Borel-Carathéodory Type Theorem for monogenic functions

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Abstract
In this paper we give a generalization of the classical Borel-Carathéodory
Theorem in complex analysis to higher dimensions in the framework of
Quaternionic Analysis.
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1 Introduction

The Borel-Carathéodory Theorem is a well known theorem about analytic func-
tions on the unit disc in the complex plane. It states that an analytic function
is essentially bounded by its real part, the proof being based on the maximum
modulus principle.

Theorem 1.1. (Borel-Carathéodory) Let a function $f$ be analytic on a
closed disk of radius $R$ centered at the origin. Suppose that $r < R$. Then,
we have the following inequality

$$
\|f\|_r \leq \frac{2r}{R-r} \sup_{|z| \leq R} \Re f(z) + \frac{R+r}{R-r}|f(0)|.
$$

We recall that the norm on the left-hand side is defined as the maximum
value of $|f(z)|$ in the closed disk, that is

$$
\|f\|_r = \max_{|z| \leq r} |f(z)| = \max_{|z|=r}|f(z)|.
$$

Since the concept of an analytic, or holomorphic, function in the complex
plane is replaced, in higher dimensions, by the one of monogenic function, it is
natural to ask whether this theorem can be generalized to monogenic functions
on a ball in the Euclidean space $\mathbb{R}^n$. In this paper we present a generalization
of this theorem to monogenic quaternionic functions.

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2 Preliminaries

Let \( \{e_0, e_1, e_2, e_3\} \) be an orthonormal basis of the Euclidean vector space \( \mathbb{R}^4 \). We consider \( e_0 \) to be the real scalar unit and \( e_1, e_2, e_3 \) the imaginary units. We introduce a multiplication of the basis vectors \( e_i \) subject to the following multiplication rules

\[
e_i e_j + e_j e_i = -2\delta_{i,j} e_0, \quad i, j = 1, 2, 3
\]

\[
e_0 e_i = e_i e_0 = e_i, \quad i = 0, 1, 2, 3.
\]

This non-commutative product, together with the extra condition \( e_1 e_2 = e_3 \), generates the algebra of real quaternions denoted by \( \mathbb{H} \). The real vector space \( \mathbb{R}^4 \) will be embedded in \( \mathbb{H} \) by means of the identification

\[
a := (a_0, a_1, a_2, a_3) \in \mathbb{R}^4 \quad \text{with} \quad a = a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \in \mathbb{H},
\]

where \( a_i (i = 0, 1, 2, 3) \) are real numbers. Remark that the vector \( e_0 \) is the multiplicative unit element of \( \mathbb{H} \). From now on, we will identify \( e_0 \) with 1.

We denote by \( \Re(a) := a_0 \) the scalar part of \( a \) and by \( \text{Vec} a := a_1 e_1 + a_2 e_2 + a_3 e_3 \) the vector part of \( a \). As in the complex case, the conjugate element of \( a \) is the quaternion \( \overline{a} := a_0 - a_1 e_1 - a_2 e_2 - a_3 e_3 \). The norm of \( a \) is given by \( |a| = \sqrt{a \overline{a}} \) and coincides with the corresponding Euclidean norm of \( a \), as vector in \( \mathbb{R}^4 \).

Let us consider the subset \( \mathcal{A} := \text{span}_{\mathbb{R}} \{1, e_1, e_2\} \) of \( \mathbb{H} \). The real vector space \( \mathbb{R}^3 \) is to be embedded in \( \mathcal{A} \) via the identification of each element \( x = (x_0, x_1, x_2) \in \mathbb{R}^3 \) with the reduced quaternion

\[
x = x_0 + x_1 e_1 + x_2 e_2 \in \mathcal{A}.
\]

As a consequence, no distinction will be made between \( x \) as point in \( \mathbb{R}^3 \) or its correspondent reduced quaternion. Also, we emphasize that \( \mathcal{A} \) is a real vectorial subspace, but not a sub-algebra, of \( \mathbb{H} \).

Let now \( \Omega \) be an open subset of \( \mathbb{R}^3 \) with piecewise smooth boundary. A quaternion-valued function on \( \Omega \) is a mapping \( f : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{H} \), with \( f(x) = \sum_{i=0}^{3} f_i(x) e_i \), where the coordinate-functions \( f_i \) are real-valued functions in \( \Omega \), \( i = 0, 1, 2, 3 \). Properties such as continuity, differentiability or integrability are ascribed coordinate-wisely.

