Successive-Cancellation Decoding of Binary Polar Codes Based on Symmetric Parametrization

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Abstract

This paper introduces algorithms for the successive-cancellation decoding and the successive-cancellation list decoding of binary polar source/channel codes. By using the symmetric parametrization of conditional probability, we reduce both space and time complexity compared to the original algorithm introduced by Tal and Vardy.

Index Terms

binary polar codes, source coding with decoder side information, channel coding, successive-cancellation decoding

I. INTRODUCTION

Polar source/channel codes were introduced by Arıkan [1], [2], [3]. When these codes are applied to source coding with decoder side information for joint stationary memoryless sources, the coding rate achieves a fundamental limit called the conditional entropy. When applied to the channel coding of a symmetric channel, the coding rate achieves a fundamental limit called the channel capacity. Arıkan introduced successive-cancellation decoding, which can be implemented with computational complexity of $O(N \log_2 N)$ where $N$ is block length.

In this paper, we introduce algorithms for successive-cancellation decoding and successive-cancellation list decoding based on the work of Tal and Vardy [10]. Our constructions can be applied to both polar source codes and polar channel codes. Furthermore, the proposed list decoding algorithm reduces the space and time complexity compared to [10].

II. DEFINITIONS AND NOTATIONS

Throughout this paper, we use the following definitions and notations.

For a given $n$, let $N \equiv 2^n$ denote the block length. We assume that the number $n$ is given as a constant, which means that all algorithms have access to this number. We use the bit-indexing approach introduced in [1]. The indexes of a $N$-dimensional vector are represented by $n$-bit sequences as $X_N \equiv (X_0^n, \ldots, X_1^n)$, where $0^n/1^n$ denotes the $n$-bit all zero/one sequence. To represent an interval of integers, we use the following notations

$[0^n : b^n] \equiv \{0^n, \ldots, b^n\}$
$[0^n : b^n] \equiv \{0^n : b^n\} \setminus \{b^n\}$.
$[b^n : 1^n] \equiv \{0^n : 1^n\} \setminus [0^n : b^n]$.
$[b^n : 1^n] \equiv \{0^n : 1^n\} \setminus [0^n : b^n]$.

For a given subset $I$ of $[0^n : 1^n]$, we define the sub-sequences of $X_N$ as

$X_I \equiv \{X_{b^0} \}_{b^0 \in I}$.

Let $c^l b^k \in \{0,1\}^{l+k}$ be the concatenation of $b^k \in \{0,1\}^k$ and $c^l \in \{0,1\}^l$. For given $b^k \in \{0,1\}^k$ and $c^l \in \{0,1\}^l$, we define subsets $c^l[0 : b^k]$ and $c^l[0 : b^k)$ of $\{0,1\}^{k+l}$ as

$c^l[0 : b^k] \equiv \{c^l d^k : d^k \in [0^k : b^k]\}$
$c^l[0 : b^k] \equiv \{c^l d^k : d^k \in [0^k : b^k)\}$.

The bipolar-binary conversion, $\mp_b$, of $b \in \{0,1\}$ is defined as

$\mp_b \equiv \begin{cases} - & \text{if } b = 1 \\ + & \text{if } b = 0. \end{cases}$

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III. BINARY POLAR CODES

In this section, we revisit the binary polar source/channel codes introduced in previous works [1], [2], [3], [8]. Assume that \{0, 1\} is the binary finite field. For given positive integer \( n \), polar transform \( G \) is defined as

\[
G \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus_n \Pi_{BR},
\]

where \( \oplus_n \) denotes the \( n \)-th Kronecker power and \( \Pi_{BR} \) is the bit-reversal permutation matrix [1]. Next, vector \( u \in \{0, 1\}^N \) is defined as \( u \equiv xG \) for given vector \( x \in \{0, 1\}^N \). For completeness, an algorithm that computes \( u \) is given in Appendix A.

Let \( \{I_0, I_1\} \) be a partition of \( [0^n : 1^n] \), satisfying \( I_0 \cap I_1 = \emptyset \) and \( I_0 \cup I_1 = [0^n : 1^n] \). We define \( \{I_0, I_1\} \) later.

Let \( X \equiv (X_0, \ldots, X_n) \) and \( Y \equiv (Y_0, \ldots, Y_n) \) be random variables and let \( U \equiv (U_0, \ldots, U_n) \) be a random variable defined as \( U \equiv XG \). Then \( P_{U_{I_0}, U_{I_1}, Y} \), the joint distribution of \( (U_{I_0}, U_{I_1}, Y) \), is defined using the joint distribution \( P_{XY} \) of \( (X, Y) \) as

\[
P_{U_{I_0}, U_{I_1}, Y}(u_{I_0}, u_{I_1}, y) = P_{XY}((u_{I_1}, u_{I_0})G^{-1}, y)
\]

where the elements in \((u_{I_1}, u_{I_0})\) are sorted in index order before operation \( G^{-1} \). We refer to \( u_{I_1} \) and \( u_{I_0} \) as frozen bits and unfrozen bits, respectively.

Let \( P_{U_{I_0}|U_{I_0},u_{I_0},y} \) be the conditional probability distribution, defined as

\[
P_{U_{I_0}|U_{I_0},u_{I_0},y}(u_{I_0}|u_{I_0},b_{I_0},y) = \frac{\sum_{u_{I_0}} P_{U_{I_0}, U_{I_1}, Y}(u_{I_0}, u_{I_1}, y)}{\sum_{u_{I_0}} P_{U_{I_0}, U_{I_1}, Y}(u_{I_0}, u_{I_1}, y)}
\]

For vector \( u_{I_1} \) and side information \( y \in Y^N \), output \( \hat{u} \equiv f(u_{I_1}, y) \) of successive-cancellation (SC) decoder \( f \) is defined recursively as

\[
\hat{u}_{I_0} = \begin{cases} f_{I_0}(\hat{u}_{I_0}|u_{I_0}, b_{I_0}), y & \text{if } b_{I_0} \in I_0 \\ u_{I_0} & \text{if } b_{I_0} \in I_1 \end{cases}
\]

using function \( 
\]

\[
f_{I_0}(u_{I_0}|u_{I_0}, b_{I_0}, y) \equiv \arg \max_{u \in \{0, 1\}} P_{U_{I_0}|U_{I_0},u_{I_0},y}(u|u_{I_0}, b_{I_0}, y)
\]

which is the maximum a posteriori decision rule after an observation \((u_{I_0},u_{I_0},y))\).

For a polar source code (with decoder side information), \( x \equiv (X_0, \ldots, X_n) \) is a source output, \( u_{I_1} \) is a codeword, and \( y \in Y^N \) is a side information output. The decoder reproduces source output \( \hat{x} \equiv f(u_{I_1}, y)G^{-1} \) from codeword \( u_{I_1} \) and \( y \). The (block) decoding error probability is given as \( \text{Prob}(f(U_{I_1}, Y)G^{-1} \neq X) \).

