On the corona theorem on smooth curves

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Abstract

We prove the corona theorem for domains whose boundary lies in certain smooth quasicircles. These curves, which are not necessarily Dini-smooth, are defined by quasiconformal mappings whose complex dilatation verifies certain conditions. Most importantly we do not assume any “thicknes” condition on the boundary domain. In this sense, our results complement those obtained by Garnett and Jones (1985) and C. Moore on $C^{1+\alpha}$ curves (1987).

1. Introduction

Let $\Omega \subset \mathbb{C}$ be a domain in the complex plane and let $H^\infty(\Omega)$ be the space of bounded analytic functions in $\Omega$.

The first corona theorem was proved by Carleson for simply connected domains \cite{1}. Denote by $\mathbb{D}$ the open unit disk.

\textbf{Theorem (Carleson).} Let $f_1(z), \ldots, f_n(z)$ be given functions in $H^\infty(\mathbb{D})$ and verifying that

$$|f_1(z)| + |f_2(z)| + \ldots + |f_n(z)| \geq \delta > 0,$$

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for some $0 < \delta \leq 1/2$. Suppose that $\|f_k\|_\infty \leq 1$, $k = 1, 2, \ldots, n$. Then, there exist $\{g_k\}_{k=1}^n \in H^\infty(\mathbb{D})$ so that:

$$\sum_{k=1}^n f_k g_k = 1 \quad \text{and} \quad \|g_k\|_\infty \leq C(n, \delta).$$

The functions $\{f_k\}$ and $\{g_k\}$ are called corona data and corona solutions respectively, and $\delta$ and $n$ are the corona constants.

In his proof, Carleson introduced what is known as Carleson measures, a fundamental tool in complex and harmonic analysis.

The next breakthrough for infinitely connected domains [3] is also due to Carleson. He used the relation between interpolating sequences, boundary thickness and the Cauchy transform to prove the corona theorem for homogeneous Denjoy domains, that is, for domains with boundary $E \subset \mathbb{R}$ (Denjoy domain) such that:

$$|(x - r, x + r) \cap E| > \varepsilon_0 r \quad \text{for all } x \in E \quad \text{and all } r > 0.$$

Newdelman extended this result to domains $\Omega = \mathbb{C} \setminus E$ where $E$ is a homogeneous set contained in a Lipschitz graph [13]. The idea was to divide $\Omega$ into two overlapping simply connected regions, $\tilde{\Omega}^+$ and $\tilde{\Omega}^-$. On each region, he used Carleson’s simply connected result to obtain regional corona solutions by an iterating method. On each iteration, a particular $\bar{\partial}$ equation was solved to modify $\{g_j^\pm\}$ so that $\max |g_j^+(z) - g_j^-(z)|$ was reduced in the overlap of the regions. See more results on corona theorem in [10] and the references within.

The first result for domains $\Omega = \mathbb{C} \setminus E$ that did not assume the homogeneous condition on the set $E$ was proved by Garnett and Jones on Denjoy domains, that is $E \subset \mathbb{R}$ [3].

Moore [12] extended the corona theorem for domains $\Omega = \mathbb{C} \setminus E$, with $E$ lying in a $C^{1+\alpha}$ curve. For that, he first proved that Cauchy integrals on a $C^{1+\alpha}$ curve behave locally like Cauchy integrals along straight lines and then used Garnett and Jones’ solutions on Denjoy domains.

Moore’s result was proved again in [4] by considering quasiconformal mappings. In fact, if $f$ is a conformal mapping from the upper plane $\mathbb{R}^2_+$ onto the
complex plane, then $\Gamma = f(\mathbb{R})$ is a $C^{1+\alpha}$ curve if and only if $f$ extends to a global quasiconformal map whose dilatation $\mu$ satisfies that $\frac{|\mu|^2}{|y|^{1+\epsilon}}dxdy$ is a Carleson measure relative to $\mathbb{R}$ \([3]\). This characterization is then used to show that $H^\infty(\Omega)$ functions are close to $H^\infty$ functions on Denjoy domains and to obtain local solutions from the Garnett and Jones’ solutions. Both proofs of corona theorem for $C^{1+\alpha}$ curves \([12], [4]\) can be extended to Dini-smooth curves with slight modifications.

A natural question would be to extend these results, where no condition on the homogeneity of the set $E$ is required, to more general curves, such as smooth curves, that is, to Jordan curves, $\Gamma$, for which there is a parametrization $f : \mathbb{R} \to \mathbb{C}$, with $f'$ continuous and $\neq 0$.

This paper presents the corona theorem for domains $\Omega = \mathbb{C}\setminus E$, where $E$ lies in certain smooth curves $\Gamma$. More precisely, we will consider quasicircles which are images of $\mathbb{R}$ under a global quasiconformal mapping of the the complex plane, $\rho$, whose complex dilatation, $\mu$, has compact support and verifies one of these two conditions:

1. The complex dilatation $\mu$ satisfies condition 1 if:

   $$\int_0^{\mu^*(t)} \frac{\log \left( \frac{1}{|t|} \right)}{|t|} dt < \infty, \quad (1)$$

   where $\mu^*(t) = \text{esssup}\{|\mu(z)| : 0 < |\text{Im}(z)| < |t|\}$ is the monotonic majorant of $\mu$.