We introduce the first order operator

\[
D = \partial x_0 + e_1 \partial x_1 + e_2 \partial x_2
\]

acting on \( C^1 \) functions. This operator will be denoted as generalized Cauchy-Riemann operator on \( \mathbb{R}^3 \). The corresponding conjugate generalized Cauchy-Riemann operator is defined ad

\[
\overline{D} = \partial x_0 - e_1 \partial x_1 - e_2 \partial x_2.
\]
A function \( f : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{H} \) of class \( C^1 \) is said to be left (resp. right) monogenic in \( \Omega \) if it verifies

\[
Df = 0 \quad \text{in} \quad \Omega \quad (\text{resp.}, \quad fD = 0 \quad \text{in} \quad \Omega).
\]

The generalized Cauchy-Riemann operator \( \mathbf{1} \) and its conjugate \( \mathbf{2} \) factorize the Laplace operator in \( \mathbb{R}^3 \). In fact, it holds

\[
\Delta_3 = DD = \overline{DD}
\]

and it implies that every monogenic function is also harmonic.

At this point we would like to remark that, in general, left (resp. right) monogenic functions are not right (resp. left) monogenic. From now on, we refer only to left monogenic functions. For simplicity, we will call them monogenic. However, all results achieved to left monogenic functions can also be adapted to right monogenic functions.

Through the remaining of this paper, we will consider the following notations: \( B := B_1(0) \) will denote the unit ball in \( \mathbb{R}^3 \) centered at the origin, \( S = \partial B \) its boundary and \( d\sigma \) the Lebesgue measure on \( S \). In what follows, we will denote by \( L_2(S;X;F) \) (resp. \( L_2(B;X;F) \)) the \( F \)-linear Hilbert space of square integrable functions on \( S \) (resp. \( B \)) with values in \( X \) (\( X = \mathbb{R} \) or \( A \) or \( \mathbb{H} \)), where \( F = \mathbb{H} \) or \( \mathbb{R} \). For any \( f, g \in L_2(S;A;\mathbb{R}) \) the real-valued inner product is given by

\[
\langle f, g \rangle_{L_2(S)} = \int_S \Re(fg)d\sigma. \tag{3}
\]

Each homogeneous harmonic polynomial \( P_n \) of order \( n \) can be written in spherical coordinates as

\[
P_n(x) = r^n P_n(\omega), \quad \omega \in S, \tag{4}
\]

its restriction, \( P_n(\omega) \), to the boundary of the unit ball is called spherical harmonic of degree \( n \). From \( \mathbf{4} \), it is clear that a homogeneous polynomial is determined by its restriction to \( S \). Denoting by \( \mathcal{H}_n(S) \) the space of real-valued spherical harmonics of degree \( n \) in \( S \), it is well-known (see \( \mathbf{3} \) and \( \mathbf{10} \)) that

\[
\dim \mathcal{H}_n(S) = 2n + 1.
\]

It is also known (see \( \mathbf{3} \) and \( \mathbf{16} \)) that if \( n \neq m \), the spaces \( \mathcal{H}_n(S) \) and \( \mathcal{H}_m(S) \) are orthogonal in \( L_2(S;\mathbb{R};\mathbb{R}) \).

Homogeneous monogenic polynomial of degree \( n \) will be denoted in general by \( H_n \). In an analogously way to the spherical harmonics, the restriction of \( H_n \) to the boundary of the unit ball is called spherical monogenic of degree \( n \). We denote by \( \mathcal{M}_n(\mathbb{H};F) \) the subspace of \( L_2(B;\mathbb{H};F) \cap \ker D(B) \) of all homogeneous monogenic polynomials of degree \( n \). Sudbery proved in \( \mathbf{17} \) that the dimension of \( \mathcal{M}_n(\mathbb{H};\mathbb{H}) \) is \( n + 1 \). In \( \mathbf{5} \), it is proved that the dimension of \( \mathcal{M}_n(\mathbb{H};\mathbb{R}) \) is \( 4n + 4 \).
3 Homogeneous Monogenic Polynomials

