I. INTRODUCTION

Among the theories introduced to describe the late-time acceleration of the universe, the modified-gravity paradigm has attracted much interest, because it explicitly states that the reason for the acceleration of the universe is due to a modified gravity law which is mostly felt at very large scales. The exploration of different ways of modifying gravity have started since the pioneeristic works in the so-called $f(R)$ gravity. Many other theories have been proposed since then. Among others, let us mention a few of them here: the extension of $f(R)$ theories to $f(R,G)$ theories where $G$ stands for the Gauss-Bonnet term, the DGP model motivated by the possible existence of spatial extra-dimensions, Galileon theories and general scalar-tensor theories of the Horndeski Lagrangian with second order differential equations. All these theories generalize the Einstein-Hilbert Lagrangian by introducing second order Lagrangians (or to macroscopic gravity). Among them, let us mention a few of them here: the extension of $f(R)$ theories to $f(R,G)$ theories where $G$ stands for the Gauss-Bonnet term, the DGP model motivated by the possible existence of spatial extra-dimensions, Galileon theories and general scalar-tensor theories of the Horndeski Lagrangian with second order differential equations. All these theories generalize the Einstein-Hilbert Lagrangian by introducing second order Lagrangians (or to macroscopic gravity).

Some people take the point of view (see e.g. [4]), that, in order to make it sensible, the non-local Lagrangian (1) as equivalent to another, local Lagrangian which can be derived by introducing auxiliary scalar fields, and study the degrees of freedom in the theory. In all these subcases we find a finite number of ghost degrees of freedom except for the $n = 1$ case. These ghosts are unavoidable, in the sense that they cannot be gauged away. Therefore their presence would make these models, in general, unviable, unless one tunes the mass of these modes to values larger than the cut-off of the theory.

This paper is organized as follows. In Sec. II we rewrite the general non-local Lagrangian in the form of a localized Lagrangian as given by Eq. (2) and analyze its physical degrees of freedom. In Sec. III, we focus on a special case where the Lagrangian is linear in the Ricci scalar, that is, the case $\partial^2 f/\partial \sigma^2 = 0$ in Eq. (2). Section IV is devoted to conclusion and discussions.

\[ \mathcal{L} = \sqrt{-g} f(R, \Box^{-1} R, \cdots, \Box^{-n} R), \quad \text{with} \quad n < +\infty, \quad (1) \]

and we will try to understand its content. The meaning of such terms in the Lagrangian is obscure, and not very well understood. Some people take the point of view (see e.g. [4]), that, in order to make it sensible, the $\Box^{-1}$ operator, must be replaced by the operator $\Box^{-1}_{\text{ret}}$ where the subscript “ret” means the retarded boundary condition. This point of view is non-standard in the conventional context of variational calculus where setting initial data is a defining constituent of a theory at level of the action. Further it is unconformable to the usual quantization procedure known for known theories based on the Lagrangian formalism.

In this paper, we take a different approach. Pursuing the standard picture of classical/quantum field theory, we interpret the non-local Lagrangian (1) as equivalent to another, local Lagrangian which can be derived by introducing auxiliary fields. The resulting Lagrangian can be studied with the usual tools of field theory. Namely we consider the Lagrangian,

\[ \mathcal{L} = \sqrt{-g} \left[ f(\sigma, U_1, U_2, \cdots, U_n) + \frac{\partial f}{\partial \sigma} (R - \sigma) + \lambda_1 (R - \Box U_1) + \lambda_2 (U_1 - \Box U_2) + \cdots + \lambda_n (U_{n-1} - \Box U_n) \right]. \quad (2) \]
II. GENERAL NON-LOCAL GRAVITY ACTION

Let us consider the general action,
\[ S = \int d^4x \sqrt{-g} f; \]
\[ f = f_1(R, \Box^{-1}R, \Box^{-2}R, \cdots, \Box^{-n}R) + f_2(\Box^{-1}R, \Box^{-2}R, \cdots, \Box^{-m}R), \tag{3} \]
where \( f \) is a general function of \( \Box^{-k}R \) (\( k = 0, 1, 2, \cdots, \max(n, m) \)), where \( n \) and \( m \) are positive integers, i.e. \( 1 \leq (m, n) < \infty \), and the function \( f_1 \) is chosen by the condition that it satisfies
\[ \frac{\partial^2 f}{\partial R \partial (\Box^{-n}R)} = \frac{\partial^2 f_1}{\partial R \partial (\Box^{-n}R)} \neq 0. \tag{4} \]
Thus \( n \) is the largest integer for which this inequality holds. Note that the choice of \( f_1 \) is not unique, given the function \( f \), but this ambiguity does not affect our discussion below.

As already mentioned in the Introduction, on allowing ourselves to interpret the action (1) as a model which can be redefined in terms of a local action (without e.g. assuming the d’Alambertian operators restricted on particular or prior-given boundary conditions, which would result in considering different theories), we can rewrite the action as
\[ S_{m \leq n} = \int d^4x \sqrt{-g} \left[ f_1(\sigma, U_1, U_2, \cdots, U_n) + \frac{\partial f_1}{\partial \sigma} (R - \sigma) + f_2(U_1, \cdots, U_m) \right. \]
\[ + \lambda_1(R - \Box U_1) + \lambda_2(U_1 - \Box U_2) + \cdots + \lambda_n(U_{n-1} - \Box U_n) \]
\[ + \lambda_1(R - \Box U_1) + \lambda_2(U_1 - \Box U_2) + \cdots + \lambda_n(U_{n-1} - \Box U_n) + \cdots + \lambda_m(U_{m-1} - \Box U_m) \right], \tag{5} \]
or
\[ S_{m > n} = \int d^4x \sqrt{-g} \left[ f_1(\sigma, U_1, U_2, \cdots, U_n) + \frac{\partial f_1}{\partial \sigma} (R - \sigma) + f_2(U_1, \cdots, U_m) \right. \]
\[ + \lambda_1(R - \Box U_1) + \lambda_2(U_1 - \Box U_2) + \cdots + \lambda_n(U_{n-1} - \Box U_n) + \cdots + \lambda_m(U_{m-1} - \Box U_m) \right]. \tag{6} \]
On taking the equations of motion for the fields \( \sigma \), and \( \lambda_i \) (\( i = 1, \cdots, n \)), we find
\[ \frac{\partial^2 f_1}{\partial \sigma^2} (R - \sigma) = 0, \tag{7} \]
\[ R = \Box U_1, \tag{8} \]
\[ U_1 = \Box U_2, \tag{9} \]
\[ \cdots \]
\[ U_{n-1} = \Box U_n, \tag{10} \]
for \( m \leq n \), and the additional equations,
\[ U_n = \Box U_{n+1}, \tag{11} \]
\[ \cdots \]
\[ U_{m-1} = \Box U_m, \tag{12} \]
for \( m > n \). Therefore provided that \( \partial^2 f_1/\partial \sigma^2 \neq 0 \), we obtain
\[ \sigma = R, \tag{13} \]
\[ U_1 = \Box^{-1}R, \tag{14} \]
\[ U_2 = \Box^{-1}U_1 = \Box^{-2}R, \tag{15} \]
\[ \cdots \]
\[ U_n = \Box^{-1}U_{n-1} = \Box^{-n}R, \tag{16} \]
for \( m \leq n \), and additionally
\[
U_{n+1} = \Box^{-1}U_n = \Box^{-n-1}R,
\]
\[
\ldots
\]
\[
U_m = \Box^{-1}U_{m-1} = \Box^{-m}R,
\]
for \( m > n \). We regard the original non-local Lagrangian (3) as equivalent to the new one, (5) or (6).

