Critical behaviors and local transformation properties of wave function

Jian Cui*, Jun-Peng Cao*, and Heng Fan
Beijing National Laboratory for Condensed Matter Physics, Institute of Physics,
Chinese Academy of Sciences, Beijing 100190, People’s Republic of China
(Dated: December 23, 2010)

We investigate crossing behavior of ground state entanglement Rényi entropies of quantum critical systems. We find a novel property that the ground state in one quantum phase cannot be locally transferred to that of another phase, that means a global transformation is necessary. This also provides a clear evidence to confirm the long standing expectation that entanglement Rényi entropy contains more information than entanglement von Neumann entropy. The method of studying crossing behavior of entanglement Rényi entropies can distinguish different quantum phases well. We also study the excited states which still give interesting results.

PACS numbers: 03.67.Mn, 73.43.Nq, 03.65.Ud, 74.40.Kb

Introduction.— Methods developed in quantum information have proven to be very useful in studying the state of many-body system [1]. At the same time experience in condensed matter physics is helping in finding novel protocols for quantum computation and communication. At the interface between many-body system and quantum information, the analysis of entanglement in quantum critical models has been attracting a great deal of interests [1,2].

Entanglement, one intriguing feature of quantum theory and a main resource for quantum information processing, is generally quantified by entanglement von Neumann entropy (EvNE). It measures how closely entangled the two subsystems are. It is shown that the behavior of the critical EvNE is analogous to that of entropy in conformal field theories which are for the quantum critical systems [3]. In addition, this EvNE contains the topological entropy, a universal constant term and a topological order, for a topological ordered state [5,6]. A natural generalization of EvNE is the entanglement Rényi entropy (ERE) which is believed to contain more information. However, it is shown that ERE does not provide more information than EvNE for the case of topological order [2]. The question arising is: what is the additional information in ERE not contained in EvNE and how to use it in studying quantum critical phenomena? In this Letter, we will provide an answer to this question.

For a pure bipartite state, $|\Psi_{AB}\rangle$, the ERE is defined with respect to a parameter $\alpha > 0$ as,

$$S_\alpha(\rho_A) = \frac{1}{1-\alpha} \log_2[\text{Tr}\rho_A^\alpha],$$

(1)

where $\rho_A$ is the reduced density operator of subsystem $A$ by tracing out another subsystem $B$, $\rho_A = \text{Tr}_B(|\Psi_{AB}\rangle \langle \Psi_{AB}|)$. Note that $\rho_A$ gives the same result. In the limit, $\alpha \rightarrow 1$, ERE recovers the definition of EvNE, $S_1(\rho_A) = S(\rho_A) \equiv -\text{Tr}[\rho_A \log_2 \rho_A]$. Entanglement is invariant for local unitary transformations in subsystems $A$ or $B$, so what matters in EvNE is the eigenvalue spectrum of the reduced density operator. Actually the entanglement spectrum, a redefinition of eigenvalue spectrum, reveals much more information than EvNE, a single number $\tau_1$. This motivated us still to consider an approach to extract the extra information from ERE, a natural generalization of EvNE.

In studying quantum phase transitions by tools developed in quantum information, generally the ground state properties, in particular entanglement [1] or fidelity [11], are studied. We may consider the EvNE, or similarly concurrence, with various partitions. The critical points can correspond to peaks or nonanalytic points with different orders for quantities like EvNE, concurrence or fidelity. Those methods, though approved to be powerful and successful, have drawbacks. One drawback might be that there is no unified standard to determine whether there exists a critical point or not. Another drawback might be that the ground state property, in particular from quantum information point of view, for different quantum phases may not be completely revealed. Next in this Letter, we will try to present a powerful and novel approach different from known methods, but without the drawbacks mentioned above.

