Total H-irregularity strength of ladder graphs

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Abstract. H-irregular total labeling and total H-irregularity strength of graphs were not complete. This research aimed to determine the total H-irregularity strength of ladder graph \( L_n \) with subgraphs \( C_8 \) and \( C_{10} \). To determine the total H-irregularity strength of ladder graphs, we have to determine the greatest lower bound and the smallest upper bound. The lower bound was analyzed based on graph characteristics and other supporting theorems, while the upper bound was analyzed by construct a function H-Irregular total labeling. The results reveal that the total H-irregularity strength of ladder graph \( L_n \) with subgraph \( C_8 \) is \( \left\lceil \frac{n-2i+14}{16} \right\rceil \leq ths(L_n, C_8) \leq \left\lceil \frac{n+14}{18} \right\rceil \). For \( 18i - 15 \leq n \leq 18i + 2 \), where \( i = 1, 2, \ldots, N \) and the total H-irregularity strength of ladder graph \( L_n \) with subgraph \( C_{10} \) is \( \left\lceil \frac{n-3i+10}{20} \right\rceil \leq ths(L_n, C_{10}) \leq \left\lceil \frac{n+10}{23} \right\rceil \). For \( 23i - 20 \leq n \leq 23i + 2 \), where \( i = 1, 2, \ldots, N \).

1. Introduction

Graph theory was first introduced by Leonhard Euler in 1736. Euler wrote about the problem of the Konisberg bridge in Europe. To solve the problem, Euler considered the land as vertex and bridge as edge or line. The story of Konisberg bridge has become a history of the birth of graph theory.

One of the results of the development of graph theory is the emergence of the concept of labeling. In principle of, graph labeling is a marking of vertex, edge or both. Graph labeling was first introduced by Sadlack (1964), Stewart (1966), then Kotzig and Rossa (1967).

Wallis [8] defines labeling on a graph as a function that maps elements of an element to a graph (vertex or edge) to set integer number. If the domain of mapping is a vertex, then labeling is vertex labeling, if the domain is the edge then labeling is edge labeling, and if the domain is the vertex and edge then the labeling is called the total labeling.

Chartrand [5] was introduced irregular labeling on a graph. For an edge \( k \)-labeling \( \delta: E(G) \to \{1, 2, \ldots, k\} \), where \( k \) is a positive integer, the weight of a vertex \( x \in V(G) \) is \( w_\delta(x) = \sum_{xy \in E(G)} \delta(xy) \), where the sum is over all vertices adjacent to \( x \). The interesting thing in irregular labeling is the smallest parameter \( k \) so that graph \( G \) has an irregular labeling. The parameter is called the irregularity strength denoted by \( s(G) \).

Bača [3] introducing other irregular labeling based on total labeling that is vertex irregular total labeling and edge irregular total labeling. The total labeling \( \varphi: V(G) \cup E(G) \to \{1, 2, \ldots, k\} \) to be vertex irregular total labeling of the graph \( G \) if for every two distinct vertices \( x \) and \( y \) of \( G \) there is \( wt(x) \neq wt(y) \) where \( wt(x) = f(x) + \sum_{xy \in E(G)} f(xy) \). The total labeling \( \varphi : V(G) \cup E(G) \to \)
{1, 2, ..., k} to be edge irregular total labeling of the graph G if for every two distinct edges xy and x'y' of G there is \( wt_\varphi (xy) = \varphi (x) + \varphi (y) \neq \varphi (x') + \varphi (y') \).

The minimum k for which the graph G has a vertex irregular total k-labeling is called the total vertex irregularity strength of the graph G and denoted by \( tvs(G) \) and the minimum k for which the graph G has an edge irregular total k-labeling is called the total edge irregularity strength of the graph G and denoted by \( tes(G) \). In 2015, Hinding determining the tes of subdivision of star graph [6]. Recently, Nurdin determining the tvs and tes of butterfly network on level two [7].

Motivated by the irregularity strength, the total vertex irregularity strength and the total edge irregularity strength of graph G. Ashraf (2017) introduced total H-irregularity strength. An edge-covering of G is a family of subgraph \( H_1, H_2, ..., H_t \) such that each edge of E(G) belongs to at least one of the subgraphs \( H_i \) (i = 1, 2, ..., t). If every subgraph \( H_i \) (i = 1, 2, ..., t) isomorphic to a given graph H, then the graph G admits an H-covering.

Let \( G \) be a graph admitting H-covering. For the subgraph \( H \subseteq G \) under the total k-labeling \( \alpha, \alpha : V(G) \cup E(G) \rightarrow \{1, 2, ..., k\} \), define the associated H-weight as

\[
wt_\alpha (H) = \sum_{v \in V(H)} \alpha (v) + \sum_{e \in E(H)} \alpha (e).
\]

A total k-labeling \( \alpha \) is called an H-irregular total k-labeling of the graph G if for every two different subgraphs \( H' \) and \( H'' \) isomorphic to H there is \( wt_\alpha (H') \neq wt_\alpha (H'') \). The smallest integer k such that G has an H-irregular total k-labeling is called the total H-irregularity strength of a graph G, denoted \( ths(G, H) \).

