On the Cohomology of a Smash Product of Hopf Algebras

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Abstract
A five term sequence for the low degree cohomology of a smash product of (cocommutative) Hopf algebras is obtained, generalizing that of Tahara for a semi-direct product of groups

0 Introduction
Our aim is to obtain a five term sequence for the cohomology of a smash product of (cocommutative) Hopf algebras, which generalizes the Tahara sequence for a semi-direct product of groups [Ta].

If the finite group $G$ is a semi direct product of subgroups $N$ and $T$ and $A$ is an abelian group, which is also a trivial $\mathbb{Z}G$ module then Tahara’s exact sequence is

$$1 \to H^1(T, H^1(N, A)) \to \tilde{H}^2(G, A) \xrightarrow{res} H^2(N, A)^T \xrightarrow{d_2} H^2(T, H^1(N, A)) \to \tilde{H}^3(G, A).$$

In our setting $N$ and $T$ are cocommutative Hopf algebras (hence $G$ a smash product of $N$ and $T$) and $A$ is replaced by a commutative algebra which is also a trivial $G$-module. The cohomology in question is the Sweedler cohomology of an algebra over a Hopf algebra. The Tahara sequence then generalizes to

$$0 \to H^1_{\text{meas}}(T, \text{Hom}(N, A)) \xrightarrow{j} \tilde{H}^2(G, A) \xrightarrow{res} H^2(N, A)^T \xrightarrow{d_2} H^2_{\text{meas}}(T, \text{Hom}(N, A)) \xrightarrow{j} \tilde{H}^3(G, A),$$

where $H^2(N, A)^T$ is the Hopf algebra analogue for the $T$-stable part of cohomology and $H^i_{\text{meas}}$ denotes the measuring cohomology (generated by the regular maps that make $T$ measure $N$ to $A$). We can identify $H^2_{\text{meas}}(T, \text{Hom}(N, A))$ with
$H^2_m(T, M(N, A))$, the multiplication part of cohomology of a Singer pair $[M]$, and $H^1_{\text{meas}}(T, \text{Hom}(N, A))$ with a certain quotient of $H^1(T, M(N, A))$ ($M(N, A)$ is the universal measuring Hopf algebra). This sequence is a generalization of the Tahara sequence in the sense that $H^i(kH, A) \simeq H^i(H, U(A))$, where $kH$ is a group algebra and $U(A)$ denotes the abelian group of units of $A$.

1 Preliminaries

All vector spaces are over a fixed ground field $k$.

1.1 Smash Product of Hopf Algebras

Let $T$ and $N$ be cocommutative Hopf algebras and let $T$ act on $N$ via $@: T \otimes N \to N$, we shall abbreviate $@({t} \otimes {n}) = {t}(n)$, so that $N$ becomes a $T$-module bialgebra.

The smash product $H = N \rtimes T$ (also called crossed product), a special case of a bicrossed product (or bismash product) arising from a matched pair (see [Kas, IX.2]), is the tensor product coalgebra $N \otimes T$ (to avoid confusion we shall write $n#t$ for $n \otimes t$) with multiplication given by:

$$(n#t)(n'#t') = nt(n')#tt'.$$

We identify $N$ and $T$ with $N#1$ and $1#T$ respectively as Hopf subalgebras of $H$. In this manner we have the equality $n#t = nt$ for $n \in N$ and $t \in T$.

1.2 The Sweedler cohomology

The Sweedler cohomology is a cohomology theory for algebras that are modules over a given Hopf algebra [Sw1]. Let $H$ be a cocommutative Hopf algebra, and let $A$ be a commutative algebra, that is also a (right) $H$-module. Although it is more usual to consider a left action, we work with a right one in this paper, for the sake of convenience (one can make a left action out of a right one by using the antipode). If the $H$-module structure is given by a right action $\Psi: A \otimes H \to A$, $a \otimes h \mapsto a^h$, then the standard (normalized) complex for computing the Sweedler cohomology is as follows:

$$\ldots \to \text{Reg}(H^{\otimes q-1}, A) \xrightarrow{\delta^{q-1}} \text{Reg}(H^{\otimes q}, A) \to \ldots,$$

where the differential is given by $\delta^{q-1}(f) = (\varepsilon \otimes f) \ast f^{-1}(m \otimes \text{id} \otimes \ldots \otimes \text{id}) \ast \ldots \ast f^\oplus(\text{id} \otimes \ldots \otimes \text{id} \otimes m) \ast \Psi(f^\top \otimes \text{id})$. Here $\text{Reg}(H^{\otimes q}, A)$ denotes the abelian group of convolution invertible linear maps $f: H^{\otimes q} \to A$ with the property that $f(h_1 \otimes \ldots \otimes h_q) = \varepsilon(h_1)\ldots \varepsilon(h_q)1_A$ whenever some $h_i \in k$. The cocycles and coboundaries for the degree 1 and degree 2 cohomology groups are described as follows:

$$Z^1(H, A) = \{ f \in \text{Reg}(H, A) | f(xy) = f(y)\big(f(x)^{y_1}\big) \},$$

$$B^1(H, A) = \{ f \in \text{Reg}(H, A) | \exists a \in U(A), s.t. f(x) = a(a^{-1})^x \}.$$
\[ Z^2(H, A) = \{ f \in \text{Reg}(H \otimes H, A) | f(x_1 \otimes y_1)z_1 f(x_2y_2 \otimes z_2) = f(y_1 \otimes z_1) f(x \otimes y_2z_2) \}, \]
\[ B^2(H, A) = \{ f \in \text{Reg}(H \otimes H, A) | \exists t \in \text{Reg}(H, A), \text{ s.t.} \]
\[ f(x \otimes y) = t(y_1)t^{-1}(x_1y_2)t(x_2)^{y_1} \}. \]

The second Sweedler cohomology group \( H^2(H, A) = Z^2(H, A)/B^2(H, A) \) classifies equivalence classes of cleft comodule algebra extensions \( A \rightharpoonup C \rightharpoonup H \) [Sw1, Mo]. These are sequences of \( H \)-comodule algebra maps with the property that \( A = C^{\text{coH}} = \{ c \in C | \rho(c) = c \otimes 1 \} \) (here \( \rho: C \rightarrow C \otimes H \) denotes the \( H \)-comodule structure on \( C \)) and that there exists a convolution invertible \( H \)-comodule map \( \chi: H \rightarrow C \) (which is automatically a splitting of \( \pi \)). The isomorphism between extensions and cohomology is induced by \( \chi \mapsto f_\chi \), where \( f_\chi \in Z^2(H, A) \) is the Sweedler cocycle given by \( f_\chi(h \otimes h') = \chi(h_1)\chi(h'_1)\chi^{-1}(h_2h'_2) \).

