Exact Resurgent Trans-series and Multi-Bion Contributions to All Orders

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The full resurgent trans-series is found exactly in near-supersymmetric CP^1 quantum mechanics. By expanding in powers of the SUSY breaking deformation parameter, we obtain the first and second expansion coefficients of the ground state energy. They are absolutely convergent series of nonperturbative exponentials corresponding to multi-bions with perturbation series on those backgrounds. We obtain all multi-bion exact solutions for finite time interval in the complexified theory. We sum the semi-classical multi-bion contributions that reproduce the exact result supporting the resurgence to all orders. We also discuss the similar resurgence structure in CP^{N-1} with N > 2 models. This is the first result in the quantum mechanical model where the resurgent trans-series structure is verified to all orders in nonperturbative multi-bion contributions.

I. INTRODUCTION

Path-integral has been extremely useful in many areas of quantum physics through perturbative and nonperturbative analysis. It is crucial to understand contributions from all the complex saddle points based on the thimble analysis in the path integral in order to give a proper foundation of quantum theories. The resurgence theory gives a stringent relation between a divergent perturbation series and a nonperturbative exponential term, which often allows reconstruction from each other [1–3]. Resurgence is originally developed in studying ordinary differential equations and provides a trans-series, containing infinitely many nonperturbative exponentials and divergent perturbation series [4]. The intimate relation between these infinitely many nonperturbative contributions and perturbative ones is expected to provide an unambiguous definition of quantum theories. A mathematically rigorous foundation of path integral is now envisaged [5,8]. Resurgence has been most precisely studied recently in quantum mechanics (QM) to yield relations between nonperturbative and perturbative contributions systematically [10,20]. 2D quantum field theories (QFT) [50,11], 4D QFT [12,48], supersymmetric (SUSY) gauge theories [49,53], the matrix models and topological string theory [54,62].

In the resurgent trans-series for theories with degenerate vacua, one needs to take account of configurations called ”bions” consisting of an instanton and an anti-instanton [2,10], which give imaginary ambiguities cancelling those of non-Borel-summable perturbation series. Recently single bion configurations are identified as saddle points in the complexified path integral [21]. Exact solutions of the holomorphic equations of motion (complex and real bion solutions) are found in the complexified path integral of double-well, sine-Gordon and CP^1 quantum mechanical models with fermionic degrees of freedom (incorporated as the parameter \( \epsilon \)) [21,23]. CP^1 quantum mechanics is a dimensional reduction of the two-dimensional CP^1 sigma model, which shows asymptotic freedom, dimensional transmutation and the existence of instantons akin to four-dimensional QCD. Contributions from these solutions are evaluated based on Lefschetz-thimble integrals and it is shown that the combined contributions vanish for the SUSY case \( \epsilon = 1 \), in conformity with the exact results of SUSY [27]. On the other hand, for the non-SUSY case \( \epsilon \neq 1 \), the result contains the imaginary ambiguity, which is expected to be cancelled by that arising from the Borel resummation of perturbation series.

Trans-series generically contain high powers of nonperturbative exponential, which may correspond to multiple bions. Non-SUSY models including CP^{N-1} quantum mechanics have been worked out explicitly to several low orders, but it was difficult to reveal explicitly the full trans-series to all powers of nonperturbative exponential and to ascertain their resurgence structure [2,13,20]. Localization in SUSY models helped to uncover the full trans-series, but so far their resurgence structures are found to be trivial without imaginary ambiguities [10,53].

The purpose of this work is to present and to verify the complete resurgence structure of the trans-series in CP^1 QM (and partly CP^{N-1} QM), focusing on the near-SUSY regime \( \epsilon \approx 1 \) where we can obtain exact results which exhibit resurgence structure to infinitely high powers of nonperturbative exponential. We will show that the contributions from an infinite tower of multi-bion solutions yield all these nonperturbative exponentials. This
is the first result revealing the thimble structure of all the complex saddle points, which is useful not only to understand the resurgence structure in quantum theories but also to study complex path integrals including real-time formalism and finite-density systems in condensed and nuclear matters [63–68].

II. EXACT GROUND-STATE ENERGY

We first consider the (Lorentzian) CP\(^1\) quantum mechanics described by the Lagrangian

\[ g^2 L = G \left[ \partial_\tau \varphi \right]^2 - m^2 |\varphi|^2 + g \bar{\psi} \partial_\tau \psi \]  

where \( \varphi \) is the inhomogeneous coordinate, \( G = \partial_\varphi \partial_\bar{\varphi} \log (1 + |\varphi|^2) \) is the Fubini-Study metric, \( D_\tau = \partial_\tau + i \varphi \partial_\varphi \log G \) is the pull back of the covariant derivative and \( \mu = m^2 |\varphi|^2 / (1 + |\varphi|^2) \) is the moment map associated with the \( U(1) \) symmetry \( \varphi \to e^{i\theta} \varphi \). The parameter \( \epsilon \) is the boson-fermion coupling and the Lagrangian becomes supersymmetric at \( \epsilon = 1 \). Since the fermion number \( F = G \bar{\psi} \psi \) commutes with the Hamiltonian, the Hilbert space can be decomposed into two subspaces with \( F = 1 \) and \( F = 0 \). By projecting quantum states onto the subspace which contains the ground state \( (F = 1) \), we obtain the bosonic Lagrangian

\[ L = \frac{|\partial_\tau \varphi|^2}{(g^2 (1 + |\varphi|^2)^2)} - V, \]

with the potential

\[ V = \frac{1}{g^2} \frac{m^2 |\varphi|^2}{(1 + |\varphi|^2)^2} - \epsilon m^2 \frac{1 - |\varphi|^2}{1 + |\varphi|^2}. \]

We note that \( \theta(\equiv -2\arctan |\varphi|) = 0, \pi \) are global and metastable vacua respectively.

For \( \epsilon = 1 \), the ground state wave function \( \Psi_0 \) preserving the SUSY is given as a zero energy solution of the Schrödinger equation

\[ H_{\epsilon=1} \Psi_0 = \left[ -g^2 (1 + |\varphi|^2)^2 \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \bar{\varphi}} + V_{\epsilon=1} \right] \Psi_0 = 0. \]

It is exactly solved as

\[ \Psi_0 = \langle \varphi | 0 \rangle = \exp(-\mu/g^2) \]

For \( \epsilon \approx 1 \), the leading order correction to the ground state wave function can be obtained by expanding the Schrödinger equation with respect to small \( \delta \epsilon \equiv \epsilon - 1 \) as \( \langle \varphi | \delta \Psi \rangle \). Correspondingly, the ground state energy \( E \) can also be expanded

\[ E = \delta \epsilon E^{(1)} + \delta \epsilon^2 E^{(2)} + \cdots. \]

