On the Mean Stability of a Class of Switched Linear Systems

Masaki Ogura and Clyde F. Martin

Abstract—This paper investigates the mean stability of a class of discrete-time stochastic switched linear systems using the $L^p$-norm joint spectral radius of the probability distributions governing the switched systems. First we prove a converse Lyapunov theorem that shows the equivalence between the mean stability and the existence of a homogeneous Lyapunov function. Then we show that, when $p$ goes to $\infty$, the stability of the $p$th mean becomes equivalent to the absolute asymptotic stability of an associated deterministic switched system. Finally we study the mean stability of Markovian switched systems. Numerical examples are presented to illustrate the results.

I. INTRODUCTION

This paper studies the discrete-time stochastic switched linear system of the form

$$\Sigma: x(k+1) = A_k x(k)$$

where $x(k)$ represents a finite-dimensional state vector and $(A_k)_{k=0}^m$ is a stochastic process taking values in the set of square matrices of an appropriate dimension. One of its most natural stability notions is almost sure stability [16], which requires that the state $x(k)$ converges to the origin with probability one as $k \to \infty$. However this difficulty is difficult to check in practice because it is characterized by a quantity called the top Lyapunov exponent, whose computation is in general a hard problem [22]. For example, the necessary and sufficient condition in [6] is one of the most tractable conditions but cannot necessarily be checked with finite computation.

This difficulty has motivated many authors to study another stability, called $p$th mean stability, which requires that the expected value of the $p$th power of the norm of the state $x(k)$ converges to 0. It is well known [9] that, roughly speaking, the $p$th mean stability becomes equivalent to almost sure stability in the limit of $p \to 0$. As a counterpart of this fact we show that, in the limit of $p \to \infty$, the $p$th mean stability is equivalent to the stability of an associated deterministic switched system.

Finally we extend the results in [18] to Markovian switched systems, where $A_k$ in (1) is not necessarily identically and independently distributed. Again assuming the invariance property of the Markov process we will give a characterization of the $p$th mean stability.

This paper is organized as follows. After preparing necessary mathematical notation and conventions, in Section II we review the basic facts of the stability of discrete-time linear switched systems. Section III proves a converse Lyapunov theorem. Then in Section IV we study the limiting behavior of the $p$th mean stability. Finally Section V studies the mean stability of Markovian switched systems.

A. Mathematical Preliminaries

Let $\mathbb{R}_+$ denote the set of nonnegative numbers. The spectral radius of a square matrix is denoted by $\rho(\cdot)$. A subset $K \subset \mathbb{R}^d$ is called a cone if $K$ is closed under multiplication by nonnegative scalars. The cone is said to be solid if it possesses a nonempty interior. We say that a cone is pointed if it contains no line; i.e., if $x, -x \in K$ then $x = 0$. We say that $K$ is proper if it is closed, convex, solid, and pointed. For example the positive orthant $\mathbb{R}_+^d$ of $\mathbb{R}^d$ is a proper cone. The dual cone $K^*$ is defined by

