Existence, Uniqueness, Analyticity, and Borel Summability for Boussinesq Equations

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Abstract

Through Borel summation methods, we analyze the Boussinesq equations for coupled fluid velocity and temperature fields:

\begin{align}
    u_t - \nu \Delta u &= -P[u \cdot \nabla u - ae_2 \Theta] + f \\
    \Theta_t - \mu \Delta \Theta &= -u \cdot \nabla \Theta.
\end{align}

We prove that an equivalent system of integral equations in the Borel variable $p \in \mathbb{R}^+$ dual to $1/t$ has a unique solution in a class of exponentially bounded functions, implying the existence of a classical solution to (1) in a complex $t$-region that includes a real positive time axis segment. For analytic initial data and forcing, it is shown that the solution is Borel summable, implying that that formal series in powers of $t$ is Gevrey-1 asymptotic, and within the time interval of existence, the solution remains analytic with the same analyticity strip width as the initial data and forcing. We also determine conditions on the integral equation solution that improve the estimate for existence time.

Keywords: Boussinesq equation, Borel summability

1. Introduction

We consider the Boussinesq equations for coupled fluid velocity and temperature fields derived under the assumption that the temperature induced density has negligible effect on momentum but causes a significant buoyant

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force. The corresponding evolution equations for $u: \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ and $\Theta: \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$ for dimension $d = 2, 3$ in non-dimensional form are:

$$u_t - \nu \Delta u = -P[u \cdot \nabla u - ae_2 \Theta] + f, \quad u(x, 0) = u_0(x)$$

$$\Theta_t - \mu \Delta \Theta = -u \cdot \nabla \Theta, \quad \Theta(x, 0) = \Theta_0(x)$$

where $P = I - \nabla \Delta^{-1}(\nabla \cdot)$ is the Hodge projection operator to the space of divergence free vector fields, $e_2$ is the unit vector aligned opposite to gravity, the parameter $a$ is proportional to gravity, and $(u, \Theta)$ are the non-dimensional fluid velocity and temperature fields. We assume the initial conditions $u_0$ and the forcing $f$ are divergence free and, for the sake of simplicity, assume $f$ to be time independent, although time dependence with some restrictions can be accommodated in a similar framework. Using standard energy methods, see for instance [22], existence of Leray type solutions in $L^\infty(0, T, L^2(\mathbb{R}^d)) \cap L^2(0, T, H^1(\mathbb{R}^d))$ follows easily for any $T > 0$. In $\mathbb{R}^2$ a unique classical global solution can be shown to exist for all time. In $\mathbb{R}^3$, local existence and uniqueness for Boussinesq equation are shown in $L^p(0, T, L^q(\mathbb{R}^d))$ for $d < p < \infty$ and $\frac{d}{p} + \frac{2}{q} \leq 1$. In $\mathbb{R}^3$ there is a unique solution under the additional assumption that the solution lies in $L^\infty(0, T, H^1(\mathbb{R}^3))$, see [4]. The case where $\mu = 0$ has also been considered in the literature, and global well-posedness is proved proved in [17] for $2 - d$.

In the problem above, the existence of classical solutions, globally in time, remains an open problem as it is for the limiting ($a \to 0$) Navier Stokes equation (NSE) in 3-D. Control of a higher order energy norm (like the $H^1$ norm of velocity) has remained a serious impediment despite extensive study of NSE. This motivates one to look for alternate formulations of existence that do not rely on energy bounds.

The primary purpose of this paper is to show that the Borel based methods, developed earlier in [10] and [13] in the context of Navier-Stokes equation, can be extended to other evolutionary PDEs (partial differential equations) such as the Boussinesq equation. This provides an alternate existence and uniqueness theory for a class of nonlinear PDEs. In this formulation, the question of global existence of solution to the PDE becomes one of asymptotics for known solution to the associated nonlinear integral equations. While the asymptotics are still difficult, it is interesting to note that an accelerated representation [13] (see [5] in the ensuing) for the related NSE results in a positive limiting kernel as $n \to \infty$, where majorization may be possible in terms of solution to a simpler integral equation. We also show
(Thm 2.3) here how information about solution to the integral equation on a finite interval in the dual variable for specific initial condition and forcing may be used to obtain better exponential bounds in the Borel plane implying a longer existence time for classical solutions to the associated PDEs.

Borel summability has been an active area of research. A vast literature has emerged recently in Borel summability theory, starting with the fundamental contributions of Ecalle (see e.g. [14] and [15]) whose consequences are far from being fully explored, and it is impossible to give a quick account of the breadth of this field (See for example [5] for more references). There has also been work in characterizing all small solutions for a generic system of ODEs [6] or difference equations [3]. There has been work on PDEs as well, starting with linear equations [19], [2] followed by general results for a class of nonlinear system of PDEs in complex sectors [7], [8]. A Borel based approach has also led to analysis of complex singularities for a specific PDE [9]. Recent developments include Navier-Stokes initial value problem (see [13], [12], [11]). Recently [20], numerical schemes have been suggested for nonlinear PDEs, based on a Borel plane reformulation. Thus, it is clear that the Borel based approach of the present paper is likely to have both theoretical and practical value. A bi-product of the present Borel based approach is that many analyticity properties of the PDE solution readily follow without additional arguments. For instance, the time analyticity for \( \Re t > \alpha \) follows from (4) after noting the solution to the integral equation is exponentially bounded in \( p \). While such analyticity results may also be obtained through other methods, see [18] and [16], it follows more readily from the current method. We also prove that the classical \( H^2(\mathbb{R}^d) \) solution, which is unique, has the Laplace transform representation given here, provided initial data and forcing in the Fourier-space are in \( L^1 \cap L^\infty \). Furthermore, for analytic initial data and forcing, we prove that the formal expansion in powers of \( t \) is Borel summable and hence Gevrey-1 asymptotic for small \( t \). As far as we know, these results are new and have not been obtained earlier for the Boussinesq equations though it is likely that these results can also be obtained through other methods. In the latter case, it is also shown that the associated power series in the Borel plane has a radius of convergence independent of size of initial data and forcing when initial data and forcing have a fixed number of Fourier modes; this is useful in computing the solution in the Borel plane.
2. Main Results

We first write the equations as integral equations in time in Fourier space. We denote the Fourier transform operator by $F$, the Fourier transform of $f$ by $\hat{f}$, and $\ast$ the Fourier convolution. As usual, a repeated index $j$ denotes the sum over $j$ from 1 to $d$. $P_k$ is the Fourier transform of the Hodge projection and has the representation

$$P_k \equiv \left(1 - \frac{k(k\cdot)}{|k|^2}\right).$$

Formal derivation based on inversion of the heat operator in Fourier space in (2) leads to the following integral equations:

\begin{align*}
\hat{u}(k, t) &= -\int_0^t e^{-\nu|k|^2(t-\tau)} \left(ik_j P_k[\hat{u}_j \hat{\ast} \hat{u} - ae_2 \hat{\Theta}](k, \tau) - \hat{f}(k)\right) d\tau \\
&\quad + e^{-\nu|k|^2t} \hat{u}_0(k) \\
\hat{\Theta}(k, t) &= -\int_0^t e^{-\mu|k|^2(t-\tau)} \left(ik_j [\hat{u}_j \hat{\ast} \hat{\Theta}](k, \tau)\right) d\tau + e^{-\mu|k|^2t} \hat{\Theta}_0(k).
\end{align*}

Definition 2.1. We introduce the norm $\| \cdot \|_{\gamma,\beta}$ for some $\beta \geq 0$ and $\gamma > d$:

$$\|\hat{f}\|_{\gamma,\beta} = \sup_{k \in \mathbb{R}^d} (1 + |k|)^\gamma e^{\beta|k|} |\hat{f}(k)|,$$

where $\hat{f}(k) = F[f(\cdot)](k)$.

Definition 2.2. We also use the space $L^1 \cap L^\infty$ with the norm defined by

$$\|\hat{f}\|_{L^1 \cap L^\infty} = \max \left\{ \int_{\mathbb{R}^d} |\hat{f}(k)| dk, \sup_{k \in \mathbb{R}^d} |\hat{f}(k)| \right\}.$$

In cases when results hold either for $\| \cdot \|_{\gamma,\beta}$ or $\| \cdot \|_{L^1 \cap L^\infty}$ norm, we will use $\| \cdot \|_N$ for brevity of notation.

We assume $\|(1 + |k|)^2(\hat{u}_0, \hat{\Theta}_0)\|_N < \infty$ and $\|\hat{f}\|_N < \infty$ in what follows. If $\| \cdot \|_N = \| \cdot \|_{\gamma,\beta}$ and $\beta > 0$ then the initial condition and forcing are real analytic in $x$ in a strip of width at least $\beta$.

\footnote{While at this stage derivation is formal, in the space of functions where existence is proved, it will become clear that the integral and differential formulations are equivalent.}
Theorem 2.1. (Boussinesq Existence and Uniqueness)

If \( \| (1 + | \cdot |)^2 (\hat{u}_0, \hat{\Theta}_0) \|_N < \infty \) and \( \| \hat{f} \|_N < \infty \), then the following statements hold:

i) The Boussinesq equation (3) has a solution \((\hat{u}, \hat{\Theta})(k, t)\) such that \( \| (\hat{u}, \hat{\Theta})(\cdot, t) \|_N < \infty \) for \( R^{\frac{1}{2}}_t > \omega \) for \( \omega \) sufficiently large\(^2\).

ii) The solution has the Laplace transform representation

\[
(\hat{u}, \hat{\Theta})(k, t) = (\hat{u}_0, \hat{\Theta}_0)(k) + \int_0^\infty (\hat{H}, \hat{S})(k, p)e^{-pt}dp
\]

where \((\hat{H}, \hat{S})\) is the unique solution to a set of integral equations in the space where \( \| (\hat{H}, \hat{S})(\cdot, p) \|_N e^{-\omega p} \in L^1(0, \infty) \). The corresponding \((u, \Theta)(x, t) = \mathcal{F}^{-1}[(\hat{u}, \hat{\Theta})(k)](x, t)\) is analytic in \( t \) for \( R^{\frac{1}{2}}_t > \omega \) and for \( \beta > 0 \) is analytic in \( x \) for any \( t \in \left[0, \frac{1}{\beta}\right)\) in a strip of width \( \beta \), where initial data and forcing are analytic.

iii) Further, this solution satisfies \( \| (1 + | \cdot |)^2 (\hat{u}, \hat{\Theta})(\cdot, t) \|_N < \infty \) for \( t \in (0, \omega^{-1}) \), implying at least the same regularity as initial conditions. Moreover, \((u, \Theta)(x, t)\) solves (2) and is the unique Boussinesq solution in \( L^\infty(0, T; H^2(\mathbb{R}^d)) \) when initial data and forcing in Fourier space satisfy given assumptions.

iv) A sufficient condition for global existence of smooth solution is that \( e^{-\omega p}\| (\hat{H}, \hat{S})(\cdot, p) \|_N \in L^1(0, \infty) \) for any \( \omega > 0 \).

Remark 2.3. If instead we assume \( \| (1 + | \cdot |)^2 (\hat{u}_0, \hat{\Theta}_0) \|_{L^1(\mathbb{R}^d)} < \infty \) and \( \| \hat{f} \|_{L^1(\mathbb{R}^d)} < \infty \), then we have a unique solution to (3) for which \( \| (\hat{u}, \hat{\Theta}) \|_{L^1(\mathbb{R}^d)} < \infty \) for \( t \in (0, \omega^{-1}) \). Using the arguments of Lemma 3.13 for \( L^1(\mathbb{R}^d) \) norm alone, the solution is shown to be in the space where \( \| (1 + | \cdot |^2)(\hat{u}(\cdot, t), \hat{\Theta}(\cdot, t)) \|_{L^1(\mathbb{R}^d)} \) is finite for \( t \in [0, T] \) and solves (2) as well. What is not known is whether the corresponding \((u, \Theta)\) in the physical \( x\)-space is in \( L^\infty(0, T; H^2(\mathbb{R}^d)) \).

Remark 2.4. The guaranteed existence time \( T = \omega^{-1} \) depends on \( \| (1 + | \cdot |)^2 (\hat{u}_0, \hat{\Theta}_0)(\cdot) \|_N \) and \( \| \hat{f} \|_N \). This condition is likely to be weakened using an accelerated version of the Borel transform as in [13], i.e. using an alternate representation for \( n > 1 \):

\[
(\hat{u}, \hat{\Theta})(k, t) = (\hat{u}_0, \hat{\Theta}_0)(k) + \int_0^\infty (\hat{H}, \hat{S})(k, q)e^{-q/(tn)}dq
\]

\(^2\) \( \omega \) is large enough so that [27] in the ensuing holds, where \((\hat{u}_1, \hat{\Theta}_1)\), defined in (12), depends on the initial data and forcing.
Further, we expect to prove, that in the periodic case \((x \in \mathbb{T}^d)\) without forcing, for any specific initial condition, global solutions of the PDE implies that there exists \(n\) sufficiently large so that \(\omega\) for the associated integral equation solution is arbitrarily small, a result already known \[13\] for the 3-d Navier-Stokes.

**Theorem 2.2.** (Borel Summability)

i) For \(\beta > 0\), i.e. for analytic initial data and forcing, the Boussinesq solution \((u, \Theta)\) is Borel summable in \(t^{-1}\), i.e. there exists \((H, S)(x, p)\) analytic in \(p\) in a neighborhood of \(\{0\} \cup \mathbb{R}^+\), exponentially bounded for large \(p\) and analytic in \(x\) for \(|\text{Im } x_j| < \beta\) for \(j = 1, \cdots, d\) such that

\[
(u, \Theta)(x, t) = (u_0, \Theta_0)(x) + \int_0^\infty (H, S)(x, p)e^{-p/t}dp.
\]

(6)

In particular, as \(t \to 0^+\),

\[
(u, \Theta)(x, t) \sim (u_0, \Theta_0)(x) + \sum_{m=1}^\infty (u_m, \Theta_m)(x)t^m,
\]

where \(|(u_m, \Theta_m)(x)| \leq m!A_0D_0^m\) with constants \(A_0\) and \(D_0\) generally dependent on the initial condition and forcing through Lemma 4.4.

ii) Further, if initial data and forcing have a finite finite number of Fourier modes, then the solution \((\hat{H}, \hat{S})(k, p)\) has a power series in \(p\) with radius of convergence \(D_0^{-1}\) is independent of the size of initial data and forcing.

**Remark 2.5.** In the case \(\beta > 0\), we do not need the restriction \(\gamma > d\). If \(||\hat{u}||_{\gamma, \beta} < \infty\), then for \(\beta' \in (0, \beta)\) we have for any \(n \in \mathbb{N}\), \(||\hat{u}||_{\gamma'+d, \beta'} < \infty\).

**Remark 2.6.** When the the radius of convergence \(D_0^{-1}\) is independent of size of initial condition and forcing, as is definitely the case for initial conditions and forcing with finite Fourier modes, the solution can be found conveniently on \([0, p_0]\) through a power series. More generally, for specific initial conditions and forcing, the solution in \([0, p_0]\) may be obtained numerically with rigorous error bounds similar to NSE \[13\]. In the following Theorem 2.3, we obtain revised estimates on \(\omega\) and therefore existence time of PDE solution, based on integral equation solution on \([0, p_0]\).
Let \((\hat{H}, \hat{S})(k, p)\) be the solution to (17) provided by Lemma 3.11. Define

\[
(\hat{H}, \hat{S})^{(a)}(k, p) = \begin{cases} (\hat{H}, \hat{S})(k, p) & \text{for } p \in (0, p_0] \subset \mathbb{R}^+ \\ 0 & \text{otherwise} \end{cases}
\] (7)

and

\[
\hat{H}^{(s)}(k, p) = \frac{ik_j \pi}{2k|\nu p|} \int_0^{\min(p, 2p_0)} \mathcal{G}(z, z') \hat{G}_j^{[1], (a)}(k, p') dp' + 2\hat{u}_1(k) \frac{J_1(2k|\nu p|)}{2k|\nu p|}
\]

\[
+ \frac{a \pi}{2k|\nu p|} \int_0^{\min(p, p_0)} \mathcal{G}(z, z') P_k[e_2 \hat{S}^{(a)}(k, p')] dp'
\]

\[
\hat{S}^{(s)}(k, p) = \frac{ik_j \pi}{2k|\mu p|} \int_0^{\min(p, 2p_0)} \mathcal{G}(\zeta, \zeta') \hat{G}_j^{[2], (a)}(k, p') dp' + 2\Theta_1(k) \frac{J_1(2k|\mu p|)}{2k|\mu p|}
\] (8)

where

\[
\hat{G}_j^{[1], (a)}(k, p) = -P_k[\hat{u}_{0,j} \hat{H}^{(a)} + \hat{H}_j^{(a)} \hat{u}_0 + \hat{H}^{(a)} \hat{S}^{(a)}]
\]

\[
\hat{G}_j^{[2], (a)}(k, p) = -[\hat{u}_{0,j} \hat{S}^{(a)} + \hat{H}_j^{(a)} \hat{\Theta}_0 + \hat{S}_j^{(a)} \hat{S}^{(a)}].
\]

Notice if \((\hat{H}, \hat{S})^{(a)}(k, p)\) is known, then \(\hat{H}^{(s)}(k, p), \hat{S}^{(s)}(k, p), \hat{G}_j^{[1], (a)}(k, p), \) and \(\hat{G}_j^{[2], (a)}(k, p)\) are also known functions. Also, recall \(\hat{u}_1\) and \(\hat{\Theta}_1\) are quantities based on the initial condition and forcing given in (12).

