Remarks on geometric entropy

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Abstract

The recently discussed notion of geometric entropy is shown to be related to earlier calculations of thermal effects in Rindler space. The evaluation is extended to de Sitter space and to a two-dimensional black hole.

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1. Introduction

A number of recent works [1–3] have been concerned with the first quantum correction to the Bekenstein-Hawking black hole entropy. The calculations involve, among other things, the statistical mechanics of fields in Rindler space-time as a simply analysed case that might throw light on the black hole system. Field theory in Rindler space-time has been extensively investigated over the past 20 years [4,5]. Complete references are not possible in this note but the fact that tracing over fields restricted to half Minkowski space yields a thermal average was early known, cf [6], and is sometimes called the thermalisation theorem [7,8].

Because of the infinite red-shift at the horizon, the integrated (total) thermodynamical quantities diverge. A regularisation can be effected by integrating up to a finite distance from the horizon, cf [9].

As a byproduct of an analysis of quantum field theory around a cosmic string, the finite-temperature energy density in Rindler space was determined in [10]. The basic fact is that the conical part of the cosmic string metric is a euclideanised part of Rindler space-time, cf [11]. In this paper we wish to investigate the relevance of this calculation and of an earlier one [12].

2. Some thermal averages in Rindler space-time.

We write the Rindler metric (in four dimensions) as

\[ ds^2 = Z^2 dv^2 - dZ^2 - dx^2 - dy^2 \]  \hspace{1cm} (2.1)

which is compared with the conical cosmic string metric

\[ ds^2 = dt^2 - dr^2 - r^2 d\phi^2 - dz^2 \]  \hspace{1cm} (2.2)

giving the identifications \( \phi \approx -iv, \ r \approx Z, \ t \approx ix, \ z \approx y \). One then has the equality

\[ \langle T^0_0 \rangle_{\text{Rindler}} = \langle T^2_2 \rangle_{\text{string}}. \]

The periodicity of \( \beta \) in the angle \( \phi \) of the string translates into a temperature of \( 1/\beta \) in Rindler space and the finite-temperature expressions in the Rindler case can be read off from the zero temperature cosmic string results which have been derived over the years by various means.
The details of our evaluation are in the cited references. The basic idea is to reperiodise the polar angle from $2\pi$ to $\beta$ in the Green function by a complex contour method. That is to say, we introduce a conical singularity into the manifold.

A point that should be made is that the vacuum averages were rendered finite in [10] by subtraction of the Minkowski expressions. This can be done at the local heat-kernel or Green function level.

We find for $\langle T^0_0 \rangle$ the expressions

$$\frac{\pi^2 T^4}{30} - \frac{1}{480\pi^2 Z^4}$$

for spin zero

$$\frac{7\pi^2 T^4}{120} + \frac{T^2}{48Z^2} - \frac{17}{1920\pi^2 Z^4} \quad (2.3)$$

for spin 1/2 and

$$\frac{\pi^2 T^4}{15} + \frac{T^2}{6Z^2} - \frac{11}{240\pi^2 Z^4}$$

for spin 1. All fields are massless.

$T$ is the local Tolman temperature $T = T_0/Z = 1/\beta Z$. The zero temperature values were evaluated by Candelas and Deutsch [13] using the same Minkowski subtraction. When $\beta = 2\pi$, the Rindler finite-temperature Green function equals the standard Minkowski one and the thermal averages are zero. The usual statement is that the Minkowski vacuum is a thermal Rindler state at temperature $T_0 = 1/2\pi$.

The thermal averages diverge as the horizon is approached just as the ordinary Casimir vacuum averages diverge when a spatial boundary is approached. The quick way of seeing this is when the method of images applies to the construction of the Green function. All this is well known. Its relevance regarding the calculations in [1–3] depends on the significance of the Minkowski subtraction.

For the time being, we assume that the global quantities are obtained by integration of the local ones. Thus,

$$E(\beta) = \int \langle T^0_0 \rangle ZdZdx dy.$$ 

Integrating over the range $\epsilon < Z < \infty$, and letting $x$ and $y$ run over a region of area $A$, we see, trivially, that $E(\beta)$ will be proportional to $A$ and will diverge as $1/\epsilon^2$. For spin zero for example

$$E(\beta, \epsilon, A) = \frac{A\pi^2}{60\epsilon^2 \beta^4} - \frac{A}{960\pi^2 \epsilon^2} = \frac{A\pi^2}{60\epsilon^2 \beta^4} + E(\infty, \epsilon, A)$$

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and of course this vanishes when $\beta = 2\pi$, corresponding to the Minkowski subtraction.

The $T$-dependent terms in (2.3) are the finite-temperature corrections to the zero temperature quantities. We shall denote such quantities by a prime, e.g. $\langle T^0_0 \rangle'$ and $E'((\beta)$ and so on.

To obtain the entropy, we need the free energy, related to $E$ by

$$E = \frac{\partial}{\partial \beta}(\beta F)$$

so that, quite generally,

$$\beta F(\beta) = \int_0^\beta E(\beta)d\beta + C = \int_0^\beta E'(\beta)d\beta + \beta E(\infty) + C$$

where $C$ is a temperature independent constant. Then

$$S = \beta E' - \int_0^\beta E'(\beta)d\beta - C \quad (2.4)$$

For spin zero

$$S(\beta, \epsilon, A) = \beta E'(\beta, \epsilon, A) + \frac{A\pi^2}{180\epsilon^2\beta^3} - C = \frac{A\pi^2}{45\epsilon^2\beta^3} - C.$$ 

Therefore

$$S(2\pi, \epsilon, A) = \frac{A}{360\epsilon^2\pi} - C \quad (2.5)$$

The significance of the Minkowski subtraction must now be fairly faced since it means that at $\beta = 2\pi$ all thermodynamical quantities like $E$, $F$ and $S$ vanish, the constant $C$ being adjusted to make this so.

