CUTOFF PROFILES FOR QUANTUM LÉVY PROCESSES AND QUANTUM RANDOM TRANSPositionS

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Abstract. We establish the existence of a cutoff phenomenon for a natural analogue of the Brownian motion on free orthogonal quantum groups. We compute in particular the cutoff profile, whose type is different from the previously known examples and involves free Poisson laws and the semi-circle distribution. We prove convergence in total variation (and even in $L^p$-norm for all $p$ greater than 1) at times greater than the cutoff time and convergence in distribution for smaller times. We also study a similar process on quantum permutation groups, as well as the quantum random transposition walk. The latter yields in particular a quantum analogue of a recent result of the second-named author on random transpositions.

1. Introduction

Let $(X_N)_{N \in \mathbb{N}}$ be a sequence of irreducible aperiodic finite state Markov chains, $\mu_N(t)$ the distribution of $X_N$ after $t$ steps, and $\mu_N(\infty)$ the stationary measure of $X_N$. Let,

$$d_N(t) = d_{\text{TV}}(\mu_N(t), \mu_N(\infty))$$

be the distance of the process to equilibrium at time $t$, where the total variation distance $d_{\text{TV}}(\mu, \nu)$ between two probability measures $\mu, \nu$ on a finite set $E$ is defined by the formula

$$d_{\text{TV}}(\mu, \nu) = \frac{1}{2} \sum_{x \in E} |\mu(x) - \nu(x)|.$$

Let $(t_N)_{N \in \mathbb{N}}$ a sequence of times. We say that $(X_N)_{N \in \mathbb{N}}$ exhibits a cutoff in total variation distance at time $(t_N)_{N \in \mathbb{N}}$ if for all $\epsilon > 0$,

$$d_N((1 - \epsilon)t_N) \xrightarrow{N \to \infty} 1 \text{ and } d_N((1 + \epsilon)t_N) \xrightarrow{N \to \infty} 0.$$

It means that the convergence to equilibrium occurs through a sharp phase transition, falling rapidly from 1 to 0 around time $t_N$.

To get a better understanding of this phenomenon, one may try to zoom on the window where the “fall” occurs. The cutoff phenomenon tells us that the width of this window is negligible with respect to the sequence $(t_N)_{N \in \mathbb{N}}$, and the next step is therefore to find the next significant “higher order term”. Here is a way to formalize this. If there exists a sequence $(w_N)_{N \in \mathbb{N}}$ and a continuous function $f$ decreasing from 1 to 0 such that for all $c \in \mathbb{R}$,

$$d_N(t_N + cw_N) \xrightarrow{N \to \infty} f(c),$$

then we say that $f$ is the cutoff profile or limit profile of $(X_N)_{N \in \mathbb{N}}$.

Computing the cutoff profile is a difficult task in general, but this was done for some important families of Markov chains and commonly involves important probability distributions shaping the profile. For instance, for the lazy random walk on the hypercube (which is equivalent to the Ehrenfest Urn) we have by [33, 28]

$$d_N \left( \frac{1}{2} N \ln(N) + cN \right) \xrightarrow{N \to \infty} d_{\text{TV}} \left( \mathcal{N} \left( e^{-c}, 1 \right), \mathcal{N} \left( 0, 1 \right) \right),$$
involving Gaussian distributions. Similar profiles were found for the dovetail shuffle [5], simple exclusion process on the circle [21], Ehrenfest Urn with multiple urns [26], or Gibbs Sampler [26]. For random transpositions, we have by [31]
\[
d_N \left( \frac{1}{2} (N \ln(N) + cN) \right) \xrightarrow{N \to \infty} d_{TV} \left( \text{Pois} \left( 1 + e^{-c} \right), \text{Pois} \left( 1 \right) \right),
\]
involving Poisson distributions. The same profile appears also for \(k\)-cycles [26].

This last result on random transpositions, by the second-named author, is one of the motivations of the present article, where we endeavour to compute the cutoff profile for some specific processes. One important difference however is that we will not work with finite classical groups, but with infinite compact quantum groups.

Compact quantum groups were introduced by S.L. Woronowicz as a generalization of classical compact groups. In particular many results from the representation theory of compact groups carry on to this setting, providing tools similar to those used in the study of random transpositions. A recent work of the first-named author [19] showed that indeed, there are natural quantum Markov chains on compact quantum groups exhibiting a cutoff phenomenon in a way paralleling the classical case. However, the cutoff profile was not studied there.

In the present paper, we will push further the study of the cutoff phenomenon for stochastic processes on compact quantum groups in two ways. First, we will consider continuous processes instead of discrete ones and second, we will study and describe the cutoff profiles.

The most natural continuous process on a simple compact Lie group is certainly the Brownian motion. Recall that this is the process whose diffusion kernel is the heat kernel corresponding to the canonical Riemannian structure on the group. Unfortunately, for quantum analogues of compact Lie groups there is to our knowledge no canonical Riemannian-like structure available to provide an analogue of the heat kernel. However, a result of M. Liao in [22] shows that if \((g_t)_{t \in \mathbb{R}_+}\) is a Lévy process on a simple compact Lie group which is invariant under the adjoint action, then its infinitesimal generator is the sum of the Laplace-Beltrami operator (which is the infinitesimal generator of the Brownian motion) and a “jump part” given by a so-called Lévy measure. It turns out that a similar decomposition also holds for some compact quantum groups. Indeed, F. Cipriani, U. Franz and A. Kula proved in [11] that on the quantum orthogonal group \(O_N^c\), there exists a distinguished process \((\psi_t)_{t \in \mathbb{R}_+}\) such that for any Lévy process which is invariant under the adjoint action, the corresponding infinitesimal generator splits as the sum of the generator of \((\psi_t)_{t \in \mathbb{R}_+}\) and a “jump part” characterized by a Lévy measure. As a consequence, \((\psi_t)_{t \in \mathbb{R}_+}\) can be seen as an analogue of the Brownian motion.

Our main result is the computation in Section 3 of the cutoff profile for this Brownian motion on the quantum orthogonal group \(O_N^c\), a compact quantum group which can be thought of as analogue of the group \(SO(N)\), for which the cutoff phenomenon was proven by P.-L. Méliot in [24]. More precisely, we prove in Theorem 3.9 that for any \(c > 0\),
\[
d_N \left( N \ln(N) + cN \right) \xrightarrow{N \to \infty} d_{TV} \left( \text{Pois}^+ \left( e^{2c}, -e^{-c} \right) \ast \delta_{e^{c}+e^{-c}}, \nu_{SC} \right),
\]
where \(\nu_{SC}\) denotes the semi-circle distribution and \(\text{Pois}^+\) denotes the free Poisson distribution. It is known that the correspondence between \(SO(N)\) (or rather \(O(N)\)) and \(O_N^c\) has to do, at the probabilistic level, with the Bercovici-Pata bijection. From that point of view, the appearance of the semi-circle distribution in the cutoff profile is quite satisfying. On the contrary, the appearance of the free Poisson distribution is surprising because it is not a priori a “deformation” of the semi-circle distribution. The picture becomes clearer when written in terms of free Meixner distributions (see Section 3.2 for the definition):
\[
d_N \left( N \ln(N) + cN \right) \xrightarrow{N \to \infty} d_{TV} \left( \text{Meix}^+ \left( -e^{-c}, 0 \right) \ast \delta_{e^{-c}}, \text{Meix}^+ \left( 0, 0 \right) \right).
\]

Let us briefly comment on the proof. On the one hand, the quantum group \(O_N^c\) is easier to study than \(SO(N)\), because its representation theory is simpler (the underlying combinatorics is essentially that of the representation theory of \(SU(2)\)). This enables to reduce the problem to the comparison of some probability measures on the interval \([-N, N]\). But this is compensated by analytic issues which prevented us from describing the cutoff profile for negative \(c\) in a satisfying way. These issues first appear as a failure of absolute continuity of the process with respect to the Haar measure, which is a purely quantum phenomenon (see Proposition 3.7). They then translate into difficulties in the computation of the total variation distance.
between the measures on $[N, N]$ alluded to before. As a consequence, we were not able to prove a convergence in total variation distance for negative values of $c$. We nevertheless obtain weak convergence in Proposition 3.12. The discrepancy between positive and negative values of $c$ is not a surprise in the sense that even for the classical examples for which convergence is known for all values of $c$, it is observed that for $c < 0$ the convergence is much slower.

In the end of Section 3, we investigate other types of convergence and prove that the convergence to the cutoff profile for $c > 0$ also occurs in $L^p$-norm for all $1 \leq p \leq \infty$. Let us mention that for $c < 0$, the aforementioned analytic issues again enter the picture, making the very definition of the $L^p$-norm unclear.

We furthermore investigate analogues of the Brownian motion on some homogeneous spaces for $O_N^+$ called free real spheres, the computations essentially boiling down to the previous ones for $O_N^+$.

The article concludes in Section 4 with a second family of examples called the quantum permutation groups and denoted by $S_N^+$. Despite bearing strong analogies with the classcal permutation group $S_N$ justifying its name, $S_N^+$ is an “infinite” quantum group. In particular, it has a well-defined Brownian motion, given by a Lévy-Khintchine decomposition similar to that of $O_N^+$. After computing its cutoff profile, we turn to a problem which was left open in [19]: the quantum random transposition walk. Here, the methods used in the previous cases break down due to a lack of absolute continuity of the random walk with respect to the Haar state at all time. We therefore have to resort to new ideas to be able to prove the existence of a cutoff phenomenon and to compute the cutoff profile. More precisely, we show in Corollary 4.6 that

$$d_N\left(\frac{1}{2}(N \ln(N) + cN)\right)_{N \to \infty} \overset{d_{TV}}{\to} D\left(\frac{1}{\sqrt{1+e^{-c}}} \text{Meix}^+\left(\frac{1-e^{-c}}{\sqrt{1+e^{-c}}}, -e^{-c}\right) \ast \delta_{-e^{-c}}, \text{Meix}^+(1,0)\right).$$

Because $\text{Meix}^+(1,0) = \text{Pois}^+(1,1) \ast \delta_{-1}$ is the standard free Poisson distribution, this provides a quantum analogue of the result of [31]. The main idea of the proof is that the random walk asymptotically coincides with the pure quantum transposition walk. This is a specifically quantum phenomenon connected to the fact that the pure quantum transposition walk has no periodicity issue because there is no quantum alternating group. We then show that the cutoff profile of the latter is the same as for the Brownian motion on $S_N$.

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2. Preliminaries

Compact quantum groups will be one of our main objects of studies in this work, and the one the probabilist reader may be least acquainted with. We will therefore devote this preliminary section to some definitions and fundamental results concerning them. In order to keep things simple, we will only introduce free orthogonal quantum groups for the moment, as well as some results concerning Lévy processes on them. Details on other compact quantum groups will be given when needed later on.

2.1. Free orthogonal quantum groups. Free orthogonal quantum groups are examples of compact quantum groups in the sense of S.L. Woronowicz [36] which were first introduced by Sh. Wang in [34]. The original definition uses C*-algebras, as may be expected for objects of noncommutative topological nature. We will nevertheless use a different definition which we believe may be easier to understand for the non-expert reader, by focusing first on the purely algebraic aspects. We refer to the books [25] and [32] for a comprehensive treatment of the theory and proofs of the main results.

2.1.1. Definition and representation theory. We recall that a $*$-algebra is an algebra $A$ endowed with an involution $x \mapsto x^*$, i.e. an antimultiplicative linear map such that $(x^*)^* = x$ and $(\lambda x)^* = \overline{\lambda} x^*$ for all $x \in A$ and $\lambda \in \mathbb{C}$. Also, a $*$-ideal $B$ of $A$ is a $*$-subalgebra of $A$ such that $\{ba, ab\} \subset B$ for all $a \in A$ and $b \in B$.

