Abstract

We obtain explicit formulas for enumerating 3-regular one-face maps on orientable and non-orientable surfaces of a given genus $g$ up to all symmetries. We use recent analytical results obtained by Bernardi and Chapuy for counting rooted precubic maps on non-orientable surfaces together with more widely known formulas for counting precubic maps on orientable surfaces. To take into account all symmetries we use a result of Krasko and Omelchenko that allows to reduce this problem to the problem of counting rooted quotient maps on orbifolds.

Keywords: map; surface; orbifold; enumeration; 3-regular maps; sensed maps; unsensed maps
1 Introduction

By a one-face (or unicellular) topological map $M$ on a surface $X$ we will mean a 2-cell imbedding of a connected graph $G$, loops and multiple edges allowed, into a compact connected 2-dimensional manifold $X$ without boundary, such that the only one connected component of $X - G$ is a 2-cell. The $0$-, and $1$–dimensional cells of the map $M$ are its vertices and edges, respectively [1]. In this paper we consider both orientable and non-orientable surfaces without boundary. Every such surface can be characterized by its genus $g$. An orientable surface of genus $g$ is a sphere with $g$ handles. A non-orientable surface of genus $g$ is a sphere with $g$ holes (removed discs) glued with crosscaps (or Mobius bands). Sometimes for a surface $X$ instead of its $g$ we will use its Euler characteristic $\chi$, equal to $2 - 2g$ in the case of an orientable surface $X^+$ and $2 - g$ in the case of a non-orientable surface $X^-$.

Two topological maps $M_1$ and $M_2$ on a surface $X$ are said to be isomorphic if there is a homeomorphism $h$ of $X$ that induces an isomorphism of the underlying graphs $G_1$ and $G_2$. Map isomorphism splits the set of all maps on $X$ into equivalence classes, and each such class is called an unlabelled map. For orientable surfaces we have two types of homeomorphisms, orientation-preserving and orientation-reversing. Unlabelled maps on an orientable surface $X^+$ up to only orientation-preserving homeomorphisms are called sensed maps. Unlabelled maps on an orientable or a non-orientable surface up to all homeomorphisms are called unsensed maps.

A general technique for counting sensed maps was developed by Liskovets [2] (sensed maps on the sphere) and by Mednykh and Nedela [3] (sensed maps on orientable surfaces of a given genus $g$). Their approach reduces the enumerating problem for sensed maps on a surface to counting quotient maps on orbifolds, rooted maps on quotients of this surface under a finite group of automorphisms. Their ideas were further developed in a series of papers devoted to enumeration of sensed hypermaps [4], one-face regular sensed maps [5], one-face maximal unsensed maps [6], regular sensed maps on the torus [7] and regular sensed maps on orientable surfaces of a given genus $g$ [8].

In the past few years appeared some new important results regarding enumeration of unsensed maps on orientable and non-orientable surfaces of a given genus $g$. In the paper [9] unsensed orientable
maps on surfaces regardless of genus were enumerated. In the paper [10] analytical formulas for
the numbers of unsensed \( r \)-regular maps on the torus were obtained. Finally, in [11] the problem
of enumeration of unlabelled maps on genus \( g \) surfaces was solved in the most general formulation for
the first time. The obtained general formulas express the numbers of such maps in the form of a
linear combination of numbers of quotient maps on cyclic orbifolds with integer coefficients. These
coefficients were expressed through the numbers of epimorphisms from fundamental groups \( \pi_1(O) \) of
orbifolds to cyclic groups \( \mathbb{Z}_d \), and exact analytical expressions were derived for these numbers.

In [11] it was pointed out that with the help of these results one can enumerate different types of
unsensed maps on surfaces of a given genus \( g \) (regular maps, one-face maps, etc.) assuming that
we are able to enumerate quotient maps on orbifolds. The main problem with this approach in the
general case is that these orbifolds are surfaces with \( r \) branch points, \( h \) boundary components and \( g \)
handles or cross-caps, and recurrence relations for the numbers of quotient maps on them depend on
a large number of additional parameters. Fortunately, for 3-regular (or cubic) one-face maps most
of these problems can be avoided.

It turns out that in the case of 3-regular maps the orbifolds can be described quite simply. They
are either closed orientable surfaces with several branch points (as in the case of sensed maps), or
surfaces with a single boundary and with possibly some branch points of index 2. In both cases it is
possible to reduce the problem of enumerating quotient maps on the torus to the enumeration of precubic maps — maps that have vertices of degree only 1 and 3 — on surfaces of a given genus. Enumerating
precubic maps on orientable surfaces is a relatively old and well-known problem — it was solved in
the works of Walsh and Lehman [12]. Using Tutte’s approach for enumerating planar maps [13, 14],
Walsh and Lehman obtained an explicit expression for the number \( \varepsilon_g(n) \) of one-face maps with \( n \)
edges on an orientable surface of genus \( g \), as well as a formula for the number of one-face maps of
genus \( g \) with predefined vertex degrees. In a recent work [15] Chapuy obtained a new recurrence
relation for the numbers \( \varepsilon_g(n) \) and gave an elegant combinatorial interpretation of it. In the same
paper he showed how to use this technique to enumerate some special kinds of maps, in particular,
cubic and precubic one-face maps.

Along with maps on orientable surfaces we also need to be able to enumerate maps, both cubic
and precubic, on non-orientable surfaces. In the recently published paper [16] Bernardi and Chapuy
using the approach similar to [15] obtained exact formulas for counting cubic and precubic maps on
non-orientable surfaces. These results along with the results of the work [11] allowed us to solve the
problem of enumerating 3-regular one-face maps on both orientable and non-orientable surfaces up
to all symmetries completely. To the best of our knowledge there are no published analytical results
on enumerating such maps for arbitrary values of \( g \).

2 The basic principles of unsensed map enumeration. Orbifolds and quotient maps

The Burnside’s lemma is typically used as the main tool for enumerating combinatorial objects up
to their symmetry group (see, for example, [17]). This lemma reduces the problem of enumerating
such objects to enumeration of labelled objects that have a trivial symmetry group. As it was noted
in [13], for maps on surfaces it is convenient to consider so-called rooted maps as labelled objects. A
map is called rooted if one of its edges is distinguished, oriented, and assigned a left and a right side
(see, for example, [1], [12]). For enumerating maps on orientable surfaces it is sufficient to distinguish
one edge-end, called a dart.

