\[ \Delta N \] formalism and conserved currents in cosmology

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Abstract. The \[ \Delta N \] formalism, based on the counting of the number of e-folds during inflation in different local patches of the Universe, has been introduced several years ago as a simple and physically intuitive approach to calculate (non-linear) curvature perturbations from inflation on large scales, without resorting to the full machinery of (higher-order) perturbation theory. Later on, it was claimed the equivalence with the results found by introducing a conserved fully non-linear current \[ \zeta_{\mu} \], thereby allowing to directly connect perturbations during inflation to late-Universe observables. We discuss some issues arising from the choice of the initial hyper-surface in the \[ \Delta N \] formalism. By using a novel exact expression for \[ \zeta_{\mu} \], valid for any barotropic fluid, we find that it is not in general related to the standard uniform density curvature perturbation \[ \zeta \]; such a result conflicts with the claimed equivalence with \[ \Delta N \] formalism. Moreover, a similar analysis is done for the proposed non-perturbative generalization \[ R_{\mu} \] of the comoving curvature perturbation \[ \mathcal{R} \].

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1 Introduction

Thanks to technologies development and satellite missions such as WMAP and Planck, an unprecedented level of precision [1] has been achieved in measurements of Cosmic Microwave Background (CMB) temperature anisotropies and polarization. The main properties of these fluctuations are well explained by the inflationary paradigm; however a strong model degeneracy persists, which will need to be discriminated by future generation experiments able to measure signatures of primordial non-Gaussianities [2–6], which are sensitive to the specific properties of the considered inflationary model (see, e.g. [7–9]). It is well known that the quest for non-Gaussian signatures requires the study of equations beyond the canonical first-order perturbative approximation, based on considering small metric fluctuations around the Friedmann-Lemaître-Robertson-Walker (hereafter FLRW) background. In principle, this perturbative approach is the most suitable one in order to give high precision phenomenological predictions, however, dealing with at least second-order perturbative equations can be really tricky. This difficulty led a non-negligible part of the scientific community to search for alternative methods which are able to overcome the difficulty of manipulating higher-order equations [10–12]. In this note we will focus on a reanalysis of the so-called $\Delta N$ formalism [13–18], pointing out some issues in its implementation. Usually, in the $\Delta N$ formalism the metric is taken in the ADM decomposition performing a gradient expansion instead of the conventional perturbative one based on small deviations from a homogenous background metric. The key quantity is the local number of e-folds $N$, that can be defined as the integral of the expansion of a velocity field defined in terms of an initial and final hyper-surface $S$. A certain level of ambiguity exists on the choice of $S$ [14, 18–20] and the role of the initial hyper-surface $S_0$ has been often overlooked. An alternative approach to the $\Delta N$ formalism was based on the current $\zeta_\mu$, proposed some years ago by Langlois and Vernizzi [21–25]
as a suitable non-perturbative generalization of the scalar quantity $\zeta$.\footnote{Notice that such an approach is different from the formalism reviewed in length in the textbook [26].} The link between these two approaches is the local number of e-folds which enters in both the definition of the $\Delta N$ formula and the current $\zeta_\mu$. The equivalence of $\zeta_\mu$ and $\Delta N$ formalisms was claimed in [19, 27]. Although a close relation between $\zeta_\mu$ and $\Delta N$, because of the role of $N$, is not surprising, a full equivalence of these two approaches is far from being trivial. Indeed, while $\zeta$ of the standard $\Delta N$ formula is only conserved at super-horizon scales, it has a nontrivial sub-horizon dynamics. On the contrary, $\zeta_\mu$ is conserved at all scales, in the sense that its Lie derivative along the flow is exactly zero in the adiabatic case. Therefore an equivalence between the two approaches is hardly achievable.

The outline of the paper is as follows. In section 2 the local number of e-folds $N$ is defined and computed up to first order in perturbation theory. Section 3 is devoted to the study of the currents $\zeta_\mu$ and $R_\mu$ introduced in [21], reconsidering their relation with the standard curvature perturbation of constant density hyper-surfaces $\zeta$ and the comoving curvature perturbation $R$. In section 4, the influence of the choice of the class of hyper-surfaces on the $\Delta N$ formula and the relation with the current $\zeta_\mu$ is reanalysed. Finally, in section 5 we consider in detail as an example a scalar field. Our main conclusions are drawn in section 6.

\section{The local number of e-folds}

One of the main physical quantities crucial for the $\Delta N$ formula, is the so-called local number of e-folds $\mathcal{N}(\eta, x^i)$, which generalizes the number of e-folds in a de Sitter spacetime. We consider a perturbed FLRW universe with metric $g_{\mu\nu}$ in the presence of a perfect fluid, with 4-velocity $u^\mu$; focusing on scalar modes only, at linear order in perturbation theory we have

$$
g_{00} = -a^2(1 + 2A), \quad g_{0i} = a^2 \partial_i B, \quad g_{ij} = a^2 \gamma_{ij} (1 - 2\psi) + 2a^2 \partial_i \partial_j E, \quad u_\mu = a(1 - (1 + A), \partial_i v) ; \quad (2.1)$$

where latin indices run from 1 to 3 and are raised/lowered with the unperturbed 3-metric $\gamma_{ij}$, while greek indices are used to describe space-time coordinates. Notice that the time coordinate $x^0 \equiv \eta$ is the background conformal time and the perturbed metric is given in a generic gauge. In general, by a suitable choice of gauge, only three out of the five scalars correspond to physical degrees of freedom. An important physical quantity, often used in this paper is the volume expansion scalar, defined as the 4-divergence of the 4-velocity:

$$\theta = \nabla^\mu u_\mu = 3 \mathcal{N} ; \quad (2.2)$$

where, in general, for any scalar quantity $f$ we define $f = u_\mu \nabla^\mu f = \frac{df}{d\tau}$, where $\tau$ is the fluid’s proper time. The local number of e-folds $\mathcal{N}$ can be defined by integrating the expansion $\theta$ of the fluid velocity along its world-lines; namely

$$\mathcal{N}(\eta, x^i) = \frac{1}{3} \int_{S_0}^{S} \theta \, d\tau = -\frac{1}{3} \int_{S_0}^{S} \frac{d\rho}{(\rho + p)} ; \quad (2.3)$$

the congruence of $u^\mu$ is supposed to pierce the hyper-surfaces $S_0$ and $S$ only once. Finally, in the last relation we have used the energy-momentum tensor (EMT) conservation of the fluid

$$u^\nu \nabla^\mu T_{\mu\nu} = -\dot{\rho} - \theta (\rho + p) = 0 . \quad (2.4)$$
Figure 1. The congruence of $u^\mu$ which intersects two generic space-like initial and final hyper-surfaces. Once the vector field is fixed, the point $x^\mu$ on the final hyper-surface is uniquely determined by the initial point $x^\mu_0$ on the initial hyper-surface.

