Hyperbolic Manifolds of Dimension \( n \) with Automorphism Group of Dimension \( n^2 - 1 \)

A. V. Isaev

We consider complex Kobayashi-hyperbolic manifolds of dimension \( n \geq 2 \) for which the dimension of the group of holomorphic automorphisms is equal to \( n^2 - 1 \). We give a complete classification of such manifolds for \( n \geq 3 \) and discuss several examples for \( n = 2 \).

0 Introduction

Let \( M \) be a connected complex manifold and \( \text{Aut}(M) \) the group of holomorphic automorphisms of \( M \). If \( M \) is Kobayashi-hyperbolic, \( \text{Aut}(M) \) is a Lie group in the compact-open topology \( [Ko, Ka] \). Let \( d(M) := \dim \text{Aut}(M) \). It is well-known (see \( [Ko, Ka] \)) that \( d(M) \leq n^2 + 2n \), and that \( d(M) = n^2 + 2n \) if and only if \( M \) is holomorphically equivalent to the unit ball \( B^n \subset \mathbb{C}^n \), where \( n := \dim_{\mathbb{C}} M \). In \( [Kra] \) we studied lower automorphism group dimensions and showed that, for \( n \geq 2 \), there exist no hyperbolic manifolds with \( n^2 + 3 \leq d(M) \leq n^2 + 2n - 1 \), and that the only manifolds with \( n^2 < d(M) \leq n^2 + 2 \) are, up to holomorphic equivalence, \( B^{n-1} \times \Delta \) (where \( \Delta \) is the unit disc in \( \mathbb{C} \)) and the 3-dimensional Siegel space (the symmetric bounded domain of type (III\(_2\)) in \( \mathbb{C}^3 \)). Further, in \( [Pi] \) all manifolds with \( d(M) = n^2 \) were determined (for partial classifications in special cases see also \( [GIK] \) and \( [KV] \)). The classification in this situation is substantially richer than that for higher automorphism group dimensions.

Observe that a further decrease in \( d(M) \) almost immediately leads to unclassifiable cases. For example, no good classification exists for \( n = 2 \) and \( d(M) = 2 \), since the automorphism group of a generic Reinhardt domain in \( \mathbb{C}^2 \) is 2-dimensional (see also \( [Pi] \) for a more specific statement). While it is possible that there is some classification for \( d(M) = n^2 - 2, n \geq 3 \) as well as for particular pairs \( d(M), n \) with \( d(M) < n^2 - 2 \) (see \( [GIK] \) in this

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regard), the case $d(M) = n^2 - 1$ is probably the only remaining candidate to investigate for the existence of a reasonable classification for every $n \geq 2$. It turns out that all hyperbolic manifolds with $d(M) = n^2 - 1$, $n \geq 2$ indeed can be explicitly described and that the case $n = 2$ substantially differs from the case $n \geq 3$. In this paper we obtain a classification for $d(M) = n^2 - 1$, $n \geq 3$ and give examples that demonstrate some of the specifics of the case $n = 2$. Our main result is the following theorem.

**THEOREM 0.1** Let $M$ be a connected hyperbolic manifold of dimension $n \geq 3$ with $d(M) = n^2 - 1$. Then $M$ is holomorphically equivalent to one of the following manifolds:

(i) $B^{n-1} \times S$, where $S$ is a hyperbolic Riemann surface with $d(S) = 0$;

(ii) the tube domain

$$T := \left\{ (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : \ (\text{Im} \ z_1)^2 + (\text{Im} \ z_2)^2 + (\text{Im} \ z_3)^2 - (\text{Im} \ z_4)^2 < 0, \ \text{Im} \ z_4 > 0 \right\}.$$

(here $n = 4$).

For $n = 2$ in addition to the direct products specified in (i) of Theorem 0.1 many other manifolds occur. They arise, in particular, from gluing together certain homogeneous strongly pseudoconvex real hypersurfaces in 2-dimensional complex manifolds with 3-dimensional groups of CR-automorphisms. All such hypersurfaces were determined by E. Cartan [C], and our considerations for $n = 2$ required an appropriate interpretation of Cartan’s results (see [I2]). Obtaining the classification for $n = 2$ is quite lengthy, and therefore the author has decided to publish it in a separate paper. Some non-trivial examples of hyperbolic domains in $\mathbb{C}^2$ and $\mathbb{C}P^2$ with 3-dimensional automorphism groups are given in Section 5.

The proof of Theorem 0.1 is organized as follows. In Section 1 we determine the dimensions of the orbits of the action on $M$ of $G(M) := \text{Aut}(M)^c$, the connected component of the identity of $\text{Aut}(M)$. It turns out that, unless $M$ is homogeneous, every $G(M)$-orbit is either a real or complex hypersurface in $M$, every real hypersurface orbit is spherical and every complex hypersurface orbit is holomorphically equivalent to $B^{n-1}$ (see Proposition 1.1). Note that Proposition 1.1 also contains some information about $G(M)$-orbits for
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$n = 2$, in particular, it allows in this case for some real hypersurface orbits to be either Levi-flat or Levi non-degenerate non-spherical, and for some 2-dimensional orbits to be totally real rather than complex submanifolds of $M$. It turns out that such orbits indeed exist; the corresponding examples are given in Section 5.

Next, in Section 2 we show that real hypersurface orbits in fact cannot occur (see Proposition 2.1). First, we prove that there may be three possible kinds of such orbits and that the presence of an orbit of a particular kind determines $G(M)$ as a Lie group. Further, when we attempt to glue real hypersurface orbits together, it turns out that for any resulting hyperbolic manifold $M$, the dimension $d(M)$ is always greater than $n^2 - 1$. Hence all orbits are in fact complex hypersurfaces unless the manifold in question is homogeneous. Parts of the arguments in Section 2 apply in the case $n = 2$ as well.

In Section 3 we prove Theorem 0.1 in the non-homogeneous case and obtain manifolds in (i) of Theorem 0.1 (see Proposition 3.1).

In Section 4 homogeneous manifolds are considered. We show that in this case $n = 4$ and obtain the tube domain in (ii) of Theorem 0.1 (see Proposition 4.1). Note that Proposition 4.1 holds for any $n \geq 2$, hence no additional homogeneous manifolds occur when $n = 2$.

1 Dimensions of Orbits

The action of $G(M) = \text{Aut}(M)^c$ on $M$ is proper (see Satz 2.5 of [Ka]), and therefore for every $p \in M$ its orbit $O(p) := \{f(p) : f \in G(M)\}$ is a closed submanifold of $M$ and the isotropy subgroup $I_p := \{f \in G(M) : f(p) = p\}$ of $p$ is compact (see [Ko], [Ka]). In this section we will obtain an initial classification of the $G(M)$-orbits.

Let $L_p := \{d_pf : f \in I_p\}$ be the linear isotropy subgroup, where $d_pf$ is the differential of a map $f$ at $p$. The group $L_p$ is a compact subgroup of $GL(T_p(M), \mathbb{C})$ isomorphic to $I_p$ by means of the isotropy representation

$$\alpha_p : I_p \to L_p, \quad \alpha_p(f) = d_pf$$

(see e.g. Satz 4.3 of [Ka]). We will now prove the following proposition.

**Proposition 1.1** Let $M$ be a connected hyperbolic manifold of dimension $n \geq 2$ with $d(M) = n^2 - 1$, and $p \in M$. Then the following holds:
(i) Either \( M \) is homogeneous, or \( O(p) \) is a real or complex closed hypersurface in \( M \), or, for \( n = 2 \), the orbit \( O(p) \) is a totally real 2-dimensional closed submanifold of \( M \).

(ii) If \( O(p) \) is a real hypersurface, the identity component \( I^c_p \) of the isotropy subgroup \( I_p \) is isomorphic to \( SU_{n-1} \), and \( I_p \) is isomorphic to a subgroup of \( \mathbb{Z}_2 \times U_{n-1} \) by means of the isotropy representation \( \alpha_p \). If \( n \geq 3 \), the orbit \( O(p) \) is spherical and \( I_p \) is isomorphic to a subgroup of \( U_{n-1} \). If \( n = 2 \) and \( O(p) \) is strongly pseudoconvex, then it is spherical, provided \( I_p \) contains more than two elements; if \( n = 2 \) and \( O(p) \) is Levi-flat, it is foliated by complex curves holomorphically equivalent to the unit disk \( \Delta \).

(iii) If \( O(p) \) is a complex hypersurface, it is holomorphically equivalent to \( B^{n-1} \). If \( n \geq 3 \), then \( I^c_p \) is isomorphic, by means of the isotropy representation \( \alpha_p \), to the group \( H_{k_1,k_2}^n \) of all matrices of the form

\[
\begin{pmatrix}
a & 0 \\
0 & B
\end{pmatrix},
\]

(1.1)

where \( B \in U_{n-1} \) and \( a \in (\det B)^{\frac{k_1}{k_2}} \), for some \( k_1, k_2 \in \mathbb{Z}, \ (k_1, k_2) = 1, k_2 \neq 0 \). If \( n = 2 \), then either \( I^c_p \) is isomorphic, by means of the isotropy representation \( \alpha_p \), to the group \( H_{k_1,k_2}^2 \) for some \( k_1, k_2 \in \mathbb{Z} \), or \( L^c_p \) acts trivially on the tangent space to \( O(p) \) at \( p \) and \( I^c_p \) is isomorphic to \( U_1 \) by means of the isotropy representation \( \alpha_p \). If \( I^c_p \) is isomorphic to \( H_{k_1,k_2}^n \) for some \( k_1 \neq 0 \), there is a real hypersurface orbit in \( M \).

(iv) If \( n = 2 \) and \( O(p) \) is totally real, then \( I^c_p \) is isomorphic to \( SO_2(\mathbb{R}) \) by means of the isotropy representation \( \alpha_p \).

**Proof:** Let \( V \subset T_p(M) \) be the tangent space to \( O(p) \) at \( p \). Clearly, \( V \) is \( L_p \)-invariant. We assume now that \( O(p) \neq M \) (and therefore \( V \neq T_p(M) \)) and consider the following three cases.

**Case 1.** \( d := \dim_\mathbb{C}(V+iV) < n \).

Since \( L_p \) is compact, one can choose coordinates in \( T_p(M) \) such that \( L_p \subset U_n \). Further, the action of \( L_p \) on \( T_p(M) \) is completely reducible and
the subspace \( V + iV \) is invariant under this action. Hence \( L_p \) can in fact be embedded in \( U_{n-d} \times U_d \). Since \( \dim O(p) \leq 2d \), it follows that

\[
n^2 - 1 \leq (n - d)^2 + d^2 + 2d,
\]

and therefore either \( d = 0 \) or \( d = n - 1 \).

If \( d = 0 \), then \( p \) is a fixed point for the action of \( G(M) \) on \( M \). Then \( I_p = G(M) \) and \( L_p \) is isomorphic to \( G(M) \). Since \( \dim L_p = n^2 - 1 \), we have \( L_p = SU_n \). The group \( SU_n \) acts transitively on directions in \( T_p(M) \). Since \( d(M) > 0 \), the manifold \( M \) is non-compact. Then, by [GK], \( M \) is holomorphically equivalent to \( B^n \), which is clearly impossible.

