A SAUSAGE BODY IS A UNIQUE SOLUTION FOR A REVERSE ISOPERIMETRIC PROBLEM

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ABSTRACT. We consider the class of $\lambda$-concave bodies in $\mathbb{R}^{n+1}$; that is, convex bodies with the property that each of their boundary points supports a tangent ball of radius $1/\lambda$ that lies locally (around the boundary point) inside the body. In this class we solve a reverse isoperimetric problem: we show that the convex hull of two balls of radius $1/\lambda$ (a sausage body) is a unique volume minimizer among all $\lambda$-concave bodies of given surface area. This is in a surprising contrast to the standard isoperimetric problem for which, as it is well-known, the unique minimizer is a ball. We solve the reverse isoperimetric problem by proving a reverse Bonnesen-style inequality, the second main result of this paper.

1. INTRODUCTION

The classical isoperimetric inequality states that if $K$ is an arbitrary domain in $\mathbb{R}^{n+1}$ with volume $\text{Vol}_{n+1}(K)$ and surface area $\text{Vol}_n(\partial K)$, then

\begin{equation}
\text{Vol}_{n+1}(K) \leq \frac{\text{Vol}_n(\partial K)^{n+1}}{(\omega_{n+1})^{\frac{n}{n+1}} (n+1)^{\frac{n+1}{n}}},
\end{equation}

where $\omega_{n+1}$ is the volume of the unit ball in $\mathbb{R}^{n+1}$. It is known that equality in (1.1) holds if and only if $K$ is a ball. In other words, the classical isoperimetric inequality asserts that among all domains with a given surface area, the ball has the smallest possible volume.

Inequality (1.1) has a long and beautiful history, and has been generalized to a variety of different settings (see, for example, surveys [BZ, Ro]). The distinctive point of almost all of these generalizations is that the extreme object is always a ball, as the most symmetric body. On the other hand, the problem can be looked at from a different point of view: under which conditions can one minimize the volume among all domains of a given constraint (such as of a given surface area, etc.)? Questions of such type are known as reverse isoperimetric problems, and have been actively studied recently.

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The naive attempt of minimizing volume among all sets of a given surface area will clearly lead to a trivial result: the \((n+1)\)-dimensional volume is zero for every set with empty interior. Therefore, we must consider a family of sets with additional conditions imposed in order to obtain a well-posed reverse isoperimetric problem. One of the natural conditions is convexity or strict convexity.

One of the first results on the reverse isoperimetric problem is due to Keith Ball. In his celebrated works [Bal1, Bal2] he showed that among all convex bodies in \(\mathbb{R}^{n+1}\) (modulo affine transformations), the standard simplex has the smallest volume for a given surface area; if the bodies are additionally assumed to be symmetric, then the cube is an extreme object. The equality case in Ball’s reverse isoperimetric inequalities was completely settled later by Barthe [Ba]. Observe that for Ball’s approach the minimizers are no longer balls.

Another approach towards obtaining a reverse isoperimetric inequality was recently taken in [PZh], where the authors provided a lower bound on the area enclosed by a convex curve \(\gamma \subset \mathbb{R}^2\) in terms of its length and the area of the domain enclosed by the locus of curvature centers of \(\gamma\). The authors also showed that equality is attained only for a disk. In this respect, the results in [PZh] do not follow the philosophy of a reverse isoperimetric problem. See also [XXZZ], but again these results, although called ‘reverse’, do not follow the philosophy of a reverse isoperimetric problem.

A different approach towards reversing the classical isoperimetric inequality is by assuming some curvature constraints for the boundary. It was pioneered by Howard and Treibergs [HTr] who proved a sharp reverse isoperimetric inequality on the Euclidean plane for closed embedded curves whose curvature \(k\), in a weak (or viscosity) sense, satisfies \(|k| \leq 1\), and whose lengths are in \([2\pi; 14\pi/3]\). In [Ga], Gard extended this result to surfaces of revolution that lie in \(\mathbb{R}^3\) and whose principal curvatures, again in a weak sense, are bounded in absolute value by 1, and the surface areas are not too big. Note that the mentioned curvature restrictions do not imply convexity.

At the same time, motivated by the study of strictly convex hypersurfaces in Riemannian spaces (see, for instance, [Bor1, BDr1]), Borisenko and Drach in a series of papers [BDr2, BDr3, Dr1] obtained two-dimensional reverse isoperimetric inequalities for so-called \(\lambda\)-convex curves, i.e. curves whose curvature \(k\), in a weak sense, satisfies \(k \geq \lambda > 0\). Recently, these results were generalized in [Bor2] for \(\lambda\)-convex curves in Alexandrov metric spaces of curvature bounded below. The result of Borisenko completely settles the reverse isoperimetric problem for \(\lambda\)-convex curves.

\(\lambda\)-convexity is a notion that can be easily transferred to higher dimensions. In particular, a convex body in \(\mathbb{R}^{n+1}\) is \(\lambda\)-convex if the principal curvatures \((k_i)_{i=1}^n\) of the boundary of the body are uniformly bounded, in a weak sense, by \(\lambda\), i.e. \(k_i \geq \lambda > 0\) for all \(i \in \{1, \ldots, n\}\) (we refer to [BM, BGR, Dr2] for various results concerning the geometry of multidimensional \(\lambda\)-convex bodies). It is worth pointing out that the reverse isoperimetric problem for \(\lambda\)-convex bodies has a non-trivial solution in any dimension, although for dimensions greater than two it is a hard problem that is still widely open.

In this paper we consider a notion, in a sense dual to the notion of \(\lambda\)-convexity. In particular, we consider so-called \(\lambda\)-concave bodies in \(\mathbb{R}^{n+1}\). These are the convex sets such that the principal curvatures of their boundaries satisfies \(\lambda \geq k_i \geq 0\) for all \(i \in \{1, \ldots, n\}\) (in
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For $\lambda$-concave bodies we completely solve the reverse isoperimetric problem in $\mathbb{R}^{n+1}$, besides the celebrated results of Ball and their various extensions, where the inequality is not restricted to curves or surfaces. Moreover, our methods allow us to prove the full family of sharp inequalities involving quermassintegrals of a convex body.

1.1. Motivation. Part of our motivation, besides previously mentioned work on the reverse isoperimetric problem for $\lambda$-convex domains due to Borisenko and Drach, came from results on the so-called Will’s conjecture.