These expansion coefficients can be determined by the standard Rayleigh-Schrödinger perturbation theory as

\[ E^{(1)} = \frac{\langle 0 | \delta H | 0 \rangle}{\langle 0 | 0 \rangle}, \]

\[ E^{(2)} = -\frac{\langle \delta \Psi | H_{\epsilon=1} | \delta \Psi \rangle}{\langle 0 | 0 \rangle}, \]

with \( \delta H = H - H_{\epsilon=1} \). We find that these coefficients \( E^{(i)} \) are real without imaginary ambiguities and can be expanded in absolutely convergent power series with respect to the nonperturbative exponential \( \exp(-2m/g^2) \)

\[ E^{(i)} = \sum_{p=0}^{\infty} E^{(i)}_p \exp(-2pm/g^2), \]

where the zero-th term \( E^{(i)}_0 \) corresponds to the perturbative contributions on the trivial vacuum (perturbative vacuum). The coefficients of \( E^{(1)} \) \( [25] \) are

\[ E^{(1)}_0 = -m + g^2, \quad E^{(1)}_p = -2m, \quad (p \geq 1). \]

If the coefficients of \( E^{(2)} \) are expanded in powers of \( g^2 \), they give factorially divergent asymptotic series, which can be Borel-resummed. Hence we rewrite the coefficient in the form of the Borel transform (See Appendix A for the details of calculations.) as

\[ E^{(2)}_0 = g^2 + 2m \int_0^\infty dt \frac{e^{-t}}{t - \frac{2m}{g^2 + i0}}, \]

\[ E^{(2)}_p = 2m \int_0^\infty dt e^{-t} \left\{ \frac{(p + 1)^2}{t - \frac{2m}{g^2 + i0}} + \frac{(p - 1)^2}{t + \frac{2m}{g^2}} \right\} + 4mp^2 \left( \gamma + \log \frac{2m}{g^2} \pm \frac{\pi i}{2} \right), \quad (p \geq 1). \]

Note that the imaginary ambiguities associated to the Borel resummation is manifest in the first term of \( E^{(2)}_p \) with \( g^2 \pm i0 \), which is compensated by the imaginary part \( \pm \pi i/2 \) in the last term of \( E^{(2)}_{p+1} \), reproducing the original real \( E^{(2)} \) precisely. In the present case, we have only poles in the Borel plane while cuts are expected for general cases. We also note that in \( [27] \) the perturbation series on 0-bion background including the level number information has been shown to give all p-bion contributions.

We can now recognize the full resurgence structure to all orders of nonperturbative exponential: imaginary ambiguity of the non-Borel summable divergent perturbation series on the p-bion background in the first term of \( E^{(2)}_p \) is cancelled by the imaginary ambiguity of the classical contribution of \( (p + 1) \)-bion contribution in the last term of \( E^{(2)}_{p+1} \). We note the absence of powers of \( g^2 \) in the imaginary ambiguity, which will allow us to recover non-Borel summable perturbation series on the p-bion background completely from the \( (p + 1) \)-bion contribution.
through the dispersion relation, without computing perturbative corrections around the multi-bion background explicitly. Moreover, if we observe that \( E^{(2)}/m \) is an even function of \( m/g^2 \), we can also understand the presence of Borel-summable part (second term of the first line in Eq. (12)). Thus all the terms can now be reproduced through resurgence relation and the sign change of \( m/g^2 \); if we can compute all the semi-classical \( p \)-bion contributions.

### III. MULTI-BION SOLUTIONS

Nonperturbative contributions to the ground state energy come from the saddle points of the path integral
\[
Z = \int D\varphi D\tilde{\varphi} e^{-S_E} \sim e^{-\beta S_E} \quad \text{(for large } \beta) \quad \text{are given by}
\]
where we have complexified the degrees of freedom by regarding \( \varphi = \varphi_R^c + i\varphi_I^c \) and \( \tilde{\varphi} = \varphi_R^c - i\varphi_I^c \) as independent holomorphic variables, and imposed the periodic boundary condition \( \varphi(\tau + \beta) = \varphi(\tau) \) and for \( \tilde{\varphi} \). The Euclidean action
\[
S_E = \int_0^\beta d\tau [\partial_\tau \varphi \partial_\tau \tilde{\varphi}/(g^2(1 + \varphi^2)^2) + V(\varphi, \tilde{\varphi})],
\]
has two conserved Noether charges associated with the complexification of the Euclidean time translation \( \tau \rightarrow \tau + a \) and the phase rotation \( (\varphi, \tilde{\varphi}) \rightarrow (e^{ib}\varphi, e^{-ib}\tilde{\varphi}) \) \( (a, b \in \mathbb{C}) \). Using the corresponding conservation laws, we can obtain the following solution of the equation of motion with nontrivial contribution in a \( \beta \rightarrow \infty \) limit,
\[
\varphi = e^{i\phi_c} \frac{f(\tau - \tau_c)}{\sin \alpha}, \quad \tilde{\varphi} = e^{-i\phi_c} \frac{f(\tau - \tau_c)}{\sin \alpha},
\]
where \((\tau_c, \phi_c)\) are complex moduli parameters associated with the symmetry and \( f(\tau) \) is the elliptic function
\[
f(\tau) = \cos(\Omega \tau, k) \equiv \cosh(\Omega \tau, k) / \sinh(\Omega \tau, k),
\]
which satisfies the differential equation
\[
(\partial_\tau f)^2 = \Omega^2 (f^2 + 1)(f^2 + 1 - k^2).
\]
Solutions are characterized by two integers \((p, q)\) for the period
\[
\beta = \frac{(2pK + 4iqK')}{\Omega}
\]
with \( 2K(k) \) and \( 4iK'(K' \equiv K(\sqrt{1 - k^2})) \) as the period of the doubly periodic function \( cs \). The parameters \((\alpha, \Omega, k)\) are given in terms of the period \( \beta \), and their asymptotic forms for large \( \beta \) (See Appendix. B for the details of calculations.) are given by
\[
k \approx 1 - 8e^{-\frac{\omega^2 - 2\pi\alpha}{\omega}} \quad \text{and} \quad \Omega \approx \omega \left(1 + 8\frac{\omega^2 + m^2}{\omega^2 - m^2} e^{-\frac{\omega^2 - 2\pi\alpha}{\omega}}\right),
\]
\[
\cos \alpha \approx \frac{m}{\omega} \left(1 - \frac{8m^2}{\omega^2 - m^2} e^{-\frac{\omega^2 - 2\pi\alpha}{\omega}}\right).
\]

### IV. MULTI-BION CONTRIBUTIONS

The contributions from the \( p \)-bion solutions can be calculated by performing the Lefschetz thimble integral associated with the saddle points. In the weak coupling limit \( g \rightarrow 0 \), we can use the Gaussian approximation for

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{fig1.png}
\caption{Multi-bion solution: kink profile of \( \Sigma(\tau) = (1 - \varphi \tilde{\varphi})/(1 + \varphi \tilde{\varphi}) \) for \((p, q) = (3, 1)\), \( \epsilon = 1 \), \( m = 1 \), \( g = 1/200 \), \( \beta = 100 \) and \( \epsilon_c = 0 \). \( \Sigma = \pm 1 \) (dashed lines) correspond to north and south poles (global and local minima) of \( \mathbb{C} P^1 \).}
\end{figure}
the fluctuation modes from the saddle points except the nearly massless modes parameterized by the quasi moduli parameters \((\tau_i, \phi_i)\). Thus, we can simplify the Lefschetz thimble analysis by reducing the degrees of freedom onto the quasi moduli space.