Suppose that \( d = n - 1 \). Then we have

\[
n^2 - 1 = \dim L_p + \dim O(p) \leq n^2 - 2n + 2 + \dim O(p).
\]

Hence \( \dim O(p) \geq 2n - 3 \), that is, either \( \dim O(p) = 2n - 2 \), or \( \dim O(p) = 2n - 3 \).

Suppose first that \( \dim O(p) = 2n - 2 \). In this case we have \( iV = V \), hence \( O(p) \) is a complex hypersurface. Then \( \dim L_p = (n - 1)^2 \). It now follows from the proof of Lemma 2.1 of [IKru1] that \( L_p^c \) is either \( U_1 \times SU_{n-1} \), or, for some \( k_1, k_2 \), the group \( H^n_{k_1,k_2} \) defined in (1.1). Therefore, if \( n \geq 3 \) or \( n = 2 \) and \( L_p^c = H^n_{k_1,k_2} \) for some \( k_1, k_2 \), then \( L_p \) acts transitively on directions in \( V \), and [GK] implies that \( O(p) \) is holomorphically equivalent to \( B^{n-1} \).

Let \( n \geq 3 \) and \( L_p^c = U_1 \times SU_{n-1} \). It then follows (see, for example, Satz 4.3 of [Ka]) that \( I_p^c := \alpha_p^{-1}(U_1) \) is the kernel of the action of \( G(M) \) on \( O(p) \), in particular, \( I_p^c \) is normal in \( G(M) \). Therefore, the factor-group \( G(M)/I_p^c \) acts effectively on \( O(p) \). Clearly, \( \dim G(M)/I_p^c = n^2 - 2 \). Thus, the group \( \text{Aut}(O(p)) \) is isomorphic to \( \text{Aut}(B^{n-1}) \) (in particular, its dimension is \( n^2 - 1 \) and has a codimension 1 (possibly non-closed) subgroup. However, the Lie algebra \( \mathfrak{su}_{n-1,1} \) of the group \( \text{Aut}(B^{n-1}) \) does not have codimension 1 subalgebras, if \( n \geq 3 \) (see, e.g., [Ea1]). Thus, we have shown that if \( n \geq 3 \), then \( L_p^c = H^n_{k_1,k_2} \) for some \( k_1, k_2 \).

Next, if \( n = 2 \) and \( L_p^c = U_1 \times SU_1 = U_1 \), then the above argument shows that \( O(p) \) is a hyperbolic 1-dimensional manifold with automorphism group of dimension at least 2. Hence \( O(p) \) is holomorphically equivalent to \( \Delta \) if \( L_p^c = U_1 \) as well.

Suppose that \( I_p^c \) is isomorphic to \( H^n_{k_1,k_2} \) where \( k_1 \neq 0 \). Then \( L_p^c \) acts as \( U_1 \) on the orthogonal complement to \( V \). Therefore, in this case there are real hypersurface orbits in \( M \) arbitrarily close to \( O(p) \).
Suppose now that \( \dim O(p) = 2n - 3 \). In this case \( \dim I_p = n^2 - 2n + 2 \). Since \( L_p \) can be embedded in \( U_1 \times U_{n-1} \), we obtain \( L_p = U_1 \times U_{n-1} \). In particular, \( L_p \) acts transitively on directions in \( V + iV \). This is, however, impossible since \( V \) is of codimension 1 in \( V + iV \) and is \( L_p \)-invariant.

**Case 2.** \( T_p(M) = V + iV \) and \( r := \dim_C(V \cap iV) > 0 \).

As above, \( L_p \) can be embedded in \( U_{n-r} \times U_r \) (clearly, we have \( r < n \)). Moreover, \( V \cap iV \neq V \) and since \( L_p \) preserves \( V \), it follows that \( \dim L_p < r^2 + (n-r)^2 \). We have \( \dim O(p) \leq 2n - 1 \), and therefore

\[
n^2 - 1 < (n-r)^2 + r^2 + 2n - 1,
\]

which shows that either \( r = 1 \), or \( r = n - 1 \). It then follows that \( \dim L_p < n^2 - 2n + 2 \). Therefore, we have

\[
n^2 - 1 = \dim L_p + \dim O(p) < n^2 - 2n + 2 + \dim O(p).
\]

Hence \( \dim O(p) > 2n - 3 \). Thus, \( \dim O(p) = 2n - 1 \), or \( \dim O(p) = 2n - 2 \).

Suppose that \( \dim O(p) = 2n - 1 \). Let \( W \) be the orthogonal complement to \( V \cap iV \) in \( T_p(M) \). Clearly, in this case \( r = n - 1 \) and \( \dim_C W = 1 \). The group \( L_p \) is a subgroup of \( U_n \) and preserves \( V \), \( V \cap iV \), and \( W \); hence it preserves the line \( W \cap V \). Therefore, it can act only as \( \pm \text{Id} \) on \( W \), that is, \( L_p \subset \mathbb{Z}_2 \times U_{n-1} \). Since \( \dim L_p = (n-1)^2 - 1 \), we have \( L_p^c = SU_{n-1} \). In particular, \( L_p \) acts transitively on directions in \( V \cap iV \), if \( n \geq 3 \). Hence, the orbit \( O(p) \) is either Levi-flat or strongly pseudoconvex for all \( n \geq 2 \).

Suppose first that \( n \geq 3 \) and \( O(p) \) is Levi-flat. Then \( O(p) \) is foliated by connected complex manifolds. Let \( M_p \) be the leaf passing through \( p \). Denote by \( g \) the Lie algebra of vector fields on \( O(p) \) arising from the action of \( G(M) \), and let \( I_p \subset g \) be the subspace consisting of all vector fields tangent to \( M_p \) at \( p \). Since vector fields in \( I_p \) remain tangent to \( M_p \) at each point in \( M_p \), the subspace \( I_p \) is in fact a Lie subalgebra of \( g \). It follows from the definition of \( I_p \) that \( \dim I_p = n^2 - 2 \). Denote by \( H_p \) the (possibly non-closed) connected subgroup of \( G(M) \) with Lie algebra \( I_p \). It is straightforward to verify that the group \( H_p \) acts on \( M_p \) by holomorphic transformations and that \( I_p^c \subset H_p \). If some non-trivial element \( g \in H_p \) acts trivially on \( M_p \), then \( g \in I_p \), and corresponds to the non-trivial element in \( \mathbb{Z}_2 \) (recall that \( L_p \subset \mathbb{Z}_2 \times U_{n-1} \)). Thus, either \( H_p \) or \( H_p/\mathbb{Z}_2 \) acts effectively on \( M_p \) (the
former case occurs if \( g_p \not\in H_p \), the latter if \( g_p \in H_p \). The group \( L_p \) acts transitively on directions in the tangent space \( V \cap iV \) to \( M_p \), and it follows from [GK] that \( M_p \) is holomorphically equivalent to \( B^{n-1} \). Therefore, the group \( \text{Aut}(M_p) \) is isomorphic to \( \text{Aut}(B^{n-1}) \) (in particular, its dimension is \( n^2 - 1 \)) and has a codimension 1 (possibly non-closed) subgroup. However, as we noted above, the Lie algebra of \( \text{Aut}(B^{n-1}) \) does not have codimension 1 subalgebras, if \( n \geq 3 \). Thus, \( O(p) \) is strongly pseudoconvex. Hence, \( L_p \) acts trivially on \( W \) and therefore \( L_p \subset U_{n-1} \). Since \( L_p^c = SU_{n-1} \), the dimension of the stability group of \( O(p) \) at \( p \) is greater than or equal to \( (n-1)^2 - 1 \), which for \( n \geq 3 \) implies that \( p \) is an umbilic point of \( O(p) \) (see e.g. [EzhI]). The homogeneity of \( O(p) \) now yields that \( O(p) \) is spherical, if \( n \geq 3 \). For \( n = 2 \) the above argument shows that \( O(p) \) is foliated by connected hyperbolic complex curves with automorphism group of dimension at least 2, that is, by complex curves holomorphically equivalent to \( \Delta \).

If \( n = 2 \), the orbit \( O(p) \) is Levi non-degenerate and \( I_p \) contains more than two elements, then arguing as in the proof of Lemma 3.3 of [IKru2], we obtain that \( O(p) \) is spherical. Alternatively, this fact can be derived from the classification in [C].

Suppose now that \( \dim O(p) = 2n - 2 \). Since \( T_p(M) = V + iV \), the orbit \( O(p) \) is not a complex hypersurface. Therefore, \( r = n - 2 \), which is only possible for \( n = 3 \) (recall that we have either \( r = 1 \), or \( r = n - 1 \)). In this case \( \dim L_p = 4 \) and, arguing as in the proof of Lemma 2.1 of [IKru1], we see that \( L_p \) acts transitively on directions in the orthogonal complement \( W \) to \( V \cap iV \) in \( T_p(M) \). This is, however, impossible since \( L_p \) must preserve \( W \cap V \).

**Case 3.** \( T_p(M) = V \oplus iV \).

In this case \( \dim V = n \) and \( L_p \) can be embedded in the real orthogonal group \( O_n(\mathbb{R}) \), and therefore

\[
\dim L_p + \dim O(p) \leq \frac{n(n-1)}{2} + n.
\]

Hence, for \( n \geq 3 \), we have \( \dim L_p + \dim O(p) < n^2 - 1 \) which is impossible.

Assume now that \( n = 2 \). If \( \dim L_p = 0 \), we get a contradiction as above. Hence \( \dim L_p = 1 \) and \( L_p^c = SO_2(\mathbb{R}) \).

The proof of the proposition is complete. \( \square \)
2 Real Hypersurface Orbits

In this section we will deal with real hypersurface orbits and eventually show that they do not occur. Our goal is to prove the following proposition.

Proposition 2.1 Let $M$ be a connected hyperbolic manifold of dimension $n \geq 3$ with $d(M) = n^2 - 1$. Then no orbit in $M$ is a real hypersurface.

Proof: Recall that every real hypersurface orbit is spherical. First, we narrow down the class of all possible spherical orbits.

