Sign balance for finite groups of Lie type

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Abstract

A product formula for the parity generating function of the number of 1’s in invertible matrices over $\mathbb{Z}_2$ is given. The computation is based on algebraic tools such as the Bruhat decomposition. The same technique is used to obtain a parity generating function also for symplectic matrices over $\mathbb{Z}_2$. We present also a generating function for the sum of entries of matrices over an arbitrary finite field $\mathbb{F}_q$ calculated in $\mathbb{F}_q$. These formulas are new appearances of the Mahonian distribution.

1 Introduction

Let $G$ be a subgroup of $GL_n(\mathbb{Z}_2)$. For every $K \in G$ define $o(K)$ to be the number of 1’s in $K$. A natural problem is to find the number of matrices with a given number of 1’s, or in other words, to compute the following generating function:

$$O(G, t) = \sum_{K \in G} t^{o(K)}.$$ 

In the case $G = GL_n(\mathbb{Z}_2)$, by considering the free action of $S_n$ by permuting rows on the subset of $G$ containing the matrices having a fixed number of 1’s, it is not hard to see that $O(G, t)$ has $n!$ as a factor but the complete generating function can be rather hard to compute. A weaker variation of this problem is to evaluate $O(G, -1)$. This is equivalent to determining the
difference between the numbers of even and odd matrices, where a matrix is
called even if it has an even number of 1’s and odd otherwise. The number
$O(G, -1)$ will be called the parity difference or the imbalance of $G$. A group
$G$ is called sign-balanced if $O(G, -1) = 0$.

The notion of sign-balance has recently reappeared in a number of con-
texts. Simion and Schmidt [7] proved that the number of 321-avoiding even
permutations is equal to the number of such odd permutations if $n$ is even,
and exceeds it by the Catalan number $C_{\frac{n}{2}}(n-1)$ otherwise. Adin and Roich-
man [1] refined this result by taking into account the maximum descent. In
a recent paper [8], Stanley established the importance of the sign-balance.

In this paper we calculate the parity difference for $G = GL_n(\mathbb{Z}_2)$. We
also generalize the problem of sign-balance to matrix groups over arbitrary
finite fields $\mathbb{F}_q$ where $q$ is a power of a prime $p$. It turns out that the appro-
priate parameter for these fields is the sum of nonzero entries of the matrix
(mod $p$) rather than just the number of nonzero elements.

In the corresponding formulas there is an appearance of the number
$[n]_q = \frac{1}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}$. The most important instance of this
number is in the following theorem of MacMahon:

**Proposition 1.1.**

$$\sum_{\pi \in S_n} q^{inv(\pi)} = \sum_{\pi \in S_n} q^{maj(\pi)} = [n]_q!$$

where $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$, $inv(\pi)$ is the number of inversions in
$\pi$ (see definition in section 2.1). (The definition of $maj(\pi)$ is
$maj(\pi) = \sum_{i=1}^{n} i \cdot \chi\{\pi(i + 1) < \pi(i)\}$.)

A combinatorial parameter on permutations which has such a distribu-
tion is called Mahonian after MacMahon.

Our results can also be seen as an example of the cyclic sieving phe-
nomenon (See [5] for details). We will expound on this subject in the appendix.

We prove the following (See Theorems 3.1 and 4.1)

**Theorem.**

$$\sum_{K \in GL_n(\mathbb{Z}_2)} (-1)^{o(K)} = -2(\begin{matrix}n \\ 2 \end{matrix})[n-1]_2!$$
Theorem. 
\[ \sum_{K \in S_{p2n}(Z_2)} (-1)^{o(K)} = -2^{n^2} \cdot [2]_2[4]_2 \cdots [2n - 2]_2 \]

We also generalize the problem of sign-balance to matrix groups over arbitrary finite fields. For \( q \) a prime power the parameter we work with is the sum of the entries of the matrix, calculated in \( F_q \), rather than just the number of 1-s. We order the elements of \( F \), i.e., choose a bijection between \( F \) and the set \{0, ..., q - 1\} such that 0 of \( F \) goes to 0. If we denote by \( s(K) \) the sum of the images of the entries of \( K \) in \( F_q \) under this bijection then the generating function we are interested in is 
\[ S(G, t) = \sum_{K \in G} t^{s(K)}. \] (1)

For \( G = GL_n(F_q) \), instead of substituting \(-1\) we substitute the \( q - th \) root of the unity in (1) to get: (see theorem 3.1):

Theorem. 
\[ \sum_{K \in GL_n(F_q)} \omega_q^{s(K)} = (q - 1)^{n-1} q^{n(n-1)/2} [n-1]_q!. \]

We note also that our results were obtained using algebraic tools such as the Bruhat decomposition. This approach enables us to present the generating function as a multiplicative formula which turns out to have a Mahonian distribution, meanwhile giving a new interpretation of the Mahonian distribution. We note that two other approaches to the case of type A were proposed to us by Alex Samorodnitzky and by the referee of the doctorate thesis of the first author.