2. The complex dilatation $\mu$ satisfies condition 2 if:

   $$\int_{\mathbb{R}} \frac{\sigma(y)}{|y|^{3/2}} dy < \infty, \quad (2)$$

   where $\sigma(y)$ is defined a.a. $y \in \mathbb{R}$ as $\sigma(y) = \left( \int_{\mathbb{R}} |\mu(x+iy)|^2 dx \right)^{1/2}$, and there exists $C > 0$ so that:

   $$|\mu(z_0)| \lesssim \int_{|z-z_0|<C|\text{Im}(z_0)|} |\mu(z)| dxdy, \quad \forall z_0 \in \mathbb{C}\setminus \mathbb{R}. \quad (3)$$

We will show that, in both cases, there exists $M > 0$ such that the Teiehmüller-Wittich-Belinski integral

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\[
\int_{\mathbb{C}} \frac{|\mu(z + t)|}{|z|^2} \, dx \, dy < M,
\]
for every \( t \in \mathbb{R} \) and, therefore, \( \Gamma = \rho(\mathbb{R}) \) is a smooth curve ([11], Corollary 1.6).

To prove our results, we will follow a similar argument as in [5]. This approach, which had been previously developed by Semmes in [16], will allow us to relate \( H^\infty(\Omega) \) to \( H^\infty \) on Denjoy domains.

**Theorem 1.** Let \( \rho \) be a global quasiconformal mapping of the complex plane, conformal at \( \infty \) and with complex dilatation \( \mu \) verifying either condition 1 or condition 2. Denote \( \Gamma = \rho(\mathbb{R}) \). Then, given a function \( f \in L^\infty(\Gamma) \), the Cauchy integral \( C_{\Gamma}(f) \in L^\infty(\mathbb{C}) \) if and only if \( C_{\Gamma}(g) \in L^\infty(\mathbb{C}) \), where \( g \) is the pullback of \( f \) under the quasiconformal mapping, \( g = f \circ \rho \).

We can now state our main result on corona theorem for both sets of curves. For that, we consider domains \( \Omega = \mathbb{C} \setminus E \) where \( E \) is a compact set with positive length contained in a quasicircle \( \Gamma = \rho(\mathbb{R}) \), analytic at \( \infty \), such that the complex dilatation of the quasiconformal mapping \( \rho \) satisfies either [11] or [2] and [3].

**Theorem 2.** With the notation above, let \( f_1, f_2, \ldots, f_n \in H^\infty(\Omega) \) so that \( \delta \leq \max_k |f_k(\omega)| \leq 1 \) for all \( \omega \in \Omega \) and some \( \delta > 0 \). Then, there exist \( g_1, g_2, \ldots, g_n \in H^\infty(\Omega) \) such that \( f_1g_1 + f_2g_2 + \ldots + f_ng_n = 1 \) on \( \Omega \).

Note that no condition on the homogeneity of the set \( E \) is assumed in Theorem 2.

The paper is structured as follows. In section 2, we review some basic definitions and facts. We prove theorem 1 in section 3 and theorem 2 in section 4. Finally, in section 5, an example of this sort of smooth curves which is not Dini-smooth is presented.
2. Preliminaries

Let us denote complex variables by \( z = x + iy \) and \( \omega = u + iv \). We shall use the following notation throughout this article: \( \mathbb{D} = \{ z : |z| < 1 \} \), \( |E| \) represents the Lebesgue measure of any set \( E \), \( \delta_{\Gamma}(\omega) \) the distance from the point \( \omega \) to the curve \( \Gamma \), \( \text{diam}(E) \) the diameter of a set \( E \) and \( H^\infty(\Omega) \) is the space of bounded analytic functions on \( \Omega \). Also, we shall write \( \overline{\partial} = \partial/\partial \bar{z} = 1/2(\partial_x - i\partial_y) \) and \( \partial = \partial/\partial z = 1/2(\partial_x + i\partial_y) \). For a square \( Q \), we will denote by \( l(Q) \) its length and we will use \( x \lesssim y \) as shorthand for the inequality \( x \leq Cy \) for some constant \( C \).

Given a function \( f \) on a rectifiable curve \( \Gamma \), define its Cauchy integral \( F(z) = C_\Gamma(f)(z) \) off \( \Gamma \) by:

\[
F(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\omega)}{\omega - z} d\omega, \quad z \notin \Gamma.
\]

We define the jump of \( F \) across \( \Gamma \) at a point \( z \), \( j(F)(z) \), as \( F_+(z) - F_-(z) \), where \( F_\pm \) denote the boundary values of \( F \). As the classical Plemelj formula states,

\[
F_\pm(z) = \pm \frac{1}{2} f(z) + \frac{1}{2} P.V. \int_\Gamma \frac{f(\omega)}{\omega - z} d\omega, \quad z \in \Gamma.
\]

Hence, \( F_+(z) - F_-(z) = j(F)(z) = f(z) \).