Lemma 2.2 Let $M$ be a connected hyperbolic manifold of dimension $n \geq 3$ with $d(M) = n^2 - 1$. Assume that for a point $p \in M$ its orbit $O(p)$ is spherical. Then $O(p)$ is $CR$-equivalent to one of the following hypersurfaces:

\begin{enumerate}[(i)]
    \item a lens manifold $\mathcal{L}_m := S^{2n-1}/\mathbb{Z}_m$ for some $m \in \mathbb{N}$,
    \item $\sigma := \{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : Re z_n = |z'|^2\}$,
    \item $\delta := \{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z_n| = \exp(|z'|^2)\}$,
    \item $\omega := \{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'|^2 + \exp(Re z_n) = 1\}$, \hfill (2.1)
    \item $\varepsilon_\alpha := \{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'|^2 + |z_n|^\alpha = 1, z_n \neq 0\}$, for some $\alpha > 0$.
\end{enumerate}

Proof of Lemma 2.2 The proof is similar to that of Proposition 3.1 of [11]. For a connected Levi non-degenerate $CR$-manifold $Q$ denote by $\text{Aut}_{CR}(Q)$ the Lie group of its $CR$-automorphisms. Let $\tilde{O}(p)$ be the universal cover of $O(p)$. The connected component of the identity $\text{Aut}_{CR}(O(p))^c$ of $\text{Aut}_{CR}(O(p))$ acts transitively on $O(p)$ and therefore its universal cover $\tilde{\text{Aut}}_{CR}(O(p))^c$ acts transitively on $\tilde{O}(p)$. Let $G$ be the (possibly non-closed) subgroup of $\tilde{\text{Aut}}_{CR}(\tilde{O}(p))$ that consists of all $CR$-automorphisms of $\tilde{O}(p)$ generated by this action. Observe that $G$ is a Lie group isomorphic to the factor-group of $\tilde{\text{Aut}}_{CR}(O(p))^c$ by a discrete central subgroup. Let $\Gamma \subset \text{Aut}_{CR}(\tilde{O}(p))$ be the discrete subgroup whose orbits are the fibers of the covering $\tilde{O}(p) \to O(p)$. The group $\Gamma$ acts freely properly discontinuously on $\tilde{O}(p)$, lies in the centralizer of $G$ in $\text{Aut}_{CR}(\tilde{O}(p))$ and is isomorphic to $H/H^c$, with $H = \pi^{-1}(I_p)$, where $\pi : \tilde{\text{Aut}}_{CR}(O(p))^c \to \text{Aut}_{CR}(O(p))^c$ is the covering map.

The manifold $\tilde{O}(p)$ is spherical, and there is a local $CR$-isomorphism $\Pi$ from $\tilde{O}(p)$ onto a domain $D \subset S^{2n-1}$. By Proposition 1.4 of [12], $\Pi$ is a
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covering map. Further, for every \( f \in \text{Aut}_{CR}(\tilde{O}(p)) \) there is \( g \in \text{Aut}(D) \) such that
\[
g \circ \Pi = \Pi \circ f. \tag{2.2}
\]
Since \( \tilde{O}(p) \) is homogeneous, (2.2) implies that \( D \) is homogeneous as well, and 
\[\dim \text{Aut}_{CR}(\tilde{O}(p)) = \dim \text{Aut}_{CR}(D).\]

Clearly, \( \dim \text{Aut}_{CR}(O(p)) \geq n^2 - 1 \) and therefore we have \( \dim \text{Aut}_{CR}(D) \geq n^2 - 1 \). All homogeneous domains in \( S^{2n-1} \) are listed in Theorem 3.1 in [BS].

It is not difficult to exclude from this list all the domains with automorphism group of dimension less than \( n^2 - 1 \). This gives that \( D \) is \( CR \)-equivalent to one of the following domains:

- (a) \( S^{2n-1} \),
- (b) \( S^{2n-1} \setminus \{\text{point}\} \),
- (c) \( S^{2n-1} \setminus \{z_n = 0\} \).

Thus, \( \tilde{O}(p) \) is respectively one of the following manifolds:

- (a) \( S^{2n-1} \),
- (b) \( \sigma \),
- (c) \( \omega \).

If \( \tilde{O}(p) = S^{2n-1} \), then by Proposition 5.1 of [BS] the orbit \( O(p) \) is \( CR \)-equivalent to a lens manifold as in (i) of (2.1).

Suppose next that \( \tilde{O}(p) = \sigma \). The group \( \text{Aut}_{CR}(\sigma) \) consists of all maps of the form
\[
\begin{align*}
z' & \mapsto \lambda Uz' + a, \\
z_n & \mapsto \lambda^2 z_n + 2\lambda(Uz', a) + |a|^2 + i\alpha,
\end{align*}
\]
where \( U \in U_{n-1} \), \( a \in \mathbb{C}^{n-1} \), \( \lambda \in \mathbb{R}^* \), \( \alpha \in \mathbb{R} \), and \( \langle \cdot , \cdot \rangle \) is the inner product in \( \mathbb{C}^{n-1} \). It then follows that \( \text{Aut}_{CR}(\sigma) = CU_{n-1} \ltimes N \), where \( CU_{n-1} \) consists of all maps of the form (2.3) with \( a = 0 \), \( \alpha = 0 \), and \( N \) is the Heisenberg group consisting of the maps of the form (2.3) with \( U = \text{id} \) and \( \lambda = 1 \).

Further, description (2.3) implies that \( \dim \text{Aut}_{CR}(\sigma) = n^2 + 1 \), and therefore \( n^2 - 1 \leq \dim G \leq n^2 + 1 \). If \( \dim G = n^2 + 1 \), then we have \( G = \text{Aut}_{CR}(\sigma)^c \), and hence \( \Gamma \) is a central subgroup of \( \text{Aut}_{CR}(\sigma)^c \). Since the center of \( \text{Aut}_{CR}(\sigma)^c \) is trivial, so is \( \Gamma \). Thus, in this case \( O(p) \) is \( CR \)-equivalent to the hypersurface \( \sigma \).

Assume now that \( n^2 - 1 \leq \dim G \leq n^2 \). Since \( G \) acts transitively on \( \sigma \), we have \( N \subset G \). Furthermore, since \( G \) is of codimension 1 or 2 in \( \text{Aut}_{CR}(\sigma) \), it
either contains the subgroup $SU_{n-1} \ltimes N$, or $n = 3$ and $G$ contains a subgroup of the form $L \ltimes N$, where $L$ is conjugate to $U_1 \times U_1$ in $U_2$. By Proposition 5.6 of [BS], we have $\Gamma \subset U_{n-1} \ltimes N$. The centralizer of $SU_{n-1} \ltimes N$ in $U_{n-1} \ltimes N$ and that of $L \ltimes N$ in $U_2 \ltimes N$ consist of all maps of the form

\begin{align*}
z' &\mapsto z', \\
z_n &\mapsto z_n + i\alpha,
\end{align*}

(2.4)

where $\alpha \in \mathbb{R}$. Since $\Gamma$ acts freely properly discontinuously on $\sigma$, it is generated by a single map of the form (2.4) with $\alpha = \alpha_0 \in \mathbb{R}^\ast$. The hypersurface $\sigma$ covers the hypersurface

\[
\left\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z_n| = \exp \left(\frac{2\pi}{\alpha_0}|z'|^2\right)\right\}
\]

(2.5)

by means of the map

\begin{align*}
z' &\mapsto z', \\
z_n &\mapsto \exp \left(\frac{2\pi}{\alpha_0}z_n\right),
\end{align*}

(2.6)

and the fibers of this map are the orbits of $\Gamma$. Hence $O(p)$ is $CR$-equivalent to hypersurface (2.5). Replacing if necessary $z_n$ by $1/z_n$ we obtain that $O(p)$ is $CR$-equivalent to the hypersurface $\delta$.

Suppose finally that $\tilde{O}(p) = \omega$. First, we will determine the group $\text{Aut}_{CR}(\omega)$. The general form of a $CR$-automorphism of $S^{2n-1} \setminus \{z_n = 0\}$ is given by the formula

\begin{align*}
z' &\mapsto \frac{Az' + b}{cz' + d}, \\
z_n &\mapsto \frac{e^{i\beta}z_n}{cz' + d},
\end{align*}

where

\[
\left(\begin{array}{cc}A & b \\ c & d\end{array}\right) \in SU_{n-1,1}, \quad \beta \in \mathbb{R},
\]

and the covering map $\Pi$ by the formula

\begin{align*}
z' &\mapsto z', \\
z_n &\mapsto \exp \left(\frac{z_n}{2}\right).
\end{align*}
Using (2.2) we then obtain the general form of a CR-automorphism of ω as follows

\[ z' \mapsto \frac{Az + b}{cz' + d}, \]

\[ z_n \mapsto z_n - 2 \ln(cz' + d) + i\beta, \]

where

\[ \left( \begin{array}{cc} A & b \\ c & d \end{array} \right) \in SU_{n-1,1}, \quad \beta \in \mathbb{R}. \]

In particular, \( \text{Aut}_{CR}(\omega) \) is a connected group of dimension \( n^2 \), and therefore \( n^2 - 1 \leq \dim G \leq n^2 \).

Thus, either \( G = \text{Aut}_{CR}(\omega) \), or \( G \) coincides with the subgroup of \( \text{Aut}_{CR}(\omega) \) given by the condition \( \beta = 0 \) in formula (2.7). In either case, the centralizer of \( G \) in \( \text{Aut}_{CR}(\omega) \) consists of all maps of the form (2.4). Hence \( \Gamma \) is generated by a single such map with \( \alpha = \alpha_0 \in \mathbb{R} \). If \( \alpha_0 = 0 \), the orbit \( O(p) \) is CR-equivalent to \( \omega \). Let \( \alpha_0 \neq 0 \). The hypersurface \( \omega \) covers the hypersurface

\[ \left\{ (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'|^2 + |z_n|^\alpha = 1, \ z_n \neq 0 \right\} \quad (2.8) \]

by means of map (2.6). Since the fibers of this map are the orbits of \( \Gamma \), it follows that \( O(p) \) is CR-equivalent to hypersurface (2.8). Replacing if necessary \( z_n \) by \( 1/z_n \), we obtain that \( O(p) \) is CR-equivalent to the hypersurface \( \varepsilon_\alpha \) for some \( \alpha > 0 \).

The proof of Lemma 2.2 is complete. \( \square \)

**Remark 2.3** For \( n = 2 \) there is an additional possibility for \( D \) that has to be taken into the account. Namely, \( S^3 \setminus \mathbb{R}^2 \) has a 3-dimensional automorphism group arising from the natural transitive action of \( O_{2,1}(\mathbb{R}) \) by fractional-linear transformations (see Section 5).

We will now show that in most cases the presence of a spherical orbit of a particular kind in \( M \) determines the group \( G(M) \) as a Lie group. Suppose that for some \( p \in M \) the orbit \( O(p) \) is spherical, and let \( m \) be the manifold from list (2.1) to which \( O(p) \) is CR-equivalent (we say that \( m \) is the model for \( O(p) \)). Since \( G(M) \) acts effectively on \( O(p) \), the CR-equivalence induces an isomorphism between \( G(M) \) and a (possibly non-closed) connected \( (n^2 - 1) \)-dimensional subgroup \( R_m \) of \( \text{Aut}_{CR}(m) \) (this subgroup a priori depends on the choice of the CR-equivalence).
We need the following lemma.

**Lemma 2.4**

(i) $R_{S^{2n-1}}$ is conjugate to $SU_n$ in $\text{Aut}(B^n)$, and $R_{L_m} = SU_n/(SU_n \cap \mathbb{Z}_m)$ for $m > 1$;

(ii) $R_\sigma = SU_{n-1} \rtimes N$;

(iii) $R_\delta$ consists of all maps of the form

\[
\begin{align*}
  z' & \mapsto Uz' + a, \\
  z_n & \mapsto e^{i\beta} \exp \left( 2 \langle Uz', a \rangle + |a|^2 \right) z_n,
\end{align*}
\]

where $U \in SU_{n-1}$, $a \in \mathbb{C}^{n-1}$, $\beta \in \mathbb{R}$;

(iv) $R_\omega$ consists of all maps of the form (2.7) with $\beta = 0$;

(v) $R_\varepsilon\alpha$ consists of all maps of the form

\[
\begin{align*}
  z' & \mapsto \frac{Az' + b}{cz' + d}, \\
  z_n & \mapsto \frac{z_n}{(cz' + d)^{2/\alpha}},
\end{align*}
\]

where $\left( \begin{array}{cc} A & b \\ c & d \end{array} \right) \in SU_{n-1,1}$.

