A.A. Stanislavsky\textsuperscript{1} and K. Weron\textsuperscript{2}

Subdiffusive transport in intergranular lanes on the Sun. The Leighton model revisited

Received: date / Accepted: date

Abstract In this paper we consider a random motion of magnetic bright points (MBP) associated with magnetic fields at the solar photosphere. The MBP transport in the short time range [0 \textendash 20 minutes] has a subdiffusive character as the magnetic flux tends to accumulate at sinks of the flow field. Such a behavior can be rigorously described in the framework of a continuous time random walk leading to the fractional Fokker-Planck dynamics. This formalism, applied for the analysis of the solar subdiffusion of magnetic fields, generalizes the Leighton’s model.

Keywords diffusion \textendash Sun: magnetic fields

PACS 96.60.-j \textendash 96.60.Hv \textendash 05.40.Fb

1 Introduction

Much of the solar surface phenomena is caused by photospheric convective motions of magnetic flux elements. Due to a complicated character of the solar convection, the heliospheric magnetic field contains a random component. Leighton (1964) suggested to consider the migration of magnetic regions on the Sun as a simple Brownian random walk on the solar surface. Until the precision of experiments was not high, this model was good enough as an approximation to the diffusion of magnetic elements. However, the recent experimental studies have shown that the model is too simple to envelop the phenomenon in full. The MDI magnetogram data (Hagenaar et al., 1999) from the SOHO spacecraft have noticed that the diffusion coefficients in tracking magnetic elements vary in time. Therefore, Cadavid et al. (1999) gave a refinement of the Leighton’s model. In their examination the walkers (magnetic elements) stick before the next jump, i.e. there are traps at stagnation points slowing down the walkers’ motion. The statistical analysis of the MBP data approves such a subdiffusive transport in the intergranular lanes within time interval to about 20 minutes. Nevertheless, this point of view produces an impression of an unfinished work. The Leighton’s model clearly leads to a diffusion equation. It is hence reasonable to ask whether it is possible to find a similar equation concerning the subdiffusive case. This question remained without any answer in the work of Cadavid et al. (1999). Perhaps, therefore Giacalone and Jokipii (2004) stressed that they do not see evidence of anomalous diffusion in random walks of magnetic footpoints. However, one may observe that the analysis of Giacalone and Jokipii (2004) is related to supergranular scales, with times being the order of days. Their casual comment about the results of Cadavid et al. (1999) is somewhat misleading since they did not investigate the same phenomena. After all, the analysis of Cadavid et al. (1999) supports the normal diffusion for larger time scales. Following Simon et al. (1995), the migration of the magnetic flux from a point to a point tends

\textsuperscript{1} Institute of Radio Astronomy, 4, Chervonoproporna St., Kharkov, 61002 Ukraine, E-mail: alexstan@ri.kharkov.ua
\textsuperscript{2} Institute of Physics, Wroclaw University of Technology, Wyb. Wyspianskiego 27, 50-370 Wroclaw, Poland, E-mail: Karina.Weron@pwr.wroc.pl
to accumulate at sinks of the flow field. The sinks displace randomly. Zimbardo, Veltri and Pommois (2000) have studied magnetic field line transport in 3D magnetic turbulence with anisotropy in the parallel and perpendicular directions to the magnetic field. The transport regime of magnetic field lines depends on parameters like the magnetic fluctuation level $\delta B/B_0$, the correlation lengths of magnetic turbulence $l_x, l_y, l_z$ and the dimensionality of turbulence. The numerical study (Pommois, Veltri and Zimbardo, 1999) shows that the transport can be anomalous (subdiffusive or superdiffusive). The various transport regimes are conveniently classified in terms of the Kubo number (Pommois, Veltri and Zimbardo, 2001). For magnetic turbulence this number is defined as

$$R = (\delta B/B_0)(l_\parallel/l_\perp)$$

where $l_\parallel = l_z$ is the correlation length parallel to the average field $B_0 = B_0 e_z$, $l_\perp = l_x = l_y$ the correlation length perpendicular to $B_0$. For $R \leq 0.2$ there are anomalous non-Gaussian transport regimes, whereas for $0.2 < R \leq 1$ there is an approximately quasilinear Gaussian diffusive regime. From this point of view the analysis of macroscopic diffusion equations for the motion of magnetic field lines represents a great interest in physics and astrophysics.

