Global existence and blow-up of solutions to some quasilinear wave equation in one space dimension

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Abstract
We consider the global existence and blow up of solutions of the Cauchy problem of the quasilinear wave equation:
\[
\partial_t^2 u = \partial_x (c(u)^2 \partial_x u),
\]
which has richly physical backgrounds. Under the assumption that \(c(u(0,x)) \geq \delta\) for some \(\delta > 0\), we give sufficient conditions for the existence of global smooth solutions and the occurrence of two types of blow-up respectively. One of the two types is that \(L^\infty\)-norm of \(\partial_t u\) or \(\partial_x u\) goes up to the infinity. The other type is that \(c(u)\) vanishes, that is, the equation degenerates.

1 Introduction
In this paper, we consider the Cauchy problem of the following wave equation:
\[
\begin{align*}
\partial_t^2 u &= \partial_x (c(u)^2 \partial_x u), & (t, x) \in (0, T] \times \mathbb{R}, \\
u(0, x) &= u_0(x), & x \in \mathbb{R}, \\
\partial_t u(0, x) &= u_1(x), & x \in \mathbb{R},
\end{align*}
\]
(1.1)
where \(u(t, x)\) is an unknown real valued function. The equation in (1.1) has some physical backgrounds including vibrations of a string.
We assume that \(c \in C^\infty((\theta_0, \infty))\) for some \(\theta_0 \in (-\infty, 0)\) satisfies that
\[
\begin{align*}
\lim_{\theta \searrow \theta_0} c(\theta) &= 0, \\
c(\theta) &> 0 \quad \text{for all } \theta > \theta_0, \\
c'(\theta) &\geq 0 \quad \text{for } \theta > \theta_0.
\end{align*}
\]
(1.2)  (1.3)  (1.4)
We denote Sobolev space \((1 - \partial_x^2)^{-\frac{s}{2}} L^2(\mathbb{R})\) for \(s \in \mathbb{R}\) by \(H^s(\mathbb{R})\). For a Banach space \(X\), \(C^j([0, T]; X)\) denotes the set of functions \(f : [0, T] \to X\) such that \(f(t)\) and its \(k\) times derivatives for \(k = 1, 2, \ldots, j\) are continuous. \(L^\infty([0, T]; X)\) denotes the set of functions \(f : [0, T] \to X\) such that the norm \(\|f\|_{L^\infty([0, T]; X)} := \text{ess. sup}_{[0, T]} \|f(t)\|_X\) is finite. Various positive constants are simply denoted by \(C\).

By dividing the both side of (1.1) by \(c(u(t, x))^2\), (1.1) is formed to
\[
\frac{1}{c(u(t, x))^2} \partial_t^2 u(t, x) - \partial_x^2 u(t, x) = \frac{2c'(u(t, x)) (\partial_x u(t, x))^2}{c(u(t, x))}.
\]
(1.5)
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Let \( T_0 \) be the blow-up time of the solution \( u \) of the Cauchy problem (1.1) by \( T^* \), that is,
\[
T^* := \sup \{ T > 0 \mid \sup_{[0,T]} \{ \| \partial_t u(t) \|_{L^\infty} + \| \partial_x u(t) \|_{L^\infty} \} < \infty, \inf_{[0,T] \times \mathbb{R}} u(t,x) > \theta_0 \}.
\]

**Theorem 1.** Let \( c(\cdot) \in C^\infty((\theta_0, \infty)) \) and initial data \((u_0, u_1) \in H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R})\) for \( s > \frac{1}{2} \). Suppose \( c(\cdot) \) and \((u_0, u_1) \) satisfy (1.2), (1.3), (1.4) and
\begin{align*}
(1.8) & \quad u_0(x) > \theta_0 \text{ for } x \in \mathbb{R}, \\
(1.9) & \quad u_1(x) \pm c(u_0(x)) \partial_x u_0(x) \leq 0 \text{ for } x \in \mathbb{R}, \\
(1.10) & \quad -\int_{\mathbb{R}} u_1(x) dx < \int_{\theta_0}^\infty c(\theta) d\theta.
\end{align*}

Then (1.1) has a unique global solution such that \( u \in \bigcap_{j=0,1,2} C^j([0, \infty); H^{s-j+1}(\mathbb{R})) \).

