Interplay between interaction and chiral anomaly: Anisotropy in the electrical resistivity of interacting Weyl metals

Yong-Soo Jho\textsuperscript{1} and Ki-Seok Kim\textsuperscript{1,2}

\textsuperscript{1}Department of Physics, POSTECH, Hyoja-dong, Namgu, Pohang, Gyeongbuk 790-784, Korea
\textsuperscript{2}Institute of Edge of Theoretical Science (IES), Hogil Kim Memorial building 5th floor, POSTECH, Hyoja-dong, Namgu, Pohang, Gyeongbuk 790-784, Korea

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We predict that long-range interactions give rise to anisotropy in the electrical resistivity of Weyl metals at low temperatures, where the electrical resistivity becomes much reduced when electric fields are applied to the direction of the momentum vector to connect two paired Weyl points. Performing the renormalization group analysis, we find that the distance between two Weyl points becomes enhanced logarithmically at low temperatures although the coupling constant of such interactions vanishes inverse-logarithmically. Considering the Adler-Bell-Jackiw anomaly, scattering between these two Weyl points becomes suppressed to increase electrical conductivity in the “longitudinal” direction, counter-intuitive in the respect that interactions are expected to reduce metallicity. We also propose that the anomalous contribution in the Hall effect shows the logarithmic enhancement as a function of temperature, originating from the fact that the anomalous Hall coefficient turns out to be proportional to the distance between two paired Weyl points. Correlations with topological constraints allow unexpected and exotic transport properties.

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I. INTRODUCTION

Nontrivial global structures of ground states sometimes violate classically respected conservation laws at quantum levels, referred to as (quantum) anomalies [1]. When anomalies are associated with breaking of local (gauge) symmetries, it means that their corresponding quantum theories are not consistent and such anomalies should be cancelled, introducing meaningful quantum fields. Indeed, the standard model and string theories are constructed consistently, cancelling gauge and gravitational (also conformal) anomalies, respectively [2]. On the other hand, when such anomalies are related with breaking of global symmetries, they give rise to various interesting physical properties. In particular, various types of topological terms associated with quantum anomalies arise to play essential roles in quantum criticality of quantum matter [3]. In addition, they turn out to be responsible for quantum number fractionalization, given by Goldstone-Wilczek currents [4]. Actually, an emergent non-abelian chiral anomaly has been proposed to cause so called deconfined quantum criticality in low dimensional spin systems [5, 6]. Furthermore, such topological terms sometimes give rise to anomalous (quantized) electrical or thermal Hall effects [7, 8]. Quantum anomalies govern quantum criticality, quantum number fractionalization, and anomalous transport phenomena [9].

In this study we focus on the role of the Adler-Bell-Jackiw anomaly or chiral anomaly [1] in anomalous transport phenomena. This anomaly means that classically conserved chiral currents, that is, currents of right-handed (Weyl) fermions minus those of left-handed fermions, are not preserved at quantum levels due to nontrivial global configurations of gauge (or electric and magnetic) fields. Applying magnetic fields to gapless semi-conductors, a Dirac point described by the four-component Dirac spinor splits into two Weyl points governed by the two-component Weyl spinors with opposite chiralities, where the distance between two Weyl points is proportional to the applied magnetic field [10]. The chiral anomaly gives rise to a topological constraint in dynamics of Weyl fermions, where right-handed Weyl fermions at one Weyl point should scatter into left-handed Weyl fermions at the other Weyl point, when currents are driven to the same direction as the momentum to connect these two Weyl points [11]. Even if short-range scatterers are taken into account, scattering between these two Weyl points becomes suppressed due to the finite distance in the momentum space. As a result, the longitudinal ($E \parallel B$) magneto-conductivity is enhanced, which turns out to be proportional to the square of the applied magnetic field or the distance of two Weyl points [11, 12].

In this paper we investigate effects of interactions on the “longitudinal” “magneto”-transport in Weyl metals. Here, “” will be clarified later. It is almost trivial to observe that local four-fermion interactions are irrelevant at low energies in a perturbative sense since the density of states vanishes at zero energy. Long-range Coulomb interactions have been investigated both extensively and intensively for transport phenomena in graphene [13]. In addition, transverse gauge interactions have been also discussed in Weyl- or Dirac-type systems [14]. Recently, the chiral anomaly has been calculated in the Weyl system [15]. However, the interplay between long-range interactions and the chiral anomaly has not been investigated clearly. In particular, it remains mysterious how this combination gives rise to anomalous “longitudinal” “magneto”-transport phenomena.

Performing the renormalization group analysis, we re-
veal that the distance between two Weyl points becomes enhanced logarithmically at low temperatures although the coupling constant for transverse long-range interactions vanishes inverse-logarithmically (expected in three dimensions). This is in contrast with “conventional” Weyl metals without interactions [10–12], where the distance between two corresponding Weyl points remains finite. As a result, scattering between two Weyl points becomes suppressed much more than the case of noninteracting Weyl metals, which increases electrical conductivity in the direction to connect the momentum vector between two Weyl points. We predict that anisotropic metallicity arises, where the electrical resistivity becomes much reduced for the longitudinal direction while normal metallic behaviors result for other directions. Furthermore, we propose that the anomalous contribution in the Hall effect becomes enhanced as a function of temperature, originating from the fact that the anomalous Hall coefficient turns out to be proportional to the distance between two paired Weyl points [16]. We discuss this interaction-enhanced anisotropy in the longitudinal resistivity and the increase of the anomalous contribution in the Hall effect, based on the quantum Boltzmann equation approach in the presence of both long-range transverse interactions and the chiral anomaly.

II. INTERPLAY BETWEEN LONG-RANGE TRANSVERSE INTERACTIONS AND THE CHIRAL ANOMALY

We start from quantum electrodynamics with a topological \( \theta \)-term in three spatial dimensions (\( \theta \)-QED)  
\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\psi} \gamma^\mu \partial_\mu \psi + \frac{e^2 \theta}{8\pi^2} F_{\mu\nu} F^{\mu\nu},
\]
where \( \bar{\psi} \) is a four-component Dirac spinor to take both chirality (associated with either orbital or sublattice indices) and spin quantum numbers and \( A_\mu \) is an electromagnetic vector potential regarded as a quantum field.\[ D_\mu = \partial_\mu + ie A_\mu \] in \( \partial_\mu = \gamma^\mu D_\mu \) is a covariant derivative with an electric charge \( e \), where \( \gamma^\mu \) is the Dirac gamma matrix satisfying the Clifford algebra with \( \mu = 0, 1, 2, 3 \). \( F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \) is a magnetic dual tensor of the electromagnetic field strength tensor. One may consider that this field theory results from certain lattice models with spin-orbit interactions for topological insulators except for long-range transverse interactions [17]. However, we would like to emphasize that even gauge fluctuations can emerge in some lattice models, supporting so called topological spin liquids [18]. The \( \theta \)-term is the fingerprint of the topological insulator in three dimensions, the source of the (longitudinal) magnetoelectric effect or equivalently, the half-quantized Hall conductance on its surface [8].

The Adler-Bell-Jackiw anomaly states that the classically conserved chiral current is not conserved at quantum levels [1], given by
\[
\partial_\mu J_5^\mu = \frac{e^2}{8\pi^2} F_{\mu\nu} F^{\mu\nu}
\]
with the chiral current \( J_5^\mu = \bar{\psi} \gamma^\mu \gamma_5 \psi = J_{5R}^\mu - J_{5L}^\mu \) mentioned before, where the \( \gamma_5 \) matrix is a Dirac matrix that anti-commutes with other Dirac matrices. In other words, when electric fields are applied in parallel with magnetic fields, the chiral current is not conserved. It is important to realize that the \( \theta \)-term is a boundary term, implying that the coefficient \( \theta \) cannot be renormalized by interactions. However, if inhomogeneous magnetic fields can be applied to this topological insulating state, the \( \theta \) coefficient depends on position [17, 19]. As a result, this term is not a boundary term any more, which can be renormalized by interactions.

