A premouse inheriting strong cardinals from \( V \)

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Abstract

We identify a premouse inner model \( L[E] \), such that for any coarsely iterable background universe \( R \) modelling ZFC, \( L[E]^R \) is a proper class premouse of \( R \) inheriting all strong and Woodin cardinals from \( R \), and iteration trees on \( L[E]^R \) lift to coarse iteration trees on \( R \).

We also prove that a slight weakening of \((k+1)\)-condensation follows from \((k,\omega_1+1)\)-iterability in place of \((k,\omega_1,\omega_1+1)\)-iterability. We also prove that full \((k+1)\)-condensation follows from \((k,\omega_1+1)\)-iterability and \((k+1)\)-solidity.

We also prove general facts regarding generalizations of bicephali; these facts are needed in the proofs of the results above.

Keywords: bicephalus, condensation, normal iterability, inner model, strong cardinal

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1. Introduction

Consider fully iterable, sound premice \( M, N \), such that \( \rho^M = \rho = \rho^N \).

Under what circumstances can we deduce that either \( M \sqsubseteq N \) or \( N \sqsubseteq M \)? This conclusion follows if \( \rho \) is a cutpoint of both models. By \cite[Lemma 3.1]{2}, the conclusion follows if \( \rho \) is a regular uncountable cardinal and there is no premouse with a superstrong extender. We will show that if

\[
M \models (\rho^+)^M = N \models (\rho^+)^N
\]

and \( M, N \) have a certain joint iterability property, then \( M = N \).

The joint iterability required, and the proof that \( M = N \), is motivated by the bicephalus argument of \cite[§9]{3}. The bicephali of \cite{3} are structures of the form \( B = (P, E, F) \), where both \( (P, E) \) and \( (P, F) \) are active premice. If \( B \) is an iterable bicephalus and there is no iterable superstrong premouse then \( E = F \) (see \cite[§9]{3} and \cite{5}); the proof is by comparison of \( B \) with itself. In §3, we will consider a more general form of bicephali, including, for example, the structure \( C = (\rho, M, N) \), where \( \rho, M, N \) are as in the previous paragraph. Given that \( C \)

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\footnote{The paper \cite{2} literally deals with premice with Jensen indexing, whereas we deal with Mitchell-Steel indexing. However, the same result still holds.}
is iterable, a comparison of \( C \) with itself will be used to show that, in the above case, \( M = N \).

Hugh Woodin also noticed that generalizations of bicephali can be used in certain fine structural arguments, probably before the author did; see [12]. The bicephali used in [12] have more closure than those considered here, but of course, the premise of [12] are long extender premise. So while there is some overlap, it seems that things are quite different.

We will also consider bicephali \((\rho', M', N')\) in which \( M' \) or \( N' \) might fail to be fully sound. However, we will assume that both \( M', N' \) project to \( \rho' \), are \( \rho' \)-sound, and \( M', N' \) agree below their common value for \((\rho')^+\). If such a bicephalus is iterable, it might be that \( M' \neq N' \), but we will see that in this situation, \( M' \) is an ultrapower of some premouse by an extender in \( \mathbb{E}_+(N') \), or vice versa.

We will also prove similar results regarding cephalanxes, a blend of bicephali and phalanxes. The presence of superstrong premice makes cephalanxes somewhat more subtle than bicephali.

We will give two applications of these results. First, in §4, we consider proving condensation under a normal iterability hypothesis. Let \( k < \omega \), let \( H, M \) be \( k \)-sound premice, let \( \pi : H \rightarrow M \) be a near \( k \)-embedding\(^2\), let \( \rho_{k+1}^H \leq \rho < \rho_k^H \), and suppose that \( H \) is \( \rho \)-sound and \( \rho \leq \text{cr}(\pi) \). We wish to prove \((k+1)\)-condensation holds.\(^3\) Recall that the standard (phalanx-based) proof of condensation relies on the \((k,\omega_1,\omega_1 + 1)\)-iterability of \( M \), through its appeal to the weak Dodd-Jensen property. We wish to reduce this assumption to \((k,\omega_1 + 1)\)-iterability. Given the latter, and the \((k+1)\)-solidity of \( M \), we will deduce the usual conclusion of condensation. Also, without assuming any solidity of \( M \), we will show that a slight weakening of \((k+1)\)-condensation still follows from \((k,\omega_1 + 1)\)-iterability. (But note that the assumption that \( H \) is \( \rho \)-sound entails that \( \rho_{k+1}^H \setminus \rho \) is \((k+1)\)-solid for \( H \).) Since we do not have \((k,\omega_1,\omega_1 + 1)\)-iterability, it is natural to consider the circumstance that \( M \) fail to be \((k+1)\)-solid. (Though on the other hand, the author believes that, at least if \( M \) has no superstrong initial segments, then it is likely that the \((k+1)\)-solidity of \( M \) follows from \((k,\omega_1 + 1)\)-iterability; see §6.) Our proof makes use of bicephali and cephalanxes in place of phalanxes, and avoids using (weak) Dodd-Jensen.\(^4\)

Next, let \( N \) be the output of a typical fully backgrounded \( L[\mathbb{E}] \)-construction. Assuming that various structures associated to the construction are sufficiently iterable, every Woodin cardinal \( \kappa \) is Woodin in \( N \). However, it seems that \( \kappa \) might be strong, but not strong in \( N \). In [10], Steel defined the local \( K^c\)-

\(^2\)Actually we will work with the more general class of \( k \)-lifting embeddings; see 2.1.

\(^3\)Approximately, that is, the “version . . . with \( \rho_{k+1}^H \) replacing \( \rho_k^H \)” in [3, pp. 87–88], or [2, Lemma 1.3], though this uses Jensen indexing, or [13, Theorem 9.3.2], though this uses Jensen indexing and \( \Sigma^*\)-fine structure.

\(^4\)The way we have presented our proof, we do make use of the standard proof of condensation, in proving 2.13, but in circumstances in which Dodd-Jensen is not required. This appeal to the standard proof can, however, be removed, by arranging things more inductively and using the main structure of the proof of 4.2 to prove 2.13.
construction, whose output $M$ inherits both Woodin and strong cardinals. But this construction requires $V$ to be a premouse, and (an important feature which helps ensure that strong cardinals are inherited is that) the background extenders used can be partial. As a consequence, when one lifts iteration trees on $M$ to iteration trees $\mathcal{U}$ on $V$, the tree $\mathcal{U}$ might have drops. In §5, we identify a new form of $L[\mathbb{E}]$-construction $C$, uniquely definable in any ZFC universe $V$ which is coarsely iterable in some larger universe $W$. Letting $L[\mathbb{E}]$ be the final model of $C$, (a) $L[\mathbb{E}]$ is a proper class premouse, (b) if $\delta$ is strong (Woodin), then $\delta$ is strong (Woodin) in $L[\mathbb{E}]$, and (c) working in $W$, $L[\mathbb{E}]$ is iterable, with iteration trees on $L[\mathbb{E}]$ lifting to (coarse) trees on $V$. Thus, we achieve many of the properties of the local $K^c$-construction, but with the advantages that $V$ need not be a premouse, and that trees $\mathcal{U}$ on $V$ resulting from lifting trees on $L[\mathbb{E}]$ are such that for all $\alpha + 1 < \text{lh}(\mathcal{U})$, $E^\mathcal{U}_\alpha$ is a total extender in $M^{\mathcal{U}}_\alpha$. (In the case of the local $K^c$-construction, locally strong cardinals are also inherited, but this does not seem to hold for $C$.)

**Conventions & Notation.**

**General:** The reverse of a finite sequence $\sigma = (x_0, \ldots, x_{n-1})$ is $\sigma^* = (x_{n-1}, \ldots, x_0)$.

The universe $N$ of a first-order structure $M = (N, \ldots)$ is denoted $\lfloor M \rfloor$.

Regarding premice and fine structure, we mostly follow [3] and [11], with some modifications as described below. We also make use of generalized solidity witnesses; see [13, §1.12].

**Premice:** We deal with premice and related structures with Mitchell-Steel indexing, but with extenders of superstrong type permitted on their extender sequence. That is, a super-fine extender sequence $\vec{E}$ is a sequence such that for each $\alpha \in \text{dom}(\vec{E})$, $\vec{E}$ is acceptable at $\alpha$, and if $E_\alpha \neq \emptyset$ then either:

- $E_\alpha$ is a $(\kappa, \alpha)$ pre-extender over $\mathcal{J}_\alpha^{\vec{E}}$ and $E_\alpha$ is the trivial completion of $E_\alpha \upharpoonright \nu(E_\alpha)$ and $E_\alpha$ is not type Z, or
- $\mathcal{J}_\alpha^{\vec{E}}$ has largest cardinal $\nu$ and $E_\alpha$ is a $(\kappa, \nu)$ pre-extender over $\mathcal{J}_\alpha^{\vec{E}}$ and $i_{E_\alpha}(\kappa) = \nu = \nu(E_\alpha)$,

and further, properties 2 and 3 of [11, Definition 2.4] hold. We then define premouse in terms of super-fine extender sequences, in the usual manner. Likewise for related terms, such as segmented-premouse (see [5, §5]). See [9, 2.1–2.6, 2.14] for discussion of the modifications of the general theory needed to deal with these changes.

Let $P$ be a segmented-premouse with active extender $F \neq \emptyset$. We say that $F$, or $P$, has superstrong type iff $i_F(\text{cr}(F)) < \text{lh}(F)$. (So if $F$ has superstrong type then $i_F(\text{cr}(F))$ is the largest cardinal of $P$, and then $P$ is a premouse iff the initial segment condition holds for $P$.) In [5], all premice are assumed to be below superstrong type, but certain results there (in particular, [5, 2.17, 2.20]) hold in our context (allowing superstrong type), by the same proofs, and when we cite these results, we literally refer to these generalizations. (However, [5, Theorem 5.3] does not go through as stated at the superstrong level; Theorem 3.32 of the
present paper generalizes that result at the superstrong level.) At certain points we will explicitly restrict our attention to premice below superstrong type.

Let $P$ be a segmented-premouse. We write $F^P = F(P)$ for the active extender of $P$ (possibly $F^P = \emptyset$), $E^P = E(P)$ for the extender sequence of $P$, excluding $F^P$, and $\mathbb{E}_+^P = \mathbb{E}_+(P) = \mathbb{E}^P \cap F^P$. If $F^P \neq \emptyset$ we write $\ellh(F^P) = OR^P$. (So $\ellh(F^P)$ is the length of $F^P$ when $F^P$ is not of superstrong type.) Given $\alpha \leq OR^P$, we write $P|\alpha$ for the $Q \trianglelefteq P$ such that $OR^Q = \alpha$, and write $P|\alpha = (\{Q\}, \mathbb{E}^Q, 0)$. (We use the same notation for cephalic $P$, given that $\alpha \leq \rho^P$; see 3.5.) If $P$ has a largest cardinal $\delta$, $\lgcd(P)$ denotes $\delta$. If $P$ is active, $\iota(P)$ and $\iota(F^P)$ both denote $\max(\lgcd(P), \nu(F^P))$. So if $P$ is an active premouse then $\iota(P) = \nu(F^P)$. A premouse extender is an extender $F^P$ for some active premouse $P$.

Given two segmented-premice $P, R$ and an ordinal $\alpha \leq \min(OR^P, OR^R)$, define

\[(P \sim R)|\alpha \iff P|\alpha = R|\alpha.\]

We also use the same notation with more than two structures, and also with """ replacing """". (We use the same notation for cephalic $P$; see 3.5.)

**Fine structure:** We officially use Mitchell-Steel fine structure, (a) as modified in [8], and (b) as further modified by using $k$-lifting embeddings in place of weak $k$-embeddings. (Modification (a) involves dropping the objects $u_n$, and defining standard parameters without regard to these objects. The reader who prefers the original Mitchell-Steel fine structure simply need put the relevant $u_n$’s into various hulls and theories. Modification (b) is described in §2.) Let $m < \omega$ and let $Q$ be an $m$-sound premouse. For $i \leq m + 1$ we write $\vec{p}_i^Q = (p_1^Q, \ldots, p_i^Q)$. Let $q \in OR_Q(Q)\subseteq \omega$. We say that $q$ is $(m + 1)$-solid for $Q$ iff for each $\alpha \in q$,

\[Th_{\Sigma_{m+1}}^Q(\alpha \cup (q\backslash(\alpha + 1)) \cup \vec{p}_m^Q) \in Q.\]

Let $\rho \leq OR_Q$. We say $Q$ is $\rho$-sound iff either (i) $\rho_0^Q \leq \rho$ or (ii) $Q$ is $\omega$-sound or (iii) there is $k < \omega$ be such that $Q$ is $k$-sound and $\rho_k^Q \leq \rho < \rho_k^Q$ and $\rho_{k+1}^Q \\rho$ is $(k + 1)$-solid for $Q$ and $Q = \Hull_{k+1}^Q(\rho \cup \{p_k^Q\})$.

ISC stands for “initial segment condition”.

**Extenders:** Given a (long) extender $E$ we write $ms(E)$ for the measure space of $E$; that is, the supremum of all $\kappa + 1$ such that for some $\alpha < \ellh(E)$, $i_E(\kappa) > \alpha$.

See [5, 2.1] for the definition of semi-close (extender).

**Ultrapowers:** Let $E$ be a (possibly long) extender over a segmented-premice $M$. We write $\Ult(M, E)$ for the ultrapower formed by using functions in $M$, without squashing (so $F^{\Ult(M, E)}$ is defined as when $M$ is a type 2 premouse). A ultrapower of $M$ formed in this way is simple.

For $M$ an $n$-sound premouse, we write $\Ult_n(M, E)$ for the degree $n$ ultrapower, with $\Ult_n(M, E) = \Ult_n(M^{seq}, E)^{\text{unseq}}$ if $M$ is type 3.

For $M$ an active segmented-premice, $\Ult^M$ and $\Ult(M)$ both denote $\Ult(M, F^M)$, and $\Ult_k = \Ult_k(M)$ denotes $\Ult_k(M, F^M)$.

For a type 3 premouse $M$, let $\mathcal{E}_{-1}(M) = \mathcal{E}_0(M)$, and for an extender $E$ over $\mathcal{E}_0(M)$, let and $\Ult_{-1}(M, E) = \Ult_0(M, E)$. 


Embeddings: Given structures $X, Y$, if context determines an obvious natural embedding $i: X \to Y$ we write $i_{X,Y}$ for $i$.

Let $M, N$ be segmented-premice. A simple embedding $\pi: M \to N$ is a function $\pi$ with $\text{dom}(\pi) = |M|$ and $\text{cod}(\pi) = |N|$, such that $\pi$ is $\Sigma_0^\text{simple}$-elementary. (Note that if $M$ is active then $\pi(\text{lgcd}(M)) = \text{lgcd}(N)$, because the amenable predicates for $F^M$ and $F^N$ specify the largest cardinal.) If $M, N$ are type $3$ premice, a squashed embedding $\pi: M \to N$ is, literally, a function $\pi$ with $\text{dom}(\pi) = |\mathcal{E}_0(M)|$ and $\text{cod}(\pi) = |\mathcal{E}_0(N)|$, such that $\pi$ is $\Sigma_0^\text{squared}$-elementary (more correctly, $\Sigma_0^\text{q-squared}$-elementary, but we just use the notation “$\Sigma^\text{q}$” for both cases).

Let $\pi: M \to N$ be simple. If $M$ is passive then $\psi_\pi$ denotes $\pi$. If $M$ is active then $\psi_\pi: \text{Ult}(M, F^M) \to \text{Ult}(N, F^N)$ denotes the simple embedding induced by $\pi$ (using the Shift Lemma).

Let $\pi: M \to N$ be squashed. Then $\psi_\pi: \text{Ult}_0(M, F^M) \to \text{Ult}_0(N, F^N)$ denotes the squashed embedding induced by $\pi$.

So $\pi \subseteq \psi_\pi$ in all cases.

We say $\pi: M \to N$ is $\nu$-preserving iff either $M, N$ are passive or $\psi_\pi(\nu(F^M)) = \nu(F^N)$. We say $\pi$ is $\iota$-preserving iff either $M, N$ are passive or $\psi_\pi(\iota(F^M)) = \iota(F^N)$. We say $\pi$ is $\iota$-preserving iff for all $\alpha$, if $\alpha$ is a cardinal of $M$ then $\pi(\alpha)$ is a cardinal of $N$. We say $\pi$ is $p_j$-preserving iff $\pi(p_j^M) = p_j^N$. We say $\pi$ is $p_j$-preserving iff $\pi(p_j^M) = p_j^N$.

Iteration trees: Let $T$ be an iteration tree. Then $T$ is maximal iff it is $k$-maximal for some $k \leq \omega$.

2. Fine structural preliminaries

2.1 Definition. Let $H, M$ be $k$-sound premice with $\rho_k^H, \rho_k^M > \omega$. Let $\mathcal{L}$ be the language of $H$ (here if $H$ is passive, we take $\mathcal{L} = \{\in, \mathcal{E}\}$). We say an embedding $\pi: H \to M$ is $k$-lifting if $\pi$ is $\Sigma_0^\text{elementary}$ with respect to $\mathcal{L}$, and if $k > 0$ then $\pi^n T_k^H \subseteq T_k^M$.

A $k$-lifting embedding is similar to a $\Sigma_0^{(k)}$-preserving embedding of [13].

2.2 Lemma. Let $H, M, k, \mathcal{L}$ be as in 2.1 and let $\pi: H \to M$. Then:

1. $\pi$ is $k$-lifting iff for every $\Sigma_k^{k+1}$ formula $\varphi$ and $x \in H$, if either $\varphi \in \mathcal{L}$ or $k > 0$ then $H \models \varphi(x) \implies M \models \varphi(\pi(x))$.
2. If $\pi$ is $k$-lifting and $H, M$ have different types then $k = 0$, $H$ is passive and $M$ is active.
3. If $k > 0$ and $\pi$ is $k$-lifting then $\pi$ is $\Sigma_k$ elementary, $(k - 1)$-lifting and $c$-preserving.
4. If \( k > 1 \) and \( \pi \) is \( \text{r} \Sigma_k \) elementary then \( \pi \) is \( p_{k-2} \)-preserving and \( \rho_{k-2} \)-preserving, and if \( \rho_{k-1}^H < \rho_0^H \) then
\[
\begin{align*}
- \pi(p_{k-1}^H) &= \pi(p_{k-1}^M) \setminus \pi(p_{k-1}^H) \\
- \sup \pi^u \rho_{k-1}^H &\leq \rho_{k-1}^M \leq \pi(\rho_{k-1}^H).
\end{align*}
\]
5. If \( k > 0 \) and \( \pi \) is \( \text{r} \Sigma_k \) elementary and \( p_{k-1} \)-preserving, \( \pi(p_{k}^H) \leq \rho_{k}^M \).
6. The Shift Lemma holds with weak \( k \)- replaced by \( k \)-lifting, or by \( k \)-lifting c-preserving.

**Proof.** Parts 1–3 are straightforward. For part 4, use \((k-1)\)-solidity witnesses for \( p_{k-1} \). For part 5 use the fact that if \( t \) is a \( k \)-solidity witness for \((H, p_{k}^H)\), then \( \pi(t) \) is a generalized \( k \)-solidity witness for \((M, \pi(p_{k}^H))\).

Part 6: We adopt the notation of [3, Lemma 5.2], but with ‘\( n \)’ replaced by ‘\( k \)’. Let \( F = F^N \) and \( \bar{U} = \text{Ult}_k(M, F) \) and \( U = \text{Ult}_k(M, F^N) \). Define
\[
\sigma : \mathcal{C}_0(\bar{U}) \to \mathcal{C}_0(U)
\]
as there. It is straightforward to see that \( \sigma \) is \( \text{r} \Sigma_k \)-elementary. Suppose \( k > 0 \).
Let us observe that \( \sigma^{-1}(T^U_k) \subseteq T^U_k \). Let \( t \in T^U_k \). Let \( x \in \bar{U} \) and \( \alpha < \rho_k^M \) be such that
\[
t = \text{Th}_{\text{r} \Sigma_k}^\bar{U}(\alpha \cup \{x\}).
\]
Let \( y \in \bar{M} \) and \( a \in \nu(F)^{<\omega} \) be such that
\[
x \in \text{Hull}^U_k(i_F^M(y) \cup a).
\]
Let \( \beta < \rho_k^\bar{M} \) be such that \( \beta \geq \text{cr}(\bar{F}) \) and \( i_F^M(\beta) \geq \alpha \). Let
\[
u = \text{Th}_{\text{r} \Sigma_k}^\bar{U}(\beta \cup \{y\}).
\]
Then \( t \) is easily computed from \( u' = i_F^\bar{M}(u) \), and by commutativity, \( \sigma(u') \in T^U_k \). It follows that \( \sigma(t) \in T^U_k \), as required. \( \square \)

**2.3 Remark.** Clearly for \( k < \omega \), any \( \text{r} \Sigma_{k+1} \)-elementary embedding is \( k \)-lifting.
However, the author does not know whether “weak \( k \)” implies “\( k \)-lifting”, or vice versa. We will not deal with weak \( k \)-embeddings in this paper.

Standard arguments show that the copying construction propagates \( k \)-lifting c-preserving embeddings. (But this may be false for weak \( k \)-embeddings; see [7].) Almost standard arguments show that \( k \)-lifting embeddings are propagated. That is, suppose \( \pi : H \to M \) is \( k \)-lifting, and let \( T \) be a \( k \)-maximal iteration tree on \( H \). We can define \( \mathcal{U} = \pi T \) as usual, assuming it has wellfounded models. Let \( H_\alpha = M^T_\alpha \) and \( M_\alpha = M^U_\alpha \). Using the Shift Lemma as usual, we get
\[
\pi_\alpha : H_\alpha \to M_\alpha
\]
for each \( \alpha < \text{lh}(T) \), and \( \pi_\alpha \) is \( \deg^\alpha(T) \)-lifting, and if \( \pi \) is c-preserving, then so is \( \pi_\alpha \). Let us just mention the extra details when \( \pi \) fails to be c-preserving. In
this case, \( k = 0 \) and \( H \) is passive. Suppose that \( E_0^T \) is total over \( H \), and let \( \kappa = \text{cr}(E_0^T) \). Suppose that \( (\kappa^+)^H < \text{OR}^H \) but \( \pi((\kappa^+)^H) \) is not a cardinal of \( M \). Then \( \mathcal{U} \) drops in model at 1, but \( \mathcal{T} \) does not. Note though that \( \text{rg}(\pi) \subseteq M_1^\mathcal{U} \) and \( \pi : H \to M_1^\mathcal{U} \) is 0-lifting (even if \( M_1^\mathcal{U} \) is active). So we can still produce \( \pi_1 : H_1 \to M_1 \) via the Shift Lemma. This situation generalizes to an arbitrary \( \alpha \) in place of 0, when \( \mathcal{T} \) does not drop in model along \([0, \alpha + 1]_\mathcal{T}\). In all other respects, the details are as usual. Moreover, if (i) \([0, \alpha]_\mathcal{T}\) drops in model or (ii) \( \text{deg}^T(\alpha) \leq k - 2 \) or (iii) \( \text{deg}^T(\alpha) = k - 1 \) and \( \pi \) is \( p_{k-1} \)-preserving, then \( \pi_\alpha \) is a near \( \text{deg}^T(\alpha) \)-embedding; this uses the argument in [4].

2.4 Lemma. Let \( k \geq 0 \), let \( \pi : H \to M \) be \( k \)-lifting, let \( \rho_{k+1} \leq \rho \leq \rho_k^H \). Then:

1. If \( p_{k-1}^H, p_k^M \in \text{rg}(\pi) \) and \( \rho_k^M = \sup \pi^\alpha \rho_k^H \) then \( \pi \) is a \( k \)-embedding.

2. If \( H \) is \( \rho \)-sound and \( (\pi \restriction \rho) \in M \) and \( \pi \) is not a \( k \)-embedding, then \( H, (\pi \restriction \rho_k^H) \in M \).

Proof. Part 1: This is fairly routine. By 2.2, we have \( \pi(p_{k-1}^H) = p_k^M \). The \( r\Sigma_{k+1} \) elementarity of \( \pi \) follows from this, together with the facts that \( \pi \) is \( k \)-lifting, \( p_k^M \in \text{rg}(\pi) \) and \( \pi^\alpha \rho_k^H \) is unbounded in \( \rho_k^M \). Now let \( \pi(q) = p_k^M \). Then \( \rho_k^H \leq q \) by 2.2, and \( q \leq \rho_k^H \) by \( r\Sigma_k \) elementarity.

Part 2: If \( \pi^\alpha \rho_k^H \) is bounded in \( \rho_k^M \), use a stratification of \( r\Sigma_{k+1} \) truth like that described in [3, §2]. Given the reader is familiar with this, here is a sketch. Let \( \alpha = \sup \pi^\alpha \rho_k^H \). Then the theory

\[
t = 0 \cup (\beta \cup \pi(\bar{p}_{k-1}^H))
\]

is in \( M \). Moreover, for any \( r\Sigma_{k+1} \) formula \( \varphi \) and \( \bar{\gamma} \in \rho^{<\omega} \),

\[
H \models \varphi(\bar{\gamma}, p_{k+1}^H)
\]

iff there is \( \beta < \alpha \) such that

\[
t \restriction (\beta \cup \pi(p_{k-1}^H))
\]

is "above" a witness to \( \varphi(\pi(\bar{\gamma}), \pi(\bar{p}_{k+1}^H)) \) (see [3, §2]). So the relation in line (1) is computable from \( t \) and \( \pi \restriction \rho \). So \( H \in M \), and a little more work gives that \((\pi \restriction \rho_k^H) \in M \).

Suppose now that \( \pi(p_{k-1}^H) = p_{k-1}^M \) but \( \pi(p_k^H) \neq p_k^M \). Then \( \pi(p_k^H) \leq p_k^M \) by 2.2, so suppose that \( \pi(p_k^H) < p_k^M \). Then we again get that \( t \in M \) (where \( t \) is defined as above), because \( t \) is computable from some \( k \)-solidity witness. The rest of the argument is the same.

Now assume that \( k > 1 \) and \( \pi(p_{k-1}^H) \neq p_{k-1}^M \). By 2.2, we therefore have \( \pi(p_{k-1}^H) < p_{k-1}^M \). Let \( \alpha = \sup \pi^\alpha p_{k-1}^H \).

Claim. Let \( \varphi \) be an \( r\Sigma_k \) formula, let \( x \in H \) and \( \bar{\gamma} \in \alpha^{<\omega} \). If \( M \models \varphi(x, \bar{\gamma}) \) then there is \( \varepsilon < \rho_k^M \), with \( \max(\bar{\gamma}) < \varepsilon \), such that the theory

\[
\text{Th}_{r\Sigma_{k+1}}(\varepsilon \cup \{\pi(x, p_{k-1}^H)\})
\]

is "above" a witness to \( \varphi(\pi(x), \bar{\gamma}) \).
Proof. Let $\delta < \rho^H_k$ be such that $\pi(\delta) > \max(\bar{\gamma})$. Let
\[
v = \text{Th}_H^\Sigma_k(\delta \cup \{x\} \cup \bar{p}_{k-1}^H).
\]
Note then that for all $\bar{\xi} \in \delta^{<\omega}$,
\[
(\varphi, (\bar{\xi}, x, \bar{p}_{k-1}^H)) \in v \implies ((\phi, (\bar{\xi}, x, \bar{p}_{k-1}^H)) \in v,
\]
where $\psi(\bar{\xi}, x, \bar{p}_{k-1}^H)$ asserts ‘There is $\varepsilon < \rho_{k-1}$, with $\max(\bar{\xi}) < \varepsilon$, such that the $r\Sigma_{k-1}$ theory in parameters $\varepsilon \cup \{x\} \cup \bar{p}_{k-1}^H$ is “above” a witness to $\varphi(\bar{\xi}, x, \bar{p}_{k-1}^H)$.’ But then the same fact holds regarding $\pi(v)$, and since $\pi$ is $k$-lifting, this proves the claim.

Now by $(k - 1)$-solidity, we have $u \in M$ where
\[
u = \text{Th}_{\Sigma_{k-1}}(\rho_{k-1}^M \cup \pi(\bar{p}_{k-1}^M)).
\]
Let $t$ be defined as before. By the claim, from $u$ we can compute $t$, so $t \in M$. Now the rest is as before.

2.5 Definition. Let $Q$ be a $k$-sound premouse. Let $\tilde{C}_0(Q) = C_0(Q)$, and for $k > 0$, let
\[
\tilde{C}_k(Q) = (Q||\rho_k(Q), T'),
\]
where $T = \text{Th}_H^\Sigma_k(\rho_k \cup \bar{p}_k^Q)$, and $T'$ is given from $T$ by substituting $p_{k}^Q$ for a constant symbol $c$.

2.6 Definition. Let $k \geq 0$. Let $Q$ be a $k$-sound premouse with $\omega < \rho_k^Q$. We say that $(U, \sigma^*)$ is $k$-suitable for $Q$ iff:
- $U, \sigma^* \in Q||\rho_k^Q$,
- $U$ is a $k$-sound premouse with $\rho_k^U > \omega$, and
- $\sigma^* : \tilde{C}_k(U) \to \tilde{C}_k(Q)$ is $\Sigma_0$-elementary.

2.7 Remark. Clearly, if $(U, \sigma^*)$ is $k$-suitable for $Q$ then $\sigma^*$ extends uniquely to a $\bar{p}_k$-preserving $k$-lifting $\sigma : U \to Q$, and moreover,
\[
\sup \sigma^u \rho_k^U < \rho_k^Q.
\]
 Conversely, if $\sigma : U \to Q$ is $\bar{p}_k$-preserving $k$-lifting and $\sup \sigma^u p_{k}^U < \rho_k^Q$ and $\sigma^* = \sigma \upharpoonright (U||\rho_k^U)$ is in $Q$, then $(U, \sigma^*)$ is $k$-suitable for $Q$.

2.8 Lemma. Let $k \geq 0$. Then there is an $r\Sigma_{k+1}$ formula $\varphi_k$ such that for all $k$-sound premice $Q$ with $\omega < \rho_k^Q$, and all $U, \sigma^* \in Q$,
\[
Q \models \varphi_k(U, \sigma^*, p_k^Q)
\]
iff $(U, \sigma^*)$ is $k$-suitable for $Q$. 8
Proof. We assume $k > 0$ and leave the other case to the reader.

The most complex clause of $\varphi_k$ says "There is $\alpha < \rho_k^Q$ such that letting 
\[ t = \text{Th}_{\Sigma_k}^Q(\alpha \cup \vec{p}_k^Q), \]
then for each $\beta < \rho_k^U$, letting 
\[ u = \text{Th}_{\Sigma_k}^U(\beta \cup \vec{p}_k^U), \]
and letting $t', u'$ be given from $t, u$ by substituting $\vec{p}_k^Q, \vec{p}_k^U$ for the constant $c$, we have $\sigma^*(u') \subseteq t''$, and this is $r\Sigma_{k+1}$. The rest is clear.

2.9 Definition. Let $m \geq 0$ and let $M$ be a segmented-premouse. Then $M$ is $m$-sound iff either $m = 0$ or $M$ is an $m$-sound premouse. \(\sqsubset\)

2.10 Definition. Let $r \geq 0$ and let $R$ be an $r$-sound premouse. Then we say that suitable condensation holds at $(R, r)$ iff for every $(H, \pi^*)$, if $(H, \pi^*)$ is $r$-suitable for $R$, $H$ is $(r+1)$-sound and 
\[ cr(\pi) \geq \rho = \rho_{r+1}, \]
then either $H \triangleleft R$, or $R|\rho$ is active with extender $F$ and $H \triangleleft \text{Ult}(R|\rho, F)$. 

Let $m \geq 0$ and let $M$ be an $m$-sound segmented-premouse. We say that suitable condensation holds below $(M, m)$ iff for every $R \subseteq M$ and $r < \omega$ such that either $R \triangleleft M$ or $r < m$, suitable condensation holds at $(R, r)$. We say that suitable condensation holds through $(M, m)$ iff $M$ is a premouse\(^5\) and suitable condensation holds below and at $(M, m)$. \(\sqsubset\)

2.11 Lemma. Let $m \geq 0$. Then there is an $r\Pi_{\max(m, 1)}$ formula $\Psi_m$ such that for all $m$-sound segmented-premice $M$, suitable condensation holds below $(M, m)$ iff $M \models \Psi_m(p_{m-1}^M)$, where $p_{m-1}^M = \emptyset$. Moreover, if $M$ is a premouse, then suitable condensation holds through $(M, m)$ iff $M \models \Psi_{m+1}(p_m^M)$.\(^6\)

Proof. This follows easily from 2.8. \(\square\)

2.12 Remark. Our proof of condensation from normal iterability (see 4.2) will use our analysis of bicephali and cephalanxes (see §3). This analysis will, in turn, depend on the premice involved satisfying enough condensation, at lower levels (that is, lower in model or degree). We will only have normal iterability for those premice, so we can’t appeal to the standard condensation theorem for this. One could get arrange everything inductively, proving condensation and analysing bicephali and cephalanxes simultaneously. However, it is simpler to avoid this by making use of the following lemmas, which are easy to prove directly. We will end up generalizing them in 4.2.

\(^5\)We could have formulated this more generally for segmented-premice, but doing so would have increased notational load, and we do not need such a generalization.

\(^6\)This clause only adds something because we do not assume that $M$ is $(m+1)$-sound.
2.13 Lemma (Condensation for $\omega$-sound mice). Let $h \leq m < \omega$ and let $H, M$ be premice. Suppose that:
- $H$ is $(h+1)$-sound.
- $M$ is $(m+1)$-sound and $(m, \omega_1 + 1)$-iterable.
- Either $\rho^M_{m+1} = \omega$ or $m \geq h + 5$.
- There is an $h$-lifting $\vec{p}_h$-preserving embedding $\pi : H \rightarrow M$ with $cr(\pi) \geq \rho = \rho^H_{h+1}$.

