Foliations induced by metallic structures

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Abstract. We give necessary and sufficient conditions for the real distributions defined by a metallic pseudo-Riemannian structure to be integrable and geodesically invariant, in terms of associated tensor fields to the metallic structures and of adapted connections. In the integrable case, we prove a Chen-type inequality for these distributions and provide conditions for a metallic map to preserve these distributions. If the structure is metallic Norden, we describe the complex metallic distributions in the same spirit.

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1. Introduction

Let $M$ be a smooth manifold and let $J$ be a $(1, 1)$-tensor field on $M$. If $J^2 = pJ + qI$, for some $p$ and $q$ real numbers, then $J$ is called a metallic structure on $M$ and $(M, J)$ is called a metallic manifold. If $g$ is a pseudo-Riemannian metric on $M$ such that $J$ is $g$-symmetric, then $(J, g)$ is called a metallic pseudo-Riemannian structure on $M$.

The aim of this paper is to consider the complementary distributions associated to a metallic pseudo-Riemannian structure and study their integrability and geodesically invariance in terms of associated tensor fields to the metallic structure and of adapted connections. In this sense, we consider the Schouten-van Kampen, Vranceanu and Vidal connections, which seem to be the most important connections for the study of foliations of a pseudo-Riemannian manifold [1]. Moreover, for these distributions, we prove a Chen-type inequality giving a relation between the squared norm of the mean curvature and the Chen
first invariant. We also prove a leaf correspondence theorem between the leaves of two metallic pseudo-Riemannian manifolds when there is given a metallic map between them with certain properties.

The sign of \( p^2 + 4q \) is important in the study of foliations induced by metallic structures; if it is positive, then \( J \) has two real eigenvalues, if it is negative, \( J \) has two complex eigenvalues. In the real case, \( J \) can be related to almost product structures and in the complex case, to Norden structures. We remark that some properties of metallic distributions have also been studied in [11]. In this paper we consider both of these cases and we describe some similarities and differences between them. In particular, in the complex case, we compute the \( \delta \)-operator in terms of \( J \). Moreover, we construct the metallic complex cohomology and homology groups.

2. Preliminaries

2.1. Metallic pseudo-Riemannian structures.

**Definition 2.1.** [3] Let \((M, g)\) be a pseudo-Riemannian manifold and let \( J \) be a metallic structure on \( M \). We say that the pair \((J, g)\) is a metallic pseudo-Riemannian structure on \( M \) if \( J \) is \( g \)-symmetric. In this case, \((M, J, g)\) is called a metallic pseudo-Riemannian manifold. If \( p^2 + 4q < 0 \), then \((J, g)\) is called a metallic Norden structure and \((M, J, g)\) is called a metallic Norden manifold.

**Remark 2.2.** Let \((M, g)\) be a pseudo-Riemannian manifold and let \( J \) be a metallic structure on \( M \) such that \( J^2 = pJ + qI \). If we require that \( J \) is \( g \)-skew-symmetric, then we obtain that \( p = 0 \). Namely, if we assume \( g(JX, Y) = -g(X, JY) \), for any \( X, Y \in C^\infty(TM) \), then we get \( g(JX, JY) = -g(X, J^2Y) = -pg(X, JY) - qg(X, Y) = pg(X, JY) - qg(X, Y) \). On the other hand, \( g(JX, JY) = -g(J^2X, Y) = -pg(JX, Y) - qg(X, Y) \), therefore \( p = 0 \). In particular, for \( p \neq 0 \), it is not possible to define the concept of metallic Hermitian structure.

**Definition 2.3.** [3] (i) A linear connection \( \nabla \) on \( M \) is called a \( J \)-connection if \( J \) is covariantly constant with respect to \( \nabla \), i.e. \( \nabla J = 0 \).

(ii) A metallic pseudo-Riemannian manifold \((M, J, g)\) such that the Levi-Civita connection \( \nabla \) with respect to \( g \) is a \( J \)-connection is called a locally metallic pseudo-Riemannian manifold.

2.2. Associated tensors to a metallic pseudo-Riemannian structure. For a metallic pseudo-Riemannian structure \((J, g)\) on the smooth manifold \( M \) with \( \nabla \) the Levi-Civita connection of \( g \), we introduce some tensor fields [7] to characterize the properties of the metallic distributions defined by \( J \):

1. the \( J \)-bracket

\[
[X, Y]_J := [JX, Y] + [X, JY] - J([X, Y]),
\]

where \([\cdot, \cdot]\) is the Lie bracket, \([X, Y] = \nabla_X Y - \nabla_Y X\)
2. the Nijenhuis tensor associated to \( J \)

\[
N_J(X, Y) := J([X, Y]_J) - [JX, JY]
\]
(3) the Jordan bracket associated to $J$

$\{X, Y\}_J := \{JX, Y\} + \{X, JY\} - J(\{X, Y\})$,

where $\{\cdot, \cdot\}$ is the Jordan bracket, $\{X, Y\} = \nabla_X Y + \nabla_Y X$

(4) the Jordan tensor associated to $J$

$M_J(X, Y) := J(\{X, Y\}_J) - \{JX, JY\}$

(5) the deformation tensor associated to $J$

$H_J(X, Y) := (J \circ \nabla_X J - \nabla_{JX} J)(Y)$,

which satisfies $2H_J = N_J + M_J$.

**Remark 2.4.** The $J$-bracket and the associated Nijenhuis tensor can be defined for any $(1, 1)$-tensor field on a smooth manifold $M$, the Jordan bracket, the associated Jordan tensor and the deformation tensor can be defined for $(1, 1)$-tensor fields on a pseudo-Riemannian manifold $(M, g)$.

Assume that $J$ satisfies $J^2 = pJ + qI$ with $p^2 + 4q > 0$. We denote by $\sigma_{\pm} := \frac{p \pm \sqrt{p^2 + 4q}}{2}$ and consider the projection operators $\mathcal{P}$ and $\mathcal{P}'$ [8]:

$$\mathcal{P} := -\frac{1}{\sqrt{p^2 + 4q}} J + \frac{\sigma_+}{\sqrt{p^2 + 4q}} I, \quad \mathcal{P}' := \frac{1}{\sqrt{p^2 + 4q}} J - \frac{\sigma_-}{\sqrt{p^2 + 4q}} I$$

satisfying

$$\mathcal{P}^2 = \mathcal{P}, \quad \mathcal{P}'^2 = \mathcal{P}', \quad \mathcal{P} + \mathcal{P}' = I, \quad \mathcal{P} \circ \mathcal{P}' = 0, \quad \mathcal{P}' \circ \mathcal{P} = 0.$$ 

By a direct computation, we get the following:

**Proposition 2.5.** For the two projection operators $\mathcal{P}$ and $\mathcal{P}'$, we have:

1. $N_{\mathcal{P}} = N_{\mathcal{P}'} = \frac{1}{p^2 + 4q} N_J$;
2. $M_{\mathcal{P}} = M_{\mathcal{P}'} = \frac{1}{p^2 + 4q} M_J$;
3. $H_{\mathcal{P}} = H_{\mathcal{P}'} = \frac{1}{p^2 + 4q} H_J$.

Consider now the deformation tensors $H$ and $H'$:

$$H(X, Y) := \mathcal{P}'(\nabla_{g_X} \mathcal{P} Y) = \mathcal{P}'((\nabla_{g_X} \mathcal{P}) Y),$$

$$H'(X, Y) := \mathcal{P}(\nabla_{g_X} \mathcal{P}' Y) = \mathcal{P}((\nabla_{g_X} \mathcal{P}') Y)$$

the twisting tensors $L$ and $L'$:

$$L(X, Y) := \frac{1}{2} [H(X, Y) - H(Y, X)], \quad L'(X, Y) := \frac{1}{2} [H'(X, Y) - H'(Y, X)]$$

and the extrinsic curvature tensors $K$ and $K'$:

$$K(X, Y) := \frac{1}{2} [H(X, Y) + H(Y, X)], \quad K'(X, Y) := \frac{1}{2} [H'(X, Y) + H'(Y, X)],$$

for any $X, Y \in C^\infty(TM)$. 

Remark 3.1. The distributions $\mathcal{D}$ and $\mathcal{D}'$ are $J$-invariant and, if $q \neq 0$, then $\mathcal{D}$ and $\mathcal{D}'$ are also $g$-orthogonal.
**Definition 3.2.** A distribution $\mathcal{D} \subset TM$ on a smooth manifold $M$ is called
(i) involutive if $X, Y \in \Gamma(\mathcal{D})$ implies $[X, Y] \in \Gamma(\mathcal{D})$;
(ii) integrable if for any $x \in M$, there exists a submanifold $N_x$ which admits
$\mathcal{D}|_{N_x}$ as tangent bundle.