Being the final point of the world-line chosen to be $x^\mu = (\eta, x^i)$ and the congruence fixed, the point $x^\mu_0 \in S_0$ is uniquely determined by tracing back the world-line until it intersects $S_0$, as shown in figure 1. In general, changing $x^i$ is equivalent to changing $x^\mu_0$ and the world-line which crosses $S_0$.

At the linear order, see appendix B.1, one gets

$$N = \int_{\text{wl}} d\eta' a \left[ 1 + A + \cdots \right] \left[ \bar{\theta} + \theta^{(1)} + \cdots \right]$$

$$= \ln \left( \frac{a(\eta)}{a(\bar{\eta}_0)} \right) + \frac{1}{3} \nabla^2 \left( E(\eta, x^i) + \int_{\eta_0}^{\eta} d\eta' \left( v - B \right) \right) - \psi(\eta, x^i)$$

(2.5)

where $\bar{\theta} = 3a'/a$ is the background value of $\theta$. We stress that there is a one-to-one relationship between the final point $x^\mu = (\eta, x^i)$ in $S$ and $x^\mu_0 \in S_0$. It is natural to impose that the initial and final space-like hyper-surfaces are homogenous at the background level. If the hyper-surface is defined as $f = \text{const.}$, where $f$ is a 4-dimensional scalar function, the following relation holds

$$\partial_\beta (f|_{S_0}) = \partial_\mu_0 (f|_{S_0}) \partial_\beta x^\mu_0,$$

(2.6)

and according to our assumptions $f = f(\eta) + f^{(1)}(\eta, x^i)$ and we have in perturbation theory

$$\partial_i (\eta|_{S_0}) = - \left( \frac{\partial_i f^{(1)}}{f'} \right) |_0 + O(2), \quad \partial_\eta (\eta|_{S_0}) = O(2),$$

(2.7)

where with 0 we denote a generic point of $S_0$. Therefore

$$\eta_0 = \bar{\eta}_0 - \left( \frac{f^{(1)}}{f'} \right) |_0.$$  

(2.8)

Notice that in the seemingly background term $\ln \left( \frac{a(\eta)}{a(\bar{\eta}_0)} \right)$ a non-trivial spatial dependence due to $\eta_0$, which can be further expanded to give

$$N = \ln \left( \frac{a(\eta)}{a(\bar{\eta}_0)} \right) + \mathcal{H}(\bar{\eta}_0) \frac{f^{(1)}}{f'} |_0 + \frac{1}{3} \nabla^2 \left( E(\eta, x^i) + \int_{\eta_0}^{\eta} d\eta' \left( v - B \right) \right)$$

$$- \psi(\eta, x^i) + \left( \psi - \frac{1}{3} \nabla^2 E \right) |_0.$$  

(2.9)
is present. By construction $N$ transforms as the perturbation of a spacetime scalar function
with a time dependent background; namely
\[ N^{(1)} \rightarrow N_0^{(1)} - \delta x^0 \partial_\eta N, \quad \bar{N} = \ln \left( \frac{a(\bar{\eta})}{a(\bar{\eta}_0)} \right). \] (2.10)

Moreover, the two hyper-surfaces needed in the definition of $N$ are defined in terms of a spacetime scalar function $f = \text{const}$. For instance, in the case $S_0$ corresponds to a constant-proper-time of the fluid, the initial hyper-surface will be of the form $\eta_0 = \bar{\eta}_0 + v(\bar{\eta}_0, x)$, see appendix A, where $\bar{\eta}_0$ is the arbitrary value of the conformal time in the gauge where the spatial velocity of the fluid $v$ is set to zero.

\section{Non-perturbative currents}

In this section we will re-examine the non-perturbative currents $\zeta_\mu$ and $R_\mu$ introduced in [21] by Langlois and Vernizzi and the relation with the constant-$\rho$ curvature perturbation [15, 28] (for a review see for instance [29])

\[ \zeta = -\psi + \frac{\rho^{(1)}}{3(\rho + p)}, \] (3.1)
and the gauge-invariant comoving curvature perturbation [30, 31]

\[ R = -\psi + H v. \] (3.2)

See also [32] for an alternative definition of vectors related to $\zeta$ or $R$ at the non-linear level and their compatibility with the $\Delta N$ formalism, after a suitable initial hyper-surface choice [33, 34].

Given a fluid with four-velocity $u^\mu$, $\zeta_\mu$ is defined as

\[ \zeta_\mu = \partial_\mu N - \frac{\dot{N}}{\rho} \partial_\mu \rho, \] (3.3)

where $N$ is precisely the local number of e-folds computed between two generic hyper-surfaces, as discussed in the previous section. The quantity $\zeta_\mu$ is fully non-perturbatively defined and, in the case of an adiabatic fluid, one can show that [21] it does not change along the fluid lines, in other words, its Lie derivative along $u^\mu$ vanishes

\[ \mathcal{L}_u \zeta_\mu = 0. \] (3.4)

The previous relation can be considered as a non-perturbative conservation law for $\zeta_\mu$ and is valid at any scale. In the case of a barotropic and irrotational perfect fluid $\zeta_\mu$ can be computed exactly, showing that it depends only on the choice of $S_0$. By using the $\theta$ definition and eq. (2.4) one gets

\[ \zeta_\mu = -\frac{1}{3} \partial_\mu \int_{\tau_0}^{\tau} d\tau' \frac{\dot{\rho}}{(\rho + p)} + \frac{1}{3} \frac{\partial_\mu \rho}{(\rho + p)}; \] (3.5)

or alternatively, by introducing the 1-form $\chi$

\[ \chi = \chi_\mu dx^\mu = \frac{\partial_\mu \rho}{(\rho + p)} dx^\mu, \] (3.6)
as a 1-form \( \zeta \) whose components are given by
\[
\zeta_\mu = \frac{1}{3} \left( \chi_\mu - \partial_\mu \int \chi_\nu dx^\nu \right). \tag{3.7}
\]
For a barotropic fluid, for which \( p = p(\rho) \), the 1-form \( \chi \) is closed, namely
\[
d\chi = -\frac{d\rho \wedge dp}{(\rho + p)^2} = 0; \tag{3.8}
\]
equivalently, in components, \( \partial_\mu \chi_\nu \equiv 0 \). By using the Poincaré lemma, one can find, at least locally, a function \( \beta \) such that \( \chi_\mu = \partial_\mu \beta \). One can get
\[
\beta(\rho) = \int^\rho \frac{dx}{x + p(x)} , \quad d\beta = \chi. \tag{3.9}
\]
Thus, neglecting any topological complication, for a barotropic fluid we can compute \( \zeta_\mu \), exactly arriving at the following simple expression
\[
\zeta_\mu = \frac{1}{3} [\partial_\mu \beta - \partial_\mu (\beta - \beta_0)] = \frac{\partial_\mu \beta_0}{3} = \frac{\partial_\mu \rho(0)}{3(\rho + p)|_0}, \tag{3.10}
\]
where \( 0 \) indicates that the relevant quantity is evaluated on the initial 3-surface \( S_0 \). One can easily see that the Lie derivative along \( u \) of eq. (3.10) is given by
\[
\mathcal{L}_u \zeta = d(u^\mu \partial_\mu \beta_0) = 0, \tag{3.11}
\]
being, by definition, \( \beta_0 \) evaluated on \( S_0 \); the above result is in agreement with (3.4). As a result, in the case of a barotropic fluid, (3.10) shows that \( \zeta \) depends exclusively on the initial hyper-surface \( S_0 \) and in this sense it is trivial as a dynamical quantity. An alternative interpretation of (3.10) is that for a barotropic fluid, when the local number of e-folds is computed on a constant \( \rho \) hyper-surface, then \( \zeta \equiv 0 \); such a result is non-perturbative.

