The Simulation of One-Step Algorithms for Treating Higher Order Initial Value Problems

J. Sabo¹, A. M. Ayinde², A. A. Ishaq³ and G. Ajileye⁴

¹Department of Mathematics, Adamawa State University, Mubi, Nigeria.
²Department of Mathematics, School of Physical Sciences, Modibbo Adamu University of Technology, Yola, Nigeria.
³Department of Physical Sciences, Al-Hikmah University, Ilorin, Nigeria.
⁴Department of Mathematics and Statistics, Federal University Wukari, Nigeria.

Abstract

The simulation of one-step methods using interpolation and collocation for the treatment of higher order initial value problems is proposed in this paper. The new approach is derived using interpolation and collocation as a basic function through power series polynomial, where the basic properties are also analyzed. The derived method is used to treat some highly stiff linear problems. The new approach compute clearly showed that the method is reliable, efficient and gives faster convergence when compared with those in literature.

Keywords: Algorithms; higher order initial value problems; one-step; and simulation.

1 Introduction

The numerical solution to the third-order ordinary differential equation is utilized in strategizing issues emerging in several areas of applied science such as chemical engineering, ethereal and quantum mechanics.

*Corresponding author: Email: sabojohn630@yahoo.com
and so on [1]. Since the invention of the computer, numerical methods have been key instruments for the approximate solution of differential equations (DEs).

Conventional, higher order ordinary differential equations of the form

$$y^d = f(x, y(x), y'(x), \ldots, y^{d-1}(x), y^{d}(x))$$

(1)

with initial conditions

$$y^s(a) = \eta_s, s = 0(1)d - 1$$

(2)

are solved by a reduction to a system of first order ordinary differential equation of the form

$$y' = f(x, y), y(a) = y_0, a \leq x \leq b, x, y \in \mathbb{R}$$

(3)

Then some appropriate numerical methods can be used to solve the resulting equation. This approach is extensively discussed by scholars such as [2-6]. It was noticed that this reduction process has a lots of setbacks such as difficulties in writing computer program for the method, wastage of human effort and computational burden which affects the accuracy of the method in terms of error. Therefore, in order to overcome these challenges, it will be appropriate and more efficient if direct method of solving (1) is applied as suggested by [7-11]. This paper looks at using interpolation and collocation to directly simulate higher-order initial value problems (specifically, initial value problem of third-order) of the form:

$$y'''(g) = f(g, y(g)), y(g_0) = y_0, y'(g_0) = y'_0, y''(g_0) = y''_0$$

(4)

The direct approaches for the solution of (1) have been established in literature to be superior to the approach of reducing in terms of approximation, the period of execution and implementation cost [12-14]. The linear multistep approaches are most commonly used to solve IVPs as a single formula, and they promote the numerical integration of ordinary differential equation but not self-starting. Therefore, it is required to develop a numerical method that eliminates the utilization of predictor with improved accuracy, efficiency and that is self-starting. Quite a number of researchers afterwards created block methods that addressed some of the mishaps of the predictor-corrector methods, [15-17]. Individuals worked on block methods using various approximation solution, which proved that the block methods are more reliable, efficient and give better accuracy.

Some researchers such as, [12, 14, 18-22] employ power series approximate as a basic function to construct continuous linear multistep approaches spanning from predictor-corrector method to hybrid block method. In the light of these, we derived the third derivative block algorithms to simulate third order IVPs on one-step using interpolation and collocation.

2 Formulation of the Methods in Mathematical Form

Consider using power series as a basic function of the form:

$$y(g) = \sum_{j=0}^{r+s} a_j g^j$$

(5)

is consider where \( r + s \) denotes the total number of interpolation and collocation points.

Substituting equation (5) into equation (4) after differentiating equation (5) three times to yield,
\[ f(g, y, y', y'') = \sum_{j=0}^{\lfloor \frac{n+1}{2} \rfloor} j(j-1)(j-2)a_j g^{j-3} \]  

(6)

A grid of one-step length is considered with a constant step size \( h \) given by \( h = g_{n+j} - g_n, \ j = 0,1 \) and four off-step points at \( g_{\frac{n+1}{2}}, g_{\frac{n+1}{4}}, g_{\frac{n+3}{8}} \) and \( g_{\frac{n+1}{2}} \).

