Maximum Likelihood Imputation

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Abstract

Maximum likelihood (ML) estimation is widely used in statistics. The h-likelihood has been proposed as an extension of Fisher’s likelihood to statistical models including unobserved latent variables of recent interest. Its advantage is that the joint maximization gives ML estimators (MLEs) of both fixed and random parameters with their standard error estimates. However, the current h-likelihood approach does not allow MLEs of variance components as Henderson’s joint likelihood does not in linear mixed models. In this paper, we show how to form the h-likelihood in order to facilitate joint maximization for MLEs of whole parameters. We also show the role of the Jacobian term which allows MLEs in the presence of unobserved latent variables. To obtain MLEs for fixed parameters, intractable integration is not necessary. As an illustration, we show one-shot ML imputation for missing data by treating them as realized but unobserved random parameters. We show that the h-likelihood bypasses the expectation step in the expectation-maximization (EM) algorithm and allows single ML imputation instead of multiple imputations. We also discuss the difference in predictions in random effects and missing data.
1 Introduction

Missing data are prevalent in statistical problems, but ignoring them can lead to erroneous results (Little and Rubin 2019; Kim and Shao 2021). Imputation is a popular technique for dealing with missing data. However, if imputed data are treated as observed, the use of the standard statistical procedure could result in erroneous inference, giving a biased estimator with an underestimated standard error estimator. Multiple imputation has been proposed by Rubin (1987) to address the uncertainty associated with imputation. However, it requires the self-consistency conditions (Wang and Robins 1998; Meng 1994; Yang and Kim 2016), which may not necessarily hold. An alternative method by Kim (2011) is fractional imputation.

ML estimation of Fisher (1922) is widely accepted in estimating fixed parameters. Missing data can be viewed as unobserved random parameters (Lee et al. 2017) so that imputation can be viewed as a prediction of random parameters, namely missing data. It necessitates an extension of the Fisher likelihood to statistical models that include unobserved random variables (Berger and Wolpert 1984; Butler 1986). Lee and Nelder (1996) intended an extension of ML estimation to models with unobserved random parameters via h-likelihood, defined on a particular scale of random parameters in the linear predictor. However, they confronted severe objections due to difficulties as Bayarri et al. (1988) showed that ML estimation of extended likelihood often provides nonsensical estimation for both fixed and random parameters. Furthermore, Firth (2006) noted that the linear predictor to form the h-likelihood might not be necessarily well defined. All the counterexamples against the h-likelihood, for examples in Little and Rubin (2002), are associated with a wrong choice of scale to form h-likelihood. Little and Rubin (2019) described the current status of h-likelihood “Unlike maximization of the marginal likelihood of Fisher (1922), maximization of an extended likelihood does not generally give consistent estimates of the parameters (Breslow and Lin 1995) ... Lee and Nelder (2001) and Lee et al. (2006) propose maximizing a “modification” ... which is the correct ML approach. For more details, see Lee and Nelder (2009) and the discussion, particularly
Meng (2009).” The success of h-likelihood approach looks coincidental, so that Meng (2009) tried a rigorous theoretical justification for the use of h-likelihood by showing its Bartlett identities. But he ended up highlighting the difficulty caused by the difference between fixed and random parameter estimations. Thus, the benefit of using h-likelihood has not been well accepted yet. This paper establishes the original aim of the h-likelihood whose maximization without any modification provides correct ML estimation and ML imputation by giving rigorous justifications.

Lee et al. (2006) defined h-likelihood precisely, but they have not fully exploited its usefulness. For example, an immediate drawback of the current h-likelihood is that it does not allow MLEs of variance components as Henderson’s joint likelihood does not. So Lee et al. (2017) use a modification to obtain MLEs of variance components, etc. We need to reformulate the h-likelihood in a thoroughly consistent way to avoid modification. Jacobian terms do not play any role in Fisher’s (1922) ML estimation of fixed parameters. However, in models with random parameters, as we shall show, Jacobian terms play a key role in ML estimation. This property has not been well known yet in literature. We clarify the role of the Jacobian term in defining h-likelihood. Currently, the h-likelihood has been defined mainly for random effect models, where linear predictors are defined (Lee and Nelder, 1996). To illustrate our proposal for a much wider class of models, we consider the imputation problem, which does not require a linear predictor, as noted by Firth (2006), and encounters difficulties in ML estimation of random parameters, as noted by Meng (2009). We clarify that the definitions of canonical scale and canonical function are keys to leading valid ML estimation on both fixed and random parameters without any modification in h-likelihood.

In Section 2 we describe the basic setup for missing data problem. In Section 3 we define the h-likelihood by using canonical scale and canonical function in terms of Jacobian term. Moreover, properties of MLEs for fixed and random parameters by using the h-likelihood are examined. In Section 4 we propose the weak canonical scale based on the Laplace approximation. The weak canonical scale can give proper ML imputation when the canonical scale is unknown. In Section 5 we propose the ML imputation by using
the MLE for random parameters. Illustrative examples in Section 6 show the usefulness of the h-likelihood in the missing data problem.

2 Basic Setup

Assume that we have a study variable $Y$ with dominating measure $\mu$ and a covariate vector $X$. The study variable $Y$ is subject to missingness and the covariates are always observed. Assume further that there are $n$ independent and identically distributed realizations of $(X, Y, \delta)$, denoted by $\{(x_i, y_i, \delta_i) : i = 1, \ldots, n\}$, where $\delta_i$ is the missingness indicator defined by $\delta_i = 1$ if $y_i$ is observed and $\delta_i = 0$ otherwise. We are interested in estimating $\eta = E(Y)$ from the observed data.

Under existence of missing data, an imputation estimator of $\eta$ can be written as

$$\hat{\eta} = \frac{1}{n} \sum_{i=1}^{n} \{\delta_i y_i + (1 - \delta_i) \hat{y}_i\}$$

where $\hat{y}_i$ is the imputed value of $y_i$. To predict realized values $y_i$ of unobserved missing data, we consider a frequentist approach using the ML imputation. The current procedure for ML imputation can be described as follows:

Step 1: Estimate $\psi$ by maximizing the observed likelihood

$$L_m(\psi) = f_\psi(y_{\text{obs}}, \delta | x) = \int f_\psi(y_{\text{obs}}, y_{\text{mis}}, \delta | x) dy_{\text{mis}},$$

(1)

where $f_\psi(y_{\text{obs}}, y_{\text{mis}}, \delta | x)$ is the joint density function of $(y_{\text{obs}}, y_{\text{mis}}, \delta)$ given $x$ with fixed unknown parameter $\psi$ and $(y_{\text{obs}}, y_{\text{mis}})$ is the observed and missing part of the complete data $y_{\text{com}} = (y_1, \ldots, y_n)$, respectively.

Step 2: For each $i$ with $\delta_i = 0$, obtain a predictor of $y_i$

$$\hat{y}_i = \int y f(y | x_i, \delta_i = 0; \hat{\psi}) d\mu(y) = E_{\hat{\psi}}(Y_i | x_i, \delta_i = 0),$$

(2)

where $\hat{\psi}$ is the MLE of $\psi$ obtained from Step 1.

We use subscript $m$ in the observed likelihood in (1) to emphasize that the likelihood is developed from the marginal density of the observed data. Robins and Wang (2000) and
Kim and Shao (2021) present some asymptotic properties of the imputation estimator under ML imputation. The above two-step imputation procedure, however, is computationally involved as the ML estimation of the fixed parameter $\psi$ is often based on the iterative procedure such as EM algorithm (Dempster et al., 1977). However, such a conditional mean imputation in (2) does not necessarily give the best prediction in terms of maximizing the predictive distribution. For example, if $y$ is categorical, the conditional mean is not necessarily categorical.

In this paper, instead of using the conditional mean imputation in (2), we propose using conditional mode of the h-likelihood given by

$$\hat{y}_{\text{mis}} = \arg \max_{y_{\text{mis}}} H(\hat{\psi}, y_{\text{mis}})$$

in the next section. In many practical situations, the conditional mode imputation is attractive as it respects the “maximum likelihood” principle by treating the unobserved $y$ values as realized random parameters. By treating $y_{\text{mis}}$ as the random parameters and applying the usual ML procedure, we can obtain imputed values, namely ML imputation, that adhere to the frequentist principle to the greatest extent possible. An immediate practical advantage is that one-shot imputation directly allows the ML estimation of fixed parameters. For one-shot imputation to be meaningful, as we shall show, it estimates the canonical function to predict future (or missing) variable, which resolves summarizability problem raised by Meng (2009).

Naively treating the missing observations as unknown parameters will be subject to biased estimation, which is well known as pointed out by Neyman and Scott (1948). Thus, we employ a technique known as h-likelihood (Lee and Nelder, 1996), to circumvent this issue and obtain valid inferences. Yun et al. (2007) studied the h-likelihood approach to estimate fixed parameters in missing data problems. We introduce the ML imputation of missing data and conduct a more systematic investigation, elucidating the mysteries of h-likelihood in general.
3 H-likelihood

In this paper, we rearrange the indices as $\delta_i = 1$ for $i = 1, \ldots, n_{\text{obs}}$ and $= 0$ for $i = n_{\text{obs}} + 1, \ldots, n$ where $n_{\text{obs}} = \sum_{i=1}^{n} \delta_i$, i.e., the first $n_{\text{obs}}$ responses are observed and remaining $n_{\text{mis}} = n - n_{\text{obs}}$ responses are not observed. Missing data can be viewed as prediction of future data which are not observed yet. By treating $y_{\text{mis}}$ as random parameters, the complete-data log-likelihood is an extended log-likelihood

$$
\ell_e(\psi, y_{\text{mis}}) = \log L_e(\psi, y_{\text{mis}}) = \log f_\psi(y_{\text{obs}}, y_{\text{mis}}, \delta | \mathbf{x})
$$
$$
= \sum_{i=1}^{n_{\text{obs}}} \log f_\psi(y_i, \delta_i = 1 | \mathbf{x}_i) + \sum_{i=n_{\text{obs}}+1}^{n} \log f_\psi(y_{\text{mis},i}, \delta_i = 0 | \mathbf{x}_i).
$$

Extended likelihood principle ([Bjørnstad, 1996]) states that $L_e(\psi, y_{\text{mis}})$ carries all the information in the data about unknown parameters $\psi$ and $y_{\text{mis}}$.