The leading order contributions come from the region around the saddle points, where all the kinks are well-separated in the weak coupling limit. Therefore, the effective potential can be approximated by that for well-separated kinks

\[
S_E \rightarrow V_{\text{eff}} = -m\epsilon \beta + \sum_{i=1}^{2p} \frac{m}{g^2} + V_i,
\]

where \(V_i\) is the asymptotic interaction potential between neighboring kink-antikink pair \([34]\)

\[
\frac{V_i}{m} = \epsilon_i(\tau_i - \tau_{i-1}) - \frac{4}{g^2} e^{-m(\tau_i - \tau_{i-1})} \cos(\phi_i - \phi_{i-1}),
\]

with \(\tau_2n = \tau_i^-, \tau_2n = \tau_i^+, \tau_0 = \tau_{2p} - \beta, \phi_0 = \phi_{2p} \pmod{2\pi}, \epsilon_{2n} = 0\) and \(\epsilon_{2n+1} = 2\epsilon\). We find that the saddle points of \(V_{\text{eff}}\) are consistent with \(\tau_n^\pm\) in Eq. (20) for large \(\beta\) and small \(g\).

We introduce a Lagrange multiplier \(\sigma\) to impose the periodicity as

\[
2\pi\delta \left( \sum_i \tau_i - \beta \right) = m \int d\sigma \exp \left[ im\sigma \left( \sum_i \tau_i - \beta \right) \right].
\]

By generalizing the Lefschetz thimble analysis in \([25]\) to the multi-bion contribution

\[
Z_p \propto \int \prod_{i=1}^{2p} d\tau_i d\phi_i, \exp(-V_{\text{eff}}),
\]

we obtain the following \(p\)-bion contribution to the partition function (See Appendix. C for the details of calculations.)

\[
\frac{Z_p}{Z_0} \approx -\frac{2im\beta}{p} e^{-\frac{2im\beta}{g^2} \sum_{i=1}^{2p} I_i},
\]

with

\[
I_i = \frac{2m}{g^2} \left( \frac{2m}{g^2} e^{\pm i\pi/2} \right)^{i\sigma - \epsilon_i} \frac{\Gamma((\epsilon_i - i\sigma)/2)}{\Gamma(1 - (\epsilon_i - i\sigma)/2)}.
\]

The sign \(\pm\) is associated with \(\arg(g^2) = \pm 0\). This gives a polynomial of \(\beta\), whose leading term is of order \(\beta^p\)

\[
Z_p/Z_0 \approx \frac{1}{p!} \left[ \frac{2m\beta \Gamma(\epsilon)}{\Gamma(1 - \epsilon)} e^{-\frac{2m}{g^2} \pi\tau_\epsilon} \left( \frac{2m}{g^2} \right)^{2(1-\epsilon)} \right]^p,
\]

consistent with the dilute gas approximation: \(Z_p/Z_0 = (Z_1/Z_0)^p/p! + \mathcal{O}(\beta^{p-1})\). From the \(p\)-bion contribution \([20]\) and the perturbative contribution \((p = 0)\), the ground state energy \(E = -\lim_{\beta \rightarrow \infty} \frac{1}{\beta} \log Z\) can be obtained as

\[
E = E_0 - \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \log \left( 1 + \sum_{p=1}^{\infty} \frac{Z_p}{Z_0} \right).
\]

By taking the logarithm, contributions of high powers of \(\beta\) such as \(\beta^p\) for \(p > 1\) should be cancelled, and the ground state energy is obtained from the remaining contributions of order \(\beta\). Fortunately, most of these contributions with high powers of \(\beta\) disappear near SUSY case thanks to the zero in \(1/\Gamma(1 - (\epsilon_i - i\sigma)/2)\). As a result, we find that the first derivative is proportional to \(\beta\) and gives the near-SUSY ground state energy \(E^{(1)}\)

\[
E^{(1)} = -\frac{e^{2m}}{\beta} \lim_{\epsilon \rightarrow 1 - \beta \rightarrow \infty} \frac{1}{\beta} \frac{\partial}{\partial \epsilon} \frac{Z_p}{Z_0} = -2m,
\]

verifying the exact result \([11]\). The second derivative in \(\epsilon\) turn out to be quadratic in \(\beta\), and

\[
E^{(2)} = -\frac{e^{2m}}{\beta} \lim_{\epsilon \rightarrow 1 - \beta \rightarrow \infty} \frac{1}{\beta} \left[ \partial^2 \frac{Z_p}{Z_0} - \sum_{i=1}^{p-1} \partial_i \frac{Z_{p-i}}{Z_0} \frac{Z_{\alpha}}{Z_0} \right],
\]

is calculated as

\[
E^{(2)} = 4mp^2 \left( \gamma + \log \frac{2m}{g^2} \pm \frac{\pi i}{2} \right),
\]

in complete agreement with the exact result \([12]\). We have obtained the classical contributions to all orders of multi-bions, which provides all terms needed for the full resurgence structure of our model, although it is difficult to check the divergent perturbation series on \(p\)-bion background directly, except for the trivial vacuum \((p = 0)\).
V. PERTURBATION SERIES ON TRIVIAL VACUUM

We obtain the perturbation series on the trivial background \((p = 0)\) by using the Bender-Wu method\[^{24, 69}\]. We first expand the energy and the wave function as

\[
E = m \sum_l A_l \eta^{2l}, \quad \Psi = \exp(-x^2) \sum_{l,k} B_{l,k} \eta^{2l} x^{2k},
\]

with \(|\varphi| = \eta x\) and \(\eta^2 = \frac{g^2}{m}\). Then, the Schrödinger equation reduces to a (Bender-Wu) recursive equation for \(A_l\) and \(B_{l,k}\), which gives the leading asymptotic behavior (See Appendix. D for the details of calculations.) as

\[
A_l \sim -\frac{\Gamma(l + 2(1 - \epsilon))}{2^{l-1} \Gamma(1 - \epsilon)^{2}}, \quad \text{for large } l.
\]

Since the coefficients \(A_l\) grow factorially for large \(l\), we obtain the perturbative part of the ground state energy by using the Borel resummation

\[
E_0 \sim \frac{2m}{\Gamma(1 - \epsilon)^2} \int_0^\infty dt e^{-t} t^{2(1 - \epsilon)} (t - \frac{2m}{g^2})^{-1}.
\]

The Borel resummation gives a finite result with the imaginary ambiguity

\[
\text{Im } E_0 = \pm \frac{2\pi m}{\Gamma(1 - \epsilon)^2} \left( \frac{g^2}{2m} \right)^{2(\epsilon - 1)} e^{-\frac{2m}{g^2}},
\]

with \((- (+)) in the right hand side for \(\text{Im } g^2 = +0 (-0)\). This imaginary ambiguity of the perturbation series in the trivial vacuum \((p = 0)\) cancels that of the single bion sector \((28)\) with \(p = 1\). Therefore, combining these two contributions gives unambiguous real result. This result verifies the resurgence for arbitrary values of \(\epsilon\) including the non-SUSY case explicitly, although only to the leading order of nonperturbative exponential.

