QUADRATIC ORDER CONDITIONS FOR AN EXTENDED WEAK MINIMUM IN OPTIMAL CONTROL PROBLEMS WITH INTERMEDIATE AND MIXED CONSTRAINTS

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Abstract. We consider a general optimal control problem with intermediate and mixed constraints. Using a natural transformation (replication of the state and control variables), this problem is reduced to a standard optimal control problem with mixed constraints, which makes it possible to obtain quadratic order conditions for an “extended” weak minimum. The conditions obtained are applied to the problem of light refraction.

1. Introduction. Let \( t_0 < t_1 < \ldots < t_\nu \) be real numbers. For any \( n \)-dimensional continuous function \( x(t) \) on the interval \([t_0, t_\nu]\), define a vector

\[
p = \left( (t_0, x(t_0)), (t_1, x(t_1)), \ldots, (t_\nu, x(t_\nu)) \right) \in \mathbb{R}^{(1+\nu)(1+n)}.
\]

On the interval \([t_0, t_\nu]\) consider the following optimal control problem:

\[
\begin{align*}
\dot{x} &= f(t, x, u), \\
g(t, x, u) &= 0, \\
\eta_j(p) &= 0, \quad j = 1, \ldots, q, \\
\varphi_i(p) &\leq 0, \quad i = 1, \ldots, h, \\
J &= \varphi_0(p) \to \min,
\end{align*}
\]

where \( t_0, t_1, \ldots, t_\nu \) are not fixed, \( x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^r \), the function \( x(\cdot) \) is absolutely continuous, \( u(\cdot) \) measurable and bounded, \( g(t, x, u) \in \mathbb{R}^d \). The number \( \nu \) is fixed.

The problem \( A \) contains intermediate constraints, i.e., equality and inequality constraints involving the state values not only at the endpoints of the interval \([t_0, t_\nu]\), but also at intermediate points \( t_1, t_2, \ldots, t_{\nu-1} \). Moreover, there are mixed constraints of equality type \( g(t, x, u) = 0 \). If \( \nu = 1 \), i.e., there are no intermediate...
points, then problem \( A \) is the Lagrange problem of classical calculus of variations stated in the Pontryagin form with additional mixed equality constraints.

We assume that

A1) the functions \( f \) and \( g \) are defined and continuous on an open set \( Q \subset \mathbb{R}^{1+n+r} \) together with their partial derivatives in \( t, x, u \) up to the second order;

A2) the functions \( \varphi_i \) and \( \eta_j \) are defined and twice differentiable on an open set \( \mathcal{P} \subset \mathbb{R}^{(1+\nu)(1+n)} \);

A3) at any point \((t, x, u)\) such that \( g(t, x, u) = 0 \), the matrix \( g_u(t, x, u) \) has rank \( d \).

**Definition 1.1.** An arbitrary pair \( w = (x, u) \) of functions of the above class defined on an interval \( \Delta = [t_0, t_\nu] \), together with a vector \( \theta = (t_0, t_1, \ldots, t_\nu) \), where \( t_0 < t_1 < \ldots < t_\nu \), will be called a process of problem \( A \).

**Definition 1.2.** A process \((x(t), u(t), \theta)\), \( t \in \Delta = [t_0, t_\nu] \), is admissible if the corresponding vector \( p \in \mathcal{P} \), the triple \((t, x(t), u(t)) \) \( \in Q \) a.e. on \( \Delta \), and all the constraints are satisfied.

Our aim is to obtain quadratic order conditions for a weak minimum in problem \( A \). To this end, we reduce this problem to a standard optimal control problem without intermediate constraints.

As a standard optimal control problem, we take the following canonical optimal control problem with mixed equality type constraints on a fixed time interval \([0, T]\) (see e.g. [13, 14]):

\[
\begin{aligned}
\dot{x} &= f(t, x, u), \\
g(t, x, u) &= 0, \\
\eta_j(p) &= 0, \quad j = 1, \ldots, q, \\
\varphi_i(p) &\leq 0, \quad i = 1, \ldots, h, \\
J &= \varphi_0(p) \rightarrow \min,
\end{aligned}
\]

Problem \( S \):

Here \( f \) and \( g \) are, as before, defined on an open set \( Q \subset \mathbb{R}^{1+n+r} \), the functions \( \varphi_i \) and \( \eta_j \) are defined on an open set \( \mathcal{P} \subset \mathbb{R}^{2n} \), the vector \( p = p(x) = (x(0), x(T)) \in \mathcal{P} \) represents the endpoints of the trajectory \( x(t) \).

Recall the definition of a weak minimum in problem \( S \).

**Definition 1.3.** An admissible process \( w^0 = (x^0(t), u^0(t)) \) provides a weak minimum in problem \( S \) if there exists an \( \varepsilon > 0 \) such that, for any admissible process \( w = (x(t), u(t)) \) satisfying the inequalities \( \|x - x^0\|C < \varepsilon, \|u - u^0\|\infty < \varepsilon \), one has \( J(p(x^0)) \leq J(p(x)) \).

In problem \( A \), however, the interval \( \Delta = [t_0, t_\nu] \) and all subintervals \( \Delta_k = [t_{k-1}, t_k] \) are not fixed, so the definition of a weak minimum should be modified.

We propose first some auxiliary notions.

Let \( f : \Delta_1 \rightarrow \mathbb{R}^n \) be an arbitrary continuous function defined on a closed interval \( \Delta_1 \) and let be given another closed interval \( \Delta_2 \). Extend \( f \) by a constant to the whole real line outside \( \Delta_1 \) preserving its continuity, and then take its restriction to \( \Delta_2 \). The obtained function \( \tilde{f} : \Delta_2 \rightarrow \mathbb{R}^n \) will be called translation of the function \( f \) from the interval \( \Delta_1 \) to the interval \( \Delta_2 \).
Using the notion of translation, we introduce the notion of $\varepsilon-$closeness of functions. Let be given a continuous function $f^0(t)$ defined on an interval $\Delta^0 = [t^0_0, t^0_{11}]$, and a number $\varepsilon > 0$.

**Definition 1.4.** We will say that a measurable function $f(t)$ defined on an interval $\Delta = [t_0, t_1]$ is $\varepsilon-$close to $f^0(t)$ if $|t^0_k - t_k| < \varepsilon$, $|t^0_0 - t_0| < \varepsilon$, and the translation $\tilde{f}^0$ of the function $f^0$ to the interval $\Delta$ satisfies the inequality $|\tilde{f}^0(t) - f(t)| < \varepsilon$ a.e. on $\Delta$.

Define also the notion of $\varepsilon-$closeness of processes in problem $A$.

For any admissible process $w = (x(t), u(t), \theta)$, $t \in \Delta$ of problem $A$ and any natural $k = 1, \ldots, \nu$, the restriction $w_k = (x_k, u_k)$ of the process $w$ on any subinterval $\Delta_k = [t_{k-1}, t_k]$ will be called a partial process.

**Definition 1.5.** We will say that a process $w = (x(t), u(t), \theta)$, $t \in \Delta$ is $\varepsilon-$close to a process $w^0 = (x^0(t), u^0(t), \theta^0)$, $t \in \Delta^0$, where $u^0(t)$ is a piecewise-continuous function with possible discontinuities only at points $t^0_k$ if all the partial processes $w_k$ are $\varepsilon-$close to the partial processes $w^0_k$, i.e., $|t^0_k - t_k| < \varepsilon$, $k = 0, \ldots, \nu$, and the translation of any partial process $w^0_k$ from the interval $\Delta^0_k$ to the interval $\Delta_k$ satisfies the inequalities $\|\tilde{w}^0_k - u_k\|_\infty < \varepsilon$, $\|\tilde{x}^0_k - x_k\|_C < \varepsilon$.

Note that, due to the smoothness of $g$, definition 5 implies that, for any $\varepsilon-$close admissible process $w = (x(t), u(t), \theta)$, $t \in \Delta$, there exists $C > 0$ such that all its partial processes $w_k$ satisfy the inequality $|g(t, \tilde{x}^0(t), \tilde{u}^0(t)) - g(t, x(t), u(t))| < C\varepsilon$ for almost all $t \in \Delta$.

Now we can give a notion of extended weak minimum in problem $A$.

**Definition 1.6.** We will say that an admissible process $w^0 = (x^0(t), u^0(t), \theta^0)$, $t \in \Delta^0$, where $u^0(t)$ is a piecewise continuous function with possible discontinuities at points $t^0_k$, provides an extended weak minimum in problem $A$ if there exists $\varepsilon > 0$ such that, for any $\varepsilon-$close admissible process $w = (x(t), u(t), \theta)$, $t \in \Delta$, with a measurable function $u(t)$, one has $J(p_0(x)) \geq J(p_0(\tilde{x}^0))$.

Thus, the base process $w^0$ has a piecewise-continuous control, but it is subject to comparison with all admissible $\varepsilon-$ close processes with arbitrary measurable controls. Note that after the translation of partial processes $w^0_k$ to $\Delta_k$, the resulting process $\tilde{w}^0$ can fail to be admissible. The procedure of translation is used here only for estimating the closeness of the processes.

**2. First and second order conditions in problem $S$.** Recall here the known first and second order conditions for a weak minimum in problem $S$. Note that we consider $x, u, g, \varphi, \eta$ as column vectors, while the corresponding multipliers as row vectors.

Let a process $w^0 = (x^0(t), u^0(t))$, $t \in [0, T]$, provide a weak minimum in problem $S$. Then, according to [9] (see also [10, 14]), there exists a collection of Lagrange multipliers $\lambda = (\alpha, \beta, \psi, m)$, where $\alpha = (\alpha_0, \ldots, \alpha_\nu) \geq 0$, $\beta = (\beta_1, \ldots, \beta_\nu)$, $\psi(\cdot)$ is an $n-$dimensional Lipschitz function, and $m(\cdot) \in L^\infty_{[0,T]}$ (all are row vectors), that generate the Pontryagin function $H = \psi f(t, x, u)$, the extended Pontryagin function $\overline{H} = H - mg(t, x, u)$, and the endpoint Lagrange function $l(p) = \alpha \varphi(p) + \beta \eta(p)$, such that the following conditions hold:
a) normalization: $|\alpha| + |\beta| = 1$,  
b) complementary slackness: $\alpha_i \varphi_i(p^0) = 0$, $i = 1, \ldots, h$;  
c) adjoint equation: a.e. on $[0, T]$  
$$-\dot{\psi}(t) = \overline{H}_x = \psi(t) f'_x(t) - m(t) g'_x(t),$$  
d) transversality:  
$$\psi(0) = l_x(0)(p^0), \quad \psi(T) = -l_x(T)(p^0),$$  
e) stationarity of $\overline{H}$ with respect to $u$: for a.e. $t \in [0, T]$  
$$\overline{H}^0_u = \psi(t)f'_x(t) - m(t) g'_x(t) = 0,$$

where all the derivatives of $f$ and $g$ are taken at the point $(t, x_0(t), u_0(t))$.