### 1.3 Cohomology of a Singer pair

A Singer pair (in literature also the term matched pair is used [Ho, Si], the term Singer pair was introduced by A. Masuoka [Ma]) consists of a pair of Hopf algebras \((B, A)\) together with an action and a coaction that are compatible in some sense. For the sake of convenience we shall consider right Singer pairs, i.e. the ones given by a right action \( \mu: A \otimes B \rightarrow A \) and a right coaction \( \rho: B \rightarrow A \otimes B \).

In the special case when the coaction is trivial, a (right) Singer pair is given by a right action \( \mu: A \otimes B \rightarrow A \) which makes \( A \) into a \( B \)-module bialgebra. If the Hopf algebra \( A \) is commutative and the Hopf algebra \( B \) is cocommutative then the Singer pair is called abelian. In this paper we consider (right) abelian Singer pairs with the trivial coaction exclusively.

The cohomology of an abelian Singer pair \((B, A)\) is computed by the total complex of a certain double complex [Ho]. Cocycles \( Z^n(B, A) \) (coboundaries \( B^n(B, A) \)) are \( n \)-tuples of linear maps \( f_i^n \) \( i = 1, \ldots, n \) that satisfy cocycle (coboundary) conditions.

In order to compare measuring cohomology (to be defined in Section 2) to cohomology of Singer pairs, we need to introduce subgroups \( Z^n_i(B, A) \) of \( Z^n(B, A) \), that are spanned by \( n \)-tuples in which the \( f_j \)'s are trivial for \( j \neq i \) and subgroups \( B^n_i = Z^n_i \cap B^n \leq B^n \). These give rise to subgroups of cohomology groups \( H^n_i = Z^n_i/B^n_i \simeq (Z^n_i + B^n)/B^n \leq Z^n/B^n = H^n \). If \( n = 2 \) then \( H^2_i(B, A) = H^2_i(B, A) \) is the comultiplication part and \( H^2_2(B, A) = H^2(B, A) \) is the multiplication part of cohomology [M].

We are mainly interested in subgroups \( H^n_2(B, A) \) of \( H^n(B, A) \). If \( n \geq 2 \) they can be computed by a subcomplex of the complex in Section 1.2. This is easily seen by comparing that complex to the complex for computing cohomology of a Singer pair [Ho, Ma] (recall that the coaction \( \rho: B \otimes A \rightarrow B \) is trivial).

**Proposition 1.1** If \( n \geq 2 \) then we have canonical isomorphisms
\[ Z^n_2(B, A) \simeq \{ f \in \text{Coalg}(B, A) | \delta^n f = \varepsilon \} , \]
where \( \delta^n f : \text{Reg}(B^\otimes n, A) \rightarrow \text{Reg}(B^\otimes n+1, A) \) is the Sweedler differential (see Section 1.2), and

\[
B^n_n(B, A) \simeq \{ f : B^\otimes n \rightarrow A \mid \exists g \in \text{Coalg}(B^\otimes n-1, A), \text{ s.t. } f = \delta^{n-1}g \}.
\]

The second cohomology group of a Singer pair \((B, A, \mu, \rho)\) classifies equivalence classes of those Hopf algebra extensions that induce the same Singer pair \([Ho]\). Hopf algebra extensions are sequences of Hopf algebra maps

\[
A \xrightarrow{\iota} C \xrightarrow{\pi} B,
\]

such that

\[
A = C^{\text{co}B} = \{ c \in C \mid (\text{id} \otimes \pi)\Delta c = c \otimes 1 \} \quad \text{and there exists a } B-\text{comodule map } \chi : B \rightarrow A \text{ or equivalently } B \simeq C_A = C / \varepsilon^{-1}(0)C \text{ and there exists an } A-\text{module map } \xi : C \rightarrow A.
\]

The group \( H^2_2(B, A) \) classifies cocentral Hopf algebra extensions \( A \rightarrow C \rightarrow B \) for which there exists an \( A-\text{module coalgebra map } \xi : C \rightarrow A \).

**Remark.** We can make a left Singer pair out of the right one with the use of antipodes. The cohomology groups for such pairs would remain the same (isomorphic).

### 1.4 Universal measuring coalgebra

Here we introduce the (contravariant) universal coalgebra functor \( M(\_ , A) : \text{Alg}^{\text{op}} \rightarrow \text{Coalg} \) from the category of algebras to the category of coalgebras, that is adjoint to \( \text{Hom}(\_ , A) : \text{Coalg} \rightarrow \text{Alg}^{\text{op}} \). For more details, we refer to [Sw2, Chapter VII] and [GP].

Let \( A, B, C \) be algebras, \( C \) a coalgebra.

**Proposition 1.2 (Sw2, 7.0.1)** A map \( \psi : C \otimes B \rightarrow A \) corresponds to an algebra map \( \rho : B \rightarrow \text{Hom}(C, A) \), \( \rho(b)(c) = \psi(c \otimes b) \) if and only if

1. \( \psi(c \otimes b b') = \psi(c_1 \otimes b) \psi(c_2 \otimes b') \),
2. \( \psi(c \otimes 1_B) = \varepsilon(c)1_A \).

If the equivalent conditions from the proposition above are satisfied, we say that \((\psi, C)\) measures \( B \) to \( A \).

Given algebras \( B \) and \( A \) there is a measuring \((\theta, M(B, A))\) with the following universal property.

**Theorem 1.3 (Sw2, 7.0.4)** The universal measuring \( \theta : M(B, A) \otimes B \rightarrow A \) has the following universal property:

for any measuring \( f : C \otimes B \rightarrow A \) there exists a unique coalgebra map \( \overline{f} : C \rightarrow M(B, A) \), s.t. \( f = \theta(\overline{f} \otimes \text{id}) \).