According to the Frobenius theorem, a distribution $\mathcal{D}$ on $M$ is involutive if
and only if it is integrable. In this case, it defines a foliation whose leaves are
the maximal connected submanifolds $N_x$ of $M$ which admit $\mathcal{D}|_{N_x}$ as tangent
bundle.

**Definition 3.3.** We say that the metallic pseudo-Riemannian manifold $(M, J, g)$
is doubly foliated if both of the distributions $\mathcal{D}$ and $\mathcal{D}'$ given by (1) are integrable
and singly foliated if only one of them is integrable.

**Remark 3.4.** The distribution $\mathcal{D}$ (resp. $\mathcal{D}'$) given by (1) is integrable if and only
if $(\nabla_X J)Y - (\nabla_Y J)X = 0$, for any $X, Y \in \Gamma(\mathcal{D})$ (resp. $X, Y \in \Gamma(\mathcal{D}')$), with $\nabla$
a torsion-free linear connection on $M$. Indeed, for $X, Y \in \Gamma(\mathcal{D})$ we have $JX = \sigma X, JY = \sigma Y$
and $J(\nabla_X Y - \nabla_Y X) = - (\nabla_X J)Y + (\nabla_Y J)X + \sigma (\nabla_X Y - \nabla_Y X)$
which implies that $[X, Y] \in \Gamma(\mathcal{D})$ if and only if $(\nabla_X J)Y - (\nabla_Y J)X = 0$.

In particular, in a locally metallic pseudo-Riemannian manifold, the two dis-
tributions $\mathcal{D}$ and $\mathcal{D}'$ given by (1) are both integrable.

**Proposition 3.5.** If $(M, J, g)$ is a metallic pseudo-Riemannian manifold, then
the distribution $\mathcal{D}$ is integrable if and only if:

$$ J \circ N_f(X, Y) = \sigma - N_f(X, Y), \quad \text{for any } X, Y \in C^\infty(TM), $$

respectively, $\mathcal{D}'$ is integrable if and only if:

$$ J \circ N_f(X, Y) = \sigma - N_f(X, Y), \quad \text{for any } X, Y \in C^\infty(TM). $$

In particular, both $\mathcal{D}$ and $\mathcal{D}'$ are integrable if and only if $N_f = 0$.

**Proof.** The distribution $\mathcal{D}$ is integrable if and only if

$$ \mathcal{P}'([\mathcal{P}X, \mathcal{P}Y]) = 0, $$

for any $X, Y \in C^\infty(TM)$. Therefore, from a direct computation and using
Proposition 2.5, we obtain that a necessary and sufficient condition for $\mathcal{D}$ to be
integrable is:

$$ 0 = \mathcal{P}'([\mathcal{P}X, \mathcal{P}Y]) = -\mathcal{P}'(N_{\mathcal{D}}(X, Y)) = -\frac{1}{p^2 + 4q} \mathcal{P}'(N_f(X, Y)) = $$

$$ = -\frac{1}{(p^2 + 4q)^2 + 4q} [J \circ N_f(X, Y) - \sigma - N_f(X, Y)]. $$

□

**Definition 3.6.** Given a linear connection $\nabla$ on a smooth manifold $M$, we say
that a distribution $\mathcal{D} \subset TM$ is $\nabla$-geodesically invariant if $X, Y \in \Gamma(\mathcal{D})$ implies
$\nabla_X Y + \nabla_Y X \in \Gamma(\mathcal{D})$.

In particular, if $\nabla$ is the Levi-Civita of the pseudo-Riemannian manifold
$(M, g)$, then $\mathcal{D}$ is geodesically invariant.
We remark that the above condition is equivalent to the following: the distribution $\mathcal{D}$ is $\mathcal{V}$-geodesically invariant if $X \in \Gamma(\mathcal{D})$ implies $\nabla_X X \in \Gamma(\mathcal{D})$.

**Remark 3.7.** For a linear connection $\nabla$ on $M$, the distribution $\mathcal{D}$ (resp. $\mathcal{D}'$) given by (1) is $\mathcal{V}$-geodesically invariant if and only if we have $(\nabla_X Y)(\nabla_Y X) = 0$, for any $X, Y \in \Gamma(\mathcal{D})$ (resp. $X, Y \in \Gamma(\mathcal{D}')$). Indeed, for $X, Y \in \Gamma(\mathcal{D})$ we have $JX = \sigma_X X, JY = \sigma_Y Y$ and $J(\nabla_X Y + \nabla_Y X) = -(\nabla_X Y)(\nabla_Y X) + \sigma_X(\nabla_X Y + \nabla_Y X)$ which implies $\nabla_X Y + \nabla_Y X \in \Gamma(\mathcal{D})$ if and only if $(\nabla_X Y)(\nabla_Y X) = 0$.

In particular, for any $J$-connection $\nabla$, the distributions $\mathcal{D}$ and $\mathcal{D}'$ are $\mathcal{V}$-geodesically invariant.

**Proposition 3.8.** If $(M, J, g)$ is a metallic pseudo-Riemannian manifold, then the distribution $\mathcal{D}$ is geodesically invariant if and only if:

$$J_0 M_f(X, Y) = \sigma_+ M_f(X, Y), \text{ for any } X, Y \in C^\infty(TM),$$

respectively, $\mathcal{D}'$ is geodesically invariant if and only if:

$$J_0 M_f(X, Y) = \sigma_- M_f(X, Y), \text{ for any } X, Y \in C^\infty(TM).$$

In particular, both $\mathcal{D}$ and $\mathcal{D}'$ are geodesically invariant if and only if $M_f = 0$.

**Proof.** The distribution $\mathcal{D}$ is geodesically invariant if and only if

$$\mathcal{D}'(\mathcal{P}X, \mathcal{P}Y) = 0,$$

for any $X, Y \in C^\infty(TM)$. Therefore, from a direct computation and using Proposition 2.5, with a similar computation like in Proposition 3.5, we obtain the conclusion. \hfill $\square$

**Remark 3.9.** $J_p := \mathcal{P} - \mathcal{P}'$ is an almost product structure on $M$ and

$$J_pX = -\frac{1}{\sqrt{p^2 + 4q}}(2J_p - pI)X,$$

for any $X \in C^\infty(TM)$.

Direct computations provide the following relationship between $J$ and $J_p$-brackets, $J$ and $J_p$ Nijenhuis tensors, Jordan bracket and Jordan tensors of the two structures. Precisely, we have the following:

**Proposition 3.10.**

$$[X, Y]_f = -\frac{\sqrt{p^2 + 4q}}{2}[X, Y]_p + \frac{p}{2}[X, Y],$$

$$N_f(X, Y) = \frac{p^2 + 4q}{4}N_p(X, Y),$$

$$\{X, Y\}_f = -\frac{\sqrt{p^2 + 4q}}{2}\{X, Y\}_p + \frac{p}{2}\{X, Y\},$$

$$M_f(X, Y) = \frac{p^2 + 4q}{4}M_p(X, Y).$$
In particular, the deformation tensors are related as follows:

\[ H_J(X, Y) = \frac{p^2 + 4q}{4} H_{J_p}(X, Y). \]

The product conjugate connection of a linear connection \( \nabla \) is [2]:

\[ \nabla'^{(J_p)} = \mathcal{P}(\nabla_X X') - \mathcal{P}(\nabla_X X') - \mathcal{P}'(\nabla_X X') + \mathcal{P}'(\nabla_X X') \tag{2} \]

and we have:

**Proposition 3.11.** [2] If \( \nabla'^{(J_p)} \) is torsion-free, then \( J_p \) is integrable, which means that \( \mathcal{D} \) and \( \mathcal{D}' \) are integrable distributions.

**Definition 3.12.** We say that a linear connection \( \nabla \) restricts to a distribution \( \mathcal{D} \subset TM \) on a metallic pseudo-Riemannian manifold \( (M, J, g) \) if \( Y \in \Gamma(\mathcal{D}) \) implies \( \nabla_X Y \in \Gamma(\mathcal{D}) \), for any \( X \in C^\infty(TM) \).

We have:

1) \( \nabla \) restricts to \( \mathcal{D} \) means \( \mathcal{P}'(\nabla_X X') = 0 \) and \( \mathcal{P}(\nabla_X X') = \nabla_X X' \),

2) \( \nabla \) restricts to \( \mathcal{D}' \) means \( \mathcal{P}(\nabla_X X') = 0 \) and \( \mathcal{P}'(\nabla_X X') = \nabla_X X' \).

