On Feasibility of Interference Alignment in MIMO Interference Networks

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Abstract—We explore the feasibility of interference alignment in signal vector space — based only on beamforming — for K-user MIMO interference channels. Our main contribution is to relate the feasibility issue to the problem of determining the solvability of a multivariate polynomial system, considered extensively in algebraic geometry. It is well known, e.g., from Bezout’s theorem, that generic polynomial systems are solvable if and only if the number of equations does not exceed the number of variables. Following this intuition, we classify signal space interference alignment problems as either proper or improper based on the number of equations and variables. Rigorous connections between feasible and proper systems are made through Bernstein’s theorem for the case where each transmitter uses only one beamforming vector. The multi-beam case introduces dependencies among the coefficients of a polynomial system so that the system is no longer generic in the sense required by both theorems. In this case, we show that the connection between feasible and proper systems can be further strengthened (since the equivalency between feasible and proper systems does not always hold) by including standard information theoretic outer bounds in the feasibility analysis.

Index Terms—Degrees of freedom, interference alignment, interference channel, MIMO, Newton polytopes, mixed volume

I. INTRODUCTION

The degrees of freedom (DoF) of wireless interference networks represent the number of interference-free signaling-dimensions in the network. In a network with K transmitters and K receivers and non-degenerate channel conditions, it is well known that K non-interfering spatial signaling dimensions can be created if the transmitters or the receivers are able to jointly process their signals. Until recently it was believed that with distributed processing at transmitters and receivers, it is not possible to resolve these signaling dimensions so that only one degree of freedom is available. However, the discovery of a new idea called interference alignment has shown that the DoF of wireless interference networks can be much higher [1].

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A. Evolution of Interference Alignment

Interference alignment refers to the consolidation of multiple interfering signals into a small subspace at each receiver so that the number of interference-free dimensions remaining for the desired signal can be maximized. The idea evolved out of the DoF studies for the 2-user X channel [2,3] and has since been applied to a variety of networks in increasingly sophisticated forms. The majority of interference alignment schemes proposed so far, fall into one of two categories — (1) signal space alignment and (2) signal level alignment.

1) Interference Alignment in Signal Vector Space: The potential for overlapping interference spaces was first pointed out by Maddah-Ali et. al. in [4,5] where iterative schemes were formulated for optimizing transmitters and receivers in conjunction with dirty paper coding/ successive decoding schemes. The idea of interference alignment was crystallized in a report by Jafar [6] where the first explicit (closed form, non-iterative) and linear (no successive-decoding or dirty paper coding) interference alignment scheme in signal vector space was presented. The explicit linear approach introduced by Jafar in [6] was adopted by Maddah-Ali et. al. in their subsequent report and journal paper [3,7], while [6] developed into the journal paper by Jafar and Shamai [2]. Interference alignment was also independently discovered by Weingarten et. al. [8] in the context of the compound multiple input single output (MISO) broadcast channel (BC).

Following the early success on the X channel and the compound MISO BC, signal space interference alignment schemes were introduced for the K-user interference channel with equal (unequal) number of antennas at all transmitters and receivers by Cadambe and Jafar (Gou and Jafar) in [1] ([9]), for X networks with arbitrary number of users by Cadambe and Jafar in [10], for cellular networks by Suh and Tse in [11], for MIMO bidirectional relay networks (Y channel) by Lee and Lim in [12], for ergodic fading interference networks by Nazer et. al. in [13], and for interference networks with secrecy constraints in [14]. Interference networks with constant channel coefficients posed a barrier for signal space interference alignment schemes because they did not provide distinct rotations of vector spaces on each link that were needed for linear interference alignment. The problem was circumvented to a certain extent for complex interference channels in [15], where phase rotations were exploited in a similar manner through the use of asymmetric complex signaling. However, for constant and real channel coefficients, these linear alignment schemes were not sufficient and a
different class of alignment schemes based on structured (e.g., lattice) codes that align interference in signal scale were introduced.

2) Interference Alignment in Signal Scale: The first interference alignment scheme in signal scale was introduced for the many-to-one interference channel by Bresler et. al. in [16] and for fully connected interference networks by Cadambe et. al. in [17]. Unlike random codes for which decoding the sum of interfering signals is equivalent to decoding each of the interfering signals, these schemes rely on codewords with a lattice structure, which opens the possibility that the sum of interfering signals can be decoded even when the individual interfering signals are not decodable. This is because the sum of lattice points is another lattice point, and hence may be decoded as a valid codeword. Lattice based alignment schemes were further investigated for interference networks by Sridharan et. al. in [18,19] and for networks with secrecy constraints by He and Yener in [20]. An interesting interference alignment scheme in signal scale was introduced by Etkin and Ordentlich in [21]. This work used fundamental results from diophantine approximation theory to prove that the rational and irrational scaled versions of a lattice “stood apart” from each other, and thus could be separated. The result was extended to almost all irrational numbers by Maddah-Ali et. al. in [3] by translating the notion of linear independence (exploited in linear interference alignment schemes) into the notion of rational independence in signal scale. With this new insight, the asymptotic alignment scheme of Cadambe and Jafar from [1] was essentially adopted in [3] to achieve interference alignment in signal scale and following the approach in [1], was shown to approach the DoF outer bound.

In spite of the obvious advantages of signal scale alignment schemes (especially those based on rational independence [3,21]) for obtaining DoF characterizations, a downside to these schemes is that they seem to bring to light primarily the artifacts of the infinite SNR regime and offer little in terms of useful insights for the practical setting with finite SNR and finite precision channel knowledge, where the notion of rational independence loses its relevance. Signal space alignment schemes on the other hand, are desirable both for their analytical tractability as well as the useful insights they offer for finite SNR regime where they may be naturally combined with selfish approaches [22]. Within the class of signal vector space interference alignment schemes, alignment in spatial dimension through multiple antennas (MIMO) is found to be more robust to practical limitations such as frequency offsets than alignment in time or frequency dimensions [23]. However, the feasibility of linear interference alignment for general MIMO interference networks remains an open problem [24]. It is this problem - the feasibility of linear interference alignment for MIMO interference networks - that we address in this paper. We explain our objective through the following examples.

B. The Feasibility Question - Examples

1) Symmetric Systems: Let \((M \times N, d)^K\) denote the \(K\)-user MIMO interference network, where every transmitter has \(M\) antennas, every receiver has \(N\) antennas and each user wishes to achieve \(d\) DoF. We call such a system a symmetric system. Consider the following examples.

- \((2 \times 2, 1)^3\) - It is shown in [1] that for the 3-user interference network with 2 antennas at each node, each user can achieve 1 DoF by presenting a closed form solution for linear interference alignment, i.e., by linear beamforming at the transmitters and linear combining at the receivers. However, is there a way to analytically determine the feasibility of this system without finding a closed form solution?

- \((5 \times 5, 2)^4\) - Consider the 4-user interference network with 5 antennas at each user and we wish to achieve 2 DoF per user for a total of 8 DoF. A theoretical solution to this problem is not known but numerical evidence in [24] clearly indicates that a linear interference alignment solution exists. Numerical algorithms are one way to determine the feasibility of linear interference alignment. However, is there a way to analytically determine the feasibility of alignment without running the numerical simulation?

2) Asymmetric Systems: Let us introduce the notation \((M^{[1]} \times N^{[1]}, d^{[1]}) \cdots (M^{[K]} \times N^{[K]}, d^{[K]})\) to denote the \(K\)-user MIMO interference network, where the \(k^{th}\) transmitter and receiver have \(M^{[k]}\) and \(N^{[k]}\) antennas, respectively and the \(k^{th}\) user demands \(d^{[k]}\) DoF. We call such a system an asymmetric system. Consider the following examples.

- Consider the simple system \((2 \times 1, 1, 2)\), which is clearly feasible (proper) because simple zero-forcing is enough for achievability. However, now consider the \((2 \times 1, 1)(1 \times 2, 1)\) system, where the same total number of DoF is desired. Although these systems have the same number of total antennas, is the latter system still achievable?

- Consider the 2-user interference network \((2 \times 3, 1)(3 \times 2, 1)\), where a total of 2 DoF is desired. The achievable scheme for this system is presented in [25]. Now, consider the same scheme with increased number of users; that is, the 4-user interference network \((2 \times 3, 1)^2(3 \times 2, 1)^2\), where a total of 4 DoF is desired. Is this system still achievable, where DoF is doubled by simply going from two users to four users?

In this paper, we address all these questions. Our approach is to consider the signal space interference alignment problem as the solvability of a multivariate polynomial system, and then place it into perspective with classical results in algebraic geometry where these problems are extensively studied.