In [5] and [6], \( \mathbb{R} \)-linear and \( \mathbb{H} \)-linear complete orthonormal systems of \( \mathbb{H} \)-valued homogeneous monogenic polynomials in the unit ball of \( \mathbb{R}^3 \) are constructed. The main idea of these constructions is based on the factorization of the Laplace operator. We take a system of real-valued homogeneous harmonic polynomials and we apply the \( D \) operator in order to obtain systems of \( \mathbb{H} \)-valued homogeneous monogenic polynomials. For an easier description, we introduce the spherical coordinates

\[
x_0 = r \cos \theta, \quad x_1 = r \sin \theta \cos \varphi, \quad x_2 = r \sin \theta \sin \varphi,
\]

where \( 0 < r < \infty \), \( 0 < \theta \leq \pi \), \( 0 < \varphi \leq 2\pi \). Each point \( x = (x_0, x_1, x_2) \in \mathbb{R}^3 \) admits a unique representation \( x = rw \), where \( r = |x| \) and \( |w| = 1 \). Therefore, \( w_i = \frac{x_i}{r} \) for \( i = 0, 1, 2 \). We will apply the operator \( \frac{i}{2} \overline{D} \) to each homogeneous harmonic polynomial of the family

\[
\{r^{n+1}U_{n+1}^0, r^{n+1}U_{n+1}^m, r^{n+1}V_{n+1}^m, m = 1, \ldots, n+1\}_{n \in \mathbb{N}_0}, \tag{5}
\]

in order to obtain a system of spherical monogenic polynomials.

The elements of the previous family (5) are homogenous extensions to the ball of the spherical harmonics (see e.g. [18]),

\[
\begin{align*}
U_{n+1}^0(\theta, \varphi) &= P_{n+1}(\cos \theta) \\
U_{n+1}^m(\theta, \varphi) &= P_{n+1}^m(\cos \theta) \cos m\varphi \\
V_{n+1}^m(\theta, \varphi) &= P_{n+1}^m(\cos \theta) \sin m\varphi, m = 1, \ldots, n+1.
\end{align*}
\] \tag{6}

Here, \( P_{n+1} \) stands for the standard Legendre polynomial of degree \( n+1 \), while the functions \( P_{n+1}^m \) are its associated Legendre functions,

\[
P_{n+1}^m(t) := (1 - t^2)^{m/2} \frac{d^m}{dt^m} P_{n+1}(t), \quad m = 1, \ldots, n+1. \tag{7}
\]

Notice that the Legendre polynomials together with the associated Legendre functions satisfy several recurrence formulae. We point out only the ones necessary for what follows in the next section. Following [2], Legendre polynomials and its associated Legendre functions satisfy the recurrence formulae

\[
(1 - t^2)(P_{n+1}^m(t))' = (n + m + 1)P_{n+1}^m(t) - (n + 1)tP_{n+1}^m(t), \quad m = 0, \ldots, n+1, \tag{8}
\]

and

\[
P_{m}(t) = (2m - 1)!!(1 - t^2)^{m/2}, \quad m = 1, \ldots, n+1. \tag{9}
\]

Finally, these functions are mutually orthogonal in \( L_2([-1, 1]) \), that is,

\[
\int_{-1}^1 P_{n+1}^m(t)P_{k+1}^m(t)dt = 0, \quad n \neq k
\]
and their $L_2$-norms are given by

\[ \int_{-1}^{1} (P_{n+1}^m(t))^2 dt = \frac{2}{2n + 3} \frac{(n + 1 + m)!}{(n + 1 - m)!}, \quad m = 0, \ldots, n + 1. \]  

(10)

For a detailed study of Legendre polynomials and associated Legendre functions we refer, for example, [2] and [18].