Consider \( \rho \) a global quasiconformal mapping of the complex plane with complex dilatation \( \mu \). Thus, \( \rho \) is a homeomorphism with locally integrable distributional derivatives verifying that \( \overline{\partial}\rho - \mu\partial\rho = 0 \), \( \mu \in L^\infty(\mathbb{C}) \) and \( \|\mu\|_\infty < 1 \). Suppose that \( \rho \) is conformal at \( \infty \), with \( \rho(\mathbb{R}) = \Gamma \) a rectifiable quasicircle. Let \( \Omega = \mathbb{C}\setminus E \), where \( E \subset \Gamma \) is a compact set with positive length. Define \( \Omega_0 = \rho^{-1}(\Omega) \) and \( E_0 = \rho^{-1}(E) \). Note that \( E_0 \) is a compact set of positive measure ([14], Theorem 6.8).

Define the space:

\[
H^\infty(\Omega_0, \mu) = \{ f \circ \rho : f \in H^\infty(\Omega) \}.
\]

Observe that if \( g \in H^\infty(\Omega_0, \mu) \), then \( \overline{\partial}f = 0 \) on \( \Omega \) translates into \( (\overline{\partial} - \mu\partial)g = 0 \) on \( \Omega_0 \) and the jump of \( g \) across \( E_0 \) is given by \( j(g) = j(f) \circ \rho \).
We review some facts which follow from Semmes’ approach in \cite{16}. Let \( f \in H^\infty(\Omega) \) and \( g = f \circ \rho \). Consider the jump of \( g \), \( j(g) \), and set \( \tilde{g} = C_\mathbb{R}(j(g)) \). If we define \( H = g - \tilde{g} \), then \( \partial H = \mu \partial g \) on \( \Omega_0 \) and since \( H \) has no jump across \( E_0 \), we can consider that this equation holds on all \( \mathbb{C} \) in the sense of distributions. For more details see \cite{3}. We can then apply Cauchy’s formula to obtain:

\[
H(z_0) = -\frac{1}{\pi} \int_\mathbb{C} \frac{\partial H}{z - z_0} \, dx \, dy = -\frac{1}{\pi} \int_\mathbb{C} \frac{\mu(z) \partial g(z)}{z - z_0} \, dx \, dy \quad \text{for all } z_0 \in \mathbb{C}.
\]

The modulus of continuity of a function \( f \) on \( \mathbb{R} \) is defined by:

\[
\omega_f(\delta) = \sup\{ |f(x_1) - f(x_2)| : x_1, x_2 \in \mathbb{R}, |x_1 - x_2| \leq \delta \}.
\]

The function \( f \) is called Dini-continuous if

\[
\int_0^T \frac{\omega_f(t)}{t} \, dt < \infty.
\]

We say that a closed Jordan curve \( \Gamma \) is Dini-smooth if it has a parametrization \( f(\tau), 0 \leq \tau \leq 2\pi \), such that \( f'(\tau) \) is Dini-continuous and \( \neq 0 \) (see \cite{14}, section 3.3 for further results).

Recall that for \( f : \Omega \to \Omega' \) a quasiconformal mapping between domains \( \Omega \) and \( \Omega' \) in \( \mathbb{R}^2 \), the well known Ghering’s result \cite{9} ensures that the Jacobian \( J_f \) of \( f \) satisfies the reverse Hölder inequality:

\[
\left( \frac{1}{|Q|^p} \int_Q J_f^p \, dx \, dy \right)^{1/p} \leq C \int_Q J_f \, dx \, dy,
\]

for some \( p > 1 \), where \( Q \) is a cube in \( \Omega \) such that \( 2Q \subset \Omega \) and where \( \int_Q \) stands for \( \frac{1}{|Q|} \int_Q \).

3. Proof of Theorem 1

We begin this section by proving that the quasicircles defined by quasiconformal mappings with complex dilatation verifying condition 1 or 2 are, indeed, smooth curves. By \cite{14} this result is an immediate consequence of the following proposition:
Proposition. If \( \mu \) verifies (1) or (2), then there exists \( M > 0 \) such that for all \( a \in \mathbb{R} \)
\[
\int_{\mathbb{C}} \frac{|\mu(z)|}{|z - a|} \frac{dx \, dy}{|y|} < M.
\]

