Introduction. Characterization of quantum chaos [1–3] has attracted considerable interest since the late 1970s [4–8]. While quantum chaos has often been studied through comparison of the eigenvalue/vector statistics of a Hamiltonian with those of random matrices [4, 6, 9–22], it can also be characterized by its dynamics as in classical chaos [23]. A prime indicator of such dynamical characterization is irreversibility in chaotic motion. As Loschmidt once stated [24], by reversing the velocity of all particles after the state of the system becomes stationary, the state eventually returns to the initial state. However, such reversal cannot be perfect due to imperfection of the control of microscopic degrees of freedom [25]. To be specific, let us consider a time-reversal test [26–28] in which a system evolves forward and then backward in time for the same time period by adding a small perturbation upon or during the return process. If the dynamics is chaotic, the final state deviates significantly from the initial state however small the perturbation is [28, 29]. References [26, 27, 30, 31] discussed the irreversibility in quantum chaos measured by expectation values of observables under the time-reversal test with a unitary perturbation added upon reversal [32]. Note that localized states are usually assumed as initial states because such states are suitable for the study of irreversible delocalization of the state under the time-reversal test [5, 26, 27, 29–31, 33].

Recently, the growth of quantum noncommutativity of two unequal-time operators has been proposed as yet another dynamical probe of quantum chaos. In fact, an expectation value of a squared commutator of two unequal-time observables has actively been investigated in fields ranging from high-energy [34–39] to condensed-matter physics [40–52]. For semiclassical chaotic models before the Ehrenfest time [35, 53–60], the semiclassical approximation ensures that the squared commutator grows exponentially reflecting the instability of phase-space trajectories.

Then the following question naturally arises: how are the two fundamental dynamical chaotic indicators, irreversibility and noncommutativity, related to each other? Several studies have suggested a close connection between the squared commutator and the Loschmidt echo [55, 57, 61, 62], but quantitative understanding has remained elusive. Recently, Schmitt et al. [31] have discussed that the irreversibility measured by observables after the time-reversal test can be written as a certain type of commutators. However, it is unclear how their results are related to previous discussions on noncommutativity growth based on the squared commutator.

In this Letter, we argue that noncommutativity and irreversibility are essentially equivalent to each other for initially localized states. Namely, the squared commutator $C_{AB}(t) := -\langle [\hat{A}(t), \hat{B}]^2 \rangle$ of two unequal-time observables $\hat{A}(t)$ and $\hat{B} = \hat{B}(0)$ is equivalent to $I_{AB}(t) := \langle \hat{A}(t)\hat{B}^\dagger\hat{B}\hat{A}(t) \rangle$, which is interpreted as the irreversibility measured through $\hat{B}$ (we assume $\langle \hat{B} \rangle = 0$) under the time-reversal test against perturbation $\hat{A}$ at time $t$. In fact, we prove the following relation:

$$\frac{C_{AB}(t)}{I_{AB}(t)} = 1 + \alpha_t(2\sqrt{s_{t1}r_t} + s_{t1}r_t),$$

(1)

where $\alpha_t$ is a time-dependent numerical factor satisfying $|\alpha_t| \leq 1$, $s_t$ measures the stability of $\hat{A}(t)\hat{A}(t)$ against the initial perturbation of $\hat{B}$, and $r_t$ represents the reversibility of $\hat{B}^\dagger\hat{B}$ under the time-reversal test (see Eq. (3)). We argue that $s_t = O(1)$ and $r_t \ll 1$ for initially localized states in chaotic systems, and hence $C_{AB}(t) \simeq I_{AB}(t)$ (see Fig. 1(a)). We test the validity of this conjecture for two prototypical examples of quantum chaos, namely, a single-particle kicked rotor and interacting nonintegrable many-body systems. For the latter systems, we also argue that $r_t \ll 1$ no longer holds for thermal initial states if the time-reversal test does not macroscopically change the energy.

Equation (1) reveals the importance of an out-of-time-ordered correlator (OTOC) in the form of three-point rather than four-point correlators [63]. The squared commutator can be decomposed as $C_{AB}(t) = I_{AB}(t) + D_{AB}(t) - 2\text{Re}[F_{AB}(t)]$, where $D_{AB}(t) := \langle \hat{B}^\dagger\hat{A}(t)\hat{A}(t)\hat{B} \rangle$ is a time-ordered correlator and $F_{AB}(t) := \langle \hat{A}(t)\hat{B}^\dagger\hat{B}\hat{A}(t) \rangle$ is an OTOC [64]. Previous studies mainly considered thermal-equilibrium initial states, which are delocalized, and argued that the dynam-
ics of four-point OTOC (4-OTOC) $F_{AB}(t)$ contributes to a nontrivial growth of $C_{AB}(t)$ around a timescale $t_*$, while $I_{AB}(t)$ and $D_{AB}(t)$ rapidly decay to constant values much before $t_*$. In this case, $[H, \rho] = 0$ and thus $I_{AB}(t)$ is (anti-)time-ordered (Fig. 1(c)). For nonequilibrium states satisfying $[H, \rho] \neq 0$, the three-point correlator $I_{AB}(t)$ becomes an OTOC, which we refer to as a three-point OTOC (3-OTOC). From Eq. (1), we can show that the 3-OTOC $I_{AB}(t)$ rather than the 4-OTOC $F_{AB}(t)$ dominates the growth of $C_{AB}(t)$ for initially localized states (Fig. 1(a)) [66].

Irreversibility and noncommutativity. We show how $I_{AB}(t)$ measures the system's irreversibility for an initially localized state. We define localized initial states with respect to $\hat{B}$ as states that satisfy $\langle B^\dagger B \rangle \ll \langle \hat{B}^\dagger(t) \hat{B} \rangle$ for sufficiently large $t$. We first decompose $D_{AB}(t)$ and $I_{AB}(t)$ as

$$D_{AB}(t) = \langle \hat{B}^\dagger \hat{A}(t)^\dagger \hat{A}(t) \hat{B} \rangle = Tr[\hat{\rho}_t \hat{A}(t)^\dagger \hat{A}(t) Tr[\hat{\rho}_t \hat{B}^\dagger \hat{B}]]$$

$$I_{AB}(t) = \langle \hat{A}(t)^\dagger \hat{B} \hat{B}(t) \hat{A}(t) \rangle = Tr[\hat{\rho}_t \hat{A}^\dagger(t) Tr[\hat{\rho}_t \hat{B}^\dagger \hat{B}]]$$

