Existence results for viscosity solutions of weakly coupled Hamilton-Jacobi systems

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Abstract

In this paper, we consider the existence of viscosity solutions of weakly coupled Hamilton-Jacobi systems. An important quantity denoted by $\chi$ is proposed, which measures the strength of coupling. A new existence result for viscosity solutions of

$$H_i(x, Du_i(x), u_i(x), u_j(x)) = 0, \quad i \neq j \in \{1, 2\}$$

is obtained in this paper when the classical monotone condition does not hold and $\chi < 1$. Each $H_i(x, p, u_i, u_j)$ is either strictly decreasing or strictly decreasing in $u_i$. For the linear coupling case, the vanishing discount method is used to get the solvability of the weakly coupled system when $\chi = 1$. At last, several remaining problems are discussed.

Keywords. Hamilton-Jacobi systems; viscosity solutions; vanishing discount method

1 Introduction and main results

The present paper focuses on the so-called weakly coupled systems of Hamilton-Jacobi equations. Assume $M$ is a connected, closed (compact without boundary) and smooth Riemannian manifold. In this paper, we denote by $T^*M$ the cotangent bundle over $M$. $\text{diam}(M)$ denotes the diameter of $M$. $| \cdot |_x$ denotes the norms induced by the Riemannian metric $g$ on both tangent and cotangent spaces of $M$. $B(0, \delta)$ denotes the closed ball given by the norm $| \cdot |_x$ centered at 0 with radius $\delta$. $D$ denotes the spacial gradient with respect to $x \in M$. $C(M)$ and Lip$(M)$ stand for the space of continuous functions on $M$ and the space of Lipschitz continuous functions on $M$ respectively. Let $u \in \mathbb{R}^m$ and $u_i$ denotes the $i$-th component of $u$. For $i \in \{1, \ldots, m\}$, let $H_i: T^*M \times \mathbb{R}^m \to \mathbb{R}$ be a continuous function. We will deal with the viscosity solutions of

$$H_i(x, Du_i, u) = 0, \quad x \in M, \quad 1 \leq i \leq m$$

(1.1)
in this paper and thus we mean by “solutions” viscosity solutions. The system is weakly coupled in the sense that every $i$th equation depends only on $Du_i$, but not on $Du_j$ for $j \neq i$.

**Definition 1.1. (Viscosity solution).** The continuous function $u : M \to \mathbb{R}^m$ is called a viscosity subsolution (resp. supersolution) of (1.1), if for each $i \in \{1, \ldots, m\}$ and test function $\phi$ of class $C^1$, when $u_i - \phi$ attains its local maximum (resp. minimum) at $x$, then

$$H_i(x, D\phi(x), u(x)) \leq 0, \quad \text{(resp. } H_i(x, D\phi(x), u(x)) \geq 0).$$

A function is called a viscosity solution of (1.1) if it is both a viscosity subsolution and a viscosity supersolution.

The standard assumption for (1.1) is so-called the monotonicity condition, which means $H_i$ is increasing in $u_i$ and nonincreasing in $u_j$ for $j \neq i$. More precisely, for any $(x, p) \in T^*M$ and $u, v \in \mathbb{R}^m$, if $u_k - v_k = \max_{1 \leq i \leq m} (u_i - v_i) \geq 0$, then $H_k(x, p, u) \geq H_k(x, p, v)$. When the coupling is linear, that is, when $H_i$ has the form

$$H_i(x, p, u) = h_i(x, p) + \sum_{j=1}^m \lambda_{ij}(x)u_j,$$

the monotonicity condition holds if and only if

$$\lambda_{ij}(x) \leq 0 \text{ if } i \neq j \quad \text{and} \quad \sum_{j=1}^m \lambda_{ij}(x) \geq 0 \text{ for all } i \in \{1, \ldots, m\}. \quad (1.2)$$

By [5, Proposition 1.2], if the coupling matrix $(\lambda_{ij}(x))$ is irreducible, then $\lambda_{ii}(x) > 0$ for every $i \in \{1, \ldots, m\}$. About the weakly coupled systems, there are several topics of concern:

- The stationary weakly coupled systems. For the existence theorems and the comparison results, one can refer to [9, 13, 18]. For the weak KAM theory, one can refer to [5]. For the vanishing discount problem, one can refer to [7, 14, 15].

- The evolutionary weakly coupled systems. The representation formula is provided in the deterministic setting by [17] and in the random setting by [6]. For the large time behavior, one can refer to [2, 19, 20]. For the homogenization theory, one can refer to [11].

The main differences between the present paper and the previous results are

- Theorem 1 goes beyond the classical monotonicity conditions as mentioned above. For each $i \in \{1, 2\}$, the Hamiltonian $H_i(x, p, u_i, u_j)$ is assumed to be either strictly increasing or strictly decreasing in $u_i$. The proof of the later case is highly based on the results obtained in [21]. Moreover, the Hamiltonian $H_i(x, p, u_i, u_j)$ can be nonmonotone in $u_j$.

- In the proof of Theorem 2 we deal with a situation different from those in the previous vanishing discount results. The vanishing discount term only appearing in the second equation, see (1.6) below. Theorem 2 also generalises [5, Theorem 2.12] with $m = 2$. 

1.1 Nonlinear coupling

Consider the following weakly coupled system of Hamilton-Jacobi equations

\[ H_i(x, Du_i(x), u_i(x), u_j(x)) = 0, \quad x \in M, \ i \neq j \in \{1, 2\}. \quad (1.3) \]

We give our main assumptions on the Hamiltonians \( H_i : T^* M \times \mathbb{R}^2 \to \mathbb{R} \) with \( i \in \{1, 2\} \):

(H0) \( H_i(x, p, u_i, u_j) \) is continuous.

(H1) \( H_i(x, p, u_i, u_j) \) is superlinear in \( p \), i.e. there exists a function \( \theta : \mathbb{R} \to \mathbb{R} \) satisfying

\[ \lim_{r \to +\infty} \frac{\theta(r)}{r} = +\infty, \quad \text{and} \quad H_i(x, p, u_i, u_j) \geq \theta(|p|_x) \quad \text{for every} \quad (x, p, u_i, u_j) \in T^* M \times \mathbb{R}^2. \]

(H2) \( H_i(x, p, u_i, u_j) \) is convex in \( p \).

(H3) \( H_i(x, p, u_i, u_j) \) is uniformly Lipschitz in \( u_i \) and \( u_j \), i.e., there is \( \Theta > 0 \) such that

\[ |H_i(x, p, u_i, u_j) - H_i(x, p, v_i, v_j)| \leq \Theta \max\{|u_i - v_i|, |u_j - v_j|\}. \]

(H4) there exists \( \lambda_{ii} > 0 \) such that for all \( (x, p, v) \in T^* M \times \mathbb{R} \) we have

\[ H_i(x, p, u_i, v) - H_i(x, p, v_i, v) \geq \lambda_{ii}(u_i - v_i), \quad \forall u_i \geq v_i. \]

(H4') there exists \( \lambda_{ii} > 0 \) such that for all \( (x, p, v) \in T^* M \times \mathbb{R} \) we have

\[ H_i(x, p, u_i, v) - H_i(x, p, v_i, v) \leq -\lambda_{ii}(u_i - v_i), \quad \forall u_i \geq v_i. \]

(H5) there is a constant \( \lambda_i > 0 \) such that for all \( (x, v) \in M \times \mathbb{R} \) we have

\[ |(H_i(x, 0, 0, u_j) - H_i(x, 0, 0, v_j))(u_i - v_i)| \leq \lambda_i |(H_i(x, 0, u_i, v) - H_i(x, 0, v_i, v))(u_j - v_j)|, \quad \forall u_i, u_j, v_i, v_j \in \mathbb{R}. \]

Remark 1.1. Let \( H_i(x, p, u_i, u_j) \) satisfy either (H4) or (H4'). If there is a constant \( \lambda_{ij} > 0 \) such that for all \( (x, p, u) \in T^* M \times \mathbb{R} \) we have

\[ |H_i(x, p, u, u_j) - H_i(x, p, u, v_j)| \leq \lambda_{ij}|u_j - v_j|, \quad \forall u_j, v_j \in \mathbb{R}, \]

then \( \lambda_i \leq \lambda_{ij}/\lambda_{ii} \). Define

\[ \chi := \lambda_1\lambda_2. \]

When \( \chi \) is small, the coupling is thought to be weak. In the linear coupling case (1.5), we set

\[ \lambda_i := \max_{x \in M} \frac{\lambda_{ij}(x)}{\lambda_{ii}(x)}. \]
Throughout this paper, we call (I) the conditions (H0)(H1)(H3)(H4)(H5), and (D) the conditions (H0)-(H3)(H4')(H5). It is worth to mention that, when $H_i(x, p, u_i, u_j)$ is increasing in $u_i$, the convexity assumption (H2) is not needed.

