A NEW BASIS FOR THE COMPLEX $K$-THEORY COOPERATIONS ALGEBRA

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ABSTRACT. A classical theorem of Adams, Harris, and Switzer states that the 0th grading of complex $K$-theory cooperations, $KU_0$ is isomorphic to the space of numerical polynomials. The space of numerical polynomials has a basis provided by the binomial coefficient polynomials, which gives a basis of $KU_0$.

In this paper, we produce a new $p$-local basis for $KU_0(p)$ using the Adams splitting. This basis is established by using well known formulas for the Hazewinkel generators. For $p = 2$, we show that this new basis coincides with the classical basis modulo higher Adams filtration.

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1. Introduction

The cooperations algebra $KU_*KU$ was originally computed by Adams, Harris, and Switzer in [1]. They show that $KU_*KU$ is torsion free, and hence the map

$$KU_*KU \to KU_*KU \otimes Q \simeq Q[u^{\pm 1}, v^{\pm 1}]$$

is monic. They determine the image of this map, described in the following theorem.

Theorem 1 (Adams-Harris-Switzer, [1]). The map

$$KU_*KU \to KU_*KU \otimes Q$$
gives an isomorphism between $KU, KU$ and the ring of finite Laurent series $f(u, v)$ which satisfy the following condition: for any nonzero integers $h, k$ we have

$$f(h\beta, k\beta) \in \mathbb{Z}[\beta^{\pm 1}, h^{-1}, k^{-1}].$$

If we are working with the 2-local complex $K$-theory spectrum $KU$, then we can rewrite this condition as

$$KU_0 KU_2 \simeq \{ f(w) \in \mathbb{Q}[w^{\pm 1}] \mid f(k) \in \mathbb{Z}(2) \text{ for all } k \in \mathbb{Z}_2^\times \}$$

where $w := v/u$. Since $KU$ is an even periodic ring spectrum, this determines the entire algebra $KU, ku$. An elegant proof of this fact using an arithmetic square can be found in [3]. In particular, this method allows one to calculate

$$KU_0 ku_2 = \{ g(w) \in \mathbb{Q}[w] \mid g(k) \in \mathbb{Z}(2) \text{ for all } k \in \mathbb{Z}_2^\times \}$$

which is known as the space of 2-local semistable numerical polynomials. This is related to the space of 2-local numerical polynomials:

$$A := \{ h(x) \in \mathbb{Q}[x] \mid h(k) \in \mathbb{Z}(2) \text{ for all } k \in \mathbb{Z}(2) \}$$

via the following change of coordinates

$$\mathbb{Z}(2) \to \mathbb{Z}_2^\times; k \mapsto 2k + 1$$

A classical result is that the ring $A$ of numerical polynomials is a free $\mathbb{Z}(2)$-module with basis given by the binomial coefficient polynomials

$$p_n(x) := \binom{x}{n} = \frac{x(x - 1) \cdots (x - n + 1)}{n!}.$$ 

Via the change of coordinates above, we obtain a basis for $KU_0 ku$,

$$g_n(w) = \frac{(w - 1)(w - 3) \cdots (w - (2n - 1))}{2^n n!}.$$ 

At any prime $p$, another basis for $KU_0 ku_{(p)}$ is discussed by Baker in [3] and [4]. In these papers, Baker gives a different basis for $KU_0 ku_{(p)}$ where the role of the polynomials $p_n(x)$ are replaced by a sequence of Teichmüller characters, and he recovers a recursive formula.

When localizing at an odd prime $p$, $KU$ splits as a wedge of suspensions of the Johnson-Wilson theory $E(1)$. The homotopy groups of this spectrum are

$$\pi_*(E(1)) = \mathbb{Z}_{(p)}[v_1^{\pm 1}].$$
The connective cover $ku(p)$ splits as a wedge of suspensions of the truncated Brown-Peterson spectrum $BP\langle 1 \rangle$. The homotopy groups of this spectrum are

$$\pi_*(BP\langle 1 \rangle) = \mathbb{Z}_{(p)}[v_1].$$

When the prime is 2, then the spectra $E(1)$ and $KU_2$ are equivalent, as are the spectra $BP\langle 1 \rangle$ and $ku_2$. Using the Künneth spectral sequence, it is shown in [5] that

$$E(1)_* BP\langle 1 \rangle \cong E(1)_* \otimes_{BP} BP_+ \otimes_{BP} BP\langle 1 \rangle_*$$

$$\cong E(1)_*[t_1, t_2, \ldots] / (\eta_R(v_2), \eta_R(v_3), \ldots)$$

where the $v_i$ denote the Hazewinkel generators for $BP_+$ and $\eta_R$ denotes the right unit for the Hopf algebroid $(BP_+, BP_+BP)$. The splitting of $KU_p$ at odd primes $p$ gives a map

$$(1) \quad \phi : E(1)_* BP\langle 1 \rangle \to KU_* ku_p$$

obtained by including the summand. At the prime 2, this map is an isomorphism.