Method.—Entanglement does not increase under local transformations. So one key property of a entanglement measure is that it does not increase under local quantum operations and classical communication (LOCC). Thus for pure bipartite quantum states, it is only possible that a state with higher entanglement be transferred by LOCC to a state with lower entanglement though it is not always successful [12]. The entanglement measure, such as the well accepted EvNE, however, is not unique. In particular, ERE with parameter $\alpha$ is also a entanglement measure. Therefor, it is not surprising that the following case is possible: For two bipartite pure states $|\psi_{AB}\rangle$ and $|\phi_{AB}\rangle$ with reduced density operators denoted as $\rho_{\psi_A}$ and $\rho_{\phi_A}$, when $\alpha = \alpha_1$, ERE of $|\psi_{AB}\rangle$ is larger than that of $|\phi_{AB}\rangle$, $S_{\alpha_1}(\rho_{\psi_A}) > S_{\alpha_1}(\rho_{\phi_A})$; while on the other hand when $\alpha = \alpha_2$, we have the opposite direction, $S_{\alpha_2}(\rho_{\psi_A}) < S_{\alpha_2}(\rho_{\phi_A})$. That means neither state $|\psi_{AB}\rangle$ be transferred locally to state $|\phi_{AB}\rangle$ nor the op-
The above results can be applied to study the quantum critical phenomena. We suppose when a quantum phase transition occurs, the behavior of ground state ERE as well as the local transformation property of the ground state wave function changes, and the different quantum phases boundaries can be determined by the ERE. By carefully analyzing the behavior of ERE, we find two cases. (i) In some phases, the ground state EREs are crossing, while in other phases, the ground state EREs are not crossing, please see the Table I (left) and example model I. (ii) The ground states EREs do not cross with others in the same phase, but they are crossing in the different phases, please see the Table I (right) and example model II. From the view of local transformation the above results are explained as follows. (i) In some phases, ground states can not be locally transferred into each other, i.e., a global transformation is necessary; while in the other phases, ground states can be locally transferred. (ii) The ground state can be transferred into each other in the same phase with local transformation. However, the ground state can not be transferred locally in the different phases. These properties can be used to distinguish the quantum phases transitions and the critical points can be found. We should notice that the information about crossing can not be obtained from the EVNE which is only the \( \alpha = 1 \) case.

### Example model I: XY spin chain.

The Hamiltonian of a 1D spin-1/2 XY chain takes the form of

\[
H = -\sum_i [(1 + \gamma)\sigma_i^x \sigma_{i+1}^x + (1 - \gamma)\sigma_i^y \sigma_{i+1}^y + h\sigma_i^z],
\]

where \( \sigma_i^{x,y,z} \) are Pauli matrices at site \( i \), \( \gamma \) and \( h \) are coupling and field parameters. Here periodical boundary condition is assumed. The phase diagram of XY chain is presented in Fig. 2. When the parameter \( \gamma \) equals to one, the system (1) degenerates into the 1D Ising model with transverse field, \( H_I = -\sum_i (\sigma_i^x \sigma_{i+1}^x + g\sigma_i^z) \), where \( h = 2g \). It is well-known, a phase transition takes place at \( g = 1 \) which is gapless, while \( g < 1 \) and \( g > 1 \) are two gapped phases.

In order to make a clear description of our method, we firstly study the 1D Ising model with transverse field by the method of ERE. In our method, we first find the ground state and partition it as two parts \( A:B \), \( |G_{AB}\rangle \), then calculate the ERE of this ground state. In our numerical calculations, the total site \( N \) is taken to be 10, and the chain is cut into two blocks with each 5 sites respectively.

The results are presented in Figs. 3, 4, 5. We see that our method works well for 1D transverse Ising model. The quantum phases are clearly distinguished by different crossing behavior of EREs of the ground state. Inter-
region II (red line) with the boundary interval. First, the step size of sharpened to approach the critical point by narrowing boundaries of I and III. Step by step, region II will be shown in Fig. 4, and we can fix that the phase transition (black line) is the no crossing region with from 0
g <
I (blue line) is the crossing region with
g >
region II (right-up), for

g <
1, EREs are crossing (right-down) and

g >
1, EREs are non-crossing (right-down) and

TABLE II: The crossing points with parameter \( g \) for the transverse Ising model. For clearance, we list the table into separate parts, \( g \leq 0.98 \), \( g \geq 1 \) and \( 0.98 \leq g \leq 1 \).