Lee and Nelder (1996) proposed the h-likelihood for ML estimation on both fixed and random parameters. Due to a Jacobian term, unlike a transformation of fixed parameter $\psi$, a nonlinear transformation of random parameter $v = g(y_{\text{mis}})$ changes the extended likelihood

$$
L_e(\psi, v) = L_e(\psi, y_{\text{mis}}) \left| \frac{\partial y_{\text{mis}}}{\partial v} \right|.
$$

Here if the joint maximization of $L_e(\psi, v)$ gives the MLE of $\psi$, that of $L_e(\psi, y_{\text{mis}})$ cannot give the MLE of $\psi$. It means that specifying the scale of a random parameter in defining the h-likelihood is important to obtain MLEs via its maximization. In this paper, we elaborate on how to use the Jacobian term to form such an h-likelihood.

Following Lee et al. (2017), the predictive likelihood of random parameter $v$ can be defined as

$$
L_p(v | \mathcal{D}; \psi) \equiv f_\psi(v | \mathcal{D}, \mathbf{x}) = f_\psi(v, \mathcal{D} | \mathbf{x}) / f_\psi(\mathcal{D} | \mathbf{x}),
$$

where $\mathcal{D} = \{y_{\text{obs}}, \delta\}$ and subscript $p$ is used to emphasize the predictive likelihood for $v$. Thus, the marginal likelihood is expressed as

$$
L_m(\psi) = \frac{L_e(\psi, v)}{L_p(v | \mathcal{D}; \psi)}.
$$
Given $\psi$, let

$$\tilde{v} = \tilde{v}(\psi, D, x) = \arg\max_v L_e(\psi, v) = \arg\max_v L_p(v \mid D; \psi)$$

(4)

be the common mode of the extended likelihood and the predictive likelihood. Note that the common mode $\tilde{v}(\psi, D, x)$ is a function of both parameter and data. However, we denote it as $\tilde{v}$ for notational convenience. Evaluating the marginal likelihood in (3) at $v = \tilde{v}$ leads to

$$L_m(\psi) = \frac{L_e(\psi, \tilde{v})}{L_p(\tilde{v} \mid D; \psi)}.$$  

(5)

If both $L_e(\psi, v)$ and $L_p(v \mid D; \psi)$ are explicitly available, at least at the mode $\tilde{v}$, the MLE for $\psi$ is immediately obtained from (5). However, both are not often available.

**Definition 3.1.** If a scale $v = g(y_{mis})$ satisfies

$$L_e(\psi, \tilde{v}) \propto L_m(\psi),$$

(6)

the $v$-scale is called the canonical scale and the mode $\tilde{v}$ is called the canonical function. The extended likelihood defined on the canonical scale $v$ is called the h-likelihood,

$$H(\psi, v) = L_e(\psi, v).$$

By combining (3) and (6), $L_p(\tilde{v} \mid D; \psi)$ does not depend on $\psi$ if $v$-scale is canonical, i.e. information neutral with respect to $\psi$ at the mode $\tilde{v}$.

Here, we emphasize defining the h-likelihood with different parametrization of a random parameter. Let $\hat{\zeta}$ be the MLE of $\zeta = k(\psi)$ under the transformation $k(\cdot)$. Then, the MLE $\hat{\psi} = k^{-1}(\hat{\zeta})$ is invariant with respect to the transformation. Similarly, the MLE of a parameter from the h-likelihood is transformation invariant. That is, we can treat $v$ as if it is a fixed parameter after defining the h-likelihood in the sense that

$$H(\psi, y_{mis}) = H\{\psi, g^{-1}(y_{mis})\} = H(\psi, v)$$

(7)

[Lee and Nelder 2005]. Here, we denote $H(\psi, y_{mis})$ the h-likelihood in terms of $y_{mis}$ as (7), whereas $L_e(\psi, y_{mis})$ indicates the extended likelihood in which the canonical scale is
yet unknown. From (7), the conditional mode of $y_{mis}$ is defined by

$$\tilde{y}_{mis} = \arg \max_{y_{mis}} H(\psi, y_{mis}) = g^{-1}(\tilde{v}).$$

(8)

If the transformation $g(\cdot)$ is not linear, we get $\tilde{y}_{mis} \neq \arg \max_{y_{mis}} L_e(\psi, y_{mis})$. Thus, under the canonical condition (6), MLEs of both fixed and random parameters can be obtained by maximizing $H(\psi, v) = L_e(\psi, v)$.

Lee et al. (2017) gave a correct definition of canonical scale above, but have not exploited it to form the h-likelihood. We now state a sufficient condition for the canonical property in (6) as follows.

**Proposition 3.1.** If a transformation $v = g(y_{mis})$ with bijective, differentiable function $g(\cdot)$ satisfies

$$\left| \frac{\partial v}{\partial y_{mis}} \right|_{v=\tilde{v}} \propto L_p(\tilde{y}_{mis} | \mathcal{D}; \psi),$$

where $\tilde{y}_{mis} = g^{-1}(\tilde{v})$ and $\tilde{v}$ is defined in (4), the canonical property in (6) is satisfied.

Proposition 3.1 gives further interpretation about Definition 3.1.

$$L_m(\psi) = \frac{L_e(\psi, \tilde{y}_{mis})}{L_p(\tilde{y}_{mis} | \mathcal{D}; \psi)} \propto L_e(\psi, \tilde{y}_{mis}) \left| \frac{\partial y_{mis}}{\partial v} \right|_{v=\tilde{v}} = L_e(\psi, \tilde{v}) = H(\psi, \tilde{v}).$$

(9)

Moreover, it shows how the canonical scale allows ML estimation. Now, we first study the ML estimation of the fixed parameter using h-likelihood.

### 3.1 MLE of Fixed Parameter

Equation (9) characterizes the canonical scale which allows the ML estimation.

**Theorem 3.1.** Suppose that the predictive likelihood $L_p(y_{mis} | \mathcal{D}; \psi)$ is unimodal with respect to $y_{mis}$. Then, there exists the canonical scale to form the h-likelihood.

Theorem 3.1 states a sufficient condition for the existence of a canonical scale. When an explicit form of the canonical scale is not available, we present a way of defining a weak canonical scale based on the Laplace approximation in Section 4. For now, we assume that an explicit form of the canonical scale $v = g(y_{mis})$ is known. The following theorem shows how to obtain the MLE of fixed parameter and also its variance estimator.
Theorem 3.2. (i) The MLE of $\psi$ can be obtained by solving the score equation

$$\frac{\partial \ell_m}{\partial \psi} = \frac{\partial}{\partial \psi} h(\psi, \tilde{v}) = \frac{\partial h}{\partial \psi} \bigg|_{v=\tilde{v}} = 0,$$

where $h = \log H(\psi, v)$ and $\ell_m = \ell_m(\psi) = \log L_m(\psi)$.

(ii) The variance estimator of the MLE can be obtained from the Hessian matrix of the $h$-likelihood as

$$\hat{I}_{\psi\psi} = I_{\psi\psi} \bigg|_{\psi = \hat{\psi}}, \quad I_{\psi\psi} = \left( -\frac{\partial^2 \ell_m}{\partial \psi \partial \psi^T} \right)^{-1},$$

where the definition of $I_{\psi\psi}$ is in Appendix.

To compare the $h$-likelihood approach with the EM algorithm, note that

$$\frac{\partial \ell_m(\psi)}{\partial \psi} = E_{\psi(t)} \left\{ \frac{\partial}{\partial \psi} \ell_e(\psi, y_{\text{mis}}) \bigg| D, x \right\}.$$

This equality is called the mean score theorem [Louis, 1982]. The EM algorithm [Dempster et al., 1977] obtains the solution to $\frac{\partial \ell_m(\psi)}{\partial \psi} = 0$ by

$$\psi^{(t+1)} \leftarrow \text{solve } E_{\psi(t)} \left\{ \frac{\partial}{\partial \psi} \ell_e(\psi, y_{\text{mis}}) \bigg| D, x \right\} = 0. \quad (10)$$

The $h$-likelihood approach gives the MLE of the fixed parameter without requiring the E-step in (10) which is often computationally intensive.

3.2 MLE of Random Parameter

If we let $y_{\text{mis}}$ be the unobserved part of the data, the missing data problem becomes a prediction problem. To understand Meng’s point in [Meng, 2009], assume that $y_{\text{obs}}$ and $y_{\text{mis}}$ are independent and the scale $v = g(y_{\text{mis}})$ is the canonical scale. Prediction of future data can be viewed as missing data problem where $y_{t+1}, \ldots, y_{t+n_{\text{mis}}}$ are future data at the present time $t = n_{\text{obs}}$. [Meng, 2009] showed that

$$\hat{v} - v = g(\tilde{y}_{\text{mis}}) - g(y_{\text{mis}}) = g'(\tilde{y}_{\text{mis}})(\tilde{y}_{\text{mis}} - y_{\text{mis}}) + R_{n_{\text{obs}}},$$

where

$$R_{n_{\text{obs}}} = O_p(1) \text{ and } g'(\tilde{y}_{\text{mis}})(\tilde{y}_{\text{mis}} - y_{\text{mis}}) = O_p(1).$$
Meng (2009) claimed that $\hat{v} - v$ is not summarizable because of the nonnegligibility of the remainder term $R_{n_{obs}}$, i.e., consistency and asymptotic normality for the MLE $\hat{v}$ from the h-likelihood are not guaranteed.

