\[ h_0(f) = 1/d_0(f) \]

holds for real analytic \( f \). More generally, in the \( C^\infty \) case, M. Greenblatt [17] obtains a sharp result which generalizes the above two-dimensional result of Varchenko.

**Theorem 2.2** ([17]). \( c_0(f) = h_0(f) \) holds for every non-flat \( f \in C^\infty(U) \).

From the above result, our holomorphic extension problem is completely understood even in the \( C^\infty \) case. It is important that \( h_0(f) \) is determined by information of the formal Taylor series of \( f \) only.

On the other hand, the situation of the meromorphic extension is quite different from the holomorphic one.
2.3. Meromorphic extension problem. Corresponding to (2.5), (2.6) in the holomorphic continuation case, we analogously define the following quantities:

\[(2.9) \quad m_0(f, \varphi) := \sup \left\{ \rho > 0 : \text{The domain to which } Z(f, \varphi) \text{ can be meromorphically continued contains the half-plane } \Re(s) > -\rho \right\},\]

\[m_0(f) := \inf \{ m_0(f, \varphi) : \varphi \in C_0^\infty(U) \}.\]

It is easy to see that \(m_0(f)\) is invariant under the change of coordinates and that \(h_0(f, \varphi) \leq m_0(f, \varphi)\) and \(h_0(f) \leq m_0(f) \leq m_0(f, \varphi)\) always hold. As mentioned in the Introduction, if \(f\) is real analytic, then \(m_0(f) = \infty\) always holds; while there exist specific (non-real analytic) \(C^\infty\) functions \(f\) such that \(m_0(f) < \infty\). Indeed, it is shown in [30] (see also [17]) that when

\[(2.11) \quad f(x, y) = x^ay^b + x^aqe^{-1/|x|^p},\]

and \(\varphi\) satisfies the condition (2.2), \(Z(f, \varphi)\) has a non-polar singularity at \(s = -1/b\), which implies \(m_0(f) = 1/b\). Here, \(p\) is a positive real number and \(a, b, q \in \mathbb{Z}_+\) satisfy that \(a < b, b \geq 2, 1 \leq q \leq b\) and \(q\) is even. Note that \(d(f) = \delta_0(f) = b\) in this case. At present, properties of the singularity at \(s = -1/b\) are not well understood (see Section 14.2). In order to understand how wide the meromorphically extendible region of a given local zeta function is, we consider the following problem.

**Problem 2.1.** For a given \(f \in C^\infty(U)\), describe (or estimate) the value of \(m_0(f)\) in terms of appropriate information of \(f\).

In [30], the above problem is investigated in the case where \(f\) has the following form which is a natural generalization of (2.11).

\[(2.12) \quad f(x, y) = u(x, y)x^ay^b + (a \text{ flat function}),\]

where \(a, b\) are nonnegative integers with \(a \leq b\) and \(u(x, y) \in C^\infty(U)\) satisfies \(u(0, 0) \neq 0\). It is shown in [30] that \(m_0(f) \geq 1/b\). Note that \(\delta_0(f) = b\) in this case.

**Remark 2.3.** Since \(x^ay^b\) with \(a, b \in \mathbb{Z}_+\) is a real analytic function, \(m_0(x^ay^b) = \infty\) holds. On the other hand, \(m_0(f) = 1/b\) holds if \(f\) is as in (2.11). From this observation, we see that \(m_0(f)\) is not always determined by the formal Taylor series of \(f\).

2.4. The quantity \(\mu_0(f)\). Let us introduce an important quantity \(\mu_0(f)\), which will be used in the statement of the main theorem.

Let \(\bar{f}(x, y) \in \mathbb{R}[[x, y]]\) be the formal Taylor series of a non-flat \(C^\infty\) function \(f(x, y)\) at the origin. It is known (c.f. [45], Corollary 2.4.2, p.32) that \(\bar{f}(x, y)\) can be
expressed as in the following factorization in terms of the formal Puiseux series

\[
\overline{f}(t^N, y) = \overline{u}(t^N, y) t^{N m_0} \prod_{j=1}^{r} (y - \overline{\phi}_j(t))^{m_j},
\]

where \( N \) is a positive integer, \( m_0 \) is a nonnegative integer, \( m_j \) are positive integers, \( \overline{u}(x, y) \in \mathbb{C}[x, y] \) has a non-zero constant term and \( \overline{\phi}_j(t) \in \mathbb{C}[t] \) are distinct (i.e., \( \overline{\phi}_j(t) \neq \overline{\phi}_k(t) \) if \( j \neq k \)). Let \( \mathcal{R}(f) \) be the subset of \( \{0, 1, \ldots, r\} \) defined by

\[
j \in \mathcal{R}(f) \iff j = 0 \text{ or } \overline{\phi}_j(t) \in \mathbb{R}[t].
\]

The case \( r = 0 \) is possible; when \( \overline{f}(x, y) \) is expressed as \( \overline{u}(x, y)x^{m_0} \), we set \( \mathcal{R}(f) = \{0\} \). The quantity \( \mu_0(f) \) is defined by

\[
\mu_0(f) = \max\{m_j : j \in \mathcal{R}(f)\}.
\]

**Remark 2.4.** (1) It is obvious from the definition that the quantity \( \mu_0(f) \) is determined by the formal Taylor series of \( f \) only, as well as the height \( \delta_0(f) \) in (2.4).

(2) We define \( \mu_0(f) \) for a \( C^\infty \) function \( f \) not satisfying the conditions (2.1) as follows. When \( f(0, 0) \neq 0 \), \( \mathcal{R}(f) = \{0\} \) with \( m_0 = 0 \), which gives \( \mu_0(f) = 0 \). When \( f(0, 0) = 0 \) and \( \nabla f(0, 0) \neq 0 \), \( \mathcal{R}(f) = \{0\} \) with \( m_0 = 1 \) or \( \mathcal{R}(f) = \{0, 1\} \) with \( m_0 = 0 \) and \( m_1 = 1 \), which gives \( \mu_0(f) = 1 \), by the implicit function theorem.

(3) The quantity \( \mu_0(f) \) is invariant under the change of coordinates, which will be shown in Section 8.3.

(4) When \( f \) is real analytic and \( \mu_0(f) \geq 1 \), \( \mu_0(f) \) is equal to the maximal order of vanishing of \( f \) along the set \( \{(x, y) \in \mathbb{R}^2 : |x|^2 + |y|^2 = \gamma\} \) with sufficiently small \( \gamma > 0 \) (see [24]).

(5) If a real analytic function \( f \) satisfies \( f(x, y) > 0 \) away from the origin, then \( \mu_0(f) = 0 \). But, in the \( C^\infty \) case, the above implication is not true. For example, consider the \( C^\infty \) function \( f(x, y) = y^{2k} + e^{-1/x^2} \) with \( k \in \mathbb{N} \). In this case, \( \mu_0(f) = 2k \).

More detailed properties of \( \mu_0(f) \) will be investigated in Section 8.

2.5. **Main theorem.** Now let us state a main theorem in this paper, which gives an answer to Problem 2.1. Indeed, we show that the meromorphically extendible region can be described by using the quantity \( \mu_0(f) \).

**Theorem 2.5.** Let \( f \) be a non-flat \( C^\infty \) function defined in a neighborhood of the origin in \( \mathbb{R}^2 \). Then we have

(i) If \( \mu_0(f) = 0, 1 \), then \( m_0(f) = \infty \) holds;

(ii) If \( \mu_0(f) \geq 2 \), then \( m_0(f) \geq 1/\mu_0(f) \) holds.

Furthermore, when \( \mu_0(f) < \delta_0(f) \), the poles of the extended local zeta function on \( \text{Re}(s) > -1/\mu_0(f) \) exist in the finitely many arithmetic progressions that are constructed from negative rational numbers.
Remark 2.6. (1) The assumption of the theorem does not need the condition (2.1).

(2) Recalling Theorem 2.1 given by Greenblatt [17], we can see \( \mu_0(f) \leq \delta_0(f) \) for \( f \in \mathcal{C}_1^\infty(U) \) by using the above theorem with the obvious inequality \( m_0(f) \leq h_0(f) \).

(3) Since the equality \( m_0(f) = 1/\mu_0(f) \) holds for \( f \) in (2.11), the estimate in (ii) is optimal in the uniform sense for \( f \). From the obvious inclusion \( \mathcal{C}^\omega(U) \subset \mathcal{C}_1^\infty(U) \), there are many \( \mathcal{C}^\infty \) functions \( f \) such that \( \mu_0(f) \geq 2 \) and \( m_0(f) = \infty \) (in particular, \( m_0(f) > 1/\mu_0(f) \)). The optimality of the estimate in (ii) will be more precisely discussed in Section 14.2.

(4) At present, very few properties of non-polar singularities of local zeta functions are known. We will also discuss these issues in Section 14.

3. Almost desingularization theorem for \( \mathcal{C}^\infty \) functions

In the discussion below in Sections 3-7, local properties of every function are essentially important and we do not care how small the domain of definition of each function is. Thus, it will be convenient to formulate our results for function-germs rather than functions. An identity involving several function-germs is defined to be
true if there exist functions from the equivalence classes of these germs such that in the intersection of their domains of definition the identity is true in the usual sense. We will make use of the following rings of germs of complex-valued function in Sections 3–7 and 14:

- \( C((x)) \) —— the set of germs of continuous functions at the origin of \( \mathbb{R} \).
- \( C^\infty((x)) \) and \( C^\infty((x,y)) \) —— the rings of germs of \( C^\infty \) functions at the origin of \( \mathbb{R} \) and \( \mathbb{R}^2 \), respectively.

The rings of germs of real-valued functions will be denoted by adding an \( \mathbb{R} \) to the above notation, e.g. \( \mathbb{R}C^\infty((x,y)) \).

For a given ring \( R \), an element of \( R \) which has an inverse is called a unit. An element of the ring of the formal power series is a unit if and only if it has non-zero constant term.

In Sections 3–7, we always assume that \( F(x,y) \in \mathbb{R}C^\infty((x,y)) \) satisfies that its formal Taylor series can be expressed as in a factorization form

\[
F(x,y) = \bar{u}(x,y) \prod_{j=1}^r (y - \bar{\Phi}_j(x))^{m_j},
\]

where \( m_j \) are positive integers, \( \bar{\Phi}_j(x) \in \mathbb{C}[x] \) are distinct (i.e., \( \bar{\Phi}_j(x) \neq \bar{\Phi}_k(x) \) if \( j \neq k \)) and \( \bar{u}(x,y) \in \mathbb{R}[x,y] \) is a unit. Let \( n := \sum_{j=1}^r m_j \). Corresponding to the general case \([2,13]\), we now give the additional assumptions: \( m_0 = 0 \) and \( N = 1 \). However, these assumptions do not essentially restrict any properties of \( C^\infty \) functions dealt with in our analysis of local zeta functions (see Sections 7–11).

Since \( F \) is a \( C^\infty \) function, it is impossible to resolve the singularities of the zero variety of \( F \) in general by using algebraic transforms only. However, we attempt to do so as much as possible by using a composition of a finite number of blowings up, which will be explained in Section 4. As a result, we succeed in giving an “almost” resolution of singularities of the zero variety, which locally expresses \( F(x,y) \) in the “almost” normal crossings form. The exact meaning of “almost” is as follows.

**Definition 3.1.** Let \( f(x,y) \in C^\infty((x,y)) \).

1. \( f(x,y) \) is said to be expressed in the normal crossings form if it is locally expressed as

\[
f(x,y) = u(x,y)x^ay^m,
\]

where \( a, m \) are nonnegative integers and \( u(x,y) \in C^\infty((x,y)) \) satisfies \( u(0,0) \neq 0 \).

2. \( f(x,y) \) is said to be expressed in the almost normal crossings form if it is locally expressed as

\[
f(x,y) = u(x,y)x^a(y^m + \varepsilon_1(x)y^{m-1} + \cdots + \varepsilon_m(x)),
\]

where \( a, m \) are nonnegative integers, \( u(x,y) \in C^\infty((x,y)) \) satisfies \( u(0,0) \neq 0 \) and \( \varepsilon_j(x) \in C^\infty((x)) \) are flat at the origin.
Considering the case where $\varepsilon_j \equiv 0$ for all $j$, we see that the concept of “normal crossings” is a special case of the concept of “almost normal crossings”. This subtle difference gives a serious influence in the analytic continuation of local zeta functions.

3.1. Almost resolution of singularities. The following theorem is the most important result in this paper from a geometrical point of view. After preparing many kinds of tools in Sections 4–6, we will give a proof of this theorem in Section 7.

Recall $R(F) = \{ j : \overline{f}_j(x) \in \mathbb{R}[x]\}$, where $\overline{f}_j(x)$ is as in (3.1) (see Section 2.4).

**Theorem 3.2.** Let $F(x,y)$ be a real-valued $C^\infty$ function defined near the origin in $\mathbb{R}^2$. If $F(x,y)$ satisfies that its Taylor series admits the factorization (3.1), then there exist an open neighborhood $U$ of the origin in $\mathbb{R}^2$, a two-dimensional $C^\infty$ real manifold $Y$ and a proper map $\pi$ from $Y$ to $U$ such that

(i) $\pi$ is a local diffeomorphism from $Y - \pi^{-1}(O)$ to $U - \{O\}$;

(ii) For each $j \in R(F)$, there exist a point $P_j$ on $\pi^{-1}(O)$ and a local $C^\infty$ coordinate $(x,y)$ centered at $P_j$ so that the following (a), (b) hold:

(a) $(F \circ \pi)(x,y)$ can be locally expressed in the almost normal crossings form. To be more specific,

$$ (F \circ \pi)(x,y) = u_j(x,y)x^{a_j}(y^{m_j} + \varepsilon_{j1}(x)y^{m_j-1} + \cdots + \varepsilon_{jm_j}(x)), $$

where $a_j$ is a nonnegative integer, $m_j \in \mathbb{N}$ is as in (3.1), $\varepsilon_{jk}(x) \in \mathbb{R}C^\infty((x,y))$ are flat functions at the origin and $u_j(x,y) \in \mathbb{R}C^\infty((x,y))$ satisfies $u_j(0,0) \neq 0$;

(b) The Jacobian of $\pi$ is locally expressed as

$$ J_\pi(x,y) = x^{M_j}, $$

where $M_j$ is a nonnegative integer;

(iii) For each $Q \in \pi^{-1}(O) - \{P_j : j \in R(F)\}$, there exists a local $C^\infty$ coordinate $(x,y)$ centered at $Q$ so that the following locally hold:

$$ (F \circ \pi)(x,y) = u_Q(x,y)x^{A_Q}y^{B_Q} \quad \text{and} \quad J_\pi(x,y) = x^{C_Q}, $$

where $u_Q(x,y) \in \mathbb{R}C^\infty((x,y))$ satisfies $u_Q(0,0) \neq 0$ and $A_Q, B_Q, C_Q$ are nonnegative integers.

**Remark 3.3.** Let us consider the case where the singularities of the zero variety of $F$ at the origin can be completely resolved; i.e., $F \circ \pi$ can be locally expressed in the normal crossings form at any point on $\pi^{-1}(O)$. (In this case, every local zeta function always admits the meromorphic extension to the whole complex plane.)

(i) When $F$ is real analytic, the functions $\varepsilon_{jk}(x)$ in (3.3) must be identically zero. Therefore, $F \circ \pi$ can be locally expressed in the normal crossings form at any point on $\pi^{-1}(O)$, which is a particular version of the desingularization theorem of Hironaka (see [3]).

(ii) In the case of $\mu_0(F) = 0$, the case (ii) does not occur. (In this case, Theorem 3.2 is the same as Proposition 6.14, below.)
(iii) In the case of $\mu_0(F) = 1$, $F \circ \pi$ in (3.3) can be also expressed in the normal crossings form $u(x, y)x^ay^b$ after a slight change of local coordinates.

Note that some cases in (ii), (iii) satisfy the $R$-nondegeneracy condition in the sense of Kouchnirenko ([1], see also Section 8.4 in this paper). It has been shown in [28] that (toric) resolution of singularities can be constructed under the $R$-nondegeneracy condition and the meromorphic continuation of local zeta functions can be precisely understood.

