Optimal Encodings for Range Min-Max and Top-$k$

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Abstract. In this paper we consider various encoding problems for range queries on arrays. In these problems, the goal is that the encoding occupies the information theoretic minimum space required to answer a particular set of range queries. Given an array $A[1..n]$ a range top-$k$ query on an arbitrary range $[i, j] \subseteq [1, n]$ asks us to return the ordered set of indices $\{\ell_1, \ldots, \ell_k\}$ such that $A[\ell_m]$ is the $m$-th largest element in $A[i..j]$. We present optimal encodings for range top-$k$ queries, as well as for a new problem which we call range min-max, in which the goal is to return the indices of both the minimum and maximum element in a range.

1 Introduction

Many important algorithms and data structures make use of range queries over arrays of values as subroutines [13,19]. As a prime example, many text indexes that support pattern matching queries often maintain an array storing the lengths of the longest common prefixes between consecutive suffixes of the text. During a search for a pattern this array is queried in order to find the position of the minimum value in a given range. That is, a subroutine is needed that can preprocess an array $A$ in order to answer range minimum queries. Formally, as input to such a query we are given a range $[i, j] \subseteq [1, n]$, and wish to return the index $k = \arg\min_{i \leq \ell \leq j} A[\ell]$. In text indexing applications memory is often the constraining factor, so the question of how many bits are needed to answer range minimum queries has been studied heavily. After a long line of research [2,18], it has been determined that such queries can be answered in constant time, by storing a data structure of size $2n + o(n)$ bits [7]. Furthermore, this space bound is optimal to within lower order terms (see [7] Sec. 1.1.2). The interesting thing about this result is that the space bound is independent of the number of bits required to store each value in the array $A$. After constructing the data structure we can discard the array $A$, while still retaining the ability to answer range minimum queries.

Results of this kind, where it is shown that we can return the solutions to all queries using less space than is required to store the original array, fall into the category of encoding data structures, and, more generally, succinct data structures [10]. Specifically, given a set of combinatorial objects $\chi$ we wish to represent an arbitrary member of this set succinctly using $\lg |\chi| + o(\lg |\chi|)$ bits\footnote{We use $\lg x$ to denote $\log_2 x$.}. We use $\lg x$ to denote $\log_2 x$. 
while still supporting queries, if possible. For the case of range minimum queries or range maximum queries, the set $\chi$ turns out to be Cartesian trees, which were introduced by Vuillemin [20]. For a given array $A$, the Cartesian tree can be used to find the solution to any range minimum query, and similarly, if two arrays have the same solutions to all range minimum queries, then their Cartesian trees are identical [7].

Recently there has been a lot of research on the topic of encodings for range queries over arrays [6,7,8,9,14,15,16]. We provide a non-exhaustive list with brief descriptions and references. The input to each of the following problem is an array $A[1..n]$, and the value $k \geq 1$ is fixed prior to computing the encoding.

1. **Range top-$k$:** given a query range $[i,j] \subseteq [1,n]$ and $k' \leq k$, return the indices of the $k'$ largest values in $[i,j]$. This problem is the natural generalization of range maximum queries and asymptotically optimal lower and upper bounds encodings of $\Omega(n \log k)$ and $O(n \log k)$ bits have been proposed recently by Grossi et al. [9] and Navarro, Raman, and Rao [14], respectively. The latter upper bound can be used to answer range top-$k$ queries in $O(k')$ time.

2. **Range $k$-selection:** given a query range $[i,j] \subseteq [1,n]$ and $k' \leq k$, return the index of the $k'$-th largest value in $[i,j]$. Note that any encoding that allows us to answer range top-$k$ queries also allows us to answer range $k$-selection queries (though the question of how much time is required during a query remains unclear) [14]. The encoding of Navarro, Raman, and Rao [14] can be used to answer range $k$-selection queries in optimal $O(\log \log k / \log n + 1)$ time.

3. **Range majority:** given a query range $[i,j] \subseteq [1,n]$ return the indices of all $(1/k)$-majorities in $[i,j]$. A $(1/k)$-majority is any element $A[\ell]$ which appears more than $j - i + 1/k$ times in $[i,j]$. Asymptotically optimal encodings have been proposed by Navarro and Thankachan [15,16].

4. **Range max-segment sum:** given a query range $[i,j] \subseteq [1,n]$ return the range $[i',j'] \subseteq [i,j]$ such that $\sum_{\ell \leq i} A[\ell]$ is maximized. This is only non-trivial if $A$ contains both positive and negative numbers. It was recently shown that an encoding using $O(n)$ bits can be used to answer such queries in constant time [8].

5. **Previous and Next Smaller Value:** given an index $i$ return the nearest index to the left $j_l < i$ and right $j_r > i$ such that $A[j_l] > A[i]$ and $A[j_r] > A[i]$. Fischer [6] considered the problem when the array elements are non-distinct and gave an optimal encoding for this problem.