**Theorem 2.3.** (Revised Exponential Estimates). For some \(\omega_0 \geq 0\), assume \(\epsilon_1, B_3\) and \(b\) are functionals of the forcing \(f\), initial condition \((\hat{u}_0, \Theta_0)\), and the solution \((\hat{H}, \hat{S})\) to the set of integral equations (18) on a finite interval \([0, p_0]\), determined from the relations:

\[
b = \omega_0 \int_{p_0}^{\infty} e^{-\omega_0 p} \|((\hat{H}, \hat{S})^{(s)}(\cdot, p))_N dp
\]

\[
\epsilon_1 = B_1 + B_4 + \int_0^{p_0} e^{-\omega_0 p} B_2(p) dp,
\] (9) (10)

where

\[
B_0(k) = C_0 \sup_{p_0 \leq p' \leq p} |G(z, z')/z|, \quad B_1 = 2 \sup_{k \in \mathbb{R}^d} |k|B_0(k)||\hat{u}_0, \Theta_0||_N,
\]
\[ B_2 = 2 \sup_{k \in \mathbb{R}^d} |k| B_0(k) \| (\tilde{H}, \tilde{S})^{(a)}(\cdot, p) \|_N, \quad B_3 = \sup_{k \in \mathbb{R}^d} |k| B_0(k), \quad B_4 = a \sup_{k \in \mathbb{R}^d} B_0(k). \]

Then, over an extended interval \( \mathbb{R}^+ \), the solution satisfies the relation

\[ \| (\tilde{H}(\cdot, p), \tilde{S}(\cdot, p)) \|_N e^{-\omega p} \in L^1(0, \infty) \]

for any \( \omega \geq \omega_0 \) satisfying

\[ \omega > \epsilon_1 + 2\sqrt{B_3 b}. \]

**Remark 2.7.** The implication of the above theorem is that if solution \((\tilde{H}, \tilde{S})\), restricted to \([0, p_0]\) is known, through computation of power series in \( p \) or otherwise, and if the corresponding functionals \( \epsilon \) and \( B_3 b \) are small, as is the case for sufficiently rapidly decaying \((\tilde{H}, \tilde{S})\) over a large enough interval \([0, p_0]\), then existence for Boussinesq PDE solution in a long interval \((0, \omega^{-1})\) is guaranteed. It is to be noted that rigorous error control of computed solution in \([0, p_0]\) is expected as for 3-d NSE \([13]\); this leads to a revised bound on \( \omega \) that can translate to a longer existence time.

### 3. Local Existence and Uniqueness of Solution

#### 3.1. Formulation of Integral Equation: Borel Transform

Our goal is to take the Borel transform and create equivalent integral equations. To ensure smallness in \( t \) for small \( t \) and avoid dealing with delta distribution in Borel transform, it is convenient to define \( h \) and \( \tilde{w} \) so that

\[ (\tilde{u}, \tilde{\Theta})(k, t) = (\tilde{u}_0, \tilde{\Theta}_0)(k) + (\tilde{h}, \tilde{s})(k, t). \]

For \[(3)\], we define

\[ g_j^{[1]} := P_k [\hat{h}_j \hat{\tilde{h}} + \hat{h}_j \hat{\tilde{u}}_0 + \hat{u}_{0,j} \hat{h}] \text{ and } g_j^{[2]} := [\hat{h}_j \hat{\tilde{s}} + \hat{h}_j \hat{\tilde{\Theta}_0} + \hat{u}_{0,j} \hat{\tilde{s}}] \]

and

\[ \hat{u}_1(k) := -\nu |k|^2 \hat{u}_0 - i k_j P_k [\hat{u}_{0,j} \hat{\tilde{u}}_0] + a P_k [e_2 \hat{\tilde{\Theta}_0}] + \hat{f} \]

\[ \hat{\Theta}_1(k) := -\mu |k|^2 \hat{\tilde{\Theta}_0} - i k_j (\hat{u}_{0,j} \hat{\tilde{\Theta}_0}). \]
Using these in (3), we obtain integral equations:

\[
\hat{h}(k, t) = \int_0^t e^{-\nu|k|^2(t-s')} (-ik \hat{g}_j^{[1]} - P_k[ae_2 \hat{s}]) (k, s') ds' + \left(1 - e^{-\nu|k|^2t}\right) \hat{u}_1.
\]

\[
\hat{s}(k, t) = -ik \int_0^t e^{-\mu|k|^2(t-s')} \hat{g}_j^{[2]}(k, s') ds' + \left(1 - e^{-\mu|k|^2t}\right) \hat{\Theta}_1.
\]  \hspace{1cm} (13)

We seek a solution as a Laplace transform,

\[
(\hat{h}, \hat{s})(k, t) = \int_0^\infty (\hat{H}, \hat{S})(k, p)e^{-p/t} dp.
\]

With this goal, we take the formal inverse Laplace transform in \(1/t\):

\[
[L^{-1} f](p) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s)e^{sp} ds,
\]

where \(c\) is chosen so that for \(\text{Re } s \geq c\), \(f\) is analytic and has suitable asymptotic decay. We define

\[
\mathcal{H}^{(\nu)}(p, p', k) := \int_1^{\frac{1}{p'}} \left\{ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tau^{-1} \exp[-\nu|k|^2 \tau^{-1}(1-s) + (p-p's^{-1})\tau] d\tau \right\} ds.
\]

Then (13) becomes

\[
\hat{H}(k, p) = \int_0^p \mathcal{H}^{(\nu)}(p, p', k) \left(-ik \hat{g}_j^{[1]}(k, p') dp' + P_k[ae_2 \hat{s}](k, p')\right) dp' + \hat{u}_1(k)L^{-1}\left(\frac{1 - e^{-\nu|k|^2t}}{\nu|k|^2}\right)(p)
\]

\[
\hat{S}(k, p) = -ik \int_0^p \mathcal{H}^{(\mu)}(p, p', k) \hat{g}_j^{[2]}(k, p') dp' + \hat{\Theta}_1(k)L^{-1}\left(\frac{1 - e^{-\mu|k|^2t}}{\mu|k|^2}\right)(p).
\]  \hspace{1cm} (14)

While the derivation of the integral equation in \(p\) is formal, we prove later (Lemma 3.12) that the unique solution to the integral equation in the Borel plane generates a solution to the Boussinesq equation through Laplace transform.
In the above, $\hat{G}_{j}^{1,2} = \mathcal{L}^{-1}[\hat{g}_{j}^{1,2}]$. Specifically,

$$\hat{G}_{j}^{[1]} = P_{k}[\hat{u}_{0,j} \hat{H} + \hat{H}_{j} \hat{u}_{0} + \hat{H}_{j} \hat{\Theta}]$$

and

$$\hat{G}_{j}^{[2]} = [\hat{u}_{0,j} \hat{S} + \hat{H}_{j} \hat{\Theta}_{0} + \hat{H}_{j} \hat{\Theta}]$$  \hspace{1cm} (16)

where $\ast$ denotes the Laplace convolution followed by Fourier convolution (order is unimportant). We now make the observation that our kernel $\mathcal{H}(\nu)(p, p', k)$ has a representation in terms of Bessel functions. Namely,

$$\mathcal{H}(\nu)(p, p', k) = \pi \frac{2J_{1}(z)}{z} \{ -J_{1}(z)Y_{1}(z') + Y_{1}(z)J_{1}(z') \}$$

where $J_{1}$ and $Y_{1}$ are the Bessel functions of order 1, $z = 2|k|\sqrt{\nu p}$, and $z' = 2|k|\sqrt{\nu p'}$. In similar spirit, we have

$$\frac{2J_{1}(z)}{z} = \mathcal{L}^{-1} \left( \frac{1 - e^{-\nu|k|^{2}t}}{\nu|k|^{2}} \right)(p).$$

These assertions are proved in the appendix in Appendix A.1 and Appendix A.2.

Thus, our integral Boussinesq equation becomes

$$\hat{H}(k, p) = \pi \int_{0}^{p} \frac{G(z, z')}{z} \left( ik_{j} \hat{G}_{j}^{[1]}(k, p' + aP_{k}[e_{2}\hat{S}(k, p')] \right) dp' + 2\hat{u}_{1}(k) \frac{J_{1}(z)}{z}$$

$$\hat{S}(k, p) = \frac{ik_{j} \pi}{2|k|\sqrt{\mu p}} \int_{0}^{p} \hat{G}(\zeta, \zeta') \hat{G}_{j}^{[2]}(k, p') dp' + 2\hat{\Theta}_{1}(k) \frac{J_{1}(z)}{z}$$  \hspace{1cm} (17)

where $\zeta = 2|k|\sqrt{\nu p}$, and $\zeta' = 2|k|\sqrt{\nu p'}$. Abstractly, we may write the set of equations (17) as

$$(\hat{H}, \hat{S})(k, p) = \mathcal{N}[(\hat{H}, \hat{S})](k, p).$$  \hspace{1cm} (18)

**Remark 3.1.** By properties of Bessel functions $|G(z, z')|$ is bounded for all real nonnegative $z' \leq z$. (The approximate bound is 0.6, see [10]). The asymptotic properties of Bessel functions for small $z$ also show $|G(z, z')/z|$ is bounded for all real nonnegative $z' \leq z$.

To prove Theorem 2.1 we will show $\mathcal{N}$ is contractive in a suitable space, so $(\hat{H}, \hat{S})$ is Laplace transformable in $1/t$. Then from Lemma 3.12,

$$(\hat{u}, \hat{\Theta})(k, t) = (\hat{u}_{0}, \hat{\Theta}_{0})(k) + \int_{0}^{\infty} (\hat{H}, \hat{S})(k, p)e^{-\rho t} dp$$

satisfies (3) for $\Re(1/t)$ large enough. Furthermore, we show $(u, \Theta)(x, t) = \mathcal{F}^{-1}[(\hat{u}, \hat{\Theta})(\cdot, t)](x)$ is a classical solution to the Boussinesq problem.
3.2. Norms in $p$

Recall the norm $|| \cdot ||_N$ in $k$ is either the $(\gamma, \beta)$ norm given in Definition 2.1 for some $\beta \geq 0$ and $\gamma > d$ or the $L^1 \cap L^\infty$ norm.

**Definition 3.2.** For $\alpha \geq 1$, we define 

$$||\hat{f}||^{(\alpha)} = \sup_{p \geq 0} (1 + p^2)e^{-\alpha p}||\hat{f}(\cdot, p)||_N.$$ 

**Definition 3.3.** We define $A^\alpha$ to be the Banach space of continuous function of $(k, p)$ for $k \in \mathbb{R}^d$ and $p \in \mathbb{R}^+$ for which $|| \cdot ||^\alpha$ is finite. In similar spirit, we define the space $A^\alpha_1$ of locally integrable functions for $p \in [0, L)$, and continuous in $k$ such that 

$$||\hat{f}||^\alpha_1 = \int_0^L e^{-\alpha p}||\hat{f}(\cdot, p)||_N dp < \infty.$$ 

**Definition 3.4.** Finally, we also define $A^\alpha_L$ to be the Banach space of continuous functions in $(k, p)$ for $k$ in $\mathbb{R}^d$ and $p \in [0, L]$ such that 

$$||\hat{f}||^\infty_L = \sup_{p \in [0, L]} ||\hat{f}(\cdot, p)||_N < \infty.$$ 

3.3. Existence of a Solution in Dual Variable

We need some preliminary lemmas. Recall, $d = 2$ or $d = 3$ denotes the dimension in $x$ or its dual $k$. Often constants appearing in subalgebra bounds will depend on dimension. We will explicitly state the dependence when defining them and suppress the dependence elsewhere.

**Lemma 3.5.** If $||\hat{v}||_{\gamma, \beta}$ and $||\hat{w}||_{\gamma, \beta} < \infty$ for $\gamma > d$ and $k \in \mathbb{R}^d$, then 

$$||\hat{v} \hat{w}||_{\gamma, \beta} \leq \tilde{C}_0(d) ||\hat{v}||_{\gamma, \beta} ||\hat{w}||_{\gamma, \beta},$$

where 

$$\tilde{C}_0(2) = 2^{\gamma+1} \int_{k' \in \mathbb{R}^2} \frac{1}{(1 + |k'|)\gamma} dk' = \frac{\pi 2^{\gamma+2}}{(\gamma - 1)(\gamma - 2)}$$

and 

$$\tilde{C}_0(3) = 2^{\gamma+1} \int_{k' \in \mathbb{R}^3} \frac{1}{(1 + |k'|)\gamma} dk' = \frac{\pi 2^{\gamma+4}}{(\gamma - 1)(\gamma - 2)(\gamma - 3)}.$$
The $d = 3$ case can be found in [10] and the $d = 2$ case is basically the same. From the definition of $|| \cdot ||_{\gamma, \beta}$ and the fact that $e^{-\beta(|k'|+|k-k'|)} \leq e^{-\beta|k|}$, we have

$$|\hat{v} \hat{w}^*| \leq e^{-\beta|k|} ||\hat{v}||_{\gamma, \beta} ||\hat{w}||_{\gamma, \beta} \int_{k' \in \mathbb{R}^2} (1 + |k'|)^{-\gamma} (1 + |k-k'|)^{-\gamma} dk'.$$

Split the integral into two domains $|k'| \leq |k|/2$ and its complement to show

$$\int_{k' \in \mathbb{R}^2} \frac{1}{(1 + |k'|)^\gamma (1 + |k-k'|)^\gamma} dk' \leq \frac{2^{\gamma+1}}{(1 + |k|)^\gamma (1 + |k-k'|)^\gamma} \left( \frac{1}{2^{\gamma+2}} \int_{k' \in \mathbb{R}^2} \frac{1}{(1 + |k'|)^\gamma} dk' \right) = \frac{1}{(1 + |k|)^\gamma (\gamma - 1)(\gamma - 2)},$$

where polar coordinates and integration by parts are used to evaluate the last integral.

**Corollary 3.6.** If $||\hat{v}||_N, ||\hat{w}||_N < \infty$, then for $C_0 = C_0(d)$ chosen such that $C_0 = \tilde{C}_0$ for $N = (\gamma, \beta), \gamma > d$ and $C_0 = 1$ for $N = L^1 \cap L^\infty$, we have

$$||\hat{v} \hat{w}^*||_N \leq C_0 ||\hat{v}||_N ||\hat{w}||_N.$$

**Lemma 3.7.** Also, notice that

$$\left\| \left( P_k(\hat{f}), P_k(\hat{g}) \right) \right\|_N \leq \|(\hat{f}, \hat{g})\|_N$$

**Proof.** $P_k$ is the projection of a vector onto $k^\perp$.

**Lemma 3.8.** With $C_0$ as defined in Corollary 3.6, appropriately modified for $d = 2$ or 3, and constants

$$C_2 = \frac{\pi C_0}{\min(\sqrt{\nu}, \sqrt{\mu})} \sup_{z \in \mathbb{R}^+, 0 \leq z' \leq z} |G(z, z')|$$

and $C_3 = \pi a \sup_{z \in \mathbb{R}^+, 0 \leq z' \leq z} \left| \frac{G(z, z')}{z} \right|$, we have the following bounds on the norm in $k$ for the operator $\mathcal{N}$ defined in (18). Let $\phi := (\hat{H}, \hat{S})$. Then

$$||\mathcal{N}[\phi(\cdot, p)]||_N \leq \frac{C_2}{\sqrt{\nu}} \int_0^p (||\phi(\cdot, p')||_N * ||\phi(\cdot, p')||_N)$$

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+ \left| |\hat{u}_0, \hat{\Theta}_0|\right|_N \|\phi(\cdot, p')\|_N \right) dp' + \left| |\hat{u}_1, \hat{\Theta}_1|\right|_N + C_3 \int_0^p \|\hat{S}(\cdot, p')\|_N dp' \quad (19)

and

\begin{align}
|N[\phi^1](\cdot, p) - N[\phi^2](\cdot, p)|_N & \leq C_2 \frac{\sqrt{p}}{p} \int_0^p \left( \|\phi^1(\cdot, p')\|_N + \|\phi^2(\cdot, p')\|_N \right) \\
* \|\phi^1[1] - \phi^2[1](\cdot, p')\|_N + \|\hat{u}_0, \hat{\Theta}_0\|_N ||\phi^1[1] - \phi^2[1](\cdot, p')||_N dp' \\
+ C_3 \int_0^p \|\hat{S}^1[1] - \hat{S}^2[1](\cdot, p')\|_N dp' \quad (20)
\end{align}

**Proof.** From [1], \(|J_1(z)/z| \leq 1/2\) for \(z \in \mathbb{R}^+\) and

\[
\left\| 2 \left( \frac{\hat{u}_1(k)}{z} \frac{J_1(z)}{\hat{\Theta}_1(k)} \frac{\hat{\zeta}(z)}{\zeta} \right) \right\|_N \leq \left| |\hat{u}_1, \hat{\Theta}_1|\right|_N.
\]

From Corollary 3.6, we have

\[
\left| |\hat{u}_0\hat{H}(\hat{\Theta}, \hat{S}) + |\hat{H}|(\hat{\Phi}, \hat{\Theta}) + \hat{H} + (\hat{\Phi}, \hat{\Theta}) \right|_N \leq \\
\left[ 2C_0 \left( \|\hat{u}_0, \hat{\Theta}_0\|_N \|\hat{H}(\hat{\Phi}, \hat{\Theta})\|_N + C_0 \|\hat{H}(\hat{\Phi}, \hat{\Theta})\|_N + \|\hat{u}_0, \hat{\Theta}_0\|_N \|\phi(\cdot, p')\|_N \right) \right].
\]

Then using Lemma 3.7 and Schwartz inequality, we obtain

\[
|k_j(G^1_j, \hat{G}^2_j)|_N \leq 2C_0|k_j| \left( \|\phi(\cdot, p')\|_N |\hat{H}(\hat{\Phi}, \hat{\Theta})\|_N + \|\hat{u}_0, \hat{\Theta}_0\|_N \|\phi(\cdot, p')\|_N \right).
\]

Now (19) follows. To obtain (20), notice that

\[
\hat{H}^1_j \hat{\phi}^1_j - \hat{H}^2_j \hat{\phi}^2_j = \hat{H}^1_j \hat{\phi}^1_j - \hat{H}^2_j \hat{\phi}^2_j + \hat{H}^1_j - \hat{H}^2_j \hat{\phi}^2_j. \quad (21)
\]

From (21) we get

\[
\left\| \hat{H}^1_j \hat{\phi} - \hat{H}^2_j \hat{\phi}^2 \right\|_N \leq C_0 \left( \|\phi^1 - \phi^2\|_N \right) \left( \|\phi^1|_N + \|\phi^2|_N \right).
\]

Combining this bound and using Lemma 3.7 as in the first part of the proof, we get (20).
Lemma 3.9. For $\hat{f}, \hat{g} \in \mathcal{A}^\alpha, \mathcal{A}^\alpha_0$ or $\mathcal{A}^\infty_L$

$$||\hat{f} \ast \hat{g}||^{(\alpha)} \leq M_0 C_0 ||\hat{f}||^{(\alpha)} ||\hat{g}||^{(\alpha)}$$
$$||\hat{f} \ast \hat{g}||^1 \leq C_0 ||\hat{f}||^1 ||\hat{g}||^1$$
$$||\hat{f} \ast \hat{g}||^\infty_L \leq L C_0 ||\hat{f}||^\infty_L ||\hat{g}||^\infty_L,$$

where $M_0 \approx 3.76 \cdots$ is large enough so

$$\int_0^p \frac{(1 + p^2)}{(1 + s^2)(1 + (p-s)^2)} ds \leq M_0.$$

This means the Banach spaces listed in the norms section form subalgebras under the operation $\ast$. The properties listed are independent of dimension except for a change in $C_0$ showing up due to the Fourier convolution. The proof is in [10]. The basic idea is that $k$ and $p$ act separately in the norm. So, we need only consider how the $p$ portion of the norm effects $\int_0^p u(p) v(p-s) ds$.

