On nonconvex optimal control problems

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Abstract. The paper addresses the general optimal control (OC) problem with inequality constraints and a cost functional of Bolza given by d.c. functions with respect to the state in the terminal and integrand parts of the functionals. First, we reduce the original OC problem with inequality constraints to the one without constraints with the help of the Exact Penalization Theory. Further, we show that the auxiliary (penalized) problem also possesses the state-DC-structure. Employing this property, we develop the new Global Optimality Conditions (GOCs) and discuss some its features allowing to construct the new schemes of local and global searchers. Finally we elucidate the relations of the GOCs to the classical OC theory, in particular, to the Pontryagin’s maximum principle.

1. Introduction

Nowadays, many optimal control (OC) problems from various application areas [1]–[5] are (implicitly or explicitly) nonconvex in the different sense. Often one can meet the nonlinear control systems that generate a huge number of local pitfalls from which it is impossible ”to jump out”, moreover, to reach a set of globally optimal solutions using the standard OC tools [6]–[11].

On the other hand, such objectives as a search for equilibria [17] (say, of Nash), the hierarchy structures of problem statements [4, 17], finally, the inverse problem [12] from various applied fields, produce generic nonconvexities, which are difficult to overcome as to finding an approximate global solution [1]–[11], [13, 14].

These situations explain the popularity of such methods, as ”direct approach”, B&B family [16] and ”bioiniciated” family methods, which have no any mathematical foundation (in particular, the convergence results). In this paper we continue to develop the theory and methods for solving the nonconvex OC problems and, this time, with inequality constraints given by the Bolza (integro-terminal) functionals with DC-state decomposition [17]–[25].

First, we reduce the original OC Problem (P) with constraints to Problem (Pσ) without constraints with the help of Exact Penalty Theory [27]–[32], and show that the auxiliary Problem (Pσ) possesses the DC-state property, as well [16]. The latter feature allows us to develop the so-called Global Optimality Conditions (GOCs) for the penalized Problem (Pσ) which under some assumptions become GOCs for (P). After discussing some properties of GOCs we establish some relations with classical results, in particular, with the Pontryagin’s principle [1]–[3], [6]–[13], [18, 19], largely known in the world and very powerful tool to generate numerical methods for various classes of OC problems [7]–[11].
2. Statement of the problem
Consider the following control system

\[ \dot{x}(t) = \varphi(x(t), u(t), t), \quad \forall t \in T := [t_0, t_1[, \quad x(t_0) = x_0 \in \mathbb{R}^n; \]
\[ u(\cdot) \in \mathcal{U} := \{ u(\cdot) \in L^r_\infty(T) \mid u(t) \in U(t) \quad \forall t \in T \}, \]

(1)

(2)

(3)

(4)

(5)

(6)

(7)

(8)

(9)

(10)

(11)

(12)

(13)

(14)

(15)

(16)

(17)

(18)

(19)

(20)

(21)

(22)

(23)

(24)

(25)

(26)

(27)

(28)

(29)

(30)
On the other hand, thanks to the state-convexity of the functions above, in particular, the following inequalities \( (\nabla := \nabla_x) \) hold true \([7, 14, 15, 20]\)

\[
\begin{align*}
(a) & : \quad \langle \nabla h_{1i}(y), x - y \rangle \leq h_{1i}(x) - h_{1i}(y) \quad \forall x, y \in \Omega_i, \\
(b) & : \quad \langle \nabla h_{i}(y(t), t), x(t) - y(t) \rangle \leq h_{i}(x(t), t) - h_{i}(y(t), t), \\
& \quad \forall x(t), y(t) \in \Omega(t), \quad t \in T, \quad i \in \{0 \} \cup I.
\end{align*}
\]

(6)

Let us now show that under the above assumptions every functional \( J_i(u) = J_i(x, u) \), defined in (3), can be represented in the form

\[
J_i(u) = G_i(x, u) - H_i(x), \quad i \in \{0 \} \cup I,
\]

(7)

where \( G_i, H_i \) possess the state-convexity properties similar to described above.

Indeed, using the notations below together with (3)–(5)

\[
\begin{align*}
(a) & : \quad G_i(x, u) := g_{1i}(x(t_1)) + \int_{T}^{T} g_{i}(x(t), u(t), t) \, dt, \\
(b) & : \quad H_i(x) := h_{1i}(x(t_1)) + \int_{T}^{T} h_{i}(x(t), t) \, dt, \quad i \in \{0 \} \cup I;
\end{align*}
\]

(7')

we obtain the desirable state-convexity properties.

