Covariant path integral for chiral p-forms

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The covariant path integral for chiral bosons obtained by McClain, Wu and Yu is generalized to chiral p-forms. In order to handle the reducibility of the gauge transformations associated with the chiral p-forms and with the new variables (in infinite number) that must be added to eliminate the second class constraints, the field-antifield formalism is used.

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I. INTRODUCTION

Chiral p-forms play a central role in supergravity and in string theory [12]. In particular, they contribute to the 'miraculous' cancellation of the gravitational anomaly in type-IIB supergravity or superstring theory, making these theories quantum-mechanically consistent.

The calculation of the gravitational anomaly for chiral p-forms was performed first in [3] without using a Lagrangian but by guessing suitable Feynman rules that incorporate the chirality condition. A Lagrangian that leads to the correct equations of motion for chiral p-forms was given later in [4,5] both in flat and in curved spacetimes. This Lagrangian generalizes to chiral p-forms the one constructed in [6] for chiral bosons in two dimensions, a model that has been extensively analysed during the last years [7]. Using the Lagrangian of [4], the authors of [8] recalculated the gravitational anomaly for chiral p-forms and found agreement with the work of [3] even though their Feynman rules turned out to be different.

One feature of the Lagrangian given in [4] for chiral bosons, and of its generalization given in [5,11] for chiral p-forms, is that it is not manifestly covariant. Furthermore, it leads to second class constraints in the hamiltonian formalism, which imply non usual commutation relations between the field variables.

In order to cure these difficulties, an infinite number of auxiliary fields were introduced in [8]. These auxiliary variables do not carry physical degrees of freedom of their own and enable one to replace the second class constraints enforcing the chirality condition by an infinite number of first class ones, along the lines of [10] and [8]. These first class constraints generate a new gauge freedom and by fixing the gauge through canonical methods one falls back on the original description of the chiral boson.

Using this new formulation, the authors of [8] were able to derive a covariant path integral for chiral bosons and to show that it reproduces the correct physical amplitudes and anomalies.

The purpose of this paper is to generalize the path integral derivation of [8] for chiral p-forms. The procedure is not entirely trivial because chiral p-forms have already a gauge invariance of their own - contrary to chiral bosons -, which is furthermore reducible. Moreover, it turns out that the new gauge invariance associated with the infinite number of auxiliary fields added to achieve covariance, mixes in a non-trivial way with the original invariance of the p-forms, leading to even more reducibility. This requires the presence of further ghosts of ghosts.

The more convenient way to handle this problem is to follow the lines of the field-antifield formalism [12–14] as we do here.

Our paper is organized as follows. First we reproduce the results of [8] through the antifield approach. We then go to the p-form case. We derive the explicit form of the pure first class action, introducing an infinite number of auxiliary fields. This first class description is verified to be physically equivalent to the second class description of [8], as in the chiral boson case [8]. We then derive the solution of the master equation and provide a gauge fixing fermion leading to the desired manifestly covariant path integral. We close our paper with some comments on the applications of this work.

II. COVARIANT PATH INTEGRAL FOR A CHIRAL BOSON

A. Classical analysis

It was recognized in [3] that a (1+1)-dimensional chiral boson (= chiral 0-form) could be consistently formulated in terms of the action

$$S_0 = \int d^2x \left( \sum_{n=0}^{\infty} \pi_n \dot{\phi}_n - \mathcal{H} \right) + \int d^2x \sum_{n=1}^{\infty} \lambda_n T_n$$

(1)

where $\mathcal{H}$ is the Hamiltonian density

$$\mathcal{H} = \frac{1}{2} \sum_{n=0}^{\infty} (\pi_n^2 + \dot{\phi}_n^2) (-1)^n.$$

(2)

The $\phi_n$ ($\phi_n \equiv \phi_n(x_0, x_1) \equiv \phi_n(\tau, \sigma)$, $\dot{\phi}_n \equiv \partial_0 \phi_n$, $\dot{\phi}_n' \equiv \partial_1 \phi_n'$) are an infinite collection of scalar fields, the $\pi_n$ are their conjugate momenta. The $T_n$ constitute an infinite set of first class constraints and are explicitly given by

$$T_m = \pi_{m-1} - \phi_m' + \pi_m + \phi_m' \approx 0 \quad m \geq 1.$$

(3)
One has for the equal time brackets
\[ [T_i(\sigma), T_m(\sigma')] = 0 \] as well as
\[ [H, T_m(\sigma)] = (-1)^{m+1} T_m'(\sigma) \quad H = \int d\sigma' \mathcal{H} \] (4)
(5)
The \( \lambda_n \) are Lagrange multipliers for the constraints (3).
The system above was obtained in [9] for the model of a single chiral boson.
It is straightforward to integrate over the momenta \( \mathcal{P} \) and leads to an
infinite tower of constraints (see also [15] for a related discussion).
To show the equivalence of (1) with the Floreanini-Jackiw action
\( S = \int d^2 x (\phi'_0 \phi_0 - \phi'_0 \phi'_0) \), one observes that (3) is invariant under the following gauge symmetries
generated by the first class constraints (3).

\[ \delta \phi_n = \epsilon_{n+1} + \epsilon_n \quad (n \geq 1), \quad \delta \phi_0 = \epsilon_1, \] (6a)

\[ \delta \pi_n = -\epsilon_{n+1} + \epsilon_n \quad (n \geq 1), \quad \delta \pi_0 = -\epsilon_1, \] (6b)

\[ \delta \lambda_n = -\epsilon_n + (-1)^n \epsilon_n \quad (n \geq 1), \] (6c)