**Theorem 2.** Let \( \theta_0 \neq -\infty \). Under the same assumption as in Theorem 1 without (1.10), we assume that
\begin{align*}
(1.11) & \quad \text{supp } u_0, \text{ supp } u_1 \subset [-K,K] \text{ for some } K > 0, \\
(1.12) & \quad -\int_{\mathbb{R}} u_1(x) dx > -2\theta_0 c(0).
\end{align*}

Then \( T^* < \infty \) and the solution \( u \in \bigcap_{j=0,1,2} C^j([0, T^*); H^{s-j+1}(\mathbb{R})) \) of (1.1) satisfies that
\[
\lim_{t \uparrow T^*} u(t, x_0) = \theta_0 \text{ for some } x_0 \in \mathbb{R}.
\]

**Theorem 3.** Let \( c \in C^\infty((\theta_0, \infty)) \) and initial data \((u_0, u_1) \in H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R}) \setminus \{0\}\) for \( s > \frac{1}{2} \). Suppose \( c(\cdot) \) and \((u_0, u_1) \) satisfy that there exists a constant \( \delta > 0 \) such that
\begin{align*}
(1.13) & \quad u_0(x) > \theta_0 \text{ for } x \in \mathbb{R}, \\
(1.14) & \quad c'(\theta) > 0 \text{ for all } \theta > \theta_0, \\
(1.15) & \quad \text{supp } u_0, \text{ supp } u_1 \subset [-K,K] \text{ for some } K > 0, \\
(1.16) & \quad u_1(x) \pm c(u_0(x)) \partial_x u_0(x) \geq 0 \text{ for } x \in \mathbb{R}.
\end{align*}

Then \( T^* < \infty \) and the solution \( u \in \bigcap_{j=0,1,2} C^j([0, T^*); H^{s-j+1}(\mathbb{R})) \) of (1.1) satisfies
\[
\lim_{t \uparrow T^*} \| \partial_t u(t) \|_{L^\infty} + \| \partial_x u(t) \|_{L^\infty} = \infty.
\]
Remark 4. Let $c(\cdot) \in C^\infty(\mathbb{R})$ and initial data $(u_0, u_1) \in H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R}) \setminus \{0\}$ for $s > \frac{1}{2}$. Suppose that there exists a constant $c_1 > 0$ such that $c(\theta) \geq c_1$ for all $\theta \in \mathbb{R}$ instead of the assumption (1.2) and (1.3). If (1.4) and (1.9) hold, then (1.1) has a unique global solution such that $u \in \bigcap_{j=0,1,2} C^j([0, \infty); H^{s-j+1}(\mathbb{R}))$.

Remark 5. In Theorem 1, if $\theta = -\infty$, then we do not need the assumption (1.10).

Remark 6. The equation in (1.1) does not degenerate for the global solution which is constructed by Theorem 1 that is, the global solution $u$ in Theorem 1 satisfies that there exists a constant $\theta_1 > \theta_0$ such that

$$u(t, x) \geq \theta_1,$$

for $(t, x) \in [0, \infty) \times \mathbb{R}$.

The equation in (1.1) has richly physical backgrounds (e.g. the flow of a one dimensional gas, the shallow water waves, the longitudinal wave propagation on a moving threadline, the dynamics of a finite nonlinear string, the elastic-plastic materials or the electromagnetic transmission line). In [2], Ames, Lohner and Adams study the group properties of the equation in (1.1) by using the Lie algebra and introduce physical backgrounds. In [20], Zabusky introduce the equation

$$\partial_t^2 v = (1 + \partial_x v)^a \partial_x^2 v,$$

which describes the standing vibrations of a finite, continuous and nonlinear string for $a > 0$. Setting $u = \partial_x v$ for the solution $v$ to (1.17), $u$ is a solution to the equation:

$$\partial_t^2 u = \partial_x ((1 + u)^a \partial_x u).$$

In author’s previous work [11], the author show a global existence theorem for (1.1) under some conditions on the function $c$ and initial data. However, we can not apply the global existence theorem of [11] to (1.18) since the theorem requires the condition that there exists a constant $c_0 > 0$ such that

$$c(\theta) \geq c_0 \text{ for all } \theta \in \mathbb{R}.$$

Our global existence theorem (Theorem 1) can yield a global solvability for some equations including (1.18).