Restricting the functions of the set (5) to the sphere, we obtain the spherical monogenics

\[ X_0^n, X_m^n, Y_m^n, m = 1, \ldots, n + 1, \]  

(11)

given by

\[ X_0^n := \left( \frac{1}{2} \right) (r^{n+1} U_0^{n+1}) \bigg|_{r=1} = A_0^{0,n} + B_0^{0,n} \cos \varphi e_1 + B_0^{0,n} \sin \varphi e_2, \]  

where

\[ A_0^{0,n} = \frac{1}{2} \left( \sin^2 \theta \frac{d}{dt} \left[ P_{n+1}^m(t) \right]_{t=\cos \theta} + (n + 1) \cos \theta P_{n+1}^m(\cos \theta) \right) \]  

(13)

\[ B_0^{0,n} = \frac{1}{2} \left( \sin \theta \cos \theta \frac{d}{dt} \left[ P_{n+1}^m(t) \right]_{t=\cos \theta} - (n + 1) \sin \theta P_{n+1}^m(\cos \theta) \right), \]  

(14)

while for the remaining polynomials we have

\[ X_m^n := \left( \frac{1}{2} \right) (r^{n+1} U_m^{n+1}) \bigg|_{r=1} = A_m^{n,m} \cos(m \varphi) \]  

\[ + (B_m^{m,n} \cos \varphi \cos m \varphi - C_m^{n,n} \sin \varphi \sin m \varphi) e_1 \]  

\[ + (B_m^{m,n} \sin \varphi \cos m \varphi + C_m^{n,n} \cos \varphi \sin m \varphi) e_2 \]  

(15)

\[ Y_m^n := \left( \frac{1}{2} \right) (r^{n+1} V_m^{n+1}) \bigg|_{r=1} = A_m^{n,m} \sin(m \varphi) \]  

\[ + (B_m^{m,n} \sin \varphi \sin m \varphi + C_m^{n,n} \cos \varphi \cos m \varphi) e_1 \]  

\[ + (B_m^{m,n} \cos \varphi \sin m \varphi - C_m^{n,n} \sin \varphi \cos m \varphi) e_2 \]  

(16)

with

\[ A_m^{n,m} = \frac{1}{2} \left( \sin^2 \theta \frac{d}{dt} \left[ P_{n+1}^{m}(t) \right]_{t=\cos \theta} + (n + 1) \cos \theta P_{n+1}^{m}(\cos \theta) \right) \]  

\[ B_m^{m,n} = \frac{1}{2} \left( \sin \theta \cos \theta \frac{d}{dt} \left[ P_{n+1}^{m}(t) \right]_{t=\cos \theta} - (n + 1) \sin \theta P_{n+1}^{m}(\cos \theta) \right) \]  

\[ C_m^{n,n} = \frac{1}{2} \frac{1}{\sin \varphi} P_{n+1}^{m}(\cos \varphi), \]
for $m = 1, ..., n + 1$.

For each fixed $n \in \mathbb{N}_0$, we obtain the set of homogeneous monogenic polynomials

$$\{ r^n X^0_n, \ r^n X^m_n, \ r^n Y^m_n : m = 1, ..., n + 1 \}$$

by taking the homogeneous monogenic extension of the previous spherical monogenics into the ball.

For future use in this paper we will need norm estimates of the spherical monogenics described in (11) and of its real part.

**Proposition 3.1.** (see [13]) For $n \in \mathbb{N}$ the homogeneous monogenic polynomials satisfy the following inequalities:

$$|r^n X^0_n(x)| \leq r^n (n + 1)^2 \frac{\pi (n + 1)}{2n + 3}$$

$$|r^n X^m_n(x)| \leq r^n (n + 1)^2 \frac{\pi (n + 1) (n + m)!}{2 (2n + 3) (n + 1 - m)!}$$

$$|r^n Y^m_n(x)| \leq r^n (n + 1)^2 \frac{\pi (n + 1) (n + 1 + m)!}{2 (2n + 3) (n + 1)!} \frac{(n + 1 + m)! (n + 1)!}{(n + 1 - m)!}$$

with $m = 1, ..., n + 1$.

**Proposition 3.2.** Given a fixed $n \in \mathbb{N}_0$, the norms of the spherical harmonics $\Re (X^0_n)$, $\Re (X^m_n)$ and $\Re (Y^m_n)$ are given by

$$\| \Re (X^0_n) \|_{L^2(S)} = (n + 1) \sqrt{\frac{\pi}{2n + 1}}$$

and

$$\| \Re (X^m_n) \|_{L^2(S)} = \| \Re (Y^m_n) \|_{L^2(S)} = \sqrt{\frac{\pi (n + 1 + m)(n + 1 + m)!}{2 (2n + 1) (n - m)!}}$$

for $m = 1, ..., n + 1$.