The rest of this paper is organized as follows: In Section 2 we survey the Bruhat decomposition for type A and for type C. In Section 3 we prove our main theorem about the sign balance for type A in the most generality i.e. for the groups \( GL_n(F_q) \). In Section 4 we present our results for type C and in the appendix we extend about the connection of our work to the Cyclic sieving phenomenon.

2 Preliminaries

2.1 Finite groups of Lie type A

Let \( F \) be any field and let \( G = GL_n(F) \) be the group of invertible matrices over \( F \). Let \( H \) be the subgroup of \( G \) consisting of the diagonal matrices.
This is a choice of a torus for $G$. It is easy to show that the normalizer of $H$, $N(H)$, is the group of monomial matrices (i.e. each row and column contains exactly one non-zero element). The quotient $N(H)/H$ is called the Weyl group of type $A$, and is isomorphic to $S_n$, the group of permutations on $n$ letters. The Borel subgroup $B^+$ of the group $G$ consists of the upper triangular matrices in $G$. The opposite Borel subgroup, consisting of the lower triangular matrices, is denoted by $B^-$. We denote by $U^+$ and $U^-$ the groups of upper and lower triangular matrices (respectively,) with 1-s along the diagonal.

The Weyl group $S_n$ has a set of Coxeter generators $\{s_1, ..., s_{n-1}\}$, where $s_i$ can be realized as the transposition $(i, i+1)$. We define also the length of a permutation $\pi \in S_n$ with respect to the Coxeter generators to be:

$$\ell(\pi) = \min\{r \in \mathbb{N} : \pi = s_{i_1} \cdots s_{i_r}, \text{for some } i_1, ..., i_r \in [0, n-1]\}.$$ 

It is well known that for every $\pi \in S_n$

$$\ell(\pi) = inv(\pi)$$

where

$$inv(\pi) = |\{(i, j)| \pi(i) > \pi(j), 1 \leq i < j \leq n\}|.$$

**Proposition 2.1.** For every finite field $\mathbb{F}$ with $q$ elements the order of $GL_n(\mathbb{F})$ is

$$q\binom{n}{2}(q-1)^n[n]_q!$$

### 2.2 The Bruhat Decomposition for type A

The Bruhat decomposition is a way to write every invertible matrix as a product of two triangular matrices and a permutation matrix. We start with the following definitions:

For every permutation $\pi \in S_n$ we identify $\pi$ with the matrix:

$$[\pi]_{i,j} = \begin{cases} 1 & i = \pi(j) \\ 0 & \text{otherwise} \end{cases}$$

Define for every $\pi \in S_n$:

$$U_\pi = U^- \cap (\pi U^- \pi^{-1}).$$
\( \pi \) consists of the matrices with 1-s along the diagonal and zeros in place \((i, j)\) whenever \(i < j\) or \(\pi^{-1}(i) < \pi^{-1}(j)\). This is an affine space of dimension \(\binom{n}{2} - \ell(\pi)\) over \(\mathbb{F}\). (\(\ell(\pi)\) is the length of \(\pi\) with respect to the Coxeter generators).

Now, given \(g \in G\), we can column reduce \(g\) by multiplying on the right by Borel matrices in order to get an element \(gb^{-1}\) satisfying the following condition:

The right most nonzero entry in each row is 1
and it is the first nonzero entry in its column. (*)

Those "leading entries" form a permutation matrix corresponding to \(\pi \in S_n\).

Now we can use \(\pi^{-1}\) to rearrange the columns of \(gb^{-1}\) in order to get \(gb^{-1}\pi^{-1} = u \in \mathbb{U}_\pi\), i.e., \(g = u\pi b\). This is called the Bruhat decomposition of the matrix \(g\). One can prove that this decomposition is unique, and thus we have a partition of \(G\) into double cosets indexed by the elements of the Weyl group \(S_n\).

If \(\pi \in S_n\) then the double coset indexed by \(\pi\) decomposes into left \(\mathbb{B}^+\)-cosets in the following way: For every choice of \(u \in \mathbb{U}_\pi\), \(u\pi\) is a representative of the left coset \(u\pi\mathbb{B}^+\). Thus a general representative of the double coset \(\mathbb{U}_\pi\) can be taken as matrix of the form \((\ast)\), with every column filled with free parameters beyond the leading 1.

We summarize the information we gathered about the Bruhat decomposition for type \(A\) in the following:

**Proposition 2.2.** The group \(GL_n(\mathbb{F})\) is a disjoint union of double cosets of the form \(\mathbb{U}_\pi\pi\mathbb{B}^+\), where \(\pi\) runs through \(S_n\). Every double coset decomposes into cosets of the form \(A\mathbb{B}^+\) where \(A\) is a general representative of the form \((\ast)\). The number of free parameters in \(A\) is equal to \(\binom{n}{2} - \ell(\pi)\).