**Theorem 1.** The system (1.3) admits viscosity solutions if each $H_i(x, p, u_i, u_j)$ satisfies either (I) or (D), and $\chi < 1$.

**Remark 1.2.** We want to give some remarks on the above result.

1. The condition (H1) can be relaxed to the coercivity condition, see Proposition 2.1 below. The condition $\chi < 1$ must be replaced by a more complicated one, for the unboundedness of the Lagrangian corresponding to $H_i(x, p, u_i, u_j)$.

2. If $H_i(x, p, u_i, u_j)$ satisfies the assumptions in Theorem 1, then $H_i(x, Du_i(x), u_i(x), u_j(x)) = c_i$ (1.4) admits viscosity solutions for each $(c_1, c_2) \in \mathbb{R}^2$, since $H_i(x, p, u_i, u_j) - c_i$ also satisfies these assumptions. It was shown in [15, Theorem 21] that there exists $(c_1, c_2) \in \mathbb{R}^2$ such that (1.4) admits viscosity solutions under assumptions (H0)(H1)(H3).

3. Consider the linear coupling case, i.e. (1.3) reduces to (1.5). Let the standard assumptions (1.2) hold, and $\sum_{j=1}^{2} \lambda_{ij}(x) > 0$ for some $i \in \{1, 2\}$ (i.e., the coupling matrix $\lambda_{ij}(x)$ is invertible), then (1.5) admits viscosity solutions by Theorem 1. The case $\sum_{i=1}^{2} \lambda_{ij}(x) = 0$ for all $i \in \{1, 2\}$ is considered in Theorem 2 as a critical case. In [5], the critical case is called degenerate.

4. We still consider the case (1.5). Assume that $\lambda_{ii}(x) > 0$ and $\lambda_{ij}(x) < 0$ for each $i \neq j \in \{1, 2\}$, but $\chi \leq 1$ holds instead of $\sum_{j=1}^{2} \lambda_{ij}(x) \geq 0$. Let $(u_1, u_2)$ be a viscosity solution of (1.5). Define $v_1 := u_1/\lambda_1$, then $(v_1, u_2)$ is a viscosity solution of (1.5) with (1.2) holds. Furthermore, one can consider the evolutionary equation associated to (1.5) when $\chi \leq 1$, see for example [2, Theorem 5.4 and Remark 5.7].

### 1.2 Linear coupling

Consider the following weakly coupled system of Hamilton-Jacobi equations

\[
H_i(x, Du_i(x)) + \sum_{j=1}^{2} \lambda_{ij}(x)u_j(x) = 0, \quad x \in M, \ i \in \{1, 2\}. \tag{1.5}
\]

Here $\lambda_{ij}(x) \in C(M)$ and $\lambda_{ii}(x) \neq 0$ for all $x \in M$. For each $i \in \{1, 2\}$, we assume that $H_i(x, p)$ satisfies the following assumptions:

1. **(h0)** $H_i(x, p)$ is locally Lipschitz continuous.

2. **(h1)** $H_i(x, p)$ is coercive in $p$, i.e. $\lim_{|p|, x \to +\infty} (\inf_{x \in M} H_i(x, p)) = +\infty$. 


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(h2) $h_i(x, p)$ is strictly convex in $p$.

The assumptions (h0)(h2) guarantee the semiconcavity of viscosity solutions. In the following, we assume $\lambda_1(x)$ and $\lambda_2(x)$ are two positive functions belonging to Lip($M$).

**Remark 1.3.** Let $\varepsilon > 0$, $c \in \mathbb{R}$ and $(u_i^{\varepsilon, c})_{i \in \{1, 2\}}$ be a viscosity solution of

$$
\begin{align*}
&h_1(x, Du_1(x)) + \lambda_1(x)(u_1(x) - u_2(x)) = c, \\
&h_2(x, Du_2(x)) + \varepsilon u_2(x) + \lambda_2(x)(u_2(x) - u_1(x)) = 0.
\end{align*}
$$

(1.6)

In order to obtain the existence of viscosity solutions of (1.6), according to the Perron’s method, we have to construct a supersolution $\Psi_i$ and a subsolution $\psi_i$ with $\psi_i(x) \leq \Psi_i(x)$. However, Proposition 2.1 implies the existence of viscosity solutions of (1.6) directly. By [5, Proposition 2.10], the viscosity solution $(u_i^{\varepsilon, c})_{i \in \{1, 2\}}$ is unique.

The system (1.6) can be thought as a discounted equation. We then use the vanishing discount method corresponding to (1.6) to get the existence of viscosity solutions of the critical equation

$$
\begin{align*}
&h_1(x, Du_1(x)) + \lambda_1(x)(u_1(x) - u_2(x)) = c, \\
&h_2(x, Du_2(x)) + \lambda_2(x)(u_2(x) - u_1(x)) = d.
\end{align*}
$$

(1.7)

**Theorem 2.** There is a unique constant $d = \alpha(c)$ such that (1.7) admits viscosity solutions. The function $c \mapsto \alpha(c)$ is nonincreasing with the Lipschitz constant $\max_{x \in M} \lambda_2(x)/\min_{x \in M} \lambda_1(x)$.

**Corollary 1.1.** If $\lambda_1(x)$ and $\lambda_2(x)$ are constant functions, then $\alpha(c) = \alpha(0) - (\lambda_2/\lambda_1)c$.

By the continuity, there is a unique $c_0 \in \mathbb{R}$ such that $\alpha(c_0) = c_0$. We have

**Corollary 1.2.** There is a unique constant $c_0 \in \mathbb{R}$ such that

$$
h_i(x, Du_i(x)) + \lambda_i(x)(u_i(x) - u_j(x)) = c_0, \quad x \in M, \ i \neq j \in \{1, 2\}
$$

admits viscosity solutions.

**Remark 1.4.** The result in Corollary 1.2 is contained in [5, Theorem 2.12] when $m = 2$. In fact, Corollary 1.2 implies Corollary 1.1. Let $c_0$ be the constant such that there is a viscosity solution $(u_1, u_2)$ of (1.7) with $c = d = c_0$. Then for each $c \in \mathbb{R}$, the pair $(u_1, u_2 - (c - c_0)/\lambda_1)$ is a viscosity solution of (1.7) with $d = c_0 - (\lambda_2/\lambda_1)(c - c_0)$.

This paper is organized as follows. Theorem 1 is proved in Section 2. Theorem 2 and Corollaries 1.1 are proved in Section 3. Appendix A provides some facts about viscosity solutions of single Hamilton-Jacobi equations with Lipschitz dependence on $u$. 
2 Proof of Theorem 1

In this section, we relax the condition (H1) to the condition (H1'). We call (I') the conditions (H0)(H1')(H3)(H4)(H5), and (D') the conditions (H0)(H1')(H2)(H3)(H4')(H5).