Interpolating (5) at point \( g_{n+i}, r = \frac{1}{8} \left( \frac{1}{3} \right) \) and collocating (6) at points \( g_{n+i}, \ s = 0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, 1 \), gives a system of nonlinear equation of the form,

\[ AG = U \]  

(7)

where,

\[ A = [a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8]^T \]

\[ U = \begin{bmatrix}
  y_{\frac{n+1}{8}} & y_{\frac{n+1}{4}} & y_{\frac{n+3}{8}} & f_n & f_{\frac{n+1}{8}} & f_{\frac{n+1}{4}} & f_{\frac{n+3}{8}} & f_{\frac{n+1}{2}} & f_{n+1}
\end{bmatrix}^T \]

\[ G = \begin{bmatrix}
  1 & g_{\frac{n+1}{8}} & g^{2}_{\frac{n+1}{8}} & g^{3}_{\frac{n+1}{8}} & g^{4}_{\frac{n+1}{8}} & g^{5}_{\frac{n+1}{8}} & g^{6}_{\frac{n+1}{8}} & g^{7}_{\frac{n+1}{8}} & g^{8}_{\frac{n+1}{8}} \\
  1 & g_{\frac{n+1}{4}} & g^{2}_{\frac{n+1}{8}} & g^{3}_{\frac{n+1}{4}} & g^{4}_{\frac{n+1}{4}} & g^{5}_{\frac{n+1}{4}} & g^{6}_{\frac{n+1}{4}} & g^{7}_{\frac{n+1}{4}} & g^{8}_{\frac{n+1}{4}} \\
  1 & g_{\frac{n+3}{8}} & g^{2}_{\frac{n+1}{8}} & g^{3}_{\frac{n+1}{4}} & g^{4}_{\frac{n+1}{4}} & g^{5}_{\frac{n+1}{4}} & g^{6}_{\frac{n+1}{4}} & g^{7}_{\frac{n+1}{4}} & g^{8}_{\frac{n+1}{4}} \\
  0 & 0 & 0 & 0 & 6 & 24g_n & 60g^2_n & 120g^3_n & 210g^4_n & 336g^5_n \\
  0 & 0 & 0 & 0 & 6 & 24g_{\frac{n+1}{8}} & 60g^2_{\frac{n+1}{8}} & 120g^3_{\frac{n+1}{8}} & 210g^4_{\frac{n+1}{8}} & 336g^5_{\frac{n+1}{8}} \\
  0 & 0 & 0 & 0 & 6 & 24g_{\frac{n+1}{4}} & 60g^2_{\frac{n+1}{4}} & 120g^3_{\frac{n+1}{4}} & 210g^4_{\frac{n+1}{4}} & 336g^5_{\frac{n+1}{4}} \\
  0 & 0 & 0 & 0 & 6 & 24g_{\frac{n+3}{8}} & 60g^2_{\frac{n+3}{8}} & 120g^3_{\frac{n+3}{8}} & 210g^4_{\frac{n+3}{8}} & 336g^5_{\frac{n+3}{8}} \\
  0 & 0 & 0 & 0 & 6 & 24g_{\frac{n+1}{2}} & 60g^2_{\frac{n+1}{2}} & 120g^3_{\frac{n+1}{2}} & 210g^4_{\frac{n+1}{2}} & 336g^5_{\frac{n+1}{2}} \\
  0 & 0 & 0 & 0 & 6 & 24g_{n+1} & 60g^2_{n+1} & 120g^3_{n+1} & 210g^4_{n+1} & 336g^5_{n+1}
\end{bmatrix} \]

Solving (7) for \( a_j, j = 0(1)8 \) which are constants to be obtained and substitute into (5) to give a one-step continuous block method of the scheme,

\[ y(g) = \alpha/g \ y_{\frac{n+1}{2}}^g + \alpha/g \ y_{\frac{n+1}{4}}^g + \alpha/g \ y_{\frac{n+3}{8}}^g + h \left[ \sum_{j=0}^{\lfloor \frac{n+1}{2} \rfloor} \beta_j(g)f_{n+j} + \beta_s(g)f_{n+s} \right] \]

(8)

where \( \alpha_r(g), \beta_j(g) \) and \( \beta_s(g) \) are presented as functions of \( x \) with

\[ x = \frac{g - g_s}{h} \]

(9)
where

\[
\begin{align*}
\alpha_1 &= 32r^2 - 20r + 3 \\
\alpha_2 &= 64r^2 + 32r - 3 \\
\alpha_3 &= 32r^2 - 12r + 1 \\
\beta_1 &= \frac{32r}{63} - \frac{64}{35} - \frac{23}{9} + \frac{1}{6} - \frac{53}{72} + \frac{1}{6} - \frac{51529}{7280480} + \frac{1787}{17203320} - \frac{7}{655360} \\
\beta_2 &= \frac{1024}{2205} + \frac{448}{45} + \frac{1856}{21} + \frac{32}{21} + \frac{16027}{188160} + \frac{3679}{225792} - \frac{401}{430080} \\
\beta_3 &= \frac{1024}{315} - \frac{315}{105} + \frac{105}{105} + \frac{448}{45} + \frac{64}{15} + \frac{32}{15} - \frac{757}{80640} + \frac{3}{17920} + \frac{1}{20480} \\
\beta_4 &= \frac{64}{63} + \frac{128}{45} + \frac{118}{45} - \frac{47}{45} + \frac{1}{6} - \frac{893}{430080} + \frac{1}{73728} - \frac{13}{983040} \\
\beta_5 &= \frac{32}{2205} - \frac{64}{2205} + \frac{1}{45} - \frac{45}{126} + \frac{1}{840} + \frac{1}{6021120} + \frac{11}{36126720} + \frac{1}{13762560} \\
\end{align*}
\]

