The minimum-error discrimination via Helstrom family of ensembles and Convex Optimization

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Abstract

Using the convex optimization method and Helstrom family of ensembles introduced in Ref. [1], we have discussed optimal ambiguous discrimination in qubit systems. We have analyzed the problem of the optimal discrimination of N known quantum states and have obtained maximum success probability and optimal measurement for N known quantum states with equiprobable prior probabilities and equidistant from center of the Bloch ball, not all of which are on the one half of the Bloch ball and all of the conjugate states are pure. An exact solution has also been given for arbitrary three known quantum states. The given examples which use our method include: 1. Diagonal N mixed states; 2. N equiprobable states and equidistant from center of the Bloch ball which their corresponding Bloch vectors are inclined at the equal angle from z axis; 3. Three mirror-symmetric states; 4. States that have been prepared with equal prior probabilities on vertices of a Platonic solid.

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1 Introduction

In quantum information, the problem of detecting information stored in the state of a quantum system is of fundamental interest. In the simplest case, two dimensional systems or
qubits can be used to store quantum information. We assume that a quantum system is prepared in a certain state that is drawn with known previous probability from a finite set of known possible states and we want to find the best possible measurement that can be used to determine the actual state of the quantum system. If the states are mutually orthogonal, then they can be distinguished perfectly. But because of the quantum interference, it is impossible to discriminate quantum states by measurement. There are two basic approaches to accomplish state discrimination. In one approach, which is called the minimum-error discrimination and is the ambiguous discrimination, measurement outcomes are not allowed to be inconclusive but there is the possibility that the state is identified incorrectly. In this case, the probability of successful discrimination is made maximum by the optimum measurement. In the other basic approach, which is called optimum unambiguous discrimination, no error occurs, but there exists a measurement outcome which gives an inconclusive result where we fail to identify the state. In this approach, it is tried to minimize the failure probability by appropriate measurements. The topic of quantum state discrimination was firmly established in the 1970s by the pioneering work of Helstrom [2], who considered a minimum error discrimination of two known quantum states and unambiguous state discrimination was originally formulated and analyzed by Ivanovic, Dieks, and Peres [3, 4, 5] in 1987. In this paper, we deal with minimum-error discrimination discrimination. We will use convex optimization [7] as a tool for reaching this aim, which has many other different applications where some of them have been seen in previous papers [6, 8, 9, 10, 11, 12, 13, 14].

In the present work, we have used convex optimization for discrimination of known quantum states with the aid of Helstrom family of ensemble idea in qubit systems. Applied optimization problem is minimization of upper bound of optimal success probability. Minimum upper bound is equal with maximum success probability because there exist POVM elements that are orthogonal to conjugate states (by definition, any conjugate state have same convex combination with its corresponding known state) [1]. In this case, the POVM elements give an optimal measurement to discriminate given states. It has been proved, at least two conjugate states can be pure. If conjugate state is pure, then its corresponding optimal measurement is orthogonal to it and if conjugate state is mixed, then its corresponding optimal measurement is zero operator. We have also shown, it is impossible for all of Lagrange multipliers associated with the inequality constraints to take on the value zero. Optimal measurement elements and optimal success probability have been obtained for discriminating: 1. N equiprobable quantum states, which distance between each of them and center of Bloch ball is equal and, not all of which lie on the one half of Bloch ball, and all of the conjugate states are pure; 2. Arbitrary three known quantum states; 3. Diagonal N mixed states; 4. N equiprobable quantum states which distance between each of them and center of Bloch ball is equal and their corresponding Bloch vectors are inclined at the equal angle from z axis.

The organization of this paper is as follows. In Sec. II, in summary we illustrate the measurement operators of constituting POVM for minimum-error discrimination of quantum states and Helstrom family of ensembles as a strategy for carrying out of discrimination. Then in Sec. III with transformation of the problem format to optimization problem we obtain the KKT conditions [7] and problem formulation, and then by using them we determine optimal measurement operators and obtain optimal success probability for N equiprobable quantum states located at equal distance from center of Bloch ball, which all of the states
are not on the one half of the Bloch ball and all of the conjugate states are on the boundary of Bloch ball and then we solve the problem exactly for two arbitrary quantum states, three arbitrary quantum states and some examples in Sec. IV which comprise: 1. Diagonal N mixed states; 2. N quantum states with equal prior probability and equidistant from center of the Bloch ball which their corresponding Bloch vectors are inclined at the equal angle from z axis; 3. The symmetrical mirror states \( |\psi_1\rangle, |\psi_2\rangle \) and \( |\psi_3\rangle \) that subject to transformation \( |0\rangle \rightarrow |0\rangle, |1\rangle \rightarrow -|1\rangle \) change into \( |\psi_2\rangle, |\psi_1\rangle \) and \( |\psi_3\rangle \), respectively; 4. Quantum states that have been prepared with equal prior probabilities on vertices of a Platonic solid. Finally, a brief conclusion and two appendices have been provided.