**Proof.** For simplicity sake, we just present the proof for the case of $\Re (X^0_n)$, the proof for $\Re (X^m_n)$ and $\Re (Y^m_n)$ being similar.

Using the definition of real-valued inner product [8] and [12-14], we get

$$\| \Re (X^0_n) \|_{L^2(S)}^2 = \frac{\pi}{2} \int_0^\pi \sin^4 \theta \left( \frac{d}{dt} [P_{n+1}(t)]_{t=\cos \theta} \right)^2 + (n + 1)^2 \cos^2 \theta (P_{n+1}(\cos \theta))^2$$

$$+ 2 \sin^2 \theta (n + 1) \cos \theta \frac{d}{dt} [P_{n+1}(t)]_{t=\cos \theta} P_{n+1}(\cos \theta) \right) \sin \theta d\theta.$$

6
Making the change of variable $t = \cos \theta$ and using the recurrence formula, the last expression becomes

$$\| \Re(X_0^n) \|^2_{L_2(S)} = \frac{\pi}{2} \int_{-1}^{1} (1 - t^2)^2 (P_{n+1}'(t))^2 dt - (n+1)^2 \int_{-1}^{1} t^2 (P_{n+1}(t))^2 dt$$

$$+ 2(n+1)^2 \int_{-1}^{1} t P_n(t) P_{n+1}(t) dt$$

$$= \frac{\pi}{2} (n+1)^2 \int_{-1}^{1} (P_n(t))^2 dt.$$  

Due to (10) we get

$$\| \Re(X_0^n) \|^2_{L_2(S)} = \frac{\pi (n+1)^2}{2n+1}.$$  

\[\blacksquare\]

**Proposition 3.3.** Given a fixed $n \in \mathbb{N}_0$, the spherical harmonics $\Re(X_{n+1}^n e_1)$ and $\Re(Y_{n+1}^n e_1)$ are orthogonal to each other (w. r. t. (8)). Moreover, their norms satisfy

$$\| \Re(X_{n+1}^n e_1) \|_{L_2(S)} = \| \Re(Y_{n+1}^n e_1) \|_{L_2(S)} = \frac{1}{2} \sqrt{\pi (n+1)(2n+2)!}. $$

**Proof.** Again, we only present the proof for the spherical harmonics $\Re\{X_{n+1}^n e_1\}$, the one for $\Re\{Y_{n+1}^n e_1\}$ being similar. Using (see [15])

$$\Re\{X_{n+1}^n e_1\} = \frac{n+1}{2} \frac{1}{\sin \theta} P_{n+1}^n(\cos \theta) \cos n\varphi,$$

the definition of real-valued inner product and (12) and (14), we obtain

$$\| \Re(X_0^n e_1) \|_{L_2(S)}^2 = \pi \left( \frac{n+1}{2} \right)^2 \int_{0}^{\pi} \frac{1}{\sin \theta} (P_{n+1}^n(\cos \theta))^2 d\theta.$$  

We make the change of variable $t = \cos \theta$ and, by (9), we get

$$\| \Re(X_0^n e_1) \|_{L_2(S)}^2 = \pi \left( \frac{n+1}{2} \right)^2 \int_{-1}^{1} (1 - t^2)^{-1} (P_{n+1}^n(t))^2 dt.$$  

Now, due to the equality (10) we finally get

$$\| \Re(X_0^n e_1) \|_{L_2(S)}^2 = \frac{\pi}{4} (n+1)(2n+2)!.$$  

\[\blacksquare\]
4 Borel-Carathéodory’s Theorem

We will denote by $X^0_n, X^m_n, Y^m_n$ the normalized basis functions in $L^2(S; H; H)$.

**Theorem 4.1.** (see [5]) Let $M_n(\mathbb{R}^3; A)$ be the space of $A$-valued homogeneous monogenic polynomials of degree $n$ in $\mathbb{R}^3$. For each $n$, the set of $2n + 3$ homogeneous monogenic polynomials

$$\{\sqrt{2n + 3}^n X^0_n, \sqrt{2n + 3}^n X^m_n, \sqrt{2n + 3}^n Y^m_n : m = 1, ..., n + 1\}$$  \hspace{1cm} (18)

forms an orthonormal basis in $M_n(\mathbb{R}^3; A)$.

