Distillation of a Bound Entangled State Under the Influence of Different Interactions

Suprabhat Sinha

Department of Electronics,
West Bengal State University,
Barasat, Kolkata-700126, India
E-mail: suprabhatsinha64@gmail.com

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In this article, the dynamics of an open quantum system is investigated for one of the bound entangled states proposed by Bennett et al. under the influence of Heisenberg interaction, bi-linear bi-quadratic interaction, and Dzyaloshinskii-Moriya (DM) interaction. During the study, one qutrit of the selected two qutrit bound entangled state has interacted with an auxiliary qutrit through different interactions. It is observed that, although the auxiliary qutrit plays a significant role during the interaction, the probability amplitude of the qutrit does not affect the open quantum system. In the present work, the computable cross-norm or realignment (CCNR) criterion has been used to detect the bound entanglement of the state and the negativity has been applied to measure the free entanglement. In this three-fold study, it is found that the bi-linear bi-quadratic interaction performs better to free the bound entanglement.

I. INTRODUCTION

Quantum entanglement is one of the cornerstones, which plays a major role in developing the applications of quantum computation and quantum information theory [1–3]. In most of the applications of quantum computation and quantum information processing, maximally entangled quantum states are preferred for perfect execution. The quantum entangled states can be divided into two types; free entangled quantum states and bound entangled quantum states [4]. Free entangled quantum states are distillable states, and pure entanglement can be turned out easily. Due to this reason, these states can be effectively utilized for quantum computation and information processing. A wide range of applications of the free entangled quantum states have been investigated in quantum teleportation [5–7], quantum cryptography [8, 9], quantum sensing [10], quantum games [11] and so many domains. On the other side, bound entangled quantum states are noisy entangled quantum states. This type of quantum states are very hard to distill and extract pure entanglement. For this reason, bound entangled quantum states are generally avoided to use in quantum computation and information processing. In some of the recent studies, it is found that the bound entangled quantum states may be useful for secure quantum key distribution [12], quantum data hiding [13], remote quantum information concentration [14], communication complexity reduction [15], channel discrimination [16] and so on. It has also indicated that the application of any bound entangled quantum states along with some free entangled quantum states can increase the teleportation fidelity and power of the free entangled quantum states [17, 18]. After the indication of the discussed applications of the bound entangled quantum states, a new research area is unfold in front of the scientific community, and some studies have been already done on the different bound entangled states.

The term ‘Bound Entanglement’ was first introduced by Horodecki et al. and the primary bound entangled quantum state is also proposed by them [19]. After that many people like Bennett et al. [20], Jurkowski et al. [21] have contributed to construct different bound entangled quantum states. Recently, experimental formation and distillation of the bound entanglement have been realized [22–25]. From the application point of view, the time evolution dynamics and the distillation of the bound entangled quantum states are as important as the construction. Different authors propose different methods for the dynamical analysis and distillation of different bound entangled quantum states. Guo-Qiang et al. shows the time evolution and distillation of a Horodecki et al. provided bound entangled quantum state under bi-linear bi-quadratic interaction [26]. Baghbanzadeh et al. presents the distillation of two bound entangled quantum states proposed by Horodecki et al., and one of the Bennett et al. provided bound entangled quantum state under bi-linear interaction [27]. Sharma et al. also studied one of the Horodecki et al. proposed bound entangled quantum states. But they use Dzyaloshinskii-Moriya (DM) interaction to show time evolution and distillation [28]. Very recently, they have made their work one step forward and repeat their study for Jurkowski et al. provided bound entangled quantum state [29]. During these studies, different tools are used to detect and measure entanglement. At present, there are several mathematical tools available for characterization,
detection, and measurement of the free entanglement for bipartite quantum systems [30]. Negativity is one of the measures among them, which is also a good tool for the quantification of the free entanglement [31]. On the other hand, the characterization and detection of the bound entanglement is still an open problem. Although some criteria have been already developed to detect the bound entanglement, such as separability criterion, realignment criterion, computable cross-norm or realignment (CCNR) criterion [32–36] etc.

