AN OPTIMALLY CONVERGENT COUPLING APPROACH FOR INTERFACE PROBLEMS APPROXIMATED WITH HIGHER–ORDER FINITE ELEMENTS

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Abstract. In this paper, we present a new numerical method for determining an optimally convergent numerical solution of interface problems on polytopial meshes. "Extended" interface conditions are enforced in the sense of a Dirichlet–Neumann coupling by means of a pullback onto the discrete interfaces. This coupling approach serves to bypass geometric variational crimes incurred by the classical finite element method. Further, the primary strength of this approach is that it does not require that the discrete interfaces are geometrically matching to obtain optimal convergence rates. Our analysis indicates that this approach is well–posed and optimally convergent in $H^1$. Numerical experiments indicate that optimal broken $H^1$ and $L^2$ convergence is achieved.

1. Introduction

Higher order finite element methods are attractive since they bring the prospect of faster converging numerical solutions for a lower computational cost. However, in many practical situations, higher order elements (i.e. elements with polynomial order of 2 or greater) are not useful since the geometric approximation error of the polytopial mesh tends to dominate the best approximation error of the inherent polynomial approximation [16, Chapter 4]. As such, practical finite element computations are often performed using only piecewise linear or stabilized first order elements. Interface problems pose an additional difficulty since separate mesh approximation of the constituent subdomains may lead to geometrically nonmatching approximations to the interface. This commonly occurs when complicated domains must be meshed and also in cases where two different numerical codes must be merged together to compute the behavior of a coupled system, as it is often done for fluid–structure interaction problems.

The most commonly utilized approach to overcome the issue of geometric noncoincidence is to incorporate transfer operators to transfer values from one polytopial interface approximation to another [10]. These operators are used in instances of the Dirichlet–Neumann coupling method and mortar element methods [11] [12] for bridging together disjoint subdomain solutions. While these methods are simple and efficient, they suffer in the fact that the accuracy of their numerical solutions tend to be capped at second order in $L^2$ due to the geometric errors described in the previous paragraph.

As in the case of simple boundary value problems, curvilinear maps can be used to better fit the discrete interface approximation to the interface given by the continuous problem. In [4] the isoparametric finite element method was generalized to the interface problem setting. Additionally, in [2], the isogeometric analysis
was applied to arterial blood flow. While these methods can provide higher-order numerical solutions, they can be restrictive in terms of computational cost since, in both cases, higher order quadrature rules must be utilized since the basis functions are no longer simple polynomials. In addition to the additional computational expense, methods based on curvilinear mappings can be laborious to implement.

A notable method presented in [14] utilizes a similar idea to what is presented in this paper. In the approach presented there, optimal convergence rates are achieved by applying the high order finite element method presented in [9] to the interface problem setting, where the main idea of the approach is to utilize line integrals to optimally transfer data from the continuous interface onto its discrete approximations. While this approach allows for higher order numerical approximations, its biggest challenge lies in the fact that this approach is for mixed formulations of elliptic problems. Another similar method described in [13] utilizes an optimization based approach to couple the extensions numerical solutions together onto a common refinement mesh generated from the vertices of the nonmatching interface approximations. In this approach, second order accuracy has been observed with linear elements.

The purpose of this paper is to present a new numerical method for computing higher order numerical solutions for interface problems for cases when the polytopial interface approximations are not necessarily geometrically matching. This is done by extending the polynomial extension finite element method [5] to this setting by enforcing that the extension of the numerical solution and its extended co-normal derivatives are approximately weakly continuous on the interface given by the continuous problem. Since the continuous interface does not coincide with the geometry of the discrete problem, this matching condition is enforced by means of a pullback, via auxiliary variables, onto the discrete interface approximations. Because this method is based on affine-equivalent finite element approaches, it’s implementation is relatively simple and its computational expense is comparable to that of classical finite element methods. We are able to demonstrate stability and optimal broken $H^1$ convergence theoretically, and optimal broken $L^2$ and $H^1$ convergence through a numerical example.

The structure of the paper is as follows: In §2, we discuss the preliminary material required for this work. In §3, we describe the elliptic interface problem and our numerical method. In §4 we state and prove our well-posedness theorem and error estimates. In §5 we provide a numerical illustration to vindicate the results of our analysis. And finally, in §6 we provide concluding remarks.

2. Preliminaries

In this section, we will discuss the preliminary notions required for this paper.

2.1. Geometric Notions. Let $k = 2, 3, \ldots$ and $\Omega_i \subset \mathbb{R}^d$, where $d = 2, 3$ and $i = 1, 2$, denote bounded open subdomains having a $C^{k+1}$-smooth boundary $\Gamma_i$ with an associated unit outer normal vector field $n_i$. We will assume that $\Gamma_1$ and $\Gamma_2$ intersect such that $\Omega_1 \cap \Omega_2 = \emptyset$ and $|\Gamma_c| > 0$, where $\Gamma_c := \Gamma_1 \cap \Gamma_2$ is the interface between the two subdomains $\Omega_1$ and $\Omega_2$. We will then denote $\Gamma_i^0 := \Gamma_i \setminus \Gamma_c$.

We now define our discrete geometry. We will define $\Omega_{h,i}$ as the discrete polytopial approximation of $\Omega_i$ that arises from meshing. For simplicity, we shall also let $\Omega_{h,i}$ be the mesh triangulation, where $K_{h,i}$ denotes an arbitrary simplicial element that belongs to $\Omega_{h,i}$ with meshsize $h_i := \max_{K_{h,i} \subset \Omega_i} \text{diam} (K_{h,i})$. Additionally, we
will denote $h = \max\{h_1, h_2\}$. Further, we will denote $\Gamma_{h,i} := \partial \Omega_{h,i}$ and $\Gamma_{c,i}^0$ as the subsets of $\Gamma_{h,i}$ that approximate $\Gamma_{h,i}^c$ and $\Gamma_{c,i}^0$, respectively. Further, we denote $n_{h,i}$ as the outer unit normal vector field associated with $\Gamma_{h,i}$.

We remark that in the setting we consider in this paper, $\Gamma_{h,1}^c$ does not necessarily coincide with $\Gamma_{c,2}^c$. We assume however that all vertices of $\Gamma_{h,i}$ belong also to $\Gamma^c$. We shall denote $\mathcal{E}_{h,i}^j$ as an edge of $\Gamma_{h,i}$, where $j$ is an index variable. Associated with $\mathcal{E}_{h,i}^j$ is the element $K_{h,i}^j$, where $\mathcal{E}_{h,i}^j$ belongs on the edge of $K_{h,i}^j$.

Figure 1. Left: Example of a continuous domain configuration. Right: Example of a discrete domain approximation.

Because we have assumed that $\Gamma_i$ is $C^{k+1}$–smooth, we have by the implicit function theorem that there exists, for every $\mathcal{E}_{h,i}^j \subset \Gamma_{h,i}$ a $C^{k+1}$ continuous mapping $\eta_i^j : \mathcal{E}_{h,i}^j \rightarrow \Gamma_i$. We can define these mappings such that $\eta_i^j : \mathcal{E}_{h,i}^j \rightarrow \Gamma_{h,i}$ such that $\bigcup_j \eta_i^j(\xi_i^j) = \Gamma^c$ and $\bigcap_j \eta_i^j(\xi_i^j) = \emptyset$, where $\xi_i^j$ is an arbitrary point of $\mathcal{E}_{h,i}^j$. Of course, since $\Gamma_i$ is sufficiently smooth, there exists a unique inverse for each $\eta_i^j$ for we will denote as $\xi_i^j$. For the sake of notational convenience, we will denote $\eta_i : \Gamma_{h,i} \rightarrow \Gamma_i$ as the formal sum of all mappings $\eta_i^j$, and likewise $\xi_i : \Gamma^c \rightarrow \Gamma_{h,i}$ as the formal sum of all $\xi_i^j$. Additionally, we will denote $\xi_i$ as a point in $\Gamma_{h,i}$ and $\eta_i$ as a point in $\Gamma_i$.

Further, we will define $\xi_i^1$ and $\xi_i^2$ as points in $\Gamma_{h,i}$ and $\Gamma_{c,i}^0$, respectively and $\eta_i^0$ as a point of $\Gamma^c$. Following this convention, we will then denote $\xi_i^j$ as the formal sum of $\eta_i^j$ that maps $\Gamma_{h,i}^j$ to $\Gamma^c$ and $\xi_i^j$ as its inverse. Finally we will denote $\eta_i^0$ as the formal sum of $\eta_i^j$ that maps $\Gamma_{h,i}$ to $\Gamma^c$ and $\xi_i^0$ as its inverse.

We may now represent $\xi_i$ in terms of $\xi_i^1$ and vice versa by virtue of the invertibility of $\eta_i^j$. This idea is illustrated in the following schematic:

$$\begin{align*}
\xi_{i,(2)}^1 : \xi_i^1 & \xrightarrow{\eta_i^1} \eta_i^c \xrightarrow{\xi_i^c} \xi_i^2 \\
\xi_{i,(1)}^2 : \xi_i^2 & \xrightarrow{\eta_i^2} \eta_i^c \xrightarrow{\xi_i^c} \xi_i^1,
\end{align*}$$

where we have denoted $\xi_{i,(2)}^1 := \xi_i^1(\xi_i^c)$ and $\xi_{i,(1)}^2 := \xi_i^2(\xi_i^c)$. For simplicity, we will denote the pullback of a variable onto $\Gamma_{h,i}$ as $\mu^{(i)}$, e.g., if $\mu := \mu(\xi_i)$ then $\mu^{(1)} := \mu(\xi_i^1)$ and $\mu^{(2)} := \mu(\xi_i^2(\xi_i)).$

As a final remark for this subsection, we have that since $\Gamma_{h,i}$ can be seen as a piecewise linear interpolant of $\Gamma_i$, we have that

$$|\eta_i(\xi_i) - \xi_i| < \delta_{h,i} = \mathcal{O}(h_i^2),$$

where we have denoted $\delta_{h,i}$ as the formal sum of $\delta_{h,i}$ and vice versa by virtue of the invertibility of $\eta_i^j$. This idea is illustrated in the following schematic:
Duality pairings over $\Omega_{h,i}$ and in addition, from a simple computation, we the following

**Proposition 1.** Let $\mathcal{J}_{1,2}^\alpha$ denote the Jacobian of the transformation $\Gamma_{h,1}^c \to \Gamma_{h,2}^c$, then the following bound is satisfied

$$\norm{\mathcal{J}_{1,2}^\alpha - 1}_{C^{0}(\Gamma_{h,2}^c)} \leq C (h_1 + h_2) \norm{\eta_2}_{C^2(\Gamma_{h,2}^c)}.$$ 

### 2.2. Function Spaces and Discrete Lifting Operators.

Let $\alpha = (\alpha_i)_{i=1}^d$, $\alpha_i \geq 0$ denote a multi-index, $|\alpha| = \sum_{i=1}^d \alpha_i$, and $\alpha! = \prod_{i=1}^d \alpha_i!$. For $D = \Omega_i$ or $\Omega_{h,i}$ and for $m \in \mathbb{N}$, let $H^m(D)$ denote the standard Sobolev space and $(H^m(D))'$ the corresponding dual space; see [1]. For the purpose of studying the interface problems presented in this paper, we also need to consider the subspaces

$$H^1_0(\Omega_i) := \{ v \in H^1(\Omega_i) : v|_{\Gamma_i} = 0 \}$$

and

$$H^1_{\Gamma_0}(\Omega_i) := \{ v \in H^1(\Omega_i) : v|_{\Gamma_0} = 0 \}.$$

Also, for any $\xi \in \mathbb{R}^d$, let $\xi^\alpha := \xi_{1}^{\alpha_1} \xi_{2}^{\alpha_2} \cdots \xi_{d}^{\alpha_d}$ and $D^\alpha := \partial^{\alpha_1} / \partial x_1 \partial^{\alpha_2} / \partial x_2 \cdots \partial^{\alpha_d} / \partial x_d$.

