On Structural Invariants in the Energy-Based Control of
Infinite-Dimensional Port-Hamiltonian Systems with In-Domain Actuation*

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Abstract

This contribution deals with energy-based in-domain control of systems governed by partial differential equations with spatial domain up to dimension two. We exploit a port-Hamiltonian system description based on an underlying jet-bundle formalism, where we restrict ourselves to systems with 2nd-order Hamiltonian. A certain power-conserving interconnection enables the application of a dynamic control law based on structural invariants. Furthermore, we use various examples such as beams and plates with in-domain actuation to demonstrate the capability of our approach.

Keywords: Infinite-dimensional systems, Partial-differential equations, Differential geometry, Port-Hamiltonian systems, In-domain actuation, Structural invariants, Dynamic controllers

1. Introduction

For finite dimensional systems, the port-Hamiltonian (pH) framework has proven as an appropriate system representation, as the structure of the ordinary differential equations (ODEs) is related to the underlying physics. From a control-engineering point of view, in particular the fact that so-called power ports can be introduced plays an important role, because it allows the application of energy-based control schemes, see e.g. [1, 2], where the objective is to design a desired closed-loop behaviour by means of energy shaping and damping injection.

A famous control scheme exploiting the occurrence of power ports is the well-known energy-Casimir method, which has already been extended to systems governed by partial differential equations (PDEs). However, in the infinite-dimensional scenario the generation of ports strongly depends on the underlying approach, which unfortunately is – in contrast to the finite-dimensional scenario – not unique. For control-engineering purposes, in particular the so-called Stokes-Dirac scenario as well as an approach based on an underlying jet-bundle structure – where the major difference of these approaches is the choice of the variables – have been established. For the well-known Stokes-Dirac scenario, relying on the use of energy variables, boundary ports solely stem from differential operators occurring in the system description, see [3, 4] for instance. In contrast, regarding the jet-bundle approach, boundary ports basically result due to derivative variables that may occur in the Hamiltonian, but can also be generated by differential operators, see e.g. [5, 6]. While the Stokes-Dirac scenario exhibits a close relation to functional analytic methods that can be used to address the well-posedness as well as stability investigations of a problem, see [7], for the systems considered in this contribution, that allow for a variational characterisation, the jet-bundle approach is particularly suitable. At this point it should be mentioned that the focus of this paper is on exploiting geometric system properties, and thus, detailed well-posedness and stability investigations based on functional analytic methods are not

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presented. Therefore, we assume well-posedness and confine ourselves to energy considerations regarding stability investigations.

The energy-Casimir method has proven to be an effective tool, where for boundary-control systems with 1-dimensional spatial domain this method enables to derive finite-dimensional controllers, see, e.g., [8, 9] for the controller design in the Stokes-Dirac scenario and, e.g., [10, 11] for the jet-bundle approach. From a mathematical point of view, this boundary-control scheme can be interpreted as the coupling of a PDE- with an ODE-system at the actuated boundary of the plant. In general, this methodology also allows the use of infinite-dimensional controllers, see [12, 13], corresponding to the coupling of PDEs with PDEs, which would imply a further rise of complexity regarding stability investigations.

In this paper, we focus on systems with in-domain actuation, where we restrict ourselves to lumped inputs, which motivates the interconnection of the plant to a finite-dimensional controller, representing the coupling of a PDE- with an ODE-system within the spatial domain. In particular, we aim to extend the foundings of [14], where an in-domain control strategy for pH-systems with 2nd order Hamiltonian and 1-dimensional spatial domain has been developed, to systems with 2-dimensional spatial domain. This approach relies on a certain interconnection of the controller and the plant via its distributed ports that can be deduced by means of a certain power-balance relation. However, it should be noted that for the system class under consideration the determination of this power-balance relation is a non-trivial task, and thus, we employ an approach based on so-called Cartan forms proposed in [15]. As an example for this system class, we will study a plate modelled according to the Kirchhoff-Love assumptions that is actuated by two pairs of piezoelectric macro-fibre composite (MFC) patches; and we intend to develop an appropriate control law. Therefore, the main contributions of this paper are as follows: i) in Section 3, which deals with the law. Therefore, the main contributions of this paper are as follows: i) in Section 3, which deals with the

2. Notation and Mathematical Preliminaries

This paper is based on differential-geometric methods, where the notation is similar to those of [16]. To keep the formulars short and readable, we use tensor notation and apply Einsteins convention on sums, where we do not indicate the range of the indices when the are clear from the context. Furthermore, we will omit pullbacks in order to avoid exaggerated notation. The standard symbols d, ∧ and j denote the exterior derivative, the exterior wedge product and the Hook operator, allowing the natural contraction between tensor fields, respectively. The set of all smooth functions on a manifold \( \mathcal{M} \) is denoted by \( C^\infty(\mathcal{M}) \).

This contribution deals with systems governed by PDEs, and therefore, to be able to distinguish between dependent and independent variables, we introduce a so-called bundle \( \pi : \mathcal{E} \to \mathcal{B} \), with \((z^i), i = 1, \ldots, m\), denoting the independent coordinates of the base manifold \( \mathcal{B} \) and \((\bz^i, x^\alpha)\), where \( x^\alpha \), with \( \alpha = 1, \ldots, n \), are the dependent variables, those of the total manifold \( \mathcal{E} \). Next, a section of the bundle \( \pi : \mathcal{E} \to \mathcal{B} \) is given by the map \( \phi : \mathcal{B} \to \mathcal{E} \), i.e. the dependent and independent variables are related according to \( x^\alpha = \phi^\alpha(\bz^i) \). Consequently, by introducing an ordered multi index \( J = j_1 \ldots j_m \) with \( \sum j_i = k = \# J \), where an index \( j_i \) indicates that the derivative with respect to the independent variable \( z^i \) is carried out \( j_i \) times, the \( k \)-th order spatial derivatives of a section \( \phi \) can be given by

\[
\frac{\partial^k}{(\partial z^{j_1}) \ldots (\partial z^{j_m})} \phi^\alpha = \partial_{[J]} \phi^\alpha = \phi^\alpha_{[J]}
\]