Here is an example of the coset decomposition for \(GL_3(\mathbb{Z}_2)\):

\[
\mathbb{U}_1\mathbb{B}^+ = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & \gamma & 1 \end{pmatrix} \mathbb{B}^+ \mid \alpha, \beta, \gamma \in \mathbb{Z}_2 \right\} \\
\mathbb{U}_{s_1s_1}\mathbb{B}^+ = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \alpha & \beta & 1 \end{pmatrix} \mathbb{B}^+ \mid \alpha, \beta \in \mathbb{Z}_2 \right\} \\
\mathbb{U}_{s_2s_2}\mathbb{B}^+ = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 0 & 1 \\ \beta & 1 & 0 \end{pmatrix} \mathbb{B}^+ \mid \alpha, \beta \in \mathbb{Z}_2 \right\}
\]
2.3 Lie Type C

Let $J$ denote the $n \times n$ matrix

\[
\begin{pmatrix}
0 & \cdots & 1 \\
0 & \cdots & 0 \\
& \ddots & \ddots \\
1 & \cdots & 0
\end{pmatrix}
\]

and let

\[M = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \]

The Lie group of type $C$, or the symplectic group, is defined over the field $\mathbb{F}$ by:

\[Sp_{2n}(\mathbb{F}) = \{ A \in SL_{2n}(\mathbb{F}) \mid A^T MA = M \}.\]

This is the set of fixed points of the automorphism $\varphi : SL_{2n}(\mathbb{F}) \rightarrow SL_{2n}(\mathbb{F})$ given by: $\varphi(A) = M^{-1}(A^T)^{-1}M$.

An alternative way to present the symplectic group is the following: We define first a bilinear form on $\mathbb{F}^{2n}$:

**Definition 2.3.** For every $x = (x_1, \ldots, x_{2n}), y = (y_1, \ldots, y_{2n}) \in \mathbb{F}^{2n}$

\[B(x, y) = \sum_{i=1}^{n} x_i \cdot y_{2n+1-i} - \sum_{i=n+1}^{2n} x_i \cdot y_{2n+1-i} \]

Denoting by $\{x_1, \ldots, x_{2n}\}$ the set of columns of $X$ it is easy to see that $X \in Sp_{2n}(\mathbb{F})$ if and only if the columns satisfy the following set of equations:

\[B(x_i, x_j) = \begin{cases} (-1)^{i-j} & i + j = 2n + 1 \\ 0 & i + j \neq 2n + 1 \end{cases} \]
We end this section with the following well known fact:

**Proposition 2.4.** (See for example [4, p.35])
For every finite field \( \mathbb{F} \) with \( q \) elements the order of \( \text{Sp}_{2n}(\mathbb{F}) \) is:
\[
q^{n^2} (q - 1)^n [2]_q \cdots [2n]_q
\]

**2.4 Bruhat Decomposition for Type C**

In order to be able to present the Bruhat decomposition for type C, we must first define a Borel subgroup for \( \text{Sp}_{2n}(\mathbb{F}) \). We present this subject following [8]. Note that although the exposition of [8] deals with groups over algebraically closed fields, the results hold also over finite fields. Start with the Borel subgroup \( B^+ \), chosen for type A, consisting of the upper triangular matrices.

If \( X = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in B^+ \), then \( \varphi(X) = \begin{pmatrix} J(C^T)^{-1}J & J(C^T)^{-1}B^T(A^T)^{-1}J \\ 0 & J(A^T)^{-1}J \end{pmatrix} \in B^+ \). (The automorphism \( \varphi \) was defined in Section 2.3). Moreover, the automorphism \( \varphi \) keeps the Borel subgroup \( B^+ \), as well as the groups of diagonal and monomial matrices (denoted by \( H \) and \( N(H) \) respectively in Section 2.1) invariant. Thus we can take \( B^+_C = \text{Sp}_{2n}(\mathbb{F}) \cap B^+ \) and \( B^-_C = \text{Sp}_{2n}(\mathbb{F}) \cap B^- \) as the Borel subgroup and the opposite Borel subgroup of \( \text{Sp}_{2n}(\mathbb{F}) \) respectively, and similarly for \( H \) and \( N(H) \).

If we label the basis elements of the space on which \( \text{Sp}_{2n}(\mathbb{F}) \) acts by indices \( n, n-1, \ldots, 1, -1, \ldots, -n \), then the Weyl group of type C can be realized as the group of those permutations \( \pi \in S_{2n} \) such that \( \pi(-i) = -\pi(i) \). This is called the octahedral group and denoted \( B_n \). Looking at \( B_n \) as a Coxeter group, it has the following set of generators:

\[
S = \{ s_0, s_1, \ldots, s_{n-1} \}
\]

where \( s_0 \) is the transposition which permutes 1 and \(-1\) and \( s_i \) permutes \( i \) and \( i+1 \), for \( 1 \leq i < n \). Just as in the case of type A, we define here the length of a \( B_n \)-permutation as:

\[
\ell(\pi) = \min \{ r \in \mathbb{N} : \pi = s_{i_1} \cdots s_{i_r}, \text{ for some } i_1, \ldots, i_r \in [0, n-1] \}.
\]

We define also the groups \( U^+_C = U^+ \cap \text{Sp}_{2n}(\mathbb{F}) \) and \( U^-_C = U^- \cap \text{Sp}_{2n}(\mathbb{F}) \) to be the upper and lower unipotent subgroups respectively. For every \( \pi \in B_n \) we define \( U^*_C = U^-_C \cap (\pi U^- \pi^{-1}) \). \( U^*_C \) is the intersection of \( \text{Sp}_{2n}(\mathbb{F}) \) with the set of matrices with 1’s along the diagonal and zeros at entries in location
(i, j) whenever i < j or \( \pi^{-1}(i) < \pi^{-1}(j) \). This is an affine space of dimension \( n^2 - \ell(\pi) \). (Here, \( \ell(\pi) \) is the length function of \( B_n \)).