Now we investigate the summarizability properties of the MLE $\hat{v}$. In missing data problem, the ML estimation of random parameter can be called the ML imputation. Let $\psi_0$ be the true value of $\psi$. As MLE $\hat{\psi}$ is estimating $\psi_0$ and similarly the MLE $\hat{y}_{\text{mis}}$ predicts a realized value of $y_{\text{mis}}$ by estimating the conditional mode $y_{\text{mis},0} = \tilde{y}_{\text{mis}}(\psi_0, D, x)$ in (8), which is a function of data and unknown parameter $\psi_0$. This clarifies the summarizability problem raised by Meng (2009); while $\hat{y}_{\text{mis}} - y_{\text{mis}}$ is not summarizable, $\hat{y}_{\text{mis}} - y_{\text{mis},0}$ is summarizable as in Theorem 3.3 below. Note that

$$y_{\text{mis}} - \hat{y}_{\text{mis}} = y_{\text{mis},0} - \hat{y}_{\text{mis}} + \varepsilon,$$

where $\varepsilon = y_{\text{mis}} - y_{\text{mis},0}$. In missing data problem, $\varepsilon = O_p(1)$. In view of predicting unobservable future (or missing) random variable, we estimate $\varepsilon$ as null. Then, $\hat{y}_{\text{mis}}$ is estimating $y_{\text{mis},0}$ to predict $y_{\text{mis}}$. Thus, we obtain

$$\text{var}_{\psi}(\hat{y}_{\text{mis}} - y_{\text{mis}}) = \text{var}_{\psi}(\hat{y}_{\text{mis}} - y_{\text{mis},0}) + \text{var}_{\psi}(\varepsilon | D, x).$$

The first term is the variance due to estimating $y_{\text{mis},0}$ by $\hat{y}_{\text{mis}}$ and the second term is the variance due to the unidentifiable error term $\varepsilon$. The second term may decrease with a better imputation model, but it does not decrease with larger sample size. Moreover, to obtain a standard error for prediction of $y_{\text{mis}}$, we need to estimate the conditional variance of $\varepsilon$ by using

$$\text{var}_{\psi}(\varepsilon | D, x) = \text{var}_{\psi}(y_{\text{mis}} - y_{\text{mis},0} | D, x) = \text{var}_{\psi}(y_{\text{mis}} | D, x).$$

Here, we are interested in estimating $\text{var}(\hat{y}_{\text{mis}} - y_{\text{mis}})$. Thus, we write the h-likelihood with respect to $y_{\text{mis}}$ as $h = h(\psi, y_{\text{mis}}) = h\{\psi, g^{-1}(v)\}$. Note that

$$\frac{\partial \tilde{y}_{\text{mis}}^T}{\partial \psi} = -I_{\psi y_{\text{mis}}}^{-1} I_{y_{\text{mis}} y_{\text{mis}}}$$

and the variance estimator of $\hat{\psi}$ is $\hat{\sigma}_{\psi}^2$ by Theorem 3.2 where $I_{\psi y_{\text{mis}}} = -\partial^2 h/\partial \psi \partial y_{\text{mis}}^T | y_{\text{mis}} = \tilde{y}_{\text{mis}}$ and $I_{y_{\text{mis}} y_{\text{mis}}} = -\partial^2 h/\partial y_{\text{mis}} \partial y_{\text{mis}}^T | y_{\text{mis}} = \tilde{y}_{\text{mis}}$. Then, by using the delta method, we have the asymptotic normality of $\hat{y}_{\text{mis}}$ as follows.
Theorem 3.3. Under regularity conditions in Appendix, we have

\[ \sqrt{n} \left( \hat{y}_{\text{mis}} - y_{\text{mis},0} \right) \xrightarrow{d} N(0, V), \]

where \( V = \lim_{n \to \infty} n \hat{I}^{-1} \hat{y}_{\text{mis}}y_{\text{mis}} \hat{I}^\psi \hat{I}^\psi y_{\text{mis}} \hat{I}^{-1} y_{\text{mis}}y_{\text{mis}} \) and \( \hat{I}_{\psi y_{\text{mis}}} \), \( \hat{y}_{\text{mis}}y_{\text{mis}} \) are evaluated at \( \psi = \hat{\psi} \). The variance of \( \hat{y}_{\text{mis}} - y_{\text{mis},0} \) can be estimated as

\[ \hat{\text{var}}(\hat{y}_{\text{mis}} - y_{\text{mis},0}) = \text{var}(\hat{y}_{\text{mis}} - y_{\text{mis},0}) = \hat{I}^{-1} \hat{y}_{\text{mis}}y_{\text{mis}} \hat{I}^\psi y_{\text{mis}} \hat{I}^\psi y_{\text{mis}} \hat{I}^{-1} y_{\text{mis}}y_{\text{mis}}. \] (11)

If \( \mathbb{E}_\psi(\epsilon) = 0, \) \( \hat{y}_{\text{mis}} \) is an asymptotically unbiased estimator of \( y_{\text{mis}} \). However, the assumption \( \mathbb{E}_\psi(\epsilon) = 0 \) is coming from model assumption which may not be identifiable by observed data. Now, to discuss the estimation of the variance due to the model error \( \epsilon \), suppose that there exists a normalizing transformation \( z = k(v) = k\{g(y_{\text{mis}})\} = k \circ g(y_{\text{mis}}) = r(y_{\text{mis}}) \) with \( r(\cdot) = k \circ g(\cdot) \) such that \( L_p(z|\mathcal{D}; \psi) \) is from the normal density with mean \( \hat{z} = \arg\max_z L_p(z|\mathcal{D}; \psi) \) and covariance matrix \( I_{zz}^{-1} \), where \( I_{zz} = -\partial^2 h(\psi, z)/\partial z \partial z^T |_{z=\hat{z}} \).

Then, it gives the h-likelihood

\[ h(\psi, z) = \ell_m(\psi) + \frac{1}{2} \log \left| \frac{1}{2\pi} I_{zz} \right| - \frac{1}{2} (z - \hat{z})^T I_{zz} (z - \hat{z}). \]

Here, \( \hat{z} = \mathbb{E}_\psi(z|\mathcal{D}, x) = r(\hat{y}_{\text{mis}}) \) provided by the normality of the predictive likelihood \( L_p(z|\mathcal{D}; \psi) \). This leads to \( \mathbb{E}_\psi(\epsilon) = \mathbb{E}_\psi(z - z_0) = 0, \)

\[ \text{var}_\psi(\hat{z} - z) = \text{var}_\psi(\hat{z} - z_0) + \mathbb{E}_\psi \{\text{var}_\psi(z_0 - z | \mathcal{D}, x)\} \]

and \( \text{var}(z_0 - z|\mathcal{D}, x) = \hat{I}_{zz}^{-1} \), where \( \hat{z} = r(\hat{y}_{\text{mis}}) \) and \( z_0 = r(y_{\text{mis},0}) = \mathbb{E}_{\psi_0}(z|\mathcal{D}, x) \). This gives

\[ \text{var}(\hat{z} - z) = \text{var}(\hat{z} - z_0) + \text{var}(z_0 - z | \mathcal{D}, x) = \hat{I}_{zz}^{-1} \hat{I}_{z\psi} \hat{I}^\psi y_{\text{mis}} \hat{I}^{-1} y_{\text{mis}} + \hat{I}_{zz}^{-1} = \hat{I}_{zz}. \]

Therefore, if a normalizing transformation exists, the h-likelihood gives not only MLEs of both fixed and random parameters, but also their corresponding variance estimators. Moreover, if \( y_{\text{mis}} \) itself satisfies normal approximation well, then, we can have a reasonable variance estimator from the Hessian matrix of h-likelihood

\[ \text{var}(\hat{y}_{\text{mis}} - y_{\text{mis}}) = \text{var}(\hat{y}_{\text{mis}} - y_{\text{mis},0}) + \text{var}(y_{\text{mis},0} - y_{\text{mis}} | \mathcal{D}, x) \]

\[ = \hat{I}_{y_{\text{mis}}y_{\text{mis}}}^{-1} \hat{I}_{y_{\text{mis}}y_{\text{mis}}} \hat{I}^\psi y_{\text{mis}} \hat{I}^\psi y_{\text{mis}} \hat{I}^{-1} y_{\text{mis}}y_{\text{mis}} + \hat{I}_{y_{\text{mis}}y_{\text{mis}}}^{-1} = \hat{I}_{y_{\text{mis}}y_{\text{mis}}}. \]
Thus, \( \hat{y}_{\text{mis},i} \pm 1.96\sqrt{\hat{I}_{ii}^{y_{\text{mis}y_{\text{mis}}}}} \) is 95% predictive interval of \( y_{\text{mis},i} \), where \( \hat{I}_{ii}^{y_{\text{mis}y_{\text{mis}}}} \) is the \( i \)-th diagonal element of \( \hat{I}^{y_{\text{mis}y_{\text{mis}}}} \). The length of predictive interval is \( O_p(1) \) and coverage probability becomes exact as \( n \to \infty \) (Lee and Kim, 2016). However, in practice, the normalizing transformation is not known. Thus, in general, for the prediction of \( y_{\text{mis}} \), Lee and Kim (2016, 2020) proposed to use the predictive distribution after eliminating \( \psi \) defined as

\[
f(y_{\text{mis}} | D, \mathbf{x}) = \int f_{\psi}(y_{\text{mis}} | D, \mathbf{x}) c(\psi) d\psi,
\]

where \( c(\psi) \) is the confidence density (Schweder and Hjort, 2016). By using the predictive likelihood (12), we can account for the uncertainty caused by estimating \( \psi \). Via simulation studies, Lee and Kim (2016) showed that resulting predictive interval maintains the stated coverage probability well as \( n \) grows.

From Theorem 3.2, MLE \( \hat{\psi} \) from the marginal likelihood can be obtained by

\[
\frac{\partial \ell_m(\psi)}{\partial \psi} = \frac{\partial h(\psi, \tilde{v})}{\partial \psi} = 0
\]

and ML imputation \( \hat{y}_{\text{mis}} = g^{-1}(\tilde{v}) \) of \( y_{\text{mis}} = g^{-1}(v) \) from the predictive likelihood can be obtained by

\[
\frac{\partial \ell_p(v | D; \tilde{\psi})}{\partial v} = \frac{\partial h(\tilde{\psi}, v)}{\partial v} = 0,
\]

where \( \tilde{\psi} \) is solution to \( \partial h(\psi, v) / \partial \psi = 0 \). In contrast to the EM algorithm, the h-likelihood provides not only the ML estimation for fixed parameters from \( H(\psi, \tilde{v}) \), but also ML imputation on random parameters from \( H(\tilde{\psi}, v) \) as in Figure 1. Moreover, the necessary standard error estimates are also given straightforwardly.