$$K^* = \{ f \in \mathbb{R}^d : f^T x \geq 0 \text{ for every } x \in K \}.$$ A matrix $M \in \mathbb{R}^{d \times d}$ is said to leave $K$ invariant, written $M \geq_K 0$, if $MK \subset K$. A subset $\mathcal{M} \subset \mathbb{R}^{d \times d}$ is said to leave $K$ invariant if any matrix in $\mathcal{M}$ leaves $K$ invariant. For $M, N \in \mathbb{R}^{d \times d}$
Lemma 1.1: Let $K$ be a proper cone and let $M \geq K$.  
1) $M$ has a simple eigenvalue $\rho(M)$, which is greater than
the magnitude of any other eigenvalue of $M$. Moreover
the eigenvector corresponding to the eigenvalue $\rho(M)$
is in int $K$ (see, e.g., [23]).
2) $K^*$ is a proper cone and $M^\top$ is $K^*$-positive [1, 2.23].
A norm $\|\cdot\|$ on $\mathbb{R}^d$ is said to be cone absolute [20] with
respect to a proper cone $K$ if, for every $x \in \mathbb{R}^d$, 
$$\|x\| = \inf_{v,w \in K, v+w = x} \|v+w\|. \tag{2}$$
Also we say that $\|\cdot\|$ is cone linear with respect to $K$ if there
exists $f \in K^*$ such that
$$\|x\| = f^\top x \text{ for every } x \in K. \tag{3}$$
A norm that is cone absolute and cone linear with respect
to a proper cone is said to be cone linear absolute. Every
norm that is cone absolute and cone linear with respect to
$\|\cdot\|$ is defined by 
$$\|M\| = \sup_{x \in \mathbb{R}^d, \|x\|=1} \|Mx\|. \tag{4}$$
Lemma 1.2 ([20]): Let $\|\cdot\|$ be a cone linear absolute norm.
1) The induced norm of $M \in \mathbb{R}^{d \times d}$, defined by 
$$\|M\| = \|M\| = \sup_{x \in \mathbb{R}^d, \|x\|=1} \|Mx\|, \tag{5}$$
2) If $M \geq K N \geq K$ then $\|M\| \geq \|N\|$.
3) If $M_i \geq K N_i \geq K$ for all $i = 1, \ldots, k$ then
$$\|M_1,\ldots,M_k\| \geq \|N_1,\ldots,N_k\|. \tag{6}$$
Proof: The first two claims can be found in [20].
The last one immediately follows from the second one.

We denote the Kronecker product (see, e.g., [4]) of
matrices $M$ and $N$ by $M \otimes N$. For a positive integer $p$
we define the Kronecker power $M^{\otimes p}$ by 
$M^{\otimes 1} := M$ and $M^{\otimes (p+1)} = M^{\otimes p} \otimes M$
recursively for a general $p$. It holds [4] that
$$\|MN\|^{\otimes p} = \|M\|^{\otimes p} \|N\|^{\otimes p}. \tag{7}$$
Lemma 1.3: Let $\mathcal{M} \subset \mathbb{R}^{d \times d}$. If $\mathcal{M}$ leaves a proper cone $K$
invariant then $\mathcal{M}^{\otimes p}$ leaves the proper cone
$$\tilde{K}_p := \text{conv}(K^{\otimes p})$$
invariant.

Let $\mu$ be a probability distribution on $\mathbb{R}^{d \times d}$. The support
of $\mu$ is denoted by supp $\mu$. For a measurable function $f$
on $\mathbb{R}^{d \times d}$ we denote the expected value of $f$ by 
$$E_\mu[f] = \int_{\mathbb{R}^{d \times d}} f(X) d\mu(X).$$
The subscript $\mu$ will be omitted when it is
clear from the context. We define the probability distribution
$\mu^{\otimes p}$ on $\mathbb{R}^{d^p \times d^p}$ as the image of the measure $\mu$
under the mapping $(\cdot)^{\otimes p}: \mathbb{R}^{d \times d} \to \mathbb{R}^{d^p \times d^p}$.
Finally we define the operator $\text{vec}: \mathbb{R}^{m \times d} \to \mathbb{R}^{md}$ by
$$\text{vec}[M_1,\ldots,M_d] := \begin{bmatrix} M_1 \\ \vdots \\ M_d \end{bmatrix}, \quad M_1,\ldots,M_d \in \mathbb{R}^m.$$
2) $\Sigma_\mu$ is $p$th mean stable;
3) $\Sigma_\mu$ is $p$th moment stable.
Moreover it holds that
$$\rho_{p,\mu} = \left(\rho\left(E[A^{\otimes p}]\right)\right)^{1/p}.$$  

We will also need the next lemma that lists basic properties of $p$-radius.

