A dimensional reduction of an \((n+1)\) compartmental model

E Cahyono\(^1\), P Elastic\(^1\), Y Soeharyadi\(^2\) and Mukhsar\(^3\)

\(^1\) Department of Mathematics, Universitas Halu Oleo, Indonesia
\(^2\) Department of Mathematics, Institut Teknologi Bandung, Indonesia
\(^3\) Department of Statistics, Universitas Halu Oleo, Indonesia

edi_cahyono@innov-center.org (corresponding author)

Abstract. An \((n+1)\) dimensional compartmental model is studied. It is an \(n\) dimensional model sits in \((n+1)\) dimensional space, consists of \((n+1)\) variables and \((n+1)\) equations. The model is a generalization of well-known SIR model. Reduced dimensional model is introduced. The reduced model consists of \(n\) variables and \(n\) equations. The equilibrium points of the original model and reduced model are discussed. The stability of the equilibrium points are analyzed and compared. Numerical simulation is applied for \(n = 5\). The numerical result which is the evolution of the variables is presented.

1. Introduction

A system consisting of \((n + 1)\) ordinary differential equations is considered. Study on such system is still an interesting topic, both from theoretical point of view and the applications. Study on differential equations was initialized after the invention of calculus where mathematics at those times meant study of change, see [1] for a brief history, modern work can be found in the books [2, 3]. Moreover, the applications have spread not only in science and engineering, but also from public health to economics.

For the applications on public health for disease spread, one among well-known work was SIR model, a model of disease spread among compartmental populations, see [4] for detail discussion. SIR stands for susceptible, infected, recovered. A simple model applied for complex transitions in epidemics was discussed in [5]. The approach of mathematical of change was applied to study time series modeling of childhood diseases [6], for disease spread among predator prey system [7]. More recently, study on a vector-borne disease model with age of vaccination was presented in [8]. An overview on the application of system dynamics for epidemics was well narrated in [9].

In economics and finance, the use of nonlinear dynamic models has expanded rapidly since the late of 1980s, [10]. Guegan [11] discussed the use of dynamical chaotic systems in economics and finance by employing methods that could be useful in practice to detect the existence of chaotic behavior inside real data sets. Jakimowicz [12] discussed model economic systems strived for a state called “the edge of chaos”. He considered a case concerned an economy based on a two-stage accelerator, where the economic cycle adopted the form of chaotic hysteresis, a case concerned a Cournot-Puu duopoly model in which striving for the edge of chaos stems from profit maximization by entrepreneurs. It was observed that the evolution of systems at the edge of chaos could be sudden. Some recent works for application on business cycle for discrete and continuous time have been reported in [13, 14].

2. A system of \(n + 1\) state variables
Let \( t \geq 0 \), an integer \( n > 2 \) and \( x_i \) for \( i = 1, \ldots, n+1 \) be non negative functions of \( t \). Consider a system of equations in the form

\[
\begin{align*}
\dot{x}_1 &= -a_1 \frac{x_i x_{n+1}}{N} \\
\dot{x}_i &= a_{i-1} \frac{x_{i-1} x_{n+1}}{N} - a_i \frac{x_i x_{n+1}}{N}, \text{ for } i = 2, \ldots, n-1 \\
\dot{x}_n &= a_{n-1} \frac{x_{n-1} x_{n+1}}{N} - a_n x_n \\
\dot{x}_{n+1} &= a_{n+1} (N - x_{n+1} - \sum_{i=1}^{n-1} x_i)
\end{align*}
\]

(1)

defined in \( n \)-dimensional set

\[
S_{n+1} = \{ (x_1, x_2, \cdots, x_{n+1}) : \sum_{i=1}^{n+1} x_i = N \}.
\]

(2)