For the near-SUSY case, we can obtain the perturbation series on the trivial vacuum exactly to all orders in \(g^2\), by exactly solving the Bender-Wu recursion relation to the second order of \(\delta \epsilon\) as

\[
E_0 = (g^2 - m)\delta \epsilon - 2m \sum_{l=2}^\infty \Gamma(l) \left( \frac{g^2}{2m} \right)^l \delta \epsilon^2 + \cdots.
\]

This agrees completely with the exact results \(E_0^{(1)}\) in Eq. (10) and \(E_0^{(2)}\) in Eq. (11) after Borel-resummation.

VI. SUMMARY AND DISCUSSION

In conclusion,

(i) We have derived the exact expansion coefficients of the ground state energy to the second order of the SUSY breaking deformation parameter \(\delta \epsilon\). The result shows a resurgent trans-series structure to all order of nonperturbative exponential.

(ii) We have derived nonperturbative multi-bion contributions with imaginary ambiguities in the weak coupling limit and found that they agree with the corresponding parts in the exact result.

(iii) At least for near-SUSY \(\mathbb{C}P^1\) QM, by assuming the cancellation of imaginary ambiguities (resurgence structure) and an even function of \(m/g^2\), we have recovered the entire trans-series which agrees with the exact result of the near-SUSY.

(iv) With the Bender-Wu recursion relation, we have obtained the perturbation series on 0-bion vacuum to all orders, which gives an imaginary ambiguity when Borel-resummed, and have verified the cancellation with that of single bion sector for general deformation parameter \(\epsilon\) including non-SUSY case.

The exact result in Eq. (12) shows that the imaginary ambiguities have no \(g^2\) corrections in \(\mathbb{C}P^1\) QM. This fact enabled us to recover the entire trans-series from the semi-classical multi-bion contributions only. In other models such as sine-Gordon QM, imaginary ambiguities from the multi-bion contribution have perturbative corrections in powers of \(g^2\) \(^{20}\). Then these perturbative corrections are needed in order to recover the full resurgent trans-series.

The same resurgence structure exists in \(\mathbb{C}P^{N-1}\) models with \(N > 2\). Similarly to \(N = 2\), we obtain \(O(\delta \epsilon^2)\) perturbative contribution with the imaginary ambiguity

\[
\text{Im } E_0^{(2)} = \pm \frac{N^2}{2} \sum_{i=1}^{N-1} m_i A_i e^{-\frac{2m_i}{g^2}},
\]

where \(A_i = \prod_{j=1, j \neq i}^{N-1} \frac{m_j - m_i}{m_j + m_i}\) and the mass parameters \(m_i\) are reduced from the 2D \(\mathbb{C}P^{N-1}\) model with twisted boundary conditions. We also calculate the \(O(\delta \epsilon^2)\) single-bion contribution

\[
E_1^{(2)} = \sum_{i=1}^{N-1} N^2 m_i A_i e^{-\frac{2m_i}{g^2}} \left( \gamma + \log \frac{2m_i}{g^2} \pm \frac{\pi i}{2} \right).
\]

The imaginary ambiguities cancel between them. As for convergence of \(\delta \epsilon\) expansion, we observe that each of the \(p\)-bion semiclassical contributions has a convergent expansion for any \(p\).

Focusing on the near-SUSY regime can be extended to the solvable models including localizable SUSY theories \(^{49, 53}\) and quasi-solvable models \(^{28}\) by softly breaking the solvable condition and expanding the physical quantities with respect to the deformation parameter. It is because these models have a similar resurgence property to the present \(\mathbb{C}P^1\) model, where the resurgence structure becomes trivial without cancellation of imaginary ambiguity at localization-applicable or quasi-exactly-solvable regimes. We also notice that the localization technique is applicable in \(\mathbb{C}P^{N-1}\) QM to compute the first order ground state energy \(E_1^{(1)}\) but not the second order. Recent results on volume independence \(^{41}\) should be useful in extending our study to QFT, which may also require...
more refined thimble analysis as has been studied intensively \[64\] \[68\].

Regarding non-SUSY gauge theories, complex instanton solutions were discussed in gauge theories with complexified gauge groups decades ago \[70\] \[71\]. It would be of importance to discuss contributions from these complex solutions in terms of resurgence theory.

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Appendix A: Exact ground-state energy

In this section we show details of calculations in the part of “Exact ground-state energy”. The leading order correction to the ground-state wave function and energy for $\mathbb{C}P^3$ quantum mechanics in Eqs. \[11\] \[13\] can be obtained by solving the $O(\delta \epsilon)$ part of the Schrödinger equation

$$H_{\epsilon=1} |\delta \Psi\rangle = \left( E^{(1)} + m \left( \frac{1 - |\varphi|^2}{1 + |\varphi|^2} \right) \right) |0\rangle,$$  \hspace{1cm} (A1)

$$E^{(1)} = g^2 - m \coth \frac{m}{g^2} \sum_{\infty}^0 \frac{2me^{-2\mu m}}{s^2}.$$  \hspace{1cm} (A2)

From this expanded form, we can read the expansion coefficients Eq. \[10\]. The above differential equation can be exactly solved as

$$\langle \varphi | \delta \Psi \rangle = e^{-\frac{\mu^2}{m}} \int_0^\mu \frac{d\mu'}{\mu'(\mu' - m)} \left( \mu' - m \frac{1 - e^{2\mu'/m}}{1 - e^{2\mu'/m}} \right).$$  \hspace{1cm} (A3)

Then we find the second-order correction to the ground-state energy as

$$E^{(2)} = - \frac{\langle \delta \Psi | H_{\epsilon=1} | \delta \Psi \rangle}{\langle 0 | 0 \rangle} \hspace{1cm} (A4)$$

$$= g^2 - 2m \coth \frac{m}{g^2} \int_0^m d\mu \frac{\sinh^2 \frac{\mu}{g^2}}{\mu^2}.$$  

Using the hyperbolic cosine integral $\text{Chi}(z)$ defined by

$$\text{Chi}(z) = \gamma + \log z - \int_0^z \frac{dt}{t} (1 - \cosh t),$$  \hspace{1cm} (A5)

we can rewrite $E^{(2)}$ as

$$E^{(2)} = g^2 - m \cosh \frac{m}{g^2} \int \frac{\text{Chi} \left( \frac{2m}{g^2} \right) - \gamma - \log \frac{2m}{g^2}}{2}.$$  \hspace{1cm} (A6)

By using the relation

$$\text{Chi} \left( \frac{2m}{g^2} \right) = - \frac{1}{2} \int_0^\infty dt e^{-t} \left( \frac{e^{-2m}}{t - 2m e + m} + \frac{e^{-2m}}{t + 2m e} \right) + \pi i \frac{1}{2},$$  \hspace{1cm} (A7)

$E^{(2)}$ can be expanded as

$$E^{(2)} = g^2 + 2m \int_0^\infty dt e^{-t} \left( \frac{1}{t - 2m e + m} + \frac{1}{t + 2m e} \right) + 4m \sum_{p=1}^\infty e^{-\frac{2pm}{g^2}} \left[ \frac{p^2 \left( \gamma + \log \frac{2m}{g^2} \pm \pi i \frac{1}{2} \right)}{t - \frac{2m}{g^2} \pm m} \right].$$  \hspace{1cm} (A8)

From this expanded form, we can read the expansion coefficients Eq. \[11\] and Eq. \[12\].

