Robust and structure exploiting optimization algorithms: An integral quadratic constraint approach

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Abstract. We consider the problem of analyzing and designing gradient-based discrete-time optimization algorithms for a class of unconstrained optimization problems having strongly convex objective functions with Lipschitz continuous gradient. By formulating the problem as a robustness analysis problem and making use of a suitable adaptation of the theory of integral quadratic constraints, we establish a framework that allows to analyze convergence rates and robustness properties of existing algorithms and enables the design of novel robust optimization algorithms with prespecified guarantees capable of exploiting additional structure in the objective function.

1. Introduction

Optimization algorithms are of key importance in science and engineering. First order, i.e., gradient-based algorithms, are an important subclass that have proved themselves in a range of applications. In recent years, such algorithms have regained interest since they are particularly suitable for large-scale optimization. While many variants of gradient-based optimization algorithms are known in the literature, no general framework for their analysis and design exists. Still, a lot of these algorithms fall into the class of Lur’e systems [20], i.e., a given or to be designed linear system in feedback with a nonlinearity, given by the gradient in optimization. Lur’e systems and the corresponding absolute stability problem are classical control problems leading to celebrated results such as the Popov or the circle criterion, which can be seen as a pioneering contribution to robust control theory. Quite astonishingly, while not being unknown [30], the apparent relation of systems and control theory, in particular absolute stability and robust control theory, to optimization algorithm analysis and design has not yet been exploited heavily. Only recently, several works relying on this systems theoretic view on optimization algorithms have been published [3, 17, 9, 38, 34, 23, 24, 10], partly also providing different approaches for convergence rate analysis using techniques from robust control. However, convergence rates are only one side of the coin; in several applications, e.g., in a data-based setting, also robustness with respect to various kinds of disturbances is a key issue and similar analysis tools have been developed [2, 26]. In addition, the problem of designing algorithms specifically tailored to classes of structured optimization problems has only been touched upon so far [8, 11, 18]. However, in situations where a class of optimization problems needs to be solved repeatedly online, as for example in model predictive control or reinforcement learning, well-performing algorithms are key.

The present paper addresses these two issues and contributes to the existing literature by providing a systematic framework to the analysis and design of robust and structure exploiting optimization algorithms. In particular, we address the following problem: Given a class of objective functions, design a gradient-based optimization algorithm with a guaranteed convergence rate that also fulfills certain $H_2$-performance specifications. We show that these performance specifications can be related to noise rejection properties of the algorithm such as the effect of additive gradient noise on the variance of the algorithm’s output. We further address the problem of how to incorporate possible structural properties of the class of objective functions in our framework, and how this can be exploited to design novel optimization algorithms superior to standard ones in terms of convergence rates. To this end, in the spirit of [23, 17], we reformulate this design problem.
as a robust controller synthesis problem and employ integral quadratic constraints theory. By building upon and extending these well-established results, we are able to provide a general framework for algorithm analysis and design. To validate the practical applicability and relevance, we provide several numerical results illustrating our methodology.

More specifically, our main contributions are as follows: We propose a class of gradient-based optimization algorithms and derive necessary and sufficient conditions for these algorithms to be capable of solving a class of optimization problems (Theorem 2), thereby providing the basis for the design of novel algorithm that generalize those known from the literature. Embedding the problem in the framework of robust control, we then derive convex analysis tools by means of linear matrix inequalities (LMIs), both in regard to convergence rates (Theorem 5) and robustness (Theorem 6). To this end, we provide a general procedure to obtain multipliers for exponential stability results from standard ones (Lemma 2) and utilize this to derive a class of multipliers generalizing those proposed in [5, 17, 15] (Theorem 4). We further provide convex synthesis conditions allowing to design novel algorithms with specified robustness properties (Theorem 7) and show how to additionally exploit structural characteristics of the objective function (Lemma 5).

2. Preliminaries

2.1. Notation

We let $\mathbb{N}$ denote the set of non-negative integers, $\mathbb{N}_{>0}$ the set of positive integers and denote by $\mathbb{Z}$ the set of all integers. We write $\mathbb{C}^p$, $p \in \mathbb{N}$, for the set of $p$-times continuously differentiable functions. For $n, m \in \mathbb{N}_{>0}$, we denote by $I_n \in \mathbb{R}^{n \times n}$ the $n \times n$ identity matrix, by $0_n \in \mathbb{R}^{n \times n}$ the $n \times n$ matrix of zeros and by $0_{n \times m} \in \mathbb{R}^{n \times m}$ the $n \times m$ matrix of zeros. Sometimes we omit the subscript if the dimensions are obvious. We let $\mathbf{1}$ denote the column vector with all entries being equal to one. For a square matrix $A$ we let $\text{tr}(A)$ denote the trace of $A$. For any two matrices $A_1, A_2$, we let $A_1 \otimes A_2$ denote their Kronecker product. If $A_1, A_2$ are square, symmetric and of the same dimension, we write $A_1 \preceq A_2$ ($A_1 \succeq A_2$) if $A_1 - A_2$ is negative definite (negative semi-definite) and $A_1 \succ A_2$ ($A_1 \succeq A_2$) if $A_1 - A_2$ is positive definite (positive semi-definite). In the same manner, we use the relations $<_\geq, \geq,$ for elementwise comparison. We further denote by $\mathbb{T} := \{ z \in \mathbb{C} \mid |z| = 1 \}$ the unit circle in the complex plane.

We let $\ell^p_\mathbb{R} = \{(q_k)_{k \in \mathbb{N}} \mid q_k \in \mathbb{R}^p \}$ denote the subspace of all one-sided sequences and denote by $\ell^p_\mathbb{N} = \{ q \in \ell^p_\mathbb{R} \mid \exists T \in \mathbb{N} \text{ s.t. } q_k = 0 \text{ for all } k > T \}$ the set of all finitely supported sequences therein. We let $\ell^p_2 \subset \ell^p_\mathbb{R}$ denote all square summable sequences in $\ell^p_\mathbb{R}$, i.e.,

$$\ell^p_2 = \{ q \in \ell^p_\mathbb{R} \mid \sum_{k=0}^\infty |q_k|^2 < \infty \}. \quad (1)$$

We sometimes omit the dimension of the signals and write $\ell^p_\mathbb{R} \equiv \bigcup_{n \in \mathbb{N}_{>0}} \ell^p_n$, $\ell^p_\mathbb{N} \equiv \bigcup_{n \in \mathbb{N}_{>0}} \ell^p_n$, $\ell^p_2 \equiv \bigcup_{n \in \mathbb{N}_{>0}} \ell^p_n$ for the collection of all sequences of the respective type. We denote the standard inner product by $(u, y) = \sum_{i=-\infty}^\infty u_i^* y_i$, where $u, y \in \ell^p_\mathbb{R}$ and the superscript $^*$ denotes transposition. We further let $\|y\|_{\ell^p} = \sqrt{(y, y)}$ denote the corresponding induced norm. If it exists, i.e., if it is well-defined, we denote by $\hat{q} = Z(q)$ the one-sided $z$-transform of a signal $q \in \ell^p_\mathbb{R}$ which is defined by

$$\hat{q}(z) = Z(q)(z) = \sum_{k=0}^\infty q_k z^{-k}, \quad (2)$$

where $\hat{q} : \mathbb{U} \to \mathbb{C}^p$ for some set $\mathbb{U} \subseteq \mathbb{C}$.

We say that an operator $\phi : \mathbb{U} \to \mathbb{V}$, $\mathbb{U}, \mathbb{V} \subseteq \ell^p_\mathbb{R}$, is bounded (on $\mathbb{U}$) if there exists $\beta \geq 0$ such that $\|\phi(y)\|_{\ell^p_2} \leq \beta\|y\|_{\ell^p_2}$ for all $y \in \mathbb{U}$. If $\mathbb{U} = \mathbb{V} = \ell^p_2$, we call the infimal $\beta$ that fulfills this inequality the $\ell^p_2$-gain of $\phi$. We further let $\mathcal{L}(\mathbb{U}, \mathbb{V})$ denote the set of operators mapping $\mathbb{U}$ to $\mathbb{V}$ that are bounded on $\mathbb{U}$. For two operators $G_1 : \ell^p_\mathbb{R} \to \ell^p_\mathbb{R}$, $G_2 : \ell^p_\mathbb{R} \to \ell^p_\mathbb{R}$, we let $G_2 \circ G_1$ denote the composition, i.e., $G_2 \circ G_1$ maps any input $u \in \ell^p_\mathbb{R}$ to $y_2 = G_2(G_1(u)) \in \ell^p_\mathbb{R}$. For linear operators $G_i$, we often omit the brackets and simply write $y_1 = G_1 u$ instead of $y_1 = G_1(u)$. We further denote by $\text{id} : \ell^p_\mathbb{R} \to \ell^p_\mathbb{R}$, $p \in \mathbb{N}_{>0}$, the identity operator.

We denote by $\mathcal{R}\mathcal{H}^{n \times m}_{\infty}$ the set of all real-rational and proper transfer matrices of dimension $n \times m$ having all poles in the open unit disk. Similarly, we denote by $\mathcal{R}\mathcal{P}^{n \times m}$ the set of all real-rational and proper transfer matrices of dimension $n \times m$ having no poles on the unit circle. For a square transfer matrix $G$ we write $G \prec_\mathbb{S} 0$ if the matrix $G(z) + G(z)^*$ is negative definite for all $z \in \mathbb{S} \subseteq \mathbb{C}$. For any transfer matrix $G \in \mathcal{R}\mathcal{CL}^{n \times m}_{\infty}$, we let $\phi_G$ denote the corresponding linear Toeplitz operator in the time domain acting on sequences in $\ell^p_\mathbb{R}$, i.e., $\phi_G$ fulfills $Z(\phi_G u) = G\hat{u}$ for any $u \in \ell^p_\mathbb{R}$. Note that when $\phi_G$ is interpreted as an infinite-dimensional matrix acting on $u$, then $\phi_G$ has a block Toeplitz structure, where the blocks are of size $n \times m$.

For a proper transfer matrix $G(z) = \frac{C(zI - A)^{-1}B + D}{\mathbb{1}}$ we write $G \sim (A, B, C, D)$ or

$$G \sim \begin{pmatrix} A & | & B \\ C & & D \end{pmatrix} \quad (3)$$

to indicate that $G$ admits the state-space realization $(A, B, C, D)$. 

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2.2. The class of objective functions

Throughout the paper we consider the following class of (strongly) convex objective functions with Lipschitz continuous gradient:

Definition 1 (Class $S_{m,L}$). A function $H : \mathbb{R}^p \to \mathbb{R}$ is said to belong to the class $S_{m,L}$, $L \geq m \geq 0$, if $H \in C^2$ and

$$\begin{align}
(\nabla H(x) - \nabla H(y))^\top (x - y) &\geq m \|x - y\|^2 \quad (4a) \\
\|\nabla H(x) - \nabla H(y)\| &\leq L \|x - y\| \quad (4b)
\end{align}$$

for all $x, y \in \mathbb{R}^p$.

In view of a lighter notation, we do not explicitly specify the dimension of the domain but tacitly assume it to be clear from the context. Note that if $m > 0$, then $H$ is strongly convex by $(4a)$ and $H$ has a unique minimum. The so-defined class of functions is well-known in the optimization literature, see, e.g., [28] for further properties and details.

2.3. Robust stability and performance via IQCs

In the following we briefly present some standard results on robust stability and performance analysis via integral quadratic constraints (IQC). For a more detailed treatment of the subject, we refer the reader to the classical paper [22] as well as [36] in a continuous-time setup. These results literally carry over to the discrete-time setup, see, e.g., [16, 14]. In the following, we consider feedback interconnections of the form (cf. Figure 1)

\begin{align}
y &= G_{yw} w + G_{yp} w_p + y_{in} \quad (5a) \\
y_p &= G_{yp} w + G_{yp} w_p \quad (5b) \\
w &= \Delta(y), \quad (5c)
\end{align}

consisting of a linear system with transfer matrix $G_{yw} \in \mathcal{RH}_{\infty}^{ny \times nw}$ in feedback with an uncertainty $\Delta : \ell^2_{y} \to \ell^2_{yw}$ and an additional performance channel from $w_p$ to $y_p$, where $G_{yp} w_p \in \mathcal{RH}_{\infty}^{wy_p \times nw}$, $G_{yp} w \in \mathcal{RH}_{\infty}^{wy_p \times nw}$ as well as $G_{yp} w_p \in \mathcal{RH}_{\infty}^{wy_p \times nw}$. Let $\Delta \subset \{\Delta \in \mathcal{L}(\ell^2_{y}, \ell^2_{yw}), \Delta \text{ causal}\}$ denote the set of uncertainties, i.e., $\Delta \in \Delta$. It is common to impose the following assumption:

Assumption 1. If $\Delta \in \Delta$ then $\tau \Delta \in \Delta$ for all $\tau \in [0,1]$.

In many cases, this assumption can be ensured by a proper redefinition of the uncertainty. In particular, this assumption will be met by the class of uncertainties we will consider in our problem setup. Our goal is to show robust stability of the feedback interconnection without the performance channel for all $\Delta \in \Delta$ and robust performance concerning the performance channel from $w_p$ to $y_p$ for all $\Delta \in \Delta$ with respect to some (integral) quadratic performance criterion. In the remainder of this section, we briefly repeat the basic definitions and results from robust control and IQC theory. We first give a proper definition of robust stability.

Definition 2 (Robust stability). The feedback interconnection (5) is said to be robustly stable against $\Delta$ if, for $w_p = 0$, it is well-posed and the $\ell_2$-gain of the map from $y_{in}$ to $y$ is bounded for all $\Delta \in \Delta$.

Therein, well-posedness of the feedback interconnection means that the map $q \mapsto (I - G_{yw} \Delta)q$ has a causal inverse for all $\Delta \in \Delta$. We next give the definition of an integral quadratic constraint (IQC).

Definition 3 (IQC). Let a so-called multiplier $\Pi : \mathcal{RL}_{\infty}^{(p_1+1) \times (p_1+1)}, p_1, p_2 \in \mathbb{N}_{>0}$, be given. We say that two signals $q_1 \in \ell^2_{p_1}, q_2 \in \ell^2_{p_2}$ satisfy the IQC defined by $\Pi$ if

\begin{equation}
\text{IQC}(\Pi, q_1, q_2) = \int_0^{2\pi} \left[ \hat{q}_1(e^{j\omega}) \right] \Pi(e^{j\omega}) \left[ \hat{q}_2(e^{j\omega}) \right] d\omega \geq 0.
\end{equation}

We further say that an operator $\Delta : \ell^2_{p_1} \to \ell^2_{p_2}$ satisfies the IQC defined by $\Pi$ if IQC($\Pi, q_1, \Delta(q_1)$) $\geq 0$ holds for all $q_1 \in \ell^2_{p_1}$.

Remark 1. By Parseval’s Theorem, the IQC (6) can equivalently be formulated in the time domain as

\begin{equation}
\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \Phi_\Pi \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \geq 0,
\end{equation}

where $\Phi_\Pi$ is the linear operator defined by $\Pi$ as introduced in Section 2.1.
In general, IQCs can be used to i) characterize the uncertain operator \( \Delta \) as well as to ii) describe performance criteria. For the case i), we try to find a set of suitable multipliers \( \Pi \) such that each \( \Delta \in \Delta \) satisfies the IQC defined by \( \Pi \) for all \( \Pi \in \Pi \). Similarly, for ii), we let \( q_1 = w_p, q_2 = y_p \) and specify a set of multipliers \( \Pi_p \subset R \mathcal{L}_\infty^{(n_p + n_p) \times (n_p + n_p)} \) that characterize the desired performance criterion imposed on the performance channel. The corresponding performance IQC is then given by
\[
\text{IQC} (-\Pi_p, w_p, y_p) \geq 0,
\]
where \( \Pi_p \in \Pi_p \). We then define robust performance of (5) as follows.

**Definition 4 (Robust performance).** We say that the feedback interconnection (5) achieves robust performance with respect to \( \Pi \) against \( \Delta \) if the feedback interconnection is robustly stable against \( \Delta \) and, for \( y_{in} = 0 \), (8) holds for all \( \Pi_p \in \Pi_p \) and all \( \Delta \in \Delta \).

In the following we limit ourselves to performance multipliers where the block corresponding to the quadratic terms in \( y_p \) is positive semi-definite, i.e., we assume
\[
\Pi_p \subset \left\{ \begin{array}{c}
\Pi_{p,11}, \Pi_{p,12} \\
\Pi_{p,22}
\end{array} \right\} \in R \mathcal{L}_\infty^{(n_p + n_p) \times (n_p + n_p)} \quad \mid \quad \Pi_{p,22} \succeq 0.
\]

This assumption is met by the most relevant performance criteria and, in particular, it holds for all performance criteria we consider in the present manuscript. We next state the well-known IQC stability theorem.

**Theorem 1.** Consider the feedback interconnection (5). Let some set of multipliers \( \Pi \subset R \mathcal{L}_\infty^{(n_p + n_p) \times (n_p + n_p)} \) as well as some set of performance multipliers \( \Pi_p \) with (9) be given. Assume that all poles of the open unit disk and assume \( \Delta \) fulfills Assumption 1. Suppose further that

1. the interconnection (5) is well-posed for all \( \Delta \in \Delta \);
2. each \( \Delta \in \Delta \) satisfies the IQC defined by \( \Pi \) for all \( \Pi \in \Pi \).

If there exist \( \Pi \in \Pi \) and \( \Pi_p \in \Pi_p \) such that
\[
\begin{bmatrix}
\star \\
| \Pi | & 0 \\
0 & \Pi_p
\end{bmatrix}^* \begin{bmatrix}
G_{yw} & G_{ywp} \\
0 & I
\end{bmatrix} \begin{bmatrix}
G_{yw} & G_{ywp} \\
0 & I
\end{bmatrix}^\top \preceq 0,
\]

then the interconnection (5) is robustly stable against \( \Delta \) and it achieves robust performance w.r.t. \( \Pi_p \) against \( \Delta \).

The proof is literally the same as in the continuous-time setup and is included in Appendix A.2.1 for the sake of completeness. The premise in applying Theorem 1 is then to find a class of multipliers valid for the class of considered uncertainties as well as an IQC formulation of the desired performance criterion such that all assumptions in the latter Theorem are met.