A straightforward computation shows that the product conjugate connection \( \nabla'^{(J_p)} \) defined by (2) restricts to \( \mathcal{D} \) and \( \mathcal{D}' \). Moreover, if \( \nabla \) restricts to both \( \mathcal{D} \) and \( \mathcal{D}' \), then

\[ \nabla'^{(J_p)} = \nabla_X X' + \nabla_X X' = \nabla_X X \tag{3} \]

and so \( \nabla \) is an \( J_p \)-connection. Let us remark that the above connection (3) is exactly the Schouten-van Kampen connection of the pair \( (\mathcal{D}, \mathcal{D}') \):

\[ \nabla_X X' = \mathcal{P}(\nabla_X X') + \mathcal{P}'(\nabla_X X') \]

which coincides with the metallic natural connection \( \hat{\nabla} \) [3] if \( \nabla \) is the Levi-Civita connection of \( g \).

Now we can express the Kirichenko tensor fields [9] in terms of the projectors \( \mathcal{P}, \mathcal{P}' \):

**Proposition 3.13.** [2] The structural and virtual tensor fields of \( J_p = \mathcal{P} - \mathcal{P}' \) are:

\[
\begin{align*}
C^{-p-p'}_\mathcal{V}(X, Y) &= 2[\mathcal{P}(\nabla_{p'} X') + \mathcal{P}'(\nabla_{p'} X')] \\
B^{-p-p'}_\mathcal{V}(X, Y) &= -2[\mathcal{P}(\nabla_{p'} X') + \mathcal{P}'(\nabla_{p'} X')].
\end{align*}
\]

Let us recall the well-known fundamental tensor fields of O’Neill-Gray:

\[
\begin{align*}
T(X, Y) &= \mathcal{P}(\nabla_{p'} X') + \mathcal{P}'(\nabla_{p'} X') \\
A(X, Y) &= \mathcal{P}'(\nabla_{p'} X') + \mathcal{P}(\nabla_{p'} X').
\end{align*}
\]

Then, a comparison of last two equations yields

\[
\begin{align*}
C^{-p-p'}_\mathcal{V}(X, Y) &= 2[T(X, \mathcal{P}'Y) + A(X, \mathcal{P}Y)] \\
B^{-p-p'}_\mathcal{V}(X, Y) &= -2[T(X, \mathcal{P}Y) + A(X, \mathcal{P}'Y)]
\end{align*}
\]

a fact which justifies the second name of \( T \) and \( A \) as invariants of the decomposition \( TM = \mathcal{D} \oplus \mathcal{D}' \) [6].
On $\mathcal{D}$ with the induced metric $g_{\mathcal{D}}$, we consider the induced connection from the pseudo-Riemannian manifold $(M, g, \nabla)$ by [10]:

$$\nabla^{\mathcal{D}} : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D}), \quad \nabla^{\mathcal{D}}_X Y := \mathcal{P}(\nabla_X Y)$$

which preserves the metric $g_{\mathcal{D}}$ and is torsion-free w.r.t. the bracket

$$[\cdot, \cdot]_\mathcal{D} : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D}), \quad [X, Y]_\mathcal{D} := \mathcal{P}([X, Y]) .$$

The bracket $[\cdot, \cdot]_\mathcal{D}$ has the usual properties of a Lie bracket excepting the Jacobi identity which is satisfied if and only if $\mathcal{D}$ is integrable.

The integrability of $\mathcal{D}$ can also be characterized in terms of second fundamental form of $\mathcal{D}$:

$$h : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D}'), \quad h(X, Y) := \nabla_X Y - \nabla^{\mathcal{D}}_X Y,$$

and we can state:

**Proposition 3.14.** [10] The distribution $\mathcal{D}$ is integrable if and only if one of the following assertions holds: (i) $\nabla^{\mathcal{D}}$ is torsion-free; (ii) $h$ is symmetric.

Similarly, on $(\mathcal{D}', g_{\mathcal{D}'})$ we define the induced connection from $(M, g, \nabla)$ by:

$$\nabla^{\mathcal{D}'} : \Gamma(\mathcal{D}') \times \Gamma(\mathcal{D}') \to \Gamma(\mathcal{D}'), \quad \nabla^{\mathcal{D}'}_X Y := \mathcal{P}'(\nabla_X Y)$$

and consider the second fundamental form $h'$ of $\mathcal{D}'$. Then the distribution $\mathcal{D}'$ is integrable if and only if one of the following assertions holds: (i) $\nabla^{\mathcal{D}'}$ is torsion-free; (ii) $h'$ is symmetric.

We remark that the restrictions of the metallic natural connection $\tilde{\nabla}$, defined in [3], to $\mathcal{D}$ and respectively, to $\mathcal{D}'$, coincide with the two induced connections, respectively:

$$\tilde{\nabla}|_{\Gamma(\mathcal{D}) \times \Gamma(\mathcal{D})} = \nabla^{\mathcal{D}}, \quad \tilde{\nabla}|_{\Gamma(\mathcal{D}') \times \Gamma(\mathcal{D}')} = \nabla^{\mathcal{D}'} .$$

**Remark 3.15.** For $p^2 + 4q = 0$, we get only one distribution, $\ker(J - \frac{p}{2}I)$, and $J := J - \frac{p}{2}I$ is an almost subtangent structure.

### 4. Adapted connections to $(\mathcal{D}, \mathcal{D}')$

**Definition 4.1.** We say that a linear connection $\nabla$ on $M$ is *adapted* to the decomposition $TM = \mathcal{D} \oplus \mathcal{D}'$ if $Y \in \Gamma(\mathcal{D})$ implies $\nabla_X Y \in \Gamma(\mathcal{D})$, for any $X \in C^\infty(TM)$ and $Y \in \Gamma(\mathcal{D}')$ implies $\nabla_X Y \in \Gamma(\mathcal{D}')$, for any $X \in C^\infty(TM)$.

**Remark 4.2.** If $(M, J)$ is a metallic manifold such that $J^2 = pJ + qI$ with $p^2 + 4q > 0$, then a linear connection $\nabla$ is adapted to $(\mathcal{D}, \mathcal{D}')$ given by (1) if and only if $\nabla$ is a $J$-connection. Indeed, for $Y \in \Gamma(\mathcal{D})$ we have $JY = \sigma_+ Y$ and $(\nabla_X J) Y = \sigma_- \nabla_X Y - J(\nabla_X Y)$, for any $X \in C^\infty(TM)$, which implies that $\nabla_X Y \in \Gamma(\mathcal{D})$ if and only if $\nabla J = 0$. Similarly we deduce the second implication.
In [1], A. Bejancu and H. R. Farran gave the expression of all adapted connections to \((\mathcal{D}, \mathcal{D}')\), namely:

\[
\nabla^r_X Y = \mathcal{P}(\nabla_X \mathcal{P}Y) + \mathcal{P}'(\nabla_X \mathcal{P}'Y) + \mathcal{P}(S(X, \mathcal{P}Y)) + \mathcal{P}'(S(X, \mathcal{P}'Y)),
\]

(4)

for any \(X, Y \in C^\infty(TM)\), where \(\nabla\) is a linear connection and \(S\) is a \((1, 2)\)-tensor field on \(M\).

4.1. Schouten-van Kampen connection. An adapted connection to \((\mathcal{D}, \mathcal{D}')\) is the Schouten-van Kampen connection \(\tilde{\nabla}\) of the linear connection \(\nabla\), obtained from (4) for \(S := 0\):

\[
\tilde{\nabla}_X Y := \mathcal{P}(\nabla_X \mathcal{P}Y) + \mathcal{P}'(\nabla_X \mathcal{P}'Y) = \nabla_X Y + \mathcal{P}((\nabla_X \mathcal{P})Y) + \mathcal{P}'((\nabla_X \mathcal{P}'Y)).
\]

(5)

If \((M, J, g)\) is a metallic pseudo-Riemannian manifold such that \(J^2 = pJ + qI\) with \(p^2 + 4q > 0\) and \(\nabla\) is torsion-free, then \(\tilde{\nabla}\) is explicitly given by:

\[
\tilde{\nabla}_X Y = \frac{1}{p^2 + 4q}[(2J - pI)(\nabla_X JY) - (pJ - (p^2 + 2q)I)(\nabla_X Y)],
\]

(6)

for any \(X, Y \in C^\infty(TM)\). We remark that if \(\nabla\) is the Levi-Civita connection associated to \(g\), then \(\tilde{\nabla}\) is exactly the metallic natural connection defined in [3]. Moreover, \(\tilde{\nabla}\) is a metric \(J\)-connection, i.e. \(\tilde{\nabla}g = \tilde{\nabla}J = 0\), whose torsion is given by:

\[
T^{\tilde{\nabla}}(X, Y) = \frac{1}{p^2 + 4q}[(2J - pI)(\nabla_X JY - \nabla_J X) - (pJ + 2qI)(\nabla_X Y - \nabla_Y X)],
\]

for any \(X, Y \in C^\infty(TM)\).