In the general $C^\infty$ case, the above theorem with its proof shows that $F \circ \pi$ can be locally expressed in the “almost” normal crossings form at any point on $\pi^{-1}(O)$ by using a composition of a finite number of blowings up.

**Definition 3.4.** The proper map $\pi : Y \to U$ in Theorem 3.2 is called an *almost resolution of singularities* for a $C^\infty$ function $F$ at the origin.

Notice that almost resolution of singularities does not always resolve the singularities of the zero varieties of $C^\infty$ functions. This phenomenon can be only found in the $C^\infty$ case and it may be interpreted as an essential difference between the geometric properties of the zero varieties of real analytic functions and $C^\infty$ functions. From an analytical point of view, as is shown by the example (2.11) in Section 2, non-zero flat functions may give an obstruction for the meromorphic extension of local zeta functions in the $C^\infty$ case. On the other hand, as shown in Theorem 2.2, the existence of flat functions gives no influence on the determination of $h_0(f)$.

**4. Construction of blowings up**

In this section, we recall ordinary blowing up which is an important tool in the studies of algebraic geometry and so on.

After constructing an appropriate complex manifold by using blowings up, we obtain a desired real manifold by restricting this complex manifold to the real space. From this reason, the choice of local coordinates must be sufficiently cared.

**4.1. Blowing up of an open set in $\mathbb{C}^2$.** Let $P = (a, b)$ be a point on $\mathbb{C}^2$ and let $U$ be an open neighborhood of $P$. Let us recall a blowing up of $U$ with center $P$.

Let $X_1$ be the subset of $\mathbb{P}^1(\mathbb{C}) \times U$ defined by

$$X_1 := \{(c_0 : c_1, (x, y)) \in \mathbb{P}^1(\mathbb{C}) \times U : c_1(x - a) = c_0(y - b)\}$$

and let $\sigma_1 : X_1 \to U$ be a projection defined by

$$\sigma_1((c_0 : c_1), (x, y)) = (x, y).$$

Then it is well known (c.f. [34]) that

(i) $X_1$ is a two-dimensional complex manifold and $\sigma_1$ is a proper map;
(ii) $\sigma_1 : X_1 - \sigma_1^{-1}(P) \to U - \{P\}$ is isomorphism;
(iii) The exceptional curve $E_0 := \sigma_1^{-1}(P)$ is isomorphic to $\mathbb{P}^1(\mathbb{C})$ as a complex manifold.
The above proper map $\sigma_1 : X_1 \to U$ is called a blowing up of $U$ with center $P$. The complex structure of $X_1$ is specified by the set of local charts $\{(V_j, \varphi_j)\}_{j=0,1}$, where
\begin{equation}
V_j := \{(c_0 : c_1, (x, y)) \in X_1 : c_j \neq 0\}
\end{equation}
and $\varphi_j : V_j \to \varphi_j(V_j) =: \tilde{V}_j$, for $j = 0, 1$, is defined by
\begin{equation}
\varphi_0^{-1}(u, v) = ((1 : v), (u + a, uv + b)),
\varphi_1^{-1}(z, w) = ((z : 1), (zw + a, w + b)).
\end{equation}
We call $(u, v)$ (resp. $(z, w)$) the canonical coordinate on $V_0$ (resp. on $V_1$). From (4.4), a coordinate transformation $\varphi_1 \circ \varphi_0^{-1} : \tilde{V}_0 - \{v = 0\} \to \tilde{V}_1 - \{z = 0\}$ is expressed as
\begin{equation}
(\varphi_1 \circ \varphi_0^{-1})(u, v) = (1/v, uv).
\end{equation}
For example, when $U = \{(x, y) \in \mathbb{C}^2 : |x - a| < \delta, |y - b| < \delta\}$ with $\delta > 0$, $X_1$ is constructed by piecing together the following open subsets of $\mathbb{C}^2$ by using (4.5):
\begin{align*}
\tilde{V}_0 &:= \{(u, v) \in \mathbb{C}^2 : |u| < \delta, |uv| < \delta\}, \\
\tilde{V}_1 &:= \{(z, w) \in \mathbb{C}^2 : |zw| < \delta, |w| < \delta\}.
\end{align*}

4.2. A series of blowings up. Next, let us construct a series of blowings up. We say that a two-dimensional complex manifold $X$ satisfies Property $(T)$, if a complex structure of $X$ is given by a set of local charts $\{(U_j, \varphi_j)\}_j$ satisfying the condition: if $U_j \cap U_k \neq \emptyset$, then the coordinate transformation $\varphi_k \circ \varphi_j^{-1} : \varphi_j(U_j \cap U_k) \to \varphi_k(U_j \cap U_k)$ is a biholomorphic map for $j, k = 0, \ldots, n$, which is defined by a finite composition of maps of the forms
\begin{equation}
(u, v) \to (v, u), \quad (u, v) \to (u, uv), \\
(u, v) \to (1/v, uv), \quad (u, v) \to (u + a, v + b), \quad (a, b \in \mathbb{C}).
\end{equation}
In other words, the complex manifold $X$ with Property $(T)$ is constructed by piecing together open sets $\varphi_j(U_j) \subset \mathbb{C}^2$ via the above maps (4.6). It is easy to see that the complex manifold $X_1$ in (4.1) satisfies Property $(T)$.

Let $X_n$ be a complex manifold with local charts $\{(U_j, \varphi_j)\}_{j=0}^n$ having Property $(T)$. Let $P_n$ be a point on $X_n$. Now let us define a blowing up of $X_n$ with center $P_n$. There exists a local chart containing $P_n$, which may be $U_0$. Let $U_\ast(\subset U_0)$ be an open neighborhood of $P_n$ and denote $\tilde{U}_\ast := \varphi_0(U_\ast)$ and $\tilde{P}_\ast := \varphi_0(P_n)$. We obtain $\tilde{\sigma} : \tilde{X} \to \tilde{U}$, which is a blowing up of $\tilde{U}$ with center $\tilde{P}_\ast$, by the same way as in (4.1), (4.2). A new complex manifold $X_{n+1}$ is constructed by piecing together $X_n - \{P_n\}$ and $\tilde{X}$ using the equivalence of $U_\ast - \{P_n\}$ and $\tilde{U}_\ast - \{\tilde{P}_\ast\}$ via the equivalence of each with $\tilde{U}_\ast - \{\tilde{P}_\ast\}$.

Let $\{(V_k, \phi_k)\}_{k=0,1}$ be the set of local charts of $\tilde{X}$ given by the same way as in (4.3), (4.4). The complex structure of $X_{n+1}$ is specified by a set of local charts
\begin{equation}
\{(U_0 - \{P_n\}, \varphi_0^\ast)\} \cup \{(U_j, \varphi_j)\}_{j=1}^n \cup \{(V_k, \phi_k)\}_{k=0,1},
\end{equation}
where \( \varphi_0 \) is the restriction of \( \varphi \) to \( U_0 - \{ P_n \} \). The canonical coordinate can be inductively introduced on each local chart by using (4.5). Moreover, it is easy to check that each coordinate transformation is expressed by using finite composition of the maps in (4.6). Therefore, \( X_{n+1} \) also has Property (T).

The proper map \( \sigma_{n+1} : X_{n+1} \to X_n \) is defined as follows. The restriction of \( \sigma_{n+1} \) to \( V_j \) is decided by \( \varphi_0^{-1} \circ \tilde{\varphi} \) for \( j = 0, 1 \) and that of \( \sigma_{n+1} \) to \( \bigcup_{j=1}^{n} U_j \cup (U_0 - \{ P_n \}) \) is the identity map. This is well-defined and the map \( \sigma_{n+1} : X_{n+1} \to X_n \) is called a blowing up of \( X_n \) with center \( P_n \).

From the above inductive process, a series of blowings up

\[
\cdots \xrightarrow{\sigma_{n+1}} X_n \xrightarrow{\sigma_n} X_{n-1} \xrightarrow{\sigma_{n-1}} \cdots \xrightarrow{\sigma_2} X_1 \xrightarrow{\sigma_1} X_0 := U \subset \mathbb{C}^2.
\]

is constructed. Here, for each \( n \in \mathbb{N} \), let \( \sigma_n : X_n \to X_{n-1} \) be the blowing up of \( X_{n-1} \) with center a point on \( X_{n-1} \) and \( U \) is an open neighborhood of the origin in \( \mathbb{C}^2 \). We call \( E_{n-1} = \sigma_n^{-1}(P_{n-1}) \) the exceptional curve of \( \sigma_n \) for \( n \in \mathbb{N} \). For \( n \in \mathbb{N} \), the composition of the blowings up in (4.7) is written as

\[
\pi_n := \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n : X_n \to X_0.
\]

When a series of blowings up is actually constructed to provide an appropriate resolution of singularities for our purpose (see the proof of Proposition 6.15), each center is chosen as follows.

**Definition 4.1.** We say that (4.7) is a series of blowings up with real centers, if each center \( P_n = (a_n, b_n) \) with \( a_n, b_n \in \mathbb{R} \) on a canonical coordinate.

4.3. A series of real blowings up. We say that a two-dimensional \( C^\omega \) real manifold \( Y \) has property \( (T_R) \), if \( Y \) admits a set of local charts \( \{(U_j, \varphi_j)\}_j \) satisfying the condition: if \( U_j \cap U_k \neq \emptyset \), then the coordinate transformation \( \varphi_k \circ \varphi_j^{-1} : \varphi_j(U_j \cap U_k) \to \varphi_k(U_j \cap U_k) \) is an isomorphism for \( j, k = 0, \ldots, n \), which is defined by a finite composition of maps in (4.6) with \( u, v, a, b \in \mathbb{R} \).

By noticing the form of maps in (4.6) with \( a, b \in \mathbb{R} \), we can define a map in the real version analogous to the blowings up in a similar fashion to the above procedure. We denote this map by \( \hat{\sigma}_{n+1} : Y_{n+1} \to Y_n \) for \( n \in \mathbb{Z}_+ \), where \( Y_{n+1}, Y_n \) are two-dimensional \( C^\omega \) real manifolds having Property \( (T_R) \), which is called a real blowings up of \( Y_n \) with center \( P_n \in Y_n \). Furthermore, a series of real blowings up can be inductively constructed as

\[
\cdots \xrightarrow{\hat{\sigma}_{n+1}} Y_n \xrightarrow{\hat{\sigma}_n} Y_{n-1} \xrightarrow{\hat{\sigma}_{n-1}} \cdots \xrightarrow{\hat{\sigma}_2} Y_1 \xrightarrow{\hat{\sigma}_1} Y_0 =: U \subset \mathbb{R}^2,
\]

where \( U \) is an open neighborhood of the origin in \( \mathbb{R}^2 \). For \( n \in \mathbb{N} \), the composition of real blowings up is written as

\[
\hat{\pi}_n := \hat{\sigma}_1 \circ \hat{\sigma}_2 \circ \cdots \circ \hat{\sigma}_n : Y_n \to Y_0.
\]

When (4.7) is a series of blowings up with real centers, that of real blowings up (4.9) can be simultaneously defined. Then, \( \hat{\sigma}_n \) may be regarded as the restriction of
Proposition 5.1. For each \( x \), \( y \) of \( \mathbb{R} \), \( (x, y) \), there exist a \( \Phi_j(x) \in C^{\infty}((x)) \) and \( \gamma_{jk}(x) \in C((x)) \) for \( k = 1, \ldots, m_j \) such that

\begin{itemize}
  \item[(i)] \( \Phi_j(x) \) admits the formal Taylor series \( \Phi_j(x) \);
  \item[(ii)] \( \gamma_{jk}(x) = O(x^l) \) as \( x \to 0 \) for any \( l \in \mathbb{N}, k = 1, \ldots, m_j \);
  \item[(iii)] \( F(x, \Phi_j(x) + \gamma_{jk}(x)) = 0 \) for \( x \in (-\delta, \delta) \) with small \( \delta > 0 \), \( k = 1, \ldots, m_j \);
  \item[(iv)] For each \( \alpha \in \mathbb{N} \), let \( E_{j\alpha}(x) \) be a continuous function defined by
    \[
    E_{j\alpha}(x) := \sum_{k=1}^{m_j} [\gamma_{jk}(x)]^\alpha.
    \]
\end{itemize}

Then \( E_{j\alpha}(x) \) belong to \( C^{\infty}((x)) \) for any \( \alpha \in \mathbb{N} \).

(v) If we additionally assume that \( \Phi_j(x) \) belongs to \( \mathbb{R}[[x]] \), then \( \Phi_j(x) \) and \( E_{j\alpha}(x) \) belong to \( \mathbb{R}C^{\infty}((x)) \) for any \( \alpha \in \mathbb{N} \).
For $j = 1, \ldots, r$, we define

\[(5.2)\]
\[F_j(x, y) := \prod_{k=1}^{m_j} (y - \Phi_j(x) - \gamma_{jk}(x)).\]

Although the continuous functions $\gamma_{jk}(x)$ may not have the $C^\infty$ differentiable property, all the elementary symmetric polynomials in the variables $\gamma_{j_1}(x), \ldots, \gamma_{j_m}(x)$ belong to $C^\infty((x))$ from (iv). Therefore, $F_j(x, y)$ can be rewritten as

\[(5.3)\]
\[F_j(x, y) = y^{m_j} + p_{j1}(x)y^{m_j-1} + \cdots + p_{jm_j}(x),\]

where $p_{jk}(x) \in C^\infty((x))$ for $k = 1, \ldots, m_j$ (i.e., $F_j(x, y) \in C^\infty((x))[y]$). Furthermore, under the assumption $\overline{\Phi}_j(x) \in \mathbb{R}[[x]]$, $F_j(x, y)$ takes the same form as (5.3) where $p_{jk}(x) \in \mathbb{R}C^\infty((x))$ for $k = 1, \ldots, m_j$ (i.e., $F_j(x, y) \in \mathbb{R}C^\infty((x))[y]$) from the above (v).

It follows from the Malgrange preparation theorem (c.f. [13], p.95) that $F(x, y)$ can be expressed as

\[(5.4)\]
\[F(x, y) = u(x, y) \left( y^n + a_1(x)y^{n-1} + \cdots + a_n(x) \right),\]

where $u(x, y) \in \mathbb{R}C^\infty((x, y))$ satisfies $u(0, 0) \neq 0$ and $a_j(x) \in \mathbb{R}C^\infty((x))$ satisfy $a_j(0) = 0$. Since $\sum_{j=1}^r m_j = n$, $F(x, y)$ can be factorized as

\[(5.5)\]
\[F(x, y) = u(x, y) \prod_{j=1}^r F_j(x, y) = u(x, y) \prod_{j=1}^r \prod_{k=1}^{m_j} (y - \Phi_j(x) - \gamma_{jk}(x)),\]

from (5.2), (5.4).

Let $B_{jk}$ be the set in $\mathbb{R} \times \mathbb{C}$ locally defined near the origin by the parametrization

\[(5.6)\]
\[x = t, \quad y = \Phi_j(t) + \gamma_{jk}(t) \quad \text{for } t \in \mathbb{R}.\]

The union of all $B_{jk}$ for all $j, k$ is called the decisive curve defined by $F$, which is denoted by $C_F$. Each $B_{jk}$ is called a branch of the decisive curve $C_F$. Since the notions of the decisive curve and its branches are locally defined near the origin in $\mathbb{C}^2$, they can be naturally defined at any point on the two-dimensional complex manifolds.

A branch $B_{jk}$ defined by (5.6) is called a real branch if the formal Taylor series $\overline{\Phi}_j(t)$ of $\Phi_j(t)$ belongs to $\mathbb{R}[[t]]$, otherwise it is called a non-real branch. We remark that if a branch is contained in $\mathbb{R}^2$ near the origin, then it is a real branch; while the converse is not always true in the $C^\infty$ setting. For example, consider the $C^\infty$ function $F(x, y) = y^2 + e^{-2/x^2}$. In this case, there are two branches defined by $x = t, y = \pm ie^{-1/t^2}$ for $t \in \mathbb{R}$. They are not contained in $\mathbb{R}^2$ but they are real branches.