Evaluating (8) to obtain the continuous form

\[
\begin{align*}
y_{1,2} &= 3y_{1,2} - 3y_{1,2} - y_{1,2} - \frac{1}{13762560} \begin{bmatrix} 147f_r + 12832f_{-r} + 14392f_{r} - 672f_{-r} \\ + 182f_{-r} - f_{w1} \end{bmatrix} \\
y_{1,2} &= y_{1,2} - 3y_{1,2} + 3y_{1,2} - \frac{1}{13762560} \begin{bmatrix} 777f_r - 288f_{-r} + 13832f_{r} + 13216f_{-r} \\ + 42f_{-r} + f_{w1} \end{bmatrix} \\
y_{w1} &= 15y_{1,2} - 35y_{1,2} + 21y_{1,2} + \frac{1}{2752512} \begin{bmatrix} 27601f_r - 152224f_{-r} + 377832f_{r} - 337832f_{-r} \\ + 30486f_{-r} + 237202f_{r} + 2613f_{w1} \end{bmatrix}
\end{align*}
\]

Differentiating (8) once, we have

\[
y'(g) = \alpha_1'(g)y_{1,2} + \alpha_2'(g)y_{1,2} + \alpha_3'(g)y_{1,2} + \beta_1'(g)f_{w1} + \beta_2'(g)f_{w1} + \beta_3'(g)f_{w1} + \beta_4'(g)f_{w1} + \beta_5'(g)f_{w1} + \left\{ \sum_{j=0}^{s} \beta_j'(g)f_{w1} + \beta_j'(g)f_{w1} \right\} , \quad s = \frac{1}{8} \left( \frac{1}{8} \right) \frac{1}{2}
\]

Evaluating (12) at \( g_n, g_{n+1}, g_{n+2}, g_{n+3}, g_{w1} \) and \( g_{w1} \), we have

\[
\begin{align*}
hy_{w1} &= -20y_{w1} + 32y_{w1} - 12y_{w1} + \frac{1}{36126720} \begin{bmatrix} 37527f_r + 588640f_{-r} + 403144f_{r} + 6048f_{-r} \\ - 490f_{-r} + 11f_{w1} \end{bmatrix} \\
hy_{w1} &= -12y_{w1} - 16y_{w1} - 4y_{w1} + \frac{1}{72253440} \begin{bmatrix} 5411f_r - 96288f_{-r} - 29898f_{r} + 18592f_{-r} \\ - 5082f_{-r} + 31f_{w1} \end{bmatrix} \\
hy_{w1} &= -4y_{w1} + 4y_{w1} + \frac{1}{645120} \begin{bmatrix} 1827f_r - 6688f_{-r} + 281624f_{r} + 108192f_{-r} \\ - 8666f_{-r} + 31f_{w1} \end{bmatrix} \\
hy_{w1} &= 4y_{w1} - 16y_{w1} + 12y_{w1} + \frac{1}{72253440} \begin{bmatrix} 665f_r + 1824f_{-r} + 409304f_{r} + 584416f_{-r} \\ + 38682f_{-r} - 11f_{w1} \end{bmatrix} \\
hy_{w1} &= 12y_{w1} - 32y_{w1} + 20y_{w1} + \frac{1}{36126720} \begin{bmatrix} 312489f_r - 17100064f_{-r} + 38950632f_{r} - 37372384f_{-r} \\ - 22054942f_{-r} + 408543f_{w1} \end{bmatrix}
\end{align*}
\]
Differentiating (8) twice, we have

$$y''(g) = \alpha''' \left( g \right) y'' \left( g \right) + \alpha'' \left( g \right) y' \left( g \right) + \alpha' \left( g \right) y \left( g \right) + h \sum_{j=0}^{n} \beta'' \left( g \right) y_j + \beta' \left( g \right) f_{n+1}$$

$$t = \frac{1}{8} \left( \frac{3}{8} \right)$$

(14)