\( \epsilon_n \equiv \epsilon_n(x) \). These gauge symmetries enable one to gauge away
the variables \( (\phi_n, \pi_n) \quad (n \geq 1) \) added to eliminate the second
class constraints, and leaves one with a single chiral boson. To see this, it is more convenient to replace
the pairs \( (\phi_n, \pi_n) \) by the self-conjugate variables
\[ \mu_n = \pi_n - \phi_n \] (7)
and
\[ \nu_n = \pi_n + \phi_n \] (8)
with brackets
\[ [\mu_n(\sigma), \mu_m(\sigma')] = -2\delta_{nm} \frac{\partial}{\partial \sigma} \delta(\sigma - \sigma') , \] (9a)
\[ [\nu_n(\sigma), \nu_m(\sigma')] = 2\delta_{nm} \frac{\partial}{\partial \sigma} \delta(\sigma - \sigma') , \] (9b)
\[ [\mu_n(\sigma), \nu_m(\sigma')] = 0 \quad , \] (9c)
in terms of which the constraints and gauge transformations read

\[ T_m = \mu_{m-1} + \nu_m \quad (m \geq 1) \] (10a)
\[ \delta \mu_m = -2\epsilon_{m-1} \quad , \quad \delta \nu_m = 2\epsilon_m \quad (n \geq 1) \] (10b)
\[ \delta \mu_0 = -2\epsilon_1 \quad , \quad \delta \nu_0 = 0 \quad . \] (10c)

On the constraint surface, one may eliminate all the \( \mu_n \)'s \( (n \geq 0) \) in terms of the \( \nu_n \)'s. Thus, the most general function on the constraint surface may be assumed to depend only on the \( \nu_n \)'s. This function will be gauge invariant if and only if it actually does not involve the \( \nu_m \)'s
for \( m \geq 1 \), since these variables transform independently under gauge transformations. Thus, the more general
gauge invariant function may be assumed to depend only on the single variable \( \nu_0 \), which is self-conjugate. This
means that the reduced phase space (see e.g. [3], chapter 2) of the system is indeed that of a single chiral boson.
Differently put, one may impose the gauge condition
\( \nu_n = 0 \quad (n \geq 1) \) to gauge away \( \nu_n \). Once this is done, the \( \mu_n \)'s must vanish by the constraints, and only the single
chiral variable \( \nu_0 \) is left.

B. Path integral

Let us now turn to the quantization of the system. We shall adopt the path integral approach. The most expedient way to get the gauge fixed action to be path-integrated is to use the antifield formalism [12–14]. The solution of the master equation for (4) is easily
constructed to be
\[ S = S_0 + \int d^2 x \{ \sum_{n \geq 1} [\phi_n^* (\sigma_{n+1} + \sigma_n) \] (11)
where the \( \sigma_n \) are the ghosts associated with the gauge symmetry (3): \( \phi_n^*, \pi_n^*, \lambda_n \) and \( \sigma_n^* \) are the antifields. In order to fix the gauge one needs to choose an appropriate gauge fixing fermion, and our goal is to end up with a covariant path integral. The original action (4) is not manifestly covariant because of the \( \lambda_n \)-terms. If those terms were absent, we would have an infinite number of uncoupled scalar fields, with action equal to the standard, covariant Klein-Gordon action (in Hamiltonian form). This suggests imposing the gauge \( \lambda_n = 0 \). This can be achieved by interchanging the roles of \( \lambda_n \) and \( \lambda_n^* \) and by taking as gauge fixing fermion \( \psi = 0 \) (exercise 19.14). One finds then

\[ \lambda_n = -\frac{\delta \psi}{\delta \lambda_n^*} = 0 \quad , \quad \phi_n^* = \frac{\delta \psi}{\delta \phi_n} = 0 \] (12a)
\[ \pi_n^* = \frac{\delta \psi}{\delta \pi_n} = 0 \quad , \quad \sigma_n^* = \frac{\delta \psi}{\delta \sigma_n} = 0 \] (12b)

With that gauge choice, the effective action is just
\[ S_{\text{eff}} = S_0^{(\lambda=0)} + \int d^2 x \sum_{n \geq 1} \lambda_n^* (-\dot{\sigma}_n + (-1)^n \sigma_n') \] (13)
It is straightforward to integrate over the momenta \( \pi_n \). One gets
\[ S_{\text{eff}} = \int d^2 x \left[ \sum_{n=0}^{\infty} \left( (-1)^n \frac{1}{2} \left( \phi_n^2 - \phi_n^2 \right) \right) + \sum_{n \geq 1} \left( \lambda_n^* (-\partial_0 + (-1)^n \partial_1) \sigma_n \right) \right] \] (14)
This is the action of \[ \mathcal{A}_n \] if one makes the identification \( \lambda^*_n = C_{n-1} \) and \( \sigma_n = b_{n-1} \). This effective action is manifestly covariant and has been shown in \[ \text{(14)} \] to yield a path integral reproducing the correct amplitudes when properly handled. The covariance of \( \text{(14)} \) is manifest if one rewrites it as

\[
S^{\text{eff}} = \int d^2x \left[ \sum_{n=0}^{\infty} (-1)^n \frac{1}{2} \partial_\mu \phi^i \partial^\mu \phi^i + \sum_{n \geq 1} \lambda_{n}^{* \mu} \partial_\mu \sigma_n \right]
\]

(15)

where \( \lambda_{n}^{* \mu} \) is a vector obeying the covariant algebraic constraint

\[
\lambda_{n}^{* \mu} = (-1)^n \epsilon_{\mu \nu \rho} \epsilon_{\nu \rho} \lambda_n^\rho
\]

(16)

and having thus only one independent component. The partition function is

\[
Z = \int D\phi D\lambda \, D\sigma \, \exp[iS^{\text{eff}}].
\]

(17)

III. FIRST CLASS FORMULATION OF CHIRAL P-FORMS

It is possible to generalize the first class formulation of chiral bosons discussed above to chiral p-forms in \( (2p+2) \)-dimensions (with \( p \) even). The starting point is the action of \[ \text{(14)} \]

\[
S[\pi, A, \lambda] = \int d^N x \left[ \sum_{i \geq 1} \pi^{i_1 \cdots i_p} A_{j_1 \cdots j_p} (0) \right]
\]

\[
- \mathcal{H} - \chi^{(0)}_{i_1 \cdots i_p} \left( \pi (0) - \beta (0) \right)^{i_1 \cdots i_p}
\]