**Remark 5.2.** For general $C^\infty$ functions $f$ with $f(0, 0) = 0$, the decisive curves $C_f$ can be similarly defined by using the factorization of Rychkov [39]. (This generalization is not necessary in the analysis of local zeta functions below.)
6. The Transforms of a Decisive Curve via Blowings Up

In this section, we attempt to construct a series of blowings up in order to resolve the singularities of the decisive curve defined by $F$. In this section, we say that $\gamma(t) \in C((t))$ has a flat property if $\gamma(t) = O(t^l)$ as $t \to 0$ for any $l \in \mathbb{N}$.

6.1. The transforms of a branch. Let us carefully observe how a branch of the decisive curve is transformed by a series of blowings up.

Let $U$ be an open neighborhood of the origin in $\mathbb{C}^2$. Let $B$ be a branch at the origin $O =: P_0$ of the decisive curve $C_F$ in $U$, which is locally expressed as

$$x = t, \quad y = \Phi_0(t) + \gamma_0(t) \quad \text{for } t \in \mathbb{R},$$

where $\Phi_0(t) \in C^\infty((t))$ satisfies $\Phi_0(0) = 0$ and $\gamma_0(t) \in C((t))$ has a flat property. We remark that $\gamma_0(t)$ may be a complex-valued function.

Blowing up with center $P_0$ produces a complex manifold $X_1$ as in Section 4.1 and the exceptional curve $E_0 = \sigma^{-1}_1(P_0)$. Let $B^{(1)}$ be the closure of $\sigma^{-1}_1(B - \{P_0\})$, which is called the strict transform of $B$. As explained in the construction of blowings up in Section 4, $X_1$ admits the set of local charts $\{(V_j, \varphi_j)\}_{j=0,1}$ as in (L3). Denote $\tilde{V}_j = \varphi_j(V_j) \subset \mathbb{C}^2$ for $j = 1, 2$.

First, let us observe geometrical situations of the strict transform $B^{(1)}$ and the exceptional curve $E_0$ on $V_0$. From the definition of blowing up, $\sigma_1$ can be regarded as the map from $\tilde{V}_0$ to $U$ given by

$$(u, v) \mapsto (x, y) = (u, uv).$$

Then the Jacobian of $\pi_1$ satisfies $J_{\pi_1}(u, v) = u$ and the strict transform $B^{(1)}$ is locally expressed as

$$u = t, \quad v = \Phi_1(t) + \gamma_1(t) \quad \text{for } t \in \mathbb{R},$$

where $\Phi_1(t) = \Phi_0(t)/t \in C^\infty((t))$ and $\gamma_1(t) = \gamma_0(t)/t \in C((t))$. We remark that $\gamma_1(t)$ also has a flat property. The exceptional curve $E_0$ is expressed as $u = 0$, $v = \tau$ for $\tau \in \mathbb{C}$ on $\tilde{V}_0$, and the strict transform $B^{(1)}$ meets $E_0$ at a unique point $P_1 = (0, \Phi_1(0))$, where $\Phi_1(0) = \lim_{t \to 0} \Phi(t)/t$. Note that $B^{(1)}$ transversely intersects $E_0$.

Next, the geometrical situation of $B^{(1)}$ and $E_0$ on $V_1$ is as follows. From the definition of blowing up, $\sigma_1$ can be regarded as the map from $\tilde{V}_1$ to $U$ given by

$$(z, w) \mapsto (x, y) = (zw, w).$$

The exceptional curve $E_0$ is expressed as $z = \tau$, $w = 0$ for $\tau \in \mathbb{C}$ on $\tilde{V}_1$. It is easy to see that $B^{(1)}$ does not intersect $E_0$ on $\tilde{V}_1$.

Inductively, let us assume that a complex manifold $X_n$ with the set of local charts $\{(U_j, \varphi_j)\}_{j=0}^l$ having Property $(T)$ in Section 4.2 and that there exists a local chart of $X_n$, which may be $U_0$, with the canonical coordinate, on which an exceptional
curve $E_n$ of $\pi_{n-1}$ is expressed as $x = 0$, $y = \tau$ for $\tau \in \mathbb{C}$ and $B^{(n)}$ is a subset of $X_n$ locally parametrized as

\begin{equation}
(6.5) \quad x = t, \quad y = \Phi_n(t) + \gamma_n(t) \quad \text{for } t \in \mathbb{R},
\end{equation}

where $\Phi_n(t) \in C^{\infty}(t)$ and $\gamma_n(t) \in C((t))$ has a flat property. Note that $B^{(n)}$ meets $E_n$ at a unique point $P_n^* = (0, \Phi_n(0))$. Let $P_n$ be a point on $X_n$. A blowing up of $X_n$ with center $P_n$ gives a new complex manifold $X_{n+1}$ and a new map $\sigma_{n+1} : X_{n+1} \to X_n$ as explained in Section 4.3. We write $E_\ast$ again for the closure of $\sigma_{n+1}^{-1}(E_n \setminus \{P_n\})$ and $E_n$ for the exceptional curve of $\sigma_{n+1}$. Let $B^{(n+1)}$ be the closure of $\sigma_{n+1}^{-1}(B^{(n)} \setminus \{P_n\})$. For a branch $B =: B^{(0)}$ of $C_F$, we can inductively define $B^{(n)}$ on $X_n$ for each $n \in \mathbb{N}$, which is called the $(n$-th) strict transform of a branch $B$. Let $U (\subset U_0)$ be an open neighborhood of $P_n$ and let $\{(V_j, \phi_j)\}_{j=0,1}$ be the set of new local charts of $X_{n+1}$ produced in the blowing up process in Section 4. Denote $V_j := \phi_j(V_j) \subset \mathbb{C}^2$ for $j = 1, 2$. Here, $V_0$ (resp. $V_1$) admits the canonical coordinate $(u, v)$ in (4.4) (resp. $(z, w)$ in (4.4)).

First, let us consider the case where the center is the point $P_n^*$. In a similar fashion to that in the case of $n = 0, 1$, the map (6.2) implies that the strict transform $B^{(n+1)}$ can be locally expressed on $V_0$ as

\begin{equation}
(6.6) \quad u = t, \quad v = \Phi_{n+1}(t) + \gamma_{n+1}(t) \quad \text{for } t \in \mathbb{R},
\end{equation}

where $\Phi_{n+1}(t) = (\Phi_n(t) - \Phi_n(0))/t \in C^{\infty}(t)$ and $\gamma_{n+1}(t) = \gamma_n(t)/t \in C((t))$. We remark that $\gamma_{n+1}(t)$ also has a flat property. Note that $E_n$ is expressed as $x = 0$, $y = \tau$ for $\tau \in \mathbb{C}$ on $V_0$ and $E_\ast$ does not appear on $V_0$.

Next, let us consider the case where the center $P_n$ is not $P_n^*$. Choose an open neighborhood $U$ of $P_n$ such that $P_n^*$ is not contained in $U$; then $B^{(n+1)}$ is expressed in the same form as in (6.5) on a local chart $U_0 \setminus \{P_n\}$ of $X_{n+1}$.

From the above inductive process, a series of blowings up gives a series of the strict transforms $\{B^{(n)}\}_{n \in \mathbb{Z}_+}$:

\begin{equation}
(6.7) \quad \cdots \xrightarrow{\sigma_{n+1}} X_n \xrightarrow{\sigma_n} X_{n-1} \xrightarrow{\sigma_{n-1}} \cdots \xrightarrow{\sigma_2} X_1 \xrightarrow{\pi_1} X_0 := U
\end{equation}

\begin{equation}
\phantom{(6.7) \quad} \quad \cup \quad \cup \quad \cup \quad \cup
\end{equation}

\begin{equation}
\cdots \xrightarrow{\sigma_{n+1}} B^{(n)} \xrightarrow{\sigma_n} B^{(n-1)} \xrightarrow{\sigma_{n-1}} \cdots \xrightarrow{\sigma_2} B^{(1)} \xrightarrow{\pi_1} B^{(0)} := B.
\end{equation}

Now, let $J_{\pi_n}$ be the Jacobian of the composition map $\pi_n$ and let

\begin{equation}
(6.8) \quad m(n) := \# \{ k : P_k = P_n^*, k = 0, \ldots, n - 1 \},
\end{equation}

where $\# A$ denotes the cardinal number of the set $A$. Then we have the following.

**Lemma 6.1.** $J_{\pi_n}(x, y) = x^{m(n)}$ on the canonical coordinate.

**Proof.** Since the Jacobian of the map (6.2) is $u$, the Jacobian of the decomposition of maps can be expressed as in the lemma. \qed
To be more precise, the strict transform $B^{(n)}$ can be specifically expressed by using the information of the original branch $B$.

**Lemma 6.2.** Let $B$ be a branch of the decisive curve $C_F$ defined by (6.1), where $\Phi(t) \in \mathcal{C}^\infty((t))$ admits the formal Taylor series at the origin

\[
\Phi(t) = \sum_{j=1}^{\infty} c_j t^j,
\]

where $c_j$ are complex numbers. Then, the strict transform $B^{(n)}$ is locally expressed on some local chart on $X_n$ with the canonical coordinate as

\[
x = t, \quad y = \Phi_n(t) + \gamma_n(t) \quad \text{for } t \in \mathbb{R},
\]

where $\Phi_n(t) \in \mathcal{C}^\infty((t))$ admits the Taylor series

\[
\Phi_n(t) = \sum_{j=1}^{\infty} c_{j+m(n)} t^j,
\]

and $\gamma_n(t) \in C((t))$ takes the form

\[
\gamma_n(t) = \frac{\gamma_0(t)}{t^{m(n)}}.
\]

Here $m(n)$ is as in (6.8), $c_j$ are the same as in (6.9) and $\gamma_n(t)$ has a flat property.

**Proof.** The equation (6.10) is shown by induction on $n$: the case of $n = 0$ is obvious. Let us assume that $B^{(n)}$ is locally expressed by using the Taylor series (6.11). If $P_n = P_n^*$, then $m(n+1) = m(n) + 1$ and the strict transform $B^{(n+1)}$ can be expressed as $\Phi_{n+1}(t) = (\Phi_n(t) - \Phi_n(0))/t$ plus a flat term, which implies

\[
\Phi_{n+1}(t) = \sum_{j=1}^{\infty} c_{j+m(n)+1} t^j.
\]

On the other hand, if $P_n \neq P_n^*$, then we have $m(n+1) = m(n)$ and $\Phi_{n+1}(t) = \Phi_n(t)$. As a result, we see that (6.11) holds in the case of $(n + 1)$.

The equation (6.10) can be similarly shown and the flat property of $\gamma_n(t)$ is obvious. □

Next, from the inductive process of blowings up of a branch which was explained in this section, we can understand geometrical situations of the strict transform of every branch of $C_F$. Let $\mathcal{B} = \{B_1, \ldots, B_l\}$ be the set of the branches of $C_F$ and let $\mathcal{B}^{(n)} = \{B_1^{(n)}, \ldots, B_l^{(n)}\}$ be the set of their $n$-th strict transforms. We denote $\mathcal{B} = \mathcal{B}^{(0)}$ and $B_j = B_j^{(0)}$ for $j = 1, \ldots, l$. Recall that $\mathcal{E}^{(n)}$ is the set of the exceptional curves of $\pi_n$ in $X_n$, which is defined in Section 4.4.
Proposition 6.3.  
(i) Every strict transform $B_j^{(n)} \in B^{(n)}$ meets one exceptional curve of $\pi_n$ at a single point. We denote $\mathcal{P}^{(n)} = \{B_j^{(n)} \cap E : B_j^{(n)} \in B^{(n)}, E \in \mathcal{E}^{(n)}\}$.
(ii) Conversely, for a given $P \in \mathcal{P}^{(n)}$, $I(P) \subset \{1, \ldots, l\}$ and $E(P) \in \mathcal{E}^{(n)}$ are defined as follows: $B_j^{(n)} \cap E = \{P\}$ if and only if $j \in I(P)$ and $E = E(P)$. For any $P \in \mathcal{P}^{(n)}$, there exists a local chart $(U, \varphi)$ with canonical coordinate such that
  (a) $U$ contains $P$;
  (b) $E(P)$ is expressed as $x = 0$, $y = \tau$ for $\tau \in \mathbb{C}$ on $\varphi(U)$;
  (c) Each $B_j^{(n)}$ with $j \in I(P)$ is locally expressed on $\varphi(U)$ as in the form:

\[
    x = t, \quad y = \Phi_j(t) + \gamma_j(t) \quad \text{for } t \in \mathbb{R},
\]

where $\Phi_j(t) \in C^\infty(\langle t \rangle)$ and $\gamma_j(t) \in C(\langle t \rangle)$ has a flat property.

Proof. The assertion of this proposition has been essentially shown by induction in this section. We remark that when the blowing up $\sigma_{n+1} : X_{n+1} \to X_n$ is constructed, on open neighborhood $U$ must be chosen so that $U$ contains at most one point in $\mathcal{P}^{(n)}$.

6.2. Exponent of contact of the strict transforms of two branches. Let us observe how the geometrical relationship between the strict transforms of two branches of $\mathcal{C}_F$ are changed by a series of blowings up in (1.7).

Let us introduce some quantity which measures the strength of the contact of two strict transforms. Let $n$ be a nonnegative integer and let $B^{(n)}$, $\tilde{B}^{(n)}$ be branches belonging to $B^{(n)}$. When $B^{(n)} \cap \tilde{B}^{(n)} \cap \pi_n^{-1}(O) \neq \emptyset$, it follows from Proposition 6.3 that there exists a local chart $U$ of $X_n$ such that $B^{(n)}$, $\tilde{B}^{(n)}$ are locally expressed on the canonical coordinate as $x = t, y = \Phi(t) + \gamma(t)$ and $x = t, y = \tilde{\Phi}(t) + \tilde{\gamma}(t)$ where $\Phi(t), \tilde{\Phi}(t) \in C^\infty(\langle t \rangle)$ admit the formal Taylor series:

\[
    \Phi(t) = \sum_{j=0}^\infty c_j t^j, \quad \tilde{\Phi}(t) = \sum_{j=0}^\infty \tilde{c}_j t^j,
\]

with $c_j, \tilde{c}_j$ are complex numbers, and $\gamma(t), \tilde{\gamma}(t) \in C(\langle t \rangle)$ have a flat property. Note that $c_0 = \tilde{c}_0$.

Definition 6.4. For $B^{(n)}$, $\tilde{B}^{(n)} \in B^{(n)}$, we define $\mathcal{O}(B^{(n)}, \tilde{B}^{(n)})$ as follows.

(i) If $B^{(n)} \cap \tilde{B}^{(n)} \cap \pi_n^{-1}(O) = \emptyset$, then set $\mathcal{O}(B^{(n)}, \tilde{B}^{(n)}) = 0$;
(ii) If $B^{(n)} \cap \tilde{B}^{(n)} \cap \pi_n^{-1}(O) \neq \emptyset$ and $\Phi(t) = \tilde{\Phi}(t)$, then set $\mathcal{O}(B^{(n)}, \tilde{B}^{(n)}) = \infty$;
(iii) If $B^{(n)} \cap \tilde{B}^{(n)} \cap \pi_n^{-1}(O) \neq \emptyset$ and $\Phi(t) \neq \tilde{\Phi}(t)$, then set $\mathcal{O}(B^{(n)}, \tilde{B}^{(n)}) = \min\{j \in \mathbb{N} : c_j \neq \tilde{c}_j\}$, where $c_j, \tilde{c}_j$ are as in (6.14).

We call $\mathcal{O}(B^{(n)}, \tilde{B}^{(n)})$ the exponent of contact of $B^{(n)}$, $\tilde{B}^{(n)}$. 
The exponent of contact of $B^{(n)}$, $\tilde{B}^{(n)}$ stands for the strength of contact of $B^{(n)}$ and $\tilde{B}^{(n)}$ at some point on $\pi_n^{-1}(O)$. In particular, $O(B^{(n)}, \tilde{B}^{(n)}) = 0$ means that $B^{(n)}$ is separated from $\tilde{B}^{(n)}$ near $\pi_n^{-1}(O)$; while $O(B^{(n)}, \tilde{B}^{(n)}) = \infty$ means that $B^{(n)}$ is infinitely tangent to $\tilde{B}^{(n)}$ at some point in $\mathcal{P}^{(n)}$. Since the following two lemmas concerning the above quantity can be easily shown from its definition, their proofs will be left to the readers.