For $D = \Gamma^c$ or $\Gamma_{h,i}^c$, we consider the fractional Sobolev space $H^{m-\frac{1}{2}}(\mathcal{D})$. The $k$-th order Lagrange finite element space is defined by

$$V_{h,i}^k := \{ v \in C^0(\overline{\Omega}_{h,i}) : v|_{\mathcal{K}_{h,i}} \in P_k(\mathcal{K}_{h,i}) \ \forall \mathcal{K}_{h,i} \in \Omega_{h,i} \},$$

where $P_k(\mathcal{K}_{h,i})$ denotes the space of polynomials of order at most $k$ defined over a $d$-simplex $\mathcal{K}_{h,i} \in \mathbb{R}^d$, and the trace spaces

$$W_{h,i}^{c,k} := V_{h,i}^k|_{\Gamma_{h,i}^c} = \{ v \in C^0(\Gamma_{h,i}^c) : v|_{\mathcal{E}_{h,i}^j} \in P_k(\mathcal{E}_{h,i}^j) \ \forall \mathcal{E}_{h,i}^j \in \Gamma_{h,i}^c \}$$

and

$$W_{h,i}^{0,k} := V_{h,i}^k|_{\Gamma_{h,i}^0} = \{ v \in C^0(\Gamma_{h,i}^0) : v|_{\mathcal{E}_{h,i}^j} \in P_k(\mathcal{E}_{h,i}^j) \ \forall \mathcal{E}_{h,i}^j \in \Gamma_{h,i}^0 \}.$$

We also define the discontinuous finite element space

$$\mathcal{V}_{h,i}^k := \{ v \in L^2(\Omega_{h,i}) : v|_{\mathcal{K}_{h,i}} \in P_k(\mathcal{K}_{h,i}) \ \forall \mathcal{K}_{h,i} \in \Omega_{h,i} \}$$

and the discrete differential operator $D_h^\alpha : \mathcal{V}_{h,i}^k \to L^2(\Omega_{h,i})$ as follows:

$$D_h^\alpha v_h(\xi) := \begin{cases} D_v v_h(\xi) & \text{if } \xi \in \tilde{\mathcal{K}}_{h,i} \\ 0 & \text{otherwise} \end{cases}.$$ 

Duality pairings over $\Omega_{h,i}$, $\Gamma_{h,i}^c$, and $\Gamma_{h,i}^0$ are defined by

$$\langle v, w \rangle_{\Omega_{h,i}} := \sum_{\mathcal{K}_{h,i} \in \Omega_{h,i}} \int_{\mathcal{K}_{h,i}} v w d\mathcal{K}_{h,i}$$

and

$$\langle v, w \rangle_{\Gamma_{h,i}^c} := \sum_j \int_{\mathcal{E}_{h,i}^j} v w d\mathcal{E}_{h,i}^j,$$

and

$$\langle v, w \rangle_{\Gamma_{h,i}^0} := \sum_j \int_{\mathcal{E}_{h,i}^j} v w d\mathcal{E}_{h,i}^j,$$
respectively. “Broken” Sobolev norms on \( \Omega_{h,i} \), \( \Gamma_{h,i}^- \), \( \Gamma_{h,i}^0 \) are defined by
\[
\| v \|^2_{m, \Omega_{h,i}} = \sum_{k_h,i \in \Omega_{h,i}} \| v \|^2_{m, k_h,i} \quad \forall v \in V^k_{h,i}.
\]
\[
\| w \|^2_{m, \Gamma_{h,i}^-} = \sum_{\ell_h,i \in \Gamma_{h,i}^-} \| w \|^2_{m, \ell_h,i} \quad \forall w \in W^{c,k}_{h,i} \quad \text{and}
\]
\[
\| w \|^2_{m, \Gamma_{h,i}^0} = \sum_{\ell_h,i \in \Gamma_{h,i}^0} \| w \|^2_{m, \ell_h,i} \quad \forall w \in W^{0,k}_{h,i},
\]
respectively. On the discrete spaces \( V^k_{h,i} \), \( W^{c,k}_{h,i} \), and \( W^{0,k}_{h,i} \) we have the inverse inequalities involving the corresponding “broken” semi-norms given by
\[
\| v \|_{m, \Omega_{h,i}} \leq C h^{-1} \| v \|_{m-1, \Omega_{h,i}} \quad \forall v \in V^k_{h,i}, \quad m = 1, 2, \ldots
\]
\[
\| w \|_{m+1/2, \Gamma_{h,i}^-} \leq C h^{-2} \| w \|_{m, \Gamma_{h,i}^-} \quad \forall w \in W^{c,k}_{h,i}, \quad m = 0, 1, \ldots
\]
and
\[
\| w \|_{m+1/2, \Gamma_{h,i}^0} \leq C h^{-2} \| w \|_{m, \Gamma_{h,i}^0} \quad \forall w \in W^{0,k}_{h,i}, \quad m = 0, 1, \ldots
\]
For simplicity of notation, we will define the product spaces
\[
H := H^1(\Omega_{h,1}) \times H^1(\Omega_{h,2}) \times H^{1/2}(\Gamma_{h,1}) \times H^{-1/2}(\Gamma_{h,2})
\]
\[
V_h^k := V^k_{h,1} \times V^k_{h,2} \times W^{c,k}_{h,1} \times W^{c,k}_{h,2},
\]
\[
W^c_h := W^{c,k}_{h,1} \times W^{c,k}_{h,2}
\]
\[
H^{-1} := H^{-1}(\Omega_{h,1}) \times H^{-1}(\Omega_{h,2}),
\]
and their norm taken to be the \( \ell^2 \) norm of their sub-norms.

We will denote the \( L^2 \left( \Gamma_{h,i}^- \right) \) projection operator onto \( W^{c,k}_{h,i} \) as \( \pi_{i}^{c} (\cdot) : L^2 \left( \Gamma_{h,i}^- \right) \rightarrow W^{c,k}_{h,i} \), defined by
\[
\int_{\Gamma_{h,i}^-} \mu \pi_{i}^{c} \, w \, dD = \int_{\Gamma_{h,i}^-} \mu \, w \, d\Gamma_{h,i}^{c} \quad \forall \mu \in W^{c,k}_{h,i}.
\]
Lifting operators will be often used in this work. We will denote \( R_{h,i}^{c} : W^{c,k}_{h,i} \rightarrow V_{h,i}^k \cap H^1_{\Gamma_{h,i}^-} (\Omega_{h,i}) \) and \( R_{h,i}^{0} : W^{0,k}_{h,i} \rightarrow V_{h,i}^k \cap H^1_{\Gamma_{h,i}^-} (\Omega_{h,i}) \) as discrete bounded lifting operators. A simple inspection indicates that
\[
\text{Ker} \left( R_{h,i}^{c} \right) \subset \text{Im} \left( R_{h,i}^{0} \right) \quad \text{and} \quad \text{Ker} \left( R_{h,i}^{0} \right) \subset \text{Im} \left( R_{h,i}^{c} \right)
\]
For simplicity, we will denote \( R_{h,i} := R_{h,i}^{c} + R_{h,i}^{0} \).

We conclude this subsection by establishing the following proposition, which is a simple consequence of the piecewise \( C^k+1 \)-diffeomorphic equivalence property between \( \Gamma_{h,1}, \Gamma_{h,1}, \) and \( \Gamma_{h,2} \).

**Proposition 2.** There exists positive constants \( c_1, c_2, C_1, C_2 \) such that for \( m \in \mathbb{R} \) the following norm equivalence relations are satisfied
\[
c_1 \| \mu_1 \|_{m, \Gamma_{h,1}^{c}} \leq \| \mu_1^{(2)} \|_{m, \Gamma_{h,1}^{c}} \leq C_1 \| \mu_1 \|_{m, \Gamma_{h,1}^{c}} \quad \forall \mu_1 \in H^m (\Gamma_{h,1}^{c})
\]
\[
c_2 \| \mu_2 \|_{m, \Gamma_{h,2}^{c}} \leq \| \mu_2^{(1)} \|_{m, \Gamma_{h,2}^{c}} \leq C_2 \| \mu_2 \|_{m, \Gamma_{h,2}^{c}} \quad \forall \mu_2 \in H^m (\Gamma_{h,2}^{c}).
\]
2.3. An Averaged Taylor Series Extension. Recall that, for every $E_{h,i}^j \subset \Gamma_{h,i}$, $K_{h,i}^j$ is the element of $\Omega_{h,i}$ that contains $E_{h,i}^j$. We will then let $\{S_{h,i}^{j,j'}\}$ be a family of disjoint star–shaped domains with respect to the balls $\sigma_{h,i}^{j,j'} \subset K_{h,i}^j$ such that $S_{h,i}^{j,j'} \cap K_{h,i}^j = \emptyset$ if $j \neq l$, $\text{diam} \left(S_{h,i}^{j,j'}\right) = \mathcal{O}(h_i)$ and $\bigcup_{j,j'} S_{h,i}^{j,j'} \supset \Omega_i \cup \Delta \Omega_i$, where $\Omega_i \cup \Delta \Omega_i := (\Omega_i \cup \Omega_h) \setminus (\Omega_i \cap \Omega_h)$ denotes the symmetric difference of $\Omega_i$ and $\Omega_h$. We also require that $S_{h,i}^{j,j'} \cap \eta_i \left(E_{h,i}^j\right) = S_{h,i}^{j,j'} \cap \eta_i \left(E_{h,i}^l\right)$ and $S_{h,i}^{j,j'} \cap E_{h,i}^j = S_{h,i}^{j,j'} \cap E_{h,i}^l$. We refer to Figure 2 as an example of how star-shaped domains $S_{h,i}^{j,j'}$ can be built for a triangular mesh. Following [6] we define, for $x \in \mathbb{R}^d$ and $v \in L^2(\Omega_i \cup \Delta \Omega_i)$, the averaged Taylor polynomial

