ON THE REPRESENTATIONS OF
DISCONNECTED REDUCTIVE GROUPS OVER $F_q$

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INTRODUCTION

Let $G$ be a connected reductive algebraic group defined over a finite field $F_q$. One of the main tools in the study of representations of the finite group $G(F_q)$ over a field of characteristic zero is the use of certain varieties $X_w$ (see [DL1]) on which $G(F_q)$ acts (here $w$ is a Weyl group element). Now let $\sigma : G \to G$ be a quasisemisimple automorphism of $G$ and let $m \geq 1$ be an integer such that $\sigma^m = 1$. Consider the semidirect product $\hat{G}$ of $G$ with the cyclic group of order $m$ with generator $\sigma$; this is naturally an algebraic group defined over $F_q$. Now the finite group $\hat{G}(F_q)$ acts naturally on the disjoint union $\sqcup_w X_w$ and from this one can again derive information about the representations of $\hat{G}(F_q)$. For example, this observation has been used by the author in his proof of the finiteness of the number of unipotent $G$-conjugacy classes in the connected component $G\sigma$ of $\hat{G}$ (see [Sp, I,4.1]); the connection between the varieties $X_w$ and the representations of $\hat{G}(F_q)$ has been systematically investigated by Digne and Michel [DM] and by Malle [Ma]. In this paper we try to extend some results on unipotent representations established for $G(F_q)$ in [L2] to $\hat{G}(F_q)$. One of the key steps in the description [L2] of the set of unipotent representations of $G(F_q)$ is the definition of a partition of that set into subsets indexed by the two-sided cells [KL1] of the Weyl group such that certain explicit $\mathbb{Q}$-linear combinations of virtual representations of $G(F_q)$ given by the alternating sum of the cohomologies of the various $X_w$ are linear combinations of unipotent representations corresponding to a fixed two-sided cell. A conjectural extension of this statement to the case of $\hat{G}(F_q)$ was formulated by Malle in [Ma] and proved in this paper (see 2.4(ii)). Our proof is a generalization of that in [L2]; the main new ingredient is the use of the generalization given in [L1] to certain Hecke algebras with unequal parameter of the polynomials $P_{y,w}$ defined in [KL1] and of their geometric interpretation stated in [L1] (and generalizing that in [KL2]) which is proved in this paper. It turns out that these generalized polynomials appear naturally in the study of the varieties $X_w$ in connection with

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a nontrivial $G(F_q)$-coset of $G(F_q)$ and they provide the necessary tools to prove the above conjecture.

**Notation.** Let $\mathbf{k}$ be an algebraic closure of the finite field $F_p$ with $p$ elements. If $q$ is a power of $p$ we denote by $F_q$ the subfield of $\mathbf{k}$ with $q$ elements. Let $\mathbf{Q}_l$ be an algebraic closure of the field of $l$-adic numbers ($l$ is a fixed prime number $\neq p$). All algebraic varieties in this paper are over $\mathbf{k}$. For an algebraic variety $Y$ of pure dimension let $H^i(Y)$ (resp. $H^i_c(Y)$) be the $i$-th hypercohomology space (resp. hypercohomology with compact support) of $Y$ with coefficients in the intersection cohomology complex $IC(Y, \mathbf{Q}_l)$; let $H^i_c(Y) = H^i_c(Y, \mathbf{Q}_l)$. If $K$ is a complex of $l$-adic sheaves on an algebraic variety $Y$ we denote by $\mathcal{H}^iK$ the $i$-th cohomology sheaf of $K$ and by $\mathcal{H}^i_yK$ the stalk of $\mathcal{H}^iK$ at $y \in Y$. The cardinal of a finite set $S$ is denoted by $|S|$. If $S$ is a set and $f : S \to S$ is a map we set $S^f = \{s \in S; f(s) = s\}$. We set $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ where $v$ is an indeterminate.

1. Preliminaries

1.1. Let $G$ be a connected reductive algebraic group over $\mathbf{k}$. Let $F' : G \to G$ be the Frobenius map relative to an $F'_q$-rational structure on $G$ ($q'$ is a power of $p$). Let $\mathcal{B}$ the variety of Borel subgroups of $G$. Note that $F'$ induces an endomorphism $B \mapsto F'(B)$ of $\mathcal{B}$. Let $W$ be the Weyl group of $G$ viewed as an indexing set for the $G$-orbits on $\mathcal{B} \times \mathcal{B}$ (simultaneous conjugation); for $w \in W$ let $O_w$ be the $G$-orbit corresponding to $w$. We regard $W$ as a Coxeter group with set of simple reflections $\{s_i; i \in I\}$ in the standard way; let $l : W \to \mathbb{N}$ be the corresponding length function. For any $I' \subset I$ let $s_{I'}$ be the longest element of the subgroup of $W$ generated by $\{s_i; i \in I'\}$. Let $\leq$ be the standard partial order on $W$. Now $F'$ induces an automorphism $\delta : W \to W$ (compatible with the length function) by the requirement that $w \in W, (B, B') \in O_w$ implies $(F'(B), F'(B')) \in O_{\delta(w)}$. Let $\text{sgn} : W \to \{\pm 1\}$ be the homomorphism $w \mapsto (-1)^{l(w)}$.

1.2. Let $\text{Irr} W$ be a set of representatives for the isomorphism classes of irreducible representations of $W$ over $\mathbf{Q}$. For $E \in \text{Irr} W$ let $M_E$ be the set of linear maps of finite order $\Delta : E \to E$ such that $\Delta(w(e)) = \delta(w)(\Delta(e))$ for any $w \in W$, $e \in E$. Then $|M_E|$ is 0 or 2. Let $\text{Irr}_2 W = \{E \in \text{Irr} W; |M_E| = 2\}$. For $E \in \text{Irr} W$ we define $E^\dagger \in \text{Irr} W$ by $E^\dagger \cong E \otimes \text{sgn}$.

Let $H$ be the Hecke algebra over $\mathcal{A}(v)$ of $W$ with respect to the weight function $w \mapsto l(w)$ on $W$. Thus $H$ has a $\mathcal{A}(v)$-basis $(T_w)_{w \in W}$ and we have $T_w T_{w'} = T_{ww'}$ if $w, w' \in W$, $l(ww') = l(w) + l(w')$; moreover we have $(T_{s_i} - v^2)(T_{s_i} + 1) = 0$ for $i \in I$. Note that $\mathbf{Q}[W] = \mathbf{Q} \otimes \mathcal{A} H$ where $\mathbf{Q}$ is viewed as an $\mathcal{A}$-algebra via $v \mapsto 1$; $w \in \mathbf{Q}[W]$ corresponds to $1 \otimes T_w$. Let $J$ be the ring with $\mathbb{Z}$-basis $(t_w)_{w \in W}$ defined as in [L3, 18.3] in terms of $(W, l)$ and let $\Phi : H \to \mathcal{A} \otimes J$ be the $\mathcal{A}$-algebra homomorphism defined in [L3, 18.9]. After applying $\mathbf{Q} \otimes \mathcal{A} (\cdot)$ to $\Phi$ we obtain an algebra isomorphism $\Phi : \mathbf{Q}[W] \overset{\sim}{\to} \mathbf{Q} \otimes J$. Via this isomorphism any $E \in \text{Irr} W$ becomes a simple $\mathbf{Q} \otimes J$-module denoted by $E_\bullet$. Let $E \in \text{Irr}_2 W$, $\Delta \in M_E$. We define $\Delta : E_\bullet \to E_\bullet$ by $\Delta(\xi) = \Phi Q(\Delta(\Phi^{-1}(\xi)))$; we have $t_{\delta(w)}(\Delta(\xi)) = \Delta(t_w(\xi))$
for any \( w \in W, \xi \in E_\bullet \). After applying \( Q(v) \otimes \mathcal{A} \) to \( \Phi \) we obtain an algebra isomorphism \( \Phi_{Q(v)} : Q(v) \otimes H \cong Q(v) \otimes J \). Via this isomorphism the simple \( Q(v) \otimes J \)-module \( Q(v) \otimes Q E_\bullet (E \text{ as above}) \) becomes a simple \( Q(v) \otimes H \)-module denoted by \( E_{u^2} \) and the isomorphism \( 1 \otimes \Delta : Q(v) \otimes Q E_\bullet \rightarrow Q(v) \otimes Q E_\bullet \) becomes an isomorphism \( \Delta : E_{u^2} \rightarrow E_{u^2} \) such that \( T_{\delta(w)}(\Delta(\xi)) = \Delta(T_w(\xi)) \) for any \( w \in W, \xi \in E_{u^2} \).

We assume that for each \( E \in \text{Irr}_\delta W \) we have chosen an element \( \Delta \in M_E \).

Let \( a : W \rightarrow N \) be the function defined (in terms of \( l : W \rightarrow N \)) in [L2, (5.27.1)] or equivalently as in [L3, 13.6]. Let \( E \rightarrow a_E \) be the function \( \text{Irr} W \rightarrow N \) defined in [L2, (4.1.1)]. For \( E \in \text{Irr} W \) we set \( a'_E = a_E \).

For \( w \in W \) we have

\[
v^{-l(w)} \text{tr}(\Delta T_w, E_{u^2}) = c'_{w, \Delta, E} v^{a'_E} + \text{lower powers of } v
\]

where \( c'_{w, \Delta, E} \in \mathbb{Z} \) (see [L2, (5.1.23)]); by an argument similar to that in [L3, 20.10], we have \( c'_{w, \Delta, E} = \text{tr}(\Delta', E'^\dagger) \) where \( \Delta' : E'^\dagger \rightarrow E^\dagger \) is \( \Delta \otimes 1 : E \otimes \text{sgn} \rightarrow E \otimes \text{sgn} \).

Let \( \mathcal{R}_W \) be the vector space of formal linear combinations \( \sum_{E \in \text{Irr}_\delta W} r_E E, r_E \in \mathbb{Q} \). For \( w \in W \) we set

\[
\mathfrak{a}_w = \sum_{E \in \text{Irr}_\delta W} c'_{w, \Delta, E} E \in \mathcal{R}_W.
\]

(Compare [L2, (5.11.6)].)

**1.3.** Let \( w \in W \). Following [DL1] we set \( X_w = \{ B' \in \mathcal{B}; (B', F'(B')) \in \mathcal{O}_w \} \). Let

\[
\bar{X}_w = \{ B' \in \mathcal{B}; (B', F'(B')) \in \cup_{z \in W; z \leq w} \mathcal{O}_z \}
\]

be the closure of \( X_w \) in \( \mathcal{B} \). For \( x \in G^{F'} \), \( \text{Ad}(x) : \mathcal{B} \rightarrow \mathcal{B} \) (conjugation by \( x \)) leaves \( X_w, \bar{X}_w \) stable and induces linear automorphisms \( \text{Ad}(x)^* \) of \( H^i(\bar{X}_w), H_c^i(X_w) \). Then \( x \mapsto \text{Ad}(x)^* \) makes \( H^i(\bar{X}_w), H_c^i(X_w) \) into \( G^{F'} \)-modules. Let \( \mathcal{E} \) be a set of representatives for the isomorphism classes of irreducible representations of \( G^{F'} \) which appear in \( H_c^i(X_w) \) for some \( w \in W, i \in \mathbb{N} \) or equivalently in \( H^i(\bar{X}_w) \) for some \( w \in W, i \in \mathbb{N} \) (thus \( \mathcal{E} \) is the set of unipotent representations of \( G^{F'} \)). Let \( [\mathcal{E}]_\mathbb{Z} \) be the Grothendieck group of the category of representations of \( G^{F'} \) which are finite sums of unipotent representations. Then \( [\mathcal{E}] := \mathbb{Q} \otimes [\mathcal{E}]_\mathbb{Z} \) has a basis \( \{ \rho; \rho \in \mathcal{E} \} \). For \( f \in [\mathcal{E}] \) let \( (\rho : f) \in \mathbb{Q} \) be the coefficient of \( \rho \in \mathcal{E} \) in \( f \).