Proof. If \( \mu \) verifies condition 1 and \( \text{supp}(\mu) \subset B(0, R) \), where \( B(0, R) \) is the ball centered at 0 and radius \( R \) for some \( R > 0 \), then for any \( a \in \mathbb{R} \)
\[
\int_{\mathbb{C}} \frac{|\mu(z)|}{|z - a|} \frac{dx \, dy}{|y|} \lesssim \int_{-R}^{R} \frac{|\mu^*(y)|}{|y|} \left( \int_{-R}^{R} \frac{1}{|x - a| + |y|} \, dx \right) \, dy \lesssim \int_{-R}^{R} \frac{|\mu^*(y)|}{|y|} \left( \int_{-R}^{R} \frac{1}{|x| + |y|} \, dx \right) \, dy \lesssim \int_{0}^{R} \frac{|\mu^*(y)|}{|y|} \log \left( \frac{1}{|y|} \right) \, dy < \infty.
\]

Let us assume next that \( \mu \) verifies (2). Then for any \( a \in \mathbb{R} \):
\[
\int_{\mathbb{C}} \frac{|\mu(z)|}{|z - a|} \frac{dx \, dy}{|y|} = \int_{R}^{R} \frac{1}{|y|} \left( \int_{R}^{R} \frac{|\mu(z)|}{|z - a|} \, dx \right) \, dy \leq \int_{R}^{R} \frac{1}{|y|} \left( \int_{R}^{R} \frac{|\mu(z)|^2}{|y|} \, dx \right)^{1/2} \left( \int_{R}^{R} \frac{|y|}{|z - a|^2} \, dx \right)^{1/2} \, dy \lesssim \int_{R}^{R} \frac{\sigma(y)}{|y|^{3/2}} \, dy < +\infty.
\]

Consider \( f \in L^\infty(\Gamma) \) so that \( C_\Gamma(f) \in L^\infty(\mathbb{C}) \) and let \( g = f \circ \rho \) be the pullback of \( f \) under the quasiconformal mapping.

Following Semmes’ approach as described in the previous section, set \( G = C_\Gamma(f) \circ \rho \) and \( H = G - C_R(g) \). Since \( H \) has no jump across \( E_0 \) and \( \mu \) has compact support:
\[
H(z_0) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\mu(z) \partial G(z)}{z - z_0} \, dx \, dy \quad \text{for all} \ z_0 \in \mathbb{C}.
\]

We will consider the Whitney decomposition associated to \( \mathbb{R}_+^2 \) and \( \mathbb{R}_-^2 \), that is, \( \mathbb{C} \setminus \mathbb{R} = \cup_k Q_k \), where the side length of the cube \( Q_k \), \( l(Q_k) \), is proportional to its distance to \( \mathbb{R} \). Denote by \( z_k \) the center of the cube \( Q_k \).
Proof for Theorem 1. Let $Q$ be a cube so that $\text{supp}(\mu) \subset Q$ and suppose that $C_\Gamma(f) \in L^\infty(\mathbb{C})$. For any $a \in \mathbb{R}$, $|z - a| \simeq |z_k - a|$ for $z \in Q_k$. Also, by the circular distortion theorem, $\delta_\Gamma(\rho(z)) \simeq \delta_\Gamma(\rho(z_k))$ for $z \in Q_k$.

Set $G(z) = C_\Gamma(f) \circ \rho$. Since $\partial G(z) = C'_\Gamma(f)(\rho(z))\partial \rho(z)$ and $C'_\Gamma(f) \lesssim C(\|f\|_\infty)/\delta_\Gamma(\rho(z))$, we get by (8):

$$|H(a)| \lesssim \sum_k \frac{\mu^*(3y_k/2)}{|z_k - a|} \frac{1}{\delta_\Gamma(\rho(z_k))} \int_{Q_k} |\partial \rho(z)| \, dx \, dy.$$

On the other hand:

$$\int_{Q_k} |\partial \rho(z)| \, dx \, dy \lesssim \left( \int_{Q_k} |\partial \rho(z)|^2 \right)^{1/2} l(Q_k) \simeq \text{diam}(\rho(Q_k)) l(Q_k).$$

Therefore, as the diameter of $\rho(Q_k)$ is comparable to $\delta_\Gamma(\rho(z_k))$, with comparison constants depending only on $\Gamma$:

$$|H(a)| \lesssim \sum_k \frac{\mu^*(3y_k/2)}{|z_k - a|} \frac{1}{\delta_\Gamma(\rho(z_k))} \text{diam}(\rho(Q_k)) l(Q_k)$$

$$\simeq \int_Q \frac{\mu^*(y)}{|y|} \frac{1}{|z - a|} \, dx \, dy \simeq \int_{l(Q)/2}^{l(Q/2)} \frac{\mu^*(t)}{|t|} \log \left( \frac{1}{|t|} \right) \, dt < \infty.$$

This proves that $H|_{\mathbb{R}} \in L^\infty(\mathbb{R})$ if $\mu$ verifies condition 1.

Consider now that $\mu$ verifies condition 2 and denote by $B_z$ the ball centered at $z$ and radius $C|y|$, where $C$ is the constant given in (3). Then, for any $a \in \mathbb{R}$, by (8) and by (3):

$$|H(a)| \lesssim \int_C \left( \int_{B_z} |\mu(\omega)| \, du \right) \frac{|\partial G(z)|}{|z - a|} \, dx \, dy.$$

By Fubini’s theorem, we get:

$$|H(a)| \lesssim \int_C \frac{|\mu(\omega)|}{|w - a|} \left( \int_{Q_\omega} |\partial G(z)| \, dx \, dy \right) \, du \, dv,$$

where $Q_\omega$ is a cube containing a ball of size comparable to $B_\omega$. 