Now, we can use the Bruhat decomposition of \( GL_{2n}(F) \) to produce the Bruhat decomposition for \( Sp_{2n}(F) \). Let \( g \in Sp_{2n}(F) \). Consider \( g \) as an element of \( GL_{2n}(F) \) and write \( g = u\pi b \) where \( \pi \in S_{2n}, u \in U_{\pi} \) and \( b \in B^+ \).

We have:

\[
g = \varphi(g) = \varphi(u)\varphi(\pi)\varphi(b),
\]

but from the uniqueness of the decomposition in \( GL_{2n}(F) \) we have:

\[
\varphi(u) = u, \quad \varphi(\pi) = \pi h^{-1}, \quad \varphi(b) = hb
\]

where \( h \) is diagonal and thus \( \pi \in B_n \) and \( b \in B^+_C \). This gives us the Bruhat decomposition. The description of the double cosets and the coset representatives is similar to the one given for type A, with the exception that here we have to intersect with \( Sp_{2n}(F) \).

We summarize the information we gathered about the Bruhat decomposition for type C in the following:

**Proposition 2.5.** The group \( Sp_{2n}(F) \) decomposes into double cosets of the form \( U_{\pi}^C \pi B^+_C \), where \( \pi \) runs through \( B_n \). Every double coset decomposes into cosets of the form \( AB^+_C \) where \( A \) is a general representative of the form \( [\pi] \). The number of free parameters in \( A \) is equal to \( n^2 - \ell(\pi) \).

### 3 Sign Balance for Type A

Let \( p \) be a prime number and let \( q \) be a power of \( p \). Denote by \( F_q \) the field with \( q \) elements. We prove the following:

**Theorem 3.1.**

\[
\sum_{K \in GL_n(F_q)} \omega_q^{s(K)} = -(q - 1)^{n-1}q^{\binom{n}{2}}[n - 1]_q!.
\]

where \( s(K) \) is the sum in \( F_q \) of the images of the elements of the matrix \( K \) under any bijection between \( F_q \) and the set \( \{0, ..., q - 1\} \) sending 0 of \( F_q \) to 0 and \( \omega_q \) is a primitive complex \( q \)-th root of unity.

The following corollary is immediate:

**Corollary 3.2.** The number of matrices in \( GL_n(F_q) \) whose sum of entries is 0 in \( F_q \) is exactly

\[
[n - 1]_q!(q - 1)^{n-1}q^{\binom{n}{2}}(q^{n-1} - 1)
\]
while for every \(1 \leq i \leq q - 1\), the number of matrices in \(GL_n(\mathbb{F}_q)\) whose entries add up to \(i\) in \(\mathbb{F}_q\) is:

\[
[n - 1]_q! (q - 1)^{n-1} q^{(n^2)/2} + n - 1.
\]

Along this chapter we call a matrix or a column of a matrix odd if its sum of entries is not 0 in \(\mathbb{F}_q\) and even otherwise. A coset consisting entirely of odd matrices will be called an odd coset. In order to prove the theorem, we take the following direction: Instead of summing over the whole group of matrices, we sum over every coset separately. It turns out that some of the cosets are sign-balanced in the sense that they contain for each \(i\) the same number of matrices with sum of entries equals to \(i\), while the others have only odd matrices.

The following lemma identifies the sign-balanced cosets.

**Lemma 3.3.** Let \(A\) be a general representative of the double coset \(U \pi B^+\) corresponding to \(\pi \in S_n\). Make some substitution in the free parameters of \(A\) to get a coset representative and call it \(\tilde{A}\). If \(\tilde{A}\) has an odd column which is not the last one then the coset \([\tilde{A}] = \{\tilde{A}B | B \in B^+\}\) is sign-balanced, i.e.,

\[
\sum_{K \in \tilde{A}B^+} \omega_q^{s(K)} = 0.
\]

**Proof.** Denote by \(j\) the first odd column of \(\tilde{A}\). For every matrix \(X \in \tilde{A}B^+\), the \(k\)-th column of \(X\) is a linear combination of the first \(k\) columns of \(\tilde{A}\). Now, for every \(B = B^{(0)} \in B^+\), construct the matrices \(B^{(i)} \in B^+\) \((1 \leq i \leq q - 1)\), such that \(B^{(i)}\) differs from \(B\) only in the entry in location \((j, n)\) and \(B^{(i)}_{j,n} \neq B^{(k)}_{j,n}\) if \(i \neq k\). Note that the set \(\{s(\tilde{A}B^{(i)}) | 0 \leq i \leq q - 1\}\) forms a full system of representatives of \(\mathbb{F}_q\). This gives us a partition of the coset \(\tilde{A}B^+\) into \(q\) equal pieces, each of size \(\frac{q^{(n+1)(q-1)}}{q}\). (Note that \(|\tilde{A}B^+| = |B^+| = (q - 1)^n q^{(n^2)/2}\)).

