Chapter 4

FERROMAGNETIC DOMAIN WALLS IN FINITE SYSTEMS: MEAN-FIELD CRITICAL EXONENTS AND APPLICATIONS

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Abstract

The distribution of magnetic moments in finite ferromagnetic bodies was first investigated by Landau and Lifshitz in a famous paper [Phys. Z. Soviet Union, 8, 153 (1935)], where they obtained the domain structure of a ferromagnetic crystal at low temperatures, in the regime of saturated magnetization. In this article, we investigate the general properties of ferromagnetic domain walls of uniaxial crystals from the viewpoint of the Landau free energy. We present the basic ideas at an introductory level, for non-experts. Extending the formalism to the vicinity of the Curie temperature, where a general qualitative description by the Landau theory of phase transitions can be applied, we find that domain walls tend to suppress the layers, leading to a continuous vanishing of the domain structure with anomalous critical exponents. In the saturated regime, we discuss the role of domain walls in mesoscopic systems and ferromagnetic nanojunctions, relating the observed magnetoresistance with promising applications in the recent area of spintronics.

1 Introduction

The distribution of the magnetization inside a general ferromagnetic body follows a closed flux configuration which leads to the appearance of magnetic domains. For the stripe domain structure, which is common in whiskers, magnetizations of neighboring domains are oppositely oriented, separated by $180^\circ$ Bloch walls of finite width. Inside the domain walls, the change of the magnetization is not discrete but smooth. In a pioneering work, Landau
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and Lifshitz proposed for the first time a theory that quantitatively predicted the above configuration, relating domain sizes and wall width with the dimensions of the body and some phenomenological parameters associated with the crystal structure. In their analysis, the magnetic energy of the crystal is built as consisting of two terms: one is the exchange interaction of the spins, proportional to

\[ [(\nabla m_x)^2 + (\nabla m_y)^2 + (\nabla m_z)^2], \]

(1)

where \( \mathbf{m} = (m_x, m_y, m_z) \) is the magnetic moment, whose absolute value is considered constant and equal to the saturation value; the other gives the contribution of the magnetic anisotropy of the crystal, that competes with the exchange interaction. Assuming an easy direction of magnetization along the \( z \)-axis, the latter was written as

\[ \beta [m_x^2 + m_y^2], \]

(2)

with \( \beta > 0 \). The exchange term (1) considered by Landau and Lifshitz can be obtained from the classical Heisenberg model,

\[ H = -J \sum_i \sum_{\delta} \mathbf{S}_i \cdot \mathbf{S}_{i+\delta}, \]

(with \( J > 0 \), and \( \delta \) indexing the nearest neighbor sites) in the continuous approximation, replacing discrete spin variables \( \mathbf{S}_i \) by a spin density order parameter (magnetization density) \( \mathbf{m}(\mathbf{r}) \), spatially averaged in a given local configuration of spins.

To find the distribution of the magnetization inside the material, one solves a variational equation that minimizes the sum of these two contributions for the particular geometry and size of the sample, with proper boundary conditions. For the stripe geometry and the closed flux configuration, boundary conditions induce the formation of stripe domains. The solution of the problem leads to a soliton-like pattern for the magnetization near domain walls, forming a non-homogeneous phase where the order parameter (magnetization) changes sign alternately from one domain to the next. Due to the ferromagnetic interactions, the spins locally tend to be aligned, and one sees that the exchange energy is small everywhere, except in the intermediate region between domains, where the magnetic moment smoothly changes its orientation to satisfy the anisotropic energy and the boundary conditions. The overall energy is minimized, since there are no field lines outside the sample. The main assumption of this treatment is that the system is ferromagnetically ordered and close to saturation, in the low temperature phase. An extensive description of the general properties of magnetic domains and domain walls is available in Ref.

There are several important applications related to the existence of magnetically ordered domain walls, including the recent advances in spintronics for electronic devices. The purpose of this article is to discuss the structure of the domain walls in a phenomenological way, using the simple but efficient tools provided by the Landau analysis on ferromagnetism. In the first part of this article we consider the mean field critical regime of domains...
walls, while the second one is devoted to some effects in the saturated regime, where we discuss possible applications.

To understand how the domain structure behaves in the vicinity of the Curie temperature, where the magnetic order becomes paramagnetic, we extend the original Landau-Lifshitz approach within the more general framework of the Landau theory of phase transitions. The Landau free energy of a ferromagnetic system is written as a series of the magnetic moment density $\mathbf{m} = (m_x, m_y, m_z)$, which is considered as the order parameter (OP). In the neighborhood of the critical temperature $T_c$, $|\mathbf{m}|$ is assumed to be small. This power series introduces phenomenological coefficients for the exchange and anisotropy contributions, and is written following the general Landau prescription. If one writes the system degrees of freedom in terms of the magnetization density $\mathbf{m}$ and the magnetic field $\mathbf{H}$, the thermodynamics is obtained through the partition function

$$Z = \exp \left[ -\frac{G(\mathbf{H}, T)}{k_B T} \right], \quad (3)$$

which is written in terms of the Gibbs free energy $G(\mathbf{H}, T)$, and is proportional to

$$\int (D\mathbf{m}) \exp \left[ -\frac{E(\mathbf{m}, \mathbf{H})}{k_B T} \right], \quad (4)$$

where $k_B$ is the Boltzmann constant and $T$ is the temperature. The symbol $D\mathbf{m}$ means integration over all possible configurations of $\mathbf{m}$, with $E(\mathbf{m}, \mathbf{H})$ being the effective Hamiltonian

$$E(\mathbf{m}, \mathbf{H}) = \int d^d r f(\mathbf{m}, \mathbf{H}). \quad (5)$$

and $f$ the Landau functional. The latter integration is carried for a $d$-dimensional system. The expression $(4)$ yields the partition function as a functional integral of the field. Assuming that the dimensions of the stripe domains are large in comparison with the lattice constant (continuous approximation), variations of the magnetization lines inside the material are considered smooth and the term that measures the inhomogeneity contribution can be treated in first order as a gradient. In other words, we assume that the field $\mathbf{m} = (m_x, m_y, m_z)$ and its derivatives are continuous. In the mean field level, the minimization of the Gibbs free energy given by Eq. $(3)$ is done in the saddle point approximation,

