ON THE $C_0$ SEMIGROUP GENERATED BY THE OSEEN OPERATOR AROUND A STEADY FLOW EXTERIOR TO A ROTATING OBSTACLE

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Abstract. We consider the motion of an incompressible viscous fluid filling the whole space exterior to a moving with rotation and translation obstacle. We show that the Stokes operator around the steady flow in the exterior of this obstacle generates a $C_0$-semigroup in $L^p$ space and then develop a series of $L^p$-$L^q$ estimates of such semigroup. As an application, we give out the stability of such steady flow when the initial disturbance in $L^3$ and the steady flow are sufficiently small.

1. Introduction

We consider the motion of an incompressible viscous fluid filling the whole space exterior to a rigid body $B$ moving with rotation and translation. It is nature to describe the motion of the fluid from a frame of reference $S$ attached to $B$ since the region occupied by the fluid will becomes time-independent in $S$. However, since $B$ may rotate, the frame $S$ is no longer inertial, and accordingly we have to modify the classical Navier-Stokes equations in order to take into account the fictitious forces. Assume that the angular velocity $\omega$ of $B$ with respect to the initial frame is constant in time, this amounts to adding Coriolis force $2\omega \times v$ to the left side of the classical Naiver-Stokes equations when the centrifugal force $\omega \times (\omega \times x)$ is absorbed in the pressure.

In mathematics, let $v = v(x, t)$ be the velocity of the fluid with respect to $S$, and $\theta = \theta(x, t)$ be the original pressure modified by adding the $-\frac{1}{2}(\omega \times x)^2$. The motion of such fluid in the frame $S$ can be described by

$$\begin{cases}
\frac{\partial v}{\partial t} - \Delta v + 2\omega \times v + v \cdot \nabla v + \nabla \theta = f & \text{in } \Omega \times (0, \infty), \\
\text{div } v = 0 & \text{in } \Omega \times [0, \infty), \\
v|_{\partial \Omega} = w_*, & \lim_{|x| \to \infty} (v + v_\infty) = 0, \ v_\infty \triangleq v_{\text{trans}} + \omega \times x, \\
v|_{t=0} = v_0,
\end{cases} \quad (1.1)$$

where $\Omega$ is the fixed region occupied by the fluid with $\partial \Omega \in C^3$, $f$ is a external force, and $w_*$ is a prescribed velocity at $\partial \Omega$. $v_{\text{trans}}$ denotes the translation velocity of the center of mass of the rigid body with respect to $S$. A most significate situation, considered in this paper, is that both $v_{\text{trans}}$ and $\omega$ are constant vectors.

Make transformations: $v \to v + v_\infty$ and $w^* \to w^* + v_\infty$, stilled denoted by $v$ and $w^*$, respectively. After a simple calculation, the $-\omega \times v_\infty$ can be formally absorbed in the

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pressure, still denoted by $\theta$, and then system (1.1) becomes
\[
\begin{aligned}
\partial_t v - \Delta v - v_{\text{trans}} \cdot \nabla v - (\omega \times x) \cdot \nabla v + \omega \times v + v \cdot \nabla v + \nabla \theta &= f \quad \text{in } \Omega \times (0, \infty), \\
\text{div } v &= 0 \quad \text{in } \Omega \times [0, \infty), \\
v|_{\partial \Omega} &= w^*, \quad v \rightarrow 0 (|x| \rightarrow \infty), \\
v|_{t=0} &= v_0.
\end{aligned}
\]

To consider the stationary flow of (1.2), we assume that $w_*$ and $f$ only depend on spatial variable throughout the paper. The steady-state counterpart of (1.2) thus becomes
\[
\begin{aligned}
-\Delta w - w_{\text{trans}} \cdot \nabla w - (\omega \times x) \cdot \nabla w + \omega \cdot \nabla w + \nabla \theta_s &= f \quad \text{in } \Omega, \\
\text{div } w &= 0 \quad \text{in } \Omega, \\
w|_{\partial \Omega} &= w_*, \quad w \rightarrow 0 (|x| \rightarrow \infty),
\end{aligned}
\]

Roughly speaking, we call a stationary flow $w$ with a “good” decay at infinite as a \textit{physically reasonable solution}, which is first introduced by Finn in [11, 12] for $\omega = 0$. More precisely, a Leary solution $w$ satisfies
\begin{enumerate}
  \item It is unique for “small” data;
  \item $|w| = O(|x|^{-1})$ as $|x| \rightarrow \infty$. Furthermore, in both cases $v_{\text{trans}} \neq 0$ with $\omega = 0$, and $v_{\text{trans}} \cdot \omega \neq 0$, the flow $w$ must exhibit an infinite wake extending in the direction opposite to $v_{\text{trans}}$ and $(v_{\text{trans}} \cdot \omega)\omega$, respectively.
\end{enumerate}

It is natural to arise a problem when the class of the physically reasonable solutions can be seen as a limit of corresponding nonstationary solutions as $t \rightarrow \infty$, which will be called a stability problem. The problem is significant because there are some interesting feature if $\omega \neq 0$ including hyperbolic aspect caused by the presence of spin. Assume that $w$ is a physically reasonable solution of (1.3). Let $u = v - w$, $P = \theta - \theta_s$ and $u_0 = v_0 - w$. The stability of $w$ can be reduced to finding a unique solution $u(x, t)$ satisfying
\[
\begin{aligned}
\partial_t u - \Delta u - u_{\text{trans}} \cdot \nabla u - (\omega \times x) \cdot \nabla u + \omega \times u + w \cdot \nabla u + u \cdot \nabla w + u \cdot \nabla u + \nabla P &= 0 \quad \text{in } \Omega \times (0, \infty), \\
\text{div } u &= 0 \quad \text{in } \Omega \times [0, \infty), \\
u|_{\partial \Omega} &= 0, \quad u \rightarrow 0 (|x| \rightarrow \infty), \\
u|_{t=0} &= u_0.
\end{aligned}
\]

such that $u(x, t) \rightarrow 0$ as $t \rightarrow \infty$.

The stability problem was widely studied in $L^2(\Omega)$ for the irrotational case $\omega = 0$ by [5, 20, 21, 22, 20, 28], under the small assumption on $w$ and $u_0$. Roughly, Heywood [21] first proved problem (1.4) admits a unique solution $u$ converging to $0$ in $L^2(\Omega)$ and $W^{1,2}(\Omega)$ as $t \rightarrow \infty$. Masuda [26] and Heywood [22] obtained an algebraic decay in time of $L^\infty(\Omega)$ of a weak solution $u$ to (1.4). Miyakawa-Sohr [28] showed that any weak solution to (1.4) satisfying the strong energy inequality tends to $0$ in $L^2(\Omega)$ as $t \rightarrow \infty$. Borchers-Miyakawa [5] further gave an algebraic convergence rate of $||u||_{L^2(\Omega)}$ if $v_{\text{trans}} = 0$.
and a logarithmic convergence rate if \( v_{\text{trans}} \neq 0 \) as \( t \to \infty \). However, Finn [13] pointed out that \( w \in L^2(\Omega) \) if and only if

\[
\int_{\Omega} (T[w, \theta_s] - (w - v_{\text{trans}}) \otimes (w - v_{\text{trans}}) + v_{\text{trans}} \otimes (w - v_{\text{trans}})) \cdot n \mathrm{d} \sigma(x) = 0,
\]

where \( T_{j,k}[w, \theta_s] = -\delta_{j,k} \theta_s + \partial_j w_k + \partial_k w_j \). Thus, it seems more reasonable to study the perturbation from the Oseen semigroup by splitting the integral of the Duhamel term \( w \). Shibata in [31] used the classical \( L^q \)-estimates of the semigroup established in [25] and then viewed the linear term \( L \) of the steady flow \( w \), obtained in [10] and satisfying

\[
|\nabla^k w| \leq C|x|^{-1-k} \quad (k = 0, 1),
\]

under the size of \( u_0 \) and \( w \) depend on \( 3 < q < \infty \). More precisely, they proved that (1.4) admits a unique solution \( u \) in \( L^{3,\infty}(\Omega) \) satisfying

\[
\begin{align*}
\|u\|_{L^3(\Omega)} &\to 0 \quad \text{as } t \to \infty, \\
\|u\|_{L^q(\Omega)} &\lesssim t^{-\frac{3}{2} + \frac{3}{2q}}, \quad \forall t > 0, \ 3 < r < q < \infty.
\end{align*}
\]

The point of their proof was to establish the \( L^pL^q \) estimates of the semigroup generated by the Stokes operator around the steady flow \( w \) in \( \Omega \) with Dirichlet zero boundary condition. On the other hand, the case \( v_{\text{trans}} \neq 0 \) was considered by Shibata [31] and Enomoto-Shibata [7] in \( L^3(\Omega) \). Shibata [31] proved that the unique solution \( u \) of problem (1.4) in \( L^3(\Omega) \) such that

\[
\begin{align*}
\|\nabla u\|_{L^3(\Omega)} &\lesssim t^{-\frac{1}{2}}, \quad \|u(t)\|_{L^q(\Omega)} \lesssim t^{-\frac{1}{2} + \frac{3}{2q}}, \quad \forall 3 \leq q < \infty, \\
\|u(t)\|_{L^{\infty}(\Omega)} &\lesssim (t^{-\frac{1}{2}} + t^{-1 + \frac{3}{2}})
\end{align*}
\]

(1.6) if the \( L^3 \) norm of \( u_0 \) and the constant \( C_\delta \) in the following relation

\[
|\nabla^k w| \leq C_\delta |x|^{-1/2} (1 + |v_{\text{trans}}|s_{\text{trans}}(x))^{-\frac{r}{2} - \delta} \quad (k = 0, 1), \quad \forall 0 < \delta < \frac{1}{4}
\]

(1.7) are very small. Here \( s_{\text{trans}}(x) \triangleq |x| - x \cdot v_{\text{trans}}|v_{\text{trans}}| \), and \( r \) is a number satisfying \( 3 < r < \infty \). Note that, the size of the small assumption on \( u_0 \) and \( w \) no longer depends on \( q \). Unlike Borchers-Miyakawa [6], Shibata in [31] used the \( L^pL^q \) estimates of the Oseen semigroup established in [25], and then viewed the linear term \( w \cdot \nabla u + u \cdot \nabla w \) as a perturbation from the Oseen semigroup by splitting the integral of the Duhamel term on account of better properties of \( w \) and \( \nabla w \). Later, Enomoto-Shibata [7] proved that the classical \( L^\infty \)-estimate also holds for the Oseen semigroup and then used it to refine the \( L^\infty \)-decay rate in (1.6) to the sharper \( t^{-1/2} \), only needing that \( w \) satisfies the summability property

\[
w \in L^{3+}(\Omega) \cap L^{3-}(\Omega), \quad \nabla w \in L^{3+}(\Omega) \cap L^{3-}(\Omega),
\]

(1.8) which is weaker than (1.7). This result shows the stability of the steady flow obtained by [31] in \( L^3 \) which satisfies (1.7).
For the rotational case \( \omega \neq 0 \), we assume that
\[
\mathbf{v}_{\text{trans}} = v_{\text{trans}} \mathbf{e} \quad \text{and} \quad \omega = \omega \mathbf{e}_1, \quad v_{\text{trans}} \geq 0, \quad \omega > 0, \quad \mathbf{e}_1 \triangleq (1, 0, 0).
\]
Let \( \mathcal{R} \triangleq v_{\text{trans}} \mathbf{e} \cdot \mathbf{e}_1 \). By the Mozzi-Chasles transformation, systems (1.11) and (1.13) become
\[
\begin{aligned}
&\partial_t \mathbf{v} - \Delta \mathbf{v} - \mathcal{R} \partial_t \mathbf{v} - \omega ((\mathbf{e}_1 \times x) \cdot \nabla - \mathbf{e}_1 \times \mathbf{e}_1) \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla \theta = \mathbf{f} \quad \text{in} \ \Omega \times (0, \infty),
&\text{div} \mathbf{v} = 0 \quad \text{in} \ \Omega \times [0, \infty),
&\mathbf{v}|_{\partial \Omega} = \mathbf{w}^*, \quad \mathbf{v} \to \mathbf{0} \ (|x| \to \infty), \quad v|_{t=0} = v_0,
\end{aligned}
\]
and
\[
\begin{aligned}
&- \Delta \mathbf{w} - \mathcal{R} \partial_t \mathbf{w} - \omega ((\mathbf{e}_1 \times x) \cdot \nabla - \mathbf{e}_1 \times \mathbf{e}_1) \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{w} + \nabla \theta_s = \mathbf{f} \quad \text{in} \ \Omega,
&\text{div} \mathbf{w} = 0 \quad \text{in} \ \Omega,
&\mathbf{w}|_{\partial \Omega} = \mathbf{w}^*, \quad \mathbf{w} \to \mathbf{0} \ (|x| \to \infty),
\end{aligned}
\]
respectively. Accordingly, problem (1.14) can be rewritten as
\[
\begin{aligned}
&\partial_t \mathbf{u} - \Delta \mathbf{u} - \mathcal{R} \partial_t \mathbf{u} - \omega ((\mathbf{e}_1 \times x) \cdot \nabla - \mathbf{e}_1 \times \mathbf{e}_1) \mathbf{u}
&\qquad + \mathbf{w} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{w} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \mathbf{P} = 0 \quad \text{in} \ \Omega \times (0, \infty),
&\text{div} \mathbf{u} = 0 \quad \text{in} \ \Omega \times [0, \infty),
&\mathbf{u}|_{\partial \Omega} = \mathbf{0}, \quad \mathbf{u} \to \mathbf{0} \ (|x| \to \infty), \quad u|_{t=0} = u_0.
\end{aligned}
\]
When \( \mathcal{R} = 0 \), Hishida-Shibata [23] proved problem (1.11) admits a unique global solutions in \( L^{3, \infty}(\Omega) \) satisfying
\[
\| \mathbf{u} \|_{L^r(\Omega)} \lesssim t^{-\frac{1}{2} + \frac{2}{q}}, \quad \forall 3 < q < \infty
\]
with \( 3 < q < \infty \) if the \( L^{3, \infty} \)-norms of \( u_0 \) and \( \mathbf{w} \), depending on \( q \), are small enough. Such result implies the stability of the stationary flows \( \mathbf{w} \) obtained by [15] in \( L^{3, \infty} \). The key consists of two points, one is the \( L^p-L^q \) estimates of the Stokes semigroup with rotating effect generated by a principal part of the linearized operator of (1.11), the another is a clever interpolation technique due to Yamazaki [36] which enables them to deal with the term \( \text{div}(\mathbf{u} \cdot \nabla \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{u}) \) as a perturbation from this semigroup. For \( \mathcal{R} \neq 0 \), Shibata [32] obtained the same results as [7] for problem (1.11) with \( \mathbf{w} \) satisfying (1.8). This shows the stability of the steady flows with (1.8) obtained in Theorem 4.4 of [17]. For such steady flows, Galdi-Kyed [17] further proved that they satisfy anisotropic pointwise decay estimates with wake structure, that is,

**Proposition 1.1** (17). *Let \( 0 < |\mathcal{R}| \leq \mathcal{R}^* \), \( 0 < \omega < \omega^* \) and \( \Omega \subset \mathbb{R}^3 \) be an exterior domain of class \( C^2 \). There exists a constant \( \eta = \eta_{\Omega, \mathcal{R}, \omega^*} > 0 \) such that if \( \mathbf{f} = \text{div} \mathbf{F} \in L^2(\Omega) \) with compact support and \( \mathbf{w}_s \in W^{3/2, 2}(\partial \Omega) \) satisfies
\[
\sup_{x \in \Omega} ((1 + |x|)^2 |\mathbf{F}|) + \| \mathbf{f} \|_{L^2(\Omega)} + \| \mathbf{w}_s \|_{W^{3/2, 2}(\partial \Omega)} < \eta,
\]
then problem (1.10) possesses a unique solution \( \mathbf{w} \) in \( \dot{W}^{1, 2}(\Omega) \cap L^6(\Omega) \). Moreover, this solution satisfies
\[
|\nabla^k \mathbf{w}(x)| \leq C_k |x|^{-\frac{3}{2}} (1 + |\mathcal{R}| s_{\mathcal{R}}(x))^{-\frac{1}{2} + \varepsilon} (k = 0, 1), \quad \forall \varepsilon \in (0, \frac{1}{2}),
\]
where \( s_R(x) \equiv |x| + \frac{x}{|x|} x_1 \).

As can be seen from the previous discussion, the stability of steady flows strongly depends on the \( L^p-L^q \) estimates of the semigroup generated by different forms of linear operators. In this sense, to establish \( L^p-L^q \) estimate of the semigroup is of independent interest. This paper is devoted to showing that the Oseen operator around a steady flow satisfying (1.12) in the interior of a rotating obstacle generates a \( C_0 \) semigroup, that is, the solution map \( u_0 \mapsto u(t) \) satisfying the following system

\[
\begin{aligned}
\partial_t u - \Delta u - 2\Omega \partial_1 u - \omega((e_1 \times x) \cdot \nabla - e_1 \times)u + u \cdot \nabla w + w \cdot \nabla u &= -\nabla P \\
\text{div} u &= 0, \\
(\mathbf{u} |_{\partial D} = 0, \quad u \rightarrow 0 \ (|x| \rightarrow \infty) \quad \mathbf{u} |_{t=0} = \mathbf{u}_0.
\end{aligned}
\]

in \( \Omega \times (0, \infty) \) defines a \( C_0 \) semigroup. And then we establish a series of \( L^p-L^q \) estimates of this \( C_0 \) semigroup. As an application, we can show the stability of the steady flow satisfying (1.12) in the sense of \( \lim_{t \rightarrow \infty} \|u(t)\|_{L^p(\Omega)} = 0 \) and

\[
t^\frac{q-2}{2} \|u(t)\|_{L^q(\Omega)} + t^\frac{1}{2} \|\nabla u\|_{L^q(\Omega)} \leq C, \quad \forall 3 \leq q < \infty,
\]

if the constant \( C_q \) in (1.12) and the \( L^3 \)-norm of \( u_0 \) are small enough.

1.1. **Notations.** To state main results more precisely, we will outline some notations used throughout the paper. \( \overline{D} \) and \( D^c \) mean the closure and complement of the domain \( D \) of \( \mathbb{R}^3 \), respectively. Given a vector or matrix \( A \), \( A^T \) means the transpose of \( A \). We denote \( C_{a,b,\cdots} \) as a positive constant depending only on the quantities \( a, b, \cdots \), and nonessential constants \( C \) and \( C_{a,b,\cdots} \) may change from line to line. In addition, we denote \( \lesssim \) and \( \lesssim_{a,b,\cdots} \) as \( \leq C \) and \( \leq C_{a,b,\cdots} \), respectively. As usual, we use the following differential symbols:

\[
\partial_t = \partial/\partial t, \quad \partial_j = \partial/\partial x_j, \quad \nabla = (\partial_1, \partial_2, \partial_3), \quad \Delta = \partial_1^2 + \partial_2^2 + \partial_3^2,
\]

\[
\partial_2^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3}, \quad \nabla^j = \{\partial_2^\alpha, |\alpha| = j \geq 2\}, \quad \alpha = (\alpha_1, \alpha_2, \alpha_3), \quad |\alpha| \equiv \alpha_1 + \alpha_2 + \alpha_3.
\]

Moreover, we employ the following special sets:

\[
B_r = \{ x \in \mathbb{R}^3 \mid |x| \leq r \}, \quad B_{r_1, r_2} = \{ x \in \mathbb{R}^3 \mid r_1 \leq |x| \leq r_2 \}, \quad \Omega_r = \Omega \cap B_r.
\]

In particular, \( R > 0 \) denotes a fixed number such that \( \Omega^c \subset B_R \).

To distinguish with scale functions, we shall use bold-face letter to denote three dimensional vector valued functions, and the black-board bold letters to denote the corresponding function spaces, i.e.

\[
C_0^\infty(D) = \{ u = (u_1, u_2, u_3) \mid u_j \in C_0^\infty(D), \; j = 1, 2, 3 \},
\]

\[
C_0^{\infty, \sigma}(D) = \{ u \in C_0^\infty(D) \mid \text{div} u = 0 \},
\]

\[
L^p(D) = \{ u = (u_1, u_2, u_3) \mid u_j \in L^p(D), \; j = 1, 2, 3 \}, \quad \| u \|_{L^p(D)} = \sum_{j=1}^3 \| u_j \|_{L^p(D)},
\]

\[
L^p(D) \quad \text{the closure in } L^p(D) \text{ of } C_0^{\infty, \sigma}(D), \; 1 < p < \infty,
\]

\[
L^p_\ell(D) = \{ f \in L^p(D) \mid f(x) = 0 \; \text{in } B_\ell^c \}.
\]
etc, if $D$ is any domain of $\mathbb{R}^3$. For two Banach space $X$ and $Y$, $(\mathcal{L}(X,Y), \| \cdot \|_{\mathcal{L}(X,Y)})$ denotes the Banach space of all bounded linear operators from $X$ into $Y$ and we set $\mathcal{L}(X) = \mathcal{L}(X,X)$. In addition, $\mathcal{A}(I,X)$ means the set of all $X$-valued holomorphic functions in $I$, and $C(I;X) (C_b(I;X))$ the set of all $X$-valued (bounded) continuous functions in $I$.

We now recall the well-known Helmholtz decomposition in [27, 34]:

$$L^p(\Omega) = \mathcal{P}^p(\Omega) \oplus \mathbb{G}^p(\Omega), \quad \mathbb{G}^p(\Omega) \triangleq \{ \nabla \varphi \in L^p(\Omega) \mid \varphi \in L^p_{\text{loc}}(\Omega) \}. \quad (1.13)$$

Let $\mathcal{P}_\Omega$ be the projector operator from $L^p(\Omega)$ to $\mathbb{G}^p(\Omega)$. For every $\mathfrak{R}, \omega \in \mathbb{R}$, we define the Oseen operator with rotating effect

$$\mathcal{L}_{\mathfrak{R}, \omega} = \mathcal{P}_\Omega \mathcal{L}_{\mathfrak{R}, \omega, \mathfrak{R}}, \quad \mathcal{L}_{\mathfrak{R}, \omega} \triangleq -\Delta - \mathfrak{R} \partial_1 - \omega((e_1 \times x) \cdot \nabla - e_1 \times ), \quad (1.14)$$
and its perturbed operator around $w$ satisfying (1.12)

$$\mathcal{L}_{\mathfrak{R}, \omega, w} = \mathcal{P}_\Omega \mathcal{L}_{\mathfrak{R}, \omega, w} = \mathcal{P}_\Omega (\mathcal{L}_{\mathfrak{R}, \omega} + B_w), \quad B_w \triangleq (u \cdot \nabla w + w \cdot \nabla u) \quad (1.15)$$

where

$$D_p(\mathcal{L}_{\mathfrak{R}, \omega}) = D_p(\mathcal{L}_{\mathfrak{R}, \omega, w}) = \{ \mathfrak{R} \in X^2(\omega) \cap \mathbb{G}^p(\Omega) \mid \mathfrak{R}|_{\partial \Omega} = 0, (e_1 \times x) \cdot \nabla u \in L^p(\Omega) \}.$$ For the uniformity of notations, we denote $\mathcal{L}_{\mathfrak{R}, \omega}$ by $\mathcal{L}_{\mathfrak{R}, \omega, 0}$.

1.2. Main results. Let $D = \Omega$ or $\mathbb{R}^3$, and then for each $0 < \varepsilon < \frac{1}{2}$, define

$$\|g\|_{\varepsilon, D} = \sup_{x \in D} \left( (1 + |x|)|g(x)| + (1 + |x|)^{\frac{3}{2}}|\nabla g(x)| \right) (1 + |x|)^{\frac{3}{2} - \varepsilon}.$$ Shibata [33] proved that $-\mathcal{L}_{\mathfrak{R}, 0, 0}$ generates a $C_0$-semigroup $\{ T_{\mathfrak{R}, 0, 0}(t) \}_{t \geq 0}$ in $\mathbb{G}^p(\Omega)$ such that

$$\| \nabla / T_{\mathfrak{R}, 0, 0}(t) u_0 \|_{\mathbb{G}^p(\Omega)} \lesssim_\gamma e^{\gamma t} t^{-j/2} \| u_0 \|_{L^p(\Omega)}, \; j = 0, 1, 2,$$
for some $\gamma > 0$. This estimate shows

$$\int_0^\alpha \| \mathbb{P}_\Omega \mathcal{L}_{\mathfrak{R}, 0} \mathcal{L}_{\mathfrak{R}, 0}(t) \|_{\mathbb{G}^p(\Omega)} \; \mathbb{d}t \lesssim_{\gamma, \mathfrak{R}, \omega} \alpha^{\frac{1}{2}} \| w \|_{\varepsilon, \omega}, \; 0 < \alpha < 1.$$ This fact together with $D_p(\mathcal{L}_{\mathfrak{R}, 0, 0}) \subset D_p(\mathcal{P}_\Omega B_w)$ yields that $-\mathcal{L}_{\mathfrak{R}, 0, w}$ generates a $C_0$-semigroup $\{ T_{\mathfrak{R}, 0, w}(t) \}_{t \geq 0}$ in $\mathbb{G}^p(\Omega)$ by the perturbation theorem in [19]. In the same way, its dual operator $-\mathcal{L}_{\mathfrak{R}, 0, w}^*$ generates a $C_0$-semigroup $\{ T_{\mathfrak{R}, 0, w}^*(t) \}_{t \geq 0}$ in $\mathbb{G}^p(\Omega)$. Summing up, we have

**Proposition 1.2.** For every $p \in (1, \infty)$, $-\mathcal{L}_{\mathfrak{R}, 0, 0}$ and $-\mathcal{L}_{\mathfrak{R}, 0, w}$ generate $C_0$-semigroups $\{ T_{\mathfrak{R}, 0, 0}(t) \}_{t \geq 0}$ and $\{ T_{\mathfrak{R}, 0, w}^*(t) \}_{t \geq 0}$ in $\mathbb{G}^p(\Omega)$, respectively. In particular, $\{ T_{\mathfrak{R}, 0, w}(t) \}_{t \geq 0}$ and $\{ T_{\mathfrak{R}, 0, w}^*(t) \}_{t \geq 0}$ are analytic semigroups in $\mathbb{G}^p(\Omega)$.

**Remark 1.3.** When $\omega = 0$, for every $\delta > 0$ and $u \in D_p(\mathcal{P}_\Omega \Delta)$, we observe

$$\| \mathcal{P}_\Omega (\mathfrak{R} \partial_1 u + w \cdot \nabla u + u \cdot \nabla w) \|_{L^p} \leq \| \mathfrak{R} \|_{L^p(\Omega)} + \| w \|_{L^p(\Omega)} \leq \delta \| u \|_{L^2(\Omega)} + C \delta_{\mathfrak{R}} \| w \|_{L^p(\Omega)} \leq \delta \| \mathcal{P}_\Omega \Delta u \|_{L^p(\Omega)} + C \delta_{\mathfrak{R}} \| u \|_{L^p(\Omega)}$$
by Corollary to Theorem 1.7 of [27]. This, together with Theorem X.54 in [30], shows that $\mathcal{L}_{\mathfrak{R}, 0, w}$ generates a holomorphic semigroup in $\mathbb{G}^p(\Omega)$. So does $\mathcal{L}_{\mathfrak{R}, 0, w}^*$. 
**Remark 1.4.** We can not expect to control \((e_1 \times x) \cdot \nabla\) by \(-\Delta\) since the drift operator \((e_1 \times x) \cdot \nabla\) is a variable coefficient growing at large distance. This implies that the nonstationary problem associated to \(L_{\eta,w}\) contains hyperbolic features if \(\omega \neq 0\). Hence, the operator \(L_{\eta,w}(\omega \neq 0)\) can only generates a \(C_0\) semigroup in \(J^p(\Omega)\). This fact was verified rigorously by Farwig-Neustupa in [9], which proved that the essential spectrum of the operator \(L_{\eta,w,0}\) coincides with

\[
\bigcup_{\ell \in \mathbb{Z}} \left\{ \sqrt{-1} \omega \ell + \{ \lambda \in \mathbb{C} | \Re \lambda + (\Im \lambda)^2 > 0 \} \right\}. \tag{1.16}
\]

Now we are in position to state the main results.

**Theorem 1.5.** Assume that \(0 < R_* \leq |R| \leq R^*, |\omega| \leq \omega^*\) and \(1 < p < \infty\). Let \(\varepsilon \in (0, \frac{1}{2})\) if \(p \geq \frac{6}{5}\) otherwise \(\varepsilon \in (0, 3 - \frac{3}{p})\). Then there exists a constant \(\eta = \eta_{R,R^*,\omega,\omega^*} > 0\), such that if \(\|w\|_{\varepsilon,\Omega} < \eta\), then for \(f \in J^p(\Omega)\),

\[
\|T_{R_\eta,w}(t)f\|_{L^p(\Omega)}, \quad \|T_{R_\eta,w}(t)f\|_{L^p(\Omega)} \leq Ct^{-\frac{3}{2} + \frac{\varepsilon}{p} + \frac{1}{2}} \|f\|_{L^p(\Omega)},\; \; p \leq q < \infty \tag{1.17}
\]

\[
\|\nabla T_{R_\eta,w}(t)f\|_{L^q(\Omega)}, \quad \|\nabla T_{R_\eta,w}(t)f\|_{L^q(\Omega)} \leq Ct^{-\frac{3}{2} + \frac{\varepsilon}{p} + \frac{1}{2}} \|f\|_{L^p(\Omega)},\; \; p \leq q \leq 3 \tag{1.18}
\]

with \(C = C_{R,R^*,\omega,\omega^*}\).

In the light of Theorem 1.2 we reduce problem (1.11) to the integral equation

\[
u = T_{R_\eta,w}(t)u_0 + \int_0^t T_{R_\eta,w}(t - \tau)P_\Omega(u(\tau) \cdot \nabla)u(\tau) \, d\tau, \tag{1.19}
\]

With the help of Theorem 1.5 we can easily deduce the following result for (1.19) by the classical Kato method, which implies the stability in \(L^3\) of the steady flow satisfying (1.10) and (1.12).

