HYPERREFLEXIVITY OF THE SPACE OF MODULE HOMOMORPHISMS BETWEEN NON-COMMUTATIVE $L^p$-SPACES

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Abstract. Let $\mathcal{M}$ be a von Neumann algebra, and let $0 < p, q \leq \infty$. Then the space $\text{Hom}_\mathcal{M}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ of all right $\mathcal{M}$-module homomorphisms from $L^p(\mathcal{M})$ to $L^q(\mathcal{M})$ is a reflexive subspace of the space of all continuous linear maps from $L^p(\mathcal{M})$ to $L^q(\mathcal{M})$. Further, the space $\text{Hom}_\mathcal{M}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ is hyperreflexive in each of the following cases: (i) $1 \leq q < p \leq \infty$; (ii) $1 \leq p, q \leq \infty$ and $\mathcal{M}$ is injective, in which case the hyperreflexivity constant is at most 8.

Introduction

Let $\mathcal{A}$ be a closed subalgebra of the algebra $B(\mathcal{H})$ of all continuous linear operators on the Hilbert space $\mathcal{H}$. Then $\mathcal{A}$ is called reflexive if

$$\mathcal{A} = \{ T \in B(\mathcal{H}) : e^\perp Te = 0 \ (e \in \text{lat}\mathcal{A}) \},$$

where $\text{lat}\mathcal{A} = \{ e \in B(\mathcal{H}) \text{ projection} : e^\perp Te = 0 \ (T \in \mathcal{A}) \}$ is the set of all projections onto the $\mathcal{A}$-invariant subspaces of $\mathcal{H}$. The double commutant theorem shows that each von Neumann algebra on $\mathcal{H}$ is certainly reflexive. The algebra $\mathcal{A}$ is called hyperreflexive if the above condition on $\mathcal{A}$ is strengthened by requiring that there is a distance estimate

$$\text{dist}(T, \mathcal{A}) \leq C \sup \{ \| e^\perp Te \| : e \in \text{lat}\mathcal{A} \} \quad (T \in B(\mathcal{H}))$$

for some constant $C$. The inequality

$$\sup \{ \| e^\perp Te \| : e \in \text{lat}\mathcal{A} \} \leq \text{dist}(T, \mathcal{A}) \quad (T \in B(\mathcal{H}))$$

is always true and elementary. This quantitative version of reflexivity was introduced by Arveson [5] and has proven to be a powerful tool when it is available. Christensen [7, 8, 9] showed that many von Neumann algebras are hyperreflexive by relating the hyperreflexivity to the vanishing of certain

2010 Mathematics Subject Classification. Primary: 46L52, 46L10; Secondary: 47L05.

Key words and phrases. Non-commutative $L^p$-spaces, injective von Neumann algebras, reflexive subspaces, hyperreflexive subspaces, module homomorphisms.

The authors were supported by project PGC2018-093794-B-I00 (MCIU/AEI/FEDER, UE), Junta de Andalucía grant FQM-185, and Proyectos I+D+i del programa operativo FEDER-Andalucía A-FQM-48-UGR18. The third named author was supported by Contrato Predoctoral FPU, Plan propio de Investigación y Transferencia 2018, University of Granada.
cohomology group. Notably each injective von Neumann algebra $\mathcal{M}$ on the Hilbert space $\mathcal{H}$ is hyperreflexive and

$$\text{dist}(T, \mathcal{M}) \leq 4 \sup \left\{ \| e^T e \| : e \in \mathcal{M} \text{ projection} \right\} \quad (T \in B(\mathcal{H}))$$

(see [7] Theorem 2.3] and [11] p. 340).

Both notions, reflexivity and hyperreflexivity, were extended to subspaces of $B(\mathcal{X}, \mathcal{Y})$, the Banach space of all continuous linear maps from the Banach space $\mathcal{X}$ to the Banach space $\mathcal{Y}$. Following Loginov and Shulman [23], a closed linear subspace $\mathcal{S}$ of $B(\mathcal{X}, \mathcal{Y})$ is called reflexive if

$$\mathcal{S} = \left\{ T \in B(\mathcal{X}, \mathcal{Y}) : T(x) \in \overline{\{ S(x) : S \in \mathcal{S} \}} (x \in \mathcal{X}) \right\}.$$ 

In accordance with Larson [21, 22], $\mathcal{S}$ is called hyperreflexive if there exists a constant $C$ such that

$$\text{dist}(T, \mathcal{S}) \leq C \sup_{x \in \mathcal{X}} \inf_{S \in \mathcal{S}} \{ \| T(x) - S(x) \| : S \in \mathcal{S} \} \quad (T \in B(\mathcal{X}, \mathcal{Y})),$$

and the optimal constant is called the hyperreflexivity constant of $\mathcal{S}$. The inequality

$$\sup_{x \in \mathcal{X}, \| x \| \leq 1} \inf_{S \in \mathcal{S}} \{ \| T(x) - S(x) \| : S \in \mathcal{S} \} \leq \text{dist}(T, \mathcal{S}) \quad (T \in B(\mathcal{X}, \mathcal{Y})).$$

is always true.

The ultimate objective of this paper is to study the hyperreflexivity of the space $\text{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ of all (automatically continuous) right $\mathcal{M}$-module homomorphisms from $L^p(\mathcal{M})$ to $L^q(\mathcal{M})$ for a von Neumann algebra $\mathcal{M}$. The non-commutative $L^p$-spaces that we consider throughout are those introduced by Haagerup (see [14, 24, 30]). For each $0 < p \leq \infty$, the space $L^p(\mathcal{M})$ is a contractive Banach $\mathcal{M}$-bimodule or a contractive $p$-Banach $\mathcal{M}$-bimodule according to $1 \leq p$ or $p < 1$, and we will focus on the right $\mathcal{M}$-module structure of $L^p(\mathcal{M})$.

Our method relies in the analysis of a continuous bilinear map $\varphi : \mathcal{A} \times \mathcal{A} \to \mathcal{X}$, for a $C^*$-algebra $\mathcal{A}$ and a normed space $\mathcal{X}$, through the knowledge of the constant $\sup \{ \| \varphi(a, b) \| : a, b \in \mathcal{A} \text{ contractions, } ab = 0 \}$, alternatively, the constant $\sup \{ \| \varphi(e, e^+) \| : e \in \mathcal{A}_+ \text{ projection} \}$ in the case where $\mathcal{A}$ is unital and has real rank zero. This is done in Section 1.

In Section 2 we prove that, for each $0 < p, q \leq \infty$, each right $\mathcal{M}$-module homomorphism from $L^p(\mathcal{M})$ to $L^q(\mathcal{M})$ is automatically continuous and that the space $\text{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ of all right $\mathcal{M}$-module homomorphisms is a reflexive subspace of $B(L^p(\mathcal{M}), L^q(\mathcal{M}))$ (the notion of reflexivity makes perfect sense for subspaces of operators between quasi-Banach spaces).

Section 3 is devoted to study the hyperreflexivity of $\text{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ for $1 \leq p, q \leq \infty$. The space $B(L^p(\mathcal{M}), L^q(\mathcal{M}))$ is a Banach $\mathcal{M}$-bimodule for the operations specified by

$$(aT)(x) = T(xa), \quad (Ta)(x) = T(x)a$$

for all $T \in B(L^p(\mathcal{M}), L^q(\mathcal{M}))$, $a \in \mathcal{M}$, and $x \in L^p(\mathcal{M})$ (note that the left $\mathcal{M}$-module structure of both $L^p(\mathcal{M})$ and $L^q(\mathcal{M})$ is disregarded), and we will
prove that there is a distance estimate
\[
\text{dist}(T, \text{Hom}_\mathcal{M}(L^p(\mathcal{M}), L^q(\mathcal{M}))) \leq C \sup\{\|e^\dagger T e\| : e \in \mathcal{M} \text{ projection}\}
\]
for each \(T \in B(L^p(\mathcal{M}), L^q(\mathcal{M}))\) in each of the following cases:

(i) \(1 \leq q < p \leq \infty\), in which case the constant \(C\) can be chosen to depend on \(p\) and \(q\), and not on \(\mathcal{M}\);

(ii) \(1 \leq p, q \leq \infty\) and \(\mathcal{M}\) is injective, in which case the constant \(C\) can be taken to be \(8\).

Further,
\[
\sup\{\|e^\dagger T e\| : e \in \mathcal{M} \text{ projection}\} 
\leq \sup_{x \in L^p(\mathcal{M}), \|x\|_p \leq 1} \inf\{\|T(x) - \Phi(x)\|_q : \Phi \in \text{Hom}_\mathcal{M}(L^p(\mathcal{M}), L^q(\mathcal{M}))\},
\]
and thus, in both cases, it turns out that the space \(\text{Hom}_\mathcal{M}(L^p(\mathcal{M}), L^q(\mathcal{M}))\) is hyperreflexive.

It is perhaps worth remarking that most of the discussion of reflexivity and hyperreflexivity is accomplished for continuous homomorphisms between modules over a \(C^*\)-algebra.

Throughout this paper we write \(\mathcal{X}^*\) for the dual of a Banach space \(\mathcal{X}\) and \(\langle \cdot, \cdot \rangle\) for the duality between \(\mathcal{X}\) and \(\mathcal{X}^*\).

1. Analysing bilinear maps through orthogonality

Goldstein proved in \cite{12} (albeit with sesquilinear functionals) that, for each \(C^*\)-algebra \(\mathcal{A}\), every continuous bilinear functional \(\varphi: \mathcal{A} \times \mathcal{A} \to \mathbb{C}\) with the property that \(\varphi(a, b) = 0\) whenever \(a, b \in \mathcal{A}_{sa}\) satisfy \(ab = 0\) can be represented in the form \(\varphi(a, b) = \omega_1(ab) + \omega_2(ba)\) \((a, b \in \mathcal{A})\) for some \(\omega_1, \omega_2 \in \mathcal{A}^*\). Independently, it was shown in \cite{11} that if \(\mathcal{A}\) is a \(C^*\)-algebra or the group algebra \(L^1(G)\) of a locally compact group \(G\), then every continuous bilinear functional \(\varphi: \mathcal{A} \times \mathcal{A} \to \mathbb{C}\) with the property that \(\varphi(a, b) = 0\) whenever \(a, b \in \mathcal{A}\) are such that \(ab = 0\) necessarily satisfies the condition \(\varphi(ab, c) = \varphi(a, bc)\) \((a, b, c \in \mathcal{A})\), which in turn implies the existence of \(\omega \in \mathcal{A}^*\) such that \(\varphi(a, b) = \omega(ab)\) \((a, b \in \mathcal{A})\). Actually, \cite{2} gives more, namely, the norms \(\|\varphi(ab, c) - \varphi(a, bc)\|\) with \(a, b, c \in \mathcal{A}\) can be estimated through the constant \(\sup\{\|\varphi(a, b)\| : a, b \in \mathcal{A}, \|a\| = \|b\| = 1, ab = 0\}\). This property has proven to be useful to study the hyperreflexivity of the spaces of derivations and continuous cocycles on \(\mathcal{A}\) (see \cite{3, 4, 27, 28, 29}). This section provides an improvement of the above mentioned property in the case of \(C^*\)-algebras, and this will be used later to study the hyperreflexivity of the space \(\text{Hom}_\mathcal{A}(\mathcal{X}, \mathcal{Y})\) of all continuous module homomorphisms between the Banach right \(\mathcal{A}\)-modules \(\mathcal{X}\) and \(\mathcal{Y}\).

**Theorem 1.1.** Let \(\mathcal{A}\) be a \(C^*\)-algebra, let \(\mathcal{Z}\) be a normed space, let \(\varphi: \mathcal{A} \times \mathcal{A} \to \mathcal{Z}\) be a continuous bilinear map, and let the constant \(\varepsilon \geq 0\) be such that
\[
a, b \in \mathcal{A}_+, \quad ab = 0 \implies \|\varphi(a, b)\| \leq \varepsilon\|a\|\|b\|.
\]
Suppose that \((e_j)_{j \in J}\) is a net in \(\mathcal{A}\) such that \((e_j)_{j \in J}\) converges to \(1_{\mathcal{A}^{**}}\) in \(\mathcal{A}^{**}\) with respect to the weak* topology. Then, for each \(a \in \mathcal{A}\), the nets \((\varphi(a, e_j))_{j \in J}\) and \((\varphi(e_j, a))_{j \in J}\) converge in \(\mathcal{Z}^{**}\) with respect to the weak* topology and

\[
\left\| \lim_{j \in J} \varphi(a, e_j) - \lim_{j \in J} \varphi(e_j, a) \right\| \leq 8\varepsilon \|a\|.
\]

In particular, if \(\mathcal{A}\) is unital, then

\[
\|\varphi(a, 1_\mathcal{A}) - \varphi(1_\mathcal{A}, a)\| \leq 8\varepsilon \|a\| \quad (a \in \mathcal{A}).
\]

**Proof.** First, we regard \(\varphi\) as a continuous bilinear map with values in \(\mathcal{Z}^{**}\). By applying [17, Theorem 2.3] to \(\mathcal{A}\) acting on the Hilbert space of its universal representation, we obtain that \(\varphi\) extends uniquely, without change of norm, to a continuous bilinear map \(\psi: \mathcal{A}^{**} \times \mathcal{A}^{**} \to \mathcal{Z}^{**}\) which is separately weak* continuous.