2 Ambiguous quantum state discrimination and Helstrom family of ensembles

We assume that a quantum system is prepared with some known prior probability, in some state chosen from a finite collection of given known conceivable states. We want to identify the actual state of the quantum system. A state discriminating measurement determines probabilistically what the actual state of the system belongs to the set of possible states. We also assume, the state space \( s \) is a convex set in a real vector space and an operator \( e_j \) on \( s \) is defined by an affine functional \( e_j(\rho_i) = Tr(\rho_i M_j^\dagger M_j) \) from \( s \) to \([0,1]\) that \( M_j \) is quantum measurement operator and \( p(j|i) = Tr(\rho_i M_j^\dagger M_j) \) is the probability to infer from the measurement that the system is in the state \( \rho_j \) if it has been prepared in a state \( \rho_i \). The state of the system after the measurement is \( \frac{M_j \rho_i M_j^\dagger}{Tr(\rho_i M_j^\dagger M_j)} \). The measurement operators satisfy the completeness equation

\[
\sum_{j=1}^{N} M_j^\dagger M_j = I. \tag{1}
\]

Defining \( \Pi_j = M_j^\dagger M_j \), then \( \Pi_j \) is a positive operator such that \[15\]

\[
\sum_{j=1}^{N} \Pi_j = I \tag{2}
\]

and \( p(j|i) = Tr(\rho_i \Pi_j) \). Thus the set of operators \( \Pi_j \) are sufficient to determine the probabilities of the different measurement outcomes. The operators \( \Pi_j \) are known as the POVM elements associated with the measurement that are suitable for minimum-error discrimination. The error probability is expressed as

\[
P_{err} = \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} p_i Tr(\rho_i \Pi_j) \tag{3}
\]

Suppose we are given a state chosen from \( \{\rho_i\}_{i=1}^{N} \) with a prior probability distribution \( \{p_i\}_{i=1}^{N} (p_i \geq 0, \sum_i p_i = 1) \). Our goal is to find an optimal measurement to maximize the success probability to discriminate the states. It is sufficient to consider an N-valued observable.
\{\Pi_i\}_{i=1}^N \text{ from which we decide the state was in } \rho_i \text{ when obtaining the output } i. \text{ The success probability is}

\[ P = 1 - P_{err} = \sum_{i=1}^N p_i \text{Tr}(\rho_i \Pi_i). \tag{4} \]

The maximal success probability \(P_{\text{opt}}\) is caused by the best operators \(\{\Pi_i\}_{i=1}^N\). In order to use the minimum-error discrimination strategy, we have to determine the particular detection operators \(\{\Pi_i\}_{i=1}^N\) that maximize the right-hand side of the equation (4) under the constraint (2). We shall use a useful family of ensembles which have been introduced in Ref. [1] and is later shown to be closely related to an optimal state discrimination strategy. A set of N-numbers \(\{\tilde{p}_i, \rho_i; 1 - \tilde{p}_i, \tau_i\}_{i=1}^N\) is called a weak Helstrom family (of ensembles) if there exist N-numbers of binary probability discriminations \(\{\tilde{p}_i, 1 - \tilde{p}_i\}_{i=1}^N\) and states \(\{\tau_i\}_{i=1}^N\) satisfying

\[ p_i \tilde{p}_i = p \leq 1 \text{ and } \tilde{p}_i \rho_i + (1 - \tilde{p}_i) \tau_i = \tilde{p}_j \rho_j + (1 - \tilde{p}_j) \tau_j \] \tag{5}

for any \(i,j=1,...,N\).

We assume that a priori probability distribution satisfies \(p_i \neq 0,1\) in order to remove trivial cases. \(p\) and \(\tau_i\) are called Helstrom ratio and conjugate state to \(\rho_i\), respectively. It has been proved that [1]

\[ P_{\text{opt}} \leq p. \tag{6} \]

An observable \(\{\Pi_i\}_{i=1}^N\) satisfies \(P_{\text{opt}} = p\) if \(\text{Tr}(\tau_i \Pi_i) = 0\) for any \(i = 1, ..., N\). In this case, the observable \(\{\Pi_i\}_{i=1}^N\) gives an optimal measurement to discriminates \(\{\rho_i\}_{i=1}^N\) and we call the family \(\{\tilde{p}_i, \rho_i; 1 - \tilde{p}_i, \tau_i\}_{i=1}^N\) Helstrom family of ensembles [1].

With

\[ \rho_i = \frac{1}{2}(I + b_i \vec{\sigma}), \quad \tau_i = \frac{1}{2}(I + c_i \vec{\sigma}) \] \tag{7}

expression (5) can be written in terms of \(c_i\) and \(b_i\) which are the corresponding Bloch vectors to \(\rho_i\) and \(\tau_i\), respectively, as

\[ \tilde{p}_i b_i + (1 - \tilde{p}_i) c_i = \tilde{p}_j b_j + (1 - \tilde{p}_j) c_j. \] \tag{8}

### 3 Problem formulation

#### 3.1 The case of N quantum states

In this paper, we have restricted ourselves to qubit systems. In future, our method will be used in qutrit systems.