**Proof of Lemma 2.4.** Suppose first that $m = \mathcal{L}_m$, for some $m \in \mathbb{N}$. Then $O(p)$ is compact and, since $I_p$ is compact as well, it follows that $G(M)$ is compact. Assume first that $m = 1$. In this case $R_{S^{2n-1}}$ is a subgroup of $\text{Aut}_{CR}(S^{2n-1}) = \text{Aut}(B^n)$. Since $R_{S^{2n-1}}$ is compact, it is conjugate to a subgroup of $U_n$, which is a maximal compact subgroup in $\text{Aut}(B^n)$. Since both $R_{S^{2n-1}}$ is $(n^2 - 1)$-dimensional, it is conjugate to $SU_n$. Suppose now that $m > 1$. It is straightforward to determine the group $\text{Aut}_{CR}(\mathcal{L}_m)$ by lifting $CR$-automorphisms of $\mathcal{L}_m$ to its universal cover $S^{2n-1}$. This group is $U_n/\mathbb{Z}_m$ acting on $\mathbb{C}^n \setminus \{0\}/\mathbb{Z}_m$ in the standard way. Since $R_{L_m}$ is of codimension 1 in $\text{Aut}_{CR}(\mathcal{L}_m)$, we obtain $R_{L_m} = SU_n/(SU_n \cap \mathbb{Z}_m)$.

Assume now that $m = \sigma$. The group $\text{Aut}_{CR}(\sigma)$ consists of all maps of the form (2.3) and has dimension $n^2 + 1$. Since $R_\sigma$ acts transitively on $\sigma$, it contains the subgroup $N$ (see the proof of Proposition 2.2). Furthermore,
$R_\sigma$ is a codimension 2 subgroup of $\text{Aut}_{CR}(\sigma)$, and thus either is the group $SU_{n-1} \ltimes N$, or, for $n = 3$, $R_\sigma \cap (U_2 \ltimes N) = L \ltimes N$, where $L$ is conjugate to $U_1 \times U_1$ in $U_2$. By (ii) of Proposition 1.1, $I_\mu^c$ is isomorphic to $SU_{n-1}$, hence the latter case in fact does not occur.

Next, the group $\text{Aut}_{CR}(\delta)$ can be determined by considering the universal cover of $\delta$ (see the proof of Proposition 2.2) and consists of all maps of the form

$$
\begin{align*}
  z' &\mapsto Uz' + a, \\
  z_n &\mapsto e^{i\beta} \exp \left( 2\langle Uz', a \rangle + |a|^2 \right) z_n,
\end{align*}
$$

(2.10)

where $U \in U_{n-1}$, $a \in \mathbb{C}^{n-1}$, $\beta \in \mathbb{R}$. This group has dimension $n^2$, and hence $R_\delta$ is of codimension 1 in $\text{Aut}_{CR}(\delta)$. Since $R_\delta$ acts transitively on $\delta$, it consists of all maps of the form (2.10) with $U \in SU_{n-1}$.

Assume now that $m = \omega$. The only codimension 1 subgroup of $\text{Aut}_{CR}(\delta)$ is given by maps with $\beta = 0$ in formula (2.7).

Let finally $m = \varepsilon_\alpha$. The group $\text{Aut}_{CR}(\varepsilon_\alpha)$ consists of all maps of the form

$$
\begin{align*}
  z' &\mapsto \frac{Az' + b}{cz' + d}, \\
  z_n &\mapsto \frac{e^{i\beta} z_n}{(cz' + d)^{2/\alpha}},
\end{align*}
$$

(2.11)

where

$$
\begin{pmatrix}
  A & b \\
  c & d
\end{pmatrix} \in SU_{n-1,1}, \quad \beta \in \mathbb{R},
$$

and its only codimension 1 subgroup is given by $\beta = 0$.

The proof of Lemma 2.4 is complete. $\square$

We will now finish the proof of Proposition 2.1. Our argument is similar to that in Section 4 of [111]. For completeness of our exposition, we will repeat it here in detail.

Suppose that for some $p \in M$ the orbit $O(p)$ is $CR$-equivalent to a lens manifold $L_m$. In this case $G(M)$ is compact, hence there are no complex hypersurface orbits and the model for every orbit is a lens manifold. Assume first that $m = 1$. Then $M$ admits an effective action of $SU_n$ by holomorphic transformations and therefore is holomorphically equivalent to one of the manifolds listed in [IKru2]. However, none of the manifolds on the list in [IKru2] with $n \geq 3$ is hyperbolic and has $(n^2 - 1)$-dimensional automorphism group.
Assume now that $m > 1$. Let $f : O(p) \to \mathcal{L}_m$ be a CR-isomorphism. Then we have
\[ f(gq) = \varphi(g)f(q), \tag{2.12} \]
where $q \in O(p)$, for some Lie group isomorphism $\varphi : G(M) \to SU_n/(SU_n \cap Z_m)$. The CR-isomorphism $f$ extends to a biholomorphic map from a neighborhood $U$ of $O(p)$ in $M$ onto a neighborhood $W$ of $\mathcal{L}_m$ in $\mathbb{C}^n \setminus \{0\}/\mathbb{Z}_m$. Since $G(M)$ is compact, one can choose $U$ to be a connected union of $G(M)$-orbits. Then property (2.12) holds for the extended map, and therefore every $G(M)$-orbit in $U$ is taken onto an $SU_n/(SU_n \cap \mathbb{Z}_m)$-orbit in $\mathbb{C}^n \setminus \{0\}/\mathbb{Z}_m$ by this map. Thus, $W = S^R_r/\mathbb{Z}_m$ for some $0 \leq r < R < \infty$, where $S^R_r := \{z \in \mathbb{C}^n : r < |z| < R\}$ is a spherical shell.

Let $D$ be a maximal domain in $M$ such that there exists a biholomorphic map $f$ from $D$ onto $S^R_r/\mathbb{Z}_m$ for some $r, R$, satisfying (2.12) for all $q \in G(M)$ and $q \in D$. As was shown above, such a domain $D$ exists. Assume that $D \neq M$ and let $x$ be a boundary point of $D$. Consider the orbit $O(x)$. Let $\mathcal{L}_k$ for some $k > 1$ be the model for $O(x)$ and $f_1 : O(x) \to \mathcal{L}_k$ a CR-isomorphism satisfying (2.12) for $q \in G(M)$, $q \in O(x)$ and an isomorphism $\varphi_1 : G(M) \to SU_n/(SU_n \cap \mathbb{Z}_k)$ in place of $\varphi$. The map $f_1$ can be holomorphically extended to a neighborhood $V$ of $O(x)$ that one can choose to be a connected union of $G(M)$-orbits. The extended map satisfies (2.12) for $q \in G(M)$, $q \in V$ and $\varphi_1$ in place of $\varphi$. For $s \in V \cap D$ we consider the orbit $O(s)$. The maps $f$ and $f_1$ take $O(s)$ into some surfaces $r_1S^{2n-1}/\mathbb{Z}_m$ and $r_2S^{2n-1}/\mathbb{Z}_k$, respectively, with $r_1, r_2 > 0$. Hence $F := f_1 \circ f^{-1}$ maps $r_1S^{2n-1}/\mathbb{Z}_m$ onto $r_2S^{2n-1}/\mathbb{Z}_k$. Since $\mathcal{L}_m$ and $\mathcal{L}_k$ are not CR-equivalent for distinct $m, k$, we obtain $k = m$. Furthermore, every CR-isomorphism between $r_1S^{2n-1}/\mathbb{Z}_m$ and $r_2S^{2n-1}/\mathbb{Z}_m$ has the form $[z] \mapsto [r_2/r_1Uz]$, where $U \in U_n$, and $[z] \in \mathbb{C}^n \setminus \{0\}/\mathbb{Z}_m$ denotes the equivalence class of a point $z \in \mathbb{C}^n \setminus \{0\}$. Therefore, $F$ extends to a holomorphic automorphism of $\mathbb{C}^n \setminus \{0\}/\mathbb{Z}_m$.

We claim that $V$ can be chosen so that $D \cap V$ is connected and $V \setminus (D \cup O(x)) \neq \emptyset$. Indeed, since $O(x)$ is strongly pseudoconvex and closed in $M$, for $V$ small enough we have $V = V_1 \cup V_2 \cup O(x)$, where $V_j$ are open connected non-intersecting sets. For each $j$, $D \cap V_j$ is a union of $G(M)$-orbits and therefore is mapped by $f$ onto a union of the quotients of some spherical shells. If there are more than one such factored shells, then there is a factored shell such that the closure of its inverse image under $f$ is disjoint from $O(x)$, and hence $D$ is disconnected which contradicts the definition of $D$. Thus, $D \cap V_j$ is connected for $j = 1, 2$, and, if $V$ is sufficiently small, then
each $V_j$ is either a subset of $D$ or is disjoint from it. If $V_j \subset D$ for $j = 1, 2$, then $M = D \cup V$ is compact, which is impossible since $M$ is hyperbolic and $d(M) > 0$. Therefore, for some $V$ there is only one $j$ for which $D \cap V_j \neq \emptyset$. Thus, $D \cap V$ is connected and $V \setminus (D \cup O(x)) \neq \emptyset$, as required.

Setting now

$$\tilde{f} := \begin{cases} f & \text{on } D \\ F^{-1} \circ f_1 & \text{on } V, \end{cases} \quad (2.13)$$

we obtain a biholomorphic extension of $f$ to $D \cup V$. By construction, $\tilde{f}$ satisfies (2.12) for $g \in G(M)$ and $q \in D \cup V$. Since $D \cup V$ is strictly larger than $D$, we obtain a contradiction with the maximality of $D$. Thus, we have shown that in fact $D = M$, and hence $M$ is holomorphically equivalent to $S^r/\mathbb{Z}_m$. However, in this case $d(M) = n^2$, which is impossible.

The orbit gluing procedure utilized above can in fact be applied in a very general setting. We will now describe it in full generality (see also [I]), assuming that every orbit in $M$ is a real hypersurface. The procedure comprises the following steps:

(1). Start with a real hypersurface orbit $O(p)$ with model $m$ and consider a real-analytic $CR$-isomorphism $f : O(p) \to m$ that satisfies (2.12) for all $g \in G(M)$ and $q \in O(p)$, where $\varphi : G(M) \to R_m$ is a Lie group isomorphism.

(2). Verify that for every model $m'$ the group $R_{m'}$ acts by holomorphic transformations with real hypersurface orbits on a domain $D \subset \mathbb{C}^n$ that contains $m'$ and that every orbit of the action is $CR$-equivalent to $m'$.