Then either
- $H = M$, or
- $H \triangleleft M$, or
- $M \upharpoonright \rho$ is active and $H \triangleleft \text{Ult}(M|\rho, G)$.

Proof. Let $\pi$, etc, be a counterexample. Let $\pi^* = \pi \upharpoonright (H||\rho^H_h)$.

Claim. $H \in M$.

Proof. Suppose not. By 2.4, $\pi$ is an $h$-embedding, and $\rho \geq \rho^M_{h+1}$. Note that $\pi(p^H_{h+1}) \leq p^M_{h+1} \setminus \rho$ (using generalized solidity witnesses). If $\pi(p^H_{h+1}) < p^M_{h+1} \setminus \rho$ then we are done, so suppose otherwise. Then $\rho^M_{h+1} \cup p^M_{h+1} \subseteq \text{rg}(\pi)$, so $H = M$, contradiction. \[\square\]

Now we may assume that $\rho^M_{m+1} = \omega$, by replacing $M$ with $\text{cHull}^M_{m+1}(\vec{p}_m)$ if necessary: all relevant facts pass to this hull because $cr(\pi) \geq \rho$ and by the claim and by 2.2(1). We can now run almost the usual proof of condensation. However, in the comparison $(T, U)$ of the phalanx $(M, H, \rho)$ with $M$, we form an $(m, h)$-maximal tree on $(M, H, \rho)$, and an $m$-maximal tree on $M$. Because $H \in M$, and using the fine-structural circumstances in place of the weak Dodd-Jensen property, this leads to contradiction. \[\square\]

2.14 Lemma (Suitable condensation). Let $M$ be an $m$-sound, $(m, \omega_1 + 1)$-iterable segmented-premouse. Then suitable condensation holds below $(M, m)$, and if $M$ is a premouse, through $(M, m)$.

Proof. If $M$ is not a premouse this follows from 2.13. So suppose $M$ is a premouse. By 2.11, we may assume that $\rho^M_{m+1} = \omega$, by replacing $M$ with $\text{cHull}^M_{m+1}(\vec{p}_m)$ if necessary. So we can argue as at the end of the proof of 2.13. \[\square\]
3. The bicephalus & the cephalanx

3.1 Definition. An exact bicephalus is a tuple \( B = (\rho, M, N) \) such that:

1. \( M \) and \( N \) are premice.
2. \( \rho < \min(\text{OR}^M, \text{OR}^N) \) and \( \rho \) is a cardinal of both \( M \) and \( N \).
3. \( M \mid (\rho^+)^M = N \mid (\rho^+)^N \).
4. \( M \) is \( \rho \)-sound and for some \( m \in \{-1\} \cup \omega \), we have \( \rho^M_{m+1} \leq \rho \). Likewise for \( N \) and \( n \in \{-1\} \cup \omega \).

We say \( B \) is non-trivial iff \( M \neq N \). Write \( \rho_B = \rho \wedge M_B = M \) and \( N_B = N \), and \( m_B, n_B \) for the least \( m, n \) as above. Let \( \gamma \) be \( \gamma_{m_B, n_B} \). We say that \( B \) has degree \( (m_B, n_B) \). We say that \( B \) is sound iff \( M \) is \( m_B+1 \)-sound and \( N \) is \( n_B+1 \)-sound. \( \dashv \)

From now on we will just say bicephalus instead of exact bicephalus. In connection with bicephali of degree \( (m, n) \) with \( \min(m, n) = -1 \), we need the following:

3.2 Definition. The terminology/notation (near) \((-1)\)-embedding, \((-1)\)-lifting embedding, \( \text{Ult}_{-1} \), \( \mathcal{C}_{-1} \), and degree \((-1)\) iterability are defined by replacing ‘\(-1\)’ with ‘0’. For \( n > -1 \) and appropriate premice \( M \), the core embedding \( \mathcal{C}_n(M) \rightarrow \mathcal{C}_{-1}(M) \) is just the core embedding \( \mathcal{C}_n(M) \rightarrow \mathcal{C}_0(M) \). \( \dashv \)

3.3 Definition. Let \( q < \omega \). A passive right half-cephalanx of degree \( q \) is a tuple \( B = (\gamma, \rho, Q) \) such that:

1. \( Q \) is a premouse,
2. \( \gamma \) is a cardinal of \( Q \) and \( (\gamma^+)^Q \leq \rho < \text{OR}^Q \),
3. \( Q \) is \( \gamma \)-sound,
4. \( \rho^Q_{\gamma+1} \leq \gamma < \rho^Q_\gamma \).

An active right half-cephalanx (of degree \( q = 0 \)) is a tuple \( B = (\gamma, \rho, Q) \) such that:

1. \( Q \) is an active segmented-premouse,
2. \( \gamma \) is the largest cardinal of \( Q \) and \( \gamma < \rho = \text{OR}^Q \).

A right half-cephalanx \( B \) is either a passive, or active, right half-cephalanx. We write \( \gamma_B, \rho_B, Q_B, q_B \) for \( \gamma, \rho, Q, q \) as above. If \( B \) is active, we write \( S_B = R_B = \text{Ult}(Q, \mathcal{F}^Q) \). If \( B \) is passive, we write \( S_B = Q \). \( \dashv \)

Note that if \( B = (\gamma, \rho, Q) \) is a right-half cephalanx, then \( B \) is active iff \( Q \mid \rho \) is active. So it might be that \( B \) is passive but \( Q \) is active.
### 3.4 Definition.** Let \( m \in \{-1\} \cup \omega \) and \( q < \omega \). A cephalanx of degree \((m,q)\) is a tuple \( B = (\gamma, \rho, M, Q) \) such that, letting \( B' = (\gamma, \rho, Q) \), we have:

1. \((\gamma, \rho, Q)\) is a right-half cephalanx of degree \( q \),
2. \( M \) is a premouse,
3. \( \rho = (\gamma^+)^M < OR^M \),
4. \( M\langle \langle (\rho^+)^M = S^{B'}\langle \langle (\rho^+)^M \),
5. \( M \rho\) sound,
6. \( \rho_m^M = \rho < \rho^M \).

We say that \( B \) is **active** (passive) iff \( B' \) is active (passive).\(^7\) We write \( \gamma^B, \rho^B \), etc, for \( \gamma, \rho \), etc. We write \( R^B \) for \( R^{B'} \), if it is defined, and \( S^B \) for \( S^{B'} \).

Suppose \( B \) is active. Let \( R = R^B \). We say \( B \) is **non-trivial** iff \( M \not\vdash R \). If \( B \) is non-exact, let \( N^B \) denote the \( N \triangleleft R \) such that \( (\rho^+)^N = (\rho^+)^M \) and \( \rho_\omega^N = \rho \), and let \( n^B \) denote the \( n \in \{-1\} \cup \omega \) such that \( \rho_{n+1}^N = \rho < \rho_n^N \).

Now suppose \( B \) is passive. We say \( B \) is **non-trivial** iff \( M \not\vDash Q \). Let \( N^B \) denote the \( N \vdash Q \) such that \( (\rho^+)^N = (\rho^+)^M \) and \( \rho_\omega^N \leq \rho \). Let \( n^B \) be the \( n \in \{-1\} \cup \omega \) such that \( \rho_{n+1}^N \leq \rho < \rho_n^N \).

A pm-cephalanx is a cephalanx \( (\gamma, \rho, M, Q) \) such that \( Q \) is a premouse. \(\)

### 3.5 Definition.** A cephal is either a bicephalus or a cephalanx. Let \( B \) be a cephal, and let \( M = M^B \).

A short extender \( E \) is **semi-close to** \( B \) iff \( cr(E) < \rho^B \) and \( E \) is semi-close\(^8\) to \( M \).

For \( \alpha \leq \rho^B \), let \( B\langle \alpha = M\langle \alpha \), and for \( \alpha < \rho^B \), let \( B\langle \alpha = M\langle \alpha \) and \( (\alpha^+)^B = (\alpha^+)^M \). We write \( P \vdash B \) iff \( P \vdash B\langle \rho^B \). Let \( C, \alpha \) be such that \( \alpha \leq \rho^C \), and either \( C \) is a segmented-premouse and \( \alpha \leq OR^C \), or \( C \) is a cephal and \( \alpha \leq \rho^C \). Then we define

\[
(B \sim C)\langle \alpha \iff B\langle \alpha = C\langle \alpha.
\]

If also \( \alpha < \rho^B \) and either \( C \) is a segmented-premouse or \( \alpha < \rho^C \), we use the same notation with “\(\langle\)” replacing “\(\langle\)”\(^9\). We also use the same notation with more than two structures. \(\)

### 3.6 Remark.** Because of the symmetry of bicephali and the partial symmetries of cephalanxes, we often state facts for just one side of this symmetry, even though they hold for both.

The proofs of the next two lemmas are routine and are omitted. In 3.7–3.13 below, the extender \( E \) might be long.

---

\(^7\)Note that a passive cephalanx \( (\gamma, \rho, M, Q) \) might be such that \( M \) and/or \( Q \) is/are active.

\(^8\)See \([5]\).
3.7 Lemma. Let $Q$ be an active segmented-premouse. Let $E$ be an extender over $Q$ with $\text{ms}(E) \leq \text{cr}(F^Q) + 1$. Let $R = \text{Ult}^Q$ and $Q' = \text{Ult}(Q, E)$ and $R' = \text{Ult}^{Q'}$. Then $R' = \text{Ult}(R, E)$ and the ultrapower embeddings commute. Moreover, $i^R_E = \psi_q^Q$.

3.8 Lemma. Let $Q$ be an active segmented-premouse. Let $E$ be an extender over $Q$ with $(\text{cr}(F^Q)^{++})^Q < \text{cr}(E)$. Let $R = \text{Ult}^Q$ and $R^* = \text{Ult}(R, E)$ and $Q' = \text{Ult}(Q, E)$. Then $\text{Ult}(Q, F^{Q'}) = R^*$ and the ultrapower embeddings commute.\(^9\)

Let $\psi : R \to R^*$ be given by the Shift Lemma (applied to $k : Q \to Q$ and $i^Q_E$). Then $i^R_E = \psi$.

3.9 Definition. Let $E$ be a (possibly long) extender over a segmented-premouse $M$. We say that $E$ is reasonable (for $M$) iff either $M$ is passive, or letting $\kappa = \text{cr}(F^M)$, $i^M_E$ is continuous at $(\kappa^+)^M$, and if $M \models \kappa^+$ exists then $i^M_E$ is continuous at $(\kappa^+)^M$.

Given a bicephalus $B = (\rho, M, N)$, an extender $E$ is reasonable for $B$ iff $E$ is over $B \| \rho$, if $m^B \leq 0$ then $E$ is reasonable for $M$, and if $n^B \leq 0$ then $E$ is reasonable for $N$.

Given a cephalanx $B = (\gamma, \rho, M, Q)$, an extender $E$ is reasonable for $B$ iff $E$ is over $B \| \rho$, if $q^B \leq 0$ then $E$ is reasonable for $Q$, if $m^B \leq 0$ then $E$ is reasonable for $M$, and if $n^B$ is defined and $n^B \leq 0$ then $E$ is reasonable for $N^B$.

3.10 Lemma. Let $Q$ be an active segmented-premouse and let $E$ be an extender reasonable for $Q$. Let $Q' = \text{Ult}(Q, E)$ and $R = \text{Ult}^Q$ and $R' = \text{Ult}^{Q'}$ and $R^* = \text{Ult}(R, E)$. Let $\kappa = \text{cr}(F^Q)$ and $\eta = (\kappa^+)^Q$. If $\eta < \text{OR}^Q$ then let

$$
\begin{align*}
\gamma = i_{Q,R}(\eta), & \quad \gamma^* = i^R_E(\gamma), \quad \eta' = i^Q_E(\eta);
\end{align*}
$$

then $\gamma^* = i_{Q',R'}(\eta')$. If $\eta = \text{OR}^Q$ then let $\gamma = \text{OR}^R$, $\gamma^* = \text{OR}^{R^*}$ and $\eta' = \text{OR}^{Q'}$. Then in either case, $(R^* \sim R^*)|\gamma^*$ and

$$i^R_E \circ i_{Q,R}|(Q|\eta) = i_{Q',R'} \circ i^Q_E \circ (Q|\eta).
$$

Moreover, let $\psi : R|\gamma \to R'|\gamma'$ be induced by the shift lemma applied to $i_{Q,Q'} | (Q|\eta)$ and $i_{Q',Q'}$. Then $i' = i^R_E | (R|\gamma)$.

Proof. Let $G$ be the extender derived from $E$, of length $i_E(\kappa)$. Let $j : \text{Ult}(Q, G) \to \text{Ult}(Q, E)$ be the factor embedding. Then $\text{cr}(j) > (\text{OR}(\kappa)^{++})^{Q,G}$ since $E$ is reasonable. Apply 3.7 to $G$, and then 3.8 to the extender derived from $j$. \(\square\)

3.11 Definition. Let $M$ be a type 3 premouse. The expansion of $M$ is the active segmented-premouse $M_*$ such that $M_*|\text{cr}(F^{M_*}) = M|\text{cr}(F^M)$, and $F^{M_*}$ is the Jensen-indexed version of $F^M$. That is, let $F = F^M$, let $\mu = \text{cr}(F)$, let $\gamma = (\mu^+)^M$, let $\gamma' = i_F(\gamma)$, let $R = \text{Ult}^M$; then $M_*|\text{OR}(M_*) = R|\gamma$, and $F^{M_*}$ is the length $i_F(\mu)$ extender derived from $i_F$. \(\square\)

\(^9\)Note that in the conclusion, it is $\text{Ult}(Q, F^Q)$, not $\text{Ult}^{Q'}$.
The calculations in [3, §9] combined with a simple variant of 3.10 give the following:

3.12 Fact. Let \( Q \) be a type 3 premouse. Let \( E \) be an extender over \( Q^{\eta} \), reasonable for \( Q \). Let \( Q_* \) be the expansion of \( Q \), let \( U_* = \text{Ult}(Q_*, E) \) and \( U = \text{Ult}_0(Q, E) \). Suppose \( U_* \) is wellfounded. Then \( U \) is wellfounded and \( U_* \) is its expansion. Moreover, let \( i_* : Q_* \to U_* \) and \( i_0 : Q \to U \) be the ultrapower embeddings (so literally, \( \text{dom}(i_*) = Q_* \) and \( \text{dom}(i_0) = Q^{\eta} \)). Then \( i_0 = i_* | Q^{\eta} \), and \( i_* = \psi_{i_0} | Q_* \).

3.13 Remark. We will apply 3.10 and 3.12 when \( E \) is the extender of an iteration map \( i^{T} \alpha, \beta \), and if \( \alpha \) is a successor, the map \( i^{T} \alpha, \beta \circ i^{T} \alpha \), where \( (\alpha, \beta)^{T} \) does not drop and \( \text{deg}^{T}(\alpha) = 0 \).

3.14 Definition. Let \( B = (\rho, M, N) \) be a bicephalus of degree \((m, n)\) and let \( E \) be an extender reasonable for \( B \). Let \( i^M_E : M \to \text{Ult}_m(M, E) \) be the usual ultrapower map, and likewise \( i^N_E \) and \( n \). Let \( \rho' = \sup i^M_E \rho \) and define

\[
\text{Ult}(B, E) = (\rho', \text{Ult}_m(M, E), \text{Ult}_n(N, E)).
\]

We say that \( \text{Ult}(B, E) \) is wellfounded iff both \( \text{Ult}_m(N, E) \) and \( \text{Ult}_n(N, E) \) are wellfounded.

3.15 Definition. Let \( B \) be a bicephalus. The associated augmented bicephalus is the tuple

\[
B_* = (\rho, M, N, M_*, N_*)
\]

where if \( m \geq 0 \) then \( M_* = M \), and otherwise \( M_* \) is the expansion of \( M \); likewise for \( N_* \). (Note that if \( m = -1 \) then \( M \) is type 3 and \( \rho = \nu(F^M) \).)

Let \( E \) reasonable for \( B \). If \( m \geq 0 \) let \( M = \text{Ult}_m(M, E) \); otherwise let \( M = \text{Ult}(M_*, E) \). Likewise for \( N \). Then we define

\[
\text{Ult}(B_*, E) = \text{Ult}(B, E) \setminus \langle M_*, N_* \rangle.
\]

We say that \( \text{Ult}(B_*, E) \) is wellfounded iff \( \text{Ult}(B, E), M_*, N_* \) are all wellfounded.

3.16 Lemma. Let \( B = (\rho, M, N) \) be a bicephalus. Let \( E \) be reasonable for \( B \). Let \( U = \text{Ult}(B, E) \) and \( \bar{U} = \text{Ult}(B_*, E) = (\rho^U, M^U, N^U, M_*, N_*) \). Suppose that \( \bar{U} \) is wellfounded. Then:

1. \( U \) is a bicephalus of degree \((m, n)\) and \( \bar{U} = U_* \).
2. \( U \) is trivial iff \( B \) is trivial.
3. \( i^M_E(p^M_{m+1} \rho) = p^M_{m+1} \rho^U \).
is trivial when is routine.

(5) \( i^*_E \) is the ultrapower map. We claim that \((\gamma,\rho,\gamma')\) is a standard calculation using generalized

Suppose \( E \) is short and semi-close to \( B \). Then \( M^U \) is \( m+1 \)-sound iff \( M \)

Likewise regarding \( N, n, E \).

Proof. Part 6 is by [5, 2.20], (3) is a standard calculation using generalized solidity witnesses (see [13]), and (5) is by 3.12 ((5) is trivial when \( m \geq 0 \)).

Consider (4). Let \( W = \text{Ult}(B \upharpoonright (\rho^+)^B, E) \) and \( j : B \upharpoonright (\rho^+)^B \to W \) be the ultrapower map. We claim that \((\dagger)\): \( j = i^*_E \upharpoonright (\rho^+)^B \), and letting \( \bar{\rho} = j(\rho) \),

If \( m \leq 0 \) this is immediate. If \( m > 0 \), then because \((\rho^+)^B \leq \rho_m^B \), by [3, §6], all functions forming the ultrapower \( M^U \) with codomain \((\rho^+)^B \) are in fact in \( B \upharpoonright (\rho^+)^B \), which gives \((\dagger)\).

Now (4) follows from \((\dagger)\). Consider (1). By 3.12, \( \tilde{M} \) is the expansion of \( M^U \).

Now we have \( \rho^U \leq \bar{\rho} \) and by \((\dagger)\),

Also if \( m \geq 0 \) then \( \bar{\rho} < \rho_m(M^U) \). The rest of the proof of (1) is routine.

Now let us prove (2). Assume \( M \neq N \). We may assume \( m = n \). Because \( M \neq N \) and by \( \rho \)-soundness, there is some \( r \Sigma_{m+1} \) formula \( \varphi \) and \( \alpha < \rho \) such that
\n\( M \models \varphi(p^M \upharpoonright \rho, \alpha) \iff N \models \gamma \varphi(p^N_m \upharpoonright \rho, \alpha) \).
\nNow \( i^*_B \) and \( i^*_N \) are \( r \Sigma_{m+1} \)-elementary, and by \((\dagger)\), \( i^*_B(\alpha) = i^*_N(\alpha) \); let \( \alpha' = i^*_B(\alpha) \). So by (3),

and therefore \( M^U \neq N^U \). \qed

3.17 Definition (Ultrapowers of cephalanxes). Let \( B = (\gamma, \rho, M, Q) \) be a cephalanx of degree \( (m, q) \) and let \( E \) be reasonable for \( B \). Let \( i^*_B \) be the degree \( m \) ultrapower map and let \( \gamma' = i^*_B(\gamma) \) and \( \rho' = \sup i^*_B(\rho) \). If \( B \) is active then we define

\[ \text{Ult}(B, E) = (\gamma', \rho', \text{Ult}_m(M, E), \text{Ult}(Q, E)). \]

---

\(^{10}\)That is, if \((\rho^+)^B \in \text{dom}(i^*_E(\gamma))\) then \( i^*_E(\gamma) \) is continuous there; if \( m \geq 0 \) and \((\rho^+)^B = \rho_0^B\) then \( \rho_0^M \) \( = \sup i^*_E(\rho^+)^B \); if \( m = -1 \) and \((\rho^+)^B = \text{OR}(\gamma)\) then \( \text{OR}(\gamma) \) \( = \sup i^*_E(\rho^+)^B \).
(Recall that the ultrapower \( \text{Ult}(Q, E) \) is simple; it might be that \( Q \) is type 3.) If \( B \) is passive then we define
\[
\text{Ult}(B, E) = (\gamma', \rho', \text{Ult}_m(M, E), \text{Ult}_n(Q, E)).
\]

3.18 Lemma. In the context of 3.17, suppose that \( B \) is passive, and that \( U = \text{Ult}(B, E) \) is wellfounded. Let \( \mu = (\rho^+)^M \). Then:

1. \( U \) is a passive cephalanx of degree \((m, q)\).
2. \( i_E^M \restriction \mu = i_E^Q \restriction \rho. \)
3. If \( \rho \in \mathcal{C}_0(M) \) then \( \rho' = i_E^M(\rho) \); otherwise \( \rho' = \rho_0(M^U) \). Likewise for \( Q, i_E^Q, Q^U \).
4. \( \psi_i^M(\rho) = \psi_i^Q(\rho) = \rho'. \)
5. If \( (\rho^+)^M \in \text{dom}(\psi_i^M) \) then \( \psi_i^M \) is continuous at \( (\rho^+)^M \); otherwise \( M \) is passive, \( \text{OR}^M = (\rho^+)^M \) and \( \text{OR}(M^U) = \sup i_E^M \cup \text{OR}^M \).
6. \( \psi_i^M \restriction (\rho^+)^M = \psi_i^Q \restriction (\rho^+)^M. \)
7. \( i_E^M(p_{m+1}^\rho) = p_{m+1}^M \restriction \rho'. \)
8. Suppose \( E \) is short and semi-close to \( B \). Then \( M^U \) is \((m + 1)\)-sound if \( M \) is \((m + 1)\)-sound and \( \text{cr}(E) < \rho_{m+1}^M \). If \( M^U \) is \((m + 1)\)-sound then \( \rho_{m+1}^M = \sup i_E^M \cup \rho_{m+1}^M \) and \( p_{m+1}^M = i_E^M(p_{m+1}^M) \).
9. If \( B \) is non-exact then \( U \) is non-exact.
10. If \( B \) is exact \((\text{so } N^B = Q)\) but \( U \) is not, then \( 0 < n^B < q \).
11. Suppose that \( B \) is non-trivial and that suitable condensation holds below \((Q, q)\). Let \( N = N^B \) and \( n = n^B \). Then:
   
   (i) \( U \) is non-trivial,
   
   (ii) \( N^U = \text{Ult}_n(M, E) \) and \( n^U = n, \)
   
   (iii) Parts (2)–(8) hold with ‘\( M \)’ replaced by ‘\( N \)’ and ‘\( m \)’ by ‘\( n \)’.

We also have \( i_E^Q(p_{q+1}^Q \restriction \gamma) = p_{q+1}^Q \cup \gamma^U \), but we won’t need this.

Proof. Parts (2)–(8) are much as in the proof of 3.16. (For (6), note that given \( A \in P(\rho) \cap M \), the value of \( \psi_i^M(A) \) is determined by the values of \( \psi_i^M(A \cap \alpha) \) for \( \alpha < \rho \); likewise for \( \psi_i^Q(A) \).) So (1) follows. Part (9) follows from (5) and (6); part (10) is easy.

Now consider (11). We first deal with the case that \( B \) is exact, so assume this. Part (iii) is just as for \( M \), so consider (i) and (ii). Since \( B \) is exact, \( N = Q \).
By the proof of 3.16, we have $\text{Ult}_n(Q, E) \neq \text{Ult}_m(M, E)$, so it suffices to see that

$$U_n = \text{Ult}_n(Q, E) \preceq Q^U = \text{Ult}_q(Q, E) = U_q.$$ 

We may assume that $n < q$. If $n = -1$ then we easily have $U_n = U_q$, so also assume $n \geq 0$, and so $\rho \in \mathcal{C}_0(Q)$. We have

$$\rho^Q_{n+1} \leq \gamma < \rho = \rho^Q_n = \rho^Q_{n+1}.$$ 

Let $\sigma : U_n \rightarrow U_q$ be the natural factor map. Let $i_n : Q \rightarrow U_n$ and $i_q : Q \rightarrow U_q$ be the ultrapower maps. Then $\sigma \circ i_n = i_q$, $\sigma$ is $\rho^U_{n+1}$-preserving $n$-lifting and $\text{cr}(\sigma) > \rho'$. Also, $U_n, U_q$ are $(n+1)$-sound and $\rho^U_{n+1} = \rho' = \rho^U_q$.

Suppose $((\rho')^+)_{U_n} = ((\rho')^+)_{U_q} < \text{cr}(\sigma)$. Then

$$\rho^U_{n} = \sup \sigma^q \rho^U_n,$$

since otherwise, using the previous paragraph and as in the proof of 2.4, otherwise $U_n \in U_q$, collapsing $((\rho')^+)_{U_q}$ in $U_q$. So by 2.4, $\sigma$ is an $n$-embedding, and in particular, is $r\Sigma_n$-elementary. Since

$$U_n \cup U_q \subseteq \text{rg}(\sigma),$$

therefore $U_n = U_q$, which suffices.

Now suppose that $((\rho')^+)_{U_n} < ((\rho')^+)_{U_q}$. Then much as in the previous case, $\rho^U_{n} > \sup \sigma^q \rho^U_n$.

Let $\sigma^* = \sigma | (U_n || \rho^U_n)$. By 2.4 we get $U_n, \sigma^* \in U_q$ and $(U_n, \sigma^*)$ is $n$-suitable for $U_q$. Since suitable condensation holds below $(Q, q)$, and by 2.11, and since $U_q\rho'$ is passive, it follows that $U_n \triangleleft U_q$, which suffices.

Now consider the case that $B$ is non-exact. So $N \triangleleft Q$. Let $U_n = \text{Ult}_n(N, E)$, consider the factor embedding

$$\sigma : U_n \rightarrow i^Q_E(N)$$

and argue that $U_n \preceq i^Q_E(N)$, like before. This completes the proof. $lacksquare$

3.19 Lemma. In the context of 3.17, suppose that $B$ is active, and that $U = \text{Ult}(B, E)$ and $R^U$ are wellfounded. Let $\mu = (\rho^+)^M$. Then:

1. $U$ is an active cephalanx of degree $(m, 0)$.
2. If $\rho \in \mathcal{C}_0(M)$ then $\rho' = i^M_E(\rho)$; otherwise $\rho' = \rho_0(M^U)$.
3. 3.18(2), (4)–(8) hold.
4. $U$ is exact iff $B$ is exact.
5. Suppose that $B$ is non-exact and non-trivial and that suitable condensation holds below $(Q, 0)$. Let $N = N^B$ and $n = n^B$. Then:
(i) \( U \) is non-trivial,
(ii) \( N^U = \text{Ult}_{n^U}(N, E) \) and \( n^U = n \),
(iii) Parts (2)–(3) hold with ‘\( M \)’ replaced by ‘\( N \)’ and ‘\( m \)’ by ‘\( n \)’.

Proof. This follows from 3.10, 3.12 and the proof of 3.18. \( \Box \)

3.20 Lemma. Let \( C \) be a cephal of degree \((m, k)\). If \( C \) is a bicephalus let \( B = C_0 \), and otherwise let \( B = C \). Let \( \langle E_\alpha \rangle_{\alpha < \lambda} \) be a sequence of short extenders. Let \( B_0 = B \), \( B_{\alpha + 1} = \text{Ult}(B_\alpha, E_\alpha) \), and let \( B_\gamma \) be the direct limit at limit \( \gamma \). Suppose that for each \( \alpha \leq \lambda \), \( B_\alpha \) is wellfounded and if \( \alpha < \lambda \) then \( E_\alpha \) is semi-close to \( B_\alpha \).

If \( C \) is a bicephalus (passive cephalanx, active cephalanx, respectively) then the conclusions of 3.16 (3.18, 3.19, respectively) apply to \( B \) and \( B_\lambda \), together with the associated iteration embeddings, after deleting the sentence “Suppose \( E \) is short and semi-close to \( B_\alpha \)” and replacing the phrase “\( \text{cr}(E) < \rho_{\alpha + 1} \)” with “\( \text{cr}(E_\alpha) < \rho_{\alpha + 1} \)”.

Proof. If \( C \) is a bicephalus, this mostly follows from 3.16, [5, 2.20] and 3.12 by induction. At limit stages, use [5, 2.20] directly to prove 3.16(6). To see 3.16(4), replace the iteration used to define \( C_\gamma \) with a single (possibly long) extender \( E \), and apply 3.16. The cephalanx cases are similar. \( \Box \)

3.21 Definition (Iteration trees on bicephali). Let \( B = (\rho, M, N) \) be a bi-cephalus of degree \((m, n)\) and let \( \eta \in \text{OR}\{0\} \). An iteration tree on \( B \), of length \( \eta \), is a tuple
\[
\mathcal{T} = \left( <\mathcal{T}, \langle E_\alpha \rangle_{\alpha + 1 < \eta} \right),
\]
such that there are sequences of models
\[
\langle B_\alpha, M_\alpha, N_\alpha \rangle_{\alpha < \eta} \quad \& \quad \langle B^*_\alpha, M^*_\alpha, N^*_\alpha \rangle_{\alpha + 1 < \eta},
\]
and embeddings
\[
\langle i_{\alpha, \beta}, j_{\alpha, \beta} \rangle_{\alpha, \beta < \eta} \quad \& \quad \langle i^*_{\alpha + 1}, j^*_{\alpha + 1} \rangle_{\alpha + 1 < \eta},
\]
and ordinals
\[
\langle \rho_\alpha \rangle_{\alpha < \eta} \quad \& \quad \langle \text{cr}_\alpha, \nu_\alpha, \text{lh}_\alpha, \rho^*_\alpha \rangle_{\alpha + 1 < \eta},
\]
sets \( \mathcal{R}, \mathcal{M}, \mathcal{N} \subseteq \eta \) (specifying types and origins of structures), a function \( \text{deg} \) with domain \( \eta \) (specifying degrees), and a set \( \mathcal{D} \subseteq \eta \) (specifying drops in model), with the following properties:

1. \( <\mathcal{T} \) is an iteration tree order on \( \eta \), with the usual properties.
2. \( B_0 = (\rho_0, M_0, N_0) = B \) and \( \text{deg}(0) = (m, n) \) and \( i_{0, 0} = \text{id} \) and \( j_{0, 0} = \text{id} \).

\(^{11}\)In 3.10 we set \( \eta = (\kappa^+)^Q \), and the reader might wonder why we didn’t just use \( \eta = (\kappa^+)^Q \). We need the larger value here if \( Q \) has superstrong type.
3. $\mathcal{B}, \mathcal{M}, \mathcal{N}$ are disjoint and for each $\alpha < \eta$, either
   
   (a) $\alpha \in \mathcal{B}$ and $B_{\alpha} = (\rho_{\alpha}, M_{\alpha}, N_{\alpha})$ is a bicephalus of degree $(m, n) = \text{deg}(\alpha)$, or
   
   (b) $\alpha \in \mathcal{M}$ and $B_{\alpha} = M_{\alpha}$ is a segmented-premouse and $N_{\alpha} = \emptyset$, or
   
   (c) $\alpha \in \mathcal{N}$ and $B_{\alpha} = N_{\alpha}$ is a segmented-premouse and $M_{\alpha} = \emptyset$.

4. For each $\alpha + 1 < \eta$:
   
   (i) Either $E_{\alpha} \in E_{\alpha}^{\alpha}(M_{\alpha})$ or $E_{\alpha} \in E_{\alpha}(N_{\alpha})$.
   
   (ii) $\text{cr}_{\alpha} = \text{cr}(E_{\alpha})$ and $\nu_{\alpha} = \nu(E_{\alpha})$ and $\text{lh}_{\alpha} = \text{lh}(E_{\alpha})$.
   
   (iii) For all $\beta < \alpha$ we have $\text{lh}_{\beta} \leq \text{lh}_{\alpha}$.
   
   (iv) $\text{pred}_{T}(\alpha + 1)$ is the least $\beta$ such that $\text{cr}_{\alpha} < \nu_{\beta}$.

Fix $\alpha + 1 < \eta$ and $\beta = \text{pred}_{T}(\alpha + 1)$ and $\kappa = \text{cr}_{\alpha}$.

5. Suppose $\beta \in \mathcal{B}$ and $\kappa < \rho_{\beta}$ and $E_{\alpha}$ is total over $B_{\beta} || \rho_{\beta}$. Then $\text{deg}(\alpha + 1) = (m, n)$ and
   
   $$(\rho_{\alpha + 1}^*, M_{\alpha + 1}^*, N_{\alpha + 1}^*) = B_{\alpha + 1}^* = B_{\beta}$$
   
   and
   
   $$B_{\alpha + 1} = \text{Ult}(B_{\alpha + 1}^*, E_{\alpha})$$
   
   and
   
   $$i_{\alpha + 1}^* : M_{\alpha + 1}^* \to M_{\alpha + 1}$$
   
   is the ultrapower map, and likewise $j_{\alpha + 1}^*$, and $i_{\gamma, \alpha + 1}$ and $j_{\gamma, \alpha + 1}$ are defined for $\gamma \leq \tau \alpha + 1$ in the obvious manner.

6. Suppose that $E_{\beta} \in E_{\beta}(M_{\beta})$. Suppose that either $\beta \notin \mathcal{B}$, or $\kappa < \rho_{\beta}$ and $E_{\alpha}$ is not total over $B_{\beta} || \rho_{\beta}$. Then we set $N_{\alpha + 1} = N_{\alpha + 1}^* = \emptyset$, and $j_{\alpha + 1}^*$, etc, are undefined. We set $M_{\alpha + 1}^* \subseteq M_{\beta}$ and $\text{deg}(\alpha + 1)$, etc, in the manner for maximal trees. Let $k = \text{deg}(\alpha + 1)$. Then
   
   $$M_{\alpha + 1} = \text{Ult}_k(M_{\alpha + 1}^*, E_{\alpha})$$
   
   and $i_{\alpha + 1}^*$, etc, are defined in the usual manner. We set $B_{\alpha + 1}^* = M_{\alpha + 1}^*$ and $B_{\alpha + 1} = M_{\alpha + 1}$.