In this paper, we use the mod $p$ Adams spectral sequence for the spectrum $E(1) \wedge BP\langle 1 \rangle$ to determine a basis for $E(1)_0 BP\langle 1 \rangle$ in terms of the generators $t_i$. Using the map $\phi$, we find what semistable numerical polynomials these basis elements correspond to. More specifically, if we set

$$\phi_n := \phi \left( v_1^{-p^n-1} t_n \right)$$

then we determine an inductive formula determining the $\phi_n$’s. The basis for $E(1)_0 BP\langle 1 \rangle$ will then be the set of certain monomials on the $\phi_n$’s. This inductive formula stems from formulas for the right unit, $\eta_R$, on the Hazewinkel generators $v_i$. Strangely, these inductive formulas bear a striking resemblance to those of Baker in [3]. The author does not know how these bases are related.

After determining a basis for $E(1)_0 BP\langle 1 \rangle$ at all primes, we focus on the prime 2, in which case $\phi$ becomes an isomorphism, giving us a new basis for $KU_0 ku$. We compare this new basis with the one provided by the $g_n$’s. In particular, it will be shown that the $g_n$-basis and the one produced here will be the same modulo higher Adams filtration. Our basis has the advantage that it is tightly connected to $BP$-theory and the Steenrod algebra. Moreover, our techniques
furnish a basis for $E(1)_0BP(1)$ at odd primes, which could not be obtained before by the result of Adams-Switzer-Harris.

**Conventions.** We will write $\zeta_i$ for the conjugates of the polynomial generators $\xi_i$ in the dual Steenrod algebra. When given a prime $p$, we will write $H_*(-)$ for the functor $H_*(-;F_p)$. We will write $\text{Ext}_{\mathcal{A}_*}(M)$ for $\text{Ext}_{\mathcal{A}_*}(F_p, M)$ when $M$ is a comodule over the dual Steenrod algebra. We will also write $\text{Ext}_{\mathcal{E}(1)_*}(M)$ for $\text{Ext}_{\mathcal{E}(1)_*}(F_p, M)$ when $M$ is a comodule over the Hopf algebra $\mathcal{E}(1)_* = E(Q_0, Q_1)_*$. If $X$ is a spectrum, we will write $\text{Ext}_{\mathcal{A}_*}(M)$ for $\text{Ext}_{\mathcal{A}_*}(F_p, M)$ when $M$ is a comodule over the dual Steenrod algebra.

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When given a prime $p$, we will write $H_*(X; Q_i)$ for the Margolis homology groups $M_*(H_*X; Q_i)$.

2. **Adams spectral sequence calculation of $E(1)_*BP(1)$**

We begin by reviewing the calculation of $ku_*ku(2)$ in terms of the Adams spectral sequence

$$\text{Ext}_{\mathcal{A}_*}(H_*(ku \wedge ku)) \Rightarrow ku_*ku_2.$$

The details of this calculation can be found in [2]. Recall that

$$H_*(ku) \simeq \mathcal{A}_*/\mathcal{E}(1)_*$$

where $\mathcal{E}(1)$ denotes the subalgebra of the Steenrod algebra $\mathcal{A}$ which is generated by the Milnor primitives $Q_0$ and $Q_1$. Thus a change-of-rings shows that the spectral sequence is of the form

$$\text{Ext}_{\mathcal{E}(1)_*}((\mathcal{A}_*/\mathcal{E}(1)_*)_*) \Rightarrow ku_*ku_2.$$

An important invariant needed in calculating $\text{Ext}$ over the Hopf algebra $\mathcal{E}(1)$ is the *Margolis homology*. If $X$ is a module over $\mathcal{E}(1)$, then as $\mathcal{E}(1)$ is an exterior algebra on $Q_0$ and $Q_1$, the actions by $Q_i$ square to zero, so we may regard $X$ as a chain complex with differentials $Q_i$. We define the *Margolis homology group with respect to $Q_i$* to be

$$M_*(X; Q_i) := \ker Q_i/\text{im} Q_i$$

i.e., the homology of $X$ with respect to the differential $Q_i$. An easy calculation (cf. [2]) shows that

$$M_*(ku; Q_0) \simeq P(\zeta_1^2)$$

and

$$M_*(ku; Q_1) \simeq E(\zeta_1^2, \zeta_2^2, \zeta_3^2, \ldots).$$

There is a *weight filtration* on $\mathcal{A}_*$ given by setting

$$\text{wt}(\zeta_k) = 2^{k-1}.$$

and extending to general monomials by
\[ \text{wt}(xy) = \text{wt}(x) + \text{wt}(y). \]
The weight filtration gives an algebraic decomposition
\[ (\mathcal{A} / \mathcal{E}(1))^* \cong \bigoplus_{k=0}^{\infty} M_1(k) \]
where the \( M_1(k) \) denote the subspaces spanned by monomials whose weight is exactly equal to \( 2k \). These turn out to be subcomodules and they are the homology of the integral Brown-Gitler spectra. The Margolits homology of the subcomodules \( M_1(k) \) have an interesting property.