Now, since \((e_j)_{j \in J} \to 1_{\mathcal{A}^{**}}\) with respect to the weak* topology and \(\psi\) is separately weak* continuous, we see that, for each \(a \in \mathcal{A}\), the nets \((\varphi(a, e_j))_{j \in J}\) and \((\varphi(e_j, a))_{j \in J}\) converge to \(\psi(a, 1_{\mathcal{A}^{**}})\) and \(\psi(1_{\mathcal{A}^{**}}, a)\), respectively, with respect to the weak* topology of \(\mathcal{Z}^{**}\). Consequently, the proof of the theorem is completed by showing that

\[
(1.1) \quad \left\| \psi(a, 1_{\mathcal{A}^{**}}) - \psi(1_{\mathcal{A}^{**}}, a) \right\| \leq 8\varepsilon \|a\| \quad (a \in \mathcal{A}).
\]

Our next objective is to prove (1.1). We begin with the case \(a \in \mathcal{A}_+\). For this purpose, we fix \(a \in \mathcal{A}_+\) with \(\|a\| \leq 1\) and, for each \(0 < \alpha < 1\), we claim that

\[
(1.2) \quad \left\| \psi(\chi_{[0, \alpha]}(a), \chi_{[\alpha, 1]}(a)) \right\| \leq \varepsilon
\]

and

\[
(1.3) \quad \left\| \psi(\chi_{[\alpha, 1]}(a), \chi_{[0, \alpha]}(a)) \right\| \leq \varepsilon.
\]

We use the notation \(\chi_\Delta\) for the characteristic function of a subset \(\Delta\) of \([0, 1]\). We choose decreasing sequences of real numbers \((\alpha_n)\) and \((\beta_n)\) with \(\alpha < \alpha_n < \beta_n < 1\) \((n \in \mathbb{N})\) and \(\lim \alpha_n = \lim \beta_n = \alpha\). For each \(n \in \mathbb{N}\), we define continuous functions \(f_n, g_n: [0, 1] \to \mathbb{R}\) by

\[
f_n(t) = \begin{cases} 
1 & \text{if } 0 \leq t \leq \alpha, \\
\frac{t - \alpha_n}{\alpha - \alpha_n} & \text{if } \alpha \leq t \leq \alpha_n, \\
0 & \text{if } \alpha_n \leq t \leq 1,
\end{cases}
\]

and

\[
g_n(t) = \begin{cases} 
\frac{t - \alpha_n}{\beta_n - \alpha_n} & \text{if } \alpha_n \leq t \leq \beta_n, \\
1 & \text{if } \beta_n \leq t \leq 1.
\end{cases}
\]

Then the sequences \((f_n)\) and \((g_n)\) are uniformly bounded and they converge pointwise in \([0, 1]\) to \(\chi_{[0, \alpha]}\) and \(\chi_{[\alpha, 1]}\), respectively. From a basic property of the Borel functional calculus on the von Neumann algebra \(\mathcal{A}^{**}\), it follows that the sequences \((f_n(a))\) and \((g_n(a))\) converge with respect to the weak operator topology to \(\chi_{[0, \alpha]}(a)\) and \(\chi_{[\alpha, 1]}(a)\), respectively. Since the sequences \((f_n(a))\) and \((g_n(a))\) are bounded and the weak operator topology of \(\mathcal{A}^{**}\) coincides with the weak* topology on any bounded subset of \(\mathcal{A}^{**}\), we conclude that
From (1.8) and (1.9) it may be concluded that
\[
\psi(\chi_{[0,a]}(a), \chi_{[a,1]}(a)) = \lim_{n \to \infty} \lim_{m \to \infty} \varphi(f_m(a), g_n(a))
\]
and
\[
\psi(\chi_{[a,1]}(a), \chi_{[0,a]}(a)) = \lim_{n \to \infty} \lim_{m \to \infty} \varphi(g_n(a), f_m(a))
\]
On the other hand, if \(m, n \in \mathbb{N}\) with \(m \geq n\), then we have \(f_m g_n = 0\), so that
\[
f_m(a) g_n(a) = g_n(a) f_m(a) = 0.
\]
Further, \(f_m(a), g_n(a) \in A_+\) and \(\|f_m(a)\|, \|g_n(a)\| \leq 1\). Therefore, by hypothesis, we have
\[
\|\varphi(f_m(a), g_n(a))\| \leq \varepsilon
\]
and
\[
\|\varphi(g_n(a), f_m(a))\| \leq \varepsilon
\]
for all \(m, n \in \mathbb{N}\) with \(m \geq n\). From (1.4) and (1.6) we deduce (1.2), while (1.5) and (1.7) give (1.3). For each \(n \in \mathbb{N}\), let \(h_n : [0, 1] \to \mathbb{R}\) be the bounded Borel function defined by
\[
h_n = \frac{1}{n + 1} \sum_{k=1}^{n} \chi_{[k/(n+1), 1]}.
\]
We also consider the sequence in \(A^{**}\) given by \((h_n(a))\). Since the sequence \((h_n)\) converges uniformly on \([0, 1]\) to the identity map on \([0, 1]\), it follows that \((h_n(a)) \to a\) with respect to the norm topology. Thus
\[
\psi(a, 1_{A^{**}}) - \psi(1_{A^{**}}, a) = \lim_{n \to \infty} \left\{ \psi(h_n(a), 1_{A^{**}}) - \psi(1_{A^{**}}, h_n(a)) \right\}
\]
\[
= \lim_{n \to \infty} \frac{1}{n + 1} \sum_{k=1}^{n} \left( \psi(\chi_{[k/(n+1), 1]}(a), 1_{A^{**}}) - \psi(1_{A^{**}}, \chi_{[k/(n+1), 1]}(a)) \right).
\]
Further, for each \(k \in \{1, \ldots, n\}\), we have
\[
1_{A^{**}} = \chi_{[0,k/(n+1)]}(a) + \chi_{[k/(n+1), 1]}(a),
\]
so that
\[
\psi(\chi_{[0,k/(n+1)]}(a), 1_{A^{**}}) - \psi(1_{A^{**}}, \chi_{[k/(n+1), 1]}(a))
\]
\[
= \psi(\chi_{[0,k/(n+1)]}(a), \chi_{[0,k/(n+1)]}(a)) - \psi(\chi_{[0,k/(n+1)]}(a), \chi_{[k/(n+1), 1]}(a)),
\]
and the inequalities (1.2) and (1.3) then give
\[
\|\psi(\chi_{[0,k/(n+1)]}(a), 1_{A^{**}}) - \psi(1_{A^{**}}, \chi_{[k/(n+1), 1]}(a))\| \leq 2 \varepsilon.
\]
From (1.8) and (1.9) it may be concluded that
\[
\|\psi(a, 1_{A^{**}}) - \psi(1_{A^{**}}, a)\| \leq \lim_{n \to \infty} \frac{n2 \varepsilon}{n + 1} = 2 \varepsilon.
\]
Now suppose that \( a \in \mathcal{A}_{\text{sa}} \). Then we can write
\[
a = a_+ - a_-, \quad \text{where} \quad a_+, a_- \in \mathcal{A}_+ \text{ are mutually orthogonal, and (1.10) gives}
\]
\[
\left\| \psi(a, 1_{\mathcal{A}^{**}}) - \psi(1_{\mathcal{A}^{**}}, a) \right\| \leq 2\varepsilon \|a_+\| + 2\varepsilon \|a_-\| \\
\leq 4\varepsilon \max\{\|a_+\|, \|a_-\|\} = 4\varepsilon \|a\|.
\]

Finally take \( a \in \mathcal{A} \), and write
\[
(1.11) \quad a = \Re a + i\Im a,
\]
where
\[
\Re a = \frac{1}{2}(a^* + a), \quad \Im a = \frac{i}{2}(a^* - a) \in \mathcal{A}_{\text{sa}},
\]
and, further, \( \|\Re a\|, \|\Im a\| \leq \|a\| \). From what has previously been proved, it follows that
\[
\left\| \psi(a, 1_{\mathcal{A}^{**}}) - \psi(1_{\mathcal{A}^{**}}, a) \right\| \leq 4\varepsilon \|\Re a\| + 4\varepsilon \|\Im a\| \\
\leq 8\varepsilon \|a\|.
\]

This gives (1.11) and completes the proof of the theorem. \(\square\)

**Theorem 1.2.** Let \( \mathcal{A} \) be a unital \( C^\ast \)-algebra of real rank zero, let \( \mathcal{Z} \) be a topological vector space, and let \( \varphi: \mathcal{A} \times \mathcal{A} \to \mathcal{Z} \) be a continuous bilinear map.

(i) Suppose that
\[
e \in \mathcal{A} \text{ projection } \implies \varphi(e, e^\perp) = 0.
\]
Then
\[
\varphi(a, 1_{\mathcal{A}}) = \varphi(1_{\mathcal{A}}, a) \quad (a \in \mathcal{A}).
\]

(ii) Suppose that \( \mathcal{Z} \) is a normed space and let the constant \( \varepsilon \geq 0 \) be such that
\[
e \in \mathcal{A} \text{ projection } \implies \|\varphi(e, e^\perp)\| \leq \varepsilon.
\]
Then
\[
\|\varphi(a, 1_{\mathcal{A}}) - \varphi(1_{\mathcal{A}}, a)\| \leq 8\varepsilon \|a\| \quad (a \in \mathcal{A}).
\]

**Proof.** Let \( a \in \mathcal{A}_+ \), and assume that \( a \) has finite spectrum, say \( \{\alpha_1, \ldots, \alpha_n\} \). Of course, we can suppose that \( 0 \leq \alpha_1 < \cdots < \alpha_n = \|a\| \). This implies that \( a \) can be written in the form
\[
a = \sum_{k=1}^n \alpha_k p_k,
\]
where \( p_1, \ldots, p_n \in \mathcal{A} \) are mutually orthogonal projections (specifically, the projection \( p_k \) is defined by using the continuous functional calculus for \( a \) by
\( p_k = \chi_{\{\alpha_k\}}(a) \) for each \( k \in \{1, \ldots, n\} \) because \( \chi_{\{\alpha_k\}} \) is a continuous function on the spectrum of \( a \), being this set finite. In case (i), we have

\[
\varphi(a, 1_A) - \varphi(1_A, a) = \sum_{k=1}^{n} \alpha_k \left( \varphi(p_k, 1_A) - \varphi(1_A, p_k) \right)
\]

(1.12)

\[
= \sum_{k=1}^{n} \alpha_k \left( \varphi(p_k, p_k^\perp + p_k) - \varphi(p_k^\perp + p_k, p_k) \right)
\]

\[
= \sum_{k=1}^{n} \alpha_k \left( \varphi(p_k, p_k^\perp) - \varphi(p_k^\perp, p_k) \right) = 0.
\]

In case (ii), we define real numbers \( \lambda_1, \ldots, \lambda_n \in [0, \infty] \) and projections \( e_1, \ldots, e_n \in \mathcal{A} \) by

\[
\lambda_1 = \alpha_1,
\]

\[
\lambda_k = \alpha_k - \alpha_{k-1} \quad (1 < k \leq n),
\]

and

\[
e_k = p_k + \cdots + p_n \quad (1 \leq k \leq n).
\]

It is a simple matter to check that

\[
a = \sum_{k=1}^{n} \lambda_k e_k.
\]

For each \( k \in \{1, \ldots, n\} \), we have

\[
\| \varphi(e_k, 1_A) - \varphi(1_A, e_k) \| = \left\| \varphi(e_k, e_k^\perp + e_k) - \varphi(e_k^\perp + e_k, e_k) \right\|
\]

\[
= \left\| \varphi(e_k, e_k^\perp) - \varphi(e_k^\perp, e_k) \right\|
\]

\[
\leq \left\| \varphi(e_k, e_k^\perp) \right\| + \left\| \varphi(e_k^\perp, e_k) \right\| \leq 2\varepsilon.
\]

We thus get

\[
\| \varphi(a, 1_A) - \varphi(1_A, a) \| = \left\| \sum_{k=1}^{n} \lambda_k \left( \varphi(e_k, 1_A) - \varphi(1_A, e_k) \right) \right\|
\]

(1.13)

\[
\leq \sum_{k=1}^{n} \lambda_k \left\| \varphi(e_k, 1_A) - \varphi(1_A, e_k) \right\|
\]

\[
\leq \sum_{k=1}^{n} \lambda_k 2\varepsilon = 2\varepsilon \alpha_n = 2\varepsilon \|a\|.
\]

Now suppose that \( a \in \mathcal{A}_{sa} \) and that \( a \) has finite spectrum. Then we can write \( a = a_+ - a_- \), where \( a_+, a_- \in \mathcal{A}_+ \) are mutually orthogonal. Since
$a_+ = f(a)$ and $a_- = g(a)$, where $f, g : \mathbb{R} \to \mathbb{R}$ are the continuous functions defined by
\[ f(t) = \max\{t, 0\}, \quad g(t) = \max\{-t, 0\}, \quad (t \in \mathbb{R}), \]
it follows that both $a_+$ and $a_-$ have finite spectra. In case (i), (1.14) gives
\[ \varphi(a, 1_A) = (\varphi(a_+, 1_A) - \varphi(1_A, a_+)) - (\varphi(a_-, 1_A) - \varphi(1_A, a_-)) = 0. \]
In case (ii), on account of (1.13), we have
\[ (1.15) \]
and, in case (ii),
\[ \text{let } C \rightarrow Z \]
be a continuous bilinear map, and let the constant $A ightarrow Z$
\[ (1.14) \]
\[ \text{giving the result.} \]
\[ \square \]

**Corollary 1.3.** Let $\mathcal{A}$ be a $C^*$-algebra, let $\mathcal{Z}$ be a normed space, let $\varphi : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{Z}$ be a continuous bilinear map, and let the constant $\varepsilon \geq 0$ be such that
\[ a, b \in \mathcal{A}, \ ab = 0 \implies ||\varphi(a, b)|| \leq \varepsilon ||a|| ||b||. \]

Then
\[ ||\varphi(ab, c) - \varphi(a, bc)|| \leq 8\varepsilon ||a|| ||b|| ||c|| \quad (a, b, c \in \mathcal{A}). \]
Further, if either $\mathcal{A}$ is unital or $\mathcal{Z}$ is a dual Banach space, then there exists a continuous linear map $\Phi: \mathcal{A} \to \mathcal{Z}$ such that
\[
||\varphi(a, b) - \Phi(ab)|| \leq 8\varepsilon||a|| ||b|| \quad (a, b \in \mathcal{A})
\]
and $\|\Phi\| \leq \|\varphi\|$.

Proof. Set $a, c \in \mathcal{A}$, and consider the continuous bilinear map $\psi: \mathcal{A} \times \mathcal{A} \to \mathcal{Z}$ defined by
\[
\psi(u, v) = \varphi(au, vc) \quad (u, v \in \mathcal{A}).
\]
If $u, v \in \mathcal{A}$ are such that $uv = 0$, then $au, vc \in \mathcal{A}$ and $(au)(vc) = 0$, which gives
\[
\|\psi(u, v)\| = \|\varphi(au, vc)\| \leq \varepsilon||au|| ||vc|| \leq \varepsilon||a|| ||c|| ||u|| ||v||
\]
by hypothesis.