$$T^{k}_{h,i}(v)\big|_x := \sum_{j,j'} 1_{S_{h,i}^{j,j'}}(x) \int_{\sigma_{h,i}^{j,j'}} \left( \sum_{\alpha \in \mathbb{N}^d} \frac{1}{\alpha!} D^\alpha v(y)(x-y)^\alpha \phi_j(y) \right) dy$$

where $\phi_j(y)$ is a cutoff function with support over $\sigma_{h,i}^{j,j'}$ and

$$1_{S_{h,i}^{j,j'}}(x) := \begin{cases} 1 & \text{if } x \in S_{h,i}^{j,j'} \\ 0 & \text{otherwise} \end{cases}$$

is the indicator function for the set $S_{h,i}^{j,j'}$. Note that $T^{k}_{h,i}$ is meaningful only at $x \in \bigcup_{j,j'} S_{h,i}^{j,j'}$ and is zero otherwise. For any $\xi \in \Gamma_{h,i}$ and its image $\eta_i(\xi_i) \in \Gamma_i$ and $v \in L^2(\Omega_i)$ we write

$$v \circ \eta_i(\xi_i) = T^{k}_{h,i}(v)\big|_{\eta_i(\xi_i)} + R^{k}_{h,i}(v)\big|_{\eta_i(\xi_i)}.$$ 

For $v \in H^{k+1}(\mathbb{R}^d)$ we have $\|R^{k}_{h,i}(v)\|_{L^2(\Omega_i)} \leq C \delta_{h,i}^{k+1} \|v\|_{k+1,\mathbb{R}^d}$ [5 Appendix A, Lemma 2] If $v \in \nabla^{k}_{h,i}$, then in every $K_{h,i}^j$, $v$ is a polynomial of degree $k$, and therefore $T^{k}_{h,i}(v)$ reproduces exactly $v$ in any $K_{h,i}^j$ adjacent to the boundary and

\[1\] The averaged Taylor polynomials on star-shaped domains $S_{h,i}^{j,j'}$, are defined for functions in $L^1(\sigma_{h,i}^{j,j'})$; see [6 Corollary 4.1.15.],
it is equivalent to the classical Taylor polynomial. For \( v \in V^k_{h,i} \) we can therefore write, for a generic \( y \in K^j_{h,i} \):

\[
T^k_{h,i}(v)\big|_x = \sum_{j,j'} s^k_{j,j'}(x) \sum_{|\alpha|=0}^k \frac{1}{\alpha!} D^\alpha v(y)(x - y)\alpha
\]

\[
= \sum_j 1_{(j,j')}(x) \sum_{|\alpha|=0}^k \frac{1}{\alpha!} D^\alpha v(y)(x - y)\alpha
\]

We take now \( y = \xi \in \hat{E}_i \) and \( x = \eta_i(\xi_i) \), and we have that

\[
T^k_{h,i}(v)\big|_{\eta_i(\xi_i)} = \sum_{|\alpha|=0}^k \frac{1}{\alpha!} D^\alpha h v(\xi_i)(\eta_i(\xi_i) - \xi_i)\alpha
\]

which is well-defined for any \( \xi_i \in \Gamma_{h,i} \) and \( v \in P^k_{h,i} \). For convenience, we also define \( T^{k,k'}_{h,i} \) as

\[
T^{k,k'}_{h,i}(v)\big|_{\eta_i(\xi_i)} = \sum_{|\alpha|=k'} \frac{1}{\alpha!} D^\alpha h v(\xi_i)(\eta_i(\xi_i) - \xi_i)\alpha.
\]

Clearly \( T^k_{h,i} = T^{0,k}_{h,i} \). For vector functions \( v \), we introduce the vector operator

\[
T^k_{h,i}(v) = \left(T^k_{h,i} v_i\right)_{i=1}^d.
\]

We use this notation in particular for gradients of scalar functions (i.e., \( T^k_{h,i}(\nabla v) \)).

3. Problem Setting

In this section we will describe the elliptic interface problem we wish to approximate the solution to and motivate the discrete coupling formulation we will use to accomplish this task.

3.1. The Elliptic Interface Problem. The continuous problem we consider in this paper is the elliptic interface problem with discontinuous coefficients given in the following

\[
\begin{aligned}
-\nabla \cdot (p_i(x_i) \nabla u_i(x_i)) &= f_i & \text{in } \Omega_i \\
u_i &= 0 & \text{on } \Gamma^0_i
\end{aligned}
\]

\[
\begin{aligned}
p_1(x_1) \frac{\partial u_1}{\partial n_1} + p_2(x_2) \frac{\partial u_2}{\partial n_2} &= 0 & \text{on } \Gamma^c,
\end{aligned}
\]

\[
\begin{cases}
  u_1 = u_2 \\
p_1(x_1) \frac{\partial u_1}{\partial n_1} + p_2(x_2) \frac{\partial u_2}{\partial n_2} = 0
\end{cases}
\]

where \( x_i \in \Omega_i \). For the purpose of having \((u_1, u_2)\) belong to \( H^{k+1}(\Omega_1) \times H^{k+1}(\Omega_2) \), as needed for determining optimal convergence rates for our method, we make the assumption that \( p_i(x_i) \in C^{k+1}(\Omega_i) \) and \( f_i(\xi_i) \in H^{k-1}(\Omega_i) \). As such, we have by the Sobolev extension theorem [11 Chapter 5] that there exists an extension for each subdomain solution \( \tilde{u}_i \in H^{k+1}(\mathbb{R}^d) \) such that \( \tilde{u}_i|_{\Omega_i} = u_i \). Further, there exists extensions of \( p_i \) and \( f_i \) denoted as \( \tilde{p}_i \) and \( \tilde{f}_i \) respectively such that \( \tilde{p}_i \in C^{k+1}(\mathbb{R}^d) \).
and \( \tilde{f}_i \in H^{k-1}(\mathbb{R}^d) \) and also \( \tilde{p}_i|_{\Omega_i} = p_i \) and \( \tilde{f}_i|_{\Omega_i} = f_i \). Further, these extensions are bounded in their respective norms, i.e.,
\[
\|\tilde{u}_i\|_{k+1,\mathbb{R}^d} \leq C_c\|u_i\|_{k+1,\Omega_i}
\]
(11)
\[
\|\tilde{p}_i\|_{C^{k+1}(\mathbb{R}^d)} \leq C_c\|p_i\|_{C^{k+1}(\mathbb{R}^d)}
\]
\[
\|\tilde{f}_i\|_{k-1,\mathbb{R}^d} \leq C_c\|f_i\|_{k-1,\Omega_i}.
\]

3.2. PE–FEM Subproblems Over \( \Omega_{h,i} \). The Polynomial Extension Finite Element Method (PE-FEM) allows one to leverage the averaged Taylor series extensions described in Section 2.3 to obtain optimal convergence, with respect to interpolation, while retaining the generated polytopial mesh. Much of the details and explanations will be omitted for the sake of brevity. We refer the reader to [5] for more details behind the logic and intuition behind this approach. We will define two PE–FEM important problems that we will frequently refer to and state their stability properties here.

Let us define \( a_{h,i}(\cdot, \cdot) : H^1(\Omega_{h,i}) \times H^1(\Omega_{h,i}) \to \mathbb{R} \) as
\[
a_{h,i}(w, v) := \int_{\Omega_{h,i}} (\tilde{p}_i(x_i) \nabla w \cdot \nabla v) \, dx \quad \forall w, v \in H^1(\Omega_{h,i}).
\]

Next, let us define \( B_{h,D}(\cdot, \cdot) : V^k_{h,i} \times V^k_{h,i} \) as
\[
B_{h,D}(w, v) := a_{h,i}(w, v - R_{h,i}v) + \theta_{h,i} \left( T^k_{h,i}w \big|_{\eta_i(\xi_i)} , v \right)_{\Gamma_{h,i}} \quad \forall w, v \in V^k_{h,i},
\]
(12)
\[
B_{h,N}(w, v) := a_{h,i}(w, v - R^0_{h,i}v) + \theta_{h,i} \left( T^k_{h,i}w \big|_{\eta_i^0(\xi_i)} , v \right)_{\Gamma^0_{h,i}} + \tau_i(w, v) \quad \forall w, v \in V^k_{h,i},
\]
(13)
where \( \theta_{h,i} \in \mathbb{R}^+ \) is a positive constant such that \( \theta_{h,i} \sim O(h_i^{-1}) \) and \( B_{h,N}(\cdot, \cdot) : V^k_{h,i} \times V^k_{h,i} \to \mathbb{R} \)

We may now define the PE–FEM problems of interest for this paper. The importance of these problems will become apparent in the discussion of the coupling equations presented in 3.3. First, consider the following: seek \( u_{h,i} \in V^k_{h,i} \) such that
\[
B_{h,D}(u_{h,i}, v) = \left( \tilde{f}_i, v - R_{h,i}v \right)_{\Omega_{h,i}} + \theta_{h,i} \left( g_D \circ \eta_i(\xi_i), v \right)_{\Gamma_{h,i}},
\]
(15)
\[
B_{h,N}(u_{h,i}, v) = \left( \tilde{f}_i, v - R^0_{h,i}v \right)_{\Omega_{h,i}} + \left( g_N \circ \eta_i^0(\xi_i), v \right)_{\Gamma^0_{h,i}},
\]
(16)
where $g_N \in H^{-1/2}(\Gamma^c)$. The discrete problems (15) and (16) are meant to provide an optimal approximation to the solution of the following boundary value problems

$$\begin{cases}
-\nabla \cdot (p_i \nabla u_i) = f_i & \text{in } \Omega_i \\
\quad u_i = 0 & \text{on } \Gamma_i^0 \\
\quad u_i = g_D & \text{on } \Gamma_i^c
\end{cases} \quad \begin{cases}
-\nabla \cdot (p_i \nabla u_i) = f_i & \text{in } \Omega_i \\
\quad u_i = 0 & \text{on } \Gamma_i^0 \\
\quad \frac{\partial u_i}{\partial n_i} = g_N & \text{on } \Gamma_i^c
\end{cases}$$

respectively. Following the spirit of the analysis presented in [5, Appendix D] the following well-posedness results can be easily determined.

