FINITE DEPTH AND JACOBSON-BOURBaki 
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Abstract. We introduce a notion of depth three tower of three rings $C \subseteq B \subseteq A$ with depth two ring extension $A \mid B$ recovered when $B = C$. If $A = \text{End}_B C$ and $B \mid C$ is a Frobenius extension, this captures the notion of depth three for a Frobenius extension in \cite{12, 13} such that if $B \mid C$ is depth three, then $A \mid C$ is depth two (a phenomenon of finite depth subfactors, see \cite{20}). We provide a similar definition of finite depth Frobenius extension with embedding theorem utilizing a depth three subtower of the Jones tower. If $A$, $B$ and $C$ correspond to a tower of subgroups $G > H > K$ via the group algebra over a fixed base ring, the depth three condition is the condition that subgroup $K$ has normal closure $K^G$ contained in $H$. For a depth three tower of rings, there is a pre-Galois theory for the ring $\text{End}_B AC$ and coring $(A \otimes_B A)^C$ involving Morita context bimodules and left coideal subrings. This is applied in two sections to a specialization of a Jacobson-Bourbaki correspondence theorem for augmented rings to depth two extensions with depth three intermediate division rings.

1. Introduction

Depth two theory is a type of Galois theory for noncommutative ring extensions, where the Galois group in field theory is replaced by a Hopf algebroid (with or perhaps without antipode). The Galois theory of depth two ring extensions has been studied in a series of papers by the author \cite{14, 15, 16, 17} in collaboration with Nikshych \cite{12, 11}, Szlachányi \cite{13}, and Külshammer \cite{10}, with a textbook treatment by Brzeziński and Wisbauer \cite{2}. There are a number of issues that remain unexplored or unanswered in full including chirality \cite{14, 15}, normality \cite{10, 16}, a Galois inverse problem and a Galois correspondence problem \cite{25}.

The Galois correspondence problem confronts the Galois theorist with a tower of three rings $C \subseteq B \subseteq A$. With no further assumption on the rings, we should impose at least a relative condition on the tower to arrive at results. With this in mind we propose to generalize the notion of depth two (D2) ring extension $A \mid B$ to a notion of depth three (D3) tower $A \mid B \mid C$. In
more detail, the tower $A \mid B \mid C$ is right depth three (rD3) if $A \otimes_B A$ is $A$-$C$-isomorphic to a direct summand of $A \oplus \cdots \oplus A$ (finitely many times). Many depth two theoretic results generalize suitably, such as a decomposition of an endomorphism ring into a crossed product with a quantum algebraic structure. If $A \supseteq C$ is D2 and $B$ a D3 intermediate ring, we may make use of Jacobson-Bourbaki theorems pairing certain ring extensions, such as division rings or simple algebras, with their endomorphism rings, in order to obtain theorems pairing D3 intermediate rings $B$ with left coideal subrings $\text{End}_C A_B$ of the bialgebroid $\text{End}_C A_C$ over the centralizer $A^C$.

The notion of D3 tower will also serve to give a transparent and workable algebraic definition of finite depth, originally an analytic notion in subfactor theory. A finite Jones index subfactor may be thought of algebraically as a Frobenius extension, where the conditional expectation and Pimsner-Popa orthonormal bases are the Frobenius coordinate system. If $A \supseteq C$ is D2 and $B$ a D3 intermediate ring, we may make use of Jacobson-Bourbaki theorems pairing certain ring extensions, such as division rings or simple algebras, with their endomorphism rings, in order to obtain theorems pairing D3 intermediate rings $B$ with left coideal subrings $\text{End}_C A_B$ of the bialgebroid $\text{End}_C A_C$ over the centralizer $A^C$.

The paper is organized as follows. In section 2 we note that right or left D3 ring towers are characterized in terms either of the tensor-square, $H$-equivalent modules, quasibases or the endomorphism ring. We prove a Theorem 2.5 that a depth three Frobenius extension $B \mid C$ embeds in a depth two extension $A \mid C$ (where $A = \text{End}_B C$). In a more technical section 8 we extend this technique to define a finite depth Frobenius extension and prove an embedding theorem for these as well: this answers a problem raised by Nikshych and the author in [11, Remark 5.2]. In section 3 we show that a tower of subgroups $G > H > K$ of finite index with the condition that the normal closure $K^G < H$ ensures that the group algebras $F[G] \supseteq F[H] \supseteq F[K]$ are a depth three tower w.r.t. any base ring $F$. We propose that the converse is true if $G$ is a finite group and $F = \mathbb{C}$. In section 4 we study the right coideal subring $E = \text{End}_B A_C$ as well as the bimodule and co-ring $P = (A \otimes_B A)^C$, which provide the quasibases for a right D3 tower $A \mid B \mid C$. We show that right depth three towers may be characterized by $P$ being finite projective as a left module over the centralizer $V = A^C$ and a pre-Galois isomorphism $A \otimes_B A \cong A \otimes_V P$.

In section 5 we study further Galois properties of D3 towers, such as the smash product decomposition of an endomorphism ring and the invariants
as a bicommutator. In section 6, we generalize the Jacobson-Bourbaki correspondence, which associates $\text{End}_E F$ to subfields $F$ of $E$ (or skew fields), and conversely associates $\text{End}_R E$ to closed subrings $R \subseteq \text{End}_E F$. We then compose this correspondence with an anti-Galois correspondence to prove the main Theorem 6.3 viz., there is a Galois correspondence between $D_3$ intermediate division rings of a $D_2$ extension of an augmented ring $A$ over a division ring $C$, on the one hand, with Galois left coideal subrings of the bialgebroid $\text{End}_C A_C$, on the other hand. In Section 7, we apply Jacobson-Bourbaki correspondence to show that the Galois connection for separable field extensions in [25] is a Galois correspondence between weak Hopf subalgebras and intermediate fields.

2. Definition and first properties of depth three towers

Let $A$, $B$ and $C$ denote rings with identity element, and $C \to B$, $B \to A$ denote ring homomorphisms preserving the identities. We use ring extension notation $A \mid B \mid C$ for $C \to B \to A$ and call this a tower of rings: an important special case if of course $C \subseteq B \subseteq A$ of subrings $B$ in $A$ and $C$ in $B$. Of most importance to us are the induced bimodules such as $B A C$ and $C A B$. We may naturally also choose to work with algebras over commutative rings, and obtain almost identical results.

We denote the centralizer subgroup of a ring $A$ in an $A$-$A$-bimodule $M$ by $M_A = \{ m \in M \mid \forall a \in A, ma = am \}$. We also use the notation $V_A(C) = A^C$ for the centralizer subring of $C$ in $A$. This should not be confused with our notation $K^G$ for the normal closure of a subgroup $K < G$. Notation like $\text{End}_{B_C}$ will denote the ring of endomorphisms of the module $B_C$ under composition and addition. We let $N^N_R$ denote the $n$-fold direct sum of a right $R$-module $N$ with itself; let $M_R \oplus * \cong N^N_R$ denote the module $M$ is isomorphic to a direct summand of $N^N_R$. Finally, the symbol $\cong$ denotes isomorphism and occasionally will denote anti-isomorphism when we can safely ignore opposite rings (such as “two anti-isomorphisms compose to give an isomorphism,” or “two opposite rings are Morita equivalent iff the rings are Morita equivalent”).

**Definition 2.1.** A tower of rings $A \mid B \mid C$ is right depth three (rD3) if the tensor-square $A \otimes_B A$ is isomorphic as $A$-$C$-bimodules to a direct summand of a finite direct sum of $A$ with itself: in module-theoretic symbols, this becomes, for some positive integer $N$,

$$AA \otimes_B AC \oplus * \cong AA^N_C$$

By switching to $C$-$A$-bimodules instead, we similarly define a left D3 tower of rings. The theory for these is dual to that for rD3 towers; we briefly consider it at the end of this section. As an alternative to referring to a rD3 tower $A \mid B \mid C$, we may refer to $B$ as an rD3 intermediate ring of $A \mid C$, if $C \to A$ factors through $B \to A$ and $A \mid B \mid C$ is rD3.
Recall that over a ring $R$, two modules $M_R$ and $N_R$ are H-equivalent if $M_R \oplus \ast \cong N_R^n$ and $N_R \oplus \ast \cong M_R^m$ for some positive integers $n$ and $m$. In this case, the endomorphism rings $\text{End} M_R$ and $\text{End} N_R$ are Morita equivalent with context bimodules $\text{Hom} (M_R, N_R)$ and $\text{Hom} (N_R, M_R)$.

**Lemma 2.2.** A tower $A \mid B \mid C$ of rings is rD3 iff the natural $A$-$C$-bimodules $A \otimes_B A$ and $A$ are H-equivalent.

**Proof.** We note that for any tower of rings, $A \oplus \ast \cong A \otimes_B A$ as $A$-$C$-bimodules, since the epi $\mu : A \otimes_B A \rightarrow A$ splits as an $A$-$C$-bimodule arrow. □

Since for any tower of rings $\text{End}_A A_C$ is isomorphic to the centralizer $V_A(C) = A^C$ (or anti-isomorphic according to convention), we see from the lemma that the notion of rD3 has something to do with classical depth three. Indeed,

**Example 2.3.** If $B \mid C$ is a Frobenius extension, with Frobenius system $(E, x_i, y_i)$ satisfying for each $a \in A$,

$$\sum_i E(ax_i)y_i = a = \sum_i x_iE(y_i a)$$

then $B \otimes_C B \cong \text{End} B_C := A$ via $x \otimes_B y \mapsto \lambda_x \circ E \circ \lambda_y$ for left multiplication $\lambda_x$ by element $x \in B$. Let $B \rightarrow A$ be the mapping $B \hookrightarrow \text{End} B_C$ given by $b \mapsto \lambda_b$. It is then easy to show that $A_B \otimes_C B \otimes_C B_C \cong A_B \otimes_B A_C$, so that for Frobenius extensions, condition (2) is equivalent to the condition for rD3 in preprint [13], which in turn slightly generalizes the condition in [12] for depth three free Frobenius extension.