Evaluating (14) at $g_n$, $g_{n+1}$, $g_{n+2}$, and $g_{n+3}$, we have

$$h^2 y''_{n+1} = 64 y_{n+1} - 128 y_{n+1} + 64 y_{n+1} + 1 \frac{9031680}{h^2} \left[ 360703f_n + 1538592f_{n+1} + 226296f_{n+2} + 169568f_{n+3} \right]$$

$$h^2 y''_{n+1} = 64 y_{n+1} - 128 y_{n+1} + 64 y_{n+1} + 1 \frac{9031680}{h^2} \left[ 19635f_n - 465184f_{n+1} - 746088f_{n+2} + 81312f_{n+3} \right]$$

$$h^2 y''_{n+1} = 64 y_{n+1} - 128 y_{n+1} + 64 y_{n+1} + 1 \frac{9031680}{h^2} \left[ 4767f_n - 52704f_{n+1} + 12040f_{n+2} + 69216f_{n+3} \right]$$

$$h^2 y''_{n+1} = 64 y_{n+1} - 128 y_{n+1} + 64 y_{n+1} + 1 \frac{9031680}{h^2} \left[ 7091f_n - 38688f_{n+1} + 68392f_{n+2} + 50780f_{n+3} \right]$$

$$h^2 y''_{n+1} = 64 y_{n+1} - 128 y_{n+1} + 64 y_{n+1} + 1 \frac{9031680}{h^2} \left[ 31290f_n + 111f_{n+1} \right]$$

$$h^2 y''_{n+1} = 64 y_{n+1} - 128 y_{n+1} + 64 y_{n+1} + 1 \frac{9031680}{h^2} \left[ 9471f_n - 67040f_{n+1} - 375816f_{n+2} + 1436064f_{n+3} \right]$$

$$h^2 y''_{n+1} = 64 y_{n+1} - 128 y_{n+1} + 64 y_{n+1} + 1 \frac{9031680}{h^2} \left[ 1436064f_n + 2355868f_{n+1} + 5115392f_{n+2} + 104626f_{n+3} \right]$$

(15)

Combining and solving (11), (13) and (15) simultaneously, to give the explicit schemes as

$$y_{n+1} = y_n + h y_n' + \frac{1}{128} h^2 y_n'' + \frac{1}{1156055040} \left[ 222943f_n + 261920f_{n+1} - 177352f_{n+2} + 88928f_{n+3} \right]$$

$$y_{n+1} = y_n + h y_n' + \frac{1}{32} h^2 y_n'' + \frac{1}{1290240} \left[ 1289f_n + 2816f_{n+1} - 1240f_{n+2} + 640f_{n+3} - 146f_{n+4} + f_{n+5} \right]$$

$$y_{n+1} = y_n + \frac{3}{8} h y_n' + \frac{9}{128} h^2 y_n'' + \frac{1}{128450560} \left[ 34671f_n + 97056f_{n+1} - 19656f_{n+2} + 17248f_{n+3} \right]$$

$$y_{n+1} = y_n + \frac{1}{2} h y_n' + \frac{1}{8} h^2 y_n'' + \frac{1}{2} \frac{1}{282240} \left[ 1267f_n + 3968f_{n+1} - 112f_{n+2} + 896f_{n+3} - 140f_{n+4} + f_{n+5} \right]$$

$$y_{n+1} = y_n + h y_n' + \frac{1}{2} h^2 y_n'' + \frac{1}{17640} \left[ 511f_n + 256f_{n+1} + 2464f_{n+2} - 1792f_{n+3} + 1484f_{n+4} + 17f_{n+5} \right]$$

(16)
\[
y''_{x_{\text{h}}} = y''_{x} + \frac{1}{8} h y'''_{x} + \frac{1}{3612672} h^2 \left\{ 140119f_{x} + 228800f_{x+1} - 140504f_{x+2} + 69440f_{x+3} \right\} - 15722f_{x+4} + 107f_{x+5}
\]

\[
y''_{x_{\text{h}}} = y''_{x} + \frac{1}{4} h y'''_{x} + \frac{1}{282240} h^2 \left\{ 2527f_{x} + 7379f_{x+1} - 2030f_{x+2} + 1232f_{x+3} - 287f_{x+4} + 2f_{x+5} \right\}
\]

\[
y''_{x_{\text{h}}} = y''_{x} + \frac{3}{8} h y'''_{x} + \frac{3}{4014080} h^2 \left\{ 18683f_{x} + 63552f_{x+1} + 2856f_{x+2} + 11200f_{x+3} - 2226f_{x+4} + 15f_{x+5} \right\}
\]

\[
y''_{x_{\text{h}}} = y''_{x} + \frac{1}{2} h y'''_{x} + \frac{1}{70560} h^2 \left\{ 1337f_{x} + 4864f_{x+1} + 896f_{x+2} + 70f_{x+3} + f_{x+4} \right\}
\]

\[
y''_{x_{\text{h}}} = y''_{x} + \frac{1}{4410} h \left\{ 553f_{x} - 1408f_{x+1} + 4816f_{x+2} - 4480f_{x+3} + 2674f_{x+4} + 50f_{x+5} \right\}
\]