For a systematic polar channel code [8], we define \( I_0' \) and \( I_1' \) as

\[
I_0' = \{ b_0 b_1 \cdots b_{n-1} : b_{n-1} \cdots b_1 b_0 \in I_0 \}
\]

\[
I_1' = \{ b_0 b_1 \cdots b_{n-1} : b_{n-1} \cdots b_1 b_0 \in I_1 \}
\]

for given \((I_0, I_1)\). We assume that encoder and decoder share a vector \( u_{I_1} \). The encoder computes \((x_{I_1}, u_{I_1})\) from message \( x_{I_1} \) and shared vector \( u_{I_1} \) so that \((x_{I_1}, x_{I_1}) = (u_{I_1}, u_{I_1})G^{-1} \), where the elements in \((x_{I_1}, x_{I_1})\) and \((u_{I_1}, u_{I_1})\) are sorted in index order before operating \( G^{-1} \). An algorithm for this computation is introduced in Appendix B. The encoder then generates channel input \( x \equiv (x_{I_1}, x_{I_1}) \), where the elements in \((x_{I_1}, x_{I_1})\) are sorted in index order. The decoder reproduces channel input \( \hat{x} \equiv f(u_{I_1}, y)G^{-1} \) from channel output \( y \in Y^N \) and shared vector \( u_{I_1} \), where \( \hat{x}_{I_1} \) is a reproduction of the message. The (block) decoding error probability is given as \( \text{Prob}(f(U_{I_1}, Y) \neq (U_{I_0}, U_{I_1})) \).

We have the following lemmas.

Lemma 1 ([2] Theorem 2, [8] Theorem 4.10): Define \( I_0 \) as

\[
I_0 \equiv \left\{ b^n \in [0^n : 1^n] : Z(U_{b^n}|U_{b^n, y}, Y_{b^n, 1^n}) \leq 2^{-2n^0} \right\}
\]

where \( Z(U_{b^n}|U_{b^n, y}, Y_{b^n, 1^n}) \) is the source Bhattacharyya parameter introduced in [2]. Then we have

\[
\lim_{n \to \infty} \frac{|I_0|}{2^n} = H(X|Y)
\]

\[
\lim_{n \to \infty} \frac{|I_1|}{2^n} = H(X|Y)
\]
for any $\beta \in [0, 1/2]$.

Lemma 2 ([8, Proposition 2.7]):

\[
\text{Prob}(f_{b^n}(U_{[0^n:b^n]}, Y) \neq U_{b^n}) \leq Z(U_{b^n}|U_{[0^n:b^n]}, Y_{[0^n:1^n]}).
\]

We have the following lemma, which can be shown as in a previous proof [1].

Lemma 3 ([6, Lemma 2], [9, Eq. (1)]):

\[
\text{Prob}(f(U_{I_1}, Y) = X) = \text{Prob}(f(U_{I_1}, Y) \neq (U_{I_0}, U_{I_1})) \\
\leq \sum_{b^n \in I_0} \text{Prob}(f_{b^n}(U_{[0^n:b^n]}, Y) \neq U_{b^n}).
\]

From the above lemmas, we have the fact that the rate of polar codes attains the fundamental limit and the decoding error probability goes to zero as $n \to \infty$. For example, we can obtain $I_0$ by using the technique introduced in [7], [9]. In the following sections, we assume that $I_0$ is given arbitrary.

IV. SYMMETRIC PARAMETRIZATION

In this section, we introduce the polar transform based on symmetric parametrization. Given $P_U$, a probability distribution of binary random variable $U$, let $\theta$ be defined as

\[
\theta \equiv P_U(0) - P_U(1).
\]

Then we have

\[
P_U(u) = \frac{1 \mp u \theta}{2},
\]

where $\mp u$ is the bipolar-binary conversion of $u$.

In the basic polar transform, a pair of binary random variables $(U_0, U_1)$ is transformed into

\[
U'_0 \equiv U_0 \oplus U_1 \\
U'_1 \equiv U_1,
\]

where $\oplus$ denotes the addition on the binary finite field. Assume that random variables $U_0, U_1 \in \{0, 1\}$ are independent. For each $i \in \{0, 1\}$, let $\theta_i$ be defined as

\[
\theta_i \equiv P_{U_i}(0) - P_{U_i}(1).
\]

First, we have

\[
P_{U'_0}(0) = P_{U_0}(0)P_{U_1}(0) + P_{U_0}(1)P_{U_1}(1) \\
= \frac{1 + \theta_0}{2} \cdot \frac{1 + \theta_1}{2} + \frac{1 - \theta_0}{2} \cdot \frac{1 - \theta_1}{2} \\
= \frac{1 + \theta_0 \theta_1}{2},
\]

where the first equality comes from the definition of $U'_0$ and the fact that $U_0$ and $U_1$ are mutually independent. The above yields

\[
P_{U'_0}(1) = 1 - P_{U'_0}(0) \\
= 1 - \frac{1 + \theta_0 \theta_1}{2}.
\]

From (2) and (5), we have

\[
P_{U'_0}(u'_0) = \frac{1 \mp u'_0 \theta_0 \theta_1}{2},
\]

where $\mp u'_0$ is the bipolar-binary conversion of $u'_0$. Let $\theta'_0$ be defined as

\[
\theta'_0 \equiv P_{U'_0}(0) - P_{U'_0}(1).
\]

From (2)–(5), we have

\[
\theta'_0 = \theta_0 \theta_1.
\]

It should be noted that, since symmetric parametrization is a binary version of the Fourier transform of the probability distribution [5, Definitions 24 and 25], the right hand side of (6) corresponds to the Fourier transform of the convolution.

We have

\[
P_{U'_0U'_1}(u'_0, 0) = P_{U_0}(u'_0)P_{U_1}(0)
\]
From (8)–(10), we have
\begin{align*}
\theta_i & \equiv \frac{1 + \theta_i}{2} + \frac{1 + \theta_i}{2} \\
& = \frac{1 + w_i^0 \theta_0 + \theta_i + \theta_i + w_i^0 \theta_0}{4},
\end{align*}

where the first equality comes from the definition of \( U_0 \) and \( U_1 \), and the fact that \( U_0 \) and \( U_1 \) are mutually independent. Then we have
\begin{align*}
P_{U_1|U_0}(0|u_0^0) & = \frac{P_{U_1|U_0}(0, u_0^0)}{P_{U_0}(u_0^0)} \\
& = \frac{1 + \theta_i}{2} + \frac{1 + \theta_i}{2} \\
& = \frac{1 + w_i^0 \theta_0 + \theta_i + \theta_i + w_i^0 \theta_0}{4}.
\end{align*}

Let \( \theta_i^0 \) be defined as
\begin{equation}
\theta_i^0 \equiv P_{U_1|U_0} (0, u_0^0) - P_{U_1|U_0} (1, u_0^0).
\end{equation}

From (8)–(10), we have
\begin{equation}
\theta_i^0 = \frac{\theta_i}{1 + w_i^0 \theta_0},
\end{equation}

where the second equality comes from (6).