Let \( X_g^+ \) be a closed orientable surface of genus \( g \). In the paper [3] with the help of the Burnside’s lemma and some additional algebraic and topological considerations Mednykh and Nedela derived the following important formula for determining the numbers \( \tilde{\tau}_{X_g^+}(n) \) of sensed orientable maps with \( n \) edges:

\[
\tilde{\tau}_{X_g^+}(n) = \frac{1}{2n} \sum_{l|2n} \sum_{m=2n/l} \text{Epi}_r(\pi_1(O), \mathbb{Z}_l) \cdot \tau_O(m).
\]

(1)

Here \( O = X_g^+/\mathbb{Z}_l \) is a quotient of the surface \( X_g^+ \) under the action of a cyclic subgroup \( \mathbb{Z}_l \) of the group of automorphisms of \( X_g^+ \), \( \text{Epi}_r(\pi_1(O), \mathbb{Z}_l) \) are integer coefficients (the numbers of order-preserving epimorphisms from the fundamental group \( \pi_1(O) \) of the orbifold \( O \) onto the cyclic group \( \mathbb{Z}_l \)), and \( \tau_O(m) \) are the numbers of rooted quotient maps with \( m \) darts on the orbifold \( O \).

Before we move on, we illustrate these concepts with a simple example. Consider a representation of a torus \( T = X_1^+ \) as a square with its opposite sides identified pairwise (Figure 1 (a)). Rotation of this square by 90° \((l = 4)\) is a typical example of a periodic orientation-preserving homeomorphism of the torus. This homeomorphism splits the set of its points into two subsets, an infinite set of points in the general position and a finite set of singular points (see Figure 1 (a)). Points in the general position are those that lie on some orbit of length \( l = 4 \). Singular points are the remaining ones, and they necessarily lie on orbits of smaller length. In our example there are four singular points: \( a, c, b_1 \) and \( b_2 \). The former two of them are fixed, and the latter two are transformed into each other by the rotation by 90°. Identifying the points of each orbit of the rotation, we obtain an orbifold \( O \), in this case a sphere (Figure 1 (b)). Critical points of the torus get transformed into branch points of the orbifold \( O \) (points \( a, b, c \) in Figure 1 (b)). From the topological point of view the described homeomorphism generates a 4-fold branched covering of the sphere by the torus \( T \), and the orbifold \( O \) is a quotient \( T/\mathbb{Z}_4 \).

![Figure 1: A homeomorphism of the torus and the corresponding orbifold](image)

In the general case, an orientation-preserving homeomorphism generates an orbifold \( O = X_g^+/\mathbb{Z}_l \) which is a surface of genus \( g \) with a finite number \( r \) of branch points. Such orbifold is usually described by its signature

\[
O(g, [m_1, \ldots, m_r]), \quad 1 < m_1 \leq \ldots \leq m_r,
\]

where \( m_i \) are branch indices of the corresponding branch points; each \( m_i \) is equal to the period \( l \) of the homeomorphism divided by the number of preimages of the corresponding branch point. For the example of the orbifold \( O \) shown in Figure 1, the branch points \( a \) and \( c \) have branch indices equal to 4 and the branch index of \( b \) is equal to 2. Consequently, the signature of the corresponding orbifold takes the form

\[
O(0; [2, 4, 4]) \equiv O(0; [2, 4^2]).
\]


For the case of the torus there is one more periodic homeomorphism that preserves its orientation and yields an orbifold with the same signature: the rotation of the square by an angle of 270°. The coefficient $\text{Epi}_o(\pi_1(O), \mathbb{Z}_l)$ in (1) is responsible for counting all homeomorphisms leading to the same orbifold $O$.

Next, consider the case of unsensed maps on an orientable surface $X^+_\chi$ of Euler characteristic $\chi$. In this case, for counting maps with $n$ edges, instead of (1) we need to use the formula given in [11]:

$$\bar{\tau}_{X^+_\chi}(n) = \frac{1}{2} \left( \bar{\tau}_{X^+_\chi}(n) + \frac{1}{2n} \sum_{m|2n} \sum_{l \in \text{Orb}(X^+_\chi/\mathbb{Z}_l)} \tau_O(2m) \cdot \text{Epi}_o^+(\pi_1(O), \mathbb{Z}_l) \right).$$

(2)

Here $\tau_O(2m)$ is the number of quotient maps with $2m$ flags on the orbifold $O$, $\text{Orb}(X^+_\chi/\mathbb{Z}_l)$ is a set of orbifolds $O$ arising from orientation-reversing homeomorphisms of the surface $X^+_\chi$. Compared to the case of orientation-preserving homeomorphisms, such orbifolds can have additional properties.

As a typical example of an orientation-reversing homeomorphism consider a glide reflection of the torus with respect to the horizontal axis $i$ of the square representing this torus on the plane (see Figure 2). Let the ratio between the value of the ‘shift’ and the length of the side if the square be a rational number $p/q$, $0 \leq p/q < 1$, $p$ and $q$ coprime. In Figure 2 an example of a glide reflection with respect to a horizontal axis and $p/q = 1/4$ is shown. The fundamental polygon in this case is one fourth of a square. Since under the glide reflection the right side of this polygon is transformed into its left side with a flip, we may think of these sides as glued together in the reverse direction. Consequently, this homeomorphism generates a 4-fold branched covering of the Klein bottle $O$ by the torus.

Now consider an example of a glide reflection for $p/q = 1/3$ (Figure 2 (b)). For this ratio of $p$ and $q$ the fundamental polygon is one sixth of the square (shaded area in Figure 2(b)). Indeed, it would take six steps for the glide reflection to transform each point of the torus into itself. After the second step the left side $a$ of the fundamental polygon will coincide with its right side $b$, and vice versa after the fourth step. At the same time its top and bottom sides will never become coincident. Consequently this glide reflection corresponds to a rectangular fundamental region with its right and left sides glued together. In other words, in this case the orbifold $O$ is an annulus.

Finally, consider the case of non-orientable surfaces. The number $\bar{\tau}_{X^-_\chi}(n)$ of unsensed maps on a non-orientable surface $X^-_\chi$ of Euler characteristic $\chi$ is calculated by the formula [11]

$$\bar{\tau}_{X^-_\chi}(n) = \frac{1}{4n} \sum_{m|2n} \sum_{l \in \text{Orb}(X^-_\chi/\mathbb{Z}_l)} \tau_O(2m) \cdot (\text{Epi}_o(\pi_1(O), \mathbb{Z}_l) - \text{Epi}_o^+(\pi_1(O), \mathbb{Z}_l)).$$

(3)
Here Orb($X^-_X$) is a set of orbifolds arising from homeomorphisms of $X^-_X$. To understand what kind of orbifolds we can obtain in this case, consider the Klein bottle. We will use the representation of the Klein bottle as a square with its top and bottom sides glued in the forward direction, and its left and right sides glued in the reverse direction (see Figure 3(a)). The homeomorphism of this surface that shifts the upper half of the square down with a flip relatively to the vertical axis, leads to an orbifold which is a projective plane with two branch points (Figure 3(b)). Indeed, consider the cell $F$ shaded in the Figure 3(a). Under the action of this homeomorphism, the point $A$ lying on the upper left boundary of the cell is transformed into the point $A'$ on the right border. This point in the Klein bottle coincides with the point $B$. Consequently, for such homeomorphism the points $A$ and $B$ on the left boundary of the cell $F$ are glued together. Points on the right border of the square behave similarly. At the same time, the points $x_1$ and $x_2$ in Figure 3 are transformed into themselves. As a consequence, these points correspond to branch points of index 2 of the orbifold (Figure 3(b)). It remains to note that the top and bottom boundaries of the cell $F$ are glued together in the opposite direction. Representing them as a boundary of a circle (Figure 3(b)), we obtain a projective plane with two branch points.