Find \( \tilde{\psi} \) by solving \( \frac{\partial h(\psi, v)}{\partial v} = 0 \) where

\[
H(\psi, v) = L_p(v | D; \psi)L_m(\psi) \propto L_m(\psi)
\]

Find \( \tilde{\psi} \) by solving \( \frac{\partial h(\psi, v)}{\partial v} = 0 \) where

\[
H(\tilde{\psi}, v) = L_p(v | D; \tilde{\psi})L_m(\tilde{\psi}) \propto L_p(v | D; \tilde{\psi})
\]

Figure 1: Estimation procedure of the h-likelihood.
Example 3.1. Suppose that \( n \) variables are generated from the exponential distribution with mean \( \theta_0 \) but only the first \( n-1 \) variables are observed, i.e., \( n_{\text{obs}} = n-1 \) and \( y_{\text{mis}} = y_n \) is not observed. In this example, the extended likelihood defined on \( y_{\text{mis}} \)-scale is

\[
\ell_e(\theta, y_{\text{mis}}) = -n \log \theta - \frac{(n-1)\bar{y}_{\text{obs}} + y_{\text{mis}}}{\theta}.
\]

Note that \( y_{\text{mis}} \)-scale is not canonical but \( v = \log y_{\text{mis}} \) is a canonical scale which gives

\[
h(\theta, v) = \ell_e(\theta, y_{\text{mis}}) + \log |\frac{\partial y_{\text{mis}}}{\partial v}| = -n \log \theta - \frac{(n-1)\bar{y}_{\text{obs}} + e^v}{\theta} + v
\]

and

\[
h(\theta, y_{\text{mis}}) = -n \log \theta - \frac{(n-1)\bar{y}_{\text{obs}} + y_{\text{mis}}}{\theta} + \log y_{\text{mis}}.
\]

Here, the canonical function of \( y_{\text{mis}} \) is \( \bar{y}_{\text{mis}} = \theta \) which gives the MLE \( \hat{\theta} = \bar{y}_{\text{obs}} \) and ML imputation \( \hat{y}_{\text{mis}} = \hat{\theta} = \bar{y}_{\text{obs}} \). In this example, the MLE of \( \theta \), \( \hat{\theta} \), satisfies the asymptotic normality

\[
\sqrt{n_{\text{obs}}} \left( \hat{\theta} - \theta_0 \right) \xrightarrow{d} N \left( 0, \theta_0^2 \right).
\]

By Theorem 3.3, the ML imputation for \( y_{\text{mis}} \), \( \hat{y}_{\text{mis}} \), satisfies the asymptotic normality

\[
\sqrt{n_{\text{obs}}} \left( \hat{y}_{\text{mis}} - y_{\text{mis,0}} \right) \xrightarrow{d} N \left( 0, \theta_0^2 \right),
\]

where \( y_{\text{mis,0}} = \theta_0 \) and \( \text{var}(\hat{y}_{\text{mis}} - y_{\text{mis,0}}) = n_{\text{obs}}^{-1} \hat{\theta}^2 \), i.e., (11) gives valid variance estimator of \( \hat{y}_{\text{mis}} - y_{\text{mis,0}} \). Moreover,

\[
\hat{\text{var}}(y_{\text{mis}}) = \hat{\theta}^2 \left( 1 + \frac{1}{n_{\text{obs}}} \right) = \hat{\text{var}}(\hat{y}_{\text{mis}} - y_{\text{mis}}).
\]

Here \( \hat{\text{var}}(y_{\text{mis}} - y_{\text{mis}}) = \hat{\text{var}}(y_{\text{mis}} - y_{\text{mis,0}}) + \hat{\text{var}}(y_{\text{mis}} | y_{\text{obs}}) = \hat{\theta}^2 / n_{\text{obs}} + \hat{\theta}^2 \). Thus, the h-likelihood gives a correct ML imputation. In this example, \( y_{\text{mis,0}} = \theta \) is a function of parameter only so that \( y_{\text{mis}} - y_{\text{mis,0}} \) is summarizable. But \( y_{\text{mis}} \) is not identifiable since \( \varepsilon = O_p(1) \) with \( \text{E}_0(\varepsilon) = 0 \). Asymptotically correct probability statement on \( y_{\text{mis}} \) can be made based on predictive interval whose length is \( O_p(1) \).

Example 3.2. Consider a one-way mixed model

\[
y_{ij} = \mu + u_i + \epsilon_{ij}, \; i = 1, \ldots, q, \; j = 1, \ldots, n,
\]
where random effects \( u_i \) are iid \( N(0, \lambda^2) \), \( \epsilon_{ij} \) are iid \( N(0, \sigma^2) \) and \( u_i \) and \( \epsilon_{ij} \) are independent. Henderson's Henderson et al. (1959) joint likelihood is the current h-likelihood of Lee and Nelder (1996)

\[
\ell_e(\theta, u) = \sum_{i,j} \left\{ -\frac{1}{2} \log 2\pi \sigma^2 - \frac{1}{2\sigma^2} (y_{ij} - \mu - u_i)^2 \right\} + \sum_i \left( -\frac{1}{2} \log 2\pi \lambda^2 - \frac{1}{2\lambda^2} u_i^2 \right), \tag{13}
\]

where \( \theta = (\mu, \sigma^2, \lambda^2) \). However, joint maximization of (13) cannot give the MLEs of variance components \( \sigma^2 \) and \( \lambda^2 \). Consider a \( v \)-scale

\[
v_i = \left\{ -\frac{\partial^2 \ell_e(\theta, u)}{\partial u_i^2} \right\}^{0.5} u_i = \left( \frac{\sigma^2 + n\lambda^2}{\sigma^2 \lambda^2} \right)^{0.5} u_i,
\]

which leads to the extended likelihood

\[
\ell_e(\theta, v) = \ell_e(\theta, u) + \log \left| \frac{\partial u}{\partial v} \right| = -\frac{N - q}{2} \log 2\pi \sigma^2 - \frac{q}{2} \log 2\pi (\sigma^2 + n\lambda^2)
\]

\[
-\frac{1}{2\sigma^2} \sum_{i,j} \left\{ y_{ij} - \mu - \left( \frac{\sigma^2 \lambda^2}{\sigma^2 + n\lambda^2} \right)^{0.5} v_i \right\}^2 - \frac{\sigma^2}{2(\sigma^2 + n\lambda^2)} \sum_i v_i^2 - \frac{q}{2} \log 2\pi,
\]

where \( N = qn \). Since \( \ell_e(\theta, v) = \ell_m(\theta) \), where

\[
\tilde{v}_i = \tilde{v}_i(\theta, y_i) = \frac{n\lambda^2 (\bar{y}_i - \mu)}{(\sigma^2 \lambda^2 (\sigma^2 + n\lambda^2))^{0.5}},
\]

\( y_i = (y_{i1}, \ldots, y_{in}) \) and \( \bar{y}_i = n^{-1} \sum_{j=1}^n y_{ij} \), we have h-likelihood \( h = \ell_e(\theta, v) \), whose simple maximization gives MLEs of the whole parameters \( \theta \). Also, it gives the best linear unbiased predictors for realized but unobserved random parameters

\[
\hat{u}_i = \tilde{u}_i(\hat{\theta}, y_i) = E(u_i \mid y_i), \quad i = 1, \ldots, q,
\]

where

\[
\tilde{u}_i(\theta, y_i) = \left\{ \frac{\sigma^2 \lambda^2}{\sigma^2 + n\lambda^2} \right\}^{0.5} \tilde{v}_i(\theta, y_i) = \frac{n\lambda^2}{\sigma^2 + n\lambda^2} (\bar{y}_i - \mu) = E_{\theta}(u_i \mid y_i).
\]

In this example, the target of \( \hat{u}_i \) is

\[
u_i \cdot 0 = \tilde{u}_i(\theta_0, y_i) = E_{\theta_0}(u_i \mid y_i),
\]

where \( \theta_0 = (\mu_0, \sigma_0^2, \lambda_0^2) \) is the true value of \( \theta \). If the MLE \( \hat{\theta} \) converges to \( \theta_0 \),

\[
\text{var} (\hat{u}_i - u_i) = \text{var} (\hat{u}_i - u_{i\cdot 0}) + \text{var} (u_i - u_{i\cdot 0} \mid y_i) \overset{P}{\to} 0
\]

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as \((q,n) \to \infty\). Thus, in this example, we have a consistent estimator of unobserved random parameter \(u_i\), i.e., \(u_i\) is identifiable with 
\[ \varepsilon = u_i - u_{i0} = o_p(1). \]
This can also be shown that
\[
\lim_{(q,n) \to \infty} \hat{u}_i = \lim_{(q,n) \to \infty} u_{i0} = \lim_{(q,n) \to \infty} \frac{n\lambda_0^2}{\sigma_0^2 + n\lambda_0^2} (\tilde{y}_i - \mu_0) = \lim_{(q,n) \to \infty} \frac{n\lambda_0^2}{\sigma_0^2 + n\lambda_0^2} (u_i + \tilde{\epsilon}_i) = u_i,
\]
where \(\tilde{\epsilon}_i = n^{-1} \sum_{j=1}^n \epsilon_{ij}\). Model assumptions on \(u_i\) can also be checkable: for various model checking plots, see Lee et al. (2017). Furthermore, if different model assumptions on \(f_\psi(u)\) lead to an identical h-likelihood, then it leads to equivalent inferences for identifiable random effects (Lee and Nelder, 2006). In missing data problem with \(\varepsilon = y_{\text{mis}} - y_{\text{mis},0} = O_p(1)\), model assumptions \(f_\psi(y_{\text{mis}}|D, x)\) cannot be checkable from the observed data (Molenberghs et al., 2008).