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As stated before, since $|\partial G(z)| \lesssim |\partial \rho(z)|/\delta_\Gamma(\rho(z))$ and, by the circular distortion theorem, $\delta_\Gamma(\rho(\omega)) \simeq \delta_\Gamma(\rho(z))$ for $z \in Q_\omega$, then

$$|H(a)| \lesssim \int_\mathbb{C} \frac{|\mu(\omega)|}{|\omega - a| \delta_\Gamma(\rho(\omega))} \left(\int_{Q_\omega} |\partial \rho(z)| \, dx \, dy\right) \, du \, dv.$$ 

On the other hand:

$$\int_{Q_\omega} |\partial \rho(z)| \, dx \, dy \lesssim \left(\int_{Q_\omega} |\partial \rho(z)|^2 \right)^{1/2} / l(Q_\omega) \simeq \text{diam}(\rho(Q_\omega)) / l(Q_\omega).$$

Finally, as the diameter of $\rho(Q_\omega)$ is comparable to its distance to $\Gamma$, with comparison constants depending only on $\Gamma$, and $l(Q_\omega) \simeq |\text{Im}(\omega)|$, we get by (6) that:

$$|H(a)| \lesssim \int_\mathbb{C} \frac{|\mu(\omega)| \, du \, dv}{|\omega - a| |v|} < \infty.$$ 

This proves now proves that $H_{|\mathbb{R}} \in L^\infty(\mathbb{R})$ if $\mu$ verifies condition 2.

In both cases, as $H = G - C_\mathbb{R}(g)$ and $G \in L^\infty(\mathbb{C})$, we obtain that $C_\mathbb{R}(g) \in L^\infty(\mathbb{C})$.

Conversely, if $C_\mathbb{R}(g)$ were bounded, the same argument would show that $G$ is bounded on $\mathbb{R}$ and that $C_\Gamma(f) \in H^\infty(\mathbb{C})$. \qed

4. Proof of Theorem 2

Before proceeding to the proof, let us recall the notation used in our setting. We consider a domain $\Omega = \mathbb{C} \setminus E$, where $E \subset \Gamma$ is a compact set of positive length contained in a quasicircle $\Gamma = \rho(\mathbb{R})$. We assume that the quasiconformal mapping $\rho : \mathbb{C} \to \mathbb{C}$ is conformal at $\infty$ and that $\mu_\rho$ satisfies either condition 1 or condition 2. We also define $\Omega_0 = \rho^{-1}(\Omega)$, $E_0 = \rho^{-1}(E)$, the space $H^\infty(\Omega_0, \mu) = \{ f \circ \rho : f \in H^\infty(\Omega) \}$ and the jump of functions in $H^\infty(\Omega)$ and $H^\infty(\Omega_0, \mu)$ as in the preliminaries.

Next we state the following lemma that will allow us to relate corona data on $\Omega$ to corona data on the Denjoy domain $\Omega_0$.
Lemma. Suppose that supp(μ) ⊂ Q for some square Q centered at a real point and that μ verifies either condition 1 or condition 2. Let g ∈ H^∞(Ω_0, μ) and ˜g ∈ H^∞(Ω_0) so that j(˜g) = j(˜g) and set H = g − ˜g. Then, for all z_0 ∈ ℂ, |H(z_0)| ≤ δ if l(Q) is small enough.

Proof. If μ verified condition 1, and from the proof of Theorem 1, we would get that for all z_0 ∈ ℜ:

$$|H(z_0)| \lesssim \int_{-l(Q)/2}^{l(Q)/2} \frac{\mu^*(t)}{|t|} \log \left( \frac{1}{|t|} \right) dt < \infty.$$ 

Therefore, |H(z_0)| ≤ δ/2 for all z_0 ∈ ℜ if supp(μ) is small enough.

Consider now z_0 ∈ ℂ\ℜ and let Q_0 be the Whitney cube centered at z_0. Then

$$|H(z_0)| \lesssim \int_{Q \cap Q_0} \frac{|\mu(z)||\partial g(z)|}{|z - z_0|} dx dy + \int_{Q \cap Q_0 \cap \partial Q} \frac{|\mu(z)||\partial g(z)|}{|z - z_0|} dx dy.$$ 

To estimate the first integral note that, for any z ∉ Q_0, |y_0| ≲ |z - z_0| and |z - x_0| ≤ |z - z_0| + |y_0| ≲ |z - z_0|. Then,

$$\int_{Q \cap Q_0} \frac{|\mu(z)||\partial g(z)|}{|z - z_0|} dx dy \lesssim \int_{Q \cap Q_0} \frac{|\mu(z)||\partial g(z)|}{|z - x_0|} dx dy \leq \int_{-l(Q)/2}^{l(Q)/2} \frac{\mu^*(t)}{|t|} \log \left( \frac{1}{|t|} \right) dt < \delta/2.$$ 