**Lemma 6.5.** For $B^{(n)}, \tilde{B}^{(n)}, \tilde{\tilde{B}}^{(n)} \in \mathcal{B}^{(n)}$ with $n \in \mathbb{Z}_+$, then the following holds:

- (i) $O(B^{(n)}, \tilde{B}^{(n)}) = O(\tilde{B}^{(n)}, B^{(n)})$;
- (ii) $O(B^{(n)}, B^{(n)}) = \infty$;
- (iii) If $O(B^{(n)}, \tilde{B}^{(n)}) = \infty$, then $O(B^{(n)}, \tilde{\tilde{B}}^{(n)}) = O(\tilde{B}^{(n)}, \tilde{\tilde{B}}^{(n)})$;
- (iv) $O(B^{(n)}, \tilde{B}^{(n)}) \geq \min \{ O(B^{(n)}, \tilde{B}^{(n)}), O(\tilde{B}^{(n)}, \tilde{\tilde{B}}^{(n)}) \}$.

**Lemma 6.6.** For $B, \tilde{B} \in \mathcal{B}$, the following three conditions are equivalent:

- (i) $O(B, \tilde{B}) = \infty$;
- (ii) $O(B^{(n)}, \tilde{B}^{(n)}) = \infty$ for some $n \in \mathbb{Z}_+$;
- (iii) $O(B^{(n)}, \tilde{B}^{(n)}) = \infty$ for any $n \in \mathbb{Z}_+$.

From Lemma 6.5 (i), (ii), (iii), an equivalence relationship "$\sim$" can be introduced in $\mathcal{B}^{(n)}$ as follows:

(6.15) $B^{(n)} \sim \tilde{B}^{(n)} \iff O(B^{(n)}, \tilde{B}^{(n)}) = \infty$.

It follows from Lemma 6.6 that $B \sim \tilde{B}$ if and only if $B^{(n)} \sim \tilde{B}^{(n)}$ for any (or some) $n$.

The following lemma is important. Notice that if $O(B^{(n)}, \tilde{B}^{(n)})$ is a positive integer, then $B^{(n)}$ and $\tilde{B}^{(n)}$ intersect on $\pi_n^{-1}(O)$ and $B^{(n)} \not\sim \tilde{B}^{(n)}$.

**Lemma 6.7.** Let $B, \tilde{B} \in \mathcal{B}$ satisfy that $O(B^{(n)}, \tilde{B}^{(n)})$ is a positive integer for some $n \in \mathbb{Z}_+$. If $\sigma_{n+1} : X_{n+1} \to X_n$ is a blowing up with center the point $P \in \mathcal{P}^{(n)}$, at which $B^{(n)}$ and $\tilde{B}^{(n)}$ intersect, then we have

(6.16) $O(B^{(n+1)}, \tilde{B}^{(n+1)}) = O(B^{(n)}, \tilde{B}^{(n)}) - 1$.

**Proof.** Let (6.14) be the formal Taylor series characterizing the branches $B^{(n)}, \tilde{B}^{(n)}$, respectively. Then, it follows from Lemma 6.2 that $\sum_{j=0}^{\infty} c_{j+1} t^j, \sum_{j=0}^{\infty} \tilde{c}_{j+1} t^j$ are the Taylor series characterizing $B^{(n+1)}, \tilde{B}^{(n+1)}$, respectively. Therefore, we see that

$O(B^{(n+1)}, \tilde{B}^{(n+1)}) = \min \{ j : c_{j+1} \neq \tilde{c}_{j+1} \}$

$= \min \{ j : c_j \neq \tilde{c}_j \} - 1 = O(B^{(n)}, \tilde{B}^{(n)}) - 1$.

$\square$

As a corollary of Lemma 6.7, we obtain the following.
Proposition 6.8. There exists a finite series of blowings up

\[(6.17) \quad X_N \xrightarrow{\sigma_N} X_{N-1} \xrightarrow{\sigma_{N-1}} \cdots \xrightarrow{\sigma_2} X_1 \xrightarrow{\sigma_1} X_0 := U\]

such that \(\mathcal{O}(B^{(n)}, \tilde{B}^{(n)}) = 0\) for every pair of branches \(B, \tilde{B} \in \mathcal{B}\) with \(B \not\sim \tilde{B}\).

Proof. Let \(M(n)\) be a nonnegative integer defined by \(M(n) = \sum \mathcal{O}(B^{(n)}, \tilde{B}^{(n)})\), where the summation is taken over all the pairs of branches \(B, \tilde{B} \in \mathcal{B}\) with \(B \not\sim \tilde{B}\). It is easy to see that \(M(0) < \infty\) and that \(M(n + 1) \leq M(n)\) for any \(n \in \mathbb{Z}_+\). It suffices to show the existence of a positive integer \(N\) such that \(M(N) = 0\). If \(M(n) > 0\), then there exist \(B, \tilde{B} \in \mathcal{B}\) with \(B \not\sim \tilde{B}\) and a point \(P_* \in \pi^{-1}(O)\) such that \(B^{(n)}(n)\) and \(\tilde{B}^{(n)}(n)\) intersect at \(P_*\). From Lemma 6.7, the blowing up of \(X_n\) with center \(P_*\) gives \(\mathcal{O}(B^{(n+1)}, \tilde{B}^{(n+1)}) = \mathcal{O}(B^{(n)}, \tilde{B}^{(n)}) - 1\), which implies \(M(n+1) \leq M(n) - 1\). By the inductive process, the existence of a positive integer \(N\) such that \(M(N) = 0\) can be shown. \(\square\)

Recall that all the branches of \(C_F\) can be parametrized by using the Rychkov factorization formula \((5.5)\). Indeed, the set of the branches of \(C_F\) is expressed as \(\mathcal{B} = \{B_{jk} : j = 1, \ldots, r; \ k = 1, \ldots, m_j\}\), where each \(B_{jk}\) is locally parametrized as \(x = t, \ y = \Phi_j(t) + \gamma_{jk}(t)\) for \(t \in \mathbb{R}\), where \(\Phi_j(t), \gamma_{jk}(t)\) are as in Proposition 5.1. It is easy to determine the equivalence of the elements in \(\mathcal{B}\) as follows.

Lemma 6.9. The following three conditions are equivalent.

(i) \(B_{jk} \sim B_{j'k'}\);
(ii) \(B_{jk}^{(n)} \sim B_{j'k'}^{(n)}\) for any \(n \in \mathbb{Z}_+\);
(iii) \(j = j'\).

Proof. The above equivalences can be seen from Lemma 6.6 and the definition \((6.13)\) of the equivalence relationship “\(\sim\)”. \(\square\)

On the complex manifold \(X_N\) constructed in Proposition 6.8, local geometrical situations of the \(N\)-th strict transforms of the branches can be clearly understood. Recall that \(\mathcal{P}^{(N)}\) is as in Proposition 6.3.

Proposition 6.10. (i) For each \(j\), all the \(B_{jk}^{(N)}(n) \cap \pi_N^{-1}(O)\) for \(k = 1, \ldots, m_j\) are the same point on \(X_N\), which will be denoted by \(P_j^{(N)}\), i.e., \(\mathcal{P}^{(N)} = \{P_j^{(N)} : j = 1, \ldots, r\}\) (\(\mathcal{P}^{(N)}\) is as in Proposition 6.3). Moreover, if \(j \neq j'\), then \(P_j^{(N)} \neq P_j^{(N)}\).
(ii) For any \(P_j^{(N)} \in \mathcal{P}^{(N)}\), there exists a local chart \((U, \varphi)\) with the canonical coordinate such that
(a) \(U\) contains \(P_j^{(N)}\);
(b) The exceptional curve containing \(P_j^{(N)}\) is locally expressed as \(x = 0, y = \tau\) for \(\tau \in \mathbb{C}\) on \(\varphi(U)\);
For $k = 1, \ldots, m$, each $B_{jk}^{(N)}$ is locally expressed on $\varphi(U)$ as

$$
\begin{align*}
(6.18) & \quad x = t, \quad y = \Phi_j^{(N)}(t) + \gamma_{jk}^{(N)}(t) \quad \text{for } t \in \mathbb{R},
\end{align*}
$$

where $\Phi_j^{(N)}(t) \in C^\infty((t))$ and $\gamma_{jk}^{(N)}(t) \in C((t))$ has a flat property.

Proof. Applying Proposition 6.8 and Lemma 6.9 to Proposition 6.3, we obtain this proposition. \qed

Remark 6.11. From Lemma 6.2, the functions $\Phi_j^{(N)}(t)$ and $\gamma_{jk}^{(N)}(t)$ in (6.18) are specifically expressed by using the information of $\Phi_j(t)$ and $\gamma_{jk}(t)$. In particular, there exists a positive integer $L$ such that $\gamma_{jk}^{(N)}(t) = \gamma_{jk}(t)/t^L$ for any $k$. Therefore, for $\alpha \in \mathbb{N}$, $j = 1, \ldots, r$, we have

$$
(6.19) \quad \mathcal{E}_{ja}^{(N)}(t) := \sum_{k=1}^{m_j} [\gamma_{jk}^{(N)}(t)]^\alpha = \frac{1}{t^L} \sum_{k=1}^{m_j} [\gamma_{jk}(t)]^\alpha.
$$

Proposition 5.1 (iv) implies that $\mathcal{E}_{ja}^{(N)}(t)$ belong to $C^\infty((t))$ for $j = 1, \ldots, r$. Furthermore, if $\overline{\Phi}_j(t) \in \mathbb{R}[[t]]$, then we can see that $\mathcal{E}_{ja}^{(N)}(t)$ belong to $\mathbb{R}C^\infty((t))$ for $j = 1, \ldots, r$ from Proposition 5.1 (v). These properties will be used later.

6.3. Removement of non-real branches. Let $B$ be a non-real branch locally parametrized as in (6.11), where $\Phi_0(t)$ admits the formal Taylor series

$$
\overline{\Phi}_0(t) = \sum_{j=1}^\infty c_j t^j.
$$

Since $B$ is a non-real branch, the following positive integer can be decided:

$$
(6.20) \quad m := \min \{ j \in \mathbb{N} : c_j \in \mathbb{C} - \mathbb{R} \}.
$$

Let us consider a series of blowings up satisfying that the center of each blowing up $\sigma_k$ is $(0, \Phi_k(0)) = (0, c_k)$ on the canonical coordinates. When $m \geq 1$, since each $\Phi_k(0) = c_k$ is real for $k = 0, 1, \ldots, m - 1$, we can construct a finite series of blowings up with real centers and their composition map $\pi_m$ in (4.18). Simultaneously, a finite series of real blowings up (4.19) and their composition map $\hat{\pi}_m$ in (4.10) can also be constructed. Let $\pi_0, \hat{\pi}_0$ denote the identity maps.

Lemma 6.12. There exists an open neighborhood $\hat{U}$ of the origin in $\mathbb{R}^2$ such that $B^{(m)} \cap \hat{\pi}_m^{-1}(\hat{U}) = \emptyset$. Here, $\hat{\pi}_m^{-1}(\hat{U}) \subset Y_m$ is regarded as a subset in $X_m$.

Proof. It follows from Lemma 6.2 that the strict transform $B^{(m)}$ is locally expressed as (6.10) with (6.11), where $n$ in (6.10) and $m(n)$ in (6.11) are replaced by $m$. From these equations, we have $\Phi_m(t) = c_m + O(t)$. Since $c_m$ is a non-real complex number, the continuity of $\Phi_m(t)$ implies the existence of $\delta > 0$ such that $\Phi_m(t) \not\in \mathbb{R}$ for $t \in (-\delta, \delta)$. Let $\hat{U} = (-\delta, \delta) \times (-\delta, \delta)$, then we can see that $B^{(m)} \cap \hat{\pi}_m^{-1}(\hat{U}) = \emptyset$. \qed
As a corollary of the above lemma, the following proposition can be easily shown.

**Proposition 6.13.** There exist an open neighborhood $U$ of the origin in $\mathbb{C}^2$ and a finite series of blowings up with real centers:

\[(6.21) \quad X_N \xrightarrow{\sigma_N} X_{N-1} \xrightarrow{\sigma_{N-1}} \cdots \xrightarrow{\sigma_2} X_1 \xrightarrow{\sigma_1} X_0 := U\]

such that $B^{(N)} \cap \hat{\pi}^{-1}(\hat{U}) = \emptyset$ for any non-real branches $B$ of $C_F$, where $\hat{U}$ is the restriction of $U$ to $\mathbb{R}^2$.

**Proof.** Let $B_1, \ldots, B_L$ be the non-real branches of $C_F$. It suffices to show the following: For any $n \in \{1, \ldots, L\}$, there exist an $m_n \in \mathbb{N}$, a finite series of blowings up \(6.17\) with real centers with $N = m_n$ and an open neighborhood $\hat{U}_n$ in $\mathbb{R}^2$ such that

\[(6.22) \quad B_j^{(m_n)} \cap \hat{\pi}^{-1}(\hat{U}_n) = \emptyset \quad \text{for } j = 1, \ldots, n.\]

Let us show the above by induction: the case of $n = 1$ has been shown in Lemma 6.12. We assume that \(6.22\) holds. If $B_{n+1}^{(m_n)} \cap \hat{\pi}^{-1}(O) = \emptyset$, then we can choose an open neighborhood $\hat{U}_{n+1}(\subset \hat{U}_n)$ of the origin such that $B_{n+1}^{(m_n)} \cap \hat{\pi}^{-1}(U_{n+1}) = \emptyset$. In this case, it suffices to set $m_{n+1} = m_n$. If $B_{n+1}^{(m_n)} \cap \hat{\pi}^{-1}(O) \neq \emptyset$, then the strict transform $B_{n+1}^{(m_n)}$ can be locally expressed on the canonical coordinate by using a formal series of the form $\sum_{j=0}^{\infty} d_j t^j \in \mathbb{C}[[t]]$. Here, we can define a positive integer $M := \min\{j : d_j \in \mathbb{C} - \mathbb{R}\}$ from the condition $B_{n+1}^{(m_n)} \cap \hat{\pi}^{-1}(O) \neq \emptyset$. Then, by applying the same blowing up process as that in Lemma 6.12, we can see the existence of the series of blowings up with real centers such that \(6.22\) holds in the case of $(n+1)$. Here, we have $m_{n+1} = m_n + M$. \(\square\)

**6.4. In the case of $\mu_0(F) = 0$.** Let us consider the case where there is no real branches of the decisive curves $C_F$. From Proposition 6.13, we can obtain the following proposition, which implies that $F$ can be locally expressed in the normal crossings form by means of an appropriate series of blowings up with real centers.

**Proposition 6.14.** If $\mu_0(F) = 0$ (i.e., every branch of the curve $C_F$ is non-real), then there exist an open neighborhood $\hat{U}$ of the origin in $\mathbb{R}^2$, a two-dimensional $C^\infty$ real manifold $Y$ and a proper $C^\infty$ map $\hat{\pi}$ from $Y$ to $\hat{U}$ such that

(i) $\hat{\pi}$ is a local isomorphism from $Y - \hat{\pi}^{-1}(O)$ to $\hat{U} - \{O\}$.

