Sliding of a vortex solid with self-generated randomness in a frustrated Josephson junction array

Hajime Yoshino, Tomoaki Nogawa\textsuperscript{A}, Bongsoo Kim\textsuperscript{B}
Osaka Univ. Osaka, Japan, \textsuperscript{A}U. of Tokyo, Tokyo, Japan, \textsuperscript{B}Changwon Univ., Changwon, Korea
E-mail: yoshino@ess.sci.osaka-u.ac.jp

Abstract. A class of Josephson junction arrays under external magnetic field with anisotropic Josephson couplings realizes a gapless band of metastable vortex solid states with self-generated randomness. Each of the metastable states is characterized by a frozen pattern of transverse undulation of vortex stripes which run along the direction of weaker Josephson coupling. The vortex stripes can slide into the direction of stronger Josephson coupling even at zero temperature, eliminating the pinning effects of the underlying lattice structure of the array, but blocked into the direction of weaker coupling. Because of the sliding of the vortex matter, the phase of superconducting order parameter remain disordered at macroscopic level even at zero temperature. This is an extremely strong version of the spin-chirality decoupling which should be manifested in electric transport properties.

1. Introduction
Josephson junction arrays (JJA) under the magnetic field [1, 2] provide a very interesting playground to study consequences of frustration. JJA is a network of superconducting islands connected by Josephson junctions in the form, say, of a square lattice of size $N = L \times L$ as shown in Fig. 1. At low enough temperatures, the amplitude of the superconducting order parameter on each of the island becomes finite and take the same value. However the phases $\theta_i$ of the order parameter on the islands $i = 1, 2, \ldots, N$ would take any values independently if the Josephson coupling is absent. With the (ferromagnetic) Josephson coupling, the difference of the phases $\theta_i - \theta_j$ between the neighbouring islands $i$ and $j$ become minimized leading to ordering of the phases across the network just like a ferromagnetic XY model. Now if one switches on an external magnetic field of strength $B$, the JJA becomes a kind of frustrated crystal - a crystal which is forced to host a given number density $f = Ba^2/\phi_0$ of dislocations (vorticies in the phase of the superconducting order parameter). Here $a^2$ is the area of a plaquette and $\phi_0$ is the flux quantum. Because of the injected dislocations the crystal is now strongly frustrated.

Recently we found a novel state of matter in a class of frustrated JJA on a square lattice with a) \textit{anisotropic} Josephson coupling into x and y directions (see Fig. 1) and with b) \textit{irrational} number density $f$ of the vortices per plaquette [3, 4]. Here we review the results with particular emphasis on the following two prominent features. Firstly we found a band of metastable vortex solid states with self-generated randomness \textit{in the absence of quenched disorder}. Whether geometrical frustration alone can realize disordered, glassy states in the absence of extrinsic quenched disorder is a long-standing issue[6, 7]. Such a question is relevant in the context of frustrated magnets. Secondly we found an extremely strong realization of an analogue of the
spin-chirality decoupling [8] found in frustrated magnets. Here the ordering of the "chirality" is translated to a formation of a vortex solid in the JJA. We found that the vortex solid can slide into the direction of stronger Josephson coupling. This should lead to destruction of the long-ranged ordering of the phase of the superconducting order parameter even at zero temperature. In this paper we also present some numerical results of Monte Carlo simulations performed to observe the spin-chirality decoupling at finite temperatures.