where $\hat{\rho}_t := \hat{U}_t \hat{\rho} \hat{U}_t^\dagger$ is the state obtained from the initial state $\hat{\rho}$ by two protocols (a) and (b) illustrated in Fig. 2. This figure shows the coarse-grained Wigner functions on the $x$-$p$ phase space for each density matrix, which is numerically obtained by using the quantum kicked rotor as detailed later. Here, we focus on how they change for two different protocols. In protocol (a), the initial state is perturbed with $\hat{B}$ as $\hat{\rho}_t = \frac{\hat{B} \hat{\rho} \hat{B}^\dagger}{Tr[\hat{\rho} \hat{B}^\dagger \hat{B}]}$, which then evolves in time as $\hat{\rho}_t := \hat{U}_t \hat{\rho} \hat{U}_t^\dagger$. The product of the expectation value of $\hat{B}^\dagger \hat{B}$ for $\hat{\rho}_t$ and that of $\hat{\rho}_t$ for $\hat{\rho}_t$ gives $D_{AB}(t)$. For protocol (b), we let the state evolve during time $t$ as $\hat{\rho}_t = \hat{U}_t \hat{\rho} \hat{U}_t^\dagger$, and perturb the state with $\hat{A}$ as $\hat{\rho}_t := \frac{\hat{A} \hat{\rho}_t \hat{A}^\dagger}{Tr[\hat{\rho} \hat{A}^\dagger \hat{A}]}$. We then perform time reversal for time $-t$, obtaining $\hat{\rho}_t := \frac{\hat{C}_t \hat{U}_t \hat{\rho}_t \hat{U}_t^\dagger \hat{C}_t^\dagger}{Tr[\hat{\rho} \hat{C}_t^\dagger \hat{C}_t]} = \frac{\hat{A}(t) \hat{\rho}(t) \hat{A}(t)^\dagger}{Tr[\hat{\rho} \hat{A}^\dagger \hat{A}]}$ [67]. The product of the expectation value of $\hat{B}^\dagger \hat{B}$ for $\hat{\rho}_t$ and that of $\hat{A}^\dagger \hat{A}$ for
\( \hat{\rho}_t \) gives \( I_{AB}(t) \). Note that this protocol is similar to the one used in Refs. [26, 27, 30, 31], where \( \hat{A} \) is chosen to be unitary.

From Eq. (2), we obtain \( \frac{D_{AB}(t)}{I_{AB}(t)} = s_t r_t \), where

\[
 s_t := \frac{\text{Tr}[\hat{\rho}_t \hat{A}^\dagger \hat{A}]}{\text{Tr}[\hat{\rho}_t \hat{A}^\dagger \hat{A}]}, \quad r_t := \frac{\text{Tr}[\hat{\rho}_t \hat{B}^\dagger \hat{B}]}{\text{Tr}[\hat{\rho}_t \hat{B}^\dagger \hat{B}]}.
\]

Then the Cauchy-Schwarz inequality leads to \( |F_{AB}(t)| \leq \sqrt{I_{AB}(t)D_{AB}(t)} = \sqrt{s_t r_t} I_{AB}(t) \) [68]. The definition of \( C_{AB}(t) \) and the triangle inequality lead to \( |C_{AB}(t) - I_{AB}(t)| \leq D_{AB}(t) + 2|F_{AB}(t)| \). Hence \( |C_{AB}(t)| - 1 \leq s_t r_t + 2\sqrt{s_t r_t} \), which leads to Eq. (1).

Now, we assume

(i) \( s_t = O(1) \), (ii) \( r_t \ll 1 \).

The condition (i) means that time evolution of \( \hat{A}^\dagger \hat{A} \) is stable under the initial perturbation of \( \hat{B} \). The condition (ii) means that the initial state \( \hat{\rho} \) is irreversible due to the sensitivity against the time-reversal test. We argue that these two conditions hold true for chaotic dynamics with initially localized states, and hence \( C_{AB}(t) \approx I_{AB}(t) \). In the following, we test this conjecture for a kicked rotor and quantum many-body systems. For the latter systems, we also argue that the condition (ii) breaks down for initially thermal states when the time-reversal test does not macroscopically change the energy.

Quantum kicked rotor. First, we numerically confirm Eq. (4) and \( C_{AB}(t) \approx I_{AB}(t) \) for a single-particle quantum kicked rotor:

\[
\hat{H}_{\text{QKR}}(t) := \frac{\hat{p}^2}{2} + K \cos \hat{x} \sum_n \delta(t - n),
\]

where \( \hat{p} = -i \hbar \frac{\partial}{\partial \hat{x}} \) is the (angular) momentum operator and \( \hbar \) denotes the dimensionless Planck constant, which scales with \( \hbar \). We impose a periodic boundary condition on \( x \) as \( -\pi \leq x < \pi \). Then, \( \hat{p} \) has eigenvalues \( m\hbar \) and eigenvectors \( |x|_m = \frac{1}{\sqrt{2\pi}} e^{i mx} \) for each \( m \) (\( m \in \mathbb{Z} \)). Time evolutions of correlators can be expressed in terms of the Floquet operator \( \hat{F} := e^{-i \frac{\sigma^2}{2\hbar^2} t} e^{-i \frac{K \cos x}{\hbar^2} t} \). We consider an initial wave-packet state

\[
\hat{\rho}_w := |\psi_w\rangle \langle \psi_w |, \quad |\psi_w\rangle := \frac{1}{Z_w} \sum_m e^{-\frac{\hbar m^2}{2\sigma^2}} |p_m\rangle,
\]

where \( Z_w := \sqrt{\sum_m e^{-\frac{\hbar m^2}{2\sigma^2}}} \). See Appendix IA in Supplemental Material [69] for different initial states.

We consider \( \hat{A} = \hat{B} = \hat{\rho} \) (the case where \( \hat{A} \) is unitary is discussed in Appendix IC [69]) after \( t (\in \mathbb{Z}) \) periods. Note that \( \hat{\rho}_w \) satisfies \( \langle \hat{\rho} \rangle = 0 \) and \( \langle \hat{\rho}^2 \rangle \propto \langle \hat{\rho}(t)^2 \rangle \propto t \) [70] for large \( t \), namely \( \hat{\rho}_w \) is initially localized with respect to \( \hat{\rho} \). Figure 3 (a) shows the dynamics of \( C_{pp}(t) = -\langle \langle \hat{\rho}(t), \hat{\rho}\rangle^2 \rangle, I_{pp}(t) = \langle \langle \hat{\rho}(t)\hat{p}^2 \hat{\rho}(t) \rangle \rangle, |\text{Re}\{F_{pp}(t)\}| = |\text{Re}\{\langle \hat{\rho}(t)\hat{p}^2 \hat{\rho}(t) \rangle \rangle |, \text{and} D_{pp}(t) = \langle \langle \hat{\rho}(t)^2 \rangle \rangle. \) We see that \( C_{pp}(t) \) and \( I_{pp}(t) \) are almost equal and that they both asymptotically behave as \( t^2 \), while \( D_{pp}(t) \) and \( |\text{Re}\{F_{pp}(t)\}| \) behave diffusively as \( t \) [71]. Figure 3 (b) shows time evolutions of \( s_t, r_t \) and \( \sqrt{s_t r_t} \). We see that two conditions (4) are satisfied when \( t \gg 3 \). Schematically, these conditions are understood from the dynamics of the coarse-grained Wigner function [72], as shown in Fig. 2.