Appendix B: Multi-bion solutions

In this section we summarize basic properties of the multi-bion solution Eq. \[14\]

$$\varphi = e^{i\phi} \frac{f(\tau - \tau_0)}{\sin \alpha}, \hspace{1cm} \tilde{\varphi} = e^{-i\phi} \frac{f(\tau - \tau_0)}{\sin \alpha},$$  \hspace{1cm} (B1)

$$f(\tau) = \cos \Omega \tau, k,$$

where the parameters are related as

$$k^2 = 1 - \tan^2 \alpha \left( \cos^2 \alpha - \frac{m^2}{\Omega^2} \right),$$  \hspace{1cm} (B2)

$$\Omega = \omega \sqrt{1 - \left( 1 + \sec^2 \alpha \right) \left( 1 - \frac{m^2}{\omega^2} \sec^2 \alpha \right)},$$
with
\[ \omega = m \sqrt{1 + \frac{2qg^2}{m}}. \]  
(B3)

This is a periodic solution, whose period is given by
\[ \beta = \oint \frac{df}{\partial f} = \frac{1}{\Omega} \oint \frac{df}{\sqrt{(f^2 + 1)(f^2 + 1 - k^2)}}, \]  
(B4)

where we have used the relation \((\partial_f f)^2 = \Omega^2 (f^2 + 1)(f^2 + 1 - k^2)\). There are four branch points corresponding to the turning points \((\partial_{\theta f} \phi = \partial_{\theta f} \phi = 0)\).

\[ f = \pm i, \pm i \sqrt{1 - k^2}. \]  
(B5)

Let us introduce two branch cuts on the lines from \(\pm i\) to \(\pm i \sqrt{1 - k^2}\) on the complex \(f\)-plane. Let \(C_A\) be the cycle from \(\text{Re } f = -\infty\) to \(\text{Re } f = \infty\) which does not pass through the branch cuts and \(C_B\) be the cycle surrounding the two branch points \(\pm i \sqrt{1 - k^2}\). Their periods are
\[ \beta_A = \frac{2K(k)}{\Omega}, \quad \beta_B = \frac{4iK(\sqrt{1 - k^2})}{\Omega}, \]  
(B6)

where \(K(k) = F(\pi/2, k)\) is the complete elliptic integral of the first kind
\[ F(x, k) = \int_0^x \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}. \]  
(B7)

The period of the solution winding the cycle \(pC_A + qC_B\) \((p, q \in \mathbb{Z})\) is given by
\[ \beta = \frac{2pK(k) + 4qK(\sqrt{1 - k^2})}{\Omega}, \quad p, q \in \mathbb{Z}. \]  
(B8)

Solving this equation and Eq. (B3), we can determine the parameters \((\alpha, \Omega, k)\) for each pair of integers \((p, q)\). The \(\beta \to \infty\) limit of \((p, q) = (1, 0)\) solution is given by the known one-bion solution for infinite time interval with
\[ E = m \epsilon, \quad k = 1, \quad \cos \alpha = \frac{m}{\omega}, \quad \Omega = \omega. \]  
(B9)

We need the \(\beta \to \infty\) limit keeping \((p, q)\) fixed. Expanding the period with respect to \(\delta k = k - 1\), we find that
\[ \beta = \frac{1}{\omega} \left[ -p \log \left( \frac{\delta k}{\delta} \right) + 2\pi iq \right] + \mathcal{O}(\delta k \log \delta k). \]  
(B10)

Therefore, the asymptotic form of \(\delta k\) for large \(\beta\) is
\[ \delta k \approx -8 e^{-\frac{\omega \delta - 2\pi q \alpha}{\overline{g} + 2k}}. \]  
(B11)

We can also show that the asymptotic forms of other parameters are
\[ \delta \alpha \approx \left( \frac{4m^2}{\omega^2 - m^2} \right) \frac{3}{2} e^{-\frac{\omega \delta - 2\pi q \alpha}{\overline{g} + 2k}}. \]  
(B12)

\[ \delta \Omega \approx 8\omega \left( \frac{\omega^2 + m^2}{\omega^2 - m^2} \right) e^{-\frac{\omega \delta - 2\pi q \alpha}{\overline{g} + 2k}}, \]
\[ \delta E \approx \frac{8\omega^2}{g^2} \left( \frac{\omega^2}{\omega^2 - m^2} \right) e^{-\frac{\omega \delta - 2\pi q \alpha}{\overline{g} + 2k}}. \]

We read Eq. (13) from these equations. Note that Eq. (B10) implies that the solution exists only for \(0 \leq q \leq p\) in the large \(\beta\) limit.

The action for this solution is given by
\[ S_{\text{sol}} = \int_0^\beta d\tau L_{\text{sol}} = -m \epsilon \beta + \frac{\Omega}{2} \int df X(f), \]  
(B13)

where The function \(X(f)\) can be written as
\[ X = -\frac{\partial}{\partial f} \left( \omega^2 - m^2 \right) \left\{ \frac{F(x, k)}{\cos^2 \alpha} - \tan^2 \alpha \Omega(\cos^2 \alpha, x, k) \right\}, \]  
(B14)

\[ + E(x, k) - \sqrt{\frac{f^2 + 1 - k^2}{f^2 + 1}} \frac{f \cos^2 \alpha}{f^2 + \sin^2 \alpha} \]
\[ + F(y, k') - E(y, k') + i \sqrt{\frac{(f^2 + 1)(f^2 + 1 - k^2)}{f^2 + \sin^2 \alpha}}, \]

with \(x = \arcsin \sqrt{\frac{f^2 + 1}{1 - k^2}}\) and \(y = \arcsin \sqrt{\frac{f^2 + 1}{1 - k^2}}\) and \(k' = \sqrt{1 - k^2}\). Then we obtain
\[ X(f) = \frac{1}{\sqrt{(f^2 + 1)(f^2 + 1 - k^2)}} \left[ -\frac{1 - k^2}{\sin^2 \alpha} \right. \]
\[ + \left. (f^2 + 1)(f^2 + 1 - k^2) \frac{2 \sin^2 \alpha}{(f^2 + \sin^2 \alpha)^2} \right]. \]  
(B15)

There are contributions from \(C_A, C_B\) and the poles at \(f = \pm i \sin \alpha\) (more precisely, integration cycles should be defined on the torus with two punctures)
\[ S = -m \epsilon \beta + pS_A + qS_B + 2\pi iS_{\text{res}}, \quad l \in \mathbb{Z}. \]  
(B16)