$$\frac{\delta E(\mathbf{m}, \mathbf{H})}{\delta \mathbf{m}} \bigg|_{\mathbf{m} = \langle \mathbf{m} \rangle} = 0,$$

with the symbol $\delta$ representing the variation of integral $(5)$ with respect to the magnetization, calculated at the saddle point $\mathbf{m} = \langle \mathbf{m} \rangle$, where the brackets have the meaning of an ensemble average. In this approximation (which neglects the effects of fluctuations around the saddle point), the Landau functional $f(\mathbf{m}, \mathbf{H})$ plays the role of the free energy density and is the relevant quantity to be minimized. In the vicinity of the phase transition, however, the fluctuations around the saddle point $\delta \mathbf{m} = \mathbf{m} - \langle \mathbf{m} \rangle$ acquire a major importance for physical systems of low dimension and coordination number. Defining $\xi$ as the correlation
length, we notice that the mean field analysis breaks down when the amplitude of the fluctuations measured by the correlation length \( \langle \delta \mathbf{m}(r) \delta \mathbf{m}(0) \rangle \propto \xi^{2-d} \) is larger than the square of the mean field order parameter \( \langle m \rangle^2 \propto \xi^{-2} \). Since \( \xi \) diverges in the critical point, the saddle point approximation excludes the region immediately bellow \( T_c \) for \( d < 4 \), where the fluctuations predominate.

### 1.1 Phenomenological Landau Theory

It is instructive to develop a few basic ideas regarding the general theory of phase transitions in a continuous medium. The free energy \( f \) is built under symmetry considerations, observing the most general form of \( f \) which is preserved under all the symmetry operations of the OP \( \mathbf{m} \) that leave the system physically invariant. As we are in the continuum, we only need to know the point group symmetry of the OP, which for a ferromagnet depends basically on the anisotropy directions in the crystal. In general grounds, the free energy is made of three terms. The first one is a series expansion in the OP derivatives \( K(\mathbf{m}, \partial \mathbf{m} / \partial r) \), which we call “kinetic” term,

\[
K \equiv \sum_{ij} A_{ij} \frac{\partial m_i}{\partial r_j} + \sum_{ijk} B_{ijk} m_i \left( \frac{\partial m_i}{\partial r_j} - \frac{\partial m_j}{\partial r_i} \right) + \sum_{ijkl} C_{ijkl} \frac{\partial m_i}{\partial r_j} \frac{\partial m_k}{\partial r_l},
\]

where \( r_i \) are the space coordinates and \( m_i \) are the OP components. Not all of these terms are to be kept, since first order derivatives on the form \( \sum_{ij} \int_V d^d r \frac{\partial m_i}{\partial r_j} \) are non-extensive when integrated in the volume \( V \). In the same way, part of the second term above

\[
\sum_{ijk} \frac{1}{2} (B_{ijk} + B_{kji}) \left( m_k \frac{\partial m_i}{\partial r_j} + m_i \frac{\partial m_k}{\partial r_j} \right) = \sum_{ijk} \frac{1}{2} (B_{ijk} + B_{kji}) \frac{\partial}{\partial r_j} [m_k m_i]
\]

is also non-extensive in the volume. Keeping only the extensive terms, the most general form of \( K \) up to second order is \[5\]

\[
K \equiv \sum_{ijk} \frac{1}{2} (B_{ijk} - B_{kji}) \left( m_k \frac{\partial m_i}{\partial r_j} - m_i \frac{\partial m_k}{\partial r_j} \right) + \sum_{ijkl} C_{ijkl} \frac{\partial m_i}{\partial r_j} \frac{\partial m_k}{\partial r_l}. \tag{6}
\]

The first term of Eq. \( (6) \) is of special importance in the phenomenological description of commensurability transitions \[5\]. As the free energy of a ferromagnet is invariant by the inversion of the coordinate basis in the space, meaning \( r_i \to -r_i \), the antisymmetric term in \( r_i \) must be also discarded. The actual form of \( K \) is ruled by the physical properties of the system and by the point group symmetry of the OP. For a cubic crystal, the tensor \( C_{ijkl} \) is a number and the lowest order term is given by Eq. \( (1) \).

The second term of the free energy \( f \) is the potential \( U(\mathbf{m}) \), described by a power series expansion in terms of the OP components,

\[
U \equiv \sum_{i_1} v^{(1)}_{i_1} m_{i_1} + \sum_{i_1, i_2} v^{(2)}_{i_1, i_2} m_{i_1} m_{i_2} + \ldots + \sum_{i_1, i_2, \ldots, i_p} v^{(p)}_{i_1, i_2, \ldots, i_p} m_{i_1} m_{i_2} \ldots m_{i_p} + O(m^{p+1}),
\]
provided that \( m \) is small, where \( i_p = 1, \ldots, n \) runs over the components of the OP. The precise form of \( U \) also depends on the OP point group. In a ferromagnetic crystal, this term defines the energy of crystalline anisotropy. The term of first order is clearly excluded, due to the inversion symmetry of the crystal.

The ferromagnets can be classified according to their axes (or planes) of easy magnetization. For crystals with one axis of anisotropy, say along the \([001]\) direction (or the \( z \) axis), the second order term is usually written in the form

\[
v^{(2)} (m_x^2 + m_y^2) \quad \text{or} \quad -v^{(2)} m_z^2.
\]

Both forms are equivalent, because they are related by an irrelevant constant \( v^{(2)} m_0^2 \). If \( v^{(2)} > 0 \) it is said that we have an easy axis of magnetization, while for \( v^{(2)} < 0 \) we have an easy plane of magnetization, namely the \( xy \) plane. The fourth order invariants are made of free combinations of products between the two second order invariants above. We see that there are four invariant terms of fourth order entering in the free energy as a linear combination. The simplest example of uniaxial ferromagnet corresponds to a crystal with tetragonal symmetry. The hexagonal lattice of cobalt is another example of uniaxial ferromagnet, with one easy axis of magnetization perpendicular to the hexagonal lattice. In fact, the anisotropy along the hexagonal directions is very small and appears only in the sixth order terms \([6]\). For biaxial crystals, we need one more free parameter in the second order invariant of the \( xy \) plane,

\[
v^{(2)} m_x^2 + v^{(2)} m_y^2.
\]

In the particular case of isotropic cubic crystals, there are no privileged directions between the three principal directions \([100]\), \([010]\) and \([001]\). In this example, we have three equivalent easy axes of magnetization. The lowest order term \( U_2 = \alpha (m_x^2 + m_y^2 + m_z^2) = \alpha m_0^2 \) is spherically symmetric. In fourth order, the invariants can be written in two different (and equivalent) ways \([6]\),

\[
U_4 = -\frac{1}{2} v^{(4)} (m_x^4 + m_y^4 + m_z^4) \quad \text{or} \quad U_4 = v^{(4)} (m_x^2 m_y^2 + m_y^2 m_z^2 + m_z^2 m_x^2)
\]

related by the constant \( \frac{1}{2} v^{(4)} m_0^4 \). After these examples, we conclude that we may set the general form of the free energy by simple symmetry arguments only. To be more specific than this and specify the values of the parameters left, we need more information from experiments or from microscopic calculations. At the mean-field level, however, the qualitative form of the free energy is in general sufficient for drawing several important conclusions about the system.