Furthermore, it should be noted that \( 1 \) corresponds to a multi index containing only zeros except the \( i \)-th entry that is one, and consequently, an increase of the \( i \)-th entry of \( J \) by one is given by \( J + 1 \). Now, in order to introduce so-called jet variables or derivative coordinates, we consider further important geometric objects, namely \( r \)-th-order jet manifolds \( \mathcal{J}^r(\mathcal{E}) \) that are equipped with the coordinates \((z^i, x^\alpha, x^\alpha_{[j]})\), where \( x^\alpha = x^\alpha_{[0 \ldots 0]} \) holds.
Next, we introduce the tangent bundle $\tau E : T(E) \to E$, which is equipped with the coordinates $(z^i, x^\alpha, \dot{z}_i, \dot{x}^\alpha)$ and the fibre basis $\partial_i = \partial / \partial z^i$, $\partial_\alpha = \partial / \partial x^\alpha$, and hence, a vector field $v : E \to T(E)$ is a section given in local coordinates as $v = v^i \partial_i + v^\alpha \partial_\alpha$. Moreover, the vertical tangent bundle $\nu : V(E) \to E$ possessing the coordinates $(z^i, x^\alpha, \dot{v})$ is of particular interest since it allows to define a vertical vector field $v = v^i \partial_i$ as a section of it. Consequently, a vertical vector field $v$ prolonged to the $r$th-order jet manifold $J^r(E)$ reads as

$$j^r(v) = v + d[1] (v^%) \partial_\alpha^{[\alpha]} \bigg| d[2] = \left( d[1]_{[1]} \right)^2 \ldots \left( d[1]_{[m]} \right)^m$$

with $1 \leq \#J \leq r$, where we exploit the total derivative $d[1]$ with respect to the independent variable $z^i$, which is given by

$$d[1]_{[1]} = \partial_i + x^\alpha_{[i+1]} \partial_\alpha^{[\alpha]}$$

A further important geometric object is the cotangent bundle $\tau^*_E : T^*(E) \to E$, which is equipped with the coordinates $(z^i, x^\alpha, \dot{z}_i, \dot{x}^\alpha)$ and the bases $dz^i$ and $dx^\alpha$. Thus, a so-called 1-form $\omega : E \to T^*(E)$ is a section, which reads as $\omega = \omega_i dz^i + \omega_\alpha dx^\alpha$ in local coordinates. By constructing certain pullback bundles, we are able to address special densities – that are quantities that can be integrated – of the form $\delta = \Omega$, with $F \in C^\infty (J^r(E))$ implying that the coefficients may depend on derivative coordinates as well. The corresponding integrated quantity $\delta \int_B \Omega$ is called a functional. Here, we have used a volume element $\Omega$ that is defined on the base manifold $\mathcal{B}$, and consequently, we have $\Omega = dz_1 \wedge \ldots \wedge dz^m$ with $\dim(\mathcal{B}) = m$ in local coordinates. Furthermore, a boundary-volume form is denoted by $\Omega_\partial = \partial_i |\Omega$. In this contribution, it is of particular interest to determine the change of geometric objects along vector fields $v$, and therefore, we exploit the so-called Lie derivative, which, exemplarily, reads as $L_v (\omega)$ for a differential form $\omega$.

3. Infinite-Dimensional PH-Systems

In this section, an approach exploiting jet-bundle structures, see e.g. [5, 17], is used to represent infinite-dimensional pH-systems with 2nd-order Hamiltonian density, i.e. $\mathcal{H} \in C^\infty (J^2(E))$, actuated within the (1- or 2-dimensional) spatial domain. The approach is based on a certain power-balance relation which can be used to introduce power ports on the domain as well as on the boundary. Furthermore, the section is completed by examples for systems with 1- or 2-dimensional spatial domain.

First, we focus on systems with 2-dimensional spatial domain, i.e. we study Hamiltonian systems on the bundle $\pi : E \to \mathcal{B}$ with $(z^1, z^2, x^\alpha)$ denoting the coordinates of $E$. The 2nd-order Hamiltonian density is given by $\delta = H \Omega$ with $H \in C^\infty (J^4(E))$, where a volume element takes the local form $\Omega = dz^1 \wedge dz^2$. Now, we focus our interest on an evolutionary vector field $v = v^\alpha \partial_\alpha$, corresponding to the set of PDEs

$$\dot{x}^\alpha = v^\alpha \quad \text{with} \quad v^\alpha \in C^\infty (J^4(E)),$$

(2)

together with appropriate boundary conditions, where the time $t$ plays the role of the evolution parameter of the solution (well-posedness provided). Next, the evolution of the Hamiltonian functional $\mathcal{H} = \int_B H \Omega$ along solutions of (2) according to

$$\dot{\mathcal{H}} = \int_B L_{\dot{z}^i} (H \Omega)$$

(3)

is of particular interest. Basically, by considering [11] with $r = 2$, the formal change $\dot{\mathcal{H}}$ can be deduced by means of integration by parts. However, for the system class under consideration, i.e. 2nd-order Hamiltonian density and 2-dimensional spatial domain, the determination of the formal change is not straightforward due to the ambiguity of the integration by parts which may yield wrong boundary terms. To cope with that inconveniences, in [15] an approach based on certain Cartan forms is proposed, where coordinates adapted to the boundary as well as a boundary-volume form $\Omega_\partial$ adapted to the boundary are used. Hence, based on Eqs. (13) and (14) it is possible to derive the boundary operators

$$\delta^{\partial,1} = (\partial^{[\alpha]}_\alpha H - d_{[1]}^{[1]}(\partial^{[\alpha]}_\alpha H) - d_{[1]}^{[2]}(\partial^{[\alpha]}_\alpha H)) dz^\alpha \wedge \Omega_\partial,$$

(4a)

$$\delta^{\partial,2} = \partial^{[\alpha]}_\alpha H dz_{[01]} \wedge \Omega_\partial,$$

(4b)
whereas the variational derivative is given by
\[ \delta \mathcal{H} = \delta_\alpha \mathcal{H} dx^\alpha \wedge \Omega \] (5)
with
\[ \delta_\alpha = \partial_\alpha - d_{[10]} \partial_{\alpha}^{[10]} - d_{[01]} \partial_{\alpha}^{[01]} + d_{[20]} \partial_{\alpha}^{[20]} + d_{[11]} \partial_{\alpha}^{[11]} + d_{[02]} \partial_{\alpha}^{[02]} . \]
Furthermore, this approach allows to introduce a so-called decomposition theorem given in [18 Theorem 3.2], which plays a crucial role not only for the determination of the formal change of the Hamiltonian, but also for the derivation of structural invariants in Section 4.

**Theorem 1.** [18 Theorem 3.2] Let \( \mathcal{H} \in C^\infty(\mathcal{J}^2(\mathcal{E})) \) be a 2nd-order density and \( v \) an evolutionary vector field. Then, the integral \( \int_B L_j^*(v)(\mathcal{H}^\Omega) \) can be decomposed into
\[ \mathcal{H}^\Omega = \int_B v |\delta \mathcal{H} + \int_{\partial B} v j^1(v) |\delta^3 \mathcal{H} + \int_{\partial B} j^1(v) |\delta^3 \mathcal{H} \]
with the domain operator [3], as well as both the boundary operators according to (4a) and (4b).

Next, we give a pH-system representation making heavy use of a certain power-balance relation that can be introduced based on Theorem 1.