4.2. Vranceanu connection. Another adapted connection to \((\mathcal{D}, \mathcal{D}')\) is the Vranceanu connection \(\tilde{\nabla}\) of the linear connection \(\nabla\), obtained from (4) for

\[
S(X, Y) := -\mathcal{P}(\nabla_{\mathcal{P}X} \mathcal{P}Y) - \mathcal{P}'(\nabla_{\mathcal{P}X} \mathcal{P}'Y) + \mathcal{P}([\mathcal{P}X, \mathcal{P}Y]) + \mathcal{P}'([\mathcal{P}X, \mathcal{P}'Y]).
\]

If \((M, J, g)\) is a metallic pseudo-Riemannian manifold such that \(J^2 = pJ + qI\) with \(p^2 + 4q > 0\), then \(\tilde{\nabla}\) is explicitly given by:

\[
\tilde{\nabla}_X Y = \tilde{\nabla}_{\mathcal{P}X} Y + \mathcal{P}'([\mathcal{P}'X, \mathcal{P}Y]) + \mathcal{P}'([\mathcal{P}X, \mathcal{P}'Y]) = \nabla_X Y + \frac{1}{p^2 + 4q}[2J((\nabla_X J)Y) - p(\nabla_X J)Y + J((\nabla_J Y)X) + (\nabla_J Y)X - p(\nabla_J Y)X] + \\
\frac{1}{p^2 + 4q}[T^{\tilde{\nabla}}(JX, JY) + J(T^{\tilde{\nabla}}(JX, Y)) - pT^{\tilde{\nabla}}(JX, Y) - J(T^{\tilde{\nabla}}(X, JY)) - qT^{\tilde{\nabla}}(X, Y)],
\]

(7)

for any \(X, Y \in C^\infty(TM)\).

Moreover, \(\tilde{\nabla}\) is a \(J\)-connection, i.e. \(\tilde{\nabla}J = 0\), whose torsion is given by:

\[
T^{\tilde{\nabla}}(X, Y) = \frac{1}{p^2 + 4q}N_J(X, Y) + \mathcal{P}'(T^{\tilde{\nabla}}(\mathcal{P}'X, \mathcal{P}'Y)) - \mathcal{P}(T^{\tilde{\nabla}}(\mathcal{P}X, \mathcal{P}Y)),
\]

for any \(X, Y \in C^\infty(TM)\).
4.3. **Vidal connection.** Let \((M, J, g)\) be a metallic pseudo-Riemannian manifold such that \(J^2 = pJ + qI\) with \(p^2 + 4q > 0\) and let \(\nabla\) be the Levi-Civita connection of \(g\).

Another adapted connection to \((\mathcal{D}, \mathcal{D}')\) is the **Vidal connection** \(\tilde{\nabla}\) associated to \(J\), obtained from (4) for

\[
S(X, Y) := -\mathcal{P}(\nabla_{pY} \mathcal{P}^t X) - \mathcal{P}(\nabla_{pY} \mathcal{P}^t X),
\]

therefore:

\[
\tilde{\nabla}X Y = \tilde{\nabla}X Y - \mathcal{P}(\nabla_{pY} \mathcal{P}^t X) - \mathcal{P}(\nabla_{pY} \mathcal{P}^t X) X =
\]

\[
= \tilde{\nabla}X Y + \frac{1}{p^2 + 4q}[(\nabla_{JY} J)X + J((\nabla_{JY} J)X) - p(\nabla_{JY} J)X] =
\]

\[
= \nabla X Y + \frac{1}{p^2 + 4q}[2J((\nabla_{JY} J)X) - p(\nabla_{JY} J)X + J((\nabla_{JY} J)X) +
\]

\[
+ (\nabla_{JY} J)X - p(\nabla_{JY} J)X],
\]

for any \(X, Y \in C^\infty(TM)\).

Moreover, \(\tilde{\nabla}\) is a \(J\)-connection, i.e. \(\tilde{\nabla}J = 0\), whose torsion is given by:

\[
T^\tilde{\nabla}(X, Y) = \frac{1}{p^2 + 4q}N_J(X, Y),
\]

for any \(X, Y \in C^\infty(TM)\).

**Remark 4.3.** The Vrâncianu connection of the Levi-Civita connection coincides with the Vidal connection.

Moreover, we get:

\[
(\tilde{\nabla}X g)(Y, Z) = -\frac{1}{p^2 + 4q}[g((\nabla_{JY} J)X - (\nabla_{JY} J)X, Z) +
\]

\[
+ g((\nabla_{ZJ} J)X - (\nabla_{ZJ} J)X, Y) =
\]

\[
= \frac{1}{p^2 + 4q}[g(M_J(Y, X), Z) + g(M_J(Z, X), Y) + g((\nabla_{JX} J)Y + (\nabla_{JX} J)X, Z) +
\]

\[
+ g((\nabla_{JX} J)Z + (\nabla_{JX} J)X, Y)],
\]

for any \(X, Y, Z \in C^\infty(TM)\).

Since \(\tilde{\nabla}J = \tilde{\nabla}J = \tilde{\nabla}J = 0\), from Remark 3.7 we deduce:

**Proposition 4.4.** The distributions \(\mathcal{D}\) and \(\mathcal{D}'\) are \(\tilde{\nabla}\)-geodesically invariant, \(\tilde{\nabla}\)-geodesically invariant and \(\tilde{\nabla}\)-geodesically invariant.

Using the Vidal connection \(\tilde{\nabla}\), we characterize the integrability and the geodesic invariance of the metallic distributions defined by \(J\) in terms of the torsion and the covariant derivative of \(g\) w.r.t. this connection. From all the above considerations, we can state:
Theorem 4.5. If $(M,J,g)$ is a metallic pseudo-Riemannian manifold such that $J^2 = pJ + qI$ with $p^2 + 4q > 0$, then the following assertions are equivalent:
(i) the distributions $\mathcal{D}$ and $\mathcal{D}'$ are integrable;
(ii) $N_I = 0$;
(iii) $L = 0$ and $L' = 0$;
(iv) the Vidal connection given by (8) is torsion-free.

Theorem 4.6. If $(M,J,g)$ is a metallic pseudo-Riemannian manifold such that $J^2 = pJ + qI$ with $p^2 + 4q > 0$, then the following assertions are equivalent:
(i) the distributions $\mathcal{D}$ and $\mathcal{D}'$ are geodesically invariant;
(ii) $M_J = 0$;
(iii) $K = 0$ and $K' = 0$;
(iv) the Vidal connection given by (8) is metric with respect to $g$.

4.4. Leaves correspondence via metallic maps. We shall provide the condition for a metallic map between two metallic pseudo-Riemannian manifolds to preserve the metallic distributions. We recall the following:

Definition 4.7. A smooth map $\Phi : (M_1,J_1) \rightarrow (M_2,J_2)$ between two metallic manifolds is called a metallic map if:
$$\Phi_* J_1 = J_2 \circ \Phi_*.$$

Remark 4.8. If $\Phi : (M_1,J_1) \rightarrow (M_2,J_2)$ is a metallic map and $J_i^2 = p_iJ_i + q_iI$ with $p_i$ and $q_i$ real numbers, $i = 1, 2$, then:
(i) $\Phi_* J_i^{k+1} = J_2^{k+1} \circ \Phi_*$, for any $k \in \mathbb{N}$;
(ii) $\{[(p_2^2 + q_2) - (p_1^2 + q_1)]J_1 + (p_2q_2 - p_1q_1)I\}(TM_1) \subset \ker \Phi_*$;
(iii) in the particular case when one the structure is product and the other one is complex, then $Im J_1 \subset \ker \Phi_*$.

Consider a metallic map $\Phi : (M_1,J_1) \rightarrow (M_2,J_2)$ between the metallic manifolds $(M_1,J_i)$ such that $J_i^2 = p_iJ_i + q_iI$ with $p_i^2 + 4q_i > 0$, $i = 1, 2$, and assume that the distributions $\mathcal{D}_i$ and $\mathcal{D}_i'$, $i = 1, 2$, are integrable. Then they define the foliations $\mathcal{F}_i$ and $\mathcal{F}_i'$, $i = 1, 2$, whose leaves are trivial metallic pseudo-Riemannian manifolds.