In particular, for the functions \( H_i(x) \) some analogs of the convexity inequalities (6) holds. Actually, under above assumptions a differential of the functional \( H_i(\cdot) \) can be defined as follows

\[
\langle (\nabla H_i(y(\cdot)), x(\cdot)) \rangle := \langle \nabla h_{1i}(y(t_1)), x(t_1) \rangle_n + \int_{T}^{T} \langle \nabla h_{i}(y(t)), x(t) \rangle_n \, dt, \quad (6')
\]

where \( \langle \cdot, \cdot \rangle_n \) is the inner product in \( \mathbb{R}^n \), \( x(\cdot) \in AC(T) \), \( y(\cdot) \) is a piece-wise continuous function, \( y(\cdot) \in PC_n(T) \). Therefore one can look at the pair \( (\nabla h_{1i}(y(t_1)), \nabla h_{i}(y(\cdot))) \) as a gradient of \( H_i(\cdot) \) at the function \( y(\cdot): \nabla H_i(y(\cdot)) := (\nabla h_{1i}(y(t_1)), \nabla h_{i}(y(\cdot))) \), so that, due to (6) and (6'), the next inequality holds

\[
\langle (\nabla H_i(y(\cdot)), x(\cdot) - y(\cdot)) \rangle \leq H_i(x(\cdot)) - H_i(y(\cdot)) \quad \forall (x(\cdot), y(\cdot)) \in AC(T) \times PC_n(T). \quad (6'')
\]

Now we are ready to study the following optimal control (OC) problem

\[
(P):
\begin{align*}
J_0(u) & := J_0(x(\cdot), u(\cdot)) \downarrow \min_{u}, \quad u(\cdot) \in \mathcal{U}, \\
J_i(u) & := J_i(x(\cdot), u(\cdot)) \leq 0, \quad i \in I = \{1, \ldots, m\},
\end{align*}
\]

(8)

Clearly, due to nonconvexity (with respect to the state) of the terminal parts \( f_{1i} \) and of the integrand \( f_i(x, u, t) \), each functional \( J_i(x, u) \), \( i \in \{0 \} \cup I \), the feasible region of Problem \( (P) \), and Problem \( (P) \) itself, as a whole, turn out to be nonconvex. It means that Problem \( (P) \) might possess a big number of locally optimal and stationary (in the sense of PMP) processes \( (x_*(\cdot), u_*(\cdot)) \), \( x_*(t) = x(t, u_*) \), \( t \in T \), \( u_*(\cdot) \in \mathcal{U} \), which may be rather far from a set \( Sol(P) \) of global solutions (if one exists) even with respect to the value of the cost functional.

Introduce then the penalty function \( Pnt(x, u) \) for Problem \( (P) \)–(8) in the following way

\[
Pnt(x, u) := Pnt(u) := \max\{0, J_1(u), \ldots, J_m(u)\}
\]

(9)

and consider the auxiliary (penalized) problem

\[
(P_\sigma):
\begin{align*}
F_\sigma(u) & := F_\sigma(x(\cdot), u(\cdot)) \downarrow \min_{u}, \quad u(\cdot) \in \mathcal{U},
\end{align*}
\]

(10)
where the cost function of Problem \((\mathcal{P}_\sigma)\)–(10) is defined as follows

\[
F_\sigma(u) := J_0(x(\cdot, u), u(\cdot)) + \sigma \text{Pnt}(x(\cdot, u), u), \tag{11}
\]

and \(\sigma \geq 0\) is the penalty parameter.

Let us recall a few facts from the Exact Penalization Theory \([27]–[32]\), the key feature of which is the existence of the threshold value \(\sigma_\ast > 0\) of the penalty parameter, such that \(\text{Sol}(\mathcal{P}_{\sigma}) \cap \text{Sol}(\mathcal{P}) \neq \emptyset \ \forall \sigma \geq \sigma_\ast\). In other words, for \(\sigma \geq \sigma_\ast\) Problems \((\mathcal{P})\) and \((\mathcal{P}_\sigma)\) turn out to be equivalent in the sense that \(\text{Sol}(\mathcal{P}) = \text{Sol}(\mathcal{P}_\sigma)\) (see \([14\text{, Chapt. VII, Lemma 1.2.1}\) and \([27]–[32]\)).

Hence, the existence of a threshold value \(\sigma_\ast > 0\) implies that instead of solving a sequence of unconstrained problems with \(\sigma_k \to \infty\) \([7, 26]\) one needs to solve only a single unconstrained problem with a penalty parameter \(\sigma \geq \sigma_\ast\).

In addition, it is well-known that, if a process \((z(\cdot), w(\cdot))\) is a global solution to Problem \((\mathcal{P}_\sigma)\): \((z, w) \in \text{Sol}(\mathcal{P}_\sigma), z(t) = x(t, u), \ t \in T, w \in U\), and, besides, \((z, w)\) is feasible in Problem \((\mathcal{P})\), i.e. \(J_0(z, w) \leq 0, \ i \in I\), then \((z(\cdot), w(\cdot))\) is a global solution to Problem \((\mathcal{P})\). One has to mention that the inverse assertion, in general, does not hold.

