Artificial Replay: A Meta-Algorithm for Harnessing Historical Data in Bandits

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Abstract

Most real-world deployments of bandit algorithms exist somewhere in between the offline and online set-up, where some historical data is available upfront and additional data is collected dynamically online. How best to incorporate historical data to “warm start” bandit algorithms is an open question: naively initializing reward estimates using all historical samples can suffer from spurious data and imbalanced data coverage, leading to computation and storage issues—particularly for continuous action spaces. To address these challenges, we propose Artificial Replay, a meta-algorithm for incorporating historical data into any arbitrary base bandit algorithm. We show that Artificial Replay uses only a fraction of the historical data compared to a full warm-start approach, while still achieving identical regret for base algorithms that satisfy independence of irrelevant data (IIData), a novel and broadly applicable property that we introduce. We complement these theoretical results with experiments on (i) $K$-armed bandits and (ii) continuous combinatorial bandits, on which we model green security domains using real poaching data. Our results show the practical benefits of Artificial Replay in reducing computation and space complexity, including for base algorithms that do not satisfy IIData.
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1 Introduction

Multi-armed bandits model decision-making settings with repeated choices between multiple actions (known as “arms”), where the expected reward of each arm is unknown, and each arm pull yields a noisy sample from the true reward distribution. Multi-armed bandits and their variants have been used effectively to model many real-world problems. Resulting algorithms have been applied to wireless networks [Zuo and Joe-Wong, 2021], COVID testing regulations [Bastani et al., 2021], and wildlife conservation to prevent poaching [Xu et al., 2021]. Typical bandit algorithms are designed in one of two regimes: offline or online, depending on the assumptions on how data is collected. In offline settings, the entire dataset is provided to the algorithm upfront, while in the online setting the algorithm starts from scratch and dynamically collects data online. However, most practical deployments exist between these two extremes; in applying bandits to wildlife conservation, for example, we may have years of historical patrol records that could help learn poaching risk before initiating any bandit algorithm executed online to assign new patrol patterns.

A priori, it is not immediately clear how one should initialize an algorithm given historical data. As a warm-up, consider the well-studied UCB1 algorithm from Lai and Robbins [1985] for the K-armed bandit problem. For a finite set of actions $a \in [K]$ with unknown mean reward $\mu(a)$, the algorithm maintains estimates $\overline{\mu}_t(a)$ for the mean reward of action $a \in [K]$ and a count $n_t(a)$ for the number of times the action $a$ has been selected by the algorithm. At time $t$ the algorithm then selects the action $A_t$ which maximizes the so-called upper confidence bound (UCB):

$$UCB_t(a) = \overline{\mu}_t(a) + O\left(1/\sqrt{n_t(a)}\right),$$

where the $O(\cdot)$ drops polylogarithmic constants depending on the total number of timesteps $T$.

A simple approach to incorporate historical data for this algorithm would be to initialize the mean $\overline{\mu}_t(a)$ and number of samples $n_t(a)$ at $t = 1$ based on the historical dataset, assuming it was collected by the bandit algorithm itself. This approach uses that a “sufficient statistic” for UCB are the mean estimates and counts for each action. We refer to this algorithm as Full Start, which is practical and reasonable so long as the dataset is representative of the underlying model. This algorithm has indeed been proposed for the simple $K$-armed bandit setting [Shivaswamy and Joachims, 2012].

1.1 Space and computational limitations of Full Start

While Full Start is intuitive and simple, this naive approach can incur unnecessary unbounded computation and storage costs. We outline the challenges of Full Start here, then in Section 5 we prove theoretically that our approach can use arbitrarily less compute than Full Start.

In an extreme scenario, consider padding the historical data with samples from an arm $a$ whose mean reward $\mu(a)$ is lowest. Full Start will use the entire historical dataset regardless of its inherent quality at timestep $t = 1$. This is even more surprising, since the celebrated result from Lai and Robbins [1985] shows that $O(1/\Delta(a)^2)$ samples are sufficient to discern that an action $a$ is suboptimal (where $\Delta(a)$ is the gap between $\mu(a)$ and the reward of the optimal action). As the number of data points in the historical dataset from this sub-optimal action tends towards infinity, Full Start can potentially have unbounded computational cost, even though only a fraction of the data was required to appropriately rule out this action as being optimal. Clearly, Full Start might not be the optimal way to incorporate the historical data since it processes the entire dataset regardless of its informativeness.

Full Start thus suffers from issues which we call spurious data and imbalanced data coverage, arising because the algorithm unnecessarily processes and stores data regardless of its quality. These
issues are particularly salient since many applications of bandit algorithms, such as online recommender systems, may have historical data with billions of past records; processing all data points would require exceptionally high upfront compute and storage costs. Similarly, in the context of wildlife poaching, historical data takes on the form of past patrol data, which is guided by the knowledge of domain experts and geographic constraints. As we will later see in Section 6, the historical data is extremely geographically biased, so processing the entire historical dataset for regions with large amounts of historical data is costly and unnecessary. Together, these challenges highlight a key aspect of incorporating historical data to warm-start any bandit algorithm: unless the historical data is collected by an optimal approach such as following a bandit algorithm, the value of information of the historical data may not be a direct function of its size.

Lastly, while we’ve so far discussed the stylized $K$-armed bandit setup as an example, many models of real-world systems require continuous actions over potentially combinatorial spaces with constraints. For these more complex bandit settings, it may not even be clear how to instantiate Full Start, as the base algorithm may not clearly admit a low-dimensional “sufficient statistic”. Furthermore, the computation and storage costs increase dramatically with these more complex models, for example, when estimates of the underlying mean reward function are computed as a high-dimensional regression problem. In such instances, the computational complexity for generating the estimate scales superlinearly with respect to the number of data points used.

Thus motivated, this paper seeks to answer the following questions:

- Is there a computationally efficient way to use only a subset of the historical data while maintaining the same regret guarantees of Full Start?
- Can we do so across different bandit set-ups, such as continuous or combinatorial bandits?
- How can we quantify the quality or usefulness of a given set of historical data?

We will evaluate efficiency in terms of algorithm performance, computation, storage space, and number of data points accessed.

1.2 Our Contributions

We propose Artificial Replay, a meta-algorithm that modifies any base bandit algorithm to optimally harness historical data — that is, using the minimal data required to achieve the highest possible performance. Artificial Replay reduces computation and storage costs by using historical data as needed — specifically, only when recommended by the base bandit algorithm. Concretely, Artificial Replay uses the historical data as a replay buffer to artificially simulate online actions. When the base algorithm selects an action, we first check the historical data for any unused samples from the chosen action. If an unused sample exists, update the reward estimates using that historical data point and continue, without advancing to the next timestep. This allows the algorithm to obtain better estimates of the reward of that action without incurring additional regret. Otherwise, take an online step by sampling from the environment (incurring regret if that action is non-optimal), and advance to the next timestep. We will demonstrate how this meta-algorithm can be applied to a wide set of practical models: standard $K$-armed bandits, metric bandits with continuous actions, and models with semi-bandit feedback.

Given that Artificial Replay only uses a subset of the data, one might wonder if it incurs higher regret than Full Start (a generalization of the algorithm described above for $K$-armed bandits). Surprisingly, in our second contribution we prove that under a widely applicable condition, the regret of Artificial Replay (as a random variable) is identical to that of Full Start, while also guaranteeing
significantly better time and storage complexity. Specifically, we show a sample-path coupling between Artificial Replay and Full Start with the same base algorithm, as long as the base algorithm satisfies a novel (and widely applicable) independence of irrelevant data (IIData) assumption. We complement this result by also highlighting the regret improvement for Artificial Replay as a function of the used historical data. Additionally, we prove that Artificial Replay can lead to arbitrary better computational complexity. These results highlight that Artificial Replay is a simple approach for incorporating historical data with identical regret to Full Start and significantly smaller computational overhead. To summarize, Artificial Replay Pareto-dominates Full Start in terms of the three main metrics of interest: (i) regret, (ii) computational complexity, and (iii) storage.

Finally, we show the benefits of Artificial Replay by instantiating it for several classes of bandits and evaluating on real-world poaching data. To highlight the breadth of algorithms that satisfy the IIData property, we show how standard UCB algorithms can be easily modified to be IIData while still guaranteeing regret-optimality, such as for K-armmed and continuous combinatorial bandits. While the results for K-armmed bandits are fairly straightforward, the fact that these results extend to complex settings such as metric bandits with continuous actions or combinatorial bandits is not at all obvious. The surprising insight in our result is that a similarly simple approach for complex bandit settings can attain the same guarantees as full usage of the historical data with significant computational gains. Even for algorithms that do not satisfy IIData, such as Thompson sampling and Information Directed Sampling (IDS), we demonstrate on real poaching data that Artificial Replay still achieves concrete gains in storage and runtime over a range of base algorithms. We close with a case study of combinatorial bandit algorithms with continuous actions in the context of green security, using a novel adaptive discretization technique.

1.3 Related Work

Multi-armed bandit problems have a long history in the online learning literature. We highlight the most closely related works below; for more background, see our extended discussion in Appendix B or refer to Bubeck et al. [2012], Slivkins [2019], Lattimore and Szepesvári [2020].

Multi-Armed Bandits The design and analysis of bandit algorithms have been considered under a wide range of models. Lai and Robbins [1985] first studied the K-armed bandit, where the decision maker chooses between a finite set of K actions at each timestep. Numerous follow-up works have extended to continuous action spaces [Kleinberg et al., 2019] and combinatorial constraints [Chen et al., 2013, Xu et al., 2021]. Zuo and Joe-Wong [2021] propose a discrete model of the combinatorial multi-armed bandit that generalizes previous work Lattimore et al. [2014, 2015], Dagan and Koby [2018] to schedule a finite set of resources to maximize the expected number of jobs finished. We extend their model to continuous actions, tailored to the green security setting from Xu et al. [2021]. Our work provides a framework to modify existing algorithms to efficiently incorporate historical data. Moreover, we also propose and evaluate a novel algorithm to incorporate adaptive discretization for combinatorial multi-armed bandits for continuous resource allocation, extending the discrete model from Zuo and Joe-Wong [2021].

Historical Data and Bandits Several papers have started to investigate how to incorporate historical data into bandit algorithms, but current approaches in the literature only consider Full Start. Shivashwamy and Joachims [2012] began with K-armed bandits, then Oetomo et al. [2021] and Wang et al. [2017] extended to linear contextual bandits, regressing over the historical data to initialize the linear context vector. In contrast to these stylized approaches to a specific bandit model, our work provides a principled method for harnessing historical data across a variety of
bandit models. We show our meta-algorithm Artificial Replay can work with any standard bandit framework and uses historical data only as needed, leading to improved computation and storage gains.

There is also work that has built off our paper. Cheung and Lyu [2024] considers leveraging offline data to facilitate online learning under distribution shift between offline data and online rewards. They modify the UCB policy with a Full Start approach that we study here, but do not additionally consider optimally incorporating the historical data. Additionally, Agrawal et al. [2023] builds on our work to consider best-arm identification in bandits with access to offline data. They instantiate Artificial Replay with a best-arm-identification base algorithm and show it yields nearly-optimal sample complexity (only off by the leading constant). They conjecture that potentially a different base algorithm can be chosen to avoid this gap.

2 Preliminaries

We start off by defining the general bandit model and assumptions we impose on the historical data, then in Section 2.2 we specify the finite-armed and continuous combinatorial allocation settings used in our experiments.

2.1 General Stochastic Bandit Model

We consider a stochastic bandit problem with a fixed feasible action set $\mathcal{A}$. Let $\mathcal{R} : \mathcal{A} \rightarrow \Delta([0, 1])$ be a collection of independent and unknown reward distributions over $\mathcal{A}$. Our goal is to pick an action (also referred to as “arm”) $a \in \mathcal{A}$ to maximize the expected reward $\mathbb{E}[\mathcal{R}(a)]$, which we denote by $\mu(a)$. The optimal reward is $\text{OPT} = \max_{a \in \mathcal{A}} \mu(a)$ under optimal action $a^* = \arg \max_{a \in \mathcal{A}} \mu(a)$.

For now, we do not impose any additional structure on $\mathcal{A}$, but later in Section 2.2 we instantiate $\mathcal{A}$ to consider finite, continuous, and combinatorial constraints.

Historical Data We assume that the algorithm designer has access to a historical dataset, which provides reward samples from a series of past actions. This historical dataset is comprised of $H_a \in \mathbb{Z}_{\geq 0}$ independent and identically distributed samples drawn from the reward $\mathcal{R}(a)$ for each $a \in \mathcal{A}$. Let $\mathcal{H}^{\text{hist}}$ denote the set where each $a \in \mathcal{A}$ is repeated $H_a$ times, and (with slight abuse of notation) $\mathcal{R}(\mathcal{H}^{\text{hist}})$ the observed samples in the historical dataset. Lastly, we use $H$ to denote the total number of historical data points.

The assumption that $H_a$ is fixed and deterministic, as well as each sample is drawn according to $\mathcal{R}(\cdot)$, rules out several important tampering scenarios on the historical dataset. For instance, consider an adversary who sorts the reward samples for each action $a$ to be in decreasing order, or an adversary that “ignores” samples smaller than a certain threshold value. In both scenarios, the marginal reward distributions are biased for $\mathcal{R}(\cdot)$. One could relax this assumption to assume that the historical data satisfies an exchangeability assumption, but we leave this for future work.

However, we note that as a follow-up to this work, Cheung and Lyu [2024] consider the setting of a fixed bias in the sampling distribution of the historical dataset, and modify our Full Start and Artificial Replay algorithms to account for this bias.

Online Structure Since the mean reward function $\mu(a)$ is initially unknown, the algorithm must interact with the environment sequentially. At timestep $t \in [T]$, the decision maker picks an action $A_t \in \mathcal{A}$ according to their policy $\pi$. The environment then reveals a reward $R_t$ sampled from the distribution $\mathcal{R}(A_t)$. The optimal reward $\text{OPT}$ would be achieved using a policy with full knowledge
of the true distribution. We thus define regret of a policy \( \pi \) as:

\[
\text{Reg}(T, \pi, \mathcal{H}_{\text{hist}}) = T \cdot \text{OPT} - \sum_{t=1}^{T} \mu(A_t),
\]

where the dependence on \( \mathcal{H}_{\text{hist}} \) highlights that the selection of \( A_t \) can depend on the historical dataset. When taking expected regret, the expectation will be respect to the arm reward distribution for both online and offline data, but with \( (H_a)_{a \in A} \) fixed. As with standard bandit algorithms, our goal is to design algorithms with minimal regret, sublinear in both \( T \) and \( |\mathcal{H}_{\text{hist}}| \), but — departing from standard bandit literature — also has computation and storage complexity sublinear with the size of \( |\mathcal{H}_{\text{hist}}| \).

### 2.2 Finite, Continuous, and Combinatorial Action Spaces

The description in Section 2.1 with an appropriate specification of the feasible action set \( A \) encompasses most well-studied models in bandits. Here we outline several examples, including the \( K \)-armed (Section 2.2.1) and continuous multi-armed bandit with combinatorial constraints (Section 2.2.5), which are used throughout the paper and in the simulation results of Section 6. With the exception of the \( K \)-armed bandits and \( K \)-armed linear bandit models, the question of how best to incorporate historical data to warm start bandit algorithms has not been previously considered in the literature for any of the other bandits models we consider here. We present a single meta-algorithm that can be used to naturally incorporate historical data across each of these different bandit models. In Section 4 we instantiate the algorithm for these models and show that this approach leads to regret-optimal algorithms.

#### 2.2.1 \( K \)-armed Bandit

The finite-armed bandit model can be viewed in this framework by considering \( A = [K] = \{1, \ldots, K\} \). This recovers the classical model from Lai and Robbins [1985], Auer et al. [2002]. Shivaswamy and Joachims [2012] started the study of incorporating historical data into these models through the use of the standard UCB algorithm. We will later extend these ideas to cover a much wider class of base bandit algorithms, and additionally reduce computation and run-time by only using the historical data as needed.

#### 2.2.2 \( K \)-armed Linear Bandit

The finite-armed linear bandit model can be viewed in this framework by again considering \( A = [K] = \{1, \ldots, K\} \). However, here it is additionally assumed that each action \( a \) has an associated known feature vector \( \phi_a \in [0,1]^d \) and the mean reward \( \mu(a) \) satisfies \( \mu(a) = \phi_a^\top \theta \) where \( \theta \in [0,1]^d \) is an unknown latent parameter [Bastani and Bayati, 2020]. Incorporating historical data in this model was first investigated in Oetomo et al. [2021], Wang et al. [2017] whereby the historical data is used through regression to estimate the latent parameter \( \theta \).