(ii) For each $Q \in \hat{\pi}^{-1}(O)$, there are local $C^\infty$ coordinates $(x,y)$ centered at $Q$ so that the following hold:

\[(6.23) \quad (F \circ \hat{\pi})(x,y) = u_Q(x,y) x^{A_Q} y^{B_Q} \quad \text{and} \quad J_\hat{\pi}(x,y) = x^{C_Q},\]

where $u_Q \in \mathbb{R}C^\infty((x,y))$ satisfies $u_Q(0,0) \neq 0$ and $A_Q, B_Q, C_Q$ are nonnegative integers.
Proof. From Proposition 6.13, there exists a map \( \hat{\pi} : Y \to \hat{U} \) constructed from the composition of a series of blowings up, which satisfies that \( \hat{\pi}^{-1}(C_F) \) consists of only real exceptional curves in \( \hat{\pi}^{-1}(\hat{U}) \). It is easy to see that the exceptional curves is locally given by the zero locus of the function of the normal crossings on local charts. Since \( \hat{\pi} \) is a composition of the maps in (4.10) with (6.21), the Jacobian of \( \hat{\pi} \) can be written as in Lemma 6.1. \( \square \)

6.5. In the case of \( \mu_0(F) \geq 1 \). Let us consider the case where at least one real branch exists in the decisive curve \( C_F \). Recall \( \mathcal{R}(F) := \{ j : \Omega_j(t) \in \mathbb{R}[[t]] \} \), where \( \Omega_j(t) \) are as in (3.1). Note that \( \mathcal{R}(F) \neq \emptyset \) if \( \mu_0(F) \geq 1 \).

Proposition 6.15. If \( \mu_0(F) \geq 1 \), then there are an open neighborhood \( \hat{U} \) of the origin in \( \mathbb{R}^2 \), a two-dimensional \( C^\omega \) real manifold \( Y \) and a proper \( C^\omega \) map \( \hat{\pi} \) from \( Y \) to \( \hat{U} \) such that

(i) \( \hat{\pi} \) is a local isomorphism from \( Y - \hat{\pi}^{-1}(O) \) to \( \hat{U} - \{ O \} \).

(ii) For each \( j \in \mathcal{R}(F) \), there exist a point \( P_j \in \hat{\pi}^{-1}(O) \) and a local \( C^\omega \) coordinate \( (x, y) \) centered at \( P_j \) so that the following (a), (b) hold:

(a) \( (F \circ \hat{\pi})(x, y) \) is locally expressed as

\[
(F \circ \hat{\pi})(x, y) = u_j(x, y)x^{a_j} \prod_{k=1}^{m_j}(y - \tilde{\Phi}_j(x) - \tilde{\gamma}_{jk}(x)),
\]

where \( u_j(x, y) \in \mathbb{R}C^\infty((x, y)) \) satisfies \( u_j(0, 0) \neq 0 \), \( a_j \) is a nonnegative integers, \( m_j \) is as in (6.21), \( \tilde{\Phi}_j(x) \) belongs to \( \mathbb{R}C^\infty((x)) \) and \( \tilde{\gamma}_{jk}(x) \in C((x)) \) \((k = 1, \ldots, m_j) \) satisfy that \( \tilde{\gamma}_{jk}(x) = O(x^l) \) as \( x \to 0 \) for any \( l \in \mathbb{N} \).

(b) The Jacobian of \( \hat{\pi} \) is locally expressed as

\[
J_{\hat{\pi}}(x, y) = x^{M_j},
\]

where \( M_j \) is a nonnegative integer.

(iii) For each \( Q \in \hat{\pi}^{-1}(O) - \{ P_j : j \in \mathcal{R}(F) \} \), there exists a local coordinate \( (x, y) \) centered at \( Q \) so that the following locally hold:

\[
(F \circ \hat{\pi})(x, y) = u_Q(x, y)x^{A_Q}y^{B_Q}, \quad \text{and} \quad J_{\hat{\pi}}(x, y) = x^{C_Q},
\]

where \( u_Q(x, y) \in \mathbb{R}C^\infty((x, y)) \) satisfies \( u_Q(0, 0) \neq 0 \) and \( A_Q, B_Q, C_Q \) are nonnegative integers.

Proof. For a given \( F \in \mathbb{R}C^\infty((x, y)) \) with \( \mu_0(F) \geq 1 \), we can construct a series of blowings up with real centers and an open neighborhood of \( U \) of the origin in \( \mathbb{C}^2 \) as in Proposition 6.13. We denote the composition map of this series of blowings up by \( \pi_A : X_A \to U \), where \( X_A := X_N \) which is as in Proposition 6.13. Moreover, we denote by \( \hat{\pi}_A : Y_A \to \hat{U} \) the \( C^\omega \) map which are simultaneously constructed in the real version. Here \( \hat{U} \) is the restriction of \( U \) to \( \mathbb{R}^2 \) and \( Y_A = \hat{X}_A \). Since all the strict
transforms of non-real branches in $X_A$ are away from $\hat{\pi}^{-1}_A(\hat{U})$, it suffices to deal with the real branches $B_{jk}$ of $C_F$.

Next, using the series of blowings up in Proposition 6.8, we can construct a series of blowings up with real centers $X_{N'} \to X_{N'-1} \to \cdots \to X_A$ such that $\mathcal{O}(B_{jk}^{(N')}, B_{j'k'}^{(N')}) = 0$ if $j \neq j'$ and $\mathcal{O}(B_{jk}^{(N')}, B_{j'k'}^{(N')}) = \infty$ if $j = j'$, where $B_{jk}, B_{j'k'}$ are real branches of $C_F$. The composition map of this series of blowings up as $\pi_B : X_B \to X_A$, where $X_B = X_{N'}$. We can simultaneously construct the $C^\infty$ map $\hat{\pi}_B : Y_B \to Y_A$ in the real version. Here $Y_B = \hat{X}_B$.

Now, we respectively denote the maps $\pi := \pi_A \circ \pi_B$ and $\hat{\pi} := \hat{\pi}_A \circ \hat{\pi}_B$. Let us show that $\hat{\pi} : Y_B \to \hat{U}$ is a desired map in the proposition.

(i) Since $\hat{\pi}$ is a composition of finite numbers of real blowing maps, the property (i) can be easily seen.

(ii) Since each branches of the zero variety of $F \circ \hat{\pi}$ can be parameterized as in (6.18) in Proposition 6.10, we obtain the equation (6.24). Moreover, the Jacobian of $\hat{\pi}$ can be expressed as (6.25) from Lemma 6.1.

(iii) In a similar fashion to the proof of Proposition 6.14, (iii) can be shown. □

7. Proof of Theorem 3.2

In comparison to Proposition 6.15, it essentially suffices to show (ii) in Theorem 3.2. Indeed, we choose the local coordinate $(\hat{x}, \hat{y})$ around each $P_j$ defined by

\[(\hat{x}, \hat{y}) = (x, y - \hat{\Phi}_j(x)),\]

where $(x, y)$ is a local coordinate around $P_j$ as in Proposition 6.15. Then we can show the claim in (ii-a) in the theorem as follows. We remark that since the above transform may not be real analytic, the manifold $Y$ may have only the $C^\infty$ structure.

First, let us show the equation in (3.3). Recall that $\sum_{k=1}^{m_j} \gamma_{jk}(t)^\alpha$ belong to $\mathbb{R}C((t))$ for $\alpha \in \mathbb{N}$ in Remark 6.11. From this property, all the elementary symmetric polynomials in the variables $\gamma_{j1}(t), \ldots, \gamma_{jm_j}(t)$ belong to $\mathbb{R}C^\infty((t))$, which implies that $\varepsilon_{jk}(t)$ in (3.3) belong to $\mathbb{R}C^\infty((t))$ for $k = 1, \ldots, m_j$.

Next, since the transformation (7.1) does not affect the properties of the Jacobian, we can see (3.4) in Theorem 3.2.

8. Properties of $\mu_0(f)$

In this section, we investigate more detailed properties of the quantity $\mu_0(f)$ introduced in Section 2.4.

8.1. The geometric meaning. For simplicity, we only consider the case of the function $F \in \mathbb{R}C^\infty((x, y))$ satisfying that its Taylor series admits the factorization (3.1). (The argument below can be naturally extended to the general case.)

First, let us consider the case where $F(x, y)$ is real analytic. In this case, the branches $B_{jk}$ are the same object in $\mathbb{C}^2$ for $k = 1, \ldots, m_j$ if $j$ is fixed. Thus we
denote $B_j := B_{j,k}$ for $k = 1, \ldots, m_j$. By using the language of the divisors, the
decisive curve $C_F$ may be written as the sum of effective divisors, i.e.,

$$(8.1) \quad C_F = \sum_{j=1}^{r} m_j B_j = \sum_{j \in \mathbb{R}(F)} m_j B_j + \sum_{j \notin \mathbb{R}(F)} m_j B_j.$$ 

In the $C^\infty$ case, although slight gaps may be made by flat functions, it can be
interpreted that the equations (8.1) hold in the level of formal power series or in
the equivalence class introduced in (6.15). From this point of view, it might be said
that the quantity $\mu_0(F)$ indicates the maximum of the multiplicities of the formal
real branches of $C_F$.

8.2. The algebraic meaning. Let $f(x, y) \in \mathbb{R}C^\infty((x, y))$ admit the Taylor series
$\overline{f}(x, y) \in \mathbb{R}[[x, y]]$. From the algebraic property of the ring of formal power series
$\mathbb{C}[[x, y]]$, let us explore the property of $\mu_0(f)$. The zero element of the ring of formal
power series is denoted by $0$.

For $\overline{P}(x, y) \in \mathbb{C}[[x, y]]$, $S_\Re(\overline{P})$ denotes the set of the pairs $(\overline{\phi}(t), \overline{\psi}(t))$ satisfying

- $\overline{\phi}(t), \overline{\psi}(t) \in \mathbb{R}[[t]]$;
- At least one of $\overline{\phi}(t), \overline{\psi}(t)$ is different from $0$;
- $\overline{P}(\overline{\phi}(t), \overline{\psi}(t)) = 0$.

Two pairs $(\overline{\phi}(t), \overline{\psi}(t))$ and $(\overline{\phi}(t), \overline{\psi}(t))$ in $S_\Re(\overline{P})$ are equivalent if there exists a unit
$\overline{\rho}(t) \in \mathbb{R}[[t]]$ such that $(\overline{\phi}(t), \overline{\psi}(t)) = (\overline{\phi}(\overline{\rho}(t)), \overline{\psi}(\overline{\rho}(t)))$. An equivalence class in the
equivalence relation defined above in $S_\Re(\overline{P})$ is called a real root of $\overline{P}$.

**Lemma 8.1.** If $\overline{P}(x, y) \in \mathbb{C}[[x, y]]$ is irreducible, then there exists at most one real
root of $\overline{P}$. Furthermore, if $\overline{P}$ has a real root, then one of the following two conditions holds:

(i) There exist $n \in \mathbb{N}$ and $\varphi(t) \in \mathbb{R}[[t]]$ such that $\overline{P}(t^n, y)$ is divisible by $(y - \overline{\varphi}(t))$;

(ii) $\overline{P}(x, y)$ is divisible by $x$.

**Proof.** If there exists a positive integer $n$ such that $\overline{P}(0, y)$ is of order $n$ (i.e.,
$P(0, y) = cy^n + \cdots$, with $c \neq 0$), then the Puiseux theorem implies

$$(8.2) \quad \overline{P}(t^n, y) = \overline{U}(t^n, y) \prod_{k=0}^{n-1} (y - \overline{\varphi}(\omega^k t)), $$

where $\overline{U}(x, y) \in \mathbb{C}[[x, y]]$ is a unit, $\overline{\varphi}(t)$ belongs to $\mathbb{C}[[t]]$ and $\omega = e^{2\pi i/n}$. From the
above factorization, there exists at most one $\overline{\varphi}(t) \in \mathbb{R}[[t]]$ such that $\overline{P}(t^n, \overline{\varphi}(t)) = 0$,
which implies $\overline{P}$ has at most one real root. On the other hand, if $\overline{P}(0, y) = 0$, then
$\overline{P}(x, y)$ can be represented as $\overline{P}(x, y) = \overline{U}(x, y)x$, where $\overline{U}(x, y)$ is a unit. In this
case, $(0, t)$ expresses only one real root of $\overline{P}$. \qed
Since \( \mathbb{C}[x, y] \) is a unique factorization domain, \( f(x, y) \) can be expressed as

\[ f(x, y) = P_1^{m_1}(x, y) \cdots P_l^{m_l}(x, y), \tag{8.3} \]

where \( m_j \) are positive integers and the factors \( P_j(x, y) \in \mathbb{C}[x, y] \) are irreducible and distinct (we cannot write \( P_j = V \cdot P_k \) for any unit \( V \) if \( j \neq k \)).

Now let \( \mathcal{R}(f) := \{ j : P_j \text{ has a real root} \} \). Then, by applying Lemma 8.1 to (8.3), the definition of \( \mu_0(f) \) in (2.15) gives the following.

**Proposition 8.2.** \( \mu_0(f) = \max \{ m_j : j \in \mathcal{R}(f) \} \).

### 8.3. Invariance under the change of coordinates

Let \( (\Phi(x, y), \Psi(x, y)) \) be a local diffeomorphism defined near the origin in \( \mathbb{R}^2 \), where \( \Phi(x, y), \Psi(x, y) \in \mathbb{R}C^\infty((x, y)) \) with \( \Phi(0, 0) = \Psi(0, 0) = 0 \). Let \( \Phi(x, y), \Psi(x, y) \in \mathbb{C}[x, y] \) be the Taylor series of \( \Phi(x, y), \Psi(x, y) \), respectively.

**Lemma 8.3.** Suppose that \( P(x, y), Q(x, y) \in \mathbb{C}[x, y] \) are irreducible and satisfy \( P(\Phi(x, y), \Psi(x, y)) = Q(x, y) \) for the above \( \Phi(x, y), \Psi(x, y) \). Then, the existence of a real root of \( P \) is equivalent to that of \( Q \).

**Proof.** There exist real numbers \( A, B, C, D \) with \( \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} \neq 0 \) such that \( \Phi(x, y), \Psi(x, y) \in \mathbb{R}[x, y] \) take a form

\[ \Phi(x, y) = Ax + By + \cdots, \quad \Psi(x, y) = Cx + Dy + \cdots. \tag{8.4} \]

If \( \phi(t), \psi(t) \in \mathbb{R}[t] \) satisfy \( \Phi(\phi(t), \psi(t)) = \Psi(\phi(t), \psi(t)) = 0 \), then we can see that \( \phi(t) = \psi(t) = 0 \) by using \( \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} \neq 0 \). From this implication, the equivalence in the lemma can be easily seen. \( \square \)

Let us show that the quantity \( \mu_0(f) \) is invariant under the change of coordinates.

**Proposition 8.4.** Let \( (\Phi(x, y), \Psi(x, y)) \) be a diffeomorphism as above and set \( g(x, y) := f(\Phi(x, y), \Psi(x, y)) \). Then, we have \( \mu_0(f) = \mu_0(g) \).

**Proof.** Let \( \overline{f}, \overline{g} \in \mathbb{R}C^\infty((x, y)) \) be the formal Taylor series of \( f, g \) and let \( \overline{f} \) admit the factorization (8.3). Since \( \overline{g}(x, y) := \overline{f}(\overline{\Phi}(x, y), \overline{\Psi}(x, y)) \), \( \overline{g}(x, y) \) can be represented as

\[ \overline{g}(x, y) = \prod_{j=1}^l P_j(\overline{\Phi}(x, y), \overline{\Psi}(x, y))^{m_j}, \tag{8.5} \]

where \( P_j(x, y) \) and \( m_j \) are the same as in (8.3). Applying Lemmas 8.1 and 8.3 to (8.5), we have \( \mu_0(g) = \max \{ m_j : j \in \mathcal{R}(\overline{f}) \} \), which implies \( \mu_0(f) = \mu_0(g) \) from Proposition 8.2. \( \square \)
8.4. From the viewpoint of Newton data. Let us investigate properties of $\mu_0(f)$ by using the Newton data of $f \in \mathbb{R}C^\infty((x, y))$ in Section 2.1. Hereafter, let $\overline{f}(x, y) = \sum_{j,k} c_{jk}x^jy^k$ be the formal Taylor series of $f$.

A face of the Newton polygon $\Gamma_+(f)$ means an edge or a vertex of $\Gamma_+(f)$. For a compact face $\kappa$ of $\Gamma_+(f)$, the $\kappa$-part of $f$ is a polynomial defined by $f_\kappa(x, y) = \sum_{(j,k)\in\kappa} c_{jk}x^jy^k$. We say

- $f$ is convenient if $\Gamma_+(f)$ intersects every coordinate axis;
- $f$ is $\mathbb{R}$-nondegenerate if $\nabla f_\kappa \neq 0$ holds on $(\mathbb{R} - \{O\})^2$ for every compact edge $\kappa$ of $\Gamma_+(f)$.