Definition 2.1. We define $O(O_N^+)$ to be the universal $*$-algebra generated by $N^2$ self-adjoint elements $u_{ij}$ (i.e. $u_{ij}^* = u_{ij}$) such that for all $1 \leq i, j \leq N$,

$$\sum_{k=1}^{N} u_{ik}u_{jk} = \delta_{ij} = \sum_{k=1}^{N} u_{ki}u_{kj}. $$
In other words, \( \mathcal{O}(O_N^+) = \mathbb{C}\langle u_{ij} : 1 \leq i, j \leq N \rangle / I \), where \( \mathbb{C}\langle u_{ij} : 1 \leq i, j \leq N \rangle \) denotes the *-algebra of (noncommutative) polynomials in variables \( u_{ij}, u_{ij}^* \) with \( 1 \leq i, j \leq N \), and \( I \) denotes the *-ideal generated by the elements

\[
\{ u_{ij}^* - u_{ij}, \sum_{k=1}^N u_{ik}u_{jk} - \delta_{ij}, \sum_{k=1}^N u_{ki}u_{kj} - \delta_{ij}, 1 \leq i, j \leq N \}.
\]

Let \( O_N \) be the usual orthogonal group, let \( c_{ij} : O_N \rightarrow \mathbb{C} \) be the function sending a matrix to its \((i, j)\)-th coefficient and let \( \mathcal{O}(O_N) \) be the algebra of \textit{regular functions} on \( O_N \), i.e. the *-algebra generated by the functions \( c_{ij} \), where the involution corresponds to the complex conjugation: \( c_{ij}^* = \overline{c_{ij}} \). Then, quotienting \( \mathcal{O}(O_N^+) \) by its commutator ideal yields a surjection

\[
\pi : \mathcal{O}(O_N^+) \rightarrow \mathcal{O}(O_N)
\]
so that \( O_N^+ \) can be seen as a “noncommutative version” of \( O_N \). The group structure can be encoded in this setting thanks to the following remark: for any two orthogonal matrices \( g \) and \( h \),

\[
c_{ij}(gh) = \sum_{k=1}^N c_{ik}(g)c_{kj}(h) = \sum_{k=1}^N c_{ik} \otimes c_{kj}(g, h),
\]
where we identify \( \mathcal{O}(O_N \times O_N) \) with \( \mathcal{O}(O_N) \otimes \mathcal{O}(O_N) \). The “group law” of \( \mathcal{O}(O_N^+) \) will therefore be given by the unique *-homomorphism \( \Delta : \mathcal{O}(O_N^+) \rightarrow \mathcal{O}(O_N^+) \otimes \mathcal{O}(O_N^+) \), called the \textit{comultiplication}, such that

\[
\Delta(u_{ij}) = \sum_{k=1}^N u_{ik} \otimes u_{kj}.
\]

The existence of \( \Delta \) follows from the universal property of \( \mathcal{O}(O_N^+) \).

Probability measures can be generalized to this setting by identifying them with their integration linear form. They then correspond to \textit{states}, i.e. linear maps

\[
\psi : \mathcal{O}(O_N^+) \rightarrow \mathbb{C}
\]
such that \( \psi(1) = 1 \) and \( \psi(x^*x) \geq 0 \) for all \( x \). There is a particular state which plays the rôle of the uniform measure on \( O_N^+ \):

\[\textbf{Theorem 2.2} \textbf{ (Woronowicz).} \textit{There is a unique state } h \textit{ on } \mathcal{O}(O_N^+) \textit{ such that for all } x \in \mathcal{O}(O_N^+),
\]

\[ (\text{id} \otimes h) \circ \Delta(x) = h(x).1 = (h \otimes \text{id}) \circ \Delta(x).
\]

\textit{It is called the Haar state of } \mathcal{O}_N^+.

Since the founding works of P. Diaconis and his coauthors, it is known that representation theory is a powerful tool to study the asymptotic behaviour of random walks on groups (see for instance [13, Chap 4]). For \( O_N^+ \), the representation theory was computed by T. Banica in [1]. However, for our purpose we will only need to understand the subalgebra \( \mathcal{O}(O_N^+)_{\text{central}} \) generated by the characters of the irreducible representations (we refer the reader for instance to [25, Sec 1.3] for the definitions of these notions and details).

\[\textbf{Theorem 2.3} \textbf{ (Banica).} \textit{Let us set } \chi_0 = 1 \textit{ and } \chi_1 = \sum_{i=1}^N u_{ii}. \textit{Then, the irreducible representations of } O_N^+ \textit{ are labelled by the integers such that if } \chi_n \textit{ denotes the character associated to the integer } n \in \mathbb{N}, \textit{we have the recurrence relation :}
\]

\[ \forall n \geq 1, \quad \chi_1 \chi_n = \chi_{n+1} + \chi_{n-1}. \]

Note that this implies that \( \chi_n^* = \chi_n \) for all \( n \in \mathbb{N} \). This recurrence relation is reminiscent of Chebyshev polynomials, and one can indeed express \( \chi_n \) in terms of \( \chi_1 \) using them. More precisely, let \( (P_n)_{n \in \mathbb{N}} \) be the sequence of polynomials defined by \( P_0(X) = 1, P_1(X) = X \) and

\[ XP_n(X) = P_{n+1}(X) + P_{n-1}(X). \]

In particular, \( P_n(2) = n + 1, P_n(-2) = (-1)^n(n + 1), \) and for \( \theta \in (0, \pi) \) and \( n \in \mathbb{N} \),

\[ P_n(2 \cos(\theta)) = \frac{\sin((n + 1)\theta)}{\sin(\theta)}. \]
Then, the map $\chi_n \mapsto P_n$ yields an isomorphism between $O(O_N^+)_{\text{central}}$ and $\mathbb{C}[X]$. Moreover, by [2, Prop 1], the restriction of the Haar state to this subalgebra coincides with integration with respect to the semicircle distribution $\nu_{sc}$, which is the measure on $[-2,2]$ with density

$$d\nu_{sc} = \frac{1}{2\pi} \sqrt{4 - x^2} dx.$$  

The polynomials $P_n$ are exactly the orthogonal polynomials for this measure.

2.1.2. Central Lévy processes. Let us now describe what the analogue of a Lévy process is on $O_N^+$. On a classical group, this is a càdlàg stochastic process $(X_t)_{t \in \mathbb{R}_+}$ with independent and stationary increments. In particular, if $\mu_t$ is the distribution of $X_tX_0^{-1}$ then we have

- $\mu_0 = \delta_{Id}$,
- $\mu_t \ast \mu_s = \mu_{t+s}$,
- $\lim_{t \to 0} \mu_t = \mu_0$ weakly.

In other words, we have a continuous convolution semigroup of probability measures. This semigroup contains most of the probabilistic information about the process, hence in this paper a quantum Lévy process will simply be for us a continuous convolution semigroup of states, i.e. a family $(\psi_t)_{t \in \mathbb{R}_+}$ of states on $O(O_N^+)$ such that

- $\psi_0 = \varepsilon : u_{ij} \mapsto \delta_{ij}$,
- $\psi_t \ast \psi_s = (\psi_t \otimes \psi_s) \circ \Delta = \psi_{t+s}$,
- $\lim_{t \to 0} \psi_t(x) = \psi_0(x)$ for all $x \in O(O_N^+)$.

Let us mention that the theory of Lévy processes on compact quantum groups can be developed in full generality (not just restricted to marginals), see for instance the survey [15].

As mentioned in the introduction, we will be interested in the case where the Lévy process is invariant under the adjoint action. In other words we will focus on states which are central, in the sense that they are invariant by conjugation (see the beginning of [11, Sec 6] for the definitions). By [11, Prop 6.9], such states have a peculiar form. First, there is a conditional expectation

$$E : O(O_N^+) \to O(O_N^+),$$

that is to say a linear map satisfying $E(x^*x) \geq 0$ for all $x$, and $E(x) = x$ for all $x \in O(O_N^+)$central. Then, a state $\psi$ is central if and only if there exists a state $\tilde{\psi}$ on $O(O_N^+)_{\text{central}}$ such that

$$\psi = \tilde{\psi} \circ E.$$

Let us summarize the previous discussion in a definition:

**Definition 2.4.** A central Lévy process on $O_N^+$ is a continuous convolution semigroup of states $(\psi_t)_{t \in \mathbb{R}_+}$ on $O(O_N^{+})$ such that $\psi_t$ is central for all $t$.

Central Lévy processes on $O_N^+$ were classified by F. Cipriani, U. Franz and A. Kula in [11, Thm 10.2] through an analogue of the Lévy-Khinchine formula. Note that because of centrality, it is enough to know the image of $\chi_n$ for all $n \in \mathbb{N}$.

**Theorem 2.5** (Cipriani-Franz-Kula). Any central Lévy process $(\psi_t)_{t \in \mathbb{R}_+}$ on $O_N^+$ is of the form

$$\psi_t : \chi_n \mapsto P_n(N)e^{-\psi(n)}$$

where

$$\psi(n) = bP_n^c(N)P_n(N) + \frac{1}{P_n(N)} \int_{-N}^N P_n(N) - P_n(x) \frac{dx}{N-x} d\nu(x)$$

for some $b \geq 0$ and a finite measure $\nu$ on $[-N,N]$ such that $\nu(\{N\}) = 0$.

Comparing this formula with the one proved by M. Liao for classical compact Lie groups in [22], we see that the process corresponding to $\psi(n) = P_n^c(N)/P_n(N)$ plays a role analogous to the one associated to the Laplace-Beltrami operator. As a consequence, we will call this process the Brownian motion on $O_N^+$. 

2.2. The cutoff phenomenon. This work is mostly concerned with the diffusion of central Lévy processes and in particular the time needed for the process to spread all over the group. This can be rigorously defined by measuring the distance between ψ and the Haar state h. Classically, one interesting and widely used distance for this is the total variation distance
\[ d_{TV}(\mu, \nu) = \|\mu - \nu\|_{TV} = \sup_{A \subset G} |\mu(A) - \nu(A)| \]
where the supremum is taken over all Borel subsets A of the classical group G. For quantum groups, the corresponding definition requires the introduction of a suitable version of the Borel σ-algebra.

Let us define an inner product on \( O_N^+ \) by the formula \( \langle x, y \rangle = h(xy^*) \). Then, taking the completion yields a Hilbert space \( L^2(O_N^+) \), and \( O(O_N^+) \) embeds through left multiplication into \( B(L^2(O_N^+)) \) (see [25, Cor 1.7.5] and the comments thereafter). The weak closure of the image is denoted by \( L^\infty(O_N^+) \) and is a von Neumann algebra. These are known to be the noncommutative generalisations of measure spaces. Indeed, as explained in [19, Sec 2], if \( P \) is the set of orthogonal projections in \( L^\infty(O_N^+) \) (thought of as indicator functions of Borel subsets), then
\[ d_{TV}(\phi, \psi) = \|\phi - \psi\|_{TV} = \sup_{p \in P} |\phi(p) - \psi(p)| \]
is a generalisation of the total variation distance. Note that in order for this formula to make sense, the states \( \phi \) and \( \psi \) must extend to the von Neumann algebra \( L^\infty(O_N^+) \). This is not always the case due to absolute continuity issues (see for instance Proposition 3.7). To remedy this drawback of the total variation distance, one may consider instead the universal enveloping \( C^* \)-algebra (see for instance [6, Sec II.8.3]) \( C(O_N^+) \) of \( O(O_N^+) \). By definition, any state on \( O(O_N^+) \) has a unique extension to a state on \( C(O_N^+) \), hence yields an element of the Fourier-Stieltjes algebra, which is the topological dual \( C(O_N^+)^* \) of \( C(O_N^+) \). The latter is thought of as an analogue of the measure algebra in noncommutative harmonic analysis. Let us denote by \( \| \cdot \|_{FS} \) the norm on this dual space and call it the Fourier-Stieltjes norm. We will see that for central processes on \( O_N^+ \), this can indeed be seen as a classical total variation distance (see the beginning of Section 3.1). Moreover, this is a generalization of the total variation distance in the sense that if \( \varphi : O(O_N^+) \to \mathbb{C} \) is a linear functional which extends to a normal bounded map on \( L^\infty(O_N^+) \), then \( \varphi \) becomes an element of the Fourier algebra, which is the Banach space predual \( L^\infty(O_N^+)^* \) of \( L^\infty(O_N^+) \) and \( \|\varphi\|_{FS} = \|\varphi\|_{L^\infty(O_N^+)} \), (see for instance [9, Prop 3.14]), which further implies \( \|\varphi\|_{FS} = 2\|\varphi\|_{TV} \) by [19, Lem 2.6]. As a consequence, this is the norm that we will use in our cutoff statements and to use the usual normalization we will set:
\[ \| \cdot \| = \frac{1}{2} \| \cdot \|_{FS}. \]

The evolution of the distance from the process to the Haar state can exhibit various behaviours. One which is especially striking is the so-called cutoff phenomenon. Here is a precise definition of what we mean by this:

**Definition 2.6.** Let \( (\mathbb{G}_N, (\psi_t^{(N)})_{t \in \mathbb{R}^+})_{N \in \mathbb{N}} \) be a family of compact quantum groups with a Lévy process \( (\psi_t^{(N)})_{t \in \mathbb{R}^+} \) on each of them. We say that the processes exhibit a cutoff phenomenon at time \( (t_N)_{N \in \mathbb{N}} \) if for any \( \epsilon > 0 \),
\[ \lim_{N \to +\infty} \|\psi_t^{(N)}(1+\epsilon)t_N - h_N\| = 1 \text{ and } \lim_{N \to +\infty} \|\psi_t^{(N)}(1-\epsilon)t_N - h_N\| = 0. \]

One may wonder why we use the Fourier-Stieltjes norm in the definition instead of the total variation distance. As will appear in the sequel, the first limit usually does not make sense for the total variation distance \( \| \cdot \|_{TV} \). In that sense, the statements of [19] and [18] used the term “cutoff” in a slightly abusive way. Nevertheless, the methods of the present paper show that there is indeed a cutoff in the sense of Definition 2.6 in these situations.