II. Preliminaries

A. System Model

We consider the same \(K\)-user MIMO interference network as considered in [24]. The received signal at the \(n^{th}\) channel use can be written as follows:

\[
Y^{[k]}(n) = \sum_{l=1}^{K} H^{[k]}(n) X^{[l]}(n) + Z^{[k]}(n),
\]
∀k ∈ K ≜ {1, 2, ..., K}. Here, Y[k](n) and Z[k](n) are the N[k] × 1 received signal vector and the zero mean unit variance circularly symmetric additive white Gaussian noise vector (AWGN) at the kth receiver, respectively. X[l](n) is the M[l] × 1 signal vector transmitted from the lth transmitter and H[k,l](n) is the N[k] × M[l] matrix of channel coefficients between the lth transmitter and the kth receiver. E[||X[k](n)||^2] = P[l] is the transmit power of the lth transmitter. Hereafter, we omit the channel use index n for the sake of simplicity. The DoF for the kth user’s message is denoted by d[k] ≤ min(M[k], N[k]).

As defined earlier, (M × N, d)^K denotes the K-user symmetric MIMO interference network, where each transmitter and receiver has M and N antennas, respectively and each user demands d DoF; therefore, the total DoF demand is Kd. In general, let \( \Pi_{k=1}^K (M[k] \times N[k], d[k]) = (M[1] \times N[1], d[1]) \cdots (M[K] \times N[K], d[K]) \) denote the K-user asymmetric MIMO interference network, where the kth transmitter and receiver have M[k] and N[k] antennas, respectively and the kth user demands d[k] DoF. Some sample symmetric and asymmetric systems are shown in Fig. 1.

### B. Interference Alignment in Signal Space - Beamforming and Zero Forcing Formulation

In interference alignment precoding, the transmitted signal from the kth user is \( X[k] = V[k]X[k] \), where \( X[k] \) is a \( d[k] \) × 1 vector that denotes the \( d[k] \) independently encoded streams transmitted from the kth user. The \( M[k] \times d[k] \) precoding (beamforming) filters \( V[k] \) are designed to maximize the overlap of interference signal subspaces at each receiver while ensuring that the desired signal vectors at each receiver are linearly independent of the interference subspace. Therefore, each receiver can zero-force all the interference signals without zero-forcing any of the desired signals. The zero-forcing filters at the receiver are denoted by \( U[k] \). In [24], it is shown that an interference alignment solution requires the simultaneous satisfiability of the following conditions:

\[
U[k] H[k,j] V[j] = 0, \forall j \neq k \quad (1)
\]

\[
\text{rank} \left( U[k] H[k,k] V[k] \right) = d[k], \forall k \in \{1, 2, ..., K\}, \quad (2)
\]

where \( \dagger \) denotes the conjugate transpose operator. Very importantly, [24] explains how the condition (2) is automatically satisfied almost surely if the channel matrices do not have any special structure, rank \( U[k] = \text{rank}(V[k]) = d[k] \leq \min(M[k], N[k]) \) and \( U[k], V[k] \) are designed to satisfy (1), which is independent of all direct channels \( H[k,k] \). We assume that general MIMO channels have no structure and we force the transmit and receive filters to achieve the required ranks by design. Thus, (2) is automatically satisfied for us as well.

### III. Proper System

Based on classical results in algebraic geometry, like Bezout’s theorem, it is well known that a generic system of multivariate polynomial equations is solvable if and only if the number of equations does not exceed the number of variables. While the qualification “generic system of polynomials” is intended in a precise sense and limits the scope of settings where the result can be rigorously applied, the intuition behind this statement is believed to be much more widely true. This conventional wisdom forms the starting point for our work. By accurately accounting for the number of equations, \( N_e \), and the number of variables, \( N_v \), we classify a signal space interference alignment problem as either improper or proper, depending on whether or not the number of equations exceeds the number of variables.

#### A. Counting the Total Number of Equations and Variables

We rewrite the conditions in (1) as follows:

\[
u_n^{[k]} H_n^{[kj]} v_n^{[j]} = 0, \quad j \neq k, j, k \in \{1, 2, ..., K\} \quad (3)
\]

\( \forall n \in \{1, 2, ..., d[j]\} \) and \( \forall m \in \{1, 2, ..., d[k]\} \)

where \( v_n^{[j]} \) and \( u_m^{[k]} \) are the transmit and receive beamforming vectors (columns of precoding and interference suppression filters, respectively).

\( N_e \) is directly obtained from (3) as follows:

\[ N_e = \sum_{k,j \in K \atop k \neq j} d[k]d[j]. \]

However, calculating the number of variables \( N_v \) is less straightforward. In particular, we have to be careful not to
count any superfluous variables that do not help with interference alignment.

At the $k$th transmitter, the number of $M[k] \times 1$ transmit beamforming vectors to be designed is $d[k]$ \((v_n^{[k]}, \forall n \in \{1, 2, ..., d[k]\})\). Therefore, at first sight, it may seem that the precoding filter of the $k$th transmitter, $\mathbf{V}^{[k]}$, has $d[k]M[k]$ variables. However, as we argue next, we can eliminate $(d[k])^2$ of these variables without loss of generality.

The $d[k]$ linearly independent columns of transmit precoding matrix $\mathbf{V}^{[k]}$ span the transmitted signal space
\[ \mathcal{T}^{[k]} = \text{span}(\mathbf{V}^{[k]}) = \{ \mathbf{v} : \exists \mathbf{a} \in \mathbb{C}^{d[k] \times 1}, \mathbf{v} = \mathbf{V}^{[k]}\mathbf{a} \}. \]

Thus, the columns of $\mathbf{V}^{[k]}$ are the basis for the transmitted signal space. However, the basis representation is not unique for a given subspace. In particular, consider any full rank $d[k] \times d[k]$ matrix $\mathbf{B}$. Then, continuing from the last step of the above equations,
\[ \mathcal{T}^{[k]} = \{ \mathbf{v} : \exists \mathbf{a} \in \mathbb{C}^{d[k] \times 1}, \mathbf{v} = \mathbf{V}^{[k]}\mathbf{B}^{-1}\mathbf{a} \} = \text{span}(\mathbf{V}^{[k]}\mathbf{B}^{-1}). \]

Thus, post-multiplication of the transmit precoding matrix with any invertible matrix on the right does not change the transmitted signal subspace. Suppose that we choose $\mathbf{B}$ to be the $d[k] \times d[k]$ matrix that is obtained by deleting the bottom $M[k] - d[k]$ rows of $\mathbf{V}^{[k]}$. Then, we have $\mathbf{V}^{[k]}\mathbf{B}^{-1} = \tilde{\mathbf{V}}^{[k]}$, which is a $M[k] \times d[k]$ matrix with the following structure:
\[ \tilde{\mathbf{V}}^{[k]} = \begin{bmatrix} v_1^{[k]} & v_2^{[k]} & \ldots & v_{d[k]}^{[k]} \end{bmatrix}, \]

where $I_{d[k]}$ is the $d[k] \times d[k]$ identity matrix and \(v_n^{[k]} = \mathbf{V}^{[k]}v_n\), \(\forall n \in \{1, 2, ..., d[k]\}\) are $(M[k] - d[k]) \times 1$ vectors. It is easy to argue that there is no other basis representation for the transmitted signal space with fewer variables.

Therefore, by eliminating superfluous variables for the interference alignment problem, the number of variables to be designed for the precoding filter of the $k$th transmitter, $\mathbf{V}^{[k]}$, is $d[k]L_1^{[k]} \times d[k]$. Likewise, the actual number of variables to be designed for the interference suppression filter of the $k$th receiver, $\mathbf{U}^{[k]}$, is $d[k]L_2^{[k]} \times d[k]$. As a result, the total number of variables in the network to be designed is:

\[ N_v = \sum_{k=1}^{K} d[k] \left( M[k] + N[k] - 2d[k] \right). \]

B. Proper System Characterization

To formalize the definition of a proper system, we first introduce some notation. We use the notation $E_{mn}^{[kj]}$ to represent the equation
\[ u_m^{[kj]}H_{mn}^{[kj]}v_n^{[kj]} = 0. \]

The set of variables involved in an equation $E$ is indicated by the function $\text{var}(E)$. Clearly
\[ |\text{var}(E_{mn}^{[kj]})| = (M[j] - d[j]) + (N[k] - d[k]), \]

where $| \cdot |$ is the cardinality of a set.

Using this notation, we denote the set of $N_v$ equations as follows:
\[ \mathcal{E} = \{ E_{mn}^{kj} | j, k \in K, k \neq j, m \in \{1, \ldots, d[k]\}, n \in \{1, \ldots, d[j]\} \}. \]

This leads us to the formal definition of a proper system.