The following lemma expands the bounds in Lemma [3.8] to bounds in $p$ in some of our other norms.

Lemma 3.10. Let $\phi := (\hat{H}, \hat{S})$. On $\mathcal{A}_0^\alpha$, the operator $\mathcal{N}$ satisfy the following inequalities

$$||\mathcal{N}[\phi]||^\alpha_1 \leq C_2 \sqrt{\pi} \alpha^{-1/2} \left\{ (||\phi||^1_1)^2 + ||(\hat{u}_0, \hat{\Theta}_0)||_N ||\phi||^\alpha_1 \right\}$$
$$+ \alpha^{-1}||(\hat{u}_1, \hat{\Theta}_1)||_N + \alpha^{-1} C_3 ||\hat{S}||^\alpha_1 \tag{22}$$

and

$$||\mathcal{N}(\phi[1]) - \mathcal{N}(\phi[2])||^\alpha_1 \leq C_2 \sqrt{\pi} \alpha^{-1/2} \left\{ (||\phi[1]||^1_1 + ||\phi[2]||^1_1) \right\}$$
$$+ ||(\hat{u}_0, \hat{\Theta}_0)||_N ||\phi[1] - \phi[2]||^1_1 \right\} + \alpha^{-1} C_3 ||\hat{S}[1] - \hat{S}[2]||^\alpha_1, \tag{23}$$

Similarly, for $\mathcal{A}^\infty_L$, we have

$$||\mathcal{N}[\phi]||^\infty_L \leq C_2 \sqrt{L} \left\{ L (||\phi||^\infty_L)^2 + ||(\hat{u}_0, \hat{\Theta}_0)||_N ||\phi||^\infty_L \right\}$$
$$+ ||(\hat{u}_1, \hat{\Theta}_1)||_N + L C_3 ||\hat{S}||^\infty_L \tag{24}$$

and

$$||\mathcal{N}[\phi[1]] - \mathcal{N}[\phi[2]]||^\infty_L \leq C_2 \sqrt{L} \left\{ L (||\phi[1]||^\infty_L + ||\phi[2]||^\infty_L) \right\}$$
$$+ ||(\hat{u}_0, \hat{\Theta}_0)||_N ||\phi[1] - \phi[2]||^\infty_L \right\} + L C_3 ||\hat{S}[1] - \hat{S}[2]||^\infty_L, \tag{25}$$

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Proof. For the space $A_1^\alpha$ and any $L > 0$, we note that
\[
\int_0^L e^{-\alpha p} ||(\hat{u}_1, \hat{\Theta}_1)||_N dp \leq \alpha^{-1} ||(\hat{u}_1, \hat{\Theta}_1)||_N
\]
and
\[
\int_0^L e^{-\alpha p} p^{-1/2} dp \leq \Gamma \left( \frac{1}{2} \right) \alpha^{-1/2} = \sqrt{\pi} \alpha^{-1/2}.
\]
We further notice that for $y(p') \geq 0$, we have
\[
\int_0^L e^{-\alpha p} \left( \int_0^p y(p') dp' \right) \left( \int_0^L e^{-\alpha(p-p')} p^{-1/2} dp' \right) dp' \leq \int_0^L e^{-\alpha p} \left( \int_0^p y(p') dp' \right) dp' \leq \int_0^L e^{-\alpha p} \sqrt{\pi} \alpha^{-1/2} dp'. \tag{26}
\]
Similarly,
\[
\int_0^L e^{-\alpha p} \left( \int_0^p ||\hat{S}(\cdot, p')||_N dp' \right) dp' \leq \alpha^{-1} ||\hat{S}||_1^\alpha.
\]
Then, using (26) in (19) and the idea in Lemma 3.9 that
\[
\int_0^L e^{-\alpha p} \left[ (\int_0^p ||g||_N * ||h||_N)(p) dp \right] \leq ||g||_1^\alpha ||h||_1^\alpha,
\]
we have
\[
\int_0^L e^{-\alpha p} ||\mathcal{N}(\hat{H}, \hat{S})||_N dp \leq C_2 \sqrt{\pi} \alpha^{-1/2} \left\{ (||\hat{H}, \hat{S}||_1^\alpha)^2 + ||(\hat{u}_0, \hat{\Theta}_0)||_N ||(\hat{H}, \hat{S})||_1^\alpha \right\} + \alpha^{-1} ||(\hat{u}_1, \hat{\Theta}_1)||_N + \alpha^{-1} C_3 ||\hat{S}||_1^\alpha.
\]
This proves (22). Further, from (20), it also follows that
\[
\int_0^L e^{-\alpha p} ||\mathcal{N}(\phi[1]) - \mathcal{N}(\phi[2])(\cdot, p)||_N dp \leq C_2 \sqrt{\pi} \alpha^{-1/2} \left\{ (||\phi[1]||_1^\alpha + ||\phi[2]||_1^\alpha)^2 + ||(\hat{u}_0, \hat{\Theta}_0)||_N ||\phi[1] - \phi[2]||_1^\alpha \right\} + \alpha^{-1} C_3 ||\hat{S}[1] - \hat{S}[2]||_1^\alpha.
\]
This proves (23).

Now, we consider $A_\infty^\alpha$. We note that for $p \in [0, L]$, we have
\[
\left| p^{-1/2} \int_0^p y(p') dp' \right| \leq \sup_{p \in [0, L]} |y(p)| \sqrt{L}.
\]
We recall from Lemma 3.9 that
\[
\left| \int_0^p y_1(s)y_2(p-s)ds \right| \leq L \left( \sup_{p \in [0,L]} |y_1(p)| \right) \left( \sup_{p \in [0,L]} |y_2(p)| \right).
\]

Taking
\[
y(p) = ||\phi(\cdot, p)||_N \ast ||\phi(\cdot, p)||_N + ||(\hat{u}_0, \hat{\Theta}_0)||_N ||\phi(\cdot, p)||_N
\]
and
\[
y_1(p) = y_2(p) = ||\phi(\cdot, p)||_N,
\]
then (24) follows from (19). To get the bound in (25), we will choose
\[
y(p) = \left( ||\phi[1]||_N + ||\phi[2]||_N \right) \ast ||\phi[1] - \phi[2]||_N + ||(\hat{u}_0, \hat{\Theta}_0)||_N ||\phi[1] - \phi[2]||_N,
\]
\[
y_1(p) = ||\phi[1]||_N + ||\phi[2]||_N, \text{ and } y_2(p) = ||\phi[1] - \phi[2]||_N
\]
now using (20) the proof follows.

**Lemma 3.11.** Equation (17) has a unique solution in $A_1^\omega$ for any $L > 0$ in a ball of size $2\omega^{-1}||(\hat{u}_1, \hat{\Theta}_1)||_N$ for $\omega$ large enough to guarantee
\[
2C_2\sqrt{\pi}\omega^{-1/2} \left\{ 2\omega^{-1}||(\hat{u}_1, \hat{\Theta}_1)||_N + ||(\hat{u}_0, \hat{\Theta}_0)||_N + \frac{C_3}{C_2}\omega^{-1/2} \right\} < 1 \tag{27}
\]
where $(\hat{u}_1, \hat{\Theta}_1)$ is given in (12). Furthermore, the solution also belongs to $A_L^\infty$ for $L$ small enough to ensure
\[
2C_2L^{1/2} \left\{ 2L||(\hat{u}_1, \hat{\Theta}_1)||_N + ||(\hat{u}_0, \hat{\Theta}_0)||_N + \frac{C_3}{C_2}L^{1/2} \right\} < 1 \tag{28}
\]
Moreover, $\lim_{p \to 0^+}(\hat{H}, \hat{S})(k, p) = (\hat{u}_1, \hat{\Theta}_1)(k)$.

**Proof.** The estimates in Lemma 3.10 imply that $N$ maps a ball of radius $2\omega^{-1}||(\hat{u}_1, \hat{\Theta}_1)||_N$ in $A_1^\omega$ into itself and is contractive when $\omega$ is large enough to satisfy (27). Similarly, $N$ maps a ball of size $2||(\hat{u}_1, \hat{\Theta}_1)||_N$ in $A_L^\infty$ into itself and is contractive when $L$ is small enough to satisfy (28). Therefore, there is a unique solution to the Boussinesq integral system of equations in the ball. Furthermore, $A_L^\infty \subseteq A_1^\omega$, so the solutions are in fact one and the same.
Moreover, applying (25) with \((\hat{H}^{[1]}, \hat{S}^{[1]}) = (\hat{H}, \hat{S})\) and \((\hat{H}^{[2]}, \hat{S}^{[2]}) = 0\), we obtain
\[
\left\| (\hat{H}, \hat{S})(k,p) - \left( \frac{2J_1(z)}{z}, \Theta_1(k, \frac{2J_1(\zeta)}{\zeta} \right) \right\|_L \leq C_2 L^{1/2} \left\{ L \left( \| (\hat{H}, \hat{S}) \|_L^\infty \right)^2 + \| (\hat{u}_0, \hat{\Theta}_0) \|_N \| (\hat{H}, \hat{S}) \|_L^\infty \right\} + L C_3 \| \hat{S} \|_L^\infty.
\]
Since \(\| (\hat{H}, \hat{S}) \|_L^\infty\) is bounded for small \(L\), letting \(L \to 0\),
\[
\left\| (\hat{H}, \hat{S})(k,p) - \left( \frac{2J_1(z)}{z}, \Theta_1(k, \frac{2J_1(\zeta)}{\zeta} \right) \right\|_L \to 0.
\]
As \(\lim_{z \to 0} 2J_1(z)/z = 1\), for fixed \(k\), \(\lim_{p \to 0} (\hat{H}, \hat{S})(k,p) = (\hat{u}_1, \hat{\Theta}_1)(k)\).

### 3.4. Proof of Local Existence for Boussinesq PDE

We have unique solutions to our integral equation, (13). We show in the following Lemma 3.12 that the solution’s Laplace transform gives a solution to (3), which is analytic in \(t\) for \(\Re \frac{1}{t} > \omega\). Lemma 3.15 below shows that any solution of (3) with \(\| (1+|\cdot|)^2(\hat{u}, \hat{\Theta})(\cdot, t) \|_N < \infty\) is inverse Fourier transformable with \((u, \Theta)\) solving (2). Lemma 3.13 below ensures that \(\| (1+|\cdot|)^2(\hat{u}, \hat{\Theta})(\cdot, t) \|_N < \infty\). Thus, combining these results, we have \((u, \Theta)(x, t) = \mathcal{F}^{-1}(\hat{u}, \hat{\Theta})(k, t)\) is a classical solutions to (2).

**Lemma 3.12.** For any solutions \((\hat{H}, \hat{S})\) of (13) such that \(\| (\hat{H}, \hat{S})(\cdot, p) \|_N \in L^1(\mathbb{C}, pdp)\) the Laplace transform
\[
(\hat{u}, \hat{\Theta})(k, t) = (\hat{u}_0, \hat{\Theta}_0)(k) + \int_0^\infty (\hat{H}, \hat{S})(k, p) e^{-pt} dp
\]
solves (3) for \(\Re(1/t) > \omega\). Moreover, \((\hat{u}, \hat{\Theta})(k, t)\) is analytic for \(t \in (0, \omega^{-1})\).

**Proof.** Recall (13),
\[
\mathcal{H}^{(\nu)}(p, p', k) = \int_{\rho/p}^1 \left\{ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tau^{-1} e^{\nu \tau} [-\nu |k|^2 \tau^{-1} (1-s) + (p-p's^{-1})\tau] d\tau \right\} ds.
\]
Let \(\hat{G}_1 = -ik_j \hat{G}_j^{[1]} + P_k (ae \hat{S})\) and \(\hat{G}_2 = -ik_j \hat{G}_j^{[2]}\). Changing variable \(p'/s \to p'\) and applying Fubini’s theorem gives
\[
\int_0^p \left( \mathcal{H}^{(\nu)}(p, p', k) \hat{G}_1(k, p'), \mathcal{H}^{(\mu)}(p, p', k) \hat{G}_2(k, p') \right) dp' \tag{29}
\]
Recalling the integral equations for \( \hat{H}, \hat{S} \) given in (15), we have

\[
(\hat{h}, \hat{s})(k, t) = \left( \hat{u}_1(k) \left( \frac{1 - e^{-\mu|k|^2t}}{\mu|k|^2} \right), \hat{\Theta}_1(k) \left( \frac{1 - e^{-\nu|k|^2t}}{\nu|k|^2} \right) \right)
\]

\[
= t \int_0^1 \left( e^{-\nu|k|^2t(1-s)} \hat{g}_1(k, st), e^{-\nu|k|^2t(1-s)} \hat{g}_2(k, st) \right) ds
\]

\[
= \int_0^t \left( e^{-\nu|k|^2(t-s)} \hat{g}_1(k, s), e^{-\nu|k|^2(t-s)} \hat{g}_2(k, s) \right) ds.
\]

Therefore, we directly verify \((\hat{u}, \hat{\Theta})(k, t) = (\hat{u}_0, \hat{\Theta}_0)(k) + (\hat{h}, \hat{s})(k, t)\) satisfies (3). Moreover, analyticity in \(t\) follows from the representation

\[
(\hat{u}, \hat{\Theta})(k, t) = (\hat{u}_0, \hat{\Theta}_0)(k) + \int_0^\infty (\hat{H}, \hat{S})(k, p)e^{-p/t} dp.
\]
Lemma 3.13. (Instantaneous smoothing) Assume \( \| (\hat{u}_0, \hat{\Theta}_0) \|_N < \infty \) and \( \| \hat{f} \|_N < \infty \) with \( N \) either \( L^1 \cap L^\infty (\mathbb{R}^4) \) or \( (\gamma, \beta) \) with \( \gamma > d, \beta \geq 0 \). For the solution \( (\hat{u}, \hat{\Theta}) \) known to exist by Lemma 3.11 for \( t \in (0, T] \) with \( T < \omega^{-1} \), we have \( \| (1 + | \cdot |)^2 (\hat{u}, \hat{\Theta})(\cdot, t) \|_N < \infty \) for \( t \in (0, T] \).

