The maximum diameter of pure simplicial complexes and pseudo-manifolds

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Abstract
We construct \(d\)-dimensional pure simplicial complexes and pseudo-manifolds (without boundary) with \(n\) vertices whose combinatorial diameter grows as \(c_d n^{d-1}\) for a constant \(c_d\) depending only on \(d\), which is the maximum possible growth. Moreover, the constant \(c_d\) is optimal modulo a singly exponential factor in \(d\). The pure simplicial complexes improve on a construction of the second author that achieved \(c_d n^{2d/3}\). For pseudo-manifolds without boundary, as far as we know, no construction with diameter greater than \(n^2\) was previously known.

Keywords: Simplicial complex, hyper-graph, pseudo-manifold, diameter, Hirsch conjecture

1 Introduction

A pure simplicial complex of dimension \(d-1\) (or a \((d-1)\)-complex, for short) is any family \(C\) of \(d\)-element subsets of a set \(V\) (typically, \(V = [n] := \{1, \ldots, n\}\)). Elements of \(C\) are called facets and any subset of a facet is a face\(^2\). More precisely, a \(k\)-face is a face with \(k+1\) elements. Faces of dimensions 0, 1, and \(d-2\) are called, respectively, vertices, edges and ridges of \(C\). Observe that a pure \((d-1)\)-complex is the same as a uniform hypergraph of rank \(d\). Its facets are called hyperedges in the hypergraph literature.

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1 Work of F. Santos is supported in part by the Spanish Ministry of Science (MICINN) through grant MTM2014-54207P. E-mail: fcriado92@gmail.com, francisco.santos@unican.es
2 The standard usage is to consider all faces, not only facets, as elements of \(C\), and then call facets the maximal ones. Our approach is equivalent and, for our purposes, simpler.
The adjacency graph or dual graph of a pure simplicial complex $C$, denoted $G(C)$, is the graph having as vertices the facets of $C$ and as edges the pairs of facets $X, Y \in C$ that differ in a single element (that is, those that share a ridge). Complexes with a connected adjacency graph are called strongly connected. The combinatorial diameter of $C$ is the diameter, in the graph theoretic sense, of $G(C)$.

We are interested in how large can the diameter of a pure simplicial complex be in terms of its dimension and number of vertices. For this we set:

$$H_s(n, d) := \text{ maximum diameter of pure strongly connected } \quad (d - 1)\text{-complexes with } n \text{ vertices.}$$

The function $H_s(n, d)$ is known to be exponential in $d$:

**Theorem 1.1 (Santos [9, Corollary 2.12])**

$$\Omega \left( \frac{n^d}{d} \right) \leq H_s(n, d) \leq \frac{1}{d - 1} \binom{n}{d - 1} \approx \frac{n^{d-1}}{d!}.$$

The upper bound is obtained by counting the possible number of ridges, while the lower bound comes from a construction using the join operation. Another construction giving a lower bound of type $n^\frac{d}{2}$ is contained in [6, Thm. 4.4]. In this short note we show a simple and (relatively) explicit construction giving

**Theorem 1.2** For every $d \in \mathbb{N}$ there are infinitely many $n \in \mathbb{N}$ such that:

$$H_s(n, d) \geq \frac{n^{d-1}}{(d + 2)^{d-1} - 3}.$$

Observe that this matches the upper bound in Theorem 1.1, modulo a factor in $\Theta(d^{3/2}e^{-d})$, since $d! \approx e^{-d}d^d\sqrt{2\pi d}$.

**Remark 1.3** Our proof of Theorem 1.2 uses an arithmetic construction valid only when the number $n$ of vertices is of the form $q(d + 2)$ for a sufficiently large prime power $q$. But every interval $[m, 2m]$ contains an $n$ of that form, because there is a power of 2 between $m/(d + 2)$ and $m/2(d + 2))$. Hence, the theorem is also valid “for every $d$ and sufficiently large $n$”, modulo an extra factor of $2^{d-1}$ in the denominator.

Motivation for this question and relatives of it comes from the Hirsch Conjecture which, written in the language of simplicial complexes, said: The
maximum diameter of a polytopal simplicial \((d-1)\)-sphere with \(n\) vertices cannot exceed \(n - d\). Here, a simplicial sphere is a pure simplicial complex whose underlying topological space is homeomorphic to a sphere. A polytopal sphere is the simplicial complex of proper faces of a simplicial polytope. Although the original Hirsch conjecture has been disproved [8], the only counter-examples to it that we know of exceed the conjectured diameter by a small fraction. In particular, the following polynomial version of the Hirsch conjecture is open, even in the linear case (the case \(k = 1\)):

**Conjecture 1.4 (Polynomial Hirsch Conjecture)** There are constants \(c, k\) such that the diameter of every polytopal \((d-1)\)-sphere with \(n\) vertices is bounded above by \(cn^k\).