Our results demonstrate that Eq. (1) with the conditions (4) leads to nontrivial consequences. Before the Ehrenfest time \( t_E \), \( C_{pp}(t) \) grows exponentially by the semiclassical approximation [35, 53, 55, 56]. Hence, from Eq. (1), irreversibility \( I_{pp}(t) \) grows exponentially at short times [73], as demonstrated in Appendix IA [69]. Conversely, for longer times, \( I_{pp}(t) \propto t^2 \) is intuitively understood, as discussed in Appendix IB [69]. Then, Eq. (1) explains \( C_{pp}(t) \propto t^2 \), which cannot be explained from the semiclassical approximation.

Interacting quantum many-body systems. Next, we consider interacting many-body systems on \( N \) lattice sites. While we do not have a simple phase-space representation as in Fig. 2, the above protocols are well-defined. We especially focus on the case where \( \hat{H} \) and \( \hat{B} \) can be written as \( \sum_i \hat{h}_i \) and \( \sum_i \hat{b}_i \), respectively. We assume that they are translationally invariant, namely the forms of \( \hat{h}_i \) and \( \hat{b}_i \) are independent of the site \( i \). We also assume \( \langle \hat{B} \rangle = 0 \), which leads to \( \langle \hat{b}_i \rangle = 0 \). We consider a translationally invariant initial state \( \hat{\rho} \) that satisfies the cluster decomposition property [74], which means that for two distant regions \( I \) and \( J \) with \( I \cap J = \emptyset \),

\[
\langle \prod_{\{f|k_i \in I\cup J\}} \hat{a}^\dagger_{f,k_i} \rangle \approx \langle \prod_{\{f|k_i \in I\}} \hat{a}^\dagger_{f,k_i} \rangle \langle \prod_{\{f|k_i \in J\}} \hat{a}^\dagger_{f,k_i} \rangle.
\]

Here \( \hat{a}^\dagger_{f,I} \) is the \( f \)-th operator localized around site \( k_f \). From the cluster decomposition, we can show that the energies of \( \hat{\rho} \) and \( \hat{\rho}' \) are macroscopically equal (see Ap-
pendix IIA [69] for a proof).

To justify condition (i) in Eq. (4), we assume the eigenstate thermalization hypothesis (ETH) [10, 12, 75], which is expected to hold for nonintegrable systems [76]. The ETH justifies that any initial state with a given energy relaxes to a state described by the canonical ensemble at the corresponding temperature for most of the time [10, 12]. By applying the ETH, we find that \( \hat{\rho} \) and \( \hat{\rho}' \) relax to the same canonical ensemble at inverse temperature \( \beta \) in the long run because they have the same energies. Thus, \( s_t \approx 1 \) for most of the time, which justifies the condition (i) in Eq. (4).

To justify condition (ii) in Eq. (4), we assume the eigen-state thermalization after quench because. Their definition and the ETH. Then, \( \langle \hat{\rho}(t) \rangle = \rho_{\text{therm}}(t) \), which satisfies \( \langle \hat{\rho}(t) \rangle = \rho_{\text{therm}}(t) \), where \( \rho_{\text{therm}}(t) \) is close to unity (see Appendix IIB [69]). This assumption is satisfied for initially localized states, which satisfies \( \langle \hat{\rho}(t) \rangle = \rho_{\text{therm}}(t) \), where \( \rho_{\text{therm}}(t) \) denotes an average over the canonical ensemble at inverse temperature \( \beta' \). We here assume that \( \rho_{\text{therm}}(t) \) and \( \rho_{\text{therm}}(t) \) have macroscopically the same energy: namely, the time-reversal test does not change the energy of the system. This is justified when \( \hat{A} \) is written as a local operator, the sum of local operators, or a unitary operator that is close to unity (see Appendix IIB [69]). This assumption leads to \( \beta = \beta' \). Now, for initially localized states, \( \text{Tr}[\hat{\rho}\hat{B}t\hat{A}] = \langle \hat{B}(t)\hat{A}(t) \rangle \approx \langle \hat{B}(t)\hat{A}(t) \rangle \approx \langle \hat{B}\hat{A} \rangle = \rho_{\text{therm}}(t) \) because of their definition and the ETH. Then, \( r_t \approx \langle \hat{B}\hat{A} \rangle \approx 1 \) holds true. On the other hand, for initially thermal states, \( r_t \approx \langle \hat{B}\hat{A} \rangle \approx 1 \) and the condition (4) (ii) does not hold [77]. We note that Eq. (4) can hold true without the same-energy conditions (see Appendix IIC [69]).

We remark that the initially localized state defined above, which satisfies \( \langle \hat{B}\hat{A} \rangle \approx \langle \hat{B}\hat{A} \rangle \), is naturally obtained for nonequilibrium states. To see this, we assume that the canonical ensemble has the cluster decomposition property and that \( \hat{\rho}_{\text{therm}}(t) \) is nonzero in the thermodynamic limit. The former assumption is satisfied for sufficiently high-temperature systems in our setup [78]. The latter assumption means that the expectation value of \( \hat{B} \) for \( \hat{\rho}_{\text{therm}}(t) \) is macroscopically distinct from that for \( \hat{\rho} \) (note that \( \langle \hat{b}_i \rangle = 0 \), which generally holds true for localized nonequilibrium initial states after quench because.

Then, we obtain \( \langle \hat{B}\hat{A} \rangle \approx \sum_{i,j}a_{ij}\langle \hat{b}_i \hat{b}_j \rangle = O(N) \) and \( \langle \hat{B}\hat{A} \rangle \approx \sum_{i,j}a_{ij}\langle \hat{b}_i \hat{b}_j \rangle = O(N^2) \). Here, \( a_{ij} \) is a set where \( i \) and \( j \) are close. Thus, our assumptions above lead to \( \langle \hat{B}\hat{A} \rangle \approx \langle \hat{B}\hat{A} \rangle \) in the thermodynamic limit. From this discussion, we also obtain \( r_t \approx O(N^{-1}) \).

We numerically check Eq. (4) and \( C_{AB}(t) \approx I_{AB}(t) \) for a 1D transverse Ising model after a quench. The Hamiltonian is given by \( \hat{H}_\text{TI}(t) := -\sum_{i=1}^L \hat{\sigma}_i^x - 1.05 \hat{\sigma}_i^z + h \hat{\sigma}_i^x \), where \( \hat{\sigma}_i^x \) and \( \hat{\sigma}_i^z \) are obtained by using the QUSPIN package [80].