So, as we see from the examples given above, in the case of unsensed maps we can obtain both orientable and non-orientable surfaces as orbifolds, and they can be either closed or have boundary. These orbifolds may have some branch points as well. Now we need to understand what quotient maps on such orbifolds can arise.

To illustrate the concept of a quotient map on an orbifold $O$, consider, for example, a map $M$ on the torus $T$ which is symmetric under the rotation of the square by 90° (Figure 4 (a)). Identifying all points lying on each orbit of the rotation, we obtain a quotient map $\mathfrak{M}$ on $O$, shown in the Figure 4 (b). This quotient map would be a map on the sphere with the numbers of vertices, edges and faces equal to those of the original map $M$ divided by 4 if the orbifold $O$ had no branch points and the surface $X^+_X + g = T$ had no corresponding critical points. The existence of these points makes this correspondence more complicated.

Assume that a vertex $x$ of a quotient map $\mathfrak{M}$ coincides with some branch point of an index $m_i$ of the orbifold $O$ (see vertex $x$ in Figure 4(b) which coincides with $a$). Then this vertex corresponds to $l/m_i$ vertices of the map $M$ on the original surface $X^+_g$. The degree $d$ of $x$ in this case gets multiplied by $m_i$ on $X^+_g$ and becomes equal to $m_i d$. For example, the vertex $x$ of degree 1 of the quotient map $\mathfrak{M}$ shown in Figure 4(b) corresponds to a single vertex $\bar{x}$ of degree 4 of the map $M$ in Figure 4(a).

Now assume that a branch point of an orbifold $O$ falls into some face $f$ of the quotient map $\mathfrak{M}$ (see branch point $c$ in Figure 4(b))). This point will correspond to $l/m_i$ points on the surface $S_g$, $m_i$ being...
the corresponding branch index. The remaining points of \( f \) are not branch points, so each of them corresponds to \( l \) points on \( S_g \). Hence, as in the case of a vertex, the degree of the face \( f \) is multiplied by \( m_i \) when this face is lifted to the surface \( S_g \). For example, the degree of the face that contains the branch point \( c \) in Figure 4(b) is multiplied by 4 on the torus \( T \).

One more property of quotient maps is the possibility of having semiedges which end not in vertices, but in branch points of degree 2 (see branch point \( b \) in Figure 4(b)). When lifted to the surface \( X_g^+ \), any such edge gets transformed into \( l/2 \) edges of \( M \). These edges on the surface \( X_g^+ \) contain critical points \( b_i \) corresponding to the branch point \( b \) (see critical points \( b_1, b_2 \) in Figure 4(a)). If an orbifold \( O \) has no branch points of index 2, then there are no semiedges in any quotient map \( M \) on \( O \).

In the general case we have to be able to enumerate quotient maps not only on closed orientable orbifolds, but also on orientable or non-orientable orbifolds with boundary and branch points. Quotient maps on such orbifolds have some additional properties. First of all, apart from complete edges and semiedges (see a complete edge \( x_1x_2 \) and a semiedge going from \( x_1 \) to the branch point of the index \( m = 2 \) in Figure 5(a)), such quotient maps may also have so-called halfedges, i.e. edges ending on the boundary (see halfedge \( x_1a \) in Figure 5(a)), and boundary edges, i.e. edges lying on the boundary (see edge \( x_2x_3 \) in Figure 5(a)). Each semiedge, halfedge or boundary edge contributes \( 1/2 \) to the number of edges of the quotient map. Secondly, the presence of boundary adds some additional restrictions on the location of branch points. Namely, a branch point can’t be located in a face incident to the boundary of the orbifold. Thirdly, the boundary of the orbifold in some sense acts similarly to a branch point of index 2. Namely, when lifted from the orbifold \( O \) to the covering surface \( X_g \), any face, vertex or edge of a quotient map \( M \) lying on the boundary gets transformed into \( l/2 \) faces, vertices or edges of the map \( M \).

Another important concept widely used in enumeration of maps on non-orientable surfaces or surfaces with boundary is the concept of a flag. Take a map and place new vertices into the centers of its
edges (see squares in Figure 5(b)), into the centers of its faces (see triangles in Figure 5(b)), and connect neighboring vertices by new edges (see dashed lines in Figure 5(b)). This operation yields a partition of this map into triangles. These triangles are called flags of the original map. Since each edge is incident to an even number of flags regardless of its type (complete edge, semiedge etc.), the total number of flags is even for any quotient map on any orbifold.

Summing up, we can conclude that to use the formulas (2) and (3) we should solve three problems: describe the sets \( \text{Orb}^{-}(X_{g}^{+}/\mathbb{Z}_{2l}) \) and \( \text{Orb}(X_{g}^{-}/\mathbb{Z}_{l}) \) of suitable cyclic orbifolds for a given orientable \((X_{g}^{+})\) or non-orientable \((X_{g}^{-})\) surface, determine the numbers \( \text{Epi}_{o}(\pi_{1}(O), \mathbb{Z}_{l}) \) and \( \text{Epi}_{o}^{+}(\pi_{1}(O), \mathbb{Z}_{l}) \) of order-preserving epimorphisms, and find the numbers \( \tau_{o}(2m) \) of quotient maps with \( 2m \) flags on cyclic orbifolds. The next section is devoted to solving these problems for the case of 3-regular one-face maps on orientable surfaces.

### 3 Enumeration of unsensed 3-regular one-face maps on orientable surfaces

Let \( M \) be a 3-regular one-face map on an orientable surface \( X_{g}^{+} \) of genus \( g \). With the help of Euler’s formula [18, p. 268]
\[
k - n + 1 = 2 - 2g
\]
and the Handshaking lemma [18, p. 35]
\[
2n = \sum_{x \in V(M)} \deg(x) = 3k
\]
we can express the number \( n \) of edges and the number \( k \) of vertices of such map \( M \) through the genus \( g \) of the surface:
\[
n = 6g - 3, \quad k = 4g - 2.
\]
Using the technique described in detail in the paper [5], one can be obtain the following formula for counting 3-regular one-face maps on an arbitrary orientable surface \( X_{g}^{+} \) of genus \( g \) (see formula (20) in [8]):
\[
\tilde{\tau}^{(3)}(g) = \frac{\tau^{(3)}_{+}(g)}{2(6g - 3)} + \sum_{g=0}^{\lfloor g/2 \rfloor} \frac{(4g - 2 - 2g)!}{2 \cdot 3^{g} g! (2g - 1 - g)! (2g - 4g + 1)!} + \frac{(2g - 2)!}{6 \cdot (g - 1)!} \sum_{g=0}^{\lfloor (g+1)/3 \rfloor} \frac{3^{g-1} 2^{g+1-3g} + (-1)^{g-3}}{g! (g + 1 - 3g)!} + \frac{(2g - 2)!}{6 \cdot (g - 2)^{2}} \sum_{k=\lfloor g/2 \rfloor}^{\lfloor (2g - 2)/3 \rfloor} \sum_{g=0}^{\lfloor k/2 \rfloor} \frac{3^{g-2} 2^{3g-1-3k} + (-1)^{k}}{g! (k - g)! (4k + 3 - 2g - 4g)! (2g - 1 - 3k)!}.
\]