V. SUCCESSIVE-CANCELLATION DECODING

This section introduces the algorithm of successive-cancellation decoding based on that introduced in [10]. We assume that Algorithms 1–3 have access to the number of transforms, \( n \), the frozen bits \( \varepsilon_{(n)} \), and the memory space
\begin{align*}
\Theta & \equiv \left\{ \theta[k] \mid e^{n-k} \in [0^{n-k} : 1^{n-k}] \right\}, \\
\Upsilon & \equiv \left\{ \upsilon[k] \mid e^{n-k} \in [0^{n-k} : 1^{n-k}] \right\},
\end{align*}

where \( \theta[k] \mid e^{n-k} \) is a real number variable, \( \upsilon[k] \mid e^{n-k} \mid b \) is a binary variable, and \( e^0 \) denotes the null string. It should be noted that \( \Theta \) has \( \sum_{k=0}^{n} 2^{n-k} = 2^{n+1} - 1 = 2N - 1 \) variables and \( \Upsilon \) has \( 2 \sum_{k=0}^{n} 2^{n-k} = 2^{n+2} - 2 = 4N - 2 \) variables.

In the following, we assume that \( \{ U_{c^n}^{(n)} \}_{c^n \in [0^n : 1^n]} \) is a memoryless source, that is, \( P_{U_{c^n}^{(n)}} (u_{c^n}^{(0)}) \) is defined as
\begin{equation}
P_{U_{c^n}^{(n)}(u_{c^n}^{(0)})} \equiv \prod_{c^n \in [0^n : 1^n]} P_{U_{c^n}^{(n)}} (u_{c^n}^{(0)}),
\end{equation}

where \( \{ P_{U_{c^n}^{(n)}} (u_{c^n}^{(0)}) \}_{c^n \in [0^n : 1^n]} \) is given depending on the context. It should be noted that \( \{ U_{c^n}^{(n)} \}_{c^n \in [0^n : 1^n]} \) is allowed to be non-stationary. We recursively define \( U_{bn}^{(n)} \) as
\begin{equation}
U_{c^n}^{(k)} \\
= \begin{cases} 
U_{c^n-kb^n-k}^{(k-1)} \oplus U_{c^n-kb^n-k}^{(k-1)} & \text{if } b_n \in \{0, 1\} \\
U_{c^n-kb^n-k}^{(k-1)} & \text{otherwise}
\end{cases}
\end{equation}

for given \( b_n \in \{0, 1\}^n \) and \( c^n-k \in \{0, 1\}^{n-k} \). This yields \( U_{c^n}^{(n)}(0) = U_{c^n}^{(0)} | G \), which is the polar transform of \( U_{c^n}^{(0)}(0) \). The goal of \( \text{update}(\Theta, \Upsilon, n, b^n) \) at Line 3 of Algorithm 1 is to compute
\begin{equation}
\theta_{b^n}^{(n)} \equiv P_{U_{c^n}^{(n)}(u_{c^n}^{(n) | b^n})} (0 \mid u_{c^n}^{(n) | b^n} = 1 \mid u_{c^n}^{(n) | b^n})
\end{equation}

recursively starting from
\begin{equation}
\theta_{b^n}^{(0)} \equiv P_{U_{c^n}^{(n)}(0)} - P_{U_{c^n}^{(n)}(1)}.
\end{equation}
In Algorithm 2 we compute a parameter defined as
\[
\theta^{(k)}_{n-k, k} \equiv P_{U^{(k)}_{n-k, k}}[Y^{(k)}_{n-k, k}] \left( 0 \bigg| u^{(k)}_{n-k} \right) = P_{U^{(k)}_{n-k, k}}[Y^{(k)}_{n-k, k}] \left( 1 \bigg| u^{(k)}_{n-k} \right) \tag{14}
\]
for each \(n-k\) for a given \(b^n \in \{0, 1\}^n\). By using (6), (11), and (13), we have the relations
\[
\theta^{(k)}_{n-k, k-1} = \theta^{(k-1)}_{n-k, k-1} = \theta^{(k-1)}_{n-k, 0} \tag{15}
\]
\[
\theta^{(k)}_{n-k, k-1} = \theta^{(k-1)}_{n-k, k-1} \oplus \theta^{(k-1)}_{n-k, k-1} \tag{16}
\]
where \(\oplus\) is the bipolar-binary conversion of \(u \equiv u^{(k)}_{n-k, k-1}\). The goal of update \(U, n, b^{n-1}\) at Line 9 of Algorithm 1 is to compute \(u^{(k)}_{n-k, k-1} \) from \(u^{(n)}_{0, n} \) by using the relations
\[
u^{(k-1)}_{n-k, k-1} = \nu^{(k)}_{n-k, k-1} \oplus \nu^{(k)}_{n-k, k-1} \tag{17}
\]
\[
u^{(k-1)}_{n-k, k-1} = \nu^{(k)}_{n-k, k-1} \tag{18}
\]
that come from (12) and (13), where we assume that \(u^{(n)}_{0, n} \) is successfully decoded. It should be noted that (15) and (16) correspond to Lines 5 and 7 of Algorithm 2 respectively, and (17) and (18) correspond to Lines 2 and 3 of Algorithm 3 respectively, where we have relations
\[
\theta[k|c^{n-k}] = \theta^{(k)}_{n-k, k} \tag{19}
\]
\[
\mathbb{P}[k|c^{n-k}] \equiv \mathbb{P}[u^{(k)}_{n-k, k-1}] \tag{20}
\]
after completing Lines 3 and 9 of Algorithm 1 respectively. We show (19) and (20) in Section VII.A. Furthermore, Line 7 of Algorithm 1 corresponds to the maximum a posteriori probability decision defined as
\[
\hat{u}^{(n)}_{b^n} \equiv \arg \max_{u \in \{0, 1\}} P_{U^{(n)}}[Y^{(n)}_{0, n} \mid u] \left( u \mid b^{n-1} \right). \tag{21}
\]
When Algorithm 1 is used for the decoder of polar source code which has access to codeword \(y_1^{(n)} \) and side information vector \(y_{\{0,1\}^n}\), we define
\[
P_{U^{(n)}}[x] = P_{X^{(n)}}[x | y_{\{0,1\}^n}] \tag{22}
\]
for \(x \in \{0, 1\}\) and obtain the reproduction \(\tilde{x}^{(n)}_{c^{n}} \in \{0, 1\}^n\) defined as
\[
\tilde{x}^{(n)}_{c^{n}} \equiv \mathbb{P}[x] [b^{n-1}], \tag{23}
\]
where \(b_{-1} \) denotes the null string.