**Example 3.4.** The following example illustrates the bijection where \(G = GL_4(\mathbb{Z}_2)\).

Let

\[
\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \in S_4.
\]
Then a possible representative of one of the cosets of $\cup_{\pi}B^+$ is:

$$\tilde{A} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1
\end{pmatrix}.$$

Column $j = 2$ of $\tilde{A}$ is odd.

Now consider the Borel matrix:

$$B = \begin{pmatrix}
1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

which produces the element

$$\tilde{A}B = \begin{pmatrix}
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1
\end{pmatrix}.$$

Toggle the element $B_{2,4}$ to get

$$B' = \begin{pmatrix}
1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

and the corresponding element of $\tilde{A}B^+$ is

$$\tilde{A}B' = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1
\end{pmatrix}.$$

Note that $\tilde{A}B$ and $\tilde{A}B'$ have opposite parity.

Now Calculate:

$$\sum_{K \in \tilde{A}B^+} \omega_q^{s(K)} = \sum_{i=0}^{q-1} \sum_{K \in \tilde{A}B^+} \omega_q^i$$

$$= (q - 1)^n q^{\binom{n}{2} - 1} \left( \sum_{i=0}^{q-1} \omega_q^i \right) = 0$$

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Lemma 3.5. Let $\pi \in S_n$. Let $A$ be a general representative of the double coset $U_\pi \pi B^+$ corresponding to $\pi \in S_n$. Make some substitution in the free parameters of $A$ to get a coset representative, and call it $\tilde{A}$. If the first $n-1$ column sums of $\tilde{A}$ are $0 \pmod{p}$, then the imbalance calculated inside $\tilde{A}B^+$ is:

$$\sum_{K \in \tilde{A}B^+} \omega_q^{s(K)} = -(q-1)^{n-1}q^{\binom{n}{2}}.$$

Proof. The last column of $\tilde{A}$ contains only one nonzero element, which must be 1. (Indeed, by the construction of $A$, the last column of $A$ contains a '1', coming from the permutation $\pi$, say, in the row numbered $k$. The entries located in location $(l, n)$ for $l < k$ are zeroes by the definition of $A$. On the other hand, every row numbered $l > k$ must contain a '1' coming from $\pi$ which is located in some column $j < n$. All of the entries of the row $l$ to the right of $j$, including the entry $(l, n)$ must be zero).

Let $K = \tilde{A}B \in \tilde{A}B^+$. Since the first $n-1$ columns of $\tilde{A}$ add up to $0 \pmod{p}$, $s(K)$ depends uniquely on the element of $B$ located in the entry numbered $(n, n)$, which can be any element of $\mathbb{F}_q \setminus \{0\}$. For every $0 \neq \alpha \in \mathbb{F}_q$ there are $q^{\binom{n}{2}}(q-1)^{n-1}$ Borel matrices $B = (b_{ij})$ with $b_{nn} = \alpha$, and thus the imbalance of this coset is:

$$\sum_{K \in \tilde{A}B^+} \omega_q^{s(K)} = (q-1)^{n-1}q^{\binom{n}{2}}(\omega_q + \omega_q^2 + \cdots + \omega_q^{q-1}) = -(q-1)^{n-1}q^{\binom{n}{2}}$$

Lemma 3.6. Let $\pi \in S_n$. The double coset $U_\pi \pi B^+$ contains odd cosets if and only if $\pi(n) = n$.

Proof. Let $A$ be a general representative of the double coset $U_\pi \pi B^+$. If $\pi(n) = n$ then since the entries of the last row of $A$ are not located above or to the right of any '1', the last row of $A$ has no limitation on its parameters.
Hence, every column, except for the last one, has parameters which we can choose such that it will be even. These choices yield odd cosets.

On the other hand, if $\pi(n) \neq n$ then there is some $i < n$ such that $\pi(i) = n$. This forces the $i$-th column of $A$ to be odd and the corresponding coset to be sign-balanced.

We turn now to the proof of Theorem 3.1: In order to calculate the imbalance we have to count only non-balanced cosets. By Lemma 3.6 we are interested only in the double cosets corresponding to permutations $\pi \in S_{n-1}$. Every such double coset has a total of $\binom{n}{2} - \ell(\pi)$ parameters, which amounts to $q^{(n)}(\pi)$ different cosets. In order to get an odd coset with sum of entries not equal to 0 modulo $q$ we have to choose the parameters in such a way that every column except for the last one will sum up to 0 (mod $p$). Exactly $\frac{1}{q}$ of the choices in each column give an odd column and thus there are $q^{(n)}(\pi) - (n-1)$ such cosets in each double coset. By Lemma 3.5 each coset contributes $-q^{(n)}(q - 1)^{n-1}$ and we have in total:

$$\sum_{K \in \text{GL}_n(\mathbb{F}_q)} \omega^s(K) = \sum_{\pi \in S_n} \frac{-(q-1)^{n-1} q^{(n)}(n-1-\ell(\pi))}{q^{(n)}(\pi) - (n-1)}$$