Since \(u_i\) itself is the normalizing transformation in this example, variances can be estimated as
\[
\hat{I}_{u_i u_i}^{-1} = \left( -\frac{\partial^2 h}{\partial u_i^2} \right)^{-1} \bigg|_{\theta = \hat{\theta}} = \frac{\hat{\sigma}^2 \hat{\lambda}^2}{\hat{\sigma}^2 + n\hat{\lambda}^2} = \text{var} (u_i | y_i) = \text{var} (u_i - u_{i0} | y_i),
\]
\[
\hat{I}_{u_i u_i} = \hat{I}_{u_i u_i}^{-1} + \frac{\partial \hat{u}_i}{\partial \hat{\theta}} \text{var} \left( \frac{\partial \hat{u}_i}{\partial \hat{\theta}} \right) \bigg|_{\theta = \hat{\theta}} = \text{var} (\hat{u}_i - u_i),
\]
\[
\hat{I}_{\theta \theta} = \text{var} \left( \frac{\partial \hat{\theta}}{\partial \theta} \right).
\]

Thus, proper MLEs of both fixed and random parameters and their variance estimators can be obtained by the maximization of the newly defined h-likelihood, which differs from the joint likelihood of Henderson (Henderson et al., 1953). Asymptotically correct probability statement on \(u_i\) can be made from the predictive interval whose length is \(o_p(1)\). For more details about general random effect models, see Paik et al. (2015), Lee et al. (2017), and Lee and Kim (2020).

### 4 Scale for Joint Maximization

When the canonical scale is unknown, Lee et al. (2017) proposed the use of the Laplace approximation to give an approximate MLE (Tierney and Kadane, 1986), which has been implemented by various packages (Kristensen et al., 2016; Ha et al., 2019). In this
section, we study how to form an h-likelihood with a weak canonical scale whose joint
maximization provides approximate MLEs obtained by the Laplace approximation. Given
\( y_{\text{mis}} \)-scale, consider a \( b \)-scale with \( b = g_1(y_{\text{mis}}) \). Let \( \Omega_b \) be the support of \( b \) taking a rectangle
form \( \Omega_b = \prod_{i=n_{\text{obs}}+1}^{n}[l_i, u_i] \), where \( l_i \) and \( u_i \) are permitted to take the value of \(-\infty\) and
\( \infty \) with boundary set \( \partial \Omega_b \), \( \xi = (\psi, b) \) and \( f_\psi(b) \) be the density function of \( b \). Meng (2009)
studied the regularity conditions for the first and second Bartlett identities of an extended
likelihood \( \ell_e(\psi, b) \).

**Theorem 4.1** (Meng, 2009). (i) If \( f_\psi(b) = 0 \) for any \( b \in \partial \Omega_b \), the first Bartlett identity
holds.

\[
E_\psi \left[ \frac{\partial}{\partial \xi} \ell_e(\psi, b) \right] = 0.
\] (14)

(ii) Furthermore, if \( \partial f_\psi(b)/\partial b = 0 \) for any \( b \in \partial \Omega_b \), the second Bartlett identity holds.

\[
E_\psi \left[ \left( \frac{\partial}{\partial \xi} \ell_e(\psi, b) \right) \left( \frac{\partial}{\partial \xi} \ell_e(\psi, b) \right)^T \right] + E_\psi \left[ \frac{\partial^2}{\partial \xi \partial \xi^T} \ell_e(\psi, b) \right] = O.
\] (15)

Corollary below gives an easy way of having a \( b \)-scale to satisfy Bartlett identities.

**Corollary 4.1.** Let \( \Omega_b = \mathbb{R}^{n_{\text{mis}}} \). If \( E_\psi(b_i) < \infty \) for all \( i \), the \( b \)-scale satisfies Bartlett
identities.

The second Bartlett identity (15) guarantees that the predictive likelihood \( L_p(b|D; \psi) \)
is unimodal with respect to \( b \) even though \( L_p(y_{\text{mis}}|D; \psi) \) may not be unimodal. From
Theorem 3.1 if we have such an extended likelihood \( L_e(\psi, b) \) there exists the canonical
scale \( v = g(b) \) to form the h-likelihood. But, the explicit form of \( g(\cdot) \) for the canonical
scale may not be known. In this case, we may consider an approximation of canonical
scale based on the Laplace approximation, which is widely used to obtain an approximate
MLE of fixed parameter, \( \hat{\psi}_{\text{Lap}} \) (Raudenbush et al., 2000; Lee et al., 2017).

**Definition 4.1.** Suppose that \( b \)-scale satisfies the Bartlett identities and \( \ell_e(\psi, b) \) is the
corresponding extended log-likelihood. Now, consider a \( w \)-scale defined as

\[
w = g_2(b) = \bar{\Omega}_{bb}^{1/2} h,
\] (16)
where \( \tilde{b} = \tilde{b}(\psi, D, x) = \arg \max_b \ell_e(\psi, b) \) and \( \tilde{\Omega}_{bb} = -\partial^2 \ell_e(\psi, b) / \partial b \partial b^T |_{b = \tilde{b}} \). Here, we call \( w \)-scale weak canonical and

\[
H = L_e(\psi, w) = L_e(\psi, b) \left| \frac{\partial b}{\partial w} \right|
\]

the \( h \)-likelihood with weak canonical scale \( w \).

By the above definition, weak canonical scale also satisfies Bartlett identities in (14) and (15) since the transformation (16) is linear. Furthermore, we have

\[
\tilde{w} = \tilde{w}(\psi, D, x) = \arg \max_w L_e(\psi, w) = g_2(\tilde{b}(\psi, D, x))
\]
since \( \tilde{b} \) is the mode of \( L_e(\psi, b) \) and the transformation \( g_2(\cdot) \) is linear. Note that the joint maximization of the \( h \)-likelihood with weak canonical scale gives the approximate MLE for \( \psi \) based on the Laplace approximation as follows.

\[
\hat{L}_m(\psi) = L_e(\psi, \hat{b}) \left| \frac{1}{2\pi} \tilde{\Omega}_{bb} \right|^{-\frac{1}{2}} \propto L_e(\psi, \hat{b}) \left| \frac{\partial b}{\partial \tilde{w}} \right|_{\tilde{w} = \tilde{w}} = L_e(\psi, \tilde{w}).
\]

This weak canonical scale does not require the existence of linear predictor. In HGLMs, a scale satisfying additivity in the linear predictor is called a weak canonical scale (Lee et al., 2017), which satisfies Corollary 4.1. In Appendix, we show how to compute the standard error estimate of the approximate MLE obtained from \( \ell_e(\psi, \tilde{w}) = \log L_e(\psi, \tilde{w}) \).

### 5 ML Imputation

In this section, we propose the ML imputation via \( h \)-likelihood.

**Definition 5.1.** With the canonical scale \( v_i = g(y_{\text{mis}, i}) \) and the canonical function \( \tilde{v}_i(\psi, D, x) \), the ML imputation gives imputed values

\[
\hat{y}_{\text{mis}, i} = g^{-1}(\hat{v}_i), \quad \hat{v}_i = \tilde{v}_i \left( \hat{\psi}, D, x \right).
\]

(17)

Theorem 3.3 implies that the MLE of a random parameter is a consistent estimator of the canonical function. Based on the ML imputation (17), we propose to use the estimator

\[
\bar{y}_{\text{ML}} = \frac{1}{n} \left( \sum_{i=1}^{n_{\text{obs}}} y_i + \sum_{i=n_{\text{obs}}+1}^{n} \hat{y}_{\text{mis}, i} \right)
\]
as an estimator of \( \eta = \text{E}(Y) \). If the canonical scale is unknown, the ML imputation based on the weak canonical scale can be used. Weak canonical scale always exists and is known. This scale gives the estimator of \( \eta \) as

\[
\hat{y}_{\text{ML}}^{\text{Lap}} = \frac{1}{n} \left( \sum_{i=1}^{n_{\text{obs}}} y_i + \sum_{i=n_{\text{obs}}+1}^{n} \hat{y}_{\text{mis},i}^{\text{Lap}} \right),
\]

where \( \hat{y}_{\text{mis}}^{\text{Lap}} = g^{-1}(\hat{w}) \), \( \hat{w} = \tilde{w}(\hat{\psi}_{\text{Lap},D,x}) \) and \( g = g_2 \circ g_1 \). From Theorem 3.1 and the definition of the weak canonical scale (16), we see that the canonical scale is a linear transformation of the weak canonical scale \( w \). Given \( \psi \), MLEs of random parameters are invariant with respect a linear transformation (Lee and Nelder, 2005) and

\[
\ell_p(\hat{\psi}_{\text{Lap},D;\psi}) - \ell_p(\psi_{\text{Lap},D;\psi}) = \ell_p(v_{\text{Lap},D;\psi}) - \hat{\ell}_p(v_{\text{Lap},D;\psi}).
\]

Thus, the ML imputation under weak canonical scale is valid in the sense that

\[
\hat{y}_{\text{mis}}^{\text{Lap}} - \hat{y}_{\text{mis}} = O_p \left( \left| \hat{\psi}_{\text{Lap}} - \hat{\psi} \right| \right),
\]

where \( \hat{\ell}_p(\psi_{\text{Lap},D;\psi}) = \log \hat{L}_p(\psi_{\text{Lap},D;\psi}) \) and \( \hat{L}_e(\psi_{\text{Lap},D;\psi}) = L_e(\psi,\psi_{\text{Lap},D;\psi})/\hat{L}_m(\psi) \). Recently, Han and Lee (2022) developed the enhanced Laplace approximation (ELA) to obtain the MLE \( \hat{\psi} \) generally. Thus, the ML imputation can be always implemented even when the canonical scale is not known by using a weak canonical scale from the ELA. Given the MLE \( \hat{\psi} \), all the results on the ML imputation in Section 3.2 hold.