\[
y''_{x_{\text{h}}} = y''_{x} + \frac{1}{645120} h \left\{ 27167f_{x} + 76672f_{x+1} - 37128f_{x+2} + 17920f_{x+3} - 4018f_{x+4} + 27f_{x+5} \right\}
\]

\[
y''_{x_{\text{h}}} = y''_{x} + \frac{1}{40320} h \left\{ 1589f_{x} + 7104f_{x+1} + 1064f_{x+2} + 448f_{x+3} - 126f_{x+4} + f_{x+5} \right\}
\]

\[
y''_{x_{\text{h}}} = y''_{x} + \frac{1}{71680} h \left\{ 973f_{x} + 3968f_{x+1} + 2408f_{x+2} + 1792f_{x+3} - 182f_{x+4} + f_{x+5} \right\}
\]

\[
y''_{x_{\text{h}}} = y''_{x} + \frac{1}{180} h \left\{ 7f_{x} + 32f_{x+1} + 12f_{x+2} + 32f_{x+3} + 7f_{x+4} \right\}
\]

\[
y''_{x_{\text{h}}} = y''_{x} + \frac{1}{630} h \left\{ 329f_{x} - 1536f_{x+1} + 3584f_{x+2} - 3584f_{x+3} + 1764f_{x+4} + 73f_{x+5} \right\}
\]

### 3 Analysis of the Method

#### 3.1 The method’s order and error constant

\(L\{y(x): h\} \) is linear operator defined by

\[
L\{y(x): h\} = A^{(0)} y_{m}^{(i)} - \sum_{i=0}^{k} \frac{j h^{(i)}}{i!} y_{n}^{(i)} - h^{(3-i)} \left[ d_{i} f\left(y_{n}\right) + b_{i} F\left(Y_{m}\right) \right]
\]

When the Taylor series expansion of \(Y_{m} \) and \(F\left(Y_{m}\right) \) are applied and the coefficients of \(h\) are compare, the results is

\[
L\{y(x): h\} = C_{0} y_{x} + C_{1} y'_{x} + \cdots + C_{p} h^{p} y^{p}(x) + C_{p+1} h^{p+1} y^{p+1}(x) + C_{p+2} h^{p+2} y^{p+2}(x) + \cdots
\]

**Definition 1:** The linear operator \(L\) and the associate block method are said to be of order \(P\) if \(C_{0} = C_{1} = \cdots = C_{p} = C_{p+1} = C_{p+2} = 0\), \(C_{p+3} \neq 0\). \(C_{p+3}\) is called the error constant and implies that the truncation error is given by \(t_{p+3} = C_{p+3} h^{p+3} y^{p+3}(x) + O(h^{p+4})\)
Using the Taylor series to expanding the method to yields

\[
\begin{align*}
\sum_{j=0}^{(1)} \left( \frac{1}{8} \right)^j y_{n+j} - y_n - \frac{1}{8} h y'_{n} - \frac{1}{128} h^2 y''_{n} - \frac{31849}{165150720} h^3 y'''_{n} & - \sum_{j=0}^{(1)} \frac{6}{j!} y^{(j)}_{n} \\
\sum_{j=0}^{(2)} \left( \frac{1}{4} \right)^j y_{n+j} - y_n - \frac{1}{4} h y'_{n} - \frac{1}{32} h^2 y''_{n} - \frac{1289}{1290240} h^3 y'''_{n} & - \sum_{j=0}^{(1)} \frac{4}{j!} y^{(j)}_{n} \\
\sum_{j=0}^{(3)} \left( \frac{3}{8} \right)^j y_{n+j} - y_n - \frac{3}{8} h y'_{n} - \frac{9}{128} h^2 y''_{n} - \frac{44577}{18350080} h^3 y'''_{n} & - \sum_{j=0}^{(1)} \frac{6}{j!} y^{(j)}_{n} \\
\sum_{j=0}^{(4)} \left( \frac{1}{2} \right)^j y_{n+j} - y_n - \frac{1}{2} h y'_{n} - \frac{1}{8} h^2 y''_{n} - \frac{181}{40320} h^3 y'''_{n} & - \sum_{j=0}^{(1)} \frac{6}{j!} y^{(j)}_{n} \\
\sum_{j=0}^{(1)} \left( \frac{1}{2} \right)^j y_{n+j} - y_n - \frac{1}{2} h y'_{n} - \frac{1}{2} h^2 y''_{n} - \frac{73}{5250} h^3 y'''_{n} & - \sum_{j=0}^{(1)} \frac{6}{j!} y^{(j)}_{n}
\end{align*}
\]

(21)

By comparing the coefficient of \(h\), we can see that our method's order \(P\) is \(p = [4, 4, 4, 4, 4]^T\) and the error constant is given by [11].

\[
[-1.274 \times 10^{-8} \quad -7.3165 \times 10^{-9} \quad -1.0644 \times 10^{-9} \quad -4.0367 \times 10^{-9} \quad -3.2294 \times 10^{-6}]\]

3.2 The Method’s consistency or steadiness

When a procedure has more than one order, it is said to be stable or constant. As a result of the aforesaid analysis, our approach appears to be stable [11].