(18)

with \( N = 2p + 2 \) (equation (78) of \[ \text{(15)} \]). Here

\[
\mathcal{H} = \frac{1}{2} \left( \pi^2 + \beta^2 \right)
\]

(19)

is the Hamiltonian density in flat space, while \( \chi^{(0)}_{i_1 \cdots i_p} \) is the Lagrange multiplier for the chiral constraint

\[
(\pi (0) - \beta (0))^{i_1 \cdots i_p} \approx 0.
\]

(20)

In \[ \text{(18)} \], \( \pi^{i_1 \cdots i_p} \) is the momentum conjugate to \( A_{i_1 \cdots i_p} (0) \) and \( \beta^{i_1 \cdots i_p} \) is the “magnetic component” of the field strength \( F \) explicitly given by

\[
F_{i_1 \cdots i_{p+1}} = \partial_{i_1} A_{i_2 \cdots i_{p+1}} + p \text{ cyclic terms}
\]

(21a)

\[
\beta \equiv \beta^{i_1 \cdots i_p} = \frac{1}{(p+1)!} \epsilon^{i_1 \cdots i_{p+1} i_{p+2} \cdots i_p} F_{i_{p+1} \cdots i_{p+2}}
\]

(21b)

If one solves the chiral constraint in \[ \text{(18)} \], one gets the action \( S[A_{i_1 \cdots i_p} (0)] = \frac{1}{p!} \int d^N x \left( \epsilon (0) \beta (0) - \beta (0)^2 \right) \) of \[ \text{(14)} \], where \( \epsilon (0) \) is the “electric component” of the field strength \( F \).

As pointed out in \[ \text{(14)} \] the chiral constraint \[ \text{(18)} \] is no longer pure second class, contrary to what happens in the case \( p = 0 \). Rather, the divergence of \[ \text{(20)} \] is first class

\[
G_{(0)}^{i_1 \cdots i_{p-1} i_p} = \pi_{(0)}^{i_1 \cdots i_p} \beta (0) = 0.
\]

(22)

It is convenient to enlarge the set of constraints by including explicitly \[ \text{(22)} \]. This is permissible since it simply amounts to replace the original description of the constraint surface by an equivalent (but reducible) one. In this redundant description the action reads

\[
S[\pi (0), A (0), \lambda (0)] = \int d^N x \left[ \sum_{i \geq 1} \pi^{i_1 \cdots i_p} A_{j_1 \cdots j_p} (0) - H \right]
\]

\[
- \lambda^{(0)}_{1 \cdots i_p} \left( \pi (0) - \beta (0) \right)^{i_1 \cdots i_p} - \lambda^{(0)} \right]_{A_{01 \cdots i_{p-1}} G^{i_1 \cdots i_{p-1}}}
\]

(23)

where \( A_{01 \cdots i_{p-1}} \) is the Lagrange multiplier for “Gauss law” \[ \text{(22)} \].

The chiral constraint \[ \text{(20)} \] has also, as in the chiral boson case, a second class component. The first step in reaching a manifestly covariant path integral is to reformulate the system in such a way that there are only first class constraints. This can be achieved by enlarging the original phase space \[ \text{(14)} \]. As in the chiral boson case, one needs an infinite set of auxiliary variables. We shall not give here all the details of the procedure, but instead, we shall directly give the final answer and check that it is indeed correct.

The purely first class formulation of a chiral p-form in \( (2p + 2) \)-dimensions is given by the action

\[
S_0 \left[ A^{(n)}_{i_1 \cdots i_p}, \pi^{(n)}_{i_1 \cdots i_p}, \lambda^{(n)}_{i_1 \cdots i_p} \right] = S_0^{(2)} + S^{(\lambda)}
\]

(24)

where \( S_0^{(2)} \) is just the action for an infinite number of non-chiral p-forms, in Hamiltonian form,

\[
S_0^{(2)} = \int d^N x \left[ \sum_{n=0}^{\infty} \pi^{(n)}_{i_1 \cdots i_p} A_{j_1 \cdots j_p}^{(n)} - \mathcal{H} \right]
\]

\[
- \lambda^{(n)} \left( \pi^{(n)} - \beta^{(n)} \right)^{i_1 \cdots i_p} - \lambda^{(n)} \right]_{A_{01 \cdots i_{p-1}} G^{i_1 \cdots i_{p-1}} (n)}
\]

(25)

\[
\mathcal{H} \approx \frac{1}{2} \sum_{n=0}^{\infty} \left[ \pi^2 + \beta^2 \right] (-1)^n
\]

(26)

while \( S^{(\lambda)} \) is the lagrange multiplier term enforcing the infinite set of first class constraints \( T^{i_1 \cdots i_p} = 0 \),

\[
S^{(\lambda)} = \int d^N x \left( \sum_{n=1}^{\infty} \lambda^{(n)} T^{i_1 \cdots i_p} \right)
\]

(27)

\[
T^{k_1 \cdots k_p} = \pi^{k_1 \cdots k_p} - \beta^{k_1 \cdots k_p} + \pi^{k_1 \cdots k_p} + \beta^{k_1 \cdots k_p}
\]

(28)

\[ n \geq 1 \]
\[
\frac{\delta S_T}{\delta \lambda^j_{(n)}} = 0 \iff T^{j_1 \cdots j_p}_{(n)} = 0. \quad (29)
\]

The constraints \( T^{j_1 \cdots j_p}_{(n)} = 0 \) are easily verified to be first class among themselves

\[
\left[ T^{j_1 \cdots j_p}_{(n)}, T^{k_1 \cdots k_p}_{(m)} \right] = 0, \quad (30)
\]

and to commute weakly with the Hamiltonian \( \int \mathcal{H} d^N x \). Note that this holds only because \( p \) is even.