It is worth noting that under various Constraint Qualification (CQ) conditions (e.g. MFCQ, etc.), the error bound properties, the calmness of constraint systems, etc., one can prove the existence of the Exact Penalty threshold value \(\sigma_\ast > 0\) for local and global solutions \([27]–[32], \ [13, 21]\). Assume that some regularity conditions which ensure the existence of the threshold value \(\sigma_\ast\) of the penalty parameter (ThrVPP) are fulfilled.

3. DC decomposition of Problem \((\mathcal{P}_\sigma)\)

First, it can be readily seen that, due to the presentations \((7), (7')\) and also to \((9)–(11)\), the cost function \(F_\sigma(x, u) \triangleq F_\sigma(u)\) of the penalized Problem \((\mathcal{P}_\sigma)\)–(10) can be represented as follows

\[
F_\sigma(x, u) := G_0(x, u) - H_0(x) + \sigma P\text{nt}(x, u) \triangleq G_0(x, u) - H_0(x) + \sigma \max\left\{0; \left[ G_i(x, u) - H_i(x) \right], \ i \in I\right\}. \tag{11'}
\]

To this end, let us show now that the penalty function \(P\text{nt}(x, u)\) defined in \((9)\) can also be represented as a DC function: \(P\text{nt}(x, u) = G_W(x, u) - H_W(x)\), where \(G_W(\cdot)\) and \(H_W(\cdot)\) possess the state-convexity property. Then, obviously, \(F_\sigma(x, u)\) will be state-DC (i.e. \(x \mapsto F_\sigma(x, u)\) is a DC function). Indeed, it follows from \((9)\) that

\[
P\text{nt}(x, u) := \max\left\{0; \left[ G_i(x, u) - H_i(x) \right], \ i \in I\right\} \pm \sum_{j \in I} H_j(x) = \max \left\{ \sum_{j \in I} H_j(x); \left[ G_i(x, u) + \sum_{p \neq i} H_p(x) \right], \ i \in I \right\} - \sum_{j \in I} H_j(x). \tag{12}
\]

Now with the help of denotations

\[
(a) : \ G_W(x, u) := \max \left\{ \sum_{i \in I} H_i(x); \left[ G_i(x, u) + \sum_{p \neq i} H_p(x) \right], \ i \in I \right\},
\]

\[
(b) : \ H_W(x) := \sum_{i \in I} H_i(x), \tag{13}
\]

one gets the following DC decomposition of the penalty function

\[
P\text{nt}(x, u) = G_W(x, u) - H_W(x), \tag{14}
\]
where the functions $G_W(\cdot)$ and $H_W(\cdot)$ obviously conserve\cite{7, 14, 15} the state-convexity property in virtue of \eqref{7}, \eqref{7'} and \eqref{13}. Moreover, as was claimed above, the cost function $F_\sigma(x, u)$, defined in \eqref{11'}, possesses the DC-state decomposition, because (see \eqref{11'}, \eqref{13}, \eqref{14})

\[
F_\sigma(x, u) \triangleq G_0(x, u) - H_0(x) + \sigma[G_W(x, u) - H_W(x)] = [G_0(x, u) + \sigma G_W(x, u) - [H_0(x) + \sigma H_W(x)]] =: G_\sigma(x, u) - H_\sigma(x),
\]

where due to \eqref{7'} we have

\[
G_\sigma(x, u) := G_0(x, u) + \sigma G_W(x, u) = \sum_{i \in I} h_{1j}(x(t_1)) + \int_T h_j(x(t), t) dt + \sigma \max \left\{ \sum_{i \in I} h_{1j}(x(t_1)) + \int_T h_j(x(t), t) dt \right\},
\]

\[
H_\sigma(x) := H_0(x) + \sigma H_W(x) \triangleq h_{10}(x(t_1)) + \int_T h_0(x(t), t) dt + \sigma \sum_{i \in I} h_{1i}(x(t_1)) + \int_T h_i(x(t), t) dt = \sum_{i \in I} h_{1i}(x(t_1)) + \int_T h_i(x(t), t) dt.
\]

It is clear, thanks to \eqref{15}--\eqref{17}, that the functions $G_\sigma(x, u)$ and $H_\sigma(x)$ are endowed with the state-convexity property, as well \cite{7, 13, 16, 18, 20}.