### 3. Problem formulation

Consider the following unconstrained convex optimization problem
\[
\min_{z \in \mathbb{R}^p} H(z),
\]
where \( H : \mathbb{R}^p \to \mathbb{R} \), \( H \in C^2 \), \( H \) is strongly convex with convexity modulus \( m > 0 \) and has Lipschitz continuous gradient with parameter \( L \), i.e., \( H \) is in the class \( S_{m,L} \) as in Definition 1. Observe that, under these assumptions, for any fixed \( H \) in the class, (11) has a unique global minimizer which will subsequently be denoted by \( z^* \).

Our goal is to analyze and design gradient-based optimization algorithms that converge to that minimizer \( z^* \). In particular, we consider optimization algorithms of the form
\[
\begin{align}
x_{k+1} &= Ax_k + B \nabla H(Cx_k) \\
z_k &= D x_k,
\end{align}
\]
where \( x_k = [x_{k,1} \ldots x_{k,n}]^\top \in \mathbb{R}^{np}, x_{k,i} \in \mathbb{R}^p, \quad i \in \{1, \ldots, n\}, z_k \in \mathbb{R}^p \). More formally, the design problem we want to address is then as follows: Given the objective function parameters \( L \geq m > 0 \), the dimensions \( n, p \geq 1 \), and a convergence rate bound \( \rho \in (0, 1) \), find matrices \( A \in \mathbb{R}^{np \times np}, B \in \mathbb{R}^{np \times p}, C \in \mathbb{R}^{p \times np}, D \in \mathbb{R}^{p \times np} \)
\[
\text{such that, for all } H \in S_{m,L}, (12) \text{ has a unique globally asymptotically stable equilibrium at } x^* \text{ with } D x^* = z^* \text{ and there exists } \eta > 0 \text{ such that }
\]
\[
||z_k - z^*||^2 \leq \eta \rho^{2k} ||z_0 - z^*||^2
\]
for each \( x_0 \in \mathbb{R}^{np} \) with \( z_0 := Dx_0 \), i.e., \( z_k \) converges exponentially with rate \( \rho \) to \( z^* \). Note that (12) captures several popular optimization algorithms with constant step size such as Gradient Descent, the Heavy Ball Method, or Nesterov’s Method, see also (61) and Table 1. By solving the design problem, we also get a solution to the corresponding analysis problem for these algorithms as a by-product, in which we aim to find an as tight as possible convergence rate bound \( \rho \). We emphasize that (12) is an overparametrization of the class of algorithms; in fact, without loss of generality, we can fix \( C = [C_1 \ C_2 \ldots \ C_n] \) arbitrarily as long as \( C_i \) is nonsingular for some \( i \in \{1, 2, \ldots, n\} \). In particular, we often
let \( C = [I \ 0 \ldots \ 0] \). A similar optimization algorithm design problem has been addressed in [11, 17] with structured and parametrized matrices \( A, B, C, D \) in (12) and \( n = 2 \). To the best of our knowledge, the only work considering general algorithms of the form (12) is [18] which focuses on determining lower convergence rates bounds.

While convergence rate bounds are an important performance measure, also other performance specifications are key in efficiently solving optimization problems, e.g., how well an algorithm performs in the presence of noise. In this vein, we consider (12) together with an additional performance channel and address the extended design problem aiming not only for minimizing the convergence rate but also the bounds on the additional performance channel.

We formalize this problem in Section 4 and propose an adapted \( H_2 \)-performance measure. It has already been recognized in different settings [26, 2] that \( H_2 \)-performance is a measure for the robustness properties of optimization algorithms.

We are further interested in designing tailored algorithms for certain subclasses of the class of objective functions \( S_{m,L} \) with additional structural properties such as having a diagonal Hessian. Summing up, in rough words our goal is to design gradient-based algorithms of the form (12) that are fast, robust and possibly exploit additional structural properties of the objective function.

4. Reformulation in the robust control framework

Our approach relies on reformulating the problem as a robust control problem by interpreting the gradient of the objective function \( H \) as the uncertainty. The reformulation as well as the adaptation to the specific problem at hand is the main subject of this section.

4.1. Equilibrium conditions

Before we proceed, we first discuss which properties an algorithm of the form (12) must fulfill to be, in principle, a candidate for the previously posed problems. This important question, which has not been addressed in the literature so far, is answered in the following Theorem which provides necessary and sufficient conditions for an algorithm of the form (12) to possess an equilibrium at the global minimizer of \( H \) for all \( H \in S_{m,L} \).

**Theorem 2.** Let an algorithm in the form (12) described by \( A, B, C, D \) be given and assume that the pair \((A, C)\) or the pair \((A, D)\) is observable. Then, (12) has an equilibrium \( x^* \) with the property \( Dx^* = Cx^* = z^* \) for any \( H \in S_{m,L} \), \( L \geq m > 0 \) arbitrary but fixed, if and only if there exists \( D^\dagger \in \mathbb{R}^{np \times p} \) such that

\[
DD^\dagger = I_p, \quad CD^\dagger = I_p, \quad (A - I)D^\dagger = 0_{np \times p}.
\]

holds.

A proof of Theorem 2 is given in Appendix A.2.2. We emphasize that the conditions (14) are necessary and sufficient, hence assuming that these conditions hold is no restriction of the class of algorithms. Note that (14c) implies that \( A \) must have at least \( p \) eigenvalues at one. The proof also reveals that \( x^* = D^\dagger z^* \) is the unique equilibrium of (12) with the desired property. We further note that, under the assumption that \( D \) has full rank, the convergence rate bound (13) holds if and only if the full state \( x_k \) itself converges exponentially with rate \( \rho \) to \( D^\dagger z^* \). In view of this, the choice of \( D \) does not alter the achievable convergence rate. Hence, remembering the previous discussion that (12) is an overparametrization, we often let \( D = C = [I \ 0 \ldots \ 0], D^\dagger = D^\top \), such that (14a), (14b) are ensured to hold. In the remainder of the paper we limit ourselves to algorithms described by \( A, B, C, D \) that fulfill all conditions in Theorem 2.

4.2. Problem reformulation

We next embed the design problem from Section 3 in the standard setup as introduced in Section 2.3. To this end, consider (12) initialized at any \( x_0 \) together with the state transformation \( \xi_k = x_k - D^\dagger z^* \), where \( D^\dagger \in \mathbb{R}^{np \times p} \) is taken such that (14) hold. The transformed algorithm together with an additional performance channel from \( w_p \) to \( y_p \) as described in Section 3 then takes the form

\[
\begin{align*}
\xi_{k+1} &= A\xi_k + B\nabla H(C\xi_k + z^*) + B_p w_p, \\
z_k &= D\xi_k + z^* \\
y_{p,k} &= C_p \xi_k + D_p w_{p,k},
\end{align*}
\]

with initial condition \( \xi_0 = x_0 - D^\dagger z^* \) and where \( B_p \in \mathbb{R}^{np \times n_p}, C_p \in \mathbb{R}^{n_p \times np} \) and \( D_p \in \mathbb{R}^{np \times n_p}, n_{w_p}, n_{y_p} \in \mathbb{N}_{>0} \). Note that if \( \xi \) converges exponentially to zero with rate \( \rho \), then so does \( x \) converge to \( D^\dagger z^* \) in (12); in other words (13) holds.

As we are aiming for analyzing and designing \( A, B, C, D \) in (15) for a whole class of objective functions, we interpret \( \nabla H \) as an uncertainty and define the causal and bounded uncertain operator \( \Delta_H : \ell_p^p \to \ell_p^p \) as

\[
\Delta_H(y)_k := \nabla H(y_k + z^*) - my_k
\]

for any \( y = [y_0, y_1, y_2, \ldots] \in \ell_p^p \). The corresponding set of all admissible operators is then defined as

\[
\Delta(m, L) := \{ \Delta_H : H \in S_{m,L} \}.
\]
Note that the set $\Delta(m, L)$ fulfills Assumption 1. Observe further that each $\Delta \in \Delta(m, L)$ is a slope-restricted operator with slope between 0 and $L - m$. In the spirit of Section 2.3 and (5) we drop the output (15b), write (15) as

\[
\begin{align*}
\xi_{k+1} &= (A + mBC)\xi_k + B_1w_k + B_2w_{p,k} \\
y_k &= C_2\xi_k \\
y_{p,k} &= C_2p_x + D_Pw_{p,k} \\
w_k &= \Delta_H(y)_k
\end{align*}
\]

and choose the transfer functions in (5) as

\[
\begin{bmatrix}
G_{yw} & G_{ywP} \\
G_{ywP} & G_{ywP}
\end{bmatrix} \sim \begin{pmatrix} A + mBC & [B \\ C] \\
[C & 0] & [0 \\ D_P]
\end{pmatrix}.
\]

We note that the latter feedback interconnection is well-posed since $G_{yw}$ is strictly proper and $\Delta$ is a static uncertainty. The original design problem can then be formulated as follows: Given a set of operators $\Delta(m, L)$ (or a structured subclass thereof, see Section 6) and a performance channel described by $B_P, C_P, D_P$, our goal is to design $A, B, C$ in such a way that (a) the transformed algorithm dynamics (18) have a globally asymptotically stable equilibrium at $\xi^* = 0$ for all $\Delta_H \in \Delta(m, L)$ and $\xi$ converges exponentially to 0 with rate $\rho$ and (b) the performance channel defined by the map from $w_p$ to $y_p$ in (15) fulfills a specified performance bound. Note that we do no longer consider $D$ as a design variable since it does not have an effect on the goals (a), (b).

Our approach to address the problem at hand is then to make use of IQC theory, in particular Theorem 1. The key steps (S1)–(S3) that pave the way to an implementable solution of the design problem are then as follows:

(S1) Extend Theorem 1 to allow for exponential stability results (Section 4.3, in particular Theorem 3).

(S2) Derive IQCs valid for the class of uncertainties that are suitable for the exponential stability result (Section 4.4, in particular Theorem 4).

(S3) Determine state-space representations of all transfer functions occurring in the frequency domain inequality (Section 4.5).

We note that the last step is more or less standard; still, we include it here in view of our goal of presenting an easily implementable framework.

### 4.3. Exponential convergence via IQCs

We begin with the first step (S1) of extending Theorem 1 which only allows to conclude $l_2$-stability results for (18). While it is noted in [22] that $l_2$-stability implies exponential stability for some classes of systems, the resulting rate bounds are often conservative [4]. In [5, 17], the concept of $p$-IQCs is introduced to derive exponential stability results. In the present manuscript, we focus on obtaining exponential stability results by an appropriate embedding into the existing IQC framework introduced in Section 2.3. As it will get apparent, this allows for a more systematic approach to derive IQCs for robust exponential convergence guarantees from standard IQCs.

We first give a proper definition of robust exponential stability. To this end, we need to step from the input-output framework introduced in Section 2.3 to state-space descriptions. Let a state-space representation of (5) be given by

\[
\begin{align*}
x_{k+1} &= Ax_k + B_1w_k + B_2w_{p,k} \\
y_k &= C_1x_k + D_{11}w_k + D_{12}w_{p,k} \\
y_{p,k} &= C_2x_k + D_{21}w_k + D_{22}w_{p,k} \\
w_k &= \Delta(y)_k,
\end{align*}
\]

with initial condition $x_0 \in \mathbb{R}^n$ and $y_{in} = 0$. Under the assumption that $\Delta(0) = 0$, i.e., the origin is an equilibrium of (20) for $w_p = 0$, we then have the following definition:

**Definition 5 (Robust exponential stability).** We say that the origin of (20) is robustly exponentially stable against $\Delta$ with rate $\rho \in (0, 1)$ if, for $w_p = 0$, it is globally (uniformly) exponentially stable with rate $\rho$ for all $\Delta \in \Delta$, i.e., there exists $\eta > 0$ such that $\|x_k\| \leq \eta \rho^k \|x_0\|^2$ for all $k \in \mathbb{N}$, for each $x_0 \in \mathbb{R}^n$ and each $\Delta \in \Delta$.

Hence, if we can show that the origin is robustly exponentially stable against $\Delta(m, L)$ as defined in (17) with rate $\rho \in (0, 1)$ for the transformed dynamics (18), then we conclude that the algorithm (12) converges with rate $\rho$ to $x^* = D^Tz^*$ for all $H \in \mathcal{S}_{m,L}$ and (13) holds. Since robust exponential stability and additional performance specifications are independent of each other, for the following discussions we neglect the additional performance channel in (18).

The idea for obtaining exponential stability results from $l_2$-stability statements is to make use of a proper time-varying transformation of the signals. Such signal transformations have a long history in convergence rate analysis, e.g., so-called exponential weightings have already been mentioned in [7] and similar signal transformations are used in [1] in the context of optimization. In particular, as in [5], for any $\rho \in (0, 1]$ and any $p \in \mathbb{N}_{>0}$, we define two operators $\rho_+, \rho_- : \ell_p^0 \to \ell_p^0$ as

\[
(\rho_+(y))_k = \rho^k y_k, \quad (\rho_-(y))_k = \rho^{-k} y_k,
\]

where $k \in \mathbb{N}$. For $\rho \in (0, 1)$, we then introduce the signal space

\[
\ell_p^0 = \{y \in \ell_p^0 : \rho_-(y) \in \ell_2^p\} \subset \ell_2^p.
\]
Theorem 3. Consider (18) and let the transfer functions be defined according to (19). Let $L \geq m > 0$ be given and fix $\rho \in (0, 1)$. Suppose that all eigenvalues of $A + mBC$ are located in the open disk of radius $\rho$. Let some set of multipliers $\Pi(\rho) \subset \mathcal{L}_2^{2p \times 2p}$ parametrized by $\rho$ and some set of performance multipliers $\Pi_p$ in the form (9) be given. Suppose that

1. the interconnection depicted in Figure 2 is well-posed for all $\Delta_{H, \rho} \in \Delta_{\rho}(m, L)$;

2. each $\Delta_{H, \rho} \in \Delta_{\rho}(m, L)$ satisfies the IQC defined by $\Pi$ for all $\Pi \in \Pi(\rho)$.

If there exist $\Pi_1 \in \Pi(\rho)$, $\Pi_2 \in \Pi(1)$ and $\Pi_p \in \Pi_p$ such that

$$
[\begin{array}{c} * \\ \Pi_1 \end{array}] \begin{bmatrix} G_{\tilde{g}_{\rho}} & 1 \\ L & I \end{bmatrix} \succ 0,
$$

(27a)

and

$$
[\begin{array}{c} * \\ \Pi_p \end{array}] \begin{bmatrix} G_{gw} & G_{gw_{\rho}} \\ I & 0 \end{bmatrix} \succ 0,
$$

(27b)

where $G_{\tilde{g}_{\rho}}$ is defined in (24), then the origin is robustly exponentially stable against $\Delta(m, L)$ with rate $\rho$ for (18) and it achieves robust performance w.r.t. $\Pi_p$ against $\Delta(m, L)$.

A proof can be found in Appendix A.2.3. In terms of the original problem described in Section 3, the previous Theorem provides a result that allows to analyze and design optimization algorithms of the form (12) in terms of convergence rates as well as performance. We note that the frequency domain inequality (FDI) for performance (27b) is equivalent to the FDI (10) in the classical result Theorem 1 and exponential convergence is captured by the FDI (27a). Compared to Theorem 1, we then need to have an adapted set of IQCs $\Pi(\rho)$ for the set of transformed uncertainties $\Delta_{\rho}(m, L)$; determining such a set will be the main subject of the following section. Following the steps (S1)-(S3), we then also discuss how to reformulate the two FDIs (27a), (27b) as matrix inequalities, thereby paving the way to an implementable solution.

Remark 2. In principle, the approach is not limited to exponential convergence rates. By employing suitable time-varying transformations, it is expected that similar results can be obtained for other types of convergence rate specifications, for example to obtain polynomial convergence rate guarantees.

4.4. IQCs for the class $\mathcal{S}_{m,L}$

The main goal of this section is to determine a set of IQCs valid for the class of transformed uncertainties $\Delta_{\rho}(m, L)$...
(step (S2)). In the following we show how to systematically derive such IQCs from well-known classical IQCs for the original set of uncertainties \( \Delta(m, L) \). This is in contrast to existing results providing similar IQCs such as \([17, 4, 12]\). Since our approach (in particular Lemma 2) allows to directly employ results from classical IQC theory, we are able to not only recover the modified IQCs from the latter references but also easily extend them. In particular, we provide a novel class of anticausal Zames-Falb multipliers and show in Section 6 how additional structural properties of the uncertain operator can be captured by IQCs.

With \( \Delta(m, L) \) being a set of slope-restricted operators, we can employ Zames-Falb IQCs that have been extensively studied in the literature, see, e.g., \([39, 37, 21]\). While there exist other related IQCs for slope-restricted operators based on discrete-time versions of the Popov criterion such as the Tsypkin or the Jury-Lee criteria, quite recently it has been shown in \([14]\) that, in the discrete-time setup, these are included in the set of Zames-Falb IQCs; thus we concentrate on this class here. We give a brief wrap up of those classical results in the sequel. In the literature, Zames-Falb IQC are commonly stated in the time domain; following Remark 1 this amounts to classifying the corresponding operator \( \phi \). To this end, we utilize the following definition introduced in \([37]\).

**Definition 6 (Doubly hyperdominant matrix).** A matrix \( M = [m_{ij}]_{i,j \in \{0,1,\ldots,r\}, r \in \mathbb{N}} \), is said to be doubly hyperdominant if

\[
m_{ij} \leq 0 \text{ for } i \neq j \text{ and } \sum_{k=0}^{r} m_{kj} \geq 0, \sum_{k=0}^{r} m_{ik} \geq 0
\]  

for all \( i,j \in \{0,1,\ldots,r\} \).

**Remark 3.** If \( \Delta(m, L) \) has the additional property that \( \Delta(y) = -\Delta(-y) \) for any \( y \in \ell_2^p, \Delta \in \Delta(m, L) \), i.e., each \( \Delta \) is an odd operator, then (29) persists to hold for \( \tilde{M} \) being doubly dominant, see \([37]\) for a definition.

The latter result is more or less known but, to the best of our knowledge, not exactly to be found in literature. In particular, besides generalizing \([21, \text{Lemma 2}]\) to slope-restricted nonlinearities, with (29) we provide a so-called hard IQC (instead a soft IQC) and thus can omit the boundedness requirements on the operator \( M \). If \( M \) is a bounded operator, then the limit of (29) as \( T \to \infty \) is well-defined and the inequality corresponds to a soft IQC, i.e., the following inequality holds for all \( \Delta_H \in \Delta(m, L) \) and all \( y \in \ell_2^p \):

\[
\left\langle \begin{bmatrix} y \\
\Delta_H(y) \end{bmatrix}, W \begin{bmatrix} 0 & M^T \\
M & 0 \end{bmatrix} W^T \begin{bmatrix} y \\
\Delta_H(y) \end{bmatrix} \right\rangle \geq 0.
\]

(31)

These extensions are important to derive an analog of the soft IQC (31) for the transformed uncertainty \( \Delta_{H,p} \). In the next Lemma we present a general result building the basis for deriving IQCs for transformed uncertainties from standard ones. A proof is provided in Appendix A.2.5.