**Definition 1.** A $\Pi_{k=1}^{K}(M[k] \times N[k], d[k])$ system is proper if and only if
\[ \forall S \subset \mathcal{E}, |S| \leq \left| \bigcup_{E \in S} \text{var}(E) \right|. \]

In other words, for all subsets of equations, the number of variables involved must be at least as large as the number of equations in that subset.

The condition to identify a proper system can be computationally cumbersome because we have to test all subsets of equations. However, several simplifications are possible in this regard. We start with symmetric systems.

C. Symmetric Systems $(M \times N, d)^K$

For symmetric systems, simply comparing the total number of equations and the total number of variables suffices to determine whether the system is proper or improper.

**Theorem 1.** A symmetric system $(M \times N, d)^K$ is proper if and only if
\[ N_v \geq N_e \Rightarrow M + N - (K + 1)d \geq 0. \]

**Proof:** Because of the symmetry, each equation involves the same number of variables and any deficiency in the number of variables shows up in the comparison of the total number of variables versus the total number of equations. Plugging in the values of $N_v$ and $N_e$ computed earlier, we have the result of Theorem 1.

**Example 1.** Consider the $(2 \times 3, 1)^4$ system. For this system, $M + N - (K + 1)d = 2 + 3 - (5) = 0$ so that this system is proper.

**Example 2.** Consider the $(1 \times 2, 1)^3$ system, i.e., a 3-user symmetric interference network, where each transmitter has one antenna, each receiver has two antennas, and each user demands 1 DoF. For this system, $M + N - (K + 1)d = 1 + 2 - (4) < 0$ so that this system is improper.

**Remark 1.** In light of the intuition that proper systems are likely to be feasible, Theorem 1 implies that for every user to achieve $d$ DoF in a $K$-user symmetric network, it suffices to have a total of $M + N \geq (K + 1)d$ antennas between the transmitter and receiver of a user. The antennas can be distributed among the transmitter and receiver arbitrarily as long as each of them has at least $d$ antennas and as long as the symmetric nature of the system is preserved. In particular, to achieve $K$ DoF in a $K$-user symmetric network (1 DoF per user), we only need a total of $K + 1$ antennas between the transmitter and receiver of a user.

**Example 3.** Consider a 4-user symmetric network, where we wish to achieve 4 DoF. Then, 5 antennas between the
Theorem 1. The DoF of a proper \((M \times N, d)^K\) system, which is normalized by a single user’s DoF in the absence of interference, is upper bounded by:

\[
\frac{dK}{\min(M,N)} \leq 1 + \frac{\max(M,N)}{\min(M,N)} - \frac{d}{\min(M,N)}.
\]

Proof: The proof is straightforward from the condition of Theorem 1.

Remark 2. For the case \(M = N\), note that the DoF of a proper system is no more than twice the DoF achieved by each user in the absence of interference. Note that for diagonal (time-varying) channels, it is shown in [1] that the DoF of a \(K\)-user MIMO network \((M = N\) antennas at each node) is \(K/2\) times the number of DoF achieved by each user in the absence of interference. This result shows that the diagonal structure of the channel matrix is very helpful. Going from the case of no structure (general MIMO channels) to diagonal structure, the ratio of total DoF to the single user DoF increases from a maximum value of 2 to \(K/2\).

The following corollary identifies the groups of symmetric systems, which are either all proper or all improper.

Corollary 2. If \((M \times N, d)^K\) system is proper (improper) then so is the \(((M + 1) \times (N - 1), d)^K\) system as long as \(d \leq \min(M, N - 1)\). Similarly, if the \((M \times N, d)^K\) system is proper (improper) then so is the \(((M - 1) \times (N + 1), d)^K\) system as long as \(d \leq \min(M - 1, N)\).

Proof: Since the condition in Theorem 1 depends only on \(M + N\), it is clear that we can transfer transmit and receive antennas without affecting the proper (or improper) status of the system.

Example 5. The systems \((1 \times 4, 1)^4\), \((2 \times 3, 1)^4\), \((3 \times 2, 1)^4\), and \((4 \times 1, 1)^4\) are in the same group, which are formed by successively transferring an antenna between transmitters and receivers. It is easy to see that the \((4 \times 1, 1)^4\) system is proper because simple zero-forcing suffices to achieve the DoF demand. By virtue of being in the same group, the rest are proper as well.

Example 6. By similar arguments, the systems \((1 \times 3, 1)^3\), \((2 \times 2, 1)^3\), and \((3 \times 1, 1)^3\) are in the same group and are all proper.

D. Asymmetric Systems \(\Pi_{k=1}^K (M^{[k]} \times N^{[k]}, d^{[k]})\)

For asymmetric systems, if the system is improper, simply comparing the total number of equations and the total number of variables may suffice.

Theorem 2. An asymmetric system \(\Pi_{k=1}^K (M^{[k]} \times N^{[k]}, d^{[k]})\) is improper if

\[
N_v < N_e \iff \sum_{k=1}^K d^{[k]} \left( M^{[k]} + N^{[k]} - 2d^{[k]} \right) < \sum_{k,j \in K, k \neq j} d^{[k]}d^{[j]},
\]

Example 7. Consider the system \((2 \times 2, 1)(2 \times 3, 1)^3\), which is clearly infeasible when we compare it to the \((2 \times 3, 1)^4\) system in Example 4. Confirmatively, the former system is improper since it has 11 variables and 12 equations in total.

Note that we can sometimes identify the bottleneck equations in the system by checking the equations with the fewest number of variables, i.e., the equations involving the fewest number of transmitter and receiver antennas.

Example 8. Consider the simple system \((2 \times 1, 1)^2\), which is clearly feasible (proper) because simple zero-forcing is enough for achievability. However, now consider the \((2 \times 1, 1)(1 \times 2, 1)\) system, which also has the same total number of equations \(N_v\) and variables \(N_e\) as the \((2 \times 1, 1)^2\) system. Thus, only comparing \(N_v\) and \(N_e\) would mislead one to believe that this system is proper. However, suppose that we only check the equation \(E_1^{[1]}\); that is, our subset is \(S = \{E_1^{[1]}\}\) so that \(|S| = 1\). \(E_1^{[1]}\) corresponds to the link between the transmitter 2 and receiver 1, both of which have only one antenna each. Therefore, \(\{|\text{var}(E_1^{[1]})|\} = 0\). Thus, this system has an equation with zero variable, which makes the system improper; therefore, infeasible.

Example 9. Several interesting cases emerge from applying the condition (5). For example, consider the 2-user interference network \((2 \times 3, 1)(3 \times 2, 1)\), where a total of 2 DoF is desired. It is easily checked that this system is proper and the achievable scheme is described in [25]. Now, consider the 4-user interference network, which consists of two sets of these networks, all interfering with each other \((2 \times 3, 1)^2(3 \times 2, 1)^2\), where a total of 4 DoF is desired. By using (5), it is easily verified that this is a proper system. Surprisingly, by simply going from two users to four users, DoF is doubled in this case. We also present the closed form solution for interference alignment of this system in Section VII.

IV. Numerical Results

We tested numerous interference alignment problems for both symmetric and asymmetric cases by using the numerical algorithm in [24]. In every case so far, we have found the numerical results to be consistent with the guiding intuition of this work; that is, for single beam cases, proper systems are almost surely feasible and improper systems are not.

In this section, we provide numerical results for a few interesting and representative cases. The results are in terms of the interference percentage, which is defined in [24], i.e., the fraction of the interference power that is existent in the dimensions reserved for the desired signal. The interference
percentage at the $k^{th}$ receiver is evaluated as follows:

$$p_k = \frac{\sum_{j=1}^{d(k)} \lambda_j [Q[k]]}{\text{Tr}[Q[k]]},$$

where $\lambda_j$ denotes the smallest eigenvalue of a matrix, $\text{Tr}$ denotes the trace of a matrix, and $Q[k]$ denotes the interference covariance matrix at the $k^{th}$ receiver:

$$Q[k] = \sum_{j=1, j \neq k}^{K} \frac{p[j]}{d[j]} H[k] V[j] V[j]^T H[k]^T.$$

The numerator and the denominator of (6) are the interference and desired signal space powers at the $k^{th}$ receiver, respectively.

In Fig. 2 the interference percentages versus the total number of beams are shown. The total number of beams starts from the expected total DoF of each network. Therefore, after the first point on the x-axis, where excess total DoF is demanded the interference percentage of each network is not zero. The nonzero interference percentage indicates that interference alignment is not possible for the demanded total DoF.