**Other Encoding Problems** For two-dimensional (and higher) arrays of numbers, the range minimum problem has been generalized as follows: given an $N \times M$ matrix, and a query rectangle $\mathcal{R} = [i_1,i_2] \times [j_1,j_2]$, return the index of the minimum element in $\mathcal{R}$. Many results have been proved for this problem, though we omit further discussion and refer the interested reader to the recent paper of Brodal et al. [3].
Table 1. Old and new results. Both upper and lower bounds are expressed in bits. Our bounds make use of the entropy function \( H(x) = x \lg \left( \frac{1}{x} \right) + (1 - x) \lg \left( \frac{1}{1-x} \right) \). For the entry marked with a † the claimed bound holds when \( k = o(n) \).

| Ref. | Query | Lower Bound | Upper Bound | Query Time |
|------|-------|-------------|-------------|------------|
| [7]  | max   | \(2n - \Theta(\lg n)\) | \(2n - \Theta(\lg n)\) | \(\Omega(1)\) |
| [9,14]| top-k | \(\Omega(n \lg k)\) | \(O(n \lg k)\) | \(O(k')\) |
| [4]  | top-2 | \(2.656n - \Theta(\lg n)\) | \(3.272n + o(n)\) | \(O(1)\) |
| Thm. 1 | min-max | \(3n - \Theta(\text{polylog}(n))\) | \(3n + o(n)\) | \(O(1)\) |
| Thm. 3 | top-2 | \(3nH\left(\frac{1}{k}\right) - \Theta(\text{polylog}(n))\) | \(3nH\left(\frac{1}{k}\right)\) | — |
| Thm. 3 | top-k | \((k + 1)nH\left(\frac{1}{k+1}\right)(1 - o(1))\) | \((k + 1)nH\left(\frac{1}{k+1}\right)\) | — |

Our Results In this paper, we present the first optimal encodings to range top-\(k\)—and therefore range \(k\)-selection also—as well as a new problem which we call range min-max, in which the goal is to return the indices of both the minimum and maximum element in the array. We present our encoding to range min-max first, as it gives a chance to review Fischer and Heun’s technique, and sets the stage for the much more involved encodings for range top-\(k\). See Table 1 for a summary of the results. Prior to this work, the only value for which the exact coefficient of \(n\) was known for top-\(k\) queries was the case in which \(k = 1\) (i.e., range maximum queries). For even \(k = 2\) the best previous estimate was that the coefficient of \(n\) is between 2.656 and 3.272 [4]. The lower bound of 2.656 was derived using generating functions and an extensive computational search [4]. In contrast, our method of analysis is purely combinatorial and gives the exact coefficient value for all \(k \geq 2\). For \(k = 2, 3, 4\) the coefficients are approximately (rounding up) 2.755, 3.245, and 3.610, respectively. At present, one negative aspect of our encodings is that they appear to be somewhat difficult to use as the basis for a data structure: currently for our top-\(k\) encodings we know of no way to support queries efficiently. We leave the question of supporting top-\(k\) queries optimally using space matching that of our top-\(k\) encodings (to within lower order terms) as an open problem.

Discussion of Techniques Prior work for top-\(k\), for \(k \geq 2\), focused on encoding a particular decomposition of the array, called a shallow cutting [9,14]. Since shallow cuttings are a general technique used to solve many other range searching problems [12,11], these previous works [9,14] required additional information beyond just storing the shallow cutting in order to recover the answers to top-\(k\) queries. Furthermore, in these works the exact constant factor is not disclosed, though we estimate it to be at least twice as large as the bounds we present. For the range top-2 a different encoding has been proposed based on extended Cartesian trees [4], which are essentially just Cartesian trees augmented with information on how the elements represented by each node in the left spine of the right child and the right spine of the left child of any given node interleave. In contrast, our encoding is based on similar ideas to those of Fischer and Heun [7], who describe what is called a 2D min-heap (resp. max-heap) in order
to encode range minimum queries (resp. range maximum queries). We begin in
Section 3 by showing how to generalize this technique to simultaneously answer
both range minimum and range maximum queries. Our encoding provides the
answer to both using $3n + o(n)$ bits in total (compared to $4n + o(n)$ using the
trivial approach of constructing both encodings separately). We then show this
bound is optimal by proving that any encoding for range min-max queries can
be used to represent an arbitrary Baxter permutation. Then, we move on in
Section 4 to generalize Fischer and Heun’s technique in a clean and natural way
to larger values of $k$: indeed the upper bound we present is quite concise. The
main difficulty is proving that the bound achieved by this technique is optimal.
For this we enumerate a particular class of Dyck paths, via an application of the
so-called cycle lemma of Dvoretzky and Motzkin.

2 Preliminaries
We make frequent use of the following result by Raman, Raman, and Rao [17]:

Lemma 1 ([17]). Let $S$ be a bit vector of length $n$ bits, containing $m$ one bits.
In the word-RAM model with word size $\Theta(\lg n)$ bits, there is a data structure of
size $\lg \binom{n}{m} + O(\frac{n \lg n \lg \lg n}{\lg n})$ bits that supports the following
operations in $O(1)$ time, for any $i \in [1, n]$:

1. access($S, i$): return the bit at index $i$ in $S$.
2. rank$_{\alpha}(S, i)$: return the number of bits with value $\alpha \in \{0, 1\}$ in $S[1..i]$.
3. select$_{\alpha}(S, i)$: return the index of the $i$-th bit with value $\alpha \in \{0, 1\}$.

3 Optimal Encodings of Range Min-Max Queries
In this section we describe our solution for range min-max. We use $\text{RMMinMax}(A[i..j])$
to denote a range min-max query on a subarray $A[i..j]$. The solution to the
query is the ordered set of indices $\{\ell_1, \ell_2\}$ such that $
\ell_1 = \arg \max_{\ell \in [i..j]} A[\ell]$ and $\ell_2 = \arg \min_{\ell \in [i..j]} A[\ell]$. This serves as a bit of a warm up for the later sections, as
it allows us to review the encoding of Fischer and Heun, since similar ideas are
used throughout the paper.