To bound the second term consider the exponent p > 1 in (5) and set p_0 = 2p. Denote q_0 = p_0/(p_0 - 1). Proceeding as in the proof of Theorem 1 we get by Hölder’s inequality and (5) that:

$$\int_{Q \cap Q_0} \frac{|\mu(z)||\partial g(z)|}{|z - z_0|} dx dy \lesssim \frac{\mu^*(3y_0/2)}{\delta_\Gamma(\rho(z_0))} \int_{Q \cap Q_0} \frac{1}{|z - z_0|^{q_0}} dx dy \lesssim \frac{\mu^*(3y_0/2)}{\delta_\Gamma(\rho(z_0))} |Q \cap Q_0|^{1/p_0 - 1/2} \left( \int_{Q \cap Q_0} |\partial g(z)|^{p_0} dx dy \right)^{1/p_0} \left( \int_{Q \cap Q_0} \frac{1}{|z - z_0|^{q_0}} dx dy \right)^{1/q_0}.$$ 

$$\lesssim \frac{\mu^*(3y_0/2)}{\delta_\Gamma(\rho(z_0))} |Q \cap Q_0|^{1/p_0 - 1/2} \left( \int_{Q \cap Q_0} \frac{1}{|z - z_0|^{q_0}} dx dy \right)^{1/2} \left( \int_{Q \cap Q_0} \frac{1}{|z - z_0|^{q_0}} dx dy \right)^{1/2}.$$ 

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But:

\[
\int_{Q \cap Q_0} \frac{1}{|z - z_0|^{q_0}} \, dx \, dy \leq \left( \int_{Q \cap Q_0} \frac{1}{|z - z_0|^{2q_0}} \, dx \, dy \right)^{1/2} |Q \cap Q_0|^{1/2} \\
\approx |y_0|^{1-q_0} |Q \cap Q_0|^{1/2}.
\]

Consider \(2l(Q) < 1/e\) so that \(\log (1/|y|) > 1\) for \(z \in Q\). We then finally get that:

\[
\int_{Q \cap Q_0} \frac{|\mu(z)||\partial g(z)|}{|z - z_0|} \, dx \, dy \lesssim \int_{l(Q)/2}^{l(Q)/2} \frac{\mu^*(t)}{|t|} \log \left( \frac{1}{|t|} \right) dt < \delta/2.
\]

Consider now \(\mu\) verifying condition 2. From the proof of Theorem 1 and (6):

\[
|H(z_0)| \lesssim \int_{\text{supp}(\mu)} \frac{|\mu(z)|}{|z - z_0|} \, dx \, dy \leq \int_{\mathbb{R}} \frac{\sigma(y)}{|y|^{3/2}} \, dy < \infty
\]

for any \(z_0 \in \mathbb{R}\). Therefore, and as the last integral does not depend on \(z_0\), \(|H(z_0)| \leq \delta\) for all \(z_0 \in \mathbb{R}\) if \(\text{supp}(\mu)\) is small enough.

For \(z_0 \in \mathbb{C}\setminus\mathbb{R}\),

\[
|H(z_0)| \lesssim \int_{Q \setminus Q_0} \frac{|\mu(z)||\partial g(z)|}{|z - z_0|} \, dx \, dy + \int_{Q \cap Q_0} \frac{|\mu(z)||\partial g(z)|}{|z - z_0|} \, dx \, dy = I + II, \quad (9)
\]

where \(Q_0\) is the Whitney cube centered at \(z_0\).

Following the same argument as in the previous case and by (6):

\[
I = \int_{Q \setminus Q_0} \frac{|\mu(z)||\partial g(z)|}{|z - z_0|} \, dx \, dy \lesssim \int_{Q \setminus Q_0} \frac{|\mu(z)||\partial g(z)|}{|z - x_0|} \, dx \, dy \leq \int_{l(Q)/2}^{l(Q)/2} \frac{\sigma(y)}{|y|^{3/2}} \, dy < \delta/2.
\]

Note now that \(|\partial g(z)| \lesssim |\partial \rho(z)|/\delta_\Gamma(\rho(z))\) and that, by the circular distortion theorem, \(\delta_\Gamma(\rho(z)) \simeq \delta_\Gamma(\rho(z_0))\) for \(z \in Q_0\). Then, if we apply Hölder’s inequality with exponents \(p_0 > 2\) and \(q_0\) as chosen in case 1, the second integral in (9) is bounded by
II \leq \frac{1}{\delta(\rho(z_0))} \left( \int_{Q\cap Q_0} |\mu(z)|^{p_0} |\partial \rho(z)|^{p_0} \, dx \, dy \right)^{1/p_0} \left( \int_{Q\cap Q_0} \frac{1}{|z - z_0|^{q_0}} \, dx \, dy \right)^{1/q_0} \\
\simeq |y_0|^\frac{2-q_0}{q_0} \delta(\rho(z_0)) \left( \int_{Q\cap Q_0} |\mu(z)|^{p_0} |\partial \rho(z)|^{p_0} \, dx \, dy \right)^{1/p_0}.

