General method for the security analysis in a quantum direct communication protocol

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Abstract

We introduce a general approach for the analysis of a quantum direct communication protocol. The method is based on the investigation of the superoperator acting on a joint system of the communicating parties and the eavesdropper. The introduced method is more versatile than the approaches used so far as it permits to incorporate different noise models in a unified way. Moreover, it makes use of a well-grounded theory of quantum discrimination for the purpose of estimating the eavesdropper's information gain.

Keywords: quantum direct communication, quantum cryptography

1 Introduction

One of the main objectives of quantum information theory [10] is to develop new protocols for controlling the networks of quantum processing units connected by quantum channels [7]. This type of networks requires new methods allowing full utilization of the capabilities offered by quantum information processing [12]. Quantum internetworking protocols should be able to exploit the quantum effects, including quantum teleportation, dense coding and quantum direct communication. However, in order to provide efficient methods for the utilization of quantum effects in large scale networks, it is necessary to analyze the robustness of quantum information processing against the errors that occur during the transmission [8] and processing of quantum information [5].

In this paper we focus on the analysis of the security of quantum ping-pong protocol in the presence of a general form of quantum noise. The ping-pong protocol [2] has attracted a lot of attention as, contrary to quantum key distribution (QKD) schemes, it does not require prior key agreement for confidentiality provision and it is provably asymptotically secure in lossless channels. The theoretical success of the protocol has been closely followed by the experimental implementation and the proof of concept installation has been realized in the laboratory [11]. It has also been shown that the protocol variant based on higher dimensional systems and exploiting dense information coding also share features of the seminal version when some improvements are introduced [13, 16].

The ping-pong protocol, similarly to other quantum direct communication (QDC) protocols, operates in two modes: a message mode is designed for information transfer and a control mode is used for an eavesdropping detection. Although the ping-pong protocol is asymptotically
secure in perfect quantum channels, the situation looks worse in noisy environments when legitimate users tolerate some level of transmission errors and/or losses. If that level is too high compared to the quality of the channel then an eavesdropper can peek some fraction of signal particles hiding himself behind accepted quantum bit error rate (QBER) threshold \cite{14, 17}. Thus the possibility to intercept some part of the message without being detected renders the protocol insecurity. To cope with this problem an additional purely classical layer has been proposed \cite{15}. However the estimation of security improvement offered by that layer heavily depends on observed QBER and the methods of the protocol analysis used so far do not offer mathematical apparatus capable of QBER estimation in noisy channels. The purpose of the presented work is to fill in this gap.

This paper is organized as follows. In Section 2 we provide a general description of the ping-pong protocol in the language of density operators. In Section 3 we describe a general model of noise which can be used to describe the errors occurring during the execution of the protocol. In Section 4 we apply the introduced description for the purpose of the security in the passive eavesdropping scenario. Finally, in Section 5 we summarize the presented work and provide concluding remarks.

2 General description of the ping-pong protocol

Let us consider the seminal version of the ping-pong protocol \cite{2} in which the message and control modes are executed only in computational basis. The communication process is started by Bob, the recipient of information, who prepares an EPR pair

\[ |\phi^+\rangle = (|0_B\rangle|0_A\rangle + |1_B\rangle|1_A\rangle) / \sqrt{2} . \]  (1)

At the same time eavesdropping Eve controls her own system, which is initially described by state \( |\chi_E\rangle \). As the states of Bob and Eve are separated, the density matrix of the whole system reads

\[ \rho_{BAE}^{(0)} = \rho_{BA}^{(0)} \otimes \rho_{E}^{(0)} = |\phi^+\rangle\langle\phi^+| \otimes |\chi_E\rangle\langle\chi_E| . \]  (2)

Next Bob sends a signal qubit \( A \) to Alice. This qubit on its way can be influenced by two factors: quantum noise because of channel imperfection and malicious activities of Eve who may entangle it with the system controlled by herself. Let us assume that Eve is positioned close to Alice, so her action takes place on the qubit modified by the noise. The density matrix of the system just before signal qubit enters the environment controlled by Alice reads

\[ \rho_{BAE}^{(1)} = (N_{BA} \otimes I_E) \left( \rho_{BA}^{(0)} \otimes \rho_{E}^{(0)} \right) = \rho_{BA}^{(1)} \otimes \rho_{E}^{(0)} , \]  (3)

where it has been explicitly highlighted that noise operator \( N \) acts only on the EPR pair (\( I \) denotes identity operation). Before signal qubit enters Alice’s environment, Eve can entangle it with her own system

\[ \rho_{BAE}^{(2)} = (I_B \otimes \mathcal{E}_{AE}) \rho_{BAE}^{(1)} , \]  (4)

where entangling operator \( \mathcal{E}_{AE} \) acts only on qubit \( A \) of the EPR pair and system possessed by Eve. At that point of protocol execution Alice can select a control mode which serves for eavesdropping detection or continue in the information mode.