The importance of the new action, (5) or (6), is that it is now clear how many degrees of freedom are present, and their scalar nature. In fact, we can rewrite them as
\[
S_{m \leq n} = \int d^4x \sqrt{-g} \left[ \left( \frac{\partial f_1}{\partial \sigma} + \lambda_1 \right) R + \frac{g^{\alpha\beta} (\partial_\alpha \lambda_1 \partial_\beta U_1 + \partial_\alpha \lambda_2 \partial_\beta U_2 + \cdots + \partial_\alpha \lambda_n \partial_\beta U_n) + f_1(\sigma, U_1, U_2, \ldots, U_n) - \sigma \frac{\partial f_1}{\partial \sigma} + \lambda_2 U_1 + \cdots + \lambda_n U_{n-1} + f_2(U_1, \cdots, U_m) \right],
\]
and
\[
S_{m > n} = \int d^4x \sqrt{-g} \left[ \left( \frac{\partial f_1}{\partial \sigma} + \lambda_1 \right) R + \frac{g^{\alpha\beta} (\partial_\alpha \lambda_1 \partial_\beta U_1 + \partial_\alpha \lambda_2 \partial_\beta U_2 + \cdots + \partial_\alpha \lambda_m \partial_\beta U_m) + f_1(\sigma, U_1, U_2, \ldots, U_n) - \sigma \frac{\partial f_1}{\partial \sigma} + \lambda_2 U_1 + \cdots + \lambda_m U_{m-1} + f_2(U_1, \cdots, U_m) \right].
\]
Let us make a field redefinition as
\[
\frac{\partial f_1}{\partial \sigma} + \lambda_1 = \Phi,
\]
which can be solved for \( U_n \) provided
\[
\frac{\partial^2 f_1}{\partial \sigma \partial U_n} \neq 0.
\]
which is guaranteed by definition, as given by Eq. (4). Notice that Eq. (22), or, in our approach, its equivalent form (4), excludes General Relativity in this class of theories. Therefore the set of theories considered here, are those ones for which it is possible to solve Eq. (21) in terms of the field \( U_n \). In fact, the field \( U_n \) becomes a function of the other \( n+2 \) fields as
\[
U_n = U_n(\sigma, U_j, \Phi - \lambda_1); \quad j = 1, \ldots, n-1.
\]
In this section, we also assume \( \frac{\partial^2 f_1}{\partial \sigma^2} \neq 0 \). The particular case \( \frac{\partial^2 f_1}{\partial \sigma^2} = 0 \) will be discussed separately in the next section.

We find, on differentiating the constraint (21), that
\[
\frac{\partial^2 f_1}{\partial \sigma^2} d\sigma + \frac{\partial^2 f_1}{\partial \sigma \partial U_j} dU_j + \frac{\partial^2 f_1}{\partial \sigma \partial U_n} dU_n + d\lambda_1 - d\Phi = 0.
\]
Recalling that \( U_n \) is a function of the other \( n+2 \) fields, we may rewrite the above as
\[
\left( \frac{\partial^2 f_1}{\partial \sigma^2} + \frac{\partial^3 f_1}{\partial \sigma \partial U_n} \right) d\sigma + \left( \frac{\partial^2 f_1}{\partial \sigma \partial U_j} + \frac{\partial^3 f_1}{\partial \sigma \partial U_n} \right) dU_j + \left( \frac{\partial^2 f_1}{\partial \sigma \partial U_n} \right) d\lambda_1 + \left( \frac{\partial^2 f_1}{\partial \sigma \partial U_n} \right) d\Phi = 0.
\]
This constraint has solution for
\[
\frac{\partial U_n}{\partial \sigma} = -\frac{\partial^2 f_1}{\partial \sigma \partial U_n} = -\frac{f_{,\sigma \sigma}}{f_{,\sigma U_n}},
\]
\[
\frac{\partial U_n}{\partial U_j} = -\frac{\partial^2 f_1}{\partial \sigma \partial U_n} = -\frac{f_{,\sigma U_n}}{f_{,\sigma U_n}},
\]
\[
\frac{\partial U_n}{\partial U_j} = -\frac{\partial^2 f_1}{\partial \sigma \partial U_n} = -\frac{f_{,\sigma U_n}}{f_{,\sigma U_n}},
\]

\[
\frac{\partial U_n}{\partial \sigma} = -\frac{\partial^2 f_1}{\partial \sigma \partial U_n} = -\frac{f_{,\sigma \sigma}}{f_{,\sigma U_n}};
\]
\[
\frac{\partial U_n}{\partial U_j} = -\frac{\partial^2 f_1}{\partial \sigma \partial U_n} = -\frac{f_{,\sigma U_n}}{f_{,\sigma U_n}};
\]
\[
\frac{\partial U_n}{\partial U_j} = -\frac{\partial^2 f_1}{\partial \sigma \partial U_n} = -\frac{f_{,\sigma U_n}}{f_{,\sigma U_n}}.
\]
\[
\begin{align*}
\frac{\partial U_n}{\partial \lambda_1} &= -\frac{\sqrt{f}}{\sqrt{\sigma f} \partial U_n} = -\frac{1}{f \sigma U_n}, \\
\frac{\partial U_n}{\partial \Phi} &= \frac{1}{\sqrt{\sigma f} \partial U_n} = \frac{1}{f \sigma U_n},
\end{align*}
\]

where we have replaced \( f_1 \) by \( f \) for notational simplicity, which is allowed because \( \partial f_1/\partial \sigma = \partial f/\partial \sigma \) by definition.