The above two conditions depend on the choice of coordinates. The $\mathbb{R}$-nondegeneracy condition was introduced by Kouchnirenko (see [1]) and plays useful roles in the study of singularity theory. The above two conditions are related to the case of $\mu_0(f) = 0, 1$.

Lemma 8.5. 
(i) If $\mu_0(f) = 0$, then $f$ is convenient on any coordinates.
(ii) If $f$ is convenient and $\mathbb{R}$-nondegenerate on some coordinate, then $\mu_0(f) = 0$ or 1.

Proof. (i) If $f$ is not convenient on some coordinate, then $\overline{f}(x, y)$ is divisible by $x$ or $y$ on the coordinate, which implies $\mu_0(f) \geq 1$.

(ii) When $f$ is convenient, $\overline{f}$ admits the factorization of the form

$$\overline{f}(t^N, y) = \overline{\pi}(t^N, y) \prod_{j=1}^r (y - \overline{\phi}_j(t))^{m_j},$$

where $\overline{\pi}(x, y) \in \mathbb{R}[[x, y]], m_j, N \in \mathbb{N}, \overline{\phi}_j(t) \in \mathbb{C}[t]$ are the same as in (2.13) and, moreover, $\overline{\phi}_j \neq \overline{0}$ for all $j$. Now, we assume that $\mu_0(f) \geq 2$. Then, there exists $j \in \{1, \ldots, r\}$ such that $m_j \geq 2$ and $\overline{\phi}_j(t) \in \mathbb{R}[t]$ takes the form $\overline{\phi}_j(t) = ct^n + \cdots$ where $c \neq 0$ and $n \in \mathbb{N}$. Let $\kappa$ be the edge of $\Gamma_+(f)$ which is contained in the line $\{(j, k) \in \mathbb{R}^2 : Nj + nk = L\}$ where $L$ is some positive number. It is easy to see that $f_\kappa(t^N, y)$ is divisible by $(y - ct^n)^2$, which is a contradiction to the $\mathbb{R}$-nondegeneracy condition of $f$. \hfill $\Box$

In Theorem 2.5, the case of $\mu_0(f) \geq 2$ is particularly interesting in the $C^\infty$ case. Therefore, it suffices to consider the case where $f$ is not convenient or does not satisfy the $\mathbb{R}$-nondegeneracy condition from the above lemma.

Remark 8.6. The converses of the implications in the above lemma are not true. In the case of (i), consider the example $f(x, y) = y^2 - x^3$. We can see that $\mu_0(f) = 1$ but $f$ is always convenient on any coordinate. In the case of (ii), consider the example $f(x, y) = xy$. We can see that $\mu_0(f) = 1$ but $f$ is not convenient.

Let us consider the relationship between the invariant $\mu_0(f)$ and the height $\delta_0(f)$ defined in Section 2.1. For this purpose, we need a result in [24], which gives a
Lemma 8.7 ([1], [24]). The coordinates are adapted to \( f \) if and only if one of the following conditions is satisfied:

(i) \( \kappa_P \) is a compact edge and \( \mu_0(f_P) \leq d(f) \);

(ii) \( \kappa_P \) consists of a vertex;

(iii) \( \kappa_P \) is unbounded.

Moreover, in case (i) we have \( \delta_0(f) = \delta_0(f_P) = d(f_P) \).

The following proposition shows that \( \mu_0(f) \) also indicates some kind of flatness of \( f \) at the origin.

Proposition 8.8. \( \mu_0(f) \leq \delta_0(f) \) holds for every non-flat \( f \in \mathbb{R}C^\infty((x, y)) \).

Proof. When \( \kappa_P \) is a compact edge, the definition of \( \mu_0(f) \) directly gives \( \mu_0(f) \leq \mu_0(f_P) \), which implies \( \mu_0(f) \leq \delta_0(f) \) by using Lemma 8.7. It is easy to show the estimate in the case where \( \kappa_P \) satisfies (ii), (iii) in Lemma 8.7.

\[\square\]

Remark 8.9. The estimate in the proposition also holds even if \( f \) does not satisfy the condition (2.1).

From Theorems 2.2 and 2.5, the difference of \( \mu_0(f) \) and \( \delta_0(f) \) is important in the analytic continuation issue for local zeta functions. Let us consider when an equal sign is established in the estimate in Proposition 8.8.

(i) When \( \kappa_P \) is unbounded, it is easy to see that \( \mu_0(f) = \delta_0(f) \) always holds.

(ii) When \( \kappa_P \) is a vertex, both cases are possible. For example, consider the functions \( f_1(x, y) = x^2 y^2 \) and \( f_2(x, y) = x^6 + x^2 y^2 + y^6 \). We can see that \( \mu_0(f_1) = \delta(f_1) = 2 \) and \( \mu_0(f_2) = 0, \delta_0(f_2) = 2 \).

(iii) When \( \kappa_P \) is a compact edge, both cases are also possible. For example, consider the functions \( f_1(x, y) = (x^2 - y^2)^2 \) and \( f_2(x, y) = x^4 + y^4 \). We can see that \( \mu_0(f_1) = \delta(f_1) = 2 \) and \( \mu_0(f_2) = 0, \delta_0(f_2) = 2 \). Notice that the Newton polygons of the above examples are the same. Although the shapes of the Newton polygons do not always determine the establishment of the equal sign, the following lemma gives the necessary condition on the Newton polygon for this establishment.

For \( \kappa_P \), there exists a unique pair \((l_1, l_2)\) with \( l_1, l_2 > 0 \) such that \( \kappa_P \) is contained in the line \( \{(\alpha, \beta) \in \mathbb{R}_+^2 : l_1 \alpha + l_2 \beta = 1\} \). Without loss of generality, by flipping coordinates, if necessary, we may assume that \( l_2 \geq l_1 \).

Lemma 8.10. If \( \mu_0(f) = \delta_0(f) \), then \( l_2/l_1 \) is a positive integer.

Proof. If \( l_2/l_1 \) is not a positive integer, then it is shown in [24] that \( \mu_0(f_P) < d(f) \) holds, which shows the adaptedness by Lemma 8.6 (i) (i.e., \( d(f) = \delta_0(f) \)) and implies that \( \mu_0(f) < \delta_0(f) \) holds.

\[\square\]
Remark 8.11. A typical example satisfying the assumption of the lemma is \( f(x, y) = y^2 - x^3 \). We can see that \( \mu_0(f) = 1 \) and \( \delta_0(f) = 6/5 \).

9. Meromorphy of local zeta functions in model cases

Sections 9-13 are the analytic part of this paper. By using almost resolution of singularities in Theorem 3.2, we will investigate the analytic continuation of the local zeta function which is defined by the integral

\[
Z(f, \varphi)(s) = \int_{\mathbb{R}^2} |f(x, y)|^s \varphi(x, y) dxdy \quad s \in \mathbb{C},
\]

where, \( f, \varphi \) are as in the beginning of the Introduction. Moreover, we assume that \( f \) is non-flat at the origin and satisfies the condition (2.1). Since the situation of analytic continuation of (9.1) does not change under a smooth change of integral variables, we may assume that \( f(x, y) \) is regular in \( y \) of order \( n \) where \( n \) is a positive integer (i.e., \( f(0, y) = cy^n + \cdots \) with \( c \neq 0 \)).

It is easy to see that the integral \( Z(f, \varphi)(s) \) can be decomposed as

\[
Z(f, \varphi)(s) = Z_+(s) + Z_-(s),
\]

where

\[
Z_\pm(s) = \int_{\mathbb{R}} \int_0^{\infty} |f(\pm x, y)|^s \varphi(\pm x, y) dxdy.
\]

Since properties of \( Z_+(s) \) and \( Z_-(s) \) are essentially the same, we will only deal with the case of \( Z_+(s) \). Furthermore, we may assume that the Taylor series of \( f \) admits the factorization (2.13) in Section 2 and let \( N \) be a positive even integer in (2.13).

By simple change of an integral variable,

\[
Z_+(s) = N \int_{\mathbb{R}} \int_0^{\infty} |f(x^N, y)|^s \varphi(x^N, y) x^{N-1} dxdy
\]

(9.3)

where \( F(x, y) = f(x^N, y) \) and \( \Phi(x, y) = \varphi(x^N, y) \). From the definition of \( F \), we may assume that the Taylor series \( T(x, y) \) of \( f(x, y) \) admits a factorization of the form (3.1) in Section 3. From the relationship between \( f(x, y) \) and \( F(x, y) \), we can easily see \( \mu_0(f) = \mu_0(F) \).

9.1. Decomposition of integrals. It follows from Theorem 3.2 that there exists a \( C^\infty \) manifold \( Y \) and an almost resolution of singularities \( \pi : Y \to U \) for a \( C^\infty \) function \( F \). Here, an open set \( U \), on which \( F \) and \( \Phi \) are defined, must be taken to be so small that the above resolution exists. Let us prepare a smooth partition of unity on \( Y \) as follows. There exist \( C^\infty \) functions \( \chi_j \), for \( j = 1, \ldots, r \), and \( \hat{\chi}_k \), for \( k = 1, \ldots, r \), defined on \( Y \) such that the following properties:

- \( \chi_j \) and \( \hat{\chi}_k \) are \( C^\infty \) functions on \( Y \) with small supports;
The support of each $\chi_j$ contains an open neighborhood of the point $P_j$, which is as in Theorem 3.2;

- The support of each $\hat{\chi}_k$ does not contain the set $\{P_1, \ldots, P_r\}$;

- $\sum_{j=1}^r \chi_j(x, y) + \sum_{k=1}^{\hat{r}} \hat{\chi}_k(x, y) = 1$ holds on $Y$.

The integrand in (9.3) is denoted by $A(x, y; s)$, that is,

$$A(x, y; s) := \frac{N}{2} |F(x, y)|^s \Phi(x, y)x^{N-1}. \tag{9.4}$$

Then $Z_+(s)$ can be expressed as

$$Z_+(s) = \sum_{j=1}^r Z_j(s) + \sum_{k=1}^{\hat{r}} \hat{Z}_k(s), \tag{9.5}$$

where

$$Z_j(s) := \int_{\mathbb{R}^2} A(\pi(x, y); s)|J_\pi(x, y)|\chi_j(x, y)dxdy; \tag{9.6}$$

$$\hat{Z}_k(s) := \int_{\mathbb{R}^2} A(\pi(x, y); s)|J_\pi(x, y)|\hat{\chi}_k(x, y)dxdy.$$}

It follows from Theorem 3.2 (ii) that every $Z_j(s)$ can be expressed as the sum of the integrals of the form:

$$\mathcal{Z}(s) = \int_{\mathbb{R}^2_+} |G(x, y)|^s x^b \phi(x, y)dxdy, \tag{9.7}$$

where $b$ is a nonnegative integer, $\phi$ is a $C^\infty$ function defined near the origin whose support is sufficiently small and

$$G(x, y) := u(x, y)x^a \left(y^m + \varepsilon_1(x)y^{m-1} + \cdots + \varepsilon_m(x)\right), \tag{9.8}$$

where $a, m$ are positive integers and $\varepsilon_j(x)$ are real-valued flat $C^\infty$ functions defined near the origin. Note that $m$ belongs to $\{m_j : j \in \mathcal{R}(F)\}$, where $m_j$ are as in (3.1) and $\mathcal{R}(F)$ is as in (2.14). From the convergence of integral, $\mathcal{Z}(s)$ can be regarded as a holomorphic function on the region $\text{Re}(s) > 0$. The following theorem shows more precise situation of analytic continuation, which is the most important results in this paper from an analytical point of view.

**Theorem 9.1.** $\mathcal{Z}(s)$ admits the meromorphic continuation to the region $\text{Re}(s) > -1/m$. Furthermore, when $m \geq a/b$, the poles of the extended $\mathcal{Z}(s)$ on $\text{Re}(s) > -1/m$ exist in the set $\{-(b+j)/a : j \in \mathbb{N}\}$. 

After investigating the properties of $\mathcal{Z}(s)$ in Sections 10–12, we will prove the above theorem in Section 13.

On the other hand, it follows from Theorem 3.2 (iii) that every $\hat{Z}_k(s)$ can be expressed as the sum of the integrals of the form:

\[(9.9) \hat{Z}(s) = \int \int_{\mathbb{R}^2} x^{a+c} y^{b+d} v(x, y)^s \phi(x, y) dxdy,\]

where $a, b, c, d$ are nonnegative integers, $\phi(x, y)$ is as in (9.1) and $v(x, y)$ is a positive $C^\infty$ function defined near the origin. From an elementary analysis in Lemma 11.1, below, we can see that $\hat{Z}(s)$ admits the meromorphic continuation to the whole complex plane and its poles exist in the set $\{- (c + j)/a, -(d + k)/b : j, k \in \mathbb{N}\}$.

10. Proof of Theorem 2.5

In the case of $\mu_0(f) = 0, 1$, an ordinary resolution of singularities for $f$ can be constructed (see Remark 3.3), which implies $m_0(f) = \infty$ by an elementary analysis in Lemma 11.1.

In the case of $\mu_0(f) \geq 2$, we may only deal with the case of $Z_+(s)$ in (9.2). By using almost resolution of singularities for $F$ in (9.3) and an appropriate smooth partition of unity, $Z_+(s)$ can be expressed as a finite sum of the integrals $Z_j(s)$ and $\hat{Z}_k(s)$ as in (9.5), where each $Z_j(s)$ (resp. $\hat{Z}_k(s)$) can be expressed as the sum of the integrals of the form $\mathcal{Z}(s)$ in (9.7) (resp. $\hat{Z}(s)$ in (9.9)), which is explained in Section 9. From the form of $\hat{Z}(s)$, $\hat{Z}_k(s)$ can always admit the meromorphic continuation on the whole complex plane. On the other hand, since $m$ belongs to $\{m_j : j \in \mathcal{R}(F)\}$ where $m_j$ are as in (2.13) and $\mathcal{R}(F)$ is as in (2.14), every $Z_j(s)$ admits the meromorphic continuation to the region $\text{Re}(s) > - \min\{1/m_j : j \in \mathcal{R}(f)\} = -1/\mu_0(F)$ from Theorem 9.1. Moreover, since the relationship between $f(x, y)$ and $F(x, y)$ implies $\mu_0(f) = \mu_0(F)$, the claim (ii) in the theorem can be shown. We remark that a coordinate is chosen so that $f(x, y)$ is regular in $y$ in the beginning of Section 9, which gives no essential influence on the above discussion by using Proposition 8.4.

The positions of candidate poles of the integrals $Z_j(s)$ and $\hat{Z}_k(s)$ have been explained in Section 9, which implies the latter part of the theorem.

11. Auxiliary lemmas

11.1. Meromorphy of one-dimensional model. The following lemma is essentially known (see [14], [4], [1], etc.). Since we will use not only the result but also an idea of its proof in the later computation, we give a complete proof here.

**Lemma 11.1.** Let $\psi : [0, r] \times \mathbb{C} \to \mathbb{C}$, with $r > 0$, satisfy the following properties:

(a) $\psi(\cdot ; s)$ is smooth on $[0, r]$ for all $s \in \mathbb{C}$;

(b) $\frac{\partial^\alpha \psi}{\partial u^\alpha}(u; \cdot)$ is an entire function on $\mathbb{C}$ for all $u \in [0, r]$ and $\alpha \in \mathbb{Z}_+$. 
Let $L(s)$ be an integral defined by
\begin{equation}
L(s) := \int_0^r u^{As+B}\psi(u;s)du \quad s \in \mathbb{C},
\end{equation}
where $A$ is a positive integer and $B$ is a nonnegative integer.

Then the following hold.

(i) The integral $L(s)$ becomes a holomorphic function on the half-plane $\text{Re}(s) > -(B+1)/A$, which is also denoted by $L(s)$.

(ii) Furthermore, $L(s)$ admits the meromorphic continuation to the whole complex plane. Moreover, its poles are simple and they exist in the set $\{- (B+j)/A : j \in \mathbb{N}\}$.