In the current study, the time evolution dynamical analysis and the distillation of a $3 \times 3$ dimensional bipartite bound entangled quantum state provided by Bennett et al., is investigated. The analysis is explored with the help of an auxiliary qutrit under three different physical interactions; Heisenberg interaction [37–39], bi-linear bi-quadratic interaction [40, 41] and DM interaction [42–44]. The selected interactions have already shown their efficiencies in the field of quantum computation and information [45–56]. According to the best of my knowledge, this type of study under the chosen interactions, with the considered bound entangled state, is missing in the literature. During the study CCNR criterion has been selected to detect the bound entanglement of the system, and negativity has been applied for quantifying and measuring the free entanglement of the system.

The study is sketched into five sections. The discussion about the selected bound entangled state, negativity, and CCNR criterion is done in section 2. Section 3 deals with the Hamiltonian and the unitary dynamics of the open quantum system. In section 4, the dynamical outcome for different interactions is displayed and analyzed the outcomes. In the last section, the conclusion of the present study is drawn.

II. BOUND ENTANGLED STATE, NEGATIVITY AND CCNR CRITERION

In the current section, the discussion about the selected bound entangled state, negativity and CCNR criterion are demonstrated. The chosen state is a $3 \times 3$ dimensional bipartite bound entangled state formed by two qutrits which is proposed by Bennett et al. [20] It is assumed that the state is prepared by qutrit A and qutrit B. The density matrix of the state can be written as,

$$\rho_{AB} = \frac{1}{4} \left( (I \times I) - \sum_{i=0}^{4} |\psi_i\rangle \langle \psi_i| \right).$$  \hspace{1cm} (1)

Where, $I$ is the $3 \times 3$ dimensional identity matrix,

$$|\psi_0\rangle = \frac{1}{\sqrt{2}} |0\rangle (|0\rangle - |1\rangle), \quad |\psi_1\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)|2\rangle, \quad |\psi_2\rangle = \frac{1}{\sqrt{2}} |2\rangle (|1\rangle - |2\rangle),$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}} (|1\rangle - |2\rangle)|0\rangle \quad \text{and} \quad |\psi_4\rangle = \frac{1}{3}(|0\rangle + |1\rangle + |2\rangle)(|0\rangle + |1\rangle + |2\rangle).$$

Negativity and CCNR criterion are predominantly used to detect and quantify the entanglement in our current work. The negativity has been used to quantify the free entanglement while the CCNR criterion has been used to detect the bound entanglement of the system. CCNR is a very simple and strong criterion for separability of density matrix. This criterion can detect a wide range of bound entangled states where the other criterion fails. It is also one of the necessary condition for arbitrary dimensional systems. CCNR criterion can be evaluated in two different forms either by cross norms or by realignment of density matrices. The negativity ($N$) and CCNR criterion are defined as below,

$$N = \frac{(\|\rho^T_{AB}\| - 1)}{2} \hspace{1cm} (2)$$

and

$$\text{CCNR} = \left\| (\rho_{AB} - \rho_A \otimes \rho_B)^R \right\| - \sqrt{(1 - \text{Tr} \rho_A^2)(1 - \text{Tr} \rho_B^2)}. \hspace{1cm} (3)$$

Where $\|..\|$, $(..)^T$ and $(..)^R$ represent the trace norm, partial transpose and realignment matrix respectively. Further $\rho_A$, $\rho_B$ and $\rho_{AB}$ are the reduced density matrices of qutrit A, qutrit B and bound entangled state AB respectively,
and expressed as,

$$\rho_A = Tr_{BC}(\rho_{ABC}) \quad \rho_B = Tr_{AC}(\rho_{ABC}) \quad \rho_{AB} = Tr_C(\rho_{ABC}).$$

For a system, $N > 0$ or $CCNR > 0$ implies that the state is entangled, $N = 0$ and $CCNR > 0$ implies that the state is bound entangled, and $N > 0$ corresponds to the free entangled state.