Another litmus test for a correct notion of depth three is that depth two extensions should be depth three in a certain sense. Recall that a ring extension $A \mid B$ is right depth two (rD2) if the tensor-square $A \otimes_B A$ is $A$-$B$-bimodule isomorphic to a direct summand of $N$ copies of $A$ in a direct sum with itself:

$$A_A \otimes_B A_B \oplus \ast \cong A_A^N$$

Since the notions pass from ring extension to tower of rings, there are several cases to look at.

**Proposition 2.4.** Suppose $A \mid B \mid C$ is a tower of rings. We note:

1. If $B = C$ and $B \rightarrow C$ is the identity mapping, then $A \mid B \mid C$ is rD3 $\iff A \mid B$ is rD2.
2. If $A \mid B$ is rD2, then $A \mid B \mid C$ is rD3 w.r.t. any ring extension $B \mid C$.
3. If $A \mid C$ is rD2 and $B \mid C$ is a separable extension, then $A \mid B \mid C$ is rD3.
4. If $B \mid C$ is left D2, and $A = \text{End} B_C$, then $A \mid B \mid C$ is left D3.
5. If $C$ is the trivial subring, any ring extension $A \mid B$, where $B_A$ is finite projective, together with $C$ is rD3.
Proof. The proof follows from comparing eqs. (2) and (3), noting that $A \otimes_B A \otimes C A$ as natural $A \cdot A$-bimodules if $B \mid C$ is a separable extension (thus having a separability element $e = e_1 \otimes_C e_2 \in (B \otimes_C B)^B$ satisfying $e_1 e_2 = 1$), and finally from [14] that $B \mid C$ left D2 extension $\Rightarrow A \mid B$ is left D2 extension if $A = \text{End } B_C$. The last statement follows from tensoring $B A \oplus \ast \cong B B^a$ by $A A \otimes_B B = \ast$.

The next theorem is a converse and algebraic simplification of a key fact in subfactor Galois theory (the $n = 3$ case): a depth three subfactor $N \subseteq M$ yields a depth two subfactor $N \subseteq M_1$, w.r.t. its basic construction $M_1 \cong M \otimes_N M$. In preparation, let us call a ring extension $B \mid C$ rD3 if the endomorphism ring tower $A \mid B \mid C$ is rD3, where $A = \text{End } B_C$ and $A \mid B$ has underlying map $\lambda : B \rightarrow \text{End } B_C$, the left regular mapping given by $\lambda(x)(b) = xb$ for all $x, b \in B$. (This definition will extend to identify depth $n > 3$ Frobenius extensions as well in section 8.)

**Theorem 2.5.** Suppose $B \mid C$ is a Frobenius extension and $A = \text{End } B_C$. If $B \mid C$ is rD3, then the composite extension $A \mid C$ is D2.

**Proof.** Begin with the well-known bimodule isomorphism for a Frobenius extension $B \mid C$, between its endomorphism ring and its tensor-square, $B A_B \cong B_B \otimes_C B_B$. Tensoring by $A A \otimes_B C A$, we obtain $A \otimes_C A \cong A \otimes_B A \otimes_B A$ as natural $A \cdot A$-bimodules. Now restrict the bimodule isomorphism in eq. (2) on the left to $B$-modules and tensor by $A A \otimes_B C A$ to obtain $A A \otimes_C A C \oplus \ast \cong A A \otimes_B A C$ after substitution of the tensor-cube over $B$ by the tensor-square over $C$. By another application of eq. (2) we arrive at

$$A A \otimes_C A C \oplus \ast \cong A A C^2$$

Thus $A \mid C$ is right D2. Since it is a Frobenius extension as well, it is also left depth two. 

We introduce quasibases for right depth three towers.

**Theorem 2.6.** A tower $A \mid B \mid C$ is right depth three iff there are $N$ elements each of $\gamma_i \in \text{End } B A_C$ and of $u_i \in (A \otimes_B A)^C$ satisfying (for each $x, y \in A$)

$$x \otimes_B y = \sum_{i=1}^N x \gamma_i(y) u_i$$

**Proof.** From the condition (2), there are obviously $N$ maps each of

$$f_i \in \text{Hom } (A A C, A A \otimes_B A C), \ g_i \in \text{Hom } (A A \otimes_B A C, A A C)$$

such that $\sum_{i=1}^N f_i \circ g_i = \text{id}_{A A B A}$. First, we note that for any tower of rings, not necessarily rD3,

$$\text{Hom } (A A C, A A \otimes_B A C) \cong (A \otimes_B A)^C$$

via $f \mapsto f(1_A)$. The inverse is given by $p \mapsto ap$ where $p = p^1 \otimes_B p^2 \in (A \otimes_B A)^C$ using a Sweedler-type notation that suppresses a possible summation over simple tensors.
The other hom-group above also has a simplification. We note that for any tower,

$$\text{Hom}(A \otimes_B A_C, A_A) \cong \text{End}_B A_C$$

via $F \mapsto F(1 \otimes_B -)$. Given $\alpha \in \text{End}_B A_C$, we define an inverse sending $\alpha$ to the homomorphism $x \otimes_B y \mapsto x\alpha(y)$.

Let $f_i$ correspond to $u_i \in (A \otimes_B A)^C$ and $g_i$ correspond to $\gamma_i \in \text{End}_B A_C$ via the mappings just described. We compute:

$$x \otimes_B y = \sum_i f_i(g_i(x \otimes y)) = \sum_i f_i(x\gamma_i(y)) = \sum_i x\gamma_i(y)u_i,$$

which establishes the rD3 quasibases equation in the theorem, given an rD3 tower.

For the converse, suppose we have $u_i \in (A \otimes_B A)^C$ and $\gamma_i \in \text{End}_B A_C$ satisfying the equation in the theorem. Then map $\pi : A^N \to A \otimes_B A$ by

$$\pi : (a_1, \ldots, a_N) \mapsto \sum_i a_iu_i,$$

an $A$-$C$-bimodule epimorphism split by the mapping $\sigma : A \otimes_B A \hookrightarrow A^N$ given by

$$\sigma(x \otimes_B y) := (x\gamma_1(y), \ldots, x\gamma_N(y)).$$

It follows from the equation above that $\pi \circ \sigma = \text{id}_{A \otimes_B A}$. \hfill \Box

2.1. **Left D3 towers and quasibases.** A tower of rings $A | B | C$ is left D3 (or ℓD3) if the tensor-square $A \otimes_B A$ is an $C$-$A$-bimodule direct summand of $A^N$ for some $N$. If $B = C$, this recovers the definition of a left depth two extension $A | B$. There is a left version of all results in this paper: we note that $A | B | C$ is a right D3 tower if and only if $A^{\text{op}} | B^{\text{op}} | C^{\text{op}}$ is a left D3 tower (cf. [14]). Finally we define a tower $A | B | C$ to be D3 if it is both left D3 and right D3.

The notation in the theorem below refers to that in the example above.

**Theorem 2.7.** Suppose $B | C$ is a Frobenius extension with $A = \text{End}_B C$. Then $A | B | C$ is right depth three if and only if $A | B | C$ is left depth three.

**Proof.** It is well-known that also $A | B$ is a Frobenius extension. Then $A \otimes_B A \cong \text{End}_B A_B$ as natural $A$-$A$-bimodules.

Now note the following characterization of left D3 with proof almost identical with that of [16], Prop. 3.8]: If $A | B | C$ is a tower where $A_B$ if finite projective, then $A | B | C$ is left D3 $\iff \text{End}_B A_B \oplus \ast \cong A^N$ as natural $A$-$C$-bimodules. The proof involves noting that $\text{End}_B A_B \cong \text{Hom}(A \otimes_B A_A, A_A)$ as natural $A$-$C$-bimodules via

$$f \mapsto (a \otimes a' \mapsto f(a)a').$$

The finite projectivity is used for reflexivity in hom’ming this isomorphism, thus proving the converse statement.
Of course a Frobenius extension satisfies the finite projectivity condition. Comparing the isomorphisms of $\text{End}_A B$ and $A \otimes_B A$ to direct summands of finitely many copies of $A$ just above and in eq. (2), we note that the tower is $\ell D3 \iff rD3$. □

In a fairly obvious reversal to opposite ring structures in the proof of Theorem 2.6, we see that a tower $A \mid B \mid C$ is left $D3$ iff there are $N$ elements $\beta_j \in \text{End}_C A_B$ and $N$ elements $t_j \in (A \otimes_B A)^C$ such that for all $x, y \in A$, we have

$$x \otimes_B y = \sum_{j=1}^{N} t_j \beta_j(x) y$$

We note explicitly that if $A \mid B$ is a Frobenius extension with Frobenius system $(E, x, y)$, then $A \mid B \mid C$ is $rD3$ iff the tower is $\ell D3$. For example, starting with the $\ell D3$ quasibases data above, a right $D3$ quasibases is given by

$$\{E(-t^1_j)t^2_j\} \{\sum_i \beta_j(x_i) \otimes_B y_i\}$$

as one may readily compute.

We record the characterization of left $D3$, noted above in the proof, for towers satisfying a finite projectivity condition.

**Theorem 2.8.** Suppose $A \mid B \mid C$ is a tower of rings where $A_B$ is finite projective. Then this tower is left $D3$ if and only if the natural $A-C$-bimodules satisfy for some $N$,

$$\text{End}_A B \oplus \ast \cong A^N\quad (11)$$

Dually we establish that if $A \mid B \mid C$ is a tower where $B_A$ is finite projective, then $A \mid B \mid C$ is right $D3$ if and only if $\text{End}_B A \oplus \ast \cong A^N$ as natural $C-A$-bimodules.

### 3. Depth three for towers of groups

Fix a base ring $F$. Groups give rise to rings via $G \mapsto F[G]$, the functor associating the group algebra $F[G]$ to a group $G$. Therefore we can pull back the notion of depth 2 or 3 for ring extensions or towers to the category of groups when reference is made to the base ring.