If $K$ is a planar convex body with inradius $r$, then the inequality

$$\text{Vol}_2(K) \leq r \text{Vol}_1(K) - r^2 \pi$$

is called Bonnesen’s inradius inequality. Equality holds for the sausage body, that is, the Minkowski sum of a line segment and a circle with radius $r$. An extension of Bonnesen’s inradius inequality to higher dimensions was conjectured by Wills [Wi] in 1970. He conjectured that

$$\text{Vol}_{n+1}(K) \leq r \text{Vol}_n(\partial K) - n r^{n+1} \omega_{n+1},$$

for every convex body $K \subset \mathbb{R}^{n+1}$ with inradius $r$. This conjecture was proven independently by Bokowski [Bo] and Diskant [Di]. Although the same inequality with the circumradius $R$ of $K$ substituting $r$ is not true in dimensions greater than two (see [Di, He]), Bokowski and Heil [BH] showed that for higher dimensions, in fact, the inequality sign is reversed:

$$\text{Vol}_{n+1}(K) \geq \frac{2R}{n} \text{Vol}_n(\partial K) - \frac{(n+2)R^{n+1}}{n} \omega_{n+1}. \quad (1.2)$$

In [BH] inequality (1.2) was obtained as a corollary of the following more general result

**Theorem 1.1 ([BH]).** For an arbitrary convex body $K \subset \mathbb{R}^{n+1}$ with circumradius $R$, the inequalities

$$c_{ijk} R^i W_i(K) + c_{jki} R^j W_j(K) + c_{kij} R^k W_k(K) \geq 0,$$

hold for every $0 \leq i < j < k \leq n + 1$, where $c_{pqg} = (r - q)(p + 1).$ \hfill \Box

Here $W_i(K)$ is the quermassintegral of order $i$ of the convex body $K$, $i \in \{1, \ldots, n+1\}$ (see the next subsection and Section 2 for precise definitions). Quermassintegrals can be viewed as geometric quantities assigned to a convex body that are a higher-dimensional generalization of the integral curvature of a closed curve, and can be explicitly calculated in terms of the principal curvatures $k_i$ of $\partial K$, provided $\partial K$ is sufficiently smooth (see (2.1)). The quermassintegrals of different order provide a natural embedding of the volume $\text{Vol}_{n+1}(K)$, the surface area $\text{Vol}_n(\partial K)$ and the volume of the unit ball $\omega_{n+1}$ into the family $(W_i(K))_{i=0}^{n+1}$ for which (up to the constant) these are respectively, the zeroth, the first, and the $(n+1)$-th element. Therefore, (1.2) is a special case of (1.3) with $i = 0$, $j = 1$ and $k = n + 1$.

The form of the Bokowski–Heil inequality (1.2) inspired the statement of our main result, Theorem A, although we use different techniques for the proof. It appears that, having a natural inclusion of the volume, the surface area and the volume of the unit ball into the
family of quermassintegrals helps to solve the reverse isoperimetric problem for \( \lambda \)-concave bodies in \( \mathbb{R}^{n+1} \) for every \( n \geq 1 \).

1.2. The main results. Recall that a convex body in the Euclidean space \( \mathbb{R}^{n+1} \) is a compact convex set with a non-empty interior. In this paper balls will be closed sets.

**Definition 1.2** (\( \lambda \)-concave body). For a given \( \lambda > 0 \), a convex body \( K \subset \mathbb{R}^{n+1} \) is called \( \lambda \)-concave if for every \( p \in \partial K \) there exists a ball \( B_{1/\lambda,p} \) (called a supporting ball at \( p \)) of radius \( 1/\lambda \) passing through \( p \) in such a way that

\[
B_{1/\lambda,p} \cap U(p) \subseteq K \cap U(p)
\]

for some small open neighborhood \( U(p) \subset \mathbb{R}^{n+1} \) of \( p \).

Note that since \( K \) is assumed to be convex, if \( K \) is \( \lambda \)-concave then a supporting ball is unique at every point. As for the nomenclature, compare it to the notion of \( \lambda \)-convexity (see [BGR, BDr1, Dr2]), for which inclusion (1.4) is reversed (see also the discussion in Subsection 4.2).

If the boundary \( \partial K \) of a convex body \( K \) is at least \( C^2 \)-smooth, then \( K \) is \( \lambda \)-concave if and only if the principal curvatures \( k_i(p) \) for all \( i \in \{1, \ldots, n\} \) are non-negative and uniformly bounded above by \( \lambda \), i.e. \( 0 \leq k_i(p) \leq \lambda \) for every \( i \) and \( p \in \partial K \). Equivalently, in the smooth setting \( \lambda \)-concavity can be expressed in terms of uniformly bounded normal curvature. Let \( p \in \partial K \) be a point, \( v \in T_p\partial K \) be a vector, \( \nu \) be the outward pointing normal to \( \partial K \) at \( p \), and \( \pi(p,v) \) be the two-dimensional plane through \( p \) spanned by \( v \) and \( \nu \). The normal curvature \( k_n(p,v) \) of \( \partial K \subset \mathbb{R}^{n+1} \) at the point \( p \) in the direction of \( v \) is defined as

\[
k_n(p,v) := \kappa(p),
\]

where \( \kappa(p) \) is the curvature of the planar curve \( \partial K \cap \pi(p,v) \) at the point \( p \). Using this notion, a convex body \( K \) with smooth boundary is \( \lambda \)-concave if and only if \( 0 \leq k_n(p,v) \leq \lambda \) uniformly over \( p \) and \( v \). In general, \( K \) is \( \lambda \)-concave if the uniform bound on the normal curvatures is satisfied in the viscosity sense (see [BGR, Definition 2.3] for the similar approach).

Recall that for a convex body \( K \subset \mathbb{R}^{n+1} \) the quermassintegral of order \( i \) (denoted by \( W_i(K) \) with \( i \in \{0, \ldots, n+1\} \)) arises as a coefficient in the polynomial expansion

\[
\Vol_{n+1}(K + tB) = \sum_{i=0}^{n+1} \binom{n+1}{i} W_i(K)t^i
\]

known as the Steiner formula; here \( B \) is the unit Euclidean ball in \( \mathbb{R}^{n+1} \) and ‘+’ stands for the Minkowski addition; see Section 2 for details.

**Definition 1.3** (\( \lambda \)-sausage body). A \( \lambda \)-sausage body in \( \mathbb{R}^{n+1} \) is the convex hull of two balls of radius \( 1/\lambda \) (see Figure 1).

We are now ready to state the main results of the paper.
Theorem A (Reverse quermassintegrals inequality for $\lambda$-concave bodies). Let $K \subset \mathbb{R}^{n+1}$ be a convex body. If $K$ is $\lambda$-concave, then

$$ (k - j)\frac{W_i(K)}{\lambda^i} + (i - k)\frac{W_j(K)}{\lambda^j} + (j - i)\frac{W_k(K)}{\lambda^k} \geq 0 $$

for every triple $(i, j, k)$ with $0 \leq i < j < k \leq n + 1$. Moreover, equality in (1.5) holds if and only if $K$ is a $\lambda$-sausage body.