Under missing at random (MAR) of Rubin (1976), the h-likelihood becomes

\[
h = \log f_{\theta}(y_{\text{obs}} | x) + \log f_{\theta}(y_{\text{mis}} | x) + \log f_{\rho}(\delta | x) + \log \left| \frac{\partial y_{\text{mis}}}{\partial v} \right|,
\]

where \( \theta \) is the parameter for the response model and \( \rho \) is the parameter associate with the missing mechanism. Under MAR assumption, the canonical function of \( v \) depends only on \( \theta \) and \( x \) to give ML imputed values \( \hat{y}_{\text{mis},i} = \tilde{y}_{\text{mis},i}(\hat{\theta},x_i), \tilde{y}_{\text{mis},i}(\theta,x_i) = g^{-1}(\tilde{v}_i(\theta,x_i)) \).
Example 5.1. Little and Rubin (2019) considered censored exponential model, where $y_{\text{com}} = (y_{\text{obs}}, y_{\text{mis}})$ are independent exponential random variables with mean $\theta$ and the missing mechanism is set to $\delta = I(Y \leq c)$ with known $c$. Here the missing mechanism is not ignorable and the complete-data likelihood is

$$
\ell_c (\theta, y_{\text{mis}}) = -n \log \theta - \frac{1}{\theta} \sum_{i=1}^{n_{\text{obs}}} y_i - \frac{1}{\theta} \sum_{i=n_{\text{obs}}+1}^{n} y_{\text{mis},i}.
$$

They noted that joint maximization of the complete-data likelihood provides nonsensical modes $(n_{\text{obs}} \bar{y}_{\text{obs}} + n_{\text{mis}}c)/n$ for $\theta$ and $c$ for $y_{\text{mis},i}$, where $\bar{y}_{\text{obs}} = \sum_{i=1}^{n_{\text{obs}}} y_i / n_{\text{obs}}$ is the sample mean based on the observed responses. Now we know that MLEs (modes) should be obtained from the h-likelihood. Yun et al. (2007) found the canonical scale $v_i = \log(y_{\text{mis},i} - c)$ to form the h-likelihood

$$
h = \ell_c (\theta, y_{\text{mis}}) + \log \left| \frac{\partial y_{\text{mis}}}{\partial v} \right| = -n \log \theta - \frac{1}{\theta} \sum_{i=1}^{n_{\text{obs}}} y_i + \sum_{i=n_{\text{obs}}+1}^{n} \left\{ -\frac{1}{\theta} (c + e^{v_i}) + v_i \right\}.
$$

The canonical function of $v$ is $\tilde{v}(\theta) = \log \theta$ which gives

$$
h \{ \theta, \tilde{v}(\theta) \} = -n_{\text{obs}} \log \theta - \frac{1}{\theta} \sum_{i=1}^{n_{\text{obs}}} y_i - \frac{n_{\text{mis}}c}{\theta} - n_{\text{mis}} = \ell_m(\theta) - n_{\text{mis}} \propto \ell_m(\theta).
$$

This gives the true MLE $\hat{\theta} = \bar{y}_{\text{obs}} + n_{\text{mis}}c/n_{\text{obs}}$ and the ML imputed values $\hat{y}_{\text{mis},i} = \hat{\theta} + c > c$ to lead that

$$
\bar{y}_{\text{ML}} = \frac{1}{n} \left( \sum_{i=1}^{n_{\text{obs}}} y_i + \sum_{i=n_{\text{obs}}+1}^{n} \hat{y}_{\text{mis},i} \right) = \hat{\theta}
$$

and $\hat{\text{var}}(\bar{y}_{\text{ML}}) = \hat{\text{var}}(\hat{\theta}) = \hat{\theta}^2 / n_{\text{obs}}$. Little and Rubin (2019) used the EM algorithm. With the E-step

$$
\text{E}_\theta(y_{\text{mis},i}|y_{\text{mis},i} > c) = \theta + c,
$$

the M-step gives

$$
\theta^{(t+1)} = \frac{1}{n} \left[ \sum_{i=1}^{n_{\text{obs}}} y_i + n_{\text{mis}} \{ \theta^{(t)} + c \} \right].
$$

Thus, the EM algorithm gives the identical MLE $\hat{\theta}$. But, the EM algorithm does not provide the variance estimator directly.
To examine the performance of the ML imputation, we set about 22% of responses as unobserved and compare three estimators $\bar{y}_{\text{com}} = \sum_{i=1}^{n} y_i/n$, $\bar{y}_{\text{obs}} = \sum_{i=1}^{n} \delta_i y_i/n_{\text{obs}}$, and $\bar{y}_{\text{ML}}$ using random samples from $\exp(2)$ distribution. The estimator $\bar{y}_{\text{com}}$ is considered as a benchmark since it cannot be used in practice. In Figure 2, it is shown that the proposed method works well. Moreover, $\bar{y}_{\text{obs}}$ shows a non-negligible bias in amount $n_{\text{mis}}c/n_{\text{obs}} \approx 0.86$ since the missing mechanism is not ignorable.

![Figure 2: Boxplots of estimators in exponential mean model with $c = 3$. Dotted line indicates the true value of $\eta$.](image)

6 Illustrative Examples

6.1 Normal Regression Model

Consider a normal regression model $Y|x \sim N(\beta_0 + \beta_1 x, \sigma^2)$ with response probability model $\logit\{P_\rho(\delta = 1|x)\} = \rho_0 + \rho_1 x + \rho_2 x^2$ under a MAR assumption. Here, $y_{\text{mis}}$-scale itself satisfies the Bartlett identities but it is canonical scale only for $(\beta_0, \beta_1)$. Thus, the joint maximization of $\ell_e(\theta, y_{\text{mis}})$ cannot give the MLE of $\sigma^2$, where $\theta = (\beta_0, \beta_1, \sigma^2)$. However, $v$-scale defined by $v_i = y_{\text{mis},i}/\sigma$ is the canonical scale with canonical function $\tilde{v}_i(\theta, x_i) = (\beta_0 + \beta_1 x_i)/\sigma$ for $i = n_{\text{obs}} + 1, \ldots, n$. Then, the canonical function of $y_{\text{mis}}$ is $\tilde{y}_{\text{mis},i}(\theta, x_i) = \beta_0 + \beta_1 x_i = E_\theta(y_{\text{mis},i}|x_i)$ and the ML imputed values are $\hat{y}_{\text{mis},i} = \hat{\beta}_0 + \hat{\beta}_1 x_i$. Moreover,

$$\hat{I}_{y_{\text{mis},i}y_{\text{mis},i}}^{-1} = \left(-\frac{\partial^2 h}{\partial y_{\text{mis},i}^2}\right)^{-1} \bigg|_{\theta = \hat{\theta}} = \hat{\sigma}^2 = \hat{\text{var}}(y_{\text{mis},i} | D, x).$$
Since
\[ \tilde{y}_{\text{mis},i}(\theta, x_i) = E_\theta(y_{\text{mis},i}|x_i), \]
the MLEs can also be obtained by the EM algorithm.

For a simulation study, we generate \( n = 100 \) and \( n = 500 \) samples with \( \theta = (1, 2, 1), \rho = (1, 2, 0.3) \) and \( x \sim U(-1, 1) \). From Figure 3, we can see that \( \bar{y}_{\text{obs}} \) is positively biased because the covariate \( x \) increases both \( E_\theta(Y|x) \) and \( P_\rho(\delta = 1|x) \). Also, the performance of \( \bar{y}_{\text{ML}} \) is almost same as \( \bar{y}_{\text{com}} \).

**Figure 3**: Boxplots of estimators in normal regression model. Dotted line indicates the true value of \( \eta \).

### 6.2 Exponential Regression Model

Consider an exponential regression model with mean \( E_\beta(Y|x) = \exp(\beta_0 + \beta_1 x) \), \( \beta = (\beta_0, \beta_1) \) and the MAR mechanism as the Example 6.1. In this example, \( v = \log y_{\text{mis}} \) scale is the canonical scale which also satisfies Bartlett identities by Corollary 4.1. Here the canonical function of \( y_{\text{mis},i} \) is \( \tilde{y}_{\text{mis},i} = \exp(\beta_0 + \beta_1 x_i) = E_\beta(y_{\text{mis},i}|x_i) \) and the ML imputed values are \( \hat{y}_{\text{mis},i} = \exp(\hat{\beta}_0 + \hat{\beta}_1 x_i) \). Moreover,

\[ \hat{I}^{-1}_{y_{\text{mis},i}y_{\text{mis},i}} = \left( -\frac{\partial^2 h}{\partial y_{\text{mis},i}^2} \right)^{-1}|_{\theta = \hat{\theta}} = \hat{\text{var}}(y_{\text{mis},i} | D, x). \]

Figure 4 shows simulation results with \( \beta \) and \( \rho \) being the same as in Example 6.1. Compared to \( \bar{y}_{\text{com}} \), \( \bar{y}_{\text{ML}} \) gives almost the same performances, whereas \( \bar{y}_{\text{obs}} \) is biased.
Figure 4: Boxplots of estimators in exponential regression model. Dotted line indicates the true value of $\eta$.