**Lemma 2.** Let \( \Delta \subset \{ \Delta \in \mathcal{L}(\ell_2^{p}, \ell_2^p) \mid \Delta(\ell_2^{p}) \subset \ell_2^{p} \} \) be some set of operators. Suppose we have given \( \phi : \ell_2^{2p} \to \ell_2^{2p} \) such that \( \hat{\phi} = \rho_+ \circ \phi \circ \rho_+ \) is bounded on \( \ell_2^{2p} \) and

\[
\left\langle \begin{bmatrix} y \\
\Delta(y) \end{bmatrix}, O_T \phi O_T^T \begin{bmatrix} y \\
\Delta(y) \end{bmatrix} \right\rangle \geq 0
\]

(32)

for all \( y \in \ell_2^p, \Delta \in \Delta, T \in \mathbb{N} \). Then

\[
\left\langle \begin{bmatrix} y \\
\Delta_p(y) \end{bmatrix}, \phi \begin{bmatrix} y \\
\Delta_p(y) \end{bmatrix} \right\rangle \geq 0
\]

(33)

holds for all \( y \in \ell_2^p \) and any \( \Delta_p = \rho_- \circ \Delta \circ \rho_+, \Delta \in \Delta \).

Note that (32) provides a hard IQC for the transformed uncertainty as required. If \( \phi : \ell_2^{2p} \rightarrow \ell_2^{2p} \) is bounded on \( \ell_2^{2p} \)
Lemma 3. In Appendix A.2.6, then yields the following result, a proof can be found in (32), i.e.,

$$\left\langle \left[ \frac{y}{\Delta(y)} \right], \phi \left[ \frac{y}{\Delta(y)} \right] \right\rangle \geq 0$$  \hspace{1cm} (34)$$

implies (33) for \( y \in \ell^p_1 \). However, typically either only \( \phi \) or the transformed operator \( \bar{\phi} \) is bounded.

We next employ Lemma 2 to derive IQCs for the set of transformed uncertainties \( \Delta_r(m, L) \). In the sequel we limit ourselves to Toeplitz type operators \( \bar{M} \). To this end, we introduce the shorthand notation \( \text{Toep}(R) : \ell^p_1 \rightarrow \ell^p_1 \) defined as

$$\text{Toep}(R) := \begin{bmatrix} R_0 & \ldots & R_{\ell_+} & 0 & \ldots & \ldots \; 0 \ldots \; 0 \\ R_{-1} & R_0 & \ldots & R_{\ell_+} & 0 & \ldots & \ldots \; 0 \ldots \; 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \end{bmatrix},$$  \hspace{1cm} (35)$$

where \( R = [R_{-\ell_+}, R_{-\ell_+}+1, \ldots, R_{\ell_+}] \in \mathbb{R}^{p \times p(\ell_++1)} \), \( R_k \in \mathbb{R}^{p \times p} \), \( k = -\ell_-, \ldots, -\ell_+ + 1, \ldots, \ell_+ \). Note that \( \text{Toep}(R) \) is a causal operator if and only if \( \ell_+ = 0 \). A combination of Lemma 1 with Lemma 2 then yields the following result, a proof can be found in Appendix A.2.6.

Lemma 3. Let \( L \geq m > 0 \) and \( \ell_- \), \( \ell_+ \in \mathbb{N} \) be given. Let \( \rho \in (0, 1) \). Let \( \Delta_r(m, L) \) be defined according to (26) and let \( M_\rho : \ell^p_2 \rightarrow \ell^p_2 \) be the bounded linear operator with

$$M_\rho = \text{Toep}(\begin{bmatrix} M_{-\ell_-} & M_{-\ell_-}+1 & \ldots & M_{\ell_+} \end{bmatrix})$$  \hspace{1cm} (36)$$

where \( M_i \in \mathbb{R}^{p \times p} \). Then, for all \( y \in \ell^p_1 \), the inequality (31) holds for all \( \Delta_H \in \Delta_r(m, L) \) if \( M = M_\rho \), \( M_i = m_iI_p \) and

$$\text{Toep}(\begin{bmatrix} \rho^{-\ell_-} & m_{\ell_-} & \ldots & m_0 & \ldots & \rho^{\ell_+} & m_{\ell_+} \end{bmatrix})$$  \hspace{1cm} (37)$$

is doubly hyperdominant, i.e.,

$$\sum_{i=-\ell_-}^{\ell_+} m_i \rho^{-i} \geq 0,$$

$$\sum_{i=-\ell_-}^{\ell_+} m_i \rho^{i} \geq 0,$$

and \( m_i \leq 0 \) for all \( i \in \{-\ell_-, \ldots, \ell_+\}, i \neq 0 \). \hspace{1cm} (38)$$

We note that the above constraints can be formulated in a simpler fashion in terms of the parameters \( m_i \); we deliberately keep this more general version since it is better suited for the extension to the structured case discussed in Section 6, cf. (81), (82).

With frequency domain representations being more common, we next briefly sketch how to obtain such a representation of the time domain IQC (31). To this end, note that a Toeplitz operator \( M = M_\rho : \ell^p_1 \rightarrow \ell^p_1 \) as in (36) defines a corresponding transfer matrix \( E_M(z) = \Sigma_{i=-\ell_+}^{\ell_+} M_{-i}z^{-i} \) by

$$\hat{M}_y(z) = E_M(z)\hat{y}(z).$$  \hspace{1cm} (40)$$

By Parseval’s Theorem, it is then straightforward to formulate (31) equally well in the frequency domain and the conditions on the operator \( M \) (or \( M_\rho \)) then translate to conditions on the corresponding transfer matrix \( E_M \). More precisely, the corresponding class of Zames-Falb multipliers is given as

$$\Pi^\rho_{\Delta_r}(m, L) = \{ \Pi = \hat{W}^T \begin{bmatrix} 0 & E^* \\ E & 0 \end{bmatrix} \hat{W} | \begin{bmatrix} M_{-\ell_-} \ldots & M_{\ell_+} \end{bmatrix} \in \mathbb{M}(\rho, \ell_+, \ell_-) \},$$  \hspace{1cm} (41)$$

where \( \hat{W} \) is defined in (39) and \( \hat{W} \in \mathbb{R}^{2p \times 2p} \) given by

$$\hat{W} = \begin{bmatrix} (L-m)I & -I \\ 0 & I \end{bmatrix}.$$

is the \( z \)-transform corresponding to the operator defined by (30). Summing up, we then have the following result concluding Step (52) in our procedure.

Theorem 4. Let \( L \geq m > 0 \) and let \( \Delta_r(m, L) \) be defined as in (26). Then, for each \( \rho \in (0, 1) \), \( \Delta_r \) satisfies the IQC defined by \( \Pi \) for each \( \Delta_r \in \Delta_r(m, L) \) and each \( \Pi \in \Pi^\rho_{\Delta_r}(m, L) \) as defined in (41). \hspace{1cm} \bullet

Theorem 4 is a generalization of the IQCs presented in [5], [17] also allowing for anticausal multipliers \( \Pi \), i.e., anticausal transfer matrices \( E \) in (41). The class of multipliers from [5], [17] is then obtained by letting \( \ell_+ = 0 \). We note that an extension to anticausal multipliers has also been proposed in [15] using a slightly different approach. However, the conditions derived in [15] are more restrictive, thus leading to a smaller class of multipliers compared to the one proposed in Theorem 4, see Appendix A.3.1 for a detailed discussion. As another advantage, the presented approach is based on standard results and allows for easy extensions such as, e.g., incorporating structural properties of the objective function, see Section 6.
4.5. Multiplier parametrization

Having a suitable set of IQCs available as it is provided by Theorem 4, applying Theorem 3 then basically amounts to checking the FDIs (27a), (27b). In order to derive efficiently implementable conditions in terms of matrix inequalities employing the KYP-Lemma, we need to factorize the multipliers and determine state-space representations of all transfer functions in these FDIs (Step (S3)). In particular, utilizing the class of multipliers (41), for a factorization of \( \Pi \in \Pi_{\Delta \rho}^s(m, L) \) we need to find a proper and stable transfer matrix \( \psi_\Delta \) and a matrix \( M_\Delta \) such that

\[
\Pi = \psi_\Delta^s M_\Delta \psi_\Delta. \tag{43}
\]

The following procedure is standard and included here for the sake of completeness; for the multipliers proposed in [15] this step is also discussed in detail in [40]. Let \([- M_{-\ell} \ldots M_{-\ell+1} \ldots M_{-1}] \in \mathbb{M}(p, \ell, \ell, \ldots, p) \) and define

\[
M_+ = \begin{bmatrix} M_1^T & M_2^T & \ldots & M_{\ell+1}^T \end{bmatrix} \in \mathbb{R}^{p \times \ell + p}, \tag{44a}
\]

\[
M_- = [- M_{-\ell} \ M_{-\ell+1} \ldots M_{-1}] \in \mathbb{R}^{p \times \ell - p}, \tag{44b}
\]

as well as the vectors of strictly proper and stable basis functions \( \psi_+ \in \mathcal{H}_\infty^{\ell \times 1}, \psi_- \in \mathcal{H}_\infty^{\ell \times 1} \) given by

\[
\psi_+(z) = [z^{-1} \ z^{-2} \ldots z^{-\ell+1}]^T \tag{45a}
\]

\[
\psi_-(z) = [z^{-\ell} \ z^{-\ell+1} \ldots z^{-1}]^T. \tag{45b}
\]

Using these definitions, it then follows by simple calculations that \( \Pi \in \Pi_{\Delta \rho}^s(m, L) \) is represented as in (43) if choosing

\[
M_\Delta = \begin{bmatrix} 0 & M_1^T & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_- & 0 & 0 & 0 & 0 \\ 0 & 0 & M_- & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_+ & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_+ & 0 & 0 \\ \end{bmatrix}, \tag{46}
\]

\[
\psi_\Delta(z) = \begin{bmatrix} I_p & 0 & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 & 0 \\ (\psi_+(z) \otimes I_p) & 0 & 0 & 0 & 0 \\ (\psi_-(z) \otimes I_p) & 0 & 0 & 0 & 0 \\ I_p & 0 & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 & 0 \\ \end{bmatrix} \hat{W}. \tag{47}
\]

Note that \( \psi_\Delta \) is proper and stable and, in the following, we denote by \( (A_\Delta, B_\Delta, C_\Delta, D_\Delta) \) its state-space realization when using the state-space realizations of \( \psi_+, \psi_- \) provided in Appendix A.3.2.

5. Robust optimization algorithms

In the following we employ the previously developed tools to analyze optimization algorithms as well as design novel algorithms with prespecified guarantees. In particular, we apply Theorem 3 utilizing the IQCs derived in Section 4.4 and the corresponding multiplier parametrizations from Section 4.5.

5.1. Analysis

5.1.1. Convergence rate analysis

We begin with deriving convex conditions in terms of linear matrix inequalities (LMIs) for determining convergence rate bounds for a given algorithm in the form (12). In a nutshell, this amount to reformulating the FDI (27a) using the KYP-Lemma (see Lemma 6). Let \( G_{\rho \theta} \) be defined according to (24). We then introduce the following state-space realization

\[
\psi_c = \psi_\Delta \begin{bmatrix} G_{\rho \theta} \\ I_p \end{bmatrix} \sim (A_c(\rho), B_c, C_c(\rho), D_c), \tag{48}
\]

where \( A_c(\rho) \in \mathbb{R}^{n_c \times n_c}, B_c \in \mathbb{R}^{n_c \times p}, C_c(\rho) \in \mathbb{R}^{q \times n_c}, D_c \in \mathbb{R}^{q \times p} \) with \( n_c = p(n + \ell_+ + \ell_-), p_c = p, \)

\( \psi_c = p(4 + \ell_+ + \ell_-) \), are as given in Appendix A.3.2. The following result then virtually follows immediately from Theorem 3 employing the IQCs derived in Theorem 4. A proof is given in Appendix A.2.7.

Theorem 5. Consider (12). Suppose \( A \in \mathbb{R}^{np \times np}, B \in \mathbb{R}^{np \times np}, C \in \mathbb{R}^{np \times np}, D \in \mathbb{R}^{np \times np}, D^T \in \mathbb{R}^{np \times np} \) are given and fulfill (14). Let \( L \geq m > 0 \) be given. Fix \( \rho \in (0, 1) \) and assume that \( A + mBC \) has all eigenvalues in the open disk of radius \( \rho \). Let some \( \ell_+, \ell_- \in \mathbb{N} \) be given and let \( M_\Delta, \mathbb{M} \) be defined according to (46), (39). If there exist \( P = P^T \in \mathbb{R}^{n_c \times n_c}, M_+ \in \mathbb{R}^{p \times \ell + p}, M_- \in \mathbb{R}^{p \times \ell - p}, M_0 \in \mathbb{R}^{p \times p} \) such that

\[
\begin{bmatrix} \psi_c \end{bmatrix}^T \begin{bmatrix} P & 0 & 0 \\ -P & 0 & 0 \\ 0 & I & M_\Delta \\ 0 & 0 & M_0 + M_+ \end{bmatrix} \begin{bmatrix} A_c(\rho) & B_c \\ 0 & C_c(\rho) & D_c \end{bmatrix} < 0, \tag{49a}
\]

\[
\begin{bmatrix} M_- & M_0 & M_+ \end{bmatrix} \in \mathbb{M}(p, \ell_+, \ell_-, \rho), \tag{49b}
\]

then the origin is robustly exponentially stable against \( \Delta(m, L) \) for (18).

If the LMIs (49) are feasible for a given optimization algorithm (12) and some \( \rho \), then, for all \( H \in S_{mL} \), its output converges exponentially with rate \( \rho \) to the unique minimizer \( z^* \) of (11). In convergence rate analysis we are typically interested in finding the smallest \( \rho \) such that (49) is feasible. If \( \rho \) is kept as a free variable, the inequalities are no longer linear in the unknown parameters. Still, if (49) is feasible for some \( \rho \in (0, 1) \), then it also feasible for all \( \tilde{\rho} \in [\rho, 1] \). Therefore, it is convenient to do a bisection search over \( \rho \) to find the optimal convergence rate.

Many optimization algorithms admit the structure \( A = A \otimes I_p, B = B \otimes I_p, C = C \otimes I_p, D = D \otimes I_p \). In
that case we can also choose a state-space realization 
\((A_c(\rho), B_c, C_c(\rho), D_c)\) such that \(A_c = \overline{A_c(\rho)} \otimes I_p, B_c = \overline{B_c} \otimes I_p, C_c = \overline{C_c(\rho)} \otimes I_p, D_c = \overline{D_c} \otimes I_p\). Recall that also 
\(M_{\Delta}(M_+, M_-, M_0) = M_{\Delta}(m_+, m_-, m_0) \otimes I_p\). In this situation we can take \(P = \overline{P} \otimes I_p\) without loss of generality, i.e., instead of (49a) we can solve the LMI

\[
\begin{bmatrix}
\star & P & 0 & 0 \\
0 & -\overline{P} & 0 & 0 \\
0 & 0 & M_{\Delta} & 0 \\
0 & 0 & 0 & \overline{C_c(\rho)} \\
\end{bmatrix} \prec 0, \quad (50a)
\]

where now the dimension is reduced by a factor of \(p\) compared to (49a). This dimensionality reduction is lossless in the sense that (49) are feasible if and only if (50) are feasible. We provide a proof of this statement in Appendix A.2.8.

5.1.2. Performance analysis

We next derive similar convex conditions for analyzing the performance of a given algorithm by analogously reformulating the FDI (27b). While the methodology applies to a wide class of performance specifications, we concentrate on \(H_2\)-performance here and show that this can be related to noise rejection properties of the optimization algorithm.

We first recall that the \(H_2\)-norm of a stable linear time-invariant system with a strictly proper transfer matrix \(G\) is defined as

\[
\|G\|_2^2 = \frac{1}{2\pi} \text{tr} \left( \int_0^{2\pi} G(e^{i\omega})^* G(e^{i\omega}) d\omega \right). \quad (51)
\]

This definition of the \(H_2\)-norm is mainly motivated by the following two interpretations: first, the \(H_2\)-norm is a measure for the energy of the system’s impulse response; second, it can as well be interpreted as the asymptotic variance of the output when the system is driven by white noise. However, these interpretations do not directly carry over to nonlinear or time-varying systems [29]. Different extensions have been proposed, which are rather based on the actual desired performance measure instead of a certain system norm. In the following we build upon a stochastic interpretation; this will be motivated in more detail in Section 5.1.3. We next give a precise definition of the proposed performance measure.

**Definition 8 (Asymptotic \(H_2\)-performance).** Consider the state-space representation (20) of the feedback interconnection (5) and let \(w_p = (W_k)_{k \in \mathbb{N}}\) be a discrete-time white noise process. Let \((y_{p,k})_{k \in \mathbb{N}}\) denote the corresponding response of (20) with initial condition \(x_0 = 0\). We then say that the feedback interconnection (5) achieves a robust \(H_2\)-performance level of \(\gamma > 0\) against \(\Delta\) if (5) is robustly stable against \(\Delta\) and if, for all \(\Delta \in \Delta\), the performance channel fulfills

\[
\|G_{y_p w_p}\|_{2,av} \leq \gamma, \quad \text{where}
\]

\[
\|G_{y_p w_p}\|_{2,av} = \limsup_{k_{\text{max}} \to \infty} \sum_{k=0}^{k_{\text{max}}} \frac{1}{k_{\text{max}}} \cdot E(\gamma_{p,k}^r y_{p,k}^r) \quad (52)
\]

and \(E(Y)\) is the expected value of a random variable \(Y\). ⊗

A similar measure has been proposed in [29]. The motivation for this definition is that a small value of this asymptotic \(H_2\)-measure can be related to the noise rejection properties of the system under consideration. If the feedback interconnection (5) is linear time invariant, then this definition specializes to a standard \(H_2\)-norm constraint involving (51).