The finite-temperature corrections would be the only terms obtained if, instead of removing the Minkowski Green function, the zero temperature ($\beta = \infty$) one were subtracted. This was the procedure adopted in [12]. Then we would have for $\langle T^0_0 \rangle'$

$$\frac{\pi^2 T^4}{30}$$

for spin zero

$$\frac{7\pi^2 T^4}{120} + \frac{T^2}{48Z^2} \quad (2.6)$$

for spin 1/2 and

$$\frac{\pi^2 T^4}{15} + \frac{T^2}{6Z^2}$$
for spin 1.

This corresponds to setting the constant $C$ equal to zero so that the zero temperature entropy vanishes. The resulting thermodynamic quantities are those that would arise by calculating the partition function in the standard sum-over-states way. Equation (2.5) agrees with the evaluations in [1–3]. For the record we give the other spin results,

$$S'_{1/2}(2\pi, \epsilon, A) = \frac{11A}{720\pi \epsilon^2}$$

$$S'_1(2\pi, \epsilon, A) = \frac{4A}{45\pi \epsilon^2}.$$

In two dimensions [12]

$$\langle T_0^0 \rangle' = \frac{\pi T^2}{6}$$

which for $\beta = 2\pi$ is Davies’ result [14]. Substitution into (2.4) yields

$$S'(\beta, \epsilon) = \frac{\pi}{3\beta} \ln(D/\epsilon)$$

where $D$ is an upper limit to the distance from the horizon. Therefore

$$S'(2\pi, \epsilon) = \frac{1}{6} \ln(D/\epsilon),$$

which again agrees with the recent calculations.

3. de Sitter space and a two-dimensional black hole.

In [12] de Sitter space was also treated by the same thermal technique of re-periodisation. The metric can be written to exhibit a Rindler-like part,

$$ds^2 = \frac{4a^2}{(1 + Z^2)^2} \left( Z^2 d(t/a)^2 - dZ^2 \right) - a^2 \left( \frac{1 - Z^2}{1 + Z^2} \right)^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

$Z$ is related to the usual static radial coordinate $r$ by $Z^2 = (a - r)/(a + r)$, $a$ being the radius of the de Sitter sphere.

Calculation shows that, when written in terms of the local temperature $T$, $\langle T_0^0 \rangle'$ takes the same form as for Rindler space. The integrated quantity is then

$$E'(\beta, \epsilon, A) = \frac{\pi^2}{120a\beta^4} \int_{\epsilon/2a}^{1} \frac{(1 - Z^2)^2}{Z^3} dZ \int d\Omega \approx \frac{A\pi^2}{60a\beta^4 \epsilon^2}$$
and so the leading term in the entropy is

\[ S'(\beta, \epsilon, A) \approx \frac{A\pi^2}{45\epsilon^2\beta^3} \]

and

\[ S'(2\pi, \epsilon, A) \approx \frac{A}{360\epsilon^2\pi} \]

which again diverges as the (Z-radial) distance from the horizon \( \epsilon \) tends to zero. In this case, \( A = 4\pi a^2 \) is the actual area of the horizon.

The two-dimensional black-hole obtained by removing the angular dependence from the usual Schwarzschild solution is also easily treated as it is conformally Rindler. \( \langle T^0_0 \rangle' \) is again given by (2.7) but now with \( T^{-1} = 4M\beta(1 - 2M/r)^{1/2} \). Calculations similar to those already detailed give the leading forms

\[ E'(\beta, \epsilon) \approx \frac{\pi}{24\beta^2} \ln(D/\epsilon), \quad S'(\beta, \epsilon) \approx \frac{\pi}{3\beta} \ln(D/\epsilon) \]

and

\[ S'(2\pi, \epsilon) \approx \frac{1}{6} \ln(D/\epsilon) \]

as \( \epsilon \), the Z-radial distance from the horizon, tends to zero. Note that the variable corresponding to the Rindler Z is here dimensionless and equals \( (r/2M - 1)^{1/2} \ln(r/4M) \) in terms of the usual radial coordinate.

### 4. Comments

We chose to open our discussion with the cosmic string results but, of course, it is not necessary to go through these in order to obtain the finite-temperature corrections. The method of [12] is a direct thermal calculation along the lines of Brown and Maclay’s Casimir effect analysis [15]. The results take the particular form that they do because of conformal relations between certain space-times, as discussed in [16]. Actually, the detailed calculations of [12] involving coincidence limits can be bypassed by using the fact that Rindler space-time is conformal to the open Einstein Universe and that the heat-kernel expansion terminates on such a space. This leads onto the topic of the global evaluation of thermodynamic quantities, which we briefly mention, leaving a closer examination for a later communication.

We have obtained the total energy, entropy etc. by integrating the local densities and the integration had to be stopped a distance \( \epsilon \) from the horizon. Regarding the divergence of the integrated quantities, it is relevant to point out that in the
ordinary Casimir effect it can happen that, although the local energy density diverges as the boundary is approached, a finite global energy can be defined starting from a renormalised, or even a finite, effective (one-loop) action. A possible way of reconciling these two total energies is to introduce a divergent boundary energy to compensate for the infinity caused by integrating the local density right up to the boundary. The amplification of this statement involves a rather technical discussion and we refer to [17,18]. A recent, relevant implementation of this idea is [19].

Maybe it is worth adding that it is possible to allow for a chemical potential, and therefore a charged black hole, by a flux through the conical singularity.

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