Since $W_0(K) = \text{Vol}_{n+1}(K)$, $W_1(K) = \text{Vol}_n(\partial K)/(n + 1)$ and $W_{n+1}(K) = \omega_{n+1}$, inequality (1.5) for $i = 0$, $j = 1$ and $k = n + 1$ immediately implies the following result:

Theorem B (Reverse isoperimetric inequality for $\lambda$-concave bodies). Let $K \subset \mathbb{R}^{n+1}$ be a convex body. If $K$ is $\lambda$-concave (for some $\lambda > 0$), then

$$ \text{Vol}_{n+1}(K) \geq \frac{\text{Vol}_n(\partial K)}{n\lambda} - \frac{\omega_{n+1}}{n\lambda^{n+1}}, $$

where $\omega_{n+1}$ is the volume of the unit ball in $\mathbb{R}^{n+1}$. Moreover, equality holds if and only if $K$ is a $\lambda$-sausage body. $\square$

Theorem A (and hence Theorem B) for $n = 2$ and $n = 3$ was first proven using different techniques in the bachelor thesis of the first author [Ch]. It should be pointed out that Theorem B for $n = 2$ was announced earlier in [BDr3], although the authors did not provide a proof.

2. General background on quermassintegrals and convex geometry

In this section we present some background material and auxiliary lemmas towards the proof of the main result.

The Minkowski addition of two convex bodies $K$ and $L$ in $\mathbb{R}^{n+1}$ is defined by

$$ K + L := \{x + y : x \in K, y \in L\}. $$
One can rewrite the definition in the following form
\[
K + L = \bigcup_{y \in L} (K + y);
\]
that is \(K + L\) can be viewed as the set that is covered if \(K\) undergoes translations by all vectors in \(L\). Since \(K\) and \(L\) are convex, then \(K + L\) is also convex. For a parameter \(t \geq 0\), the Minkowski sum \(K + tB\), where \(B\) is the unit ball in \(\mathbb{R}^{n+1}\), is called the *outer parallel body* for \(K\). The *Minkowski difference* of convex bodies \(K\) and \(L\) is defined by
\[
K - L := \{x \in \mathbb{R}^{n+1} : L + x \subset K\}.
\]
Similarly to the operation of addition, we can rewrite the definition of Minkowski difference in the form
\[
K - L = \bigcap_{y \in L} (K - y).
\]
For a parameter \(t \geq 0\), the Minkowski difference \(K - tB\) is called the *inner parallel body*.

For a convex body \(K \subset \mathbb{R}^{n+1}\), the \((n+1)\)-dimensional volume \(\text{Vol}_{n+1}(K + tB)\) of its outer parallel body \(K + tB\) is a polynomial in \(t\), and by the classical *Steiner formula*,
\[
\text{Vol}_{n+1}(K + tB) = \sum_{i=0}^{n+1} \binom{n+1}{i} W_i(K) t^i,
\]
where \(W_i(K)\) is the *quermassintegral of order* \(i\) of the convex body \(K\). If the boundary \(\partial K\) of the body \(K\) is at least \(C^{1,1}\)-smooth, and hence the principal curvatures \(k_1, \ldots, k_n\) are well-defined almost everywhere on \(\partial K\), then
\[
W_j(K) = \frac{1}{(n+1)(n-j-1)} \int_{\partial K} \sigma_{j-1} d\mathbf{x} \quad \text{for } 1 \leq j \leq n+1,
\]
where \(\sigma_0 = 1\) and
\[
\sigma_l = \sum_{1 \leq i_1 < \cdots < i_l \leq n} k_{i_1} k_{i_2} \cdots k_{i_l}
\]
is the \(l\)-th symmetric function of principal curvatures. By convention, \(W_0(K)\) is equal to the \((n+1)\)-dimensional volume of the body. It follows from (2.1) that \((n+1)W_1(K)\) is the \(n\)-dimensional volume (surface area) of \(\partial K\), and \((n+1)W_{n+1}(K) =: s_n\) is the \(n\)-dimensional volume (surface area) of the unit sphere \(S^n\). Recall that \(s_n/(n+1) = \omega_{n+1}\), where \(\omega_{n+1}\) is the volume of a unit \(n\)-ball in \(\mathbb{R}^{n+1}\); hence \(W_{n+1}(K) = \omega_{n+1}\).

We will need the following generalization of the Steiner formula for inner parallel bodies (see [Sch, p. 225]):
\[
W_q(K - B) = \sum_{i=0}^{n+1-q} (-1)^i \binom{n+1 - q}{i} W_{q+i}(K)
\]
for every $0 \leq q \leq n + 1$. In particular, for $q = 0$ we have

$$\text{(2.2)} \quad \text{Vol}_{n+1}(K - B) = \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} W_i(K).$$

The following continuity result [G, p. 399] will be useful later on for the proof of Theorem A.

**Proposition 2.1** (Continuity of quermassintegrals). Let $(K_i)_{i=0}^\infty$ be a sequence of convex bodies. Suppose

$$\lim_{i \to \infty} K_i = K$$

in the Hausdorff metric. Then

$$\lim_{i \to \infty} W_j(K_i) = W_j(K)$$

for every $j \in \{0, 1, \ldots, n\}$. □

**Definition 2.2** (Core of a $\lambda$-concave body). A core of a $\lambda$-concave body $K$ is the set $K_c := K - B_{1/\lambda}$, where “$-$” denotes the Minkowski difference of convex sets, $B_{1/\lambda}$ is a ball of radius $1/\lambda$.

(Compare this notion to the notion of a kernel of a convex body [SY, p. 374].)

It is easy to see that $K_c$ is a convex set in $\mathbb{R}^{n+1}$; however, the core is not necessarily $\lambda$-concave, even more, $K_c$ is not necessarily a convex body in $\mathbb{R}^{n+1}$. Recall that the affine hull of a convex set $S \subset \mathbb{R}^{n+1}$ is the affine subspace of least dimension that contains $S$. We will call the dimension of the core (denoted by $\dim K_c$) to be the dimension of the affine hull of $K_c$. Clearly, $K_c$ is a convex body if its dimension is $n + 1$.

The classical result due to Blaschke implies that local condition (1.4) is in fact global (see [BS, Mi], and [Bla] for the original result of Blaschke).