A non-trivial dynamics is reintroduced when the fluid is non-barotropic, namely when
\[
\Gamma_\mu = \partial_\mu p - c^2 s \partial_\mu \rho \neq 0, \quad c^2 s = \frac{\dot{p}}{\dot{\rho}}. \tag{3.12}
\]
In this case \( \zeta_\mu \) not only depends on the final hyper-surface but also on the world-line.

This exact result (3.10) can be used as an alternative starting point for a perturbative expansion. We note that, for any function \( f \), say \( \rho \), defined in \( S_0 \) where \( \tau = \tau_0 \) we have
\[
\partial_\mu \rho(\eta_0, x_0^i) = \frac{\partial \rho(x_0)}{\partial x_0^\mu} \frac{\partial x_0^\nu}{\partial x^\mu} = \frac{d\rho(\eta_0)}{d\eta_0} \frac{\partial x^\mu}{\partial x_0^\mu} + \frac{\partial \rho^{(1)}(0)}{\partial x_0^\mu} \frac{\partial x_0^\mu}{\partial x^\mu}; \tag{3.13}
\]
in \( S_0 \) the conformal time will be a function of \( x^i \). Expanding, we have \( x^i = x_0^i + O(1) \) and \( \frac{\partial x_0^i}{\partial \eta} \approx \frac{\partial x^i}{\partial \eta} = O(1) \). Furthermore, using the relations \( \eta^{(1)}_0 = -\frac{F^{(1)}}{f} (\bar{\eta}_0, x^i) \) and \( \partial_\eta \eta_0 = O(2) \) (see section 2), we find for the first-order expansion \( \zeta^{(1)}_\mu \) of \( \zeta_\mu \)
\[
\zeta_0^{(1)} = 0, \quad \zeta_i^{(1)} = \partial_i \zeta_s^{(1)}, \quad \zeta_s^{(1)} = \frac{\mathcal{H}(\eta_0)}{f'(\eta_0)} f^{(1)} (\bar{\eta}_0, x^i) - \frac{\mathcal{H}(\eta_0)}{\rho'(\eta_0)} \rho^{(1)}(\bar{\eta}_0, x^i). \tag{3.14}
\]
From (3.14) and (3.11) it is clear that, for a barotropic fluid, \(\zeta^{(1)}_s\) is defined on the initial hyper-surface; any dependence on the final hyper-surface cancels out and thus it is not related to \(\zeta\) [21] and to the \(\Delta N\) formalism [19, 27]. Such argument is valid at any scale. Notice that \(\zeta^{(1)}_s\) is gauge invariant.

Finally, as an additional check, \(\zeta^{(1)}_\mu\) can be computed perturbatively starting from its definition (3.3). By using the results of appendix B.1, at the linear order and for a generic perfect fluid, we have that

\[
\begin{align*}
\zeta^{(1)}_0 &= 0, \\
\zeta^{(1)}_i &= \partial_i \zeta^{(1)}_s, \\
\zeta^{(1)}_s &= \zeta + \frac{1}{3} \left( \nabla^2 E + \int_{w} d\eta \nabla^2 (v - B) \right) + \left[ \psi + H \frac{f^{(1)}}{f'} - \frac{1}{3} \nabla^2 E \right] \big|_{(\eta_0, x^i)}. 
\end{align*}
\]  

(3.15)  

(3.16)

Where we suppose to find a scalar function \(\hat{\zeta}^{(1)}_s\) such that \(\zeta^{(1)}_i \equiv \partial_i \hat{\zeta}^{(1)}_s\), also in the non-adiabatic case. From the standard relation

\[
\zeta' = -\frac{1}{3} \nabla^2 (E' + v - B) - \frac{\mathcal{H}}{\bar{\rho} + \bar{p}} \Gamma^{(1)},
\]

(3.17)

we can write (3.16) as

\[
\begin{align*}
\hat{\zeta}^{(1)}_s &= \zeta - \int_{\eta_0}^{\eta} d\eta' \left[ \zeta' + \frac{\mathcal{H}}{\bar{\rho} + \bar{p}} \Gamma^{(1)} \right] + \psi (\eta_0, x^i) + H \frac{f^{(1)}}{f'} (\eta_0, x^i) \\
&= \zeta (\eta_0, x^i) + \psi (\eta_0, x^i) + H \frac{f^{(1)}}{f'} (\eta_0, x^i) - \int_{\eta_0}^{\eta} d\eta' \frac{\mathcal{H}}{\bar{\rho} + \bar{p}} \Gamma^{(1)}. 
\end{align*}
\]

(3.18)

Let us stress that \(\hat{\zeta}^{(1)}_s\) is a gauge invariant quantity which depends only on the choice of the initial and final hyper-surface. If \(\Gamma^{(1)} = 0\) (barotropic case), \(\hat{\zeta}^{(1)}_s \equiv \zeta^{(1)}_s\) is constant in time at all scales and coincides with its value at \(\eta = \eta_0\), namely only on the choice of the initial hyper-surface, confirming the result (3.14) based on the exact expression (3.10). It is worth to point out that, in order to get (3.18), the somehow hidden dependence on \(x^i\) of the \(\eta_0(x)\), parametrizing the initial hyper-surface, gives the term of the form \(H \frac{f^{(1)}}{f'}\). The lesson is that \(\zeta_\mu\) as dynamical quantity is trivial in the barotropic case. Moreover, if the hyper-surface \(f\) is taken to be a uniform density hyper-surface, namely \(f = \rho\), then \(\zeta^{(1)}_\mu = 0\). Summarizing

- \(\zeta^{(1)}_i \neq \partial_i \zeta\);
- For an adiabatic fluid, \(\zeta^{(1)}_i\) is conserved at all scales at the first order in perturbation theory and does not depend on the choice of the final hyper-surface.
- \(\zeta^{(1)}_\mu = 0\) when a uniform density hyper-surface is considered.
- \(\zeta^{(1)}_i = \partial_i \zeta (\eta_0, x^i)\) when the initial hyper-surface is that \(f = \eta\) and \(\psi = 0\), namely a flat slice is considered.