(3). Observe that $f$ can be extended to a biholomorphic map from a $G(M)$-invariant connected neighborhood of $O(p)$ in $M$ onto an $R_m$-invariant neighborhood of $m$ in $D$. First of all, extend $f$ to some neighborhood $U$ of $O(p)$ to a biholomorphic map onto a neighborhood $W$ of $m$ in $\mathbb{C}^n$. Let $W' = W \cap D$ and $U' = f^{-1}(W')$. Fix $s \in U'$ and $s_0 \in O(s)$. Choose $h_0 \in G(M)$ such that $s_0 = h_0s$ and define $f(s_0) := \varphi(h_0)f(s)$. To see that $f$ is well-defined at $s_0$, suppose that for some $h_1 \in G(M)$, $h_1 \neq h_0$, we have $s_0 = h_1s$, and show that $\varphi(h)$ fixes $f(s)$, where $h := h_1^{-1}h_0$. Indeed, for every $g \in G(M)$ identity (2.12) holds for $q \in U_g$, where $U_g$ is the connected component of $g^{-1}(U') \cap U'$ containing $O(p)$. Since $h \in I_s$, we have $s \in U_h$ and the application of (2.12) to $h$ and $s$ yields that $\varphi(h)$ fixes $f(s)$, as required. Thus,
extends to \( U'' := \bigcup_{q\in U'} O(q) \). The extended map satisfies (2.12) for all \( g \in G(M) \) and \( q \in U'' \).

(4). Consider a maximal \( G(M) \)-invariant domain \( D \subset M \) from which there exists a biholomorphic map \( f \) onto an \( R_m \)-invariant domain in \( D \) satisfying (2.12) for all \( g \in G(M) \) and \( q \in D \). The existence of such a domain is guaranteed by the previous step. Assume that \( D \neq M \) and consider \( x \in \partial D \). Let \( m_1 \) be the model for \( O(x) \) and let \( f_1 : O(x) \to m_1 \) be a real-analytic CR-isomorphism satisfying (2.12) for all \( g \in G(M) \), \( q \in O(x) \) and some Lie group isomorphism \( \varphi_1 : G(M) \to R_{m_1} \) in place of \( \varphi \). Let \( D_1 \) be the domain in \( \mathbb{C}^n \) containing \( m_1 \) on which \( R_{m_1} \) acts by holomorphic transformations with real hypersurface orbits CR-equivalent to \( m_1 \). As in (3), extend \( f_1 \) to a biholomorphic map from a connected \( G(M) \)-invariant neighborhood \( V \) of \( O(x) \) onto an \( R_{m_1} \)-invariant neighborhood of \( m_1 \) in \( D_1 \). The extended map satisfies (2.12) for all \( g \in G(M) \), \( q \in V \) and \( \varphi_1 \) in place of \( \varphi \). Consider \( s \in V \cap D \). The maps \( f \) and \( f_1 \) take \( O(s) \) onto an \( R_m \)-orbit in \( D \) and an \( R_{m_1} \)-orbit in \( D_1 \), respectively. Then \( F := f_1 \circ f^{-1} \) maps the \( R_m \)-orbit onto the \( R_{m_1} \)-orbit. Since all models are pairwise CR non-equivalent, we obtain \( m_1 = m \).

(5). Show that \( F \) extends to a holomorphic automorphism of \( D \).

(6). Show that \( V \) can be chosen so that \( D \cap V = \emptyset \). This follows from the hyperbolicity of \( M \) and the existence of a neighborhood \( V' \) of \( O(x) \) such that \( V' = V_1 \cup V_2 \cup O(x) \), where \( V_j \) are open connected non-intersecting sets. The existence of such \( V' \) follows from the strong pseudoconvexity of \( m \).

(7). Use formula (2.13) to extend \( f \) to \( D \cup V \) thus obtaining a contradiction with the maximality of \( D \). This shows that in fact \( D = M \) and hence \( M \) is biholomorphically equivalent to an \( R_m \)-invariant domain in \( D \). In all the cases below the determination of \( R_m \)-invariant domains will be straightforward.

Assume first that every orbit in \( M \) is a real hypersurface. Let first \( m = \sigma \). Clearly, the group \( R_\sigma \) acts with real hypersurface orbits on all of \( \mathbb{C}^n \), so in this case \( D = \mathbb{C}^n \). The \( R_\sigma \)-orbit of every point in \( \mathbb{C}^n \) is of the form

\[
\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : \text{Re } z_n = |z'|^2 + r\},
\]
where \( r \in \mathbb{R} \), and every \( R_\sigma \)-invariant domain in \( \mathbb{C}^n \) is given by
\[
\mathcal{S}^R_r := \left\{ (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : r + |z'|^2 < \Re z_n < R + |z'|^2 \right\},
\]
where \(-\infty \leq r < R \leq \infty\). Every \( CR \)-isomorphism between two \( R_\sigma \)-orbits is a composition of a map of the form (2.3) and a translation in the \( z_n \)-variable. Therefore, \( F \) in this case extends to a holomorphic automorphism of \( \mathbb{C}^n \). Now our gluing procedure implies that \( M \) is holomorphically equivalent to \( \mathcal{S}^R_r \) for some \(-\infty \leq r < R \leq \infty\). Therefore, \( M \) is holomorphically equivalent either to the domain
\[
\mathcal{S} := \left\{ (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : -1 + |z'|^2 < \Re z_n < |z'|^2 \right\},
\]
or (for \( R = \infty \)) to \( B^n \). The latter is clearly impossible; the former is impossible either since \( d(\mathcal{S}) = n^2 \) (see e.g. \([\text{I1}]\)).

Assume next that \( m = \delta \). Again, we have \( D = \mathbb{C}^n \). The \( R_\delta \)-orbit of every point in \( \mathbb{C}^n \) has the form
\[
\left\{ (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z_n| = r \exp \left( |z'|^2 \right) \right\},
\]
where \( r > 0 \), and hence every \( R_\delta \)-invariant domain in \( \mathbb{C}^n \) is given by
\[
D^R_r := \left\{ (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : r \exp \left( |z'|^2 \right) < |z_n| < R \exp \left( |z'|^2 \right) \right\},
\]
for \( 0 \leq r < R \leq \infty \). Every \( CR \)-isomorphism between two \( R_\delta \)-orbits is a composition of a map from of the form (2.10) and a dilation in the \( z_n \)-variable. Therefore, \( F \) extends to a holomorphic automorphism of \( \mathbb{C}^n \). Hence, we obtain that \( M \) is holomorphically equivalent to \( D^R_r \) for some \( 0 \leq r < R \leq \infty \) and therefore either to
\[
D_{r/R, 1} := \left\{ (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : r/R \exp \left( |z'|^2 \right) < |z_n| < \exp \left( |z'|^2 \right) \right\},
\]
or (for \( R = \infty \)) to
\[
D_{0, -1} := \left\{ (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : 0 < |z_n| < \exp \left( -|z'|^2 \right) \right\}.
\]
This is, however, impossible since \( d(D_{r/R, 1}) = d(D_{0, -1}) = n^2 \) (see e.g. \([\text{I1}]\)).

Assume now that \( m = \omega \). In this case \( D \) is the cylinder
\[
C := \left\{ (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'| < 1 \right\}.
\]
The \( R_\omega \)-orbit of every point in \( C \) has the form
\[
\left\{ (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'|^2 + r \exp (\Re z_n) = 1 \right\},
\]
where $r > 0$, and any $R_\omega$-invariant domain in $\mathcal{C}$ is of the form

\[
\Omega_r^R : = \left\{ (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'| < 1, \quad r(1 - |z'|^2) < \exp (\text{Re } z_n) < R(1 - |z'|^2) \right\},
\]

for $0 \leq r < R \leq \infty$. Every $CR$-isomorphism between two $R_\omega$-orbits is a composition of a map from of the form (2.7) and a translation in the $z_n$-variable. Therefore, $F$ extends to a holomorphic automorphism of $\mathcal{C}$. In this case $M$ is holomorphically equivalent to $\Omega_r^R$ for some $0 \leq r < R \leq \infty$, and hence either to

\[
\Omega_{r/R,1} : = \left\{ (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'| < 1, r/R(1 - |z'|^2) < \exp (\text{Re } z_n) < (1 - |z'|^2)^{-1} \right\},
\]

or (for $R = \infty$) to

\[
\Omega_{0,-1} : = \left\{ (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'| < 1, \exp (\text{Re } z_n) < (1 - |z'|^2)^{-1} \right\}.
\]

As before, this is impossible since $d(\Omega_{r/R,1}) = d(\Omega_{0,-1}) = n^2$ (see e.g. [11]).

Assume now that $m = \varepsilon_\alpha$ for some $\alpha > 0$. Here $\mathcal{D}$ is the domain $\mathcal{C}' : = \mathcal{C} \setminus \{ z_n = 0 \}$. The $R_\varepsilon_\alpha$-orbit of every point in $\mathcal{C}'$ is of the form

\[
\{ (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'|^2 + r|z_n|^\alpha = 1, z_n \neq 0 \},
\]

where $r > 0$, and every $R_\varepsilon_\alpha$-invariant domain in $\mathcal{C}'$ is given by

\[
\mathcal{E}_r^{R,1/\alpha} : = \left\{ (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'| < 1, \quad r(1 - |z'|^2)^{1/\alpha} < |z_n| < R(1 - |z'|^2)^{1/\alpha} \right\},
\]

for $0 \leq r < R \leq \infty$. Since every $CR$-isomorphism between $R_\varepsilon_\alpha$-orbits is a composition of a map of the form (2.11) and a dilation in the $z_n$-variable, the map $F$ extends to an automorphism of $\mathcal{C}'$. Thus, we have shown that $M$ is holomorphically equivalent to $\mathcal{E}_r^{R,1/\alpha}$ for some $0 \leq r < R \leq \infty$, and hence either to

\[
\mathcal{E}_{r/R,1/\alpha} : = \left\{ (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'| < 1, r(1 - |z'|^2)^{1/\alpha} < |z_n| < (1 - |z'|^2)^{1/\alpha} \right\},
\]
or (for $R = \infty$) to

$$\mathcal{E}_{0,-1/\alpha} := \left\{ (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'| < 1, 0 < |z_n| < (1 - |z'|^2)^{-1/\alpha} \right\}.$$ 

As above, this is impossible since $d(\mathcal{E}_{r,R,1/\alpha}) = d(\mathcal{E}_{0,-1/\alpha}) = n^2$ (see e.g. [II]).

Assume now that both real and complex hypersurface orbits are present in $M$. Since the action of $G(M)$ on $M$ is proper, it follows that the orbit space $M/G(M)$ is homeomorphic to one of the following: $\mathbb{R}$, $S^1$, $[0,1]$, $[0,1)$ (see [M, B-B, AA1, AA2]), and thus there can be no more than two complex hypersurface orbits in $M$. It follows from (iii) of Proposition [I] and Lemma [2.3] that the model for every real hypersurface orbit is $\varepsilon_\alpha$ for some $\alpha > 0$, $\alpha \in \mathbb{Q}$. Let $M'$ be the manifold obtained from $M$ by removing all complex hypersurface orbits. It then follows from the above considerations that $M'$ is holomorphically equivalent to $\mathcal{E}_{r,1/\alpha}^R$ for some $0 \leq r < R \leq \infty$.