Recent progress (Metzler and Klafter, 2000; Zaslavsky, 2002; Stanislavsky, 2004; Meerschaert and Scheffler, 2004; Magdziarz et al., 2007; Magdziarz and Weron 2007) in understanding anomalous diffusion allows one to represent a subdiffusive transport of magnetic fragments on the solar surface in a more comprehensive form. The purpose of this paper is to perform this work. A simple mathematical review on the random processes of anomalous diffusion is given in Section 2. This view will be especially useful for readers far from the probabilistic theory. In Section 3 we derive the fractional Fokker-Planck equation (FPE) in spherical coordinates. This particular equation corresponds to a rotational subdiffusion being of interest to the solar physics of magnetic fields. In next section we discuss the solution describing the migration of the MBP on the Sun. Section 5 contains conclusions.

2 Diffusive processes as a continuous limit of CTRW

The notion of continuous time random walks (CTRW) has been introduced in physics by Montroll and Weiss (1965). They generalized a simple random walk which is based on the assumption that step changes (jumps) are made through equal time intervals. In contrast, the CTRW concerns random walks with a random waiting time among subsequent random jumps. The generalization has become very popular for many physical applications of anomalous diffusion, e.g. for transport in disordered media, superslow relaxation, etc. However, this model is useless until random values (jumps and waiting times) are undefined by a probabilistic description (for example, by their probability densities or characteristic functions).

In order to explain the idea of a CTRW let us consider the simplest one-dimensional case. The position of a walker after $k$ random jumps is given by

$$R(k) = \sum_{i=1}^{k} R_i, \quad R(0) = 0,$$  \hspace{1cm} (1)

where $\{R_i\}$ are random variables representing the length and the direction of jumps. The corresponding time interval reads

$$T(k) = \sum_{i=1}^{k} T_i, \quad T(0) = 0,$$  \hspace{1cm} (2)

where $\{T_i\}$ represent the random time intervals (waiting time) between successive jumps of a walker. The number $N_t$ of jumps performed by the walker till time $t$ is a counting process given by the following relation

$$N_t = \max\{k : T(k) \leq t\},$$  \hspace{1cm} (3)

meaning that the random number $N_t$ of jumps occurred up to time $t$ is equal to the largest index $k$ for which the sum $T_1 + T_2 + \ldots + T_k = T(k)$ of $k$ random time intervals does not exceed time $t$. Consequently, the total distance reached by the walker up to time $t$ determines the stochastic process

$$R(N_t) = \sum_{i=1}^{N_t} R_i,$$  \hspace{1cm} (4)
known as the CTRW. The analysis of probabilistic properties of such a random sum as given in Eq. (4) is a core of limit theorems in probability theory. Limit theorem, under a certain necessary and sufficient mathematical conditions, yields the continuous limit of the sums (or any other operation on the sequences of random variables, following Feller, 1971). In particular, taking into account the expected values \( \langle R_i \rangle < \infty \) and \( \langle T_i \rangle \to \infty \), we get the continuous limit \( X(S_t) \) of the CTRW process \( X \) subordinated by the random time clock \( S_t \). In terms of probability density functions we can write the probability density of the subordinated process \( X(S_t) \) as an integral relation

\[
p(x, t) = \int_0^\infty f(x, \tau) g(t, \tau) \, d\tau,
\]

where \( f(x, \tau) \) and \( g(t, \tau) \) are the probability density functions of \( X \) and \( S_t \), respectively. Here, the probability density \( f(x, \tau) \) represents the probability of finding the parent process \( X(\tau) \) at \( x \) on the operational time \( \tau \), whereas \( g(t, \tau) \) describes the probability for the operational time \( \tau \) to coincide with the real time \( t \). The physical interpretation of \( S_t \) is that this process accounts for the amount of time, when a walker does not participate in motion (Baeumer et al., 2005). If the walker randomly moves all time, the operational time coincides with the physical one, and \( g(t, \tau) \) is simply the Dirac \( \delta \)-function.

