SOME EFFECTIVITY QUESTIONS FOR PLANE CREMONA
TRANSFORMATIONS IN THE CONTEXT OF SYMMETRIC
KEY CRYPTOGRAPHY

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Abstract  An effective lower bound on the entropy of some explicit quadratic plane Cremona transformations is given. The motivation is that such transformations (Hénon maps, or Feistel ciphers) are used in symmetric key cryptography. Moreover, a hyperbolic plane Cremona transformation $g$ is rigid, in the sense of [5], and under further explicit conditions some power of $g$ is tight.

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1. Introduction

The 2-dimensional Cremona group $\text{Cr}_2(k)$ is the group of $k$-birational automorphisms of the projective plane $\mathbb{P}_k^2$ over a field $k$. As such, it is an object of algebraic geometry, but it is also of interest from the viewpoints of dynamics [12] and group theory, including geometric group theory [5]. This latter paper introduces and uses a certain infinite-dimensional hyperbolic space $\mathcal{H} = \mathcal{H}_k = \mathcal{H}(\mathbb{P}_k^2)$ on which $\text{Cr}_2(k)$ acts as a group of isometries and makes it clear that this action is an important tool for studying both $\text{Cr}_2(k)$ and its individual elements. However, given an algebraic description of a Cremona transformation $g$ in terms of explicit rational functions, it is not always clear how to calculate anything about $g$ that is relevant to any of these frameworks. For example, there is no known effective procedure for determining the translation length $L(g) = L(g_\ast)$ of the isometry $g_\ast$ of $\mathcal{H}$ that is associated to $g$.

One of our goals here is to make matters more nearly effective in the very particular case where $g$ is a special quadratic transformation. That is, $g = \sigma \alpha$, where $\sigma$ is the standard quadratic transformation given in terms of homogeneous co-ordinates $x, y, z$ by $\sigma : (x, y, z) \mapsto (yz, xz, xy)$ and $\alpha$ is a linear involution. For this special class of Cremona transformations, we give (Theorem 1.1 (2)) a simple and explicit condition, in terms of a constant number of field operations (addition and multiplication in the ground field) to ensure that $g$ is hyperbolic: it is enough to take the three vertices $P, Q, R$ of the triangle $\Delta$ given by $xyz = 0$, compute the 12 points $g^i(P)$, etc., for $1 \leq i \leq 4$ and verify that they are distinct and that none of them lie on $\Delta$. In comparison, Bedford and Kim [BK, Theorem 3.2] give an exact formula for $L(g)$ when $\alpha$ is any linear transformation.
that relies on an unbounded quantity of information about the map $g$. The upper bound
$L(h) \leq \log D$ for a Cremona transformation $h$ of degree $D$ is well known \cite{5}; these bounds
together give an arbitrarily fine estimate for $L(g)$. These bounds will also determine the
type of $g$; that is, whether $g$ is elliptic, parabolic or hyperbolic (the word loxodromic is
also used for this last class). This is motivated by the fact, explained in \S\ 2, that certain
contemporary encryption algorithms are Cremona transformations and that estimates
such as those proved here can be seen as a speedy check that the key being used is not
obviously weak.

Say that points $x_1, x_2, \ldots, x_n$ in $\mathbb{P}^2$ are in general position if they are all distinct
and none of them, except for those that happen to equal one of $P, Q, R$, lies on $\Delta$. For
any natural number $n$, let $w_n$ denote the reduced word in $\sigma, \alpha$ that has length $n$ and
begins with $\alpha$ when reading from right to left. So, for example, $w_0 = 1, w_1 = \alpha, w_2 = \sigma \alpha$.
Set $P_n = w_n(P), Q_n = w_n(Q)$ and $R_n = w_n(R)$. So $P_0 = P$, etc. We say that $\alpha$, or $g$,
is in $(p,q,r)$-general position if $P_0, \ldots, P_{p-1}, Q_0, \ldots, Q_{q-1}, R_0, \ldots, R_{r-1}$ are in general
position. We abbreviate $(r,r,r)$-general position to r-general position. Of course, r-general
position implies s-general position for any $s \leq r$.

\textbf{Theorem 1.1.} (1) (= Theorem 4.21) Suppose that $g$ is in $(p,q,r)$-general position
and that $1/p + 1/q + 1/r < 1$. Then $g$ is hyperbolic. If also $p \leq q \leq r$, then $L(g) \geq \log(2 -
3.2^{-p/2})$.

(2) (= Theorem 4.22) Assume that $0 < \epsilon < 1/3$. Then, if $p \geq 12/\epsilon$ and the $3p$
points $P_0, \ldots, P_{p-1}, \ldots, R_{p-1}$ are in general position, $g$ is hyperbolic and
$$\log 2 - \epsilon < L(g) \leq \log 2.$$ 

We can also give some analogous sufficient conditions that permit the precise
determination of the type of a special quadratic transformation $g$ and its length. Here is
an example.

Assume that $g$ is in $(p,q,r)$-general position and that $P_p = P_{p-1}, Q_q = Q_{q-1}$ and
$R_r = R_{r-1}$. Let $\lambda_{\text{Lehmer}}$ denote Lehmer’s number, the smallest known algebraic integer $\lambda$
such that $|\lambda| > 1$ and every other conjugate $\lambda'$ of $\lambda$ has $|\lambda'| \leq 1$ and $|\lambda'| = 1$ for at least
one $\lambda'$. (These are the Salem numbers.)

\textbf{Theorem 1.2.} (= Theorem 4.26) (1) The transformation $g$ is biregular on the
blow-up of $\mathbb{P}^2$ at these $p + q + r$ points and is a Coxeter element in a Weyl group of type
$T_{p,q,r}$.

(2) If $1/p + 1/q + 1/r > 1$, then $g$ is elliptic and its order is the corresponding
Coxeter number. In particular, if $p = 2, q = 3$ and $r = 5$, then $g$ has order 30.

(3) If $1/p + 1/q + 1/r = 1$, then $g$ is parabolic.

(4) If $1/p + 1/q + 1/r < 1$, then $g$ is hyperbolic and $L(g)$ is the logarithm of a
Salem number of norm 1. In particular, if $p = 2, q = 3$ and $r = 7$, then $L(g) = \log \lambda_{\text{Lehmer}}$.

These results, and their proofs, can be summarized by saying that there is a
Coxeter–Dynkin diagram associated to the problem and if $P_0, \ldots, R_{r-1}$ are in general
position, then the diagram contains the standard tree $T_{p,q,r}$. Since $T_{2,3,5} = E_8$ the number
30 appears as the Coxeter number of $E_8$. This is a particular instantiation of the very
old idea of relating groups of Cremona transformations to Coxeter–Dynkin diagrams and
Weyl groups, although usually the diagrams that have arisen are of type $T_{2,3,r}$, and also of Steinberg’s idea [13] of describing Coxeter elements as a product of two involutions via a description of the Coxeter diagram as a bipartite graph. This viewpoint has also been exploited by McMullen [12], whose calculations have inspired some of those that appear here, and Blanc and Cantat [1], who use an infinite group $W_\infty$ that is something like a Coxeter group of type $E_\infty$ to prove that, for any hyperbolic Cremona transformation $h$, the spectral radius or dynamical degree $\lambda(h) = \exp L(h)$ lies in the closure of the set $\mathcal{T}$ of Salem numbers and $\lambda(h) \geq \lambda_{\text{Lehmer}}$.

The other main point of the paper is to extend the results of [5] that concern the rigidity and tightness of elements, or conjugacy classes, in the Cremona group. (The definitions of these properties are recalled later, at the start of § 5.) Some of these results also complement a recent paper by Lonjou [9], who exhibits, over any field $k$, an explicit Cremona transformation some power of which generates a proper normal subgroup of $\text{Cr}_2(k)$. She also points out an error in an earlier version of this paper; the mistake lay in overlooking the possibility that a rigid element of $\text{Cr}_2(k)$ might normalize a 2-dimensional additive subgroup of $\text{Cr}_2(k)$. In consequence, the results of § 5 in characteristic $p$ that refer to normal subgroups of $\text{Cr}_2(k)$ require the assumption that $k$ be algebraic (that is, algebraic over its prime subfield).

Fix a plane Cremona transformation $g$ over a field $k$.

**Theorem 1.3.** Assume that $g$ is hyperbolic.

1. (= Theorem 5.8) $g$ is rigid.
2. (= Theorem 5.13) Suppose that $L(g)$ is not the logarithm of a quadratic unit; if $\text{char } k = p > 0$ assume also that $k$ is algebraic and that $L(g)$ is not an integral multiple of $\log p$. Then some power of $g$ is tight. If also $n$ is sufficiently divisible, then the normal closure $\langle \langle gn \rangle \rangle$ does not contain $g$, so is a non-trivial normal subgroup of $\text{Cr}_2(k)$.

In [5], it was shown that if $g$ is a very general Cremona transformation of any degree $\geq 2$, then some power of $g$ is tight and that $g \notin \langle \langle gn \rangle \rangle$ for sufficiently divisible $n$. (If $g$ is very general, then $L(g) = \log \deg(g)$.)

In particular, this applies to special quadratic transformations $g$ if $k$ is algebraic and $\text{char } k \neq 2$ (if $\text{char } k = 2$ we must assume also that $L(g) \neq \log 2$), since it is an easy consequence of the other results that we prove about them that they satisfy the hypotheses of Theorem 1.3 (2).

In particular, these hypotheses can be realized over a finite field, since they impose three conditions on the 4-dimensional variety of involutions in $\text{PGL}_3$. In fact, over a finite field more is true.

**Theorem 1.4.** (= Theorems 5.17 and 5.18) Suppose that $k$ is a finite field and that $g$ is a hyperbolic element of $\text{Cr}_2(k)$. Then $g$ is tight and $g \notin \langle \langle gn \rangle \rangle$ for all sufficiently divisible $N$.

2. Background and motivation: dynamical systems and symmetric key cryptography

Hénon introduced certain complex quadratic plane Cremona transformations, now called Hénon maps, as models of (sections of) dynamical systems such as the Lorenz equations.
They are of the form
\[ f(x, y) = (ay + q(x), x), \]
where \( q \) is a quadratic polynomial and \( a \) is a non-zero scalar.

Then \( f \) might have sensitive dependence on initial conditions in this sense: even if initial points \( x_0 \) and \( y_0 \) are very close, their images \( f^n(x_0) \) and \( f^n(y_0) \) can be far apart for large values of \( n \).

In the context where \( f \) is a smooth self-map of a compact manifold \( X \) this, when stated precisely in terms of Lyapunov exponents, turns out to be equivalent to the topological entropy \( h(f) \) being strictly positive. (Recall that the topological entropy \( h(g) \) of a self-map \( g \) of a compact metric space \( X \) is defined by
\[ h(g) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{n} N(n, \epsilon) \right), \]
where \( N(n, \epsilon) \) is the number of \( g \)-orbit segments of length \( n \) that are at least a distance \( \epsilon \) apart. Gromov \cite{8} extended this definition to cover correspondences, which include Cremona transformations, as well.) However, this definition involves two limits, so the questions arise of finding how large \( n \) must be taken, and how small \( \epsilon \), in order to estimate it into a given accuracy in a bounded time.

On the other hand, over a finite field, especially one of characteristic 2, Feistel introduced the same kind of Cremona transformations
\[ f(x, y) = (y + q(x), x), \]
extcept that he took the parameter \( a \) to be \( a = 1 \) always and he did not demand that \( q \) be quadratic. Note that, in characteristic 2, this map \( f \) is the composite of two involutions:
\[ f = \alpha \circ \sigma, \]
where \( \alpha \) is the linear map \( (x, y) \mapsto (y, x) \) and \( \sigma(x, y) = (x, y + q(x)) \). These maps are also known as round functions and they are an essential element of Feistel ciphers such as the Data Encryption Standard (DES). The Advanced Encryption Standard (AES) uses different Cremona transformations, but otherwise both DES and AES have a similar structure. In fact, as explained below, AES is a Cremona transformation that is an element of a Galois twist of the group of standard Cremona transformations of \( \mathbb{P}^{128}_{\mathbb{F}_2} \). This (the untwisted group, that is) is the group generated by \( PGL_{128}(\mathbb{F}_2) \) and the standard non-linear birational involution \( x_i \mapsto x_i^{-1} \).

Here is a toy model of DES; it is a toy because it omits the key schedule. Taking the key schedule into account, as is done below, gives something like a noisy dynamical system, but where the noise is wholly determined in advance, as part of the infrastructure.

After Alice and Bob have established a key \( K \) (for example, by using some version of public key cryptography based, say, on elliptic curves), the key determines, according to a fixed public procedure that is part of the infrastructure of the algorithm, a round function \( f_K = \alpha \circ \sigma_K \) that is a Cremona transformation of some projective space \( \mathbb{P}^n_k \) over a finite field \( k \).

Once the key has been established, encryption of a message \( M \) is this: break \( M \) into blocks \( M_i \), each of size \( n \) (that is, \( M_i \) is a \( k \)-point of \( \mathbb{A}^n \)) and then, for a fixed integer
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that is also part of the infrastructure of the algorithm, apply the transformation \( f^N_K \) to the plaintext block \( M_i \) and transmit \( f^N_K(M_i) \). Decryption is: apply \((f^{-1}_K)^N\) to each block \( f^N_K(M_i) \) that is received. It is a basic requirement that, given possession of the key \( K \), decryption should be as fast as encryption; this is achieved by constructing \( f_K \) as the product of two involutions, and then decryption is merely the process of applying the same two involutions but in the opposite order.