The last term of the free energy is due to the inclusion of the external magnetic field \( H \),

\[
U_B \equiv -\mathbf{m} \cdot \mathbf{H} - \frac{1}{8\pi} H^2.
\]

The expression above satisfies the thermodynamic relation for the total magnetic field \( \mathbf{B} \) of the crystal,

\[
\frac{1}{4\pi} \int_V d^d r B_i(r) = \frac{1}{4\pi} \int_V d^d r \left[ H_i(r) + 4\pi (m_i(r)) \right] = -\frac{\partial G(\mathbf{m}, \mathbf{H})}{\partial H_i}.
\]
where the Gibbs potential $G$ has been defined in Eq. (3)–(5) and $M_i = \int_V d^d r \langle m_i \rangle$ is the total magnetization.

## 2 Critical Region

We will concentrate our focus in the simplest case, a crystal with one axis of easy magnetization along the $z$ direction. Anisotropies in the exchange are usually small, and (1) is a good approximation, even for axial symmetry. The dominant anisotropy effects in the spin Hamiltonian come from the admixture of the spin-orbit coupling into the crystal field. Similarly to (2), we will write the second order contribution from the crystalline anisotropy in the equivalent form

$$\left[ \alpha (m_x^2 + m_y^2) + \gamma m_z^2 \right],$$

with $\alpha$ and $\gamma$ positive constants. In the bulk, far from the domain wall, we have

$m_x = m_y = 0, \quad m_z = \pm m_0, \quad \nabla m_z = 0$.

To study the interface criticality, we write the excess free energy relative to a bulk system in the following form:

$$f(r, T) = a(T) \left[ \alpha (m_x^2 + m_y^2) + \gamma (m_z^2 - m_0^2) \right] +$$

$$+ b \left[ \gamma^2 (m_z^4 - m_0^4) + 2\alpha \gamma (m_x^2 + m_y^2) m_z^2 + \alpha^2 (m_x^2 + m_y^2)^2 \right] +$$

$$+ c \left[ |\nabla m_x|^2 + |\nabla m_y|^2 + |\nabla m_z|^2 \right].$$

(8)

As usual, the phenomenological coefficient $a(T)$ for the quadratic term changes sign at the critical temperature and is taken as a linear function of $T$, in the form

$$a(T) = \left( \frac{T - T_c}{T_c} \right) a_0 \equiv a_0 t,$$

(9)

with $a_0 > 0$ and $t = \frac{T - T_c}{T_c}$, so $a(T)$ is negative in the low temperature phase. We then assume $\alpha < \gamma$, since the $z$-axis is taken as the easy axis of magnetization. Additionally, we will assume that $H \equiv 0$.

The above Landau free energy (8) is a simple generalization of the commonly used expression to study phase transitions (see for instance Ref.[7], p. 417), with quadratic and quartic terms in the order parameter. Due to the magnetic anisotropy, the elementary invariants of the axial symmetry group are now $(m_x^2 + m_y^2)$ and $m_z^2$, and the free energy is built from them[5]. The fourth order term in the free energy can be interpreted as the contribution of the spin quadrupole interaction. To minimize the number of parameters, the quartic term is written as the square of the quadratic one. So we are left with the set $(a(T), b, c, \alpha, \gamma)$ of free parameters, where $\alpha$ and $\gamma$ are determined by the crystal field of the ferromagnet. The others are usual Landau parameters, with $b$ and $c$ slowly varying
with temperature, even at the critical point. We are assuming that they are constant. The behavior at large $|m|$ is dominated by the quartic term in (8). This requires the constant $b$ to be positive, in order to have minima of the free energy at $m \neq 0$, for the low temperature phase (where $a < 0$). The constant $c$ is also positive, since it costs some energy to create an interface.

The original treatment of Landau and Lifshitz argues that the minimization of the energy has to be done in two different regions of the crystal: i) in the intermediate region between domains (interface), where the contribution of the inhomogeneity cannot be neglected; and ii) in the region close to the surfaces, where the closed flux configuration of the field requires appropriate boundary conditions. We will use here the same argument, but for the more general free energy given by (8).

2.1 Domain Wall Magnetization

We shall concentrate first on the interface region. Without loss of generality, we assume that the crystal is infinite and the interface between domains is in the $yz$-plane. In the absence of a magnetic field, due to symmetry, the spins are all in the $yz$-plane. Far from the wall, they are aligned with the $z$-axis (parallel or anti-parallel). Close to the walls, we assume that they are deflected by an angle $\theta$ from the $z$-axis, with components:

$$
m_x = 0, \quad m_y = m_0 \sin \theta, \quad m_z = m_0 \cos \theta,
$$

where $\theta \equiv \theta(x)$ is a function of $x$ only. Note that $m_0 = |m|$ is temperature dependent, but considered uniform in space. The substitution of (10) into (8) leads to

$$
f[\theta(x), \theta'(x); T] = cm_0^2(\theta')^2 + a(T)[(\alpha + (\gamma - \alpha) \cos^2 \theta)m_0^2 - a(T)\gamma m_0^2 +
$$

$$+ b\gamma^2 \cos^4 \theta + 2\gamma \alpha \sin^2 \theta \cos^2 \theta + \alpha^2 \sin^4 \theta]m_0^4 - b\gamma^2 m_0^4.
$$