![FIG. 4. Quench dynamics of the transverse Ising model H_{TI}(h = 0.5) for L = 14. Here, A = \sum_{i=1}^L \hat{\sigma}_i^z, B = \sum_{i=1}^L \langle \hat{\sigma}_i^x - \hat{\sigma}_i^z \rangle, and the initial state is the ground state of H_{TI}(h = -5.0). The Planck constant is set to be unity. (a) Time evolutions of C_{AB}(t) = -\langle [\hat{A}(t), \hat{B}](t) \rangle, D_{AB}(t) = \langle [\hat{B}^\dagger(\hat{A}(t), \hat{B}^\dagger(\hat{A}(t)) \rangle and |\text{Re}[F_{AB}(t)]| = |\text{Re}[\langle \hat{A}(t)^\dagger \hat{B}^\dagger(\hat{A}(t)\hat{B}) \rangle]|. Equation (1) holds true for \( t \geq 1 \). (b) Time evolutions of s_t, r_t and \( \sqrt{s_t r_t} \) defined in Eq. (3). When \( t \geq 1 \), two conditions (4) are satisfied and the second term in Eq. (1) becomes small. The curves are obtained by using the QUSPIN package [80].](image)
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[1] F. Haake, *Quantum signatures of chaos*, Vol. 54 (Springer Science & Business Media, 2010).
[2] H.-J. Stöckmann, *Quantum Chaos: An Introduction* (Cambridge University Press, 2006).
[3] M. C. Gutzwiller, *Chaos in classical and quantum mechanics*, Vol. 1 (Springer Science & Business Media, 2013).
[4] M. V. Berry, Journal of Physics A: Mathematical and General 10, 2083 (1977).
[5] A. Peres, Phys. Rev. A 30, 1610 (1984).
[6] O. Bohigas, M. J. Giannoni, and C. Schmit, Phys. Rev. Lett. 52, 1 (1984).
[7] M. Berry, Physica Scripta 40, 335 (1989).
[8] A. V. Andreev, O. Agam, B. D. Simons, and B. L. Altshuler, Phys. Rev. Lett. 76, 3947 (1996).
[9] M. Feingold, N. Moiseyev, and A. Peres, Phys. Rev. A 30, 500 (1984).
[10] M. Srednicki, Phys. Rev. E 50, 888 (1994).
[11] S. Müller, S. H. Strogatz, P. Braun, F. Haake, and A. Altland, Phys. Rev. Lett. 93, 014103 (2004).
[12] M. Rigol, V. Dunjko, and M. Oshariani, Nature 452, 854 (2008).
[13] L. F. Santos and M. Rigol, Phys. Rev. E 81, 036206 (2010).
[14] A. Pal and D. A. Huse, Phys. Rev. B 82, 174411 (2010).
[15] E. Khatami, G. Pupillo, M. Srednicki, and M. Rigol, Phys. Rev. Lett. 111, 050403 (2013).
[16] W. Beugeling, R. Moessner, and M. Haque, Phys. Rev. E 89, 042112 (2014).
[17] W. Beugeling, R. Moessner, and M. Haque, Phys. Rev. E 91, 012144 (2015).
[18] L. D’Alessio, Y. Kabir, A. Polkovnikov, and M. Rigol, Advances in Physics 65, 239 (2016).
[19] D. J. Luitz and Y. Bar Lev, Phys. Rev. Lett. 117, 170404 (2016).
[20] M. Serbyn and J. E. Moore, Phys. Rev. B 93, 041424 (2016).
[21] R. Hamazaki and M. Ueda, Phys. Rev. Lett. 120, 080603 (2018).
[22] P. Kos, M. Ljubotina, and T. Prosen, Phys. Rev. X 8, 021062 (2018).
[23] S. H. Strogatz, *Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering* (CRC Press, 2015).
[24] J. Loschmidt, Wiener Ber. 73, 128 (1876).
[25] W. Thomson, in *Proc. R. Soc. Edinburgh*, Vol. 8 (1874) p. 325.

[26] S. Adachi, M. Toda, and K. Ikeda, Phys. Rev. Lett. 61, 659 (1988).
[27] H. S. Yamada and K. S. Ikeda, The European Physical Journal B-Condensed Matter and Complex Systems 85, 1 (2012).
[28] Y. Murashita, N. Kura, and M. Ueda, arXiv preprint arXiv:1802.10483 (2018).
[29] D. L. Shepelyansky, Physica D: Nonlinear Phenomena 8, 208 (1983).
[30] M. Schmitt and S. Kehrein, arXiv preprint arXiv:1711.00015 (2017).
[31] M. Schmitt, D. Sels, S. Kehrein, and A. Polkovnikov, arXiv preprint arXiv:1802.06796 (2018).
[32] This is similar to the Loschmidt echo [5, 33, 81, 82], which measures the fidelity between the initial and final states under the time-reversal test.
[33] R. A. Jalabert and H. M. Pastawski, Phys. Rev. Lett. 86, 2490 (2001).
[34] A. Kitaev, in *talk given at Fundamental Physics Prize Symposium* (2014).
[35] A. Kitaev, in *KITP strings seminar and Entanglement* (2015).
[36] S. H. Shenker and D. Stanford, Journal of High Energy Physics 2015, 132 (2015).
[37] J. Maldacena, S. H. Shenker, and D. Stanford, Journal of High Energy Physics 2016, 106 (2016).
[38] J. Maldacena and D. Stanford, Phys. Rev. D 94, 106002 (2016).
[39] J. Polchinski and V. Rosenhaus, Journal of High Energy Physics 2016, 1 (2016).
[40] Y. Gu, X.-L. Qi, and D. Stanford, Journal of High Energy Physics 2017, 125 (2017).
[41] A. Bohrdt, C. Mendl, M. Endres, and M. Knap, New Journal of Physics 19, 063001 (2017).
[42] M. Gartner, J. G. Bohnet, A. Safavi-Naini, M. L. Wall, J. J. Bollinger, and A. M. Rey, Nature Physics (2017).
[43] S. Banerjee and E. Altman, Phys. Rev. B 95, 134302 (2017).
[44] H. Shen, P. Zhang, R. Fan, and H. Zhai, Phys. Rev. B 96, 054503 (2017).
[45] R. Fan, P. Zhang, H. Shen, and H. Zhai, Science Bulletin (2017).
[46] Y. Huang, Y.-L. Zhang, and X. Chen, Annalen der Physik 529 (2017).
[47] J. Li, R. Fan, H. Wang, B. Ye, B. Zeng, H. Zhai, X. Peng, and J. Du, Phys. Rev. X 7, 031011 (2017).
[48] N. Tsuji, P. Werner, and M. Ueda, Phys. Rev. A 95, 011601 (2017).
[49] I. Kukuljan, S. Grozdanov, and T. Prosen, Phys. Rev. B 96, 060301 (2017).
[50] B. Dóra and R. Moessner, Phys. Rev. Lett. 119, 026802 (2017).
[51] E. Iyoza and T. Sagawa, Phys. Rev. A 97, 042330 (2018).
[52] A. A. Patel, D. Chowdhury, S. Sachdev, and B. Swingle, Phys. Rev. X 7, 031047 (2017).
[53] A. Larkin and Y. N. Ovchinnikov, Sov Phys JETP 28, 1200 (1969).
[54] J. Kurchan, Journal of Statistical Physics 171, 965 (2018).
[55] E. B. Rozenbaum, S. Ganeshan, and V. Galitski, Phys. Rev. Letters 1118, 086801 (2017).
[56] J. S. Cotler, D. Ding, and G. R. Penington, arXiv preprint arXiv:1704.02979 (2017).
We consider discrete Wigner functions \[ \mathcal{W}_n \]. This process generates unphysical ghost images, but they mostly vanish after coarse-graining. We note that while quantum interference patterns may vanish by coarse-graining, it does not affect our main results.