When Algorithm 2 is used to decode a systematic polar channel code, which has access to channel output vector \(y_{\{0,1\}^n}\) and shared vector \(u_{X_0^n}^{(n)} \), we define
\[
P_{U^{(n)}}[x] = \sum_{x' \in \{0, 1\}} P_{X^{(n)}}[x' | y_{\{0,1\}^n}] \tag{24}
\]
for given channel distribution \(\{P_{X^{(n)}}[x' | y_{\{0,1\}^n}]\}_{c^{n} \in \{0, 1\}^n}\), input distribution \(\{P_{X^{(n)}}[x' | y_{\{0,1\}^n}]\}_{x \in \{0, 1\}, y_{\{0,1\}^n} \in \mathcal{Y}}\). This yields a reproduction \(\tilde{x}^{(n)}_{c^{n}} \in \mathcal{X}_{\{0,1\}^n}\) defined by (22), where \(\mathcal{X}_{\{0,1\}^n}\) is defined by (1). When Algorithm 1 is used in the decoder of a non-systematic polar channel code, we have to prepare binary variables \(\{M[b^n]\}_{b^n \in \mathcal{X}_0}\) and insert
\[
M[b^n] \leftarrow \mathbb{P}[y_{\{0,1\}^n} | c^{0}] [b_{n-1} \mid 0] \tag{25}
\]
just after the renewal of \(U[n] | c^{0} | b_{n-1}\) (Line 7 of Algorithm 1). This yields reproduction \(\tilde{x}^{(n)}_{c^{n}}\) defined as
\[
\tilde{u}^{(n)}_{b^n} \equiv M[b^n]. \tag{26}
\]

**Remark 1:** When Algorithm 1 is applied to a binary erasure channel, we can assume that \(\theta[k|c^{n-k}]\) takes a value in \(-1, 0, 1\), where \(\theta[0|b^n] \leftarrow P_{U^{(n)}}[0] - P_{U^{(n)}}[1]\) in Line 1 of Algorithm 1 can be replaced by
\[
\theta[0|b^n] \leftarrow \begin{cases} 1 & \text{if } y_n = 0 \\ 0 & \text{if } y_n \text{ is the erasure symbol} \\ -1 & \text{if } y_n = 1. \end{cases} \tag{27}
\]
for given channel output \(y_{\{0,1\}^n}\). We can improve Algorithm 2 as described in Appendix C.
are stored in memory space complexity of our algorithm is around half that mentioned in [10].

\[
L \hat{=} n
\]

\[
\text{Algorithm 1 Successive-cancellation decoder}
\]

Input: \(\mathcal{I}_1, u^{(n)}_t, \left\{P_{U_i^{(n)}}\right\}_{b \in \{0^n : 1^n\}}\)

1: for \(b^n \in \{0^n : 1^n\}\) do
2: for \(b^n \in \{0^n : 1^n\}\) do
3: update\(\Theta(\mathcal{U}, U, n, b^n)\)
4: if \(b^n \in \mathcal{I}_1\) then
5: \(U[n][c^n][b_{n-1}] \leftarrow u^{(n)}_{b_{n-1}}\)
6: else
7: \(U[n][c^n][b_{n-1}] \leftarrow \begin{cases} 0 & \text{if } \Theta[n][0] > 0 \\ 1 & \text{if } \Theta[n][0] < 0 \\ 0 \text{ or } 1 & \text{if } \Theta[n][0] = 0 \end{cases}\)
8: end if
9: if \(b_{n-1} = 1\) then update\(U(\mathcal{U}, U, n, b^{n-1})\)
10: end for

\[
\text{Algorithm 2 update}\Theta(\Theta, U, k, b^k)
\]

1: if \(k = 0\) then return
2: if \(b_{k-1} = 0\) then update\(\Theta(\Theta, U, k - 1, b^{k-1})\)
3: for \(c^{n-k} \in \{0^{n-k} : 1^{n-k}\}\) do
4: if \(b_{k-1} = 0\) then
5: \(\Theta[k][c^{n-k}] \leftarrow \Theta[k - 1][c^{n-k}] - \Theta[k - 1][c^{n-k}0]\)
6: else
7: \(\Theta[k][c^{n-k}] \leftarrow \Theta[k - 1][c^{n-k}] \hat{=} \Theta[k - 1][c^{n-k}0]\)
8: end if
9: end for

VI. SUCCESSIVE-CANCELLATION LIST DECODING

This section introduces an algorithm for the successive-cancellation list decoding. It is based on that introduced in [10]. It should be noted that we use a fixed-addressing memory space instead of the stacking memory space approach used in [10]. Since the size of memory space for the computation of conditional probability is around half that used in [10], the time complexity of our algorithm is around half that mentioned in [10].

We assume that Algorithms 2–9 have access to the number of transforms, \(n\), the list size \(L\), the frozen bits \(u^{(n)}_t\), and the memory space \(\left\{\Theta[\lambda]\right\}_{\lambda=0}^{L-1}, \left\{U[\lambda]\right\}_{\lambda=0}^{L-1}, \left\{P[\lambda]\right\}_{\lambda=0}^{L-1}, \left\{\text{Active}[\lambda]\right\}_{\lambda=0}^{L-1}\), where \(\Theta[\lambda] \text{ and } U[\lambda]\) are accessed by Algorithms 2 and 3, \(P[\lambda]_{\lambda=0}^{L-1}\) are real numbers, and \(\text{Active}[\lambda]_{\lambda=0}^{L-1}\) are binary variables. After Algorithm 4 concludes, the results are stored in \(\left\{U[\lambda]\right\}_{\lambda=0}^{L-1}\) and \(\left\{P[\lambda]\right\}_{\lambda=0}^{L-1}\) satisfying

\[
\frac{P[\lambda]}{2^N} = \prod_{b^n \in \{0^n : 1^n\}} P_{U_i^{(n)}(U_t^{(n)}|b^n)} \left(\hat{\gamma}(n)_{b_{n-1}}(\lambda) \bigg| \hat{\beta}(n)_{0^n: b^n}(\lambda)\right) = P_{U_i^{(n)}(U_t^{(n)}|b^n)} \left(\hat{\gamma}(n)_{0^n: b^n}(\lambda)\right)
\]

and

\[
U[\lambda][0^n][b_{n-1}] = \hat{\gamma}(0)_{b_{n-1}}(\lambda) \quad \hat{\beta}(0)_{0^n:1^n}(\lambda) = \hat{\beta}(0)_{0^n:1^n}(\lambda)G,
\]

where \(\hat{\gamma}(n)_{0^n:1^n}(\lambda)\) is the \(\lambda\)-th surviving path. It should be noted that, at Line 5 of Algorithm 2 we select paths \(\hat{\beta}(n)_{0^n: b^n}\) that have the \(L\) largest probability

\[
\frac{P[\lambda]}{2^{[0^n: b^n]}} = \prod_{d^n \in \{0^n : b^n\}} P_{U_i^{(n)}(U_t^{(n)}|d^n)} \left(\hat{\gamma}(n)_{d^n}(\lambda) \bigg| \hat{\beta}(n)_{0^n: d^n}(\lambda)\right) = P_{U_i^{(n)}(U_t^{(n)}|d^n)} \left(\hat{\beta}(n)_{0^n: b^n}(\lambda)\right),
\]

where \(\hat{\beta}(n)_{0^n: b^n}(\lambda)\) is the \(\lambda\)-th surviving path. We show (23) and (24) in Section VII-B.
Algorithm 3 updateU(U, k, b^{k-1})