Resorting to this anomaly equation, we rewrite the effective field theory as follows
\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\partial - e A + \gamma_5 \psi),
\]
where \( c_\mu \) with the \( \gamma_5 \) Dirac matrix is a chiral gauge field, given by \( \partial_\mu \theta \). See appendix A1 for the derivation from Eq. (3) to Eq. (1) with Eq. (2). It is interesting to observe that the Dirac point splits into two Weyl points at \( K = \pm e \). In this respect our problem is to investigate the nature of the quantum critical point between a topological insulator and a band insulator in the presence of inhomogeneous magnetic fields. In other words, we study how both the interaction parameter \( e \) and the distance between two Weyl points \( c_\mu \) are renormalized to evolve at low temperatures, and reveal how these renormalization effects modify transport properties, compared with the case in the absence of interactions. Such inhomogeneous magnetic fields may be created by either ferri-magnetism [17] or some ferromagnetic clusters, given by randomly distributed magnetic ions [20].

Introducing counter terms, we rewrite this effective field theory as follows
\[
\mathcal{L} = -\frac{Z_A}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(Z_\psi i\partial + Z_\psi \gamma_5 - Z_\psi e A) \psi,
\]
where \( Z_\psi, Z_A, Z_\psi, \) and \( Z_e \) are field renormalization constants of \( \psi, A_\mu, c_\mu \) and the vertex or coupling (\( e \)) renormalization constant, respectively. See appendix A1 for details. We emphasize that the renormalization factor \( Z_e \) has never been introduced as far as we know, thus regarded as an essential aspect of this study. We alert that \( c_\mu \) is a background gauge field, not dynamical in the present study.

In order to perform the renormalization group analysis, we resort to the dimensional regularization. A subtle point arises due to the presence of the \( \gamma_5 \) matrix, which needs some care for its treatment, because its existence depends on dimensionality [1]. We point out that the presence of the \( \gamma_5 \) matrix makes our calculations much
more complicated and laborious. One nontrivial check for the validity of our calculations is that the Ward identity is respected in the one-loop level. All details are presented in appendix A2. As a result, we obtain our coupled renormalization group equations for \(e\) and \(c_\mu\),

\[
\beta_e(\mu) = \mu \frac{de}{d\mu} = -\frac{e^2}{12\pi^2}, \quad (5)
\]

\[
\beta_{c_\mu}(\mu) = \mu \frac{dc_\mu}{d\mu} = -\frac{e^2}{4\pi^2} c_\mu. \quad (6)
\]

See appendix A3 for the derivation of Eqs. (5) and (6).

The first equation is nothing but the conventional renormalization group equation of the coupling constant, which tells us that the electric charge renormalizes to vanish at zero temperature. Even if the coupling constant vanishes, the background chiral gauge field flows to go to infinity. Inserting the solution of the first equation into the second equation, we find

\[
e^2(T) = \frac{e^2_D}{1 + \frac{e^2_D}{4\pi^2} \ln \left( \frac{D}{T} \right)}, \quad (7)
\]

\[
c_\mu(T) = c^{D}_\mu \left[ 1 + \frac{e^2_D}{4\pi^2} \ln \left( \frac{D}{T} \right) \right], \quad (8)
\]

where \(c^{D}_\mu\) and \(e_D\) are chiral gauge field and coupling constant at the energy scale of the bandwidth or cutoff. See appendix A4. The chiral gauge field increases in a logarithmic way. This indicates that the distance between two Weyl points becomes “infinite” at zero temperature, implying that their scattering events are suppressed “completely”. This renormalization effect should be observed in transport coefficients.

III. ANISOTROPY IN THE LONGITUDINAL ELECTRICAL TRANSPORT AND ENHANCEMENT OF THE ANOMALOUS CONTRIBUTION IN THE HALL EFFECT

Our framework for anomalous transport is the “semi-classical” quantum Boltzmann equation approach. Here, the term “semi-classical” means that the role of both Berry curvature and chiral anomaly or the topological \(\theta\)-term is introduced from coupled semi-classical equations of motion based on the wave-packet picture in solids [7]. Benchmarking a recent transport study based on the classical Boltzmann equation [12], we incorporate this information into the quantum Boltzmann equation, which have been applied to transport dynamics in strongly correlated electrons [21]. As a result, inelastic scattering events can be taken into account naturally in the presence of the topological \(\theta\)-term. We consider the case of a finite chemical potential, more generic than the case with two Weyl points. In principle, one can derive the quantum Boltzmann equation in a matrix form, regarded as a full quantum transport theory [22, 23]. However, its derivation is much complicated and not easy to perform. We would like to emphasize that our phenomenological “quantum” transport theory with the introduction of the topological \(\theta\)-term recovers the known result for the longitudinal transport coefficient in Weyl metals, the so-called “negative magnetoresistance” proposed in Ref. [11].

We start from the quantum Boltzmann equation for a steady state [24]

\[
\dot{\rho} \cdot \frac{\partial G^<(p, \omega)}{\partial p} + \dot{\varphi} \cdot \frac{\partial G^<(p, \omega)}{\partial \omega} - \dot{\rho} \cdot \left\{ \frac{\partial \Sigma^<(p, \omega)}{\partial \omega} - \frac{\partial \Sigma^{ret}G^{ret}(p, \omega)}{\partial p} \partial \omega \right\} = -2\Gamma(p, \omega)G^<(p, \omega) + \Sigma^<(p, \omega)A(p, \omega). \quad (9)
\]

\(G^<(p, \omega)\) is the lesser Green’s function, regarded as a quantum distribution function, where \(p\) and \(\omega\) represent momentum and frequency for relative coordinates, respectively. \(\dot{\rho}\) denotes the derivative with respect to time \(t\). \(\Sigma^<(p, \omega)\) and \(G^{ret}(p, \omega)\) indicate the lesser self-energy and the retarded Green’s function, respectively, where \(\Re\) is their real part. The right hand side introduces collision terms, where \(\Gamma(p, \omega)\) and \(A(p, \omega)\) indicate the scattering rate and the spectral function.

\(\rho\) and \(p\) are governed by semi-classical equations of motion [7], given by

\[
\dot{\rho} = \frac{\partial \rho}{\partial p} + \dot{\varphi} \times \Omega_p, \\
\dot{\varphi} = eE + \frac{e}{c} \dot{\rho} \times B, \quad (10)
\]

where \(\Omega_p\) represents the Berry curvature of the momentum space. Solving these equations, one obtains

\[
\dot{\rho} = \left( 1 + \frac{e}{c} B \cdot \Omega_p \right)^{-1} \left\{ v_p + eE \times \Omega_p + \frac{e}{c} \Omega_p \cdot v_p B \right\}, \\
\dot{\varphi} = \left( 1 + \frac{e}{c} B \cdot \Omega_p \right)^{-1} \left\{ eE + \frac{e}{c} v_p \times B + \frac{e^2}{c} (E \cdot B) \Omega_p \right\}. \quad (11)
\]

An essential point is the presence of the \(E \cdot B\) term in the second equation, imposing the Adler-Bell-Jackiw anomaly [12].