Denote the set of all such collections $\lambda$ by $\Lambda$. Obviously, it is a finite-dimensional compactum, parametrized by the pair $(\alpha, \beta)$.

Now, let us pass to the quadratic order conditions. Suppose that a process $w^0 = (x^0(t), u^0(t))$ satisfy the Euler–Lagrange (EL) equation with Lagrange multipliers $\Lambda = \{\lambda\}$, where each $\lambda = (\alpha, \beta, \psi, m)$. Let $I = \{0\} \bigcup \{i : \varphi_i(p^0) = 0\}$ be the corresponding set of active indices.

For any $\lambda \in \Lambda$ define the Lagrange function  
$$\Phi[\lambda](w) = l[\lambda](p) + \int_0^T \left(\psi(t) \dot{x}(t) - \overline{H}(t, x(t), u(t))\right) dt,$$

and consider its second variation at $w^0$:

$$(\Phi''[\lambda](w^0)\BAR{w}, \BAR{u}) = (l''[\lambda](p^0)\BAR{p}, \BAR{p}) -$$

$$- \int_0^T \left((\overline{H}'_{xx}[\lambda](t)\BAR{x}(t), \BAR{x}(t)) + 2(\overline{H}'_{xu}[\lambda](t)\BAR{x}(t), \BAR{u}(t)) + (\overline{H}'_{uu}[\lambda](t)\BAR{u}(t), \BAR{u}(t))\right) dt,$$

where all the derivatives of $\overline{H}(t, x, u)$ are taken at the point $(t, x_0(t), u_0(t))$.

Define the function  
$$\Omega[\lambda](\BAR{w}) = \max_{\lambda \in \Lambda} \left(\Phi''[\lambda](w^0)\BAR{w}, \BAR{u}\right)$$

and the quadratic order  
$$\gamma(\BAR{w}) = |\BAR{x}(0)|^2 + \int_0^T |\BAR{u}(t)|^2 dt.$$

These functions should be considered on the cone of critical variations  
$$\mathcal{K} = \left\{\BAR{w} = (\BAR{x}, \BAR{u}) : \dot{x} = f_x(t, x_0, u_0)\BAR{x} + f_u(t, x_0, u_0)\BAR{u}, \right.$$  
$$g_x(t, x_0, u_0)\BAR{x} + g_u(t, x_0, u_0)\BAR{u} = 0, \right.$$  
$$\varphi_i'(p^0)\BAR{p} \leq 0, \quad i \in I, \quad \eta_j'(p^0)\BAR{p} = 0, \quad j = 1, \ldots, q\right\}.$$

Here one can consider the space $\BAR{u} \in L^r_\infty[0, T]$ instead of the original space $\BAR{u} \in L^r_\infty[0, T]$. This can be shown by the lemma on denseness from [6].

**Theorem 2.1 (Quadratic order conditions in problem S).**  

a) If the process $w^0$ provides a weak minimum in problem $S$, then $\Omega[\lambda](\BAR{w}) \geq 0$ for any $\BAR{w} \in \mathcal{K}$.  

b) If $\exists c > 0$ such that $\Omega[\lambda](\BAR{w}) \geq c\gamma(\BAR{w})$ for any $\BAR{w} \in \mathcal{K}$, then the process $w^0$ provides a strict weak minimum in problem $S$. 
The proof can be found in [14]. If the endpoint inequalities are absent, and \(\Lambda\) consists of a single collection \(\lambda\), this theorem is a well known fact of classical calculus of variations, see e.g. [3]. Note that in part b) one can actually assert more [14]: \(\exists c' > 0\) and \(\varepsilon > 0\) such that, for any pair \((x, u)\) satisfying the inequalities \(\|x - x^0\|_C < \varepsilon\), \(\|u - u^0\|_\infty < \varepsilon\), the violation function

\[
\sigma(x, u) = \int_0^T |\dot{x} - f(t, x, u)| \, dt + \max_{t \in [0, T]} |g(t, x, u)| + \sum |\eta_j(p)| + \sum \varphi_i^+(p) + (\varphi_0(p) - \varphi_0(p^0))^+,
\]

where \(a^+ = \max\{0, a\}\), satisfies the estimate \(\sigma(x, u) \geq c' \gamma(x - x^0, u - u^0)\).

3. First order conditions in problem A. To obtain optimality conditions in problem A, we use a quite natural trick, in fact, a change of variables, which was used earlier by a number of authors, both for obtaining optimality conditions (probably, the first work was [5], see also [16, 4, 2, 7, 8]) and for constructing numerical algorithms (see e.g. [12, 17]). Recently, it was used for obtaining quadratic order conditions for an extended weak minimum in problem A without mixed constraints [11].

Let \(w^0 = (x^0(t), u^0(t), \theta^0)\), \(t \in [t_0^0, t_\nu^0]\) be an admissible process in problem A. Following [7, 8], introduce a new time \(\tau \in [0, 1]\) and define the functions

\[
z_k(\tau) = t_k - t_{k-1}, \quad \rho_k : [0, 1] \to [t_{k-1}, t_k], \quad \rho_k(\tau) = t_k + z_k(\tau) \tau, \quad k = 1, \ldots, \nu.
\]

The function \(\rho_k\) plays the role of time \(t\) on the interval \(\Delta_k = [t_{k-1}, t_k]\), while \(z_k\) is the length of \(\Delta_k\).

Obviously, \(z_k(\tau)\) and \(\rho_k(\tau)\) satisfy the equations

\[
\frac{dz_k}{d\tau} = 0, \quad \frac{d\rho_k}{d\tau} = z_k, \quad \rho_k(0) = t_{k-1}, \quad \rho_k(1) = t_k.
\]

Define the functions

\[
y_k(\tau) = x(\rho_k(\tau)), \quad v_k(\tau) = u(\rho_k(\tau)), \quad k = 1, \ldots, \nu, \quad \tau \in [0, 1],
\]

which satisfy the relations:

\[
\begin{align*}
\frac{dy_k}{d\tau} &= z_k f(\rho_k, y_k, v_k), \\
\frac{d\rho_k}{d\tau} &= z_k, \\
\frac{dz_k}{d\tau} &= 0, & k = 1, \ldots, \nu, \\
g(\rho_k, y_k, v_k) &= 0, & k = 1, \ldots, \nu,
\end{align*}
\]

\[
\begin{cases}
y_2(0) - y_1(1) = 0, \\
y_3(0) - y_2(1) = 0, \\
\vdots \\
y_\nu(0) - y_{\nu-1}(1) = 0,
\end{cases}
\]
\[
\begin{align*}
\rho_2(0) - \rho_1(1) &= 0, \\
\rho_3(0) - \rho_2(1) &= 0, \\
\vdots & \quad \vdots \\
\rho_v(0) - \rho_{v-1}(1) &= 0,
\end{align*}
\]

(6)

where, for the sake of brevity, we set

\[
\begin{align*}
\eta_j(\bar{p}) &= 0, \quad j = 1, \ldots, q, \\
\varphi_i(\bar{p}) &\leq 0, \quad i = 1, \ldots, h,
\end{align*}
\]

(7)

The obtained optimal control problem will be called problem \( \tilde{A} \). Here the state variables are \( z_k, \rho_k \) and \( y_k \), while the controls are \( v_k, \quad k = 1, \ldots, v \); the time interval \([0,1]\) is fixed. The open set \( \mathcal{Q} \) consists here of all tuples \( (z_k, \rho_k, y_k, v_k) \), where \( (\rho_k, y_k, v_k) \in \mathcal{Q} \) and \( z_k > 0 \). The open set \( \mathcal{P} \) consists of all vectors \( \bar{p} \), for which the “truncated” vector \( \bar{p} \in \mathcal{P} \).

Let us find the correspondence between the admissible processes in problems \( A \) and \( \tilde{A} \).

Denote by \( F \) the mapping that maps any admissible process \( w = (x(t), u(t), \theta) \) of problem \( A \) into the admissible process \( \bar{w} = (z(\tau), \rho(\tau), y(\tau), v(\tau)) \) of problem \( \tilde{A} \), obtained by relations (1) and (2). Since \( p(t,x) = \bar{p}(\rho, y) \), we have \( J(w) = J(F(w)) \), i.e. \( F \) preserves the value of the cost.

Using the relations

\[
x(t) = y_k(\rho_k^{-1}(t)), \quad u(t) = v_k(\rho_k^{-1}(t)), \quad t \in \Delta_k,
\]

(8)

one can as well define the inverse mapping \( G = F^{-1} \), which also preserves the value of the cost.

The constructed mappings \( F \) and \( G \) possess the following properties.

**Theorem 3.1.** If a process \( w^0 = (x^0(t), u^0(t), \theta^0) \), where \( u^0(\cdot) \) is a piecewise-continuous function with possible discontinuities at points \( t^0_k \), provides an extended weak minimum in problem \( A \), then the process \( \tilde{w}^0 = F(w^0) \) provides a weak minimum in problem \( \tilde{A} \), and vice versa, if a process \( \tilde{w}^0 = (z^0(\tau), \rho^0(\tau), y^0(\tau), v^0(\tau)) \) with continuous controls \( v^0_k \) provides a weak minimum in problem \( \tilde{A} \), then the process \( w^0 = G(\tilde{w}^0) \) provides an extended weak minimum in problem \( A \).

The proof is similar to the proof of Theorem 2 in \([11]\) related to problem \( A \) without mixed constraints.