It is very important that the functional form of the probability density of the random variable \( S_t \) can be calculated explicitly. The procedure can be given in few steps. According to Bingham (1971), the Laplace transform of the probability density \( g(t, \tau) \) with respect to \( \tau \) equals

\[
\tilde{g}(t, v) = \int_0^\infty e^{-vt} g(t, \tau) \, d\tau = \langle e^{-vS_t} \rangle = E_\alpha(-vt^\alpha),
\]

where \( E_\alpha(-vt^\alpha) \) is the Mittag-Leffler function. The Mittag-Leffler function has a simple Laplace image with respect to \( t \), namely

\[
\int_0^\infty e^{-ut} \tilde{g}(t, v) \, dt = u^{\alpha-1}/(u^\alpha + v).
\]

The latter expression is easily inverted by Laplace with respect to \( v \) as an exponential function. As a consequence, the Laplace image of \( g(t, \tau) \) with respect to \( t \) is

\[
\tilde{g}(u, \tau) = \int_0^\infty e^{-ut} g(t, \tau) \, dt = u^{\alpha-1} \exp\{-u^\alpha \tau\}.
\]

By taking now the inverse Laplace transform of \( \tilde{g}(u, \tau) \) with respect to \( u \), we obtain the probability density of the process \( S_t \) in the form

\[
g(t, \tau) = t^{-\alpha} F_\alpha(\tau/t^\alpha),
\]

where the function \( F_\alpha(z) \) can be written as a Taylor series

\[
F_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma(1 - \alpha - n\alpha)}.
\]

Thus, according to Eq. (6), the probability density \( p(x, t) \) reads

\[
p(x, t) = \int_0^\infty F_\alpha(z) f(x, t^\alpha z) \, dz
\]

and its Laplace image takes the form

\[
\tilde{p}(x, u) = \int_0^\infty p(x, t) \exp\{-ut\} \, dt = u^{\alpha-1} \tilde{f}(x, u^\alpha),
\]

where

\[
\tilde{f}(x, u^\alpha) = \int_0^\infty f(x, \tau) \exp\{-u^\alpha \tau\} \, d\tau.
\]
The representation (8) will be used in the next section for derivation of the macroscopic equation of anomalous diffusion.

The above-mentioned mathematical techniques, appropriate for translational motion of a walker, can be also applied for a rotational random walk. As for the motion of MBP, associated with magnetic fields at the photosphere, subdiffusion on a sphere is of interest to us. This is nothing else but a rotational subdiffusion. Then the space jumps will be given by a polar angle $\theta$ and a longitude $\phi$ in the spherical coordinates. It is important to observe that the density $f(x, \tau)$ may be governed by a Fokker-Plank equation (FPE) with a time-independent potential, discussed in the next section.

3 Equation of rotational subdiffusion

To derive an equation of rotational subdiffusion, we start with one-dimensional case. Let $\hat{L}(x)$ be a time-independent Fokker-Planck operator well-known in the classical statistical physics. Assume that the probability density $f(x, \tau)$ describes an ordinary Brownian motion with respect to the operational time $\tau$. It will satisfy the FPE with the operational time:

$$\frac{\partial f(x, \tau)}{\partial \tau} = \hat{L}(x)f(x, \tau).$$

Applying operator $\hat{L}(x)$ to the Laplace image (8), we find the following expression

$$\hat{L}(x) \tilde{p}(x, u) = u^\alpha \tilde{p}(x, u) - q(x) u^\alpha - 1,$$

where $q(x)$ is the initial condition. The inverse Laplace transform of the latter gives the fractional integral form of the FPE

$$p(x, t) = q(x) + \frac{1}{\Gamma(\alpha)} \int_0^t d\tau (t - \tau)^{\alpha - 1} \hat{L}(x)p(x, \tau).$$