Let $f : M' \to \mathcal{E}_{r,1/\alpha}^R$ be a biholomorphic map satisfying (2.12) for all $g \in G(M)$, $q \in M'$ and some isomorphism $\varphi : G(M) \to R_{\varepsilon_\alpha}$. The group $R_{\varepsilon_\alpha}$ in fact acts on all of $\mathcal{C}$, and the orbit of any point in $\mathcal{C}$ with $z_n = 0$ is the complex hypersurface

$$c_0 := \left\{ (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'| < 1, z_n = 0 \right\}.$$ 

For a point $s \in \mathcal{C}$ denote by $J_s$ the isotropy subgroup of $s$ under the action of $R_{\varepsilon_\alpha}$. If $s_0 \in c_0$ and $s_0 = (z'_0, 0)$, $J_{s_0}$ is isomorphic to $H_{k_1,k_2}^n$, where $k_1/k_2 = 2/\alpha n$ and consists of all maps of the form (2.9) for which the transformations in the $z'$-variables form the isotropy subgroup of the point $z'_0$ in $\text{Aut}(B^{n-1})$.

Fix $s_0 = (z'_0, 0) \in c_0$ and let

$$N_{s_0} := \left\{ s \in \mathcal{E}_{r,1/\alpha}^R : J_s \subset J_{s_0} \right\}.$$ 

We have

$$N_{s_0} = \left\{ (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : z' = z'_0, r(1 - |z'_0|^2)^{1/\alpha} < |z_n| < R(1 - |z'_0|^2)^{1/\alpha} \right\}.$$ 

Thus, $N_{s_0}$ is either an annulus (possibly, with an infinite outer radius) or a punctured disk. In particular, $N_{s_0}$ is a complex curve in $\mathcal{C}'$. Since $J_{s_0}$ is a maximal compact subgroup of $R_{\varepsilon_\alpha}$, $\varphi^{-1}(J_{s_0})$ is a maximal compact subgroup of $G(M)$. Let $O$ be a complex hypersurface orbit in $M$. For $q \in O$ the subgroup $I_q$ is compact and has dimension $(n - 1)^2 = \dim J_{s_0}$. Therefore, $\varphi^{-1}(J_{s_0})$ is conjugate to $I_q$ for every $q \in O$ (in particular, $I_q$ is connected),
and hence there exists \( q_0 \in O \) such that \( \varphi^{-1}(J_{s_0}) = I_{g_0} \). Since the isotropy subgroups in \( R_{\varepsilon_0} \) of distinct points in \( c_0 \) do not coincide, such a point \( q_0 \) is unique.

Let

\[
K_{q_0} := \{ q \in M' : I_q \subset I_{q_0} \}.
\]

Clearly, \( K_{q_0} = f^{-1}(N_{s_0}) \). Thus, \( K_{q_0} \) is a \( I_{q_0} \)-invariant complex curve in \( M' \) equivalent to either an annulus or a punctured disk. By Bochner’s theorem there exist a local holomorphic change of coordinates \( F \) near \( q_0 \) on \( M \) that identifies an \( I_{q_0} \)-invariant neighborhood \( U \) of \( q_0 \) with an \( L_{q_0} \)-invariant neighborhood of the origin in \( T_{q_0}(M) \) such that \( F(q_0) = 0 \) and \( F(qg) = \alpha_{q_0}(g)F(q) \) for all \( g \in I_{q_0} \) and \( q \in U \) (here \( L_{q_0} \) is the linear isotropy group and \( \alpha_{q_0} \) is the isotropy representation at \( q_0 \)). In the proof of Proposition 1.1 (see Case 1) we have seen that \( L_{q_0} \) has two invariant subspaces in \( T_{q_0}(M) \). One of them corresponds in our coordinates to \( O \), the other to a complex curve \( C \) intersecting \( O \) at \( q_0 \). Observe that near \( q_0 \) the curve \( C \) coincides with \( K_{q_0} \cup \{ q_0 \} \). Therefore, in a neighborhood of \( q_0 \) the curve \( K_{q_0} \) is a punctured analytic disk. Further, if a sequence \( \{ q_n \} \) from \( K_{q_0} \) accumulates to \( q_0 \), the sequence \( \{ f(q_n) \} \) accumulates to one of the two ends of \( N_{s_0} \), and therefore we have either \( r = 0 \) or \( R = \infty \). Since both these conditions cannot be satisfied simultaneously due to hyperbolicity of \( M \), we conclude that \( O \) is the only complex hypersurface orbit in \( M \).

Assume first that \( r = 0 \). We will extend \( f \) to a map from \( M \) onto the domain

\[
\left\{ (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'|^2 + \frac{1}{R}|z_n|^\alpha < 1 \right\}
\]  

(2.14)

by setting \( f(q_0) = s_0 \), where \( q_0 \in O \) and \( s_0 \in c_0 \) are related as specified above. The extended map is one-to-one and satisfies (2.12) for all \( g \in G(M), q \in M \). To prove that \( f \) is holomorphic on all of \( M \), it suffices to show that \( f \) is continuous on \( O \). It will be more convenient for us to show that \( f^{-1} \) is continuous on \( c_0 \). Let first \( \{ s_j \} \) be a sequence of points in \( c_0 \) converging to \( s_0 \). Then there exists a sequence \( \{ q_j \} \) of elements of \( R_{s_0} \) converging to the identity such that \( s_j = g_j s_0 \) for all \( j \). Then \( f^{-1}(s_j) = \varphi^{-1}(g_j) q_0 \), and, since \( \{ \varphi^{-1}(g_j) \} \) converges to the identity, we obtain that \( \{ f^{-1}(s_j) \} \) converges to \( q_0 \). Next, let \( \{ s_j \} \) be a sequence of points in \( E_{s_0}^R \) converging to \( s_0 \). Then we can find a sequence \( \{ q_j \} \) of elements of \( R_{s_0} \) converging to the identity such that \( g_j s_j \in N_{s_0} \) for all \( j \). Clearly, the sequence \( \{ f^{-1}(g_j s_j) \} \) converges to \( q_0 \), and hence the sequence \( \{ f^{-1}(s_j) \} \) converges to \( q_0 \) as well. Thus, we have
shown that $M$ is holomorphically equivalent to domain (2.14) and hence to the domain

$$E_{\alpha} := \{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'|^2 + |z_n|^\alpha < 1\}.$$  

This is, however, impossible since $d(E_{\alpha}) \geq n^2$.

Assume now that $R = \infty$. Observe that the action of the group $R_{\varepsilon_{\alpha}}$ on $\mathcal{C}$ extends to an action on $\tilde{\mathcal{C}} := B^{n-1} \times \mathbb{CP}^1$ by holomorphic transformations by setting $g(z', \infty) := (a(z'), \infty)$ for every $g \in R_{\varepsilon_{\alpha}}$, where $a$ is the corresponding automorphism of $B^{n-1}$ in the $z'$-variables (see formula (2.9)). Now arguing as in the case $r = 0$, we can extend $f$ to a biholomorphic map between $M$ and the domain in $\tilde{\mathcal{C}}$

$$\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'| < 1, |z_n| > r(1 - |z'|^2)^{1/\alpha}\} \cup (B^{n-1} \times \{\infty\}).$$

This domain is holomorphically equivalent to

$$E_{-1/\alpha} := \{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'| < 1, |z_n| < (1 - |z'|^2)^{-1/\alpha}\},$$

and so is $M$. This is, however, impossible since $d(E_{-1/\alpha}) = n^2$.

The proof of Proposition 2.1 is complete. □

3 The Case of Complex Hypersurface Orbits

We will now assume that all orbits in $M$ are complex hypersurfaces. As we have shown above, this is always the case for $n \geq 3$, unless $M$ is homogeneous. We will prove the following proposition.

**Proposition 3.1** Let $M$ be a connected hyperbolic manifold of dimension $n \geq 3$ with $d(M) = n^2 - 1$, and such that for every $p \in M$ its orbit $O(p)$ is a complex hypersurface in $M$. Then $M$ is holomorphically equivalent to $B^{n-1} \times S$, where $S$ is a hyperbolic Riemann surface with $d(S) = 0$.

**Proof:** Fix $p \in M$. It then follows from (iii) of Proposition 1.1 that $I_p^c$ is isomorphic to $U_{n-1}$, moreover, one can choose coordinates $(w_1, \ldots, w_n)$ in $T_p(M)$ so that $L_p^c$ consists of all matrices of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix},$$

(3.1)
where \( B \in U_{n-1} \) and \( T_p(O(p)) = \{ w_1 = 0 \} \). Arguing as in the proof of Lemma 4.4 of [Kru1] we obtain that the full group \( L_p \) consists of all matrices of the form
\[
\begin{pmatrix}
\alpha & 0 \\
0 & B
\end{pmatrix},
\]
where \( B \in U_{n-1} \) and \( \alpha^m = 1 \) for some \( m \geq 1 \). It then follows (see e.g. Satz 4.3 of [Ka]) that the kernel of the action of \( G(M) \) on \( O(p) \) is \( J_p := \alpha_p^{-1}(Z_m) \), where we identify \( Z_m \) with the subgroup of \( L_p \) that consists of all matrices of the form (3.2) with \( B = \text{id} \). Thus, \( G(M)/J_p \) acts effectively on \( O(p) \). Since \( O(p) \) is holomorphically equivalent to \( B^{n-1} \) and \( \dim G(M) = n^2 - 1 = \dim \text{Aut}(B^{n-1}) \), we obtain that \( G(M)/J_p \) is isomorphic to \( \text{Aut}(B^{n-1}) \). It then follows that \( I_p \) is a maximal compact subgroup in \( G(M) \) since its image under the projection \( G(M) \rightarrow \text{Aut}(B^{n-1}) \) is a maximal compact subgroup of \( \text{Aut}(B^{n-1}) \). However, every maximal compact subgroup of a connected Lie group is connected whereas \( I_p \) is not if \( m > 1 \). Thus, \( m = 1 \), hence \( G(M) \) is isomorphic to \( \text{Aut}(B^{n-1}) \). In particular, \( L_p \) fixes every point of the orthogonal complement \( W_p \) to \( T_p(O(p)) \) in \( T_p(M) \). Observe that the above arguments apply to every point in \( M \).

Define
\[
N_p := \{ s \in M : I_s = I_p \}.
\]
Clearly, \( I_p \) fixes every point in \( N_p \) and \( N_{gp} = gN_p \) for all \( g \in G(M) \). Further, since for two distinct points \( s_1, s_2 \) lying in the same orbit we have \( I_{s_1} \neq I_{s_2} \), the set \( N_p \) intersects every orbit in \( M \) at exactly one point. By Bochner’s theorem there exist a local holomorphic change of coordinates \( F \) near \( p \) on \( M \) that identifies an \( I_p \)-invariant neighborhood \( U \) of \( p \) with an \( L_p \)-invariant neighborhood \( V \) of the origin in \( T_p(M) \) such that \( F(p) = 0 \) and \( F(gq) = \alpha_p(g)F(q) \) for all \( g \in I_p \) and \( q \in U \). Since \( L_p \) coincides with the group of matrices of the form (3.1), \( N_p \cap U = F^{-1}(W_p \cap V) \). In particular, \( N_p \) is a complex curve near \( p \). Since the same argument can be carried out at every point of \( N_p \), we obtain that \( N_p \) is a closed complex hyperbolic curve in \( M \).