The first class constraints \( T^{j_1 \cdots j_p}_{(n)} = 0 \) and \( G^{j_1 \cdots j_p-1}_{(n)} = 0 \) are non independent since \( G^{j_1 \cdots j_p-1}_{(m-1)} + G^{j_1 \cdots j_p-1}_{(m)} \equiv T^{j_1 \cdots j_p}_{(m)} \), \( \nu_{(p)} \). Furthermore \( G^{j_1 \cdots j_p-1}_{(n)} \) are \( 0 \). The constraints are clearly highly reducible.

To verify the equivalence of the system \((24)\) with the original system describing a single chiral \( p \)-form, one can observe that the observables ("gauge invariant functions") may be assumed to depend only on the \( \nu_{(n)} \)'s, subject to the transversality condition \( \nu_{(n)}^{1, \cdots k_p} \equiv 0 \). This function will be gauge invariant if and only if it actually does not involve the \( \nu_{(n)} \)'s for \( n \geq 1 \) since the variables can be completely gauged away by the gauge transformations \((35)\) (any \( \nu_{(n)}^{1, \cdots k_p} \) subject to \( \nu_{(n)}^{1, \cdots k_p} = 0 \) can be written as \( e^{k_1 \cdots k_p} j_1 \cdots j_p} \partial_\epsilon \lambda_{j_1 \cdots j_p} \). Thus, the reduced phase space of the system is spanned by the single variable \( \nu_{(0)}^{1, \cdots k_p} \) obeying the commutation relations \((34)\) and subject to the transversality condition \((33)\), exactly as in the original description.

A different way to say the same thing is to observe that a partial gauge fixing is given by \( \nu_{(n)}^{1, \cdots k_p} = 0 \) \( (n \geq 1) \), \( A_{(n)}^{1, \cdots k_p} = 0 \) \( (n \geq 1) \). The residual gauge freedom is just the standard gauge freedom of the \( 0 \)-th \( p \)-form. One then finds, since the gauge conditions and the constraints imply together \( A_{(n)}^{1, \cdots k_p} = 0 \) and \( \nu_{(n)}^{1, \cdots k_p} = 0 \) \( (n \geq 1) \), that the action \((24)\) reduces to the original action \((18)\), establishing again equivalence.

### IV. Minimal Solution of the Master Equation

We now proceed to the construction of the solution of the master equation. For definiteness and simplicity of notations, we consider the case of a chiral 2-form in 6 dimensions. We shall comment on the general case at the end of the paper.

The equations of motion for the canonical momenta \( \pi_{(n)}^{i_1 \cdots i_p} \) can be solved to express them in terms of the 2-form components \( A_{(n)}^{i_1 \cdots i_p} \) and the multipliers \( \lambda^{n}_{k,m} \). One says that the \( \pi^{kl}_{(m)} \) are "auxiliary fields". We shall work from now on with the action \( S_0[A_{(n)}^{i_1 \cdots i_p}, \lambda^{n}_{k,m}] \) obtained by eliminating the \( \pi^{s} \)'s using their own equations of motion, which is permissible \((14)\). We shall not need the explicit form of the action \( S_0[A_{(n)}^{i_1 \cdots i_p}, \lambda^{n}_{k,m}] \) for all \( \lambda^{s} \)'s. We shall just need the form of \( S_0[A_{(n)}^{i_1 \cdots i_p}, \lambda^{n}_{k,m}] \) for \( \lambda^{n}_{k,m} = 0 \) because we shall impose this condition when fixing the gauge. If \( \lambda^{n}_{k,m} = 0 \), the expression for the momenta in terms of the \( A_{(n)}^{i_1 \cdots i_p} \) is

\[
\pi^{kl}_{(m)} = -F^{0kl}_{(m)}, \quad (36)
\]

as for non-chiral 2-forms. Therefore \( S_0 \) reduces, when \( \lambda^{n}_{k,m} = 0 \), to the sum of the standard actions for non-chiral \( p \)-forms, one for each \( A_{(n)}^{i_1 \cdots i_p} \),

\[
S_0[A_{(n)}^{i_1 \cdots i_p}, \lambda^{n}_{k,m} = 0] =
\]
\[ -\int d^6x \left[ \sum_{n\geq 0} \left( \frac{1}{6} F_{\lambda\mu\nu}^{(n)} F_{\lambda\mu\nu}^{(n)} \right) (-1)^n \right], \quad (37) \]

\[ F_{\lambda\mu\nu}^{(n)} = 3\partial_\lambda A_{\mu\nu}^{(n)} . \quad (38) \]

This action is manifestly covariant.

The action \( S_0[A^{(n)}_\lambda, \lambda^{(n)}_k] \) (for all \( \lambda \)'s) is invariant under the usual 2-form gauge transformations, which are

\[ \delta_\lambda A_{\mu\nu}^{(n)} = \partial_\mu \epsilon_{\nu}^{(n)} - \partial_\nu \epsilon_{\mu}^{(n)} \quad (n \geq 0) \]

\[ \delta_\lambda \lambda^{(n)}_k = 0 \quad . \quad (39b) \]

These transformations are generated by the Gauss constraints \( G^k_\lambda \approx 0 \), which is also invariant under the gauge transformations associated with the chirality constraints \( T^{kl}_m \approx 0 \), which read explicitly, in covariant form,

\[ \delta_\lambda A^{(0)}_{\mu\nu} = u^{(1)}_{\mu\nu} \quad (40a) \]

\[ \delta_\lambda A^{(n)}_{\mu\nu} = u^{(n)}_{\mu\nu} + u^{(n+1)}_{\mu\nu} \quad (n \geq 1) \]

\[ \delta_\lambda \lambda^{(n)}_k = -H^{(n)}_{0kl}[u] + \frac{1}{6} (-1)^n \epsilon_{klpq} H^{pq\sigma\tau}[u] \quad , \quad (40c) \]

where the \( H^{(n)}_{\mu\nu\rho} \) are the strength tensor components for the gauge parameters \( u^{(n)}_{\mu\nu} = -u^{(n)}_{\nu\mu} \), \( n \geq 1 \)

\[ H^{(n)}_{\mu\nu\rho}[u] = 3\partial_\mu u^{(n)}_{\nu\rho} \quad . \quad (41) \]