In appendix B.2 we have computed \(\zeta_\mu\) at second order in perturbation theory, starting from eq. (3.10), verifying that (3.4) holds.
Besides $\zeta_\mu$, it is possible to define another quantity, $\mathcal{R}_\mu$, related to the curvature of comoving hyper-surfaces, defined as [21]

$$\mathcal{R}_\mu = h^\nu_{\mu} \partial_\nu \mathcal{N} = \zeta_\mu - \frac{D_\mu \rho}{3(\rho + p)}, \quad D_\mu \rho = h^\nu_{\mu} \partial_\nu \rho; \quad (3.19)$$

where $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is the projector orthogonal to $u_\mu$. In the barotropic fluid case, while $\zeta_\mu$ is exactly conserved at all scales, this is not the case for $\mathcal{R}_\mu$. Interestingly, the difference between $\mathcal{R}_\mu$ and $\zeta_\mu$ can be written as

$$\mathcal{R}_\mu - \zeta_\mu = \frac{\theta}{3\bar{\rho}} (\partial_\mu \rho + \bar{\rho} u_\mu) \equiv \frac{\theta}{3\bar{\rho}} D_\mu \rho, \quad (3.20)$$

where $D_\mu \rho$ can be read as a covariant and non-linear generalization of the comoving density perturbation. As before, it is also possible to find a perturbative expansion for $\mathcal{R}_\mu$. The first non-trivial order is the linear one, for which by perturbing (3.20) and by using (3.14), one gets

$$\mathcal{R}_0 = \bar{\mathcal{N}}' + \frac{1}{3} \bar{\theta} \bar{u}_0 + \mathcal{N}^{(1)} + \frac{\bar{u}_0 \theta^{(1)}}{3} + \frac{u_0^{(1)}}{3} = O(2) \quad (3.21)$$

$$\mathcal{R}_i = \partial_i \mathcal{R}_s, \quad \mathcal{R}_s = \zeta_s - \int_{\eta_0}^{\eta} \frac{\mathcal{H}}{\bar{\rho} + \bar{p}} \Gamma^{(1)} d\eta + \frac{\mathcal{H}}{\bar{\rho}} \Delta \rho = \bar{\zeta}_s + \frac{\mathcal{H}}{\bar{\rho}} \Delta \rho, \quad (3.22)$$

where $\Delta \rho = \bar{\rho}' v + \rho^{(1)}$ is the so-called comoving-gauge density perturbation.

Moreover, by the above analysis it is clear that when the $\mathcal{R}$ modes are not conserved, being $\Delta \rho \propto \mathcal{R} - \zeta$, the difference $\mathcal{R}_\mu - \zeta_\mu$ can be used as a tool for the study of the Weinberg theorem [35] at the non-perturbative level even for a fluid; see, for a recent discussion [36–40]. We leave the study of such violations for future work.

Hereafter, let us give a concrete and natural example for the choice of $S_0$, taking a constant-proper-time hyper-surface. In this case we can replace $-\frac{\Gamma^{(1)}}{T}$ with the scalar velocity $v$ (see appendix A)

$$\zeta_s^{(1)} = -\frac{\mathcal{H}(\eta_0)}{\bar{\rho}^{(1)}(\eta_0)} \left[ \bar{\rho}'(\eta_0) v(\bar{\eta}_0, x^i) + \rho^{(1)}(\bar{\eta}_0, x^i) \right]$$

$$\equiv -\frac{\mathcal{H}(\eta_0)}{\bar{\rho}^{(1)}(\eta_0)} \Delta \rho(\bar{\eta}_0, x^i), \quad (3.23)$$

Note that, in the Fourier space, when a constant-proper-time initial hyper-surface is used, for super-horizon modes, in the adiabatic case

$$\mathcal{R}_s, \zeta_s \rightarrow 0 \quad \text{super-horizon}; \quad (3.24)$$

where we have used that from the perturbed Einstein equations $\Delta \rho \sim k^2 \psi$. The results are different from [21].

In conclusion, the currents $\zeta_\mu$ and $\mathcal{R}_\mu$ are not in general directly related to $\zeta$ or $\mathcal{R}$, which are conserved in the super-horizon limit for an adiabatic fluid.
As we have seen in the previous section the integral of the expansion \( \theta \), at small momenta (large distances) involves the gravitational potential \( \psi \) of the perturbed metric (2.1); the \( \Delta N \) formula exploits this integral by computing how the local number of e-folds changes moving along two different hyper-surfaces. According to the Separate Universe approach [14, 15], perturbation theory can be formulated as a derivative expansion and terms with more than one spatial derivative can be neglected at large distances. Following [13, 14, 41–43], by choosing a suitable initial and final hyper-surfaces \( S_0 \) and \( S \), one can isolate the perturbation mode \( \psi \), which is proportional to the curvature of the \( \rho = \text{constant} \) hyper-surface and is conserved on super-horizon scales. A number of recipes for the choice of \( S \) have been proposed in the literature. For instance, in [14], one first computes \( N_A \) taking \( S_0 \) to be a flat constant-conformal-time hyper-surface, while \( S \) is a slicing with \( \rho^{(1)} = 0 \) hyper-surface, then the same quantity, \( N_B \), is computed, taking both \( S_0 \) and \( S \) as flat constant-conformal-time hyper-surfaces; finally the \( \Delta N \) formula is defined as \( N_A - N_B \); notice that the additional hypothesis that the two initial hyper-surfaces are tangent in the point of interest \( x^i \). From our general expression (2.9), the role of \( N_B \) is that of subtracting the background value \( \bar{N} \) to single out \( \psi \). By eq. (2.9), we get

\[
N_A = \bar{N} - \psi(\eta, x^i) \quad \Rightarrow \quad \Delta N = -\psi(\eta, x^i). \tag{4.1}
\]

Alternatively, according to [18], one should define \( \Delta N \) as the difference of \( N_A \), computed by using a flat conformal hyper-surfaces and \( N_B \) computed by gauge transforming \( N_A \) from the flat to the uniform density gauge, but only on the final hyper-surface. The result is the same of eq. (4.1). On the other hand, \( \mathcal{N} \), obtained by taking two constant-conformal-time hyper-surfaces and neglecting spatial derivatives (Separate Universe assumption), is given by (see eq. (2.9))

\[
\mathcal{N} = \bar{N} - \psi(\eta, x^i) + \psi(\bar{\eta}_0, x^i), \tag{4.2}
\]

and typically the contribution of \( \psi(\bar{\eta}_0, x^i) \) is neglected. Notice however that such a term is important; indeed, in the gauge \( \rho^{(1)} = 0 \), the energy-momentum conservation leads to \( \psi' = 0 \) and then \( \Delta N = 0 \). Other definitions can be found, see for instance [19, 20].