3.1 Review of Fischer and Heun’s Technique
We start by reviewing the encoding of Fischer and Heun for supporting either
range minimum queries or range maximum queries.

We explain the algorithm to construct the encoding which allows us to answer
range minimum (resp. maximum) queries. Consider an array $A[1..n]$ storing $n$
numbers. Without loss of generality we can alter the values of the numbers so
that they are a permutation. To construct the encoding for range minimum
queries we sweep the array from left to right
$^2$ while maintaining a stack. A

$^2$ In the original paper the sweeping process moves from right to left, but either direction yields a correct algorithm by symmetry.
string of bits \( T_{\text{min}} \) (resp. \( T_{\text{max}} \)) will be emitted in reverse order as we scan the array. Whenever we push an element onto the stack, we emit a one bit, and whenever we pop we emit a zero bit. Initially the stack is empty, so we push the position of the first element we encounter on the stack, in this case, 1. Each time we increment the current position, \( i \), we compare the value of \( A[i] \) to that of the element in the position \( t \), which is stored on the top of the stack. While \( A[t] \) is not less than (resp. not greater than) \( A[i] \), we pop the stack. Once \( A[t] \) is either empty or less than (resp. greater than) the current element, we push \( i \) onto the stack. When we reach the end of the array, we pop all the elements on the stack, emitting a zero bit for each element popped, followed by a single one bit.

Fischer and Heun showed that the string of bits output by this process can be used to encode a rooted ordinal tree in terms of its depth first unary degree sequence or DFUDS \(^7\). To extract the tree from a sequence, suppose we read \( d \) zero bits until we hit the first one bit. Based on this, we create a node \( v \) of degree \( d \), and continue building first child of \( v \) recursively. Since there are at most \( 2n \) stack operations, the tree is therefore represented using \( 2n \) bits. We omit the technical details of how a query is answered, but the basic idea is to augment this tree representation with succinct data structures supporting navigation operations.

**Lemma 2 (Corollary 5.6 \(^7\)).** Given the DFUDS representation of \( T_{\text{min}} \) (resp. \( T_{\text{max}} \)) any query \( \text{RMin}(A[i..j]) \) (resp. \( \text{RMax}(A[i..j]) \)) can be answered in constant time using an index occupying \( O\left(\frac{n \log \log n}{\log n}\right) = o(n) \) additional bits of space.
3.2 Upper Bound for Range Min-Max Queries

We propose the following encoding for a simultaneous representation of \(T_{\text{min}}\) and \(T_{\text{max}}\). Scan the array from left to right and maintain two stacks: a min-stack for range minimum queries, and a max-stack for range maximum queries. Notice that in each step except for the first and last, we are popping an element from exactly one of the two stacks. This crucial observation allows us to save space. We describe our encoding in terms of the min-stack and the max-stack maintained as above. Unlike before however, we maintain two separate bit strings, \(T\) and \(U\).

If the new element causes \(\delta \geq 1\) elements on the min-stack to be popped, then we prepend \(0^{\delta - 1}\) to the string \(T\), and prepend 0 to the string \(U\). Otherwise, if the new element causes \(\delta\) elements on the max-stack to be popped, we prepend \(0^{\delta - 1}\) to the string \(T\), and 1 to the string \(U\). Since exactly \(2n\) elements are popped during \(n\) push operations, the bit string \(T\) has length \(2^n\), and the bit string \(U\) has length \(n\), for a total of \(3n\) bits.

The remainder of this section is dedicated to proving the following theorem:

**Theorem 1.** There is a data structure that occupies \(3n + o(n)\) bits of space, such that any query \(\text{RMINMAX}(A[i..j])\) can be answered in \(O(1)\) time.

**Proof.** To prove the theorem, it is sufficient to show that there is a data structure that occupies \(3n + o(n)\) bits of space, and can recover any block of \(\lg n\) consecutive bits from both \(T_{\text{min}}\) and \(T_{\text{max}}\) in \(O(1)\) time. If we have such a structure that can extract any block from either DFUDS representation, then we can use it as an oracle to access the DFUDS representation of either tree. Thus, we need only apply Lemma 2 to complete the theorem. The data structure makes use of the bit strings \(T\) and \(U\), as well as the following auxiliary data structures:

1. We precompute a lookup table \(L\) of size \(\Theta(\sqrt{n \lg n})\) bits. The lookup table takes two bit strings as input, \(s_1\) and \(s_2\), both with length \(\frac{\lg n}{4}\), as well as a single bit \(b\). We conceptually think of the bit string \(s_1\) as having the format \(0^{\gamma_1}10^{\gamma_2}1\ldots0^{\gamma_t}1\), where each \(\gamma_i \geq 0\). The table returns a new bit string \(s_3\), of length no greater than \(\frac{\lg n}{4}\), that we will define next. Let \(\cdot\) be the concatenation operator, and define the function:

\[
f(x, y, y') = \begin{cases} 
0 \cdot x & \text{if } y = y' \\
1 & \text{otherwise.}
\end{cases}
\]

If \(u_i = 0^{\gamma_i}1\) then \(s_3 = f(u_1, s_1[1], b) \cdot f(u_2, s_2[2], b) \cdots f(u_k, s_2[k], b)\) and \(s_2[i]\) denotes the \(i\)-th bit of \(s_2\). Such a table occupies no more than the claimed amount of space, and can return \(s_3\) (as well as \(k\)) in \(O(1)\) time.