The key schedule amounts to fixing \( k \)-linear involutions \( L_1, \ldots, L_N \) as part of the infrastructure (so that, in particular, the \( L_i \) are independent of the key), defining \( f_i = f_{K,i} = f_K \circ L_i \) and then taking encryption to be the iterate \( f_N \circ \cdots \circ f_1 \). Since each \( f_i \) is a product of three involutions, decryption is merely the process of applying the same two involutions, and so is as cheap, in terms of time and memory, as encryption.

AES can be described in similar terms. First, some Galois twist \( \sigma \) of the standard nonlinear birational involution

\[
(x_0, \ldots, x_n) \mapsto (x_0^{-1}, \ldots, x_n^{-1})
\]

of \( \mathbb{P}^n_k \) is given in advance and is public. Then, after the key \( K \) has been established, as above, it is used to construct a linear transformation of \( \mathbb{P}^n_k \) and we set \( f_K = \sigma \circ L_K \). Thereafter, the algorithm runs as for DES. (If the process ever encounters a base point, meaning that it is trying to invert 0, then it maps 0 to 0. So the iterated Cremona transformation is garbled. However, the security of the scheme does not reside in this garbling.) Since \( f_K^{-1} = \sigma \circ L_K^{-1} \), decrypting is then, as with DES, as fast and cheap as encrypting (especially if \( L_K \) is an involution). However, in higher dimensions, the inverse of a general Cremona transformation \( \phi \) is of higher degree than \( \phi \), so that inversion of \( \phi \) is slower and more expensive than the execution of \( \phi \).

As with DES, so AES has a key schedule, and the structure is similar. Encryption should also mix up the points of projective space thoroughly and quickly; in other words, it is desirable that if \( x_0 \) and \( y_0 \) are distinct basepoints that are close, then the points \( f^N_K(x_0) \) and \( f^N_K(y_0) \) should be far apart for some large, but fixed, value of \( N \). In other words, the round function should be sensitive to the parameters that define it, in the sense of (1) above. That is, in highly simplified terms, over the complex numbers certain Cremona transformations serve as simple models of a process that is known to be chaotic, while over a finite field the same Cremona transformations are used to create a process that merely has a convincing appearance of chaos.

Moreover \([12]\), positive entropy does not exclude the existence of Siegel discs; Siegel discs (whatever their analogues might be over finite fields) are undesirable in a cryptographic context because they are regions consisting of plaintexts that are close and that remain close after encryption. On the other hand, the theorem of Gromov and Yomdin, that the entropy of an endomorphism \( g \) of a smooth projective variety \( X \) is the logarithm of the spectral radius of the action of \( g \) on the cohomology of \( X \), shows that, as a consequence of the Lefschetz fixed point formula, \( h(g) \) can be computed by counting fixed points of a certain number of iterates of \( g \); how many iterates are required depends on the Betti numbers of \( X \).

Despite the fact that the contemporary algorithm AES is a Cremona transformation, in this paper we consider only plane Cremona transformations. The reason is
simple: we do not know any analogous results in higher dimensions, even for the group of standard transformations.

3. Hyperbolic space

Here we review the construction and basic properties of the infinite hyperbolic space $\mathcal{H} = \mathcal{H}_k = \mathcal{H}(\mathbb{P}^2_k)$ and the action of $\text{Cr}_2(k)$ on it. This is taken from Cantat and Lamy [5]; we repeat it only in order to establish notation.

Let $V$ be any smooth projective surface over the field $k$. Set $\mathcal{Z}(V)_\mathbb{Z} = \lim_{V \rightarrow Y} \text{NS}(Y)$, where the direct limit is taken over all blow-ups $Y \rightarrow V$, and $\mathcal{Z}(V) = \overline{\mathcal{Z}(V)}_\mathbb{Z} \otimes \mathbb{R}$. There is a hyperbolic completion $\tilde{\mathcal{Z}}(V)$ of $\mathcal{Z}(V)$, given by

$$\tilde{\mathcal{Z}}(V) = \left\{ \lambda + \sum_{e_P \in \mathcal{E}} n_P e_P | \lambda \in \text{NS}(V)_\mathbb{R}, n_P \in \mathbb{R}, \sum \deg(P)n_P^2 < \infty \right\},$$

where $e_P$ is the exceptional curve associated to the closed point $P$ on some blow-up $Y$, $\mathcal{E}$ is the set of such curves, $\deg(P)$ is the degree of the field extension $k(P)/k$ and $\sum \deg(P)n_P^2 < \infty$ means that, for all $\epsilon > 0$ there is a finite subset $\mathcal{F}$ of $\mathcal{E}$ such that, for all finite subsets $\mathcal{G}$ of $\mathcal{E} - \mathcal{F}$, we have $\sum_{e_P \in \mathcal{G}} \deg(P)n_P^2 < \epsilon$. (By definition, $\sum \deg(P)n_P^2$ is the number $l$ such that for all $\epsilon > 0$ there is a finite subset $S$ of closed points $P$ such that, whenever $S \subset T$ and $T$ is a finite set of closed points, $|\sum_{e_P \in T} \deg(P)n_P^2 - l| < \epsilon$.)

On $\tilde{\mathcal{Z}}(V)$, there is a hyperbolic inner product denoted by $(x,y)$. The hyperbolic space $\mathcal{H}(V)$ is one of the two connected components of the locus $\{ x \in \tilde{\mathcal{Z}}(V) | (x,x) = 1 \}$. Note that, if $x = \lambda + \sum n_P e_P$, then $(x,x) = (\lambda,\lambda) - \sum \deg(P)n_P^2$. The distance on $\mathcal{H}(V)$ is denoted by $d$, so that $\cosh d(x,y) = (x,y)$. The isometries of $\mathcal{H}(V)$ are the continuous linear transformations of $\tilde{\mathcal{Z}}(V)$ that preserve the inner product and the connected component above.

Given any blow-up $Y \rightarrow V$, there are natural isomorphisms $\mathcal{Z}(V)_\mathbb{Z} \rightarrow \mathcal{Z}(V)_\mathbb{Z}$, $\mathcal{H}(V) \rightarrow \mathcal{H}(Y)$, etc., so that the group $\text{Bir}(V)$ of birational automorphisms of $V$ acts as a group of isometries of $\mathcal{H}(V)$ via $g \mapsto g_*$. From now on, we take $V = \mathbb{P}^2_k$ and write $\mathcal{H}(V) = \mathcal{H}_k$ and $\text{Bir}(V) = \text{Cr}_2(k)$.

**Lemma 3.1.** The formation of $\mathcal{H}_k$ is functorial in $k$ and, if $k$ is a subfield of $K$, then $\mathcal{H}_k$ is naturally a closed geodesic subspace of $\mathcal{H}_K$ and, as a subgroup of $\text{Cr}_2(K)$, $\text{Cr}_2(k)$ acts on $\mathcal{H}_k$ so as to preserve $\mathcal{H}_k$.

**Proof.** Immediate from the construction of $\mathcal{H}_k$. \hfill \square

We shall usually drop the subscript $k$ from $\mathcal{H}_k$.

Isometries $\phi$ of $\mathcal{H}$ are of three types: $\phi$ is **elliptic** if it has a fixed point in (the interior of) $\mathcal{H}$; $\phi$ is **parabolic** if it has a unique fixed point on the ideal boundary $\partial \mathcal{H}$ of $\mathcal{H}$ and no fixed point in $\mathcal{H}$; $\phi$ is **hyperbolic** if it has just two fixed points on the ideal boundary and no fixed point in $\mathcal{H}$.

Equivalently, $\phi$ is hyperbolic if the lower bound $L(\phi) = \inf d(x,\phi(x))$ is strictly positive and is attained in $\mathcal{H}$, while $\phi$ is parabolic if $L(\phi) = 0$ but is not attained in $\mathcal{H}$.

In this last case, the set $\{ x \in \mathcal{H} | d(x,\phi(x)) = L(\phi) \}$ is a geodesic in $\mathcal{H}$. It is the **axis** of $\phi$ and is denoted by $\text{Ax}(\phi)$. It is the unique geodesic preserved by $\phi$ and its endpoints.
on $\partial \mathcal{H}$ are the unique fixed points on the closure $\overline{\mathcal{H}} = \mathcal{H} \cup \partial \mathcal{H}$. The quantity $L(\phi)$ is the translation length of $\phi$. On the other hand, a parabolic isometry does not preserve any geodesic.

**Lemma 3.2.** Suppose that $\mathcal{G}$ is a closed hyperbolic subspace of $\mathcal{H}$ and $\phi$ an isometry of $\mathcal{H}$ that preserves $\mathcal{G}$.

1. $\phi$ is elliptic if and only if $\phi|_{\mathcal{G}}$ is elliptic.
2. $\phi$ is hyperbolic if and only if $\phi|_{\mathcal{G}}$ is hyperbolic, and in this case $Ax(\phi|_{\mathcal{G}}) = Ax(\phi)$.
3. $\phi$ is parabolic if and only if $\phi|_{\mathcal{G}}$ is parabolic, and in this case the unique ideal boundary point of $\mathcal{H}$ that is fixed by $\phi$ equals the ideal boundary point of $\mathcal{G}$ that is fixed by $\phi|_{\mathcal{G}}$.
4. $L(\phi|_{\mathcal{G}}) = L(\phi)$.

**Proof.**

1. If $P \in \mathcal{H} - \mathcal{G}$ and $\phi(P) = P$, then $\phi$ also fixes the unique point $Q$ on $\mathcal{G}$ that is closest to $P$.
2. Assume $\phi$ to be hyperbolic, with axis $\Gamma$. There is a map $\Gamma \to \mathcal{G} : x \mapsto y$ where $y$ is the closest point to $x$ that lies on $\mathcal{G}$. The image is a geodesic $\Delta$ which is preserved by $\phi$. Since $\phi$ does not preserve two geodesics, $\Gamma = \Delta$.
   Conversely, if $\phi|_{\mathcal{G}}$ is hyperbolic, it preserves a geodesic $\Delta$ in $\mathcal{G}$, so that, again by (1), $\phi$ is hyperbolic and its axis is $\Delta$.
3. Obvious, from (1) and (2).
4. Obvious. \qed

**Corollary 3.3.** If $g \in \text{Cr}_2(k)$, then its nature (elliptic, parabolic or hyperbolic) and its translation length can be calculated after making any extension of $k$.

**Lemma 3.4.** Suppose that $\Gamma$ is a geodesic in $\mathcal{H}$ and that $\phi$ is an isometry of $\mathcal{H}$ that preserves $\Gamma$. Then the following statements hold.

1. Either $\phi$ is elliptic and fixes a point on $\Gamma$ or $\phi$ is hyperbolic.
2. If $\phi$ is hyperbolic then $\Gamma$ is the unique geodesic preserved by $\phi$ and equals the axis of $\phi$.
3. The translation length of $\phi$ acting on $\mathcal{H}$ equals the translation length of its restriction $\phi|_{\Gamma}$ to $\Gamma$.

**Proof.** Take $\mathcal{G} = \Gamma$ in Lemma 3.2. \qed

**Lemma 3.5.** Suppose that $\sigma, \alpha$ are involutions of $\mathcal{H}$.

1. $\text{Fix}(\sigma)$ and $\text{Fix}(\alpha)$ are hyperbolic subspaces of $\mathcal{H}$.
2. $\sigma$ preserves each geodesic that is perpendicular to $\text{Fix}(\sigma)$, and the same for $\alpha$. 

(3) Fix(σ) and Fix(α) meet in \(\mathcal{H}\) if and only if \(\sigma = g\), say, is elliptic.

(4) Fix(σ) and Fix(α) are parallel if and only if they meet in a single ideal boundary point \(P\) if and only if \(g\) is parabolic.

(5) If Fix(σ) and Fix(α) are ultraparallel then there is a unique geodesic \(\Gamma\) perpendicular to both and \(g\) is hyperbolic. This geodesic is preserved by both \(\alpha\) and \(\sigma\) and is the axis of \(g\).

**Proof.** (1) is clear: both fixed loci are projectivizations of linear spaces.

(2) is a simple observation.

(3) if Fix(σ) and Fix(α) meet in an interior point \(x\) of \(\mathcal{H}\) then \(g(x) = x\) and \(g\) is elliptic. Conversely, suppose \(x\) is an interior point and \(g(x) = x\). If \(x \in \text{Fix}(\sigma) \cup \text{Fix}(\alpha)\) then it is easy to see that \(x \in \text{Fix}(\sigma) \cap \text{Fix}(\alpha)\), so suppose that this is not the case. Then there is a unique geodesic segment \(l\) from \(x\) to \(\alpha(x)\): this is perpendicular to Fix(α) and Fix(α) cuts \(l\) at its midpoint. Similarly, there is a unique geodesic segment \(m\) from \(\alpha(x)\) to \(\sigma\alpha(x)\): this is perpendicular to Fix(σ) and Fix(σ) cuts \(m\) at its midpoint. But \(\sigma\alpha(x) = x\), so \(l = m\) and the midpoint lies in \(\text{Fix}(\alpha) \cap \text{Fix}(\sigma)\) and (3) is proved.

So we can assume that Fix(σ) \(\cap\) Fix(α) contains no interior point and that \(g\) is not elliptic.