### 2 The measuring cohomology

Let \( H = N \rtimes T \) (as in Section 1.1) and let \( T \) act on \( \text{Hom}(N, A) \) via pre-composition: \( \Psi : \text{Hom}(N, A) \otimes T \rightarrow \text{Hom}(N, A), f \otimes t \rightarrow f^t \), where \( f^t(n) = f(t(n)) \). Let \( \text{Reg}_{\text{meas}}(T^\otimes q, \text{Hom}(N, A)) \) denote the subgroup of
Reg($T^\otimes q$, Hom($N, A$)) consisting of the maps $f: T^\otimes q \to \text{Hom}(N, A)$ that make $T^\otimes q$ measure $N$ to $A$, i.e. $f(t)(nn') = \sum f(t_1)(n)f(t_2)(n')$ and $f(t)(1_N) = \varepsilon(t)1_A$ for $t \in T^\otimes q$ and $n, n' \in N$. The differential
\[ \text{Reg}(T^\otimes q^{-1}, \text{Hom}(N, A)) \to \text{Reg}(T^\otimes q, \text{Hom}(N, A)) \]
described in Section 1.2 restricts to
\[ \text{Reg}_{\text{meas}}(T^\otimes q^{-1}, \text{Hom}(N, A)) \to \text{Reg}_{\text{meas}}(T^\otimes q, \text{Hom}(N, A)) \]
thus giving rise to a sub complex of the complex given in Section 1.2. We name the cohomology it produces the "measuring cohomology" and denote it $\text{H}_{\text{meas}}(T, \text{Hom}(N, A))$. We denote the measuring cocycles and coboundaries by $Z^q_{\text{meas}}(T, \text{Hom}(N, A))$ and $B^q_{\text{meas}}(T, \text{Hom}(N, A))$, respectively. If $q = 1$ or $q = 2$ these are as follows:

$Z^1_{\text{meas}} = \{ f \in \text{Reg}_{\text{meas}}(T, \text{Hom}(N, A)) | f(ts)(n) = f(s_1)(n_1)f(t)(s_2(n_2)) \}$

$B^1_{\text{meas}} = \{ f \in \text{Reg}_{\text{meas}}(T, \text{Hom}(N, A)) | \exists g \in \text{Hom}(N, A) \text{ s.t.} f(t)(n) = g(n_1)g^{-1}(t(n_2)) \}$

$Z^2_{\text{meas}} = \{ f \in \text{Reg}_{\text{meas}}(T \otimes T, \text{Hom}(N, A)) | f(t_1 \otimes s_1)(r_1(n_1)) f(t_2 \otimes s_2)(r_2(n_2)) \}$

$B^2_{\text{meas}} = \{ f \in \text{Reg}_{\text{meas}}(T \otimes T, \text{Hom}(N, A)) | \exists g \in \text{Reg}_{\text{meas}}(T, \text{Hom}(N, A)) \text{ s.t.} f(ts)(n) = f(s_1)(n_1)f(t)(s_2(n_2)) \}$

We will often identify $f(t)(n) = f(t \otimes n)$ for $f \in \text{Hom}(T^\otimes q, \text{Hom}(N, A)) \simeq \text{Hom}(T^\otimes q \otimes N, A)$.

**Remark.** Even though the notation $\text{H}^q_{\text{meas}}(T, N, A)$ would probably be more precise (since $\text{H}^q_{\text{meas}}(T, B)$ only makes sense if $B = \text{Hom}(N, A)$ and it is also possible that $\text{Hom}(N, A) = \text{Hom}(N', A')$ for different $N, N', A, A'$), we write $\text{H}^q_{\text{meas}}(T, \text{Hom}(N, A))$ to emphasize the fact that the measuring cohomology is computed by a subcomplex of the standard complex for computing Sweedler cohomology.

## 2.1 $T = kG$

In the case $T = kG$ is a group algebra we can identify

$\text{Reg}_{\text{meas}}((kG)^{\otimes i}, \text{Hom}(N, A)) \simeq \text{Map}(G^\times i, \text{Alg}(N, A)),$

where $\text{Alg}(N, A)$ denotes the group of algebra maps from $N$ to $A$ (with convolution product) and $\text{Map}(H, B)$ denotes the abelian group of unital maps from $H$ to $B$ (with pointwise multiplication). Note that this isomorphism induces an isomorphism of complexes, i.e. preserves the differentials; hence we have:

**Theorem 2.1** There is an isomorphism

$\text{H}^i_{\text{meas}}(kG, \text{Hom}(N, A)) \simeq \text{H}^i(G, \text{Alg}(N, A)).$
If \( N = Ug \) is the universal envelope of a Lie algebra then we have a natural isomorphism \( \text{Alg}(Ug, A) \simeq \text{Lie}(g, Lie(A)) \) [CE], where \( \text{Lie}(g, h) \) denotes the group of Lie algebra maps \( g \to h \) (with pointwise addition) and \( \text{Lie}(A) \) denotes the underlying Lie algebra of the algebra \( A \), that is the Lie algebra where the Lie bracket is given by \( [x, y] = xy - yx \). In our case \( \text{Lie}(A) \) is an abelian Lie algebra (since the algebra \( A \) is commutative) and hence \( \text{Lie}(g, \text{Lie}(A)) \simeq \text{Lie}(g/[g, g], \text{Lie}(A)) \simeq \text{Vect}((g/[g, g])^+, A^+) \). Here \( g/[g, g] \) is the abelianization of the Lie algebra \( g \), \( ^* \) is the underlying vector space functor and \( \text{Vect}(V, W) \) is the abelian group of linear maps from \( V \) to \( W \) (with pointwise addition). Hence we have:

**Theorem 2.2** Let \( G \) be a finite group, and \( g \) a Lie algebra. Then

\[
H^i_{\text{meas}}(kG, \text{Hom}(Ug, A)) \simeq H^i(G, \text{Vect}((g/[g, g])^+, A^+)).
\]

If \( |G|^{-1} \in k \) then \( H^i_{\text{meas}}(kG, \text{Hom}(Ug, A)) = 0 \).

**Proof.** The first equality was already explained in the paragraph preceding the theorem. Note that if \( |G|^{-1} \in k \), then \( \text{Vect}((g/[g, g])^+, A^+) \) is uniquely \( |G| \) divisible and hence \( H^i_{\text{meas}}(kG, \text{Hom}(Ug, A)) = 0 \).