An approach that has been tried often is to generalize the conjecture to more general complexes than polytopal spheres. Theorem 1.1 shows that generalizing to arbitrary pure complexes is too much, but it is plausible that simplicial manifolds still have polynomial diameter. In particular, the polymath3 project [5] was devoted to (a more abstract and generalized version of) the following conjecture, inspired by the results in [4] and which implies Conjecture 1.4 with \(k = 2\) and \(c = 1\):

**Conjecture 1.5 (Hähnle, in [5])** The diameter of every normal pure \((d-1)\)-complex with \(n\) vertices is bounded above by \(dn\).

Here a pure simplicial complex \(C\) is called normal if every two facets \(F_1\) and \(F_2\) are connected in \(G(C)\) via a path with the property that all facets in the path contain \(F_1 \cap F_2\). Another important class of pure complexes are pseudo-manifolds without boundary\(^3\): strongly connected complexes in which every ridge belongs to exactly two facets. For example, Adler and Dantzig [2] call normal pseudo-manifolds without boundary abstract polytopes. Here we prove the following:

**Theorem 1.6** For every strongly connected pure \((d-1)\)-dimensional simplicial complex with \(n\) vertices and diameter \(\delta\) there is a \((d-1)\)-dimensional pseudo-manifold without boundary with \(2n\) vertices and diameter at least \(\delta + 2\).

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\(^3\) \(C\) is a pseudo-manifold with boundary if ridges are contained in at most two facets, the boundary of \(C\) consisting of the ridges lying in only one facet. Standard usage is to say “pseudo-manifold” alone meaning “without boundary” and “pseudo-manifold with boundary” when boundary is allowed. But to avoid confusion we here insist in saying “without boundary” when boundary is forbidden.
Together with Theorem 1.2 this implies the following, where \( H_{pm}(n, d) \) denotes the maximum diameter of \((d-1)\)-dimensional pseudo-manifolds without boundary. As far as we know this is the first construction of pseudo-manifolds without boundary and of exponential diameter.

**Corollary 1.7** For every \( d \in \mathbb{N} \) there are infinitely many \( n \in \mathbb{N} \) such that:

\[
H_{pm}(n, d) \geq \frac{n^{d-1}}{(2(d + 2))^{d-1}} - 1.
\]

In a similar spirit, Todd [10] defined *semi-duoids* as the pure simplicial complexes in which every ridge lies in an even number of facets, and called *duoids* the semi-duoids that do not properly contain other semi-duoids. Semi-duoids were later called *oiks* (as a short-hand for “Euler complexes”) by Edmonds [3] (see also [11]). One of the results in [10] is the construction of duoids with quadratic diameter. Since every pseudo-manifold without boundary is a duoid, Corollary 1.7 significantly improves that construction.

## 2 Proof of Theorem 1.2

Our construction is arithmetic, and uses the following well-known result that can be found, for example, in [7, Theorem 33.16]:

**Theorem 2.1** Let \( p(x) = x^d + a_1 x^{d-1} + \cdots + a_d \) be a primitive polynomial of degree \( d \) over the field \( \mathbb{F}_q \) with \( q \) elements, for some \( d \in \mathbb{N} \) and some prime power \( q \). Consider the sequence \((u_n)_{n \in \mathbb{N}}\) defined by the linear recurrence

\[
u_n + d + a_1 u_{n+d-1} + \cdots + a_d u_n = 0,
\]

starting with any non-zero vector \((u_1, \ldots, u_d) \in \mathbb{F}_q^d\). Then, \((u_n)_{n \in \mathbb{N}}\) has period \( q^d - 1 \). In particular, its intervals of length \( d \) cover all of \( \mathbb{F}_q^d \setminus \{(0, \ldots, 0)\} \). That is:

\[
\{(u_i, \ldots, u_{i+d-1}) : i \in \{1, \ldots, q^d - 1\}\} = \mathbb{F}_q^d \setminus \{(0, \ldots, 0)\}.
\]