**Theorem 2.3** (Blaschke’s ball rolling theorem of $\lambda$-concave bodies). Let $K \subset \mathbb{R}^{n+1}$ be a $\lambda$-concave body. Then

$$B_{1/\lambda,p} \subseteq K$$

for every point $p \in \partial K$ and every supporting ball $B_{1/\lambda,p}$ at $p$. □

Using Blaschke’s ball rolling theorem it is easy to prove the following approximation result. We will say that a $\lambda$-concave body $P$ is a $\lambda$-concave polytope if $P_c$ is a polytope, possibly of lower dimension. Note that a $\lambda$-concave polytope is not a polytope, but rather an outer parallel body (at distance $1/\lambda$) for a polytope.

**Proposition 2.4** (Approximation of $\lambda$-concave bodies). Let $K$ be a $\lambda$-concave body. Then there exists a sequence $(P_i)_{i=1}^\infty$ of $\lambda$-concave polytopes such that $P_1 \subseteq P_2 \subseteq \ldots \subseteq K$, $\dim(P_i)_c = \dim K_c$ for all $i$, and

$$\text{(2.3)} \quad \lim_{i \to \infty} P_i = K$$

in the Hausdorff metric.
Proof. It is known that a convex body can be approximated from inside in the Hausdorff metric by convex polytopes with vertices on $\partial K$ [Sch, p. 67]. Let $(P_i')$ be such a sequence. Since $P_i'$ tends to $K$ as $i \to \infty$, we can assume that $\dim(P_i') = \dim K$, after possibly passing to a subsequence. Moreover, we can further refine this sequence into a sequence $(P''_i)$ which is nested, that is $P''_1 \subseteq P''_2 \subseteq P''_3 \subseteq \ldots \subseteq K$ (for example, by choosing $P''_2$ to be the convex hull of $P'_1$ and $P'_2$, $P''_3$ to be the convex hull of $P'_1$, $P'_2$ and $P'_3$, and so on).

Let $V''_i \subset \partial K$ be the vertex set of $P''_i$. By Blaschke’s ball rolling theorem (Theorem 2.3), the union of all supporting balls of radius $1/\lambda$ at points in $V''_i$ lies in $K$:

\[(2.4) \quad V_i := \bigcup_{p \in V''_i} B_{1/\lambda, p} \subseteq K.\]

Define $P_i$ to be the convex hull of $V_i$. Clearly, $P_i$ is a $\lambda$-concave polytope. Moreover, by construction, $P''_i \subset P_i$, $\dim(P_i) = \dim K$, and $P_i \subseteq K$; the later inclusion follows from (2.4). Moreover, since supporting balls are uniquely defined by the corresponding points of support, it follows that $P_i \subseteq P_{i+1}$ for all $i \geq 1$. Finally, since $P''_i \subset P_i \subseteq K$ and $\lim_{i \to \infty} P''_i = K$, we obtain (2.3). \[\square\]

3. PROOF OF THE REVERSE QUERMASSEINTEGRAL INEQUALITY FOR $\lambda$-CONCAVE BODIES (THEOREM A)

In order to prove Theorem A, we will use an approximation of a given $\lambda$-concave body with $\lambda$-concave polytopes. Therefore, we start by establishing the following proposition.

Proposition 3.1 (Reverse quermassintegral inequality for $\lambda$-concave polytopes). Theorem A holds for $\lambda$-concave polytopes.

Proof. Let $P$ be a $\lambda$-concave polytope. We want to show that inequality (1.5) for $P$ is strict unless $P$ is a $\lambda$-sausage body. Since the left-hand side of (1.5) is scale-invariant, without loss of generality we can assume $\lambda = 1$.

Consider the core $P_c$ of $P$; define

\[(3.1) \quad m := \begin{cases} 
\dim P_c - 1, & \text{if } \dim P_c = n + 1, \\
\dim P_c, & \text{if } \dim P_c \leq n.
\end{cases}\]

By construction, $P = P_c + B$, where $B$ is a unit ball in $\mathbb{R}^{n+1}$. The latter equality provides a complete description of the boundary of $P$ in terms of principal curvatures, and hence simplifies computation. In particular, $\partial P$ is decomposed as

\[(3.2) \quad \partial P = \text{Cl} \left( \bigcup_{s=0}^{m} F_{n-s} \right) \]

into open sets $F_{n-s}$ such that on $F_{n-s}$ exactly $n - s$ principal curvatures equal to 1, all the rest, namely $s$, equal to 0 (here Cl(·) stands for the closure of a set in $\mathbb{R}^{n+1}$). Each $F_{n-s}$ is an open subset of the outer parallel set of the union of $s$-dimensional facets of $P_c$, and hence each connected component of $F_{n-s}$ is a measurable subset of the cylinder $\mathbb{S}^{n-s} \times \mathbb{R}^s$ of non-zero $n$-dimensional Lebesgue measure. Note that the intersection of any two connected components of Cl $F_j$ and Cl $F_i$ has $n$-dimensional Lebesgue measure zero.
Plugging (3.2) into (2.1), we get

\[ W_i(P) = \frac{1}{(n+1)(i-1)} \int_{\partial P} \sigma_{i-1} \, d\mathbf{x} = \frac{1}{(n+1)(i-1)} \sum_{s=0}^{m} \int_{F_{n-s}} \sigma_{i-1} \, d\mathbf{x} \]

for every \( i \in \{1, \ldots, n+1\} \).

For \( s \in \{0, \ldots, m\} \) define \( a_s := \text{Vol}_s(F_{n-s}) \). Observe that it might happen that some of \( a_s \) vanish. For example, for a \( \lambda \)-sausage body its core is just a segment, and hence \( a_s = 0 \) for all \( s \geq 2 \). Since we know the values of principal curvatures at every point on \( F_{n-s} \), we obtain

\[ \int_{F_{n-s}} \sigma_{i-1} \, d\mathbf{x} = \left( \frac{n-s}{i-1} \right) a_s, \]

where, by convention, \( \left( \frac{1}{i} \right) = 0 \) if \( l > j \). Since \( P_t \) is convex, its Gauss image is equal to \( S^n \), hence \( a_0 = (n+1)\omega_{n+1} \) (recall that \( \omega_{n+1} \) is the volume of the \( (n+1) \)-dimensional unit ball in \( \mathbb{R}^{n+1} \)). Plugging (3.4) into (3.3), we obtain

\[ W_i(P) = \omega_{n+1} + \frac{1}{(n+1)(i-1)} \sum_{s=1}^{m} \left( \frac{n-s}{i-1} \right) a_s \]

for every integer \( i \) between 1 and \( n+1 \).