Inserting Eq. (11) into Eq. (9) and performing straightforward but rather tedious algebra, we reach the following expression for the longitudinal conductivity

\[
\sigma_L(T) \rightarrow (1 + K[c(T)]^2)\sigma_n(T), \quad (12)
\]

where \(K\) is a positive numerical constant and the normal conductivity \(\sigma_n(T)\) is determined by (gauge-interaction induced) intra-scattering events at one Weyl point. All details are shown in appendix B. An essential point is in the \(c(T)\) term, where \(c(T)\) is the distance between two paired Weyl points, given by our renormalization group analysis. Here, the vector index \(\mu\) is fixed and omitted for simplicity. Actually, this expression is to replace
the applied magnetic field in noninteracting Weyl metals with the distance between Weyl points in interacting Weyl metals, when electric fields are applied in parallel with “magnetic fields” or the momentum vector to connect such Weyl points, implying the reason why we call “longitudinal” in front of the conductivity. We emphasize that this expression recovers that of the original proposal [11], where the magneto-conductivity is proportional to the square of the distance between two Weyl points. Although the distance does not renormalize in the noninteracting case, the presence of transverse long-range interactions gives rise to the logarithmic enhancement at zero chemical potential.

One aspect should be pointed out carefully. We proved that the distance between two paired Weyl points increases logarithmically at low temperatures, originating from long-range transverse interactions. It should be noticed that this result appears at zero chemical potential. On the other hand, we found the longitudinal conductivity [Eq. (9)] for a finite chemical potential. Can we expect the similar enhancement of the distance between two paired Weyl points in the case of a finite chemical potential? The renormalization group analysis for the Weyl metallic state with a finite chemical potential turns out to be more complex and technically involved, originating from the treatment of four by four matrices and angular integrals along Fermi surfaces. Besides such technical difficulties, we reach the conclusion that an exotic phenomenon may appear. It is natural to expect that the Weyl metallic state with two corresponding Fermi surfaces will not be stable at low temperatures when there exist interactions. This instability originates from their perfect nesting. As a result, some types of charge or spin density waves are expected to arise at low temperatures. However, the main result of the previous section in the case of zero chemical potential is that the distance between two paired Weyl points increases to diverge, implying that scattering between these two Weyl points is suppressed. Then, we expect that the competition between two kinds of divergences, one of which comes from the perfect nesting while the other of which originates from the chiral anomaly with long-range transverse interactions, may allow an exotic balance, which gives rise to an interacting fixed point, identified with a novel non-Fermi liquid metallic state. In this respect we believe that the study in the case of a finite chemical potential should be performed more carefully and would like to leave it as a future work.

Although we cannot determine the temperature dependence of the distance between two Weyl points in the presence of Fermi surfaces, the presence of the prefactor \((1 + K'[c(T)]^2)\) in the longitudinal resistivity guarantees an anisotropic metallic behavior because such a prefactor does not exist in the transverse resistivity. Unfortunately, we cannot quantify the degree of anisotropy at present.

We propose another fingerprint of the interacting or critical Weyl metallic state with long-range transverse interactions, that is, peculiar temperature dependencies for the anomalous contribution of the Hall coefficient, depending on the chemical potential. Recently, the anomalous contribution for the Hall effect has been evaluated [15, 16], given by

\[
\sigma_{\mu\nu} = \frac{e^2}{2\pi^2} \Gamma_{\mu\nu\gamma} c_\gamma
\]

in the absence of Fermi surfaces, where \(c_\gamma\) is the distance between two paired Weyl points. Based on our renormalization group analysis [Eq. (8)], we find

\[
\sigma_{xy}(T) = \frac{e^2}{2\pi^2} c_z(D) \left[ 1 + \frac{c_z^2}{4\pi^2} \ln \frac{D}{T} \right],
\]

where \(c_z(D)\) is the distance of two paired Weyl points at \(T = D\). This expression is rather unexpected because the anomalous Hall coefficient diverges, dominated over the normal Hall effect. Here, the term “divergence” should be regarded more carefully. Since our long-wave-length effective description is valid only below the momentum cutoff, at most within the first Brillouin zone, the term “divergence” is more accurate to be replaced with enhancement at low temperatures.

We show \(\sigma_{xy}(T)/\sigma_{xy}(D)\) with \(\sigma_{xy}(D) = \frac{e^2}{2\pi^2} c_z(D)\) in Fig. 1 for clarity of physics. Interestingly, it has been also shown that Eq. (13) is not modified even in the presence of a finite chemical potential, based on the Kubo formula [15]. It will be quite interesting to reveal the temperature dependence for the anomalous Hall coefficient in the presence of Fermi surfaces near two paired Weyl points.

![FIG. 1: Temperature dependence of the anomalous contribution for the Hall effect in an interacting Weyl metal with zero chemical potential. We plot a dimensionless anomalous Hall coefficient \(\sigma_{xy}(T)/\sigma_{xy}(D) = 1 + \frac{c_z^2}{4\pi^2} \ln \frac{D}{T}\) as a function of temperature \(T\) of a unit of \(K\), using \(D = 10^4 K\) and \(\frac{c_z^2}{4\pi^2} = 1/137\). An essential feature is the logarithmic enhancement of the anomalous Hall coefficient as a result of the interplay between transverse long-range interactions and the chiral anomaly.](image)
IV. CONCLUSION AND PERSPECTIVES

In summary, we investigated the quantum critical point of the topological phase transition from a topological insulator to a band insulator in the presence of inhomogeneous ferromagnetism or under nonuniform magnetic fields, where the topological $\theta$–term gives rise to a topological constraint in dynamics of bulk fermions, referred to as the Adler-Bell-Jackiw anomaly. Such inhomogeneous magnetic fields serve background chiral gauge fields, splitting the Dirac point into two Weyl points. Introducing long-range transverse interactions and performing the renormalization group analysis, we uncovered that the distance between these two Weyl points becomes enhanced logarithmically at low temperatures although the coupling constant vanishes as expected. Resorting to the semi-classical quantum Boltzmann equation approach, we claimed that the enhancement of the distance strengthens metallic properties at low temperatures when electric fields are applied to the same direction as the momentum to connect these Weyl points because scattering between the Weyl points are suppressed due to their huge distance in the momentum space. Besides this emergent enhanced anisotropy in electrical resistivity, we predicted the logarithmically “divergent” temperature dependence for the anomalous contribution of the Hall effect. These two anomalous transport properties are proposed to be fingerprints of Weyl metals with transverse long-range interactions.

There remain three important problems in our direction. The first question is to perform the renormalization group analysis in the case of a finite chemical potential, as discussed before. Since the competition between the enhancement of the distance between two paired Weyl points and the presence of perfect nesting between two paired Fermi surfaces is expected to cause a delicate balance, we are expecting an interacting fixed point, which can be identified with a novel non-Fermi liquid metal. The second question is what happens if we take into account chiral gauge fields quantum mechanically. This situation arises when ferromagnetic phase transitions occur near the topological phase transition [17]. Is it possible to obtain a novel interacting fixed point, too? The third question is more practical thus experimentally verified. If we introduce weak anti-localization corrections in the transport theory, how is the longitudinal transport coefficient modified? This question is still meaningful even without interactions because this transport signature can be measured actually [25].