7. Suppose that $E_{\beta} \notin E_{\beta}(M_{\beta})$ (so $E_{\beta} \in E_{\beta}(N_{\beta})$) and $B_{\alpha + 1}$ is not defined through clause 5. Then we proceed symmetrically to clause 6 (interchanging “$M$” with “$N$”).

8. $\alpha + 1 \in \mathcal{D}$ iff either $\emptyset \neq M_{\alpha + 1} \subset M_{\beta}$ or $\emptyset \neq N_{\alpha + 1} \subset N_{\beta}$.

9. For every limit $\lambda < \eta$, $\mathcal{D} \cap [0, \lambda]$ is bounded in $\lambda$, and $\lambda \in \mathcal{B}$ iff $[0, \lambda] \subseteq \mathcal{B}$; the models $M_{\lambda}$, etc, and embeddings $i_{\alpha, \lambda}$, etc, are defined via direct limits.

For $\alpha < \text{lh}(T)$, $\mathcal{B}(\alpha)$ denotes $\max(\mathcal{B} \cap [0, \alpha])$. 

\[\]
3.22 Lemma. Let $T$ be an iteration tree on a bicephalus of degree $(m, n)$ and let $\alpha < \text{lh}(T)$. We write $B_\alpha = B^*_\alpha$, etc. Then:

1. If $\alpha + 1 < \text{lh}(T)$ then $E_\alpha$ is semi-close to $B^{\alpha+1}_\alpha$.
2. If $\alpha + 1 < \text{lh}(T)$ and $\alpha + 1 \notin \mathcal{B}$ then $E_\alpha$ is close to $B^{\alpha+1}_\alpha$.
3. $\mathcal{B}$ is closed downward under $<_T$ and if $\alpha \in \mathcal{M}$ then $\mathcal{N} \cap [0, \alpha]_T = \emptyset$.
4. If $\alpha \in \mathcal{M}$ and $[0, \alpha]_T \cap \mathcal{D} = \emptyset$ then $m \geq 0$.
5. If $\alpha \in \mathcal{M}$, $[0, \alpha]_T \cap \mathcal{D} = \emptyset$, $\deg(\alpha) = m$ and $\beta = \mathcal{B}(\alpha)$ then:
   - $M_\beta$ is $\rho_\beta$-sound, whereas $M_\alpha$ is $\rho_\beta$-solid but not $\rho_\beta$-sound,
   - $M_\beta$ is the $\rho_\beta$-core of $M_\alpha$ and $i_{\beta, \alpha}$ is the $\rho_\beta$-core embedding,
   - $\rho_{m+1}(M_\beta) = \rho_{m+1}(M_\alpha)$,
   - $i_{\beta, \alpha}$ preserves $p_{m+1 \setminus p}$.
6. Suppose $\alpha \in \mathcal{M}$ and $[0, \alpha]_T$ drops in model or degree. Let $k = \deg^T(\alpha)$.
   Then the core embedding $\xi_{k+1}(M_\alpha) \rightarrow M_\alpha$ relates to $T$ in the manner usual for maximal iteration trees.

Proof. Parts 1, 3 and 4 are easy. For part 2, use essentially the proof of [3, 6.1.5], combined with the following simple observation. Let $\xi + 1 < \text{lh}(T)$ be such that $[0, \xi]_T$ does not drop in model and $E_\xi = F(M_\xi)$. Let $\chi = \text{pred}^T(\xi+1)$. Then $[0, \xi+1]_T$ does not drop in model and $\chi$ is the least $\chi' \in [0, \xi]_T$ such that $\text{cr}(F(M_{\chi'})) = \text{cr}_\xi$. We omit further details of the proof of part 2.

Parts 5 and 6 now follow as usual. \hfill \Box

3.23 Definition (Iteration trees on cephalanxes). Let $B$ be a cephalanx. The notion of an iteration tree $T$ on $B$ is defined much as in 3.21. The key differences are as follows. The models of the tree are all either cephalanxes or segmented-premice$^{12}$, and if $B$ is passive, then the models are all either cephalanxes or premice. We write $(M_\alpha, i_{\alpha, \beta})$ and $(Q_\alpha, k_{\alpha, \beta})$, etc, for the models and embeddings above $M'$ and $Q'$ respectively. We write $B_\alpha = (\gamma_\alpha, \rho_\alpha, M_\alpha, Q_\alpha)$ when $B_\alpha$ is a cephalanx, and otherwise $B_\alpha = M_\alpha \neq \emptyset$ or $B_\alpha = Q_\alpha \neq \emptyset$, and write $\mathcal{D}$ for the set of $\alpha$ such that $B_\alpha = Q_\alpha$. Let $i_\alpha = \iota(E_\alpha)$. Other notation is as in 3.21.

Let $\alpha + 1 < \text{lh}(T)$. Then:

- Either $E_\alpha \in E_+(M_\alpha)$ or $E_\alpha \in E_+(Q_\alpha)$.

Let $\kappa = \text{cr}_\alpha$. Then:

- $\text{pred}^T(\alpha + 1)$ is the least $\beta$ such that $\kappa < \iota_\beta$.

$^{12}$In fact, even for the active cephalanxes $B$ we will produce (all in the proof of 4.2), the models of all trees on $B$ will be either cephalanxes or premice.
Suppose $\beta \in \mathcal{B}$. Then:

- If $E_{\beta} \in \mathcal{E}_+ (M_{\beta})$ and either $\rho_{\beta} < \kappa$ or $E_{\alpha}$ is not total over $M_{\beta}$ then $M_{\alpha+1}^* \subseteq M_{\beta}$ and $Q_{\alpha+1} = \emptyset$.

- If $E_{\beta} \notin \mathcal{E}_+ (M_{\beta})$ and either $\rho_{\beta} < \kappa$ or $E_{\alpha}$ is not total over $Q_{\beta}$ then $Q_{\alpha+1}^* \subseteq Q_{\beta}$ and $M_{\alpha+1} = \emptyset$.

Now suppose that $\kappa < \rho_{\beta}$ and $E_{\alpha}$ is total over $B_{\beta} |\langle \rho_{\beta} \rangle$ (so $\kappa \leq \gamma_{\beta}$). Then:

- Suppose either $\kappa < \gamma_{\beta}$ or $E_{\beta} \in \mathcal{E}_+ (M_{\beta})$. Then $B_{\alpha+1}^* = B_{\beta}$.\footnote{Here if $\kappa = \gamma_{\beta}$ (so $E_{\beta} \in \mathcal{E}_+ (M_{\beta})$), one might wonder why we do not just set $M_{\alpha+1}^* = \emptyset$ and $Q_{\alpha+1}^* = Q_{\beta}$. This might be made to work, but doing this, it seems that $E_{\alpha}$ might not be close to $Q_{\alpha+1}$.}

- If $\kappa = \gamma_{\beta}$ and $E_{\beta} \notin \mathcal{E}_+ (M_{\beta})$\footnote{When this situation arises with one of the active cephalanxes we will produce, $Q$ and $Q_{\beta}$ must be type 2 premice.} then $M_{\alpha+1} = \emptyset$ and $Q_{\alpha+1}^* = Q_{\beta}$.\footnote{In this situation it would have been possible to set $B_{\alpha+1}^* = B_{\beta}$, and the reader might object that we are dropping information unnecessarily here. But for the cephalanxes we will produce, our proof of iterability would break down if we set $B_{\alpha+1}^* = B_{\beta}$, and it will turn out that we have in fact carried sufficient information (at least, for our present purposes).}

The remaining details are like in 3.21. \(\dashv\)

**3.24 Lemma.** Let $\mathcal{T}$ be an iteration tree on a cephalanx $B = (\gamma, \rho, M, Q)$ of degree $(m, q)$ and let $\alpha + 1 < \lh(\mathcal{T})$. Then parts 1–6 of 3.22, replacing $'\mathcal{K}'$ with $'\mathcal{B}'$, hold. Parts 5 and 6, replacing $'\mathcal{K}'$ with $'\mathcal{B}'$, $'M'$ with $'Q'$, $'m'$ with $'q'$, and $'\rho'$ with $'\gamma'$, also hold.

**Proof.** This is proved like 3.22. \(\Box\)

**3.25 Definition.** Let $\mathcal{T}$ be an iteration tree on a cephal $B$ and $\alpha + 1 < \lh(\mathcal{T})$. We write $P^T_{\alpha}$ for the active segmented-premouse $P$ such that $E_{\alpha}^T = FP$ and either

- $B$ is a bicephalus and $P \subseteq M^T_{\alpha}$ or $P \subseteq N^T_{\alpha}$, or

- $B$ is a cephalanx and $P \subseteq M^T_{\alpha}$ or $P \subseteq Q^T_{\alpha}$. \(\dashv\)

**3.26 Definition.** Let $B$ be a cephal. A **potential tree on $B$** is a tuple

$$\mathcal{T} = \langle < \mathcal{T}, \langle E_{\alpha} \rangle_{\alpha+1 < \eta} \rangle,$$

such that if $\eta$ is a limit then $\mathcal{T}$ is an iteration tree on $B$, and if $\eta = \gamma + 1$ then $\mathcal{T} |\langle \gamma \rangle$ is an iteration tree on $B$, and $\mathcal{T}$ satisfies all requirements of 3.21, except that we drop the requirement that $B_{\gamma}$ be a cephal or premouse, and add the requirement that $M_{\gamma}$, $N_{\gamma}$, $Q_{\gamma}$, Ult$M_{\gamma}$, Ult$N_{\gamma}$, and Ult$Q_{\gamma}$ are all wellfounded (if defined). \(\dashv\)
The next lemma is easy:

3.27 Lemma. Let $\mathcal{T}$ be a potential tree on a cephal $B$. Then $\mathcal{T}$ is an iteration tree. Moreover, if $\alpha < \beta < \text{lh}(\mathcal{T})$ and $\beta \in \mathcal{B}^\mathcal{T}$ then we can apply 3.20 to $B_\alpha, B_\beta$ and the sequence of extenders used along $(\alpha, \beta)$. Further, assume that if $B$ is an active cephalanx and $\text{lgcd}(Q^B) < \nu(FQ^B)$ then $Q^B$ is a premouse. Then every model of $\mathcal{T}$ is either a cephal or a premouse.

3.28 Definition (Iterability for cephal). Let $B$ be a bicephalus and $\alpha \in \text{OR}$. The length $\theta$ iteration game for $B$ is defined in the obvious way: given $T \upharpoonright \alpha + 1$ with $\alpha + 1 < \theta$, player I must choose an extender $E_\alpha$, and given $T \upharpoonright \lambda$ for a limit $\lambda < \theta$, player II must choose $[0, \lambda]$. The first player to break one of these rules or one of the conditions of 3.21 loses, and otherwise player II wins.

The iteration game for cephalanxes is defined similarly.

We say that a cephal $B$ is $\alpha$-iterable if there is a winning strategy for player II in the length $\alpha$ iteration game for $B$.

3.29 Lemma. Let $B$ be an $(\omega_1 + 1)$-iterable cephal of degree $(m, k)$. Let $T$ be an iteration tree on $B$ and $\alpha < \text{lh}(T)$. Then:

- Suppose $M^T_\alpha \neq \emptyset$. If $\alpha \in \mathcal{B}^T$ let $d = m$; otherwise let $d = \text{deg}^T(\alpha)$. Then suitable condensation holds through $(M^T_\alpha, \max(d, 0))$.

- Suppose $B$ is a cephalanx and $Q^T_\alpha \neq \emptyset$. If $\alpha \in \mathcal{B}^T$ let $d = k$; otherwise let $d = \text{deg}^T(\alpha)$. Then suitable condensation holds below $(Q^T_\alpha, k)$, and if either $[0, \alpha]T$ drops or $Q, Q^T_\alpha$ are premice, then suitable condensation holds through $(Q^T_\alpha, k)$.

Proof. If $T$ is trivial, use 2.14 (for example, $M^B$ is $(m, \omega_1 + 1)$-iterable). This extends to longer trees $T$ by 2.11 and the elementarity of the iteration maps.

3.30 Definition. Let $m < \omega$ and let $M$ be a $\rho$-sound premouse, where $\rho^M_{m+1} \leq \rho \leq \rho^M_m$, and let $\kappa < \text{OR}^M$. We say that $M$ has an $(m, \rho)$-good core at $\kappa$ iff $\kappa < \rho$ and and letting

$$H = \text{cHull}^M_{m+1}(\kappa \cup \vec{p}^M_{m+1}),$$

$H$ is $\kappa$-sound and

$$H \upharpoonright ((\kappa^+)^H = M \upharpoonright ((\kappa^+)^M),$$

and letting $\pi: H \to M$ be the uncollapse map, $\text{cr}(\pi) = \kappa$ and $\pi(\kappa) \geq \rho$ and

$$\pi(p_{m+1}^H \setminus \kappa) = p_{m+1}^M \setminus \kappa.$$

In this context, let $H^M_{m, \kappa} = H$ and let $G^M_{m, \kappa, \rho}$ be the length $\rho$ extender derived from $\pi$.

3.31 Remark. Note that if $M$ has an $(m, \rho)$-good core at $\kappa$ then, with $\pi, H$ as above, we have $\rho^M_{m+1} \leq \kappa$, $M$ is not $(m + 1)$-sound, $G = G^M_{m, \kappa, \rho}$ is semi-close to $H$, $M = \text{Ult}_m(H, G)$ and $i^H_G = \pi$. 22
We can now state and prove a restriction on iterable bicephali.

**3.32 Theorem.** Let $B = (\rho, M, N)$ be an $(\omega_1 + 1)$-iterable non-trivial bicephalus. Then $B$ is not sound. Let $m = m^B$ and $n = n^B$. Then exactly one of the following holds:

(a) $N$ is active type 1 or type 3 with largest cardinal $\rho$, and letting $\kappa = \cf(F^N)$, then $m \geq 0$ and $M$ has an $(m, \rho)$-good core at $\kappa$, and $G^M_{m, \kappa, \rho} = F^N \upharpoonright \rho$.

(b) Vice versa.

**Proof.** Let $B$ be a counterexample. We may assume that $B$ is countable. We mimic the self-comparison argument used in [3, §9]. That is, fix an $(\omega_1 + 1)$-iteration strategy $\Sigma$ for $B$. We form a pair of padded iteration trees $(T, U)$ on $B$, each via $\Sigma$, by comparison. We will ensure that we never use compatible extenders in the process, and use this to show that the comparison terminates, using the ISC and an extra argument. Assuming that $B$ is sound, we will reach a contradiction by showing that the comparison cannot terminate. If $B$ is unsound, we will reach the desired conclusion by examining the circumstances under which the comparison must terminate.

Regarding padding, for each $\alpha$ we will have either $E^T_\alpha \neq \emptyset$ or $E^U_\alpha \neq \emptyset$. If $\alpha = \text{pred}^T(\beta + 1)$ is such that $E^T_\alpha = \emptyset$, then $\beta = \alpha$. Likewise for $U$.

At stage $\alpha$ of the comparison, given $\alpha \in \mathcal{B}^T$, we may set $E^T_\alpha = \emptyset$, and simultaneously declare that, if $T$ is to later use a non-empty extender, then letting $\beta > \alpha$ be least such that $E^T_\beta \neq \emptyset$, we will have $E^T_\beta \in E_+(M^T_\alpha) = E_+(M^U_\alpha)$. Or instead, we may declare that $E^T_\beta \in E_+(N^T_\alpha)$. Toward this, we define non-empty sets $\mathcal{M}^T_\beta \subseteq \{M^T_\beta, N^T_\beta\} \setminus \{\emptyset\}$.

We will require that if $E^T_\beta \neq \emptyset$, then $E^T_\beta \in E_+(P)$ for some $P \in \mathcal{M}^T_\beta$. All models in $\mathcal{M}^T_\beta$ will be non-empty.

We also define sets $S^T_\beta \subseteq t^T_\beta \subseteq \{0, 1\}$ for convenience. Let $0 \in t^T_\beta$ iff $M^T_\beta \neq \emptyset$, and $1 \in t^T_\beta$ iff $N^T_\beta \neq \emptyset$. Let $0 \in S^T_\beta$ iff $M^T_\beta \in \mathcal{M}^T_\beta$, and $1 \in S^T_\beta$ iff $N^T_\beta \in \mathcal{M}^T_\beta$.

(We will explicitly define either $\mathcal{M}^T_\beta$ or $S^T_\beta$, implicitly defining the other.)

The preceding definitions also extend to $U$.

We now begin the comparison. We start with $B^T_0 = B = B^U_0$ and $S^T_0 = S^U_0 = \{0, 1\} = S^U_0$.

Suppose we have defined $(T, U) \upharpoonright \lambda$ for some limit $\lambda$. Then $(T, U) \upharpoonright \lambda + 1$ is determined by $\Sigma$, and $S^T_\lambda = \lim_{\alpha < \tau} S^T_\alpha$, and $S^U_\lambda$ is likewise.

Now suppose we have defined $(T, U) \upharpoonright \alpha + 1$ and $S^T_\alpha$ and $S^U_\alpha$, we determine what to do next (at stage $\alpha$).

**Case 1.** There is $\xi \in \mathcal{O}$ such that for some $Y \in \mathcal{M}^T_\alpha$ and $Z \in \mathcal{M}^U_\alpha$, we have $\xi \leq \text{OR}^Y \cap \text{OR}^Z$ and $Y|\xi \neq Z|\xi$.

Let $\xi$ be least as above and let $\nu$ be the minimum possible value of $\min(\nu(F^Y|\xi), \nu(F^Z|\xi))$ over all choices of $Y, Z$ witnessing the choice of $\xi$. 

\[ \_]
SUBCASE 1.1. For some choice of $Y,Z$ witnessing the choice of $\xi$, $Y|\xi$ and $Z|\xi$ are both active and $\nu(F^Y|\xi) = \nu(F^Z|\xi) = \nu$. Fix such $Y,Z$. We set $E_T^\alpha = F^Y|\xi$ and $E_{\mu}^\alpha = F^Z|\xi$. This determines $(T,\mathcal{U})|\alpha + 2$. Also set $S_{\alpha+1}^T = t_{\alpha+1}^T$ and $S_{\alpha+1}^\mu = t_{\alpha+1}^\mu$.

SUBCASE 1.2. Otherwise.

Then take $Y,Z$ witnessing the choice of $\xi$ and such that either:

- $Y|\xi$ is active, $\nu(F^Y|\xi) = \nu$, and if $Z|\xi$ is active then $\nu(F^Z|\xi) > \nu$; or
- vice versa.

Say $Y|\xi$ is active with $\nu(F^Y|\xi) = \nu$. Then we set $E_T^\alpha = F^Y|\xi$ and $E_{\mu}^\alpha = \emptyset$. This determines $(T,\mathcal{U})|\alpha + 2$. Set $S_{\alpha+1}^T = t_{\alpha+1}^T$. Now suppose there is $X \in \mathcal{M}_\alpha$ with $X|\xi$ active and $\nu(F^X|\xi) = \nu$. Then $X|\xi = Y|\xi$, so we must be careful to avoid setting $E_T^\beta = F^X|\xi$ at some $\beta > \alpha$. So we set $\mathcal{M}_{\alpha+1} = \{Z\}$, and set $S_{\alpha+1}^\mu$ accordingly. If there is no such $X$ then set $S_{\alpha+1}^\mu = S_{\alpha}^\mu$. (In any case, later extenders used in $\mathcal{U}$ will be incompatible with $E_T^\alpha$.) The remaining cases are covered by symmetry.

CASE 2. Otherwise.

Then we stop the comparison at stage $\alpha$.

This completes the definition of $(T,\mathcal{U})$. For $\alpha < \text{lh}(T,\mathcal{U})$, let $S_T^\alpha(\alpha)$ be the largest $\beta \leq_T \alpha$ such that $S_T^\beta = \{0,1\}$; here if $\alpha \in \mathcal{B}^T$ then $B_T^\beta = B_T^\alpha$. Let $S^\mu(\alpha)$ be likewise.

CLAIM 1. The comparison terminates at some stage.

Proof. This follows from the ISC essentially as in the proof that standard comparison terminates (using the fact that we observe the restricting sets $\mathcal{M}_{\alpha+1}^T, \mathcal{M}_{\alpha+1}^\mu$ as described above).

So let $\alpha$ be such that the comparison stops at stage $\alpha$.

CLAIM 2. card$(S_T^\alpha) = \text{card}(S^\mu_\alpha) = 1$ and $\mathcal{M}^T_\alpha = \mathcal{M}^\mu_\alpha$.

Proof. If $\alpha \in \mathcal{B}^T$ then $B_T^\alpha$ is non-trivial, by 3.27; likewise for $\mathcal{U}$. So because Case 2 attains at stage $\alpha$, we do not have $S_T^\alpha = S^\mu_\alpha = \{0,1\}$.

It is not true that $(\dagger)$ $Q \triangleleft P$ or $P \triangleleft Q$ for some $Q \in \mathcal{M}^T_\alpha$ and $P \in \mathcal{M}^T_\alpha$. For suppose $(\dagger)$ holds; we may assume $Q \triangleleft P$. Then $Q$ is sound, so by 3.22, $\alpha \in \mathcal{B}^\mu$, so by $(\dagger)$ and Case 2 hypothesis, card$(S^\mu_\alpha) = 1$. Say $S^\mu_\alpha = \{0\}$. Let $\beta = S^\mu(\alpha)$. Then $B_T^\beta = B^\mu_\alpha$ and for all $\gamma \in [\beta,\alpha)$, $E_T^\gamma = \emptyset$, and $E_T^\beta \in \mathcal{E}_\alpha(N^\mu_\beta)$. Let $\varphi = \rho^\mu_\beta$. Then $\text{lh}_\varphi \geq (\varphi^+)B^\mu_\beta$. So $P(\varphi) \cap P = P(\varphi) \cap B^\mu_\beta$, contradicting the fact that $M^\mu_\beta = Q \triangleleft P$.

Now suppose that $S_T^\alpha = \{0,1\}$ but card$(S^\mu_\alpha) = 1$. Let $\delta$ be least such that $M^\mu_\alpha|\delta \neq N^\mu_\alpha|\delta$. Let $Q \in \mathcal{M}^\mu_\alpha$. Then $Q \triangleleft M^\mu_\alpha|\delta = N^\mu_\alpha|\delta$, so $(\dagger)$ holds; contradiction. So card$(S^\mu_\alpha) = \text{card}(S^\mu_\alpha) = 1$, and because $(\dagger)$ fails, $M^\mu_\alpha = \mathcal{M}^\mu_\alpha$.

CLAIM 3. $\alpha \in \mathcal{B}^T \triangle \mathcal{B}^\mu$.  

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Proof. By Claim 2, $\alpha \notin \mathcal{B}^T \cap \mathcal{B}^U$, so assume that $\alpha \notin \mathcal{B}^T \cup \mathcal{B}^U$. Then standard calculations using 3.22 give that $\mathcal{T}, \mathcal{U}$ use compatible extenders, a contradiction.

Using the previous claims, let us assume that $\alpha \in \mathcal{B}^T \setminus \mathcal{B}^U$, $S^T_\alpha = \{0\}$ and $S^U_\alpha = \{1\}$, so $\tilde{B} = B^T_\alpha$ is a bicephalus, $\alpha \in \mathcal{N}^U$, and $M^U_\alpha = N^U_\alpha$; the other cases are almost symmetric. We will deduce that conclusion (a) of the theorem holds; under symmetric assumptions (b) can hold instead. Let $\beta = S^T(\alpha)$. Let $\tilde{\rho} = \rho(\tilde{B})$. Then $\tilde{B} = B^T_\beta$ and for all $\gamma \in [\beta, \alpha)$, we have $E^T_\gamma = \emptyset \neq E^U_\gamma$ and $(\tilde{\rho}^+)B \leq lh^U_\gamma$.

Claim 4. $\alpha = \beta + 1$ and $lh^U_\beta = (\tilde{\rho}^+)\tilde{B}$ and $E^U_\beta$ is type 1 or type 3.

Proof. Suppose the claim fails. Then by 3.22, $N^U_\alpha$ is not $\tilde{\rho}$-sound (recall that if $\alpha > \beta + 1$ and $lh^U_{\beta+1} = lh^U_\beta$ then $E^U_{\beta+1}$ is type 2). But by 3.22, $M^T_\alpha$ is $\tilde{\rho}$-sound. So $M^T_\alpha \neq N^U_\alpha$, contradiction.

Let $\tilde{B} = (\tilde{\rho}, \tilde{M}, \tilde{N}) = B^T_\alpha = B^T_\beta$. Since $E^U_{\beta} \in E_+ (\tilde{N})$, and $lh^U_\beta = (\tilde{\rho}^+)\tilde{B}$, $\tilde{N}((\tilde{\rho}^+)\tilde{B})$ projects to $\tilde{\rho}$, so $OR(\tilde{N}) = (\tilde{\rho}^+)\tilde{B}$ and $F\tilde{N} = E^U_{\beta}$. Let $\tilde{F} = F\tilde{N}$ and $\tilde{k} = cr(\tilde{F})$. It follows that (a) of the theorem holds regarding $\tilde{B}$; using the iteration embeddings we will deduce that $B$ is not sound, and (a) holds regarding $B$. Note that either $OR(\tilde{M}) > OR(\tilde{N})$, or $OR(\tilde{M}) = OR(\tilde{N})$, $\tilde{N}$ has superstrong type and $\tilde{M}$ is type 2; in either case $m \geq 0$. Also $OR^N = (\rho^+)B$ and $N$ is active with $F = F^N$, a preimage of $\tilde{F}$. Let $\kappa = cr(\tilde{F})$; so $\kappa < \rho$.

Claim 5. $M$ is not $m+1$-sound, so $B$ is not sound.

Proof. Suppose $M$ is $m+1$-sound. Let $z = \zeta^M_{m+1}$ and $\zeta = \zeta^M_{m+1}$. By [5, 2.17], $z = p^M_{\alpha+1}$ and $\zeta = \rho^M_{m+1} \leq \rho$. So

$$\kappa \in \text{Hull}^M_{m+1} (\zeta \cup z \cup \rho^M_{m+1}).$$

Let $\tilde{\zeta} = \zeta^M_{m+1}$ and $\tilde{\zeta} = \zeta^M_{m+1}$. By [5, 2.20], $\tilde{\zeta} = i^T_{0, \alpha} (z)$ and $\tilde{\zeta} = \sup i^T_{0, \alpha} \zeta$, so $\tilde{\zeta} \leq \tilde{\rho}$ and

$$i^T_{0, \alpha} (\kappa) \in \text{Hull}^M_{m+1} (\tilde{\zeta} \cup \tilde{\zeta} \cup \rho^M_{m+1}). \quad (2)$$

Let $\tilde{H} = N^U_{\alpha}$. Then $\tilde{M} = N^U_{\alpha} = \text{Ult}_m (\tilde{H}, \tilde{F})$ and $\tilde{\zeta} = \sup i^\tilde{H}_{\tilde{F}} \zeta^\tilde{H}$, and since $\tilde{\zeta} \leq \tilde{\rho}$, therefore $\tilde{\zeta} \leq \tilde{k}$. Also, $\tilde{\zeta} = i^\tilde{H}_{\tilde{F}} (\zeta^\tilde{M}_{m+1})$. But $\tilde{k} \notin \text{rg}(i^\tilde{H}_{\tilde{F}})$, so

$$\tilde{k} \notin \text{Hull}^M_{m+1} (\tilde{\zeta} \cup \tilde{\zeta} \cup \rho^M_{m+1}). \quad (3)$$

But $i^T_{0, \alpha} \upharpoonright \rho = j^T_{0, \alpha} \upharpoonright \rho$, so $i^T_{0, \alpha} (\kappa) = \tilde{k}$, contradicting lines (2) and (3).

We can now complete the proof.

Claim 6. Conclusion (a) of the theorem holds.
Proof. Suppose $N$ is type 1. Let $\bar{\rho} = p_{m+1}^M \setminus \rho$ and 

$$\bar{H} = \text{cHull}_m^M(\tilde{\kappa} \cup \bar{\rho} \cup \tilde{\rho}_m)$$

and let $\tilde{\pi} : \bar{H} \to M$ be the uncollapse. Then $\bar{H} = N^\bar{\mu}_\alpha$, $\tilde{\pi} = j^\bar{\mu}_\alpha$, $\bar{H}$ is $\tilde{\kappa}$-sound and letting $\tilde{q} = p_{m}^\bar{H} \setminus \tilde{\kappa}$, we have $\pi(\tilde{q}) = \bar{\rho}$,

$$\tilde{H}|| (\tilde{\kappa}^+) \bar{H} = \tilde{M}|| (\tilde{\kappa}^+) \tilde{M} = \tilde{\mathcal{N}}|| (\tilde{\kappa}^+) \tilde{\mathcal{N}}$$

and $\rho_m^\bar{\kappa} = \sup \tilde{\pi}^m \rho_m^{\tilde{\rho}}$.

We have $\kappa, H, \pi$ defined as in (a); let $\bar{p} = p_{m+1}^M \setminus \rho$. Let $\langle M_\gamma \rangle_{\gamma < \rho_m}$ be the natural stratification of $\text{Hull}_m^M(\kappa \cup \rho \cup \rho_m^M)$ (the uncollapsed hull), and let $H_\gamma$ be the transitive collapse of $M_\gamma$ and $\pi_\gamma : H_\gamma \to M_\gamma$ the uncollapse map. (For example if $m = 0$ and $M$ is passive, $M_\gamma = \text{Hull}_1^M(\kappa \cup \rho)$. If $M$ is active or $m > 0$ use the stratification of $\text{rΣ}_{m+1}$ truth described in [3, §2.]) Let $\kappa' = \pi(\kappa)$ and let $E_\gamma$ be the (short) extender $E_\gamma$ of length $\kappa'$ derived from $\pi_\gamma$. Then $H_\gamma, E_\gamma \in M$. Let $\tilde{\pi}_\gamma : \tilde{H}_\gamma \to \tilde{M}_\gamma$ and $\tilde{\kappa}'$ and $\tilde{E}_\gamma$ be defined likewise over $\tilde{M}$.

We have $(\tilde{\tilde{\pi}}_\gamma \sim \tilde{\tilde{M}})|| (\tilde{\kappa}^+) \tilde{H}_\gamma$ and $\tilde{E}_\gamma \setminus \tilde{\rho} \subseteq F^\mathcal{N}$ for each $\gamma$; the former is because by 2.13, $N$ $\bar{\mathcal{N}}$“Lemma 2.13 holds for my proper segments”.

Let $i = i^\mu_{\tilde{\mu}, \alpha}$. Since $i$ is an $m$-embedding and $i(\kappa, \rho) = (\tilde{\kappa}, \bar{\rho})$, for each $\gamma < \rho_m$, $i(H_\gamma) = \tilde{H}_i(\gamma)$, and $i(\kappa') = \tilde{\kappa}'$ and $i(E_\gamma) = \tilde{E}_i(\gamma)$. Also $\rho_m^\bar{\kappa} = \sup i^\mu \rho_m^\mu$ and $\text{OR}^\mathcal{N} = \sup j^\mu \text{OR}^\mathcal{N}$ and $i, j$ are continuous at $(\tilde{\kappa}^+) \mathcal{N}$ and $j^\mu F^\mathcal{N} \subseteq F^\tilde{\mathcal{N}}$.

It follows easily that $(H_\gamma \sim \bar{M}|| (\kappa^+) \bar{H}_\gamma$ and $E_\gamma \setminus \rho \subseteq F^\mathcal{N}$ for each $\gamma < \rho_m$. Therefore $H|| (\kappa^+) \bar{H} = \bar{M}|| (\kappa^+) \bar{M}$ and $F^\mathcal{N} \setminus \rho$ is derived from $\pi$. It follows that $F^\mathcal{N}$ is semi-close to $H$, $M = \text{Ult}_m(H, F^\mathcal{N})$, and $\pi = i^\mu_{\tilde{\mu}, \alpha}$ (because we can factor the embedding $\pi : H \to M$ through $\text{Ult}_m(H, F^\mathcal{N})$, and $\nu(F^\mathcal{N}) = \rho$). So by [5], $\pi(z_{m+1}) = z_{m+1}^\kappa$, but $z_{m+1}^\kappa \setminus \rho = p_{m+1}^\kappa \setminus \rho$, and therefore $z_{m+1}^\kappa | \kappa = p_{m+1}^\kappa | \kappa$, so $H$ is $\kappa$-sound. This completes the proof assuming that $N$ is type 1.

If instead, $N$ is type 3, then almost the same argument works. □

This completes the proof of the theorem. □

We now move on to analogues of 3.32 for cephalanxes.

3.33 Definition. Let $B$ be a passive cephalanx of degree $(m, q)$ and let $N = N^B$. We say that $B$ has a good core if $m \geq 0$ and $N$ is active and letting $F = F^\mathcal{N}$, $\kappa = \text{cr}(F)$ and $\nu = \nu(F)$, we have:

- $\text{OR}^\mathcal{N} = (\kappa^+) M$ and $N$ is type 1 or 3,
- $M$ has an $(m, \nu)$-good core at $\kappa$,
- $G_{m, \nu, \kappa}^M = F \setminus \nu$, and
- if $N$ is type 1 then $H_{m, \kappa}^M = Q$ and $m = q$. □
3.34 Theorem. Let $B = (\gamma, \rho, M, Q)$ be an $(\omega_1+1)$-iterable, non-trivial, passive cephalanx. Then $B$ is not sound, and $B$ has a good core.

Proof. The proof is based on that of 3.32. The main difference occurs in the rules guiding the comparison, so we focus on these.