We next discuss how the so-defined robust \(H_2\)-performance properties can be analyzed within the IQC framework. We consider performance multipliers that admit a parametrization of the form

\[
\Pi_p = \{ \psi_p^* M_p \psi_p : M_p = \begin{bmatrix} M_{p,11} & M_{p,12} \\ M_{p,12}^T & M_{p,22} \end{bmatrix} \in M_p, M_{p,22} \succeq 0 \}, \quad (53)
\]

where \(M_p \subset \mathbb{R}^{q \times q}\) is some convex set and \(\psi_p \in \mathcal{R}(\mathbb{R}_+, n_{wp} + n_{yp})\), \(q \in \mathbb{N}_{>0}\), is a proper and stable transfer function. We emphasize that the most relevant performance measures fall into that class, e.g., \(H_{\infty}\) or the \(H_2\)-performance measure as considered here. Using the factorization of the Zames-Falb multipliers from (43), the FDI for performance (27b) is then given by

\[
\psi_p^* \begin{bmatrix} M_{\Delta} & 0 \\ 0 & M_p \end{bmatrix} \psi_p \prec 0, \quad (54)
\]

where

\[
\psi_p = \begin{bmatrix} \psi_{\Delta} & 0 \\ 0 & \psi_p \end{bmatrix}, \quad \begin{bmatrix} G_{yw} & G_{ywp} \\ I_{wp} & I_{n_{wp}} \\ G_{yp} & G_{yp} \end{bmatrix}. \quad (55)
\]

Let a state-space realization be given by

\[
\psi_{\nabla} \sim \begin{bmatrix} A_c & B_{c,1} & B_{c,2} \\ C_{c,1} & D_{c,11} & D_{c,12} \\ C_{c,2} & D_{c,21} & D_{c,22} \end{bmatrix}. \quad (56)
\]

Deriving such a state-space realization is standard. For implementation purpose, this is particularly easy making use of numerical tools such as the control systems toolbox in MATLAB. For completeness, we give an explicit state-space realization in Appendix A.3.2.
In the particular case of the considered $H_2$-performance measure, the performance multiplier is defined with

$$
\psi_p(z) = \left[ \begin{array}{cc}
0_{np \times np} & I_{np} \\
\end{array} \right], \quad M_p = \left\{ M = I_{np} \right\}
$$

(57)

and we denote the resulting state-space realization by (56). Note that $M_p$ is a singleton and the $H_2$-performance level is set by the aforementioned additional condition resulting in (58b). We then have the following result for $H_2$-performance; a proof is given in Appendix A.2.9.

**Theorem 6 (Robust $H_2$-performance).** Consider (18) and suppose $D_p = 0$. Suppose $A \in \mathbb{R}^{np \times np}$, $B \in \mathbb{R}^{np \times p}$, $C \in \mathbb{R}^{p \times np}$, $D \in \mathbb{R}^{p \times np}$, $D^T \in \mathbb{R}^{np \times p}$ are given and fulfill (14). Let $L \geq m > 0$ be given. Assume that $A + mBC$ has all eigenvalues in the open unit disk. Let some $\ell_+, \ell_- \in \mathbb{N}$ be given and let $M_\Delta$ be defined according to (46). If there exist $P_p = P_p^T \in \mathbb{R}^{R \times R}$, $M_+ \in \mathbb{R}^{p \times R}$, $M_- \in \mathbb{R}^{p \times R}$, $M_0 \in \mathbb{R}^{p \times \gamma}$, such that

$$
\begin{bmatrix}
P_0 & 0 & 0 & 0 \\
0 & -P_p & 0 & 0 \\
0 & 0 & M_\Delta & 0 \\
0 & 0 & 0 & I_{np}
\end{bmatrix}
\begin{bmatrix}
A_c & B_{c,1} \\
I & 0 \\
C_{c,1} & D_{c,1} \\
C_{c,2} & 0
\end{bmatrix}
\begin{bmatrix}
P_p & 0 & 0 & 0 \\
0 & -P_p & 0 & 0 \\
0 & 0 & M_\Delta & 0 \\
0 & 0 & 0 & I_{np}
\end{bmatrix}
< 0,

(58a)

$$
$$
\text{tr}(B_{c,2}^TNN^TP_pNN^TB_{c,2}) \leq \gamma^2,

(58b)

$$
\epsilon \text{ such that } \text{tr}(Z) < \gamma^2. \quad (58c)

P_p > 0,

(58c)

$$
\begin{bmatrix}
0_{np \times (\ell_+ + \ell_- + p)} & I_{np}
\end{bmatrix}

(58d)

$$

\text{then the interconnection (18) is robustly stable against } \Delta(m, L) \text{ and it achieves a robust } H_2-\text{performance level of } \gamma \text{ against } \Delta(m, L).

\begin{itemize}
\item Remark 4. Under (58c), it is well-known that (58b) holds if and only if there exists a matrix $Z$ of suitable dimensions such that

$$
\begin{bmatrix}
N^TP_pN & N^TP_pNN^TB_{c,2} \\
\ast & Z
\end{bmatrix}
> 0,

\text{tr}(Z) < \gamma^2.

(59)

To see this, note that, by taking Schur complements, the above conditions hold if and only if

$$
N^TP_pN > 0,

Z > B_{c,2}^TNN^TP_pNN^TB_{c,2}

(60)

$$

\text{and } \text{tr}(Z) < \gamma^2. \text{ If the latter inequalities are feasible, this implies that there exists an } \epsilon > 0 \text{ such that } Z = B_{c,2}^TNN^TP_pNN^TB_{c,2} + \epsilon I \text{ is a valid solution. Then this implies that (58b) holds.}

\begin{itemize}
\item If the LMIs (58) are feasible for a given optimization algorithm (15) and some $\gamma$, then, for all $H \in S_{m,L}$, its performance channel has an $H_2$-performance level of $\gamma$. Similar as for the convergence rate test from Theorem 5, an analogous lossless dimensionality reduction is possible for $H_2$-performance analysis for equally structured optimization algorithms whenever the matrices defining the performance channel admit the same structure, i.e., $B_p = B_p \otimes I_p$, $C_p = C_p \otimes I_p$, $D_p = D_p \otimes I_p$. This structure then also transfers to the matrices $A_c, B_c, C_c, D_c$.

\end{itemize}

\begin{itemize}
\item 5.1.3. Numerical results

In the following we use the presented results to numerically analyze existing optimization algorithms in terms of their convergence rate as well as their properties in the presence of additive gradient noise. Here, we consider the Gradient Descent algorithm (GD), Nesterov’s Method (NM) with constant step size [28], the Triple Momentum Method (TMM) [35] and the Heavy Ball Method (HB) [30].

In all cases, the matrices $A, B, C, D$ in (12) take the form

$$
A = \begin{bmatrix} 1 + v_2 & -v_2 \\ 1 & 0 \end{bmatrix} \otimes I_p, \quad B = \begin{bmatrix} -v_1 \\ 0 \end{bmatrix} \otimes I_p, \quad (61a)

C = \begin{bmatrix} 1 + v_3 & -v_3 \end{bmatrix} \otimes I_p, \quad D = \begin{bmatrix} 1 & 0 \end{bmatrix} \otimes I_p, \quad (61b)

$$

where the scalar parameters $v_1, v_2, v_3$ are as given in Table 1. Note that with $D^T = [1 \ 1]^T \otimes I_p$, the conditions (14) are fulfilled. Due to the Kronecker structure and by our previous discussion, we set $p = 1$ without loss of generality. The following and all other numerical results in the paper were obtained using MATLAB together with YALMIP [19].

\begin{itemize}
\item Convergence rate analysis

For convergence rate analysis, we use Theorem 5 together with a bisection search over $\rho$ to determine upper bounds on the convergence rates for different condition ratios $\kappa = L/m$, see Figure 3. For Nesterov’s Method, the Gradient Descent algorithm and the Triple Momentum Method, we reproduce the known convergence rate bounds. For the Heavy Ball Method, global convergence can be guaranteed for small condition numbers; for larger condition numbers the LMIs in Theorem 5 turn out to be infeasible, even for $\rho = 1$. These results are in concordance with [17]. For all considered algorithms, the addition of anticausal Zames-Falb multipliers does not lead to any improvement of the upper bound on the convergence rate and it is even sufficient to choose $\ell_+ = 0$, $\ell_- = 1$. Except for the Gradient Descent algorithm, it is still an open question whether the known as well as the numerically determined bounds are tight in the sense that there exists some objective function $H \in S_{m,L}$ such that the resulting algorithm does not converge faster than presumed by the given bound.

\end{itemize}

\begin{itemize}
\item $H_2$-performance

In the following we investigate the properties of the algorithms from Table 1 when the gradient is affected by white noise in an additive fashion,

\end{itemize}
Table 1. Parameters of several popular algorithms of the form (61) and the corresponding known upper bounds on the convergence rate in dependency of the condition ratio $\kappa = \frac{L}{m}$. Note that in the case of the Gradient Descent algorithm the parameter choice reduces (61) to a first order algorithm. While an explicit convergence rate can be given for the Heavy Ball Method in case of quadratic objective functions, it is known not to be globally convergent for general $H \in S_{m,L}$, see [17] for a counterexample.

| Parameters | $v_1$ | $v_2$ | $v_3$ | $\rho$ |
|------------|-------|-------|-------|--------|
| GD         | $\frac{2}{m+L}$ | 0     | 0     | $\frac{k-1}{k+1}$ | [30] |
| NM         | $\frac{1}{L}$ | $\frac{\sqrt{L} - \sqrt{m}}{\sqrt{L} + \sqrt{m}}$ | $v_2$ | $\sqrt{1 - \frac{1}{4\kappa^2}} - 1$ | [33] |
| TMM        | $\frac{1+\rho}{L}$ | $\frac{\rho^2}{2-\rho}$ | $\frac{\rho^2}{(1+\rho)(2-\rho)}$ | $1 - \frac{1}{\sqrt{\kappa}}$ | [35] |
| HB         | $\left(\frac{2}{\sqrt{L} + \sqrt{m}}\right)^2$ | $\frac{\sqrt{L} - \sqrt{m}}{\sqrt{L} + \sqrt{m}}$ | 0     | $-$ | [30] |

i.e., in rough words, in (12) we have $\nabla H(Cx_k) + W_k$ instead of $\nabla H(Cx_k)$, where $(W_k)_{k \in \mathbb{N}}$ is a discrete-time white noise process. Such situations occur in several applications, e.g., when the objective function and its gradient are not evaluated by means of numeric calculations but rather by measurements [32]. Likewise, in empirical risk minimization as it is utilized, e.g., in the context of learning algorithms [27], the objective function is given by an expected value, which, however, cannot be evaluated since the underlying probability distribution is unknown. It is therefore common to use a sample-based approximation of the expected value, the so-called empirical risk. According to the central limit theorem, this approximation differs from the original expected value by additive random noise, see [30]. Similar situations appear when employing Monte-Carlo methods. The $H_2$-performance as defined in Definition 8 is a measure for the noise attenuation; hence we choose the performance channel as $B_p = B$, $C_p = D$, $D_p = 0$ such that the corresponding $H_2$-performance level is a measure how additive gradient noise affects the resulting optimizer. We then use Theorem 6 to determine upper bounds on the corresponding $H_2$-performance. The results are depicted in Figure 4. These numerical results suggest that the Gradient Descent algorithm has the best properties in terms of additive noise attenuation for condition ratios larger than approximately 10, followed by Nesterov’s Method. The fastest method in terms of convergence rates, the Triple Momentum Method, however, has the worst noise attenuation. We note that these results

![Figure 3](image1.png)  
**Figure 3.** Upper bounds on the convergence rates of different algorithms obtained from Theorem 5 for condition ratios between 1.02 and 1000. The corresponding dashed lines indicate the known analytical convergence rate bounds, cf. Table 1, where, for Nesterov’s Method, the analytical bound for the modified version is depicted. The black line indicates the fundamental lower bound for any first order optimization algorithm for objective functions in $S_{m,L}$, see [28].

![Figure 4](image2.png)  
**Figure 4.** Upper bounds on the $H_2$-performance related to additive gradient noise for different optimization algorithms and condition ratios between 1.02 and 1000 obtained using Theorem 6. In all cases, the dimensions of the Zames-Falb multipliers were chosen as $\ell_+ = 0$, $\ell_- = 4$. 


5.2. Synthesis

In the following we consider the problem of designing algorithms that have desired properties specified in terms of convergence rates and $H_2$-performance bounds. To this end, roughly speaking, we need to combine Theorem 5 and Theorem 6 with the difference that the algorithm parameters $A, B, C, D$ are now to be determined as well. Motivated by our discussions in Section 3, we fix $C, D$ and $D^\top$ to

$$C = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix} \otimes I_p, \quad D = C, \quad D^\top = C^\top. \quad (62)$$

Still, having $A, B$ as additional design variables, the following main difficulties compared to the analysis problem need to be addressed: (1) The conditions (14) have to be ensured to hold; (2) nominal exponential stability of the feedback interconnection has to be guaranteed, i.e., $A + mBC$ must have all its eigenvalues in the open disk of radius $\rho$; and (3) the introduction of new design variables leads to bilinear instead of linear matrix inequalities in (49), (58), which, in general, cannot be solved efficiently. We emphasize that in the end we are interested in finding LMI conditions, which is the core difficulty of all three problems. As it turns out, (1) can be handled without much effort and, as we show next, (2) can be resolved utilizing the special structure of the problem at hand.

Lemma 4 (Nominal exponential stability). Let $\rho \in (0, 1)$ be given and consider $A, B, C, D$ as defined by (48), cf. (153). Suppose that there exists $P$ partitioned as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \end{bmatrix}, \quad P_{11} \in \mathbb{R}^{p(\ell_- + \ell_+ + 1) \times p(\ell_- + \ell_+ + 1)}, P_{22} \in \mathbb{R}^{np \times np},$$

such that (49a) holds. Then $A + mBC$ has all eigenvalues in the open disk of radius $\rho$ if and only if $P_{22} \succ 0$. \hfill \bullet

Proof. Consider the lower right block of (49a). The inequality then implies that

$$\rho^{-2} (A + mBC)^\top P_{22} (A + mBC) - P_{22}$$

$$+ \rho^{-2} C_{\psi, \gamma}^\top D_{\Delta}^\top M_\Lambda (M_+, M_-, M_0) D_{\Delta} C_{\psi, \gamma} < 0. \quad (64)$$

By straightforward calculations, it is seen that $C_{\psi, \gamma}^\top D_{\Delta}^\top M_\Lambda (M_+, M_-, M_0) D_{\Delta} C_{\psi, \gamma} = 0$; thus, by standard Lyapunov theory for linear systems, we infer that $A + mBC$ has all eigenvalues in the open disk of radius $\rho$ if and only if $P_{22} \succ 0$. \hfill \square

In order to address (3), it is convenient to first reformulate the matrix inequalities by means of Schur complements. As apparent in (64), with $A, B$ being design variables in the synthesis case, (49a) is a quadratic matrix inequality in the unknowns. The same holds true for the
performance inequalities (58a). As a first step, we utilize the positive definiteness condition for nominal exponential stability and a similar condition for performance in order to reformulate these quadratic matrix inequalities to bilinear matrix inequalities (BMIs). Note that with (58d) and (63) we have \( P_{22} = N^\top PN \); hence the positive definiteness constraint \( P_{22} > 0 \) allows us to equivalently formulate (49a) as (see Appendix A.3.3 for a derivation)

\[
\begin{bmatrix}
-P_{22} & P_{22}N^\top \\
* & U(P, M_{\Delta})
\end{bmatrix}
\begin{bmatrix}
A_c(\rho) & B_c \\
0 & I
\end{bmatrix}
< 0,
\]

where we introduced the shorthand notation

\[
U(P, M_{\Delta}) = \begin{bmatrix}
* \\
0 -P 0 \\
0 0 \bar{M}_{\Delta}
\end{bmatrix}.
\]

Note that, by the structure of the state-space representation (153), (65) is bilinear in the unknowns \( A, B, P, M_{\Delta} \).

A similar procedure applies to the problem with additional performance specifications. More precisely, assuming \( P_p \) to be structured in the same manner as \( P, (58a), (58c) \) are equivalently formulated as

\[
\begin{bmatrix}
-P_{p,22} & P_{p,22}N^\top \\
* & U_p(P_p, M_{\Delta})
\end{bmatrix}
\begin{bmatrix}
A_c & B_{c,1} \\
0 & I
\end{bmatrix}
< 0,
\]

where

\[
U_p(P_p, M_{\Delta}) = \begin{bmatrix}
* \\
0 -P_p 0 \\
0 0 \bar{M}_{\Delta}
\end{bmatrix}.
\]

In the following we consider a performance channel defined as in the example in Section 5.1.3, i.e., \( B_p = B, C_p = C, D_p = 0 \). Note that, by this choice, \( B_p \) is as well a design variable, while \( C_p, D_p \) are fixed, and (67) is bilinear in the unknowns \( A, B, P, M_{\Delta} \). Additionally, keeping in mind Remark 4 and employing the specific state-space realization from Appendix A.3.2, (58b) can be reformulated as

\[
\begin{bmatrix}
P_{p,22} & P_{p,22}N^\top B_{c,2} \\
* & Z
\end{bmatrix}
> 0,
\]

\[
\text{tr}(Z) < \gamma^2.
\]

Still, for the synthesis problem all of the former inequalities are bilinear. We next discuss two approaches to cope with that fact. The first one relies on rendering the conditions linear by imposing certain restrictions on the design variables and employing a suitable variable transformation; the second one is an often used rather hands-on heuristic based on alternately solving LMIs to obtain feasible solutions of the BMIs.

We formulate the subsequent synthesis results for full matrices \( A, B, C, D \). We emphasize that a Kronecker type structure can be enforced simply by designing the matrices for \( p = 1 \) and then adapt the obtained matrices to the required dimension \( \bar{p} \) taking Kronecker products with \( I_p \). This enables a design independent of the dimension of the optimization variable.

### 5.2.1. Convex solution

We next discuss how to obtain LMI conditions from the BMIs (65), (67), (69). Variable transformations have successfully been employed in many situations to render the inequalities linear in the new variables. However, since \( A_c(\rho), B_c, A_c, B_c \) are structured matrices here, the standard variable transformations do not directly apply. To make this more vivid, consider (65). With \( A, B, P \) being design variables, a typical approach is to define \( Q_A = P_{22}A, Q_B = P_{22}B \). However, \( U \) as defined in (66) also contains nonlinear terms of the form \( P_{12}A = P_{12}P_{22}^{-1}Q_A \), \( P_{12}B = P_{12}P_{22}^{-1}Q_B \), where \( P_{12} \) is the right upper block of \( P \), see (63). These terms cannot be handled by that approach; a simple remedy is to let \( P_{12} = 0 \), i.e., \( P \) is block diagonal, which is the idea behind the following Theorem.