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Appendix A: Renormalization group analysis in the presence of chiral anomaly

1. Introduction of counter-terms

We start from the following Lagrangian

$$
\mathcal{L}_B = i\bar{\psi}_B \gamma^\mu \partial_\mu \psi_B - e_B A_{B\mu} \bar{\psi}_B \gamma^\mu \psi_B - \frac{1}{4} F_{B\mu
u} F^{\mu\nu} + \frac{e_B^2 \theta_B}{16 \pi^2} \epsilon^{\mu\nu\rho\sigma} F_{B\mu\nu} F_{\rho\sigma},
$$

where $\psi_B$, $A_{B\mu}$, $e_B$, $\theta_B$ are bare quantities to be renormalized. Introducing renormalization factors of $\psi_B = Z_{B/\psi} \psi$, $A_{B\mu} = Z_{A/\psi} A_\mu$, $e_B Z_{Z/\psi} Z_\psi = Z_e e$, $\theta_B = Z_\theta \theta$, one can rewrite the above bare Lagrangian in terms of its renormalized part and counter-term part,

$$
\mathcal{L}_B = \mathcal{L}_r + \mathcal{L}_{c.t.},
$$

$$
\mathcal{L}_r = i\bar{\psi} \gamma^\mu \partial_\mu \psi - e A_\mu \bar{\psi} \gamma^\mu \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{e^2 \theta}{16 \pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma},
$$

$$
\mathcal{L}_{c.t.} = \delta_\psi i\bar{\psi} \gamma^\mu \partial_\mu \psi - \delta_e e A_\mu \bar{\psi} \gamma^\mu \psi - \frac{\delta A}{4} F_{\mu\nu} F^{\mu\nu} + \delta_\theta \frac{e^2 \theta}{16 \pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma},
$$

where $\delta_\psi = Z_{\psi} - 1$, $\delta_e = Z_e - 1$, $\delta_A = Z_{A} - 1$, $\delta_\theta = Z_{\theta} - 1$.

Incorporating the anomaly equation [Eq. (2)] for renormalized fields into the above expression, we obtain Eq. (4) for our renormalization group analysis. This procedure can be performed in a more formal way. This Lagrangian functional is invariant under the chiral transformation of $\psi \rightarrow e^{i\alpha(x)\gamma_5} \psi$ as long as fermions remain massless. Consider the following replacement

$$
Z = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left\{ i \int d^4 x \mathcal{L}_r + \mathcal{L}_{c.t.} \right\},
$$

$$
\rightarrow Z = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left\{ i \int d^4 x \mathcal{L}_r + \mathcal{L}_{c.t.} + \alpha(x) \left( \partial_\mu J^{\gamma\mu} + \frac{e^2}{16 \pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \right) \right\}.
$$
where gauge fixing is assumed. \( \alpha(x) \) is an arbitrary and infinitesimal local parameter. Taking \( \alpha(x) = -\theta(x) \), we see that the \( \theta F \bar{F} \) term is replaced with the chiral gauge-field term in Eq. (4). As a result, we obtain the following expression

\[
Z = \int DAD\bar{\psi}D\psi \exp \left\{ i \int d^4x \mathcal{L}_1 + \mathcal{L}_{1.c.t.} \right\}
\]

\[
\mathcal{L}_1 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\psi} \gamma^\mu (\partial_\mu - i c_\mu \gamma_5) \psi - e A_\mu \bar{\psi} \gamma^\mu \psi
\]

\[
\mathcal{L}_{1.c.t.} = -\frac{\delta_A}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\psi} \gamma^\mu (\partial_\mu - i \delta_\mu \gamma_5) \psi - \delta_e e A_\mu \bar{\psi} \gamma^\mu \psi,
\]

where gauge fixing is also assumed. It is important to notice that the chiral anomaly equation is satisfied for renormalized fields, not bare fields.

2. One-loop structure of QED\(_4\) with a background chiral gauge field

We perform one-loop renormalization group analysis in the presence of the background chiral gauge field, i.e., \( c_\mu = \text{constant} \). We obtain renormalization constants from \( \Sigma_1(p, c), \Gamma_1^\nu(p, p', c), \Pi_1^\mu(q, c) \), corresponding to one-loop fermion self-energy, one-loop vertex correction, and one-loop gauge-boson self-energy, respectively.

It is important to notice that the background chiral gauge field is taken into account non-perturbatively. In other words, our vacuum state is a state that chiral currents are flowing. Thus, the fermion propagator is modified to be

\[
\frac{1}{\not{\!p}} \to \frac{1}{\not{\!p} + \gamma_5} = \frac{(p^2 + c^2) \not{\!p} + 2(p \cdot c) \not{\!\gamma}_5 - (p^2 + c^2) \not{\!\gamma}_5 - 2(p \cdot c) \not{\!\gamma}_5}{(p - c)^2(p + c)^2}.
\]

(A5)

In this respect the key point is how self-energies of fermions and gauge bosons and vertex corrections are modified in this novel vacuum state. A subtle point arises due to the presence of the \( \gamma_5 \) matrix in the regularization procedure [1]. When dimensional regularization is used to regularize loop-integrals including \( \gamma_5 \), some anomalous terms appear. They originate from components perpendicular to physical four-dimensions. We separate out these perpendicular momentum components of \( l_\perp = l^\mu - l^\mu \) explicitly in our dimensional regularization. However, it turns out that they do not result in divergent contributions.

First, we calculate the fermion self-energy

\[
-i \Sigma_1(p, c) = (-ie)^2 \int \frac{d^d k}{(2\pi)^d} \gamma^\mu \frac{i(k^2 + c^2) \not{\!k} - 2(k \cdot c) \not{\!\gamma}_5(k \cdot c) \not{\!\gamma}_5 - (k^2 + c^2) \not{\!\gamma}_5 \gamma^\mu - i\eta_{\mu\nu}}{(k + c)^2(k - c)^2}
\]

\[
= -e^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{(p - k)^2(k - c)^2(k + c)^2} \left\{ (k^2 + c^2) \gamma^\mu \gamma_5 k_{\mu} - 2(k \cdot c) \gamma^\mu \not{\!\gamma}_5 k_{\mu} + 2(k \cdot c) \gamma^\mu \not{\!\gamma}_5 k_{\mu} - (k^2 + c^2) \gamma^\mu \not{\!\gamma}_5 k_{\mu} \right\}.
\]

(A6)

Replacing the denominator with Feynman parameters, we rewrite the above expression as follows

\[
-i \Sigma_1(p, c) = -2e^2 \int_0^1 dx dy dz \delta(x + y + z - 1) \int \frac{d^d l}{(2\pi)^d} \frac{1}{|l^2 - \Delta|^3}
\]

\[
\times \left\{ -2((l + a)^2 + c^2)(I + \not{\!\gamma}) + 4(l \cdot c + a \cdot c) \not{\!\gamma}_5 + 4(l \cdot c + a \cdot c)(I + \not{\!\gamma}) \gamma_5 - 2((l + a)^2 + c^2) \not{\!\gamma}_5 \right\},
\]

(A7)

where \( l_\mu = k_\mu - a_\mu, \ a_\mu = xp_\mu + (y - z)c_\mu, \ \Delta = (x^2 - x)p^2 + (y^2 - y + z^2 - z - 2yz)c^2 + (2xy - 2xz)(p \cdot c) \). Note that the degree of divergence in each term depends on only the power of redefined loop momenta \( l \). Since we need to calculate only divergent terms, we consider

\[
-i \Sigma_1(p, c) = -2e^2 \int_0^1 dx dy dz \delta(x + y + z - 1) \int \frac{d^d l}{(2\pi)^d} \frac{1}{|l^2 - \Delta|^3}
\]

\[
\times \left\{ -2l_\mu^2 \not{\!\gamma} + 4l_\mu \not{\!\gamma}_5 a_\lambda \gamma_5 + 4l_\mu \not{\!\gamma}_5 c_\lambda \gamma_5 \gamma_5 - 2l_\mu \not{\!\gamma}_5 \right\} + \text{finite}.
\]

\[
= -2e^2 \int_0^1 dx dy dz \delta(x + y + z - 1) \times \left\{ -2(\not{\!\gamma} + \not{\!\gamma}_5) \left( \int \frac{d^d l}{(2\pi)^d} \frac{l^2}{|l^2 - \Delta|^3} \right) - 4(a_\lambda \gamma_5 - c_\lambda \gamma_5 \gamma_5) \left( \int \frac{d^d l}{(2\pi)^d} \frac{l^3 l_\mu}{|l^2 - \Delta|^3} \right) \right\} + \text{finite}.
\]