We may assume that $B$ is countable. We define padded iteration trees $\mathcal{T}, \mathcal{U}$ on $B$, and sets $S^\mathcal{T}_\alpha, S^\mathcal{U}_\alpha, R^\mathcal{T}_\alpha, R^\mathcal{U}_\alpha$, much as before. We start with $S^\mathcal{T}_0 = S^\mathcal{U}_0 = \{0, 1\}$. At limit stages, proceed as in 3.32. Suppose we have defined $(\mathcal{T}, \mathcal{U}) \upharpoonright \alpha+1$, $S^\mathcal{T}_\alpha$ and $S^\mathcal{U}_\alpha$ and if $\operatorname{card}(S^\mathcal{T}_\alpha) = \operatorname{card}(S^\mathcal{U}_\alpha) = 1$ then $B^\mathcal{T}_\alpha \not\sqsubset B'^\mathcal{U}_\alpha \not\sqsubset B^\mathcal{T}_\alpha$ (otherwise the comparison has already terminated). We just consider enough cases that the rest are covered by symmetry.

Case 1. $\operatorname{card}(S^\mathcal{T}_\alpha) = \operatorname{card}(S^\mathcal{U}_\alpha) = 1$.

Choose extenders as usual (as in 3.32).

Case 2. $S^\mathcal{T}_\alpha = \{0, 1\}$ and if $S^\mathcal{U}_\alpha = \{0, 1\}$ then $\rho^\mathcal{T}_\alpha \leq \rho^\mathcal{U}_\alpha$.

So $B^\mathcal{T}_\alpha$ is a cephalanx; let $B^\mathcal{T}_\alpha = (\gamma^\mathcal{T}_\alpha, \rho^\mathcal{T}_\alpha, M^\mathcal{T}_\alpha, Q^\mathcal{T}_\alpha) = B^\mathcal{T}_\alpha$. Let $B^\mathcal{U}_\alpha = B'_\alpha$. We will have by induction that for every $\beta < \alpha$, $lh^\mathcal{T}_\beta \leq \rho^\mathcal{T}_\alpha$ and $lh^\mathcal{U}_\beta \leq \rho^\mathcal{U}_\alpha$. Since $B$ is passive, $B^\mathcal{T} \upharpoonright \rho^\mathcal{T}_\alpha$ and $B^\mathcal{U} \upharpoonright \rho^\mathcal{U}_\alpha$ are well-defined premice.

Subcase 2.1. $B^\mathcal{T} \upharpoonright \rho^\mathcal{T}_\alpha \neq B^\mathcal{U} \upharpoonright \rho^\mathcal{U}_\alpha$.

Choose extenders as usual.

Suppose $B^\mathcal{T} \upharpoonright \rho^\mathcal{T}_\alpha = B'^\mathcal{U} \upharpoonright \rho^\mathcal{U}_\alpha$. We say that in $\mathcal{T}$ we move into $M^\mathcal{T}$ to mean that we either set $E^\mathcal{T}_\alpha \neq \emptyset$ and $E^\mathcal{T}_\alpha \supseteq \mathcal{E}_+(M^\mathcal{T})$, or set $E^\mathcal{T}_\alpha = \emptyset$ and $S^\mathcal{T}_{\alpha+1} = \{0\}$. Likewise for move into $Q^\mathcal{T}$, and likewise with regard to $\mathcal{U}$ if $S^\mathcal{U}_\alpha = \{0, 1\}$. In each case below we will move into some model in $\mathcal{T}$. In certain cases we do likewise for $\mathcal{U}$. These choices will produce two premice $R, S$ from which to choose $E^\mathcal{T}_\alpha, E^\mathcal{U}_\alpha$, in the usual manner, given that $R \not\sqsubset S \not\sqsubset R$ (for example, if $S^\mathcal{U}_\alpha = \{1\}$ and in $\mathcal{T}$ we move into $M^\mathcal{T}$, then $R = M^\mathcal{T}$ and $S = Q^\mathcal{U}$). If $R \sqsubset S$ or $S \sqsubset R$, then we terminate the comparison, and say that the comparison terminates early. If $B'^\mathcal{U}$ is a cephalanx and we do not move into any model in $\mathcal{U}$ and $E^\mathcal{U}_\alpha = \emptyset$ then we set $S^\mathcal{U}_{\alpha+1} = \{0, 1\}$.

Subcase 2.2. $\operatorname{card}(S^\mathcal{U}_\alpha) = 1$ and $B^\mathcal{T} \upharpoonright \rho^\mathcal{T}_\alpha = B'^\mathcal{U} \upharpoonright \rho^\mathcal{U}_\alpha$.

Let $P \in \mathcal{M}_\alpha$.

If $Q^\mathcal{T} \sqsubseteq P$ then in $\mathcal{T}$ we move into $M^\mathcal{T}$.

If $Q^\mathcal{T} \not\sqsubseteq P$ then in $\mathcal{T}$ we move into $Q^\mathcal{T}$.

Subcase 2.3. $S^\mathcal{T}_\alpha = S^\mathcal{U}_\alpha = \{0, 1\}$ and $B^\mathcal{T} \upharpoonright \rho^\mathcal{T}_\alpha = B'^\mathcal{U} \upharpoonright \rho^\mathcal{U}_\alpha$.

Let $(\gamma^\mathcal{U}_\alpha, \rho^\mathcal{U}_\alpha, M^\mathcal{U}_\alpha, Q^\mathcal{U}_\alpha) = B'^\mathcal{U}$. So $\rho^\mathcal{T}_\alpha \leq \rho^\mathcal{U}_\alpha$.

Suppose $Q^\mathcal{T} = Q^\mathcal{U}$. Let $X \in \{0, 1\}$ be random. Then:16

- If $\rho^\mathcal{T}_\alpha < \rho^\mathcal{U}_\alpha$ or $X = 0$ then in $\mathcal{T}$ we move into $M^\mathcal{T}$, and if also $M^\mathcal{T} \upharpoonright \rho^\mathcal{T}_\alpha = B'^\mathcal{U} \upharpoonright \rho^\mathcal{U}_\alpha$ then in $\mathcal{U}$ we move into $Q^\mathcal{U}$.

- If $\rho^\mathcal{T}_\alpha = \rho^\mathcal{U}_\alpha$ and $X = 1$ then in $\mathcal{U}$ we move into $M^\mathcal{U}$ and in $\mathcal{T}$ we move into $Q^\mathcal{T}$.

16 We use the random variable $X$ just for symmetry.
If \( Q^T \vartriangleleft Q^U \), then in \( T \) we move into \( M^T \). (Note that \( \rho^T \vartriangleleft \rho^U \) and \( Q^T \vartriangleleft B^U\lVert \rho^U \), so we do not need to move into any model in \( U \).)

If \( Q^U \vartriangleleft Q^T \), then in \( T \) we move into \( Q^T \) and in \( U \) we move into \( M^U \). (Note that \( \rho^T \vartriangleleft \rho^U \).

Suppose \( Q^T \not\vartriangleleft Q^U \not\vartriangleleft Q^T \). Then in \( T \) we move into \( Q^T \). If also \( Q^T\lvert \rho^U = B^U\lVert \rho^U \), then in \( U \) we move into \( Q^U \).

The remaining cases are determined by symmetry.

The comparison terminates as usual. We now analyse the manner in which it terminates.

**Claim 1.** Let \( \alpha < \text{lh}(T, U) \). Then (i) the comparison does not terminate early at stage \( \alpha \); (ii) if at stage \( \alpha \), in \( T \) we move into \( R \), then for every \( \beta \in (\alpha, \text{lh}(T, U)) \), \( R \not\vartriangleleft S \) for any \( S \in \mathcal{M}_\alpha \).

**Proof.** By induction on \( \alpha \). Suppose for example that Subcase 2.2 attains at stage \( \alpha \). We have \( P \in \mathcal{M}_\alpha \).

Suppose \( Q^T \not\vartriangleleft P \), so in \( T \) we move into \( M^T \). We have \( M^T\lvert \rho^T = P\lvert \rho^T \) and \( N^T \vartriangleleft Q^T \vartriangleleft P \) and \( M^T \neq N^T \) and

\[
M^T \lvert ((\rho^T)^+)M^T = N^T \lvert ((\rho^T)^+)N^T
\]

and both \( M^T, N^T \) project \( \leq \rho^T \). So \( M^T \not\vartriangleleft P \) and letting \( \lambda \) be least such that \( M^T\lvert \lambda \neq N^T\lvert \lambda \), we have

\[
\rho^T < \lambda \leq \text{min}(\text{OR}(M^T), \text{OR}(N^T)).
\]

So the comparison does not terminate early at stage \( \alpha \), and since \( M^T \) projects \( \leq \rho^T \), for no \( \beta > \alpha \) can we have \( M^T \vartriangleleft S \in \mathcal{M}_\beta \).

Now suppose \( Q^T \not\vartriangleleft P \), so in \( T \) we move into \( Q^T \). If \( \alpha \not\in \mathbb{B}^U \) then \( P = B^U \) is unsound. Otherwise there is \( \delta < \alpha \) such that at stage \( \delta \), in \( U \) we moved into \( P \).

In either case (by induction in the latter), \( P \not\vartriangleleft Q^T \). So again, the comparison does not terminate early at stage \( \alpha \). Let \( \lambda \) be least such that \( Q^T\lvert \lambda \neq P\lvert \lambda \). Then \( \rho^T < \lambda \) and since \( Q^T \) projects \( \leq \gamma^T \), there is no \( \beta > \alpha \) such that \( Q^T \vartriangleleft S \in \mathcal{M}_\beta \).

The proof is similar in the remaining subcases. \[ \Box \]

So let \( \alpha + 1 = \text{lh}(T, U) \). As in the proof of 3.32, we have \( \text{card}(S^T_\alpha) = \text{card}(S^U_\alpha) = 1 \) and \( \alpha \in \mathbb{B}^T \Delta \mathbb{B}^U \). We may assume that \( \alpha \in \mathbb{B}^T \), so \( B^T_\alpha = (\gamma^T, \rho^T, M^T, Q^T) \) is a cephalanx and \( B^U_\alpha \) is not. Then \( B^U_\alpha \) is not sound, so letting \( P \in \mathcal{M}^U_\alpha \), we have \( P \vartriangleleft B^U_\alpha \). But by Claim 1, \( P \not\vartriangleleft B^U_\alpha \), so \( P \vartriangleleft B^U_\alpha \). Let \( \beta = S^T(\alpha) \).

**Claim 2.** \( S^T_\alpha = \{0\} \).

**Proof.** Suppose \( S^T_\alpha = \{1\} \), so \( Q^T = P = B^U_\alpha \) is \( \gamma \)-sound. At stage \( \beta \), in \( T \) we move into \( Q^T \). For all \( \xi \in [\beta, \alpha) \), \( E^T_\xi = \emptyset \), so \( E^U_\xi \neq \emptyset \), and \( \rho^T < \text{lh}^U_\xi \), because \( B^U\lvert \rho^T = B^U\lvert \rho^T \), and therefore \( \rho^T \leq \nu^U_\xi \), because \( \rho^T \) is a cardinal of \( Q^T \). But then \( B^U_\alpha \) is not \( \gamma \)-sound, contradicting the fact that \( Q^T = B^U_\alpha \). \[ \Box \]

So \( M^T = P = B^U_\alpha \). Let \( N^T = N^T_\alpha \).

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Claim 3. We have:

- OR\(N'\) = \((\rho')^+)M'\),
- \(N'\) is active type 1 or type 3,
- \(\alpha = \beta + 1\),
- \(E_{\beta}^M = F^{N'}\),
- if \(N'\) is type 1 then \(B_\alpha^{\text{ul}} = Q'\).

Proof. Assume, for example, that Subcase 2.2 attains at stage \(\beta\). So \(N' \subseteq Q' \subseteq B_\beta^{\text{ul}}\). We have \(M' \neq N'\), both \(M', N'\) project to \(\rho'\), and

\[M'\|(\rho')^+)M' = N'\|(\rho')^+)N'\.

We have \(E_{\beta}^M = \emptyset\), so \(E_{\beta}^M \neq \emptyset\) and note that \(E_{\beta}^M \in E_+(N')\) and \(lh_{\beta}^M > \rho'\). Since \(M' = B_\alpha^{\text{ul}}\) is \(\rho'\)-sound it follows that \(\alpha = \beta + 1\) and \(\nu_\beta^{M'} = \rho'\), so \(E_{\beta}^M\) is type 1 or type 3. Therefore \(N'[lh_{\delta}^M]\) projects to \(\rho'\), so \(OR(N') = lh_{\delta}^M\).

Now suppose further that \(N'\) is type 1; we want to see that \(B_\alpha^{\text{ul}} = Q'\). We have \(Q' \subseteq B_\beta^{\text{ul}}\) and \(cr(F^{N'}) = \gamma'\) and \(\rho_\omega(Q') \leq \gamma'\) and

\[\mathcal{P}(\gamma') \cap Q' = \mathcal{P}(\gamma') \cap N'\.

So it suffices to see that \(\text{pred}^M(\alpha) = \beta\). We may assume that \(lh_{\delta}^M = \rho'\) for some \(\delta < \beta\). Then \(\rho'\) is a cardinal of \(B_\delta^{\text{ul}}\), so \(Q' \neq B_\beta^{\text{ul}}\), so \(Q' = B_\beta^{\text{ul}}\). So \(B_\beta^{\text{ul}}\) is \(\gamma'\)-sound, so there is a unique \(\delta\) such that \(lh_{\delta}^M = \rho'\), and moreover, \(E_{\beta}^M\) is type 3 and \(\beta = \delta + 1\). Therefore \(\text{pred}^M(\alpha) = \beta\), as required. \(\square\)

To complete the proof, one can now argue like in Claim 6 of 3.32. \(\square\)

3.35 Remark. We next proceed to the version of 3.34 for active cephalanxes \(B = (\gamma, \rho, M, Q)\). Here things are more subtle for two reasons. First, if \(Q\) is type 3 then we can have \(\alpha\) such that \(Q_\alpha^M\) or \(Q_\alpha^{\text{ul}}\) is not a premouse, and in particular, its active extender can fail the ISC; this complicates the proof that the comparison terminates. Second, if \(Q\) is superstrong then the comparison termination proof is complicated further, and more importantly, it seems that we need not get the analogue of a good core (see 3.44), and moreover, in this case we do not see how to rule out the possibility that \(B\) is exact and \(M\) is sound with \(\rho_{m+1}^M = \rho\). In fact, it is easy enough to illustrate how the latter might occur. Let \(Q\) be a sound superstrong premouse and \(\kappa = \text{cr}(F^Q)\) and let \(J\) be a sound premouse such that \(J\|(\kappa^+)^J = Q\|(\kappa^+)Q\) and \(\rho_{m+1}^J = (\kappa^+)Q = (\kappa^+)J < \rho_m^J\.

Let \(M = \text{Ult}_n(J, F^Q)\) and \(B = (\gamma, \rho, M, Q)\), where \(\rho = OR^Q\) and \(\gamma\) is the largest cardinal of \(Q\). Suppose that \(M\) is wellfounded. Then \(B\) is an exact, sound bicephalanx. (We have \(\rho_{m+1} = \rho < \rho_m^M\) and \(M\) is \(m+1\)-sound, and \(B\) is exact because \(i_{F^Q}^Q\) and \(i_{F^Q}^O\) are both continuous at \((\kappa^+)^J\).) It seems that reasonable that such a pair \((J, Q)\) might arise from as iterates of a single model,
and so it seems that \( B \) might also be iterable. Conversely, we will show that the kind of example illustrated here is the only possibility (other than that given by good cores).

3.36 Definition. Let \( \mathcal{T} \) be an iteration tree on an active cephalanx \( B \) and \( \alpha + 1 < \text{lh}(\mathcal{T}) \). We say \( \alpha \) is \( \mathcal{T} \)-special iff \( \alpha \in \mathcal{B}^\mathcal{T} \) and \( E^\mathcal{T}_\alpha = F(Q^\mathcal{T}_\alpha) \).

3.37 Lemma. Let \( \mathcal{T} \) be an iteration tree on an active cephalanx \( B \) and \( \alpha < \text{lh}(\mathcal{T}) \). Then:

(a) If \( \alpha \in \mathcal{B}^\mathcal{T} \) then \( Q^\mathcal{T}_\alpha \) has superstrong type iff \( Q \) does.

(b) If \( \iota(Q^B) = \gamma^B \) then \( \mathcal{Q} = \emptyset \).

Suppose also that \( \alpha + 1 < \text{lh}(\mathcal{T}) \). Then:

(c) If \( \alpha \) is \( \mathcal{T} \)-special then \( \alpha + 1 \in \mathcal{B}^\mathcal{T} \) and \( E^\mathcal{T}_\alpha \) has superstrong type, \( \lambda \) is \( \text{OR}(\mathcal{B}^\mathcal{T}_{\alpha+1}) \) such that \( \text{cr}(F(Q^\mathcal{T}_\epsilon)) = \text{cr}^T_\alpha \).

(d) If \( B \) is a pm-cephalanx and \( P^\mathcal{T}_\alpha \) is not a premouse then \( \alpha \) is \( \mathcal{T} \)-special (so \( P^\mathcal{T}_\alpha = Q^\mathcal{T}_\alpha \)) and \( Q \) is type 3.

Proof. For (a), recall that in \( \mathcal{T} \), we only form simple ultrapowers of \( Q^B \) and its images.

3.38 Lemma. Let \( \mathcal{T} \) be an iteration tree on an active pm-cephalanx \( B = (\gamma, \rho, M, Q) \). Let \( \alpha < \beta < \text{lh}(\mathcal{T}) \). Let \( \lambda = \text{lh}^\mathcal{T}_\alpha \). Then either:

1. \( \beta \notin \mathcal{B}^\mathcal{T} \) and either (i) \( \lambda < \text{OR}(\mathcal{B}^\mathcal{T}_\beta) \) and \( \lambda \) is a cardinal of \( \mathcal{B}^\mathcal{T}_\beta \), or (ii) \( \beta = \alpha + 1 \), \( E^\mathcal{T}_\alpha \) has superstrong type, \( \lambda = \text{OR}(\mathcal{B}^\mathcal{T}_\beta) \) and \( \mathcal{B}^\mathcal{T}_\beta \) is an active type 2 premouse; or

2. \( \beta \in \mathcal{B}^\mathcal{T} \) and either (i) \( \lambda < \rho(\mathcal{B}^\mathcal{T}_\beta) \) and \( \lambda \) is a cardinal of \( \mathcal{B}^\mathcal{T}_\beta \), or (ii) \( \beta = \alpha + 1 \), \( E^\mathcal{T}_\alpha \) has superstrong type, \( \lambda = \rho(\mathcal{B}^\mathcal{T}_\beta) \), and letting \( \epsilon = \text{pred}^\mathcal{T}(\beta) \), \( \text{cr}^T_\alpha = \gamma(\mathcal{B}^\mathcal{T}_\beta) \).

Therefore if \( \text{lh}^\mathcal{T}_\alpha < \text{lh}^\mathcal{T}_\beta \) then \( \text{lh}^\mathcal{T}_\alpha \) is a cardinal of \( \mathcal{P}^\mathcal{T}_\beta \).

Proof. If \( \beta = \alpha + 1 \) it is straightforward to prove the conclusion. Now suppose \( \beta > \alpha + 1 \). If \( \lambda < \text{lh}^\mathcal{T}_{\alpha+1} \) it is straightforward, so suppose \( \lambda = \text{lh}^\mathcal{T}_{\alpha+1} \). Then since the lemma held for \( \beta = \alpha + 1 \), either \( E^\mathcal{T}_{\alpha+1} \) is type 2, in which case things are straightforward, or \( \alpha + 1 \) is \( \mathcal{T} \)-special, so letting \( \mu = \text{cr}^T_{\alpha+1} \) and \( \chi = \text{pred}^\mathcal{T}(\alpha+2) \), we have that \( B^\mathcal{T}_{\alpha+2} = B^\mathcal{T}_\chi \) is a cephalanx and \( \mu < \gamma(\mathcal{B}^\mathcal{T}_\chi) \), which implies that \( \lambda < \rho(\mathcal{B}^\mathcal{T}_{\alpha+2}) \) and \( \lambda \) is a cardinal of \( \mathcal{B}^\mathcal{T}_{\alpha+2} \). The rest is clear.

3.39 Definition. Let \( B = (\gamma, \rho, M, Q) \) be an active cephalanx of degree \((m, 0)\). We say that \( B \) is exceptional iff

- \( B \) is active and exact,
- \( Q \) has superstrong type, and

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3.40 Lemma. Let $M$ be an $m$-sound premouse and let $\rho_{m+1}^M \leq \gamma < \rho_m^M$. Then $M$ is $\gamma$-sound iff $M = \text{Hull}_{m+1}^M (\gamma \cup z_{m+1}^M \cup \vec{p}_m^M)$.

Proof. This follows from [5, 2.17].

3.41 Lemma. Let $B, B'$ be active cephalanxes such that $B'$ is an iterate of $B$. Then $B'$ is exceptional iff $B$ is exceptional.

Proof. By 3.27, 3.37(a) and 3.40 and [5, 2.20].

3.42 Definition. Let $B = (\gamma, \rho, M, Q)$ be an active cephalanx of degree $(m, 0)$. We say that $B$ has a exceptional core iff $Q$ has superstrong type and letting $F = F^Q, \kappa = \text{cr}(F), X = i_F^Q \cup (\kappa^+)^M, m' = \max(m, 0)$,

$$H = \text{cHull}_{m+1}^M (X \cup z_{m+1}^M \cup \vec{p}_m^M),$$

and $\pi : H \to M$ be the uncollapse, then $\pi^+(\kappa^+)^H = X$ and

$$H|((\kappa^+)^H = M|(\kappa^+)^M).$$

3.43 Lemma. Let $B = (\gamma, \rho, M, Q)$ be an active pm-cephalanx of degree $(m, 0)$. Suppose $B$ has an exceptional core. Let $F, \kappa, m', H, \pi$ be as in 3.42. Then:

1. $M = \text{Ult}_m (H, F)$ and $\pi = i_F^H$ is an $m$-embedding.
2. $\pi(z_{m+1}^H) = z_{m+1}^M$ and $\pi(p_{m+1}^H \setminus (\kappa^+)^H) = p_{m+1}^M \setminus \rho$.
3. $\rho_{m+1}^H \leq (\kappa^+)^H < \rho_m^H$ and $H$ is $(\kappa^+)^H$-sound.
4. If $\rho_{m+1}^H = (\kappa^+)^H$ then $\rho_{m+1}^M = \rho$ and $H, M$ are $(m+1)$-sound.
5. If $\rho_{m+1}^H \leq \kappa$ then $\rho_{m+1}^M = \rho_{m+1}^M$ and $H$ is not $(m+1)$-sound.
6. If $M = \text{Hull}_{m+1}^M (\alpha \cup z_{m+1}^M \cup \vec{p}_m^M)$ where $\alpha < \rho$ and $\alpha$ is least such, then $\alpha \in \text{rg}(\pi)$.

Proof. Parts 1–4 are standard. Part 5: Because $\rho_{m+1}^H \leq \kappa$, we have $m \geq 0$. Since $Q$ is a type 3 premouse and $M|((\kappa^+)^M = H|((\kappa^+)^H, F$ is close to $H$, so $\rho_{m+1}^M = \rho_{m+1}^H \leq \kappa$. Suppose $M$ is $(m+1)$-sound, so

$$M = \text{Hull}_{m+1}^M (\kappa \cup \vec{p}_{m+1}^M).$$

It follows that

$$M = \text{Hull}_{m+1}^M \left( \text{rg}(\pi) \cup q \right)$$

for some $q \in \gamma^{<\omega}$. But the generators of $F$ are unbounded in $\gamma$, a contradiction. Part 6: Suppose there is $\alpha < \rho$ such that

$$M = \text{Hull}_{m+1}^M (\alpha \cup z_{m+1}^M \cup \vec{p}_m^M).$$

(4)
Since $\pi$ is continuous at $(\kappa^+)^H$ and $\pi$ is $\mathsf{r}\Sigma_{m^+}$-elementary and $\pi(z_{m+1}^H) = z_{m+1}^M$, this fact easily reflects to $H$; i.e. there is $\beta < (\kappa^+)^H$ such that $H = \operatorname{Hull}_{m^+}^H(\beta \cup z_{m+1}^H \cup \vec{p}_m^H)$.

Let $\beta$ be least such. Then $\alpha = \pi(\beta)$ is least such that line (4) holds. \hfill \Box

3.44 Definition. Let $B = (\gamma, \rho, M, Q)$ be an active cephalax of degree $(m, 0)$, with $m \geq 0$. We say that $B$ has a good core iff the following holds. Either:

(i) $B$ is exact; let $F = F^Q$; or

(ii) $B$ is not exact and letting $N = N^B$, we have $\operatorname{OR}^N = (\rho^+)^M$ and $N$ is active type 1 or 3; let $F = F^N$.

Let $\kappa = \operatorname{cr}(F)$ and $\nu = \nu(F)$. Then:

1. $M$ has an $(m, \nu)$-good core at $\kappa$, and $G_{m, \kappa, \nu}^M = F \upharpoonright \nu$.

2. Suppose case (ii) holds and $N$ is type 1; so $\kappa = \gamma$. Then:

   - If $Q$ is type 2 then $H_{m, \kappa}^M = Q$.
   - Suppose $Q$ is not type 2, nor superstrong. Let $\mu = \operatorname{cr}(F^Q)$. Then $M$ has an $(m, \gamma)$-good core at $\mu$, and $G_{m, \mu, \gamma}^M = F^Q$.

3.45 Remark. It seems that $B$ might have an exceptional core but not a good core.

3.46 Theorem. Let $B = (\gamma, \rho, M, Q)$ be an $(\omega_1 + 1)$-iterable, non-trivial, active pm-cephalanx, of degree $(m, q)$. Then $B$ is not sound. If $B$ is non-exceptional then $m \geq 0$ and $B$ has a good core. If $B$ is exceptional then $B$ has an exceptional core.

Proof. Suppose first that $B$ is exact and $Q$ is superstrong, but $B$ is not exceptional. Then $\rho_{m+1}^M \leq \gamma$ and $M$ is $\gamma$-sound, as is $Q$. So $C = (\gamma, M, Q)$ is a non-trivial bicephalus, and note that $C$ is $(\omega_1 + 1)$-iterable. So by 3.32, $B$ has a good core. So we may now assume that:

If $B$ is exact and $Q$ is superstrong, then $B$ is exceptional. (5)

Under this assumption, the proof is based on that of 3.34. The main differences occur in the rules guiding the comparison, the proof that the comparison terminates, and when $B$ is exceptional.

Assume $B$ is countable. We define $\mathcal{T}, \mathcal{U}$ on $B$ and sets $S^T_\alpha, S^\mathcal{U}_\alpha, M^\mathcal{T}_\alpha, M^\mathcal{U}_\alpha$, much as before. Suppose we have defined $(\mathcal{T}, \mathcal{U}) \upharpoonright \alpha + 1$, $S^T_\alpha$ and $S^\mathcal{U}_\alpha$, but if $\operatorname{card}(S^T_\alpha) = \operatorname{card}(S^\mathcal{U}_\alpha) = 1$ and $R \in M^\mathcal{T}_\alpha$ and $S \in M^\mathcal{U}_\alpha$ then $R \not\subseteq S \not\subseteq R$. We will implicitly specify two segmented-premice from which to select $E^T_\alpha, E^\mathcal{U}_\alpha$, however, we minimize on $\nu(E)$, rather than $\nu(E)$, when selecting these extenders. (For example, if $E^T_\alpha \neq \emptyset \neq E^\mathcal{U}_\alpha$ then $i^T_\alpha = i^\mathcal{U}_\alpha$.) We use the terminology terminates early as before. Let $B^T = B^T_\alpha, M^T = M^T_\alpha$, etc.
CASE 1. \( \text{card}(S^T_\alpha) = \text{card}(S^U_\alpha) = 1. \)

Do the obvious thing.

CASE 2. \( S^T_\alpha = \{0, 1\} \) and if \( S^U_\alpha = \{0, 1\} \) then \( \rho^T \leq \rho^U. \)

We will have by induction that (i) for every \( \beta < \alpha \), if \( E^T_\beta \neq \emptyset \) then \( \text{lh} E^T_\beta \leq \rho^T \) and \( \nu^T_\beta \leq \gamma^T \), and if \( E^U_\beta \neq \emptyset \) then \( \text{lh} E^U_\beta \leq \rho^U \) and \( \nu^U_\beta \leq \gamma^U \). We leave the maintenance of (i) to the reader. We may assume that \( B^T || \rho^T = B^U || \rho^T. \)

We use the terminology in \( T \) we move into \( M^T \) as before. In \( T \), we will not move into \( Q^T \) (but we might set \( E^T_\alpha = F(Q^T) \)). Likewise with regard to \( U \) if \( S^U_\alpha = \{0, 1\}. \)

We say that \( \alpha \) is \((T, U)\)-unusual iff either (i) there is \( \xi < \alpha \) such that

\[
F(Q^T) | \nu(F(Q^T)) = E^T_\xi | \nu^T_\xi,
\]

or (ii) there are \( \chi < \xi < \alpha \) such that

- \( \alpha = \xi + 1, \)
- \( S^T_\chi = \{0, 1\} \) and \( E^T_\chi = \emptyset \) and \( E^U_\chi = F(Q^T) \) and \( S^T_{\chi + 1} = \{0\}, \)
- \( S^U_\chi = \{0, 1\} \) and \( E^U_\chi = \emptyset \) and \( E^T_\chi = F(Q^T) \) and \( S^U_{\chi + 1} = \{0\}, \)
- \( \chi T = \gamma(B^T_X). \)

In case (i) (respectively, (ii)) we say that \( \alpha \) is \( \text{type} \) (i) (respectively, (ii)). We define \((U, T)\)-unusual symmetrically.

SUBCASE 2.1. \( \alpha \) is not \((T, U)\)-unusual and \( \text{card}(S^U_\alpha) = 1. \)

Let \( P \in \mathcal{M}^U_\alpha. \) We have \( B^T_\alpha | \rho^T = P | \rho^T. \)

If \( Q^T \leq P \) then in \( T \) we move into \( M^T \) (so \( E^T_\alpha = \emptyset \) and \( E^U_\alpha = F(Q^T) \)).

If \( Q^T \not\leq P \) then we select extenders from \( Q^T \) and \( P \) (but do not move into \( Q^T \) in \( T). \)

SUBCASE 2.2. \( \alpha \) is not \((T, U)\)- or \((U, T)\)-unusual and \( S^T_\alpha = S^U_\alpha = \{0, 1\}. \)

If \( Q^T \leq Q^U \) then in \( T \) we move into \( M^T \), and set \( E^U_\alpha = F(Q^T) \).

If \( Q^T \not\leq Q^U \) then we select extenders from \( Q^T \) and \( Q^U. \)

SUBCASE 2.3. \( \alpha \) is \((T, U)\)-unusual.

In \( T \) we move into \( M^T. \)

Suppose also that \( S^U_\alpha = \{0, 1\} \) (otherwise we are done); so \( \alpha \) is type (i). If \( Q^U \leq M^T \) then in \( U \) we move into \( M^U \); otherwise select extenders from \( M^T \) and \( Q^U. \)

The remaining rules for the comparison are determined by symmetry. We will observe in Claim 2 below that no ordinal is both \((T, U)\)-unusual and \((U, T)\)-unusual, so the definition of \((T, U)\) is reasonably symmetric (although not completely). Now if \( B \) is active and \( Q \) is type 3, for some \( \alpha \), \( Q^T_\alpha \) might fail the ISC. So the next claim needs some argument.

\footnote{It might be that \( P(\text{OR}(Q^T)) \) is active with extender \( E \) and \( \iota(F(Q^T)) > \iota(E) \), in which case \( E^T_\alpha = \emptyset \) and \( E^U_\alpha = E \). In this case we keep \( S^T_{\alpha + 1} = \{0, 1\}. \) This is because if \( E \) is superstrong, we could end up with \( F(Q^T) \) active on some \( S \in \mathcal{M}^T_{\alpha + 1}. \)}
Claim 1. For all $\alpha + 1, \beta + 1 < \operatorname{lh}(\mathcal{T}, \mathcal{U})$, if $E_{\alpha}^T \neq \emptyset \neq E_{\beta}^T$ then

$$E_{\alpha}^T \upharpoonright \nu_{\alpha}^T \neq E_{\beta}^T \upharpoonright \nu_{\beta}^T.$$  

Proof. Suppose otherwise and let $(\alpha, \beta)$ be the lexicographically least counterexample. Let $\lambda = \operatorname{lh}_{\nu_{\alpha}^T}$.

Suppose that $\operatorname{lh}_{\nu_{\beta}^T} = \lambda$. So $E_{\alpha}^T = E_{\beta}^T$, so $\alpha \neq \beta$; so suppose $\alpha < \beta$. Note that there is $\delta \in [\alpha, \beta)$ such that $E_{\delta}^T \neq \emptyset$; let $\delta$ be least such and let $G = E_{\delta}^T$. Then $\operatorname{lh}(G) = \lambda$, and since $\operatorname{lh}_{\nu_{\beta}^T} = \lambda$, therefore $G$ has superstrong type. So $\nu_{\alpha}^T = \iota(G)$ and $\delta = \alpha$ and $E_{\alpha}^T \neq G$. Let $\varepsilon = \text{pred}^\mathcal{U}(\alpha + 1)$. By 3.38, $\alpha + 1 \in \mathcal{B}^\mathcal{U}$ and $\operatorname{cr}(G) = \gamma(B_{\varepsilon}^\mathcal{U})$. So $\alpha$ is not $\mathcal{U}$-special, so $G$ is a premouse extender. Standard arguments (for example, see [5, §5]) now show that there is $\alpha' < \alpha$ such that $E_{\alpha'}^T = G$. But $(\alpha', \alpha) <_{\text{lex}} (\alpha, \beta)$, contradiction.