Here \( \tau^{(3)}_{+}(g) \) is the number of rooted 3-regular one-face maps on the orientable surface \( X_{g}^{+} \), equal to
\[
\tau^{(3)}_{+}(g) = \frac{2(6g - 3)!}{12^{g} g! (3g - 2)!}.
\]

In order to count maps by the formula (2) it remains to enumerate maps on orbifolds for orientation-reversing homeomorphisms \( h \). It turns out that in this case the orbifolds admit a simple description. Namely, the following statement holds.
Proposition 3.1. In the case of one-face maps on an orientable surface $X_g^+$, an orbifold $O$ corresponding to any orientation-reversing homeomorphism $h$ is an orientable or a non-orientable surface with boundary and without branch points.

Proof. We will give two proofs of this important statement. The first of them relies on the connection between one-face maps on an orientable surface and chord diagrams. It is well known (see, for example, [19]) that any one-face map admits a representation in the form of a chord diagram built on $2n$ points. But any chord diagram allows only two types of symmetries — rotations and reflections. The first type of symmetry corresponds to orientation-preserving homeomorphisms and to counting sensed one-faced maps. The second type corresponds to orientation-reversing homeomorphisms with the period equal to 2.

The second proof is more formal and essentially relies on the properties of quotient maps on orbifolds. Namely, it is known that an orientation-preserving homeomorphism $h$ of $X_g^+$ generates an orientable orbifold without boundary and corresponds to sensed maps. Any orientation-reversing homeomorphism $h$ of $X_g^+$ corresponding to unsensed maps generates an orbifold $O$ which is either a non-orientable surface without boundary or an orientable or a non-orientable surface with boundary. In the case of a non-orientable orbifold without boundary we have to place the branch point of the index $l$ into the only face of the quotient map $\mathcal{M}$ on the orbifold $O$. It can be proven that the coefficients $\text{Epi}^+_0(\pi_1(O),\mathbb{Z}_2)$ in the formula (2) for this case are equal to 0, which means that in our case there are no one-face quotient maps on non-orientable orbifolds without boundary. As noted above, in the case of a surface with boundary the period of homeomorphism $h$ has to be equal to 2. This means that all branch points, if they exist, must have branch indices equal to 2. But as noted, for example, in paper [9] (see page 1198), the orbifold $O$ may not contain both branch points of index 2 and boundary components. So in our case there are only orbifolds with boundary and with no additional branch points.

\[\text{Consequence 3.2.} \text{ The number } \bar{\tau}_{X_g^+}(n) \text{ of unsensed one-face maps on an orientable surface } X_g^+ \text{ is calculated by the formula} \]

\[
\bar{\tau}_{X_g^+}(n) = \frac{1}{2} \left( \bar{\tau}_{X_g^+}(n) + \frac{1}{2n} \sum_{O \in \text{Orb}^-} \tau_O(2n) \right).
\]

Proof. From the Proposition 3.1 we have that in our case $l = 1$, $m = n$. As it was shown in [11], the coefficients $\text{Epi}^+_0(\pi_1(O),\mathbb{Z}_2)$ in (2) are equal to 1 for any orientation-reversing homeomorphism. As a result, we get (3) from the formula (2).

In the general case of $r$-regular one-face maps the boundary may consist of several components. It turns out that for 3-regular maps this boundary consists of a single component.

Proposition 3.3. In the case of 3-regular one-face maps on an orientable surface $X_g^+$, any orbifold $O$ corresponding to an orientation-reversing homeomorphism $h$ is an orientable or a non-orientable surface with a single boundary component.

Proof. In the case of a 3-regular map, on the boundary of an orbifold $O$ may lie either a halfedge (see Figure 6(a)) or a boundary edge with two distinct vertices incident to it, which have exactly one other normal edge incident to each of them (see Figure 6(b)). In both cases we cannot completely “cover” any boundary component with edges, which means that each boundary component will be incident to the face. But the face can’t be incident to more than one boundary component — otherwise after lifting the map to the original surface $X_g^+$ we would get a face that is not homeomorphic to a disk. For the same reason it is impossible that the same face will be incident to same boundary component.
several times. So in the case of a 3-regular one-faced map we have the only boundary component and the only face which almost completely covers the boundary component, except for the segment covered by a single edge (Figure 6(b)) or for the endpoint of a single halfedge (Figure 6(a)). □

Now we are ready to describe all orbifolds $O$ appearing in the formula (6).

**Proposition 3.4.** For an odd $g$ the orbifold $O$ must be a non-orientable surface of genus $\hat{g} = g$. For an even $g$ the orbifold $O$ is either an orientable surface of genus $\hat{g}/2$ or a non-orientable surface of genus $\hat{g} = g$.

**Proof.** We use the Riemann-Hurwitz formula

$$\chi = l \left[ \alpha \hat{g} - 2 + h + \sum_{i=1}^{r} \left( 1 - \frac{1}{m_i} \right) \right],$$

that connects the Euler characteristics $\chi$ of the original surface $X_\chi$ with the period $l$ of the homeomorphism and the parameters of the orbifold $O$. Here, for an orientable orbifold $\alpha = 2$ and $g$ is the number of handles. For a non-orientable orbifold $\alpha = 1$ and $g$ is the number of crosscaps. The parameter $h$ defines the number of boundary components. The numbers $m_i$ are branch indices of branch points.

In Proposition 3.3 we proved that in our case $\chi = 2 - 2g$, $l = 2$, $h = 1$, $r = 0$, so we get

$$g = \begin{cases} \frac{2g}{2}, & \text{if } O \text{ is an orientable surface,} \\ g, & \text{if } O \text{ is a non-orientable surface.} \end{cases}$$

□

**Consequence 3.5.** The formula (6) for the numbers $\bar{\tau}_+^{(3)}(n)$ of unsensed 3-regular one-face maps with $n = 6g - 3$ edges on an orientable surface $X^+_g$ can be rewritten as

$$\bar{\tau}_+^{(3)}(n) = \frac{1}{2} \left( \tau_+^{(3)}(n) + \frac{1}{2n} \left( \tau_{O^+}(n) + \tau_{O^-}(n) \right) \right),$$

where $\tau_{O^-}(n)$ is the number of quotient maps with $n$ darts on a non-orientable orbifold $O^-$ of genus $g$, $\tau_{O^+}(g/2)$ is the number of quotient maps with $n$ darts on an orientable orbifold $O^+$ of genus $g/2$ for even $g$ and 0 for odd $g$.