Using the above result, the action is further rewritten as

\[
S_{m \leq n} = \int d^4 x \sqrt{-g} \left[ \Phi R + g^{\alpha \beta} \left( \partial_\alpha \lambda_1 \partial_\beta U_1 + \partial_\alpha \lambda_2 \partial_\beta U_2 + \cdots + \partial_\alpha \lambda_{n-1} \partial_\beta U_{n-1} \right) \right. \\
+ g^{\alpha \beta} \partial_\alpha \lambda_n \left( \frac{1}{f \sigma U_n} \partial_\beta \Phi - \frac{f \sigma U_n}{f \sigma U_n} \partial_\beta \sigma - \frac{f \sigma U_n}{f \sigma U_n} \partial_\beta U_j - \frac{1}{f \sigma U_n} \partial_\beta \lambda_1 \right) \\
+ f_1(U_1, \ldots, U_n) \left. - \frac{\partial f_1}{\partial \sigma} \sigma + \lambda_2 U_1 + \cdots + \lambda_n U_{n-1} + f_2(U_1, \cdots, U_m) \right],
\]

or

\[
S_{m > n} = \int d^4 x \sqrt{-g} \left[ \Phi R + g^{\alpha \beta} \left( \partial_\alpha \lambda_1 \partial_\beta U_1 + \partial_\alpha \lambda_2 \partial_\beta U_2 + \cdots + \partial_\alpha \lambda_{n-1} \partial_\beta U_{n-1} + \partial_\alpha \lambda_{n+1} \partial_\beta U_{n+1} + \cdots + \partial_\alpha \lambda_m \partial_\beta U_m \right) \right. \\
+ g^{\alpha \beta} \partial_\alpha \lambda_n \left( \frac{1}{f \sigma U_n} \partial_\beta \Phi - \frac{f \sigma U_n}{f \sigma U_n} \partial_\beta \sigma - \frac{f \sigma U_n}{f \sigma U_n} \partial_\beta U_j - \frac{1}{f \sigma U_n} \partial_\beta \lambda_1 \right) \\
+ f_1(U_1, \ldots, U_n) \left. - \frac{\partial f_1}{\partial \sigma} \sigma + \lambda_2 U_1 + \cdots + \lambda_m U_{m-1} + f_2(U_1, \cdots, U_m) \right].
\]

On performing the following conformal transformation

\[
\bar{g}_{\alpha \beta} = \xi g_{\alpha \beta}, \\
\sqrt{-\bar{g}} = \sqrt{-g} \xi^2, \\
R = \xi \left[ \bar{R} + 3 \Box \ln \xi - \frac{3}{\xi^2} g^{\alpha \beta} \partial_\alpha \xi \partial_\beta \xi \right],
\]

where

\[
\xi = \frac{2 \Phi}{M_{Pl}^2},
\]

we can see that the new action, for \( m \leq n \), becomes, up to a total derivative,

\[
S_{m \leq n} = \int d^4 x \sqrt{-g} \left[ \frac{M_{Pl}^2}{2} \bar{R} - \frac{3 M_{Pl}^2}{4 \Phi^2} \bar{g}^{\alpha \beta} \partial_\alpha \Phi \partial_\beta \Phi + \frac{M_{Pl}^2 \bar{g}^{\alpha \beta}}{2 \Phi} \partial_\alpha \lambda_j \partial_\beta U_j - V \right. \\
+ \frac{M_{Pl}^2 \bar{g}^{\alpha \beta}}{2 \Phi} \partial_\alpha \lambda_n \left( \frac{1}{f \sigma U_n} \partial_\beta \Phi - \frac{f \sigma U_n}{f \sigma U_n} \partial_\beta \sigma - \frac{f \sigma U_n}{f \sigma U_n} \partial_\beta U_j - \frac{1}{f \sigma U_n} \partial_\beta \lambda_1 \right) \right],
\]

where \( j = 1, \ldots, n-1 \), and

\[
V = -\frac{M_{Pl}^4}{4 \Phi^2} \left[ f_1 - \frac{\partial f_1}{\partial \sigma} \sigma + \lambda_2 U_1 + \cdots + \lambda_n U_{n-1} + f_2 \right].
\]

On the other hand, for \( m > n \), we have

\[
S_{m > n} = \int d^4 x \sqrt{-g} \left[ \frac{M_{Pl}^2}{2} \bar{R} - \frac{3 M_{Pl}^2}{4 \Phi^2} \bar{g}^{\alpha \beta} \partial_\alpha \Phi \partial_\beta \Phi + \frac{M_{Pl}^2 \bar{g}^{\alpha \beta}}{2 \Phi} \partial_\alpha \lambda_j \partial_\beta U_j + \frac{M_{Pl}^2 \bar{g}^{\alpha \beta}}{2 \Phi} \partial_\alpha \lambda_k \partial_\beta U_k \right. \\
+ \frac{M_{Pl}^2 \bar{g}^{\alpha \beta}}{2 \Phi} \partial_\alpha \lambda_n \left( \frac{1}{f \sigma U_n} \partial_\beta \Phi - \frac{f \sigma U_n}{f \sigma U_n} \partial_\beta \sigma - \frac{f \sigma U_n}{f \sigma U_n} \partial_\beta U_j - \frac{1}{f \sigma U_n} \partial_\beta \lambda_1 \right) \right] \right],
\]

where \( j = 1, \ldots, n-1, k = n+1, \ldots, m \) and

\[
V = -\frac{M_{Pl}^4}{4 \Phi^2} \left[ f_1 - \frac{\partial f_1}{\partial \sigma} \sigma + \lambda_2 U_1 + \cdots + \lambda_m U_{m-1} + f_2 \right].
\]

In the following, we will consider the two cases, \( m \leq n \) and \( m > n \), separately.
A. Case $m \leq n$

Let us make a further field redefinition by constant rescaling as

$$\Phi = M_{Pl} q_1,$$  \hspace{1cm} (41)
$$\sigma = M_{Pl} q_2,$$  \hspace{1cm} (42)
$$U_j = \frac{q_j + 2}{M_{Pl}^2} \quad (j = 1, \cdots, n - 1), \quad U_n = \frac{u_n}{M_{Pl}^{2n}},$$  \hspace{1cm} (43)
$$\lambda_i = M_{Pl}^{2i-1} q_{n+1+i} \quad (i = 1, \cdots, n),$$  \hspace{1cm} (44)
$$f = M_{Pl} f, \quad f_{\alpha U_n} = M_{Pl}^2 f_{q_2 u_n},$$  \hspace{1cm} (45)
$$f_{,\alpha} = \tilde{f}_{q_2 q_3}, \quad f_{,\alpha U_j} = M_{Pl}^2 f_{q_2 q_3+2},$$  \hspace{1cm} (46)

where we have included the rescaling of $U_n$ although it is not an independent field. Notice also that this rescaling is not necessary for the function $f_2$ as this quantity only enters in the definition of the potential, i.e. it does not affect the kinetic term of any of the fields. Thus in total we have $2n+1$ fields that we have named $q_i$ where $i = 1, \cdots, 2n+1$. Then the action takes the following form:

$$S = \int d^4x \sqrt{-g} \left[ \frac{M_{Pl}^2}{2} R - \frac{1}{2} \bar{g}^{\alpha \beta} G^{\alpha \beta} \partial_\alpha q_\beta q_i - V \right],$$  \hspace{1cm} (47)

where $(k,l) = 1, \cdots, 2n+1$, and the kinetic-term metric $G^{\alpha \beta}$ is a field-dependent symmetric matrix $G^{\alpha \beta}$ whose only non-zero elements are

$$G^{11} = \frac{3M_{Pl}^2}{2q_1^2}, \quad G^{1,2n+1} = \frac{M_{Pl}}{2q_1 f_{q_2 u_n}}, \quad G^{2,2n+1} = \frac{M_{Pl} f_{q_2 u_n}}{2q_1 f_{q_2 u_n}},$$  \hspace{1cm} (48)
$$G^{j+2,j+n+1} = - \frac{M_{Pl} f_{q_2 q_3+2}}{2q_1 f_{q_2 u_n}}, \quad G^{j+2,n+1} = \frac{M_{Pl} f_{q_2 q_3+2}}{2q_1 f_{q_2 u_n}},$$  \hspace{1cm} (49)