Note that the forth equality of the last calculation follows from the obvious bijection inside $S_{n-1}$ given by multiplying by the longest permutation.

$$= -(q-1)^{n-1} q^{(n)} \sum_{\pi \in S_{n-1}} q^{(n-1)-\ell(\pi)}$$

$$= -(q-1)^{n-1} q^{(n)} \sum_{\pi \in S_{n-1}} q^{\ell(\pi)}$$

$$= -(q-1)^{n-1} q^{(n)} [n-1]_q!$$

Note that the forth equality of the last calculation follows from the obvious bijection inside $S_{n-1}$ given by multiplying by the longest permutation.

4 Sign Balance for Type $C$

In this section we prove the following result:
Theorem 4.1.
\[ \sum_{K \in Sp_{2n}(\mathbb{Z}_2)} (-1)^{o(K)} = -2^{n^2} \cdot [2]_2 [4]_2 \cdots [2n - 2]_2 \]
where \( o(K) \) is the number of 1's in \( K \).

The following corollary is immediate:

Corollary 4.2. The number of even matrices in \( Sp_{2n}(\mathbb{Z}_2) \) is exactly
\[ 2^{n^2-1} [2]_2 \cdots [2n - 2]_2 ([2n]_2 - 1) \]
while the number of odd matrices is
\[ 2^{n^2-1} [2]_2 \cdots [2n - 2]_2 ([2n]_2 + 1). \]

In proving the theorem, we use the same strategy used for type A. We sum over each coset separately and distinguish between odd and sign-balanced cosets. The following lemma identifies the sign-balanced cosets.

Lemma 4.3. Let \( A \) be a general representative of the double coset \( U \pi \pi B \) corresponding to \( \pi \in B_n \). Make some substitution in the free parameters of \( A \) to get a coset representative, and call it \( \tilde{A} \). If \( \tilde{A} \) has an odd column which is not the last one, then the coset \( [\tilde{A}] = \{ \tilde{A}B \mid B \in B \} \) is sign-balanced, i.e.,
\[ \sum_{K \in \tilde{A}B} (-1)^{o(K)} = 0. \]

Proof. An element of \( B \) is an invertible upper triangular matrix which is also symplectic. If we take \( b \) to be an upper triangular matrix with a set of columns \( \{ v_1, ..., v_{2n} \} \) then, as was stated in Section 2.3, forcing it to be symplectic is equivalent to imposing the equations (note that we are working over \( \mathbb{Z}_2 \)):
\[ B(v_i, v_j) = \begin{cases} 1 & i + j = 2n + 1 \\ 0 & i + j \neq 2n + 1 \end{cases} \]
As is easy to check, the equations of the form \( B(v_i, v_i) = 0 \) are trivial over \( \mathbb{Z}_2 \). The equations of the form \( B(v_i, v_{2n+1-i}) = 1 \) are also trivial. (Indeed, \( B(v_i, v_{2n+1-i}) = \sum_{k=1}^{2n} b_{k,i} \cdot b_{2n+1-k,2n+1-i} \) but since \( b \) is upper triangular, over \( \mathbb{Z}_2 \) we have \( b_{ii} \cdot b_{2n+1-i,2n+1-i} = 1 \) and the other summands vanish since for \( k > i \) one has \( b_{ki} = 0 \) and for \( k > 2n + 1 - i \) one has \( b_{2n+1-k,2n+1-i} = 0 \).)
Now, the only nontrivial equations involving the parameters appearing in the last column are the ones of the form:

\[ B(v_i, v_{2n}) = 0, \quad (2 \leq i \leq 2n - 1) \]

and each such equation can be written in such a way that the parameters of the last column are free while the parameters of the first row depend on them. Explicitly, we write the equation \( B(v_i, v_{2n}) = 0 \) as

\[
b_{1i} = \sum_{k=2}^{2n} b_{ki} \cdot b_{2n+1-k,2n}.
\]

Note that the elements of the last column of the matrix \( b \) have no appearance as a part of a linear combination in any place other than the first row. This is justified by the fact that every nontrivial equation, involving the first row, which we have not treated yet must be of the form \( B(v_i, v_j) = 0 \) for \( 1 \leq i < j \leq 2n - 1 \). Thanks to the upper triangularity of \( b \), the elements laying in the first row vanish in these equations.

Let us look at the following example:

\[
b = \begin{pmatrix}
1 & b_{12} & b_{13} & b_{14} \\
0 & 1 & b_{23} & b_{24} \\
0 & 0 & 1 & b_{34} \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

The only nontrivial equations involving the last column are: \( B(v_2, v_4) = 0 \) and \( B(v_3, v_4) = 0 \).