In Ref. [31], the authors discussed an exponential growth of irreversibility via the semiclassical approximation of the double commutator in the form of \( [\hat{X}(t), [\hat{X}(t), \hat{Y}]] \). In contrast, our work relates \( I_{pp}(t) \) to the better-known indicator \( C_{pp}(t) \), where the squared form rules out the vanishment of the chaotic growth due to the averaging.

D. Ruelle, *Statistical mechanics: Rigorous results* (World Scientific, 1999).

J. M. Deutsch, Phys. Rev. A 43, 2046 (1991).

Note that the ETH does not hold for integrable systems [12, 13], many-body localized systems [14, 84, 85], and systems that have certain symmetry sectors [86–88].

This argument also holds for other delocalized states that are equivalent to thermal states for macroscopic observables, such as typical pure states [89, 90].

M. Kliesch, C. Gogolin, M. J. Kastoryano, A. Riera, and J. Eisert, Phys. Rev. X 4, 031019 (2014).

H. Kim, M. C. Bañuls, J. I. Cirac, M. B. Hastings, and D. A. Huse, Phys. Rev. E 92, 012128 (2015).

P. Weinberg and M. Bukov, SciPost Physics 2, 003 (2017).

T. Gorin, T. Prosen, T. H. Seligman, and M. Žnidarič, Physix Reports 435, 33 (2006).

P. Jacquod and C. Petitjean, Advances in Physics 58, 67 (2009).

A. R. Kolovsky, Chaos: An Interdisciplinary Journal of Nonlinear Science 6, 534 (1996).

R. Nandkishore and D. A. Huse, Annual Review of Condensed Matter Physics 6, 15 (2015).

J.-y. Choi, S. Hild, J. Zeiher, P. Schauss, A. Rubio-Abadal, T. Yefsah, V. Khemani, D. A. Huse, I. Bloch, and C. Gross, Science 352, 1547 (2016).

R. Hamazaki, T. N. Ikeda, and M. Ueda, Phys. Rev. E 93, 032116 (2016).

N. Shiraishi and T. Mori, Phys. Rev. Lett. 119, 030601 (2017).

T. Mori and N. Shiraishi, Phys. Rev. E 96, 022153 (2017).

S. Goldstein, J. L. Lebowitz, R. Tumulka, and N. Zanghì, Phys. Rev. Lett. 96, 050403 (2006).

S. Popescu, A. J. Short, and A. Winter, Nature Physics 2, 754 (2006).
Supplemental Material for “Operator Noncommutativity and Irreversibility in Quantum Chaos”

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I. DETAILS OF NUMERICAL SIMULATIONS OF THE QUANTUM KICKED ROTOR

A. Short-time behavior and semiclassical representation

In this section, we consider the short-time dynamics of the quantum kicked rotor (Eq. (5) in the main text) before the Ehrenfest time $t_E$ and its semiclassical representation. We consider two localized initial states in momentum space (i.e., $\hat{B} = \hat{p}$). The first is a wave packet state $\hat{\rho}_w := |\psi_w\rangle \langle \psi_w|$, $|\psi_w\rangle := \frac{1}{Z_w} \sum_m e^{-\frac{\hbar_{\text{eff}} m^2}{2\sigma^2}} |p_m\rangle$ (S-1),

which is discussed in the main text. The second is the canonical distribution for a free Hamiltonian $\hat{H}_0 := \hat{p}^2/2$, $\hat{\rho}_T := \frac{1}{Z_T} \sum_m e^{-\frac{\hbar_{\text{eff}} m^2}{2T}} |p_m\rangle \langle p_m|$ (S-2).

This initial canonical distribution is localized with respect to $\hat{p}$ (but not $\hat{\dot{x}}$) when we consider the Floquet time evolution $\hat{F} = e^{-\frac{\hbar_{\text{eff}} \hat{p}^2}{2T}} e^{-\frac{iK \cos \hat{x}}{\hbar_{\text{eff}}}}$ for $\hat{H}(t)$ (see Eq. (5) in the main text). Note that the state is not stationary ($[\hat{F}, \hat{\rho}_T] \neq 0$) due to periodic kicks. Thus, $I_{AB}(t) = \langle \hat{A}^\dagger(t) \hat{B}^\dagger \hat{B} \hat{A}(t) \rangle$ becomes a 3-OTOC for these initial states.

For reference, we also apply a semiclassical approximation to each correlator. We consider the average of a classical function $S(x,p,t)$ over the Wigner distribution $W$ of the initial states,

$S_t := \int dx dp W(x,p) S(x,p,t)$. (S-3)

As shown below, every correlator is approximated before $t_E$ by $S_t$ for an appropriate $S(x,p,t)$. The Wigner distributions of our initial states, $\hat{\rho}_w$ and $\hat{\rho}_T$, are approximated in Gaussian forms as

$W_w(x,p) = \frac{1}{\pi \hbar_{\text{eff}}} e^{-\frac{p^2}{\hbar_{\text{eff}} \sigma^2} - \frac{x^2}{\hbar_{\text{eff}} \sigma^2}}$ (S-4)

and

$W_T(x,p) = \frac{1}{\sqrt{(2\pi)^3 T}} e^{-\frac{p^2}{2T}}$ (S-5)
FIG. S-1. Short-time dynamics of $C_{pp}(t) = -\langle [\hat{p}(t), \hat{p}]^2 \rangle$, $D_{pp}(t) = \langle \hat{p}\hat{p}(t)^2 \hat{p} \rangle$, Re [$F_{pp}(t)$] = Re [$\langle \hat{p}(t)\hat{p}\hat{p}(t)\hat{p} \rangle$], $I_{pp}(t) = \langle \hat{p}(t)^2 \hat{p}(t) \rangle$, and $\hbar^2_{\text{eff}} \left( \frac{\partial \hat{p}}{\partial x} \right)^2$ for initial states (a) $\hat{\rho}_w$ and (b) $\hat{\rho}_T$. (Upper panels) For both initial states and up to the Ehrenfest time $t \lesssim t_E \sim 6$, $C_{pp}(t)$ and $I_{pp}(t)$ agree excellently and exhibit exponential growths, and they are well approximated by $\hbar^2_{\text{eff}} \left( \frac{\partial \hat{p}}{\partial x} \right)^2$.

(Bottom panels) Compared with $I_{pp}(t)$, $D_{pp}(t)$ and Re [$F_{pp}(t)$] are well described by the classical average $\overline{p_1^2 p_2^2}$ for $t \lesssim t_E$.