Case (a): Fix(σ) \(\cap\) Fix(α) contains a boundary point \(P\). Then \(g(P) = P\). Suppose that \(g\) is hyperbolic, with axis \(\text{Ax}(g) = l\), and suppose that the boundary point \(Q\) is the other endpoint of \(l\). Since \(\sigma g \sigma = g^{-1}\) and \(\text{Ax}(g^{-1}) = \text{Ax}(g)\), \(\sigma\) preserves \(\text{Ax}(g)\) and so fixes \(Q\). Similarly, \(\alpha(Q) = Q\). Then \(\text{Fix}(\sigma) \cap \text{Fix}(\alpha)\) contains \(l\), which is absurd. So \(g\) is parabolic.

Case (b): Fix(σ) \(\cap\) Fix(α) contains no interior nor boundary point. That is, they are ultraparallel. Then there is a unique geodesic \(l\) perpendicular to both of them. Because \(\alpha, \sigma\) are involutions, each of them preserves \(l\), and so \(g\) is not elliptic but preserves a geodesic \(l\). Then \(g\) is hyperbolic and \(l\) is its axis. This proves (4) and (5). □

Suppose that \(\delta\) is a Cremona transformation of degree \(D\) (in that \(\delta\) is defined by a net of homogeneous polynomials of degree \(D\)). Then [5] \(L(\delta_*)\) equals the dynamical degree of \(\delta\), defined as \(\lim_{n \to \infty} (\deg(\delta^n))^{1/n}\). Say that \(\delta\) is elliptic, etc., if \(\delta_*\) is so and that \(\delta\) is *biregular* if there is some rational surface \(X\) on which \(\delta\) is biregular (in other words, is an automorphism).

Over an algebraically closed field if \(\delta\) is elliptic, then it is biregular, and in this case there is a rational surface \(X\) on which \(\delta\) is biregular and an integer \(n > 0\) such that \(\delta^n\) lies in the connected component \(\text{Aut}_0^0\) of the identity element in \(\text{Aut}_X\). The map \(\delta\) is parabolic if and only if it preserves a pencil of curves of genus at most 1.

Over any field, if \(\delta\) is hyperbolic, then there is no pencil of curves that is preserved by \(\delta\).

Moreover, over \(\mathbb{C}\), the entropy \(h(\delta)\) of \(\delta\) satisfies \(h(\delta) \leq L(\delta_*)\) and equality holds if \(\delta\) is biregular on some smooth projective rational surface.
4. Graphs and lattices

Fix homogeneous co-ordinates \(x, y, z\) on \(V = \mathbb{P}_k^2\). We denote by \(\sigma\) the standard quadratic involution \(\sigma : (x, y, z) \mapsto (y, x, xy)\) and by \(\alpha\) a linear involution. We put \(g = \sigma\alpha\).

Say that \(P, Q, R\) are the base points of \(\sigma\) and \(Y = \text{Bl}_{P,Q,R} V\), with exceptional curves \(e_P, e_Q, e_R\). Note that \(\sigma\) is biregular on \(Y\) and \(\alpha\) is biregular on \(V\). Let \(l\) denote the class of a line in \(V\).

Let \(E\) denote the set of all exceptional curves \(e_x\) as \(x\) runs over all closed points of all blow-ups of \(Y\).

Since \(Z(Y)_\mathbb{Z} \cong Z(V)_\mathbb{Z}\), the lattice \(Z(V)_\mathbb{Z}\) has a \(\mathbb{Z}\)-basis \(\{l\} \cup \{e_P, e_Q, e_R\} \cup E\).

From now on we shall not always be careful to distinguish between \(\alpha\) and \(\alpha_\ast\), nor between \(\sigma\) and \(\sigma_\ast\).

**Lemma 4.1.** \(\alpha\) permutes the set \(\{e_P, e_Q, e_R\} \cup E\) and \(\sigma\) permutes \(E\).

**Proof.** Immediate, from the facts that \(\alpha\) is biregular on \(V\) and that \(\sigma\) is biregular on \(Y\).

Let \(v_0\) denote the root \(v_0 = l - e_P - e_Q - e_R\).

**Lemma 4.2.** \(\alpha\) preserves \(l\) and \(\sigma\) acts on the lattice \(\mathbb{Z}\{l, e_P, e_Q, e_R\}\) as the reflection \(s_{v_0}\) in \(v_0\).

Our aim is to construct a bipartite graph \(H\) that depends upon \(\alpha\). Then \(g = \sigma\alpha\) will act as something close to a Coxeter element in the corresponding Coxeter group.

We begin by constructing subsets \(\tilde{\Gamma}_\alpha\) and \(\tilde{\Gamma}_\sigma\) of the \(\mathbb{Z}\)-lattice spanned by \(\{l, e_P, e_Q, e_R\} \cup E\), as follows. These subsets are not necessarily disjoint.

The elements (or vertices) of \(\tilde{\Gamma}_\sigma\) are \(v_0\) and one representative \(e_x - \sigma(e_x)\) of each non-zero pair \(\pm(e_x - \sigma(e_x))\) as \(e_x\) runs over \(E\). The vertices of \(\tilde{\Gamma}_\alpha\) are one representative \(e_y - \alpha(e_y)\) of each non-zero pair \(\pm(e_y - \alpha(e_y))\) as \(e_y\) runs over \(E\) \(\cup\) \(\{e_P, e_Q, e_R\}\). Finally, if \(v\) lies in \(\tilde{\Gamma}_\alpha\) and \(\pm v\) lies in \(\tilde{\Gamma}_\sigma\), then choose \(v\) rather than \(-v\) in \(\tilde{\Gamma}_\sigma\).

Put \(\tilde{\Gamma} = \tilde{\Gamma}_\alpha \cup \tilde{\Gamma}_\sigma\). Because \(\alpha\) and \(\sigma\) are involutions, two different elements of \(E \cup \{e_P, e_Q, e_R\}\) (resp., in \(E\), cannot give the same vertex in \(\tilde{\Gamma}_\alpha\) (resp., in \(\tilde{\Gamma}_\sigma\)).

We join two distinct vertices in \(\tilde{\Gamma}\) by an edge of multiplicity equal to their intersection number, if that number is non-zero. If the intersection number is zero, then the corresponding vertices remain disjoint. So every edge has multiplicity \(\pm 1\). (Since there are no loops, that is, since no vertex is joined to itself, \(-2\) does not occur. Since we never take both \(v\) and \(-v\), \(2\) does not occur either.) Define the valency of a vertex \(v\) to be the sum of the absolute values of the multiplicities of the edges meeting \(v\).

**Lemma 4.3.** If \(v\) lies in the intersection \(\tilde{\Gamma}_\alpha \cap \tilde{\Gamma}_\sigma\) then \(v\) is disjoint from all other vertices in \(\tilde{\Gamma}\).

**Proof.** Suppose that \(v = e_x - \sigma(e_x)\) and \(v = e_y - \alpha(e_y)\), where \(e_x \in E\). Note that \(e_x, \sigma(e_x), e_y\) and \(\alpha(e_y)\) are all classes of irreducible curves. We proceed to consider three cases separately.
(1) \( v \) meets \( e_z - \sigma(e_z) \). Since \( e_z \) and \( \sigma(e_z) \) are also classes of irreducible curves, either \( e_x = e_z \), and then \( v = e_z - \sigma(e_z) \), or \( e_x = \sigma(e_z) \). This latter possibility contradicts the construction of \( \bar{\Gamma}_\sigma \).

(2) \( v \) meets \( e_t - \alpha(e_t) \). We reach a similar conclusion.

(3) \( v \) meets \( v_0 = l - e_P - e_Q - e_R \). Then \( e_x \in \{ e_P, e_Q, e_R \} \), which is impossible. \( \square \)

Now define \( G_i = \bar{\Gamma}_i - (\bar{\Gamma}_\alpha \cap \bar{\Gamma}_\sigma) \) for \( i = \alpha, \sigma \). Construct a graph \( G \) whose set of vertices is \( G_\alpha \cup G_\sigma \), and then join two vertices in \( G \) if and only if they are joined in \( \bar{\Gamma} \). Note that \( v_0 \) lies in \( G_\sigma \).

Lemma 4.4.

(1) There are no edges within either \( G_\alpha \) or \( G_\sigma \).
(2) \( G \) is a bipartite graph.
(3) The vertex \( v_0 \) is of valency at most 3 and every other vertex of \( G \) is of valency at most 2.

Proof. It is enough to notice that in \( \bar{\Gamma} \) the vertex \( v_0 \) has valency at most 3 and that every other vertex has valency at most 2, so that deleting \( \bar{\Gamma}_\alpha \cap \bar{\Gamma}_\sigma \) amounts to deleting those vertices that are joined to no other vertex. Equivalently, deleting \( \bar{\Gamma}_\alpha \cap \bar{\Gamma}_\sigma \) amounts to deleting all double bonds and the corresponding vertices. \( \square \)

Now define \( \tilde{H} \) to be the connected component of \( G \) that contains \( v_0 \).

Lemma 4.5. \( \tilde{H} \) is bipartite and is either a tree \( T_{p,q,r} \) consisting of \( v_0 \) and three arms of lengths \( p, q, r \leq \infty \) attached to \( v_0 \) or the union \( \Delta_{m,r} \) of a cycle of finite length \( m \) together with an arm \( v_0, w_{r-1}, \ldots, w_1 \) of length \( r \leq \infty \) attached to the cycle at \( v_0 \).

Proof. Immediate. \( \square \)

Remark. When we speak of the length of an arm, we count the vertex to which it is joined. So, for example, \( T_{2,3,5} \) is the \( E_8 \) diagram, \( T_{p,q,r} \) has a total of \( p + q + r - 2 \) vertices and, in general, a non-trivial arm has length at least 2. So \( \Delta_{m,1} \) is a cycle of length \( m \).

Lemma 4.6. If \( \tilde{H} = \Delta_{m,r} \) then \( m \) is even.

Proof. Any subgraph of a bipartite graph is bipartite, so the cycle in \( \tilde{H} \) is bipartite, so even. \( \square \)

Write \( m = 2n \). From \( \tilde{H} \), construct a graph \( H \) with the same vertices as \( \tilde{H} \), but where every edge except at most one has multiplicity +1, by starting at \( v_0 \) and proceeding either outwards along one arm at a time (in the case of \( T_{p,q,r} \)) or around the cycle and then along the arm (in the case of \( \Delta_{2m,r} \)) as follows: at each step, change an edge of multiplicity \(-1\) into an edge of multiplicity \(+1\) by replacing a vector \( v \) by its negative,
−v. If \( H \) is of type \( T_{p,q,r} \) then all its edges have multiplicity +1, and we write \( H = T_{p,q,r} \); in the other case all edges, except possibly one edge that meets \( v_0 \) and lies in the cycle, are of multiplicity +1, and we write \( H = \Delta_{2n,r}^{\pm} \) accordingly.

**Lemma 4.7.**

1. \( H \) is either of type \( T_{p,q,r} \) with \( p, q, r \leq \infty \) or of type \( \Delta_{2n,r}^{-} \) with \( 2 \leq n < \infty \) and \( 2 \leq r \leq \infty \).

2. If \( p, q, r \) are finite and the points \( P_0, \ldots, P_{p-1}, Q_0, \ldots, Q_{q-1}, R_0, \ldots, R_{r-1} \) are in general position, then \( H \) contains the diagram \( T_{p,q,r} \).

**Proof.** (1) We use the notation of the preceding proof, and in addition, for any element \( s \) of \( \{ \alpha, \sigma \} \), let \( s' \) denote the element of \( \{ \alpha, \sigma \} \) distinct from \( s \). Then there are consecutive nodes in the cycle that are of the form \( e_{sw}(P) - e_{w}(P), e_{sw}(P) - e_{sw}(P) \) and \( e_{sw}(Q) - e_{w}(Q) \), and also \( e_{sw}(P) - e_{sw}(P) = e_{sw}(Q) - e_{sw}(Q) \). However, this is absurd.

From the definition of \( \Delta_{2n,r}^{-} \), it is clear that \( n, r \geq 2 \).

(2) is an immediate observation.

Denote by \( \Lambda(H) \) the lattice on the vertices \( v, w \) of \( H \), with pairing given by the usual intersection pairing of curves.

Put \( H_{\alpha} = H \cap G_{\alpha} \) and \( H_{\sigma} = H \cap G_{\sigma} \), so that \( v_0 \in H_{\sigma} \). For \( v \in H \), let \( s_v \) denote the reflection in \( v \).

**Lemma 4.8.** \( \sigma \) acts on \( \Lambda(H) \) via the product \( S = \prod_{v \in H_{\sigma}} s_v \) and \( \alpha \) acts on \( \Lambda(H) \) via the product \( A = \prod_{w \in H_{\alpha}} s_w \).

**Proof.** Immediate observation. Notice that because all the reflections in each product commute with each other, the order in which they are taken is immaterial. The fact that each product contains infinitely many factors is also immaterial, since there are only finitely many terms in either product that act non-trivially on any given element of \( \Lambda(H) \).

The next lemma is well known but for lack of a convenient reference we give a proof.

**Lemma 4.9.** Suppose that \( 2 \leq p \leq q \leq r < \infty \) and \( H = T_{p,q,r} \). Then \( \Lambda(H) \) is negative definite if \( 1/p + 1/q + 1/r > 1 \), degenerate if \( 1/p + 1/q + 1/r = 1 \) and hyperbolic if \( 1/p + 1/q + 1/r < 1 \).