### 3 Measuring Hopf algebra

Our aim is to interpret measuring cohomology groups as certain subgroups of Singer cohomology. We start by showing that there is a way of making the universal measuring coalgebra \( M(N, A) \) into a Hopf algebra, if \( N \) is a Hopf algebra and \( A \) is commutative.

**Proposition 3.1.** Let \( N \) be a Hopf algebra, \( A \) a commutative algebra and \( \theta: M(N, A) \otimes N \to A \) the universal measuring. There is a unique algebra structure \( (M(N, A), m, \eta) \) so that \( (N, \theta) \) measures \( M(N, A) \) to \( A \), i.e. \( \theta(m(f \otimes g) \otimes n) = \theta(f \otimes n_1)\theta(g \otimes n_2) \) and \( \theta(\eta \otimes \text{id}) = \eta_A \otimes \varepsilon_N \). Moreover \( M(N, A) \) becomes a Hopf algebra with this additional structure.

**Proof.** Note that the map \( \omega: M(N, A) \otimes M(N, A) \otimes N \to A \), defined by \( \omega(f \otimes g \otimes n) = \theta(f \otimes n_1)\theta(g \otimes n_2) \) is a measuring and hence there exists the unique coalgebra map \( \omega: M(N, A) \otimes M(N, A) \to M(N, A) \), so that \( \theta(m \otimes \text{id}) = \omega \).

Similarly we define the unit \( \eta: k \to M(N, A) \) to be the unique coalgebra map s.t. \( \eta(\eta \otimes \text{id}) = \eta_A \otimes \varepsilon_N \), where \( \tau: k \otimes N \to N \) is the natural isomorphism \( (x \otimes n \mapsto xn) \).

A routine computation shows that \( \eta \) defined above is a unit for multiplication \( m \). The following paragraph proves that \( m \) is associative.

Define a measuring \( \omega_3: M(N, A) \otimes M(N, A) \otimes M(N, A) \otimes N \to A \) by the rule \( \omega_3(f \otimes g \otimes h \otimes n) = \theta(f \otimes n_1)\theta(g \otimes n_2)\theta(h \otimes n_3) \) and note that \( \theta(m(\text{id} \otimes m) \otimes \text{id}) = \omega_3 = \theta(\eta_3 \otimes \text{id}) \otimes \text{id} \) and hence by the uniqueness \( m(\text{id} \otimes m) = m(\text{id} \otimes \text{id}) \).

Since \( \eta \) and \( m \) are coalgebra maps, \( M(N, A) \) is a bialgebra. Now we conclude the proof by showing the existence of the antipode. We define
Lemma 4.2. If \( \theta(S \otimes \text{id}) = \theta(\text{id} \otimes S_N) \) (here \( co-op \) refers to the opposite coalgebra structure, i.e. \( \Delta_{co-op}(f) = f_2 \otimes f_1 \)). We claim that \( S \) is the antipode. It is sufficient to show \( \theta(S \ast \text{id} \otimes \text{id}) = \theta(\eta \varepsilon \otimes \text{id}) = \theta(\text{id} \ast S \otimes \text{id}) \). This is observed by the following computation: \( \theta(S \ast \text{id} \otimes \text{id})(f \otimes n) = \theta(S f_1) f_2 \otimes n = \omega(S(f_1) \otimes f_2 \otimes n) = \theta(S f_1 \otimes n) \theta(f_2 \otimes n_2) = \theta(f_1 \otimes S(n_1)) \theta(f_2 \otimes n_2) = \theta(f \otimes S(n_1) n_2) = \theta(f \otimes \varepsilon(n) 1_N) = \varepsilon(f \varepsilon(n) 1_{A}) = \theta(\eta \varepsilon(f) \otimes n); \) symmetrically for \( \theta(\eta \varepsilon \otimes \text{id}) = \theta(\text{id} \ast S \otimes \text{id}). \)

**Proposition 3.2.** If \( N \) is cocommutative, then \( M(N, A) \) is commutative.

**Proof.** \( \theta(m \otimes \text{id})(f \otimes g \otimes n) = \theta(f \otimes n_1 \otimes g \otimes n_2) = \theta(f \otimes n_2 \otimes g \otimes n_1) = \theta(gf \otimes n) = \theta((\sigma m) \otimes \text{id})(f \otimes g \otimes n). \)

### 4 Measuring cohomology vs. Singer cohomology

In this section we interpret \( H^n_{\text{meas}}(T, \text{Hom}(N, A)) \) as \( H^n_T(T, M(N, A)). \)

**Proposition 4.1.** If \( N \) is left \( T \)-module bialgebra via \( \mu: T \otimes N \to N \) (\( t \otimes n \to t(n) \)), then \( M(N, A) \) is a right \( T \)-module bialgebra via \( \overline{\pi}: M(N, A) \otimes T \to M(N, A) \) (\( f \otimes t \to f^t \)), which is the unique map, such that \( \theta(\overline{\pi} \otimes \text{id}) = \theta(\text{id} \otimes \mu). \)

**Proof.** By the universal property \( \overline{\pi} \) is a coalgebra map. The following computation proves that \( M(N, A) \) is also a \( T \)-module algebra, i.e. \( \overline{\pi}(m \otimes \text{id}) = m(\overline{\pi} \otimes \overline{\pi}) \sigma_{3,2}(\text{id} \otimes \text{id} \otimes \Delta): \theta((fg)^t \otimes n) = \theta(fg \otimes t(n)) = \theta(f \otimes t_1(n_1)) \theta(g \otimes t_2(n_2)) = \theta(f^t_1 \otimes n_1) \theta(g^t_2 \otimes n_2) = \theta(f^{t_1} g^{t_2} \otimes n). \)

**Remark.** Note that \( (T, M(N, A), \overline{\pi}, \rho) \), where \( \overline{\pi} \) is as above and \( \rho \) is the trivial coaction \( \rho = \eta \otimes \text{id}: T \to M(N, A) \otimes T \) is a Singer pair.