So we may assume that $\lambda = \operatorname{lh}_{\nu_{\alpha}^T} < \operatorname{lh}_{\nu_{\beta}^T}$; so $\alpha < \beta$. Then $P_{\beta}^\mathcal{U}$ is not a premouse because, letting $\nu = \nu_{\beta}^T$, we gave $\nu < \operatorname{lh}_{\nu_{\alpha}^T}$ and by 3.38, $\operatorname{lh}_{\nu_{\alpha}^T}$ is a cardinal of $P_{\beta}^\mathcal{U}$, and $E_{\alpha}^T \upharpoonright \nu \neq P_{\beta}^\mathcal{U}$. So $\beta$ is $\mathcal{U}$-special. But then $\beta$ is $(\mathcal{U}, \mathcal{T})$-unusual (of type (i)), so $E_{\beta}^T \neq F(Q_{\beta}^\mathcal{U})$, contradiction. \hfill \square

Claim 2. Let $\alpha$ be $(\mathcal{T}, \mathcal{U})$-unusual. Then:

- $Q$ is type 3 and $Q_{\alpha}^T$ is not a premouse,
- $\alpha$ is not $(\mathcal{U}, \mathcal{T})$-unusual,
- for all $\beta \in \mathcal{B}^\mathcal{T}$, $\operatorname{cr}(F(Q_{\beta}^T)) \neq \gamma_{\alpha}^T$, and
- for all $\beta \in \mathcal{B}^\mathcal{U}$, $\operatorname{cr}(F(Q_{\beta}^\mathcal{U})) \neq \gamma_{\alpha}^T$.
- Suppose $\alpha$ is type (i), as witnessed by $\xi$. Then:
  - $Q$ is not superstrong,
  - $\alpha = \xi + 1$ and $\operatorname{lh}_{\xi}^\mathcal{U} < \gamma_{\alpha}^T$ and $E_{\alpha}^T = \emptyset$,
  - the trivial completion of $E_{\xi}^\mathcal{U} \upharpoonright \nu_{\xi}^T$ is a type 3 premouse extender, and
  - for each $P \in \mathcal{M}_{\alpha}^\mathcal{U}$, $P \upharpoonright \rho_{\alpha}^T = Q_{\alpha}^T \upharpoonright \rho_{\alpha}^T$.
- Suppose that $\alpha$ is type (ii), as witnessed by $\chi, \xi$.
  - $Q$ is superstrong and $B$ is exact,
  - $\chi = \text{pred}^\mathcal{T}(\alpha)$, and
  - $F(Q_{\alpha}^T) = E_{\xi}^T \circ E_{\xi}^\mathcal{U}$.

Proof. The proof is by induction on $\alpha$. We will prove that $\alpha$ is not $(\mathcal{U}, \mathcal{T})$-unusual at the end. Let $B^\mathcal{T} = B_{\alpha}^\mathcal{T}$, $M^\mathcal{T} = M_{\alpha}^\mathcal{T}$, etc.

First suppose that $\alpha$ is type (i). Let $F = F(Q_{\alpha}^T)$. We first show that $\operatorname{lh}_{\xi}^\mathcal{U} < \rho_{\alpha}^T$. Otherwise $\operatorname{lh}_{\xi}^\mathcal{U} = \rho_{\alpha}^T$, and so $E_{\lambda}^T = F$. It follows that $E_{\lambda}^T \neq \emptyset$ for some $\delta \in [\xi, \alpha)$, since otherwise $B_{\xi}^\mathcal{T} = B_{\alpha}^\mathcal{T}$, and since $S_{\alpha}^\mathcal{U} = \{0, 1\} = S_{\alpha}^T$, we have
$Q^T_\xi = P^H_\xi$ and $E^U_\xi \neq F$, contradiction. So let $\delta$ be least such and let $G = E^T_\delta$.
Then as in the proof of Claim 1, $G$ is a superstrong premouse extender also used in $U$, contradicting Claim 1.

Since $lh^U_\delta < \rho^T$, $Q^T$ is not a premouse, so $Q$ is type 3. It easily follows that
$lh^U_\xi < \gamma^T$, since if $\gamma$ is a successor cardinal in $Q$ then $Q^T$ is a premouse.

Now suppose $Q$ is superstrong. Then because $Q^T$ is not a premouse, there is $\delta < \tau$ such that $Q^T_\delta$ is a premouse and $cr(j^T_{\delta,\xi}) = \gamma(B^T_{\delta,\xi})$ (otherwise $j^T_{0,\xi}$ is continuous at $\gamma^T$ so $Q^T$ is a premouse). So $Q^T$ fails the ISC. So $E^U_\xi$ is not a premouse extender and $\xi$ is $U$-special. But then $F(Q^U_\xi)$ has superstrong type, so $lh^U_\xi = \rho^T$, a contradiction.

Since $Q^T$ is not superstrong, letting $\mu = cr(F)$, we have $i_F(\mu) > \rho^T$, and so $B^U_{\xi+1}\rho^T = Q^T || \rho^T$. Now suppose there is $\delta \in \{\xi, \alpha\}$ such that $E^U_\delta \neq \emptyset$. Fix such a $\delta$ with $\delta + 1 \in [0, \alpha]_\tau$. Let $\varepsilon = \text{pred}^T(\delta + 1)$. So $\varepsilon \in B^T$ and $\kappa = \text{cr}^T_\varepsilon \leq \gamma^T_\varepsilon$. If $\kappa < \nu(F(Q^T_\varepsilon))$ then easily $\nu(F) > \nu^U_\varepsilon$, contradiction. So $\kappa \geq \nu(F(Q^T_\varepsilon))$. But then standard arguments show that $E^T_\delta$ is a premouse extender used in both $U, T$, a contradiction.

It follows that $\xi + 1 = \alpha$ and $E^T_\xi = \emptyset$, so we are done.

Now suppose that $\alpha$ is type (ii). Let $F = F(Q^T_\chi) = E^U_\xi$, $\kappa = cr(F)$ and $\mu = cr^T_\chi = \gamma(B^T_\chi)$.

Suppose that $Q$ is not superstrong. Let $\delta \in [\chi, \xi]$ be such that $\delta + 1 \in [0, \xi]_U$. Let $G = E^U_\delta$ and $\theta = cr(G)$. Then $\theta < \mu < i_G(\theta)$, so $\mu \notin rg(j^U_{0,\xi})$, but $\mu = cr(F(Q^U_\xi))$, so $\mu \in rg(j^U_{0,\xi})$, contradiction.

We now observe that $\chi = \text{pred}^T(\alpha)$. Let $\beta < \alpha$ be such that $E^T_\beta \neq \emptyset$. Then $\beta \neq \chi$. If $\beta < \chi$ then $i^T_\beta \leq \nu(F(Q^T_\chi)) = \mu$

(using (i); see the begining of Case 2). Since $S^T_{\chi+1} = \{0\}$, if $\beta > \chi$ then $\rho^T_\chi < i^T_\beta$. This suffices.

We now prove that $B$ is exact. Let $\varepsilon = \text{pred}^U(\chi + 1)$. Then we claim that

$$\chi + 1 \leq_U \xi \quad \& \quad cr(F(Q^U_\varepsilon)) = \kappa. \tag{6}$$

For $\xi \in B^U$ and $cr(F(Q^U_\xi)) = \mu = i^U_\chi$. So suppose line (6) fails; then one can show that $\chi + 1 \in B^U$ and $\gamma^U_\chi = \kappa$ and $E^U_{\chi+1} = F(Q^U_{\chi+1})$: here $\gamma^U_{\chi+1} = \mu$.

Then $\varepsilon$ is not $(U, T)$-unusual, by induction and since $\gamma^T_\varepsilon = \kappa = cr(F(Q^T_\varepsilon))$.

So $E^T_\varepsilon = F(Q^T_\varepsilon)$. But then $\chi + 1$ is $(U, T)$-unusual, so $E^U_{\chi+1} \neq F(Q^U_{\chi+1})$, contradiction.

Using line (6) and since $E^U_\chi$ is total over $B^U_\chi$, $(\kappa^{++})^{Q^T_\chi} \leq (\kappa^{++})^{B^U_\chi}$ and

$$(Q^T_\chi \sim B^U_\chi) || (\kappa^{++})^{Q^T_\chi}.$$  

Since $k = i^T_{Q^T_\xi}$ is continuous at $(\kappa^{++})^{Q^T_\xi}$, therefore $$(\mu^{++})^{UH(Q^T_\xi, F)} \leq (\mu^{++})^{B^U_{\chi+1}}.$$
and
\[(\Ult(Q^T_{\kappa}, F) \sim B^U_{\kappa+1}) \mid\mid (\mu^{++})^{\Ult(Q^T_{\kappa}, F)}\]

Now suppose that \(B\) is not exact. By 3.27, neither is \(B^T_{\kappa}\). So
\[(\mu^{++})^{M^T_{\kappa}} < (\mu^{++})^{\Ult(Q^T_{\kappa}, F)}\]

and
\[(M^T_{\kappa} \sim \Ult(Q^T_{\kappa}, F) \sim B^U_{\kappa+1}) \mid\mid (\mu^{++})^{M^T_{\kappa}}\]

but by non-triviality, \(M^T_{\kappa} \not\in \Ult(Q^T_{\kappa}, F)\). So \(M^T_{\kappa} \not\in B^U_{\kappa+1}\). We have \(E^T_{\kappa} = \emptyset\) and \(S^T_{\kappa+1} = \{0\}\) and \(M^T_{\kappa+1} = M^T_{\kappa}\). So
\[(\mu^{++})^{M^T_{\kappa}} = (\mu^{++})^{\text{pred}} = (\mu^{++})^{B^U_{\kappa+1}} > (\mu^{++})^{M^T_{\kappa}},\]

contradiction. (For similar reasons, \((\kappa^{++})^{B^U_{\kappa+1}} = (\kappa^{++})^{Q^T_{\kappa}}\)

We leave to the reader the proof that for any \(\beta > \alpha\), if \(\beta \in \mathcal{B}^T\) then \(\cr(F(Q^T_{\beta})) \neq \gamma^T_{\alpha}\) and if \(\beta \in \mathcal{B}^U\) then \(\cr(F(Q^T_{\beta})) \neq \gamma^T_{\alpha}\).

Finally suppose that \(\alpha\) is both \((\mathcal{T}, \mathcal{U})\)-unusual and \((\mathcal{U}, \mathcal{T})\)-unusual. Then either \(\alpha\) is type (ii) with respect to both, or type (i) with respect to both, since this depends on whether or not \(Q\) is superstrong. But then \(\alpha = \xi + 1\) and \(E^T_{\xi} = \emptyset = E^U_{\xi}\), contradiction.

\[\text{Claim 3. Let } \xi < \alpha \text{ such that } S^U_{\xi} = \{0, 1\} \text{ and } E^U_{\xi} = \emptyset \text{ and } E^T_{\xi} = F(Q^T_{\xi})\]

and \(\xi + 1 \in \mathcal{B}^T\) and letting \(\chi = \text{pred}^T(\xi + 1)\), then \(\text{cr}^T_{\xi} = \gamma(B^T_{\kappa})\). Then \(\xi + 1\) is \((\mathcal{T}, \mathcal{U})\)-unusual of type (ii).

\[\text{Proof. Suppose not. Then note that } \chi \text{ is } (\mathcal{T}, \mathcal{U})\text{-unusual. But } \cr(F(Q^T_{\xi})) = \gamma^T_{\kappa},\]

contradicting Claim 2.

\[\text{Claim 4. The comparison terminates at some countable stage.}\]

\[\text{Proof. We may assume that } B \text{ is active and } Q \text{ is type 3, since otherwise every extender used in } (\mathcal{T}, \mathcal{U}) \text{ is a premouse extender, and so the usual argument works.}\]

Suppose that \((\mathcal{T}, \mathcal{U})\) reaches length \(\omega_1 + 1\). Let \(\eta \in \text{OR}\) be large and let \(\tau : X \to V_\eta\) be elementary with \(X\) countable and transitive, and everything relevant in \(\text{rg}(\tau)\). Let \(\kappa = \cr(\tau)\). Let \(W = B^T_{\omega_1} \upharpoonright \omega_1 = B^U_{\omega_1} \upharpoonright \omega_1\). Standard arguments show that either \(i^T_{\kappa, \omega_1}\) or \(j^T_{\kappa, \omega_1}\) is defined, and if \(i^T_{\kappa, \omega_1}\) is defined then
\[M^T_{\kappa} \upharpoonright ((\kappa^{++})^{M^T_{\kappa}} = W) \upharpoonright ((\kappa^{++})^W)\]

and
\[i^T_{\kappa, \omega_1}(W) = \tau \upharpoonright (W)\]

and likewise if \(j^T_{\kappa, \omega_1}\) is defined. Likewise for \(\mathcal{U}\).
Let $\alpha + 1 = \min((\kappa, \omega_1 | \tau)$ and $\beta + 1 = \min((\kappa, \omega_1 | \mu)$. Let us assume that $\alpha \leq \beta$; in the contrary case the proof is essentially\textsuperscript{18} the same. Let $\iota = \min(\iota^T, \iota^U)$. Then $E^T_\alpha | \iota = E^U_\beta | \iota$.

**Subclaim 4.1.** We have:

(a) The trivial completion of $E^T_\alpha | \nu^T_\alpha$ is a premouse extender.

(b) $\alpha < \beta$ and $\iota^T_\alpha < \iota^U_\beta$ and $\nu^T_\alpha < \nu^U_\beta$ and $\lh^T_\alpha < \lh^U_\beta$.

(c) $E^U_\beta | \nu^T_\alpha \notin P^U_\beta$, so $P^U_\beta$ is not a premouse and $\beta$ is $\mathcal{U}$-special.

**Proof.** Part (a): We have $\nu^T_\alpha \leq \nu^U_\beta$ because $\iota^T_\alpha \leq \iota^U_\beta$ and by compatibility. So part (a) follows from a standard argument (i.e. otherwise we get some premouse extender, which factors into $E^T_\alpha$, used in both $\mathcal{T, \mathcal{U}}$; cf. [5, §5]).

Part (b): We have $\iota^T_\alpha \leq \iota^U_\beta$. If $\iota^T_\alpha = \iota^U_\beta$ then $E^T_\alpha = E^U_\beta$ and $\alpha < \beta$, contradicting Claim 1. So $\iota^T_\alpha < \iota^U_\beta$, and therefore $\alpha < \beta$.

We have $\nu^T_\alpha \leq \nu^U_\beta$. But we can’t have $\nu^T_\alpha = \nu^U_\beta$, by Claim 1 and compatibility. So $\nu^T_\alpha < \nu^U_\beta$.

We have $lh^T_\alpha \leq lh^U_\beta$. Suppose $lh^T_\alpha = \lambda = lh^U_\beta$. Let $P = P^U_\beta$ and $\delta = \gcd(P) = \gcd(P^T_\alpha)$. Then $E^T_\alpha \notin P$. Since $\nu^T_\alpha < \nu^U_\beta$ and by part (a), therefore $P$ is not a premouse. So $\beta$ is $\mathcal{U}$-special, so $\iota^T_\alpha < \delta = \iota^U_\beta$. But $\iota^T_\alpha \geq \delta$ as $\delta = \gcd(P^T_\alpha)$, a contradiction. So $lh^T_\alpha < lh^U_\beta$.

Part (c): $lh^T_\alpha$ is a cardinal of $P = P^U_\beta$ and $E^T_\alpha \notin P$, by 3.38 and agreement between models of $\mathcal{T}$ and $\mathcal{U}$. Since $\nu^T_\alpha < lh^T_\alpha < lh^U_\beta$ and by part (a), $P$ fails the ISC, and so $\beta$ is $\mathcal{U}$-special. \(\square\)

Let $\varepsilon <_\mu \beta$ be largest such that $F(Q^U_\varepsilon) | \nu(F(Q^U_\varepsilon))$ satisfies the ISC. Let $\delta + 1 = \min((\varepsilon, \beta | \mu)$. So $\delta$ is not $\mathcal{U}$-special. Let $F = F(Q^U_\varepsilon)$; then

$$E^T_\alpha | \nu^T_\alpha = F | \nu(F).$$

(7)

Let $\chi + 1 = \min((\alpha + 1, \omega_1 + 1) | \tau)$. Let $\iota_1 = \min(\iota^U_\delta, \iota^T_\chi)$. Note that

$$E^T_\chi | \iota_1 = E^U_\delta | \iota_1.$$  

Then:

**Subclaim 4.2.** We have:

(a) $P^U_\delta$ is a premouse.

(b) $\chi > \delta$ and $\iota^T_\chi > \iota^U_\delta$ and $lh^T_\chi > lh^U_\delta$ and $\nu^T_\chi > \nu^U_\delta$.

(c) $E^T_\chi | \nu^U_\delta \notin P^T_\chi$, so $P^T_\chi$ fails the ISC and $\chi$ is $\mathcal{T}$-special.

\textsuperscript{18}Only essentially because our definition of $(\mathcal{T, \mathcal{U}})$ was not quite symmetric.
Proof. This follows from the proof of Subclaim 4.1 and the fact that δ is not $U$-special.

Since $\chi$ is $T$-special and

$$\mu = cr_\chi^T > cr_\alpha^T = \kappa$$

and $\alpha + 1 = \operatorname{pred}(\chi + 1)$, by 3.37(c), $\kappa \in B^T$ and $\kappa \leq cr(F(Q_\kappa^T))$ and

$$\mu = j_{\kappa, \alpha + 1}^T(\kappa).$$

But then by line (7), $i_F(\kappa) = \mu \leq \gamma_e^U$. So $i_F(\kappa) = \gamma_e^U$, so $Q_e^U$ and $Q$ have superstrong type.

It is now easy to see that $\beta$ is $(U, T)$-unusual of type (ii), and therefore $E_3^U \neq F(Q_3^U)$, a contradiction. This completes the proof of the claim.\qed

Now that we know the comparison terminates, we must analyse the manner in which it does. Let $\alpha + 1 = \lim(T, U)$. Let $B^T = B^T_\alpha$, etc.

Claim 5. Suppose that $B$ is non-exceptional and the comparison does not terminate early. Then:

- $\alpha \in B^T \Delta B^U$ and $\operatorname{card}(S^T) = \operatorname{card}(S^U) = 1$ and $M^T = M^U$.
- $m \geq 0$ and the cephalanx $C \in \{B^T, B^U\}$ has a good core.

Proof. We have:

Either $B$ is non-exact or $Q$ is non-superstrong, (8)

because $B$ is non-exceptional and by line (5).

We will show later that $\operatorname{card}(S^T) = \operatorname{card}(S^U) = 1$; assume this for now. Let $Y \in M^T$ and $Z \in M^U$. So $Y, Z$ are the final models of the comparison and either $Y \subseteq Z$ or $Z \subseteq Y$. As in the proof of 3.34, because $\operatorname{card}(S^T) = \operatorname{card}(S^U) = 1$ and the comparison does not terminate early, we have $\alpha \in B^T \Delta B^U$; we may assume $\alpha \in B^T \setminus B^U$. So $Z$ is unsound and $M^T = Y \subseteq Z$. We need to see that $B^T$ has a good core. We have $S^T = \{0\}$. Let $\beta = B^T(\alpha)$. We will show later that $M^T = Z$; assume this for now.

Case 1. $\beta$ is $(T, U)$-unusual.

By line (8) and Claim 2, $\beta$ is type (i), $\beta = \xi + 1$, and $E_\xi^T$ is equivalent to $F(Q^T)$, and $Q$ is type 3 but not superstrong. At stage $\beta$, in $T$ we move into $M^\beta_T = M^T$. We have $E_\beta^T = \emptyset \neq E_3^T$ and $(M^T \sim B^T_\beta)[\rho^T]$ and $\rho^T \leq \lim^T_U$ and $\rho^T$ is a successor cardinal of $B^T_3$. So $\rho^T \leq i_3^U$. Since $M^T$ is $\rho^T$-sound, it follows that there is exactly one ordinal $\delta$ such that $\delta \geq \beta$ and $\delta + 1 \leq^* \alpha$, and in fact $\delta + 1 = \alpha$. So $P^U_3$ is a premouse, as $\alpha \notin B^U$. Since $M^T$ is $\rho^T$-sound, therefore $\delta = \beta$ and $E_\beta^U$ is type 1 or type 3, with $\lim^T_U = (\rho^T)^{\operatorname{M}}$. It follows that $m \geq 0$ and $B^T$ is non-exact, and letting $F^* = F(N^{B^T})$ and $\kappa^* = cr(F^*)$, we have $E_3^U = F^*$, and $M^T$ has an $(m, \rho^T)$-good core at $\kappa^*$, and $G^U_{m, \kappa^*, \rho^T} = F^* \upharpoonright \rho^T$.
and \( H^{M_T}_{m,n,*} = B^u_{\alpha} \). Also, if \( F^* \) is type 1 then \( \text{pred}^d(\alpha) = \beta \) and \( \beta \notin B^d \) and \( B^u_{\alpha} = B^d_{\beta} \), and \( M^T \) has an \((m,\gamma^T)\)-good core at \( \text{cr}(F(Q^T)) \), etc. So \( B^T \) has a good core.

**Case II.** \( \beta \) is not \((T,U)\)-unusual.

So \( E^T_{\beta} = F' \) where \( F' = F(Q^T) \).

**Subcase II.1.** \( Q \) is not superstrong.

So \( F' \) does not have superstrong type. Things work much as in the previous case, but there are a couple more possibilities, which we just outline. If \( B \) is exact then \( \alpha = \beta + 1 \), and \( F' \) is the last extender used in \( U \). If \( B \) is non-exact then \( \alpha = \beta + 2 \) and like above, \( F^* = F(N^{B^T}) \) is type 1 or type 3 and is the last extender used in \( U \). In the latter case, if \( N^{B^T} \) is type 1 and \( Q^T \) is type 2 then \( Q^T = B^u_{\alpha} \).

**Subcase II.2.** \( Q \) is superstrong.

So \( F' \) has superstrong type, and by line (8), \( B \) is non-exact. Things work much as before, but there are some extra details. We just give the details in one example case. Let \( \varepsilon = \text{pred}^d(\beta + 1) \). So \( \kappa' = \text{cr}(F') < \varepsilon \) and

\[
((\kappa')^+)^Q < \text{lh}^U \varepsilon.
\]

Suppose for example that \( ((\kappa')^+)^Q = \text{lh}^U \varepsilon \). Then \( E^d_{\varepsilon} \) is type 2 and \( B^u_{\beta+1} = P^U_{\varepsilon} \), and \( \text{OR}(B^d_{\beta+1}) = \text{OR}(Q^T) \) and \( B^d_{\beta+1} \) is active type 2, so \( E^d_{\beta+1} = F(B^d_{\beta+1}) \). Note that

\[
(Q^T \sim B^d_{\beta+1}) \sim ((\kappa')^+)^Q,
\]

and so by 3.8,

\[
(\text{Ult}(Q^T, F') \sim B^d_{\beta+2}) \sim ((\gamma^T)^{++})^{\text{Ul}H(Q^T, F')} \sim \text{Ult}(Q^T, F') \sim B^d_{\beta+2},
\]

and so \( E^d_{\beta+2} = F(N^{B^T}) \), etc. We leave the remaining details to the reader.

This completes the proof under the assumptions made above. Now suppose the assumptions fail. Then we either have \( \emptyset \neq M^T \triangleleft Z \) for all \( Z \in \mathcal{W}^U \), or the reflection of this; suppose the former. Suppose \( B \) is non-exact. Arguing as above, if \( \beta \) is \((T,U)\)-unusual then \( N^{B^T} \triangleleft B^d_{\beta} \); otherwise \( E^d_{\beta} = F(Q^T) \) and either \( N^{B^T} \triangleleft B^d_{\beta+1} \) or \( N^{B^T} \triangleleft B^d_{\beta+2} \). But because \( M^T \neq N^{B^T} \) this contradicts the fact that \( M^T \triangleleft Z \). So \( B \) is exact, so \( Q \) is not superstrong. But then arguing again as above, \( ((\rho^T)^+)^Z = (\rho^T)^{M^T} \), again contradicting that \( M^T \triangleleft Z \). \( \square \)

The next claim follows immediately from the rules of comparison. It applies in particular if any \( \beta \) is \((T,U)\)- or \((U,T)\)-unusual.

**Claim 6.** If \( Q \) is type 1 or type 3 then for all \( \beta \), \( S^T_{\beta} \neq \{1\} \neq S^d_{\beta} \).

**Claim 7.** Suppose the comparison terminates early (so \( \alpha \) is either \((T,U)\)- or \((U,T)\)-unusual). Then:

- \( B \) is exact.
- If $\alpha$ is $(T, U)$-unusual then $S^U = \{0\}$, and vice versa.
- The final models of the comparison are $M^T = M^U$.

**Proof.** We use Claim 2 in what follows.

Suppose $\alpha$ is $(T, U)$-unusual of type (i). Since $Q_T^T$ does not have superstrong type and $B_T$ is non-trivial, it is easy to see that $M^T \not\sim B^U$, and $B^T, B$ are exact. But then if $S^U = \{0, 1\}$, the comparison in fact does not terminate early at stage $\alpha$. So $S^U = \{0\}$. Also $\alpha = \xi + 1$ and $E^U_\xi \neq \emptyset$, so $\alpha \notin B^U$. So $M^U$ is unsound, so $M^T = M^U$.

Now suppose $\alpha$ is $(T, U)$-unusual of type (ii). We adopt the notation used in the comparison rules and in the proof of Claim 2. Let $\delta = \gamma(B_T)$. We have

$$\rho^T = \rho^U = (\delta^+)^{M^T} = (\delta^+)^{M^U}$$

and $M^U = M^T$, and either $M^T \triangleleft M^U$ or $M^T \triangleleft M^U$ and $M^T, M^U$ both project $\leq \rho^T$. So it suffices to see that

$$(\delta^{++})^{M^T} = (\delta^{++})^{M^U}.$$ 

Now $Q$ is superstrong and $B$ is exact. As mentioned in the proof of Claim 2, $\kappa^T = \zeta$ where $\zeta = (\kappa^+)^{B_T}$, and $(Q_T^T \sim B^T)$. Let $i_F = i_{F^T}^{Q_T^T}$. So by exactness, (letting $v$ be)

$$v = (\mu^{++})^{M^T} = (\mu^{++})^{B_T^{x+1}} = (\mu^{++})^{B_T} = (\mu^{++})^{B_T} = \sup i_F \zeta$$

(recall that $F = F(Q_T^T)$) and

$$(M^T \sim B^T_{x+1} \sim B^T) \upharpoonright v$$

and letting $G = F(Q^U_\xi)$ and $i_G = i_{G^T}^{Q^U_\xi}$,

$$(\delta^{++})^{M^U} = \sup i_G \zeta$$

and letting $H = F(Q^T)$ and $i_H = i_{H^T}^{Q^T}$,

$$(\delta^{++})^{M^T} = \sup i_H \zeta.$$ 

We claim that

$$i_H \upharpoonright \zeta = i_G \circ i_F \upharpoonright \zeta,$$

which completes the proof. This claim holds because $H = G \circ F$ and (so)

$$i_H \upharpoonright (\kappa^{+})^{Q^T} = i_G \circ i_F \upharpoonright (\kappa^{+})^{Q^T}$$

and $i_H$ and $i_G \circ i_F$ are both continuous at $(\kappa^{+})^{Q^T}$. \qed
Claim 8. Suppose that the comparison terminates early and \( B \) is non-exceptional. Then \( B^T \) has a good core.

Proof. Because \( B \) is non-exceptional, by Claim 7 and line (5), \( \alpha \) is either \((T, U)\)- or \((U, T)\)-unusual of type (i). Now argue like in the proof of Claim 5, using Claim 7.

We now move to the case that \( B \) is exceptional.

Claim 9. Suppose \( B \) is exceptional and the comparison does not terminate early. Then one of \( B^T, B^U \) is a cephalanx with an exceptional core.

Proof. We consider a few cases:

Case I. Either (a) \( S^T = \{0\} = S^U \) and \( M^T \triangleleft M^U \), or (b) \( S^T = \{0, 1\} \).

This case is covered by the next case and symmetry.

Case II. Either (a) \( S^T = \{0\} = S^U \) and \( M^T \triangleleft M^U \), or (b) \( S^U = \{0, 1\} \).

If (b) holds, then because the comparison does not terminate early, \( S^T = \{0\} \) and \( M^T \triangleleft B^U \). So given either (a) or (b), \( M^T \triangleleft B^U \). So \( \alpha \in B^T \). Let \( \beta = S^T(\alpha) \). So \( E^U(\beta) \neq \emptyset \) and \( \text{lh}^U(\beta) \geq \rho_T \). Because \( M^T \) projects \( \leq \rho_T \), it follows that \( \text{lh}^U(\beta) = \rho_T \), so \( \alpha \) is not \((T, U)\)-unusual and \( E^U(\beta) = F' \) where \( F' = F(Q^T) \).

Also, if \( \alpha > \beta + 1 \) then \( \beta + 1 \) is not \((U, T)\)-unusual, by Claim 2 and since \( M^T \triangleleft B^U \). Let \( \kappa = \text{cr}(F') \) and \( \varepsilon = \text{pred}^U(\beta + 1) \). We split into two subcases:

Subcase II.1. \( B^U_{\beta+1} \mid \rho_T \) is active.

By the discussion above:

- \( B^U_{\beta+1} \) is type 2 and \( \rho_T = \text{OR}(B^U_{\beta+1}) \),
- \((\kappa^+)^Q = \text{OR}(B^U_{\beta+1}) \),
- \( E^U(\beta) = F(B^U_{\beta+1}) \) and \( E^U_{\beta+1} = F(B^U_{\beta+1}) \),
- \( \alpha = \beta + 2 \).

Let \( R = B^U_{\beta+1} \) and \( G = F^R = E^U_{\beta} \). Then

\[ (\kappa^+)^{\text{Ult}_0(R,G)} = (\kappa^+)^{Q^T} = \text{lh}(G) \]

and

\[ (\text{Ult}_0(R,G) \sim Q^T) || (\kappa^+)^{Q^T}, \]

but because \( B^T \) is exact and \( M^T \triangleleft B^U \) and by 3.10,

\[ (\kappa^+)^{\text{Ult}_0(R,G)} > (\kappa^+)^{Q^T}. \]

Let \( H \triangleleft \text{Ult}_0(R,G) \) and \( h \in \{-1\} \cup \omega \) be such that \( (\kappa^+)^H = (\kappa^+)^{Q^T} \) and \( \rho_H^{h+1} = \text{lh}(G) < \rho_H^h \). Let \( H^* = \text{Ult}_0(R,G)(H) \) so \( H^* \triangleleft \text{Ult}_0(\text{Ult}_0(R,G), F') \).
We claim that 

$$\text{Ult}_h(H, F') \leq H^*.$$ 

If $h = -1$ this is by the ISC, so suppose $h \geq 0$. Let

$$\sigma : \text{Ult}_h(H, F') \rightarrow H^*$$

be the factor map. Arguing as in the proof of 2.13, we get that $H, \sigma \in \text{Ult}_0(R, G)$ and the hypotheses of 2.13 hold (for $H, \sigma, h, H^*$). By 2.13, $R \models \text{"Lemma 2.13 holds for my proper segments".}$ If $\sigma \neq \text{id}$ then $\rho^H_{h+1} < \text{cr}(\sigma)$, and so line (9) holds.

So $\text{Ult}_h(H, F') \triangleleft B^U_{\beta+2}$. But

$$(\rho^T)^{\text{Ult}_h(H, F')} = ((\rho^T)^+)^M,$$

so $h = m$ and $M^T = \text{Ult}_h(H, F')$. It easily follows that $B^T$ has an exceptional core, and with $X$ as in 3.42,

$$H = c\text{Hull}^{M^T}_{m+1}(X \cup z^{M^T}_{m+1} \cup \bar{p}^{M^T}_m).$$

**Subcase II.2.** $B^U_{\beta+1} | \rho^T$ is passive.

Then $\alpha = \beta + 1$, so $M^T \triangleleft B^U_{\beta+1} = B^U$. Let $R = B^U_{\alpha | \beta}$. If $\alpha \in B^U$ then $\kappa < \gamma(R)$, so $(\kappa^{++})^R$ is well-defined. In any case, $\kappa$ is not the largest cardinal of $R$. We have $(\kappa^+)^R = (\kappa^+)^Q^T$ and

$$(R \sim Q) || (\kappa^{++})^{Q^T}.$$ 

If $(\kappa^{++})^R > (\kappa^{++})^{Q^T}$ then a simplification of the argument in the previous subcase works. Suppose then that $(\kappa^{++})^R = (\kappa^{++})^{Q^T}$. Because $M^T \triangleleft B^U$, it is easy enough to see that $\alpha \notin B^U$, so $R$ is a premouse. If $R$ is active type 3, then $(\kappa^+)^R < \nu(F^R)$, because if $(\kappa^+)^R = \nu(F^R)$ then $\text{OR}(B^U_{\beta+1}) = ((\rho^T)^+)^M$, a contradiction. Let $d = \deg^U(\beta + 1)$. Then $\nu^U_{\beta+1} \in \nu^U_{\beta+1}$ is discontinuous at $(\kappa^{++})^R$, and so $(\kappa^+)^R = \rho^R_d$, so $d > 0$. Let $r < d$ be such that $\rho^R_r < (\kappa^+)^R < \rho^R_d$. Then arguing like in the previous subcase, but using 3.29 instead of 2.13,

$$\text{Ult}_r(R, F') \triangleleft B^U_{\beta+1},$$

so $\text{Ult}_r(R, F') = M^T$, and like before, $B^T$ has an exceptional core (and $m = r$).

**Case III.** $S^T = \{0\} = S^U$ and $M^T = M^U$.

Then $\alpha \in B^T \setminus B^U$; assume $\alpha \in B^T \setminus B^U$. Let $\beta = S^T(\alpha)$.

**Subclaim 9.1.** $\beta$ is not $(T, U)$-unusual.

**Proof.** Suppose otherwise. Let $\chi < \xi < \beta$ witness this. Since the comparison does not terminate early, and $M^T$ is $\rho^T$-sound, it follows that $\alpha = \beta + 1$ and

$$\rho^T = \nu^U_{\beta} < \text{Ult}^{U}_{\beta} = ((\rho^T)^+)^M.$$
So \( \rho^T \) is not the largest cardinal in \( M^T \), and so nor in \( M^U_\beta \). So \( P^U_\beta \prec M^U_\beta \), so
\[
((\rho^T)^+)^{M^T} < ((\rho^T)^+)^{M^U_\beta}.
\]
This contradicts Claim 2. \( \square \)

So \( E^U_\beta = F' = F(Q^T) \).