**Proof.** Our goal is to bootstrap using derivatives of \( (u, \Theta) \). Consider the time interval \( [\epsilon, T] \) for \( \epsilon > 0 \) and \( T < \omega^{-1} \). Define
\[
\hat{V}_\epsilon(k) = \sup_{\epsilon \leq t \leq T} \| (\hat{u}, \hat{\Theta})(k, t) \|.
\]
Since \( \| (\hat{u}, \hat{\Theta})(k, t) \| \leq \| (\hat{u}_0, \hat{\Theta}_0)(k) \| + \int_0^\infty \| (\hat{H}, \hat{S})(k, p) \| e^{-\omega p} dp \),
\[
\| \hat{V}_\epsilon(k) \|_N \leq \| (\hat{u}_0, \hat{\Theta}_0)(k) \|_N + \| (\hat{H}, \hat{S})(k, p) \|_N < \infty.
\]
On \( [\epsilon, T] \) for \( \epsilon > 0 \),
\[
\hat{u}(k, t) = e^{-\nu|k|^2 t} \hat{u}_0(k) - \int_0^t e^{-\nu|k|^2 (t-\tau)} \left( ik_j P_k[\hat{u}_j \hat{\Theta}] + aP_k[e_2 \hat{\Theta}] - \hat{f} \right) d\tau
\]
\[
\hat{\Theta}(k, t) = e^{-\mu|k|^2 t} \hat{\Theta}_0(k) - ik_j \int_0^t e^{-\mu|k|^2 (t-\tau)} \left\{ (\hat{u}_j \hat{\Theta})(k, \tau) \right\} d\tau.
\]
Therefore,
\[
|k| |(\hat{u}, \hat{\Theta})(k, t)| \leq \left| (\hat{u}_0, \hat{\Theta}_0)(k) \right| \sqrt{\min(\nu, \mu)} \sup_{z \geq 0} ze^{-z^2} + |\hat{f}| \int_0^t |k| e^{-\min(\nu, \mu)|k|^2 (t-\tau)} d\tau
\]
\[
+ \left( \hat{V}_0 + \hat{V}_0 \ast \hat{V}_0 \right) \int_0^t |k|^2 e^{-\min(\nu, \mu)|k|^2 (t-\tau)} d\tau.
\]
Noticing that
\[
\int_0^t |k|^2 e^{-\min(\nu, \mu)|k|^2 (t-\tau)} d\tau \leq \frac{1}{\min(\nu, \mu)}
\]
and
\[
\int_0^t |k| e^{-\min(\nu, \mu)|k|^2 (t-\tau)} d\tau \leq \sup_{z \geq 0} \frac{1 - e^{-z}}{\sqrt{z}} \sqrt{\frac{T}{\min(\nu, \mu)}},
\]
it follows that
\[
\left\| \| k \| \hat{V}_\epsilon \right\|_N \leq \frac{C}{\epsilon^{1/2}} \| (\hat{u}_0, \hat{\Theta}_0) \|_N + \frac{1}{\min(\nu, \mu)} \left( C_0 \| \hat{V}_0 \|_N^2 + \| \hat{V}_0 \|_N + C \sqrt{T} \| \hat{f} \|_N \right) < \infty.
\]
In the same spirit, for \( t \in [\varepsilon, T] \), we have
\[
\hat{u}(k, t) = e^{-\nu|k|^2 t} \hat{u}(k, \varepsilon/2) - \int_{\varepsilon/2}^{t} e^{-\nu|k|^2 (t-\tau)} \left( P_k(\hat{u}_j \hat{k}_j) + ae_2 \hat{\Theta}(k, \tau) - \hat{f}(k) \right) d\tau
\]
\[
\hat{\Theta}(k, t) = e^{-\mu|k|^2 t} \hat{\Theta}(k, \varepsilon/2) - i \int_{\varepsilon/2}^{t} e^{-\frac{|k|^2 (t-\tau)}{n\sigma}} \left\{ (\hat{u}_j \hat{k}_j \hat{\Theta})(k, \tau) \right\} d\tau,
\]
where we used the divergence free conditions \( k \cdot \hat{u} = 0 \). Multiplying by \(|k|^2\) and using our previous bounds, we have for \( t \in [\varepsilon, T] \)
\[
|k|^2 |(\hat{u}, \hat{\Theta})(k, t)| \leq \left| (\hat{u}, \hat{\Theta})(k, \varepsilon/2) \right| \frac{1}{(t-\varepsilon/2) \min(\nu, \mu)} \sup_{z \geq 0} e^{-z} \cdot
\]
\[ + (\hat{V}_{\varepsilon/2} \hat{k}) |k| \hat{V}_{\varepsilon/2} + |k| |\hat{V}_{\varepsilon/2} + |\hat{f}|) \int_{\varepsilon/2}^{t} |k|^2 e^{-\min(\nu, \mu)|k|^2 (t-\tau)} d\tau \]

Hence,
\[
\left\| |k|^2 \hat{V}_t \right\|_N \leq \frac{C}{\varepsilon} \left\| \hat{u}_0, \hat{\Theta}_0 \right\|_N + \left( \frac{C_0 \left\| \hat{V}_{\varepsilon/2} \right\|_N + 1}{\min(\nu, \mu)} \left\| |k| \hat{V}_{\varepsilon/2} \right\|_N + ||\hat{f}||_N \right\).
\]

All the terms on the right hand side are bounded, which gives \( \| (1 + |k|^2) \hat{V}_t \|_N < \infty \). Further, as \( \varepsilon > 0 \) is arbitrary, it follows that \( \| (1 + \cdot)^2 (\hat{u}, \hat{\Theta})(\cdot, t) \|_N < \infty \) for \( t \in (0, T) \).

**Remark 3.14.** We note that the smoothness argument in \( x \) of the previous Lemma can be easily extended further to show \( \left\| (1 + |k|)^4 \hat{V}_t \right\|_N \) is finite provided \( \left\| (1 + |k|^2) \hat{f} \right\|_N \) is finite. Since \( \varepsilon > 0 \) is arbitrary, this implies instantaneous smoothing two orders more than the forcing.

**Lemma 3.15.** Given \((\hat{u}, \hat{\Theta})\) a solution to (2) such that \( \| (1 + |\cdot|)^2 (\hat{u}, \hat{\Theta})(\cdot, t) \|_N < \infty \) for \( t \in (0, \omega^{-1}) \), then \((u, \Theta) \in L^\infty[0, \omega^{-1}, H^2(\mathbb{R}^d)]\) solves (3).

**Proof.** Suppose \((\hat{u}, \hat{\Theta})\) is a solution to (3) such that \( \| (1 + |\cdot|)^2 (\hat{u}, \hat{\Theta})(\cdot, t) \|_N < \infty \) for \( t \in (0, \omega^{-1}) \). We notice that by our choice of norms, \((1 + |\cdot|)^2 (\hat{u}, \hat{\Theta})(\cdot, t) \in L^2(\mathbb{R}^d) \) for any \( t \in (0, \omega^{-1}) \). Indeed for \( N = (\gamma, \beta) \), we have
\[
\int (1 + |k|)^4 |(\hat{u}, \hat{\Theta})(k, t)|^2 dk \leq \| (1 + |\cdot|)^2 (\hat{u}, \hat{\Theta})(\cdot, t) \|_{\beta}^2 \int e^{-2\beta|k|} \frac{1}{(1 + |k|)^{2\gamma}} dk.
\]
As \( \gamma > \alpha \), \( \int \frac{1}{(1+|k|)^{2\alpha}} e^{-2\beta|k|} dk < \infty \). For \( N = L^1 \cap L^\infty \) we have,

\[
\int (1+|k|)^4 |(\hat{u}, \hat{\Theta})(k, t)|^2 dk \leq \int (1+|k|)^2 |(\hat{u}, \hat{\Theta})(k, t)| dk \sup_{k \in \mathbb{R}^d} (1+|k|)^2 |(\hat{u}, \hat{\Theta})(k, t)|.
\]

So, \( \|(1 + |\cdot|^2)(\hat{u}, \hat{\Theta})(\cdot, t)\|_{L^2(\mathbb{R}^d)} \leq \|(1 + |\cdot|^2)(\hat{u}, \hat{\Theta})(\cdot, t)\|_{L^1 \cap L^\infty(\mathbb{R}^d)} \). Thus, by well known properties of the Fourier transform \( (u, \Theta) = F^{-1}(\hat{u}, \hat{\Theta})(x, t) \in L^\infty(0, \omega^{-1}, H^2(\mathbb{R}^d)) \). As \( (\hat{u}, \hat{\Theta}) \) solves \( (3) \), \( (\hat{u}, \hat{\Theta}) \) is differentiable almost everywhere and

\[
\begin{align*}
\hat{u}_t + \nu |k|^2 \hat{u} &= -ik_j P_k [\hat{u}_j \hat{\Theta}] + \alpha P_k [e_2 \hat{\Theta}] + \hat{f} \\
\hat{\Theta}_t + \mu |k|^2 \hat{\Theta} &= -ik_j [\hat{u}_j \hat{\Theta}], \quad k \in \mathbb{R}^d \quad t \in \mathbb{R}^+.
\end{align*}
\]

Further, \( (\hat{u}_t, \hat{\Theta}_t)(k, t) \in L^\infty(0, \omega^{-1}, L^2(\mathbb{R}^d)) \) since \( (1 + |k|)^2 (\hat{u}, \hat{\Theta})(k, t) \in L^\infty(0, \omega^{-1}, L^2(\mathbb{R}^d)) \). Hence, \( (u, \Theta)(x, t) = F^{-1}(\hat{u}, \hat{\Theta})(x, t) \) solves

\[
\begin{align*}
u u_t - \nu \Delta u &= -P[u \cdot \nabla u - \alpha c \Theta] + f(x) \\
\Theta_t - \nu \Delta \Theta &= -u \cdot \nabla \Theta.
\end{align*}
\]

**Proof of Theorem 2.1:** Suppose \( \|(1 + |\cdot|^2)(\hat{u}_0, \hat{\Theta}_0)\|_N < \infty \) and \( \|\hat{f}\|_N < \infty \). Then from the definition of \( (\hat{u}_1, \hat{\Theta}_1) \) in \( (12) \) we see \( \|(\hat{u}_1, \hat{\Theta}_1)\|_N < \infty \), since

\[
\|(\hat{u}_1, \hat{\Theta}_1)\|_N \leq \max(\nu, \mu) \left\| |k|^2 (\hat{u}_0, \hat{\Theta}_0) \right\|_{L^N} + C_0 \|\hat{u}_0\|_N \left\| |k|(\hat{u}_0, \hat{\Theta}_0) \right\|_{L^N} + a\|\hat{\Theta}_0\|_N + \|\hat{f}\|_N.
\]

Therefore, when \( \omega \) is large enough to ensures \( (27) \), Lemma 3.11 gives \( (\hat{H}, \hat{S})(k, \cdot) \) is in \( L^1(e^{-\omega p} dp) \). Applying Lemma 3.12 we know for \( t \) such that \( \mathbb{R}^d_T > \omega \), \( (\hat{H}, \hat{S})(k, p) \) is Laplace transformable in \( 1/t \) with \( (\hat{u}, \hat{\Theta})(k, t) = (\hat{u}_0, \hat{\Theta}_0)(k) + (\hat{h}, \hat{s})(k, t) \) satisfying Boussinesq equation in the Fourier space, \( (3) \). Since \( \|((\hat{H}, \hat{S})(\cdot, p))\|_N < \infty \), we have \( \|(\hat{u}, \hat{\Theta})(\cdot, t)\|_N < \infty \) if \( \mathbb{R}^d_T > \omega \), and \( i \) is proved. Moreover, Lemma 3.12 shows that \( (\hat{u}, \hat{\Theta}) \) is analytic for \( \mathbb{R}^d_T > \omega \) and has the representation

\[
(\hat{u}, \hat{\Theta})(k, t) = (\hat{u}_0, \hat{\Theta}_0)(x) + \int_0^\infty (\hat{H}, \hat{S})(k, p)e^{-p/t} dp
\]

proving \( ii \). For \( iii \), Lemma 3.13 shows that \( \|(1 + |\cdot|^2)(\hat{u}, \hat{\Theta})(\cdot, t)\|_N < \infty \) for \( t \in [0, \omega^{-1}) \) while Lemma 3.15 shows that \( (u, \Theta)(x, t) \in L^\infty(0, T, H^2(\mathbb{R}^d)) \).

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solves (2). Moreover, \((u, \Theta)(x, t)\) is the unique solution to (2) in \(L^\infty(0, T, H^2(\mathbb{R}^d))\) as classical solutions are known to be unique, \([22]\). Finally, suppose \((\hat{H}, \hat{S})(k, \cdot)\) is in \(L^1(e^{-\omega p} dp)\) for any \(\omega > 0\). By Lemma 3.12, we know for any \(t > 0\), \((\hat{H}, \hat{S})(k, p)\) is Laplace transformable with \((\hat{u}, \hat{\Theta})(k, t) = (\hat{u}_0, \hat{\Theta}_0)(k) + (\hat{h}, \hat{s})(k, t)\) satisfying Boussinesq equation in the Fourier space, \([3]\). Further, appealing to instantaneous smoothing Lemma 3.13 the solution is smooth. Thus, if \((\hat{H}, \hat{S})(k, \cdot)\) is in \(L^1(e^{-\omega p} dp)\) for any \(\omega > 0\), then a smooth global solution exists and iv) is proved.

4. Borel-Summability

We now show Borel-summability of the solutions guaranteed by Theorem 2.1 for \(\beta > 0\). This requires us to show that the solutions \((\hat{H}, \hat{S})(k, p)\) to the Boussinesq equation in Borel space is analytic in \(p\) for \(p \in \{0\} \cup \mathbb{R}^+\). First, we will seek a solution which is a power series \((\hat{H}, \hat{S})(k, p) = (\hat{u}_1, \hat{\Theta}_1)(k) + \sum_{l=1}^\infty (\hat{H}^{[l]}, \hat{S}^{[l]})(k)p^l\). (30)

Remark 4.1. We will use induction to bound the successive terms of the power series. Many of these bounds have constants depending on the dimension in \(k\) as before. For brevity of notation the dependence on dimension is suppressed after introducing the constants.

For the purpose of finding power series solutions, (17) is not a good representation. By construction, \(\tilde{G}(z, z')\) satisfies \([p\partial_{pp} + 2\partial_p + \nu|k|^2]y = 0\) with \(\tilde{G}(z, z') \to 0\) and \(\partial_p (\tilde{G}(z, z')) \to \frac{1}{p}\) as \(p'\) approaches \(p\) from below. Hence, we have the equivalent equations

\[
[p\partial_{pp} + 2\partial_p + \nu|k|^2]\hat{H} = ik_j\hat{G}_j^{[1]} + aP_k[e_2\hat{S}]
\]

\[
[p\partial_{pp} + 2\partial_p + \mu|k|^2]\hat{S} = ik_j\hat{G}_j^{[2]}.
\]

We substitute (30) into (31) and identify powers of \(p^l\) to get a relationship for the coefficients. We will use the fact that

\[
p^l \ast n^m = \frac{l!n!}{(l+n+1)!}p^{l+n+1}.
\]

For \(l = 0\), we have

\[
2\hat{H}^{[1]} = -ik_{j}P_k[\hat{u}_{1,j} \ast \hat{u}_0 + \hat{u}_{0,j} \ast \hat{u}_1] - \nu|k|^2\hat{u}_1 + P_k[ae_2\hat{\Theta}_1]
\]

(32)
Lemma 4.4.

4.1. Estimates on the Solution in the Borel Plane

More generally, for power series representations in \( p \),

Furthermore, the solutions guaranteed to exist in Lemma (3.11) have convergent power series representations in \( p \), and for \( |p| < (4D_0)^{-1} \)

\[
(\hat{H}, \hat{S})(k, p) = (\hat{u}_1, \hat{\Theta}_1)(k) + \sum_{l=1}^{\infty} (\hat{H}^{[l]}, \hat{S}^{[l]})(k)p^l.
\]
To prove this lemma we will establish bounds for \((\hat{H}^{[l]}, \hat{S}^{[l]})\) using induction.

**Lemma 4.5.** For the base case, we have

\[
|((\hat{H}^{[1]}, \hat{S}^{[1]}))(k)| \leq \frac{e^{-\beta|k|} Q_2(\beta|k|) A_0 D_0}{(1 + |k|) \gamma^9}
\]  

for

\[
A_0 D_0 \geq \frac{9}{\beta^2} \left( C_0 \beta \left( \|\hat{u}_0, \hat{\Theta}_0\|_{\gamma, \beta} M_1 + \alpha \beta^2 \right) \right)
\]

**Proof.** From (32) and Lemma 3.7, we get

\[
|((\hat{H}^{[1]}, \hat{S}^{[1]}))(k)| \leq \frac{e^{-\beta|k|} \left( |k|^2 \left( \|\hat{u}_1, \hat{\Theta}_1\|_{\gamma, \beta} M_1 + 2C_0 |k| \left( \|\hat{u}_0, \hat{\Theta}_0\|_{\gamma, \beta} \right) \right) + 2C_0 |k| \left( \|\hat{u}_1, \hat{\Theta}_1\|_{\gamma, \beta} \right) \right) \cdot \frac{1}{(l + 1)(l + 2)} \gamma^9
\]

The result now follows after noting that \(Q_2(\beta|k|) = 4 + 2\beta|k| + \frac{1}{2}(\beta|k|)^2\).

For the general terms we will need a series of lemmas, which depend on the Fourier inequalities developed in Appendix B, bounding the terms that appear on the right side of (34).