One very useful tool to prove that such a phenomenon occurs is the following lemma originally due to P. Diaconis and M. Shahshahani in [14] for finite groups and to J.P. McCarthy in [23] for finite quantum groups. A proof for compact quantum groups can be found in [19, Lem 2.7], but we simply state it in our particular case.
Lemma 2.7. Let \((\psi_t)_{t \in \mathbb{R}_+}\) be a central Lévy process on \(O_N^+\). If for some \(t > 0\) the sum \(\sum_{n=1}^{+\infty} P_n(N)^2 e^{-2t\psi(n)}\) is finite, then the norm \(\|\psi_t\|_{TV}\) exists, and

\[
\|\psi_t - h\|^2_{TV} \leq \frac{1}{4} \sum_{n=1}^{+\infty} P_n(N)^2 e^{-2t\psi(n)}.
\]

Remark 2.8. The right-hand side is nothing but the \(L^2\)-norm of the density of \(\psi_t - h\) with respect to \(h\), computed using the Plancherel formula. We recover in that way the fact that as soon as a state has an \(L^2\)-density with respect to the Haar state, it has an extension to \(L^\infty(O_N^+)\) and the total variation distance is well-defined.

Let us end these preliminaries with a few practical comments. To lighten notations and because this number is actually the dimension of the corresponding irreducible representation, we will write \(d_n\) for \(P_n(N)\) in the sequel. There is an explicit formula to compute these numbers: setting, for \(N > 2\),

\[
q(N) = \frac{N - \sqrt{N^2 - 4}}{2},
\]

we have, writing \(q\) for \(q(N)\),

\[
d_n = \frac{q^{-n} - q^{n+1}}{q^{-1} - q}.
\]

3. The quantum orthogonal Brownian motion

In this section we study the cutoff phenomenon for the analogue of the Brownian motion on \(O_N^+\). As explained above, this means that we will take \(\nu = 0\) in Equation (1). Once that choice is made, changing the value of \(b\) is equivalent to rescaling the time, so that there is no loss in generality in fixing \(b = 1\) for all \(N\), leading to

\[
\psi(n) = \frac{P'_n(N)}{P_n(N)}.
\]

We will use an elementary but useful fact on the monotonicity of the total variation distance which is well-known in the classical case.

Lemma 3.1. For \(N\) fixed, the map \(t \mapsto \|\psi_t - h\|_{FS}\) is decreasing.

Proof. First note that for any two bounded linear forms \(\varphi, \psi\) on \(C(O_N^+)\),

\[
\|\varphi * \psi\|_{FS} = \| (\varphi \otimes \psi) \circ \Delta \|_{FS} \leq \| \varphi \otimes \psi\|_{FS} \leq \| \varphi\|_{FS} \| \psi\|_{FS}.
\]

Thus, for any \(t \geq s\),

\[
\|\psi_t - h\|_{FS} = \| (\psi_{t-s} * h) \|_{FS} \leq \|\psi_{t-s} - h\|_{FS},
\]

where we used the fact that any state \(\psi\) on a C*-algebra has norm one. \(\square\)

3.1. The cutoff phenomenon. We first want to show that the process \((\psi_t)_{t \in \mathbb{R}_+}\) exhibits a cutoff phenomenon. As explained in Section 2.2, this requires precise estimates for Fourier-Stieltjes norm. As soon as the right-hand side of Equation (2) is finite, we can use it to bound the total variation distance, which is then well-defined and coincides with the half the Fourier-Stieltjes norm. This is the strategy which was already used in [19] and the computations will be similar. To obtain the lower bound, however, we need to deal directly with the Fourier-Stieltjes norm and this will require an alternate description which we now detail.

By [8, Lem 4.2], the closure of \(O(O_N^+)_{central}\) in \(C(O_N^+)\) is a commutative C*-algebra isomorphic to \(C([-N, N])\). Moreover, if \(\varphi = \tilde{\varphi} \circ E\) is a bounded central linear form, then

\[
\|\tilde{\varphi}\|_{FS} = \|\varphi|_{O(O_N^+)_{central}}\|_{FS} \leq \|\varphi\|_{FS} = \|\tilde{\varphi} \circ E\|_{FS} \leq \|\tilde{\varphi}\|_{FS}
\]

so that the problem reduces to the computation of the norm of a bounded linear form on a commutative C*-algebra. By the Riesz Representation Theorem, there exists a measure \(\mu\) on \([-N, N]\) such that

\[
\tilde{\varphi}(x) = \int_{-N}^{N} x \, d\mu
\]

and moreover, the Fourier-Stieltjes norm of \(\tilde{\varphi}\) coincides with twice the total variation of \(\mu\). Using that observation, we can now establish that the brownian motion on \(O_N^+\) exhibits a cutoff phenomenon.
Theorem 3.2. The central Lévy process on $O_N^+$ given by $(\psi(n))_{n \in \mathbb{N}}$ exhibits a cutoff phenomenon in total variation distance at time $t_N = N \ln(N)$.

Proof. We start with the upper bound and we will obtain an estimate which is finer than what is actually needed. Applying Lemma 2.7 to $\psi_1$ yields, for $t$ such that the right-hand side is finite,

$$\|\psi_t - h\|_{TV} \leq \frac{1}{4} \sum_{n=1}^{+\infty} d_n^2 e^{-2\psi(n)}$$

and it was proven in [16, Lem 1.7] that

$$\frac{n}{N} \leq \psi(n) \leq \frac{n}{N-2}.$$  

Hence, using the estimates of [19, Lem 3.3], the sum in the right-hand side can be bounded as soon as $q^{-1} e^{-t/N} < 1$ by

$$\sum_{n=1}^{+\infty} \frac{q^{-2n}}{(1-q^2)^2} e^{-2tn/N} = \frac{1}{(1-q^2)^2} \frac{q^{-2} e^{-2t/N}}{1 - q^{-2} e^{-2t/N}} = \frac{1}{(1-q^2)^2} \frac{1}{q^2 e^{2t/N} - 1}.$$  

For $c > 0$ and $t = N \ln(N) + cN$, we get, using $1/2 > q(N) > 1/N$ (see for instance [19, Lem 3.8])

$$\frac{1}{(1-q^2)^2} \frac{1}{q^2 N^2 e^{2c} - 1} \leq \frac{4}{3} \frac{e^{-2c}}{1 - e^{-2c}}.$$  

If now $t = (1+\epsilon)t_N$ with $\epsilon > 0$, we have by Lemma 3.1 and the preceding computation (taking $c = \epsilon \ln(N)$),

$$\|\psi_t - h\|_{TV} \leq \left\|\psi_{N \ln(N) + cN} - h\right\|_{TV} \leq \frac{4}{3} \frac{e^{-2\ln(N)}}{1 - e^{-2\ln(N)}} = O(N^{-2\epsilon})$$

so that

$$\lim_{N \to \infty} \|\psi_t - h\|_{TV} = 0.$$  

For the lower bound, we will show that the character $\chi_1$ is a good witness of the distance between $\psi_t$ and $h$. To do that, let us first estimate its mean and variance. We have, for $c < 0$ and $t = N \ln(N) + cN$,

$$\psi_t(\chi_1) = Ne^{-t/N} = e^{-c}$$

and the variance $\text{var}_{\psi_t}(\chi_1) = \psi_t(\chi_1^2) - \psi_t(\chi_1)^2$ is given, using the fact that $\chi_1^2 = 1 + \chi_2$, by

$$\text{var}_{\psi_t}(\chi_1) = 1 + (N^2 - 1)e^{-2t/(N^2-1)} - N^2 e^{-2t/N} \leq 1 + N^2 e^{-2t/N} - N^2 e^{-2t/N} \leq 1.$$  

Let us now view $\chi_1$ as a continuous function on $[-N, N]$ and consider the Borel subset

$$B = \{ s \in [-N, N] \mid |\chi_1(s)| \leq e^{-c/2} \}.$$  

The indicator function $p = 1_B$ can be seen as a projection in the von Neumann algebra $L^\infty([-N, N])$ of essentially bounded functions.

If we denote by $\nu_t$ the unique Borel probability measure on $[-N, N]$ such that for any $x \in O(O_N^+)$,

$$h(x) = \int_{-N}^{N} x \, d\nu_t$$

and by $\mu_t$ the unique Borel probability measure on $[-N, N]$ such that for any $x \in O(O_N^+)$,

$$\psi_t(x) = \int_{-N}^{N} x \, d\mu_t,$$
then these formulæ provide norm-preserving extensions of the states $h$ and $\psi_t$ to $L^\infty([-N,N])$. Moreover, because $\psi_t(\chi_1) = e^{-c}$, we have $B \subset \{ s \in [-N,N] \mid |\chi_1(s) - \psi_t(\chi_1)| \geq e^{-c}/2 \}$ so that by Chebyshev’s inequality
\[
\mu_t(B) \leq \left(\frac{e^{-c}/2}{\var\mu_t(\chi_1)}\right)^2 \leq 4e^{2c}.
\]
Using again the Chebyshev inequality for $\nu_h$ with $h(\chi_1) = 0$ and $\var h(\chi_1) = 1$, we eventually get
\[
|\mu_t(B) - \nu_h(B)| \geq \nu_h(B) - \mu_t(B) \\
= 1 - \nu_h([-N,N] \setminus B) - \mu_t(B) \\
\geq 1 - 4e^{2c} - 4e^{2c} \\
= 1 - 8e^{2c}.
\]
To conclude, recall that because $\mu_t$ and $\nu_h$ are probability measures, the total variation norm of their difference coincides with twice their total variation distance, so that
\[
\|\psi_t - h\| = \frac{1}{2} |\mu_t - \nu_h|([-N,N]) = \|\mu_t - \nu_h\|_{TV} \geq |\mu_t(B) - \nu_h(B)| \geq 1 - 8e^{2c}.
\]
For $c = -\epsilon \ln(N)$ ($\epsilon > 0$ fixed), the right-hand side becomes
\[
1 - 8N^{-2\epsilon}
\]
which tends to 1 as $N \to \infty$, hence the proof is complete. \hfill \Box

Remark 3.3. In the papers [19] and [18], the lower bounds were only proven for $c$ such that the corresponding state is absolutely continuous with respect to the Haar state. As a consequence, this does not yield a cutoff phenomenon in the sense of Definition 2.6 unless one makes sure that the states are asymptotically always absolutely continuous. As we will show below, and this was already observed in the aforementioned papers, this is never true. Hence, the term “cutoff” was slightly abusive there. However, using the same argument as in the above proof together with the estimates of [19] and [18], one can easily show that the random walks studied there indeed exhibit a bona fide cutoff phenomenon for the Fourier-Stieltjes norm.

Remark 3.4. In [24], P.-L. Méliot proved that the Brownian motion on $SO(N)$ exhibits a cutoff phenomenon at time $2 \ln(N)$. The factor 2 comes from the fact that he chooses one half of the Laplace-Beltrami operator as an infinitesimal generator. As for the additional factor $N$ in our result, it could be removed through setting $b_N = N$. A scaling-free statement on our case would therefore be that the cutoff time $t_N$ satisfies $t_Nb_N = N \ln(N)$ while in the case of P.-L. Méliot we have $t_Nb_N = \ln(N)$.

