Why stratification may hurt, & how much

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Abstract.
There are circumstances under which stratified sampling is worse than simple random sampling, even if the allocation of the sample sizes is optimal. This phenomenon was discovered more than sixty years ago, but is not as widely known as one might expect. We provide it with lower and upper bounds for its badness as well as with an explanation.

Key words: proportional allocation, replacement, stratified sampling

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1. PROLOGUE

‘Stratification is a common technique,’ often necessary, but also attractive because it ‘may produce a gain in precision in the estimates of characteristics of the whole population’ (Cochran (1963), §5.1). In fact, if all sampling is done with replacement and the sample sizes are proportional to the strata sizes, stratified random sampling is at least as precise as simple random sampling. It even approaches perfection as the homogeneity inside the strata increases, i.e., as the heterogeneity of the population is more reflected by the heterogeneity between the strata and less by the heterogeneity inside the strata. Consequently, the rule of thumb with respect to stratified sampling is that it doesn’t hurt to try.

In real life, however, sampling is without replacement (cf. Cochran (1963), §2.1: ‘Sampling with replacement is entirely feasible but except in special circumstances is seldom used, since there seems little point in having the same unit twice in the sample’) — and without replacement, the rule of thumb is no longer valid. Even optimal stratified sampling may hurt then, in that the corresponding estimator can have a larger variance than the estimator based on a simple random sample (cf. Armitage (1947), Cochran (1963) §5.6, Evans
(1951), and Govindarajulu (1999) §5.5; for the obscurity of this fact, please see, for instance, the same Govindarajulu (1999) §5.5 and Wilks (1963), §10.9). The intuition might be helped here by realizing that, as equalities (1) below remind us, not replacing yields an advantage that is zero for a sample size of 1 and increases with the sample size. Thus, the advantage is larger for one sample of size \( n > 1 \) than for \( n \) samples of size 1; cf. the illustration of Theorem 3 in §2.

We only consider the simplest possible case, that of a dichotomous population. For this case, the results in Armitage (1947), Cochran (1963), and Evans (1951) are extended to what looks like a quite complete picture. The simple-is-better effect shows up in more circumstances than previously thought and is provided with exact bounds for its size.

2. LOWER AND UPPER BOUNDS

Consider an urn containing \( N \) balls, of which \( pN \) are red and \( (1 - p)N \) are black for a \( p \in [0, 1] \). We want to estimate \( p \). One approach is to take a sample of \( n \) balls from the urn and estimate \( p \) by the fraction of the red balls in the sample. A sample of \( n \) balls with replacement is a random member \((b_1, \ldots, b_n)\) of \((\text{urn})^n\), where all outcomes are equally likely; a sample of \( n \) balls without replacement is the same, except that the \( b_1, \ldots, b_n \) are all different. Let \( X \) and \( Y \) denote the number of red balls in a sample of size \( n \) with and without replacement, respectively. Then for the fraction estimators \( X/n \) and \( Y/n \) for \( p \) the truth of

\[
\text{var} \frac{X}{n} = p(1 - p) \frac{n}{N - n} \text{var} \frac{Y}{n} \tag{1}
\]

is well known; it is better never to see the same ball twice.

Now suppose the urn consists of \( m \geq 2 \) disjoint sub-urns, strata, each stratum \( j \) containing \( N_j \geq 2 \) balls, \( \sum_{j=1}^{m} N_j = N \), and that for each stratum \( j \) we know \( N_j/N \) but not its fraction \( p_j \) of red balls. Let \((n_1, \ldots, n_m)\) be an allocation, i.e., the \( n_j \) are natural numbers with \( 1 \leq n_j \leq N_j \) for all \( j \) and \( \sum_{j=1}^{m} n_j = n \), and let \( X_j, Y_j, j = 1, \ldots, m \), denote the number of red balls in a sample of size \( n_j \) from stratum \( j \) with and without replacement, respectively; then each of

\[
\frac{X}{n} \quad \text{(the simple estimator with replacement),}
\]

\[
\sum_{j=1}^{m} \frac{N_j X_j}{N n_j} \quad \text{(the stratification estimator with replacement),}
\]

\[
\frac{Y}{n} \quad \text{(the simple estimator without replacement),}
\]

and

\[
\sum_{j=1}^{m} \frac{N_j Y_j}{N n_j} \quad \text{(the stratification estimator without replacement)}
\]
Stratification

is an unbiased estimator for \( p \); for their optimality, cf. Neyman (1934). For both with and without replacement we want to compare the variance of the simple estimator to that of the stratification estimator, i.e., \( \text{var}(X/n) \) to \( \text{var}(\sum_{j=1}^{m} (N_j/N)X_j/n_j) \) and \( \text{var}(Y/n) \) to \( \text{var}(\sum_{j=1}^{m} (N_j/N)Y_j/n_j) \), under the assumption that the \( X_j \) are independent as well as the \( Y_j \).