(A8)
Integrating over the loop momenta \( l \), we obtain

\[
-i \Sigma_1(\phi, c) = \frac{e^2 i}{n_\epsilon} \int_0^1 dxdydz \, \delta(x + y + z - 1) \left\{ \frac{3}{4} \phi + \frac{1}{4} \phi \gamma_5 \right\} + \text{finite}
\]

\[
= \frac{e^2 i}{8 \pi^2 \epsilon} (\phi + \phi \gamma_5) + \text{finite}
\]

with \( \epsilon = 4 - d \).

Second, we calculate the vertex correction in the same way as the above

\[
\Gamma_\mu(p, p', c) = (-ie)^2 \int \frac{d^4k}{(2\pi)^d} \left\{ \gamma^\nu \left\{ \gamma^\mu_i (k^2 + c^2) \gamma^\nu_i (k^2 - 2k' \cdot c) \gamma^\nu_i (k^2 - c^2) \gamma^\nu_i \right\} + \text{finite} \right\}
\]

\[
\implies \Gamma_\mu(p, p', c) = -ie^2 \int \frac{d^4k}{(2\pi)^d} \left( \frac{1}{(k^2 - \Delta)^3} \right) \left\{ \gamma^\lambda \gamma^\mu \gamma^\nu \right\} \gamma^\lambda \gamma^\mu \gamma^\nu + \text{finite},
\]

where \( p^2 = p'^2 = m_\gamma^2 = 0 \), \( k_{\mu} + q_{\mu} = k_{\mu}' \), \( p_{\mu} + q_{\mu} = p_{\mu}' \). Similarly, replacing the denominator with Feynman parameters and taking only divergent terms, we obtain

\[
\Gamma_\mu(p, p', c) = -24ie^2 \int_0^1 dxdydzdudv \, \delta(x + y + z + u + v - 1) \int d^4k \frac{1}{(2\pi)^d} \left( \frac{1}{(k^2 - \Delta)^3} \right) \left\{ \gamma^\lambda \gamma^\mu \gamma^\nu \right\} \gamma^\lambda \gamma^\mu \gamma^\nu + \text{finite},
\]

where \( \Delta = k_{\mu} - a_{\mu}, \ a_{\mu} = xp_{\mu} - (y + z)q_{\mu} + (y - z + u - v)c_{\mu} \).

Third, we evaluate the vacuum polarization tensor. Although the procedure is essentially the same as before, this calculation is much more complicated, given by

\[
i \Pi_{\mu\nu}^\mu(q, c) = -(ie)^2 \int \frac{d^4k}{(2\pi)^d} \text{Tr} \left[ \gamma^\mu_i (k^2 + c^2) \gamma^\nu_i (k^2 - 2k' \cdot c) \gamma^\nu_i (k^2 + c^2) \gamma^\nu_i \right]
\]

\[
\times \Gamma^\nu_\lambda \gamma^\nu_\lambda \gamma^\nu_\mu + \text{finite},
\]

Notice that the extra \((-1)\) factor comes from the fermion loop in the diagram. Now, we have to change the loop
moments as before. Straightforward but rather tedious algebras give us the following expression

\[ i\Pi_1^{\mu\nu}(q, c) = -6e^2 \int_0^1 dx dy dz \, \delta(x + y + z - 1) \int \frac{d^4l}{(2\pi)^4} \frac{1}{|l^2 - \Delta|^4} \times \]

\[ \times \left[ (l^2)^3 \{-4\eta_{\mu\nu} + 8(l^2)^2\eta_{\mu\nu} \\
+ (l^2)^2 \{4a^\mu b^\nu + 4a^\nu b^\mu - 4\eta_{\mu\nu}(a \cdot b) - 4ie^{\mu\nu\rho\sigma}c_\rho q_\sigma + 8c^\mu c^\nu - 12\eta_{\mu\nu}c^2 - 4\eta_{\mu\nu}(a^2 + b^2)\} \\
+ l^2\eta_\lambda \cdot \eta_\gamma \{8\eta_\lambda h^\nu - 32c_\lambda c^\nu \}\right] + \text{finite.} \]

\[ \text{(A13)} \]

where

\[ l_\mu = k_\mu - a_\mu, \]
\[ a_\mu = Ac_\mu + Bq_\mu = (x - y + z - u)c_\mu + (u + z)q_\mu, \]
\[ b_\mu = a_\mu + q_\mu = Ac_\mu + (B + 1)q_\mu, \]
\[ h_\mu = a_\mu + b_\mu = 2Ac_\mu + (2B + 1)q_\mu, \]
\[ \Delta = l^2 - [x(k - c)^2 + y(k + c)^2 + z(k + q - c)^2 + u(k + q + c)^2] \]
\[ = 2(c \cdot q)(u^2 - u - xu + yu + z - xz + yz - z^2) + q^2(u^2 - u - z^2 + z + 2uz) \]
\[ + q^2(x^2 + y^2 + z^2 + u^2 - x - y - z - u - 2xy + 2xz - 2yz - 2xu + 2yu - 2zu). \]

\[ \text{(A14)} \]

Integrating over the loop momenta \( l \), we get

\[ i\Pi_1^{\mu\nu}(q, c) = -\frac{6e^2i}{\pi^2\epsilon} \int_0^1 dx dy dz \, \delta(x + y + z - 1) \times \left\{ -\eta_{\mu\nu}\Delta + c^\nu c^\sigma \left( \frac{10}{3} A^2 - \frac{2}{3} \right) + \eta^{\mu\nu}c^2 \left( -\frac{7}{3} A^2 - \frac{1}{3} \right) + q^\nu q^\sigma \left( \frac{10}{3} B^2 + \frac{10}{3} B + \frac{1}{2} \right) + \eta^{\mu\nu}q^2 \left( -\frac{7}{3} B^2 - \frac{7}{3} B - \frac{1}{2} \right) \right\} + \text{finite} \]

\[ = -\frac{e^2i}{6\pi^2\epsilon} (\eta^{\mu\nu}q^2 - q^\nu q^\sigma) + \text{finite.} \]

\[ \text{(A15)} \]

It is interesting to observe that the divergent part is not modified by the background chiral gauge field although it makes the expression much complicated in the intermediate stage. Contributions from the background chiral gauge field turn out to be cancelled exactly in the polarization contribution. It is quite laborious to check this cancellation.