Remember that a primitive polynomial of degree \( d \) is the minimal polynomial of a primitive element in the degree \( d \) extension \( \mathbb{F}_{q^d} \) of \( \mathbb{F}_q \). The number of monic primitive polynomials of degree \( d \) over \( \mathbb{F}_q \) equals \( \phi(q^d - 1)/d \), since \( \mathbb{F}_{q^d} \) has \( \phi(q^d - 1) \) primitive elements, and each primitive polynomial is the minimal polynomial of \( d \) of them. In our construction we will need the coefficients of \( p(x) \) to be all different from zero. Primitive polynomials with this property
do not exist for all \( q \), but they exist when \( q \) is sufficiently large with respect to \( d \), which is enough for our purposes:

**Lemma 2.2** For every fixed \( d \in \mathbb{N} \) and every sufficiently large prime power \( q \), there is a primitive polynomial of degree \( d \) over \( \mathbb{F}_q \) with all coefficients different from zero.

**Proof.** This follows from the fact that the number of primitive monic polynomials of degree \( d \) is greater than the number of monic polynomials of degree \( d \) with at least one zero coefficient, for \( q \) large.

Indeed, the latter is \( q^d - (q-1)^d \leq dq^{d-1} \). The former equals \( \phi(q^d-1)/d \), which is greater than \((q^d-1)^{1-\epsilon}/d\), for every \( 0 < \epsilon < 1 \) and sufficiently large \( q \). Letting \( \epsilon = \frac{1}{d^2} \) we get:

\[
\frac{\phi(q^d - 1)}{d} > \frac{(q^d - 1)^{1-\frac{1}{d^2}}}{d} > \frac{q^d(q^d - 1) - 1}{2^{1-\frac{1}{d^2}} d} > dq^{d-1}.
\]

\( \square \)

With this we can now show our first construction proving Theorem 1.2.

**Theorem 2.3** Suppose that \( p(x) \in \mathbb{F}_q[x] \) is a primitive polynomial of degree \( d-1 \) with no zero coefficients. Then, there is a pure simplicial complex \( C \) of dimension \( d-1 \), with \( n = (d+2)q \) vertices and at least \( n^d - 1 \) facets whose dual graph is a cycle.

**Proof.** Our set of vertices is \( V = \mathbb{F}_q \times [d+2] \). That is, we have as vertices the elements of \( \mathbb{F}_q \) but each comes in \( d+2 \) different “colors”. In the sequence \((u_i)_{i \in \mathbb{N}}\) of Theorem 2.1 we color its terms cyclically. That is, call

\[
v_i = (u_i, \text{ mod } (d+2)).
\]

Let \( C \) be the simplicial complex consisting of the intervals of length \( d \) in the sequence \((v_i)_{i \in \mathbb{N}}\). That is, we let:

\[
F_i = \{v_i, \ldots, v_{i+d-1}\}, \quad C = \{F_i : i \in \{1, \ldots, q^d - 1\}\}.
\]

Observe that the sequence \( \{F_i\}_{i \in \mathbb{N}} \) is periodic of period \( \text{lcm}\{q^d - 1, d + 2\} \geq q^d - 1 = \frac{n^d - 1}{(d+2)^{d-1}} - 1 \). Also, by construction, \( G(C) \) contains a Hamiltonian cycle. We claim that, in fact, \( G(C) \) equals that cycle.

For this, observe that ridges in \( C \) are of two types: some are of the form \( \{v_i, \ldots, v_{i+d-2}\} \) and some are of the form \( \{v_i, \ldots, v_j, v_{j+2}, \ldots, v_{i+d-1}\} \). We will study the facets that these types of ridges may belong to.
For a ridge \( R = \{v_i, \ldots, v_{i+d-2}\} \) to be contained in a facet \( F \) we need the color of the vertex in \( F \setminus R \) to be either \( i-1 \) or \( i + d - 1 \) (modulo \( d+2 \)).

Once we have the color \( c \) of the new vertex \( v = (u, c) \in F \setminus R \), the recurrence relation (and the fact that \( p \) has non-zero coefficients) gives us only one choice for \( u \). Thus, \( R \) is only contained in the two contiguous facets \( F_{i-1} \) and \( F_i \).