**Proposition 1.** The Margolits homology groups of \( M_1(k) \) are the subspaces of the Margolits homology of \( (\mathcal{A} / \mathcal{E}(1))^* \) spanned by the weight \( 2k \) monomials. In particular
\[ M_*(M_1(k); \mathbb{Q}) = \mathbb{F}_2\{\zeta_1^{2k}\} \]
and if the binary expansion of \( k \) is
\[ k = k_0 + k_12 + k_22^2 + \cdots \]
then
\[ M_*(M_1(k); \mathbb{Q}_1) = \mathbb{F}_2\{\zeta_1^{2k_0}, \zeta_2^{2k_1}, \zeta_3^{2k_2}, \ldots\}. \]
In particular, the Margolits homology groups of \( M_1(k) \) are one dimensional.

Adams was able to show in [2] that, since the Margolits homology of the \( M_1(k) \) are one dimensional, there is an isomorphism
\[ M_1(k)^* \cong \overline{\mathcal{E}(1)}^{\otimes k - \alpha(k)} \oplus F \]
where \( \overline{\mathcal{E}(1)} \) denotes the augmentation ideal of \( \mathcal{E}(1) \), \( F \) is some free \( \mathcal{E}(1) \)-module, and \( \alpha(k) \) denotes the number of 1’s in the dyadic expansion of \( k \). Thus,
\[ \text{Ext}_{\mathcal{E}(1)}(M_1(k)) / \text{tors} \cong \text{Ext}_{\mathcal{E}(1)}(H_*(ku^{(k - \alpha(k))})) \]
where \( ku^{(i)} \) denotes the \( i \)th Adams cover of \( ku \). From this it follows that the Adams spectral sequence (2) collapses at \( E_2 \).

The algebra \( KU_1ku(2) \) is obtained from \( ku_1ku(2) \) by inverting the element \( v_1 \), thus it is the direct sum of the modules
\[ v_1^{-1}\text{Ext}_{\mathcal{E}(1)}(M_1(k)). \]
We will now calculate these $v_1$-inverted Ext-groups. Here is an example of the Adams chart for $v_1^{-1} \text{Ext}(M_1(4))$.

**Example 1.** We will calculate $v_1^{-1} \text{Ext}_{\mathcal{E}(1)}^*(M_1(4))$ and find a $\mathbb{Z}(2)$-generator in degree 8. Here is a picture of the Adams chart.

![Adams Chart](image)

This picture is obtained by drawing the Adams chart for $\text{Ext}(M_1(4))$ and then drawing $v_1^{-1}$-towers on each dot on the 0-line. In this example, we see that the relations give $2^3 v_1^{-3} \zeta_2^2 = \zeta_8^1$. This shows that the group $v_1^{-1} \text{Ext}_{\mathcal{E}(1)}^s(M_1(4))$ is generated over $\mathbb{Z}(2)$ by $v_1^{-3} \zeta_2^8$. This also shows that the contribution of $v_1^{-1} \text{Ext}_{\mathcal{E}(1)}^*(M_1(4))$ to $KU_0 ku$ is the free $\mathbb{Z}(2)$-module generated by $v_1^{-7} \zeta_2^2$.

**Proposition 2.** Let $k = k_0 + k_1 2 + k_2 2^2 + \cdots$ be a natural number, then as a module over $\mathbb{Z}(2)[v_1^{\pm 1}]$, the modules $v_1^{-1} \text{Ext}_{\mathcal{E}(1)}^*(M_1(k))$ are generated by $v_1^{k-\alpha(k)} \zeta_1^{2k_0} \zeta_2^{2k_1} \ldots$

Recall that in the Adams spectral sequence for $BP_*BP$,

$$\text{Ext}_{\mathcal{E}(1)}(P(\zeta_1^2, \zeta_2^2, \zeta_3^2, \ldots)) \Rightarrow BP_*BP,$$

the elements $t_i \in BP_*BP$ are detected by $\zeta_i^2$. Since

$$E(1)_*BP\langle 1 \rangle \simeq E(1)_* \otimes_{BP} BP_*BP \otimes_{BP} BP\langle 1 \rangle$$

$$\simeq E(1)_*[t_1, t_2, \ldots]/(\eta_R(v_2), \eta_R(v_3), \ldots)$$

the elements $\zeta_i^2$ in the Adams spectral sequence for $E(1)_*BP\langle 1 \rangle$ detect $t_i$. With this notation, we conclude\(^1\)

\(^1\)Note since $KU_0 ku$ has no divisible summands, a set of elements of $KU_0 ku(2)$ is a basis if and only if it is a basis of $KU_0 ku(2)$. 