Therefore, by observing zero interference percentages on the DoF point in Fig. 2 we show that the numerical results are consistent with our statements in Section III that these networks are proper, and thus feasible. Note that in Fig. 2 there are numerical results also for multi-beam cases, which we discuss in Section VIII.

From the excess total DoF results in Fig. 2 we also understand that the first two systems with expected 4 total DoF have more interference percentages than other systems with expected 8 total DoF. We also observe that the system with the less total number of antennas at the receiver side has more interference percentage than other system with the same expected total DoF; e.g., $(2 \times 3, 1)^2(3 \times 2, 1)^2$ has more interference percentage than $(2 \times 3, 1)^2$.

V. BEZOUT’S AND BERNSTEIN’S THEOREMS

As explained earlier, the definition of proper systems is an intuitive generalization of the classical result in algebraic geometry known as Bezout’s theorem. While the formal statement of Bezout’s theorem is presented later, the theorem essentially states that “generic” systems of multivariate polynomial equations are solvable if and only if the number of equations does not exceed the number of variables. Since the notion of generic system is critical in our work, we first summarize what is meant by a generic system in simple terms. While the mathematical definition of “genericity” is presented in Appendix, the notion of a “generic” system refers to two aspects.

1) The supports of polynomials, which are determined by non-zero coefficients of polynomials.

2) The independence (e.g. algebraic independence) of non-zero coefficients.

“Generic” system in the sense of Bezout’s theorem refers to the system of dense polynomials (all coefficients up to the degree of each polynomial are non-zero) with independent random coefficients. According to Bezout’s theorem, these systems are solvable almost surely as long as the number of polynomial equations does not exceed the number of variables, and the number of solutions is equal to the product of the degrees of polynomials.

On the other hand, “generic” system in the sense of Bernstein’s theorem refers to the system of sparse polynomials with independent random coefficients. For dense polynomials, the result of Bezout’s theorem can be derived from Bernstein’s theorem and therefore, Bernstein generalizes Bezout’s theorem. For our feasibility problem, the system of polynomials does not always satisfy “genericity” in Bernstein’s theorem since while the coefficients are independent in the single-beam case, the same is not true for the multi-beam case.

The single beam case refers to the scenario when each user demands only one DoF, to be achieved by sending one beam from each transmitter. Therefore, for this scenario, the channel matrix $H[k]$ of each user occurs only once in the corresponding polynomial system (I). On the other hand, for a multi-beam case, consider the user who wishes to achieve more than 1 DoF. The channel matrix $H[k]$ of that user occurs more than once in the corresponding polynomial system, which leads to dependent coefficients.

As mentioned before, although we use only Bernstein’s theorem for the proofs in our work, we also summarize Bezout’s theorem as an elementary step. We briefly rephrase these two theorems insofar as required within the scope of this paper. Let us start with definitions and notations.

A. Multivariate Polynomial Systems

1) A polynomial system and its support sets: Let $\mathbb{C}[x_1, \cdots, x_n]$ denote the polynomial ring, where the coeffi-
cients are in the field \( \mathbb{C} \) and the variable \( x_i, \forall i \in \{1, 2, \ldots, n\} \) has nonnegative integer (denoted by \( \mathbb{Z}_{\geq 0} \)) exponent. The multivariate polynomial system that we are interested in consists of \( n \) variables and \( n \) equations:

\[
f_1 = 0, \ldots, f_n = 0, \tag{7}
\]

where \( f_1, \ldots, f_n \in \mathbb{C} [x_1, \ldots, x_n] \).

Let \( c^j_i \) denote the nonnegative integer exponent of the \( j^{th} \) variable \( x_i \) in the \( j^{th} \) monomial of the polynomial \( f_i, \forall i \in \{1, 2, \ldots, n\} \):

\[
f_i = \cdots + c_i^j x_1^{e_1} x_2^{e_2} \cdots x_i^{e_i} \cdots x_n^{e_n} + \cdots,
\]

where \( c_{ij} \) denotes the complex valued coefficient.

Also, let

\[
a_{ij} \triangleq (c_1^j, c_2^j, \ldots, c_m^j) \in \mathbb{Z}_{\geq 0}^n,
\]

\[\forall i \in \{1, 2, \ldots, n\} \text{ and } \forall j \in \{1, 2, \ldots, m_i\}\]

denote a nonnegative integer vector, which is also called an exponent vector.

Then, we denote the \( j^{th} \) monomial in \( f_i \) as follows:

\[
x^{a_{ij}} \triangleq x_1^{e_1} x_2^{e_2} \cdots x_i^{e_i} \cdots x_n^{e_n}.
\]

Finally, let \( A_i = \{a_{i1}, \ldots, a_{im_i}\} \subset \mathbb{Z}_{\geq 0}^n \) denote the set of exponent vectors with nonzero coefficients in \( f_i \). \( A_i \) is also called the support set of \( f_i \).

Therefore, each polynomial has the following structure with a support set \( A_i \):

\[
f_i = \sum_{j=1}^{m_i} c_{ij} x^{a_{ij}}. \tag{8}
\]

**Example 10.** Consider the following \( \ell^{th} \) polynomial:

\[
f_i = c_{11} x_1 + c_{22} x_1 x_2 + c_{33}.
\]

Then, \( a_{11} = (1, 0), a_{12} = (1, 1), \) and \( a_{33} = (0, 0) \). Accordingly, the support set \( A_i \) for this polynomial is the set of vertexes of a right triangle.

2) Common solutions of a polynomial system: Let \( S_k = \{x_1^k, \ldots, x_n^k\} \) denote the \( k^{th} \) monomial, \( \forall k \in \{1, 2, \ldots, s\} \) common solution for the \( n \) dimensional polynomial system (7), which has \( s \) common solutions in total:

\[
f_1(x_1^k, \ldots, x_n^k) = 0, \ldots, f_n(x_1^k, \ldots, x_n^k) = 0.
\]

Then, the set of all common solutions \( S_C \) that satisfies the polynomial system (7) is as follows:

\[
S_C = \{S_1, \ldots, S_s\}.
\]

In other words, there are \( s \) points in the corresponding space that satisfy the polynomial system (7), e.g., \( (x_1^k, \ldots, x_n^k) \in \mathbb{C}^n, \forall k \in \{1, 2, \ldots, s\} \).

3) The degree of a polynomial: Let \( \deg(f_i) \) denote the degree of \( f_i \), which is defined as follows:

\[
\deg(f_i) = \max \left( e_{i1} + \cdots + e_{im}, \cdots, e_{11} + \cdots + e_{im} \right).
\]

**B. Dense and Sparse Polynomial Systems**

For a dense polynomial system, in any polynomial \( f_i \), monomials with all combinations of variable exponents up to \( \deg(f_i) \) have nonzero coefficients. On the other hand, for a sparse polynomial system, in any polynomial \( f_i \), some certain monomials may have zero coefficients.

**Example 11.** For \( n = 2 \), \( \deg(f_1) = 3 \) and \( \deg(f_2) = 4 \), a dense polynomial system is as follows:

\[
f_1 = c_{11} x_1^3 + c_{12} x_2^3 + c_{13} x_1 x_2 + c_{14} x_1 x_2^2 + c_{15} x_1^2 + c_{16} x_2^2 + c_{17} x_1 x_2^2 + c_{18} x_1 + c_{19} x_2 + c_{110}
\]

\[
f_2 = c_{21} x_1^4 + c_{22} x_2^4 + c_{23} x_1 x_2 + c_{24} x_1 x_2^3 + c_{25} x_1^2 x_2^2 + c_{26} x_1^3 + c_{27} x_2^3 + c_{28} x_1 x_2^2 + c_{29} x_1 x_2 + c_{210} x_1^2 + c_{211} x_2^2 + c_{212} x_1 x_2 + c_{213} x_1 + c_{214} x_2 + c_{215}.
\]

**Example 12.** One of the sparse polynomial systems corresponding to the previous example may be as follows:

\[
f_1 = c_{11} x_1 x_2 + c_{12} x_1^2 + c_{13} x_2^3 + c_{13} x_1 x_2
\]

\[
f_2 = c_{21} x_1 x_2^2 + c_{22} x_2^3 + c_{23} x_1 x_2
\]

Now, we are ready to state Bezout’s theorem.

**C. Bezout’s Theorem**

**Theorem 3** (Bezout’s Theorem - specialized). Given dense polynomials \( f_1, \ldots, f_m \in \mathbb{C}[x_1, \ldots, x_n] \) with common solutions in \( \mathbb{C}^n \), let \( \deg(f_i) \) be the degree of polynomial \( f_i \). For independent random coefficient \( c_{ij}, \forall i \in \{1, 2, \ldots, n\} \text{ and } \forall j \in \{1, 2, \ldots, m_i\}, \) the number of common solutions is exactly equal to \( \deg(f_1) \deg(f_2) \cdots \deg(f_n) \).