The objective of the pH-system representation is to exploit the structure of the governing evolution equations (2), which are therefore rewritten in the form
\[ \dot{x} = (\mathcal{J} - \mathcal{R}) (\delta \mathcal{H}) + u_i \mathcal{G}, \] (6a)
\[ y = \mathcal{G}_i^* \delta \mathcal{H}, \] (6b)
where \( \mathcal{H} \) denotes a 2nd-order Hamiltonian. Here, the (skew-symmetric) interconnection map \( \mathcal{J} : \mathcal{T}^* (\mathcal{E}) \wedge \mathcal{T}^+ (\mathcal{B}) \to \mathcal{V} (\mathcal{E}) \), where \( \mathcal{J}^{\alpha \beta} = -\mathcal{J}^{\beta \alpha} \in C^\infty (\mathcal{J}^4 (\mathcal{E})) \) is met for the coefficients, describes the internal power flow of the system, whereas the (symmetric and positive semi-definite) map \( \mathcal{R} : \mathcal{T}^* (\mathcal{E}) \wedge \mathcal{T}^+ (\mathcal{B}) \to \mathcal{V} (\mathcal{E}) \), satisfying \( \mathcal{R}^{\alpha \beta} = \mathcal{R}^{\beta \alpha} \in C^\infty (\mathcal{J}^4 (\mathcal{E})) \) and \( |\mathcal{R}^{\alpha \beta}| \geq 0 \) for the coefficient matrix, is related to the dissipation effects. Furthermore, the input map \( \mathcal{G} : \mathcal{U} \to \mathcal{V} (\mathcal{E}) \) allows to include external inputs that may be distributed over (a part of) the spatial domain – i.e. both, the input coordinates \( v^i \in \mathcal{U} \) as well as the coefficients \( \mathcal{G}_i^\alpha \), may depend (amongst others) on the spatial variables \( z^i \) and is of great relevance as we intend to develop in-domain control strategies in this paper. Consequently, due to the distributed components \( \mathcal{G}_i^\alpha \) of the adjoint output map \( \mathcal{G}_i^* : \mathcal{T}^+ (\mathcal{E}) \wedge \mathcal{T}^* (\mathcal{B}) \to \mathcal{Y} \) (6b), can be interpreted as distributed output densities. Moreover, the fact that the input bundle \( \rho : \mathcal{U} \to \mathcal{J}^3 (\mathcal{E}) \) is dual to the output bundle \( \rho : \mathcal{Y} \to \mathcal{J}^1 (\mathcal{E}) \), see [3] Section 4 or [17] Section 3, yields the important relation
\[ (u_i \mathcal{G}) |\delta \mathcal{H} = u_i (\mathcal{G}_i^* |\delta \mathcal{H} \mathcal{H} \mathcal{G}_i^* u_i \mathcal{G} ) = y \mathcal{G}, \] (7)
which will play an important role for evaluating \( \mathcal{H} \). Furthermore, in local coordinates (6) reads as
\[ \dot{x}^\alpha = (\mathcal{J}^{\alpha \beta} - \mathcal{R}^{\alpha \beta} ) \delta_\beta \mathcal{H} + \mathcal{G}_i^\alpha u_i \mathcal{G}_i^\alpha \] (8a)
\[ y_i = \mathcal{G}_i^\alpha \delta_\alpha \mathcal{H}, \] (8b)
with \( \alpha, \beta = 1, \ldots, n \) and \( \xi = 1, \ldots, l \).

Next, it is of particular interest to reinterpret the formal change of the Hamiltonian by keeping the pH-system representation (6) in mind. By applying the decomposition Theorem 1 where we substitute \( v = \dot{x} \) with (6a) and use the relation (7), we conclude that
\[ \mathcal{H} = - \int_B \mathcal{R} |\delta \mathcal{H} \mathcal{G}, + \int_B u_i |y + \int_{\partial B} \dot{x}^\alpha |\delta^0 \mathcal{H} + \int_{\partial B} \dot{x} |\delta^0 \mathcal{H} \mathcal{G}, \]

is divided into 4 parts. The energy of the system that is dissipated – e.g., due to damping – is described by
the expression - \( \int_B \mathcal{R} |\delta \mathcal{H} \mathcal{G} \), whereas the remaining terms denote collocation on the domain as well as
on the boundary. In particular the expression $\int_\partial w\, y\, d\nu$, which follows from the in- and outputs distributed over the spatial domain, is of significant importance in this contribution. Moreover, it should be noted that the 2 different boundary-port categories $\int_{\partial B} \dot{x}^\alpha \delta^{\alpha,1}_2 \bar{H}$ and $\int_{\partial B} \dot{x}^{[01]} \delta^{\alpha,2}_2 \bar{H}$ are a consequence of the 2nd-order Hamiltonian density. For the sake of completeness, by using the calculus of variation, we can state the formal change in local coordinates according to

$$\delta \mathcal{H} = - \int_B \delta_\alpha (\bar{\mathcal{H}}) \bar{\mathcal{R}}^{\alpha,\beta} \delta_\beta (\bar{\mathcal{H}}) \bar{\Omega} + \int_B w^\alpha y_\epsilon \bar{\Omega} + \int_{\partial B} \dot{x}^\alpha \delta^{\alpha,1}_2 \mathcal{H}\bar{\Omega}_2 + \int_{\partial B} \dot{x}^{[01]} \delta^{\alpha,2}_2 \mathcal{H}\bar{\Omega}_2$$

with the variational derivative (5) and the boundary operators (4a) and (4b). At this point, it should be stressed that we confine ourselves to systems with in-domain actuation solely, implying that no power can be extracted from or delivered to the system via the boundary, and therefore, in this scenario the boundary ports $\int_{\partial B} \dot{x}^\alpha \delta^{\alpha,1}_2 \mathcal{H}\bar{\Omega}_2$ and $\int_{\partial B} \dot{x}^{[01]} \delta^{\alpha,2}_2 \mathcal{H}\bar{\Omega}_2$ vanish identically.

Remark 1. Although boundary ports only play an tangential role in this contribution, worth mentioning is the fact that the boundary terms can easily be deduced by means of (4a) and (4b). Furthermore, the boundary operators $\delta^{\alpha,1}_2$ and $\delta^{\alpha,2}_2$ are of major importance for the determination of certain Casimir conditions for the controller design treated in the next section.

Having discussed the framework for pH-systems with 2nd-order Hamiltonian and 2-dimensional spatial domain, as an example a plate that is modelled according to the Kirchhoff-Love theory and actuated by 2 pairs of piezoelectric MFC patches shall be studied, see Fig. 1. Hence, our intention is to find a pH-system representation for the governing equation of motion – that can be derived by using the calculus of variation, see [19] for instance – being useful regarding the energy-based controller design proposed in Section 4.