Corollary 1. Let \( \phi_n = v_1^{2^n+1}t_n \) in \( KU_0ku(2) \). The following monomials

\[
\phi_1^{e_1} \phi_2^{e_2} \cdots
\]

with \( e_j \in \{0,1\} \) forms a basis for the free \( \mathbb{Z}_{(2)} \)-module \( KU_0ku(2) \).

At an odd prime \( p \), the dual Steenrod algebra is given by

\[
A^* = P(\zeta_1, \zeta_2, \ldots) \otimes E(\tau_0, \tau_1, \ldots)
\]

and the mod \( p \) homology of \( BP\langle 1 \rangle \) is given by

\[
H_*(BP\langle 1 \rangle) = (A/\sslash E(Q_0, Q_1))^* \]

where the \( Q_0, Q_1 \) are the Milnor primitives. Concretely this algebra is

\[
(A/\sslash E(Q_0, Q_1))^* = P(\zeta_1, \zeta_2, \zeta_3, \ldots) \otimes E(\tau_2, \tau_3, \ldots).
\]

There is a left action of \( E(Q_0, Q_1) \) on \( (A/\sslash E(Q_0, Q_1))^* \) given by

\[
Q_i(\tau_k) = \zeta_{k-i}^p
\]

\[
Q_i(\zeta_k) = 0
\]

for all \( k \). This shows that the Margolis homology of \( BP\langle 1 \rangle \) is

\[
M_*(BP\langle 1 \rangle; Q_0) = P(\zeta_1)
\]

\[
M_*(BP\langle 1 \rangle; Q_1) = P(\zeta_1, \zeta_2, \zeta_3, \ldots) / (\zeta_{k+1}^p, \zeta_{k+2}^p, \ldots).
\]

Similar to the 2-primary case, one can put a weight filtration on \( (A/\sslash E(Q_0, Q_1))^* \) by

\[
\text{wt}(\zeta_k) = \text{wt}(\tau_k) = p^k.
\]

If we let \( M_1(k) \) denote the subcomodule spanned by the monomials of weight exactly \( pk \) then we get an algebraic decomposition

\[
(A/\sslash E(Q_0, Q_1))^* \simeq \bigoplus_{k=0}^{\infty} M_1(k).
\]

As in the 2-primary case, the Margolis homology of the subcomodules \( M_1(k) \) are both one-dimensional, which from the classification theorem shows that

\[
M_1(k)^* \simeq \mathcal{E}(1)^{k-\alpha_p(k)} \oplus F
\]
where $\alpha_p(k)$ is the sum of the digits in the $p$-adic expansion of $k$ and $F$ is a free module. In particular

$$\text{Ext}_{E(Q_0,Q_1)}(M_1(k))/\text{tors} \simeq \text{Ext}_{\mathcal{O}_k} \left( H_* \left( BP\langle 1 \rangle \frac{k_{-p}(k)}{p-1} \right) \right).$$

From this it follows that the Adams spectral sequence for $BP\langle 1 \rangle_*BP\langle 1 \rangle$ collapses at the $E_2$-page.

Recall that the Adams spectral sequence for $BP_*BP$ at an odd prime is

$$\text{Ext}_{E(\tau_0,\tau_1,\ldots)}(P(\zeta_1,\zeta_2,\ldots)) \Rightarrow BP_*BP$$

and in this spectral sequence the $\zeta_k$ detects $t_k \in BP_*BP$. Thus we shall write $t_k$ for $\zeta_k$. Then a proof similar to the proof of Proposition 2 shows that

**Proposition 3.** Let the $p$-adic expansion of $k$ be given by $k = k_0 + k_1p + k_2p^2 + \cdots$. Then over $\mathbb{Z}(p)[v^{\pm 1}]$, the module $v_1^{-1}\text{Ext}_{E(\tau_0,\tau_1)}(BP\langle 1 \rangle)$ is generated by

$$v_1^{k_{-p}(k)} t_0 t_1 t_2 t_3 \cdots.$$

**Corollary 2.** Let $\eta_n := v_1^{p^n-1} t_n$. The $\mathbb{Z}(p)$-module $E(1)_0BP\langle 1 \rangle$ is free with basis given by the monomials

$$\eta_1^{k_1} \eta_2^{k_2} \cdots$$

where each $k_i \in \{0,1,\ldots,p-1\}$.