#### 2.2.3 Metric Bandits

The metric or continuous bandit model from Kleinberg et al. [2019] can be viewed in this framework where we assume that the action set \( A \) is a compact metric space with metric \( d \). The mean reward function \( \mu(a) \) is then assumed to be Lipschitz with the metric \( d \), in that there exists a constant \( L \) such that:

\[
|\mu(a) - \mu(a')| \leq Ld(a, a') \quad \forall a, a' \in A.
\]
2.2.4 Combinatorial Finite-Armed Bandits

Here we consider a central planner who has access to a finite set of \([K]\) actions with unknown mean \(\mu(a)\) for each \(a \in [K]\) and over each round is tasked with picking at most \(B \leq K\) of them (see Chen et al. [2013] for a more general model). We denote the feasible action space as the set of allocation vectors \(\vec{\beta}\) as follows:

\[
A = \left\{ \vec{\beta} \in \{0, 1\}^K \mid \sum_{a \in [K]} \beta(a) \leq B \right\}.
\]

(2)

It is further assumed that the mean reward over \(A\) is additive, in that for any \(\vec{\beta} \in A\) we have: \(\mu(\vec{\beta}) = \sum_{a \in [K]} \beta(a) \mu(a)\), and the algorithm receives feedback of the form of \(R \sim \mathcal{R}(a)\) for each \(a \in [K]\) with \(\beta(a) = 1\). The goal of the algorithm is then to learn the top \(B\) actions in order to maximize their cumulative mean reward. Lastly, we consider the historical data \(\mathcal{H}_{\text{hist}}\) to also be decomposed, in that each element is a particular \((a, R)\) pair with \(R \sim \mathcal{R}(a)\).

2.2.5 Combinatorial Multi-Armed Bandit for Continuous Resource Allocation (CMAB-CRA)

This model combines the metric and Lipschitz structure of Section 2.2.3 with the combinatorial structure of Section 2.2.4. We consider a central planner who has access to a metric space \(S\) of resources with metric \(d_S\). They are tasked with splitting a total amount of \(B\) divisible budget across \(N\) different resources within \(S\). For example, in wildlife conservation, the space \(S\) can be considered as the protected area of a park, and the allocation budget corresponds to divisible effort, or the proportion of rangers allocated to patrol in the chosen area. We denote the feasible space of budget allocations as \(B\) and define the feasible action space as follows:

\[
A = \left\{ (\vec{p}, \vec{\beta}) \in S^N \times B^N \mid \sum_{i=1}^{N} \beta(i) \leq B, \quad d_S(p(i), p(j)) \geq \epsilon \quad \forall i \neq j \right\}.
\]

(3)

Note that we require the chosen action to satisfy the budget constraint (i.e. \(\sum_i \beta(i) \leq B\)), and that the chosen resources are distinct (e.g., \(\epsilon\)-away from each other according to \(d_S\)).

We further assume that the reward distribution \(\mathcal{R}\) decomposes additively over \(A\) as follows. Indeed, overload notation and let \(\mathcal{R} : S \times B \to \Delta([0, 1])\) be the unknown reward distribution over the resource and allocation space. The goal of the algorithm is to pick an action \(A = (\vec{p}, \vec{\beta}) \in A\) in a way that maximizes \(\sum_{i=1}^{N} \mathbb{E}[\mathcal{R}(p(i), \beta(i))]\), the expected total mean reward accumulated from the resources subject to the budget constraints. Denoting \(\mathbb{E}[\mathcal{R}(p, \beta)]\) as \(\mu(p, \beta)\), the optimization problem to determine the optimal \(a = (\vec{p}, \vec{\beta}) \in A\) is formulated below:

\[
\max_{\vec{p}, \vec{\beta}} \sum_{i=1}^{N} \mu(p(i), \beta(i))
\]

\[
\text{s.t.} \quad \sum_{i} \beta(i) \leq B, \quad d_S(p(i), p(j)) \geq \epsilon \quad \forall i \neq j.
\]

(4)

Lastly, we consider the historical data \(\mathcal{H}_{\text{hist}}\) to also be decomposed, in that each element is a particular \((p, \beta, R)\) pair with \(R \sim \mathcal{R}(p, \beta)\).
Algorithm 1 Monotone UCB (MonUCB)

1: Initialize $n_1(a) = 0$, $\bar{\mu}_1(a) = 1$, and $\text{UCB}_1(a) = 1$ for each $a \in [K]$
2: for $t = \{1, 2, \ldots \}$ do
3: Let $A_t = \arg \max_{a \in [K]} \text{UCB}_t(a)$
4: Receive reward $R_t$ sampled from $\mathcal{R}(A_t)$
5: for all $a \neq A_t$ do
6: $n_{t+1}(a) = n_t(a)$
7: $\bar{\mu}_{t+1}(a) = \mu_t(a)$
8: $\text{UCB}_{t+1}(a) = \text{UCB}_t(a)$
9: end for
10: $n_{t+1}(A_t) = n_t(A_t) + 1$
11: $\bar{\mu}_{t+1}(A_t) = (n_t(A_t)\bar{\mu}_t(A_t) + R_t)/n_{t+1}(A_t)$
12: $\text{UCB}_{t+1}(A_t) = \min\{\text{UCB}_t(A_t), \bar{\mu}_{t+1}(A_t) + \sqrt{2 \log(T)/n_{t+1}(A_t)}\}$
13: end for

Mapping to Green Security Domains The CMAB-CRA model can be used to specify green security domains from Xu et al. [2021]. $S$ is used to represent a protected area, which is a geographic space designated for wildlife conservation. $B$ is the discrete set of potential patrol efforts to allocate, such as the number of ranger hours per week, with the total budget $B$ being 40 hours. The reward function $\mu(p, \beta)$ then models the probability of observing a snare at location $p$ at effort level $\beta$.

In the simplest case, $S$ is a two-dimensional space where every point corresponds to a GPS coordinate in the park. For more precision, we could consider planning over not just the physical space, but rather the feature space. In that case, $S$ is a $k$-dimensional space over $k$ features in the park that influence green security threats (such as land cover, elevation, and distance to rivers that influence poaching). For purposes of exposition, in this paper we consider the former case, where $S$ directly corresponds to GPS coordinates. This formulation generalizes Xu et al. [2021] to a more realistic continuous space model of the landscape, enabling greater geographic precision in resource allocation than the artificial fixed discretization that was considered in prior work (consisting of $1 \times 1$ sq. km regions of the park).

3 Artificial Replay for Harnessing Historical Data

In this section we propose Artificial Replay, a simple approach to incorporate historical data into bandit algorithms. We start off with a case study of UCB for $K$-armed bandits (Section 2.2.1) for which the instantiation is more straightforward.

However, for algorithms in complex settings (such as the others discussed in Section 2.2) it is not obvious how one should initialize the algorithm given historical data. We first discuss Full Start, which initializes an algorithm by using the entire historical data upfront. We later develop Artificial Replay, a meta-algorithm to efficiently incorporate historical data which can be integrated with any base algorithm. We highlight how Artificial Replay depends on an action-similarity relationship, which captures the notion of whether data from a given action can be used to inform the reward distribution of other actions.

3.1 Case study: MonUCB for $K$-armed Bandits

To first address the problem of incorporating historical data into a bandit algorithm, we consider a modification of the standard UCB1 algorithm from Auer et al. [2002]. We first ignore the historical
data and propose MonUCB, which is identical to standard UCB1 except for two small modifications. First, we adjust the confidence terms to depend on \( \log(T) \) instead of \( \log(t) \). Second, we force the UCB estimates to be monotone decreasing over timesteps \( t \). These modifications have no effect on the regret guarantee. Under the “good event” analysis, if \( \text{UCB}_t(a) \geq \mu(a) \) with high probability, then the condition still holds at time \( t + 1 \), even after observing a new data point. We then modify the MonUCB algorithm to use either the full historical data upfront, or adaptively using the Artificial Replay framework.

MonUCB tracks the same values as UCB1: (i) \( \pi_t(a) \) for the estimated mean reward of \( a \), (ii) \( n_t(a) \) for the number of times \( a \) has been selected by the algorithm, and (iii) \( \text{UCB}_t(a) \) for an upper confidence bound estimate for the mean reward of \( a \). At every timestep \( t \), MonUCB picks the action \( A_t \) which maximizes \( \text{UCB}_t(a) \) (breaking ties deterministically). After observing \( R_t \sim \mathcal{R}(A_t) \), we update our counts \( n_{t+1}(a) \) and estimates \( \pi_{t+1}(a) \) but, critically, we only update \( \text{UCB}_{t+1}(a) \) if the new UCB estimate is less than the prior UCB estimate \( \text{UCB}_t(a) \). The full pseudocode is given in Algorithm 1.

We start by highlighting the fact that these changes to UCB1 do not affect the regret. Indeed, MonUCB achieves the same instance-dependent regret bound as the standard UCB1 algorithm.

**Theorem 3.1.** The MonUCB base algorithm has for \( \Delta(a) = \max_{a'} \mu(a') - \mu(a) \):

\[
\mathbb{E}[\text{REG}(T, \pi^{\text{MonUCB}})] = O \left( \sum_{a \neq a^*} \log(T) / \Delta(a) \right).
\]

*Proof.* See Appendix F.2. \( \square \)

We note that the monotonicity modification can be applied to more general \( \psi \)-UCB algorithms from Bubeck et al. [2012] (see details in Appendix D.1).

**Incorporating all historical data upfront** As written, MonUCB does not directly incorporate the historical data \( \mathcal{H}^{\text{hist}} \). We first consider a straightforward modification, resulting in a policy that we denote as \( \pi^{\text{Full Start}(\text{MonUCB})} \), which utilizes the entire historical data upfront. The only modification for \( \text{Full Start}(\text{MonUCB}) \) is that at \( t = 1 \) (Line 1 of Algorithm 1) we initialize: (i) \( \bar{\pi}_1(a) \) for the estimated mean reward of action \( a \) taken over the \( H_a \) samples in \( \mathcal{H}^{\text{hist}} \), (ii) \( n_1(a) \) for the number of times the action \( a \) was selected in \( \mathcal{H}^{\text{hist}} \), i.e. \( H_a \). After the first round, the remaining steps are identical, where new data is used to update the already pre-initialized \( \bar{\pi}_1(a) \) and \( n_1(a) \) values. Here we see the importance of the first modification of MonUCB: replacing \( \log(t) \) with \( \log(T) \) in the confidence radius ensures that the UCB\(_t(a) \) values are properly calibrated with a union bound [Shivaswamy and Joachims, 2012].

Unfortunately, \( \pi^{\text{Full Start}(\text{MonUCB})} \) has several drawbacks as hinted at in the introduction. Consider a sub-optimal action \( a \in [K] \). Then by padding the historical dataset with additional samples of \( a \), Full Start(MonUCB) will read in the entire dataset for \( H_a \), even though the celebrated result by Lai and Robbins [1985] shows that \( O(\log(T)/\Delta(a)^2) \) samples are sufficient to discern that \( a \) is suboptimal.

**A more natural and adaptive way to incorporate historical data** Based on this drawback to Full Start(MonUCB), the obvious question is how can we adaptively incorporate the historical data to ensure that at most \( O(\log(T)/\Delta(a)^2) \) samples from each suboptimal action are used? The issue with Full Start(MonUCB) is that the historical data is used independent of its quality. A better strategy would be to only use the historical data for an action \( a \) when MonUCB suggests to play that
action. This automatically ensures that historical data for high-performing actions is used, since by design, MonUCB plays near-optimal actions. We formalize this through the following steps. Let $\mathcal{H}_t$ be the set of online data observed by the algorithm, and $\mathcal{H}_t^{used}$ as the subset of $\mathcal{H}_t^{hist}$ containing the used historical data, where initially $\mathcal{H}_1^{used} = \emptyset$. We also abuse notation and let $\pi(\mathcal{H}, a)$ and $n(\mathcal{H}, a)$ to be the empirical mean reward and number of appearances for any $a \in \mathcal{A}$.

We construct a policy, which we denote as $\pi^{Artificial\ Replay}(\text{MonUCB})$ as follows. Initially, at $t = 1$ we have that $\overline{\pi}_t(a) = 1$, $n_t(a) = 0$, and $\text{UCB}_t(a) = 1$. Over rounds $t$, we let $\tilde{A}_t = \arg \max_{a \in [K]} \text{UCB}_t(a)$ be the proposed action. In order to only use historical data from good actions, since the algorithm proposes to play $\tilde{A}_t$ we first check if the set of unused historical data $\mathcal{H}_t^{hist} \setminus \mathcal{H}_t^{used}$ contains any additional samples from $\tilde{A}_t$. If so, we update $\overline{\pi}_t(\tilde{A}_t)$, $n_t(\tilde{A}_t)$, and $\text{UCB}_t(\tilde{A}_t)$ with this additional sample, otherwise we play $\tilde{A}_t$ and continue to the next timestep.

- **No historical data available:** If $\mathcal{H}_t^{hist} \setminus \mathcal{H}_t^{used}$ does not contain any additional samples from $\tilde{A}_t$, then the selected action is $A_t = \tilde{A}_t$. We keep $\mathcal{H}_t^{used} = \mathcal{H}_t^{used}$, set $\mathcal{H}_{t+1} = \mathcal{H}_t \cup \{(A_t, \mathcal{R}(A_t))\}$, and advance to timestep $t + 1$ by updating:

$$n_{t+1}(A_t) = n(\mathcal{H}_{t+1} \cup \mathcal{H}_{t+1}, A_t), \quad \overline{\pi}_{t+1}(A_t) = \overline{\pi}(\mathcal{H}_{t+1} \cup \mathcal{H}_{t+1}, A_t),$$

$$\text{UCB}_{t+1}(A_t) = \min \left\{ \text{UCB}_t(A_t), \overline{\pi}_{t+1}(A_t) + \sqrt{2 \log(T)/n_{t+1}(A_t)} \right\}.$$

- **Historical data available:** If $\tilde{A}_t$ is contained in $\mathcal{H}_t^{hist} \setminus \mathcal{H}_t^{used}$, then we add that data point to $\mathcal{H}_t^{used} = \mathcal{H}_t^{used} \cup \{(\tilde{A}_t, \mathcal{R}(\tilde{A}_t))\}$. We update the estimates as

$$n_t(A_t) = n(\mathcal{H}_t^{used} \cup \mathcal{H}_t, A_t), \quad \overline{\pi}_t(A_t) = \overline{\pi}(\mathcal{H}_t^{used} \cup \mathcal{H}_t, A_t),$$

$$\text{UCB}_t(A_t) = \min \left\{ \text{UCB}_t(A_t), \overline{\pi}_t(A_t) + \sqrt{2 \log(T)/n_t(A_t)} \right\},$$

and repeat by picking another proposed action $\tilde{A}_t = \arg \max_{a \in [K]} \text{UCB}_t(a)$. We remain at time $t$.

By design, $\text{Artificial\ Replay}(\text{MonUCB})$ only incorporates historical data when MonUCB itself decides it wants to play an action. We will later see in Theorem 5.5 that this avoids the issues of $\text{Full\ Start}(\text{MonUCB})$ and only reads in a subset of the historical data necessary to ensure that an action is sub-optimal. While this simple idea seems almost obvious within the context of $K$-armed bandits, the appropriate extension to general bandit models can be quite complex. For example, in a continuous bandit setting as described in Section 2.2.3, one wouldn’t expect the historical data to contain multiple data points for any single given action, and it may even be likely that the dataset contains no data at all for any of the discrete set of actions recommended by any given base bandit algorithm. However, this does not mean that the historical data contains no relevant information about this set of actions, as data from nearby actions provides weak information about the reward due to the Lipschitz assumption on the reward function. Formalizing the appropriate way to utilize this historical data will require care. Next, we provide a principled method for incorporating historical data across a variety of bandit scenarios (including those in Section 2.2).

### 3.2 Incorporating historical data: Ignorant and Full Start approaches

In order to generalize the previous discussion for $K$-armed bandits, we model algorithms for online stochastic bandits as a function mapping histories (collections of observed $(a, R)$ pairs) to a distribution over actions in $\mathcal{A}$. We let $\Pi : \mathcal{D} \to \Delta(\mathcal{A})$ denote an arbitrary base algorithm where $\mathcal{D}$ denotes the collection of possible unordered histories (i.e., $\mathcal{D} = \cup_{i \geq 0} (\mathcal{A} \times \mathbb{R}_+)$). This definition ensures that
the algorithm is designed based on a sufficient statistic of the data (for example, the empirical mean reward estimate for the case of MonUCB). It also importantly assumes that the algorithm’s action selection is identical regardless of the order of observations. This also requires that the action selection does not depend on \( t \). Again, in MonUCB, the order of observations does not impact the final decision, since the decision is determined based on the empirical average reward per action and number of times each action was played; those two aggregate values are sufficient statistics. While this restricts the set of algorithms considered, in Section 4 we will provide algorithms that satisfy the required assumptions across the bandit models discussed in Section 2.2.

With these definitions in hand, we now define the **ignorant** algorithm, which ignores the historical data, and the **full warm start** algorithm, which includes all the historical data upfront.

**Definition 3.2 (Ignorant algorithm).** The **ignorant** algorithm (\textit{Ignorant}) obtained by a base algorithm \( \Pi \) that does not incorporate historical data takes actions according to the policy:

\[
\pi_t^{\text{Ignorant}(\Pi)} = \Pi(\mathcal{H}_t)
\]

where \( \mathcal{H}_t \in \mathcal{D} \) is the online data observed by timestep \( t \).

**Definition 3.3 (Full warm start).** The **full warm start** algorithm (\textit{Full Start}), which uses the full historical data upfront, takes actions according to the policy:

\[
\pi_t^{\text{Full Start}(\Pi)} = \Pi(\mathcal{H}^{\text{hist}} \cup \mathcal{H}_t).
\]

### 3.3 Equivalence Classes over Actions

As we presented in Section 3.1, a key step of Artificial Replay applied to MonUCB is to check whether the historical data contains unused samples for the current proposed action \( \tilde{A}_t \). While this is simple in the \( K \)-armed bandit setup, with continuous or combinatorial actions it is unlikely that each individual action is multiply represented within the historical dataset. However, “similar” actions can still be used in order to infer additional information on the reward distribution for \( \tilde{A}_t \). We thus introduce the notion of equivalence relations over actions, which is key to establishing Artificial Replay for more general base bandit algorithms.