Example 1 (Piezo-actuated Kirchhoff-Love plate). Let $B = \{(z^1, z^2) | 0 \leq z^1 \leq L_1 \land 0 \leq z^2 \leq L_2 \}$ be the spatial domain of a rectangular plate modelled according to the Kirchhoff-Love hypothesis, where the plate is clamped at the edge $\partial B_1 = \{(z^1, z^2) | z^1 = 0 \land 0 \leq z^2 \leq L_2 \}$, while the remaining edges $\partial B_2 = \{(z^1, z^2) | 0 \leq z^1 \leq L_1 \land z^2 = 0 \}$, $\partial B_3 = \{(z^1, z^2) | z^1 = L_1 \land 0 \leq z^2 \leq L_2 \}$ and $\partial B_4 = \{(z^1, z^2) | 0 \leq z^1 \leq L_1 \land z^2 = L_2 \}$ are free. Moreover, the considered plate is actuated by 2 pairs of piezoelectric patches, each pair consists of 2 single patches that are placed symmetrically on the upper and lower side of the plate. Thus, the equation of motion for the system under consideration is given by

$$\mu (z^1, z^2) \ddot{w} = -d_{[20]} (\Xi (z^1, z^2) (w_{[20]} + \nu w_{[02]})) - d_{[02]} (\Xi (z^1, z^2) (\nu w_{[20]} + w_{[02]}))$$

$$- 2d_{[11]} (\Xi (z^1, z^2) (1 - \nu) w_{[11]}) - \Lambda_k u^k_{\text{in}}, \quad (9)$$

with Poisson’s ratio $\nu$ and $w_{[20]} = \frac{\partial^2 w}{\partial z_1^2}$ denoting the 2nd-order derivative of the transverse plate deflection $w$ with respect to $z^1$ for instance. Furthermore, the voltages $u^k_{\text{in}}$ with $k = 1, 2$, which are applied to the
MFC patches, shall serve as manipulated variables, where \( \Lambda_k = \Psi_p (a^1 d_{20} \Gamma_k (z^1, z^2) + a^2 d_{02} \Gamma_k (z^1, z^2)) \), with \( a^1, a^2, \Psi_p \) comprising several piezo-parameters, states the spatial distribution of the inputs; see \[20\] for a similar model where also dissipation effects are considered. It is worth stressing that in \[9\] the mass density \( \mu(z^1, z^2) = \rho_h c + 2 \rho_h h_p (\Gamma_1(z^1, z^2) + \Gamma_2(z^1, z^2)) \), with \( \rho_h c \) and \( \rho_h h_p \) denoting the mass densities of the carrier layer and of the MFC pachcs, as well as the flexural rigidity \( \Xi(z^1, z^2) = E_c I_c + 2 \Xi_p (\Gamma_1(z^1, z^2) + \Gamma_2(z^1, z^2)) \), with the flexural rigidity of the carrier layer \( E_c I_c \) and those of the MFC patches \( \Xi_p \), are spatially dependent due to the incorporation of the piezoelectric patches, which can be included by the spatial characteristic functions

\[
\Gamma_k (z^1, z^2) = \left( h (z^1 - z^1_p) - h (z^1 - z^1_p - L^1_p) \right) \times \left( h (z^2 - z^2_p) - h (z^2 - z^2_p - L^2_p) \right),
\]

with \( k = 1, 2 \), the heaviside function \( h(\cdot) \) and the geometric dimensions that are depicted in Fig. 7. At this point it should be mentioned that in \[9\] spatial derivatives of the characteristic functions \( \Gamma_k(z^1, z^2) \) arise. Consequently, the use of the characteristic function \[10\] would require a weak formulation of the equation of motion. To be able to exploit the strong formulation nonetheless, we approximate the characteristic function \[10\] by the spatially differentiable function

\[
\Gamma_k (z^1, z^2) = \left( \frac{1}{2} \tanh (\sigma (z^1 - z^1_p)) - \frac{1}{2} \tanh (\sigma (z^1 - z^1_p - L^1_p)) \right) \times \left( \frac{1}{2} \tanh (\sigma (z^2 - z^2_p)) - \frac{1}{2} \tanh (\sigma (z^2 - z^2_p - L^2_p)) \right),
\]

where \( \sigma \in \mathbb{R}_+ \) denotes a scaling factor. By virtue of the plate configuration, see Fig. 7 regarding the boundary conditions we have

\[
w = 0 \quad \text{for} \quad \partial B_1, \quad (11a)
\]

whereas the shear force and the bending moment vanish along free edges, i.e.

\[
Q_2 = 0 \quad M_2 = 0 \quad \text{for} \quad \partial B_3, \quad (11b)
\]

\[
Q_1 = 0 \quad M_1 = 0 \quad \text{for} \quad \partial B_2, \partial B_4, \quad (11c)
\]

with \( Q_1 = w_{[03]} + (2 - \nu) w_{[21]} \), \( Q_2 = -w_{[03]} - (2 - \nu) w_{[12]} \), \( M_1 = -w_{[02]} - \nu w_{[20]} \) and \( M_2 = w_{[20]} + \nu w_{[02]} \) denoting the shear force and the bending moment on the particular edge.

Now, we focus our interest in finding a proper pH-system representation for the piezo-actuated Kirchhoff-Love plate. To this end, we introduce the momentum \( p = \mu (z^1, z^2) \dot{w} \), and consequently, the total kinetic energy density of the underlying system reads as

\[
T = \frac{1}{\mu (z^1, z^2)} p^2.
\]

If the plate is modelled based on the Kirchhoff-Love hypothesis – i.e straight lines perpendicular to the midplane are supposed to remain straight and perpendicular to the midplane during motion, and, the transverse normal stress can be neglected as it is sufficient small compared to the other normal stresses –, and it is assumed that the piezoelectric material can be described by linear constitutive relations, see \[21\], Section 3.3.1, the total potential energy density follows to

\[
V = \frac{1}{2} \Xi (z^1, z^2) \left( (w_{[20]})^2 + (w_{[02]})^2 \right) + \frac{1}{2} \Xi (z^1, z^2) \left( 2 \nu w_{[20]} w_{[02]} + 2 (1 - \nu) (w_{[11]})^2 \right) + \Psi_p (a^1 w_{[20]} + a^2 w_{[02]}) \Gamma_k (z^1, z^2) w_{in}^k,
\]

where the constants \( \Xi_p \) and \( \Psi_p \) comprise material parameters of the MFC patches and \( a^1, a^2 \) stem from the linear constitutive relations.Basically, with regard to boundary-control systems, the total-energy density
$T + V$ is used to obtain a proper pH-system representation. However, due to the fact that we focus on systems with in-domain actuation, we have included an input part in $\mathcal{I}$. Therefore, the aim is to find a Hamiltonian density such that an evaluation of $\delta_{\gamma} \mathcal{H}$ yields the right-hand side of (9), but without the input part comprising $u^{\alpha}_{in}$. To incorporate the input part in the pH-system representation, we set the input-map components to $g_{21}(z^1, z^2) = -\Lambda_1(z^1, z^2)$ and $g_{22}(z^1, z^2) = -\Lambda_2(z^1, z^2)$ describing the spatial distribution of the inputs $u^{\alpha}_{in}$. Consequently, if we choose the Hamiltonian density according to

\[
\mathcal{H} = \frac{1}{2\mu(z^1, z^2)} \dot{p}^2 + \frac{1}{2} \mathcal{E}(z^1, z^2)((w_{[20]}^1)^2 + (w_{[02]}^2)^2) + \frac{1}{2} \mathcal{E}(z^1, z^2)(2\nu w_{[20]} w_{[02]} + 2(1 - \nu)(w_{[11]}))^2,
\]

a suitable pH-system description for the piezo-actuated Kirchhoff-Love plate reads as

\[
\begin{bmatrix}
\dot{w} \\
\dot{p}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
\delta_{\gamma} \mathcal{H} \\
\delta_{\gamma} \mathcal{H}
\end{bmatrix} +
\begin{bmatrix}
0 & 0 \\
g_{21} & g_{22}
\end{bmatrix}
\begin{bmatrix}
u w_{[1]} \\
u w_{[2]}
\end{bmatrix},
\]