Proposition 2.1. The system (1.3) admits viscosity solutions if one of the following conditions holds:

(a) each \( H_i(x, p, u_i, u_j) \) satisfies (I'), and

\[ \lambda_1 \lambda_2 < 1. \]

(b) each \( H_i(x, p, u_i, u_j) \) satisfies (D'), and there is \( \mu > 0 \) as mentioned in Lemma A.1 such that

\[ \left( (1 + \Theta \mu e^{\Theta \mu}) \lambda_1 + \Theta \mu e^{\Theta \mu} \right) \left( (1 + \Theta \mu e^{\Theta \mu}) \lambda_2 + \Theta \mu e^{\Theta \mu} \right) < 1. \]

(c) \( H_i(x, p, u_i, u_j) \) satisfies (I') and \( H_j(x, p, u_i, u_j) \) satisfies (D') for some \( i \neq j \in \{1, 2\} \), and there is \( \mu > 0 \) as mentioned in Lemma A.1 such that

\[ \lambda_i \left( (1 + \Theta \mu e^{\Theta \mu}) \lambda_j + \Theta \mu e^{\Theta \mu} \right) < 1. \]

Remark 2.1. Theorem 1 can be obtained from Proposition 2.1 directly. If \( H_i(x, p, u_i, u_j) \) satisfies (H1) instead of (H1'), then \( L_i(x, \dot{x}, u_i, u_j) \) is finite for all \( \dot{x} \in T_x M \). If \( \lambda_1 \lambda_2 < 1 \), then one can take \( \mu > 0 \) as mentioned in Lemma A.1 sufficiently small such that the conditions in Proposition 2.1 hold.

Let \( u_0^0 \equiv 0 \) and consider the following iteration procedure for \( n = 0, 1, 2, \ldots \)

\[
\begin{align*}
H_1(x, Du_1^n(x), u_1^n(x), u_2^n(x)) &= 0, \\
H_2(x, Du_2^n+1(x), u_2^{n+1}(x), u_1^n(x)) &= 0.
\end{align*}
\]  

(2.1)

Remark 2.2. Suppose that both \( u_1^n \) and \( u_2^n \) are uniformly bounded with respect to \( n \). By (H1'), both \( u_1^n \) and \( u_2^n \) are uniformly Lipschitz continuous with respect to \( n \). By the Arzela-Ascoli theorem, there is a subsequence of \( (u_1^n, u_2^n) \) converges uniformly to a pair \( (u, v) \). By the stability of viscosity solutions [10, Theorem 8.1.1], the limit \( (u, v) \) is a solution of (1.3).

Let \( \Theta, C, \mu \) and \( \lambda_i \) be the constants defined in the basic assumptions (H3)(H5) and Lemma A.1. In this section, we define

\[ H_i := \|H_i(x, 0, 0, 0)\|_\infty, \quad A := \Theta \mu e^{\Theta \mu}, \quad B := C \mu e^{\Theta \mu}, \quad \bar{\lambda}_i := (1 + A) \lambda_i + A. \]

and

\[ \kappa := \lambda_1 \lambda_2, \quad \bar{\kappa} := \bar{\lambda}_1 \bar{\lambda}_2, \quad \tilde{\kappa} := \lambda_1 \bar{\lambda}_2. \]
2.1 Case (a)

In this section, we will prove Proposition 2.1 with \( H_i(x, p, u_i, u_j) \) satisfying (I') for each \( i = 1, 2 \).

**Lemma 2.1.** For each \( i \in \{1, 2\} \) and \( v(x) \in C(M) \), there is a unique viscosity solution \( u(x) \) of

\[
H_i(x, Du, u, v(x)) = 0. \tag{2.2}
\]

Moreover, we have \( \|u(x)\|_\infty \leq \frac{1}{\lambda_{ii}} \mathbb{H}_i + \lambda_i \|v\|_\infty \).

**Proof.** By (H5) we have

\[
|H_i(x, 0, 0, v(x)) - H_i(x, 0, 0, 0)| \leq \lambda_i |v(x)|.
\]

Therefore, we have

\[
|H_i(x, 0, 0, v(x)) - H_i(x, 0, 0, 0)| \leq H_i(x, 0, 0, v(x)) - H_i(x, 0, 0, v(x)). \tag{2.3}
\]

Similarly, we have

\[
|H_i(x, 0, 0, v(x)) - H_i(x, 0, 0, 0)| \leq H_i(x, 0, 0, v(x)) - H_i(x, 0, -\lambda_i \|v\|_\infty, v(x)). \tag{2.4}
\]

By (H4) and (2.3) we have

\[
H_i(x, 0, \frac{1}{\lambda_{ii}} \mathbb{H}_i + \lambda_i \|v\|_\infty, v(x)) - H_i(x, 0, 0, v(x)) + H_i(x, 0, 0, v(x)) - H_i(x, 0, 0, 0) \geq \mathbb{H}_i.
\]

By (H4) and (2.4) we have

\[
H_i(x, 0, 0, 0) - H_i(x, 0, 0, v(x)) + H_i(x, 0, 0, v(x)) - H_i(x, 0, -\lambda_i \|v\|_\infty, v(x)) + H_i(x, 0, -\lambda_i \|v\|_\infty, v(x)) - H_i(x, 0, 0, 0) \geq \mathbb{H}_i,
\]

Thus for every \( x \in M \), we have

\[
H_i(x, 0, \frac{1}{\lambda_{ii}} \mathbb{H}_i + \lambda_i \|v\|_\infty, v(x)) \geq 0,
\]

and

\[
H_i(x, 0, -\frac{1}{\lambda_{ii}} \mathbb{H}_i - \lambda_i \|v\|_\infty, v(x)) \leq 0,
\]

which implies that \( \frac{1}{\lambda_{ii}} \mathbb{H}_i + \lambda_i \|v\|_\infty \) (resp. \( -\frac{1}{\lambda_{ii}} \mathbb{H}_i - \lambda_i \|v\|_\infty \)) is a supersolution (resp. subsolution) of (2.2). Therefore, the viscosity solution \( u(x) \) of (2.2) exists by the Perron’s method [11]. The uniqueness of \( u(x) \) and \( \|u(x)\|_\infty \leq \frac{1}{\lambda_{ii}} \mathbb{H}_i + \lambda_i \|v\|_\infty \) is given by the comparison principle. \( \square \)
Lemma 2.2. For \( n = 1, 2, \ldots \), we have

\[
\|u_1^n\|_\infty \leq \frac{1}{\lambda_{11}} \sum_{l=0}^{n} \kappa^l \mathbb{H}_1 + \frac{\lambda_1}{\lambda_{22}} \sum_{l=0}^{n-1} \kappa^l \mathbb{H}_2,
\]

and

\[
\|u_2^{n+1}\|_\infty \leq \frac{1}{\lambda_{22}} \sum_{l=0}^{n} \kappa^l \mathbb{H}_2 + \frac{\lambda_2}{\lambda_{11}} \sum_{l=0}^{n} \kappa^l \mathbb{H}_1.
\]