We shall find optimal success probabilities and optimal measurements for discrimination of states \(\rho_i, \ i=1,...,N\) which have been prepared with prior probabilities \(p_i\). We will see that minimum Helstrom ratio equals optimal success probabilities.

Our problem is

\[ \text{to minimize } \ p, \]

subject to \( |c_i|^2 - 1 \leq 0, \quad i = 1, ..., N; \)

\[ \tilde{p}_i b_i + (1 - \tilde{p}_i) c_i - \tilde{p}_i b_i - (1 - \tilde{p}_i) c_i = 0, \quad i = 1, ..., N \]
that have been formulated as an optimization problem (see Appendix A). It follows that this problem has the Lagrangian

\[ L = p + \sum_{i=1}^{N} \lambda_i (x_i^2 + y_i^2 + z_i^2 - 1) \]

\[ + \sum_{i=1}^{N-1} \nu_{3i} (\tilde{\rho}_1 b_{ix} + (1 - \tilde{\rho}_1) x_{i1} - \tilde{\rho}_{i+1} b_{(i+1)_{x}} - (1 - \tilde{\rho}_{i+1}) x_{i+1}) \]

\[ + \sum_{i=1}^{N-1} \nu_{3i-1} (\tilde{\rho}_1 b_{iy} + (1 - \tilde{\rho}_1) y_{i1} - \tilde{\rho}_{i+1} b_{(i+1)_{y}} - (1 - \tilde{\rho}_{i+1}) y_{i+1}) \]

\[ + \sum_{i=1}^{N-1} \nu_{3i} (\tilde{\rho}_1 b_{iz} + (1 - \tilde{\rho}_1) z_{i1} - \tilde{\rho}_{i+1} b_{(i+1)_{z}} - (1 - \tilde{\rho}_{i+1}) z_{i+1}) \] (9)

where \( b_i = (b_{ix}, b_{iy}, b_{iz}) \) and \( c_i = (x_i, y_i, z_i) \).

The partial derivative of the Lagrangian with respect to \( p \) and \( x_i, y_i, z_i, (1 \leq i \leq N) \) must vanish. Thus, the KKT conditions with respect to \( \vec{\nu}_i = (\nu_{3i-5}, \nu_{3i-4}, \nu_{3i-3}), (2 \leq i \leq N) \) are

\[ |c_i|^2 - 1 \leq 0, \quad i = 1, \ldots, N; \] (10)

\[ \tilde{\rho}_1 b_1 + (1 - \tilde{\rho}_1) c_1 - \tilde{\rho}_i b_i - (1 - \tilde{\rho}_i) c_i = 0, \quad i = 2, \ldots, N; \] (11)

\[ \lambda_i \geq 0, \quad i = 1, \ldots, N; \] (12)

\[ 1 + \sum_{i=2}^{N} \vec{\nu}_i (c_1 - c_i) = 0; \] (13)

\[ 2 \lambda_1 c_1 + (1 - \tilde{\rho}_1) \sum_{i=2}^{N} \vec{\nu}_i = 0; \] (14)

\[ 2 \lambda_i c_i - (1 - \tilde{\rho}_i) \vec{\nu}_i = 0, \quad i = 2, \ldots, N; \] (15)

\[ \lambda_i (|c_i|^2 - 1) = 0, \quad i = 1, \ldots, N. \] (16)

By the relations (13), (14) and (15) we can conclude that it is impossible which \( \lambda_i = 0, \quad i = 1, \ldots, N \). Also, The KKT conditions conclude there can be at least two of \( c_i, 1 \leq i \leq N \), so \( |c_i| = 1 \).

Now our aim will be to solve KKT conditions to find optimal POVM elements. From (13), (14) and (15)

\[ \sum_{i=1}^{N} \lambda_i c_i = 0 \] (17)

and

\[ \sum_{i=1}^{N} \lambda_i |c_i|^2 \frac{1}{1 - \tilde{\rho}_i} = \frac{1}{2} \] (18)
and by calculating \( c_i \) from (8) and then substituting it into (17) we arrive at

\[
c_j = \sum_{i=1}^{N} \frac{\lambda_i (p_i b_i - p_j b_j)}{(p - p_i)^2}, \quad j = 1, ..., N. \tag{19}
\]

When \( |c_i| = 1 \), we choose its corresponding measurement operator orthogonal to \( \tau_i \). Thus, with the aid of (17) and (18) the POVM elements are found as

\[
\Pi_j = \frac{4 p \lambda_j}{p - p_j} |\chi_j\rangle\langle \chi_j|, \quad j = 1, ..., N \tag{20}
\]

where

\[
|\chi_j\rangle\langle \chi_j| = \frac{1}{2} (I - c_j \vec{\sigma}) \tag{21}
\]

and when \( |c_i| < 1 \), the state \( \tau_i \) is mixed and \( \Pi_i \) corresponding to \( c_i \) is considered zero operator (and therefore \( \lambda_i = 0 \)) in order that \( Tr(\tau_i \Pi_i) = 0 \) is satisfied for all \( i = 1, ..., N \) and condition of \( P_{opt} = p \) is provided. The terms corresponding to all states \( \rho_i \)'s that conjugate states to them are mixed states do not have contributions to the sum in the relation (11).