According to this theorem, a monogenic $L^2$-function $f : \Omega \subset \mathbb{R}^3 \rightarrow A$ can be decomposed into

$$f = f(0) + f_1 + f_2$$  \hspace{1cm} (19)

where the components $f_1$ and $f_2$ have Fourier series

$$f_1(x) = \sum_{n=1}^{\infty} \sqrt{2n + 3}^n r^n \left( X^0_n(x) \alpha^0_n + \sum_{m=1}^{n} [X^m_n(x) \alpha^m_n + Y^m_n(x) \beta^m_n] \right)$$

$$f_2(x) = \sum_{n=1}^{\infty} \sqrt{2n + 3}^n r^n \left[ X^{n+1}_n(x) \alpha^{n+1}_n + Y^{n+1}_n(x) \beta^{n+1}_n \right].$$

Moreover, we remark that the associated Fourier coefficients are real-valued.

In what follows, we prove that a monogenic $L^2$-function $f : \Omega \subset \mathbb{R}^3 \rightarrow A$ function can be bounded by its real part. For this purpose, we must find relations between the Fourier coefficients and the real part of $f$.

**Lemma 4.1.** Given a fixed $n \in \mathbb{N}_0$, the spherical harmonics

$$\{\Re(X^0_n), \Re(X^m_n), \Re(Y^m_n) : m = 1, ..., n\}$$

are orthogonal to each other with respect to the inner product $\langle \rangle$.

The proof is immediate if one takes in consideration [12], [15] and [16].

**Lemma 4.2.** Given a fixed $n \in \mathbb{N}_0$, the set of spherical harmonics

$$\{\Re(X^0_n e_1), \Re(X^m_n e_1), \Re(Y^m_n e_1) : m = 1, ..., n\}$$

is orthogonal to the set

$$\{\Re(X^{n+1}_n e_1), \Re(Y^{n+1}_n e_1)\}$$

with respect to the inner product $\langle \rangle$. 


Lemma 4.3. Let $f$ be a square integrable $\mathcal{A}$-valued monogenic function. Then, the Fourier coefficients are given by

$$
\sqrt{2n + 3} \alpha_n^0 = \frac{\|X_n^0\|_{L_2(S)}}{\|\Re(X_n^0)\|_{L_2(S)}^2} \int_S \Re(f)\Re(X_n^0) d\sigma
$$

$$
\sqrt{2n + 3} \alpha_n^p = \frac{\|X_n^p\|_{L_2(S)}}{\|\Re(X_n^p)\|_{L_2(S)}^2} \int_S \Re(f)\Re(X_n^p) d\sigma
$$

$$
\sqrt{2n + 3} \beta_n^m = \frac{\|Y_n^m\|_{L_2(S)}}{\|\Re(Y_n^m)\|_{L_2(S)}^2} \int_S \Re(f)\Re(Y_n^m) d\sigma, \quad m = 1, \ldots, n
$$

$$
\sqrt{2n + 3} \alpha_{n+1}^n = \frac{\|X_n^{n+1}\|_{L_2(S)}}{\|\Re(X_n^{n+1})\|_{L_2(S)}^2} \int_S \Re(f)e_1\Re(X_n^{n+1}e_1) d\sigma
$$

$$
\sqrt{2n + 3} \beta_{n+1}^m = \frac{\|Y_n^{n+1}\|_{L_2(S)}}{\|\Re(Y_n^{n+1})\|_{L_2(S)}^2} \int_S \Re(f)e_1\Re(Y_n^{n+1}e_1) d\sigma.
$$

Proof. According to Theorem 4.4, a monogenic $L_2$-function $f : \Omega \subset \mathbb{R}^3 \rightarrow \mathcal{A}$ can be written as Fourier series

$$
f(x) = f(0) + \sum_{n=1}^{\infty} \sqrt{2n + 3} \ r^n \left( X_n^{0,*}(x) \alpha_n^0 + \sum_{m=1}^{n+1} [X_n^{m,*}(x) \alpha_n^m + Y_n^{m,*}(x) \beta_n^m] \right).
$$

We will present the proof for the coefficients $\alpha_n^0$ of $f_1$, the remaining coefficients $\alpha_n^m$ and $\beta_n^m$ ($m = 1, \ldots, n + 1$) being obtained in a similar way.