In the former case she measures the received qubit in computational basis, \( i.e. \) performs von Neumann measurement using projectors \( M_{x,A} = I_B \otimes |x_A\rangle\langle x_A| \otimes I_E , \ x = 0, 1 \). The probability that she finds qubit under investigation in state \( |x\rangle \) (measures \( \pm 1 \)) is given by

\[ p_A(x) = \text{Tr} \left( \rho_{BAE}^{(2)} M_{x,A} \right) \]  (5)
After the measurement the state of the whole system is described by

\[ \sigma_{xBAE}^{(2)} = \frac{M_{x,AP_{BAE}^{(2)}M_{x,A}}}{\text{Tr}(\rho_{BAE}^{(2)}M_{x,A})}. \]  

(6)

Subsequently Bob measures his qubit in computational basis using projectors

\[ M_{y,B}^A = |y_B\rangle\langle y_B| \otimes I_A \otimes I_E, \quad y = 0, 1. \]

The probability that Bob finds his qubit in state |y⟩ provided that Alice has found his qubit in state |x⟩ is given by

\[ p_{B|A}(y|x) = \text{Tr} \left( \sigma_{xBAE}^{(2)}M_{y,B} \right). \]  

(7)

From the above, it follows that errors in control mode appear with probability

\[ P_{EC} = p_{B|A}(1|0)p_A(0) + p_{B|A}(0|1)p_A(1). \]  

(8)

In information mode, Alice encodes a classic bit \( \mu \) applying \( (\mu = 1) \) or not \( (\mu = 0) \) operator \( Z_A \) to the possessed qubit. The system state after encoding is given by

\[ \rho_{\mu BA}^{(3)} = (I_B \otimes Z_\mu^A \otimes I_E) \rho_{BAE}^{(2)} \left( I_B \otimes (Z_\mu^A)^\dagger \otimes I_E \right). \]  

(9)

The qubit \( A \) is sent back to Bob after the encoding operation. Eve’s task is to discriminate between states \( \rho_{\mu BA}^{(3)} = \text{Tr}_B \left( \rho_{\mu BAE}^{(3)} \right) \) with maximal confidence. The system states after Bob’s reception of qubit \( A \) traveling back from Alice and in the absence of Eve measurements are given by

\[ \rho_{\mu BA}^{(4)} = (N_{BA} \otimes I_E) \rho_{\mu BA}^{(3)}, \]  

(10)

so Bob has to distinguish the states

\[ \rho_{\mu BA}^{(4)} = \text{Tr}_E \left( \rho_{\mu BAE}^{(4)} \right). \]  

(11)

When Eve performs measurements, the same quantum discrimination strategy is used but Bob is unconscious that measured states are of the form

\[ \tau_{\mu,\alpha BA}^{(4)} = \text{Tr}_E \left( N_{BA} \otimes I_E \right) \frac{M_{\alpha,E}\rho_{\mu BAE}^{(3)}M_{\alpha,E}^\dagger}{\text{Tr}(\rho_{\mu BAE}^{(3)}M_{\alpha,E})}. \]  

(12)

The analysis of the protocol should determine Eve’s information gain \( I_E \) and the probability of erroneous Bob’s decoding \( QBER \) as functions of probability of error observed in control mode \( P_{EC} \) and, optionally, parameters describing noise operator \( N \).

### 3 Model of the noise

Any interaction with the environment observed from the perspective of the principal system can be given as operator sum (Kraus) representation \[10\] \[3\]

\[ \rho \rightarrow \rho' = N\rho = \sum_k K_k \rho K_k^\dagger \]  

(13)

provided that \( \sum_k K_k K_k^\dagger = I \). Such an approach hides the details of the interaction of the system under investigation with the environment, but these details are not of immediate relevance.
in analysis of many quantum information processing related tasks. In such situations Kraus representation proved to be useful because it provides a unified description of many, seemingly different, physical processes.

It follows from the description of the protocol operation that the signal qubit only interacts with the environment. If that qubit was separated from the home qubit possessed by Bob, the noise operator would be simply decomposed as \( N_{BA} = I_B \otimes N_A \), where \( N_A \) describes the interaction of the signal qubit with the environment. However, the signal and home qubits are maximally entangled in the ping-pong protocol and such decomposition is not obvious. In this section we introduce a general procedure for deriving Kraus operator for such a situation and exemplify it with the depolarizing channel.