**Theorem 1** (Well-Posedness of (15)). Let $B^i_{h,D}(\cdot, \cdot)$ be defined as in (12) with $\tilde{p}_i(x_i) > 0$ everywhere in $\Omega_{h,i}$. Assume that $\theta_{h,i} \leq C_0 h_i$ with $C_0$ chosen large enough and also assume that $\delta_{h,i} \sim O(h_i^{2/3})$. Then for $h_i$ small enough and $k = 1, 2, \ldots$, we have that

$$B^i_{h,D}(u, v) \leq M_{D,i} (1 + \theta_{h,i}) \|u\|_{1, \Omega_{h,i}} \|v\|_{1, \Omega_{h,i}} \quad \forall u, v \in V^k_{h,i}$$

and

$$B^i_{h,D}(u, u) \geq \gamma_{D,i} \|u\|^2_{1, \Omega_{h,i}} \quad \forall u \in V^k_{h,i}.$$ 

If $g_D \circ \eta^i_i(\xi^c_i) \in H^{1/2}(\Gamma^c_{h,i})$, then (15) has a unique solution and the solution satisfies the following stability bounds

$$\|u_{h,i}\|_{1, \Omega_{h,i}} \leq \Lambda_{D,i} \left\{ \|\tilde{f}_i\|_{-1, \Omega_{h,i}} + \|g_D \circ \eta^i_i(\xi^c_i)\|_{1/2, \Gamma^c_{h,i}} \right\}$$

$$\|u_{h,i}\|_{1, \Omega_{h,i}} \leq \Lambda'_{D,i} \left\{ \|\tilde{f}_i\|_{-1, \Omega_{h,i}} + h_i^{-2} \|g_D \circ \eta^i_i(\xi^c_i)\|_{0, \Gamma^c_{h,i}} \right\}.$$ 

**Theorem 2** (Well-Posedness of (16)). Let $B^i_{h,N}(\cdot, \cdot)$ be defined as in (13) with $\tilde{p}_i(x_i) > 0$ everywhere in $\Omega_{h,i}$. Then for $h_i$ small enough and $k = 1, 2, \ldots$, we have that

$$B^i_{h,N}(u, v, \mu_i) \leq M_{N,i} \|u_{h,i}\|_{1, \Omega_{h,i}} \|v\|_{1, \Omega_{h,i}} \quad \forall v \in V^k_{h,i},$$

if $u_{h,i} \in V^k_{h,i}$ satisfies (16),

$$B^i_{h,N}(u_{h,i}, \mathcal{R}^c_{h,i} \mu_i) \leq M'_{N,i} \|u_{h,i}\|_{1, \Omega_{h,i}} \|\mu_i\|_{1/2, \Gamma^c_{h,i}} \quad \forall \mu_i \in W_{r,h,i}^c,$$

and

$$B^i_{h,N}(u, u) \geq \gamma_{N,i} \|u\|^2_{1, \Omega_{h,i}} \quad \forall u \in V^k_{h,i}.$$ 

If $g_N \circ \eta^i_i(\xi^c_i) \in H^{-1/2}(\Gamma^c_{h,i})$, then (16) has a unique solution and the solution satisfies the following stability bound

$$\|u_{h,i}\|_{1, \Omega_{h,i}} \leq \Lambda_{N,i} \left\{ \|\tilde{f}_i\|_{-1, \Omega_{h,i}} + \|g_N \circ \eta^i_i(\xi^c_i)\|_{-1/2, \Gamma^c_{h,i}} \right\}.$$ 

**Remark 1.** The constant $M_{N,i}$ in the continuity bound (20) is independent of $\theta_{h,i}$ because we have enforced that the polynomial extension from $\Gamma^0_{h,i}$ onto $\Gamma^c_i$ is zero weakly, whereas the constant $M'_{N,i}$ in the continuity bound (21) is independent of $\theta_{h,i}$ because $\mathcal{R}^0_{h,i} \left( \mathcal{R}^c_{h,i} \mu_i \right) = 0$.

**Remark 2.** The stability constants $\Lambda_{D,i}, \Lambda'_{D,i}$ and $\Lambda_{N,i}$ are independent of $\theta_{h,i}$. 

The comprehensive set of variational equations is to seek \( (u_{h,2}, u_{h,2}, \lambda_{h}) \in V_{h}^{k} \) that satisfies

\[
\begin{align*}
B_{h,1}(u_{h,1}, v_{1}) - \theta_{h,1} \left( \lambda_{h}, v_{1} \right)_{\Gamma_{h,1}} &= \left( f_{1}, v_{1} - R_{h,1}v_{1} \right)_{\Omega_{h,1}}, \\
B_{h,2}(u_{h,2}, v_{2}) - \theta_{h,2} \left( \lambda_{h}^{(2)}, v_{2} \right)_{\Gamma_{h,2}} &= \left( f_{2}, v_{2} - R_{h,2}v_{2} \right)_{\Omega_{h,2}}, \\
B_{h,1}(u_{h,1}, R_{h,1}^{c}\mu_{1}) + \left( \rho_{h}^{(1)}, \mu_{1} \right)_{\Gamma_{h,1}} &= \left( f_{1}, R_{h,1}^{c}\mu_{1} \right)_{\Omega_{h,1}}, \\
B_{h,2}(u_{h,2}, R_{h,2}^{c}\mu_{2}) - \langle \rho_{h}, \mu_{2} \rangle_{\Gamma_{h,2}} &= \left( f_{2}, R_{h,2}^{c}\mu_{2} \right)_{\Omega_{h,2}}, \\
\forall (v_{1}, v_{2}, \mu_{1}, \mu_{2}) \in V_{h}^{k},
\end{align*}
\]

From a simple inspection, it is easily determined that

\[
\lambda_{h} = \Pi_{W_{h,1}}^{c,k} \left( T_{h,1}^{k}u_{h,1}, \eta_{1}(\xi^{c}) \right),
\]

and the second equation of (24) implies that

\[
T_{h,2}^{k}u_{h,2} \bigg|_{\eta_{2}^{c}(\xi^{c})} \approx T_{h,1}^{k}u_{h,1} \bigg|_{\eta_{2}^{c}(\xi_{1,2}^{c})}.
\]

and hence, the extension of the subdomain solutions match approximately on the continuous interface \( \Gamma^{c} \). Further inspection implies that

\[
\langle \rho_{h}, \mu_{2} \rangle_{\Gamma_{h,2}} \approx \left( \tilde{p}_{i} \circ \eta_{2}^{c}(\xi^{c}) T_{h,i}^{k-1}\nabla u_{h,i} \bigg|_{\eta_{2}^{c}(\xi^{c})} \cdot n_{i}, \mu_{2} \right)_{\Gamma_{h}^{c}} \quad \forall \mu_{2} \in W_{h,2}^{c,k}.
\]

We refer the reader to [5, Section 3] for a more detailed explanation. From this, it becomes apparent from the third equation of (24) that

\[
\sum_{i=1,2} \tilde{p}_{i} \circ \eta_{i}^{c}(\xi^{c}) T_{h,i}^{k-1}\nabla u_{h,i} \bigg|_{\eta_{i}^{c}(\xi^{c})} \cdot n_{i} \approx 0.
\]

For narrative simplicity, we will refer to \( \tilde{p}_{i} \circ \eta_{i}^{c}(\xi^{c}) T_{h,i}^{k-1}\nabla u_{h,i} \bigg|_{\eta_{i}^{c}(\xi^{c})} \cdot n_{i} \) as the extended co-normal derivative of \( u_{h,i} \). It then becomes clear from (25) and (26) that (24) approximates (10) by enforcing that the polynomial extensions of \( u_{h,i} \) match weakly and that the extended co-normal derivatives are approximately balanced. Of course, the higher order convergence rates obtained by this method is due to the inclusion of the extension operators used to approximate the Dirichlet and Neumann interface conditions given by (10b).

4. Analysis

In this section, we will present the necessary theoretical tools for the analysis of our discrete coupling formulation. We will then state and prove the well-posedness and \( H^{1} \)-optimality results.
4.1. Dirichlet PE–FEM Solution Operators. We introduce three operators for the purpose of the analysis of (24). First, we define $G_{h,i}^i : H^{-1}(\Omega_{h,i}) \to V_{h,i}^k$ as the solution operator for the following problem: Given $\chi_i \in H^{-1}(\Omega_{h,i})$, seek $\phi_{h,i} \in V_{h,i}^k$ such that
\[
B_{h,D}^i(\phi_{h,i}, v_i) = \langle \chi_i, v_i - R_{h,i} v_i \rangle_{\Omega_{h,i}} \quad \forall v_i \in V_{h,i}^k.
\]
We will denote $G_{h,i}^i \chi_i := \phi_{h,i}$. Next, we define $H_{h,i}^c(\cdot) : H^{-1/2}(\Gamma_{h,i}) \to V_{h,i}^k$ as the solution operator of: seek $\psi_{h,i}^c \in V_{h,i}^k$ such that
\[
B_{h,D}^i(\psi_{h,i}^c, v_i) = \theta_{h,i} \langle \nu, v_i \rangle_{\Gamma_{h,i}^0} \quad \forall v_i \in V_{h,i}^k.
\]
We will similarly denote $H_{h,i}^c(\cdot, \cdot) : \psi_{h,i}^c, \nu \in H^{-1/2}(\Gamma_{h,i})$. Finally, we will denote $\hat{H}_{h,i}^c : H^{-1/2}(\Gamma_{h,i}) \to V_{h,i}^k \cap H^{1}_{0,h,i}(\Omega_{h,i})$ as the solution operator of the variational problem: seek $\hat{\psi}_{h,i}^c \in V_{h,i}^k$ such that
\[
B_{h,D}^i(\hat{\psi}_{h,i}^c, v_i) = \theta_{h,i} \langle \nu, v_i \rangle_{\Gamma_{h,i}^0} \quad \forall v_i \in V_{h,i}^k.
\]
Likewise, we denote $\hat{H}_{h,i}^c(\cdot, \cdot) : \hat{\psi}_{h,i}^c, \nu \in H^{-1/2}(\Gamma_{h,i})$. An inspection of (29) indicates that $\hat{\psi}_{h,i}^c = 0$ on $\Gamma_{h,i}^0$ by means of the weak enforcement
\[
\theta_{h,i} \left\langle \hat{\psi}_{h,i}^c, v_i \right\rangle_{\Gamma_{h,i}^0} = 0 \quad \forall v_i \in V_{h,i}^k.
\]
(29) can easily be determined to be well–posed since the perturbation on $B_{D,h}^i(\cdot, \cdot)$ is miniscule. Through a modified set of steps presented in the analysis found in [29 Appendix B], we are able to demonstrate that the following stability bound is satisfied:
\[
\left\| \hat{H}_{h,i}^c(\cdot, \cdot) \right\|_{H^{1/2}(\Gamma_{h,i}) \to H^1(\Omega_{h,i})} \leq \bar{\Lambda}_{D,i} \| \nu \|_{1/2, \Gamma_{h,i}}.
\]
This stability bound then allows us to prove the following.