According to Bezout’s theorem, the number of common solutions is \( \deg(f_1) \deg(f_2) = 12 \) for the Example 11 that is, \( s = 12 \).

When the polynomial system is sparse, Bezout’s theorem gives a loose upper bound, which is still 12 for the Example 12. On the other hand, Bernstein’s theorem gives a tighter result 9 (this result is exact when the coefficients are independent random variables) as we will show next.

**D. Bernstein’s Theorem**

Chapter 7 of [26] (hereafter, we briefly refer as [26]) is recommended for an excellent introduction and for further information for Bernstein’s theorem. Here, we first briefly summarize the rudiments of this theorem.

1) Newton Polytopes: Let \( \mathbb{C}^n \) denote the complex field excluding zeros, \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \). A polytope is the convex hull of a finite set in \( \mathbb{R}^n \) and a polytope with integer coordinates is called lattice polytope. A Newton polytope is a lattice polytope defined for a polynomial, which is based on the exponent vectors of monomials with nonzero coefficients:

\[
P_i = \text{Conv}(A_i),
\]

\footnote{1Also called generic choices of coefficients or almost all specializations of coefficients in mathematics terminology, which we explain in the Appendix.}
The mixed volume of Newton polytopes has the following general formula:

$$\text{MV}(P_1, \cdots, P_n) = \sum_{k=1}^{n} (-1)^{n-k} \sum_{\substack{I \subseteq \{1, \cdots, n\} \atop |I| = k}} \text{Vol}(\sum_{i \in I} P_i),$$

where $\text{Vol}(\cdot)$ and $\text{MV}(\cdot)$ denote the volume and mixed volume operators, respectively. $\sum_{i \in I} P_i$ denotes the Minkowski sum of Newton polytopes. It can be shown that mixed volume always has a nonnegative value [26].

As a simple example, consider the mixed volume of two Newton polytopes:

$$\text{MV}(P_1, P_2) = -\text{Vol}(P_1) - \text{Vol}(P_2) + \text{Vol}(P_S),$$

where $P_S = P_1 + P_2$.

Therefore, the mixed volume for the system in Example 13 is found as follows:

$$\text{MV}(P_1, P_2) = -3 - 3 + 15 = 9.$$ 

**Theorem 4 (Bernstein’s Theorem - specialized).** Given polynomials $f_1, \cdots, f_n \in \mathbb{C}[x_1, \cdots, x_n]$ with common solutions in $(\mathbb{C}^*)^n$, let $P_i$ be the Newton polytope of $f_i$ in $\mathbb{R}^n$. For independent random coefficients $c_{ij}$, $\forall i \in \{1, 2, \cdots, n\}$ and $\forall j \in \{1, 2, \cdots, m_i\}$, the number of common solutions is exactly equal to the mixed volume of Newton polytopes, $\text{MV}(P_1, \cdots, P_n)$.

It can be shown that Bezout’s theorem is a special case of Bernstein’s theorem. That is, mixed volume for an $n$ dimensional dense polynomial system is equal to $\deg(f_1)\deg(f_2)\cdots\deg(f_n)$ [26].

**E. High Dimensional Polynomial Systems**

For high dimensional polynomial systems, there is a nice connection between the facets (an $n - 1$ dimensional face is called facet for an $n$ dimensional polytope) and the volumes of polytopes, which significantly simplifies the computation of mixed volume. For example, by using facets, the mixed volume for the following simple 3 dimensional polynomial system is easily found to be 0:

$$f_1 = c_{11}x_1^2 + c_{12}x_2^2 + c_{13}x_3 + c_{14} = 0,$$

$$f_2 = c_{21}x_1^2 + c_{22} = 0,$$

$$f_3 = c_{31}x_1 + c_{32} = 0,$$

where clearly, there is no solution when the coefficients are random variables. We leave the details of facet approach to [26] since this is a further detail beyond our scope.

Computing the mixed volume by using facets is still cumbersome when the system is a little more complicated even for 3 dimensional polynomial systems. Therefore, there are several theoretical approaches in the mathematics literature that lead to algorithms to compute the mixed volumes, e.g., [27]. These softwares provide rigorous mixed volume results for polynomial systems. In the next section, we use the softwares mentioned in [26] to compute the mixed volumes for some important cases.

**VI. RIGOROUS CONNECTION BETWEEN PROPER AND FEASIBLE SYSTEMS - BEZOUT’S AND BERNSTEIN’S THEOREMS**

As mentioned in the previous sections, we can use Bernstein’s theorem in order to indirectly show that the corresponding polynomial system for a single beam case is solvable.

**Example 13.** The support sets of $f_1$ and $f_2$ in Example 12 are

$$A_1 = \{(1, 2), (2, 0), (0, 2), (0, 0)\}$$

$$A_2 = \{(3, 1), (0, 4), (1, 1)\},$$

respectively. The corresponding Newton polytopes (also called the supports of polynomials) are shown in Fig. 3.

**2) Mixed Volume and Minkowski Sum:**

The mixed volume of Newton polytopes includes Minkowski sum operation of Newton polytopes, which can be carried on by summing two Newton polytopes at a time. For example, the Minkowski sum of three Newton polytopes $P_S = P_1 + P_2 + P_3$ can be evaluated in two steps, e.g., $P_S = P_{S12} + P_3$, where $P_{S12} = P_1 + P_2$. The Minkowski sum of two Newton polytopes is based on the Minkowski sum of their support sets, i.e., $A_S = A_1 + A_2$. Minkowski sum of two sets is basically adding every element of $A_1 = \{a_{11}, \cdots, a_{1m_1}\}$ to every element of $A_2 = \{a_{21}, \cdots, a_{2m_2}\}$:

$$A_S = \{a_{1j} + a_{2k} : a_{1j} \in A_1 \text{ and } a_{2k} \in A_2\}.$$ 

**Example 14.** The Minkowski sum of two sets in Example 13 is as follows:

$$A_S = \{(4, 3), (1, 6), (2, 3), (5, 1), (2, 4), (3, 1), (3, 3), (0, 6), (1, 3), (3, 1), (0, 4), (1, 1)\}.$$ 

Therefore, the Minkowski sum of corresponding two Newton polytopes $P_S = P_1 + P_2$ is found as follows:

$$P_S = \text{Conv}(A_S),$$

which is also shown in Fig. 3.

Fig. 3. Minkowski sum of two Newton polytopes.

where $\text{Conv}(\cdot)$ denotes the convex hull of a finite set.
(not solvable) almost surely if the mixed volume for that system is nonzero (zero). Once again, note that the coefficients must be generic in order to use Bernstein’s theorem. Next, we apply this theorem for some important systems.

Example 15. For the systems $(2 \times 3, 1)^4$ and $(2 \times 3, 1)^2(3 \times 2, 1)^2$, the mixed volumes are 9 and 8, respectively. In other words, these polynomial systems with independent random coefficients are solvable almost surely since for each system, mixed volume is nonzero.

Example 16. Now, consider the system $(2 \times 2, 1)^3(3 \times 5, 1)$, which is infeasible according to the simulation result. Since the subset of equations, which is obtained by shutting down the fourth receiver has 9 equations and 8 variables, this system is improper. The mixed volume for this system is 0. In other words, the corresponding polynomial system with independent random coefficients is not solvable almost surely.

Note once again that we only provide mixed volumes for only some important cases and also note that mixed volume computation is #P-complete [26].

Bernstein’s theorem applies to a polynomial system with independent random coefficients and with equal number of equations and variables ($N_e = N_v$). When the number of equations is greater than the number of variables ($N_e > N_v$), it can be argued that there is no solution almost surely by using Bernstein’s theorem as follows. First, note that the equations in the polynomial system are independent for a single beam case (the coefficients are independent random variables) and with general MIMO channels (the polynomial system has no structure). Second, suppose that we apply Bernstein’s theorem to only $N_v$ polynomials of all $N_e$ polynomials. Since the mixed volume for these $N_v$ polynomials is finite, the number of solutions for these $N_v$ polynomials is also finite. If the mixed volume is equal to zero, then there is no solution for these polynomials, and hence there is no solution for the whole polynomial system. If there are finite number of solutions for these polynomials, then these solutions cannot satisfy the rest of the polynomials with probability one since these polynomials are independent with the rest of the polynomials. Therefore, there is no solution for the whole polynomial system almost surely.