The action of the quantum channel [13] can be represented as a supermatrix \( M_N \) [9, Eq. (5)]

\[
\text{Res}(\rho) = \sum_k \text{Res} \left( K_k \rho K_k^\dagger \right) = \sum_k K_k^* \otimes K_k \text{Res}(\rho) = M_N \text{Res}(\rho),
\]

where \( \text{Res}(\cdot) \) denotes reshape transformation which maps a density matrix into a column vector row wise. The Kraus operators can be recovered from \( M_N \) with singular matrix decomposition of the matrix \( D_N \) which is related to \( M_N \) via

\[
M_N = \sum_{k,l} \{ D_N \}_{k,l} \epsilon_k \otimes \epsilon_l,
\]

where \( \epsilon_k \) denotes \( k \)-th element of the canonical base of \( \rho \) space [9]. When two quantum channels \( N_B \) and \( N_A \) are applied to the parts of the composite system the supermatrix of the composite channel can be found as

\[
M_{N_B \otimes N_A} = M_R(N_B \otimes N_A)M_R,
\]

where \( M_R \) is the matrix representing the change of base in the product space and it is commonly called the reshuffle matrix [9]. In the considered case \( N_B = I_B \) and our task is to derive Kraus operators for \( M_{I_B \otimes N_A} \) and subsequently a map for density operators analogous to [13].

The single qubit depolarizing channel with reliability \( r \), which is commonly used to model white noise, is described by the map [10]

\[
N_A^{(D)}(\rho_A) = r\rho_A + \frac{1-r}{4} \sum_{k=0}^{3} \sigma_k \rho_A \sigma_k^\dagger,
\]

where \( \sigma_k \) are Pauli matrices. The supermatrix of this map is given as

\[
M^{(D)}_{N_A} = \begin{pmatrix}
\frac{1-r}{4} & 0 & 0 & 0 \\
0 & \frac{1-r}{4} & 0 & 0 \\
0 & 0 & \frac{1-r}{4} & 0 \\
0 & 0 & 0 & \frac{1-r}{4}
\end{pmatrix}.
\]

Using the formula [13], one gets the explicit form of the supermatrix for the extended channel

\[
M^{(D)}_{I_B \otimes N_A} = \begin{pmatrix}
\frac{1-r}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1-r}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1-r}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1-r}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1-r}{4} & 0 & 0 & 0 & \frac{1-r}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1-r}{4} & 0 & 0 & \frac{1-r}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1-r}{4} & 0 & 0 & \frac{1-r}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1-r}{4} & 0 & 0 & \frac{1-r}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1-r}{4} & 0 & 0 & \frac{1-r}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1-r}{4} & 0 & 0 & 0 & 0 & 0 & \frac{1-r}{4} & 0 & 0 & \frac{1-r}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-r}{4} & 0 & 0 & \frac{1-r}{4} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-r}{4} & 0 & 0 & \frac{1-r}{4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-r}{4} & 0 & 0 & \frac{1-r}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-r}{4} & 0 & 0 & \frac{1-r}{4} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-r}{4} & 0 & 0 & \frac{1-r}{4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-r}{4} & 0 & 0 & \frac{1-r}{4}
\end{pmatrix}.
\]
Kraus operators for the above extended channel can be obtained using eigendecomposition of the dynamical matrix [15] corresponding to the supermatrix $M^{(D)}_{I_B \otimes N_A}$:

$$K^{(D)}_{I_B \otimes N_A} = \left\{ \frac{\sqrt{1-r}}{2} I_B \otimes \sigma_z, \frac{\sqrt{1-r}}{\sqrt{2}} I_B \otimes [0 1], \frac{\sqrt{1-r}}{\sqrt{2}} I_B \otimes [1 0], \frac{\sqrt{1+3r}}{2} I_B \otimes I_A \right\}. \quad (20)$$

Kraus representation [20] implies that the action of the extended channel on the second subsystem is independent from the action on the first subsystem. Using the representation of the composite system density matrix

$$\rho_{BA} = \left[ \begin{array}{cccc} \rho_{00,00} & \rho_{00,01} & \rho_{01,00} & \rho_{01,01} \\ \rho_{00,10} & \rho_{00,11} & \rho_{01,10} & \rho_{01,11} \\ \rho_{10,00} & \rho_{10,01} & \rho_{11,00} & \rho_{11,01} \\ \rho_{10,10} & \rho_{10,11} & \rho_{11,10} & \rho_{11,11} \end{array} \right],$$

one can show that it implies the following map

$$\mathcal{N}_{BA} \rho_{BA} = (I_B \otimes I_A) \rho_{BA} = r \rho_{BA} + \frac{1-r}{2} \text{Tr}_A (\rho_{BA}) \otimes I_A. \quad (22)$$