If states have been prepared with equal prior probabilities, then (17), (18) and (19) become

\[
\sum_{i=1}^{N} \lambda_i c_i = 0; \tag{22}
\]

\[
\sum_{i=1}^{N} \lambda_i |c_i|^2 = \frac{N p - 1}{2N p}; \tag{23}
\]

\[
c_j = \frac{D}{(N p - 1) \sum_{i=1}^{N} \lambda_i} - \frac{b_j}{N p - 1}, \quad j = 1, ..., N, \tag{24}
\]

respectively, that we have defined \( D = \sum_{i=1}^{N} \lambda_i b_i \). Therefore,

\[
|c_k|^2 - |c_j|^2 = \frac{b_k^2 - b_j^2}{(N p - 1)^2} + \frac{2D (b_j - b_k)}{(N p - 1)^2 \sum_{i=1}^{N} \lambda_i}, \quad j, k = 1, ..., N. \tag{25}
\]

As a special case we suppose, all of the states \( \rho_1, ..., \rho_N \) are not on the one half of the Bloch ball \( (N \geq 4) \) and their corresponding Bloch vectors have equal length of \( b \). We also suppose \( |c_i| = 1 \) for all of the vectors \( c_i, i = 1, ..., N \) and then we can result \( D = 0 \). Therefore,

\[
|\chi_j\rangle\langle \chi_j| = \frac{1}{2} (I + \frac{b_j \vec{\sigma}}{N p - 1}), \quad j = 1, ..., N \tag{26}
\]

and

\[
P_{opt} = p = \frac{1}{N} (1 + b) \tag{27}
\]

where we have used \( |c_i| = 1 \) for some \( i \).

In the next two subsections, we precisely work out the maximum success probability and the optimal POVM elements for ambiguously discriminating between any two-states, with prior probabilities \( p_1, p_2 \) and among any three-states, with prior probabilities \( p_1, p_2, p_3 \).
3.2 The case of two quantum states

Although the case of two qubit states is studied \[16, 17\] we find instructive to see how the known solution follows from our method. Equations (14), (15) and (18) in this case are simply

\[
2\lambda_1 c_1 + (1 - \tilde{p}_1)\vec{\nu}_2 = 0, \tag{28}
\]
\[
2\lambda_2 c_2 - (1 - \tilde{p}_2)\vec{\nu}_2 = 0 \tag{29}
\]
and

\[
(1 - \tilde{p}_1)(1 - \tilde{p}_2) - 2\lambda_1(1 - \tilde{p}_2) - 2\lambda_2(1 - \tilde{p}_1) = 0, \tag{30}
\]
respectively.

The considerations $|c_1| = 1,$ $|c_2| = 1$ and equations (28), (29) and (30) can be used to drive

\[
\lambda_1 = \frac{1 - \tilde{p}_1}{4}, \quad \lambda_2 = \frac{1 - \tilde{p}_2}{4} \tag{31}
\]
and substituting these values into equation (19) gives

\[
c_1 = \frac{p_2 b_2 - p_1 b_1}{2p - 1}, \quad c_2 = -c_1. \tag{32}
\]

Using $|c_1| = 1$ and paying attention to $p \geq p_1$ and $p \geq p_2$ we obtain

\[
P_{opt} = p = \frac{1}{2}(1 + |p_2 b_2 - p_1 b_1|). \tag{33}
\]

Note that

\[
\Pi_1 = \frac{1}{2}(I - \frac{p_2 b_2 - p_1 b_1}{2p - 1}.\vec{\sigma}), \quad \Pi_2 = \frac{1}{2}(I + \frac{p_2 b_2 - p_1 b_1}{2p - 1}.\vec{\sigma}). \tag{34}
\]

The minimum error probability $p_{err}^{\text{min}}$ is found to be

\[
p_{err}^{\text{min}} = \frac{1}{2}(1 - |p_2 b_2 - p_1 b_1|). \tag{35}
\]

It can be written as

\[
p_{err}^{\text{min}} = \frac{1}{2}(1 - Tr|p_2 \rho_2 - p_1 \rho_1|) \tag{36}
\]
which was originally found by Helstrom [16] where $\frac{1}{2}Tr|p_2 \rho_2 - p_1 \rho_1|$ is the trace distance.