Denoting by $\Phi^* \mathcal{D}_2$ the pull-back distribution, i.e.:
$$\mathcal{D}_2_x := \ker \Phi : \{X_x \in TM : \Phi_* x(\mathcal{D}_2_x) \in \mathcal{D}_2(\Phi(x))\},$$
since $\Phi$ is a metallic map, we get:
$$\mathcal{D}_2_x = \ker \Phi : \{X_x \in TM : (J_1 - \sigma x)(X_x) \in \ker \Phi_* x\},$$
where $\sigma_{i+} = \frac{p_i + \sqrt{p_i^2 + 4q_i}}{2}$, $i = 1, 2$ and
$$\mathcal{D}_2'_x = \ker \Phi : \{X_x \in TM : (J_1 - \sigma x')(X_x) \in \ker \Phi_* x\},$$
where $\sigma_{i-} = \frac{p_i - \sqrt{p_i^2 + 4q_i}}{2}$, $i = 1, 2$.

From the above considerations, we obtain a sufficient condition for the pull-back distribution $\Phi^* \mathcal{D}_2$ to coincide with one of the distributions $\mathcal{D}_1$ or $\mathcal{D}_1'$:
Proposition 4.9. If ker $\Phi_\ast = (J_1 - \sigma_2 I)(\ker(J_1 - \sigma_1 I))$, then $\Phi^\ast \mathcal{D}_2 = \mathcal{D}_1$. Moreover, if $\Phi$ is a surjective submersion with connected fibers, then a leaf of $\mathcal{F}_2$ corresponds to a leaf of $\mathcal{F}_1$.

5. A Chen-type inequality for the metallic distributions

A fundamental problem in the theory of submanifolds is the problem posed by B. Y. Chen [4], namely, to find relations between the main intrinsic and extrinsic invariants of a submanifold. In this sense, the Chen’s inequalities for submanifolds in real space forms was proved by B. Y. Chen [4], in complex space forms by Y. Doğru [5], in quaternionic space forms by G. E. Vilcu [12] etc. In the same spirit, we shall prove a Chen-type inequality in the metallic case, for an integrable distribution defined by the metallic structure.

Let $(M, J, g)$ be an $m$-dimensional metallic Riemannian manifold and assume that the distribution $\mathcal{D}$ is integrable. In this case, the Riemann curvature tensors of $\mathcal{D}$ (computed with respect to the induced connection $\nabla^\mathcal{D}$ on $\mathcal{D}$ and the Lie bracket $\lbrack \cdot, \cdot \rbrack^\mathcal{D}$) and $M$ satisfy [10]:

$$R^\mathcal{D}(X, Y, Z, W) = R^M(X, Y, Z, W) -$$

$$-g(h(X, Z), h(Y, W)) + g(h(X, W), h(Y, Z)),
$$

for any $X, Y, Z, W \in \Gamma(\mathcal{D})$.

The relation between the mean curvature (the main extrinsic invariant) and the Chen first invariant (an intrinsic invariant), in a particular case of constant $J$-sectional curvature, is given in the following.

From a direct computation we obtain:

Proposition 5.1. Let $(M, J, g)$ be an $m$-dimensional metallic Riemannian manifold $(m > 2)$ such that $J^2 = pJ + qI$ with $p^2 + 4q > 0$, whose Riemann curvature tensor is given by

$$R^M(X, Y, Z, W) = c[g(X, FW)g(Y, FZ) - g(X, FZ)g(Y, FW)],
$$

for any $X, Y, Z, W \in C^\infty(TM)$, where $F := aJ + bI$ with $a$ and $b$ real numbers satisfying $qa^2 - pab - b^2 = 1$. Then the $J$-sectional curvature of $M$ is constant equal to $c$.

Denote by $H := \frac{1}{n} \text{tr}(h)$ the mean curvature and by $\delta_\mathcal{D} := \tau^\mathcal{D} - \inf K^\mathcal{D}$ the Chen first invariant of $\mathcal{D}$, where $\tau^\mathcal{D}$ denotes the scalar curvature of $\mathcal{D}$ and $K^\mathcal{D}$ its sectional curvature.

Theorem 5.2. Let $(M, J, g)$ be an $m$-dimensional metallic Riemannian manifold $(m > 2)$ such that $J^2 = pJ + qI$ with $p^2 + 4q > 0$, whose Riemann curvature tensor is given by (10) and let $\mathcal{D}$ be an $n$-dimensional integrable distribution. Then:

$$\delta_\mathcal{D} \leq \frac{c(a \sigma_2 + b)^2(n^2 - n + 2)}{2} + \frac{2n^2(n - 2)}{2(n - 1)} ||H||^2.$$
**Proof.** Consider an orthonormal frame field \( \{e_1, \ldots, e_n\} \) for \( D \), \( \{f_1, \ldots, f_{m-n}\} \) an orthonormal frame field for \( D' \) and denote by
\[
h^k_{ij} := g(h(e_i, e_j), f_k).
\]
From (9) and (10) we get
\[
2\tau^D = c(a\sigma_+ + b)^2n(n - 1) - ||h||^2 + n^2||H||^2.
\]
Moreover
\[
K^D(e_1, e_2) = -c(a\sigma_+ + b)^2 - \sum_{k=1}^{m-n} h^k_{11} h^k_{22} + \sum_{k=1}^{m-n} (h^k_{12})^2
\]
and
\[
\tau^D - K^D(e_1, e_2) = \frac{c(a\sigma_+ + b)^2(n^2 - n + 2)}{2} + \sum_{k=1}^{m-n} \left[ \sum_{3 \leq i < j \leq n}(h^k_{ij}h^k_{ij} - (h^k_{ij})^2) + \sum_{j=3}^{n}(h^k_{11}h^k_{22})h^k_{ij} - \sum_{j=3}^{n}((h^k_{1j})^2 + (h^k_{2j})^2) \right] \leq \frac{c(a\sigma_+ + b)^2(n^2 - n + 2)}{2} + \frac{n^2(n - 2)}{2(n - 1)} \sum_{k=1}^{m-n} (h^k_{11})^2 - \sum_{k=1}^{m-n} (h^k_{12})^2 \leq \frac{c(a\sigma_+ + b)^2(n^2 - n + 2)}{2} + \frac{n^2(n - 2)}{2(n - 1)} ||H||^2.
\]

**Remark 5.3.** If \( p = 0 \) and \( q = 1 \), i.e. \( J \) is an almost product structure, then the inequality from Theorem 5.2 becomes
\[
\delta_D \leq \frac{c(a - b)^2(n^2 - n + 2)}{2} + \frac{n^2(n - 2)}{2(n - 1)} ||H||^2.
\]
In particular, if \( a = 1 \) and \( b = 0 \), i.e. \( F = J \), we get
\[
\delta_D \leq \frac{c(n^2 - n + 2)}{2} + \frac{n^2(n - 2)}{2(n - 1)} ||H||^2.
\]

6. Metallic Norden structures

6.1. Complex metallic distributions. Let \((M, J, g)\) be a metallic Norden manifold such that \( J^2 = pl + ql \) with \( p^2 + 4q < 0 \) and let \( T^C \mathcal{M} := TM \otimes_{\mathbb{R}} \mathbb{C} \) be the complexified tangent bundle. Then we can define the complexified metallic pseudo-Riemannian structure:
\[
J^C(X + iY) := JX + iJY,
\]
\[
g^C(X_1 + iY_1, X_2 + iY_2) := g(X_1, X_2) - g(Y_1, Y_2) + i[g(X_1, Y_2) + g(Y_1, X_2)],
\]
for any \( X, X_1, X_2, Y, Y_1, Y_2 \in C^\infty(TM) \).
Denote by $\sigma_\pm^C := \frac{p \pm \sqrt{p^2 + 4q}}{2}$ and consider the projection operators $\mathcal{P}^C$ and $\mathcal{P}^{C'}$:

$$
\mathcal{P}^C := -\frac{1}{\sqrt{p^2 + 4q}} J^C + \frac{\sigma_+^C}{\sqrt{p^2 + 4q}} I^C, \quad \mathcal{P}^{C'} := \frac{1}{\sqrt{p^2 + 4q}} J^C - \frac{\sigma_-^C}{\sqrt{p^2 + 4q}} I^C
$$

satisfying

$$
\mathcal{P}^C = \mathcal{P}^{C'}, \quad \mathcal{P}^C + \mathcal{P}^{C'} = I^C, \quad \mathcal{P}^C \circ \mathcal{P}^{C'} = 0, \quad \mathcal{P}^{C'} \circ \mathcal{P}^C = 0
$$

and define the complementary distributions:

$$
\mathcal{D}^C := \ker \mathcal{P}^{C'}, \quad \mathcal{D}^{C'} := \ker \mathcal{P}^C
$$

which we shall call the complex metallic distributions defined by $J$.