**Lemma 1.** Let $H_{h,i}^c(\cdot)$ and $\hat{H}_{h,i}^c(\cdot)$ be the solution operators for the problems defined in (28) and (29) respectively, then if $\delta \sim \mathcal{O}(h_i^2)$ we have that
\[
\left\| H_{h,i}^c(\cdot) - \hat{H}_{h,i}^c(\cdot) \right\|_{H^{1/2}(\Gamma_{h,i}) \to H^1(\Omega_{h,i})} \leq C h_i.
\]

**Proof.** We begin by seeing that $\left( H_{h,i}^c(\cdot) - \hat{H}_{h,i}^c(\cdot) \right) : H^{1/2}(\Gamma_{h,i}) \to V_{h,i}^k$ is the solution operator for the following: seek $\psi_{h,i}^c - \hat{\psi}_{h,i}^c \in V_{h,i}^k$ such that
\[
B_{D,h}^i(\psi_{h,i}^c - \hat{\psi}_{h,i}^c, v_i) = \theta_{h,i} \left\langle T_{h}^1, k, \psi_{h,i}^c \mid \eta_i^0(\xi) \right\rangle_{\Gamma_{h,i}^0} \quad \forall v_i \in V_{h,i}^k.
\]
By taking the difference between (28) and (29), Then (196) of Theorem 1 implies that
\[
\left\| \left( H_{h,i}^c(\cdot) - \hat{H}_{h,i}^c(\cdot) \right) \nu \right\|_{1, \Omega_{h,i}} \leq \Delta_{D,i} \| \nu \|_{H^{1/2}(\Gamma_{h,i})} \leq C \Delta_{D,i} \| \nu \|_{1, \Omega_{h,i}}.
\]
after applying [5] Appendix A, Lemma 4] and [30]. The result of this lemma results from the definition of the operator norm.

We now prove that $\mathcal{H}_{h,i}^e(\cdot)$ is invertible. This bound will be essential in the stability analysis.

**Lemma 2.** Let $\mathcal{H}_{h,i}^e(\cdot)$ be defined as in (28), then the following bound is satisfied

$$
\|\mathcal{H}_{h,i}^e\|_{1,\Omega_{h,i}} \geq C \|\mu\|_{1/2,\Gamma_{h,i}^c}, \quad \forall \mu \in W_{h,i}^{c,k},
$$

under the assumption that $\delta_{h,i} \sim O(h_i^2)$.

**Proof.** Let $\psi_{h,i}^e = \mathcal{H}_{h,i}^e \mu$. Then we have that $\mu := \pi_k^e \left( T_{h,i}^k \psi_{h,i}^e \right)$. It then follows that

$$
\|\psi_{h,i}^e\|_{1,\Omega_{h,i}} \geq C \|\psi_{h,i}^e\|_{1/2,\Gamma_{h,i}^c},
$$

and

$$
\|\mu\|_{1/2,\Gamma_{h,i}^c} - h_i^{-\frac{1}{2}} \left\| T_{h,i}^{1,k} \psi_{h,i}^e \right\|_{0,\Omega_{h,i}}^e
$$

$$
\geq C \left( \|\mu\|_{1/2,\Gamma_{h,i}^c} - \sum_{|\alpha| \leq 1} \delta_{h,i}^{\alpha} h_i^{-|\alpha|+\frac{1}{2}} \|\psi_{h,i}^e\|_{1,\Omega_{h,i}} \right)
$$

$$
\geq C \|\mu\|_{1/2,\Gamma_{h,i}^c},
$$

after applying [5] Appendix A, Lemma 4], [19a], and the assumption that $\delta_{h,i} \sim O(h_i^2)$.

### 4.2. Stability Analysis.

We will now analyze the well–posedness of our coupling formulation. First, we begin by seeing that, by a change of variables for integrals, (24) is equivalent to seeking $(u_{h,1}, u_{h,2}, \lambda_h, \rho_h) \in V_h^k$ such that

$$
B_{1,D}^1(u_{h,1}, v_1) - \theta_{h,1} \langle \lambda_h, v_1 \rangle_{\Gamma_{h,1}^c} = \left\langle \bar{f}_1, v_1 - R_{h,1} v_1 \right\rangle_{\Omega_{h,1}},
$$

$$
B_{1,D}^2(u_{h,2}, v_2) - \theta_{h,2} \langle \lambda_h(2), v_2 \rangle_{\Gamma_{h,2}^c} = \left\langle \bar{f}_2, v_2 - R_{h,2} v_2 \right\rangle_{\Omega_{h,2}},
$$

$$
\sum_{i=1,2} B_{h,N}^i (u_{h,i}, R_{h,i}^c \mu_i) + \left\langle \rho_h, \mathcal{J}_{i,(2)}^c \mu_1(2) - \mu_2 \right\rangle_{\Gamma_{h,2}^c} = \sum_{i=1,2} \left\langle \bar{f}_i, R_{h,i}^c \mu_i \right\rangle_{\Omega_{h,i}},
$$

$$
\forall (v_1, v_2, \mu_1, \mu_2) \in V_h^k.
$$

The first two equations in (33) implies that

$$
u_{h,i} = \mathcal{H}_{h,1}^c \lambda_h + \mathcal{G}_{h,i}^c \bar{f}_i$$

and thus the third equation in (33) can be written as

$$
\sum_{i=1,2} B_{h,N}^i \left( \mathcal{H}_{h,i}^c \lambda_h, R_{h,i}^c \mu_i \right) + \left\langle \rho_h, \mathcal{J}_{i,(2)}^c \mu_1(2) - \mu_2 \right\rangle_{\Gamma_{h,2}^c} = \sum_{i=1,2} \left\langle \bar{f}_i, R_{h,i}^c \mu_i \right\rangle_{\Omega_{h,i}},
$$

$$
\forall (\mu_1, \mu_2) \in W_h^k.
$$

This equation will be of paramount importance in the following analysis, as it allows us to determine a bound $\lambda_h$. We now prove the following well–posedness result.
Theorem 3. Assume that $\bar{p}_1 > 0$ everywhere on $\Omega_{h,i}$ and $\tilde{f}_i \in H^{-1}(\Omega_{h,i})$ and $\theta_{h,i} \geq C_0 h_i^{-1}$ with $C_0$ large enough. Then if $\delta_{h,i} \sim O(h_i^2)$ with $h_i$ small enough, the following stability bound is satisfied

$$
\|(w_{h,1}, w_{h,2}, \lambda_h, \rho_h)\|_H \leq C \left\| \left( \tilde{f}_1, \tilde{f}_2 \right) \right\|_{H^{-1}}.
$$

for $k = 1, 2, \ldots, d = 2, 3$.

Remark 3. This stability result implies that the solution to (33) is unique, since the equations are linear.

Proof. First, we begin by recalling that

$$
\langle \rho_h, (\mu_2) \rangle_{\Gamma_{h,2}} = B_{h,N}^2 (u_{h,2}, R_{h}^\mu_{h,2}) - \left( \tilde{f}_2, R_{h}^\mu_{h,2} \right)_{\Gamma_{h,2}}.
$$

By applying (20), the solution decomposition (34), and (19), we then have that

$$
\|\rho_h\|_{-1/2, \Gamma_{h,2}} \leq C \left( \|\lambda_h\|_{1/2, \Gamma_{h,1}} + \|\tilde{f}_2\|_{-1, \Omega_{h,2}} \right).
$$

Next, we proceed by choosing $R_{h,i}^\mu (\cdot) = \tilde{H}_{h,i}^\mu (\cdot)$, $\mu_1 = \lambda_h$, $\mu_2 = \pi_2 \lambda_h^{(2)}$. The third equation in (33) then becomes

$$
\sum_{i=1,2} B_{h,N}^i \left( \mathcal{H}_{h,i}^{c}, \mathcal{H}_{h,i}^{c \lambda} \right) = \sum_{i=1,2} \left( \tilde{f}_i, \tilde{H}_{h,i}^{c \lambda} \right)_{\Omega_{h,i}} - B_{h,N}^i \left( \mathcal{G}_{h,i}^c \tilde{f}_i, \mathcal{H}_{h,i}^{c \lambda} \right)
$$

$$
- \left( \rho_h, \left( \mathcal{J}_{1,2}^c - 1 \right) \lambda_h^{(2)} \right)_{\Gamma_{h,2}},
$$

where we have utilized the projection theorem to see that

$$
\left. \langle \nu, \lambda_h - \pi_2 \lambda_h^{(2)} \rangle_{\Gamma_{h,2}} \right| = 0 \quad \forall \nu \in W_{h,2}^{c, k},
$$

and the identity $\tilde{H}_{h,2}^\mu (\pi_2 \lambda_h^{(2)}) = \tilde{H}_{h,2}^\mu \lambda_h^{(2)}$. Using (20), (37), and Proposition 2 we have that

$$
\left. \langle \rho_h, \left( \mathcal{J}_{1,2}^c - 1 \right) \lambda_h^{(2)} \rangle_{\Gamma_{h,2}} \right| = Ch \|\rho_h\|_{-1/2, \Gamma_{h,2}} \|\lambda_h\|_{1/2, \Gamma_{h,1}} \leq Ch \left( \|\lambda_h\|_{1/2, \Gamma_{h,1}} + \|\tilde{f}_1\|_{-1, \Omega_{h,1}} \|\lambda_h\|_{1/2, \Gamma_{h,1}} \right),
$$

and hence

$$
\sum_{i=1,2} B_{h,N}^i \left( \mathcal{H}_{h,i}^{c \lambda}, \mathcal{H}_{h,i}^{c \lambda} \right) = Ch \|\lambda_h\|_{2, \Gamma_{h,1}} \leq Ch \sum_{i=1,2} \|\tilde{f}_i\|_{-1, \Omega_{h,i}} \|\lambda_h\|_{1/2, \Gamma_{h,1}},
$$

after seeing that $\mathcal{H}_{h,i}^{c \lambda} = 0$ on $\Gamma_0^{h,i}$, and applying (20). We then have that

$$
B_{h,N}^i \left( \mathcal{H}_{h,i}^{c \lambda}, \mathcal{H}_{h,i}^{c \lambda} \right)
$$

$$
= B_{h,N} \left( \mathcal{H}_{h,i}^{c \lambda}, \mathcal{H}_{h,i}^{c \lambda} \right) + B_{h,N} \left( \mathcal{H}_{h,i}^{c \lambda}, \left( \mathcal{H}_{h,i}^{c \lambda} - \mathcal{H}_{h,i}^{c \lambda} \right) \lambda_h^{(2)} \right)
$$

$$
\geq C \left( \gamma_{N,i} - h_i \right) \|\lambda_h\|_{1/2, \Gamma_{h,1}},
$$

by means of (22), Lemma 1 (32), and applying (20). It then follows that

$$
\|\lambda_h\|_{1/2, \Gamma_{h,1}} \leq C \sum_{i=1,2} \|\tilde{f}_i\|_{-1, \Omega_{h,i}},
$$

(38).
From (34), we have that

\[ \sum_{i=1,2} \|u_{h,i}\|_{1,\Omega_{h,i}} \leq C \left( \|\lambda_h\|_{1/2,\Gamma_{h,1}} + \sum_{i=1,2} \|\tilde{f}_i\|_{-1,\Omega_{h,i}} \right). \]  

The proof is thus concluded by substituting \( \|\lambda_h\|_{1/2,\Gamma_{h,1}} \) in the above with (38), and subsequently adding (38) and (37) to the resulting bound. \( \square \)

4.3. **Error Analysis.** Here, we will present the analysis for the error of (24).