In the paper [10], a depth two subgroup w.r.t. the complex numbers is shown to be equivalent to the notion of normal subgroup for finite groups. This consists of two results. The easier result is that over any base ring, a normal subgroup of finite index is depth two by exhibiting left or right $D2$ quasibases via coset representatives and projection onto cosets. This proof suggests that the converse hold as well. The second result is a converse for complex finite dimensional $D2$ group algebras where normality of the subgroup is established using character theory and Mackey’s subgroup theorem.
In this section, we will similarly do the first step in showing what group-theoretic notion corresponds to depth three tower of rings. Let $G > H > K$ be a tower of groups, where $G$ is a finite group, $H$ is a subgroup, and $K$ is a subgroup of $H$. Let $A = F[G]$, $B = F[H]$ and $C = F[K]$. Then $A \mid B \mid C$ is a tower of rings, and we may ask what group-theoretic notion on $G > H > K$ will guarantee, with fewest possible hypotheses, that $A \mid B \mid C$ is rD3.

**Theorem 3.1.** The tower of groups algebras $A \mid B \mid C$ is D3 if the corresponding tower of groups $G > H > K$ satisfies

\[ K^G < H \]

where $K^G$ denotes the normal closure of $K$ in $G$.

**Proof.** Let $\{g_1, \ldots, g_N\}$ be double coset representatives such that $G = \bigsqcup_{i=1}^N Hg_iK$. Define $\gamma_i(g) = 0$ if $g \not\in Hg_iK$ and $\gamma_i(g) = g$ if $g \in Hg_iK$. Of course, $\gamma_i \in \text{End}_{BA_C}$ for $i = 1, \ldots, N$.

Since $K^G \subseteq H$, we have $gK \subseteq Hg$ for each $g \in G$. Hence for each $k \in K$, $g_jk = hg_j$ for some $h \in H$. It follows that

\[ g_j^{-1} \otimes_B g_jk = g_j^{-1}h \otimes_B g_j = kg_j^{-1} \otimes_B g_j. \]

Given $g \in G$, we have $g = hg_jk$ for some $j = 1, \ldots, N$, $h \in H$, and $k \in K$. Then we compute:

\[ 1 \otimes_B g = 1 \otimes_B hg_jk = hg_jg_j^{-1} \otimes_B g_jk = hg_jkg_j^{-1} \otimes_B g_j \]

so $1 \otimes_B g = \sum_i \gamma_i(g)g_i^{-1} \otimes_B g_i$ where $g_i^{-1} \otimes_B g_i \in (A \otimes_B A)^C$. By theorem then, $A \mid B \mid C$ is an rD3 tower.

The proof that the tower of group algebras is left D3 is entirely symmetrical via the inverse mapping. \qed

The theorem is also valid for infinite groups where the index $[G : H]$ is finite, since $HgK = Hg$ for each $g \in G$.

Notice how the equivalent notions of depth two and normality for finite groups over $\mathbb{C}$ yields the Proposition 2.4 for groups. Suppose we have a tower of groups $G > H > K$ where $K^G \subseteq H$. If $K = H$, then $H$ is normal (D2) in $G$. If $K = \{e\}$, then it is rD3 together with any subgroup $H < G$. If $H \triangleleft G$ is a normal subgroup, then necessarily $K^G \subseteq H$. If $K \triangleleft G$, then $K^G = K < H$ and the tower is D3.

Question: Can the character-theoretic proof in [10] be adapted to prove that a D3 tower $\mathbb{C}[G] \supseteq \mathbb{C}[H] \supseteq \mathbb{C}[K]$ where $G$ is a finite group satisfies $K^G < H$?

4. **Algebraic structure on $\text{End}_{BA_C}$ and $(A \otimes_B A)^C$**

In this section, we study the calculus of some structures definable for an rD3 tower $A \mid B \mid C$, which reduce to the dual bialgebroids over the centralizer
of a ring extension in case $B = C$ and their actions/coactions. Throughout the section, $A | B | C$ will denote a right depth three tower of rings,

$$P := (A \otimes_B A)^C, \quad Q := (A \otimes_C A)^B,$$

which are bimodules with respect to the two rings familiar in depth two theory,

$$T := (A \otimes_B A)^B, \quad U := (A \otimes_C A)^C.$$

Note that $P$ and $Q$ are isomorphic to two $A$-$A$-bimodule Hom-groups:

$$P \cong \text{Hom}(A \otimes_C A, A \otimes_B A), \quad Q \cong \text{Hom}(A \otimes_B A, A \otimes_C A).$$

Recall that $T$ and $U$ have multiplications given by $tt' = t^1 t'^1 \otimes_B t'^2 t'^2$, $uu' = u^1 u'^1 \otimes_C u'^2 u'^2$, where $1_T = 1_A \otimes 1_A$ and a similar expression for $1_U$. Namely, the bimodule $TP_U$ is given by

$$TP_U : t \cdot p \cdot u = u^1 p^1 t^1 t'^1 \otimes_B t'^2 p'^2 u'^2,$$

The bimodule $UQ_T$ is given by

$$UQ_T : u \cdot q \cdot t = t^1 q^1 u^1 \otimes_C u'^2 q'^2 t'^2$$

We have the following result, also mentioned in passing in [15] with several additional hypotheses.

**Proposition 4.1.** The bimodules $P$ and $Q$ over the rings $T$ and $U$ form a Morita context with associative multiplications

$$P \otimes_U Q \to T, \quad p \otimes q \mapsto pq = q^1 p^1 \otimes_B p'^2 q'^2$$

$$Q \otimes_T P \to U, \quad q \otimes p \mapsto qp = p^1 q^1 \otimes_C q'^2 p'^2$$

If $B | C$ is an H-separable extension, then $T$ and $U$ are Morita equivalent rings via this context.

**Proof.** The equations $p(qp') = (pq)p'$ and $q(pq') = (qp)q'$ for $p, p' \in P$ and $q, q' \in Q$ follow from the four equations directly above.

Note that

$$T \cong \text{End}_A A \otimes_B A, \quad U \cong \text{End}_A A \otimes_C A$$

as rings. We now claim that the hypotheses on $A | B$, $A | C$ and $B | C$ imply that the $A$-$A$-bimodules $A \otimes_B A$ and $A \otimes_C A$ are H-equivalent. Then the endomorphism rings above are Morita equivalent via context bimodules given by eqs. [13], which proves the proposition.

Since $B | C$ is H-separable, it is in particular separable, and the canonical $A$-$A$-epi $A \otimes_C A \to A \otimes_B A$ splits via an application of a separability element. Thus, $A \otimes_B A \oplus * \cong A \otimes_C A$. The defining condition for H-separability is $B \otimes_C B \oplus * \cong B^N$ as $B$-$B$-bimodules for some positive integer $N$. Therefore, $A \otimes_C A \oplus * \cong A \otimes_B A^N$ as $A$-$A$-bimodules by an application of the functor $A \otimes_B - \otimes_A A$. Hence, $A \otimes_B A$ and $A \otimes_C A$ are H-equivalent $A^e$-modules (i.e., $A$-$A$-bimodules).
We denote the centralizer subrings $A^B$ and $A^C$ of $A$ by
\begin{equation}
R := V_A(B) \subseteq V_A(C) := V
\end{equation}

From $R \cong \text{Hom}(A \otimes_B A, A)$ and $V \cong \text{Hom}(A \otimes_C A, A)$ and composition with eq. (13), we obtain the generalized anchor mappings (cf. [15]),
\begin{equation}
R \otimes_T P \longrightarrow V, \quad r \otimes p \mapsto p_1^1 r p_2^2
\end{equation}
\begin{equation}
V \otimes_U Q \longrightarrow R, \quad v \otimes q \mapsto q_1^1 v q_2^2
\end{equation}

**Proposition 4.2.** The two generalized anchor mappings are bijective if $B \mid C$ is $H$-separable.

**Proof.** Denote $r \cdot p := p_1^1 r p_2^2$ and $v \cdot q := q_1^1 v q_2^2$. From the previous proposition, there are elements $p_i \in P$ and $q_i \in Q$ such that $\sum p_i q_i = 1_T$; in addition, $p_j' \in P$ and $q_j' \in Q$ such that $1_U = \sum_j q_j' p_j'$. Let $v \in V$, then
\begin{equation*}
v = v \cdot 1_U = \sum_j v \cdot (q_j' p_j') = \sum_j (v \cdot q_j') \cdot p_j'
\end{equation*}
and a similar computation starting with $r = r \cdot 1_T$ shows that the two generalized anchor mappings are surjective.

In general, we have the corestriction of the inclusion $T \subseteq A \otimes_B A$,
\begin{equation}
\tau T \hookrightarrow \tau P
\end{equation}
which is split as a left $T$-module monic by $p \mapsto e^1 p e^2$ in case there is a separability element $e = e^1 \otimes_C e^2 \in B \otimes_C B$. Similarly,
\begin{equation}
\tau U \hookrightarrow \tau Q
\end{equation}
is a split monic in case $B \mid C$ is separable. If $\sum_i v_i \otimes_U q_i \in \ker(V \otimes_U Q \rightarrow R)$ then $\sum_i v_i \otimes_U q_i \mapsto \sum_i v_i \cdot q_i \otimes_U 1_U = 0$ via an injective mapping, whence $\ker(V \otimes_U Q \rightarrow R) = \{0\}$.

Of course, if $B \mid C$ is $H$-separable, we note from Proposition 4.1 and Morita theory that $P$ and $Q$ are projective generators on both sides, (and faithfully flat). If $K := \ker(R \otimes_T P \rightarrow V)$, then $K \otimes_U Q = 0$, since $\sum_j r_j \cdot p_j = 0$ implies
\begin{equation*}
\sum_j r_j \otimes_T p_j \otimes q \otimes U 1_U = 0
\end{equation*}
via an injective mapping. It follows from faithful flatness of $U Q$ that $K = \{0\}$. \hfill \Box

Note that $P$ is a $V$-$V$-bimodule (via the commuting homomorphism and antihomomorphism $V \rightarrow U \leftarrow V$):
\begin{equation}
v P_v : \quad v \cdot p \cdot v' = v p_1^1 \otimes B p_2^2 v'
\end{equation}
Note too that $E = \text{End}_{B \otimes C}$ is an $R$-$V$-bimodule via
\begin{equation}
R E_v : \quad r \cdot \alpha \cdot v = r \alpha (-) v
\end{equation}
Note the subring and over-ring
\begin{equation}
\text{End}_B A_B \subseteq E \subseteq \text{End}_C A_C
\end{equation}
which are the total algebras of the left $R$- and $V$-bialgebroids in depth two theory \cite{13,14,15}.