(12a)

with the boundary conditions (11), where it should be stressed that $Q_1$, $Q_2$, $M_1$ and $M_2$ can be deduced by evaluating (4a) and (22), c.f. Rem. 4. Next, to be able to introduce the power-ports for the system under consideration, we determine the formal change of the Hamiltonian functional $\mathcal{H} = \int_B \mathcal{H} \Omega$, where the decomposition Theorem 7 shall be used. As we do not have any boundary actuation, see the boundary conditions (11), it becomes obvious that the boundary ports vanish. Furthermore, dissipation effects are neglected in this example at all, and therefore, the formal change reduces to

\[
\mathcal{H}_{\alpha} = \int_B (g_{21}(z^1, z^2)\dot{w} u^{\alpha}_{in} + g_{22}(z^1, z^2)\dot{w} u^{\alpha}_{in}) \Omega.
\]

(13)

At this point it should be mentioned that a careful investigation of the actuator parameters hidden in $\Psi_{\alpha}$ shows that the unit of the distributed output densities (12b) is $\frac{\text{N}}{\text{m}}$. Hence, one can conclude that the formal change (13) corresponds to an electrical power-balance relation. By means of Ex. 4 we have demonstrated that due to the incorporation of in-domain actuators, which basically always exhibit a spatial distribution, power ports that are distributed over (a part of) the spatial domain can arise. In Section 4 these distributed power ports shall be used for the controller design. However, from a control-engineering point of view, it can also be of particular interest to investigate distributed-parameter systems with actuators that can be modelled – at least approximately – with an infinitesimal distribution, where for the sake of simplicity we focus on distributed-parameter systems with 1-dimensional spatial domain, i.e. we equip $\mathcal{B}$ with the independent coordinate $z^1$ solely. Consequently, a volume form on $\mathcal{B}$ reads as $\Omega = dz^1$ implying that the corresponding boundary-volume form follows to $\Omega_1 = \delta_{11} dz^1$. For pH-systems with 1-dimensional spatial domain and 2nd-order Hamiltonian density, the variational derivative in local coordinates is given by

\[
\delta_{\alpha} = \partial_{\alpha} - d_{[1]} \delta^{[1]}_{\alpha} - d_{[2]} \delta^{[2]}_{\alpha},
\]

whereas the boundary operators locally read as

\[
\delta^{[\gamma]}_{\alpha} \mathcal{H} = \partial^{[\gamma]}_{\alpha} \mathcal{H} - d_{[1]} \left( \delta^{[\gamma]}_{\alpha} \mathcal{H} \right), \quad \delta^{[\gamma]}_{\alpha} \mathcal{H} = \partial^{[\gamma]}_{\alpha} \mathcal{H}.
\]

In light of the aforementioned aspect, we introduce a specific form of pH-systems according to

\[
\begin{align*}
\mathcal{E} & = (\mathcal{F}^{\alpha} - \mathcal{R}^{\alpha}) \delta_{\alpha} \mathcal{H} + \mathcal{G}^{\alpha}_{\xi} \mathcal{Y} \\
\mathcal{Y} & = \mathcal{G}^{\alpha}_{\xi} \delta_{\alpha} \mathcal{H}
\end{align*}
\]

(15a)

(15b)

with

\[
\begin{align*}
\mathcal{G}^{\alpha}_{\xi} & = 0 & \text{for } & \gamma = 1, \ldots, n_1 \\
\mathcal{G}^{\rho}_{\xi} & = \delta (z^1 - A_{\rho}) & \text{for } & \rho = n_1 + 1, \ldots, n
\end{align*}
\]

(15c)
where $\delta (z_1 - A_1)$ denotes the Dirac delta function at the position $z_1 = A_1$ indicating that the inputs exhibit an infinitesimal spatial distribution. As a consequence, the formal change of $\mathcal{H}$ follows to

$$
\dot{\mathcal{H}} = - \int_B \delta_\alpha (H) R^{\alpha \beta} \delta_\beta (H) \Omega + u^\xi \xi + (\dot{x}^\alpha \delta_\alpha^{0,1} H) |_{\partial B} + (\dot{x}^\alpha \delta_\alpha^{0,2} H) |_{\partial B},
$$

with the 0-dimensional boundary ports $(\dot{x}^\alpha \delta_\alpha^{0,1} H) |_{\partial B}$ and $(\dot{x}^\alpha \delta_\alpha^{0,2} H) |_{\partial B}$ that vanish again if systems with in-domain actuation solely are considered. Consequently, it becomes obvious that we only have collocation located pointwise on the domain, which is visualised by the following example.

**Example 2 (Pointwise actuated beam).** Now, we consider an Euler-Bernoulli beam with the length $L$ actuated at $z_1 = A_1$ and $z_1 = A_2$ by means of the forces $u_1$ and $u_2$, where the governing PDE is given by

$$
\rho A \ddot{w} = -EI w_{[4]} + \delta (z_1 - A_1) u_1 + \delta (z_1 - A_2) u_2.
$$

Furthermore, both ends of the beam $\partial B_1 = 0$ and $\partial B_2 = L$ are free, and therefore, the boundary conditions read as

$$
\begin{align*}
Q &= 0 \\
M &= 0
\end{align*}
$$

for $\partial B_1, \partial B_2$

with the shear force $Q = -EI w_{[3]}$ and the bending moment $M = EI w_{[2]}$. Consequently, if we use the momentum $p = \rho A \dot{w}$ and the Hamiltonian density

$$
\mathcal{H} = \frac{1}{2} \rho A p^2 + \frac{1}{2} EI (w_{[2]})^2,
$$

the system under consideration can be written as

$$
\begin{bmatrix}
\dot{\hat{w}} \\
\dot{\hat{p}}
\end{bmatrix} = 
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
\delta_\alpha H \\
\delta_\rho H
\end{bmatrix} + 
\begin{bmatrix}
0 & \delta (z_1 - A_1) \\
\delta (z_1 - A_2) & 0
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix},
$$

where due to $y = \mathcal{G}^* | \delta \mathcal{H}$ the collocated outputs corresponds to $y_1 = \hat{w}|_{A_1}$ and $y_2 = \hat{w}|_{A_2}$, i.e. to velocities at defined positions. Thus, the formal change (16) follows to

$$
\dot{\mathcal{H}} = u_1 \hat{w}|_{A_1} + u_2 \hat{w}|_{A_2},
$$

which states a mechanical power-balance relation.

In this section, different classes of pH-systems, which also exhibit different types of power ports, have been considered. These power ports shall be used for the proposed control by interconnection methodology in the following.