**Theorem 7.** Let \( L \geq m > 0 \) as well as \( n \in \mathbb{N}_{>0}, p \in \mathbb{N}_{>0} \) be given. Let \( C \in \mathbb{R}^{p \times np}, D \in \mathbb{R}^{np \times p}, D^\top \in \mathbb{R}^{p \times np} \) be defined as in (62). Fix \( \rho \in (0, 1) \). Let some \( \ell_- \leq \ell_+ \in \mathbb{N} \) be given and let \( M_{\Delta}, M_b \) be defined according to (46), (39). Suppose there exist \( P_{11} = P_{11}^\top \in \mathbb{R}^{(\ell_-+\ell_+)p \times (\ell_-+\ell_+)p}, P_{22} = P_{22}^\top \in \mathbb{R}^{np \times np}, Q_A \in \mathbb{R}^{np \times np}, Q_B \in \mathbb{R}^{np \times p}, M_+ \in \mathbb{R}^{p \times \ell_-+\ell_+}, M_b \in \mathbb{R}^{p \times np} \) such that

\[
\begin{bmatrix}
P_{22} & 0 & \rho^{-1}Q_A + mp^{-1}Q_B & C \\
* & U(P, M_{\Delta})
\end{bmatrix}
\begin{bmatrix}
A_c & B_{c,1} \\
I & 0
\end{bmatrix}
< 0
\]

(70a)

\[
P = \text{blkdiag}(P_{11}, P_{22})
\]

(70b)

\[
(Q_A - P_{22})D^\top = 0
\]

(70c)

\[
[M_- \quad M_0 \quad M_+] \in \mathbb{M}(\rho, \ell_-, \ell_+, p).
\]

(70d)

Then, with \( A = P_{22}^{-1}Q_A, B = P_{22}^{-1}Q_B \), the equilibrium \( x^* = Dz^* \) is globally robustly exponentially stable against \( \Delta(m, L) \) with rate \( \rho \) for \( 18 \).

**Proof.** The result follows directly from Theorem 5 noting that (70a), (66), (70d) imply (49a), (70a) implies \( P_{22} > 0 \) and thus nominal exponential stability by Lemma 4, and (70c) together with the choice of \( C, D^\top \) implies (14). 

We note that the latter result provides a convex solution to the original design problem introduced in Section 3;
however, as it turns out in numerical examples, the restrictions on $P$ are conservative and we trade off convexity for slower convergence rates.

When it comes to the synthesis problem with additional performance specifications, similar restrictions on $P_p$ can be employed. Building upon that, the following Theorem provides a convex solution to this extended problem. A proof is provided in Appendix A.2.10.

**Theorem 8.** Let $L \geq m > 0$ as well as $n \in \mathbb{N}_{>0}$, $p \in \mathbb{N}_{>0}$ be given. Let $C \in \mathbb{R}^{p \times np}$, $D \in \mathbb{R}^{p \times np}$, $D^T \in \mathbb{R}^{p \times np}$ be defined as in (62). Fix $\rho \in (0,1)$. Let some $\ell_- , \ell_+ \in \mathbb{N}$ be given and let $M_{\Delta}, M$ be defined according to (46), (39). Let further $B_p = B$, $C_p = C$, $D_p = 0$. Suppose there exist $P_{11} = P_{11}^{T} \in \mathbb{R}^{p (\ell_- + \ell_+) \times p (\ell_- + \ell_+)}$, $P_{22} = P_{22}^{T} \in \mathbb{R}^{np \times np}$, $Q_A \in \mathbb{R}^{np \times np}$, $Q_B \in \mathbb{R}^{np \times np}$, $P_{p,11} = P_{p,11}^{T} \in \mathbb{R}^{(\ell_- + \ell_+) \times p (\ell_- + \ell_+)}$, $Z = Z^T \in \mathbb{R}^{p \times p}$, $M_+, M_{p,0} \in \mathbb{R}^{p \times \ell_-}$, $M_-, M_{p,-} \in \mathbb{R}^{p \times \ell_-}$, $M_0, M_{p,0} \in \mathbb{R}^{p \times p}$ such that (70) holds and

\[
\begin{bmatrix}
-P_{22} & [0 \ Q_A + m Q_B C \ Q_B^T]
\end{bmatrix}
\prec 0
\]

(71a)

\[P_p = \text{blkdiag}(P_{p,11}, P_{22})\]

(71b)

\[P_{22} \ [0 \ Q_B^T]
\prec 0\]

(71c)

\[
\text{tr}(Z) < \gamma^2,
\]

(71d)

\[
[M_{p,0} \ M_{p,-}] \in \mathbb{M}_1(1, \ell_- , \ell_- , p),\]

(71e)

where $M_{p,\Delta}$ is defined as $M_{\Delta}$ in (46) replacing $M_+, M_-, M_0$ by their counterparts $M_{p,0}, M_{p,+}, M_{p,-}$.

Then, with $A = P_{22}^{-1} Q_A$, $B = P_{22}^{-1} Q_B$, the equilibrium $x^* = D z^*$ is globally robustly exponentially stable against $\Delta(m,L)$ with rate $\rho$ for (18) and (18) achieves a robust $H_2$-performance level of $\gamma$ against $\Delta(m,L)$. * }

5.2.2. **Synthesis based on BMI optimization techniques**

While Theorem 7, Theorem 8 provide a convex synthesis procedure, the block diagonal structure of $P$ as well as the assumptions on $P_p$ are restrictive and result in convergence rates inferior to what can be achieved, see Section 5.2.3. As an alternative, BMI optimization techniques, which directly try to solve (65), (67), (69) or variants thereof, can be employed. However, while many of the approaches perform reasonably well as we will also illustrate in the subsequent example section, the non-convex nature of the problem does not allow for any guarantees of finding the global optimizer in general. Here, we employ the so-called alternating method which alternates between solving two different semi-definite programs obtained from fixing two subsets of the set of all decision variables in the BMI. In particular, assuming that $C, D, D^T$ are chosen as in (62), we alternate between finding the best algorithm $A, B$ in terms of $\rho$ and $\gamma$ for fixed $P, P_p$ and solving for $P, P_p$ for this $A, B, \rho, \gamma$. Building upon an iterative scheme, having a good initial feasible solution is key for successfully applying this procedure. As it turns out in numerical examples, certain types of parametrized algorithms are better suited for initialization than making use of Theorem 7. In particular, we let $A, B$ be parametrized by the free matrices $K_i \in \mathbb{R}^{p \times p}$, $i = 1, 2, \ldots, n$, as

\[
A = A_1 + I_{np} + B_1 [0 \ K_2 \ldots \ K_n]
\]

\[
B = B_1 K_s
\]

(72a)

(72b)

where $A_1 \in \mathbb{R}^{np \times np}$, $B_1 \in \mathbb{R}^{np \times p}$ are defined as

\[
A_1 = \begin{bmatrix}
0 & I_p & \cdots & \cdots & \cdots
0 & 0 & I_p & \cdots & \cdots & \cdots
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots
\vdots & \vdots & \cdots & \cdots & \ddots & \ddots
0 & \cdots & \cdots & \cdots & \cdots & I_p
0 & \cdots & \cdots & \cdots & \cdots & 0
\end{bmatrix}, B_1 = \begin{bmatrix}
0 & \cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}.
\]

(73)

The particular structure is motivated by [23], [24] and corresponds to a Euler discretization of the $n$-th order heavy ball method presented in the latter references. For the initialization of the BMI iteration we then fix $C, D$ as in (62) and utilize Lemma 7 to find suitable parameters $K_i$, $i = 1, 2, \ldots, n$, see Appendix A.3.4.

5.2.3. **Numerical results**

In this section we employ the presented results to design novel optimization algorithms that guarantee a priori specified convergence rates and that are robust against additive noise, i.e., the performance channel is defined as explained in Section 5.1.3. More precisely, for a fixed condition ratio $\kappa = L/m$, we choose different desired convergence rates and then design algorithms that ensure this convergence rate and additionally have a minimal bound on the $H_2$-performance level. The numerical results for two specific condition ratios (left: $\kappa = 50$, right: $\kappa = 100$) are depicted in Figure 6. This figure also shows the convergence rate and $H_2$-performance levels obtained from analyzing the Triple Momentum Method, Nesterov’s Method and the Gradient Descent algorithm using Theorem 5, Theorem 6. Apparently, we are not only able to recover the convergence rate and robustness properties of these existing algorithm but our design approach enables us to weigh up these two performance specifications against each other. More specifically, as already observed in our analysis in Figure 4, there is a trade-off between convergence rates and robustness against noise and the newly generated algorithms lie on the corresponding Pareto frontier. The numeric results also show that higher order algorithms (i.e., a larger dimension $n$ of the state in (12)) can yield better $H_2$-performance levels while providing the same convergence rate guarantees. Still, we
6. Structure exploiting algorithms

Up to now the only assumption on the objective function was that $H \in S_{m,L}$. However, in many situations it is possible to further characterize the objective function in terms of its structure. In such cases, it is to be expected that a tailored algorithm that exploits these properties has much better guaranteed convergence rates than a standard algorithm for the class $S_{m,L}$. In this section we elaborate on how additional structural knowledge can be incorporated in the presented framework. To this end, for any $L \geq m > 0$, we define the following two sets of functions:

\[
S_{m,L}^{n-rep} = \{H \in S_{m,L} \mid \exists \phi_i : \mathbb{R} \to \mathbb{R}, \phi_i \in C^0, i = 1,2,\ldots, p, \text{ such that } \nabla H(x) = [\phi_1(x_1) \ldots \phi_p(x_p)]^\top \text{ for all } x = [x_1 \ x_2 \ \ldots \ x_p]^\top \in \mathbb{R}^p \}
\]

\[
S_{m,L}^{rep} = \{H \in S_{m,L} \mid \exists \phi : \mathbb{R} \to \mathbb{R}, \phi \in C^0, \text{ such that } \nabla H(x) = [\phi(x_1) \ldots \phi(x_p)]^\top \text{ for all } x = [x_1 \ x_2 \ \ldots \ x_p]^\top \in \mathbb{R}^p \}.
\]

Note that $S_{m,L}^{n-rep}$ is the set of all functions with a diagonal Hessian and $S_{m,L}^{rep}$ is the set of all functions with a diagonally repeated Hessian. Objective functions of this form appear, e.g., in distributed optimization problems where the objective function is a sum of the agents’ individual objective functions. We discuss how this information can be translated to IQCs in Section 6.1. It is straightforward to apply the same procedure to the case where each $\phi_i$, $i = 1,2,\ldots, p$, in (74) is a map from $\mathbb{R}^{q_i}$ to $\mathbb{R}^{q_i}, q_i \geq 1$ (or $\phi : \mathbb{R}^{q_i} \to \mathbb{R}^{q_i}$ in (75), respectively). However, this will yield the same matrix inequalities for algorithm analysis and synthesis after a dimension reduction; thus we directly consider the reduced case here. In other words, the resulting matrix inequalities are scalable with respect to the dimensions $q_i$ (or $q$, respectively). Additionally, in Section 6.2, we consider parametrized objective functions consisting of a known quadratic and a specifically structured unknown part.

6.1. IQCs for the classes $S_{m,L}^{n-rep}$, $S_{m,L}^{rep}$

In the following we want to derive IQCs for uncertainties of the form (16) and the transformed version (101) thereof under the additional assumption that the objective function $H$ is in $S_{m,L}^{n-rep}$ or $S_{m,L}^{rep}$, respectively. At this point, we benefit from embedding our approach in the standard IQC framework rendering the following derivation straightforward. In the spirit of our previous discussions, we define
the following sets of uncertainties

$$\Delta_{\text{rep}}(m, L) = \{ \Delta_H \mid H \in \mathcal{S}_{m,L}^n \},$$ (76a)

$$\Delta_{\text{rep}}(m, L) = \{ \Delta_H \mid H \in \mathcal{S}_{m,L}^n \},$$ (76b)

with \( \Delta_H(y)_k := \nabla H(y_k + z^*) - my_k \), see (16). Observe that \( \Delta_{\text{rep}}(m, L) \subseteq \Delta_{\text{rep}}(m, L) \subseteq \Delta(H, L) \). Operators of this type have as well been studied in the literature. In fact, the following modification of Lemma 1 holds, see [14].

Lemma 5. Let \( L \geq m \geq 0 \) and \( \ell_-, \ell_+ \in \mathbb{N} \) be given. Let \( \Delta_{\text{rep}}(m, L) \), \( \Delta_{\text{rep}}(m, L) \) be defined according to (76a), (76b) and let \( M : \ell^2 \to \ell^2 \) be a bounded linear operator defined as

$$M = \text{Toep}\left(\begin{bmatrix} M_{-\ell_-} & M_{-\ell_-+1} & \cdots & M_{\ell_+} \end{bmatrix}\right),$$ (77)

where \( M_i \in \mathbb{R}^{p \times p} \). Then (29) persists to hold

(a) for all \( \Delta \in \Delta_{\text{rep}}(m, L) \) if, for \( i \in \{-\ell_-, \ldots, \ell_+\} \),

$$M_i = \text{blkdiag}(m_{i,1}, m_{i,2}, \ldots, m_{i,p})$$ (78)

and, for each \( k = 1, 2, \ldots, p \), the operator

$$\text{Toep}\left(\begin{bmatrix} m_{-\ell_-,k} & \cdots & m_{0,k} & \cdots & m_{\ell_+,k} \end{bmatrix}\right)$$ (79)

is doubly hyperdeterminant.

(b) for all \( \Delta \in \Delta_{\text{rep}}(m, L) \) if \( M \) is doubly hyperdeterminant.

By our discussions in Section 4.3, it is not difficult to derive IQCs for exponential stability analysis. We first define the corresponding sets of transformed uncertainties as

$$\Delta_{\text{rep},\rho}(m, L) = \{ \rho_+ \circ \Delta \circ \rho_+ \mid \Delta \in \Delta_{\text{rep}}(m, L) \}$$ (80a)

$$\Delta_{\text{rep},\rho}(m, L) = \{ \rho_+ \circ \Delta \circ \rho_+ \mid \Delta \in \Delta_{\text{rep}}(m, L) \}. \quad (80b)$$

Following the same steps, again we demand \( M_{\rho} = \rho_+ \circ M \circ \rho_- \) to adhere to condition (a) in Lemma 5 if \( \Delta_\rho \in \Delta_{\text{rep},\rho}(m, L) \) or to (b) if \( \Delta_\rho \in \Delta_{\text{rep},\rho}(m, L) \). The sets of admissible \( \{ M_i \}_{i \in \{-\ell_-, \ldots, \ell_+\}} \) are then defined as

$$\mathcal{M}_{\text{rep}}(\rho, \ell_+, \ell_-, p) = \{ \begin{bmatrix} M_{-\ell_-} & M_{-\ell_-+1} & \cdots & M_{\ell_+} \end{bmatrix} \mid M_i \in \mathbb{R}^{p \times p}, M_i \leq 0 \text{ for all } i \neq 0, M_i \geq 0, \text{ and } M_{\ell_+} \geq 0 \}$$ (81)

in the repeated case and as

$$\mathcal{M}_{\text{rep}}(\rho, \ell_+, \ell_-, p) = \{ \begin{bmatrix} M_{-\ell_-} & M_{-\ell_-+1} & \cdots & M_{\ell_+} \end{bmatrix} \mid M_i \leq 0 \text{ for all } i \neq 0, M_i \geq 0, \text{ and } M_{\ell_+} \geq 0 \}$$ (82)

for the non-repeated case. The corresponding Zames-Falb multipliers \( \Pi_{\Delta_\rho,\text{rep}}(m, L) \), \( \Pi_{\Delta_\rho,\text{rep}}(m, L) \) are obtained from (41) simply by replacing \( M \) by \( \mathcal{M}_{\text{rep}}(\rho, \ell_+, \ell_-, p) \) and \( M_{\text{rep}}(\rho, \ell_+, \ell_-, p) \), respectively. Analogously to Theorem 4, we then have the following result.

Theorem 9. Let \( L \geq m > 0 \) and let \( \Delta_{\text{rep},\rho}(m, L) \), \( \Delta_{\text{rep},\rho}(m, L) \) be defined as in (80a), (80b). Then, for each \( \rho \in (0, 1], \Delta_\rho \) satisfies the IQC defined by \( \Pi \)

1. for each \( \Delta_\rho \in \Delta_{\text{rep},\rho}(m, L) \) and each \( \Pi \in \Pi_{\Delta_\rho,\text{rep}}(m, L) \);

2. for each \( \Delta_\rho \in \Delta_{\text{rep},\rho}(m, L) \) and each \( \Pi \in \Pi_{\Delta_\rho,\text{rep}}(m, L) \).

It is then straightforward to include additional structural assumptions in the results from Section 5. In a nutshell, we only need to replace the constraint on \([M_- \quad 0 \quad M_+]\) by the respective structured counterpart.

### 6.2. Parametrized objective functions

In the following we assume that the gradient of the objective function \( H : \mathbb{R}^p \to \mathbb{R} \) admits the form

$$\nabla H(z) = H_1 z + T^\top \nabla H_2(Tz),$$ (83)

where \( H_1 \in \mathbb{R}^{p \times p}, H_1 > 0, T \in \mathbb{R}^{q \times p} \) are known whereas \( H_2 : \mathbb{R}^q \to \mathbb{R} \) is unknown but fulfills \( H_2 \in \mathcal{S}_{m,L} \) for some known constants \( L_2 \geq m_2 > 0 \). Objective functions of this specific form arise in the context of linear relaxed logarithmic barrier function based model predictive control [13], where fast converging optimization algorithms are crucial for the practical applicability of the control scheme. It is clear that \( H \in \mathcal{S}_{m,L} \) with \( m = \lambda_{\min}(H_1), L = \lambda_{\max}(H_1 + T^\top T L_2) \), where \( \lambda_{\min}, \lambda_{\max} \) denote the minimal and maximal eigenvalue, respectively. Again, we denote by \( z^* \) the minimizer of \( H \). Note that \( m, L \) can both be computed under the assumption that \( H_1 \) and \( T \) are known. Hence, we can make use of any algorithm for the class \( \mathcal{S}_{m,L} \); in particular, we can make use of the fastest known algorithm in the class of considered algorithms, i.e., the Triple Momentum Method [35]. However, with the structure of \( H \) being known and taking the form (83), it is possible to obtain algorithms with improved convergence rate guarantees in many cases using the presented framework.