1: for \( c^{n-k} \in [0^{n-k} : 1^{n-k}] \) do
2: \( U[k-1][c^{n-k}0][b_{k-2}] \leftarrow U[k][c^{n-k}0] \oplus U[k][c^{n-k}1] \)
3: \( U[k-1][c^{n-k}1][b_{k-2}] \leftarrow U[k][c^{n-k}1] \)
4: end for
5: if \( b_{k-2} = 1 \) then updateU(U, k − 1, b^{k-2})

When Algorithm 4 is used in the decoder of polar source code that has access to the codeword \( u_{\mathcal{X}} \) and side information vector \( y_{[0:n-1]} \), we define \( P_{U_{\mathcal{X}}}^{(0)} \) by (21) and obtain reproduction \( \{\hat{x}_{c^n}(l)\}_{c^n \in [0^{n-1}:1]} \) defined as
\[
\hat{x}_{c^n}(l) \equiv U[l][0][c^n][b_{n-1}],
\]
where
\[
l \equiv \arg \max_{\lambda} P[l].
\]
When the outer parity check function \( \text{parity} \) generates an extension \( s \equiv \text{parity}(x^n) \) to the codeword \( u_{\mathcal{X}} \), the corresponding reproduction is defined as (25) for an \( l \) satisfying \( \text{parity}\{\{\hat{x}_{c^n}(l)\}_{c^n \in [0^{n-1}:1]}\} = s \).

When Algorithm 4 is used in the decoder of systematic polar channel code that has access to channel output vector \( y_{[0:n-1]} \), and shared vector \( u_{\mathcal{X}} \), we obtain reproduction \( \{\hat{x}_{c^n}(l)\}_{c^n \in \mathcal{I}_0} \) defined by (25) and (26), where \( \mathcal{I}_0 \) is defined by (1). When we use the outer parity check function \( \text{parity} \) (e.g. polar code with CRC [10]) with check vector \( s \) satisfying \( s = \text{parity}(x^n) \) for all channel inputs \( x^n \), the resulting reproduction \( \{\hat{x}_{c^n}(l)\}_{c^n \in \mathcal{I}_0} \) is defined as (25) for an \( l \) satisfying \( \text{parity}\{\{\hat{x}_{c^n}(l)\}_{c^n \in [0^{n-1}:1]}\} = s \).

When Algorithm 4 is used in the decoder of non-systematic polar channel code, we have to prepare binary variables \( \{M[\lambda][b^n]\}_{\lambda \in \{0, \ldots, L-1\}, b^n \in \mathcal{I}_0} \) and insert
\[
M[\lambda][b^n] \leftarrow U[\lambda][n][c^n][b_{n-1}],
\]
\[
M[A + \lambda][b^n] \leftarrow U[A + \lambda][n][c^n][b_{n-1}],
\]
just after the renewal of \( U[\lambda][n][c^n][b_{n-1}] \) and \( U[A + \lambda][n][c^n][b_{n-1}] \), respectively (Lines 5 and 6 of Algorithm 6 and Lines 8, 11, 15, and 26 of Algorithm 7). This yields reproduction \( \{\hat{u}_{b^n}(l)\}_{b^n \in \mathcal{I}_0} \) defined as
\[
\hat{u}_{b^n}(l) \equiv M[l][b^n],
\]
where \( l \) is defined by (24). When we use the outer parity check function \( \text{parity} \) (e.g. polar code with CRC [10]) with check vector \( s \) satisfying \( s = \text{parity}(x^n) \) for all channel inputs \( x^n \), the resulting reproduction \( \{\hat{u}_{b^n}(l)\}_{b^n \in \mathcal{I}_0} \) of the non-systematic code is defined as (27) for an \( l \) such that corresponding channel input \( \{\hat{x}_{c^n}(l)\}_{c^n \in [0^{n-1}:1]} \) defined by (25) satisfies \( \text{parity}\{\{\hat{x}_{c^n}(l)\}_{c^n \in [0^{n-1}:1]}\} = s \).

For completeness, we introduce an algorithm for Line 5 of Algorithm 7 in Appendix D.

Remark 2: Line 17 of Algorithm 4 is unnecessary if we use the infinite precision real number variables. We assumed the use of the finite precision (floating point) real number variables to prevent \( P[\lambda] \) from vanishing as \( b^n \) increases. We can skip Line 17 of Algorithm 4 and Line 2 of Algorithm 5 while \( b^n \in \mathcal{I}_0 \) is satisfied continuously from the beginning (\( b^n = 0^0 \)). It should be noted that this type of technique is used in [10] Algorithm 10, Lines 20–25, where this technique is repeated \( Nn \) times. In contrast, Algorithm 4 uses this technique outside the renewal of parameters \( \{\Theta[\lambda]\}_{\lambda = 0}^{L-1} \) (Algorithm 2), where magnify \( P[\lambda] \) is repeated \( N \) times.

Remark 3: When we assume that \( L \) is a power of 2, \( \lambda = L \) is always satisfied at Line 13 of Algorithm 4. Accordingly, we can omit Line 14 of Algorithm 4 and Lines 9–12 of Algorithm 7 because \( \lambda + l \geq L \) is always satisfied.

VII. PROOFS

A. Proof of (19) and (20)

Here, we check that we can compute \( u_{b^n0}^{(k)} \) from \( u_{b^n0}^{(n)} \). We introduce the following theorems. In the proof of theorems, we write \( d^n < b^n \) when the corresponding integers satisfy the same relation.

Theorem 1: For a given \( b^n \equiv (b_0, b_1, \ldots, b_{n-1}) \), we have
\[
U[k][c^{n-k}][b_{k-1}] = u_{c^{n-k}b^n}^{(k)}
\]
for all \( k \) and \( c^{n-k} \in \{0, 1\}^{n-k} \) after the operations
\[
U[n][c^n][0] \leftarrow u_{b^n0}^{(n)},
\]
\[
U[n][c^n][1] \leftarrow u_{b^n1}^{(n)}.
\]
Algorithm 4 Successive-cancellation list decoder

**Input:** $I_1$, $u^{(n)}_{I_1}$, $\{P_{u,v}^{(n)}\}_{b,v\in[0^n:1^n]}$, $L$

1. $\Lambda \leftarrow 1$
2. for $b^n \in [0^n : 1^n]$ do $\Theta[0][0][b^n] \leftarrow P_{u,v}^{(n)}(0) - P_{u,v}^{(n)}(1)$
3. $P[0] \leftarrow 1$
4. for $b^n \in [0^n : 1^n]$ do
   5. for $\lambda \in \{0, ..., \Lambda - 1\}$ do $\Theta[\Theta[0][0][b^n]]$, $U[\lambda][n], n, b^n$
   6. if $b^n \in I_1$ then
      7. extendPath($b^n, A$)
   8. else
      9. if $2 \cdot \Lambda \leq L$ then
         10. splitPath($b^n, A$)
      11. $\Lambda \leftarrow 2 \cdot \Lambda$
      12. else
         13. prunePath($b^n, A$)
      14. $\Lambda \leftarrow L$
      15. end if
   16. end if
   17. magnifyP($\Lambda$)
   18. if $b_{n-1} = 1$ then
      19. for $\lambda \in \{0, ..., \Lambda - 1\}$ do updateU($U[\lambda][n], n, b^{n-1}$)
   20. end if
21. end for

Algorithm 5 extendPath($b^n, A$)

1. for $\lambda \in \{0, ..., \Lambda - 1\}$ do
2. $P[\lambda] \leftarrow P[\lambda] \cdot (1 + a \Theta[\lambda][n][b^n])$, where $u \equiv u^{(n)}_{b^n}$
3. $U[\lambda][n][c^m][b_{n-1}] \leftarrow u^{(n)}_{b^n}$
4. end for

updateU($U, n, d^{n-1}$) \hspace{1cm} (31)

employed for each $d^{n-1} \in [0^n : b^{n-1}]$. In particular, after the operations \(29\)--\(31\) for each $d^{n-1} \in [0^n : 1^{n-1}]$, \hspace{1cm} (32)

for all $c^n \in [0^n : 1^n]$.