**Remark 6.1.** If $(M, J, g)$ is a metallic Norden manifold such that $J^2 = pJ + qI$ with $p^2 + 4q < 0$, then $\mathcal{D}^C$ and $\mathcal{D}^{C'}$ are $J^C$-invariant.

**Lemma 6.2.**

$$
\mathcal{D}^{C'} = \overline{\mathcal{D}^C}
$$

**Proof.** It follows from $\sigma_+^C = \overline{\sigma_-^C}$. \qed

In particular, if $J$ is not trivial, that it admits two complex eigenvalues, or the two distributions are both different from 0, then the complexified tangent bundle splits as a direct sum of two conjugate subbundles:

$$
T^C M = \mathcal{D}^C \oplus \overline{\mathcal{D}^C}.
$$

Extending the Lie bracket to:

$$
[X_1 + iY_1, X_2 + iY_2]^C := [X_1, X_2] - [Y_1, Y_2] + i([X_1, Y_2] + [Y_1, X_2]),
$$

for any $X_1, X_2, Y_1, Y_2 \in C^\infty(TM)$, we say that:

**Definition 6.3.** A distribution $\mathcal{D}^C \subset T^C M$ is called integrable if $X, Y \in \Gamma(\mathcal{D}^C)$ implies $[X, Y]^C \in \Gamma(\mathcal{D}^C)$.

**Lemma 6.4.** The distribution $\mathcal{D}^C$ is integrable if and only if

$$
\mathcal{P}^{C'}([\mathcal{P}^C X, \mathcal{P}^C Y]^C) = 0,
$$

for any $X, Y \in C^\infty(T^C M)$.

**Proposition 6.5.** The distribution $\mathcal{D}^C$ (resp. $\mathcal{D}^{C'}$) given by (11) is integrable if and only if $N_J = 0$.

Extending the Levi-Civita connection $\nabla$ of $g$ to:

$$
\nabla^C_{X_1 + iY_1} (X_2 + iY_2) := \nabla_{X_1} X_2 - \nabla_{Y_1} Y_2 + i(\nabla_{X_1} Y_2 + \nabla_{Y_1} X_2),
$$

for any $X_1, X_2, Y_1, Y_2 \in C^\infty(TM)$, we pose the following:
**Proposition 6.11.** Given a complex linear connection $\nabla^C$ on a smooth manifold $M$, a distribution $\mathcal{D}^C \subset T^C M$ is called $\nabla^C$-geodesically invariant if $X, Y \in \Gamma(\mathcal{D}^C)$ implies $\nabla^C_X Y + \nabla^C_Y X \in \Gamma(\mathcal{D}^C)$.

In particular, if $\nabla^C$ is the Levi-Civita connection of the pseudo-Riemannian manifold $(M, g^C)$, then $\mathcal{D}^C$ is called geodesically invariant.

**Lemma 6.7.** The distribution $\mathcal{D}^C$ is geodesically invariant if and only if

$$\mathcal{P}^C(\{\mathcal{P}^C X, \mathcal{P}^C Y\}^C) = 0,$$

for any $X, Y \in C^\infty(T^C M)$, where $\{X, Y\}^C := \nabla^C_X Y + \nabla^C_Y X$.

**Proposition 6.8.** The distribution $\mathcal{D}^C$ (resp. $\mathcal{D}^{C'}$) given by (11) is geodesically invariant if and only if $M_f = 0$.

**Remark 6.9.** For a complex linear connection $\nabla^C$ on $M$, the distribution $\mathcal{D}^C$ (resp. $\mathcal{D}^{C'}$) given by (11) is $\nabla^C$-geodesically invariant if and only if $(\nabla^C_X Y + (\nabla^C_Y X) = 0$, for any $X, Y \in \Gamma(\mathcal{D}^C)$ (resp. $X, Y \in \Gamma(\mathcal{D}^{C'})$). Indeed, for $X, Y \in \Gamma(\mathcal{D}^C)$ we have $J^C X = \sigma^C_X X, J^C X = \sigma^C_X Y$ and $J^C(\nabla^C_X Y + \nabla^C_Y X) = -\nabla^C_X Y + \nabla^C_Y X + \nabla^C(\nabla^C_X Y + \nabla^C_Y X)$ which implies that $\nabla^C_X Y + \nabla^C_Y X \in \Gamma(\mathcal{D}^C)$ if and only if $\nabla^C(\nabla^C_X Y + (\nabla^C_Y X) = 0$.

In particular, for any $J^C$-connection $\nabla^C$, the distributions $\mathcal{D}^C$ and $\mathcal{D}^{C'}$ are $\nabla^C$-geodesically invariant.

**Remark 6.10.** $J_c := i(\mathcal{P}^C - \mathcal{P}^{C'})$ is a Norden structure on $M$ and

$$J_c X = -\frac{1}{\sqrt{-p^2 - 4q}}(2J - pl) X,$$

for any $X \in C^\infty(TM)$.

By a direct computation we get:

**Proposition 6.11.** The Nijenhuis tensors of $J_c$ and $J$ are related as follows:

$$N_{J_c}(X, Y) = -\frac{4}{p^2 - 4q}N_j(X, Y),$$

for any $X, Y \in C^\infty(TM)$.

Moreover, if

$$T^C M = T^{(1,0)} M \oplus T^{(0,1)} M$$

is the decomposition of the complexified tangent bundle into $(1, 0)$ and $(0, 1)$ parts, with respect to the almost complex structure $J_c$, we have:

$$\mathcal{D}^{C'} = T^{(1,0)} M, \quad \mathcal{D}^C = T^{(0,1)} M.$$

**Definition 6.12.** We say that a complex linear connection $\nabla^C$ on $M$ is adapted to the decomposition $T^C M = \mathcal{D}^C \oplus \mathcal{D}^{C'}$ if $Y \in \Gamma(\mathcal{D}^C)$ implies $\nabla^C_X Y \in \Gamma(\mathcal{D}^C)$, for any $X \in C^\infty(T^C M)$, and $Y \in \Gamma(\mathcal{D}^{C'})$ implies $\nabla_X Y \in \Gamma(\mathcal{D}^{C'})$, for any $X \in C^\infty(T^C M)$. 
Remark 6.13. If \((M, J, g)\) is a metallic Norden manifold such that \(J^2 = pJ + qI\) with \(p^2 + 4q < 0\), then a complex linear connection \(\nabla^C\) is adapted to \((\mathcal{D}^C, \mathcal{D}^C')\) given by (11) if and only if \(\nabla^C\) is a \(J^C\)-connection. Indeed, for \(Y \in \Gamma(\mathcal{D}^C)\) we have \(J^C Y = \sigma^C X Y\) and \((\nabla^C_X Y)^C = \sigma^C \nabla^C_X Y - J^C (\nabla^C_X Y)\), for any \(X \in C^\infty(T^C M)\), which implies that \(\nabla^C_X Y \in \Gamma(\mathcal{D}^C)\) if and only if \(\nabla^C J^C = 0\). Similarly we deduce the second implication.

Proposition 6.14. All adapted connections to \((\mathcal{D}^C, \mathcal{D}^C')\) are of the form:

\[
(\nabla^C)^C_X Y = \mathcal{D}^C(\nabla^C_X Y) + \mathcal{D}^C(\nabla^C_X \nabla^C Y) + \mathcal{D}^C(\nabla^C_X \nabla^C Y') + \mathcal{D}^C(\nabla^C_X \nabla^C Y') + \\
+ \mathcal{D}^C(\nabla^C_X \nabla^C Y) \quad \text{(12)}
\]

for any \(X, Y \in C^\infty(T^C M)\), where \(\nabla^C\) is a complex linear connection and \(S\) is a complex \((1, 2)\)-tensor field on \(M\).

Proof. We follow the same steps like in the real case [1].