Proof. We prove by induction. We first prove the Lemma when \( n = 1 \). By Lemma 2.1, we have

\[
\|u_1^0(x)\|_\infty \leq \frac{\mathbb{H}_1}{\lambda_{11}},
\]

\[
\|u_1^1(x)\|_\infty \leq \frac{1}{\lambda_{22}} \mathbb{H}_2 + \lambda_2 \|u_1^0(x)\|_\infty \leq \frac{1}{\lambda_{22}} \mathbb{H}_2 + \frac{\lambda_2 \mathbb{H}_1}{\lambda_{11}},
\]

\[
\|u_1^1(x)\|_\infty \leq \frac{1}{\lambda_{11}} \mathbb{H}_1 + \lambda_1 \|u_2^0(x)\|_\infty \leq \frac{1}{\lambda_{11}} (1 + \kappa) \mathbb{H}_1 + \frac{\lambda_1 \mathbb{H}_2}{\lambda_{22}},
\]

\[
\|u_2^1(x)\|_\infty \leq \frac{1}{\lambda_{22}} \mathbb{H}_2 + \lambda_2 \|u_1^1(x)\|_\infty \leq \frac{1}{\lambda_{22}} (1 + \kappa) \mathbb{H}_2 + \frac{\lambda_2}{\lambda_{11}} (1 + \kappa) \mathbb{H}_1.
\]

Assume the assertion holds true for \( n = k - 1 \). We prove the Lemma when \( n = k \). By Lemma 2.1, we have

\[
\|u_1^k(x)\|_\infty \leq \frac{1}{\lambda_{11}} \mathbb{H}_1 + \lambda_1 \|u_2^k(x)\|_\infty
\]

\[
\leq \frac{1}{\lambda_{11}} (1 + \lambda_1 \lambda_2 \sum_{l=0}^{k-1} \kappa^l) \mathbb{H}_1 + \lambda_1 \frac{1}{\lambda_{22}} \sum_{l=0}^{k-1} \kappa^l \mathbb{H}_2
\]

\[
= \frac{1}{\lambda_{11}} \sum_{l=0}^{k} \kappa^l \mathbb{H}_1 + \frac{\lambda_1}{\lambda_{22}} \sum_{l=0}^{k-1} \kappa^l \mathbb{H}_2,
\]

and

\[
\|u_2^{k+1}(x)\|_\infty \leq \frac{1}{\lambda_{22}} \mathbb{H}_2 + \lambda_2 \|u_1^k(x)\|_\infty
\]

\[
\leq \frac{1}{\lambda_{22}} (1 + \lambda_2 \lambda_1 \sum_{l=0}^{k-1} \kappa^l) \mathbb{H}_2 + \lambda_2 \frac{1}{\lambda_{11}} \sum_{l=0}^{k} \kappa^l \mathbb{H}_1
\]

\[
= \frac{1}{\lambda_{22}} \sum_{l=0}^{k} \kappa^l \mathbb{H}_2 + \frac{\lambda_2}{\lambda_{11}} \sum_{l=0}^{k} \kappa^l \mathbb{H}_1.
\]

The proof is now completed. 

By assumption we have \( \kappa < 1 \). Then both \( u_1^n \) and \( u_2^n \) are uniformly bounded with respect to \( n \). By Remark 2.2, there exists a viscosity solution of (1.3).
2.2 Case (b)

In this section, we will prove Proposition 2.1 with $H_i(x, p, u_i, u_j)$ satisfying (D’) for each $i = 1, 2$. If $H_i(x, p, u_1, u_2)$ satisfies (D’), the comparison principle does not hold. Therefore, one can not obtain similar results as in Lemma 2.1 directly.

**Lemma 2.3.** For each $i \in \{1, 2\}$ and $v(x) \in C(M)$, the viscosity solutions of (2.2) exist. For each viscosity solution $u(x)$ of (2.2), we have

$$\|u(x)\|_\infty \leq \frac{1 + A}{\lambda_{ii}} \|F_i(x, 0, 0, 0)\|_\infty + B,$$

where $F_i(x, p, u_i, u_j) := H_i(x, -p, -u_i, u_j)$. By the definition of $F_i$, one can easily check that $\|F_i(x, 0, 0, 0)\|_\infty = \|H_i(x, 0, 0, 0)\|_\infty$ and

$$F_i(x, p, u_i, u_j) - F_i(x, p, v_i, u_j) \geq \lambda_{ii}(u_i - v_i), \quad \forall u_i \geq v_i, \ (x, p, u) \in T^*M \times \mathbb{R},$$

$$|F_i(x, 0, 0, u_j) - F_i(x, 0, v_i, u_j)| \leq \lambda_{ii}|(F_i(x, 0, u_i, v) - F_i(x, 0, v_i, v))(u_j - v_j)|, \quad \forall u_i, v_i, u_j, v_j \in \mathbb{R}, \ (x, v) \in M \times \mathbb{R}.$$

By Lemma 2.1, $\frac{1}{\lambda_{ii}} F_i + \lambda_i \|v\|_\infty$ (resp. $-\frac{1}{\lambda_{ii}} F_i - \lambda_i \|v\|_\infty$) is a supersolution (resp. subsolution) of (2.5). By the Perron’s method, the viscosity solution $u_-$ of (2.5) exists, which implies the existence of viscosity solutions of (2.2) by Proposition A.3. Moreover, we have $\|u_--\|_\infty \leq \frac{1}{\lambda_{ii}} F_i + \lambda_i \|v\|_\infty$.

Combining Propositions A.2 and A.4, we conclude that

$$\|u(x)\|_\infty = \|v_+\|_\infty \leq (1 + \Theta \mu e^{\Theta \mu}) \|u_-\|_\infty + C \mu e^{\Theta \mu} + \Theta \mu e^{\Theta \mu} \|v(x)\|_\infty$$

$$\leq (1 + A)(\frac{1}{\lambda_{ii}} F_i + \lambda_i \|v\|_\infty) + B + A \|v(x)\|_\infty = \frac{1 + A}{\lambda_{ii}} F_i + \lambda_i \|v\|_\infty + B,$$

where $v_+$ is a forward weak KAM solution of (2.5). 

**Lemma 2.4.** For $n = 1, 2, \ldots$, we have

$$\|u_1^n\|_\infty \leq \frac{1 + A}{\lambda_{11}} \sum_{l=0}^n \kappa^l F_1 + \frac{(1 + A) \bar{\lambda}_1}{\lambda_{22}} \sum_{l=0}^{n-1} \kappa^l F_2 + (\sum_{l=0}^n \kappa^l + \bar{\lambda}_1 \sum_{l=0}^{n-1} \kappa^l) B,$$

and

$$\|u_2^{n+1}\|_\infty \leq \frac{1 + A}{\lambda_{22}} \sum_{l=0}^n \kappa^l F_2 + \frac{(1 + A) \bar{\lambda}_2}{\lambda_{11}} \sum_{l=0}^n \kappa^l F_1 + (\sum_{l=0}^n \kappa^l + \bar{\lambda}_2 \sum_{l=0}^n \kappa^l) B.$$
\textbf{Proof.} We prove by induction. We first prove the Lemma when \( n = 1 \). By Lemma 2.3, we have

\[
\| u_1^0(x) \|_\infty \leq \frac{1 + A}{\lambda_{11}} \mathbb{H}_1 + B, \\
\| u_2^1(x) \|_\infty \leq \frac{1 + A}{\lambda_{22}} \mathbb{H}_2 + \tilde{\lambda}_2 \| u_1^0(x) \|_\infty + B \leq \frac{1 + A}{\lambda_{22}} \mathbb{H}_2 + \frac{1 + A}{\lambda_{11}} \mathbb{H}_1 + (1 + \tilde{\lambda}_2)B, \\
\| u_1^1(x) \|_\infty \leq \frac{1 + A}{\lambda_{11}} \mathbb{H}_1 + \tilde{\lambda}_1 \| u_2^1(x) \|_\infty + B \\
\leq \frac{1 + A}{\lambda_{11}} (1 + \bar{u}) \mathbb{H}_1 + \frac{(1 + A)\tilde{\lambda}_1}{\lambda_{22}} \mathbb{H}_2 + (1 + \bar{u} + \tilde{\lambda}_1)B, \\
\| u_2^1(x) \|_\infty \leq \frac{1 + A}{\lambda_{11}} \mathbb{H}_1 + \tilde{\lambda}_1 \| u_2^1(x) \|_\infty + B \\
\leq \frac{1 + A}{\lambda_{11}} (1 + \bar{u}) \mathbb{H}_1 + \frac{(1 + A)\tilde{\lambda}_2}{\lambda_{22}} \mathbb{H}_2 + (1 + \bar{u} + \tilde{\lambda}_2(1 + \bar{u}))B.
\]