**Subclaim 9.2.** \( \beta + 1 \) is not \((\mathcal{U}, \mathcal{T})\)-unusual.

**Proof.** This is like the proof of Subclaim 9.1. \( \square \)

By the subclaim and Claim 3, one of the following holds:

(a) \( \alpha = \beta + 1 \).

(b) \( \alpha = \beta + 2 \) and \( \text{lh}_U^{\mathcal{M}}(\beta + 1) = \rho^T \) and \( E^M_\beta \) is type 2.

(c) \( \alpha = \beta + 2 \) and \( \text{lh}_U^{\mathcal{M}}(\beta + 1) = ((\rho^T)^+)^{M^T} \) and \( E^M_\beta \) is either (i) type 1 or (ii) type 3.

(d) \( \alpha = \beta + 3 \) and \( \text{lh}_U^{\mathcal{M}}(\beta + 2) = \rho^T \) and \( E^M_\beta \) is type 2 and \( \text{lh}_U^{\mathcal{M}}(\beta + 2) = ((\rho^T)^+)^{M^T} \) and \( E^M_{\beta + 2} \) is either (i) type 1 or (ii) type 3.

The same general argument works in each case, but the details vary. We just discuss cases (a), (b), (c)(i) and sketch (d)(i). In each case let \( \varepsilon = \text{pred}^\mathcal{U}(\beta + 1) \) and \( R = B^M_\beta \) and \( \kappa = \text{cr}(F') \).

Consider case (a). We first observe that
\[
\rho^R_{m+1} = (\kappa^+)^R < \rho^R_m.
\]
For if \( \rho^R_{m+1} > (\kappa^+)^R \) then
\[
\rho^R_{m+1}(M^U) > \rho^T \succ \rho_m(M^T);
\]
if \( \rho^R_{m+1} \leq \kappa \) then \( \rho_m(M^U) < \rho^T \) and \( M^U \) is \( \gamma^T \)-sound, so \( B^T \) is not exceptional, contradicting 3.41.

Let \( d = \deg^\mathcal{U}(\alpha) \). Note that \( (\kappa^{++})^R = (\kappa^{++})B^T \) and
\[
((\rho^T)^+)_{\text{Ult}_m(R, F')} = ((\rho^T)^+)^{M^T},
\]
so arguing like in the proof of 2.13, it follows that
\[
\text{Ult}_m(R, F') = \text{Ult}_d(R, F') = M^T
\]
and the factor map between the ultrapowers is the identity. (We don’t need to use any condensation here.) Letting \( \pi = i^R_{\mathcal{U}} \) and \( H = R \), then \( H, \pi \) are as in 3.42.

Now consider case (b). Note that \( R = F^\mathcal{U}_\varepsilon \), so \( R \) is active type 2, and \( \text{OR}^R = (\kappa^+)B^T \). Note that \( \deg^\mathcal{U}(\beta + 2) = m \) and \( \text{cr}(F^R) = \text{cr}^M_{\beta + 1} \), so \( \text{pred}^\mathcal{U}(\beta +
\[2) = \text{pred}^U(\varepsilon + 1) \text{ and } B_{\varepsilon+1}^U = B_{\beta+2}^U \text{ and } \deg^U(\varepsilon + 1) = m. \text{ Let } H = B_{\varepsilon+1}^U. \]

Then

\[\text{Ult}_m(H, F') = M^U\]

and letting \( \pi = i^H, \) then \( H, \pi \) are as in \( 3.42. \)

Now consider case \( (c)(i). \)

**SUBCLAIM 9.3.** In case \( (c)(i), \) \( E^U_\varepsilon \) is the preimage of \( E^U_{\beta+1} \) and

\[ \text{lh}^U_\varepsilon = (\kappa^+)B^T. \]

**Proof.** We have \( P_\varepsilon \subseteq R \) and

\[(\kappa^+)B^T = (\kappa^+)P_\varepsilon = (\kappa^+)P_\varepsilon = (\kappa^+)R < \text{lh}^U_\varepsilon. \]

We also have \( (\kappa^+)R \geq (\kappa^+)B^T \) and

if \( \beta + 1 \in \mathcal{B}^U \) then \( (\kappa^+)R > (\kappa^+)B^T; \)

the latter is because \( P^U_{\beta+1} \not\subseteq M^T \) and \( P^U_{\beta+1} \) projects \( \leq \rho^T. \)

Let \( P \subseteq R \) and \( p \in \{-1\} \cup \omega \) be such that \( (\kappa^+)B^T = (\kappa^+)P \) and

\[ \rho^P_{p+1} \leq (\kappa^+)B^T < \rho^P_{p}. \]

Then like before, using condensation or the ISC, we have

\[ \text{Ult}_p(P, F') \subseteq M^U_{\beta+1}. \]

But

\[ (\rho^T)^+_{\beta+1} = (\rho^T)^+ M^T, \]

and since \( i^U_{\beta+1} = \rho^T, \) therefore \( \text{Ult}_p(P, F') = P^U_{\beta+1}. \) So \( P \) is type 1 and \( \text{OR}^P = (\kappa^+)B^T. \) Therefore \( E^U_\varepsilon = F^P. \) Now \( i^U_{\beta+1} \) is continuous at \( (\kappa^+)B^T. \) So if \( P < R \) then \( i^U_{\beta+1} \) is continuous at \( \text{OR}^P, \) and so \( i^U_{\beta+1}(P) = P^U_{\beta+1}. \) If \( P = R \) then \( \text{Ult}_p(P, F') = M^U_{\beta+1} \) (even if \( p < \deg^U(\beta + 1)). \)

Since \( E^U_\varepsilon = F^P \) and \( \text{cr}(F^P) = \text{cr}(F^U), \) \( \text{pred}^U(\varepsilon + 1) = \varepsilon \) and \( B_{\varepsilon+1}^U = R \) and \( \deg^U(\varepsilon + 1) = \deg^U(\beta + 1). \) Also, \( \text{pred}^U(\beta + 2) = \beta + 1 \) and \( m = \deg^U(\beta + 2) = \deg^U(\varepsilon + 1). \) Using this, and letting \( H = B_{\varepsilon+1}^U, \) we get

\[ \text{Ult}_m(H, F') = M^T \]

and letting \( \pi = i^H, \) then \( H, \pi \) are as in \( 3.42. \)

Finally consider case \( (d)(i). \) For illustration, assume that \( \beta + 2 \not\in \mathcal{B}^U. \) Let \( \chi = \text{pred}^U(\beta + 2) \) and \( S = B_{\beta+2}^U \) and \( j = \deg^U(\beta + 2). \) A combination of the preceding arguments gives the following:

- \( P^U_\varepsilon \) is the type 2 preimage of \( P^U_{\beta+1}. \)
\[ \text{pred}^U(\varepsilon + 1) = \chi \] and \[ B_{\varepsilon+1} = S \] and \[ \deg^U(\varepsilon + 1) = j, \]

- \( \varepsilon = \text{pred}^U(\varepsilon + 2) \) and \( \deg^U(\varepsilon + 2) = 0, \)
- \( \beta + 1 = \text{pred}^U(\beta + 3) \) and \( m = \deg^U(\beta + 3) = 0. \)

Let \( J = B_{\varepsilon+1}^U \) and \( H = B_{\varepsilon+2}^U. \) Those arguments also give that

\[ \text{Ult}_{\chi}^j(J,F') = B_{\beta+2}^U \]

and letting \( \sigma = i_{j'}^U, \) then \( \sigma(P_{\varepsilon+1}^U) = P_{\beta+2}^U \) (as mentioned above), and

\[ \text{Ult}_0(H,F') = M^T, \]

and etc.

Cases (b)(ii) and (c)(ii) are fairly similar to the preceding cases. \( \square \)

There is just one case left:

**Claim 10.** Suppose that \( B \) is exceptional and the comparison terminates early. Then \( \alpha \in \mathcal{B}^T \cap \mathcal{B}^U \) and one of \( \mathcal{B}^T, \mathcal{B}^U \) has an exceptional core.

**Proof.** We may assume that \( \alpha \) is \((\mathcal{U}, T)\)-unusual. Let \( \chi < \xi < \alpha \) witness this. So \( E_{\xi}^U = F^I = F(Q^T). \) We have either \( M^T \not\subseteq M^U \) or \( M^U \not\subseteq M^T. \) By Claim 2, therefore \( M^T = M^U. \) Let \( H = B_{\varepsilon+1}^U. \) Then

\[ \text{Ult}_m(H,F') = M^T, \]

and etc. \( \square \)

If \( B \) is non-exceptional (exceptional) then by some of the preceding claims, we have an iterate \( B' \) of \( B \) such that \( B' \) has a good (exceptional) core. In the non-exceptional case, the proof of Claim 6 of 3.32 then shows that \( B \) has a good core. So the following claim completes the proof of the theorem:

**Claim 11.** Suppose that \( B \) is exceptional and let \( B' \) be a non-dropping iterate of \( B. \) Then \( B \) has an exceptional core iff \( B' \) does.

**Proof.** The proof similar to 3.32, but with some extra argument. We assume that \( m \geq 0 \) and leave the other case to the reader (the main distinction in that case is that even though \( m = -1, \) all ultrapower embeddings are at least \( r\Sigma_1 \) elementary). Fix \( H, \kappa, F, X \) as in 3.42. Let \( B' = (\gamma', \rho', M', Q') \) and fix \( H', \kappa', F', X' \) as in 3.42 with respect to \( B'. \) Let \( i : M \to M' \) and \( j : Q \to Q' \) be the iteration maps. So \( i \upharpoonright (B||\rho) = j \upharpoonright (B||\rho). \) Note that for each \( \alpha < \rho, \) we have \( X \cap \alpha \in B||\rho \) and

\[ i(X \cap \alpha) = X' \cap i(\alpha). \quad (10) \]

Also, \( i(\kappa, x_{m+1}^M) = (\kappa', x_{m+1}^{M'}). \) Because \( B' \) has an exceptional core, we have

\[ \text{Hull}_{m+1}^{M'}(X' \cup x_{m+1}^{M'} \cup \bar{\rho}^M) \cap \rho' = X'. \]
From these facts, it follows easily that
\[ \text{Hull}^M_{m+1}(X \cup z^M_{m+1} \cup \bar{p}^M_m) \cap \rho = X. \] (11)

It remains to see that
\[ H\|((\kappa^+)^H = M|((\kappa^+)^M. \]

Let
\[ Y = \text{Hull}^M_{m+1}(X \cup z^M_{m+1} \cup \bar{p}^M_m) \cap (\rho^+)^M, \]

let \( \sigma = {}_I^M |(\kappa^+)^M \) and \( Z = \text{rg}(\sigma) \). It suffices to see that \( Y = Z \). Let \( Y', \sigma', Z' \) be defined analogously from \( B' \). Because \( B' \) has an exceptional core, \( Y' = Z' \). So the subclaims below immediately give that \( Y = Z \), completing the proof. They are proven by breaking \( Y \) and \( Z \) into unions of small pieces.

**Subclaim 11.1.** For any \( \alpha < (\rho^+)^M \), \( \alpha \in Y \) iff \( i(\alpha) \in Y' \).

**Proof.** If \( \alpha \in Y \) then \( i(\alpha) \in Y' \) because \( i^\prime X \subseteq X' \) and \( i(z^M_{m+1}) = z^M ' \).

Suppose \( \alpha \notin Y \). For \( \beta < \rho \) and \( \delta < \rho^M \) let \( Y_{\beta, \delta} \) be the set of all \( \xi < (\rho^+)^M \) such that
\[ \xi \in \text{Hull}^M_{m+1}((X \cap \beta) \cup z^M_{m+1} \cup \bar{p}^M_m), \]
as witnessed by some theory below
\[ \text{Th}_{z_{m+1}^n}(\delta \cup \{ \bar{p}^M_m \}). \]

(See [3, \S2], in particular, the stratification of \( r\Sigma_{m+1} \) described there, for more details. If \( m = 0 \) or this should be modified appropriately; for example, if \( m = 0 \) and \( M \) is passive and \( \Omega^M \) is divisible by \( \omega^2 \) then instead, the \( r\Sigma_1 \) fact should be true in \( M(\delta) \). Then \( Y_{\beta, \delta} \in M \). Define \( Y_{\beta, \delta}' \) analogously over \( M' \). Let \( I = \rho^M_m \times \rho \). Using line (10), we have
\[ i(Y_{\beta, \delta}) = Y_{i(\beta), i(\delta)}, \]
and an easy calculation gives
\[ Y' = \bigcup_{(\beta, \delta) \in I} i(Y_{\beta, \delta}). \]
The fact that \( i(\alpha) \notin Y' \) follows easily. \qed

**Subclaim 11.2.** For any \( \alpha < (\rho^+)^M \), \( \alpha \in Z \) iff \( i(\alpha) \in Z' \).

**Proof.** Suppose \( \alpha \in Z \). Let \( \beta = \sigma^{-1}(\alpha) \). We claim that \( \sigma'(i(\beta)) = i(\alpha) \), which suffices. For let \( C \subseteq (\kappa^+)^M \) be a wellorder of \( (\kappa^+)^M \) in ordertype \( \beta \), with \( C \in M \). Then \( \sigma(C) \in M \) and is a wellorder of \( \rho^B \) in ordertype \( \alpha \). Therefore \( i(C), i(\beta) \) and \( i(\sigma(C)), i(\alpha) \) are likewise. So it suffices to see that \( \sigma'(i(C)) = i(\sigma(C)) \). But for any \( D \in \mathcal{P}(\kappa) \cap M \), we have \( \sigma'(i(D)) = i(\sigma(D)) \), and so the continuity of the various maps at \( (\kappa^+)^M \) then easily gives what we want.
Now suppose $\alpha \notin Z$; we want to see that $i(\alpha) \notin Z'$. The proof is similar in nature to the proof of Subclaim 11.1; we just describe the decomposition of $Z$. For $\beta < (\kappa^+)^M$ let $f_\beta : (\kappa^+)^M \to \beta$ be a surjection in $M$. For $\beta < (\kappa^+)^M$ and $\delta < \rho$ let

$$Z_{\beta, \delta} = \sigma(f_\beta)''(X \cap \delta).$$

Then $Z_{\beta, \delta} \in M$. Now $\sigma$ is continuous at $(\kappa^+)^M$ and note that

$$\text{rg}(\sigma) = \bigcup_{\beta, \delta} Z_{\beta, \delta}.$$

Define $f'_\beta$ and $Z'_{\beta, \delta}$ analogously from $B'$; we may assume that $i(f_\beta) = f'_{i(\beta)}$. Then we have

$$i(Z_{\beta, \delta}) = Z'_{i(\beta), i(\delta)},$$

and it follows that $i(\alpha) \notin Z'$. This completes the proof of the theorem. \qed

4. Condensation from normal iterability

Standard $(k+1)$-condensation\(^{19}\) gives the following. Let $H, M$ be $(k+1)$-sound premice, where $M$ is $(k, \omega_1, \omega_1 + 1)$-iterable, and let $\pi : H \to M$ be a near $k$-embedding, with $\text{cr}(\pi) \geq \rho = \rho_H^{k+1}$. Then either:

- $H \subseteq M$, or
- $M \models \rho$ is active with extender $F$ and $H \triangleleft \text{Ult}(M|\rho, F)$. \hfill (12)

As discussed in the introduction, the standard proof uses the $(k, \omega_1, \omega_1 + 1)$-iterability of $M$. We now give a proof of the above statement, reducing the iterability hypothesis to just $(k, \omega_1 + 1)$-iterability. In our proof, we will replace the phalanx used in the standard proof with a cephal, and avoid any use of Dodd-Jensen. Much as in [13, 9.3.2], we will also weaken the fine structural assumptions on $\pi, H, M$ somewhat from those stated above. In particular, as discussed earlier, we will not assume that $M$ is $(k+1)$-solid. Because we drop this assumption, it seems that we need to weaken a little the conclusion of condensation in the case that $H \notin M$ (cf. 4.2(1)), compared to the version stated in [13]. So in this sense we cannot quite prove full condensation.\(^{20}\) In order to state the weakened conclusion, we need the following definition.

4.1 Definition. Let $M$ be a $k$-sound premouse and let $\rho \in [\zeta_{k+1}^M, \rho_k^M]$. The $\rho$-solid-core of $M$ is

$$H = \text{cHull}_{k+1}^M(\rho \cup z_{k+1}^M \cup \bar{p}_k^M),$$

and the $\rho$-solid-core map is the uncollapse map $\pi : H \to M$. \hfill \dag

\(^{19}\)Cf. [3, pp. 87–88] or [13, Theorem 9.3.2].

\(^{20}\)But see §6.
Note that the $\rho$-solid-core map is a $k$-embedding, since $H \notin M$ and by 2.4.

4.2 Theorem (Condensation). Let $M$ be a $k$-sound, $(k,\omega_1+1)$-iterable premouse. Let $H$ be a $\rho$-sound premouse where $\rho$ is a cardinal of $H$ and $\rho_{k+1}^H \leq \rho < \rho_k^H$. Suppose there is a $k$-lifting embedding $\pi : H \rightarrow M$ such that $\text{cr}(\pi) \geq \rho$. Let $\gamma = \text{card}^M(\rho)$. Then either:

1. $H \notin M$ and:
   
   (a) $\zeta^H_{k+1} = \zeta^M_{k+1} \leq \rho$ and $\pi(\zeta^H_{k+1}) = \zeta^M_{k+1}$,
   
   (b) $H$ is the $\rho$-solid-core of $M$ and $\pi$ is the $\rho$-solid-core map,
   
   (c) $\rho_{k+1}^H \notin [\gamma, \rho)$,
   
   (d) if $\rho_{k+1}^H = \rho$ and $(\rho^+)^H < (\rho^+)^M$ then $M|\rho$ is active with a superstrong extender with critical point $\kappa$ and $\rho_{k+1}^M \leq (\kappa^+)^M < \rho$,
   
   (e) $\rho_{k+1}^H \geq \rho_{k+1}^M$,
   
   (f) if $M$ is $(k+1)$-solid then $\rho_{k+1}^H = \rho_{k+1}^M$,
   
   (g) if $\rho_{k+1}^H = \rho_{k+1}^M$ then $H$ is the $\rho$-core of $M$, $\pi$ is the $\rho$-core map and $\pi(p_{k+1}^H) = p_{k+1}^M$.

   or

2. $H \in M$ and exactly one of the following holds:
   
   (a) $H \not< M$, or
   
   (b) $M|\rho$ is active with extender $F$ and $H \not< \text{Ult}(M|\rho, F)$, or
   
   (c) $M|\rho$ is passive, $M|(\rho^+)^H$ is active with a type 1 extender $G$ and $H = \text{Ult}_\rho(Q, G)$, where $Q \not< M$ is such that $(\gamma^+)^Q = \rho$ and $\rho_{k+1}^Q = \gamma < \rho_{k+1}^Q$, or
   
   (d) $k = 0$ and $H, M$ are active type 2 and $M|\rho$ is active with a type 2 extender $F$ and letting $R = \text{Ult}(M|\rho, F)$, $R|(\rho^+)^H$ is active with a type 1 extender $G$ and $H = \text{Ult}_0(M|\rho, G)$.

4.3 Remark. It is easy to see that if we add the assumption that $H, M$ are both $(k+1)$-sound, then line (12) holds. In fact, it suffices to add the assumption that if $H \notin M$ then $M$ is $\rho$-sound, and if $H \in M$ then $H$ is $(k+1)$-sound.

Proof. Recall that by 2.4, if $H \notin M$ then $\pi$ is a $k$-embedding. This gives:

Claim 1. If $(\rho^+)^H = (\rho^+)^M$ or $\rho_{k+1}^H < \gamma$ then $H \notin M$ and $\pi$ is a $k$-embedding.

An easy calculation using the $\rho$-soundness of $H$ gives (cf. [5, 2.17]):

Claim 2. $\zeta^H_{k+1} \leq \rho$ and $p_{k+1}^H \setminus \rho = \zeta_{k+1}^H \setminus \rho$.

Claim 3. Suppose that $H \notin M$ and (1)(a),(c) hold. Then so do (1)(b),(e), (f),(g).

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Proof. We know $\pi$ is a $k$-embedding. So part (b) follows easily from Claim 2 and (a). Consider (e). Let $\kappa = \rho^H_{k+1}$. If $P(\kappa)^H = P(\kappa)^M$ then (e) is clear. If $P(\kappa)^H \neq P(\kappa)^M$ then by (c), $\kappa = \rho$, so because $H \notin M$, (e) holds.

For (f), because $\rho^M_k \leq \rho^H_{k+1}$, the fact that $\rho^M_{k+1} = \rho^H_{k+1}$ follows from [5, 2.17] and (a). For (g), suppose $\rho^M_{k+1} = \rho^H_{k+1}$. We have $\rho^H_{k+1} \leq \rho$. If $\rho^M_{k+1} = \rho^H_{k+1} = \rho$ then because $H \notin M$, and by the solidity of $p^H_{k+1} = p^H_{k+1} \setminus \rho$, we then have $p^M_{k+1} = \pi(p^H_{k+1})$. Suppose $\rho^M_{k+1} = \rho^H_{k+1} = \kappa < \rho$, so by (c), $\kappa < \gamma$. So $P(\kappa)^M = P(\kappa)^H$, so $p^M_{k+1} \leq \pi(p^H_{k+1})$, and so using the solidity of $p^H_{k+1} \setminus \rho$, it follows that $\pi(p^H_{k+1} \setminus \rho) = p^M_{k+1} \setminus \rho$, and since $\pi \mid \rho = \text{id}$, that $\pi(p^H_{k+1}) = p^M_{k+1}$. Now (g) easily follows.

Now there are two main cases overall.

Case 1. $(\rho^+)^H = (\rho^+)^M$.

We will establish (1). By Claim 1, $H \notin M$ and $\pi$ is a $k$-embedding. Note that $\zeta^H_{k+1} \leq \rho$. Using generalized solidity witnesses and because $P(\rho)^H = P(\rho)^M$, (a) easily follows. Parts (c),(d) are trivial. Now use Claim 3.

Case 2. $(\rho^+)^H < (\rho^+)^M$.

In this case we produce an iterable cephal $C$, which we use to deduce the required facts. Let $\eta = (\rho^+)^H$. Clearly either $\text{cr}(\pi) = \rho$ or $\text{cr}(\pi) = \eta$. If $M \not\models \rho$ is passive then let $J \leq M$ be least such that $\rho^J \leq \rho$ and $\eta \leq \text{OR}^J$. If $M \not\models \rho$ is active and $\eta < (\rho^+)_{\text{Ult}(M \mid \rho)}$ then let $J = \text{Ult}(M \mid \rho)$ be least such that $\eta \leq \text{OR}^J$ and $\rho^J = \rho$. Otherwise leave $J$ undefined. We may assume that if $J$ is defined then $H \neq J$, since otherwise (2) holds (it is easy to see that the four options are mutually exclusive). This ensures that the cephal $C$ we define next is non-trivial.

If $\rho$ is a cardinal of $M$, let $C = (\rho, H, J)$. Then $C$ is a bicephalus. Here the fact that $H \models [\eta = J] \land \eta$, and therefore that $\eta = (\rho^+)^J$, follows from condensation for $\omega$-sound mice, 2.13. If instead $\rho$ is not a cardinal of $M$ (and so $\gamma < \rho < (\gamma^+)^M$), let $C = (\gamma, \rho, H, Q)$, where $Q \not\models \rho$ is least such that $\rho \leq \text{OR}^Q$ and $\rho^Q = \gamma$; here $C$ is a cephalanx, again using 2.13.

Claim 4. $C$ is a non-trivial, $(\omega_1 + 1)$-iterable cephal.

Assume this claim for now; we will use it to finish the proof.

Claim 5. Suppose that either:

(i) $\rho$ is a cardinal of $M$, so $C = (\rho, H, J)$ is a bicephalus; or
(ii) $\rho$ is not a cardinal of $M$ and $C = (\gamma, \rho, H, Q)$ is a passive cephalanx, OR$^J = \eta$ and $J$ is type 3.

Then (1) holds.

Proof. Note that in case (ii), $N^C = J$. Using 3.32 or 3.34, and since $H \neq J$ and $J$ is sound, we have that OR$^J = \eta$, $J$ is active type 1 or type 3, and letting
\[ F = F^J \text{ and } \kappa = \text{cr}(F), \text{ we have } \kappa < \gamma \text{ (as in case (i), } \gamma = \rho, \text{ and in case (ii), } J \text{ is type 3), and letting } N \text{ be the } \kappa \text{-core of } H, N \text{ is } \kappa \text{-sound}, N \| (\kappa^+)^N = H \| (\kappa^+)^H \text{ (so } F \text{ is semi-close to } N) \text{ and } H = \text{Ult}_k(N, F). \] So \[ N \| (\kappa^+)^N = H \| (\kappa^+)^H = M \| (\kappa^+)^M. \quad (13) \]

It follows that \( \rho^H_k < \gamma \leq \kappa < \gamma \), so \( H \not\in M \) and \( \pi \) is a \( k \)-embedding and \( \rho^{M_k} \leq \rho^H_k \).

Now \( \zeta^H_k \leq \zeta^N_k \leq \kappa \) since \( N \) is \( \kappa \)-sound. So \( \zeta^H_k = \zeta^N_k \leq \kappa \). But \( \kappa < \gamma \), so \( \mathcal{P}(\kappa)^H = \mathcal{P}(\kappa)^M \), so (a) follows. Since \( \rho^H_k \leq \zeta^H_k \) we have (c), and (d) is trivial.

**Claim 6.** Suppose that \( \rho \) is not a cardinal of \( M \) and \( C = (\gamma, \rho, H, Q) \) is a passive cephalanx (so \( N^C = J \)), and if \( \text{OR}^J = \eta \) then \( J \) is not type 3. Then (2)(c) holds.

**Proof.** Using 3.34, \( \text{OR}^J = \eta, J \) is active type 1, \( \rho^Q_k \leq \text{cr}(F^J) = \gamma < \rho^Q \), \( H = \text{Ult}_k(Q, F^J), \) and since \( J \subseteq Q \), therefore \( \rho^H_k = \rho^Q_k = \gamma < \rho \).

**Claim 7.** Suppose that \( \rho \) is not a cardinal of \( M \) and \( C \) is an active cephalanx. Then either (1) or (2)(d) holds.

**Proof.** We have \( C = (\gamma, \rho, H, Q) \) where \( Q = M \restriction \rho \) is active. Let \( F = F^Q \). Apply 3.46 to \( C \).

If \( C \) is non-exceptional then \( C \) has a good core, and the arguments from before give that either:

- \( \rho^H_k < \gamma \) and (1) holds, or
- \( \rho^H_k = \gamma \) and (2)(d) holds.

Now suppose that \( C \) is exceptional, so \( C \) has an exceptional core. Let \( K = \text{cHull}^H_{k+1}(X \cup z^H_k \cup p^H_k) \), where \( X \) is defined as in 3.42. Let \( \kappa = \text{cr}(F) \). By 3.43, \( K \) is \((\kappa^+)^K\)-sound, and \( \rho^K_{k+1} \leq (\kappa^+)^K \). Since \((\kappa^+)^K = (\kappa^+)^M \), therefore \( K \not\in M \). Since \( Q \in M \) and \( \text{Th}^K_{k+1}(\rho^K_{k+1} \cup (\kappa^+)^K) \) can be computed from \( F^Q \) and \( \text{Th}^H_{k+1}(\rho^H_{k+1} \cup \rho) \), it follows that \( H \not\in M \), so \( \pi \) is a \( k \)-embedding, as is \( i_K \). So we must verify (1).

**Subclaim 7.1.** If \( \rho^K_{k+1} = (\kappa^+)^K \) then (1) holds.
Proof. The argument here is similar to that used to illustrate the failure of solidity for long extender premice. By 3.43, we have $\rho_{k+1}^H = \rho$ and $i_{F}^K(p_{k+1}^K) = p_{k+1}^H$ and both $K, H$ are $(k + 1)$-sound. Moreover,

$$p_{k+1}^M \leq \pi(p_{k+1}^H) \upharpoonright (\rho)$$

because $K \notin M$ and by the calculation above. Since $H$ is $(k + 1)$-solid, therefore

$$p_{k+1}^M \setminus \rho = \pi(p_{k+1}^H).$$

But for $\alpha \leq \rho$,

$$\alpha < \rho \iff \text{Th}^M_{\Sigma_{k+1}}(\pi(p_{k+1}^H) \cup \alpha) \in M,$$

(14)

because (in the case that $\alpha = \rho$) $H \notin M$, and (in the case that $\alpha < \rho$) cr$(\pi) = \rho = p_{k+1}^H$. But line (14) implies that

$$p_{k+1}^M = \pi(p_{k+1}^H) \upharpoonright (\rho).$$

Now $z_{k+1}^H = p_{k+1}^H$ and $\zeta_{k+1}^H = \rho$, and (1)(a),(c),(d) follow. \qed

Note that in the above case, $M$ is not $(k + 1)$-solid.

Subclaim 7.2. If $\rho_{k+1}^K \leq \kappa < \zeta_{k+1}^K$ then (1) holds.

Proof. Suppose $\rho_{k+1}^K \leq \kappa$. Then $\zeta_{k+1}^K < (\kappa^+)^K$, since otherwise,

$$\text{Th}^K_{\Sigma_{k+1}}(\rho_{k+1}^K \cup z_{k+1}^K) \in K,$$

a contradiction. So suppose that $\kappa < \zeta_{k+1}^K < (\kappa^+)^K$. Then $z_{k+1}^H = i_{F}^K(z_{k+1}^K)$ and $\zeta_{k+1}^H = \sup i_{F}^K(z_{k+1}^K)$, by [5, 2.20]. So $\gamma < \zeta_{k+1}^H < \rho$. So to see that (a) holds, it suffices to see that

$$\text{Th}^M_{\Sigma_{k+1}}(\pi(z_{k+1}^H) \cup \zeta_{k+1}^H \cup p_{k}^M) \notin M,$$

so suppose otherwise. Then because $F^Q \in M$, we get

$$\text{Th}^K_{\Sigma_{k+1}}(\zeta_{k+1}^K \cup z_{k+1}^K \cup \tilde{p}_{k}^K) \in M.$$

But $P(\kappa)^K = P(\kappa)^M$, so the above theory is in $K$, a contradiction.

We also have $p_{k+1}^H \leq \rho_{k+1}^K \leq \kappa$, so (c) holds, and (d) is trivial. \qed

Subclaim 7.3. If $\zeta_{k+1}^K \leq \kappa$ then (1) holds.

Proof. This follows as before since $P(\kappa)^K = P(\kappa)^H = P(\kappa)^M$. \qed

This completes the proof of the claim.
Proof of Claim 4. The basic approach is to lift iteration trees on $C$ to iteration trees on $M$. There are some details here that one must be careful with. For illustration, we assume that $C = (\gamma, \rho, H, M | \rho)$ is an active cephalanx. The other cases are similar (the bicephalus case a little different, but simpler). In order to keep focus on the main points, we also assume that $\pi$ is $c$-$\nu$-preserving (see [6]). By the calculations in [6], this will ensure that all lifting maps we encounter are $c$-$\iota$-preserving, keeping the copying process smooth. (If instead, $\pi$ is not $\nu$-preserving, one should just combine the copying process to follow with that given in [6]. In the next section we do provide details of a copying process, with resurrection, which incorporates those extra details. If $\pi$ is not $c$-preserving, one can incorporate the changes sketched in 2.3.)

For a tree $T$ on $C$ and $\alpha + 1 < \text{lh}(T)$, we say $T$ lift-drops at $\alpha + 1$ iff $\alpha + 1 \in \mathcal{B}^T$, $\text{pred}^T(\alpha + 1) \in \mathcal{B}^T$ and $[0, \alpha + 1]_T$ does not drop in model.

If $T$ lift-drops at $\alpha + 1$ then $Q$ is type 2, and letting $\beta = \text{pred}^T(\alpha + 1)$, we have $E^T_\beta = F(Q^T_\beta)$ and $cr(j^T_{\beta, \alpha + 1}) = \text{l}g\text{cd}(Q^T_\beta)$.

Let $\Sigma$ be a $(k, \omega_1 + 1)$-iteration strategy for $M$. Consider building an iteration tree $T$ on $C$, and lifting this to a $k$-maximal tree $U$ on $M$, via $\Sigma$, inductively on $\text{lh}(T)$. Having defined $(T, U)| \lambda + 1$, then for each $\alpha \leq \lambda$, letting $B_\alpha, M_\alpha, Q_\alpha$ be the models of $T$, and $S_\alpha = M^U_\alpha$, we will have also defined embeddings $\pi_\alpha$ and $\sigma_\alpha$, such that:

1. We have $<T = <U$. The drop structure of $U$ matches that of $T$, except for the following exceptions:
   - If $\alpha \in \mathcal{B}^T$ then $[0, \alpha]_U$ does not drop in model or degree (so $\deg^U(\alpha) = k$).
   - If $T$ lift-drops at $\alpha$ then $U$ drops in model at $\alpha$.
   Moreover, if $\alpha \notin \mathcal{B}^T$ then $\deg^U(\alpha) \geq \deg^T(\alpha)$.