3. **Relationship to numerical polynomials**

We will now determine the map

$$\varphi : E(1)_0BP\langle 1 \rangle \rightarrow KU_0ku$$

in terms of numerical polynomials. Recall that the homotopy groups of the integral complex $K$-theory spectrum are

$$\pi_* KU = \mathbb{Z}[v^{\pm 1}]$$

and thus the rational homotopy groups are

$$\pi_* (KU_Q) = \mathbb{Q}[v^{\pm 1}].$$

Thus we get

$$\pi_* (KU \wedge KU_Q) = \mathbb{Q}[v^{\pm 1}, u^{\pm 1}].$$
where we let \( u \) denote the Bott element coming from the right hand side \( KU \). Similarly, the rational homotopy groups of \( KU \wedge ku \) is given by

\[
\pi_* (KU \wedge ku \mathbb{Q}) = \mathbb{Q}[v^{\pm 1}, u].
\]

Given a prime \( p \), the rational homotopy groups of \( E(1) \wedge BP\langle 1 \rangle \) are given by

\[
\pi_* (E(1) \wedge BP\langle 1 \rangle \mathbb{Q}) = \mathbb{Q}[v^{\pm 1}, u].
\]

Moreover, at a prime \( p \), there is a topological splitting

\[
KU_p \simeq E(1) \lor \Sigma^2 E(1) \lor \cdots \lor \Sigma^{2(p-2)} E(1)
\]

and the inclusion

\[
E(1) \rightarrow KU
\]

is given in homotopy by

\[
\pi_* E(1) \rightarrow \pi_* KU_p; \quad v_1 \mapsto v^{p-1}.
\]

Thus the morphism

\[
\phi : E(1) \wedge BP\langle 1 \rangle \rightarrow KU \wedge ku_p
\]

is given in rational homotopy by

\[
\phi Q : E(1)_* BP\langle 1 \rangle \rightarrow KU_* ku_p; \quad v_1 \mapsto v^{p-1}, \quad u_1 \mapsto u^{p-1}.
\]

Let \( w_1 := u_1 / v_1 \), then under \( \phi Q \), we have that

\[
w_1 \mapsto w^{p-1}.
\]

We will now determine the value of \( \phi \) on the monomials

\[
\phi_n := \phi v_1^{p^n-1} t_n.
\]

To do this, we will need the following formula which determines the Hazewinkel generators

\[
p \lambda_n = \sum_{0 \leq i < n} \lambda_i v_{n-i}^{p^i}
\]

and the formula for the right unit on \( \lambda_n \)

\[
\eta_R(\lambda_n) = \sum_{0 \leq i \leq n} \lambda_i t_{n-i}^{p^i}.
\]

One can find proofs of these formulas in part 2 of [2] and in [7]. Here the \( \lambda_n \) is the coefficient of \( x_n^{p^n} \) in the logarithm for the universal \( p \)-typical formal group law. We will show
Theorem 2. The semistable polynomials $\varphi_n$ are given recursively by

$$\varphi_1 = \frac{wp - 1}{p}$$

and

$$\varphi_n = \frac{wp^{n-1} - p^{n-1}\varphi_{n-1} - \cdots - p\varphi_1^{p-1} - 1}{p^n}.$$

We will work out a few examples explicitly and then prove the theorem. Firstly, one has

$$p\lambda_1 = v_1$$

and so

$$\lambda_1 = \frac{v_1}{p}.$$

We will write $u_n$ for $\eta_R(v_n)$. This is justified because in $E(1)_*, E(1)$, $\eta_R(v_1)$ is $u_1$. Applying $\eta_R$ gives

$$\eta_R(v_1/p) = \eta_R(\lambda_1) = t_1 + \lambda_1$$

and so

$$u_1 = \eta_R(v_1) = pt_1 + v_1$$

Thus

$$t_1 = \frac{u_1 - v_1}{p}$$

and so

$$\varphi_1 = \frac{wp - 1}{p}.$$

To get at $\varphi_2$, we need to compute $\eta_R(v_2)$. We have

$$p\lambda_2 = v_2 + \lambda_1v_1^p$$

and so

$$v_2 = p\lambda_2 - \frac{v_1^{p+1}}{p}.$$

Applying $\eta_R$ we get

$$u_2 = p(t_2 + \lambda_1t_1^p + \lambda_2) - \frac{u_1^{p+1}}{p}.$$

Rewriting this, we get

$$u_2 = pt_2 + v_1t_1^p + v_2 + \frac{v_1^{p+1}}{p} - \frac{u_1^{p+1}}{p}.$$
Tensoring with $BP\langle 1 \rangle_*$ produces the following relation in $E(1)_*BP\langle 1 \rangle$:

$$0 = pt_2 + v_1 t_1^p + \frac{v_1^{p+1}}{p} - \frac{u_1^{p+1}}{p}$$

and hence

$$t_2 = \frac{u_1^{p+1} - v_1^{p+1}}{p^2} - \frac{v_1 t_1^p}{p}.$$  

Multiplying by $v_1^{-p-1}$ gives

$$v_1^{-p-1} t_2 = \frac{w_1^{p+1} - p(v_1^{-1} t_1)^p - 1}{p^2}$$

which shows that

$$\varphi_2 = \frac{w p^{p-1} - p \varphi_1^p - 1}{p^2}.$$  

We will need the following lemma

**Lemma 1.** In $E(1)_*BP\langle 1 \rangle$ there is the following equality

$$\lambda_n = \frac{v_1^{p^n-1}}{p^n}$$

**Proof.** This follows from the identity

$$p \lambda_n = \sum_{0 \leq i < n} \lambda_i v_1^{p^i}$$

and the fact that in $E(1)_*BP\langle 1 \rangle$, $v_k = 0$ for $k > 1$. Thus $p \lambda_n = \lambda_{n-1} v_1^{p^{n-1}}$. Proceeding inductively gives the identity

$$\lambda_n = \frac{v_1^{p^{n-1} + p^{n-2} + \ldots + p + 1}}{p^n} = \frac{v_1^{p^n-1}}{p^n}. $$

We will prove our theorem from the following proposition.

**Proposition 4.** In $E(1)_*BP\langle 1 \rangle$, there is the relation

$$pt_n + \sum_{1 \leq i \leq n} \frac{v_1^{p^i-1} t_1^{n-i}}{p^{i-1}} = \frac{u_1^{p^n-1}}{p^{n-1}}.$$
Proof. The formula for the Hazewinkel generators is
\[ p\lambda_n = v_n + \sum_{1 \leq i \leq n-1} \lambda_i v_{n-i}^p, \]
whereby
\[ v_n = p\lambda_n - \sum_{1 \leq i \leq n-1} \lambda_i v_{n-i}^p. \]
Applying \( \eta_R \) then gives
\[ u_n = p \sum_{0 \leq i \leq n} \lambda_i t_{n-i}^p = \sum_{0 \leq j \leq n-1} \left( \sum_{0 \leq j \leq n} \lambda_j t_{n-1-j}^p \right) u_{n-1}^p. \]
In \( E(1)_*BP(1) \), the \( u_k \) are zero for \( k > 1 \). So this gives
\[ p \sum_{0 \leq i \leq n} \lambda_i t_{n-i}^p = \sum_{0 \leq j \leq n-1} \lambda_j t_{n-1-j}^p u_{n-1}^p. \]
Using the previous lemma, we can rewrite this as
\[ p \sum_{0 \leq i \leq n} \frac{v_{n-i}^{p-1}}{p^i} t_{n-i}^p = \left( \sum_{0 \leq j \leq n-1} \frac{v_{n-1-j}^{p-1}}{p^i} t_{n-1-j}^p \right) u_{n-1}^{p-1}. \]
We will proceed inductively, the base case being trivial to check. Suppose that we have shown the formula for \( n-1 \). To complete the induction, it is enough to show that
\[ \sum_{0 \leq j \leq n-1} \frac{v_{n-1-j}^{p-1}}{p^j} t_{n-1-j}^p = \frac{u_1^{p-1}}{p^{n-1}}. \]
Plugging in our inductive formula for \( t_{n-1} \):
\[ t_{n-1} = \frac{u_1^{p-1}}{p^{n-1}} - \sum_{1 \leq k \leq n-1} \frac{v_{n-1-k}^{p-1}}{p^k} t_{n-1-k}^p, \]
into the right hand side of equation (3) yields
\[ \frac{u_1^{p-1}}{p^{n-1}} - \sum_{1 \leq k \leq n-1} \frac{v_{n-1-k}^{p-1}}{p^k} t_{n-1-k}^p + \sum_{1 \leq k \leq n-1} \frac{v_{n-1-j}^{p-1}}{p^j} t_{n-1-j}^p = \frac{u_1^{p-1}}{p^{n-1}} \]
which completes the proof. \( \square \)

We now prove the theorem.
Proof of Theorem. By definition,

$$\varphi_n := \varphi \left( v_1^{-p^{n-1}/p-1} t_n \right).$$

Observe that

$$p^n - 1 = p^j \left( p^{n-j} - 1 \right) + \frac{p^j - 1}{p - 1}. \quad \text{for } 1 \leq j \leq n.$$ 