### 4. In-Domain Control using Structural Invariants

The aim of this section is to develop a control strategy based on structural invariants being suitable for the different classes of pH-systems treated in Section 3. These mentioned categories mainly differ in the dimension of the spatial domain and the spatial distribution of the actuators; however, they have in common that the inputs themselves are lumped, and consequently, in light of this aspect, the application of a finite-dimensional controller is motivated. Furthermore, in the Subsections 4.2 and 4.3 we derive casimir conditions which differ due to the assumptions concerning the spatial distribution of the actuators.
4.1. Control by Interconnection

Next, we adapt the control by interconnection strategy based on structural invariants, which utilises damping injection and energy shaping in order to stabilise certain equilibria, to in-domain actuated pH-system with spatial domain up to dimension two. To achieve the damping-injection part, the passivity of a pH-controller, coupled by a power-conserving interconnection structure (PCIS) to the infinite-dimensional plant, shall be exploited. Moreover, since the aim is to shape the energy of the closed-loop system, we are interested in a relation between the plant and the controller, which shall be obtained by means of Casimir functionals.

Motivated by the lumped input of the considered plants, we use a finite-dimensional pH-controller given in local coordinates as

\[ \dot{x}_c^{\alpha_c} = (J_c^{\alpha_c\beta_c} - R_c^{\alpha_c\beta_c}) \partial_{\beta_c} H_c + C_c^{\alpha_c} y_c^\xi, \]  
\[ y_c,\xi = G_c^{\beta_c} \partial_{\beta_c} H_c, \]  

(18a)  
(18b)

with \( \alpha_c, \beta_c = 1, \ldots, n_c \). As already mentioned, the main idea is to couple the finite-dimensional controller to the infinite-dimensional plant in a power-conserving manner, and therefore, we choose the dimension of the controller in- and outputs according to \( \dim(u_c) = \dim(y_c) = l \). As the outputs of the plant may (in general) be distributed over (a part of) the spatial domain, cf. Ex. 1, to allow for a coupling the distributed output densities must be integrated over \( B \), and therefore, a power-conserving interconnection can be given by

\[ u_c^\xi \int_B y_c^\eta \Omega + u_c^{\xi \eta} y_c,\xi = 0. \]  

(19)

Here, it should be mentioned that we did not make any restriction concerning the spatial dimension of the plant yet, i.e. \( \dim(B) = 1, 2 \). If we choose the feedback structure according to

\[ u_c^\xi = K_c^{\xi \eta} \int_B y_c^\eta \Omega, \quad u_c^{\xi \eta} = -K_c^{\xi \eta} y_c,\eta, \]  

(20)

where \( K_c^{\xi \eta} \) denotes the components of an appropriate map \( K \), a PCIS meeting (19) is obtained. Furthermore, the closed loop, that results by using the coupling (20), is again a (mixed-dimensional) pH-system characterised by the Hamiltonian \( \mathcal{H}_c = \mathcal{H} + H_c \). Next, by taking the coupling (20) into account – and keeping in mind that we consider systems with in-domain actuation solely –, a straightforward calculation yields the formal change of \( \mathcal{H}_c \) according to

\[ \dot{\mathcal{H}}_c = -\int_B \delta_\alpha(H) R_c^{\alpha \beta} \delta_\beta(H) \Omega - \partial_{\alpha_c}(H_c) R_c^{\alpha_c \beta_c} \partial_{\beta_c}(H_c) \leq 0. \]  

(21)

Equ. (21) clearly highlights that we are able to inject damping into the closed-loop system by means of the pH-controller (18).

**Remark 2.** It should be stressed again that detailed stability investigations based on functional-analytic methods are not in the scope of this contribution. Instead, we focus on energy considerations, where \( \dot{\mathcal{H}}_c \leq 0 \) implies that the total energy is non-increasing along closed-loop solutions (provided they exist). Hence, by using \( \mathcal{H}_c \) as Lyapunov candidate, the relations \( \dot{\mathcal{H}}_c > 0 \) and \( \dot{\mathcal{H}}_c \leq 0 \) serve as necessary conditions for stability investigations in the sense of Lyapunov.

However, we are not content with damping injection only; in particular, we additionally aim to shape the energy of the closed-loop system. To this end, it is necessary to find a relation between the plant and (some of) the controller states. Therefore, in accordance with [10, 11], we are interested in Casimir functionals of the form

\[ \mathcal{C}^\lambda = x_c^\lambda + \int_B \mathcal{C}^\lambda \Omega, \quad \mathcal{C}^\lambda \in C^\infty(\mathcal{F}^2(\mathcal{E})), \]  

(22)

with \( \lambda = 1, \ldots, \bar{n} \leq n_c \); however, it should be stressed that in this contribution \( \dim(B) = 1, 2 \) is valid. To serve as conserved quantity, the functionals (22) have to fulfil \( \mathcal{C}^\lambda = 0 \) independently of \( \mathcal{H} \) and \( H_c \). Apart
from that, the requirement \( \dot{\mathcal{E}}^\lambda = 0 \) of course depends on the plant under consideration. Consequently, in
the following we distinguish between plants with actuators distributed over (a part of) the spatial domain
and plants with actuators modelled with an infinitesimal distribution. In light of this aspect, we derive
conditions for structural invariants depending on the particular plant category and demonstrate the
applicability of the proposed approach by deriving controllers for the examples treated in Section 3.

4.2. Controller Scenario I

This subsection deals with the controller design for infinite-dimensional pH-systems with in-domain
actuators that exhibit a spatial distribution. Based on the findings of the previous subsection, in the following
proposition necessary conditions regarding the controller design for the system class under consideration shall
be given.

Proposition 2. Let the interconnection of the plant \([8]\), where \( \dim(B) = 2 \) is valid, and the controller \([18]\)
be given by \([20]\). Then, if the functionals \([22]\) meet the conditions

\[
(J_c^\lambda \beta_c - R_c^\lambda \alpha_c) = 0 \quad (23a)
\]

\[
\delta_\alpha C^\lambda (J^\alpha \beta_c - R^\alpha \beta_c) + G^\lambda_{c,\xi} K^\xi \partial_{c,\eta} C^\alpha_c = 0 \quad (23b)
\]

\[
\delta_\alpha C^\lambda(\partial_{c,\eta} K^\xi \partial_{c,\eta} C^\alpha_c) = 0 \quad (23c)
\]

\[
(\dot{x}^\alpha \delta_\alpha^1 C^\lambda + \dot{x}^\alpha_{[01]} \delta_\alpha^2 C^\lambda)_{\partial B} = 0 \quad (23d)
\]

for \( \lambda = 1, \ldots, \tilde{n} \leq n_c \), they qualify as structural invariants of the closed loop.

Proof 1. To prove the requirement \( \mathcal{E}^\lambda = 0 \), we exploit the decomposition Theorem \([7]\) where we use \( C^\lambda \Omega \)
instead of \( \mathcal{H} \Omega \) now –, and consequently, the formal change of \([22]\) follows to

\[
\dot{\mathcal{E}}^\lambda = \dot{x}_c^\lambda + \int_B \dot{x}_c^\alpha C^\lambda \Omega + \int_{\partial B} (\dot{x}^\alpha \delta_\alpha^1 C^\lambda + \dot{x}^\alpha_{[01]} \delta_\alpha^2 C^\lambda) \Omega_{\partial B}.
\]

Then, by taking into account the dynamics of the plant \([8]\) and the controller \([18]\), as well as the coupling
\([20]\), we end up with

\[
\dot{\mathcal{E}}^\lambda = (J_c^\alpha \beta_c - R_c^\alpha \beta_c) \partial_{\beta_c} \mathcal{H}_c + \int_B \delta_\alpha C^\lambda (J^\alpha \beta_c - R^\alpha \beta_c) + \bar{G}^\lambda_{c,\xi} K^\xi \partial_{c,\eta} C^\alpha_c \delta_\beta \mathcal{H}_c
\]

\[
- \int_B \delta_\alpha C^\lambda(\partial_{c,\eta} K^\xi \partial_{c,\eta} C^\alpha_c) \partial_{\alpha_c} \mathcal{H}_c + \int_{\partial B} (\dot{x}^\alpha \delta_\alpha^1 C^\lambda + \dot{x}^\alpha_{[01]} \delta_\alpha^2 C^\lambda) \Omega_{\partial B} = 0 \quad (24)
\]

which yields exactly the conditions given in Prop. 2.