Many authors [5, 6, 18, 19, 8, 9, 11] study the Cauchy problem of the equation

$$\partial_t^2 u = c(u)^2 \partial_x^2 u + \lambda c(u)c'(u)(\partial_x u)^2,$$

for $0 \leq \lambda \leq 2$. (1.20) with $\lambda = 2$ is the equation in (1.1).

Kato and Sugiyama [9] and Sugiyama [9] show that the same theorem as Theorem 2 holds for (1.20) for $0 \leq \lambda < 2$ without the restriction $\int_{\mathbb{R}} u_1(x) \, dx$ (the assumption (1.12)).

The equation in (1.1) is related to equations

$$\partial_t v = \pm c(v) \partial_x v \text{ and } \partial_x^2 v = c(\partial_x v) \partial_x^2 v.$$
equation in (1.1). Lax [10] and John [3] study the blow up for the first and the second equations of (1.21) respectively. In [16], MacCamy and Mizel study the Dirichlet problem for the second equation in (1.21).

The blow up of the 2 and 3 dimensional versions of the equation in (1.1):

\[ \partial^2_t u = \text{div}(c(u)^2 \nabla u), \]

is studied by Li, Witt and Yin [14] and Ding and Yin [4] respectively.

We prove Theorem 1 by using Zhang and Zheng’s idea in [18] and an estimate which ensure that the equation does not degenerate. In [18], Zhang and Zheng show the global existence of solution to (1.20) with \( \lambda = 1 \) under some conditions on \( c \) and initial data including (1.19).

The proof of Theorem 2 is based on the method in [9, 11] which give a sufficient condition that the equation (1.20) for \( 0 \leq \lambda < 2 \) and \( c(u) = u + 1 \) degenerates in finite time.

In the proof of Theorem 3 we use the Riemann invariants and the method of characteristic.

This paper is organized as follows: In Section 2, we introduce the local existence and the uniqueness of solutions of (1.1). In Sections 3, 4 and 5, we show Theorems 1, 2 and 3 respectively.

**2 Local existence and uniqueness**

In this section, we introduce the local existence and the uniqueness of solutions of (1.1). The local well-posedness of some class of second order quasilinear strictly hyperbolic equations including the equation (1.1) is established by Hughes, Kato and Marsden [7]. Their proofs are based on the Energy method. Furthermore, by the Moser type inequality, the above local well-posedness results are sharpened (e.g. Majda [15] and Taylor [17]). Roughly speaking, the results in [15] and [17] state that the solution \( u \) of (1.1) persists as long as \( \| \partial_t u \|_{L^\infty} \) and \( \| \partial_x u \|_{L^\infty} \) are bounded.

The following theorem is obtained by applying Theorem 2.2 in [15] and Proposition 5.3.B in [17] to the Cauchy problem (1.1).

**Proposition 7.** Suppose that \( c(\theta) \) and \( (u_0, u_1) \in H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R}) \) for \( s > 1/2 \) and \( c \in C^\infty(\mathbb{R}) \) satisfy (1.8). Then there exist \( T > 0 \) and a unique solution \( u \) of (1.1) with

\[ u \in \bigcap_{j=0,1,2} C^j([0, \infty); H^{s-j+1}(\mathbb{R})) \]

and

(2.1) \[ u(t, x) > \theta_0 \quad \text{for} \quad (t, x) \in [0, T] \times \mathbb{R}. \]

Furthermore, if (1.1) does not have a global solution \( u \) satisfying (2.1) and (2.2), then the solution \( u \) satisfies

(2.3) \[ \lim_{t \to T} \| \partial_t u(t) \|_{L^\infty} + \| \partial_x u(t) \|_{L^\infty} = \infty. \]
or

\( (2.4) \quad \lim_{t \to T} \inf_{(s,y) \in [0,t] \times \mathbb{R}} u(s,y) = \theta_0, \)

for some \( T > 0.\)

### 3 Proof of Theorem [1]

We set the Riemann invariants \( R_1(t, x) \) and \( R_2(t, x) \) as follows

\[
R_1 = \partial_x u + c(u) \partial_x u, \\
R_2 = \partial_x u - c(u) \partial_x u.
\]

By (1.1), \( R_1 \) and \( R_2 \) are solutions to the system of the following first order equations

\[
\begin{align*}
\partial_t R_1 - c(u) \partial_x R_1 &= \frac{c'(u)}{2c(u)} (R_1^2 - R_2 R_1), \\
\partial_t u &= \frac{1}{2} (R_1 + R_2), \\
\partial_t R_2 + c(u) \partial_x R_2 &= \frac{c'(u)}{2c(u)} (R_2^2 - R_1 R_2).
\end{align*}
\]

For the proof of Theorem [1] we prove some lemma.