Observe that \( S_{n+1} \) is an \( n \)-dimensional set which sits in \((n + 1)\)-dimensional ambient space. This problem is a generalization of the results discussed in [15], where its application is for knowledge dissemination [16] for \( n = 3 \). The analysis was conducted to the system (1) directly, to the reduced system, and also the relation of the two systems. For the case \( n = 3, 5 \) system (1) has also directly been studied in [17]. For the case \( n = 2 \), see [15], the system is in the form

\[
\begin{align*}
\dot{x}_1 &= -a_1 \frac{x_1 x_3}{N} \\
\dot{x}_2 &= a_1 \frac{x_1 x_3}{N} - a_2 \frac{x_2 x_3}{N} \\
\dot{x}_3 &= a_2 x_2
\end{align*}
\]

Transforming system (1) into a system consisting \( n \) equations can be done by applying the restriction of \( S_{n+1} \), i.e. \( \sum_{i=1}^{n+1} x_i = N \) to eliminate \( x_n \). Hence, the corresponding system is in the form

\[
\begin{align*}
\dot{x}_1 &= -a_1 \frac{x_i x_{n+1}}{N} \\
\dot{x}_i &= a_{i-1} \frac{x_{i-1} x_{n+1}}{N} - a_i \frac{x_i x_{n+1}}{N}, \text{ for } i = 2, \ldots, n-1 \\
\dot{x}_{n-1} &= a_{n-2} \frac{x_{n-1} x_{n+1}}{N} - a_{n-1} x_{n-1} \\
\dot{x}_{n+1} &= a_{n+1} \left( N - x_{n+1} - \sum_{i=1}^{n-1} x_i \right)
\end{align*}
\]

(3)

Proposition 1

System (3) has an equilibrium point \( P(x_i; x_i = 0, \text{ for } i = 1, \ldots, n-1, x_{n+1} = N) \) and an equilibrium manifold \( M = \{ (x_i; i = 1, \ldots, n-1, n+1); x_{n+1} = 0, \sum_{i=1}^{n-1} x_i = N \} \).

The proof is obvious.

Proposition 2

The equilibrium point \( P(x_i; x_i = 0, \text{ for } i = 1, \ldots, n-1, x_{n+1} = N) \) is stable, and the equilibrium manifold \( M = \{ (x_i; i = 1, \ldots, n-1, n+1); x_{n+1} = 0, \sum_{i=1}^{n-1} x_i = N \} \) is unstable.

Proof

This Jacobian matrix of (3) evaluated at \( P \) is

\[
J_P = \begin{bmatrix}
-a_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
a_1 & -a_2 & 0 & 0 & 0 & 0 & 0 \\
0 & a_2 & -a_3 & 0 & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & -a_{n-2} & 0 & 0 & 0 \\
0 & 0 & 0 & a_{n-2} & -a_{n-1} & 0 & 0 \\
-a_n & -a_n & -a_n & \cdots & -a_n & a_n & -a_n
\end{bmatrix}_{n \times n}
\]
Eigenvalues of \( J_p \) are
\[
\lambda_{p,i} = -a_i, \text{ for } i = 1, \ldots, n.
\]
Observe that all the eigenvalues are negative.

On the other hand, let \( Q(x_i^*; i = 1, \ldots, n-1, n+1) \) be any point in \( M \). The Jacobian matrix of the system (3) evaluated at \( Q \) is
\[
J_Q = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & -\beta_1 \\
0 & 0 & 0 & \cdots & 0 & \beta_1 - \beta_2 \\
0 & 0 & 0 & \cdots & 0 & \beta_2 - \beta_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -a_n & \beta_{n-2} - \beta_{n-1} \\
-a_n & -a_n & -a_n & \cdots & -a_n & -a_n
\end{bmatrix}_{n \times n}
\]
where \( \beta_i = \frac{a_i x_i^*}{N} \), for \( i = 1, \ldots, n-1 \), and \( \sum_{i=1}^{n-1} x_i^* = N \). Eigenvalues of \( J_Q \) are
\[
\lambda_{Q,i} = 0, \text{ for } i = 1, \ldots, n-2, \text{ and }
\lambda_{Q,n-1} = -\frac{a_n}{2} - \frac{\sqrt{a_n^2 + 4 a_n \beta_{n-1}}}{2} < 0,
\]
\[
\lambda_{Q,n} = -\frac{a_n}{2} + \frac{\sqrt{a_n^2 + 4 a_n \beta_{n-1}}}{2} > 0.
\]
Eigenvalue \( \lambda_{Q,n} \) guarantees that the equilibrium manifold \( M \) is not stable. Hence, \( P \) is a stable point and \( M \) is an unstable manifold.