On the other hand, in can be readily seen that the function $H_\sigma(\cdot)$ is differentiable in the sense that $\forall y(\cdot) \in PC(T)$, in particular, we have

\[
\langle \langle \nabla H_\sigma(y(\cdot)), x(\cdot) \rangle \rangle \triangleq \langle x(t_1), \nabla h_{10}(y(t_1)) + \sigma \sum_{i \in I} \nabla h_{1i}(y(t_1)) \rangle + \int_T \langle x(t), \nabla h_0(y(t), t) + \sigma \sum_{i \in I} \nabla h_{1i}(y(t), t) \rangle dt.
\]

Hence, due to the state-convexity property of $H_\sigma(\cdot)$, the next inequality takes place $\forall u \in \mathcal{U}$,

\[
\langle \langle \nabla H_\sigma(y(\cdot)), x(\cdot, u) - y(\cdot) \rangle \rangle \leq H_\sigma(x(\cdot, u)) - H_\sigma(y(\cdot)).
\]

4. Global optimality conditions (GOCs)

Let us return now to Problem \eqref{P}--\eqref{8}, assuming that the feasible set is not empty, i.e.

\[
\mathcal{F} := \{(x(\cdot), u(\cdot)) \mid x(t) = x(t, u), t \in T, u \in \mathcal{U}; J_i(u) \leq 0, i \in I \} \neq \emptyset;
\]

and the optimal value $\mathcal{V}(\mathcal{P})$ of Problem \eqref{P} is finite, i.e.

\[
\mathcal{V}(\mathcal{P}) := \inf_u \{J_0(u) \mid u \in \mathcal{U}, (x(\cdot, u), u) \in \mathcal{F} \} > -\infty.
\]

Assume, in addition, that the set of globally optimal processes is not empty, as well:

\[
\text{Sol}(\mathcal{P}) := \{(x(\cdot), u(\cdot)) \in \mathcal{F} \mid J_0(u) = \mathcal{V}(\mathcal{P})\} \neq \emptyset.
\]
We will say equivalently that (22) amounts

\[ \text{Sol}(P) := \{ u(\cdot) \in U \mid (x(\cdot),u(\cdot)) \in F, J_0(u) = \mathcal{V}(P) \} \neq \emptyset. \]  

(22')

In other words, a process \((z(\cdot),w(\cdot)) \in F \) \((z(t) = x(t,w),t \in T, w(\cdot) \in U, P_{nt}(z(\cdot),w(\cdot)) = 0)\) \(\) (or a control \(w(\cdot) \in U\) ) is said to be globally optimal to Problem \((P)\) \(\) \((z(\cdot),w(\cdot)) \in \text{Sol}(P)\) \(\) or \(w(\cdot) \in \text{Sol}(P)\) \(\), if one has

\[ J_0(z,w) \overset{\triangle}{=} J_0(w) \leq J_0(u) \overset{\triangle}{=} J_0(x(\cdot),u(\cdot)) \forall (x(\cdot),u(\cdot)) \in F \text{ (or } \forall u(\cdot) \in U). \]

Employing the following denotation for \((z(\cdot),w(\cdot)) \in F\)

\[ \zeta := J_0(w) \overset{\triangle}{=} J_0(z(\cdot),w(\cdot)) = F_\sigma(w) \overset{\triangle}{=} J_0(w) + \sigma P_{nt}(z(\cdot),w(\cdot)) \]  

(23)

we can present the main result of Section 4 (cf. [18]–[24]).

**Theorem 1.** Assume that the feasible (in Problem \((P)\) ) process \((z(\cdot),w(\cdot))\) is globally optimal to Problem \((P)\) \(\) \(\) and \(\sigma \geq \sigma_*\) \(\), where \(\sigma_* > 0\) \(\) is the threshold value of the penalty parameter, so that \(\text{Sol}(P) = \text{Sol}(P_\sigma)\).

Then, for every pair \((y(\cdot),\beta) \in PC(T) \times IR, y(\cdot) : T \rightarrow IR,\) \(\) satisfying the equality (see (17))

\[ H_\sigma(y(\cdot)) \overset{\triangle}{=} H_0(y(\cdot)) + \sigma H_W(y(\cdot)) = \beta - \zeta, \]  

(24)

the following inequality holds

\[ G_\sigma(x(\cdot),u) - \beta \geq \langle \nabla H_\sigma(y(\cdot)),x(\cdot),u \rangle - y(\cdot) \forall u \in U. \]  

(25)

**Proof.** Since \(w(\cdot) \in \text{Sol}(P) = \text{Sol}(P_\sigma)\) \(\) and, on account of (23), we have that \(\forall u \in U\)

\[ \zeta = F_\sigma(w) \overset{\triangle}{=} G_\sigma(z,w) - H_\sigma(z) \leq G_\sigma(x(\cdot),u) - H_\sigma(x(\cdot),u). \]

Then, thanks to the equality (24), we get

\[ G_\sigma(x(\cdot),u) \geq \beta - H_\sigma(y(\cdot)) + H_\sigma(x(\cdot),u)). \]

Whence, with the help of the state-convexity of \(H_\sigma(\cdot)\) \(\) and the inequality (19), we derive the inequality (25).