Thus, the study of the extended weak minimality of a process \( w^0 \) in problem \( A \) is reduced to the study of the weak minimality of the process \( \tilde{w}^0 = F(w^0) \) in problem \( \tilde{A} \).
Problem $\tilde{A}$ is a particular case of the canonical optimal control problem $S$. If a process $\tilde{w}^0$ provides a weak minimum in problem $\tilde{A}$, it satisfies the Euler–Lagrange equation, i.e., there exists a collection of Lagrange multipliers

$$\hat{\lambda} = (\alpha, \beta, \gamma, \delta, \psi_i(\cdot), \psi_\rho(\cdot), \psi_z(\cdot), \tilde{m}(\cdot)),$$

where $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_k) \in \mathbb{R}^{h+1}$, $\alpha \geq 0$, $\beta = (\beta_1, \beta_2, \ldots, \beta_q) \in \mathbb{R}^q$, $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{\nu-1}) \in \mathbb{R}^{\nu-1}$, $\delta = (\delta_1, \delta_2, \ldots, \delta_{\nu-1}) \in \mathbb{R}^{\nu-1}$, $\psi_i = (\psi_{i_1}, \ldots, \psi_{i_v})$, and $\psi_z = (\psi_{z_1}, \ldots, \psi_{z_{\nu}})$ are Lipschitz functions on $[0,1]$, and $\tilde{m} = (\tilde{m}_1, \ldots, \tilde{m}_\nu) \in L^\infty_\nu[0,1]$, that generate the Pontryagin function

$$\mathcal{H} = \sum_{k=1}^{\nu} z_k \left( \psi_{y_k} f(\rho_k, y_k, v_k) + \psi_\rho_k \right) = \sum_{k=1}^{\nu} z_k \Pi_k(\rho_k, y_k, v_k),$$

(9)

where $\Pi_k(\rho_k, y_k, v_k) = \psi_{y_k} f(\rho_k, y_k, v_k) + \psi_\rho_k$, the extended Pontryagin function

$$\overline{\mathcal{H}} = \sum_{k=1}^{\nu} z_k \left( \Pi_k(\rho_k, y_k, v_k) - \tilde{m}_k g(\rho_k, y_k, v_k) \right),$$

and the endpoint Lagrange function

$$\tilde{l}(\tilde{v}) = l(\tilde{v}) + \sum_{k=1}^{\nu-1} \gamma_k (y_{k+1}(0) - y_k(1)) + \sum_{k=1}^{\nu-1} \delta_k (\rho_{k+1}(0) - \rho_k(1)),$$

where

$$l(\tilde{v}) = \sum_{i=0}^{h} \alpha_i \phi_i(\tilde{v}) + \sum_{j=1}^{q} \beta_j \eta_j(\tilde{v}),$$

in terms of which the following conditions should hold:

a) nontriviality: $(\alpha, \beta, \gamma, \delta) \neq (0,0,0,0)$;

b) complementary slackness: $\alpha_i \phi_i(\tilde{v}^0) = 0$, $i = 1, \ldots, h$;

c) adjoint equations: for all $k = 1, \ldots, \nu$,

$$-\dot{\psi}_{y_k}(\tau) = \overline{\mathcal{H}}_{y_k} = z_k^0 (\psi_{y_k} f_x - \tilde{m}_k g_x),$$
$$-\dot{\psi}_\rho_k(\tau) = \overline{\mathcal{H}}_{\rho_k} = z_k^0 (\psi_{y_k} f_t - \tilde{m}_k g_t),$$
$$-\dot{\psi}_{z_k}(\tau) = \overline{\mathcal{H}}_{z_k} = \Pi_k = \psi_{y_k} f + \psi_\rho,$$

(10)

(11)

d) stationarity in $v_k$: for all $k = 1, \ldots, \nu$,

$$\overline{\mathcal{H}}_{v_k} = z_k^0 (\psi_{y_k} f_u - \tilde{m}_k g_u) = 0,$$

(12)

(all the derivatives of $f$ and $g$ are taken at the point $(\rho_k^0, y_k^0, v_k^0)$);

e) transversality:

$$\begin{cases}
\psi_{y_1}(0) = l_{y_1}(0), & \psi_{y_1}(1) = \gamma_1, \\
\psi_{y_2}(0) = l_{y_2}(0) + \gamma_1, & \psi_{y_2}(1) = \gamma_2, \\
\vdots & \vdots \\
\psi_{y_{\nu-1}}(0) = l_{y_{\nu-1}}(0) + \gamma_{\nu-2}, & \psi_{y_{\nu-1}}(1) = \gamma_{\nu-1}, \\
\psi_{y_{\nu}}(0) = l_{y_{\nu}}(0) + \gamma_{\nu-1}; & \psi_{y_{\nu}}(1) = -l_{y_{\nu}}(1);
\end{cases}$$
\[
\begin{aligned}
\psi_{p_1}(0) &= l_{p_1}(0), & \psi_{p_1}(1) &= \delta_1, \\
\psi_{p_2}(0) &= l_{p_2}(0) + \delta_1, & \psi_{p_2}(1) &= \delta_2, \\
\vdots & & \\
\psi_{p_{\nu-1}}(0) &= l_{p_{\nu-1}}(0) + \delta_{\nu-2}, & \psi_{p_{\nu-1}}(1) &= \delta_{\nu-1}, \\
\psi_{p_\nu}(0) &= l_{p_\nu}(0) + \delta_{\nu-1}, & \psi_{p_\nu}(1) &= -l_{p_\nu}(1),
\end{aligned}
\]

(all the derivatives of \(l(p)\) are taken at the point \(\bar{p}^0\), and
\[
\psi_{z_k}(0) = \psi_{z_k}(1) = 0, \quad k = 1, \ldots, \nu.
\]

Let us analyze these conditions. First, we note that the last equalities together with (11) give the integral relations
\[
\int_0^1 \Pi_k(\rho_k^0, y_k^0, v_k^0) \, d\tau = 0, \quad k = 1, \ldots, \nu. \tag{13}
\]

The condition of nontriviality can be simplified by the following

**Lemma 3.2.** \(|\alpha| + |\beta| + |\gamma| + |\delta| > 0\) if and only if \(|\alpha| + |\beta| > 0\).

**Proof.** Note that the functions \(\tilde{m}_k(\tau)\) can be linearly expressed through \(\psi_{y_k}(\tau)\) from the relations (12) in view of nondegeneracy of the matrix \(g_w(y_k^0, y_k^0, v_k^0)\), so \(\psi_{y_k}(\tau)\) satisfies a linear homogeneous ODE. If \(\alpha = 0\) and \(\beta = 0\), then \(l(p) \equiv 0\), and the transversality gives \(\psi_{y_1}(0) = 0\), hence \(\psi_{y_1}(\tau) \equiv 0\), and so \(\gamma_1 = \psi_{y_1}(1) = 0\). Then \(\psi_{y_2}(0) = 0\), hence \(\psi_{y_2}(\tau) \equiv 0\), and so on. We obtain that all \(\psi_{y_k}(\tau) \equiv 0\), \(\tilde{m}_k(\tau) \equiv 0\), and \(\gamma_k = 0\). Therefore, \(\psi_{y_k}(\tau) = 0\), and again, the transversality gives that all \(\psi_{y_k}(\tau) \equiv 0\) and \(\delta_k = 0\).

Note that the multipliers \(\gamma_k\) and \(\delta_k\) come only into the transversality conditions e), which can be rewritten in the equivalent form without these multipliers:

\[
\begin{aligned}
\psi_{y_1}(0) &= l_{y_1}(0)(\bar{p}^0), \\
\psi_{y_{k+1}}(0) - \psi_{y_k}(1) &= l_{y_{k+1}}(0)(\bar{p}^0), & k &= 1, \ldots, \nu - 1, \\
\psi_{y_\nu}(1) &= -l_{y_\nu}(1)(\bar{p}^0), \\
\psi_{p_1}(0) &= l_{p_1}(0)(\bar{p}^0), \\
\psi_{p_{k+1}}(0) - \psi_{p_k}(1) &= l_{p_{k+1}}(0)(\bar{p}^0), & k &= 1, \ldots, \nu - 1, \\
\psi_{p_\nu}(1) &= -l_{p_\nu}(1)(\bar{p}^0).
\end{aligned}
\]

The process \(\tilde{u}^0 = (\rho^0(\tau), y^0(\tau), v^0(\tau), z^0(\tau))\) in problem \(\tilde{A}\) corresponds to a process \(u^0 = (x^0(t), u^0(t), \theta^0)\), \(t \in \Delta^0 = [t^0_0, t^0_\nu]\) in problem A, which (if a priori unknown) can be recovered in the following way. Define the moments \(t^0_{k-1} = \rho^0_k(0), k = 1, \ldots, \nu, \quad t^0_\nu = \rho^0_\nu(1)\), and the intervals \(\Delta^0_k = [t^0_{k-1}, t^0_k], k = 1, \ldots, \nu\). On each interval, consider the inverse function \(\sigma^0_k(t) = (\rho^0_k)^{-1}(t)\) and define the functions
\[
\begin{aligned}
x^0(t) &= \psi^0_k(\sigma^0_k(t)), \\
u^0(t) &= v^0_k(\sigma^0_k(t)), \\
\psi_1(t) &= \psi_{y_k}(\sigma^0_k(t)), \\
\psi_2(t) &= \psi_{y_k}(\sigma^0_k(t)), \\
m_k(t) &= \tilde{m}_k(\sigma^0_k(t)).
\end{aligned}
\]
Obviously, these functions are defined on the whole interval $\Delta^0$, and so, the process $w^0 = (x^0(t), u^0(t), \theta^0)$, where $\theta^0 = (t^0_0, \ldots, t^0_\nu)$, is completely determined.

Since $p(t, x) = \hat{p}(\rho, y)$, we have
\[
l_{yx(0)}(p^0) = l_{x(t_0)}(p^0), \quad l_{px(0)}(p^0) = l_{t_k}(p^0), \quad k = 1, \ldots, \nu.
\]

Note that $x(t)$ is Lipschitz continuous on $\Delta^0$, while $\psi_x(t)$ and $\psi_t(t)$ are Lipschitz continuous on each separate $\Delta_k^0$, possibly having jumps at $t^0_k$. We will call the last class of functions piecewise Lipschitz. Define also the functions
\[
H(t, x, u) = \psi_x f(t, x, u) + \psi_t \quad \text{and} \quad \overline{H} = H(t, x, u) - mg(t, x, u).
\]

Since on each separate $\Delta_k^0$ we have $dt = s_k^0 d\tau$, the first state equation in (3) turns into $\dot{x}^0(t) = f(t, x^0(t), u^0(t))$ a.e. on $\Delta^0$, and the adjoint equations (10) into
\[
\dot{\psi}_x(t) = -\overline{H}_x(t, x^0(t), u^0(t)), \quad \dot{\psi}_t(t) = -\overline{H}_t(t, x^0(t), u^0(t)).
\]

Moreover, we have
\[
H(t, x^0(t), u^0(t)) = \Pi_k(\rho^0_k(\sigma^0(t)), y^0_k(\sigma^0(t)), \nu^0_k(\sigma^0(t)))
\]
on each $\Delta_k$, and so, relations (13) take the form
\[
\int_{\Delta_k} H(t, x^0(t), u^0(t)) dt = 0, \quad k = 1, \ldots, \nu.
\]

Thus, the above considerations yield the following result.