**Theorem 1.6.** Assume that \(0 < R_* \leq |R| \leq R^*, |\omega| \leq \omega^*\) and \(\varepsilon \in (0, \frac{1}{2})\). Let \(u_0 \in J^3(\Omega)\). Then there exists a constant \(\eta = \eta_{R,R^*,\omega,\omega^*} > 0\) such that if

\[
\|u_0\|_{L^3(\Omega)} + \|w\|_{\varepsilon,\Omega} \leq \eta \tag{1.20}
\]

then problem (1.19) admits a unique global solution \(u\) satisfying

\[
u \in C_b([0, \infty); J^3(\Omega)), \quad t^{\frac{3}{2}} \nabla u(t) \in C_b([0, \infty); L^3(\Omega))
\]

such that

\[
\|u(t)\|_{L^3(\Omega)} \to 0, \; \; \text{as} \; \; t \to \infty. \tag{1.21}
\]

\[
t^{\frac{3}{2} - \frac{\varepsilon}{2}} \|u(t)\|_{L^3(\Omega)} + t^{\frac{3}{4}} \|\nabla u\|_{L^3(\Omega)} \leq C, \; \; \forall \; \; 3 \leq q < \infty. \tag{1.22}
\]

We would like to give the sketch proof of Theorem 1.5. Due to (1.16), the traditional way to establish \(L^p - L^q\) estimates of semigroups no longer hold for \(T_{R_\eta,w}(t)\). So we will adopt the domain decomposition to study

\[
\begin{aligned}
\partial_t u + L_{R_\eta,w} u + \nabla P = 0, \; \; &\text{div } u = 0 \; \; \text{in } \Omega \times (0, \infty), \\
u|_{\partial \Omega} = 0, \; \; u|_{t=0} = u_0 \in J^p(\Omega)
\end{aligned} \tag{1.23}
\]
and then establish $L^p$-$L^q$ estimates of $T_{\mathfrak{K},\omega,w}(t)$. Roughly speaking, let $\varphi \in C_0^\infty(B_{R+2})$ be a bump function with $\varphi = 1$ in $B_{R+1}$, and define

$$\tilde{v}_0 = (1 - \varphi)u_0 + B[\nabla \varphi \cdot u_0] \in \mathbb{H}(\mathbb{R}^3)$$

where $B$ is a Bogovskiǐ's operator, see Lemma 4.1 below. Let $\psi \in C_0^\infty(B_{R+1})$ satisfy $0 \leq \psi \leq 1$ and $\psi = 1$ in $B_{R+1/2}$. Suppose that $(\tilde{v}(t), \tilde{\theta}(t))$ solves

$$\begin{cases}
\partial_t \tilde{v} + L_{\mathfrak{K},\omega,w} \tilde{v} + \nabla \tilde{\theta} = 0, \quad \text{div} \tilde{v} = 0 \quad \text{in} \quad \mathbb{R}^3 \times (0, \infty), \\
\tilde{v}\big|_{t=0} = \tilde{v}_0, \quad \int_{\Omega_{R+3}} \tilde{\theta}(t) \, dx = 0
\end{cases}$$

(1.24)

where $\overline{w} = (1 - \psi)w + B[\nabla \psi \cdot w]$ is an extension of $w$ to $\mathbb{R}^3$ such that

$$\overline{w} = w \quad \text{in} \quad B_{R+1}, \quad \text{div} \overline{w} = 0, \quad \|\overline{w}\|_{\mathbb{H}, \mathbb{R}^3} \leq C\|w\|_{\mathbb{H}, \Omega}.$$ 

(1.25)

We decompose initial data $u_0$ as follows:

$$u_0 = v_0 + \tilde{u}_0, \quad v_0 = (1 - \varphi)\tilde{v}_0 + B[\nabla \varphi \cdot \tilde{v}_0].$$

This yields the following decomposition of the solution $u$

$$\begin{cases}
u = v + \tilde{u}, \quad v(t) = (1 - \varphi)\tilde{v}(t) + B[\nabla \varphi \cdot \tilde{v}(t)], \\
P = \theta + \tilde{P}, \quad \theta = (1 - \varphi)\tilde{\theta},
\end{cases}$$

where $(v, \theta)$ and $(\tilde{u}, \tilde{P})$ satisfy

$$\begin{cases}
\partial_t v + L_{\mathfrak{K},\omega,w} v + \nabla \theta = F(t), \quad \text{div} v = 0 \quad \text{in} \quad \Omega \times (0, \infty), \\
v|_{\partial \Omega} = 0, \quad v|_{t=0} = v_0
\end{cases}$$

and

$$\begin{cases}
\partial_t \tilde{u} + L_{\mathfrak{K},\omega,w} \tilde{u} + \nabla \tilde{P} = -F(t), \quad \text{div} \tilde{u} = 0 \quad \text{in} \quad \Omega \times (0, \infty), \\
\tilde{u}|_{\partial \Omega} = 0, \quad \tilde{u}|_{t=0} = \tilde{u}_0 \in L^p_{R+2}(\Omega),
\end{cases}$$

(1.26)

respectively. Here

$$F(t) = -\tilde{\theta} \nabla \varphi + (\Delta \varphi) \tilde{v} + 2(\nabla \varphi) \cdot \tilde{v} + R(\partial_t \varphi) \tilde{v} + \omega((e_1 \times x) \cdot \nabla \varphi) \tilde{v} + (w \cdot \nabla \varphi) \tilde{v} - B[\nabla \varphi \cdot (L_{\mathfrak{K},\omega,w} v + \nabla \tilde{\theta})] + L_{\mathfrak{K},\omega,w} B[\nabla \varphi \cdot \tilde{v}].$$

Hence, to prove the $L^p$-$L^q$ estimates of $T_{\mathfrak{K},\omega,w}(t)$, it suffices to show the decay estimates with respect to $t$ of the solution maps of (1.24) and (1.26).

We observe that equations (1.24) with $\omega \neq 0$ can be reduced to the case $\omega = 0$ under the rotation transformation (2.9). Hence, we only prove the $L^p$-$L^q$ estimates of the solution map associated to the case $\omega = 0$, which can be obtained by making use of $L^p$-$L^q$ estimates of Oseen semigroups and the decay estimates (1.25) of $\overline{w}$ and viewing the additional linear term invoking $\overline{w}$ as a perturbation from the Oseen semigroup by splitting the integral of the Duhamel term on account of (1.25).
Now we turn to the study of decay estimates on $t$ of the solution map of \((1.26)\). For this, we only to study the following problem
\[
\begin{aligned}
\dot{u} + L_{g;\omega,w} u + \nabla P &= 0, \quad \text{div } u = 0 \quad \text{in } \Omega \times (0,\infty), \\
\left. u \right|_{\partial \Omega} &= 0, \quad u|_{t=0} = P_{\Omega} u_0, \quad u_0 \in L^p_{R+2}(\Omega) 
\end{aligned}
\]  
(1.27)

by the homogenization principle since $u_0, F(t) \in L^p_{R+2}(\Omega)$. From $u_0 \in L^p_{R+2}$, we can expect that $(\lambda I + L_{g;\omega,w})^{-1} P_{\Omega} u_0$ and corresponding pressure operator both process decay properties with respect to $\text{Re}\lambda > 0$ in $L^p(\Omega)$ such that
\[
\mathbf{u} = T_{g;\omega,w}(t) P_{\Omega} u_0 = \lim_{t \to \infty} \int_{\gamma - i\theta} e^{\lambda t} (\lambda I + L_{g;\omega,w})^{-1} P_{\Omega} u_0 \, d\lambda, \quad \gamma \geq 1. 
\]  
(1.28)

In addition, we can prove $(\lambda I + L_{g;\omega,w})^{-1} P_{\Omega} u_0$ in $L^p(\Omega_{R+3})$ has some decay estimates with respect to $\text{Re}\lambda \geq 0$, which enable us show the key estimates, the local energy decay of $(\lambda I + L_{g;\omega,w})^{-1} P_{\Omega}$ acting on $L^p_{R+2}(\Omega)$.

For this propose, we adopt the “splitting-gluing” argument to study the resolvent problem
\[
(\lambda I + L_{g;\omega,w}) u + \nabla P = f \in L^p_{R+2}(\Omega), \quad \text{div } u = 0 \quad \text{in } \Omega, \quad \left. u \right|_{\partial \Omega} = 0. 
\]  
(1.29)

Roughly speaking, we first split \((1.29)\) into a resolvent problem in a bounded domain
\[
\begin{cases}
(\lambda I + L_{g;\omega,w}) u + \nabla P = f \in L^p(\Omega_{R+3}), \\
\text{div } u = 0 \quad \text{in } \Omega_{R+3}, \quad \left. u \right|_{\partial \Omega_{R+3}} = 0
\end{cases}
\]  
(1.30)

and a resolvent problem in whole space
\[
(\lambda I + L_{g;\omega,w}) u + \nabla P = f \in L^p_{R+2}(\mathbb{R}^3), \quad \text{div } u = 0 \quad \text{in } \mathbb{R}^3. 
\]  
(1.31)

For \((1.30)\), we can view it as a perturbation of the resolvent problem of the usual Stokes operator in $\Omega_{R+3}$. Thus we construct the solution operators $(\mathcal{R}_{g;\omega,w}^I(\lambda), \mathcal{Q}_{\Omega_{R+3}} + \mathcal{H}_{g;\omega,w}^I(\lambda))$ and establish their decay estimates on $\lambda$, see Theorem 3.1 for details below.

For \((1.31)\), we don’t deal with it by the perturbation argument since $(e^t x) \cdot \nabla$ does not subordinate to $\Delta$. Fortunately, we observe that the solution map to the corresponding nonstationary problem defines a $C_0$-semigroup \(\{T_{g;\omega,w}^G(t)\}_{t \geq 0} \) in $L^p(\mathbb{R}^3)$. This enables us define the resolvent operator $\mathcal{R}_{g;\omega,w}^G(\lambda)$ of \((1.31)\) via the Laplace transform of $T_{g;\omega,w}^G(t) P_{\mathbb{R}^3}$ and corresponding pressure operator $(\mathcal{Q}_{\mathbb{R}^3} + \mathcal{H}_{g;\omega,w}^G(\lambda))$ by Helmholtz decomposition. To investigate the behavior on $\text{Re}\lambda \geq 0$ of $\mathcal{R}_{g;\omega,w}^G(\lambda)$ acting on $L^p_{R+2}(\mathbb{R}^3)$, we formally write
\[
\mathcal{R}_{g;\omega,w}^G(\lambda) = \sum_{j=0}^{\infty} (\mathcal{R}_{g;\omega,w}^G(\lambda) B_{w_j})^j \mathcal{R}_{g;\omega,w}^G(\lambda). 
\]

Making use of the multiplier associated with $T_{g;\omega,w}^G(t) P_{\mathbb{R}^3}$ and the better pointwise estimates \((1.25)\) of $w$, we give out decay estimates of $\mathcal{R}_{g;\omega,w}^G(\lambda)$ with respect to $\text{Re}\lambda > 0$ in $L(L^p_{R+2}(\mathbb{R}^3), W^{2,p}(\mathbb{R}^3))$.

To study the behavior near $\text{Re}\lambda = 0$ of $\mathcal{R}_{g;\omega,w}^G(\lambda)$ acting on $L^p_{R+2}(\mathbb{R}^3)$, we first use the pointwise estimates of the kernel function of $T_{g;\omega,w}^G(t) P_{\mathbb{R}^3}$ to construct an iterative scheme
on account of the domain decomposition, and so prove $L^p$ estimates in small scale and decay estimates in large scale of $\mathcal{R}_{\text{reg},\omega,\overline{\nu}}^G(\lambda)$ acting on $L^p_{R+2}(\mathbb{R}^3)$ which are uniformly with respect to $\text{Re}\lambda > 0$. Further, we establish a “tree self-similar” iteration by the so called “self-similar iteration” and then obtain the decay estimates on $\text{Re}\lambda \geq 0$ of $\mathcal{R}_{\text{reg},\omega,\overline{\nu}}^G(\lambda)$ and $(\mathcal{Q}_{\mathbb{R}^3} + \mathcal{P}_{\text{reg},\omega,\overline{\nu}}^G(\lambda))$ from $L^p_{R+2}(\mathbb{R}^3)$ to local $L^p$ spaces.

Next, we glue the solutions of problems \([\text{1.30}]\) and \([\text{1.31}]\) to obtain the solution of problem \([\text{1.29}]\). More precisely, Let $f \in L^p_{\text{in}2}(\Omega)$ and denote $f_0$ by the zero extension to $\mathbb{R}^3$ of $f$ and $f_{\Omega_{R+3}}$ by the restriction on $\Omega_{R+3}$ of $f$. By the Bogovskiĭ operator, we construct the parametrix $(\Phi_{\text{reg},\omega,\overline{\nu}}(\lambda)f, \Psi_{\text{reg},\omega,\overline{\nu}}(\lambda)f)$ to \([\text{1.29}]\)

\[
\begin{cases}
\Phi_{\text{reg},\omega,\overline{\nu}}(\lambda)f = (1 - \varphi)\mathcal{R}_{\text{reg},\omega,\overline{\nu}}^G(\lambda)f_0 + \varphi\mathcal{R}_{\text{reg},\omega,\overline{\nu}}^I(\lambda)f_{\Omega_{R+3}} + \mathbb{B}[\nabla \varphi \cdot (\mathcal{R}_{\text{reg},\omega,\overline{\nu}}^G(\lambda)f_0 - \mathcal{R}_{\text{reg},\omega,\overline{\nu}}^I(\lambda)f_{\Omega_{R+3}})], \\
\Psi_{\text{reg},\omega,\overline{\nu}}(\lambda)f = (1 - \varphi)(\mathcal{Q}_{\mathbb{R}^3} + \mathcal{P}_{\text{reg},\omega,\overline{\nu}}^G(\lambda))f_0 + \varphi(\mathcal{Q}_{\Omega_{R+3}} + \mathcal{P}_{\text{reg},\omega,\overline{\nu}}^I(\lambda))f_{\Omega_{R+3}}
\end{cases}
\]

such that

\[
(\lambda + \mathcal{L}_{\text{reg},\omega,\overline{\nu}})(\Phi_{\text{reg},\omega,\overline{\nu}}(\lambda)f + \nabla \Psi_{\text{reg},\omega,\overline{\nu}}(\lambda)f) = (I + T + \mathcal{K}_{\text{reg},\omega,\overline{\nu}}(\lambda))f \quad \text{in } \Omega,
\]

\[
\text{div } \Phi_{\text{reg},\omega,\overline{\nu}}(\lambda)f = 0 \quad \text{in } \Omega, \quad \Phi_{\text{reg},\omega,\overline{\nu}}(\lambda)f |_{\partial \Omega} = 0.
\]

where $T + \mathcal{K}_{\text{reg},\omega,\overline{\nu}}(\lambda)$ is a compact operator in $L(L^p_{\text{in}2}(\Omega))$ and $K_{\text{reg},\omega,\overline{\nu}}(\lambda)$ tends to zero as $|\lambda| \to \infty$. So we can show the operator $(I + T + \mathcal{K}_{\text{reg},\omega,\overline{\nu}}(\lambda))$, from $L^p_{\text{in}2}(\Omega)$ to itself, is reversible. Then, the resolvent operator $(\lambda + \mathcal{L}_{\text{reg},\omega,\overline{\nu}})^{-1}P_{\Omega} \triangleq \mathcal{R}_{\text{reg},\omega,\overline{\nu}}(\lambda)$ to \([\text{1.29}]\) and corresponding pressure operator $\mathcal{P}_{\text{reg},\omega,\overline{\nu}}(\lambda)$ can be given by the following formulas:

\[
\mathcal{R}_{\text{reg},\omega,\overline{\nu}}(\lambda) = \Phi_{\text{reg},\omega,\overline{\nu}}(\lambda)(I + T + \mathcal{K}_{\text{reg},\omega,\overline{\nu}}(\lambda))^{-1},
\]

\[
\mathcal{P}_{\text{reg},\omega,\overline{\nu}}(\lambda) = \Psi_{\text{reg},\omega,\overline{\nu}}(\lambda)(I + T + \mathcal{K}_{\text{reg},\omega,\overline{\nu}}(\lambda))^{-1}.
\]

These equalities help us give out the decay estimates of $\mathcal{R}_{\text{reg},\omega,\overline{\nu}}(\lambda)f$ and $\mathcal{P}_{\text{reg},\omega,\overline{\nu}}(\lambda)f$ with respect to $\text{Re}\lambda > 0$, and of $\mathcal{R}_{\text{reg},\omega,\overline{\nu}}(\lambda)f$ with respect to $\text{Re}\lambda = 0$ in $L^p(\Omega_{R+3})$.

The rest of the paper is organized as follows. In Section 2, we show the $L^p-L^q$ estimates of $T_{\text{reg},\omega,\overline{\nu}}^G(t)$ and decay estimates of $(\mathcal{R}_{\text{reg},\omega,\overline{\nu}}^G(\lambda), \mathcal{P}_{\text{reg},\omega,\overline{\nu}}^G(\lambda))$ with respect to $\text{Re}\lambda > 0$ acting on $L^p_{R+2}(\mathbb{R}^3)$. In Section 3, we investigate the behavior of $(\mathcal{R}_{\text{reg},\omega,\overline{\nu}}^G(\lambda), \mathcal{P}_{\text{reg},\omega,\overline{\nu}}^G(\lambda))$ with regard to $\lambda$. In Section 4, we study the solvability of \([\text{1.29}]\) via constructing its parametrix. In Section 5, we show the behavior on $t$ of $T_{\text{reg},\omega,\overline{\nu}}^G(t)P_{\Omega}$ acting on $L^p_{R+2}(\Omega)$. Section 6 is devoted to the proof of Theorem \([\text{1.5}]\). In the section 7, we prove Theorem \([\text{1.6}]\). For the sake of readers, we give some useful technique lemmas or the well known results in Section 8.

2. The resolvent problem in $\mathbb{R}^3$

In this section, we mainly study the resolvent problem

\[
\lambda u + L_{\text{reg},\omega,\overline{\nu}} u + \nabla P = f, \quad \text{div } u = 0 \quad \text{in } \mathbb{R}^3.
\]

First, we define $P_{\mathbb{R}^3}$ (Leray projection operator) and $Q_{\mathbb{R}^3}$ as follows:

\[
(P_{\mathbb{R}^3}f)_i \triangleq F^{-1}(\mathbb{P}(\xi)\hat{f})_i = \sum_{j=1}^3 F^{-1}\left(\left(\delta_{ij} - \frac{\xi_i \xi_j}{|\xi|^2}\right)\hat{f}_j\right), \quad i = 1, 2, 3,
\]
\[ Q_{R^3} f \triangleq F^{-1} \left( \sum_{j=1}^{3} \xi_j \hat{f}(\xi) \right). \] (2.3)

It is well-known that \( \text{div} \mathcal{P}_{R^3} f = 0 \) and

\[ \| \mathcal{P}_{R^3} f \|_{L^p(R^3)} + \| \nabla Q_{R^3} f \|_{L^p(R^3)} \leq \| f \|_{L^p(R^3)}. \] (2.4)

For \( f \in L^p(R^3) \), we have the Helmholtz decomposition

\[ f = \mathcal{P}_{R^3} f + \nabla \tilde{Q}_{R^3} f, \] (2.5)

which is the unique in the sense of \( \int_{\Omega_{R^3}} \tilde{Q}_{R^3} f \, dx = 0 \). Here

\[ \tilde{Q}_{R^3} f \triangleq Q_{R^3} f - \frac{1}{|\Omega_{R^3}|} \int_{\Omega_{R^3}} Q_{R^3} f \, dx. \] (2.6)

We define the operator associated to problem (2.1) as follows

\[ \mathcal{L}_{R^3} = \mathcal{P}_{R^3} L_{R^3,0} + \mathcal{P}_{R^3} B_{R^3}, \]

\[ D_p(L_{\mathfrak{M}_{R^3} R^3}) = \{ u \in W^{2,p}(R^3) \cap L^p(R^3) \mid (e_1 \times x) \cdot \nabla u \in L^p(R^3) \}. \] (2.7)

Here, we used the fact that \( \mathcal{P}_{R^3} L_{\mathfrak{M}_{R^3} R^3} = L_{\mathfrak{M}_{R^3} 0} \mathcal{P}_{R^3} \).

As we all know, \( L_{\mathfrak{M}_{R^3} 0 R^3} \) generates a \( C_0 \)-semigroup \( \{ T_{\mathfrak{M}_{R^3} 0}^G(t) \}_{t \geq 0} \) in \( L^p(R^3) \) such that

\[ \| \nabla^j T_{\mathfrak{M}_{R^3} 0}^G(t) \|_{L^p(R^3)} \leq C_{p,q} t^{-j \left( \frac{3}{p} - \frac{1}{q} \right)} \left( \frac{1}{q} \right)^j, \quad 1 \leq q \leq p \leq \infty, \quad j \leq 2, \]

see \cite{33} for details. As a consequence of the perturbation theorem in \cite{19}, we deduce that \( L_{\mathfrak{M}_{R^3} R^3} \) and its dual operator \( L_{\mathfrak{M}_{R^3} R^3}^{*} \) generate \( C_0 \)-semigroups \( \{ T_{\mathfrak{M}_{R^3} R^3}^G(t) \}_{t \geq 0} \) and \( \{ T_{\mathfrak{M}_{R^3} R^3}^{G*}(t) \}_{t \geq 0} \) in \( L^p(R^3) \), respectively.

### 2.1. \( L^p-L^q \) estimates of \( T_{\mathfrak{M}_{R^3}}^G(t) \) and \( T_{\mathfrak{M}_{R^3}}^{G*}(t) \).

Consider

\[ \begin{cases} \partial_t u + L_{\mathfrak{M}_{R^3}} u + \nabla P = 0, & \text{div } u = 0 \quad \text{in } R^3 \times (0, \infty), \\ u|_{t=0} = u_0 \in L^p(R^3). \end{cases} \] (2.8)

Set

\[ \begin{cases} y = \mathcal{O}(\omega)x, & \tilde{u}(y,t) = \mathcal{O}(\omega t)u(\mathcal{O}^T(\omega t)y,t), \\ \tilde{w}(y,t) = \mathcal{O}(\omega t)\mathfrak{M}(\mathcal{O}^T(\omega t)y), & \tilde{P}(y,t) = P(\mathcal{O}^T(\omega t)y,t) \end{cases} \] (2.9)

with

\[ \mathcal{O}(\omega t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \omega t & -\sin \omega t \\ 0 & \sin \omega t & \cos \omega t \end{pmatrix}. \]

Note that

\[ \| \tilde{w}(t) \|_{L^p(R^3)} = \| \mathfrak{M} \|_{L^p(R^3)} \leq C \| \tilde{u} \|_{L^p(\omega)} \quad \text{for every } t \geq 0, \] (2.10)

then problem (2.8) is equivalent to the nonautonomous system

\[ \begin{cases} \partial_t \tilde{u} + L(t) \tilde{u} + \nabla \tilde{P} = 0, & \text{div } \tilde{u} = 0 \quad \text{in } R^3 \times (0, \infty), \\ \tilde{u}|_{t=0} = u_0 \in L^p(R^3) \end{cases} \] (2.11)

where

\[ L(t) = L_{\mathfrak{M}_{R^3} 0} + B(t), \quad B(w(t)) \triangleq B_{\mathfrak{M}} \text{ by replacing } \mathfrak{M} \text{ by } w(t). \]
From the parabolic evolution system theory in [11,2], we conclude that $-\mathcal{P}_{0}(R^{3}) L(t)$ and its dual operator generate unique evolution operators $\{G(t,s)\}_{0 \leq s \leq t}$ and $\{G^\ast(t,s)\}_{0 \leq s \leq t}$ in $\mathcal{L}^p(R^3)$, respectively, satisfying for $j \leq 2$
\[\|\nabla^j G(t,s)\|_{\mathcal{L}^p(R^3),\mathcal{L}^q(R^3)} \leq e^{C_\omega (t-s)} (t-s)^{-\frac{j}{2}}. \] (2.12)
See Lemma 8.1 below for details. This gives
\[
\begin{align*}
\mathcal{P}_{0}(R^{3}) t \mathcal{u} &= T^G_{(\mathcal{P}(R^3))}(t) \mathcal{u}(0) = \mathcal{O}^T(\omega t) (G(t,0) \mathcal{u}(0)) (\mathcal{O}(\omega t) x, t), \\
T^G_{(\mathcal{P}(R^3))}(t) \mathcal{v}_0 &= Q^T (-\omega t) (G^\ast(t,0) \mathcal{v}_0) (Q(-\omega t) x, t).
\end{align*}
\] (2.13)
So it suffices to show the $L^p$-$L^q$ estimates of $G(t,s)$ and $G^\ast(t,s)$. It is well known that
\[
T^G_{(\mathcal{P}(R^3))}(t) f(x) = (4\pi t)^{-\frac{j}{2}} \int_{R^3} e^{i x \cdot y - \frac{t}{4} \|y\|^2} f(y) dy
\[
= (2\pi)^{-\frac{3}{2}} \int_{R^3} e^{i |\xi|^2 - i \xi \cdot x} \hat{f}(\xi) e^{ix \cdot \xi} d\xi
\] (2.14)
satisfies
\[
\|\nabla^j T^G_{(\mathcal{P}(R^3))}(t)\|_{\mathcal{L}^p(R^3),\mathcal{L}^q(R^3)} \leq Ct^{\frac{j}{2} \left(\frac{1}{p} - \frac{1}{q}\right) - \frac{j}{2}}, \quad j \leq 2, \quad 1 \leq q \leq p \leq \infty,
\] (2.15)
for details see Chapter VIII in [16]. This helps us to deduce the following propositions.

**Proposition 2.1.** Let $\varepsilon \in (0, \frac{1}{2})$, $p \in (1, \infty)$ and $f \in \mathcal{L}^p(R^3)$. Then, there exists a constant $\eta = \eta_{p,\varepsilon} > 0$ such that if $\|f\|_{\varepsilon, \Omega} < \eta$, then
\[
\|G(t,s)f\|_{\mathcal{L}^p(R^3)} \leq C (t-s)^{-\frac{3}{2} \left(\frac{1}{p} - \frac{1}{q}\right) - \frac{j}{2}} \|f\|_{\mathcal{L}^p(R^3)}, \quad p \leq q \leq \infty,
\] (2.16)
\[
\|\nabla G(t,s)f\|_{\mathcal{L}^p(R^3)} \leq C (t-s)^{-\frac{1}{2} - \frac{3}{2} \left(\frac{1}{p} - \frac{1}{q}\right) - \frac{j}{2}} \|f\|_{\mathcal{L}^p(R^3)}, \quad p \leq q \leq 3.
\] (2.17)

**Proof.** By Duhamel principle, we write $\tilde{u}(t,s) = G(t,s)f$ as
\[
\tilde{u}(t,s) = T^G_{(\mathcal{P}(R^3))}(t-s)f + \int_{s}^{t} T^G_{(\mathcal{P}(R^3))}(t-\tau) \mathcal{P}_{0}(R^3) B_\varepsilon(\tau) \tilde{u}(\tau,s) d\tau
\]
\[
= T^G_{(\mathcal{P}(R^3))}(t-s)f + L\tilde{u}(t,s).
\]
It is obvious that form (2.15)
\[
\|\nabla^k T^G_{(\mathcal{P}(R^3))}(t-s)f\|_{\mathcal{L}^p(R^3)} \leq (t-s)^{-\frac{3}{2} - \frac{3}{2} \left(\frac{1}{p} - \frac{1}{q}\right) - \frac{j}{2}} \|f\|_{\mathcal{L}^p(R^3)}, \quad k = 0, 1, 1 < p \leq q \leq \infty.
\] (2.18)
Thus, we next to prove this proposition by splitting the integral of the Duhamel term $L\tilde{u}(t,s)$ on account of (2.10).

**Proof of (2.16).** We first estimate $\|L\tilde{u}(t,s)\|_{\mathcal{L}^p(R^3)}$. By Hölder’s inequality and Lemma 8.3, we observe from (2.10) that
\[
\|\tilde{w}(\tau)\|_{\mathcal{L}^p(R^3)} + \|\nabla \tilde{w}(\tau)\|_{\mathcal{L}^p(R^3)} \lesssim_{\eta} \|\tilde{w}\|_{\varepsilon, \Omega}, \quad \forall \eta \in \left(\frac{6}{3 - 2\eta}, \infty\right], \quad \ell \in \left(\frac{3}{2 - \eta}, \infty\right].
\] (2.19)
Hence we deduce by (2.15) that for $q_0 \in (\max(p, \frac{3}{2}), \infty)$ satisfying $\frac{1}{p} - \frac{1}{q_0} < \frac{1}{3}$,
\[
\|L\tilde{u}(t,s)\|_{\mathcal{L}^p(R^3)} \lesssim \int_{s}^{t} (t-\tau)^{-\frac{3}{2} \left(\frac{1}{p} - \frac{1}{q_0}\right)} \|\tilde{w}(\tau)\|_{\mathcal{L}^p(R^3)} \|\tilde{u}(\tau,s)\|_{\mathcal{L}^{q_0}(R^3)} d\tau
\] (2.20)
\[
\lesssim \|\tilde{w}\|_{\varepsilon, \Omega} \sup_{s \leq \tau \leq t} (\tau-s)^{-\frac{3}{2} \left(\frac{1}{p} - \frac{1}{q_0}\right)} \|\tilde{u}(\tau,s)\|_{\mathcal{L}^{q_0}(R^3)}
\] (2.21)
where we have used the fact that
\[ T_{\infty,0,0}^{G}(t-\tau)P_{R}^{3}B_{\tilde{w}(\tau)}\tilde{u}(\tau, s) = \nabla \cdot T_{\infty,0,0}^{G}(t-\tau)P_{R}^{3}(\tilde{w}(\tau) \otimes \tilde{u}(\tau, s) + \tilde{u}(\tau, s) \otimes \tilde{w}(\tau)). \] (2.22)

Next, we estimate \( (t-s)\frac{3}{p}||L\tilde{u}(t, s)||_{L^\infty(R^3)} \). When \( t-s < 2 \), by [2.15], (2.19) and (2.22), we obtain for \( q_1 \in (\text{max}(3, p), \infty) \) satisfying \( \frac{1}{p} - \frac{1}{q_1} < \frac{2}{3} \),
\[
(t-s)\frac{3}{p}||L\tilde{u}(t, s)||_{L^\infty(R^3)} \leq \rho(t-s)^{\frac{3}{p}} \int_{s}^{t} (t-\tau)^{-\frac{1}{2} - \frac{3}{2q_1}} ||\tilde{w}(\tau)||_{L^\infty(R^3)} ||\tilde{u}(\tau, s)||_{L^{q_1}(R^3)} d\tau
\]
\[
\leq \rho(t-s)^{\frac{3}{p}} \||w||_{\infty, \Omega} \sup_{s \leq \tau \leq t} ||(t-s)^{\frac{3}{2}\left(\frac{1}{p} - \frac{1}{q_1}\right)}||\tilde{u}(\tau, s)||_{L^{q_1}(R^3)}
\]
\[
\leq \rho ||w||_{\infty, \Omega} \sup_{s \leq \tau \leq t} ||(t-s)^{\frac{3}{2}\left(\frac{1}{p} - \frac{1}{q_1}\right)}||\tilde{u}(\tau, s)||_{L^{q_1}(R^3)}. \] (2.23)

When \( t-s > 2 \), we make the following decomposition
\[
L\tilde{u}(t, s) = \left[ \int_{s}^{\frac{t+s}{2}} \right] + \left[ \int_{\frac{t+s}{2}}^{t-1} \right] + \left[ \int_{t-1}^{t} \right] T_{\infty,0,0}^{G}(t-\tau)P_{R}^{3}B_{\tilde{w}(\tau)}\tilde{u}(\tau, s) d\tau.
\]
and then get by (2.15), (2.19) and (2.22)
\[
(t-s)^{\frac{3}{p}}||L\tilde{u}(t, s)||_{L^\infty(R^3)} \leq \rho(t-s)^{\frac{3}{p}} \left( \int_{s}^{\frac{t+s}{2}} \right) (t-\tau)^{-1 - \frac{3}{2q_1}} ||\tilde{w}(\tau)||_{L^q(R^3)} ||\tilde{u}(\tau, s)||_{L^{q_1}(R^3)} d\tau
\]
\[
+ \int_{\frac{t+s}{2}}^{t-1} (t-\tau)^{-\frac{1}{2} - \frac{3}{2q_1}} ||\tilde{w}(\tau)||_{L^1(R^3)} ||\tilde{u}(\tau, s)||_{L^{\infty}(R^3)} d\tau
\]
\[
+ \int_{t-1}^{t} (t-\tau)^{-\frac{1}{2} - \frac{3}{2q_1}} ||\tilde{w}(\tau)||_{L^{2}(R^3)} ||\tilde{u}(\tau, s)||_{L^{\infty}(R^3)} d\tau
\]
\[
\leq \rho ||w||_{\infty, \Omega} \left( \left( t-s \right)^{-1 + \frac{3}{2}\left(\frac{1}{p} - \frac{1}{q_1}\right)} \int_{s}^{\frac{t+s}{2}} (t-s)^{-\frac{3}{2}\left(\frac{1}{p} - \frac{1}{q_1}\right)} d\tau \sup_{s \leq \tau \leq t} ||(t-s)^{\frac{3}{2}\left(\frac{1}{p} - \frac{1}{q_1}\right)}||\tilde{u}(\tau, s)||_{L^{q_1}(R^3)}
\]
\[
+ \left( \int_{\frac{t+s}{2}}^{t-1} (t-\tau)^{-\frac{1}{2} - \frac{3}{2q_1}} d\tau + \int_{t-1}^{t} (t-\tau)^{-\frac{1}{2} - \frac{3}{2q_1}} d\tau \right) \sup_{s \leq \tau \leq t} \|\tilde{u}(\tau, s)\|_{L^{\infty}(R^3)}
\]
\[
\leq \rho ||w||_{\infty, \Omega} \left( \left( t-s \right)^{-\frac{3}{2}\left(\frac{1}{p} - \frac{1}{q_1}\right)} ||\tilde{u}(\tau, s)||_{L^{q_1}(R^3)} + \sup_{s \leq \tau \leq t} \left( t-s \right)^{\frac{3}{p}} \|\tilde{u}(\tau, s)\|_{L^{\infty}(R^3)} \right)
\]

with \( r_1 \in (\frac{6}{3q_2}, 3) \) and \( r_2 \in (3, \infty) \). This estimate, together with (2.18), (2.21) and the interpolation inequality between \( L^p(R^3) \) and \( L^\infty(R^3) \), yields that
\[
\sup_{s \leq t \leq \infty} \|\tilde{u}(t, s)\|_{L^p(R^3)} + \sup_{s \leq t \leq \infty} (t-s)^{\frac{3}{p}} \|\tilde{u}(t, s)\|_{L^\infty(R^3)}
\]
\[
\leq C_p \|f\|_{L^p(R^3)} + C_{\varepsilon, p} \|w\|_{\infty, \Omega} \left( \sup_{s \leq t \leq \infty} \|\tilde{u}(t, s)\|_{L^p(R^3)} + \sup_{s \leq t \leq \infty} (t-s)^{\frac{3}{p}} \|\tilde{u}(t, s)\|_{L^\infty(R^3)} \right).
\]
Hence, we deduce (2.16) if \( C_{\varepsilon, p} \|w\|_{\infty, \Omega} < 1 \).
Proof of (2.17). Thanks to (2.16) and the fact: $G(t, s) = G(t, s_0)G(s_0, s)$ for all $s \leq s_0 \leq t$, we only need prove (2.17) with $1 < p = q \leq 3$.