Our next step is to obtain exact expressions for the numbers $\tau_{O^+}(g/2)$ and $\tau_{O^-}(g)$. 

![Figure 6](image-url)
Proposition 3.6. The problem of enumerating quotient maps on the orbifold $O$ is reduced to the problem of enumerating rooted 3-regular maps on an orientable or a non-orientable surface without boundary.

Proof. Take a quotient map with $n$ darts (see Figure 6) and contract the boundary component into a point. In the case of an edge lying on the boundary (see Figure 6(b)) we obtain a vertex of the degree 2 as a result of such contraction. For the map shown in Figure 6(a) we get a halfedge going from the vertex of the degree 3. This halfedge can be contracted again to get a vertex of degree 2 as in the previous case. So in both cases we obtain a vertex $x$ of degree 2 as a result of contracting the boundary component. Then we can get rid of this vertex $x$ by replacing it with a root edge and obtain a rooted 3-regular map with $n' = (n - 3)/2$ edges on an orientable surface of genus $g = g/2$ (if $g$ is even) or a non-orientable surface $Y$ of genus $g = g$ (g can be arbitrary) without boundary.

Vice versa, if we take a rooted 3-regular map with $n'$ edges on a surface $Y$ of genus $g$, place a vertex of the degree 2 on its root edge, add a boundary component in 2 ways, select the root dart in $n = 2n' + 3$ ways, we will obtain either a rooted quotient map with $n$ darts with one halfedges going to the boundary (Figure 6(a)) or a rooted quotient map with $n$ darts with an edge lying on the boundary (Figure 6(b)).

Considering that the numbers $\tau_{O^+}(n)$ and $\tau_{O^-}(n)$ of quotient maps on orbifolds $O^+$ and $O^-$ are expressed through the numbers $\tau_+^{(3)}(g/2)$ and $\tau_-^{(3)}(g)$ of 3-regular rooted maps on the surfaces of the corresponding genera by the formulas

$$
\tau_{O^+}(n) = 2n \cdot \tau_+^{(3)}(g/2), \quad \tau_{O^-}(n) = 2n \cdot \tau_-^{(3)}(g),
$$

we obtain from the formula (8) the following result.

Theorem 3.1. The numbers $\bar{\tau}_+^{(3)}(g)$ of unsensed 3-regular one-face maps on orientable surfaces $X_g^+$ are equal to

$$
\bar{\tau}_+^{(3)}(g) = \frac{1}{2} \left( \tau_+^{(3)}(g) + \tau_+^{(3)}(g/2) + \tau_-^{(3)}(g) \right),
$$

where $\bar{\tau}_+^{(3)}(g)$ is the number of sensed 3-regular one-face maps on an orientable surface $X_g^+$ calculated by the formula (4), $\tau_+^{(3)}(g/2)$ is the number of rooted 3-regular one-face maps on an orientable surface $X_{g/2}^+$ calculated by the formula (5) in the case of even $g$ and equal to 0 in the case of odd $g$, and $\tau_-^{(3)}(g)$ is the number of rooted 3-regular one-face maps on a non-orientable surface $X_g^-$ calculated by the formula (see [16])

$$
\tau_-^{(3)}(g) = \begin{cases} 
\frac{2^{2h-2} h! (6h - 2)!}{3^{h-1} (2h)! (3h - 1)!} \sum_{i=0}^{h-1} \binom{h}{i} 16^{-i}, & h = g/2, g \text{ is even}, \\
\frac{2^h (3h)!}{3^h h!}, & h = (g - 1)/2, g \text{ is odd}.
\end{cases}
$$

4 Enumeration of unsensed 3-regular one-face maps on non-orientable surfaces

Consider a 3-regular one-face map $M$ on a non-orientable surface $X_g^-$ of genus $g$. For this map from Euler’s formula and the Handshaking lemma we have the following equalities connecting the genus $g$ of the surface, the number $n$ of edges and the number $k$ of vertices:

$$
n = 3g - 3, \quad k = 2g - 2.
$$
The following statement is analogous to the Proposition 3.1.

**Proposition 4.1.** In the case of one-face maps on a non-orientable surface $X_g^-$, any orbifold is either an orbifold $O$ with boundary and possibly with some branch points of index 2, coinciding either with vertices or with free ends of semiedges, or a non-orientable orbifold $O^-$ without boundary and with $r > 0$ branch points. One of these branch points is located in the only face of the quotient map and has a branch index equal to 1, and the others coincide with vertices or free ends of semiedges of the quotient map.

**Proof.** In the case of an orbifold $O$ with boundary, for the map to have one face after lifting, the period $l$ of the corresponding homeomorphism $h_2$ has to be equal to 2. This means that all branch points, if they exist, must have branch indices equal to 2. For non-orientable surfaces the presence of boundary does not interfere with the existence of such branch points. These branch points, however, can not be located in the only face of the quotient map $M$, so they can be located either in its vertices or in its ends of semiedges.

There are no orientable orbifolds without boundary in our case [11]. In the case of a non-orientable orbifold $O^-$ without boundary corresponding to a homeomorphism $h_l$ of period $l$, we have to place the branch point of index $l$ into the only face of the quotient map $M$. This is also to ensure that when the quotient map $M$ is lifted from the orbifold $O^-$ to the original surface $X_g^-$, the corresponding map $M$ on $X_g^-$ will have a single face. The remaining branch points should be placed either into the vertices or into the ends of semiedges of the quotient map $M$. In the latter case these branch points must have indices equal to 2.

We begin with the case of an orbifold $O$ with boundary, corresponding to a homeomorphism $h_2$ of period 2. It is easy to see that in this case the Proposition 3.3 still holds true. Together with the Proposition 4.1 and the Riemann-Hurwitz formula (7) this fact allows to derive the following statement.

**Proposition 4.2.** In the case of a 3-regular one-face map $M$ on a non-orientable surface $X_g^-$, the orbifold $O$ corresponding a homeomorphism $h_2$ is either an orientable surface of genus $g \in [0, \lceil g/4 \rceil]$ with $r = g - 4g$ branch points of index 2 or a non-orientable surface of genus $g \in [1, \lceil g/2 \rceil]$ with $r = g - 2g$ branch points of index 2.

**Proof.** Indeed, in this case the parameters in the Riemann-Hurwitz formula (7) are $h = 1$, $\chi = 2 - g$, $m_i = 2$, $i = 1, \ldots, r$, so from (7) we have

$$g = \begin{cases} 4g + r, & O \text{ is an orientable surface}, \\ 2g + r, & O \text{ is a non-orientable surface}. \end{cases}$$

The next step in using the formula (3) is determining the numbers $\text{Epi}_o(\pi_1(O^+), \mathbb{Z}_2)$ and $\text{Epi}_o^+(\pi_1(O^+), \mathbb{Z}_2)$.