**Lemma 2.4 ([18]):**
1) $\rho_{p,\mu}$ is non-decreasing with respect to $p$.
2) For all $p$ and $k$ it holds that
$$\rho_{p,\mu} = \left[\rho_{p/k,\mu^{1/k}}\right]^{1/k}.$$  

Let us also review the notion of joint spectral radius [13]. Let $\mathcal{M}$ be a subset of $\mathbb{R}^{d \times d}$. The joint spectral radius of $\mathcal{M}$ is defined by
$$\rho(\mathcal{M}) := \limsup \sup_{k \to \infty} \|A_k \cdots A_1\|^{1/k}.$$  
Again this quantity is independent of the norm $\|\cdot\|$. The joint spectral radius is known to characterize the stability of the deterministic switched system
$$\Sigma_{\mathcal{M}}: x(k+1) = A_kx(k), \ A_k \in \mathcal{M}.$$  
$\Sigma_{\mathcal{M}}$ is said to be absolutely asymptotically stable if $x(k) \to 0$ as $k \to \infty$ for every possible switching pattern. The next proposition is well known (for its proof see, e.g., [21]).

**Proposition 2.5:** $\Sigma_{\mathcal{M}}$ is absolutely asymptotically stable if and only if $\rho(\mathcal{M}) < 1$.

### III. CONVERSE LYAPUNOV THEOREM

The aim of this section is to show a converse Lyapunov theorem for the switched system $\Sigma_\mu$. Let us begin by defining Lyapunov functions for $\Sigma_\mu$.

**Definition 3.1:** A continuous and positive definite function $V: \mathbb{R}^d \to \mathbb{R}$ is said to be a Lyapunov function for $\Sigma_\mu$ if there exists $0 \leq \gamma < 1$ such that
$$E[V(Ax)] \leq \gamma V(x)$$  
for every $x \in \mathbb{R}^d$.

The next theorem is the main result of this section.

**Theorem 3.2:** Assume that either
a) $p$ is even or
b) $\text{supp } \mu$ leaves a proper cone $K$ invariant and moreover
$$E[A^{\otimes p}] > \delta p 0.$$  
Then $\Sigma_\mu$ is $p$th mean stable if and only if it admits a homogeneous Lyapunov function of degree $p$.

**Remark 3.3:** The assumptions in Theorem 3.2 are needed because the theorem relies on Proposition 2.3. The relaxation of those assumptions is left as an open problem. A possible approach can be found in [15], where the authors propose a method to approximately compute $p$-radius without such assumptions.

It is straightforward to prove sufficiency.

**Proof of sufficiency in Theorem 3.2:** Assume that $\Sigma_\mu$ admits a homogeneous Lyapunov function of degree $p$. Let $x_0 \in \mathbb{R}^d$ be arbitrary. Using induction we can show $E[V(x(k); x_0)] \leq \gamma^k V(x_0)$. Since $V$ is continuous, homogeneous with degree $p$, and positive definite, there exist constants $C_1, C_2 > 0$ such that $C_1\|x\|^p \leq V(x) \leq C_2\|x\|^p$ for every $x \in \mathbb{R}^d$. Therefore $E[\|x(k); x_0\|^p] \leq (C_2/C_1)\gamma^k\|x_0\|^p$. Thus $\Sigma_\mu$ is $p$th mean stable.

The rest of this section is devoted for the proof of necessity. The proof for the case a) depends on its special case $p = 2$, which is proved in [18] and is quoted as the next proposition for ease of reference.

**Proposition 3.4 ([18]):** $\Sigma_\mu$ is mean square stable if and only if it admits a quadratic Lyapunov function of the form $x^\top Hx$, where $H$ is a positive definite matrix. Moreover such an $H$ can be obtained by solving a linear matrix inequality.

To prove the second case b) we will need the next proposition.

**Proposition 3.5:** Let $K \subset \mathbb{R}^d$ be a proper cone and assume that $M > K 0$. Then there exists $f \in \text{int}(K^*)$ that induces the cone linear absolute norm $\|\cdot\|_f$ such that $\|M\|_f = \rho(M)$.

