Model of a motion of substance in a channel of a network consisting of two arms

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Abstract

We study the problem of the motion of substance in a channel of a network for the case of channel having two arms. Stationary regime of the flow of the substance is considered. Analytical relationships for the distribution of the substance in the nodes of the arms of the channel are obtained. The obtained results are discussed from the point of view of technological applications of the model (e.g., motion of substances such as water in complex technological facilities).

1 Introduction

In the last decades the researchers realized the importance of dynamics of complex systems and this lead to intensive studies of such systems, especially in the area of social dynamics and population dynamics [1] - [31]. In the course of this research the networks have appeared as important part of the structure of many complex systems [32] - [35]. And an important part of the processes in a network are the network flows. Research on network flows has many roots and some of them are in the studies on transportation problems [36] or in the studies on migration flows [37] - [43]. At the beginning of the research the problems of interest have been, e.g., minimal cost flow problems or possible maximal flows in a network. Today one uses the methodology from the theory of network flows [44] to solve problems connected to, e.g., just in time scheduling, facility layout and location or electronic route guidance in urban traffic networks [45].

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Below we shall consider a network consisting of nodes and edges - Fig. 1. Some of the nodes of the network (pictured by circles in Fig. 1) and the corresponding edges (represented by solid lines in Fig. 1) form a channel. This channel has a single arm up to the node \( n \) where the channel splits to two arms. We shall consider a flow of substance through such a channel. For an example the channel can consists of chains of cells connected by transportation systems that transport the substance from one cell to the next cell of the channel. We can think about such a situation as a situation reflecting the case of logistic channel consisting of storing facilities and each of these facilities supplies a city from a network of cities. Another possible interpretation of such a channel is motion of some substance (e.g. water) through cells of complex technological system. In this case the edges can be water pipes.

The following processes can be observed in a node of the studied channel: exchange (inflow and outflow) of substance with the previous node of the channel; exchange (outflow and inflow) of substance with the next node of the channel; exchange (inflow and outflow) of substance with the environment; "leakages": outflow and inflow of substance to and from the corresponding node of the network. Below we shall consider the stationary regime of the functioning of the channel (i.e., the regime where the amount of the substance in the cells doesn't depend explicitly on the time: the quantities of the substance that enter and leave the channel are the same). Our interest will be focused on distributions of the substance in the nodes of the studied channel for the case of this stationary regime. Studied model has numerous applications, e.g. in migration channels, in scientometrics, in logistics, etc. [4], [43].

2 Mathematical formulation of the problem

Specific feature of the study below is that the channel splits to two arms at the \( n \)-th node. Thus there are two special nodes in our case. The first node of the network (called also the entry node and labeled by the number 0) is the only node of the network where the substance may enter the channel. The second special node of the network is the node where the channel splits in two arms. We assume that the substance moves only in one direction along the channel (from nodes labeled by smaller numbers to nodes labeled by larger numbers). The "leakage" is also only in the direction from the channel to the network (and not in the opposite direction).

Let us now consider the situation where the single arm of the channel contains \( n + 1 \)-nodes (the nodes labeled by the numbers from 0 to \( n \)) and
Figure 1: Part of a network and the two arms of the studied channel. The nodes of the arms are represented by circles and the edges that connect the nodes of the arms are represented by solid lines. At a selected node the channel splits to two arms. This node is labelled \( n/0 \). \( n \) is the number of the splitting node from the point of view of the first arm of the channel (the horizontal chain of nodes). 0 is the number of the node from the point of view of the second arm of the channel. The nodes of the network that are not a part of the studied channel are connected by edges that are represented by dashed lines. Note that the entry nodes of the two arms of the channel are labelled with 0.

then the channel splits to two arms. From the point of view of Fig. 1 this is the part of the channel that is shown in the left-hand side of the picture before the node where the channel splits to two arms. This part of the channel consists of a chain of nodes of a network. The nodes are connected by edges and each node is connected only to the two neighboring nodes of the channel exclusive for the first node of the channel that is connected only to the neighboring node. We study a model of the motion of substance through such a channel which is an extension of the model discussed in [46] and [43]. We consider each node as a cell (box), i.e., we consider an array of infinite
number of cells indexed in succession by non-negative integers. We assume that an amount \( x \) of some substance is distributed among the cells and this substance can move from one cell to another cell. Let \( x_i \) be the amount of the substance in the \( i \)-th cell. Then

\[
x = \sum_{i=0}^{\infty} x_i
\]