Remark 3.5. Thanks to the work of F. Cipriani, U. Franz and A. Kula [11], it is possible to construct a Dirac operator and a non-commutative Riemannian structure (a spectral triple) on $O_N^+$ out of a Lévy process. However, it was already noted in [11, Sec 10] that in our case, and independently from the choice of $b$, the dimension of the resulting object is infinite.

We mentioned earlier that the use of the Fourier-Stieltjes norm was necessary because $\psi_t$ cannot be extended to $L^\infty(O_N^+)$ in general. This can be thought of as an absolute continuity issue in the following sense. Let us denote by $L^1(O_N^+)$ the completion of $L^\infty(O_N^+)$ with respect to the norm $||x||_1 = h(|x|)$, where $|x|$ is obtained by functional calculus. A state $\psi_t$ is then said to be absolutely continuous (with respect to the Haar state) if there exists $a_t \in L^1(O_N^+)$ such that $\psi_t(x) = h(a_t x)$. It follows from the general theory (see [29, Thm V.2.18]) that a state on $O(N)_N$ is absolutely continuous with respect to the Haar state if and only if it extends to a normal linear map on $L^\infty(O_N^+)$. We now want to give a precise result about the absolute continuity window, and this requires a better upper bound on $\psi(n)$ than that of [16, Lem 1.7].

Lemma 3.6. Let $N \geq 4$ and $n \geq 1$, then
\[
\left|\psi(n) - \frac{n}{N} - \frac{2n - 2}{N^3}\right| \leq \frac{16n}{N^4}.
\]
Proof. recall that
\[
\psi(n) = \frac{P_n(N)}{P_n(N)} = \sum_{k=1}^n \frac{1}{N - x_k},
\]
where \( x_k = 2 \cos \left( \frac{kn}{n+1} \right) \) for \( 1 \leq k \leq n \) are the roots of the polynomial \( P_n \). Observe moreover that as for every \( k \), \( |x_k| \leq 2 \),

\[
\left| \psi(n) - \frac{1}{N} \sum_{k=1}^{n} \left( 1 + \frac{x_k}{N} + \left( \frac{x_k}{N} \right)^2 \right) \right| \leq \sum_{i=3}^{\infty} \left( \frac{2}{N} \right)^i = \left( \frac{2}{N} \right)^3 \frac{1}{1 - 2/N} \leq \frac{16n}{N^4}.
\]

Since \( \sum_{k=1}^{n} x_k = 0 \), we only have to compute \( \sum_{k=1}^{n} (x_k)^2 \) to conclude. Using \( \cos(x)^2 = \frac{1}{2} (1 + \cos(2x)) \) yields

\[
\sum_{k=1}^{n} (x_k)^2 = 2^2 \sum_{k=1}^{n} \frac{1}{2} \left( 1 + \cos \left( \frac{2k\pi}{n+1} \right) \right) = 2n + 2 \left( -1 + \sum_{k=0}^{n} \cos \left( \frac{2k\pi}{n+1} \right) \right) = 2n - 2.
\]

We can now give a precise criterion for absolute continuity.

**Proposition 3.7.** Let \( N \geq 4 \) and consider the central Lévy process on \( O_N^+ \) given by \( (\psi_t)_{t \in \mathbb{R}_+} \). Then there exists a positive time \( t_{\text{abscont}}(N) \) such that

- If \( t < t_{\text{abscont}}(N) \), then \( \psi_t \) is not absolutely continuous with respect to the Haar state,
- If \( t > t_{\text{abscont}}(N) \), then \( \psi_t \) is absolutely continuous with respect to the Haar state.

Moreover, as \( N \to \infty \),

\[
t_{\text{abscont}}(N) = N \ln(N) - \frac{2 \ln(N)}{N} + O \left( \frac{1}{N} \right) = N \ln(N) + o(1).
\]

**Proof.** Let \( N \geq 4 \), and set

\[
a_t = \sum_{n=0}^{+\infty} d_n e^{-t\psi(n)} \chi_n.
\]

If this series converges in \( L^1(O_N^+) \), then \( \psi_t \) is absolutely continuous with respect to the Haar state with density \( a_t \). Moreover, if it converges in \( L^2(O_N^+) \), then it converges also in \( L^1(O_N^+) \) (as the \( L^1 \)-norm is dominated by the \( L^2 \)-norm), and

\[
\|a_t\|_2^2 = \sum_{n=0}^{+\infty} d_n^2 e^{-2t\psi(n)}.
\]

Recall that \( d_n \leq q(N)^{-n}/(1-q(N)^2) \). Using the precise bound from the previous proposition, we compute:

\[
\ln \left( d_n^2 e^{-2t\psi(n)} \right) \leq 2n \ln(1/q(N)) - \ln(1 - q(N)^2) - 2t\psi(n)
\]

\[
\leq 2n \left( \ln(1/q(N)) - t \left( \frac{1}{N} + \frac{2}{N^3} - \frac{16}{N^4} \right) \right) + g(N),
\]

where \( g(N) \) is some function, that we can explicit, depending only on \( N \). Consequently, \( \|a_t\|_2 \) is finite (so \( \psi_t \) is absolutely continuous with respect to the Haar state) as soon as

\[
\ln(1/q(N)) - t \left( \frac{1}{N} + \frac{2}{N^3} - \frac{16}{N^4} \right) < 0.
\]

As for the second point, observe that

\[
\frac{\psi_t(\chi_n)}{\|\chi_n\|_\infty} = \frac{d_n e^{-t\psi(n)}}{n+1}.
\]

If the right-hand side is not uniformly bounded with respect to \( n \), then \( \psi_t \) cannot extend to \( L^\infty(O_N^+) \). Using the previous lemma and the fact (see for instance [19, Lem 3.8]) that \( d_n \geq Nq(N)^{-(n-1)} \), and proceeding as above, we see that \( \frac{d_n e^{-t\psi(n)}}{n+1} \) will not be bounded as soon as

\[
\ln(1/q(N)) - t \left( \frac{1}{N} + \frac{2}{N^3} + \frac{16}{N^4} \right) > 0.
\]

The existence of \( t_{\text{abscont}}(N) \) is then guarantied by the fact that if \( \psi_t \) is absolutely continuous with respect to \( h \) for some \( t \), then it is also for all \( t' > t \). Indeed, we may write \( \psi_{t'} = \psi_t + \psi_{t-t} \), and \( \psi_{t'} \) belongs to the predual.
$L^\infty(O^+_N)_*$ of $L^\infty(O^+_N)$ if so does $\psi_t$, since the Fourier algebra $L^\infty(O^+_N)_*$ is an ideal of the Fourier-Stieltjes algebra $C(O^+_N)^*$ according to [9, Prop 3.15]. From the previous bounds, we deduce that

$$\ln(1/q(N)) - t_{abscont}(N) \left(\frac{1}{N} + \frac{2}{N^3} + O\left(\frac{1}{N^4}\right)\right) = 0,$$

and to obtain a good estimate on $t_{abscont}(N)$, the only thing left to be done is to give a Taylor expansion of $\ln(1/q(N))$. Observe that

$$q(N) = \frac{N}{2} \left(1 - \sqrt{1 - \frac{4}{N^2}}\right) = \frac{1}{N} + O\left(\frac{1}{N^3}\right),$$

so that

$$q(N)^{-1} = N \left(1 + O\left(\frac{1}{N^2}\right)\right).$$

From this we obtain that

$$\ln(1/q(N)) = \ln(N) + O\left(\frac{1}{N^2}\right).$$

This allows us to write

$$t_{abscont}(N) = \ln(1/q(N))/\left(\frac{1}{N} + \frac{2}{N^3} + O\left(\frac{1}{N^4}\right)\right) = N \ln(N) - \frac{2\ln(N)}{N} + O\left(\frac{1}{N}\right),$$

which concludes the proof. $\square$

**Remark 3.8.** This is in sharp contrast with the classical case, where any non-degenerate (a condition analogous to requiring $b > 0$) Lévy process automatically has an $L^2$-density with respect to the Haar measure by [22, Thm 1], and is thus absolutely continuous.

In particular, we see that for $\psi_N N \ln(N) + cN$ to be absolutely continuous, one must have $c \geq -2\ln(N)/N^2 + O\left(1/N^2\right)$, which goes to 0 as $N$ goes to infinity. Thus the total variation distance is asymptotically only defined for $c \geq 0$.

### 3.2. Cutoff profile.

We will now try to get a better understanding of the cutoff phenomenon by computing the corresponding **cutoff profile**, that is to say the limit of the distance between the process at time $t = N \ln(N) + cN$ and the Haar state as $N$ goes to infinity, $c$ being fixed. Our main result is an expression of this limit as the distance between two explicit probability measures. Before stating it, let us give some heuristics.

In the proof of Theorem 3.2, we saw that it was enough to consider the element $\chi_1$ to obtain a lower bound of the correct order for the mixing time. In the case of a classical compact matrix group, $\chi_1$ is nothing but the trace function, and this would mean that the trace of the matrices is the last thing to be mixed by the Brownian motion. In the case of $O^+_N$, we know that the distribution of $\chi_1$ under the Haar state is the semi-circle distribution $\nu_{SC}$, so that we may expect the cutoff profile to be given by the distance between $\nu_{SC}$ and a “deformation” of it. The whole problem of course lies in the vague meaning of the word “deformation”.

We will show in the first part of Theorem 3.9 that the profile indeed appears as the distance between $\nu_{SC}$ and a family of closely related laws called the **free Poisson distributions**. Let us recall that the free Poisson distribution with rate $\lambda$ and jump size $\alpha$ is given, for $\lambda > 1$, by

$$d\text{Poiss}^+(\lambda, \alpha)(t) = \frac{1}{2\pi at^2} \sqrt{4\lambda^2 - (t - \alpha(1 + \lambda))^2} dt$$

(see [27, Def 12.12] for details). Unfortunately, there is no value of the parameters for which the free Poisson distribution equals the semi-circle one.

One can nevertheless write things differently using a larger family of probability distribution called the **free Meixner distributions**. Let us denote by $\text{Meix}^+(a, b)$ the standardised (i.e. with mean 0 and variance 1) free Meixner law with parameters $a$ and $b$ (see for instance [7, Sec 2.2] for details). Its absolutely continuous part with respect to the Lebesgue measure is given by

$$d\text{Meix}^+(a, b)(t) = \frac{\sqrt{4(1 + b) - (t - a)^2}}{2\pi(bt^2 + at + 1)} 1_{|a - 2\sqrt{1+b}| < 2\sqrt{1+b} + 2\sqrt{1+|a|}} dt.$$  

For $a = b = 0$, the formula reduces to the density of the semi-circular distribution, while for $b = 0$ it yields the density of a free Poisson distribution with mean 0 and variance 1.
We will now state our result using both the free Poisson and the free Meixner settings, after introducing some extra notations. If $X$ is a random variable with law $\mu$, then we denote by $D_r(\mu)$ the $r$-dilation of $\mu$ (that is to say the law of $rX$) and by $\mu \ast \delta_a$ its translation by $a$ (that is to say the law of $X + a$).

**Theorem 3.9.** Let $c > 0$, and recall $t_c = N \ln(N) + cN$. Then

$$d_{TV}(\psi_{t_c}, h) \xrightarrow{N \to \infty} f(c) := d_{TV}(\text{Pois}^+(e^{2c}, -e^{-c}) \ast \delta_{e^{c} + e^{-c}}, \nu_{\text{SC}})$$

$$= d_{TV}(\text{Meix}^+(-e^{-c}, 0) \ast \delta_{e^{-c}}, \text{Meix}^+(0, 0)),$$

where $d_{TV}$ denotes the usual total variation distance for Borel measures on $\mathbb{R}$.