### III. HAMILTONIAN AND UNITARY DYNAMICS

In this section, the interaction between the closed system and auxiliary qutrit, the unitary dynamics of the system, and the Hamiltonians of the different interactions are discussed. During the present study, it is considered that the auxiliary qutrit (C) interacts with one of the qutrits of the pair in the closed system through one of the previously discussed interactions. The closed system is the bound entangled state provided by Bennett et al. and consists of two qutrits ($A$ and $B$) as discussed before. In the current article, it is assumed that the interaction occurs only in the $Z$-direction between qutrit $A$ and auxiliary qutrit $C$. The state vector of the additional auxiliary qutrit ($C$) can be expressed as,

$$|C\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle \quad (4)$$

with the normalization condition,

$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1. \quad (5)$$

The density matrix of the qutrit $C$ reads as,

$$\rho_C = \begin{bmatrix} |\alpha|^2 & \alpha\beta & \alpha\gamma \\ \alpha\beta & |\beta|^2 & \beta\gamma \\ \alpha\gamma & \beta\gamma & |\gamma|^2 \end{bmatrix}. \quad (6)$$

Using the Eq.5 the above density matrix can be written as,

$$\rho_C = \begin{bmatrix} |\alpha|^2 & \alpha\beta & \alpha\sqrt{1 - |\alpha|^2 - |\beta|^2} \\ \alpha\beta & |\beta|^2 & \beta\sqrt{1 - |\alpha|^2 - |\beta|^2} \\ \alpha\sqrt{1 - |\alpha|^2 - |\beta|^2} & \beta\sqrt{1 - |\alpha|^2 - |\beta|^2} & 1 - |\alpha|^2 - |\beta|^2 \end{bmatrix}. \quad (7)$$

After interaction, the initial density matrix of the open system can be expressed as below,

$$\rho_{ABC}(0) = \rho_{AB} \otimes \rho_C. \quad (8)$$

Since it is assumed that in the current article the auxiliary qutrit ($C$) interacts with the qutrit $A$ of the bound entangled closed system. So, the Hamiltonian of the interacted system can be written as,

$$H = H_{AB} + H_{AC}^{int}. \quad (9)$$

Where $H_{AB}$ is the Hamiltonian of qutrit $A$ and qutrit $B$ and $H_{AC}^{int}$ is the interaction Hamiltonian of qutrit $A$ and qutrit $C$. Here it is considered that qutrit $A$ and qutrit $B$ are uncoupled, so $H_{AB}$ is zero. Now the Hamiltonian of the open quantum system becomes,

$$H = H_{AC}^{int}. \quad (10)$$

According to the postulate of quantum mechanics, the unitary time evolution of a physical system is obtained from the time-dependent Schrödinger equation given below,

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = E|\psi(t)\rangle. \quad (11)$$
Where $E$ is the real energies of the physical system. The solution of the above equation is expressed as,

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle.$$  \hfill (12)

For the application of density matrix, Eq.12 can be framed as,

$$\rho(t) = U(t) \cdot \rho(0) \cdot U(t)^\dagger.$$  \hfill (13)

Where $U(t) = e^{-iHt/\hbar}$ is the unitary matrix, known as ‘Time Evolution Operator’, includes the Hamiltonian ($H$) in exponential. To simplify the present study $\hbar$ is assumed as 1 ($\hbar = 1$) and using the Eqs.8 and 13 time evolution density matrix of the open system can be written as,

$$\rho_{ABC}(t) = U(t) \cdot \rho_{ABC}(0) \cdot U(t)^\dagger.$$  \hfill (14)

In the very next section, this time evolution density matrix is used to explain the dynamics of the open system for different interactions.