**Lemma 8.** Suppose that \( c(\theta) \in C^\infty([\theta_0, \infty)) \) and initial data \((u_0, u_1) \in H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R})\) with \( s > 1/2 \) satisfy (1.8) and that \( R_1 \) and \( R_2 \) are the functions in (3.1) for the solution \( u \) of (1.1) such that \( u \in \bigcap_{j=0,1,2} C^j([0, T^*); H^{s-j+1}(\mathbb{R})].\)

- If \( R_1(0, x) \geq 0 \) for all \( x \), then \( R_1(t, x) \geq 0 \) for all \( (t, x) \in [0, T^*) \times \mathbb{R}.\)
- If \( R_1(0, x) \leq 0 \) for all \( x \), then \( R_1(t, x) \leq 0 \) for all \( (t, x) \in [0, T^*) \times \mathbb{R}.\)
- If \( R_2(0, x) \geq 0 \) for all \( x \), then \( R_2(t, x) \geq 0 \) for all \( (t, x) \in [0, T^*) \times \mathbb{R}.\)
- If \( R_2(0, x) \leq 0 \) for all \( x \), then \( R_2(t, x) \leq 0 \) for all \( (t, x) \in [0, T^*) \times \mathbb{R}.\)

**Proof.** We show that \( R_1(t, \cdot) \geq 0 \) with \( R_1(0, 0) \geq 0 \) only.

For any point \((t_0, x_0) \in [0, T) \times \mathbb{R},\) let \( x_+(t) \) denote the plus and minus characteristic curves on the first and third equations of (3.2) through \((t_0, x_0)\) respectively as follows,

\[
(3.3) \quad \frac{dx_{\pm}(t)}{dt} = \pm u(t, x_{\pm}(t)), \quad x_{\pm}(t_0) = x_0.
\]

From (3.2), \( R_1(t, x_-(t)) \) is a solution to

\[
(3.4) \quad \frac{d}{dt} R_1(t, x_-(t)) = \frac{c'(u)}{2c(u)} (R_1(t, x_-(t))^2 - R_2(t, x_-(t)) R_1(t, x_-(t))).
\]

By the uniqueness of the differential equation (3.4), we have \( R_1(t, x_-(t)) = 0 \) for \( t \in [0, T^*) \) with \( R_1(0, x_-(0)) = 0,\) which implies that \( R_1(t, \cdot) \geq 0 \) with \( R_1(0, \cdot) \geq 0.\) \(\qed\)
Lemma 9. Let \( p \in [1, \infty) \). Suppose that \( c(\theta) \in C^\infty((\theta_0, \infty)) \) and initial data \((u_0, u_1) \in H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R}) \) with \( s > \frac{1}{2} \) satisfy (1.4), (1.8) and (1.9). Then we have

\[
\|R_1(t)\|_{L^p}^p + \|R_2(t)\|_{L^p}^p \leq \|R_1(0)\|_{L^p}^p + \|R_2(0)\|_{L^p}^p, \quad \text{for } t \in [0, T^*) ,
\]

where \( R_1 \) and \( R_2 \) are the functions in (3.1) for the solution \( u \) of (1.1) such that \( u \in \bigcap_{j=0,1,2} C^j([0, T^*]; H^{s-j+1}(\mathbb{R})) \).

Proof. The proof is almost the same as in the proof of Lemma 5 in Zhang and Zheng’s paper [18]. We give the proof of this lemma for reader’s convenience.

We denote \( \tilde{R}_1 := -R_1 \) and \( \tilde{R}_2 := -R_2 \). Lemma 3 implies that \( \tilde{R}_1(t) \geq 0 \) and \( \tilde{R}_2(t) \geq 0 \) for all \( t \). By the first equation of (3.2), we have

\[
\partial_t \tilde{R}_1 - c(u)\partial_x \tilde{R}_1 = -\frac{c'(u)}{2c(u)} (\tilde{R}_1^2 - \tilde{R}_2 \tilde{R}_1).
\]