From now on assume \( T \) and \( N \) are both cocommutative. In this case we can talk about the differentials \( \text{Reg}(T^\otimes p, M(N, A)) \to \text{Reg}(T^\otimes p+1, M(N, A)) \) for computing Sweedler cohomology \( H^p(T, U(M(N, A))) \) (here \( U: \text{Hopf} \to \text{Alg} \) denotes the underlying algebra functor). The following lemma compares them to Sweedler differentials \( \delta: \text{Reg}(T^n \otimes N, A) \to \text{Reg}(T^{n+1} \otimes N, A) \) (recall that we are identifying \( \text{Hom}(T^n \otimes N, A) \simeq \text{Hom}(T^n, \text{Hom}(N, A)). \)

**Lemma 4.2.** If \( \overline{\pi}: T^\otimes n \to M(N, A) \) is the coalgebra map corresponding to the measuring \( \alpha: T^\otimes n \otimes N \to A \) then, \( \overline{\alpha} = \overline{\alpha} \) i.e. the unique coalgebra map corresponding to the measuring \( \delta \alpha: T^\otimes n+1 \otimes N \to A \) is \( \overline{\alpha}: T^\otimes n+1 \to M(N, A). \)

**Proof.** By the universal property it is sufficient to prove that \( \theta(\overline{\delta \alpha} \otimes \text{id}) = \delta \alpha \). Since \( \theta(\overline{\alpha} \ast \overline{\beta} \otimes \text{id}) = \overline{\alpha} \ast \overline{\beta} \) and also \( \overline{\alpha}^{-1} = \overline{\alpha} (\text{id}_{T^\otimes p} \otimes S) = \overline{S \alpha} = \overline{\alpha}^{-1} \) (it is a coalgebra map, since \( T \) is cocommutative), it is enough to see that \( \theta(d_{n+1} \overline{\alpha} \otimes \text{id}_N) = d_{n+1} \alpha, \) where \( d_{n+1} \overline{\alpha} = \overline{\alpha} (\text{id}_{T^\otimes n} \otimes m_{n} \otimes \text{id}_{T^\otimes n-1} \otimes \Delta_{n-1}^1): T^\otimes n+1 \to M(N, A), \) \( d_{n+1} \alpha = \alpha (\text{id}_{T^\otimes n} \otimes m_{n} \otimes \text{id}_{T^\otimes n-1} \otimes \Delta_{n-1}^1): T^\otimes n+1 \otimes N \to A. \) for \( i = 0, \ldots, n \) and \( d_{n+1} \overline{\alpha} = \overline{\alpha} (\text{id}_{T^\otimes n} \otimes m_{n} \otimes \text{id}_{T^\otimes n-1} \otimes \Delta_{n-1}^1): T^\otimes n+1 \otimes N \to A. \) for \( i = 0, \ldots, n \) and \( d_{n+1} \alpha = \alpha (\text{id}_{T^\otimes n} \otimes m_{n} \otimes \text{id}_{T^\otimes n-1} \otimes \Delta_{n-1}^1): T^\otimes n+1 \otimes N \to A. \)
id_N: T^{⊗n+1} ⊗ N → T^{⊗n} ⊗ N for i = 0, ..., n and D_{n+1} = id_N ⊗ μ and note that 
\[d_i\alpha = \alpha D_i\] and hence \[\theta(\pi \otimes id_N)D_i = d_i\alpha.\]

Observe that \[\eta_{\mathcal{A}^{⊗p} \otimes N} = \eta_{M(N,A)} \epsilon_{T^{⊗p}}\] and hence \[\alpha: T^{⊗p} \otimes N \to A|\alpha \text{ measures} \to \text{Coalg}(T^{⊗p}, M(N, A))\]
gives an isomorphism of complexes \[\text{(Reg}_{meas}(T^{⊗p}, \text{Hom}(N, A)), \delta) \to \text{(Coalg}(T^{⊗p}, M(N, A), 0).\]

**Theorem 4.3** \[H^n(T, M(N, A)) \simeq H^n_{meas}(T, \text{Hom}(N, A)) \text{ for } n \geq 2.\]

**Proof.** Apply Proposition 1.1 and the lemma above. □

**Remark.** Hence the degree two measuring cohomology characterizes those Hopf algebra extensions \[M(N, A) \to H \to T,\] for which there exists an \[M(N, A)-\text{module coalgebra map } \xi: H \to M(N, A).\] It is also easy to see that \[Z_1^1(T, M(N, A)) = Z_1^1(T, M(N, A)) = Z_{meas}^1(T, \text{Hom}(N, A))\] and that \[B_1^1(T, M(N, A)) = B_1^1(T, M(N, A)) = 0.\] Hence \[H^1_{meas}(T, \text{Hom}(N, A))\] is a quotient of \[H^1_1(T, M(N, A)) = H^1(T, M(N, A)).\]

### 5 Five term exact sequence for a smash product

The purpose of this section is to prove the following theorem by explicitly describing the maps involved.

**Theorem 5.1** Let \(H = N \rtimes T\) be a smash product of cocommutative Hopf algebras (more precisely, we are given an action \(\mu: T \otimes N \to N\), that makes \(N\) into a \(T\)-module bialgebra) and let the commutative algebra \(A\) be a trivial \(H\)-module. Then we have the following exact sequence:

\[
0 \to H^1_{meas}(T, \text{Hom}(N, A)) \xrightarrow{\delta} \tilde{H}^2(H, A) \xrightarrow{\text{res}} H^2(N, A) \xrightarrow{\text{coh}} H^3(H, A).
\]

We prove the above theorem by transporting some arguments from [Ta] into our more general setting.

First we have to define the Hopf algebra analog of the stable part of cohomology.

**Definition 5.2** Let \(N, T, \mu, A\) be as above. We say that a cohomology class \([f] \in H^i(N, A)\), where \(f \in Z^i(N, A)\), is \(T\)-stable if there exists a convolution invertible linear map \(g: T \otimes N^{⊗i-1} \to A\), such that \(f \star (f^{-1})^t = \delta' g(t \otimes \_).\) Here \(\delta'\) denotes the Sweedler differential (see Section 1.2) from \(\text{Reg}(T \otimes N^{⊗i-1}, A) \simeq \text{Reg}(N^{⊗i-1}, \text{Hom}(T, A))\) to \(\text{Reg}(T \otimes N^{⊗i}, A) \simeq \text{Reg}(N^{⊗i}, \text{Hom}(T, A)).\) The subgroup of \(H^i(N, A)\) consisting of all \(T\)-stable elements is called the \(T\)-stable part of cohomology and is denoted by \(H^i(N, A)^T\).
Note that if $T = kG$ then $H^i(N, A)^T = H^i(N, A)^G$.