2. If $\alpha \in \mathcal{B}^T$ then
   $$\pi_\alpha : \mathfrak{C}_0(M_\alpha) \to \mathfrak{C}_0(S_\alpha)$$
   is a $c$-$\iota$-preserving $k$-lifting embedding and letting $W_\alpha = i^T_{0, \alpha}(Q)$,
   $$\sigma_\alpha : Q_\alpha \to W_\alpha$$
   is an $r\Sigma_0$-elementary simple embedding. Moreover,
   $$\pi_\alpha| \rho(B_\alpha) = \sigma_\alpha.$$

3. If $\alpha \in \mathcal{M}^T$ then $\sigma_\alpha$ is undefined and
   $$\pi_\alpha : \mathfrak{C}_0(M_\alpha) \to \mathfrak{C}_0(S_\alpha)$$
   is a $c$-$\iota$-preserving $\deg^T(\alpha)$-lifting embedding.
4. If $\alpha \in \mathcal{E}^T$ then $\pi_\alpha$ is undefined and
\[ \sigma_\alpha : \mathcal{C}_0(Q_\alpha) \to \mathcal{C}_0(S_\alpha) \]
is a $c$-$i$-preserving deg$^T(\alpha)$-lifting embedding.

5. Suppose $\alpha < \lambda$. Let $\beta \in (\alpha, \lambda]$. If $E_\alpha^T \in \mathcal{E}_+(M_\alpha^T)$ let $\psi_\alpha = \psi_{\pi_\alpha}$; otherwise let $\psi_\alpha = \psi_{\sigma_\alpha}$. Let $\tau \in \{\pi_\beta, \sigma_\beta\}$. Then
\[ \psi_\alpha | \text{lh}_\alpha^T \subseteq \tau \]
and
\[ \tau(i_\alpha^T) = \psi_\alpha(i_\alpha^T) = i_{\alpha}^T. \]

6. Suppose $\alpha < \lambda$ and let $\delta = \text{pred}^T(\alpha + 1) = \text{pred}^U(\alpha + 1)$.

(a) Suppose $\mathcal{T}$ drops in model at $\alpha + 1$. Then so does $\mathcal{U}$. If $\alpha + 1 \in \mathcal{M}^T$ then
\[ \psi_\delta(M_{\alpha + 1}^T) = S_{\alpha + 1}^U \]
and
\[ \sigma_{\alpha + 1} \circ j_{\alpha + 1}^T = i_{\alpha + 1}^U \circ \psi_\delta(\mathcal{C}_0(M_{\alpha + 1}^T)). \]

(b) Suppose $\mathcal{T}$ lift-drops at $\alpha + 1$. Then $\mathcal{U}$ drops in model at $\alpha + 1$ (but note that $[0, \delta]_{\mathcal{U}}$ does not drop in model or degree),
\[ S_{\alpha + 1}^U = i_{\alpha + 1}^U, \]
and
\[ \sigma_{\alpha + 1} \circ j_{\alpha + 1}^T = i_{\alpha + 1}^U \circ \sigma_\delta. \]

7. Suppose $\alpha < \lambda$ and $\alpha < T \beta \leq \lambda$ and $(\alpha, \beta]_T$ neither drops in model nor lift-drops. If $M_\beta$ is defined then
\[ \pi_\beta \circ i_{\alpha, \beta}^T = i_{\alpha, \beta}^U \circ \pi_\alpha, \]
and if $Q_\beta$ is defined then
\[ \sigma_\beta \circ j_{\alpha, \beta}^T = i_{\alpha, \beta}^U \circ \sigma_\alpha. \]

This completes the inductive hypotheses.

We now start the construction. We start with $\pi_0 = \pi$ and $\sigma_0 = \text{id}$. Since $\text{cr}(\pi_0) = \rho$, we have $\sigma_0 \subseteq \pi_0$.

Now let $E_\lambda = E_\lambda^T$ be given. We define $F_\lambda = E_\lambda^U$ by copying in the usual manner. That is:

(i) Suppose $E_\lambda \in \mathcal{E}_+(M_\lambda)$. Then:

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- If $E_\lambda = F(M_\lambda)$ then $F_\lambda = F(S_\lambda)$.
- If $E_\lambda \neq F(M_\lambda)$ then $F_\lambda = \psi_{\pi_\lambda}(E_\lambda)$.

(ii) Suppose $E_\lambda \notin \mathcal{E}_+(M_\lambda)$; so $E_\lambda \in \mathcal{E}_+(Q_\lambda)$. Then:

- If $E_\lambda = F(Q_\lambda)$ and $[0, \lambda]_T$ does not drop or lift-drop then $F_\lambda = F(W_\lambda)$.
- If $E_\lambda = F(Q_\lambda)$ and $[0, \lambda]_T$ drops or lift-drops then $F_\lambda = F(S_\lambda)$.
- If $E_\lambda \neq F(Q_\lambda)$ then $F_\lambda = \psi_{\sigma_\lambda}(E_\lambda)$.

The agreement hypotheses and the fact that $\pi_\lambda$ and $\sigma_\lambda$ are $\iota$-preserving (if defined) ensures that this choice of $F_\lambda$ is legitimate.

Let $\beta = \text{pred}^T(\lambda + 1)$ and $\kappa = \text{cr}^T_\beta$. It is routine to propagate the inductive hypotheses to $(T, \mathcal{U}) \mid \lambda + 2$ unless $\beta \in \mathcal{B}^T$ and $\kappa \leq (B^T_\beta)$ and $T$ does not drop in model at $\lambda + 1$. So suppose this is the case. We have $\beta = \text{pred}^\mathcal{U}(\lambda + 1)$ by property 5.

Case I. $\lambda + 1 \in \mathcal{B}^T$.

In this case $[0, \lambda + 1]_\mathcal{U}$ does not drop in model or degree; this is because $\pi_\lambda$ is $c$-preserving and because if $E_\beta = F(Q_\beta)$ then $\text{cr}^T_\beta < \gamma(B_\beta)$. By 2.2 and properties 2 and 5, we can apply (essentially)\footnote{We say essentially because if $Q$ is type 3, $\sigma_\beta$ is a simple embedding, not an embedding between squashed preimage.} the Shift Lemma to $(\pi_\beta, \psi_\lambda \mid P^T_\lambda)$ and $(\sigma_\beta, \psi_\lambda \mid P^T_\lambda)$, to produce $\pi_{\lambda + 1}$ and $\sigma_{\lambda + 1}$. For the latter, we have

$$\sigma_\beta : Q_\beta \rightarrow W_\beta \triangleleft S_\beta = S^{*}_{\lambda + 1},$$

and we set

$$\sigma_{\lambda + 1}([a, f]^{Q_\lambda}) = [\psi_\lambda(a), \sigma_\beta(f)]^{S^*_\beta}.$$ 

It follows easily that $\sigma_{\lambda + 1} \subseteq \pi_{\lambda + 1}$. See [6] for the proof that $\pi_{\lambda + 1}$ is $\iota$-preserving (using the fact that $\pi_\beta$ is). (Here $\iota$-preservation for $\sigma_{\lambda + 1}$ is immediate because this embedding is simple). The remaining properties are established as usual.

Case II. $\lambda + 1 \in \mathcal{A}^T$.

This case is routine, using the fact that $E_\beta \in \mathcal{E}_+(M_\beta)$.

Case III. $\lambda + 1 \in \mathcal{A}^T$.

So $T$ lift-drops at $\lambda + 1$, and so $E_\beta = F(Q_\beta)$ and $\text{cr}^T_\lambda = \gamma(B_\beta)$. Therefore $F_\beta = F(W_\beta)$ and $\text{cr}^H_\lambda = \sigma_\beta(\gamma(B_\beta))$ is the largest cardinal of $W_\beta$. Therefore

$$S^*_\lambda = W_\beta \triangleleft S_\beta,$$

and in particular, $\mathcal{U}$ drops in model at $\lambda + 1$. This is precisely enough to define $\sigma_{\lambda + 1}$. Everything else is routine in this case.

This completes the propagation of the properties to $(T, \mathcal{U}) \mid \lambda + 2$.

For limit $\lambda$, everything is routine.

This completes the proof that $C$ is iterable, and so the proof of the theorem. \qed
5. A premouse inner model inheriting strong cardinals

In this section we define a proper class premouse $L[E]$ which inherits all Woodin and strong cardinals from $V$, assuming that $V$ is sufficiently iterable in some larger universe $W$. Note here that $V$ need not be a premouse, so in this sense, our construction is more general than that in [10]. The model will be produced by a variant of a full background extender construction, in which we allow certain types of partial background extenders. However, all background extenders will be total in some ultrapower of $V$, and moreover, we will be able to lift iteration trees on $L[E]$ to (non-dropping) iteration trees on $V$.

Instead of using rank to measure the strength of extenders, we use:

5.1 Definition. Let $E$ be an extender. The strength of $E$, denoted $\text{str}(E)$, is the largest $\rho$ such that $H_\rho \subseteq \text{Ult}(V,E)$.

So $\text{str}(E)$ is always a cardinal. The backgrounding we use is described as follows:

5.2 Definition. Let $\lambda \leq \text{OR}$. An ultra-backgrounded construction (of length $\lambda$) is a sequence $\langle S_\alpha \rangle_{\alpha < \lambda}$ such that:

1. Each $S_\alpha$ is a premouse.

2. Given a limit $\beta < \lambda$, $S_\beta = \lim \inf_{\alpha < \beta} S_\alpha$.

3. Given $\beta = \alpha + 1 < \lambda$, either:
   (a) $S_{\alpha+1} = J_\omega(\mathcal{C}_\omega(S_\alpha))$; or
   (b) $S_\alpha$ is passive and there is $F$ and an extender $G$ such that $S_{\alpha+1} = (S_\alpha,F)$ and $F \upharpoonright \nu(F) \subseteq G$ and $\text{str}(G) \geq \nu(F)$; or
   (c) $\alpha$ is a limit, $S_\alpha$ has a largest cardinal $\rho$, and there is an extender $G$ such that letting $\kappa = \text{cr}(G)$, we have:
      i. $\text{str}(G) \geq \rho$,
      ii. $\kappa \leq \rho \leq i_G(\kappa)$,
      iii. $\rho$ is a cardinal in $i_G(S_\alpha)$,
      iv. $(S_\alpha \sim i_G(S_\alpha))|\text{OR}(S_\alpha)$,
      v. $S_{\alpha+1} \triangleleft i_G(S_\alpha)$,
      vi. $\rho_\omega(S_{\alpha+1}) = \rho$,
      vii. $\text{OR}(S_\alpha) = (\rho^+)^{S_{\alpha+1}}$.

5.3 Definition. Suppose that $V$ is a premouse. A pm-ultra-backgrounded construction is a sequence $\langle S_\alpha \rangle_{\alpha < \lambda}$ as in 5.2, except that in (3b) and (3c) we also require that $G \in \mathbb{E}^V$ and $\nu(G)$ is a cardinal.
5.4 Remark. When we refer to, for example, 5.3(3c), we mean the analogue of 5.2(3c) for 5.3. We will mostly work explicitly with ultra-backgrounded constructions; the adaptation to pm-ultra-backgrounded is mostly obvious, so we mostly omit it. For all definitions to follow, we either implicitly or explicitly make the pm-ultra-backgrounded analogue, denoted by the prefix pm-.

5.5 Definition. Let $C = \langle S_\alpha \rangle_{\alpha < \lambda}$ be an ultra-backgrounded construction. Let $\beta < \lambda$. Then we say that $\beta$, or $S_\beta$, is $C$-standard iff 5.2(2), (3a) or (3b) holds (for $\beta$). We say that $\beta$ is $C$-strongly standard iff 5.2(3c) does not hold. Given also $n \leq \omega$, we say that $(\beta, n)$ is $C$-relevant iff either (i) $\beta$ is $C$-standard, or (ii) $\beta = \alpha + 1$ and $\rho_n(N_{\alpha+1}) = \rho_\omega(N_{\alpha+1})$.

Clearly $C$-strongly standard implies $C$-standard. The next two lemmas are easy to see:

5.6 Lemma. Let $C = \langle S_\alpha \rangle$ be an ultra-backgrounded construction. Let $(\beta, n)$ be $C$-relevant. Let $\rho$ be a cardinal of $S_\beta$ such that $\rho \leq \rho^n_\beta$. Let $P \in S_\beta$ be such that $P^C_\beta = \rho$. Then there is $\alpha < \beta$ such that $C_\alpha (P) = C_\omega (S_\alpha)$.

5.7 Lemma. Let $C = \langle S_\alpha \rangle$ be an ultra-backgrounded construction. Suppose that $S_{\alpha+1}$ is active type 1 or type $\beta$ and $\rho_\omega(S_{\alpha+1}) = \nu(F(S_{\alpha+1}))$. Then $\alpha + 1$ is $C$-standard, so $F(S_{\alpha+1})$ is backgrounded by a $V$-extender.

5.8 Definition. Let $C = \langle S_\alpha \rangle$ be an ultra-backgrounded construction. Suppose that $\alpha + 1$ is not $C$-standard, and let $\rho = \rho_\omega(S_{\alpha+1})$. An extender $G$ is a $C$-nice witness for $\alpha + 1$ iff $G$ witnesses 5.2(3c), $i_G(\text{cr}(G)) > \rho$, and $S_{\alpha+1}$ is $i_G(C)$-strongly standard (in $\text{Ult}(V, G)$).

In 5.8, there is $\xi$ such that $S_{\alpha+1} = C_\omega(S^G_{\xi+1}(C))$, by 5.6 and because $\alpha$ is a limit and $\rho$ is a cardinal of $i_G(S_\alpha)$.

5.9 Lemma. Let $C = \langle S_\alpha \rangle$ be an ultra-backgrounded construction. Suppose that $\alpha + 1$ is not $C$-standard and let $\rho = \rho_\omega(S_{\alpha+1})$. Then there is a $C$-nice witness for $\alpha + 1$.

Let $G$ be a $C$-nice witness for $\alpha + 1$. Then:

- If $\text{cr}(G) < \rho$ then $\text{str}(G)$ is the the least cardinal $\geq \rho$.
- If $\text{cr}(G) = \rho$ then $\text{str}(G) = \rho^+$.
- If condensation for $\omega$-sound mice holds for all proper segments of $S_\alpha$ then $\rho$ is not measurable in $\text{Ult}(V, G)$.

Proof. Because $V$ is linearly iterable and $\alpha + 1$ is not $C$-standard, there is an extender $H$ witnessing 5.2(3c) and such that $\xi + 1$ is $i_H(C)$-strongly standard, where $\xi + 1$ is defined as in 5.8. Letting $G = i_H(H) \circ H$, then $G$ is a nice witness $(S_{\alpha+1} \triangleleft i_H(S_\alpha))$ because in $\text{Ult}(V, H)$, $i_H(H)$ coheres $i_H(S_\alpha))$.

Now let $G$ be a nice witness. The facts regarding str($G$) are easy. Suppose $F$ is a measure on $\rho$ in $U = \text{Ult}(V, G)$. Then by condensation, $S_{\alpha+1} \triangleleft i^U_F(S_{\alpha+1})$, contradicting the niceness of $G$. \qed
For pm-ultra-backgrounding, we need to modify the notion of nice witness a little:

5.10 Definition. Suppose $V$ is a premouse and let $\mathbb{C} = \langle S_\alpha \rangle$ be a pm-ultra-backgrounded construction. Suppose that $\alpha + 1$ is not pm-$\mathbb{C}$-standard, and let $\rho = \rho_\omega(S_{\alpha + 1})$. The pm-$\mathbb{C}$-nice witness for $\alpha + 1$ is the extender $G$ such that, letting $G_1$ be the least witness to 5.3(3c) (that is, the witness with $lh(G_1)$ minimal), either:

(i) $S_{\alpha + 1}$ is pm-$i_{G_1}(\mathbb{C})$-strongly standard and $G = G_1$, or

(ii) $S_{\alpha + 1}$ is not pm-$i_{G_1}(\mathbb{C})$-strongly standard and letting $G_2$ be the least witness to 5.3(3c) for $(i_{G_1}(\mathbb{C}), S_{\alpha + 1})$, then $G = G_2 \circ G_1$.

5.11 Lemma. Suppose $V$ is a premouse and let $\mathbb{C} = \langle S_\alpha \rangle$ be a pm-ultra-backgrounded construction. Suppose that $\alpha + 1$ is not pm-$\mathbb{C}$-standard, let $\rho = \rho_\omega(S_{\alpha + 1})$ and let $G$ be the pm-$\mathbb{C}$-nice witness for $\alpha + 1$. Suppose that condensation for $\omega$-sound mice holds for all proper segments of $S_\alpha$. Then:

- $S_{\alpha + 1}$ is pm-$i_G(\mathbb{C})$-strongly standard.
- $\rho$ is not measurable in $\text{Ult}(V, G)$, so $i_G(\text{cr}(G)) > \rho$.
- If 5.10(i) attains and $\rho$ is not a cardinal then $\nu(G) = \rho^+$.
- If 5.10(i) attains $\rho$ is a cardinal then either $\nu(G) = \rho$, or $G$ is type 1 and $\text{cr}(G) = \rho$.
- If 5.10(ii) attains then $\rho$ is a cardinal and letting $G_1, G_2$ be as there, $\nu(G_1) = \text{cr}(G_2) = \rho$ and $G_2$ is type 1.

Proof. By coherence and the ISC, and using condensation as in 5.9.

5.12 Definition. The ultra-stack construction is the sequence $\langle R_\alpha \rangle_{\alpha \leq \text{OR}}$ such that $R_0 = V_\omega$, the sequence is continuous at limits, and for each $\alpha < \text{OR}$ we have the following. Let $\rho = \text{OR}(R_\alpha)$. Then $R_{\alpha + 1}$ is the stack of all sound premice $R$ such that $R_\alpha \triangleleft R$ and $\rho^{R_\alpha}_R = \rho$ and $R = \mathcal{C}_\omega(S^R_{\gamma})$ for some ultra-backgrounded construction $\mathbb{C}$ and $\gamma < lh(\mathbb{C})$, assuming this stack forms a premouse (if it does not, the construction not well-defined).

Clearly if the ultra-stack construction is well-defined then $R_{\text{OR}}$ has height $\text{OR}$, and for all $\alpha < \beta \leq \text{OR}$, $R_\alpha \triangleleft R_\beta$, and $\rho$ is a cardinal of $R_{\text{OR}}$ iff $\rho = \text{OR}(R_\alpha)$ for some $\alpha \in \text{OR}$.

In order to prove that the ultra-stack construction inherits strong and Woodin cardinals, we will need to prove that certain pseudo-premice are in fact premice, just like in [3]. So we make one further definition:

5.13 Definition. Let $\lambda < \text{OR}$. An ultra-backgrounded pseudo- construction (of length $\lambda + 2$) is a sequence $\mathbb{C} = \langle S_\alpha \rangle_{\alpha < \lambda + 2}$ such that:

- $\mathbb{C} \upharpoonright \lambda + 1$ is an ultra-backgrounded construction and $S_\lambda$ is passive,
– For some $F$, $S_{\lambda+1} = (S_{\lambda}, F)$ is an active pseudo-premouse, and there is an extender $G$ such that $F \upharpoonright \nu(F) \subseteq G$ and $\text{str}(G) \geq \nu(F)$.

5.14 Definition. An almost normal iteration tree $\mathcal{U}$ on a premouse $P$ is an iteration tree as defined in [1],\textsuperscript{22} such that for all $\alpha + 1 < \beta + 1 < \text{lh}(\mathcal{U})$, we have $\nu(E^\mathcal{U}_\beta) \leq \nu(E^\mathcal{U}_\alpha)$.

5.15 Remark. It is easy to see that if $P$ is a normally iterable premouse then $P$ is iterable with regard to almost normal trees.

5.16 Theorem. Suppose that $V$ is a class of some universe $W$, and $W \models \text{“}V$ is $(\omega_1 + 1)$-iterable for arbitrary coarse trees”. Then:

(a) If $C = \langle S_\alpha \rangle$ is an ultra-backgrounded construction then for each $\alpha < \text{lh}(C)$ and $n < \omega$, $\mathcal{C}_n(S_\alpha)$ exists and is $(n, \omega_1, \omega_1 + 1)$-iterable in $W$.

(b) The ultra-stack construction is well-defined. Let $L[\mathbb{E}]$ be its final model.

(c) $\kappa$ is strong iff $L[\mathbb{E}] \models \text{“}\kappa$ is strong”. If $\kappa$ is Woodin then $L[\mathbb{E}] \models \text{“}\kappa$ is Woodin”.

5.17 Theorem. Suppose that $V$ is a premouse, is a class of some universe $W$, and $W \models \text{“}V$ is $(\omega, \omega_1, \omega_1 + 1)$-iterable”. Then the conclusions of 5.16 hold, with ultra replaced by pm-ultra.

5.18 Remark. Part (c) also holds for $A$-strong cardinals $\kappa$, for $A \subseteq \text{OR}$ such that $A$ is a class of $L[\mathbb{E}]$. (Here $\kappa$ is $A$-strong iff for every $\eta$ there is an $\eta$-strong extender $G$ such that $i_G(A) \cap \eta = A \cap \eta$.)

However, (c) does not seem to apply to local strength: it seems that we might have $\kappa$ being $\eta$-strong (some $\eta \in \text{OR}$) but $L[\mathbb{E}] \models \text{“}\kappa$ is not $\eta$-strong”.

Proof. Each part will depend on the sufficient iterability of certain structures in $W$, which we will establish in Claim 3 below. Part (a) then follows as usual.

Assuming (a), let us prove (b). Suppose (b) fails. Then it is easy to see that we have ultra-backgrounded constructions

$C = \langle S_\alpha \rangle_{\alpha \leq \lambda^C} \upharpoonright \langle S^C_\alpha \rangle_{\lambda^C \beta}$

and

$\tilde{C} = \langle S_\alpha \rangle_{\alpha \leq \lambda^C} \upharpoonright \langle S^C_\alpha \rangle_{\lambda^C \beta}$

and $\rho \in \text{OR}$ such that letting $M^C = S^C_{\lambda^C}$ and $N^C = S^C_{\lambda^C}$:

$- M = \mathcal{C}_\omega(M')$ and $N = \mathcal{C}_\omega(N')$ both exist,

$- \rho^M = \rho = \rho^N$.

\textsuperscript{22}The only difference between these and normal trees is that it is not required that $\text{lh}(E^\mathcal{U}_\beta) < \text{lh}(E^\mathcal{U}_\alpha)$ for $\alpha < \beta$. 

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- $S'_\chi = M||\langle \rho^+ \rangle^M = N||\langle \rho^+ \rangle^N = M'||\langle \rho^+ \rangle^{M'} = N'||\langle \rho^+ \rangle^{N'}$, but $M \neq N$.

It follows that $C = (\rho, M, N)$ is a sound, non-trivial bicephalus. By Claim 3 below, we have a contradiction to 3.32, completing the proof of (b).

Next, assuming (a) and (b), let us prove (c). The fact that every strong cardinal of $L[\mathbb{E}]$ is strong is by 5.7. So suppose that $\kappa$ is strong; we want to see that $\kappa$ is strong in $L[\mathbb{E}]$. Let $\eta > \kappa$ be a cardinal of $L[\mathbb{E}]$. Let $M_\alpha$ be such that $\eta = \text{OR}(M_\alpha)$. We will show that $\kappa$ is $H_\eta$-strong in $L[\mathbb{E}]$. The key is:

Claim 1. There is $\chi > \eta$ such that if $F$ is any extender with $\text{str}(F) > \chi$ then $i_F(M_\alpha)||\eta = M_\alpha$.

This follows readily from the definitions (but this claim seems to fail for the traditional fully backgrounded $L[\mathbb{E}]$-construction). Using the claim, together with a slight variant of the proof of [3, Lemma 11.4], one can show that $\kappa$ is strong in $N$, as witnessed by restrictions of extenders in $V$. The details of the argument relating to the uniqueness of the next extender are somewhat different, so we describe the differences. Let $F$ be as in the claim. We adopt the notation $(\rho$ and $G)$ of [3, 11.4].

Claim 2. [3, 11.4] holds for all $\rho < \eta$ such that $G$ is not type Z.

**Proof.** Let $\sigma : \text{Ult}(L[\mathbb{E}], G) \rightarrow \text{Ult}(L[\mathbb{E}], F)$ be the natural factor map. Let $\xi = \langle \rho^+ \rangle^{\text{Ult}(L[\mathbb{E}], G)}$. By 5.16(a), condensation holds for segments of $L[\mathbb{E}]$, and so because of the existence of $\sigma$, either:

(i) $L[\mathbb{E}]|\rho$ is passive and $\text{Ult}(L[\mathbb{E}], G)||\xi = L[\mathbb{E}]||\xi$, or

(ii) $L[\mathbb{E}]|\rho$ is active and $\text{Ult}(L[\mathbb{E}], G)||\xi = \text{Ult}(L[\mathbb{E}], F^{L[\mathbb{E}]|\rho})||\xi$.

Suppose first that $\rho$ is a cardinal of $L[\mathbb{E}]$, and so (i) holds. Then there is an ultra-backgrounded construction with last model $P = (L[\mathbb{E}]||\xi, G)$. It follows that $\rho^P = \rho$, so $P$ is fully sound, and therefore that $P \subseteq L[\mathbb{E}]$.

Now suppose that $\rho$ is not a cardinal of $L[\mathbb{E}]$. Let $\gamma = \text{card}^{L[\mathbb{E}]}(\rho)$. If $\rho$ is not a generator of $F$ then the previous argument adapts easily. So suppose $\rho$ is a generator of $F$. So $\text{cr}(\sigma) = \rho = (\gamma^+)^{\text{Ult}(L[\mathbb{E}], G)}$. In this case it seems that there might not be an ultra-backgrounded construction with last model $\text{Ult}(L[\mathbb{E}], G)||\xi$. Let $G'$ be the trivial completion of $F \upharpoonright (\rho + 1)$. Let $\xi' = (\rho^+)^{\text{Ult}(L[\mathbb{E}], G')}$, then $\text{Ult}(L[\mathbb{E}], G')||(\xi' = L[\mathbb{E}]||\xi'$ and $\gamma$ is the largest cardinal of $L[\mathbb{E}]||\xi'$. So there is an ultra-backgrounded construction with last model $L[\mathbb{E}]||\xi'$. Let $P = (L[\mathbb{E}]||\xi', G')$. Then there is a pseudo-ultra-backgrounded construction with last model $P$. By Claim 3 below, $P$ is (0, $\omega_1$, $\omega_1 + 1$)-iterable in $W$. So by [3, §10] (combined with the generalization of the latter using the weak Dodd-Jensen property), $P$ is a premouse. Therefore either $G \in \mathbb{E}$, or $L[\mathbb{E}]|\rho$ is active and $G \in \mathbb{E}(\text{Ult}(L[\mathbb{E}]|\rho, F^{L[\mathbb{E}]|\rho}))$, as required.

The fact that every Woodin cardinal is Woodin in $L[\mathbb{E}]$ is proved similarly.

We now prove the iterability we have used above.
Claim 3. We have:

- For any ultra-backgrounded construction $C$, $\alpha < \text{lh}(C)$ and $n < \omega$, $\mathcal{E}_n(S^C_\alpha)$ exists and is $(n, \omega, 1, \omega + 1)$-iterable in $W$.

- The bicephalus $C$ defined in the proof of (b) is $\omega_1 + 1$-iterable in $W$.

- For any ultra-backgrounded pseudo-construction $C$, with last model $P$, $P$ is $(0, \omega, 1, \omega + 1)$-iterable in $W$.

Proof. We will just prove the iterability of $C$; the others are simplifications of this proof. The main difference between the present iterability proof and that for a standard $L[\mathbb{E}]$-construction is in the resurrection process. The details of this process will be dealt with in a manner similar to that in $[7]$, and moreover, the resurrection process of $[7]$ will need to be folded into the present one. We follow the iterability proof of $[7]$ closely. In one regard, the present proof is slightly simpler, because in $[7]$, arbitrary standard trees were considered, whereas here we deal with a more restricted class of trees (roughly, normal trees). In the pm-ultra-backgrounded setting, i.e. the proof of 5.17, the natural adaptation of the proof lifts a tree on $C$ to an almost normal tree $U$ on $V$. We leave the verification of this to the reader. Likewise, its adaptation to stacks of normal trees on $C$ produces stacks of almost normal trees on $V$. This ensures that we only use the $(\omega, \omega, 1, \omega + 1)$-iterability of $V$ in this context, though at the cost of increasing a little the work involved in our proof of 5.16.

In the adaptations for 5.17, one should use background extenders $G$ with $\text{lh}(G)$ minimal (when witnessing 5.3(3b)), and use pm-nice witnesses (but when the pm-nice witness is as in 5.10(ii), one must literally use the two extenders $G_1$ and $G_2$ in $U$).

Let $\Sigma_V \in W$ be an iteration strategy for $V$. We will describe a strategy $\Sigma_B$ for player II in the $(\omega_1 + 1)$-iteration game on $B$. Let $T$ be an iteration tree on $B$ which is via $\Sigma_B$. Then by induction, we can lift $T$ to a tree $U$ on $V$ (as to be defined), via $\Sigma_V$, and if $T$ has limit length, use $\Sigma_V(U)$ to define $\Sigma_B(T)$. Let us say that an iteration tree $U'$ is neat iff $U'$ is non-overlapping and such that

$$\alpha + 1 < \beta + 1 < \text{lh}(U') \implies \text{str}^{M^U_{\alpha}}(E_{\alpha}^{U'}) \leq \text{str}^{M^T_{\beta}}(E_{\beta}^{T'})$$

The tree $U$ may use padding, but the tree $U'$ given by removing all padding from $U$ will be neat. (So in the adaptation to the proof of 5.17, $U'$ will be almost normal.)

We will have $\text{lh}(U) \geq \text{lh}(T)$, but in general may have $\text{lh}(U) > \text{lh}(T)$. For each node $\alpha$ of $T$, $(\alpha, 0)$ will be a node of $U$, and the model $M^U_{\alpha}$ will correspond directly to $B^T_{\alpha}$. However, there may also be a further finite set of nodes $(\alpha, i)$ of $U$, and models $M^U_{\alpha i}$ associated to initial segments of $M^T_{\alpha}$ or $N^T_{\alpha}$. For indexing, let $\text{OR}^* = \text{OR} \times \omega$; we order $\text{OR}^*$ lexicographically. We index the nodes of $U$ with elements of a set $\text{dom}(U) \subseteq \text{OR}^*$, such that for some sequence $\langle k_\alpha \rangle_{\alpha < \text{lh}(T)}$ of integers $k_\alpha \geq 1$, we have:

$$\text{dom}(U) = \{ (\alpha, i) \mid \alpha < \text{lh}(T) \land i < k_\alpha \}.$$
So if \( \text{lh}(\mathcal{T}) > 1 \) then \( \text{dom}(\mathcal{U}) \) will form a set not closed downward under \( < \).

For notational convenience we also allow \( \mathcal{U} \) to use padding. If \( E = E^M_{\alpha_i} = \emptyset \) we consider \( \text{str}_{\alpha_i}^M(E) = \text{OR}(M^M_{\alpha_i}) \); we do allow \( \text{pred}_U(\beta, j) = (\alpha, i) \) in this case. Let \( \text{str}_{\alpha_i} = \text{str}_{\alpha_i}^M(E^M_{\alpha_i}) \). If \( E^M_{\alpha_i} \neq \emptyset \) we will also associate an ordinal \( s_{\alpha_i} \leq \text{str}_{\alpha_i} \), to be defined below.

We now fix some notation pertaining to \( \mathcal{T} \) and lifting maps. Let \( \alpha < \text{lh}(\mathcal{T}) \). We write \( B_\alpha = B^T_\alpha \), etc. If \( \alpha \in \mathcal{B}^T \) let \( (m_\alpha, n_\alpha) = \text{deg}^T(\alpha) \). If \( \alpha \in \mathcal{M}^T \) let \( n_\alpha = \text{deg}^T(\alpha) \). Let \( \mathcal{C}_{\alpha_i} = \mathcal{I}_{\text{lh}(\mathcal{C}_{\alpha_i})}^M(\mathcal{C}) \) and \( \Gamma_{\alpha_i} = \text{lh}(\mathcal{C}_{\alpha_i}) \). Let \( \check{\mathcal{C}}_{\alpha_i} = \mathcal{I}_{\text{lh}(\mathcal{C}_{\alpha_i})}^M(\mathcal{C}) \) and \( \check{\Gamma}_{\alpha_i} = \text{lh}(\check{\mathcal{C}}_{\alpha_i}) \). When we say, for example, \( \mathcal{C}_{\alpha_i}\text{-standard} \) we literally mean \( \mathcal{C}_{\alpha_i}\text{-standard in } M^M_{\alpha_i} \).

**5.19 Definition.** Let \( M \) be a premouse and \( \gamma \leq \text{OR}^M \). The \( \gamma\)-**dropout sequence** of \( M \) is the sequence \( \sigma = \langle (M_i, \vartheta_i) \rangle_{i < n} \) of maximum length such that \( \gamma = \text{OR}^M \) then \( \sigma = \emptyset \), and if \( \gamma < \text{OR}^M \) then \( M_0 = M|\gamma \), and for each \( i < n \), \( \vartheta_i = \rho_{\omega}(M_i) \), and if \( i + 1 < n \) then \( M_{i+1} \) is the least \( N \) such that \( M_i \triangleleft N \triangleleft M \) and \( \rho_{\omega}(N) < \vartheta_i \).

Suppose \( M_\alpha \neq \emptyset \) and let \( \gamma \leq \text{OR}(M_\alpha) \). The \( \langle \mathcal{T}, \alpha, \gamma \rangle\)-**dropout sequence** \( \tau \) of \( M_\alpha \) is defined as follows. Let \( \sigma \) be the \( \gamma\)-dropout sequence of \( M_\alpha \). If either \( \alpha \in \mathcal{M}^T \), or \( \alpha \in \mathcal{B}^T \) and \( \gamma < \rho(B_\alpha) \), then

\[
\tau = \sigma^* \langle (M_\alpha, 0) \rangle.
\]

Otherwise

\[
\tau = \sigma^* \langle (M_\alpha, \rho(B_\alpha)), (M_\alpha, 0) \rangle.
\]

If \( N_\alpha \neq \emptyset \) then for \( \gamma \leq \text{OR}(N_\alpha) \), we define the \( \langle \mathcal{T}, \alpha, \gamma \rangle\)-**dropout sequence** of \( N_\alpha \) analogously.

We now state some intentions and introduce more notation. Let \( \alpha < \text{lh}(\mathcal{T}) \).