**Proof.** If \( 1/p + 1/q + 1/r > 1 \) then \( H \) is one of the Dynkin diagrams classified in Bourbaki [GrLie4-6] and \( \Lambda(H) \) is the corresponding root lattice, twisted by \((-1)\).

If \( 1/p + 1/q + 1/r = 1 \) then \( H \) is an affine Dynkin diagram of type \( \tilde{E}_{n-1} \), where \( n = p + q + r - 2 \), and is degenerate. The radical \( R \) is of rank 1 and \( \Lambda(H)/R \) is isomorphic to the root lattice \( E_{n-1} \).

If \( 1/p + 1/q + 1/r < 1 \) then \( H \) contains \( H' = T_{p,q,r-1} \). By induction, \( \Lambda(H') \) is either degenerate or hyperbolic, and then \( \Lambda(H) \) is hyperbolic.
Lemma 4.10. \( \Lambda(H) \) is degenerate only when \( H \) is of finite type \( T_{p,q,r} \) with 
\[
1/p + 1/q + 1/r = 1.
\]

Proof. Suppose that \( H = \Delta_{2n,r} \), that \( r \geq 2 \) and that the diagram is

\[
\begin{array}{c}
\bullet v_0 \\
\bullet v_1 \\
\bullet w_{r-2} \\
\bullet \cdots \\
\bullet v_{2n-1}
\end{array}
\]

where the vertices \( v_0, \ldots, v_{2n-1} \) are arranged in a cycle of length \( 2n \). Suppose that \( \eta = \sum_{i=0}^{2n-1} a_i v_i + \sum_{j=0}^{r-2} b_j w_j \) is in the radical, so that \( \eta v_i = \eta w_j = 0 \) for all \( i, j \). Put \( w_{r-1} = v_0 \) and \( b_{r-1} = a_0 \).

By letting \( i \) run from 0 to \( 2n - 1 \), we see that \( a_i \) is a linear function of \( i \); say \( a_i = \lambda i + a_0 \) for \( i = 0, \ldots, 2n - 1 \). By letting \( j \) run from 0 to \( r - 1 \), we see in a similar way that \( b_j = \mu j + b_0 \) for \( j = 0, \ldots, r - 1 \).

Since \( v_0 = w_{r-1} \), we get \( a_0 = b_{r-1} = \mu(r-1) + b_0 \). Also, \( b_1 = 2b_0 \), so that \( \mu = b_0 \), \( a_0 = \mu r \).

From \( \eta v_{2n-1} = 0 \), we get \( a_{2n-2} - 2a_{2n-1} - a_0 = 0 \), so that \( 2n\lambda + a_0 = 0 \) and \( 2n\lambda + \mu r = 0 \).

From \( \eta v_0 = 0 \) we get \( a_1 - a_{2n-1} - 2a_0 + b_{r-2} = 0 \), so that
\[
0 = -2(n+1)\lambda - 2a_0 + \mu(r-1).
\]

Then \( 0 = 2\lambda - \mu \), so that \( n = r = 0 \), which is absurd.

Finally, suppose that \( r = 1 \) (that is, \( H \) is a cycle) and that \( \eta = \sum a_i v_i \) is in the radical. Then the equations \( v_i \eta = 0 \) for \( i = 1, \ldots, 2n - 2 \) give
\[
2a_1 = a_0 + a_2, \ldots, 2a_{2n-2} = a_{2n-3} + a_{2n-1},
\]
so that there is some \( \lambda \) such that \( a_j = a_0 + \lambda j \) for every \( j = 0, \ldots, 2n - 1 \). But \( v_0.\eta = 0 \) gives \( a_1 = 2a_0 + a_{2n-1} \), so that \( 0 = 2a_0 + (2n - 2)\lambda \), while \( v_{2n-1}.\eta = 0 \) gives \( 0 = 2a_0 + 2n\lambda \). Then \( a_0 = \lambda = 0 \) and \( \eta = 0 \).

There is an obvious natural homomorphism \( \beta : \Lambda(H) \to \mathcal{Z}(V)_{\mathbb{Z}} \) of lattices.

Lemma 4.11. \( \beta \) is injective.

Proof. Inspection. \( \square \)

Let \( \Lambda(T_{p,q,r}^{(\lambda)}) \) and \( \Lambda(\Delta_{2n,r}^{-\lambda}) \) denote the lattices corresponding to the diagrams \( T_{p,q,r} \) and \( \Delta_{2n,r}^{-\lambda} \), but where each vertex \( v \) has \( v^2 = -\lambda \) and the other intersection numbers are unchanged. So, for example, \( \Lambda(H^{(2)}) = \Lambda(H) \).
Lemma 4.12. $\Lambda(\Delta_{2n,r}^{-}(2))$ is hyperbolic if $2/n + 1/r < 1$.

Proof. $\Lambda(\Delta_{2n,r}^{-}(2))$ is non-degenerate, by Lemma 4.10. Deleting $v_0$ leaves a negative definite lattice, and so $\Lambda(\Delta_{2n,r}^{-}(2))$ is either negative definite or hyperbolic. However, deleting the vertex opposite $v_0$ in the cycle leaves a $T_{n,n,r}$ diagram, which is hyperbolic. □

Say that a lattice $\Lambda$ is affine if $\Lambda$ is degenerate, its radical $R(\Lambda)$ is of rank 1 and $\Lambda/R(\Lambda)$ is negative definite. For example, $\Lambda(T_{p,q,r}^{(2)})$ is affine if and only if $1/p + 1/q + 1/r = 1$.

As $\lambda$ varies, so $\Lambda(T_{p,q,r}^{(\lambda)})$ sweeps out a line in the space of real quadratic forms in $n = p + q + r - 2$ variables. Moreover, $\Lambda(T_{p,q,r}^{(\lambda)})$ is negative definite if $\lambda \gg 0$ and positive definite if $\lambda \ll 0$ and the signature of $\Lambda(T_{p,q,r}^{(\lambda)})$ is a non-increasing function of $\lambda$. (We have adopted the convention that the signature of a positive definite form of rank $n$ is $n$ and the signature of a negative definite form of rank $n$ is $-n$.) For critical values of $\lambda$, the form will be degenerate.

In particular therefore, there exists $\mu = \mu(p, q, r)$ such that $\Lambda(T_{p,q,r}^{(\mu)})$ is negative definite if $\lambda > \mu$, while $\Lambda(T_{p,q,r}^{(\mu)})$ is negative semi-definite and degenerate. For example, $\mu(2, 3, 6) = \mu(2, 4, 4) = \mu(3, 3, 3) = 2$. Also, define $m = m(p, q, r)$ by

$$\mu^2 = m + m^{-1} + 2$$

and $m \geq 1$.

Lemma 4.13. (1) $\mu(p, q, r)$ is a strictly increasing function of each of $p, q, r$. That is, $\mu(p, q, r) > \mu(p, q, r - 1)$ and the same for $p$ and $q$.

(2) If $1/p + 1/q + 1/r < 1$ then $\mu(p, q, r) > 2$.

Proof. (1) Say $s = \mu(p, q, r - 1)$, so that $\Lambda(T_{p,q,r-1}^{(s)})$ is affine. Then there is a non-zero vector $\sum n_i v_i$ in the radical of $\Lambda(T_{p,q,r-1}^{(s)})$, and it is easy to see that we can take every $n_i$ to be positive.

Say that $w$ is the vertex adjoined in passing from $T_{p,q,r-1}$ to $T_{p,q,r}$ and that $w$ meets $v_{r-1}$ in $T_{p,q,r-1}$. Then

$$\left(\sum n_i v_i + \epsilon w\right)^2 = 2n_{r-1} \epsilon - \epsilon^2 s > 0$$

for $0 < \epsilon \ll 1$, so that $\Lambda(T_{p,q,r}^{(s)})$ is hyperbolic. By the discussion above, the act of increasing $s$ will lead to the lattice becoming negative definite; there is therefore a critical point $t = \mu(p, q, r)$ at which the lattice becomes degenerate, and $t > s$.

(2) is a consequence of (1) and the equalities $\mu(2, 3, 6) = \mu(2, 4, 4) = \mu(3, 3, 3) = 2$. □

Fix $p \leq q \leq r < \infty$, with $1/p + 1/q + 1/r < 1$. Say $\mu(p, q, r) = \mu$. Write $\mu(p, p, p) = \mu_p$ and $m(p, p, p) = m_p$. 

Lemma 4.14. \( \lim_{p,q,r \to \infty} \mu(p,q,r) = 3/\sqrt{2} \) and \( \mu_p \geq \frac{3}{\sqrt{2}}(1 - 2^{-p/4}) \).

Proof. Say that \( f_1, \ldots, f_{p-1}; g_1, \ldots, g_{q-1}; h_1, \ldots, h_{r-1} \) are the vertices of \( T_{p,q,r} \), reading outwards along the arms from the central vertex \( v_0 \). We can suppose that \( p = q = r \) and then, by symmetry, that

\[
\xi = b_p(0)v_0 + \sum_{i=1}^{p-1} b_p(i)(f_i + g_i + h_i)
\]
is in the radical of \( \Lambda(T^{(\mu_p)}_{p,p,p}) \).

Set \( b_p(0) = 3 \) and \( b_p(p) = 0 \). Then \( 0 = \xi.v_0 = -3\mu_p + 3b_p(1) \) and we have a recurrence relation

\[
0 = \xi.f_n = b_p(n-1) + b_p(n+1) - \mu_p b_p(n) \quad \forall n \geq 1.
\]

Therefore, by the well-known formula for the solution of such recurrence relations, there are constants \( C_p, D_p \) such that

\[
b_p(n) = C_p \theta^n_p + D_p \delta^n_p,
\]

where \( \theta_p = (\mu_p + \sqrt{\mu_p^2 - 4})/2 \) and \( \delta_p = (\mu_p - \sqrt{\mu_p^2 - 4})/2 = \theta_p^{-1} \leq \theta_p \). Since \( b_p(p) = 0 \), this can be written as

\[
b_p(n) = E_p(\theta^n_p - \delta^n_p)
\]

for some \( E_p \) that is independent of \( n \). Since \( b_p(0) = 3 \) and \( b_p(1) = \mu_p \), we get

\[
\frac{\mu_p}{3} = \frac{b_p(1)}{b_p(0)} = \frac{\theta^{2p-1}_p - \theta_p}{\theta^{2p}_p - 1}.
\]

Say \( \mu = \lim_{p \to \infty} \mu_p \) and \( \theta = \lim_{p \to \infty} \theta_p \); then

\[
\frac{\mu}{3} = \frac{1}{\theta}.
\]

This leads to \( \mu = \sqrt{3}/2 \). Since \( \mu_p \) is an increasing function of \( p \), we get \( \mu_p \to (\sqrt{3}/2)^{-} \).

For the speed of convergence, note that \( \mu_p = m^{1/2}_p + m^{-1/2}_p \) and \( \mu_p = \theta_p + \theta_p^{-1} \). So \( m_p = \theta^2_p \). The formula for \( \mu_p/3 \) above then gives

\[
m^0_p = \frac{2m_p - 1}{2 - m_p}.
\]

The same equation is satisfied by \( m^{-1}_p \). \( \square \)

Lemma 4.15. \( m_p \to 2^{-} \) and \( m_p > 2 - 3.2^{-p/2} \), while \( m^{-1}_p \to (1/2)^{+} \) and \( m^{-1}_p < 1/2 + 2^{-p} \).

Proof. Since \( \mu_p \to (3/\sqrt{2})^{-} \), \( m_p \to 2^{-} \).
Solving the above equation for $m_p$ when $p = 4$ gives $m_4 \approx 1.72208$, so $m_4 > \sqrt{2}$. Since $m_p$ is an increasing function of $p$, $m_p > \sqrt{2}$ for all $p \geq 4$.

Say $2 - m_p = \eta$. Then the equation above gives

$$\eta = (3 - 2\eta)(2 - \eta)^{-p}.$$ 

Since $m_p > \sqrt{2}$, this leads to

$$\eta < (3 - 2\eta)2^{-p/2} < 3.2^{-p/2}. $$

The proof for $m_p^{-1}$ is similar. Since $\mu_p^2 = m_p + m_p^{-1} + 2$, we get

$$\mu_p^2 > 2 - 3.2^{-p/2} + \frac{1}{2} + 2 = 3\left(\frac{3}{2} - 2^{-p/2}\right).$$

So $\mu_p > \sqrt{3^2/2} (1 - 2^{-p/2})$ and Lemma 4.14 is proved. \(\Box\)

Fix $p \leq q \leq r < \infty$, with $1/p + 1/q + 1/r < 1$. Say $\mu(p, q, r) = \mu$.

Then there is a unique totally isotropic vector $\xi = \xi(p, q, r) \in \Lambda(T_{p,q,r})$ where the coefficient of the branch vertex is 1.

Fix a subset $E$ of $\{p, q, r\}$ and let the members of $\{p, q, r\} - E$ tend to $\infty$. Let $H_E$ denote the corresponding infinite graph.

**Lemma 4.16.** $\mu \to \mu_E$ for some $\mu_E \leq 3/\sqrt{2}$ and in the completed hyperbolic vector space generated by the vector space $\Lambda(H_E) \otimes \mathbb{R}$ there is an eigenvector $\xi$, with an eigenvalue $\mu_E$, of the adjacency matrix of $H_E$.