**Lemma 4.3.** The modules $\nu P$ and $E_V$ are finitely generated projective. In case $A \mid C$ is left $D_2$, the subring $E$ is a right coideal subring of the left $V$-bialgebroid $\text{End}_C A_C$.

**Proof.** This follows from eq. (5), since $p \in P \subseteq A \otimes B A$, so
\begin{equation}
p = \sum_i p^i \gamma_i(p^2)u_i
\end{equation}
where $u_i \in P$ and $p \mapsto p^i \gamma_i(p^2)$ is in $\text{Hom}(\nu P, \nu V)$, thus dual bases for a finite projective module. The second claim follows similarly from
\begin{equation}
\alpha = \sum_i \gamma_i(-)u_i \alpha(u^2)
\end{equation}
where $\gamma_i \in E$ and $\alpha \mapsto u^i \alpha(u^2)$ are mappings in $\text{Hom}(E_V, V_V)$.

Now suppose $\beta_j \in S := \text{End}_C A_C$ and $t_j \in (A \otimes_C A)^C$ are left $D_2$ quasibases of $A \mid C$. Recall that the coproduct $\Delta : S \to S \otimes_V S$ given by ($\beta \in S$)
\begin{equation}
\Delta(\beta) = \sum_j \beta(-t_j^1)\beta_j \otimes \beta_j
\end{equation}
makes $S$ a left $V$-bialgebroid \cite{13}. Of course this restricts and corestricts to $\alpha \in E$ as follows: $\Delta(\alpha) \in E \otimes_V S$. Hence, $E$ is a right coideal subring of $S$.

In fact, if $A \mid B$ is also $D_2$, and $S = \text{End}_B A_B$, then $E$ is similarly shown to be an $S$-$S$-bicomodule ring For we recall the coaction $E \to S \otimes_S E$ given by
\begin{equation}
\alpha(-1) \otimes_R \alpha(0) = \sum_i \tilde{\gamma_i} \otimes \tilde{u}_i \alpha(\tilde{u}_i^2 -)
\end{equation}
where $\tilde{\gamma_i} \in S$ and $\tilde{u}_i \in (A \otimes_B A)^B$ are right $D_2$ quasibases of $A \mid B$ (restriction of \cite{13} eq. (19)).

Twice above we made use of a $V$-bilinear pairing $P \otimes E \to V$ given by
\begin{equation}
\langle p, \alpha \rangle := p^i \alpha(p^2), \quad (p \in P = (A \otimes_B A)^C, \, \alpha \in E = \text{End}_B A_C)
\end{equation}

**Lemma 4.4.** The pairing above is nondegenerate. It induces $E_V \cong \text{Hom}((\nu P, \nu V)$ via $\alpha \mapsto \langle -, \alpha \rangle$.

**Proof.** The mapping has the inverse $F \mapsto \sum_i \gamma_i(-)F(u_i)$ where $\gamma_i \in E, u_i \in P$ are $\text{rD}_3$ quasibases for $A \mid B \mid C$. Indeed, $\sum_i \langle p, \gamma_i \rangle F(u_i) = F(\sum_i \gamma_i(-)u_i) = F(p)$ for each $p \in P$ since $F$ is left $V$-linear, and for each $\alpha \in E$, we note that $\sum_i \gamma_i(-)\langle u_i, \alpha \rangle = \alpha$. \qed
Proposition 4.5. There is a $V$-coring structure on $P$ left dual to the ring structure on $E$.

Proof. We note that

\[ P \otimes V \cong (A \otimes_B A \otimes_B A)^C \]

via $p \otimes p' \mapsto p^1 \otimes p^2 p'^1 \otimes p'^2$ with inverse

\[ p = p^1 \otimes p^2 \otimes p^3 \mapsto \sum_i (p^1 \otimes_B p^2 \gamma_i(p^3)) \otimes_V u_i. \]

Via this identification, define a $V$-linear coproduct $\Delta : P \to P \otimes V P$ by

\[ \Delta(p) = p^1 \otimes_B 1_A \otimes_B p^2. \]

Alternatively, using Sweedler notation and rD3 quasibases,

\[ p(1) \otimes_V p(2) = \sum_i (p^1 \otimes_B \gamma_i(p^2)) \otimes_V u_i. \]

Define a $V$-linear counit $\varepsilon : P \to V$ by $\varepsilon(p) = p^1 p^2$. The counital equations follow readily [2].

Recall from Sweedler [24] that the $V$-coring $(P, V, \Delta, \varepsilon)$ has left dual ring $^*P := \text{Hom}_V(VP, V^V)$ given by Sweedler notation by

\[ (f \ast g)(p) = f(p(1))g(p(2)) \]

with $1 = \varepsilon$. Let $\alpha, \beta \in E$. If $f = \langle -, \alpha \rangle$ and $g = \langle -, \beta \rangle$, we compute $f \ast g = \langle -, \alpha \circ \beta \rangle$ below, which verifies the claim:

\[ f(p(1))g(p(2)) = \sum_i \langle p^1 \otimes_B \gamma_i(p^2), u_i, \alpha \rangle = \langle p^1 \otimes_B \beta(p^2), \alpha \rangle = \langle p, \alpha \circ \beta \rangle. \]

In addition, we note that $P$ is $V$-coring with grouplike element

\[ g_P := 1_A \otimes_B 1_A \]

since $\Delta(g_P) = 1 \otimes 1 \otimes 1 = g_P \otimes_V g_P$ and $\varepsilon(g_P) = 1$.

There is a pre-Galois structure on $A$ given by the right $P$-comodule structure $\delta : A \to A \otimes V P$, $\delta(a) = a(0) \otimes_V a(1)$ defined by

\[ \delta(a) := \sum_i \gamma_i(a) \otimes_V u_i. \]

The pre-Galois isomorphism $\beta : A \otimes_B A \cong A \otimes_V P$ given by

\[ \beta(a \otimes a') = aa'_{(0)} \otimes_V a'_{(1)} \]

is utilized below in another characterization of right depth three towers.

Theorem 4.6. A tower of rings $A | B | C$ is right depth three if and only if $V P$ is finite projective and $A \otimes_V P \cong A \otimes_B A$ as natural $A$-$C$-bimodules.
Proof. (⇐) If $V \oplus \ast \cong V^N$ and $A \otimes_V P \cong A \otimes_B A$, then tensoring by $A \otimes_V -$, we obtain $A \otimes_B A \oplus \ast \cong A^N$ as natural $A$-$C$-bimodules, the rD3 defining condition on a tower.

(⇒) By lemma $V^P$ is f.g. projective. Map $A \otimes_V P \to A \otimes_B A$ by $a \otimes p \mapsto ap \otimes Bp^2$, clearly an $A$-$C$-bimodule homomorphism. The inverse is the “pre-Galois” isomorphism, 

(36) \[ \beta: A \otimes_B A \to A \otimes_V P, \quad \beta(a \otimes_B a') = \sum_i a \gamma_i(a') \otimes_V u_i \]

since $\sum_i a \gamma_i(p^2) \otimes_V u_i = a \otimes_V p$ and $\sum_i a \gamma_i(a') u_i = a \otimes a'$ for $a, a' \in A, p \in P$. □

If $B \mid C$ is H-separable, there is more to say about the structure of the Morita equivalent total rings for the bialgebroids $T$ and $U$ and bijective anchor maps in propositions 4.1 and 4.2. This stems from the fact that for a $B$-bimodule $M$, we have an Azumaya-type condition for the centralizers, 

$M^C \cong M^B \otimes_{Z(B)} B^C$ via $m \otimes c \mapsto mc$ in one direction. This may now be applied to each of the cases $M = A, A \otimes_B A$, and $A \otimes_C A$ to obtain formulas relating $V$ and $R$, $T$ and $P$, as well as $Q$ and $U$. We will study the relationship of these remarks to monoidal functors and Takeuchi’s $\sqrt{\text{Morita}}$ base change outlined in the paper [22] in another paper.

5. Further Galois properties of depth three

We will show here that the smaller of the endomorphism rings of a depth three tower decomposes tensorially over the overalgebra and the mixed bimodule endomorphism ring studied above. In case the composite ring extension is depth two, this is a smash product decomposition in terms of a coideal subring of a bialgebroid. Finally, we express the invariants of this coideal subring acting on the overalgebra in terms of a bicommutator.

**Theorem 5.1.** If $A \mid B \mid C$ is left D3, then

(37) \[ \text{End } A_B \cong A \otimes_V \text{End } C_A B \]

via the homomorphism $A \otimes_V \text{End } C_A B \to \text{End } A_B$ given by $a \otimes_V \alpha \mapsto \lambda_a \circ \alpha$.

**Proof.** Given a left D3 quasibases $\beta_j \in \text{End } C_A B$ and $t_j \in (A \otimes_B A)^C$, note that the mapping $\text{End } A_B \to A \otimes_V \text{End } C_A B$ given by

(38) \[ f \mapsto \sum_j f(t_j^1) t_j^2 \otimes_V \beta_j \]

is an inverse to the homomorphism above. □

**Corollary 5.2.** If $A \mid C$ is additionally D2, then $\text{End } C_A B$ a left coideal subring of $\text{End } C_A C$ and there is a ring isomorphism with a smash product ring,

(39) \[ \text{End } A_B \cong A \times \text{End } C_A B \]
Proof. Recall from depth two theory [13] that the $V$-bialgebroid $\text{End}_C A_C$ acts on the module algebra $A$ by simple evaluation, $\beta \triangleright a = \beta(a)$. That the action is measuring is not hard to see from the formula for the coproduct on $\text{End}_C A_C$ given by

$$\Delta(\beta) = \beta(1) \otimes_V \beta(2) := \sum_k \tilde{\gamma}_k \otimes_V \tilde{u}_k \beta(\tilde{u}_k^2 - )$$

where $\tilde{\gamma}_k \in \text{End}_C A_C$ and $\tilde{u}_k \in (A \otimes_C A)^C$ are right D2 quasibases for the composite ring extension $A|C$. Note then that for $\alpha \in \text{End}_C A_B \subseteq \text{End}_C A_C$, the equation yields $\alpha(1) \otimes_V \alpha(2) \in \text{End}_C A_C \otimes_V \text{End}_C A_B$. Hence, $\text{End}_C A_B$ is a left coideal subring. The details and verifications of the definition of such an object, over a smaller base ring than that of the bialgebroid, are rather straightforward and left to the reader.