In order to apply the Euler variational principle to the action (5), one notes that the ‘Lagrangian’ $f[\theta(x), \theta'(x); T]$ does not explicitly depend on the variable $x$. In this case, the Euler-Lagrange equation can be written as

$$
\frac{d}{dx} \left[ \theta'(x) \frac{\partial f}{\partial \theta'} - f \right] = 0,
$$

which reduces to

$$
cm_0^2(\theta')^2 = K + a(T)[(\alpha + (\gamma - \alpha) \cos^2 \theta)m_0^2 - a(T)\gamma m_0^2 +
$$

$$+ b\gamma^2 \cos^4 \theta + 2\gamma \alpha \sin^2 \theta \cos^2 \theta + \alpha^2 \sin^4 \theta]m_0^4 - b\gamma^2 m_0^4,
$$

where $K$ is a constant to be evaluated using the boundary conditions (BC). Far from the interface between domains, we set the spins asymptotically aligned with the $z$-axis, remembering that $z$ is our easy magnetization direction. To satisfy the closure configuration, we impose
\[ \theta = \begin{cases} 0, & x \to -\infty, \\ \pi, & x \to \infty, \end{cases} \]  

(13)

and

\[ \theta' = 0, \to \pm \infty, \]  

(14)

which result in \( K = 0 \). The equation to be solved now is

\[
 cm_0^2 (\theta')^2 = a(T) \left[ (\alpha - \gamma) \sin^2 \theta \right] m_0^2 + b m_0^4 \left[ 2 \gamma \alpha \sin^2 \theta \cos^2 \theta + \alpha^2 \sin^4 \theta + \gamma^2 (\cos^4 \theta - 1) \right],
\]

(15)

which means that the free energy is minimized when the exchange energy density is equal to the anisotropy one. Equation (15) can be written in the form

\[
 (\theta')^2 = \sin^2 \theta \left( A + B \cos^2 \theta \right),
\]

(16)

whose solution can be given as

\[
 \sqrt{\frac{1 + B}{A}} \cos \theta \sqrt{\frac{1 + B}{A}} \cos^2 \theta = -\tanh \left[ x \sqrt{A + B} \right],
\]

(17)

where \( A \) and \( B \) are

\[
 A = \frac{a(T)}{c} (\alpha - \gamma) \left[ 1 + \frac{b}{a(T)} m^2 (\alpha + \gamma) \right]
\]

(18)

\[
 B = \frac{a(T)}{c} (\alpha - \gamma) \left[ -\frac{b}{a(T)} m^2 (\alpha - \gamma) \right].
\]

The above equations give the distribution of the magnetic moment density in a general crystal between two neighboring domains. There are two equivalent solutions with opposite helicities. The width of the interface (Bloch wall) is given by

\[
 \lambda = \frac{1}{\sqrt{A + B}}.
\]

(19)

To check the consistency of (17), we remind the reader that \( a(T) \) is a negative scalar function for \( T < T_c \). Noting that \( \alpha < \gamma \) (easy \( z \)-direction), the condition \( A + B > 0 \), will impose limitations for the lower bound of \( \alpha \). In our discussion, we are considering the limit of small anisotropy, for which \( \alpha \lesssim \gamma \). As we will show below, mean field implies that \( bm_0^2/a(T) \) is finite at the critical point, which determines the critical exponent of the magnetization. We will discuss those points later on.
The symmetry of the magnetization planes in the presence of an uniaxial anisotropy is represented by the $C_{2v}$ point group, which admits two invariant terms, $m_0^2$ and $m_0^2 \cos 2\theta$ in the expansion of the free energy [5]. This is easily noticed by decomposing the anisotropy term $U_2, m_0^2 (\alpha \sin^2 \theta + \gamma \cos^2 \theta)$ [see (7) and (10)], into the polynomial basis formed by the two invariants, giving

$$\frac{1}{2} \left[ (\alpha + \gamma) m_0^2 + (\gamma - \alpha) m_0^2 \cos 2\theta \right].$$

Clearly, the full symmetry term $m_0^2$ due to the paramagnetic phase ($O(2)$ point group) competes with the other term representing the symmetry of the domain ordered phase ($C_{2v}$ point group). The stability of the domain walls is ruled by the anisotropy parameter $\epsilon$, whose proper expression is

$$0 < \epsilon \equiv \frac{\gamma - \alpha}{\gamma + \alpha} < 1.$$

Assuming that the anisotropy is small ($\epsilon \ll 1$), the ratio $\frac{B}{A}$ is also small provided that $(A + B) > 0$,

$$\frac{B}{A} \approx \frac{2\alpha bm_0^2}{a(T)} \cdot \frac{\epsilon}{1 + 2\alpha bm_0^2 a(T)},$$

and the solution can be expanded in terms of $\frac{B}{A}$

$$\left(1 + \frac{B}{2A}\right) \cos \theta \left[1 - \frac{B}{2A} \cos^2 \theta\right] = -\tanh[\sqrt{A} (1 + \frac{B}{2A}) x]. \quad (20)$$

Fig. 1 shows the distribution of the magnetization angle $\theta$ through the interface. The quantity $(A + B)$ vanishes at the critical point yielding the limit $\lambda \to \infty$ and the solutions $\theta = \pm \frac{\pi}{2}$ everywhere. At first sight, this solution may seem inconsistent with the boundary condition imposed. However, we observe that the magnetization intensity goes to zero at $T_c$. In any case, this behavior signals that something odd is happening at the critical point and questions the validity of mean field solutions there. The limit $\lambda \to \infty$ means that magnetic fluctuations are paramount at $T_c$.

### 2.2 Surface Energy

Next we consider a finite crystal, where the domain structure is organized in layers, as shown in Fig. 2. We already know the spin distribution through the Bloch wall, and want to calculate the width of magnetic domains in a finite volume. This is done in a variational way, as in the original contribution by Landau and Lifshitz [11]. We note that the flux closure condition induces the formation of small domains near the surfaces, where the magnetization points perpendicular to the easy direction.