The same argument applies to a ridge \( \{v_i, \ldots, v_j, v_{j+2}, \ldots, v_{i+d-1}\} \). Now the color of the new vertex must be \( j+1 \mod d+2 \) and the recurrence relation implies the vertex to be precisely \( v_{j+1} \).

\[ \square \]

**Proof of Theorem 1.2.** Delete a facet in the complex \( C \) of Theorem 2.3. \( \square \)

A complex whose dual graph is a path, such as the one in this proof, is called a *corridor* in [9]. It is a general fact that the maximum diameter \( H(n, d) \) is always attained at a corridor ([9, Corollary 2.7]). That is to say, \( H(n, d) \) equals the maximum length of an induced path in the *Johnson graph* \( J_{n,d} \); the dual graph of the complete complex of dimension \( d-1 \) with \( n \) vertices. Induced paths in graphs are sometimes called *snakes*. In this language Theorem 1.2 can be restated as:

**Theorem 2.4** There is a constant \( c > 0 \) such that for every fixed \( d \) and sufficiently large \( n \) the Johnson graph \( J_{n,d} \) contains snakes passing through a fraction \( c^{-d} \) of its vertices.

A stronger statement is known for the graph of a \( d \)-dimensional hypercube: it contains snakes passing through a positive, independent of \( d \), fraction of the vertices [1].

### 3 Proof of Theorem 1.6

Let \( C \) be the simplicial complex in the statement and \( V \) its vertex set. By [9, Corollary 2.7] there is no loss of generality in assuming that \( C \) is a *corridor*. That is, its dual graph is a path, so its facets come with a natural order \( F_0, \ldots, F_\delta \).

We now construct a simplicial complex \( C' \) in the vertex set \( V' = V \times \{1, 2\} \). For a vertex \( v \in V \) we denote \( v^1 \) and \( v^2 \) the two copies of it in \( V' \), and refer to the superscripts as “colors”. Let \( a_i \) and \( b_i \) be the unique vertices in \( F_i \setminus F_{i+1} \) and \( F_i \setminus F_{i-1} \), respectively. (For \( F_0 \) and \( F_\delta \) we choose \( a_0 \) and \( b_\delta \) arbitrarily, but different from \( b_0 \) and \( a_\delta \)). We define \( C' \) as the complex containing, for each \( F_i \), the \( 2^{d-1} \) colored versions of it in which \( a_i \) and \( b_i \) have the same color. The diameter of \( C' \) is at least the same as that of \( C \). Let us see that \( C' \) is almost
a pseudo-manifold:

- If a ridge $R$ in $C'$ is obtained from a colored version of $F_i$ by removing a vertex $v$ different from $a_i$ or $b_i$, then the only other facet containing $R$ is the copy of $F_i$ in which the color of $v$ is changed to the opposite one. This is so because the “uncolored” version of $R$ is a ridge of only the facet $F_i$ of $C$, by assumption.

- If a ridge $R$ in $C'$ is obtained from a colored version of $F_i$ ($i < \delta$) by removing $a_i$ then the only other facet containing $R$ is obtained by adding to it the vertex $b_{i+1}$ with the same color as $a_{i+1}$ has in $R$.

- Similarly, if a ridge $R$ in $C'$ is obtained from a colored version of $F_i$ ($i > 0$) by removing $b_i$ then the only other facet containing $R$ is obtained by adding to it the vertex $a_{i-1}$ with the same color as $b_{i-1}$ has in $R$.

That is, the only ridges of $C'$ that do not satisfy the pseudo-manifold property are the $2^{d-1}$ colored versions of $R_1 := F_0 \setminus \{b_0\}$ and the $2^{d-1}$ colored versions of $R_2 := F_{\delta} \setminus \{a_\delta\}$, which form two $(d-2)$-spheres, (each with the combinatorics of a cross-polytope). Choose a vertex $a$ in $R_1$ and a vertex $b \in R_2$, different from one another (which can be done since $R_1 \neq R_2$). Consider the complex $C''$ obtained from $C'$ adding to it all the colored versions of $R_1 \setminus a$ joined to $\{a^1, a^2\}$ and all the colored versions of $R_2 \setminus b$ joined to $\{b^1, b^2\}$. The effect of this is gluing two $(d-1)$-balls with boundary the two $(d-2)$-spheres we wanted to get rid off, so that $C''$ is now a pseudo-manifold. (Observe that the new ridges introduced in $C''$ all contain either $\{a^1, a^2\}$ or $\{b^1, b^2\}$ so they were not already in $C'$).

**Remark 3.1** In some contexts it may be useful to apply Theorem 1.6 to closed corridors, that is, pure complexes whose dual graph is a cycle. The construction in the proof works exactly the same except now $C''$ is already a pseudo-manifold, with no need to glue two additional balls to it as we did in the final step of the proof.

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