In order to prove (1.5), we partly follow ideas in [BH, Theorem 2]. We claim that it suffices to show that (1.5) is true for any three consecutive indices, i.e. to show that

\[ E_i(P) := W_{i-1}(P) - 2W_i(P) + W_{i+1}(P) \]

is non-negative for every \( i \in \{1, \ldots, n\} \). Indeed, if (1.5) is true for every triple \((i-1, i, i+1)\), then for a triple \((i, j, k)\) applying it repeatedly, we get

\[ W_k - W_{k-1} \geq W_{k-1} - W_{k-2} \geq \cdots \geq W_j - W_{j-1} \geq \cdots \geq W_i - W_i. \]

Estimating the sum of the first \( k-j \) and the last \( j-i \) differences in this sequence, we get

\[
(W_k - W_{k-1}) + \cdots + (W_{j+1} - W_j) \geq (k-j)(W_j - W_{j-1}),
\]

\[
(W_j - W_{j-1}) + \cdots + (W_i+1 - W_i) \leq (j-i)(W_j - W_{j-1})
\]

(he here we simplify our notation by setting \( W_i = W_i(P) \)). Performing cancellation and dividing both inequalities by \( k-j \) and \( j-i \) respectively, we obtain

\[
\frac{W_k - W_j}{k-j} \geq \frac{W_j - W_{j-1}}{j-i},
\]

and hence \((W_k - W_j)/(k-j) \geq (W_j - W_i)/(j-i)\) for every ordered triple \((i, j, k)\) with \( 1 \leq i < j < k \leq n+1 \). This is equivalent to (1.5), and the claim follows.

Let us now turn to showing that

\[ E_i(P) \geq 0 \text{ for every } i \in \{1, \ldots, n\}. \]

We consider two cases: \( i \geq 2 \) and \( i = 1 \); this is done because (3.3) does not cover the latter case, and Steiner formula (2.2) should be used instead.
Case 1: $i \geq 2$. Using (3.5) and performing cancellation along the way, we obtain

$$E_i(P) = \frac{1}{n+1} \sum_{s=1}^{m} a_s \cdot b_{s,i}, \quad \text{where } b_{s,i} := \frac{(n-s)}{(i-2)} - 2\frac{(n-s)}{(i-1)} + \frac{(n-s)}{i}.$$  

In order to establish (3.6), it suffices to show that

\begin{equation}
(3.7) \quad b_{s,i} \geq 0 \quad \text{for all } i \in \{1, \ldots, n\} \text{ and } s \in \{1, \ldots, m\}
\end{equation}

(recall that $m \leq n$). There are three possibilities: either $i \leq n-s$, and hence all three terms in $b_{s,i}$ are non-zero, or $i = n-s+1$, and hence the last term in $b_{s,i}$ is zero, or, finally, $i > n-s+1$. For the last possibility it is clear that $b_{s,i} \geq 0$. In the second to last possibility we have

$$b_{s,i} = b_{s,n-s+1} = \frac{(n-s)}{(n-s-1)} - 2\frac{(n-s)}{n} = \frac{s-1}{s},$$

which is non-negative for every $s \geq 1$. Therefore, (3.7) holds true for $i \geq n-s+1$, and hence we are left with the first possibility $i \leq n-s$. In this case we compute:

$$b_{s,i} = \frac{(n-s)!}{n!} \left( \frac{(n-i+2)!}{(n-i+2-s)!} - 2\frac{(n-i+1)!}{(n-i+1-s)!} + \frac{(n-i)!}{(n-i-s)!} \right)$$

$$= \frac{(n-s)!}{n!(n-i+2-s)!} \left( (n-i+2)(n-i+1) - 2(n-i+1)(n-i+2) \right)$$

$$+ (n-i+1-s)(n-i+2-s) = \frac{(n-s)!}{n!(n-i+2-s)!} s(s-1).$$

Therefore, $b_{s,i} \geq 0$ for $s \geq 1$ (assuming $i \leq n-s$, that is when the first possibility occurs). Claim (3.7) follows; this claim implies (3.6) for $i \geq 2$, as desired. Note that $b_{s,i} > 0$ unless $s = 1$. We will use this observation below to treat the equality case.

Case 2: $i = 1$. Since $W_0(P)$, by definition, is the volume of $P$, we will use Steiner formula (2.2) to estimate $E_1(P)$. Because $\text{Vol}_{n+1}(P-B) = \text{Vol}_{n+1}(P_c) \geq 0$, from (2.2) we obtain

\begin{equation}
(3.8) \quad W_0(P) + \sum_{j=1}^{n+1} (-1)^j \binom{n+1}{j} W_j(P) \geq 0
\end{equation}

Plugging (3.5) into (3.8) we get

$$0 \leq W_0(P) + \omega_{n+1} \cdot \sum_{j=1}^{n+1} (-1)^j \binom{n+1}{j} + \sum_{j=1}^{n+1} \frac{(-1)^j}{j} \sum_{s=1}^{m} \frac{(n-s)}{(j-1)} a_s$$

$$= W_0(P) - \omega_{n+1} + \sum_{s=1}^{m} \left( \sum_{j=1}^{n+1} \frac{(-1)^j}{j} \binom{n-s}{j} \right) a_s,$$
where in the last step we used the identity \( \sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j} = 0 \). Hence, again using (3.5),

\[
E_1(P) = W_0(P) - 2W_1(P) + W_2(P) \geq \sum_{s=1}^{m} a_s \cdot c_s,
\]

where

\[
c_s := -\sum_{j=1}^{n+1} \frac{(-1)^j}{j} \binom{n-s}{j-1} - \frac{2}{n+1} + \frac{n-s}{n(n+1)}
\]

\[
= -\sum_{j=1}^{n+1} \frac{(-1)^j}{j} \binom{n-s}{j-1} - \frac{n+s}{n(n+1)}.
\]

Similar to (3.7), in order to prove (3.6) for \( i = 1 \) it suffices to show that

\[
c_s \geq 0 \text{ for all } s \in \{1, \ldots, m\}.
\]

Observe that, in fact,

\[
\sum_{j=1}^{n+1} \frac{(-1)^j}{j} \binom{n-s}{j-1} = \sum_{j=1}^{n-s+1} \frac{(-1)^j}{j} \binom{n-s}{j-1}
\]

because the remaining terms vanish. Moreover,

\[
\sum_{j=1}^{n-s+1} \frac{(-1)^j}{j} \binom{n-s}{j-1} = \frac{1}{n-s+1} \sum_{j=1}^{n-s+1} (-1)^j \binom{n-s+1}{j} = -\frac{1}{n-s+1}.
\]

Therefore, putting all together we obtain

\[
c_s = \frac{1}{n-s+1} - \frac{n+s}{n(n+1)} = \frac{s(s-1)}{n(n+1)(n-s+1)},
\]

which implies (3.9) for all \( s \in \{1, \ldots, m\} \) (recall that \( m \leq n \)). Claim (3.9) yields (3.6) for \( i = 1 \). This finishes the proof of Case 2.