**Proof:** By Lemma 1.1 the matrix $M$ admits the Jordan canonical form $J = V^{-1}AMV$ where $V \in \mathbb{R}^{d \times d}$ is an invertible matrix whose columns are the generalized eigenvectors of $M$ and $J \in \mathbb{R}^{d \times d}$ is of the form
$$J = \begin{bmatrix} J_0 & 0 \\ 0 & \rho(A) \end{bmatrix},$$  
for some upper diagonal matrix $J_0$. Define $f \in \mathbb{R}^d$ by
$$V^{-1} = f^\top x.$$  
We can easily see that $f$ is an eigenvector of $M^\top$ corresponding to the eigenvalue $\rho(M)$. Since $K^*$ is proper, Lemma 1.1 shows $f \in \text{int}(K^*)$. Thus $f$ gives a cone linear absolute norm with respect to $K$. Since for every $x \in K$ we have
$$\|Mx\|_f = f^\top Mx = \rho(M)f^\top x = \rho(M)\|x\|_f,$$
the equation (4) immediately shows $\|M\|_f = \rho(M)$.

Now we can complete the proof of Theorem 3.2.

**Proof of necessity in Theorem 3.2:** First consider the case a). Assume that $\Sigma_\mu$ is $p$th mean stable for $p = 2q$ where $q$ is a positive integer. Then Proposition 2.3 gives $\rho_{p,\mu} < 1$ and therefore $\rho_{2,\mu^{1/2}} < 1$ by (8). Hence, by Proposition 3.4, $\Sigma_{\mu^{1/2}}$ admits a homogeneous Lyapunov function $V$ of degree 2. Now define $W: \mathbb{R}^d \to \mathbb{R}$ by $W(x) := V(x^{\otimes q})$. Since $V$ is a Lyapunov function and is homogeneous of degree 2, $W$ is continuous, positive definite, and is homogeneous of degree $2q = p$. Moreover, by (5), if $B$ follows $\mu^{1/2}$ then
$$E[W(Ax)] = E[V(A^{\otimes p}x^{\otimes q})]$$
$$= E[V(Bx^{\otimes q})]$$
$$\leq \gamma V(x^{\otimes q})$$
$$= \gamma W(x)$$
for some $\gamma < 1$ since $V$ is a Lyapunov function for $\Sigma_{\mu^{1/2}}$. This shows that $W$ is a Lyapunov function for $\Sigma_\mu$.

Then let us turn to the second case b). We only consider the special case $p = 1$. Notice that $K_1 = K$. Assume that
support $\mu$ leaves a proper cone $K$ invariant and $E[A] > K 0$. If $\Sigma_\mu$ is (first) mean stable then $\gamma := \rho(E[A]) < 1$ by Proposition 2.3. Also, by Proposition 3.5, there exists a cone linear absolute norm $\|\cdot\|$ on $\mathbb{R}^d$ such that $\|E[A]\| = \gamma$. Now we define $V(x) := \|x\|$. We need to show (9). Let $x \in \mathbb{R}^d$ and $\varepsilon > 0$ be arbitrary. Since $\|\cdot\|$ is cone absolute there exist $x_1, x_2 \in K$ such that $x = x_1 - x_2$ and $\|x_1 + x_2\| \leq \|x\| + \varepsilon$. Notice that, by the linearity of $\|\cdot\|$ on $K$, we have $E[\|Ax_i\|] = \|E[A]x_i\| \leq \gamma \|x_i\|$. Therefore

$$E[V(Ax)] = E[\|Ax_1 - Ax_2\|]$$

$$\leq E[\|Ax_1\|] + E[\|Ax_2\|]$$

$$\leq \gamma (\|x_1\| + \|x_2\|)$$

$$\leq \gamma (\|x\| + \varepsilon)$$

again by the linearity of $\|\cdot\|$ on $K$. Since $\varepsilon > 0$ was arbitrary the inequality (9) actually holds and this finishes the proof for $p = 1$. The case for a general $p$ can be proved in the same way as the first half of this proof with Lemma 1.3.