This and the proposition then show that

$$v_1^{-p^{n-1}/p-1} t_n = w_1^{-p^{n-1}/p-1} \sum_{0 < j \leq n} \left( v_1^{-p^{n-j-1}/p-1} t_{n-j} \right)^{p^j}. \quad \text{for } 1 \leq j \leq n.$$ 

Applying $\varphi$ now shows that $\varphi_n$ satisfies the recursive formula, by induction. □

4. Comparison of the $\varphi_n$ and the $g_n$

In this section we will let $p = 2$, so that $E(1)$ is equivalent to $KU(2)$ and $BP(1)$ is equivalent to $ku(2)$. Thus the map $\varphi$ is an isomorphism providing $KU_0ku(2)$ with the basis provided by the $\varphi_n$’s. In this section we compare this basis with the basis provided by the $g_n$’s. In particular we show that the bases are the same modulo higher Adams filtration.

In the Adams spectral sequence computing $\pi_*BP$:

$$\text{Ext}_{\mathcal{A}}(H_*BP) \Rightarrow \pi_*BP^\wedge_2$$

the elements $v_i$ have Adams filtration 1. Also, in the ASS computing $BP_*BP$,

$$\text{Ext}_{\mathcal{A}}(H_*(BP \wedge BP)) \Rightarrow \pi_*(BP \wedge BP)^\wedge_2$$

the elements detecting $t_i$ have Adams filtration 0. Moreover, the map

$$\varphi : E(1)_*BP(1) \rightarrow KU_*ku(2)$$

preserve Adams filtration. Therefore, as $\varphi_n$ is the image of $v_1^{-2^n+1} t_n$ under $\varphi$, we can conclude:

**Proposition 5.** The Adams filtration of $\varphi_n$ is given by

$$\text{AF}(\varphi_n) = -(2^n - 1).$$
The Adams filtration of the semistable numerical polynomial $g_n$ is given by (cf. section 2.3 of [6])

$$AF(g_n) = \alpha(n) - 2n$$

where $\alpha(n)$ denotes the number of 1’s in the binary expansion of $n$. We will equate the $g_n$ with products of $\varphi_n$ modulo elements of higher Adams filtration. Write out $n$’s binary expansion

$$n = n_0 + n_1 2 + n_2 2^2 + \cdots + n_\ell 2^\ell,$$

then

$$AF(\varphi_1^{n_0} \varphi_2^{n_1} \cdots \varphi_\ell^{n_\ell}) = \sum_{i=0}^\ell n_i (1 - 2^{i+1}) = \alpha(n) - 2n$$

so $g_n$ and $\varphi_1^{n_0} \varphi_2^{n_1} \cdots \varphi_\ell^{n_\ell}$ have the same Adams filtration. We will prove the following.

**Proposition 6.** Given $n$ and its dyadic expansion

$$n = n_0 + n_1 2 + n_2 2^2 + \cdots$$

we have that

$$g_n \equiv \varphi_1^{n_0} \varphi_2^{n_1} \cdots \mod \text{higher Adams filtration}.$$ 

To prove this proposition, we will need to prove several lemmas, which is done below.

**Lemma 2.** We have

$$\varphi_n \equiv \frac{\varphi_1^{2^{n-1}}}{2^{2^{n-1}-1}} \mod \text{higher Adams filtration}.$$ 

**Proof.** The map $\varphi$ preserves Adams filtration. Moreover, from Proposition 2 in $v_1^{-1} \text{Ext}(M_1(2^{n-1}))$, there is the relation

$$2^{2^{n-1}-1} v_1^{-2^{n-1}-1} t_n = t_1^{2^{n-1}}.$$ 

Multiplying by $v_1^{-2^{n-1}}$ and applying $\varphi$ gives the desired relation. \qed