**Proof:** For a given $b^n \in [0^n : 1^n]$, we have

\[
U[n][c^n][b_{n-1}] = u^{(n)}_{b^n} = u^{(n)}_{c^n b_{n-1}}
\]

after the operations \(29\) and \(30\).

From Line 4 of Algorithm 3, updateU($U, k, b^{k-1}$) is called only when $(b_{k-1}, ..., b_{n-1}) = 1^{n-k}$. Let us assume that $(b_{k-1}, ..., b_{n-1}) = 1^{n-k}$. Since $b^01^{n-k-1} < b^n$, we have the fact that updateU($k + 1, b^k$) is called and $U[k][c^{n-k-1}][0][0]$ and $U[k][c^{n-k-1}][0][0]$ are defined. Here, let us assume that $U[k][c^{n-k}][b] = u^{(k)}_{c^n b_{n-1}b}$ for all $c^{n-k}$ and $b \in \{0, 1\}$. Then we have

\[
U[k-1][c^{n-k}][b_{k-2}] = U[k][c^{n-k}][0][0] \oplus U[k][c^{n-k}][1]
\]

where the third equality comes from \(17\) and \(18\). In addition, we have

\[
U[k-1][c^{n-k}] = U[k][c^{n-k}][1] = u^{(k)}_{c^n b_{n-1}b}
\]

(33)
Algorithm 6 splitPath(b^n, A)
1: for λ ∈ {0, . . . , A − 1} do
2:   P[A + λ] ← P[λ] · (1 - Θ[λ][b^n])
3:   P[λ] ← P[λ] · (1 + Θ[λ][b^n])
4: end for

Algorithm 7 prunePath(b^n, A)
1: for λ ∈ {0, . . . , A − 1} do
2:   P[A + λ] ← P[λ] · (1 - Θ[λ][b^n])
3:   P[λ] ← P[λ] · (1 + Θ[λ][b^n])
4: end for
5: Define {Active[λ]}_A-1^λ=0 such that Active[λ] = 1 iff P[λ] is one of the L largest values of {P[λ]}_A-1^λ=0 (ties are broken arbitrarily).
6: for λ ∈ {0, . . . , A − 1} do
7:   if Active[λ] = 1 then
8:     U[λ][n][c^0][b_{n-1}] ← 0
9:   if Active[A + λ] = 1 and A + λ < L then
10:    copyPath(A + λ, λ)
11:    U[λ + λ][n][c^0][b_{n-1}] ← 1
12:    end if
13:   else if Active[A + λ] = 1 then
14:     copyPath(A + λ, λ)
15:     U[λ][n][c^0][b_{n-1}] ← 1
16:    Active[λ] ← 1
17:    Active[A + λ] ← 0
18: end if
19: end for
20: λ' ← 0
21: for λ ∈ {L, . . . , 2 · A - 1} do
22:   if Active[λ] = 1 then
23:     while Active[λ] = 1 do λ' ← λ' + 1
24:     P[λ'] ← P[λ]
25:     copyPath(λ', λ - A)
26:     U[λ][n][c^0][b_{n-1}] ← 1
27:     λ' ← λ' + 1
28:   end if
29: end for

where the last equality comes from 18. The above yields
\[ U[k-1][c^{n-k+1}][b_{k-2}] = u^{(k-1)}_{\omega^{n-k+1}b_{k-2}}, \]
for all c^{n-k+1}. Induction yields the relation 28 for all k and c^{n-k} for a given b^n. By letting k = 0 and b^n = 1^n, we have the fact that updateU(U, 0, b_{-1}) is called and U[n][c^0][b_{-1}] satisfies 32 for all c^n.

Theorem 2: Assume that
\[ \Theta[0][c^n] = \theta^{(0)}_{c^n} \]
for all c^n ∈ [0^n : 1^n]. Then, for a given b^n ≡ (b_0, b_1, . . . , b_{n-1}), we have
\[ \Theta[k][c^{n-k}] = \theta^{(k)}_{c^{n-k}b^k} \]
for all k and c^{n-k} ∈ {0, 1}^{n-k} after the operations
\[ \text{update}\Theta(\theta, n, d^{n-1}) \]
\[ U[n][c^0][0] ← u^{(n)}_{d^{n-1}0} \]
from Theorem 1. Then we have comes from (15).

Algorithm 8 copyPath($\lambda', \lambda$)

1: for $k \in \{0, \ldots, n\}$ do
2: for $c^k \in \{0^k, \ldots, 1^k\}$ do
3: $\Theta[\lambda'][k][c^k] \leftarrow \Theta[\lambda][k][c^k]$
4: for $b \in \{0, 1\}$ do
5: $U[\lambda'][k][c^k][b] \leftarrow U[\lambda][k][c^k][b]$
6: end for
7: end for
8: end for

Algorithm 9 magnifyP($A$)

1: $\maxP \leftarrow 0$
2: for $\lambda \in \{0, \ldots, A - 1\}$ do
3: if $\maxP < P[\lambda]$ then $\maxP \leftarrow P[\lambda]$
4: end for
5: for $\lambda \in \{0, \ldots, A - 1\}$ do $P[\lambda] \leftarrow P[\lambda]/\maxP$
6: $U[n][\theta][0][1] \leftarrow u_{(n)}$
7: $\text{update}(u, n, d^{n-1})$

for each $d^{n-1} \in [0^n : b^{n-1}]$ and

$\text{update}(\theta, n, b^{n-1})$.

**Proof:** We have the fact that $\text{update}(\theta, k - 1, b^{k-1})$ is called only when $(b_k, \ldots, b_{n-1}) = 0^{n-k}$. Let $b^0$ denote the null string.

Let us assume that (36) is satisfied for all $k \in \{1, \ldots, n - 1\}$ and $c^{n-k}$ and

$$\Theta[k - 1][c^{n-k+1}] = \Theta^{(k-1)}_{c^{n-k+1}b^{k-1}}$$

(37)

for all $c^{n-1}$ and $b^n$, where this equality is satisfied when $k = 1$ from assumption (35).