**Proposition 4.3.** The coefficients in (3) corresponding to the numbers of epimorphisms are calculated by the formulas

$$\text{Epi}_o(\pi_1(O^+), \mathbb{Z}_2) - \text{Epi}_o^+(\pi_1(O^+), \mathbb{Z}_2) =: \varepsilon^+(g) = \begin{cases} 2^g, & r > 0, \\ 2^g - 1, & r = 0 \end{cases} \quad (11)$$
in the case of an orientable orbifold $O^+$ and

$$\text{Epi}_o(\pi_1(O^-), \mathbb{Z}_2) - \text{Epi}_o^+(\pi_1(O^-), \mathbb{Z}_2) =: \varepsilon^-(g) = \begin{cases} 2^g, & r > 0, \\ 2^g - 1, & r = 0 \end{cases} \tag{12}$$

in the case of a non-orientable orbifold $O^-$. 

**Proof.** For calculating these numbers we can use the results obtained in the paper [11]. Namely, in [11] it was proved that the numbers of order-preserving and orientation-and-order-preserving epimorphisms for the case of an orientable orbifold $O^+$ with $h > 0$ boundary components are equal to

$$\text{Epi}_o(\pi_1(O^+), \mathbb{Z}_2) = (m')^{2g+h-1} \cdot J_{2g+h-1} \left( \frac{l}{m'} \right) \cdot \prod_{i=1}^r \varphi(m_i), \quad m' = \text{lcm}(2, m_1, \ldots, m_r), \quad l \text{ even}, \tag{13}$$

$$\text{Epi}_o^+(\pi_1(O^+), \mathbb{Z}_2) = m^{2g+h-1} \cdot J_{2g+h-1} \left( \frac{l}{m} \right) \cdot \prod_{i=1}^r \varphi(m_i), \quad m = \text{lcm}(m_1, \ldots, m_r), \quad l \text{ odd}, \tag{14}$$

where $J_k(n)$ is the Jordan’s totient function (see formulas (17)–(18) in the paper [11]). Here the number of epimorphisms is equal to zero if the argument $n$ of $J_k(n)$ is not an integer. In our case the values in (13) are the following: $l = 2$, $h = 1$, $m_i = 2$, $i = 1, \ldots, r$, $\varphi(m_i) = 1$, $m' = 2$, $J_{2g}(1) = 1$. Consequently, $\text{Epi}_o(\pi_1(O^+), \mathbb{Z}_2) = 2^g$. In the formula (14) we have $l = 1$, so $J_{2g}(1)$ is nonzero only if $r = 0$. In this case $m = 1$ and $\text{Epi}_o^+(\pi_1(O^+), \mathbb{Z}_2) = 1$. 

For the case of a non-orientable orbifold $O^-$ with $h > 0$ boundary components, the numbers of epimorphisms are as follows:

$$\text{Epi}_o(\pi_1(O^-), \mathbb{Z}_2) = (m')^{g+h-1} \cdot J_{g+h-1} \left( \frac{l}{m'} \right) \cdot \prod_{i=1}^r \varphi(m_i), \quad m' = \text{lcm}(2, m_1, \ldots, m_r), \quad l \text{ even},$$

$$\text{Epi}_o^+(\pi_1(O^-), \mathbb{Z}_2) = m^{g+h-1} \cdot J_{g+h-1} \left( \frac{l}{m} \right) \cdot \prod_{i=1}^r \varphi(m_i), \quad m = \text{lcm}(m_1, \ldots, m_r), \quad l \text{ odd}$$

(see formulas (19)–(20) in the paper [11]). Arguments analogous to the previous case allow us to conclude that here we have $\text{Epi}_o(\pi_1(O^-), \mathbb{Z}_2) = 2^g$, and $\text{Epi}_o^+(\pi_1(O^-), \mathbb{Z}_2)$ is equal to 1 if $r = 0$ and to 0 otherwise. 

Now consider a quotient map $\mathfrak{M}$ on an orbifold $O$ of genus $g$ which corresponds to a 3-regular map $M$ on a non-orientable surface $X_g^-$. Such quotient map has $n = 3g - 3$ halfedges and $r$ semiedges ending in branch points of index 2. Putting a vertex of degree 1 in each such branch point we obtain a precubic map with $n' = n + r = 3g - 3 + r$ halfedges on the orbifold $O$ with the only boundary component. Getting rid of it in the same way that was described in the proof of Theorem 3.6 we obtain a precubic map with $n' - 3 = 3g - 6 + r$ halfedges on a surface of genus $g$ without boundary. 

In the case of an orientable orbifold $O^+$ we have $r = g - 4g$, so the quotient map has $n = 2g - 2g - 3$ edges. In the case of a non-orientable orbifold $O^-$ we have $r = g - 2g$, so such quotient map has $n = 2g - g - 3$ edges. 

The number of precubic maps with $n = 2m + 1$ edges on an orientable surface $O^+$ of genus $g$ is calculated by the formula (see, for example, [15], Corollary 7)

$$\tau_+^{(1+3)}(g, m) = \frac{2(2m + 1)!}{12g!g!(m + 2 - 3g)!m!}, \quad m = g - g - 2. \tag{15}$$
For enumerating quotient maps on a non-orientable orbifold $O^-$ we can use the formulas for precubic one-face maps obtained in \[16\] (see Corollary 8 and 9). It follows from Euler’s formula and the Handshaking lemma that any precubic map has $e = 2m + 1$ edges in the case of even $g = 2h$ and $e = 2m$ edges in the case of odd $g = 2h + 1$. The number $k$ of leaves of such map is equal to $k = m - 3h + 2$ in the case of even $g = 2h$ and to $k = m - 3h$ in the case of odd $g = 2h + 1$. The number $v$ of vertices of degree 3 in both cases is equal to $m + h$.

In the paper \[16\] is given a formula for the number of precubic one-face maps with the root incident to a vertex of degree 1. For our case we need formulas for precubic one-face maps with the root incident to an arbitrary vertex. To obtain the desired formula we divide the formulas given in \[16\] by the number of leaves and multiply by twice the number of all edges of the precubic map. As the result we get the following proposition.