Multiplying the both side of the above equation by \((\tilde{R}_1)^{p-1}\), we obtain

\[
\frac{1}{p} \{ \partial_t (\tilde{R}_1)^p - c \partial_x (\tilde{R}_1)^p \} = -\frac{c'}{2c} ((\tilde{R}_1)^{p+1} - \tilde{R}_2 (\tilde{R}_1)^p),
\]

By the third equation of (3.2), we have

\[
\frac{1}{p} c \partial_x (\tilde{R}_1)^p = \frac{1}{p} \partial_x (c (\tilde{R}_1)^p) + \frac{1}{p} c' (\tilde{R}_1 - \tilde{R}_2),
\]

from which, (3.6) yields that

\[
\frac{1}{p} \{ \partial_t (\tilde{R}_1)^p - c \partial_x (\tilde{R}_1)^p \} = -\left( \frac{1}{2} - \frac{1}{2p} \right) \frac{c'}{c} (\tilde{R}_1)^{p+1} + \frac{c'}{2c} \tilde{R}_2 (\tilde{R}_1)^p - \frac{c'}{2pc} \tilde{R}_1 (\tilde{R}_2)^p.
\]

By the similar computation as above, we have

\[
\frac{1}{p} \{ \partial_t (\tilde{R}_2)^p - c \partial_x (\tilde{R}_2)^p \} = -\left( \frac{1}{2} - \frac{1}{2p} \right) \frac{c'}{c} (\tilde{R}_2)^{p+1} + \frac{c'}{2c} \tilde{R}_1 (\tilde{R}_2)^p - \frac{c'}{2pc} \tilde{R}_2 (\tilde{R}_1)^p.
\]

By summing up (3.8) and (3.9) and integration over \( \mathbb{R} \), we have

\[
\frac{1}{p} \frac{d}{dt} \left( \int_{\mathbb{R}} (\tilde{R}_1)^p + (\tilde{R}_2)^p \right) = -\left( \frac{1}{2} - \frac{1}{2p} \right) \int_{\mathbb{R}} \frac{c'}{c} ((\tilde{R}_1)^{p+1} - \tilde{R}_1 (\tilde{R}_2)^p) dx + \left( \frac{1}{2} - \frac{1}{2p} \right) \int_{\mathbb{R}} \frac{c'}{c} ((\tilde{R}_2)^{p+1} - \tilde{R}_2 (\tilde{R}_1)^p) dx
\]

\[
= -\left( \frac{1}{2} - \frac{1}{2p} \right) \int_{\mathbb{R}} \frac{c'}{c} (\tilde{R}_1 - \tilde{R}_2)((\tilde{R}_1)^p - (\tilde{R}_2)^p) dx \leq 0.
\]

Therefore, integrating the both side of (3.10) over \([0, t]\), we have (3.5). \( \square \)
Lemma 10. Under the same assumption as in Lemma 9, we have

\[ ||R_1(t)||_{L^\infty} + ||R_2(t)||_{L^\infty} \leq 2(||R_1(0)||_{L^\infty} + ||R_2(0)||_{L^\infty}), \text{ for } t \in [0, T^*) \]  \hspace{1cm}(3.11)

Proof. Noting inequalities \( a^p + b^p \leq (a + b)^p \) and \( (a + b)^p \leq 2^p (a + b)^p \) for \( a, b \geq 0 \), by raising the both side of (3.5) to the \( \frac{1}{p} \) power, we have

\[ ||R_1(t)||_{L^p} + ||R_2(t)||_{L^p} \leq 2(||R_1(0)||_{L^p} + ||R_2(0)||_{L^p}). \]

From the fact that \( \lim_{p \to \infty} ||u||_{L^p} = ||u||_{L^\infty} \) with \( u \in H^s(\mathbb{R}) \) (e.g. Lemma 11 in [11]), we have (3.11). \( \square \)

Lemma 11. Suppose that \( c(\theta) \in C^\infty((\theta_0, \infty)) \) and initial data \( (u_0, u_1) \in H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R}) \) with \( s > \frac{1}{2} \) satisfy (1.4), (1.8), (1.9) and (1.10). Then there exists \( \theta_1 > \theta_0 \) such that

\[ u(t, x) \geq \theta_1, \text{ for } (t, x) \in [0, T^*) \times \mathbb{R}. \]  \hspace{1cm}(3.12)

where \( R_1 \) and \( R_2 \) are the functions which defined in (3.1) for the solution \( u \) of (1.1) such that \( u \in \bigcap_{j=0,1,2} C^j([0, T^*); H^{s-j+1}(\mathbb{R})) \).