The invariance of the action under \( (40) \) is most easily verified in the Hamiltonian formalism. If one takes \( u^{(n)}_{0k} = 0 \), the transformation \( (40) \) (together with \( \delta_\lambda \pi^{(n)kl} = \left[ \pi^{(n)kl} \right] \sum_m \int d^6x \left( u^{(n)}_{pq} T^{pq}_{m}(m) \right) \)) are just the standard gauge transformations generated by the constraints \( T^{pq}_m \), whereas the transformations \( (40) \) with \( u^{(n)}_{mll} = 0 \) arise because the constraints \( (G^k_\lambda, T^{kl}_m) \) are not independent (see [13], chapter 3). These transformations leave the Hamiltonian action \( (22) \) invariant and thus also the action \( S_0 \left[ A^{(n)}_\lambda, \lambda^{(n)}_k \right] \) obtained by eliminating the auxiliary fields \( \pi^{(n)kl} \) .

The gauge transformations \( (39) \), \( (40) \) form a complete set. However, they are not independent. If one takes

\[ u^{(n)}_{\mu\nu} = \partial_\mu k^{(n)}_\nu - \partial_\nu k^{(n)}_\mu \quad (n \geq 1) \]

\[ \epsilon^{(n)}_{\nu} = -k^{(n+1)}_\nu - k^{(n)}_\nu + \partial_\nu \lambda^{(n)} \quad (n \geq 1) \]

\[ \epsilon^{(0)}_{\nu} = -k^{(1)}_\nu + \partial_\nu \lambda^{(0)} \quad , \quad (42c) \]

one gets zero field variations for any choice of \( k^{(n)}_\mu \) \( (n \geq 1) \) and \( \lambda^{(n)}_k \) \( (n \geq 0) \).

These are the basic “reducibility identities” and they are not, in turn, independent. If one takes

\[ k^{(n)}_\nu = \partial_\nu \varphi^{(n)} \]

\[ \Lambda^{(0)} = \varphi^{(1)} , \quad \Lambda^{(n)} = \varphi^{(n+1)} + \varphi^{(n)} , \quad (n \geq 1) \]

one gets identically vanishing gauge parameters in \( (42) \). There is no further “reducibility of the reducibility”.

Since the gauge transformation are abelian and the reducibility identities linear and holding off shell, the minimal solution of the master equation is easy to work out.

One gets, following the well-known procedure,

\[ S^{\text{min}} = S_0[A, \lambda] = + \int d^6x \left\{ \sum_{n \geq 0} \left[ A^{\ast \mu\nu}_n \left( \partial_\mu C^{\ast (n)}_\nu - \partial_\nu C^{\ast (n)}_\mu \right) \right. \right. \]

\[ + \left. \left. \sum_{n \geq 1} \left[ A^{\ast \mu\nu}_n \left( \eta^{(n)}_{\mu\nu} + u^{(n+1)}_{\mu\nu} \right) - \lambda^{* \ast \nu}_k \left( B^{kl}_n(\eta)_n \right) \right. \right. \]

\[ -C^{\ast \nu}_n(\sigma^{(n)} - \sigma^{(n+1)} - \varphi^{\ast \nu}_n \partial_\nu \sigma^{(n)} - \partial_\nu \varphi^{\ast \nu}_n) \]

\[ + \sigma^{\ast \nu}_n \partial_\nu \lambda^{(n)} + \rho^{\ast \nu}_n \left( \lambda^{(n)} + \lambda^{(n+1)} \right) \]

\[ + A^{\ast \nu}_0(\eta^{(1)}_n - C^{\ast \nu}_0(\sigma^{(1)}_n + \rho^{\ast}_n) \lambda^{(1)}_n \right) \quad (44) \]

The ghosts \( C^{\ast \nu}_n \) \( (n \geq 0) \) are associated with the 2-form gauge symmetry \( (39) \) and have ghost number one. Their antifields are \( C^{\ast \nu}_n \) and have ghost number -2. The ghosts \( \eta^{\ast \mu}_{\nu} \) \( (n \geq 1) \) are associated with the gauge symmetry \( (40) \) and have also ghost number one. Their antifields \( \eta^{\ast \mu}_{\nu} \) have ghost number -2. We have defined \( \Theta^{(n)}_{\rho\sigma} (\eta) \) to be the field strengths of the \( \eta \)’s

\[ \Theta^{(n)}_{\rho\sigma}(\eta) = 3\partial_\nu \eta^{\ast \rho\sigma} \quad (45) \]

and

\[ \bar{B}^{kl}_n = \Theta^{(n)}_{0kl} - \frac{1}{6} \epsilon_{klpq} \Theta^{(n)\rho\sigma\tau} (-1)^n \quad . \quad (46) \]

Finally we have the following ghosts of ghosts and antifields corresponding to the various reducibilities

\[ \rho^{\ast \nu}, \ g\rho^{\ast \nu}_n = 2, \ \rho^{\ast \nu}, \ g\rho^{\ast \nu}_n = -3, \ n \geq 0 \quad (47a) \]

\[ \sigma^{\ast \nu}_n, \ g\sigma^{\ast \nu}_n = 2, \ \sigma^{\ast \nu}_n, \ g\sigma^{\ast \nu}_n = -3, \ n \geq 1 \quad (47b) \]

\[ \chi^{\ast \nu}_n, \ g\chi^{\ast \nu}_n = 3, \ \chi^{\ast \nu}_n, \ g\chi^{\ast \nu}_n = -4, \ n \geq 1 \quad . \quad (47c) \]