As shown in Fig. S-1, we first consider the short-time behaviors of $C_{pp}(t) = -\langle [\hat{p}(t), \hat{p}]^2 \rangle$, $I_{pp}(t) = \langle \hat{p}(t)^2 \hat{p}(t) \rangle$ (\(\hat{p}(t) := (\hat{F}^\dagger)^n \hat{p} \hat{F}^n\)), Re [$F_{pp}(t)$] = Re [$\langle \hat{p}(t)\hat{p}\hat{p}(t)\hat{p} \rangle$], $D_{pp}(t) = \langle \hat{p}\hat{p}(t)^2 \hat{p} \rangle$, the classical average neglecting the noncommutativity $\overline{p_1^2 p_2^2}$, and the initial sensitivity $\hbar^2_{\text{eff}} \left( \frac{\partial \hat{p}}{\partial x} \right)^2$. The left and right figures correspond to $\hat{\rho}_w$ and $\hat{\rho}_T$, respectively. For $t \lesssim t_E \simeq 6$, $D_{pp}(t)$ and Re [$F_{pp}(t)$] are well described by $\overline{p_1^2 p_2^2}$, whereas the 3-OTOC $I_{pp}(t)$ grows exponentially. The exponential growth of $I_{pp}(t)$ represents the initial sensitivity of classical chaos because it is close to $C_{pp}(t)$ (i.e., Eq. (1) with the conditions (4) in the main text holds true), which reduces to $\hbar^2_{\text{eff}} \left( \frac{\partial \hat{p}}{\partial x} \right)^2$ [2–4] in the semiclassical limit.
FIG. S-2. Momentum distributions $P(p)$ of (a) $\hat{\rho}_t$ and (b) $\hat{\tilde{\rho}}_t$ for $t = 50, 100$ and $200$. For $\hat{\rho}_t$, $P(p)$ is close to a Gaussian form and spreads with increasing $t$. For $\hat{\tilde{\rho}}_t$, $P(p)$ is not Gaussian, but spreads as $t$ increases. (insets) Dynamical scaling of $P(p)$ for $\hat{\rho}_t$ and $\hat{\tilde{\rho}}_t$. All the curves collapse to a single curve after rescaling according to $P(p, t) = f(p/\sqrt{t})/\sqrt{t}$ for both of the states. For both figures, we take the initial state $\hat{\rho}_w$ of the wave packet with $\hbar_{\text{eff}} = 2^{-6}$, $\sigma = 4$, and $K = 10$.

B. Origin of the anomalous quadratic scaling in the long-time behavior

As we have seen in Fig. 4 (a) in the main text, $D_{pp}(t)$ grows diffusively in the long-time regime as $\propto t$ (the dynamical localization [5] does not occur within the time scale of our interest). Indeed, from Eq. (2) in the main text, $D_{pp}(t)$ is the product of $\text{Tr}[\hat{\rho}\hat{p}^2]$ (where we use the wave-packet state $\hat{\rho} = \hat{\rho}_w$) and $\text{Tr}[\hat{\rho}'\hat{p}^2]$. The former does not depend on time and the latter behaving diffusively [5], so this time-ordered correlator grows as $\propto t$.

On the other hand, $I_{pp}(t)$ is proportional to $t^2$, which is different from the classical diffusive behavior. Thanks to Eqs. (1) and (4) in the main text, $C_{pp}(t)$ also follows a $t^2$ power law [4]. The anomalous quadratic scaling for $I_{pp}(t)$ originates from the fact that the momentum distribution of $\hat{\tilde{\rho}}_t$ spreads as much as that of $\hat{\rho}_t$. As we have seen in Eq. (2) in the main text, $I_{pp}(t)$ is the product of $\text{Tr}[\hat{\tilde{\rho}}\hat{p}^2]$ and $\text{Tr}[\hat{\rho}_t\hat{p}^2]$. Figures S-2 (a) and (b) plot the coarse-grained momentum distribution

$$P(p) := \frac{1}{\Delta p} \sum_{p_m \in |p - \Delta p/2, p + \Delta p/2)} \langle p_m | \hat{\rho} | p_m \rangle$$

(S-6)

for $\hat{\rho}_t$ and $\hat{\tilde{\rho}}_t$, respectively. Figure S-2 (a) shows a diffusive, Gaussian profile in quantum chaos [6]. After the time-reversal test, $\hat{\tilde{\rho}}_t$ will remain extended in momentum space, especially
for large $t$. In this time evolution, $P(p)$ for $\hat{\rho}_t$ and $\hat{\tilde{\rho}}_t$ obeys a dynamical scaling relation

$$P(p, t) = \frac{1}{\sqrt{t}} f\left(\frac{p}{\sqrt{t}}\right), \quad \text{(S-7)}$$

as shown in the insets of Fig. S-2(a) and (b) [7]. Note that $P(p)$ for $\hat{\tilde{\rho}}_t$ obeys the above-mentioned diffusive scaling, even though it is not Gaussian. Such a delocalization, which obeys the scaling in Eq. (S-7), leads to

$$\text{Tr}[\hat{\tilde{\rho}}_t \hat{p}^2] \simeq \int dp p^2 P(p, t) \propto t. \quad \text{(S-8)}$$

Thus, Eq. (2) in the main text and the above-mentioned diffusive behavior of $\langle \hat{p}(t)^2 \rangle$ give $\langle \hat{p}(t)\hat{p}^2(t) \rangle \propto t^2$. This clearly shows that the 3-OTOC $I_{pp}(t)$ gives the measure of irreversibility that explains the anomalous power-law growth of $C_{pp}(t)$ in the long-time regime.

**C. Unitary perturbations**

Here, we consider unitary perturbations, which can often be implemented experimentally [8]. We take $\hat{A} = \hat{V} = e^{i\hat{p}t_{\text{eff}}} / \hbar$, which translates the state by $\epsilon$ in the $x$ direction, and $\hat{B} = \hat{p}$ [9, 10]. Similarly to the case for $\hat{A} = \hat{p}$, the short-time dynamics of

$$C_{Vp}(t) = -\langle [e^{i\hat{p}(t)_{\epsilon}} \hat{p}]^2 \rangle \quad \text{(S-9)}$$

exhibits an exponential growth that corresponds to $-\hbar^2 e^{i\hat{p}(t)_{\epsilon}} \left. \frac{\partial}{\partial x} e^{i\hat{p}(t)_{\epsilon}} \right|_{x=\epsilon} = \epsilon^2 \left( \frac{\partial \hat{p}}{\partial x} \epsilon \right)^2$ before $t_E$ (data not shown). On the other hand, as shown in Fig. S-3, $C_{Vp}(t)$ for large $t$ grows as $\propto t^2$ and $\propto t$ for small and large perturbations $\epsilon$, respectively.