**Proof.** Recall that as $c > 0$, $\psi_{t_c}$ has an $L^1$-density given by

$$a_{t_c} = \sum_{n=0}^{+\infty} d_n e^{-t_c \psi(n)} \chi_n$$

Moreover, we know from Lemma 3.6 that

$$\psi(n) = \frac{n}{N} + O \left( \frac{1}{N^3} \right),$$

and an easy computation yields $d_n \sim N^n$. In particular, for each $n$, $d_n e^{-t_c \psi(n)}$ converges to $e^{-cn}$ as $N$ goes to $+\infty$. Moreover,

$$\|\sum_{n=1}^{+\infty} d_n e^{-t_c \psi(n)} \chi_n\|_1 \leq \|\sum_{n=1}^{+\infty} d_n e^{-t_c \psi(n)} \chi_n\|_2 = d_n e^{-t_c \psi(n)}$$

and because $qN \geq 1$, for $N \geq 3$,

$$d_n e^{-t_c \psi(n)} \leq \frac{q^{-n}}{(1 - q^2)^n} N^{-n} e^{-nc} \leq \frac{3}{2} e^{-nc}.$$ 

The latter being summable and independent of $N$, we can exchange the sum over $n$ and the limit in $N$. This yields (recall that there is an isomorphism between $O(\mathcal{O}_N)$central and $\mathbb{C}[X]$ sending $\chi_n$ to $P_n$ and sending the measure associated to $h$ to the semi-circle distribution)

$$\lim_{N \to +\infty} \|\psi_{t_c} - h\|_{TV} = \frac{1}{2} \left\| \sum_{n=1}^{+\infty} e^{-cn} P_n \right\|_1,$$

where the $L^1$-norm is computed with respect to the standard semi-circular distribution. Using the generating series of the Chebyshev polynomials of the second kind (which is easily computed, multiplying by $t$ and using the recursion relation), we get for every $t \in [-2, 2]$,

$$\sum_{n=1}^{+\infty} e^{-cn} P_n(t) = \frac{1}{1 - e^{-c} + e^{-2c}} - 1$$

$$= \frac{1}{1 + \beta^2 (1 - \gamma t) - 1} = F_c(t) - 1,$$

where $\beta = e^{-c}$ and $\gamma = \beta/(1 + \beta^2) < 1/2$. Thus, the cutoff profile is equal to

$$\lim_{N \to +\infty} \|\psi_{t_c} - h\|_{TV} = \frac{1}{2} \int_{-2}^{2} |F_c(t) - 1| d\nu_{\text{SC}}(t).$$

Performing the change of variables $u = 1 - \gamma t$,

$$F_c(t) d\nu_{\text{SC}}(t) = F_c(t) \frac{\sqrt{4 - \gamma^2} - 1}{2\pi} 1_{[-2,2]}(t) dt = \frac{1}{2\pi(1 + \gamma^2)} \sqrt{4 - \left(\frac{1 - u}{\gamma}\right)^2} 1_{[1-\gamma,1+\gamma]}(u) \frac{du}{\gamma}$$

$$= \frac{1}{2\pi\gamma^2(1 + \gamma^2)} \sqrt{4\gamma^2 - (1 - u)^2} 1_{[1-\gamma,1+\gamma]}(u) du.$$ 

Setting $\alpha = \beta \gamma = \gamma^2(1 + \beta^2)$ and $\lambda = \beta^{-2} > 1$, this density becomes

$$\frac{1}{2\pi\alpha u} \sqrt{4\lambda u^2 - (u - \alpha(1 + \lambda))^2 1_{[\alpha(1 - \sqrt{\lambda})^2, \alpha(1 + \sqrt{\lambda})^2]}(u) du}.$$
This is exactly the free Poisson distribution with rate \( \lambda = e^{2c} \) and jump size \( \alpha = e^{-2c}/(1 + e^{-2c}) \). Reversing the change of variables, we see that \( F_c(t)d\nu_{SC}(t) \) is the density of the law
\[
D_{-1/\gamma} \left( \text{Poiss}^+ \left( \beta^{-2}, -\beta \gamma \right) \ast \delta_{-1} \right) = \text{Poiss}^+ \left( \beta^{-2}, -\beta \right) \ast \delta_{1/\gamma} = \text{Poiss}^+ \left( e^{2c}, -e^{-c} \right) \ast \delta_{e^{c}+e^{-c}},
\]
hence the result. Using the facts that \( \text{Poiss}^+ \left( a^{-2}, a \right) \ast \delta_{-a} = \text{Meix}^+ (a, 0) \) and that \( \nu_{SC} = \text{Meix}^+(0, 0) \), the second formula follows.

As explained heuristically at the beginning of this subsection, the fact that the Brownian motion is not completely mixed is witnessed by the “trace” it can attain, and the cutoff profile gives a precise quantitative description of this phenomenon. In particular, it shows that the “trace” of the Brownian motion is averagely shifted to the right and more concentrated around its mean. Here is a plot of the density of \( \text{Meix}^+(-e^{-c}, 0) \ast \delta_{e^{-c}} \) with respect to the Lebesgue measure for values of \( c \) between 0 and 5:

For \( c = 0 \) we get a free Poisson law (this is the curve with a peak on the right) while for \( c = 5 \) the density is already indistinguishable to the naked eye from that of the semi-circle distribution.
Remark 3.10. The integral giving the total variation distance can be computed explicitly in terms of $c$, yielding the formula:

$$f(c) = \left| \frac{e^{-2c} - 1}{2\pi e^{-2c}} \right| \arcsin \left( \frac{e^{-3c} - 3e^{-c}}{2} \right) + \frac{e^{-2c} - 1}{2\pi e^{-2c}} \arcsin \left( \frac{e^{-c}}{2} \right) + \frac{e^{-2c} + 2}{4\pi e^{-c}} \sqrt{4 - e^{-2c}}.$$

By Proposition 3.7, the computations above can only make sense for $c \geq 0$. However, the free Poisson distribution makes sense even for $c < 1$, with the only difference that some mass is carried by an atom:

$$d\text{Poiss}^+(\lambda, \alpha)(t) = (1 - \lambda) \delta_0 + \frac{\lambda}{2\pi \alpha t} \sqrt{4\lambda \alpha^2 - (t - \alpha(1 + \lambda))^2} 1_{\alpha(1 - \sqrt{\lambda})^2, \alpha(1 + \sqrt{\lambda})^2}(t)dt.$$

As a consequence, the formula

$$f(c) = d_{TV}\left(\text{Poiss}^+(e^{2c}, -e^{-c}) \ast \delta_{e^{c}+e^{-c}}, \nu_{SC}\right)$$

does indeed make sense for $c < 0$. Plotting it shows that it is a reasonable candidate for a complete cutoff profile:

![Profile](image)

We will now show that this is indeed the profile in a weak sense. Let us start with a characterization of the limit distribution in terms of "Chebyshev moments".

Lemma 3.11. For any $c \in \mathbb{R}$, the measure

$$\mu_c = \text{Poiss}^+(e^{2c}, -e^{-c}) \ast \delta_{e^{c}+e^{-c}}$$

is the unique probability measure on $\mathbb{R}$ such that for any $n \in \mathbb{N},$

$$\int_{\mathbb{R}} P_n d\mu_c = e^{-cn}.$$

Proof. Let us first recall that the free cumulants (see [27, Def 11.3] for the definition of free cumulants and [27, Prop 12.11] for the free Poisson case) of the free Poisson distribution $\text{Poiss}^+(\lambda, \alpha)$ are given by

$$\kappa_n = \lambda \alpha^n.$$

As a consequence, the free cumulants of $\text{Poiss}^+(e^{2c}, -e^{-c})$ are Laurent polynomials in $e^c$. The free additive convolution with $\delta_{e^{c}+e^{-c}}$ only modifies the first cumulant $\kappa_1$ by adding $e^c + e^{-c}$ to it, hence the free cumulants of $\mu_c$ are Laurent polynomials in $e^c$. Because the moments are polynomial functions of the free cumulants (by virtue of the moment-cumulant formula, see [27, Prop 11.4]), we conclude that there exist Laurent polynomials $L_n$ such that for any $c \in \mathbb{R},$

$$\int_{\mathbb{R}} P_n(x) d\mu_c(x) = Q_n(e^c).$$

Let us now assume that $c > 0$. Then, we know from the proof of Theorem 3.9 that $\mu_c$ is absolutely continuous with respect to the semi-circle distribution $\nu_{SC}$ with density $F_c$. Using the fact that the polynomials $P_n$ are orthonormal for the semi-circle distribution, we get

$$\int_{\mathbb{R}} P_n d\mu_c = \int_{\mathbb{R}} P_n \left( \sum_{n=0}^{+\infty} e^{-cn} P_n \right) d\nu_{SC} = e^{-cn}.$$

As a consequence, $Q_n(x) = x^{-n}$ for any $x > 1$, so that by uniqueness of the decomposition of a Laurent polynomial, $Q_n(X) = X^{-n}$ and the formula for the integral of Chebyshev polynomials also holds for $c \leq 0$.

As for the uniqueness assertion, it simply follows from the fact that $\mu_c$ has compact support, hence is determined by its moments. \qed
With this in hand, we can at least prove that the difference $\psi_{t_c} - h$ converges to the difference $\mu_c - h$ for certain topologies on measures. More precisely, let us consider the probability measure $m^{(N)}$ on $\mathbb{R}$ given by

$$
\int_{-N}^{N} P_n dm^{(N)} = \varphi_n(\chi_n)
$$

for all $n \in \mathbb{N}$. Using the centrality of the process and the Riesz representation theorem (see the discussion at the beginning of Section 3.1), we know that

$$
\|\psi_{t_c} - h\| = \left\| m^{(N)} - \nu_{SC} \right\|_{TV}.
$$

Of course, the convergence of $m^{(N)} - \nu_{SC}$ is equivalent to the convergence of $m^{(N)}$ and we can prove that this takes place, though not in total variation distance.

**Proposition 3.12.** For any $c \in \mathbb{R}$, the following holds in the sense of weak convergence of measures:

$$
m^{(N)} \xrightarrow{N \to +\infty} \text{Pois}^{+} (e^{2c}, -e^{-c}) * \delta_{e^{c} + e^{-c}}.
$$

**Proof.** We have already computed that for any $c \in \mathbb{R}$,

$$
\int_{\mathbb{R}} P_n dm^{(N)} \xrightarrow{N \to +\infty} e^{-cn}.
$$

By Lemma 3.11, the limit is the integral of $P_n$ with respect to Pois$^{+} (e^{2c}, -e^{-c}) * \delta_{e^{c} + e^{-c}}$, hence we have convergence in moments because the Chebyshev polynomials form a basis of $C[X]$. Since moreover the sequence is uniformly bounded in total variation, it is tight so that by Prokhorov’s criterion, it is relatively weakly compact. In particular, it has a weakly converging subsequence. The limit being determined by its moments, any converging subsequence has the same limit, hence the whole sequence must converge weakly to that limit, concluding the proof. \qed

We were not able to upgrade this result to a convergence in Fourier-Stieltjes norm, but we strongly believe that it should hold. We therefore state it as a conjecture:

**Conjecture.** For any $c \in \mathbb{R}$,

$$
\|\psi_{t_c} - h\| \xrightarrow{N \to +\infty} \| \text{Pois}^{+} (e^{2c}, -e^{-c}) * \delta_{e^{c} + e^{-c}} - \nu_{SC} \|_{TV}.
$$

3.3. **Further results.** Let us complete this section with some additional remarks and results concerning various generalizations of the original problem.

3.3.1. **Other norms.** We have worked so far with the Fourier-Stieltjes norm, because it is the only natural norm available which makes sense for all $t \in \mathbb{R}_+$. However, the upper bound was computed using the total variation distance, and one may wonder whether the cutoff upper bound also occurs with respect to other distances. It turns out that the answer is yes.

**Corollary 3.13.** The central Lévy process on $O^+_N$ given by $(\psi(n))_{n \in \mathbb{N}}$ satisfies, for all $1 \leq p \leq \infty$ and $c > 0$, with $t_c = N \ln(N) + cN$,

$$
\lim_{N \to +\infty} \|\psi_{t_c} - h\|_{L^p} = \| \text{Meix}^+ (-e^{-c}, 0) * \delta_{e^{-c}} - \text{Meix}^+(0, 0) \|_{L^p}.
$$

**Proof.** Recall that the density of the process at time $t$ is, if the series makes sense,

$$
a_t = \sum_{n=0}^{+\infty} d_n e^{-t\psi(n)} \chi_n.
$$

Using the fact that $\|\chi_n\|_{\infty} = P_n(2) = n + 1$, we see that the density converges in $L^\infty$-norm at $t_c$ as soon as $c > 0$ since (for $N \geq 3$)

$$
\left\| d_n e^{-t\psi(n)} \right\|_{\infty} \leq (n + 1) d_n e^{-t\psi(n)} \leq \frac{3}{2} (n + 1) e^{-nc}.$$
Moreover, using this bound the same strategy as for Theorem 3.9 yields the cutoff profile in the case \( p = \infty \). As for finite \( p \), it follows from the noncommutative Hölder inequality (see for instance [30, Thm 2.13.iv]) that for any \( 1 \leq p \leq \infty \),
\[
\|d_n e^{-t_{\psi_{\infty}(n)}} \chi_n\|_p \leq \|d_n e^{-t_{\psi_{\infty}(n)}} \chi_n\|_\infty
\]
hence we can once again resort to the same argument. \( \square \)

Let us compare this with the classical case. P.-L. Méliot proved in [24, Thm 7], building on results of G. Chen and L. Saloff-Coste in [10], that the cutoff phenomenon for the Brownian motion on \( SO(N) \) indeed occurs for all \( 1 \leq p \leq \infty \) and that the cutoff time is the same as for the \( L^1 \)-norm for all \( 1 \leq p < \infty \). However, for \( p = \infty \), the cutoff is doubled and becomes \( 4 \ln(N) \). It is therefore quite surprising that in the quantum case, the difference between the case of finite and infinite parameter \( p \) disappears.