3.3 The method's zero stability

If the roots of the first characteristics polynomial \(\rho(r) = 0\) meet \(\left[ \sum A^0 R^{k-1} \right] \leq 1\), and those roots with \(R = 1\) must be simple, a block method is said to be zero stable as \(h \to 0\).

Now,

\[
\rho(r) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix} - \begin{bmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
r & 0 & 0 & 0 & -1 \\
0 & r & 0 & 0 & -1 \\
0 & 0 & r & 0 & -1 \\
0 & 0 & 0 & r & -1 \\
0 & 0 & 0 & 0 & r
\end{bmatrix} = r^4(r-1)
\]
Then, solving for $z$ in
\[ r^5 - r^4 \]
\[ r = 0 \] is the result. As a result, the approach is described as zero stable.

$r = 0, 0, 0, 0, 1$ is the results. The approach is described as zero stable.

### 3.4 Block method of the convergence

Theorem 1: the method is said to be convergent if it is consistent and zero-stable. Therefore, our method is consistent [12].

### 3.5 Our method’s absolute stability region

The stability polynomial is obtained using the boundary locus method as follows:

\[ \tau_h(w) = h(w) \left( -\frac{523}{25979622060800}w^5 - \frac{1}{6061057848152000}w^3 + \frac{10705061}{227289668432000}w^0 - \frac{8429}{340945039648000}w^7 \right) + h(w) \left( \frac{463317839}{2841120860400}w^5 - \frac{2311}{526133493800}w^3 + \frac{7441243}{5284823040}w^0 - \frac{11102000}{880803840}w^7 \right) + h(w) \left( \frac{184343}{860160}w^5 - \frac{27}{28672}w^3 + \frac{5}{2}w^0 \right) \]

We achieve the region of absolute stability in Fig. 1 below, by applying the stability polynomial.

\[ (20) \]

![Fig. 1. Showing the region of absolute stability of the method](image)

### 4 Results and Discussion

Here, we simulate the method on a third-order linear problem of the form (1) to see how effective and valid it is. Our results are compared with that of existing methods in literature.
Problem 1: Consider the highly non-stiff third order problem
\[ y'''(g) = 3\cos(g), \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 2, \]
with the exact solution: \[ y(g) = g^2 - 3\sin(g) + 3g + 1 \]
Source: [22, 23].

Table 1. Showing the results for problem 1

| g  | Exact Result | approximate Result | Error in our Method in [22] | Error in [23] |
|----|--------------|--------------------|----------------------------|---------------|
| 0.1| 1.01049975005951553510 | 1.01049975005951553520 | 1.9030e-16 | 6.5972e-11 |
| 0.2| 1.04399200761481635360 | 1.04399200761480949040 | 6.8632e-15 | 9.9109e-12 |
| 0.3| 1.10343938001598127470 | 1.10343938001595164050 | 2.9634e-14 | 2.1387e-10 |
| 0.4| 1.19174497307404852500 | 1.19174497307397051660 | 7.9636e-14 | 4.9842e-10 |
| 0.5| 1.31172338418739099920 | 1.31172338418722993730 | 1.6106e-13 | 9.6309e-10 |
| 0.6| 1.46607257981489392840 | 1.46607257981460612970 | 2.8780e-13 | 1.6495e-09 |
| 0.7| 1.65734693828692683900 | 1.65734693828646022620 | 4.6661e-13 | 2.5970e-09 |
| 0.8| 1.88793172730143171510 | 1.88793172730072628640 | 7.0543e-13 | 3.8425e-09 |
| 0.9| 2.16001927111754983460 | 2.16001927111653825740 | 1.0116e-12 | 5.4197e-09 |
| 1.0| 2.47558704557631048000 | 2.47558704557491875790 | 1.3917e-12 | 7.3591e-09 |

Source: [22, 23]