Let us analyze the kinetic matrix $G$. In order to examine whether the fields $q$ have positive kinetic terms, we need to study whether $G$ is positive-definite or not. For this purpose, we notice that it enters in the Lagrangian in the form of $\mathcal{L} \equiv v^T \cdot G \cdot v$ where $v$ is a $(2n+1)$-dimensional vector. Hence it suffices to look for a linear transformation of $v$ in the form $v = A \cdot w$ which diagonalizes the matrix $G$. Such a transformation is found as

$$v_1 = w_1 + \frac{q_1 f_{q_2 u_n}}{3M_{Pl} f_{q_2 q_3} - q_1} w_2 - \frac{q_1 f_{q_2 q_3}}{6M_{Pl} f_{q_2 q_3} - q_1} w_{2n+1},$$  \hspace{1cm} (50)
$$v_2 = w_{2n+1},$$  \hspace{1cm} (51)
$$v_{j+2} = \frac{w_{2j+1}}{2} w_{2j+1} + \frac{\delta_{j+1,1}}{f_{q_2 u_n}} w_{2n} - \frac{3M_{Pl} f_{q_2 q_3} \delta_{j+1,1}}{6M_{Pl} f_{q_2 q_3} - q_1} w_{2n+1},$$  \hspace{1cm} (52)
$$v_{j+n+1} = \frac{w_{2j+1}}{2} w_{2j+1} + \frac{\tilde{f}_{q_2 q_3+2}}{f_{q_2 u_n}} w_{2n} - \frac{3M_{Pl} f_{q_2 q_3+2} \tilde{f}_{q_2 q_3+2}}{6M_{Pl} f_{q_2 q_3} - q_1} w_{2n+1},$$  \hspace{1cm} (53)
$$v_{2n+1} = \frac{w_{3M_{Pl} f_{q_2 q_3} \tilde{f}_{q_2 u_n}} - q_1}{6M_{Pl} f_{q_2 q_3} - q_1} w_{2n+1}.$$

Then the new kinetic matrix $\tilde{G} = A^T \cdot G \cdot A$ becomes diagonal with the elements given by

$$\tilde{G}^{11} = \frac{3M_{Pl}^2}{2q_1^2}, \quad \tilde{G}^{2j,2j} = - \frac{M_{Pl}}{q_1}, \quad \tilde{G}^{2j+1,2j+1} = \frac{M_{Pl}}{4q_1},$$  \hspace{1cm} (55)
$$\tilde{G}^{2n+2n} = \frac{M_{Pl} f_{q_2 q_3} - q_1}{6q_1 f_{q_2 u_n}^2}, \quad \tilde{G}^{2n+1,2n+1} = - \frac{3M_{Pl} f_{q_2 q_3}^2}{2q_1 (6M_{Pl} f_{q_2 q_3} - q_1)}.$$

As clear from the above, for this theory, we conclude that there always exist $n$ ghosts independently of the sign of $q_1$.

B. Case $m > n$

In this case we perform the field redefinition,

$$\Phi = M_{Pl} q_1,$$  \hspace{1cm} (57)
\[ \sigma = M^{11} q_2, \]
\[ U_j = \frac{q_j + 2}{M^{2j-1}} (j = 1, \ldots, n-1), U_n = \frac{u_n}{M^{2n-1}}, \]
\[ U_r = \frac{q_{r+1}}{M^{2r-1}} (r = n+1, \ldots, m), \]
\[ \lambda_i = M^{2i-1} q_{m+1+i} (i = 1, \ldots, m), \]
\[ f = M^{2i} f_i, f_{,\sigma U_n} = M^{2i} f_{i, q_{2n}}; \]
\[ f_{,\sigma\sigma} = \tilde{f}_{i, q_{2i+2}}, f_{,\sigma U_i} = M^{2i} \tilde{f}_{i, q_{2i+2}}, \]

where again we have rescaled the dependent field \( U_n \) as well. Thus we have \( 2m + 1 \) fields in total, and the action takes the form,

\[ S = \int d^4x \sqrt{-g} \left[ \frac{M^2}{2} R - \frac{1}{2} g^{\alpha\beta} G^{kl} \partial_\alpha q_k \partial_\beta q_l - V \right], \]

where \((k,l) = 1, \ldots, 2m + 1\). The only non-zero elements of the kinetic-term metric \( G^{kl} \) are

\[ G^{11} = \frac{3M^2}{2q_1^2}, \quad G^{1,m+1} = -\frac{M}{2q_1 f_{i, q_{2u_n}}}, \quad G^{2, m+1} = \frac{M}{2q_1 f_{i, q_{2u_n}}}, \]
\[ G^{j+2, m+j+1} = \frac{M}{2q_1}, \quad G^{j+2, m+n+1} = \frac{M}{2q_1 f_{i, q_{2u_n}}}, \quad G^{m+2, m+n+1} = \frac{M}{2q_1 f_{i, q_{2u_n}}}, \]

where \( j = 1, \ldots, n-1 \) and \( r = n + 1, \ldots, m \).

Once again the kinetic matrix enters the Lagrangian in the form \( v^T \cdot G \cdot v \), and whether it is positive definite or not can be examined by diagonalizing the matrix by a transformation of the form \( v = A \cdot w \). We find that the transformation,

\[ v_1 = w_1 + \frac{q_1}{3M^{11} f_{i, q_{2u_n}}} w_{2m} - \frac{q_1 \tilde{f}_{i, q_{2u}}}{6M^{11} f_{i, q_{2u_n}} - q_1} w_{2m+1}, \]
\[ v_2 = w_{2m+1}, \]
\[ v_{j+2} = w_{2j} - \frac{1}{2} w_{2j+1} + \frac{\delta_{j, 1}}{f_{i, q_{2u_n}}} w_{2m} - \frac{3M^{11} \tilde{f}_{i, q_{2u}} \delta_{j, 1}}{6M^{11} f_{i, q_{2u_n}} - q_1} w_{2m+1}, \]
\[ v_{k+1} = w_{2k-2} - \frac{1}{2} w_{2k-1}, \]
\[ v_{m+j+1} = w_{2j} + \frac{1}{2} w_{2j+1} + \frac{\tilde{f}_{i, q_{2u_n}}}{f_{i, q_{2u_n}}} w_{2m} - \frac{3M^{11} \tilde{f}_{i, q_{2u}} \tilde{f}_{i, q_{2u_n}} + 2}{6M^{11} f_{i, q_{2u_n}} - q_1} w_{2m+1}, \]
\[ v_{m+n+1} = w_{2m} - \frac{3M^{11} \tilde{f}_{i, q_{2u}} \tilde{f}_{i, q_{2u_n}}}{6M^{11} f_{i, q_{2u_n}} - q_1} w_{2m+1}, \]
\[ v_{m+k+1} = w_{2k-2} + \frac{1}{2} w_{2k-1}, \]

diagonalizes the matrix \( G \). The new kinetic matrix \( \tilde{G} = A^T \cdot G \cdot A \) becomes diagonal with the elements given by

\[ \tilde{G}^{11} = \frac{3M^2}{2q_1^2}, \quad \tilde{G}^{2j, 2j} = -\frac{M}{q_1}, \quad \tilde{G}^{2j+1, 2j+1} = \frac{M}{4q_1}, \]
\[ \tilde{G}^{2k-2, 2k-2} = \frac{M}{q_1}, \quad \tilde{G}^{2k-1, 2k-1} = \frac{M}{4q_1}, \]
\[ \tilde{G}^{2m, 2m} = \frac{6M^{11} \tilde{f}_{i, q_{2u_n}} - q_1}{6q_1 f_{i, q_{2u_n}}}, \quad \tilde{G}^{2m+1, 2m+1} = -\frac{3M^2}{2q_1 (6M^{11} \tilde{f}_{i, q_{2u_n}} - q_1)}. \]

Therefore, similar to the previous case, there always exist \( m \) ghosts independently of the sign of \( q_1 \).