We aim to compare each Fourier coefficient $\alpha_n^0$ with $\Re(f)$. In fact, multiplying both sides of the expression

$$
\Re(f) = \sum_{n=0}^{\infty} \sqrt{2n + 3} \ r^n \left\{ \Re(X_n^{0,*})\alpha_n^0 + \sum_{m=1}^{n} [\Re(X_n^{m,*})\alpha_n^m + \Re(Y_n^{m,*})\beta_n^m] \right\}
$$

by the real part of the homogeneous monogenic polynomials described in (17) and integrating over the sphere, we get the desired relations. In particular, multiplying both sides of (20) by $\text{Sc}\{X_0^k\}$ $k = 1, \ldots$ and integrating over the sphere, we obtain

$$
\sqrt{2k + 3} \alpha_k^0 = \frac{\|X_k^0\|_{L_2(S)}}{\|\Re(X_k^0)\|_{L_2(S)}^2} \int_S \Re(f)\Re(X_k^0) d\sigma.
$$

We now study the coefficients $\alpha_{n+1}^n$ and $\beta_{n+1}^m$. Multiplying $f$ at right by $e_1$ we get

$$
\tilde{f} := fe_1
$$

$$
= \sum_{n=0}^{\infty} \sqrt{2n + 3} \ r^n \left[ \Re(X_n^{0,*}e_1)\alpha_n^0 + \sum_{m=1}^{n} [\Re(X_n^{m,*}e_1)\alpha_n^m + \Re(Y_n^{m,*}e_1)\beta_n^m] \right].
$$
Again, we compare the unknown coefficients $\alpha_n^{n+1}$ and $\beta_n^{n+1}$, with $\Re(\hat{f})$. Multiplying

$$
\Re(\hat{f}) = \sum_{n=0}^{\infty} \sqrt{2n+3} \, r^n \left[ \Re(X_n^0 \cdot e_1) \alpha_n^0 + \sum_{m=1}^{n+1} [\Re(X_n^m \cdot e_1) \alpha_n^m + \Re(Y_n^m \cdot e_1) \beta_n^m] \right]
$$

(21)

by the homogeneous harmonic polynomials $\Re(X_k^{k+1} e_1)$ (resp. $\Re(Y_k^{k+1} e_1)$), using Lemma 4.2 and integrating over the sphere carries our results

$$
\sqrt{2k+3} \, \alpha_k^{k+1} = \frac{\|X_k^0 \|_{L^2(S)}}{\|\Re(X_k^0 e_1)\|_{L^2(S)}^2} \int_S \Re(f e_1) \Re(X_k^{k+1} e_1) d\sigma
$$

$$
\sqrt{2k+3} \, \beta_k^{k+1} = \frac{\|Y_k^{k+1} e_1\|_{L^2(S)}}{\|\Re(Y_k^{k+1} e_1)\|_{L^2(S)}^2} \int_S \Re(f e_1) \Re(Y_k^{k+1} e_1) d\sigma.
$$

\[ \square \]

**Corollary 4.1.** Let $f$ be a square integrable $A$-valued monogenic function. Then, the Fourier coefficients satisfy the following inequalities:

$$
\sqrt{2n+3} \, |\alpha_n^n| \leq \frac{\|X_n^0\|_{L^2(S)}}{\|\Re(X_n^0 e_1)\|_{L^2(S)}^2} \|\Re(f)\|_{L^2(S)}
$$

$$
\sqrt{2n+3} \, |\alpha_n^n| \leq \frac{\|X_n^m\|_{L^2(S)}}{\|\Re(X_n^m e_1)\|_{L^2(S)}^2} \|\Re(f)\|_{L^2(S)}
$$

$$
\sqrt{2n+3} \, |\beta_n^n| \leq \frac{\|X_n^m\|_{L^2(S)}}{\|\Re(Y_n^m e_1)\|_{L^2(S)}^2} \|\Re(f)\|_{L^2(S)}, \quad m = 1, \ldots, n
$$

$$
\sqrt{2n+3} \, |\alpha_n^{n+1}| \leq \frac{\|X_n^{n+1} e_1\|_{L^2(S)}}{\|\Re(X_n^{n+1} e_1)\|_{L^2(S)}^2} \|\Re(f e_1)\|_{L^2(S)}
$$

$$
\sqrt{2n+3} \, |\beta_n^{n+1}| \leq \frac{\|Y_n^{n+1} e_1\|_{L^2(S)}}{\|\Re(Y_n^{n+1} e_1)\|_{L^2(S)}^2} \|\Re(f e_1)\|_{L^2(S)}
$$

The proof follows directly from Lemma 4.3 and Schwarz inequality.