Case 1: $p < 3$. When $t - s \leq 2$, by (2.14), (2.15) and Lemma 8.4, we deduce

$$
(t - s)^\frac{1}{2} \| \nabla L \tilde{u}(t, s) \|_{L^p(\mathbb{R}^3)} \lesssim \| \nabla L \tilde{u}(t, s) \|_{L^p(\mathbb{R}^3)}
$$

$$
\lesssim \int_s^t (t - \tau)^{-\frac{1}{2}} \left( \| \tilde{w}(\tau) \|_{L^\infty(\mathbb{R}^3)} \| \nabla \tilde{w}(\tau, s) \|_{L^p(\mathbb{R}^3)} + \| x \| \| \nabla \tilde{w}(\tau) \|_{L^\infty(\mathbb{R}^3)} \| \frac{\tilde{u}(\tau, s)}{|x|} \|_{L^p(\mathbb{R}^3)} \right) d\tau
$$

$$
\lesssim \| w \|_{L^p(\Omega)} \sup_{s \leq \tau \leq t} (\tau - s)^\frac{1}{2} \| \nabla \tilde{w}(\tau, s) \|_{L^p(\mathbb{R}^3)}. \tag{2.24}
$$

Similarly, when $t - s > 2$, we have for $\alpha \in (0, \min(\frac{1}{3}, \frac{1}{2} - \varepsilon))$, $r > \frac{2}{3}$,

$$
(t - s)^\frac{1}{2} \| \nabla L \tilde{u}(t, s) \|_{L^p(\mathbb{R}^3)}
$$

$$
\lesssim \int_s^t \left( t - \tau \right)^{-\frac{1}{2}} \left( \| \tilde{w}(\tau) \|_{L^\infty(\mathbb{R}^3)} \| \nabla \tilde{w}(\tau, s) \|_{L^p(\mathbb{R}^3)} + \| x \| \| \nabla \tilde{w}(\tau) \|_{L^\infty(\mathbb{R}^3)} \| \frac{\tilde{u}(\tau, s)}{|x|} \|_{L^p(\mathbb{R}^3)} \right) d\tau
$$

$$
\lesssim \| w \|_{L^p(\Omega)} \sup_{s \leq \tau \leq t} (\tau - s)^\frac{1}{2} \| \nabla \tilde{w}(\tau, s) \|_{L^p(\mathbb{R}^3)},
$$

where we have used the fact that for every $\delta > 1$ and $0 < \rho < 1$

$$
\int_s^t (t - s)^{-\delta} (t - s)^{-\rho} d\tau \leq (t - s)^{-\delta} \int_s^t (\tau - s)^{-\rho} d\tau + (t - s)^{-\rho} \int_{t-s}^{t-\frac{1}{2}} (t - s)^{-\delta} d\tau
$$

$$
\lesssim (t - s)^{1-\delta - \rho} \lesssim (t - s)^{-\rho}, \quad \forall t - s > 2. \tag{2.25}
$$

This, together with (2.18) and (2.24), yields

$$
\sup_{s \leq t \leq \infty} (t - s)^\frac{1}{2} \| \nabla \tilde{u}(t, s) \|_{L^p(\mathbb{R}^3)} \leq C \| f \|_{L^p(\mathbb{R}^3)} + C_{\varepsilon,p} \| w \|_{L^p(\mathbb{R}^3)} \tag{2.26}
$$

which proves (2.17) with $1 < p = q < 3$ if $C_{\varepsilon,p} \| w \|_{L^p(\mathbb{R}^3)} < 1$.

Case 2: $p = 3$. When $t - s \leq 2$, by (2.18) and (2.19), we have

$$
(t - s)^\frac{1}{2} \| \nabla L \tilde{u}(t, s) \|_{L^3(\mathbb{R}^3)} \lesssim \| \nabla L \tilde{u}(t, s) \|_{L^3(\mathbb{R}^3)}
$$

$$
\lesssim \int_s^t (t - \tau)^{-\frac{1}{2}} \left( \| \tilde{w}(\tau) \|_{L^\infty(\mathbb{R}^3)} \| \nabla \tilde{w}(\tau, s) \|_{L^3(\mathbb{R}^3)} + \| \nabla \tilde{w}(\tau) \|_{L^3(\mathbb{R}^3)} \| \tilde{u}(\tau, s) \|_{L^\infty(\mathbb{R}^3)} \right) d\tau
$$

$$
\lesssim \| w \|_{L^3(\Omega)} \sup_{s \leq \tau \leq t} (\tau - s)^\frac{1}{2} \| \nabla \tilde{w}(\tau, s) \|_{L^3(\mathbb{R}^3)} + \sup_{s \leq \tau \leq t} (\tau - s)^\frac{1}{2} \| \tilde{u}(\tau, s) \|_{L^\infty(\mathbb{R}^3)}.
$$

When $t - s > 2$, by (2.18), (2.19) and (2.25), we have for $\ell_0 \in (6/(3 - 2\varepsilon), 3)$,

$$
(t - s)^\frac{1}{2} \| \nabla L \tilde{u}(t, s) \|_{L^3(\mathbb{R}^3)} \lesssim (t - s)^\frac{1}{2} \left[ \int_s^t (t - \tau)^{-\frac{1}{2}} \| \tilde{w}(\tau) \|_{L^{\ell_0}(\mathbb{R}^3)} \| \tilde{u}(\tau, s) \|_{L^\infty(\mathbb{R}^3)} d\tau
$$

$$
+ \int_s^t (t - \tau)^{-\frac{1}{2}} \left( \| \tilde{w}(\tau) \|_{L^\infty(\mathbb{R}^3)} \| \nabla \tilde{w}(\tau, s) \|_{L^3(\mathbb{R}^3)} + \| \nabla \tilde{w}(\tau) \|_{L^3(\mathbb{R}^3)} \| \tilde{u}(\tau, s) \|_{L^\infty(\mathbb{R}^3)} \right) d\tau \right]
$$

$$
\lesssim \| w \|_{L^3(\Omega)} \left( \sup_{s \leq \tau \leq t} (\tau - s)^\frac{1}{2} \| \nabla \tilde{w}(\tau, s) \|_{L^3(\mathbb{R}^3)} + \sup_{s \leq \tau \leq t} (\tau - s)^\frac{1}{2} \| \tilde{u}(\tau, s) \|_{L^\infty(\mathbb{R}^3)} \right).
$$
Thus, collecting the above two estimates, (2.16) and (2.18), we deduce
\[ \sup_{s \leq t \leq \infty} (t-s)^{\frac{1}{2}} \| \nabla \tilde{u}(t, s) \|_{L^1(R^3)} \leq C(1 + \| w \|_{\varepsilon, \Omega}) \| f \|_{L^p(R^3)} + C'_{\varepsilon} \| w \|_{\varepsilon, \Omega} \sup_{s \leq t \leq \infty} (t-s)^{\frac{1}{2}} \| \nabla \tilde{u}(t, s) \|_{L^1(R^3)}. \]

This yields (2.17) with \( p = q = 3 \) if \( C'_{\varepsilon} \| w \|_{\varepsilon, \Omega} < 1 \) and so ends proof of this proposition. \( \Box \)

**Proposition 2.2.** Let \( \varepsilon \in (0, \frac{1}{p}) \), \( p \in (1, \infty) \) and \( q \in [p, \infty] \). Then there exists a constant \( \eta = \eta_{p, \varepsilon} > 0 \) such that if \( \| w \|_{\varepsilon, \Omega} < \eta \), then for \( f \in L^p(R^3) \), \( k = 0, 1, \)
\[ \| \nabla^k G^*(t, s) f \|_{L^q(R^3)} \leq C(t-s)^{-\frac{k}{2} - \frac{3}{p} \left( \frac{1}{p} - \frac{1}{q} \right)} \| f \|_{L^p(R^3)}, \quad (k, q) \neq (1, \infty). \]

**Proof.** Thanks to Proposition 2.1 and the duality argument, we only need to prove
\[
\| \nabla G^*(t, s) f \|_{L^p(R^3)} \leq C(t-s)^{-\frac{1}{2} - \frac{3}{p} \left( \frac{1}{p} - \frac{1}{q} \right)} \| f \|_{L^p(R^3)}, \quad (p, q) = (3, 3).
\]

Let \( B_{\tilde{w}(t)}^* \) be the dual operator of \( B_{\tilde{w}(t)} \). We write \( \tilde{v}(t, s) \triangleq \nabla^* G^*(t, s) f \) into
\[
\tilde{v}(t, s) = T_{G_{\tilde{w}(0), 0}(t-s)} f + \int_0^t T_{G_{\tilde{w}(0), 0}(t-s)} \mathcal{P}_{R, 1} B_{\tilde{w}(\tau)}^* \tilde{v}(\tau, s) d\tau,
\]
where we used \( T_{G_{\tilde{w}(0), 0}(t-s)} = T_{G_{\tilde{w}(0), 0}(t-s)} \). Note that, for every \( \varphi \in C_0^\infty (R^3) \),
\[ \langle \mathcal{P}_{R, 1} B_{\tilde{w}(\tau)}^* \tilde{v}(\tau, s), \varphi \rangle = \langle \nabla \tilde{v}(\tau, s), (\tilde{w}(\tau) \otimes \varphi + \varphi \otimes \tilde{w}(\tau)) \rangle. \]

So, by (2.19) we get for \( m, q \in (1, \infty) \) and \( r \in \left( \frac{6}{3 - 2q}, \infty \right) \) satisfying \( \frac{1}{m} = \frac{1}{r} + \frac{1}{q} < 1 \)
\[ \| \mathcal{P}_{R, 1} B_{\tilde{w}(\tau)}^* \tilde{v}(\tau, s) \|_{L^m(R^3)} \lesssim \| \tilde{w}(\tau) \|_{L^r(R^3)} \| \nabla \tilde{v}(\tau, s) \|_{L^q(R^3)} \]
\[
\lesssim p \| f \|_{L^p(R^3)} + \| w \|_{\varepsilon, \Omega} (t-s)^{\frac{3}{p} - \frac{3}{q_0}} \| \mathcal{P}_{R, 1} B_{\tilde{w}(\tau)}^* \tilde{v}(\tau, s) \|_{L^m(R^3)} d\tau \lesssim p \| f \|_{L^p(R^3)}
\]
\[
\leq p \| f \|_{L^p(R^3)} \quad \text{and} \quad \| \nabla \tilde{v}(t, s) \|_{L^p(R^3)} \leq p \| f \|_{L^p(R^3)}
\]
with \( \frac{1}{m} = \frac{1}{r} + \frac{1}{q_0} \), which proves (2.27).

Next, we prove (2.26). Let \( q_0' \) be the number such that \( \frac{1}{q_0} + \frac{1}{q_0'} = 1 \). When \( t - s \leq 2 \), by (2.15) and (2.28) with \( (r, q) = (\infty, q_0') \), we have
\[
(t-s)^{\frac{1}{2} + \frac{3}{p} \left( \frac{1}{p} - \frac{1}{q} \right)} \| \nabla \tilde{v}(t, s) \|_{L^q(R^3)} \lesssim (t-s)^{\frac{1}{2} \left( \frac{1}{p} - \frac{1}{q} \right)} \| \nabla \tilde{v}(t, s) \|_{L^q(R^3)}
\]
\begin{align*}
\lesssim & (t-s)^{\frac{3}{2}(\frac{1}{p} - \frac{1}{m})} \int_s^t (t-\tau)^{-\frac{1}{2}} \|P_{R(t)} B_{w(t)}^{e} \tilde{v}(\tau, s)\|_{L^{\infty}(\mathbb{R}^3)} \, d\tau \\
\lesssim & (t-s)^{\frac{3}{2}(\frac{1}{p} - \frac{1}{m})} \int_s^t (t-\tau)^{-\frac{1}{2}} \|w\|_{\varepsilon, \Omega} \|\nabla \tilde{v}(\tau, s)\|_{L^{\infty}(\mathbb{R}^3)} \, d\tau \\
\lesssim & \|w\|_{\varepsilon, \Omega} \sup_{s \leq \tau \leq t} (t-s)^{\frac{1}{2} + \frac{3}{2}(\frac{1}{p} - \frac{1}{m})} \|\nabla \tilde{v}(\tau, s)\|_{L^{\infty}(\mathbb{R}^3)}. \quad (2.29)
\end{align*}

When $t - s > 2$, making the decomposition:

\begin{align*}
L \tilde{v}(t, s) = & \left[ \int_s^{t-1} + \int_{t-1}^t \right] T_{-0,0,0}^G(t-\tau) \mathcal{P}_{R(t)} B_{w(t)}^{e} \tilde{v}(\tau, s) \, d\tau,
\end{align*}

by (2.15) and (2.28) with $q = q_0$ and $r = r_0 \in \{ \max(\frac{6}{2\varepsilon}, q'_0), 3 \}$ or $r = \infty$, we get from (2.25)

\begin{align*}
& (t-s)^{\frac{1}{2} + \frac{1}{2}(\frac{1}{p} - \frac{1}{m})} \|\nabla L \tilde{v}(t, s)\|_{L^0(\mathbb{R}^3)} \\
\lesssim & c_{e,p} (t-s)^{\frac{1}{2} + \frac{3}{2}(\frac{1}{p} - \frac{1}{m})} \left[ \int_s^{t-1} (t-\tau)^{-\frac{1}{2} - \frac{3}{2m}} \|\mathcal{P}_{R(t)} B_{w(t)}^{e} \tilde{v}(\tau, s)\|_{L^m(\mathbb{R}^3)} \, d\tau \\
+ & \int_{t-1}^t (t-\tau)^{-\frac{1}{2}} \|\mathcal{P}_{R(t)} B_{w(t)}^{e} \tilde{v}(\tau, s)\|_{L^{\infty}(\mathbb{R}^3)} \, d\tau \right]
\end{align*}

\begin{align*}
\lesssim & c_{e,p} \|w\|_{\varepsilon, \Omega} \sup_{s \leq \tau \leq t} (t-s)^{\frac{1}{2} + \frac{3}{2}(\frac{1}{p} - \frac{1}{m})} \|\nabla \tilde{v}(\tau, s)\|_{L^{\infty}(\mathbb{R}^3)}
\end{align*}

with $\frac{1}{m} = \frac{1}{r_0} + \frac{1}{q_0}$. This estimate, combining with (2.18) and (2.29), gives that

\begin{align*}
\sup_{s \leq t \leq \infty} (t-s)^{\frac{3}{2} + \frac{3}{2}(\frac{1}{p} - \frac{1}{m})} \|\nabla \tilde{v}(t, s)\|_{L^{\infty}(\mathbb{R}^3)} 
\leq & C \|f\|_{L^p(\mathbb{R}^3)} + c_{e,p} \|w\|_{\varepsilon, \Omega} \sup_{s \leq t \leq \infty} (t-s)^{\frac{3}{2} + \frac{3}{2}(\frac{1}{p} - \frac{1}{m})} \|\nabla \tilde{v}(t, s)\|_{L^{\infty}(\mathbb{R}^3)}.
\end{align*}

Thus, we obtain

\begin{align*}
\sup_{s \leq t \leq \infty} (t-s)^{\frac{3}{2} + \frac{3}{2}(\frac{1}{p} - \frac{1}{m})} \|\nabla \tilde{v}(t, s)\|_{L^{\infty}(\mathbb{R}^3)} 
\leq & C \|f\|_{L^p(\mathbb{R}^3)}. \quad (2.30)
\end{align*}

if $c_{e,p} \|w\|_{\varepsilon, \Omega} < 1$. Hence, we have by (2.15) and (2.28) with $(r, q) = (3, q_0)$

\begin{align*}
(t-s)^{\frac{3}{2}} \|\nabla \tilde{v}(t, s)\|_{L^{p}(\mathbb{R}^3)} 
\lesssim & \|f\|_{L^p(\mathbb{R}^3)} + (t-s)^{\frac{3}{2}} \int_s^t (t-\tau)^{-1 + \frac{3}{2}(\frac{1}{p} - \frac{1}{m})} \|\mathcal{P}_{R(t)} B_{w(t)}^{e} \tilde{v}(\tau, s)\|_{L^m(\mathbb{R}^3)} \, d\tau \\
\lesssim & \|f\|_{L^p(\mathbb{R}^3)} + \|w\|_{\varepsilon, \Omega} (t-s)^{\frac{3}{2}} \int_s^t (t-\tau)^{-1 + \frac{3}{2}(\frac{1}{p} - \frac{1}{m})} \|\nabla \tilde{v}(t, s)\|_{L^{\infty}(\mathbb{R}^3)} \, d\tau 
\lesssim & \|f\|_{L^p(\mathbb{R}^3)}.
\end{align*}

with $\frac{1}{m} = \frac{1}{3} + \frac{1}{q_0}$. This proves (2.27). So Proposition 2.2 is proved. \qed
2.2. The resolvent estimates. Now we consider the resolvent problem (2.1). Let

$$\mathcal{R}_{\delta_{\omega}, \pi}(\lambda) = \int_0^\infty e^{-\lambda t} T_{\delta_{\omega}, \pi}(t) P_{\Omega} dt, \quad \mathcal{R}_{\delta_{\omega}, \pi}^*(\lambda) = \int_0^\infty e^{-\lambda t} T_{\delta_{\omega}, \pi}(t) \mathcal{P}_{\Omega} dt.$$  

For every \( f \in L^p(\mathbb{R}^3) \), \( u = \mathcal{R}_{\delta_{\omega}, \pi}(\lambda) f \) and \( v = \mathcal{R}_{\delta_{\omega}, \pi}^*(\lambda) f \) solve

\[
(\lambda I + L_{\delta_{\omega}, \pi, \Omega}) u = P_{\Omega} f, \quad (\lambda I + L_{\delta_{\omega}, \pi, \Omega}^*) v = P_{\Omega} f
\]

uniquely, respectively. Thus, we have

$$\mathcal{R}_{\delta_{\omega}, \pi}(\lambda) = (\lambda I + L_{\delta_{\omega}, \pi, \Omega})^{-1} P_{\Omega}, \quad \mathcal{R}_{\delta_{\omega}, \pi}^*(\lambda) = (\lambda I + L_{\delta_{\omega}, \pi, \Omega}^*)^{-1} P_{\Omega}. \tag{2.34}$$

Set

$$\Pi_{\delta_{\omega}, \pi}(\lambda) f = Q_{\Omega} (B_{\pi} \mathcal{R}_{\delta_{\omega}, \pi}(\lambda) f), \quad \Pi_{\delta_{\omega}, \pi}^*(\lambda) f = Q_{\Omega} (B_{\pi} \mathcal{R}_{\delta_{\omega}, \pi}^*(\lambda) f). \tag{2.35}$$

By the Helmholtz decomposition (2.3), we conclude

$$u = \mathcal{R}_{\delta_{\omega}, \pi}(\lambda) f, \quad P = \hat{Q}_{\Omega} f + \Pi_{\delta_{\omega}, \pi}(\lambda) f$$

solve problem (2.1) with \( \int_{\Omega_{R^3}} P dx = 0 \) uniquely.

For \( \theta \in (0, \pi/2) \) and \( \ell, \gamma > 0 \), define

\[
\Sigma_{\theta} = \{ \lambda \in \mathbb{C} \mid \arg \lambda \leq \pi - \theta \}, \quad \Sigma_{\theta, \ell} = \{ \lambda \in \Sigma_{\theta} \mid |\lambda| \geq \ell \}, \\
C_+ = \{ \lambda \in \mathbb{C} \mid \Re \lambda > 0 \}, \quad C_{+\gamma} = \{ \lambda \in C_+ \mid \Re \lambda \geq \gamma \}.
\]

The resolvent operator \( \mathcal{R}_{\delta_{\omega}, \pi}(\lambda) \) has the following properties.

Theorem 2.4. Assume that \( 0 < |\Re| \leq |\Re^*|, |\omega| \leq \omega^* \) and \( p \in (1, \infty) \). Let \( \theta \in (0, \pi/2) \), \( \gamma > 0 \) and \( \varepsilon \in (0, \frac{\pi}{2}) \) if \( p \geq \frac{6}{5} \) otherwise \( \varepsilon \in (0, \frac{3p-3}{p}) \). Then, there exists a constant \( \eta = \eta_{p, \theta, \gamma, \delta_{\omega}, \pi} > 0 \) such that if \( \|w\|_{e, \Omega} < \eta \), then

$$\mathcal{R}_{\delta_{\omega}, \pi}(\lambda) \in \mathcal{A}(C_+, L(\mathbb{L}_{R^3}^p(\mathbb{R}^3), \mathbb{W}^2_p(\mathbb{R}^3)))$$

with the decomposition

$$\mathcal{R}_{\delta_{\omega}, \pi}(\lambda) = (\lambda I - \Delta - \Re \partial) \mathcal{R}_{\Omega}^{-1} + \mathcal{R}_{\delta_{\omega}, \pi}^{G, 1}(\lambda) + \mathcal{R}_{\delta_{\omega}, \pi}^{G, 2}(\lambda) \tag{2.36}$$

such that

\[
\begin{align*}
\mathcal{R}_{\delta_{\omega}, \pi}^{G, 1}(\lambda) &\in \mathcal{A}(\Sigma_{\theta, \ell_0}, L(\mathbb{L}_{R^3}^{p-2}(\mathbb{R}^3), \mathbb{W}^2_p(\mathbb{R}^3))), \\
\mathcal{R}_{\delta_{\omega}, \pi}^{G, 2}(\lambda) &\in \mathcal{A}(C_+, L(\mathbb{L}_{R^3}^p(\mathbb{R}^3), \mathbb{W}^2_p(\mathbb{R}^3))), \\
(\lambda I - \Delta - \Re \partial) \mathcal{R}_{\Omega}^{-1} &\in \mathcal{A}(\Sigma_{\theta, \ell_0}, L(\mathbb{L}^p(\mathbb{R}^3), \mathbb{W}^2_p(\mathbb{R}^3))).
\end{align*}
\]
Moreover, we have for every $|\beta| \leq 2$

\[
\begin{align*}
&\|\partial_2^2 R_{\vartheta, \omega, \overline{w}}^G(\lambda)\|_{L^p(R^4 \setminus \mathbb{R}^4)} \leq C_{\theta, R, \vartheta, \omega, \omega} |\lambda|^{\frac{3-|\beta|}{2}}, \quad \lambda \in \Sigma_{\theta, \ell_0}, \\
&\|\partial_2^2 R_{\gamma, R, \omega, \omega}^G(\lambda)\|_{L^p(R^4 \setminus \mathbb{R}^4)} \leq C_{\gamma, R, \vartheta, \omega, \omega} |\lambda|^{\frac{5-|\beta|}{2} + \delta}, \quad \lambda \in C_{\gamma, \delta}, \ 0 < \delta < \frac{1}{2}, \\
&\|\partial_2^2 (\lambda I - \Delta - \mathcal{R}_1)^{-1} \mathcal{P}_R \|_{L^p(R^4)} \leq C_{\theta, R} |\lambda|^{-\frac{2-|\beta|}{2}}, \quad \lambda \in \Sigma_{\theta, \ell_0}.
\end{align*}
\] (2.37) (2.38) (2.39)

**Proof.** In view of (2.34), we formally write

\[
\mathcal{R}_{\vartheta, \omega, \overline{w}}^G(\lambda) = \sum_{j=0}^{\infty} (- (\lambda I + \mathcal{L}_{\vartheta, \omega, R^4})^{-1} \mathcal{P}_R \mathcal{B}_{\overline{w}})^j (\lambda I + \mathcal{L}_{\vartheta, \omega, R^4})^{-1} \mathcal{P}_R.
\] (2.40)

We will divide into two steps to prove Theorem 2.34

**Step 1.** Analysis of $(\lambda I + \mathcal{L}_{\vartheta, \omega, R^4})^{-1} \mathcal{P}_R g$. Since $G(t, 0) = T_{\vartheta, 0, 0}^G(t)$ provided $\overline{w} = 0$, we have from (2.33)-(2.41) and (2.33)-(2.34) that for $g \in L^p(R^3)$

\[
\begin{align*}
(\lambda I + \mathcal{L}_{\vartheta, \omega, R^4})^{-1} \mathcal{P}_R g &= \int_0^\infty e^{-\lambda t} \mathcal{O}^T(\omega^T)(T_{\vartheta, 0, 0}^G(t) \mathcal{P}_R g)(\mathcal{O}(\omega^T)x) \, dt \\
&= \frac{1}{(2\pi)^3} \int_0^\infty \int_{R^3} e^{-\lambda t} \mathcal{O}^T(\omega^T)\mathcal{P}(\mathcal{O}(\omega^T)\xi) \mathbf{g}(\mathcal{O}(\omega^T)\xi) e^{ix \cdot \xi} \, d\xi \, dt.
\end{align*}
\] (2.41)

Integrating by parts $N$-times, we get

\[
\begin{align*}
(\lambda I + \mathcal{L}_{\vartheta, \omega, R^4})^{-1} \mathcal{P}_R g &= \frac{1}{(2\pi)^3} \sum_{j=0}^{N-1} \int_{R^3} \frac{e^{ix \cdot \xi}}{(\lambda + |\xi|^2 + i \mu_1)^j} \partial_1^j \left( \mathcal{O}^T(\omega^T)\mathcal{P}(\mathcal{O}(\omega^T)\xi) \mathbf{g}(\mathcal{O}(\omega^T)\xi) \right) |_{t=0} \, d\xi \\
&\quad + \frac{1}{(2\pi)^3} \int_{R^3} \int_0^\infty \frac{e^{-\lambda t + |\xi|^2 - i \mu_1 t}}{(\lambda + |\xi|^2 + i \mu_1)^N} \partial_1^N \left( \mathcal{O}^T(\omega^T)\mathcal{P}(\mathcal{O}(\omega^T)\xi) \mathbf{g}(\mathcal{O}(\omega^T)\xi) \right) \, d\xi \, dt \\
&= A_{\mu_1}^N(\lambda) g + A_{\mu_1}^N(\lambda) g.
\end{align*}
\] (2.42)

By Leibniz rule, we have

\[
\begin{align*}
&\partial_1^j \left( \mathcal{O}^T(\omega^T)\mathcal{P}(\mathcal{O}(\omega^T)\xi) \mathbf{g}(\mathcal{O}(\omega^T)\xi) \right) \\
&= \omega^j \sum_{k=0}^j \sum_{|\alpha|=k, \alpha_1=0} d_\alpha^j \sin(\omega t, \cos(\omega t, \xi / |\xi|)) (\partial_1^\alpha \mathbf{g})(\mathcal{O}(\omega^T)\xi)
\end{align*}
\]

where $d_\alpha^j(a, b, v)$ are some $3 \times 3$ matrices of polynomials with respect to $a, b$ and $v = (v_1, v_2, v_3)$. This equality gives

\[
\begin{align*}
&\partial_1^j \left( \mathcal{O}^T(\omega^T)\mathcal{P}(\mathcal{O}(\omega^T)\xi) \mathbf{g}(\mathcal{O}(\omega^T)\xi) \right) |_{t=0} = \omega^j \sum_{k=0}^j \sum_{|\alpha|=k, \alpha_1=0} d_\alpha^j(0, 1, \xi / |\xi|) (\partial_1^\alpha \mathbf{g})(\xi).
\end{align*}
\]
Thus, we can rewrite
\[ A_1^N(\lambda)g = \sum_{j=0}^{N-1} \omega^j \sum_{k=0}^N \sum_{|\alpha|=k, \alpha_1=0} \frac{1}{(2\pi)^j} \int_{\mathbb{R}^3} \frac{|\xi|^k d_\alpha^j(0, 1, \xi/|\xi|)}{(\lambda + |\xi|^2 - i9\xi_1)^{j+1}} (\partial_\xi^\alpha \hat{g})(\xi) e^{ix \cdot \xi} d\xi, \]
\[ A_2^N(\lambda)g = \omega^N \sum_{k=0}^N \sum_{|\alpha|=k, \alpha_1=0} \frac{1}{(2\pi)^j} \int_{\mathbb{R}^3} \int_{0}^{\infty} \frac{|\xi|^k e^{-(\lambda + |\xi|^2 - 9\xi_1)t}}{(\lambda + |\xi|^2 - 9\xi_1)^N} d^N_\alpha(\sin \omega t, \cos \omega t, \mathcal{O}(\omega t)|\xi|) (\partial_\xi^\alpha \hat{g})(\xi) e^{i\omega(x \cdot \xi)} d\xi dt. \]

Obviously,
\[ A_1^N(\lambda) = (\lambda I - \Delta - 9\partial_t)^{-1} \mathcal{P}_{\mathbb{R}^3}. \]  
(2.43)

From Lemma 1 in \[ \text{[33]}, \] one has
\[ |\lambda + |\xi|^2 - 9\xi_1| \geq C_{\theta, \gamma, \omega^*}(|\lambda| + |\xi|^2), \quad \lambda \in \Sigma_{\theta, \ell_0} \cup C_{+\gamma} \]
which implies
\[ |\partial_\xi^\alpha(\lambda + |\xi|^2 - 9\xi_1)^{-m}| \leq C_{\theta, \gamma, \omega^*}(|\lambda| + |\xi|^2)^{-m-(|\nu|/2)}, \quad \lambda \in \Sigma_{\theta, \ell_0} \cup C_{+\gamma}. \]  
(2.44)

This, combining with that
\[ |\partial_\xi^\alpha d_\alpha(\sin \omega t, \cos \omega t, \mathcal{O}(\omega t)|\xi|)| \leq C|\xi|^{-|\nu|}, \]  
(2.45)
gives for \(|\beta| \leq 2\)
\[ \left| \partial_\xi^\beta \left( \frac{(i\xi)^\beta |\xi|^k d_\alpha^j(0, 1, \xi/|\xi|)}{(\lambda + |\xi|^2 - 9\xi_1)^{j+1}} \right) \right| \leq \frac{C_{j,k,\theta,\gamma,\omega^*}}{|\lambda|^{j+1-((|\beta|+k)/2)}} |\xi|^{-|\nu|}, \quad \lambda \in \Sigma_{\theta, \ell_0} \cup C_{+\gamma}. \]  
(2.46)

Thus, we obtain by Fourier multiplier theorem that for every \(\lambda \in \Sigma_{\theta, \ell_0} \cup C_{+\gamma}\)
\[ \|x^\alpha A_1^N(\lambda)g\|_{L^p(\mathbb{R}^3)} \leq \frac{C_{\theta, \gamma, \omega^*}}{|\lambda|^{1-|\beta|/2}} \|g\|_{L^p(\mathbb{R}^3)}, \]  
(2.47)
\[ \|x^\beta (A_1^N(\lambda) - A_1(\lambda))g\|_{L^p(\mathbb{R}^3)} \leq \frac{C_{N, \theta, \gamma, \omega^*}}{|\lambda|^{(3-|\beta|)/2}} \sum_{0 \leq |\alpha| \leq N-1} \|\hat{x}^\alpha g\|_{L^p(\mathbb{R}^3)}, \quad N \geq 2. \]  
(2.48)

Further, we observe for \(|\mu| = 1\)
\[ x^\mu \partial_\xi^\beta \int_{\mathbb{R}^3} \frac{|\xi|^k d_\alpha^j(0, 1, \xi/|\xi|)}{(\lambda + |\xi|^2 - 9\xi_1)^{j+1}} (\partial_\xi^\alpha \hat{g})(\xi) e^{ix \cdot \xi} d\xi \]
\[ = i \int_{\mathbb{R}^3} \partial_\xi^\beta \left( \frac{(i\xi)^\beta |\xi|^k}{(\lambda + |\xi|^2 - 9\xi_1)^{j+1}} d_\alpha^j(0, 1, \xi/|\xi|) \right) (\partial_\xi^\alpha \hat{g})(\xi) e^{ix \cdot \xi} d\xi \]
\[ + i \int_{\mathbb{R}^3} \frac{(i\xi)^\beta |\xi|^k}{(\lambda + |\xi|^2 - 9\xi_1)^{j+1}} d_\alpha^j(0, 1, \xi/|\xi|) (\partial_\xi^{\alpha+\mu} \hat{g})(\xi) e^{ix \cdot \xi} d\xi. \]