Proof. From Lemma 8 we have

\[ \int_0^{u(t,x)} c(\theta) d\theta \leq -\partial_t u(t, x), \] \hspace{1cm}(3.13)

from which, a simple computation yields that

\[ \left| \int_0^{u(t,x)} c(\theta) d\theta \right| = \left| \int_{-\infty}^{x} c(u) \partial_x u(t, y) dy \right| \leq \int_{\mathbb{R}} |c(u) \partial_x u(t, y)| dy \leq -\int_{\mathbb{R}} \partial_t u(t, y) dy. \] \hspace{1cm}(3.14)

While, by the equation in (1.1), we have

\[ \frac{d}{dt} \int_{\mathbb{R}} \partial_t u(t, y) dy = 0. \] \hspace{1cm}(3.15)

By (1.10), (3.14) and (3.15) we have

\[ \left| \int_0^{u(t,x)} c(\theta) d\theta \right| \leq \int_{\mathbb{R}} u_1(x) dx < \int_{\theta_0}^{0} c(\theta) d\theta. \] \hspace{1cm}(3.16)

From (3.16), (1.2) and (1.3), we have (3.12). \( \square \)
Proof of Theorem 1

From Lemma 11, (2.4) does not occur.

The estimates (3.11) and (3.12) yield the uniform boundedness of \( \| \partial_x u \|_{L^\infty} \) and \( \| \partial_t u \|_{L^\infty} \) with \( t \in [0, T^*) \). So (2.3) does not occur.

Therefore, we complete the proof of Theorem 1.

\[\Box\]

Proof of Remark 5

Suppose \( T^* < \infty \).

By a simple computation, we have

\[
\| u(t) \|_{L^\infty} \leq \| u_0 \|_{L^\infty} + T^* \sup_{[0,T^*)} \{ \| \partial_t u(t) \|_{L^\infty} \}.
\]

By Lemma 11, we obtain the boundedness of \( \| u(t) \|_{L^\infty} \), \( \| \partial_t u(t) \|_{L^\infty} \) and \( \| \partial_x u(t) \|_{L^\infty} \) for \( t \in [0, T^*) \), which implies that the blow up (2.3) and (2.4) does not occur, which is contradiction to \( T^* < \infty \).

\[\Box\]

4 Proof of Theorem 2

First, we proof \( T^* < \infty \). For this purpose, we use the following lemma.

Lemma 12. Suppose that \( c(\theta) \in C^\infty((\theta_0, \infty)) \) and initial data \( (u_0, u_1) \in H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R}) \) with \( s > \frac{1}{2} \) satisfy (1.8) and (1.11). Then the solution \( u \in \bigcap_{j=0,1,2} C^j([0, T^*); H^{s-j+1}(\mathbb{R})) \) satisfies that

\[
\text{supp } u(t, x) \subset [-c(0)t - K, c(0)t + K],
\]

where \( K > 0 \) is a constant in (1.11).

Lemma 12 is proved in many text book (e.g. p. 16 in Sogge’s book [12]). Sogge Prove the same assertion as in Lemma 12 for the \( C^2 \) solution \( u \). By the standard approximation argument, Lemma 12 can be proved in the same way as in the proof in [12].

Set \( F(t) = -\int_{\mathbb{R}} u(t, x)dx \) for \( 0 \leq t < T^* \).

By the equation in (1.1), we have

\[
\frac{d^2 F}{dt^2}(t) = 0,
\]

which implies that

\[
F(t) = F(0) + t F'(0).
\]

By Lemma 8 and the fact that \( u(t, \cdot) > \theta_0 \) for \( t \in [0, T^*) \), we have

\[
F(t) = -\int_{-c(0)t-K}^{c(0)t+K} u(t, x)dx \leq -\int_{-c(0)t-K}^{c(0)t+K} \theta_0 dx = -2\theta_0(c(0)t + K).
\]