Some experts have determined the total H-irregularity strength of some graphs. Ashraf [2] determined on the total H-irregularity strength of path graph, ladder graph for subgraph \( C_4 \) and \( C_6 \), and fan graph. Ika Hesti Agustin [1] determined the total H-irregularity strength of some graphs using shackle and amalgamation of any graph, and Bača [4] determined on the total H-irregularity strength of plane graphs. Beside that, Ashraf [2] have determined the total H-irregularity strength of ladder graph for \( C_8 \) and \( C_6 \), but has not yet determined the total H-irregularity strength of ladder graph for \( C_6 \) and \( C_{10} \). Therefore, the research will be focused H-irregularity strength of ladder graph for \( C_8 \) and \( C_{10} \).

2. Main Results

**Theorem 1.** Let \( L_n \cong P_n \times P_2 \), for \( 18i - 15 \leq n \leq 18i + 2 \), where \( i \geq 1 \) be a ladder a \( C_8 \)-covering. then :

\[
\left\lfloor \frac{n - 2i + 14}{16} \right\rfloor \leq ths(L_n, C_8) \leq \left\lceil \frac{n + 14}{18} \right\rceil.
\]

**Proof:**

Let \( L_n \cong P_n \times P_2 \), for \( 18i - 15 \leq n \leq 18i + 2 \), where \( i \geq 1 \) be a ladder with the vertex set \( V(L_n) = \{u_i, v_i| i = 1, 2, ..., n\} \) and the edge set \( E(L_n) = \{u_i u_{i+1}, v_i v_{i+1}|i = 1, 2, ..., n - 1\} \cup \{u_i v_i| i = 1, 2, ..., n\} \). The ladder \( L_n \), for \( 18i - 15 \leq n \leq 18i + 2 \), where \( i = 1, 2, ..., N \), admits a \( C_8 \)-covering with exactly \( n - (2i + 1) \) cycles \( C_8 \). To show that k is lower bound for the total \( C_8 \)-irregularity strength of \( L_n \), with \( |V(H)| + |E(H)| = 16 \). we get \( |V(H)| + |E(H)| + n - (2i + 1) \leq (|V(H)| + |E(H)|)k \) and \( k \geq \frac{n - 2i + 14}{16} \). To show that k is an upper bound for the total \( C_8 \)-irregularity strength of \( L_n \), we define a \( C_8 \)-irregular total k-labeling \( \alpha : V(L_n) \cup E(L_n) \rightarrow \{1, 2, ..., k\} \), in the following way :

\[
f(u_i) = \left\lfloor \frac{i + 10}{18} \right\rfloor, \text{ for } i = 1, 2, ..., n
\]
\[ f(v_i) = \begin{cases} \frac{14+i}{18}, & \text{for } i = 1, 2, ..., n \\ \frac{18+i}{18}, & \text{for } i = 1, 2, ..., n-1 \\ \frac{16+i}{18}, & \text{for } i = 1, 2, ..., n-1 \\ \frac{18+i}{18}, & \text{for } i = 1, 2, ..., n \end{cases} \]

Under the labeling \( \alpha \) all vertex labels and edge labels are at most \( \left\lceil \frac{n+14}{18} \right\rceil \) for the \( C_8 \)-weight, we get

\[ wt(H_i) = \sum_{n=i}^{i+3} (f(u_n) + f(v_n)) + \sum_{n=i}^{i+2} (f(u_nu_{n+1}) + f(v_nv_{n+1})) + f(u_i) + f(u_{i+3}v_{i+3}). \]

Therefore it is enough to prove that \( wt(H_i) \neq wt(H_{i+1}) \).

For every \( i = 1, 2, ..., n-2i-2 \), we have

\[
wt(H_{i+1}) = 1 + \sum_{n=i}^{i+3} \left( \frac{n+10}{18} \right) + \left( \frac{n+14}{18} \right) \right) + \sum_{n=i}^{i+2} \left( \frac{n+4}{18} \right) + \left( \frac{n+7}{18} \right) + \left( \frac{i}{18} \right) + \left( \frac{i+1}{18} \right)
\]

Thus, \( wt(H_i) \neq wt(H_{i+1}) \).