Hence, by (2.44)-(2.45), we deduce for every \(|\mu| = 1\) and \(|\beta| = 1, 2\)
\[ \left| \partial_\xi^\beta (i\xi)^\beta \frac{|\xi|^k d_\alpha^j(0, 1, \xi/|\xi|)}{(\lambda + |\xi|^2 - 9\xi_1)^{j+1}} \right| \leq \frac{C_{j,k,\theta,\gamma,\omega^*}}{|\lambda|^{j+1-((|\beta|+1+k)/2)}} |\xi|^{-|\nu|}, \quad \lambda \in \Sigma_{\theta, \ell_0} \cup C_{+\gamma}. \]

Combining this estimate with (2.46), we get by Fourier multiplier theorem
\[ \left\| x^\mu \partial_\xi^\beta \int_{\mathbb{R}^3} \frac{|\xi|^k d_\alpha^j(0, 1, \xi/|\xi|)}{(\lambda + |\xi|^2 - 9\xi_1)^{j+1}} (\partial_\xi^\alpha \hat{g})(\xi) e^{ix \cdot \xi} d\xi \right\|_{L^p(\mathbb{R}^3)} \]
\[
\begin{align*}
\|x^\alpha \partial_x^\beta A_1^N(\lambda)g\|_{L^p(\mathbb{R}^3)} & \leq \frac{C_{\lambda, k, \theta, \gamma, \rho_1} \|x^\alpha g\|_{L^p(\mathbb{R}^3)}}{|\lambda|^{j + (3/2) - (\langle|\beta| + k\rangle/2)}} + \frac{C_{\lambda, k, \theta, \gamma, \rho_1} \|x^{\alpha + \mu} g\|_{L^p(\mathbb{R}^3)}}{|\lambda|^{j + 1 - (\langle|\beta| + k\rangle/2)}}, \quad \lambda \in \Sigma_{\theta, \rho_0} \cup C_{+\gamma}.
\end{align*}
\]

This equality implies for every \(|\mu| = 1\) and \(|\beta| = 1, 2\)
\[
\|x^\alpha \partial_x^\beta A_1^N(\lambda)g\|_{L^p(\mathbb{R}^3)} \leq \frac{C_{\lambda, \gamma, \rho_1, \rho_2, \omega^*}}{|\lambda|^{1 - (\langle|\beta|\rangle/2)}} \sum_{0 \leq |\alpha| \leq N} \|x^\alpha g\|_{L^p(\mathbb{R}^3)}, \quad \lambda \in \Sigma_{\gamma, \rho_0} \cup C_{+\gamma}.
\] (2.49)

On the other hand, since
\[
|\partial^\mu_x e^{-(\lambda + |\xi|^2 - i\beta \xi_1)}| \leq \sum_{\ell = 0}^{|\mu|} \ell! (|\xi|^2 + \Re)^{\ell} e^{-(\Re \lambda + |\xi|^2)t}
\]
and \(r^s e^{-r} \leq C_s, \ s \geq 0\), we have
\[
|\partial^\mu_x e^{-(\lambda + |\xi|^2 - i\beta \xi_1)}| \leq C_{2\gamma, \gamma - 1}|\xi|^{-|\nu|} e^{-(\Re \lambda + |\xi|^2)t}, \quad \lambda \in C_{+\gamma}
\]
which, together with (2.44)-(2.45), yields for \(|\beta| \leq 2\)
\[
|\partial^\mu_x \left( \frac{i\mathcal{O}^T(\omega \xi) \delta e^{-(\lambda + |\xi|^2 - i\beta \xi_1)}|\xi|^k d_{\alpha}^N(\sin \omega t, \cos \omega t, \mathcal{O}^T(\omega \xi)|/|\xi|)\right)\right| \leq \begin{cases} 
C_{2\gamma, \gamma - 1, \frac{\ell - (\langle|\beta| + k\rangle/2)}{|\lambda|} e^{-t \Re \lambda} |\xi|^{-|\nu|}, & \text{if } |\beta| + k \leq 1, \\
C_{2\gamma, \gamma - 1, \frac{\ell - 1 + \delta}{|\lambda|^{n + 2 - (\langle|\beta| + k\rangle/2)}} e^{-t \Re \lambda} |\xi|^{-|\nu|}, & 0 < \delta < \frac{1}{2}, \text{ if } |\beta| + k \geq 2.
\end{cases}
\]
Hence, by Fourier multiplier theorem and the fact that
\[
\int_0^\infty t^{-s} e^{-t \Re \lambda} dt = (\Re \lambda)^{-1 + s} T(1 - s) \quad \text{if } s < 1 \text{ and } \lambda \in C_+,
\]
we deduce for \(0 < \delta < \frac{1}{2}\) and \(\lambda \in C_{+\gamma}\)
\[
\|\partial^\beta_x A_1^N(\lambda)g\|_{L^p(\mathbb{R}^3)} \leq \sum_{0 \leq |\alpha| \leq 1} \|x^\alpha g\|_{L^p(\mathbb{R}^3)} \begin{cases} 
C_{2\gamma, \gamma - 1, \rho_1} |\lambda|^{-1}, & |\beta| = 0 \\
C_{2\gamma, \gamma - 1, \rho_1} |\lambda|^{-\frac{3 - |\beta|}{2} + \delta}, & |\beta| = 1, 2.
\end{cases}
\] (2.50)
and
\[
\|\partial^\beta_x A_2^N(\lambda)g\|_{L^p(\mathbb{R}^3)} \leq \frac{C_{N, \gamma, \rho_1, \rho_2, \omega^*}}{|\lambda|^{N + 2 - (\langle|\beta| + k\rangle/2)}} \sum_{0 \leq |\alpha| \leq N} \|x^\alpha g\|_{L^p(\mathbb{R}^3)}, \quad N \geq 2.
\] (2.51)

**Step 2.** Proof of (2.36)-(2.39) Here, we argue that \(|\beta| \leq 2, f \in L^p_{R + 2}(\mathbb{R}^3)\) and the constants appearing in this step depends on \(R, \Re^*, \omega^*\). Let
\[
B_{\overline{\omega}} f \triangleq B_{1, \overline{\omega}} f + B_{2, \overline{\omega}} f, \quad B_{1, \overline{\omega}} f \triangleq \overline{\omega} \cdot \nabla f, \quad B_{2, \overline{\omega}} f \triangleq f \cdot \nabla \overline{\omega}.
\] (2.52)
We rewrite by (2.42)
\[
\mathcal{R}_{\overline{\omega}, \overline{\omega}}^G(\lambda) = (\lambda I - \Delta - \Re \partial_1)^{-1} \mathcal{P}_{\mathbb{R}^3} + \mathcal{R}_1(\lambda) + \mathcal{R}_2(\lambda) + \mathcal{R}_3(\lambda) + \mathcal{R}_4(\lambda) + \mathcal{R}_5(\lambda),
\]
where
\[
\mathcal{R}_1(\lambda) \triangleq \sum_{j=0}^\infty \left( ( - A_1^1(\lambda) - A_2^1(\lambda)) B_{\overline{\omega}} \right)^j A_2^j(\lambda),
\]
Meanwhile, we have

\[ R_2(\lambda) \triangleq A_1^4(\lambda) - A_1^1(\lambda) + \sum_{j=1}^{\infty} (- A_1^1(\lambda) B_{\mathcal{F}}) J^j A_1^1(\lambda), \]

\[ R_3(\lambda) \triangleq \sum_{j=3}^{\infty} \sum_{\alpha \in \{1, 2\}, \alpha \neq (1, 1, 1)} (-1)^j (\prod_{i=1}^{j} (- A_1^{\alpha_i}(\lambda) B_{\mathcal{F}})) A_1^1(\lambda), \]

\[ R_4(\lambda) \triangleq (A_1^1(\lambda) + A_1^4(\lambda)) B_{\mathcal{F}} A_1^1(\lambda) A_1^4(\lambda) + A_1^2(\lambda) A_1^4(\lambda) B_{\mathcal{F}} A_1^1(\lambda) \]

\[ R_5(\lambda) \triangleq A_1^4(\lambda) B_{\mathcal{F}} A_1^1(\lambda) - A_1^2(\lambda) B_{\mathcal{F}} A_1^1(\lambda) - A_1^2(\lambda) B_{\mathcal{F}} A_1^1(\lambda). \]

In the course of the proof, we will repeatedly use

\[ \| B_{\mathcal{F}} g \|_{L^p(R^3)} + \| x B_{\mathcal{F}} g \|_{L^p(R^3)} \leq C \| \omega \|_{C^1,\Omega} (\| g \|_{L^p(R^3)} + \| \nabla g \|_{L^p(R^3)}), \]

\[ \| x^s f \|_{L^p(R^3)} \leq C_{R,s} \| f \|_{L^p(R^3)}, \quad s > 0, \quad f \in L^p_{R+1}(R^3). \] (2.53)

These estimates together with (2.47)-(2.48) and (2.50)-(2.51) with \( \delta = \frac{1}{4} \) yield

\[ \| \partial_x^2 R_1(\lambda) f \|_{L^p(R^3)} \lesssim \sum_{j=0}^{\infty} (C_{\gamma} \| \omega \|_{C^1,\Omega})^j |\lambda|^{3-j/2} \| f \|_{L^p(R^3)}, \quad \lambda \in C_{+\gamma}, \]

\[ \| \partial_x^2 R_2(\lambda) f \|_{L^p(R^3)} \lesssim \sum_{j=0}^{\infty} (C_{\theta} \| \omega \|_{C^1,\Omega})^j |\lambda|^{3-j/2} \| f \|_{L^p(R^3)}, \quad \lambda \in \Sigma_{\theta,\delta}, \]

\[ \| \partial_x^2 R_3(\lambda) f \|_{L^p(R^3)} \lesssim \sum_{j=0}^{\infty} (C_{\gamma} \| \omega \|_{C^1,\Omega})^j |\lambda|^{3-j/2} \| f \|_{L^p(R^3)}, \quad \lambda \in C_{+\gamma}. \]

Hence we conclude

\[ \| \partial_x^2 R_1(\lambda) f \|_{L^p(R^3)} + \| \partial_x^2 R_2(\lambda) f \|_{L^p(R^3)} \lesssim \gamma \| \lambda \|^{-5/2} \| f \|_{L^p(R^3)}, \quad \lambda \in C_{+\gamma}, \]

\[ \| \partial_x^2 R_2(\lambda) f \|_{L^p(R^3)} \lesssim \theta \| \lambda \|^{-3/2} \| f \|_{L^p(R^3)}, \quad \lambda \in \Sigma_{\theta,\delta}, \]

provided that

\[ (C_{\gamma} + C_{\theta}) \| \omega \|_{C^1,\Omega} \leq 1. \]

Meanwhile, we have

\[ \| \partial_x^3 R_4(\lambda) f \|_{L^p(R^3)} \lesssim \gamma \delta \| \lambda \|^{-5/2} \| f \|_{L^p(R^3)}, \quad \lambda \in C_{+\gamma}, \quad 0 < \delta < \frac{1}{2}. \]

Now we are in position to estimate \( R_5(\lambda) \). Observing by (2.42)

\[ A_1^1(\lambda) + A_1^4(\lambda) = A_1^2(\lambda) + A_1^3(\lambda) \]

we decompose \( R_5(\lambda) = R_{5,1}(\lambda) + R_{5,2}(\lambda) + R_{5,3}(\lambda) \), where

\[ R_{5,1}(\lambda) \triangleq (A_1^4(\lambda) - A_1^1(\lambda)) B_{\mathcal{F}} A_1^1(\lambda) A_1^4(\lambda) \]

\[ R_{5,2}(\lambda) \triangleq A_1^2(\lambda) B_{\mathcal{F}} A_1^1(\lambda) A_1^4(\lambda) - A_1^2(\lambda) B_{\mathcal{F}} A_1^1(\lambda), \]

\[ R_{5,3}(\lambda) \triangleq - A_1^3(\lambda) B_{\mathcal{F}} A_1^1(\lambda). \]
By (2.47)-(2.51) and (2.53), we easily get
\[ \left\| \partial_x^2 R_{5,1}(\lambda) f \right\|_{L^p(\mathbb{R}^3)} \lesssim_\delta |\lambda|^{-2+([\beta]/2)} \left\| f \right\|_{L^p(\mathbb{R}^3)}, \quad \lambda \in \Sigma_{0,\ell_0}, \]
\[ \left\| \partial_x^2 R_{5,2}(\lambda) f \right\|_{L^p(\mathbb{R}^3)} \lesssim_{\delta, \gamma} |\lambda|^{-((5-|\beta|)/2)+\delta} \left\| f \right\|_{L^p(\mathbb{R}^3)}, \quad \lambda \in C_{+\gamma}, \ 0 < \delta < \frac{1}{2}. \]

For \( \mathcal{R}_{5,3}(\lambda) \), we observe
\[ (\lambda I - \Delta - \mathfrak{R}\partial_1) P_{R^3} = \lambda^{-1} P_{R^3} + \lambda^{-1} (\Delta + \mathfrak{R}\partial_1) (\lambda I - \Delta - \mathfrak{R}\partial_1)^{-1} P_{R^3}. \]
which together with (2.43) implies
\[ \mathcal{R}_{5,3}(\lambda) = -\lambda^{-1} A_2^2(\lambda) B_{2,\mathfrak{R}} P_{R^3} - \lambda^{-1} A_2^2(\lambda) B_{2,\mathfrak{R}}(\Delta + \mathfrak{R}\partial_1) A_1^1(\lambda). \quad (2.54) \]
By (2.49) and (2.51) with \( \delta = \frac{1}{4} \), we have
\[ \left\| \partial_x^2 A_2^2(\lambda) B_{2,\mathfrak{R}}(\Delta + \mathfrak{R}\partial_1) A_1^1(\lambda) f \right\|_{L^p(\mathbb{R}^3)} \leq C_{\gamma} |\lambda|^{-3-|\beta|/2} \left\| f \right\|_{L^p(\mathbb{R}^3)}, \quad \lambda \in C_{+\gamma}, \quad (2.55) \]
In addition, since the kernel function of \( \mathcal{P}_{R^3} \) is bounded by \( |x|^{-3} \) and \( f = 0 \) in \( B_{R+2} \), we get
\[ \left\| \mathcal{P}_{R^3} f \right\| (x) \leq C |x|^{-3} \int_{\mathbb{R}^3} |f(y)| \, dy \leq C_R |x|^{-3} \left\| f \right\|_{L^p(\mathbb{R}^3)}, \quad |x| \geq 3(R + 2). \]
This inequality gives
\[ \left\| x^2 B_{2,\mathfrak{R}} P_{R^3} f \right\|_{L^p(\mathbb{R}^3)} \leq \left\| x^2 B_{2,\mathfrak{R}} P_{R^3} f \right\|_{L^p(B_{3(R+2)})} + \left\| x^2 B_{2,\mathfrak{R}} P_{R^3} f \right\|_{L^p(B^{3(R+2)})} \]
\[ \leq C_R \left\| f \right\|_{L^p(\mathbb{R}^3)} \left\| x, \Omega \right\| \left( 1 + \left\| \cdot \right\|_{-5/2} (1 + s_\Omega(\cdot))^{-(1/2)+\varepsilon} \right) \left\| f \right\|_{L^p(\mathbb{R}^3)}, \]
for \( \varepsilon \in (0, \min(\frac{1}{2}, \frac{3p-3}{p})) \). Hence we conclude by (2.51) with \( \delta = \frac{1}{4} \)
\[ \left\| \partial_x^3 A_2^2(\lambda) B_{2,\mathfrak{R}} P_{R^3} f \right\|_{L^p(\mathbb{R}^3)} \lesssim_{\gamma} \left\| w \right\|_{x, \Omega} |\lambda|^{-3-|\beta|/2} \left\| f \right\|_{L^p(\mathbb{R}^3)}. \quad (2.56) \]
This, combining with (2.54)-(2.55), yields for \( \varepsilon \in (0, \min(\frac{1}{2}, \frac{3p-3}{p})) \)
\[ \left\| \partial_x^3 \mathcal{R}_{5,3} f \right\|_{L^p(\mathbb{R}^3)} \lesssim_{\gamma} \left\| w \right\|_{x, \Omega} |\lambda|^{-5-3|\beta|/2} \left\| f \right\|_{L^p(\mathbb{R}^3)}, \quad \lambda \in C_{+\gamma}. \quad (2.57) \]

Finally, we define
\[ \mathcal{R}_{G,\mathfrak{R},\omega, \mathfrak{R}}(\lambda) = \mathcal{R}_2(\lambda) + \mathcal{R}_{5,1}(\lambda), \]
\[ \mathcal{R}_{G,\mathfrak{R},\omega, \mathfrak{R}}(\lambda) = \mathcal{R}_1(\lambda) + \mathcal{R}_3(\lambda) + \mathcal{R}_{4}(\lambda) + \mathcal{R}_{5,2}(\lambda) + \mathcal{R}_{5,3}(\lambda). \]
Collecting (2.47) and the estimates of \( \mathcal{R}_1(\lambda)-\mathcal{R}_5(\lambda) \), we prove (2.37)-(2.39) and so complete the proof of this theorem. \( \square \)

Let \( D = \mathbb{R}^3 \), \( \Omega \) or \( \Omega_{R+3} \) and define
\[ \hat{W}^{1,p}(D) = \left\{ \pi \in L^p_{\text{loc}}(D) \mid \nabla\pi \in L^p(D) \right\}, \quad \hat{W}^{1,\infty}(D) = \left\{ \pi \in \hat{W}^{1,p}(D) \mid \int_{\Omega_{R+3}} \pi \, dx = 0 \right\}. \]
In the light of (2.35)-(2.36), we split \( \hat{\Pi}_{\mathfrak{R},\omega, \mathfrak{R}}^G(\lambda) \) into two parts:
\[ \begin{cases} \hat{\Pi}_{\mathfrak{R},\omega, \mathfrak{R}}^{G,1}(\lambda) f = \hat{Q}_{R^3}(B_{\mathfrak{R}}((\lambda - \Delta - \mathfrak{R}\partial_1)^{-1} P_{R^3} + \mathcal{R}_{G,\mathfrak{R},\omega, \mathfrak{R}}(\lambda)) f), \\
\hat{\Pi}_{\mathfrak{R},\omega, \mathfrak{R}}^{G,2}(\lambda) f = \hat{Q}_{R^3}(B_{\mathfrak{R}} \mathcal{R}_{G,\mathfrak{R},\omega, \mathfrak{R}}(\lambda) f). \end{cases} \quad (2.58) \]
As a consequence of Theorem [2.4] we have

**Corollary 2.5.** Under the assumption of Theorem [2.4] there exists a positive constant 
\( \eta = \eta_{\gamma,R,R^*,\omega^*} \) such that if \( \| u \|_{\varepsilon,\Omega} < \eta \), then

\[
\Pi_{\mathcal{R}_{\omega,\omega}}^G(\lambda) \in \mathcal{A}(C_+, \mathcal{L}(L^p_{R+2}(\mathbb{R}^3), W^{1,p}(\mathbb{R}^3)))
\]

with the decomposition

\[
\Pi_{\mathcal{R}_{\omega,\omega}}^G(\lambda) = \Pi_{\mathcal{R}_{\omega,\omega}}^{G,1}(\lambda) + \Pi_{\mathcal{R}_{\omega,\omega}}^{G,2}(\lambda)
\]

(2.59)

where

\[
\begin{align*}
\Pi_{\mathcal{R}_{\omega,\omega}}^{G,1}(\lambda) & \in \mathcal{A}(\Sigma_{\theta,\lambda_0}, \mathcal{L}(L^p_{R+2}(\mathbb{R}^3), W^{1,p}(\mathbb{R}^3))), \\
\Pi_{\mathcal{R}_{\omega,\omega}}^{G,2}(\lambda) & \in \mathcal{A}(C_+, \mathcal{L}(L^p_{R+2}(\mathbb{R}^3), W^{1,p}(\mathbb{R}^3))).
\end{align*}
\]

Moreover for \( f \in L^p_{R+2}(\mathbb{R}^3) \) and \( 0 < \delta < 1/2 \), we have

\[
\| \nabla \Pi_{\mathcal{R}_{\omega,\omega}}^{G,1}(\lambda)f \|_{L^p(\mathbb{R}^3)} + \| \Pi_{\mathcal{R}_{\omega,\omega}}^{G,1}(\lambda)f \|_{L^p(\Omega_{R,\lambda})} \leq \frac{C_{\theta,\gamma,R,\omega^*}}{\| \lambda \|_{\frac{1}{2}}} \| f \|_{L^p(\mathbb{R}^3)}, \quad \lambda \in \Sigma_{\theta,\lambda_0},
\]

(2.60)

\[
\| \nabla \Pi_{\mathcal{R}_{\omega,\omega}}^{G,2}(\lambda)f \|_{L^p(\mathbb{R}^3)} + \| \Pi_{\mathcal{R}_{\omega,\omega}}^{G,2}(\lambda)f \|_{L^p(\Omega_{R,\lambda})} \leq \frac{C_{\gamma,\lambda_0,R,\omega^*}}{\| \lambda \|_{\frac{2}{2-\delta}}} \| f \|_{L^p(\mathbb{R}^3)}, \quad \lambda \in C_{+\gamma}.
\]

(2.61)

In the rest part in this section, we will study the behavior of operators \( \mathcal{R}_{\mathcal{R}_{\omega,\omega}}^G(\lambda) \) and \( \Pi_{\mathcal{R}_{\omega,\omega}}^G(\lambda) \) acting on \( L^p_{R+2}(\mathbb{R}^3) \) near \( R \lambda = 0 \) and \( |x| = \infty \). Set \( \Lambda^s f = \mathcal{F}^{-1}(\xi^s \hat{f}(\xi)) \). We start from some preliminary lemmas.

**Lemma 2.1.** Let \( \Gamma_{ij}(x,t) \) be the kernel function of \( T_{\mathcal{R}_{\omega,\omega}}^G(t)(\mathcal{P}_{R^2})_{ij} \). Then

\[
|\Lambda^s \Gamma_{ij}(x,t)| \leq \frac{C_s}{(t + |x + \Re e_1|^2)^{\frac{2s}{2}}, \quad (x,t) \in \mathbb{R}^3 \times [0, \infty) \setminus (0,0), \quad s \geq 0.
\]

(2.62)

**Proof.** Let \( \Gamma_0^s(x,t) \) be the kernel function of \( e^{-\Delta}(\mathcal{P}_{R^2})_{ij} \). Since \( \Gamma_{ij}(x,t) = \Gamma_0^s(x + t \Re e_1, t) \), we conclude (2.62) from the following classical estimate

\[
|\Lambda^s \Gamma_0^s(x,t)| \leq C_s(t + |x|^2)^{-\frac{2s}{2}}, \quad (x,t) \in \mathbb{R}^3 \times [0, \infty) \setminus (0,0), \quad s \geq 0.
\]

\[\square\]

**Lemma 2.2.** Let \( s, r \geq 0 \) with \( s + 2 - r > 0 \). Then

\[
J_{s,r}(x) \triangleq \int_0^\infty \frac{t^r}{(t + |x + \Re e_1|^2)^{(s+2)/2}} dt
\]

satisfies for \( \theta \in (0, \frac{1}{7}) \)

\[
J_{s,r}(x) \leq \mathcal{B}(\frac{r+1}{2}, 1 + \frac{s-r}{2}) \left\{ \begin{array}{ll}
2^{-1} |\mathcal{R}|^{-r-1}|x|^{-2-s+r}, & \text{if } |x| \leq \theta |\mathcal{R}|, \\
C_{\theta,r} |\mathcal{R}|^{2(r-s)-2} \frac{1}{|x|^{1+r/2}+(1+2|\mathcal{R}|s_{\mathcal{R}}(x))^{1+r/2}}, & \text{if } |x| > \theta |\mathcal{R}|,
\end{array} \right.
\]

(2.63)

where \( \mathcal{B}(\cdot, \cdot) \) denotes the Beta function. In particular, we have for \( |x| \leq \theta |\mathcal{R}|^{-1} \)

\[
J_{s,r}(x) \leq C_{\theta,r} \mathcal{B}(r + 1, \frac{s+1-2r}{2}) |x|^{1-s+2r}, \quad 1 + s - 2r > 0.
\]

(2.64)

To prove Lemma 2.2 we begin with some basic integral identities.
Lemma 2.3. Assume that $a, b > 0$ and $s \geq 0$, then
\[
\int_0^\infty \frac{t^r}{(at^2 + b)^{(3+s)/2}} \, dt = 2^{-1}b^{-\frac{2+s}{2}}a^{-\frac{s+1}{2}}B\left(\frac{r+1}{2}, \frac{s+2-r}{2}\right), \quad 0 \leq r < s + 2, \tag{2.65}
\]
\[
\int_0^\infty \frac{t^r}{(at + b)^{(3+s)/2}} \, dt = 2^{-1}b^{-\frac{4+s}{2}}a^{-\frac{s+1}{2}}B\left(\frac{r+1}{2}, \frac{s+2-r}{2}\right), \quad 0 \leq r < 1 + s. \tag{2.66}
\]

Proof. Since (2.65) implies (2.66), it suffices to prove (2.65). A simple computation yields that if $s + 2 - r > 0$,
\[
\int_0^\infty \frac{t^r}{(at^2 + b)^{(3+s)/2}} \, dt = b^{-\frac{2+s}{2}}a^{-\frac{s+1}{2}}\int_0^\pi/2 (\sin \theta)^r (\cos \theta)^{1+s-r} \, d\theta
\]
by making use of $t = \sqrt{a} \tan \theta$ in the first equality. 

Proof of Lemma 2.2. Obviously,
\[
t + |x + Re_1|^2 = R^2 t^2 + (1 + 2Rx_1) t + |x|^2.
\]

We will divide $x \in \mathbb{R}^3$ into three regions to prove this lemma.

**Region 1:** $|x| \leq \theta |\mathcal{R}|^{-1}$. In this region,
\[
t + |x - Re_1|^2 \geq R^2 t^2 + (1 - 2\theta) t + |x|^2.
\]

This inequality, together with Lemma 2.3 yields
\[
J_{s,r}(x) \leq \begin{cases} 2^{-1}B\left(\frac{r+1}{2}, 1 + \frac{s-r}{2}\right)|\mathcal{R}|^{-r-1}|x|^{-2-s+r}, \\
\quad C_{\theta,r}B\left(r + 1, \frac{s+1-2r}{2}\right)|x|^{-1-s+2r}, \quad r < s/2 + 1/2. 
\end{cases}
\]

**Region 2:** $|x| > \theta |\mathcal{R}|^{-1}$ and $1 + 2Rx_1 \geq 0$. Observing in this region that
\[
t + |x + Re_1|^2 \geq R^2 t^2 + |x|^2, \quad 1 + 2|R|s_R(x) \leq (\theta^{-1} + 4)|\mathcal{R}||x|,
\]
we get by (2.65)
\[
J_{s,r}(x) \leq 2^{-1}B\left(\frac{r+1}{2}, 1 + \frac{s-r}{2}\right)|\mathcal{R}|^{-r-1}|x|^{-2-s+r}
\]
\[
\quad \leq C_{\theta,s}B\left(\frac{r+1}{2}, 1 + \frac{s-r}{2}\right)|\mathcal{R}|^{\frac{r}{2}}|x|^{-1-s+r}(1 + 2|R|s_R(x))^{-1-\frac{r}{2}}.
\]

**Region 3:** $|x| > \theta |\mathcal{R}|^{-1}$ and $1 + 2Rx_1 < 0$. We have
\[
t + |x + Re_1|^2 = \left(|\mathcal{R}| t + \frac{1 + 2Rx_1}{2|R|}\right)^2 + \frac{4R^2 |x|^2 - (1 + 2Rx_1)^2}{4R^2}
\]
\[
\geq \left(|\mathcal{R}| t + \frac{1 + 2Rx_1}{2|R|}\right)^2 + \frac{(1 + 2|R|s_R(x)) |x|}{2|R|}.
\]

With the help of the above relation, we obtain
\[
J_{s,r}(x) \leq \int_0^\infty \left(\frac{t^r}{(|\mathcal{R}| t + \frac{1 + 2Rx_1}{2|R|})^2 + \frac{(1 + 2R|x|s_R(x))}{2|R|}}\right)^{(3+s)/2} \, dt.
\]
\[
= \frac{1}{|R|^{r+1}} \int_0^\infty \left( \frac{\tau - \frac{1+2[n]}{2|R|}}{\frac{1+2[n]}{2|R|}} \right)^r \frac{d\tau}{(\tau^2 + \frac{1+2[n]}{2|R|} |s_R(x)|^2)^{(3+s)/2}} \leq \frac{C_r}{|R|^{r+1}} \int_0^\infty \left( \frac{\tau - \frac{1+2[n]}{2|R|}}{\frac{1+2[n]}{2|R|}} \right)^r \frac{d\tau}{(\tau^2 + \frac{1+2[n]}{2|R|} |s_R(x)|^2)^{(3+s)/2}} + \frac{|x|^r}{\frac{1+2[n]}{2|R|}} \frac{d\tau}{(\tau^2 + \frac{1+2[n]}{2|R|} |s_R(x)|^2)^{(3+s)/2}}.
\]

Thus, we deduce by (2.65)
\[
J_{s,r}(x) \leq \frac{C_{r,2}(r+1, s+1-2r)/2}{|R|^{r+1}} \left( \frac{|R|^{1+\frac{s-r}{2}}}{(1+2[n])} \frac{d\tau}{\frac{1+2[n]}{2|R|} |s_R(x)|^{1+s-r/2}} + \frac{|R|^{1+\frac{s}{2}} \frac{d\tau}{\frac{1+2[n]}{2|R|} |s_R(x)|^{1+s/2}} \right)
\]
\[
\leq \frac{C_{\delta,r} |R|^{(s/2)-r} |x|^{-1(s/2)+r} (1 + 2 |\Re| s_R(x))^{-1(s/2)}}.
\]

Summing up, we complete the proof of Lemma 2.2

To briefly state our results, we define
\[
\|g\|_{X^{r}_{s,\ell}} \triangleq \sup_{|x| \geq \ell} |1+\frac{s-r}{2} (1 + s \Re(x))| g(x)|, \quad s, r \geq 0, \quad \ell > 0
\]
and denote by \((\Delta_h \ g)(\lambda) = g(\lambda + h) - g(\lambda)\) the difference quotient.