Assume the assertion holds true for \( n = k - 1 \). We prove the Lemma when \( n = k \). By Lemma 2.3 we have

\[
\| u_1^k(x) \|_\infty \leq \frac{1 + A}{\lambda_{11}} \mathbb{H}_1 + \tilde{\lambda}_1 \| u_2^k(x) \|_\infty + B \\
\leq \frac{1 + A}{\lambda_{11}} (1 + \bar{u}) \mathbb{H}_1 + \frac{1 + A}{\lambda_{22}} \mathbb{H}_2 + \frac{k}{\lambda_{11}} \quad \text{and}
\]

\[
\| u_2^k(x) \|_\infty \leq \frac{1 + A}{\lambda_{22}} \mathbb{H}_2 + \tilde{\lambda}_2 \| u_2^k(x) \|_\infty + B \\
\leq \frac{1 + A}{\lambda_{22}} (1 + \bar{u}) \mathbb{H}_2 + \frac{k}{\lambda_{11}} \quad \text{and}
\]

The proof is now completed. \( \square \)

By assumption we have \( \bar{u} < 1 \), both \( u_1^n \) and \( u_2^n \) are uniformly bounded with respect to \( n \). By Remark 2.2 there exists a viscosity solution of (4.3).

\section{Case (c)}

Without any loss of generality, we assume \( H_1(x, p, u_1, u_2) \) satisfies (I') and \( H_2(x, p, u_2, u_1) \) satisfies (D') in this section.
Lemma 2.5. For \( n = 1, 2, \ldots \), we have
\[
\|u^n_1\|_\infty \leq \frac{1}{\lambda_{11}} \sum_{l=0}^{n} \tilde{\kappa}_l^l \Phi_1 + \frac{(1 + A)\lambda_1}{\lambda_{22}} \sum_{l=0}^{n-1} \tilde{\kappa}_l^l \Phi_2 + \lambda_1 \sum_{l=0}^{n-1} \tilde{\kappa}_l^l B.
\]
and
\[
\|u^n_2\|_\infty \leq \frac{1 + A}{\lambda_{22}} \sum_{l=0}^{n} \tilde{\kappa}_l^l \Phi_2 + \frac{\tilde{\lambda}_2}{\lambda_{11}} \sum_{l=0}^{n} \tilde{\kappa}_l^l \Phi_1 + \sum_{l=0}^{n} \tilde{\kappa}_l^l B.
\]

The proof of Lemma 2.5 is quite similar to that of Lemmas 2.2 and 2.4, we omit it here for brevity. By assumption we have \( \tilde{\kappa} < 1 \), both \( u^n_1 \) and \( u^n_2 \) are uniformly bounded with respect to \( n \). By Remark 2.3 there exists a viscosity solution of (1.3).

3 Proof of Theorem 2

Step 1. We first prove that for given \( c \in \mathbb{R} \), there is a unique constant \( d = \alpha(c) \in \mathbb{R} \) such that (1.7) admits viscosity solutions. Without any loss of generality, we may assume \( c = 0 \) after a translation. By Remark 1.3 we denote by \( u^c \) the unique viscosity solution of (1.6) with \( c = 0 \).

Lemma 3.1. The family \( \{\varepsilon u^c_\varepsilon(x)\}_{\varepsilon > 0} \) is uniformly bounded with respect to \( \varepsilon \).

Proof. Let \( x_1 \) and \( x_2 \) be the minimal points of \( u^c_1(x) \) and \( u^c_2(x) \) respectively. Since \( u^c_\varepsilon \) is Lipschitz continuous by (h1), \( u^c_\varepsilon \) is semiconcave for each \( i \in \{1, 2\} \) by [3, Theorem 5.3.7]. Therefore, for each \( i \in \{1, 2\} \), \( u^c_\varepsilon \) is differentiable at \( x_1 \), and \( Du^c_\varepsilon(x_1) = 0 \). Plugging into (1.6) we have
\[
h_1(x_1, 0) + \lambda_1(x_1)(u^c_1(x_2) - u^c_2(x_2)) \geq h_1(x_1, 0) + \lambda_1(x_1)(u^c_1(x_1) - u^c_2(x_1)) = 0, \quad (3.1)
\]
and
\[
h_2(x_2, 0) + \varepsilon u^c_2(x_2) + \lambda_2(x_2)(u^c_2(x_2) - u^c_1(x_2)) = 0. \quad (3.2)
\]
Multiplying (3.1) with \( \lambda_2(x_2)/\lambda_1(x_1) \) and adding with (3.2), we get
\[
\frac{\lambda_2(x_2)}{\lambda_1(x_1)} h_1(x_1, 0) + h_2(x_2, 0) + \varepsilon u^c_2(x_2) \geq 0,
\]
which implies
\[
\varepsilon u^c_2(x) \geq \varepsilon u^c_2(x_2) \geq - (\iota \|h_1(x, 0)\|_\infty + \|h_2(x, 0)\|_\infty).
\]

where \( \iota := \max_{x \in M} \lambda_2(x)/\min_{x \in M} \lambda_1(x) \). Since both \( u^c_1 \) and \( u^c_2 \) are semiconcave, for all \( \varepsilon > 0 \) and each \( i \in \{1, 2\} \), there is a point \( y_i \) such that \( u^c_\varepsilon \) is differentiable at \( y_i \), and
\[
u^c_i(y_i) \geq \max_{x \in M} u^c_\varepsilon(x) - \varepsilon.
\]
Plugging into (1.6), we have
\[
\begin{align*}
    h_1(y_1, Du_1^\varepsilon(y_1)) + \lambda_1(y_1)(u_1^\varepsilon(y_2) - u_2^\varepsilon(y_2)) \\
    \leq h_1(y_1, Du_1^\varepsilon(y_1)) + \lambda_1(y_1) \max_{x \in M} u_1^\varepsilon(x) - \lambda_1(y_1)(u_2^\varepsilon(y_1) - \varepsilon) \\
    \leq h_1(y_1, Du_1^\varepsilon(y_1)) + \lambda_1(y_1)(u_1^\varepsilon(y_1) - u_2^\varepsilon(y_1)) + 2\lambda_1(y_1)\varepsilon = 2\lambda_1(y_1)\varepsilon,
\end{align*}
\]
(3.3)
and
\[
    h_2(y_2, Du_2^\varepsilon(y_2)) + \varepsilon u_2^\varepsilon(y_2) + \lambda_2(y_2)(u_2^\varepsilon(y_2) - u_1^\varepsilon(y_2)) = 0.
\]
(3.4)
Multiplying (3.3) with \( \lambda_2(y_2)/\lambda_1(y_1) \) and adding with (3.4), we get
\[
\frac{\lambda_2(y_2)}{\lambda_1(y_1)}h_1(y_1, Du_1^\varepsilon(y_1)) + h_2(y_2, Du_2^\varepsilon(y_2)) + \varepsilon u_2^\varepsilon(y_2) \leq 2\lambda_2(y_2)\varepsilon,
\]
which implies
\[
\varepsilon u_2^\varepsilon(x) \leq \varepsilon \max_{x \in M} u_2^\varepsilon(x) \leq \varepsilon \left( \min_{(x,p) \in T^*M} h_1(x,p) \right) + \min_{(x,p) \in T^*M} h_2(x,p) + (2\|\lambda_2(x)\|_\infty + \varepsilon)\varepsilon.
\]
Let \( \varepsilon \to 0^+ \), we get
\[
\varepsilon u_2^\varepsilon(x) \leq \varepsilon \left( \min_{(x,p) \in T^*M} h_1(x,p) \right) + \min_{(x,p) \in T^*M} h_2(x,p).
\]
By (h1), \( \min_{(x,p) \in T^*M} h_i(x,p) \) is finite for each \( i \in \{1, 2\} \). Therefore, \( \varepsilon u_2^\varepsilon(x) \) is uniformly bounded with respect to \( \varepsilon \).