The fractions \( y_i = x_i/x \) can be considered as probability values of distribution of a discrete random variable \( \zeta \)

\[
y_i = p(\zeta = i), \quad i = 0, 1, \ldots
\]

The content \( x_i \) of any cell may change as consequence of the following 3 processes:

1. Some amount \( s \) of the substance \( x \) enters the system of cells from the external environment through the 0-th cell;
2. Rate \( f_i \) from \( x_i \) is transferred from the \( i \)-th cell into the \( i+1 \)-th cell;
3. Rate \( g_i \) from \( x_i \) leaks out the \( i \)-th cell into the external environment.

We assume that the process of the motion of the substance is continuous in the time. Then the process can be modeled mathematically by the system of ordinary differential equations:

\[
\begin{align*}
\frac{dx_0}{dt} &= s - f_0 - g_0; \\
\frac{dx_i}{dt} &= f_{i-1} - f_i - g_i, \quad i = 1, 2, \ldots.
\end{align*}
\]

There are two regimes of functioning of the channel: stationary regime and non-stationary regime. What we shall discuss below is the stationary regime of functioning of the channel. In the stationary regime of the functioning of the channel \( \frac{dx_i}{dt} = 0, \quad i = 0, 1, \ldots \) Let us mark the quantities for the stationary case with *.

Then from Eqs. (3) one obtains

\[
\begin{align*}
f_0^* &= s^* - g_0^*; \\
f_i^* &= f_{i-1}^* - g_i.
\end{align*}
\]

This result can be written also as

\[
f_i^* = s^* - \sum_{j=0}^{i} g_j^*
\]
Hence for the stationary case the situation in the channel is determined by the quantities \( s^* \) and \( g^*_j, j = 0, 1, \ldots \). In this paper we shall assume the following forms of the amount of the moving substances in Eqs. (3) (\( \alpha, \beta, \gamma_i, \sigma \) are constants)

\[
\begin{align*}
    s &= \sigma_0 x_0; \quad \sigma_0 > 0 \\
    f_i &= (\alpha_i + \beta_i) x_i; \quad \alpha_i > 0, \ \beta_i \geq 0 \\
    g_i &= \gamma^*_i x_i; \quad \gamma^*_i \geq 0 \to \text{non-uniform leakage in the nodes (6)}
\end{align*}
\]

\( \gamma^*_i \) is a quantity specific for the present study. \( \gamma^*_i = \gamma_i + \delta_i \) describes the situation with the leakages in the nodes of the channel. We shall assume that \( \delta_i = 0 \) for all \( i \) except for \( i = n \). This means that in the \( n \)-th node (where the second arm of the channel splits from the first arm of the channel) in addition to the usual leakage \( \gamma_i \) there is additional leakage of substance given by the term \( \delta_n x_n \) and this additional leakage supplies the substance that then begins its motion along the second arm of the channel.

On the basis of all above the model system of differential equations for this arm of the channel becomes

\[
\begin{align*}
    \frac{dx_0}{dt} &= \sigma_0 x_0 - \alpha_0 x_0 - \gamma^*_0 x_0 \\
    \frac{dx_i}{dt} &= [\alpha_{i-1} + (i - 1)\beta_{i-1}] x_{i-1} - (\alpha_i + i\beta_i + \gamma^*_i) x_i; \quad i = 1, 2, \ldots
\end{align*}
\]

We shall consider the stationary regime of functioning of the channel. Then \( dx_0/dt = 0 \) and from the first of the Eqs. (7) it follows that \( \sigma_0 = \alpha_0 + \gamma_0 \). This means that \( x_0 \) (the amount of the substance in the 0-th cell of the channel) is free parameter. In principle the solution of Eqs. (7), \( i = 1, 2, \ldots \) is

\[
x_i = x^*_i + \sum_{j=0}^{i} b_{ij} \exp[-(\alpha_j + j\beta_j + \gamma^*_j)t]
\]

where \( x^*_i \) is the stationary part of the solution. For \( x^*_i \) one obtains the relationship (just set \( dx/dt = 0 \) in the second of Eqs. (7))

\[
x^*_i = \frac{\alpha_{i-1} + (i - 1)\beta_{i-1}}{\alpha_i + i\beta_i + \gamma^*_i} x^*_{i-1}, \quad i = 1, 2, \ldots
\]