**Proof.** (i) Since the integral $L(s)$ locally uniformly converges on the half-plane $\text{Re}(s) > -(B+1)/A$, the assumption and the Lebesgue convergence theorem imply that the integral becomes a holomorphic function there.

(ii) Let $N$ be an arbitrary natural number. The Taylor formula implies
\begin{equation}
\psi(u;s) = \sum_{\alpha=0}^N \frac{\partial^\alpha \psi(0;s)}{\alpha!} u^\alpha + u^{N+1} R_N(u;s)
\end{equation}
with
\[ R_N(u;s) = \frac{1}{N!} \int_0^1 (1-t)^N \frac{\partial^{N+1} \psi}{\partial u^{N+1}}(tu;s)dt. \]

Here $R_N(u;s)$ satisfies the following.

- $R_N(\cdot;s)$ is smooth on $[0,r]$ for all $s \in \mathbb{C}$.
- $R_N(u;\cdot)$ is an entire function on $\mathbb{C}$ for all $u \in [0,r]$.

Substituting (11.2) into the integral (11.1), we have
\begin{equation}
L(s) = \sum_{\alpha=0}^N \frac{r^{As+B+\alpha+1}}{\alpha!(As+B+\alpha+1)} \frac{\partial^\alpha \psi(0;s)}{\alpha!} + \int_0^r u^{As+B+N+1} R_N(u;s)du
\end{equation}
on $\text{Re}(s) > -(B+1)/A$. From (i), the integral in (11.3) becomes a holomorphic function on the half-plane $\text{Re}(s) > -(B+N+2)/A$. Therefore, $L(s)$ can be analytically continued as a meromorphic function to the half-plane $\text{Re}(s) > -(B+N+2)/A$. Moreover, all the poles of the above meromorphic function are simple and they are contained in the set $\{- (B+j)/A : j = 1, \ldots, N+1\}$. Letting $N$ tend to infinity, we have the assertion. \qed

### 11.2. Meromorphy of important integrals.

Let $p$ be a positive even integer and let $r, R$ be positive real numbers satisfying $r^p \leq R$. Let $U$ be an open neighborhood of the set $\{(u, v) \in \mathbb{R}^2 : |u| \leq r, |v| \leq R\}$. We define
\begin{equation}
D_p := \{(u, v) \in U : |u| \leq r, u^p < v \leq R\}.
\end{equation}

The following lemma will play a useful role in the proof of Theorem 9.1.
Lemma 11.2. Let $\Phi : U \times \mathbb{C} \to \mathbb{C}$ satisfy the following properties.

(a) $\Phi(\cdot, s)$ is a $C^\infty$ function on $U$ for all $s \in \mathbb{C}$;
(b) $\frac{\partial^{\alpha + \beta} \Phi}{\partial u^\alpha \partial v^\beta}(u, v; \cdot)$ is an entire function for all $(u, v) \in D_p$ and $(\alpha, \beta) \in \mathbb{Z}_+^2$.

Let $H(s)$ be an integral defined by

\[
H(s) := \int_{D_p \cap \mathbb{R}_+^2} u^{a+b} v^m s \Phi(u, v; s) dudv \quad s \in \mathbb{C},
\]

where $a, m$ are positive integers and $b$ is a nonnegative integer.

Let $N$ be an arbitrary natural number. By the Taylor formula,

\[
\Phi(u, v; s) = \sum_{(\alpha, \beta) \in [0, \ldots, N]^2} \frac{\Phi^{(\alpha, \beta)}(0, 0; s)}{\alpha! \beta!} u^\alpha v^\beta + \sum_{\alpha=0}^{N} u^\alpha v^{N+1} A^{(N)}(v; s) + \sum_{\beta=0}^{N} u^{N+1} v^\beta B^{(N)}(v; s) + \sum_{\alpha=0}^{N} u^\alpha v^{N+1} C^{(N)}(u, v; s),
\]

and

\[
\lim_{p \to \infty} \frac{b + p + 1}{a + pm} = \frac{1}{m}.
\]
where
\[
\tilde{A}_\alpha^{(N)}(v; s) := \frac{1}{\alpha!N!} \int_0^1 (1 - t)^N \psi^{(\alpha,N+1)}(0, tv; s) dt \quad \text{for } \alpha \in \{0, \ldots, N\},
\]
\[
\tilde{B}_\beta^{(N)}(u; s) := \frac{1}{\beta!N!} \int_0^1 (1 - t)^N \psi^{(N+1,\beta)}(tu, 0; s) dt \quad \text{for } \beta \in \{0, \ldots, N\},
\]
\[
\tilde{C}^{(N)}(u, v; s) := \frac{1}{(N!)^2} \int_0^1 \int_0^1 (1 - t_1)^N (1 - t_2)^N \psi^{(N+1,N+1)}(t_1 u, t_2 v; s) dt_1 dt_2.
\]

(11.8)

Note that
• \(\tilde{A}_\alpha^{(N)}(v; s)\) is a \(C^\infty\) function on \([-R, R]\) for all \(s \in \mathbb{C}\),
• \(\tilde{B}_\beta^{(N)}(u; s)\) is a \(C^\infty\) function on \([-r, r]\) for all \(s \in \mathbb{C}\),
• \(\tilde{C}^{(N)}(u, v; s)\) is a \(C^\infty\) function on \(U\) for all \(s \in \mathbb{C}\),
• \(\tilde{A}_\alpha^{(N)}(v; \cdot), \tilde{B}_\beta^{(N)}(u; \cdot), \tilde{C}^{(N)}(u, v; \cdot)\) are entire functions for all \((u, v) \in U\).

Substituting (11.7) into (11.5), we have
\[
H(s) = \sum_{(\alpha, \beta) \in \{0, \ldots, N\}^2} \frac{\psi^{(\alpha,\beta)}(0, 0; s)}{\alpha!\beta!} H_{\alpha,\beta}(s)
\]
\[
+ \sum_{\alpha=0}^N A_{\alpha}^{(N)}(v; s) + \sum_{\beta=0}^N B_{\beta}^{(N)}(v; s) + C^{(N)}(s)
\]
(11.9)

with
\[
H_{\alpha,\beta}(s) = \int_{D_p \cap \mathbb{R}_+^2} u^{as+b+\alpha} v^{ms+\beta} dudv \quad \text{for } (\alpha, \beta) \in \{0, \ldots, N\}^2,
\]
\[
A_{\alpha}^{(N)}(s) = \int_{D_p \cap \mathbb{R}_+^2} u^{as+b+\alpha} v^{ms+N+1} \tilde{A}_\alpha^{(N)}(v; s) dudv \quad \text{for } \alpha \in \{0, \ldots, N\},
\]
\[
B_{\beta}^{(N)}(s) = \int_{D_p \cap \mathbb{R}_+^2} u^{as+b+N+1} v^{ms+\beta} \tilde{B}_\beta^{(N)}(u; s) dudv \quad \text{for } \beta \in \{0, \ldots, N\},
\]
\[
C^{(N)}(s) = \int_{D_p \cap \mathbb{R}_+^2} u^{as+b+N+1} v^{ms+N+1} \tilde{C}^{(N)}(u, v; s) dudv.
\]

Now let us consider the meromorphy of the above integrals.
The integral $H_{\alpha,\beta}(s)$.

A simple calculation gives

$$H_{\alpha,\beta}(s) = \int_0^r u^{as+b+\alpha} \left( \int_{u^p}^R v^{ms+\beta} dv \right) du$$

$$= \frac{1}{ms + \beta + 1} \left( \frac{R^{ms+\beta+1}u^{as+b+\alpha+1}}{as + b + \alpha + 1} - \frac{r^{(a+mp)s+b+p(\beta+1)+1}}{(a + mp)s + \alpha + b + p(\beta + 1) + 1} \right),$$

which implies that every $H_{\alpha,\beta}(s)$ becomes a meromorphic function on the whole complex plane and has three poles which are contained in the set

$$(11.10) \left\{ -\frac{b + j}{a}, -\frac{k}{m}, -\frac{b + j + pk}{a + mp} : j, k \in \mathbb{N} \right\}.$$

The integral $A_{(N)}^{(N)}(s)$.

A simple calculation gives

$$A_{(N)}^{(N)}(s) = \left( \int_0^{r^p} \int_0^{u^{1/p}} + \int_0^R \int_{r^p}^{u^p} \right) u^{as+b+\alpha} v^{ms+N+1} \tilde{A}_{(N)}^{(N)}(v; s) du dv$$

$$= \frac{1}{as + b + \alpha + 1} \left( \int_{r^p}^{R} u^p (a+mp)s+b+pN+p+\alpha+1 \tilde{A}_{(N)}^{(N)}(v; s) dv \right.$$

$$\left. + r^{as+b+\alpha+1} \int_{r^p}^{R} v^{ms+N+1} \tilde{A}_{(N)}^{(N)}(v; s) dv \right).$$

(11.11)

Lemma 11.1 (i) implies that the first integral in (11.11) becomes a holomorphic function on the half-plane $\text{Re}(s) > -(pN + 2p + b + \alpha + 1)/(a + mp)$. Moreover, it is easy to check that the second integral is an entire function. Hence, $A_{(N)}^{(N)}(s)$ can be analytically continued as a meromorphic function to the half-plane $\text{Re}(s) > -(pN + 2p + b + \alpha + 1)/(a + mp)$ and has one pole which is contained in the set

$$(11.12) \left\{ -\frac{b + j}{a} : j \in \mathbb{N} \right\}.$$

The integral $B_{(N)}^{(N)}(s)$.

A simple calculation gives

$$B_{(N)}^{(N)}(s) = \int_0^r \left( \int_{u^p}^R v^{ms+\beta} dv \right) u^{as+b+N+1} \tilde{B}_{(N)}^{(N)}(u; s) du$$

$$= \frac{1}{ms + \beta + 1} \left( R^{ms+\beta+1} \int_0^r u^{as+b+N+1} \tilde{B}_{(N)}^{(N)}(u; s) du \right.$$

$$\left. - \int_0^r u^{(a+mp)s+b+p(\beta+1)+N+1} \tilde{B}_{(N)}^{(N)}(u; s) du \right).$$

(11.13)
Lemma 11.1 (i) implies that the first (resp. the second) integral in (11.13) becomes a holomorphic function on the half-plane $\Re(s) > -(b + N + 2)/a$ (resp. $\Re(s) > -(b + p\beta + p + N + 2)/(a + pm)$). Thus, $B^{(N)}_{\beta}(s)$ can be analytically continued as a meromorphic function to the half-plane $\Re(s) > \max\{-b + N + 2)/a, -(b + p\beta + p + N + 2)/(a + pm)\}$ and has one pole which is contained in the set

\[
\left\{-\frac{k}{m} : k \in \mathbb{N}\right\}.
\]

(When $N$ is sufficiently large, the above maximum is $-(b + p\beta + p + N + 2)/(a + pm)$.)

The integral $C^{(N)}(s)$.

It follows from the proof of (i) in this lemma that the integral $C^{(N)}(s)$ converges on the half-plane $\Re(s) > \max\{-b + N + 2)/a, -(N + 2)/b\}$, which implies that $C^{(N)}(s)$ can be analytically continued as a holomorphic function there.

From the above, letting $N$ tend to infinity in (11.9), we can see that $H(s)$ becomes a meromorphic function on the whole complex plane and that its poles are contained in the set (11.10), (11.12), (11.14).

\[\square\]

11.3. A van der Corput-type lemma.

**Lemma 11.4** ([17]). Let $f$ be a $C^k$ function on an interval $I$ in $\mathbb{R}$. If $|f^{(k)}| > \eta$ on $I$, then for $\sigma \in (-1/k, 0)$ there is a positive constant $C(\sigma, k)$ depending only on $\sigma$ and $k$ such that

\[
\int_I |f(x)|^\sigma dx < C(\sigma, k)\eta^\sigma |I|^{1+k\sigma},
\]

where $|I|$ is the length of $I$.

The above van der Corput-type lemma plays the most important role in our analysis. This lemma has been shown in [17], [31].

12. Analytic properties of $G(x, y)$ in (9.8)

In this section, let us investigate analytic properties of the $C^\infty$ function

\[
G(x, y) = u(x, y)x^a\left(y^{m} + \varepsilon_1(x)y^{m-1} + \cdots + \varepsilon_m(x)\right),
\]

where $a, m$ are positive integers, $u(x, y) \in C^\infty(U)$ satisfies $u(0, 0) > 0$ and $\varepsilon_j$ are real-valued flat $C^\infty$ functions defined near the origin.

12.1. The decisive curve $C_G$ and its branches. In order to understand the analytic continuation of the function $Z(s)$, it is important to understand geometric properties of the decisive curve $C_G$ defined by $G$. From the argument in Section 6, there exists a small open neighborhood $U$ of the origin such that

\[
G(x, y) = u(x, y)x^a\prod_{k=1}^m(y - \gamma_k(x))\quad \text{on } U,
\]
where each $\gamma_k \in C(U)$ satisfies that $\gamma_k(x) = O(x^l)$ for any $l \in \mathbb{N}$ and $u \in C^\infty(U)$ satisfies $u(0,0) > 0$.

From [12.1], the decisive curve $C_G$ is composed of $(m+1)$ branches parametrized as: for $t \in \mathbb{R}$ with small $|t|$, 

\[
B_0 : \quad x = 0, \quad y = t, \\
B_k : \quad x = t, \quad y = \gamma_k(t) \quad \text{for} \quad k = 1, \ldots, m.
\]

Roughly speaking, for the meromorphic extension of $Z(s)$, the branch $B_0$ gives good influence; while the branches $B_k$ with $\gamma_k \not\equiv 0$ may give bad one. In the analysis of $Z(s)$, the integral region is divided into two parts: one is avoided from the bad branches as large as possible, while the other is its complement. Of course, their shapes must be chosen so that they will be effective for the computation. Actually, we use the two regions with a parameter $p$ which is a large even integer:

\[
U_1^{(p)} = \{(x,y) \in U : |x| \leq r_p, \quad y > x^p\}, \\
U_2^{(p)} = \{(x,y) \in U : |x| \leq r_p, \quad 0 < y \leq x^p\},
\]

where $r_p > 0$ will be appropriately decided later in (12.5).

### 12.2. Two expressions of $G$

In order to understand important properties of $G$, we express $G$ by using the two functions $G_1 : U \setminus \{y = 0\} \to \mathbb{R}$ and $G_2 : U \to \mathbb{R}$ defined by

\[
G_1(x,y) = u(x,y) \left(1 + \sum_{j=1}^{m} \frac{\tilde{\varepsilon}_{m-j}(x)}{y^j}\right), \\
G_2(x,y) = u(x,y) \left(y^m + \sum_{j=0}^{m-1} \tilde{\varepsilon}_j(x)y^j\right),
\]

where $\tilde{\varepsilon}_j(x) := \varepsilon_j(x)/x^a$ for $j = 0, \ldots, m-1$. Note that each $\tilde{\varepsilon}_j$ is a flat $C^\infty$ function defined near the origin. Then $G$ can be expressed as

\[
G(x,y) = x^a y^m G_1(x,y) \quad \text{on} \quad U \setminus \{y = 0\}, \\
G(x,y) = x^a G_2(x,y) \quad \text{on} \quad U.
\]

In order to investigate $G$, we use $G_1$ on $U_1^{(p)}$ and $G_2$ on $U_2^{(p)}$.

### 12.3. Properties of $G_1$

Let $\gamma$ be the real-valued function defined near the origin by

\[
\gamma(x) = \max\{|\gamma_k(x)| : k = 1, \ldots, a\},
\]

where $\gamma_k$ are as in [12.1]. Since each $\gamma_k(x)$ satisfies that $\gamma_k(x) = O(x^l)$ for any $l \in \mathbb{N}$, it is easy to see that for each $p \in \mathbb{N}$ there exists a positive number $r_p$ such that

\[
\gamma(x) \leq \frac{1}{2} |x|^p \quad \text{for} \quad x \in [-r_p, r_p].
\]
We take the value of $r_p$ in (12.2) such that (12.5) holds.

**Lemma 12.1.** The function $G_1$ satisfies the following properties.

(i) There exists a positive number $c$ independent of $p$ such that $G_1(x,y) \geq c$ on $U_1^{(p)}$.