We summarize leading divergent contributions of fermion one-loop self-energy, one-loop gauge-fermion vertex, and gauge-boson one-loop self-energy as follows

\[ -i\Sigma_1(p, c) = \frac{e^2i}{8\pi^2\epsilon} p + \frac{e^2i}{8\pi^2\epsilon} \gamma_5 + \text{finite}, \]

\[ \Gamma_1^\mu(p, p', c) = \frac{e^2}{8\pi^2\epsilon} \gamma^\mu + \text{finite}, \]

\[ i\Pi_1^{\mu\nu}(q, c) = -\frac{e^2i}{6\pi^2\epsilon} (\eta^{\mu\nu}q^2 - q^\mu q^\nu) + \text{finite.} \]

\[ \text{(A16)-(A18)} \]

As a result, we obtain

\[ \delta_\phi = -\frac{e^2}{8\pi^2\epsilon} + \text{finite}, \]

\[ \delta_\epsilon = -\frac{e^2}{8\pi^2\epsilon} + \text{finite}, \]

\[ \delta_c = -\frac{e^2}{8\pi^2\epsilon} + \text{finite}, \]

\[ \delta_A = -\frac{e^2}{6\pi^2\epsilon} + \text{finite}. \]

\[ \text{(A19)-(A22)} \]
3. Renormalization group equations

We would like to point out that the Ward identity of $Z_\psi = Z_e$ is satisfied, guaranteeing the gauge invariance. We emphasize that this serves quite a nontrivial check for our renormalization group analysis, where complex dependencies for chiral gauge fields are all cancelled to give rise to the Ward identity. Recalling our perturbative analysis in the one-loop level, the satisfaction of the Ward identity implies that the renormalization group equation for the coupling constant does not change, compared with the case in the absence of the background chiral gauge field. On the other hand, the renormalization group equation for the chiral gauge field is an essential point of our study.

The beta function for the chiral gauge field is given by

$$\beta_{c_\nu}(\mu) = \mu \frac{dc_\nu}{d\mu}. \quad (A23)$$

Considering that the bare quantity $c_{B\nu} = Z_e c_\nu$ is independent of the scale parameter of $\mu$, we obtain the renormalization group equation for the chiral gauge field

$$0 = \frac{d}{d\ln \mu} \ln c_{B\nu} = \frac{dM_e}{de} \frac{de}{d\ln \mu} + \frac{d}{d\ln \mu} \ln c_\nu, \quad (A24)$$

where

$$M_e = \ln Z_e = \sum_{n=1}^{\infty} \frac{m_n(e,c)}{\epsilon^n} = -\frac{\epsilon^2}{\epsilon} + O(\epsilon^4) + O(\frac{1}{\epsilon^2}). \quad (A25)$$

In the one-loop level we obtain $m_1(e,c) = -\frac{\epsilon^2}{8\pi^2}$. Inserting $\frac{de}{d\ln \mu} = \beta_e(\mu) - \epsilon e$ into the above expression, we reach the following formula

$$0 = \left( -\frac{\epsilon}{4\pi^2} + O(\epsilon^3) \right) \frac{1}{\epsilon} + O\left( \frac{1}{\epsilon^2} \right) (\beta_e(\mu) - \epsilon e) + \frac{1}{c_\nu} \beta_{c_\nu}(\mu). \quad (A26)$$

Renormalizability guarantees the cancellation in higher negative orders. As a result, we obtain

$$\beta_{c_\nu}(\mu) = \mu \frac{dc_\nu}{d\mu} = -\frac{\epsilon^2}{4\pi^2} c_\nu + O(\epsilon^4). \quad (A27)$$

4. Low temperature behaviors for background chiral gauge fields

Solving the renormalization group equation for the coupling constant

$$\beta_e(\mu) = \frac{de}{d\ln \mu} = \frac{\epsilon^3}{12\pi^2}, \quad (A28)$$

we obtain

$$\epsilon^2(\mu) = \frac{\epsilon^2_D}{1 - \frac{\epsilon^2}{4\pi^2} \ln \left( \frac{\mu}{D} \right)}. \quad (A29)$$

Substituting this solution into the renormalization group equation for the chiral gauge field

$$\beta_{c_\nu}(\mu) = \mu \frac{dc_\nu}{d\mu} = -\frac{\epsilon^2}{4\pi^2} c_\nu, \quad (A30)$$

we obtain

$$\ln \left( \frac{c_\nu(\mu)}{c_\nu(D)} \right) = -\frac{1}{4\pi^2} \int_D^\mu d(\ln \mu) \frac{\epsilon^2_D}{1 - \frac{\epsilon^2}{4\pi^2} \ln \left( \frac{\mu}{D} \right)}$$

$$= -\int_{\ln D}^{\ln \mu} dx \frac{1}{\alpha^{-1}_D + \ln D} - x$$

$$= \int_0^{\ln \frac{\mu}{D}} dx \frac{1}{x - \alpha^{-1}_D}$$

$$= \ln \left| \frac{\ln \frac{\mu}{D} - \alpha^{-1}_D}{\alpha^{-1}_D} \right|, \quad (A31)$$
which consists of the equilibrium part (the first term) and its correction term (the second term). We call \( \Lambda(p, \omega) \) the "vertex distribution function" although it sounds somewhat confusing.

Inserting the solutions [Eq. (11)] of semi-classical equations [Eq. (10)] into the quantum Boltzmann equation [Eq. (9)] and performing some algebra, we obtain the following expression

\[
\begin{align*}
\left(1 + \frac{e}{c} \mathbf{B} \cdot \Omega_p\right)^{-1} \frac{e}{c} v_p \cdot \left(\mathbf{B} \times \frac{\partial G^<}{\partial p}\right) & + \left(1 + \frac{e}{c} \mathbf{B} \cdot \Omega_p\right)^{-2} \left(\frac{e}{c} v_p \times \mathbf{B}\right) \cdot (e \mathbf{E} \times \Omega_p) \frac{\partial G^<}{\partial \omega} \\
-\left[A(p, \omega)\right]^2 \left(-\frac{\partial f(\omega)}{\partial \omega}\right) & + \frac{e}{c} \mathbf{B} \cdot \Omega_p \left\{e \mathbf{E} + \frac{e^2}{c}(\mathbf{E} \cdot \mathbf{B})\Omega_p\right\} \cdot \left\{v_p + \frac{e}{c}(\Omega_p \cdot v_p)\mathbf{B}\right\} \Gamma(p, \omega) \\
& = -i[2\Gamma G^< - \Sigma^< A],
\end{align*}
\]

where the argument of \((p, \omega)\) is omitted for simplicity. The lesser self-energy is given by \cite{[21]}

\[
\Sigma^<(p, \omega) = \sum_q \int_0^\infty d\nu \left\{\frac{p \times \mathbf{q}}{m}\right\}^2 D_a(q, \nu) \left\{[n(\nu) + 1]G^<(p + q, \omega + \nu) + n(\nu)G^<(p + q, \omega - \nu)\right\}.
\]

Here, we consider gauge interactions for example. Thus, \(D_a(q, \nu)\) represents the Green function of gauge fluctuations. \(n(\nu)\) is the Bose-Einstein distribution function. One can replace the gauge-boson propagator with some other types of fluctuations such as phonons, spin fluctuations, and etc. One may consider the diffusion-mode propagator for weak anti-localization, where the form of its vertex should be changed, of course.

We write down the lesser Green’s function in the following way \cite{[21]}

\[
G^<(p, \omega) = if(\omega)A(p, \omega) + i\left(-\frac{\partial f(\omega)}{\partial \omega}\right)A(p, \omega)v_p \cdot \Lambda(p, \omega),
\]

which consists of the equilibrium part (the first term) and its correction term (the second term). We call \(\Lambda(p, \omega)\) “vertex distribution function” although it sounds somewhat confusing. \(f(\omega)\) is the Fermi-Dirac distribution function.