For \( w \in W \) let \( R_w = \sum_i (-1)^i H_c^i(X_w) \) viewed as an element of \( [\mathcal{E}] \). As in [L2, 3.7], for any \( E \in \text{Irr}_\delta W \) we define

\[
R_E = |W|^{-1} \sum_{w \in W} \text{tr}(\Delta w, E) R_w \in [\mathcal{E}].
\]

For any \( \xi = \sum_{E \in \text{Irr}_\delta W} r_E E \in \mathcal{R}_W \) we set \( R_\xi = \sum_{E \in \text{Irr}_\delta W} r_E R_E \in [\mathcal{E}] \). In particular, for \( w \in W, R_{\mathfrak{a}_w} \) is defined.
1.4. Let $\leq_{LR}$ be the preorder on $W$ defined in [KL1] and let $\sim_{LR}$ be the corresponding equivalence relation on $W$ (the equivalence classes are the two-sided cells of $W$; they form a set $\mathcal{C}$). For $w, w' \in W$ we write $w' <_{LR} w$ when $w' \leq_{LR} w$ and $w' \not<_{LR} w$. The relation $\leq_{LR}$ on $W$ induces a relation on $\mathcal{C}$ denoted again by $\leq_{LR}$. It is known [KL1] that if $c \in \mathcal{C}$ then $c^* := cs_I = s_I c \in \mathcal{C}$ and that $c \mapsto c^*$ reverses the preorder $\leq_{LR}$.

If $E \in \text{Irr} W$ then there is a unique $c \in \mathcal{C}$ such that $t_w : E_{c^*} \to E_{c^*}$ is nonzero for some $w \in c$ and is zero for all $w \in W - c$; we then write $E \vdash c$. From the definitions we see that

(a) If $c \in \mathcal{C}, w \in c$ then $\mathfrak{A}_w$ is a linear combination of $E$ such that $E \vdash c^*$.

The following result is contained in [L2, 4.23].

**Theorem 1.5.** Let $\rho \in \mathcal{E}$. There exists a unique $c \in \mathcal{C}$ such that $(\rho : R_E) \neq 0$ for some $E \in \text{Irr}_3 W$ with $E \vdash c$ and $(\rho : R_E) = 0$ for all $E \in \text{Irr}_3 W$ with $E \not\vdash c$. We then write $\underline{\rho} = c$.

Using the theorem and 1.4(a) we see that for $c \in \mathcal{C}, w \in c$ we have

(a) $R_{\mathfrak{A}_w} = \sum_{\rho \in \mathcal{E} : \rho = c^*} Q \rho$.

The following result is contained in [L2, 6.15].

**Theorem 1.6.** Let $w \in W$. We have

(a) $H^{l(w) - a(w)}(\bar{X}_w) = (-1)^{l(w) - a(w)} R_{\mathfrak{A}_w} + \sum_{w' \in W : w' <_{LR} w} n_{w', w} R_{\mathfrak{A}_{w'}} \in \mathcal{E}$

where $n_{w', w} \in \mathbb{Q}$.

For $w \in W, i \in \mathbb{N}$ and $\rho \in \mathcal{E}$ we denote by $H^i(\bar{X}_w)_{\rho}$ the $\rho$-isotypic component of the $G_{F'}$-module $H^i(\bar{X}_w)$ and we set

$$H^{l(w) - a(w)}(\bar{X}_w)_{c^*} = \sum_{\rho : \rho = c^*} H^{l(w) - a(w)}(\bar{X}_w)_{\rho}$$

where $c \in \mathcal{C}$ is defined by $w \in c$.

Let $c \in \mathcal{C}$. Using the theorem and 1.5(a) we see that if $w \in c$ then

(b) $H^{l(w) - a(w)}(\bar{X}_w)$ is a $\mathbb{Q}$-linear combination of $\rho \in \mathcal{E}$ such that $c^* \leq_{LR} \rho$.

Hence projecting the equality (a) onto the subspace generated by the $\rho \in \mathcal{E}$ such that $\underline{\rho} = c^*$ we see that for any $w \in c$ we have

(c) $H^{l(w) - a(w)}(\bar{X}_w)_{c^*} = (-1)^{l(w) - a(w)} R_{\mathfrak{A}_w}$.

(This projection maps each $R_{\mathfrak{A}_{w'}}$ with $w' <_{LR} w$ to zero, see 1.5(a).) In particular, for any $w \in W$,

(d) $(-1)^{l(w) - a(w)} R_{\mathfrak{A}_w}$ is an actual $G_{F'}$-module.
Corollary 1.7. Let $\rho \in \mathcal{E}$. We have $H^{l(w)-a(w)}(X_w)_{\rho} \neq 0$ for some $w \in \rho^*$. If $c' \in C, w' \in c$ and $H^{l(w')-a(w')}((X_w))_{\rho} \neq 0$ then $\rho^* \leq_{LR} c'$.

Let $c = \rho$. We can find $E \in \text{Irr}_q W$ such that $(\rho : R_E) \neq 0$, $E \vdash c$. By [L2, 5.13(ii)], $E$ is a $Q$-linear combination of elements $\mathfrak{A}_x \ (x \in c^*)$. Hence $R_E$ is a $Q$-linear combination of elements $R_{\mathfrak{A}_x} \ (x \in c^*)$. It follows that $(\rho : \mathfrak{A}_x) \neq 0$ for some $x \in c^*$. Hence by 1.6(c) we have $H^{l(x)-a(x)}((X_x))_{\rho} \neq 0$. The last sentence in the corollary follows from 1.6(b).

2. The main results

2.1. We preserve the setup of 1.1. In addition, we fix an automorphism $\sigma : G \rightarrow G$ such that for some $(B, B^*) \in O_{s_I}$ we have $\sigma(B) = B$, $\sigma(B^*) = B^*$ and such that $\sigma F' = F' \sigma : G \rightarrow G$. We also fix an integer $m \geq 1$ such that $\sigma^m = 1$. Now $\sigma$ induces a (length preserving) automorphism of $W$ denoted again by $\sigma$; thus, for $w \in W, (B_1, B_2) \in O_w$ we have $(\sigma(B_1), \sigma(B_2)) \in O_{\sigma(w)}$; moreover $\sigma \delta = \delta \sigma : W \rightarrow W$. For $i \in I$ we have $\sigma(s_i) = s_{\sigma(i)}$ where $\sigma : I \rightarrow I$ is a bijection. Let $\hat{G}$ be the semidirect product of $G$ with the cyclic group of order $m$ with generator $\sigma$ so that in $\hat{G}$ we have $\sigma g \sigma^{-1} = \sigma(g)$ for all $g \in G$. Note that $\hat{G}$ is naturally an affine algebraic group with identity component $G$. We extend $F'$ to a homomorphism $\hat{G} \rightarrow \hat{G}$ (denoted again by $F'$) by $\sigma^i \mapsto \sigma^j F'(g)$ for $i \in [0, m-1], g \in G$. This is the Frobenius map on an $F_q^r$-rational structure on $\hat{G}$. Note that $\hat{G}'$ is the semidirect product of $G'$ with the cyclic group of order $m$ with generator $\sigma$.

Let $d \geq 1$ be an integer with the following property: there exists $(B, B^*) \in O_{s_I}$ such that $\sigma(B) = B$, $\sigma(B^*) = B^*$, $F'^d(B) = B$, $F'^d(B^*) = B^*$, $F'^d$ acts on $B \cap B^*$ as $t \mapsto t^{d^d}$ (clearly, such $d$ exists). Let $r \geq 1$ be a multiple of $d$ and let $F = F'^r \sigma : G \rightarrow G$, a Frobenius map relative to an $F_q^r$-rational structure on $G$ where $q = q^r$. Note that $w \in W, (B_1, B_2) \in O_w$ implies $(F(B_1), F(B_2)) \in O_{\sigma(w)}$.

Let $\hat{W} = \{w \in W; \sigma(w) = w\}$. It is known that $\hat{W}$ is itself a Weyl group with standard generators $s_\omega$ where $\omega$ runs over the set $I$ of $\sigma$-orbits on $I$. Let $\hat{l} : \hat{W} \rightarrow N$ be the length function of $\hat{W}$; thus $\hat{l}(s_\omega) = 1$ for any $\omega \in I$. The restriction $l|_{\hat{W}}$ of $l$ to $\hat{W}$ is a weight function (in the sense of [L3, 3.1]) on the Coxeter group $\hat{W}$. We define $\text{sgn} : \hat{W} \rightarrow \{\pm 1\}$ by $\text{sgn}(w) = (-1)^{\hat{l}(w)}$.

2.2. Let $\text{Irr} \hat{W}$ be a set of representatives for the isomorphism classes of irreducible representations of $\hat{W}$ over $Q$. Now $\delta : W \rightarrow W$ restricts to an automorphism of $\hat{W}$ (denoted again by $\delta$) preserving the set $\{s_\omega, \omega \in I\}$ and the weight function $l|_{\hat{W}}$. For $E \in \text{Irr} \hat{W}$ let $M_E$ be the set of linear maps of finite order $\Delta : E \rightarrow E$ such that $\Delta(w(e)) = \delta(w)(\Delta(e))$ for any $w \in \hat{W}, e \in E$. Then $|M_E|$ is 0 or 2. Let $\text{Irr}_q \hat{W} = \{E \in \text{Irr} \hat{W}; |M_E| = 2\}$. For $E \in \text{Irr} \hat{W}$ we define $E^\dagger \in \text{Irr} \hat{W}$ by $E^\dagger \cong E \hat{\otimes} \text{sgn}$.

Let $\hat{H}$ be the Hecke algebra over $A$ of $\hat{W}$ with respect to the weight function $l|_{\hat{W}}$. Thus $\hat{H}$ has an $A$-basis $(\hat{T}_w)_{w \in \hat{W}}$ and we have $\hat{T}_w \hat{T}_{w'} = \hat{T}_{ww'}$ if $w, w' \in \hat{W}$,
$l(wv') = l(w) + l(v')$; moreover we have $(\hat{T}_{sv} - v^{2l(sv)})(\hat{T}_{sv} + 1) = 0$ for $\omega \in \hat{I}$. Note that $Q[\hat{W}] = Q \otimes_A \hat{H}$ where $Q$ is viewed as an $A$-algebra via $v \mapsto 1$; $w \in Q[\hat{W}]$ corresponds to $1 \otimes \hat{T}_w$.

Let $^\sim : A \to A$ be the ring involution such that $v \mapsto v^{-1}$. Let $^- : \hat{H} \to \hat{H}$ be the ring homomorphism such that $\tilde{aT}_w = \tilde{a}T_{w^{-1}}^{-1}$ for $w \in \hat{W}, a \in A$. For any $x, y \in \hat{W}$ we define a polynomial $\hat{R}_{x,y}(X) \in \mathbb{Z}[X], (X$ is an indeterminate) by the equality

$$\hat{T}_y = \sum_{x \in \hat{W}} \hat{R}_{x,y}(v^2)v^{2l(x)}\hat{T}_{x^{-1}}^{-1}.$$

(See [KL1,(2.0.a)], [L3, 4.3].) Note that $\hat{R}_{x,y} = 0$ unless $x \leq y$. It follows that

(a) $$\hat{T}_y \hat{T}_{y_I} = \sum_{x \leq y} \hat{R}_{x,y}(v^2)v^{2l(x)}\hat{T}_{x_{y_I}}^{-1}.$$

For any $w \in \hat{W}$ there is a unique element $\hat{C}_w \in \hat{H}$ such that

$$\hat{C}_w = \sum_{y \leq w} \text{sgn}(yw)\hat{P}_{y,w}(v^{-2})v^{l(w) - 2l(y)}\hat{T}_y$$

$$= \sum_{y \leq w} \text{sgn}(yw)\hat{P}_{y,w}(v^2)v^{l(w) + 2l(y)}\hat{T}_{y^{-1}}^{-1}.$$

where $\hat{P}_{y,w}(X) \in \mathbb{Z}[X]$ has degree $\leq (l(w) - l(y) - 1)/2$ if $y < w$ and $\hat{P}_{w,w}(X) = 1$. (See [L3,§5]; compare [KL1].) For $y \leq w$ in $\hat{W}$ we have from the definitions