To summarize, for the Lagrangian of the form (3), there always exist \( \max(n, m) \) ghosts provided \( f_1 \) is nonlinear in the Ricci scalar \( R \).
C. Case \( n = 0 \)

In this case we want to discuss here the model described by the Lagrangian (6) where \( n = 0 \), that is the action can be written as

\[
S_{n=0} = \int d^4x \sqrt{-g} \left[ f_1(\sigma) + \frac{\partial f_1}{\partial \sigma} (R - \sigma) + f_2(U_1, \cdots, U_m) \right.
\]
\[
+ \lambda_1 (R - \Box U_1) + \lambda_2 (U_1 - \Box U_2) + \cdots + \lambda_m (U_{m-1} - \Box U_m) \left. \right],
\]

(78)

and we will assume

\[
\frac{\partial f_2}{\partial U_m} \neq 0, \quad \text{and} \quad \frac{\partial^2 f_1}{\partial \sigma^2} \neq 0.
\]

(79)

which can be described as a non-local term correction to an \( f_2(R) \) gravity theory. The term \( f_2 \) alone is known not to introduce any ghosts, provided that \( \partial f_1/\partial \sigma > 0 \), i.e. \( \partial f_1/\partial \sigma > 0 \). We can rewrite the action in the following form

\[
S_{n=0} = \int d^4x \sqrt{-g} \left[ \left( \lambda_1 + \frac{\partial f_2}{\partial \sigma} \right) R + f_1 + f_2 - \frac{\partial f_2}{\partial \sigma} \sigma - \lambda_1 \Box U_1 + \lambda_2 (U_1 - \Box U_2) + \cdots + \lambda_m (U_{m-1} - \Box U_m) \right],
\]

(80)

and define

\[
\Phi = \lambda_1 + \frac{\partial f_2}{\partial \sigma},
\]

(81)

which, because of Eq. (79), can be inverted for the field \( \sigma \), as \( \sigma = \sigma(\Phi, \lambda_1) \), so that the action becomes

\[
S_{n=0} = \int d^4x \sqrt{-g} \left[ \Phi R + \nabla_\alpha \lambda_1 \nabla^\alpha U_1 + \nabla_\alpha \lambda_2 \nabla^\alpha U_2 + \cdots + \nabla_\alpha \lambda_m \nabla^\alpha U_m \right.
\]
\[
+ f_1 + f_2 (\sigma(\Phi, \lambda_1)) - (\Phi - \lambda_1) \sigma(\Phi, \lambda_1) + \lambda_2 U_1 + \cdots + \lambda_m U_{m-1} \].

(82)

On performing the conformal transformation we find

\[
S_{n=0} = \int d^4x \sqrt{-g} \left[ \frac{M_{Pl}^2}{2} R - \frac{3 M_{Pl}^2}{4 \Phi^2} g^{\alpha \beta} \partial_\alpha \Phi \partial_\beta \Phi + \frac{M_{Pl}^2 \bar{g}^{\alpha \beta}}{2 \Phi} \sum_{j=1}^m \partial_\alpha \lambda_j \partial_\beta U_j - V \right],
\]

(83)

where the potential \( V \) is defined as

\[
V = \frac{M_{Pl}^4 Q_1}{4 \Phi^2} [(\Phi - \lambda_1) \sigma(\Phi, \lambda_1) - f_1 - f_2 - \lambda_2 U_1 - \cdots - \lambda_m U_{m-1}] .
\]

(84)

Also in this case, we can perform the following field redefinition,

\[
\Phi = M_{Pl} q_1, \quad \text{and} \quad U_j = \frac{q_{j+1}}{M_{Pl}^2}, \quad (j = 1, \cdots, m),
\]

(85)

\[
\lambda_j = M_{Pl}^2 q_{m+1+j} \quad (j = 1, \cdots, m),
\]

(86)

so that the action becomes

\[
S_{n=0} = \int d^4x \sqrt{-g} \left[ \frac{M_{Pl}^2}{2} R - \frac{3 M_{Pl}^2}{4 q_1^2} \bar{g}^{\alpha \beta} \partial_\alpha q_1 \partial_\beta q_1 + \frac{M_{Pl}^2 \bar{g}^{\alpha \beta}}{2 q_1} \sum_{j=1}^m \partial_\alpha q_{m+1+j} \partial_\beta q_{j+1} - V \right],
\]

(88)

or, the equivalent form,

\[
S = \int d^4x \sqrt{-g} \left[ \frac{M_{Pl}^2}{2} R - \frac{1}{2} \bar{g}^{\alpha \beta} G^{\alpha \beta} q_1 - V \right],
\]

(89)
where \((k,l) = 1, \cdots, 2m + 1\). The only non-zero elements of the symmetric kinetic-term metric \(G^{kl}\) are

\[
G^{11} = \frac{3M_{Pl}^2}{2q_1^2}, \quad G^{j+1,j+1+m} = -\frac{M_{Pl}}{2q_1},
\]

where \(j = 1, \cdots, m\). On making the following final field redefinition

\[
q_1 = Q_1, \quad q_{j+1} = Q_{2j} - \frac{Q_{2j+1}}{2}, \quad q_{j+1+m} = Q_{2j} + \frac{Q_{2j+1}}{2},
\]

then the new kinetic matrix becomes diagonal with elements

\[
\dot{G}^{11} = \frac{3M_{Pl}^2}{2q_1^2}, \quad \dot{G}^{2j,2j} = -\frac{M_{Pl}}{q_1}, \quad \dot{G}^{2j+1,2j+1} = \frac{M_{Pl}}{4q_1}.
\]

Therefore, for this theory, independently of the sign of \(q_1\), there will always exist \(m\) ghosts, independently of the sign of \(q_1\).

### III. LINEAR-\(R\) NON-LOCAL GRAVITY THEORY

Let us consider a subcase of the theory where the Lagrangian is linear in \(R\), which corresponds to the case \(\partial^2 f_1/\partial \sigma^2 = 0\) in the action (3) studied in the previous section. Namely we consider

\[
S = \int d^4x \sqrt{-g} [R f_1(\box^{-1} R, \box^{-2} R, \cdots, \box^{-n} R) + f_2(\box^{-1} R, \box^{-2} R, \cdots, \box^{-m} R)],
\]

where we suppose \(n \geq 2\). For completeness, the special case \(n = 1\), which has been already studied in the literature, will be discussed separately.