In Section 3.1 the equivalence is clear by taking \( a = a' \) if and only if \( a = a' \). However, in more complicated bandit models, such as continuous or combinatorial bandits (Sections 2.2.3 to 2.2.5), we will need to use a coarser relation. For example, consider the metric bandit model of Section 2.2.3. Under the Lipschitz assumption on the mean reward we have that:

\[
|\mu(a) - \mu(a')| \leq Ld(a, a'),
\]

where \( d \) is the metric over \( \mathcal{A} \). Then incorporating historical data from actions \( a \) where \( d(a, \tilde{A}_t) \leq \epsilon \) would allow us to infer some information about \( \tilde{A}_t \), which can be exploited to learn the optimal action faster. We could safely ignore historical data from actions \( a \) where \( d(a, \tilde{A}_t) \) is large, which would maintain that only a necessary subset of the historical data is used. To formalize this idea, we assume that the action set \( \mathcal{A} \) is endowed with a relation as defined below.

**Definition 3.4.** We define the **information relation** over \( \mathcal{A} \) as a set of relations indexed over \( \mathcal{H} \in \mathcal{D} \) for any set of histories, which we denote as \( \doteq_{\mathcal{H}} \).

When measuring the computational complexity of the algorithms, we will assume that checking the relation \( \doteq_{\mathcal{H}} \) can be done in constant time.
Algorithm 2 Artificial Replay

1: **Input:** Historical dataset $H^{hist}$, base algorithm $\Pi$, relation over $A \bowtie_{\mathcal{H}}$
2: Initialize set of used historical data points $H^{used} = \emptyset$ and set of online data $H_{1} = \emptyset$
3: for $t = \{1, 2, \ldots \}$ do
4: Initialize flag to be True
5: while flag is True do
6: Pick action $\tilde{A}_{t} \sim \Pi(H^{used}_{t} \cup H_{t})$
7: /* If an unused and relevant sample does not exist in the historical dataset */
8: if $(H^{hist}_{t} \setminus H^{used}_{t}) \cap \{a \in A : a \bowtie_{H_{t}^{used} \cup H_{t}} \tilde{A}_{t}\} = \emptyset$ then
9: Set online action $A_{t} = \tilde{A}_{t}$
10: Execute $A_{t}$ and observe reward $R_{t} \sim \mathcal{R}(A_{t})$
11: Update $H_{t+1} = H_{t} \cup \{(A_{t}, R_{t})\}$ and $H^{used}_{t+1} = H^{used}_{t}$
12: Update flag to be False
13: else
14: Update $H^{used}_{t}$ to include a sample for any $a$ in $(H^{hist}_{t} \setminus H^{used}_{t}) \cap \{a \in A : a \bowtie_{H_{t}^{used} \cup H_{t}} \tilde{A}_{t}\}$
15: end if
16: end while
17: end for

The action set $A$ is endowed with several vacuous relations. For example, the trivial relation is one where $a \bowtie a'$ for any $a, a' \in A$. Another standard relation would be to assume that all actions are only related to themselves (i.e. $a \bowtie_{\mathcal{H}} a'$ if and only if $a = a'$). This captures the model from Section 3.1, but does not incorporate the Lipschitz assumption from the metric bandits in Section 2.2.3. We also allow the information relation to depend on the history $\mathcal{H}$. This need arises since the choice of relation should be algorithm (and potentially data) dependent, as we will later see in Section 4. When there is no dependence on $\mathcal{H}$ we will abuse notation and omit it.

While we will give concrete instantiations of base algorithms alongside their information relation later in Section 4, we briefly give some relations under different action spaces as follows.

**Continuous Actions** Typical algorithms for continuous action spaces work by constructing a discretization over $A$ according to a fixed bandwidth parameter $\gamma$, and running a standard UCB algorithm over the (now-discrete) approximate action set [Kleinberg et al., 2019]. A natural information relation would be to set $a \bowtie a'$ whenever $d(a, a') \leq \gamma$. However, an adaptive discretization algorithm requires a history-dependent information relation which encodes the discretization at the time of the proposed action. We will see this in Section 4.

**Semi-Bandit Feedback** In Sections 2.2.4 and 2.2.5 we introduced bandit models which observe semi-bandit feedback. Suppose that actions can be written as $a = (a_{1}, \ldots , a_{N})$ and the reward function decomposes additively with independent rewards for each subcomponent $a_{i}$ (sometimes referred to as a “subarm” in combinatorial bandits). A natural information relation would be to set $a \bowtie a'$ whenever $d(a, a') \leq \gamma$. However, an adaptive discretization algorithm requires a history-dependent information relation which encodes the discretization at the time of the proposed action. We will see this in Section 4.

### 3.4 Artificial Replay

The Artificial Replay meta-algorithm takes as input an equivalence relation $\bowtie$ over $A$ and simulates the base algorithm using historical data as a replay buffer, resulting in a policy which we denote $\pi_{\text{Artificial Replay}}(\Pi)$. The algorithm keeps track of a set $H^{used}_{t}$ of historical data points used by the
start of time $t$. Initially, $H^\text{used}_t = \emptyset$. For an arbitrary base algorithm $\Pi$, let $\tilde{A}_t \sim \Pi(H^\text{used}_t \cup H_t)$ be the proposed action. There are two possible cases: whether or not the current set of unused historical data points $H^\text{hist} \setminus H^\text{used}_t$ contains any additional samples from actions related to the proposed action $\tilde{A}_t$. Specifically, we test if \((H^\text{hist} \setminus H^\text{used}_t) \cap \{a \in A \mid a \doteq H^\text{used}_t \cup H_t \tilde{A}_t\} = \emptyset\). If so, the historical data is uninformative of the current proposed action, otherwise some action in the historical data can be used to update the information about $\tilde{A}_t$.

- No historical data available: If there does not exist an $a \doteq H^\text{used}_t \cup H_t \tilde{A}_t$ contained in $H^\text{hist} \setminus H^\text{used}_t$, then the selected action is:

  \[ \pi_t^{\text{Artificial Replay}(\Pi)}(H^\text{used}_t \cup H_t) = A_t = \tilde{A}_t. \]

  We also keep $H^\text{used}_{t+1} = H^\text{used}_t$ and advance to timestep $t+1$.

- Historical data available: If there exists an $a \doteq H^\text{used}_t \cup H_t \tilde{A}_t$ contained in $H^\text{hist} \setminus H^\text{used}_t$, add that data point to $H^\text{used}_t = H^\text{used}_t \cup \{(a, \mathcal{R}(a))\}$ and repeat by picking another proposed action:

  \[ \tilde{A}_t = \pi_t^{\text{Artificial Replay}(\Pi)}(H^\text{used}_t \cup H_t). \]

  We remain at time $t$.

Each step is guaranteed to result in a selected action $A_t$ since the historical data is finite. We provide pseudocode in Algorithm 2 and use $\pi^{\text{Artificial Replay}(\Pi)}$ to denote the resulting policy, omitting the dependence on the relation $\doteq$.

An important step of Artificial Replay is to check whether the unused historical data contains additional samples relevant to the proposed action $\tilde{A}_t$. Since this sampling aligns with the base algorithm $\Pi$, it inherently focuses on high-performing actions. Under the trivial relation where $a \doteq a'$ for all $a, a' \in A^2$, the Artificial Replay algorithm becomes equivalent to Full Start, reading the entire dataset before the first action. The computational benefits of Artificial Replay hinge on balancing the relation: it must be fine enough to limit data usage, but coarse enough to maintain meaningful informativeness.

## 4 IIData Algorithms

Theorems 5.2 and 5.3 show that for IIData algorithms, Artificial Replay achieves identical regret as Full Start and offers a regret improvement with respect to the amount of used historical data. The regret performance of Artificial Replay therefore hinges on the choice of the base algorithm and information relation; as a result, we suggest using a regret-optimal base algorithm and relation that satisfies IIData.

In addition to Artificial Replay being optimal for the MonUCB algorithm as we proved in Section 5.2, here we show that the widely used regret-optimal bandit algorithms which use the upper confidence bound approach (i.e., choosing greedily from optimistic estimates) can be adapted to satisfy IIData while retaining their regret guarantees. We provide example regret-optimal algorithms for metric bandits and CMAB-CRA (Sections 2.2.3 and 2.2.5). Other algorithms, such as for linear response bandits [Bastani and Bayati, 2020, Goldenshluger and Zeevi, 2013], can be similarly modified to satisfy IIData. Additional details are in Appendix D.
4.1 IIData in Metric Bandits

We first consider the standard metric bandit set-up of Section 2.2.3. We assume that $\mathcal{A}$ is a compact metric space with metric $d$, and that the reward function $\mu$ is $L$-Lipschitz continuous with respect to the metric, i.e.:

$$|\mu(a) - \mu(a')| \leq L d(a, a').$$

Incorporating historical data efficiently is difficult in continuous action spaces. The key issues are that the computation and storage costs grow with the size of the historical dataset, and the estimation and discretization are done independently of the quality of the reward. Two natural approaches to address continuous actions are to (i) discretize the action space based on the data using nearest-neighbor estimates, or (ii) learn a regression of the mean reward using available data. When using Full Start, both approaches are compute-suboptimal in light of spurious data — when excessive data is collected from poor-performing actions. Discretization-based algorithms will unnecessarily process and store a large number of discretizations in low-performing regions of the space (Fig. 7); regression-based methods require additional compute resources to learn an accurate predictor of the mean reward in irrelevant regions.

We consider fixed and adaptive discretization algorithms, establishing that a straightforward modification of existing UCB algorithms are regret-optimal and satisfy IIData under an appropriate information relation [Kleinberg et al., 2019]. Here we use the natural information relation, where at a high level $a \equiv_H a'$ so long as they both fall within the same discretized region.

The algorithm maintains a collection of regions $\mathcal{P}_t$ of $\mathcal{A}$ which covers $\mathcal{A}$. For fixed discretization, $\mathcal{P}_t$ is fixed at the start of learning as a $\gamma$-covering of the action set $\mathcal{A}$; with adaptive discretization, it is refined over the course of learning based on observed data. We use $\mathcal{R} \in \mathcal{P}_t$ to denote a region of the action space. The monotone discretization algorithms track the following: (i) $\overline{\mu}_t(\mathcal{R})$ for the estimated mean reward of region $\mathcal{R}$, (ii) $n_t(\mathcal{R})$ for the number of times $\mathcal{R}$ has been selected, and (iii) $\text{UCB}_t(\mathcal{R})$ for an upper confidence bound estimate of the reward. At each timestep $t$, our algorithm performs three steps. First for the action selection we select the region which maximizes $\text{UCB}_t(\mathcal{R})$ and pick any $a \in \mathcal{R}$ to play arbitrarily. Afterwards, we update parameters by incrementing $n_t(\mathcal{R})$ by one, update $\overline{\mu}_t(\mathcal{R})$ based on observed data, and set

$$\text{UCB}_{t+1}(\mathcal{R}) = \min\{\text{UCB}_t(\mathcal{R}), \overline{\mu}_t(\mathcal{R}) + b(n_t(\mathcal{R}))\}$$

for some appropriate bonus term $b(\cdot)$. The UCB update enforces monotonicity in the UCB values similar to MonUCB and is required to preserve IIData. Lastly, in the case of adaptive-discretization, we potentially re-partition the space. We split a region when the confidence in its estimate $b(n_t(\mathcal{R}))$ is smaller than the diameter of the region and replace it with new regions of half the diameter. This condition may seem independent of the quality of a region, but since it is incorporated into a learning algorithm, the number of samples in a region is correlated with its reward.

Under appropriately defined bonus term $b(\cdot)$ and selection of $\gamma$ (see Kleinberg et al. [2019], Sinclair et al. [2023] and Appendix D.2 for further details), it can be shown that these algorithms are regret-optimal. However, we also have:

**Theorem 4.1.** The fixed discretization algorithm under the relation where $a \equiv_H a'$ whenever $d(a, a') \leq \gamma$ is IIData. The adaptive discretization algorithm under the relation where $a \equiv_H a'$ whenever $a$ and $a'$ belong to the same region $\mathcal{R}$ over data $\mathcal{H}$ satisfies IIData.

Theorem 4.1 allows us match the regret performance of Full Start while simultaneously avoiding reading in the entire historical dataset.
4.2 IIData in CMAB-CRA

Now we extend the algorithms from Section 4.1 to the CMAB-CRA set-up (see Section 2.2.5) which are used in the computational results in Section 6. First, we outline two assumptions on the underlying problem. The first is the standard nonparametric assumption, highlighting that the resource space $S$ is a compact metric space with metric $d_S$, diameter $d_{\text{max}}$, and that the true underlying reward function $\mu$ is Lipschitz with respect to $d_S$. Indeed we have:

$$|\mu(p, \beta) - \mu(p', \beta)| \leq L d_S(p, p').$$

The next assumption assumes access to an oracle for solving optimization problems of the form of Eq. (4) for arbitrary choice of reward functions $\mu(p, \beta)$. We can relax this assumption to instead assume that there exists a randomized approximation oracle by appropriately shifting the regret benchmark. See Zuo and Joe-Wong [2021] which considers this in a simpler discrete setting.

Our algorithms directly modify those in Section 4.1 to maintain a separate discretization for each allocation $\beta \in \mathcal{B}$. Indeed, the algorithms are UCB style, where the selection rule optimizes over the combinatorial action space (Eq. (3)) through a discretization of $S$. For each allocation $\beta \in \mathcal{B}$, the algorithm maintains a collection of regions $\mathcal{P}_t^\beta$ of $S$ which covers $S$. For fixed discretization, $\mathcal{P}_t^\beta$ is fixed at the start of learning; with adaptive discretization it is refined throughout learning based on observed data.

The algorithm tracks the following: (i) $\bar{\pi}_t(\mathcal{R}, \beta)$ for the estimated mean reward of region $\mathcal{R}$ at allocation $\beta$, (ii) $n_t(\mathcal{R}, \beta)$ for the number of times $\mathcal{R}$ has been selected at allocation $\beta$, and (iii) $\text{UCB}_t(\mathcal{R}, \beta)$ for an upper confidence bound estimate of the reward. At each timestep $t$, our algorithm performs three steps:

1. **Action selection**: Greedily select at most $N$ regions in $\mathcal{P}_t^\beta$ to maximize $\text{UCB}_t(\mathcal{R}, \beta)$ subject to the budget constraint (see Eq. (11) in the appendix). Note that we must ensure that each region is selected at only a single allocation value. This is solved using a standard knapsack formulation.

2. **Update parameters**: For each of the selected regions, increment $n_t(\mathcal{R}, \beta)$ by one, update $\bar{\pi}_t(\mathcal{R}, \beta)$ based on observed data, and set

$$\text{UCB}_{t+1}(\mathcal{R}, \beta) = \min\{\text{UCB}_t(\mathcal{R}, \beta), \bar{\pi}_t(\mathcal{R}, \beta) + b(n_t(\mathcal{R}, \beta))\}$$

for some appropriate bonus term $b(\cdot)$. The UCB update enforces monotonicity in the UCB estimates similar to MonUCB and is required to preserve IIData.

3. **Re-partition**: We split a region when the confidence in its estimate $b(n_t(\mathcal{R}, \beta))$ is smaller than the diameter of the region. In Fig. 4 (appendix) we highlight how the adaptive discretization algorithm hones in on high-reward regions without knowing the reward function before learning.

Due to space, we describe the algorithms at a high level here and defer details to Appendix D.3. These algorithms modify existing approaches applied to CMAB-CRA in the bandit and reinforcement learning literature, which have been shown to be regret-optimal [Xu et al., 2021, Sinclair et al., 2023]. Additionally, these approaches are IIData under the natural discretization-based information relation (where we additionally account for the semi-bandit feedback in the style of Section 3.3).

**Theorem 4.2.** The fixed and adaptive discretization algorithms when using a “greedy” solution to solve the action selection rule have property IIData.
Figure 1: (K-armed) Increasing the number of historical samples \( H \) leads Full Start to use unnecessary data, particularly as \( H \) gets very large. Artificial Replay achieves equal performance in terms of regret (plot a) while using less than half the historical data (plot b). In plot (c) we see that with \( H = 1,000 \) historical samples, Artificial Replay uses (on average) 117 historical samples before taking its first online action. The number of historical samples used increases at a decreasing rate, using only 396 of 1,000 total samples by the horizon \( T \). Results are shown on the K-armed bandit setting with \( K = 10 \) and horizon \( T = 1,000 \).

Figure 2: (CMAB-CRA) Holding \( H = 10,000 \) constant, we increase the fraction of historical data samples on bad arms (bottom 20% of rewards). The plots show (a) regret, (b) % of unused historical data, and (c) number of discretized regions in partition \( \mathcal{P} \). Artificial Replay enables significantly improved runtime and reduced storage while matching the performance of Full Start. Results on the CMAB-CRA setting with adaptive discretization on the quadratic domain.