Let $(e_i)_{i \in I}$ be an approximate identity for $\mathcal{A}$ of bound one. Since the net $(e_i)_{i \in I}$ is bounded, it has a subnet $(e_j)_{j \in J}$ which converges to an element $E \in \mathcal{A}^{**}$ with respect to the weak* topology. We claim that $E = 1_{\mathcal{A}^{**}}$. Indeed, let $a \in \mathcal{A}$. Then $(ae_j)_{j \in J} \to a$ with respect to the norm topology and, further, $(ae_j)_{j \in J} \to aE$ with respect to the weak* topology. Thus $aE = a$. From the weak* density of $\mathcal{A}$ in $\mathcal{A}^{**}$ and the separate weak* continuity of the product of $\mathcal{A}^{**}$, we conclude that $\mathcal{A}E = \mathcal{A}$ for each $a \in \mathcal{A}^{**}$, hence that $E = 1_{\mathcal{A}^{**}}$, as claimed.

From Theorem 1.4 we deduce that, for each $b \in \mathcal{A}$, the nets $(\psi(b, e_j))_{j \in J}$ and $(\psi(e_j, b))_{j \in J}$ converge in $\mathcal{Z}^{**}$ with respect to the weak* topology and
\[
\lim_{J \in J} \psi(b, e_j) = \lim_{J \in J} \psi(e_j, b) = \varphi(ab, c).
\]
(1.16)
Since $(ae_j)_{j \in J}$ converges to $a$ and $(e_j c)_{j \in J}$ converges to $c$ in norm and $\varphi$ is continuous, it follows that
\[
\lim_{J \in J} \psi(b, e_j) = \lim_{J \in J} \varphi(ab, e_j c) = \varphi(ab, c)
\]
and
\[
\lim_{J \in J} \psi(e_j, b) = \lim_{J \in J} \varphi(ae_j, bc) = \varphi(a, bc)
\]
in norm for each $b \in \mathcal{A}$, which establishes the required inequality when combined with (1.15).

Now suppose that $\mathcal{A}$ is unital, and define $\Phi: \mathcal{A} \to \mathcal{Z}$ by
\[
\Phi(a) = \varphi(a, 1_{\mathcal{A}}) \quad (a \in \mathcal{A}).
\]
Then $\Phi$ is a continuous linear map, and clearly $\|\Phi\| \leq \|\varphi\|$. For each $a, b \in \mathcal{A}$, we have
\[
\|\varphi(a, b) - \Phi(ab)\| = \|\varphi(a, b1_{\mathcal{A}}) - \varphi(ab, 1_{\mathcal{A}})\| \leq 8\varepsilon||a|| ||b||,
\]
as claimed.

Finally suppose that $\mathcal{Z}$ is the dual of a Banach space $\mathcal{Z}_*$. Let $\mathcal{U}$ be an ultrafilter on $J$ containing the order filter on $J$. It follows from the Banach-Alaoglu theorem that each bounded subset of $\mathcal{Z}$ is relatively compact with respect to the weak* topology on $\mathcal{Z}$. Consequently, each bounded net
\((z_j)_{j \in J}\) in \(Z\) has a unique limit with respect to the weak* topology along the ultrafilter \(U\), and we write \(\lim_U z_j\) for this limit. Further, it is worth noting that

\[
\left\| \lim_U z_j \right\| \leq \lim_U \| z_j \|. 
\]

Indeed, for each \(\zeta \in Z_*\) such that \(\| \zeta \| = 1\), we have \(\langle z_j, \zeta \rangle \leq \| z_j \| \) (\(j \in J\)), and hence

\[
\left\| \lim_U z_j, \zeta \right\| = \lim_U \langle z_j, \zeta \rangle \leq \lim_U \| z_j \|,
\]

which establishes (1.17).

For each \(a \in Z\), we have

\[
\| \varphi(a, e_j) \| \leq \| \varphi \| \| a \| \quad (j \in J),
\]

and hence the net \((\varphi(a, e_j))_{j \in J}\) is bounded. Consequently, we can define the map \(\Phi: A \rightarrow Z\) by

\[
\Phi(a) = \lim_U \varphi(a, e_j) \quad (a \in A).
\]

The linearity of the limit along an ultrafilter on a topological linear space gives the linearity of \(\Phi\). Take \(a \in A\). From (1.17) and (1.18) we deduce that \(\| \Phi(a) \| \leq \| \varphi \| \| a \|\), which gives the continuity of \(\Phi\) and \(\| \Phi \| \leq \| \varphi \|\). Now take \(a, b \in A\). We have

\[
\| \varphi(ab, e_j) - \varphi(a, be_j) \| \leq 8\varepsilon \| a \| \| b \| \quad (j \in J).
\]

Since \((be_j)_{j \in J} \rightarrow b\) in norm, the continuity of \(\varphi\) gives \((\varphi(a, be_j))_{j \in J} \rightarrow \varphi(a, b)\) in norm. Since \(U\) refines the order filter on \(J\) we see that \(\lim_U \varphi(a, be_j) = \varphi(a, b)\). Taking the limit in (1.19) along the ultrafilter \(U\), and using (1.17), we obtain \(\| \Phi(ab) - \varphi(a, b) \| \leq 8\varepsilon \| a \| \| b \|\), as required. \(\square\)

It is worth noting that Corollary 1.3 gives a sharper estimate for the constant of the strong property \(B\) of a \(C^*\)-algebra to the one given in [29, Theorem 3.4], where our constant 8 is replaced by \(384\pi^2(1 + \sqrt{2})\). The hyperreflexivity constant given in [29, Theorem 4.4] for \(C^*\)-algebras can be sharpened as well accordingly.

2. Primary estimates and reflexivity

2.1. Homomorphisms between modules over a \(C^*\)-algebra. Let \(A\) be a \(C^*\)-algebra, and let \(X\) and \(Y\) be quasi-Banach right \(A\)-modules. For a linear map \(T: X \rightarrow Y\) and \(a \in A\), define linear maps \(aT, Ta: X \rightarrow Y\) by

\[
(aT)(x) = T(ax), \quad (Ta)(x) = T(x)a \quad (x \in X).
\]

Then the space \(B(X, Y)\) of all continuous linear maps from \(X\) to \(Y\) is a quasi-Banach \(A\)-bimodule for the operations specified by (2.1). For \(T \in B(X, Y)\), let \(\text{ad}(T): A \rightarrow B(X, Y)\) denote the inner derivation implemented by \(T\), so that

\[
\text{ad}(T)(a) = aT - Ta \quad (a \in A).
\]
It is clear that $T$ is a right $\mathcal{A}$-module homomorphism if and only if $\text{ad}(T) = 0$, and, in the case where $\mathcal{X}$ and $\mathcal{Y}$ are Banach $\mathcal{A}$-modules, the constant
\begin{equation}
\| \text{ad}(T) \| \tag{2.2}
\end{equation}
is intended to estimate the distance from $T$ to the space $\text{Hom}_\mathcal{A}(\mathcal{X}, \mathcal{Y})$ of all continuous module homomorphisms from $\mathcal{X}$ to $\mathcal{Y}$. This is actually very much in the spirit of [7, 8, 9]. We will use several additional alternative ways to estimate the distance $\text{dist}(T, \text{Hom}_\mathcal{A}(\mathcal{X}, \mathcal{Y}))$ that equally make sense, namely
\begin{equation}
\alpha(T) = \sup \{\| e^* Te \| : e \in \mathcal{A} \text{ projection} \}, \tag{2.3}
\end{equation}
\begin{equation}
\beta(T) = \sup \{\| eTf \| : e, f \in \mathcal{A} \text{ projections, } ef = 0 \}, \tag{2.4}
\end{equation}
\begin{equation}
\gamma(T) = \sup \{\|aTb\| : a, b \in \mathcal{A}_+ \text{ contractions, } ab = 0 \}, \tag{2.5}
\end{equation}
\begin{equation}
\delta(T) = \sup_{x \in \mathcal{X}, \|x\| \leq 1} \inf \{\|T(x) - \Phi(x)\| : \Phi \in \text{Hom}_\mathcal{A}(\mathcal{X}, \mathcal{Y}) \}. \tag{2.6}
\end{equation}
For [2.3], the algebra is supposed to be unital, and this constant is actually very much in the spirit of the celebrated Arveson’s distance formula [6]. The significance of the constants (2.3) and (2.4) for our purposes requires that the $C^*$-algebra $\mathcal{A}$ to be sufficiently rich in projections (as does a $C^*$-algebra of real rank zero) and they are in the spirit of [19, 20]. We take the constant (2.6) in accordance with [21, 22].

Throughout we suppose that the Banach $\mathcal{A}$-modules are contractive. The statements of our results can be easily adapted to non-contractive Banach modules.

**Proposition 2.1.** Let $\mathcal{A}$ be a $C^*$-algebra, let $\mathcal{X}$ and $\mathcal{Y}$ be Banach right $\mathcal{A}$-modules, and let $T : \mathcal{X} \to \mathcal{Y}$ be a continuous linear map.

(i) $\beta(T) \leq \gamma(T) \leq \delta(T) \leq \text{dist}(T, \text{Hom}_\mathcal{A}(\mathcal{X}, \mathcal{Y}))$.

(ii) $\gamma(T) \leq \| \text{ad}(T) \| \leq 2 \text{dist}(T, \text{Hom}_\mathcal{A}(\mathcal{X}, \mathcal{Y}))$.

(iii) If $\mathcal{A}$ is unital, then $\alpha(T) = \beta(T)$.

(iv) If $\mathcal{A}$ is unital and has real rank zero, then $\beta(T) = \gamma(T)$.

**Proof.** (i) This is immediate.

(ii) Let $a, b \in \mathcal{A}_+$ contractions with $ab = 0$. Then $aTb = (aT - Ta)b$ and therefore $\|aTb\| \leq \|aT - Ta\| \leq \|\text{ad}(T)\|$. We thus get $\gamma(T) \leq \|\text{ad}(T)\|$.

Now take a sequence $(\Phi_n)$ in $\text{Hom}_\mathcal{A}(\mathcal{X}, \mathcal{Y})$ such that
\[
\text{dist}(T, \text{Hom}_\mathcal{A}(\mathcal{X}, \mathcal{Y})) = \lim_{n \to \infty} \|T - \Phi_n\|.
\]

Let $x \in \mathcal{X}$ and $a \in \mathcal{A}$ with $\|x\| = \|a\| = 1$. Then
\[
\|T(xa) - T(x)a\| = \|T(xa) - T(x)a - (\Phi_n(xa) - \Phi_n(x)a)\|
\leq \|T(xa) - \Phi_n(xa)\| + \|(T(x) - \Phi_n(x))a\|
\leq 2\|T - \Phi_n\|,
\]
which gives $\|(aT - Ta)(x)\| \leq 2 \text{dist}(T, \text{Hom}_\mathcal{A}(\mathcal{X}, \mathcal{Y}))$, and the second inequality is proved.
(iii) It suffices to prove that $\beta(T) \leq \alpha(T)$. Let $e, f \in \mathcal{A}$ mutually orthogonal projections. Then $e \leq f^\perp$, so that $eTf = e(f^\perp Tf)$ and therefore

$$
\|eTf\| = \|e(f^\perp Tf)\| \leq \|f^\perp Tf\| \leq \alpha(T),
$$

which implies that $\beta(T) \leq \alpha(T)$, as required.

(iv) It suffices to show that $\gamma(T) \leq \beta(T)$. Take $a, b \in \mathcal{A}_+$ mutually orthogonal contractions, and the task is to show that $\|aTb\| \leq \beta(T)$.

First, assume that both $a$ and $b$ have finite spectrum, say $\{\alpha_1, \ldots, \alpha_m\}$ and $\{\beta_1, \ldots, \beta_n\}$ with $0 \leq \alpha_1 < \cdots < \alpha_m = \|a\| \leq 1$ and $0 \leq \beta_1 < \cdots < \beta_n = \|b\| \leq 1$, and write

$$
a = \sum_{j=1}^m \alpha_j p_j, \quad b = \sum_{k=1}^n \beta_k q_k,
$$

where $p_1, \ldots, p_m, q_1, \ldots, q_n \in \mathcal{A}$ are mutually orthogonal projections, and, further,

$$
a = \sum_{j=1}^m \lambda_j e_j, \quad b = \sum_{k=1}^n \mu_k f_k,
$$

where $\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_n \in [0, \infty[$ and $e_1, \ldots, e_m, f_1, \ldots, f_n \in \mathcal{A}$ are defined by

$$
\lambda_1 = \alpha_1, \quad \mu_1 = \beta_1,
\lambda_j = \alpha_j - \alpha_{j-1} \ (1 < j \leq m), \quad \mu_k = \beta_k - \beta_{k-1} \ (1 < k \leq n),
\lambda_j = p_j + \cdots + p_m \ (1 \leq j \leq m), \quad f_k = q_k + \cdots + q_n \ (1 \leq k \leq n),
$$
as in the proof of Theorem 1.2. Since $e_j f_k = 0$ for all $j \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, n\}$, it follows that

$$
\|aTb\| = \left\| \sum_{j=1}^m \sum_{k=1}^n \lambda_j \mu_k e_j T f_k \right\| \leq \sum_{j=1}^m \sum_{k=1}^n \lambda_j \mu_k \|e_j T f_k\|
$$

$$
\leq \sum_{j=1}^m \sum_{k=1}^n \lambda_j \mu_k \beta(T) = \|a\| \|b\| \beta(T) \leq \beta(T),
$$
as required.

Now consider the general case. Since $\mathcal{A}$ has real rank zero, it follows that there exists a sequence $(c_n)$ in $\mathcal{A}_{sa}$ such that each $c_n$ has finite spectrum and $(c_n)$ converges to $a - b$ in norm. For each $n \in \mathbb{N}$, set

$$
a_n = \frac{1}{2} (|c_n| + c_n), \quad b_n = \frac{1}{2} (|c_n| - c_n).
$$

Since $(|c_n|) \to a + b$, we see that $(a_n) \to a$ and $(b_n) \to b$. Further, for each $n \in \mathbb{N}$, $a_n, b_n \in \mathcal{A}_+$, have finite spectra, and $a_n b_n = 0$. From the previous step we deduce that

$$
\|a_n T b_n\| \leq \beta(T) \|a_n\| \|b_n\| \quad (n \in \mathbb{N}),
$$
and so, taking limits on both sides of the above inequality, we obtain \( \|aTb\| \leq \beta(T) \), which completes the proof.