First, we state some bounds that will become ubiquitous throughout the error analysis. First, we will denote \( L_i(\cdot) := -\nabla \cdot (\tilde{p}_i(x) \nabla (\cdot)), \quad \tilde{f}_i := L_i \tilde{u}_i, \quad u_{p,i} \in V_{h,i}^k \) as \( u_{p,i} := H_{h,i}^c (\tilde{u}_i \circ \eta_t^c (\xi_t^c)) + \eta_t^c, \quad e_{p,i} := \tilde{u}_i - u_{p,i}. \) Since \( \tilde{f}_i \) is a bonafide extension of \( f_i \), we may apply [5, Appendix B, Lemma 7] to determine that

\[ \|\tilde{f}_i - f_i\|_{-1,\Omega_{h,i}} \leq C h_i^{2k-2} \|f_i\|_{k-1,\Omega_i}, \]

under the assumption that \( \delta_{h,i} \sim O(h_i^2) \). The error analysis for Dirichlet PE–FEM [5 Theorem 3] imply the following bound

\[ \|e_{p,i}\|_{m,\Omega_{h,i}} \leq C h_i^{k-m+1} |u_i|_{k+1,\Omega_i}, \]

for \( m = 1, \ldots, k + 1. \) Finally, we also have from

\[ \|L_i e_{p,i}\|_{-1,\Omega_{h,i}} = \sup_{v \in H_{h,i}^c(\Omega_{h,i})} \frac{\langle \tilde{p}_i \nabla e_{p,i}, \nabla v \rangle_{\Omega_{h,i}}}{\|v\|_{1,\Omega_{h,i}} = 1} \leq \|\tilde{p}_i\|_{CN(\Omega_{h,i})} \|e_{p,i}\|_{1,\Omega_{h,i}}, \]

that the following is bound satisfied

\[ \|L_i e_{p,i}\|_{-1,\Omega_{h,i}} \leq C h_i^k \|u_i\|_{k+1,\Omega_i}. \]

Next, we will decompose \( u_{h,i} := u_{p,i} + z_{h,i}, \) where \( z_{h,i} \in V_{h,i}^k \) is the discrete error term. Using this decomposition, we are able to write (33) in the following from: seek \( (z_{h,1}, z_{h,2}, u_h, \omega_h) \in V_h^k \) such that

\[ B_{h,1}^1 (z_{h,1}, v_1) - \theta_{h,1} \langle u_h, v_1 \rangle_G \Gamma_{h,1}^c = \langle \kappa_{h,1}, v_1 - R_{h,1} v_1 \rangle_{\Gamma_{h,1}^c}, \]

\[ B_{h,2}^2 (z_{h,2}, v_2) - \theta_{h,2} \langle u^{(2)}_h, v_2 \rangle_{\Gamma_{h,2}^c} = \langle \kappa_{h,2}, v_2 - R_{h,2} v_2 \rangle_{\Gamma_{h,2}^c}, \]

\[ \sum_{i=1,2} B_{h,N}^i (z_{h,i}, R_{h,i}^c, \mu_i) + \langle \omega_h, J_{1,(2)}^c (\mu_1^{(2)} - \mu_2) \rangle_{\Gamma_{h,2}^c} = M_h (\mu_1, \mu_2), \]

\[ \forall (v_1, v_2, \mu_1, \mu_2) \in V_h^k, \]

where we have defined

\[ t_h := \lambda_h - \pi_t^c (\tilde{u}_i \circ \eta_t^c (\xi_t^c)), \quad \kappa_{h,i} := \tilde{f}_i - f_i + L_i e_{p,i}, \]

\[ \omega_h := \rho_h - \tilde{p}_2 \circ \eta_t^c (\xi_t^c) T_{h,1}^{-1} (\nabla u_{p,i}) \big|_{\eta_t^c (\xi_t^c) \cdot n_2}, \]
and

\[ M_h(\mu_1, \mu_2) := \sum_{i=1,2} \left( \langle \tilde{f}_i - \tilde{f}_i, \mathcal{R}_{h,i}^c \rangle \right)_{\Omega_{h,i}} + B_{h,N}^k \left( e_{p,i}, \mathcal{R}_{h,i}^c \right) + \langle \tilde{\eta}_2 \circ \tilde{\eta}_2(\xi_1) T_{h,2}(\nabla e_{p,2}) \rangle \eta_2(\xi_1) \cdot \mathbf{n}_2, J_{h,2}^c(\mu_1) - \mu_2 \rangle_{\Gamma_{h,2}^c} + \langle \tilde{\eta}_1 \circ \tilde{\eta}_1(\xi_1) R_{h,1}(\nabla \tilde{u}_1) \rangle \eta_1(\xi_1) \cdot \mathbf{n}_1, J_{h,1}^c(\mu_2) \rangle_{\Gamma_{h,1}^c} + \langle \tilde{\eta}_2 \circ \tilde{\eta}_2(\xi_2) R_{h,2}(\nabla \tilde{u}_2) \rangle \eta_2(\xi_2) \cdot \mathbf{n}_2, J_{h,2}^c(\mu_1) \rangle_{\Gamma_{h,2}^c}. \]

To streamline the exposition of the error analysis, we shall relegate the derivation of (43) to Appendix B.

**Theorem 4.** Assume that \( \tilde{f}_i \in H^{k-1}(\mathbb{R}^d) \), and that the hypotheses of Theorem 3 hold. Then, the following error bound is satisfied

\[
\| (\tilde{u} - u_{h,1}, \tilde{u} - u_{h,2}) \|_{H^1(\Omega_{h,1}) \times H^1(\Omega_{h,2})} \leq C \sum_{i=1,2} \left( h^k \| u_i \|_{k+1, \Omega_i} + h^{2k-1} \| f_i \|_{k-1, \Omega_i} \right).
\]

**Proof.** For notational convenience, we will conform to the notation used in Appendix B. We begin our error analysis by seeing that \( z_{h,i} \) may be written in the form

\[
z_{h,i} = \mathcal{H}_{h,i}^c(\mu_1) + \mathcal{G}_{h,i}^c(\kappa_{h,i}).
\]

Then, it follows from (45) and Proposition 2 that

\[
\| z_{h,i} \|_{1, \Omega_{h,i}} \leq C \left( \| h_i \|_{1/2, \Gamma_{h,i}^c} + \| \kappa_{h,i} \|_{-1, \Omega_{h,i}} \right) \leq C \left( \| u_i \|_{1/2, \Gamma_{h,i}^c} + h_i^{2k-1} \| f_i \|_{-1, \Omega_i} + h_i^k \| u_i \|_{k+1, \Omega_i} \right),
\]

after seeing that

\[
\| \kappa_{h,i} \|_{-1, \Omega_{h,i}} \leq C \left( h_i^{2k-1} \| f_i \|_{-1, \Omega_i} + h_i^k \| u_i \|_{k+1, \Omega_i} \right),
\]

by virtue of (40) and (42).

We now derive a bound for \( \omega_h \). From (55) we have that

\[
\langle \omega_h, \mu_2 \rangle \Gamma_{h,2}^c = B_{h,N}^2 (z_{h,2}, \mathcal{R}_{h,2}^c \mu_2) - \langle \tilde{f}_2 - \tilde{f}_2, \mathcal{R}_{h,2}^c \mu_2 \rangle_{\Omega_{h,2}} + a_{h,2} (e_{p,2}, \mathcal{R}_{h,2}^c \mu_2) - \langle E_{h,2}^c \mu_2 \rangle_{\Gamma_{h,2}^c} \leq C \left( \| u_i \|_{1/2, \Gamma_{h,1}^c} + h_i^{2k-1} \| f_2 \|_{k-1, \Omega_i} + \| u_2 \|_{k+1, \Omega_2} \right) \| \mu_2 \|_{1/2, \Gamma_{h,2}^c},
\]

after applying (20), (52) of Lemma 3 (40), and (44). It then follows from the definition of the dual norm that

\[
\| \omega_h \|_{-1/2, \Gamma_{h,2}^c} \leq C \left( \| u_i \|_{1/2, \Gamma_{h,1}^c} + h_i^{2k-1} \| f_2 \|_{k-1, \Omega_i} + h_i^{k-1/2} \| u_2 \|_{k+1, \Omega_2} \right).
\]
Next, we derive a bound for $\ell_h$. Let us set $\mu_1 = \ell_h$, $\mu_2 = \pi^c_2 \ell_h$, $\mathcal{R}_{h,i}^c(\cdot) = \hat{H}_{h,i}^c(\cdot)$ in (43). It follows that the third equation of (43) can be written as

$$\sum_{i=1,2} B_{i,N}^h \left( \mathcal{H}_{h,i}^c(\cdot), \hat{H}_{h,i}^c(\cdot) \right) + \left( \omega_h, \mathcal{F}_{1,2}^c \ell_h - \pi^c_2 \ell_h \right)_{\Gamma_{h,2}}$$

(47)

$$= \mathcal{M}_h \left( \ell_h, \pi^c_2 \ell_h \right) - \sum_{i=1,2} B_{i,N}^h \left( G_{h,i} \kappa_{h,i}, \hat{H}_{h,i}^c(\cdot) \right).$$

First, we have that

$$B_{i,N}^h \left( G_{h,i} \kappa_{h,i}, \hat{H}_{h,i}^c(\cdot) \right) \leq C \left( h^{2k-1} \left\| \tilde{F} \right\|_{k-1, \Omega_i} + h^k \| u_i \|_{k+1, \Omega_i} \right).$$

by (21), (19a), and (15). Then, utilizing (2), Proposition 2 and the projection theorem, implies that

$$\left( \omega_h, \mathcal{F}_{1,2}^c \ell_h - \pi^c_2 \ell_h \right)_{\Gamma_{h,2}} = \left( \rho_h - F_{p,2}, \mathcal{F}_{1,2}^c \ell_h - \pi^c_2 \ell_h \right)_{\Gamma_{h,2}}$$