V. TEMPORAL GAUGE

One can verify the correctness of the minimal solution of the master equation by writing the path integral in the “temporal gauge” \( A^{(n)}_{k0} = 0, C^{(n)}_0 = 0 \ (n \geq 0), \lambda^{(n)}_k = 0, \eta^{(n)}_0 = 0, \sigma^{(n)}_0 = 0 \ (n \geq 1) \). This gauge fixing can be reached without need for non minimal variables, by exchanging the roles of the fields that are set equal to zero for their antifields, and by taking \( \psi = 0 \) [13].
antifields conjugate to the fields \( A^{(n)}_{kl}, C^{(n)}, \lambda^{(n)}_k, \eta^{(n)}_k \) and \( \sigma^{(n)}_k \) play the role of antighosts and will be denoted in the remainder of this section as

\[ A^{(n)}_{kl} \equiv \tilde{C}^{k}_{(n)}, C^{(n)}_k \equiv \tilde{C}^{k}_{(n)}, \eta^{(n)}_k \equiv \tilde{\mu}^{k}_{(n)}, \sigma^{(n)}_k \equiv \tilde{\mu}^{(n)}. \quad (48) \]

The partition function is then

\[
Z = \int D\lambda^{(n)}_{kl} D\sigma^{(n)}_k D\chi^{(n)} D\tilde{\mu}^{(n)} D\tilde{\mu}^{(n)} \exp[iS_{\text{eff}}^{(n)}] \quad (49)
\]

with

\[
S_{\text{term}}^{\text{eff}} = \int d^6x \left[ \sum_{n \geq 0} \left( -\frac{1}{2} F^{(n)}_{klm} F^{(n)}_{klm} - \frac{1}{6} F^{(n)}_{klm} F^{(n)}_{klm} \right) + \tilde{C}^{k}_{(n)} \partial_0 C^{(n)}_k + C^{(n)}_k \partial_0 \rho^{(n)} \right] - \sum_{n \geq 1} \left( \chi^{(n)}_k \tilde{B}^{k}_{(n)} - \tilde{\mu}^{k}_{(n)} \partial_0 \rho^{(n)} - \tilde{\mu}^{(n)} \partial_0 \chi^{(n)} \right) \quad (50)
\]

The partition function (49) is equal to an infinite product of determinants which can be evaluated as follows. The second order differential operator \( D \) acting on the \( A_{kl}^{(n)} \)'s in the Euler-Lagrange equations following from the gauge fixed action (42) can be written as a product of first order differential operators,

\[
D = D_+ D_- \quad (51a)
\]

\[
D_+ A^{(n)}_{kl} = \partial_0 A^{(n)}_{kl} + \frac{1}{2} \epsilon_{klmpq} \partial^m A^{(n)pqr} \quad (51b)
\]

\[
D_- A^{(n)}_{kl} = \partial_0 A^{(n)}_{kl} - \frac{1}{2} \epsilon_{klmpq} \partial^m A^{(n)pqr} \quad (51c)
\]

If \( A^{(n)}_{kl} \) is “longitudinal” (\( \partial^m A^{(n)}_{kl} = 0 \)), the operators \( D_+ \) and \( D_- \) reduce to \( \partial_0 \), while \( D \) becomes \( \partial_0^2 \). There are 4 longitudinal modes among the 10 \( A_{kl}^{(n)} \)'s, and 6 modes transverse to them. If one denotes by \( D_+ \) and \( D_- \) the operators induced in the transverse subspace, one has formally

\[
det D_+ = det D_+ \quad (det \partial_0)^4, \quad det D_- = det D_- \quad (det \partial_0)^4 \quad (52)
\]

Thus, \( det D = det D_+ \cdot det D_- \quad (det \partial_0)^8 \) and the integration over \( A^{(n)}_{kl} \) yields for each \( n \) the factor

\[
(det D)^{-\frac{1}{2}} = (det D_+)^{-\frac{1}{2}} (det D_-)^{-\frac{1}{2}} (det \partial_0)^{-4} \quad (53)
\]

The integration over the 5 anticommuting ghost pairs \( C_k^{(n)} \) and \( \tilde{C}^{k}_{(n)} \) (\( n \) fixed) clearly yields \( (det \partial_0)^5 \), while the integration over the single commuting ghost pair \( \chi^{(n)} \) and \( \sigma^{(n)}_k \) gives \( (det \partial_0)^{-1} \). Accordingly, the integration over \( A^{(n)}_{kl}, C^{(n)}_k, \tilde{C}^{k}_{(n)}, \tilde{\mu}^{k}_{(n)}, \tilde{\mu}^{(n)} \) yields, for each given \( n \),

\[
(det D_+)^{-\frac{1}{2}} (det D_-)^{-\frac{1}{2}} \quad (54)
\]

Consider now the integration over the sector \( (\chi^{(n)}_{kl}, \tilde{\mu}^{(n)}_k, \sigma^{(n)}_k, \tilde{\mu}^{(n)}_k, \chi^{(n)}_k) \). The \( \chi \eta \) term can be written as \( -\chi^{(1)}_k D^2 \eta^{(1)}_k \), thus we get from \( \int D\chi^{(1)}_k D\eta^{(1)}_k \) the determinant

\[
det D_+ \cdot det D_- \quad (det \partial_0)^4 \quad (55)
\]

At the same time, the integration over the commuting ghost pairs \( (\tilde{\mu}^{(1)}_k, \sigma^{(1)}_k) \) brings in \( (det \partial_0)^{-5} \) and the integration over the anticommuting ghost pair \( (\tilde{\mu}^{(1)}_k, \chi^{(1)}_k) \) gives \( (det \partial_0)^{1} \). Accordingly, the integration over \( (\chi^{(1)}_{kl}, \tilde{\mu}^{(1)}_k, \sigma^{(1)}_k, \tilde{\mu}^{(1)}_k, \chi^{(1)}_k) \) brings in the factor \( (det D_+) \). The same argument applies to the integration for the other indices \( n \) with \( n \) odd, while for \( n \) even one gets \( (det D_-) \).