Now let \( \mathcal{A} \) be a \( C^* \)-algebra, and let \( \mathcal{X} \) a Banach right \( \mathcal{A} \)-module. Then \( \mathcal{X}^* \) is a Banach left \( \mathcal{A} \)-module with respect to the module operation specified by
\[
\langle x, a\phi \rangle = \langle xa, \phi \rangle \quad (\phi \in \mathcal{X}^*, a \in \mathcal{A}, x \in \mathcal{X}).
\]
This module has the property that the map \( \phi \mapsto a\phi \) from \( \mathcal{X}^* \) to \( \mathcal{X}^* \) is weak* continuous for each \( a \in \mathcal{A} \). Similarly, if \( \mathcal{Y} \) is a Banach left \( \mathcal{A} \)-module, then \( \mathcal{X}^* \) is a Banach right \( \mathcal{A} \)-module with respect to the module operation specified by
\[
\langle x, \phi a \rangle = \langle ax, \phi \rangle \quad (\phi \in \mathcal{X}^*, a \in \mathcal{A}, x \in \mathcal{X}),
\]
and the map \( \phi \mapsto \phi a \) from \( \mathcal{X}^* \) to \( \mathcal{X}^* \) is weak* continuous for each \( a \in \mathcal{A} \).

**Theorem 2.2.** Let \( \mathcal{A} \) be a \( C^* \)-algebra, let \( \mathcal{X} \) and \( \mathcal{Y} \) be Banach right \( \mathcal{A} \)-modules with \( \mathcal{X} \) essential, and let \( T: \mathcal{X} \to \mathcal{Y} \) be a continuous linear map.

(i) Suppose that \( \{ y \in \mathcal{Y} : yA = 0 \} = \{ 0 \} \) and that
\[
a, b \in \mathcal{A}_+, \quad ab = 0 \implies aTb = 0.
\]
Then \( T \) is a right \( \mathcal{A} \)-module homomorphism.

(ii) Suppose that \( \mathcal{Y} \) satisfies the condition
\[
\|y\| = \sup \{ \|ya\| : a \in \mathcal{A}, \|a\| = 1 \}
\]
for each \( y \in \mathcal{Y} \). Then
\[
\|\text{ad}(T)\| \leq 8 \sup \{ \|aTb\| : a, b \in \mathcal{A}_+ \text{ contractions}, ab = 0 \},
\]
Further, the above condition holds in each of the following cases:

(a) \( \mathcal{Y} \) is essential;

(b) \( \mathcal{Y} \) is the dual of an essential Banach left \( \mathcal{A} \)-module.

**Proof.** Define the continuous bilinear map \( \varphi: \mathcal{A} \times \mathcal{A} \to B(\mathcal{X}, \mathcal{Y}) \) by
\[
\varphi(a, b) = aTb \quad (a, b \in \mathcal{A}),
\]
and set \( \varepsilon = \sup \{ \|aTb\| : a, b \in \mathcal{A}_+ \text{ contractions}, ab = 0 \} \). Then \( \|\varphi(a, b)\| \leq \varepsilon \|a\| \|b\| \) whenever \( a, b \in \mathcal{A}_+ \) are such that \( ab = 0 \).

Let \( (e_i)_{i \in I} \) be an approximate identity for \( \mathcal{A} \) of bound one. As in the proof of Corollary 1, we see that \( (e_i)_{i \in I} \) has a subnet \( (e_j)_{j \in J} \) which converges to \( 1_{\mathcal{A}^{**}} \) in \( \mathcal{A}^{**} \) with respect to the weak* topology.

From Theorem 1 it follows that, for each \( a \in \mathcal{A} \), the nets \( (\varphi(a, e_j))_{j \in J} \) and \( (\varphi(e_j, a))_{j \in J} \) converge in \( B(\mathcal{X}, \mathcal{Y})^{**} \) with respect to the weak* topology and
\[
\left\| \lim_{j \in J} \varphi(a, e_j) - \lim_{j \in J} \varphi(e_j, a) \right\| \leq 8\varepsilon \|a\|,
\]
whence
\[
\left\| \lim_{j \in J} aTe_j - \lim_{j \in J} e_j Ta \right\| \leq 8\varepsilon \|a\|.
\]
In particular, for each \( x \in \mathcal{X} \), \( a, b \in \mathcal{A} \), and \( \phi \in \mathcal{Y}^* \), the net
\[
\{ \langle b\phi, (aTe_j - e_j Ta)(x) \rangle \}_{j \in J} = \{ \langle \phi, T(xa) - T(xe_j ab) \rangle \}_{j \in J}
\]
converges and

\[
\lim_{j \in J} \langle \phi, T(xa)e_j b - T(xe_j ab) \rangle \leq 8\varepsilon \|a\|\|b\|\|x\|\|\phi\|.
\]

Since $X$ is essential, it follows that $(xe_j)_{j \in J} \to x$ in norm for each $x \in X$. Thus \eqref{eq:2.7} gives

\[
\|T(xa) - T(xa)b\| \leq 8\varepsilon \|a\|\|b\|\|x\|
\]

for all $x \in X$, $a, b \in A$, and $\phi \in Y^*$, and hence

\[
\|T(xa) - T(x)ab\| \leq 8\varepsilon \|a\|\|b\|\|x\|
\]

for all $x \in X$, and $a, b \in A$.

(i) In this case, we have $\varepsilon = 0$ and, for each $x \in X$ and $a \in A$, \eqref{eq:2.8} gives

\[
(T(xa) - T(x)ab) = 0 \quad (b \in A),
\]

which yields $T(xa) = T(x)ab$. Hence $T$ is a right $A$-module homomorphism.

(ii) In this case, for each $x \in X$ and $a \in A$, \eqref{eq:2.8} gives

\[
\|T(xa) - T(xa)b\| \leq 8\varepsilon \|a\|\|b\|\|x\|
\]

so that $\|\text{ad}(T)\| \leq 8\varepsilon$, as claimed.

Now suppose that $Y$ satisfies either of the additional assumptions (a) or (b); we will prove that $\|y\| = \sup \{\|ya\| : a \in A, \|a\| = 1\}$ for each $y \in Y$. Take $y \in Y$, and set $\alpha = \sup \{\|ya\| : a \in A, \|a\| = 1\}$. It is clear that $\alpha \leq \|y\|$

In case (a), since $(ye_j)_{j \in J} \to y$, it follows that $(\|ye_j\|)_{j \in J} \to \|y\|$, and consequently $\|y\| \leq \alpha$.

In case (b), $Y$ is the dual of an essential Banach left $A$-module $Y^*$. Take $\varepsilon > 0$, and let $\phi \in Y^*$ with $\|\phi\| = 1$ and $\|y\| - \varepsilon < \|\langle \phi, y \rangle\|$. Then $(e_j \phi)_{j \in J} \to \phi$ and the continuity of $\phi$ gives $(\langle \phi, ye_j \rangle)_{j \in J} = (\langle e_j \phi, y \rangle)_{j \in J} \to \langle \phi, y \rangle$, which implies that there exists $j \in J$ such that $\|y\| - \varepsilon < \|\langle \phi, ye_j \rangle\| \leq \alpha$. We thus get $\|y\| \leq \alpha + \varepsilon$ for each $\varepsilon > 0$, and hence $\|y\| \leq \alpha$, which completes the proof.

\section*{Theorem 2.3}

Let $A$ be a unital $C^*$-algebra of real rank zero, let $X$ and $Y$ be unital quasi-Banach right $A$-modules, and let $T : X \to Y$ be a continuous linear map.

(i) Suppose that

\[
e \in A \text{ projection} \implies e^\perp Te = 0.
\]

Then $T$ is a right $A$-module homomorphism.

(ii) Suppose that $X$ and $Y$ are Banach modules. Then

\[
\|\text{ad}(T)\| \leq 8 \sup \{\|e^\perp Te\| : e \in A \text{ projection}\}.
\]

\section*{Proof}

Define the continuous bilinear map $\varphi : A \times A \to B(X, Y)$ by

\[
\varphi(a, b) = aTb \quad (a, b \in A).
\]
(i) In this case, \( \varphi(e, e^\perp) = 0 \) for each projection \( e \in \mathcal{A} \). Consequently, Theorem 1.2(i) shows that \( \varphi(a, 1_\mathcal{A}) = \varphi(1_\mathcal{A}, a) \) for each \( a \in \mathcal{A} \), which gives \( aT = Ta \), and this is precisely the assertion of (i).

(ii) Set \( \varepsilon = \sup \{ \|e^\perp Te\| : e \in \mathcal{A} \text{ projection} \} \). Then \( \|\varphi(e, e^\perp)\| \leq \varepsilon \) for each projection \( e \in \mathcal{A} \), and Theorem 1.2(ii) now shows that

\[
\|aT - Ta\| = \|\varphi(a, 1_\mathcal{A}) - \varphi(1_\mathcal{A}, a)\| \leq 8\varepsilon\|a\|
\]

for each \( a \in \mathcal{A} \). We thus get \( \|\text{ad}(T)\| \leq 8\varepsilon \), as claimed. \( \square \)

**Theorem 2.4.** Let \( \mathcal{A} \) be a C*-algebra of real rank zero, let \( \mathcal{X} \) and \( \mathcal{Y} \) be Banach right \( \mathcal{A} \)-modules with \( \mathcal{X} \) essential, and let \( T : \mathcal{X} \to \mathcal{Y} \) be a continuous linear map.

(i) Suppose that \( \{y \in \mathcal{Y} : y\mathcal{A} = 0\} = \{0\} \) and that \( e, f \in \mathcal{A} \) projections, \( ef = 0 \implies eTf = 0 \).

Then \( T \) is a right \( \mathcal{A} \)-module homomorphism.

(ii) Suppose that \( \mathcal{Y} \) satisfies the condition

\[
\|y\| = \sup \{\|ya\| : a \in \mathcal{A}, \|a\| = 1\}
\]

for each \( y \in \mathcal{Y} \). Then

\[
\|\text{ad}(T)\| \leq 8\sup \{\|eTf\| : e, f \in \mathcal{A} \text{ projections, } ef = 0\}.
\]

Further, the above condition holds in each of the following cases:

(a) \( \mathcal{Y} \) is essential;
(b) \( \mathcal{Y} \) is the dual of an essential Banach left \( \mathcal{A} \)-module.

**Proof.** Take an approximate identity \( (e_j)_{j \in J} \) for \( \mathcal{A} \) consisting of projections. Fix \( j \in J \), define the continuous bilinear map \( \varphi_j : e_j\mathcal{A}e_j \times e_j\mathcal{A}e_j \to B(\mathcal{X}, \mathcal{Y}) \) by

\[
\varphi_j(a, b) = aTb \quad (a, b \in e_j\mathcal{A}e_j),
\]

and set \( \varepsilon = \sup \{ \|eTf\| : e, f \in \mathcal{A} \text{ projections, } ef = 0 \} \). Then \( e_j\mathcal{A}e_j \) is a unital C*-algebra (with unit \( e_j \)) and has real rank zero. Further, \( \|\varphi(e, e_j - e)\| \leq \varepsilon \) for each projection \( e \in e_j\mathcal{A}e_j \). From Theorem 1.2(ii) it follows that

\[
\|aTe_j - e_jTa\| = \|\varphi(a, e_j) - \varphi(e_j, a)\| \leq 8\varepsilon\|a\|
\]

for each \( a \in e_j\mathcal{A}e_j \). Hence

\[
(2.9) \quad \|T(xe_jae_j)e_jb - T(xe_j)e_jae_jb\| \leq 8\varepsilon\|x\||a||b|
\quad (j \in J, x \in \mathcal{X}, a, b \in \mathcal{A}).
\]

For each \( x \in \mathcal{X} \) and \( a, b \in \mathcal{A} \), we have

- \( (e_jae_j)_{j \in J} \to a \) and \( (e_jb)_{j \in J} \to b \) in norm, so that (using the continuity of \( T \)) \( (T(xe_jae_j)e_jb)_{j \in J} \to T(xa)b \) in norm;
- \( (xe_j)_{j \in J} \to x \) in norm, because \( \mathcal{X} \) is essential, and \( (e_jae_jb)_{j \in J} \to ab \) in norm, and hence (using the continuity of \( T \)) \( (T(xe_j)e_jae_jb)_{j \in J} \to T(x)ab \) in norm.
Thus, taking the limit in (2.9) we see that
\[ \|T(x)a - T(x)ab\| \leq 8\varepsilon\|x\|\|a\||\|b\| \quad (x \in \mathcal{X}, \ a, b \in \mathcal{A}). \]
The rest of the proof goes through as for Theorem 2.2 (from inequality (2.8) on).

**Corollary 2.5.** Let \( \mathcal{A} \) be a \( C^* \)-algebra, and let \( \mathcal{X} \) and \( \mathcal{Y} \) be Banach right \( \mathcal{A} \)-modules. Suppose that \( \mathcal{X} \) is essential and that \( \{y \in \mathcal{Y} : y\mathcal{A} = 0\} = \{0\} \). Then the space \( \text{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}) \) is reflexive.

**Proof.** Take \( T \in B(\mathcal{X}, \mathcal{Y}) \) such that
\[ T(x) \in \{ \Phi(x) : \Phi \in \text{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}) \} \quad (x \in \mathcal{X}). \]
Let \( a, b \in \mathcal{A} \) be such that \( ab = 0 \). We claim that \( aTb = 0 \). For each \( x \in \mathcal{X} \), there exists a sequence \( (\Phi_n) \) in \( \text{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}) \) such that \( (\Phi_n(xa)) \to T(xa) \) in norm, and hence
\[ (aTb)(x) = T(xa)b = \lim_{n \to \infty} \Phi_n(xa)b = \lim_{n \to \infty} \Phi_n(xab) = 0, \]
which proves our claim.