3.3.2. The free real sphere. In [24], P.-L. Méliot did not only prove the cutoff phenomenon for compact simple Lie groups, but also for their homogeneous spaces. In the quantum setting, there is no structure theory of compact homogeneous spaces paralleling the classical one, but there are nevertheless some explicit examples. We will now consider the simplest of them, which is an analogue of the real sphere on which the classical orthogonal group acts. The same idea of “liberation” as for the definition of free orthogonal quantum groups suggest that the free analogue of the real sphere should be described by the universal \(*\)-algebra generated by \( N \) self-adjoint elements \((x_i)_{1 \leq i \leq N}\) such that
\[
\sum_{i=1}^{N} x_i^2 = 1.
\]
Denoting by \( \mathcal{O}(S^N_{+}^{-1}) \) this object, it is endowed with an action of \( O^+_N \) through the map
\[
\alpha : x_i \mapsto \sum_{i=1}^{N} u_{ij} \otimes x_j.
\]
Note that the abelianization of \( \mathcal{O}(S^N_{+}^{-1}) \) is exactly the algebra of polynomial functions on the \( N - 1 \) dimensional sphere in \( \mathbb{R}^N \) and that the formula defining \( \alpha \) also defines the usual action of \( O_N \) on that sphere.

Intuitively, the Brownian motion on such a space should be a Lévy process invariant under the action \( \alpha \), and the analogue of the uniform measure should be the unique probability measure invariant under \( \alpha \). Such an \( \alpha \)-invariant state does indeed exist and can be constructed in the following way. Consider the subalgebra \( \mathcal{O}(X_N) \subset \mathcal{O}(O^+_N) \) generated by the elements \( u_{i1} \) for \( 1 \leq i \leq N \). Then, there is a surjective \(*\)-homomorphism \( \pi : \mathcal{O}(S^N_{+}^{-1}) \to \mathcal{O}(X_N) \) sending \( x_i \) to \( u_{i1} \). Moreover, one has
\[
(\pi \otimes \text{id}) \circ \alpha = (\Delta \otimes \text{id}) \circ \pi
\]
so that the state \( \omega = h \circ \pi \) is invariant under the action \( \alpha \). As a consequence, we will only consider the “concrete” model \( \mathcal{O}(X_N) \) instead of \( \mathcal{O}(S^N_{+}^{-1}) \).

The Brownian motion considered in [24] on a homogeneous space is then the projection of the Brownian motion coming from the group. In our case, this simply amounts to restricting \( \psi_t \) to \( \mathcal{O}(X_N) \). Before giving the expression, let us first recall that by [12, Lem 7.3], one may find a basis for the carrier Hilbert space of each irreducible representation \( u^n \) of \( O^+_N \) such that
\[
\mathcal{O}(X_N) = \text{Span}\{u^n_{i1} \mid 1 \leq i \leq N, n \in \mathbb{N}\}.
\]

**Proposition 3.14.** The Lévy process given by the restriction of \( \psi_t \) to \( \mathcal{O}(X_N) \) exhibits a cutoff phenomenon at time \( t_N = \frac{1}{2} N \ln(N) \).

**Proof.** We will only sketch the proof, since it is similar to that of Theorem 3.2. For \( t \) large enough, the density of \( \psi_t - h \) is
\[
\sum_{n=1}^{+\infty} d_n e^{-t_{P_n^i}(N)/P_n(N)} u^n_{11}\]
whose \( L^2 \)-norm squared is
\[
\sum_{n=1}^{+\infty} d_n e^{-2t_{P_n^i}(N)/P_n(N)}.
\]
The difference with the previous case is that the dimension $d_n$ is not squared, due to the fact that coefficients of irreducible representations form an orthogonal but not orthonormal basis. This accounts for the factor $1/2$ in the cutoff time, exactly as in [24].

As for the lower bound, it is obtained by using the element $x = \sqrt{N}u_{11}$ and the computations are the same as in the proof of Theorem 3.2. \hfill \Box

**Remark 3.15.** Let us consider, for $c > 0$, the process at time $\frac{1}{2}(N \ln(N) + cN)$. Then, it is clear that the limit of the distance as $N$ goes to infinity yields the same result as for the Brownian motion on $O_N^+$. There is however another candidate for a Brownian motion on the real free sphere. Lévy processes on the later quantum space were classified by B. Das, U. Franz and X. Wang in [12, Thm 7.5] using a formula similar to Equation (1), i.e. involving a positive constant $b$ and a Lévy measure $\nu$. Taking as before $b = 1$ and $\nu = 0$ yields a reasonable notion of a Brownian motion on $X_N$ which is not the projection of the one on $O_N^+$. The convolution semigroup of states $(\varphi_t)_{t \in \mathbb{R}_+}$ we are interested in is then given by:

$$\varphi_t : u_{11}^n \mapsto \delta_{11}e^{-tR_n(1)},$$

where the polynomials $(R_n)_{n \in \mathbb{N}}$ are the orthogonal polynomials associated to the spectral measure of $u_{11}$ (see [4] for details and explicit computations).

**Proposition 3.16.** The $O_N^+$-invariant Lévy process on $X_N$ given by $(\varphi_t)_{t \in \mathbb{R}_+}$ exhibits a cutoff phenomenon at time $t_N = \frac{1}{2} \ln(N)$.

**Proof.** For $t$ large enough, $\varphi_t$ has an $L^2$-density with respect to $\omega = h \circ \pi$ given by

$$\sum_{n=0}^{+\infty} a_n e^{-tR_n(1)} u_{11}^n$$

so that

$$\|\varphi_t - \omega\|_{TV}^2 \leq \frac{1}{4} \|a_t - 1\|_1^2 \leq \frac{1}{4} \|a_t - 1\|_2^2 = \frac{1}{4} \sum_{n=1}^{+\infty} d_n^2 e^{-2tR_n(1)} \|u_{11}^n\|^2_2 = \frac{1}{4} \sum_{n=1}^{+\infty} d_n^2 e^{-2tR_n(1)}.$$

Now, we know from [12, Cor 7.14] that

$$n \leq R_n(1) \leq \frac{N - 1}{N - 2} n,$$

and combining this with [19, Lem 3.3] shows that $t = \frac{1}{2}(\ln(N) + c)$ is enough to ensure the existence of the $L^2$-density and that

$$\|\varphi_t - \omega\|_{TV}^2 \leq \frac{1}{4} \frac{1}{(1 - q^2)} \sum_{n=1}^{+\infty} q^{-n} e^{-2t_n} \leq \frac{1}{4} \frac{1}{(1 - q^2)} q e^{2t} - 1 \leq \frac{1}{2} e^{-c}$$

and the upper bound follows.

The lower bound is proven as in Proposition 3.14. \hfill \Box

**Remark 3.17.** Note that there is an abuse of notations since the polynomials $R_n$ also depend on the integer $N$. This is different from the case $O_N^+$ where for all $N$, the orthogonal polynomials were always the same Chebyshev polynomials $P_n$. That fact, combined with the cumbersome available descriptions of $R_n$ (see for instance [12, Sec 7.3]), make it difficult to compute the cutoff profile. However, because $\sqrt{N}u_{11}$ becomes semi-circular when $N$ goes to infinity by [3, Thm 6.1], it is reasonable to conjecture that $\sqrt{N}R_n$ converges to $P_n$, in which case we would obtain the same cutoff profile as for $O_N^+$ when considering time $t = \frac{1}{2}(\ln(N) + c)$.

4. Quantum permutations

Our second family of examples will be quantum permutations. The quantum permutation groups $S_N^+$ were introduced by Sh. Wang in [35]. The corresponding $*$-algebra $\mathcal{O}(S_N^+)$ is the quotient of $\mathcal{O}(O_N^+)$ by the relations $u_{ij}^2 = u_{ij}$. The coproduct factors through this and yields the compact quantum group structure. The connection to classical permutation may seem loose from that definition, but one easily shows that if $c_{ij} : S_N \to \mathbb{C}$ is the function sending a permutation $\sigma$ to $\delta_{\sigma(i)j}$, then there is a surjective $*$-homomorphism $\mathcal{O}(S_N^+) \to \mathcal{O}(S_N)$ sending $u_{ij}$ to $c_{ij}$ and that the latter is in fact the abelianization of the former. Thus, $S_N^+$ is
a quantum version of $S_N^+$ somehow like $O_N^+$ is the quantum version of $O_N$. Beyond this fact which motivated the original definition, several strong connections between classical and quantum permutations have emerged which strongly back the idea that $S_N^+$ is the correct generalization of $S_N$. An example of particular interest from the probabilistic point of view is the free De Finetti theorem of C. Köstler and R. Speicher [20].

The representation theory of $S_N^+$ is close to that of $O_N^+$, with the difference that when multiplying two characters (which are still indexed by the integers with $\chi_0 = 1$ and $\chi_1 = \sum_{i=1}^N u_i$),

$$\chi_1 \chi_n = \chi_{n+1} + \chi_n + \chi_{n-1}.$$ 

The corresponding orthogonal polynomials are then given by the restriction to $[0, 4]$ of $Q_n(t) = P_{2n}(\sqrt{t})$, yielding the free Poisson law $\text{Poiss}^+(1, 1)$ as spectral measure of $\chi_1$ under the Haar state. We are now going to study two examples of processes on $S_N^+$, one continuous and one discrete.

### 4.1. Brownian motion

The natural candidate for the Brownian motion on $S_N^+$ can be constructed exactly as in the case of $O_N^+$. Indeed, U. Franz, A. Kula and A. Skalski proved in [17, Thm 10.10] a decomposition result for central Lévy processes on $S_N^+$ involving as before a positive constant $b$ and a Lévy measure $\nu$. Setting $b = 1$ and $\nu = 0$ leads to a central Lévy process. We will again denote by $(\psi(n))_{n \in \mathbb{N}}$ the sequence determining the process, which is in this case given by

$$\psi(n) = \frac{Q_n'(N)}{Q_n(N)}.$$ 

The previous arguments carry on almost verbatim to yield the cutoff phenomenon and one can once again describe the cutoff profile as a distance between two free Meixner laws.

**Theorem 4.1.** The central Lévy process defined above exhibits a cutoff phenomenon at time $N \ln(N)$. Moreover, setting again $t_c = N \ln(N) + cN$ for every $c \in \mathbb{R}$, we have for $c > 0$,

$$\lim_{N \to +\infty} \frac{n}{N} \leq \frac{Q_n'(N)}{Q_n(N)} \leq \frac{n}{\sqrt{N(N - 2)}}.$$ 

Proof. The proof of the cutoff phenomenon is very similar to the case of $O_N^+$, the estimate for the derivative of $P_n$ yielding

$$\frac{n}{N} \leq \frac{Q_n'(N)}{Q_n(N)} \leq \frac{n}{\sqrt{N(N - 2)}}.$$ 

Moreover, from Proposition 3.6 we can get a more precise asymptotic expansion: for every $n \geq 1$,

$$\psi(n) = \frac{Q_n'(N)}{Q_n(N)} = \frac{1}{2\sqrt{N}} \frac{P_{2n}(\sqrt{N})}{P_{2n}(\sqrt{N})} = n \left( \frac{1}{N} + O \left( \frac{1}{N^2} \right) \right).$$ 

As for the lower bound, we can bound from below the expectation of $\chi_1$ : for any $c$,

$$\psi_{t_c}(\chi_1) = (N - 1)e^{-t_c/(N-1)} = e^{-c} \left( 1 + O \left( \frac{\ln(N)}{N} \right) \right).$$ 

We also need an upper bound on the variance. As in this case $\chi_1^2 = \chi_0 + \chi_1 + \chi_2$,

var\,\psi_{t_c}(\chi_1) = \psi_{t_c}(\chi_1^2) - \psi_{t_c}(\chi_1)^2

$$= 1 + Q_1(N)e^{-t_c\psi(1)} + Q_2(N)e^{-t_c\psi(2)} - Q_1(N)^2e^{-2t_c\psi(1)}$$

$$= 1 + (N - 1)e^{-t_c\left( \frac{1}{N} + O \left( \frac{1}{N^2} \right) \right)} + (N^2 - 3N + 1)e^{-2t_c\left( \frac{1}{N} + O \left( \frac{1}{N^2} \right) \right)} - (N - 1)^2e^{-2t_c\left( \frac{1}{N} + O \left( \frac{1}{N^2} \right) \right)}$$

$$= 1 + e^{-c} + O \left( \frac{\ln(N)}{N} \right).$$ 

As this variance stays bounded as $N \to \infty$, the same strategy as in the proof of Theorem 3.2 finishes the proof.