The perturbation-dependent behavior is understood, by using Eq. (1) with the conditions (4) in the main text, from the behavior of the following 3-OTOC:

$$I_{Vp}(t) = \langle e^{i\hat{p}(t)_{\epsilon}} \hat{p}^2 e^{-i\hat{p}(t)_{\epsilon}} \rangle = \langle \tilde{\psi}_t | \hat{p}^2 | \tilde{\psi}_t \rangle \left( | \tilde{\psi}_t \rangle = e^{i\hat{p}(t)_{\epsilon}} | \psi \rangle \right). \quad \text{(S-10)}$$

When the perturbation is so small that

$$\frac{\epsilon^2 \langle \hat{p}(t)^2 \rangle}{\hbar_{\text{eff}}^2} \simeq \frac{\epsilon^2 t}{\hbar_{\text{eff}}^2} \ll 1 \text{ for a given } t, \quad \text{(S-11)}$$

$$| \tilde{\psi}_t \rangle \simeq \left( 1 + \frac{i \hat{p}(t) \epsilon}{\hbar_{\text{eff}}} \right) | \psi \rangle \quad \text{(S-12)}$$
follows and the dynamics is almost reversible in terms of fidelity \((\langle \psi | \tilde{\psi}_t \rangle \simeq 1)\). However, 
\(I_{Vp}(t)\) can be approximated as
\[
\langle \tilde{\psi}_t | \hat{p}^2 | \tilde{\psi}_t \rangle \simeq \frac{\epsilon^2}{\hbar^2_{\text{eff}}} \langle \hat{p}(t) \hat{p}^2 \hat{p}(t) \rangle ,
\]
which grows in proportion to \(t^2\) as can be seen from the results in the previous section [11]. In this case, \(I_{Vp}(t)\) becomes sufficiently large, providing a measure of irreversibility which is more sensitive than fidelity [12]. On the other hand, for the large perturbation \(\frac{\epsilon^2 t}{\hbar^2_{\text{eff}}} \simeq 1\), the completely irreversible (diffusive) delocalization of \(|\tilde{\psi}_t\rangle\) occurs, leading to
\[
\langle e^{-i\hat{p}(t)} \hat{p}^2 e^{i\hat{p}(t)} \rangle \propto t .
\]
(S-14)
Note that we find a crossover into this regime even for small \(\epsilon\) if we wait for a long time (i.e., large \(t\)). For both cases, \(C_{Vp}(t) \simeq I_{Vp}(t)\) (data not shown) holds true, which leads to results in Fig. S-3.

II. DETAILS FOR QUANTUM MANY-BODY SYSTEMS

A. Unchanged energy after the perturbation of \(\hat{B}\) on the initial state

In this subsection, we show that the energies of \(\hat{\rho}\) and \(\hat{\rho}' = \frac{\hat{B} \hat{\rho} \hat{B}^\dagger}{\text{Tr}[\hat{\rho} \hat{B} \hat{B}^\dagger]}\) are macroscopically equal when \(\hat{B}\) can be written as a sum of local operators and \(\hat{\rho}\) satisfies the cluster decomposition property. We first note \(\text{Tr}[\hat{\rho}' \hat{H}] = \frac{\langle \hat{B}^\dagger \hat{H} \hat{B} \rangle}{\langle \hat{B}^\dagger \hat{B} \rangle}\). Using the cluster decomposition, we can decompose \(\langle \hat{B}^\dagger \hat{H} \hat{B} \rangle = \sum_{i,j,k} \langle \hat{b}_i^\dagger \hat{h}_j \hat{b}_k \rangle\) into
\[
\sum_{(i,j,k) \in A_1} \langle \hat{b}_i^\dagger \hat{h}_j \hat{b}_k \rangle + \sum_{(i,j,k) \in A_2} \langle \hat{b}_i^\dagger \hat{b}_k \rangle \langle \hat{h}_j \rangle .
\]
(S-15)
Here, \(A_1\) is a set of the pair \((i,j,k)\) where \(i, j\) and \(k\) are close to one another, and \(A_2\) is a set where \(i\) and \(k\) are close to each other but neither of them is close to \(j\). Note that the contributions from other pairs vanish due to \(\langle \hat{b}_i \rangle = 0\). The second term is a leading term of the order \(N^2\), which is approximated by
\[
\langle \hat{H} \rangle \sum_{(i,k) \in A_0} \langle \hat{b}_i^\dagger \hat{b}_k \rangle .
\]
(S-16)
Here, \(A_0\) is a set where \(i\) and \(k\) are close to each other and we have safely replaced \(\sum_{(i,j,k) \in A_2} \) with \(\sum_j \sum_{(i,k) \in A_0} \) without changing the leading contribution. Similarly, the denominator is
FIG. S-3. Long-time dynamics of the squared commutator for different strengths of perturbation: $\epsilon/\hbar = 0.01, 0.02, 0.05, 1, \text{ and } 20$. For small perturbations, the growth depends on $\epsilon/\hbar$ and exhibits a quadratic scaling. For strong perturbations, the growth is independent on $\epsilon/\hbar$ and exhibits a linear scaling.

approximated by $\sum_{(i,k) \in A_0} \langle \hat{b}_i^\dagger \hat{b}_k \rangle$ up to the leading order, which leads to

$$\text{Tr}[\hat{\rho}^\prime \hat{H}] \simeq \langle \hat{H} \rangle = \text{Tr}[\hat{\rho} \hat{H}] .$$

(B) Unchanged energy after the perturbation of $\hat{A}$ on the time-evolved state

Next, we argue that $\hat{\rho}_t$ and $\frac{\hat{A}_{\hat{\rho}_t} \hat{A}^\dagger_{\hat{\rho}_t}}{\text{Tr}[\hat{\rho}_t \hat{A}^\dagger_{\hat{\rho}_t} \hat{A}_{\hat{\rho}_t}]}$ have macroscopically the same energy when $\hat{A}$ is written as a local operator, a sum of local operators, or a unitary operator that is close to unity. First, by assuming the ETH, we can write the energy of each state as

$$\text{Tr}[\hat{\rho}_t \hat{H}] = \langle \hat{H} \rangle_{\beta},$$

(S-18)

$$\text{Tr} \left[ \frac{\hat{A}_{\hat{\rho}_t} \hat{A}^\dagger_{\hat{\rho}_t}}{\text{Tr}[\hat{\rho}_t \hat{A}^\dagger_{\hat{\rho}_t} \hat{A}_{\hat{\rho}_t}]} \hat{H} \right] \simeq \frac{\langle \hat{A}^\dagger_{\beta} \hat{H} \hat{A} \rangle_{\beta}}{\langle \hat{A}^\dagger_{\beta} \hat{A} \rangle_{\beta}}$$

(S-19)
for sufficiently large $t$.

Let us first consider the case where $\hat{A}$ can be written as a sum of local operators as $\hat{A} = \sum_i \hat{a}_i$, where $\hat{a}_i$’s have the same form for any $i$. We assume the cluster decomposition property for the canonical ensemble, which is justified for sufficiently high-temperature systems in our setup [13]. Then, if $\langle \hat{a}_i \rangle_\beta = 0$, $\frac{\langle \hat{A}^\dagger \hat{H} \hat{A} \rangle_\beta}{\langle \hat{A}^\dagger \hat{A} \rangle_\beta} \simeq \langle \hat{H} \rangle_\beta$ in a manner similar to the discussion in the previous subsection. If $\langle \hat{a}_i \rangle_\beta \neq 0$, the leading term of the denominator, which is $O(N^3)$, becomes

$$\langle \hat{A}^\dagger \hat{H} \hat{A} \rangle_\beta \simeq \sum_{(i,j,k) \in A_3} \langle \hat{a}_i^\dagger \rangle_\beta \langle \hat{h}_j \rangle_\beta \langle \hat{a}_k \rangle_\beta . \quad \text{(S-20)}$$

$A_3$ is a set of $(i, j, k)$ where none of the three sites are close to one another, and we can safely replace $\sum_{(i,j,k) \in A_3}$ with $\sum_j \sum_{(i,k) \notin A_0} \langle \hat{a}_i^\dagger \rangle_\beta \langle \hat{a}_k \rangle_\beta$ without changing the leading order. Similarly, the numerator, which is $O(N^2)$, becomes

$$\langle \hat{A}^\dagger \hat{A} \rangle_\beta \simeq \sum_{(i,k) \notin A_0} \langle \hat{a}_i^\dagger \rangle_\beta \langle \hat{a}_k \rangle_\beta . \quad \text{(S-22)}$$

Thus, we obtain $\frac{\langle \hat{A}^\dagger \hat{H} \hat{A} \rangle_\beta}{\langle \hat{A}^\dagger \hat{A} \rangle_\beta} \simeq \langle \hat{H} \rangle_\beta$ in this case as well. We note that similar discussions are applicable to the case in which $\hat{A}$ is a local operator.