2. Each bit in \(T\) corresponds to at least one bit in \(T_{\text{min}}\) or \(T_{\text{max}}\). Also recall that at each step during preprocessing we append the value \(\delta - 1\) in unary to \(T\) rather than \(\delta\) (as in the representation of Fischer and Heun). Thus, we can treat each push operation (with the exception of the first and last) corresponding to a single one bit in \(T\) as representing three bits: two bits in \(T_{\text{min}}\) and one bit in \(T_{\text{max}}\) or two bits in \(T_{\text{max}}\) and one bit in \(T_{\text{min}}\), depending
on the corresponding value in $U$. We store a bit vector $B_{\text{min}}$ of length $2n$ which marks the position in $T$ of the bit corresponding to the $(i \lg n + 1)$-th bit of $T_{\text{min}}$, for $0 \leq i \leq \lfloor \frac{2n}{\lg n} \rfloor$. We do the analogous procedure for $T_{\text{max}}$ and call the resulting bit vector $B_{\text{max}}$.

Suppose now that we support the operations rank and select on $B_{\text{min}}$, $B_{\text{max}}$, and $T$. We use the data structure of Lemma 1 that for $B_{\text{min}}$ and $B_{\text{max}}$ will occupy $O \left( \lg \left( \frac{n}{\lg n} \right) + \frac{n \lg \lg n}{\lg n} \right) = O \left( \frac{n \lg \lg n}{\lg n} \right)$ bits, and for $T$ will occupy no more than $2n + O \left( \frac{n \lg \lg n}{\lg n} \right)$ bits. Thus, our data structures at this point occupy $3n + o(n)$ bits in total, counting the space for $U$. We will describe how to recover $\lg n$ consecutive bits of $T_{\text{min}}$; the procedure for $T_{\text{max}}$ is analogous. Consider the distances between two consecutive 1 bits having indices $x_i$ and $x_{i+1}$ in $B_{\text{min}}$. Suppose $x_{i+1} - x_i \leq c \lg n$ in $B_{\text{min}}$, for some constant $c \geq 9$. In this case we call the corresponding block $\beta_i$ of $\lg n$ consecutive bits of $B_{\text{min}}$ min-good, and otherwise we call $\beta_i$ min-bad. We also define similar notions for max-good and max-bad blocks. The problem now becomes recovering any block (good or bad), since if the $\lg n$ consecutive bits we wish to extract are not on block boundaries we can simply extract two consecutive blocks which overlap the desired range, then recover the bits in the range using bit shifting and bitwise arithmetic.

If $\beta_i$ is min-good, then we can recover it in $O(c) = O(1)$ time, since all we need to do is scan the corresponding segment of $T$ between the two 1s, as well as the segment of $U$ starting at $\text{rank}_1(T, x_i)$. We process the bits of $T$ and $U$ together in blocks of $\frac{\lg n}{4}$ each, using the lookup table $L$: note that we can advance in $U$ correctly by determining $t$ by counting the number of 1 bits in either in $s_1$ or $s_3$. This can be done using either an additional lookup table of size $\Theta(\sqrt{n})$ using constant time, or by storing the answer explicitly in $L$. When we do this, there is one border case which we must handle, which occurs when the last bit in $s_1$ is not a 1. However, we can simply append a 1 to end of $s_1$ in this case, and then delete either 1 or 01 from the end of $s_3$, depending on the value of $s_2[t]$. This correction can be done in $O(1)$ time using bit shifting and bitwise arithmetic.

If $\beta_i$ is min-bad, then we store the answer explicitly. This can be done by storing the answer for each bad $\beta_i$ in an array of size $z \lg n$ bits, where $z$ is the number of bad blocks. Since $z \leq \lfloor \frac{n}{\frac{\lg \lg n}{\lg n}} \rfloor$ this is $\lfloor \frac{n}{c} \rfloor$ bits in total. We also must store yet another bit vector, encoded using Lemma 1 marking the start of the min-bad blocks, which occupies another $O \left( \frac{n \lg \lg n}{\lg n} \right)$ bits by a similar calculation as before. Thus, we can recover any block in $B_{\text{min}}$ using $3n + \lfloor \frac{n}{c} \rfloor + o(n)$ bits in $O(c)$ time, for any constant $c > 1$.

In fact, by examining the structure of Lemma 1 in more detail we can argue that it compresses $T$ slightly for each bad block, to get a better space bound than $2n + o(n)$ bits. Consider all the min-bad blocks $\beta_1, \ldots, \beta_z$ in $B_{\text{min}}$ and the
max-bad blocks $\beta_1',...,\beta_r'$ in $B_{\max}$. For a given min-bad block $\beta_i$, any max-bad block $\beta_j'$ can only overlap its first or last $2\lg n$ bits in $T$. This follows since each bit in $T$ corresponds to at least one bit in either $T_{\min}$ or $T_{\max}$, and because less than half of these $2\lg n$ bits can correspond to bits in $T_{\min}$ (since the block is min-bad). Thus, each bad block has a middle part of at least $(c-4)\lg n$ bits, which are not overlapped by any other bad block. We furthermore observe that these $(c-4)\lg n$ middle bits are highly compressible, since they contain at most $\lg n$ one bits, by the definition of a bad block. Since these $(c-4)\lg n$ middle bits are compressed to their zeroth-order entropy in chunks of $\frac{\lg n}{c}$ consecutive bits by Lemma [1], we get that the space occupied by each of them is at most

$$\left\lfloor \lg \left(\frac{(c-4)\lg n}{\lg n}\right)\right\rfloor + \Theta(c) \leq (c-4)H\left(\frac{1}{c-4}\right)\lg n + \Theta(c).$$