(9)

The kernel of this integral is a power function resulting in long-term memory effects for the process $X(S_t)$. This memory is a direct consequence of subordination of the space variable $X(\tau)$. For $\alpha = 1$, equation (9) reduces to the ordinary FPE without any memory effects. It should be mentioned that the integral form of equation (9) corresponds to the equivalent differential form

$$\frac{\partial^\alpha p(x, t)}{\partial t^\alpha} - \frac{q(x) t^{-\alpha}}{\Gamma(1 - \alpha)} = \hat{L}(x)p(x, t),$$

where $\partial^\alpha / \partial t^\alpha$ denotes the Liouville-Riemann fractional differential operator of order $\alpha$ (Samko et al., 1993). The Liouville-Riemann fractional differential operator of integer order is simply the ordinary derivative.

As a consequence, the equation of rotational subdiffusion in the simplest case of motion on a circle reads

$$\frac{\partial^\alpha W_\alpha(\theta, t)}{\partial t^\alpha} - \frac{W_\alpha(\theta, 0) t^{-\alpha}}{\Gamma(1 - \alpha)} = \frac{D_\theta}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial W_\alpha(\theta, t)}{\partial \theta} \right),$$

(10)

where $D_\theta$ is the diffusion coefficient and $W_\alpha(\theta, 0)$ the initial condition. Here, the probability density $W(\theta, t)$ depends on the polar angle $\theta$ only. The solution of Eq. (10) may be expressed in terms of the following integral transformation

$$W_\alpha(\theta, t) = \int_0^\infty F_\alpha(z) W_1(\theta, t^\alpha z) dz.$$

This implies that the averaged value of $\cos \theta$ equals

$$\langle \cos \theta \rangle = \int_0^\pi \cos \theta W_\alpha(\theta, t) \sin \theta d\theta = E_\alpha(-2D_\theta t^\alpha).$$

For large $t$ all the directions of the walker motion become equiprobable. However, in contrast to the normal rotational diffusion this equilibrium is reached slowly because of a power asymptotics of the Mittag-Leffler function.
4 Transverse subdiffusion of magnetic footpoints at the Sun

We now write down the equation of subdiffusion associated with the observed motion of magnetic footpoints embedded in the transverse flows on the solar surface. The distribution \( g_n(\theta, \phi, t) \) of magnetic footpoints obeys the two-dimensional equation in spherical coordinates

\[
\frac{\partial^\alpha g_\alpha}{\partial t^\alpha} = g_\alpha(\theta, \phi, 0) t^{-\alpha} - \frac{1}{R_\odot \sin \theta} \frac{\partial}{\partial \theta} \left( \frac{\kappa \sin \theta}{R_\odot} \frac{\partial g_\alpha}{\partial \theta} \right) + \frac{1}{R_\odot \sin \theta} \frac{\partial}{\partial \phi} \left( \frac{\kappa}{R_\odot \sin \theta} \frac{\partial g_\alpha}{\partial \phi} \right),
\]

where \( \kappa \) is the subdiffusion coefficient with the following dimension \([\kappa] = 1/\text{time}^\alpha\). The solution of Eq. (11) can be obtained via a separation ansatz in terms of the spherical harmonics \( Y_{mn}(\theta, \phi) \), namely

\[
g_\alpha(\theta, \phi, t) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{mn} Y_{mn}(\theta, \phi) E_\alpha \left( -\frac{\kappa}{R_\odot} n(n+1) t^\alpha \right).
\]