**Lemma 3.2.** Both \( u_1^\varepsilon \) and \( u_2^\varepsilon \) are equi-Lipschitz continuous with respect to \( \varepsilon \).

**Proof.** Multiplying the first equality of (1.6) with \( \lambda_2(x)/\lambda_1(x) \) and adding with the second equality of (1.6), we get
\[
\frac{\lambda_2(x)}{\lambda_1(x)}h_1(x, Du_1^\varepsilon(x)) + h_2(x, Du_2^\varepsilon(x)) = -\varepsilon u_2^\varepsilon(x),
\]
which implies
\[
h_1(x, Du_1^\varepsilon(x)) \leq \bar{\varepsilon} \left( \|h_1(x,0)\|_\infty + \|h_2(x,0)\|_\infty + \min_{(x,p) \in T^*M} h_2(x,p) \right),
\]
and
\[
h_2(x, Du_2^\varepsilon(x)) \leq \bar{\varepsilon} \|h_1(x,0)\|_\infty + \|h_2(x,0)\|_\infty + \bar{\varepsilon} \min_{(x,p) \in T^*M} h_1(x,p),
\]
where \( \bar{\varepsilon} := \max_{x \in M} \lambda_1(x)/\min_{x \in M} \lambda_2(x) \). Thus, \( \|Du_1^\varepsilon\|_\infty \) and \( \|Du_2^\varepsilon\|_\infty \) are uniformly bounded with respect to \( \varepsilon \). \( \square \)
Fix $x_0 \in M$ and define
\[
\begin{align*}
\tilde{u}_1^\varepsilon(x) &= u_1^\varepsilon(x) - u_2^\varepsilon(x), \\
\tilde{u}_2^\varepsilon(x) &= u_2^\varepsilon(x) - u_1^\varepsilon(x).
\end{align*}
\]
The pair $(\tilde{u}_1^\varepsilon, \tilde{u}_2^\varepsilon)$ satisfies
\[
\begin{align*}
\begin{cases}
h_1(x, D\tilde{u}_1^\varepsilon(x)) + \lambda_1(x)(\tilde{u}_1^\varepsilon(x) - \tilde{u}_2^\varepsilon(x)) = 0, \\
h_2(x, D\tilde{u}_2^\varepsilon(x)) + \varepsilon \tilde{u}_2^\varepsilon(x) + \lambda_2(x)(\tilde{u}_1^\varepsilon(x) - \tilde{u}_2^\varepsilon(x)) + \varepsilon u_2^\varepsilon(x) = 0.
\end{cases}
\end{align*}
\]
Since $\|Du_1^\varepsilon\|_\infty$ is uniformly bounded, according to the first equality of (1.6), $\tilde{u}_1^\varepsilon(x_0) = u_1^\varepsilon(x_0) - u_2^\varepsilon(x_0)$ is uniformly bounded with respect to $\varepsilon$. We also have $\tilde{u}_2^\varepsilon(x_0) = 0$. Thus, both $\tilde{u}_1^\varepsilon(x)$ and $\tilde{u}_2^\varepsilon(x)$ are uniformly bounded and equi-Lipschitz continuous with respect to $\varepsilon$. We have also proved that $-\varepsilon u_2^\varepsilon(x_0)$ is bounded. Therefore, there exists a sequence $\varepsilon_k \to 0^+$ such that the pair $(\tilde{u}_1^\varepsilon, \tilde{u}_2^\varepsilon)$ converges to $(u, v)$ uniformly, and $-\varepsilon u_2^\varepsilon(x_0)$ converges to $d_0 \in \mathbb{R}$. According to the stability of viscosity solutions, the limit $(u, v)$ is a viscosity solution of (1.7) with $c = 0$ and $d = d_0$.

We now prove that $d_0$ is the unique constant such that (1.7) admits solutions. Here we still set $c = 0$. Assume there are two different constants $d_1 < d_2$ such that (1.7) has solutions $(u_1, u_2)$ and $(v_1, v_2)$ with $d_1 = d_1$ and $d = d_2$ respectively. When $\varepsilon > 0$ is small enough, we have $d_1 + \varepsilon u_2 \leq d_2 + \varepsilon v_2$. Noticing that for $k \in \mathbb{R}$, the pair $(u_1 + k, u_2 + k)$ also solves (1.7) with $c = 0$ and $d = d_1$. Let $k > 0$ large enough such that $u_1 > v_1, u_2 > v_2$. Since
\[
\begin{align*}
\begin{cases}
h_1(x, Du_1) + \lambda_1(x)(u_1 - u_2) = 0, \\
h_2(x, Du_2) + \varepsilon u_2 + \lambda_2(x)(u_2 - u_1) = d_1 + \varepsilon u_2 \leq d_2 + \varepsilon v_2.
\end{cases}
\end{align*}
\]
According to [5 Proposition 2.10], we have $u_1 \leq v_1$ and $u_2 \leq v_2$, which contradicts $u_1 > v_1$ and $u_2 > v_2$.

**Step 2.** We now prove the properties of the function $\alpha(c)$. Let $\{(u_i^\varepsilon,c_i)\}_{i \in \{1,2\}}$ be the viscosity solution of (1.6). Take $c_1 > c_2$. By [5 Proposition 2.10], $u_2^\varepsilon,c_1 \geq u_2^\varepsilon,c_2$. By Step 1, $\alpha(c)$ is the limit of a converging sequence $-\varepsilon_k u_2^\varepsilon,k,c(x)$. Therefore, we have $\alpha(c_1) \leq \alpha(c_2)$. Define
\[
K_1 := \frac{\max_{x \in M} \lambda_2(x) + \varepsilon}{\varepsilon \min_{x \in M} \lambda_1(x)}(c_1 - c_2), \\
K_2 := \frac{\max_{x \in M} \lambda_2(x)}{\varepsilon \min_{x \in M} \lambda_1(x)}(c_1 - c_2).
\]
Then we have
\[
\begin{align*}
\begin{cases}
h_1(x, Du_1^\varepsilon,c_1) + \lambda_1(x)((u_1^\varepsilon,c_1) - K_1) - (u_2^\varepsilon,c_1 - K_2) = c_1 + \lambda_1(x)(K_2 - K_1) \leq c_2, \\
h_2(x, Du_2^\varepsilon,c_1) + (\lambda_2(x) + \varepsilon)((u_2^\varepsilon,c_1) - K_2) - \lambda_2(x)(u_2^\varepsilon,c_1 - K_1) = \lambda_2(x)(K_1 - K_2) - \varepsilon K_2 \leq 0.
\end{cases}
\end{align*}
\]
which implies $u_2^\varepsilon,c_1 - u_2^\varepsilon,c_2 \leq K_2$ by [5 Proposition 2.10]. Consider the following identity
\[
\alpha(c_1) - \alpha(c_2) = [\alpha(c_1) + \varepsilon u_2^\varepsilon,c_1] + [\varepsilon u_2^\varepsilon,c_2 - \varepsilon u_2^\varepsilon,c_1] - [\varepsilon u_2^\varepsilon,c_2 + \alpha(c_2)].
\]
Since both $\varepsilon u_2^\varepsilon,c_1$ and $\varepsilon u_2^\varepsilon,c_2$ are sequentially compact, there is a sequence $\varepsilon_k \to 0^+$ such that $\varepsilon_k u_2^\varepsilon,k,c_1$ tends to $-\alpha(c_1)$ and $\varepsilon_k u_2^\varepsilon,k,c_2$ tends to $-\alpha(c_2)$. For each $\varepsilon > 0$, we have
\[
\varepsilon u_2^\varepsilon,c_2 - \varepsilon u_2^\varepsilon,c_1 \geq -\varepsilon K_2 = -\frac{\max_{x \in M} \lambda_2(x)}{\min_{x \in M} \lambda_1(x)}(c_1 - c_2).
\]
We conclude that
\[
-\frac{\max_{x \in M} \lambda_2(x)}{\min_{x \in M} \lambda_1(x)} (c_1 - c_2) \leq \alpha(c_1) - \alpha(c_2) \leq 0, \quad \forall c_1 > c_2.
\]