These equations can be written as:

\[
b_{12} = b_{34} \]
\[
b_{13} = b_{24} + b_{23} \cdot b_{34}
\]

so after intersecting with \( Sp_{2n}(\mathbb{Z}_2) \), the matrix \( b \) looks like:

\[
b = \begin{pmatrix}
1 & b_{34} & b_{24} + b_{23} \cdot b_{34} & b_{14} \\
0 & 1 & b_{23} & b_{24} \\
0 & 0 & 1 & b_{34} \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

The elements of the last column appear only in the first row and in the equations of the form \( B(v_i, v_j) = 0 \) the elements located in the first row vanish.
Note that in this case all of the parameters outside the first row are free. This doesn’t hold in general. Nevertheless, as we have proven, we can arrange the parameters such that the elements of the last column reappear only in the first row.

Returning now to the proof, we have two cases:

- The first column of $\tilde{A}$ is odd. In this case we can use the element located in place $(1,2n)$ to construct a bijection between odd and even matrices inside the coset $\tilde{A}B_C^+$. This is done in the same way described earlier for type A: Divide $B_C^+$ into two disjoint subsets:

$$B_{C0}^+ = \{ T = (t_{i,j}) \in B_C^+ \mid t_{1,2n} = 0 \}$$
$$B_{C1}^+ = \{ T = (t_{i,j}) \in B_C^+ \mid t_{1,2n} = 1 \}.$$  

For every matrix $X \in \tilde{A}B_C^+$, the $k$-th column of $X$ is a linear combination of the first $k$ columns of $\tilde{A}$. Now, due to the fact that the parameter appearing in the location $(1,2n)$ has no other appearance, for every $B \in B_{C0}^+$ there is some $B' \in B_{C1}^+$ such that $B$ and $B'$ differ only in the entry numbered $(1,2n)$.

Note that $\tilde{A}B$ and $\tilde{A}B'$ are obtained from $\tilde{A}$ by the same sequence of column operations except for the first column which was used in producing $AB$ but was not used in producing $\tilde{A}B'$. Hence $\tilde{A}B$ and $\tilde{A}B'$ have opposite parity. This gives us a bijection between the odd and the even matrices of the coset $\tilde{A}B_C^+$.

- The first column of $\tilde{A}$ is even. Denoting by $j$ the number of the first odd column of $\tilde{A}$, we use the element located in place $(j,2n)$ to construct a bijection between the odd and even matrices inside $A B_C^+$ in the same way as in the previous case. Note that since the element located in place $(j,2n)$ in the matrices of the Borel subgroup can reappear only in the first row, it affects only the first column of $\tilde{A}$, which is even.

We turn now to treat the odd cosets.

**Lemma 4.4.** Let $\pi \in B_n$. Let $A$ be a general representative of the double coset $U_{\pi}B_C^+$ corresponding to $\pi \in B_n$. Make some substitution in the free parameters of $A$ to get a coset representative, and call it $\tilde{A}$. If all of the first
2n − 1 columns of \( \tilde{A} \) are even then all of the matrices belonging to the coset \( \tilde{A}B^{+}_C \) are odd. The imbalance calculated inside this coset is:

\[
\sum_{K \in \tilde{A}B^{+}_C} (-1)^{o(K)} = -|B^{+}_C| = -2n^2.
\]

**Proof.** The last column of \( \tilde{A} \) is always odd and thus since all other columns of \( \tilde{A} \) are even, \( \tilde{A} \) itself is an odd matrix and the same holds for \( \tilde{A}B \) for every \( B \in B^{+}_C \). The size of the coset \( \tilde{A}B^{+} \) is \( 2n^2 \), and the result follows. \( \square \)

**Lemma 4.5.** Let \( \pi \in B_n \). The double coset \( U^{C}_\pi \pi B^{+}_C \) contains odd cosets if and only if \( \pi(2n) = 2n \).

**Proof.** Let \( A \) be a general representative of the double coset \( U^{C}_\pi \pi B^{+}_C \). Write \( U = A\pi^{-1} \). Then \( U \in U^{C}_\pi \) is a lower triangular matrix and since \( \pi(2n) = 2n \) (which implies also \( \pi(1) = 1 \)), the first column as well as the last row of \( U \) contain \( 2n - 1 \) parameters. Note that \( U^T \in B^{+}_C \) and thus by the considerations described in Lemma 4.3, the parameters appearing in the last column of \( U^T \) can reappear only in the first row of \( U^T \). We conclude that the parameters of the last row of \( U \) can reappear only in the first column of \( U \). Now, for every column numbered \( 2 \leq k \leq 2n - 1 \) in \( U \) and for every choice of the first elements of the column numbered \( k \), we are free to choose the parameter located in the bottom of this column, \( (2n,k) \), in such a way that the column will be even. The parameter located in the place \( (2n,1) \) has no other appearance and thus we can choose all of the first \( 2n - 1 \) columns of \( U \) to be even. Getting back to the general representative \( A \), since \( \pi(2n) = 2n \), we have also \( \pi(1) = 1 \) and thus \( A \) and \( U \) differ only in the columns \( 1 < k < 2n \) so that the proof works also for \( A \).