If one takes an impulse release of footpoints at the location \((\theta_0, \phi_0)\), the initial condition is determined through the Dirac \( \delta \)-function

\[
g_\alpha(\theta, \phi, 0) = \frac{1}{\sin \theta} \delta(\phi - \phi_0) \delta(\theta - \theta_0).
\]

Using the completeness relationship of spherical harmonics states, one obtains the amplitude coefficients \( a_{mn} \) expressed by the complex conjugate of the spherical harmonics \( Y_{mn}^*(\theta_0, \phi_0) \). It is interesting to consider an asymptotic behavior of the solution (12) as \( t \to \infty \). The distribution approaches asymptotically the limit

\[
\lim_{t \to \infty} g_\alpha(\theta, \phi, t) = \frac{1}{4\pi},
\]

which simply corresponds to the homogeneous distribution on the sphere in equilibrium. After integrating expression (12) from \( \phi = 0 \) to \( 2\pi \), we get the solution of Eq. (11) which will depend only on the polar angle \( \theta \) and time \( t \):

\[
g_\alpha(\theta, t) = \sum_{n=0}^{\infty} \frac{2n+1}{2} P_n(\cos \theta_0) P_n(\cos \theta) E_\alpha \left( -\frac{\kappa}{R_\odot} n(n+1) t^\alpha \right),
\]

where \( P_n(y) \) is the Legendre polynomial of degree \( n \). All the odd moments of the distribution are equal to zero, but the even moments are not. So the first even moment takes the form

\[
\langle \sin^2 \theta \rangle = \frac{2}{3} \left( 1 - E_\alpha \left( -\frac{6\kappa}{R_\odot} t^\alpha \right) \right).
\]

Since the subdiffusion motion of magnetic footpoints is characterized by a small value \( \kappa/R_\odot^2 \) (about 0.001 in order of magnitude), we arrive at the relation

\[
\langle \theta^2 \rangle \approx \frac{4\kappa}{T(1+\alpha) R_\odot^2} t^\alpha.
\]

It is obvious that for \( \alpha = 1 \) this description leads to the Leighton’s model.

In the framework of the fractional FPE approach to the transport of magnetic fields on the Sun we are able to recover the experimental evidence which demonstrates the change of diffusion properties during the life time of the MBP, from subdiffusion to normal one. From the theoretical point of view the difference between subdiffusion and ordinary diffusion is related with an evolution of the value of the parameter \( \alpha \), from \( \alpha < 1 \) to \( \alpha = 1 \). This change corresponds to different properties of the distribution of the interjump time intervals of the MBPs. If the expected value of the waiting time is infinite \((0 < \alpha < 1)\), then bright points have to be trapped. This effect is connected with the long-tailed properties of the waiting-time distribution, the necessary condition to obtain the subdiffusion from the CTRW scheme. For \( \alpha=1 \) there are no traps, since the expected waiting-time value is finite or time variable is deterministic. Due to the traps the diffusion of the MBPs has a mixed character of
random stops and motion, whereas in the case of normal diffusion the motion of the MBPs continues all time. From the astrophysical point of view the appearance of subdiffusion in the short-time range means that the life time of the traps is shorter than the life line of the MBP. Following the magnetic turbulence studies on laboratory and astrophysical plasmas, the anomalous diffusion of magnetic field lines may be associated with the existence of closed magnetic surfaces. Probably, the surfaces serve as traps for the MBP. The slope of the experimental data variance in time is a criterion for revealing the peculiarity. The comparison of the subdiffusive model with experimental data may be carried out in the same manner as this is the case for the work of Cadavid et al. (1999).

5 Conclusions

We have presented an approach to anomalous diffusion which follows from an intuitive concept of sticking times suggested in the paper of Cadavid et al. (1999). We have derived the fractional FPE in spherical coordinates appropriate for describing the subdiffusive migration of the MBP. As a special case, the normal diffusion of the MBPs on the Sun can be obtained. Therefore, the Leighton's model has been generalized. It should be noticed that the consideration of two different diffusive regimes in the MBP motion, as resulting from a simple sum of an ordinary first temporal derivative and a fractional temporal one in the FPE, will not lead to the expected phenomenon. In this case for short-time region such a temporal operator would lead to a normal diffusion, whereas for the large-time scale it does a subdiffusion (see, as an example, the work of Schumer et al., 2003).