**Remark 1.** It can be readily seen that Theorem 1 reduces the nonconvex OC Problem \((P_\sigma)\) \(\) to a study of the family of the (partially) linearized problems as follows

\[ (P_\sigma L(y)) : \Phi_{\sigma y}(x(\cdot),u) := G_\sigma(x(\cdot),u) - \langle \nabla H_\sigma(y(\cdot)),x(\cdot) \rangle \downarrow \min_u, u \in U, \]  

(26)

depending on the pair \((y(\cdot),\beta) \in PC(T) \times IR,\) \(\) fulfilling the equation (24) \(\) with the functions given by (16)–(19). It is not difficult to notice that the linearization is carried out with respect to the “united” nonconvexity of Problem \((P)\),–(8) \(\) generated only by functionals \(J_0(u),...,J_m(u)\) \(\) (but not by the nonlinear control system \((1)\)–(2),) \(\) and accumulated by the function \(H_\sigma(\cdot)\) \(\) (see \((P)\)–(8) and (17).

Clearly, in the case when the system \((1)\)–(2) \(\) (which is implicitly present in (26)) \(\) is state-linear \(\) (see below), then the problem \((P_\sigma L(y))\)–(26) \(\) becomes convex in the sense that the Pontryagin principle turns out to be the sufficient conditions for a global solution to the problem \((P_\sigma L(y))\)–(26). Thus, the verification of the principal inequality (25) can be reduced to the solution
of the linearized problems \((P_\sigma L(y))\) together with the simultaneous varying the parameters \((y(\cdot), \beta) \in PC(T) \times \mathbb{R}\) satisfying (24).

Moreover, employing the idea of consecutive solutions of the linearized problem of the kind \((P_\sigma L(x^\sigma(\cdot)))\) (with the linearization at a current iterate \(x^\sigma(t) = x(t, u^\sigma(\cdot)), u^\sigma(\cdot) \in U, x^{\sigma+1}(t) \in \text{Sol}(P_\sigma L(x^\sigma(\cdot)))\) one can construct a Local Search Method with very good convergence properties (see [17]–[20], [24, 25]). Hence, the idea of linearization with respect to the basic nonconvexity of a problem under study has shown itself to be effective and beneficial from the view-point of finding global solutions to applied nonconvex optimization problems [17]–[25].

**Remark 2.** Suppose, one found a tuple \((y(\cdot), \beta, u(\cdot))\): \(y(\cdot) \in PC(T), \beta \in \mathbb{R}, u(\cdot) \in U, H_\sigma(y(\cdot)) = \beta - \zeta, \zeta := F_\sigma(w),\) which violates the principal inequality (25) of Theorem 1:

\[
0 > G_\sigma(x(\cdot, u), u) - \beta - \langle \langle \nabla H_\sigma(y(\cdot)), x(\cdot, u) - y(\cdot) \rangle \rangle.
\]

Then, due to (19) and (24) we derive

\[
0 > G_\sigma(x(\cdot, u), u) - \beta - H_\sigma(x(\cdot, u)) + H_\sigma(y(\cdot)) = F_\sigma(x(\cdot, u), u) - \beta - F_\sigma(x(\cdot, u), u) - H_\sigma(z(\cdot, w(\cdot))),
\]

which yields \(F_\sigma(x(\cdot, u), u) < F_\sigma(z(\cdot, w(\cdot))),\) which means, the process \((z(\cdot), w(\cdot))\) (the control \(w(\cdot)\)) is not a solution to problem \((P_\sigma).\)

If, in addition, the process \((x(\cdot, u), u)\) is feasible in \((P): (x(\cdot, u), u) \in \mathcal{F}\) (i.e. \(\text{Pnt}(x(\cdot, u), u) = 0\)), then the control \(w(\cdot)\) is not globally optimal in \((P): w \notin \text{Sol}(P).\) Moreover, the control \(u(\cdot) \in U\) turns out to be better than \(w(\cdot) \in U,\) since

\[
J_0(u) = F_\sigma(u) < F_\sigma(w) = J_0(w).
\]

In other words, the conditions (24)–(25) of Theorem 1 are gifted with the classical constructive (algorithmic) property, which means that, if the conditions are violated, then one can find a feasible process (control) which is better than this one under study (see examples in [2, 3], [7]–[10], [20, 26, 32]).