**Theorem 3.3 (Euler–Lagrange equation for problem A).** Let a process $w^0 = (x^0(t), u^0(t), \theta^0)$, $t \in \Delta^0 = [t^0_0, t^0_\nu]$, where $u^0(t)$ is a piecewise continuous function with possible discontinuities at points $t^0_k$, provide an extended weak minimum in problem A. Then there exists a collection $\lambda = (\alpha, \beta, \psi_x(\cdot), \psi_t(\cdot), m(\cdot))$, where $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_h) \in \mathbb{R}^{h+1}$, $\alpha \geq 0$, $\beta = (\beta_1, \beta_2, \ldots, \beta_q) \in \mathbb{R}^q$, $m = (m_1, m_2, \ldots, m_q) \in L_{\infty}^q(\Delta^0)$, and $\psi_x$, $\psi_t$ are piecewise Lipschitz functions on $\Delta^0$, that generate the Pontryagin function
\[
H(t, x, u) = \psi_x f(t, x, u) + \psi_t,
\]
the extended Pontryagin function
\[
\overline{H}(t, x, u) = H(t, x, u) - mg(t, x, u),
\]
and the terminal Lagrange function
\[
l(p) = \sum_{i=0}^{h} \alpha_i \varphi_i(p) + \sum_{j=1}^{q} \beta_j \eta_j(p),
\]
in terms of which the following conditions must hold:

a) nontriviality: $(\alpha, \beta) \neq (0, 0)$;

c) complementary slackness: $\alpha_i \varphi_i(p^0) = 0$, $i = 1, \ldots, h$;

d) adjoint equations: almost everywhere on $\Delta^0$
\[
-\dot{\psi}_x(t) = \overline{H}_x = \psi_x f_x(t) - m(t) \theta_x^0(t), \quad -\dot{\psi}_t(t) = \overline{H}_t = \psi_t f_t(t) - m(t) \theta_t^0(t);
\]

e) transversality at the endpoints:
\[
\psi_x(t^0_0) = l_x(t^0_0)(p^0), \quad \psi_x(t^0_\nu) = -l_x(t^0_\nu)(p^0), \quad \psi_t(t^0_0) = l_t(p^0), \quad \psi_t(t^0_\nu) = -l_t(p^0);
\]
f) jump conditions for $\psi_x$ and $\psi_t$ at the intermediate points:

$$
\Delta \psi_x(t^0_0) = \psi_x(t^0_0 + 0) - \psi_x(t^0_0 - 0) = l_x(t_0^0)(p^0),
$$

$$
\Delta \psi_t(t^0_0) = \psi_t(t^0_0 + 0) - \psi_t(t^0_0 - 0) = l_x(t_0^0)(p^0);
$$

g) stationarity of $\mathcal{H}$ with respect to $u$: for all $t \in \Delta^0$

$$
\mathcal{H}_{u}^0 = \psi_x(t) f_u^0(t) - m(t) g_u^0(t) = 0;
$$

h) integral relations: for all $k = 1, \ldots, \nu$

$$
\int_{\Delta^0_k} H(t, x^0(t), u^0(t)) \, dt = 0. \quad (19)
$$

We claim that actually one can write

$$
H(t, x^0(t), u^0(t)) \, dt = 0 \quad \text{on } \Delta^0. \quad (20)
$$

This can be obtained by introducing more intermediate points $t^0_j, \ j = 1, \ldots, N$, including the “old” $t^0_k$, so that the new intervals $\Delta^0_j = [t^0_{j-1}, t^0_j]$ are arbitrarily small. Then the integral relations

$$
\int_{\Delta^0_j} H(t, x^0(t), u^0(t)) \, dt = 0 \quad \text{for all } \Delta^0_j \text{ readily yield (20).}
$$

Since the control $u^0(t)$ is continuous on every “old” interval $\Delta^0_k$, the notion of extended weak minimum would not change. The jump conditions at all additional $t^0_j$ would only say that $\psi_x$ and $\psi_t$ are continuous at these points, i.e. no new conditions to e) and f) would appear.

4. Quadratic order conditions in problem $\tilde{A}$. Denote by $\pi = (z, \rho, \psi, v)$ the quadruple of variables in problem $\tilde{A}$, and by $\tilde{\pi} = (\tilde{z}, \tilde{\rho}, \tilde{\psi}, \tilde{v})$ its variation. Let $\tilde{q}$ be a variation of $\tilde{p}$.

The second variation of the Lagrange function in problem $\tilde{A}$ at the point $\tilde{w}^0$ is

$$
\left(\tilde{\Phi}''[\tilde{\lambda}](\tilde{w}^0), \tilde{\pi}, \tilde{\pi}\right) = \left(\tilde{t}''[\tilde{\lambda}](\tilde{p}^0), \tilde{q}, \tilde{q}\right) - \int_0^1 \left(\mathcal{H}_{\tilde{n}}^0[\tilde{\lambda}](\tau), \tilde{\pi}(\tau), \tilde{\pi}(\tau)\right) \, d\tau.
$$

Consider the functional

$$
\tilde{\Omega}[\tilde{\lambda}](\tilde{\pi}) = \max_{\lambda \in \Lambda} \left(\tilde{\Phi}''[\tilde{\lambda}](\tilde{w}^0), \tilde{\pi}, \tilde{\pi}\right),
$$

and define the quadratic order

$$
\tilde{\gamma}(\tilde{\pi}) = \sum_{k=1}^\nu \tilde{z}_k^2(0) + \sum_{k=1}^\nu \tilde{\rho}_k^2(0) + \sum_{k=1}^\nu |\tilde{y}_k(0)|^2 + \sum_{k=1}^\nu \int_0^1 |\tilde{\psi}_k(\tau)|^2 \, d\tau.
$$

These functions should be considered on the cone of critical variations

$$
\tilde{K} = \left\{ \tilde{\pi} = (\tilde{z}(\tau), \tilde{\rho}(\tau), \tilde{y}(\tau), \tilde{v}(\tau)) : \tilde{v} \in L_2[0, 1], \right. 
\tilde{z}_k = 0, \quad \tilde{\gamma}_k = \tilde{z}_k, \quad \tilde{y}_k = f^0_i \tilde{\rho}_k + f^0_a \tilde{y}_k + f^0_u \tilde{v}_k + f^0_a \tilde{\psi}_k, \\
g^0_{sx} \tilde{\rho}_k + g^0_{xy} \tilde{y}_k + g^0_{su} \tilde{v}_k = 0, \quad s = 1, \ldots, d, \\
\varphi''(\tilde{p}^0) \tilde{q} \leq 0, \quad i \in I, \quad \eta''(\tilde{p}^0) \tilde{q} = 0, \quad j = 1, \ldots, q, \\
\tilde{\rho}_{k+1}(0) - \tilde{\rho}_k(1) = 0, \quad \tilde{y}_{k+1}(0) - \tilde{y}_k(1) = 0, \quad k = 1, \ldots, \nu - 1 \right\}.
$$
where all the derivatives of $f$ and $g$ are taken at $(\rho_k^0(\tau), y_k^0(\tau), v_k^0(\tau))$, and $I$ is the set of active indices.

Theorem 2.1 says that

a) if the process $\bar{w}^0$ provides a weak minimum in problem $\bar{A}$, then $\tilde{\Omega}(\bar{\pi}) \geq 0$ for any $\bar{\pi} \in \bar{K}$;

b) if $\exists c > 0$ such that $\tilde{\Omega}(\bar{\pi}) \geq c \bar{\gamma}(\bar{\pi})$ for any $\bar{\pi} \in \bar{K}$, then the process $\bar{w}^0$ provides a strict weak minimum in problem $\bar{A}$ (and moreover, the corresponding violation function is bounded from below by $\bar{\gamma}$).

**Analysis of quadratic conditions for problem $\bar{A}$.**

For convenience, define the triple $\mu = (\rho, y, v)$, so that $\mu = (\mu_1, \ldots, \mu_\nu)$, where $\mu_k = (\rho_k, y_k, v_k)$, and $\pi = (z, \mu)$. Similarly, we set $\bar{\pi} = (\bar{z}, \bar{\mu})$, where $\bar{\mu} = (\bar{\rho}, \bar{y}, \bar{v})$.

In this notation, the second variation takes the form

$$
\left(\tilde{\Phi}''[\tilde{\lambda}] (\bar{w}^0) \bar{\pi}, \bar{\pi}\right) = \left(l''[\tilde{\lambda}] \bar{q}, \bar{q}\right) - \sum_{k=1}^\nu \int_0^1 \left(\left(\bar{H}^{(\rho)}_{\mu_k}(\tilde{\lambda}(\tau), \bar{\rho}_k(\tau), \bar{v}_k(\tau)) + 2 \bar{z}_k (\bar{H}^{(\mu_k)}_{\rho_k, \mu_k}(\tilde{\lambda}(\tau), \bar{\rho}_k(\tau)) \bar{\mu}_k(\tau))\right) d\tau
$$

(here we took into account that $\bar{H}_{zz} = 0$). Using formula (9), rewrite the last expression as

$$
\left(\tilde{\Phi}''[\tilde{\lambda}] (\bar{w}^0) \bar{\pi}, \bar{\pi}\right) = \left(l''[\tilde{\lambda}] \bar{q}, \bar{q}\right) - \sum_{k=1}^\nu \int_0^1 \left(z^0_k \bar{H}^{(\rho)}_{\mu_k}(\tilde{\lambda}(\tau), \bar{\rho}_k(\tau), \bar{v}_k(\tau)) + 2 \bar{z}_k (\bar{H}^{(\mu_k)}_{\rho_k, \mu_k}(\tilde{\lambda}(\tau), \bar{\rho}_k(\tau)) \bar{\mu}_k(\tau))\right) d\tau,
$$

where $\Pi^k(\mu_k) = \Pi_k(\mu_k) - \bar{m}_k g(\mu_k)$, and its derivatives are taken at the point $\mu_k^0(\tau)$.

Now, consider the process $w^0 = (x^0, u^0) = G(\bar{w}^0)$ of problem $A$, corresponding to the process $\bar{w}^0$, that is defined by formulas (16), and the Lagrange multipliers $\lambda = (\alpha, \beta, \psi_x, \psi_t, m)$, defined by formulas (17). Define the variations

$$
\bar{t}(t) = \bar{\rho}_k (\sigma_k^0(t)), \quad \bar{x}(t) = \bar{y}_k (\sigma_k^0(t)), \quad \bar{u}(t) = \bar{v}_k (\sigma_k^0(t)),
$$

and rewrite the cone $\bar{K}$, the functional $\tilde{\Omega}$, and the order $\bar{\gamma}$ in these new variables.