From Theorem 2.2(i), it follows that \( T \in \text{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}) \). \( \square \)

**Corollary 2.6.** Let \( \mathcal{A} \) be a unital \( C^* \)-algebra of real rank zero, and let \( \mathcal{X} \) and \( \mathcal{Y} \) be unital quasi-Banach right \( \mathcal{A} \)-modules. Then the space \( \text{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}) \) is reflexive.

**Proof.** This follows by the same method as in Corollary 2.5 with Theorem 2.2(i) replaced by Theorem 2.3(i). \( \square \)

### 2.2. Homomorphisms between non-commutative \( L^p \)-spaces

Let \( \mathcal{M} \) be a von Neumann algebra. For each \( 0 < p \leq \infty \), the space \( L^p(\mathcal{M}) \) is a contractive Banach \( \mathcal{M} \)-bimodule or a contractive \( p \)-Banach \( \mathcal{M} \)-bimodule according to \( p \geq 1 \) or \( p < 1 \). More generally, if \( 0 < p, q, r \leq \infty \) are such that \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r} \) (we adopt throughout the convention that \( \frac{1}{\infty} = 0 \)), then
\[ x \in L^p(\mathcal{M}), \ y \in L^q(\mathcal{M}) \implies xy \in L^r(\mathcal{M}) \quad \text{and} \quad ||xy||_r \leq ||x||_p ||y||_q. \]
This is the non-commutative Hölder’s inequality. From now on we confine attention to the right \( \mathcal{M} \)-module structure of \( L^p(\mathcal{M}) \).

**Theorem 2.7.** Let \( \mathcal{M} \) be a von Neumann algebra, let \( 0 < p, q \leq \infty \), and let \( T : L^p(\mathcal{M}) \to L^q(\mathcal{M}) \) be a linear map. Suppose that the map \( e^\perp T e : L^p(\mathcal{M}) \to L^q(\mathcal{M}) \) is continuous for each projection \( e \in \mathcal{M} \). Then \( T \) is continuous.

**Proof.** We first observe that \( eT - Te = eTe^\perp - e^\perp Te \) is continuous for each projection \( e \in \mathcal{M} \).

Now we consider the separating space of \( T \), which is defined by
\[ S(T) = \{ y \in L^q(\mathcal{M}) : \text{there exists } (x_n) \to 0 \text{ in } L^p(\mathcal{M}) \text{ with } (T(x_n)) \to y \}. \]
It is an immediate restatement of the closed graph theorem that \( T \) is continuous if and only if \( S(T) = 0 \).
We claim that \( S(T) \) is a closed right submodule of \( L^q(M) \). By \([10] \) Proposition 5.1.2, \( S(T) \) is a closed subspace of \( L^q(M) \). Let \( y \in S(T) \), and let \( e \) be a projection in \( M \). Take a sequence \( (x_n) \) in \( L^p(M) \) with \( \lim x_n = 0 \) and \( \lim T(x_n) = y \). Then \( \lim x_ne = 0 \) and, using the first observation,
\[
T(x_ne) = (eT - Te)(x_n) + T(x_n)e \to ye.
\]
Thus \( ye \in S(T) \). Now let \( a \in M \) be an arbitrary element. Then there exists a sequence \( (a_n) \) in \( M \) such that each \( a_n \) is a linear combination of projections and \( \lim a_n = a \). Since \( S(T) \) is a closed subspace of \( L^q(M) \), it follows that \( ya_n \in S(T) \) \( (n \in \mathbb{N}) \) and that \( ya = \lim ya_n \in S(T) \). Hence \( S(T) \) is a right submodule of \( L^q(M) \), as claimed.

We now consider the subspace \( I(T) \) defined by
\[
I(T) = \{ a \in M : S(T)a = 0 \}.
\]
It is clear that \( I(T) \) is a closed right ideal of \( M \). Further, since \( S(T) \) is a right submodule of \( L^q(M) \), it follows immediately that \( I(T) \) is also a left ideal of \( M \). Our next goal is to prove that \( I(T) \) is weak* closed in \( M \). Take \( y \in L^q(M) \), and let \( s_r(y) \) be the right support projection of \( y \). Then
\[
\{ a \in M : ya = 0 \} = \{ a \in M : s_r(y)a = 0 \}
\]
(see \([26] \) Fact 1.2(ii)) and, since \( s_r(y) \in M \), this latter set is clearly weak* closed in \( M \). Since
\[
I(T) = \bigcap_{y \in S(T)} \{ a \in M : ya = 0 \},
\]
we conclude that \( I(T) \) is weak* closed.

Since \( I(T) \) is a weak* closed two-sided ideal of \( M \), it follows that there exists a central projection \( e \in M \) such that
\[
I(T) = eM.
\]
We now claim that \( \dim e^\perp M < \infty \). Assume towards a contradiction that \( \dim e^\perp M = \infty \). Then we can take a sequence \( (e_n) \) of non-zero mutually orthogonal projections in \( e^\perp M \). For \( n \in \mathbb{N} \), we define the projection \( f_n \in M \) by
\[
f_n = \bigvee_{k \geq n} e_k,
\]
and consider the maps \( R_n \in B(L^p(M), L^p(M)) \) and \( S_n \in B(L^q(M), L^q(M)) \) defined by
\[
R_n(x) = xf_n, \quad S_n(y) = yf_n \quad (x \in L^p(M), \quad y \in L^q(M)).
\]
Our next objective is to apply a fundamental result about the separating space: the so-called stability lemma. By hypothesis, \( TR_n - S_nT \) is continuous for each \( n \in \mathbb{N} \), and hence, by \([10] \) Corollary 5.2.7, \( (S_1 \cdots S_n(S(T))) \) is a nest in \( L^q(M) \) which stabilizes. Since \( f_{n+1} \leq f_n \) for each \( n \in \mathbb{N} \), it follows that \( S_1 \cdots S_n = S_n \), and hence that
\[
\overline{S_1 \cdots S_n(S(T))} = S(T)f_n = S(T)f_n
\]
for each $n \in \mathbb{N}$. Thus there exists $N \in \mathbb{N}$ such that
\[ S(T)f_N = S(T)f_n \quad (N \leq n). \]
In particular, since $f_N e_N = e_N$ and $f_{N+1} e_N = 0$, we have
\[ S(T)e_N = (S(T)f_N)e_N = (S(T)f_{N+1})e_N = 0. \]
Hence $e_N \in \mathcal{I}(T) = e\mathcal{M}$. But this is a contradiction of the facts that $e_N \in e\mathcal{M}$ and $e_N \neq 0$.

Our next claim is that the map $T e^\bot : L^p(\mathcal{M}) \to L^q(\mathcal{M})$ is continuous. Since the projection $e^\bot$ is central, we see that $e^\bot x = xe^\bot$ for each $x \in L^p(\mathcal{M})$, and hence $e^\bot L^p(\mathcal{M}) = e^\bot L^p(\mathcal{M})e^\bot$. Moreover, \cite{Fact1.4} shows that the subspace $e^\bot L^p(\mathcal{M})e^\bot$ is isometrically isomorphic to $L^p(e^\bot \mathcal{M} e^\bot)$. Since $\dim e^\bot \mathcal{M} < \infty$, it follows that $\dim L^p(e^\bot \mathcal{M} e^\bot) < \infty$, so that $\dim e^\bot L^p(\mathcal{M}) < \infty$. Thus the restriction of $T$ to the subspace $e^\bot L^p(\mathcal{M})$ is continuous, and hence the map
\[ e^\bot T : L^p(\mathcal{M}) \to L^q(\mathcal{M}), \ x \mapsto T(xe^\bot) \]
is continuous. On the other hand,
\[ Te^\bot = e^\bot T - (e^\bot T - Te^\bot), \]
which implies that $Te^\bot$ is continuous, as claimed.

Finally, we are in a position to prove the continuity of $T$. From the above claim we deduce that $S(T)e^\bot = 0$, and hence that $e^\bot \in \mathcal{I}(T) = e\mathcal{M}$. This implies that $e^\bot = 0$, whence $1_{\mathcal{M}} = e \in \mathcal{I}(T)$, which gives $S(T) = S(T)1_{\mathcal{M}} = 0$ and $T$ is continuous.

**Corollary 2.8.** Let $\mathcal{M}$ be a von Neumann algebra, let $0 < p,q \leq \infty$, and let $T : L^p(\mathcal{M}) \to L^q(\mathcal{M})$ be a right $\mathcal{M}$-module homomorphism. Then $T$ is continuous.

**Proof.** It is clear that $T$ satisfies the requirement in Theorem 2.7 and hence $T$ is continuous. \qed

Suppose that $0 < p,q,r \leq \infty$ are such that $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$. By Hölder’s inequality, for each $\xi \in L^r(\mathcal{M})$, we can define the left composition map $L_\xi : L^p(\mathcal{M}) \to L^q(\mathcal{M})$ by
\[ L_\xi(x) = \xi x \quad (x \in L^p(\mathcal{M})). \]
Further $L_\xi$ is continuous with $\|L_\xi\| \leq \|\xi\|_r$, and it is obvious that $L_\xi$ is a right $\mathcal{M}$-module homomorphism.

**Theorem 2.9.** Let $\mathcal{M}$ be a von Neumann algebra, and let $0 < p,q \leq \infty$.

(i) Suppose that $p \geq q$, and let $0 < r \leq \infty$ be such that $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$. Then the map
\[ \xi \mapsto L_\xi, \ L^r(\mathcal{M}) \to \text{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M})) \]
is an isometric linear bijection.
Proof. (i) By \cite[Theorem 2.5]{[18]}, this map is a surjection. We proceed to show that it is an isometry. Let $\xi \in L^r(\mathcal{M}) \setminus \{0\}$. We have already seen that $\|L_\xi\| \leq \|\xi\|_r$. We now establish the reverse inequality by considering three cases.

Assume that $p = \infty$. Then $r = q$ and

$$\|\xi\|_r = \|L_\xi(1_{\mathcal{M}})\|_q \leq \|L_\xi\|\|1_{\mathcal{M}}\| = \|L_\xi\|,$$

as required.

Now assume that $p < \infty$ and that $r = \infty$. Then $p = q$, and, for each $x \in L^p(\mathcal{M})$, we have

$$\|L_\xi(x)\|_p = \|\xi x\|_p = \|\langle \xi x \rangle^* (\xi x)\|^{1/2}_{p/2} = \|x^* \xi x\|^{1/2}_{p/2}$$

$$= \|x^*\|^{1/2}_{p/2} \|\langle \xi x \rangle^* (\xi x)\|^{1/2}_{p/2} = \|\langle \xi x \rangle\|_p = \|L_\xi(x)\|_p.$$

Thus $\|L_\xi\| = \|L_{|\xi|}\|$, and \cite[Lemma 2.1]{[18]} shows that $\|L_{|\xi|}\| = \|\xi\| = \|\xi\|_r$.

Finally, assume that $p, r < \infty$. Then $|\xi|^{r/p} \in L^p(\mathcal{M})$, and we have

$$\|L_\xi(|\xi|^{r/p})\|_q = \|\xi|x|^{r/p}\|_q = \|\xi|^{r/p} x^{*} \xi|^{r/p}\|^{1/2}_{q/2}$$

$$= \|\xi|^{2(1-r/p)}\|^{1/2}_{q/2} = \|\xi|^{2r/q}\|^{1/2}_{q/2} = \|\xi\|^{r/q}.$$

On the other hand, we have

$$\|L_\xi(|\xi|^{r/p})\|_q \leq \|L_\xi\|\|\xi|^{r/p}\|_p = \|L_\xi\|\|\xi\|^{r/p},$$

and hence

$$\|\xi\|_r = \|\xi|^{r/q-r/p}\| \leq \|L_\xi\|,$$

as required.

(ii) \cite[Corollary 2.7]{[18]}. \hfill $\square$

**Corollary 2.10.** Let $\mathcal{M}$ be a von Neumann algebra, let $0 < p, q \leq \infty$, and let $T : L^p(\mathcal{M}) \to L^q(\mathcal{M})$ be a continuous linear map.

(i) Suppose that

$$e \in \mathcal{M} \text{ projection} \implies e^T e = 0.$$

Then $T$ is a right $\mathcal{M}$-module homomorphism.

(ii) Suppose that $1 \leq p, q \leq \infty$. Then

$$\|\text{ad}(T)\| \leq 8 \sup\{\|e^T e\| : e \in \mathcal{M} \text{ projection}\}.$$

**Proof.** This follows from Theorem \cite{[2.3]}. \hfill $\square$

By Corollary \cite{[2.6]}, the space $\text{Hom}_\mathcal{M}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ is reflexive. However we next show that this space is not merely reflexive.
Corollary 2.11. Let $\mathcal{M}$ be a von Neumann algebra, let $0 < p, q \leq \infty$, and let $T: L^p(\mathcal{M}) \to L^q(\mathcal{M})$ be a linear map such that

$$T(x) \in \{ \Phi(x) : \Phi \in \mathrm{Hom}_\mathcal{M}(L^p(\mathcal{M}), L^q(\mathcal{M})) \} \quad (x \in L^p(\mathcal{M})).$$

Then $T \in \mathrm{Hom}_\mathcal{M}(L^p(\mathcal{M}), L^q(\mathcal{M}))$. In particular, the space $\mathrm{Hom}_\mathcal{M}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ is reflexive.

Proof. Take a projection $e \in \mathcal{M}$. For each $x \in L^p(\mathcal{M})$, there exists a sequence $(\Phi_n)$ in $\mathrm{Hom}_\mathcal{M}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ such that $T(xe^\perp) = \lim \Phi_n(xe^\perp)$ in norm, and hence

$$(e^\perp Te)(x) = T(xe^\perp)e = \lim_{n \to \infty} \Phi_n(xe^\perp)e = \lim_{n \to \infty} \Phi_n(x)e^\perp e = 0.$$ 

This shows that $e^\perp Te = 0$.