Putting things together, one finds that the partition function \( Z \) is equal to the infinite product

\[
(det D_+) \quad (56)
\]

The first two factors \( (det D_+) \) and \( (det D_-) \) come from the integration over \( A_{kl} \) and its companion variables, the next factor \( det D_+ \) comes from the integration over \( \chi^{(1)}_k \) and its companion variables, the next two factors \( (det D_+) \) and \( (det D_-) \) come from the integration over \( A_{kl} \) and its companion variables etc... In order to regularize the expression \( (56) \), we regroup the factors along the ideas of \( (57) \) (formula (4.11)), which follows the way the extra variables have been progressively added. More precisely, we rewrite \( (56) \) as

\[
(det D_-)^{-\frac{1}{2}} \left( (det D_+)^{-\frac{1}{2}} (det D_-)^{-\frac{1}{2}} \right) \quad (57)
\]

By regrouping the factors in this manner, one finds that the partition function reduces to

\[
Z = (det D_-)^{-\frac{1}{2}} 1.1.1.... \quad (58a)
\]

\[
= (det D_-)^{-\frac{1}{2}} \quad (58b)
\]

as it should.

VI. COVARIANT PATH INTEGRAL

While the temporal gauge \( A_{0k}^{(n)} = 0 \), \( \lambda_0^{(n)} = 0 \) does not lead to a manifestly Lorentz invariant effective action, one may devise gauge conditions that do achieve this goal. For instance, one may impose the Lorentz gauge

\[
\partial_\mu A^{(n)}_{\mu k} = 0 \quad (n \geq 0) \quad (59)
\]

for the ordinary 2-form gauge symmetries, together with

\[
\lambda^{(n)}_{kl} = 0 \quad (n \geq 1) \quad (60)
\]
for the gauge transformation arising from the introduction of the auxiliary variables. This second condition is intended to eliminate the non-covariant Lagrange multiplier term $\sum_{kl}^{(n)} T_{(n)}^{kl}$ from the action, as in the chiral boson case. The gauge conditions (63), (64), (65) and (66) do not require the introduction of non minimal variables. It can again be implemented by exchanging the roles of $\lambda_{kl}^{(n)}$ and $\lambda_{kl}^{(n)}$ and by taking a gauge fixing fermion $\psi$ that does not depend on $\lambda_{kl}^{(n)}$: so that $\lambda_{kl}^{(n)} = -\frac{1}{2} \lambda_{kl}^{(n)}$ indeed vanishes.

By contrast, the gauge conditions (59), (60), (61), (62) do need a non-minimal sector. The non-minimal sector required to the 2-form gauge symmetry is well known (12, chapter 19) and is given by the antighosts $\bar{C}_n^{(n)}$, $\tilde{C}_n^{(n)}$ together with the auxiliary variables $\bar{b}_n^{(n)}$, $b_n^{(n)}$, $\pi_n^{(n)}$ and $\eta_n^{(n)}$, with ghost number assignments

$$
\begin{align*}
gh \bar{C}_n^{(n)} &= 1, \quad gh \tilde{C}_n^{(n)} = 0 \quad (63a) \\
gb_n^{(n)} &= 0, \quad gh \bar{b}_n^{(n)} = 0 \quad (63b) \\
gb_n^{(n)} &= 1, \quad gh \bar{b}_n^{(n)} = -1 \quad (63c) \\
gh \eta_n^{(n)} &= 0, \quad gh \eta_n^{(n)} = -1 \quad (63d)
\end{align*}
$$

The non-minimal term in the solution of the master equation required for freezing covariantly $A_{\mu}^{(n)} \rightarrow A_{\mu}^{(n)} + \partial_\mu A_\nu - \partial_\nu A_\mu$ is then

$$
\int d^6x \sum_{n=0}^{\infty} \left( \bar{C}_n^{(n)} b_n^{(n)} + \tilde{C}_n^{(n)} b_n^{(n)} + \eta_n^{(n)} \pi_n^{(n)} \right) .
$$

Since the gauge conditions and the structure of the minimal solution of the master equation for the ghost variables $\eta_n^{(n)}$ is quite similar to that for $A_n^{(n)}$ with mere shift in the ghost number, we also add to $S$ similar non-minimal terms for imposing the conditions $\partial_{\mu} d_n^{(n)} = 0$, $\partial_{\mu} \sigma_n^{(n)} = 0$,

$$
\int d^6x \sum_{n=1}^{\infty} \left( \bar{d}_n^{(n)} + \sigma_n^{(n)} d_n^{(n)} + \mu_n^{(n)} \theta_n^{(n)} \right) .
$$

The complete, non-minimal solution of the master equation appropriate to the problem at hand is thus

$$
S = S^{\text{min}} + \int d^6x \sum_{n=0}^{\infty} \left( \bar{C}_n^{(n)} b_n^{(n)} + \tilde{C}_n^{(n)} b_n^{(n)} + \eta_n^{(n)} \pi_n^{(n)} \right) \\
+ \sum_{n=1}^{\infty} \left( \bar{d}_n^{(n)} + \sigma_n^{(n)} d_n^{(n)} + \mu_n^{(n)} \theta_n^{(n)} \right)
$$

where $\zeta_n^{(n)}$ is any field or ghost present in the gauge fixing fermion $\Psi$ (but $\lambda_n^{(n)}$ and $C_n^{(n)}$ are not frozen there).