**Lemma 4.6.** Assume that \((\hat{H}^{[l]}, \hat{S}^{[l]})\) satisfies (36) for \(l \geq 1\). Then we have,

\[
\frac{|k|^2 \left( \|\hat{H}^{[l]}, \hat{S}^{[l]}\| \right)}{(l + 1)(l + 2)} \leq \frac{6A_0 D_0 e^{-\beta|k|} Q_{2l+2}(\beta|k|)}{\beta^2 (1 + |k|)^\gamma (2l + 3)^2}
\]

**Proof.** The proof follows from (36) directly by noting that for \(y \geq 0\)

\[
\frac{y^2 Q_{2l}(y)}{(2l + 2)(2l + 1)} \leq Q_{2l+2}(y) \text{ and } \frac{(2l + 2)(2l + 3)^2}{(l + 1)(l + 2)(2l + 1)} \leq 6.
\]

**Lemma 4.7.** Suppose \((\hat{H}^{[l]}, \hat{S}^{[l]})\) satisfies (36) for \(l \geq 1\). Then both

\[
|k_j \left( P_k(\hat{u}_0, \hat{\Theta}_0, \hat{H}^{[l]}, \hat{S}^{[l]}) \right) \left( \hat{u}_0, \hat{\Theta}_0, \hat{H}^{[l]}, \hat{S}^{[l]} \right) | \text{ and } \left| k_j \left( P_k(\hat{H}^{[l]} \hat{u}_0, \hat{\Theta}_0, \hat{H}^{[l]} \hat{\Theta}_0) \right) \right|
\]

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are bounded by

\[ 2^\gamma ||(\hat{u}_0, \hat{\Theta}_0)||_{\gamma, \beta} \frac{9C_7\pi A_0 D_0 e^{-\beta|k|(l + 1)(l + 2)}}{2\beta^d(2l + 3)^2(1 + |k|)^\gamma} Q_{2l+2}(|\beta k|). \]

Similarly, suppose \((\hat{H}^{[l-1]}, \hat{S}^{[l-1]})\) satisfies (36) for \(l \geq 2\). Then both

\[ |k_j \left( P_k(\hat{u}_{1,j} \hat{\star} \hat{H}^{[l-1]}), \hat{u}_{1,j} \hat{\star} \hat{S}^{[l-1]} \right) | \] and
\[ |k_j \left( P_k(\hat{H}_{j}^{[l-1]} \hat{\star} \hat{u}_1), \hat{H}_{j}^{[l-1]} \hat{\star} \hat{\Theta}_1 \right) | \]

are bounded by

\[ 2^\gamma ||(\hat{u}_1, \hat{\Theta}_1)||_{\gamma, \beta} \frac{9C_7\pi A_0 D_0 e^{-\beta|k|(l + 1)(l + 2)}}{2\beta^d(2l + 1)^2(1 + |k|)^\gamma} Q_{2l+2}(|\beta k|). \]

**Proof.** We use the estimate (36) on \((\hat{H}^{[l]}, \hat{S}^{[l]})\) and Appendix B.5 in \(\mathbb{R}^d\) with \(n = 0\) to get

\[ |k_j \hat{u}_{0,j} \hat{\star} (\hat{H}^{[l]}, \hat{S}^{[l]})| \leq ||\hat{u}_0||_{\gamma, \beta} \frac{A_0 D_0^l}{(2l + 1)^2} \left( \int_{k' \in \mathbb{R}^d} \frac{e^{-\beta|k'|+|k-k'|}}{(1 + |k'|)^\gamma (1 + |k - k'|)^\gamma} Q_{2l+2}(|\beta k'|) dk' \right) \]

\[ \leq \frac{||\hat{u}_0||_{\gamma, \beta} A_0 D_0^l}{(2l + 1)^2} \sum_{m=0}^{2l} \frac{2^{2l-m}}{m!} \left( \int_{k' \in \mathbb{R}^d} e^{-\beta|k'|+|k-k'|} (1 + |k'|)^\gamma (1 + |k - k'|)^\gamma |\beta k'|^m dk' \right) \]

\[ \leq \frac{C_7\pi ||\hat{u}_0||_{\gamma, \beta} A_0 D_0^l}{(2l + 1)^2} 2^{2l-m} (m + 2) Q_{m+2}(\beta|k|) \]

\[ \leq \frac{2^\gamma C_7\pi ||\hat{u}_0||_{\gamma, \beta} A_0 D_0 e^{-\beta|k|}}{(2l + 1)^2} \left( \frac{1}{(1 + |k|)^\gamma} \right) (l + 2) Q_{2l+2}(\beta|k|). \]

The first part of the lemma now follows noting \(\frac{2(2l+3)^2}{(2l+1)(l+1)} \leq 9\) for \(l \geq 1\). The second parts are proved similarly.

**Lemma 4.8.** Let \(l \geq 3\). Suppose \((\hat{H}^{[l_1]}, \hat{S}^{[l_1]})\) and \((\hat{H}^{[l-1-l_1]}, \hat{S}^{[l-1-l_1]})\) satisfy (36) for \(l_1 = 1, \ldots, l - 2\). Then

\[ |k_j \left[ \sum_{l_1=1}^{l-2} \frac{l_1!(l - 1 - l_1)!}{(l + 2)!} \left( P_k(\hat{H}_{j}^{[l_1]} \hat{\star} \hat{H}^{[l-1-l_1]}), \hat{H}_{j}^{[l_1]} \hat{\star} \hat{S}^{[l-1-l_1]} \right) \right] | \]

is bounded by

\[ 2^{\gamma+3}C_7 A_0^2 D_0^{l-1}(1 + |k|)^{-\gamma} e^{-\beta|k|} \frac{Q_{2l}(\beta|k|)}{\beta^d(2l + 3)^2}. \]
Lemma 4.9. For \( l = 2 \) we have,

\[
|\left( \hat{H}^2, \hat{S}^2 \right)| \leq \frac{e^{-\beta|k|}Q_4(\beta|k|)}{5^2(1 + |k|)^\gamma} \left( \frac{6A_0D_0M_1}{\beta^2} + \frac{2\gamma 9C_7\pi A_0D_0\|\hat{u}_0, \hat{\Theta}_0\|_{\gamma, \beta}}{\beta^d} + A_0D_0a + \frac{C_0}{\beta}\|\hat{u}_1, \hat{\Theta}_1\|_{\gamma, \beta}^2 \right).
\]

Thus, \((\hat{H}^2, \hat{S}^2)\) satisfies (36) for

\[
D_0^2 \geq \frac{6D_0M_1}{\beta^2} + D_0a + \frac{2\gamma 9C_7\pi D_0}{\beta^d}\|\hat{u}_0, \hat{\Theta}_0\|_{\gamma, \beta} + \frac{C_0}{A_0^\beta}\|\hat{u}_1, \hat{\Theta}_1\|_{\gamma, \beta}^2. \tag{39}
\]

Proof. The proof is similar to that in [10] with \( \hat{W}^{[lz]} \) replaced by \((\hat{W}^{[lz]}, \hat{Q}^{[lz]})\). For more details see [10] and [21].

Proof of Lemma 4.4. The base case is proved picking \( D_0 \) large enough so (39) and (37) hold. For general \( l \geq 2 \) suppose \((\hat{H}^{[m]}, \hat{S}^{[m]})\) satisfies (36) for \( m = 1, \ldots, l \). We estimate terms on the right of (34) and (35), using Lemma 4.6, 4.7, and 4.8 and the fact that \( Q_2(y) \leq 1/4Q_{2l+2}(y) \), to get

\[
|\left( \hat{H}^{[l+1]}, \hat{S}^{[l+1]} \right)| \leq \frac{A_0D_0^{-1}Q_{2l+2}(\beta|k|)}{(2l + 3)^2(1 + |k|)^\gamma} \left\{ \frac{6D_0M_1}{\beta^2} + \frac{aD_0}{2} + \frac{2\gamma 9C_7\pi D_0}{\beta^d}\|\hat{u}_0, \hat{\Theta}_0\|_{\gamma, \beta} \right. \\
\left. + \frac{2\gamma 9C_7\pi (2l + 3)^2}{4(l + 2)(2l + 1)^2\beta^d}\|\hat{u}_1, \hat{\Theta}_1\|_{\gamma, \beta} + \frac{2\gamma 3C_7A_0}{4\beta^d} \right\} \\
\leq \frac{A_0D_0^{-1}e^{-\beta|k|}}{(1 + |k|)^\gamma(2l + 3)^2}Q_{2l+2}(\beta|k|)
\]

where \( D_0 \) has been chosen large enough so

\[
\left\{ \frac{6D_0M_1}{\beta^2} + \frac{aD_0}{2} + \frac{2\gamma 9C_7\pi D_0}{\beta^d}\|\hat{u}_0, \hat{\Theta}_0\|_{\gamma, \beta} + \frac{2\gamma 9C_7\pi D_0}{4\beta^d}\|\hat{u}_1, \hat{\Theta}_1\|_{\gamma, \beta} \right. \\
\left. + \frac{2\gamma 3C_7A_0}{4\beta^d} \right\} \leq D_0^2.
\]

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We also used $\frac{(2l+3)^2}{(2l+1)^3(l+2)} \leq 1$ in the above. Thus, by induction, we have (36) satisfied for any $l \geq 1$. So, $\sum_{l=1}^{\infty}(\hat{H}^{[l]}, \hat{S}^{[l]})(k)p^l$ is convergent for $|p| \leq \frac{1}{4D_0}$ since $Q_{2l}(\beta|k|) \leq 4'e^{\beta|k|/2}$. By construction of the iteration, $(\hat{H}, \hat{S}) - (\hat{u}_1, \hat{\Theta}_1) = \sum_{l=1}^{\infty}(\hat{H}^{[l]}, \hat{S}^{[l]})(k)p^l$ is a solution to (31) which is zero at $p = 0$. However, we know there is a unique solutions to (31) which is zero and $p = 0$ in the space $A^{\infty}_L$, which includes analytic functions at the origin for $L$ sufficiently small. Thus, for $(\hat{H}, \hat{S})$ the solution guaranteed by Lemma 3.11, we have

$$(\hat{H}, \hat{S})(k, p) = (\hat{u}_1, \hat{\Theta}_1)(k) + \sum_{l=1}^{\infty}(\hat{H}^{[l]}, \hat{S}^{[l]})(k)p^l.$$ 

4.2. Estimates on $\partial_p^l(\hat{H}, \hat{S})(k, p)$

We now want to develop estimates on $\partial_p^l(\hat{H}, \hat{S})(k, p)$ in order to show that we can analytically extend our solutions along $\mathbb{R}^+$ with a radius of convergence independent of center $p_0$ along $\mathbb{R}^+$. Combining this with the fact that the solutions are exponentially bounded will give Borel summability.

**Definition 4.10.** For $l \geq 1$ we define,

$$(\hat{H}^{[l]}, \hat{S}^{[l]})(k, p) = \frac{1}{l!} \partial_p^l(\hat{H}, \hat{S})(k, p)$$

$$(\hat{H}^{[0]}, \hat{S}^{[0]})(k, p) = (\hat{H}, \hat{S})(k, p) - (\hat{u}_1, \hat{\Theta}_1).$$

**Lemma 4.11.** If $||(\hat{u}_0, \hat{\Theta}_0)||_{\gamma+2, \beta} < \infty$ for and $\beta > 0$, then there are constants $A, D > 0$ not depending on $l, k$ or $p$ such that

$$|(\hat{H}^{[l]}, \hat{S}^{[l]})(k, p)| \leq \frac{e^{\omega_p}e^{-\beta|k|}AD^l}{(1 + p^2)(1 + |k|)^\gamma} \frac{Q_{2l}(\beta|k|)}{(2l + 1)^2}$$

where $\omega' = \omega + 1$ for $\omega$ chosen as in Lemma 3.11. We will prove the lemma by induction, and as before we will develop several lemmas to establish the bound.

For $l = 0$, we use Lemma 3.11 which says that for $\omega$ sufficiently large

$$|(\hat{H}, \hat{S})(k, p)| \leq \frac{2e^{-\beta|k|+\omega p}}{(1 + |k|)^\gamma}||(\hat{u}_1, \hat{\Theta}_1)||_{\gamma, \beta}.$$ 

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We chose \( \omega' = \omega + 1 \) and recall Definition 4.10 to get

\[
\|(\hat{H}^{[l]}, \hat{S}^{[l]})(k, p)\| \leq \frac{3e^{-|k|+\omega'}\|((\hat{u}_1, \hat{\Theta}_1))\|_{\gamma, \beta}}{(1 + p^2)(1 + |k|)^\gamma},
\]

and the base cases of (40) is proved for \( A = 3\|((\hat{u}_1, \hat{\Theta}_1))\|_{\gamma, \beta}. \)

For the general case \( (l \geq 1) \) we take \( \partial_p^l \) in (31) and divide by \( l! \), to obtain

\[
p\hat{H}^{[l]}_{pp} + (l+2)\hat{H}^{[l]}_p + \mu |k|^2 \hat{H}^{[l]} = (-ik_j P_k [\hat{u}_{0,j} \hat{\Theta}_1 + \hat{u}_{1,j} \hat{\Theta}_0] - \nu |k|^2 \hat{u}_1) \delta_{l,0}
\]

\[
-ik_j \left[ \int_0^p \hat{H}^{[l]}_j (\cdot, p-s) \hat{S}^{[l]}(\cdot, s) ds + \sum_{l_1=1}^{l-1} \frac{l_1!(l-l_1-1)!}{l!} \hat{H}^{[l_1]}_j (\cdot, 0) \hat{S}^{[l-l_1-1]} (\cdot, p) \right]
\]

\[
-ik_j \left[ \int_0^p \hat{H}^{[l]}_j (\cdot, p-s) \hat{S}^{[l]}(\cdot, s) ds + \sum_{l_1=1}^{l-1} \frac{l_1!(l-l_1-1)!}{l!} \hat{H}^{[l_1]}_j (\cdot, 0) \hat{S}^{[l-l_1-1]} (\cdot, p) \right]
\]

\[
-ik_j \left[ \int_0^p \hat{H}^{[l]}_j (\cdot, p-s) \hat{S}^{[l]}(\cdot, s) ds + \sum_{l_1=1}^{l-1} \frac{l_1!(l-l_1-1)!}{l!} \hat{H}^{[l_1]}_j (\cdot, 0) \hat{S}^{[l-l_1-1]} (\cdot, p) \right]
\]

Denote the right hand side of these four equations by \( \hat{R}^{[l]}_m \) for \( m = 1 \) and 2 respectively.

**Lemma 4.12.** For any \( l \geq 0 \) and for some absolute constant \( C_6 \), if \( (\hat{H}^{[l]}, \hat{S}^{[l]}) \) satisfies (40), and is bounded at \( p = 0 \) then

\[
| (\hat{H}^{[l+1]}, \hat{S}^{[l+1]}) (k, p) | \leq \frac{C_6}{(l+1)^{5/3}} \sup_{p \in [0,1]} | (\hat{R}^{[1]}_1, \hat{R}^{[2]}_2) | + \frac{M_1 |k|^2 |(\hat{H}^{[l]}, \hat{S}^{[l]}) (k, 0) |}{(l+1)(l+2)}.
\]

**Proof.** The proof is in (10) under Lemma 4.4. The lemma is dependent only on the operator \( \mathcal{D} \) which is the same in our case.

**Lemma 4.13.** Suppose \( (\hat{H}^{[l]}, \hat{S}^{[l]}) \) satisfies (40) for \( l \geq 1 \). Then

\[
| k_j \left( P_k (\hat{u}_{0,j} \hat{H}^{[l]}_j), \hat{u}_{0,j} \hat{S}^{[l]} \right) | \text{ and } | k_j \left( P_k (\hat{H}^{[l]}_j \hat{u}_0), \hat{H}^{[l]}_j \hat{S} \right) |
\]
are bounded by

\[ C_1 \left\| \left( \hat{u}_0, \hat{\Theta}_0 \right) \right\|_{\gamma, \beta} \frac{(l + 1)^{2/3} AD' e^{-\beta |k| + \omega' p}}{(2l + 1)(1 + p^2)(1 + |k|)^\gamma} Q_{2l+2}(|\beta k|), \]

\[ \left| \frac{k_j}{l} \left( P_k(\hat{u}_{1,j} \hat{H}^{[l-1]}), \hat{u}_{1,j} \hat{S}^{[l-1]} \right) \right| \quad \text{and} \quad \left| \frac{k_j}{l} \left( P_k(\hat{H}^{[l-1]} \hat{u}_1), \hat{H}^{[l-1]} \hat{\Theta}_1 \right) \right| \]

are bounded by

\[ C_1 \left\| \left( \hat{u}_1, \hat{\Theta}_1 \right) \right\|_{\gamma, \beta} \frac{l^{2/3} AD^{l-1} e^{-\beta |k| + \omega' p}}{l(2l - 1)(1 + p^2)(1 + |k|)^\gamma} Q_{2l}(|\beta k|), \]

and

\[ |P_k(ae \hat{S}^{[l]})| \leq a \frac{e^{\omega' p} e^{-\beta |k|} AD^l}{(1 + p^2)(1 + |k|)^\gamma} Q_{2l+2}(|\beta k|). \]

In the above, \( C_1 = C_1(d) \) is defined in \( \text{Appendix B.8} \).

**Proof.** For the first inequality, we use (40) and then apply \( \text{Appendix B.8} \) to get

\[ (1 + p^2) e^{-\omega' p} |k_j \hat{u}_0, \hat{S}(\hat{H}^{[l]}, \hat{S}^{[l]})| \]

\[ \leq \left\| \hat{u}_0 \right\|_{\gamma, \beta} \frac{AD'}{(2l + 1)^2 |k|} \int_{k' \in \mathbb{R}^d} e^{-\beta (|k'| + |k-k'|)} Q_{2l}(|\beta k'|) dk' \]

\[ \leq C_1 (l + 1)^{2/3} \left\| \hat{u}_0 \right\|_{\gamma, \beta} \frac{AD' e^{-\beta |k|}}{(2l + 1)(1 + |k|)^\gamma} Q_{2l+2}(|\beta k|). \]

The other inequalities are proved similarly and the last is simply the statement of the assumed bound.