**Proof.** The existence of $\mu_E$ follows from Lemmas 4.13 and 4.14.

To prove that $\xi$ exists, we write it down. Define $\theta = (\mu_E + \sqrt{\mu_E^2 - 4})/2$. Suppose, for the sake of explicitness, that $E = \{p, q\}$. Then

$$\xi = \sum_{0}^{p} a_ie_i + \sum_{0}^{q} b_j f_j + \sum_{0}^{\infty} c_k g_k,$$

where $e_0 = f_0 = g_0 = v_0$, the branch vertex, $a_p = b_q = 0$ and $a_0 = b_0 = c_0 = 1$, and we require that $\xi$ be totally isotropic in $\Lambda(T_{p,q,\infty})$. This is achieved by taking

$$a_i = A(\theta^i - \theta^{-i}) + \theta^{-i}, \quad b_j = B(\theta^j - \theta^{-j}) + \theta^{-j}, \quad c_k = \theta^{-k},$$

where $A = 1/(1 - \theta^{2p})$ and $B = 1/(1 - \theta^{2q}).$ \(\Box\)

**Corollary 4.17.** If $p, q$ are fixed then $\lim_{r \to \infty} m(p, q, r)$ exists and $\leq 2$. There is a similar limit if $p$ is fixed and $q, r \to \infty$.

Now suppose that $H = \Delta_{2n, r}$ and that $2/n + 1/r < 1$. Define $\mu(2n, r)$ in the same way that $\mu(p, q, r)$ is defined for $T_{p,q,r}$; that is, $\mu(2n, r)$ is the critical value $t$ such that $\Lambda(H(t))$ is degenerate.
Lemma 4.18. (1) $\mu(n - 1, n - 1, r) \leq \mu^\Delta(2n, r)$.

(2) $\mu^\Delta(2n, r)$ is a strictly increasing function of both $n$ and $r$.

(3) $\lim_{r \to \infty} \mu^\Delta(2n, r)$ exists and $\leq 3/\sqrt{2}$.

Proof. (1) Delete the three vertices $u_1, u_2, u_3$ consisting of the vertex opposite $v_0$ and the two vertices adjacent to it. This shows that $\Lambda(T_{n-1,n-1,r}) \subseteq \Lambda(\Delta_{2n,r}^-)$. Say $\mu(n - 1, n - 1, r) = \lambda$, and pick $\xi$ in the radical of $\Lambda(T_{n-1,n-1,r})$. Then there is a vector $\eta = \xi + \sum_{i=1}^3 \alpha u_i$ in $\Lambda(\Delta_{2n,r}^-)$ with $\eta^2 > 0$. So $\mu(n - 1, n - 1, r) \leq \mu^\Delta(2n, r)$.

(2) and (3) are proved by the same kind of calculation used in the proof of Lemma 4.14. □

Corollary 4.19. $\mu^\Delta(2p, p) > 2 - 3.2^{-p+1}$.

Proof. Apply Lemma 4.15. □

Lemma 4.20. If $n$ is fixed and $r = \infty$, then there is an eigenvector $\xi$, with eigenvalue $\lim_{r \to \infty} \mu^\Delta(2n, r)$, of the infinite adjacency matrix.

Proof. Put $\mu_E = \mu^\Delta(2n, \infty)$ and $\theta = (\mu_E + \sqrt{\mu_E^2 - 4})/2$. As in the proof of Lemma 4.16, the easiest thing is to write $\xi$ down:

$$\xi = \sum_{i=0}^{2n-1} a_i v_i + \sum_{j=0}^{\infty} b_j w_j,$$

where $b_j = \theta^{-j}$, $a_i = A_1 \theta^i + A_2 \theta^{-i}$ and $A_1, A_2$ are determined by the equations

$$0 = -s + A_1(\theta - \theta^{2n-1}) + A_2(\theta^{-1} - \theta^{-(2n-1)}) + \theta^{-1},$$

$$0 = A_1(\theta^{2n-2} - s\theta^{2n-1}) + A_2(\theta^{-(2n-2)} - s\theta^{-(2n-1)}) - 1.$$ □

Suppose now that $1/p + 1/q + 1/r < 1$ or $2/n + 1/r < 1$, as appropriate. Then at this point we have shown, whether $H$ is finite or infinite, the existence of a totally isotropic vector $\xi$ in $\Lambda(H^{(\mu)})$ for some $\mu > 2$. Moreover, $\xi$ generates the radical $R(\Lambda(H^{(\mu)}))$.

Let $M$ denote the adjacency matrix of $H$; then the matrix $B = -\mu 1 + M$ is the Gram matrix of $\Lambda(H^{(\mu)})$ and $\xi$ is an eigenvector, with an eigenvalue of 0, of $B$. The adjacency matrix is, according to the bipartite decomposition $H = H_\sigma \sqcup H_\alpha$, of the form

$$M = \begin{bmatrix} 0 & tC \\ C & 0 \end{bmatrix},$$

where each row and column of $C$ contains at most three non-zero entries, and each non-zero entry is 1. (In the case of $\Delta_{2n,r}^\pm$, some entries will be $-1$.)

Since $\xi$ generates $R(\Lambda(H^{(\mu)}))$, the other eigenvalues of $B$ are real and strictly negative. In particular, $\mu$ is the maximum eigenvalue of $M$ and is of multiplicity 1.

Write $\xi = [cu\ z]$. Then $tCz = \mu u$ and $Cu = \mu z$. 
Recall that $g = σα$ and notice that, from the bipartite description of $H$, $α$ acts as the matrix $A = \begin{bmatrix} cμ & 0 \\ 0 & 1 \end{bmatrix}$, while $σ$ acts as the matrix $S = \begin{bmatrix} c−1μ & 0 \\ 0 & 1 \end{bmatrix}$. Then

$$S \begin{bmatrix} u \\ 0 \end{bmatrix} = \begin{bmatrix} −u \\ 0 \end{bmatrix}, \quad S \begin{bmatrix} 0 \\ z \end{bmatrix} = \begin{bmatrix} μu \\ z \end{bmatrix},$$
$$A \begin{bmatrix} u \\ 0 \end{bmatrix} = \begin{bmatrix} u \\ μz \end{bmatrix}, \quad A \begin{bmatrix} 0 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ −z \end{bmatrix}.$$  

Thus, $S$ and $A$ preserve the real 2-plane $Π$ based by $\{[cμ \ 0], [cμ \ 0]\}$ and act on $Π$ with respect to this basis as the matrices $\begin{bmatrix} c−1μ & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} cμ & 0 \\ −1 & 0 \end{bmatrix}$, respectively.

The matrix of $g_*$ is then $SA = \begin{bmatrix} cμμ−1 & −μ \\ μ−1 & 1 \end{bmatrix} = δ^2$, where $δ = \begin{bmatrix} cμ & 0 \\ −1 & 1 \end{bmatrix}$. Since $Tr δ = μ > 2$, it follows that $δ$, and so $g_*$, is hyperbolic.

The projectivization of $Π$ is a geodesic $ℓ$ in the hyperbolic space $ℋ$ and $ℓ$ is preserved by $σ$, $α$ and $g_*$. Therefore, by Lemma 3.4, the translation length of $g_*$ acting on $ℋ$ equals the translation length of $g_*$ acting on $ℓ$. We shall calculate this from the preceding discussion.

Write $μ^2 = m + m^{-1} + 2$, so that

$$log m = 2 cosh^{-1}(μ/2).$$

The next result contains the first part of Theorem 1.1.

**Theorem 4.21.** Suppose that $g$ is in $(p, q, r)$-general position and that $1/p + 1/q + 1/r < 1$. Then $g$ is hyperbolic. If also $4 ≤ p ≤ q ≤ r$ then $L(g) ≥ log(2 − 3.2^{−p/2})$.

**Proof.** The hypotheses imply that $H$ is either $T_{p.q.r}$ with $1/p + 1/q + 1/r < 1$ or $Δ_{2n,r}$ with $2/n + 1/r < 1$. We now use the notation of the preceding discussion.

The vector $x = [cμ] \in Π$ satisfies $(x,x) = 1$ and $(x,δ(x)) = μ/2$, so that $d(x,δ(x)) = cosh^{-1}(μ/2)$. That is, $L(δ) = cosh^{-1}(μ/2)$, so that $g_*$ preserves the geodesic $ℓ$. Therefore, $g_*$ is hyperbolic and

$$L(g) = 2 cosh^{-1}(μ/2) = log m > 0.$$  

For the second part, note that, since $μ_E$ is an increasing function of each argument and, in the notation of Lemma 4.15 and the formula immediately preceding it, $μ_{p}^2 = m_p + m_{p−1} + 2$, we have $L(g) ≥ log m_p > log(2 − 3.2^{−p/2})$, by Lemma 4.15. □

Finally, we give the proof of the rest of Theorem 1.1.

**Theorem 4.22.** (= Theorem 1.1 (2)) Assume that $0 < ϵ < 1/3$. Then, if $p ≥ 12/ϵ$ and the 3p points $P_0, \ldots, P_{p−1}, \ldots, R_{p−1}$ are in general position, the Cremona transformation $g = σα$ is hyperbolic and

$$log 2 − ϵ < L(g) ≤ log 2.$$  

**Proof.** We know that $L(g) ≥ log(2 − 3.2^{−p/2})$, so that it is enough to verify that $log(2 − 3.2^{−p/2}) ≥ log 2 − ϵ$ if $p > 12/ϵ$. This is elementary. □
These techniques also give Cremona transformations with prescribed properties, as follows.

For example, let us calculate \( \mu(2, 3, 7) \). In terms of the \( T_{2,3,7} \) diagram

we have to find the value of \( s \) for which the lattice \( \Lambda(T_{s}^{2,3,7}) \) has a non-trivial totally isotropic vector \( \xi = \sum_{i=1}^{10} b_{i}u_{i} \). Then \( \mu(2, 3, 7) = s \). That is, we must eliminate the variables \( b_{1}, \ldots, b_{10} \) from the equations

\[
\xi \cdot u_{i} = 0
\]

for all \( i = 1, \ldots, 10 \) where \( u_{i}^{2} = -s \) and the other intersection numbers are given by the diagram. We can normalize \( \xi \) by assuming that \( b_{10} = 1 \).

This process of elimination is mechanical and yields

\[
s^{10} - 9s^{8} + 27s^{6} - 31s^{4} + 12s^{2} - 1 = 0.
\]

Since \( \mu(2, 3, 7) > 2 \) it follows that \( m(2, 3, 7) > 1 \) and is a zero of Lehmer’s polynomial. Therefore,

\[
m(2, 3, 7) = \lambda_{\text{Lehmer}} \approx 1.17628.
\]

(Recall that \( \mu^{2} = m + m^{-1} + 2 \).)

Exactly similar calculations show that \( \mu(2, 4, 5) \) is a solution of

\[
s^{8} - 8s^{6} - 20s^{4} + 17s^{2} - 3 = 0,
\]

so that \( m(2, 4, 5) \) is a zero of

\[
m^{8} - m^{5} - m^{4} - m^{3} + 1;
\]

\( \mu(3, 3, 4) \) is a zero of

\[
s^{6} - 6s^{4} + 8s^{2} - 1
\]

so that \( m(3, 3, 4) \) is a zero of

\[
m^{6} - m^{4} - m^{3} - m^{2} + 1;
\]

\[
\mu(4, 4, 4) = \sqrt{(5 + \sqrt{13})/2};
\]

\[
\mu(5, 5, 5) = \sqrt{3 + \sqrt{2}}.
\]

The approximate values are given in this table.

| \((p, q, r)\) | \((2,3,7)\) | \((2,4,5)\) | \((3,3,4)\) | \((4,4,4)\) | \((5,5,5)\) | \((\infty, \infty, \infty)\) |
|-----------|-----------|-----------|-----------|-----------|-----------|----------------|
| \(\mu\)   | 2.0066    | 2.0153    | 2.0285    | 2.0743    | 2.101     | 3/\sqrt{2}    |
| \(m\)     | \(\lambda_{\text{Lehmer}}\) | 1.28064   | 1.40127   | 1.6644    | 1.8832    | 2             |

Fix (finite) integers \( p, q, r \). Then we can force \( H \) to be equal to the finite tree \( T_{p,q,r} \) by putting stringent conditions on the linear involution \( \alpha \) as follows:
$P_0, \ldots, P_{p-1}, \ldots, R_{r-1}$ must be in general position while $P_p = P_{p-1}, Q_q = Q_{q-1}, R_r = R_{r-1}$.

Note that, since $\sigma$ has only finitely many fixed points while $\alpha$ has a line of fixed points, and (if char $k \neq 2$) one isolated fixed point, these are two conditions on $\alpha$ (which moves in a 4-dimensional subvariety of the 8-dimensional group $PGL_{3,k}$) for each of $p, q, r$ that is odd, and one condition on $\alpha$ for each that is even. As before, $l$ is the class of a line in $\mathbb{P}^2$.

**Lemma 4.23.** $H$ is a finite diagram of type $T_{p,q,r}$ and its vertices are $v_0 = l - e_{P_0} - e_{Q_0} - e_{R_0}$; $e_{P_0} - e_{P_1}$, $e_{P_{p-2}} - e_{P_{p-1}}$; $e_{Q_0} - e_{Q_1}$, $\ldots$, $e_{Q_{q-2}} - e_{Q_{q-1}}$; $e_{R_0}$ $e_{R_1}$, $\ldots$, $e_{R_{r-2}} - e_{R_{r-1}}$.