Proof sketch. In this proof, we require the algorithm to use the standard “greedy approximation”, which is a knapsack problem in the CMAB-CRA set-up [Williamson and Shmoys, 2011]. In general, this introduces additional approximation ratio limitations. However, under additional assumptions on the mean reward function \( \mu(p, \beta) \), the greedy solution is provably optimal. For example, optimality of the greedy approximation holds when \( \mu(p, \beta) \) is piecewise linear and monotone, or more broadly when \( \mu(a) \) is submodular. See Appendix D.3 for more discussion.

Note that these assumptions hold for our application of green security to prevent wildlife poaching that we present in Section 6.
5 Theoretical Analysis of Artificial Replay

In this section, we highlight the theoretical benefits of Artificial Replay. We start by proving that for the broad class of base algorithms that satisfy independence of irrelevant data (IIData), a novel property we propose, Artificial Replay incurs identical regret to Full Start—thereby reducing computation costs without compromising performance. We then highlight a regret improvement, emphasizing that Artificial Replay improves the regret over Ignorant based on the amount of used historical data. Finally, we offer a case study of Artificial Replay applied to the MonUCB algorithm from Section 3.1 to (i) establish that Artificial Replay uses arbitrarily less compute than Full Start, and (ii) quantify the value of the historical data. However, we believe these results extend more generally to arbitrary base algorithms under a variety of bandit models.

5.1 IIData and Regret Couplings

As defined, it is not immediately clear how to analyze the regret of Artificial Replay, or even compare its performance to Full Start. To enable regret analysis, we introduce a new property for bandit algorithms, independence of irrelevant data (IIData), which states that when an algorithm is about to take an action, providing additional data about other unrelated actions will not change the algorithm’s decision.

Definition 5.1 (Independence of irrelevant data). A deterministic algorithm \( \Pi \) together with an information relation \( = \) satisfies the independence of irrelevant data (IIData) property if whenever action \( A \) is proposed by the algorithm \( \Pi \) based on the current data \( (A = \Pi(H)) \) then

\[
\Pi(H) = \Pi(H \cup H')
\]

for any dataset \( H' \) satisfying \( H' \cap \{a \in A : a \neq_H \Pi(H)\} = \emptyset \), that is, containing data from actions which are not related to \( \Pi(H) \).

While it is not a priori clear whether IIData is satisfied by existing algorithms in the literature, in Section 4 we highlight regret-optimal algorithms under a variety of domains which satisfy this property (including MonUCB from Section 3.1). However, it is easy to see that any base algorithm satisfies IIData under the trivial relation where \( a \equiv a' \) for any two actions \( a, a' \).

More generally, IIData is a natural robustness property for an algorithm to satisfy and is conceptually analogous to the independence of irrelevant alternatives (IIA) axiom in computational social choice, often cited as a desiderata used to evaluate voting rules [Arrow, 1951]. In the existing bandit literature, there has been a narrow focus on only finding regret-optimal algorithms. We propose that IIData is another desirable property that implies ease and robustness for optimally and efficiently incorporating historical data. We moreover conjecture that IIData is necessary for ensuring that the regret gains from incorporating historical data is monotone increasing, since without the property the adversary can, in an algorithm-dependent way, augment \( H_{hist} \) to include samples and influence the algorithm to select a sub-optimal action. We show that IIData avoids these issues by ensuring Artificial Replay and Full Start have identical regret.

Theorem 5.2 (Regret Coupling of Artificial Replay to Full Start). Suppose that base algorithm \( \Pi \) with information relation \( = \) satisfies IIData. Then for any problem instance, horizon \( T \), and historical dataset \( H_{hist} \) we have the following:

\[
\pi^{Artificial\,Replay(\Pi)}_t \overset{d}{=} \pi^{Full\,Start(\Pi)}_t
\]

\[
REG(T, \pi^{Artificial\,Replay(\Pi), H_{hist}}) \overset{d}{=} REG(T, \pi^{Full\,Start(\Pi), H_{hist}})
\]
Proof sketch. The proof of this result uses the reward stack model for analyzing the performance of bandit algorithms developed in Lattimore and Szepesvári [2020] and follows the spirit of Wilson [1996]. We provide a sample path coupling over the reward observations for $\pi_{Artificial \ Replay}$ and $\pi_{Full \ Start}$ to show that $\pi_{Artificial \ Replay} = \pi_{Full \ Start}$ for all $t$, and hence have the same distribution.

We establish this result by induction over $t$, where we explicitly use the IIData property to prove that although Artificial Replay uses a subset of the historical data, its selected action is identical to that of Full Start. Indeed, consider the case when $t = 1$, then Artificial Replay builds up a set $H_{1}^{used}$ from the historical data such that $(H_{1}^{hist} \setminus H_{1}^{used}) \cap \{a \doteq_{H_{1}^{used}} \Pi(H_{1}^{used})\} = \emptyset$. However, by the fact that $\Pi$ satisfies IIData it must be that $\pi_{Full \ Start} = \Pi(H_{hist}) = \Pi(H_{1}^{used}) = \pi_{1}^{Artificial \ Replay}$.

The remainder of the proof proceeds similarly and is delegated to Appendix F.1.

Next we show a regret improvement for using Artificial Replay. By design, Artificial Replay takes as input a base algorithm and information relation and simulates the historical data in a replay buffer. As such, it seems natural that we can bound the regret of Artificial Replay (as a random variable), to a counterfactual simulation of the algorithm applied to a longer time horizon (where the length corresponds to the amount of used historical data). Indeed, we can show the following:

**Theorem 5.3** (Regret Improvement of Artificial Replay). Let $\Pi$ be any base algorithm, and denote by $\Delta_{\min} = \min_{a \in H_{hist}, a \neq \star} OPT - \mu(a)$ the smallest suboptimality gap in the historical dataset. Let $H_{1}^{used}$ be a random variable corresponding to the subset of $H_{1}^{hist}$ used by $\pi_{Artificial \ Replay}$ after the $T$ timesteps. Further let $H_{T}^{used}(\dagger)$ correspond to the subset of $H_{T}^{used}$ containing data for actions which are not optimal. Then we have that:

$$
\mathbb{E}\left[\text{REG}\left(T, \pi_{Artificial \ Replay}, H_{hist}\right)\right] \leq \mathbb{E}\left[\text{REG}\left(T + |H_{T}^{used}|, \pi_{Ignorant}\right)\right] - \Delta_{\min}|H_{T}^{used}(\dagger)|.
$$

Proof sketch. The proof of Theorem 5.3 again follows using a regret coupling and is detailed in Appendix F.

Note here that the base algorithm $\Pi$ needs to be identical on the $T$-length and $T + |H_{T}^{used}|$ problems, but this is avoided in UCB-style algorithms through using $T + H$ in the logarithmic terms. The expectation is taken over the randomness in the reward observations for both offline and online data. Note that Eq. (6) depends on both $|H_{T}^{used}|$ and $|H_{T}^{used}(\dagger)|$, which are random variables and are a non-trivial function of the choice of $\Pi$ alongside the information relation. In Theorem 5.6 we instantiate this theorem for MonUCB to obtain a more interpretable regret improvement bound. However, Theorem 5.3 already highlights the benefits of incorporating historical data into a regret-optimal bandit algorithm. In a typical regret-optimal algorithm, REG$(T, \pi_{Ignorant})$ is a sublinear function of $T$ for any choice of $T > 0$. Thus the first term in Eq. (6) scales sublinearly with respect to $|H_{T}^{used}|$, while the second term scales linearly with respect to $|H_{T}^{used}|$. This highlights the benefit of incorporating historical data for a general regret-optimal base algorithm.

**5.2 Case Study: MonUCB**

We now offer a case study of Artificial Replay applied to MonUCB from Section 3.1. We show that MonUCB satisfies IIData and hence Artificial Replay has identical regret guarantees to Full Start, and Artificial Replay uses arbitrarily less compute than Full Start. We then quantify the value of the historical data in terms of regret improvement. Many of the results will extend to arbitrary base algorithms in more general settings.
For the rest of this section, we consider the relation where $a \equiv a'$ if and only if $a = a'$. We start by recalling Theorem 5.2 which establishes that Full Start and Artificial Replay have identical regret for IIData algorithms. Indeed, by showing MonUCB satisfies IIData, we obtain the following:

**Theorem 5.4.** MonUCB satisfies IIData. Hence

$$
\text{REG}\left(T, \pi^{\text{Artificial Replay}(\text{MonUCB})}, \mathcal{H}^{\text{hist}}\right) \overset{d}{=} \text{REG}\left(T, \pi^{\text{Full Start}(\text{MonUCB})}, \mathcal{H}^{\text{hist}}\right).
$$

This is the finest relation such that MonUCB satisfies IIData. Not all $K$-armed bandit algorithms satisfy IIData under this relation. For example, Thompson Sampling (TS) [Russo et al., 2018] samples arms according to the posterior probability that they are optimal. TS does not satisfy IIData: data from other actions other than the one chosen will adjust the posterior distribution, and hence will adjust the selection probabilities as well. However, as we show in Fig. 8, Artificial Replay still achieves empirical gains over Full Start with Thompson sampling, despite not satisfying IIData.

We next highlight that Artificial Replay is robust to spurious data, where the historical data has excess samples coming from poor-performing actions. Spurious data imposes computational challenges since the Full Start approach will pre-process the full historical dataset regardless of the observed rewards or the inherent value of the historical data. In contrast, Artificial Replay will only use the amount of data useful for learning. This allows Artificial Replay to use arbitrarily less compute than Full Start!

Note that Artificial Replay imposes minimal computational and storage overhead on top of existing algorithms, simply requiring a data structure to verify whether $(\mathcal{H}^{\text{hist}} \setminus \mathcal{H}^{\text{used}}_t) \cap \{a \in \mathcal{A} : a \not\equiv \mathcal{H}^{\text{used}}_t, \Pi(\mathcal{H}^{\text{used}}_t) = \emptyset\} = \emptyset$. However, this can be done efficiently with hashing techniques. We make this formal in the following theorem:

**Theorem 5.5.** For every $H \in \mathbb{N}$ there exists a historical dataset $\mathcal{H}^{\text{hist}}$ with $|\mathcal{H}^{\text{hist}}| = H$ where the runtime of $\pi^{\text{Full Start}(\text{MonUCB})}$ is linear in $\Omega(H + T)$ whereas the runtime of $\pi^{\text{Artificial Replay}(\text{MonUCB})}$ is only $O(T + \min\{\sqrt{T}, \log(T)/\min_a \Delta(a)^2\})$.

**Proof.** Theorem 5.5 is a corollary of Theorem D.2 under the selection of $\psi(\lambda) = \lambda^2/8$. 

For the $K$-armed bandit, Full Start requires $O(K)$ storage to maintain estimates for each arm, while a naive implementation of Artificial Replay requires $O(K + H)$ storage since the entire historical dataset needs to be stored. However, effective hashing can address the extra $H$ factor. We also note that most practical bandit applications (including content recommendation systems and the poaching prevention setting discussed) incorporate historical data obtained from database systems. This historical data will be stored regardless of the algorithm being employed, and so the key consideration is computational requirements.

Lastly, to complement the computational improvements of Artificial Replay applied to MonUCB, we also characterize the regret improvement due to incorporating historical data relative to a data ignorant algorithm. As the regret coupling implies that Artificial Replay achieves the same regret as Full Start, we can simply analyze the regret improvement gained by Full Start to characterize the regret of Artificial Replay.

**Theorem 5.6.** Let $H_a$ be the number of data points in $\mathcal{H}^{\text{hist}}$ for each action $a \in [K]$. Then the regret of Monotone UCB with historical dataset $\mathcal{H}^{\text{hist}}$ is:

$$
\mathbb{E}\left[\text{REG}\left(T, \pi^{\text{Artificial Replay}(\text{MonUCB})}, \mathcal{H}^{\text{hist}}\right)\right] \leq O\left(\sum_{a \in [K]: \Delta_a \neq 0} \max\{0, \frac{\log(T)}{\Delta(a)} - H_a \Delta(a)\}\right).
$$

**Proof.** Theorem 5.6 is a corollary of Theorem D.3 under the choice of $\psi(\lambda) = \lambda^2/8$. 

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Theorem 5.5 together with Theorem 5.6 helps demonstrate the advantage of using Artificial Replay over Full Start: we improve computational complexity while maintaining an equally improved regret guarantee. Moreover, it highlights the impact historical data can have on the regret. If $|H_a| \geq \log(T)/\Delta(a)^2$ for each action $a$, then the regret of the algorithm will be constant not scaling with $T$. This result also reduces to the standard MonUCB guarantee (Theorem 3.1) when $H_{hist} = \emptyset$. We remark that there are no existing regret lower bounds for incorporating historical data in bandit algorithms. Our regret guarantees seem likely to be optimal, as $\log(T)/\Delta(a)$ has been shown to be the minimax regret for gathering sufficient information to eliminate a suboptimal arm $a$, and $H_a\Delta(a)$ naturally represents a reduction in regret of $\Delta(a)$ for each pull of arm $a$ in the dataset. Theorem 5.6 also highlights interesting insights into a measure of usefulness of the historical data. Historical data for the optimal action has no impact on regret (since playing that action online does not incur any regret). Moreover, it is better to have more data on more suboptimal actions.

It remains an interesting direction for future work to understand the optimality of Full Start as well as how to optimally incorporate historical data in settings when Full Start may not be optimal, for example, due to contaminated or stale historical data. Our key contribution is to highlight a simple, intuitive, and implementable approach through Artificial Replay which matches the performance of Full Start while simultaneously requiring less compute.

6 Empirical Benefits of Artificial Replay

We complement the theoretical advantages of Artificial Replay by showing that, in practice, our meta-algorithm achieves identical performance to Full Start while significantly reducing runtime and storage. Experiment details and additional results are available in Appendix E. We conduct experiments on two of the bandit models described in Section 2.2: finite $K$-armed bandits and CMAB-CRA, using both fixed and adaptive discretization. For the continuous combinatorial setting,
we provide two stylized domains, a piecewise-linear and a quadratic reward function.

**Green Security Domain** To emphasize the practical benefit of Artificial Replay, we evaluate on a real-world resource allocation setting for biodiversity conservation. Here, we introduce a new model for green security games with continuous actions by using adaptive discretization. The CMAB-CRA model can be used to specify the green security setting from Xu et al. [2021], where the space $S$ represents a protected area and $B$ is the discrete set of patrol resources to allocate, such as number of ranger hours per week with a budget $B$ of 40 hours. This formulation generalizes to a continuous-space model of the landscape, instead of the artificial fixed discretization that was considered in prior work consisting of $1 \times 1$ sq. km regions of the park. The reward function $\mu(p, \beta)$ then models the probability of observing a snare at location $p$ at effort level $\beta$.

We study real ranger patrol data from Murchison Falls National Park, shared as part of a collaboration with the Uganda Wildlife Authority and the Wildlife Conservation Society. We use historical patrol observations to build the history $H_{\text{hist}}$; we analyze these historical observations in detail in Appendix E to show that this dataset exhibits both spurious data and imbalanced coverage.

**Baselines** We compare Artificial Replay against Ignorant and Full Start approaches. In the $K$-armed model, we use MonUCB as the base algorithm. In CMAB-CRA we use fixed and adaptive discretization as well as Regressor, a neural network learner that is a regression-based approach analogue to Full Start. Regressor is initially trained on the entire historical dataset, then iteratively retrained after 128 new samples are collected. We compute for each setting the performance of an Optimal action based on the true rewards and a Random baseline that acts randomly while satisfying the budget constraint.

**Results** The results in Fig. 3 empirically validate our theoretical result from Theorem 5.2: the performance of Artificial Replay is identical to that of Full Start, and reduces regret considerably compared to the naive Ignorant approach. We evaluate the regret (compared to Optimal) of each approach across time $t \in [T]$. Concretely, we consider the three domains of piecewise-linear reward, quadratic reward, and green security with continuous space $S = [0, 1]^2$, $N = 5$ possible action components, a budget $B = 2$, and 3 levels of effort. We include $H = 300$ historical data points. See Fig. 9 (appendix) for regret and analysis of historical data use on the $K$-armed bandit. Results are averaged over 60 iterations with random seeds, with standard error plotted.

Not only does Artificial Replay achieve equal performance, but its computational benefits over Full Start are clear even on practical problem sizes. As we increase historical data from $H = \{10; 100; 1,000; 10,000\}$ in Fig. 1, the proportion of irrelevant data increases. Our method achieves equal performance, overcoming the previously unresolved challenge of spurious data, while Full Start suffers from arbitrarily worse storage complexity (Theorem 5.5). With 10,000 historical samples and a time horizon of 1,000, we see that 58.2% of historical samples are irrelevant to producing the most effective policy.

When faced with imbalanced data coverage, the benefits of Artificial Replay become clear—most notably in the continuous action setting with adaptive discretization. In Fig. 2, as we increase the number of historical samples on bad regions (bottom 20th percentile of reward), the additional data require finer discretization, leading to arbitrarily worse storage and computational complexity for Full Start with equal regret. In Fig. 2(c), we see that with 10% of data on bad arms, Artificial Replay requires only 446 regions $R$ compared to 688 used by Full Start; as we get more spurious data and that fraction increases to 90%, then Artificial Replay requires only 356 regions while Full Start still stores 614 regions.
7 Conclusion

We present Artificial Replay, a meta-algorithm that modifies any base bandit algorithm to efficiently harness historical data. As we show, under a widely applicable IIData condition, the regret of Artificial Replay is distributionally identical to that of a full warm-start approach, while also guaranteeing significantly better time complexity.