We proceed to the calculation of the wall energy. If $l$, $l_x$ and $l_y$ are the dimensions of the crystal in the $z$, $x$ and $y$ directions respectively, the energy associated with one interface (wall) between two domains is
Figure 1: Magnetization profile in the vicinity of a domain wall. The solid curve represents the solution in the saturated regime at low temperature (asymptotic case for $B \to 0$ and $A = 8$, now normalized to adimensional parameters). The dashed and dot-dashed curves were obtained solving (20), keeping the anisotropy fixed and varying the temperature in direction to $T_c$, where both, $A$ and $B$ vanish (dashed for $B = 0.1$, $A = 1$ and dot-dashed for $B = 0.05$, $A = 0.5$, respectively).

\[
E_{wall} = l_l \int_{-\infty}^{\infty} dx \int_{0}^{\theta} f(m,T) = \\
= l_l \int_{-\infty}^{\infty} dx \left[ cm_0^2 (\theta')^2 + a(T)[\alpha + (\gamma - \alpha) \cos^2 \theta]m_0^2 - a(T) \gamma m_0^2 + \\
b \gamma^2 \cos^4 \theta + 2\gamma \alpha \sin^2 \theta \cos^2 \theta + \alpha^2 \sin^4 \theta \right]m_0^2 - b \gamma^2 m_0^4 \right] \tag{21}
\]

where the limits of integration have been extended to $] -\infty, \infty[$, considering that the layer width is much larger that the wall region. The dominant contribution to this integral is concentrated inside domain walls, where $\theta$ is close to $\pi/2$. This way, we may neglect the cubic term in $\cos(\theta)$ in equation (20). Substituting expressions (15) into (21) yields an expression only in terms of the anisotropy

\[
E_{wall} = 2ll_s (\alpha - \gamma) \int_{-\infty}^{\infty} dx \sin^2 \theta \left[ a(T)m_0^2 + \\
b [(\alpha + \gamma) - (\alpha - \gamma) \cos \theta]m_0^4 \right],
\]

where we have used the form given by (16). To lowest order in the anisotropy $\varepsilon$ we get the result

\[
E_{wall} = -4ll_s \alpha m_0^2 \varepsilon \left[ a(T) + 2b \alpha m_0^2 \right] \int_{-\infty}^{\infty} \frac{dx}{\cosh^2 \left( x/\sqrt{A} \right)}
\]
It then follows that the total energy of \((l_x/d)\) walls is
\[
E_1 = 4 \frac{ll_xl_y}{d} c m_0^2 \sqrt{A} ,
\]
where \(d\) is the layer width.

Now, we turn to the surfaces for the flux closure configuration. The ferromagnetic phase of the body produces some amount of field given by the Maxwell equation \(\nabla \cdot (\mathbf{H} + 4\pi \mathbf{m}) = 0\) inside the sample, and \(\nabla \cdot \mathbf{H} = 0\) outside. Near the surface and far from domain walls, the exchange contribution to the energy is zero. The closed flux configuration, with no field lines outside the sample, is favored when the anisotropy is small. Under the above conditions, we get
\[
\mathbf{H} = 0, \quad \text{and} \quad \nabla \cdot \mathbf{m} = 0 ,
\]
what means that the spin distribution satisfies the boundary condition \(\mathbf{m} \cdot \mathbf{n} = 0\), with \(\mathbf{n}\) normal to the surface. This way, surface poles are avoided and the global spin field inside the crystal has no singular points as shown in Fig.2. For surface domains, the main contribution to the energy density comes from the anisotropy. At the surface, we get
\[
m_x = \pm m_0, \quad m_y = 0 , \quad m_z = 0 ,
\]
and going back to \((8)\) we get the energy density
\[
f_{\text{surface}} = a(T)m_0^2(\alpha - \gamma) \left[ 1 + \frac{bm_0^2}{a(T)} (\alpha + \gamma) \right] ,
\]
which is again proportional to the anisotropy \(\varepsilon\). For a finite sample, we have two opposite surfaces at \(z = 0\) and \(z = -l\). Associating the volume \((l_xd^2/4)\) to a single surface domain, and summing over \((2l_x/d)\) of such domains, the total surface energy is given by
\[
E_2 = \frac{1}{2} l_xl_yd m_0^2 c A .
\]

We then minimize the total energy \(E = E_1 + E_2\) simultaneously in relation to \(d\) and \(m_0\)
\[
\begin{align*}
\frac{\partial E}{\partial d} &= 0 , \\
\frac{\partial E}{\partial m_0} &= 0 ,
\end{align*}
\]
yielding the results
\[
\begin{align*}
d &= 2\sqrt{2l_x} \sqrt{\sqrt{\frac{A}{\varepsilon}}} , \\
\frac{bm_0^2}{a(T)} &= - \frac{4}{7 \left( \alpha + \gamma \right)} ,
\end{align*}
\]
\((22)\)
The quantity \( A \), defined in (18), vanishes at the critical point, thus causing the divergence of the domain width at \( T_c \). The associated critical exponent is different from the one that gives the divergence of the wall width \( \lambda \) in (19).

\[
\lambda^{-1} = \sqrt{A + B} = \sqrt{\left(\frac{\alpha - \gamma}{c}\right)a(T) \left[1 + \frac{2\gamma b m_0^2}{a(T)}\right]}
\]

In the expression above we encounter the quantity

\[D \equiv 1 + \frac{2\gamma b m_0^2}{a(T)},\]

that has to be positive in order to get real solutions. From (22), we see that this is the case for \( \gamma / 7 < \alpha < \gamma \). So, we are on safe grounds for small anisotropy, \( \alpha \lesssim \gamma \). We note that the critical exponent for the magnetization is the same as in the homogeneous case (\( \beta = 1/2 \)), but the full expression is different from the one obtained in conventional mean field theory. It is worth to note that the isotropic, homogeneous case, satisfies \( D = 0 \), i.e. no domains are present.

From (22), we get the result

\[
m_0 = \begin{cases} 
0 & , T > T_c \\
\sqrt{-\frac{4a(T)}{b(\alpha + \gamma)}}, & T < T_c
\end{cases}
\]

The two scales, \( d \) and \( \lambda \), diverge at the critical point, but the divergence of the wall width is faster (exponent \(-1/2\)) than the one for the domain width (exponent \(-1/4\)), thus showing that the critical point marks the onset of strong magnetic fluctuations.

For magnetic thin films, the Bloch walls produce magnetostatic poles in the surface of the film, enhancing enormously the energy of the crystal when the thickness is smaller.
than \( \sim 10^{-6} \text{cm} \) [8]. In general, the crystal prefers to deflect the spins of the wall along the directions parallel to the thin film surface, giving rise to a Neél wall, when the easy magnetization direction is in the plane of the film, as shown schematically in Fig. 3.