### 5. Relations with the classical optimal control theory

Let us consider a feasible in Problem \((P)\) process \((z(\cdot), w(\cdot)), z(t) = x(t, w), t \in T, w \in U,\) which satisfies the conditions (24)–(25) of Theorem 1 and let \(y(\cdot)\) be equal to \(z(\cdot),\) i.e. \(y(t) = z(t), t \in T,\) in (24)–(25). Then we immediately get from (24) that

\[
\beta = H_\sigma(z(\cdot)) + \zeta = H_\sigma(z(\cdot)) + F_\sigma(z(\cdot), w(\cdot)) = G_\sigma(z(\cdot), w(\cdot)).
\]

Moreover from (25) and the latter equality it follows that \(\forall u \in U\) we have

\[
G_\sigma(x(\cdot, u), u) - G_\sigma(z(\cdot), w) \geq \langle \langle \nabla H_\sigma(z(\cdot)), x(\cdot, u) - z(\cdot) \rangle \rangle.
\]

It means that the process \((z(\cdot), w(\cdot))\) is a solution to the following linearized problem

\[
(P_\sigma L(z)): \quad \Phi_{\sigma z}(u) \overset{\Delta}{=} G_\sigma(x(\cdot, u), u) - \langle \langle \nabla H_\sigma(z(\cdot)), x(\cdot, u) \rangle \rangle \downarrow \min_{\hat{u}} u \in U.
\]  

Henceforth, such a process \((z(\cdot), w(\cdot))\) (a control \(w(\cdot) \in U\) is said to be critical.

It is not difficult to point out that, the function \(G_\sigma(x(\cdot, u), u)\) is not smooth, because of the operation ”max” in (13) and (16), meanwhile the entries of the original problem \((P)–(8))\) are assumed to be differentiable.

To overcome this shortcoming, let us apply Lemma 4.1 from [22] (see also [26]), according to which Problem \((P_\sigma L(z))–(8)\) amounts the parametric OC problem (see (7), (13), (16), (17)):
From the first glance, it might seem that the parametric optimal control (OC) problem (28) is state-convex, because all the functionals $G_i(x,u), H_i(x), i \in \{0\} \cup I$, are state-convex. However, one has to mention that notation $\hat{x}(\cdot), u \in \mathcal{U}$ means that $x(t) = x(t,u)$, $t \in T$, $u \in \mathcal{U}$, is the solution of the nonlinear system (1)–(2) corresponding to the control $u \in \mathcal{U}$.

Hence, the problem (28) is nonconvex in the sense that the Pontryagin’s principle (PMP) is not sufficient for the control $w(\cdot)$ (i.e. the process $(z(\cdot), w(\cdot))$) and $\gamma_z \in \mathbb{R}$, such that

$$
\gamma_z = \max \left\{ \sum_{p \in I} H_p(z(\cdot)); \ G_i(z(\cdot), w(\cdot)) + \sum_{p \notin I} H_p(z(\cdot)) \right\}, \ i \in I
$$

(29)

satisfying the PMP, would be a global solution to (28).

Clearly, when the control system (1) is state-linear, i.e. instead of (1) we have

$$
\dot{x}(t) = A(t)x(t) + B(u(t), t) \quad \forall t \in T, \ x(t_0) = x_0,
$$

(1′)

then the problem (28) becomes state-convex in the sense above explained.

In what follows we assume that the process $(x(\cdot), u(\cdot))$ is steered by the state-linear system (1′), and therefore the problem (28) turns out to be state-convex.

In both cases the reachable set $\mathcal{R}(t), t \in T$ stays bounded and, therefore, $\forall u \in \mathcal{U}$ there exists a number $\gamma_z \in \mathbb{R}$, such that the inequality constraints in (28) are strongly fulfilled, so that

$$
G_i(x(\cdot), u(\cdot)) + \sum_{p \notin I} H_p(x(\cdot), u) < \gamma, \ i \in I
$$

$$
\sum_{j \in I} H_j(x(\cdot), x) < \gamma.
$$

It means that in the parametric control problem (28) the Slater’s conditions takes place.

Therefore, we employ the Lagrange function of the problem (28) with the Lagrange multiplier $\mu_0 = 1$, and consider the following optimal control problem which amounts (28) (due to convexity of the problem (28))

$$
\mathcal{L}(x(\cdot), u(\cdot), \gamma, \mu) \downarrow \min _{(u, \gamma)} u(\cdot) \in \mathcal{U}, \gamma \in \mathbb{R}, \mu = (\mu_1, \ldots, \mu_m, \mu_{m+1}) \in \mathbb{R}^{m+1}.
$$

(28′)