But by \cite{3} and as the Jacobian of $\rho$ satisfies the reverse Hölder inequality \cite{4}:

$$\left( \int_{Q\cap Q_0} |\mu(z)|^{p_0} |\partial \rho(z)|^{p_0} \, dx \, dy \right)^{1/p_0} \lesssim \left( \int_{2Q_0} |\mu(z)| \, dx \, dy \right)^{1/p_0} |Q\cap Q_0|^{1/p_0-1/2} \text{diam}(\rho(Q_0)),$$

and:

$$II \lesssim |y_0|^\frac{2-q_0}{q_0} \cdot |y_0|^{\frac{2}{p_0} - 1} \left( \int_{2Q_0} |\mu(z)| \, dx \, dy \right) = \int_{2Q_0} |\mu(z)| \, dx \, dy \\
\simeq \frac{1}{|2Q_0|} \int_{y_0+l(Q_0)}^{y_0-l(Q_0)} \left( \int_{x_0-l(Q_0)}^{x_0+l(Q_0)} |\mu(z)| \, dx \right) \, dy \lesssim \frac{1}{y_0^2} \int_{y_0-l(Q_0)}^{y_0+l(Q_0)} \sigma(y) \cdot l(Q_0)^{1/2} \, dy \\
\simeq \int_{y_0-l(Q_0)}^{y_0+l(Q_0)} \sigma(y)/|y|^{3/2} \, dy \leq \int_{|y|=l(Q_0)/2}^{l(Q_0)/2} \sigma(y)/|y|^{3/2} \, dy < \frac{\delta}{2}$$

as long as $Q$ is small enough.

We now prove Theorem 2 for the two settings. We will follow the same steps as in \cite{4} but for the sake of completeness we will reproduce all the details.

**Theorem 2.** Let $f_1, f_2, \ldots, f_n \in H^\infty(\Omega)$ so that $\delta \leq \max_k |f_k(\omega)| \leq 1$ for all $\omega \in \Omega$ and some $\delta > 0$. Then, there exist $g_1, g_2, \ldots, g_n \in H^\infty(\Omega)$ such that $f_1g_1 + f_2g_2 + \ldots + f_ng_n = 1$ on $\Omega$.

**Proof.** Gamelin proved that it is sufficient to solve the corona problem locally \cite{7}, i.e., that for any $\zeta \in \Gamma$ there exists a neighborhood of $\zeta$ on which it is true and such that the size of the neighborhood is determined by $\delta, n$ and other parameters concerning $\Gamma$ (see also \cite{8}, page 358).

We can then assume that $\mu(z) = 0$ outside a small enough square centered at a real point.
Let \( f_k^* = f_k \circ \rho \) be quasiregular functions defined on \( \Omega_0 \). Then, the jump of \( f_k^* \), \( j(f_k^*) \), is indeed, the pullback of \( j(f_k) \) under the mapping \( \rho \), that is, \( j(f_k^*) = j(f_k) \circ \rho \), where \( j(f_k) \) is the jump of \( f_k \) across \( E \). Note that \( f_1^*, \ldots, f_n^* \in H^\infty(\Omega_0, \mu) \).

Set the analytic functions \( \tilde{f}_k = C_H(j(f_k^*)) \). By theorem 1, \( \tilde{f}_k \in H^\infty(\Omega_0) \). To show that \( \{\tilde{f}_k\} \) are indeed corona data, define \( H_k = \tilde{f}_k^* - \tilde{f}_k \) and fix \( z_0 \in \Omega_0 \). Then, there exists \( 1 \leq j \leq n \) such that \( \delta \leq |f_j^*(z_0)| \leq |H(z_0)| + |\tilde{f}_j(z_0)| \).

By lemma 1, \( |H(z_0)| \leq \delta/2 \) if \( \text{supp}(\mu) \) is sufficiently small and, therefore, \( \delta/2 \leq |\tilde{f}_j(z_0)| \).

According to Garnett and Jone's theorem for Denjoy domains [8], there exist \( \tilde{p}_1, \ldots, \tilde{p}_n \in H^\infty(\Omega_0) \) such that \( \tilde{f}_1 \tilde{p}_1 + \cdots + \tilde{f}_n \tilde{p}_n = 1 \) on \( \Omega_0 \) with \( \|\tilde{p}_k\|_\infty \leq C(n, \delta) \).

Define \( P_k^* = j(\tilde{p}_k) \). Then, \( P_k^* \in L^\infty(\mathbb{R}) \) and \( \tilde{p}_k = C_H(P_k^*) \). Set \( P_k = P_k^* \circ \rho^{-1} \) on \( \Gamma \) and define the bounded analytic functions \( p_k = C_H(P_k) \) on \( \Omega \).