We rewrite the action, along the same lines as the previous section, as

\[
S_{m \leq n} = \int d^4x \sqrt{-g} [R f_1(U_1, \cdots, U_n) + \lambda_1 (R - \box U_1) + \lambda_2 (U_1 - \box U_2) + \cdots
\]

\[
+ \lambda_n (U_{n-1} - \box U_n) + f_2(U_1, \cdots, U_m)],
\]

for \(m \leq n\), and

\[
S_{m > n} = \int d^4x \sqrt{-g} [R f_1(U_1, \cdots, U_n) + \lambda_1 (R - \box U_1) + \lambda_2 (U_1 - \box U_2) + \cdots
\]

\[
+ \lambda_n (U_{n-1} - \box U_n) + \cdots + \lambda_m (U_{m-1} - \box U_m) + f_2(U_1, \cdots, U_m)],
\]

for \(m > n\). These can be cast into the form,

\[
S_{m \leq n} = \int d^4x \sqrt{-g} [(f_1 + \lambda_1) R + g^{\alpha \beta} (\partial_\alpha \lambda_1 \partial_\beta U_1 + \partial_\alpha \lambda_2 \partial_\beta U_2 + \cdots + \partial_\alpha \lambda_n \partial_\beta U_n)
\]

\[
+ \lambda_2 U_1 + \cdots + \lambda_n U_{n-1} + f_2(U_1, \cdots, U_m)],
\]

and

\[
S_{m > n} = \int d^4x \sqrt{-g} [(f_1 + \lambda_1) R + g^{\alpha \beta} (\partial_\alpha \lambda_1 \partial_\beta U_1 + \cdots + \partial_\alpha \lambda_n \partial_\beta U_n + \cdots + \partial_\alpha \lambda_m \partial_\beta U_m)
\]

\[
+ \lambda_2 U_1 + \cdots + \lambda_n U_{n-1} + f_2(U_1, \cdots, U_m)],
\]

respectively.

Let us make the field redefinition as

\[
f_1(U_1, \cdots, U_n) + \lambda_1 = \Phi,
\]

and let us use this equation to express \(U_n\) in terms of the other fields as

\[
U_n = U_n(\Phi - \lambda_1, U_j); \quad j = 1, \cdots, n - 1.
\]
In this case the derivative of Eq. (98) gives
\[
\frac{\partial f_1}{\partial U_j} dU_j + \frac{\partial f_1}{\partial U_n} dU_n + d\lambda_1 - d\Phi = 0,
\] (100)
or
\[
\left( \frac{\partial f_1}{\partial U_j} + \frac{\partial f_1}{\partial U_n} \frac{\partial U_n}{\partial U_j} \right) dU_j + \left( 1 + \frac{\partial f_1}{\partial U_n} \frac{\partial U_n}{\partial \lambda_1} \right) d\lambda_1 + \left( \frac{\partial f_1}{\partial U_n} \frac{\partial U_n}{\partial \Phi} - 1 \right) d\Phi = 0.
\] (101)
This implies
\[
\frac{\partial U_n}{\partial U_j} = -\frac{\partial f_1}{\partial U_n} = -\frac{f_1, U_n}{f_1, U_n},
\] (102)
\[
\frac{\partial U_n}{\partial \lambda_1} = \frac{1}{f_1, U_n},
\] (103)
\[
\frac{\partial U_n}{\partial \Phi} = \frac{1}{f_1, U_n}.
\] (104)
Therefore, the actions for \(m \leq n\) and \(m > n\) can be rewritten, respectively, as
\[
S_{m \leq n} = \int d^4 x \sqrt{-g} \left[ \Phi R + g^{\alpha \beta} \left( \partial_\alpha \lambda_1 \partial_\beta U_1 + \cdots + \partial_\alpha \lambda_{n-1} \partial_\beta U_{n-1} - \frac{f_1, U_n}{f_1, U_n} \partial_\alpha \lambda_n \partial_\beta U_j \right) + \lambda_2 U_1 + \cdots + \lambda_n U_{n-1} \right],
\] (105)
and
\[
S_{m > n} = \int d^4 x \sqrt{-g} \left[ \Phi R + g^{\alpha \beta} \left( \partial_\alpha \lambda_1 \partial_\beta U_1 + \cdots + \partial_\alpha \lambda_{n-1} \partial_\beta U_{n-1} - \frac{f_1, U_n}{f_1, U_n} \partial_\alpha \lambda_n \partial_\beta U_j \right) \right.
\]
\[
+ \partial_\alpha \lambda_{n+1} \partial_\beta U_{n+1} + \cdots + \partial_\alpha \lambda_m \partial_\beta U_m - \frac{1}{f_1, U_n} \partial_\alpha \lambda_n \partial_\beta \lambda_1 + \frac{1}{f_1, U_n} \partial_\alpha \lambda_n \partial_\beta \Phi
\]
\[
\left. + \lambda_2 U_1 + \cdots + \lambda_{n+1} U_{n+1} (U_j, \Phi - \lambda_1) + \cdots + \lambda_m U_{m-1} + f_2 (U_1, \cdots, U_m) \right].
\] (106)
Let us now perform a conformal transformation to the Einstein frame, as in the previous section. For \(m \leq n\) we find
\[
S_{m \leq n} = \int d^4 x \sqrt{-g} \left[ \frac{M_{Pl}^2}{2} \bar{R} - \frac{3M_{Pl}^2}{4\Phi^2} \partial_\alpha \Phi \partial_\beta \Phi + \frac{M_{Pl}^2 \partial \alpha \beta}{2\Phi} \left( \partial_\alpha \lambda_1 \partial_\beta U_1 + \partial_\alpha \lambda_2 \partial_\beta U_2 + \cdots + \partial_\alpha \lambda_{n-1} \partial_\beta U_{n-1} \right)
\]
\[
- \frac{f_1, U_n}{f_1, U_n} \partial_\alpha \lambda_n \partial_\beta U_j - \frac{1}{f_1, U_n} \partial_\alpha \lambda_n \partial_\beta \lambda_1 + \frac{1}{f_1, U_n} \partial_\alpha \lambda_n \partial_\beta \Phi \right] - V_1,
\] (107)
where
\[
V_1 = -\frac{M_{Pl}^4}{4\Phi^2} \left[ \lambda_2 U_1 + \cdots + \lambda_n U_{n-1} + f_2 \right].
\] (108)
For \(m > n\) we find
\[
S_{m > n} = \int d^4 x \sqrt{-g} \left[ \frac{M_{Pl}^2}{2} \bar{R} - \frac{3M_{Pl}^2}{4\Phi^2} \partial_\alpha \Phi \partial_\beta \Phi + \frac{M_{Pl}^2 \partial \alpha \beta}{2\Phi} \left( \partial_\alpha \lambda_1 \partial_\beta U_1 + \partial_\alpha \lambda_2 \partial_\beta U_2 + \cdots + \partial_\alpha \lambda_{n-1} \partial_\beta U_{n-1} \right)
\]
\[
+ \partial_\alpha \lambda_{n+1} \partial_\beta U_{n+1} - \frac{f_1, U_n}{f_1, U_n} \partial_\alpha \lambda_n \partial_\beta U_j - \frac{1}{f_1, U_n} \partial_\alpha \lambda_n \partial_\beta \lambda_1
\]
\[
+ \frac{1}{f_1, U_n} \partial_\alpha \lambda_n \partial_\beta \Phi + \partial_\alpha \lambda_{n+1} \partial_\beta U_{n+1} + \cdots + \partial_\alpha \lambda_m \partial_\beta U_m \right) - V_2,
\] (109)
where
\[ V_2 = -\frac{\mathcal{M}_p^4}{4\Phi^2} \left[ \lambda_2 U_1 + \cdots + \lambda_n U_{m-1} + f_2 \right] . \] (110)