Here the Lagrange function of Problem (28) has the following form (see (7), (28′), (15)–(19))

$$
\mathcal{L}(x, u, \gamma, \mu) := G_0(x, u) + \sum_{p \in I} H_p(x) - \gamma + \sum_{i \in I} \mu_i \left( G_i(x, u) + \sum_{p \notin I} H_p(x) - \gamma \right) + \mu_{m+1} \left( \sum_{j \in I} H_j(x) - \gamma \right) =
$$

$$
g_{i0}(x(t_1)) + \int_T g_0(x(t), u(t), t)dt + \sigma \gamma - \beta \langle x(t_1), \nabla h_{10}(z(t_1))\rangle + \sigma \sum_{i \in I} \nabla h_i(z(t_1)) - \gamma
$$

$$
- \beta \langle x(t), \nabla h_{10}(z(t), t) + \sigma \sum_{i \in I} \nabla h_i(z(t), t)dt + \sum_{i \in I} \mu_i \left( g_{i1}(x(t_1)) + \int_T g_i(x(t), u(t), t)dt + \sum_{p \in I} h_{1p}(x(t_1)) + \int_T h_p(x(t), t)dt - \gamma \right) =
$$

$$
+ \sum_{p \in I} \left( h_{1p}(x(t_1)) + \int_T h_p(x(t), t)dt - \gamma \right) + \mu_{m+1} \left( \sum_{j \in I} h_{1j}(x(t_1)) + \int_T h_j(x(t), t)dt - \gamma \right).
$$

(30)
Now due to the optimality conditions for Problem (28'), we obtain

\[ \frac{\partial L(z, w, \gamma; \mu)}{\partial \gamma} = \sigma - \sum_{i \in I} \mu_i - \mu_{m+1} = 0, \text{ i.e. } \sum_{i \in I} \mu_i + \mu_{m+1} = \sigma. \]  

(31)

On the other hand, it is now clear that the OC problem (28) has the following form

\[ \varphi(x(t_1)) + \int_{T} F(x(t), u(t), t) dt \quad \min_{u \in U} \quad x(\cdot) = x(\cdot, u), \quad \gamma \in IR. \]  

(28'')

(over the control system (1' )–(2)). Therefore, the corresponding Pontryagin’s Principle (PMP) for the process \((z(\cdot), w(\cdot))\) holds, because \((z(\cdot), w(\cdot))\) is the solution to the following problems \((P_L z)–(26), (28), (28')\) and \((28'')\).

**Proposition 1.** The process \((z(\cdot), w(\cdot))\) (the control \(w(\cdot) \in U\)) is the solution to the convex OC problem (28) (over the state-linear control system (1')–(2)) if and only if the following Pontryagin’s principle holds

\[ H(z(t), w(t), \psi(t), t) = \min_{v \in \Gamma(t)} H(z(t), v, \psi(t), t) \quad \hat{\psi} t \in T, \]  

(32)

with the Hamiltonian \(H(x, u, \psi, t)\) defined according to the Optimal Control Theory [1]–[11]

\[ H(x, u, \psi, t) = \langle \psi, f_L(x, u, t) \rangle + F(x, u, t), \]  

(33)

(where \(f_L(x, u, t) = A(t)x(t) + B(u(t), t)\), and the conjugate state \(\psi(\cdot) \in AC(T)\) is defined by the adjoint system of ODE

\[ \begin{cases} 
\dot{\psi}(t) = -A^\top(t)\psi(t) - \nabla_x F(z(t), w(t), t) \\
\psi(t_1) = \nabla \varphi(z(t_1)).
\end{cases} \]  

(34)

The conclusion is obvious: the PMP (32)–(34) follows from GOCs (24)–(25) of Theorem 1. In other words, GOCs (24)–(25) are connected with the classical OC Theory.

Now we intend to develop further the foundation for constructing some numerical procedures (based on the constructive property of GOCs) allowing to escape any local pitfall of Problem (P) and answering the natural question: whether one can find a triple \((y(\cdot), \beta, u) \in AC(T) \times IR \times U\) (satisfying (24)) which violates inequality (25).

To this end, we present the next result (see [18]–[23]).

**Theorem 2.** Assume that a feasible in Problem (P)–(8) process \((z(\cdot), w(\cdot)), \quad z(t) = x(t, w), \quad t \in T, \quad w \in U\) is not an \(\varepsilon\)-solution to (P):

\[ \inf(J_0, F) + \varepsilon \triangleq V(P) + \varepsilon < \zeta := J_0(z(\cdot), w(\cdot)). \]  

(35)

In addition, let a feasible control \(v \in U\) satisfy the following inequality

\[ (A_0): \quad J_0(x(\cdot, v), v) > \zeta - \varepsilon. \]  

(36)

Then, for any penalty parameter \(\sigma > 0\) one can find a tuple \((y, \beta, u), \quad y(\cdot) \in AC(T), \beta \in IR, \quad u(\cdot) \in U\), such that the following conditions hold

\[ \begin{cases} 
(a): \quad H_\sigma(y(\cdot)) = \beta - \zeta + \varepsilon; \quad (b): \quad G(y(\cdot), \bar{v}(\cdot)) \leq \beta, \\
(c): \quad G_\sigma(x(\cdot, u), u) < \beta + \langle \nabla H_\sigma(y(\cdot), x(\cdot, u) - y(\cdot)) \rangle.
\end{cases} \]  