Despite it is tempting to generalize our previous conclusions to this system, the long-range dipolar interactions (which we have neglected so far) are relevant in an infinite slab of small thickness, not mattering how weak the dipolar interaction is in comparison to the nearest neighbor exchange interaction [9]. To illustrate how important this interaction might be, in some ferromagnets, the dipolar interactions are responsible for the appearance of domains in the complete absence of crystalline field effects, when the free energy is fully isotropic in a given plane of magnetization, for example. In these systems, the exchange competes with the dipolar interactions, stabilizing a non-homogeneous phase with a finite wave-vector in an arbitrary direction of the magnetization plane. Because of the large phase space of all the degenerate wave-vector directions allowed by symmetry, the fluctuations are strongly enhanced at \( T_c \), giving rise to the fluctuation induced first order transitions studied by Brazovskii [10]. In this specific case, the study of the phase transition is totally out of the scope of the mean field analysis. Here, we have focused our study in ordinary 3D ferromagnets, where the dipolar contribution is not a relevant interaction.

![Figure 3: Illustration of a 180° Neél wall.](image)

### 2.3 Critical Exponents

When temperature approaches \( T_c \) from below, \( a(T) \) as defined in [10], goes to zero linearly, and the entire system is affected by long range fluctuations. At \( T_c \), the correlation length diverges and the system is scale invariant[7]. In our model, this is signaled by the divergence of the wall width \( \lambda \), which plays the role of the length scale and goes to infinity under the power law \( \lambda \propto |t|^{-\frac{1}{2}} \). Since the domain width behaves as \( d \propto |t|^{-\frac{1}{4}} \), we find that the
wall region enlarges and ‘compresses’ the domains as temperature raises in direction to $T_c$, meaning that domains firstly lose their identity and finally disappear at the critical point. We observe that all the critical exponents in our treatment refer to behaviors below $T_c$ (in critical phenomena, one distinguishes the behaviors above and below $T_c$).

In what follows, we advance conclusions derived from the previous calculation. To calculate the heat capacity, we compute the total internal energy of the system as

$$E = 2c\sqrt{2l_xl_y}m_0^2A^\frac{1}{2},$$

once the minimization process (22) is done. Note that this is the excess energy relative to the bulk system, associated with surfaces and interfaces. From this, we calculate the specific heat $C = \frac{T\partial^2 E}{\partial T^2}$, and find that $C \propto |t|^{-\alpha}$ diverges at $t = 0$, with the critical exponent $\alpha = 1/4$, which is different from the standard mean field approximation $\alpha_B = 0$ for homogeneous systems (bulk magnetization) and also different from standard mean field results for interfaces ($\alpha_S = 1/2$) [11]. The different critical behavior comes here from the criticality of the domain width $d$.

We also know from experiments that the static magnetic susceptibility diverges at the critical point. In our calculation we get

$$\chi = \left(\frac{\partial^2 E}{\partial m_x^2}\right)^{-1} \propto |t|^{-\frac{3}{4}}, \text{ for } t \ll 1,$$

and therefore $\gamma = \frac{3}{4}$, which also disagrees with the standard mean field value $\gamma = 1$. Applying the Josephson, Fisher and Widom scaling laws [7, 12]

$$\nu d = 2 - \alpha \quad \text{and} \quad \gamma = \nu(2 - \eta) = \beta(\delta - 1)$$

for the correlation length $\xi \propto t^{-\nu}$, for the equation of state $M = \int_V d^d x \langle m \rangle \propto H^{-\frac{1}{\delta}}$, and for the correlation function ($\Gamma$) power law $p = d - 2 + \eta$, with

$$\Gamma \propto r^{-p} e^{-r/\xi},$$

we find that $\nu = \frac{7}{12}, \eta = \frac{5}{7}$ and $\delta = \frac{5}{7}$.

The different exponents and the singularity in the heat capacity are explained by the finite size effects manifested in the geometry of the ferromagnet surfaces, which lead to the formation of Bloch walls between magnetic domains. This statement can be easily tested by removing the closed flux boundary conditions in the thermodynamic limit. In this limit, the magnetization will be homogeneously distributed as if the crystal had a single domain with the size of the system, i.e. $d \sim l_x$ not mattering the temperature. We regain all the standard mean field exponents in this situation.

3 Mesoscopic Systems

The main effect resulting from the introduction of an external magnetic field $H$ in the ferromagnet is the displacement of the domain walls from the zero field equilibrium position.
As the bulk energy of each domain is enhanced or reduced by the magnetic coupling of the magnetization with the magnetic field,

\[-m \cdot H,\]

depending on the orientation of the spins with respect to \(H\), the domain walls are dislocated in the direction that reduces the size of energetically unfavorable domains, enlarging the favorable ones and reestablishing the equilibrium. As shown by Landau and Lifshitz [1], the dynamical interaction of the magnetization with the magnetic field is driven by the Zeeman effect, where the spins start to precess around \(H\) as free moments, and by the relativistic interaction (with coupling \(\eta\)), following the equation of motion

\[\dot{m} = \mu_0 \left( H \times m + \eta \left( H - (H \cdot m) \frac{m}{m_0^2} \right) \right),\]

where \(\mu_0 = e/mc\), with \(e\) the electron charge, \(m\) the electronic mass and \(c\) the speed of light. The dot represents a time derivative. Considering the case where \(H\) is oriented along the easy magnetization direction, they found the domain walls speed of displacement to be

\[v \approx \frac{\mu_0 m_0^2}{\eta} H \sqrt{\left( \frac{\alpha - \gamma}{c} \right) a},\]

employing the notation of sec. 2 for the saturated regime \((a = \text{const. and } b = 0)\). If the magnetic field is constant in time, the velocity will be reduced to zero by the magnetic pressure of the bulk in a second moment, as the system approaches the equilibrium. On the other hand, if we orient the external magnetic field transversely to the easy magnetization axis, the domain walls will remain in equilibrium with the magnetization of the bulk. The application of external magnetic fields on domain walls may have important applications in the fabrication of switches for bulk spin polarized electrons in mesoscopic systems.