It is immediate that if the strata are homogeneous but the whole population is not, i.e., \( p \in (0,1) \) and each \( p_j \) is equal to 0 or 1, then the stratification estimators are perfect while, with replacement, the simple estimator is not, and, without replacement, the simple estimator is only perfect when it is exhaustive, so that \( 0 = \text{var}(\sum_{j=1}^{m} (N_j/N)X_j/n_j) < \text{var}(X/n) \) and \( 0 = \text{var}(\sum_{j=1}^{m} (N_j/N)Y_j/n_j) \leq \text{var}(Y/n) \).

For arbitrary \( p_j \), the stratification estimator \( \sum_{j=1}^{m} (N_j/N)X_j/n_j \) is still not worse than the simple estimator \( X/n \) as long as the allocation is proportional, i.e., \( n_j = (N_j/N)n \) for every \( j \), because in that case

\[
\text{var} \sum_{j=1}^{m} \frac{N_jX_j}{Nn_j} = \text{var} \frac{X}{n} - \frac{1}{n} \sum_{j=1}^{m} \frac{N_j}{N} (p_j - p)^2
\]

holds, as one easily verifies. Thus, if all sampling is with replacement and the allocation is proportional, then stratified sampling is seen to reduce the variance, unless all the \( p_j \) are equal. And where it doesn’t help, it doesn’t harm either. However, this reassurance no longer holds as soon as we change the allocation:

**Theorem 1.** If \( p_1 = \cdots = p_m = p \in (0,1) \) and the allocation is not proportional, then simple is better in that

\[
\text{var} \sum_{j=1}^{m} \frac{N_jX_j}{Nn_j} > \text{var} \frac{X}{n},
\]

i.e., the variance of the stratification estimator with replacement is greater than the variance of the simple estimator with replacement.

Nor does it hold if all samples are drawn without replacement:

**Theorem 2.** If \( p_1 = \cdots = p_m = p \in (0,1) \) and \( n < N \), then simple is better in that

\[
\text{var} \sum_{j=1}^{m} \frac{N_jY_j}{Nn_j} > \text{var} \frac{Y}{n},
\]

i.e., the variance of the stratification estimator without replacement is greater than the variance of the simple estimator without replacement.
Stratification

This is (viib) in Armitage (1947) for a dichotomous population, i.e., for the case where Armitage’s ‘variable x’ has only 2 different values, except that we do not need his condition that (in the dichotomous case) all the $N_j$ are equal. It is also the dichotomous case of what is proved at the end of §5.6 in Cochran (1963), except that our condition $p_j = p$ is replaced by the condition that all the ‘mean square errors within strata’ $p_j(1 - p_j)N_j/(N_j - 1)$ are equal and larger than the ‘mean square error among strata’ $\sum_{j=1}^{m} N_j(p_j - p)^2/(m-1)$. In Hájek (1981), the observation after (20.31) that simple is better if $p_j = p$ only refers to proportional allocation, not necessarily to all allocations.

Under additional conditions inequality (3) may be sharpened:

**Theorem 3.** Let

$$B := \frac{N - 1}{N - m} \text{var} \frac{Y}{n} = \frac{N - n}{N - m} \frac{p(1 - p)}{n}.$$  

If $p_1 = \cdots = p_m = p \in (0, 1)$, $n < N$, and

(c1) $n_j \leq \frac{3}{4} N_j$ for $j = 1, \ldots, m$, or
(c2) the allocation is proportional, or
(c3) $N_1 = N_2 = \ldots = N_m = N/m$,

then

$$\text{var} \sum_{j=1}^{m} \frac{N_j Y_j}{N} \frac{n_j}{n_j} \geq B,$$

i.e., the variance of the stratification estimator without replacement is at least $(N - 1)/(N - m) \times$ the variance of the simple estimator without replacement, and

$$\text{var} \sum_{j=1}^{m} \frac{N_j Y_j}{N} \frac{n_j}{n_j} = B \iff n_j = n/m, \ N_j = N/m \ \forall j.$$

Theorem 3(c3) follows from Evans (1951) (12a, c).