Our experimental results highlight the advantage of using Artificial Replay over Full Start on a variety of base algorithms, applied to K-armed and continuous combinatorial bandit models. These advantages even hold for base algorithms such as Thompson sampling and Information Directed Sampling (IDS) that do not exhibit IIData.

Directions for future work include (i) find IIData algorithms in other bandit domains such as linear contextual bandits, (ii) incorporate the Artificial Replay approach into reinforcement learning, (iii) motivate IIData by highlighting it is robust to adversarial data generation, and (iii) provide theoretical bounds showing that Artificial Replay has optimal data usage when incorporating historical data.

Acknowledgments

Part of this work was done while Sid Banerjee, Sean Sinclair, and Christina Lee Yu were visiting the Simons Institute for the Theory of Computing for the semester on Data-Driven Decision Processes. This work was supported, in part, by the NSF under grants ECCS-1847393, DMS-1839346, CCF-1948256, and CNS-1955997, and the ARL under grant W911NF-17-1-0094. Lily Xu acknowledges the support of a Google PhD Fellowship.

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# Table of Notation

| Symbol | Definition |
|--------|------------|
| $\mathcal{A}$ | Feasible action space |
| $\Re$ | Reward distribution, i.e., $\Re: \mathcal{A} \rightarrow \Delta([0,1])$ |
| $T$ | Time horizon |
| $\mathcal{D}$ | An arbitrary set of $(a,r)$ pairs |
| $\mathcal{H}^{hist}$, $H$ | Historical data available to algorithm and number of historical datapoints |
| $\text{REG}(\pi,T,\mathcal{H}^{hist})$ | Cumulative regret for an algorithm $\pi$ on $T$ timesteps with historical data $\mathcal{H}^{hist}$ |
| $a$ | Generic action $a \in \mathcal{A}$ |
| $a^j_H$ | Historical action at index $j$ in history $\mathcal{H}^{hist}$ |
| $A_t$ | Selected action chosen at timestep $t$ |
| $R_t$ | Reward observed at timestep $t$ from action $A_t$ |
| $\Pi$ | Base algorithm that maps ordered $(a,R)$ pairs to a distribution over $\mathcal{A}$ |
| $\equiv^\mathcal{H}$ | Information relation over $\mathcal{A}$ based on the data $\mathcal{H}$ |
| $\pi_{\text{Artificial Replay}}$ | Artificial Replay framework and its resulting policy |
| $\pi_{\text{Full Start}}$ | Full Start: Full warm starting operation and its resulting policy |
| $\pi_{\text{Ignorant}}$ | Ignorant: Original policy ignoring historical data |

**CMAB-CRA specification**

- $\mathcal{CMAB-CRA}$: Combinatorial Multi-Armed Bandit for Continuous Resource Allocation
- $\mathcal{S}, \mathcal{B}$: The continuous resource space, and (discrete) space of allocation values
- $d_\mathcal{S}, d_{\text{max}}$: Metric over $\mathcal{S}$ and the diameter of $\mathcal{S}$
- $N, \epsilon$: Maximum number of regions which can be selected, and the minimum distance
- $\Re(p, \vec{\beta})$: Reward distribution for a particular allocation $(p, \vec{\beta})$
- $\mu(p, \vec{\beta})$: Mean reward for a particular allocation $(p, \vec{\beta})$

**Monotone UCB**

- $\mu_t(a), n_t(a)$: Mean reward estimates and number of samples for action $a \in [K]$
- $\text{UCB}_t(a)$: Upper confidence bound estimate of action $a$

**Fixed Discretization**

- $\mathcal{P}$: Fixed $\epsilon$ covering of $\mathcal{S}$
- $\mu_t(\mathcal{R}, \beta), n_t(\mathcal{R}, \beta)$: Mean reward estimates and number of samples for region $\mathcal{R} \in \mathcal{P}$
- $\text{UCB}_t(\mathcal{R}, \beta)$: Upper confidence bound estimate of $\mu(\mathcal{R}, \beta)$
- $b(t)$: Bonus term (confidence radius) for a region which has been selected $t$ times

**Adaptive Discretization**

- $\mathcal{P}_t^\beta$: Partition of space $\mathcal{S}$ at timestep $t$ for allocation $\beta \in \mathcal{B}$
- $\mu_t(\mathcal{R}, \beta), n_t(\mathcal{R}, \beta)$: Mean reward estimates and number of samples for region $\mathcal{R} \in \mathcal{P}_t^\beta$
- $\text{UCB}_t(\mathcal{R}, \beta)$: Upper confidence bound estimate of $\mu(\mathcal{R}, \beta)$
- $r(\mathcal{R})$: Diameter of a region $\mathcal{R}$
- $b(t)$: Bonus term for a region which has been selected $t$ times

Table 1: List of common notations
B Detailed Related Work

Multi-armed bandit problems and its sub-variants (including the finite-armed and CMAB-CRA model discussed here) have a long history in the online learning and optimization literature. We highlight the most closely related works below, but for more extensive references see Bubeck et al. [2012], Slivkins [2019], Lattimore and Szepesvári [2020].

Multi-Armed Bandit Algorithms The design and analysis of bandit algorithms have been considered under a wide range of models. These were first studied in the so-called $K$-Armed Bandit model in Lai and Robbins [1985], Auer et al. [2002], where the algorithm has access to a finite set of $K$ possible actions at each timestep. The algorithm is characterized by its Optimistic Upper Confidence Bound approach, where exploration is garnered by acting greedily with respect to optimistic estimates of the mean reward of each action. Numerous follow-up works have considered similar approaches when designing algorithms in continuous action spaces [Kleinberg et al., 2019], linear reward models [Chu et al., 2011], and with combinatorial constraints [Xu et al., 2021, Zuo and Joe-Wong, 2021]. Our work provides a framework for taking existing algorithms to additionally harness historical data. Moreover, in Section 4.2 we also propose a novel algorithm incorporating data-driven adaptive discretization for combinatorial multi-armed bandits for continuous resource allocation.

Incorporating Historical Data Several papers have started to investigate techniques for incorporating historical data into bandit algorithms. Shivaswamy and Joachims [2012] started by considering a $K$-armed bandit model where each arm has a dataset of historical pulls. The authors develop a Warm Start UCB algorithm where the confidence term of each arm is initialized based on the full historical data, prior to learning. Similar techniques were extended to models where there are pre-clustered arms, where the authors provide regret guarantees depending on the cluster quality of the fixed clusters [Bouneffouf et al., 2019]. These techniques were extended to Bayesian and frequentist linear contextual bandits where the linear feature vector is updated by standard regression over the historical data [Oetomo et al., 2021, Wang et al., 2017]. The authors show empirically that these approaches perform better in early rounds, and applied the set-up to recommendation systems. A second line of work has considered warm-starting contextual bandit models with fully supervised historical data and online bandit interaction [Swaminathan and Joachims, 2015, Zhang et al., 2019]. In Zuo et al. [2020] consider augmented data collection schemes where the decision maker can “pre sample” some arms before decisions in the typical bandit set-up. We lastly note that Wagenmaker and Pacchiano [2022] consider how best to use offline data to minimize the number of online interactions necessary to learn a near optimal policy in the linear RL setting.

All of the prior bandit literature considers a Full Start approach in specific bandit models to incorporate historical data. In contrast, we provide an efficient meta-algorithm for harnessing historical data in arbitrary stochastic bandit models. We show our approach has improved runtime and storage over a naive full-start approach. Additionally, we provide, to the best of our knowledge, the first application of incorporating historical data to combinatorial bandit models.

Bandit Algorithms for Green Security Domains Green security focuses on allocating defender resources to conduct patrols across protected areas to prevent illegal logging, poaching, or overfishing [Fang et al., 2015, Plumptre et al., 2014]. These green security challenges have been addressed with game theoretic models [Yang et al., 2014, Nguyen et al., 2016]; supervised machine learning [Kar et al., 2017, Xu et al., 2020]; and multi-armed bandits, including restless [Qian et al.,
Combinatorial Bandits for Resource Allocation

The CMAB-CRA model is a continuous extension of the combinatorial multi-armed bandit for discrete resource allocation (CMAB-DRA) problem studied in Zuo and Joe-Wong [2021]. They propose two algorithms which both achieve logarithmic regret when the allocation space is finite or one-dimensional. We extend their upper confidence bound algorithmic approach to consider both fixed and adaptive data-driven discretization of the continuous resource space, and additionally consider the impact of historical data in learning.

Adaptive Discretization Algorithms

Discretization-based approaches to standard multi-armed bandits and reinforcement learning have been explored both heuristically and theoretically in different settings. Adaptive discretization was first analyzed theoretically for the standard stochastic continuous multi-armed bandit model, where Kleinberg et al. [2019] developed an algorithm which achieves instance-dependent regret scaling with respect to the so-called “zooming dimension” of the action space. This approach was later extended to contextual models in Slivkins [2011]. In Elmahdou et al. [2017] the authors improve on the practical performance and scalability by considering decision-tree instead of dyadic partitions of the action space. Similar techniques have been applied to reinforcement learning, where again existing works have studied the theoretical challenges of designing discretization-based approaches with instance-specific regret guarantees [Sinclair et al., 2023], and heuristic performance under different tree structures [Uther and Veloso, 1998, Pyeatt and Howe, 2001]. However, none of these algorithms have been extensively studied within the concept of including historical data, or applied to the combinatorial bandit model, with the exception of Xu et al. [2021]. Our work builds upon theirs through a novel method of incorporating historical data into an algorithm, and by additionally considering adaptive instead of fixed discretization.

Experience Replay in Reinforcement Learning

Lastly, we note that the Artificial Replay approach has relations to experience replay in the reinforcement learning literature [Schaul et al., 2017, Mnih et al., 2013]. In contrast to Artificial Replay, which is designed to use historical data collected independently of the algorithm, experience replay uses online observations (i.e., data points in $H_t$) and requires using off-policy estimation procedures to incorporate the information in learning.

C Full Preliminary Details

In this section we restate and give further assumptions for the general stochastic bandit model described in Section 2.

C.1 K-Armed Bandit

The finite-armed bandit model can be viewed in this framework by considering $\mathcal{A} = [K] = \{1, \ldots, K\}$. This recovers the classical model from Lai and Robbins [1985], Auer et al. [2002].
C.2 Combinatorial Multi-Armed Bandit for Continuous Resource Allocation (CMAB-CRA)

A central planner has access to a metric space $S$ of resources with metric $d_S$. They are tasked with splitting a total amount of $B$ divisible budget across $N$ different resources within $S$. For example, in a wildlife conservation domain, the space $S$ can be considered as the protected area of a park, and the allocation budget corresponds to divisible effort, or proportion of rangers allocated to patrol in the chosen area. We denote the feasible space of allocations as $B$ and define the feasible action space as follows:

$A = \left\{ (\mathbf{p}, \beta) \in S^N \times B^N \mid \sum_{i=1}^{N} \beta^{(i)} \leq B, \quad d_S(p^{(i)}, p^{(j)}) \geq \epsilon \quad \forall i \neq j \right\}.$  \hspace{1cm} (7)

Note that we require the chosen action to satisfy the budgetary constraint (i.e. $\sum_i \beta^{(i)} \leq B$), and that the chosen resources are distinct (aka $\epsilon$-away from each other according to $d_S$).

Further, let $\mathcal{R} : S \times B \to \Delta([0,1])$ be the unknown reward distribution over the resource and allocation space. The goal of the algorithm is to pick an action $A = (\mathbf{p}, \beta) \in A$ in a way that maximizes $\sum_{i=1}^{N} \mathbb{E}[\mathcal{R}(p^{(i)}, \beta^{(i)})]$, the expected total mean reward accumulated from the resources subject to the budget constraints. Denoting $\mathbb{E}[\mathcal{R}(\mathbf{p}, \beta)]$ as $\mu(\mathbf{p}, \beta)$, the optimization problem is formulated below:

$$\begin{align*}
\max_{\mathbf{p}, \beta} & \quad \sum_{i=1}^{N} \mu(p^{(i)}, \beta^{(i)}) \\
\text{s.t.} & \quad \sum_i \beta^{(i)} \leq B \\
& \quad d_S(p^{(i)}, p^{(j)}) \geq \epsilon \quad \forall i \neq j .
\end{align*}$$  \hspace{1cm} (8)

Lastly, we consider the historical data $\mathcal{H}^{\text{hist}}$ to also be decomposed, in that each element is a particular $(\mathbf{p}, \beta, R)$ pair with $R \sim \mathcal{R}(\mathbf{p}, \beta)$.

C.2.1 Assumptions

We make two assumptions on the underlying problem. The first is a standard nonparametric assumption, highlighting that the resource space $S$ is a metric space and that the true underlying reward function $\mu$ is Lipschitz with respect to $d_S$. This assumption is common in the continuous bandit literature; see Kleinberg et al. [2019] for more discussion.

**Assumption 1.** $S$ is a compact metric space endowed with a metric $d_S$ with diameter $d_{\text{max}}$, and $B$ is a discrete space, both known to the decision maker. We assume that the mean reward function $\mu(\mathbf{p}, \beta)$ is Lipschitz with respect to the metric $d_S$ over $\mathbf{p}$ with known Lipschitz constant $L$.

The next assumption assumes access to an oracle for solving optimization problems of the form of Eq. (4) for arbitrary choice of reward functions $r(\mathbf{p}, \beta)$. We can relax this assumption to instead assume that there exists a randomized approximation oracle by appropriately shifting the regret benchmark. However, in Section 6 we run experiments with exact solution oracles and omit this discussion from this work.

**Assumption 2.** The optimization problem formulated in Eq. (4) can be solved for arbitrary reward functions $\mu(\mathbf{p}, \beta)$.
C.2.2 Mapping to Green Security Domains

The CMAB-CRA model can be used to specify green security domains from Xu et al. [2021]. $S$ is used to represent the “protected region”, or geographic region of the park, and $B$ is the discrete set of potential patrol efforts to allocate, such as the number of ranger hours per week, with the total budget $B$ being 40 hours. This formulation generalizes Xu et al. [2021] to a more realistic continuous space model of the landscape, instead of the artificial fixed discretization that was considered in prior work consisting of $1 \times 1$ sq. km regions on park).

C.3 Historical Dataset

We assume that the algorithm designer has access to a historical dataset $\mathcal{H}_{\text{hist}} = \{a_j^H, R_j^H\}_{j \in [H]}$ containing $H$ historical points with actions $a_j^H$ and rewards $R_j^H$ sampled according to $\mathcal{R}(a_j^H)$.

We briefly discuss the required assumptions on how the dataset is chosen in order for Full Start and Artificial Replay to be a reasonable algorithm for incorporating historical data in the first place. We first give two examples:

- The historical dataset is sampled according to a fixed, non-adaptive policy. In this case we have that $H_a$, the number of samples for each arm is selected apriori, and each sample $a_j^H$ has rewards $R_j^H$ sampled according to $\mathcal{R}(a_j^H)$.
- The historical dataset is sampled using an IIData algorithm. Then the number of samples from each action $H_a$ is a random variable, but each observed reward has the correct marginal distribution by being sampled according to $\mathcal{R}(a_j^H)$.

However, there are many scenarios on how the historical dataset is generated that could lead to Full Start or Artificial Replay to be the right algorithm in the first place. For a simple example, suppose that the samples for each arm $a$ were sorted in decreasing order. In this case, Artificial Replay will learn biased estimates of the underlying mean reward by not using the entire historical dataset. Similarly, suppose that the historical dataset was sampled according to an algorithm that “ignored” samples smaller than a certain threshold value. Both Full Start and Artificial Replay would again learn biased estimates of the mean reward for the actions from this historical data.

We leave the interesting question of when Full Start and Artificial Replay are the optimal heuristics for incorporating dataset due to tampering of the historical data for future work.

D IIData Algorithms

D.1 MonUCB Algorithms for $K$-Armed Bandits

In this section we detail the Monotone UCB (denoted MonUCB) style algorithms for the $K$-Armed Bandit problem of Section 2.2.1, which are regret-optimal and satisfy the IIData property under the relation where $a \equiv a'$ if and only if $a = a'$. These algorithms are derived from the $\psi$-UCB based algorithms from Bubeck et al. [2012]. For full pseudocode of the algorithm see Algorithm 3. We describe the algorithm without incorporating historical data (where its counterpart involving historical data are derived by treating this as the base algorithm $\Pi$ and appealing to Artificial Replay or Full Start described in Section 3).