To this end, consider an algorithm of the form (12) under the structural assumption (83) which then takes the form

$$x_{k+1} = (A + BH_1C)x_k + B T^\top \nabla H_2(TCx_k)$$ (84a)

$$z_k = D x_k.$$ (84b)
By the state transformation $\xi_k = x_k - D^Tz^*$ and under the assumptions (14) we then obtain the transformed dynamics

$$\begin{align*}
\tilde{\xi}_{k+1} &= (A + BH_1C)\tilde{\xi}_k + BT^T\nabla H_2(TC\tilde{\xi}_k + Tz^*) \\
&\quad + BH_1z^* \\
z_k &= D\tilde{\xi}_k + z^*.
\end{align*}$$

(85a)

Since $z^*$ is the minimizer of $H$, we have $\nabla H(z^*) = H_1z^* + T^T\nabla H_2(Tz^*) = 0$. Since $H_1 \succeq mL > 0$, $H_1$ is non-singular and hence the previous equality implies that $z^* = -H_1^{-1}T^T\nabla H_2(Tz^*)$. Using this in (85), we obtain

$$\begin{align*}
\tilde{\xi}_{k+1} &= (A + BH_1C)\tilde{\xi}_k \\
&\quad + BT^T(\nabla H_2(TC\tilde{\xi}_k + Tz^*) - \nabla H_2(Tz^*)) \\
z_k &= D\tilde{\xi}_k + z^*.
\end{align*}$$

(85b)

In the same spirit as in the unstructured case, we hence write the feedback interconnection in the standard form as

$$\begin{align*}
\tilde{\xi}_{k+1} &= (A + BH_1C + m_2BT^TC)\tilde{\xi}_k + w_k \\
y_k &= TC\tilde{\xi}_k \\
w_k &= \Delta_{H_2}(y)_k
\end{align*}$$

(87a)

(87b)

(87c)

with the new uncertainty $\Delta_{H_2} : \ell_2^q \rightarrow \ell_2^d$ defined as

$$\Delta_{H_2}(y)_k = \nabla H_2(y_k + Tz^*) - \nabla H_2(Tz^*) - m_2y_k$$

(88)

for any $y = [\ldots, y_0, y_1, y_2, \ldots] \in \ell_2^q$ and the corresponding class of uncertainties

$$\Delta = \{\Delta_{H_2} : H_2 \in S_{m,L_2-m_2}\}.$$  

(89)

Note that, again, the so-defined $\Delta \in \Delta$ is a slope-restricted operator in the sector $[0, L_2 - m_2]$. Hence we can make use of all of the previously derived IQCs for the new uncertainty. In a nutshell, this means that we can employ the very same techniques with the substitutions $A + mBC \mapsto A + BH_1C + m_2BT^TC$, $B \mapsto BT^T$, $C \mapsto TC$, $L - m \mapsto L_2 - m_2$. We emphasize that this also applies to the convex synthesis procedures presented in Section 5.2.1 using corresponding substitutions for $Q_A$, $Q_B$.

In the following numerical example we show that exploiting structural knowledge about the objective function can lead to a significant improvement in convergence rate guarantees. We choose $H_1 = \text{blkdiag}(1, 2, 10, 4)$, i.e., $m = 1$, and further let $T$ be given by

$$T = \begin{bmatrix}
2 & -7 & 0 & 5 \\
-1 & 4 & -3 & 2 \\
0 & -2 & 1 & 0
\end{bmatrix}$$

(90)

$m_2 = 1$, and let $L_2$ vary from 1 to 20, resulting in $L$ ranging from 89.677 to 1774.5. For the problem at hand, we design tailored algorithms based on the convex synthesis procedure from Section 5.2.1 and the BMI optimization approach described in Section 5.2.2. The convergence rates of the respective algorithms are depicted in Figure 7. In the considered example, structure exploiting algorithms provide much better guaranteed convergence rates compared to the fastest known standard algorithm, the Triple Momentum Method, using only $m$ and $L$ as known parameters but neglecting any structural properties of $H$. This effect is most considerable for small values of $L_2$ resulting in a small class of uncertainties $\Delta$ in (89). In particular, for $L_2 = m_2 = 1$, there is no uncertainty and (87) is linear, hence resulting in arbitrarily small convergence rates. Still, we emphasize that the achievable improvements heavily depend on the specific problem. The numerical results also show that additionally assuming that $H_2 \in S_{m,L}$ employing the results from Section 6.1 can further improve the guaranteed convergence rates.

7. Conclusions and Outlook

We presented a novel and general framework for analyzing and designing robust and structure exploiting opti-
mization algorithms suitable for solving a class of unconstrained optimization problems with strongly convex cost and possible additional structure. Building upon the well-studied field of robust control theory based on integral quadratic constraints and adapting it to our needs, we provide an approach to the design of robust and structure exploiting optimization algorithms specifically tailored to the class of optimization problems at hand. Several numerical examples illustrate that tailored algorithms designed following the presented methodology can outperform standard algorithms in terms of robustness against noise and guaranteed convergence rates in the considered scenarios.

One key advantage of the approach is that it allows for systematic extensions by suitable adaptations of existing results from robust control theory. In particular, it is to be expected that further characterizations of the class of objective functions as well as performance characterizations, both stated in terms of suitable integral quadratic constraints, can be embedded in the presented framework. Further, extending the class of optimization algorithms to primal dual dynamics or utilizing a (relaxed) barrier function based approach, it might be possible to establish a similar framework applicable to constrained optimization problems.

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A. Appendix

A.1. Some standard results

For the sake of completeness, we state some well-known results in the following that can be found in standard textbooks.
Lemma 6 (Discrete-time KYP-Lemma). Let $G$ be a real rational and proper transfer matrix and let $(A, B, C, D)$ denote a state-space representation of $G$. Suppose that $A$ has no eigenvalues on the unit circle and let $M$ be a real symmetric matrix. Then the following frequency domain inequality

$$G(z)^* M G(z) \succeq 0$$  \hspace{1cm} (91)

holds if and only if there exists a $P = P^\top$ such that

$$
\begin{bmatrix}
A & B \\
I & 0 \\
C & D
\end{bmatrix}^\top
\begin{bmatrix}
P & 0 & 0 \\
0 & -P & 0 \\
0 & 0 & M
\end{bmatrix}
\begin{bmatrix}
A & B \\
I & 0 \\
C & D
\end{bmatrix} \prec 0. \hspace{1cm} (92)
$$

A.2. Proofs

We collect all proofs of the presented results in the following.

A.2.1. Proof of Theorem 1

We follow the lines of the proof of [36, Corollary 3] and first show robust stability. Let $\Pi_p$ be partitioned according to (9) and observe that (10) implies that

$$
\begin{bmatrix}
G_{yw}^* \\
I
\end{bmatrix}^* \Pi
\begin{bmatrix}
G_{yw} \\
I
\end{bmatrix} + G_{yw}^* \Pi_{p,22} G_{yw} \prec 0 \hspace{1cm} (93)
$$

Since $\Pi_{p,22} \succeq 0$ according to (9), we have

$$
\begin{bmatrix}
G_{yw}^* \\
I
\end{bmatrix}^* \Pi
\begin{bmatrix}
G_{yw} \\
I
\end{bmatrix} \prec 0. \hspace{1cm} (94)
$$

Thus, with Assumption 1 and 1.2, robust stability follows from the standard IQC-Theorem, see, e.g., [16, 14]. For robust performance, let $w_p \in L_2^{n_p}$ and let $w$ denote the corresponding signal resulting from the feedback interconnection (5). Observe that $w \in L_2^{n_w}$ due to robust stability which implies that $y \in L_2^{n_y}$. Hence, the $z$-transforms of $w$ and $w_p$ exist almost everywhere on the unit circle and multiplying (10) from left by $[\hat{w}(z)^* \  \hat{w}_p(z)^*]$ and from right by its transposed we obtain

$$
\begin{bmatrix}
\hat{w}(z)^* \\
\hat{w}_p(z)^*
\end{bmatrix}^* \Pi_2 \begin{bmatrix}
\hat{y}(\Delta(y)) \\
\hat{y}_p
\end{bmatrix}^\top \leq 0. \hspace{1cm} (95)
$$

Integrating on both sides over the set $T$ yields

$$\text{IQC} (\Pi_2, y, \Delta(y)) + \text{IQC} (\Pi_p, w_p, y_p) \leq 0. \hspace{1cm} (96)$$

Since $\text{IQC} (\Pi_2, y, \Delta(y)) \geq 0$ by assumption, we conclude that the performance criterion defined by $\Pi_p$ is fulfilled.

A.2.2. Proof of Theorem 2

We first show sufficiency and then necessity.

A). Sufficiency. Let some $D^\top$ be given that fulfills (14) and let $x^* = D^\top z^*$. Then $x^*$ fulfills $Dx^* = Cx^* = z^*$ by (14a), (14b). The condition that (12) has an equilibrium at $x^*$ implies that

$$D^\top z^* = AD^\top z^* + B \nabla H(CD^\top z^*). \hspace{1cm} (97)$$

Using (14b), (14c), this holds if $B \nabla H(Cx^*) = Ax^*$ (98)

since $Cx^* = z^*$ is the unique minimizer of $H$. Hence,

$$z^* = CA^i z^* = DA^i x^* \hspace{1cm} (99)$$

for any $i = 0, 1, \ldots$, and we infer that

$$\begin{bmatrix}
I \\
I \\
\vdots \\
I
\end{bmatrix}
\begin{bmatrix}
C \\
CA \\
CA^2 \\
\vdots \\
CA^{n_p-1}
\end{bmatrix}
\begin{bmatrix}
x^* \\
D
\end{bmatrix} =: Q_C =: Q_D \hspace{1cm} (100)$$

Since the pair $(A, C)$ or the pair $(A, D)$ is observable by assumption, the matrix $Q_C \in \mathbb{R}^{n_p \times n_p}$ or $Q_D \in \mathbb{R}^{n_p \times n_p}$ has full rank and we infer that there exists some matrix $U \in \mathbb{R}^{n_p \times n_p}$ independent of $x^*, z^*$ such that $x^* = U z^*$. We next show that $D^\top = U$ fulfills (14). To this end, note that

$$z^* = CA^i z^* = CA^i U z^*$$

and, equally well, $z^* = Dx^* = Du z^*$. Since these two equation need to hold for all $z^*$, this implies that $CU = DU = I$ and we infer that (14a), (14b) hold for $D^\top = U$. The third condition (14c) follows directly from (98) using $x^* = U z^*$.

A.2.3. Proof of Theorem 3

In order to apply Theorem 1 to the transformed feedback interconnection, we first need to show boundedness of the transformed uncertainty as well as shift the non-zero initial conditions of (23) to the signal $y_{in}$ in Figure 2.

**Boundedness of $\Delta_{H, \rho}$.** Consider the transformed uncertainty

$$\Delta_{H, \rho} = \rho_- \circ \Delta_H \circ \rho_+. \hspace{1cm} (101)$$

We next show that $\Delta_{H, \rho}$ is a bounded operator mapping $L_2^2$ to $L_2^2$ since $\Delta_{H}$ itself is bounded, static and time-invariant.
To this end, first note that, there exists $\beta \in \mathbb{R}_{\geq 0}$ such that $\| (\Delta H(y))_k \| = \| \Delta H(y_k) \| \leq \beta \| y_k \|$ for any $k \in \mathbb{N}$ and any $y \in \ell^p_2$. With $\ell^p_{2,\rho} \subset \ell^p_2$, we then infer that for any $y \in \ell^p_{2,\rho}$ we have

$$\| \rho_-(\Delta H(y))_k \| = \rho^{-k} \| \Delta H(y_k) \| \leq \rho^{-k} \beta \| y_k \|. \quad (102)$$

With $y \in \ell^p_{2,\rho}$, i.e., $\sum_{k=0}^{\infty} \rho^{-2k} \| y_k \|^2$ is finite, we conclude that $\rho_-(\Delta H(y)) \in \ell^p_2$ for any $y \in \ell^p_{2,\rho}$, hence $\Delta H : \ell^p_{2,\rho} \to \ell^p_2$, and $\Delta H_{\rho} : \ell^p_2 \to \ell^p_2$ as required, see also Figure 8 for an overview of the relations. We next show that $\Delta H_{\rho}$ is bounded on $\ell^p_2$. For $\tilde{y} \in \ell^p_2$, we indeed have $y := \rho_+ (\tilde{y}) \in \ell^p_{2,\rho}$ and thus, using (102),

$$\| \Delta H_{\rho}(\tilde{y}) \| = \| \rho_-(\Delta H(y)) \| \leq \beta \| \rho_-(y) \| = \beta \| \tilde{y} \|. \quad (103)$$

Shift of initial conditions. We next need to shift the non-zero initial conditions of (23) to the signal $\tilde{y}_m$ in Figure 2. To this end, let $\tilde{y}_k$, $k \in \mathbb{N}$, denote the output of (23) for $\tilde{x}_0 = 0$. We then have that $\tilde{y}_k$ also is the output of

$$\begin{align*}
\tilde{x}_{k+1} &= \rho^{-1}(A + mB)\tilde{x}_k + \rho^{-1}B\tilde{w}_k \\
\tilde{y}_k &= C\tilde{x}_k + \tilde{y}_{m,k} \\
\tilde{w}_k &= \rho^{-k}\Delta H(A^{\rho}k\tilde{y}_k)
\end{align*} \quad (104)$$

with initial condition $\tilde{x}_0 = 0$ and $\tilde{y}_{m,k} = \rho^{-k}C(A + mB)\tilde{x}_0$, $k \in \mathbb{N}$. We infer that $\tilde{y}_m \in \ell^p_2$ since $\rho^{-1}(A + mB)$ has all eigenvalues in the open unit disk.

We are now ready to apply Theorem 1. Robust performance follows directly from Theorem 1. For robust exponential stability, we note that, by Theorem 1, (27a) implies robust stability of the transformed loop from Figure 2 against $\Delta_p(m, L)$. Hence, $\tilde{y}$ and $\tilde{w}$ in (23) reside in $\ell_2$. Together with the assumption that all eigenvalues of $A + mB$ are located in the open disk of radius $\rho$, we infer that also $\tilde{x} \in \ell^{2p}_2$ in (23). With $y = \rho_+ (\tilde{y})$, $w = \rho_+ (\tilde{w})$, $\xi = \rho_+ (\tilde{x})$, and since $\rho_+$ maps $\ell_2$-signals to $\ell_2$-signals, we conclude that $y, w, \xi$ in (18) reside in $\ell_{2,\rho}$, hence the exponential decay follows.

A.2.4. Proof of Lemma 1

The proof boils down to extending the result for monotone nonlinearities from [21, Lemma 2] to the case of slope-restricted nonlinearities. The following argumentations are similar to those provided in the proof of [6, Theorem 1] where only scalar nonlinearities are considered. We first repeat the definition of a monotone function.

**Definition 9 (Monotone function).** A continuously differentiable function $f : \mathbb{R}^p \to \mathbb{R}^p$ is said to be **monotone** if (1) it is conservative, i.e., there exists $F : \mathbb{R}^p \to \mathbb{R}$ such that $\nabla F(x) = f(x)$ for all $x \in \mathbb{R}^p$ and (2) it fulfills

$$\langle f(x) - f(y), x - y \rangle \geq 0 \quad (105)$$

for all $x, y \in \mathbb{R}^p$.

It is well-known that a function is monotone if it is the gradient of some convex function. Monotone operators $\phi : \ell^p_2 \to \ell^p_2$ are then defined as operators of the form

$$\langle \phi(y) \rangle_k = f(y_k), \quad (106)$$

where $y = [y_0, y_1, \ldots] \in \ell^p_2$, $k \in \mathbb{N}$, and $f : \mathbb{R}^p \to \mathbb{R}^p$ is a monotone function. If $M$ adheres to the conditions given in Lemma 1, by [21, Lemma 2] it is then known that for any monotone operator $\phi : \ell^p_2 \to \ell^p_2$ we have

$$\langle O_T y, M^T \phi(O_T y) \rangle \geq 0 \quad (107)$$

for all $y \in \ell^p_2$, $T \in \mathbb{N}$. To use this result, we next rewrite (29) in the form (107). To this end, consider (29), (30) with $L$ replaced by $\tilde{L} := L + \epsilon$, $\epsilon > 0$. By simple calculations as well as the fact that $\Delta H$ is static and time-invariant, (29) is

$$2\langle (\tilde{L} - m)O_T y - \Delta H(O_T y), M^T \Delta H(O_T y) \rangle \geq 0. \quad (108)$$

In view of (107), for any $y \in \ell^p_2$ we define the auxiliary sequence $\tilde{y} \in \ell^p_2$ by

$$\tilde{y}_k := (\tilde{L} - m)y_k - \Delta H(y_k), \quad k \in \mathbb{N}. \quad (109)$$

Let $\phi : \mathbb{R}^p \to \mathbb{R}$ be defined as

$$\phi(y) = \frac{1}{2}(\tilde{L} - m)y^\top y - (H(y) - \frac{1}{2}my^\top y) = \frac{1}{2}\tilde{L}y^\top y - H(y), \quad (110)$$

which gives $\tilde{y} = \nabla \phi(y)$ and

$$\Delta H(y) = \nabla \phi(y) - (\tilde{L} - m)y. \quad (111)$$

Then (108) reads as

$$2\langle O_T \tilde{y}, M^T \Delta H(O_T y) \rangle \geq 0. \quad (112)$$

Note that $H \in S_{m,L}$ implies $\phi \in S_{m,\tilde{L} - m}$. We then infer from [31, Theorem 26.6, Lemma 26.7] that $\nabla \phi$ has a well-defined inverse $(\nabla \phi)^{-1}$ and $(\nabla \phi)^{-1}$ is itself the gradient.
of a strictly convex function, namely the Legendre conjugate of \( \phi \), cf. [31, Theorem 26.5]. Thus, we can write (112) as

\[
2(OT\bar{y}, M^T \Delta_H ((\nabla \phi)^{-1}(OT\bar{y})) \geq 0. \tag{113}
\]

Comparing with (107), we aim to show that the operator \( \varphi := \Delta_H \circ (\nabla \phi)^{-1} \) is monotone. To this end, first note that, since \((\nabla \phi)^{-1}\) is the gradient of a strictly convex function, there exists some function \( F : \mathbb{R}^p \to \mathbb{R} \) such that \((\varphi(\bar{y}))_k = \nabla F(\bar{y}_k)\). We next show that \( \nabla F \) fulfills (105), hence \( \bar{w} \) is a monotone operator. By (111) we infer that

\[
\varphi(\bar{y}) = \Delta_H\left((\nabla \phi)^{-1}(\bar{y})\right) = \bar{y} - (L - m) (\nabla \phi)^{-1}(\bar{y}). \tag{114}
\]

Thus, the Hessian of \( F \) is given by

\[
\nabla^2 F(\bar{y}_k) = (L - m) \left( \nabla^2 \phi (\nabla \phi)^{-1}(\bar{y}_k) \right)^{-1} - I. \tag{115}
\]

Now, since \( \phi \in \mathcal{S}_{\epsilon L - m} \), we have by [28, Theorem 2.1.6, Theorem 2.1.11] that \( \epsilon I \leq \nabla^2 \phi(y) \leq (L - m + \epsilon)I \) for all \( y \in \mathbb{R}^p \), and hence \( \nabla^2 F(\bar{y}_k) \succeq \left( \frac{L - m}{L - m} - 1 \right)I = 0 \) for all \( \bar{y}_k \in \mathbb{R}^p \). Consequently, \( F \) is convex [28, Theorem 2.1.4] and we conclude that \( \varphi \) is a monotone operator, thus (108) holds for all \( y \in \ell_2^p, T \in \mathbb{N} \), \( \epsilon > 0 \). For arbitrary but fixed \( y \in \ell_2^p, T \in \mathbb{N} \), we can then take the limit \( \epsilon \to 0 \) in (108) since the left-hand side is continuous in \( \epsilon \) and infer that (29) holds, thus concluding the proof.