Now, it is of interest to interpret the results of Prop. 2 where we have the remarkable fact that – in
general – the conditions \([23a]-[23c]\) holds for systems with 1- or 2-dimensional spatial domain, cf. \([14]\) Eqs.
\((21a)-(21c)]\). Nevertheless, the differences are hidden in the geometric objects and operators that of course
strongly depend on the dimension of the spatial domain. Furthermore, condition \([23d]\) – and this is a major
difference compared to boundary-control schemes – enables to relate controller states with plant states within
the spatial domain. However, condition \([23d]\) – making heavy use of the boundary operators \( \delta_\alpha^1 \) and \( \delta_\alpha^2 \),
cf. Rem. 4 – describes the fact that we are not able to find relations restricted to the boundary, which is a
consequence of the circumstance that systems with in-domain actuation solely are considered. Next, the
applicability of the proposed control strategy shall be demonstrated by developing a Casimir-controller for
the piezo-actuated Kirchhoff-Love plate.

Example 3 (Energy-Casimir controller for Ex. 1). Now, we intend to exploit the pH-system represen-
tation of the piezo-actuated Kirchhoff-Love plate given in \([73]\) in order to derive an energy-based control law.
The aim is to move the plate from the initial position \( w_0(z^1, z^2) = 0 \) to the special rest position

\[
w^d = \begin{cases}
   a(z^1)^2 k(z^2) & \text{for } 0 \leq z^1 < z_b^1 \\
   (b(z^1 - z_b^1) + a(z_b^1)^2) k(z^2) & \text{for } z_b^1 \leq z^1 \leq L^1
\end{cases}
\]

(25)
with \( k(z^2) = -c + dz^2 \), \( z^1_0 = z^1_p + \frac{L_p}{z} \) and \( a, b, c, d \in \mathbb{R} \). To this end, by considering the dimension of the output densities \( \{\mathbf{z}\} \), 2 controller states shall be related to the plant. To fulfill the conditions \( (23) \), we choose \( C^1 = -g_{21}(z^1, z^2)w \) and \( C^2 = -g_{22}(z^1, z^2)w \) fixing a part of the controller mappings \( J_c, R_c \) and \( G_c \) because we set \( \mathbf{z} = \delta \mathbf{z}^\eta \) with the Kronecker-Delta symbol meeting \( \delta_\eta = 1 \) for \( \xi = \eta \) and \( \delta_\eta = 0 \) for \( \xi \neq \eta \). Furthermore, this ansatz allows for a relation between the plant and the first 2 controller states as it yields

\[
x_c^1 = \int_B g_{21}(z^1, z^2)w \Omega, \tag{26a}
\]

\[
x_c^2 = \int_B g_{22}(z^1, z^2)w \Omega, \tag{26b}
\]

by choosing appropriate initial states for the controller. Compared to boundary-control schemes, the relations \( (26) \) are a major difference as the controller states \( x^1_c \) and \( x^2_c \) correspond to a plant state that is weighted and integrated over the 2-dimensional spatial domain, whereas boundary controller exploit plant states restricted to the actuated boundary. Note that we have not determined the dimension of the controller, which can be interpreted as degree of freedom, yet. In this regard, damping shall be injected into the closed-loop system by means of 2 further controller states, and therefore, we set \( n_c = 4 \). Keeping the preceding facts in mind, we find that the controller dynamics are restricted to the mappings

\[
J_c - R_c = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -R^{33}_c & J^{34}_c - R^{34}_c & -R^{44}_c \ \\
0 & -J^{34}_c - R^{34}_c & J^{34}_c - R^{34}_c & -R^{44}_c \\
\end{bmatrix}, \tag{27a}
\]

\[
G_c = \begin{bmatrix}
1 & 0 & G^3_{c,1} & G^3_{c,1} \\
0 & 1 & G^3_{c,2} & G^3_{c,2} \end{bmatrix}^T. \tag{27b}
\]

Next, it remains to assign the Hamiltonian of the controller. Here, it should be mentioned that the equilibrium \( (25) \) is one that requires non-zero power, and therefore, we have to include an appropriate term in the controller Hamiltonian. Furthermore, we intend to obtain a minimum of the closed-loop Hamiltonian \( \mathcal{H}_c = \int_B \Omega H_c + H_c \) that involves the desired equilibrium \( (25) \). To this end, we exploit the relations

\[
x_c^{1,d} = \int_B g_{21}(z^1, z^2)w d \Omega, \\
x_c^{2,d} = \int_B g_{22}(z^1, z^2)w d \Omega,
\]

which are a consequence of \( (26) \), and choose

\[
H_c = \frac{c_1}{2}(x_c^1 - x_c^{1,d} - \frac{u_c^1}{c_1})^2 + \frac{c_2}{2}(x_c^2 - x_c^{2,d} - \frac{u_c^2}{c_2})^2 + \frac{1}{2}M_{c,\mu,\nu_c} x_c^{\mu_c, \nu_c},
\]

with the positive definite matrix \( [\mathbf{M}_c] \), \( M_{c,\mu,\nu_c} \in \mathbb{R} \) for \( \mu_c, \nu_c = 3, 4 \) and the positive constants \( c_1, c_2 > 0 \). As already mentioned, we have chosen \( \mathbf{z} = \delta \mathbf{z}^\eta \), which yields the PCIS

\[
\begin{align*}
\dot{u}_c^1 &= \int_B g_{21}(z^1, z^2)w \Omega, \\
\dot{u}_c^2 &= \int_B g_{22}(z^1, z^2)w \Omega,
\end{align*}
\]

and consequently, the formal change of \( \mathcal{H}_c \) follows to

\[
\dot{\mathcal{H}}_c = -x_c^{\mu_c} M_{c,\mu,\nu_c} x_c^{\mu_c, \nu_c} M_{c,\nu,\nu_c} x_c^{\nu_c} \leq 0,
\]

with \( \rho_c, \vartheta_c = 3, 4 \). As we do not carry out extensive stability investigations, cf. Rem. \( 3 \), the simulation results given in the Figs. \( 3 \) and \( 4 \) are used to verify the applicability of the proposed approach. In Fig. \( 4 \) the final plate deflection \( w \) is depicted over the spatial domain \( (z^1, z^2) \). Here, for the sake of simplicity, all plate
parameters are set to 1, except for the Poisson’s ratio \( \nu = 0.2 \). Furthermore, regarding the desired equilibrium (25) we have chosen \( a = 0.16, b = 0.12, c = 1 \) and \( d = 2 \). The MFC patches with \( L_1^p = L_2^p = 0.25 \) are placed at \( z_{1p} = 0.25, z_{2p,1} = 0.1 \) and \( z_{2p,2} = 0.65 \), see Fig. 1. The controller parameters have been chosen as \( J_{34}^c = 1, R_{34}^c = -1, R_{33}^c = 200, M_{33}^c = M_{44}^c = 10^4, M_{34}^c = 0, G_{31}^c = G_{41}^c = 100, G_{32}^c = G_{42}^c = 0 \) and \( c_1 = c_2 = 0.1 \). Worth stressing is the fact that the finite difference-coefficient method has been applied as discretisation scheme, where each direction of the plate have been divided into 20 intervals.