Fig. 2. Graphical solution of problem 1

Problem 2: Consider the highly non-stiff third order problem
\[ y'''(g) = 3\sin(g), \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2, \]
with the exact solution: \[ y(g) = 3\cos(g) + \frac{g^2}{2} - 2 \]
Source: [14, 21, 24].
Table 2. Showing the results for problem 2

| g   | Exact Result | approximate Result | Error in our Method | Error in [14] | Error in [21] | Error in [24] |
|-----|--------------|--------------------|---------------------|---------------|---------------|---------------|
| 0.1 | 0.99001249583407729830 | 0.99001249583407729028 | 8.0200e-18         | 8.8818e-16    | 3.3307e-16    | 1.7282e-12    |
| 0.2 | 0.96019973352372489340 | 0.96019973352372457434 | 3.1906e-16         | 4.4409e-16    | 3.3307e-16    | 6.3179e-12    |
| 0.3 | 0.91100946737681805890 | 0.91100946737681605408 | 2.0048e-15         | 6.6613e-16    | 3.3307e-16    | 1.4295e-11    |
| 0.4 | 0.84318298200865524840 | 0.84318298200864815661 | 7.0918e-15         | 1.6653e-16    | 1.1102e-16    | 2.5020e-11    |
| 0.5 | 0.75774768567111814840 | 0.75774768567109960777 | 1.8541e-14         | 1.9984e-15    | 1.1102e-16    | 3.8928e-11    |
| 0.6 | 0.65600684472903489170 | 0.65600684472899467507 | 4.0217e-14         | 3.1086e-15    | 4.4409e-16    | 5.5360e-11    |
| 0.7 | 0.53952651853465278800 | 0.5395265185338842715 | 7.6852e-14         | 3.9968e-15    | 5.5511e-16    | 7.4644e-11    |
| 0.8 | 0.41012012804149626280 | 0.41012012804136226690 | 1.3400e-13         | 4.6074e-15    | 5.5511e-16    | 9.6128e-11    |
| 0.9 | 0.26982990481199336940 | 0.26982990481177540620 | 2.1796e-13         | 5.2181e-15    | 7.2164e-16    | 1.2002e-10    |
| 1.0 | 0.12090691760441915220 | 0.12090691760408338428 | 3.3577e-13         | 5.8703e-15    | 1.0547e-15    | 1.4570e-10    |

Source: [14, 21, 24]

Fig. 3. Graphical solution of problem 2
Problem 3: Consider the highly non-stiff third order problem
\[ y'''(g) = e^g, \quad y(0) = 3, \quad y'(0) = 1, \quad y''(0) = 5, \]
with the exact solution: \[ y(g) = 2 + 2g^2 + e^g \]

Source: [24, 14, 21].

Table 3. Showing the results for problem 3

| g    | Exact Result | approximate Result | Error in our Method | Error in [24] | Error in [14] | Error in [21] |
|------|--------------|--------------------|---------------------|---------------|---------------|---------------|
| 0.1  | 3.12517091807564762480 | 3.12517091807564769120 | 6.6400e-17         | 0.0000e-00    | 2.66454e-15   | 6.34270e-13   |
| 0.2  | 3.30140275816016983390 | 3.30140275816017223430 | 2.4004e-15         | 2.8422e-13    | 4.44089e-16   | 2.32882e-12   |
| 0.3  | 3.52985880757600310400 | 3.52985880757601371930 | 1.0615e-14         | 1.6729e-12    | 3.10862e-15   | 5.44348e-12   |
| 0.4  | 3.81182469764127031780 | 3.81182469764129902240 | 2.8705e-14         | 2.9983e-11    | 6.66134e-15   | 9.85317e-12   |
| 0.5  | 4.14872127070012814680 | 4.14872127070018922840 | 6.1082e-14         | 3.1673e-11    | 9.76996e-15   | 1.59974e-11   |
| 0.6  | 4.54211880039050897490 | 4.54211880039062159880 | 1.1262e-13         | 9.1899e-11    | 2.04281e-14   | 2.37223e-11   |
| 0.7  | 4.99375270747047652160 | 4.99375270747066524390 | 1.8872e-13         | 8.9531e-11    | 2.13163e-14   | 3.35679e-11   |
| 0.8  | 5.50554092849246760460 | 5.50554092849276293860 | 2.9533e-13         | 1.9168e-10    | 1.86516e-14   | 4.53443e-11   |
| 0.9  | 6.07960311115694966380 | 6.07960311115738870660 | 4.3904e-13         | 2.1110e-10    | 2.22045e-14   | 5.97084e-11   |
| 1.0  | 6.71828182845904523540 | 6.71828182845967236080 | 6.2713e-13         | 4.9398e-10    | 2.13163e-14   | 7.64322e-11   |

Source: [24, 14, 21]
5 Summary and Conclusion

The simulation of one-step algorithms for higher order initial value problems collocation is proposed. The method was derived using interpolation and collocation as a basic function. After examining the basic property, numerical analysis revealed that the proposed approach is steady, convergence, and zero stable.