Heisenberg interaction is one of the commonly considered physical interaction in the scientific community. The entanglement dynamics of several quantum states have been explored under this interaction. The Hamiltonian of the open system under this interaction is considered to be XXX model in one dimension without any external magnetic field and following no boundary conditions. Under the above consideration, the Heisenberg Hamiltonian (Hamiltonian of Heisenberg interaction) of a system can be expressed as,

$$H_1 = -J \cdot \sum_{k=1}^{N} S_k^z \cdot S_{k+1}. \quad (15)$$

As per the previous assumption, the interaction happens in the Z-direction between qutrit $A$ and auxiliary qutrit $C$. So, under this assumption, the Heisenberg Hamiltonian of Eq.15 can be framed as,

$$H_1 = -J \cdot (S_A^z \otimes S_C^z). \quad (16)$$

Where $J$ is the coupling constant, and $S_A^z$ and $S_C^z$ are the Gell-Mann matrices of qutrit $A$ and qutrit $C$ respectively.

Bi-linear bi-quadratic interaction is the non-linear extension of Heisenberg interaction. Under the same consideration of the Heisenberg Hamiltonian, the bi-linear bi-quadratic Hamiltonian of a system can be written as,

$$H_2 = -J \cdot \sum_{k=1}^{N} \left[ (S_k^z \cdot S_{k+1}^z) + \left( S_k^z \cdot S_{k+1}^z \right)^2 \right]. \quad (17)$$

According to the interaction assumption of the study, this interaction also occurs in the Z-direction between qutrit $A$ and auxiliary qutrit $C$. So, for our study bi-linear bi-quadratic Hamiltonian of Eq.17 can be rewritten as,

$$H_2 = -J \cdot \left[ (S_A^z \otimes S_C^z) + \left( S_A^z \otimes S_C^z \right)^2 \right]. \quad (18)$$

Here $J$ is the coupling constant, and $S_A^z$ and $S_C^z$ are the Gell-Mann matrices of qutrit $A$ and qutrit $C$ respectively similar to the Heisenberg Hamiltonian. It is also noted that $(S_A^z \otimes S_C^z)^2$ is the non-linear term arrived in the bi-linear bi-quadratic Hamiltonian as an extension over Heisenberg Hamiltonian.

DM interaction is also a well-known physical interaction in scientific society for the last few decades. This interaction has a significant contribution to the determination of the total magnetic exchange between two neighboring magnetic spins. For this, it is also known as an anti-symmetric exchange. The DM interaction Hamiltonian of any system can be written as,

$$H_3 = \vec{D} \cdot (\vec{S}_1 \times \vec{S}_2). \quad (19)$$

Where $\vec{D}$ is a constant vector that represents interaction strength along the direction of the interaction and $\vec{S}_1$ and
$\vec{S}$ denote the vectors along the perpendicular of the interaction. As per the previous assumption regarding the interaction between qutrit A and auxiliary qutrit C, the DM interaction Hamiltonian of Eq. 19 can be expressed as,

$$H_3 = D \cdot (S_A^x \otimes S_C^x - S_A^y \otimes S_C^y).$$

(20)

Where $D$ is the interaction strength along Z-direction and $S_A^x$, $S_A^y$ and $S_C^x$, $S_C^y$ are the Gell-Mann matrices of qutrit $A$ and qutrit $C$ respectively.

IV. DYNAMICS OF THE OPEN SYSTEM

In this section, the dynamics of the open system is explored under the negativity and CCNR criterion using the time evolution density matrix of the system, given in Eq. 14. The open system is formed through the interaction between qutrit $A$ of the bound entangled state and auxiliary qutrit $C$, as discussed before. In the current article, three different interactions are considered between qutrits and discussed the results in three consecutive cases, which are given in the successive subsections.
Case 1: Dynamics under Heisenberg interaction

In this case, it is considered that qutrit $A$ of the bound entangled state and auxiliary qutrit $C$ interacts through Heisenberg interaction in the time interval $0 \leq t \leq 20$. The open system dynamics of the bound entangled state is examined under the Heisenberg Hamiltonian for different coupling constant ($J$) in the range $0 \leq J \leq 1$ and shown the outcomes in the figure 1. In the figure, the solid red line indicates the negativity (N), and the dotted blue line represents the CCNR criterion of the system, which will be followed throughout this article.