Assume that $b_{k-1} = 0$. Since $b^{k-2}0^{n-k+2} < b^n$, then $\text{update}(\theta, k - 1, b^{k-1})$ is called and

$$\Theta[k][c^{n-k}] = \Theta[k - 1][c^{n-k-1}] \cdot \Theta[k - 1][c^{n-k-0}]$$

$$= \Theta^{(k-1)}_{c^{n-k+1}b^{k-1}} \cdot \Theta^{(k-1)}_{c^{n-k+0}b^{k-1}}$$

$$= \Theta^{(k)}_{c^{n-k}b^{k-1}}$$

(38)

for all $c^{n-1}$, the first equality comes from Line 5 of Algorithm 3, the second equality comes from (37), and the last equality comes from (35).

Assume that $b_{k-1} = 1$. Since $b^{k-2}0^{n-k+2} < b^n$, $\text{update}(\theta, k - 1, b^{k-1})$ is called and we have (38). Furthermore, since $b^{k-2}0^{1+n-k+1} < b^n$, then $\text{update}(U, k, b^{k-1})$ is called and we have

$$U[k][c^{n-k}][0] = u^{(k)}_{c^{n-k}b^{k-1}}$$

(39)

from Theorem 11. Then we have

$$\Theta[k][c^{n-k}] = \Theta[k - 1][c^{n-k-1}] \oplus u \cdot \Theta[k - 1][c^{n-k-0}]$$

$$= \Theta^{(k)}_{c^{n-k+1}b^{k-1}} \oplus u \cdot \Theta^{(k-1)}_{c^{n-k+0}b^{k-1}}$$

$$= \Theta^{(k)}_{c^{n-k}b^{k-1}}$$

(40)

for all $c^{n-k}$, where

$$u \equiv U[k][c^{n-k}][0]$$

$$= u^{(k)}_{c^{n-k}b^{k-1}}$$

(41)

from (39), where the first equality comes from Line 7 of Algorithm 3 and (38), the second equality comes from (37), and the last equality comes from (16).

From (38) and (40), we have (36) for all $c^{n-1}$ and $b^n$ by induction.

$\blacksquare$
B. Proof of (43) and (44)

Here, we show (43) by proving the following theorem, where (43) is shown by letting $b^n \equiv 1^n$.

**Theorem 3:** Let $\hat{u}_{[0^n : b^n]}^{(n)}(\lambda)$ be the $\lambda$-th surviving path after employing one of Algorithms 5–7 (at Line 17 of Algorithm 4). We have

$$
\frac{P[\lambda]}{2^{[0^n : b^n]}} = \prod_{d^n \in [0^n : b^n]} P_{U_{d^n}^{(n)} | U_{[0^n : b^n]}^{(n)}} \left( \hat{u}_{d^n}^{(n)}(\lambda) \mid \hat{u}_{[0^n : b^n]}^{(n)}(\lambda) \right) = P_{U_{[0^n : b^n]}^{(n)}} \left( \hat{u}_{[0^n : b^n]}^{(n)}(\lambda) \right).
$$

**Proof:** After employing one of Algorithms 5–7 we have the substitution

$$
P[\lambda] \leftarrow P[\lambda] \cdot \left( 1 \mp \theta[\lambda][n][b^n] \right),
$$

where

$$
\hat{u}(\lambda) \equiv \hat{u}_{[0^n : b^n]}^{(n)}(\lambda)
$$

is the $b^n$-th symbol of the $\lambda$-th surviving binary path $\hat{u}_{[0^n : b^n]}^{(n)}(\lambda)$. Then we have

$$
\frac{P[\lambda]}{2^{[0^n : b^n]}} = \prod_{d^n \in [0^n : b^n]} \frac{1 \mp \theta_d^{(n)}(\lambda)}{2} = \prod_{d^n \in [0^n : b^n]} P_{U_{d^n}^{(n)} | U_{[0^n : b^n]}^{(n)}} \left( \hat{u}_{d^n}^{(n)}(\lambda) \mid \hat{u}_{[0^n : b^n]}^{(n)}(\lambda) \right) = P_{U_{[0^n : b^n]}^{(n)}} \left( \hat{u}_{[0^n : b^n]}^{(n)}(\lambda) \right),
$$

(44)

where

$$
\theta_d^{(n)}(\lambda) \equiv P_{U_d^{(n)} | U_{[0^n : b^n]}^{(n)}} \left( 0 \mid \hat{u}_{[0^n : b^n]}^{(n)}(\lambda) \right) - P_{U_d^{(n)} | U_{[0^n : b^n]}^{(n)}} \left( 1 \mid \hat{u}_{[0^n : b^n]}^{(n)}(\lambda) \right),
$$

(45)

the first equality is shown by induction, the second equality comes from Theorem 2, and the third equality comes from (43) and (44).

**APPENDIX**

A. Algorithm for Polar Transform

This section introduces the polar transform defined by (12) and (13). We assume that the following algorithm have access to memory space

$$
\left\{ u[k][b^n] : k \in \{0, 1\}, b^n \in [0^n : 1^n] \right\}
$$

and function $\lambda b(k)$ outputs the least significant bit of $k$, which equals $k \mod 2$. The result is stored in $\{u[1\lambda b(n)][b^n]\}_{b^n \in [0^n : 1^n]}$. It should be noted that, from the relation $G = G^{-1}$, we have the fact that $u = xG$ is equivalent to $x = uG$. This implies that we can obtain $x$ from $u$ so that $x = uG^{-1} = uG$ is satisfied.

**Algorithm 10** Polar transform

**Input:** $u^{(0)}_{[0^n : 1^n]}$

1: for $b^n \in [0^n : 1^n]$ do $u[0][b^n] \leftarrow u_{b^n}^{(0)}$
2: for $k \in \{0, \ldots, n - 1\}$ do
3: for $c^{n-k-1} \in [0^{n-k-1} : 1^{n-k-1}]$ do
4: for $b^k \in [0^k : 1^k]$ do
5: $u[\lambda b(k + 1)][c^{n-k-1}b^k0] = u[\lambda b(k)][c^{n-k-1}0b^k] \oplus u[\lambda b(k)][c^{n-k-1}1b^k]$
6: $u[\lambda b(k + 1)][c^{n-k-1}b^k1] = u[\lambda b(k)][c^{n-k-1}1b^k]$
7: end for
8: end for
9: end for
B. Algorithm for Systematic Channel Encoder

To find elements in \( \Theta \).

We introduce an improvement for Algorithm 2 by assuming \( \Theta \).

For simplicity, we assume that \( \Theta \).

The following algorithm finds \( X \) for the systematic encoder of a polar channel code based on [3].

The result is stored in \( \{ X[b^n] : b^n \in [0^n : 1^n] \} \), which is also used to refer to the value \( u_{\mathcal{I}_1} \) in Algorithm 12. For a given \( b^n \equiv (b_0, \ldots, b_{n-1}) \), let \( br(b^n) \) be defined as \( br(b^n) \equiv (b_{n-1}, \ldots, b_0) \).