Acknowledgements The work was partly realized within the framework of the project INTAS-03-5727. A.A.S. is grateful to A. C. Cadavid for fruitful remarks.

References

1. Baeumer, B., Benson, D. A., & Meerschaert, M. M.: Advectionand dispersion in time and space, Physica A 350, 245(2005)
2. Bingham, N.: Limit theorems for occupation times of Markov processes., Z. Wahrsch. verw. Geb. 17, 1(1971)
3. Cadavid, A. C., Lawrence, J. K., & Ruzmaikin, A. A.: Anomalous diffusion of solar magnetic elements, Astrophys. J. 521, 844(1999)
4. Feller, W.: An Introduction to Probability Theory and Its Applications. Wiley, New York (1971)
5. Giacalone, J. & Jokipii, J. R.: Magnetic footpoint diffusion at the Sun and its relation to the heliospheric magnetic field, Astrophys. J. 616, 573(2004)
6. Hageman, H. J., Schrijver, C. J., Title, A. M., & Shine, R. A.: Dispersal of magnetic flux in the quiet solar photosphere, Astrophys. J. 511, 932(1999)
7. Leighton, R. B.: Transport of magnetic field on the Sun, Astrophys. J. 140, 1547(1964)
8. Magdziarz, M. & Weron, K.: Anomalous diffusion schemes underlying the Cole-Cole relaxation: The role of the inverse-time α-stable subordinator, Physica A367, 1(2006)
9. Magdziarz M., Weron A., & Weron K.: Fractional Fokker-Planck dynamics: Stochastic representation and computer simulation, Phys. Rev. E75, 016708 (2007)
10. Magdziarz M. & Weron A.: Competition between subdiffusion and Lvy flights: A Monte Carlo approach, Phys. Rev. E75, 056702 (2007)
11. Montroll, E. W., Weiss, G. H.: Random walks on lattices. II, J. Math. Phys. 6, 167-181(1965).
12. Meerschaert M. M. & Scheffler H.-P.: Limit theorems for continuous-time random walks with infinite mean waiting time, J. Appl. Probab. 41, 623(2004)
13. Metzler, R. & Klafter, J.: The random walk's guide to anomalous diffusion: A fractional dynamics approach, Phys. Rep. 339, 1(2000)
14. Pommis P., Velti P., & Zimbardo G.: Anomalous and Gaussian transport regimes in anisotropic three-dimensional magnetic turbulence, Phys. Rev. E59, 2244(1999)
15. Pommis P., Velti P., & Zimbardo G.: Kubo number and Magnetic field line diffusion coefficient for anisotropic magnetic turbulence, Phys. Rev. E63, 066405(2001)
16. Schumer R., Benson D.A., Meerschaert M. M., & Baeumer B., Water Resources Research 39, No. 10, 1296(2003)
17. Samko, S. G., Kilbas, A. A., & Marichev, O. I.: Fractional Integrals and Derivatives – Theory and Applications, Gordon and Breach, New York (1993)
18. Simon, G. W., Title, A. M., & Weiss, N. O.: Kinematic model of supergranular diffusion on the Sun, Astrophys. J. 442, 886(1995)
19. Stanislavsky, A. A.: Probability interpretation of the integral of fractional order, Theor. and Math. Phys. 138, 418(2004)
20. Zaslavsky, G. M.: Chaos, fractional kinetics, and anomalous transport, Phys. Rep. 371, 461(2002)
21. Zimbardo G., Pommois P., & Veltri P.: Anomalous, quasilinear, and percolative regimes for magnetic-field-line transport in axially symmetric turbulence, Phys. Rev. E61, 1940(2000)