In the limit \( f \to 1/2 \), the system is called fully frustrated XY (FFXY) model \([10]\). In this case the charge of the vortices take half-integer values \( \cdots, -3/2, -1/2, 1/2, 3/2, \cdots \) (see below) and the vortices become literally equivalent to the chiralities which play important roles in frustrated magnets (see \([8]\) for a recent review). Recent extensive numerical studies on the FFXY model \([11, 12, 13]\) as well as vectorial spin-glasses \([14, 15]\) show convincingly that the chirality exhibit long-ranged order at a transition temperature \( T_c \) above the spin transition temperature \( T_s > 0 \). Quite remarkably the decoupling can become much more stronger in the present system with irrational \( f \) in the sense that \( T_s = 0 \) due to the free sliding of the vortex solid at zero temperature while the vortex solid itself appear to be developed at a finite temperature \( T_c > 0 \). (See sec.4)

In the next section we define our model. In sec. 3 we review our resent work \([4]\). In sec. 4 we present numerical results on the “spin-chirality decoupling” at finite temperatures. Finally we conclude this paper in sec. 5 with additional discussions on the possibility of experiments.

**Figure 1.** A schematic picture of a Josephson junction array (JJA). The superconducting islands (circles) are connected via Josephson junctions (links). Anisotropic JJA with different strength of the Josephson coupling into \( x \) and \( y \) directions with the ratio \( \lambda \) can be created in laboratories by lithography techniques with which the spacings between the superconducting islands can be controlled \([9]\). The squares in the plaquettes represent vortices. (Figure taken from \([4]\))

2. Model
The effective Hamiltonian of the JJA is given by \([1]\),

\[
H = - \sum_{<i,j> \parallel x-axis} \cos(\psi_{ij}) - \lambda \sum_{<i,j> \parallel y-axis} \cos(\psi_{ij})
\]  
(1)

with the gauge-invariant phase difference, \( \psi_{ij} \equiv \theta_i - \theta_j - A_{ij} \). We measure temperature \( T \) in a unit with \( k_B = 1 \). Here \( \lambda \) is the ratio of the strength of the Josephson couplings along \( x \) and \( y \)-axis. In the following we only examine \( \lambda \geq 1 \), which is obviously sufficient by symmetry. The vector potential \( A_{ij} (= -A_{ji}) \) is defined such that directed sum of them around each cell is \( 2\pi f \).

Vortex charge \( v_i \) of the vortex at the plaquette associated with the \( i \)-th vertex is defined by taking directed sum of \( \langle \psi_{ij} / 2\pi - [\psi_{ij} / 2\pi]_n \rangle \) on the junctions around the plaquette. Here \( [x]_n \) denotes the nearest integer of the real variable \( x \). It takes values \( \ldots, -1-f, -f, 1-f, \ldots \).

Any irrational number \( f \) can be approximated by a series of rational numbers \( p/q \) with integer \( p \) and \( q \). For instance we used a series of rational numbers \( p/q = 5/13, 8/21, 13/34, 21/55, 34/89, 55/144, 89/233 \) which approximate \((3 - \sqrt{5})/2 \approx 0.38196601 \ldots \). This allows us to use the periodic boundary conditions for both \( x \) and \( y \) directions in systems of size \( L \times L \). With this convention the ratio \( f = p/q \) converges to the target irrational number in \( L \to \infty \) limit.
Figure 2. Vortex patterns in an irrationally frustrated JJA under external magnetic field with anisotropic coupling. Here $\lambda = 1.5$ so that the coupling is stronger along $y$ direction. The system size is $L = 55$. A fraction $f = 21/55$, which approximates an irrational number $(3-\sqrt{5})/2 = 0.381966...$, of the plaquettes are occupied by vortices with charge $1-f$ represented by filled squares. The left panel shows an equilibrium vortex pattern at $T = 0.2$ and the right panel displays that at a nearby energy minimum. (Figure taken from [4])

3. Vortex stripes in the strongly anisotropic limit

Let us now review our recent theoretical results [4] focusing on the formation of vortex stripe states with frozen transverse undulations in strongly anisotropic case $\lambda \gg 1$.