As a final comment we point out that, although eq. (2.9) can be extended to hyper-surfaces \( \eta = \text{const.} \), by setting \( f^{(1)}(\eta) = 0 \), such a choice is ambiguous, being the surface defined by using the unperturbed coordinated time \( \eta \) of a perturbed universe and \( \psi \) is in an unspecified gauge. The choice of \( \eta = \text{const.} \) in a perturbed universe does not identify uniquely the metric perturbations; indeed, by an infinitesimal change of coordinates the metric takes a physically equivalent form, leaving two scalars to be gauge fixed.

Let us now compare \( \zeta_\mu \) with \( \Delta N \). In [19, 27] the former was claimed to be equivalent to the \( \Delta N \) formalism. The starting point is the relation \( \zeta_\mu^{(1)} = (0, \partial_i \zeta) \), which we have shown that is not correct. According to their reasoning, by using the same hyper-surfaces for the computation of \( \Delta N \), they find

\[
\zeta_i = \partial_i (\psi(\eta, x) - \psi(\bar{\eta}_0, x)) \equiv \partial_i \Delta N. \tag{4.3}
\]

However, this result is based on the results presented in [21] obtained without taking into account the subtle issues previously analyzed, and it is not coherent with eq. (3.14) or with the general extension for non-barotropic fluids, eq. (B.6) (see appendix B.1).
Indeed, from the above analysis it is clear that \( \zeta_\mu \), and the related quantity \( \zeta_s^{(1)} \) are conceptually rather different from the constant-\( \rho \) curvature perturbation \( \zeta \). Indeed, in the barotropic case, \( \zeta \) is a quantity that depends on the initial hyper-surface only as we have shown in eq. (3.10), while on the contrary, by construction \( \Delta N \) is sensitive to the final hyper-surface, see eq. (4.1). At the linear order in perturbation theory this shows up from the fact that \( \zeta_s^{(1)} \) is a function of the spatial coordinates only, before the super-horizon limit is taken and thus no genuine sub-horizon dynamics is present, in sharp contrast with \( \zeta \). A rather formal comparison can be made by choosing the initial hyper-surface for the computation of \( \zeta_\mu \) to be the same as the one used for the \( \Delta N \) computation; namely we set in (3.14) \( f^{(1)} = 0 \) and take \( \psi(\eta_0, x^i) = 0 \), thus we get
\[
\zeta_s^{(1)} = \zeta(\eta_0, x^i) \quad \eta = \eta_0 \text{ and flat}. \tag{4.4}
\]
On the other hand, for the final uniform density hyper-surface in (4.1) we have that \( \Delta N = -\psi(\eta, x^i) \equiv \zeta(\eta, x^i) \). Thus, if we are interested in super-horizon scales only, being \( \zeta \) conserved, we get the somehow accidental relation
\[
\zeta_s^{(1)} = \Delta N. \tag{4.5}
\]
In spite of the previous relation, the two objects are intrinsically different. Taking a standard scenario with adiabatic initial conditions, \( \zeta_\mu \) is completely determined at all scales by its initial value when inflation starts. On the contrary, \( \Delta N \) has a non-trivial dynamics and is constant only at the zero order of the gradient expansion \( (k \to 0) \).

We point out that the relation \( \zeta_i = \partial_i \zeta_s^{(1)} \) can be extended beyond perturbation theory thanks to the non-perturbative nature of (3.10); indeed for a barotropic fluid, choosing in addition a set of adapted coordinates such that \( u^\mu = (u^0, 0) \) (comoving threading) we have that
\[
\zeta_i = \frac{1}{3} \frac{\partial_i \rho}{(\rho + p)}|_{(\eta_0, x^i)} = \frac{1}{3} \partial_i \left( \frac{\int_{\rho(\eta_0, x^i)} \rho(\eta_0, x^i) dx}{\rho(\eta_0, x^i)} \right), \tag{4.6}
\]
which is completely non linear and time independent at all scales, in sharp contrast with the \( \Delta N \) formalism. In general, (4.6) does not hold for a generic choice of threadings where \( x^i_0 \neq x^i \).

## 5 Scalar field

As an explicit example let us consider a scalar field. One should first emphasize that a scalar field and a perfect fluid are inequivalent when thermodynamics is taken into account, unless a shift symmetry is present [44]. For a real scalar field with a canonical kinetic term and a potential \( V(\phi) \), we have
\[
\rho = K + V, \quad p = K - V, \tag{5.1}
\]
\[
K = -\frac{1}{2} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi;
\]
the velocity is
\[
u_\mu = -\frac{\partial_\mu \phi}{\sqrt{-g^{\alpha \beta} \partial_\alpha \phi \partial_\beta \phi}}. \tag{5.2}
\]
Of course, in general the relation between $p$ and $\rho$ will be non-barotropic, indeed $p = \rho - 2V$; as a result, the quantity $\Gamma_\mu$, see (3.12), which measures that effect will be non-zero and given by [22]

$$\Gamma_\mu = \left(1 - \frac{\dot{\rho}}{\rho}\right) D_\mu \rho - 2D_\mu V = 2V_\phi \frac{\dot{\phi}}{\rho} D_\mu \rho; \tag{5.3}$$

where we have used that $D_\mu \phi = D_\mu V = 0$; in particular, setting $\partial K/\partial (\partial_\mu \phi) = K^\mu$, we arrive at the following expression

$$\dot{\rho} = K^\mu \mathcal{L}_\mu \rho + V_\phi \dot{\phi}. \tag{5.4}$$

Expanding at linear order around a homogeneous cosmological background for which $\dot{\phi} = \ddot{\phi}(t) + \phi^{(1)} + \cdots$, we obtain for the linear perturbation of $\Gamma_\mu$

$$\Gamma_\mu^{(1)} = 0; \quad \Gamma_i^{(1)} = \partial_i \Gamma^{(1)}, \quad \Gamma^{(1)} = 2V_\phi \frac{\ddot{\phi}}{\rho} \Delta \rho^{(1)} = -2V_\phi \left[-A \dddot{\phi} + \phi^{(1)} + \phi^{(1)} \left(3 \mathcal{H} + \frac{3H^2}{a^2} + \ddot{V}_\phi\right)\right]. \tag{5.5}$$

In particular

$$\Delta \rho = -A \frac{\ddot{\phi}^2}{a^2} + \phi^{(1)} \frac{\dddot{\phi}}{a^2} + \phi^{(1)} \left(3 \mathcal{H} + \frac{3H^2}{a^2} + \ddot{V}_\phi\right) \tag{5.6}$$

For what concerns $R_s$, it is given by

$$R_s = \zeta_s - 2 \int_{t_0}^t d\eta \frac{\mathcal{H} \ddot{\phi} \dot{V}_\phi}{(\dot{\rho} + \ddot{\rho})} \Delta \rho + \mathcal{H} \frac{\dot{\rho}^2}{\rho} \Delta \rho. \tag{5.7}$$

Finally, remember that $\zeta_s$ is given by

$$\zeta_s = \mathcal{H} f^{(1)} - \frac{\mathcal{H}}{\dot{\rho}} \rho_s^{(1)} \bigg|_{(\eta_0, x^i)} , \tag{5.8}$$

with $\rho_s^{(1)} = \Delta \rho - \rho' v^{(1)} = \Delta \rho + \rho' \frac{\dot{\rho}^{(1)}}{a^2}$. Our result differs from the ones found in [22]. Notice that on super-horizon scales

$$\zeta = R = -\psi - \frac{\mathcal{H}}{\ddot{\phi}} \phi^{(1)}, \tag{5.9}$$

therefore as shown by this simple example, is quite evident that there is no direct correlation between the currents and the standard curvature perturbations $\zeta$ and $R$.