**Proposition 4.4.** In the case of even $g = 2h$, the number of precubic maps with $e = 2m + 1$ edges, with $k$ leaves and with the root incident to an arbitrary vertex is calculated by the formula

$$\tau_{(1+3)}(2h, k) = \frac{2c_h(2m + 1)!}{m!(m + 2 - 3h)!} = \frac{2c_h(2k + 6h - 3)!}{k!(k + 3h - 2)!}, \quad c_h = \frac{2^{2h-2}h!}{3^h-1(2h)!} \sum_{i=0}^{h-1} \left(\frac{2h}{i}\right) 16^{-i}. \tag{16}$$

In the case of odd $g = 2h + 1$ the number of such maps with $e = 2m$ edges and $k$ leaves is calculated by the formula

$$\tau_{(1+3)}(2h + 1, k) = \frac{2^{2m}m!}{3^h h!(m - 3h)!} = \frac{2^{6h+2k}(k + 3h)!}{3^h h! k!}. \tag{17}$$

In our case we have a precubic map with $n = 2g - g - 3$ edges. Consequently, from the formulas (16) and (17) we have the following expressions for the numbers $\tau_{(1+3)}(g, g)$:

$$\tau_{(1+3)}(g, g) = \begin{cases} \frac{2c_h(2g - 2h - 3)!}{(g - h - 2)!(g - 4h)!}, & h = g/2, \ g \text{ is even,} \\ \frac{2^{2g-2h-4}(g - h - 2)!}{3^h h!(g - 4h)!}, & h = (g - 1)/2, \ g \text{ is odd.} \end{cases} \tag{18}$$

Substituting them into the formula (3) and considering that $\tau_0(g, g) = 2n\tau_{(1+3)}(g, g)$, we obtain the following explicit formula for the terms in the formula (3) corresponding to a homeomorphism $h_2$:

$$\tilde{\tau}^{(3)}(g)\big|_{h_2} = \frac{1}{2} \sum_{g=0}^{[g/4]} \varepsilon^+(g)\tau_{(1+3)}^+(g, g) + \frac{1}{2} \sum_{g=1}^{[g/2]} \varepsilon^-(g)\tau_{(1+3)}^-(g, g). \tag{19}$$

Here the numbers $\tau_{(1+3)}^+(g, g)$ and $\tau_{(1+3)}^-(g, g)$ are calculated by the formulas (15) and (18) respectively, and the coefficients $\varepsilon^+(g)$ and $\varepsilon^-(g)$ are calculated by the formulas (11) and (12) respectively.

Now consider the case of a non-orientable orbifold $O^-$ without boundary, corresponding to a homeomorphism $h_l$ of period $l > 1$. As noted above, in this case one of $r$ branch points falls into the only face of the quotient map and has a branch index equal to $l$. Others branch points coincide with vertices (these branch points have branch indices equal to 3) or with free ends of semiedges (these branch points have branch indices equal to 2). As before, to use the formula (2) we have to describe all orbifolds $O^-$ corresponding to such homeomorphisms, determine the number of epimorphisms $\text{Epi}_c(\pi_1(O^-), Z_l)$ and $\text{Epi}_c^+(\pi_1(O^-), Z_l)$, and then enumerate quotient maps on the corresponding orbifolds $O^-$. 
We begin with the first subproblem. Signatures of the corresponding orbifolds \(O^-\) have the following form:

\[
O^-(g, [\underbrace{2, \ldots, 2, 3, \ldots, 3, l}]_{n_s, n_v}).
\]

(20)

From the Riemann–Hurwitz formula (7) for the orbifolds with the signature (20) we have:

\[
g - 1 = l \left( g - 1 + \frac{n_s}{2} + \frac{2n_v}{3} \right) \iff 6g - 6 = l [6g - 6 + 3n_s + 4n_v].
\]

(21)

We need to find all solutions of this equation for parameters \(l, g, n_s\) and \(n_v\) that satisfy some additional constraints. First of all, as follows from [?], for non-orientable surfaces \(X_g^-\) with \(g \geq 3\) the period \(l\) can be bounded from above:

\[
l \leq 2g - 2, \quad g \text{ even; } \quad l \leq 2g, \quad g \text{ odd.}
\]

Using the Riemann–Hurwitz formula (7) it can be shown that for the Klein bottle \((g = 2)\) orbifolds with signature (20) may correspond only to \(l = 2\), so the same bound still holds. For the projective plane \((g = 1)\) there are no 3-regular maps at all [16], so from now on we will assume that \(g \geq 2\). For any fixed values of \(g\) and \(l\) the value of \(g\) must be in the range \([1, (g + l - 1)/l]\). Finally, the parameter \(n_s\) can be non-zero only if \(2 \mid l\), and \(n_v\) can be non-zero only if \(3 \mid l\).

The next subproblem is to calculate the coefficients \(\text{Epi}_o(\pi_1(O^-), \mathbb{Z}_l)\) and \(\text{Epi}_o^+(\pi_1(O^-), \mathbb{Z}_l)\) in (3) for a given set of parameters \(g, l, g, n_s\) and \(n_v\) that satisfies the equation (21) and the constraints formulated above.

**Proposition 4.5.** The coefficients

\[
\text{Epi}_o(\pi_1(O^-), \mathbb{Z}_l) - \text{Epi}_o^+(\pi_1(O^-), \mathbb{Z}_l) =: \varepsilon(l, g, n_v)
\]

in (3) corresponding to the numbers of epimorphisms can be calculated by the formulas

\[
\varepsilon(l, g, n_v) = \begin{cases} 
   l^{g-1} \cdot \varphi(l) \cdot 2^{n_v}, & l \text{ is odd;} \\
   2 \cdot l^{g-1} \cdot \varphi(l) \cdot 2^{n_v}, & l \text{ is even and } \frac{1}{2} \cdot n_s + \frac{1}{3} \cdot n_v + 1 \text{ is even;}
\end{cases}
\]

(22)

otherwise.

Proof. In the paper [11] it was proved that the number of order-preserving epimorphisms for a non-orientable orbifold \(O^-\) without boundary in the case of odd \(l\) is equal to

\[
\text{Epi}_o(\pi_1(O^-), \mathbb{Z}_l) = m^{g-1} \cdot J_{g-1} \left( \frac{l}{m} \right) \cdot \prod_{i=1}^{r} \varphi(m_i), \quad m = \operatorname{lcm}(m_1, \ldots, m_r).
\]

(23)

For \(l = 2^q k\), where \(k\) is odd and \(q > 1\), the corresponding number of epimorphisms is equal to

\[
\text{Epi}_o(\pi_1(O^-), \mathbb{Z}_l) = 2^{(m')^{g-1}} \cdot J_{g-1} \left( \frac{l}{m'} \right) \cdot \prod_{i=1}^{r} \varphi(m_i), \quad m' = \operatorname{lcm}(2, b, m_1, \ldots, m_r),
\]

(24)

where \(b\) is the denominator of the fraction \(\sum \frac{1}{2m_i}\) after simplification. Finally, for \(l = 2k\) where \(k\) is odd, we have

\[
\text{Epi}_o(\pi_1(O^-), \mathbb{Z}_l) = 2^{(m')^{g-1}} \cdot J_{g-1} \left( \frac{l}{m'} \right) \cdot \prod_{i=1}^{r} \varphi(m_i) - m^{g-1} J_{g-1} \left( \frac{l}{2m} \right) \prod_{i=1}^{r} \varphi(m_i),
\]
where $m$ and $m'$ are defined as in the equations (23) and (24).