Example 3.6: Consider the probability distribution

$$\mu = \begin{bmatrix} 0,1.5 & 0,1.8 \\ 0,0.15 & 0,1.2 \end{bmatrix}$$

where each closed interval indicate that the corresponding element of $\mu$ is the uniform distribution on the interval. Clearly $\text{supp}\mu$ leaves the proper cone $\mathbb{R}_+^2$ invariant. Also the mean $E[A]$ has only positive entries so that it is $\mathbb{R}_+^2$-positive. Therefore Theorem 3.2 shows that $\Sigma_\mu$ admits a homogeneous Lyapunov function of degree 1. From the proof of the theorem, the Lyapunov function is given as a cone linear absolute norm $\|x\|_f$ and we can obtain the $f$ by following the proof of Proposition 3.5 as $f = \begin{bmatrix} 0.3838 & 1 \end{bmatrix}^T$.

We generate 200 sample paths of $\Sigma_\mu$ with the initial state $x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T$. Fig. 1 shows the sample means of the Lyapunov function $\|x(k)\|_f$ and the Euclidean norm $\|x(k)\|$. While the sample mean of our Lyapunov function is almost decreasing, that of the Euclidean norm shows oscillation. Fig. 2 shows the average of the sample paths and the contour plot of our Lyapunov function and the Euclidean norm.

Remark 3.7: Since $\text{supp}\mu$ is infinite in Example 3.6, we cannot use the methods to construct a Lyapunov function for positive systems proposed in the literature (see, e.g., [8]).

IV. LIMITING BEHAVIOR OF $p$TH MEAN STABILITY

This section studies the limiting behavior of $p$th mean stability as $p \to \infty$. We start with the following observation. Let $\mu$ be a probability distribution on $\mathbb{R}^{d \times d}$ and let $M = \text{supp}\mu$. Then the definitions of $p$-radius and joint spectral radius show $p_{p,M} \leq \hat{\rho}(M)$. Since $p_{p,M}$ is non-decreasing with respect to $p$ by Lemma 2.4 we have

$$\lim_{p \to \infty} p_{p,M} \leq \hat{\rho}(M).$$

It is then natural to ask when the equality holds in this inequality. We will show that the equality still holds under the following assumption, which is weaker than the one in [2].

Assumption 4.1:

a) $M$ leaves a proper cone invariant;

b) The singular part $\mu_s$ of $\mu$ consists of only point measures, i.e., $\mu_s = \sum_{i=0}^{N_1} p_i \delta_{M_i}$ for some positive numbers $p_1, \ldots, p_N$ and matrices $M_1, \ldots, M_N$.

The next theorem is the main result of this section.

Theorem 4.2: If $\mu$ satisfies Assumption 4.1 then

$$\lim_{p \to \infty} p_{p,M} = \hat{\rho}(M).$$

This theorem has two corollaries, both of which can be proved easily using Proposition 2.3. The first one shows a novel relationship between the stability of the deterministic switched system $\Sigma_{\mu}$ and the stochastic switched system $\Sigma_{\mu}$.

Corollary 4.3: Assume that $\mu$ satisfies Assumption 4.1. Then $\Sigma_{\mu}$ is absolutely asymptotically stable if and only if there exists $\gamma < 1$ such that $\Sigma_{\mu}$ is $p$th mean stable with the decay rate at most $\gamma$, i.e., the expectation $E[\|x(k)\|^p]^{1/p}$ is of order $O(\gamma^k)$ for every $p$.

The second corollary gives an expression of joint spectral radius.

Corollary 4.4: If $\mu$ satisfies Assumption 4.1 then

$$\hat{\rho}(M) = \lim_{p \to \infty} \rho(E[A \odot p])^{1/p}.$$
for every $M \in \mathcal{M}$.

Let us prove Theorem 4.2.

Proof of Theorem 4.2: By Proposition 2.3 and the inequality (10) it is sufficient to show \( \lim_{p \to \infty} \rho_{p,M} \geq \hat{\rho}(\mathcal{M}) \). Let \( \gamma := \hat{\rho}(\mathcal{M}) \). Let us take any cone linear absolute norm \( ||·|| \) with respect to the invariant cone. By Proposition 2.5, we can show that there exist \( C > 0 \) and \( \{M_k\}_{k=1}^\infty \subset \mathcal{M} \) such that \( ||M_1 \cdots M_k|| > C\gamma^k \) for infinitely many \( k \).