**Theorem 2.6.** Let \(1 < p < \infty, \epsilon \in (0, \frac{1}{2}), p \in (0, \frac{1}{2}), 0 < \Re \leq |\Re| \leq \Re^*\) and \(|\omega| \leq \omega^*\). Then, there exists a constant \(\eta = \eta_{R,p,\rho,\Re,\omega} > 0\) such that if \(\|\omega\|_{L_{p,\Omega}} \leq \eta\), then
\[
R^G_{\Re,\omega,\Omega}(\lambda) \subset C(\overline{C_+}, \mathcal{L}(L_{P,2}(R^3), \mathcal{W}^{p,2}(B_0(B_{R+2})))
\]
Moreover, for \(\lambda, \lambda + h \in \overline{C_+}\) and \(f \in L_{P,2}(R^3)\), we have for \(s \in [0, 2]
\]
\[
\|\Lambda^s \partial_l \mathcal{R}^G_{\Re,\omega,\Omega}(\lambda) f\|_{L_{p}(B_{R+2})} \leq C_{s,\Re,\rho,\Re^*,\omega} \|f\|_{L_{p}(R^3)}, \quad k = 0, 1, \quad (2.67)
\]
\[
\|\Lambda^s (\Delta_h \partial_l \mathcal{R}^G_{\Re,\omega,\Omega}(\lambda) f\|_{L_{p}(B_{R+2})} \leq C_{s,\Re,\rho,\Re^*,\omega} \|h^k\|_{L_{p}(R^3)}, \quad (2.68)
\]
and for \(s \in [0, 2]
\]
\[
\|\Lambda^s \partial_l \mathcal{R}^G_{\Re,\omega,\Omega}(\lambda) f\|_{X_{R+2}^{s,k}} \leq C_{s,\Re,\rho,\Re^*,\omega} \|f\|_{L_{p}(R^3)}, \quad k = 0, 1, \quad (2.69)
\]
\[
\|\Lambda^s (\Delta_h \partial_l \mathcal{R}^G_{\Re,\omega,\Omega}(\lambda) f\|_{X_{R+2}^{s,k}} \leq C_{s,\Re,\rho,\Re^*,\omega} \|h^k\|_{L_{p}(R^3)}, \quad (2.70)
\]
In particular, we have for \(j \leq 2, 0 < \rho \ll 1/2\) and \(0 < |h| \leq h_0\)
\[
\|\nabla^j \mathcal{R}^G_{\Re,\omega,\Omega}(\lambda) f\|_{L_{p}(B_{R+2})} \leq C_{\Re,\rho,\Re^*,\omega} \left( 1 + |\lambda| \right)^{-1+\frac{j}{2}} \|f\|_{L_{p}(R^3)}, \quad (2.71)
\]
\[
\|\nabla^j \partial_l \mathcal{R}^G_{\Re,\omega,\Omega}(\lambda) f\|_{L_{p}(B_{R+2})} \leq C_{\Re,\rho,\Re^*,\omega} \left( 1 + |\lambda| \right)^{-2+\frac{j}{2}} \|f\|_{L_{p}(R^3)}, \quad (2.72)
\]
\[
\|\nabla^j (\Delta_h \partial_l \mathcal{R}^G_{\Re,\omega,\Omega}(\lambda) f\|_{L_{p}(B_{R+2})} \leq C_{\Re,\rho,\Re^*,\omega} \left( 1 + |\lambda| \right)^{-2+\frac{j}{2}} \|f\|_{L_{p}(R^3)}. \quad (2.73)
\]

**Proof.** We always assure that \(\rho \in (0, \frac{1}{2}), \lambda, \lambda + h \in \overline{C_+}\) and \(f \in L_{P,2}(R^3)\). Define
\[
\mathcal{K}_j(\lambda) \triangleq (\lambda I + \mathcal{L}_{\Re,0,0,\Omega})^{-1} \mathcal{P}(\lambda I + \mathcal{L}_{\Re,0,0,\Omega})^{-1} \mathcal{P}_{\Omega} f, \quad j \geq 0
\]
Obviously,
\[
\mathcal{R}^G_{\Re,\omega,\Omega}(\lambda) = \sum_{j=0}^{\infty} \mathcal{K}_j(\lambda), \quad \mathcal{K}_j(\lambda) = -\mathcal{K}_0(\lambda) B_{\Omega} \mathcal{K}_{j-1}(\lambda), \quad j \geq 1
\]
In the course of the proof, we shall repeatedly use the fact
\[ \|g\|_{L^1(\mathbb{R}^3)} \leq C(q, R)\|g\|_{L^p(\mathbb{R}^3)}, \quad g \in L^q_{R+2}(\mathbb{R}^3), \ 1 \leq q \leq \infty. \]

**Part 1: Proof of (2.67) - (2.70).** It suffices to prove that there exist two positive constants
\[ M_1 = C_{R, R^3, \mathfrak{A}^*}, \quad M_2 = C_{\rho, R, \mathfrak{A}^*} \]
such that for every \( s \in [0, 2], \ k = 0, 1 \) and \( \lambda \geq 0, \)
\[
\|\Lambda^*(\partial_{\lambda}^s K_\lambda)(\lambda) f\|_{L^p(B_{\mathfrak{A}^*}(\mathbb{R}^3))} \leq C_{s, R, \mathfrak{A}^*, \mathfrak{A}^*}(M_1\|w\|_{e, \Omega})^2\|f\|_{L^p(\mathbb{R}^3)},
\]
\[
\|\Lambda^*(\partial_{\lambda}^s K_\lambda)(\lambda) f\|_{X_{s, k}^{R}(\mathbb{R}^3)} \leq C_{s, R, \mathfrak{A}^*, \mathfrak{A}^*}(M_1\|w\|_{e, \Omega})^2\|f\|_{L^p(\mathbb{R}^3)}, \quad s \neq 2,
\]
\[
\|\Lambda^*(\Delta_h \partial_{\lambda} K_\lambda)(\lambda) f\|_{L^p(B_{\mathfrak{A}^*}(\mathbb{R}^3))} \leq C_{\rho, R, \mathfrak{A}^*, \mathfrak{A}^*}(M_2\|w\|_{e, \Omega})^2\|f\|_{L^p(\mathbb{R}^3)},
\]
\[
\|\Lambda^*(\Delta_h \partial_{\lambda} K_\lambda)(\lambda) f\|_{X_{s, 1, \mathfrak{A}^*}^{R}(\mathbb{R}^3)} \leq C_{\rho, R, \mathfrak{A}^*, \mathfrak{A}^*}(M_2\|w\|_{e, \Omega})^2\|f\|_{L^p(\mathbb{R}^3)}, \quad s \neq 2. \tag{2.76}
\]

In fact, (2.67) - (2.70) follow from (2.75) - (2.76) provided that
\[ (M_1 + M_2)\|w\|_{e, \Omega} < 1. \tag{2.77} \]

Now we will establish the iterative scheme on account of the idea of the scale decomposition, to prove (2.76). We agree that \( \lesssim \) and \( \lesssim_{a, b, \ldots} \) both depends on \( R, \mathfrak{A}, \mathfrak{A}^* \).

**Step 1: Preliminary analysis of \( K_j(\lambda) \).**

**Case 1: \( j = 0 \).** Define
\[ [I_{s, r}g](x) \triangleq \int_{\mathbb{R}^3} K_{s, r}(x, y)g(y)dy, \quad K_{s, r}(x, y) \triangleq \int_0^\infty \frac{t^r}{(t + |Q(x) y - \mathfrak{A} e_1|e^x)} dt. \]

Thanks to the facts: \( K_0(\lambda) = (\lambda I + \mathcal{L}_{\mathfrak{A}, \mathfrak{A}^*, \mathfrak{A}^*})^{-1}P_{\mathbb{R}^3} \) and
\[ |e^{-\lambda t} - e^{-\lambda x}| \leq 2^{1-\delta}\delta|\lambda_1 - \lambda_2\delta|, \quad \lambda_1, \lambda_2 \in \mathbb{C}_+, \quad \delta \in [0, 1], \]
we get by (2.41) and Lemma 2.1 that, for every \( s \in [0, 2] \) and \( k = 0, 1 \)
\[ |\Lambda^*(\partial_{\lambda}^s K_\lambda)(\lambda) g(x)| \leq C_0|I_{s, 1} g(x)|, \quad |\Lambda^*(\partial_{\lambda} \partial_{\lambda}^s K_\lambda)(\lambda) g(x)| \leq C_0|\partial_{\lambda} g|I_{s, k+2} g(x). \tag{2.78} \]

Hence, we deduce (2.76) with \( j = 0 \) except the case \( (s, k) = (2, 0) \) by making use of the following claim which will be proved in Step 3 below.

**Claim 1:** For every \( s \in [0, 2], \ r \in [0, \frac{3}{2}] \) and \( g \in L^q_{\ell_1}(\mathbb{R}^3) \) with \( \ell_1 \geq 1 \) and \( q \in (1, \infty), \)
\[
\|I_{s, r} g\|_{L^p(B_{\ell_2})} \leq C_1\|g\|_{L^q(\mathbb{R}^3)}, \quad \ell_2 > 0, \quad (s, r) \neq (2, 0), \tag{2.79}
\]
\[
\|I_{s, r} g\|_{X_{s, r, \ell_3}} \leq C_2\|g\|_{L^q(\mathbb{R}^3)}, \quad \ell_3 > \ell_1, \quad s \neq 2. \tag{2.80}
\]

with \( C_1 = C_{s, r, \ell_1+\ell_2, a_*, \mathfrak{A}^*} \) and \( C_2 = C_{s, r, \ell_2/2, a_*, \mathfrak{A}^*}. \)

For case \( (s, k) = (2, 0) \), we adopt the idea in (23) to get
\[ \|\Lambda^2 K_0(\lambda) g\|_{L^p(B_{\ell_2})} \leq C_3\|g\|_{L^p(\mathbb{R}^3)}, \quad g \in L^p_{\ell_1}(\mathbb{R}^3), \quad \ell_1, \ell_2 > 0 \tag{2.81} \]
for some positive constant \( C_3 = C_{\ell_1+\ell_2, \mathfrak{A}^*, \mathfrak{A}^*} \), see also \( [32, 33] \).

**Case 2:** \( j \geq 1 \). In spite of \( K_j(\lambda) = K_0(\lambda)B_{\mathfrak{A}^*} K_{j-1}(\lambda) \) for \( j \geq 1 \), we can not iterate \( K_{j-1}(\lambda) f \) to estimate \( K_j(\lambda) f \) by (2.79) - (2.81) since \( K_{j-1}(\lambda) f \) does not have compact
support. But, we can do it by the decay estimate \( K_{j-1} (\lambda) f \) at large scale of \(|x|\). Precisely, we decompose \( K_j (\lambda) \) with \( \ell_i \geq R + 2 \) as

\[
K_j (\lambda) = -K_0 (\lambda) \chi_{B_{\ell_i}} B_{\overline{\nu}} K_{j-1} (\lambda) - K_0 (\lambda) \chi_{B_{\ell_i}} B_{\overline{\nu}} K_{j-1} (\lambda) \triangleq K_{j-1}^{\text{in}, \ell_i} (\lambda) + K_{j-1}^{\text{out}, \ell_i} (\lambda). \quad (2.82)
\]

We first deal with \( K_{j-1}^{\text{in}, \ell_i} (\lambda) \). By the Leibniz rule and \( 2.75 \), we have

\[
\left\{ \begin{array}{l}
|\Lambda^2 K_j^{\text{in}, \ell_i} (\lambda) f | \leq \left| [\Lambda^2 K_0 (\lambda) (\chi_{B_{\ell_i}} B_{\overline{\nu}} K_{j-1} (\lambda)) (\lambda) f ] (x) \right|,
|\Lambda^2 (\partial_\lambda K_j^{\text{in}, \ell_i} (\lambda) f | \leq \left| [\Lambda^2 K_0 (\lambda) (\chi_{B_{\ell_i}} B_{\overline{\nu}} (\partial_\lambda K_{j-1} (\lambda)) (\lambda) f ]) (x) \right| \\
+ C_0 [I_{2,1} (\chi_{B_{\ell_i}} B_{\overline{\nu}} K_{j-1} (\lambda) f )] (x),
\end{array} \right.
\]

\[
|\Lambda^2 (\Delta_h \partial_\lambda K_j^{\text{in}, \ell_i} (\lambda) f | \leq \left| [\Lambda^2 K_0 (\lambda) (\chi_{B_{\ell_i}} B_{\overline{\nu}} (\Delta_h \partial_\lambda K_{j-1} (\lambda)) (\lambda) f )] (x) \right| \\
+ C_0 [I_{2,1} (\chi_{B_{\ell_i}} B_{\overline{\nu}} (\Delta_h K_{j-1} (\lambda) f )] (x),
\]

and for every \( s \in [0,2) \) and \( k = 0,1, \)

\[
\left\{ \begin{array}{l}
|\Lambda^s (\partial_\lambda^k K_j^{\text{in}, \ell_i} (\lambda) f | \leq C_0 \left( \sum_{r=0}^k [I_{s,r} (\chi_{B_{\ell_i}} B_{\overline{\nu}} (\partial_\lambda^{k-r} K_{j-1} (\lambda) (\lambda) f ))] (x) \right),
|\Lambda^s (\Delta_h \partial_\lambda^k K_j^{\text{in}, \ell_i} (\lambda) f | \leq C_0 \left( \sum_{r=0}^k \left[ |h|^\rho [I_{s,r} (\chi_{B_{\ell_i}} B_{\overline{\nu}} (\partial_\lambda^{k-r} K_{j-1} (\lambda) (\lambda) f ))] (x) \right] \\
+ [I_{s,r} (\chi_{B_{\ell_i}} B_{\overline{\nu}} (\Delta_h \partial_\lambda^{k-r} K_{j-1} (\lambda) (\lambda) f ))] (x) \right). \end{array} \right.
\]

These estimates help us to iterate \( K_{j-1} (\lambda) \) to estimate \( K_j^{\text{in}, \ell_i} (\lambda) \) by \( 2.79 \)-\( 2.81 \).

Now we consider \( K_{j-1}^{\text{out}, \ell_i} (\lambda) \). Denote \( C_4 = C_{\ell_2} \) the uniformly constant such that

\[
\|g\|_{L^q (B_{\ell_2})} \leq C_4 \|g\|_{L^\infty (B_{\ell_2})}, \quad q \in [1, \infty].
\]

For every \( s \in [0,2], \ r_1, r_2 \geq 0 \) with \( r_1 + r_2 < \frac{3}{2} \), define

\[
\Pi_{s,r_1,r_2}^\ell (x) \triangleq \int_{|y| \geq \ell_1} \frac{J_{s,r_1} (x-y)}{\| y \|^{(5/2)-r_2} (1 + s \| y \|)^{(3/2)-\varepsilon}} \, dy, \quad \ell_1 > 0, \ \varepsilon \in (0, \frac{1}{2}).
\]

We observe that for every radial function \( g \),

\[
[I_{s,r} h] (x) \leq \int_{R^3} |J_{s,r} (x-y)| g (y) \, dy, \quad |h (x)| \leq |g (x)|.
\]

Hence, we have for \( k = 0,1 \)

\[
\left\{ \begin{array}{l}
|\Lambda^s (\partial_\lambda^k K_j^{\text{out}, \ell_i} (\lambda) f | \leq C_0 \|w\|_{L^\infty} \sum_{m=0}^k \sum_{r=0}^k \| \nabla^m (\partial_\lambda^k K_{j-1} (\lambda) (\lambda) f ) \|_{X_{\ell_1}^{m-r} \Pi_{s,r_1,r_2}^\ell (x)},
|\Lambda^s (\Delta_h \partial_\lambda^k K_j^{\text{out}, \ell_i} (\lambda) f | \leq \sum_{m=0}^k \sum_{r=0}^k \left[ |h|^\rho \| \nabla^m (\partial_\lambda^k K_{j-1} (\lambda) (\lambda) f ) \|_{X_{\ell_1}^{m-r} \Pi_{s,r_1,r_2}^\ell (x)} + \| \nabla^m (\Delta_h \partial_\lambda^k K_{j-1} (\lambda) (\lambda) f ) \|_{X_{\ell_1}^{m-r} \Pi_{s,r_1,r_2}^\ell (x)} \right]. \end{array} \right.
\]

(2.85)
This implies that we can iterate $K_{j-1}(\lambda)$ to estimate $K_{j}^{\text{out, fil}}(\lambda)$ by the following claim which will be proved later in Step 3.

**Claim 2** For $0 \leq r_1 + r_2 < \frac{3}{2}$,
\[ \Pi_{s,r_1,r_2}^\ell (x) \leq C_5 (1 + |x|)^{-\beta + r_1 + r_2 (1 + s_\beta(x))^{-1}}, \quad s \in [0, 2), \quad x \in \mathbb{R}^3, \quad (2.86) \]
with $C_5 = C_{s,r_1,r_2,\mathcal{A}_s,\mathcal{A}_r}$, and
\[ \Pi_{2,1,1}^\ell (x) \leq C_6 = C_{r_1,\ell_1,\ell_2/s_\beta,\mathcal{A}_s,\mathcal{A}_r}, \quad |x| \leq \ell_2 < \ell_1. \quad (2.87) \]

**Step 2: Iterative scheme and the proof of (2.70)**

For $j = 0$, according to (2.78), we get (2.76) by (2.79)-(2.81) with $\ell_1 = R + 2$ and $\ell_2 = 9(R + 2)$.

Next, we are in position to prove (2.76) for $j \geq 1$. When $s \in [0, 2)$, in the light of (2.82) and (2.84)-(2.85) with $\ell_1 = 7(R + 2)$, we deduce by (2.79)-(2.80) and (2.86) with $\ell_2 = 9(R + 2)$, that for $s \in [0, 2)$ and $k = 0, 1$
\[ \left\{ \begin{array}{l}
\| \Lambda^s (\partial_{\Lambda}^k K_j)(\lambda) f \|_{L^p(B_{9(R+2)})} + \| \Lambda^s (\partial_{\Lambda}^k K_j)(\lambda) f \|_{X_{\mathcal{A}_s,9(R+2)}} \\
\lesssim s \| w \|_{\ell,\Omega} \sum_{r=0}^{k} \left[ \| (\partial_{\Lambda}^k K_{j-1})(\lambda) f \|_{W^{1,p}(B_{7(R+2)})} + \sum_{m=0}^{1} \| \nabla^m (\partial_{\Lambda}^k K_{j-1})(\lambda) f \|_{X_{\mathcal{A}_s,7(R+2)}} \right], \\
\| \Lambda^s (\Delta_h \partial_{\Lambda}^k K_j)(\lambda) f \|_{L^p(B_{9(R+2)})} + \| \Lambda^s (\Delta_h \partial_{\Lambda}^k K_j)(\lambda) f \|_{X_{\mathcal{A}_s,9(R+2)}} \\
\lesssim s \| w \|_{\ell,\Omega} \sum_{r=0}^{k} \left[ \| (\Delta_h \partial_{\Lambda}^k K_{j-1})(\lambda) f \|_{W^{1,p}(B_{7(R+2)})} \right] + \sum_{m,r=0}^{1} \left( |h|^{r} \| \nabla^m (\partial_{\Lambda}^k K_{j-1})(\lambda) f \|_{X_{\mathcal{A}_s,7(R+2)}} + \| \nabla^m (\Delta_h \partial_{\Lambda}^k K_{j-1})(\lambda) f \|_{X_{\mathcal{A}_s,7(R+2)}} \right) \right\}. \quad (2.88)\]

When $s = 2$, we choose $\ell_1 = 10(R+2)$ and $\ell_2 = 9(R+2)$. Then, according to (2.82)-(2.85) and (2.87), we get by (2.79), (2.81), (2.87) and the fact
\[ \| g \|_{X_{10(R+2)}^{s,r}} \leq \| g \|_{X_{7(R+2)}^{s,r}}, \quad \| g \|_{L^p(B_{10(R+2)})} \leq \| g \|_{L^p(B_{7(R+2)})} + C_R \| g \|_{X_{7(R+2)}^{s,r}}, \]
that for $k = 0, 1$
\[ \left\{ \begin{array}{l}
\| \Lambda^2 (\partial_{\Lambda}^k K_j)(\lambda) f \|_{L^p(B_{9(R+2)})} \\
\lesssim \omega^s \| w \|_{\ell,\Omega} \sum_{r=0}^{k} \left[ \| (\partial_{\Lambda}^k K_{j-1})(\lambda) f \|_{W^{1,p}(B_{7(R+2)})} + \sum_{m=0}^{1} \| \nabla^m (\partial_{\Lambda}^k K_{j-1})(\lambda) f \|_{X_{\mathcal{A}_s,7(R+2)}} \right], \\
\| \Lambda^2 (\Delta_h \partial_{\Lambda}^k K_j)(\lambda) f \|_{L^p(B_{9(R+2)})} \\
\lesssim \rho \omega^s \| w \|_{\ell,\Omega} \sum_{r=0}^{k} \left[ \| (\Delta_h \partial_{\Lambda}^k K_{j-1})(\lambda) f \|_{W^{1,p}(B_{7(R+2)})} \right] + \sum_{m,r=0}^{1} \left( |h|^{r} \| \nabla^m (\partial_{\Lambda}^k K_{j-1})(\lambda) f \|_{X_{\mathcal{A}_s,7(R+2)}} + \| \nabla^m (\Delta_h \partial_{\Lambda}^k K_{j-1})(\lambda) f \|_{X_{\mathcal{A}_s,7(R+2)}} \right) \right\}. \quad (2.89)\]
In view of iterative inequalities (2.88)-(2.89), the proof of (2.76) for $j \geq 1$ can be reduced to proving that, there exist $\widetilde{M}_1 = C_{R,R_\ast,\Omega_*}$ and $\widetilde{M}_2 = C_{\rho,R,R_\ast,\Omega_*}$ such that for all $k, m = 0, 1,$

\[
\begin{align*}
\left\{ \begin{array}{l}
\| (\partial^k_1 K_j)(\lambda) f \|_{W^{1,p}(B_7)} \leq C_R \widetilde{M}_1 \| w \|_{\mathcal{E},\Omega}^j \| f \|_{L^p(R^3)}, \\
\| \nabla^m (\partial^k_1 K_j)(\lambda) f \|_{X^{m,k}_{7}(B_{7/2})} \leq C_R \widetilde{M}_1 \| w \|_{\mathcal{E},\Omega}^j \| f \|_{L^p(R^3)}, \\
\| (\triangle \partial^k_1 K_j)(\lambda) f \|_{W^{1,p}(B_{7/2})} \leq C_R \widetilde{M}_2 (\| \nabla w \|_{\mathcal{E},\Omega}^j) \| f \|_{L^p(R^3)}, \\
\| \nabla^m (\triangle \partial^k_1 K_j)(\lambda) f \|_{X^{m,k}_{7}(B_{7/2})} \leq C_R \widetilde{M}_2 (\| \nabla w \|_{\mathcal{E},\Omega}^j) \| f \|_{L^p(R^3)},
\end{array} \right. \\
\leq \widetilde{M}_1 \| w \|_{\mathcal{E},\Omega}^j \sum_{k=0}^1 \left[ \| (\partial^k_1 K_j)(\lambda) f \|_{W^{1,p}(B_{7/2})} + \sum_{m=0,1} \| \nabla^m (\partial^k_1 K_j)(\lambda) f \|_{X^{m,k}_{7}(B_{7/2})} \right],
\end{align*}
\]

which inserted into (2.88)-(2.89), implies (2.76) with $M_1 = 2\widetilde{M}_1 + M_2$ and $M_2 = 2\widetilde{M}_1 + M_2$.

To prove (2.90), we give out the following claim which will be proved in Step 3.

**Claim 3:** There exists a constant $C_7 = C_{r,\ell_1+\ell_2,\Omega_*,\Omega^*} > 0$ such that for $g \in \mathbb{L}^q_{\ell_1}(R^3)$

\[
\begin{align*}
\left\{ \begin{array}{l}
\| I_{s,r} g \|_{L^q_{\ell_1}(B_{\ell_2})} \leq C_7 \| g \|_{L^q(R^3)}, \quad (s, r) \in [0, 1] \times [0, 3/2), \quad 1 < q < 4, \\
\| I_{s,r} g \|_{L^q_{\ell_1}(B_{\ell_2})} \leq C_7 \| g \|_{L^q(R^3)}, \quad (s, r) \in [0, 1] \times [0, 3/2), \quad 4 \leq q \leq \infty.
\end{array} \right.
\]

Then we choose a sequence $\{(\ell_{1,j}, \ell_{2,j})\}_{j=0}^{\infty}$ such that $R + 2 = \ell_{1,0} \leq \ell_{1,j} < \ell_{2,j}$ for $j \geq 0$. Let $p_{-1} \triangleq p$ and construct a sequence $\{p_j\}_{j=0}^{\infty}$ with

\[
p_j = \frac{4p_{j-1}}{4-p_{j-1}} \quad \text{if} \quad 1 < p_{j-1} < 4; \quad p_j = \infty, \quad \text{if} \quad 4 \leq p_{j-1} \leq \infty.
\]

Set

\[
C_{2,j} \triangleq C_2|_{\ell_1=\ell_{1,j},\ell_2=\ell_{2,j}}, \quad C_{4,j} \triangleq C_4|_{\ell_2=\ell_{2,j}}, \quad C_{7,j} \triangleq C_7|_{\ell_1=\ell_{1,j},\ell_2=\ell_{2,j}}.
\]

Inserting (2.80), (2.86) and (2.91) into (2.78) and (2.84)-(2.85), we obtain for $k, m = 0, 1$

\[
\begin{align*}
\sum_{k=0}^1 \left[ \| (\partial^k_1 K_0)(\lambda) f \|_{W^{1,p_0}(B_{2,\ell_2})} + \sum_{m=0,1} \| \nabla^m (\partial^k_1 K_0)(\lambda) f \|_{X^{m,k}_{2,0}} \right] \leq \widetilde{M}_{1,j} \| f \|_{L^p(R^3)}, \\
\sum_{k=0,1} \left[ \| (\partial^k_1 K_j)(\lambda) f \|_{W^{1,p_j}(B_{\ell_2})} + \sum_{m=0,1} \| \nabla^m (\partial^k_1 K_j)(\lambda) f \|_{X^{m,k}_{\ell_2}} \right] \\
\leq \widetilde{M}_{1,j} \| w \|_{\mathcal{E},\Omega}^j \sum_{k=0}^1 \left[ \| (\partial^k_1 K_{j-1})(\lambda) f \|_{W^{1,p_{j-1}}(B_{1,j})} + \sum_{m=0,1} \| \nabla^m (\partial^k_1 K_{j-1})(\lambda) f \|_{X^{m,k}_{1,j}} \right],
\end{align*}
\]
and
\[
\begin{align*}
\sum_{k=0,1} \left[ \sum_{m=0,1} \left[ \left( \Delta_h \partial_h^J \mathcal{K}_0(\lambda) f \right) \|_{W^{1,p}(B_{t_2,0})} + \sum_{m=0,1} \left| \nabla^m (\Delta_h \partial_h^J \mathcal{K}_0(\lambda) f) \|_{X^{m,k+r}_{t_2,0}} \right] \right] \right]
\leq \tilde{M}_{2,j} \left[ \left| \sum_{k=0,1} \left[ \sum_{m=0,1} \left[ \left( \left( \Delta_h \partial_h^J \mathcal{K}_j(\lambda) f \right) \|_{W^{1,p}(B_{t_2,0})} + \sum_{m=0,1} \left| \nabla^m (\Delta_h \partial_h^J \mathcal{K}_j(\lambda) f) \|_{X^{m,k+r}_{t_2,0}} \right) \right] \right] \right] \right],
\end{align*}
\]
(2.94)

where
\[
\begin{align*}
\tilde{M}_{1,j} &= 4C_0 \sup_{s \in [0,1]} \sum_{k=0,1} \left( (C_{7,j} + C_{2,j})_{r=1-k} + (1 + C_{4,j})(C_5 |_{r_1=k} + C_5 |_{r_2=k}) \right), \\
\tilde{M}_{2,j}(\rho) &= 4C_0 \sup_{s \in [0,1]} \sum_{k=0,1} \left( (C_{7,j} + C_{2,j})_{r=1-k} + (C_{7,j} + C_{2,j})_{r=1-k+\rho} \right) + (1 + C_{4,j})(C_5 |_{r_1=k} + C_5 |_{r_1=k+\rho} + C_5 |_{r_2=0} + C_5 |_{r_2=k+\rho}),
\end{align*}
\]

Since \( p_j = \infty \) if \( j \geq 3 \), invoking that
\[
\| g \|_{X_{s,r,t_1,j}} \leq \| g \|_{X_{s,r,t_2,j}} + C_{8,j} \| g \|_{L^\infty(B_{t_1,j},t_2,j)}, \quad C_{8,j} \triangleq C_{\ell_2,j},
\]
we improve (2.93)-(2.94) for \( j \geq 4 \) to be
\[
\begin{align*}
\sum_{k=0,1} \left[ \sum_{m=0,1} \left[ \left( \left( \Delta_h \partial_h^J \mathcal{K}_j(\lambda) f \right) \|_{W^{1,p}(B_{t_2,0})} + \sum_{m=0,1} \left| \nabla^m (\Delta_h \partial_h^J \mathcal{K}_j(\lambda) f) \|_{X^{m,k}_{t_2,0}} \right) \right] \right] \right]
\leq 2(1 + C_{8,j}) \tilde{M}_{1,j} \| g \|_{e,\Omega} \sum_{r=0,1} \left[ \sum_{m=0,1} \left[ \left( \left( \Delta_h \partial_h^J \mathcal{K}_{r-1}(\lambda) f \right) \|_{W^{1,p-1}(B_{t_2,0})} + \sum_{m=0,1} \left| \nabla^m (\Delta_h \partial_h^J \mathcal{K}_{r-1}(\lambda) f) \|_{X^{m,k}_{t_2,0}} \right) \right] \right] \right]
\end{align*}
\]
(2.95)

and
\[
\begin{align*}
\sum_{k=0,1} \left[ \sum_{m=0,1} \left[ \left( \Delta_h \partial_h^J \mathcal{K}_j(\lambda) f \right) \|_{W^{1,p}(B_{t_2,0})} + \sum_{m=0,1} \left| \nabla^m (\Delta_h \partial_h^J \mathcal{K}_j(\lambda) f) \|_{X^{m,k+r}_{t_2,0}} \right) \right] \right]
\leq 2(1 + C_{8,j}) \tilde{M}_{2,j} \| \|_{e,\Omega} \sum_{r=0,1} \left[ \sum_{m=0,1} \left[ \left( \left( \Delta_h \partial_h^J \mathcal{K}_{r-1}(\lambda) f \right) \|_{W^{1,p-1}(B_{t_2,0})} + \sum_{m=0,1} \left| \nabla^m (\Delta_h \partial_h^J \mathcal{K}_{r-1}(\lambda) f) \|_{X^{m,k+r}_{t_2,0}} \right) \right] \right] \right]
\end{align*}
\]
(2.96)

To close the iterative scheme (2.93)-(2.96), we choose
\[
\begin{align*}
\ell_{1,j} &= (2j + 1)(R + 2) \quad \text{if } j \leq 3, \quad \ell_{1,j} = 7(R + 2) \quad \text{if } j \geq 4, \\
\ell_{2,j} &= \ell_{1,j} + 2(R + 2),
\end{align*}
\]
such that \( \widetilde{M}_{1,j} \) and \( \widetilde{M}_{2,j} \) are uniformly bounded in \( j \). So we denote
\[
\widetilde{M}_1 = 2 \max_{j \geq 0} (1 + C_{8,j}) \widetilde{M}_{1,j}, \quad \widetilde{M}_2(\rho) = 2 \max_{j \geq 0} (1 + C_{8,j}) \widetilde{M}_{2,j}.
\]

From (2.93) and (2.95), we deduce by induction
\[
\begin{align*}
\sum_{k=0}^3 \left[ \| (\partial_k^h \mathcal{K}_j)(\lambda) f \|_{W^{1,p_j}(B_{\ell_{2,j}})} + \sum_{m=0,1} \| \nabla^m (\partial_k^h \mathcal{K}_j)(\lambda) f \|_{X^{m,k}_{\ell_{2,j}}} \right] \\
\leq \widetilde{M}_1(\widetilde{M}_i \| \omega \|_{\varepsilon,\Omega})^j \| f \|_{L^p(\mathbb{R}^3)}
\end{align*}
\tag{2.97}
\]
with
\[
\begin{align*}
p_j &= \infty, \quad \forall j \geq 0, \quad \text{if} \quad 4 \leq p < \infty, \\
p_0 &= \frac{4p}{4-p}, \quad p_j = \infty, \quad \forall j \geq 1, \quad \text{if} \quad 2 \leq p < 4, \\
p_0 &= \frac{4p}{4-2p}, \quad p_1 = \infty, \quad \forall j \geq 2, \quad \text{if} \quad \frac{3}{2} \leq p < 2, \\
p_0 &= \frac{4p}{4-2p}, \quad p_2 = \frac{4p}{4-3p}, \quad p_j = \infty, \quad \forall j \geq 3, \quad \text{if} \quad 1 < p < \frac{4}{3}.
\end{align*}
\]

Inserting (2.97) into (2.94) and (2.96), we get for \( j \geq 1 \)
\[
\begin{align*}
\sum_{k=0}^3 \left[ \| (\Delta_h \partial_k^h \mathcal{K}_j)(\lambda) f \|_{W^{1,p_j}(B_{\ell_{2,j}})} + \sum_{m=0,1} \| \nabla^m (\Delta_h \partial_k^h \mathcal{K}_j)(\lambda) f \|_{X^{m,k}_{\ell_{2,j}}} \right] \\
\leq \widetilde{M}_2|\omega|^p(\widetilde{M}_1 \| \omega \|_{\varepsilon,\Omega})^j \| f \|_{L^p(\mathbb{R}^3)} + \widetilde{M}_2 \| \omega \|_{\varepsilon,\Omega} \sum_{r=0,1} \| (\Delta_h \partial_r^h \mathcal{K}_{j-1})(\lambda) f \|_{W^{1,p_j-1}(B_{\ell_{3,j}})} \\
+ (1/2) \widetilde{M}_2 \| \omega \|_{\varepsilon,\Omega} \sum_{k,m=0,1} \| \nabla^m (\Delta_h \partial_k^h \mathcal{K}_{j-1})(\lambda) f \|_{X^{m,r}_{\ell_{3,j}}},
\end{align*}
\]
with \( \ell_{3,j} = \ell_{1,j} \) if \( j \leq 3 \) and \( \ell_{3,j} = \ell_{1,j} \) if \( j \geq 4 \). This iteration scheme, together with the first inequality of (2.94), yields that for \( k = 0, 1 \) and \( j \geq 0 \)
\[
\begin{align*}
\sum_{k=0}^3 \left[ \| (\Delta_h \partial_k^h \mathcal{K}_j)(\lambda) f \|_{W^{1,p_j}(B_{\ell_{2,j}})} + \sum_{m=0,1} \| \nabla^m (\Delta_h \partial_k^h \mathcal{K}_j)(\lambda) f \|_{X^{m,k}_{\ell_{2,j}}} \right] \\
\leq 2\widetilde{M}_2(\widetilde{M}_1 + \widetilde{M}_2) \| \omega \|_{\varepsilon,\Omega} \| f \|_{L^p(\mathbb{R}^3)},
\end{align*}
\]
which together with (2.97) implies (2.90).