Having demonstrated the capability of the proposed control scheme for pH-systems with 2-dimensional spatial domain, in the following subsection a dynamic controller for pH-systems, actuated pointwise within the (1-dimensional) spatial domain, is derived.

4.3. Controller Scenario II

In this subsection, we restrict ourselves to systems described by (15), which has the consequence that modified conditions, being suitable for the system class under consideration, can be deduced. Furthermore, an energy-based controller stabilising a certain rest position shall be derived, see [13, Subsection 3.2.5].
Proposition 3. For the closed-loop system that is obtained by an interconnection of the certain system class (15) and the controller (18) via the coupling (20), the functionals (22) have to meet the conditions

\[ (J^\lambda c - R^\lambda c) = 0 \]  \hspace{1cm} (28a)
\[ \delta_\lambda \sigma^\lambda (J^{\rho \gamma} - R^{\rho \gamma}) = 0 \]  \hspace{1cm} (28b)
\[ \delta_\alpha C^\lambda (J^{\rho \rho} - R^{\rho \rho}) + (C^\lambda_{c \xi} K^{\xi \rho})|_{A^\prime_c} = 0 \]  \hspace{1cm} (28c)
\[ \delta_\alpha C^\lambda K^{\xi \eta} G^{\eta \eta} = 0 \]  \hspace{1cm} (28d)
\[ (\dot{\lambda}^{\delta \beta} C^\lambda + \dot{x}_c^\beta C^{\delta \lambda})|_{AB} = 0 \]  \hspace{1cm} (28e)

for \( \gamma = 1, \ldots, n_1 \) and \( \rho = n_1 + 1, \ldots, n \) to qualify as structural invariants.

Proof 2. The proof follows the intention of those of Prop. 3 and consequently, we have

\[ \mathcal{E}^\lambda = (J^{\rho c \beta c} - R^{\rho c \beta c}) \partial_{\beta c} H_c + \int_B \left( \delta_\alpha C^\lambda (J^{\alpha \beta} - R^{\alpha \beta}) + G^\lambda_{c \xi} K^{\xi \rho} G^{\eta \rho} \right) \delta_\beta H \Omega 
- \int_B \delta_\alpha C^\lambda G_{c}^{\xi \eta} G^{\eta \eta} \partial_{\alpha c} H_c \Omega + (\dot{\lambda}^{\delta \beta} C^\lambda + \dot{x}_c^\beta C^{\delta \lambda})|_{AB}, \]  \hspace{1cm} (29)

now. Consequently, by substituting the specific input restrictions (15c) in (29), the proof follows immediately.

Next, it remains to draw conclusions to the findings of Prop. 2. Similar to (23b), the condition (28c) allows for a relation between the controller and the plant that is now restricted to a certain position of the spatial domain due to the specific input assignment we made. Furthermore, this assignment implies condition (28d) for the system states where no input is acting.

With the preceding findings in mind, the pointwise actuated beam of Ex. 2 shall be used to demonstrate the proposed approach, see also [18, Subsection 3.2.5].

Example 4 (Energy-Casimir controller for Ex. 2). Now, a controller that stabilises the desired equilibrium

\[ w^d = az^1 + b, \quad w^d|_{A^1} = a \]  \hspace{1cm} (30)

for the pointwise actuated beam of Ex. 3 shall be derived. There, we consider actuators and sensors with infinitesimal distribution, and for the particular example the outputs are velocities at defined positions. As a consequence, the coupling (20) reduces to \( u^\xi = \delta^\xi y^\eta_c \), \( u^\xi = -\delta^\xi y^\eta_c \) with \( \delta^\xi \) denoting the Kronecker-Delta symbol for \( \xi, \eta = 1, 2 \). Moreover, due to the fact that two pointwise outputs are present, we aim to relate 2 controller states to the plant. To be able to inject damping into the closed-loop system, we choose the controller dimension to \( n_c = 4 \). With regard to our control objectives, we set \( C^1 = \delta(z^1 - A^1)\omega \) and \( C^2 = \omega(z^1 - A^1)\omega \), where straightforward calculations show that they satisfy the conditions (29) and yield the important relations \( x^1_c = w|_{A^1}, \quad x^2_c = w|_{A^2} \), implying the remarkable fact that we have the same structure for the controller dynamics as in Ex. 3, see [27], even though the problem is quite different. If we consider \( x^1_c = w|_{A^1} = aA^1 + b, \quad x^2_c = w|_{A^2} = aA^2 + b \) and choose

\[ H_c = \frac{c_1}{2}(x_c^1 - x^1)\omega^2 + \frac{c_2}{2}(x_c^2 - x^2)\omega^2 + \frac{1}{2}M_c \mu \nu \omega \omega - M_c \omega \omega, \]

with the positive definite matrix \( [M_c], M_{\mu \nu} \in \mathbb{R} \) for \( \mu, \nu = 3, 4 \), and the positive constants \( c_1, c_2 > 0 \), the equilibrium (37) becomes a part of the minimum of

\[ \mathcal{H}_c = \int_B \left( \frac{1}{2}EI(w^2) + \frac{1}{2\rho A^2}p^2 \right) + H_c \]  \hspace{1cm} (31)

The positive definiteness of (37) together with

\[ \mathcal{H}_c = -x_c^\mu M_c \mu \nu \omega R_c^\mu \nu \omega M_c \omega \omega - M_c \omega \omega \leq 0 \]

for \( \mu, \nu = 3, 4 \), yield necessary conditions for the stability of the desired equilibrium (37), cf. Rem 5.
5. Summary and Outlook

In this paper, a control methodology based on structural invariants, that is able to cope with in-domain actuated pH-systems with spatial domain up to dimension two, has been presented. We restricted ourselves to the scenario of lumped inputs and exploited a certain PCIS to deal with the distributed output densities that (may) arise due to the spatial distribution of the actuators. Furthermore, as discussed in Ex. 3 as discretisation scheme the finite difference-quotient method has been applied as it allows to easily include in-domain inputs. However, this discretisation method also has some drawbacks like the quadratically rising complexity, and therefore, in future investigations we shall adapt more sophisticated – like e.g. structure preserving – discretisation schemes for spatially higher dimensional systems with in-domain actuation to our framework.

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