We will now construct a biholomorphic map \( \Phi : M \rightarrow B^{n-1} \times N_p \). Let \( \Psi : O(p) \rightarrow B^{n-1} \) be a biholomorphism. For \( q \in M \) let \( r \) be the (unique) point where \( N_p \) intersects \( O(q) \). Let \( g \in G(M) \) be such that \( q = gr \). Then we set \( \Phi(q) := (F(gp), r) \). By construction, \( \Phi \) is biholomorphic. Since \( M \) is holomorphically equivalent to \( B^{n-1} \times N_p \), we have \( d(N_p) = 0 \).

The proof is complete.
4 The Homogeneous Case

In this section we will prove the following proposition.

**Proposition 4.1** If $M$ is a homogeneous connected hyperbolic manifold of dimension $n \geq 2$ with $d(M) = n^2 - 1$, then $n = 4$ and $M$ is holomorphically equivalent to the tube domain

$$T = \left\{ (w_1, w_2, w_3, w_4) \in \mathbb{C}^4 : (\text{Im } w_1)^2 + (\text{Im } w_2)^2 + (\text{Im } w_3)^2 - (\text{Im } w_4)^2 < 0, \text{Im } w_4 > 0 \right\}.$$

**Proof:** The proof is similar to that of Proposition 5.1 of [1]. Since $M$ is homogeneous, by [N], [P-S], it is holomorphically equivalent to a Siegel domain $U$ of the second kind in $\mathbb{C}^n$. For $n = 2$, this gives that $M$ is equivalent to either $B^2$ or $\Delta^2$, which is impossible since $d(B^2) = 8$ and $d(\Delta^2) = 6$. For $n = 3$ we obtain that $M$ is equivalent to one of the following domains: $B^3$, $B^2 \times \Delta$, $\Delta^3$, $S$, where $S$ is the 3-dimensional Siegel space. None of these domains has an automorphism group of dimension 8.

Assume now that $n \geq 4$. The domain $U$ has the form

$$U = \left\{ (z, w) \in \mathbb{C}^{n-k} \times \mathbb{C}^k : \text{Im } w - F(z, z) \in C \right\},$$

where $1 \leq k \leq n$, $C$ is an open convex cone in $\mathbb{R}^k$ not containing an entire affine line and $F = (F_1, \ldots, F_k)$ is a $\mathbb{C}^k$-valued Hermitian form on $\mathbb{C}^{n-k} \times \mathbb{C}^{n-k}$ such that $F(z, z) \in \overline{C} \setminus \{0\}$ for all non-zero $z \in \mathbb{C}^{n-k}$.

We will show first that in most cases we have $k \leq 2$. As we noted in [Kra]

$$d(U) \leq 4n - 2k + \text{dim } \mathfrak{g}_0(U). \quad (4.1)$$

Here $\mathfrak{g}_0(U)$ is the Lie algebra of all vector fields on $\mathbb{C}^n$ of the form

$$X_{A,B} = Az \frac{\partial}{\partial z} + Bw \frac{\partial}{\partial w},$$

where $A \in \mathfrak{gl}_{n-k}(\mathbb{C})$, $B$ belongs to the Lie algebra $\mathfrak{g}(C)$ of the group $G(C)$ of linear automorphisms of the cone $C$, and the following holds

$$F(Az, z) + F(z, Az) = BF(z, z), \quad (4.2)$$

for all $z \in \mathbb{C}^{n-k}$. By the definition of Siegel domain, there exists a positive-definite linear combination $R$ of the components of the Hermitian form $F$. 
Then, for a fixed matrix $B$ in formula (4.2), the matrix $A$ is determined at most up to a matrix that is skew-Hermitian with respect to $R$. Since the dimension of the algebra of matrices skew-Hermitian with respect to $R$ is equal to $(n - k)^2$, we have

$$\dim g_0(U) \leq (n - k)^2 + \dim g(C).$$

(4.3)

In Lemma 3.2 of [IKra] we showed that

$$\dim g(C) \leq \frac{k^2}{2} - \frac{k}{2} + 1. \quad (4.4)$$

It now follows from (4.3) and (4.4) that the following holds

$$\dim g_0(U) \leq \frac{3k^2}{2} - k \left(2n + \frac{1}{2}\right) + n^2 + 1,$$

which together with (4.1) for gives

$$d(U) \leq \frac{3k^2}{2} - k \left(2n + \frac{5}{2}\right) + n^2 + 4n + 1. \quad (4.5)$$

It is straightforward to check that the right-hand side of (4.5) is strictly less than $n^2 - 1$ if $k \geq 3$ for $n \geq 5$, and does not exceed 15 for $n = 4$. Furthermore, for $n = 4$ the right-hand side of (4.5) is equal to 15 only if $k = 3$ or $k = 4$ and $\dim g(C) = k^2/2 - k/2 + 1$.

Suppose that $n = 4$ and the right-hand side of (4.5) is equal to 15. In this case for every point $x_0 \in C$ there exist coordinates in $\mathbb{R}^k$ such that the isotropy subgroup of $x_0$ in $G(C)$ contains $SO_{k-1}(\mathbb{R})$ (see the proof of Lemma 3.2 in [IKra]). Then after a linear change of coordinates the cone $C$ takes the form

$$\{ x = (x_1, \ldots, x_k) \in \mathbb{R}^k : \langle x, x \rangle < 0, \ x_k > 0 \},$$

where $\langle x, x \rangle := x_1^2 + \ldots + x_{k-1}^2 - x_k^2$. In these coordinates the algebra $g(C)$ is generated by the subalgebra of scalar matrices in $gl_k(\mathbb{R})$ and the algebra of pseudo-orthogonal matrices $\mathfrak{o}_{k-1,1}(\mathbb{R})$. Assume first that $k = 3$. Then we have $F = (v_1|z|^2, v_2|z|^2, v_3|z|^2)$ for some vector $v := (v_1, v_2, v_3) \in C$. It follows from (4.2) that $v$ is an eigenvector of the matrix $B$ for every $X_{A,B} \in g_0(U)$, which implies that $\dim g_0(U) = 3$. Hence by (4.1) we have $\dim \text{Aut}(U) \leq 13$, which is impossible.
Suppose now that \(k = 4\). In this case \(U\) is holomorphically equivalent to the tube domain \(T\). Let \(g(T)\) be the Lie algebra of \(\text{Aut}(T)\). It follows from the results of [KMO] that \(g(T)\) is a graded Lie algebra

\[
g(T) = g_{-1}(T) \oplus g_0(T) \oplus g_1(T),
\]

where \(g_{-1}\) is spanned by \(i\partial/\partial w_j, j = 1, 2, 3, 4\), and \(\dim g_1(T) \leq 4\). Clearly, \(g_0(T)\) is isomorphic to \(\mathbb{R} \oplus o_{3,1}(\mathbb{R})\) and thus has dimension 7. The component \(g_1(T)\) also admits an explicit description (see e.g. p. 218 in [S]). It follows from this description that \(g_1(T)\) consists of all vector fields of the form

\[
Z_{\alpha, \beta, \gamma, \delta} := \left(\alpha(w_1^2 - w_2^2 - w_3^2 + w_4^2) + 2(\beta w_1 w_2 + \gamma w_1 w_3 + \delta w_1 w_4)\right)\frac{\partial}{\partial w_1} + \\
\left(\beta(-w_1^2 + w_2^2 - w_3^2 + w_4^2) + 2(\alpha w_1 w_2 + \gamma w_2 w_3 + \delta w_2 w_4)\right)\frac{\partial}{\partial w_2} + \\
\left(\gamma(-w_1^2 - w_2^2 + w_3^2 + w_4^2) + 2(\alpha w_1 w_3 + \beta w_2 w_3 + \delta w_3 w_4)\right)\frac{\partial}{\partial w_3} + \\
\left(\delta(w_1^2 + w_2^2 - w_3^2 + w_4^2) + 2(\alpha w_1 w_4 + \beta w_2 w_4 + \gamma w_3 w_4)\right)\frac{\partial}{\partial w_4},
\]

where \(\alpha, \beta, \gamma, \delta \in \mathbb{R}\), and thus has dimension 4. Therefore, \(\dim \text{Aut}(T) = 15\).

It is also clear that \(T\) is homogeneous under affine automorphisms.

Assume now that \(n \geq 4\) is arbitrary and \(k \leq 2\). If \(k = 1\), the domain \(U\) is equivalent to \(B^n\) which is impossible. Hence \(k = 2\). It follows from (4.2) that the matrix \(A\) is determined by the matrix \(B\) up to a matrix \(L \in \mathfrak{gl}_{n-2}(\mathbb{C})\) satisfying

\[
F(Lz, z) + F(z, Lz) = 0,
\]

for all \(z \in \mathbb{C}^{n-2}\). Let \(s\) be the dimension of the subspace of all such matrices \(L\). Then

\[
\dim \mathfrak{g}_0(U) \leq s + \dim \mathfrak{g}(C),
\]

and (4.4) yields

\[
\dim \mathfrak{g}_0(U) \leq s + 2,
\]

which together with (4.1) implies

\[
s \geq n^2 - 4n + 1. \tag{4.6}
\]
By the definition of Siegel domain, there exists a positive-definite linear combination of the components of $F$, and we can assume that $F_1$ is positive-definite. Further, applying an appropriate linear transformation of the $z$-variables, we can assume that $F_1$ is given by the identity matrix and $F_2$ by a diagonal matrix.

Suppose first that the matrix of $F_2$ is scalar. If $F_2 \equiv 0$, then $U$ is holomorphically equivalent to $B^{n-1} \times \Delta$ which is impossible. If $F_2 \not\equiv 0$, then $U$ is holomorphically equivalent to the domain

$$V := \{(z, w) \in \mathbb{C}^{n-2} \times \mathbb{C}^2 : \text{Im } w_1 - |z|^2 > 0, \text{Im } w_2 - |z|^2 > 0\}.$$  

It was shown in [IKra] that $d(V) \leq n^2 - 2n + 3$ and hence $d(V) < n^2 - 1$. Thus, the matrix of $F_2$ is not scalar. Inequality (4.6) now yields that the matrix of $F_2$ can have at most one pair of distinct eigenvalues, and therefore $n = 4$ and $U$ is holomorphically equivalent to $B^2 \times B^2$. This is clearly impossible, and the proof of the proposition is complete. \hfill \Box

5 Examples for the Case $n = 2$, $d(M) = 3$

In this section we give examples of families of hyperbolic domains in $\mathbb{C}^2$ and $\mathbb{C}P^2$ with automorphism groups of dimension 3 whose orbit structure is different from that observed above for $n \geq 3$. Define

$$\Omega_t := \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 - 1 < t|z|^2 + w^2 - 1\},$$

where $0 < t \leq 1$. Clearly, $\Omega_t$ is bounded if $0 < t < 1$. Further, $\Omega_1$ is hyperbolic since it is contained in the hyperbolic product domain

$$\{(z, w) \in \mathbb{C}^2 : z, w \not\in (-\infty, -1] \cup [1, \infty)\}.$$

The group $\text{Aut}(\Omega_t)$ for every $t$ consists of the maps

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

where

$$Q := \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ c_1 & c_2 & d \end{pmatrix} \in SO_{2,1}(\mathbb{R}),$$

(5.1)
and thus is 3-dimensional. The group $\text{Aut}(\Omega_t)$ has two connected components (that correspond to the connected components of $SO_{2,1}(\mathbb{R})$), and its identity component $G(\Omega_t)$ is given by the condition $a_{11}a_{22} - a_{12}a_{21} > 0$. The orbits of $G(\Omega_t)$ on $\Omega_t$ are as follows:

$$O^\Omega_\alpha := \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 - 1 = \alpha |1 + z^2 - w^2| \} \setminus \{(x, u) \in \mathbb{R}^2 : x^2 + u^2 = 1\}, \quad -1 < \alpha < t,$$

$$\Delta_\mathbb{R} := \{(x, u) \in \mathbb{R}^2 : x^2 + u^2 < 1\}.$$

Note that $O^\Omega_0$ is the only spherical real hypersurface orbit in $\Omega_t$ and that $\Delta_\mathbb{R}$ is a totally real orbit. All the orbits are pairwise $CR$ non-equivalent.