The scattering of an electron through a 180° domain wall barrier comprehends two cases of technological interest for the fabrication of switches. Consider for example one electron of spin up, aligned with the magnetization of the bulk, flowing in direction to a domain wall. If the time of flight of the electron through the wall is long enough, its spin will be adiabatically deflected and the scattered electron will be transmitted to the other side of the barrier with the spin flipped down, causing no additional cost of energy to the bulk. On the other hand, if the time of flight is too short, (considering that the electron is too fast or the barrier is too narrow) the spin will not have time enough to be deflected by the spiral of magnetization and the electron (in the classical picture) will be reflected back from the barrier, in order to save energy from the bulk.

As shown by Cabrera and Falicov [13], the domain walls may respond for a large change in the resistivity of metallic ferromagnets when the difference in the density of states between majority and minority spins at the Fermi surface is also large. Consider that we have two spherical Fermi surfaces of different sizes, for example, one for spin up electrons and
another for spin down ones. In the situation where the radius of one of the bulk Fermi surfaces is very small (minority spin) and the other very large (majority one), the majority spin electrons in one side of the barrier will find few channels to tunnel ballistically to the other side, where they would occupy the minority spin states. This way, if the wall is sufficiently narrow in comparison to the electron mean free path, the electrons will most probably be backscattered by the barrier. The change in the bulk resistivity predicted, however, is not so large as to induce giant and colossal magnetoresistances. In contrast, the magnetoresistance can grow several orders of magnitude when the width of the walls becomes small in comparison to its size in bulk, as it has been observed experimentally in magnetic nanojunctions.

A very interesting physics shows up in mesoscopic systems when a domain wall is geometrically constrained by a small constriction separating two ferromagnetic bulks of wider cross section. Bruno [14] derived the somehow remarkable result that when the constriction cross section is much smaller than the bulk one, the domain walls are practically independent on the specific characteristics of the material, including crystalline anisotropy and exchange stiffness, and depend only on the geometry of the constriction. The free energy of the system is in the form

$$f[\theta(x), \theta'(x)] = [cm_0^2(\theta')^2 + U(\theta)] S(x),$$

where $S(x)$ defines the geometry of the constriction and $U$ is the crystalline anisotropy term. Proceeding with the minimization, one finds through the usual Euler-Lagrange equation

$$\frac{d}{dx} \frac{\partial f}{\partial \theta'} - \frac{\partial f}{\partial \theta} = 0$$

that

$$\theta'' + \frac{\theta'}{S} - \frac{1}{2cm_0^2} \frac{\partial U}{\partial \theta} = 0.$$  

Bruno proposes that if the second term is much larger than the third and in addition if $S$ is integrable, meaning if $\int_{-\infty}^{\infty} S(x) dx$ is finite, then we may drop the crystalline contribution. Defining the wall width according to the criterion

$$\lambda = 4 \left[ \int_{-\infty}^{\infty} d\theta (\theta')^2 \right]^{-1} = 4 \left[ \int_{-\pi/2}^{\pi/2} d\theta \theta' \right]^{-1},$$

he finds the simple results:

$$\theta(x) = \pi \int_{-\infty}^{x} \frac{S^{-1}(u) du}{\int_{-\infty}^{\infty} S^{-1}(x) dx} - \frac{\pi}{2},$$

and

$$\lambda = \frac{4}{\pi^2} \left[ \int_{-\infty}^{\infty} dx S^{-2}(x) \right]^2 \frac{\left[ \int_{-\infty}^{\infty} dx S^{-1}(x) \right]^2}{\int_{-\infty}^{\infty} dx S^{-2}(x)}.$$
which are defined by the constriction only. In the most general case, where the crystalline term \( U \) is present, he finds for different models of constriction that the domain wall is still driven by \( S \), and not by \( U \) or the exchange stiffness \( c \).

When a magnetic field is applied along the easy magnetization axis, the domain wall inside the constriction suffers a finite displacement proportional to the applied field, where the new position of equilibrium is achieved thanks to the magnetostatic pressure played by the constriction geometry [15]. The localization of domain walls in nanojunctions are a theoretical possibility for the fabrication of fast switches demanded for spintronic devices.

In a wide sense, the word spintronics means manipulation and control of the spin degrees of freedom in condensed matter systems, and to use this knowledge to make useful devices [16]. Here, we focus on spin transport properties in ferromagnetic metals. We have already discussed the magnetoresistance effect (MR) due to the domain structure and the scattering of electrons from Bloch walls. The resistance of a ferromagnetic sample changes with the application of a small magnetic field. The field reorients the domains, and for a large enough value, it saturates the sample. The saturated configuration has a smaller resistance (negative magnetoresistance) than the one with domains, where the magnetization is not homogeneous (for a 180° wall, the magnetization of one domain is antiparallel to the magnetization of the adjacent domain) [13]. This simple system suggests the invention of “artificial” devices where one could monitor the magnetic configuration through the application of small magnetic fields. Based on the same physics, magnetic tunneling junctions (MTJ) [17] and metallic multilayers (MML) structures [18] were conceived, in order to operate as spin valves, due to the ‘giant’ magnetoresistance (GMR) effect displayed by those systems. In a typical MTJ, two ferromagnetic metallic electrodes are separated by a narrow nonmagnetic insulating layer. For MML, two or more ferromagnetic layers are separated by nonmagnetic metallic spacers. In both structures, one can pin the magnetization of one of the electrodes using an additional magnetic layer which is strongly exchanged-biased with the ferromagnet [19]. The other ferromagnetic electrode is left ‘free’ to orient its magnetization with small applied fields. One can then change the resistance of the device by manipulating the relative orientation of the magnetization of both electrodes, phenomenon that has been named as spin-valve effect and is currently used in commercial applications for magnetic recording [20, 21]. For technological applications, ideal devices should have a GMR as large as possible, obtained at room temperature with small applied fields. Experimental measurements of GMR in magnetic multilayers are usually performed for the current-in-plane (CIP) geometry, where the electric current flows parallel to the layers. For this setup, the resistance is fairly large and can be measured using standard techniques. The other possibility, much more complicated for experimental implementation, is to do the measurement with the current perpendicular to the layer plane, the so called CPP geometry (CPP stands for current-perpendicular-to-the-plane). In this case, the resistance is very low

\[1\] The MR effect is called GMR for values exceeding 10%, which are one order of magnitude larger than the typical values of the anisotropic magnetoresistance used commercially. MTJ prototypes currently produced nowadays present a GMR in the range 25%-30% at room temperature. In MML systems, the MR may be as large as 50% - 65% , but saturating fields are substantially higher (about 1 T in the original system of Ref. [18]).
and can only be measured using extremely sensitive techniques[22].