In the case of an orbifold of the form (20) we have $m = l$, so for odd values of $l$ the numbers $\text{Epi}_o(\pi_1(O^-), \mathbb{Z}_l)$ are equal to $l^{g-1} \cdot \varphi(l) \cdot 2^{n_v}$. For an even $l$ the Jordan function $J_{g-1}\left(\frac{l}{m'}\right)$ is non-zero only if $l = m'$. For our case this is equivalent to the condition $b \leq l$. Since

$$\sum \frac{1}{2m_i} = \frac{n_s}{4} + \frac{n_v}{6} + \frac{1}{2l} = \frac{1}{2l} \left[ \frac{l}{2} \cdot n_s + \frac{l}{3} \cdot n_v + 1 \right],$$

then the condition $b \leq l$ is satisfied if the number $\frac{l}{2} \cdot n_s + \frac{l}{3} \cdot n_v + 1$ is even. In this case the numbers $\text{Epi}_o(\pi_1(O^-), \mathbb{Z}_l)$ are equal to $2 \cdot l^{g-1} \cdot \varphi(l) \cdot 2^{n_v}$. Otherwise they are equal to 0.

Finally, we can use the formulas for the numbers of orientation-and-order-preserving epimorphisms for the case of a non-orientable orbifold $O^-$ without boundary obtained in [11]:

$$\text{Epi}_o^+(\pi_1(O^-), \mathbb{Z}_l) = 0, \quad l \text{ is odd},$$

$$\text{Epi}_o^+(\pi_1(O^-), \mathbb{Z}_l) = m^{g-1} \cdot J_{g-1}\left(\frac{l}{2m'}\right) \cdot \prod_{i=1}^r \varphi(m_i), \quad m = \text{lcm}(m_1, \ldots, m_r), \quad l = 2k, \text{ } k \text{ is odd},$$

$$\text{Epi}_o^+(\pi_1(O^-), \mathbb{Z}_l) = 2 \cdot (m')^{g-1} \cdot J_{g-1}\left(\frac{l}{2m'}\right) \cdot \prod_{i=1}^r \varphi(m_i), \quad m' = \text{lcm}(2, m_1, \ldots, m_r), \quad 4 \mid l.$$
\[ (17) \], multiply it by the number \( \binom{n_s + n_v}{n_s} \) of ways to choose \( n_s \) leaves from \( n_s + n_v \) leaves. Then we use double counting to correctly recalculate the number of possible root positions: divide the result by the number \((6g - 6)/l + n_s\) of darts of the map \( M^{(1+3)} \) and multiply it by the number \((6g - 6)/l\) of darts of the quotient map \( \mathcal{M} \). As the result we obtain the following explicit formula for the terms of (3) corresponding to homeomorphisms \( h_l \):

\[
\tilde{\tau}_-(3)(g)_{h_1} = \frac{1}{4} \sum_{l=2}^{2g-2} \frac{(g+l-1)/l}{l} \sum_{n_s,n_v} \varepsilon(l, g, n_v) \binom{n_s + n_v}{n_s} \frac{\tau_-^{(1+3)}(g, n_s + n_v)}{3g - 3 + ln_s/2}. \tag{25}
\]

Here the numbers \( \tau_-^{(1+3)}(g, n_s + n_v) \) are calculated by the formulas (16) and (17), the coefficients \( \varepsilon(l, g, n_v) \) are calculated by the formulas (22), and summing over \( n_s, n_v \) is done over all solutions of the equation (21).

Summing up, from (3) we obtain the following formula for counting 3-regular one-face maps on a non-orientable surface \( X_g^- \):

\[
\tilde{\tau}_-(3)(g) = \frac{1}{4(3g - 3)} \tau_-^{(3)}(g) + \tilde{\tau}_-(3)(g)_{h_2} + \tilde{\tau}_-(3)(g)_{h_1}. \tag{26}
\]

Here the numbers \( \tau_-^{(3)}(g) \), \( \tilde{\tau}_-(3)(g)_{h_2} \) and \( \tilde{\tau}_-(3)(g)_{h_1} \) are calculated by the formulas (10), (19) and (25) respectively.

**Conclusion**

The results presented in this article allowed us to enumerate unsensed 3-regular one-face maps on orientable and non-orientable surfaces of a given genus \( g \). In the Table 2 we provide the results for rooted, sensed and unsensed maps on orientable surfaces of genus \( g \in [1, 10] \). In the Table 3 we provide the results for rooted and unsensed maps on non-orientable surfaces of genus \( g \in [2, 20] \).

To verify the obtained analytical results we also implemented an algorithm for generating maps on orientable or non-orientable surfaces based on the ideas formulated in [20]. The numerical results obtained by generating such maps coincided with the first terms obtained by analytical formulas (10) and (26).

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Table 2: The numbers of 3-regular one-face maps on orientable surfaces $X_g^+$

| $g$ | $\tau_+^{(3)}(g)$ | $\tau_-^{(3)}(g)$ | $\varphi_+^{(3)}(g)$ |
|-----|-------------------|-------------------|-------------------|
| 1   | 1                 | 1                 | 1                 |
| 2   | 105               | 9                 | 8                 |
| 3   | 50050             | 1726              | 927               |
| 4   | 56581525          | 1349005           | 676445            |
| 5   | 11712376570       | 2169056374        | 1084610107        |
| 6   | 386078943500250   | 5849686966988     | 2924847922929     |
| 7   | 1857039718236202500 | 23808202021448662 | 11904101304325611 |
| 8   | 122773583718909378125 | 136415042681045401661 | 68207521363461659373 |
| 9   | 106815706684397824557193750 | 1047212810636411989605202 | 5236064053201747218125584 |
| 10  | 1183197582943074702620035168750 | 1037892616616792737980819918 | 5189463083084174721816125584 |

Table 3: The numbers of 3-regular one-face maps on non-orientable surfaces $X_g^-$

| $g$ | $\tau_+^{(3)}(g)$ | $\tau_-^{(3)}(g)$ |
|-----|-------------------|-------------------|
| 2   | 6                 | 2                 |
| 3   | 128               | 11                |
| 4   | 3780              | 144               |
| 5   | 163840            | 3627              |
| 6   | 8828820           | 149288            |
| 7   | 587202560         | 8170800           |
| 8   | 4582135560        | 545671762         |
| 9   | 4133900622400     | 43063046307       |
| 10  | 421946699674500   | 3906934079662     |
| 11  | 48151737348915200 | 401264673924343  |
| 12  | 607054485920582700 | 45988979036528440 |
| 13  | 83822544376991580160 | 5821010056777072838 |
| 14  | 12578768914952672932500 | 806331341176441101980 |
| 15  | 2038564279248435229491200 | 121343111865634574938768 |
| 16  | 3548258423062128958589960000 | 19712546794881999409462482 |
| 17  | 660168656191813264718430208000 | 3438378471666873290074260643 |
| 18  | 130746566969439734302274293825000 | 64091453759785062325259175158 |
| 19  | 274630166975794318122866966652800000 | 1271435930443495008040170430994988 |
| 20  | 6098023559259606741021710317037175000 | 26745717365173718867249062116990380 |

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