The corresponding relationships for the coefficients \( b_{ij} \) are \( (i = 1, \ldots) \):

\[
b_{ij} = \frac{\alpha_{i-1} + (i - 1)\beta_{i-1}}{(\alpha_i - \alpha_j) + (i\beta_i - j\beta_j) + (\gamma^*_i - \gamma^*_j)} b_{i-1,j}, \quad j = 0, 1, \ldots, i - 1
\]
From Eq. (9) one obtains

\[
x_i^* = \frac{\prod_{j=0}^{i-1} \left[ \alpha_i-j-1 + (i-j-1)\beta_i-j-1 \right]}{\prod_{j=0}^{i-1} \alpha_i-j + (i-j)\beta_i-j + \gamma_i^*} x_0^*
\]

(11)

The form of the corresponding stationary distribution \( y_i^* = x_i^*/x^* \) (where \( x^* \) is the amount of the substance in all of the cells of the arm of the channel) is

\[
y_i^* = \frac{\prod_{j=0}^{i-1} \left[ \alpha_i-j-1 + (i-j-1)\beta_i-j-1 \right]}{\prod_{j=0}^{i-1} \alpha_i-j + (i-j)\beta_i-j + \gamma_i^*} y_0^*
\]

(12)

To the best of our knowledge the distribution presented by Eq. (12) was not discussed up to now outside our research group. Let us show that this distribution contains as particular cases several famous distributions, e.g., Waring distribution, Zipf distribution, and Yule-Simon distribution. In order to do this we consider the particular case when \( \beta_i \neq 0 \) and write \( x_i \) from Eq. (11) as follows

\[
x_i^* = \frac{\prod_{j=0}^{i-1} \tilde{b}_i-j[k_i-j-1 + (i-j-1)]}{\prod_{j=0}^{i-1} [k_i-j + a_i-j + (i-j)]} x_0^*
\]

(13)

where \( k_i = \alpha_i/\beta_i; \ a_i = \gamma_i^*/\beta_i; \ \tilde{b}_i = \beta_i-1/\beta_i \). The form of the corresponding stationary distribution \( y_i^* = x_i^*/x^* \) is

\[
y_i^* = \frac{\prod_{j=0}^{i-1} \tilde{b}_i-j[k_i-j-1 + (i-j-1)]}{\prod_{j=0}^{i-1} [k_i-j + a_i-j + (i-j)]} y_0^*
\]

(14)

Let us now consider the particular case where \( \alpha_i = \alpha \) and \( \beta_i = \beta \) for \( i = 0, 1, 2, \ldots \). Then from Eqs. (13) and (14) one obtains

\[
x_i^* = \frac{[k + (i-1)]!}{(k-1)! \prod_{j=1}^{i} (k + j + a_j)} x_0^*
\]

(15)
where \( k = \alpha / \beta \) and \( a_j = \gamma_j / \beta \). The form of the corresponding stationary distribution \( y_i^* = x_i^* / x^* \) is

\[
y_i^* = \frac{[k + (i - 1)]!}{(k - 1)! \prod_{j=1}^{i} (k + j + a_j)} y_0^*
\]

Let us consider the particular case where \( a_0 = \cdots = a_N \). In this case the distribution from Eq. (16) is reduced to the distribution:

\[
P(\zeta = i) = P(\zeta = 0) \frac{(k - 1)^{|i|}}{(a + k)^{|i|}}; \quad k^{[i]} = \frac{(k + i)!}{k!}; \quad i = 1, 2, \ldots
\]

\( P(\zeta = 0) = y_0^* = x_0^* / x^* \) is the percentage of substance that is located in the first cell of the channel. Let this percentage be

\[
y_0^* = \frac{a}{a + k}
\]