The appropriate gauge fixing fermion that enforces the gauge conditions (59), (60), (61), (62) and (63) is

$$
\Psi = \int d^6x \sum_{n=0}^{\infty} \left( \bar{C}_n^{(n)} \left( \partial_\nu A_\mu^{(n)} \right) + C_n^{(n)} \partial_\nu \bar{C}_n^{(n)} \right) \\
+ \bar{C}_n^{(n)} \partial_\nu \eta_n^{(n)} + \bar{A}_n^{(n)} \partial_\nu \sigma_n^{(n)} + \bar{\pi}_n^{(n)} \partial_\nu \mu_n^{(n)}
$$

(see [12, 13, 14] chapter 9). So the final expression for the solution of the master equation is, taking (68) into account,

$$
S_\Psi = \int d^6x \left\{ \sum_{n=0}^{\infty} \left[ -\frac{1}{6} F_{\mu\nu}^2 (-1)^n \right] \\
+ \frac{1}{2} \left( \partial_\nu C_n^{(n)} - \bar{C}_n^{(n)} \right) \left( \partial_\mu C_n^{(n)} - \bar{C}_n^{(n)} \right) \\
+ \partial_\mu \pi_n^{(n)} + \eta_n^{(n)} + \partial_\nu A_\mu^{(n)} b_n^{(n)} + \partial_\nu \pi_n^{(n)} + \mu_n^{(n)} \theta_n^{(n)} \right) \\
+ \sum_{n=1}^{\infty} \left[ -\lambda_n^{(n)} \left( \bar{B} \right)_n \right] \\
+ \frac{1}{2} \left( \partial_\nu \bar{C}_n^{(n)} - \partial_\nu \bar{C}_n^{(n)} \right) \left( \partial_\mu \pi_n^{(n)} - \eta_n^{(n+1)} \right) \\
+ \partial_\mu \bar{C}_n^{(n)} \sigma_n^{(n)} - \sigma_n^{(n+1)} \right) \\
+ \frac{1}{2} \left( \partial_\nu \bar{C}_n^{(n)} - \partial_\nu \bar{C}_n^{(n)} \right) \left( \partial_\mu \pi_n^{(n)} - \eta_n^{(n+1)} \right) \\
+ \partial_\mu \bar{C}_n^{(n)} \sigma_n^{(n)} - \sigma_n^{(n+1)} \right) \\
+ \partial_\mu \bar{C}_n^{(n)} \sigma_n^{(n)} - \sigma_n^{(n+1)} \right) \\
+ \partial_\mu \bar{C}_n^{(n)} \sigma_n^{(n)} - \sigma_n^{(n+1)} \\
+ \partial_\mu \bar{C}_n^{(n)} \sigma_n^{(n)} - \sigma_n^{(n+1)} \right).
$$
The action is completely gauge fixed, as one easily verifies. All the terms are manifestly covariant, including the term $\lambda^k_{\mu n} B_{kl}^\eta$; which can be written as $\tau^\mu_{\alpha n} B_{\rho}^\alpha$, where the three-rank antisymmetric tensor $\tau^\mu_{\alpha n}$ is subject to the algebraic constraint

$$\tau^\mu_{\alpha n} = \frac{1}{6} (-1)^n \epsilon^{\mu\nu\lambda\alpha\beta\gamma} \tau_{(n)\alpha\beta\gamma},$$

(71)

which reduces its number of independent components to the 10 independent $\lambda^{kli}$. [Recall that the $\Theta$'s are the field strengths of the ghosts $\eta_{\mu
u}$, formula (13).]

Finally, the same analysis can be repeated along identical lines for higher rank chiral p-forms in $2p + 2$ dimensions ($p = 2k, k \geq 1$). One simply needs more ghosts of ghosts. The procedure follows the standard pattern of the antifield formalism. The details are left to the reader.

VII. CONCLUSIONS AND PROSPECTS

In this paper, we have obtained a manifestly Lorentz invariant path integral for a chiral p-form in $(2p+2)$-dimensional Minkowskian space-time. Our approach generalizes the calculations of McClain, Wu and Yu [3] performed for chiral bosons. The generalization presents new non-trivial features because the gauge symmetries are now reducible. The gauge symmetries that enable one to gauge away the auxiliary fields necessary for replacing the second class constraints by first class ones (leading to the standard covariant two-point functions) are not independent from the standard p-form gauge symmetries, which are themselves already reducible. The correct handling of this difficulty requires ghosts of ghosts, absent in the 0-form case, and is most easily carried out in the framework of the antifield formalism.

One of the striking features of the manifestly covariant formulation is that it involves an infinite number of auxiliary field variables, as in the chiral boson treatment. Of course, the manipulation of an infinite number of variables can be tricky and even misleading in some calculations, as the attempts to derive a manifestly covariant formulation of the superparticle through the introduction of an infinite number of auxiliary variables have shown [17,18]. A prescription must be given on how to compute with the infinite number of variables. For instance, the terms in the infinite sums or infinite products that arise should be grouped in a manner compatible with the actual way the new variables have been progressively added in order to reach the covariant formulation, as in formula (13) above. More covariant regularizations may be desirable, however. Let us briefly comment on the gravitational anomaly in this context.

The advantage of the manifestly covariant formulation is that it enables a direct coupling to gravity along the standard lines of ordinary tensor calculus. The coupling to gravity in the original non-manifestly covariant formulation has been actually worked out first in [3] (see also [5] and [14]), but it does not follow the familiar pattern.

Now, all the terms in the final gauge fixed action written in an arbitrary covariant background are chirally invariant, except the terms $\sum_n \lambda^{kli} B_{kl}^\eta(n)$. These terms are the only sources of the gravitational anomaly. Let us denote by $A$ the anomaly due to a single chiral 2-form (as evaluated in [3] and [19]), but it does not follow the familiar pattern. The next fermionic form $\lambda^{kli} B_{kl}^\eta(2)\eta$ has the same chirality as $A_{\mu\nu}^0$ and contributes -2A. Going on in the same fashion for higher n's one finds that the total contribution (due to the infinite number of $(\lambda, \eta)$ pairs) to the anomaly is given by the infinite sum

$$A' = 2A(1 - 1 + 1 - 1...),$$

(72)

This sum is equal to $A$ if one regularizes it as $\lim_{k \rightarrow -1} (1 + k + k^2 + ...) = \frac{1}{2}$. We have not attempted to justify this particular regularization in the present framework but we believe that the above heuristic derivation indicates the potential usefulness of our approach.

It is hoped to return to this question in the future. It is also hoped to analyse in detail the BRST cohomology and the physical spectrum in the covariant formulation.

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