Take arbitrary \( \gamma_1 \) and \( \gamma_2 \) such that \( \gamma > \gamma_1 > \gamma_2 \). Define \( \varepsilon := (\gamma - \gamma_1)/\gamma \) and take the corresponding \( \delta > 0 \) given by Lemma 4.5. Observe that, by Lemma 1.2, if \( X_i \geq \delta \) \( (1 - \varepsilon)M_i = (\gamma_1/\gamma)M_i \) then \( ||X \cdots X|| \geq (\gamma_1/\gamma)^k ||M_1 \cdots M_k|| > C\gamma^k \). Therefore,

\[
E[||A_1 \cdots A_1||^p] > \mu^k \left( \left[ \frac{C\gamma^k}{\delta^k} \right] \right)^p \geq \mu^k \left( \left[ \frac{C\gamma^k}{\delta^k} \right] \right)^p \geq \delta^k C\gamma^k \gamma_1^p
\]

and hence we have \( E[||A_1 \cdots A_1||^p]^{1/p} > C^{1/k} \delta^{1/p} \gamma_1 \). Choose a sufficiently large \( p \) such that \( \delta^{1/p} \gamma_1 = \gamma_2 \). Then \( E[||A_1 \cdots A_1||^p]^{1/p} > C^{1/k} \gamma_2 \) for infinitely many \( k \). This implies \( \rho_{p,M} \geq \gamma \) and therefore \( \lim_{p \to \infty} \rho_{p,M} \geq \gamma \). This completes the proof since \( \gamma \) can be made arbitrary close to \( \gamma = \hat{\rho}(\mathcal{M}) \).

The next example gives a simple illustration of Theorem 4.2.

Example 4.6: Let \( \mu \) be the uniform distribution on \([0, \gamma]\) for some \( \gamma > 0 \). We can see that \( \mu \) satisfies Assumption 4.1. Also clearly \( \hat{\rho}(\mathcal{M}) = \gamma \). On the other hand we have \( \rho_{p,M} = \rho(E[\gamma ||\gamma||]) = \gamma p/(p+1) \). Therefore \( \rho_{p,M} - \gamma = p/(p+1) \gamma \) and hence \( \lim_{p \to \infty} \rho_{p,M} = \gamma = \hat{\rho}(\mathcal{M}) \), as expected. We remark that, since \( \sup \mu = \infty \), we cannot apply the result in [2] to this example.

Finally the next theorem generalizes another characterization of joint spectral radius in [25], which does not need the existence of an invariant cone.

Theorem 4.7: If \( \mu \) satisfies the condition b) of Assumption 4.1 then

\[
\hat{\rho}(\mathcal{M}) = \limsup_{p \to \infty} \rho(E[\gamma ||\gamma||])^{1/p} \tag{11}
\]

Sketch of the proof: Using Theorem 4.2 and the semidefinite lifting [2] of matrices one can show \( \hat{\rho}(\mathcal{M}) = \lim_{p \to \infty} \rho(E[\gamma ||\gamma||]^{1/2})^{1/2(p)} \). Also it is possible to see that the sequence \( \{\rho(E[\gamma ||\gamma||]^{1/2})^{1/p}\}_{p=1}^\infty \) is not decreasing. These observations yield (11).

V. MARKOVIAN CASE

So far we have restricted our attention to the case when the random variables \( \{A_k\}_{k=0}^\infty \) are identically and independently distributed. In this section we study a more practical case of when the process \( \{A_k\}_{k=0}^\infty \) is a time-homogeneous Markov chain. Let \( \{A_k\}_{k=0}^\infty \) be a time-homogeneous Markov chain taking values in \( \{1, \ldots, N\} \) with the transition probability matrix \( P = [p_{ij}] \) and the constant initial state \( \sigma_0 \in \{1, \ldots, N\} \).