(49)

$$= \left( \rho_h - \pi^c_2 F_{p,2}, \left( \mathcal{F}_{1,2}^c - 1 \right) \ell_h \right)_{\Gamma_{h,2}} + \left( \pi^c_2 F_{p,2} - F_{p,2}, \mathcal{F}_{1,2}^c \ell_h \right)_{\Gamma_{h,2}}$$

$$= \left( \omega_h, \left( \mathcal{F}_{1,2}^c - 1 \right) \ell_h \right)_{\Gamma_{h,2}} + \left( \pi^c_2 F_{p,2} - F_{p,2}, \ell_h \right)_{\Gamma_{h,2}}$$

$$\leq C \left( h \|\ell_h\|_{1/2, \Gamma_{h,1}} + h^{2k} \|h_{1/2, \Omega_2} + h^k \| u_2 \|_{1/2, \Omega_2} \right) \|\ell_h\|_{1/2, \Gamma_{h,1}},$$

since

$$\left\| \ell_h^c - \pi^c_2 \ell_h \right\|_{0, \Gamma_{h,2}} \leq C h_{\Gamma_{h,1}}^2 \|\ell_h\|_{1/2, \Gamma_{h,1}},$$

from standard approximation arguments, and

$$\|\pi^c_2 F_{p,2} - F_{p,2}\|_{0, \Gamma_{h,2}} = \|\pi^c_2 F_{p,2} - F_{2}\|_{0, \Gamma_{h,2}} + \|E_{p,2}\|_{0, \Gamma_{h,2}}$$

$$= \|\pi^c_2 E_{p,2}\|_{0, \Gamma_{h,2}} + \|\pi^c_2 F_{2} - F_{2}\|_{0, \Gamma_{h,2}} + \|E_{p,2}\|_{0, \Gamma_{h,2}}$$

$$\leq C h^{k-\frac{1}{2}} \|u_2\|_{k+1, \Omega_2},$$

after applying (63) of Lemma 3 with $\delta_{h,2} \sim O(h_{\Gamma_{h,2}}^2)$, and seeing that $F_2 \in H^{k-\frac{1}{2}} (\Gamma_{h,2})$. Next, we proceed to bound $\mathcal{M}_h \left( \ell_h, \pi^c_2 \ell_h \right)$; let us denote

$$\hat{E}_{p,2} := \tilde{F}_{2} \circ \eta^c_{2} (\xi_{2}) \mathcal{T}^{-1}_{h,2} (\nabla e_{p,2}) \eta_{2} (\xi_{2}) \cdot n_2,$$

then the second term in $\mathcal{M}_h \left( \ell_h, \pi^c_2 \ell_h \right)$ can be bounded as follows

$$\left( \hat{E}_{p,2}, \mathcal{F}_{1,2}^c \ell_h - \pi^c_2 \ell_h \right)_{\Gamma_{h,2}}$$

$$\leq \|\hat{E}_{p,2}\|_{0, \Gamma_{h,2}} \left( \left\| \ell_h^c - \pi^c_2 \ell_h \right\|_{0, \Gamma_{h,2}} + \left\| \left( \mathcal{F}_{1,2}^c - 1 \right) \ell_h \right\|_{1/2, \Gamma_{h,2}} \right)$$

$$\leq C h^{k} \|u_2\|_{k+1, \Omega_2} \|\ell_h\|_{1/2, \Gamma_{h,1}},$$
after applying (62) of Lemma 3 [2], and a standard approximation argument. Subsequently, we have that

\[ B_{h,N}^i \left( e_{p,i, \hat{H}_{h,i}^{(i)}} \right) \leq C h_k^k \| u_i \|_{k+1, \Omega_i} \]

by virtue of Lemma 4 and (41). It then follows easily from [5, Appendix A, Lemma 2], [40], [29], and [41] that

\[ \mathcal{M}_h \left( \bar{t}_h, \pi_2 \bar{t}_h^{(2)} \right) \leq C \sum_{i=1,2} \left( h^{2k-1} \| f_i \|_{k-1, \Omega_i} + h^k \| u_i \|_{k+1, \Omega_i} \right) \| t_h \|_{1/2, \Gamma_{h,i}}. \]

Finally, we have that

\[ B_{h,N}^i \left( \mathcal{H}_{h,i}^{(i)}, \hat{H}_{h,i}^{(i)} \right) = B_{h,N}^i \left( \mathcal{H}_{h,i}^{(i)}, \hat{H}_{h,i}^{(i)} \right) \]

\[ + B_{h,N}^i \left( \mathcal{H}_{h,i}^{(i)}, \left( \mathcal{H}_{h,i}^{(i)} - \hat{H}_{h,i}^{(i)} \right) \right) \]

\[ \geq C (\gamma_{N,i} - h_i) \| t_h \|_{1/2, \Gamma_{h,i}} \]

from Lemma 11 [22], [32], and [20]. The above bound together with (49), (50), and (48) yields the following

\[ \| t_h \|_{1/2, \Gamma_{h,i}} \leq C \sum_{i=1,2} \left( h^k \| u_i \|_{k+1, \Omega_i} + h^{2k-1} \| f_i \|_{k-1, \Omega_i} \right). \]

The proof is therefore concluded by recalling (44),

\[ \| \bar{u}_i - u_{h,i} \|_{1, \Omega_{h,i}} \leq \| e_{p,i} \|_{1, \Omega_{h,i}} + \| z_{h,i} \|_{1, \Omega_{h,i}}, \]

and applying (41), (44), and (51).

5. Numerical Illustration

In this section, we present an illustrative numerical example for this coupling approach. We consider the interface problem where \( \Omega_1 \) is a disk of radius \( \frac{1}{4} \) and \( \Omega_2 \) is an annulus with an inner radius of \( \frac{1}{4} \) and an outer radius of \( \frac{1}{2} \). \( \Omega_{h,1} \) and \( \Omega_{h,2} \) are discretized so that each element on \( \Gamma_{h,1} \) and \( \Gamma_{h,2} \) respectively have the same length. Additionally, we fix the ratio of elements between \( \Gamma_{h,1} \) and \( \Gamma_{h,2} \) to be 1 : 1, 1 : 2, and 2 : 1. The manufactured solution we consider is the following

\[
\begin{align*}
    u_1 &= e^{-5(x^2+y^2)} \\
    u_2 &= \frac{e^{-5(x^2+y^2)} + e^{-\frac{5}{2}(x^2+y^2)}}{2},
\end{align*}
\]

This solution corresponds to (10) with \( p_1 = 1, \ p_2 = 2, \ f_1 = f_2 = -100(x^2 + y^2) + 20e^{-5(x^2+y^2)} \). The numerical solutions are computed using quadratic, cubic, and quartic Lagrange elements on triangles. We present the convergence histories in the following tables.

6. Concluding Remarks

In this paper, we have presented a new method, based on PE–FEM, for coupling numerical solutions together on geometrically nonmatching discrete interface approximations. Stability and optimal \( H^1 \) error bounds was proven, and the numerical illustration confirms our theoretical findings. Additionally, the numerical results imply that our method is optimally convergent in \( L^2 \)–as well. In future
| $h$   | $L^2$-Error $H^1$-Error | $L^2$-Error $H^1$-Error | $L^2$-Error $H^1$-Error |
|-------|-------------------------|-------------------------|-------------------------|
| 0.3827 | 4.7000E-3 5.7280E-2   | 4.9120E-4 7.33919E-3   | 1.7986E-4 1.4585E-3   |
| 0.2199 | 6.4000E-4 1.5427E-2   | 1.7007E-5 1.04039E-3   | 5.1535E-6 9.0318E-6   |
| 0.1200 | 7.6813E-5 3.6798E-3   | 1.0265E-6 1.29739E-4   | 1.2813E-7 5.3379E-6   |
| 0.0700 | 1.0906E-5 1.0265E-3   | 9.1517E-8 2.03711E-5   | 5.4741E-9 4.0525E-7   |
| 0.0349 | 1.3666E-6 2.5706E-4   | 2.5686E-9 2.55939E-6   | 1.4585E-3 4.0525E-7   |
| Rate   | 3.424       2.274       | 4.699       3.340       | 6.120       4.803     |

Table 1. Convergence histories of numerical tests. We observe optimal convergence in the broken $L^2$ and $H^1$ norms.

works, we will apply and analyze generalizations of this coupling approach to other interface phenomena, such as groundwater flows and fluid–structure interaction.

**Appendix A. Technical Lemmas**

**Lemma 3.** Let $e_{p,i} \in H^1(\Omega_{h,i})$, $H^{k+1}(K_{h,i})$ be defined as in Section 4.3, then the following bounds are satisfied

\[
\| \tilde{p}_i(\xi^c_i) \nabla e_{p,i} \cdot n_{h,i} \|_{0, \Gamma_{h,i}^c} \leq C \| \tilde{p}_i \|_{C^0(\Omega_{h,i})} h_i^{k-\frac{1}{2}} \| u_i \|_{k+1, \Omega_i}
\]

(52)

\[
\| \tilde{p}_i \circ \eta_i(\xi^c_i) T_{h,i}^{k-1}(\nabla e_{p,i}) \eta_i(\xi^c_i) \cdot n_i \|_{0, \Gamma_{h,i}^c} \leq C h_i^{k-\frac{1}{2}} \| u_i \|_{k+1, \Omega_i}.
\]