Explicitly, \(S_{\text{res}} = \epsilon S_A \) and \(S_B\) are given by
\[ S_A = \frac{2\Omega}{g^2} \left( \frac{\omega^2 - m^2}{\omega^2} \right) \left\{ \frac{K(k)}{\cos^2 \alpha} - \tan^2 \alpha \Omega(\cos^2 \alpha, k) \right\}, \]
\[ + E(k), \]
\[ S_B = \frac{4i\Omega}{g^2} \left( \frac{\omega^2 - m^2}{\omega^2} \right) \left\{ \frac{K(k') - \Pi \left( \frac{1 - k^2}{\sin^2 \alpha}, k' \right)}{f^2 + \sin^2 \alpha} \right\} \]
\[ + K(k') - E(k') \],

(B17)

where \(E(k) = E(\frac{\pi}{2}, k)\) and \(\Pi(a, k) = \Pi(a, \frac{\pi}{2}, k)\) are the complete elliptic integrals of the second and third kind
\[ E(x, k) = \int_0^x d\theta \sqrt{1 - k^2 \sin^2 \theta}, \]  
(B18)

\[ \Pi(a, x, k) = \int_0^x d\theta \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} \frac{1}{1 - a \sin^2 \theta}, \]
and $k' = \sqrt{1-k^2}$. For large $\beta$,

$$S \approx -mc\beta + p \left[ \frac{2\omega}{g^2} + 2\epsilon \log \omega + m \right] + 2\pi i c,$$

from which we read Eq. (B9). This implies that the integer $p$ corresponds to the number of bions.

Focusing on the region around

$$\tau \approx \frac{n\beta}{p} \quad (n = 0, 1, \ldots, p - 1),$$

we can approximate the solution for large $\beta$ as

$$h(\tau) \approx \sqrt{\frac{\omega^2}{\omega^2 - m^2}} \left[ \sinh \left\{ \omega \left( \tau - \frac{n\beta}{p} \right) + \frac{2\pi inq}{p} \right\} \right]^{-1},$$

where we have used $\cosh(x) = 1/ \sinh x$. Therefore, the solution in this region looks like the single bion configuration

$$\varphi \approx \left( e^{\omega(\tau - y_n^+)} + e^{-\omega(\tau + y_n^-)} \right)^{-1},$$

$$\tilde{\varphi} \approx \left( e^{\omega(\tau - \tilde{y}_n^+)} + e^{-\omega(\tau + \tilde{y}_n^-)} \right)^{-1},$$

with

$$\omega y_n^\pm = \omega \tau_c + i\phi_c + \frac{n\omega}{p} - \frac{2\pi inq}{p} \pm \log \sqrt{\frac{4\omega^2}{\omega^2 - m^2}} \quad (\text{mod } 2\pi i),$$

$$\omega \tilde{y}_n^\pm = \omega \tau_c - i\phi_c + \frac{n\omega}{p} - \frac{2\pi inq}{p} \pm \log \sqrt{\frac{4\omega^2}{\omega^2 - m^2}} \quad (\text{mod } 2\pi i).$$

From this asymptotic form, we can read off the positions $\tau_n^\pm = (y_n^\pm + \tilde{y}_{n+1}^\pm) / 2$ and phases $\phi_n^\pm = (y_{n+1}^\pm - \tilde{y}_n^-) / 2i$, of the component kinks. The $n$-th kink (+) and antikink (-) locations Eq. (20) are given by

$$\tau_n^\pm = \tau_c + \frac{n - 1}{\omega p} \left( \omega^2 - 2\pi i q \right) \pm \frac{1}{2} \log \frac{4\omega^2}{\omega^2 - m^2} \quad (\text{mod } 2\pi i).$$

The poles of the Lagrangian are located at

$$\omega_{\text{pole},n}^\pm \approx \omega \tau_c + \frac{n\omega}{p} - \frac{2\pi inq}{p} \pm \arccosh \left( \frac{\omega^2}{\omega^2 - m^2} \right) \quad (\text{mod } \pi i).$$

These poles pass through the real $\tau$ axis for certain values of $\Im \tau_0$, at which the value of the action jumps discontinuously. When one of the poles, for example $\tau_{\text{pole},n^+}$, is on the real $\tau$ axis, then $\tau_{\text{pole},n^+}$ with $n^+ = n + kp / \gcd(p, 2q)$ ($k = 0, \ldots, \gcd(p, 2q) - 1$) are also on the real $\tau$ axis, where $\gcd(p, 2q)$ is the greatest common divisor of $p$ and $2q$. Therefore, the discontinuity of the action when the poles pass through the real $\tau$ axis is

$$\Delta S = \pm 2\pi i \gcd(p, 2q).$$

Appendix C: Multi-bion contributions

In this section we explicitly evaluate the quasi moduli integral for the chain of $p$ kinks and $p$ anti-kinks alternately aligned on $S^1$ with period $\beta$. The effective potential consists of the nearest neighbor interactions

$$V_{\text{eff}} = -mc\beta + \sum_{i=1}^{2p} \left( \frac{m}{g^2} + V_i \right),$$

where $V_i$ is the interaction potential

$$V_i = m \epsilon_i (\tau_i - \tau_{i-1}) - \frac{4m}{g^2} e^{-m(\tau_i - \tau_{i-1})} \cos(\phi_i - \phi_{i-1}),$$

where $(\tau_i, \phi_i)$ are quasi moduli parameters corresponding to the position and phase of the $i$-th (anti)kink ($\tau_0 = \tau_{2p} - \beta$, $\phi_0 = \phi_{2p} \mod 2\pi$) and

$$\epsilon_i = \begin{cases} 2\epsilon & \text{for } i \in \mathbb{Z}^2 \\ 0 & \text{for } i \in \mathbb{Z}^2 + 1 \end{cases}.$$ 

It is convenient to redefine the relative quasi moduli parameters as

$$z_i = m(\tau_i - \tau_{i-1}) + i(\phi_i - \phi_{i-1}),$$

$$\tilde{z}_i = m(\tau_i - \tau_{i-1}) - i(\phi_i - \phi_{i-1}).$$

Note that the imaginary parts of $z_i$ and $\tilde{z}_i$ are phases defined modulo $2\pi$. The complex variables $z_i$ and $\tilde{z}_i$ are subject to the following constraints

$$\sum_{i=1}^{2p} \frac{z_i + \tilde{z}_i}{2} = m\beta, \quad \sum_{i=1}^{2p} \frac{z_i - \tilde{z}_i}{2i} = 0 \quad (\text{mod } 2\pi),$$

which are expressed by the integral forms of delta functions as functions of $\sigma$ and $s$

$$\delta \left( \sum_{i=1}^{2p} \frac{z_i + \tilde{z}_i}{2} - m\beta \right) = \int \frac{d\sigma}{2\pi} \exp \left[ i\sigma \left( \sum_{i=1}^{2p} \frac{z_i + \tilde{z}_i}{2} - m\beta \right) \right],$$

$$\sum_{n=-\infty}^{\infty} \delta \left( \sum_{i=1}^{2p} \frac{z_i - \tilde{z}_i}{2i} - 2\pi n \right) = \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} \exp \left( s \tilde{z}_i - \tilde{z}_i \right).$$