Concretely, MonUCB tracks the following, akin to standard UCB approaches: (i) $\bar{\pi}_t(a)$ for the estimated mean reward of action $a \in [K]$, (ii) $n_t(a)$ for the number of times the action $a$ has been selected by the algorithm prior to timestep $t$, and (iii) $\text{UCB}_t(a)$ for an upper confidence bound estimate for the reward of action $a$. At every timestep $t$, the algorithm picks the action $A_t$ which
Algorithm 3 Monotone $\psi$-UCB (MonUCB)

1: **Input**: Convex function $\psi$
2: Initialize $n_1(a) = 0$, $\pi_1(a) = 1$, and $\text{UCB}_1(a) = 1$ for each $a \in [K]$
3: for $t = \{1, 2, \ldots \}$ do
4: Let $A_t = \arg \max_{a \in [K]} \text{UCB}_t(a)$
5: Receive reward $R_t$ sampled from $\mathcal{R}(A_t)$
6: for all $a \neq A_t$ do
7: $n_{t+1}(a) = n_t(a)$
8: $\pi_{t+1}(a) = \pi_t(a)$
9: $\text{UCB}_{t+1}(a) = \text{UCB}_t(a)$
10: end for
11: $n_{t+1}(A_t) = n_t(A_t) + 1$
12: $\pi_{t+1}(A_t) = (n_t(A_t)\pi_t(A_t) + R_t)/n_{t+1}(A_t)$
13: $\text{UCB}_{t+1}(A_t) = \min \left\{ \text{UCB}_t(A_t), \pi_{t+1}(A_t) + (\psi^*)^{-1} \left( \frac{2 \log(T)}{n_{t+1}(A_t)} \right) \right\}$
14: end for

maximizes $\text{UCB}_t(a)$ (breaking ties deterministically). After observing $R_t$, we update our counts $n_{t+1}(a)$ and estimates $\pi_{t+1}(a)$.

Similar to Bubeck et al. [2012], we assume the algorithm has access to a convex function $\psi$ satisfying the following assumption:

**Assumption 3.** For all $\lambda \geq 0$ and $a \in \mathcal{A}$ we have that:

$$
\log \left( \mathbb{E} \left[ e^{\lambda (\mathcal{R}(a) - \mu(a))} \right] \right) \leq \psi(\lambda)
$$
$$
\log \left( \mathbb{E} \left[ e^{\lambda (\mu(a) - \mathcal{R}(a))} \right] \right) \leq \psi(\lambda).
$$

Note that this is trivially satisfied when $\psi(\lambda) = \lambda^2/8$ using Hoeffding’s lemma and the assumption that the rewards are bounded in $[0, 1]$.

We further denote by $\psi^*$ as the Legendre-Fenchel transform of $\psi$, defined via:

$$
\psi^*(\epsilon) = \sup_{\lambda} \lambda \epsilon - \psi(\epsilon).
$$

With this, we are now ready to define how the confidence bounds are computed as follows:

$$
\text{UCB}_{t+1}(A_t) = \text{UCB}_t(A_t), \quad \pi_{t+1}(A_t) + (\psi^*)^{-1} \left( \frac{2 \log(T)}{n_{t+1}(A_t)} \right)
$$

Note that again when $\psi(\lambda) = \lambda^2/8$ then we again recover the standard UCB1 algorithm (as presented in Section 4). Next we state the more general version of the theorems from Section 3.1 and Section 5 applied to MonUCB with arbitrary convex function $\psi$. We omit all proofs from the results here, and defer them to Appendix F.

First we highlight that MonUCB for an arbitrary convex function $\psi$ satisfying Assumption 3 satisfies the IIData property under the relation where $a \overset{d}{=} a'$ if and only if $a = a'$. We similarly establish that the base algorithm is regret-optimal, recovering the results established in Bubeck et al. [2012].
Theorem D.1. The MonUCB base algorithm with arbitrary convex function $\psi$ satisfying Assumption 3 and under the relation where $a \sim a'$ if and only if $a = a'$ satisfies the IIDData property. Moreover, for $\Delta(a) = \max_{a'} \mu(a') - \mu(a)$ then

$$E \left[ \text{REG}(T, \pi_{\text{Ignorant} \text{(MonUCB)}}), \mathcal{H}^{\text{hist}} \right] = O(\sum_a \Delta(a) \log(T)/\psi^*(\Delta(a))).$$

Note that Theorems 3.1 and 5.4 are corollaries of Theorem D.1 under the selection of $\psi(\lambda) = \lambda^2/8$. The next result highlights that there exists a historical dataset such that Full Start has unbounded computational complexity (as the number of datapoints goes to infinity), whereas Artificial Replay has bounded computational complexity. Combined with the regret coupling, this highlights that Artificial Replay Pareto-dominates Full Start in terms of regret and computational complexity.

Theorem D.2. Consider the relation where $a \sim a'$ if and only if $a = a'$. For every $H \in \mathbb{N}$ there exists a historical dataset $\mathcal{H}^{\text{hist}}$ with $|\mathcal{H}^{\text{hist}}| = H$ where the runtime of $\pi_{\text{Full Start} \text{(MonUCB)}} = \Omega(H + T)$ whereas the runtime of $\pi_{\text{Artificial Replay} \text{(MonUCB)}} = O(T + \min\{\sqrt{T}, \log(T)/\min_a \Delta(a)^2\}).$

We again note that Theorem 5.5 is a corollary of Theorem D.2 under the selection of $\psi(\lambda) = \lambda^2/8$. The final result establishes a regret improvement for Artificial Replay scaling as the size of the historical data used.

Theorem D.3. Consider the relation where $a \sim a'$ if and only if $a = a'$. Let $H_a$ be the number of datapoints in $\mathcal{H}^{\text{hist}}$ for each action $a \in [K]$. Then the regret of Monotone UCB with historical dataset $\mathcal{H}^{\text{hist}}$ is:

$$E \left[ \text{REG}(T, \pi_{\text{Artificial Replay} \text{(MonUCB)}}), \mathcal{H}^{\text{hist}} \right]$$

$$\leq O\left( \sum_{a \in [K]} \max \left\{ 0, \frac{\Delta(a) \log(T)}{\psi^*(\Delta(a))} - H_a \Delta(a) \right\} \right).$$

We again note that Theorem 5.6 is a corollary of Theorem D.3 under the choice of $\psi(\lambda) = \lambda^2/8$.

D.2 Discretization Algorithms for Metric Bandits

In this section we detail a fixed and adaptive discretization algorithm for metric bandits from Section 4.1 which satisfies the IIDData property. These serve as a straightforward modification to the algorithms from Kleinberg et al. [2019], where similar to MonUCB we adjust the confidence terms to be monotone decreasing. We start by summarizing the main algorithm sketch, before highlighting the key details and differences between the two. Full pseudocode is omitted, since in Appendix D.3 we will extend the approach to the more complicated CMAB-CRA set-up. We here describe the algorithm without incorporating historical data, since its counterpart involving historical data can be used by treating this as the base algorithm II and appealing to Artificial Replay or Full Start described in Section 3.

Our algorithms are Upper Confidence Bound (UCB) style as the selection rule picks an action $a \in \mathcal{A}$ over a discretization of $\mathcal{A}$. Both algorithms are parameterized by the time horizon $T$ and a value $\delta \in (0,1)$. The algorithm maintains a collection of regions $\mathcal{P}_t$ of $\mathcal{A}$. Each element $\mathcal{R} \in \mathcal{P}_t$ is a region (i.e. subset of $\mathcal{A}$) with diameter $r(\mathcal{R})$. For the fixed discretization variant, $\mathcal{P}_t$ is fixed at the start of $t = 1$. In the adaptive discretization algorithm, this partitioning is refined over the course of partitioning.

For each time period $t$ the algorithm maintains three tables linear with respect to the number of regions in the partition. This includes an upper confidence value $\text{UCB}_t(\mathcal{R})$ for the true $\mu(a)$ value.
for points $a$ in $\mathcal{R}$ (which is initialized to be one), determined based on an estimated mean $\overline{\nu}_t(\mathcal{R})$ and $n_t(\mathcal{R})$ for the number of times $\mathcal{R}$ has been selected by the algorithm in timesteps up to $t$. The latter is incremented every time $\mathcal{R}$ is played and is used to construct the bonus term. At a high level, our algorithms perform two steps in each iteration $t$: select an action via the selection rule and then update parameters. In addition, the adaptive discretization algorithm will re-partition the space. In order to define the steps, we first introduce some definitions and notation.

The confidence radius (or bonus) of region $\mathcal{R}$ is defined:

$$b(n_t(\mathcal{R})) = 2\sqrt{\frac{2\log(T/\delta) + 2L}{n_t(\mathcal{R})}}$$

(9)

corresponding to the uncertainty in estimates due to the stochastic nature of the rewards. Then the steps of the algorithm are defined as follows:

1. **Selection rule**: Pick the action $\mathcal{R}_t = \arg \max_{\mathcal{R} \in \mathcal{P}_t} \text{UCB}_t(\mathcal{R})$. Play any action $A_t \in \mathcal{R}_t$.

2. **Update parameters**: Increment $n_t(\mathcal{R}_t)$ by 1, update $\overline{\nu}_t(\mathcal{R}_t)$ based on observed data, and update $\text{UCB}_t(\mathcal{R}_t)$ while ensuring monotonicity, i.e.:

$$\text{UCB}_{t+1}(\mathcal{R}_t) = \min(\text{UCB}_t(\mathcal{R}_t), \mu_{t+1}(\mathcal{R}_t) + b(n_{t+1}(\mathcal{R}_t))).$$

In the fixed discretization algorithm we set $\mathcal{P}_1$ to be a $\gamma$-covering of $\mathcal{A}$. For the adaptive discretization, the algorithm additionally decides whether to update the partition. This is done via:

3. **Re-partition the space**: Let $\mathcal{R}_t$ denote the selected region and $r(\mathcal{R}_t)$ denote its radius. We split when $n_t(\mathcal{R}_t) \geq (1/r(\mathcal{R}_t))^2$. We then cover $\mathcal{R}$ with new regions $\mathcal{R}^1, \ldots, \mathcal{R}^m$ which form an $\frac{1}{2}r(\mathcal{R}_t)$-Net of $\mathcal{R}_t$. We call $\mathcal{R}_t$ the parent of these new balls and each child ball inherits all values from its parent. We then add the new balls $\mathcal{R}^1, \ldots, \mathcal{R}^m$ to $\mathcal{P}_t$ to form the partition for the next timestep $\mathcal{P}_{t+1}$.

**Information relations** Under the fixed discretization algorithm we take the natural information relation where $a \equiv a'$ whenever $d(a,a') \leq \gamma$, where $\gamma$ is bandwidth parameter for the discretization $\mathcal{P}_t$. For the adaptive discretization algorithm, we use a history-dependent information ordering. Here, $a \equiv_H a'$ whenever $a$ and $a'$ fall into the same region in the adaptive partition of $\mathcal{A}$ determined based on $\mathcal{H}$. Here we emphasize that the adaptive discretization generated is only a function of the sequence of selected actions (with no dependence on $t$ or the order for which they are read).

### D.3 Discretization Algorithms for CMAB-CRA

In this section we detail the fixed and adaptive discretization algorithms for CMAB-CRA which satisfy the IIData property, and serve as a combinatorial extension to the algorithms developed in Appendix D.2. We start off by summarizing the main algorithm sketch, before highlighting the key details and differences between the two. For pseudocode of the adaptive discretization algorithm see Algorithm 4. We describe the algorithm without incorporating historical data (where its counterpart involving historical data can be used by treating this as the base algorithm $\Pi$ and appealing to Artificial Replay or Full Start described in Section 3).

Our algorithms are Upper Confidence Bound (UCB) style as the selection rule maximizes Eq. (4) approximately over a discretization of $\mathcal{S}$. Both algorithms are parameterized by the time horizon $T$ and a value $\delta \in (0, 1)$. For each allocation $\beta \in \mathcal{B}$ the algorithm maintains a collection of regions $\mathcal{P}_t^\beta$
Figure 4: Comparison of a fixed (middle) and adaptive (right) discretization on a two-dimensional resource set $S$ for a fixed allocation level $\beta$. The underlying color gradient corresponds to the mean reward $\mu(p, \beta)$ with red corresponding to higher value and blue to lower value (see figure on left for legend). The fixed discretization algorithm is forced to explore uniformly across the entire resource space. In contrast, the adaptive discretization algorithm is able to maintain a data efficient representation, even without knowing the underlying mean reward function a priori.

of $S$. Each element $R \in P_\beta^t$ is a region with diameter $r(R)$. For the fixed discretization variant, $P_\beta^t$ is fixed at the start of learning. In the adaptive discretization algorithm, this partitioning is refined over the course of learning in a data-driven manner.

For each time period $t$, the algorithm maintains three tables linear with respect to the number of regions in the partitions $P_\beta^t$ and size of $B$. For every region $R \in P_\beta^t$ we maintain an upper confidence value $UCB_t(R, \beta)$ for the true $\mu(R, \beta)$ value for points in $R$ (which is initialized to be one), determined based on an estimated mean $\overline{\mu}_t(R, \beta)$ and $n_t(R, \beta)$ for the number of times $R$ has been selected by the algorithm in timesteps up to $t$. The latter is incremented every time $R$ is played, and is used to construct the bonus term. At a high level, our algorithms perform two steps in each iteration $t$: select an action via the selection rule and then update parameters. In addition, the adaptive discretization algorithm will re-partition the space. In order to define the steps, we first introduce some definitions and notation.

Let $t_R = n_t(R, \beta)$ be the number of times the algorithm has selected region $R \in P$ at allocation $\beta$ by time $t$. The confidence radius (or bonus) of region $R$ is defined:

$$b(t_R) = 2\sqrt{\frac{2\log(T/\delta)}{t_R}}$$

(10)

corresponding to the uncertainty in estimates due to stochastic nature of the rewards. Lastly, the UCB value for a region $R$ is computed as $UCB_t(R, \beta) = \min\{UCB_{t-1}(R, \beta), \overline{\mu}_t(R, \beta) + b(n_t(R, \beta))\}$. This enforces monotonicity in the UCB estimates, similar to $\text{MonUCB}$, and is required for the iIData property to hold.

At each timestep $t$, the algorithm selects regions according to the following optimization procedure:

$$\max_{z(R, \beta) \in \{0, 1\}} \sum_{\beta \in B} \sum_{R \in P_\beta^t} UCB_t(R, \beta) \cdot z(R, \beta)$$

(11)

s.t. $\sum_{\beta \in B} \sum_{R \in P_\beta^t} \beta \cdot z(R, \beta) \leq B$

$$\sum_{\beta \in B} \sum_{R \in P_\beta^t} z(R, \beta) \leq N$$
\[ z(\mathcal{R}, \beta) + \sum_{\beta' \neq \beta} \sum_{\tilde{\mathcal{R}} \in \mathcal{P}_t^\beta} z(\tilde{\mathcal{R}}, \beta') \leq 1 \quad \forall \beta, \mathcal{R} \in \mathcal{P}_t^\beta \]

The objective encodes the goal of maximizing the upper confidence bound terms. The first constraint encodes the budget limitation, and the second that at most \(N\) regions can be selected. The final constraint is a technical one, essentially requiring that for each region \(\mathcal{R} \in \mathcal{S}\), the same region is not selected at different allocation amounts. In the supplementary code base we provide an efficient implementation which avoids this step by “merging” the trees appropriately so that each region contains a vector of estimates of \(\pi_t(\mathcal{R}, \beta)\) for each \(\beta \in \mathcal{B}\). Based on the optimal solution, the final action \(A^*\) is taken by picking \((p, \beta)\) for each \(\mathcal{R}\) such that \(p \in \mathcal{R}\) and \(z(\mathcal{R}, \beta) = 1\). We lastly note that this optimization problem is a well-known “knapsack” problem with efficient polynomial-time approximation guarantees. It also has a simple “greedy” solution scheme, which iteratively selects the regions with largest UCB terms. See Williamson and Shmoys [2011] for more discussion.

After subsequently observing the rewards for the selected regions, we increment \(t_\mathcal{R} = n_t(\mathcal{R}, \beta)\) by one for each selected region, update \(\pi_t(\mathcal{R}, \beta)\) accordingly with the additional datapoint, and compute \(UCB_t(\mathcal{R}, \beta)\). Then the two rules are defined as follows:

1. **Selection rule:** Greedily select at most \(N\) regions subject to the budgetary constraints which maximizes \(UCB_t(\mathcal{R}, \beta)\) following Eq. (11).

2. **Update parameters:** For each of the selected regions \(\mathcal{R}\), increment \(n_t(\mathcal{R}, \beta)\) by 1, update \(\pi_t(\mathcal{R}, \beta)\) based on observed data, and update \(UCB_t(\mathcal{R}, \beta)\) while ensuring monotonicity.

### D.3.1 Fixed Discretization

This algorithm is additionally parameterized by a discretization level \(\gamma\). The algorithm starts by maintaining a \(\gamma\)-covering of \(\mathcal{S}\), which we denote as \(\mathcal{P}_t^\beta = \mathcal{P}\) for all \(t \in [T]\) and \(\beta \in \mathcal{B}\). For the information relation, we define:

\[ ((p_1, \beta_1), \ldots, (p_N, \beta_N)) \equiv ((p'_1, \beta'_1), \ldots, (p'_N, \beta'_N)) \]

if there exists an index \(j\) with \(\beta_j = \beta'_j\) and \(d_S(p_j, p'_j) \leq \gamma\). Note that this combines the continuous and semi-bandit relations described in Section 3.3.

**Information relation** For the information relation, we define:

\[ ((p_1, \beta_1), \ldots, (p_N, \beta_N)) \equiv ((p'_1, \beta'_1), \ldots, (p'_N, \beta'_N)) \]

if there exists an index \(j\) with \(\beta_j = \beta'_j\) and \(d_S(p_j, p'_j) \leq \gamma\). This combines the continuous and semi-bandit relation models outlined in Section 3.3.