The cost of explicitly storing the answer for the bad block was $\lg n$ bits. Since $c \geq 9$, and assuming $n$ is sufficiently large, we get that this additional $\lg n$ bits of space can be added to the cost of storing the middle part of the bad block in compressed form, without exceeding the cost of storing the middle part of the bad block in uncompressed form. The value of $c \geq 9$ came from a numeric calculation by finding the first value of $c$ such that $(c-4)H\left(\frac{1}{c-4}\right) + 1 < (c-4)$. Thus, the total space bound is $3n + o(n)$ bits.

3.3 Lower Bound for Range Min-Max Queries

Given a permutation $\pi = (p_1,...,p_n)$, we say $\pi$ contains the permutation pattern $s_1s_2...s_m$ if there exists a subsequence $\pi$ whose elements have the same relative ordering as the elements in the pattern. That is, there exist some $x_1 < x_2 < ... < x_m \in [1,n]$ such that for all $i,j \in [1,m]$ we have that $\pi(x_i) < \pi(x_j)$ if and only if $s_i < s_j$. For example, if $\pi = (1,4,2,5,3)$ then $\pi$ contains the permutation pattern 1-3-4-2: we use this hyphen notation to emphasize that the indices need not be consecutive. In this case, the series of indices in $\pi$ matching the pattern are $x_1 = 1$, $x_2 = 2$, $x_3 = 4$ and $x_4 = 5$. If no hyphen is present between elements $s_i$ and $s_{i+1}$ in the permutation pattern, then the indices $x_i$ and $x_{i+1}$ must be consecutive: i.e., $x_{i+1} = x_i + 1$. In terms of the example, $\pi$ does not contain the permutation pattern 1-34-2.

A permutation $\pi = (p_1,...,p_n)$ is a Baxter permutation if there exist no indices $1 \leq i < j < k \leq n$ such that $\pi(j+1) < \pi(i) < \pi(k) < \pi(j)$ or $\pi(j) < \pi(k) < \pi(i) < \pi(j+1)$. Thus, Baxter permutations are those that do not contain 2-41-3 and 3-14-2. For permutations on 4 elements, the non-Baxter permutations are exactly (2,4,1,3) and (3,1,4,2). Baxter permutations are well studied, and their asymptotic behaviour is known (see, e.g., OEIS A001181 [1]).

We have the following lemma:

Lemma 3. Suppose we are given a Baxter permutation $\pi$, stored in an array $A[1,n]$ such that $A[i] = \pi(i)$. If we construct an encoding data structure that answers range minimum and maximum queries on $A$, then our encoding data structure can fully recover $\pi$.
Proof. In order to recover the permutation, it suffices to show that we can perform pairwise comparisons on any two elements in $A$ using range minimum and range maximum queries. The proof follows by induction on $n$.

For the base case, for $n = 1$ there is exactly one permutation, so there is nothing to recover. Thus, let us assume that the lemma holds for all permutations on less than $n$ elements. For a permutation on $n$ elements, consider the subpermutation induced by the array prefix $A[1..(n-1)]$ and suffix $A[2..n]$. These subpermutations must be Baxter permutations, since deleting elements from the prefix or suffix of a Baxter permutation cannot create a 2-41-3 or a 3-14-2. Thus, it suffices to show that we can compare $A[1]$ and $A[n]$, as all the remaining pairwise comparisons can be performed by the induction hypothesis.

Let $x = \text{RMin}(A[1..n])$ and $y = \text{RMax}(A[1..n])$ be the indices of the minimum and maximum elements in the array, respectively. If $x \in \{1, n\}$ or $y \in \{1, n\}$ we can clearly compare $A[1]$ and $A[n]$, so assume $x, y \in [2, n-1]$. Without loss of generality we consider the case where $x < y$: the opposite case is symmetric (i.e., replacing 3-14-2 with 2-41-3), and $x \neq y$ because $n \geq 2$. Consider an arbitrary index $i \in [x, y]$, and the result of comparing $A[1]$ to $A[i]$ and $A[i]$ to $A[n]$ (which can be done by the induction hypothesis, as $i \in [2, n-1]$). The result is either:

1. One of the two chains $A[1] < A[i] < A[n]$ or $A[n] < A[i] < A[1]$, in which case we are done; or
2. A partial order in which $A[i]$ is the minimum or maximum element, and $A[1]$ is incomparable with $A[n]$.

In the second case let $f(i) = 0$ if $A[i]$ is the minimum element in this partial order, and $f(i) = 1$ otherwise. Because of how $x$ and $y$ were chosen, $f(x) = 0$ and $f(y) = 1$. Therefore, if we consider the values of $f(i)$ for all $i \in [x, y]$, there must exist two indices $i, i+1 \in [x, y]$ such that $f(i) = 0$ and $f(i+1) = 1$, and then $1, i, i+1, n$ form the forbidden pattern 3-14-2, unless $A[1] < A[n]$. \hfill $\Box$

**Theorem 2.** Any data structure that encodes range minimum and maximum queries simultaneously must occupy $3n - \Theta(\log n)$ bits, for sufficiently large values of $n$.