At last, we prove Corollary 1.1. When \(\lambda_1(x)\) and \(\lambda_2(x)\) are constants, we have \(u_1^{\varepsilon}c_1 - u_2^{\varepsilon}c_2 = K_2\) with \(K_2 = \frac{\lambda_2}{\lambda_1}(c_1 - c_2)\). Therefore, the function \(\alpha(c)\) is linear, with the slope \(-\lambda_2/\lambda_1\).

4 Remaining problems

At last, we list several remaining problems:

**The generalised critical case.** Let \(H_i(x, p, u_i, u_j)\) satisfies either (I) or (D). By Theorem 1 if \(\chi < 1\), the system (1.3) admits viscosity solutions. The case \(\chi = 1\) is thought to be critical. Theorem 2 deals with a very special case. For general cases, the proof of Theorem 2 is not applicable. A natural question is how to deal with the critical case \(\chi = 1\).

**The vanishing discount problem.** In the proof of Theorem 2 we showed that there is a sequence \(\varepsilon_k\) converging to zero such that \((\tilde{u}_i^{\varepsilon_k})_{i \in \{1,2\}}\) converges. Without any loss of generality, we assume \(c = \alpha(c) = 0\) up to a translation. Let \(u_i^c\) be the viscosity solution of (1.6) with \(c = 0\). Following [8], we want to know whether \(u_i^c\) converges to a viscosity solution of (1.7) uniformly as \(\varepsilon \to 0^+\). The argument may be related to the Mather measures defined in [15].

**The coupling vanishing problem.** Let \(H_i(x, p, u_i, u_j)\) satisfy either (I) or (D). For \(\delta > 0\), consider
\[
H_i(x, Du_i(x), u_i(x), \delta u_j(x)) = 0, \quad x \in M, \quad i \neq j \in \{1, 2\}. \tag{4.1}
\]
For \(\delta\) small enough, the viscosity solutions of (4.1) exist by Theorem 1. We denote it by the pair \((u_1^\delta, u_2^\delta)\). Fix \(\delta_0 > 0\) small, By Lemmas 2.2, 2.4 and 2.5, the family \(\{(u_1^\delta, u_2^\delta)\}_{\delta \in (0,\delta_0]}\) is uniformly bounded with respect to \(\delta\). By (H1) it is also equi-Lipschitz with respect to \(\delta\). By the Arzela-Ascoli theorem, there is a subsequence \((u_1^{\delta_k}, u_2^{\delta_k})\) converges to a viscosity solution of the uncoupled system
\[
H_i(x, Du_i(x), u_i(x), 0) = 0, \quad x \in M, \quad i \in \{1, 2\}. \tag{4.2}
\]
as \(\delta_k \to 0^+\). It is natural to ask when does the family \((u_1^\delta, u_2^\delta)\) converge uniformly to a viscosity solution of (4.2) as \(\delta \to 0^+\).

**The long time behavior of solution semigroup.** This problem is called non-trivial by [5]. In [2], the authors dealt with a very special case, where the solution semigroup has nonexpansion property. By Remark 1.2(4), this case is equivalent to the case \(\chi \leq 1\) when \(\lambda_i(x) > 0\) and \(\lambda_{ij}(x) < 0\). For single Hamilton-Jacobi equations, let \(T_t^-\) be the semigroup defined in Proposition A.1. Given \(\varphi \in C(M)\), if \(T_t^-\varphi(x)\) has a bound independent of \(t\), then the lower half limit
\[
\check{\varphi}(x) = \lim_{r \to 0^+} \inf \{T_t^- \varphi(y) : d(x, y) < r, \ t > 1/r\}
\]
is a Lipschitz continuous viscosity solution of (A.3) (see [16, Theorem 6.1]). If $H(x, p, u)$ satisfies the dynamical assumptions: $C^3$, strictly convex and superlinear, then the function $\lim_{t \to +\infty} T_t^{-1} \varphi$ is well-defined, and is a viscosity solution of (A.3) (see [22]). For weakly coupled systems, one can not prove similar conclusions directly.

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**A Single Hamilton-Jacobi equations**

Instead of (H1), for each $i \in \{1, 2\}$, we assume (H0)(H2) and

(H1') $H_i(x, p, u_i, u_j)$ is coercive in $p$, i.e. $\lim_{|p| \to +\infty} (\inf_{x \in M} H_i(x, p, 0, 0)) = +\infty$;

The Lagrangian associated to $H_i(x, p, u_i, u_j)$ is defined by

$$L_i(x, \dot{x}, u_i, u_j) := \sup_{p \in T_x^* M} \{\langle \dot{x}, p \rangle - H_i(x, p, u_i, u_j)\},$$

(A.1)

where $\langle \cdot, \cdot \rangle$ represents the canonical pairing between the tangent space and cotangent space. Similar to [12, Proposition 2.1], one can prove the local boundedness of $L_i(x, \dot{x}, 0, 0)$:

**Lemma A.1.** Let $H_i(x, p, 0, 0)$ satisfy (H0)(H1')(H2) for each $i \in \{1, 2\}$, then there exist constants $\delta > 0$ and $C > 0$ independent of $i$ such that for each $i \in \{1, 2\}$, the corresponding Lagrangian $L_i(x, \dot{x}, 0, 0)$ satisfies

$$L_i(x, \dot{x}, 0, 0) \leq C, \quad \forall (x, \dot{x}) \in M \times \bar{B}(0, \delta).$$

Define $\mu := \text{diam}(M)/\delta$.

Take $v \in C(M)$. We collect some facts given by [21] in view of the Hamiltonian defined by

$$H(x, p, u) := H_i(x, p, u, v(x))$$

for sake of completeness. In fact, if $H_i(x, p, u_i, u_j)$ satisfies (H0)(H1')(H2)(H3), then $H(x, p, u)$ satisfies the basic assumptions in [21], i.e. continuous in $(x, p, u)$, convex and coercive in $p$, and uniformly Lipschitz in $u$.