**Proposition 3**
Let \( (x_i(t); i = 1, \ldots, n-1, n+1) \) be a solution of system (3), then, \( x_{n+1}(t) \) is a non-decreasing function.

The proof is obvious, since \( \dot{x}_{n+1} \) is not negative.

**Proposition 4**
Let \( (x_i(t); i = 1, \ldots, n-1, n+1) \) be a solution of system (3). If \( x_i(0) = 0 \), for \( i = 1, \ldots, n-1, n+1 \), then for \( x_i(t) = 0 \), for \( i = 1, \ldots, n-1 \) and \( t > 0 \). Moreover,
\[
x_{n+1}(t) = C_0 \exp(-a_n t) + N,
\]
where \( C_0 = x_{n+1}(0) - N \).

**Proof**
Let \( x_i(0) = 0 \), for \( i = 1, \ldots, n-1, n+1 \). Hence, \( \dot{x}_i(0) = 0 \), for \( i = 1, \ldots, n-1 \). This implies that
\[
x_i(t) = 0, \text{ for } i = 1, \ldots, n-1 \text{ and } t > 0.
\]
On the other hand, applying \( \sum_{i=1}^{n-1} x_i = 0 \), then (3) gives
\[
\dot{x}_{n+1} = a_n N - a_n x_{n+1} > 0.
\]
This implies
\[
x_{n+1}(t) = C_0 \exp(-a_n t) + N,
\]
where \( C_0 = x_{n+1}(0) - N \).

**3. Numerical simulation**
For numerical simulation purposes, the case of \( n = 5 \) is considered. Four cases are taken into account. Each case is related to the parameters listed in the Table 1. The sum of the state variables is normalized \( (N = 1) \). The initial conditions of each variable are listed in Table 2. Variables \( x_i \), for \( i = 1,2,3,4,6 \) are computed using (3). Variable \( x_5 \) is computed by applying
\[
x_5 = N - x_1 - x_2 - x_3 - x_4 - x_6.
\]
Table 1: Parameters for numerical simulation

| Case  | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|-------|-------|-------|-------|-------|-------|
| Case 1 | 0     | 0     | 0.01  | 0.5   | 0.5   |
| Case 2 | 0     | 0.75  | 0.01  | 0.5   | 0.5   |
| Case 3 | 0.75  | 0     | 0.01  | 0.5   | 0.5   |
| Case 4 | 0.75  | 0.75  | 0.01  | 0.5   | 0.5   |

Table 2: Initial conditions of each variables

| Variable | Value at $t = 0$ | Variable | Value at $t = 0$ |
|----------|------------------|----------|------------------|
| $x_1$    | 0.15             | $x_4$    | 0.2              |
| $x_2$    | 0.2              | $x_5$    | 0.15             |
| $x_3$    | 0.15             | $x_6$    | 0.15             |