(37)

with some \(\bar{v}(\cdot) \in L^\infty_x(T), \quad \bar{v}(t) \in convU(t) \quad \hat{\psi} t \in T.\)
References

[1] Pontryagin L S, Boltyanskii V G, Gamkrelidze R V and Mishchenko E F 1976 The mathematical theory of optimal processes (New York: Interscience)
[2] Chernousko F L, Ananievski I M and Reshmin S A 2008 Control of nonlinear dynamical systems: methods and applications (Berlin: Springer)
[3] Phedorenko R P 1978 Approximate solution of optimal control problems (Moscow: Nauka) [in Russian]
[4] Pang J-S 2010 Three modelling paradigms in mathematical programming Math. Program. Ser. B 125 297–323
[5] Kurzhanski A B and Varaiya P 2014 Dynamics and control of trajectory tubes: theory and computation (Boston: Birkhauser)
[6] Kurzansky A B 1977 Control and observation under uncertainty conditions (Moscow: Nauka) [in Russian]
[7] Vasil’ev J P 2002 Optimization methods (Moscow: Factorial Press) [in Russian]
[8] Gabasov R and Kirillova F M 1979 Optimization of linear systems (New York: Plenum Press)
[9] Gabasov R and Kirillova F M 1974 Maximum’s principle in the optimal control theory (Minsk: Nauka i Technika) [in Russian]
[10] Vasiliev O V 1996 Optimization methods (Atlanta: Word Federation Publishing Company)
[11] Srochko V A 2000 Iterative solution of optimal control problems (Moscow: Fizmatlit) [in Russian]
[12] Tihonov A N and Arsenin V J 1979 Solution methods of incorrect problems (Moscow: Nauka) [in Russian]
[13] Clarke F 1990 Optimization and nonsmooth analysis (Philadelphia: SIAM) 2nd edn.
[14] Hiriart-Urruty J-B and Lememarshal C 1993 Convex analysis and minimization algorithms (Berlin: Springer)
[15] Hiriart-Urruty J-B 1995 Generalized differetiality, duality and optimization for problem dealing with difference of convex functions Convexity and Duality in Optimization ed J Ponstein (Berlin: Springer) pp 37–69
[16] Tuy H 1995 D.C. optimization: theory, methods and algorithms Handbook of Global Optimization eds R Horst and P M Pardalos (Dordrecht: Kluwer Academic Publishers) pp 149–216
[17] Strekalovsky A S 2014 On solving optimization problems with hidden nonconvex structures Optimization in Science and Engineering eds M Themistocles, Ch A Floudas and S Butenko (New York: Springer) pp 465–502
[18] Strekalovsky A S 2013 Global optimality conditions for optimal control problems with functions of A.D. Alexandrov J. Optim. Theory Appl. 159 297–321
[19] Strekalovsky A S 2012 Maximizing a state convex Lagrange functional in optimal control Autom. Remote Control 73(6) 949–61
[20] Strekalovsky A S 2003 Elements of nonconvex optimization (Novosibirsk: Nauka) [in Russian]
[21] Strekalovsky A S 2019 Global optimality conditions and exact penalization Optim. Lett. 13 597–615
[22] Strekalovsky A S 2017 Global optimality conditions in nonconvex optimization J. Optim. Theory Appl. 173(3) 770–92
[23] Strekalovsky A S 2019 New global optimality conditions in a problem with d.c. constraints Trudy Inst. Mat. i Mehk. UrO RAN 25 245–61 [in Russian]
[24] Strekalovsky A S and Yanulevich M V 2016 On global search in nonconvex optimal control problems J. Global Optim. 65(1) 119–35
[25] Strekalovsky A S and Yanulevich M V 2008 Global search in the optimal control problem with a terminal objective functional represented as a difference of two convex functions Comput. Math. Math. Phys. 48(7) 1119–32
[26] Nocedal J and Wright St 2006 Numerical optimization (New York: Springer) 2nd edn
[27] Eremin I 1966 The penalty method in convex programming Soviet Math. Dokl. 8 459–62
[28] Dolgopolik M V and Foninhyh A V 2019 Exact penalty functions for optimal control problems I: Main theorem and free-endpoint problems Optimal Control Appl. Meth. 40 1018–44
[29] Gugat M and Zuazua E 2016 Exact penalization of terminal constraints for optimal control problems Optimal Control Appl. Meth. 37 1329–54
[30] Li, B., Yu, C.J., Teo, K.L. et al. An Exact Penalty Function Method for Continuous Inequality Constrained Optimal Control Problem. J Optim Theory Appl 151, 260 (2011).
[31] Demyanov V F, Giannessi F and Karelin V V 1998 Optimal control problems via exact penalty functions J. Global Optim. 12(3) 215–23
[32] Demyanov V 2005 Extremum’s conditions and variational calculus (Moscow: High school edition) [in Russian]