### D.3.2 Adaptive Discretization

For each effort level \(\beta \in \mathcal{B}\) the algorithm maintains a collection of regions \(\mathcal{P}_t^\beta\) of \(\mathcal{S}\) which is refined over the course of learning for each timestep \(t\). Initially, when \(t = 1\), there is only one region in each partition \(\mathcal{P}_1^\beta\) which has radius 1 containing \(\mathcal{S}\).

The key differences from the fixed discretization are two-fold. First, the confidence radius or bonus of region \(\mathcal{R}\) is defined via:

\[ b(t) = 2\sqrt{\frac{2 \log(T/\delta)}{t}} + \frac{2Ld_{\text{max}}}{\sqrt{t}}. \]
Algorithms 4 and 5 are presented below:

**Algorithm 4 Adaptive Discretization for CMAB-CRA (AdaMonUCB)**

1. **Input**: Resource set $S$, effort set $B$, timesteps $T$, and probability of failure $\delta$
2. Initiate $|B|$ partitions $\mathcal{P}_t^\beta$ for each $\beta \in B$, each containing a single region with radius $d_{\text{max}}$ and $\overline{\mu}_t^\beta$ estimate equal to 1.
3. **for** each timestep $\{t \leftarrow 1, \ldots, T\}$ do
4. Select the regions by the selection rule Eq. (11).
5. For each selected region $R$ (regions where $z(R, \beta) = 1$), add $(a, \beta)$ to $A_t$ for any $a \in R$.
6. Play action $A_t$ in the environment.
7. Update parameters: $t = n_{t+1}(R_{\text{sel}}, \beta) \leftarrow n_t(R_{\text{sel}}, \beta) + 1$ for each selected region $R_{\text{sel}}$ with $z(R_{\text{sel}}, \beta) = 1$, and update $\overline{\mu}_t(R_{\text{sel}}, \beta)$ accordingly with observed data.
8. **if** $n_{t+1}(R, \beta) \geq \left(\frac{d_{\text{max}}}{r(R)}\right)^2$ and $r(R) \geq 2\epsilon$ **then**
9. **end if**
10. **end for**

**Algorithm 5 Split Region (Sub-Routine for AdaMonUCB)**

1. **Input**: Region $R$, allocation amount $\beta$, timestep $t$.
2. Set $R_1, \ldots, R_n$ to be a $\frac{1}{2}r(R)$-packing of $R$, and add each region to the partition $\mathcal{P}_t^\beta$.
3. Initialize parameters $\overline{\mu}_t(R_i, \beta)$ and $n_t(R_i, \beta)$ for each new region $R_i$ to inherent values from the parent region $R$.

The first term corresponds to uncertainty in estimates due to stochastic nature of the rewards, and the second is the discretization error by expanding estimates to all points in the region.

Second, after selecting an action and updating the estimates for the selected regions, the algorithm additionally decides whether to update the partition. This is done via:

**3 Re-partition the space**: Let $R$ denote any selected ball and $r(R)$ denote its radius. We split when $r(R) \geq 2\epsilon$ and $n_t(R, \beta) \geq (d_{\text{max}}/r(R))^2$. We then cover $R$ with new regions $R_1, \ldots, R_n$ which form an $\frac{1}{2}r(R)$-Net of $R$. We call $R$ the parent of these new balls and each child ball inherits all values from its parent. We then add the new balls $R_1, \ldots, R_n$ to $\mathcal{P}_t^\beta$ to form the partition for the next timestep $\mathcal{P}_{t+1}^\beta$.

**Information relation** For the information relation, we define:

$$((p_1, \beta_1), \ldots, (p_N, \beta_N)) \equiv_h ((p'_1, \beta'_1), \ldots, (p'_N, \beta'_N))$$

if there exists an index $j$ with $\beta_j = \beta'_j$ and $p_j$ and $p'_j$ both belong to the same region in the adaptive discretization dictated by $H$. Note that this combines the continuous and semi-bandit relations described in Section 3.3.

**Benefits of Adaptive Discretization** The fixed discretization algorithm cannot adapt to the underlying structure in the problem since the discretization is fixed prior to learning. This causes increased computational, storage, and sample complexity since each individual region must be explored in order to obtain optimal regret guarantees. Instead, our adaptive discretization algorithm adapts the discretization in a data-driven manner to reduce unnecessary exploration. The algorithms keep a fine discretization across important parts of the space, and a coarser discretization across unimportant regions. See Fig. 4 for a sample adaptive discretization observed, and Kleinberg et al. [2010], Sinclair et al. [2023] for more discussion on the benefits of adaptive discretization over fixed.
**Implementation of Adaptive Discretization**  In the attached code base we provide an efficient implementation of AdaMonUCB, the adaptive discretization algorithm for the CMAB-CRA domain. Pseudocode for AdaMonUCB is in Algorithm 4.

We represent the partition $P^\beta_t$ as a tree with leaf nodes corresponding to active balls (i.e. ones which have not yet been split). Each node in the tree keeps track of $n_t(R, \beta)$, $\mu_t(R, \beta)$, and $\text{UCB}_t(R, \beta)$. While the partitioning works for any compact metric space, we implement it in $S = [0,1]^2$ under the infinity norm metric. With this metric, the high level implementation of the three steps is as follows:

- **Selection rule:** In this step we start by “merging” all of the trees for each allocation level $\beta$ (i.e. $P^\beta_t$ for $\beta \in B$) into a single tree, with estimates $\text{UCB}_t(R, \cdot)$ represented as a vector for each $\beta \in B$ rather than a scalar. On this merged partition of $S$ we solve Eq. (11) over the leaves to avoid modelling the constraint which ensures that each region is selected at a single allocation amount.

- **Update estimates:** Updating the estimates simply updates the stored $n_t(R, \beta), \mu_t(R, \beta)$, and $\text{UCB}_t(R, \beta)$ for each of the selected regions based on observed data.

- **Re-partition the space:** In order to split a region $R$, we create four new subregions corresponding to splitting the two dimensions in half. For example, the region $[0,1]^2$ will be decomposed to

$$[0, \frac{1}{2}] \times [0, \frac{1}{2}] \quad [0, \frac{1}{2}] \times [\frac{1}{2}, 1] \quad [\frac{1}{2}, \frac{1}{2}] \times [0, \frac{1}{2}] \quad [\frac{1}{2}, \frac{1}{2}] \times [\frac{1}{2}, 1].$$

We add on the children to the tree with links to its parent node, and initialize all estimates to that of its parent.

Lastly, we comment that in order to implement the Full Start and Artificial Replay versions of the adaptive discretization algorithm we pre-processed the entire historical data into a tree structure. This allowed us to check whether the given action has historical data available by simply checking the corresponding node in the pre-processed historical data tree.

**E  Experiment Details**

We provide additional details about the experimental domains, algorithm implementation, and additional results. The additional results include experiments to evaluate the performance of Artificial Replay on algorithms for combinatorial finite-armed bandit [Chen et al., 2013] as well as the standard $K$-armed bandit. For the later we include simulations with Thompson sampling [Russo et al., 2018], and information-directed sampling [Russo and Van Roy, 2018], which do not satisfy the IIData property, but still experience empirical improvements when using Artificial Replay against Full Start.

**E.1 Domain Details**

**E.1.1 Finite $K$-armed bandit**

For the $K$-armed bandit, we generate mean rewards for each arm $a \in [K]$ uniformly at random.
E.1.2 CMAB-CRA

**Piecewise-linear** This synthetic domain has a piecewise-linear reward function to ensure that the greedy approximation solution is optimal, as discussed in Section 4.2. As a stylized setting, this reward function uses a very simple construction:

\[
\mu(p, \beta) = \beta \cdot \left( \frac{p_1}{2} + \frac{p_2}{2} \right).
\]  

(12)

We visualize this reward in Fig. 6. The optimal reward is at (1, 1).

**Quadratic** The quadratic environment is a synthetic domain with well-behaved polynomial reward function of the form:

\[
\mu(p, \beta) = \beta\left(1 - (p_1 - 0.5)^2 + (p_2 - 0.5)^2\right)
\]

which we visualize in Fig. 5. The optimal reward at is achieved at (0.5, 0.5).

**Green Security Domain** For the green security domain, we wish to predict the probability that poachers place snares throughout a large protected area using ranger patrol observations. We use real-world historical patrol data from Murchison Falls National Park. The historical data are continuous-valued GPS coordinates (longitude, latitude) marking trajectories with locations automatically recorded every 30 minutes. Between the years 2015 and 2017, we have 180,677 unique GPS waypoints.

We normalize the space of the park boundary to the range [0, 1] for both dimensions. For each point \( p \in [0, 1]^2 \) in this historical data, we compute “effort” or allocation by calculating straight-line trajectories between the individual waypoints to compute the distance patrolled, allocating to each waypoint one half the sum of the line segments to which it is connected. We then associate with each point a binary label \{0, 1\} representing the observation. Direct observations of poaching are rather rare, so to overcome strong class imbalance for the purposes of these experiments, we augment the set of instances we consider a positive label to include any human-related or wildlife observation. Everything else (e.g., position waypoint) gets a negative label.

To generate a continuous-action reward function, we build a neural network to learn the reward function across the park and use that as a simulator for reward. The neural network takes three inputs, \( a = (p_1, p_2, \beta) \), and outputs a value \([0, 1]\) to indicate probability of an observation. This probability represents \( \mu(a) \) for the given point and allocation.
E.2 Adaptive Discretization

We offer a visual demonstration of the adaptive discretization process in Fig. 7, using 10,000 real samples of historical patrol observations from Murchison Falls National Park. This discretization is used to iteratively build the dataset tree used to initialize the Full Start algorithm with adaptive discretization.

E.3 Additional Experimental Results

In Fig. 8 we evaluate the performance of Artificial Replay compared to Full Start and Ignorant using two multi-armed bandit algorithms that do not have the IIData property, Thompson Sampling (TS) and Information-Directed Sampling (IDS). Although our theoretical regret guarantees do not apply to Thompson sampling or IDS as base algorithms, these results demonstrate that empirically Artificial Replay still performs remarkably well, matching the performance of Full Start with Thompson sampling and avoiding the exploding regret that Full Start suffers with IDS.

We note that with imbalanced data, Full Start is converging on a suboptimal action with more historical data. This is because the IDS algorithm, when warm-started with historical data, maintains a near-‘zero entropy’ posterior distribution over a sub-optimal action which is over-represented in the historical dataset. Since the selection procedure takes the action that maximizes expected return divided by posterior variance, the algorithm continuously picks this sub-optimal action at each timestep.

In Fig. 9 we consider the combinatorial bandit setting with a set of $K = 10$ discrete arms and a budget $B = 3$ over $T = 1,000$ timesteps. This setting is similar to Fig. 1 but instead here we consider a combinatorial setting (where multiple arms can be pulled at each timestep) rather than a standard stochastic $K$-armed bandit. Across different values of $H$, Artificial Replay matches the performance of Full Start (Fig. 9(a)) despite using an increasing smaller fraction of the historical dataset $H_{\text{hist}}$ (Fig. 9(b)). The regret plot in Fig. 9(c) shows that the regret of our method is coupled with that of Full Start across time. Fig. 9(d) tracks the number of samples from history $H_{\text{hist}}$, with $H = 1,000$, used over time: Artificial Replay uses 471 historical samples before taking its first online action. The number of historical samples used increases at a decreasing rate and ends with using 740 samples.

E.4 Experiment Execution

Each experiment was run with 60 iterations where the relevant plots are taking the mean of the related quantities. All randomness is dictated by a seed set at the start of each simulation for verifying results. The experiments were conducted on a personal laptop with a 2.4 GHz Quad-Core Intel Core i5 processor and 16 GB of RAM.

F Omitted Proofs

F.1 Section 5.1 Proofs

Proof of Theorem 5.2. We start off by showing that $\pi_{\text{Artificial Replay}(\Pi)} \equiv \pi_{\text{Full Start}(\Pi)}$ using the reward stack model for a stochastic bandit instance introduced in Lattimore and Szepesvári [2020]. Due to the fact that the observed rewards are independent (both across actions but also across timesteps), consider a sample path where $(R_{a,t})_{a \in A, t \in [T]}$ are pre-sampled according to $\mathcal{R}(a)$. Upon pulling arm $a$ in timestep $t$, the algorithm is given feedback $R_{a,t}$.
Figure 7: Adaptive discretization in the $\mathcal{S} = [0, 1]^2$ space using 10,000 samples of real historical patrol observations from Murchison Falls National Park. Each row depicts the distribution of historical samples and the space partition after 500, 1,500, 6,000, and 10,000 samples are added to the dataset tree. Each column visualizes the dataset tree for each of three levels of effort $\beta \in \mathcal{B} = \{0, 0.5, 1\}$. As shown, these real-world historical samples exhibit strong imbalanced data coverage, leading to significantly fine discretizations in areas with many samples and very coarse discretization in other regions.
Figure 8: Cumulative regret (y-axis; lower is better) across time $t \in [T]$. Artificial Replay performs competitively across all domain settings, with both Thompson sampling [Russo et al., 2018] (left) and information-directed sampling [Russo and Van Roy, 2018] (right). In Full Start applied to information-directed sampling with $H = 1,000$ the algorithm converges on a sub-optimal arm since its posterior variance is low (due to more data), resulting in poor regret performance due to spurious data.
Figure 9: We consider a combinatorial bandit setting with finite actions: $K = 10$ arms, $B = 3$ budget, and horizon $T = 1,000$. Increasing the number of historical samples $H$ leads Full Start to use unnecessary data, particularly as $H$ gets very large. Artificial Replay achieves equal performance in terms of regret (plot a) while using less than half the historical data (plot b). In (plot c) we see that with $H = 1,000$ historical samples, Artificial Replay uses 471 historical samples before taking its first online action. The number of historical samples used increases at a decreasing rate, using 740 total samples by the horizon $T$.

It is important to note that the resulting probability space generated in the reward stack model is identical in distribution to any sequence of histories observed by running a particular algorithm. More specifically, it preserves the following two properties:

(a) The conditional distribution of the action $A_t$ given the sequence $(A_1, R_{A_1}, 1), \ldots, (A_{t-1}, R_{A_{t-1}}, t-1)$ is $\pi_t(\cdot \mid H_t)$ almost surely.

(b) The conditional distribution of the reward $R_t$ is $\mathcal{R}(A_t)$ almost surely.

Based on this reward stack model, we show by induction on $t$ that $\pi_t^{\text{Artificial Replay}(\Pi)} = \pi_t^{\text{Full Start}(\Pi)}$. Since this is true on an independent sample path, it results in a probabilistic coupling between the two algorithms, implying that the chosen online actions and collected rewards have the same distribution.

**Base Case:** $t = 1$.

By definition of $\pi^{\text{Full Start}(\Pi)}$, we know that

$$\pi_1^{\text{Full Start}(\Pi)} = \Pi(H^{\text{hist}}).$$

However, consider $\pi^{\text{Artificial Replay}(\Pi)}$. The Artificial Replay meta-algorithm will keep selecting actions until it creates a dataset $H_1^{\text{used}} \subset H^{\text{hist}}$ such that $\Pi(H_1^{\text{used}})$ has no more unused samples in $H^{\text{hist}}$, i.e.
\( \mathcal{H}_{\text{hist}} \setminus \mathcal{H}_{\text{used}} \cap \{ a \sim \mathcal{H}_{\text{used}} \} = \emptyset \). Denoting \( A_1 = \Pi(\mathcal{H}_{\text{used}}) \), the unused samples \( \mathcal{H}_{\text{hist}} \setminus \mathcal{H}_{\text{used}} \) contains no data on \( A_1 \). As a result, by the independence of irrelevant data property for \( \Pi \) we have that \( \Pi(\mathcal{H}_{\text{used}}) = \Pi(\mathcal{H}_{\text{hist}}) \) and so \( \pi_1^{\text{Full Start}(II)} = \pi_1^{\text{Artificial Replay}(II)} \). Note that this shows that the observed online data for the algorithms as denoted by \( H_2 = \{ A_1, R_{A_1,1} \} \) are also identical (due to the reward stack model).

**Step Case**: \( t - 1 \to t \)

Since we know that \( \pi_\tau^{\text{Full Start}(II)} = \pi_\tau^{\text{Artificial Replay}(II)} \) for \( \tau < t \), both algorithms have access to the same set of observed online data \( \mathcal{H}_t \). By definition of \( \pi_\tau^{\text{Full Start}(II)} \):

\[
\pi_t^{\text{Full Start}(II)} = \Pi(\mathcal{H}_{\text{hist}} \cup \mathcal{H}_t).
\]

However, the **Artificial Replay** algorithm continues to use offline samples until it generates a subset \( \mathcal{H}_t^{\text{used}} \subset \mathcal{H}_{\text{hist}} \) such that \( \Pi(\mathcal{H}_t^{\text{used}} \cup \mathcal{H}_t) \) has no further samples in \( \mathcal{H}_{\text{hist}} \), i.e.

\[
(\mathcal{H}_{\text{hist}} \setminus \mathcal{H}_t^{\text{used}}) \cap \{ a \in \mathcal{A} \mid a \sim \mathcal{H}_t^{\text{used}} \cup \mathcal{H}_t, \Pi(\mathcal{H}_t^{\text{used}} \cup \mathcal{H}_t) \} = \emptyset.
\]

Hence, by the independence of irrelevant data property again:

\[
\Pi(\mathcal{H}_{\text{hist}} \cup \mathcal{H}_t) = \Pi(\mathcal{H}_t^{\text{used}} \cup \mathcal{H}_t),
\]

and so \( \pi_t^{\text{Full Start}(II)} = \pi_t^{\text{Artificial Replay}(II)} \). Again we additionally have that \( \mathcal{H}_{t+1} = \mathcal{H}_t \cup \{ (A_t, R_{A_t,t}) \} \) are identical for both algorithms.