Algorithm 11 Systematic Channel Encoder

Input: \( \mathcal{I}_1, x_{\mathcal{I}_2}, u_{\mathcal{I}_1} \)

1: for \( b^n \in [0^n : 1^n] \) do
2: \( \text{if} \ b^n \in \mathcal{I}_1 \) then
3: \( X[b^n] \leftarrow u_{b^n} \)
4: else
5: \( V[b^n] \leftarrow x_{br(b^n)} \)
6: end if
7: end for
8: \( \text{calcV}(n, b^n) \)
9: for \( b^n \in [0^n : 1^n] \) do \( X[br(b^n)] = V[b^n] \)

Algorithm 12 \( \text{calcV}(k, b^{n-k}) \)

1: if \( k = 0 \) then
2: \( \text{if} \ b^n \in \mathcal{I}_1 \) then \( V[b^n] \leftarrow X[b^n] \)
3: else
4: \( \text{calcV}(k-1, b^{n-k}) \)
5: \( \text{for} \ c^{k-1} \in [0^{k-1} : 1^{k-1}] \) do
6: \( \text{if} \ b^{n-k}c^{k-1} \notin \mathcal{I}_1 \) then \( V[b^{n-k}0c^{k-1}] \leftarrow V[b^{n-k}0c^{k-1}] \oplus V[b^{n-k}1c^{k-1}] \)
7: end for
8: \( \text{calcV}(k-1, b^{n-k}) \)
9: \( \text{for} \ c^{k-1} \in [0^{k-1} : 1^{k-1}] \) do \( V[b^{n-k}0c^{k-1}] \leftarrow V[b^{n-k}0c^{k-1}] \oplus V[b^{n-k}1c^{k-1}] \)
10: end if

C. Improvement of Algorithm 2 by Assuming \( \Theta[k][c^{n-k}] \in \{-1, 0, 1\} \)

We introduce an improvement for Algorithm 2 by assuming \( \Theta[k][c^{n-k}] \in \{-1, 0, 1\} \).

For simplicity, we assume that \( \Theta[k][c^{n-k}] \) is represented by a 3-bit signed integer consisting of a sign bit and two bits representing an absolute value.

Algorithm 2' update \( \Theta(U, U, k, b^k) \)

1: if \( k = 0 \) then return
2: \( \text{if} \ b_{k-1} = 0 \) then \( \text{update}\Theta(U, U, k - 1, b^{k-1}) \)
3: \( \text{for} \ c^{n-k} \in [0^{n-k} : 1^{n-k}] \) do
4: \( \text{if} \ b_{k-1} = 0 \) then
5: \( \Theta[k][c^{n-k}] \leftarrow \Theta[k-1][c^{n-k}] \cdot \Theta[k-1][c^{n-k}] \)
6: else
7: \( \Theta[k][c^{n-k}] \leftarrow \Theta[k-1][c^{n-k}] + u \Theta[k-1][c^{n-k}], \) where \( u \equiv \Theta[k][c^{n-k}][0] \)
8: \( \text{if} \ \Theta[k][c^{n-k}] < 0 \) then \( \Theta[k][c^{n-k}] \leftarrow -1 \)
9: \( \text{if} \ \Theta[k][c^{n-k}] > 0 \) then \( \Theta[k][c^{n-k}] \leftarrow 1 \)
10: end if
11: end for
D. Algorithm for Line 5 of Algorithm 7

We can implement Line 5 of Algorithm 7 by markPath(λ) defined as Algorithm 13.

We assume that Algorithms 13, 14, and 15 can access the memory space \( \{P[\lambda]\}_{\lambda=0}^{2L-1} \), \( \{\text{Index}[\lambda]\}_{\lambda=0}^{2L-1} \), and \( \{\text{Active}[\lambda]\}_{\lambda=0}^{2L-1} \), where \( \text{Index}[\lambda] \in \{0, \ldots, 2L - 1\} \) is an integer variable. The result is stored in \( \{\text{Active}[\lambda]\}_{\lambda=0}^{2L-1} \).

Algorithm 13 markPath(λ)

1: for \( \lambda \in \{0, \ldots, \Lambda - 1\} \) do
2: \( \text{Index}[\lambda] = \lambda \)
3: \( \text{Active}[\lambda] = 0 \)
4: end for
5: selectPath(0, 2 · \Lambda - 1)
6: for \( \lambda \in \{0, \ldots, \Lambda - 1\} \) do \( \text{Active}[\text{Index}[\lambda]] = 1 \)

Algorithm 14 selectPath(left, right)

1: if \( \text{left} < \text{right} \) then
2: \( \lambda \leftarrow \text{partition}(\text{left}, \text{right}) \)
3: if \( \lambda > \Lambda \) then selectPath(\text{left}, \lambda - 1)
4: if \( \lambda < \Lambda \) then selectPath(\lambda + 1, \text{right})
5: end if

Algorithm 15 partition(left, right)

1: Let \( \lambda \) be one of the values in \( \{\text{left}, \ldots, \text{right}\} \) selected uniformly at random and call swapIndex(\lambda, \text{right}).
2: \( p \leftarrow P[\text{right}] \)
3: \( \lambda \leftarrow \text{left} - 1 \)
4: \( \lambda' \leftarrow \text{right} \)
5: loop
6: \( \lambda \leftarrow \lambda + 1 \)
7: \( \lambda' \leftarrow \lambda' - 1 \)
8: while \( P[\text{Index}[\lambda]] \geq p \) and \( \lambda \leq \lambda' \) do \( \lambda \leftarrow \lambda + 1 \)
9: while \( P[\text{Index}[\lambda']] < p \) and \( \lambda \leq \lambda' \) do \( \lambda' \leftarrow \lambda' - 1 \)
10: if \( \lambda \geq \lambda' \) then break
11: swapIndex(\lambda, \lambda').
12: end loop
13: swapIndex(\lambda, \text{right})
14: return \( \lambda \)

Algorithm 16 swapIndex(\lambda, \lambda')

1: \( i \leftarrow \text{Index}[\lambda] \)
2: \( \text{Index}[\lambda] \leftarrow \text{Index}[\lambda'] \)
3: \( \text{Index}[\lambda'] \leftarrow i \)

Remark 4: As mentioned in [10], we can simply sort \( \{\text{Index}[\lambda]\}_{\lambda=0}^{\Lambda-1} \) so that \( P[\text{Index}[0]] \geq P[\text{Index}[1]] \geq \cdots \geq P[\text{Index}[\Lambda-1]] \) instead of calling selectPath(0, 2 · \Lambda - 1) at Line 5 of Algorithm 13. Although the time complexity of sorting is \( O(\Lambda \log \Lambda) \), it could be faster than selectPath(0, 2 · \Lambda - 1) when \( \Lambda \) is small.

Remark 5: Line 1 of Algorithm 15 which can be omitted, guarantees that the average time complexity of selectPath(0, 2 · \Lambda - 1) is \( O(\Lambda) \). We can replace this line by selecting the index corresponding to the median of \( \{P[\lambda]\}_{\lambda=\text{left}}^{\text{right}} \) to guarantee worst-case time complexity \( O(\Lambda) \) (see [4]).

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