From Theorem 2.7, it follows that $T$ is continuous, and Corollary 2.10 then gives $T \in \mathrm{Hom}_\mathcal{M}(L^p(\mathcal{M}), L^q(\mathcal{M})).$ \hfill \(\Box\)

3. Distance estimates

3.1. Homomorphisms between modules over a $C^*$-algebra. Let $\mathcal{M}$ be a von Neumann algebra, and let $\mathcal{X}$ a Banach right $\mathcal{M}$-module. Then the Banach left $\mathcal{M}$-module $\mathcal{X}^*$ is called normal if the map $a \mapsto a\phi$ from $\mathcal{M}$ to $\mathcal{X}^*$ is weak* continuous for each $\phi \in \mathcal{X}^*$. Similarly, if $\mathcal{X}$ is a Banach left $\mathcal{M}$-module, then the Banach right $\mathcal{M}$-module $\mathcal{X}^*$ is called normal if the map $a \mapsto \phi a$ from $\mathcal{M}$ to $\mathcal{X}^*$ is weak* continuous for each $\phi \in \mathcal{X}^*$.

Theorem 3.1. Let $\mathcal{M}$ be an injective von Neumann algebra, let $\mathcal{X}$ and $\mathcal{Y}$ be Banach right and left $\mathcal{M}$-modules, respectively, and let $T: \mathcal{X} \to \mathcal{Y}^*$ be a continuous linear map.

(i) The subset $W(T)$ of $\mathcal{X}$ consisting of the elements $x \in \mathcal{X}$ with the property that the bilinear map $(a, b) \mapsto T(xa)b$ from $\mathcal{M} \times \mathcal{M}$ to $\mathcal{Y}^*$ is separately weak* continuous is a closed submodule of $\mathcal{X}$. Further, if the modules $\mathcal{X}^*$ and $\mathcal{Y}^*$ are normal, then $W(T) = \mathcal{X}$.

(ii) There exists a continuous linear map $\Phi: \mathcal{X} \to \mathcal{Y}^*$ such that:

(a) $\|\Phi\| \leq \|T\|$;

(b) $\Phi(xa) = \Phi(x)a$ for all $x \in W(T)$ and $a \in \mathcal{M}$; in particular, if both $\mathcal{X}^*$ and $\mathcal{Y}^*$ are normal, then $\Phi$ is a right $\mathcal{M}$-module homomorphism;

(c) $\|1_\mathcal{M}T - \Phi\| \leq \|\mathrm{ad}(T)\|$; in particular, if the module $\mathcal{X}$ is unital, then $\|T - \Phi\| \leq \|\mathrm{ad}(T)\|$;

(d) $\|T1_\mathcal{M} - \Phi\| \leq \|\mathrm{ad}(T)\|$; in particular, if the module $\mathcal{Y}$ is unital, then $\|T - \Phi\| \leq \|\mathrm{ad}(T)\|$.

Proof. (i) Routine verifications show that $W(T)$ is a submodule of $\mathcal{X}$. To show that $W(T)$ is closed, take a sequence $(x_n)$ in $W(T)$ and $x \in \mathcal{X}$ such that $(x_n) \to x$ in norm. We define continuous bilinear maps $\tau, \tau_n: \mathcal{M} \times \mathcal{M} \to \mathcal{Y}^*$ by

$$\tau(a, b) = T(xa)b, \quad \tau_n(a, b) = T(x_na)b \quad (a, b \in \mathcal{M}, \ n \in \mathbb{N}).$$
Then \((\tau_n) \to \tau\) in norm, and each \(\tau_n\) is separately weak* continuous. This implies that \(\tau\) is separately weak* continuous, which shows that \(x \in \mathcal{W}(T)\).

Suppose that both \(\mathcal{X}^*\) and \(\mathcal{Y}^*\) are normal, and take \(x \in \mathcal{X}\). For each \(a \in \mathcal{M}\), \(T(xa) \in \mathcal{Y}^*\), so that, by definition, the map \(b \mapsto T(xa)b\) from \(\mathcal{M}\) to \(\mathcal{Y}^*\) is weak* continuous. For each \(b \in \mathcal{M}\) and each \(y \in \mathcal{Y}\), define \(\phi_{b,y} \in \mathcal{X}^*\) by

\[
\langle x, \phi_{b,y} \rangle = \langle y, T(x) b \rangle \quad (x \in \mathcal{X}).
\]

Then

\[
\langle y, T(xa)b \rangle = \langle x, a \phi_{b,y} \rangle \quad (x \in \mathcal{X}, a \in \mathcal{M}),
\]

and, since the map \(a \mapsto a \phi_{b,y}\) is weak* continuous, it follows that the functional \(a \mapsto \langle y, T(xa)b \rangle\) is weak* continuous for each \(x \in \mathcal{X}\).

(ii) Let \(G\) be the discrete semigroup of the isometries of \(\mathcal{M}\). A mean on \(G\) is a state \(\mu\) on \(\ell^\infty(G)\) and, for a given mean \(\mu\), we use the formal notation

\[
\int_G \phi(u) \, d\mu(u) := \langle \phi, \mu \rangle \quad (\phi \in \ell^\infty(G)).
\]

By [15, Theorem 2.1], there exists a mean \(\mu\) on \(G\) with the property that

\[
\int_G \tau(au^*, u) \, d\mu(u) = \int_G \tau(u^*, ua) \, d\mu(u)
\]

for each separately weak* continuous bilinear functional \(\tau: \mathcal{M} \times \mathcal{M} \to \mathbb{C}\) and each \(a \in \mathcal{M}\).

Define \(\Phi: \mathcal{X} \to \mathcal{Y}^*\) by

\[
\langle y, \Phi(x) \rangle = \int_G \langle y, T(xu^*)u \rangle \, d\mu(u) \quad (x \in \mathcal{X}, y \in \mathcal{Y}).
\]

Then \(\Phi\) is well-defined and linear. Further, for each \(x \in \mathcal{X}, y \in \mathcal{Y},\) and \(u \in G\), we have

\[
|\langle y, T(xu^*)u \rangle| \leq \|y\| \|T(xu^*)u\| \leq \|y\| \|T(xu^*)\| \leq \|y\| \|T\| \|xu^*\| \leq \|y\| \|T\| \|x\|
\]

which implies that

\[
\left| \int_G \langle y, T(xu^*)u \rangle \, d\mu(u) \right| \leq \|T\| \|x\| \|y\|
\]

and hence that \(\|\Phi(x)\| \leq \|T\| \|x\|\). Thus \(\Phi\) is continuous and (a) holds.

Let \(x \in \mathcal{W}(T), a \in \mathcal{M},\) and \(y \in \mathcal{Y}\). Since, by definition, the bilinear functional \((u, v) \mapsto \langle y, T(xu)v \rangle\) is separately weak* continuous, it follows from [31] that

\[
\langle y, \Phi(xa) \rangle = \int_G \langle y, T(xau^*)u \rangle \, d\mu(u) = \int_G \langle y, T(xu^*)ua \rangle \, d\mu(u)
\]

\[
= \int_G \langle ay, T(xu^*)u \rangle \, d\mu(u) = \langle ay, \Phi(x) \rangle = \langle y, \Phi(x)a \rangle.
\]

This establishes (b).
Now let $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ with $\|x\| = \|y\| = 1$. Then
\[
\langle y, T(x1_M) - \Phi(x) \rangle = \left| \int_G \langle y, T(x1_M) - T(xu^*)u \rangle \, d\mu(u) \right|
= \left| \int_G \langle y, T(xu^*)u - T(xu^*)u \rangle \, d\mu(u) \right|
\leq \int_G \|T(xu^*)u - T(xu^*)u\| \, d\mu(u)
\leq \int_G \|\text{ad}(T)||xu^*||u\| \, d\mu(u)
\leq \int_G \|\text{ad}(T)||x||u^*||u\| \, d\mu(u) = \|\text{ad}(T)\|.
\]
This gives (c). We also have
\[
\langle y, T(x)1_M - \Phi(x) \rangle = \left| \int_G \langle y, T(x)1_M - T(xu^*)u \rangle \, d\mu(u) \right|
= \left| \int_G \langle y, T(xu^*)u - T(xu^*)u \rangle \, d\mu(u) \right|
\leq \int_G \|T(xu^*)u - T(xu^*)u\| \, d\mu(u)
\leq \int_G \|\text{ad}(T)||xu^*||u\| \, d\mu(u)
\leq \int_G \|\text{ad}(T)||x||u^*||u\| \, d\mu(u) = \|\text{ad}(T)\|,
\]
and this gives (d). \qed

**Theorem 3.2.** Let $\mathcal{A}$ be a nuclear $C^*$-algebra, let $\mathcal{X}$ and $\mathcal{Y}$ be Banach right and left $\mathcal{A}$-modules, respectively, and let $T: \mathcal{X} \to \mathcal{Y}^*$ be a continuous linear map. Then there exists a continuous right $\mathcal{A}$-module homomorphism $\Phi: \mathcal{X} \to \mathcal{Y}^*$ such that:

(a) $\|\Phi\| \leq \|T\|$;
(b) $\|aT - a\Phi\| \leq \|\text{ad}(T)||a\| \ (a \in \mathcal{A})$; moreover, if the module $\mathcal{X}$ is essential, then $\|T - \Phi\| \leq \|\text{ad}(T)\|$;
(c) $\|Ta - \Phi u\| \leq \|\text{ad}(T)||a\| \ (a \in \mathcal{A})$; moreover, if the module $\mathcal{Y}$ is essential, then $\|T - \Phi\| \leq \|\text{ad}(T)\|$.

**Proof.** Consider the projective tensor product $\mathcal{A} \widehat{\otimes} \mathcal{A}$, and let $\pi: \mathcal{A} \widehat{\otimes} \mathcal{A} \to \mathcal{A}$ be the continuous linear map defined through
\[
\pi(a \otimes b) = ab \quad (a, b \in \mathcal{A}).
\]
The Banach space $\mathcal{A} \mathcal{A}^\otimes \mathcal{A}$ is a contractive Banach $\mathcal{A}$-bimodule with respect to the operations defined through

$$(a \otimes b)c = a \otimes bc, \quad c(a \otimes b) = ca \otimes b \quad (a, b, c \in \mathcal{A}).$$

By [15, Theorem 3.1], there exists a virtual diagonal for $\mathcal{A}$ of norm one. This is an element $M \in (\mathcal{A} \mathcal{A}^\otimes \mathcal{A})^{**}$ with $\|M\| = 1$ such that, for each $a \in \mathcal{A}$, we have

$$aM = Ma \quad \text{and} \quad \pi^{**}(M)a = a.$$ 

Here, both $(\mathcal{A} \mathcal{A}^\otimes \mathcal{A})^{**}$ and $\mathcal{A}^{**}$ are considered as dual $\mathcal{A}$-bimodules in the usual way. For each continuous bilinear functional $\tau: \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ there exists a unique element $\hat{\tau} \in (\mathcal{A} \mathcal{A}^\otimes \mathcal{A})^*$ such that

$$\hat{\tau}(a \otimes b) = \tau(a, b) \quad (a, b \in \mathcal{A}),$$

and we use the formal notation

$$\int_{\mathcal{A} \times \mathcal{A}} \tau(u, v) dM(u, v) := \langle \hat{\tau}, M \rangle.$$ 

Using this notation, the defining properties of $M$ can be written as

$$\int_{\mathcal{A} \times \mathcal{A}} \tau(au, v) dM(u, v) = \int_{\mathcal{A} \times \mathcal{A}} \tau(u, va) dM(u, v) \quad (3.2)$$

and

$$\int_{\mathcal{A} \times \mathcal{A}} \langle auv, \phi \rangle dM(u, v) = \langle a, \phi \rangle \quad (3.3)$$

for each continuous bilinear functional $\tau: \mathcal{A} \times \mathcal{A} \to \mathbb{C}$, each $a \in \mathcal{A}$, and each $\phi \in \mathcal{A}^*$; further, it will be helpful noting that

$$\int_{\mathcal{A} \times \mathcal{A}} \tau(u, v) dM(u, v) \leq \|M\|\|\hat{\tau}\| = \|\tau\| \quad (3.4).$$

Define $\Phi: \mathcal{X} \to \mathcal{Y}^{*}$ by

$$\langle y, \Phi(x) \rangle = \int_{\mathcal{A} \times \mathcal{A}} \langle y, T(xu)v \rangle dM(u, v) \quad (x \in \mathcal{X}, \ y \in \mathcal{Y}).$$

Then $\Phi$ is well-defined and linear. For each $x \in \mathcal{X}$, $y \in \mathcal{Y}$, and $u, v \in \mathcal{A}$, we have

$$|\langle y, T(xu)v \rangle| \leq \|T(xu)v\|\|y\| \leq \|T(xu)\|\|v\|\|y\| \leq \|T\|\|xu\|\|v\|\|y\| \leq \|T\|\|x\|\|u\|\|v\|\|y\|.$$ 

Then, using (3.4), we have

$$\int_{\mathcal{A} \times \mathcal{A}} \langle y, T(xu)v \rangle dM(u, v) \leq \|T\|\|x\|\|y\|,$$

which implies that $\|\Phi(x)\| \leq \|T\|\|x\|$. Thus $\Phi$ is continuous and (a) holds.
We claim that $\Phi$ is a right $\mathcal{A}$-module homomorphism. Indeed, for $x \in \mathcal{X}$, $a \in \mathcal{A}$, and each $y \in \mathcal{Y}$, (3.2) gives

$$\langle y, \Phi(xa) \rangle = \int_{\mathcal{A} \times \mathcal{A}} \langle y, T(xau) \rangle \, dM(u, v) = \int_{\mathcal{A} \times \mathcal{A}} \langle y, T(xuva) \rangle \, dM(u, v) = \int_{\mathcal{A} \times \mathcal{A}} \langle ay, T(xu) \rangle \, dM(u, v) = \langle ay, \Phi(x) \rangle = \langle y, \Phi(xa) \rangle.$$ 

Our next objective is to prove (b). Take $x \in \mathcal{X}$, $a \in \mathcal{A}$, and $y \in \mathcal{Y}$ with $\|x\| = \|a\| = \|y\| = 1$, and define $\phi \in \mathcal{A}^*$ and $\tau: \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ by

$$\langle u, \phi \rangle = \langle y, T(xu) \rangle \quad (u \in \mathcal{A}),$$

$$\tau(u, v) = \langle y, T(xauv) - T(xauv) \rangle \quad (u, v \in \mathcal{A}).$$

For each $u, v \in \mathcal{A}$, we have

$$|\tau(u, v)| \leq \|T(xauv) - T(xauv)\| \leq \|\text{ad}(T)||xauv||v||$$

$$\leq \|\text{ad}(T)||x||au||v|| \leq \|\text{ad}(T)||u||v||,$$

so that $\|\tau\| \leq \|\text{ad}(T)||$. By (3.3),

$$\langle y, T(xa) \rangle = \langle u, \phi \rangle = \int_{\mathcal{A} \times \mathcal{A}} \langle auw, \phi \rangle \, dM(u, v) = \int_{\mathcal{A} \times \mathcal{A}} \langle y, T(xauv) \rangle \, dM(u, v),$$

and, using the definition of $\Phi$, we obtain

$$\langle y, T(xa) - \Phi(xa) \rangle = \int_{\mathcal{A} \times \mathcal{A}} \tau(u, v) \, dM(u, v).$$

By (3.4), $\|\langle y, T(xa) - \Phi(xa) \rangle\| \leq \|\text{ad}(T)||$. Since this inequality holds for each $y \in \mathcal{Y}$ with $\|y\| = 1$, it follows that

$$\|T(xa) - \Phi(xa)\| \leq \|\text{ad}(T)||.$$

Now assume that $\mathcal{X}$ is essential. Take an approximate identity $(e_j)_{j \in J}$ for $\mathcal{A}$ of bound 1. Then $(e_j)_{j \in J}$ is a right approximate identity for $\mathcal{X}$ and, for each $x \in \mathcal{X}$ with $\|x\| = 1$,

$$\|T(xe_j) - \Phi(xe_j)\| \leq \|\text{ad}(T)|| \quad (j \in J),$$

so that, using the continuity of $T$ and $\Phi$, we see that $\|T(x) - \Phi(x)\| \leq \|\text{ad}(T)||$. Thus $\|T - \Phi\| \leq \|\text{ad}(T)||$.