**Step 3. Proof of Claim 1-Claim 3.**

**Proof of (2.79) and (2.91).** Set \( g \in L^p_{\ell_1} \) and \((s, r) \in [0, 2] \times [0, \frac{3}{2}] \) with \((s, r) \neq (2, 0)\). Since \( g = 0 \) in \( B_{\ell_2} \), we can rewrite
\[
K_{s,r}(x, y) = K^1_{s,r}(x, y) + K^2_{s,r}(x, y)
\]
\[
\Delta \left[ \int_{2(\ell_1 + \ell_2)/|\mathcal{R}|}^{t \chi_{|y| \leq \ell_1}} \int_0^{2(\ell_1 + \ell_2)/|\mathcal{R}|} (t + |Q(\omega t)x - y + \Re e_1|^2)^{(3+s)/2} dt \right].
\]

Obviously, when \( t > 2(\ell_1 + \ell_2)/|\mathcal{R}| \)
\[
|Q(\omega t)x - y + \Re e_1|^2 \geq (x_1 - y_1 + \Re t)^2 \geq \Re^2 t^2/4, \quad x \in B_{\ell_2}, \quad y \in B_{\ell_1}.
\]
Hence, we have for \( x \in B_{\ell_2} \) and \( y \in B_{\ell_1} \),
\[
K_{s,r}^1(x, y) \leq C \int_{2(\ell_1 + \ell_2)/|\mathbb{R}|}^{\infty} \left( \frac{t^r}{(t + |\mathbb{R}|^2t^2)^{3s/2}} \right) dt \leq \frac{C|\mathbb{R}|^{-r-1}}{(\ell_1 + \ell_2)^{2s-r}}.
\]
This inequality implies
\[
\sup_{x \in B_{\ell_2}} \|K_{s,r}^1(x, \cdot)\|_{L^q(B_{\ell_1})} \leq C \frac{|\mathbb{R}|^{-1-\ell_1}}{(\ell_1 + \ell_2)^{2s-\ell_1}}.
\]
For \( K_{s,r}^2(x, y) \), by the Minkowski inequality, we have
\[
\|K_{s,r}^2(x, \cdot)\|_{L^q(B_{\ell_1})} \leq \int_0^{2(\ell_1 + \ell_2)/|\mathbb{R}|} \left( \int_{|y| \leq \ell_1} \left( t + |\mathbb{Q}(\omega t)x - y + \mathbb{R}te_1|^2 \right)^{-\frac{3s}{2}} dy \right)^{\frac{1}{q}} dt.
\]
Observing for \( x \in B_{\ell_2} \) and \( t \leq 2(\ell_1 + \ell_2)/|\mathbb{R}| \)
\[
\left( \int_{|y| \leq \ell_1} \left( t + |\mathbb{Q}(\omega t)x - y + \mathbb{R}te_1|^2 \right)^{-\frac{3s}{2}} dy \right)^{\frac{1}{q}} \leq \left( \int_{|y| \leq 2(\ell_1 + \ell_2)} \left( t + |z|^2 \right)^{-\frac{3s}{2}} dy \right)^{\frac{1}{q}} \leq C_{s,q} \frac{t^{\frac{3s}{2} - \frac{3s}{2}}}{r}
\]
we have for \( q \in [1, \infty) \) with \( s < 2r + \frac{3}{q} - 1 \)
\[
\sup_{x \in B_{\ell_2}} \|K_{s,r}^2(x, \cdot)\|_{L^q(B_{\ell_1})} \leq C_{s,q} \int_0^{2(\ell_1 + \ell_2)/|\mathbb{R}|} t^{r + \frac{3s}{2} - \frac{3s}{2}} dt \leq C_{s,q,r} |\mathbb{R}|^{\frac{s+1}{2} - r - \frac{3}{2}} (\ell_1 + \ell_2)^{\frac{3}{2} - \frac{1-s}{2}}.
\]
Hence, in the light of \( K_{s,r}(x, y) = K_{s,r}(y, x) \), we obtain that (2.79) and (2.91) by Lemma 8.2 via choosing \( q = 1 \) or \( q = \frac{4}{3} \), respectively.

**Proof of (2.80).** Let \( g \in L^p_{\ell_1}(\mathbb{R}^3) \). We observe that for every \( x, y \in \mathbb{R}^3 \)
\[
|\mathbb{O}(\omega t)x - y + \mathbb{R}te_1| = |x - \mathbb{O}^T(\omega t)y + \mathbb{R}te_1| \geq |x - y^* + \mathbb{R}te_1| \quad \text{for all } t \geq 0
\]
where \( y^* = (y_1^*, y_2^*, y_3^*) \) satisfies
\[
|y^*| = |y|, \quad y_1^* = y_1, \quad x_2y_3^* = x_3y_2^*.
\]
Hence, we have
\[
[I_{s,r}g](x) \leq \int_{\mathbb{R}^3} J_{s,r}(x - y^*)|g(y)| dy, \quad (s, r) \in [0, 2) \times [0, 2).
\]
Since
\[
|x - y^*| \geq \ell_2 - \ell_1 |x| \geq \ell_2 - \ell_1, \quad x \in B_{\ell_2}^c, \quad y \in B_{\ell_1}, \quad \ell_2 > \ell_1,
\]
we get by (2.61)
\[
[I_{s,r}g](x) \leq \int_{B_{\ell_1}} \frac{C_{s,r,\ell_2,\ell_1,\ell_1-1} |\mathbb{R}|^{s-2-r} |g(y)|}{|x - y^*|^{1+\frac{s}{2} - r}(1 + 2|\mathbb{R}|s\mathbb{Q}(x - y^*))^{1+\frac{s}{2}}} dy, \quad x \in B_{\ell_2}^c.
\]
This inequality, combining with
\[
s\mathbb{Q}(x) \leq s\mathbb{Q}(x - y^*) + s\mathbb{Q}(y^*) \leq s\mathbb{Q}(x - y^*) + 2|y^*|,
\]
yields for all \( x \in B_{\ell_2}^c \)
\[
(1 + s\mathbb{Q}(x))[I_{s,r}g](x)
\]
Collecting the above two estimates, we obtain for
\[ x \mid s \]
Next, we deal with II 
\[ s,r \]
\[ 1 + \left( 1 + s(q(x - y)) \right) + 2 \ell_1 \frac{(1 + s(q(x - y)))}{|y|^{\frac{d}{2} - r}} \frac{1}{(1 + s(q(x - y)))^{\frac{d}{2} - r}} |y|^{\frac{1}{2} - r} \|g\| \leq C_{s,r,\mathfrak{H},q,\varepsilon,2,(\ell_2 - \ell_1)^{-1}} \|g\| \leq C_{s,r,\mathfrak{H},q,\varepsilon,2,(\ell_2 - \ell_1)^{-1}} \|g\|_{L^p(\mathbb{R}^3)}.
\]
This ends the proof of (2.80).

**Proof of (2.80).** Let \( s \in [0, 2) \), \( r_1 + r_2 \in (0, 3/2) \) and \( \varepsilon \in (0, \frac{1}{2}) \). We decompose
\[ \Pi_{s,r_1,r_2}^{1}(x) = \Pi_{s,r_1,r_2}^{1,1}(x) + \Pi_{s,r_1,r_2}^{1,2}(x) \]
\[ \Delta = \int_{B_{x_1} \cap B_{(2\|x\|^{-1})}(x)} J_{s,r_1}(-x) \frac{C_{s,r_1,\mathfrak{H},q,\varepsilon}}{|x - y|^{2 + s - r_1}} \frac{1}{|y|^{\frac{d}{2} - r_2} (1 + s(q(x - y)))^{\frac{d}{2} - r_2}} dy.
\]
Let us begin with \( \Pi_{s,r_1,r_2}^{1,1}(x) \). When \( 2 + s - r_1 < 3 \), by (2.63) and the fact that
\[ 1 + s(q(x)) \leq 1 + s(q(x - y)) + s(q(y)) \leq (1 + 1/|\mathfrak{H}|)(1 + s(q(y))), \quad |x - y| \leq \frac{1}{2|\mathfrak{H}|}, \]
we have
\[ \Pi_{s,r_1,r_2}^{1,1}(x) \leq \int_{B_{x_1} \cap B_{(2|\mathfrak{H}|^{-1})}(x)} C_{s,r_1,\mathfrak{H},q,\varepsilon} \frac{1}{|x - y|^{2 + s - r_1}} \frac{1}{|y|^{\frac{d}{2} - r_2} (1 + s(q(x)))^{\frac{d}{2} - r_2}} dy
\]
which implies that
\[ \|\Pi_{s,r_1,r_2}^{1,1}\|_{L^\infty(\mathbb{R}^3)} \leq \int_{|x| \leq 2|\mathfrak{H}|^{-1}} C_{s,r_1,r_2,1,\mathfrak{H},q,\varepsilon} \frac{1}{|x|^{2 + s - r_1}} dy \leq C_{s,r_1,r_2,1,\mathfrak{H},q,\varepsilon}.
\]
Further, according to that \( \frac{3|y|}{2} \leq |y| \leq \frac{3|y|}{2} \) if \( |y| \leq \frac{1}{\mathfrak{H}} \), we get from (2.98)
\[ \Pi_{s,r_1,r_2}^{1,1}(x) \leq \int_{|y| \leq 2|\mathfrak{H}|^{-1}} C_{s,r_1,r_2,\mathfrak{H},q,\varepsilon} \frac{1}{|y|^{\frac{d}{2} - r_2} (1 + s(q(x)))^{\frac{d}{2} - r_2}} \frac{1}{|x - y|^{2 + s - r_1}} dy
\]
\[ \leq C_{s,r_1,r_2,\mathfrak{H},q,\varepsilon} \frac{1}{|y|} |x|^{-(5/2) + r_2} (1 + s(q(x)))^{-(3/2) + \varepsilon}.
\]
Collecting the above two estimates, we obtain for \( x \in \mathbb{R}^3 \)
\[ \Pi_{s,r_1,r_2}^{1,1}(x) \leq C_{s,r_1,r_2,\mathfrak{H},q,\varepsilon} (1 + |x|)^{-\frac{d}{2} + r_2} (1 + s(q(x)))^{-\frac{d}{2} + \varepsilon}, \quad 2 + s - r_1 < 3. \]
When \( 2 + s - r_1 \geq 3 \), we get by (2.64)
\[ \Pi_{s,r_1,r_2}^{1,1}(x) \leq \int_{B_{x_1} \cap B_{(2|\mathfrak{H}|^{-1})}(x)} C_{s,r_1,\mathfrak{H},q,\varepsilon} \frac{1}{|x - y|^{1 + s - r_2}} \frac{1}{|y|^{\frac{d}{2} - r_2} (1 + s(q(y)))^{\frac{d}{2} - r_2}} dy.
\]
Hence, in same way as deducing (2.99), we get for \( x \in \mathbb{R}^3 \)
\[ \Pi_{s,r_1,r_2}^{1,1}(x) \leq C_{s,r_1,r_2,\mathfrak{H},q,\varepsilon} (1 + |x|)^{-(5/2) + r_2} (1 + s(q(x)))^{-(3/2) + \varepsilon}, \quad 2 + s - r_1 \geq 3. \]
Next, we deal with \( \Pi_{s,r_1,r_2}^{1,2}(x) \). Noting that
\[ \frac{1 + |y|}{1 + s(q(y))} \leq \min\{1, |\mathfrak{H}|\} \quad \text{for} \quad y \in \mathbb{R}^3; \quad \frac{1 + |x - y|}{|x - y|} \leq 1 + 2|\mathfrak{H}| \quad \text{for} \quad |x - y| \geq \frac{1}{2|\mathfrak{H}|},
\]
we have by (2.63)
\[ \Pi_{s,r_1,r_2}^{1,2}(x) \]
\[
\leq \int_{B_{\ell_1}^c \cap B_{\ell_2/(2|y|)}^c(x)} \frac{C_{s,r_{1,\mathcal{R}_{1,\mathcal{R}^*}}}}{(1 + |x-y|)^{1+s_0(x-y)}(1 + s_0(x-y))^{1+s_0(y))^{1-\frac{1}{2}}} \, dy.
\]

This, combining with the fact that
\[
\frac{|x-y|}{2} \leq |y| \leq \frac{3|x-y|}{2} \quad \text{for } |x-y| \geq 2|x|,
\]
and Lemma 8.3 yields
\[
\|P_{s,r_{1,2}} \|_{L^\infty(B_{1/(4|y|)})} \leq \int_{B_{\ell_1}^c} \frac{C_{s,r_{1,\mathcal{R}_{1,\mathcal{R}^*}}}}{|y|^{1+s_0(x-y)}(1 + s_0(x-y))^{1+s_0(y))^{1-\frac{1}{2}}} \, dy \leq C_{s,r_{1,2,\mathcal{R}_{1,\mathcal{R}^*}}}
\]

On the other hand, for \( |x| \geq \frac{1}{4|y|} \), we make the following decomposition
\[
P_{s,r_{1,2}}(x) \leq \left[ \int_{B_{\ell_1}^c \cap B_{\ell_2/(2|y|)}^c(x)} + \int_{B_{\ell_1}^c \cap B_{\ell_2/(2|y|),2|y|}^c(x)} \right] \frac{1}{(1 + |x-y|)^{1+s_0(x-y)}(1 + s_0(x-y))^{1+s_0(y))^{1-\frac{1}{2}}} \, dy \\
\leq J_1(x) + J_2(x).
\]

Since \( s_0(x) \leq s_0(x-y) + s_0(y) \), we obtain by (2.101) and Lemma 8.3
\[
(1 + s_0(x))J_1(x) \leq \int_{B_{\ell_1}^c \cap B_{\ell_2/(2|y|)}^c(x)} \frac{C_{s,r_{1,2,\mathcal{R}_{1,\mathcal{R}^*}}}}{|y|^{1+s_0(x-y)}(1 + s_0(x-y))^{1+s_0(y))^{1-\frac{1}{2}}} \, dy
\]
\[
+ \int_{B_{\ell_1}^c \cap B_{\ell_2/(2|y|),2|y|}^c(x)} \frac{C_{s,r_{1,2,\mathcal{R}_{1,\mathcal{R}^*}}}}{|y|^{1+s_0(x-y)}(1 + s_0(x-y))^{1+s_0(y))^{1-\frac{1}{2}}} \, dy
\]
\[
\leq C_{s,r_{1,2,\mathcal{R}_{1,\mathcal{R}^*}}} |x|^{-1+s_0(x)}.
\]

For \( J_2(x) \), with the help of Lemma 3.1 in [3], we have
\[
J_2(x) \leq \int_{B_{\ell_1}^c \cap B_{\ell_2/(2|y|),2|y|}^c(x)} \frac{C_{s,r_{1,2,\mathcal{R}_{1,\mathcal{R}^*}}}}{|x|^{1+s_0(x-y)}(1 + s_0(x-y))^{1+s_0(y))^{1-\frac{1}{2}}} \, dy
\]
\[
\leq C_{s,r_{1,2,\mathcal{R}_{1,\mathcal{R}^*}}} |x|^{-1+s_0(x)}.
\]

Summing up, we get
\[
P_{s,r_{1,2}}(x) \leq C_{s,r_{1,2,\mathcal{R}_{1,\mathcal{R}^*}}}(1 + |x|)^{-1+s_0(x)}(1 + s_0(x))^{-1}, \quad x \in \mathbb{R}^3.
\]

Collecting (2.99)-(2.100) and (2.102), we prove (2.86).

**Proof of (2.87).** Noting that
\[
\frac{\ell_1 - \ell_2}{\ell_1} |y| \leq |y - x| \leq \frac{\ell_1 + \ell_2}{\ell_1} |y|, \quad |x| \leq \ell_2, \quad |y| \leq \ell_1, \quad 0 < \ell_2 < \ell_1,
\]
we get by Lemma 2.2 and Lemma 8.3 that for \( r_1 + r_2 \in [0, 3/2) \)
\[
P_{s,r_{1,2}}(x) \leq \int_{|y| \geq \ell_1} \frac{C_{s,r_{1,\mathcal{R}_{1,\mathcal{R}^*}}}}{|x - y|^{2+s_0(x-y)}(1 + s_0(x-y))^{1+s_0(y))^{1-\frac{1}{2}}} \, dy
\]
\[
\leq \int_{|y| \geq \ell_1} \frac{C_{s,r_{1,2,\mathcal{R}_{1,\mathcal{R}^*}}}}{|y|^{1+s_0(x-y)}(1 + s_0(y))^{1+s_0(y))^{1-\frac{1}{2}}} \, dy \leq C_{s,r_{1,2,\mathcal{R}_{1,\mathcal{R}^*}}}.
\]

Hence, we conclude (2.87), and so complete the proof of Claim 1-Claim 3.
Part 2: Proof of (2.71) - (2.73). In view of (2.67) - (2.68), we only consider the case $|\lambda| \geq 2h_0$. For the sake of statement, we agree that $\lesssim$ and $\lesssim_{a,b,\ldots}$ both still depend on $p, R, R_s, \mathcal{R}^*, \omega^*$ and set

$$E_j \triangleq (M_1 \|w\|_{\epsilon, \Omega})^j, \quad F_j \triangleq ((M_1 + M_2)\|w\|_{\epsilon, \Omega})^j$$

where $M_1$ and $M_2$ are the constants in (2.76). Thanks to (2.75) and (2.77), we only need to prove that for all $|\beta| \leq 2$ and $0 < \delta \ll 1/2$

$$(2.103)$$

$$
\begin{cases}
\|\partial^2_x \mathcal{K}_j(\lambda) f\|_{L^p(B_{2h_0}(R^3))} \leq C_{R, R_s, R^*, \omega^*} |\lambda|^{-1 + \beta/2} E_j \|f\|_{L^p(R^3)}, \\
\|\partial^2_x (\partial_{\lambda} \mathcal{K}_j)(\lambda) f\|_{L^p(B_{2h_0}(R^3))} \leq C_{R, R_s, R^*, \omega^*} |\lambda|^{-2 + \beta/2 + \delta} E_j \|f\|_{L^p(R^3)}, \\
\|\partial^2_x (\Delta_h \partial_{\lambda} \mathcal{K}_j)(\lambda) f\|_{L^p(B_{2h_0}(R^3))} \leq C_{R, R_s, R^*, \omega^*} |h|^\delta |\lambda|^{-2 + \beta/2 + \delta} F_j \|f\|_{L^p(R^3)}.
\end{cases}
$$

To prove (2.103), we will adopt the "tree self-similar iterative" idea.

Step 1: Preliminary analysis of $\partial^2_x \mathcal{K}_j(\lambda)$.

Case 1: $|\beta| = 0$. Let $\mathcal{M} \triangleq \mathcal{L}_{9, \omega, 0, R^3} - \omega [e_1 \times] = -\Delta - R \partial_1 - \omega (e_1 \times x) \cdot \nabla$, where $[e_1 \times]^k \triangleq e_1 \times e_1 \times \cdots e_1 \times$, for every integer $k > 0$.

In the light of (2.72) and the fact that

$$(\lambda I + \mathcal{L}_{9, \omega, 0, R^3})^{-1} \mathcal{P}_{R^3} = \lambda^{-1} \mathcal{P}_{R^3} - \lambda^{-1} \mathcal{L}_{9, \omega, 0, R^3} (\lambda + \mathcal{L}_{9, \omega, 0, R^3})^{-1} \mathcal{P}_{R^3},$$

we can rewrite $\mathcal{K}_j(\lambda)$ as

$$\mathcal{K}_j(\lambda) = \begin{cases}
\frac{1}{\lambda} \left( \mathcal{P}_{R^3} - \mathcal{M} \mathcal{K}_0(\lambda) \right) - \frac{\omega}{\lambda} e_1 \times \mathcal{K}_0(\lambda), & j = 0, \\
\frac{1}{\lambda} \left( \mathcal{P}_{R^3} B_{\omega} \mathcal{K}_{j-1}(\lambda) + \mathcal{M} \mathcal{K}_j(\lambda) \right) - \frac{\omega}{\lambda} e_1 \times \mathcal{K}_j(\lambda), & j \geq 1.
\end{cases} \quad (2.104)$$

Iterating (2.104), we obtain

$$\mathcal{K}_j(\lambda) = \begin{cases}
\left( \frac{1}{\lambda} - \frac{\omega |e_1 \times|^2}{\lambda^2} \right) \left( \mathcal{P}_{R^3} - \mathcal{M} \mathcal{K}_0(\lambda) \right) + \frac{\omega^2 |e_1 \times|^2}{\lambda^2} \mathcal{K}_0(\lambda), & j = 0, \\
\frac{\omega^2 |e_1 \times|^2}{\lambda^2} \mathcal{K}_j(\lambda) - \left( \frac{1}{\lambda} - \frac{\omega |e_1 \times|^2}{\lambda^2} \right) \left( \mathcal{P}_{R^3} B_{\omega} \mathcal{K}_{j-1}(\lambda) + \mathcal{M} \mathcal{K}_j(\lambda) \right), & j \geq 1.
\end{cases}$$

Hence, by making use of (2.76) and the mean value theorem, we deduce

$$\begin{cases}
\|\mathcal{K}_0(\lambda) f\|_{L^p(B_{2h_0}(R^3))} \lesssim |\lambda|^{-1} \|f\|_{L^p(R^3)}, \\
\|\partial_{\lambda} \mathcal{K}_0(\lambda) f\|_{L^p(B_{2h_0}(R^3))} \lesssim |\lambda|^{-2} \|f\|_{L^p(R^3)} + |\lambda|^{-1} \|\mathcal{M} (\partial_{\lambda} \mathcal{K}_0(\lambda) f)\|_{L^p(B_{2h_0}(R^3))}, \\
\|\Delta_h \partial_{\lambda} \mathcal{K}_0(\lambda) f\|_{L^p(B_{2h_0}(R^3))} \lesssim |h| |\lambda|^{-2} \|f\|_{L^p(R^3)} + |\lambda|^{-1} \|\mathcal{M} (\Delta_h \partial_{\lambda} \mathcal{K}_0(\lambda) f)\|_{L^p(B_{2h_0}(R^3))}.
\end{cases} \quad (2.105)$$
and for all $j \geq 1$

\[
\begin{align*}
&\left\| K_j(\lambda) f \right\|_{L^p(B_{\theta}(R^{r+2}))} \\
\leq & |\lambda|^{-1} E_j \left\| f \right\|_{L^p(R^3)} + |\lambda|^{-1} \left\| \mathcal{P}_{R^3} B_{\bar{\nu}} K_{j-1}(\lambda) f \right\|_{L^p(B_{\theta}(R^{r+2}))} \\
&\left\| (\partial_\lambda K_j)(\lambda) f \right\|_{L^p(B_{\theta}(R^{r+2}))} \\
\leq & |\lambda|^{-2} E_j \left\| f \right\|_{L^p(R^3)} + |\lambda|^{-2} \left\| \mathcal{P}_{R^3} B_{\bar{\nu}} K_{j-1}(\lambda) f \right\|_{L^p(B_{\theta}(R^{r+2}))} \\
&+ |\lambda|^{-2} \left\| (\mathcal{M}(\partial_\lambda K_j)(\lambda) f, \mathcal{P}_{R^3} B_{\bar{\nu}} (\partial_\lambda K_{j-1})(\lambda) f) \right\|_{L^p(B_{\theta}(R^{r+2}))},
\end{align*}
\]

(2.106)

Next, we deal with the nonlocal terms invoking $B_{\bar{\nu}}$ in (2.106). Since

\[B_{\bar{\nu}}(\partial_\lambda K_{j-1})(\lambda) f, \ B_{\bar{\nu}}(\Delta h \partial_\lambda K_{j-1})(\lambda) f \not\in L^p(R^3),\]

here we will adopt the “divide and conquer” argument, such as dividing $g$ into

\[g = \chi_{B_{\theta}(R^{r+2})} g + \chi_{B_{\theta}(R^{r+2})^c} g,\]

to obtain for every $T \in \mathcal{L}(L^p(R^3)) \cap \mathcal{L}(L^q(R^3))$

\[
\left\| T g \right\|_{L^p(B_{\theta}(R^{r+2}))} \leq C_{\rho, q, R} \left\| T \right\|_{\mathcal{L}(L^p(R^3)) \cap \mathcal{L}(L^q(R^3))} \left\| g \right\|_{L^p(R^3)},
\]

(2.107)

where $L^p_q(R^3) \triangleq L^p(B_{\theta}(R^{r+2})) \cap L^q(B_{\theta}(R^{r+2}))$.

From (1.25) and (2.76), we have for $q_1 \triangleq \max\{p, 3\}$

\[
\begin{align*}
&\left\| B_{\bar{\nu}} K_{j-1}(\lambda) f \right\|_{L^p_q(R^3)} + \left\| B_{\bar{\nu}} (\partial_\lambda K_{j-1})(\lambda) f \right\|_{L^p_q(R^3)} \lesssim E_j \left\| f \right\|_{L^p(R^3)}, \\
&\left\| B_{\bar{\nu}} (\Delta h \partial_\lambda K_{j-1})(\lambda) f \right\|_{L^p_q(R^3)} \lesssim |h|^p E_j \left\| f \right\|_{L^p(R^3)}.
\end{align*}
\]

(2.108)

This, together with (2.107), implies

\[
\begin{align*}
&\left\| \mathcal{P}_{R^3} (\partial_\lambda K_{j-1})(\lambda) f \right\|_{L^p_q(R^3)} \lesssim E_j \left\| f \right\|_{L^p(R^3)}, \quad k = 0, 1, \\
&\left\| \mathcal{P}_{R^3} B_{\bar{\nu}} (\Delta h \partial_\lambda K_{j-1})(\lambda) f \right\|_{L^p_q(R^3)} \lesssim |h|^p E_j \left\| f \right\|_{L^p(R^3)}.
\end{align*}
\]

(2.109)

On the other hand, in the light of (2.104), we know that

\[B_{2\bar{\nu}} K_{j-1}(\lambda) = \begin{cases} 
\frac{1}{2} B_{2\bar{\nu}} P_{R^3} - \frac{1}{2} B_{2\bar{\nu}} L_{\theta, \omega} \omega_0 R^3 K_0(\lambda) + \frac{1}{2} B_{2\bar{\nu}} \Delta K_0(\lambda), & j = 1, \\
- \frac{1}{2} B_{2\bar{\nu}} P_{R^3} B_{2\bar{\nu}} K_{j-2}(\lambda) - \frac{1}{2} B_{2\bar{\nu}} L_{\theta, \omega} \omega_0 R^3 K_{j-1}(\lambda), & j \geq 2.
\end{cases}
\]

(2.110)

Set $q_2 \triangleq \max\{6, p\}$. We conclude by (1.25) and (2.76) that

\[
\begin{align*}
&\left\| B_{2\bar{\nu}} (\Delta (\partial_\lambda K_{j-1})(\lambda) f) \right\|_{L^p_{q_2}(R^3)} \lesssim E_j \left\| f \right\|_{L^p(R^3)}, \quad k = 0, 1, \\
&\left\| B_{2\bar{\nu}} (\Delta (\partial_\lambda K_{j-1})(\lambda) f) \right\|_{L^p_{q_2}(R^3)} \lesssim |h|^p E_j \left\| f \right\|_{L^p(R^3)}.
\end{align*}
\]

(2.111)
In view of (2.105) and (2.112)-(2.113), the proof of (2.103) for 

This, combining with (2.108), gives from (2.107)

Plugging the above set of inequalities and (2.109) into (2.106), we conclude for \( j \geq 1 \)

and

In view of (2.105) and (2.112)-(2.113), the proof of (2.103) for \( |\beta| = 0 \) is reduced to the case \( |\beta| \neq 0 \). For details, see Step 3 below.

**Case 2:** \( |\beta| = 1 \). We know

where \( \mathcal{A}_{i}^{N}(\lambda) (i = 1, 2) \) is defined in (2.42). By a simple calculation, we have

By Leibniz rule, we have for \( \ell \geq 1 \),

\[
\partial_{t}^{\ell} \left( \mathcal{O}^{T}(\omega t) \mathcal{P}(\xi) \hat{g}(\xi) e^{i \mathcal{O}(\omega t) x} \xi \right) = \sum_{k=0}^{\ell} \binom{k}{\ell} (-\omega)^{k} \mathcal{O}^{T}(\omega t) [e_{1} \times]^{k} \mathcal{P}(\xi) \hat{g}(\xi) \partial_{t}^{\ell-k} (e^{i \mathcal{O}(\omega t) x} \xi). 
\]
In addition, we get for $k \geq 1$,
\[
\partial_t^k (e^{iO(\omega)t}x) = \omega^k e^{iO(\omega)t}x \left((i(e_1 \times O(\omega)t) \cdot \xi)^k + \sum_{k_1 + \cdots + k_n = k, n \geq 2 \atop k_{n-1} + \cdots + k_1 \geq 2} d_{k_1, \ldots, k_n} (i([e_1 \times]^{k_1}O(\omega)t) \cdot \xi) \cdots (i([e_1 \times]^{k_{n-1}}O(\omega)t) \cdot \xi) (i([e_1 \times]^{k_n}O(\omega)t) \cdot \xi)^k_n \right).
\]
Hence, setting
\[
G_j(\lambda, \xi) = (\lambda + |\xi|^2 - i\Re \xi_1)^{-j}P(\xi), \quad j \geq 1,
\]
\[
H(\lambda, \xi) = e^{-\lambda T}(at)F(T_{\Re,0}(t)P_{\Re}g)(O(atx)) (\xi),
\]
we deduce that
\[
A_1^N(\lambda)g = \sum_{\ell=0}^{N-1} \sum_{|\alpha|=|\gamma|=\ell} \sum_{j=0}^{j} \frac{i^\ell |\omega|^j}{(2\pi)^3} c_{\alpha,\gamma}^j \int_{\mathbb{R}^3} x^\alpha \xi^\gamma G_j+1(\lambda, \xi)g(\xi)e^{ix \cdot \xi} d\xi
\]
\[
= \sum_{\ell=0}^{N-1} \sum_{|\alpha|=|\gamma|=\ell} |\omega|^\ell c_{\alpha,\gamma}^\ell x^\alpha \partial_x^\gamma \frac{1}{(\lambda - \Delta - \Re \partial_1)^{\ell+1}} P_{\Re}^3 g
\]  \hspace{1cm} (2.114)

and
\[
A_2^N(\lambda)g = \sum_{\ell=0}^{N} \sum_{|\alpha|=|\gamma|=\ell} \frac{i^m |\omega|^N}{(2\pi)^3} c_{\alpha,\gamma}^N \int_{0}^{\infty} \int_{\mathbb{R}^3} x^\alpha \xi^\gamma G_N(\lambda, \xi)H(\lambda, \xi)e^{ix \cdot \xi} d\xi dt
\]
\[
= |\omega|^N \sum_{\ell=0}^{N} \sum_{|\alpha|=|\gamma|=\ell} c_{\alpha,\gamma}^N x^\alpha \partial_x^\gamma \frac{1}{(\lambda - \Delta - \Re \partial_1)^N} \eta (\lambda + \mathcal{L}_{\Re,0,\Re}^{-1}) P_{\Re}^3 g.
\]
So, we decompose
\[
\mathcal{K}_j(\lambda) = \mathcal{K}_{1,j}^N(\lambda) + \mathcal{K}_{2,j}^N(\lambda), \quad j \geq 0, \quad N \geq 1, \hspace{1cm} (2.115)
\]
with
\[
\mathcal{K}_{1,j}^N(\lambda) \triangleq A_1^N(\lambda)(-B_{\mathcal{L}}(\lambda + \mathcal{L}_{\Re,0,\Re}^{-1}) P_{\Re}^3)^j,
\]
\[
\mathcal{K}_{2,j}^N(\lambda) \triangleq A_2^N(\lambda)(-B_{\mathcal{L}}(\lambda + \mathcal{L}_{\Re,0,\Re}^{-1}) P_{\Re}^3)^j
\]
\[
= |\omega|^N \sum_{\ell \leq N} \sum_{|\alpha|=|\gamma|=\ell} c_{\alpha,\gamma}^N x^\alpha \partial_x^\gamma \frac{1}{(\lambda - \Delta - \Re \partial_1)^N} \mathcal{K}_j(\lambda).
\]
Let us begin with $\mathcal{K}_{2,j}^N(\lambda)$. Since $|\beta| = 1, 2$, we observe that for all $j \geq 0$
\[
\partial^\beta_x \mathcal{K}_{2,j}^N(\lambda)f = |\omega|^N \sum_{\ell=0}^{N} \sum_{|\alpha|=|\gamma|=\ell+|\beta|} c_{\alpha,\gamma}^N x^\alpha \partial_x^\gamma (\lambda - \Delta - \Re \partial_1)^{-N} \mathcal{K}_j(\lambda)f
\]
\[
= |\omega|^N \sum_{|\alpha|=0, |\gamma|=|\beta|} c_{\alpha,\gamma}^N \partial_x^\gamma \Lambda^{-\max\{|\beta|-2\phi,1\}} (\lambda - \Delta - \Re \partial_1)^{-N} \Lambda^{\max\{|\beta|-2\phi,1\}} \mathcal{K}_j(\lambda)f
\]
\[ + |\omega|^N \sum_{\ell=1}^{N} \sum_{[\alpha]=\ell,|\gamma|=\ell+|\beta|} c_{\alpha,\gamma}^N x^\alpha \partial_x^\gamma \Lambda^{-2+2\theta}(\lambda - \Delta - 2\mathfrak{R}\partial_1)^{-N} \Lambda^{2-2\theta} \mathcal{K}_J(\lambda) f \]

\[ \triangle T_{1,\beta,\beta}^N(\lambda) \Lambda^{\max\{1,\beta\} - 2\theta} \mathcal{K}_J(\lambda) f + T_{2,\beta,\beta}^N(\lambda) \Lambda^{2-2\theta} \mathcal{K}_J(\lambda) f. \]