### A.2.5. Proof of Lemma 2

Let \( \phi_T = OT \phiOT, T \in \mathbb{N} \). Since \( \rho_+(\ell_2^p) = \ell_2^p \), we infer from (32) that

\[
\left\langle \begin{bmatrix} \rho_+(y) \\ \Delta(\rho_+(y)) \end{bmatrix}, \phi_T \begin{bmatrix} \rho_+(y) \\ \Delta(\rho_+(y)) \end{bmatrix} \right\rangle \geq 0 \tag{116}
\]

for all \( y \in \ell_2^p, \Delta \in \Delta, T \in \mathbb{N} \). With \( \rho_+ \circ \rho_- = \text{id}_{\ell_2^p} \), this is equivalently formulated as

\[
\left\langle \begin{bmatrix} \rho_+(y) \\ \rho_+(\Delta(\rho_+(y))) \end{bmatrix}, \phi_T \begin{bmatrix} \rho_+(y) \\ \rho_+(\Delta(\rho_+(y))) \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} y \\ \Delta(\rho_+(y)) \end{bmatrix}, \phi_T \begin{bmatrix} y \\ \Delta(\rho_+(y)) \end{bmatrix} \right\rangle \geq 0 \tag{117}
\]

for all \( y \in \ell_2^p, \Delta \in \Delta, T \in \mathbb{N} \). Since \( \Delta(\ell_2^p) \subset \ell_2^p \), we have \( \Delta(\ell_2^p) \subset \ell_2^p \); hence, by the assumption that \( \phi \) is bounded on \( \ell_2^p \), we can take the limit \( T \to \infty \) in (117) to obtain that (33) holds.

### A.2.6. Proof of Lemma 3

Observe that \( M_\rho = \bar{M}_\rho \otimes Ip \) with \( \bar{M}_\rho = \text{Toep}(\begin{bmatrix} m_{-\ell} & \ldots & m_{-\ell+1} \\ m_{\ell-1} & \ldots & m_{\ell} \end{bmatrix}) \) and define \( M = \rho_- M_\rho \rho_- = (\rho_- M_\rho \rho_-) \otimes Ip \). We compute

\[
\rho_- M_\rho \rho_- = \begin{bmatrix} \bar{m}_0 & \rho_1 & \rho_2 & \cdots \\ \rho_1 & \bar{m}_0 & \rho_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}. \tag{118}
\]

Thus, \( \rho_- M_\rho \rho_- \in \mathcal{M} \) if, for each finite \( k \in \mathbb{N} \),

\[
\rho^{-k} i \bar{m}_i \leq 0 \text{ for } i \in \{ -\ell, \ldots, \ell \}, i \neq 0 \tag{119a}
\]

\[
\rho^{-2k} \sum_{i=-\ell}^{\ell} \bar{m}_i \rho^{-i} \geq 0, \rho^{-2k} \sum_{i=-\ell}^{\ell} \bar{m}_i \rho^{-i} \geq 0. \tag{119b}
\]

With \( \rho > 0 \), we can get rid of the dependency on \( k \) such that the above conditions are equivalent to requiring that the Toeplitz operator in (37) is doubly hyperdeterminant. We hence conclude by Lemma 1 that under these conditions the inequality (32) holds with

\[
\phi = W^T \begin{bmatrix} 0 & M^T \\ M & 0 \end{bmatrix} W, \tag{120}
\]

where \( W \) is defined in (30) and \( M = (\rho_- M_\rho \rho_-) \otimes Ip \).

Additionally, \( \rho_+ \circ \phi \circ \rho_- \) is bounded on \( \ell_2^p \), since \( M \) is bounded on \( \ell_2^p \) by assumption. Hence, using also that \( \Delta_{H,\rho} : \ell_2^p \to \ell_2^p \) as discussed in Section 4.3, we may apply Lemma 2 and conclude that

\[
\left\langle \begin{bmatrix} y \\ \Delta_{H,\rho}(y) \end{bmatrix}, \rho_+ W^T \begin{bmatrix} 0 & M^T \\ M & 0 \end{bmatrix} W \begin{bmatrix} y \\ \Delta_{H,\rho}(y) \end{bmatrix} \right\rangle \geq 0 \tag{121}
\]

for any \( y \in \ell_2^p, \Delta_{H,\rho} \in \Delta_{p}(m, L) \), thus concluding the proof.

### A.2.7. Proof of Theorem 5

The result virtually is a direct consequence of Theorem 3 employing the class of multipliers introduced in Theorem 4. We first note that under the assumption that \( A + mBC \) has all eigenvalues in the open disk of radius \( \rho \), the same holds for \( A_\epsilon \) by (48) since \( \psi_\Delta \) has all its poles at zero. With \( \psi_{\Delta} M_{\Delta}(M_+, M_-, M_0) \psi_{\Delta} = I \) and by the KYP-Lemma we then observe that (49a) is equivalent to the frequency domain inequality (27a) for exponential stability in Theorem 3. Further, by Theorem 4 and (49b), we infer that \( IQC(\Pi_{\rho}, \bar{y}, \Delta_{H,\rho}(\bar{y})) \geq 0 \) holds for all \( \Pi_{\rho} \in \Pi_{\Delta,\rho}^0 \) and all \( \Delta_{H,\rho} \in \Delta_{p}(m, L) \). Hence, by Theorem 3, we conclude that the origin is globally robustly exponentially
stable against $\Delta(m, L)$ with rate $\rho$ for (18), which in turn implies the last claim in Theorem 5, thus concluding the proof.

### A.2.8. Proof of dimensionality reduction in Theorem 5

We follow the arguments of [17] to prove this statement. It is clear that if (50a) has a solution $\bar{P}, m_+, m_-, m_0$, then $P = \bar{P} \otimes I_p, M_+ = m_+ \otimes I_p, m_- \otimes I_p, m_0 \otimes I_p$. Now suppose that (49a) has a solution $P, M_+, M_-, M_0$. Multiply (49a) from right and left by $\text{blkdiag}(I_{n_c} \otimes e_1, I_{n_c} \otimes e_1)$ and its transpose, where $e_1 \in \mathbb{R}^{p \times 1}$ is the first unit vector. Observing that for any $U = \bar{U} \otimes I_p, \bar{U} \in \mathbb{R}^{n_c \times r}$, we have $U(I_n \otimes e_1) = (\bar{U} \otimes I_p)(I_{n_c} \otimes e_1) = \bar{U} \otimes e_1 = (I_n \otimes e_1)(U \otimes 1)$, we obtain

$$
\begin{bmatrix}
\bar{A}_c & \bar{B}_c \\
I & 0
\end{bmatrix}^T
\begin{bmatrix}
\bar{P} & 0 \\
0 & \bar{P}
\end{bmatrix}
\begin{bmatrix}
\bar{A}_c & \bar{B}_c \\
I & 0
\end{bmatrix}
+ \begin{bmatrix}
\bar{C}_c & \bar{D}_c
\end{bmatrix}^T(I_{3p} \otimes e_1)^T M_{\Delta}(I_{3p} \otimes e_1) \begin{bmatrix}
\bar{C}_c & \bar{D}_c
\end{bmatrix} < 0
$$

with $\bar{P} = (I_n \otimes e_1)^T P(I_n \otimes e_1)$. Let $M_{\Delta}(M_+, M_-, M_0)$ be defined as in (46) explicitly including the dependency on $M_+, M_-, M_0$, and note that

$$
\begin{align*}
(I_{3p} \otimes e_1)^T M_{\Delta}(M_+, M_-, M_0)(I_{3p} \otimes e_1) &= (I_{3p} \otimes e_1)^T (M_{\Delta}(m_+, m_-, m_0) \otimes I_p)(I_{3p} \otimes e_1) \\
&= (I_{3p} \otimes e_1)^T (M_{\Delta}(m_+, m_-, m_0) \otimes e_1) \\
&= M_{\Delta}(m_+, m_-, m_0).
\end{align*}
$$

Hence, $\bar{P} = (I_n \otimes e_1)^T P(I_n \otimes e_1)$, $m_+, m_-, m_0$ is a solution to (50a) which concludes the proof.

### A.2.9. Proof of Theorem 6

We first observe that robust stability follows directly from Theorem 5 noting that $A_c = A_c(1), B_{c,1} = B_{c,1}, C_{c,1} = C_{c,1}, D_{c,11} = D_c$ by (158) and $C_{c,1} \geq 0$; hence (58a) implies that $\rho = 1$. Consider the complete transfer matrix $\Phi_e$ as defined in (55) and let a state-space realization be given as in (56) specifically stated in (158) in Appendix A.3.2. Let $x_c$ denote the solution of

\begin{align}
\dot{x}_{c,k+1} &= A_c x_{c,k} + B_{c,1} w_k + B_{c,2} w_{p,k} \quad (124a) \\
w_k &= \Delta(CN^T x_{c,k}) \quad (124b)
\end{align}

with initial condition $x_{c,0} = 0$. Note that with $\bar{x}_{c,k} = N^T x_{c,k}$ we have

$$
\bar{x}_{c,k+1} = (A + mBC)\bar{x}_{c,k} + B\Delta(C\bar{x}_{c,k}) + B_p w_{p,k}.
$$

(125)

i.e., $\bar{x}_{c,k}$ follows the same dynamics as (18) and the performance output is given by $y_{p,k} = C_p \bar{x}_{c,k} = C_{c,2} x_{c,k}$.

Consequently, multiplying (58a) from left by $[x_{c,k}^T \ w_k^T]$ and from right by its transpose, we infer

\begin{align*}
(\ast)^T P_p (A_c x_{c,k} + B_{c,1} w_k) - x_{c,k}^T P_p x_{c,k} \\
\leq y_{p,k}^T y_{p,k} - (\ast)^T M_{\Delta}(C_{c,1} x_{c,k} + D_{c,11} w_k).
\end{align*}

(126)

Let

$$
P_p' = N^T P_p N, \quad P_{p'} = P - N N^T P_p N N^T,
$$

(127)

i.e., $P_p = P_{p'} + N P_{p'} N^T$ and $N^T P_{p'} N = 0$. By (158), we have $B_{c,2} = NB_0$; hence we infer that $B_{c,2}^T P_p B_{c,2} = B_{c,2}^T N P_{p'} N^T B_{c,2}$. We then calculate

\begin{align*}
x_{c,k+1}^T P_{p} x_{c,k+1} - x_{c,k}^T P_{p} x_{c,k} \\
= (\ast)^T P_p (A_c x_{c,k} + B_{c,1} w_k) - x_{c,k}^T P_p x_{c,k} \\
+ 2 x_{c,k+1}^T P_{p} B_{c,2} w_{p,k} + w_{p,k}^T B_{c,2}^T N P_{p'} N^T B_{c,2} w_{p,k}.
\end{align*}

(128)

Using (126), (128) and summing from $k = 0$ to $k_{\text{max}} \in \mathbb{N}$, we obtain

$$
\sum_{k=0}^{k_{\text{max}}} x_{c,k+1}^T P_{p} x_{c,k+1} - x_{c,k}^T P_{p} x_{c,k} \\
\leq \sum_{k=0}^{k_{\text{max}}} \left(- y_{p,k}^T y_{p,k} - (\ast)^T M_{\Delta}(C_{c,1} x_{c,k} + D_{c,11} w_k) \right) \\
+ 2 \sum_{k=0}^{k_{\text{max}}} x_{c,k+1}^T P_{p} B_{c,2} w_{p,k} + w_{p,k}^T B_{c,2}^T N P_{p'} N^T B_{c,2} w_{p,k}.
$$

(129)

We note that $w_{p,k}$ is a discrete-time white noise process with independent components; hence, $w_{p,k}$ and $x_{c,k+1}$ are independent for any $k \in \mathbb{N}$, $E(w_{p,k}) = 0$ and $E(w_{p,k}^T X w_{p,k}) = \text{tr}(X)$ for any $X \in \mathbb{R}^{p_{\text{rep}} \times p_{\text{rep}}}$. Taking expectations, we hence infer

$$
\text{E} \left( \sum_{k=0}^{k_{\text{max}}} x_{c,k+1}^T P_{p} x_{c,k+1} - x_{c,k}^T P_{p} x_{c,k} \right) \\
\leq \text{E} \left( \sum_{k=0}^{k_{\text{max}}} - y_{p,k}^T y_{p,k} - (\ast)^T M_{\Delta}(C_{c,1} x_{c,k} + D_{c,11} w_k) \right) \\
+ k_{\text{max}} \text{tr}(B_{c,2}^T N P_{p'} N^T B_{c,2}).
$$

(130)

With $x_{c,0} = 0$ we further note that for any $k_{\text{max}} \in \mathbb{N}$ we have

$$
\sum_{k=0}^{k_{\text{max}}} x_{c,k+1}^T P_{p} x_{c,k+1} - x_{c,k}^T P_{p} x_{c,k} = x_{c,k_{\text{max}}+1}^T P_{p} x_{c,k_{\text{max}}+1}.
$$

(131)
Combining (130) with (131) and using (58b) we then obtain
\[
\frac{1}{k_{max}} \sum_{k=0}^{k_{max}} E(y_p^T y_p k) + \frac{1}{k_{max}} E(x_{c,kmax+1}^T P_x x_{c,kmax+1}) \leq -\frac{1}{k_{max}} E\left( \sum_{k=0}^{k_{max}} (\cdot)^T M_{\Delta} (C_{c,1} x_{c,k} + D_{c,11} w_k) \right) + \gamma^2
\]

Now note that, for a fixed realization \( w_{p,k} \),
\[
\sum_{k=0}^{\infty} (\cdot)^T M_{\Delta} (C_{c,1} x_{c,k} + D_{c,11} w_k)
\]

is a time-domain representation of IQC \( \psi_{\Delta} M_{\Delta} \psi_{\Delta}, y, \Delta_H(y) \). Since (29) is a hard IQC, we infer that
\[
\sum_{k=0}^{k_{max}} (\cdot)^T M_{\Delta} (C_{c,1} x_{c,k} + D_{c,11} w_k) \geq 0
\]

for any realization \( w_{p,k} \) and any \( k_{max} \), and the latter inequality persists to hold after taking expectations. Additionally, with \( P_p \) being positive definite, \( E(x_{c,k}^T P_x x_{c,k}) \) is positive as well. Thus, if we let \( k_{max} \) tend to infinity, we finally obtain from (132)
\[
\lim_{k_{max} \to \infty} \frac{1}{k_{max}} \sum_{k=0}^{k_{max}} E(y_p^T y_p k) \leq \gamma^2,
\]

hence concluding the proof.

### A.2.10. Proof of Theorem 8

Robust exponential stability follows directly from Theorem 7. For robust performance, first note that (71a), (71b) implies that (67) holds with \( P_{22} = P_{22}, Q_A = P_{22} A, Q_B = P_{22} B \), and making use of the specific state-space realization from Appendix A.3.2, thus (71a), (71b) implies (58a). Note further that \( U_p \) as defined in (68) is independent of \( A, B \) by the specific structure of \( P_p \). We further note that (71c) holds if and only if \( P_{22} > 0 \) and \( Z > Q_B P_{22}^{-1} Q_B \). Together with the trace condition (71d) then implies that \( \text{tr}(B^\top P_{22} B) < \gamma^2 \). Now note that with the specific state-space realization Appendix A.3.2 and by the definition of \( P_p \) we have that \( \text{tr}(B_{c,2}^\top P_p B_{c,2}) = \text{tr}(B^\top P_{22} B) \), hence (71d) together with (71d) implies (58b), thus concluding the proof.