Three highly stiff linear problems are solved using the developed approach. The results obtained in Tables 1-3 clearly shown that the derived method is reliable and efficient computationally and gave better performance compared with the existing results. This is because the computed solution matches the exact solution as shown graphically. The method is also efficient because form the graphical solutions, the numerical solution matched the exact solution. This showed that the method generates results very fast.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Ejaz ST, Mustafa GA. Subdivision based iterative collocation algorithm for nonlinear third-order boundary value problems. Adv. Math. Phys. 2016;1-14.

[2] Spiegel RM. Theory and Problems of Advance Mathematics for Engineers and scientist, McGraw Hill, Inc. New York; 1971.

[3] Lambert JD. Computational methods in ordinary differential equations. Introductory Mathematics for Scientists and Engineers. Wiley; 1973.

[4] Fatunla SO. Numerical methods for initial value problems in ordinary differential equations. Academic press inc. Harcourt Brace Jovanovich Publishers, New York; 1988.

[5] Sarafyan D. New algorithms for the continuous approximate solutions of ordinary differential equations and the upgrading of the order of the processes. Computers & Mathematics with Applications. 1990;20(1):77-100.
[6] Awoyemi DO. A class of Continuous methods for general second order initial value problems in ordinary differential equations. International Journal of Computer Mathematics. 1999;72(1):29-37.

[7] Dahlquist G. Convergence and stability in the numerical integration of ordinary differential equations. Mathematica Scandinavia. 1959;4:33-53.

[8] Hall G, Suleiman M B. Stability of Adams-type formulae for second order ordinary differential equations. IMA Journal of Numerical Analysis. 1981;1(4):427-438.

[9] Omar Z. Developing parallel 3-point implicit block method for solving second order ordinary differential equations directly. IJMS. 2004;11(1):91-103.

[10] Kayode SJ. A class of maximal order linear multistep collocation methods for direct solution of ordinary differential equations. Unpublished doctoral dissertation, Federal University of Technology, Akure, Nigeria; 2004.

[11] Sabo J, Skwame Y, Kyagya TY, Kwanamu JA. The direct simulation of third order linear problems on single step block method. Asian Journal of Research in Computer Science. 2021;12(2):1-12.

[12] Adeniyi RB, Adeyefa EO. On Chebyshev collocation approach for continuous formulation of implicit hybrid block method for IVPs in second order ordinary differential equations. IOSR-J. Math. 2013;6(4):09-12.

[13] Omar Z, Abdelrahim R. Application of single step with three generalized hybrid points block method for solving third order ordinary differential equations. J. Nonlinear Sci. Appl. 2016;9, 2705-2717.

[14] Omar Z, Abdullahi YA, Kuboye JO. Predictor-Corrector block method of order seven for solving third order ordinary differential equations. Int. J. Math. Anal. 2016;10(5):223-235.

[15] Kuboye JO, Omar Z. Numerical solution of third order ordinary differential equations using a seven-step block method. Int. J. Math. Anal. 2015;9(15):743-754.

[16] Odekunle MR, Egwurube MO, Adesanya AO, Udo MO. Five steps block predictor-block corrector method for the solution of \( y'' = f(x, y, y') \). Appl. Math. 2014;5:1252-1266.

[17] Ogunware BG, Awoyemi DO, Adoghe LO, Olanegean OO, Omole EO. Numerical treatment of general third order ordinary differential equations using Taylor series as predictor. Phys. Sci. Int. J. 2018;17(3):1-8.

[18] Jikantoro YD, Ismail F, Senu N, Ibrahim ZB. A new integrator for special third order differential equations with application to thin film flow problem. Indian J. Pure Appl. Math. 2018;49(1):151-167.

[19] Adeyeye O, Omar Z. New self-starting approach for solving special third order initial value problems. Int. J. Pure Appl. Math. 2018;118(3):511-517.

[20] Awoyemi DO, Kayode, SJ, Adoghe, LO. A five-step p-stable method for the numerical integration of third order ordinary differential equations. Amer. J. Comput. Math. 2014;4:119-126.

[21] Skwame Y, Dalatu PI, Sabo J, Mathew M. Numerical application of third derivative hybrid block methods on third order initial value problem of ordinary differential equations. Int. J. Stat. Appl. Math. 2019;4(6):90-100.

[22] Yakusak NS, Emmanuel S, John D, Taiwo EO. Orthogonal collocation technique for the construction of continuous Hybrid method for second order Initial Value Problems. IJEAM. 2015;2(1):177-181.
[23] Taparki RM, Gurah D, Simon S. An implicit Runge-Kutta method for solution of third order initial value problem in ordinary differential equations. Int. J. Numer. Math. 2010;6:174-189.

[24] Awoyemi DO, Kayode SJ, Adoghe LO. A five-step p-stable method for the numerical integration of third order ordinary differential equations, Amer. J. Comput. Math. 2014;4:119-126.

© 2021 Sabo et al.; This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
https://www.sdiarticle4.com/review-history/75912