Inserting this ansatz into the quantum Boltzmann equation with the expression of the lesser self-energy and performing some straightforward algebra, we obtain

\[
\begin{align*}
& i\left(1 + \frac{e}{c} \mathbf{B} \cdot \Omega_p\right)^{-1} \frac{e}{mc} v_p \cdot \left(\mathbf{B} \times \frac{\partial \Lambda}{\partial p}\right) \Lambda(p, \omega) - i\left(1 + \frac{e}{c} \mathbf{B} \cdot \Omega_p\right)^{-2} \left(\frac{e}{c} v_p \times \mathbf{B}\right) \cdot (e \mathbf{E} \times \Omega_p) \\
& -\left[A(p, \omega)\right]^2 \left(-\frac{\partial f(\omega)}{\partial \omega}\right) + \frac{e}{c} \mathbf{B} \cdot \Omega_p \left\{e \mathbf{E} + \frac{e^2}{c}(\mathbf{E} \cdot \mathbf{B})\Omega_p\right\} \cdot \left\{v_p + \frac{e}{c}(\Omega_p \cdot v_p)\mathbf{B}\right\} \Gamma(p, \omega) \\
& = 2\Gamma(p, \omega)v_p \cdot \Lambda(p, \omega) - \sum_q \int_0^\infty d\nu \left\{\frac{p \times \mathbf{q}}{m}\right\}^2 D_a(q, \nu) \left\{[n(\nu) + f(\omega + \nu)]A(p + q, \omega + \nu)v_{p+q} \cdot \Lambda(p + q, \omega + \nu) \\
& -[n(\nu) + f(\omega - \nu)]A(p + q, \omega - \nu)v_{p+q} \cdot \Lambda(p + q, \omega - \nu)\right\},
\end{align*}
\]

where we have used the following relation

\[
2\Gamma(p, \omega) = \sum_q \int_0^\infty d\nu \left\{\frac{p \times \mathbf{q}}{m}\right\}^2 D_a(q, \nu) \left\{[n(\nu) + f(\omega + \nu)]A(p + q, \omega + \nu) - [n(\nu) + f(\omega - \nu)]A(p + q, \omega - \nu)\right\}.
\]
Writing down the quantum Boltzmann equation in terms of components and focusing on dynamics near the Fermi surface, we reach the following expression for each component,

$$\frac{\Lambda^p_F(\omega)}{\tau_{tr}(\omega)} + \left(1 + \frac{e}{c} B \cdot \Omega_F\right)^{-1} \frac{e B_y}{mc} \Lambda^p_F(\omega) - i \left(1 + \frac{e}{c} B \cdot \Omega_F\right)^{-1} \frac{e B_y}{mc} \Lambda^p_F(\omega)
$$

$$= -i \left(1 + \frac{e}{c} B \cdot \Omega_F\right)^{-2} \left\{ \frac{e}{c} B_y (eE \times \Omega_F)_y - \frac{e}{c} B_z (eE \times \Omega_F)_z \right\}
$$

$$- A(p_F, \omega) \Gamma(p_F,\omega) \left(1 + \frac{e}{c} B \cdot \Omega_F\right)^{-2} \left\{ \frac{e E_y + \frac{e^2}{c} (E \cdot B) \Omega^y_F + \frac{e^2}{c} \left(1 + \frac{e}{c} B \cdot \Omega_F\right) (E \cdot B) \Omega^y_F \right\},$$

(B6)

and

$$\frac{\Lambda^y_F(\omega)}{\tau_{tr}(\omega)} - i \left(1 + \frac{e}{c} B \cdot \Omega_F\right)^{-1} \frac{e B_z}{mc} \Lambda^y_F(\omega) - i \left(1 + \frac{e}{c} B \cdot \Omega_F\right)^{-1} \frac{e B_z}{mc} \Lambda^y_F(\omega)
$$

$$= -i \left(1 + \frac{e}{c} B \cdot \Omega_F\right)^{-2} \left\{ -\frac{e}{c} B_y (eE \times \Omega_F)_x + \frac{e}{c} B_z (eE \times \Omega_F)_y \right\}
$$

$$- A(p_F, \omega) \Gamma(p_F,\omega) \left(1 + \frac{e}{c} B \cdot \Omega_F\right)^{-2} \left\{ \frac{e E_x + \frac{e^2}{c} (E \cdot B) \Omega^x_F + \frac{e^2}{c} \left(1 + \frac{e}{c} B \cdot \Omega_F\right) (E \cdot B) \Omega^x_F \right\},$$

(B7)

where the transport time is given by

$$\frac{1}{\tau_{tr}(\omega)} = \sum_q \int_0^\infty d\nu \left| \frac{p_e \times \hat{q}}{m} \right|^2 \Im D_a(q, \nu)(1 - \cos \theta) \left\{ [n(\nu) + f(\nu + \pi)] A(p_F + q, \omega + \nu) - [n(-\nu) + f(\omega - \nu)] A(p_F + q, \omega - \nu) \right\}.$$  

(B9)

We note the $1 - \cos \theta$ factor in this expression, which extracts out back scattering contributions.

### 2. Current formulation

It is natural to define a current in the following way [12]

$$J = -e \frac{1}{\beta} \sum_{i\omega} \int \frac{d^3p}{(2\pi\hbar)^3} \left(1 + \frac{e}{c} B \cdot \Omega_p\right)^{-1} \left\{ v_p + eE \times \Omega_p + \frac{e}{c} (\Omega_p \cdot v_p) B \right\} [-iG^c(p, i\omega)].$$

(B10)

We note the $\dot{v}$ term in the integral expression.

Inserting the ansatz for the lesser Green’s function into the above expression, we obtain

$$J = -e \frac{1}{\beta} \sum_{i\omega} \int \frac{d^3p}{(2\pi\hbar)^3} \left(1 + \frac{e}{c} B \cdot \Omega_p\right)^{-1} (E \times \Omega_p) f(\omega) A(p, \omega)
$$

$$- e \frac{1}{\beta} \sum_{i\omega} \int \frac{d^3p}{(2\pi\hbar)^3} \left(1 + \frac{e}{c} B \cdot \Omega_p\right)^{-1} \left\{ v_p + \frac{e}{c} (\Omega_p \cdot v_p) B \right\} \left( \frac{\partial f(\omega)}{\partial \omega} \right) A(p, \omega) v_p \cdot \Lambda(p, \omega).$$

(B11)
Then, each component is given by

\[ J_z = -e^2 \frac{1}{\beta} \sum_{\omega} \int \frac{d^3 p}{(2\pi \hbar)^3} \left( 1 + \frac{e}{c} B \cdot \Omega_p \right)^{-1} \left( E_y \Omega_p^z - E_z \Omega_p^y \right) f(\omega) A(p, \omega) \]

\[ - \frac{1}{\beta} \sum_{\omega} \int \frac{d^3 p}{(2\pi \hbar)^3} \left( 1 + \frac{e}{c} B \cdot \Omega_p \right)^{-1} (v_p^y)^2 \left( -\frac{\partial f(\omega)}{\partial \omega} \right) A(p, \omega) \Lambda_x(p, \omega) \]

\[ - \frac{1}{\beta} \sum_{\omega} \int \frac{d^3 p}{(2\pi \hbar)^3} \left( 1 + \frac{e}{c} B \cdot \Omega_p \right)^{-1} \frac{e}{c} B_z \left\{ (v_p^y)^2 \Omega_p^x \Lambda_x(p, \omega) + (v_p^y)^2 \Omega_p^y \Lambda_y(p, \omega) + (v_p^y)^2 \Omega_p^z \Lambda_z(p, \omega) \right\} \left( -\frac{\partial f(\omega)}{\partial \omega} \right) A(p, \omega) \]