The conclusions of Case 1 and Case 2 together imply claim (3.6). In turn, as we showed above, (3.6) implies inequality (1.5) for an arbitrary ordered triple \((i,j,k)\), as desired. The inequality part of Proposition 3.1 is proven.

Let us now turn to showing that equality in (1.5) is possible if and only if \( P \) is a 1-sausage body. The ‘if’ part is trivial, thus we focus on the ‘only if’ part.

**Equality Case.** Suppose we have equality in (1.5) for some ordered triple \((i,j,k)\). From our reduction to three consecutive indices in the proof of inequality for \( P \) it follows that we must have equality for the triple \((i, i+1, i+2)\), that is we must have \( E_{i+1}(P) = 0 \).

If \( i \geq 1 \), then we must have equality throughout our computation in Case 1. In particular, we must have \( a_s \cdot b_{s,i} = 0 \) for the given \( i \) and all \( s \in \{1, \ldots, m\} \). But as we showed (see the remark at the end of Case 1), \( b_{s,i} > 0 \) for \( s > 1 \). Therefore, we must have \( a_s = 0 \) for all \( s \in \{2, \ldots, m\} \), which immediately implies that \( \partial P \) does not contain open
regions where at least two principal curvatures are zero. This is only possible if $P_c$ is a segment (possibly degenerate to a point), and hence $P$ is a 1-sausage body.

If $i = 0$, then similarly we must have $a_s \cdot c_s = 0$ for every $s \in \{1, \ldots, m\}$, and moreover, we must have equality in (3.8). The latter is possible only if the $(n+1)$-dimensional volume of the core of $P$ is zero (hence dim $P_c$ is at most $n$). Again, as we showed, for $s \geq 2$ we have $c_s > 0$, thus $a_s$ must necessarily vanish for $s \geq 2$. The same conclusion as in the previous paragraph follows.

Summing up the cases $i \geq 1$ and $i = 0$ we obtain that for $P$ inequality (1.5) turns into equality if and only if $P$ is a 1-sausage body. This finishes the proof of the equality part of Proposition 3.1, and together with the proven inequality part concludes the proof. □

The proof of Theorem A, similarly to the proof of Proposition 3.1, naturally splits into two parts: proving the inequality (the inequality part), and proving that equality is attained only for λ-sausage bodies (the equality part). As we already mentioned, we will establish the inequality part by using an approximation of a λ-concave body by λ-concave polytopes (Proposition 2.4), and then applying the continuity result of Proposition 2.1. This is a straightforward part of the proof. Establishing the equality case, on the other hand, involves a more delicate argument. For it we will need the following technical lemma, which gives us control over the $n$-volume (area) of ‘the least curved’ domains on the boundary of a given λ-concave polytope. Recall that if $P$ is a λ-concave polytope, then the boundary $\partial P$ admits decomposition (3.2) into open sets $F_{n-s}$ comprised of points at which $s$ principal curvatures equal to 0 and $n - s$ principal curvatures equal to 1.

**Lemma 3.2** (Volume dependence for facets of highest dimension). Let $P \subset \mathbb{R}^{n+1}$ be a 1-convex polytope, and let $d = \dim P_c - 1 \leq n$. Then

$$a_d = \text{Vol}_n (F_{n-d}) = \frac{1}{2} \text{Vol}_d (G_d) \cdot \text{Vol}_{n-d} \left( S^{n-d} \right),$$

where $G_d$ is the union of all $d$-dimensional facets of $P_c$.

**Remark.** In Lemma 3.2 we assume that $S^0$ is a pair of points at distance two and $\text{Vol}_0$ is the counting measure; hence $\text{Vol}_0 (S^0) = 2$.

**Remark.** It is useful to connect $d$ and $m$: if $\dim P_c = n + 1$, then $d = n$ and $m = n$; however, if $\dim P_c \leq n$, then $d = \dim P_c - 1$, while $m = \dim P_c$.

**Proof.** Let $x \in \partial P_c$ be a point, and $\mathcal{H}(x)$ be the set of all supporting hyperplanes to $P_c$ at $x$. Since $P_c$ is convex, we have $\mathcal{H}(x_1) = \mathcal{H}(x_2)$ if and only if $x_1$ and $x_2$ lie in the interior of a common face of $P_c$ (in the topology of the affine hull of this face). Therefore, if $g$ is the interior of an $d$-dimensional facet of $P_c$, then we can write $\mathcal{H}(g) := \mathcal{H}(x)$, $x \in g$, and this set is well-defined. The set $\mathcal{H}(g)$ comprises of all supporting (to $P_c$) hyperplanes that contain $g$.

Write $N(g)$ for the set of unit outward pointing normal vectors for the hyperplanes in $\mathcal{H}(g)$; the set $N(g)$ can be identified with a closed subset of $S^{n-d}$ (for example, by introducing the coordinate system $(x_1, \ldots, x_{n+1})$ such that $g$ lies in the plane $x_{d+1} = x_{d+2} = \ldots = x_{n+1} = 0$ and $P_c$ lies in the set $\{(x_1, \ldots, x_{n+1}) : x_{d+2} = \ldots = x_{n+1} = 0, x_{d+1} \leq 0\}$; in this coordinate system the coordinates of the vectors in $N(g)$ will have first $d$ entries equal to zero; the
set is closed because each vector in $N(g)$ gives a point in $\partial P$, taking limits preserves this property). Finally, put $f(g) := \{x + \nu: x \in g, \nu \in \text{int}(N(g))\}$ with $\text{int}(\cdot)$ being the interior of a set in the topology of $\mathbb{S}^{n-d}$; $f(g)$ is an open subset of $F_{n-d}$. Since $P_c$ is convex, it follows that $f(g_1) \cap f(g_2) \neq \emptyset$ if and only if $g_1 = g_2$. From the definition of the Minkowski addition of $P_c$ and the unit ball (recall that $P = P_c + B$) it follows that

$$F_{n-d} = \bigcup_g f(g),$$

where the union is taken over all $d$-dimensional facets of $P_c$.