A special case will illustrate Theorem 3 and bring out a weak point of the stratification estimator; the factor $(N - 1)/(N - m)$ in $B$ appearing here was met in (1) (take $n = m$). Imagine that all strata not only have the same composition, i.e., $p_j = p$, but also the same size, i.e., $N_j = N/m$, and that from each stratum only one ball is taken, so $n_j = 1$ and $n = m$. Then $\sum_{j=1}^{m} (N_j/N) Y_j/n_j = (1/n) \sum_{j=1}^{m} Y_j$, which is distributed as $X/n$, so that with (1)

$$\text{var} \sum_{j=1}^{m} \frac{N_j Y_j}{N} \frac{n_j}{n_j} = \text{var} \frac{X}{n} = \frac{N - 1}{N - n} \text{var} \frac{Y}{n} = B. $$

Splitting the sample over the strata reduces the without-replacement bonus from (1).
The need for extra conditions in Theorem 3 such as (c1), (c2), or (c3), and the room there is for Theorem 2 are demonstrated by considering $m = N_1 = n_1 = 2$, $n = N - 1$, and $N > 5$.

Further, one may ask if the condition ‘$p_1 = \cdots = p_m$’ is only the beginning: are there other distributions of the red balls among the strata, for which there are theorems similar to Theorem 3 but with lower bounds that are even higher than $B$ in Theorem 3? The answer is ‘no’, as long as $N_j := N/m$ and $n_j := n/m$ are feasible choices, because for these choices $B$ is an upper bound (over varying distributions, given $N$, $n$, $m$, and $p$) for the variance of the stratification estimator $\sum_{j=1}^{m} (N_j/N)Y_j/n_j$ (corresponding to $p_1 = \cdots = p_m$; cf. Theorem 3):

**Theorem 4.** If $N_j = N/m \geq 2$, $n_j = n/m$ for all $j$, and $n < N$ with $B$ as in Theorem 3, then

$$\text{var} \sum_{j=1}^{m} \frac{N_j Y_j}{N n_j} \leq B,$$

i.e., the variance of the stratification estimator without replacement is at most $(N - 1)/(N - m) \times$ the variance of the simple estimator without replacement, and

$$\text{var} \sum_{j=1}^{m} \frac{N_j Y_j}{N n_j} = B \Leftrightarrow p_1 = \cdots = p_m.$$

Cf. ‘the worst result to be anticipated’ on p. 99 of Evans (1951). (Namely, ‘for a second’ variable of interest; strata that are good, i.e., different, with respect to the first variable of interest need not be so for a second.) In the situation of Theorem 4, the worst is not that bad: in practice, $(N - 1)/(N - m)$ will be close to 1, and it also follows that if for every stratum the sample size is increased by 1, the new variance will not exceed the simple random sample variance corresponding to the old sample size.

Theorem 4 shows that if we want to curb the badness of stratified sampling and proportional allocation is an option, then it works, at least for strata of the same size. This makes us realize that it always works, even for arbitrary strata sizes, because independence, (1), and (2) imply

$$\text{var} \sum_{j=1}^{m} \frac{N_j Y_j}{N n_j} \leq \text{var} \sum_{j=1}^{m} \frac{N_j X_j}{N n_j} \leq \frac{p(1 - p)}{n}$$

under proportional allocation. Our final results essentially show how the upper bound $p(1 - p)/n$ for $\text{var} \sum_{j=1}^{m} (N_j/N)Y_j/n_j$ can be improved.

**Theorem 5.** If $(N_j/N)n$ and $pN_j$ are integers and $0 < pN_j < N_j$ for all $j$,
then

$$B \leq \min_{\text{Statistician}} \max_{\text{Nature}} \var \sum_{j=1}^{m} \frac{N_j Y_j}{N} n_j$$

$$\leq B + \frac{N-n}{4(N-m)nN^2} \sum_{j=1}^{m} \frac{N-mN_j}{N_j-1},$$

(4)

where ‘Statistician’ means an allocation \((n_1, \ldots, n_m)\) that satisfies \(n_j \leq 3N_j/4\) for all \(j\) or is proportional, ‘Nature’ means a distribution \((p_1, \ldots, p_m)\) of the \(pN\) red balls among the strata (so \(p_j \in [0,1]\), \(\sum_{j=1}^{m} p_j N_j = pN\), and \(p_j N_j \in \{0,1, \ldots, N_j\}\)), and \(B\) is as in Theorem 3; in fact, \(\max_{\text{Nature}}\) does not exceed the upper bound in (4) if the allocation is proportional.