It is well known that

\[ \| \partial_2^\alpha \Lambda^{-\gamma} \partial_\lambda^\beta (\lambda - \Delta - 2\mathfrak{R}\partial_1) \|_{L^p(R^3)} \leq C_{2\gamma} |\lambda|^{-k - \ell + \frac{|\alpha| - \gamma}{2}}, \quad 0 \leq |\alpha| - \gamma \leq 2\ell. \quad (2.116) \]

This estimate gives for \( k \leq 2 \)

\[ \left\{ \begin{array}{l}
\| (\partial_\lambda^k T_{1,\beta,\beta}^N(\lambda)) \|_{L^p(R^3)} \leq C_{N,2\gamma} |\lambda|^{-k - N + \theta}, \\
\| (1 + |x|)^{-N} (\partial_\lambda^k T_{2,\beta,\beta}^N(\lambda)) \|_{L^p(R^3)} \leq C_{N,2\gamma} |\lambda|^{-k - N + \theta}.
\end{array} \right. \]

Moreover, from (2.76), we observe for \( s \in \{1, 2\} \)

\[ \left\{ \begin{array}{l}
\| \Lambda^s \mathcal{K}_J(\lambda) f \|_{L^p(R^3)} + \| \Lambda^s (\partial_\lambda^k \mathcal{K}_J(\lambda) f) \|_{L^p(R^3)} \lesssim_{s,p} E_j f \|_{L^p(R^3)}, \\
\| \Lambda^s (\partial_\lambda^k \mathcal{K}_J(\lambda) f) \|_{L^p(R^3)} \lesssim_{s,p} |h|^p F_j f \|_{L^p(R^3)}.
\end{array} \right. \quad (2.117) \]

with \( q_3 \triangleq \max\{p, 6\} \) and \( q_4 \triangleq \max(p, \frac{3}{(1/2)-p}) \). Thus by (2.107) we get for \( j \geq 0 \),

\[ \left\{ \begin{array}{l}
\| T((1 + |x|)^{-N} \partial_\lambda^k (\partial_\lambda^k \mathcal{K}_J^N(\lambda) f)) \|_{L^p(B_{\rho}(R^3))} \\
\lesssim_{N,\theta} |\lambda|^{-(N+2-|\beta|)/2+\theta} \| T \|_{L^p(B_{\rho}(R^3)) \cap L^p(L^1(R^3))} E^N f \|_{L^p(R^3)}, \quad k = 0, 1, \\
\| T((1 + |x|)^{-N} \partial_\lambda^k (\partial_\lambda \mathcal{K}_J^N(\lambda) f)) \|_{L^p(B_{\rho}(R^3))} \\
\lesssim_{N,\theta} |h|^p |\lambda|^{-(N+2-|\beta|)/2+\theta} \| T \|_{L^p(B_{\rho}(R^3)) \cap L^p(L^1(R^3))} F^N f \|_{L^p(R^3)}.
\end{array} \right. \quad (2.118) \]

Now we consider \( \mathcal{K}^N_{J,1}(\lambda) \). In view of (2.114) and (2.116), we deduce for \( k \in \mathbb{N} \)

\[ \left\{ \begin{array}{l}
\| T((1 + |x|)^{-N} \partial_\lambda^k (\partial_\lambda^k \mathcal{K}^N_{J,1}(\lambda) f)) \|_{L^p(B_{\rho}(R^3))} \\
\lesssim_{N,\theta} |\lambda|^{-(N+1+|\beta|)/2} \| T \|_{L^p(B_{\rho}(R^3)) \cap L^p(L^1(R^3))} E^N f \|_{L^p(R^3)}, \\
\| T((1 + |x|)^{-N} \partial_\lambda^k (\partial_\lambda \mathcal{K}^N_{J,1}(\lambda) f)) \|_{L^p(B_{\rho}(R^3))} \\
\lesssim_{N,\theta} |\lambda|^{-(N+1+|\beta|)/2} \| T \|_{L^p(B_{\rho}(R^3)) \cap L^p(L^1(R^3))} F^N f \|_{L^p(R^3)}
\end{array} \right. \quad (2.119) \]

When \( j \geq 1 \), since \( \mathcal{K}^N_{J,1}(\lambda) = -A^N(\lambda) B_{\mathfrak{R}} \mathcal{K}_{J-1}(\lambda) \), we have by (2.107), (2.108) and (2.119)

\[ \left\{ \begin{array}{l}
\| T((1 + |x|)^{-N+1} \partial_\lambda^k \mathcal{K}^N_{J,1}(\lambda) f)) \|_{L^p(B_{\rho}(R^3))} \\
\lesssim_{N,\theta} |\lambda|^{-1+|\beta|/2} \| T \|_{L^p(B_{\rho}(R^3)) \cap L^p(L^1(R^3))} E^N f \|_{L^p(R^3)}, \\
\| T((1 + |x|)^{-N+1} \partial_\lambda^k (\partial_\lambda \mathcal{K}^N_{J,1}(\lambda) f)) \|_{L^p(B_{\rho}(R^3))} \\
\lesssim_{N,\theta} |\lambda|^{-1+|\beta|/2} \| T \|_{L^p(B_{\rho}(R^3)) \cap L^p(L^1(R^3))} F^N f \|_{L^p(R^3)} \\
+ \| T((1 + |x|)^{-N+1} \partial_\lambda^k A^N(\lambda) B_{\mathfrak{R}} (\partial_\lambda \mathcal{K}_{J-1}(\lambda) f)) \|_{L^p(B_{\rho}(R^3))}, \\
\| T((1 + |x|)^{-N+1} \partial_\lambda^k (\partial_\lambda^k \mathcal{K}^N_{J,1}(\lambda) f)) \|_{L^p(B_{\rho}(R^3))} \\
\lesssim_{N,\theta} |h|^p |\lambda|^{-2+(|\beta|/2)} \| T \|_{L^p(B_{\rho}(R^3)) \cap L^p(L^1(R^3))} F^N f \|_{L^p(R^3)} \\
+ \| T((1 + |x|)^{-N+1} \partial_\lambda^k A^N(\lambda) B_{\mathfrak{R}} (\partial_\lambda^k \mathcal{K}_{J-1}(\lambda) f)) \|_{L^p(B_{\rho}(R^3))},
\end{array} \right. \quad (2.120) \]
Meanwhile, we have
\[
\begin{align*}
& \left\| T((1 + |x|)^{- N + 1} \partial_x^\beta A_1^N(\lambda) B_{j, R}(\partial_x \mathcal{K}_{j - 1})(\lambda \mathbf{f}) \right\|_{L^p(B_0(R + 2))} \\
& \leq N |\lambda|^{-1 - (\beta/2)} \| T \|_{L^p(\mathbb{R}^3)} \| E_j \|_{L^p(\mathbb{R}^3)} \\
& \leq N, \| h \|^{1/2} \| T \|_{L^p(\mathbb{R}^3)} \| E_j \|_{L^p(\mathbb{R}^3)} \\
& \leq N, \| h \|^{1/2} \| T \|_{L^p(\mathbb{R}^3)} \| E_j \|_{L^p(\mathbb{R}^3)}.
\end{align*}
\] (2.121)

Further, in the light of (2.110), we have by (2.107)-(2.108), (2.111) and (2.119)
\[
\begin{align*}
& \| T((1 + |x|)^{- N + 1} \partial_x^\beta A_1^N(\lambda) B_{2, R}(\partial_x \mathcal{K}_{j - 1})(\lambda \mathbf{f}) \right\|_{L^p(B_0(R + 2))} \\
& \leq N |\lambda|^{-2 + ((\beta/2))} \| T \|_{L^p(\mathbb{R}^3)} \| E_j \|_{L^p(\mathbb{R}^3)} \\
& + |\lambda|^{-1} \sum_{k=0}^{1} \| T((1 + |x|)^{- N + 1} \partial_x^\beta A_1^N(\lambda) B_{2, R} \Delta \partial_x^k \mathcal{K}_{j - 1})(\lambda \mathbf{f}) \right\|_{L^p(B_0(R + 2))} \\
& \leq N, \| h \|^{1/2} \| T \|_{L^p(\mathbb{R}^3)} \| E_j \|_{L^p(\mathbb{R}^3)} \\
& + |\lambda|^{-2} \sum_{k=0}^{1} \| T((1 + |x|)^{- N + 1} \partial_x^\beta A_1^N(\lambda) B_{2, R} \Delta \partial_x^k \mathcal{K}_{j - 1})(\lambda \mathbf{f}) \right\|_{L^p(B_0(R + 2))} \\
& + |\lambda|^{-2} \| T((1 + |x|)^{- N + 1} \partial_x^\beta A_1^N(\lambda) B_{2, R} \Delta \partial_x^k \mathcal{K}_{j - 1})(\lambda \mathbf{f}) \right\|_{L^p(B_0(R + 2))} \\
& + |\lambda|^{-1} |\lambda|^{-1} \| T((1 + |x|)^{- N + 1} \partial_x^\beta A_1^N(\lambda) B_{2, R} \Delta \partial_x^k \mathcal{K}_{j - 1})(\lambda \mathbf{f}) \right\|_{L^p(B_0(R + 2))} \\
& + |\lambda|^{-1} \| T((1 + |x|)^{- N + 1} \partial_x^\beta A_1^N(\lambda) B_{2, R} \Delta \partial_x^k \mathcal{K}_{j - 1})(\lambda \mathbf{f}) \right\|_{L^p(B_0(R + 2))} \\
& + |\lambda|^{-1} \| T((1 + |x|)^{- N + 1} \partial_x^\beta A_1^N(\lambda) B_{2, R} \Delta \partial_x^k \mathcal{K}_{j - 1})(\lambda \mathbf{f}) \right\|_{L^p(B_0(R + 2))} \\
& + |\lambda|^{-1} \| T((1 + |x|)^{- N + 1} \partial_x^\beta A_1^N(\lambda) B_{2, R} \Delta \partial_x^k \mathcal{K}_{j - 1})(\lambda \mathbf{f}) \right\|_{L^p(B_0(R + 2))}.
\end{align*}
\] (2.123)

Based on the above analysis in case $|\beta| = 1, 2,$ we know that (2.118)-(2.119) give the required estimates of $\partial_x^\beta \mathcal{K}_0(\lambda)$ and $\partial_x^\beta \mathcal{K}_2(\lambda),$ and (2.123) helps us to reduce the estimate of $\partial_x^\beta \mathcal{K}_j(\lambda)$ with $j \geq 1$ to those of $\mathcal{K}_{j - 1}(\lambda).$ Hence, we explore a “tree self-similar” iterative mechanism on account of (2.115). Roughly speaking, for $N \geq 3,$
\[
\begin{align*}
\mathcal{K}_j(\lambda) &= \mathcal{K}_N^N(\lambda) + \mathcal{K}_N^N(\lambda) \\
\mathcal{K}_j(\lambda) &= \mathcal{K}_{1, j - 1}(\lambda) + \mathcal{K}_{2, j - 1}(\lambda) + \mathcal{K}_{2, j - 1}(\lambda) \\
\mathcal{K}_j(\lambda) &= \mathcal{K}_{1, j - 1}(\lambda) + \mathcal{K}_{2, j - 1}(\lambda) + \mathcal{K}_{2, j - 1}(\lambda) + \mathcal{K}_{2, j - 1}(\lambda).
\end{align*}
\]

This iterative progress will help us to obtain the required estimates of $\partial_x^\beta \mathcal{K}_j(\lambda).$

**Step 2. Proof of (2.103) for $1 \leq |\beta| \leq 2.$**

First, we adopt the decomposition (2.115) with $N = 3.$ When $j = 0,$ as a consequence of (2.118) with $T = \chi_{B_0(R + 2)}(1 + |x|)^3$ and (2.119) with $T = \chi_{B_0(R + 2)}(1 + |x|)^2,$ we deduce (2.103) for $|\beta| = 1, 2.$ When $j \geq 1,$ by (2.118) with $T = \chi_{B_0(R + 2)}(1 + |x|)^3,$ we obtain
\[
\begin{align*}
& \left\{ \| \partial_x^\beta \mathcal{K}_j^3(\lambda) \mathbf{f} \|_{L^p(B_0(R + 2))} \leq |\lambda|^{-2 + ((\beta/2))} E_j \| \mathbf{f} \|_{L^p(\mathbb{R}^3)} \right. \\
& \left. \| \partial_x^\beta (\partial_x \mathcal{K}_2^3)(\lambda) \mathbf{f} \|_{L^p(B_0(R + 2))} \leq |\lambda|^{-2 + ((\beta/2))} E_j \| \mathbf{f} \|_{L^p(\mathbb{R}^3)} \right. \\
& \left. \| \partial_x^\beta (\Delta_h \partial_x \mathcal{K}_2^3)(\lambda) \mathbf{f} \|_{L^p(B_0(R + 2))} \leq |\lambda|^{-2 + ((\beta/2))} F_j \| \mathbf{f} \|_{L^p(\mathbb{R}^3)} \right. \\
& \right. \| \partial_x^\beta (\Delta_h \partial_x \mathcal{K}_2^3)(\lambda) \mathbf{f} \|_{L^p(B_0(R + 2))} \right. \leq |\lambda|^{-2 + ((\beta/2))} F_j \| \mathbf{f} \|_{L^p(\mathbb{R}^3)}.
\end{align*}
\] (2.124)
In addition, utilizing (2.120) and (2.123) with \( T = \chi_{B_0(R+2)}(1 + |x|)^2 \), we deduce

\[
\| \partial^2_x K_{1,j}^3(\lambda) f \|_{L^p(B_0(R+2))} \lesssim |\lambda|^{-1+((\beta/2))} E_j \| f \|_{L^p(R^3)},
\]

(2.125) and

\[
\begin{align*}
\| \partial^2_x (\partial_{\lambda} K_{1,j}^3(\lambda) f) \|_{L^p(B_0(R+2))} & \lesssim |\lambda|^{-2+((\beta/2))} E_j \| f \|_{L^p(R^3)} + \| \partial^2_x A_1^3(\lambda) B_1,\overline{\pi}(\partial_{\lambda} K_{j-1})(\lambda) f \|_{L^p(B_0(R+2))} \\
+ |\lambda|^{-1} \sum_{k=0}^1 \| \partial^2_x A_1^3(\lambda) B_2,\overline{\pi} \Delta(\partial_{\lambda}^k K_{j-1})(\lambda) f \|_{L^p(B_0(R+2))} & \\
\| \partial^2_x (\Delta_h \partial_{\lambda} K_{1,j}^3(\lambda) f) \|_{L^p(B_0(R+2))} & \lesssim |h|^{\rho} |\lambda|^{-2+((\beta/2))} E_j \| f \|_{L^p(R^3)} + \| \partial^2_x A_1^3(\lambda) B_1,\overline{\pi}(\Delta_h \partial_{\lambda} K_{j-1})(\lambda) f \|_{L^p(B_0(R+2))} \\
+ |\lambda|^{-1} \sum_{k=0}^1 \left[ |h|^{\rho} \| \partial^2_x A_1^3(\lambda) B_2,\overline{\pi} \Delta(\partial_{\lambda}^k K_{j-1})(\lambda) f \|_{L^p(B_0(R+2))} \right],
\end{align*}
\]

(2.126)

Next, we use the decomposition (2.115) with \( N = 1 \) for \( K_{j-1}(\lambda) \) to proceed the estimate (2.126). In the light of (2.125) and (2.119) with \( T = \chi_{B_0(R+2)}(1 + |x|)^2 \), we know

\[
\| (\chi_{B_0(R+2)} \partial^2_x A_1^3(\lambda) \nabla \overline{w}(1 + |x|), \chi_{B_0(R+2)} \partial^2_x A_1^3(\lambda) \overline{w}(1 + |x|) \|_{L^p(R^3)} \lesssim C_{R, R^*, \rho, \omega} \| \overline{w} \|_{L^p(\Omega)} |\lambda|^{-1+((\beta/2))}.
\]

Hence, by making use of

\[
\begin{align*}
(2.115), \quad & \text{with } T = \chi_{B_0(R+2)} \partial^2_x A_1^3(\lambda) \nabla \overline{w}(1 + |x|), \\
(2.119) - (2.121), \quad & \text{with } T = \chi_{B_0(R+2)} \partial^2_x A_1^3(\lambda) \overline{w},
\end{align*}
\]

we deduce for \( j \geq 1 \)

\[
\begin{align*}
\| \partial^2_x A_1^3(\lambda) B_2,\overline{\pi} \Delta K_{j-1}(\lambda) f \|_{L^p(B_0(R+2))} & \lesssim |\lambda|^{-1+((\beta/2))} E_j \| f \|_{L^p(R^3)}, \\
\| \partial^2_x A_1^3(\lambda) B_2,\overline{\pi} \Delta(\partial_{\lambda} K_{j-1})(\lambda) f \|_{L^p(B_0(R+2))} & \lesssim |\lambda|^{-1+((\beta/2))} E_j \| f \|_{L^p(R^3)}, \\
\| \partial^2_x A_1^3(\lambda) B_2,\overline{\pi} \Delta(\Delta_h \partial_{\lambda} K_{j-1})(\lambda) f \|_{L^p(B_0(R+2))} & \lesssim |h|^{\rho} |\lambda|^{-1+((\beta/2))} F_j \| f \|_{L^p(R^3)}.
\end{align*}
\]

Moreover, with the help of

\[
\begin{align*}
(2.118), \quad & \text{with } T = \chi_{B_0(R+2)} \partial^2_x A_1^3(\lambda) \overline{w}(1 + |x|), \\
(2.119) - (2.120), (2.122) - (2.123), \quad & \text{with } T = \chi_{B_0(R+2)} \partial^2_x A_1^3(\lambda) \overline{w},
\end{align*}
\]

we calculate

\[
\begin{align*}
\| \partial^2_x A_1^3(\lambda) B_1,\overline{\pi}(\partial_{\lambda}(\partial_{\lambda} K_0))(\lambda) f \|_{L^p(B_0(R+2))} & \lesssim |\lambda|^{-2+(\beta/2)+\rho} E_1 \| f \|_{L^p(R^3)}, \\
\| \partial^2_x A_1^3(\lambda) B_1,\overline{\pi}(\Delta_h \partial_{\lambda}(\partial_{\lambda} K_0))(\lambda) f \|_{L^p(B_0(R+2))} & \lesssim |h|^{\rho} |\lambda|^{-2+(\beta/2)+\rho} F_1 \| f \|_{L^p(R^3)},
\end{align*}
\]

and for all \( j \geq 2 \),

\[
\begin{align*}
\| \partial^2_x A_1^3(\lambda) B_1,\overline{\pi}(\partial_{\lambda}(\partial_{\lambda} K_{j-1}^1))(\lambda) f \|_{L^p(B_0(R+2))} & \lesssim |\lambda|^{-2+(\beta/2)+\rho} E_j \| f \|_{L^p(R^3)}, \\
\| \partial^2_x A_1^3(\lambda) B_1,\overline{\pi}(\Delta_h \partial_{\lambda}(\partial_{\lambda} K_{j-1}^1))(\lambda) f \|_{L^p(B_0(R+2))} & \lesssim |h|^{\rho} |\lambda|^{-2+(\beta/2)+\rho} F_j \| f \|_{L^p(R^3)}.
\end{align*}
\]
and
\[
\begin{align*}
&\left\| \partial_x^\beta A_1^4(\lambda) B_1,\mathbb{R}(\partial_x K_{1,j-1}^1)(\lambda) f \right\|_{L^p(B_{(R+2)})} \\
&\lesssim |\lambda|^{-2+|\beta|/2} E_j f \left\|_{L^p(\mathbb{R}^3)} + \| \partial_x^\beta A_1^4(\lambda) B_1,\mathbb{R}(\partial_x K_{j-2}^1)(\lambda) f \right\|_{L^p(B_{(R+2)})} \\
&\quad + |\lambda|^{-1} \sum_{k=0}^1 \| \partial_x^\beta A_1^1(\lambda) B_1,\mathbb{R}(\partial_x K_{j-2}^1)(\lambda) f \|_{L^p(B_{(R+2)})},
\end{align*}
\]

Then, plugging the above four sets of estimates into (2.126), we deduce
\[
\begin{align*}
&\left\| \partial_x^\beta (\partial_x K_{1,1}^3)(\lambda) f \right\|_{L^p(B_{(R+2)})} \lesssim |\lambda|^{-2+|\beta|/2+\epsilon E_j} f \left\|_{L^p(\mathbb{R}^3)},
\end{align*}
\]
and for \( j \geq 2 \),
\[
\begin{align*}
&\left\| \partial_x^\beta (\partial_x K_{1,1}^3)(\lambda) f \right\|_{L^p(B_{(R+2)})} \\
&\lesssim |\lambda|^{-2+|\beta|/2+\epsilon E_j} f \left\|_{L^p(\mathbb{R}^3)} + \| \partial_x^\beta A_1^4(\lambda) B_1,\mathbb{R}(\partial_x K_{j-2}^1)(\lambda) f \|_{L^p(B_{(R+2)})} \\
&\quad + \sum_{k=0}^1 |\lambda|^{-1} \| \partial_x^\beta A_1^1(\lambda) B_1,\mathbb{R}(\partial_x K_{j-2}^1)(\lambda) f \|_{L^p(B_{(R+2)})},
\end{align*}
\]

Thus (2.103) for \( j = 1 \) and \(|\beta| = 1, 2\) is obtained by (2.121) and (2.127).

Finally, we use (2.115) with \( N = 1 \) again for \( K_{j-2}(\lambda) \) in (2.128). We observe from (2.125) and (2.119) that
\[
\begin{align*}
&\left\| \chi_{B_{(R+2)}} \partial_x^\beta A_1^4(\lambda) B_1,\mathbb{R}(1 + |x|) \right\|_{L^p(\mathbb{R}^3)} \lesssim C_{R,\mathbb{R}^3, \omega} \| w \|_{L^\infty, \Omega} |\lambda|^{-3-|\beta|/2},
\end{align*}
\]

Hence, we get from (2.128) that for all \( j \geq 2 \)
\[
\begin{align*}
&\left\| \partial_x^\beta (\partial_x K_{1,1}^3)(\lambda) f \right\|_{L^p(B_{(R+2)})} \lesssim |\lambda|^{-2+|\beta|/2+\epsilon E_j} f \left\|_{L^p(\mathbb{R}^3)},
\end{align*}
\]
by making use of

\[
\begin{align*}
(2.115) & \quad \text{with } T = \chi_{B_0(R+2)} \partial_\lambda^\beta A_1^t(\lambda) B_1, \bar{w} A_1^t(\lambda) \bar{w}(1 + |x|), \\
(2.116) & \quad \text{with } T = \chi_{B_0(R+2)} \partial_\lambda^\beta A_1^t(\lambda) B_1, \bar{w} A_1^t(\lambda) \nabla \bar{w}(1 + |x|), \\
(2.119) - (2.123) & \quad \text{with } T = \chi_{B_0(R+2)} \partial_\lambda^\beta A_1^t(\lambda) B_1, \bar{w} A_1^t(\lambda) \nabla \bar{w}, \\
(2.119) - (2.123) & \quad \text{with } T = \chi_{B_0(R+2)} \partial_\lambda^\beta A_1^t(\lambda) B_1, \bar{w} A_1^t(\lambda) \nabla \bar{w},
\end{align*}
\]

Collecting (2.124) and (2.125), we prove (2.103) for \( j \geq 2 \) and \( |\beta| = 1, 2 \), and so end the proof of (2.103) for all \( j \geq 0 \) and \( |\beta| = 1, 2 \).

**Step 3: Proof of (2.103) for \( |\beta| = 0 \).**

Using (2.105) and the estimate (2.103) for \( |\beta| = 1, 2 \), we easily verify (2.103) for \( j = 0 \) and \( |\beta| = 0 \). Meanwhile, for \( j \geq 1 \), we obtain from (2.112)-(2.113)

\[
\|K_j(\lambda)f\|_{L^p(B_0(R+2))} \lesssim |\lambda|^{-1}E_j \|f\|_{L^p(\mathbb{R}^3)}
\]

and

\[
\begin{align*}
&\|\partial_\lambda K_j(\lambda)f\|_{L^p(B_0(R+2))} \\
\lesssim &\|h|^{\rho}E_j \|f\|_{L^p(\mathbb{R}^3)} + |\lambda|^{-1}\|P_{\mathbb{R}^3} B_1, \bar{w} (\partial_\lambda K_{j-1})(\lambda)f\|_{L^p(B_0(R+2))} \\
&+ |\lambda|^{-2}\|P_{\mathbb{R}^3} B_2, \bar{w} \Delta K_{j-1}(\lambda)f; P_{\mathbb{R}^3} B_2, \bar{w} \Delta (\partial_\lambda K_{j-1})(\lambda)f\|_{L^p(B_0(R+2))} \\
&+ \|P_{\mathbb{R}^3} B_2, \bar{w} (\Delta \partial_\lambda K_{j-1})(\lambda)f; P_{\mathbb{R}^3} B_2, \bar{w} \Delta \partial_\lambda K_{j-1}(\lambda)f\|_{L^p(B_0(R+2))},
\end{align*}
\]

(2.131)

Now, we are going to further estimate (2.131) by the “tree self-similar” iterative mechanism used in Step 2. We first use (2.115) with \( N = 1 \) for \( K_{j-1}(\lambda) \). Hence we obtain

\[
\begin{align*}
&\|\partial_\lambda K_j(\lambda)f\|_{L^p(B_0(R+2))} \lesssim |\lambda|^{-2+\rho}E_j \|f\|_{L^p(\mathbb{R}^3)}, \\
&\|\partial_\lambda K_j(\lambda)f\|_{L^p(B_0(R+2))} \lesssim |h|^{\rho}E_j \|f\|_{L^p(\mathbb{R}^3)},
\end{align*}
\]

(2.132)

and for \( j \geq 2 \)

\[
\begin{align*}
&\|\partial_\lambda K_j(\lambda)f\|_{L^p(B_0(R+2))} \\
\lesssim & |\lambda|^{-2+\rho}E_j \|f\|_{L^p(\mathbb{R}^3)} + |\lambda|^{-1}\|P_{\mathbb{R}^3} B_1, \bar{w} A_1(\lambda) B_1, \bar{w} (\partial_\lambda K_{j-2})(\lambda)f\|_{L^p(B_0(R+2))} \\
&+ |\lambda|^{-2}\sum_{k=0,1} \|P_{\mathbb{R}^3} B_2, \bar{w} A_1^t(\lambda) B_2, \bar{w} \Delta (\partial_\lambda K_{j-2})(\lambda)f\|_{L^p(B_0(R+2))} \\
&\lesssim |\lambda|^{-2+\rho}E_j \|f\|_{L^p(\mathbb{R}^3)} + |\lambda|^{-1}\|P_{\mathbb{R}^3} B_1, \bar{w} A_1(\lambda) B_1, \bar{w} (\partial_\lambda K_{j-2})(\lambda)f\|_{L^p(B_0(R+2))} \\
&+ |\lambda|^{-2}\sum_{k=0,1} \|P_{\mathbb{R}^3} B_2, \bar{w} A_1^t(\lambda) B_2, \bar{w} \Delta (\partial_\lambda K_{j-2})(\lambda)f\|_{L^p(B_0(R+2))}.
\end{align*}
\]
Corollary 2.7. Under the assumption of Theorem 2.6, there exists a positive constant completes the proof of Theorem 2.6. □

Since the kernel function of in particular, for into the above set of inequalities, we deduce for every finally, using with \(|\beta| = 0\). This completes the proof of Theorem 2.6. □

For the following, we have the following result.

**Corollary 2.7.** Under the assumption of Theorem 2.6, there exists a positive constant \(\eta = \eta_{R,p,\vartheta,R,\vartheta^*,\omega^*} > 0\) such that if \(\|w\|_{\varepsilon,\Omega} \leq \eta\), then

\[
\Pi_{\vartheta,\omega}^G(\lambda) \in C(C_+, L^p(B_{R}(R^2), W^1,p(B_{R}(R + 2)))
\]

Moreover, for \(\lambda, \lambda + h \in \mathbb{C}_+\) and \(f \in L^p_{R^2}(\mathbb{R})\), we have

\[
\|\nabla(\partial_1^k \Pi_{\vartheta,\omega}^G(\lambda)f)\|_{L^p(B_{R}(R^2))} \leq C_{R,p,\vartheta,R^*,\omega^*} \|f\|_{L^p(\mathbb{R})}, \quad k = 0, 1, \quad (2.134)
\]

\[
\|\nabla(\Delta_h \partial_1 \Pi_{\vartheta,\omega}^G(\lambda)f)\|_{L^p(B_{R}(R^2))} \leq C_{R,p,\vartheta,R^*,\omega^*} |h|^\rho \|f\|_{L^p(\mathbb{R})}, \quad (2.135)
\]

\[
\|\Pi_{\vartheta,\omega}^G(\lambda)f\|_{L^p(B_{R}(R^2))} \leq C_{R,p,\vartheta,R^*,\omega^*} |x|^{-3/2} \|f\|_{L^p(\mathbb{R})}, \quad x \in B_{10}(R+2). \quad (2.136)
\]

In particular, for \(0 < |h| \leq h_0\), we have for \(0 < \rho \leq 1/2\)

\[
\|\nabla(\partial_1^k \Pi_{\vartheta,\omega}^G(\lambda)f)\|_{L^p(B_{R}(R^2))} \leq C_{R,p,\vartheta,R^*,\omega^*} |\lambda|^{-1+\rho} \|f\|_{L^p(\mathbb{R})}, \quad k = 0, 1, \quad (2.137)
\]

\[
\|\nabla(\Delta_h \partial_1 \Pi_{\vartheta,\omega}^G(\lambda)f)\|_{L^p(B_{R}(R^2))} \leq C_{R,p,\vartheta,R^*,\omega^*} |h|^\rho \|f\|_{L^p(\mathbb{R})}, \quad (2.138)
\]

**Proof.** From (1.23) and (2.64) - (2.70), we have for \(q = \max(p, 3)\)

\[
\|B_{\Pi}^* \partial_1^k R_{\vartheta,\omega}^G(\lambda)f\|_{L^p_{\varepsilon}\Omega} \leq C_{R,p,\vartheta,R^*,\omega^*} \|w\|_{\varepsilon,\Omega} \|f\|_{L^p(\mathbb{R})}, \quad k = 0, 1, \quad (2.134)
\]

\[
\|B_{\Pi}^* (\Delta_h \partial_1 R_{\vartheta,\omega}^G(\lambda)f)\|_{L^p_{\varepsilon}\Omega} \leq C_{R,p,\vartheta,R^*,\omega^*} \|w\|_{\varepsilon,\Omega} \|f\|_{L^p(\mathbb{R})}. \quad (2.135)
\]

Hence we easily prove (2.134) - (2.135) by (2.1) and (2.107).