Therefore,

$$a_d = \text{Vol}_n(F_{n-d}) = \sum_g \text{Vol}_n(f(g)).$$

Since the induced Riemannian metric on $f(g)$ is a product metric of the flat metric on $g$ and the round metric on $N(g) \subset \mathbb{S}^{n-d}$, we get

$$a_d = \sum_g \text{Vol}_d(g) \cdot \text{Vol}_{n-d}(N(g)).$$

We claim that $N(g)$ is a closed hemisphere in $\mathbb{S}^{n-d}$. To establish this, first observe that $N(g)$ contains the vector $e = (0, \ldots, 1, \ldots, 0)$ with the only non-zero entry in the $(d+1)$-st coordinate, and moreover, $-e \notin N(g)$. If $d = n$, there is nothing to prove further, and we are done. Therefore, we can assume $d < n$. Then $N(g)$ contains the unit ‘equatorial’ sphere $\mathbb{S}^{n-d-1} \subset \mathbb{S}^{n-d}$: it is given by all unit vectors which in the coordinate system specified above have first $d + 1$ entries equal to zero. Finally, let $\nu_0, \nu_1$ be two distinct points in $N(g)$ with $\nu_0 \neq -\nu_1$, $H_0, H_1 \in \mathcal{H}(g)$ be the supporting hyperplanes corresponding to $\nu_0, \nu_1$, respectively, and let $R$ be the intersection of the closed half-spaces with respect to $H_0$ and $H_1$ that contain $P_c$ (if it happens that $P_c \subset H_0 \cap H_1$, then we assume that the half-spaces are chosen to be the lower half-spaces with respect to the normal vectors of $H_0$ and $H_1$). Then for every $t \in [0, 1]$ the hyperplane $H_t$ through $H_0 \cap H_1$ with the unit normal vector $\nu_t := t\nu_1 + (1-t)\nu_0/||\nu_1 + (1-t)\nu_0||$ is supporting to $R$ and hence to $P_c$ (here $|| \cdot ||$ stands for the norm in $\mathbb{R}^{n+1}$). By construction, $g \subset H_t$, and we conclude $H_t \in \mathcal{H}(g)$ and $\nu_t \in N(g)$ for every $t \in [0, 1]$. Applying this reasoning to $\nu_0 = e$ and each $\nu_1 \in \mathbb{S}^{n-d-1}$, we obtain that $N(g)$ contains the closed hemisphere of $\mathbb{S}^{n-d}$; for convenience, call this hemisphere ‘upper’. Let us show that, in fact, $N(g)$ is equal to this hemisphere.

Assume the contrary, and let $v \in N(g)$ be a point in the open lower hemisphere. Let $\gamma \subset \mathbb{S}^{n-d}$ be the big circle through $v$ and $-e$, and let $v_0$ and $v_1$ be a pair of points in $\gamma \cap \partial N(g)$ (where the boundary is taken with respect to the topology in $\mathbb{S}^{n-d}$). Since $-e$ is not in $N(g)$, this pair exists; and since the whole closed upper hemisphere is in $N(g)$, this pair necessarily lies in the closed lower hemisphere. Moreover, since $v$ is in the open lower hemisphere, one of the points $v_0$ or $v_1$ can be chosen to lie in the open lower hemisphere as well (for instance, by picking a pair of points in $\gamma \cap \partial N(g)$ that are closest, in the spherical metric, to $-e$). Hence, $v_0 \neq -v_1$ and the arc of $\gamma$ connecting $v_0$ to $v_1$ through $-e$ is the unique minimizing geodesic segment between $v_0$ and $v_1$. By our construction above, all
points in this segment belong to $N(g)$. But this is a contradiction because $-e \notin N(g)$. Therefore, $N(g)$ is equal to a closed upper hemisphere, as was claimed.

The claim implies that $\text{Vol}_{n-d}(N(g)) = \text{Vol}_{n-d}(S^{n-d}) / 2$, which together with (3.11) yields (3.10).

**Proof of Theorem A.** Again, by scale-invariance of the left-hand side of (1.5), we can assume $\lambda = 1$. By Proposition 2.4, there exists a nested sequence $(P_l)_{l=1}^{\infty}$ of 1-concave polytopes approximating $K$ from inside. For each of the polytopes $P_l$ in the sequence we know that (1.5) holds true. By Proposition 2.1, the left-hand side of (1.5) depends continuously on the body. From this the inequality for $K$ follows by passing to the limit as $l \to \infty$ in the inequality for $P_l$. The inequality case is proven.

Now we do a careful treatment of the equality case for general 1-concave bodies. This is where Lemma 3.2 and the properties of our approximation will come into play.

**Equality case.** As usual, assume that we have equality for a triple $(i,j,k)$, that is

$$(k-j)W_i(K) + (i-k)W_j(K) + (j-i)W_k(K) = 0,$$

and let $(P_l)_{l=1}^{\infty}$ be a sequence of 1-concave polytopes that approximate $K$; this sequence is given by Proposition 2.4.

Similar to the proof of the equality case in Proposition 3.1, if we have equality for the triple $(i,j,k)$, then we must have equality for the triple $(i,i+1,i+2)$, that is

$$E_{i+1}(K) = W_i(K) - 2W_{i+1}(K) + W_{i+2}(K) = 0$$

(here we use the notation from the proof of Proposition 3.1).

We claim that if there exists an index $l_0$ such that for $P_{l_0}$ we have equality in (1.5) for the triple $(i,i+1,i+2)$, then $K$ is necessarily a 1-sausage body. Indeed, if

$$W_i(P_{l_0}) - 2W_{i+1}(P_{l_0}) + W_{i+2}(P_{l_0}) = 0,$$

then $P_{l_0}$ is a 1-sausage body by Proposition 3.1. By Proposition 2.4, $\dim(P_{l_0})_c = \dim K_c$. Therefore, $\dim K_c \leq 1$, and since $K_c$ is convex, it follows that $K$ is a 1-sausage body as well.

Let us show that such an index $l_0$ exists. Assume the contrary, that is $E_{i+1}(P_l) > 0$ for all $l$. By continuity, it then follows that

$$E_{i+1}(P_l) \xrightarrow{l \to \infty} E_{i+1}(K) = 0.$$  \hspace{1cm} (3.12)

Since none of the non-negative quantities $b_s$ (for the case $i = 0$) and $c_s$ (for the case $i > 0$) depend on the geometry of the actual polytope (in other words, do not depend on $l$), and are non-zero by assumption $E_{i+1}(P_l) > 0$, it then follows from (3.12) that we must have

$$a_d^l \xrightarrow{l \to \infty} 0,$$

where $d = \dim (P_l)_c - 1$, and the index $l$ indicates the corresponding quantity defined for $P_l$. Note here that $d$ is well-defined, since the dimension of the cores of $P_l$ is the same. The second remark after Lemma 3.2 describes the relation between $d$ and $m$. 


By Proposition 2.4 we also know that the sequence of approximating polytopes is nested as

\[ P_1 \subseteq P_2 \subseteq P_3 \subseteq \ldots, \]

hence their cores are also nested as

\[ (P_1)_c \subseteq (P_2)_c \subseteq (P_3)_c \subseteq \ldots \]

and have common affine hull.

But since all the cores are convex, the \( d \)-dimensional Lebesgue measure of the union \( G^l_d \) of all \( d \)-dimensional facets of a given polytope \( P_l \) monotonically increasing as \( l \) goes to \( \infty \).