As a consequence of the smash product formula $\text{End}_A C \cong A \rtimes \text{End}_C A_C$ over the centralizer $V$, we restrict to $\text{End}_A B \subseteq \text{End}_A C$, apply the theorem above, to obtain the equation for $\alpha, \beta \in \text{End}_A B$,

$$\alpha \circ \beta = \alpha(1) \triangleright \beta \triangleright \alpha(2) \in A \otimes_V \text{End}_C A_B$$

where $a, b \in A$, and $\triangleright$, $\circ$ are used interchangeably. \hfill $\square$

In case $A|C$ continues to be a D2 extension, the theorem below will characterize the subring $A^S$ of invariants of $S = \text{End}_C A_C$ as well as $A^J$ where $J := \text{End}_C A_B$, the coideal subring of $S$, in terms of $A$ as the natural module over $E := \text{End}_A B$. The endomorphism ring $\text{End}_E A$ is familiar from the Jacobson-Bourbaki theorem in Galois theory [7, 21].

**Theorem 5.3.** Let $A|B|C$ be left D3 and

$$A^J = \{ x \in A| \forall \alpha \in J, \alpha(x) = \alpha(1)x \}.$$  

Then $A^J \cong \text{End}_E A$ via the anti-isomorphism $x \mapsto \rho_x$.

**Proof.** We first note that $A^J = \{ x \in A| \forall f \in E, y \in A, f(yx) = f(y)x \}$. The inclusion $\supseteq$ easily follows from letting $y = 1_A$ and $\alpha \in J \subseteq E$. The reverse inclusion follows from Theorem 5.1. Since $E \cong A \otimes_V J$, note that $f \circ \lambda_y \in E$ decomposes as $\sum_j f(yt_j^1) t_j^2 \beta_j \in A \otimes_V J$ for an arbitrary $y \in A$. Given $x \in A$ such that $\alpha(x) = \alpha(1)x$ for each $\alpha \in J$, then

$$f(yx) = \sum_j f(yt_j^1) t_j^2 \beta_j(x) = \sum_j f(yt_j^1) t_j^2 \beta_j(1)x = f(y)x.$$

It follows from these considerations that $\rho_x \in \text{End}_E A$ for $x \in A^J$, since $\rho_x(f(a)) = f(\rho_x(a))$ for each $f \in E, a \in A$.

Now an inverse mapping $\text{End}_E A \rightarrow A^J$ is given by $G \mapsto G(1)$. Of course $\rho_x(1) = x$. Note that $G(1) \in A^J$, since for $\alpha \in J$, we have $\alpha(G(1)) = G(\alpha(1)) = \lambda_\alpha G(1)$, since $\lambda_\alpha \in E$ for all $a \in A$. Finally, we note that $G(a) = G \circ \lambda_a(1) = aG(1)$, whence $G = \rho_{G(1)}$ for each $G \in \text{End}_E A$. \hfill $\square$

The following clarifies and extends part of [13, 4.1]. Let $S$ denote the bialgebroid $\text{End}_B A_B$ below and $E$ as before is $\text{End}_A B$. 

...
Corollary 5.4. If $A \mid B$ is left $D2$, then $A^S \cong \text{End}_E A$. Thus if $A_B$ is balanced, $A^S = B$.

Proof. Follows by Prop. 2.4 and from the theorem by letting $B = C$. We note additionally from its proof that

$$A^S = \{ x \in A \mid \forall \alpha \in S, \alpha(x) = x\alpha(1) \}$$

since $\rho_\alpha(1) \in E$ in this case.

If $A_B$ is balanced, $\text{End}_E A = \rho(B)$ by definition. This recovers the result in [13, Section 4].

In other words, this corollary states that the invariant subring of $A$ under the action of the bialgebroid $S$ is (anti-isomorphic to) the bicommutator of the natural module $A$. Sugano studies the derived ring extension $A^* \mid B^*$ of bicommutants of a ring extension $A \mid B$, where $M_A$ is a faithful module, $E := \text{End} M_A$, $E : = \text{End} M_B$, $A^* = \text{End}_E M$, $B^* = \text{End}_E M$ and there are natural monomorphisms $A \rightarrow A^*$ and $B \rightarrow B^*$ commuting with the mappings $B \rightarrow A$ and $B^* \rightarrow A^*$ [23]: in these terms, $A^S \subseteq A$ is then the bicommutator of $A_A$ over the depth two extension $A \mid B$.

6. A JACOBSON-BOURBAKI CORRESPONDENCE FOR AUGMENTED RINGS

The Jacobson-Bourbaki correspondence is usually given between subfields $F$ of finite codimension in a field $E$ on the one hand, and their linear endomorphism rings $\text{End}_E F$ on the other hand. A subring of $\text{End}_E F$ which is itself an endomorphism ring of this form is characterized by containing $\lambda(E)$ and being finite dimensional over this. The inverse correspondence associates to such a subring $R \subseteq \text{End}_E F$, the subfield $\text{End}_R E$, since $\text{End}_R E$ is simple as a module. (The centralizer or commutant of $R$ in $\text{End}_E Z$ in other words.) The correspondences are inverse to one another by the Jacobson-Chevalley density theorem, and may be extended to division rings [7, Section 8.2].

Usual Galois theory follows from this correspondence, for if $E^G = F$ where $G$ is a finite group of automorphisms of $E$, then $\text{End}_E F \cong E \# G$ and subrings of the form $\text{End}_E H$ correspond to the subrings $E \# H$ where $H$ is a subgroup of $G$ such that $E^H = K$ for an intermediate field $K$ of $F \subseteq E$. In this section, we will use a similar idea to pass from the Jacobson-Bourbaki correspondence to the correspondence $A \mid B \mapsto \text{End}_B A_B$ and inverse $S \mapsto A^S$ for certain Hopf subalgebroids $S$ of $\text{End}_B A_B$ for certain depth two extensions $A \mid B$. First, we will give an appropriate generalization of the Jacobson-Bourbaki correspondence to noncommutative algebra, with a proof similar to Winter [26, Section 2].

For the purposes below, we say an augmented ring $(A, D)$ is a ring $A$ with a ring homomorphism $A \rightarrow D$ where $D$ is a division ring. Examples are division rings, local rings, Hopf algebras and augmented algebras. A subring $R$ of $\text{End}_A := \text{End} A_Z$, containing $\lambda(A)$, left finitely generated over this, where $R A$ is simple, is said to be a Galois subring.
Theorem 6.1 (Jacobson-Bourbaki correspondence for noncommutative augmented rings). Let $(A, D)$ be an augmented ring. There is a one-to-one correspondence between the set of division rings $B$ within $A$, where $B$ is a subring of $A$ and $A_B$ is a finite dimensional right vector space, and the set of Galois subrings of $A$. The correspondence is given by $B \mapsto \text{End}_{A_B}$ with inverse correspondence $R \mapsto \text{End}_{R}A$.

Proof. We first show that if $B$ is a division ring and subring of $A$ of finite right codimension, then $E = \text{End}_{A_B}$ is a Galois subring and $\text{End}_{E}A \cong B$. We will need a theory of left or (dually) right vector spaces over a division ring as for example to be found in [8, chap. 4]. Suppose $[A : B]_r = d$.

Since $\text{End}_{A_B}$ is isomorphic to square matrices of order $d$ over the division ring $B$, it follows that $\text{End}_{A_B}$ is finitely generated over the algebra $\lambda(A)$ of left multiplications of $A$. Also $E$ is simple, since $E = \text{End}_{A_B}$ acts transitively on $A$. Hence $\text{End}_{E}A$ is a division ring. Since

\begin{equation}
A_B = B_B \oplus W_B
\end{equation}

for some complementary subspace $W$ over $B$, it follows from Morita’s lemma (“generator modules are balanced”) that in fact $B \cong \text{End}_{E}A$.

Conversely, let $R$ be a Galois subring. Let $F^{op} = \text{End}_{R}A$ be the division ring (by Schur’s lemma) contained in $A^{op}$ (since $A \subseteq R$ and $\text{End}_{A}A \cong A^{op}$). To finish the proof we need to show that $[A : F]_r < \infty$ and $R = \text{End}_{A_F}$.

Since $R$ is finitely generated over $A$, we have $s_1, \ldots, s_n \in R$ such that

\begin{equation}
R = As_1 + \cdots + As_n.
\end{equation}

Let $e_1, \ldots, e_m \in A$ be linearly independent in the right vector space $A$ over $F$. Since $\pi A$ is simple, the Jacobson-Chevalley density theorem ensures the existence of elements $r_1, \ldots, r_m \in R$ such that for all $i$ and $k$,

\begin{equation}
r_i(e_k) = \delta_{ik}1_A.
\end{equation}

By the lemma below and the hypothesis that $A$ is an augmented ring, $m \leq n$. With a maximal linear independent set of vectors $e_i$ in $A$, we may assume $e_1, \ldots, e_m$ a basis for $A_F$. By definition of $F$, we have $R \subseteq \text{End}_{A_F}$. Let $E_{ij} := e_i r_j$ for $1 \leq i, j \leq m$ in $R$. Since $E_{ij}(e_k) = \delta_{jk}e_i$, these are matrix units which span $\text{End}_{A_F}$. Hence $\text{End}_{A_F} = R$. \hfill \Box

Lemma 6.2. Let $s_1, \ldots, s_n \in \text{End}_{A_Z}$, where $(A, D)$ is an augmented ring. Suppose that

\begin{equation}
r_1, \ldots, r_m \in As_1 + \cdots + As_n
\end{equation}

and there are elements $e_1, \ldots, e_m \in A$ such that $r_i(e_k) = \delta_{ik}1_A$ for $1 \leq i, k \leq m$. Then $m \leq n$.