Consider the following adapted connection to \((\mathcal{D}^C, \mathcal{D}^C')\):

1) The complex Schouten-van Kampen connection \(\nabla^C\) of the complex linear connection \(\nabla^C\), obtained from (12) for \(S := 0\):

\[
\nabla^C_X Y := \mathcal{D}^C(\nabla^C_X Y) + \mathcal{D}^C(\nabla^C_X \nabla^C Y).
\]

If \((M, J, g)\) is a metallic Norden manifold such that \(J^2 = pJ + qI\) with \(p^2 + 4q < 0\) and \(\nabla^C\) is torsion-free, then \(\nabla^C\) is explicitly given by:

\[
\nabla^C_X Y = \frac{1}{p^2 + 4q}[2(J^C - pI^C)\nabla^C_X JY - (pJ^C - (p^2 + 2q)I^C)(\nabla^C_X Y)] = \\
= \nabla^C_X Y + \frac{1}{p^2 + 4q}[2(J^C - pI^C)\nabla^C_X JY - (pJ^C - (p^2 + 2q)I^C)(\nabla^C_X Y)] = \\
\text{for any } X, Y \in C^\infty(T^C M).
\]

We remark that if \(\nabla^C\) is the Levi-Civita connection associated to \(g^C\), then \(\nabla^C\) is a metric \(J^C\)-connection, i.e. \(\nabla^C g^C = \nabla^C J^C = 0\), whose torsion is given by:

\[
T^{\nabla^C}(X, Y) = \frac{1}{p^2 + 4q}[2(J^C - pI^C)\nabla^C_X JY - \nabla^C_X JY) - \\
- (pJ^C + 2qI^C)(\nabla^C_X Y - \nabla^C_X Y)],
\]

for any \(X, Y \in C^\infty(T^C M)\).

2) The complex Vrănceanu connection \(\nabla^C\) of the complex linear connection \(\nabla^C\), obtained from (12) for

\[
S(X, Y) := -\mathcal{D}^C(\nabla^C_X \nabla^C Y) - \mathcal{D}^C(\nabla^C_X \nabla^C Y') + \\
+ \mathcal{D}^C(\nabla^C_X \nabla^C Y) + \mathcal{D}^C(\nabla^C_X \nabla^C Y') + \\
+ \mathcal{D}^C(\nabla^C_X \nabla^C Y) + \mathcal{D}^C(\nabla^C_X \nabla^C Y') + \\
+ \mathcal{D}^C(\nabla^C_X \nabla^C Y') + \mathcal{D}^C(\nabla^C_X \nabla^C Y'),
\]

If \((M, J, g)\) is a metallic Norden manifold such that \(J^2 = pJ + qI\) with \(p^2 + 4q < 0\), then \(\nabla^C\) is explicitly given by:

\[
\nabla^C_X Y = \nabla^C_X Y + \mathcal{D}^C(\nabla^C_X \nabla^C Y) + \mathcal{D}^C(\nabla^C_X \nabla^C Y'),
\]

(14)
for any $X, Y \in C^\infty(T^C M)$.

Moreover, $\tilde{\nabla}^C$ is a $J^C$-connection, i.e. $\tilde{\nabla}^C J^C = 0$, whose torsion is given by:

$$T^{\tilde{\nabla}^C}(X, Y) = \frac{1}{p^2 + 4q}N_{J^C}(X, Y) + \mathfrak{p}^{C'}(T^{\tilde{\nabla}^C}(\mathfrak{p}^C X, \mathfrak{p}^C Y)) -$$

$$-\mathfrak{p}^{C'}(T^{\tilde{\nabla}^C}(\mathfrak{p}^C X, \mathfrak{p}^C Y)),$$

for any $X, Y \in C^\infty(T^C M)$.

3) The complex Vidal connection $\tilde{\nabla}^C$ associated to the metallic Norden structure $(J, g)$, obtained from (12) for

$$S(X, Y) := -\mathfrak{p}^{C'}(\nabla_{p^C Y} \mathfrak{p}^{C'}) X - \mathfrak{p}^{C'}(\nabla_{p^C Y} \mathfrak{p}^{C}) X,$$

therefore:

$$\tilde{\nabla}^C_{X} Y = \tilde{\nabla}^C_{X} Y - \mathfrak{p}^{C'}(\nabla_{p^C Y} \mathfrak{p}^{C'}) X - \mathfrak{p}^{C'}(\nabla_{p^C Y} \mathfrak{p}^{C}) X =$$

$$= \tilde{\nabla}^C_{X} Y + \frac{1}{p^2 + 4q}[(\nabla_{p^C Y} J^C)X + J^C((\nabla_{p^C Y} J^C)X) - p(\nabla_{p^C Y} J^C)X],$$

for any $X, Y \in C^\infty(T^C M)$, where $\nabla^C$ is the Levi-Civita connection of $g^C$.

Moreover, $\tilde{\nabla}^C$ is a $J^C$-connection, i.e. $\tilde{\nabla}^C J^C = 0$, whose torsion is given by:

$$T^{\tilde{\nabla}^C}(X, Y) = \frac{1}{p^2 + 4q}N_{J^C}(X, Y),$$

for any $X, Y \in C^\infty(T^C M)$.

Moreover, we get:

$$(\tilde{\nabla}^C_{X} g^C)(Y, Z) = -\frac{1}{p^2 + 4q}[g^C((\tilde{\nabla}^C_{J^C Y} J^C)X - (\tilde{\nabla}^C_{J^C Y} J^C)X) +$$

$$+g^C((\tilde{\nabla}^C_{J^C Z} J^C)X - (\tilde{\nabla}^C_{J^C Z} J^C)X)Y] =$$

$$= \frac{1}{p^2 + 4q}[g^C(M_{J^C Y}(X, Y), Z) + g^C(M_{J^C Z}(X, Z), Y) +$$

$$+g^C((\tilde{\nabla}^C_{J^C X} J^C)Y + (\tilde{\nabla}^C_{J^C X} J^C)X, Z) + g^C((\tilde{\nabla}^C_{J^C X} J^C)Z + (\tilde{\nabla}^C_{J^C X} J^C)X, Y)],$$

for any $X, Y, Z \in C^\infty(T^C M)$.

Since $\tilde{\nabla}^C J^C = \tilde{\nabla}^C_{J^C} = \tilde{\nabla}^C J^C = 0$, from Remark 6.9 we deduce:

**Proposition 6.15.** The distributions $D^C$ and $D^C'$ are $\tilde{\nabla}^C$-geodesically invariant, $\tilde{\nabla}^C$-geodesically invariant and $\tilde{\nabla}^C$-geodesically invariant.

From all the above considerations, we can state:

**Theorem 6.16.** If $(M, J, g)$ is a metallic Norden manifold such that $J^2 = pJ + qI$ with $p^2 + 4q < 0$, then the following assertions are equivalent:

(i) the distributions $D^C$ and $D^C'$ are integrable;

(ii) $(M, J_c)$ is a complex manifold;

(iii) the complex Vidal connection given by (15) is torsion-free.
Theorem 6.17. If $(M, J, g)$ is a metallic Norden manifold such that $J^2 = pJ + qI$ with $p^2 + 4q < 0$, then the following assertions are equivalent:

(i) the distributions $\mathcal{D}$ and $\mathcal{D}^\prime$ are geodesically invariant;
(ii) the complex Vidal connection given by (15) is metric with respect to $g^C$.

6.2. The $\delta^c$-operator of a metallic complex structure.

Definition 6.18. A metallic manifold $(M, J)$ such that $J^2 = pJ + qI$ with $p^2 + 4q < 0$ and $J$ integrable is called metallic complex manifold.

Let $(M, J)$ be a metallic complex manifold and let $J_c = -\frac{1}{\sqrt{-p^2 - 4q}}(2J - pI)$ be the associated complex structure. Consider its dual map $J^*_c : T^* M \rightarrow T^* M$, defined by $(J^*_c \alpha)(X) := \alpha(J_c X)$, for any $\alpha \in C^\infty(T^* M)$ and for any $X \in C^\infty(TM).

We shall define the real differential operator $d^c$ acting on forms:

$$d^c := J^*_c \circ d \circ J^*_c,$$

where $d$ is the real differential operator.

If $(M, J, g)$ is an integrable metallic Norden manifold, we can consider the real codifferential operator $\delta^c$ acting on forms:

$$\delta^c := \ast \circ d^c \circ \ast,$$

where $\ast$ is the Hodge-star operator with respect to the metric $g$.

We obtain

$$d^c \circ d^c = 0, \quad d \circ d^c + d^c \circ d = 0,$$

$$\delta^c \circ \delta^c = 0, \quad \delta \circ \delta^c + \delta^c \circ \delta = 0,$$

where $\delta$ is the codifferential operator, and with respect to the scalar product $\langle \cdot, \cdot \rangle$ induced by $g$, the operators $d^c$ and $\delta^c$ are adjoint, i.e.

$$\langle d^c \alpha, \beta \rangle = \langle \alpha, \delta^c \beta \rangle,$$

for any $\alpha, \beta \in C^\infty(T^* M)$.