Although \( \{p_k\} \in H^\infty(\Omega) \) with \( \|p_k\|_\infty \leq C(n, \delta, \Gamma) \) by theorem 1, they are not corona solutions as they do not verify that \( \sum f_k p_k = 1 \) on \( \Omega \).

Consider the functions \( g_k(\omega) = p_k(\omega)/(\sum f_j(\omega)p_j(\omega)) \), \( 1 \leq k \leq n \), on \( \Omega \). They clearly satisfy that \( \sum_k f_k g_k = 1 \). To prove they are, indeed, corona solutions, it is sufficient to show that \( \sum_j f_j p_j \) is close to 1 and therefore bounded away from 0.

Let us denote \( p_k^* = p_k \circ \rho \in H^\infty(\Omega_0, \mu) \). Note that, again, \( j(p_k^*) = j(\tilde{p}_k) = P_k^* \). Consider \( \omega \in \Omega \) and \( z \in \Omega_0 \) so that \( \omega = \rho(z) \). Then

\[
| \sum_{j=1}^n f_j(\rho(z))p_j(\rho(z)) - 1 | = | \sum_{j=1}^n f_j(\rho(z))p_j(\rho(z)) - \sum_{j=1}^n \tilde{f}_j(z)\tilde{p}_j(z) | \\
\leq \sum_{j=1}^n |f_j^*(z)||p_j^*(z) - \tilde{p}_j(z)| + \sum_{j=1}^n |\tilde{p}_j(z)||f_j^*(z) - \tilde{f}_j(z)|
\]

is small enough as \( f_j^* \) and \( \tilde{p}_j \) are bounded and \( |p_j^*(z) - \tilde{p}_j(z)|, |f_j^*(z) - \tilde{f}_j(z)| \) are also small enough due to lemma 1. \qed
5. An example of a smooth but not Dini-smooth curve

In this section, we provide an example of a smooth quasicircle \( \Gamma = \rho(\mathbb{R}) \) with \( \mu_r \) satisfying condition 2 and such that \( \Gamma \) is not a Dini-smooth curve.

Let \( h \) be the conformal map taking \( \mathbb{D} \) onto the ball \( B(9/10, 1/10) \), \( h(z) = (9 + z)/10. \) Consider:

\[
g(z) = 2z + \frac{1 - z}{\log (1 - z)},
\]
and set \( f = g \circ h \). Then \( f \) defines an analytic function on \( \mathbb{D} \).

Since \( f' \neq 0 \) in \( \mathbb{D} \), then \( f \) is locally univalent. Also, \( (1 - |z|^2)|zf''(z)/f'(z)| \leq 1 \) and Becker’s univalence criteria ([14], Theorem 1.11) shows that \( f \) is indeed an univalent function.

A simple computation shows \( \lim_{z \to 1} (1 - |z|)|f''(\bar{z})|/|f'(\bar{z})| < 1 \). Therefore, by Becker and Pommerenke result [1]:

\[
f(z) = f(1/\bar{z}) + f'(1/\bar{z})(z - 1/\bar{z}), \quad \text{for } |z| > 1.
\]
defines a quasiconformal extension of \( f \) in a neighbourhood of the unit circle and

\[
|\mu(1/\bar{z})| \asymp (1 - |z|)|f''(z)|/|f'(z)| \quad \text{for } z \in \mathbb{D},
\]
where

\[
f'(z) = \frac{1}{10} \left( 2 - \frac{1}{\log (1 - h(z))} + \frac{1}{(\log (1 - h(z)))^2} \right)
\]
and

\[
|f''(z)| \asymp \frac{1}{|\log (1 - h(z))|^2 |1 - h(z)|} \asymp \frac{1}{|\log \frac{10}{1 - z}|^2 |1 - z|}
\]
in \( \mathbb{D} \).

Consider polar coordinates \( z = re^{i\theta} \). As for \( r > 1, |z - 1| \asymp \theta + r - 1 \), then:

\[
|\mu(re^{i\theta})| \lesssim (r - 1) \left( \frac{1}{\log \frac{10}{|1 - z|}} \right)^2 \frac{1}{|1 - z|}.
\]
and a simple calculation yields $\sigma(r) \lesssim (r - 1)^{1/2}/(\log \frac{1}{r-1})^2$, when $r \to 1^+$.

Then

$$
\int_1^r \frac{\sigma(r)}{(r - 1)^{3/2}} dr \lesssim \int_1^r \frac{1}{(r - 1)(\log \frac{1}{r-1})^2} dr \lesssim \infty.
$$

Since estimate $3$ obviously holds for $\mu$, we have proved that $\mu$ satisfies condition 2.

On the other hand, the modulus of continuity of $f'$ verifies that $\omega_{f'}(t) \simeq 1/\log (1/t)$ and

$$
\int_0^1 \frac{1}{t \log (1/t)} dt = \infty.
$$

Therefore, see [15, Theorem 3.5] the curve $\Gamma = f(T)$ is not Dini-smooth.

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