### 6.3 Tobit Regression Model

Suppose that responses are generated from the normal regression model in Example 6.1. In addition, missing data are created by $y_{\text{mis}} > c$ at a known censoring point $c$. The extended likelihood

$$
\ell_e(\theta, y_{\text{mis}}) = -\frac{n}{2} \log 2\pi\sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^{n_{\text{obs}}} (y_i - \tilde{x}_i^T \beta)^2 - \frac{1}{2\sigma^2} \sum_{i=n_{\text{obs}}+1}^{n} (y_{\text{mis},i} - \tilde{x}_i^T \beta)^2,
$$

where $\theta = (\beta, \sigma^2)$, $\beta = (\beta_0, \beta_1)$ and $\tilde{x} = (1, x)$. Here a $b$-scale

$$
b_i = g_1(y_{\text{mis},i}) = \log (y_{\text{mis},i} - c),
$$

satisfies Bartlett identities by Corollary 4.1 but it is not canonical. Now, consider a $w$-scale with $w_i = g_2(b_i) = \tilde{\Omega}^{0.5}_{b_i} b_i$ by (16). Then, we have the approximate MLE $\hat{\theta}^{\text{Lap}}$ and approximate ML imputed values $\hat{y}_{\text{mis}}^{\text{Lap}}$ by jointly maximizing $\ell_e(\theta, w)$. However, the exact marginal log-likelihood is available in Tobit regression model.

$$
\ell_m(\theta) = -\frac{n_{\text{obs}}}{2} \log \sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^{n_{\text{obs}}} (y_i - \tilde{x}_i^T \beta)^2 + \sum_{i=n_{\text{obs}}+1}^{n} \log \left\{ \Phi \left( \frac{\tilde{x}_i^T \beta - c}{\sigma} \right) \right\}.
$$

This means that explicit form of the predictive likelihood $L_e(b_i|y_{\text{obs}}; \theta)$ is available to give the canonical scale

$$
v_i = L_e \left( b_i \mid y_{\text{obs}}; \theta \right) b_i, \quad (18)
$$
where
\[
\tilde{b}_i = \log \left\{ \tilde{x}_i^T \beta - c + \sqrt{(\tilde{x}_i^T \beta - c)^2 + 4\sigma^2} \right\} - \log 2.
\]

Thus, all MLEs are computed directly by simple maximization of the h-likelihood.

In the simulation study, we examine the performance of ML imputations by using two estimators \( \bar{y}_{\text{ML}} \) using the MLE and \( \bar{y}_{\text{Lap}}^{\text{ML}} \) using the approximate MLE. From (18), we see that both \( b \) and \( w \) are linear transformations of \( v \). Thus, approximate ML imputation works well as approximate MLE does. Given MLE for fixed parameters, weak canonical scale gives an exact ML imputation.

For simulation, we set \( \theta = (1, 3, 1) \), \( c = 3 \) and \( x_i = -1 + 2i/n \) for \( i = 1, \ldots, n \). In Figure 5, we see that the difference between \( \bar{y}_{\text{ML}} \) and \( \bar{y}_{\text{Lap}}^{\text{ML}} \) is negligible because \( \hat{\theta} \) and \( \hat{\theta}_{\text{Lap}} \) are very close. Therefore, we can use the weak canonical scale and approximate MLE when canonical scale is unknown.

### 7 Conclusion

Firth (2006) and Meng (2009) raised two important reservations about the use of the h-likelihood. Firth (2006) noted that the linear predictor in HGLM may not be well-defined to form the h-likelihood. Lee et al. (2006) resolved his question by defining the canonical scale. Meng (2009) claimed the asymptotic theory for the prediction of the future data would be impossible because the consistency cannot be achieved for the predicted
values from the h-likelihood. In this paper, we have answered their queries on the h-likelihood in the context of imputation for missing data. Specifically, we have shown that prediction becomes an estimation of canonical function of the h-likelihood whose consistent estimation and asymptotic normality can be justifiable. We further showed that standard errors of prediction can be directly obtained from the h-likelihood.

Little and Rubin (2019) pointed out that the current h-likelihood procedure achieves the correct ML estimation by modifying h-likelihood. In this paper, we achieve the true ML approach via h-likelihood without any modification by reformulating the h-likelihood. We present the meaning of the canonical scale and canonical function in detail, which allow ML estimation of fixed parameters and ML imputation of random parameters, namely missing data. The Jacobian term is a key to finding the canonical scale.

The ML imputation using the h-likelihood estimates the conditional mode, rather than the conditional mean of the missing value. We call this conditional mode imputation the ML imputation for the random parameters. The h-likelihood used for ML imputation provides an efficient algorithm because resampling procedure for multiple imputations or expectation steps in EM algorithm is not compulsory.

References

Bayarri, M. J., DeGroot, M. H., and Kadane, J. B. (1988). Statistical Decision Theory and Related Topics IV. Vol. 1, eds S.S. Gupta and J.O. Berger. New York: Springer.

Berger, J. O. and Wolpert, R. (1984). The Likelihood Principle. Hayward: Institute of Mathematical Statistics Monograph Series.

Bjørnstad, J. F. (1996). On the generalization of the likelihood function and likelihood principle. Journal of the American Statistical Association, 91:791–806.

Breslow, N. E. and Lin, X. (1995). Bias correction in generalised linear mixed models with a single component of dispersion. Biometrika, 82:81–91.
Butler, R. W. (1986). Predictive likelihood inference with applications. Journal of the Royal Statistical Society: Series B, 48:1–38.

Dempster, A. P., Laird, N. M., and Rubin, D. B. (1977). Maximum likelihood from incomplete data via the EM algorithm. Journal of the Royal Statistical Society: Series B, 39:1–37.

Firth, D. (2006). Invited discussion (seconder of the vote of thanks) of ‘double hierarchical generalized linear models’ by lee and nelder. Journal of the Royal Statistical Society: Series C, 55:168–170.

Fisher, R. A. (1922). On the mathematical foundations of theoretical statistics. Philos. Trans. Roy. Soc. London Ser. A, 222:309–368.

Ha, I., Noh, M., Kim, J., and Lee, Y. (2019). frailtyHL: Frailty models via hierarchical likelihood. R package version 2.3.

Han, J. and Lee, Y. (2022). Enhanced laplace approximation. Manuscript prepared.

Henderson, C. R., Kempthorne, O., Searle, S. R., and Von Krosigk, C. M. (1959). The estimation of genetic and environmental trends from records subject to culling. Biometrics, 15:192–218.

Kim, J. K. (2011). Parametric fractional imputation for missing data analysis. Biometrika, 98:119–132.

Kim, J. K. and Shao, J. (2021). Statistical Methods for Handling Incomplete Data. CRC press, 2nd edition.

Kristensen, K., Nielsen, A., Berg, C. W., Skaug, H., and Bell, B. M. (2016). Tmb: Automatic differentiation and laplace approximation. Journal of Statistical Software, 70:1–21.

Lee, Y. and Kim, G. (2016). H-likelihood predictive intervals for unobservables. International Statistical Review, 84:487–505.
Lee, Y. and Kim, G. (2020). Properties of h-likelihood estimators in clustered data. International Statistical Review, 88:380–395.

Lee, Y. and Nelder, J. A. (1996). Hierarchical generalised linear models (with discussion). Journal of the Royal Statistical Society, Series B, 58:619–678.

Lee, Y. and Nelder, J. A. (2001). Hierarchical generalized linear models: a synthesis of generalised linear models, random effects models and structured dispersions. Biometrika, 88:987–1006.

Lee, Y. and Nelder, J. A. (2005). Likelihood for random-effect models (with discussion). Statistics and Operations Research Transactions, 29:141–164.

Lee, Y. and Nelder, J. A. (2006). Fitting via alternative random effect models. Statistics and Computing, 16:69–75.

Lee, Y. and Nelder, J. A. (2009). Likelihood inference for models with unobservables: Another view. Statistical Science, 24:255–269.

Lee, Y., Nelder, J. A., and Pawitan, Y. (2006). Generalized Linear Models with Random Effects: Unified Analysis via H-likelihood. Chapman & Hall/CRC, 1st edition.

Lee, Y., Nelder, J. A., and Pawitan, Y. (2017). Generalized Linear Models with Random Effects: Unified Analysis via H-likelihood. Chapman & Hall/CRC, 2nd edition.

Little, R. J. and Rubin, D. B. (2019). Statistical Analysis with Missing Data. John Wiley & Sons, 3rd edition.

Little, R. J. A. and Rubin, D. B. (2002). Statistical Analysis with Missing Data. John Wiley & Sons, 2nd edition.

Louis, T. A. (1982). Finding the observed information matrix when using the EM algorithm. Journal of the Royal Statistical Society: Series B, 44:226–233.
Meng, X. L. (1994). Multiple-imputation inferences with un congenial sources of input (with discussion). *Statistical Science, 9*:538–573.

Meng, X. L. (2009). Decoding the h-likelihood. *Statistical Science, 24*:280–293.

Molenberghs, G., Beunckens, C., and Kenward, M. G. (2008). Every missingness not at random has a missingness at random counterpart with equal fit. *Journal of the Royal Statistical Society: Series B, 70*:371–388.

Neyman, J. and Scott, E. (1948). Consistent estimates based on partially consistent observations. *Econometrica, 16*:1–32.

Paik, C. M., Lee, Y., and Ha, I. (2015). Frequentist inference on random effects based on summarizability. *Statistica Sinica, 25*:1107–1132.

Raudenbush, S. W., Yang, M., and Yosef, M. (2000). Maximum likelihood for generalized linear models with nested random effects via high-order, multivariate laplace approximation. *Journal of Computational and Graphical Statistics, 9*:141–157.

Robins, J. M. and Wang, N. (2000). Inference for imputation estimators. *Biometrika, 87*:113–124.

Rubin, D. B. (1976). Inference and missing data. *Biometrika, 63*(3):581–592.

Rubin, D. B. (1987). *Multiple Imputation for Nonresponse in Surveys*. New York: Wiley.

Schweder, T. and Hjort, N. L. (2016). *Confidence, Likelihood and Probability*. Statistical Inference with Confidence Distributions. Cambridge University Press.

Tierney, L. and Kadane, J. B. (1986). Accurate approximations for posterior moments and marginal densities. *Journal of the American Statistical Association, 81*:82–86.

Wang, N. and Robins, J. M. (1998). Large-sample theory for parametric multiple imputation procedures. *Biometrika, 85*:935–948.
A note on multiple imputation for method of moments estimation. *Biometrika,* 103:244–251.

Using h-likelihood for missing observations. *Biometrika,* 94:905–919.

**Appendix: Supplementary Materials for “Maximum Likelihood Imputation”**

**A1 Regularity Conditions**

In this paper, we assume the following regularity conditions in developing the proposed method.