**Theorem 4.2.** Let $f$ be a square integrable $A$-valued monogenic function in $B$. Then, for $0 \leq r < \frac{1}{2}$ we have the following inequality:

$$
|f| \leq |f(0)| + \frac{4r}{(2r-1)^2} \left( \|\Re(f)\|_{L^2(S)} A_1(r) + \|\Re(f e_1)\|_{L^2(S)} A_2(r) \right)
$$

where

$$
A_1(r) = \frac{3(3 - 4r) + 8r^2(2 - r)}{(2r - 1)^2}
$$

$$
A_2(r) = 3(1 - r).
$$
Proof. Considering $f$ written as in (19) we have

$$|f| \leq |f(0)| + |f_1| + |f_2|.$$ 

We start now to study the function $f_1$. Using the previous corollary it follows that

$$|f_1| = \|\Re(f)\|_{L_2(S)} \sum_{n=1}^{\infty} \left[ |X_n^0| \frac{\|X_n^0\|_{L_2(S)}}{\|\Re(X_n^0)\|_{L_2(S)}} + \sum_{m=1}^{n} \left( |X_m^{n, s}| \frac{\|X_m^n\|_{L_2(S)}}{\|\Re(X_m^n)\|_{L_2(S)}} + |Y_m^{n, s}| \frac{\|Y_m^n\|_{L_2(S)}}{\|\Re(Y_m^n)\|_{L_2(S)}} \right) \right]$$

and, due to the Proposition 3.1

$$|f_1| \leq \|\Re(f)\|_{L_2(S)} \sum_{n=1}^{\infty} (2r)^n (n + 1) \left( \frac{\|X_n^0\|_{L_2(S)}}{\|\Re(X_n^0)\|_{L_2(S)}} + 2 \sum_{m=1}^{n} \frac{\|X_m^n\|_{L_2(S)}}{\|\Re(X_m^n)\|_{L_2(S)}} \right)$$

Now, using the estimates given by Proposition 3.2

$$|f_1| \leq \frac{1}{2} \|\Re(f)\|_{L_2(S)} \sum_{n=1}^{\infty} (2r)^n (n + 1)(n + 2)(2n + 1).$$

Note that the previous inequality is also based on [3] where the following relations are proved

$$\|X_n^0\|_{L_2(S)} = \sqrt{\pi(n + 1)}$$

$$\|X_m^n\|_{L_2(S)} = \|Y_m^n\|_{L_2(S)} = \sqrt{\frac{\pi}{2}(n + 1)(n + 1 + m)!} \frac{(n + 1)(n + 1 + m)!}{(n + 1 - m)!} m = 1, \ldots, n + 1.$$

In the same way, we can study the function $f_2$. In fact it follows

$$|f_2| \leq 3\|\Re(f e_1)\|_{L_2(S)} \sum_{n=1}^{\infty} (2r)^n (n + 1).$$

Finally

$$|f| \leq |f(0)| + 3\|\Re(f e_1)\|_{L_2(S)} \sum_{n=1}^{\infty} (2r)^n (n + 1)$$

$$+ \frac{1}{2} \|\Re(f)\|_{L_2(S)} \sum_{n=1}^{\infty} (2r)^n (n + 1)(n + 2)(2n + 1).$$

Now, note that the last series are convergent for $0 \leq r < \frac{1}{2}$. \hfill \Box

As a immediate consequence of the previous theorem we can state a type of Schwartz Lemma as follows:
Corollary 4.2. Let \( f \) be a square integrable \( A \)-valued monogenic function in \( B \). If \( f(0) = 0 \) and \( \| \Re(f) \|_{L^2(S)} A_1(r) + \| \Re(fe_1) \|_{L^2(S)} A_2(r) \leq \frac{4}{(2r-1)^2} \), then

\[
|f| \leq r, \quad \text{for} \quad 0 \leq r < \frac{1}{2}.
\]

The proof follows directly from the previous theorem.

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