The results show that lower bound the total \( C_8 \)-irregularity strength of ladder graph is \( h\sigma(L_n, C_8) \geq \frac{n-2i+14}{16} \), while the upper bound the total \( C_8 \)-irregularity strength of ladder graph is \( th\sigma(L_n, C_8) \leq \frac{n+14}{18} \). Therefore show that the total \( C_8 \)-irregularity strength of ladder graph is

\[
\frac{n-2i+14}{16} \leq th\sigma(L_n, C_8) \leq \frac{n+14}{18}.
\]

**Theorem 2.** Let \( L_n \equiv P_n \times P_2 \), for \( 23i-20 \leq n \leq 23i+2 \), where \( i = 1, 2, ..., N \) be a ladder a \( C_{10} \)-covering, then \( \frac{n-3i+18}{20} \leq th\sigma(L_n, C_{10}) \leq \frac{n+18}{23} \).

**Proof.**

Let \( L_n \equiv P_n \times P_2 \), for \( 23i-20 \leq n \leq 23i+2 \), where \( i = 1, 2, ..., N \) be a ladder with the vertex set \( V(L_n) = \{u_i, v_i | i = 1, 2, ..., n\} \) and the edge set \( E(L_n) = \{u_iu_{i+1}, v_i v_{i+1} | i = 1, 2, ..., n-1\} \cup \{u_i v_i | i = 1, 2, ..., n\} \). The ladder \( L_n \), for \( 23i-20 \leq n \leq 23i+2 \), where \( i = 1, 2, ..., N \), admits a \( C_{10} \)-covering with exactly \( n - (3i + 1) \) cycles \( C_{10} \). To show that \( k \) is lower bound for the total \( C_{10} \)-irregularity strength of \( L_n \), with \( |V(H)| + |E(H)| = 20 \), we get \( |V(H)| + |E(H)| + n - (3i + 1) \leq (|V(H)| + |E(H))k \) and \( k \geq \frac{n-3i+18}{20} \). To show that \( k \) is an upper bound for the total \( C_{10} \)-irregularity strength of \( L_n \), we define a \( C_{10} \)-irregular total \( k \)-labeling \( \beta: V(L_n) \cup E(L_n) \to \{1, 2, ..., k\} \), in the following way:

\[
\begin{align*}
f(u_i) &= \frac{14+i}{23}, & \text{for } i = 1, 2, ..., n \\ f(v_i) &= \frac{18+i}{23}, & \text{for } i = 1, 2, ..., n \\ f(u_iu_{i+1}) &= \frac{14+i}{23}, & \text{for } i = 1, 2, ..., n-1 \\ f(v_iv_{i+1}) &= \frac{18+i}{23}, & \text{for } i = 1, 2, ..., n-1 \\ f(u_iv_i) &= \frac{18+i}{23}, & \text{for } i = 1, 2, ..., n. 
\end{align*}
\]
Under the labeling \( \alpha \) all vertex labels and edge labels are at most \( \left\lfloor \frac{n+18}{23} \right\rfloor \) for the \( C_{10} \)-weight, we get

\[
wt(H_i) = \sum_{n=i}^{i+4} (f(u_n) + f(v_n)) + \sum_{n=i}^{i+3} (f(u_n u_{n+1}) + f(v_n v_{n+1})) + f(u_i v_i) + f(u_{i+4} v_{i+4})
\]

Therefore it is enough to prove that \( wt(H_i) \neq wt(H_{i+1}) \).

For every \( i = 1, 2, \ldots, n - 2i - 2 \), we have

\[
wt(H_{i+1}) = \sum_{n=i+1}^{i+4} \left( \left\lfloor \frac{n+13}{23} \right\rfloor + \left\lfloor \frac{n+18}{23} \right\rfloor \right) + \sum_{n=i+1}^{i+4} \left( \left\lfloor \frac{n+5}{23} \right\rfloor + \left\lfloor \frac{n+9}{23} \right\rfloor + \frac{i+1}{23} + \frac{i+5}{23} \right)
\]

\[
= 1 + \sum_{n=i}^{i+4} (f(u_n) + f(v_n)) + \sum_{n=i}^{i+3} (f(u_n u_{n+1}) + f(v_n v_{n+1})) + \sum_{n=i}^{i+1} (f(u_i v_i))
\]

thus with \( wt(H_i) \neq wt(H_{i+1}) \).

The results show that lower bound the total \( C_{10} \)-irregularity strength of ladder graph is \( hs(L_n, C_0) \geq \left\lfloor \frac{n-3i+18}{20} \right\rfloor \), while the upper bound the total \( C_{10} \)-irregularity strength of ladder graph is \( ths(L_n, C_{10}) \leq \left\lfloor \frac{n+18}{23} \right\rfloor \). Therefore, we show that the total \( C_{10} \)-irregularity strength of ladder graphs is \( \left\lfloor \frac{n-3i+18}{20} \right\rfloor \leq ths(L_n, C_{10}) \leq \left\lfloor \frac{n+18}{23} \right\rfloor \).

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