VII. NEW CLOSED FORM SOLUTIONS

The closed form solutions of interference alignment for MIMO interference networks with constant channel coefficients are known for the cases including 3-user interference network with $M = N$ antennas at each node [1] and the symmetric $(4 \times 8, 3) (4 \times 8, 2)^3$ system [9]. These closed form solutions share a common structure: If two interference vectors are aligned at two different receivers, then there exists an eigenvector solution. Motivated by this structure, we provide the solutions for the systems $(2 \times 3, 1)^3 (3 \times 2, 1)^2$ and $(2 \times 3, 1)^4$ in this section. Note that we proved these systems are feasible by computing the mixed volumes in the previous section. Since for both systems, each transmitter sends only one beam, we hereafter drop the subscript “1” for convenience, i.e., $\mathbf{v}^{[i]}$ and $\mathbf{u}^{[i]}$ denote the beamforming vectors of the $i^{th}$ transmitter and receiver, respectively.

A. Asymmetric $(2 \times 3, 1)^2 (3 \times 2, 1)^2$ System

We first consider receiver 3 and 4. At receiver 3, the interference is nulled by the receive filter $\mathbf{u}^{[3]}$, i.e.,

$$\mathbf{u}^{[5]} \mathbf{H}^{[31]} \mathbf{v}^{[1]} = \mathbf{u}^{[5]} \mathbf{H}^{[32]} \mathbf{v}^{[2]} = \mathbf{u}^{[5]} \mathbf{H}^{[34]} \mathbf{v}^{[4]} = 0. \quad (9)$$

To satisfy the above condition, we align the interference from transmitter 1 and 2 along the same dimension at receiver 3, i.e.,

$$\text{span} \left( \mathbf{H}^{[31]} \mathbf{v}^{[1]} \right) = \text{span} \left( \mathbf{H}^{[32]} \mathbf{v}^{[2]} \right) \quad (10)$$

$$\Rightarrow \text{span} \left( \left( \mathbf{H}^{[32]} \right)^{-1} \mathbf{H}^{[31]} \mathbf{v}^{[1]} \right) = \text{span} \left( \mathbf{v}^{[2]} \right),$$

where $\text{span}(\cdot)$ denotes the space spanned by the columns of a matrix.

Similarly, at receiver 4, the interference is nulled by the receive filter $\mathbf{u}^{[4]}$, i.e.,

$$\mathbf{u}^{[4]} \mathbf{H}^{[41]} \mathbf{v}^{[1]} = \mathbf{u}^{[4]} \mathbf{H}^{[42]} \mathbf{v}^{[2]} = \mathbf{u}^{[4]} \mathbf{H}^{[43]} \mathbf{v}^{[3]} = 0. \quad (11)$$

To satisfy the above condition, we align the interference from transmitter 1 and 2 along the same dimension at receiver 4, i.e.,

$$\text{span} \left( \mathbf{H}^{[41]} \mathbf{v}^{[1]} \right) = \text{span} \left( \mathbf{H}^{[42]} \mathbf{v}^{[2]} \right) \quad (12)$$

$$\Rightarrow \text{span} \left( \left( \mathbf{H}^{[42]} \right)^{-1} \mathbf{H}^{[41]} \mathbf{v}^{[1]} \right) = \text{span} \left( \mathbf{v}^{[2]} \right).$$

Notice that both $\mathbf{v}^{[1]}$ and $\mathbf{v}^{[2]}$ have to satisfy (10) and (12), from which we obtain

$$\text{span} \left( \mathbf{v}^{[1]} \right) = \text{span} \left( \left( \mathbf{H}^{[41]} \right)^{-1} \mathbf{H}^{[42]} \left( \mathbf{H}^{[32]} \right)^{-1} \mathbf{H}^{[31]} \mathbf{v}^{[1]} \right).$$

Therefore, we find that $\mathbf{v}^{[1]}$ is the eigenvector of $\left( \mathbf{H}^{[41]} \right)^{-1} \mathbf{H}^{[42]} \left( \mathbf{H}^{[32]} \right)^{-1} \mathbf{H}^{[31]}$. Then, we determine $\mathbf{v}^{[2]}$ from (12). Since $\mathbf{v}^{[1]}$ and $\mathbf{v}^{[2]}$ are determined, we can determine the receive filters $\mathbf{u}^{[3]}$ and $\mathbf{u}^{[4]}$ from (9) and (11). The interference alignment we presented so far is shown in Fig. 4.

Now, we consider the interference from transmitter 4 at receiver 3. Since $\mathbf{u}^{[3]}$ is already fixed (designed), we need
As a result, we determined $a_2$ instead of designing a filters of receiver 1 and 2. $v_1$ and $v_2$.

Therefore, following the same approach as above, we have considered all interference at receiver 3 and 4. As a result, we determined $v_1$, $v_2$, $u_1^3$, and $u_1^4$. We left $v_3$ and $v_4$ to be determined later. Now, let us consider receiver 1 and 2.

At receiver 1 and 2, the interference is nulled by the receive filters $u_1^1$ and $u_1^2$, i.e.,

$$u_1^1 \mathbf{H}^{[12]} v_2 = u_1^1 \mathbf{H}^{[13]} v_3 = u_1^1 \mathbf{H}^{[14]} v_4 = 0 \quad (16)$$

and

$$u_1^2 \mathbf{H}^{[21]} v_1 = u_1^2 \mathbf{H}^{[23]} v_3 = u_1^2 \mathbf{H}^{[24]} v_4 = 0, \quad (17)$$

respectively. As we did previously, one may directly want to determine $v_3$, $v_4$, $u_1^1$ and $u_1^2$ from the above equations by writing similar alignment conditions in (10) and (12). However, notice that $v_1^1$ and $v_2^2$ are already fixed (designed). Therefore, we need to null the interference from transmitter 2 and 1, i.e.,

$$u_1^1 \mathbf{H}^{[12]} v_2 = 0$$

and

$$u_1^2 \mathbf{H}^{[21]} v_1 = 0,$$

respectively. The above equations imply that $u_1^1$ and $u_1^2$ lay in the left null space of $\mathbf{H}^{[12]}$ and $\mathbf{H}^{[21]}$, respectively. Let $P_1$ be a $2 \times 3$ matrix whose two rows are orthogonal to $\mathbf{H}^{[12]} v_2$. Then, $u_1^1$ can be expressed as

$$u_1^1 = u_1^1 P_1,$$

where $u_1^1$ is a $1 \times 2$ vector. As a result, instead of designing a $1 \times 3$ vector $u_1^1$, we need to design a $1 \times 2$ vector $u_1^2$. Equivalently, we can also think that receiver 1 loses 1 antenna, which is illustrated in Fig. 5(a). We leave $v_4^1$ to be determined later.

Now, we consider the interference from transmitter 3 at receiver 4. Since $u_4^3$ is already fixed (designed), we need to zero-force the interference from transmitter 3 at receiver 4, i.e.,

$$u_4^3 \mathbf{H}^{[43]} v_3 = 0, \quad (15)$$

which implies that $v_3$ lies in the null space of $u_4^3 \mathbf{H}^{[43]}$. Therefore, following the same approach as $v_4^1$, $v_3^1$ can be expressed as

$$v_3^1 = N_3 v_3^2,$$

where $N_3$ is a $3 \times 2$ matrix whose columns form the basis of the null space of $u_4^3 \mathbf{H}^{[43]}$ and $v_3^2$ is a $2 \times 1$ vector. Again, instead of designing a $3 \times 1$ vector $v_3^1$, now we need to design a $2 \times 1$ vector $v_3^2$. Similarly, we can also think that transmitter 3 loses 1 antenna.

So far, we have considered all interference at receiver 3 and 4. As a result, we determined $v_1^1$, $v_2^2$, $u_3^3$, and $u_4^4$. We left $v_3^1$ and $v_4^1$ to be determined later. Now, let us consider receiver 1 and 2.