It is obvious that, the minimum error probability is achieved when $\Pi_1$ and $\Pi_2$ are the projectors onto eigenstates of $p_2 \rho_2 - p_1 \rho_1$ that belong to eigenvalues $\frac{p_2 - p_1}{2} - \frac{|p_2 b_2 - p_1 b_1|}{2}$ and $\frac{p_2 - p_1}{2} + \frac{|p_2 b_2 - p_1 b_1|}{2}$, respectively [18].
3.3 The case of three quantum states

We now want to obtain the exact solution for discrimination of three arbitrary known mixed states. We place \( N=3 \) on the equations (14), (15) and (18) then we have

\[
2\lambda_1 c_1 + (1 - \tilde{p}_1)(\bar{\nu}_2 + \bar{\nu}_3) = 0,
\]

\[
2\lambda_2 c_2 - (1 - \tilde{p}_2)\bar{\nu}_2 = 0,
\]

\[
2\lambda_3 c_3 - (1 - \tilde{p}_3)\bar{\nu}_3 = 0
\]

and

\[
(1 - \tilde{p}_1)(1 - \tilde{p}_2)(1 - \tilde{p}_3) - 2\lambda_1(1 - \tilde{p}_1)(1 - \tilde{p}_3)|c_1|^2 - 2\lambda_2(1 - \tilde{p}_1)(1 - \tilde{p}_2)|c_2|^2 - 2\lambda_3(1 - \tilde{p}_1)(1 - \tilde{p}_2)|c_3|^2 = 0.
\]

If now we make the assumption \(|c_1| = 1\), \(|c_2| = 1\), \(|c_3| \neq 1\) then (39) becomes \(\bar{\nu}_3 = 0\) and (37), (38) and (40) are led back to (28), (29) and (30), respectively. In analogy to the two states case, the results are given by

\[
\lambda_1 = \frac{1 - \tilde{p}_1}{4}, \quad \lambda_2 = \frac{1 - \tilde{p}_2}{4}, \quad \lambda_3 = 0;
\]

\[
c_1 = \frac{p_2b_2 - p_1b_1}{2p - p_1 - p_2}, \quad c_2 = -c_1, \quad c_3 = \frac{p_1b_1 - p_3b_3}{p - p_3} + \frac{(p - p_1)(p_2b_2 - p_1b_1)}{(p - p_3)(2p - p_1 - p_2)};
\]

\[
P^{opt} = p = \frac{1}{2}(p_1 + p_2 + |p_2b_2 - p_1b_1|);
\]

\[
\Pi_1 = \frac{1}{2}(I - \frac{p_2b_2 - p_1b_1}{2p - p_1 - p_2}, \bar{\sigma}), \quad \Pi_2 = \frac{1}{2}(I + \frac{p_2b_2 - p_1b_1}{2p - p_1 - p_2}, \bar{\sigma}), \quad \Pi_3 = 0.
\]

The cases of \(|c_1| \neq 1\), \(|c_2| = 1\), \(|c_3| = 1\) and \(|c_1| = 1\), \(|c_2| \neq 1\), \(|c_3| = 1\) have similar results for optimal success probability and the optimal POVM elements.

Now we consider \(|c_1| = 1\), \(|c_2| = 1\), \(|c_3| = 1\). Using \([\tilde{p}_i b_i - \tilde{p}_j b_j]^2 = [(1 - \tilde{p}_j)c_j - (1 - \tilde{p}_i)c_i]^2\), we arrive at

\[
c_i, c_j = \frac{(p - p_i)^2 + (p - p_j)^2 - (p_i b_i - p_j b_j)^2}{2(p - p_i)(p - p_j)}, \quad i, j = 1, 2, 3.
\]

Substituting (15) for \(c_1, c_2\), \(c_1, c_3\) and \(c_2, c_3\) into (see Appendix B)

\[
(c_1, c_2)^2 + (c_1, c_3)^2 + (c_2, c_3)^2 = 2(c_1, c_2)(c_1, c_3)(c_2, c_3) + 1,
\]

we obtain

\[
P^{opt} = p = \frac{-M + \sqrt{M^2 - 4LN}}{2L}
\]

that

\[
L = 4(-p_1p_2 + p_1p_3 + p_2p_3 - p_3^2)I + 4(p_1p_2 - p_1p_3 + p_2p_3 - p_3^2)J + 4(p_1p_2 + p_1p_3 - p_2p_3 - p_3^2)K
\]

\[+ 2IJ + 2IK + 2JK - I^2 - J^2 - K^2,
\]

8
Examples

1. We will consider the situation in which given states are

\[ \rho_i = \frac{1}{2}(I + b_{iz}\sigma_z) = \frac{1}{2}[(1 + b_{iz})|0\rangle\langle 0| + (1 - b_{iz})|1\rangle\langle 1|], \quad i = 1, ..., N. \]  

(48)

Then relation (13) follows that the x- and y- components of vectors \( c_i, \) \( i = 1, ..., N \) are zero, and thus there are only two vectors of \( c_i, \) \( i = 1, ..., N \) which have length 1. While the last \( N - 2 \) of POVM elements can be chosen zero operator, the relation (17) leads to the requirement

\[ \frac{\lambda_1}{p - p_1} = \frac{\lambda_2}{p - p_2}. \]  

(49)

Using the z-component of the condition (8) and combining the relations (49), (18) and (20), we can obtain the following optimal results.

\[ \Pi_1 = |0\rangle\langle 0|, \quad \Pi_2 = |1\rangle\langle 1| \quad \text{if} \quad p_1b_{iz} \leq p_2b_{2z} \]  