**Proof.** Immediate. □

**Lemma 4.24.** Both $\alpha$ and $\sigma$ are biregular on the blow-up $Y$ of $\mathbb{P}^2$ at the $p + q + r$ points $P_0, \ldots, R_{r-1}$.

**Proof.** Also immediate. □

Of course, $g$ is then also biregular on $Y$.

**Lemma 4.25.** The orthogonal complement $\Lambda(H)^\perp$ of $\Lambda(H)$ in $NS(Y)$ has a $\mathbb{Z}$-basis $(x_P, x_Q, x_R)$ given by $x_P = l - \sum e_{P_i}$, etc. Each of $\alpha, \sigma$ acts trivially on $\Lambda(H)^\perp$.

**Theorem 4.26.** (= Theorem 1.2) The biregular quadratic map $g$ acts on $NS(Y)$ as a Coxeter element in the Weyl group of the lattice $\Lambda(T_{p,q,r})$. In particular, if $(p, q, r) = (2, 3, 5)$ then $g$ has order 30; if $(p, q, r) = (2, 3, 6)$ then $g$ is parabolic; while if $(p, q, r) = (2, 3, r)$ with $r \geq 7$ then $g$ is hyperbolic and $L(g) = \log \lambda_{r+3}$. In particular, if $r = 7$ then $L(g) = \log \lambda_{Lehmer}$.

**Proof.** This follows from the discussion above. □

**Remark.** Blanc and Cantat prove [1] that $L(g) \geq \log \lambda_{Lehmer}$ for any hyperbolic element $g$ of $Cr_2(k)$, while, in [12], McMullen proves the existence of biregular Cremona transformations $g$ with $L(g) = \log \lambda_{r+3}$ for all $r \geq 7$. His examples also preserve a cuspidal cubic curve but are less explicit than ours. For example, sufficient conditions on the linear involution $\alpha$ for $g$ to be biregular of type $(2, 3, 7)$ are that $\alpha(P)$ should be a fixed point of $\sigma$ and that $\sigma \alpha(Q)$ and $(\sigma \alpha)^3(R)$ should lie on the line that is the 1-dimensional part of $\text{Fix}\alpha$. These are four explicit polynomial conditions on the 4-dimensional family of involutions in $PGL_3$.

**Remark.** There is a much shorter argument that suffices to prove merely that $g$ is hyperbolic if the diagram $H$ is hyperbolic, without any estimates, as follows.

Consider the action of the involutions $\sigma, \alpha$ on the completed hyperbolic space $\mathcal{H}$ associated to $\Lambda(H)$.

**Lemma 4.27.** $\text{Fix}(\alpha_*)$ and $\text{Fix}(\sigma_*)$ are ultraparallel.
Proof. Fix(\(\alpha_\ast\)) \(\cap\) Fix(\(\sigma_\ast\)) is spanned by common non-zero eigenvectors \(p = [x \ y]^\top\) of \(\alpha_\ast\) and \(\sigma_\ast\) such that \((p, p) \geq 0\). Here, \(x\) (resp., \(y\)) lies in the Hilbert space completion of the negative definite vector space \(\Lambda(H_\alpha) \otimes \mathbb{R}\) (resp., \(\Lambda(H_\sigma) \otimes \mathbb{R}\)), each of which is naturally embedded in the hyperbolic completion of \(\Lambda(H) \otimes \mathbb{R}\).

We shall check that no such vectors \(p\) exist. There are three cases to consider.

(1) \(\alpha_\ast(p) = p\) and \(\sigma_\ast(p) = p\). Then \(Cx = 2y\) and \(^t\!Cy = 2x\). So \((-2 + M)p = 0\). Since \(-2 + M\) is the Gram matrix of \(\Lambda(H)\), \(p\) is then orthogonal to every vector in \(\Lambda(H)\). However, \(\Lambda(H)\) is hyperbolic, so non-degenerate.

(2) \(\alpha_\ast(p) = -p\). Then \(x = 0\) and \(y \neq 0\), so that \((p, p) = -2^t yy < 0\).

(3) \(\sigma_\ast(p) = -p\). Then \(y = 0\) and \(x \neq 0\), so that \((p, p) = -2^t xx < 0\). \(\square\)

Lemma 3.5 now finishes this shorter argument.

5. Rigidity and tightness

Recall from Cantat and Lamy [5], the crucial notions of rigidity and tightness: given \(\epsilon, B > 0\), a hyperbolic conjugacy class \(C\) in \(G = Cr_2(k)\) is \((\epsilon, B)\)-rigid if

\[
\text{diam}(\text{Tub}_\epsilon \text{Ax}(g) \cap \text{Tub}_\epsilon \text{Ax}(h)) \leq B,
\]

whenever \(g, h \in C\) and \(\text{Ax}(g) \neq \text{Ax}(h)\). If \(\epsilon' > \epsilon\) and \(C\) is \((\epsilon, B)\)-rigid then it is also \((\epsilon', B')\)-rigid for some explicit \(B' = B'(\epsilon, \epsilon', B)\) ([5], 2.3.2). So we can speak of rigidity without reference to the precise values of \(\epsilon\) and \(B\).

A hyperbolic element is rigid if its conjugacy class is rigid.

Lemma 5.1. If \(g\) is rigid then so is \(g^n\), for every \(n \neq 0\).

Proof. Taking \(n\)th powers of elements in a conjugacy class gives a new conjugacy class but does not change the set of axes. \(\square\)

A rigid hyperbolic conjugacy class \(C\) is tight if whenever \(g, h \in C\) and \(\text{Ax}(g) = \text{Ax}(h)\), then \(h = g^{\pm1}\). An element \(g\) is tight if its conjugacy class is tight.

For the rest of this section, \(g\) will denote a fixed hyperbolic element of \(Cr_2(k)\).

Suppose that, for all \(i \in \mathbb{N}\), \(\Sigma_i\) is a segment of \(\text{Ax}(g)\) of length \(i\) and that all the \(\Sigma_i\) have the same midpoint. Denote this midpoint by \(x_0\). For \(\epsilon > 0\) define

\[
V_{\Sigma_i, \epsilon} = V_{i, \epsilon} = \{f \in Cr_2(k) \mid d(x, f(x)) < \epsilon \ \forall \ x \in \Sigma_i\}.
\]

Remark. Note that, if \(g\) is not rigid, then, by Prop. 3.3 of [5], for all bounded segments \(\Sigma\) of \(\text{Ax}(g)\) and for all \(\epsilon > 0\), there exists \(f \in Cr_2(k)\) such that \(d(x, f(x)) < \epsilon\) for all \(x \in \Sigma\) while \(f\) does not preserve \(\text{Ax}(g)\). So, if \(g\) is not rigid, then, for every \(i\) and every \(\epsilon > 0\) the set \(V_{i, \epsilon}\) contains elements \(f\) of \(Cr_2(k)\) that do not preserve \(\text{Ax}(g)\). In particular, if \(g\) is not rigid then every \(V_{i, \epsilon}\) is infinite.

We now isolate the main part of the argument and formulate it as a separate result.
Proposition 5.2. Assume that \( k \) is algebraically closed and that \( V_{i,\epsilon} \) is infinite for all \( i \) and for all \( \epsilon > 0 \). There is a positive-dimensional affine algebraic group variety \( S \) over \( k \) acting biregularly and effectively on a \( k \)-rational surface \( Y \) such that, when \( S(k) \) is identified with a subgroup of \( \text{Cr}_2(k) \),

1. for all sufficiently large \( i \) and for all \( \epsilon < 1 \) \( V_{i,\epsilon} \) is a Zariski dense subset of \( S(k) \) and
2. \( g \) normalizes \( S(k) \).

Proof. Fix \( i \in I \) and \( \epsilon \) with \( 0 < \epsilon < 1 \). Put \( V_{i,\epsilon,\leq r} = \{ f \in V_{i,\epsilon} \mid \deg f \leq r \} \). Let \( \ell \in S_1 \) be the class of a line in \( \mathbb{P}^2 \) and suppose that \( f \in V_{i,\epsilon} \). Then

\[
d(\ell, f(\ell)) \leq d(\ell, x_0) + d(x_0, f(x_0)) + d(f(x_0), f(\ell)) \leq 2d(\ell, x_0) + \epsilon < 2d(\ell, x_0) + 1.
\]

That is, the degree of \( f \) is bounded independently of \( i \) and \( \epsilon \). In other words, there is an integer \( D \) such that \( V_{i,\epsilon} = V_{i,\epsilon,\leq D} \) for all \( i, \epsilon \).

Note that \( V_{i,\epsilon,\leq D} \supseteq V_{i,\epsilon} \) when \( i^+ \geq i \) and \( \epsilon^- \leq \epsilon \), so that \( Z_{i,\epsilon,\leq D} \supseteq Z_{i,\epsilon} \).

Now suppose that \( f, h \in V_{i,\epsilon,\leq D} \). Then \( fh \in V_{i,2\epsilon} \), so that \( d(\ell, fh(\ell)) \leq 2d(\ell, x_0) + 2\epsilon \). Since \( \cosh^{-1}(\mathbb{N}) \) is discrete in \( \mathbb{R} \), it follows that \( \deg(fh) \leq D \) if, as we now assume, \( 0 < \epsilon \ll 1 \). So multiplication gives a map

\[
V_{i,\epsilon,\leq D} \times V_{i,\epsilon,\leq D} \to V_{i,2\epsilon,\leq D}.
\]

According to Blanc and Furter [2], the set of Cremona transformations whose degree is exactly \( d \) is naturally the set of \( k \)-points of a reduced quasi-projective scheme \( (\text{Cr}_2)_d \), while \( (\text{Cr}_2(k))_{\leq D} = \bigcup_{d \leq D} (\text{Cr}_2)_d(k) \) has no natural structure as a scheme or algebraic space, although \( (\text{Cr}_2(k))_{\leq D} \) is naturally a noetherian topological space. Let \( Z_{i,\epsilon,\leq D} \) denote the closure of \( V_{i,\epsilon,\leq D} \) in \( (\text{Cr}_2(k))_{\leq D} \), so that multiplication defines a map

\[
m : Z_{i,\epsilon,\leq D} \times Z_{i,\epsilon,\leq D} \to Z_{i,2\epsilon,\leq D}.
\]

By the noetherian property, \( Z_{i,2\epsilon,\leq D} = Z_{i,\epsilon,\leq D} \) for \( 0 < \epsilon \ll 1 \), so that \( m \) is a map

\[
m : Z_{i,\epsilon,\leq D} \times Z_{i,\epsilon,\leq D} \to Z_{i,\epsilon,\leq D}.
\]

We can, and do, suppose that \( D \) is minimal with respect to the two properties

1. \( Z_{i,\epsilon,\leq D} \) is infinite and
2. \( m(Z_{i,\epsilon,\leq E} \times Z_{i,\epsilon,\leq E}) \) is not contained in \( Z_{i,\epsilon,\leq E} \) for any \( E \leq D - 1 \).

We can write \( Z_{i,\epsilon,\leq D} = \bigcup_{d \leq D} Z_{i,\epsilon,d} \), where the \( k \)-scheme \( Z_{i,\epsilon,d} \) is the Zariski closure of \( V_{i,\epsilon,d} \) in \( (\text{Cr}_2)_d \). Then \( Z_{i,\epsilon,d} \supseteq Z_{i,\epsilon-,d} \). \( \Box \)

Lemma 5.3. \( m(Z_{i,\epsilon,D}(k) \times Z_{i,\epsilon,D}(k)) \) meets \( Z_{i,\epsilon,D}(k) \) non-trivially.

Proof. Assume otherwise, so that \( m(Z_{i,\epsilon,D}(k) \times Z_{i,\epsilon,D}(k)) \) lies in \( Z_{i,\epsilon,\leq D-1} \). We use the notation of [2]: \( W_d \) is the projectivized space of triples of homogeneous degree \( d \) polynomials in three indeterminates, \( H_d \) is the locally closed subscheme of \( W_d \) consisting of triples that define a Cremona transformation of degree at most \( d \), and, for
any $e \leq d$, $H_{e,d}$ is the locally closed subscheme of $H_d$ of triples that define a Cremona transformation of degree exactly $e$. There are surjections $\pi_{e,d} : H_{e,d} \to (\text{Cr}_2)_{e}$ and $\pi_d : H_d(k) \to (\text{Cr}_2(k))_{\leq d}$. The morphism $\pi_{d,d}$ is an isomorphism.

Set $Z'_{i,e,D} = \pi_{D,D}^{-1}(Z_{i,e,D})$, so that there is a commutative diagram

$$
\begin{array}{ccc}
Z'_{i,e,D}(k) & \subseteq & H_{D,D}(k) \subseteq \text{open} \subseteq H_D(k) \\
\downarrow \cong & \downarrow \cong & \downarrow \pi_D \\
Z_{i,e,D}(k) & \subseteq & (\text{Cr}_2)_D(k) \subseteq (\text{Cr}_2(k))_{\leq D}
\end{array}
$$

From our assumption, there exists $E \leq D - 1$ and a non-trivial open subscheme $U$ of $Z_{i,e,D} \times Z_{i,e,D}$ such that $m(U(k)) \subset Z_{i,e,E}(k)$. Write $U' = \pi_{D,D}^{-1}(U)$ and let $m_D : H_D \to H_D$ denote the multiplication. Then $m_D(U') \subset H_E \times D$.