| \( g \)  | 0.94 | 0.95 | 0.96 | 0.97 | 0.98 | 0.99 | 1.00 | 1.01 | 1.02 | 1.03 | 1.04 |
|------|------|------|------|------|------|------|------|------|------|------|------|
| 0.95 | N    | N    | N    | N    | N    | N    | N    | N    | N    | N    | N    |
| 0.96 | N    | N    | N    | N    | N    | N    | N    | N    | N    | N    | N    |
| 0.97 | N    | N    | N    | N    | N    | N    | N    | N    | N    | N    | N    |
| 0.98 | N    | N    | N    | N    | N    | N    | N    | N    | N    | N    | N    |
| 0.99 | N    | N    | N    | N    | N    | N    | N    | N    | N    | N    | N    |
| 1.00 | N    | N    | N    | N    | N    | N    | N    | N    | N    | N    | N    |
| 1.01 | N    | N    | N    | N    | N    | N    | N    | N    | N    | N    | N    |
| 1.02 | N    | N    | N    | N    | N    | N    | N    | N    | N    | N    | N    |
| 1.03 | N    | N    | N    | N    | N    | N    | N    | N    | N    | N    | N    |
| 1.04 | N    | N    | N    | N    | N    | N    | N    | N    | N    | N    | N    |

FIG. 5: (Color online). The finite size scaling behavior of the ground state EREs.

happens at around \( g = 1 \). Next, the step size of \( g \) is fixed to 0.01 to be more accurate and we let \( g \) run from 0.94 to 1.04. For this case, the crossing figure is not quite clear, so we use table to show where the crossing points are. For example, in the first row of Table II, we find that \( g = 0.94 \) get crossed with \( g = 0.95, 0.96, ..., \) at \( \alpha = 0.6, 0.5, .... \). For region III, no crossing exists which is denoted as N. By Table II, we find region II is 0.98 \leq g \leq 1.00. We can go on investigating this phase transition more accurately by the same method and we list the result here: When step size of \( g \) is 0.001, the critical region obtained by this method is 0.987 \leq g \leq 0.989; When step size of \( g \) is 0.0001, the critical region is 0.9883 \leq g \leq 0.9885. We can see that region II is sharpened as parameter \( g \) becomes more accurate. The finite size scaling analysis is shown in Fig. 5. We see that the critical point obtained by this method is 0.9949, which is very close to the actual value.

Here we present also the interesting crossing phenomena of EREs between ground state and the first excited state in different phases for Ising model, see Fig. 6. In the ferromagnetic phase (\( g < 1 \)), the ground state and the first excited state can not transfer locally, while in the paramagnetic phase (\( g > 1 \)), the first excited state can locally transfer to the ground state. This result generalizes our previous ones and we can determine the zero temperature quantum phase transition even using the finite temperature properties.

Then, we consider the general case. Without loss of generality, we consider the the red dashed line in Fig.2 where \( \gamma = \sqrt{3}/2 \). Then the phase changes from \( 1B \) to \( 1A \).
At $h = 1$ and then from 1A to phase 2 at $h = 2$. We find that the ground state EREs of the system with different magnetic field $h$ are crossed in phase 1B, not crossed in phase 1A, and crossed again in phase 2. Table III gives a summary of the crossing results. The details of crossing properties show that the phase transitions take place in regions $0.999 \leq h \leq 1.000$ and $2.010 \leq h \leq 2.012$, which are very close to the actual values $h = 1$ and $h = 2$.

**Example model II: XXZ spin chain.**—The Hamiltonian of spin-1/2 XXZ model is,

$$H_{XXZ} = \sum_{i} \sigma_{i}^{x} \sigma_{i+1}^{x} + \sigma_{i}^{y} \sigma_{i+1}^{y} + \Delta \sigma_{i}^{z} \sigma_{i+1}^{z},$$

(3)

where $\Delta$ is the anisotropic parameter. There are two critical points: $\Delta = -1$ corresponds to a first order phase transition, $\Delta = 1$ is a continuous phase transition. In particular phase transition at $\Delta = 1$ is a Kosterlitz-Thouless like transition, the entanglements and their arbitrary order of derivatives are analytic. We next try to identify the critical point $\Delta = 1$ by the ERE method.