The next family of domains is associated with a different action of $SO_{2,1}(\mathbb{R})$ on a part of $\mathbb{C}^2$. Define

$$D_t := \{(z, w) \in \mathbb{C}^2 : 1 + |z|^2 - |w|^2 > t|1 + z^2 - w^2|, \quad \text{Im}(1 + \overline{w}) > 0\},$$

where $t \geq 1$. All these domains lie in the hyperbolic product domain

$$\{(z, w) \in \mathbb{C}^2 : \text{Im} z > 0, \quad w \not\in \langle -\infty, -1 \rangle \cup \langle 1, \infty \rangle\},$$

hence they are hyperbolic as well. For every matrix $Q \in SO_{2,1}(\mathbb{R})$ as in (5.1) consider the map

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} a_{22} & b_2 \\ c_2 & d \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} + \begin{pmatrix} a_{21} \\ c_1 \end{pmatrix},$$

$$a_{12}z + b_1w + a_{11}.$$

The group $\text{Aut}(D_t) = G(D_t)$ for every $t$ consists of all such maps. The orbits of $G(D_t)$ on $D_t$ are the following non-spherical hypersurfaces

$$O^D_\alpha := \{(z, w) \in \mathbb{C}^2 : 1 + |z|^2 - |w|^2 = \alpha |1 + z^2 - w^2|, \quad \text{Im} z(1 + \overline{w}) > 0\}, \quad \alpha > t.$$

All the orbits are pairwise $CR$ non-equivalent.

The next family of domains is associated with an action of $SO_3(\mathbb{R})$ on $\mathbb{CP}^2$. Define

$$E_t := \{(z : w : \zeta) \in \mathbb{CP}^2 : |z|^2 + |w|^2 + |\zeta|^2 < t|z^2 + w^2 + \zeta^2|\},$$
where $t > 1$. The domain $E_t$ is hyperbolic for each $t$ since it is covered in a 2-to-1 fashion by the manifold

$$\left\{(z, w, \zeta) \in \mathbb{C}^3 : |z|^2 + |w|^2 + |\zeta|^2 < t, z^2 + w^2 + \zeta^2 = 1\right\},$$

which is clearly hyperbolic; the covering map is $(z, w, \zeta) \mapsto (z : w : \zeta)$. The group $\text{Aut}(E_t) = G(E_t)$ for every $t$ is given by applying matrices from $SO_3(\mathbb{R})$ to vectors of homogeneous coordinates. The action of the group $G(E_t)$ on $E_t$ has the totally real orbit $\mathbb{R}P^2$, and the rest of the orbits are the following non-spherical hypersurfaces

$$O_\alpha^E := \left\{(z : w : \zeta) \in \mathbb{CP}^2 : |z|^2 + |w|^2 + |\zeta|^2 = \alpha |z^2 + w^2 + \zeta^2|\right\}, \quad 1 < \alpha < t.$$

All the orbits are pairwise $CR$ non-equivalent.

Next, define

$$S_t := \{(z, w) \in \mathbb{C}^2 : (\Re z)^2 + (\Re w)^2 < t\},$$

where $t > 0$. All these domains are clearly hyperbolic and the group $\text{Aut}(S_t)$ for every $t$ consists of all maps of the form

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto C \begin{pmatrix} z \\ w \end{pmatrix} + i \begin{pmatrix} p \\ q \end{pmatrix},$$

where $C \in O_2(\mathbb{R})$ and $p, q \in \mathbb{R}$. The group $G(S_t)$ is given by matrices $C \in SO_2(\mathbb{R})$. The action of the group $G(S_t)$ on $S_t$ has the totally real orbit

$$\{(z, w) \in \mathbb{C}^2 : \Re z = 0, \Re w = 0\},$$

and the rest of the orbits are the following non-spherical tube hypersurfaces

$$O_\alpha^S := \{(z, w) \in \mathbb{C}^2 : (\Re z)^2 + (\Re w)^2 = \alpha\}, \quad 0 < \alpha < t.$$  

Every non-spherical orbit is clearly $CR$-equivalent to $O_1^S$.

Now fix $a \in \mathbb{R}$ such that $|a| > 1$, $a \neq 1, 2$, and consider the following family of tube domains

$$R_{a,t} := \{(z, w) \in \mathbb{C}^2 : \Re z < t (\Re w)^a, \Re w > 0\},$$

where $t > 0$. All these domains are obviously hyperbolic and the group $\text{Aut}(R_{a,t}) = G(R_{a,t})$ consists of all the maps

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} \lambda^a z \\ \lambda w \end{pmatrix} + i \begin{pmatrix} p \\ q \end{pmatrix},$$
where $\lambda > 0$ and $p, q \in \mathbb{R}$. The action of this group on $R_{a,t}$ has the Levi-flat orbit
$$\{(z, w) \in \mathbb{C}^2 : \text{Re } z = 0, \text{Re } w > 0\},$$
which is foliated by the half-planes
$$\{(z, w) \in \mathbb{C}^2 : z = ic, \text{Re } w > 0\}, \quad c \in \mathbb{R}.$$ All other orbits are the following non-spherical hypersurfaces

$$O^{R}_{a,\alpha} := \{(z, w) \in \mathbb{C}^2 : \text{Re } z = \alpha (\text{Re } w)^a, \text{Re } w > 0\}, \quad \alpha < t, \alpha \neq 0.$$ Every non-spherical orbit is $CR$-equivalent to $O^{R}_{a,1}$.

Further, define
$$U_t := \{(z, w) \in \mathbb{C}^2 : \text{Re } z < \text{Re } w \cdot \ln (t \text{Re } w), \text{Re } w > 0\},$$
where $t > 0$. All these domains are clearly hyperbolic and the group $\text{Aut}(U_t) = G(U_t)$ consists of all the maps
$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} \lambda z + (\lambda \ln \lambda) w \\ \lambda w \end{pmatrix} + i \begin{pmatrix} p \\ q \end{pmatrix},$$
where $\lambda > 0$ and $p, q \in \mathbb{R}$. The orbits of $G(U_t)$ on $U_t$ are the following non-spherical hypersurfaces

$$O^{U}_{\alpha} := \{(z, w) \in \mathbb{C}^2 : \text{Re } z = \text{Re } w \cdot \ln (\alpha \text{Re } w), \text{Re } w > 0\}, \quad 0 < \alpha < t.$$ Every orbit is $CR$-equivalent to $O^{U}_{1}$.

Finally, fix $a > 0$ and consider
$$V_{a,t,s} := \{(z, w) \in \mathbb{C}^2 : se^{a\varphi} < r < te^{a\varphi}\},$$
where $t > 0$, $e^{-2\pi a t} < s < t$, and $(r, \varphi)$ denote the polar coordinates in the $(\text{Re } z, \text{Re } w)$-plane with $\varphi$ varying from $-\infty$ to $\infty$ (thus, the boundary of $V_{a,t,s}$ consists of two infinite spirals). All these domains are hyperbolic and $\text{Aut}(V_{a,t,s}) = G(V_{a,t,s})$ consists of all maps of the form
$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto e^{a\varphi} \begin{pmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} + i \begin{pmatrix} p \\ q \end{pmatrix},$$
where $\beta, p, q \in \mathbb{R}$. The orbits under the action of $G(V_{a,t,s})$ on $V_{a,t,s}$ are the following non-spherical hypersurfaces

$$O_{V,a}^V := \{(z, w) \in \mathbb{C}^2 : r = e^{a \varphi} \}, \quad s < \alpha < t.$$ 

Clearly, every orbit is $CR$-equivalent to $O_{V,a}^V$.

The orbits $O_{\alpha}^O$ with $-1 < \alpha < 1$ and $\alpha \neq 0$, $O_{\alpha}^D$ with $\alpha > 1$, $O_{\alpha}^E$ with $\alpha > 1$, $O_{1}^S$, $O_{a,1}^R$ with $|a| > 1$ and $a \neq 1, 2$, $O_{V}^V$, and $O_{V,a}^V$ with $a > 0$ are part of E. Cartan’s classification of homogeneous hypersurfaces in the non-spherical case (see [C]). They are pairwise $CR$ non-equivalent, both locally and globally, and give a complete classification from the local point of view. To obtain a global classification, one has to additionally consider all possible covers of these hypersurfaces (see [I2]).

We will now give an example of a hyperbolic domain in $\mathbb{C}^2$, for which almost every orbit is spherical. Define

$$W_t := \{(z, w) \in \mathbb{C}^2 : \text{Re } w > |z|^2 + t \text{Re } z^2, \text{Re } z > 0\},$$

where $t \in \mathbb{R}$. This domain is hyperbolic since for $t \geq -2$ it is equivalent to a subdomain of the hyperbolic product domain

$$\{(z, w) \in \mathbb{C}^2 : \text{Re } z > 0, \text{Re } w > 0\},$$

and for $t < -2$ it is equivalent to the hyperbolic domain

$$\{(z, w) \in \mathbb{C}^2 : \text{Re } w < |z|^2, \text{Re } z > 0\}.$$ 

The group $\text{Aut}(W_t) = G(W_t)$ consists of the maps

$$z \mapsto \lambda z + ia,$$

$$w \mapsto \lambda^2 w - 2i\lambda az + a^2 + i\beta,$$

where $\lambda > 0, a, \beta \in \mathbb{R}$ (cf. (2.3)). The action of this group on $W_t$ has the Levi-flat orbit

$$\{(z, w) \in \mathbb{C}^2 : \text{Re } (w + z^2) = 0, \text{Re } z > 0\},$$

which is foliated by the complex curves

$$\{(z, w) \in \mathbb{C}^2 : w + z^2 = ic, \text{Re } z > 0\}, \quad c \in \mathbb{R}.$$ 

All other orbits are the following spherical hypersurfaces

$$O_{a}^W := \{(z, w) \in \mathbb{C}^2 : \text{Re } (w - az^2/2) = |z|^2(1 + \alpha/2), \text{Re } z > 0\}, \quad \alpha > t.$$ 

Clearly, every spherical orbit is $CR$-equivalent to $O_{0}^W$. 

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Department of Mathematics
The Australian National University
Canberra, ACT 0200
AUSTRALIA
E-mail: alexander.isaev@maths.anu.edu.au