Domain walls are thought to play a relevant role in recent GMR experiments in magnetic nanocontacts. Large values of MR, of the order of 300%-3000% at room temperature and for fields of about 100 Oe, have been reported in the literature[23, 24]. The MR values found are attributed to strong electron scattering from narrow domain walls which are formed in the contact region. Due to the constricted geometry and the rapid variation of the magnetization across the domain wall, the electron spin cannot follow adiabatically the local magnetization, as it is the case in bulk ferromagnets[13].

4 Conclusion

We have developed an illustrative application of the Landau mean field theory for phase transitions in ferromagnetic bodies. The advantage of this procedure resides in the phenomenological nature of the free energy parameters, allowing us to calculate equations of state without stating a specific microscopic model. The classical Landau approach can then be extended to describe the formation of domains, with the corresponding change of the critical behavior, still within the mean field approximation. Despite this theory is in fact very crude to describe the neighborhood of the critical point for systems whose specific heat diverges at the phase transition [12], the theory is also known for making remarkable qualitative predictions, with deep insights on the physics of critical phenomena, even in situations that extrapolate its region of validity[25]. In particular, the theory provides its own criterion of failure and suggests new procedures to correctly describe the physics at the critical point, where fluctuations must be included[26].

We have shown that the problem of domains in ferromagnets is driven by the presence of two different length scales competing in the vicinity of the critical point. On one side, there is the bulk correlation length which drives the system to a classical mean field ferromagnetic behavior, and on the other, the length scale of the interfaces between domains. The competition between them produces two major effects: i) a new critical behavior, where the specific heat is strongly enhanced near $T_c$ from below in comparison to the bulk homogeneous limit, but is still small in comparison to the limit of surfaces, giving an intermediary critical exponent $\alpha_B < \alpha < \alpha_S$; and ii) a raise in the magnetoresistance in the ordered phase, specially for ferromagnets with a layered domain wall geometry. Despite the study of magnetic domains is a rather old subject, it has recently opened a very promising field in technology. The transport properties of ferromagnetic domains suggest several interesting applications in spintronic devices, which we have briefly discussed.

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References

[1] L. D. Landau and E. M. Lifshitz, *Phys. Z. Soviet Union*, 8, 153 (1935), reprinted in *Men of Physics: L. D. Landau* by D. ter Haar (Pergamon Press, Oxford, 1965), pp. 178-194.

[2] C. Herring and C. Kittel, *Phys. Rev.* 81(5), 869-880 (1951); C. Kittel and J. K. Galt, “Ferromagnetic Domains”, in *Solid State Physics*, Vol. 3, F. Seitz and D. Turnbull, Eds. (Academic Press, New York, 1956), pp. 439-564.

[3] A. Hubert, and R. Schafer, *Magnetic Domains: the analysis of magnetic microstructures* (Springer, Berlin, 1998).

[4] P. M. Chaikin and T. C. Lubenski, *Principles of condensed matter physics* (Cambridge University Press, Cambridge, 1997), chap. 4.

[5] J. C. Tolédano and P. Tolédano, *The Landau Theory of Phase Transitions* (World Scientific, Singapore, 1987).

[6] L. D. Landau and E. M. Lifshitz, *Electrodinamique des millieux continus* (MIR, Moscou, 1990), 2nd. ed., chap. 5.

[7] K. Huang, *Statistical Mechanics* (John Wiley and Sons, New York, 1987), 2nd. ed., chap. 17.

[8] S. Chikasumi, *Physics of ferromagnetism* (Oxford Press, New York, 1997), 2nd. ed., p. 430.

[9] T. Garel, and S. Doniach, *Phys. Rev. B* 26, 325 (1982).

[10] S. A. Brazovskii, *Sov. Phys. JETP* 41, 85 (1975).

[11] K. Binder, “Critical Behavior at Surfaces”, in *Phase Transitions and Critical Phenomena*, C. Domb and J. L. Lebowitz, Eds., Vol. 8 (Academic Press, London, 1983), pp. 1-144.

[12] H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Oxford University Press, New York and Oxford, 1971).

[13] G. G. Cabrera and L. M. Falicov, *Phys. Status Solidi B* 61, 539 (1974).

[14] P. Bruno, *Phys. Rev. Lett.* 83, 2425 (1999).

[15] N. Garcia, V. V. Osipov, and E. V. Ponizovskaya, *Phys. Rev. B* 64, 184412 (2001).

[16] I. Zutic, J. Fabian, and S. Das Sarma, *Rev. Mod. Phys.* 76, 323 (2004).
[17] W. J. Gallagher, S. S. P. Parkin, Yu Lu, X. P. Bian, A. Marley, K. P. Roche, R. A. Altman, S. A. Rishton, C. Jahnes, T. M. Shaw, and Gang Xiao, *J. Appl. Phys.* **81**, 3741 (1997); J. S. Moodera, L. R. Kinder, T. M. Wong, and R. Meservey, *Phys. Rev. Lett.* **74**, 3273 (1995); J. S. Moodera and L. R. Kinder, *J. Appl. Phys.* **79**, 4724 (1996); T. Miyazaki and N. Tezuka, *J. Magn. Magn. Mater.* **139**, L231 (1995).

[18] M. N. Baibich et al., *Phys. Rev. Lett.* **61**, 2472 (1988).

[19] J. Nogues and I. K. Schuller, *J. Mag. Mag. Mat.* **192**, 203 (1998).

[20] L. M. Falicov et al., *J. Mater. Res.* **5**, 1299 (1990).

[21] S. S. P. Parkin et al., *Proc. IEEE* **91**, 661 (2003).

[22] M. A. M. Gijs and G. E. W. Bauer, *Adv. Phys.* **46**, 285 (1997).

[23] N. Garcia, M. Muñoz, G.G. Qian, H. Rohrer, L. G. Saveliev, and Y-W Zhao, *Appl. Phys. Lett.* **79**, 4550 (2001).

[24] H. D. Chopra and S. Z. Hua, *Phys. Rev. B* **66**, 020403(R) (2002).

[25] Leo P. Kadanoff *et al.*, *Rev. Mod. Phys.* **39**(2), 395-431 (1967).

[26] J. Als-Nielsen and R. J. Birgenau, *Am. J. Phys.* **45**(6), 554-560 (1977).