If \( M_\alpha \neq \emptyset \) we will define:

- \( D_{\alpha_0} \in \{ \mathcal{C}_{\alpha_0}, \check{\mathcal{C}}_{\alpha_0} \} \),
- \( \Delta_{\alpha_0} = \text{lh}(D_{\alpha_0}) \),
- \( \xi_{\alpha_0} < \Delta_{\alpha_0} \), where if \( \alpha \notin \mathcal{B}^T \) then \( \xi_{\alpha_0} \) is \( D_{\alpha_0}\text{-standard} \),
- \( Q_{\alpha_0} = S_{\xi_{\alpha_0}}^{D_{\alpha_0}} \),
- and a \( \text{c}\)-preserving \( m_{\alpha_0}\text{-lifting} \) \( \pi_{\alpha_0} : \mathcal{C}_{0}(M_{\alpha_0}) \rightarrow \mathcal{C}_{m_{\alpha_0}}(Q_{\alpha_0}) \),

such that if \( [0, \alpha]_{\mathcal{B}^T} \) does not drop in model then \( D_{\alpha_0} = \mathcal{C}_{\alpha_0} \) and \( \xi_{\alpha_0} = i_{M_{(0\alpha)}(\lambda)}^{(0\alpha)}(\lambda) \).

If \( N_\alpha \neq \emptyset \) we will define \( \check{D}_{\alpha_0}, \check{\Delta}_{\alpha_0}, \check{\xi}_{\alpha_0}, \check{Q}_{\alpha_0} \) and \( \check{\pi}_{\alpha_0} \) analogously.

Now suppose \( \alpha + 1 < \text{lh}(\mathcal{T}) \). Let \( E = E_{\alpha_0} \).

Suppose that \( E \in \mathcal{E}_+(M_{\alpha_0}) \). Let \( \sigma \) be the \( \langle \mathcal{T}, \alpha, \text{lh}(\mathcal{E}) \rangle\)-dropout sequence of \( M_{\alpha_0} \) and let \( \sigma^* \) be its reverse. Let \( u_\alpha + 1 = \text{lh}(\sigma) \). Let \( \sigma^* = \langle (M_{\alpha_i}, \vartheta_{\alpha_i}) \rangle_{i < u_\alpha} \).

We will have

\[
k_{\alpha} = 2\text{lh}(\sigma) - 1 = 2u_\alpha + 1.
\]

Fix \( i \leq u_\alpha \). Let \( m_{\alpha_i} = m_{\alpha} \) if \( M_{\alpha_i} = M_{\alpha} \) and \( m_{\alpha_i} = \omega \) otherwise. If \( i > 0 \) then for each \( j \in \{ 2i - 1, 2i \} \) we will define:
We will maintain the following conditions by induction on initial segments of \( (T, U) \):

- \( D_{\alpha j} \in \{C_{\alpha j}, \tilde{C}_{\alpha j}\} \),
- \( \Delta_{\alpha j} = \text{lh}(D_{\alpha j}) \),
- \( \xi_{\alpha, j} < \Delta_{\alpha, j} \), such that \( \xi_{\alpha, 2i} \) is \( D_{\alpha, 2i} \)-standard,
- \( R_{\alpha i} = S_{\xi_{\alpha, i}}^{D_{\alpha, i-1}} \),
- \( Q_{\alpha i} = S_{\xi_{\alpha, i}}^{D_{\alpha, i}} \),
- and a c-preserving \( m_{\alpha i} \)-lifting embedding \( \pi_{\alpha i} : \mathfrak{C}_0(M_{\alpha i}) \to \mathfrak{C}_{m_{\alpha i}}(Q_{\alpha i}) \).

For \( m \leq n \leq m_{\alpha i} \) let

\[ \tau_{\alpha i}^{nm} : \mathfrak{C}_n(Q_{\alpha i}) \to \mathfrak{C}_m(Q_{\alpha i}) \]

be the core embedding. Let \( Q^*_\alpha = Q_{\alpha u_\alpha} \) and

\[ \pi^*_\alpha : \mathfrak{C}_0(P^T_\alpha) \to \mathfrak{C}_0(Q^*_\alpha), \]

where letting \( m = m_{\alpha u_\alpha} \),

\[ \pi^*_\alpha = \tau_{\alpha u_\alpha}^{m0} \circ \pi_{\alpha u_\alpha}. \]

Let \( c_\alpha \) be the set of infinite cardinals \( \kappa < \nu(E) \) of \( P^T_\alpha \). Fix \( \kappa \in c_\alpha \). Let \( i_{\alpha \kappa} \) be the largest \( i \) such that \( \bar{g}_i \leq \kappa \). Let \( i = i_{\alpha \kappa} \). Let \( m_{\alpha \kappa} \) be the least \( m \) such that either

- \( M_{\alpha i} = M_\alpha \) and \( m = m_\alpha \), or
- \( \rho_{m+1}(M_{\alpha i}) \leq \kappa. \)

Let \( M_{\alpha \kappa} = M_{\alpha i} \). We define the c-preserving \( m_{\alpha \kappa} \)-lifting embedding

\[ \pi_{\alpha \kappa} : \mathfrak{C}_0(M_{\alpha \kappa}) \to \mathfrak{C}_{m_{\alpha \kappa}}(Q_{\alpha i}) \]

by \( \pi_{\alpha \kappa} = \tau_{\alpha i}^{m_{\alpha \kappa}} \circ \pi_{\alpha i} \), where \( n = m_{\alpha i} \) and \( m = m_{\alpha \kappa} \). If \( \alpha \in \mathcal{B}^T \) and \( \kappa < \rho_\alpha \) and \((\kappa^+)_{\mathcal{B}_\alpha} < \text{lh}(E)\) then we also define \( N_{\alpha \kappa} = N_\alpha, n_{\alpha \kappa} = n_\alpha \), and \( \tilde{\pi}_{\alpha \kappa} = \tilde{\pi}_{\alpha 0}. \)

Now suppose instead that \( E \in \mathcal{E}_+(N_\alpha) \setminus \mathcal{E}_+(M_\alpha) \). Then we make symmetric definitions by analogy to the preceding ones. (So for example, we let \( \sigma \) be the \((T, \alpha, \text{lh}(E))-\text{dropdown sequence of } N_\alpha \), and set \( u_\alpha + 1 = \text{lh}(\sigma) \), and for \( i \leq u_\alpha \) we define \( N_{ai} \) and \( n_{ai} \), and also define \( \tilde{\xi}_{ai}, \tilde{Q}_{ai} \), etc.)

Let \( \omega^*_\alpha = \pi^*_\alpha \) or \( \omega^*_\alpha = \tilde{\pi}^*_\alpha \), whichever is defined.

Let \( \text{OR}^1 = \text{OR} \cup \{!\} \), where \( ! \notin \text{OR} \) ("!" should be interpreted as \emph{undefined}).

Let \( \xi, \zeta \in (\text{OR}^1)^2 \), and let \( \tilde{\xi} = (\xi, \tilde{\xi}) \) and \( \tilde{\zeta} = (\zeta, \tilde{\zeta}) \). We write \( \xi < \zeta \) iff \( \xi = \zeta \) and \( \xi < \tilde{\zeta} \) and \( \xi < \tilde{\xi} \). For \( \gamma \in \text{OR} \) let \( \max(\gamma, !) = \max(\gamma, !) = \gamma. \) We write \( \xi < \zeta \) iff \( \in \xi \) and \( \max(\xi) \leq \max(\zeta) \) and either \( \max(\xi) < \max(\zeta) \) or \( \notin \zeta. \)

Let \( \xi_{\alpha i} = (\xi_{\alpha i}, \xi_{\alpha i}). \)

We will maintain the following conditions by induction on initial segments of \((T, U)\):
1. Let $\alpha < \lh(T)$ be such that $B_\alpha$ is a bicephalus. Then
\[
\pi_{\alpha 0} \upharpoonright \rho_\alpha = \tilde{\pi}_{\alpha 0} \upharpoonright \rho_\alpha
\]
and so if $\alpha + 1 < \lh(T)$ then for all $\kappa \in c_\alpha \cap \rho_\alpha$ such that $(\kappa^+)^{B_\alpha} < \lh_\alpha^T$, 
\[
\pi_{\alpha \kappa} \upharpoonright (\kappa^+)^{B_\alpha} = \tilde{\pi}_{\alpha \kappa} \upharpoonright (\kappa^+)^{B_\alpha}.
\]

2. Let $\alpha < \beta < \lh(T)$ and $\alpha < \beta' < \lh(T)$ and $\kappa \in c_\alpha$. Then:
- If $\pi_{\alpha \kappa}$ is defined then $\pi_{\alpha \kappa} \upharpoonright (\kappa^+)^{M_{\alpha \kappa}} \subseteq \omega^*_\alpha$.
- If $\tilde{\pi}_{\alpha \kappa}$ is defined then $\tilde{\pi}_{\alpha \kappa} \upharpoonright (\kappa^+)^{N_{\alpha \kappa}} \subseteq \omega^*_\alpha$.
- If $\pi_{\beta 0}$ is defined then $\omega^*_\alpha \subseteq \pi_{\beta 0}$ and $\pi_{\beta 0}(\nu^T_\alpha) \geq \nu(F(Q^*_\alpha))$.
- If $\tilde{\pi}_{\beta 0}$ is defined then $\omega^*_\alpha \subseteq \tilde{\pi}_{\beta 0}$ and $\tilde{\pi}_{\beta 0}(\nu^T_\alpha) \geq \nu(F(Q^*_\alpha))$.
- If $\pi_{\beta,0}$ and $\tilde{\pi}_{\beta',0}$ are both defined then they agree over $P^T_\alpha$.

We write $\omega_\alpha$ for the restriction of $\pi_{\alpha+1,0}$ or $\tilde{\pi}_{\alpha+1,0}$ to $P^T_\alpha$, whichever is defined. Then $\omega_\infty = \bigcup_{\alpha+1 < \lh(T)} \omega_\alpha$ is a function.

3. Let $\beta + 1 < \lh(T)$ and $\alpha = \pred^T(\beta + 1)$ and suppose that $\beta + 1 \in \mathcal{B}^T$.
Then $\pred^T(\beta + 1, 0) = (\alpha, 0)$.

4. Let $\beta + 1 < \lh(T)$ and $\alpha = \pred^T(\beta + 1)$ and suppose that $\beta + 1 \in K^T$.
Let $i < u_\alpha$ be such that $M^T_{\beta + 1} = M_{\alpha i}$, and if $\alpha \in \mathcal{B}^T$ then $i \geq 1$. Then $\pred^T(\beta + 1, 0) = (\alpha, 2i)$. Likewise if $\beta + 1 \in \mathcal{N}^T$.

5. Let $\beta + 1 < \lh(T)$ and suppose that $E^\beta_\beta \in \mathcal{E}^+(M^\beta_\beta)$. If $0 \leq i < u_\beta$ then letting $n = m_{\beta i}$,
\[
\tau^0_{\beta i} \circ \pi_{\beta i} \upharpoonright n_{\beta, i + 1} \subseteq \omega^*_\beta.
\]

6. The tree given by removing padding from $U$ is neat. Let $\alpha + 1 < \lh(T)$ and $i < k_\alpha$ be such that $E^T_\alpha \neq \emptyset$. Let $(\beta, k) \in \text{dom}(U)$. Then:
- $s_{\alpha i} \leq \text{str}_{\alpha i}$ and $\text{Ult}(M^T_{\alpha i}, E^T_{\alpha i})$ has no measurables in $[s_{\alpha i}, \text{str}_{\alpha i})$.
- If $(\beta, j) < (\alpha, i)$ and $E^T_{\beta j} \neq \emptyset$ then $s_{\beta j} < s_{\alpha i}$.
- If $(\alpha, i) < (\beta, j)$ and $P_{\alpha i} = Q_{\alpha i}$ if $i$ is even and $P_{\alpha i} = R_{\alpha i}$ if $i$ is odd and $P_{\beta j}$ is likewise then $(P_{\alpha i} \sim P_{\beta j}) \upharpoonright s_{\alpha i}$.

7. Let $\alpha < \lh(T)$. Let $\nu = \sup_{\beta < \alpha} \nu^T_\beta$. Let $(\beta, j) \in \text{dom}(U)$ with $\beta < \alpha$. If $\alpha$ is a limit then
\[
s_{\beta j} < \sup \omega_\infty \nu
\]
(and note that $\omega_\infty \upharpoonright \nu = \pi_{\alpha 0} \upharpoonright \nu$ or $\tilde{\pi}_{\alpha 0} \upharpoonright \nu$, whichever is defined). If $\alpha = \gamma + 1$, so $\nu = \nu^T_\gamma$, then
\[
s_{\beta j} \leq \omega_\infty(\nu)
\]
(and note that $\omega_\infty(\nu) = \pi_{\alpha 0}(\nu)$ or $\tilde{\pi}_{\alpha 0}(\nu)$, whichever is defined).
8. Let \( \alpha + 1 < \text{lh}(T) \) and suppose that \( E_\alpha \in E_+(M_\alpha) \). Then \( E_{\alpha,2u_\alpha}^T \neq \emptyset \) and

\[
\sup \omega_\alpha \nu_\alpha^T \leq s_{\alpha,2u_\alpha} \leq \omega_\alpha (\nu_\alpha^T).
\]

Now suppose also that \( u_\alpha > 0 \). Let \( \nu \) be as before. Note that

\[
\nu \leq \varrho_\alpha < \varrho_\alpha 2 < \ldots < \varrho_\alpha u_\alpha \leq \nu_\alpha^T.
\]

(Here if \( u_\alpha \leq 1 \) and \( \alpha \) is a limit we could have \( \nu = \nu_\alpha^T \).) Let \( i < u_\alpha \) and let \( j \in \{2i, 2i + 1\} \). If \( E_{\alpha j} \neq \emptyset \) then

\[
\sup \omega_\alpha \iota_{\alpha,i+1} \leq s_{\alpha,j} \leq \omega_\alpha (\iota_{\alpha,i+1}).
\]

9. Let \( \alpha \leq \beta < \text{lh}(T) \) and \( j < k \) and \( k < k_\beta \) and suppose that \( (\alpha, j) = \text{pred}^T(\beta, k) \). Then

\[
\xi_\beta k \leq \iota_{\alpha,j,k}(\xi_\alpha j)
\]

and if \( \alpha < \beta \) and \( k \neq 0 \) then

\[
\xi_\beta k \leq \iota_{\alpha,j,k}(\xi_\alpha j).
\]

Suppose \( k = 0 \); so \( \alpha = \text{pred}^T(\beta) \). Then \( j = 2i \) is even and

\[
(\Xi_{\beta 0}, \tilde{\Xi}_{\beta 0}, \xi_\beta 0) = \iota_{\alpha,j,0}^T(\Xi_{\alpha j}, \tilde{\Xi}_{\alpha j}, \xi_\alpha 0)
\]

and if \( M_\beta \neq \emptyset \) then

\[
\pi_{\beta 0} \circ i_{\beta}^T = \iota_{\alpha,j,0}^T \circ \tau_{\alpha i,m_\alpha m_\beta} \circ \pi_{\alpha i},
\]

and if \( N_\beta \neq \emptyset \) then \( \tilde{\pi}_{\beta 0} \circ j_{\beta}^T \) is likewise.

10. Let \( \lambda < \text{lh}(T) \) be a limit and let \( \alpha \rightarrow^T \lambda \) be such that \( (\alpha, \lambda)_T \) does not drop in model and if \( \alpha \in B^T \) then \( \lambda \in B^T \). Then for all \( \beta, i, (\alpha, 0) \leq_T (\beta, i) \leq_T (\lambda, 0) \) iff \( i = 0 \) and \( \alpha \leq_T \beta \leq_T \lambda \). Moreover,

\[
\iota_{(\alpha_0), (\lambda_0)}^T(\Xi_{\alpha_0}, \tilde{\Xi}_{\alpha_0}, \xi_\alpha 0) = (\Xi_{\lambda_0}, \tilde{\Xi}_{\lambda_0}, \xi_\lambda)
\]

and if \( M_\alpha \neq \emptyset \) then letting \( m = m_\alpha \) and \( n = m_\lambda \),

\[
\pi_{\lambda_0} \circ \iota_{\alpha_0, \lambda_0}^T = \iota_{\alpha_0, \lambda_0}^T \circ \tau_{\alpha_0 m_\alpha m_\lambda} \circ \pi_{\alpha_0},
\]

and likewise if \( N_\alpha \neq \emptyset \).

We now begin. Let \( \xi_{00} = \lambda^C \) and

\[
\pi_{00} : C_0(M_0) \rightarrow C_{m_0}(Q_{00})
\]

be the core embedding. (Note that \( M_0 = M \) and \( Q_{00} = M' \); in the notation that assumes \( 1 < \text{lh}(T) \), the core embedding is \( \tau_{00}^{m_0} \).) We define \( \xi_{00} \) and \( \pi_{00} \).
analogously. Then $\pi_{00} \upharpoonright \rho_0 = \text{id} = \pi_{00} \upharpoonright \rho_0$, so the inductive hypotheses are immediate for $T \upharpoonright 1$ and $U \upharpoonright (0, 1)$ (in place of $T$ and $U$).

Now let $J$ be a limit ordinal and suppose that the inductive hypotheses hold of $T \upharpoonright J$ and $U \upharpoonright (\lambda, 0)$; we will define $U \upharpoonright (\lambda, 1)$ and $T \upharpoonright \lambda + 1$ and verify that the hypotheses still hold.

Note that $U \upharpoonright (\lambda, 0)$ has limit length and is cofinally non-padded. Let $c = \Sigma_V(U \upharpoonright (\lambda, 0))$. Let $\Sigma_x(T \upharpoonright \lambda)$ be the unique branch $b$ such that for eventually all $\alpha \in b$, we have $(\alpha, 0) \in c$. The inductive hypotheses ensure that $b$ is indeed a well-defined $T \upharpoonright \lambda$-cofinal branch, and there are only finitely many drops in model along $b$, and there are unique choices for $\pi_{00}$, etc, maintaining the requirements.

Now let $J = \delta + 1$ and suppose that the inductive hypotheses hold for $T \upharpoonright \delta + 1$ and $U \upharpoonright (\delta, 1)$. We will define $U \upharpoonright (\delta + 1, 1)$ and show that they hold for $T \upharpoonright \delta + 2$ and $U \upharpoonright (\delta + 1, 1)$.

**Case 1.** $u_\delta = 0$ and $\delta \in \mathcal{M}^T$.

So $E_\delta = F(M^{\delta}_{\mathcal{T}})$ and $\xi_\delta$ is $\mathcal{D}_{\mathcal{M}}$-standard. Let $\lambda = \sup_{\beta < \delta} \text{lh}_T \delta$. Then $\lambda = \rho_{\alpha}(\delta)$, so $\pi_{00}^{\mathcal{M}}(\sup \pi_{00}^{\mathcal{M}}(\lambda)) = \text{id}$, so $\omega_\delta \upharpoonright \lambda = \pi_{00} \upharpoonright \lambda$. (15)

Now if $\nu^T_\delta$ is not a limit cardinal of $P^T_\delta$ then we choose $E^\mathcal{T}_\delta$ to be some $E^* \in M^\mathcal{T}_{\mathcal{D}}$ witnessing $5.2(3b)$ for $(\mathcal{D}_\mathcal{M}, Q_{\mathcal{M}})$; we take $E^*$ of minimal rank in the Mitchell order, and set $s_{\mathcal{M}} = nu(Q_{\mathcal{M}})$. Suppose $\nu^T_\delta$ is a limit cardinal of $P^T_\delta$; in particular $E_\delta$ is type 3. Let $\nu' = \sup \omega_\delta^{\alpha} \nu_\delta$. Note that $\nu'$ is a limit cardinal of $Q_{\mathcal{M}}$ and so a limit of generators of $F(Q_{\mathcal{M}})$. Let $Q' \subseteq Q_{\mathcal{M}}$ be such that $F^Q = F(Q_{\mathcal{M}}) \upharpoonright \nu'$. Because $\nu' = \nu(F^Q)$ is a cardinal of $Q_{\mathcal{M}}$ and by 5.6 and 5.7, $Q' = S^\mathcal{T}_{\mathcal{M}}$ for some $\mathcal{D}_{\mathcal{M}}$-standard $\gamma$. So like before, we can let $E^T_{\delta}$ be some $E^* \in M^\mathcal{T}_{\mathcal{M}}$ witnessing $5.2(3b)$ for $Q'$, taking $E^*$ Mitchell-minimal, and set $s_{\mathcal{M}} = \nu'$. Let $\kappa = c_\delta$ and $\alpha = \text{pred}^T(\delta + 1)$ and $i = i_{\alpha, \kappa}$. Note that $M^T_{\delta+1} = M_{\alpha}$ and $m_{\alpha} = m_{\delta+1}$ and $N^{*}_{\delta+1} = N_{\alpha}$ and $n_{\alpha} = n_{\delta+1}$ (with each of these equalities, it is included that the object on the left is defined iff the one on the right is).

We can do set $\text{pred}^T(\delta + 1, 0) = (\alpha, 2i)$, by properties 6–8. The identities of $\mathcal{D}_{\delta+1, 0}, \mathcal{D}_{\delta+1, 0}, \xi_{\delta+1, 0}, \xi_{\delta+1, 0}, \xi_{\delta+1, 0}, \xi_{\delta+1, 0}$ are determined by property 9. We define $\pi_{\delta+1, 0}$ and/or $\pi_{\delta+1, 0}$ as usual. It is routine to show that the inductive hypotheses are maintained; we just make a couple of remarks. The fact, for example if $\pi_{\delta+1, 0}$ is defined, that $\omega_{\delta}(\nu^T_{\delta}) = \pi_{\delta+1, 0}(\nu^T_{\delta}) \geq s_{\delta}$, follows from our choice of $E^*$ (this is why we introduced $Q'$ earlier). Also, by line (15), and because $\omega_\delta \subseteq \pi_{\delta+1, 0}$ and/or $\omega_\delta \subseteq \pi_{\delta+1, 0}$, we have maintained the well-definedness of $\omega_{\delta}$.

**Case 2.** $u_\delta = 0$ and $\delta \in \mathcal{N}^T$.

By symmetry with the previous case.

**Case 3.** $u_\delta > 0$ and $\delta \in \mathcal{M}^T$.

Let $q = q_{\delta 1}$; then $q$ is a cardinal of $M_\delta$, so $q \leq \rho_0(M_\delta)$. 65
Subcase 3.1. \( q < \rho_0(M_{\delta_1}) \).

Set \( E_{80}^M = \emptyset \); so \( M_{\delta_1}^M = M_{80}^M \). Set \( \mathbb{D}_{\delta_1} = \mathbb{D}_{80} \). Let

\[ \varphi : \mathcal{C}_0(M_{\delta_1}) \to \mathcal{C}_0(Q_{\delta_0}) \]

be \( \varphi = \tau_{80}^{M_{\delta_1}} \circ \pi_{\delta_0} \). Let \( R = \varphi(M_{\delta_1}) \). Then \( \varphi(q) \) is a cardinal of \( Q_{\delta_0} \) (\( \varphi \) is c-preserving) and \( \rho_{\alpha}^R = \varphi(q) \) and \( \xi_{\delta_0} \) is \( \mathbb{D}_{80} \)-standard. So we can let \( \xi = \xi_{\delta_1} \) be such that

\[ \mathcal{C}_0(R) = \mathcal{C}_\omega(S_{\xi_{\delta_1}}^\mathbb{D}_{\delta_1}). \]

We will set \( \pi_{\delta_1} = \varphi \restriction \mathcal{C}_0(M_{\delta_1}) \) and will have that \( \mathcal{C}_\omega(Q_{\delta_1}) = \mathcal{C}_0(R) \). (Recall though that \( Q_{\delta_1} = S_{\xi_{\delta_1}}^\mathbb{D}_{\delta_1} ^{\mathbb{D}_{\delta_1}} \); we will define \( M_{\delta_2}^M, \mathbb{D}_{\delta_2} \) and \( \xi_{\delta_2} \) below.)

If \( \xi_{\delta_1} \) is \( \mathbb{D}_{\delta_1} \)-standard we set \( E_{\delta_1}^M = \emptyset, \mathbb{D}_{\delta_2} = \mathbb{D}_{\delta_1} \) and \( \xi_{\delta_2} = \xi_{\delta_1} \).

Suppose that \( \xi_{\delta_1} \) is not \( \mathbb{D}_{\delta_1} \)-standard. So \( R = S_{\xi_{\delta_1}}^{\mathbb{D}_{\delta_1}} \). We set \( E_{\delta_1}^M \) to be some \( G \in M_{\delta_1} \) such that \( G \) is a \( \mathbb{D}_{\delta_1} \)-nice witness for \( R \), and set \( s_{\delta_1} = \rho_{\alpha}^R \). This is okay, as by 5.9, if \( s = \text{str}(G) > \rho_{\alpha}^R \), then \( \text{Ult}(M_{\delta_1}, G) \) has no measurables in \( (\rho_{\alpha}^R, s) \). Let \( (\alpha, j) = \text{pred}^M(\delta, 2) \) be least such that either (i) \( (\alpha, j) = (\delta, 1) \), or (ii) \( E_{\alpha j}^M \neq \emptyset \) and \( \kappa < s_{\alpha j} \). If \( E_{\alpha} \in \mathbb{E}_\delta(M_{\alpha}) \) then let \( F = \mathbb{D}_{\alpha j} \) and \( \zeta = \xi_{\alpha j} \), and if \( 2i = j \) then let \( P = Q_{\alpha i} \), and if \( 2i - 1 = j \) then \( P = R_{\alpha i} \). Otherwise let \( F = \mathbb{D}_{\alpha j} \), \( \zeta = \xi_{\alpha j} \) and \( P = \bar{Q}_{\alpha i} \) or \( P = \bar{R}_{\alpha i} \). Let \( \kappa = \text{cr}(G) \) and \( f = \iota_{(\alpha j), (\delta, 2)} \).

By property 6, \( (P \approx R) \restriction \kappa \), and note that \( P \restriction \kappa = S_{\xi_{\delta_2}}^\mathbb{D}_{\delta_2} \) for some \( \mathbb{F} \)-standard \( \gamma < \zeta \). Since \( G \) is a nice witness, \( f(\kappa) > \varphi(q) \), and so \( Q_{\delta_1} \triangleleft f(P) \restriction \kappa \), and note that \( \varphi(q) \) is a cardinal of \( f(P) \restriction \kappa \). We set \( \mathbb{D}_{\delta_2} = f(\mathbb{F}) \), and let \( \xi_{\delta_2} \) be the \( \xi < f(\zeta) \) such that

\[ \mathcal{C}_0(R) = \mathcal{C}_\omega(S_{\xi_{\delta_2}}^{\mathbb{D}_{\delta_2}}). \]

Because \( G \) is a nice witness, the agreement between \( M_{\delta_2}^M \) and \( \text{Ult}(M_{\delta_1}^M, G) \) implies that \( \xi_{\delta_2} \) is \( \mathbb{D}_{\delta_2} \)-standard. We defined \( \pi_{\delta_1} \) earlier.

Subcase 3.2. \( q = \rho_0(M_\delta) \).

So \( M_{\delta} \) is active type 3. Let \( \nu : \mathcal{C}_0(M_{\delta}) \to \mathcal{C}_0(Q_{\delta_0}) \) be \( \nu = \tau_{80}^{M_{\delta_1}} \circ \pi_{\delta_0} \). Let \( \psi = \psi_\nu \).

Subsubcase 3.2.1. \( \psi(q) \leq \nu(F(Q_{\delta_0})) \).

Proceed as in Subcase 3.1, but using \( \varphi = \psi \) instead.

Subsubcase 3.2.2. \( \psi(q) > \nu(F(Q_{\delta_0})) \).

Here we proceed as in [7]. Set \( E^* = E_{80}^M \) to be a Mitchell-minimal witness to 5.2(3b) for \( (\mathbb{D}_{80}, Q_{80}) \) and set \( s_{80} = \nu(F(Q_{80})) \). Let \( F = F(M_{\delta}) \) and let \( T \) be the putative iteration tree on \( C \) of the form \( (T \restriction \delta + 1) \uparrow F \). Then \( M_{\delta_1} \triangleleft M^T_{\delta_1 + 1} \). Let \( \alpha = \text{pred}^T(\delta + 1) \) and \( \kappa = \text{cr}(F) \) and \( i = i_{\alpha \kappa} \). Let \( j = 2i \) and \( \text{pred}^M(\delta, 1) = (\alpha, j) \): as in Case 1 this works. Let \( \mathbb{F}, \zeta, P, f \) be defined from \( (\alpha, j) \) as in Subcase 3.1. Let \( \mathbb{D}_{\delta_1} = f(\mathbb{F}) \) and \( R = \psi(M_{\delta_1}) \). Then like in Subcase 3.1, \( R \triangleleft f(P) \) and \( R = S_{\xi_{\delta_2}}^{\mathbb{D}_{\delta_2}} \) for some \( \xi < f(\zeta) \); let \( \xi_{\delta_1} \) be this \( \xi \). Let \( \pi_{\delta_1} = \psi \restriction \mathcal{C}_0(M_{\delta_1}) \).

Now if \( \xi \) is \( \mathbb{D}_{\delta_1} \)-standard, we set \( E_{\delta_1}^M = \emptyset \), etc. Otherwise, proceed as in Subcase 3.1. Note that in the latter case,

\[ s_{80} = \nu(F(Q_{\delta_0})) < \psi(q) = s_{\delta_1}. \]
This completes the definition of $U | (\delta, 2)$ in all subcases. If $u_\delta = 1$ we set $E_5^{\delta 2}$ to be a Mitchell-minimal background for $Q^\delta_5$, and $s_{52} = \nu(F(Q^\delta_5))$. We claim that if $E_5^{\delta 1} \neq \emptyset$ then $s_{51} < s_{52}$. For certainly $s_{51} \leq \rho_\omega(R_{51}) = \rho_\omega(Q_{51}) \leq s_{52}$. But if $s_{51} = s_{52}$ then note that $R_{51} = Q_{51}$ is type 1 or type 3, and by 5.7, $\xi_{51}$ is $D_{51}$-standard, so $E_5^{\delta 1} = \emptyset$, contradiction. Also if $E_5^{\delta 1} \neq \emptyset = E_5^{\delta 1}$, then $s_{50} < \psi(\g) \leq s_{52}$.

If $u_\delta > 1$ then we now repeat the subcases, working with $M_{52}, \pi_{51},$ etc, in place of $M_{51}, \pi_{50},$ etc. We continue in this manner until producing $\omega^\delta_1, Q^\delta_1$ and $E_5^{\delta 2u_\delta}$. This completes the definition of $U | (\delta + 1, 1)$. It is straightforward to see that the inductive hypotheses are maintained.

CASE 4. $u_\delta > 0$ and $\delta \in \mathcal{N}^T$.

By symmetry.

CASE 5. $u_\delta > 0$ and $\delta \in \mathcal{N}^T$ and $E_\delta \in E_+(M_\delta)$.

This case proceeds mostly like the preceding cases, but the first step is a little different. We set $E_5^{\delta 0} = \emptyset$ and $D_{\delta 1} = D_{50}, \xi_{\delta 1} = \xi_{50},$ etc. If $\xi_{\delta 1}$ is $D_{51}$-standard then we also set $E_5^{\delta 1} = \emptyset$, etc. Suppose otherwise. We have $R_{\delta 1} = Q_{\delta 0} = i^{\mu}_{(00), (\delta 1)}(M)$. We set $E_5^{\mu}_{\delta 1}$ to be a $D_{\delta 1}$-nice witness $G$ for $R_{\delta 1}$, set $s_{\delta 1} = \rho_\omega(R_{\delta 1})$, set pred$^{\mu}(\delta, 2), F, f$ like usual, set $D_{\delta 2} = f(F)$, and set $\xi_{\delta 2}$ to be the $\xi$ such that $R_{\delta 1} = C_\omega(S^\delta_\xi)$. So in either case, $\xi_{\delta 2}$ is $D_{\delta 2}$-standard. After this we proceed as before.

CASE 6. $u_\delta > 0$ and $\delta \in \mathcal{N}^T$ and $E_\delta \notin E_+(M_\delta)$.

By symmetry.

This completes the proof of the claim and the theorem. 

\begin{enumerate}
\item \textbf{5.20 Remark.} Suppose $V$ is a premouse, iterable in a larger universe. Let $L[E]$ be the output of the pm-ultra stack construction. Then we have the usual partial converse to the fact that $L[E]$ inherits Woodins. That is, let $\delta$ be Woodin in $L[E]$. Then $V | \delta$ is generic for the extender algebra of $L[E]$ at $\delta$, and $\delta$ is Woodin in $L[E][V | \delta]$.

It is also easy to see that stationarity of $L[E]$-constructions (see, for example, \cite{9}) goes through for the ultra-stack construction, assuming that sufficient extenders cohere the relevant iteration strategies.

\section{Questions}

Given that condensation follows from normal iterability, it is natural to ask the following questions:

- Let $m < \omega$ and let $M$ be an $m$-sound, $(m, \omega + 1)$-iterable premouse. Is $M$ $(m + 1)$-universal? Is $M$ $(m + 1)$-solid?

- Let $M$ be an active, 1-sound, $(0, \omega + 1)$-iterable premouse. Is $M$ Dodd-solid?
We conjecture that the answer in each case is “yes”, at least if $M$ has no superstrong initial segments. However, it appears less clear how to prove these things than it is condensation; if one attempts an approach similar to the proof of condensation (from normal iterability) then, at least naively, structures arise similar to bicephali $B$, but the premise involved may fail to be $\rho(B)$-sound. Such generalizations of cephalanxes also arise. This lack of soundness makes the analysis of these structures less clear than those considered in this paper.

One also uses $(0,\omega_1,\omega_1+1)$-iterability of pseudo-premice to prove that they satisfy the ISC. It seems that one might get around this by avoiding pseudo-premice entirely (in the proof of 5.16), using bicephali and cephalanxes instead. Extra difficulties also seem to arise here with superstrong premice.

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