6 Conclusions

In [21] a generalization of the curvature perturbation of the constant $\rho$ hyper-surfaces $\zeta$ was proposed, based on a non-perturbative approach. We have computed $\zeta_\mu$ at the full non-perturbative level in the case of a barotropic fluid, showing that the relation with the standard quantity $\zeta$ is non-trivial. By matching the expansion of our non-perturbative expression for $\zeta_\mu$, we have found that, although at the linear level, $\zeta_i^{(1)} = \partial_i \zeta_s^{(1)}$, $\zeta_s^{(1)} \neq \zeta$. In particular, while $\zeta$ is time-independent only on super-horizon scales, the time derivative of $\zeta_s^{(1)}$ vanishes identically. That $\zeta_s^{(1)} = 0$ at all scales can also be deduced by expanding $\mathcal{L}_\mu \zeta_\mu = 0$. These facts profoundly change the physical interpretation of the $\zeta_\mu$ 4-vector, which is conserved on all scales and cannot be compared with the gradient of the curvature perturbation $\zeta$, which is instead
conserved only on large super-horizon scales. Similar considerations apply to second order: while \( \zeta^{(1)}_s \) is a genuine gauge-invariant quantity likewise \( \zeta \), this is not the case for \( \zeta^{(3)}_s \). We have also studied the non-perturbative generalization \( R_\mu \) of the comoving curvature perturbation \( R \) proposed in [21]. While at leading non-trivial order \( \zeta_\mu \) has no dynamics, \( R_\mu = \partial_\mu R_s \) is a genuine dynamical quantity. However from our analysis it follows that \( R_\mu - \zeta_\mu \) can be used as a tool for studying the violation of the Weinberg theorem.

We have also critically reconsidered the \( \Delta N \) formalism and its relation with the covariant vector \( \zeta_\mu \) proposed by Langlois and Vernizzi. Concerning the \( \Delta N \) formalism, we have clarified some ambiguities arising from the choice of the initial and final space-like hyper-surfaces \( S_0 \) and \( S \), respectively. Using the prescription defined in [14], we recover the standard result \( \Delta N = -\psi(\eta, x^i) \), which coincides with \( \zeta(\eta, x^i) \) under the condition \( \rho^{(1)} = 0 \). Elaborating on the result in [21], where \( \zeta_i \) was claimed to reduce to the spatial gradient of \( \zeta \) in the barotropic case, it was put forward the equivalence between the current \( \zeta_\mu \) and the \( \Delta N \) formalism [19, 27] once the same prescription of spacetime slicing and threading is applied. According to the our result, bases on both perturbation theory and on the novel exact expression for \( \zeta_\mu \) in the barotropic case, a number of issues exist.

- The current \( \zeta^{(1)}_\mu \) depends only on the initial hyper-surface, on the contrary, in the \( \Delta N \) formalism the role of such a surface is to avoid any initial contribution,
- By using a comoving threading and flat initial hyper-surface, \( \zeta^{(1)}_s \) accidentally reduces to \( \zeta(\eta_0, x^i) \), while \( \zeta_i \) is exactly time independent at all scales.

Hence, we conclude that the \( \Delta N \) formalism, which represents a genuine dynamical quantity which reduces to \( \zeta \) only on super-horizon scales, cannot be fully equivalent to \( \zeta_\mu \).

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A Parametrization of the constant-proper-time hyper-surfaces

Given an irrotational fluid with four-velocity \( u^\mu \), consider the constant-proper-time \( \tau \) hyper-surface \( S \), with normal vector \( u_\mu \propto \partial_\mu \tau \). Suppose that \( \tau \) is a differentiable function, that in a given point \( (\bar{\eta}, \bar{x}^i) \) gives

\[
\tau(\bar{\eta}, \bar{x}^i) = \tau_0 = 0,
\]

(A.1)

with \( \tau_0 \) constant and \( \partial_\eta \tau(\bar{\eta}, \bar{x}^i) \neq 0 \). Thus, thanks to the implicit function theorem, we have

\[
\frac{\partial \eta_0(x_0)}{\partial x_0^i} = -\frac{\partial \tau}{\partial \eta \tau} \bigg|_0 = -\frac{u_i}{u_0} \bigg|_0,
\]

(A.2)

where the subscript \( |_0 \) denotes the set of points \( (\eta_0, x_0^i) \), which describes the initial surface \( S_0 \). Once \( u^\mu \) is fixed, there is a one-to-one correspondence between the point \( x_0^\mu = (\eta_0, x_0^i) \) on the initial constant \( \tau \) 3-surface, with the final point taken to be a generic space-time point \( x^\mu \), as illustrated in figure 1. Thus

\[
\eta_0 = \eta_0(\eta, x) , \quad x_0^i = x_0^i(\eta, x) ,
\]

(A.3)
and using the rule for the differentiation of composite functions we get
\[ \partial_{i}\eta_0 = \frac{\partial \eta_0(x_0)}{\partial x^j_0} \partial_{i} x^j_0, \quad \partial_{i} \eta_0 = \frac{\partial \eta_0(x_0)}{\partial x^j_0} \partial_{i} x^j_0. \] (A.4)

Finally we can substitute eq. (A.2) obtaining
\[ \partial_{i} \eta_0 = \frac{-u_i}{u_0} \partial_{i} x^j_0. \] (A.5)

One can compute \( \partial_{i} x^j_0 \) from the definition of the \( u \) congruence
\[ x^\mu = x^\mu_0 + \int_{\tau_0}^{\tau} u^\mu (\tau') d\tau'. \] (A.6)

Therefore
\[ \partial_{i} x^j_0 = \delta^j_i + \partial_{i} x^j_0^{(1)} = \delta^j_i - \partial_{i} \int_{\tau_0}^{\tau} u^j (\tau') d\tau', \] (A.7)

Substituting these relations in eq. (A.5) and expanding up to second order we obtain
\[ \partial_{i} \eta_0^{(1)} = - \frac{u_i^{(1)}}{u_0} \big|_0 = \partial_{i} v \big|_0, \]
\[ \partial_{i} \eta_0^{(2)} = - \frac{u_i^{(2)}}{u_0} \big|_0 - \left( \frac{u_i^{(1)}}{u_0} \right)^{\prime} \big|_0 + \frac{u_i^{(1)} u_0^{(1)}}{u_0} |_0 + \left[ \partial_j \left( \frac{u_j^{(1)}}{u_0} \right) \big|_0 + \frac{u_j^{(1)}}{u_0} |_0 \partial_{i} \right] \int w^j (1) d\tau, \] (A.8)
\[ \partial_{i} \eta_0^{(1)} = 0, \]
\[ \partial_{i} \eta_0^{(2)} = - \frac{u_i^{(1)}}{u_0} \big|_0 \left( \partial_{i} x^j_0 \right)^{(1)}, \]
where with the subscript \( |_0 \) we label the point \((\bar{\eta}_0, x^i)\).