If \(n \leq (3/4)N\), then the upper bound in (4) does not exceed \(p(1-p)/n\).

For the lower bound we observe that it follows from Theorem 3 and that, also by Theorem 3, if not \(N_j = N/m\) for all \(j\), then ‘\(B \leq\)’ may be replaced by ‘\(B <\)’. The difference between upper and lower bound is bounded by \(1/4N\) because \(\sum_{j=1}^{m} (N-mN_j)/(N_j-1) \leq \sum_{j=1}^{m} N/(2-1) = mN\); it reduces to 0 in case all strata have the same size (cf. Theorem 4).

The circumstances under which stratified sampling will hurt, have been called ‘very unusual’ and ‘extreme’ (Evans, 1951), ‘an academic curiosity’, which will happen only ‘mathematically’ (Cochran, 1963), as well as ‘quite conceivable’ (Govindarajulu, 1999).

3. JUSTIFICATIONS

Proof of Theorem 1. By Jensen’s inequality we obtain

$$\var \sum_{j=1}^{m} \frac{N_j X_j}{N} n_j = p(1-p) \var \sum_{j=1}^{m} \frac{1}{n_j N_j / N}$$

$$\geq p(1-p) \frac{1}{\sum_{j=1}^{m} n_j N_j / N}$$

$$= \frac{p(1-p)}{n} \var \frac{X}{n},$$

with ‘=’ instead of ‘\(\geq\)’ if and only if \(n_j/N_j\) is constant, i.e., the allocation is proportional.

Proof of Theorem 2. Let \(m \geq 2, 1 \leq n_j \leq N_j, N_j \geq 2, j = 1, \ldots, m\) be
integers with $N = \sum_{j=1}^{m} N_j$, $n = \sum_{j=1}^{m} n_j < N$. In order to prove

$$\sum_{j=1}^{m} \frac{N_j^2}{N_j - 1} \frac{N_j - n_j}{n_j} > \frac{N^2}{N - 1} \frac{N - n}{n},$$

(5)
it suffices to prove it for $m = 2$. Indeed, by splitting off one stratum from the urn at a time, applying (5) with $m = 2$ each time, and observing that one still has ‘≥’ instead of ‘>’ if $n = N$, one obtains (5) for the general case. For $1 \leq k \leq K$ and $1 \leq \ell \leq L$, $K, L > 1$, $k + \ell < K + L$ we will prove

$$\frac{(K + L)^2}{K + L - 1} \left( \frac{K + L}{k + \ell} - 1 \right) - \frac{K^2}{K - 1} \left( \frac{K}{k} - 1 \right) - \frac{L^2}{L - 1} \left( \frac{L}{\ell} - 1 \right) < 0.$$  

(6)

To this end we rewrite the LHS of (6) as $S + T$ with

$$S = \frac{K + L}{K + L - 1} \left( \frac{(K + L)^2}{k + \ell} - \frac{K^2}{k} - \frac{L^2}{\ell} \right) = \left( \frac{K + L}{K + L - 1} \right) U,$$

$$T = -\frac{(K + L)^2}{K + L - 1} + \left( \frac{1}{K + L - 1} - \frac{1}{K - 1} \right) \frac{K^2}{k} + \frac{K^2}{K - 1} + \left( \frac{1}{K + L - 1} - \frac{1}{L - 1} \right) \frac{L^2}{\ell} + \frac{L^2}{L - 1}.$$

Note that $\partial U/\partial k = -(K + L)^2 / (k + \ell)^2 + K^2 / k^2 \geq 0$ iff $k \leq (K/L)\ell$. Consequently, $U$ is maximal for $k = (K/L)\ell$ and

$$U \leq \frac{1}{T} \left( \frac{(K + L)^2}{K/L + 1} - KL - L^2 \right) = 0.$$

Clearly, $T$ is strictly increasing in $k$ and $\ell$ and hence

$$T < \frac{1}{K + L - 1} \left( -(K + L)^2 + K + L \right) + \frac{K^2 - K}{K - 1} + \frac{L^2 - L}{L - 1} = 0,$$

which completes the proof.