**a) The cone of critical variations $K$.** Since $K$ involves the relations $\bar{\rho}_{k+1}(0) - \bar{\rho}_k(1) = 0$, $\bar{y}_{k+1}(0) - \bar{y}_k(1) = 0$, the functions $\bar{t}(t)$ and $\bar{x}(t)$ are continuous at $t_k^0$. Moreover, they satisfy the equations on each $\Delta_k^0$:

$$
\frac{d\bar{t}}{dt} = \frac{\bar{z}_k}{\sigma_k^0}, \quad \frac{d\bar{x}}{dt} = f_1^0 \bar{t} + f_x^0 \bar{x} + f_u^0 \bar{u} + \frac{1}{\sigma_k^0} f_u^0 \bar{z}_k,
$$

$$
g_t^0 \bar{t} + g_x^0 \bar{x} + g_u^0 \bar{u} = 0.
$$

Since

$$
\hat{\rho}(\rho, y) = p(t, x) \quad \text{and} \quad \hat{q}(\rho, y) = \bar{p}(t, x),
$$

we have

$$
\varphi'_i (p^0) \bar{q} = \varphi'_i (p^0) \bar{p}, \quad \eta'_j (p^0) \bar{q} = \varphi'_j (p^0) \bar{p}.
$$
Thus, the cone $\mathcal{K}$ in the new variables has the form:

$$\mathcal{K} = \left\{ \bar{w} = (\bar{z}, \bar{t}, \bar{x}, \bar{u}) : \dot{\bar{z}}_k = 0, \quad \dot{\bar{t}} = \dot{\bar{z}}_{k_0} \right\},$$

$$\dot{\bar{x}}(t) \text{ and } \dot{\bar{t}}(t) \text{ are continuous on the whole } \Delta^0,$$

$$g_0^k \bar{t} + g_0^0 \bar{x} + g_0^0 \bar{u} = 0,$$

$$\varphi_i'(\bar{u}) \bar{p} \leq 0, \quad i \in I, \quad \eta_j'(\bar{u}) \bar{p} = 0, \quad j = 1, \ldots, q \}.$$

b) Quadratic form $\Omega$. Relation (22) implies that $(l''(\bar{\lambda}) q, q) = (l''(\lambda) \bar{p}, \bar{p})$.

Define the Pontryagin function for problem $A$: $H(\lambda) = \psi_x f + \psi_t$ and the extended Pontryagin function $\overline{H}(\lambda) = H(\lambda) - mg$. Relation (18) says that $H^0(\lambda)(t) = \Pi_{k_0}[\bar{\lambda}](\sigma^0(t))$ for $t \in \Delta^0_k$.

Set $w = (t, x, u)$ and $\bar{w}(t) = (\bar{t}(t), \bar{x}(t), \bar{u}(t))$, $t \in [t^0_0, t^0_\nu]$. Then, the variation process $w^0$ can be represented as $\bar{w}(t) = \bar{w}(\sigma^0_k(t))$, $k = 1, \ldots, \nu$, and so, the $k$-th term in the integral part of (21) can be written as

$$\int_0^1 \left( z^0_k \Pi^k_{\mu}[\bar{\lambda}](\tau) \bar{w}(\tau) + 2z^0_k (\Pi^k_{\mu}[\bar{\lambda}](\tau), \bar{w}(\tau)) \right) d\tau =$$

$$= \int_{\Delta^0_k} \left( \overline{H}^0_{\bar{w}}(\lambda)(t) \bar{w}(t), \bar{w}(t) \right) dt + 2z^0_k \int_{\Delta^0_k} \left( \overline{H}^0_{\bar{w}}(\lambda)(t) \bar{w}(t) \right) dt,$$

where all the derivatives of $\overline{H}$ are taken at the point $(t, x^0(t), u^0(t))$.

The stationarity condition yields $\Pi_{\overline{w}}[\lambda](t) = \Pi_{\overline{v}}[\bar{\lambda}](\tau(t)) = 0$ for all $t \in [t^0_0, t^0_\nu]$.

Thus, the quadratic form in the new variables is

$$\Omega[\lambda](\bar{w}) = (l''(\bar{\lambda}) \bar{p}, \bar{p}) - \int_{t^0_0}^{t^0_\nu} \left( \overline{H}^0_{\bar{w}}[\lambda](t) \bar{w}(t), \bar{w}(t) \right) dt -$$

$$- 2 \sum_{k=1}^{\nu} z^0_k \int_{\Delta^0_k} \left( \overline{H}^0_{x\bar{w}}[\lambda](t) \bar{x}(t) + \overline{H}^0_{\bar{w}\bar{x}}[\lambda](t) \bar{t}(t) \right) dt.$$

Note that variations $\bar{z}_k$ come in the cone and the quadratic form with the multiplier $1/z^0_k$. Making the change $\bar{z}_k = z_k / z^0_k$, we obtain an equivalent quadratic order $\gamma'$, so the fraction $\bar{z}_k / z^0_k$ in the cone $\mathcal{K}$ and the quadratic form $\Omega$ can be harmlessly changed by the term $\bar{z}_k$, which we will still denote by $\bar{z}_k$.

d) Order $\gamma$. In the new variables, the quadratic order $\gamma$ on the cone $\mathcal{K}$ takes the form:

$$\gamma(\bar{w}) = \sum_{k=1}^{\nu} z^2_k + \sum_{k=0}^{\nu-1} \bar{t}^2(t^0_k) + \sum_{k=0}^{\nu-1} |\bar{x}(t^0_k)|^2 + \int_{t^0_0}^{t^0_\nu} |\bar{u}(t)|^2 dt.$$

Since $\bar{x}$ satisfies on $\Delta^0_k$ an ODE that is linear in $\bar{z}_k, \bar{t}, \bar{x}, \bar{u}$ and the function $\bar{t}(t)$ is linear in $t$, we get the estimate

$$|\bar{x}(t^0_k)|^2 \leq \text{const} \left( |x(t^0_{k-1})|^2 + \bar{z}^2_k + \bar{t}^2(t^0_{k-1}) + \int_{\Delta^0_k} |\bar{u}(t)|^2 dt \right),$$

hence, the terms $|\bar{x}(t^0_k)|^2$, $k = 1, \ldots, \nu - 1$, can be excluded from $\gamma$, remaining only $|\bar{x}(t^0_0)|^2$ (or any one $|\bar{x}(t^0_{k_0})|^2$, excluding all other terms with $k \neq k_0$).
For any \( k = 1, \ldots, \nu \) the value of \( \bar{t}(t) \) at \( t_k^0 \) can be represented as
\[
\bar{t}(t_k^0) = \bar{t}(t_0^0) + \sum_{r=1}^{k} \bar{z}_r t_r^0.
\]
Therefore, \(|\bar{t}(t_k)| \leq \text{const} (|\bar{t}(t_0)| + \sum_{r=1}^{k} |\bar{z}_r|)\), and so, the terms \( \bar{t}^2(t_k^0) \), \( k = 1, \ldots, \nu - 1 \) can also be excluded from \( \gamma \), remaining only \( \bar{t}^2(t_0^0) \) (or any one \( |\bar{t}(t_k^0)|^2 \)), excluding all other terms with \( k \neq k_0 \).

Thus, the order \( \gamma \) on \( \mathcal{K} \) can be taken in the form
\[
\gamma(\bar{w}) = \sum_{k=1}^{\nu} \bar{z}_k^2 + \bar{t}^2(t_0^0) + |\bar{x}(t_0^0)|^2 + \int_{t_0^0}^{t_k^0} |\bar{u}(t)|^2 \, dt. \tag{23}
\]

5. Quadratic conditions for problem A. The above analysis of conditions for problem \( \bar{A} \) gives the following result for problem \( A \).

Let a process \( u^0(t,'u,\theta_0) \), \( t \in [t_k^0, t_0^0] \), where \( u^0(\cdot) \) is a piecewise continuous function with possible discontinuities at points \( t_k^0 \), satisfy the EL equation for problem \( A \) (see theorem 3.3). Let \( \Lambda \) be the corresponding set of collections of normalized Lagrange multipliers \( \lambda = (\alpha, \beta, \psi, \psi, m) \), where \( \alpha = (\alpha_0, \ldots, \alpha_h) \), \( \beta = (\beta_1, \ldots, \beta_q) \), \( m = (m_1, \ldots, m_d) \in L^2_u(\Delta^0) \).

Define the cone of critical variations
\[
\mathcal{K} = \left\{ \bar{w} = (\bar{z}, \bar{t}, \bar{x}, \bar{u}) : \bar{u} \in L_2(\Delta^0), \quad \bar{z}_k = 0, \quad \bar{t} = \bar{z}_k, \quad \bar{x} = f^0 \bar{t} + g^0 \bar{x} + h^0 \bar{u} + f^0 \bar{z}_k, \quad t \in \Delta^0_k = [t_k^0, t_0^0], \quad k = 1, \ldots, \nu, \quad \bar{t}(t) \text{ and } \bar{x}(t) \text{ are continuous on the whole } \Delta^0, \quad g^0 \bar{t}(t) + g^0 \bar{x}(t) + g^0 \bar{u}(t) = 0 \quad \text{a.e. on } \Delta^0, \quad \varphi'(\rho^0)\bar{p} \leq 0, \quad i \in I \quad \eta'_j(\rho^0)\bar{p} = 0, \quad j = 1, \ldots, q \right\},
\]

where \( I \) is the set of active indices, and the derivatives of \( f \) and \( g \) are taken at the point \((t, x^0(t), u^0(t))\).

For any \( \lambda \in \Lambda \) and \( \bar{w} = (\bar{t}, \bar{x}, \bar{u}) \) define the quadratic form
\[
\Omega[\lambda](\bar{w}) = (l''[\lambda](\rho^0) \bar{p}, \bar{p}) - \int_{\Delta^0} \left( \overline{\mathcal{P}}_{uw}[\lambda](t) \bar{w}(t), \bar{w}(t) \right) \, dt - 2 \sum_{k=1}^{\nu} \bar{z}_k \int_{\Delta^0_k} \left( \overline{\mathcal{P}}_{w}[\lambda](t) \bar{x}(t) + \overline{\mathcal{P}}_{u}[\lambda](t) \bar{t}(t) \right) \, dt \tag{24}
\]
(here we took into account that \( \overline{\mathcal{P}}_{w} = 0 \), where
\[
H = \psi_x f(t, x, u) + \psi_t, \quad \overline{H} = H - mg(t, x, u), \quad (\overline{\mathcal{P}}_{uw}[\lambda]\bar{w}, \bar{w}) = (\overline{\mathcal{P}}_{wx}[\lambda]\bar{x}, \bar{x}) + 2(\overline{\mathcal{P}}_{wx}[\lambda]\bar{x}, \bar{u}) + (\overline{\mathcal{P}}_{wu}[\lambda]\bar{u}, \bar{u}) + 2(\overline{\mathcal{P}}_{w}[\lambda]\bar{t}, \bar{t}) + 2(\overline{\mathcal{P}}_{tx}[\lambda]\bar{x}\bar{t} + 2(\overline{\mathcal{P}}_{tu}[\lambda]\bar{u}\bar{t}),
\]
and \( l[\lambda](p) = \alpha \varphi(p) + \beta \eta_j(p) \). Define the function
\[
\Omega[\lambda](\bar{w}) = \max_{\lambda \in \Lambda} \Omega[\lambda](\bar{w}),
\]
and the quadratic order (23).
Theorem 5.1 (Quadratic order conditions for an extended weak minimum in problem A).

a) If the process $w^0$ provides an extended weak minimum in problem A, then $\Omega[\Lambda](\bar{w}) \geq 0$ for any $\bar{w} \in \mathcal{K}$.

b) If $\exists c > 0$ such that $\Omega[\Lambda](\bar{w}) \geq c\gamma(\bar{w})$ for any $\bar{w} \in \mathcal{K}$, then the process $w^0$ provides a strict extended weak minimum in problem A. (Moreover, the corresponding violation function satisfies the below estimate. We do not formulate it in detail.)