In Fig. 2 we show an example of the vortex pattern observed numerically in an equilibrium at finite temperature (left panel) and in a nearly energy minimum (right panel). The equilibrium configuration is generated during a Monte Carlo simulation (see next section for the details). The energy minimum is obtained by an energy descent algorithm starting from the thermalized configuration. Apparently the vortices exhibit stripes running parallel to the direction of weaker coupling. The formation of the stripes itself may be not so surprising since the effective repulsive interactions between the vortices becomes anisotropic with the anisotropic Josephson coupling. Remarkable features are that I) the stripes are disordered by transverse undulations (in the absense of quenched disorder) but that II) the undulated patterns are stacked regularly along the stronger coupling. Starting from different thermalized configurations, different energy minima with different realizations of the stacked undulations can be sampled [4].

In [4], we obtained explicit analytical forms which describe the ground state (vortex stripes without the undulation) and the metastable states (vortex stripes with the undulation). The strategy is to start from an ansatz that the gauge invariant phase differences $\psi$ across the Josephson junctions along the $x$ and $y$ directions connecting neighbouring islands (vertexes) can be written as,

$$
\psi(x,y)(x+1,y) = \phi_x[y + \alpha(x)] \quad \psi(x,y)(x,y+1) = \phi_y[y + \alpha(x)] \quad (2)
$$

Here $\phi_x[y]$ and $\phi_y[y]$ are unknown functions defined on the “folded coordinate” $[y] = fy - \text{int}(fy)$ where $\text{int}(x)$ is the floor function. The folded coordinate takes values limited in the range $0 \leq [y] < 1$. The crucial point is that if $f$ is irrational, the vertexes of the JJA uniformly fill the entire range of the folded coordinate $[y]$ in the limit $N \to \infty$ (even with, e. g., $f = 1/2 + \epsilon$ with infinitesimal irrational $\epsilon$). Thus we can treat $[y]$ as a continuous variable. Quite amusingly the set of functions can be regarded as a two dimensional generalization of the so called hull function which play central roles in the analysis of the sliding/jamming in the one-dimensional Frenkel-Kotorova model extensively studied by S. Aubry and coworkers [16] and related one-dimensional friction models such as the Matsukawa-Fukuyama model [17, 18]. (See [5] for a review on the relation among the seemingly unrelated problems.)
The functions $\phi_x[y]$ and $\phi_y[y]$ obey two equations (see [4] for the details). One follows from the constraint that directed sum over $\psi_{ij}$ around each plaquette must be $-2\pi f$. The other follows from the conservation of the Josephson currents at each vertex (force balance condition) in each energy minimum. By a perturbative analysis in series of $1/\lambda$ starting from infinite anisotropy limit $\lambda = \infty$, we solved the equations and computed the functions $\phi_x[y]$ and $\phi_y[y]$ explicitly.

Now let us focus on the parameter $\alpha(x)$ which appears in Eq. (2). Physically it describes the transverse undulation of vortices such as the one shown in the right panel of Fig. 2. In the ground state we find $\alpha(x)$ does not depend on $x$, i.e. $\alpha(x) = \text{const}$ and the solution for the hull functions $\phi_x[y]$ and $\phi_y[y]$ describe a simple horizontal stripe of vortices running along the direction of weaker coupling.

Quite remarkably we found a continuous spectrum of solutions with non-zero transverse undulation parametrized by an arbitrary continuous function $\alpha(x)$. By computing their energies we find that they constitute a continuous, gapless band of the energy. Although they have higher energies than the ground state, they must be regarded as metastable states since they themselves satisfy the force balance (current conservation) condition. Thus the irrationally frustrated JJA admits a continuous, gapless spectrum metastable states with self-generated randomness.