The saddle points ($q = 0, 1, \ldots, p - 1$) which give non-trivial contributions to the ground state energy are lo-
cated at
\[
\begin{aligned}
\bar{z}_i = \left\{ \begin{array}{l}
- \log \left( \frac{\epsilon g^2}{4m} \pi^2 + e^{-\frac{m\beta - 2\pi i q}{p}} \right) \approx \\
\log \frac{2m}{g^2} \quad \text{(for } i \in 2\mathbb{Z})
\end{array} \right.
\end{aligned}
\]
\[
- \log \left( \frac{\epsilon g^2}{4m} \pi^2 + e^{-\frac{m\beta - 2\pi i q}{p}} \right) \approx \\
\log \frac{2m}{g^2} + \frac{m\beta - 2\pi i q}{p} \quad \text{(for } i \in 2\mathbb{Z} + 1)
\]
\]
(C7)

We note that the Lagrange multiplier \( \sigma \) is expressed in terms of the other parameters on the saddle points. This is consistent with the weak coupling limit \( \gamma^2 \to 0 \) of Eq. (20) with \( z_{2n}/\omega = \tau_n^* - \tau_n \) and \( z_{2n+1}/\omega = \tau_{n+1}^* - \tau_n^* \).

The \( p \)-bion contribution to the partition function is given by
\[
\frac{Z_p}{Z_0} = \frac{1}{p} \prod_{i=1}^{2p} \left[ d\tau_i \wedge d\phi_i \pi^2 g^2 \exp \left( \frac{-m\gamma^2}{\pi^2} - V_i \right) \right],
\]
where the factor \( \frac{2m^2}{g^2} \) is the 1-loop determinant from the massive modes around each kink and the factor \( 1/p \) is inserted since the bions are indistinguishable. The integration measure can be rewritten as
\[
\prod_{i=1}^{2p} d\tau_i \wedge d\phi_i = m d\tau_c \wedge d\phi_c \wedge \left( \prod_{i=1}^{2p} \frac{i}{2m} dz_i \wedge d\bar{z}_i \right)
\times \delta \left( \sum_{i=1}^{2p} \frac{z_i + \bar{z}_i}{2} - m\beta \right) \delta \left( \sum_{i=1}^{2p} \frac{z_i - \bar{z}_i}{2} - 2\pi n \right),
\]
(C9)

where \( \tau_c \) and \( \phi_c \) are the overall moduli parameters. We can rewrite the \( p \)-bion contribution as
\[
\frac{Z_p}{Z_0} = \frac{2m\beta}{\pi g^2} e^{-\frac{2m\gamma^2}{\pi^2}} \sum_{s=-\infty}^{\infty} \frac{1}{4\pi^2} \int d\sigma \, e^{-i m\beta \sigma} \prod_{i=1}^{2p} I_i,
\]
(C10)

where
\[
I_i = \left. \frac{im}{\pi g^2} \int dz_i \wedge d\bar{z}_i \exp(-V_i) \exp(-\bar{V}_i) \right|_{s=0},
\]
(C11)

with
\[
V_i = -\frac{2m}{g^2} e^{-z_i} + \frac{1}{2}(\epsilon_i - s - i\sigma)z_i,
\]
\[
\bar{V}_i = -\frac{2m}{g^2} e^{-\bar{z}_i} + \frac{1}{2}(\epsilon_i + s - i\sigma)\bar{z}_i.
\]
(C12)

We can show that the \( p \)-bion contribution satisfies the following differential equation
\[
\prod_{i=1}^{2p} \left( s^2 - (\epsilon_i - i\sigma)^2 \right) \left( \frac{2m}{g^2} \right)^{4p} e^{-2m\beta} \left( \frac{1}{Z_p} \right) \approx 0,
\]
\[
\bar{\sigma} = \frac{i}{m} \partial \beta.
\]
(C13)

There are \( 4p \) linearly independent solutions, whose asymptotic forms for large \( \beta \) are given by
\[
\frac{Z_p}{Z_0} \approx \beta^p e^{-(2\pi s x)\beta} \text{ or } \beta^p e^{(2\pi s x)\beta},
\]
(C14)

where we have used the reflection formula for the gamma function
\[
\sin(\pi x) \Gamma(x) = \frac{\pi}{\Gamma(1 - x)}.
\]
(C19)

Then, the contour integral for the \( p \)-bion contribution
\[
\frac{Z_p}{Z_0} \approx \frac{m\beta}{p} e^{-\frac{2m\pi^2}{\gamma^2}} \int d\sigma \, e^{-i\sigma m\beta} \prod_{i=1}^{2p} I_i |_{s=0},
\]
(C20)
can be evaluated by picking up the poles at \( \sigma = -2ik \) and \( \sigma = -2i(\epsilon + k) \) \((k \in \mathbb{Z}_{\geq 0})\). In the \( \beta \to \infty \) limit, the \( p \)-th order pole at \( \sigma = 0 \) gives the leading order term Eq. (20)

\[
\frac{Z_p}{Z_0} \approx -\frac{im\beta}{p} e^{-\frac{2m}{\sigma^2}} \text{Res}_{\sigma=0} \left[ e^{-im\sigma} \prod_{i=1}^{2p} I_i|_{\sigma=0} \right]
\]

\[
= -\frac{im\beta}{p!} e^{-\frac{2m}{\sigma^2}} \lim_{\sigma \to 0} \left( \frac{\partial}{\partial \sigma} \right)^{p-1} \left[ \frac{8i m^2}{g^4} e^{-\frac{i m \sigma}{\sigma^2}} \frac{2m}{g^2} e^{\frac{i \epsilon}{2}} \right]^{2(i\sigma - \epsilon)} \times \left[ \frac{\Gamma \left( 1 - \frac{i \epsilon}{2} \right) \Gamma \left( 1 - \frac{i \epsilon}{2} \right)}{\Gamma \left( 1 - \epsilon + \frac{i \sigma}{2} \right) \Gamma \left( 1 + \frac{i \sigma}{2} \right)} \right]^p.
\]

(C21)

The leading order term Eq. (28) is

\[
\frac{Z_p}{Z_0} \approx \frac{1}{p!} \left[ \frac{2m\beta \Gamma(\epsilon)}{\Gamma(1-\epsilon)} \right] e^{-\frac{2m}{\sigma^2}} \left[ \frac{2m}{g^2} \right]^{2(1-\epsilon)}. \quad (C22)
\]

This is consistent with the dilute gas approximation. In the supersymmetric case \( \epsilon = 1 \), \( Z_p/Z_0 \) vanishes due to the factor \( 1/\Gamma(1-\epsilon) \). In the near SUSY case, we obtain

\[
\lim_{\epsilon \to 1} \frac{\partial}{\partial \epsilon} \frac{Z_p}{Z_0} \approx 2m\beta e^{-\frac{2m}{\sigma^2}}, \quad (C23)
\]

where we have used

\[
\lim_{\epsilon \to 1} \frac{\partial}{\partial \epsilon} \frac{1}{\Gamma(1-\epsilon)} = 1. \quad (C24)
\]