Proof. Let $L(n)$ be the number of Baxter permutations on $n$ elements. It is known (cf. [1]) that

$$
\lim_{n \to \infty} \frac{L(n)\sqrt{3n^4}}{2^{3n+5}} = 1.
$$

Since we can encode and recover each one by the procedure discussed in Lemma 3, our encoding data structure must occupy at least $\lg L(n) = 3n - \Theta(\log n)$ bits, for sufficiently large values of $n$. \hfill $\Box$

### 4 Optimal Encodings for Top-k Queries

In this section we use $\text{RTopK}(A[i..])$ to denote a range top-$k$ query on the subarray $A[i..]$. The solution to such a query is an ordered list of indices $\{\ell_1, \ldots, \ell_k\}$ such that $A[\ell_m]$ is the $m$-th largest element in $A[i..]$. 


4.1 Upper Bound for Encoding Top-k Queries

Like the encoding for range min-max queries, our encoding for range top-k queries is based on representing the changes to a certain structure as we scan through the array $A$. Each prefix in the array will correspond to a different structure. We denote the structure, which we will soon describe, for prefix $A[1..j]$ as $S_k(j)$, for all $1 \leq j \leq n$. The structure $S_k(j)$ will allow us to answer $\text{RTOPK}(A[i..j])$ for any $i \in [1, j]$. Our encoding will store the differences between $S_k(j)$ and $S_k(j + 1)$ for all $j \in [1, n - 1]$. Let us begin by defining a single instance for an arbitrary $j$.

We first define the directed graph $G_j = (V, E)$ with vertices labelled $\{1, ..., j\}$, and where an edge $(i', j') \in E$ iff both $i' < j'$ and $A[i'] < A[j']$ for all $1 \leq i' < j' \leq j$. We call $G_j$ the dominance graph of $A[1..j]$, and say $j'$ dominates $i'$, or $i'$ is dominated by $j'$, if $(i', j') \in E$. Next consider the degree $d_j(\ell)$ of the vertex labelled $\ell \in [1, j]$ in $G_j$. We define an array $S[1..j]$, where $S[\ell] = d_j(\ell)$ for $1 \leq \ell \leq j$. The structure $S_k(j)$ is defined as follows: take the array $S[1..j]$, and for each entry $\ell \in [1, j]$ such that $S[\ell] > k$, replace $S[\ell]$ with $k$. We refer to the indices in $S_k(j)$ to be active iff $S[\ell] < k$, and as inactive otherwise.

**Lemma 4.** Suppose we are given the structure $S_k(j)$, and let $\{i_1, ..., i_{j'}\}$ be the active indices. We can recover the total order of the elements $A[i_1], ..., A[i_{j'}]$ by examining only $S_k(j)$.

**Proof.** We scan the structure $S_k(j)$ from index $j$ down to 1, maintaining a total ordering on the active elements seen so far. Initially, we have an empty total ordering. At each active location $\ell$ the value $S[\ell]$ indicates how many active
elements in locations \([\ell + 1, j]\) are larger than \(A[\ell]\). This follows since a non-active element cannot dominate an active element in the graph \(G_j\). Thus, we can insert \(A[\ell]\) into the current total ordering of active elements.

We define the size of \(S_k(j)\) as follows: \(|S_k(j)| = \sum_{i=1}^{j} (k - S[i])\). The key observation is that the structure \(S_k(j + 1)\) can be constructed from \(S_k(j)\) using the following procedure:

1. Compute the value \(\delta_j = |S_k(j)| - |S_k(j + 1)| + k\). This quantity is always non-negative, as we add one new element to the large staircase, which increases the size by at most \(k\).
2. Find the \(\delta_j\) indices among the active elements in \(S_k(j)\) such that their values in \(A\) are the smallest via Lemma 4. Denote this set of indices as \(I\).
3. Increment \(S[i]\) for each \(i \in I\).
4. Add the new element at the end of the array by setting \(S[j + 1] = 0\).

Thus, all we need to construct \(S_k(j + 1)\) is \(S_k(j)\) and the value \(\delta_j\). This implies that by storing \(\delta_j\) for \(j \in [1, n-1]\) we can recover any \(S_k(j)\).

**Theorem 3.** We can encode solutions to all queries \(\text{RTopK}(A[i..j])\) using at most \((k + 1)\frac{1}{(k+1)}H(\frac{1}{k+1})\) bits of space.

**Proof.** Suppose we store the bitvector \(0^{k}10^{k+1}1\ldots0^{n-1}1\). This bitvector contains no more than \(kn\) zero bits. This follows since each active counter can be incremented \(k\) times before it becomes inactive. Thus, storing the bitvector requires no more than \(\lfloor (k+1)n \rfloor \leq (k + 1)\frac{1}{(k+1)}H(\frac{1}{k+1})\) bits.

Next we prove that this is all we need to answer a query \(\text{RTopK}(A[i..j])\). We use the encoding to construct \(S_k(j)\). We know that for every element at inactive index \(\ell\) in \(S_k(j)\) there are at least \(k\) elements with larger value in \(A[\ell + 1..j]\). Consequently, these elements need not be returned in the solution, and it is enough to recover the indices of the top-\(k\) values among the elements at active indices at least \(i\). We apply Lemma 4 on \(S_k(j)\) to recover these indices and return them as the solution.