(b) $$v^{2l(w)}\hat{P}_{y,w}(v^{-2}) = \sum_{z \leq w} v^{2l(y)}\hat{R}_{y,z}(v^2)\hat{P}_{z,w}(v^2).$$

We set $\hat{P}_{y,w} = 0$ if $y \not\leq w$. We define for $y, w$ in $\hat{W}$ a polynomial $\hat{Q}_{y,w}(X) \in \mathbb{Z}[X]$ by the requirement that for any $y, w$ in $\hat{W}$,

(c) $$\sum_{z \in \hat{W}} \text{sgn}(zw)\hat{P}_{y,z}(X)\hat{Q}_{z,w}(X) = 1 \text{ if } y = w \text{ and is 0 if } y \neq w.$$

Let $\hat{J}$ be the ring with $\mathbb{Z}$-basis $(\hat{t}_w)_{w \in \hat{W}}$ defined as in [L3, 18.3] in terms of $\hat{W}$ and the weight function $l|_{\hat{W}}$ and let $\Phi : \hat{H} \to A \otimes \hat{J}$ be the $A$-algebra homomorphism defined in [L3, 18.9]. (The definitions and results of [L3] that were just quoted are applicable in view of [L3, §16].) After applying $Q \otimes_A ()$ to $\Phi$ we obtain an algebra isomorphism $\Phi_Q : Q[\hat{W}] \sim Q \otimes \hat{J}$. Via this isomorphism any $E \in \text{Irr} \hat{W}$ becomes a simple $Q \otimes J$-module denoted by $E$. Let $E \in \text{Irr} \hat{W}, \Delta \in M_E$. We define $\Delta : E \to E$ by $\Delta(\xi) = \Phi_Q(\Delta(\xi))$. We have $\hat{t}_{\delta(w)}(\Delta(\xi)) = \Delta(\hat{t}_w(\xi))$.
for any \( w \in \hat{W}, \xi \in E_{\bullet} \). After applying \( Q(v) \otimes A \) to \( \Phi \) we obtain an algebra isomorphism \( \Phi_{Q(v)} : Q(v) \otimes A \tilde{H} \cong Q(v) \otimes Z \hat{J} \). Via this isomorphism the simple \( Q(v) \otimes \hat{J} \)-module \( Q(v) \otimes Q E_{\bullet} \) (\( E \) as above) becomes a simple \( Q(v) \otimes A \hat{H} \)-module denoted by \( E_{v^2} \) and the isomorphism \( 1 \otimes \Delta : Q(v) \otimes Q E_{\bullet} \rightarrow Q(v) \otimes Q E_{\bullet} \) becomes an isomorphism \( \Delta : E_{v^2} \rightarrow E_{v^2} \) such that \( T_{\delta(w)}(\Delta(\xi)) = \Delta(T_w(\xi)) \) for any \( w \in \hat{W}, \xi \in E_{v^2} \).

We assume that for each \( E \in \text{Irr}_s \hat{W} \) we have chosen an element \( \Delta \in M_E \).

2.3. Let \( w \in \hat{W} \). Now \( X_w \) and \( \bar{X}_w \) are stable under \( \phi := F^d : B \rightarrow B \) (since \( \phi \) commutes with \( F' \) and it acts trivially on \( W \)) and under \( \sigma : B \rightarrow B \) (since \( \sigma \) commutes with \( F' \) and \( \sigma(w) = w \)); hence they are \( F \)-stable and there are induced automorphisms \( \phi^*, (F^d)^*, \sigma^*, F^* \) of \( H^*(X_w) \) and \( H_c^*(X_w) \). Hence \( X_w, \bar{X}_w \) are stable under the \( \hat{G}^F' \)-action \( \sigma^i g : B \rightarrow \sigma^i(\text{Ad}(g)B) \) (\( i \in [0, m-1] \), \( g \in \hat{G}^F' \)) and \( \hat{G}^F' \) acts on \( H^*(X_w) \) and \( H_c^*(X_w) \) by \( \sigma^i g \rightarrow (\sigma^*)^{-1}\text{Ad}(g^{-1})^* \). Let \( \hat{E} \) be a set of representatives for the isomorphism classes of irreducible representations of \( \hat{G}^F' \) whose restriction to \( G^F' \) is a direct sum of unipotent representations. Let \( \mathcal{E}_0 \) be the set of all \( \rho \in \mathcal{E} \) such that \( \rho \) extends to a \( \hat{G}^F' \)-module; let \( \hat{E}_0 \) be the set of all \( \hat{\rho} \in \hat{E} \) such that \( \hat{\rho} |_{G^F'} \) is irreducible. Let \( [\hat{E}^\mathbb{Z}] \) be the Grothendieck group of the category of representations of \( \hat{G}^F' \) whose restriction to \( G^F' \) are finite sums of unipotent representations. Then \( [\hat{E}] := Q \otimes [\hat{E}^\mathbb{Z}] \) has a basis \( \{ \hat{\rho} \in \hat{E} \} \). For \( \hat{\rho} \in \hat{E} \), \( f \in [\hat{E}] \) we denote by \( (\hat{\rho} : f) \) the coefficient of \( \hat{\rho} \) in \( f \). Let \( [\hat{E}] \) be the quotient of \( [\hat{E}] \) by the subspace consisting of all elements whose character restricted to \( G^F' \sigma \) is zero. Note that for \( h \in [\hat{E}] \), \( \text{tr}(x\sigma, h) \) makes sense for any \( x \in G^F' \) and we can define for \( \hat{\rho} \in \hat{E} \),

\[
(\hat{\rho} : h) = |G^F'|^{-1} \sum_{x \in G^F'} \text{tr}((x\sigma)^{-1}, \hat{\rho})\text{tr}(x\sigma, h).
\]

We show that if \( f \in [\hat{E}] \), \( h \) is its image in \([\hat{E}]\) and \( \hat{\rho} \in \hat{E} \), then

(a) \( (\hat{\rho} : h) = \sum_{\chi \in L} (\hat{\rho} \otimes \chi : f)\chi(\sigma)^{-1} \)

where \( L \) is the set of one dimensional characters of \( \hat{G}^F' \), trivial on \( G^F' \). Indeed,

\[
\begin{align*}
\sum_{\chi \in L} (\hat{\rho} \otimes \chi : f)\chi(\sigma)^{-1} &= |G^F'|^{-1}m^{-1} \sum_{\chi \in L} \sum_{x \in G^F', j \in [0, m-1]} \text{tr}(x\sigma^j, \hat{\rho} \otimes \chi)\text{tr}((x\sigma^j)^{-1}, f)\chi(\sigma)^{-1} \\
&= |G^F'|^{-1}m^{-1} \sum_{x \in G^F', j \in [0, m-1]} \text{tr}(x\sigma^j, \hat{\rho})\text{tr}((x\sigma^j)^{-1}, f) \sum_{\chi \in L} \chi(\sigma^j)^{-1} \\
&= |G^F'|^{-1} \sum_{x \in G^F'} \text{tr}(x\sigma, \hat{\rho})\text{tr}((x\sigma)^{-1}, f) = (\hat{\rho} : h),
\end{align*}
\]
as desired.

For any $\rho \in \mathcal{E}_0$ let $[\hat{\mathcal{E}}]^\rho$ be the (one dimensional) subspace of $[\hat{\mathcal{E}}]$ generated by the elements $\hat{\rho} \in \hat{\mathcal{E}}_0$ such that $\rho = \hat{\rho}|_{\mathcal{G}_F}$. From the definitions we have a direct sum decomposition

$$[\hat{\mathcal{E}}] = \bigoplus_{\rho \in \mathcal{E}_0} [\hat{\mathcal{E}}]^\rho.$$ 

For any $c \in \mathcal{C}$ we set $[\hat{\mathcal{E}}]^c = \bigoplus_{\rho \in \mathcal{E}_0} [\hat{\mathcal{E}}]^\rho$. We have

$$[\hat{\mathcal{E}}] = \bigoplus_{c \in \mathcal{C}} [\hat{\mathcal{E}}]^c.$$ 

The following result is clear from the definitions.

(b) Let $h \in [\hat{\mathcal{E}}]$ be such that $(\hat{\rho} : h) = 0$ for any $\hat{\rho} \in \hat{\mathcal{E}}$. Then $h = 0$.

For $w \in \hat{W}$ let

$$\hat{R}_w = \sum_i (-1)^i H^i_c(X_w)$$

viewed as an element of $[\hat{\mathcal{E}}]$. For any $E \in \text{Irr}_\delta \hat{W}$ we set

$$\hat{R}_E = |\hat{W}|^{-1} \sum_{w \in W} \text{tr}(\Delta w, E) \hat{R}_w \in [\hat{\mathcal{E}}].$$

For any $\xi \in \hat{H}$ any $E \in \text{Irr}_\delta \hat{W}$ and any $i \in \mathbb{Z}$ we define $\text{tr}(\Delta h, E_{v^2}; i) \in \mathbb{Z}$ by

$$\text{tr}(\Delta h, E_{v^2}) = \sum_i \text{tr}(\Delta h, E_{v^2}; i)v^i.$$ 

Part (i) of the following theorem is a partial generalization of [L2, 3.8(ii)]; part (ii) (a partial generalization of [L2, 4.23], see also 1.5) provides an affirmative answer to a conjecture of G. Malle [Ma] (which he proved in the case where $\text{rank}(G) \leq 6$).

**Theorem 2.4.** (i) For any $w \in \hat{W}$ the following equality holds in $\mathbb{Q}[v] \otimes [\hat{\mathcal{E}}]$:

(a) $$\sum_i (-1)^i H^i(\bar{X}_w)v^i = \sum_{E \in \text{Irr}_\delta \hat{W}} \text{tr}(\Delta) \sum_{y \in W} \hat{P}_{y,w}(v^2)\hat{T}_y, E_{v^2}) \hat{R}_E.$$ 

Equivalently, for any $i \in \mathbb{Z}$ we have

$$(-1)^i H^i(\bar{X}_w) = \sum_{E \in \text{Irr}_\delta \hat{W}} \text{tr}(\Delta) \sum_{y \in W} \hat{P}_{y,w}(v^2)\hat{T}_y, E_{v^2}; i) \hat{R}_E.$$ 

(ii) Let $E \in \text{Irr}_\delta \hat{W}$. There exists $c \in \mathcal{C}$ such that $\hat{R}_E \in [\hat{\mathcal{E}}]^c$.

The proof of (i) (which has much in common with that of [L2, 3.8(ii)]) is given in 2.15. The proof of (ii) is given in 2.19 where the two-sided cell $c$ in (ii) is explicitly described in terms of $E$ (see 2.19(c)).
2.5. We fix a square root \( q^{1/2} \) of \( q' \) in \( \mathbb{Q}_l \); for any integer \( n \) we set \( q^{n/2} = (q^{1/2})^n \), \( q^{n/2} = q^{mr/2} \). Let \( \hat{H}_q = \hat{\mathbb{Q}}_l \otimes \hat{A} \hat{H} \) where \( \mathbb{Q}_l \) is viewed as an \( A \)-algebra via \( v \mapsto q^{1/2} \).

Let \( \mathcal{F} \) be the vector space of functions \( B^F \to \mathbb{Q}_l \). For any \( w \in \hat{W} \) we define a linear map \( \hat{T}_w : \mathcal{F} \to \mathcal{F} \) by \( \hat{T}_w(f) = f' \) for \( f \in \mathcal{F} \) where \( f'(B) = \sum_{B' \in B^F} f(B') \) for \( B \in B^F \). In this way \( \mathcal{F} \) becomes a \( \hat{H}_q \)-module. Note that the linear maps \( \hat{T}_w : \mathcal{F} \to \mathcal{F} \) are linearly independent. For \( w, w' \in \hat{W} \) we have \( \hat{T}_w \hat{T}_{w'} = \sum_{w'' \in \hat{W}} N_{w,w',w''} \hat{T}_{w''} \) where

\[ N_{w,w',w''} = |\{B \in B^F; (B_1, B) \in \mathcal{O}_w, (B, B_2) \in \mathcal{O}_{w'}\}| \]

for any \((B_1, B_2) \in \mathcal{O}^{F}_{w''} \). Using this and 2.2(a) we see that

\[ \sum_{z \in \hat{W}} N_{y,s_1,z} \hat{T}_z = \sum_{x \in \hat{W}} \hat{R}_{x,y}(q) q^{l(x)} \hat{T}_{xs_1} \]

for any \( y \in \hat{W} \) as linear maps \( \mathcal{F} \to \mathcal{F} \). Using the linear independence of the linear maps \( \hat{T}_z \) we deduce

(a) \[ \hat{R}_{x,y}(q) q^{l(x)} = N_{y,s_1,xs_1} \]

for any \( x, y \) in \( \hat{W} \).