We are now ready to discuss the problem. As before we consider the two cases, \( m \leq n \) and \( m > n \), separately.

**A. Case \( m \leq n \)**

We perform the field redefinition,
\[ \Phi = M_{\text{Pl}} q_1 , \]
\[ U_j = \frac{q_{j+1}}{M_{\text{Pl}}^{2j-1}} \quad (j = 1, \cdots, n-1) , \quad U_n = \frac{u_n}{M_{\text{Pl}}^{2n-1}} , \]
\[ \lambda_i = M_{\text{Pl}}^{2i-1} q_{n+i} \quad (i = 1, \cdots, n) , \]
\[ f_1 = M_{\text{Pl}} \bar{f}_1 , \]
\[ f_{1,u_n} = M_{\text{Pl}}^{2n} \bar{f}_{1,u_n} ; \]
\[ f_{1,u_{n+1}} = M_{\text{Pl}}^{2j} \bar{f}_{1,u_{n+1}} . \]

Note that \( u_n \) is a function of the other \( q_k \) fields. Then the Lagrangian becomes
\[ S = \int d^4 \sqrt{-g} \left[ \frac{M_{\text{Pl}}^2}{2} R - \frac{1}{2} \bar{g}^{\alpha \beta} G^{\alpha \beta} \partial_\alpha q_k \partial_\beta q_l - V \right] , \] (117)
where \( k, l = 1, \cdots, 2n \). Therefore this Lagrangian has one degree of freedom less than the previous general case, as expected.

The non-vanishing elements of the kinetic (symmetric) matrix \( G^{\alpha \beta} \) are
\[ G^{11} = \frac{3M_{\text{Pl}}^2}{2q_1^2} , \quad G^{1,2n} = \frac{M_{\text{Pl}}}{2q_1 f_{1,u_n}} , \quad G^{j+1,j+n} = \frac{M_{\text{Pl}}}{2q_1 f_{1,u_n}} , \quad G^{j+1,2n} = \frac{M_{\text{Pl}} f_{1,q_{j+1}}}{2q_1 f_{1,u_n}} . \] (118)
\[ G^{n+1,2n} = \frac{M_{\text{Pl}}}{2q_1 f_{1,u_n}} . \] (119)

As before we examine whether \( v^T \cdot G \cdot v \) for a general \( 2n \)-dimensional vector \( v \) is positive definite or not. We find that a transformation of the form \( v = A \cdot w \) that diagonalizes the matrix \( G \) is given by
\[ v_1 = w_1 + \frac{q_1}{3M_{\text{Pl}}} \frac{1}{f_{1,u_n}} w_{2n} , \] (120)
\[ v_{j+1} = w_{2j} - \frac{1}{2} \frac{\delta_{j+1,1}}{f_{1,u_n}} w_{2n} , \] (121)
\[ v_{j+n} = w_{2j} + \frac{1}{2} \frac{f_{1,q_{j+1}}}{f_{1,u_n}} w_{2n} , \] (122)
\[ v_{2n} = w_{2n} . \] (123)

In fact in this case the new kinetic matrix \( \tilde{G} \equiv A^T \cdot G \cdot A \), is diagonal with the non-zero elements given by
\[ \tilde{G}^{11} = \frac{3M_{\text{Pl}}^2}{2q_1^2} , \quad \tilde{G}^{2j,2j} = -\frac{M_{\text{Pl}}}{q_1} , \quad \tilde{G}^{2j+1,2j+1} = \frac{M_{\text{Pl}}}{4q_1} , \quad \tilde{G}^{2n,2n} = \frac{1}{f_{1,u_n}^2} \frac{M_{\text{Pl}} f_{1,q_{j+1}} - 1}{6} . \] (124)

Thus we conclude that this theory contains, in general, at least \( n - 1 \) ghosts independently of the sign of \( q_1 \).

**B. Case \( m > n \)**

In this case, we perform the field redefinition,
\[ \Phi = M_{\text{Pl}} q_1 , \] (125)
where $\lambda_i = M_{2i-1} q_{m+i}$ \,(i = 1, \cdots, m),
\begin{equation}
\begin{aligned}
U_j &= \frac{q_{j+1}}{M_{2j-1}^2} \quad (j = 1, \cdots, n-1), \\
U_r &= \frac{q_r}{M_{2r-1}^2} \quad (r = n+1, \cdots, m), \\
\lambda_i &= M_{2i-1} q_{m+i} \\
f_1 &= M_{2i} \bar{f}_1, \\
f_{1,u} &= M_{2i} \bar{f}_{1,u}, \\
f_{1,uj} &= M_{2j} \bar{f}_{1,uj}, \\
\end{aligned}
\end{equation}

where $u_n$ is a function of the other fields. In total, there are $2m$ fields. The kinetic matrix for the scalar fields is in the form,
\begin{equation}
S = \int d^4x \sqrt{-g} \left[ \frac{M_{Pl}^2}{2} R - \frac{1}{2} \tilde{\gamma}^{\alpha \beta} G^{kl} \partial_\alpha q_k \partial_\beta q_l - V \right],
\end{equation}

where $k, l = 1, \cdots, 2m$. The non-zero elements of the matrix $G^{kl}$ are
\begin{equation}
\begin{aligned}
G^{11} &= \frac{3M_{Pl}^2}{2q_1^2}, \\
G^{1,m+n} &= -\frac{M_{Pl}}{2q_1 \bar{f}_{1,u},} \\
G^{j+1,m+j} &= -\frac{M_{Pl}}{2q_1} \\
G^{k,m+k} &= -\frac{M_{Pl}}{2q_1} \\
\end{aligned}
\end{equation}

where $j = 1, \cdots, n-1$, $k = n+1, \cdots, m$.