(53)
Proof. First, we have that
\[ \| \bar{p}_i (\xi_i^e) \| \nabla e_{p,i} \cdot n_{\Gamma_h,i} \|_{0,\Gamma_h,i} \leq C \| \bar{p}_i \| C^0 (\bar{\Omega}_{h,i}) \| \nabla e_{p,i} \|_{0,\Omega_h} \]
\[ \leq C \| \bar{p}_i \| C^0 (\bar{\Omega}_{h,i}) \| e_{p,i} \|_{1,\Omega_h,i} \| \| e_{p,i} \|_{2,\Omega_h,i} \]
\[ \leq Ch_i^{k-1/2} \| \bar{p}_i \| C^0 (\bar{\Omega}_{h,i}) \| u_u \|_{k+1,\Omega} \]
by virtue of (41), thus we have (42).
Next, we have that
\[ \| \bar{p}_i \circ \eta_i^e (\xi_i^e) \| T_h^{k-1} (\nabla e_{p,i}) \| \eta_i^e (\xi_i^e) \cdot n_i \|_{0,\Gamma_h,i} \leq \| \bar{p}_i \| C^0 (\bar{\Omega}_{h,i}) \| T_h^{k-1} (\nabla e_{p,i}) \| \eta_i^e (\xi_i^e) \|_{0,\Gamma_h,i} \]
It then follows that
\[ \| T_h^{k-1} (\nabla e_{p,i}) \| \eta_i^e (\xi_i^e) \|_{0,\Gamma_h,i} \leq \| T_h^{k-1} (\nabla e_{p,i}) \| \eta_i^e (\xi_i^e) - \nabla e_{p,i} \|_{0,\Gamma_h,i} + \| e_{p,i} \|_{0,\Gamma_h,i} \]
\[ \leq C \left( h_i \| e_{p,i} \|_{2,\Omega_h,i} + h_i^{k-1} \| e_{p,i} \|_{k+1,\Omega_h,i} \right) + \| \nabla e_{p,i} \|_{0,\Gamma_h,i} \]
\[ \leq Ch_i^{k-1/2} \| u_u \|_{k+1,\Omega} \]
by virtue of (5), Appendix A, Lemma 3. Thus we have (53). □

Lemma 4. Let \( v \in L^2(\Omega_{h,i}), H^{k+1}(K_{h,i}) \) and assume that \( \delta_{h,i} \sim O(h_i^2) \), then for \( k = 1, 2, \ldots \)
\[ \tau_i (v, \mu) \leq C \| \bar{p}_i \| C^1 (\bar{\Omega}_{h,i}) \left( h_i \| v \|_{2,\Omega_h,i} + h_i^{2k-1} \| v \|_{k+1,\Omega_h,i} \right) \| \mu \|_{1/2,\Gamma_h,i} \]
\[ \forall \mu \in H^{1/2}(\Gamma_h,i) \]
(54)

Proof. The definition of \( \tau_i (\cdot, \cdot) \), see (14), and a triangle inequality, we have that
\[ \tau_i (v, \mu) \leq \| \mu \|_{1/2,\Gamma_h,i} \left( \| \bar{p}_i \circ \eta_i^e (\xi_i^e) \| \nabla v \|_{0,\Gamma_h,i} \| \eta_i^e (\xi_i^e) - \nabla v (\xi_i^e) \cdot n_i \|_{0,\Gamma_h,i} \right) \]
\[ + \| (\bar{p}_i \circ \eta_i^e (\xi_i^e) - \bar{p}_i (\xi_i^e)) \| \nabla v (\xi_i^e) \cdot n_i \|_{0,\Gamma_h,i} \]
\[ + \| \bar{p}_i (\xi_i^e) \| \nabla v (\xi_i^e) \cdot n_i \|_{0,\Gamma_h,i} \| \bar{p}_i (\xi_i^e) - n_{h,i} \|_{0,\Gamma_h,i} \)
From (5) Appendix A, Lemma 3 we have that
\[ \| \bar{p}_i \circ \eta_i^e (\xi_i^e) \| \nabla v \|_{0,\Gamma_h,i} \| \eta_i^e (\xi_i^e) - \nabla v (\xi_i^e) \cdot n_i \|_{0,\Gamma_h,i} \]
\[ \leq \| \bar{p}_i \| C^0 (\bar{\Omega}_{h,i}) \left( h_i \| v \|_{2,\Omega_h,i} + h_i^{2k-1} \| v \|_{k+1,\Omega_h,i} \right) . \]
We then also have that
\[ \| (\bar{p}_i \eta_i^e (\xi_i^e) - \bar{p}_i (\xi_i^e)) \nabla v (\xi_i^e) \cdot n_i \|_{0,\Gamma_h,i} \leq C h_i^2 \| \bar{p}_i \| C^1 (\bar{\Omega}_{h,i}) \| \nabla v \|_{0,\Gamma_h,i} \]
\[ \leq C h_i^2 \| \bar{p}_i \| C^1 (\bar{\Omega}_{h,i}) \| v \|_{2,\Omega_h,i} \]
after applying Taylor’s theorem for continuous functions and the trace inequality. Finally, since \( |n_i - n_{h,i}| \sim O(h_i) \), we have that
\[ \| \bar{p}_i (\xi_i^e) \nabla v (\xi_i^e) \cdot n_i \|_{0,\Gamma_h,i} \leq C h_i \| \bar{p}_i \| C^0 (\bar{\Omega}_{h,i}) \| v \|_{2,\Omega_h,i} \]
Hence (54) is established. □
Appendix B. Derivation of (43)

We begin our derivation by seeing that

\[ B_{h,D}^i (u_{h,i}, v_i) = B_{h,D}^i (z_{h,i}, v_i) + \theta_{h,i} \left\langle T_{h,i}^k u_{p,i} | \eta^c_i (\xi_i) \right\rangle_{\Gamma^0_{h,i}} \]

\[ + \theta_{h,i} \left\langle T_{h,i}^k u_{p,i} | \eta^c_i (\xi_i^e) \right\rangle_{\Gamma^0_{h,i}} + \langle L_i u_{p,i}, v_i \rangle_{\Omega_{h,i}} \quad \forall v_i \in V^k_{h,i}, \]

by the definition of \( B_{h,D}^i (\cdot, \cdot) \) given in (12) and an application of Green’s identity.

We then have that

\[ \left\langle f_i, v_i - R_{h,i} v_i \right\rangle_{\Omega_{h,i}} = \left\langle f_i - f_i + L_i \tilde{u}_i, v_i - R_{h,i} v_i \right\rangle_{\Omega_{h,i}} \]

by definition of \( \tilde{f}_i \). It then follows that

\[ B_{h,D}^i (z_{h,i}, v_i) - \theta_{h,i} \left\langle (i), v_i \right\rangle_{\Gamma^c_{h,i}} = \langle \kappa_{h,i}, v_i - R_{h,i} v_i \rangle_{\Omega_{h,i}} \quad \forall v_i \in V^k_{h,i}, \]

since by the definition of \( u_{p,i} \), we have that

\[ \left\langle T_{h,i}^k u_{p,i} | \eta^c_i (\xi_i) - \tilde{u}_i \circ \eta^c_i (\xi_i^e), \mu_i \right\rangle_{\Gamma^c_{h,i}} = 0 \quad \forall \mu_i \in W^c_{h,i} \]

and

\[ \left\langle T_{h,i}^k u_{p,i} | \eta^c_i (\xi_i^e), \mu_i \right\rangle_{\Gamma^0_{h,i}} = 0 \quad \forall \mu_i \in W^{0,k}_{h,i}. \]

Hence, the first two equations of (43) are derived.

Next, we will derive the third equation of (43). For notational convenience, we will denote

\[ F_1 := \tilde{p}_i \circ \eta^c_i (\xi_i) (\nabla \tilde{u}_i \circ \eta^c_i (\xi_i^e)) \cdot n_i, \]

\[ F_{p,i} := \tilde{p}_i \circ \eta^c_i (\xi_i) \left. T_{h,i}^{k-1} (\nabla u_{p,i}) \right|_{\eta^c_i (\xi_i^e)} \cdot n_i, \]

\[ E_{p,i} := \tilde{p}_i (\xi_i^e) \nabla e_{p,i} \cdot n_{h,i}, \]

and

\[ E_{p,i} := \tilde{p}_i \circ \eta^c_i (\xi_i) \left( T_{h,i}^{k-1} (\nabla e_{p,i}) \right|_{\eta^c_i (\xi_i^e)} + \left. \right|_{\eta^c_i (\xi_i^e)} \cdot n_i. \]

Then, from a similar set of steps used in the previous paragraph, we have that the last two equations of (24) can be written as

\[ B_{h,N}^1 (z_{h,1}, R_{h,1}^c \mu_1) + \left\langle p_{h,1}^{(1)}, F_{p,1}, \mu_1 \right\rangle_{\Gamma^c_{h,1}} \]

\[ = \left\langle f_1 - f_1, R_{h,1}^c \mu_1 \right\rangle_{\Gamma^c_{h,1}} + a_{h,1} (e_{p,1}, R_{h,1}^c \mu_1) - \left\langle E_{p,1}, \mu_1 \right\rangle_{\Gamma^c_{h,1}} \quad \forall (\mu_1, \mu_2) \in W^k \]

(55)

\[ B_{h,N}^2 (z_{h,2}, R_{h,2}^c \mu_2) - \langle \rho_h - F_{p,2}, \mu_2 \rangle_{\Gamma^c_{h,2}} \]

\[ = \left\langle f_2 - f_2, R_{h,2}^c \mu_2 \right\rangle_{\Gamma^c_{h,2}} + a_{h,2} (e_{p,2}, R_{h,2}^c \mu_2) - \left\langle E_{p,2}, \mu_2 \right\rangle_{\Gamma^c_{h,2}} \quad \forall (\mu_1, \mu_2) \in W^k \]
We now see that

\[
\langle \rho_h^{(1)} + F_{p,1}, \mu_1 \rangle_{\Gamma_{h,1}} - \langle \rho_h - F_{p,2}, \mu_2 \rangle_{\Gamma_{h,2}}
\]

\[
= \langle \rho_h + F_{p,1}^{(2)} + J_{1,(2)}^c \mu_1^{(2)} \rangle_{\Gamma_{h,2}} - \langle \rho_h - F_{p,2}, \mu_2 \rangle_{\Gamma_{h,2}}
\]

\[
= \langle F_{p,1}^{(2)} + J_{1,(2)}^c \mu_1^{(2)} \rangle_{\Gamma_{h,2}} + \langle \rho_h - F_{p,2}, J_{1,(2)}^c \mu_1^{(2)} - \mu_2 \rangle_{\Gamma_{h,2}}
\]

\[
= - \langle E_{p,1}, \mu_1 \rangle_{\Gamma_{h,1}} - \langle E_{p,2}, \mu_2 \rangle_{\Gamma_{h,2}} - \langle E_{p,2} + \rho_h - F_{p,2}, J_{1,(2)}^c \mu_1^{(2)} - \mu_2 \rangle_{\Gamma_{h,2}}
\]

where we have added the productive zero in the third equality, i.e.,

\[
-p_1 \nabla_1 \cdot \mathbf{n}_1 - p_2 \nabla_2 \cdot \mathbf{n}_2 = - \sum_{i=1,2} F_i = 0.
\]

Adding together the equations \([55]\), and applying the change of variables formula on the trace duality term over \(\Gamma_{h,1}\), substituting \([56]\), and seeing that

\[
\langle E_{p,i} - E_{p,i}^{\tau}, \mu_i \rangle_{\Gamma_{h,i}} = \tau_i(e_{p,i}, \mu_i)
\]

yields the third equation in \([53]\).

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