Table IV shows the crossing points near $\Delta = 1$. We can see that each state in either region $\Delta \geq 1.0$ or $\Delta \leq 1.0$ never cross with any of the states in the same region, but get crossed with at least one state from the other region. The critical region can be found to be $0.9 \leq \Delta \leq 1.1$. By raising the accuracy, this critical point can be found exactly. In all, our ERE method also works well for the infinite order phase transition in XXZ spin chain.

**Summary.**—We propose a new method concerning about the local transformation property of ground state to study quantum phase transitions. Further, we have shown ERE contains more information than EvNE in that by ERE we known deterministically whether two ground states can be transferred locally to each other. As example models, our method works well for 1D transverse Ising model and XY spin chain. Interestingly, ground states similarly in two gapped regions may possess different local transformation properties. Our method also works well for the elusive critical point of XXZ spin chain. This simple and general method is worth (a) generalizing to study finite temperature phase transitions (b) generalizing based on the majorization scheme [12] and (c) applying to other systems.

We thank Zhi-Hao Xu and Zhao Liu for helpful discussions. This work is supported by NSFC grants (10934010, 10974233, 10974247), Knowledge Innovation Project of Chinese Academy of Sciences, and “973” program (2010CB922904, 2011CB921500, 2011CB921704).

| $\Delta$ | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 | 1.1 | 1.2 | 1.3 | 1.4 |
|---------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\Delta = 1$ | N | N | N | N | N | N | N | N | N | N | N | N |
| $\Delta = 2$ | N | N | N | N | N | N | N | N | N | N | N | N |
| $\Delta = 3$ | N | N | N | N | N | N | N | N | N | N | N | N |
| $\Delta = 4$ | N | N | N | N | N | N | N | N | N | N | N | N |
| $\Delta = 5$ | N | N | N | N | N | N | N | N | N | N | N | N |
| $\Delta = 6$ | N | N | N | N | N | N | N | N | N | N | N | N |

* Electronic address: [cuijian; junpencao; hfan@iphy.ac.cn]

[1] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys. 80, 517 (2008).
[2] A. Osterloh, L. Amico, G. Falci, and R. Fazio, Nature 416, 608 (2002).
[3] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Phys. Rev. Lett. 90, 227902 (2003).
[4] J. Cui, J.P. Cao, H. Fan, Phys. Rev. A 82, 022319 (2010); H. Fan, V. Korepin, V. Roychowdhury, Phys. Rev. Lett. 93, 227203 (2004).
[5] A. Kitaev and J. Preskill, Phys. Rev. Lett. 96, 110404 (2006).
[6] M. Levin and X. G. Wen, Phys. Rev. Lett. 96, 110405 (2006).
[7] S. T. Flammia, A. Hamma, T. L. Hughes, and X. G. Wen, Phys. Rev. Lett. 103, 261601 (2009).
[8] M. B. Hastings, I. Gonzalez, A. B. Kallin, and R. G. Melko, Phys. Rev. Lett. 104, 157201 (2010).
[9] C. Nadal, S. N. Majumdar, and M. Vergassola, Phys. Rev. Lett. 104, 115001 (2010).
[10] H. Li and F. D. M. Haldane, Phys. Rev. Lett. 101, 010504 (2008).
[11] H. T. Quan, Z. Song, X. F. Liu, P. Zanardi, and C. P. Sun, Phys. Rev. Lett. 96, 140604 (2006); P. Zanardi and N. Paunkovic, Phys. Rev. E 74, 031123 (2006); S. J. Gu, arXiv:0811.3127; P. Buonsante and A. Vezzani, Phys. Rev. Lett. 98, 110601 (2007); P. Zanardi, P. Giorda, and M. Cozzini, Phys. Rev. Lett. 99, 100603 (2007); H. Q. Zhou, R. Orus, and G. Vidal, Phy. Rev. Lett. 100, 080601 (2008).
[12] M. A. Nielsen, Phys. Rev. Lett. 83, 436 (1999).
[13] D. Jonathan and M. Plenio, Phys. Rev. Lett. 83, 3566 (1999).
[14] S. Turgut, J. Phys. A 40, 12185 (2007); M. Klimesh, Eprint arXiv:0709.3680.
[15] In this Letter, the local transformation includes the case of "catalyst", that means the transformation may be assisted by an ancillary entangled state.