Let \( \mathcal{M} = \{M_1, \ldots, M_N\} \) be a set of \( d \times d \) square matrices. We define the stochastic process \( A = \{A_k\}_{k=0}^\infty \) by \( A_k = M_{\sigma_k} \) and the switched system \( \Sigma_A \) by

\[
\Sigma_A : x(k+1) = A_k(x(k)).
\]

The next corollary of Theorem 5.1 enables us to compute \( p \)-radius efficiently.

Corollary 5.3: If \( p \) and \( \mu \) satisfies the assumption in Theorem 5.1 then it holds that

\[
\rho_{p,A} = \rho(T_p)^{1/p}.
\]
Finally let us apply the results obtained in this section to the stabilization of Markovian switched systems. Let

\[
N = 3, \quad P = \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.5 & 0.3 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0.32 & 0.49 \\ 0.24 & 0.33 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0.53 & 0.65 \\ 0.75 & 0.85 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 1.5 & 0.51 \\ 0.18 & 0.69 \end{bmatrix}.
\]

Corollary 5.3 gives \( \rho_{1,A} = 1.221 \) and therefore \( \Sigma_A \) is not first mean stable. Let us consider the stabilization of \( \Sigma_A \). Define the switched system with input by \( x(k+1) = A_k x(k) + b_k u(k) \) where \( b_k = n_k \) with

\[
n_1 = \begin{bmatrix} -0.56 \\ 0.39 \end{bmatrix}, \quad n_2 = \begin{bmatrix} 0.40 \\ -1.70 \end{bmatrix}, \quad n_3 = \begin{bmatrix} -0.37 \\ -0.49 \end{bmatrix}.
\]

As an input we use the static state feedback \( u(k) = f(x(k)) \) for some \( f \in \mathbb{R}^{1 \times 2} \). This yields the controlled system

\[
\Sigma_{A+bf} : x(k+1) = (A_k + b_k f) x(k)
\]

Let \( f = \begin{bmatrix} 0.36 & 0.50 \end{bmatrix} \). Then all the matrices \( M_i + n_i f \) \( (i = 1, 2, 3) \) have only positive entries. Therefore we can use Corollary 5.3 to find \( \rho_{1,A+bf} = 0.9554 \). Therefore the controlled system \( \Sigma_{A+bf} \) is first mean stable by Theorem 5.1. Fig. 3 shows the 20 sample paths of the original switched system and the stabilized switched system. Finding a systematic way to obtain a stabilizing feedback gain is left as an open problem.

VI. Conclusion

We investigated the mean stability of a class of discrete-time stochastic switched systems. First we presented the equivalence between mean stability and the existence of a homogeneous Lyapunov function. Then we showed that, in the limit of \( p \to \infty \), the \( p \)th mean stability becomes equivalent to the absolute asymptotic stability of an associated deterministic switched system. Finally the characterization of the stability of a class of Markovian switched systems was given. Throughout the paper \( L^p \)-norm joint spectral radius has played a key role.

**References**

[1] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*. Philadelphia: SIAM, 1979.

[2] V. D. Blondel and Y. Nesterov, “Computationally efficient approximations of the joint spectral radius,” *SIAM Journal on Matrix Analysis and Applications*, vol. 27, no. 1, pp. 256–272, 2005.

[3] V. I. Bogachev, *Measure Theory*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2007.

[4] J. Brewer, “Kronecker products and matrix calculus in system theory,” *IEEE Transactions on Circuits and Systems*, vol. 25, no. 9, pp. 772–781, 1978.

[5] O. Costa and M. Fragoso, “Comments on “Stochastic stability of jump linear systems”,” *IEEE Transactions on Automatic Control*, vol. 49, no. 8, pp. 1414–1416, 2004.

[6] X. Dai, Y. Huang, and M. Xiao, “Almost sure stability of discrete-time switched linear systems: a topological point of view,” *SIAM Journal on Control and Optimization*, vol. 47, no. 4, pp. 2137–2156, 2008.