Remark 1. Problem A is a generalization of the canonical problem S. Let us show that, for a piecewise-continuous control $u^0(t)$, the quadratic order conditions for an extended weak minimum in problem A (theorem 5.1) generalize the quadratic order conditions in the canonical problem S (see e.g. [14]).

Since the interval $[t_0, t_1]$ in problem S is fixed, we have $\bar{t}(t_0) = 0$ and $\bar{t}(t_1) = 0$. In view of linearity of the function $\bar{t}(t)$ we then obtain $\bar{t}(t) \equiv 0$, hence also $\bar{z} = 0$. One can easily check that theorem 5.1 for $\bar{t} = \bar{z} = 0$ transforms into the quadratic order conditions in problem S.

Thus, for a piecewise-continuous control $u^0(t)$, theorem 5.1 can be considered as a generalization of the quadratic order conditions for a weak minimum in problems with mixed constraints to problems with mixed and intermediate constraints.

Remark 2. In [1] the authors consider a problem of type A with mixed constraints but nonvariable intermediate times $t_k$. The proposed quadratic necessary conditions are: $\Omega[\Lambda_{h+q}](\bar{w}) \geq 0$ on $\mathcal{K}$, where $\Lambda_{h+q}$ is the set of $\lambda \in \Lambda$ such that the index of quadratic form $\Omega[\lambda](\bar{w})$ (the dimension of a subspace on which it is negative) is not greater than $h + q$.

The paper [11] deals with a problem of type A without mixed constraints. For this class of problems, the obtained results are equivalent to theorem 5.1 (since, in the absence of mixed constraints, we have $H_u = 0$). Because of this, theorem 5.1 can be considered as a generalization of the quadratic order conditions for an extended weak minimum in problems with intermediate constraints to problems with intermediate and mixed equality type constraints.

In [15] and [14, Ch.2], Osmolovskii considered the so-called broken extremals in problems of calculus of variations. It is a particular case of problem A, where the control system is $\dot{x} = u$, the mixed constraints are absent, and the functions $\varphi, \eta$ depend only on the endpoints and do not depend on the intermediate points (so, the last ones only mark the intervals of continuity of $u^0(t)$). The author obtains quadratic conditions of a $\theta-$minimum which is just a bit stronger than our extended weak minimum. Though the results have rather different form than theorem 5.1, we believe that on the class of mutual applicability they are equivalent. A more detailed comparison will be given in a later paper.

Remark 3. The above transformation (replication of state and control variables) can be also used for obtaining quadratic order conditions for an extended weak minimum in the following generalizations of problem A.

a) Consider a problem of type A in which, on every subinterval $\Delta_k$, the process should satisfy its own mixed constraints: $g^k_s(t, x, u) = 0$, $s = 1, \ldots, d_k$ (problem $A'$). Obviously, this statement does not satisfy Assumption A3 on the whole interval $\Delta$.

Note that, in derivation of Euler–Lagrange equation, we needed the fulfillment of Assumption A3 only on every subinterval $\Delta_k$, because each mixed constraint
g_s = 0 in problem \( A \) decomposes into \( \nu \) mixed constraints in accordance with the number of subintervals. By this reason, theorems 3.3 and 5.1 remain valid also in this case, with only alteration that now the extended Pontryagin function is

\[
\mathcal{P} = H - \sum_{s=1}^{d_k} m_k g_s^k, \quad t \in \Delta_k^0,
\]

and the definition of the cone \( K \) now includes the relations

\[
g_{s,t}^{k0} \dot{t} + g_{s,t}^{k0} \ddot{x} + g_{s,u}^{k0} \ddot{u} = 0, \quad t \in \Delta_k^0.
\]

b) Consider the problem \( A \), in which on every subinterval \( \Delta_k \), the trajectory \( x(t) \) should satisfy the differential equation with its own right hand side \( f_k \) and its own mixed constraint i.e.

\[
\dot{x} = f_k(t, x, u), \quad g_k^s(t, x, u) = 0, \quad t \in \Delta_k, \quad s = 1, \ldots, d_k.
\]

(A problem with switched or variable structure.) Call it problem \( B \).

Note that, in the proof of theorem 5.1, we never used the fact that the control system is the same on all subintervals \( \Delta_k \). Therefore, using the above transformation, we can obtain quadratic order conditions for an extended weak minimum in problem \( B \) (theorem 5.2), which, in view of sec. a), essentially coincide with theorem 5.1. The only differences are:

- the cone of critical variations involves the piecewise differential equation
  \[
  \dot{x} = f_k^\nu(t, x, u) + f_k^\nu(t, x, u) + f_k^\nu(t, x, u)
  \]
  on \( \Delta_k^0 \),

- the Pontryagin function is now
  \[
  H(\psi_t, \psi_x, t, x, u) = \psi_x f_k(t, x, u) + \psi_t, \quad t \in \Delta_k^0,
  \]
  and correspondingly, \( \mathcal{P} = \psi_x f_k + \psi_t - \sum_{s=1}^{d_k} m_k g_s^k \) on \( \Delta_k^0 \), whence the integrals of the derivatives of \( \mathcal{P} \) in the quadratic form decompose into the sums of integrals over the subintervals \( \Delta_k^0 \), each of which is equipped with its own control system.

The other conditions of theorem 5.1 come to theorem 5.2 without changes.

c) As a further generalization of problem \( A \) consider a problem coinciding in its form with problem \( B \), but having on each \( \Delta_k \) its own control system \( \dot{x}_k = f_k(t, x_k, u_k) \), where \( x_k \in \mathbb{R}^{n_k} \), \( u_k \in \mathbb{R}^{r_k} \) of its own dimensions (problem \( C \)). In the case of coinciding dimensions of \( x \) on neighboring subintervals \( \Delta_k \) this statement allows for discontinuities of the trajectory at the points \( t_k \).

Applying the procedure of obtaining quadratic order conditions, similar to that used for problem \( B \) (here we do not need to replicate the variables \( x, u \), because from the outset, they are different for each \( \Delta_k \) and it only remains to redefine them on a common time interval), one can obtain conditions for an extended weak minimum also in problem \( C \) (theorem 5.3).

The difference of theorem 5.3 from theorem 5.2 is that now the cone \( K \) includes, instead of one function \( \ddot{x}(t) \), a tuple of functions \( \ddot{x}_k(t) \in \mathbb{R}^{n_k} \) defined on their respective intervals \( \Delta_k^0 \), \( k = 1, \ldots, \nu \), each satisfying its own differential equation. Moreover, since the original trajectory admits discontinuities, no junction conditions at the points \( t_k \) for \( \ddot{x}_k \) will appear on the cone \( K \). In the order \( \gamma \), the term \( |\ddot{x}(t_k^0)|^2 \) should be replaced by \( \sum_{k=0}^{\nu-1} |\ddot{x}(t_k^0 + 0)|^2 \). The other conditions of theorem 5.2 come to theorem 5.3 without changes.

6. Example. To demonstrate the application of the obtained quadratic order conditions, consider the classical problem on the refraction of the light ray.
Let in the space $\mathbb{R}^n$ there be given two isotropic optical media separated by a smooth surface $S = \{ x \in \mathbb{R}^n : g(x) = 0 \}$ without singular points. The speed of light in each medium is constant. A light ray emanates from a point $x_0$ in the first medium and comes to a point $x_2$ in the second medium. Since the media are different, the light ray breaks at a point $x_1$ where it intersects the surface $S$. According to the Fermat’s principle, the trajectory from $x_0$ to $x_2$ corresponds to a minimal time. It is required to find the trajectory.

This problem can be stated as the following time-optimal control problem with an intermediate and a mixed equality type constraints (like e.g. in [7]):

\[
\begin{align*}
\dot{x} &= c_1 u & \text{on} & \Delta_1 = [t_0, t_1], \\
\dot{x} &= c_2 u & \text{on} & \Delta_2 = [t_1, t_2], \\
(u, u) &= 1, \\
x(t_0) &= a, & g(x(t_1)) &= 0, & x(t_2) &= b, \\
t_0 &= 0, & J &= t_2 \to \min,
\end{align*}
\]

where $g(a) < 0$, $g(b) > 0$, the moments $0 < t_1 < t_2$ are not fixed, and $c_k > 0$, $k = 1, 2$ are given speeds of light in the both media. (Since we use the scalar product, we identify the column and row vectors.)

Here the Pontryagin function is

\[ H = c_k (\psi_x, u) + \psi_t, \quad t \in \Delta_k, \]

the extended Pontryagin function is

\[ \bar{H} = H - \frac{m}{2} (u, u - 1), \]

and the terminal Lagrange function is

\[ l = \alpha_0 t_2 + \beta_{t_0} t_0 + \beta_{x_0} (x(0) - a) - \beta_{x_2} (x(t_2) - b) + \delta g(x(t_1)), \]

where the collection $\{\alpha_0, \beta_{t_0}, \beta_{x_0}, \beta_{x_2}, \delta\}$ is nontrivial.

Let us write out the Euler–Lagrange equation.

- The adjoint equations are:
  \[-\dot{\psi}_x = 0, \quad -\dot{\psi}_t = 0 \quad \text{a.e. on} \ \Delta_1 \ \text{and} \ \Delta_2,\]

- The transversality conditions:
  \[
  \begin{align*}
  \psi_x(0) &= \beta_{x_0}, & \psi_x(t_2) &= \beta_{x_2}, \\
  \psi_t(0) &= \beta_{t_0}, & \psi_t(t_2) &= -\alpha_0;
  \end{align*}
  \]

- The jump conditions:
  \[
  \Delta \psi_x(t_1) = \delta g'(x(t_1)), \quad \Delta \psi_t(t_1) = 0;
  \]

- The stationarity of $\bar{H}$ with respect to $u$:
  \[
  \bar{H}_u(t) = c_k \psi_x(t) - m(t) u(t) = 0; \tag{25}
  \]

- $H^0(t) = c_k (\psi_x, u) + \psi_t = 0$ on $\Delta$. 
The adjoint equations and transversality conditions imply that
\[ \psi_x(t) \equiv \beta_{x_0} \text{ on } \Delta_1, \quad \psi_x(t) \equiv \beta_{x_2} \text{ on } \Delta_2, \quad \psi_t(t) \equiv -\alpha_0 \text{ on the whole } \Delta. \tag{26} \]

Multiplying equality (25) scalarly by \( u \), we get
\[ m = c_k(\psi_x, u) \quad \text{on } \Delta_k. \tag{27} \]

But the condition \( H^0 = 0 \) implies that
\[ c_k(\psi_x(t), u(t)) = -\psi_t(t) = -\alpha_0. \]

Hence \( m(t) \equiv \alpha_0 \), and in view of (25) we get
\[ c_k \psi_x = \alpha_0 u. \tag{28} \]

Let us show that \( \alpha_0 \neq 0 \). Indeed, if \( \alpha_0 = 0 \), then \( m = -\psi_t = 0 \). From (28) it follows that \( \psi_x = 0 \) on the whole \( \Delta \). Hence \( \delta = 0 \) and all the Lagrange multipliers vanish, a contradiction.