Let us now compute the cutoff profile. Setting

$$G_c(t) = \sum_{n=0}^{+\infty} e^{-cn}Q_n(t),$$
the cutoff profile equals \( \|G_{2c}(t) - 1\|_1 \),

where the \( L^1 \) norm is computed with respect to the spectral measure of \( \chi_1 \) with respect to the Haar state, which is \( \text{Poiss}^+(1,1) \). Note that because \( P_{2n} \) is an even function and \( P_{2n+1} \) an odd one,

\[
G_{2c}(t) = \sum_{n=0}^{\infty} e^{-2cn} P_{2n}(\sqrt{t}) = \frac{F_c(\sqrt{t}) + F_c(-\sqrt{t})}{2}.
\]

Setting \( \beta = e^{-c} \) and \( \gamma = \beta/(1 + \beta^2) \) are as in Subsection 3.2, this leads to the formula

\[
G_{2c}(t) = \frac{1}{2(1 + \beta^2)} \left( \frac{1}{1 - \gamma \sqrt{t}} + \frac{1}{1 + \gamma \sqrt{t}} \right) = \frac{1}{1 + \beta^2} \frac{1}{1 - \gamma^2 t}.
\]

Let us also set

\[
\eta = \frac{1 - \sqrt{1 - 4\gamma^2}}{2\gamma^2} = 1 + \beta^2.
\]

Then, making the changes of variables \( u = t - \eta \) and \( v = u/\sqrt{\eta} \), and observing that \( \gamma^2 = (\eta - 1)\eta^{-2} \), the density of \( G_{2c}(t)d\text{Poiss}^+(1,1)(t) \) becomes

\[
\frac{1}{\eta} \frac{1}{1 - \gamma^2 t} \frac{1}{2\pi t} \sqrt{4 - (t - 2)^2} \mathbf{1}_{[0,4]}(t) dt
\]

\[
= \frac{1}{2\pi \eta} \frac{1}{(1 - \gamma^2(\eta + \eta))(\eta + \eta)} \sqrt{4 - (\eta - 2)^2} \mathbf{1}_{[-\eta,4-\eta]}(\eta) du
\]

\[
= \frac{1}{2\pi \eta} \frac{1}{(1 - \eta - 1)\eta^{-2}(v \sqrt{\eta} + \eta)(v \sqrt{\eta} + \eta)} \sqrt{4 - (v \sqrt{\eta} - 2)^2} \mathbf{1}_{[-\sqrt{\eta},4-\eta]}(v) \sqrt{\eta} dv
\]

\[
= \frac{1}{2\pi} \frac{1}{1 + v(2 - \eta)/\sqrt{\eta} + v^2(1 - \eta)/\eta} \sqrt{4/\eta - (v - 2)^2} \mathbf{1}_{[-\sqrt{\eta},4-\eta]}(v) dv.
\]

Setting \( a = (2 - \eta)/\sqrt{\eta} \), and \( b = (1 - \eta)/\eta \), this is exactly the density of the standardised free Meixner law with parameters \( a \) and \( b \),

\[
\frac{1}{2\pi} \sqrt{4(1 + b) - (v - a)^2} \mathbf{1}_{a-2\sqrt{1+b},a+2\sqrt{1+b}} dv.
\]

Thus, \( G_{2c}(t)d\text{Poiss}^+(1,1)(t) \) is the density of the law

\[
D_{\sqrt{\eta}} \left( \text{Meix}^+ \left( \frac{2 - \eta}{\sqrt{\eta}}, \frac{1 - \eta}{\eta} \right) \right) * \delta_\eta.
\]

Writing \( \text{Poiss}^+(1,1) = \text{Meix}^+(0,1) * \delta_1 \), applying * \( \delta_{-1} \) on both sides and replacing \( 2c \) by \( c \) now yields the desired result.

\[ \square \]

We can give an interpretation of this result similar to the one for Theorem 3.2. Indeed, the function giving the number of fixed points of a permutation is, in terms of the generators of \( O(S_N) \), \( F = \sum c_i t_i \). Therefore, the elements \( \chi_1 = \sum u_i t_i \) is the quantum version of the number of fixed points. In particular, its law with respect to the Haar state, which is \( \text{Poiss}^+(1,1) \), can be considered as the “fixed points law for quantum permutations”. As a consequence, the difference between the Brownian motion and the uniform measure on \( S_N^+ \) is asymptotically due to the fact that the Brownian motion has “too many fixed points”.

As for the cutoff profile on the left (for negative \( c \)), the same argument as for the orthogonal case in Proposition 3.12 yields

**Proposition 4.2.** For any \( c \in \mathbb{R} \), the following holds in the sense of weak convergence of measures:

\[
m^{(N)}_c \overset{N \to +\infty}{\longrightarrow} D_{\sqrt{1+e^{-c}}} \left( \text{Meix}^+ \left( \frac{1 - e^{-c}}{\sqrt{1+e^{-c}}}, \frac{-e^{-c}}{1 + e^{-c}} \right) \right) * \delta_{1+e^{-c}}.
\]
4.2. Quantum random transpositions. We will conclude with a discrete example, namely the quantum random transposition walk on the symmetric group. The reason for this is that the second-named author recently computed the cutoff profile for the classical version of that walk, while nothing is known in the quantum case.

Recall that if $\mu_{tr}$ is the uniform measure on the set of transpositions, then the classical random transposition walk has distribution

$$\mu = \frac{N - 1}{N} \mu_{tr} + \frac{1}{N} \delta_e.$$ 

One of the first results in the theory of the cutoff phenomenon was the proof by P. Diaconis and M. Shahshahani in [14] that the random transposition walk exhibits a cutoff phenomenon at $\frac{1}{2} N \ln(N)$ steps. The second named author proved in [31] that the cutoff profile has the following form: for any $c \in \mathbb{R}$,

$$d_{TV} \left( \mu^* \frac{1}{2} \left( \frac{N \ln(N) + c N}{N} \right), h \right) \xrightarrow{N \to \infty} d_{TV} \left( \text{Poiss} \left( 1 + e^{-c} \right), \text{Poiss}(1) \right).$$

Note that $\text{Poiss}(1) * \delta_{-1}$ is the asymptotic law of the number of fixed points of a uniformly distributed permutation, which is the same as the law of the trace of a permutation matrix under the Haar measure, i.e. the law of $\chi_1$.

The $\delta_e$-part in the definition of $\mu$ is used to rule out periodicity issues, but translates into an analytical problem on the quantum side. Indeed, there is a natural analogue of $\mu_{tr}$ introduced in [19] and denoted by $\varphi_{tr}$. This is a central state given on the characters by

$$\varphi_{tr}(\chi_n) = \frac{Q_n(N - 2)}{Q_n(N)}.$$ 

It was proven in [19] that the corresponding random walk on $S_N^+$ exhibits a cutoff phenomenon (with the same caveat as in Remark 3.3), and that there is no periodicity issue. However, the state $\varepsilon : u_{ij} \mapsto \delta_{ij}$, which is the analogue of $\delta_e$, is never absolutely continuous with respect to the Haar state on $S_N^+$. As a consequence, the previous methods do not apply to the state $\varphi = \frac{N - 1}{N} \varphi_{tr} + \frac{1}{N} \varepsilon$.

To go round the problem, we will prove a stronger result, namely:

**Theorem 4.3.** Let $c > 0$ be fixed and let $k = \lceil \frac{1}{2} (N \ln(N) + c N) \rceil$. Then,

$$\| \varphi^k - \varphi_{tr}^k \|_{FS} \xrightarrow{N \to +\infty} 0.$$ 

The idea behind this statement is that in the binomial sum defining $\varphi^k$, only the last term $\varphi_{tr}^k$ asymptotically matters, so that it should determine the long-time behaviour. Proving this is however not straightforward and requires some computations, part of which will be done in separate lemmata. It will be convenient in the proofs to use an intermediate time between $N \ln(N)$ and $k = \lceil N \ln(N) + c N \rceil$. For that purpose we set $k_1 = \lceil \frac{1}{2} N (\ln(N) + c/2) \rceil$. The order of magnitude of $k - k_1 \sim cN/4$ is certainly not optimal, but it is enough for the arguments to work and proves practical for the proofs. We will consider the following truncated version of $\varphi^k$:

$$\phi^{(k_1)}_k = \sum_{p = k_1 + 1}^k \binom{k}{p} \left( \frac{N - 1}{N} \right)^p \frac{1}{N^{k - p}} \varphi_{tr}^p$$

and use it as an intermediate step between $\varphi^k$ and $\varphi_{tr}^k$. The first thing to prove is that it has the same asymptotic behaviour as $\varphi^k$.

**Lemma 4.4.** With the notations above,

$$\| \varphi^k - \phi_k^{(k_1)} \|_{FS} \xrightarrow{N \to +\infty} 0.$$
Proof. We are interested in the quantity

\[
\left\| \varphi_k^{*k} - \phi_k^{(k_1)} \right\|_{FS} \leq \sum_{p=0}^{k_1} \binom{k}{p} \left( \frac{N-1}{N} \right)^p \frac{1}{N^{k-p}} \left\| \varphi_{tr}^p \right\|_{FS}
\]

\[
\leq \sum_{p=0}^{k_1} \binom{k}{p} \left( \frac{N-1}{N} \right)^p \frac{1}{N^{k-p}} = A.
\]

This can be bounded by a probabilistic argument. Indeed, if \((X_i)_{0 \leq i \leq k}\) are i.i.d Bernoulli random variables taking the value 1 with probability \((N-1)/N\) and if \(X\) is their sum, then

\[A = \mathbb{P}(X \leq k_1).\]

Setting \(\delta = 1 - N(k_1 + 1)/(N-1)k\), the Chernoff bound then yields

\[A = \mathbb{P}(X < (1 - \delta)\mathbb{E}(X)) \leq e^{-\mathbb{E}(X)\delta^2/2} = e^{-\frac{(N-1)k}{2N} \left(1 - \frac{N(k_1+1)}{(N-1)k} \right)^2}.\]

Thus, \(A \leq e^{-\alpha}\) with

\[\alpha = \frac{(N-1)k}{2N} \left(1 - \frac{N(k_1+1)}{(N-1)k} \right)^2 \xrightarrow{N \to \infty} \frac{c^2N}{16 \ln(N)},\]

which tends to \(+\infty\), hence the result. \(\square\)

To conclude we now have to prove that

\[\left\| \varphi_{tr}^k - h \right\|_{FS} - \left\| \phi_k^{(k_1)} - h \right\|_{FS} \leq \left\| \varphi_{tr}^k - \phi_k^{(k_1)} \right\|_{FS} \xrightarrow{N \to +\infty} 0.\]

Since we are now considering two states which have an \(L^2\)-density with respect to the Haar state, it suffices to show that the difference of the densities converges to 0 in \(L^2\)-norm, and this boils down to making sure that we can exchange the limit and summation symbols for the corresponding series. In order to make computations clear, we first establish the existence of a dominating series. In the sequel, we denote by \(d_n\) the dimension of the irreducible representation corresponding to \(\chi_n\), which is explicitly given by

\[d_n = Q_n(N).\]

Lemma 4.5. For any \(N \geq 16\) and for all \(n \in \mathbb{N}\) and \(k = \frac{1}{2}(N \ln(N) + cN),\)

\[d_n |\phi_k^{(k_1)}(n) - \varphi_{tr}(n)^k| \leq e^{-cn}\]