**Lemma 4.14.** Suppose \((\hat{H}^{[l]}, \hat{S}^{[l]})\) satisfies (40) for \( l \geq 1 \). Then

\[ \left| \frac{k_j}{l} \left( P_k(\hat{H}^{[l-1]} \cdot, \hat{H}^{[0]} \cdot, \hat{H}^{[l-1]} \cdot, \hat{S}^{[0]} \cdot, p) \right) \right| \]

is bounded by

\[ C_1 (l + 1)^{2/3} \frac{\tilde{A}^2 \tilde{D}^{l-1} e^{-\beta |k| + \alpha' p}}{l(2l - 1)(1 + |k|)^\gamma(1 + p^2)} Q_{2l}(|\beta k|). \]
Proof. Using (40) with \( p = 0 \) and (111) with \( A = 3||(\hat{u}_1, \hat{\Theta}_1)||_{\gamma, \beta} \) along with Appendix B.8, we get
\[ (1 + p^2)e^{-\omega p} | \frac{k_j}{l} [\hat{H}^{[l-1]}_j(\cdot, 0)\hat{s}(\hat{H}^{[0]}_j, \hat{S}^{[0]}_j)(\cdot, p)] | \]
\[ \leq \frac{A^2 D^{l-1}}{l(2l-1)^2} |k| \int_{k' \in \mathbb{R}^d} \frac{e^{-\beta(|k'| + |k - k'|)}}{(1 + |k'|)^\gamma (1 + |k - k'|)^\gamma} Q_{2l-2}(\beta |k'|)dk' \]
\[ \leq C_1 \frac{l^{2/3} A^2 D^{l-1} e^{-\beta |k|}}{l(2l-1)(1 + |k|)^\gamma} Q_{2l}(\beta |k|) \]

From this the lemma follows after using Lemma 3.7.

Lemma 4.15. Suppose \((\hat{H}^{[l]}_1, \hat{S}^{[l]}_1)\) and \((\hat{H}^{[l-l_1-1]}_j, \hat{S}^{[l-l_1-1]}_j)\) satisfies (40) for \( l_1 = 1, \ldots, l - 2 \) where \( l \geq 2 \). Then for \( C_8 = 82 \) and \( C_7 = C_7(d) \) given in Appendix B.7, we have
\[ \left| k_j \sum_{l_1=1}^{l-2} \frac{l_1!(l - l_1 - 1)!}{l!} \left( P_k(\hat{H}^{[l]}_j(\cdot, 0)\hat{s}(\hat{H}^{[l-l_1-1]}_j(\cdot, p)), \hat{H}^{[l]}_j(\cdot, 0)\hat{s}(\hat{S}^{[l-l_1-1]}_j(\cdot, p)) \right) \right| \]

is bounded by
\[ C_8 C_7^2 2^{\gamma} \pi A^2 D^{l-1} \frac{e^{-\beta |k| + \omega p}}{3\beta^4(1 + p^2)(1 + |k|)^\gamma} \frac{l Q_{2l}(\beta |k|)}{(2l + 3)^2}. \]

The proof is the same as in [10], the only difference is a change in the constants arising when Appendix B.7 in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) is applied.

Lemma 4.16. Suppose \((\hat{H}^{[l]}_j, \hat{S}^{[l]}_j)\) satisfies (40) for \( l \geq 0 \). Then
\[ \left| k_j \int_0^p \left( P_k(\hat{H}^{[l]}_j(\cdot, p-s)\hat{s}(\hat{H}^{[0]}_j(\cdot, s)), \hat{H}^{[l]}_j(\cdot, p-s)\hat{s}(\hat{S}^{[0]}_j(\cdot, s)) \right) ds \right| \]
\[ \leq C_1 M_0 A^2 D^l \frac{(l + 1)2^{3/2} e^{-\beta |k| + \omega p}}{(2l + 1)(1 + |k|)^\gamma (1 + p^2)} Q_{2l+2}(\beta |k|) \]

In the above, \( M_0 \), defined in Lemma 3.5, is such that
\[ \int_0^p \frac{1}{(1 + (p - s)^2)(1 + s^2)} ds \leq \frac{M_0}{1 + p^2}. \]
Proof. Using (40) for the first inequality and Appendix B.8 and Lemma 3.9 for the second, we have
\[
    \left| k \int_0^p \left( \hat{P}_k(\hat{H}_j^l(\cdot, p - s)\hat{s}^0(\cdot), s), (\hat{H}_j^l(\cdot, p - s)\hat{s}^0(\cdot, s)) \right) ds \right| \leq \\
    |k| \frac{A^2D_l}{2l + 1} \int_0^p \int_{k' \in \mathbb{R}^d} e^{-\beta|k'| + |k' - k|} e^{\omega'(p-s) + \omega's} Q_{2l}(\beta|k'|) ds dk' \\
    \leq C_1M_0A^2D_l \frac{(l + 1)^{2/3}e^{-\beta|k| + \alpha p}}{(2l + 1)(1 + |k|)^\gamma(1 + p^2)} Q_{2l+2}(\beta|k|).
\]

Lemma 4.17. We have
\[
    \left| k \left( P_k(\hat{u}_0,j\hat{s}u_1), \hat{u}_0,j\hat{s}\Theta_1 \right) + k \left( P_k(\hat{u}_1,j\hat{s}u_0), \hat{u}_1,j\hat{s}\Theta_0 \right) \right| \\
    \leq 2C_0|k|e^{-\beta|k|} \frac{1}{(1 + |k|)^\gamma} ||(\hat{u}_0, \hat{\Theta}_0)||_{\gamma, \beta} ||(\hat{u}_1, \hat{\Theta}_1)||_{\gamma, \beta} \\
    \left| k \left( P_k(\hat{u}_1,j\hat{s}u_1), \hat{u}_1,j\hat{s}\Theta_1 \right) \right| \leq |k|e^{-\beta|k|} C_0 \frac{1}{(1 + |k|)^\gamma} ||\hat{u}_1, \hat{\Theta}_1||_{\gamma, \beta}^2.
\]

Proof. This follows directly from Corollary 3.6 and Lemma 3.7.

Lemma 4.18. For the case $l = 1$, we have
\[
    |(\hat{H}_1^l, \hat{S}_1^l)(k, p)| \leq \frac{e^{\omega_p e^{-\beta|k|} AD}}{(1 + p^2)(1 + |k|)^\gamma} Q_2(\beta|k|),
\]
where
\[
    AD \geq C_6 \left( \frac{C_0}{\beta} ||(\hat{u}_0, \hat{\Theta}_0)||_{\gamma, \beta} ||(\hat{v}_1, \hat{\Theta}_1)||_{\gamma, \beta} + M_1 \frac{2}{\beta^2} ||(\hat{u}_1, \hat{\Theta}_1)||_{\gamma, \beta} \\
    + C_1M_0A^2 + 2C_1A ||(\hat{u}_0, \hat{\Theta}_0)||_{\gamma, \beta} + \frac{aA}{4} \right).
\]

Proof. Lemma 4.12 with $l = 0$ tells us that
\[
    |(\hat{H}_1^l, \hat{S}_1^l)(k, p)| \leq C_6 \sup_{p' \in [0, p]} |(\hat{R}_1^0, \hat{R}_2^0)(k, p')|
\]
since $(\hat{H}_1^l, \hat{S}_1^l)(k, 0) = 0$. We use Lemma 4.13, Lemma 4.16 and Lemma 4.17 to bound the terms appearing in $R_m$s.
\[
    |(\hat{R}_1^0, \hat{R}_2^0)(k, p)| \leq \frac{2C_0|k|e^{-\beta|k|}}{(1 + |k|)^\gamma} ||(\hat{u}_0, \hat{\Theta}_0)||_{\gamma, \beta} ||(\hat{u}_1, \hat{\Theta}_1)||_{\gamma, \beta}
\]

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\[
+ M_1 \left| k \right|^2 e^{-\beta \left| k \right|} \left( \hat{v}_1, \hat{\Theta}_1 \right)_{\gamma, \beta} + C_1 M_0 A^2 \frac{e^{-\beta \left| k \right| + \omega' p}}{(1 + |k|)\gamma(1 + p^2)} Q_2(\beta |k|)
+ 2C_1 \left| (\hat{u}_0, \hat{\Theta}_0) \right|_{\gamma, \beta} A e^{-\beta \left| k \right| + \omega' p} \frac{Q_2(\beta |k|)}{(1 + p^2)(1 + |k|)} + A e^\omega p e^{-\beta \left| k \right| A} \frac{Q_2(\beta |k|)}{(1 + p^2)(1 + |k|)}
\]

The lemma now follows since \(4|k| \leq \frac{2Q_2}{\beta}\) and \(|k|^2 \leq \frac{2Q_2}{\beta^2}\).

**Proof of Lemma 4.11** Lemma 4.18 and (41) prove the base case. Suppose, for the purpose of induction, that for \(l \geq 1\) (40) holds. Then by Lemma 4.12 we need only prove a bound for \(|(\hat{R}_1^l, \hat{R}_2^l)|\) whose terms we bounded in the previous lemmas.

\[
\left| (\hat{R}_1^l, \hat{R}_2^l) \right| \leq \frac{AD^l - 1 e^{-\beta \left| k \right| + \omega' p}}{(2l + 3)^2(1 + p^2)(1 + |k|)\gamma} Q_{2l+2}(\beta |k|) \left\{ C_1 M_0 A D (l + 1)^{2/3} (2l + 3)^2 \frac{1}{(2l + 1)} + \frac{C_1 A (l + 1)^{2/3} (2l + 3)^2}{4l(2l - 1)} + \frac{C_S C_7 2^{\gamma} \pi Al}{12\beta d} + \frac{C_1 l^{2/3} \left| (\hat{u}_1, \hat{\Theta}_1) \right|_{\gamma, \beta} (2l + 3)^2}{2l(2l - 1)} + 2C_1 D \left| (\hat{u}_0, \hat{\Theta}_0) \right|_{\gamma, \beta} \frac{(l + 1)^{2/3} (2l + 3)^2}{2l + 1} + 25\delta_{l, 1} \frac{C_0}{A^\beta} \left| (\hat{u}_1, \hat{\Theta}_1) \right|_{\gamma, \beta} + aD (2l + 3)^2 \frac{1}{4(2l + 1)^2} \right\}.
\]

We also note that as \((\hat{H}^l, \hat{S}^l)\) satisfies (40),

\[
\left| \left| k \right|^2 (\hat{H}^l, \hat{S}^l)(k, 0) \right| \leq \frac{\left| k \right|^2 e^{-\beta \left| k \right|} AD^l Q_{2l}(\beta |k|)}{(l + 1)(l + 2)} \leq \frac{\left| k \right|^2 e^{-\beta \left| k \right|} AD^l Q_{2l}(\beta |k|)}{(l + 1)(l + 2)(1 + |k|)\gamma(2l + 1)^2} \leq \frac{\left| k \right|^2 e^{-\beta \left| k \right|} AD^l e^{-\beta \left| k \right| + \alpha' p} Q_{2l+2}(\beta |k|) 6}{\beta^2}.
\]

Here, we used the following two facts

\[
\frac{y^2 Q_{2l}(y)}{(2l + 2)(2l + 1)} \leq Q_{2l+2}(y) \text{ and } \frac{(2l + 2)(2l + 3)^2}{(l + 1)(l + 2)(2l + 1)} \leq 6.
\]

Thus, for \(D\) chosen, independently of \(l, k, \) and \(p,\) large enough so

\[
D^2 \geq C_6 \left\{ \frac{C_1 M_0 AD (2l + 3)^2}{(l + 1)(2l + 1)} + \frac{C_1 A (2l + 3)^2}{4(l + 1)l(2l - 1)} + \frac{C_S C_7 2^{\gamma} \pi Al}{12\beta d (l + 1)^{5/3}} + \frac{C_1 \left| (\hat{u}_1, \hat{\Theta}_1) \right|_{\gamma, \beta} (2l + 3)^2}{2(l + 1)^{5/3} l^{1/3} (2l - 1)} + 2C_1 D \left| (\hat{u}_0, \hat{\Theta}_0) \right|_{\gamma, \beta} \frac{(2l + 3)^2}{(l + 1)(2l + 1)} \right\}.
\]
\[ + 25 \delta \beta \frac{C_0}{A} e_{l,1} \left\| |(\hat{u}_1, \hat{\Theta}_1)||_{7,\beta}^2 + \frac{aD(2l + 3)^2}{4(l + 1)^{5/3}(2l + 1)} \right\} + M_1 \frac{6D}{\beta^2}, \]

(40) holds and the lemma is proved.

As \( Q_2(\beta |k|) \leq 4e^{\beta|k|/2} \),

\[
(\hat{H}, \hat{S})(k, p; p_0) = \sum_{l=0}^{\infty} (\hat{H}^l, \hat{S}^l)(k, p_0)(p - p_0)^l
\]

is convergent for \(|p - p_0| < \frac{1}{4B}\) where \( D \) is independent of \( p_0 \). Moreover, the following lemma proved in [10] says that these series are indeed local representations of the solution \((\hat{H}, \hat{S})(k, p)\).

**Lemma 4.19.** The unique solution to (31) satisfying \( (\hat{H}, \hat{S})(k, 0) = 0 \) guaranteed in Lemma 3.11 has a local representation given by \((\hat{H}, \hat{S})(k, p; p_0)\) for \( p_0 \in \mathbb{R}^+ \). So, the solution is analytic on \( \mathbb{R}^+ \cup \{0\} \).

**Proof of Theorem 2.2 i)** Using Lemma 4.11 and the fact that \( ||g||_{L^\infty} \leq ||\hat{g}||_{L^1} \) we know that

\[
|(H^l, S^l)(x, p_0)| \leq \frac{8\pi A(4B)^l e^{\omega p_0}}{\beta(2l + 1)^2(1 + p_0^2)}
\]

\[
|D(H^l, S^l)(x, p_0)| \leq \frac{8\pi A(4B)^l e^{\omega p_0}}{\beta(2l + 1)^2(1 + p_0^2)}
\]

\[
|D^2(H^l, S^l)(x, p_0)| \leq \frac{16\pi A(4B)^l e^{\omega p_0}}{\beta^2(2l + 1)^2(1 + p_0^2)}
\]

and the series (44) converges for \(|p - p_0| < \frac{1}{4B}\). By Lemma 4.19 the series is the local representation of the solution guaranteed to exist by Lemma 3.11 which is zero at \( p = 0 \). Combining this with the facts that the solution is analytic in a neighborhood of zero and exponentially bounded for large \( p \), recall \((\hat{H}, \hat{S}) \in A^\omega\), implies Borel summability in \( 1/t \). Watson’s Lemma then implies as \( t \rightarrow 0^+ \)

\[
(u, \Theta)(x, t) \sim (u_0, \Theta_0)(x) + \sum_{m=1}^{\infty} (u_m, \Theta_m)(x)t^m
\]

where \( |(u_m, \Theta_m)(x)| \leq m!A_0D_0^m \) with constants \( A_0 \) and \( D_0 \) generally dependent on the initial condition and forcing through Lemma 4.3.
5. Extension of Existence Time

We have shown by Theorem 2.1 that there is a unique solution to (17) within the class of locally integrable functions, which are exponentially bounded in \( p \), uniformly in \( x \). Further, the solution \( (\hat{H}, \hat{S})(k, p) \) generates a smooth solution to the Boussinesq equation for \( t \in [0, \omega^{-1}] \) where \( \omega \) is the exponential growth rate of the integral equation (17), and we showed that the solution is Borel summable. The question of global existence is then reduced to a question of exponential growth for the integral equation solution. If \( (\hat{H}, \hat{S})(k, p) \) grows subexponentially, then global existence follows. The exponential growth rate \( \omega \) previously found is suboptimal and ignores possible cancellations in the integrals. If we improve the estimates, we get a longer interval of existence. Here we present two examples of cases which can result in longer interval of existence.

5.1. Improved Radius of Convergence

When the initial data and forcing are analytic Borel summability given in Theorem 2.2 implies that

\[
(\hat{H}, \hat{S})(k, p) = \sum_{m=0}^{\infty} (\hat{u}^{[m]}, \hat{\Theta}^{[m]})(k) \frac{p^{m}}{m!} = \sum_{m=0}^{\infty} (\hat{u}^{[m+1]}, \hat{\Theta}^{[m+1]})(k) \frac{p^{m}}{m!}
\]

has a finite radius of convergence depending on the size of the initial data and forcing. However, in the special case when the initial data and forcing have only a finite number of Fourier modes the radius of convergence is in fact independent of the size of the initial data or \( f \). The argument allows forcing to be time dependent.