So, we can set \( \alpha_0 = 1 \) and determine uniquely all the multipliers. Indeed, \( \psi_t \equiv -1, \ m(t) \equiv 1, \) and (28) yields
\[ c_k \psi_x = u, \quad t \in \Delta_k. \tag{29} \]

This implies that \( u \) is a piecewise-constant function with a possible discontinuity at the point \( t_1 \), and since \( |u| = 1 \), it can be represented in the form:
\[ u = u_1 = \frac{x(t_1) - a}{|x(t_1) - a|} \text{ on } \Delta_1, \quad u = u_2 = \frac{b - x(t_1)}{|b - x(t_1)|} \text{ on } \Delta_2. \tag{30} \]

Calculating the duration of motion on each interval \( \Delta_k \) as the ratio between the distance and the speed, we obtain
\[ t_1 = \frac{|x(t_1) - a|}{c_1}, \quad t_2 - t_1 = \frac{|b - x(t_1)|}{c_2}. \tag{31} \]

The unknown vector \( x(t_1) \) can be found from the system of equations
\[
\begin{aligned}
\psi_x(t_1 + 0) - \psi_x(t_1 - 0) &= \delta g'(x(t_1)), \\
g(x(t_1)) &= 0,
\end{aligned}
\]

where \( \psi_x(t_1 \pm 0) \) are expressed through \( x(t_1) \) by the formulas (29) and (30).

In the generic case this system has a solution which determines the whole extremal. Thus, the obtained trajectory of the light is a concatenation of two straight line segments with a common point at the boundary of the given two media. (This fact can be as well obtained without an advanced theory, but the condition \( (u_2/c_2 - u_1/c_1) \parallel g'(x(t_1)) \) require some optimization arguments.)

Now, let us write out the quadratic order conditions of an extended weak minimum for the found extremal.
The cone of critical variations $\mathcal{K}$ (here a subspace) is the set of $\bar{w} = (\bar{z}_1, \bar{z}_2, \bar{t}, \bar{x}, \bar{u})$ satisfying on the interval $[0, t_2]$ the following conditions:
\[
\begin{aligned}
\dot{x} &= c_k (\bar{u} + u \bar{z}_k), \\
\dot{t} &= \bar{z}_k, \\
\dot{z}_k &= 0, \\ t &\in \Delta_k, \\ k = 1, 2, \\
\bar{t}(0) &= 0, \\ \bar{x}(0) &= 0, \\ \bar{x}(t_2) &= 0, \\
\bar{x} \text{ and } \bar{t} \text{ are continuous on the whole } \Delta, \\
g'(x(t_1)) \bar{x}(t_1) &= 0, \\
(u, \bar{u}) &= 0.
\end{aligned}
\] (33)

The inequality $\bar{t}(t_2) \leq 0$ can be dropped, because $\alpha_0 > 0$, so the the index $i = 0$ is “rigid” and this inequality actually turns into equality, whence in view of the EL equation one equality in the definition of the subspace $\mathcal{K}$ can be removed.

On this subspace, the quadratic form is
\[
\Omega(\bar{w}) = \delta \left( g''(x(t_1)) \bar{x}(t_1), \bar{x}(t_1) \right) + \int_0^{t_2} |\bar{u}(t)|^2 \, dt, 
\] (34)
and the quadratic order is
\[
\gamma(\bar{w}) = \bar{z}_1^2 + \bar{z}_2^2 + |\bar{x}(t_1)|^2 + \int_0^{t_2} |\bar{u}(t)|^2 \, dt
\]
(here in $\gamma$ we take for convenience $|\bar{x}(t_1)|^2$ instead of $|\bar{x}(t_0)|^2$).

Let us analyze the quadratic order conditions.

The order $\gamma$ contains $\bar{z}_1^2$ and $\bar{z}_2^2$, but $\Omega$ does not depend explicitly on $\bar{z}_1$ and $\bar{z}_2$. We claim that both $\bar{z}_k^2 \leq \text{const } |\bar{x}(t_1)|^2$.

Indeed, among the relations on $\mathcal{K}$ we have the differential equation
\[
\dot{x} = c_k (\bar{u} + u \bar{z}_k), \quad t \in \Delta_k. 
\] (35)

Multiply scalarly both sides of this equation by $u$ and then integrate over the interval $\Delta_k$. Since the function $u(t) = u_k$ is constant on each $\Delta_k$, moreover, $|u| = 1$ and $(u, \bar{u}) = 0$ on $\mathcal{K}$, we get
\[
(\bar{x}(t_k) - \bar{x}(t_{k-1}), u_k) = c_k \bar{z}_k |\Delta_k|. 
\] (36)

In view of the endpoint relations $\bar{x}(0) = \bar{x}(t_2) = 0$, these equalities give
\[
(\bar{x}(t_1), u_1) = c_1 t_1 \bar{z}_1, \quad (-\bar{x}(t_1), u_2) = c_2 (t_2 - t_1) \bar{z}_2, 
\] (37)
which implies that both $\bar{z}_k^2 \leq \text{const } |\bar{x}(t_1)|^2$, so the claim is proved.

The obtained estimates allow us to exclude the terms $\bar{z}_1^2$ and $\bar{z}_2^2$ from $\gamma$.

Then, the quadratic order can be taken in a reduced form:
\[
\gamma(\bar{w}) = |\bar{x}(t_1)|^2 + \int_0^{t_2} |\bar{u}(t)|^2 \, dt.
\]

Thus, to verify the extended weak minimality of the given process, one has to check the sign definiteness of quadratic form (34) with respect to $\gamma$ on the subspace $\mathcal{K}$ defined in (33). Looking at (34), one can notice that the first part of $\Omega$ makes this task not obvious, in general. This depends on the specificity of $g(x)$. Let us consider the following
Particular case. Let \( x \in \mathbb{R}^2 \), the surface \( g(x) = 0 \) be a parabola, e.g.
\[
g(x) = g(x_1, x_2) = x_2 + \sigma x_1^2 = 0,
\]
and the initial and terminal positions of the light ray lay on the straight line \( x_1 = 0 \), i.e. \( a = (0, h_1) \) and \( b = (0, -h_2) \), where \( h_1 > 0, \ h_2 > 0, \ \sigma \in \mathbb{R} \).

Consider the process \( u^0 \) joining the points \( a \) and \( b \) by the straight line coming through the intermediate point \( x(t_1) = (0, 0) \). The corresponding control \( u(t) \equiv (0, -1) \), \( g'(0, 0) = (0, 1) \),
\[
z_k = |\Delta_k| = \frac{h_k}{c_k}, \quad \psi_x = -\frac{1}{c_k} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ on } \Delta_k, \quad k = 1, 2.
\] (38)

Setting \( \psi_t(t) \equiv -1, \ m(t) \equiv \alpha_0 = 1 \), we see that this process satisfies the EL equation for any \( \sigma \). Condition \( \Delta \psi_x(t_1) = \delta g'(x(t_1)) \) takes here the form
\[
\frac{1}{c_1} - \frac{1}{c_2} = \delta.
\]

Let us check the quadratic order conditions. Denote \( \bar{u}(t) = (\alpha(t), \beta(t)), \ t \in \Delta. \)
Since \( u(t) = (0, -1) \) and \( (u, \bar{u}) = 0 \), we have \( \beta(t) = 0 \), while \( \alpha(t) \in L_2[0, t] \) is an arbitrary function, i.e. \( \bar{u}(t) = (\alpha(t), 0) \).

Since \( g'(0, 0) \bar{x}(t_1) = 0 \), we have \( \bar{x}(t_1) = (s, 0) \) with arbitrary \( s \in \mathbb{R} \). Then \( (\bar{x}(t_1), u) = 0 \), and (37) gives \( \bar{z}_1 = \bar{z}_2 = 0 \). Moreover, \( \dot{\bar{x}}_1 = c_k \alpha(t) \) on \( \Delta_k \), \( \dot{\bar{x}}_2 = 0 \), whence \( \bar{x}_2 \equiv 0 \) and
\[
s = c_1 \int_0^{t_1} \alpha(t) \, dt = -c_2 \int_{t_1}^{t_2} \alpha(t) \, dt.
\] (39)

Thus, the subspace \( K \) actually consists of the pairs \( (\alpha(t), s) \) satisfying these last relations.

The quadratic form in the new notation is:
\[
\Omega(\alpha, s) = Ps^2 + \int_0^{t_2} \alpha^2(t) \, dt, \text{ where } P = \left( \frac{1}{c_1} - \frac{1}{c_2} \right) 2\sigma.
\] (40)

We have to study its sign definiteness on \( K \). First, let us find \( \inf \Omega \) over \( u \) subject to constraints (39), regarding \( s \) as a parameter. One can easily show that this problem always has a solution. We consider it in a more general setting.

Let on an interval \([0, T]\) be given numbers \( 0 = t_0 < t_1 < \ldots < t_{\nu-1} < t_\nu = T \). In the space \( L_2[0, T] \times \mathbb{R}^m \) let be given a quadratic form of the type
\[
\Omega(u, s) = \sum_{j=1}^{m} P_j s_j^2 + \int_0^T |u(t)|^2 \, dt,
\] (41)
which is considered under the constraints
\[
\int_{t_{k-1}}^{t_k} u(t) \, dt = b_k(s), \quad k = 1, \ldots, \nu,
\] (42)
where \( P_j \) are given numbers, \( u \in L_2[0, T], \ s = (s_1, \ldots, s_m) \in \mathbb{R}^m \), and \( b_k(s) \) are given scalar linear functions of \( s \), i.e. \( b_k(s) = \sum_{i=1}^{m} c_{ki} s_i, \ c_{ki} \in \mathbb{R} \).