**4.2 Lower Bound for Encoding Top-\(k\) Queries**

The goal of this section is to show that the encoding from Section 4.1 is, in fact, optimal. The first observation is that all structures \(S_k(j)\) for \(j \in [1, n]\) can be reconstructed with \(\text{RTopK}\) queries.

**Lemma 5.** Any \(S_k(j)\) can be reconstructed with \(\text{RTopK}\) queries.

**Proof.** To reconstruct \(S_k(j)\), we execute the query \(\text{RTopK}(A[i..j])\) for each \(i \in [1, j]\). If index \(i\) is returned as the \(k'\)-th largest element in \([i, j]\), then by definition there are exactly \(k' - 1\) elements in locations \(A[i + 1..j]\) with value larger than \(A[i]\). Thus, \(i\) is an active location and \(S[i] = k' - 1\). If \(i\) is not returned by the query, then it is inactive and we set \(S[i] = k\).

\[\Box\]
Recall that we encode all structures by specifying \( \delta_1, \delta_2, \ldots, \delta_{n-1} \). We call an \((n-1)\)-tuple of nonnegative integers \((\delta_1, \delta_2, \ldots, \delta_{n-1})\) valid if it encodes some \( S_k(1), S_k(2), \ldots, S_k(n) \), i.e., if there exists at least one array \( A[1..n] \) consisting of distinct integers such that the structure constructed for \( A[1..j] \) is exactly the encoded \( S_k(j) \), for every \( j = 1, 2, \ldots, n \). Then the number of bits required by the encoding is at least as large as logarithm of the number of valid \((n-1)\)-tuples \((\delta_1, \delta_2, \ldots, \delta_{n-1})\). Our encoding from Section 4.1 shows that this number is at most \((\frac{k+1}{n})^n\), but here we need to argue in the other direction, which is far more involved.

Recall that the size of a particular \( S_k(j) \) is \( |S_k(j)| = \sum_{i=1}^{j} (k - S[i]) \). We would like to argue that there are many valid \((n-1)\)-tuples \((\delta_1, \delta_2, \ldots, \delta_{n-1})\).

This will be proven in a series of transformations.

**Lemma 6.** If \((\delta_1, \delta_2, \ldots, \delta_{n-1})\) is valid, then for any \( \delta_n \in \{0, 1, \ldots, \left\lfloor \frac{M}{k} \right\rfloor \} \) where \( M = \sum_{i=1}^{n-1} (k - \delta_i) \), the tuple \((\delta_1, \delta_2, \ldots, \delta_{n-1}, \delta_n)\) is also valid.

**Proof.** Let \( A[1..n] \) be an array such that the structure constructed for \( A[1..j] \) is exactly \( S_k(j) \), for every \( j = 1, 2, \ldots, n \). By definition of \( \delta_j \), we have that \( M = \sum_{i=1}^{n-1} (k - \delta_i) < |S_k(n)| \). Denote the number of active elements in \( S_k(j) \) with \( \alpha \) and apply Lemma 6. Now because \( |S_k(n)| = \sum_{\alpha=0}^{k-1} (k - \alpha) m_\alpha \leq k \sum_{\alpha=0}^{k-1} m_\alpha \), we have \( \sum_{\alpha=0}^{k-1} m_\alpha \geq \left\lceil \frac{|S_k(n)|}{k} \right\rceil \), which gives the claim. □

Every valid \((n-1)\)-tuple \((a_1, a_2, \ldots, a_{n-1})\) corresponds in a natural way to a walk of length \( n-1 \) in a plane, where we start at \((0, 0)\) and perform steps of the form \((1, a_i)\), for \( i = 1, 2, \ldots, n-1 \). We consider a subset of all such walks. Denoting the current position by \((x_i, y_i)\), we require that \( a_i \) is an integer from \([k - \left\lfloor \frac{M}{k} \right\rfloor, k]\). Under such conditions, any walk corresponds to a valid \((n-1)\)-tuple \((\delta_1, \delta_2, \ldots, \delta_{n-1})\), because we can choose \( \delta_i = k - a_i \) and apply Lemma 6. Therefore, we can focus on counting such walks.

The condition \([k - \left\lfloor \frac{M}{k} \right\rfloor, k]\) is not easy to work with, though. We will count more restricted walks instead. A \( Y \)-restricted nonnegative walk of length \( n \) starts at \((0, 0)\) and consists of \( n \) steps of the form \((1, a_i)\), where \( a_i \in Y \) for \( i = 1, 2, \ldots, n \), such that the current \( y \)-coordinate is always nonnegative. \( Y \) is here an arbitrary set of integers.

**Lemma 7.** The number of valid \((n-1)\)-tuples is at least as large as the number of \([k - \Delta, k]\)-restricted nonnegative walks of length \( n - 1 - \Delta \).

**Proof.** We have already observed that the number of valid \((n-1)\)-tuples is at least as large as the number of walks consisting of \( n-1 \) steps of the form \((1, a_i)\), where \( a_i \in [k - \left\lfloor \frac{M}{k} \right\rfloor, k] \) for \( i = 1, 2, \ldots, n-1 \). We distinguish a subset of such walks, where the first \( \Delta \) steps are of the form \((1, k)\), and then we always stay
above (or on) the line \( y = k\Delta \). Under such restrictions, \( a_i \in [k - \Delta, k] \) implies \( a_i \in [k - \lceil \frac{y}{\Delta} \rceil, k] \), so counting \([k - \Delta, k]\)-restricted nonnegative walks gives us a lower bound on the number of valid \((n - 1)\)-tuples.