(B12)

and

\[ J_y = e^2 \frac{1}{\beta} \sum_{\omega} \int \frac{d^3 p}{(2\pi \hbar)^3} \left( 1 + \frac{e}{c} B \cdot \Omega_p \right)^{-1} \left( E_z \Omega_p^y - E_x \Omega_p^z \right) f(\omega) A(p, \omega) \]

\[ + \frac{1}{\beta} \sum_{\omega} \int \frac{d^3 p}{(2\pi \hbar)^3} \left( 1 + \frac{e}{c} B \cdot \Omega_p \right)^{-1} (v_p^x)^2 \left( -\frac{\partial f(\omega)}{\partial \omega} \right) A(p, \omega) \Lambda_y(p, \omega) \]

\[ + \frac{1}{\beta} \sum_{\omega} \int \frac{d^3 p}{(2\pi \hbar)^3} \left( 1 + \frac{e}{c} B \cdot \Omega_p \right)^{-1} \frac{e}{c} B_y \left\{ (v_p^x)^2 \Omega_p^y \Lambda_x(p, \omega) + (v_p^x)^2 \Omega_p^y \Lambda_y(p, \omega) + (v_p^x)^2 \Omega_p^z \Lambda_z(p, \omega) \right\} \left( -\frac{\partial f(\omega)}{\partial \omega} \right) A(p, \omega) \]

(B13)

3. Longitudinal magneto-transport

Solving the quantum Boltzmann equation in the unconventional setup of \( B = B_x \hat{x} \) and \( E = E_x \hat{x} \), we find

\[ \Lambda^x_F(\omega) = -e A(p_F, \omega) \frac{\tau_{tr}(\omega)}{\tau_{sc}(\omega)} E_x, \tag{B14} \]

\[ \Lambda^y_F(\omega) = mc \frac{\omega^x \tau_{tr}(\omega)}{(1 + \frac{e}{c} B_x \Omega^x_F)^2} \left( -\Omega^x_F + \Omega^y_F \frac{\omega^x \tau_{tr}(\omega)}{1 + \frac{e}{c} B_x \Omega^x_F} \right) E_x \]

\[ -A(p_F, \omega) \frac{\tau_{tr}(\omega)}{(1 + \frac{e}{c} B_x \Omega^x_F)^2} \left( -\Omega^y_F + \Omega^x_F \frac{\omega^x \tau_{tr}(\omega)}{1 + \frac{e}{c} B_x \Omega^x_F} \right) E_x B_z, \tag{B15} \]

and

\[ \Lambda^z_F(\omega) = mc \frac{\omega^x \tau_{tr}(\omega)}{(1 + \frac{e}{c} B_x \Omega^x_F)^2} \left( -\Omega^z_F + \Omega^x_F \frac{\omega^x \tau_{tr}(\omega)}{1 + \frac{e}{c} B_x \Omega^x_F} \right) E_x \]

\[ -A(p_F, \omega) \frac{\tau_{tr}(\omega)}{(1 + \frac{e}{c} B_x \Omega^x_F)^2} \left( -\Omega^z_F + \Omega^x_F \frac{\omega^x \tau_{tr}(\omega)}{1 + \frac{e}{c} B_x \Omega^x_F} \right) E_x B_z, \tag{B16} \]

where \( \omega^x = \frac{eB_x}{mc} \) is the “cyclotron” frequency associated with the \( B_x \) field. We notice that there are \( E \cdot B = E_x B_x \) terms, which are topological in their origin.

Inserting these vertex distribution functions into the current formula, we obtain a rather complicated expression...
for the $x$- component of the current,

$$J_x = e^2 \beta \sum_{\omega} \frac{d^3 p}{(2\pi\hbar)^3} \left( -\frac{\partial f(\omega)}{\partial \omega} \right) A(p_F, \omega) \frac{\tau_{tr}(\omega)}{\tau_{sc}(\omega)} E_x$$

$$+ e^3 \frac{1}{c \beta} \sum_{\omega} \frac{d^3 p}{(2\pi\hbar)^3} \left( -\frac{\partial f(\omega)}{\partial \omega} \right) A(p_F, \omega) \frac{\tau_{tr}(\omega)}{\tau_{sc}(\omega)} B_x E_x$$

$$- \frac{m e^3}{c} \sum_{\omega} \frac{d^3 p}{(2\pi\hbar)^3} \left( -\frac{\partial f(\omega)}{\partial \omega} \right) A(p_F, \omega) \frac{\tau_{tr}(\omega)}{\tau_{sc}(\omega)} B_x E_x$$

Expanding the above expression up to the second order for the Berry curvature and keeping only even-power contributions [12], we obtain

$$J_x \approx e^2 \beta \sum_{\omega} \frac{d^3 p}{(2\pi\hbar)^3} \left( -\frac{\partial f(\omega)}{\partial \omega} \right) A(p_F, \omega) \frac{\tau_{tr}(\omega)}{\tau_{sc}(\omega)} E_x$$

$$- 2 \frac{m e^3}{c} \sum_{\omega} \frac{d^3 p}{(2\pi\hbar)^3} \left( -\frac{\partial f(\omega)}{\partial \omega} \right) A(p_F, \omega) \frac{\tau_{tr}(\omega)}{\tau_{sc}(\omega)} B_x E_x$$

$$+ e^3 \frac{1}{c \beta} \sum_{\omega} \frac{d^3 p}{(2\pi\hbar)^3} \left( -\frac{\partial f(\omega)}{\partial \omega} \right) A(p_F, \omega) \frac{\tau_{tr}(\omega)}{\tau_{sc}(\omega)} B_x E_x$$

$$= CN_F e^2 v_F^2 \tau_{tr}(T) E_x + 2C' e^4 N_F v_F^2 \frac{\tau_{tr}(T)}{1 + [\omega_x^2 \tau_{tr}(T)]^2} B_x^2 E_x - 2C'' \frac{me^3}{c} N_F v_F^2 \frac{\tau_{sc}(T)}{1 + [\omega_x^2 \tau_{tr}(T)]^2} B_x E_x. \quad (B18)$$

The first term is also the conventional contribution near the Fermi surface, but there is no dependence for magnetic fields. This is certainly expected because the magnetic field is in the same direction as the electric field. On the other hand, the second contribution originates from the topological $E \cdot B$ term. The third term is also anomalous, which results from the Berry curvature but not from the $E \cdot B$ term.

4. Discussion

The longitudinal magnetoconductivity is

$$\sigma_L(B_x, T) = CN_F e^2 v_F^2 \tau_{tr}(T) + 2C' e^4 N_F v_F^2 \frac{\tau_{tr}(T)}{1 + [\omega_x^2 \tau_{tr}(T)]^2} B_x^2 - 2C'' \frac{me^3}{c} N_F v_F^2 \frac{\tau_{sc}(T)}{1 + [\omega_x^2 \tau_{tr}(T)]^2} B_x. \quad (B19)$$

If we limit our discussion on low magnetic fields, we are allowed to neglect the last contribution. Then, the above expression can be rewritten as follows

$$\sigma_L(B_x, T) = (1 + C_W B_x^2)\sigma_0(T), \quad (B20)$$
where $\sigma_n(T) = C N_F e^2 \nu_F^2 \tau_n(T)$ is the normal conductivity and $C_W = 2(C'/C)(c^2/c^2)$ is a positive constant.

Our proposal is to replace the $B_2^s$ term with $|c(T)|^2$, where $c(T)$ represents the distance between two Weyl points. Then, the final expression for the "longitudinal" conductivity becomes

$$\sigma_L(T) \rightarrow (1 + K|c(T)|^2)\sigma_n(T),$$

(B21)

where $K$ is a positive numerical constant and the normal conductivity is determined by intra-scattering events at one Weyl point.

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