**Lemma 6.1.** For any \( s \in \mathbb{R}^m \) the minimum of \( \Omega(u, s) \) over \( u \in L_2[0, T] \) satisfying (42) is attained and is equal to the value of the finite-dimensional quadratic
form
\[ \Phi(s) = \sum_{j=1}^{m} P_j s_j^2 + \sum_{k=1}^{\nu} \frac{b_k^2(s)}{t_k - t_{k-1}}. \] (43)

Moreover, the positive definiteness of \( \Omega(u, s) \) with respect to \( (u, s) \) from the subspace (42) is equivalent to the positive definiteness of \( \Phi(s) \) with respect to \( s \in \mathbb{R}^m \).

Proof. Since the values of \( u(t) \) on different intervals \( \Delta_k = [t_{k-1}, t_k] \) are independent (related only by the integral equalities (42)), the problem of minimization of \( \Omega \) for a given \( s \) decomposes into \( \nu \) following problems of minimization over \( u \in L_2(\Delta_k) \):

\[
\begin{cases}
\Omega_k = \int_{\Delta_k} |u(t)|^2 \, dt \to \min, \\
\int_{\Delta_k} u(t) \, dt = b_k(s).
\end{cases}
\] (44)

It is easily seen that, for each \( k = 1, \ldots, \nu \), the solution of problem (44) is \( u^0(t) = b_k(s)/|\Delta_k|, \ t \in \Delta_k \) (this follows, e.g. from the Cauchy–Bunyakowski inequality, or, alternatively, from the stationarity of the Lagrange function). Then the function \( u^0(t) \) defined for all \( t \in [0, T] \) minimizes \( \Omega(u, s) \) over \( u \) under the constraints (42). Dividing the integral of \( |u(t)|^2 \) over the interval \([0, T] \) into the sum of integrals over the intervals \( \Delta_k \) and substituting the found value of \( u^0(t) \) on each interval \( \Delta_k \) into \( \Omega \), we obtain the required value (43).

Let us prove the second part of lemma. Since \( \Omega(u, s) \) is a Legendre form, its positive definiteness is equivalent to its simple positivity: \( \Omega(u, s) > 0 \) for all \( (u, s) \neq (0, 0) \) satisfying (42).

Let \( \Phi(s) > 0 \) for all \( s \neq 0 \). Then \( \Omega(u, s) \geq \Phi(s) > 0 \) for \( s \neq 0 \). If \( s = 0 \) and \( u \neq 0 \), then \( \Omega(u, 0) = \int_0^T |u(t)|^2 \, dt > 0 \). Thus, for all \( (u, s) \neq (0, 0) \) we have \( \Omega(u, s) > 0 \), i.e. the form \( \Omega(u, s) \) is positive.

Conversely, let \( \Omega(u, s) > 0 \) for all \( (u, s) \neq (0, 0) \). Then for any \( s \neq 0 \) we have \( \Phi(s) = \min_u \Omega(u, s) > 0 \) (since the minimum is attained), i.e. the form \( \Phi(s) \) is positive.

From lemma 6.1 it follows that the positive definiteness of quadratic form (40) is equivalent to the positive definiteness of the quadratic form
\[ \Phi(s) = R(\sigma)s^2, \text{ where } R(\sigma) = \left( \frac{1}{c_1} - \frac{1}{c_2} \right) 2\sigma + \frac{1}{c_1^2 h_1} + \frac{1}{c_2^2 h_2}. \]

If \( R(\sigma) > 0 \), then \( \Phi(s) \) is positive definite, and then by lemma 6.1 \( \Omega \) is also positive definite, i.e. the sufficient conditions for the extended weak minimum are satisfied. If \( R(\sigma) < 0 \), then \( \Omega < 0 \), so the necessary conditions are not satisfied. This case is realized, e.g. when \( c_1 > c_2 \) and \( h_1, h_2 \gg 1 \). If \( R(\sigma) = 0 \), then \( \Phi(s) = 0 \) for all \( s \), and so, only the necessary conditions for the extended weak minimum are satisfied.

Remark 4. If the speed of light in both the media is the same: \( c_1 = c_2 \) (so, in fact, we totally have just one medium), then
\[ R_\sigma = \left( \frac{1}{c_1^2 h_1} + \frac{1}{c_2^2 h_2} \right) > 0. \]
In this case, the given extremal always satisfies the sufficient conditions for the extended weak minimum, which is well in accordance with the common physical sense.

**Example with a double refraction of the light.** Consider the case when the light ray comes through two surfaces:

\[ g_1(x_1, x_2) = x_2 + \sigma_1 x_1^2 + r_1 = 0 \quad \text{and} \quad g_2(x_1, x_2) = x_2 + \sigma_2 x_1^2 + r_2 = 0. \]

Here the problem has the form

\[
\begin{aligned}
\dot{x} &= c_1 u \quad \text{on} \quad \Delta_1 = [t_0, t_1], \\
\dot{x} &= c_2 u \quad \text{on} \quad \Delta_2 = [t_1, t_2], \\
\dot{x} &= c_3 u \quad \text{on} \quad \Delta_3 = [t_2, t_3], \\
t_0 &= 0, \quad x(t_0) = a, \quad x(t_3) = b, \\
g_1(x(t_1)) &= 0, \quad g_2(x(t_2)) = 0, \\
(u, u) &= 1, \\
J &= t_3 \to \min.
\end{aligned}
\]

Let us analyze the extended weak minimality of the process \( w^0 \) that connects the four points \( a = (0, h_1), \ x(t_1) = (0, r_1), \ x(t_2) = (0, r_2), \ b = (0, h_2) \) lying on one straight line (here \( h_1 > r_1 > r_2 > h_2 \)).

Here again \( u(t) \equiv (0, -1), \ g'(0, r_1) = g'(0, r_2) = (0, 1) \). One can easily check that this process satisfies the EL equation with the parameters: \( m(t) \equiv \alpha_0 = 1, \)

\[
z_1 = |\Delta_1| = \frac{h_1 - r_1}{c_1}, \quad z_2 = |\Delta_2| = \frac{r_1 - r_2}{c_2}, \quad z_3 = |\Delta_3| = \frac{r_2 - h_2}{c_3},
\]

\[\psi_x = -\frac{1}{c_k} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad t \in \Delta_k, \quad k = 1, 2, 3, \quad (45)\]

The jump conditions for \( \psi_x \) at the points \( t_1, t_2 \) give

\[
\frac{1}{c_1} - \frac{1}{c_2} = \delta_1, \quad \frac{1}{c_2} - \frac{1}{c_3} = \delta_2. \quad (46)
\]

Consider the quadratic order conditions. We will follow the line of the above case with one intermediate point. Defining the subspace \( K \) similar to (33), we again have \( (u, \tilde{u}) = 0 \), so \( \tilde{u}(t) = (\alpha(t), 0) \) with arbitrary \( \alpha \in L_2[0, t_3] \), and also have equations (35), (36), now for \( k = 1, 2, 3 \), which imply

\[
(\tilde{x}(t_1), u) = c_1|\Delta_1| \tilde{z}_1, \quad (\tilde{x}(t_2) - \tilde{x}(t_1), u) = c_2|\Delta_2| \tilde{z}_2, \quad (-\tilde{x}(t_2), u) = c_3|\Delta_3| \tilde{z}_3.
\]

Since \( \tilde{x}(t_1), \tilde{x}(t_2) \perp g' = (0, 1) \), the left hand sides here vanish, whence \( \tilde{z}_1 = \tilde{z}_2 = \tilde{z}_3 = 0 \). Therefore, \( \tilde{x}_2 = 0 \), hence \( \tilde{x}_2 = 0 \), while \( \tilde{x}_1 = c_k \alpha(t) \) on \( \Delta_k \). Denoting \( \tilde{x}_1(t_k) = s_k, \quad k = 1, 2 \), we get the restrictions on \( \alpha \):

\[
s_1 = c_1 \int_{t_0}^{t_1} \alpha(t) \, dt, \quad s_2 - s_1 = c_2 \int_{t_1}^{t_2} \alpha(t) \, dt, \quad -s_2 = c_3 \int_{t_2}^{t_3} \alpha(t) \, dt. \quad (47)
\]
The subspace $\mathcal{K}$ consists of the triples $(\alpha(t), s_1, s_2)$ satisfying these relations, and the quadratic order is
\[
\gamma = s_1^2 + s_2^2 + \int_0^{t_3} \alpha^2(t) \, dt.
\]

The quadratic form (24) here is
\[
\Omega(\bar{w}) = \delta_1(g''_1 \bar{x}(t_1), \bar{x}(t_1)) + \delta_2(g''_2 \bar{x}(t_2), \bar{x}(t_2)) + \int_0^{t_3} |\bar{u}(t)|^2 \, dt.
\]

In the new notation and in view of relations (46), we have
\[
\Omega(\alpha, s_1, s_2) = P_1 s_1^2 + P_2 s_2^2 + \int_0^{t_3} \alpha^2(t) \, dt,
\]
where
\[
P_1 = \left( \frac{1}{c_1} - \frac{1}{c_2} \right) 2\sigma_1, \quad P_2 = \left( \frac{1}{c_2} - \frac{1}{c_3} \right) 2\sigma_2.
\]

Let us fix arbitrary $s_1, s_2$ and find the minimum of $\Omega$ over $\alpha \in L_2[0, t_3]$ subject to constraints (47). By lemma 6.1, the sign definiteness of $\Omega(\alpha, s_1, s_2)$ is equivalent to that of the quadratic form
\[
\Phi(s_1, s_2) = P_1 s_1^2 + P_2 s_2^2 + \frac{s_1^2}{c_1^2 |\Delta_1|} + \frac{(s_2 - s_1)^2}{c_2^2 |\Delta_2|} + \frac{s_2^2}{c_3^2 |\Delta_3|}.
\]

Setting
\[
Q_{11} = P_1 + \frac{1}{c_1^2 |\Delta_1|} + \frac{1}{c_2^2 |\Delta_2|}, \quad Q_{12} = -\frac{1}{c_2^2 |\Delta_2|}, \quad Q_{22} = P_2 + \frac{1}{c_2^2 |\Delta_2|} + \frac{1}{c_3^2 |\Delta_3|},
\]
and
\[
Q_{11} Q_{22} - Q_{12}^2 < Q_{22},
\]
we get
\[
\Phi(s_1, s_2) = Q_{11} s_1^2 + 2Q_{12} s_1 s_2 + Q_{22} s_2^2.
\]

The sign definiteness of this form can be verified by the standard Silvester criterion.

For example, given positive $c_k$ and $\Delta_k$, for all sufficiently small $\sigma_1$ and $\sigma_2$ the form $\Phi$ is positive, and so, the straight line between the given points is time-optimal in the extended weak sense. On the other hand, if $c_1 \neq c_2$ and $c_2 \neq c_3$, one can find the curvatures $\sigma_1$ and $\sigma_2$ such that either $Q_{11} < 0$ or $Q_{22} < 0$ or $Q_{11} Q_{22} < Q_{12}^2$, and so, the straight line is not time-optimal.

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