(50)

and

\[ \Pi_1 = |1\rangle\langle 1|, \quad \Pi_2 = |0\rangle\langle 0| \quad \text{if} \quad p_1b_{iz} \geq p_2b_{2z} \]  

(51)

and

\[ P^{opt} = \frac{1}{2}(p_1 + p_2 + |p_2b_{2z} - p_1b_{z1}|), \]  

(52)

where \( b_{iz} \) and \( b_{2z} \) are components that maximize the right side of the relation (52) over all \( b_{iz}. \)

Similar to the two state case, the operators \( \Pi_1 \) and \( \Pi_2 \) are the projectors onto eigenstates of \( p_2\rho_2 - p_1\rho_1. \)

2. Let us consider \( N \) mixed states \( \rho_j, \) \( j = 1, ..., N \) with the corresponding Bloch vectors as

\[ \mathbf{b}_j = (b \sin \theta \cos \varphi_j, b \sin \theta \sin \varphi_j, b \cos \theta), \quad j = 1, ..., N \]  

(53)
with uniform prior probability distribution \((p_j = \frac{1}{N}, j = 1, \ldots, N)\). By (22) we see

\[
\sum_{j=1}^{N} \lambda_j z_j = 0,
\]

which implies that vectors \(c_1, \ldots, c_N\) are in the \(z = 0\) plane. Furthermore relation (11) concludes that \(b_j - b_i\) is parallel with \(c_i - c_j\) and

\[
p = \frac{1}{N} + \frac{|b_j - b_i|}{N|c_j - c_i|}, \quad i \neq j.
\]

Therefore \(p\) is minimum if \(|c_i| = 1, i = 1, \ldots, N\).

Equation (25) concludes, \(D\) is orthogonal to all \(b_i - b_j, i, j = 1, \ldots, N, i \neq j\). Thus

\[
D = (0, 0, b \cos \theta \sum_{i=1}^{N} \lambda_i).
\]

By applying (24), we can show

\[
c_j = \frac{b \cos \theta k - b_j}{Np - 1} = -\frac{b \sin \theta}{Np - 1} (\cos \varphi_j, \sin \varphi_j, 0)
\]

and note that \(|c_i| = 1\) and \(p \geq p_i, \quad i = 1, \ldots, N\), we can obtain

\[
p = \frac{1}{N} (1 + b \sin \theta).
\]

The corresponding *conjugate states* to \(\rho_j, j = 1, \ldots, N\) are

\[
|\phi_j\rangle = \cos \left(\frac{\pi}{4}\right) |0\rangle + \sin \left(\frac{\pi}{4}\right) e^{i(\varphi_j + \pi)} |1\rangle, \quad j = 1, \ldots, N.
\]

By substituting (58) in equation (20) we obtain

\[
\Pi_j = \frac{4\lambda_j (1 + b \sin \theta)}{b \sin \theta} |\chi_j\rangle \langle \chi_j|, \quad j = 1, \ldots, N.
\]

We must have

\[
|\chi_j\rangle = \cos \left(\frac{\pi}{4}\right) |0\rangle + \sin \left(\frac{\pi}{4}\right) e^{i\varphi_j} |1\rangle
\]

because \(\text{Tr}(\tau_j |\chi_j\rangle \langle \chi_j|) = 0\) must be satisfied for all \(j = 1, \ldots, N\). Hence \(P_{\text{opt}} = p\) and \(|\chi_j\rangle \langle \chi_j|, j = 1, \ldots, N\) are optimal measurement operator for discriminating the states \(\rho_j, j = 1, \ldots, N\).

These results accord with given results in example 4 of Ref. [1].

3. For now we will consider the example in which the input states are given by

\[
|\psi_1\rangle = \cos \theta |0\rangle + \sin \theta |1\rangle
\]

\[
|\psi_2\rangle = \cos \theta |0\rangle - \sin \theta |1\rangle
\]
\(|\psi_3\rangle = |0\rangle\), \hspace{1cm} (64)

with prior probabilities \(p_1, p_2(= p_1)\) and \(p_3 = 1 - 2p_1\). On the one hand, if \(|c_1| = 1, |c_2| = 1\) and \(|c_3| \neq 1\) then the relation (43) gives

\[ P_{\text{opt}} = p_1(1 + \sin 2\theta), \]  

on the other hand if \(|c_1| = 1, |c_2| = 1\) and \(|c_3| = 1\), we can use (45) to show

\[ c_1 \cdot c_2 = \frac{(p - p_1)^2 - p_1^2(1 - \cos 4\theta)}{(p - p_1)^2}, \]  

\[ c_1 \cdot c_3 = c_2 \cdot c_3 = \frac{p^2 - p_1 p - p + p_1(1 - 2p_1) \cos 2\theta}{(p - p_1)(p + 2p_1 - 1)} \]  

and (46) evidently leads to

\[ (c_1 \cdot c_3)^2 = \frac{1 + c_1 \cdot c_2}{2} \]  

and by substituting (66) and (67) in (68), the optimal success probability can be attained. It is

\[ P_{\text{opt}} = \frac{(1 - 2p_1)(p_1 \sin^2 \theta + 1 - 2p_1 - p_1 \cos^2 \theta)}{1 - 2p_1 - p_1 \cos^2 \theta}. \]  

Therefore, there are two regimes depending on \(p_1\) and \(\theta\) which coincide at \(p_1' = \frac{1}{2 + \cos \theta(\sin \theta + \cos \theta)}\).