The figure shows that at the initial condition when there is no interaction in the system ($J = 0$), the negativity (N) of the state is zero, but the CCNR criterion exists. This result implies that at the initial condition, the state is bound entangled, and no free entanglement exists in the state. But as the Heisenberg interaction introduces in the system, the negativity of the state increases in oscillatory pattern, and free entanglement produces in the state. The CCNR criterion also follows the negativity and increases in the oscillatory pattern due to the interaction. It is also noted that this oscillatory pattern does not exhibit the exact sinusoidal behavior. The frequency of this oscillation depends on the value of the coupling constant ($J$), and as the value of $J$ increases, the frequency of the oscillation increases, which can be seen in the figure.

Case 2: Dynamics under bi-linear bi-quadratic interaction

Current case deals with the assumption that the interaction between qutrit $A$ and auxiliary qutrit $C$ occurs through a bi-linear bi-quadratic Hamiltonian in the interval $0 \leq t \leq 20$. The dynamical behavior of the open system under
FIG. 3. Plot of Negativity (N) and CCNR vs. t

bi-linear bi-quadratic interaction is displayed in the figure 2 for some selective values of coupling constant ($J$) in the range $0 \leq J \leq 1$. The figure exhibits that when there is no interaction in the system ($J = 0$), the state has zero negativity (N), and there is no free entanglement in the state. But due to the existence of the CCNR criterion bound entanglement exists in the state as similar to the previous case. When the interaction is applied in the system, it repeats the case 1 behavior, i.e. negativity and CCNR criterion increases with oscillatory manner due to the bi-linear bi-quadratic interaction and free entanglement produced in the state. But in this case, the oscillation shows exact sinusoidal behavior. Besides this, the oscillation frequency also follows the previous case and increases with the increment of the coupling constant ($J$), which is displayed in the figure.

Case 3: Dynamics under DM interaction

This case is framed by considering the DM interaction between qutrit $A$ and auxiliary qutrit $C$ in the time interval $0 \leq t \leq 20$. The dynamics of the open system under DM interaction is discussed for the different interaction strength ($D$) in the range $0 \leq D \leq 1$ and plotted the results in the figure 3. Analyzing the results it is noticed that for the starting condition ($D = 0$), the system shows a similar result with the previous two cases, i.e. there is no free entanglement in the state, but bound entanglement exist. After introducing DM interaction in the system, it is found that the oscillatory free entanglement produces in the state due to the interaction. Although the non-sinusoidal behavior of the entanglement is observed in the system, for the current case, the oscillatory behavior is more distorted than the case 1. Holding the attitude of the previous case, the frequency of this system also increases with the increment of interaction strength ($D$), which can be analyzes from the figure.
In the current article, the open system dynamics of the $3 \times 3$ dimensional bipartite bound entangled state, proposed by Bennett et al., have been studied. The study has been explored the dynamical analysis under the influence of three different physical interactions; Heisenberg interaction, bi-linear bi-quadratic interaction, and DM interaction. The impact of each interaction has been discussed in the respective cases of the interaction. After analyzing all the cases, it has been found that all the interactions produced oscillatory free entanglement in the bound entangled state with the negativity measurement. Here, it is noted that although the interaction of the auxiliary state with the closed bound entangled state has been produced free entanglement in the state, the probability amplitude of the auxiliary state has not played any role in the production of this free entanglement. In short, it can be narrated that the bound entangled state chosen in the paper is quite rigid, and free entanglement can easily be produced in the state. But in the matter of quality, one can have to be specific. The study can be continued on this bound entangled state to explore the influence of other interactions.

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