The following lemma is the main tool in establishing this result. It is a generalization of the essential part of Proposition 1 from [Ta]:

**Lemma 5.3** Let the Hopf algebra $H$ be a smash product of cocommutative Hopf algebras $N$ and $T$. Furthermore assume that $H$ acts trivially on a cocommutative algebra $A$. Then every cocycle $f: H \otimes H \to A$ is cohomologous to a cocycle $f': H \otimes H \to A$, which is trivial on $N \otimes T$, i.e. $f'(n \otimes t) = \varepsilon(n)\varepsilon(t)1_A$.

**Proof** (of Lemma 5.3). Let the extension

$$A \xrightarrow{\chi} K \xrightarrow{\pi} H$$

be an $H$-comodule algebra extension with associated 2-cocycle $f$ (see Section 1.1). We shall denote the $H$-comodule structure on $K$ by $\rho: K \to K \otimes H$. We will show that $\chi$ can be “repaired” into a $\chi': H \to K$ that satisfies the equality $\chi'(nt) = \chi'(n)\chi'(t)$, for $n \in N$, $t \in T$. Then it is easy to see that the cocycle $f' = f'_{\chi': H \otimes H \to A}$ associated to $\chi'$ satisfies the desired condition.

Let $\{u_i\}_{i \in I}$ be a basis for $N$ and let $\{v_j\}_{j \in J}$ be a basis for $T$. Then $\{u_i \otimes v_j\}_{(i,j) \in I \times J}$ is a basis for $H$. Define a linear map $\chi': H \to K$ by the rule $\chi'(u_i \otimes v_j) = m_K(\chi(u_i) \otimes \chi(v_j))$. The following calculation shows that $\chi'$ is an $H$-comodule map, i.e. $\rho \chi' = (\chi' \otimes 1)\Delta_H$:

$$(\chi' \otimes 1)\Delta_H(u_i \otimes v_j) = \sum \chi'((u_i)_1 \otimes (v_j)_1) \otimes (u_i)_2 \otimes (v_j)_2$$

$$= \sum \chi(((u_i)_1) \otimes (v_j)_1) \otimes (u_i)_2 \otimes (v_j)_2$$

$$= \sum m_{K \otimes H}(\chi((u_i)_1) \otimes (u_i)_2 \otimes (v_j)_1 \otimes (v_j)_2)$$

$$= m_{K \otimes H}(\rho \chi(u_i) \otimes \rho \chi(v_j)) = \rho m_K(\chi(u_i) \otimes \chi(v_j))$$

$$= \rho \chi'(u_i \otimes v_j).$$

Now observe $\chi'(u_i) = \chi(u_i)$ and $\chi'(v_j) = \chi(v_j)$ and hence $\chi'(nt) = \chi'(n \otimes t) = \chi'(n)\chi'(t)$ for $n \in N$ and $t \in T$. \[\Box\]

A cocycle $f'$ that satisfies the condition of Lemma 5.3 will be called a normalized cocycle.

**Corollary 5.4** Let $f: H \otimes H \to A$ be a normalized cocycle, where $H$ is a smash product of $N$ and $T$ acting trivially on the cocommutative algebra $A$. Then $f$ satisfies the following equations:

$$f(nt \otimes h') = \sum f(t_1 \otimes h'_1) f(n \otimes t_2 h'_2)$$

$$f(nt \otimes t') = \varepsilon(n)f(t \otimes t')$$

$$f(h \otimes nt') = \sum f(h_1 \otimes n'_1) f(h_2 n'_2 \otimes t')$$

$$f(n \otimes n') = \sum f(n \otimes n')\varepsilon(t')$$

$$f(nt \otimes n') = \sum f(t_1 \otimes t')f(t_2 \otimes n'_1)f(n \otimes t_3(n'_2))$$

for $n, n' \in N$, $t, t' \in T$ and $h, h' \in H$. 

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Proof. The equations (1) and (3) are just special cases of the cocycle condition. Equations (2) are (4) are special cases of (1) and (3) respectively and (5) follows from (1)-(4). □

Corollary 5.5 A map \( f: H \otimes H \rightarrow A \) is a normalized 2-cocycle if and only if the following are satisfied:

1. \( f|_{N \otimes T} = \varepsilon \)
2. \( f|_{N \otimes N} \) is a 2-cocycle on \( N \)
3. \( f|_{T \otimes T} \) is a 2-cocycle on \( T \)
4. \( f(t t' \otimes n') = \sum f(t_1' \otimes n_1') f(t \otimes t_2' (n_2')) \), where \( n' \in N \) and \( t, t' \in T \)
5. \( \sum f(n_1 \otimes n_1') f^{-1}(t_1 (n_2) \otimes t_2 (n_2')) = \sum f(t_1 \otimes n_1) f^{-1}(t_2 (n_2') \otimes t_3 (n_2') f(t (n_2' \otimes t_3)) \), where \( n, n' \in N \) and \( t \in T \).

Moreover, the data \( f|_{N \otimes N}, f|_{T \otimes T}, f|_{T \otimes N} \) satisfying the conditions above determine a unique normalized cocycle.

Proof. First assume that \( f \) is a normalized 2-cocycle on \( H \). Then clearly \( f \) is also a 2-cocycle on both \( N \) and \( T \). The equations in conditions 3 and 5 are obtained from the cocycle condition together with the equations (2) and (3) from the previous corollary.