If \(p_1 \geq p_1'\) (\(p_1 \leq p_1'\)), the optimal success probability is given by the relation (69) (the relation (68) \cite{19}).

In Ref. \cite{19}, the measurement strategy of minimum-error discrimination has been used for obtaining the relations (65) and (69).

4. In this example, we consider states \(\rho_i, i = 1, ..., N\) with equal prior probabilities, which form the vertices of a Platonic solid centered at the origin with length of edge \(a\). By using relations (26) and (27), we have given optimal success probability and the Bloch vectors of conjugate states.

Pyramid:

\[ c_i = -\frac{b_i}{4P_{\text{opt}} - 1}, \quad i = 1, ..., 4; \quad P_{\text{opt}} = \frac{1}{4}(1 + \sqrt{\frac{3}{8}}a) \]  

Cube:

\[ c_i = -\frac{b_i}{8P_{\text{opt}} - 1}, \quad i = 1, ..., 8; \quad P_{\text{opt}} = \frac{1}{8}(1 + \sqrt{\frac{3}{2}}a) \]  

Octahedron:

\[ c_i = -\frac{b_i}{6P_{\text{opt}} - 1}, \quad i = 1, ..., 6; \quad P_{\text{opt}} = \frac{1}{6}(1 + \sqrt{\frac{2}{2}}a) \]  

Dodecahedron:

\[ c_i = -\frac{b_i}{20P_{\text{opt}} - 1}, \quad i = 1, ..., 20; \quad P_{\text{opt}} = \frac{1}{20}(1 + \frac{1}{3}a) \]  

Icosahedron:

\[ c_i = -\frac{b_i}{12P_{\text{opt}} - 1}, \quad i = 1, ..., 12; \quad P_{\text{opt}} = \frac{1}{12}(1 + \sqrt{\frac{5 + \sqrt{5}}{2\sqrt{2}}}a) \]  

11
5 Conclusion

Using the idea of Helstrom family, there is one method for ambiguous discrimination. In this method, Helstrom ratio is considered to be the cost function, subject to resulted constraints of Helstrom family of ensembles. If KKT conditions associated with the optimization problem are satisfied, minimum Helstrom ratio will be equal with maximum success probability. At least, two conjugate states to known states can be pure. Every optimal non-zero POVM element is orthogonal to its corresponding pure conjugate state and all of the optimal POVM elements corresponding to all mixed conjugate states are zero operators. It is not possible for all of Lagrange multipliers associated with the inequality constraints to take on the value zero. Our method has been restricted in qubit systems.

Form of state space is not a sphere for qutrit systems and the located states in boundary of state space are not necessarily pure, therefore using this way for qutrit systems seems different and it will be investigated in the future.

Appendix A

A summary of convex optimization

An optimization problem [7] has the standard form:

\[ \text{minimize } f_0(x), \]
\[ \text{subject to } f_i(x) \leq 0, \quad i = 1, \ldots, m, \]
\[ h_i(x) = 0, \quad i = 1, \ldots, p, \]  

where the vector \( x = (x_1, \ldots, x_n) \) is called the optimization variable and the function \( f_0 : \mathbb{R}^n \rightarrow \mathbb{R} \) the cost function. The inequalities \( f_i(x) \leq 0 \) are called inequality constraints, and the equations \( h_i(x) = 0 \) are called the equality constraints.

The Lagrangian \( L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R} \) is

\[ L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x). \]  

We refer to \( \lambda_i \) as the Lagrange multiplier associated with the ith inequality constraint \( f_i(x) \geq 0 \); similarly we refer to \( \nu_i \) as the Lagrange multiplier associated with the ith equality constraint \( h_i(x) \). The vectors \( \lambda \) and \( \nu \) are \( (\lambda_1, \ldots, \lambda_m) \) and \( (\nu_1, \ldots, \nu_p) \), respectively.

The dual function \( g : \mathbb{R}^{m+p} \rightarrow \mathbb{R} \) is defined as the minimum value of the Lagrangian over \( x \) that is written as

\[ g(\lambda, \nu) = \inf_{x \in D} L(x, \lambda, \nu) \]  

The dual function yields lower bounds on the optimal value \( p^* \) of the problem [A-i]: for any \( \lambda \geq 0 \) and any \( \nu \) we have

\[ g(\lambda, \nu) \leq p^*. \]
A natural question is: what is the best lower bound that can be obtained from the Lagrange dual function? This leads to the optimization problem

\[
\begin{align*}
\text{maximize} & \quad g(\lambda, \nu) \\
\text{subject to} & \quad \lambda \succeq 0
\end{align*}
\]  
(A-v)

This problem is called the Lagrange dual problem associated with the problem (A-i). The problem (A-i) is sometimes called the primal problem. We refer to \((\lambda, \nu)\) as dual optimal if they are optimal for the problem (A-v).