Proof. By the hypothesis, there are elements $a_{ij} \in A$ such that $r_i = \sum_{j=1}^{n} a_{ij} s_j$ for each $i = 1, \ldots, m$. Then for $1 \leq i, k \leq m$,

\begin{equation}
\sum_{j=1}^{n} a_{ij} s_j(e_k) = r_i e_k = \delta_{ik}1_A.
\end{equation}
Applying the ring homomorphism $A \to D$ into the division ring $D$, where $a_{ij} \mapsto d_{ij}$, $s_j(e_k) \mapsto z_{jk}$, we obtain the matrix product equation,

$$
\begin{pmatrix}
  d_{11} & \cdots & d_{1n} \\
  \vdots & \ddots & \vdots \\
  d_{m1} & \cdots & d_{mn}
\end{pmatrix}
\begin{pmatrix}
  z_{11} & \cdots & z_{1m} \\
  \vdots & \ddots & \vdots \\
  z_{n1} & \cdots & z_{nm}
\end{pmatrix} =
\begin{pmatrix}
  1_D & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & 1_D
\end{pmatrix}
$$

This shows in several ways that $m \leq n$; for example, by the rank + nullity theorem for right vector spaces [8, Ch. 4, corollary 2.4].

Let $A \supseteq C$ be a D2 ring extension, so that $S := \text{End}_CA_C$ is canonically a left bialgebroid over the centralizer $A^C$. Any D2 subextension $A \supseteq B$ has sub-$R$-bialgebroid $S := \text{End}_BA_B$ where $R = A^B \subseteq A^C$. If all extensions are balanced, as in the situation we consider above, we recover the intermediate D2 subring $B$ by $S \sim A^S = B$. Whence $B \sim S$ is a surjective correspondence and Galois connection between the set of intermediate D2 subrings of $A \supseteq C$ and the set of sub-$R$-bialgebroids of $S$ where $R$ is a subring of $A^C$. We widen our perspective to include D3 intermediate subrings $B$, i.e. D3 towers $A \supseteq B \supseteq C$, and left coideal subrings of $S$ in order to pass from surjective Galois connection to Galois correspondence.

The Galois correspondence given by $B \sim \text{End}_CA_B$ and $\mathcal{J} \sim A^\mathcal{J}$ factors through the Jacobson-Bourbaki correspondence sketched in the theorem above. We apply Theorems 5.3, 5.2 and 2.8 below to do this. We need a notion of Galois left coideal subring $\mathcal{J}$ of a left $V$-bialgebroid $S$. For this we require of the left coideal subring $\mathcal{J} \subseteq \text{End}_CA_C$ that

1. the module $V \mathcal{J}$ is finitely generated projective where $V = A^C$;
2. $A$ has no proper $\mathcal{J}$-stable left ideals.

**Theorem 6.3.** Let $A \supseteq C$ be a D2 extension of an augmented ring $A$ over a division ring $C$, with centralizer $A^C$ denoted by $V$ and left $V$-bialgebroid $\text{End}_CA_C$ by $S$. Suppose $A_V$ is faithfully flat. Then the left D3 intermediate division rings of $A \supseteq C$ are in Galois correspondence with the Galois left coideal subrings of $S$.

**Proof.** Since $C \subseteq A$ is D2 and left or right split (as in eq. [13]), we may apply a projection $CA \to CC$ to the left D2 quasibases eq. to see that $A_C$ is a finite dimensional right vector space. For the same reasons, each extension $A \supseteq B$ (for an intermediate division ring $B$) is balanced by Morita’s lemma. If $B$ is additionally a left D3 intermediate ring, with $\mathcal{J} = \text{End}_CA_B$ a left coideal subring of the bialgebroid $S$ by Corollary 5.2 we have by Theorem 5.3 that the invariant subring $A^\mathcal{J} = B$. We just note that $V \mathcal{J}$ is f.g. projective by the dual of Lemma 4.3 and that a proper $\mathcal{J}$-stable left ideal of $A$ would be a proper $\text{End}_BA_B$-stable left ideal in contradiction of the transitivity argument in Theorem 6.1. Thus $B \leftrightarrow \text{End}_CA_B$ is a surjective order-reversing correspondence between the set of left D3 intermediate division rings $A \supseteq B \supseteq C$ into the set of Galois left coideal subrings of the $V$-bialgebroid $S$. 

Suppose we are given a Galois left coideal subring \( I \) of \( S = \text{End}_CA_C \). Then the smash product ring \( A \rtimes I \) has image we denote by \( R \) in \( \text{End}_CA_C \) via \( a \rtimes \alpha \mapsto \lambda_a \circ \alpha \) that is clearly a Galois subring, since \( \lambda(A) \subseteq R \) and is a finitely generated extension; also the module \( RA \) is simple by hypothesis (2) above. Then \( B = \text{End}_RA \) is an intermediate division ring between \( C \subseteq A \), and \( R = \text{End}_CA_B \) by Theorem 6.1. Since \( I \subseteq S \) and \( \nu I \) is flat, it follows from \( A \rtimes V S = \text{End}_CA_C \) that \( \text{End}_CA_B = A \rtimes V I \) via the mapping above. Note that \( I \subseteq \text{End}_CA_B \cap S = \text{End}_CA_B \) and let \( Q \) be the cokernel. Since \( A \rtimes \nu I = R \subseteq A \rtimes \text{End}_CA_B \) it follows that \( A \rtimes \nu Q = 0 \). Since \( A \nu \) is faithfully flat, \( Q = 0 \), whence \( I = \text{End}_CA_B \). Finally, \( \text{End}_CA_B \) is isomorphic to an \( A \)-\( C \)-bimodule direct summand of \( A^N \), since \( \nu I \otimes *= V^N \) for some \( N \), to which we apply the functor \( A \text{End}_CA \rtimes V \). Since \( A_B \) is finite free, it follows from Theorem 2.8 that \( A \supseteq B \supseteq C \) is left D3. \( \Box \)

If \( A \) or \( V \) is a division ring, the faithful flatness hypothesis in the theorem is clearly satisfied. In connection with this theorem we note the following criterion for a depth three tower of division algebras.

**Proposition 6.4.** Suppose \( C \subseteq B \subseteq A \) is a tower of division rings where the right vector space \( A_B \) has basis \( \{a_1, \ldots, a_n\} \) such that

\[
Ca_i \subseteq a_i B \quad (i = 1, \ldots, n)
\]

Then \( A \mid B \mid C \) is left D3.

*Proof.* It is easy to compute that \( x \otimes_B 1 = \sum_i a_i \otimes_B a_i^{-1} \beta_i(x) \) for all \( x \in A \). Here \( \beta_i \) is the rank one projection onto the right \( B \)-span of the basis element \( a_i \) along the span of \( a_1, \ldots, a_i, \ldots, a_n \), and \( a_i^{-1} \otimes_B a_i \in (A \otimes_B A)^C \) for each \( i \). Of course, \( \beta_i \in \text{End}_CA_B \), so \( A \mid B \) is left D3. \( \Box \)

We may similarly prove that the tower is rD3 if \( B \) has basis \( \{a_i\} \) satisfying \( a_i C \subseteq B a_i \). When \( B = C \) we deduce the following criterion for a depth two subalgebra pair of division rings. For example, the real quaternions \( A = \mathbb{H} \), and subring \( B = C \) meet this criterion.

**Corollary 6.5.** Suppose \( B \subseteq A \) is a subring pair of division rings where the left vector space \( _BA \) has basis \( \{a_1, \ldots, a_n\} \) such that

\[
a_i B = B a_i \quad (i = 1, \ldots, n).
\]

Then \( A \mid B \) is depth two.

We remark that if the centralizer \( V \) of a depth two proper extension \( A \mid C \) is contained in \( C \) (as in the example \( C = \mathbb{C} \) and \( A = \mathbb{H} \) just mentioned above), then \( \text{End}_CA_C \) is a skew Hopf algebra over the commutative base ring \( V \) [17]. Any intermediate ring \( B \) of \( A \mid C \), for which \( A \mid B \) is D2, has skew Hopf algebra \( \text{End}_B A_B \) over \( R = A^B \) for the same reason, since \( R \subseteq V \subseteq C \subseteq B \). It is interesting to determine under what conditions these are skew Hopf subalgebras, i.e., the antipodes are compatible under the sub-\( R \)-bialgebroid structures.
7. Application to field theory

Given a separable finite field extension $F \subseteq E$ Szlachányi shows that there is a Galois connection between intermediate fields and weak Hopf subalgebras of $\text{End}_E F$. A weak Hopf algebra $H$ the reader will recall from the already classic [3] is a weakening of the notion of Hopf algebra to include certain non-unital coproducts, non-homomorphic counits with weakened antipode equations. There are certain canonical coideal subalgebras $H^L$ and $H^R$ that are separable algebras and anti-isomorphic copies of one another via the antipode. Nikshych and Etingof [5] have shown that $H$ is a Hopf algebroid over the separable algebra $H^L$, and conversely the author and Szlachányi [13] have shown that Hopf algebroids over a separable algebra are weak Hopf algebras. Let’s revisit one of the important, motivating examples.

Example 7.1. Let $G$ be a finite groupoid with $x, y \in G_{\text{obj}}$ the objects and $g, h \in G_{\text{arrows}}$ the invertible arrows (with sample elements). Let $s(g)$ and $t(g)$ denote the source and target objects of the arrow $g$. Suppose $k$ is a field. Then the groupoid algebra $H = kG$ (defined like a quiver algebra, where $gh = 0$ if $t(h) \neq s(g)$) is a weak Hopf algebra with coproduct $\Delta(g) = g \otimes k g$, counit $\varepsilon(g) = 1$, and antipode $S(g) = g^{-1}$. Since the identity is $1_H = \sum_{x \in G_{\text{obj}}} \text{id}_x$, we see that $\Delta(1_H) \neq 1_H \otimes 1_H$ if $G_{\text{obj}}$ has two or more objects. Notice too that $\varepsilon(gh) \neq \varepsilon(g)\varepsilon(h)$ if $gh = 0$.