Together this shows that \( \pi^{\text{Full Start}(II)} \overset{d}{=} \pi^{\text{Artificial Replay}(II)} \). Lastly we note that the definition of regret is \( \text{REG}(T, \pi, \mathcal{H}_{\text{hist}}) = T \cdot \text{OPT} - \sum_{t=1}^{T} \mu(A_t) \) where \( A_t \) is sampled from \( \pi \). Hence the policy-based coupling implies that \( \text{REG}(T, \pi^{\text{Artificial Replay}(II)}, \mathcal{H}_{\text{hist}}) \overset{d}{=} \text{REG}(T, \pi^{\text{Full Start}(II)}, \mathcal{H}_{\text{hist}}) \) as well.

**Proof of Theorem 5.3.** By definition, we have that the regret of \( \pi^{\text{Artificial Replay}(II)} \) is given by:

\[
\text{REG}(T, \pi^{\text{Artificial Replay}(II)}) = \sum_{t=1}^{T} \text{OPT} - \mu(\Pi(\mathcal{H}_t \cup \mathcal{H}_t^{\text{used}})),
\]

where \( \mathcal{H}_t \) corresponds to the online data observed by the algorithm at the start of round \( t \), and \( \mathcal{H}_t^{\text{used}} \subset \mathcal{H}_{\text{hist}} \) the set of used data points by **Artificial Replay** by the time the final action is selected. However, within each round \( t \), **Artificial Replay** will propose a sequence of selected actions \( \tilde{A}_t \) where \( \tau \) indexes over the number of selected actions considered before finally picking an action which has no related samples in the historical dataset. Denote by \( \mathcal{H}_t^{\text{used}}(\tau) \subset \mathcal{H}_t^{\text{used}} \) as the set of historical data points considered by **Artificial Replay** upon selecting the proposed action \( \tilde{A}_t \). We note that \( \mathcal{H}_t^{\text{used}}(\tau) \) are nested, and that if \( I_t \) is the number of proposed actions selected by the algorithm in round \( t \), \( \mathcal{H}_t^{\text{used}}(I_t + 1) = \mathcal{H}_t^{\text{used}} \). Hence we have that:

\[
\text{REG}(T, \pi^{\text{Artificial Replay}(II)})
\]

\[
= \sum_{t=1}^{T} \text{OPT} - \mu(\Pi(\mathcal{H}_t \cup \mathcal{H}_t^{\text{used}}))
\]

\[
= \sum_{t=1}^{T} \sum_{\tau \in [I_t+1]} \text{OPT} - \mu(\Pi(\mathcal{H}_t \cup \mathcal{H}_t^{\text{used}}(\tau))) - \sum_{t=1}^{T} \sum_{\tau \in [I_t]} \text{OPT} - \mu(\Pi(\mathcal{H}_t \cup \mathcal{H}_t^{\text{used}}(\tau)))
\]

45
arg max that there is a unique action the UCB any information on dataset will have its constructed bound values.

definition of decomposition, which notes that optimality gap for any other action and let H

Proof of Theorem D.1.

F.2 Section 3.1, Section 5.2, Appendix D.1 Proofs for MonUCB

Proof of Theorem D.1. Suppose that the base algorithm II is MonUCB for any convex function ψ, and let H be an arbitrary dataset. Using the dataset, MonUCB will construct upper confidence bound values UCB(a) for each action a ∈ [K]. The resulting policy is to pick the action II(H) = arg maxa∈[K] UCB(a). Let A_H be the action which maximizes the UCB(a) value.

Additionally, let H’ be an arbitrary dataset containing observations from actions other than A_H. Based on the enforced monotonicity of the indices from MonUCB, any a ∈ [K] with a ≠ A_H will have its constructed UCB(a) no larger than its original one constructed with only using the dataset H. Moreover, UCB(A_H) will be unchanged since the additional data H’ does not contain any information on A_H. As a result, the policy will take II((H∪H’) = A_H since it will still maximize the UCB(a) index.

Next we provide a regret analysis for Ignorant(MonUCB). We assume without loss of generality that there is a unique action a∗ which maximizes μ(a). We let Δ(a) = μ(a∗) − μ(a) be the sub-optimality gap for any other action a. To show the regret bound we follow the standard regret decomposition, which notes that E[REG(T, π, Hhist)] = ∑a Δ(a)E[nT(a)]. To this end, we start off with the following Lemma, giving a bound on the expected number of pulls of MonUCB in the “ignorant” setting (i.e., without incorporating any historical data).

Lemma F.1. The expected number of pulls for any sub-optimal action a of MonUCB with convex function ψ satisfies

\[ \mathbb{E}[n_T(a)] \leq \frac{2\log(2TK)}{\psi^*(\Delta(a)/2)} + 1. \]

Proof of Lemma F.1. Denote by S_{a,τ} to be the empirical sum of τ samples from action a. Note that via an application of Markov’s inequality along with Assumption 3:

\[ \mathbb{P}\left( \left| \mu(a) - \frac{S_{a,τ}}{τ} \right| \geq (\psi^*)^{-1}\left( \frac{2\log(2TK)}{τ} \right) \right) \leq \frac{1}{2T^2K^2}. \]

A straightforward union bound with this fact shows that the following event occurs with probability at least 1 − 1/T:

\[ \mathcal{E} = \left\{ \forall a \in [K], 1 \leq k \leq T, |\mu(a) - S_{a,k}/k| \leq (\psi^*)^{-1}\left( \frac{2\log(2TK)}{τ} \right) \right\}. \]
Now consider an arbitrary action $a \neq a^*$. We start by showing that on the event $\mathcal{E}$ that $n_t(a) \leq 4 \log(T)/\Delta(a)^2$. If action $a$ was taken over $a^*$ at some timestep $t$ then:

$$UCB_t(a) > UCB_t(a^*) .$$

However, using the reward stack model and the definition of $UCB_t(a)$ we know that

$$UCB_t(a) = \min_{\tau \leq t} \frac{S_{a,\tau}(a)}{n_\tau(a)} + (\psi^*)^{-1} \left( \frac{2 \log(2TK)}{n_\tau(a)} \right)$$

by definition of monotone UCB

$$\leq \frac{S_{a,n_t}(a)}{n_\tau(a)} + (\psi^*)^{-1} \left( \frac{2 \log(2TK)}{n_\tau(a)} \right).$$

Moreover, under the good event $\mathcal{E}$ we know that $\mu(a^*) \leq UCB_t(a^*)$ and that

$$\mu(a) \geq \frac{S_{a,n_t}(a)}{n_t(a)} - (\psi^*)^{-1} \left( \frac{2 \log(2TK)}{n_\tau(a)} \right).$$

Combining this together gives

$$\mu(a^*) \leq \mu(a) + 2(\psi^*)^{-1} \left( \frac{2 \log(2TK)}{n_\tau(a)} \right).$$

Rearranging this inequality gives that $n_t(a) \leq \frac{2\log(2TK)}{\psi^*(\Delta(a)/2)}$. Lastly, the final bound comes from the law of total probability and the bound on $P(\mathcal{E})$. 

Lastly, Lemma F.1 can be used to obtain a bound on regret via:

$$\mathbb{E}[\text{REG}(T, \pi_{\text{MonUCB}}, H_{\text{hist}})] \leq O\left( \sum_a \frac{2\Delta(a) \log(2TK)}{\psi^*(\Delta(a)/2)} \right)$$

using the previous regret decomposition, recovering the regret bound of standard $\psi$ UCB.

In order to show Theorems D.2 and D.3 we start by showing a lemma similar to Lemma F.1 but for Full Start(MonUCB).

**Lemma F.2.** Let $H_a$ be the number of data points in $H_{\text{hist}}$ for an action $a \in [K]$. The expected number of pulls for any sub-optimal action $a$ of Full Start (MonUCB) satisfies

$$\mathbb{E}[n_T(a) \mid H_a] \leq 1 + \max \left\{ 0, \frac{2 \log(2TK)}{\psi^*(\Delta(a)/2)} - H_a \right\} .$$

**Proof of Lemma F.2.** We replicate the proof of Lemma F.1 but additionally add $H_a$ samples to the estimates of each action $a \in [K]$. Denote by $S_{a,\tau}$ the empirical sum of $\tau + H_a$ samples from action $a$. Note that via an application of Hoeffding’s inequality:

$$\mathbb{P}\left( \left| \mu(a) - \frac{S_{a,\tau}}{\tau + H_a} \right| \geq (\psi^*)^{-1} \left( \frac{2 \log(2TK)}{\tau + H_a} \right) \right) \leq \frac{1}{2T^2K^2} .$$

A straightforward union bound with this fact shows that the following event occurs with probability at least $1 - 1/T$:

$$\mathcal{E} = \left\{ \forall a \in [K], 1 \leq k \leq T, |\mu(a) - S_{a,k}/(k + H_a)| \leq (\psi^*)^{-1} \left( \frac{2 \log(2TK)}{\tau + H_a} \right) \right\} .$$
Now consider an arbitrary action \( a \neq a^* \). If action \( a \) was taken over \( a^* \) at some timestep \( t \) then:

\[
\text{UCB}_t(a) > \text{UCB}_t(a^*).
\]

By definition of the \( \text{UCB}_t(a) \) term we know

\[
\text{UCB}_t(a) = \min_{\tau \leq t} \frac{S_{a,n_t(a)}(\tau)}{n_t(a) + H_a} + (\psi^*)^{-1}\left(\frac{2 \log(2TK)}{n_t(a) + H_a}\right) \quad \text{(by definition of monotone UCB)}
\]

\[
\leq \frac{S_{a,n_t(a)}(\tau)}{n_t(a) + H_a} + (\psi^*)^{-1}\left(\frac{2 \log(2TK)}{n_t(a) + H_a}\right).
\]

Moreover, under the good event \( \mathcal{E} \) we know that \( \mu(a^*) \leq \text{UCB}_t(a^*) \) and that

\[
\mu(a) \geq \frac{S_{a,n_t(a)}(\tau)}{n_t(a) + H_a} - (\psi^*)^{-1}\left(\frac{2 \log(2TK)}{n_t(a) + H_a}\right).
\]

Combining this together gives

\[
\mu(a^*) \leq \mu(a) + 2(\psi^*)^{-1}\left(\frac{2 \log(2TK)}{n_t(a) + H_a}\right).
\]

Rearranging this inequality gives that \( n_t(a) \leq \frac{2 \log(2TK)}{\psi^*(\Delta(a)/2)} - H_a \). Lastly, the final bound comes from the law of total probability and the bound on \( \mathbb{P}(\mathcal{E}) \).

Using the previous two lemmas we are able to show Theorems D.2 and D.3.

**Proof of Theorem D.2.** Without loss of generality we consider a setting with \( K = 2 \). Let \( a^* \) denote the optimal arm and \( a \) the other arm. Consider the historical dataset as follows: \( \mathcal{H}^{\text{hist}} = (a, R_j^H)_{j \in [H]} \) where each \( R_j^H \sim \mathcal{R}(a) \). By definition of \( \text{Full Start(MonUCB)} \) we know that the time complexity of the algorithm is at least \( T + H \) since the algorithm will process the entire historical dataset.

In contrast, \( \text{Artificial Replay(MonUCB)} \) will stop playing action \( a \) after \( O(\log(T)/\psi^*(\Delta(a))) \) timesteps via Lemma F.1. Hence the time complexity of \( \text{Artificial Replay(MonUCB)} \) can be upper bound by \( O(T + \log(T)) \).

**Proof of Theorem D.3.** First via Theorem 5.2 we note that in order to analyze \( \pi^{\text{Artificial Replay(MonUCB)}} \) it suffices to consider \( \pi^{\text{Full Start(MonUCB)}} \). Using Lemma F.2 and a standard regret decomposition we get that:

\[
\mathbb{E}\left[\text{REG}(T, \pi^{\text{Full Start(MonUCB)}}, \mathcal{H}^{\text{hist}}) \mid H_a\right] = \sum_{a \neq a^*} \Delta(a)\mathbb{E}[n_T(a) \mid H_a] \quad \text{(by regret decomposition)}
\]

\[
\leq \sum_{a \neq a^*} \Delta(a)O\left(\max\left\{0, \frac{\log(TK)}{\psi^*(\Delta(a))} - H_a\right\}\right) \quad \text{(by Lemma F.2)}
\]

\[
= \sum_{a \neq a^*} O\left(\max\left\{0, \frac{\Delta(a)\log(TK)}{\psi^*(\Delta(a))} - H_a\Delta(a)\right\}\right).
\]
F.3 Appendix D.2 Proofs

Proof of Theorem 4.1. We first consider the fixed discretization algorithm. Denote by $\bar{\mu}(H, R)$ as the average reward for samples accumulated in a region $R$ over dataset $H$, and $n(H, R)$ the number of times points in $R$ have been selected based on dataset $H$. Lastly we let $\text{UCB}(H, R)$ be the monotone UCB value for a region $R$ based on dataset $H$ (which is ensured to be monotone by taking the minimum value). By definition of the fixed discretization algorithm, we have that

$$II(H) = \arg\max_{R \in P_1} \text{UCB}(H, R).$$

Let $R^*$ denote the region that maximizes the UCB value and $a^* \in R^*$ arbitrary. To check for the IIData property, let $H'$ be any other dataset containing samples for actions $a \in A$ such that $d(a, a^*) > \gamma$. Then by construction of the discretization, it must follow that $a$ is contained in regions other than $R^*$. Thus for regions $R$ other than $R^*$ we have $\text{UCB}(H \cup H', R) \leq \text{UCB}(H, R)$, and $\text{UCB}(H \cup H', R^*) = \text{UCB}(H, R^*)$. Thus, $R^*$ remains as the region which maximizes $\text{UCB}(H \cup H', R)$.

The proof for the adaptive discretization follows similarly. Let $H'$ be any other dataset containing samples for actions $a$ in regions other than $R$ based on the current partition $P(H)$ (note that while in the algorithm description we indexed the partition by $t$, it is solely a function of the observed data thus far). We emphasize that $P(H \cup H') \subset P(H)$. Moreover, $R^* \in P(H \cup H')$ since $H'$ contains data for actions other than those contained in $R^*$. Hence we have that:

$$\text{UCB}(H \cup H', R) \leq \text{UCB}(H \cup H', R')$$

for any $R \in P(H \cup H')$ with $R \neq R^*$ since the UCB values are enforced to be monotone decreasing, and child regions inherit the values from their parents. Moreover, $\text{UCB}(H \cup H', R^*) = \text{UCB}(H, R^*)$. Thus, $R^*$ again remains as the region which maximizes $\text{UCB}(H \cup H', R)$ over all $R \in P(H \cup H')$.

Here we see the necessity that the information relation is history-dependent, since the discretization changes as a function of the observed data.

F.4 Appendix D.3 Proofs

Proof of Theorem 4.2. We first consider the fixed discretization algorithm. Using the dataset, the algorithm constructs UCB values $\text{UCB}(R, \beta)$ for each $R$ and $\beta \in B$ in the discretization $P_1$ of $S$. The resulting policy is to pick the action which solves an optimization problem of the form:

$$\max_z \sum_{\beta \in B} \sum_{R \in P_1^d} \text{UCB}(R, \beta) z(R, \beta)$$

subject to

$$\sum_{\beta \in B} \sum_{R \in P_1^d} \beta z(R, \beta) \leq B$$

$$\sum_{\beta \in B} \sum_{R \in P_1^d} z(R, \beta) \leq N$$

$$z(R, \beta) + \sum_{\tilde{R} \notin \tilde{R}} \sum_{R \subset \tilde{R}} z(\tilde{R}, \beta') \leq 1 \ \forall \beta, R \in P_1^d$$

where $\text{UCB}(R, \beta)$ is constructed based on $H$. Let $A_H = ((R_1, \beta_1), \ldots, (R_N, \beta_N))$ be the resulting combinatorial action from solving the optimization problem. Note here again we abuse notation slightly and assume the algorithm plays a fixed point $p \in R$ for each region $R_i$. By definition of the information relation, let $H'$ be an arbitrary dataset containing observations of the form $(p, \beta)$ for
any \((p, \beta)\) such that for each \(i\) either (i) \(p \notin \mathcal{R}_i\) or (ii) \(\beta \neq \beta_i\). After updating the UCB estimates with \(\mathcal{H}'\) we have that \(\text{UCB}(\mathcal{R}, \beta)\) decreases for any \(\mathcal{R}, \beta\) with \(\mathcal{R} \neq \mathcal{R}_i\) or \(\beta \neq \beta_i\). Hence, since the algorithm is solving the expression using a greedy selection rule, we know the resulting action is the same since the selected regions UCB estimates were unchanged and the rest decreased.

The proof for the adaptive discretization algorithm follows similarly using the idea from the proof of Theorem 4.1 in Appendix F.3, where we again exploit the fact that Eq. (14) is solved greedily.