Proof of Theorem 3. The statements corresponding to (c2) and (c3) follow straightforwardly from the fact that if terms $t_j > 0$ have sum $\sum_{j=1}^{m} t_j = t$, then for all $a_j \geq 0$, not every $a_j = 0$, we have

$$\sum_{j=1}^{m} \frac{a_j}{t_j} = \left( \sum_{j=1}^{m} \sqrt{\frac{a_j}{t_j}} \right) \sum_{j=1}^{m} \left( \sqrt{\frac{t_j}{t}} \right)^2 \geq \frac{1}{l} \left( \sum_{j=1}^{m} \sqrt{a_j} \right)^2.$$
with ‘=’ instead of ‘≥’ if and only if \( t_j = t\sqrt{a_j} / \sum_{i=1}^{m} \sqrt{a_i} \), by Cauchy-Schwarz. With \( a_j = N_j^2 \) and \( t_j = N_j - 1 \) we obtain

\[
\sum_{j=1}^{m} \frac{N_j^2}{N} \frac{p(1-p)}{n} \frac{N_j - \frac{N}{n} n}{N_j - 1} = \frac{p(1-p)(N - n)}{nN^2} \sum_{j=1}^{m} \frac{N_j^2}{N_j - 1} \geq \frac{p(1-p)(N - n)}{nN^2} \frac{N^2}{N - m},
\]

which proves the statements corresponding to (c2), and with \( a_j = 1 \) and \( t_j = n_j \) we obtain

\[
\sum_{j=1}^{m} \frac{1}{m^2} \frac{p(1-p)}{n_j} \frac{n_j - n_j}{m - 1} = \frac{p(1-p)}{m^2(N - m)} \left( m \sum_{j=1}^{m} \frac{1}{n_j} - m^2 \right) \geq \frac{p(1-p)}{m^2(N - m)} \left( \frac{m^2}{n} - m^2 \right) = \frac{p(1-p)}{n} \frac{N - n}{N - m},
\]

which proves the statements corresponding to (c3).

In order to prove the statements corresponding to (c1), we observe that the function

\[
\psi(x, y) = \frac{y - 1}{1 - x}
\]

is strictly convex on \((0, 1) \times (0, \frac{3}{4}]\), because it is strictly convex on any segment in \((0, 1) \times (0, \frac{3}{4}]\). On any segment \( \{ t(x_1, y_1) + (1 - t)(x_2, y_2) \} \), namely, the strict convexity is clear, while for a segment not contained in \( y = \frac{3}{4} \) we have

\[
\left| \frac{\partial^2}{\partial x^2} \psi(x, y) \frac{\partial^2}{\partial y^2} \psi(x, y) - \frac{\partial^2}{\partial x \partial y} \psi(x, y) \right| = \left| \frac{2(1-y)}{y(1-x)^2} - \frac{1}{y^2(1-x)^2} \right| = \frac{3 - 4y}{y^4(1-x)^4}.
\]

The Hessian of \( \psi \), therefore, of which the determinant is the product of the eigenvalues and the sum of the diagonal elements is the sum of the eigenvalues, is positive definite outside \( y = \frac{3}{4} \), so the second derivative of \( t \in [0, 1] \mapsto \psi(t(x_1, y_1) + (1 - t)(x_2, y_2)) \) is positive on \((0, 1)\).

Consequently, applying Jensen’s inequality to the random 2-vector \( \begin{pmatrix} X \\ Y \end{pmatrix} \): \( j \in \{1, \ldots, m\} \mapsto \begin{pmatrix} 1/N_j \\ n_j/N_j \end{pmatrix} \in \mathbb{R}^2 \) with \( P(\{j\}) = N_j/N, j = 1, \ldots, m \), gives

\[
\sum_{j=1}^{m} \frac{1/N_j - 1}{1 - 1/N_j} \frac{N_j}{N} = E\psi(X, Y) \geq \psi(EX, EY)
\]
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\[ = \psi^*(m, n) = N \frac{N}{N-m} \left( \frac{N}{n} - 1 \right), \]

with '=' instead of '≥' if and only if \(\frac{1}{N_j} \text{ and } \frac{n_j}{N_j}\) are constant. This proves the statements corresponding to condition (c1).