2.6. Now let \( B, B^* \) be as in 2.1. For any \( w \in \hat{W} \) we define \( B_w \in \mathcal{B} \) by the conditions \( (B, B_w) \in \mathcal{O}_w, (B, B^*) \in \mathcal{O}_{w-1,s_1} \). Note that \( B_w \) is fixed by \( F^{tr} : \mathcal{B} \to \mathcal{B} \) and by \( \sigma : \mathcal{B} \to \mathcal{B} \) hence by \( F : \mathcal{B} \to \mathcal{B} \). For \( z \in \hat{W} \) let

\[ B_z = \{B' \in \mathcal{B}; (B, B') \in \mathcal{O}_z\}, \]

\[ \mathcal{B}_z = \{B' \in \mathcal{B}; (B, B') \in \bigcup_{z' \in \hat{W}; z' \leq z} \mathcal{O}_{z'}\}, \]

\[ A^z = \{B' \in \mathcal{B}; (B', B_{zs_1}) \in \mathcal{O}_{zs_1}\}; \]

note that \( B_z, \mathcal{B}_z, A^z \) are stable under \( F^{tr} : \mathcal{B} \to \mathcal{B} \) and under \( \sigma : \mathcal{B} \to \mathcal{B} \) hence under \( F : \mathcal{B} \to \mathcal{B} \).

Let \( w \in \hat{W} \). Let \( K_w = IC(\mathcal{B}_w, \mathbb{Q}_l) \) be the intersection cohomology complex of \( \mathcal{B}_w \). For any \( y \in \hat{W} \) such that \( y \leq w \), \( \sigma \) induces an isomorphism of local systems \( \sigma^*(\mathcal{H}^i K_w|_{B_y}) \to \mathcal{H}^i K_w|_{B_y} \); this induces an automorphism \( \sigma^*: \mathcal{H}^i_{B_y} K_w \rightarrow \mathcal{H}^i_{B_y} K_w \) since \( B_y \in \mathcal{B}_y \) is fixed by \( \sigma \). The trace of this automorphism is denoted by \( n_{i,y,w,\sigma} \).

The following generalization of [KL2, 4.3] was stated without proof in [L1, (8.1)]; its proof (which uses [KL2, 4.2]) is a generalization of that of [KL2, 4.3].
Theorem 2.7. For any \( y, w \in \hat{W} \) such that \( y \leq w \) we have
\[
\hat{P}_{y, w}(X) = \sum_{i \in \mathbb{N}} n_{2i, y, w, \sigma} X^i.
\]

From the definitions, for \( z, y \in \hat{W} \) we have \( |(B_z \cap A^y)| = N_{z, s_l, y s_l} \); using 2.5(a), it follows that
\[
(a) \quad |(B_z \cap A^y)| = \hat{R}_{y, z}(q) q^l(y).
\]

Note that \( K_w|_{\hat{B}_w \cap A^y} \) is the intersection cohomology complex \( IC(\hat{B}_w \cap A^y, \mathbb{Q}) \) of \( \hat{B}_w \cap A^y \) since \( \hat{B}_w \cap A^y \) is open in \( \hat{B}_w \). Now the Grothendieck-Lefschetz trace formula for \( F : \hat{B}_w \cap A^y \to \hat{B}_w \cap A^y \) gives:
\[
\sum_i (-1)^i \text{tr}(F^*, H^i_c(\hat{B}_w \cap A^y)) = \sum_{B' \in (\hat{B}_w \cap A^y)^F} \sum_i (-1)^i \text{tr}(F^*, H^i_{B'} K_w).
\]

By specifying \( z \in W \) such that \( B' \in \mathcal{O}^F_z \) (so that \( z \) is necessarily in \( \hat{W} \)) we obtain
\[
\sum_i (-1)^i \text{tr}(F^*, H^i_c(\hat{B}_w \cap A^y)) = \sum_{z \in W : z \leq w} \sum_{B' \in (B_z \cap A^y)^F} \sum_i (-1)^i \text{tr}(F^*, H^i_{B'} K_w).
\]

Note that \( \text{tr}(F^*, H^i_{B'} K_w) \) is independent of \( B' \) when \( B' \) runs through \( (B_z \cap A^y)^F \) and even when \( B' \) runs through \( B^F_z \) (since the local system \( H^1 K_w|_{B_z} \) is \( B \)-equivariant for the obvious transitive \( B \)-action on \( B_z \) with connected isotropy groups); moreover we have \( B_z \in B^F_z \) and we deduce that
\[
\sum_i (-1)^i \text{tr}(F^*, H^i_c(\hat{B}_w \cap A^y)) = \sum_{z \in W : z \leq w} |(B_z \cap A^y)| \sum_i (-1)^i \text{tr}(F^*, H^i_{B_z} K_w);
\]

using (a) and the fact that \( \hat{R}_{y, z}(X) = 0 \) unless \( y \leq z \), we deduce that
\[
\sum_i (-1)^i \text{tr}(F^*, H^i_c(\hat{B}_w \cap A^y)) q^l(y) \sum_i (-1)^i \text{tr}(F^*, H^i_{B_z} K_w).
\]

By Poincaré duality for intersection cohomology we have
\[
\sum_i (-1)^i \text{tr}(F^*, H^i_c(\hat{B}_w \cap A^y)) = q^l(w) \sum_i (-1)^i \text{tr}(F^{*\neg 1}, H^i(\hat{B}_w \cap A^y))
\]
since $B_u \cap A^y$ is of pure dimension $l(w)$ (or is empty). By [KL2, 1.5, 4.5] we have

$$\text{tr}(F^{*^{-1}}, H^i(B_u \cap A^y)) = \text{tr}(F^{*^{-1}}, H^i_{B_y} K_w)$$

hence

$$q^{l(w)} \sum_i (-1)^i \text{tr}(F^{*^{-1}}, H^i_{B_y} K_w)$$

$$= \sum_{z \in \hat{W}; y \leq z \leq w} R_{y,z}(q) q^{l(y)} \sum_i (-1)^i \text{tr}(F^*, H^i_{B_z} K_w).$$

By [KL2, 4.2], $H^i_{B_z} K_w$ is zero if $i$ is odd while if $i$ is even, $(F'^r)^*$ acts on $H^i_{B_z} K_w$ as $q^{i/2}$ times a unipotent transformation hence $F^*$ acts on $H^i_{B_z} K_w$ as $q^{i/2}$ times a unipotent transformation times the action of $\sigma^*$ (which commutes with the unipotent transformation, since $\sigma F'^r = F'^r \sigma$). Thus we have

$$q^{l(w)} \sum_{i \in 2\mathbb{N}} q^{-i/2} \text{tr}(\sigma^{*-1}, H^i_{B_y} K_w)$$

$$= \sum_{z \in \hat{W}; y \leq z \leq w} R_{y,z}(q) q^{l(y)} \sum_{i \in 2\mathbb{N}} q^{i/2} \text{tr}(\sigma^*, H^i_{B_z} K_w).$$

Now for any $z \in \hat{W}$ such that $z \leq w$ we set:

$$\tilde{P}_{z,w}(X) = \sum_{i \in 2\mathbb{N}} \text{tr}(\sigma^*, H^i_{B_z} K_w) X^{i/2} \in R[X],$$

$$\tilde{P}'_{z,w}(X) = \sum_{i \in 2\mathbb{N}} \text{tr}(\sigma^{*-1}, H^i_{B_z} K_w) X^{i/2} \in R[X]$$

where $R$ is the subring of $\bar{Q}_l$ generated by the $m$-th roots of 1. By the definition of intersection cohomology, $\tilde{P}_{z,w}, \tilde{P}'_{z,w}$ are polynomials in $X$ of degree $\leq (l(w) - l(z) - 1)/2$ (if $z < w$) and are equal to 1 if $z = w$. We have

$$q^{l(w)} \tilde{P}'_{y,w}(q^{-1}) = \sum_{z \in \hat{W}; y \leq z \leq w} q^{l(y)} \hat{R}_{y,z}(q) \tilde{P}_{z,w}(q).$$

Since here $q$ is an arbitrary power of $q^{1d}$ it follows that

$$X^{l(w)} \tilde{P}'_{y,w}(X^{-1}) = \sum_{z \in \hat{W}; y \leq z \leq w} X^{l(y)} \hat{R}_{y,z}(X) \tilde{P}_{z,w}(X).$$

Note also that

$$X^{l(w)} \tilde{P}_{y,w}(X^{-1}) = \sum_{z \in \hat{W}; y \leq z \leq w} X^{l(y)} \hat{R}_{y,z}(X) \tilde{P}_{z,w}(X).$$
We show by induction on \( l(w) - l(y) \) that

\[
\tilde{P}_{y,w}(X) = \tilde{P}'_{y,w}(X) = \tilde{P}_{y,w}(X)
\]

for any \( y \in \hat{W} \) such that \( y \leq w \). If \( l(y) = l(w) \) we have \( y = w \) and all three terms in (d) are 1. Now assume that \( y < w \) and that the result is known when \( y \) is replaced by \( z \in \hat{W} \) where \( y < z \leq w \). Subtracting (c) from (b) we then find (using that \( \tilde{R}_{y,y} = 1 \)):

\[
\sum_{i}(-1)^i \text{tr}(F^*\text{Ad}(x)^*,H^i(\tilde{X}_w)) = \sum_{y \in \hat{W}; y \leq w} \tilde{P}_{y,w}(q) \sum_{i}(-1)^i \text{tr}(F^*\text{Ad}(x)^*,H^i_c(X_y)).
\]

By the Grothendieck-Lefschetz trace formula we have

\[
\sum_{i}(-1)^i \text{tr}(F^*\text{Ad}(x)^*,H^i(\tilde{X}_w)) = \sum_{B' \in \tilde{X}_w: \text{Ad}(x)F(B') = B'} \sum_{i}(-1)^i \text{tr}(F^*\text{Ad}(x)^*,\mathcal{H}^i_{B'},L_w).
\]

(Note that \( \text{Ad}(x)F = \text{Ad}(x)\sigma F^{tr} : \tilde{X}_w \to \tilde{X}_w \) is a Frobenius map relative to an \( F_q \)-rational structure since \( \text{Ad}(x)\sigma \) is an automorphism of finite order of \( \tilde{X}_w \) commuting with the Frobenius map \( F^{tr} : \tilde{X}_w \to \tilde{X}_w \).) In the sum over \( B' \) in (a) we can specify \( y \in W \) such that \( B' \in X_y \); we have necessarily \( y \in \hat{W} \). (Indeed
assume that $B' \in X_y$. Then $Ad(x)F(B') \in X_{\sigma(y)}$; hence if $Ad(x)F(B') = B'$ then $\sigma(y) = y$. Thus the right hand side of (a) becomes

$$\sum_{y \in \hat{W} ; y \leq w} \sum_{B' \in X_y ; Ad(x)F(B') = B'} (-1)^i \text{tr}(F^*Ad(x)^*, H^i_{B', L_w})$$

and it is enough to show that for any $y \in \hat{W}$ such that $y \leq w$ we have

$$\sum_{B' \in X_y ; Ad(x)F(B') = B'} (-1)^i \text{tr}(F^*Ad(x)^*, H^i_{B', L_w})$$

$$= \hat{P}_{y,w}(q) \sum_{i} (-1)^i \text{tr}(F^*Ad(x)^*, H^i(X_y)).$$

By the Grothendieck-Lefschetz trace formula we have

(b) \[ \sum_i (-1)^i \text{tr}(F^*Ad(x)^*, H^i(X_y)) = \{ \{ B' \in X_y ; Ad(x)F(B') = B' \} \}. \]