Now suppose that we have the data from conditions 1-5. Then the Equation (5) of the Corollary 5.4 gives a formula for a map \( f: H \otimes H \rightarrow A \) (it is well defined, since it is linear in each of the variables). An elementary (but lengthy) computation shows that the cocycle condition is satisfied. □

Let \( \tilde{H}^i(H, A) \) be the kernel of the restriction homomorphism \( H^i(H, A)_{\text{reg}} \rightarrow H^i(T, A) \). Since the inclusion \( T \rightarrow H \) splits we can conclude that

Proposition 5.6

\[ H^i(H, A) \simeq H^i(T, A) \oplus \tilde{H}^i(H, A). \]

We shall denote the group of normalized cocycles \( H \otimes H \rightarrow A \) that are trivial when restricted to \( T \) by \( Z^2(H, A) \), i.e. \( Z^2(H, A) = \{ f \in Z^2(H, A) | f(n \otimes t) = \varepsilon(n) \varepsilon(t) \) and \( f(t \otimes t') = \varepsilon(t) \varepsilon(t') \}, n \in N, t, t' \in T \). Furthermore, let \( B^2(H, A) = B^2(H, A) \cap Z^2(H, A) \). Using the canonical map \( H \rightarrow T \) together with Corollaries 5.4 and 5.5 we can show that there is an injective map \( B^2(T, A) \rightarrow B^2(H, A) \) and hence

Proposition 5.7 \( H^2(H, A) = Z^2(H, A)/B^2(H, A) \simeq \tilde{H}^2(H, A) \).

We proceed by defining the homomorphisms involved in the generalized Tahara sequence, and also prove exactness at the same time.

The injective homomorphism \( H^1_{\text{meas}}(T, \text{Hom}(N, A)) \rightarrow \tilde{H}^2(H, A) \):
Define a homomorphism $\iota: \mathbb{Z}^1_{\text{meas}}(H, \text{Hom}(N, A)) \to \mathbb{Z}^2(H, A)$ by the rule $\iota(f)(nt \otimes n't') = f(t(n')\varepsilon(n)\varepsilon(t'))$. This homomorphism induces an injective homomorphism $\mathbb{H}^1_{\text{meas}}(T, \text{Hom}(N, A)) \to \mathbb{H}^2(H, A)$. 

**The exactness at $\mathbb{H}^2(H, A)$:**

We claim, that the image of the homomorphism just described equals the kernel of the restriction homomorphism $\mathbb{H}^2(H, A) \to \mathbb{H}^2(N, A)$. 

Clearly $\text{rest} = 0$. Suppose the cocycle $f \in \mathbb{Z}^2(H, A)$ is such that $f|_{N \otimes N} \in \mathbb{B}^2(N, A)$, that is there exists $g: N \to A$ such that $f(n \otimes n') = \delta g(n \otimes n')$. Extend $g$ to a linear map $g: H \to A$ by the rule $g(n \times t) = g(n)\varepsilon(t)$. Now define $f' \in \mathbb{Z}^2_{\text{meas}}(T, \text{Hom}(N, A))$ by the rule $f'(t(n')) = \sum f(t_1 \otimes n'_1)g^{-1}(n'_2)g(t_2(n'_3))$. A routine calculation shows that $f * \delta g^{-1} = \iota(f')$ (we use Equation (5) of Corollary 5.5 to expand $f(nt \otimes n't')$ and take into account that $f|_{N \otimes N} = \delta g$ and that $f|_{T \otimes T} = \varepsilon$) and hence $[f] = \iota([f'])$. 

**The homomorphism $\mathbb{H}^2(N, A)^T \xrightarrow{d_0} \mathbb{H}^2_{\text{meas}}(T, \text{Hom}(N, A))$ and the exactness at $\mathbb{H}^2_{\text{meas}}(T, \text{Hom}(N, A))$:**

Take $[f] \in \mathbb{H}^2(N, A)^T$, for $f \in \mathbb{Z}^2(N, A)$. Then there is a $g: T \otimes N \to A$ s.t. $f * (f^{-1})^t = \delta g(t \otimes \lambda)$, i.e. $\sum f(n_1 \otimes n'_1)f^{-1}(t_1(n_2) \otimes t_2(n'_2)) = \sum g(t_1 \otimes n_1)g^{-1}(t_2 \otimes n_2)(n'_1)g(t_3 \otimes n'_2)$. Now define $d$ by $(df)(t \otimes t' \otimes n) = \sum g(t'_1 \otimes n_1)g^{-1}(t_1(t'_2 \otimes n_2))g(t_2 \otimes t'_3(n_3))$. Clearly $df \in \mathbb{Z}^2_{\text{meas}}(T, \text{Hom}(N, A))$. We claim that the class $[df]$ is independent of the choice of $g$, which in turn also implies that $d(\mathbb{B}^2(N, A)) \subset \mathbb{B}^2(T, \text{Hom}(N, A))$ and hence $d$ gives rise to a homomorphism $\mathbb{H}^2(N, A)^T \to \mathbb{H}^2_{\text{meas}}(T, \text{Hom}(N, A))$. Suppose there is a $g': T \otimes N \to A$ such that $\sum f(n_1 \otimes n'_1)f^{-1}(t_1(n_2) \otimes t_2(n'_2)) = \sum g'(t_1 \otimes n_1)g^{-1}(t_2 \otimes n_2)(n'_1)g(t_3 \otimes n'_2)$. We need to show that there exists $w \in \text{Reg}_{\text{meas}}(T, \text{Hom}(N, A))$ such that $w(t'_1 \otimes n_1)g^{-1}(t_1(t'_2 \otimes n_2))g(t_2 \otimes t'_3(n_3))g^{-1}(t'_4 \otimes n_4)g(t_4t'_5 \otimes n_5)g^{-1}(t_5 \otimes t'_6(n_6)) = \sum w(t'_1 \otimes n_1)w^{-1}(t_1t'_2 \otimes n_2)w(t_2 \otimes t'_3(n_3))$. Define $w$ by $w(t \otimes n) = \sum g(t_1 \otimes n_1)g^{-1}(t_2 \otimes n_2)$ and observe that it does the trick.

It is clear that $dw = 0$. Suppose $df \in \mathbb{B}^2(T, \text{Hom}(N, A))$. Then there exists a $w \in \text{Reg}_{\text{meas}}(T, \text{Hom}(N, A))$ s.t. $(df)(t \otimes t' \otimes n) = \sum w(t'_1 \otimes n_1)w^{-1}(t_1t'_2 \otimes n_2)w(t_2 \otimes t'_3(n_3))$. Define $z: T \otimes N \to A$ by $z(t \otimes n) = \sum h(t_1 \otimes n_1)w^{-1}(t_2 \otimes n_2)$ and note that it gives rise to a normalized cocycle $z \in \mathbb{Z}^2(H, A)$ given by $z(nt \otimes n't') = z(t \otimes n)$ (see Lemma 5.5) and that $[z] = [f]$.