At receiver 1 and 2, the interference is nulled by the receive filters $u_1^1$ and $u_2^2$, i.e.,

$$u_1^1 \mathbf{H}^{[12]} v_2^1 = u_1^1 \mathbf{H}^{[13]} v_3^2 = u_1^1 \mathbf{H}^{[14]} v_4^1 = 0 \quad (16)$$

and

$$u_2^2 \mathbf{H}^{[21]} v_1^2 = u_2^2 \mathbf{H}^{[23]} v_3^2 = u_2^2 \mathbf{H}^{[24]} v_4^1 = 0, \quad (17)$$

respectively. As we did previously, one may directly want to determine $v_3^2$, $v_4^2$, $u_1^1$ and $u_2^2$ from the above equations by writing similar alignment conditions in (10) and (12). However, notice that $v_1^1$ and $v_2^2$ are already fixed (designed). Therefore, we need to null the interference from transmitter 2 and 1, i.e.,

$$u_1^1 \mathbf{H}^{[12]} v_2^1 = 0$$

and

$$u_2^2 \mathbf{H}^{[21]} v_1^2 = 0,$$

respectively. The above equations imply that $u_1^1$ and $u_2^2$ lay in the left null space of $\mathbf{H}^{[12]}$ and $\mathbf{H}^{[21]}$, respectively. Let $P_1$ be a $2 \times 3$ matrix whose two rows are orthogonal to $\mathbf{H}^{[12]} v_2$. Then, $u_1^1$ can be expressed as

$$u_1^1 = u_1^1 P_1,$$

where $u_1^1$ is a $1 \times 2$ vector. As a result, instead of designing a $1 \times 3$ vector $u_1^1$, we need to design a $1 \times 2$ vector $u_1^2$. Equivalently, we can also think that receiver 1 loses 1 antenna, which is illustrated in Fig. 5(b). Similarly, $u_2^2$ can be expressed as

$$u_2^2 = u_2^2 P_2,$$

where $u_2^2$ is a $1 \times 2$ vector and $P_2$ is a $2 \times 3$ matrix whose two rows are orthogonal to $\mathbf{H}^{[21]} v_1^2$.

Now, the interference alignment conditions (16) and (17) are equivalent to

$$u_1^1 P_1 \mathbf{H}^{[13]} N_3 v_3^3 = u_1^1 P_1 \mathbf{H}^{[14]} N_3 v_4^4 = 0, \quad (18)$$

and

$$u_2^2 P_2 \mathbf{H}^{[23]} N_3 v_3^5 = u_2^2 P_2 \mathbf{H}^{[24]} N_3 v_4^4 = 0, \quad (19)$$

respectively where $\mathbf{H}^{[13]}$, $\mathbf{H}^{[14]}$, $\mathbf{H}^{[23]}$, and $\mathbf{H}^{[24]}$ are $2 \times 2$ matrices.

Similar to $v_1^1$, we find that $v_3^3$ is the eigenvector of $(\mathbf{H}^{[23]})^{-1} \mathbf{H}^{[24]} (\mathbf{H}^{[14]})^{-1} \mathbf{H}^{[13]}$. Then, we determine $v_4^4$ and $u_4^4$ from (18) and $u_2^2$ from (19). The interference alignment we presented for receiver 1 and 2 is shown in Fig. 6.

As a result, we completed designing all transmit and receive beamforming filters in the $(2 \times 3, 1)^4$ system.

**B. Symmetric $(2 \times 3, 1)^4$ System**

It is difficult to directly express the closed form solution for the $(2 \times 3, 1)^4$ system. However, by first presenting a closed form solution for the $(2 \times 4, 1)(2 \times 3, 1)^3$ system, where there is an extra freedom (i.e., an extra receive antenna), we show the solution for the $(2 \times 3, 1)^4$ system.

For the $(2 \times 4, 1)(2 \times 3, 1)^3$ system, suppose that we randomly pick the beamforming vector $v_1^1$ at transmitter 1. To eliminate the interference caused by transmitter 1 at receiver 2, 3 and 4, each receiver needs to discard the dimension occupied by this interference. In other words, each receiver only uses the 2 dimensional subspace, which is orthogonal to the direction of the interference caused by transmitter 1. Equivalently, we can think that each of the receivers 2, 3, and 4 loses 1 antenna as...
choose the receive beamforming vector at receiver 1: \( \text{(iteratively solve the equations from transmitter 2, 3, and 4 to receiver 1). We}
\)
An extra variable at receiver 1 due to an extra antenna gives for which we previously showed that there exists a closed form extra variable. However, instead of choosing a closed form solution by using the \( \text{brevity concern of the paper.}
\)
Next, we intuitively show that the \( (2 \times 3, 1)^4 \) system has also a closed form solution by using the \( (2 \times 4, 1)(2 \times 3, 1)^3 \) system for which we previously showed that there exists a closed form solution. Now, for the \( (2 \times 4, 1)(2 \times 3, 1)^3 \) system, consider the receive beamforming vector at receiver 1:
\[
\mathbf{u}^{[1]} = \begin{bmatrix}
1 \\
u_1 \\
u_2 \\
u_3
\end{bmatrix}.
\]
An extra variable at receiver 1 due to an extra antenna gives us a freedom to choose it arbitrarily. We pick \( u_3 \) as this extra variable. However, instead of choosing \( u_3 \) arbitrarily, we choose \( u_3 \) in terms of other variables by iteratively solving it from the equations that it is involved.
\( \mathbf{u}^{[1]} \) is involved in 3 equations, which are
\[
E_{11}^{12}, E_{11}^{13}, \text{ and } E_{11}^{14}
\]
(the equations from transmitter 2, 3, and 4 to receiver 1). We iteratively solve \( u_3 \) in terms of other variables, which are
\[
\mathbf{v}^{[2]} = \begin{bmatrix}
1 \\
v_2
\end{bmatrix}, \mathbf{v}^{[3]} = \begin{bmatrix}
1 \\
v_3
\end{bmatrix}, \text{ and } \mathbf{v}^{[4]} = \begin{bmatrix}
1 \\
v_4
\end{bmatrix}.
\]
As a result, eliminating the extra variable \( u_3 \) by solving it iteratively in terms of other variables provides us the solution for the \( (2 \times 3, 1)^4 \) system, although it is difficult to express it in a closed form.

\( \text{The iterative solution for the variable } u_3 \text{ can be clearly seen in the corresponding polynomial system of the } (2 \times 4, 1)(2 \times 3, 1)^3 \text{ system.}
\)

VIII. MULTI-BEAM CASES

The solvability of polynomial systems for the multi-beam cases is more involved as explained in Section [V]. Only the proper system definition itself cannot state the feasibility of these cases since this definition does not consider the dependency of coefficients. Note that even the current advancements in algebraic geometry are insufficient for these cases. At this point, we can use information theoretic outer bounds (general and cooperative outer bounds), which we explain next, in addition to our proper system condition to test the feasibility of these cases.

It is well known that for a point to point MIMO channel with \( M \) transmit and \( N \) receive antennas, DoF is \( \min(M, N) \) [25,28]. In addition, for a 2-user MIMO interference channel with \( M^{[1]}, M^{[2]} \) and \( N^{[1]}, N^{[2]} \) antennas at transmitters and receivers, respectively, it is well known that DoF is \( \min \left(M^{[1]} + M^{[2]}, N^{[1]} + N^{[2]}, \max(M^{[1]}, N^{[2]}), \max(M^{[2]}, N^{[1]}) \right) \) [25]. These two results serve as general DoF outer bounds for a \( K \)-user MIMO interference network:
\[
d^{[i]} \leq \min(M^{[i]}, N^{[i]}) \quad (20)
\]
\[
d^{[i]} + d^{[j]} \leq \min \left(M^{[i]} + M^{[j]}, N^{[i]} + N^{[j]}, \max(M^{[i]}, N^{[j]}), \max(M^{[j]}, N^{[i]}) \right) \quad (21)
\]
for all \( i, j \in K \).

As mentioned before, we assume that the first condition \( (20) \) is always satisfied even if it is not explicitly stated every time throughout the paper.

Example 17. Although the \( (3 \times 3, 2)^2 \) system is proper, this system is almost surely infeasible since it does not satisfy the general outer bound \( (21) \).

Another outer bound, which is trivially obtained by using \( (21) \) is the cooperative outer bound. That is, the general outer bound \( (21) \) can also be used for all combinations of cooperation within transmitters and within receivers in a \( K \)-user MIMO interference network. If the general outer bound \( (21) \) is not satisfied for any of these combinations, then the system is almost surely infeasible.

Example 18. Consider a 4-user MIMO interference network with \( M^{[3]}, \ldots, M^{[4]} \) antennas and \( N^{[3]}, \ldots, N^{[4]} \) antennas at transmitters and receivers, respectively. The general outer bound \( (21) \) can also be used for the 3-user cooperative case of this network with \( M^{[1]} + M^{[2]}, M^{[3]} + M^{[4]} \) and \( N^{[1]} + N^{[2]}, N^{[3]} + N^{[4]} \) antennas at transmitters and receivers, respectively. In addition, it can also be used for the 2-user cooperative case of this network with \( M^{[1]} + M^{[2]}, M^{[3]} + M^{[4]} \) and \( N^{[1]} + N^{[2]}, N^{[3]} + N^{[4]} \) antennas at transmitters and receivers, respectively. These are the only 2 cooperative cases of the original 4-user network and the general outer bound \( (21) \) can be checked for all cooperative cases.