Let $Z'_{i,e,D}$ be the closure of $Z'_{i,e,D}$ in $H_D$. Since $Z'_{i,e,D}$ maps onto $Z_{i,e,D}$ under $\pi_D$, it follows that $m(Z_{i,e,D} \times Z_{i,e,D}) \subset Z_{i,e,E}$. In particular, $m(Z_{i,e,D} \times Z_{i,e,D}) \subset Z_{i,e,E}$, so that, by the minimality assumption, $Z_{i,e,E}$ is finite. But then $m(Z_{i,e,D} \times Z_{i,e,D})$ is finite, which contradicts the fact that $Z_{i,e,D}$ is infinite. This completes the proof of Lemma 5.3.

Since $Z_{i,e,D} \supset Z_{i+e,-D}$ and $Z'_{i,e,D} = \pi_{D,D}^{-1}(Z_{i,e,D})$, it follows that $Z'_{i,e,D} \supset Z'_{i+e,-D}$. By the noetherian property it follows that $Z'_{i,e,D} = Z'_{i+e,-D}$ for all $i \geq 0$ and all $e \ll 1$.

Now define $W$ as the closure of $Z'_{i,e,D} \cup \{1\}$ in $H_D$. By what we have proved so far, $m$ then defines a rational map $\overline{m} : W \times W \to W$. Since $W$ has an identity element and is preserved under taking inverses and since multiplication is associative in $\text{Cr}_2$, $\overline{m}$ is a group chunk. By the theorem of Weil and Rosenlicht, there is then a rational surface $Y$ over $k$, a positive-dimensional group variety $S$ over $k$ that is $k$-birational to $W$ and an embedding of $S$ into the group scheme $\text{Aut}_Y$. Since $Y$ is rational, the group variety $S$ has no abelian part, and so is a linear algebraic group. There is a Zariski open subvariety of $S$ that is identified with a Zariski open subvariety of $Z'_{i,e,D}$, and now (1) of Proposition 5.2 is proved.

For (2), note first that, if $j \geq i$, then $V_{i,e} \supset V_{j,e}$ and $V_{i,e}$ is Zariski dense in $S(k)$. Now suppose that $j \geq i + 2L(g)$. Let $f \in V_{j,e}$; then $f \in V_{i,e}$ and then $gfg^{-1} \in V_{i,e}$, since $L(g) \leq (j - i)/2$. Therefore, there is a Zariski dense subset $T$ of $S(k)$ such that $gTg^{-1} \subset S(k)$. It follows that $g$ normalizes the subgroup $S(k)$ of $\text{Cr}_2(k)$.

This concludes the proof of Proposition 5.2. \hfill \Box

**Lemma 5.4.** Assume that $k$ is algebraically closed and that every $V_{i,e}$ is infinite.

1. Every positive-dimensional algebraic subgroup $B$ of $S$ that is normalized by $g$ has an open orbit in $Y$.

2. Every positive-dimensional algebraic subgroup of $S$ has an open orbit in $Y$. 

\textbf{Proof.} (1) Suppose that \( B \) has no open orbit in \( Y \). Then its orbits form a pencil of curves. This pencil is then preserved by \( g \), which is impossible since \( g \) is hyperbolic.

(2) follows at once. \( \square \)

Note that, if we regard \( g \) as a birational map \( Y \to Y \), then \( S \) preserves the base loci \( Bs(g^{\pm 1}) \) of \( g^{\pm 1} \).

Recall that a connected linear algebraic group \( G \) over an algebraically closed field is \textit{reductive} if its unipotent radical is trivial and \textit{semi-simple} if also every homomorphism \( G \to G \) is constant.

\textbf{Lemma 5.5.} Assume that \( k \) is algebraically closed and that every \( V_{i, \epsilon} \) is infinite. Then the linear algebraic group \( S \) is not semi-simple.

\textbf{Proof.} Assume that \( S \) is semi-simple. There is an \( S \)-equivariant desingularization \( \tilde{Y} \to Y \). Then there is an \( S \)-equivariant blowing-down \( \tilde{Y} \to Y' \) where \( Y' \) is a minimal rational surface. That is, \( Y' \) is either \( \mathbb{P}^2 \) or a Hirzebruch surface \( \Sigma_n \) (a \( \mathbb{P}^1 \)-bundle \( \Sigma_n \to \mathbb{P}^1 \) where \( n \geq 0 \), \( n \neq 1 \) and, if \( n \geq 2 \), then \( \Sigma_n \) has a unique negative section \( \sigma \) with \( \sigma^2 = -n \).

We can now assume that \( Y' = Y \). Note also that \( g \) normalizes the connected component \( S^0 \) of \( S \).

The connected semi-simple part \( G^{ss} \) of the automorphism group of \( \Sigma_n \) is isogenous to \( SL_2 \) if \( n \geq 2 \) and is \( PGL_2 \times PGL_2 \) if \( n = 0 \), while if \( Y = \mathbb{P}^2 \) then \( G^{ss} = PGL_3 \).

Recall that \( \Sigma_n \) has either one or two rulings (that is, structures as a \( \mathbb{P}^1 \)-bundle over \( \mathbb{P}^1 \)) according as \( n \geq 2 \) or \( n = 0 \).

If \( Y = \Sigma_n \) and \( n \geq 2 \) then \( S^0 = G^{ss} \), so that \( Y \) has only two \( S^0 \)-orbits, namely, \( \sigma \) and \( Y - \sigma \). It follows that \( Bs(g) \) is empty, which contradicts the fact that \( g \) is hyperbolic.

If \( Y = \Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1 \) then \( S^0 \) preserves at least one fibre in each ruling (the fibre through a point of \( Bs(g) \)). This is impossible since \( S^0 \) is semi-simple.

If \( Y = \mathbb{P}^2 \) then \( S^0 \) preserves the pencil of lines through a point \( P \) of \( Bs(g) \). Since \( S^0 \) is semi-simple it acts transitively on the copy of \( \mathbb{P}^1 \) that parametrizes this pencil. Let \( \pi : X \to \mathbb{P}^2 \) be the blow-up at \( P \), with exceptional curve \( E \). Then \( S \) acts on \( X \), and acts transitively on \( E \), so that \( g \) has no base points on \( E \), and so none on \( X \). That is, \( g \) is biregular on \( X \), which contradicts the fact that \( g \) is hyperbolic. \( \square \)

\textbf{Lemma 5.6.} If a torus \( T \) acts effectively on a rational surface \( Y \), then \( \dim T \leq 2 \).

\textbf{Proof.} As in the proof of the previous lemma, we reduce to an action of \( T \) on \( \mathbb{P}^2 \) or \( \Sigma_n \). Then \( T \) lies in \( PGL_3 \) or \( PGL_2 \) or \( PGL_2 \times PGL_2 \) or a group isogenous to \( SL_2 \). \( \square \)

\textbf{Lemma 5.7.} Assume that \( k \) is algebraically closed and that every \( V_{i, \epsilon} \) is infinite. In \( \text{Cr}_2(k) \), the element \( g \) normalizes a 2-dimensional connected commutative group \( A \) which is isomorphic to either \( \mathbb{G}_a^2 \) or \( \mathbb{G}_m^2 \).

\textbf{Proof.} Suppose first that \( S \) is not reductive. Then take \( A \) to be the connected component of the centre of the unipotent radical of \( S^0 \); since \( A \) acts effectively on \( Y \) and has an open orbit on \( Y \), it follows that \( \dim A \geq 2 \).
Suppose that \( \dim A \geq 3 \). Let \( y \in Y \) be a point in the open orbit. The connected component \( \text{Stab}_y^0 \) of its stabilizer is normal, since \( A \) is commutative, and so fixes every point \( a(y) \) for \( a \in A \). Then \( \text{Stab}_y^0 \) fixes every point in a dense open subset of \( Y \), while \( A \) acts effectively. This contradiction shows that \( A \) is a 2-dimensional unipotent commutative algebraic group.

If \( A \) is not isomorphic to \( G_2 \), then \( \text{char } k = p > 0 \) and \( A \) has a non-trivial subgroup \( B \) that is characteristic in \( S \); for example, take \( B \) to consist of the points in \( A \) that are of order \( p \). However, \( B \) then also acts on \( Y \) with a dense orbit, so that \( \dim B \geq 2 \), and it follows that \( A \cong \mathbb{G}_a^2 \).

If \( S \) is reductive, then take \( A \) to be the soluble radical of \( S \). This is a torus, and the previous argument applies. \( \square \)

**Theorem 5.8.** \( g \) is rigid.

**Proof.** Suppose not, and assume first that \( k \) is algebraically closed. So every \( V_{i,\epsilon} \) is infinite. Note that each set \( V_{i,\epsilon} \) contains a subset \( V_{i,\epsilon,0} \) that is Zariski dense in \( S(k) \), where \( S \) and \( Y \) are the objects provided by Proposition 5.2.

Regard \( Y \) as the rational surface on which \( \text{Cr}_2(k) \) acts. In the inverse system of all blow-ups \( X \to Y \), there is an inverse subsystem of \( S \)-equivariant blow-ups; the normalizer \( N \) of \( S(k) \) in \( \text{Cr}_2(k) \) acts on this subsystem. Note that \( g \) lies in \( N \).

Taking the direct limit of the Néron–Severi groups of the surfaces in this system and then constructing infinite-dimensional hyperbolic spaces gives a closed hyperbolic subspace \( \mathfrak{H}_N \) of \( \mathfrak{H} \) that is preserved by \( N \). Now \( g \) acts hyperbolically on \( \mathfrak{H}_N \) and therefore acts hyperbolically on every \( g \)-invariant closed hyperbolic subspace of \( \mathfrak{H} \). Therefore, \( \text{Ax}(g) \subset \mathfrak{H}_N \). However, \( S(k) \) acts trivially on the Néron–Severi group of every \( S \)-surface in the subsystem above, so acts trivially on \( \mathfrak{H}_N \) and so on \( \text{Ax}(g) \). Then \( S(k) \), and so \( V_{i,\epsilon} \), preserves \( \text{Ax}(g) \), which, as in the Remark preceding Proposition 5.2, contradicts our assumption.

So we have proved that \( g \) is rigid when \( k \) is algebraically closed.

Now suppose that \( k \) is an arbitrary field and that \( g \) is not rigid. Fix an algebraic closure \( K \) of \( k \). Then the hyperbolic space \( \mathfrak{H}_K \) is, from its definition, a closed geodesic subspace of \( \mathfrak{H}_K \) that is preserved by \( g \). By assumption, \( g \) acts hyperbolically on \( \mathfrak{H}_K \), so preserves \( \text{Ax}(g) \), which is a geodesic \( \gamma \) in \( \mathfrak{H}_K \) and so a geodesic in \( \mathfrak{H}_K \).

Suppose now that \( g \) is not hyperbolic on \( \mathfrak{H}_K \); then \( g \) is not parabolic, since it preserves \( \gamma \), so is elliptic. Suppose that \( P \in \mathfrak{H}_K \) is a fixed point; then \( g \) preserves the hyperbolic plane \( \Pi \) spanned by \( \gamma \) and \( P \). However, no non-trivial isometry of \( \Pi \) can both fix a point and preserve a geodesic. So \( g \) is hyperbolic on \( \mathfrak{H}_K \) and preserves \( \gamma \). It follows that \( \gamma \) is the axis of \( g \) when \( g \) is regarded as an isometry of \( \mathfrak{H}_K \).

Pick \( \epsilon, B > 0 \) such that \( g \) is \((\epsilon, B)\)-rigid as an element of \( \text{Cr}_2(K) \). Suppose that \( f, h \) lie in the conjugacy class of \( g \) in \( \text{Cr}_2(k) \). Then

\[
\text{diam} \left( \text{Tub}_\epsilon \text{Ax}(f) \cap \text{Tub}_\epsilon \text{Ax}(h) \right) < B,
\]

where the tubular neighbourhoods are taken in \( \mathfrak{H}_K \), and then the same inequality holds in the subspace \( \mathfrak{H}_k \). That is, \( g \) is rigid. \( \square \)
Theorem 5.9. If \( k \) is algebraically closed and no power of \( g \) is tight then \( g \) normalizes a copy of either \( \mathbb{G}_m^2 \) or \( \mathbb{G}_a^2 \).

Proof. We know that \( g \) is rigid. Fix an orientation of \( \text{Ax}(g) \) and consider the subgroup
\[
N = \{ f \in \text{Cr}_2(k) | f(\text{Ax}(g)) = \text{Ax}(g) \}
\]
of \( \text{Cr}_2(k) \) and its subgroup \( N_+ \) of index at most 2 consisting of transformations that preserve the orientation. Then there is a group homomorphism \( \pi : N_+ \to (\mathbb{R}, +) \), where \( h \in N_+ \) acts on \( \text{Ax}(g) \) as a shift by \( \pi(h) \). Note that an element \( h \) of \( N_+ \) is hyperbolic if and only if \( \pi(h) \neq 0 \), and then \( |\pi(h)| = L(h) \). Set \( H = \ker \pi \), so that \( H = \{ f \in \text{Cr}_2(k) | f(x) = x \ \forall \ x \in \text{Ax}(g) \} \).