(4.2)
From (4.1) and (4.2), we obtain that
\[ F(0) + 2\theta_0 c(0) \geq t. \]
\[ \int_{\mathbb{R}} u_1(x)dx - 2\theta_0 c(0) \geq t. \]
\[ 2\theta_0 c(0) \geq t. \]
We note that the left hand side of the above inequality is finite by (1.12).
Since \( t \) can be chosen for all \([0, T^*]\), we have \( T^* \leq \frac{F(0) + 2\theta_0 c(0)}{c(0)} < \infty \).
Next, we show that
\[ \lim_{t \rightarrow T^*} \inf_{(s, y) \in [0, t) \times \mathbb{R}} u(s, y) = \theta_0. \]
Suppose that (4.3) does not occur. So there exists a constant \( \delta > 0 \) such that \( c(u(t, x)) \geq \delta \), for all \((t, x) \in [0, T^*) \times \mathbb{R}\).
By Lemma 11, we have the boundedness of \( \|\partial_t u(t)\|_{L^\infty} \) and \( \|\partial_x u(t)\|_{L^\infty} \) on \([0, T^*)\), which is contradiction to the fact that \( T^* < \infty \). Hence we have (4.3).
Finally, we show that
\[ \lim_{t \rightarrow T^*} u(t, x_0) = \theta_0 \quad \text{for some } x_0 \in \mathbb{R}. \]
Since \( u(t, x) \) is a monotone decreasing function of \( t \) for fixed \( x \), we have
\[ \lim_{t \rightarrow T^*} \inf_{(s, y) \in [0, t) \times \mathbb{R}} u(s, y) = \lim_{t \rightarrow T^*} \inf_{t \in t \rightarrow T^*} u(t, x) \]
\[ = \inf_{x \in \mathbb{R}} \lim_{t \rightarrow T^*} u(t, x). \]
The right hand side of (4.5) is equivalent to (4.4) since \( \lim_{t \rightarrow T^*} u(t, x) \) is compactly supported.

Remark 13. The same theorem as Theorem 2 holds for the equation (1.20) for \( 0 \leq \lambda \leq 2 \).

5 Proof of Theorem 3

We define functions \( R_1, R_2 \) and characteristic lines \( x_{\pm} \) as (3.1) and (3.3) respectively.
By \( u_1(x) \neq 0 \), we have \( R_1(0, \cdot) \neq 0 \) or \( R_2(0, \cdot) \neq 0 \). We assume that \( R_1(0, x_0) \neq 0 \).
Suppose that \( T^* = \infty \).
From
\[ \frac{d}{dt} u(t, x_-(t)) = R_2(t, x_-(t)), \]
and the assumption \( R_2(0, x) \geq 0 \), Lemma 8 yields that \( u(t, x_-(t)) \) is a monotone increasing function with \( t \). By (1.4), there exists a \( \delta > 0 \) such that
\[ c(u(t, x_-(t))) \geq \delta. \]
In the same way as in the proof of Lemma 8, we obtain

\[ R_2(t, x_+(t)) = 0 \quad \text{for } t \geq 0, \]

with \( x_+(0) \notin \text{supp } R_2(0, \cdot). \)

Since \( R_2(0, \cdot) \) is compactly supported, there exists \( T_0 > 0 \) such that

(5.3) \[ R_2(t, x_-(t)) = 0 \quad \text{for } t \geq T_0. \]

By (5.1) and (5.3), we have

(5.4) \[ u(0, x_-(0)) \leq u(t, x_-(t)) \leq C, \]

for some constant \( C > 0. \)

By (1.14), (5.2) and (5.4), we obtain

(5.5) \[ \delta \leq c(u(t, x_-(t))) \leq C_1 \quad \text{and} \quad C_2 \leq c'(u(t, x_-(t))) \leq C_3 \]

for some constant \( C_j > 0 \) for \( j = 1, 2 \) and 3.

We chose \( x_-(0) \) such that \( R_1(0, x_-(0)) > 0. \)

Noting that \( R_1(t, x_-(t)) > 0 \) for \( t \geq 0, \) by (5.1) and (5.2), \( R_1(t, x_-(t)) \) satisfies that

(5.6) \[ \frac{d}{dt} R_1(t, x_-(t)) \geq CR_1(t, x_-(t))^2, \quad \text{for } t \geq T_0. \]

From \( R(T_0, x_-(T_0)) > 0, \) \( R(t, x_-(t)) \) is going to infinity in finite time, which is contradiction to \( T^* = \infty. \)

Since the first estimate in (5.5) holds on \([0, T^*), \) we have

\[ \lim_{t \to T^*} \| \partial_t u(t) \|_{L^\infty} + \| \partial_x u(t) \|_{L^\infty} = \infty. \]

\[ \square \]

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