Thus it is enough to show that for any $B' \in X_y$ such that $Ad(x)F(B') = B'$ we have

$$\sum_i (-1)^i \text{tr}(F^*Ad(x)^*, H^i_{B', L_w}) = \hat{P}_{y,w}(q).$$

Let $\bar{w} \in G$ be such that $\bar{w}(B \cap B^*)\bar{w}^{-1} = B \cap B^*$ and $(B, \bar{w}B\bar{w}^{-1}) \in O_w$. Let $G_w = B\bar{w}B$, $G'_w = \{ g \in G ; g^{-1}F'(g) \in G_w \}$. Let $\bar{G}_w = \bigcup_{z \in \hat{W} ; z \leq w} G_z$, $\bar{G}'_w = \bigcup_{z \in \hat{W} ; z \leq w} G'_z$ be the closures of $G_w, G'_w$ in $G$. Let $\bar{K}_w$ (resp. $\bar{L}_w$) be the intersection cohomology complex of $\bar{G}_w$ (resp. $\bar{G}'_w$) with coefficients in $G$. Define $\alpha : \bar{G}'_w \to \bar{K}_w$ by $g \mapsto g\bar{B}g^{-1}$ (a principal $B$-bundle). Define $\Phi : \bar{G}'_w \to \bar{G}'_w$ by $\Phi(g) = xF(g)$; note that $\Phi$ is a Frobenius map for an $F_q$-rational structure on $\bar{G}_w$ and $\alpha(\Phi(g)) = Ad(x)F(\alpha(g))$ for any $g \in \bar{G}_w$. Hence $\alpha^{-1}(B')$ is $\Phi$-stable. Since $\alpha^{-1}(B')$ is a homogeneous $B$-space with a compatible $F_q$-rational structure given by $\Phi$ we can find $g' \in \alpha^{-1}(B')$ such that $\Phi(g') = g'$. We have canonically $H^i_{B', L_w} = H^i_{\bar{G}_w \to \bar{K}_w}$ and $\text{tr}(F^*Ad(x)^*, H^i_{B', L_w}) = \text{tr}(\Phi^*, H^i_{\bar{G}_w \to \bar{K}_w})$; moreover we have $g' \in (G')^\Phi$. Define $\mathcal{L} : \bar{G}'_w \to \bar{G}_w$ by $g \mapsto g^{-1}F'(g)$ (an étale covering). Define $\Phi' : \bar{G}_w \to \bar{G}_w$ by $g \mapsto F(g)$; note that $\mathcal{L}(\Phi(g)) = \Phi'(\mathcal{L}(g))$ for any $g \in \bar{G}_w$. We have canonically $H^i_{\bar{G}_w \to \bar{K}_w} = \mathcal{L}^i_{\mathcal{L}(g)}\bar{K}_w$ and $\text{tr}(\Phi^*, H^i_{\bar{G}_w \to \bar{K}_w}) = \text{tr}(\Phi'^*, H^i_{\mathcal{L}(g)}\bar{K}_w)$; moreover, $\mathcal{L}(g) \in G^\Phi_{\bar{G}_w}$. Define $\beta : \bar{G}_w \to \bar{B}_w$ by $g \mapsto g\bar{B}g^{-1}$ (a principal $B$-bundle). Note that $\beta(\Phi'(g)) = F(\beta(g))$ for any $g \in \bar{G}_w$. We have canonically $H^i_{\mathcal{L}(g)\bar{K}_w} = \mathcal{L}^i_{\beta(\mathcal{L}(g))}\bar{K}_w$ and $\text{tr}(\Phi'^*, H^i_{\mathcal{L}(g)\bar{K}_w}) = \text{tr}(F^*, H^i_{\beta(\mathcal{L}(g))\bar{K}_w})$; moreover, $\beta(\mathcal{L}(g)) \in B^F_y$. By the proof of 2.7 we have

$$\text{tr}(F^*, H^i_{\beta(\mathcal{L}(g))\bar{K}_w}) = \text{tr}(F^*, H^i_{B^F_y\bar{K}_w}) = n_{i,y,w,o}q^{i/2}.$$ 

It remains to use the equality

$$\hat{P}_{y,w}(q) = \sum_{i \in \mathbb{N}} (-1)^i n_{i,y,w,o}q^{i/2}$$

which follows from 2.7. The proposition is proved.
2.10. Let \( w \in \hat{W} \). The equality in Proposition 2.9 can be rewritten as
\[
\sum_{i} (-1)^i \text{tr}( (\phi^*)^{i'} \sigma^* \text{Ad}(x)^*, H^i(X_w)) \\
= \sum_{y \in W; y \leq w} \dot{P}_{y,w}(q_0') \sum_{i} (-1)^i \text{tr}( (\phi^*)^{i'} \sigma^* \text{Ad}(x)^*, H^i_c(X_y))
\]
for \( r' \in \{1, 2, 3, \ldots\} \) where \( q_0 = (q')^d \). Here we can make formally \( r' \) tend to zero and we obtain
\[
\sum_{i} (-1)^i \text{tr}( \sigma^* \text{Ad}(x)^*, H^i(X_w)) \\
= \sum_{y \in W; y \leq w} \dot{P}_{y,w}(1) \sum_{i} (-1)^i \text{tr}( \sigma^* \text{Ad}(x)^*, H^i_c(X_y)).
\]

2.11. If \( x \in G^F \) we have \( F'(x) \in G^F \) since \( FF' = F'F \); moreover we have \( F'^{rm}(x) = x \) since \( F'^{rm} = F^m \). Let \( \hat{G}^F \) be the semidirect product of \( G^F \) with the cyclic group of order \( rm \) with generator \( \theta \) in which we have \( \theta z \theta^{-1} = F'(z) \) for all \( z \in G^F \).

For any \( z \in G^F \) we define a linear map \( \hat{z} : \mathcal{F} \to \mathcal{F} \) (\( \mathcal{F} \) as in 2.2) by \( \hat{z}(f) = f' \) \( (f \in \mathcal{F}) \) where \( f'(B_1) = f(z^{-1}B_1 z) \) for \( B_1 \in \mathcal{B}^F \). Now \( z \mapsto \hat{z} \) makes \( \mathcal{F} \) into a \( \hat{G}^F \)-module. We define a linear map \( \theta : \mathcal{F} \to \mathcal{F} \) by \( f \mapsto f' \) \( (f \in \mathcal{F}) \) where \( f'(B_1) = f(F'^{-1}B_1) \) for \( B_1 \in \mathcal{B}^F \) (this is well defined since \( FF' = F'F \)). This linear map together with the \( G^F \)-module structure define a \( \hat{G}^F \)-module structure on \( \mathcal{F} \).

Following Shintani, for each \( x \in G^F' \) we define (noncanonically) an element \( \hat{x} \in G^F \) as follows. We write \( x = aF(a^{-1}) \) with \( a \in G \) and we set \( \hat{x} = a^{-1}F'(a) \). Note that \( \hat{x} \in G^F \) (since \( FF' = F'F \)) hence \( \hat{x} : \mathcal{F} \to \mathcal{F} \) is well defined. We have the following result (compare [L2, 2.10]):

**Proposition 2.12.** For any \( w \in \hat{W} \) and any \( x \in G^F' \) we have (with notation of 2.11):
\[
\sum_{i} (-1)^i \text{tr}( F'^* x^*, H^i_c(X_w)) = \text{tr}(\theta^{-1} \hat{x}^{-1} \dot{T}_{w^{-1}}, \mathcal{F}).
\]

By 2.9(b) with \( y \) replaced by \( w \), the left hand side of the equality above is equal to \( |\{B' \in X_w; \text{Ad}(x)F(B') = B'\}| \) hence to
\[
|\{B' \in B; (B', F'(B')) \in \mathcal{O}_w, aF(a^{-1})F(B')F(a)a^{-1} = B'\}|.
\]

Setting \( B_1 = a^{-1}B'a \), we see that the last number is equal to
\[
|\{B_1 \in B; (aB_1a^{-1}, F'(a)F'(B_1)F(a^{-1})) \in \mathcal{O}_w; F(B_1) = B_1\}| \\
= |\{B_1 \in \mathcal{B}^F; (B_1, \hat{x}F'(B_1) \hat{x}^{-1} \in \mathcal{O}_w\}|.
\]
For any $B_1 \in B^F$ we denote by $f_{B_1}$ the function $B^F \to \bar{Q}_l$ such that $f_{B_1}(B')$ is 1 if $B' = B_1$ and 0 if $B' \neq B_1$. Let $f_1 = \hat{T}_{w^{-1}}(f_{B_1})$. We have $f_1(B'') = 1$ if $(B'', B_1) \in \mathcal{O}_{w^{-1}}$, $f_1(B'') = 0$ if $(B'', B_1) \notin \mathcal{O}_{w^{-1}}$. Let $f_2 = \hat{x}^{-1}(f_1)$. We have $f_2(B'') = 1$ if $(\hat{x}B''\hat{x}^{-1}, B_1) \in \mathcal{O}_{w^{-1}}$, $f_2(B'') = 0$ if $(\hat{x}B''\hat{x}^{-1}, B_1) \notin \mathcal{O}_{w^{-1}}$. Let $f_3 = \theta^{-1}(f_2)$. We have $f_3(B'') = 1$ if $(\hat{x}F''(B'')\hat{x}^{-1}, B_1) \in \mathcal{O}_{w^{-1}}$, $f_2(B'') = 0$ if $(\hat{x}F''(B'')\hat{x}^{-1}, B_1) \notin \mathcal{O}_{w^{-1}}$. We see that

$$
\text{tr}(\theta^{-1}\hat{x}^{-1}\hat{T}_{w^{-1}}, \mathcal{F}) = |\{B_1 \in B^F; (\hat{x}F'(B_1)\hat{x}^{-1}, B_1) \in \mathcal{O}_{w^{-1}}\}|
$$

This completes the proof.

2.13. Let $w \in \check{w}$, $x \in G^{F'}$. Combining 2.9 and 2.12 we obtain

(a) \[\sum_i (-1)^i \text{tr}(F^*\text{Ad}(x)^*, H^i(\check{X}_w)) = \sum_{y \in \check{W}; y \leq w} \hat{P}_{y,w}(q) \text{tr}(\theta^{-1}\hat{x}^{-1}\hat{T}_{y^{-1}}, \mathcal{F}).\]

By [L2, 2.20, 3.8], for \(\rho \in \mathcal{E}, i \in \mathbf{N}, \phi^* : H^i(\check{X}_w) \to H^i(\check{X}_w)\) (notation of 2.10) leaves stable $H^i(\check{X}_w)_\rho$ (notation of 1.6) and acts on it as $\lambda_{\rho}q^{id/2}U$ where $U$ is a unipotent transformation of $H^i(\check{X}_w)_\rho$ and $\lambda_{\rho}$ is a root of 1 depending only on $\rho$ (not on $w, i$); also if $\rho \in \mathcal{E} - \mathcal{E}_0$, $\sigma^*$ maps $H^i(\check{X}_w)_\rho$ to $H^i(\check{X}_w)_{\rho'}$ where $\rho' \neq \rho$, while if $\rho \in \mathcal{E}_0$ then $\sigma^*$ leaves stable $H^i(\check{X}_w)_\rho$ and the $\sigma^{-1}$ action makes $H^i(\check{X}_w)_\rho$ into a $G^{F'}$-module. It follows that

\[\sum_i (-1)^i \text{tr}(F^*x^*, H^i(\check{X}_w)) = \sum_{\rho \in \check{E}_0} (-1)^i \sum_{\rho \in \check{E}_0} \lambda_{\rho}^{id/2} q^{i/2} \text{tr}(\sigma^*x^*, H^i(\check{X}_w)_{\rho}).\]