The case described by Eq. (18) corresponds to the situation where the amount of substance in the first cell is proportional of the amount of substance in the entire channel. In this case Eq. (16) is reduced to:

\[
P(\zeta = i) = \frac{a}{a + k} \frac{(k - 1)^{|i|}}{(a + k)^{|i|}}; \quad k^{[i]} = \frac{(k + i)!}{k!}; \quad i = 1, 2, \ldots
\]

The distribution (19) is exactly the Waring distribution (probability distribution of non-negative integers named after Edward Waring - the 6th Lucasian professor of Mathematics in Cambridge from the 18th century) \[47\] - \[49\]. The mean \( \mu \) (the expected value) and the variance \( V \) of the Waring distribution are

\[
\mu = \frac{k}{a - 1} \text{ if } a > 1; \quad V = \frac{ka(k + a - 1)}{(a - 1)^2(a - 2)} \text{ if } a > 2
\]

\( \rho \) is called the tail parameter as it controls the tail of the Waring distribution. Waring distribution contains various distributions as particular cases. Let \( i \to \infty \) Then the Waring distribution is reduced to the frequency form of the Zipf distribution \[50\]

\[
P(\zeta = i) \approx \frac{1}{i^{(1 + \alpha)}}.
\]

If \( k \to 0 \) the Waring distribution is reduced to the Yule-Simon distribution \[51\]

\[
P(\zeta = i) = aB(a + 1, i)
\]
where $B$ is the beta-function.

Let us now consider the stationary regime of functioning of the second arm of the channel. Here we shall denote as 0-th node the node where the second arm of the channel splits from the first arm. We assume that an amount $z$ of the substance becomes distributed among the cells of the second arm of the channel and this substance can move from one cell to another cell. Let $z_i$ be the amount of the substance in the $i$-th cell. Then

$$z = \sum_{i=0}^{\infty} z_i$$

(23)

The fractions $y_i = z_i/z$ can be considered as probability values of distribution of a discrete random variable $\zeta$

$$y_i = p(\zeta = i), \ i = 0, 1, \ldots$$

(24)

The content $z_i$ of any cell may change due to the same three processes that govern the motion of the substance in the first arm of the channel. The process of the motion of the substance is continuous in the time. Then the process can be modeled mathematically by the system of ordinary differential equations:

$$\frac{dz_0}{dt} = \hat{s} - \hat{f}_0 - \hat{g}_0;$$

$$\frac{dz_i}{dt} = \hat{f}_{i-1} - \hat{f}_i - \hat{g}_i, \ i = 1, 2, \ldots.$$

(25)

The relationships for the quantities of the above equations are ($\hat{\alpha}, \hat{\beta}, \hat{\gamma}_i$ are constants)

$$\hat{s} = \delta_n x_n;$$

$$\hat{f}_i = (\hat{\alpha}_i + \hat{\beta}_i i) x_i; \ \ \hat{\alpha}_i > 0, \ \hat{\beta}_i \geq 0$$

$$\hat{g}_i = \hat{\gamma}_i x_i; \ \ \hat{\gamma}_i \geq 0$$

(26)

Thus the system of equations for the motion of the substance in this arm of the channel is

$$\frac{dz_0}{dt} = \delta_n x_n - \hat{\alpha}_0 z_0 - \hat{\gamma}_0 z_0$$

$$\frac{dz_i}{dt} = [\hat{\alpha}_{i-1} + (i - 1)\hat{\beta}_{i-1}]z_{i-1} - (\hat{\alpha}_i + i\hat{\beta}_i + \hat{\gamma}_i)z_i; \ \ i = 1, 2, \ldots$$

(27)

In this article we shall discuss the situation in which the stationary state is established in the entire channel (in the two arm of the channel). In this case $x_n \to x_n^*; \ \frac{dx_0}{dt} \to 0$ and $\frac{dx_i}{dt} \to 0$. Then

$$z_0^* = \frac{\delta_n x_n^*}{\hat{\alpha}_0 + \hat{\gamma}_0}; \ \ \ \ z_i^* = \frac{\hat{\alpha}_{i-1} + (i - 1)\hat{\beta}_{i-1}}{\hat{\alpha}_i + i\hat{\beta}_i + \hat{\gamma}_i} z_{i-1}^*, \ i = 1, 2, \ldots$$