**Lemma 3.** We have

$$g_n \equiv \frac{\varphi_1^n}{n!} \mod \text{higher Adams filtration}.$$ 

**Proof.** We prove this by induction on $n$. Note that $g_1 = \psi_1$. Suppose that we have shown that

$$g_n \equiv \frac{\varphi_1^n}{n!} \mod \text{higher Adams filtration}.$$
Note that
\[ g_{n+1} = g_n \cdot \frac{w - (2n + 1)}{2(n + 1)}. \]
Even though \( \frac{w - (2n + 1)}{2(n + 1)} \) is not an element of \( KU_0 ku \otimes \mathbb{Q} \), it is an element of \( KU_0 ku \otimes \mathbb{Q} \). We will show that in \( KU_0 ku \otimes \mathbb{Q} \), the element \( g_{n+1} \) is congruent to \( \phi_1^{n+1} \) modulo higher Adams filtration in \( KU_0 ku \otimes \mathbb{Q} \), where Adams filtration is extended to \( KU_0 ku \otimes \mathbb{Q} \) by setting
\[ \text{AF} \left( \frac{x}{2^i} \right) = \text{AF}(x) - i. \]
This will complete the induction process because the map
\[ KU_* ku \to KU_* ku \otimes \mathbb{Q} \]
preserves Adams filtration and is monic, inducing a monomorphism on associated graded spaces
\[ E^0 KU_* ku \to E^0 KU_* ku \otimes \mathbb{Q}. \]
Note that
\[ \text{AF}(g_{n+1}) = \alpha(n + 1) - 2n - 2 \]
and also that
\[ \text{AF} \left( \frac{w - (2n + 1)}{2(n + 1)} \right) = \alpha(n + 1) - \alpha(n) - 2. \]
From the formula
\[ v_2(n!) = n - \alpha(n) \]
we find
\[ v_2(n + 1) = v_2((n + 1)!) - v_2(n!) = 1 - \alpha(n + 1) - \alpha(n). \]
Thus
\[ \text{AF} \left( \frac{w - (2n + 1)}{2(n + 1)} \right) = \text{AF} \left( \frac{\phi_1}{n + 1} \right) = -1 - v_2(n + 1). \]
This suggests that these numerical polynomials might be equivalent modulo higher Adams filtration. Indeed,
\[ \frac{w - (2n + 1)}{2(n + 1)} = \frac{w - 1}{2(n + 1)} = \frac{2n}{2(n + 1)} = \frac{n}{n + 1} \]
and
\[ \text{AF} \left( \frac{n}{n + 1} \right) = v_2(n) - v_2(n + 1) > -1 - v_2(n + 1). \]
and so, in $E^0(KU, ku \otimes \mathbb{Q})$,
\[
\frac{w - (2n + 1)}{2(n + 1)} \equiv \frac{\varphi_1}{n + 1} \pmod{\text{higher Adams filtration}}
\]
which implies that
\[
g_{n+1} \equiv \frac{\varphi_1^{n+1}}{(n + 1)!} \pmod{\text{higher Adams filtration}}
\]
which completes the induction. \hfill \Box

Corollary 3. We have the following congruence
\[
g_n \equiv \frac{\varphi_1^n}{2^{n-\alpha(n)}} \pmod{\text{higher Adams filtration}}.
\]
Proof. This is because the 2-adic valuation of $n!$ is
\[
\nu_2(n!) = n - \alpha(n).
\]
\hfill \Box

Lemma 4. There is the following congruence
\[
g_{2^n} \equiv \varphi_{n+1} \pmod{\text{higher Adams filtration}}.
\]
Proof. The previous corollary gives that
\[
g_{2^n} \equiv \frac{\varphi_1^{2^n}}{2^{2^n-\alpha(2^n)}} \pmod{\text{higher Adams filtration}}
\]
and by Lemma 2
\[
\varphi_{n+1} \equiv \frac{\varphi_1^{2^n}}{2^{2^n-1}} \pmod{\text{higher Adams filtration}}.
\]
Since $\alpha(2^n) = 1$, this proves the lemma. \hfill \Box

We can now prove Proposition 6.

Proof of Proposition 6. First observe that if we take the binary expansion of $n$
\[
n = n_0 + n_12 + n_22^2 + \cdots
\]
then
\[
g_n \equiv g_1^{n_0} g_2^{n_1} g_2^{n_2} \cdots \pmod{\text{higher Adams filtration}}
\]
Indeed, by Corollary 3 we have the congruence
\[
g_n \equiv \frac{\varphi_1^n}{n!} \pmod{\text{higher Adams filtration}}
\]
and
\[ S_1 S_2 S_2 \equiv \left( \frac{\varphi_1}{2!} \right)^{n_0} \left( \frac{\varphi_1}{2!} \right)^{n_1} \left( \frac{\varphi_2}{2!} \right)^{n_2} \cdots \mod \text{higher Adams filtration}. \]

In this last expression, the right hand side is equal to
\[ \frac{\varphi_1^n}{(2!)^{n_0} (2!)^{n_1} (2^2!)^{n_2} \cdots}. \]

So in order to show (4), it needs to be shown that
\[ \nu_2(n!) = \nu_2((2^0!)^{n_0} (2^1!)^{n_1} (2^2!)^{n_2} \cdots). \]

The right hand is equal to
\[ \sum_i n_i \nu_2(2^i) = \sum_i n_i 2^i - n_i = n - \alpha(n) = \nu_2(n!) \]
and this proves the congruence (4).

To prove the proposition, apply Corollary 4 to the right hand side of (4). This gives
\[ S_n \equiv \varphi_1^{n_0} \varphi_2^{n_1} \varphi_3^{n_2} \cdots \]
completing the proof of Proposition 6. \( \square \)

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