The optimal value of the Lagrange dual problem, which we denote \(d^*\), is, by definition, the best lower bound on \(p^*\) that can be obtained from the dual function. We have

\[
d^* \leq p^*
\]  
(A-vi)

We refer to the difference \(p^* - d^*\) as the optimal duality gap of the original problem (A-i).

We now assume that functions \(f_0, \ldots, f_m, h_1, \ldots, h_p\) are differentiable. Let \(x^*\) and \((\lambda^*, \nu^*)\) be any primal and dual optimal points with zero duality gap. Then we have

\[
\begin{align*}
f_i(x^*) & \leq 0, \quad i = 1, \ldots, m \\
h_i(x^*) & = 0, \quad i = 1, \ldots, p \\
\lambda^*_i & \geq 0, \quad i = 1, \ldots, m \\
\nabla f_0(x^*) + \sum_{i=1}^m \lambda^*_i \nabla f_i(x^*) + \sum_{i=1}^p \nu^*_i \nabla h_i(x^*) & = 0 \\
\lambda^*_i f_i(x^*) & = 0, \quad i = 1, \ldots, m
\end{align*}
\]  
(A-vii)

which are called the Karush-Kuhn-Tucker (KKT) conditions. The condition \(\lambda^*_i f_i(x^*) = 0, \quad i = 1, \ldots, m\) is known as complementary slackness; it holds for any primal optimal \(x^*\) and any dual optimal \((\lambda^*, \nu^*)\) (when duality gap is zero).

The converse holds, if the primal problem is convex. In other words, if \(\tilde{x}, \tilde{\lambda}, \tilde{\nu}\) are any points that satisfy the KKT conditions and \(f_i\) are convex and \(h_i\) are affine, then \(\tilde{x}\) and \((\tilde{\lambda}, \tilde{\nu})\) are primal and dual optimal, with zero duality gap.

Appendix B

Proof of (46)

By taking dot products of \(c_1\) by \(c_1\) and \(c_2\), and then using them and equations (38) and (13), the Lagrange multipliers associated with inequality constraints can be expressed as

\[
\lambda_1 = \frac{(1 - \tilde{p}_1)[(c_1, c_2)(c_2, c_3) - (c_1, c_3)]}{2[1 + (c_1, c_2) - (c_1, c_3) - (c_2, c_3)][1 - (c_1, c_2)]},
\]  
(B-i)
\[
\lambda_2 = \frac{(1 - \tilde{p}_2)[(c_1 \cdot c_2)(c_1 \cdot c_3) - (c_2 \cdot c_3)]}{2[1 + (c_1 \cdot c_2) - (c_1 \cdot c_3) - (c_2 \cdot c_3)][1 - (c_1 \cdot c_2)]},
\]  
(B-ii)

\[
\lambda_3 = \frac{(1 - \tilde{p}_3)[1 + (c_1 \cdot c_2)]}{2[1 + (c_1 \cdot c_2) - (c_1 \cdot c_3) - (c_2 \cdot c_3)]}.
\]  
(B-iii)

Taking dot products of (37) by \(c_2\) and \(c_3\) and in the same manner, we find

\[
\lambda_1 = \frac{(1 - \tilde{p}_1)[(c_2 \cdot c_3)^2 - 1]}{2[2(c_1 \cdot c_2)(c_1 \cdot c_3)(c_2 \cdot c_3) - (c_1 \cdot c_3)(c_2 \cdot c_3) - (c_1 \cdot c_2)(c_2 \cdot c_3) - (c_1 \cdot c_2)^2 - (c_1 \cdot c_3)^2 + (c_1 \cdot c_2) + (c_1 \cdot c_3)]},
\]  
(B-iv)

\[
\lambda_2 = \frac{(1 - \tilde{p}_2)[(c_1 \cdot c_2) - (c_1 \cdot c_3)(c_2 \cdot c_3)]}{2[2(c_1 \cdot c_2)(c_1 \cdot c_3)(c_2 \cdot c_3) - (c_1 \cdot c_3)(c_2 \cdot c_3) - (c_1 \cdot c_2)(c_2 \cdot c_3) - (c_1 \cdot c_2)^2 - (c_1 \cdot c_3)^2 + (c_1 \cdot c_2) + (c_1 \cdot c_3)]},
\]  
(B-v)

\[
\lambda_3 = \frac{(1 - \tilde{p}_3)[(c_1 \cdot c_3) - (c_1 \cdot c_2)(c_2 \cdot c_3)]}{2[2(c_1 \cdot c_2)(c_1 \cdot c_3)(c_2 \cdot c_3) - (c_1 \cdot c_2)(c_2 \cdot c_3) - (c_1 \cdot c_3)(c_2 \cdot c_3) - (c_1 \cdot c_2)^2 - (c_1 \cdot c_3)^2 + (c_1 \cdot c_2) + (c_1 \cdot c_3)]}.
\]  
(B-vi)

These equations lead to the equation (46).
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