We move to counting \(Y\)-restricted nonnegative walks of length \(n\). Again, counting them directly is not trivial, so we introduce a notion of \(Y\)-restricted returning walk of length \(n\), where we ignore the condition that the current \(y\)-coordinate should be always nonnegative, but require that the walk ends at \((n, 0)\).

\[
\text{Lemma 8. The number of } Y \text{-restricted nonnegative walks of length } n \text{ is at least as large as the number of } Y \text{-restricted returning walks of length } n \text{ divided by } n.
\]

\[
\text{Proof. This follows from the so-called cycle lemma [5], but we prefer to provide a simple direct proof. We consider only } Y \text{-restricted nonnegative walks of length } n \text{ ending at } (n, 0), \text{ and denote their set by } W_1. \text{ The set of } Y \text{-restricted returning walks of length } n \text{ is denoted by } W_2. \text{ The crucial observation is that a cyclic rotation of any walk in } W_2 \text{ is also a walk in } W_2. \text{ Moreover, there is always at least one such cyclic rotation which results in the walk becoming nonnegative (see Figure 3). Therefore, we can define a total function } f : W_2 \to W_1, \text{ which takes a walk } w \text{ and rotates it cyclically as to make it nonnegative. Because there are just } n \text{ cyclic rotations of a walk of length } n, \text{ any element of } W_1 \text{ is the image of at most } n \text{ elements of } W_2 \text{ through } f. \text{ Therefore, } |W_1| \geq \frac{|W_2|}{n} \text{ as claimed.} \]

The only remaining step is to count \([k - \Delta, k]\)-restricted returning walks of length \(n - 1 - \Delta\). This is equivalent to counting ordered partitions of \(k(n - 1 - \Delta)\) into parts \(a_1, a_2, \ldots, a_{n-1-\Delta}\), where \(a_i \in [0, \Delta]\) for every \(i = 1, 2, \ldots, n-1-\Delta\).

\[
\text{Lemma 9. The number of ordered partitions of } N \text{ into } g \text{ parts, where every part is from } [0, B], \text{ is at least } \binom{N - 3g' + g - 1}{g - 1}, \text{ where } g' = \left\lfloor \frac{N}{B} \right\rfloor.
\]

\[
\text{Proof. The number of ordered partitions of } N \text{ into } g \text{ parts, where there are no restrictions on the sizes of the parts, is simply } \binom{N + g - 1}{g - 1}. \text{ To take the restrictions into the account, we first split } N \text{ into blocks of length } B \text{ (except for the last block of size } B - N\text{). The number of ways to partition these blocks into } g \text{ parts is } \binom{N - (B - N) + g - 1}{g - 1} = \binom{N - 3g' + g - 1}{g - 1} \text{ as claimed.} \]

\[
\text{Fig. 3. Left: a } Y \text{-restricted walk ending at } (n, 0). \text{ Right: a cyclic rotation of the walk on the left such that the walk is always nonnegative.}
\]
block, which might be shorter). This creates \( g' + 1 \) blocks. Then, we additionally split the blocks into smaller parts, which ensures that all parts are from \([0, B]\). We restrict the smaller parts, so that the last smaller part in every block is strictly positive. This ensures that given the resulting partition into parts, we can uniquely reconstruct the blocks. Therefore, we only need to count the number of ways we can split the blocks into such smaller parts, and by standard reasoning this is \( \binom{N - 3g' + g - 1}{g - g' - 1} \). This follows by conceptually fusing together the last two elements in block \( i \) with the first element in block \( i + 1 \), deleting the first element in the block 1, and the last two elements in the block \( g' \), and then partitioning the remaining set into \( g - g' \) pieces. \( \Box \)

We are ready to combine all the ingredients. Setting \( N = k(n - 1 - \Delta) \), \( g = n - 1 - \Delta \), \( g' = \left\lceil \frac{k(n-1-\Delta)}{\Delta} \right\rceil = \left\lceil \frac{k(n-1)}{\Delta} \right\rceil - k \) and substituting, the number of bits required by the encoding is at least:

\[
\log \left( \frac{N - 3g' + g - 1}{g - g' - 1} \right) = \log \left( \frac{(k+1)(n - 1 - \Delta - g') - 1}{n - 2 - \Delta - g'} \right) \geq \log \left( \frac{(k+1)(n - 2 - \Delta - g')}{n - 2 - \Delta - g'} \right).
\]

Using the entropy function as a lower bound, this is at least \((k+1)n'H(\frac{1}{k+1}) - \Theta(\log n')\), where \( n' = n - 2 - \Delta - g' \geq n(1 - \frac{k}{3}) + \frac{k}{3} - k - 2 - \Delta \). Thus, we have the following theorem:

**Theorem 4.** For sufficiently large values of \( n \), any data structure that encodes range top-\( k \) queries must occupy \((k+1)n'H(\frac{1}{k+1}) - \Theta(\log n')\) bits of space, where \( n' \geq n(1 - \frac{k}{3}) + \frac{k}{3} - k - 2 - \Delta \), and we are free to select any positive integer \( \Delta \geq 1 \). If \( k = o(n) \), then we can choose \( \Delta \) such that \( \Delta = \omega(k) \) and \( \Delta = o(n) \), yielding that the lower bound is \((k+1)n'H(\frac{1}{k+1})(1 - o(1))\) bits.

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