Proof of Theorem 4. Under \(n_j = m/N, N_j = N/m\) the variance of the stratification estimator becomes

\[ \sum_{j=1}^{m} \frac{1}{m^2} \frac{p_j(1-p_j)}{m} \frac{N}{m} - \frac{n}{m} \]

and \(\sum p_j N_j = pN\) becomes \(\sum p_j = mp\), while by Cauchy-Schwarz

\[ \sum p_j(1-p_j) = mp - \frac{\sum p_j^2 \cdot \sum 1^2}{m} \leq mp - \frac{1}{m} \left( \sum (p_j \cdot 1) \right)^2 = mp(1-p). \]

This proves Theorem 4. If, in the situation of Theorem 4, for every stratum the sample size is increased by 1, we have \(n_{\text{new}} = n + m\) and the new variance will not exceed

\[ B_{\text{new}} = N - m - n \frac{p(1-p)}{n + m} \leq N - n \frac{p(1-p)}{n}. \]

Rest of Proof of Theorem 5. For the upper bound, let \(\alpha_j, \beta_j, 1 \leq j \leq m\), be positive reals with \(\sum_{j=1}^{m} \beta_j = 1\). By the multiplier method of Lagrange we see that \(\sum_{j=1}^{m} \alpha_j p_j(1-p_j)\) attains its maximum over \(p_j\) under the side condition \(\sum_{j=1}^{m} \beta_j(p_j - \frac{1}{2}) = 0\) at

\[ p_j = \frac{1}{2} + \frac{\beta_j(p - \frac{1}{2})}{\alpha_j \sum_{k=1}^{m} \alpha_k^{-1} \beta_k^2} \]

with maximum value equal to

\[ \frac{1}{4} \left( \sum_{j=1}^{m} \alpha_j - \frac{(2p - 1)^2}{\sum_{k=1}^{m} \alpha_k^{-1} \beta_k^2} \right). \]

With

\[ \alpha_j = \frac{N_j^2(n_j - n_j)}{N^2(N_j - 1)n_j}, \quad \beta_j = \frac{N_j}{N}, \quad 1 \leq j \leq m, \]

this shows that under proportional allocation, \(n_j = (N_j/N)n\),

\[ \max_{\text{Nature}} \ \frac{\sum_{j=1}^{m} N_j Y_j}{N n_j} \leq \frac{N - n}{4(N-m)n} \left( \frac{N - m}{N^2} \sum_{j=1}^{m} \frac{N_j^2}{N_j - 1} - (2p - 1)^2 \right) \] (7)
Stratification

holds. The right-hand side of (7) is equal to the upper bound in (4). Finally, as

\[ B = \frac{N - n}{N - m} \frac{p(1 - p)}{n}, \]

the fact that the upper bound in (4) does not exceed \( p(1 - p)/n \) is equivalent to

\[ \frac{N - n}{4(N - m)nN^2} \sum_{j=1}^{m} \frac{N - mN_j}{N_j - 1} \leq \frac{n - m}{N - m} \frac{p(1 - p)}{n}. \]

Suppose \( N_1 \leq N_2 \leq \cdots \leq N_m \). As \( 1 \leq pN_1 \leq N_1 - 1 \), we have

\[ \frac{n - m}{N - m} \frac{p(1 - p)}{n} \geq \frac{n - m}{(N - m)n} \frac{1}{N_1} \left( 1 - \frac{1}{N_1} \right). \]

Consequently, it is sufficient to prove

\[ \sum_{j=1}^{m} \frac{N - mN_j}{N_j - 1} \leq \frac{4(N - m)nN^2}{N - n} \cdot \frac{(n - m)(N_1 - 1)}{(N - m)nN^2}, \]

whose left-hand side is equal to the left-hand side in

\[ (N - m) \sum_{j=1}^{m} \frac{1}{N_j - 1} - m^2 \leq (N - m) \frac{m}{N_1 - 1} - m^2 = \frac{m(N - mN_1)}{N_1 - 1} \]

(remember \( N_j \geq N_1 \)), so that it is sufficient to prove

\[ m(N - mN_1) \left( \frac{N_1}{N_1 - 1} \right)^2 \leq 4(n - m) \frac{N^2}{N - n}. \]

This is true if

\[ m \left( N - m \frac{N}{n} \right) \left( \frac{N/n}{N/n - 1} \right)^2 \leq 4(n - m) \frac{N^2}{N - n} \]

(as \( N_1 \geq N/n \)), which is equivalent to

\[ mN(n - m) \leq 4n(N - n)(n - m), \]

which is true if \( n \leq (3/4)N \), because \( m \leq n \).

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