We can find $d' \geq 1$ such that $\lambda_{\rho}^{id'} = 1$ for all $\rho \in \mathcal{E}$. From now on we assume that $r$ is a multiple of $dd'$. Then (b) becomes

\[\sum_i (-1)^i \text{tr}(F^*x^*, H^i(\check{X}_w)) = \sum_{\rho \in \check{E}_0} (-1)^i \sum_{\rho \in \check{E}_0} q^{i/2} (\hat{\rho} : H^i(\check{X}_w) \to \text{tr}((x\sigma)^{-1}, \hat{\rho}). \]

2.14. After applying $Q_l \otimes_A ()$ to $\Phi$ in 2.2 (where $Q_l$ is viewed as an $A$-algebra via $v \mapsto q^{1/2}$) we obtain an algebra isomorphism $\Phi_q : H_q \sim Q_l \otimes \check{J}$. If $E \in \text{Irr}\check{W}$ then the simple $Q_l \otimes \check{J}$-module $Q_l \otimes E_\bullet$ (see 2.2) can be viewed via $\Phi_q$ as a simple $H_q$-module denoted by $E_q$. If $E \in \text{Irr}\check{W}$ we define $\Delta : E_q \to E_q$ to correspond under $\Phi_q$ to $1 \otimes \Delta : Q_l \otimes E_\bullet \to Q_l \otimes E_\bullet$; we have $\hat{T}_{\delta(w)}(\Delta(\xi)) = \Delta(\hat{T}_w(\xi))$ for any $w \in \check{W}, \xi \in E_q$.\]
For \( E \in \text{Irr} \hat{\mathcal{W}} \) let \( \mathcal{F}_E \) be the \( E_q \)-isotypic component of the \( H_q \)-module \( \mathcal{F} \); thus we have \( \mathcal{F} = \bigoplus_{E \in \text{Irr} \hat{\mathcal{W}}} \mathcal{F}_E \). Note that each \( \mathcal{F}_E \) is a \( G^F \)-submodule of \( \mathcal{F} \). From the definitions, for any \( w \in \hat{\mathcal{W}} \) we have \( \hat{T}_w \theta^{-1} = \theta^{-1} \hat{T}_{\delta(w)} : \mathcal{F} \to \mathcal{F} \). It follows that \( \theta^{-1} : \mathcal{F} \to \mathcal{F} \) permutes the summands \( \mathcal{F}_E \) of \( \mathcal{F} \) according to the permutation of \( \text{Irr} \hat{\mathcal{W}} \) induced by \( \delta : \mathcal{W} \to \hat{\mathcal{W}} \). In particular only the summands corresponding to \( E \in \text{Irr}_s \hat{\mathcal{W}} \) are mapped into themselves. We see that for \( z \in G^F, y \in \hat{\mathcal{W}} \), we have

\[
\text{tr}(\theta^{-1} z^{-1} \hat{T}_{y^{-1}}, \mathcal{F}) = \sum_{E \in \text{Irr}_s \hat{\mathcal{W}}} \text{tr}(\theta^{-1} z^{-1} \hat{T}_{y^{-1}}, \mathcal{F}_E).
\]

For \( E \in \text{Irr} \hat{\mathcal{W}} \) we set \( V_E = \text{Hom}_{H_q}(E_q, \mathcal{F}) \); this is an irreducible \( G^F \)-module (the \( G^F \)-module structure comes from that of \( \mathcal{F} \)); we have a canonical isomorphism of \((H_q, G^F)\)-modules \( E_q \otimes V_E \cong \mathcal{F}_E \). Via this isomorphism, assuming \( E \in \text{Irr}_s \hat{\mathcal{W}} \), the map \( \theta^{-1} : \mathcal{F}_E \to \mathcal{F}_E \) becomes an isomorphism \( X_E : E_q \otimes V_E \to E_q \otimes V_E \) necessarily of the form \( X_E = X'_E \otimes X''_E \) where \( X'_E : E_q \to E_q \), \( X''_E : V_E \to V_E \) are isomorphisms such that \( \hat{T}_w X'_E = \hat{T}_w X'_E : E_q \to E_q \) and \( z X''_E = X''_E F''(z) : V_E \to V_E \) for any \( z \in G^F \). Thus \( X'_E \) acts on \( E_q \) as a nonzero constant times \( \Delta^{-1} : E_q \to E_q \); we may assume that the constant is 1 (by absorbing it into \( X''_E \)) so that \( X'_E = \Delta^{-1} \). Since \( X''_E^r = 1 \) and \( (X''_E)^r = 1 \) we see that \( (X''_E)^r = 1 \) so that the \( G^F \)-module structure on \( V_E \) extends to a \( \hat{\mathcal{G}}^F \)-module structure in which \( \theta^{-1} \) acts as \( X''_E \). We see that for \( z \in G^F, y \in \hat{\mathcal{W}} \), we have

\[
\text{tr}(\theta^{-1} z^{-1} \hat{T}_{y^{-1}}, \mathcal{F}) = \sum_{E \in \text{Irr}_s \hat{\mathcal{W}}} \text{tr}(\theta^{-1} z^{-1}, V_E)\text{tr}(\Delta^{-1} \hat{T}_{y^{-1}}, E_q).
\]

We now substitute this (with \( z = \hat{x} \)) into 2.13(a), taking into account 2.13(c) and using that \( \text{tr}(\Delta^{-1} \hat{T}_{y^{-1}}, E_q) = \text{tr}(\hat{T}_{y^{-1}}, E_q) = \text{tr}(\hat{T}_{y}, E_q) \) we obtain

\[
\sum_i (-1)^i \sum_{\rho \in \hat{\xi}_0} q^{i/2} \rho(H^i(X_w)) \text{tr}((x\sigma)^{-1}, \rho)
\]

(a)

\[
= \sum_{y \in \hat{\mathcal{W}}} \hat{P}_{y,w}(q) \sum_{E \in \text{Irr}_s \hat{\mathcal{W}}} \text{tr}(\theta^{-1} \hat{x}^{-1}, V_E)\text{tr}(\Delta \hat{T}_{y}, E_q)
\]

for any \( w \in \hat{\mathcal{W}}, x \in G^{F''} \).

2.15. Multiplying both sides of 2.14(a) by \( \text{sgn}(wu)\hat{Q}_{w,u}(q) \) with \( u \in \hat{\mathcal{W}} \) and summing over all \( w \in \hat{\mathcal{W}} \) we obtain

\[
\sum_i (-1)^i \sum_{w \in \hat{\mathcal{W}}} \sum_{\rho \in \hat{\xi}_0} q^{i/2} \rho(H^i(X_w)) \times \text{sgn}(wu)\hat{Q}_{w,u}(q)\text{tr}((x\sigma)^{-1}, \rho)
\]

\[
= \sum_{E \in \text{Irr}_s \hat{\mathcal{W}}} \text{tr}(\theta^{-1} \hat{x}^{-1}, V_E)\text{tr}(\Delta \hat{T}_{u}, E_q)
\]
for any $u \in \hat{W}, \quad x \in G^{F'}$. Multiplying both sides of the last equality by $|G^{F'}|^{-1} \text{tr}(x \sigma, \hat{\rho}')$ with $\hat{\rho}' \in \hat{E}_0$ and summing over all $x \in G^{F'}$ we obtain

$$(a) \quad \sum_i (-1)^i \sum_{u \in \hat{W}} \sum_{x \in L} q^{i/2}(\hat{\rho}' \otimes \chi : H^i(\hat{X}_u))_{\text{sgn}(wu)} \hat{\chi}_w(u(q)\chi(\sigma)^{-1}$$

$$(b) \quad = \sum_{u \in \hat{W}} q^{l(u)} \text{tr}(\Delta \hat{T}_u, E_q') \text{tr}(\Delta \hat{T}_u, E_q)$$

for any $u \in \hat{W}, \hat{\rho}' \in \hat{E}_0$. Here $L$ is as in 2.3(a). We have used that, if $\hat{\rho}, \hat{\rho}' \in \hat{E}_0$, then $|G^{F'}|^{-1} \sum_{x \in G^{F'}} \text{tr}((x \sigma)^{-1}, \hat{\rho}) \text{tr}(x \sigma, \hat{\rho}') = \chi(\sigma)^{-1}$ if $\hat{\rho} = \hat{\rho}' \otimes \chi$ for some $\chi \in L$ and is 0, otherwise. Next we note that for $E, E' \in \text{Irr}_{\delta} \hat{W}$,

$$\sum_{u \in \hat{W}} q^{l(u)} \text{tr}(\Delta \hat{T}_u, E_q') \text{tr}(\Delta \hat{T}_u, E_q)$$

is equal to 0 if $E \neq E'$ and is equal to $D(E)(q)$ for some polynomial $D(E)(X) \in \mathbb{Q}[X]$ if $E = E'$; moreover, $D(E)(q)$ is a nonzero rational number.

Multiplying both sides of (a) by $D(E')(q)^{-1} q^{l(u)} \text{tr}(\Delta \hat{T}_u, E_q')$ with $E' \in \text{Irr}_{\delta} \hat{W}$ and summing over all $u \in \hat{W}$ we obtain

$$(b) \quad \sum_i (-1)^i \sum_{u \in \hat{W}} \sum_{x \in L} q^{i/2}(\hat{\rho}' \otimes \chi : H^i(\hat{X}_u))_{\text{sgn}(wu)} \hat{\chi}_w(u(q)\chi(\sigma)^{-1}$$

$$\times \text{tr}(\Delta \hat{T}_u, E_q') q^{1/2} \text{tr}(\Delta \hat{T}_u, E_q)$$

for any $\hat{\rho}' \in \hat{E}_0, \quad E' \in \text{Irr}_{\delta} \hat{W}$.

Let $h(r)$ be the common value of the two sides of (b). We now show that (for fixed $\hat{\rho}', E'$) $h(r)$ has the following properties:

(i) $|G^{F'}| h(r)$ is a cyclotomic integer;

(ii) when $r$ varies (as a multiple of $dd'$), $h(r)$ is in a fixed (cyclotomic) subfield of $\mathbb{Q}_l$ of finite degree over $\mathbb{Q}$;

(iii) all complex conjugates of $h(r)$ have absolute value $\leq 1$.

Now (i) is obvious from the right hand side of (b) since the character values of a finite group (namely $\hat{G}^{F'}$ and $\hat{G}^{F'}$) are cyclotomic integers. Also (ii) is obvious from the left hand side of (b); we use that there are only finitely many $\chi(\sigma)$ (roots of 1) and that $\text{tr}(\Delta \hat{T}_{u-1}, E_q') \in \mathbb{Q}(q^{1/2})$. To prove (iii) we set:

$$c(r) = |G^{F'}|^{-1} \sum_{x \in G^{F'}} \text{tr}(\theta^{-1}\hat{x}^{-1}, V_{E'}) \text{tr}(\hat{x} \theta, V_{E'})$$
\[ e'(r) = |G^{F'}|^{-1} \sum_{x \in G^{F'}} \text{tr}(\sigma^{-1}x^{-1}, \rho') \text{tr}(x\sigma, \rho'). \]

By the Cauchy-Schwarz inequality, the absolute value square of \( h(r) \) (in the form given by the right hand side of (b)) is \( \leq \) than \( e(r)e'(r) \). Hence it is enough to show that \( e(r) = e'(r) = 1 \). Now \( e'(r) = 1 \) by the orthogonality relations for irreducible characters of \( \hat{G}^{F'} \) which remain irreducible when restricted to \( G^{F'} \). It remains to show that \( e(r) = 1 \). Setting \( x = aF(a^{-1}), a \in G \) in the definition of \( e(r) \) we have

\[
e(r) = |G^F|^{-1} |G^{F'}|^{-1} \sum_{a \in G; F'(aF(a^{-1})) = aF(a^{-1})} \text{tr}((a^{-1}F'(a)\theta)^{-1}, V_{E'}) \text{tr}(a^{-1}F'(a)\theta, V_{E'}).\]

(We use that, if \( a \) is in the sum and \( b \in G^F \) then

\[
\text{tr}(a^{-1}F'(a)\theta, V_{E'}) = \text{tr}(b^{-1}a^{-1}F'(a)F'(b)\theta, V_{E'}).\]

It is enough to show that if \( z \in G^F \) then \( \text{tr}(z\theta, V_{E'}) = \text{tr}(b^{-1}zF'(b)\theta, V_{E'}) \) which follows from \( F'(b)\theta = \theta b \). Setting \( z = a^{-1}F'(a) \in G^F \) we obtain

\[
e(r) = |G^F|^{-1} \sum_{z \in G^F} \text{tr}((z\theta)^{-1}, V_{E'}) \text{tr}(z\theta, V_{E'})\]

and this equals 1 by the orthogonality relations for irreducible characters of \( \hat{G}^F \) which remain irreducible when restricted to \( G^F \). This proves (iii).