A very important consequence of the existence of the analytic hull functions $\phi_x[y]$ and $\phi_y[y]$ is the sliding. Starting from any metastable state parametrized by an arbitrary continuous function $\alpha(x)$ (including the ground state $\alpha(x) = \text{const}$), there is a flat band of infinitely many states with exactly the same energy which can be obtained by a uniform translation $\alpha(x) \to \alpha(x) + \Delta \alpha$ with arbitrary shift $\Delta \alpha$. Physically this means sliding of the vortices into the direction of stronger coupling. This would not be surprising at all in the case of clean bulk superconductors without any pinning centers [1]. However it must be recalled that there is an underlying lattice structure in the present system due to the array itself which naturally gives rise to pinning effects. Indeed there are non-zero critical currents in the cases of rational $f$ [19]. The presence of the sliding mode across the flat band of the states mean that this pinning effect is effectively cancelled out in the case of irrational $f$. Moreover we must note that the sliding mode is present only along the direction of stronger coupling and absent along the weaker coupling. Thus the vortex solid can flow without energy dissipation along the stronger coupling but completely blocked along the weaker coupling. This strong anisotropy of the mobility of the vortex solid should be directly reflected in the transport properties of the system as we discuss in the end of this paper.

4. “Spin-chirality decoupling”

Sliding of the vortices discussed at the end of the preceding section immediately means extremely strong spin-chirality decoupling. The point is that configuration of the phase (spin) $\theta_i$ is not fixed only by the structure of the vortices. The uniform translation of the vortices along one axis changes the phase differences across the system along the orthogonal axis. The presence of the sliding mode along the stronger coupling means the phase differences along the weaker coupling can be changed by an arbitrary amount. The phase cannot establish long-ranged order even at zero temperature.

Now let us present some numerical results to look for the signature of the spin-chirality decoupling at finite temperatures. We performed Monte Carlo (MC) simulations on systems with $L = 13 - 89$ using $20 - 120$ temperatures in the temperature range $T = 0.2 - 0.4$ by the exchange MC method [20]. More precisely each MC step consists of one sweep over the system by the Metropolis updates of each phase $\theta_i$, one step of the over-relaxation [21] and one step of the exchange MC. We used $10^5 - 10^6$ MC steps for the equilibration and observations.

In Fig. 3 we display the scaled correlation length $\xi\sqrt{}/L$ of the vortices measured by the same method as in [22]. (See [22] for the details of the method, which is a standard technique for spin-glasses [14, 15].) Here the correlation length along the direction of stronger coupling is shown.
Figure 3. Scaled correlation lengths in equilibrium observed by Monte Carlo simulations. a) Correlation length of the vortex $\xi_V$ and b) phase $\xi_P$. Here $\lambda = 1.75$ for which we estimate the critical temperature of the vortices $T_c \sim 0.26$ by the linear fit shown in c).

The scaled correlation length of vortices $\xi_V/L$ of different sizes exhibit crossings suggesting a 2nd order phase transition at some finite critical temperature $T_c$. To estimate the critical temperature $T_c$, we extracted the crossing temperature $T_{\text{cross}}(L)$ of the scaled correlation length of the vortices $\xi_V$ between two system sizes $(L, L') = (13, 21), (21, 34), (34, 55)$ and $(55, 89)$. By fitting the crossing temperatures $T_{\text{cross}}(L)$ to a fitting formula $T_{\text{cross}}(L) = T_c + \text{const}/L$ we estimated the critical temperature $T_c(\lambda)$. As shown in the panel c) of Fig. 3, we find non-zero critical temperature $T_c(\lambda = 1.75) \sim 2.6$. This is at marked variance with the isotropic $\lambda = 1$ case [22] in which $\xi_V \sim T^{-2.2}$ suggesting $T_c(\lambda = 1) = 0$.

On the other hand the scaled correlation length $\xi_p$ of the phases exhibit no hint of phase transition at finite temperatures, as expected from the analysis on the low lying states discussed in the preceding section. To sum up, we find clear numerical evidences that the vortex (chiral) solid state appears at a finite temperature $T_c > 0$ while the phase (spin) remains disordered at finite temperatures, i.e. $T_s = 0$.