[7] W. Dayawansa and C. Martin, “A converse Lyapunov theorem for a class of dynamical systems which undergo switching,” *IEEE Transactions on Automatic Control*, vol. 44, no. 4, pp. 751–760, 1999.

[8] T. S. Doan, A. Kalauch, and S. Siegmund, “A constructive approach to linear Lyapunov functions for positive switched systems using Collatz-Wielandt sets,” *IEEE Transactions on Automatic Control*, vol. 58, no. 3, pp. 748–751, 2013.

[9] Y. Fang and K. A. Loparo, “On the relationship between the sample path and moment Lyapunov exponents for jump linear systems,” *IEEE Transactions on Automatic Control*, vol. 47, no. 9, pp. 1556–1560, 2002.

[10] ——, “Stochastic stability of jump linear systems,” *IEEE Transactions on Automatic Control*, vol. 47, no. 7, pp. 1204–1208, 2002.

[11] Y. Fang, K. A. Loparo, and X. Feng, “Almost sure and \( \delta \)-moment stability of jump linear systems,” *International Journal of Control*, vol. 59, no. 5, pp. 1281–1307, 1994.

[12] M. D. Fragoso and O. L. V. Costa, “A unified approach for stochastic and mean square stability of continuous-time linear systems with Markovian jumping parameters and additive disturbances,” *SIAM Journal on Control and Optimization*, vol. 44, no. 4, pp. 1165–1191, 2005.

[13] R. M. Jungers, *The Joint Spectral Radius*, ser. Lecture Notes in Control and Information Sciences. Berlin, Heidelberg: Springer Berlin Heidelberg, 2009, vol. 385.

[14] R. M. Jungers and V. Y. Protasov, “Weak stability of switching dynamical systems and fast computation of the \( p \)-radius of matrices,” in *49th IEEE Conference on Decision and Control*, 2010, pp. 7328–7333.

[15] ——, “Fast methods for computing the \( p \)-radius of matrices,” *SIAM Journal on Scientific Computing*, vol. 33, no. 3, pp. 1246–1266, 2011.

[16] E. Kozin, “A survey of stability of stochastic systems,” *Automatica*, vol. 5, no. 1, pp. 95–112, 1969.

[17] A. Molchanov and Y. Pyatnitskiy, “Criteria of asymptotic stability of differential and difference inclusions encountered in control theory,” *Systems & Control Letters*, vol. 13, pp. 59–64, 1989.

[18] M. Ogura and C. Martin, “Generalized joint spectral radius and stability of switching system,” *Linear Algebra and its Applications*, vol. 439, no. 8, pp. 2222–2239, 2013.

[19] V. Y. Protasov, “The generalized joint spectral radius. A geometric approach,” *Evestiua: Mathematics*, vol. 61, no. 5, pp. 995–1030, 1997.

[20] T. I. Seidman, H. Schneider, and M. Arav, “Comparison theorems using general cones for norms of iteration matrices,” *Linear Algebra and its Applications*, vol. 399, pp. 169–186, 2005.

[21] J. Theys, “Joint Spectral Radius: theory and approximations,” Ph.D. dissertation, Université Catholique de Louvain, 2005.

[22] J. N. Tsitsiklis and V. D. Blondel, “The Lyapunov exponent and joint spectral radius of pairs of matrices are hard – when not impossible – to compute and to approximate,” *Mathematics of Control, Signals, and Systems*, vol. 10, no. 1, pp. 31–40, 1997.

[23] J. S. Vandergraft, “Spectral properties of matrices which have invariant cones,” *SIAM Journal on Applied Mathematics*, vol. 16, no. 6, pp. 1208–1222, 1968.

[24] A. N. Vargas and J. B. R. do Val, “Average cost and stability of time-varying linear systems,” *IEEE Transactions on Automatic Control*, vol. 55, no. 3, pp. 714–729, 2010.

[25] J. Xu and M. Xiao, “A characterization of the generalized spectral radius with Kronecker powers,” *Automatica*, vol. 47, no. 7, pp. 1530–1533, 2011.