Then we obtain

\[
\lim_{\epsilon \to 1} \frac{\partial}{\partial \epsilon} E = \lim_{\epsilon \to 1} \frac{\partial}{\partial \epsilon} \left[ E_{\text{pert}} - \lim_{\beta \to \infty} \frac{1}{\beta} \log \left( 1 + \sum_{p=1}^{\infty} \frac{Z_p}{Z_0} \right) \right]
\]

\[
= \lim_{\epsilon \to 1} \frac{\partial}{\partial \epsilon} E_{\text{pert}} - \sum_{p=1}^{\infty} e^{-\frac{2m}{\sigma^2}} (2m + O(g^2)).
\]

(C25)

This is consistent with the exact result. Using the relation

\[
\frac{1}{p!} \lim_{\epsilon \to 1} \frac{\partial^2}{\partial \sigma^2} \lim_{\sigma \to 0} \left( \frac{X}{\Gamma(1-\epsilon + i\sigma/2)} \right)^p
\]

\[
= \lim_{\epsilon \to 1} \lim_{\sigma \to 0} \left( \frac{i}{2} \right)^{p-1} \left( \frac{X}{\Gamma(1-\epsilon + i\sigma/2)} \right)^p \left[ (p+1)\gamma - 2(p-1)i\sigma - 2\partial_{\sigma} \right] X,
\]

we can show that

\[
\frac{1}{2} \lim_{\epsilon \to 1} \frac{\partial^2}{\partial \epsilon^2} \frac{Z_p}{Z_0} \approx -2m\beta e^{-\frac{2m}{\sigma^2}} \left[ 2p^2 \left( \gamma + \log \frac{2m}{g^2} \pm \frac{\pi i}{2} \right) - (p-1)m\beta \right].
\]

Therefore, the second order coefficient of the ground state energy in Eq. (20) is given by

\[
\frac{1}{2} \lim_{\epsilon \to 1} \frac{\partial^2}{\partial \epsilon^2} E = \frac{1}{2} \lim_{\beta \to \infty} \frac{1}{\beta} \left( -\frac{1}{\beta} \frac{\partial^2}{\partial \sigma^2} Z - (\partial_{\sigma} Z)^2 \right)
\]

\[
= \sum_{p} 4me^{-\frac{2m}{\sigma^2}} p^2 \left[ \gamma + \log \frac{2m}{g^2} \pm \frac{\pi i}{2} + O(g^2) \right]. \quad (C28)
\]

Appendix D: Perturbation series on trivial vacuum

In this section we derive the perturbative part of the ground state energy by using the Bender-Wu method. Since the ground state is invariant under the phase rotation \( \varphi \to e^{i\phi} \varphi \), the corresponding wave function \( \Psi \) a function of \( |\varphi| \). By redefining the wave function and the coordinate as

\[
\Psi = e^{-x^2} \psi(x), \quad |\varphi| = \eta x, \quad \eta \equiv \frac{g}{\sqrt{m}}. \quad (D1)
\]

The Schrödinger equation can be rewritten as

\[
m \left[ -\frac{1}{4}(1 + \eta^2 x^2)^2 \left\{ \frac{\partial^2}{\partial x^2} + (1 - 4x^2)^2 \frac{1}{x \partial x} \right\} \right] + V(x) \psi = E \psi, \quad (D2)
\]

where the potential is

\[
V(x) = (1 - x^2)(1 + \eta^2 x^2)^2 + \frac{x^2}{(1 + \eta^2 x^2)^2} - \frac{1}{1 + \eta^2 x^2}. \quad (D3)
\]

Let us expand the energy and the wave function with respect to \( \eta \)

\[
\frac{E}{m} = \sum_{l=0}^{\infty} A_l \eta^{2l}, \quad \psi = \sum_{l=0}^{\infty} \psi_l(x) \eta^{2l}. \quad (D4)
\]

Then, the Schrödinger equation \( \left( \hat{H} - E \right) \psi = 0 \) can be expanded as

\[
0 = \frac{1}{4} \sum_{i=0}^{4} \left( \frac{4}{i} \right) x^{2i} \left[ \psi''_{l-i} + (1 - 4x^2)^2 \frac{1}{x} \psi'_{l-i} \right.
\]

\[
\left. - 4(1 - x^2) \psi_{l-i-1} \right] + \sum_{l=0}^{l} A_l \left( \psi_{l-i} + 2x^2 \psi_{l-i-1} + x^4 \psi_{l-i-2} \right)
\]

\[
+ (\epsilon - x^2) \psi_{l-2} - x^4 \epsilon \psi_{l-2}, \quad (D5)
\]

where \( \psi_l = 0 \) for \( l < 0 \). Setting \( \psi_0 = 1 \), we can solve these equations order by order. It is not difficult to show that \( \psi_l \) are polynomials of the form

\[
\psi_l = \sum_{k=0}^{l} B_{l,k} x^{2k}. \quad (D6)
\]
We can always fix the normalization of the wave function as \( \Psi(x = 0) = 1 \), i.e. \( B_{0,0} = 1 \), \( B_{l,0} = 0 \) \((l \neq 0)\). The Schrödinger equation reduces to

\[
0 = 4 \sum_{i=0}^{4} \binom{4}{i} [(k-i+1)^2 B_{l-i,k-i+1} - (2k-2i+1) B_{l-i,k-i} + B_{l-i,k-i-1}] \\
+ \sum_{i=1} A_i (B_{l-i,k} + 2B_{l-i-1,k-1} + B_{l-i-2,k-2}) \\
- B_{l,k-1} + \epsilon(B_{l,k} - B_{l,k-2}),
\]

where \( B_{l,k} = 0 \) if \( l < 0, k < 0, k > 2l \). As shown in Fig. 3, the asymptotic behavior Eq. (44) for \( 0 < \epsilon < 2 \) is consistent with

\[
A_l \sim -\frac{1}{2^{l-1}} \frac{\Gamma(l+2(1-\epsilon))}{\Gamma(1-\epsilon^2)}.
\]

The Borel resummation of the right hand side gives

\[
\text{Borel} \equiv -\frac{g^2}{\Gamma(1-\epsilon)^2} \int_0^\infty dt e^{-t} \sum_{l=0}^\infty t^{2(1-\epsilon)} \left( \frac{\eta^2}{2} t \right)^l \\
= \frac{2m}{\Gamma(1-\epsilon)^2} \int_0^\infty dt e^{-t} \frac{t^{2(1-\epsilon)}}{t - \frac{2m}{\eta^2}}.
\]

Therefore the imaginary ambiguity Eq. (22) from the perturbative part is

\[
\text{Im} E_0 = \mp \frac{2\pi m}{\Gamma(1-\epsilon)^2} \left( \frac{g^2}{2m} \right)^{2(\epsilon-1)} e^{-\frac{2m}{\eta^2}}.
\]

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