**Proof of Theorem 2.2 ii)** For small time

\[
(u, \Theta)(k, t) = (\hat{u}^{[0]}, \hat{\Theta}^{[0]})(k) + \sum_{m=0}^{\infty} (\hat{u}^{[m]}, \hat{\Theta}^{[m]})(k) t^m
\]

\[
\hat{f}(k, t) = \hat{f}^{[0]} + \sum_{m=1}^{\infty} \hat{f}^{[m]}(k) t^m,
\]

where by (3) for \( m \geq 0 \)

\[
(m + 1)\hat{u}^{[m+1]} = \hat{f}^{[m]} - \nu |k|^2 \hat{u}^{[m]} - i k_j P_k \left( \sum_{l=0}^{m} \hat{u}_j^{[l]} \hat{u}^{[m-l]} \right) + a P_k (e_2 \hat{\Theta}^{[m]})
\]

(46)
\[(m+1)\hat{\Theta}^{[m+1]} = -\mu|k|^2\hat{\Theta}^{[m]} - ik_j \left( \sum_{l=0}^{m} \hat{a}_j^{[l]} \hat{\Theta}^{[m-l]} \right).\]

Suppose the initial data and forcing have a finite number of Fourier modes. Let \(K_1 = \max(\sup_{k \in \text{supp}(\hat{u}^{[0]}, \hat{\Theta}^{[0]})} |k|, \sup_{k \in \text{supp}(\hat{f})} |k|).\) Then by induction on \(k\) we have \(\sup_{k \in \text{supp}(\hat{u}^{[m]}, \hat{\Theta}^{[m]})} |k| \leq (m+1)K_1.\) Taking the \(\| \cdot \|_{\gamma, \beta}\) norm of both sides of (46) with respect to \(k\) and writing

\[a_m = \|(\hat{u}^{[m]}, \hat{\Theta}^{[m]})\|_{\gamma, \beta}, \quad b_m = \|\hat{f}^{[m]}\|_{\gamma, \beta},\]

we obtain

\[a_{m+1} \leq \frac{1}{m+1} \left[ b_m + M_1 \left\| |k|^2 |(\hat{u}^{[m]}, \hat{\Theta}^{[m]})| \right\|_{\gamma, \beta} + \sum_{l=0}^{m} \left\| |k| |\hat{u}^{[l]}| \hat{\Theta}^{[m-l]}| \right\|_{\gamma, \beta} + a a_m \right]
\leq \frac{b_m}{m+1} + \frac{a a_m}{m+1} + K_1^2 M_1 (m+1) a_m + 2K_1C_0 \sum_{l=0}^{m} a a_{m-l}.
\]

Consider the formal power series \(y_0(t) := \sum_{m=1}^{\infty} \tilde{a}_m t^m,\) where \(\tilde{a}_0 = a_0\) and

\[\tilde{a}_{m+1} = \frac{b_m}{m+1} + \frac{a a_m}{m+1} + K_1^2 M_1 (m+1) \tilde{a}_m + 2K_1C_0 \sum_{l=0}^{m} \tilde{a}_l \tilde{a}_{m-l}. \quad (47)\]

Clearly, \(a_m \leq \tilde{a}_m,\) so \(y_0(t)\) majorizes \(\|(\hat{u}, \hat{\Theta})(\cdot, t)\|_{\gamma, \beta}.\) If we multiply both sides of (47) by \(t^m\) and sum over \(m,\) then

\[\sum_{m=0}^{\infty} \tilde{a}_m t^m = \sum_{m=0}^{\infty} \left( \frac{b_m}{m+1} + \frac{a a_m}{m+1} + K_1^2 M_1 (m+1) \tilde{a}_m + 2K_1C_0 \sum_{l=0}^{m} \tilde{a}_l \tilde{a}_{m-l} \right) t^m.
\]

In other words, \(y_0(t)\) is a formal power series solution to

\[\frac{1}{t}(y - \tilde{a}_0) = w + \frac{a}{t} \int_0^t y(\tau)d\tau + K_1^2 M_1 (ty') + 2K_1C_0 y^2,
\]

where \(w(t) = \sum_{m=0}^{\infty} \frac{b_m}{m+1} t^m.\) With the change of variables \(s = 1/t,\) we have

\[-K_1^2 M_1 y' + 2K_1C_0 s^{-1} y^2 + (K_1^2 M_1 s^{-1} - 1)y + (s^{-1} w + \tilde{a}_0) + a \int_0^{1/s} y(\tau)d\tau = 0.
\]

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A singularity of $B(y(s))$ in the Borel plane exhibits itself as an exponential small correction to $y_0$. So, we let $y = y_0 + \delta$ and construct the equation for $\delta$:

$$-K_1^2 M_1 \delta' + 2K_1 C_0 s^{-1}(\delta^2 + 2y_0 \delta) + (K_1^2 M_1 s^{-1} - 1)\delta + a s \int_0^{1/s} \delta(\tau) d\tau = 0.$$ 

If we assume $\delta$ is exponentially small, then to leading order the equation is

$$-K_1^2 M_1 \delta' + [(4K_1 C_0 s^{-1}\tilde{\alpha}_0 + (K_1^2 M_1) s^{-1} - 1) \delta = 0,$$

which yields

$$\delta \sim e^{-K_1^{-2} M_1^{-1} s} s^{\tilde{\alpha}_0} C_0 K_1^{-1} M_1^{-1} + 1.$$ 

So, the radius of convergence of $B(y)$ is at least $K_1^{-2} M_1^{-1}$ which is independent of the size of initial data as claimed. As $y$ majorizes our solution $(\hat{u}, \hat{\Theta})(k, t)$ the radius of convergence of (45) is independent of the size of initial data or forcing as well.

5.2. Improved Growth Estimates Based on Knowledge of the Solution in $[0, p0]$.

Let $(\hat{H}, \hat{S})(k, p)$ be the solution to (17) provided by Theorem 2.1. Recall the definitions of $(\hat{H}, \hat{S})^{(a)}$ and $(\hat{H}, \hat{S})^{(s)}$ given by (41) and (42) and the functionals in (9) and (10). Now, let $(\hat{H}, \hat{S})^{(b)} = (\hat{H}, \hat{S}) - (\hat{H}, \hat{S})^{(a)}$. It is convenient to write the integral equation for $(\hat{H}, \hat{S})^{(b)}$ for $p > p_0$,

$$\hat{H}^{(b)}(k, p) = \frac{\pi}{z} \int_{p_0}^p G(z, z') (ik_j \hat{G}_j^{[1], (b)}(k, p') + P_k [e_2 \hat{s}^{(b)}(k, p')]) dp' + \hat{H}^{(a)}(k, p)$$

$$\hat{S}^{(b)}(k, p) = \frac{ik_j \pi}{2|k| \sqrt{hp}} \int_{p_0}^p G(\zeta, \zeta') \hat{G}_j^{[2], (b)}(k, p') dp' + \hat{S}^{(s)}(k, p),$$

where

$$\hat{G}_j^{[1], (b)}(k, p) = -P_k [\hat{u}_{0, j} \hat{s}^{(b)} + \hat{H}_j^{(b)} \hat{s} \hat{u}_0 + \hat{H}_j^{(a)} \hat{H}^{(b)} + \hat{H}_j^{(b)} \hat{H}^{(a)} + \hat{H}_j^{(b)} \hat{H}^{(b)}]$$

$$\hat{G}_j^{[2], (b)}(k, p) = -[\hat{u}_{0, j} \hat{s}^{(b)} + \hat{H}_j^{(b)} \hat{s} \hat{u}_0 + \hat{H}_j^{(a)} \hat{S}^{(b)} + \hat{H}_j^{(b)} \hat{S}^{(a)} + \hat{H}_j^{(b)} \hat{S}^{(b)}].$$

We also define

$$\hat{R}^{(b)}(k, p) = ik_j (\hat{G}_j^{[1]}, \hat{G}_j^{[2], (b)}(k, p) + a P_k [e_2 \hat{s}^{(b)}(k, p)].$$

(49)
Proof of Theorem 2.3 We note that
\[
|R^{(b)}(k, p)| \leq \left| \left| k \left[ |\hat{u}_0| (\hat{H}, \hat{S})^{(b)} | + |\hat{H}(b)| \right| + 2 |(\hat{H}, \hat{S})^{(a)}| + |(\hat{H}, \hat{S})^{(b)}| \right| + a|\hat{H}(b)| \right) (k, p),
\]
where $| \cdot |$ is the usual euclidean norm. Let $\psi(p) = \| (\hat{H}, \hat{S})^{(b)}(\cdot, p) \|_{\gamma, \beta}$. Then
\[
\left\| \left( \frac{G(z, z')}{z} (ik_j \hat{G}^{[1]}(b), k, p) + aP_k e^{2 \hat{S}^{(b)}(k, p)} \right), \frac{G(z, z')}{z} (ik_j \hat{G}^{[2]}(b), k, p) \right\|_{\gamma, \beta}
\leq B_0(k) \cdot \left| \left| k \right| \left| \hat{u}_0 \right| \right|_{\gamma, \beta} \psi(p) \| \psi(p) \|_{\gamma, \beta} + 2 \left| (\hat{H}, \hat{S})^{(a)} \right|_{\gamma, \beta} \psi(p)
\]
where $\psi(s) = \| (\hat{H}, \hat{S})^{(s)}(\cdot, p) \|_{\gamma, \beta}$. Recalling that $\psi = 0$ on $[0, p_0]$, we note that for any $u$
\[
\int_{p_0}^{p} e^{-wp} (\psi \ast u)(p) dp = \int_{p_0}^{p} \psi(s) e^{-ws} \int_{0}^{s} e^{-wp} u(p) dp ds.
\]
Using this, we obtain
\[
L_{p_0, M} \leq \frac{1}{\omega} \left\{ (b_1 + \int_{0}^{M-p_0} e^{-wp} B_2(p) dp) L_{p_0, M} + B_2 L_{p_0, M}^2 + B_4 L_{p_0, M} \right\} + b \omega^{-1}
\leq \omega^{-1} \left\{ \epsilon L_{p_0, M} + B_2 L_{p_0, M}^2 \right\} + b \omega^{-1}.
\]

For \( \epsilon_1 < \omega \) and \( (\epsilon_1 - \omega)^2 > 4B_3b \), we get an estimate for \( L_{p_0,M} \) that is independent of \( M \). Namely,

\[
L_{p_0,M} \leq \frac{1}{2B_3} \left[ \omega - \epsilon_1 - \sqrt{(\epsilon_1 - \omega)^2 - 4B_3b} \right].
\]

So, \( ||(\hat{H},\hat{S})(\cdot,p)||_{\gamma,\beta} \in L^1(e^{-\omega p dp}) \), and the solution to the Boussinesq exists for \( t \in (0,\omega^{-1}) \) for \( \omega \) sufficiently large so that

\[
\omega \geq \omega_0 \quad \text{and} \quad \omega > \epsilon_1 + 2\sqrt{B_3b}.
\]

Equivalently, we could choose our original \( \omega_0 \) large enough so that \( \omega_0 > \epsilon_1 + 2\sqrt{B_3b} \). This completes the proof of Theorem 2.3.

Appendix A. Bessel Function Representation of the Kernel

Lemma Appendix A.1. The kernel \( G(z,z') \) given by

\[
G(z,z') = z'(-J_1(z)Y_1(z') + Y_1(z)J_1(z')), \quad \text{where} \quad z = 2|k|\sqrt{np} \quad \text{and} \quad z' = 2|k|\sqrt{np'},
\]

satisfies \( \pi z\dot{G}(z,z') = H^{(\nu)}(p,p',k) \) with \( H^{(\nu)} \) given by (14).

Proof. We will show that \( H^{(\nu)}(p,p',k) \) solves \( (p\partial_{pp} + 2\partial_p + \nu|k|^2)H^{(\nu)} = 0 \) for \( 0 < p' < p \) with the condition that \( H^{(\nu)}(p,p',k) \to 0 \) and \( H^{(\nu)}(p,p',k) \to \frac{1}{p} \) as \( p' \) approaches \( p \) from below.

First, we notice that

\[
H^{(\nu)}(p,p',k) = \frac{p'}{p} \int_1^{p/p'} F(\eta) d\eta,
\]

where

\[
\eta = \nu|k|^2p \left( 1 - \frac{sp'}{p} \right) \left( 1 - \frac{1}{s} \right), \quad F(\eta) = \frac{1}{2\pi i} \int_C \zeta^{-1}e^{\zeta-\nu\zeta^{-1}} d\zeta,
\]

and \( C \) is the contour starting and \( \infty e^{-\pi i} \) turning around the origin in counterclockwise direction and ending at \( \infty e^{\pi i} \). In the appendix of [13], it is shown that \( F \) is entire, \( F(0) = 1 \), and \( F \) satisfies \( \eta F''(\eta) + F'(\eta) + F(\eta) = 0 \). We will use these facts as given. As \( F \) is continuous and the interval of
integration shrinks to length zero, \( \mathcal{H}^{(\nu)}(p, p', k) \to 0 \) as \( p' \) tends to \( p \) from below. For \( p > p' \), \( \mathcal{H}^{(\nu)} \) is twice differentiable in \( p \) as \( F \) is twice continuously differentiable. Moreover, we have

\[
\mathcal{H}_p^{(\nu)}(p, p', k) = -\frac{1}{p} \mathcal{H}^{(\nu)}(p, p', k) + \frac{1}{p} F(0) + \frac{p'}{p} \int_1^{p/p'} F'(\eta) \frac{d\eta}{dp} \, ds,
\]

\[
(p \mathcal{H}_p^{(\nu)})_p = -\mathcal{H}_p^{(\nu)} + F'(0) \nu|k|^2 (1 - \frac{p'}{p}) + p' \int_1^{p/p'} F''(\eta) \left( \frac{d\eta}{dp} \right)^2 \, ds,
\]

where the second equality uses that \( \frac{d\eta}{dp} = \nu|k|^2 \left( 1 - \frac{1}{s} \right) \) is \( p \) independent. Thus, as \( F(0) = 1 \), we have \( \mathcal{H}_p^{(\nu)}(p, p', k) \to 1 \) as \( p' \) tends to \( p \) from below.

We notice that \( (d\eta/dp)^2 = \frac{\eta \nu|k|^2}{p} - \frac{\nu|k|^2(s - 1) d\eta}{ds} \).

So, integrating by parts and using \( \eta F''(\eta) + F'(\eta) + F(\eta) = 0 \), we have

\[
(p \mathcal{H}_p^{(\nu)})_p + \mathcal{H}_p^{(\nu)} = F'(0) \nu|k|^2 (1 - \frac{p'}{p}) + p' \int_1^{p/p'} F''(\eta) \left( \frac{\eta \nu|k|^2}{p} \right) \, ds
\]

\[
- p' \int_1^{p/p'} \frac{d}{ds}(F'(\eta)) \frac{\nu|k|^2(s - 1)}{p} \, ds
\]

\[
= \frac{\nu|k|^2 (1 - \frac{p'}{p})}{p} \int_1^{p/p'} \eta F''(\eta) \, ds + \frac{p' \nu|k|^2}{p} \int_1^{p/p'} F'(\eta) \, ds = -\nu|k|^2 \mathcal{H}^{(\nu)}.
\]

In other words, \( p \mathcal{H}_p^{(\nu)} + 2 \mathcal{H}_p^{(\nu)} + \nu|k|^2 \mathcal{H}^{(\nu)} = 0 \), and the lemma is proved.

**Lemma Appendix A.2.** We also have the representation in terms of Bessel functions

\[
\mathcal{L}^{-1}\left( \frac{1 - e^{-\nu|k|^2 \tau^{-1}}}{\nu|k|^2} \right)(p) = \frac{2 J_1(z)}{z}.
\]

**Proof.** Notice that by contour deformation the contribution from \( \frac{1}{\nu|k|^2} \) is zero. Factoring out \( |k|\sqrt{\nu p} \) in the exponent and using the change of variables \( \frac{\tau \sqrt{\nu p}}{|k| \sqrt{\nu p}} \to w \), we have

\[
\mathcal{L}^{-1}\left( \frac{1 - e^{-\nu|k|^2 \tau^{-1}}}{\nu|k|^2} \right)(p) = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{\nu|k|\sqrt{\nu p}(w-w^{-1})}}{|k|\sqrt{\nu p}} \, dw = \frac{2 J_1(z)}{z}.
\]
Appendix B. Fourier Inequalities in Two Dimensions

In the appendix of [10], Fourier inequalities are developed in $\mathbb{R}^3$. We present the counterparts to those inequalities for $\mathbb{R}^2$ here. Where a lemma is referenced from this section, we use either the $\mathbb{R}^2$ or $\mathbb{R}^3$ version as appropriate. The basic idea is that in 2-d Appendix B.4 below differs by a constant from 3-d case. All other lemmas are basically the same for $\mathbb{R}^2$ or $\mathbb{R}^3$ once the change in Appendix B.4 is taken into account.

**Definition Appendix B.1.** Define the polynomial

$$P_n(z) = \sum_{j=0}^{n} \frac{n!}{j!} z^j.$$

**Lemma Appendix B.2.** For all $y \geq 0$ and integers $n \geq m \geq 0$, we have

$$y^{m+1} \int_{0}^{\infty} e^{-y(\rho-1)(1+\text{sgn}(\rho-1))} \rho^m P_n(y|1-\rho|) d\rho \leq m! n! Q_{m+n+1}(y).$$

Proof can be found in [10].

**Proposition Appendix B.3.** Let $n$ be an integer no less than 0 and $r \geq 0$ and $\rho \geq 0$ fixed. Then

$$\int_{0}^{2\pi} e^{-|\rho-re^{i\theta}|} |\rho - re^{i\theta}|^n d\theta \leq 2\pi e^3 e^{-|\rho-r|} P_n(|r-\rho|) + \frac{4e}{\rho} e^{-|\rho-r|} P_{n+1}(|\rho-r|).$$

**PROOF.** Case 1. Suppose $0 \leq r \leq 2$. Then for all $\theta$,

$$|\rho - r| \leq |\rho - re^{i\theta}| \leq |\rho - r| + |r - re^{i\theta}| \leq |\rho - r| + 4.$$

We also notice, for $x \geq 0$,

$$(x + 1)^n = \sum_{j=0}^{n} x^j \frac{n!}{j!(n-j)!} \leq \sum_{j=0}^{n} x^j \frac{n!}{j!} = P_n(x). \quad (B.1)$$

Further, for $x, a \geq 0$,

$$\sum_{j=0}^{n} \frac{(x + a)^j}{j!} = \sum_{m=0}^{n} \sum_{j=m}^{n} \frac{x^{j-m} a^m}{m!(j-m)!} \leq \sum_{m=0}^{n} \frac{a^m}{m!} \sum_{j=0}^{n} \frac{x^j}{m!} \leq e^a \sum_{j=0}^{n} \frac{x^j}{m!}. \quad (B.2)$$
Thus, 
\[ |\rho - r e^{i\theta}|^n \leq \sum_{j=0}^{n} (|\rho - r| + 3)^j \frac{n!}{j!} \leq e^{3P_n(|\rho - r|)} \]

and 
\[ \int_{0}^{2\pi} e^{-|\rho - r e^{i\theta}|} |\rho - r e^{i\theta}|^n d\theta \leq 2\pi e^{-|\rho - r|} P_n(|r - \rho|). \]

So, the proposition holds in this case.