Since [1] the spectrum of \( \text{Cr}_2(k) \) (the set of lengths of its hyperbolic elements) is bounded away from zero, the image \( \text{im}\pi \) of \( \pi \) is discrete, so infinite cyclic. Choose \( g_0 \in N_+ \) such that \( \pi(g_0) \) generates \( \text{im}\pi \), so that \( N_+ \) is a semi-direct product \( N_+ = H \rtimes \langle g_0 \rangle \).

If \( H \) is not finite then every \( V_{i,t} \) is infinite, so that, by Proposition 5.2 and results following, there is a \( k \)-group variety \( S \) normalized by \( g \) such that \( S(k) \subset H \) and \( S \) contains a subgroup isomorphic to either \( \mathbb{G}_m^2 \) or \( \mathbb{G}_a^2 \) that is also normalized by \( g \). So we may, and do, assume that \( H \) is finite.

We can write \( g = hg_0^m \) for some \( h \in H \), \( m \in \mathbb{Z} \). Consider the conjugation action of \( g_0 \) on the finite group \( H \); then \( g_0 \) centralizes \( H \) for some \( s > 0 \). That is, \( g_0^s \) lies in the centre \( Z(N_+) \). Also, \( g^t = c.g_0^{ms} \) for some \( c \in H \), so that \( g^{st} = c^tg_0^{mst} \) for all \( t \); choosing \( t \) to be divisible by the order of \( H \) leads to \( g^{st} \in Z(N_+) \). Then \( fg^{st}f^{-1} = g^{st} \) whenever \( f \in N_+ \).

If \( f \in N \setminus N_+ \), put \( g_1 = fgf^{-1} \). Then \( g_1 = h_1g_0^m \) and \( g_1^t = c_1g_0^{-m} \) for some \( h_1, c_1 \in H \). It follows that \( g_1^{st} = c_1^tg_0^{-nst} \), choosing \( t \) as before gives \( g_1^{st} = g_0^{-nst} = g^{-st} \) and \( fg^{st}f^{-1} = g^{-st} \), so that \( g^{st} \) is tight.

Proposition 5.10. Suppose that \( k \) is algebraically closed and that \( g \) normalizes a copy \( A \) of \( \mathbb{G}_a^2 \) in \( \text{Cr}_2(k) \). Then \( A \) acts biregularly on \( \mathbb{A}^2 \) via the standard additive action.

Proof. As before, we can find a smooth minimal projective surface \( Y \) on which \( A \) acts effectively and biregularly and \( Y \) is either \( \mathbb{P}^2 \) or \( \Sigma_n \) with \( n \neq 1 \).

It remains to show that, whether \( Y = \Sigma_n \) or \( \mathbb{P}^2 \), there is an open subvariety, homogeneous under \( A \), which is isomorphic to \( \mathbb{A}^2 \) on which \( A \) acts additively.

If \( Y = \mathbb{P}^2 \) then \( A \subset PGL_3 \); conjugating \( A \) into the subgroup represented by strictly upper triangular matrices shows that \( A \) acts additively on \( \mathbb{A}^2 \).

If \( Y = \Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1 \) then conjugating \( A \) into a suitable subgroup of \( PGL_2 \times PGL_2 \) shows the same thing.

Recall that, if \( n \geq 1 \), then the automorphism group of \( \Sigma_n \) is isogenous to a semi-direct product \( L_n \rtimes GL_2 \) where \( L_n = H^0(\mathbb{P}^1, O(n)) \) and \( A \) is conjugate to a subgroup of \( L_n \rtimes U \) where \( U \subset GL_2 \) is the group of strictly upper triangular matrices. In terms of suitable inhomogeneous co-ordinates \((x, y)\) on \( \Sigma_n \), \( L_n \) is the space of polynomials in \( x \) of degree \( \leq n \), and the element \((v, 0)\) of \( L_n \times \{0\} \) acts via \((v, 0)(x, y) = (x, y + v(x))\) while the element \((0, \alpha)\) of \( \{0\} \times U \cong \mathbb{G}_a \) acts via \((0, \alpha)(x, y) = (x + \alpha, y)\). Since \( A \) has a dense orbit in \( Y \), \( A \) is not contained in \( L_n \). It is clear that \((v, 0)\) and \((0, \alpha)\) commute for all \( \alpha \in \mathbb{G}_a \) if and only if \( v \) is a constant polynomial, and the result is proved.
Proposition 5.11. Suppose that $k = \mathbb{F}_p$ and that $g$ normalizes a copy $A$ of $\mathbb{G}^2_a$ in $\text{Cr}_2(k)$. Then $L(g)$ is an integral multiple of $\log p$.

Proof. We know that $A$ acts additively on $\mathbb{A}^2$. That is, $a(x) = a + x$. Since $g$ normalizes this action, $g$ is biregular on $\mathbb{A}^2$ and we can write $gag^{-1} = h(a)$ for $a \in A$. The equation $h(a)(g(x)) = g(a + x)$ can be re-written as $g(a + x) = h(a) + g(x)$. Pick an origin $0 \in \mathbb{A}^2$; this provides an isomorphism $A \to \mathbb{A}^2$.

Setting $x = 0$ gives $g(a) = h(a) + g(0)$. Set $g^n(0) = r_n$; then

$$g^n(a) = h^n(a) + r_n.$$  

Since $r_n$ is constant, it follows that $g^n$ and $h^n$ are defined by polynomials of the same degrees, so that $\lambda(g) = \lambda(h)$. (Recall that $L(g) = \log \lambda(g)$ and $\lambda(g) = \lim_{n \to \infty}(\deg(g^n)^{1/n})$.) Therefore, it is enough to prove the proposition for automorphisms $g$ of the group variety $\mathbb{G}^2_a$.

Choose a finite subfield $\mathbb{F} = \mathbb{F}_q = \mathbb{F}_{p^m}$ of $k$ over which $A$ and $g$ are defined. Take the non-commutative polynomial ring $\mathbb{F}[V]$, the quotient of $W(\mathbb{F})[F,V]$ (the Dieudonné ring of $\mathbb{F}$) by the ideal $(F)$. Then, after fixing an identification $A = \mathbb{G}^2_{a,F}$, every $\mathbb{F}$-endomorphism of $A$ is a $\mathbb{F}[V]$-linear endomorphism $\Phi$ of a free $\mathbb{F}[V]$-module $M$ of rank 2. Moreover, $\deg(g) = p^{\deg(\Phi)}$, where $\deg(\Phi)$ is the maximum of the degrees, with respect to $V$, of the entries of a matrix that represents $\Phi$; this is independent of any choice of basis.

Let $R$ denote the restriction of scalars $R = R_{\mathbb{F}/\mathbb{F}_p} A$ and let $r : R \to R$ be the automorphism induced by $g$. Then $\deg r^m = \deg g^m$ for all $m$ and, after fixing a $\mathbb{F}_p$-isomorphism $R \to \mathbb{G}^2_{a,F}$, we can identify $r$ with a $2n \times 2n$ matrix $\Phi$ over the commutative polynomial ring $\mathbb{F}_p[V]$.

Taking degrees of polynomials, with respect to $V$, defines a discrete valuation $\deg : \mathbb{F}_p(V) \to \mathbb{Z}$. We define a norm $v$ on $\mathbb{F}_p(V)$ by $v(f) = p^{-\deg(f)}$. Let $K = \mathbb{F}_p((V))$ be the $v$-adic completion of $\mathbb{F}_p(V)$, $\overline{K}$ an algebraic closure of $K$ and $\mathbb{C}_\overline{K}$ the $v$-completion of $\overline{K}$; then $\mathbb{C}_\overline{K}$ is algebraically closed, and $v$ extends to a norm on it.

We can regard matrices over $\mathbb{F}_p[V]$ as having entries lying in $\mathbb{C}_\overline{K}$. Then $v$ defines a norm on such matrices $\Psi, \Omega$ with the properties that $v(\Psi \Omega) \leq v(\Psi) v(\Omega)$ and $v(\Psi + \Omega) \leq \max\{v(\Psi), v(\Omega)\}$.

Then, by definition and by Gel’fand’s theorem on the spectral radius,

$$\lambda(g) = \lambda(r) = \lim_{m \to \infty} (v(\Phi^m)^{1/m}) = \sup v(\alpha),$$

where $\alpha \in \mathbb{C}_\overline{K}$ runs over the spectrum of $\Phi$. Since $\Phi$ has finite $\mathbb{F}_p[V]$-rank, this supremum is achieved by an eigenvalue $\alpha$ that is algebraic over $\mathbb{F}_p(V)$. Then $\deg(\alpha) \in \mathbb{Q}$, so that $\lambda(g) = p^{\deg(\alpha)}$ is a rational power $p^{a/b} \geq 1$ of $p$. On the other hand, by Diller and Favre [7] (see also [1, Th. 1.2]) $\lambda(g)$ is either a Salem number or a Pisot number, so is a positive integral power of $p$. \hfill $\square$

The case where $\text{char} \ k = 0$, $k$ is algebraically closed and $g$ normalizes a copy of $\mathbb{G}^2_a$ does not arise. For then $g$ would be linear, so not hyperbolic.

Proposition 5.12. If $g$ normalizes a 2-torus, then its length $L(g)$ equals its length as an isometry of the upper half-plane and so is the logarithm of a real quadratic unit.
Proof. The same argument as in the additive case above shows that it is enough to prove this for the action of \( g \) on the torus itself. This appears in [1, p. 4]. □

The next result is now immediate. As we have already observed, given any overfield \( K \) of \( k \), \( L(g) \) is the same, whether taken in \( \text{Cr}_2(k) \) or in \( \text{Cr}_2(K) \).

**Theorem 5.13.** Assume that \( \text{char } k = 0 \) or that \( k \) is algebraic.
Suppose that \( L(g) \) is not the logarithm of a quadratic unit; if \( \text{char } k = p \) suppose also that \( L(g) \) is not an integral multiple of \( \log p \).

1. Some power of \( g \) is tight.
2. For all sufficiently divisible \( n \), the normal subgroup \( \langle \langle g^n \rangle \rangle \) of \( \text{Cr}_2(k) \) is proper.

**Proof.** (1) The previous results can be applied to show tightness over \( k \). But enlarging the ground field is irrelevant, and the result follows.

(2) now follows from Theorem 2.10 of [5]. (Note the typo in loc. cit.: the phrase ‘either \( h \) is a conjugate of \( g \), or…’ should read ‘either \( h \) is a conjugate of \( g^{\pm 1} \), or…’.) □

**Corollary 5.14.** If \( g \) is a hyperbolic quadratic Cremona transformation and \( L(g) \neq \log((1 + \sqrt{5})/2) \) (if \( \text{char } k = 2 \) then assume also that \( L(g) \neq \log 2 \)) then some power of \( g \) is tight and, for all sufficiently divisible \( n \), the normal closure \( \langle \langle g^n \rangle \rangle \) does not contain \( g \) and so is a non-trivial normal subgroup of \( \text{Cr}_2(k) \).

**Proof.** It is only necessary to check that \((1 + \sqrt{5})/2\) is the unique quadratic unit between 1 and 2. □

5.1. Now suppose that the ground field \( k \) is finite

**Proposition 5.15.** For every point \( z \in \mathfrak{H} \) and every \( r > 0 \), the set of \( f \in \text{Cr}_2(k) \) such that \( d(z, f(z)) < r \) is finite. In particular, \( \text{Cr}_2(k) \) acts properly discontinuously on \( \mathfrak{H} \).

**Proof.** Fix \( z, r \) and suppose that \( d(z, f(z)) < r \). Let \( \ell \) be the class of a line in \( \mathbb{P}^2 \). Then

\[
d(\ell, f(\ell)) \leq d(\ell, z) + d(z, f(z)) + d(f(z), f(\ell)),
\]

so that \( d(\ell, f(\ell)) < 2d(\ell, z) + r \). Since \( \cosh d(\ell, f(\ell)) = (\ell, f(\ell)) = \deg(f) \), it follows that \( \deg(f) \) is bounded. Because \( k \) is finite, we are done. □

We know that, over any field, there exist hyperbolic elements \( g \) of \( \text{Cr}_2(k) \); for example, a hyperbolic element of \( SL_2(\mathbb{Z}) \) acting on \( \mathbb{G}^2_m \) will do. Fix a hyperbolic element \( g \).

Let \( M = \text{Fix}(\text{Ax}(g)) \) denote the set of elements \( f \) of \( \text{Cr}_2(k) \) that fix every point on \( \text{Ax}(g) \).

**Lemma 5.16.** \( M \) is finite.
Proof. The argument involving the triangle inequality that was used in the proof of Proposition 5.2 shows that \((t, f(t))\) is bounded independently of \(f \in M\), and now finiteness follows again.

Theorem 5.17. Some power of \(g\) is tight.

Proof. We know that all powers of \(g\) are rigid. Note that \(g\) acts on \(M\) by conjugation; choose \(n > 0\) such that the conjugation action of \(g^n\) is trivial.

Put \(N = \text{Stab}(\text{Ax}(g))\); then \(N\) is a semi-direct product \(N = M \rtimes \langle \gamma \rangle\), where \(\gamma\) is a hyperbolic element of \(N\) whose length is minimal, and \(g^n\) is a central element of \(N\). So, for every \(h \in N\), we have \(hg^n h^{-1} = g^n\), so that \(g^n\) is tight. □

Theorem 5.18. If \(k\) is finite then each sufficiently divisible power of \(g\) generates a normal subgroup of \(\text{Cr}_2(k)\) that does not contain \(g\).

Proof. Apply Theorem 2.10 from [5] once more. □

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