The Hopf algebroid structure has total algebra $H$, and has base algebra the separable algebra $kG_{\text{obj}}$, which is a product algebra $k^N$ where $N = |G_{\text{obj}}|$. The source and target maps of the Hopf algebras $s_L, t_L : R \rightarrow H$ are simply $s_L = t_L : x \mapsto \text{id}_x$. The resulting bimodule structure $R H R = s_L t_L H$ is given by $x \cdot g \cdot y = g$ if $x = y = t(g)$, 0 otherwise. The coproduct is $\Delta(g) = g \otimes R g$, counit $\varepsilon(g) = t(g)$, and antipode $S(g) = g^{-1}$. This defines a Hopf algebroid in the sense of Lu and Xu. That this is also a Hopf algebroid in the sense of Böhm-Szlachányi may be seen by defining a right bialgebroid structure on $H$ via the counit $\varepsilon_r(g) = s(g)$.

If $G$ is the finite set $\{1, \ldots, n\}$ with singleton hom-groups, suggestively denoted by $\text{Hom}(i, j) = \{e_{ij}\}$ for all $1 \leq i, j \leq n$, the groupoid algebra considered above is the full matrix algebra $H \cong M_n(k)$ and $R$ is subalgebra of diagonal matrices. Note that the projection $\Pi^L = \varepsilon_l$ in [5] defined as $\Pi^L(x) = \varepsilon(1_{(1)} x)1_{(2)}$ is given here by $e_{ij} \mapsto e_{ii}$. Similarly, $\Pi^R(e_{ij}) = e_{jj}$.

In [25], Szlachányi shows that although Hopf-Galois separable field extensions do not have a universal Hopf algebra as “Galois quantum group,” they have a universal weak Hopf algebra or ”Galois quantum groupoid.” For example, the field $E = \mathbb{Q}(\sqrt{2})$ is a four dimensional separable extension of $F = \mathbb{Q}$ which is Hopf-Galois with respect to two non-isomorphic Hopf algebras, $H_1$ and $H_2$ [6]. However, the endomorphism ring $\text{End}_E F$ is then a smash product in two ways, $E \# H_i, i = 1, 2$, and is a weak Hopf algebra over the separable $F$-algebra $E$. It is universal in a category of weak Hopf
algebras viewed as left bialgebroids [25, Theorem 2.2], with modifications to the definition of the arrows resulting (see [25, Prop. 1.4] for the definition of weak left morphisms of weak bialgebras). The separable field extensions that are Hopf-Galois may then be viewed as being weak Hopf-Galois with a uniqueness property.

The following corollary addresses an unanswered question in [25, Section 3.3]. Namely, there is a Galois connection between intermediate fields $K \subseteq F \subseteq E$ of a separable (finite) field extension $E \mid K$ and weak Hopf subalgebras of the weak Hopf algebra $A := \text{End} E_K$ that include $E$ as left multiplications. The correspondences are denoted by

$$\text{Sub}_{WA/K}(A) \xrightarrow{\text{Fix}} \text{Sub}_{Alg/K}(E)$$

which associates to a weak Hopf subalgebra $W$ of $\text{End} E_K$ the subfield

$$\text{Fix}(W) = \{x \in E \mid \forall \alpha \in W, \alpha(x) = \alpha(1)x\},$$

in other words, $E^W$, and the correspondence

$$\text{Sub}_{Alg/K}(E) \xrightarrow{\text{Gal}} \text{Sub}_{WA/K}(A)$$

where the intermediate subfield $K \subseteq F \subseteq E$ gets associated to its Galois algebra

$$\text{Gal}(E) = \{\alpha \in A \mid \forall x \in E, y \in F, \alpha(xy) = \alpha(x)y\}.$$ 

Clearly $\text{Gal}(E) = \text{End} E_F$.

Szlachányi [25, 3.3] notes that $\text{Gal}$ is a surjective correspondence, since $E = \text{Fix}(\text{Gal}(F))$ for each intermediate subfield (e.g. since $E_F$ is a generator module, it is balanced by Morita’s lemma). $\text{Gal}$ is indeed a one-to-one correspondence by

**Corollary 7.2.** $\text{Gal}$ and $\text{Fix}$ are inverse correspondences between intermediate fields of a separable field extension $E \mid K$ and weak Hopf subalgebras of the full linear endomorphism algebra $\text{End} E_K$.

**Proof.** We just need to apply the Jacobson-Bourbaki correspondence with a change of notation. Before changing notation, first note that if $A \supseteq B$ is a depth two extension where $B$ is a commutative subring of the center of $A$, then the centralizer $A^B = A$ and the left bialgebroid $\text{End}_B A_B = \text{End} A_B$ over $A$. Indeed, a faithfully flat $B$-algebra $A$ is depth two iff it is finite projective. If $A$ and $B$ are fields, this reduces to: depth two extension $A \mid B$ $\iff$ finite extension $A \mid B$. If $A \mid B$ is a Frobenius extension (as are separable extensions of fields), there is an antipode on $\text{End}_B A_B$ defined in terms of the Frobenius homomorphism (such as the trace map of a separable field extension [18]) and its dual bases [4]. Now, changing notation, we have a bialgebroid $\text{End} E_K$ over the separable $F$-algebra $E$, or equivalently a weak bialgebra — which becomes a weak Hopf algebra via an involutive antipode given in terms of the trace map and its dual bases [25 eq. (3.5)]).

Given a weak Hopf subalgebra $W$ of $\text{End} E_K$ containing $\lambda(E)$, it is automatically finite dimensional over $E$ and $WE$ is simple since a submodule
is a $W$-stable ideal, but $E$ is a field. Hence, $W$ is a Galois subring and the Theorem 6.1 shows that $\text{End}_{W}E \cong E^{W}$ is an intermediate field $F$ between $K \subseteq E$, such that $\text{End}_{E}F = W$. But $\text{Gal}(F) = \text{End}_{E}F$ has been noted above. Hence, $\text{Gal}(\text{Fix}(W)) = W$. □

The only reason we need restrict ourselves to separable field extensions above is to acquire a fixed base algebra that is a separable algebra, so that we acquire antipodes from Frobenius extensions, and Hopf algebroids become weak Hopf algebras. Let us be clear on what happens when we drop this hypothesis. For the purpose of the next corollary, we define a sub-$R$-bialgebroid of bialgebroid $(H, R, s_L, t_L\Delta, \varepsilon)$ to be a subalgebra $V$ of the total algebra $H$ with the same base algebra $R$, source $s_L$ and target $t_L$ maps having image within $V$, and $V$ is a sub-$R$-coring of $(H, \Delta, \varepsilon)$.

Corollary 7.3. Let $E \supseteq K$ be a finite field extension. Then the poset of intermediate subfields is in Galois correspondence with the poset of sub-$E$-bialgebroids of End$_E^K$.

Proof. This follows from the Jacobson-Bourbaki correspondence, where intermediate field $F \mapsto \text{End}_{E}F$ with inverse, Galois subring $R \mapsto \text{End}_{R}E$, with the same proof as in the previous corollary. Note from the proof of Jacobson-Bourbaki in the field context that any subring of $\text{End}_{E}^K$ containing $\lambda(E)$ is indeed of the form $\text{End}_{E}F$ for some intermediate $K \subseteq F \subseteq E$, and therefore the left bialgebroid of the depth two (= finite) field extension $F \subseteq E$, and sub-$E$-bialgebroid of $\text{End}_{E}^K$. □

The Jacobson-Bourbaki correspondence also exists between subfields of a finite dimensional simple algebra $A$ and subalgebras of the linear endomorphism algebra which contain left and right multiplications [21, sect. 12.3], a theorem related to the topic of Brauer group of a field. By the same reasoning, we arrive at Galois correspondences between subfields and bialgebroids over $A$. Namely, let $A^{'}$ denote the image of $A \otimes_F A^{\text{op}}$ in the linear endomorphism algebra $\text{End}_{A^{\text{op}}}A^{\text{op}}$ via left and right multiplication $x \otimes y \mapsto \lambda_x \circ \rho_y$, and $Z(A)$ denote the center of $A$, which is a field since $Z(A) \cong \text{End}_{A^{\text{op}}}A$. We note that $\text{End}_{A^{\text{op}}}A^{\text{op}}$ is a bialgebroid over $A$ for any intermediate field $F \subseteq E \subseteq Z(A)$ with Lu structure [19], and a Hopf algebroid in the special case $E = Z(A)$ where $A$ becomes Azumaya so $A \otimes_E A^{\text{op}} \cong \text{End}_{A^{\text{op}}}A^{\text{op}}$. The proof is quite the same as above and therefore omitted.

Corollary 7.4. Let $A$ be a simple finite dimensional $F$-algebra. Then the fields that are intermediate to $F \subseteq Z(A)$ are in Galois correspondence to the sub-$A$-bialgebroids of $\text{End}_{A^{\text{op}}}A^{\text{op}}$. In case $A$ is a separable $F$-algebra, the intermediate fields are in Galois correspondence to weak Hopf subalgebras of $\text{End}_{A^{\text{op}}}A^{\text{op}}$.

8. AN EMBEDDING THEOREM FOR FINITE DEPTH FROBENIUS EXTENSIONS

In this section we define finite depth Frobenius extension using the notion of depth three tower by choosing a suitable three-ring sub-tower of the Jones
tower. We show that this definition is consistent with previous definitions of finite depth for subfactors and free Frobenius extensions. We show in Theorem 3.3.4 that any finite depth Frobenius extension extends to a depth two extension somewhere further along in its Jones tower.