Proof. It is shown in [19, Thm 4.4] that for \(N \geq 16,\)

\[q(\sqrt{N})^{-1}\sqrt{N}q(\sqrt{N}-2)^2(1 - q(\sqrt{N}-2)^2) \geq e^{2/N},\]
and together with [19, Lemma 3.8], for $p > k_0 = \left\lceil \frac{1}{2} N \ln(N) \right\rceil$ this yields
\[
d_n \varphi_{tr}(n)^p = Q_n(N) \left( \frac{Q_n(N - 2)}{Q_n(N)} \right)^p \\
\leq (\sqrt{N} q(\sqrt{N})^{-2} - (2n - 1))^{-(p-1)} \left( q(\sqrt{N} - 2)^{-2n} \right)^p \\
= \left( \frac{q(\sqrt{N})^{2p-2}}{q(\sqrt{N} - 2)^{2p}} \right) \frac{q(\sqrt{N})^{p-1}}{\sqrt{N}} q(\sqrt{N} - 2)^{2p} (1 - q(\sqrt{N} - 2)^2)^p \\
\leq \left( \frac{q(\sqrt{N})^{2p-2}}{q(\sqrt{N} - 2)^{2p}} \right) \frac{q(\sqrt{N})^{-1}}{\sqrt{N}} e^{-2p/N} \\
\leq \frac{\sqrt{N}}{q(\sqrt{N})^{2p-1}} e^{-2p/N} \\
\leq N^{n} e^{-2(2n \ln(N)/2 + p - k_0)/N} \\
= e^{-2n(p-k_0)/N}.
\]

We can now use this to infer that for $p \geq k_1 + 1$, $d_n \varphi_{tr}(n)^p \leq e^{-cn/2}$ so that for $k \geq k_1$,
\[
d_n (\phi_k^{(k)}(n) - \varphi_{tr}(n)^k) \leq d_n \sum_{p=k_1+1}^{k} \binom{k}{p} \left( \frac{N - 1}{N} \right)^p \frac{1}{N^{k-p}} (\varphi_{tr}(n)^p - \varphi_{tr}(n)^k) \\
\leq d_n \sum_{p=k_1+1}^{k} \binom{k}{p} \left( \frac{N - 1}{N} \right)^p \frac{1}{N^{k-p}} \varphi_{tr}(n)^p \\
\leq e^{-cn/2} \sum_{p=k_1+1}^{k} \binom{k}{p} \left( \frac{N - 1}{N} \right)^p \frac{1}{N^{k-p}} \\
\leq e^{-cn/2}.
\]

We are ready to complete the proof of the main result of this section:

**Proof of Theorem 4.3.** Let us consider the densities
\[
a_k = \sum_{n=0}^{+\infty} d_n \varphi_{tr}(n)^k \chi_n \quad \text{and} \quad b_k = \sum_{n=0}^{+\infty} d_n \phi_k^{(k)}(n) \chi_n
\]
of $\varphi_{tr}^k$ and $\phi_k^{(k)}$ respectively. We want to consider
\[
\left\| \varphi_{tr}^k - h \right\|_{FS} - \left\| \phi_k^{(k)} - h \right\|_{FS} \leq \left\| \varphi_{tr}^k - \phi_k^{(k)} \right\|_{FS} = \left\| \varphi_{tr}^k - \phi_k^{(k)} \right\|_{TV} = \frac{1}{2} \left\| a_k - b_k \right\|_1 \leq \frac{1}{2} \left\| a_k - b_k \right\|_2.
\]

We will prove that this converges to 0 as $N$ goes to infinity. Using Lemma 4.5, it is enough to prove that
\[
\lim_{N \to +\infty} d_n |\varphi_{tr}(n)^k - \phi_k^{(k)}(n)| = 0
\]
to be able to conclude by exchanging the limit and summation symbols.
Let us start by using [19, Lem 3.3] to get:

\[
\frac{Q_n(N)}{Q_n(N-2)} = \frac{P_{2n}(\sqrt{N})}{P_{2n}(\sqrt{N-2})} = \frac{q(\sqrt{N})-2n}{1-q(\sqrt{N-2})^2} \frac{q(\sqrt{N})^{2n-1}}{\sqrt{N-2}} = \left(\frac{q(\sqrt{N})}{q(\sqrt{N-2})}\right)^{2n} \frac{1}{1-q(\sqrt{N}) \sqrt{N-2q(\sqrt{N-2})}} \leq \left(\frac{N}{N-2}\right)^n \frac{1}{1-q(\sqrt{N})^2},
\]

where we used in the last line [19, Lem 3.8] together with \( q(x) > 1/x \). Using \( (1-q(x)^2)^{-1} = 1 + O(1/x^2) \) eventually yields

\[
\varphi_{tr}(n)^{-1} = \frac{Q_n(N)}{Q_n(N-2)} \leq \left(\frac{N}{N-2}\right)^n + O\left(\frac{1}{N}\right).
\]

Using this and the Mean Value Theorem for the function \( x \mapsto (x/(N-2))^n \), we can compute

\[
d_n\left(\frac{N-1}{N} \varphi_{tr}(n) + \frac{1}{N}\right)^k \leq d_n\varphi_{tr}(n)^k \left(1 + \frac{\varphi_{tr}(n)^{-1} - 1}{N}\right)^k \leq d_n\varphi_{tr}(n)^k e^{k(\varphi_{tr}(n)^{-1} - 1)/N} \leq d_n\varphi_{tr}(n)^k e^{(N^n/(N-2)^n - 1)/N + O(k/N^2)} \leq d_n\varphi_{tr}(n)^k e^{2kn(N/(N-2))^{-n-1}}/N + O(k/N^2) \leq d_n\varphi_{tr}(n)^k e^{2n \ln(N+1)/N-2} \left(\frac{N}{N-2}\right)^{-n-1} + O(k/N^2)
\]

and conclude that

\[
\limsup_{N} d_n\left(\frac{N-1}{N} \varphi_{tr}(n) + \frac{1}{N}\right)^k \leq e^{-nc}.
\]

Note that this does not contradict the atomicity of \( \varphi \) because the rate of convergence depends on \( n \).

Applying now the Mean Value Theorem to the function \( x \mapsto ((1-x)\varphi_{tr}(n) + x)^k \), we get

\[
d_n(\varphi(n)^k - \varphi_{tr}(n)^k) = d_n\left(\frac{N-1}{N} \varphi_{tr}(n) + \frac{1}{N}\right)^k - \varphi_{tr}(n)^k \leq d_n k\left(\frac{N-1}{N} \varphi_{tr}(n) + \frac{1}{N}\right)^{k-1} (1 - \varphi_{tr}(n))/N.
\]

Using similar estimates, we conclude that

\[
\limsup_{N} d_n(\varphi(n)^k - \varphi_{tr}(n)^k) \leq e^{-nc} \limsup_{N} k \left(1 - \frac{(N-2)^n}{N^{n}}\right) \frac{1}{N} = 0.
\]

On the other hand, with the notation \( A \) from Lemma 4.4,

\[
d_n(\phi_{k}^{(k)}(n) - \varphi(n)^k) \leq 2N^n \sum_{p=0}^{k_0} \binom{k}{p} \left(\frac{N-1}{N}\right)^p \frac{1}{N^{k-p}} \varphi_{tr}(n)^p \leq 2e^{\ln N} A \leq 2e^{\ln N} N^{-\alpha} = 2e^{\ln N}(1-\alpha/n \ln(N))
\]
and the right-hand side goes to 0 since $\alpha/\ln(N) \to +\infty$. Combining these computations then yields
\[
\lim_{N \to +\infty} \|a_k - b_k\|_1 = 0
\]
and the proof is complete. \qed

As a consequence of this result, we get the upper bound part of the cutoff phenomenon as well as the cutoff profile one the right. Completing the proof of the cutoff phenomenon is not a difficult task, hence we state everything as a corollary.

**Corollary 4.6.** The random walk associated to $\varphi$ exhibits a cutoff phenomenon at $N \ln(N)/2$ steps in Fourier-Stieltjes norm. Moreover, the cutoff profile of the random transposition walk is given, for $c > 0$, by
\[
\lim_{N \to +\infty} \|\varphi^{*k(N \ln(N)+cN)} - h\|_{FS} = \left\| D_{\sqrt{1+e^{-c}}} \left( \frac{1-e^{-c}}{\sqrt{1+e^{-c}}} \right) \ast \delta_{e^{-c}} - \text{Meix}^+(0,1) \right\|_{TV}.
\]

**Proof.** All that is left to prove is the lower bound part of the cutoff phenomenon and this will be done by considering the value of $\varphi$ on $\chi_1$. Let $c < 0$ such that $-c \leq \ln(N)$. (In this proof $c$ is not fixed, we will take it to converge to $-\infty$ as a $-\epsilon \ln(N)$.) Recall that $k = \frac{1}{2}(N \ln(N) + cN)$. First,
\[
\varphi^{*k}(\chi_1) = (N - 1) \left( \frac{N - 1}{N} - 3 + \frac{1}{N} \right)^k = (N - 1) \left( 1 - \frac{2}{N} \right)^k = e^{-c} \left( 1 + O \left( \frac{\ln(N)}{N} \right) \right).
\]
Second,
\[
\varphi^{*k}(\chi_2) = ((N - 1)(N - 2) - 1) \left( \frac{(N - 3)(N - 4) - 1}{(N - 1)(N - 2) - 1} \right)^k = (N^2 + O(N)) \left( 1 - 4N + O \left( \frac{1}{N^2} \right) \right)^k = e^{-2c} \left( 1 + O \left( \frac{\ln(N)}{N} \right) \right).
\]
We deduce that
\[
\text{var}_{\varphi^{*k}}(\chi_1) = 1 + \varphi^{*k}(\chi_1) + \varphi^{*k}(\chi_2) - \varphi^{*k}(\chi_1)^2 = 1 + e^{-c} \left( 1 + O \left( \frac{\ln(N)}{N} \right) \right) + e^{-2c} \cdot O \left( \frac{\ln(N)}{N} \right).
\]
Thus, noting that $1 < e^{-c} < e^{-2c}$, there exist an absolute constant $K \geq 2$ (independant of $c$ an $N$) such that
\[
\text{var}_{\varphi^{*k}}(\chi_1) \leq 1 + e^{-c} + Ke^{-2c} \frac{\ln(N)}{N} \leq Ke^{-c} \left( 1 + \frac{e^{-c}\ln(N)}{N} \right).
\]
We now apply the same technique as in the proof of Theorem 3.2, except that the closure of $\mathcal{O}(S_N^+)$central is naturally isomorphic to $C([0, N])$ instead of $C([-N, N])$. Set $B = \{ s \in [0, N] \mid |\chi_1(s)| \leq e^{-c}/10 \}$ and apply Chebyshev’s inequality to get
\[
\varphi^{*k}(p) \leq 100e^{2c} \text{var}_{\varphi^{*k}}(\chi_1) \leq 100Ke^c \left( 1 + \frac{e^{-c}\ln(N)}{N} \right).
\]
Combining this with $h(\chi_1) = 0$ and $\text{var}_h(\chi_1) = 1$, we get
\[
|\mu(B)| = |\varphi^{*k}(p) - h(p)| \geq 1 - 100e^{2c} - 100K \left( e^c + \frac{\ln(N)}{N} \right).
\]
Let $\epsilon > 0$ and let $c = -\epsilon \ln(N)$. Then right-hand side tends to 1, and we can hence conclude, as $\|\varphi^{*k} - h\| \geq |\varphi^{*k}(p) - h(p)|$, that
\[
\|\varphi^{*(1-\epsilon)\frac{1}{2}N\ln(N)} - h\| \xrightarrow{N \to \infty} 1.
\]
For the explicit determination of the cutoff profile of $\varphi^k_{\text{tr}}$, the bounds given in [19, Thm 4.4] show that we can exchange the sum and limit symbols. It is therefore enough to see that
\[
d_n \varphi^k_{\text{tr}} N^{\ln(N)/2 + cN} = Q_n(N) \left( \frac{Q_n(N - 2)}{Q_n(N)} \right)^{1/2} N^{\ln(N) + cN} \\
\sim N^n \left( 1 - \frac{2}{N} \right)^{\frac{c}{2} N^{\ln(N) + cN}} \\
\sim e^{-cn}
\]

to conclude that the cutoff profile is the same as for the Brownian motion on $S^*_N$.

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