Note on the Unruh Effect

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Abstract: It was suggested by Unruh that a uniformly accelerated detector in vacuum would perceive a noise with a thermal distribution. We obtain a representation of solutions of the wave equation in two dimensions suitable for the Rindler regions. The representation includes the dependence on a parameter. The Unruh field corresponds to a singular limit of the representation.

Keywords: thermalization, quantum noise, Unruh effect.
1. Introduction

It was suggested that a uniformly accelerated detector in vacuum in the Minkowski space would perceive a thermal bath of particles \[1\]. This phenomenon is known as the Unruh effect, see \[2, 3, 4\] for review. It is supposed that the Unruh quantum field in the Rindler region is the ordinary quantum field but only expended in terms of another modes related with the original ones by means of the Bogolyubov transformation.

We first consider the classical field. We obtain a representation of the two-dimensional massless field in Minkowski space by using the Mellin transform. There is a parameter \(\lambda\) in our representation. If \(\lambda > 0\) then we have the ordinary field. The formal limit \(\lambda \to 0\) corresponds to the Unruh field. In fact one can perform the limit \(\lambda \to 0\) for classical solutions with the special boundary condition, but not for arbitrary solutions. Therefore the classical version of the Unruh field is not equal to the ordinary field.

Our representation after the standard quantization produces a set of modes that also depend on \(\lambda\). From these modes in the singular limit \(\lambda \to 0\) one can get the thermal vacuum distribution.
2. Classical Field

2.1 Wave equation

Let us discuss first solutions of the classical two-dimensional wave equation

\[(\partial_t^2 - \partial_x^2) \Phi(x, t) = 0 \quad (2.1)\]

We will consider smooth solutions \(\Phi(x, t) \in C^\infty(\mathbb{R}^2)\) belonging to the space \(S(\mathbb{R}^1)\) of fast decreasing functions over the variable \(x\). By using the Fourier transform one can write the solution in the standard form

\[\Phi(x, t) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} \left[ a(k)e^{ikx - t|k|} + a^*(k)e^{-ikx - t|k|} \right] \frac{dk}{\sqrt{|k|}} \quad (2.2)\]

where \(a(k)\) is a fast decreasing function. This formula can be written in the form

\[\Phi(x, t) = F(u) + G(v), \quad u = x - t, \; v = x + t \quad (2.3)\]

where

\[F(u) = \frac{1}{\sqrt{4\pi}} \int_0^{\infty} \left[ a(k)e^{iku} + a^*(k)e^{-iku} \right] \frac{dk}{k} \quad (2.4)\]

\[G(v) = \frac{1}{\sqrt{4\pi}} \int_0^{\infty} \left[ a(-k)e^{-ikv} + a^*(-k)e^{ikv} \right] \frac{dk}{k} \quad (2.5)\]

Note that \(F, G \in S(\mathbb{R}^1)\). Writing \(G(v) = \theta(v)G_+(v) + \theta(-v)G_-(v)\) and \(F(u) = \theta(u)F_+(u) + \theta(-u)F_-(u)\) one gets the representation of the solution of Goursat problem for the wave equation in the \(R, L, F\) and \(P\) regions (in \(L\) and \(R\) region one interchanges time and space coordinates).

2.2 Mellin transform

The plane wave in terms of the Rindler coordinate \(\xi = \log v\) reads \(e^{i\omega \xi} = v^{i\omega}\) that corresponds to the Mellin transform.

For a function \(G \in S(\mathbb{R}^1)\) the Mellin transform of its restriction \(G_+\) to the positive half-line \(v > 0\) is defined by

\[\tilde{G}_+(s) = \int_0^{\infty} G_+(v) v^{s-1} dv, \; \Re s > 0. \quad (2.6)\]

\(\tilde{G}_+(s)\) admits an analytic continuation to a meromorphic function in the whole complex plane with simple poles at \(s = 0, -1, \ldots\). For small \(s\) one has \(\tilde{G}(s) = G_+(0)/s + \ldots\)
The inverse Mellin transform is given by

\[ G_+(v) = \frac{v^{-\lambda}}{2\pi} \int_{-\infty}^{\infty} \tilde{G}_+(\lambda+i\omega)v^{-i\omega} d\omega, \quad v > 0 \quad (2.7) \]

where \( \lambda \) is an arbitrary positive number, we will assume \( 0 < \lambda < 1/2 \).