Figure 1 shows the evolution of $x_i$ for $i = 1, 2, \ldots, 6$ graphically; (a) for case 1, (b) for case 2, (c) for 3 and (d) for case 4. As $a_1 = a_2 = 0$ in case 1, hence $x_1$ and $x_2$ are constant. Variables $x_4$ and $x_5$ decrease, and $x_3$ decreases very slowly because the small positive value of $a_3$. On the other hand, $x_6$ significantly increases. This is presented in Figure 1(a). In Figure 1(b) where $a_1 = 0$ and other parameters are positive, variable $x_1$ is constant. State variables $x_3$ and $x_6$ increase, while $x_2$, $x_4$ and $x_5$ decrease. Observe that the evolution of $x_3$ is much different than in case 1, Figure 1(a). In Figure 1(c) $a_2 = 0$ and other parameters are positive. State variables $x_2$ and $x_6$ increase, while $x_1$, $x_4$ and $x_5$ decrease. Observe that the evolution of $x_3$ decrease slowly as also indicated in case 1, Figure 1(a). In Figure 1(c) all parameters are positive. State variables $x_3$ and $x_6$ increase, while $x_1$, $x_2$, $x_4$ and $x_5$ decrease.

4. Conclusion and further research

An $n$ dimensional and compartmental model that sits in $n + 1$ dimensional space has been discussed. The model is a dynamical system that consists of $n + 1$ variables. The system has been reduced to a system of $n$ variables. The reduced system has an equilibrium point which is stable, and an equilibrium manifold that is unstable. Future work will be on the analysis of the equilibrium computed directly from $n + 1$ dimensional compared to the one from its reduced system.
Figure 1: Evolution of the state variables. (a) case 1, (b) case 2, (c) case 3, (d) case 4.

Acknowledgments
This research of the first author has been supported by Kemenristek Dikti of Republik Indonesia for hibah Penelitian Dasar Universitas Halu Oleo 2018 – 2020 [Contract No. 447/UN29.20/PPM/2018 and Contract No. 511b/UN29.20/PPM/2019].

References
[1] Devlin K 1997 Mathematics: The Science of Patterns, 2nd Ed., (New York: Scientific American Library)
[2] Guckenheimer J and Holmes P 1983 Nonlinear Oscillations, Dynamical Systems and Bifurcation of Vector Fields, (Berlin: Springer-Verlag)
[3] Kuznetsov Y A 1995 Elements of Applied Bifurcation Theory, 2nd Ed. (New York: Springer-Verlag)
[4] Hethcote H W 1976 Mathematical Biosciences 28 335-356
[5] Earn D J D, Rohani P, Bolker B M and Grenfell B T 2000 Science 287 667 (DOI: 10.1126/science.287.5453.667)
[6] Finkenstadt B F and Grenfell B T 2000 Applied Statistics 49 (2) 187-205
[7] Sani A, Cahyono E, Mukhsar, Rahman G A, Hewindati Y T, Faeldog F A A and Abdullah F A 2014 Advanced Studies in Biology 6 (4), 169 – 179
[8] Ouaro S and Traoré A 2018 International Journal of Differential Equations (Article ID
[9] Homer J and Hirsch G 2006 *American Journal of Public Health* **96** (3), 452-458, 2006.

[10] Diks C, Hommes C, Panchenko V and van der Weide R 2008 *Computational Economics* **32**, 221–244 (DOI:10.1007/s10614-008-9130-x)

[11] Guegan D 2009 *Annual Reviews in Control*, vol., **33** (1), 89-93

[12] Jakimowicz A 2013 *Chaotic Modeling and Simulation* **4**, 657-667

[13] Bashkirtseva I, Ryashko L and Sysolyatina A 2016 *Communication in Nonlinear Science and Numerical Simulation* **36**, 446–456

[14] Bashkirtseva I, Ryashko L and Ryazanova T 2018 *Communication in Nonlinear Science and Numerical Simulation* **54**, 174–184

[15] Cahyono E, Soeharyadi Y and Mukhsar 2018 “Dimensional reduction for a SIR type model,” *AIP Conference Proceedings*, **1937** (Article ID 020005, DOI 10.1063/1.5026077)

[16] Cahyono E, Mukhsar and Elastic P 2017 *Far East Journal of Maathematical Sciences* **102** (6), 1065–1076

[17] Elastic P 2016 *Model matematika diseminasi pengetahuan pada manusia* (Kendari: Universitas Halu Oleo)