This time, the transformation $v = Au$ that diagonalizes the matrix $G$ is
\begin{equation}
\begin{aligned}
v_1 &= w_1 + \frac{q_1}{3M_{Pl} \bar{f}_{1,u},} w_{2m}, \\
v_{j+1} &= w_{j+1} - \frac{1}{2} w_{2j+1} + \frac{q_{j+1}}{\bar{f}_{1,u},} w_{2m}, \\
v_k &= w_{2k-2} - \frac{1}{2} w_{2k-1}, \\
v_{m+j} &= w_{2j+1} + \frac{1}{2} w_{2j+1} + \frac{\bar{f}_{1,uj}}{\bar{f}_{1,u},} w_{2m}, \\
v_{m+n} &= w_{2m}, \\
v_{m+k} &= w_{2k-2} + \frac{1}{2} w_{2k-1}.
\end{aligned}
\end{equation}

The new diagonal kinetic matrix $\bar{G} = A^T \cdot G \cdot A$ has the elements,
\begin{equation}
\begin{aligned}
\bar{G}^{11} &= \frac{3M_{Pl}^2}{2q_1^2}, \\
\bar{G}^{2j,2j} &= -\frac{M_{Pl}}{q_1}, \\
\bar{G}^{2j+1,2j+1} &= \frac{M_{Pl}}{4q_1}, \\
\bar{G}^{2k-2,2k-2} &= -\frac{M_{Pl}}{q_1}, \\
\bar{G}^{2k-1,2k-1} &= \frac{M_{Pl}}{4q_1}, \\
\bar{G}^{2m,2m} &= \frac{6M_{Pl} \bar{f}_{1,uj} - q_1}{6q_1 \bar{f}_{1,u},}.
\end{aligned}
\end{equation}

Therefore, in this case there always exist $m-1$ ghosts, independently of the sign of $q_1$.

To summarize, in the case the Lagrangian is linear in $R$, and provided that $n \geq 2$, there always exist at least $\max(m,n)-1$ ghosts in the theory.

\begin{center}
C. No ghost case
\end{center}

The only sub-theory which can be made free from ghosts is the case $n = 1$ and $m = 0 \,(f_2 = 0)$. The Lagrangian in this case reads
\begin{equation}
S = \int d^4x \sqrt{-g} \, f_1(\Box^{-1}R),
\end{equation}
Following the same procedure used in the previous section, we can rewrite the action as
\[ S = \int d^4x \sqrt{-g}\left[ R f_1(U_1) + \lambda_1(R - \Box U_1) \right], \tag{145} \]
or
\[ S = \int d^4x \sqrt{-g}\left[ (f_1 + \lambda_1) R + g^{\alpha\beta} \partial_{\alpha}\lambda_1 \partial_{\beta} U_1 \right]. \tag{146} \]
Let us make the field redefinition,
\[ f_1(U_1) + \lambda_1 = \Phi, \tag{147} \]
and use this equation to express \( U_1 \) in terms of the other two fields as
\[ U_1 = U_1(\Phi - \lambda_1). \tag{148} \]
The derivative of Eq. (147) gives
\[ \frac{df_1}{dU_1} du_1 + d\lambda_1 - d\Phi = 0, \tag{149} \]
or
\[ \left( 1 + \frac{df_1}{dU_1} \frac{\partial U_1}{\partial \lambda_1} \right) d\lambda_1 + \left( \frac{df_1}{dU_1} \frac{\partial U_1}{\partial \Phi} - 1 \right) d\Phi = 0. \tag{150} \]
This implies
\[ \frac{\partial U_1}{\partial \lambda_1} = -\frac{1}{f_1(U_1)}, \tag{151} \]
\[ \frac{\partial U_1}{\partial \Phi} = \frac{1}{f_1(U_1)}. \tag{152} \]
Therefore, the action (146) can be rewritten as
\[ S = \int d^4x \sqrt{-g}\left[ \frac{M_{Pl}^2}{2} R - \frac{3M_{Pl}^2}{4\Phi^2} g^{\alpha\beta} \partial_{\alpha}\Phi \partial_{\beta}\Phi + \frac{M_{Pl}^2 g^{\alpha\beta}}{2\Phi f_1(U_1)} (\partial_{\alpha}\lambda_1 \partial_{\beta}\Phi - \partial_{\alpha}\lambda_1 \partial_{\beta}\lambda_1) \right], \tag{153} \]
which may be transformed to the Einstein frame as
\[ S = \int d^4x \sqrt{-\bar{g}}\left[ \frac{M_{Pl}^2}{2} \bar{R} - \frac{1}{2} G^{ij} \bar{g}^{\alpha\beta} \partial_{\alpha}q_i \partial_{\beta}q_j \right], \tag{154} \]
where \( i, j = 1, 2 \), and
\[ G = \begin{pmatrix} \frac{3M_{Pl}^2}{2f_1(U_1)} & -\frac{M_{Pl}}{2q_1 f_1(U_1)} \\ -\frac{M_{Pl}}{2q_1 f_1(U_1)} & \frac{M_{Pl} q_1 f_1(U_1)}{2q_1 f_1(U_1)} \end{pmatrix}. \tag{155} \]
This matrix is positive definite when
\[ \frac{6M_{Pl}}{q_1} f_1(U_1) > 1. \tag{160} \]
Only when this condition is satisfied, the theory can be made free from ghost [28].
IV. CONCLUSION

We considered a class of non-local gravity where the Lagrangian is a general function of $\Box^{-k} R$ ($k = 1, 2, \cdots, n$) where $R$ is the Ricci scalar, and studied its formally equivalent local Lagrangian by introducing auxiliary fields. Taking the viewpoint that the physical degrees of freedom in thus localized theory properly represent those in the original non-local theory, we examined the kinetic term of the localized Lagrangian to see whether there are ghosts or not.

We found that, for a theory which contains a nonlinear function of $R$, there always exist $n$ ghost fields for $n \geq 1$, while for a theory linear in $R$, there always exist $n-1$ ghost fields for $n \geq 2$. The case of $n = 1$ with linear $R$ has been already studied and it is known that one may or may not make the theory ghost-free depending on the choice of the parameters.

Thus except for the special case, this class of non-local gravity always suffers from the presence of a ghost. This result makes these theories problematic to be used as effective theories to describe the evolution of the universe at all times. The only possible way-out seems to be the case when the masses of these ghosts are individually tuned to be larger than the cut-off of the theory, e.g., larger than the Planck mass (in some other contexts, the cut-off mass can be lowered, but a careful choice of the functions $f_1$ and/or $f_2$ is needed in order to achieve large masses).

Of course, if we abandon our localization approach used to discuss the degrees of freedom, this ghost proliferation may be avoided. For example, one could regard the operator $\Box^{-1}$ in the Lagrangian as $\Box_{\text{ret}}^{-1}$. One would need a new formalism to deal with such a Lagrangian. In particular, it is not clear at all how to quantum the theory in this case. After all, a ghost field is fatally dangerous in quantum theory. Thus avoiding ghosts by using $\Box_{\text{ret}}^{-1}$ in the Lagrangian may simply mean avoiding quantization.

Our result leads to discussion on the fundamentals of theories of non-local gravity with $\Box^{-k} R$ ($k = 1, 2, \cdots, n$), about how it is possible to make sense of it in terms of understanding the physical degrees of freedom. We hope this discussion will stimulate other studies on conditions for healthy extensions of a field theory and the quantization procedure for such non-standard Lagrangians.

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