Next, we consider the case where $\hat{A}$ is a unitary operator that is close to unity. We focus on the case for which $\hat{A}$ can be written as $\hat{A} = e^{i\phi\hat{C}}$, where $\hat{C} = \sum_i \hat{c}_i$ and $\phi$ is a constant. Then, the numerator on the right-hand side of Eq. (S-19) can be expanded as

$$\langle \hat{A}^\dagger \hat{H} \hat{A} \rangle_\beta = \langle \hat{H} \rangle_\beta - \frac{\phi^2}{2} \langle [\hat{C}, [\hat{C}, \hat{H}]] \rangle_\beta + \cdots , \quad \text{(S-23)}$$

where we used $\langle [\hat{C}, \hat{H}] \rangle_\beta = 0$ because $[\hat{H}, \hat{\rho}_\beta] = 0$. Since $[\hat{C}, [\hat{C}, \hat{H}]]$ can be written as a sum of local operators, $\langle [\hat{C}, [\hat{C}, \hat{H}]] \rangle_\beta$ is at most $O(N)$ due to the cluster decomposition property. Thus, if $\phi$ decays faster than $O(N^{-1/2})$, we obtain $\langle \hat{A}^\dagger \hat{H} \hat{A} \rangle_\beta \simeq \langle \hat{H} \rangle_\beta$. Since $\langle \hat{A}^\dagger \hat{A} \rangle_\beta = 1$, we obtain $\frac{\langle \hat{A}^\dagger \hat{H} \hat{A} \rangle_\beta}{\langle \hat{A}^\dagger \hat{A} \rangle_\beta} \simeq \langle \hat{H} \rangle_\beta$ in this case as well.
C. Energy change due to strong perturbation of $\hat{A}$

We finally consider the case where the perturbation of $\hat{A}$ is so strong that $\frac{\langle \hat{A}^\dagger \hat{H} \hat{A} \rangle_\beta}{\langle \hat{A}^\dagger \hat{A} \rangle_\beta}$ deviates significantly from $\langle \hat{H} \rangle_\beta$. Then, we need to evaluate the reversibility

$$r_t \simeq \frac{\langle \hat{B}^\dagger \hat{B} \rangle}{\langle \hat{B}^\dagger \hat{B} \rangle_{\beta'}}$$

(S-24)

with the condition $\beta \neq \beta'$. With an argument similar to that in the main text, the cluster decomposition property for the canonical ensemble at an inverse temperature $\beta'$ leads to $r_t = O(N^{-1})$ if we assume $\langle \hat{b}_i \rangle_{\beta'} \neq 0$. We note that the condition $\langle \hat{b}_i \rangle_{\beta'} \neq 0$ can be achieved even when the initial state is a non-localized, thermal state at an inverse temperature $\beta(\neq \beta')$ if $\langle \hat{b}_i \rangle_{\beta'} \neq \langle \hat{b}_i \rangle_\beta$, where $\langle \hat{b}_i \rangle_\beta = 0$ by our assumption $\langle \hat{B} \rangle = \langle \hat{B} \rangle_\beta = 0$. Thus, in this case, Eq. (4) in the main text and thus $C_{AB}(t) \simeq I_{AB}(t)$ hold true even though $I_{AB}(t)$ becomes anti-time-ordered. We stress that we cannot use such an argument for the perturbation that does not macroscopically change energy. In this sense, nonequilibrium localized initial states and the 3-OTOC are important because they satisfy Eq. (4) in the main text and $C_{AB}(t) \simeq I_{AB}(t)$ even with the same-energy condition.

[1] Here we ignore the discreteness of $p$, which is justified for small $\hbar_{\text{eff}}$. Because of this, the periodicity about $x$ is lost. In particular, unphysical ghost images in Ref. [14] are lost, which are expected not to change the results for small $\hbar_{\text{eff}}$.

[2] A. Larkin and Y. N. Ovchinnikov, Sov Phys JETP 28, 1200 (1969).

[3] A. Kitaev, in KITP strings seminar and Entanglement (2015).

[4] E. B. Rozenbaum, S. Ganeshan, and V. Galitski, Phys. Rev. Letters 118, 086801 (2017).

[5] B. Chirikov, F. Izrailev, and D. Shepelyansky, Physica D: Nonlinear Phenomena 33, 77 (1988).

[6] A. Altland, Phys. Rev. Lett. 71, 69 (1993).

[7] Precisely speaking, we have found a peak at $p = 0$, which is not scaled diffusively. However, this peak does not affect the main discussion.

[8] M. Gärttner, J. G. Bohnet, A. Safavi-Naini, M. L. Wall, J. J. Bollinger, and A. M. Rey, Nature Physics (2017).
[11] The zeroth-order term in $\epsilon$ is time-independent. The first-order terms can be evaluated as

$$|\frac{i\epsilon}{\hbar_{\text{eff}}}(\langle \hat{p}(t)\hat{p}^2 \rangle + \langle \hat{p}^2\hat{p}(t) \rangle)| \leq \frac{2\epsilon\sqrt{\langle \hat{p}(t)^2 \rangle \langle \hat{p}^4 \rangle}}{\hbar_{\text{eff}}} \approx \frac{2\epsilon\sqrt{t\langle \hat{p}^4 \rangle}}{\hbar_{\text{eff}}},$$

which is small by our assumption.

[12] In Ref. [15], the authors expand $\langle \hat{V}^\dagger(t)\hat{X}\hat{V}(t) \rangle$ ($\hat{V} = e^{-i\hat{H}\epsilon}$) up to the second order in $\epsilon$, especially before $t_E$. Although the expansion series have the same form as ours, we argue that the expansion radius is determined by the expansion for the state, not for the correlator as they discuss.

[13] M. Kliesch, C. Gogolin, M. J. Kastoryano, A. Riera, and J. Eisert, Phys. Rev. X 4, 031019 (2014).

[14] A. R. Kolovsky, Chaos: An Interdisciplinary Journal of Nonlinear Science 6, 534 (1996).

[15] M. Schmitt, D. Sels, S. Kehrein, and A. Polkovnikov, arXiv preprint arXiv:1802.06796 (2018).