4 Passive eavesdropping in a noisy channel

As the application of the introduced description, we consider a simple model of quantum noise described by the depolarizing channel. Let us consider the situation in which Eve does not entangle with a signal qubit i.e. $\mathcal{E}_{AE} = I_{AE}$. Such assumption results in the separation of the system controlled by Eve, so it is not taken into account in further expressions. From the map [22] it follows that the system state after the reception of the signal qubit by Alice reads

$$\rho^{(2)}_{BA} = \frac{r}{2} [0_B 0_A] \langle 1_B 1_A | + | 1_B 1_A \rangle \langle 0_B 0_A | +$$

$$\quad + \frac{1-r}{4} [0_B 0_A] \langle 0_B 0_A | + | 0_B 1_A \rangle \langle 0_B 1_A | +$$

$$\quad + \frac{1+r}{4} [0_B 0_A] \langle 0_B 0_A | + | 1_B 1_A \rangle \langle 1_B 1_A |. \quad (23)$$

It follows from [8] that the probability of error occurrence in control mode is equal to

$$P_{EC} = (1-r)/2. \quad (24)$$

In the information mode the encoding operation leads to states

$$\rho^{(3)}_{pBA} = (-1)^{\mu} \frac{r}{2} [0_B 0_A] \langle 1_B 1_A | + | 1_B 1_A \rangle \langle 0_B 0_A | +$$

$$\quad + \frac{1-r}{4} [0_B 0_A] \langle 0_B 0_A | + | 0_B 1_A \rangle \langle 0_B 1_A | +$$

$$\quad + \frac{1+r}{4} [0_B 0_A] \langle 0_B 0_A | + | 1_B 1_A \rangle \langle 1_B 1_A |. \quad (25)$$

However, Eve’s observation capabilities are limited only to travelling qubit $A$, thus she has to distinguish between states $\rho^{(3)}_{pA} = \text{Tr}_B (\rho^{(3)}_{pBA})$. But $\rho^{(3)}_{pA} = \rho^{(3)}_{1A}$ and, in consequence, $I_E = 0$. Thus the noise gives no additional advantage to passively eavesdropping Eve.
Figure 1: Probabilities of a particle loss (QLOSS) or an erroneous decoding (QBER) as a function of control mode failure probability (PEC) in protocol operation over depolarizing channel.

The qubit A in its way back to Bob again interacts with the environment so he receives the states

\[ \rho^{(4)}_{\mu BA} = r \rho^{(3)}_{\mu BA} + \frac{1-r^2}{2} \text{Tr}_{A} \left( \rho^{(3)}_{\mu BA} \right) \otimes I_A = \]

\[ = (-1)^\mu \frac{r^2}{2} [\langle 0_B 0_A \rangle \langle 1_B 1_A \rangle | 1_B 1_A \rangle + | 0_B 0_A \rangle \langle 0_B 0_A \rangle ] + \]

\[ + \frac{1-r^2}{4} [\langle 1_B 0_A \rangle \langle 1_B 0_A \rangle | 0_B 1_A \rangle + | 1_B 1_A \rangle \langle 0_B 1_A \rangle ] + \]

\[ + \frac{1+r^2}{4} [\langle 0_B 0_A \rangle \langle 0_B 0_A \rangle + | 1_B 1_A \rangle \langle 1_B 1_A \rangle ] . \] (26)

If Bob uses unambiguous discrimination, the bits are lost (measurement fails) with a probability [6]

\[ QLOSS = 1 - P_{s}^{\text{max}} = F \left( \rho^{(4)}_{0 BA}, \rho^{(4)}_{1 BA} \right), \] (27)

where the overlap of \( \rho^{(4)}_{\mu BA} \) is given by

\[ F \left( \rho^{(4)}_{0 BA}, \rho^{(4)}_{1 BA} \right) = \frac{1}{2} (1 - r^2) + \frac{1}{2} \sqrt{(1 - r^2)(1 + 3r^2)}. \] (28)

On the other hand, if Bob uses minimum error discrimination the observed bit error rate is equal to [4]

\[ QBER = \frac{1}{2} \left( 1 - \frac{1}{2} \text{Tr} \left( \left| \rho^{(4)}_{0 BA} - \rho^{(4)}_{1 BA} \right| \right) \right) = (1 - r^2)/2. \] (29)

Quantities QBER and QLOSS as a function of control mode failure probability [8], which is a parameter directly accessible to communicating parties, are shown in Figure 1. Both QBER
and QLOSS do not scale linearly with $P_{EC}$. Moreover, the functional form of the obtained scaling heavily depends on the parameters of the noise model used, thus the correct modeling of noise is of prime importance in the estimation of the protocol operation over non-perfect quantum channels.

5 Concluding remarks

The usefulness of the general method based on density operator analysis for ping-pong protocol operation has been presented. As the proof of concept the example of its application to the analysis of the protocol execution over depolarizing channel has been given. The analysis of a more complicated case of an active eavesdropping is left for future research. Although the method is more cumbersome than the approach used so far, it is more versatile as it permits an incorporation of different models of noise in a unified way and makes use of a well grounded theory of quantum discrimination in estimation of eavesdropper’s information gain.

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