(ii) $\frac{\partial^{\alpha+\beta} G_1}{\partial^x \partial^y} (x,y)$ can be continuously extended to $\overline{U_1^{(p)}}$, the closure of the set $U_1^{(p)}$, for all $\alpha, \beta \in \mathbb{Z}_+$. 

**Proof.** (i) From (12.1) and (12.4), $G_1$ takes the following form on $U_1^{(p)}$:

\[
G_1(x,y) = u(x,y) \prod_{k=1}^m \left( 1 - \frac{\gamma_k(x)}{y} \right).
\]

From (12.5), we easily see that $|\gamma_k(x)| \leq \frac{x^p}{2y} \leq \frac{1}{2}$ for $(x,y) \in U_1^{(p)}$, which implies that $G_1(x,y) \geq \frac{u(0,0)}{2^a}$ for $(x,y) \in U_1^{(p)}$.

(ii) Since $G_1$ is a $C^\infty$ function on $U \setminus \{y = 0\}$, it suffices to show that every partial derivative of $G_1$ can be continuously extended to the origin. Moreover, from the first equation in (12.3), it suffices to show that every partial derivative of $\psi_j(x,y) := \tilde{\varepsilon}_{m-j}(x)/y^j$ can be continuously extended to the origin.

Let $(\alpha, \beta) \in \mathbb{Z}_+^2$ be arbitrarily given. A direct calculation gives

\[
\frac{\partial^{\alpha+\beta} \psi_j}{\partial^x \partial^y} = (-1)^\alpha \frac{(j+\alpha-1)! \tilde{\varepsilon}^{(\beta)}_{m-j}(x)}{(j-1)!} \frac{\varepsilon^{(\beta)}_{m-j}(x)}{y^{j+\alpha}},
\]

where $\tilde{\varepsilon}^{(\beta)}_{m-j}$ is the $\beta$-th derivative of $\tilde{\varepsilon}_{m-j}$. Since $\tilde{\varepsilon}^{(\beta)}_{m-j}$ is flat at the origin, there exists $r_{p,\alpha,\beta} > 0$ such that $|\tilde{\varepsilon}^{(\beta)}_{m-j}(x)| \leq \frac{(j-1)!}{(j+\alpha-1)!} x^{p(j+\alpha+1)}$ for $x \in [-r_{p,\alpha,\beta}, r_{p,\alpha,\beta}]$ and $j = 1, \ldots, m$. Considering the shape of the set $U_1^{(p)}$, we have

\[
\left| \frac{\partial^{\alpha+\beta} \psi_j}{\partial^x \partial^y} \right| \leq y \text{ on } U_1^{(p)} \cap \{|x| \leq r_{p,\alpha,\beta}\} \text{ for } j = 1, \ldots, m.
\]

Therefore,

\[
\lim_{U_1^{(p)} \ni (x,y) \to (0,0)} \frac{\partial^{\alpha+\beta} \psi_j}{\partial^x \partial^y} (x,y) = 0 \text{ for } j = 1, \ldots, m.
\]

In particular, $\frac{\partial^{\alpha+\beta} \psi_j}{\partial^x \partial^y}$ can be continuous up to the origin for $j = 1, \ldots, m$. ☐

**Lemma 12.2.** There exists a $C^\infty$ function $\tilde{G}_1$ on $U$ such that
(i) the restriction of $\tilde{G}_1$ to $U_1^{(p)}$ is $G_1$;
(ii) $\tilde{G}_1 \geq c/2$ on $U$, where $c$ is as in Lemma 12.1.

Proof. It follows from the property (ii) in Lemma 12.2 that $G_1$ can be smoothly extended to the region $U$ by using the Whitney extension theorem ([46], [40]). Furthermore, since $G_1 \geq c$ on $U_1^{(p)}$, the above $C^\infty$ extension can be performed so that $\tilde{G}_1 \geq c/2$ on $U$. □

12.4. Properties of $G_2$. When a van der Corput-type lemma in Lemma 11.4 is applied in the next section, the following lemma is important.

**Lemma 12.3.** There exist positive real numbers $R$ and $\mu$ such that
$$\frac{\partial^m}{\partial y^m} G_2(x,y) \geq \mu \quad \text{on } [-R, R]^2.$$

Proof. A direct computation gives
$$\frac{\partial^m}{\partial y^m} G_2(0,0) = m! u(0,0) \geq 0,$$
which implies the lemma. □

13. Proof of Theorem 9.1

For $p \in \mathbb{N}$, let $U_1^{(p)}$ ($j = 1, 2$) be as in (12.2) and let $r_p$ be a positive constant determined by (12.5).

13.1. A decomposition of $Z(s)$. Let $\chi_p : \mathbb{R} \to [0, 1]$ be a cut-off function satisfying that $\chi_p(x) = 1$ if $|x| \leq r_p/2$ and $\chi_p(x) = 0$ if $|x| \geq r_p$. By using $\chi_p$, the integral $Z(s)$ in (9.7) can be decomposed as

$$Z(s) = I_1^{(p)}(s) + I_2^{(p)}(s) + J^{(p)}(s)$$

with

$$I_j^{(p)}(s) = \int_{U_j^{(p)} \cap \mathbb{R}^2} |G(x,y)|^s x^b \phi(x,y) \chi_p(x) dx dy \quad \text{for } j = 1, 2,$$

$$J^{(p)}(s) = \int_{\mathbb{R}^2} |G(x,y)|^s x^b \phi(x,y)(1 - \chi_p(x)) dx dy.$$

13.2. Meromorphy of the integrals $I_1^{(p)}(s), I_2^{(p)}(s), J^{(p)}(s)$. In order to prove Theorem 9.1, it suffices to show the following.

**Lemma 13.1.** Let $p \in \mathbb{N}$. If the support of $\phi$ is contained in $[-R, R]^2$ where $R > 0$ is as in Lemma 11.2, then the following hold:
(i) The integral $I_1^{(p)}(s)$ becomes a meromorphic function to the whole complex plane. Moreover, its poles exist in the set
\[ \left\{ -\frac{b+j}{a}, -\frac{k}{m}, -\frac{b+j+pk}{a+pm} : j, k \in \mathbb{N} \right\}. \]

(ii) The integral $I_2^{(p)}(s)$ becomes a holomorphic function to the half-plane $\text{Re}(s) > -(p+b+1)/(mp+a)$.

(iii) The integral $J^{(p)}(s)$ becomes a holomorphic function to the half-plane $\text{Re}(s) > -1/m$.

Proof. (i) From (12.4), $I_1^{(p)}(s)$ can be expressed as
\[ I_1^{(p)}(s) = \int_{U_1^{(p)} \cap \mathbb{R}_+^2} x^{as+b} y^{ms} \Psi(x,y;s) dxdy, \]
where $\Psi : U \times \mathbb{C} \to \mathbb{C}$ is defined by
\[ \Psi(x,y;s) = \tilde{G}_1(x,y)^s x^b \phi(x,y) \chi_p(x). \]
We remark that $\tilde{G}_1 = G_1$ on $U_1^{(p)}$.

Lemma 13.2. The function $\Psi$ satisfies the following properties.

(i) $\Psi(\cdot, s)$ is a $C^\infty$ function on $U$ for all $s \in \mathbb{C}$.

(ii) $\frac{\partial^{\alpha+\beta} \Psi}{\partial x^\alpha \partial y^\beta}(x,y;\cdot)$ is an entire function for all $(x,y) \in U_1^{(p)}$ and $(\alpha, \beta) \in \mathbb{Z}_+^2$.

Proof. Since $x^b \phi(x,y) \chi_p(x)$ does not give any essential influence, it suffices to show that the function $\tilde{G}_1(x,y)^s$ satisfies the properties (i), (ii) in the lemma.

Every partial derivative of $\tilde{G}_1(x,y)^s$ with respect to $x, y$ can be expressed as the sum of $s(s-1) \cdots (s-k+1)\tilde{G}_1(x,y)^{s-k}$ for $k \in \mathbb{N}$ multiplied by polynomials of the partial derivatives of $\tilde{G}_1(x,y)$ with respect to $x, y$. Applying Lemma 12.2 to this expression, we can easily see that $\tilde{G}_1(x,y)^s$ satisfies the properties in the lemma.

Since $\Psi$ satisfies the same properties as those of $\Phi$ in Lemma 11.2, the integral $I_1^{(p)}(s)$ can be analytically continued as a meromorphic function to the whole complex plane and, moreover, its poles are contained in the set
\[ \left\{ -\frac{b+j}{a}, -\frac{k}{m}, -\frac{b+j+pk}{a+pm} : j, k \in \mathbb{N} \right\}. \]

(ii) From (12.4), $I_2^{(p)}(s)$ can be expressed as
\[ I_2^{(p)}(s) = \int_{U_2^{(p)} \cap \mathbb{R}_+^2} x^{as+b} |G_2(x,y)|^s \varphi(x,y) \chi_p(x) dxdy. \]
It is easy to see that

\begin{equation}
|I_2^{(p)}(s)| \leq C_p \int_0^{R_p} x^{a \text{Re}(s)+b} \left( \int_0^{x_p} |G_2(x, y)|^{\text{Re}(s)} dy \right) dx,
\end{equation}

where \( C_p := \sup_{(x, y) \in U_2^{(p)}} (|\varphi(x, y)\chi_p(y)|). \) Since Lemma 11.4 can be applied to the integral with respect to the variable \( x \) in (13.4) from Lemma 12.3, if \( \text{Re}(s) > -1/m \), then

\begin{equation}
|I_2^{(p)}(s)| < C_p \mathcal{C}(\text{Re}(s), a)^{\mu_{\text{Re}(s)}} \int_0^{R_p} x^{a \text{Re}(s)+b} (x^p)^{1+m\text{Re}(s)} dx
\end{equation}

where \( \mathcal{C}(\cdot, \cdot) \) is as in Lemma 11.4 and \( \mu \) is as in Lemma 12.3. The second integral in (13.5) converges on the half-plane \( \text{Re}(s) > -(p + b + 1)/(mp + a) \), on which \( I_2^{(p)}(s) \) becomes a holomorphic function.

(iii) In a similar fashion to the case of integral \( I_2^{(p)}(s) \), we have

\begin{equation}
|J^{(p)}(s)| \leq \tilde{C}_p \int_{R_p/2}^{R} x^{a \text{Re}(s)+b} \left( \int_0^{x} |G_2(x, y)|^{\text{Re}(s)} dy \right) dx,
\end{equation}

where \( \tilde{C}_p := \sup_{(x, y) \in [0, R] \times [R_p/2, R]} (|\varphi(x, y)(1-\chi_p(x))|) \). Applying Lemma 11.4, we can show that if \( \text{Re}(s) > -1/m \), then

\begin{equation}
|J^{(p)}(s)| \leq \tilde{C}_p \mathcal{C}(\text{Re}(s), a)^{\mu_{\text{Re}(s)}} R^{1+m\text{Re}(s)} \int_{R_p/2}^{R} x^{a \text{Re}(s)+b} dx.
\end{equation}

Since the above integral converges for any \( s \in \mathbb{C} \), \( J^{(p)}(s) \) can be analytically continued as a holomorphic function to the half-plane \( \text{Re}(s) > -1/m \). \( \square \)

13.3. Proof of Theorem 9.1. From (13.1), (13.2), Lemma 13.1 gives Theorem 9.1 by letting \( p \) tend to infinity.

14. Some comments

14.1. Revisited Section 2 by means of formal power series. Since many concepts related to \( f \in \mathbb{R} C^\infty((x, y)) \) in Section 2 can be determined by its formal Taylor series only, we will rewrite these concepts and the related results by using the formal series \( f \in \mathbb{R}[[(x, y)]] \), where \( f \) is the Taylor series of \( f \).

For \( f \in \mathbb{R} C^\infty((x, y)) \), let \( \rho(f) \) be the Taylor series of \( f \). From this, the map \( \rho : \mathbb{R} C^\infty((x, y)) \to \mathbb{R}[[(x, y)]] \) is defined, which is called the Borel map. It follows from E. Borel’s theorem that the map \( \rho \) is surjective. On the other hand, the restriction of \( \rho \) to \( \mathbb{R}\{x, y\} \) is injective, but \( \rho \) itself is not so. This subtle difference is important in our investigation.
The Newton polygon $\tilde{\Gamma}_+(f)$ of $f \in \mathbb{R}[[x, y]]$ can be well-defined by $\tilde{\Gamma}_+(f) = \Gamma_+(f)$ where $\rho(f) = f$. Furthermore, the maps $\tilde{\delta}_0, \tilde{\mu}_0 : \mathbb{R}[[x, y]] \to \mathbb{R}_+$ can be also well-defined by

$$\tilde{\delta}_0(f) := \delta_0(f), \quad \tilde{\mu}_0(f) := \mu_0(f) \quad \text{for } f \in \mathbb{R}[[x, y]],$$

where $\rho(f) = f$.

Next, let us consider the quantities $h_0(f)$ and $m_0(f)$. Theorem 2.2, given by Greenblatt [17], shows that $h_0(f)$ is determined by the formal Taylor series of $f$ only, which implies that the map $\tilde{\delta}_0$ can be well-defined by

$$\tilde{\delta}_0(f) := \delta_0(f), \quad \tilde{\mu}_0(f) := \mu_0(f) \quad \text{for } f \in \mathbb{R}[[x, y]],$$

where $\rho(f) = f$. Theorem 2.2 can be rewritten as Theorem 14.1.

$$\tilde{\delta}_0(f) = 1 / \tilde{\mu}_0(f) \text{ holds for every } f \in \mathbb{R}[[x, y]] \text{ with } f \neq 0.$$

On the other hand, the example (2.11) implies that $m_0(f)$ is not always determined by the formal Taylor series of $f$ only (see Remark 2.3). Now, we define the map $\tilde{m}_0 : \mathbb{R}[[x, y]] \to \mathbb{R}_+$ by

$$\tilde{m}_0(f) := \inf \{ m_0(f) : \rho(f) = f \} \quad \text{for } f \in \mathbb{R}[[x, y]].$$

From Theorem 2.5 (ii), the following problem is naturally raised.

**Problem 14.1.** Does $\tilde{m}_0(f) = 1 / \tilde{\mu}_0(f)$ hold, if $\tilde{\mu}_0(f) \geq 2$?

If the above problem is affirmatively solved, then a strong optimality of the inequality in Theorem 2.5 (ii) is shown. It is shown in [30], [33] that if the Newton polygon of $f$ takes the form \{$(\alpha, \beta) : \alpha \geq a, \ beta \geq b$\} where $a, b \in \mathbb{N}$ satisfy $2 \leq a < b$, then the above equality holds.

14.2. **Non-polar singularity of local zeta functions.** From Theorem 2.5 and the example (2.11), it is possible that local zeta functions have a singularity different from the pole if $\mu_0(f) \geq 2$. It is an interesting issue to investigate what kinds of singularities these local zeta functions have. At present, there seems to be no investigation of this issue except one in specific cases in [30], [33]. Let us roughly explain the situation which has been known in the above papers.

In [30], the behavior at $s = -1 / \mu_0(f)(= -1 / b)$ of the restriction of the local zeta function associated with (2.11) to the real axis is exactly computed. (We assume that the parameters $a, b, p, q$ in (2.11) satisfy the conditions in Section 2.3.) Note that this example satisfies the condition $\mu_0(f) = \delta_0(f)(= b)$. Though it depends on the parameters, the above behavior is different from that at the poles in any cases, which implies that the local zeta function associated with (2.11) has a non-polar singularity at the point $s = -1 / b$. Unfortunately, the above information cannot determine whether this singularity is isolated or not. If this singularity was isolated, then it must be an essential singularity. On the other hand, we cannot deny that the line $\text{Re}(s) = -1 / \mu_0(f)$ is the natural boundary for its local zeta function.
More recently, T. Nose shows the existence of a $C^\infty$ function $f$ with $\mu_0(f) < \delta_0(f)$ such that its local zeta function has a non-polar singularity at the point $s = -1/\mu_0(f)$.

It is expected to understand more detailed properties of non-polar singularities of local zeta functions in the future.

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