### B Perturbative computation for \( \zeta_\mu \)

#### B.1 1-st order, generic fluid

In this appendix we will compute perturbatively \( \zeta_\mu \) at first order, in the case of a generic perfect fluid, starting directly from the \( \zeta_\mu \) definition (3.3) and verify that it is coherent with the perturbative expansion of (3.10), when \( \Gamma_\mu = 0 \) (barotropic fluid). By using the definition of the expansion scalar \( \theta \), we find at first order
\[ \theta = \nabla_\mu u^\mu = \bar{\theta} + \theta^{(1)} + O(2) = \frac{1}{a} \left[ 3 H (1 - A) + \frac{1}{2} (-6 \psi + 2 \nabla^2 E)^{\prime} + \nabla^2 (v - B) \right] + O(2). \] (B.1)

\( x^j \) and \( \eta \) are independent coordinates therefore \( \partial_{\eta} x^j = 0. \)
At this point, we have all the ingredients to compute \( \zeta^{(1)}_i \)

\[
\zeta^{(1)}_i = \partial_i \left( -\mathcal{H}(\bar{\eta}_0)\eta^{(1)}_0 - \mathcal{H}^{(1)}_\rho - \frac{1}{3}(3\psi - \nabla^2 E)|^{(\eta,x)} - \frac{1}{3} \int_{\bar{\eta}_0}^{\eta} \nabla^2 (v - B) \, d\eta \right),
\]

\[
\zeta_0 = -\mathcal{H}(\bar{\eta}_0)\partial_t \eta_0 + O(2) = O(2).
\]

Using the \( \zeta \) definition

\[
\zeta_i = \partial_i \left( \zeta + \frac{1}{3} \nabla^2 E + \frac{1}{3} \int_{\bar{\eta}_0}^{\eta} \nabla^2 (v - B) \, d\eta + \psi(\bar{\eta}_0, x^i) + \mathcal{H}(\bar{\eta}_0) \frac{f^{(1)}}{f'}(\bar{\eta}_0, x^i) - \frac{1}{3} \nabla^2 E(\bar{\eta}_0, x^i) \right)
\]

\[
= \partial_i \zeta_s, \quad (B.3)
\]

thus, also in the case of a generic perfect fluid we find a scalar \( \zeta^{(1)}_s \) such that \( \zeta_i = \partial_i \zeta_s \). At this point we can show that in the case of a perfect and barotropic fluid, the \( \zeta_i \) time dependence is completely fictitious. Indeed, from the standard relation

\[
\zeta' = -\frac{1}{3} \nabla^2 (E' + v - B) - \frac{\mathcal{H}}{\rho + p} \Gamma^{(1)}, \quad (B.4)
\]

integrating we get

\[
\zeta + \frac{1}{3} \nabla^2 E + \frac{1}{3} \int_{\bar{\eta}_0}^{\eta} \nabla^2 (v - B) \, d\eta' = \zeta(\bar{\eta}_0, x^i) + \frac{1}{3} \nabla^2 E(\bar{\eta}_0, x^i) - \int_{\bar{\eta}_0}^{\eta} \frac{\mathcal{H}}{\rho + p} \Gamma^{(1)} \, d\eta'. \quad (B.5)
\]

Substituting eq. (B.5) in eq. (B.3) we get

\[
\zeta^{(1)}_s = \mathcal{H}(\bar{\eta}_0) \frac{f^{(1)}}{f'}(\bar{\eta}_0, x^i) + \psi(\bar{\eta}_0, x^i) + \zeta(\bar{\eta}_0, x^i) - \int_{\bar{\eta}_0}^{\eta} \frac{\mathcal{H}}{\rho + p} \Gamma^{(1)} \, d\eta'
\]

\[
= \zeta^{(1)}_s - \int_{\bar{\eta}_0}^{\eta} \frac{\mathcal{H}}{\rho + p} \Gamma^{(1)} \, d\eta' \quad (B.6)
\]

In the case of a barotropic perfect fluid (\( \Gamma^{(1)} = 0 \), eq. (B.6) coincides with eq. (3.14), showing that there is no time dependence and the perturbative approach is coherent with our result (3.10).

The same conclusion is reached proceeding as in [21]; by expanding the definition \( \theta = 3u^\rho \partial_\rho \mathcal{N} \), we get

\[
\theta = \frac{1}{a} \left[ 3 \mathcal{H}(1 - A) + 3 \mathcal{N}^{(1)} \right]. \quad (B.7)
\]

By comparison with eq. (B.1), one can check that \( \mathcal{N}^{(1)} \) has the following form

\[
\mathcal{N}^{(1)} = -\psi' + \frac{1}{3} \nabla^2 (E' + v - B); \quad (B.8)
\]

thus

\[
\mathcal{N}^{(1)} = \int \left( \frac{1}{3} \nabla^2 (E' + v - B) - \psi' \right) \, d\eta' + L(x^i)
\]

\[
= -\psi + \frac{1}{3} \nabla^2 E + \frac{1}{3} \int_{\bar{\eta}_0}^{\eta} \nabla^2 (v - B) \, d\eta' + \psi(\bar{\eta}_0, x^i) - \frac{1}{3} \nabla^2 E(\bar{\eta}_0, x^i) - \mathcal{H}(\eta_0)\eta^{(1)}_0. \quad (B.9)
\]
In the second line of eq. (B.9), we set the arbitrary function of the spatial coordinates $L$, equal to $-\mathcal{H}(\bar{\eta}_0)\eta^{(1)}_0$. This time-independent function is a first-order contribution coming from perturbing the background number of e-folds in $\ln\left(\frac{a(q)}{a(0)}\right)$. With such a choice we recover the expression (B.3).