### A.3. Additional material

#### A.3.1. Comparison to [15]

In the following we compare the set of multipliers derived in [15] to the multipliers defined in (41). For the sake of simplicity we limit ourselves to the case of monotone uncertainties \( \Delta \); we emphasize that the same applies to general slope-restricted uncertainties with minor adaptations. Let \( M = \text{Toep}(\{m_j\}_{j=\ell_-, \ldots, \ell_+}) \), \( \bar{M} = \text{Toep}(\{m_j\}_{j=\ell_-, \ldots, \ell_+}) \), be two Toeplitz operators \( M, \bar{M} : \ell_1 \to \ell_2 \). In contrast to [15], we consider only finite Toeplitz operators here; the following arguments also apply to infinite operators. In [15], the author considers operators \( \bar{M} \) with the property \( \bar{M} y = y - h \ast y \) for all \( y \in \ell_1^\ell \), where \( \ast \) is the convolution operator and \( h(z) = \sum_{j=\ell_-}^{\ell_+} h_j z^{-j} \), i.e., \( \bar{M} \) defines the transfer function \( E_{\bar{M}}(z) = 1 - h(z) \). It is then shown that if
\[
h_j \geq 0 \quad \text{for } j \in \{-\ell_-, \ldots, \ell_+\}
\]

then
\[
\langle M y, \Delta(y) \rangle = \langle \rho_- \bar{M} y, \rho_- \Delta(y) \rangle \geq 0
\]

for all \( y \in \ell_2 \), where \( (x, y)_w = \langle \rho_- x, \rho_- y \rangle \) denotes the weighted inner product on \( \ell_2 \). Condition (135) is equivalently formulated in the parameters \( \bar{m}_j, j \in \{-\ell_-, \ldots, \ell_+\} \), as
\[
\bar{m}_j \leq 0 \quad \text{for } j \in \{-\ell_-, \ldots, \ell_+\} \setminus \{0\} \quad \text{and} \quad \bar{m}_0 \geq -1
\]

Similarly, in Lemma 3 we have shown that if
\[
m_j \leq 0 \quad \text{for } j \in \{-\ell_-, \ldots, \ell_+\} \setminus \{0\}
\]

then
\[
\langle M y, \Delta(y) \rangle = \langle \rho_- M y, \rho_- \Delta(y) \rangle \geq 0
\]

for all \( y \in \ell_2 \), i.e., equivalently,
\[
\langle M \rho_- y, \rho_- \Delta(y) \rangle \geq 0
\]

for all \( y \in \ell_2 \). We hence note that the operators \( \bar{M} \) and \( \bar{M} \) in (136) and (140) are related by \( \rho_- \bar{M} = M \rho_- \) and, therefore,
\[
\bar{m}_j = \rho_- m_j \quad \text{for } j \in \{-\ell_-, \ldots, \ell_+\}
\]

We are now ready to compare the conditions (137) from [15] with the proposed conditions (138). To this end,
we express (137) in terms of \( m_j \) instead of \( \bar{m}_j \), i.e., we have

\[
\rho^{-j} m_j \leq 0 \text{ for } j \in \{-\ell_-, \ldots, \ell_+\} \setminus \{0\} \text{ and } m_0 \geq -1
\]

(142a)

\[
\sum_{j=-\ell_-}^{\ell_+} \rho^j m_j \max\{1, \rho^{-2j}\} \geq 0.
\]

(142b)

Note that with \( \rho \in (0, 1) \) this holds if and only if

\[
m_j \leq 0 \text{ for } j \in \{-\ell_-, \ldots, \ell_+\} \setminus \{0\}
\]

(143a)

\[
\sum_{j=-\ell_-}^{\ell_+} \rho^j m_j + \sum_{j=-\ell_-}^{\ell_+} \rho^{-j} m_j = \sum_{j=-\ell_-}^{\ell_+} \rho^{-|j|} m_j \geq 0.
\]

(143b)

We first note that (143a) is the same as (138a) and next show that (143b) implies (138b), (138c). To this end, observe that for any \( j \in \mathbb{Z} \setminus \{0\} \) we have

\[
\rho^{\ell_j} m_j \geq m_j \geq \rho^{-\ell_j} m_j
\]

(144)

since \( \rho \in (0, 1) \) and \( m_j \leq 0 \). Consequently,

\[
\sum_{j=1}^{\ell_+} \rho^{-j} m_j \leq \sum_{j=1}^{\ell_+} \rho^j m_j
\]

(145a)

\[
\sum_{j=-\ell_-}^{0} \rho^j m_j = \sum_{j=-\ell_-}^{0} \rho^{-|j|} m_j \leq \sum_{j=1}^{\ell_-} \rho^j m_j = \sum_{j=1}^{\ell_-} \rho^{-j} m_j.
\]

(145b)

Thus, (143b) implies (138b) by (145b) and (138c) by (145a). The converse is in general not true as we will show next. For the sake of a clearer presentation, we suppose that \( \ell_- = \ell_+ \). Summing up the two inequalities (138b), (138c), we obtain

\[
m_0 + \frac{1}{2} \sum_{i=1}^{\ell_-} (\rho^i + \rho^{-i})(m_i + m_{-i}) \geq 0.
\]

(146)

Note further that (143b) is equivalently formulated as

\[
m_0 + \sum_{i=1}^{\ell_-} \rho^{-i} (m_i + m_{-i}) \geq 0.
\]

(147)

and hence \( m_0 \) does fulfill (147) only if \( m_i = m_{-i} = 0 \) for \( i = 1, 2, \ldots, \ell_- \). We hence conclude that the conditions on the anticausal parts proposed in the present paper are less restrictive than those derived in [15]. We illustrate the potential benefits by means of an example. Since anticausal multipliers do not yield an improvement of convergence rate guarantees for the optimization algorithm analysis problem, we consider a numerical example unrelated to the problem at hand. More precisely, we consider a stable linear system described by the following transfer function

\[
G(z) = \frac{z + 0.8111}{z^4 + 1.552z^3 + 0.6995z^2 + 0.06042z - 0.01241}
\]

(150)

in feedback with a static, slope-restricted uncertainty \( \Delta \) in the sector \((0, 1)\), i.e., \( \Delta \in \Delta(0, 1) \). Employing Theorem 5, we then determine upper bounds on the exponential convergence rates. The resulting convergence rates are displayed in Table 2. Compared to the multipliers introduced in [15], the multipliers introduced in the present paper lead to an improvement of 10.85%.

### A.3.2. State-space realizations

In the following we provide explicit state-space realizations of all transfer functions as required for implementation; an overview is given in Table 3.

#### State-space realizations of \( \psi_- \), \( \psi_+ \). The state-space realization of \( \psi_+ \) as defined in (45) is given by

\[
A_+ = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \cdots & 1 \\ 0 & \cdots & 0 & \cdots & 0 \\ 1 & \cdots & 0 & \cdots & 0 \end{bmatrix}, \quad B_+ = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix},
\]

(151a)

\[
C_+ = \begin{bmatrix} 0 \\ \cdots \\ 1 \\ \cdots \\ 0 \end{bmatrix}, \quad D_+ = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 0 \end{bmatrix}.
\]

(151b)

For \( \psi_- \) as defined in (45), \( A_-, B_- \) have exactly the same structure as \( A_+, B_+ \) but may be of different size, while \( C_- = I_{\ell_-} \) and \( D_- = 0 \).
State-space realization of \( \psi_\Delta \). The state-space realization of \( \psi_\Delta \) as defined in (47) is given by

\[
A_\Delta = \begin{bmatrix} A_- & 0 \\ 0 & A_+ \end{bmatrix} \otimes I_p, \quad B_\Delta = \begin{bmatrix} B_- \otimes I_p & 0 \\ 0 & B_+ \otimes I_p \end{bmatrix} \tilde{W}_p, 
\]

\[
C_\Delta = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ C_- & 0 \\ 0 & 0 \\ 0 & C_+ \end{bmatrix} \otimes I_p, \quad D_\Delta = \begin{bmatrix} I_p & 0 \\ 0 & I_p \\ 0 & I_p \\ 0 & D_- \\ 0 & I_p \\ 0 & 0 \end{bmatrix}. 
\]  
(152a)

Note that \( D_{\psi_p} = [0_{n_{\psi_p} \times n_{\psi_p}} \quad I_{n_{\psi_p}}] \) in the case of \( H_2 \)-performance. By standard rules for series connections, the state-space realization of \( \psi_c \) as defined in (55) is then given by

\[
A_c = \begin{bmatrix} A_2 & B_2 C_1 \\ 0 & A_1 \end{bmatrix}, \quad B_c = \begin{bmatrix} B_2 D_1 \\ B_1 \end{bmatrix}, 
\]

\[
A_c = A_c(1), \quad B_c = \begin{bmatrix} 0 \\ B_p \end{bmatrix}, 
\]

\[
C_c = \begin{bmatrix} C_2 \\ D_2 C_1 \end{bmatrix}, \quad D_c = \begin{bmatrix} C_c(1) \\ D_c \end{bmatrix}, 
\]

\[
= \begin{bmatrix} 0 & D_{\psi_p} & 0 \\ 0 & C_p \end{bmatrix} \tilde{W}_p, \quad = \begin{bmatrix} 0 \\ D_{\psi_p} \end{bmatrix} [I_{n_{\psi_p}} 0]. 
\]

(158b)

A.3.3. Derivation of (65)

Consider (49a) and note that we may rewrite its left-hand side as

\[
U(A_c(\rho), B_c, C_c(\rho), D_c, P, M_\Lambda) - [\star]^T \Gamma \Gamma^T P \Gamma \Gamma^T [A_c(\rho) \quad B_c] \prec 0. 
\]

(159)

Utilizing that \( P_{22} = N^T P N > 0 \) and employing Schur complements, (65) immediately follows.

A.3.4. Design of parametrized optimization algorithms

In this section we consider algorithms of the form (12) where \( A, B \) are parametrized by \( K_i \in \mathbb{R}^{p \times p}, i = 1, 2, \ldots, n \), as in (72) and \( C, D \) are given by (62). Similar to the present paper, we interpret the gradient \( \nabla H \) as an uncertainty and embed the design problem into a robust state-feedback synthesis problem. We follow the procedure from [23], [24] in a discrete-time setting and extend it to structured objective functions (83). To this end, consider an algorithm of the form (12) with \( A, B \) as in (72) together with the following state transformation

\[
\xi_{k+1} := \nabla H(C x_k), \quad \xi_{k,i} = x_{k,i} \text{ for } i = 2, 3, \ldots, n. 
\]

(160)

This yields the transformed dynamics

\[
\xi_{k+1} = A_2 \xi_k + B_1 K \xi_k + B_2 \left( \nabla H(C(A_1 + I)x_k + CB_1 K \xi_k) - \nabla H(C x_k) \right), 
\]

(161)

where

\[
A_2 = A_\Delta, \quad B_2 = \begin{bmatrix} B_2 & 0 \\ 0 & I \end{bmatrix}, 
\]

\[
C_2 = \begin{bmatrix} C_\Delta \\ 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} D_\Delta & 0 \\ 0 & D_{\psi_p} \end{bmatrix}. 
\]

(162)
We prove this result by showing that the equilibrium as a standard robust state-feedback problem and then class of objective functions.

Lemma 7. bounds for the uncertain terms in (161). also includes the case when no structural assumptions are known constants where \( H \) is unknown but fulfills \( H_2 \in S_{m_2,L_2} \) for some known constants \( L_2 \geq m_2 > 0 \). Note that this trivially also includes the case when no structural assumptions are taken on the objective function, simply by letting \( H_1 = 0 \), \( T = 1 \). The following result then follows by using sector bounds for the uncertain terms in (161).

Lemma 7. Let \( L_2 \geq m_2 > 0 \), \( n \in \mathbb{N}_{>0} \), \( p \in \mathbb{N}_{>0} \) and \( H_1 \in \mathbb{R}^{p \times p}, m_1 I \leq H_1 \leq L_1 I, L_1 \geq m_1 > 0, T \in \mathbb{R}^{q \times p}, q \in \mathbb{N}_{>0}, \) be given. Let \( C \in \mathbb{R}^{p \times n_p} \) be defined as in (62). Fix \( \rho \in (0,1) \). Suppose there exist \( M \in \mathbb{R}^{p \times n_p}, Q \in \mathbb{R}^{p \times p}, \) such that

\[
\begin{bmatrix}
Q & 0 & \bar{A}Q + \bar{B}M & \frac{1}{2}(L_2 - m_2)\bar{G}
\end{bmatrix}
\begin{bmatrix}
I
0
\tilde{p}^2Q
0
\end{bmatrix}
> 0,
\]

where

\[
\bar{A} = A_2 + B_2H_1C_1 + \frac{1}{2}(L_2 + m_2)B_2T^TCB_1;
\]

\[
\bar{B} = B_1 + B_2H_1CB_1 + \frac{1}{2}(L_2 + m_2)B_2T^TCB_1;
\]

\[
\bar{G} = B_2T^T.
\]

Then, with \( K = MQ^{-1} \) and \( A, B \) given by (72) and \( C, D \) as in (62), the equilibrium \( x^* = Dz^* \) is globally exponentially stable with rate \( \rho \) for (12) for all \( H \) in the form (163) with \( H_2 \in S_{m_2,L_2} \).

Proof. We prove this result by showing that the equilibrium \( \xi^* = 0 \) is globally exponentially stable with rate \( \rho \) for the transformed dynamics (161) for the considered class of objective functions \( H \). To this end, we write (161) as a standard robust state-feedback problem and then use quadratic Lyapunov functions together with an S-procedure argument. We first calculate for the uncertain terms in (161)

\[
\nabla H(C(A_1 + I)x_k + CB_1\xi_k) - \nabla H(Cx_k)
= H_1(C(A_1 + I)x_k + CB_1\xi_k) - H_1Cx_k
+ T^T\nabla H_2(TC(A_1 + I)x_k + TCB_1\xi_k) - T^T\nabla H_2(TCx_k)
= H_1(CA_1\xi_k + CB_1\xi_k) + T^T(\nabla H_2(TC(A_1 + I)x_k + TCB_1\xi_k) - \nabla H_2(TCx_k))
\]

and then define the uncertainty as

\[
\nu_k = \nabla H_2(TC(A_1 + I)x_k + TCB_1\xi_k) - \nabla H_2(TCx_k) - \beta TC(A_1 + B_1K)\xi_k
\]

with \( \beta = \frac{1}{2}(L_2 + m_2) \). The motivation behind this definition is that the uncertainty, seen as a function of \( TC(A_1 + B_1K)\xi_k \), is sector bounded in the sector \([-L_2 - \beta], L_2 - \beta] \). More precisely, for any \( \xi_k \in \mathbb{R}^{n_p} \), it fulfills the inequality

\[
\begin{bmatrix}
\xi_k

w_k
\end{bmatrix}^T
\begin{bmatrix}
(L_2 - \beta)\Gamma^T\Gamma
0
-
\frac{1}{L_2 - \beta}I
\end{bmatrix}
\begin{bmatrix}
\xi_k

w_k
\end{bmatrix} \geq 0,
\]

where \( \Gamma = TC(A_1 + B_1K) \). The transformed dynamics (161) then read as

\[
\tilde{\xi}_{k+1} = (\bar{A} + B_2H_1C_1 + \beta B_2T^TCA_1)\xi_k + (B_1 + B_2H_1CB_1 + \beta B_2T^TCB_1)K\xi_k + B_2T^Tw_k.
\]

Using the shorthand notation (165), (169) can equivalently be written as

\[
\tilde{\xi}_{k+1} = (\bar{A} + BK)\xi_k + \bar{G}w_k.
\]

Hence the problem of finding parameters \( K_i \) is a standard robust state-feedback problem and the proof is standard from now on. Consider a quadratic Lyapunov function candidate \( V : \mathbb{R}^{n_p} \rightarrow \mathbb{R} \) defined as \( V(\xi) = \xi^TP\xi \) with \( P \succ 0 \). For exponential convergence with rate \( \rho \) of (170) for all \( w \) that fulfill (168) – and hence exponential convergence

| Transfer function | Definition | State-space realization |
|-------------------|-----------|------------------------|
| \( \psi_+ \in \mathcal{RH}_2^{p \times 1} \) | (45) | \((A_+, B_+, C_+, D_+)\) |
| \( \psi_- \in \mathcal{RH}_2^{p \times 1} \) | (45) | \((A_-, B_-, C_-, D_-)\) |
| \( \psi_\Delta \in \mathcal{RH}_2^{p \times (n_p + 4) \times 2p} \) | (47) | \((A_\Delta, B_\Delta, C_\Delta, D_\Delta)\) |
| \( \psi_c \in \mathcal{RH}_2^{p \times (n_p + 4) \times p} \) | (48) | \((A_c, B_c, C_c, D_c)\) |
| \( \psi_e \in \mathcal{RH}_2^{p \times (n_p + 4) \times q \times 2(p + n_p)} \) | (55) | \((A_c, B_c, C_c, D_c)\) |

Table 3. Overview of all required state-space realizations.
with rate \( \rho \) of the original algorithm to the minimizer of \( H \) for all \( H \in S_{m,L} \) – it is sufficient that there exists a \( P > 0 \) such that

\[
V(\xi_{k+1}) - \rho^2 V(\xi_k) =
\begin{bmatrix}
\star \\
\star
\end{bmatrix}^T
\begin{bmatrix}
0 & -\rho^2 P \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\bar{A} + \bar{B}K & G \\
I & 0
\end{bmatrix}
\begin{bmatrix}
\xi_k \\
w_k
\end{bmatrix} < 0
\tag{171}
\]

for all \( [\xi_k^T \ w_k^T]^T \neq 0 \) that fulfill (168). By the \( S \)-procedure it is hence sufficient that there exist \( P > 0 \) and \( \lambda \geq 0 \) such that

\[
\begin{bmatrix}
\bar{A} + \bar{B}K & G \\
I & 0
\end{bmatrix}^T
\begin{bmatrix}
P & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\bar{A} + \bar{B}K & G \\
I & 0
\end{bmatrix}
+ \lambda
\begin{bmatrix}
(L_2 - \beta)^T & 0 \\
0 & -\frac{1}{L_2 - \beta} I
\end{bmatrix} < 0.
\tag{172}
\]

We exclude the case \( \lambda = 0 \) and thus set \( \lambda = 1 \) without loss of generality. We can write the above inequality equivalently as

\[
\begin{bmatrix}
\star \\
\star
\end{bmatrix}^T
\begin{bmatrix}
P^{-1} & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
P(\bar{A} + \bar{B}K) & P\bar{G} \\
\sqrt{L_2 - \beta} & 0
\end{bmatrix}
- \begin{bmatrix}
\rho^2 P & 0 \\
0 & \frac{1}{L_2 - \beta} I
\end{bmatrix} < 0.
\tag{173}
\]

Since \( P \) is assumed to be a positive definite matrix, we can employ Schur complements such that the above matrix inequality holds if and only if

\[
\begin{bmatrix}
P & 0 & P(\bar{A} + \bar{B}K) & P\bar{G} \\
* & I & \sqrt{L_2 - \beta} & 0 \\
* & * & \rho^2 P & 0 \\
* & * & * & \frac{1}{L_2 - \beta} I
\end{bmatrix} > 0.
\tag{174}
\]

We multiply from left and right by the positive definite block diagonal matrix \( \text{blkdiag}(\sqrt{L_2 - \beta} P^{-1}, I, \sqrt{L_2 - \beta} P^{-1}, \sqrt{L_2 - \beta} I) \) such that we obtain

\[
\begin{bmatrix}
(L_2 - \beta) P^{-1} & 0 & (L_2 - \beta)(\bar{A} + \bar{B}K) P^{-1} & (L_2 - \beta) \bar{G} \\
* & I & (L_2 - \beta) P^{-1} & 0 \\
* & * & \rho^2 (L_2 - \beta) P^{-1} & 0 \\
* & * & * & I
\end{bmatrix} > 0.
\tag{175}
\]

We note that the additional scaling is not required and included for numerical purpose only. The variable transformation \( Q = (L_2 - \beta) P^{-1}, M = (L_2 - \beta) K P^{-1} \) then gives (164), thus concluding the proof. \( \square \)