Therefore, \( a^l_d \) also monotonically increasing as \( l \to \infty \) due to Lemma 3.2, which is clearly in contradiction with (3.13). We came to the contradiction, which proves the equality case for general 1-concave bodies. Theorem A is proven. \( \Box \)

4. Concluding remarks

4.1. The reverse isodiametric inequality. In this subsection we want to extend a bit our philosophy of a reverse isoperimetric problem to a so-called isodiametric inequality. Recall that a diameter of a convex body \( K \subset \mathbb{R}^{n+1} \), denoted as \( \text{diam}(K) \), is the following quantity:

\[ \text{diam}(K) = \max_{p,q \in K} |pq|. \]

In other words, the diameter is the length of the largest segment that connects two points in \( K \). The classical isodiametric inequality for convex bodies in \( \mathbb{R}^{n+1} \) asserts that for a given diameter \( D \) the ball of radius \( D/2 \) has the largest volume among all convex bodies of diameter \( D \) (see [Sch, p. 383]).

One simple observation allows us to prove the reverse isodiametric inequality.

**Theorem 4.1** (Reverse isodiametric inequality for \( \lambda \)-concave bodies). Let \( K \subset \mathbb{R}^{n+1} \) be a convex body. Suppose \( K \) is \( \lambda \)-concave, and let \( S_\lambda \subset \mathbb{R}^{n+1} \) be the \( \lambda \)-sausage body with 

\[ \text{diam}(K) = \text{diam}(S_\lambda). \]

Then

\[ W_i(K) \geq W_i(S_\lambda) \]

for every \( i \in \{0, 1, \ldots, n\} \). Moreover, equality holds if and only if \( K \) is a \( \lambda \)-sausage body.

**Proof.** Let \( p \) and \( q \) be a pair of points in \( K \) realizing the diameter of \( K \). It is easy to see that necessarily \( p, q \in \partial K \), and moreover, both tangent planes to \( \partial K \) at \( p \) and \( q \) are perpendicular to the segment \( pq \). Therefore, if \( B_{1/\lambda,p} \) and \( B_{1/\lambda,q} \) are the supporting balls at \( p \) resp. \( q \) of radius \( 1/\lambda \), then \( B_{1/\lambda,p} \subseteq K \) and \( B_{1/\lambda,q} \subseteq K \) (by Blaschke’s ball rolling theorem (Theorem 2.3)) and the convex hull of \( B_{1/\lambda,p} \cup B_{1/\lambda,q} \) is the \( \lambda \)-sausage body \( S_\lambda \) of diameter \( |pq| = \text{diam}(K) \). Inequality (4.1) and equality case then follow by monotonicity of quermassintegrals with respect to inclusion (see [Sch, p. 282]). \( \Box \)
Remark. Theorem 4.1 implies the following sharp estimate on the $i$-th quermassintegral $W_i = W_i(K)$ of a 1-concave body $K$ in terms of its diameter $D = \text{diam}(K)$:

$$W_i \geq \omega_{n+1} + \frac{n-i+1}{n+1} (D-2) \omega_n$$

for every $i \in \{0, \ldots, n\}$.

The estimate follows by a direct computation of $W_i(S_1)$.

4.2. The reverse isoperimetric problem for $\lambda$-convex domains. We conclude with a surprising difference between the reverse isoperimetric problems for $\lambda$-convex and $\lambda$-concave bodies. For simplicity we restrict ourselves to the Euclidean space, although everything written below makes perfect sense for constant curvature spaces and even general Riemannian manifolds (with appropriate adjustments).

Recall that a convex body $K \subset \mathbb{R}^{n+1}$ is $\lambda$-convex if for every $p \in \partial K$ there exists a ball $B_{1/\lambda, p}$ of radius $1/\lambda$ with the boundary sphere passing through $p$ in such a way that

$$(4.2) \quad B_{1/\lambda, p} \cap U(p) \supseteq K \cap U(p)$$

for some small open neighborhood $U(p) \subset \mathbb{R}^{n+1}$ of $p$ (see [BDr1, Dr2]).

Although $\lambda$-convexity and $\lambda$-concavity seem to be two notions dual to each other, methods and difficulties in solving the reverse isoperimetric problem in each of these classes are quite distinct. In our paper we completely solved the reverse isoperimetric problem for $\lambda$-concave bodies in $\mathbb{R}^{n+1}$. The same question for $\lambda$-convex bodies appears to be harder, and only partial results are currently available. In particular, the two-dimensional case of the reverse isoperimetric problem for $\lambda$-convex curves, as we already mentioned in the introduction, is completely solved, see [Bor2, BDr2, BDr3, Dr1]. For higher dimensions the following conjecture is due to Alexander Borisenko (private communication).

Conjecture (Borisenko). A $\lambda$-convex lens in $\mathbb{R}^{n+1}$, that is an intersection of two balls of radius $1/\lambda$, is the unique body that minimizes the volume among all $\lambda$-convex bodies of given surface area.

Remark. A similar conjecture can be stated for all model spaces of constant curvature. In this case the balls are substituted with convex bodies whose boundary is of constant normal curvature equal to $\lambda$.

Apart from the case $n = 1$, so far this conjecture was verified only in $\mathbb{R}^3$ for $\lambda$-convex surfaces of revolution, see the research announcement in [Dr3], and [Dr4].

Finally, it is interesting to point out numerous results concerning the so-called ball-polyhedron (see, for example, the paper of Bezdek et al. [BLNP] and references therein). A ball-polyhedron is the intersection of finitely many balls of the same radius. Therefore, this is a dual notion to a $\lambda$-concave polytope. In our terminology we would call them $\lambda$-convex polytopes, and a $\lambda$-convex lens is one of them. Following the ideas of Bezdek et al., Fodor, Kurusa and Vigh [FKV] introduce a notion of $r$-hyperconvexity, which is $1/r$-convexity in the sense of definition above. In the same paper the authors prove that a two-dimensional $\lambda$-convex lens is a solution of the reverse isoperimetric problem for $\lambda$-convex curves in $\mathbb{R}^2$ [FKV, Theorem 1.3], which was proven earlier in a sharper version in [BDr2]. Besides, Fodor, Kurusa and Vigh [FKV] state a conjecture (attributed to Bezdek) which in our
language asserts that the intersection of all balls of radius $1/\lambda$ containing a pair of given points (a $\lambda$-convex spindle) is a unique body with smallest volume among all $\lambda$-convex bodies of given surface area. This conjecture is clearly false, at least in $\mathbb{R}^3$, as the results in [Dr3, Dr4] indicate (it is also not hard to check by a direct comparison of volumes of the conjectural solutions).

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