(R1) Let $\psi_0 = \arg \max_\psi E_\psi \{\ell_m(\psi)\}$ be the true value of $\psi$. Here, the number of fixed parameters does not depend on $n_{\text{obs}}$. Then, the MLE $\hat{\psi} = \arg \max_\psi \ell_m(\psi)$ satisfies the asymptotic normality with mean $\psi_0$ and variance $I_0^{-1} = I^{-1}(\psi_0)$, where

$$I(\psi) = \lim_{n_{\text{obs}} \to \infty} \frac{1}{n_{\text{obs}}} \left( -\frac{\partial^2 \ell_m(\psi)}{\partial \psi \partial \psi^T} \right) \bigg|_{\psi = \psi_0}$$

is the expected Fisher information.

(R2) The support of missing values $\Omega_{y_{\text{mis}}} = \left\{ y_{\text{mis}} \in \mathbb{R}^{n_{\text{mis}}} : \prod_{i=n_{\text{obs}}+1}^{n} f_\psi (y_{\text{mis},i}, \delta_i = 0 \mid x_i) > 0 \right\} \subset \mathbb{R}^{n_{\text{mis}}}$ does not depend on fixed parameter $\psi$.

**A2 Proofs**

**A2.1 Proof of Theorem 3.1**

*Proof.* By assumption, there exists $\tilde{y}_{\text{mis}} = \arg \max_{y_{\text{mis}}} \ell_e(\psi, y_{\text{mis}})$. Now, consider a $v$-scale defined by

$$v_i = g(y_{\text{mis},i}) = \{L_p (\tilde{y}_{\text{mis}} \mid D; \psi)\}^{1/n_{\text{mis}}} y_{\text{mis},i}, \quad i = n_{\text{obs}} + 1, \ldots, n,$$
with the predictive likelihood $L_p(y_{mis} \mid D; \psi) = f_\psi(y_{mis} \mid D, x)$. Here, the transformation $g(\cdot)$ is bijective and differentiable since it is linear. The predictive likelihood on $v$-scale is also well-defined with the Jacobian term

$$L_p(v \mid D; \psi) = L_p(y_{mis} \mid D; \psi) \left| \frac{\partial y_{mis}}{\partial v} \right|_{v = \tilde{v}} = L_p(y_{mis} \mid D; \psi),$$

where $\tilde{v}_i = g(\tilde{y}_{mis,i})$. Note that $\tilde{v}$ is also the mode of $L_p(v \mid D; \psi)$ since $\tilde{y}_{mis}$ is the mode of $L_p(y_{mis} \mid D; \psi)$ and the transformation $g(\cdot)$ is linear. Therefore, there exists a canonical scale which satisfies (12). \hfill \Box

### A2.2 Proof of Theorem 3.2

**Proof.** Let $v$-scale be the canonical scale and $\tilde{v} = \tilde{v}(\psi, D, x)$. Then, the h log-likelihood can be written as

$$h(\psi, \tilde{v}) = \ell_m(\psi) + c,$$

where $c$ is a constant which is free of $\psi$. Then, we can prove the first equality

$$\frac{\partial}{\partial \psi} h(\psi, \tilde{v}) = \frac{\partial h}{\partial \psi} \bigg|_{v = \tilde{v}} + \tilde{v}^T \frac{\partial h}{\partial \psi} \bigg|_{v = \tilde{v}} = \frac{\partial h}{\partial \psi} \bigg|_{v = \tilde{v}} = \frac{\partial \ell_m}{\partial \psi},$$

where $h = h(\psi, v)$ and $\ell_m = \ell_m(\psi)$. To show the second equality, recall that

$$\frac{\partial h}{\partial v} \bigg|_{v = \tilde{v}} = 0. \quad (19)$$

By differenciating (19) with respect to $\psi$,

$$\frac{\partial^2 h}{\partial \psi \partial v^T} \bigg|_{v = \tilde{v}} + \frac{\partial \tilde{v}^T}{\partial \psi} \left\{ \frac{\partial^2 h}{\partial v \partial v^T} \right\} \bigg|_{v = \tilde{v}} = O. \quad \Rightarrow \quad \frac{\partial \tilde{v}^T}{\partial \psi} = -\frac{\partial \ell_m}{\partial \psi} = I_{\psi v} I_{v v}^{-1} \bigg|_{v = \tilde{v}}. \quad (20)$$

Therefore, from (20), we can prove the required result.

$$\frac{\partial \ell_m}{\partial \psi} = \frac{\partial}{\partial \psi} h(\psi, \tilde{v}).$$

$$\Rightarrow -\frac{\partial^2 \ell_m}{\partial \psi \partial \psi^T} = -\frac{\partial^2 h}{\partial \psi \partial \psi^T} \bigg|_{v = \tilde{v}} - \frac{\partial \tilde{v}^T}{\partial \psi} \frac{\partial^2 h}{\partial v \partial \psi^T} \bigg|_{v = \tilde{v}} = I_{\psi v} - I_{\psi v} I_{v v}^{-1} I_{v \psi} = (I_{\psi v})^{-1}.$$
Here,
\[
\begin{pmatrix}
I \psi \psi & I \psi v \\
I \psi v & I v v
\end{pmatrix} = \begin{pmatrix}
I \psi \psi & I \psi v \\
I \psi v & I v v
\end{pmatrix}^{-1},
\quad
\begin{pmatrix}
I \psi \psi & I \psi v \\
I \psi v & I v v
\end{pmatrix} = \begin{pmatrix}
-\partial^2 h / \partial \psi \partial \psi^T & -\partial^2 h / \partial \psi \partial v^T \\
-\partial^2 h / \partial v \partial \psi^T & -\partial^2 h / \partial v \partial v^T
\end{pmatrix}_{v=\tilde{v}}.
\]

A2.3 Proof of Corollary 4.1

Proof. It suffices to show that the case \(n_{mis} = 1\). If \(E_\psi(b) < \infty\), then
\[
\lim_{|b| \to \infty} b f_\psi(b, \delta = 0 | x) = 0 \Rightarrow \lim_{|b| \to \infty} f_\psi(b, \delta = 0 | x) = 0.
\]
Since \(f_\psi\) is continuous, \(f_\psi(b, \delta = 0 | x) = 0\) for \(b \in \partial \Omega_b = \{-\infty, \infty\}\). Moreover, \(f_\psi\) is bounded since \(f_\psi\) is a density function of a continuous random variable whose support is the whole real line with finite mean. This guarantees that \(f_\psi\) is uniformly continuous which implies
\[
\lim_{|b| \to \infty} f'_\psi(b, \delta = 0 | x) = \lim_{|b| \to \infty} \lim_{t \to 0} f_\psi(b + t, \delta = 0 | x) - f_\psi(b, \delta = 0 | x) = 0,
\]
i.e., \(f'_\psi(b, \delta = 0 | x) = 0\) for \(b \in \partial \Omega_b = \{-\infty, \infty\}\). Then, the first and second Bartlett identities hold by the result of Theorem 4.1.

A2.4 Score and Hessian of \(\hat{\ell}_m(\psi)\) and \(\ell_e(\psi, \tilde{w})\)

By the definition of \(\hat{\ell}_m(\psi)\), the score and Hessian can be expressed as
\[
\begin{align*}
\frac{\partial}{\partial \psi_j} \hat{\ell}_m(\psi) &= \frac{\partial}{\partial \psi_j} \ell_e(\psi, b)|_{b=\tilde{b}} - \frac{1}{2} \text{tr} \left\{ \left( I^b_{bb} \right)^{-1} \left( \frac{\partial}{\partial \psi_j} I^b_{bb} \right) \right\}, \\
-\frac{\partial^2}{\partial \psi_j \partial \psi_k} \hat{\ell}_m(\psi) &= I^b_{\psi_j \psi_k} - I^b_{\psi_j \psi_k} (I^b_{bb})^{-1} I^b_{\psi_k \psi_k} \\
&\quad + \frac{1}{2} \text{tr} \left\{ \left( I^b_{bb} \right)^{-1} \left( \frac{\partial^2}{\partial \psi_j \partial \psi_k} - I^b_{bb} \right)^{-1} \left( \frac{\partial}{\partial \psi_k} I^b_{bb} \right)^{-1} \left( \frac{\partial}{\partial \psi_k} I^b_{bb} \right)^{-1} \left( \frac{\partial}{\partial \psi_k} I^b_{bb} \right)^{-1} \left( \frac{\partial}{\partial \psi_k} I^b_{bb} \right)^{-1} \right\}.
\end{align*}
\]
for \(1 \leq j, k \leq p\), where \(I^b_{xy} = \left. - \frac{\partial^2}{\partial x \partial y} \ell_e(\psi, b) \right|_{b = \tilde{b}}\). On the other hand, with \(\ell_e = \ell_e(\psi, w)\),

\[
\frac{\partial}{\partial \psi} \ell_e(\psi, \tilde{w}) = \left. \frac{\partial \ell_e}{\partial \psi} \right|_{w = \tilde{w}},
\]

\[
\left\{ - \frac{\partial^2}{\partial \psi \partial \psi^T} \ell_e(\psi, \tilde{w}) \right\}^{-1} = I^\psi_e,
\]

where

\[
\begin{pmatrix}
I_{e,\psi} & I_{e,\psi w} \\
I_{e,\psi w} & I_{e,w}
\end{pmatrix} = \begin{pmatrix}
I_{e,\psi} & I_{e,\psi w} \\
I_{e,\psi w} & I_{e,w}
\end{pmatrix}^{-1},
\]

\[
\begin{pmatrix}
I_{e,\psi} & I_{e,\psi w} \\
I_{e,\psi w} & I_{e,w}
\end{pmatrix} = \begin{pmatrix}
- \frac{\partial^2 \ell_e}{\partial \psi \partial \psi^T} & - \frac{\partial^2 \ell_e}{\partial \psi \partial w^T} \\
- \frac{\partial^2 \ell_e}{\partial w \partial \psi^T} & - \frac{\partial^2 \ell_e}{\partial w \partial w^T}
\end{pmatrix}_{w = \tilde{w}}.
\]