Suppose $M_{-1} \hookrightarrow M_0$ is a Frobenius extension; e.g. a (type II$_1$) subfactor of finite index [9, 12, 13] or a Frobenius algebra in the tensor category of bimodules over $M$ of finite index [9, 12, 13]. Let $M_1$ denote its basic construction $M_1 = Me_1 M$ which is isomorphic to End $M_N$ and to $M \otimes_N M$ where $M := M_0$ and $N := M_{-1}$. The ring extension $M_1 | M$ is itself a Frobenius extension with $M$-bimodule Frobenius homorphism $E_1 : M_1 \to M$ defined by $E_1(e_1) = 1$. The Jones element $e_1$ maps isomorphically into the Frobenius homomorphism and into the cyclic generator $1_M \otimes_N 1_M$. Iterate this to obtain the Jones tower,

\[(46) \quad N = M_{-1} \hookrightarrow M = M_0 \hookrightarrow M_1 \hookrightarrow M_2 \hookrightarrow \cdots \hookrightarrow M_n \hookrightarrow \cdots,\]
e.g., $M_2 = M_1 e_2 M_1$ where $e_2$ maps into the Frobenius homomorphism $E_1 : M_1 \to M_0$ and into $1_{M_1} \otimes_M 1_{M_1}$, $1_{M_1} = \sum_i x_i e_1 y_i$. Note that $M_n \cong M \otimes N \cdots \otimes N M$ ($n + 1$ times $M$). Each ring extension $M_n | M_{n-1}$ is a Frobenius extension by the endomorphism ring theorem (iterated). Each composite ring extension $M_n | M_{n-k}$ is a Frobenius extension by composing Frobenius homomorphisms, and $M_{n+k}$ is isomorphic to the basic construction of $M_n | M_{n-k}$ by [11] appendix]. While the $e_i$ may not be idempotents or projections, they satisfy $e_i e_{i \pm 1} e_i = e_i$, $e_i y e_i = E_{i-1}(y)e_i = e_i E_{i-1}(y)$ and $e_i x = e_i E_i(e_i x)$ for all $y \in M_{i-1}$, $x \in M_i$ with Frobenius homomorphisms $E_i : M_i \to M_{i-1}$. For more details on the Jones tower over a Frobenius extension, please see [13, section 6] and [9, chapter 3].

**Definition 8.1** (Finite depth Frobenius extensions). The Frobenius extension $N \hookrightarrow M$ is said to be of depth $n > 1$ if the composite tower $M_{n-2} | M_{n-3} | M_{-1}$ is a right or left depth three tower.

The definition allows for the possibility of a depth $n$ extension being at the same time depth $n + 1$, something we note to be true below. In subfactor theory, one speaks of depth $n$ subfactor as the least $n$ for which the relative commutant $M^N_n$ is a basic construction of the two previous semisimple algebras in the derived tower, which we introduce next.

Let $M^N_n$ denote the centralizer of $N$ in $M_i$. We note the derived tower of eq. (46) above:

\[(47) \quad N^N \hookrightarrow M^N \hookrightarrow M^N_1 \hookrightarrow \cdots \hookrightarrow M^N_{n-1} \hookrightarrow M^N_n \hookrightarrow \cdots\]

In classical subfactor theory, depth $n$ is characterized by the least $n$ for which $M^N_n$ is isomorphic to the basic construction of $M^N_{n-1}$ over $M^N_{n-2}$. We compute that this is so with our new definition, which we also show to be consistent with the definition in [12, 3.1] of depth $n$ free Frobenius extension.
Proposition 8.2. A depth $n$ Frobenius extension $N \hookrightarrow M$ has $n$-step centralizer $M_n^N \cong M_{n-1}^N \otimes_{M_{n-2}^N} M_{n-1}^N$. The Frobenius homomorphism $E_{n-1}$ has dual bases elements in $M_{n-1}^N$.

Proof. Let $A = M_{n-2}$, $B = M_{n-3}$ and $C = M_{n-1}$. Then subtower $A | B | C$ of the Jones tower $[46]$ is $D3$. Whence the $A$-$C$-bimodules $A \otimes_B A$ and $A$ are $H$-equivalent, and their endomorphism rings are Morita equivalent.

Note that $\text{End}_A A \otimes_B A_C \cong \text{End}_A(M_{n-1})_N \cong M_n^N$, since $M_{n-1}$ is isomorphic to the basic construction of $M_{n-2} | M_{n-3}$ and anti-isomorphic to $\text{End}_B A$. We use as well that the $C$-centralizer $(\text{End}_B A)^C \cong \text{End}_B A_C$.

On the other hand, we note

$$\text{End}_A A \cong A^C \cong M_{n-2}^N,$$

so we conclude $M_n^N$ and $M_{n-2}^N$ are Morita equivalent rings. The Morita context bimodules are the $A$-$C$-bimodule hom-groups $\text{Hom}(A \otimes_B A, A)$ and $\text{Hom}(A, A \otimes_B A)$. In section 2 we saw that first of all,

$$\text{Hom}(A \otimes_B A, A) \cong \text{End}_B A_C \cong M_{n-1}^N,$$

since $M_{n-1}$ is anti-isomorphic to the left endomorphism ring of $M_{n-2} | M_{n-3}$. Since the Frobenius extension $A | B$ satisfies $\text{End}_B A \cong A \otimes_B A$, we also obtain that the $C$-centralizers,

$$\text{End}_B A_C \cong (A \otimes_B A)^C \cong \text{Hom}(A, A \otimes_B A)$$

the last step following from section 2. In other words, $M_{n-1}^N$ doubles as both of the Morita context bimodules between $M_n^N$ and $M_{n-2}^N$.

By Morita theory, it follows that the Morita context bimodules satisfy $M_n^N \cong M_{n-1}^N \otimes_{M_{n-2}^N} M_{n-1}$.

The last statement is proven as $[10]$ Theorem 2.1, part (5) and $[13]$ Prop. 6.4]. Toward this end, we note that $M_{n-1} \cong A e_{n-1} A$ and $M_n^N \cong (A \otimes_B A)^C \cong \text{End}_C A B$. Suppose a left $D3$ quasibases given by $t_i = t_i^1 e_{n-1} t_i^2$ and $\beta_i \in \text{End}_C A B = (\text{End}_B A)_C$; with $\sum_j x_j \otimes_B y_j \in (A \otimes_B A)^A$ dual bases for the Frobenius homomorphism $E_{n-2} : A \rightarrow B$. Then $E_{n-1} : M_{n-1} = A e_{n-1} A \rightarrow A$ has dual bases $t_i = t_i^1 e_{n-1} t_i^2$ and $\sum_j \beta_i(x_j) e_{n-1} y_j$, both in $M_{n-1}^N$.

We note that depth $n$ Frobenius extensions are depth $n + 1$ as follows.

Lemma 8.3 (Endomorphism ring lemma for $D3$ towers). Suppose $A | B | C$ is $D3$ where $A | B$ is a Frobenius extension. Let $D = \text{End}_B A$ and $D | A$ denote the left regular mapping $a \mapsto \lambda_a$. Then the tower $D | A | C$ is $D3$.

Proof. Since $A \otimes_B A$ is a Frobenius extension, we note that $D D_A \cong D A \otimes_B A A$ given by $d \mapsto \sum_i d(x_i) \otimes_B y_i$ in the Frobenius system notation above. Hence $D D_A D_C \cong D A \otimes_B A \otimes_B A C$. To $A A \otimes_B A C \otimes \ast \cong A A_C^n$ we apply the functor
Suppose the tower $A \mid B \mid C$ is rD3. Let $C \rightarrow B$ factor through a Frobenius extension $D \rightarrow B$ such that $A \cong \text{End} B_D$. Then $A \mid D \mid C$ is rD3.

**Proof.** We have $A \cong B \otimes_D B$, from which we obtain $A \otimes_B A \otimes_B A \cong A \otimes_D A$. Tensoring $A \otimes_B A \oplus \ast \cong A^n$ by $A \otimes_B -$, we obtain this fact. □

The next theorem shows that a depth $n = 2^m + 1 = 3, 5, 9, 17, \ldots$ Frobenius extension $N \leftarrow M$ is embedded in (“is a factor of”) the depth two extension $N \leftarrow M_{n-2}$.

**Theorem 8.5.** If $N \leftarrow M$ is a depth $n$ Frobenius extension, where $n = 2^m + 1$ for a positive integer $m$, then $N \leftarrow M_{n-2}$ is a depth two Frobenius extension.

**Proof.** By hypothesis, the Jones subtower $M_{n-2} \mid M_{n-3} \mid M_{-1}$ is D3. Since $M_{n-2} \cong M_{n-3} \otimes M_{n-4} M_{n-3}$, it follows from the tunneling lemma that $M_{n-2} \mid M_{n-4} \mid M_{-1}$ is D3. Iterating use of the lemma, also $M_{n-2} \mid M_{n-6} \mid M_{-1}$ is D3. Continuing $m-1$ steps, $M_{n-2} \mid M_{2m-1-1} \mid M_{-1}$ is D3. But $M_{n-2} \cong M_{2m-1-1} \otimes N M_{2m-1-1}$ since $n - 2 = 2^m - 1$ (and recalling $M_i = M^\otimes N i + 1$). By Theorem 2.5 $M_{n-2} \mid N$ is D2. □

Suppose a Frobenius extension $N \leftarrow M$ is depth $n$, where $2^{m-1} < n \leq 2^m$. Then by the endomorphism ring lemma this extension is also depth $2^m + 1$, so $N \leftarrow M_{2m-1}$ is D2 by the theorem. This proves:

**Corollary 8.6.** Any finite depth Frobenius extension $N \leftarrow M$ embeds into a depth two extension of $N$ in an $n$’th iterated endomorphism ring $M_n$.

This result may be viewed as an algebraic version of a result in [20] and an answer to a question in [11, appendix]. The theory above seems to indicate that one of a variety of extensions of Jacobson-Bourbaki correspondence to pairs of simple algebras by the Japanese school of ring theory (Hirata, Müller, Onodera, Sugano, Szeto, Tominaga, and others) would adapt via depth two extensions and depth three towers to an algebraic version of the Galois theory for subfactors in Nikshych and Vainerman [20]. This will be investigated in a future paper.

Since group algebras and their finite index subgroups form Frobenius extensions, we pose the following question in character theory and group theory in extension of the discussion and results in sections 3 and 8 of this paper.

**Question:** what precisely are the group-theoretic conditions on a subgroup of a finite group $H < G$ that its Frobenius algebra extension $N = \mathbb{C} [H] \subseteq M = \mathbb{C} [G]$ be depth $n$?
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