### B.2 2-nd order, barotropic fluid

In this appendix the second-order $\zeta_\mu$ expression in the case of perfect and barotropic fluid is computed starting from our exact result (3.10), using initial constant-proper-time hypersurfaces for which we have found second order contributions in appendix A. Let us start by denoting with $g$ a generic first-order physical quantity, such that

$$g|_0 = g(\bar{\eta}_0, \delta \eta, x^j + \delta x^j) = g(\bar{\eta}_0, x^j) + (\partial_\mu g)|_0 \delta \mu, \quad \delta \mu = (\delta \eta_0, \delta x^j). \quad (B.10)$$

Where $\delta \eta_0 = \eta^{(1)}_0 + \eta^{(2)}_0 + O(3)$ and $\delta x^j = x^j_0 - x^j = -\int_{a(t)} d\tau u^j$. Using this simple Taylor expansion of quantities computed on the initial hyper-surface and simply remembering that we are dealing with composed functions

$$\partial_\mu g(x_0(x)) = \partial_\mu g|_0 \partial_\mu x_0^\alpha, \quad (B.11)$$

we can analyse the second-order terms inside eq. (3.10). Indeed, perturbing up to the second order the $\partial_\mu \rho(x_0)$ term we find

$$\zeta_s = \zeta_s^{(1)} + \zeta_s^{(2)}$$

$$= \zeta_s^{(1)} - \frac{1}{3(\bar{\rho} + \bar{\rho}^2)} \left( \frac{\partial}{\partial \eta} \right) \left( \rho^{(1)} + \rho^{(1)} + v^{(1)}(\bar{\rho} + \bar{\rho}) \right) \left( \partial_i \rho^{(1)} + \bar{\rho}' \partial_i v^{(1)} \right) |_0 \quad (B.12)$$

$$+ \frac{1}{3(\bar{\rho} + \bar{\rho}^2)} \left[ \partial_i \left( \partial_j \rho^{(1)} \delta x_j^{(1)} \right) + \bar{\rho}' \partial_i \eta^{(2)}_0 + \partial_j \rho^{(2)} + \partial_i \left( \rho^{(1)} v^{(1)} \right) + \frac{1}{2} \bar{\rho}'' v^{(1)} \right] |_0.$$

Using the fluid barotropicity $p^{(1)} = w(\rho)\rho^{(1)}$, and putting into evidence a spatial gradient, we get that in the case of a barotropic perfect fluid, up to the second perturbative order we can write $\zeta_s = \partial_i(\zeta_s^{(1)} + \zeta_s^{(2)})$, where

$$\zeta_s^{(2)} = -\frac{1}{2} \left\{ 3(1 + w)\zeta_{s_1}^{(1)} + \frac{\mathcal{H}}{\bar{\rho}} \left[ \left( \rho^{(2)} + \rho^{(1)} v^{(1)} + \frac{1}{2} \bar{\rho}'' v^{(1)} \right) |_0 + \bar{\rho}' \eta^{(2)}_0 + \partial_j \rho^{(1)} |_0 \delta x_j^{(1)} \right] \right\}$$

and

$$\delta x_j^{(1)} = -\int_{u^0} d\eta' a(\eta') w^{(1)}, \quad (B.14)$$

while $\eta^{(2)}_0$ is a function characterized by eq. (A.8). The $\zeta_s^{(2)}$ expression is particularly simple at large scales, where neglecting terms with two spatial derivatives we find

$$\zeta_s^{(2)} = \frac{\mathcal{H}}{\bar{\rho}'} \left[ \left( \rho^{(2)} + \rho^{(1)} v^{(1)} + \frac{1}{2} \bar{\rho}'' v^{(1)} \right) |_0 + \bar{\rho}' \eta^{(2)}_0 \right], \quad (B.15)$$

which has a very compact form in a gauge where $v = 0$

$$\zeta_s^{(2)} = \frac{\mathcal{H}}{\bar{\rho}'} \eta^{(2)}_0. \quad (B.16)$$

---

3In the original paper [21], actually it was set $L = 0$. 

---
The $\zeta_0$ computation is completely analogous, noting that:

$$\partial_\eta(\rho)|_0 = \partial_{\eta_0} \rho \partial_\eta \eta_0 + \partial_{j_0} \rho \partial_\eta x^j_0$$

$$= \rho'|_0 \partial_\eta(\eta_0^{(2)}) + \left( \partial_j \rho^{(1)} \right) |_0 \partial_\eta x^j_0 + O(3), \quad (B.17)$$

therefore

$$\zeta_0^{(2)} = -\mathcal{H}(\bar{\eta}_0) \partial_\eta \left( \eta_0^{(2)} + \frac{\partial_j \rho^{(1)}}{\rho} |_0 x^j_0 \right). \quad (B.18)$$

Using the $\partial_\eta \eta_0$ and $\partial_\eta x^j_0$ expressions obtained in appendix A and substituting in $\zeta_0^{(2)}$

$$\zeta_0^{(2)} = \mathcal{H}(\bar{\eta}_0) \partial_\eta \left[ \frac{v \rho'}{\rho} |_0 + \rho^{(1)} |_0 \right] \partial_\eta x^j_0$$

$$= \zeta_j^{(1)} \partial_\eta x^j_0$$

$$= \bar{u}_0 \, v^j (1) \zeta_j^{(1)}. \quad (B.19)$$

Notice that this relation holds also in the case of a generic perfect fluid. Indeed, starting from the identity

$$D_0 N = u_0 u^i \partial_i N - u_i u^i N', \quad (B.20)$$

we get

$$\zeta_0^{(2)} = D_0 N - \frac{N}{\rho} D_0 \rho$$

$$= \bar{u}_0 u^i (1) \left[ \partial_i N^{(1)} - \frac{N'}{\rho'} \partial_i \rho^{(1)} \right]$$

$$= \bar{u}_0 u^i (1) \zeta_i^{(1)}, \quad (B.21)$$

which is the same result obtained in eq. (B.19).

Finally, as a further check, let us show that our result (3.10) is consistent with $\mathcal{L}_u \zeta_\mu = 0$, order-by-order in perturbation theory. At linear order we have

$$\zeta_\mu^{(1)'} = 0. \quad (B.22)$$

We have

$$\mathcal{L}_u \zeta_\mu = u^\nu \partial_\nu \zeta_\mu + \zeta_\mu \partial_\mu u^\nu$$

$$= \bar{u}^\mu \zeta^{(1)}_\mu + u^0 \zeta^{(1)}_\mu + u^0 \zeta^{(1)}_\mu + u^i (1) \partial_i \zeta^{(1)}_\mu + \zeta_0^{(2)} \partial_\mu \bar{u}^0 + \zeta^{(1)}_i \partial_\mu u^i (1) + O(3) \quad (B.23)$$

$$= \bar{u}^\mu \zeta^{(2)}_\mu + u^i (1) \partial_i \zeta^{(1)}_\mu + \zeta^{(2)}_0 \partial_\mu \bar{u}^0 + \zeta^{(1)}_i \partial_\mu u^i (1) + O(3) = 0.$$  

Therefore

$$\bar{u}^\mu \zeta^{(2)}_\mu = -u^i (1) \partial_i \zeta^{(1)}_i - \zeta^{(1)}_j \partial_\mu u^j (1), \quad (B.24)$$

$$\bar{u}^\mu \zeta^{(2)}_0 = -u^i (1) \zeta^{(2)}_0 - u^i (1) \zeta^{(1)}_i. \quad (B.25)$$

As a matter of fact, (B.13) and (B.19) satisfy the above relations.
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