5. Discussions

To conclude we presented a review on our recent theoretical results on the irrationally frustrated anisotropic Josephson junction arrays. In particular we focused on the nature of the low lying metastable states which exhibit several novel features, i.e. self-generated randomness without quenched disorder, sliding and spin-chirality decoupling. We also presented a set of numerical results which strongly suggests formation of the vortex (chiral) solid state a a finite temperature $T_c > 0$ while the phase (spin) remains disordered at finite temperature $T_s = 0$. This is an extremely strong form of the spin-chirality decoupling. We plan to present more details of the numerical simulations at finite temperatures elsewhere.

Finally let us discuss possible experimental measurements concerning our problem. In [3] we reported numerical simulations of the transport properties of the present system by the method of RCSJ (Resistively and Capacitively Shunted Junction) model performed at zero temperature. We found clear evidences of sliding/jamming of the vortex solid along the direction of stronger/weaker coupling. This agrees well with the analysis of the low lying states discussed in sec. 3. Thus the standard transport measurement, i.e characterization of the current-voltage relation, should yield superconducting/normal behavior with respect to the electric current injected along the direction of stronger/weaker coupling. The coexistence of the macroscopic superconducting/normal behaviour would be a proof the spin (phase) - chirality (vortex) decoupling. Let us recall again that anisotropic JJAs can be created in laboratory by
standard lithography techniques [9].

Similar frustrated crystals in which the frustration is brought into the system in the form of gauge field would be found in also other systems. An example is the charge density wave system on ring crystal [23, 24]. Another example is oxide high-Tc superconductors under external magnetic field parallel to the ab-plane [25]. It would be very interesting to investigate these systems from the viewpoint similar to our works.

Acknowledgement We gratefully acknowledge Hikaru Kawamura and Hiroshi Matsukawa for interesting discussions over many years on related problems which inspired our works. We thank Toshihito Osada for useful discussions on the experiments on JJAs. We thank the Supercomputer Center, ISSP, University of Tokyo for the use of the facilities. This work is supported by Grant-in-Aid for Scientific Research on Priority Areas "Novel States of Matter Induced by Frustration" (1905200*) and Grant-in-Aid for Scientific Research (C) (21540386).

References
[1] Tinkham M. 2004 Introduction to Superconductivity, Courier Dover Publications.
[2] van der Zant H. S. J., Rijken H. A., and Mooij J. E. 1991 J. Low. Temp. Phys., 82 67.
[3] Yoshino H., Nogawa T. and Kim Bongsoo 2009 New J. Phys. 11 013010.
[4] Yoshino H., Nogawa T. and Kim Bongsoo 2010 Phys. Rev. Lett. 105 257004.
[5] Yoshino H., Nogawa T. and Kim Bongsoo 2010 Prog. Theor. Phys. Supplement 184 153.
[6] Sadoc J. F. Sadoc, Mosseri R. 1999 Geometrical frustration, Cambridge University Press.
[7] Tarjus G., Kivelson S. A., Nussinov Z. and Viot P. 2005 J. Phys. Condens. Matters 17 R 1143.
[8] Kawamura H. 2010 J. of Phys. Soc. Jpn. 79 011007.
[9] Saito S. and Osada T. 2000, Physica B: Condensed Matter 284-288 614.
[10] Teitel S. and Aubry S. 1983 J. Phys. C 16 1593.
[11] Matsukawa H. and Fukuyama H. 1994 Phys. Rev. B 49, 17286.
[12] Kawaguchi T. and Matsukawa H. 1997 Phys. Rev. B 56 13932.
[13] Creutz M. 1987, Phys. Rev. D 36, 515.
[14] Granato E. 2008 Phys. Rev. Lett. 101 027004.
[15] Tanda S., Tsunetsu T., Okajima Y., Inagaki K., Ya-aya K. and Hatakenaka N. 2002 Nature 417 397.
[16] Nogawa T. and Nemoto K. 2006, Phys. Rev. B 73 184504.
[17] Hu X. and Tachiki M. 1998 Phys. Rev. Lett. 80 4044.