For real valued functions \( G_+ \) by using (2.5), (2.6), (2.7) and the relation

\[ \int_0^{\infty} v^{s-1} e^{-ikv} dv = k^{-s} e^{-\frac{is\pi}{2}} \Gamma(s) \quad (2.8) \]

one gets the representation

\[ G_+(v) = \frac{v^{-\lambda}}{2\sqrt{\pi}} \int_0^{\infty} [B_+(\omega, \lambda)v^{-i\omega} + B^*_+(\omega, \lambda)v^{i\omega}] \frac{d\omega}{\sqrt{\omega}}, \quad v > 0, \lambda > 0. \quad (2.9) \]

Here

\[ B_+(\omega, \lambda) = \Gamma(\lambda + i\omega) \frac{\sqrt{\omega}}{2\pi} \int_0^{\infty} dk \, k^{-i\omega-\frac{1}{2}-\lambda} \left[ a(-k) e^{-\frac{\pi i\omega}{2}} + a^*(-k) e^{-\frac{\pi i\omega}{2}} \right] \quad (2.10) \]

We can take the limit \( \lambda \to 0 \) in (2.10) to get

\[ B_+(\omega, 0) = \Gamma(i\omega) \frac{\sqrt{\omega}}{2\pi} \int_0^{\infty} dk \, k^{-i\omega-\frac{1}{2}} \left[ a(-k) e^{\frac{\pi i\omega}{2}} + a^*(-k) e^{-\frac{\pi i\omega}{2}} \right] \quad (2.11) \]

This corresponds to the Unruh-DeWitt modes \[3\]. However we cannot just substitute (2.11) into the formula (2.9) for the inverse Mellin transform due to the singularity \( \Gamma(i\omega) \sim \frac{1}{i\omega} \) as \( \omega \to 0 \). The singularity in the inverse Mellin transform is related with the boundary condition \( G_+(0) = 0 \).

The importance of the boundary condition in the Unruh problem is stressed in \[3\].

If \( \lambda > 0 \) then we have the ordinary classical field. Analogously to (2.9) one can write a Mellin representation for the field (2.3) for the functions \( G_\pm(v) \) and \( F_\pm(u) \).

3. Quantum field

Quantization of the field \( \Phi(x, t) \) (2.2) can be performed by postulating the canonical commutation relations:

\[ [a(k), a^*(k')] = \delta(k - k') \quad (3.1) \]

To cure the infrared divergence we introduce the cut-off \( \kappa > 0 \).
From (2.10) and (3.1) we obtain

\[
[B_+(\omega, \lambda), B^*_+(\omega', \lambda)] = \Gamma(\lambda+i\omega)\Gamma(\lambda-i\omega') \frac{\sqrt{\omega\omega'}}{2\pi^2} \sinh\left(\frac{\pi(\omega + \omega')}{2}\right) \int_\kappa^\infty dk k^{-i(\omega-\omega')-1-2\lambda}.
\]  

(3.2)

The integral is

\[
\int_\kappa^\infty dk k^{-1-2\lambda-i(\omega-\omega')} = \frac{\kappa^{-2\lambda-i(\omega-\omega')}}{2\lambda + i(\omega - \omega')}
\]  

(3.3)

In the limit \( \lambda \to 0 \) and \( \kappa \to 0 \) the integral goes to the \( 2\pi \) times the \( \delta \)-function and one can write

\[
[B_+(\omega, 0), B^*_+(\omega', 0)] = |\Gamma(i\omega)|^2 \frac{\omega}{2\pi^2} \sinh(\pi\omega) 2\pi \delta(\omega - \omega') = \delta(\omega - \omega').
\]

However as it was discussed earlier we should keep \( \lambda > 0 \) for the ordinary field.

The formula (2.11) provides a Bogolyubov transformation for the Unruh field. For the ordinary field we have the formula (2.10).

For the vacuum \( \langle a(k)|0 \rangle = 0 \) expectation value we obtain

\[
\langle 0 | B^*_+(\omega, \lambda) B_+(\omega, \lambda) | 0 \rangle = \Gamma(\lambda - i\omega)\Gamma(\lambda + i\omega) \frac{\omega}{4\pi^2} e^{-\pi\omega} \frac{\kappa^{-2\lambda}}{2\lambda}
\]  

(3.4)

Since

\[
\Gamma(-i\omega)\Gamma(i\omega)e^{-\pi\omega} = \frac{2\pi}{e^{2\pi\omega} - 1}
\]  

(3.5)

the formula (3.4) in the limit \( \lambda \to 0 \) (one ignores the divergent constant) describes the thermal distribution with the temperature \( T = 2\pi \).

4. Conclusions

In this note the representation of the solutions of the classical two-dimensional massless field is obtained. The representation includes the dependence on the parameter \( \lambda > 0 \). It is shown that after quantization in the singular limit \( \lambda \to 0 \) one can get the thermal vacuum distribution.

In quantum field theory and in quantum optics various quantum noises are occurred if one uses an appropriate limiting procedure such as the stochastic limit, see [6]. It is important to note that the limiting procedure can be performed only for functionals of special form. In this note the thermal vacuum noise in the limit \( \lambda \to 0 \) for special functionals is obtained.

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