Consider the evolutionary equation:

$$\begin{cases}
\partial_t u(x, t) + H(x, Du(x, t), u(x, t)) = 0, & (x, t) \in M \times (0, +\infty), \\
u(x, 0) = \varphi(x), & x \in M.
\end{cases}$$

(A.2)

and the stationary equation:

$$H(x, Du(x), u(x)) = 0.$$ 

(A.3)
Correspondingly, one has the Lagrangian associated to $H$:
\[
L(x, \dot{x}, u) := \sup_{p \in T^*_x M} \left\{ \langle \dot{x}, p \rangle - H(x, p, u) \right\}.
\]

**Remark A.1.** The Lagrangian $L(x, \dot{x}, u)$ equals to $L_i(x, \dot{x}, u, v(x))$ which is defined in (A.1). One can check that $L_i(x, \dot{x}, u_i, u_j)$ is also uniformly Lipschitz in $u_j$ with Lipschitz constant $\Theta$. Let $\delta > 0$ and $C > 0$ be the constants defined in Lemma [A.7] then
\[
L(x, \dot{x}, 0) = L_i(x, \dot{x}, 0, v(x)) \leq C + \Theta \|v(x)\|_{\infty}, \quad \forall (x, \dot{x}) \in M \times \bar{B}(0, \delta).
\]

**Proposition A.1.** [27] Theorem 1 The backward solution semigroup
\[
T^{-}_t \varphi(x) = \inf_{\gamma(t) = x} \left\{ \varphi(\gamma(0)) + \int_{0}^{t} L(\gamma(\tau), \dot{\gamma}(\tau), T^{-}_\tau \varphi(\gamma(\tau))) d\tau \right\}
\]
is well-defined. The infimum is taken among absolutely continuous curves $\gamma : [0, t] \to M$ with $\gamma(t) = x$. If $\varphi$ is continuous, then $u(x, t) := T^{-}_t \varphi(x)$ represents the unique continuous viscosity solution of (A.2). If $\varphi$ is Lipschitz continuous, then $u(x, t) := T^{-}_t \varphi(x)$ is also Lipschitz continuous on $M \times [0, +\infty)$. Define
\[
F(x, p, u) := H(x, -p, -u),
\]
and let $T^+_t$ be the backward solution semigroup corresponding to $F$, then the forward solution semigroup $T^+_t \varphi := -T^-_{-t} (-\varphi)$ is also well-defined, and satisfies
\[
T^+_t \varphi(x) = \sup_{\gamma(0) = x} \left\{ \varphi(\gamma(t)) - \int_{0}^{t} L(\gamma(\tau), \dot{\gamma}(\tau), T^+_{\tau-t} \varphi(\gamma(\tau))) d\tau \right\}. \quad (A.4)
\]

Following Fathi [10], one can extend the definition of weak KAM solutions of equation (A.3) by using absolutely continuous calibrated curves instead of $C^1$ curves.

**Definition A.1.** A function $u \in C(M)$ is called a backward (resp. forward) weak KAM solution of (A.3) if

1. For each absolutely curve $\gamma : [t', t] \to M$, we have
   \[
u(\gamma(t)) - u(\gamma(t')) \leq \int_{t'}^{t} L(\gamma(s), \dot{\gamma}(s), u(\gamma(s))) ds.
   \]
   The above condition reads that $u$ is dominated by $L$ and denoted by $u \prec L$.

2. For each $x \in M$, there exists an absolutely continuous curve $\gamma_- : (-\infty, 0] \to M$ with $\gamma_- (0) = x$ (resp. $\gamma_+ : [0, +\infty) \to M$ with $\gamma_+ (0) = x$) such that
   \[
u(x) - u(\gamma_-(t)) = \int_{t}^{0} L(\gamma_-(s), \dot{\gamma}_-(s), u(\gamma_-(s))) ds, \quad \forall t < 0.
   \]
   \[
u(x) - u(\gamma_+(t)) = \int_{0}^{t} L(\gamma_+(s), \dot{\gamma}_+(s), u(\gamma_+(s))) ds, \quad \forall t > 0.
   \]
   The curves satisfying the above equality are called $(u, L, 0)$-calibrated curves.
Proposition A.2. \cite[Proposition D.4]{21} Let $u_- \in C(M)$. The following statements are equivalent: $u_-$ is a fixed point of $T^+_t$; $u_-$ is a backward weak KAM solution of (A.3); $u_-$ is a viscosity solution of (A.3). Similarly, let $v_+ \in C(M)$. The following statements are equivalent: $v_+$ is a fixed point of $T^-_t$; $v_+$ is a forward weak KAM solution of (A.3); $-v_+$ is a viscosity solution of (A.3):

$$F(x, Du(x), u(x)) = 0.$$ (A.5)

Proposition A.3. \cite[Theorem 3]{21} Assume there is a viscosity solution $u_-$ of (A.3), then $T^+_t u_-$ is nonincreasing in $t$, and converges to $u_+$ uniformly. Here, the function $-u_+$ is a viscosity solution of (A.3). Thus, the existence of viscosity solutions of (A.3) is equivalent to the existence of viscosity solutions of (A.3). Moreover, the set $\{ x \in M : u_- = \lim_{t \to +\infty} T^+_t u_- \}$ is nonempty.

In the following, we assume that $H_i(x, p, u, u_j)$ is strictly increasing in $u$, then so is $H(x, p, u)$.

If there exists a viscosity solution $u_-$ of (A.3), then by \cite[Theorem 3.2]{4}, it is unique.

Proposition A.4. Let $v_+$ be a forward weak KAM solution of (A.3), then the set

$$\mathcal{I}_{v_+} := \{ x \in M : u_-(x) = v_+(x) \}$$

is nonempty. Moreover, we have

$$u_-(y) = (C + \Theta \|v(x)\|_\infty + \Theta \|u_-\|_\infty ) \mu e^{\Theta \mu} \leq v_+(x) \leq u_-(x), \quad y \in \mathcal{I}_{v_+},$$

where $C > 0$ and $\mu > 0$ are constants given in Lemma A.1.

Proof. Since $u_-$ is unique, by Proposition A.3 $u_- = \lim_{t \to +\infty} T^-_t v_+$ for each forward weak KAM solution $v_+$, and $\mathcal{I}_{v_+}$ is nonempty.

Since $v_+ \leq u_-$, we only need to show that $v_+$ has a lower bound. Take $y \in \mathcal{I}_{v_+}$ and $x \in M$, then $v_+(y) = u_-(y)$. Let $\alpha : [0, \mu] \to M$ be a geodesic satisfying $\alpha(0) = x$ and $\alpha(\mu) = y$ with constant speed, then $||\dot{\alpha}|| \leq \delta$. If $v_+(x) \geq u_-(y)$, then the proof is finished. If $v_+(x) < u_-(y)$, since $v_+(y) = u_-(y)$, there is $\sigma \in (0, \mu]$ such that $v_+(\alpha(\sigma)) = u_-(y)$ and $v_+(\alpha(\sigma)) < u_-(y)$ for all $s \in [0, \sigma)$. Since $v_+ \leq L$, we have

$$v_+(\alpha(\sigma)) - v_+(\alpha(s)) \leq \int_s^\sigma L(\alpha(\tau), \dot{\alpha}(\tau), v_+(\alpha(\tau)))d\tau,$$

which implies

$$u_-(y) - v_+(\alpha(s)) \leq \int_s^\sigma L(\alpha(\tau), \dot{\alpha}(\tau), v_+(\alpha(\tau)))d\tau$$

$$\leq \int_s^\sigma L(\alpha(\tau), \dot{\alpha}(\tau), u_-(y))d\tau + \Theta \int_s^\sigma (u_-(y) - v_+(\alpha(\tau)))d\tau$$

$$\leq L_0 \mu + \Theta \int_s^\sigma (u_-(y) - v_+(\alpha(\tau)))d\tau,$$

where

$$L_0 := C + \Theta \|v(x)\|_\infty + \Theta \|u_-\|_\infty.$$
Let $G(\sigma - s) = u_-(y) - v_+(\alpha(s))$, then
\[
G(\sigma - s) \leq L_0 \mu + \Theta \int_0^{\sigma - s} G(\tau) d\tau,
\]
By Gronwall inequality we get
\[
u_+(x) \geq \left[ u_-(y) - v_+(\alpha(s)) \right] = G(\sigma - s) \leq L_0 \mu e^{\Theta(\sigma - s)} \leq L_0 \mu e^{\Theta \mu}, \quad \forall s \in [0, \sigma).
\]
Therefore $v_+(x) \geq u_-(y) - L_0 e^{\lambda}$. \qed

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