On the other hand, if \( \pi(2n) \neq 2n \) then \( \pi \) contains a column numbered \( k < 2n \) which has only one nonzero element, located in place \( (2n,k) \). By the construction of the general representative \( A \), there are only zeros above the 1 coming from the permutation and thus this odd column appears also in \( A \). By the previous lemma, the coset \( \{ AB \mid B \in B^{+}_C \} \) is sign-balanced. \( \square \)

Now, we have to count the imbalance on the odd cosets. By Lemma 4.5, we are interested only in the double cosets corresponding to the permutations \( \pi \in B_{n-1} \). The following lemma shows how to count.

**Lemma 4.6.** Let \( \pi \in B_n \) such that \( \pi(n) = n \). The double coset \( U^{C}_\pi \pi B^{+}_C \) contains exactly \( 2^{(n-1)2 - \ell(\pi)} \) odd cosets.
Proof. Let \( A \) be representative of the double coset \( \mathcal{U}_C^\pi \mathcal{B}_C^\pi \). As was shown in the previous lemma, the parity of each one of the first \( 2n - 1 \) columns of \( A \) is determined by the free parameter in its bottom. Since there are a total of \( n^2 - \ell(\pi) \) free parameters and exactly \( 2n - 1 \) 'bottom parameters', the number of substitutions of parameters giving all of the \( 2n - 1 \) first columns even is \( 2^{n^2-\ell(\pi)-(2n-1)} \). This is also the number of odd cosets in the double coset \( \mathcal{U}_C^\pi \).

We turn now to the proof of Theorem 4.1. In order to calculate the imbalance we have to count only odd cosets. By Lemma 4.5 we are interested only in the double cosets corresponding to permutations \( \pi \in \mathcal{B}_{n-1} \). By Lemma 4.6 every such double coset contains \( 2^{(n-1)^2-\ell(\pi)} \) odd cosets. By Lemma 4.4 each odd coset contributes \(-2n^2\) to the imbalance, and we have in total:

\[
\sum_{K \in S_{p2n}(\mathbb{Z}_2)} (-1)^{o(K)} = \sum_{\pi \in \mathcal{B}_{n-1}^{\pi \cdot n = n}} -2n^2 \cdot 2^{(n-1)^2-\ell(\pi)}
\]

\[
= -2n^2 \sum_{\pi \in \mathcal{B}_{n-1}} 2^{(n-1)^2-\ell(\pi)}
\]

\[
= -2n^2 \sum_{\pi \in \mathcal{B}_{n-1}} 2^{\ell(\pi)}
\]

\[
= -2n^2 [n-1]_2!
\]

\[
= -2n^2 \cdot [2]_2[4]_2 \cdots [2n-2]_2
\]

5 Appendix

Our results can be seen as an example of the Cyclic sieving phenomenon introduced in the paper of Reiner, Stanton and White [5]. In this section we present this point of view. We start with the definition of the cyclic sieving phenomenon, following [5].

Let \( X \) be a finite set and let \( C_n \) be the cyclic group acting on \( X \). Let \( X(t) \) be a polynomial in \( t \) having nonnegative integer coefficients, with the property that \( X(1) = |X| \). One can think of \( X(t) \) as a generating function for \( X \). Fix an isomorphism \( \omega \) of \( C_n \) with the complex \( n \)-th roots of unity. Then it is easy to see that the following are equivalent:
1. For every $c \in C_n$:

$\left[ X(t) \right]_{t=\omega(c)} = |\{ x \in X \mid c(x) = x \}|.$

2. The coefficient $a_l$ defined uniquely by the expansion

$$X(t) = \sum_{l=0}^{n-1} a_l t^l \mod t^n - 1$$

has the following interpretation: $a_l$ counts the number of $C_n$-orbits on $X$ for which the stabilizer-order divides $l$. In particular, $a_0$ counts the total number of $C_n$-orbits on $X$, and $a_1$ counts the number of free $C_n$ orbits on $X$.

When either of these two conditions holds, one says that $(X, X(t), C_n)$ has the Cyclic sieving phenomenon.

In our case, $X$ is either the set $GL_n(F_q)$ or $Sp_{2n}(Z_2)$ and

$$X(t) = \sum_{K \in G} t^{o(K)}$$

where $G$ is one of the above groups while the group acting on $X$ is $C_q$ in the first case and $C_2$ in the second. We describe now the action of $C_q$ in the type A case. The other case is very similar.

Note first that if $q$ is not a prime number then we have to define a linear order on $F_q$. For example, consider $F_q$ as a vector space over $Z_p$, $p$ prime and take the lexicographic order of the coordinates).

Now, for every matrix $K \in GL_n(F_q)$, we have two cases:

- $K = \tilde{A}B$ where $\tilde{A}$ is a representative of an odd coset. In this case the action of $C_q$ is trivial.

- $K = \tilde{A}B$ where $\tilde{A}$ is a representative of a sign-balanced coset. In this case we denote by $j$ the first odd column of $\tilde{A}$ which is not the first one and define for the generator $c$ of $C_q$:

$$c \cdot \tilde{A}B = AB'$$

where $B'$ is obtained from $B$ by replacing $B_{j,n}$ by the successor of this element with respect to the prescribed order.
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