(28)
From the second of Eqs. (28) one obtains

\[ z_i^* = \frac{\prod_{j=0}^{i-1} [\hat{\alpha}_{i-j-1} + (i-j-1)\hat{\beta}_{i-j-1}]}{\prod_{j=0}^{i-1} \hat{\alpha}_{i-j} + (i-j)\hat{\beta}_{i-j} + \hat{\gamma}_{i-j}} z_0^* \]  

(29)

The form of the corresponding stationary distribution \( y_i^* = z_i^* / z^* \) (where \( z^* \) is the amount of the substance in all of the cells of second arm of the channel) is

\[ y_0^* = \frac{1}{1 + \sum_{i=1}^{\infty} \frac{\prod_{j=0}^{i-1} [\hat{\alpha}_{i-j-1} + (i-j-1)\hat{\beta}_{i-j-1}]}{\prod_{j=0}^{i-1} \hat{\alpha}_{i-j} + (i-j)\hat{\beta}_{i-j} + \hat{\gamma}_{i-j}}}; \quad y_i^* = \frac{\prod_{j=0}^{i-1} [\hat{\alpha}_{i-j-1} + (i-j-1)\hat{\beta}_{i-j-1}]}{\prod_{j=0}^{i-1} \hat{\alpha}_{i-j} + (i-j)\hat{\beta}_{i-j} + \hat{\gamma}_{i-j}}, \]  

\( i = 1, 2, \ldots \)  

(30)

On the basis of the analogy between Eqs. (12) and (29) one can easily see that the Waring distribution is a particular case also for the distribution given by Eq. (30) that describes the distribution of the substance in the second arm of the channel. One has just to repeat the calculations starting from Eq. (13) and finishing at Eq. (22).

### 3 Concluding remarks

In this article we obtain analytical relationships for the distribution of the substance in the two arm of a channel of a network for the case of the stationary regime of the functioning of the channel. On the basis of these relationships we can make numerous conclusions. Let us discuss just one of these conclusions: the presence of the second arm of the channel changes the distribution of the substance in the first arm of the channel. In order to discuss this let us denote as \( y_i^{*(1)} \) the distribution of the substance in the cells of the first arm of the channel for the case of lack of second arm of the channel. Let \( y_i^{*(2)} \) be the distribution of the substance in the cells of the first arm of the channel for the case of presence of second arm of the channel. From the theory in the previous section one easily obtains the relationship

\[ \frac{y_i^{*(1)}}{y_i^{*(2)}} = \prod_{j=0}^{i-1} \frac{\alpha_{i-j} + (i-j)\beta_{i-j} + \gamma_{i-j}^*}{\alpha_{i-j} + (i-j)\beta_{i-j} + \gamma_{i-j}} = \prod_{j=0}^{i-1} \left[ 1 + \frac{\delta_{i-j}}{\alpha_{i-j} + (i-j)\beta_{i-j} + \gamma_{i-j}} \right] \]  

(31)
When \( i < n \) then \( \delta_i = 0 \) and there is no difference between the distribution of the substances in the channel with single arm and in the channel with two arms. The difference arises at the splitting cell (the \( n \)-th cell from the point of view of the numbering of the first arm of the channel). As it can be easily calculated for \( i \geq n \) Eq. (31) reduces to

\[
\frac{y_i^{(2)}}{y_i^{(1)}} = \frac{1}{1 + \frac{\delta_n}{\alpha_n + \beta_n + \gamma_n}}, \quad i \geq n
\]

Eq. (32) shows clearly that splitting of the channel affects the tail of the distribution of the substance in the first arm of the channel. If the second arm of the channel don’t exists then \( \delta_n = 0 \) and the distribution of the substance is a long-tail distribution that contains the long-tail Waring distribution as a particular case. The ”leakage” of the substance to second arm of the channel may reduce much the tail of the distribution of substance in the first arm of the channel. This reduction can even be kink-wise at the \( n \)-th node for large value of \( \delta_n \).

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