Remark that $J^* \circ \ast = \ast \circ J^*$ (and $J^*_c \circ \ast = \ast \circ J^*_c$) implies $\delta^c = J^*_c \circ \delta \circ J^*_c$ and

$$d^c \circ J^*_c = -J^*_c \circ d, \quad J^*_c \circ d^c = -d \circ J^*_c,$$

$$\delta^c \circ J^*_c = -J^*_c \circ \delta, \quad J^*_c \circ \delta^c = -\delta \circ J^*_c.$$

From the above relations, we can state:

Proposition 6.19. Let $\alpha$ be a real form on $M$.

(i) If $\alpha$ is $d^c$-closed (resp. $\delta^c$-coclosed), then $J^*_c \alpha$ is closed (resp. coclosed).
(ii) If $\alpha$ is closed (resp. coclosed), then $J^*_c \alpha$ is $d^c$-closed (resp. $\delta^c$-coclosed).
(iii) If $\alpha$ is $J^*_c$-invariant, i.e. $J^*_c \alpha = \alpha$, then $\alpha$ is $d^c$-closed (resp. $\delta^c$-coclosed) if and only if it is closed (resp. coclosed).

Therefore, the $d^c$-closed (resp. $\delta^c$-coclosed) forms are the $J^*_c$-invariant closed (resp. coclosed) forms. Then

$$\ker(d^c) = \ker(d) \cap \{J^*_c - \text{invariant forms}\}, \quad \text{Im}(d^c) = J^*_c(\text{Im}(d)),$$

$$\ker(\delta^c) = \ker(\delta) \cap \{J^*_c - \text{invariant forms}\}, \quad \text{Im}(\delta^c) = J^*_c(\text{Im}(\delta)).$$
Then we can consider the metallic cohomology groups
\[ H^r(M) := \ker(d_r^c)/\text{Im}(d_{r-1}^c), \]
where
\[ d_r^c : C^\infty(\Lambda^r(M)) \to C^\infty(\Lambda^{r+1}(M)) \]
and the metallic homology groups
\[ H_r(M) := \ker(\delta_r^c)/\text{Im}(\delta_{r+1}^c), \]
where
\[ \delta_r^c : C^\infty(\Lambda^r(M)) \to C^\infty(\Lambda^{r-1}(M)). \]

Now we can introduce the metallic Hodge-Laplace operator
\[ \Delta^c : C^\infty(\Lambda^r(M)) \to C^\infty(\Lambda^r(M)), \quad \Delta^c := d^c \circ \delta^c + \delta^c \circ d^c, \]
which is symmetric and self-adjoint w.r.t. \( \langle \cdot, \cdot \rangle \). Remark that
\[ \Delta^c = -J^*_c \circ \Delta \circ J^*_c, \]
where \( \Delta = d \circ \delta + \delta \circ d \) is the Hodge-Laplace operator, and \( \Delta^c \) satisfies
\[ \Delta^c \circ J^*_c = J^*_c \circ \Delta, \quad J^*_c \circ \Delta^c = \Delta \circ J^*_c. \]

**Definition 6.20.** A real form \( \alpha \) is called \( J \)-harmonic if it belongs to the kernel of the metallic Hodge-Laplace operator, i.e. \( \Delta^c \alpha = 0 \).

From the above relations, we get:

**Proposition 6.21.** Let \( \alpha \) be a real form on \( M \).
(i) If \( \alpha \) is \( J \)-harmonic, then \( J^*_c \alpha \) is harmonic.
(ii) If \( \alpha \) is harmonic, then \( J^*_c \alpha \) is \( J \)-harmonic.
(iii) If \( \alpha \) is \( J^*_c \)-invariant, i.e. \( J^*_c \alpha = \alpha \), then \( \alpha \) is \( J \)-harmonic if and only if it is harmonic.
(iv) \( \alpha \) is \( J \)-harmonic if and only if it is \( d^c \)-closed and \( \delta^c \)-coclosed.

Therefore, the \( J \)-harmonic forms are the \( J^*_c \)-invariant harmonic forms. Then
\[ \ker(\Delta^c) = \ker(\Delta) \cap \{ J^*_c \text{-invariant forms} \}, \quad \text{Im}(\Delta^c) = J^*_c(\text{Im}(\Delta)). \]

Let
\[ T^CM = T^{(1,0)}M \oplus T^{(0,1)}M = \mathcal{D}^c \oplus \mathcal{D}^c \]
be the decomposition of the complexified tangent bundle into \( (1,0) \) and \( (0,1) \) parts, with respect to the complex structure \( J_c \) or, equivalently, with respect to the distributions defined by \( J \).

The \( \delta \)-operator and \( \bar{\delta} \)-operator acting on \( (r,s) \)-forms on \( M \) are defined as follows:
\[ \delta : C^\infty(\Lambda^{(r,s)}(M)) \to C^\infty(\Lambda^{(r,s+1)}(M)), \quad \delta := \frac{1}{2}(d - id^c), \]
\[ \bar{\delta} : C^\infty(\Lambda^{(r,s+1)}(M)) \to C^\infty(\Lambda^{(r,s)}(M)), \quad \bar{\delta} := \frac{1}{2}(\delta - i\delta^c). \]
Remark that the integrability of $J$ (which is equivalent to the integrability of $J_u$) implies
\[ \tilde{\delta} \circ \tilde{\delta} = 0, \quad \tilde{\delta} \circ \tilde{\delta} = 0, \]
therefore we can consider the metallic complex cohomology groups
\[ H^{(r,s)}(M) := \ker(\tilde{\delta}_{(r,s)}) / \text{Im}(\tilde{\delta}_{(r,s-1)}), \]
where
\[ \tilde{\delta}_{(r,s)} : C^\infty(\Lambda^{(r,s)}(M)) \to C^\infty(\Lambda^{(r,s+1)}(M)) \]
and the metallic complex homology groups
\[ H_{(r,s)}(M) := \ker(\tilde{\delta}_{(r,s)}) / \text{Im}(\tilde{\delta}_{(r,s+1)}), \]
where
\[ \tilde{\delta}_{(r,s)} : C^\infty(\Lambda^{(r,s)}(M)) \to C^\infty(\Lambda^{(r,s-1)}(M)). \]

Now, if
\[ T^* C M = D^* C \oplus \overline{D^* C} \]
is the decomposition of the complexified cotangent bundle defined by $J^*$, then we get the following:

**Proposition 6.22.** Let $(M, J)$ be a metallic complex manifold such that $J^2 = pJ + qI$ with $p^2 + 4q < 0$. Then the $\tilde{\delta}$-operator:
\[ \tilde{\delta} = \frac{1}{2(p^2 + 4q)} \left( (p^2 + 4q)d + i(4J^* \circ d \circ J^* - 2pd \circ J^* - 2pJ^* \circ d + p^2 d) \right) \]
is acting on $C^\infty(\Lambda'(D^*)) \otimes C^\infty(\Lambda^2(D^* C))$.

**Proof.** We have:
\[ d^c = \left[ -\frac{1}{\sqrt{-p^2 - 4q}}(2J^* - pI) \right] \circ d \circ \left[ -\frac{1}{\sqrt{-p^2 - 4q}}(2J^* - pI) \right] = \]
\[ = -\frac{1}{p^2 + 4q} \left( (4J^* \circ d \circ J^* - 2pd \circ J^* - 2pJ^* \circ d + p^2 d) \right). \]
Then the statement follows. \(\square\)

Similarly, we prove that:

**Proposition 6.23.** Let $(M, J, g)$ be a metallic Norden manifold such that $J^2 = pJ + qI$ with $p^2 + 4q < 0$. Then the $\tilde{\delta}$-operator:
\[ \tilde{\delta} = \frac{1}{2(p^2 + 4q)} \left( (p^2 + 4q)d + i(4J^* \circ d \circ J^* - 2pd \circ J^* - 2pJ^* \circ d + p^2 d) \right) \]
is acting on $C^\infty(\Lambda'(D^*)) \otimes C^\infty(\Lambda^2(D^* C))$.

**Remark 6.24.** The operators $d^c$ and $\tilde{\delta}$ can be defined on metallic complex manifolds and $\delta^c$, $\Delta^c$ and $\tilde{\delta}$ only on metallic Norden manifolds.
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