Example 19. Consider the \( (3 \times 4, 2)(1 \times 3, 1)(10 \times 4, 2) \) system, which is proper and which satisfies the general outer bound \( (21) \). Now, consider the cooperative case between the first and second users; that is, consider the \( (4 \times 7, 3)(10 \times 4, 2) \) system. Since the general outer bound \( (21) \) for this cooperative
case (briefly, cooperative outer bound) is not satisfied, this system is almost surely infeasible.

Next, we list some examples for \( K \)-user interference networks with more than \( K \) DoF, all of which satisfy the general outer bound \( \ref{eq:outer-bound-general} \) and the cooperative outer bound. We discuss the feasibility or infeasibility of these systems depending on the proper system condition. These examples highlight the usefulness of inequalities from Theorems \( \ref{eq:generic-bound} \) and \( \ref{eq:cooperative-bound} \) and the usefulness of grouping.

**Example 20.** Consider the \((5 \times 5, 2)^4\) system. There are \( N_e = 48 \) equations in total; therefore, there are \( 2^{48} - 1 \) subsets of equations. Testing each of them could be very challenging due to the proper system definition \( \ref{eq:proper-system} \). However, the system is easily seen to be proper from Theorem \( \ref{eq:generic-bound} \) since \( M + N - (K + 1)d = 5 + 5 - 10 = 0 \). Note that \((2 \times 8, 2)^4, (3 \times 7, 2)^4, (4 \times 6, 2)^4, (5 \times 5, 2)^4, (6 \times 4, 2)^4, (7 \times 3, 2)^4, \) and \((8 \times 2, 2)^4\) all belong to the same group, where any system in the group can be obtained by successively transferring an antenna between transmitters and receivers.

**Example 21.** Consider the \((5 \times 5, 3)(5 \times 5, 2)^3\) system. There are \( N_e = 60 \) equations in total; therefore, there are \( 2^{60} - 1 \) subsets of equations. Testing each of them could be very challenging due to the proper system definition \( \ref{eq:proper-system} \). However, the system is easily seen to be improper from Theorem \( \ref{eq:cooperative-bound} \) since the total number of variables \( N_e = 48 \) is less than the total number of equations \( N_e = 60 \).

Finally, note that there are two important features in a polynomial system that can lead the polynomial system to solvability or non-solvability: The coefficients and the structure of polynomial system. That is, one can lead the system to solvability or non-solvability by deliberately selecting the coefficients or by deliberately introducing a structure to the polynomial system. Bernstein’s theorem captures the structure of a multivariate polynomial system by finding the mixed volume of the Newton polytopes of the multivariate polynomial system. Therefore, Bernstein’s theorem only requires the independency of coefficients. Otherwise, if the coefficients are dependent, Bernstein’s theorem provides only an upper bound for the number of solutions of a multivariate polynomial system. Following a similar argument, due to the nature of our proper system definition, the proper system condition cannot handle the cases when the coefficients are dependent or when the polynomial system has a certain structure. For the latter case, consider the example mentioned in Remark \( \ref{remark:restriction} \).

For diagonal (time-varying) channels, the DoF of a \( K \)-user MIMO network \((M = N \text{ antennas at each node})\) is \( K/2 \) times the number of DoF achieved by each user in the absence of interference \( \ref{eq:multivariate-polynomial} \). Note that for the corresponding polynomial system, \( N_e > N_w \). Although interference networks with diagonal channels are improper (because \( N_e > N_w \)), the interference alignment is feasible since the diagonal channels obviously bring a structure to the polynomial system, which leads the system to solvability.

**IX. Conclusion**

In this paper, we explore the feasibility of interference alignment through beamforming in MIMO interference networks. Accordingly, we consider the alignment problem for an interference network as the solvability of its corresponding multivariate polynomial system. Ideally, we would like to find the conditions that would show the direct link between the feasibility (infeasibility) of an interference network and the solvability (non-solvability) of its corresponding polynomial system. For single beam cases, our results indicate that the solvability of corresponding polynomial systems is based on counting the number of equations and variables in the polynomial systems. We support our intuition by providing numerical results for a variety of cases, by presenting closed form solutions for new systems, and by providing rigorous proofs for some important cases.

On the other hand, for multi-beam cases, the current advancements in algebraic geometry are insufficient to prove the solvability of corresponding polynomial systems. Based on numerical results, we show that the connection between feasible and proper systems can be further strengthened by including information theoretic (general and cooperative) outer bounds to our proper system condition. In addition, based on numerical results, we also observe that if the system is improper, then it is infeasible.

**Appendix**

**Genericity**

The term genericity in algebraic geometry has a mathematical explanation, which is beyond our scope (see “generic property” from Wikipedia). In Chapter 7 of \( \ref{dejong2000} \), the proof that the polynomial system \( f_1 = 0, \ldots, f_n = 0 \) has generically \( \text{MV}(P_1, \ldots, P_n) \) number of common solutions is shown by induction on \( n \), which denotes the dimension of a polynomial system. Here, we will show that genericity implies independent random coefficients, which matters for our scope in this paper. For this purpose, we first start with the definition of coefficient polynomial. Note that mathematics literature does not directly and simply present the genericity in the matter of our scope as we present in this Appendix.

Let \( p(c) \) denote the coefficient polynomial, which is dependent on the coefficients of polynomials \( f_1, \ldots, f_n \), where

\[
   c \in C = \{ c_{ij} | i \in \{1, \ldots, n\}, j \in \{1, \ldots, m_i\} \},
\]

is the subset of all coefficients of polynomial system. Next, we define the term algebraic independence of coefficients, which is originally related to the term algebraic independence in algebraic geometry.

**Definition 2.** Let \( c \) denote the subset of all coefficients of polynomials \( f_1, \ldots, f_n \). The subset is called algebraically dependent if there is a coefficient polynomial satisfying the equality \( p(c) = 0 \). Otherwise, it is called algebraically independent.

**Example 22.** Consider the polynomial \( f = c_1 x^2 + c_2 x + c_3 \).

- \( c = \{ c_1 \} \) is algebraically dependent if there is a coefficient polynomial satisfying \( p(c_1) = 0 \), e.g., \( p(c_1) = c_1^2 + 2c_1 = 0 \). That is, \( c_1 \) is not transcendental.
- \( c = \{ c_1, c_2 \} \) is algebraically independent if there is a coefficient polynomial satisfying \( p(c_1, c_2) = 0 \), e.g., \( p(c_1, c_2) = c_1^2 + c_2 = 0 \).
If a polynomial is equal to zero, it is also called a “vanishing polynomial” in mathematics, e.g., \( p(c) = 0 \).

Next, we define \textit{genericity}, which we rephrase from the definitions 5.6 and 5.3 in Chapter 3 and 7 of [26], respectively. Note once again that Bezout’s and Bernshtein’s theorems give the exact number of common solutions when the coefficients are generic.

\textbf{Definition 3.} A property is said to hold generically for the polynomials \( f_1, \ldots, f_n \) if there is a coefficient polynomial \( p(c) \) such that the nonvanishing of \( p(c) \) implies that this property holds.

Intuitively, this definition means that the property for all polynomials holds for most of the coefficients; that is, for those coefficients satisfying \( p(c) \neq 0 \).

\textbf{Example 23.} Consider \( f = c_1 x^2 + c_2 x + c_3 = 0 \), which has a mixed volume 2. One can claim that the property “\( f \) has two (distinct) solutions” holds generically. To prove this, we must find a coefficient polynomial, whose nonvanishing implies this desired property. The condition is easily seen to be the nonvanishing discriminant of polynomial, \( p(c) = \text{Disc}(f) = c_1 (c_2^2 - 4 c_1 c_3) \neq 0 \), which is satisfied always if the set of all coefficients is algebraically independent. \textit{Some cases do not require the set of all coefficients to be algebraically independent.} For example, for the same polynomial, one can also claim that the property “\( f \) has two solutions with multiplicities counted” (i.e., the solutions may not be distinct this time) holds generically. The coefficient polynomial \( p(c) = c_1 c_3 \neq 0 \) implies this desired property. As a result, we briefly say that \( f \) has \textit{generically} 2 solutions.

As mentioned before, the proof that the polynomial system has generically MV(\( P_1, \ldots, P_n \)) number of common solutions is shown by induction on \( n \) in [26]. We leave the further details to be inquired in [26].

As a result, based on Definition 5, we can simply argue that independent random coefficients are almost surely generic since the set of independent random coefficients is algebraically independent; that is, \( p(c) \neq 0 \) almost surely, where \( c \) is the set of independent random coefficients.

The proof that the genericity in Bernshtein’s theorem implies independent random coefficients is also shown from an algebraic point of view in Section 2 of [29].

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