If we combine the various imbeddings of the number field in (ii) into \( C \) we get an imbedding of that field into some \( C^n \). By (i),(iii) the set \( \{h(r)\} \) is carried by that imbedding into a discrete and bounded subset of \( C^n \). Hence the set \( \{h(r)\} \) is finite. This holds for any \( \rho', E' \) which are in finite number. We see that there exists an infinite subset \( \mathcal{I} \) of \( \{dd', 2dd', 3dd', \ldots\} \) such that for any \( \rho', E' \), the expression (b) is a constant \( c_{\rho', E'} \) when \( r \) runs through \( \mathcal{I} \). Thus we have

\[
\sum_{i}(-1)^i \sum_{w \in W} \sum_{x \in L} \sum_{u \in W} q^{l(u)} D(E')(q)^{-1} \text{tr}(\Delta T_u, E_q') \times q^{i/2}(\rho' \otimes \chi : H^i(\bar{X}_w))\text{sgn}(wu)\bar{Q}_{w,u}(q)\chi(\sigma)^{-1} = c_{\rho', E'} \]

for any \( \rho' \in \hat{E}_0, E' \in \text{Irr}_\phi \bar{W}, r \in \mathcal{I} \).

We multiply the two sides of 2.14(a) by \( |G^{F'}|^{-1} \text{tr}(x\sigma, \rho') \) with \( \rho' \in \hat{E}_0 \) and sum over all \( x \in G^{F'} \); we obtain

\[
\sum_{i}(-1)^i \sum_{\chi \in L} q^{i/2}(\rho' \otimes \chi : H^i(\bar{X}_w))\chi(\sigma)^{-1} = \sum_{y \in W} \sum_{E \in \text{Irr}_\phi \bar{W}} \sum_{x \in G^{F'}} |G^F|^{-1} \text{tr}(x\sigma, \rho') \text{tr}(x^{-1} \hat{x}^{-1}, V_E) \text{tr}(\Delta \hat{T}_y, E_q) \]

\[= \sum_{y \in W} P_{y,w}(q) \sum_{E \in \text{Irr}_\phi \bar{W}} \sum_{x \in G^{F'}} |G^F|^{-1} \text{tr}(x\sigma, \rho') \text{tr}(\theta^{-1} \hat{x}^{-1}, V_E) \text{tr}(\Delta \hat{T}_y, E_q) \]
that is
\[ \sum_i (-1)^i \sum_{\chi \in L} q^{i/2} (\rho' \otimes \chi : H^i(\overline{X}_w)) \overline{\chi}(\sigma)^{-1} = \sum_{y \in W} \hat{P}_{y,w}(q) \sum_{E \in \text{Irr}_W} c_{\rho',E} \text{tr}(\Delta \hat{T}_y, E_q) \]
for any \( w \in \hat{W} \), \( \rho' \in \hat{E}_0 \), \( r \in {\mathcal I} \). Since \( r \) runs through an infinite set we have an equality of polynomials in \( v \):
\[ (d) \sum_i (-1)^i \sum_{\chi \in L} (\rho' \otimes \chi : H^i(\overline{X}_w)) \overline{\chi}(\sigma)^{-1} v^i = \sum_{y \in W} \hat{P}_{y,w}(v^2) \sum_{E \in \text{Irr}_W} c_{\rho',E} \text{tr}(\Delta \hat{T}_y, E_{v^2}) \]
for any \( w \in \hat{W} \), \( \rho' \in \hat{E}_0 \). Similarly, since (c) holds for \( r \) running through an infinite set, we have an equality of polynomials in \( v \):
\[ \sum_i (-1)^i \sum_{w \in W} \sum_{\chi \in L} \sum_{u \in W} v^{2l(u)} D(E')(v^2)^{-1} \times \text{tr}(\Delta \hat{T}_u, E_{u^2})(\rho' \otimes \chi : H^i(\overline{X}_w)) \overline{\chi}(\sigma)^{-1} v^i = c_{\rho',E'} \]
for any \( \rho' \in \hat{E}_0 \), \( E' \in \text{Irr}_W \). Setting \( v = 1 \) in the last equality and using the identity \( D(E')(1) = |\hat{W}| \) we obtain
\[ (e) c_{\rho',E'} = \sum_i (-1)^i \sum_{w \in W} \sum_{\chi \in L} \sum_{u \in W} |\hat{W}|^{-1} \times \text{tr}(\Delta u, E')(\rho' \otimes \chi : H^i(\overline{X}_w)) \overline{\chi}(\sigma)^{-1}. \]

By 2.3(a) we have for fixed \( \rho' \):
\[ \sum_i (-1)^i \sum_{\chi \in L} (\rho' \otimes \chi : H^i(\overline{X}_w)) \overline{\chi}(\sigma)^{-1} = \sum_i (-1)^i (\rho' : H^i_c(X_w)) \]
which by 2.10(a) is equal to
\[ \sum_{y \in W : y \leq w} \hat{P}_{y,w}(1) \sum_i (-1)^i (\rho' : H^i_c(X_y)). \]

Substituting this into (e) we obtain
\[ c_{\rho',E'} = \sum_{w \in W} \sum_{u \in W} |\hat{W}|^{-1} \text{tr}(\Delta u, E') \overline{\chi}(\sigma)^{-1} \times |G'|^{-1} \sum_{y \in W : y \leq w} \hat{P}_{y,w}(1) \sum_i (-1)^i (\rho' : H^i_c(X_y)). \]
Using the definition of $\hat{Q}_{w,u}(1)$ we deduce

$$c_{\hat{\rho}',E'} = \sum_{u \in W} |\hat{W}|^{-1} \text{tr}(\Delta u, E')|G_{\hat{E}'}|^{-1}(\hat{\rho}' : \hat{R}_u) = (\hat{\rho}' : \hat{R}_{E'}$$

for any $\hat{\rho}' \in \hat{\mathcal{E}}_0$, $E' \in \text{Irr}_s \hat{W}$. Substituting this into (d) and using the equality

$$\sum_{\chi \in L} (\hat{\rho}' \otimes \chi : H^i(\bar{X}_w))\chi^{-1} = (\hat{\rho}' : H^i(\bar{X}_w))$$

(see 2.3(a)), we obtain

$$\sum_{i} (-1)^i (\hat{\rho}' : H^i(\bar{X}_w))\nu^i = \sum_{y \in W} \bar{\rho}_{y,w}(v^2) \sum_{E \in \text{Irr}_s \hat{W}} (\hat{\rho}' : \hat{R}_{E'})\text{tr}(\Delta \bar{T}_y, E_{\nu^2})$$

for any $w \in \hat{W}$, $\hat{\rho}' \in \hat{\mathcal{E}}_0$. The previous equality also holds for $\hat{\rho}' \in \hat{\mathcal{E}} - \hat{\mathcal{E}}_0$ (in that case both sides are zero). This proves 2.4(a) in view of 2.3(b).

2.16. Let $\preceq_{\text{LR}}$ be the preorder on $\hat{W}$ (with the weight function $l|_{\hat{W}}$) defined in [L3, 8.1] where it is denoted by $\preceq_{\text{LR}}$ and let $\prec_{\text{LR}}$ be the corresponding equivalence relation on $\hat{W}$ (the equivalence classes are the two-sided cells of $\hat{W}$ with the weight function above; they form a set $\hat{\mathcal{C}}$). For $w, w' \in \hat{W}$ we write $w' \prec_{\text{LR}} w$ when $w' \preceq_{\text{LR}} w$ and $w' \not\sim_{\text{LR}} w$. The relations $\preceq_{\text{LR}}$, $\prec_{\text{LR}}$ on $\hat{W}$ induce relations on $\hat{\mathcal{C}}$ denoted again by $\preceq_{\text{LR}}$, $\prec_{\text{LR}}$. It is known [L3, §10] that if $c \in \hat{\mathcal{C}}$ then $c^* := cs_I = s_Ic \in \hat{\mathcal{C}}$ and that $c \mapsto c^*$ reverses the preorder $\preceq_{\text{LR}}$. If $E \in \text{Irr}\hat{W}$ then there is a unique $c \in \hat{\mathcal{C}}$ such that $\hat{t}_w : E^\bullet \to E^\bullet$ is nonzero for some $w \in c$ and is zero for all $w \in \hat{W} - c$; we then write $c = c_E$.

Let $a : \hat{W} \to \mathbb{N}$ is the function defined as in [L3, 13.6] in terms of the weight function $l|_{\hat{W}}$; this function is in fact the restriction to $\hat{W}$ of the function $a : W \to \mathbb{N}$ in 1.2, see [L3, 16.5]. Let $E \mapsto a_E$ be the function $\text{Irr}\hat{W} \to \mathbb{N}$ defined in [L3, 20.6] (in terms of the weight function of $\hat{W}$). For $E \in \text{Irr}\hat{W}$ we set $a'_E = a_{E^\dagger}$.

Using [L3, §20] we see that for $w \in \hat{W}$, $E \in \text{Irr}_s \hat{W}$ we have

$$v^{-l(w)}\text{tr}(\Delta \hat{T}_w, E_{\nu^2}) = c'_{w,\Delta,E}v^{a'E} + \text{lower powers of } v$$

where $c'_{w,\Delta,E} \in \mathbb{Z}$; by an argument similar to that in [L3, 20.10], we have $c'_{w,\Delta,E} = \text{tr}(\Delta', E^\dagger)$ where $\Delta' : E^\dagger \to E^\dagger$ is $\Delta \otimes 1 : E \otimes s_{\text{sgn}} \to E \otimes s_{\text{sgn}}$.

Let $\hat{\mathcal{A}}_W$ be the vector space of formal linear combinations $\sum_{E \in \text{Irr}_s \hat{W}} r_E E$, $r_E \in \mathbb{Q}$. For $w \in \hat{W}$ we set

$$\hat{\mathcal{A}}_w = \sum_{E \in \text{Irr}_s \hat{W}} c'_{w,\Delta,E} E \in \hat{\mathcal{A}}_W.$$
From the definitions we see that 
(a) If \( c \in \hat{\mathcal{C}}, w \in c \) then \( \hat{\mathfrak{A}}_w \) is a linear combination of \( E \in \text{Irr}_Q \hat{W} \) such that \( c_E = c^w. \)

Conversely, if \( E \in \text{Irr}_Q \hat{W} \) then

(b) \( E \) is a \( \mathbb{Q} \)-linear combination of elements \( \hat{\mathfrak{A}}_x \) \( (x \in c_E^w). \)

The proof is along the lines of that of \([L2, 5.13(ii)]\) (which is the analogous result for \( W \) instead of \( \hat{W} \)).

For any \( \xi = \sum_{E \in \text{Irr}_Q \hat{W}} r_E E \in \hat{\mathcal{R}}_W \) we set \( \hat{R}_\xi = \sum_{E \in \text{Irr}_Q \hat{W}} r_E \hat{R}_E \in [\hat{\mathcal{E}}]. \) In particular, for \( w \in \hat{W}, \hat{R}_{\hat{\mathfrak{A}}_w} \in [\hat{\mathcal{E}}] \) is defined.