Finally, we proceed to prove (c). Take $x \in \mathcal{X}$, $a \in \mathcal{A}$, and $y \in \mathcal{Y}$ with $\|x\| = \|a\| = \|y\| = 1$, and define $\phi \in \mathcal{A}^*$ and $\tau: \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ by

$$\langle u, \phi \rangle = \langle y, T(xu) \rangle \quad (u \in \mathcal{A}),$$

$$\tau(u, v) = \langle y, T(xauv) - T(xauv) \rangle \quad (u, v \in \mathcal{A}).$$

For each $u, v \in \mathcal{A}$, we have

$$|\tau(u, v)| \leq \|T(xauv) - T(xauv)\| \leq \|T(xauv) - T(xauv)\||v||$$

$$\leq \|\text{ad}(T)||x||au||v|| \leq \|\text{ad}(T)||u||v||,$$
so that $\|\tau\| \leq \|\text{ad}(T)\|$. By (3.3),
\[
\langle y, T(x)a \rangle = \langle a, \phi \rangle = \int_{A \times A} \langle auv, \phi \rangle \, dM(u, v) = \int_{A \times A} \langle y, T(x)auv \rangle \, dM(u, v),
\]
and, using the definition of $\Phi$, we obtain
\[
\langle y, T(x)a - \Phi(x)a \rangle = \langle y, T(x)a - \Phi(xa) \rangle = \int_{A \times A} \tau(u, v) \, dM(u, v).
\]
From (3.4) we see that $|\langle y, T(x)a - \Phi(x)a \rangle| \leq \|\text{ad}(T)\|$. Thus
\[
\|T(x)a - \Phi(x)a\| \leq \|\text{ad}(T)\|.
\]
Assume that $\mathcal{Y}$ is essential, and take an approximate identity $(e_j)_{j \in J}$ for $\mathcal{A}$ of bound 1. For each $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ with $\|x\| = \|y\| = 1$, we have
\[
|\langle e_j y, T(x) - \Phi(x) \rangle| = |\langle y, T(x)e_j - \Phi(x)e_j \rangle| \leq \|\text{ad}(T)\| \quad (j \in J).
\]
Since $\mathcal{Y}$ is essential, it follows that $(e_j)_{j \in J}$ is a right approximate identity for $\mathcal{Y}$ and hence, taking limit, we see that $|\langle y, T(x) - \Phi(x) \rangle| \leq \|\text{ad}(T)\|$. Therefore $\|T(x) - \Phi(x)\| \leq \|\text{ad}(T)\|$, and the proof is complete.

**Corollary 3.3.** Let $\mathcal{M}$ be an injective von Neumann algebra, and let $\mathcal{X}$ and $\mathcal{Y}$ be unital Banach right and left $\mathcal{M}$-modules, respectively, with both $\mathcal{X}^*$ and $\mathcal{Y}^*$ normal. Then
\[
\text{dist}(T, \text{Hom}_\mathcal{M}(\mathcal{X}, \mathcal{Y}^*)) \leq 8 \sup \{\|e^\perp Te\| : e \in \mathcal{M} \text{ projection}\}
\]
for each $T \in B(\mathcal{X}, \mathcal{Y}^*)$. In particular, the space $\text{Hom}_\mathcal{A}(\mathcal{X}, \mathcal{Y}^*)$ is hyperreflexive.

**Proof.** Take $T \in B(\mathcal{X}, \mathcal{Y}^*)$. Then Theorem 3.1 gives $\Phi \in \text{Hom}_\mathcal{M}(\mathcal{X}, \mathcal{Y}^*)$ such that $\|\Phi\| \leq \|T\|$ and $\|T - \Phi\| \leq \|\text{ad}(T)\|$. Theorem 2.3(ii) now shows that
\[
\|T - \Phi\| \leq 8 \sup \{\|e^\perp Te\| : e \in \mathcal{M} \text{ projection}\},
\]
which establishes our estimate of the distance to $\text{Hom}_\mathcal{M}(\mathcal{X}, \mathcal{Y}^*)$.

The hyperreflexivity follows from the estimates in Proposition 2.1.

**Corollary 3.4.** Let $\mathcal{A}$ be a nuclear $\mathcal{C}^*$-algebra, and let $\mathcal{X}$ and $\mathcal{Y}$ be essential Banach right and left $\mathcal{A}$-modules, respectively. Then
\[
\text{dist}(T, \text{Hom}_\mathcal{A}(\mathcal{X}, \mathcal{Y}^*)) \leq 8 \sup \{\|ab\| : a, b \in \mathcal{A}_+ \text{ contractions, } ab = 0\}
\]
for each $T \in B(\mathcal{X}, \mathcal{Y}^*)$. In particular, the space $\text{Hom}_\mathcal{A}(\mathcal{X}, \mathcal{Y}^*)$ is hyperreflexive.

**Proof.** The estimate follows from Theorem 2.2(ii) and Theorem 3.2 as in Corollary 3.3. The hyperreflexivity follows from the estimates in Proposition 2.1.

**Corollary 3.5.** Let $\mathcal{A}$ be a unital nuclear $\mathcal{C}^*$-algebra of real rank zero, and let $\mathcal{X}$ and $\mathcal{Y}$ be unital Banach right and left $\mathcal{A}$-modules, respectively. Then
\[
\text{dist}(T, \text{Hom}_\mathcal{A}(\mathcal{X}, \mathcal{Y}^*)) \leq 8 \sup \{\|e^\perp Te\| : e \in \mathcal{A} \text{ projection}\}
\]
for each $T \in B(\mathcal{X}, \mathcal{Y}^*)$. 

Proof. The estimate follows from Theorem 2.3(ii) and Theorem 3.2 as in Corollary 3.3.

Corollary 3.6. Let \( \mathcal{A} \) be a nuclear \( C^\ast \)-algebra of real rank zero, and let \( \mathcal{X} \) and \( \mathcal{Y} \) be essential Banach right and left \( \mathcal{A} \)-modules, respectively. Then
\[
\text{dist}(T, \text{Hom}_\mathcal{A}(\mathcal{X}, \mathcal{Y}^\ast)) \leq 8 \sup \{\|eTf\| : e, f \in \mathcal{A} \text{ projections}, ef = 0\}
\]
for each \( T \in B(\mathcal{X}, \mathcal{Y}^\ast) \).

Proof. The estimate follows from Theorem 2.3(ii) and Theorem 3.2 as in Corollary 3.3.

3.2. Homomorphisms between non-commutative \( L^p \)-spaces. Let \( \mathcal{M} \) be a von Neumann algebra. For each \( 1 \leq p \leq \infty \), define \( 1 \leq p^\ast \leq \infty \) by the requirement that \( \frac{1}{p} + \frac{1}{p^\ast} = 1 \). There exists a natural isomorphism \( \omega \mapsto x_\omega \) from \( \mathcal{M}_\ast \) onto \( L^1(\mathcal{M}) \) (this isomorphism preserves the adjoint operation, positivity, and polar decomposition), and hence the space \( L^1(\mathcal{M}) \) is equipped with a distinguished contractive positive linear functional \( \text{Tr} \) defined by \( \text{Tr}(x_\omega) = \omega(1_{\mathcal{M}}) \) (\( \omega \in \mathcal{M}_\ast \)). This functional implements, for each \( 1 \leq p \leq \infty \), the duality \( \langle \cdot, \cdot \rangle : L^p(\mathcal{M}) \times L^{p^\ast}(\mathcal{M}) \to \mathbb{C} \) defined by
\[
\langle x, y \rangle = \text{Tr}(xy) = \text{Tr}(yx) \quad (x \in L^p(\mathcal{M}), y \in L^{p^\ast}(\mathcal{M})).
\]
In the case where \( p \neq \infty \), the above duality gives an isometric isomorphism from \( L^p(\mathcal{M}) \) onto \( L^p(\mathcal{M})^\ast \). Moreover the duality satisfies the following properties:
\begin{align}
(3.5) \quad & \langle ax, y \rangle = \langle x, ya \rangle, \quad \langle xa, y \rangle = \langle x, ay \rangle, \\
(3.6) \quad & \langle ax, y \rangle = \langle xy, a \rangle, \quad \langle xa, y \rangle = \langle yx, a \rangle
\end{align}
for all \( x \in L^p(\mathcal{M}), y \in L^{p^\ast}(\mathcal{M}) \), and \( a \in \mathcal{M} \). Condition (3.5) shows that, for \( p \neq \infty \), the identification of \( L^{p^\ast}(\mathcal{M}) \) with \( L^p(\mathcal{M})^\ast \) is an isomorphism of \( \mathcal{M} \)-bimodules, and, further, condition (3.6) shows that \( L^p(\mathcal{M})^\ast \) is a normal \( \mathcal{M} \)-bimodule.

Theorem 3.7. Let \( \mathcal{M} \) be a von Neumann algebra, and let \( 1 \leq p, q \leq \infty \). Then
\[
\text{dist}(T, \text{Hom}_\mathcal{M}(L^\infty(\mathcal{M}), L^q(\mathcal{M}))) \leq 8 \sup \{\|eT\| : e \in \mathcal{M} \text{ projection}\}
\]
for each \( T \in B(L^\infty(\mathcal{M}), L^q(\mathcal{M})) \), and
\[
\text{dist}(T, \text{Hom}_\mathcal{M}(L^p(\mathcal{M}), L^1(\mathcal{M}))) \leq 8 \sup \{\|eT\| : e \in \mathcal{M} \text{ projection}\}
\]
for each \( T \in B(L^p(\mathcal{M}), L^1(\mathcal{M})) \). In particular, the spaces \( \text{Hom}_\mathcal{M}(L^\infty(\mathcal{M}), L^q(\mathcal{M})) \) and \( \text{Hom}_\mathcal{M}(L^p(\mathcal{M}), L^1(\mathcal{M})) \) are hyperreflexive.

Proof. Suppose that \( T \in B(L^\infty(\mathcal{M}), L^q(\mathcal{M})) \). Define \( \xi = T(1_{\mathcal{M}}) \in L^q(\mathcal{M}) \).

Then, for each \( x \in \mathcal{M} \), we have
\[
\| (T - L\xi)(x) \|_q = \| T(1_{\mathcal{M}})x - T(1_{\mathcal{M}})x \|_q \leq \| \text{ad}(T) \| \| 1_{\mathcal{M}} \| \| x \|,
\]
so that \( \|T - L_\xi\| \leq \|\text{ad}(T)\| \). Corollary 2.10 now gives
\[
\|T - L_\xi\| \leq 8 \sup \{\|e^T e\| : e \in \mathcal{M} \text{ projection}\},
\]
which establishes the required inequality.

Now suppose that \( T \in B(L^p(\mathcal{M}), L^q(\mathcal{M})) \). In order to get the desired inequality, we are reduced to consider the case \( p \neq \infty \). Consider the continuous linear functional \( \phi \) on \( L^p(\mathcal{M}) \) defined by
\[
\langle x, \phi \rangle = \text{Tr}(T(x)) \quad (x \in L^p(\mathcal{M})).
\]
Then there exists \( \xi \in L^{p^*}(\mathcal{M}) \) such that \( \|\xi\| = \|\phi\| \leq \|T\| \) and
\[
\text{Tr}(\xi x) = \langle x, \phi \rangle = \text{Tr}(T(x)) \quad (x \in L^p(\mathcal{M})).
\]
For each \( x \in L^p(\mathcal{M}) \) and \( a \in \mathcal{M} \), we see that
\[
|\text{Tr}((T - L_\xi)(x)a)| = |\text{Tr}(T(x)a - \xi xa)| = |\text{Tr}(T(x)a - T(x)a)| \leq \|T(x)a - T(x)a\|_1 \leq \|\text{ad}(T)\|_p \|a\|
\]
This implies that \( \|(T - L_\xi)(x)\|_1 \leq \|\text{ad}(T)\|_p \|x\|_p \), whence \( \|T - L_\xi\| \leq \|\text{ad}(T)\|_p \). Corollary 2.10 now gives that
\[
\|T - L_\xi\| \leq 8 \sup \{\|e^T e\| : e \in \mathcal{M} \text{ projection}\}.
\]
The hyperreflexivity follows from the estimates in Proposition 2.1. □

**Theorem 3.8.** Let \( \mathcal{M} \) be an injective von Neumann algebra, and let \( 1 \leq p, q \leq \infty \). Then
\[
\text{dist}(T, \text{Hom}_\mathcal{M}(L^p(\mathcal{M}), L^q(\mathcal{M}))) \leq 8 \sup \{\|e^T e\| : e \in \mathcal{M} \text{ projection}\}
\]
for each \( T \in B(L^p(\mathcal{M}), L^q(\mathcal{M})) \). In particular, the space \( \text{Hom}_\mathcal{M}(L^p(\mathcal{M}), L^q(\mathcal{M})) \) is hyperreflexive and the hyperreflexivity constant is at most 8.