Case 2. Suppose \( r > 2 \). Let \( \theta_1 \in (0, \frac{\pi}{3}) \) be such that \( |r - r e^{i\theta_1}| = 1 \). We split our integral into three pieces. For \( \theta \in [0, \theta_1] \),

\[ |\rho - r| \leq |\rho - r e^{i\theta}| \leq |\rho - r| + |r - r e^{i\theta}| \leq |\rho - r| + 1. \]

Applying (B.1) with \( x = |\rho - r| \) gives,

\[ 2 \int_{\theta_1}^{\pi} e^{-|\rho - r e^{i\theta}|} |\rho - r e^{i\theta}|^n d\theta \leq \frac{2\pi}{3} e^{-|\rho - r|} P_n(|r - \rho|). \] (B.3)

Suppose \( \theta \in [\theta_1, \pi - \theta_1] \). Let \( z = |\rho - r e^{i\theta}| = \sqrt{(\rho - r)^2 + 2pr(1 - \cos \theta)} \). Then \( d\theta = \frac{zd\theta}{\rho r \sin \theta} \). However, since \( \theta \in [\theta_1, \pi - \theta_1] \),

\[ \frac{1}{r \sin \theta} \leq \frac{1}{r \sin \theta_1} = \frac{\theta_1}{r \sin \theta_1}. \] (B.4)

Now, notice that \( \theta_1 r \geq |r - r e^{i\theta_1}| = 1 \) and \( \frac{\theta_1}{\sin \theta_1} \leq \frac{\pi/2}{\sin(\pi/3)} < 2 \) since \( \theta_1 \in [0, \frac{\pi}{3}] \).

Hence,

\[ d\theta = \frac{zd\theta}{\rho r \sin \theta} \leq \frac{2zd\theta}{\rho} \]

and

\[ 2 \int_{\theta_1}^{\pi-\theta_1} e^{-|\rho - r e^{i\theta}|} |\rho - r e^{i\theta}|^n d\theta \leq \frac{4}{\rho} \int_{|\rho - r e^{i(\pi - \theta_1)}|}^{|\rho - r e^{i(\pi - \theta_1)}|} e^{-z z^{n+1}} dz \] (B.5)

\[ = \frac{4}{\rho} P_{n+1}(|\rho - r e^{i\theta_1}|) e^{-|\rho - r e^{i\theta_1}|} - P_{n+1}(|\rho - r e^{i(\pi - \theta_1)}|) e^{-|\rho - r e^{i(\pi - \theta_1)}|}. \]

We bound the positive contribution as in (B.2) by

\[ \frac{4}{\rho} P_{n+1}(|\rho - r e^{i\theta_1}|) e^{-|\rho - r e^{i\theta_1}|} \leq \frac{4}{\rho} P_{n+1}(|\rho - r| + 1) e^{-|\rho - r|} \leq \frac{4e}{\rho} P_{n+1}(|\rho - r|) e^{-|\rho - r|}. \] (B.6)
For $\theta \in [\pi - \theta_1, \pi]$, we again use (B.1) and get
\[ |\rho - re^{i\theta}|^n \leq (|\rho - re^{i(\pi - \theta_1)}| + 1)^n \leq P_n(|\rho - re^{i(\pi - \theta_1)}|) \]
and
\[ |\rho - e^{i\theta}| = \sqrt{\rho^2 - 2\rho r \cos(\pi - \theta_1) + r^2 + 2pr(\cos(\pi - \theta_1) - \cos \theta)} \geq |\rho - re^{i(\pi - \theta_1)}|. \]
So,
\[ 2 \int_{\pi - \theta_1}^{\pi} e^{-|\rho - re^{i\theta}|} |\rho - re^{i\theta}|^n d\theta \leq \frac{2\pi}{3} e^{-|\rho - re^{i(\pi - \theta_1)}|} P_n(|\rho - re^{i(\pi - \theta_1)}|). \quad (B.7) \]
Now, we notice that $|\rho - re^{i(\pi - \theta_1)}| > \rho$, so
\[ \frac{P_{n+1}(|\rho - re^{i(\pi - \theta_1)}|)}{\rho} \geq (n + 1)! \left( \frac{1}{\rho} + \sum_{j=1}^{n+1} \frac{|\rho - re^{i(\pi - \theta_1)}|^{j-1}}{j!} \right) \geq n! \sum_{j=0}^{n} \frac{n + 1}{j + 1} \frac{|\rho - re^{i(\pi - \theta_1)}|^{j}}{j!} \geq P_n(|\rho - re^{i(\pi - \theta_1)}|). \]
Thus,
\[ e^{-|\rho - re^{i(\pi - \theta_1)}|} \left( -\frac{2}{\rho} P_{n+1}(|\rho - re^{i(\pi - \theta_1)}|) + \frac{\pi}{3} P_n(|\rho - re^{i(\pi - \theta_1)}|) \right) < 0. \quad (B.8) \]
Adding the contributions from (B.3), (B.5), and (B.7) and using (B.6) and (B.8) gives
\[ 2 \int_0^{\pi} e^{-|\rho - re^{i\theta}|} |\rho - re^{i\theta}|^n d\theta \leq \frac{2\pi}{3} e^{-|\rho - r|} P_n(|r - \rho|) + \frac{4e}{\rho} e^{-|\rho - r|} P_{n+1}(|r - \rho|). \quad (B.9) \]
As all values of $r$ fall into one of these three cases, the proposition is proved.

**Lemma Appendix B.4.** If $m$ and $n$ are integers no less than $-1$, then
\[ |q| \int_{q' \in \mathbb{R}^d} e^{||q| - |q'| - |q' - q||m} |q - q'|^n dq' \leq C_\gamma(d)(m + 1)!(n + 1)! Q_{m+n+3}(|q|), \]
where $C_\gamma(2) = 6\pi e^3 + 4e$ and $C_\gamma(3) = 2$. 

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Proof. We note that we may assume without loss of generality that $m \leq n$ since a change of variables $q' \rightarrow q - q'$ switches the roles of $m$ and $n$. Write $q = \rho e^{i\phi}$, $q' = r e^{i\phi}$ and $\theta = \varphi - \phi$. Let $I$ be the integral on the left hand side. Then switching to polar coordinates gives

$$I = \rho \int_{0}^{\infty} \int_{0}^{2\pi} e^{\rho r - |\rho - r e^{i\phi}|^m} r - r e^{i\theta}|^nrdrd\theta.$$  

For $n \geq 0$, using Proposition [Appendix B.3] above gives,

$$I \leq \rho \int_{0}^{\infty} e^{\rho r - |\rho - r|} (2\pi e^3 P_n(|\rho - r|) + 4e \frac{P_{n+1}(|\rho - r|)}{\rho})dr.$$  

Now, we let $\tilde{\rho} = \frac{\rho}{r}$. Then $d\tilde{\rho} = \frac{dr}{\rho}$ and $-|\rho - r| = -\rho(\tilde{\rho} - 1)sgn(\tilde{\rho} - 1)$, so

$$I \leq \rho^{m+3} \int_{0}^{\infty} e^{-\rho(\tilde{\rho}-1)(1+sgn(\tilde{\rho}-1))} \tilde{\rho}^{m+1} (2\pi e^3 P_n(\rho|\tilde{\rho} - 1|)) + 4e \frac{P_{n+1}(\rho|\tilde{\rho} - 1|)}{\rho}) d\tilde{\rho}.$$  

Applying [Appendix B.2] gives

$$I \leq 2\pi e^3 \rho (m + 1)! Q_{m+n+2}(\rho) + 4e(m + 1)!Q_{m+n+3} \leq (6\pi e^3 + 4e)(m + 1)!Q_{m+n+3}(\rho),$$

where the last inequality follows as $m \leq n$, so

$$\rho \sum_{j=0}^{m+n+2} \frac{2^{m+n+2-j} \rho^j}{j!} \leq \sum_{j=1}^{m+n+3} \frac{2^{m+n+2-j} \rho^j}{(j-1)!} \leq Q_{m+n+3}(\rho)(m + n + 3) \leq 3(n + 1)Q_{m+n+3}(\rho).$$

For $n = m = -1$, we use a slightly different approach. Assuming $q$ is not zero, we split the integral over two regions, a ball of radius $3|q|/2$ centered at zero and its compliment. For the compliment region we have $|q - q'| \geq |q|/2$, so

$$|q| \int_{|q'| \geq 3|q|/2} e^{|q| - |q'| - |q - q'|} \frac{1}{|q'||q - q'|} dq' \leq 2e^{|q|/2} \int_{0}^{2\pi} \int_{3|q|/2}^{\infty} e^{-r dr d\theta} = 4\pi e^{-|q|} \leq 4\pi.$$
For the interior region we have
\[
|q| \int_{|q'| \leq 3|q|/2} e^{\|q| - |q'|-|q-q'|} \frac{1}{|q'||q - q'|} dq' \leq |q| \int_{|q'| \leq 3|q|/2} \frac{1}{|q'||q - q'|} dq'.
\]

We now note that \( \int_{|q'| \leq 3|q|/2} \frac{1}{|q'||q - q'|} dq' \) is bounded. Without trying to be precise we can bound the integral by \( 13\pi \) by splitting the region into two disks of radius \( |q|/2 \) centered at 0 and \( q \) and the compliment, call the compliment \( D \). We have
\[
\int_{|q'| \leq |q|/2} \frac{1}{|q'||q - q'|} dq' \leq \frac{2}{|q|} \int_{|q'| \leq |q|/2} \frac{1}{|q'|} dq' \leq 2\pi.
\]

Similarly,
\[
\int_{|q'-q| \leq |q|/2} \frac{1}{|q'||q - q'|} dq' \leq 2\pi.
\]

Finally,
\[
\int_D \frac{1}{|q'||q - q'|} dq' \leq \frac{4}{|q|^2} \int_D dq' \leq \frac{4}{|q|^2} \int_{|q'| \leq 3|q|/2} dq' \leq 9\pi.
\]

Thus,
\[
|q| \int e^{\|q| - |q'|-|q-q'|} \frac{1}{|q'||q - q'|} dq' \leq 13\pi|q| + 4\pi \leq 13\pi(|q| + 2) = 13\pi Q_1(|q|)
\]
for all nonzero \( q \). Hence, the lemma is proved with \( C_7(2) = 6\pi e^3 + 4e \).

**Lemma Appendix B.5.** For any \( \gamma \geq 1 \) and nonnegative integers \( m \) and \( n \), we have
\[
|k| \int_{k' \in \mathbb{R}^d} \frac{e^{-\beta(k' + |k - k'|)}(\beta|k'|)^m(\beta|k - k'|)^n}{(1 + |k'|)^\gamma(1 + |k - k'|)^\gamma} dk' \\
\leq \frac{C_7\pi^{2\gamma}e^{-\beta|k|}m!n!}{\beta^d(1 + |k|)^\gamma(m + n + 2)Q_{m+n+2}(\beta|k|)}.
\]

**Proof.** The proof is the same as the proof for 3-d given in [10] after using our new bound in **Appendix B.4**.
Lemma Appendix B.6. For any $\gamma \geq 2$ and $n \in \mathbb{N} - 0$, we have

$$|k| \int_{k' \in \mathbb{R}^d} \frac{e^{-\beta(|k'|+|k-k'|)}}{(1 + |k'|)^\gamma (1 + |k - k'|)^\gamma} |\beta(k - k')|^n dk'
\leq \frac{C_7 \pi 2^\gamma e^{-\beta|k|}}{\beta^{d-1}(1 + |k|)^\gamma} \left\{ (n - 1)! Q_{n+1}(\beta|k|) + \frac{3(n + 1)! (\beta|k|)^{2/3}}{2^{2/3}} \sum_{j=0}^{n+1} (\beta|k|)^j \right\}. $$

PROOF. We split the region into two integrals $\int_{|k'| \leq |k|/2} + \int_{|k'| \geq |k|/2}$. In the outer region, we have $(1 + |k'|)^{-\gamma} \leq 2^{\gamma} (1 + |k|)^{-\gamma}$, and in the inner, we have $(1 + |k-k'|)^{-\gamma} \leq 2^{\gamma} (1 + |k|)^{-\gamma}$. We use this and $\gamma \geq 2$ for the first inequality and Appendix B.4 for the second to get a bound for the outer region

$$|k| \int_{|k'| \geq |k|/2} \frac{e^{-\beta(|k'|+|k-k'|)}}{(1 + |k'|)^\gamma (1 + |k - k'|)^\gamma} |\beta(k - k')|^n dk'
\leq \frac{2^\gamma e^{-\beta|k|}}{\beta^{d-1}(1 + |k|)^\gamma} |q| \int_{q' \in \mathbb{R}^d} e^{||q'| - |q||} |q - q'|^{n-2} dq'
\leq \frac{C_7 \pi 2^\gamma e^{-\beta|k|}}{\beta^{d-1}(1 + |k|)^\gamma} (n - 1)! Q_{n+1}(|q|).$$

In the inner region, we also use $(1 + |k'|)^{-\gamma} \leq (|k'|)^{-2+2/3}$, a change to polar coordinates as in the proof of Appendix B.4, and integration by parts to get

$$|k| \int_{|k'| \leq |k|/2} \frac{e^{-\beta(|k'|+|k-k'|)}}{(1 + |k'|)^\gamma (1 + |k - k'|)^\gamma} |\beta(k - k')|^n dk'
\leq \frac{2^\gamma e^{-\beta|k|}}{\beta^{d-1+2/3}(1 + |k|)^\gamma} |q| \int_{|q'| \leq |q|/2} e^{||q'| - |q||} |q - q'|^{n-2+2/3} dq'
= \frac{2^\gamma e^{-\beta|k|}}{\beta^{d-1+2/3}(1 + |k|)^\gamma} \rho \int_0^{\rho/2} \int_0^{\rho/2} 2^\gamma e^{-\rho - r - re^{i\theta}} |\rho - r e^{i\theta}| r^{n-2+2/3} rd\theta \rho dr
\leq \frac{2^\gamma e^{-\beta|k|}}{\beta^{d-1+2/3}(1 + |k|)^\gamma} \int_0^{\rho/2} r^{-1+2/3} (2\pi e^3 \rho P_n(\rho - r) + 4e P_{n+1}(\rho - r)) dr
\leq \frac{2^\gamma e^{-\beta|k|}}{\beta^{d-1+2/3}(1 + |k|)^\gamma} \left( 2\pi e^3 n! r^{1+2/3} \sum_{j=0}^{n} \frac{\rho^j}{j!} \int_0^1 r^{-1+2/3} (1 - r)^j \, dr \right) 4e(n + 1)! r^{2/3} \sum_{j=0}^{n+1} \frac{\rho^j}{j!} \int_0^1 r^{-1+2/3} (1 - r)^j \, dr.$$
\[
\leq \frac{2^{\gamma} e^{-\beta |k|}}{\beta^{d-1+2/3}(1+|k|)^\gamma} \frac{C_7 3}{2 \pi (n+1)!} \rho^{2/3} \sum_{j=0}^{n+1} \frac{\rho^j}{j!}.
\]

The proof of the remaining lemmas is the same in 2-d as in 3-d after the change in bound given in Appendix B.6 and can be found in [10]. Whenever Lemma 6.8. is invoked in [10] the 2-d proofs use Appendix B.6.

Lemma Appendix B.7. For any \( \gamma \geq 1 \) and nonnegative integers \( l_1, l_2 \geq 0 \), we have

\[
|k| \int_{k' \in \mathbb{R}^d} \frac{e^{\beta(|k'|-|k-k'|)}}{(1+|k'|)^\gamma (1+|k-k'|)^\gamma} Q_{2l_1}(\beta |k'|) Q_{2l_2}(\beta |k-k'|) dk' \\
\leq \frac{C_7 \pi^{2/3} e^{-\beta |k|}}{3 \beta^d (1+|k|)^\gamma} (2l_1 + 2l_2 + 1)(2l_1 + 2l_2 + 2)(2l_1 + 2l_2 + 3) Q_{2l_1+2l_2+2}(\beta |k|).
\]

Lemma Appendix B.8. If \( \gamma \geq 2 \) and \( l \geq 0 \), then

\[
|k| \int_{k' \in \mathbb{R}^d} \frac{e^{-\beta(|k'|+|k-k'|)}}{(1+|k'|)^\gamma (1+|k-k'|)^\gamma} Q_{2l}(\beta |k-k'|) dk' \\
\leq \frac{C_1 e^{-\beta |k|}}{(1+|k|)^\gamma (2l + 1)} Q_{2l+2}(\beta |k|),
\]

where

\[
C_1 = C_1(d) = 6 C_7 \pi^{2/3} \beta^{-d+1/3} + C_7 \pi^{2/3} \beta^{-d+1} + \frac{1}{2} C_0 \beta^{-1}.
\]

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