**Proposition 2.17.** Let \( w \in \hat{W}. \) We have

\[
(-1)^{l(w)-a(w)} H^{l(w)-a(w)}(X_w) = (-1)^{l(w)+a(w)} H^{l(w)+a(w)}(X_w)
\]

\[(a) \quad = \hat{R}_{\hat{\mathfrak{A}}_w} + \sum_{w' \in \hat{W}, w' \prec_{LR} w} n_{w',w} \hat{R}_{\hat{\mathfrak{A}}_{w'}} \in [\hat{\mathcal{E}}] \]

for certain rational numbers \( n_{w',w}. \)

The first equality follows from the Lefschetz hard theorem in intersection cohomology applied to \( \hat{X}_w \) which has pure dimension \( l(w). \) In the rest of the proof we concentrate on the second equality. Using the second part of 2.4 with \( i = l(w) + a(w) \) we see that it is enough to show that

\[
\sum_{E \in \text{Irr}_Q \hat{W}} \text{tr}(\Delta \sum_{y \in \hat{W}} \hat{P}_{y,w}(v^2) \hat{T}_y, E_{v^2}; l(w) + a(w)) E
\]

\[(b) \quad = \hat{\mathfrak{A}}_w + \sum_{w' \in \hat{W}, w' \prec_{LR} w} \hat{\mathfrak{A}}_{w'} \]

(equality in \( \hat{\mathcal{R}}_W \)). Let \( c \in \hat{\mathcal{C}} \) be such that \( w \in c. \) From the definitions we see that, for \( E \in \text{Irr}_Q \hat{W} \), the operator \( \sum_{y \in \hat{W}} \hat{P}_{y,w}(v^2) \hat{T}_y : E_{v^2} \to E_{v^2} \) is zero unless \( c_E^w \preceq_{LR} c. \) Thus the sum in the left hand side of (b) can be restricted to the \( E \) such that \( c_E^w \preceq_{LR} c. \) Next we show that for any \( E \in \text{Irr}_Q \hat{W} \) we have

\[(b) \quad \text{tr}(\Delta \sum_{y \in \hat{W}} \hat{P}_{y,w}(v^2) \hat{T}_y, E_{v^2}) = \hat{c}'_{w,\Delta,E} v^{l(w)+a_E} + \text{lower powers of } v. \]

(Using the definition of \( \hat{c}'_{w,\Delta,E} \), it is enough to show that for any \( y \in \hat{W}, y < w \) we have

\[
\hat{P}_{y,w}(v^2) \text{tr}(\Delta \hat{T}_y, E_{v^2}) \in v^{l(y)+a_E-1} Z[v^{-1}];
\]

since

\[
\text{tr}(\Delta \hat{T}_y, E_{v^2}) \in v^{l(y)+a_E} Z[v^{-1}],
\]
it is enough to note that \( \hat{P}_{y,w}(v^2) \in v^{l(w)-l(y)-1}Z[v^{-1}] \).

It follows that any \( E \) which appears with nonzero coefficient in the first sum in (b) satisfies \( c_E^* \leq_{LR} c \) and that the contribution to that sum of the \( E \) such \( c_E^* = c \) (hence \( a_E^* = a(w) \), see [L3, 20.6(c)]) is

\[
\sum_{E \in \text{Irr}_w \delta; c_E^* = c} c_E' w_{E, \Delta, E} E = \mathfrak{A}_w
\]

(see 2.16(a)). It remains to show that if \( E \) satisfies \( c_E^* \leq_{LR} c \) then \( E \) is a linear combination of elements \( \mathfrak{A}_w \) with \( w' \in \hat{W}, w' \leq_{LR} w \). This follows from 2.16(b).

2.18. For any \( c \in \mathcal{C} \) we set

\[
[\hat{\mathcal{E}}]^{\leq c} = \oplus_{c' \leq c; c' \leq_{LR} c} [\hat{\mathcal{E}}]^{c'}, \quad [\hat{\mathcal{E}}]^{\geq c} = \oplus_{c' \geq c; c' \leq_{LR} c} [\hat{\mathcal{E}}]^{c'}.
\]

By [L3, 23.5] any \( c \in \hat{C} \) is contained in a unique two-sided cell \( c^! \in \mathcal{C} \); moreover, the map \( \hat{C} \to \mathcal{C}, c \mapsto c^! \) is injective (we have \( c^! \cap \hat{W} = c \)). Also \((c^!)^! = (c^!)^* \). (Indeed, if \( w \in c \) so that \( w \in c^! \), we have \( ws_I \in c^* \) so that \( ws_I \in c^! \) and \( ws_I \in (c^!)^! \); but we have also \( ws_I \in c^! s_I = (c^!)^* \). Hence the desired equality.)

We show:

(a) If \( c \in \hat{C} \) and \( w \in c \) then \( \hat{R}_{\hat{\mathfrak{A}}_w} \in [\hat{\mathcal{E}}]^{\leq c} \).

We can assume that this is true if \( w \) is replaced by \( w' \) with \( w' \in \hat{W}, w' \leq_{LR} w \). By 1.7, the \( GF' \)-module \( H^{l(w)-a(w)}(\hat{X}_w) \) is a sum of representations isomorphic to \( \rho \) such that \( \rho^* \leq_{LR} c^! \). Hence \( H^{l(w)-a(w)}(\hat{X}_w) \in [\hat{\mathcal{E}}]^{\leq c} \). Using now 2.17 we see that

(b) \( \hat{R}_{\hat{\mathfrak{A}}_w} + \sum_{w' \in \hat{W}, w' \leq_{LR} w} n_{w',w} \hat{R}_{\hat{\mathfrak{A}}_{w'}} \in [\hat{\mathcal{E}}]^{\leq c} \)

with \( n_{w',w} \in \mathbb{Q} \). By the induction hypothesis for any \( w' \) in the last sum (with \( w' \in c^!, c^! \in \hat{C} \)) we have \( \hat{R}_{\hat{\mathfrak{A}}_{w'}} \in [\hat{\mathcal{E}}]^{\leq c^!} \). By arguments in [L3, §16] (see especially 16.6, 16.13(a)), from \( w' \leq_{LR} w \) we deduce that \( w' \leq_{LR} w \) that is \( c^! \leq c^! \). Hence \([\hat{\mathcal{E}}]^{\leq c^!} \subset [\hat{\mathcal{E}}]^{\leq c} \). Thus \( \hat{R}_{\hat{\mathfrak{A}}_{w'}} \in [\hat{\mathcal{E}}]^{\leq c} \). Introducing this in (b) we see that \( \hat{R}_{\hat{\mathfrak{A}}_w} \in [\hat{\mathcal{E}}]^{\leq c} \). This proves (a).

We show:

(c) If \( E \in \text{Irr}_\delta \hat{W} \) then \( \hat{R}_E \in [\hat{\mathcal{E}}]^{\leq (c_E^*)^!} \).

This follows from (a) using 2.16(b).

2.19. Recall the operation of “duality” [DL2, DL3] \( D : [\mathcal{E}] \rightarrow [\mathcal{E}'] \) which is a linear map such that \( D(\rho) = \hat{\rho} \) where \( \rho \mapsto \hat{\rho} \) is an involution of \( \mathcal{E} \) and such that \( D(R_w) = \text{sgn}(w)R_w \) for any \( w \in W \). It follows that if \( E \in \text{Irr}_\delta W \) then \( D(R_E) = \pm R_{E'} \). Hence if \( \rho \in \mathcal{E} \), then \( \hat{\rho} = \rho^* \).

The operator \( D \) is generalized in [DM, §3] to a linear map \( \hat{D} : [\hat{\mathcal{E}}] \rightarrow [\hat{\mathcal{E}}] \) with the following properties: (i) if \( \hat{\rho} \in \hat{\mathcal{E}}_0 \) and \( \hat{\rho}|_{GF'} = \rho \in \mathcal{E}_0 \) then \( \hat{D}(\hat{\rho}) \) is up to scalar of the form \( \hat{\rho}' \in \hat{\mathcal{E}}_0 \) where \( \hat{\rho}'|_{GF'} = \hat{\rho} \) (as above); (ii) \( \hat{D}(\hat{R}_w) = \text{sgn}(w)\hat{R}_w \) for any \( w \in \hat{W} \). It follows that if \( E \in \text{Irr}_\delta \hat{W} \) then \( \hat{D}(\hat{R}_E) = \pm \hat{R}_{E'} \).

We show:
If $c \in \mathcal{C}$ then $\hat{D}([\hat{\mathcal{E}}]^{\leq c}) = [\hat{\mathcal{E}}]^{\geq c}$.

It is enough to show that, if $\rho \in \mathcal{E}_0, \hat{\rho} \in \mathcal{E}_0$ are as above, then $\hat{\rho}^* \leq_{LR} c$ implies $\hat{\rho}^* \geq_{LR} c^*$, that is $\rho \geq_{LR} c^*$. But this follows from the fact $*$ reverses the preorder $\leq_{LR}$.

Applying $\hat{D}$ to 2.18(c) with $E$ replaced by $E'$ (as above) and using (a) we obtain

$\hat{\mathcal{E}} \in [\hat{\mathcal{E}}]^{\geq (c_{E'})^*}$.

We have $((c_{E'})^*)^* = (c'_E)^* = (c_{E'}^*)$. Hence

(b) $\hat{\mathcal{E}} \in [\hat{\mathcal{F}}]^{\geq (c_{E'})^*}$.

From (b) and 2.18(b) we deduce that $\hat{\mathcal{E}} \in [\hat{\mathcal{E}}]^{\leq (c_{E'})^*} \cap [\hat{\mathcal{E}}]^{\geq (c_{E'})^*}$. Hence

(c) $\hat{\mathcal{E}} \in [\hat{\mathcal{E}}]^{(c_{E'})^*}$.

This proves 2.4(ii).

From (c) we deduce, using 2.16(a), that for any $c \in \hat{\mathcal{C}}$ and any $w \in c$ we have

(d) $\hat{\mathcal{E}}_w \in [\hat{\mathcal{E}}]^c$.

Let $c \in \hat{\mathcal{C}}, w \in c$. Let $\pi_{c'} : [\hat{\mathcal{C}}] \rightarrow [\hat{\mathcal{C}}]^c$ be the canonical projection. We apply $\pi_{c'}$ to the equality between the first and third member of 2.17(a) and we use 2.19(d). We obtain

$$(-1)^{l(w)-a(w)} \pi_{c'}(\hat{H}^{l(w)-a(w)}(\hat{X}_w)) = \hat{\mathcal{E}}_w.$$

We have used that if $w' \in \hat{W}, w' <_{LR} w$ then $\pi_{c'}(\hat{\mathcal{E}}_w) = 0$. (Indeed, let $c' \in \hat{\mathcal{C}}$ be such that $w' \in c'$. We have $c' \neq c$. Using 2.19(d) for $w'$ instead of $w$ it is enough to show that $c'^* \neq c^*$; this follows from the injectivity of the map $c \mapsto c^*$.) Note that $\hat{H}^{l(w)-a(w)}(\hat{X}_w)$ (see 1.6) is a $\hat{G}'$-submodule of $\hat{H}^{l(w)-a(w)}(\hat{X}_w)$; moreover from the definitions we have $\pi_{c'}(\hat{H}^{l(w)-a(w)}(\hat{X}_w)) = \hat{H}^{l(w)-a(w)}(\hat{X}_w)$ (equality in $[\hat{\mathcal{C}}]$). Thus we have the following generalization of 1.6(c):

(a) $$(-1)^{l(w)-a(w)} \hat{H}^{l(w)-a(w)}(\hat{X}_w) = \hat{\mathcal{E}}_w.$$

In particular, the following generalization of 1.6(d) holds:

(b) $$(-1)^{l(w)-a(w)} \hat{\mathcal{E}}_w$$

can be represented by an actual $\hat{G}'$-module.

2.21. Assume now that $G$ is an even special orthogonal group over $F_{q'}$ and that $\sigma$ is conjugation by a reflection defined over $F_{q'}$ so that $\hat{G}$ is an even full orthogonal group over $F_{q'}$. Using the results of this paper (especially 2.20(b)) one can show that if $E \in \text{Irr}_\delta \hat{\mathcal{W}}$ then there exists $c \in \hat{\mathcal{C}}$ such that

$$2^n \hat{\mathcal{E}}_c = \sum_{\rho \in \mathcal{E}; \rho = c} \tilde{\rho}$$

where $n$ is defined by $|\{\rho \in \mathcal{E}; \rho = c\}| = 2^n$ and for each $\rho$ in the sum, $\tilde{\rho}$ denotes a certain extension of $\rho$ to a $G'_{F'}$-module. The proof will be given elsewhere.
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