**Proof.** By Theorem 3.7, we need only to consider the case where \( p \neq \infty \) and \( q \neq 1 \), and then the result follows from Corollary 3.3, because both modules \( L^p(\mathcal{M})^* \) and \( L^q(\mathcal{M}) (= L^q(\mathcal{M})^*) \) are normal. □

At the expense of replacing the condition \( 1 \leq p, q \leq \infty \) by \( 1 \leq q < p \leq \infty \) and losing the bound 8 on the distance estimate, we may remove the injectivity of the von Neumann algebra \( \mathcal{M} \) in Theorem 3.8. To this end, we will be involved with the ultraproduct of non-commutative \( L^p \)-spaces. We summarize some of its main properties.

Let \( (\mathcal{X}_n) \) be a sequence of Banach spaces and let \( \mathcal{U} \) be an ultrafilter on \( \mathbb{N} \). Let \( \prod \mathcal{X}_n \) be the \( \ell^\infty \)-sum of the sequence \( (\mathcal{X}_n) \) and take
\[
N_{\mathcal{U}} = \{(x_n) \in \prod \mathcal{X}_n : \lim_{\mathcal{U}} \|x_n\| = 0\}.
\]
Then the ultraproduct \( \prod_{\mathcal{U}} \mathcal{X}_n \) of the sequence \( (\mathcal{X}_n) \) along \( \mathcal{U} \) is the quotient Banach space \( \prod \mathcal{X}_n / N_{\mathcal{U}} \). Given \((x_n) \in \prod \mathcal{X}_n \), we write \((x_n)_{\mathcal{U}}\) for its corresponding equivalence class in \( \prod_{\mathcal{U}} \mathcal{X}_n \). The norm on \( \prod_{\mathcal{U}} \mathcal{X}_n \) is given by
\[
\|[(x_n)_{\mathcal{U}}]\| = \lim_{\mathcal{U}} \|x_n\|.
for each \((x_n)_{\mathcal{U}} \in \prod_{\mathcal{U}} \mathcal{X}_n\). Let \((\mathcal{Y}_n)\) be another sequence of Banach spaces and let \((T_n) \in \prod B(\mathcal{X}_n, \mathcal{Y}_n)\). Then we define \(\prod_{\mathcal{U}} T_n : \prod_{\mathcal{U}} \mathcal{X}_n \to \prod_{\mathcal{U}} \mathcal{Y}_n\) by

\[
\prod_{\mathcal{U}} T_n((x_n)_{\mathcal{U}}) = (T_n(x_n))_{\mathcal{U}}
\]

for each \((x_n)_{\mathcal{U}} \in \prod_{\mathcal{U}} \mathcal{X}_n\). Of course, it can be checked that the definition we make is independent of the choice of the representative of the equivalence class. Moreover, \(\prod_{\mathcal{U}} T_n\) is continuous and

\[
\|\prod_{\mathcal{U}} T_n\| = \lim_{\mathcal{U}} \|T_n\|.
\]

All the above statements are also valid for quasi-Banach spaces. We refer the reader to [16] for the basics of ultraproducts.

If \((\mathcal{A}_n)\) is a sequence of C*-algebras, then \(\prod_{\mathcal{U}} \mathcal{A}_n\) is again a C*-algebra. The ultraproduct of a sequence \((\mathcal{M}_n)\) of von Neumann algebras is not as straightforward as the C*-algebra case. According to [13, 25], it is known that \(\prod_{\mathcal{U}} L^1(\mathcal{M}_n)\) is isometrically isomorphic to the predual of a von Neumann algebra \(\mathcal{M}_{\mathcal{U}}\). Further, it is shown in [25] that \(\mathcal{M}_{\mathcal{U}}\) has such a nice behaviour as \(\prod_{\mathcal{U}} L^p(\mathcal{M}_n)\) is isometrically isomorphic to \(L^p(\mathcal{M}_{\mathcal{U}})\) for each \(p < \infty\). Specifically,

- there exists an isometric \(*\)-homomorphism

\[
\iota : \prod_{\mathcal{U}} \mathcal{M}_n \to \mathcal{M}_{\mathcal{U}}
\]

from the C*-algebra \(\prod_{\mathcal{U}} \mathcal{M}_n\) into the von Neumann algebra \(\mathcal{M}_{\mathcal{U}}\) such that \(\iota(\prod_{\mathcal{U}} \mathcal{M}_n)\) is weak* dense in \(\mathcal{M}_{\mathcal{U}}\), and,

- for each \(p < \infty\), there exists an isometric isomorphism

\[
\Lambda_p : \prod_{\mathcal{U}} L^p(\mathcal{M}_n) \to L^p(\mathcal{M}_{\mathcal{U}})
\]

such that

\[
\Lambda_p((a_n)_{\mathcal{U}}(x_n)_{\mathcal{U}}(b_n)_{\mathcal{U}}) = \iota((a_n)_{\mathcal{U}}) \Lambda_p((x_n)_{\mathcal{U}}) \iota((b_n)_{\mathcal{U}})
\]

and, for \(0 < p, q, r < \infty\) with \(\frac{1}{p} + \frac{1}{q} = \frac{1}{r}\),

\[
\Lambda_r((x_n)_{\mathcal{U}}(y_n)_{\mathcal{U}}) = \Lambda_p((x_n)_{\mathcal{U}}) \Lambda_q((y_n)_{\mathcal{U}})
\]

for all \((a_n)_{\mathcal{U}}, (b_n)_{\mathcal{U}} \in \prod_{\mathcal{U}} \mathcal{M}_n\), \((x_n)_{\mathcal{U}} \in \prod_{\mathcal{U}} L^p(\mathcal{M}_n)\), and \((y_n)_{\mathcal{U}} \in \prod_{\mathcal{U}} L^q(\mathcal{M}_n)\).

Actually, [25] is concerned with the ultrapower of \(L^p(\mathcal{M})\) for a given von Neumann algebra, but it is also emphasized there that the results are equally valid for the above situation.

**Theorem 3.9.** Let \(1 \leq q < p \leq \infty\). Then there exists a constant \(C_{p, q} \in \mathbb{R}^+\) with the property that, for each von Neumann algebra \(\mathcal{M}\) and each continuous linear map \(T : L^p(\mathcal{M}) \to L^q(\mathcal{M})\), we have

\[
\text{dist}(T, \text{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))) \leq C_{p, q} \sup\{\|e^{\perp} Te\| : e \in \mathcal{M} \text{ projection}\}.
\]

In particular, the space \(\text{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))\) is hyperreflexive.
Proof. In the case where either $p = \infty$ or $q = 1$, we apply Theorem 3.7 to obtain the result.

Suppose that $1 < q < p < \infty$, and take $1 < r < \infty$ such that $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$. Our objective is to prove that there exists a constant $c_{p,q} \in \mathbb{R}^+$ with the property that for each von Neumann algebra $\mathcal{M}$ and each $T \in B(L^p(\mathcal{M}), L^q(\mathcal{M}))$, we have

$$\dist(T, \Hom_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))) \leq c_{p,q} \|\text{ad}(T)\|.$$  

(3.8)

Assume towards a contradiction that the clause is false, and there is no such constant $c_{p,q}$. Then, for each $n \in \mathbb{N}$, there exists a von Neumann algebra $\mathcal{M}_n$ and a continuous linear map $R_n : L^p(\mathcal{M}_n) \rightarrow L^q(\mathcal{M}_n)$ such that

$$\delta_n := \dist(R_n, \Hom_{\mathcal{M}_n}(L^p(\mathcal{M}_n), L^q(\mathcal{M}_n))) > n \|\text{ad}(R_n)\|.$$  

For each $n \in \mathbb{N}$, set $S_n = \delta_n^{-1} R_n$. Then

$$\|\text{ad}(S_n)\| < \frac{1}{n} \quad (n \in \mathbb{N})$$  

(3.9)

and

$$\dist(S_n, \Hom_{\mathcal{M}_n}(L^p(\mathcal{M}_n), L^q(\mathcal{M}_n))) = 1 \quad (n \in \mathbb{N}).$$  

(3.10)

Since the sequence $(S_n)$ need not to be bounded, we replace it with a bounded one that still satisfies both (3.9) and (3.10). For this purpose, for each $n \in \mathbb{N}$, we take $\Psi_n \in \Hom_{\mathcal{M}_n}(L^p(\mathcal{M}_n), L^q(\mathcal{M}_n))$ such that $\|S_n - \Psi_n\| < 1 + 1/n$ and consider the map $T_n = S_n - \Psi_n$. Then $(T_n)$ is bounded. It is clear that $\|\text{ad}(T_n)\| = \|\text{ad}(S_n)\|$ and that

$$\dist(T_n, \Hom_{\mathcal{M}_n}(L^p(\mathcal{M}_n), L^q(\mathcal{M}_n))) = \dist(S_n, \Hom_{\mathcal{M}_n}(L^p(\mathcal{M}_n), L^q(\mathcal{M}_n)))$$  

for each $n \in \mathbb{N}$, so that (3.9) and (3.10) give

$$\|\text{ad}(T_n)\| < \frac{1}{n} \quad (n \in \mathbb{N}),$$  

(3.11)

$$\dist(T_n, \Hom_{\mathcal{M}_n}(L^p(\mathcal{M}_n), L^q(\mathcal{M}_n))) = 1 \quad (n \in \mathbb{N}).$$  

(3.12)

Take a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Consider the ultraproduct von Neumann algebra

$$\mathcal{M}_\mathcal{U} = (\prod_{\mathcal{U}} L^1(\mathcal{M}_n))^*$$

and the maps

$$\iota : \prod_{\mathcal{U}} \mathcal{M}_n \rightarrow \mathcal{M}_\mathcal{U},$$

$$\Lambda_p : \prod_{\mathcal{U}} L^p(\mathcal{M}_n) \rightarrow L^p(\mathcal{M}_\mathcal{U}),$$

$$\Lambda_q : \prod_{\mathcal{U}} L^q(\mathcal{M}_n) \rightarrow L^q(\mathcal{M}_\mathcal{U}),$$

$$\Lambda_r : \prod_{\mathcal{U}} L^r(\mathcal{M}_n) \rightarrow L^r(\mathcal{M}_\mathcal{U})$$

introduced in the preliminary remark. Further, take the ultraproduct map

$$\prod_{\mathcal{U}} T_n : \prod_{\mathcal{U}} L^p(\mathcal{M}_n) \rightarrow \prod_{\mathcal{U}} L^q(\mathcal{M}_n).$$
We claim that $\prod_{U} T_n$ is a right $\prod_{U} \mathcal{M}_n$-module homomorphism. Take elements $(x_n)_{U} \in \prod_{U} L^p(\mathcal{M}_n)$ and $(a_n)_{U} \in \prod_{U} \mathcal{M}_n$. Then (3.11) gives
\[
\| \prod_{U} T_n ((x_n)_{U}(a_n)_{U}) - \prod_{U} T_n ((x_n)_{U})(a_n)_{U} \| = \lim_{U} \| T_n (x_n a_n) - T_n (x_n) a_n \|
\leq \lim_{U} (\| \text{ad}(T_n) \| \| x_n \| \| a_n \|)
\leq \lim_{U} (\frac{1}{n} \| x_n \| \| a_n \|) = 0.
\]

Define $T: L^p(\mathcal{M}_U) \to L^q(\mathcal{M}_U)$ by
\[
T = \Lambda_q \circ \prod_{U} T_n \circ \Lambda_p^{-1}.
\]

Then $T$ is a right $\iota(\prod_{U} \mathcal{M}_n)$-module homomorphism. We now note that:

- $\iota(\prod_{U} \mathcal{M}_n)$ is weak* dense in $\mathcal{M}_U$;
- the module maps $a \mapsto xa$ and $a \mapsto ya$ are weak*-weak* continuous for all $x \in L^p(\mathcal{M}_U)$ and $y \in L^q(\mathcal{M}_U)$;
- the map $T$ is weak*-weak* continuous, since both $L^p(\mathcal{M}_U)$ and $L^q(\mathcal{M}_U)$ are reflexive (being $1 < p, q < \infty$).

The above conditions imply that $T$ is a right $\mathcal{M}_U$-module homomorphism. By Theorem 2.9 there exists $\Xi \in L^r(\mathcal{M}_U)$ such that
\[
T(x) = \Xi x \quad (x \in L^p(\mathcal{M}_U)).
\]

Set $(\xi_n)_{U} = \Lambda_r^{-1}(\Xi) \in \prod_{U} L^r(\mathcal{M}_n)$, and, for each $n \in \mathbb{N}$, take the left composition map $L_{\mathcal{M}_n}: L^p(\mathcal{M}_n) \to L^q(\mathcal{M}_n)$. Then, for each $x \in L^p(\mathcal{M}_U)$, we have
\[
T(x) = \Xi x = \Lambda_r ((\xi_n)_{U}) \Lambda_p (\Lambda_p^{-1}(x)) = \Lambda_q ((\xi_n)_{U} \Lambda_p^{-1}(x))
= (\Lambda_q \circ \prod_{U} L_{\mathcal{M}_n} \circ \Lambda_p^{-1})(x),
\]

whence $\prod_{U} T_n = \prod_{U} L_{\mathcal{M}_n}$, so that (3.7) gives $\lim_{U} \| T_n - L_{\mathcal{M}_n} \| = 0$ and hence
\[
\lim_{U} \text{dist} (T_n, \text{Hom}_{\mathcal{M}_n} (L^p(\mathcal{M}_n), L^q(\mathcal{M}_n))) \leq \lim_{U} \| T_n - L_{\mathcal{M}_n} \| = 0,
\]
contrary to (3.12).

Finally, (3.8) and Corollary 2.10 give the desired inequality with $C_{p,q} = 8c_{p,q}$.

The hyperreflexivity follows from the estimates in Proposition 2.1. \qed

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