**The homomorphism $\mathbb{H}^2_{\text{meas}}(T, \text{Hom}(N, A)) \to \mathbb{H}^3(H, A)$ and the exactness at $\mathbb{H}^2_{\text{meas}}(T, \text{Hom}(N, A))$:**

Let $f \in \mathbb{Z}^2_{\text{meas}}(T, \text{Hom}(N, A))$. Define a map $jf: H \otimes H \to A$ by $jf(nt \otimes n't' \otimes n''t'') = f(t \otimes t' \otimes n''t'')$. A straightforward calculation shows that $jf$ is a 3-cocycle on $H$. Suppose that $f$ is a measuring 2-coboundary. Then there exists $v \in \text{Reg}_{\text{meas}}(T, \text{Hom}(N, A))$ s.t. $f(t \otimes t' \otimes n'') = \sum v(t'_1 \otimes n''_1)v^{-1}(t_1t'_2 \otimes n''_2)v(t_2 \otimes t'_3(n''_3))$. Now let $v': H \otimes H \to A$ be defined by $v'(nt \otimes n't') = f(t \otimes n')$ and show $jfv' = \delta'v'$. This proves that the homomorphism $j$ is well defined.

Suppose $[h] \in \mathbb{H}^2(N, A)^T$ and let $w: T \otimes N \to A$ be s.t. $h * (h^{-1})^t = \delta'(u(t \otimes \lambda))$. Define $v: H \otimes H \to A$ by $v(tn \otimes t'n') = \sum u(t_1 \otimes n'_1)h(n \otimes t_2(n'_2))$.
and observe that \( jd_1 = \delta^2 v \in B^3(H, A) \). This shows that \( jd = 0 \).

Now suppose the measuring 2-cocycle \( f \) is such that \( jf \) is a 3coboundary. Then there exists \( v \in \mathrm{Reg}(H \otimes H, A) \) s.t. \( jf = \delta^2 v \). Define a \( u: T \otimes N \to A \) by \( u(t \otimes n) = \sum v(t_1 \otimes n_1) v^{-1}(t_2(n_2) \otimes t_3) \) and \( h: N \otimes N \to A \) by \( h(n \otimes n') = v(n \otimes n') \) and note that \( \delta^1(h) = \eta \varepsilon \) (hence \( h \in Z^2(N, A) \)) and that \( h * (h^{-1})^t = \delta^1 u(t \otimes \omega) \), so that \([h] \in H^2(N, A)^T \). Observe also that \( f(t \otimes t') = \sum u(t'_1) u'^{-1}(t_1 t'_2) u'^{-1}(t_2) \), i.e. \([f] = d[h] \).

**Corollary 5.8** If \( H = N \otimes T \), i.e. if the action of \( T \) on \( N \) is trivial, then we have a canonical isomorphism

\[
H^2(H, A) \cong H^2(T, A) \oplus H^1_{\text{meas}}(T, \mathrm{Hom}(N, A)) \oplus H^2(N, A).
\]

**Proposition 5.9** If the action of \( T \) on \( N \) is trivial, then

\[
H^1_{\text{meas}}(T, \mathrm{Hom}(N, A)) \cong \mathrm{P}(T, N, A),
\]

where \( \mathrm{P}(T, N, A) \) denotes the abelian group of maps \( f: T \otimes N \to A \) that measure in both variables, i.e. correspond to algebra maps \( T \to \mathrm{Hom}(N, A) \) and \( N \to \mathrm{Hom}(T, A) \).

**Remark.** The isomorphism \( H^2(N \otimes T, A) \cong H^2(T, A) \oplus H^2(N, A) \oplus \mathrm{P}(T, N, A) \) has a description similar to that in the case of group cohomology (for the group cohomology case see for instance [Kar]): \( [f] \mapsto ([f]_{\mathrm{T}} , [f]_{\mathrm{N}} , \hat{f}) \), where \( \hat{f}(t \otimes n) = \sum f(t_1 \otimes n_1) f^{-1}(n_2 \otimes t_2) \).

### 6 On \( H^2(kG \rtimes U \mathfrak{g}, A) \)

Here we illustrate how the generalized Tahara sequence sheds some light on the Sweedler cohomology, when the cocommutative Hopf algebra in question is a smash product of a group algebra \( T = kG \) and the universal envelop of a Lie algebra \( N = U \mathfrak{g} \). This is always the case when \( k \) is algebraically closed and of characteristic 0 [Gr].

**Theorem 6.1** Let \( G \) be a finite group acting on a Lie algebra \( \mathfrak{g} \), furthermore let \( A \) be a commutative algebra which is also a trivial \( U \mathfrak{g} \rtimes kG \) module and assume the ground field \( k \) contains \( |G|^{-1} \). Then

\[
H^2(U \mathfrak{g} \rtimes kG) \cong H^2(\mathfrak{g}, A^+)^G \oplus H^2(G, A).
\]

**Proof.** If \( |G|^{-1} \in k \) then \( H^1_{\text{meas}}(kG, \mathrm{Hom}(U \mathfrak{g}, A)) \) is trivial and hence the restriction homomorphism \( \text{res}: H^2(N \rtimes T) \to H^2(N, A)^T \) is an isomorphism. So we get the isomorphism \( H^2(N \rtimes T, A) \cong H^2(T, A) \oplus H^2(N, A)^T \). Now \( H^2(T, A) = H^2(G, U(\mathfrak{a})) \), \( H^2(N, A) \cong H^2(G, A^+) \) (see [Sw1]) and \( H^2(N, A)^T \cong H^2(\mathfrak{g}, A^+)^G \).
Example 6.2 Assume the ground field $k$ has characteristic 0, let $\mathfrak{g} = \mathfrak{sl}_n(k)$ be a Lie algebra consisting of trace zero $n \times n$ matrices (with Lie bracket given by a commutator) and let $G \cong C_n \leq \text{GL}_n(k)$ be a group generated by the standard $n$-cycle permutation matrix acting on $\mathfrak{g}$ by conjugation. Then $H^2(U\mathfrak{g} \rtimes kG, k) \cong k^*/(k^*)^n$.

**Proof.** Apply Theorem 6.1 and note that $H^2(\mathfrak{g}, A^+)$ is trivial by the Whitehead’s second lemma [We] and that $H^2(C_n, k^*) = k^*/(k^*)^n$ [We]. □

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