Next, we prove (2.136). Rewrite

\[
\Pi_{\vartheta,\omega}^G(\lambda)f = Q_{R^3}(x B_{R+2}) B_{\Pi}^* R_{\vartheta,\omega}^G(\lambda)f + Q_{R^3}(x B_{R+2}) B_{\Pi}^* R_{\vartheta,\omega}^G(\lambda)f.
\]

Since the kernel function of \(Q_{R^3}\) is bounded by \(|x|^{-2}\), we have by (2.67) and (2.68)

\[
\|\Pi_{\vartheta,\omega}^G(\lambda)f\|_{L^p(\mathbb{R})} \leq \|w\|_{\varepsilon,\Omega} \|f\|_{L^p(\mathbb{R})} \times \left(\frac{1}{|x|^2} + \int_{B_{R+2}} \frac{1}{|x-y|^2} \frac{1}{|s_{R^3}(y)|} \frac{1}{y^2} dy\right), \quad x \in B_{10}(R+2).
\]
We observe for every \( x \in B_{10}^{R,2} \)
\[
\int_{B_{9}^{R,2}} \frac{1}{|x-y|^2} \frac{1}{|y|^\frac{5}{2} (1 + sQ(y))} \, dy \lesssim \frac{1}{|x|^2} \left[ \int_{|y| < \frac{9}{20}} + \int_{|y| \geq \frac{9}{20}} \right] \frac{1}{|y|^\frac{5}{2} (1 + sQ(y))} \, dy
\]
\[
+ \frac{1}{|x|} \frac{1}{5} \int_{|y|/10 \leq |y| < 3|x|/2} \frac{1}{|x-y|^2} \, dy \lesssim |x|^{-\frac{3}{2}}.
\]
where we used Lemma [8.3] in the last inequality. Hence, we prove (2.136).

Finally, we prove (2.137)-(2.138). In view of (2.75), we write
\[
\nabla P_{G,\mu}^R(\lambda) f = \sum_{j=0}^\infty \left( \nabla Q_{2,\mu}^R(B_1,\mu)K_j(\lambda) f + \nabla Q_{2,\mu}^R(B_2,\mu)K_j(\lambda) f \right).
\]
We first use (2.107) with \( T = \nabla Q_{2,\mu}^R \), (2.108) and (2.110)-(2.111) to reduce the estimate of \( \nabla Q_{2,\mu}^R(B_2,\mu)K_j(\lambda) f \) to the estimate of \( \nabla Q_{2,\mu}^R(B_2,\mu)\Delta K_j(\lambda) f \). Then, utilizing the self-similar iterative mechanism (2.115) with \( N = 1 \) two times, we finally deduce (2.137)-(2.138) by (2.118)-(2.123) \( \square \)

Set
\[
\Pi_{G,\mu}^R(\lambda) = Q_{2,\mu}^R(B_2,\mu)G_{\mu}^R(\lambda), \quad \Pi_{G,\mu}^R(\lambda) = \tilde{Q}_{2,\mu}^R(B_{\infty}^\infty)G_{\mu}^R(\lambda).
\]
Then, we have for \( f \in L^p(R^3) \)
\[
(\lambda + L_{G,\mu}^R)R_{G,\mu}^R(\lambda) f + \nabla (Q_{2,\mu}^R f + \Pi_{G,\mu}^R(\lambda) f) = f, \quad \text{div} \, R_{G,\mu}^R(\lambda) f = 0. \tag{2.139}
\]

In the same way as treating \( (R_{G,\mu}^R(\lambda), \Pi_{G,\mu}^R(\lambda)) \), we obtain

**Corollary 2.8.** Let \( f \in L^p_{R+3}(R^3), p \in (1, \infty) \). Then, Theorem 2.4 and Corollary 2.5 hold for \( R_{G,\mu}^R(\lambda) f \) and \( \Pi_{G,\mu}^R(\lambda) f \), or \( \Pi_{G,\mu}^R(\lambda) f \).

3. INTERIOR RESOLVENT PROBLEM

In this section we will discuss the resolvent problem associated to \( L_{G,\mu}^R \) in \( \Omega_{R+3} \)
\[
(\lambda I + L_{G,\mu}^R)u + \nabla P = f, \quad \text{div} \, u = 0 \quad \text{in} \ \Omega_{R+3}, \quad u|_{\partial \Omega_{R+3}} = 0. \tag{3.1}
\]
with the addition condition \( \int_{\Omega_{R+3}} P \, dx = 0 \). From the Helmholtz decomposition in [10] [14], given \( f \in L^p(\Omega_{R+3}) \), there exist unique \( h \in \mathbb{P}(\Omega_{R+3}) \) and \( g \in \tilde{W}^{1,p}(\Omega_{R+3}) \) such that
\[
f = h + \nabla g.
\]
This leads us to define operators \( \mathcal{P}_{G,\mu}^R f = h \) and \( \tilde{Q}_{G,\mu}^R f = g \) satisfying
\[
\mathcal{P}_{G,\mu}^R \in \mathcal{L}(L^p(\Omega_{R+3}), \tilde{W}^{1,p}(\Omega_{R+3})), \quad \tilde{Q}_{G,\mu}^R \in \mathcal{L}(L^p(\Omega_{R+3}), \tilde{W}^{1,p}(\Omega_{R+3})).
\]
Define
\[
\begin{aligned}
L_{G,\mu}^R(\lambda) & = \mathcal{P}_{G,\mu}^R + L_{G,\mu}^R, \\
D_p(L_{G,\mu}^R, \Omega_{R+3}) & = \{ u \in \mathbb{W}^{2,p}(\Omega_{R+3}) \cap \tilde{W}^{1,p}(\Omega_{R+3}) \mid u|_{\partial \Omega_{R+3}} = 0 \}.
\end{aligned}
\]
Then, system (3.1) is equivalent to
\[
(\lambda I + L_{G,\mu}^R)u = \mathcal{P}_{G,\mu}^R f \tag{3.3}
\]
with $P = \tilde{Q}_{\Omega_{R+3}} f + \tilde{Q}_{\Omega_{R+3}} (L_{\mathcal{R},\omega,w} u)$. Denote $\mathcal{P}_{\Omega_{R+3}} \Delta$ by $\Delta_{\Omega_{R+3}}$. Since $D_p(-\Delta_{\Omega_{R+3}}) \subset D_p(L_{\mathcal{R},\omega,w,\Omega_{R+3}} + \Delta_{\Omega_{R+3}})$, we can view (3.3) as a perturbation of the classical Stokes resolvent problem

$$\left(\lambda I - \Delta_{\Omega_{R+3}}\right) v = \mathcal{P}_{R^3} f. \tag{3.4}$$

It is well known from \textbf{[11] [8]} that, for every $f \in \mathcal{P}(\Omega_{R+3})$ and $\lambda \in \Sigma_\theta \cup \{0\}$ with $0 < \theta < \pi/2$,

$$\|\nabla^j(\lambda I - \Delta_{\Omega_{R+3}})^{-1} f\|_{L^p(\Omega_{R+3})} \leq C_\theta(1 + |\lambda|)^{-1+j/2} \|f\|_{L^p(\Omega_{R+3})}, \quad j \leq 2. \tag{3.5}$$

Hence, we have the following theorem for problem (3.1).

**Theorem 3.1.** Let $p \in (1, \infty)$, $\theta \in (0, \frac{\pi}{2})$, $\varepsilon \in (0, \frac{1}{2})$, $0 < |\mathcal{R}| \leq \mathcal{R}^*$, and $|\omega| \leq \omega^*$. Then there exist two constants $\eta, \eta_1 > 0$, depending only on $R, p, \theta, \mathcal{R}^*, \omega^*$, such that if

$$\|\mathbf{w}\|_{L^\infty \Omega} \leq \eta,$$

then for every $f \in \mathcal{P}(\Omega_{R+3})$, problem (3.1) admits a unique solution

$$(u, P) = (\mathcal{R}^I_{\mathcal{R},\omega,w}(\lambda) f, \tilde{Q}_{\Omega_{R+3}} f + \tilde{\Pi}^I_{\mathcal{R},\omega,w}(\lambda) f)$$

satisfying

$$\mathcal{R}^I_{\mathcal{R},\omega,w}(\lambda) \in \mathcal{S}(\Sigma_{\theta,\varepsilon} \cup \overline{C}_+), \mathcal{L}(L^p(\Omega_{R+3}), W^{2,p}(\Omega_{R+3}) \cap \mathcal{P}(\Omega_{R+3})), \tilde{\Pi}^I_{\mathcal{R},\omega,w}(\lambda) \in \mathcal{S}(\Sigma_{\theta,\varepsilon} \cup \overline{C}_+), \mathcal{L}(L^p(\Omega_{R+3}), W^{1,p}(\Omega_{R+3})).$$

Moreover, for every $k \geq 0$ and $\lambda \in \Sigma_{\theta,\varepsilon} \cup \overline{C}_+$, one has

$$\|\nabla^j(\partial_k^{\lambda} \mathcal{R}^I_{\mathcal{R},\omega,w}(\lambda))\|_{L^p(\Omega_{R+3})} \leq C_{\theta,\lambda,R} (1 + |\lambda|)^{-k+j/2}, \quad j = 1, 2, \tag{3.6}$$

$$\|\nabla(\partial_k^{\lambda} \tilde{\Pi}^I_{\mathcal{R},\omega,w}(\lambda))\|_{L^p(\Omega_{R+3})} \leq C_{\theta,\lambda,R} (1 + |\lambda|)^{-k}, \tag{3.7}$$

$$\|\partial_k^{\lambda} \tilde{\Pi}^I_{\mathcal{R},\omega,w}(\lambda)\|_{L^p(\Omega_{R+3}), L^{p}(\Omega_{R+3})} \leq C_{\theta,\lambda,R} (1 + |\lambda|)^{-k+(1/2)-1/(2p)}. \tag{3.8}$$

**Proof.** It suffices to prove that for every $\lambda \in \Sigma_{\theta,\varepsilon} \cup \overline{C}_+$,

$$\left(I + (L_{\mathcal{R},\omega,w,\Omega_{R+3}} + \Delta_{\Omega_{R+3}})(\lambda I - \Delta_{\Omega_{R+3}})^{-1}\right)^{-1} \in L(\mathcal{P}(\Omega_{R+3})), \tag{3.9}$$

$$\|\left(I + (L_{\mathcal{R},\omega,w,\Omega_{R+3}} + \Delta_{\Omega_{R+3}})(\lambda I - \Delta_{\Omega_{R+3}})^{-1}\right)^{-1}\|_{L(\mathcal{P}(\Omega_{R+3}))} \leq C_{\theta,R,\mathcal{R}^*, \omega^*}. \tag{3.10}$$

In fact, we have from (3.5) and (3.3)-(3.10) that for all $\lambda \in \Sigma_{\theta,\varepsilon} \cup \overline{C}_+$

$$(\lambda I + L_{\mathcal{R},\omega,w,\Omega_{R+3}})^{-1} = (\lambda I - \Delta_{\Omega_{R+3}})^{-1} \left(I + (L_{\mathcal{R},\omega,w,\Omega_{R+3}} + \Delta_{\Omega_{R+3}})(\lambda I - \Delta_{\Omega_{R+3}})^{-1}\right)^{-1},$$

$$\|\nabla^j(\lambda I + L_{\mathcal{R},\omega,w,\Omega_{R+3}})^{-1}\|_{L^p(\Omega_{R+3})} \leq C_{\theta,\lambda,R^*, \omega^*} (1 + |\lambda|)^{-1+j/2}, \quad j \leq 2.$$ 

Hence, $(u, P) \triangleq (\mathcal{R}^I_{\mathcal{R},\omega,w}(\lambda) f, \tilde{Q}_{\Omega_{R+3}} f + \tilde{\Pi}^I_{\mathcal{R},\omega,w}(\lambda) f)$ with

$$\mathcal{R}^I_{\mathcal{R},\omega,w}(\lambda) \triangleq (\lambda I + L_{\mathcal{R},\omega,w,\Omega_{R+3}})^{-1} \mathcal{P}_{R^3}, \quad \tilde{\Pi}^I_{\mathcal{R},\omega,w}(\lambda) \triangleq \tilde{Q}_{\Omega_{R+3}} L_{\mathcal{R},\omega,w} \mathcal{R}^I_{\mathcal{R},\omega,w}(\lambda),$$

solve problem (3.1) uniquely, and satisfies (3.6)-(3.7) since

$$\partial_k^{\lambda} \mathcal{R}^I_{\mathcal{R},\omega,w}(\lambda) = (-1)^k k! (\lambda I + L_{\mathcal{R},\omega,w,\Omega_{R+3}})^{-1-k} \mathcal{P}_{R^3},$$

$$\partial_k^{\lambda} \tilde{\Pi}^I_{\mathcal{R},\omega,w}(\lambda) = \tilde{Q}_{\Omega_{R+3}} L_{\mathcal{R},\omega,w} \partial_k^{\lambda} \mathcal{R}^I_{\mathcal{R},\omega,w}(\lambda).$$
To prove (3.8), we adopt the known result in [10], that is, for every \( \phi \in C_0^\infty(\Omega_{R+3}) \), there exists a \( \Phi \in W^{2,p'}(\Omega_{R+3}) \) solves

\[
\Delta \Phi = \tilde{\phi} \quad \text{in} \; \Omega_{R+3}, \quad \partial_\nu \Phi|_{\partial \Omega_{R+3}} = 0
\]

where \( \nu \) is the unit outer normal to \( \partial \Omega_{R+3} \) and \( \tilde{\phi} = \phi - |\Omega_{R+3}|^{-1} \int_{\Omega_{R+3}} \phi \, dx \), such that

\[
\| \Phi \|_{W^{2,p'}(\Omega_{R+3})} \leq C\| \tilde{\phi} \|_{L^{p'}(\Omega_{R+3})} \leq C\| \phi \|_{L^{p'}(\Omega_{R+3})}.
\]  

This yields

\[
\begin{aligned}
\langle \hat{Q}_{\Omega_{R+3}} \Delta \partial_\lambda^k \mathcal{R}_{\mathfrak{g},\omega,w}^I (\lambda) f, \tilde{\phi} \rangle_{\Omega_{R+3}} &= \langle \Delta \partial_\lambda^k \mathcal{R}_{\mathfrak{g},\omega,w}^I (\lambda) f, \nabla \tilde{\phi} \rangle_{\Omega_{R+3}} \\
&= -\sum_{j=1}^3 \langle \partial_\nu (\partial_\lambda^k \mathcal{R}_{\mathfrak{g},\omega,w}^I (\lambda) f)_j, \partial_\lambda^k \Phi \rangle_{\partial \Omega_{R+3}} + \sum_{j,m=1}^3 \langle \partial_m (\partial_\lambda^k \mathcal{R}_{\mathfrak{g},\omega,w}^I (\lambda) f)_j, \partial_m \partial_\lambda^k \Phi \rangle_{\partial \Omega_{R+3}}
\end{aligned}
\]

where \( \langle \cdot, \cdot \rangle_{\partial \Omega_{R+3}} \) and \( \langle \cdot, \cdot \rangle_{\Omega_{R+3}} \) denotes the inner-product in \( \partial \Omega_{R+3} \) and \( \Omega_{R+3} \). Hence, by (3.6), (3.11) and the interpolation inequality

\[
\| g \|_{L^p(\partial \Omega_{R+3})} \leq \| \nabla g \|_{L^p(\Omega_{R+3})}^{1-(1/p)} \| g \|_{L^p(\Omega_{R+3})}^{1/p} + \| g \|_{L^p(\Omega_{R+3})},
\]

we have

\[
\| \hat{Q}_{\Omega_{R+3}} \Delta \partial_\lambda^k \mathcal{R}_{\mathfrak{g},\omega,w}^I (\lambda) f \|_{L^p(\Omega_{R+3})} \leq C_{\theta,p,\omega^*}(1 + |\lambda|)^{-(1/2)-(1/2p)}. \tag{3.12}
\]

This inequality together with

\[
\| \hat{Q}_{\Omega_{R+3}} (L_{\mathfrak{g},\omega,w} + \Delta) \partial_\lambda^k \mathcal{R}_{\mathfrak{g},\omega,w}^I (\lambda) f \|_{L^p(\Omega_{R+3})} \\
\leq \| \nabla \hat{Q}_{\Omega_{R+3}} (L_{\mathfrak{g},\omega,w} + \Delta) \partial_\lambda^k \mathcal{R}_{\mathfrak{g},\omega,w}^I (\lambda) f \|_{L^p(\Omega_{R+3})} \\
\leq \| \partial_\lambda^k \mathcal{R}_{\mathfrak{g},\omega,w}^I (\lambda) f \|_{W^{1,p}(\Omega_{R+3})} \leq C_{\theta,p,\omega^*}(1 + |\lambda|)^{-(1/2)-(1/2)p}\| f \|_{L^p(\Omega_{R+3})},
\]

yields (3.8).

Now, we are going to prove (3.9)-(3.10).

**Case 1:** \( \lambda \in \Sigma_{\theta,\ell_1} \). Observe from (3.5) that

\[
\begin{aligned}
\langle (\mathcal{L}_{\mathfrak{g},\omega,w,\Omega_{R+3}} + \Delta_{\Omega_{R+3}})(\lambda I - \Delta_{\Omega_{R+3}})^{-1} \rangle_{L^p(\Omega_{R+3})} \\
\leq C_{\theta,p}(1 + |\lambda|)^{-1/2}(\mathfrak{R}^* + \omega^*) + \| w \|_{\mathcal{L}^\infty(\Omega_{R+3})} \| g \|_{L^p(\Omega_{R+3})},
\end{aligned}
\]

\( \lambda \in \Sigma_\theta \cup \{0\} \). \tag{3.13}

Hence, choosing \( \ell_1, \eta > 0 \) such that \( C_{\theta,\ell,p}(\mathfrak{R}^* + \omega^*)(1 + \ell_1)^{-1/2} < \frac{1}{2} \) and \( C_{\theta,\ell,p}(1 + \ell_1)^{-1/2} \eta < \frac{1}{2} \), we deduce that

\[
\| (\mathcal{L}_{\mathfrak{g},\omega,w,\Omega_{R+3}} + \Delta_{\Omega_{R+3}})(\lambda I - \Delta_{\Omega_{R+3}})^{-1} \|_{L^p(\Omega_{R+3})} < 1, \quad \lambda \in \Sigma_{\theta,\ell_1}
\]

only if \( \| w \|_{\mathcal{L}^\infty(\Omega_{R+3})} \leq \eta \). This, together with the Neumann series proves (3.9)-(3.10) for every \( \lambda \in \Sigma_{\ell,\ell_1} \).

**Case 2:** \( \lambda \in \Sigma_+ \) and \( |\lambda| \leq \ell_1 \). Assume that (3.9) holds, then we have from (3.5)

\[
(I + (\mathcal{L}_{\mathfrak{g},\omega,w,\Omega_{R+3}} + \Delta_{\Omega_{R+3}})(\lambda I - \Delta_{\Omega_{R+3}})^{-1})^{-1} \in C(\| \lambda \|_{\Sigma_+}; \mathcal{L}(\mathcal{L}^p(\Omega_{R+3}))).
\]

This deduces (3.10) for \( \lambda \in \Sigma_+ \) and \( |\lambda| \leq \ell_1 \).

In what follows, we are going to prove (3.9). Thanks to (3.5), we easily verify that \( (\lambda I - \Delta_{\Omega_{R+3}})^{-1} \) is compact from \( \mathcal{L}^p(\Omega_{R+3}) \) to \( W^{1,p}(\Omega_{R+3}) \cap \mathcal{L}^p(\Omega_{R+3}) \). This implies that \( (\mathcal{L}_{\mathfrak{g},\omega,w,\Omega_{R+3}} + \Delta_{\Omega_{R+3}})(\lambda I - \Delta_{\Omega_{R+3}})^{-1} \) is compact from \( \mathcal{L}^p(\Omega_{R+3}) \) to itself since \( \mathcal{L}_{\mathfrak{g},\omega,w,\Omega_{R+3}} + \Delta_{\Omega_{R+3}} \) is compact from \( \mathcal{L}^p(\Omega_{R+3}) \) to itself.
$$\Delta_{\Omega^{R+3}} \in L(\mathbb{W}^{1,p}(\Omega^{R+3}) \cap J^p(\Omega^{R+3}), \mathbb{J}^p(\Omega^{R+3}))$$. Hence, to prove (3.9), it suffices to prove the injectivity of \( I + (\mathcal{L}_{\Omega^{R+3}} + \Delta_{\Omega^{R+3}})(\lambda I - \Delta_{\Omega^{R+3}})^{-1} \) by the Fredholm alternative theorem.

Let \( g \in \mathbb{J}^p(\Omega^{R+3}) \) satisfy \( (I + (\mathcal{L}_{\Omega^{R+3}} + \Delta_{\Omega^{R+3}})(\lambda I - \Delta_{\Omega^{R+3}})^{-1})g = 0 \). Obviously, \( v = (\lambda I - \Delta_{\Omega^{R+3}})^{-1}g \) satisfies

$$((\lambda I - \Delta_{\Omega^{R+3}})v) = -(\mathcal{L}_{\Omega^{R+3}} + \Delta_{\Omega^{R+3}})v.$$  \hspace{1cm} (3.14)

If \( p \geq 2 \), then \( v \in \mathbb{W}^{2,2}(\Omega^{R+3}) \) and \( \theta \in \mathbb{W}^{1,2}(\Omega^{R+3}) \). If \( p < 2 \), by the classical theory for the Stokes system in bounded domain with Dirichlet boundary and the bootstrap argument, we deduce \( v \in \mathbb{W}^{2,2}(\Omega^{R+3}) \). Thus, thanks to that \( \text{div} \, v = 0 \), we deduce

$$0 = \lambda \| v \|_{L^2(\Omega^{R+3})}^2 + \| \nabla v \|_{L^2(\Omega^{R+3})}^2 + \langle (L_{\Omega^{R+3}} + \Delta) v, v \rangle_{\Omega^{R+3}}.$$

By a simple calculation, we obtain

$$\text{Re} (\langle L_{\Omega^{R+3}} + \Delta, v \rangle_{\Omega^{R+3}}) = - (\text{Re} \, v \cdot \nabla \text{Re} \, v + \text{Im} \, v \cdot \nabla \text{Im} \, v, \omega).$$

Since \( v|_{\partial \Omega^{R+3}} = 0 \), we have by Poincare’s inequality

$$|\text{Re} \, (\langle L_{\Omega^{R+3}} + \Delta, v \rangle_{\Omega^{R+3}}) \leq C_R \| w \|_{L^2(\Omega^{R+3})}^2 \| \nabla v \|_{L^2(\Omega^{R+3})}^2.$$

This implies \( \| \nabla v \|_{L^2(\Omega^{R+3})} = 0 \) only if \( C_R \| w \|_{L^2(\Omega^{R+3})} < 1 \). Since \( v|_{\partial \Omega^{R+3}} = 0 \), we have \( v = 0 \), and so \( g = 0 \) in \( \Omega^{R+3} \). Thus, \( I + (\mathcal{L}_{\Omega^{R+3}} + \Delta_{\Omega^{R+3}})(\lambda I - \Delta_{\Omega^{R+3}})^{-1} \) is an injection from \( \mathbb{J}^p(\Omega^{R+3}) \) to itself. This completes the proof of Theorem 3.1.

Consider the resolvent problem associated with the dual operator \( L^*_{\Omega^{R+3}} \) of \( L_{\Omega^{R+3}} \),

$$((\lambda I + L^*_{\Omega^{R+3}}) v + \nabla (\hat{\Theta} + \hat{Q}_{\Omega^{R+3}} f) = f, \quad \text{div} \, v = 0 \text{ in } \Omega^{R+3}, \quad v|_{\partial \Omega^{R+3}} = 0,$$

\hspace{1cm} (3.15)

In the same way as proving Theorem 3.1, we have

**Theorem 3.2.** Under the assumption of Theorem 3.1, there exist two constant \( \eta, \ell_1 > 0 \), depending on \( R, p, \theta, \mathbb{R}, \omega^* \), such that if \( \| w \|_{L^2(\Omega^{R+3})} \leq \eta \), then for every \( f \in L^p(\Omega^{R+3}) \), problem (3.15) admits a unique solution \( (v, \hat{\Theta}) = (R^*_{\Omega^{R+3}}(\lambda)f, \hat{\Pi}^*_{\Omega^{R+3}}(\lambda)f) \)

satisfying

$$R^*_{\Omega^{R+3}}(\lambda) \in \mathcal{A}(\Sigma_{\theta, \ell_1} \cup C^+_+, \mathcal{L}(L^p(\Omega^{R+3}), \mathbb{W}^{2,2}(\Omega^{R+3}) \cap \mathbb{J}^p(\Omega^{R+3}))),$$

$$\hat{\Pi}^*_{\Omega^{R+3}}(\lambda) \in \mathcal{A}(\Sigma_{\theta, \ell_1} \cup C^+_+, \mathcal{L}(L^p(\Omega^{R+3}), \mathbb{W}^{1,2}(\Omega^{R+3}))),$$

and for every \( k \geq 0 \) and \( \lambda \in \Sigma_{\theta, \ell_1} \cup C^+_+ \),

$$\| \nabla^j (\partial^k \mathcal{R}^*_{\Omega^{R+3}}(\lambda)) \|_{L^p(\Omega^{R+3})} \leq C_{k, \theta, R, \mathbb{R}, \omega^*} (1 + |\lambda|)^{-1-k+1/2}, \quad j = 0, 1, 2,$$

$$\| \nabla (\partial^k \hat{\Pi}^*_{\Omega^{R+3}}(\lambda)) \|_{L^p(\Omega^{R+3})} \leq C_{k, \theta, R, \mathbb{R}, \omega^*} (1 + |\lambda|)^{-k},$$

$$\| (\partial^k \hat{\Pi}^*_{\Omega^{R+3}}(\lambda)) \|_{L^p(\Omega^{R+3})} \leq C_{k, \theta, R, \mathbb{R}, \omega^*} (1 + |\lambda|)^{-k-(1/2)-(1/2p)}.$$
4. RESOLVENT PROBLEM IN AN EXTERIOR DOMAIN

In this section, we will construct the solution operators of the resolvent problem in the exterior domain $\Omega$:

\[(\lambda I + L_{\partial\Omega,w})u + \nabla p = f \in L^p_{\text{div}}(\Omega), \quad \text{div } u = 0 \text{ in } \Omega, \quad u|_{\partial\Omega} = 0 \quad (4.1)\]

by means of the cut-off technique. To recover the divergence free on $u$ destroyed by cut-off technique, we will invoke the Bogovskiĭ operator given in [3, 4]. To state it, we define

\[W_0^{\alpha}(\Omega) = L^p(\Omega), \quad W_0^{m,p}(\Omega) = \{ g \in W^{m,p}(\Omega) \mid \partial_\alpha g|_{\partial\Omega} = 0, \ |\alpha| \leq m - 1 \} \]

**Lemma 4.1** (Bogovskiĭ operators). Let $p \in (1, \infty)$ and $E \subset \mathbb{R}^3$.

1. Assume that $E$ is a bounded Lipschitz domain, then there exists a bounded linear operator $B$ from $W_0^{m,p}(E)$ to $W^{m+1,p}(\mathbb{R}^3)$ such that $\text{supp } B[f] \subset E$ and

\[\|B[g]\|_{W^{m+1,p}(\mathbb{R}^3)} \leq C_{m,p}\|g\|_{W^{m,p}(E)} \quad (4.2)\]

for every integer $m \geq 0$. In addition, if $g \in W_0^{m,p}(E)$ satisfies $\int_E g \, dx = 0$, then

\[\text{div } B[g] = g \text{ in } E \quad \text{and} \quad \text{div } B[g] = 0 \text{ in } \mathbb{R}^3 \setminus E.\]

2. Let $m$ be a positive integer. Let $\phi \in C^\infty(\mathbb{R}^3)$ such that $\nabla \phi$ has a compact support and $\text{supp } \nabla \phi \subset E$. If $u \in W_0^{m,p}(E)$ satisfies $\text{div } u = 0$ in $E$ and $v \cdot u|_{\partial E} = 0$ where $v$ is the unit outer normal vector of $E$, then $\nabla \phi \cdot u \in W_0^{m,p}(\text{supp } \nabla \phi)$ and

\[\int_{\text{supp } \nabla \phi} \nabla \phi \cdot u \, dx = 0.\]

3. Let $\phi$ be the same function given in (2), then

\[\|B(\nabla \phi \cdot u)\|_{W^{j,p}(\mathbb{R}^3)} \leq C_{j,p}\|u\|_{W^{j-1}(\text{supp } \nabla \phi)}, \quad j = 1, 2, \]

\[\|B(\nabla \phi \cdot \nabla g)\|_{W^{j,p}(\mathbb{R}^3)} \leq C_{j,p,R}\|g\|_{W^{j-1}(\text{supp } \nabla \phi)}, \quad j = 0, 1, 2. \quad (4.3)\]

Let $\varphi \in C^\infty_0(\mathbb{R}^3)$ satisfy $0 \leq \varphi \leq 1$, $\varphi = 1$ in $B_{r+1}$ and $\varphi = 0$ in $B^c_{r+2}$. Denote $f_{\Omega_{R+3}}$ and $f_0$ by the restriction of $f$ on $\Omega_{R+3}$ and the zero extension of $f$ to $\mathbb{R}^3$, respectively. Then, we can construct a parametrix $(\Phi_{\partial\Omega,w}(\lambda)f, \Psi_{\partial\Omega,w}(\lambda)f)$ of (4.1) as follows:

\[\begin{cases}
\Phi_{\partial\Omega,w}(\lambda)f = (1 - \varphi)R^G_{\partial\Omega,w}(\lambda) f_0 + \varphi R^I_{\partial\Omega,w}(\lambda) f_{\Omega_{R+3}} + B[\nabla \varphi \cdot D(\lambda) f], \\
\Psi_{\partial\Omega,w}(\lambda)f = (1 - \varphi)(\partial_{R^3} + \Pi^G_{\partial\Omega,w}(\lambda)) f_0 + \varphi(\partial_{\Omega_{R+3}} + \Pi^I_{\partial\Omega,w}(\lambda)) f_{\Omega_{R+3}},
\end{cases} \quad (4.4)\]

with

\[D(\lambda)f = R^G_{\partial\Omega,w}(\lambda) f_0 - R^I_{\partial\Omega,w}(\lambda) f_{\Omega_{R+3}}.\]

One easily verifies that

\[\begin{cases}
(\lambda I + L_{\partial\Omega,w})\Phi_{\partial\Omega,w}(\lambda)f + \nabla \Psi_{\partial\Omega,w}(\lambda)f = (I + T + K_{\partial\Omega,w}(\lambda)) f \quad \text{in } \Omega, \\
\text{div } \Phi_{\partial\Omega,w}(\lambda)f = 0 \quad \text{in } \Omega, \quad \Phi_{\partial\Omega,w}(\lambda)f|_{\partial\Omega} = 0,
\end{cases} \quad (4.5)\]

where

\[T f = -\nabla \varphi \cdot (\partial_{R^3} f_0 - \partial_{\Omega_{R+3}} f_{\Omega_{R+3}}) - B[\nabla \varphi \cdot \nabla (\partial_{R^3} f_0 - \partial_{\Omega_{R+3}} f_{\Omega_{R+3}})], \quad (4.6)\]

\[K_{\partial\Omega,w}(\lambda)f = (\partial_{\Omega_{R+3}} D(\lambda)f + 2\nabla \varphi \cdot \nabla D(\lambda)f + R(\partial \varphi) D(\lambda)f + \omega((e_1 x) \cdot \nabla) D(\lambda)f - (u \cdot \nabla \varphi) D(\lambda)f + L_{\partial\Omega,w} B[\nabla \varphi \cdot D(\lambda)f]
\]

\[+ B[\nabla \varphi \cdot (L_{\partial\Omega,w} D(\lambda)f)] - B[\nabla \varphi \cdot \nabla \Xi(\lambda)f] - \nabla \varphi \cdot \Xi(\lambda)f, \quad (4.7)\]
Hence, we have from the Neumann series expansion that

\[ \Xi(\lambda) f = \prod_{j=0}^{\infty} \left( \mathcal{K}_{\lambda,\omega,\varnothing}(\lambda) f_0 - \prod_{j=0}^{\infty} \mathcal{K}_{\lambda,\omega,\varnothing}(\lambda) f_{\Omega_{R+3}} \right) \]

We see that \( T + K_{\lambda,\omega,\varnothing}(\lambda) \) is a compact operator from \( L^{p}_{R+2}(\Omega) \) to itself, and

\[
\begin{aligned}
\left\{ \mathcal{K}_{\lambda,\omega,\varnothing}(\lambda) \in &\mathscr{A}(C_{+}, L(L^{p}_{R+2})) \cap C(C_{+}, L(L^{p}_{R+2}(\Omega))) , \\
\| \mathcal{K}_{\lambda,\omega,\varnothing}(\lambda) \|_{L(L^{p}_{R+2}(\Omega))} \leq & C_{R,\lambda,\varnothing,\omega} (1 + |\lambda|)^{-1/2+1/(2p)}, \quad \lambda \in C_{+}.
\end{aligned}
\] (4.8)

In fact, by Poincare’s inequality and Lemma 4.1, we know \( T \) is a compact operator from \( L^{p}_{R+2}(\Omega) \) to itself. In addition, from Lemma 4.1, we have

\[
\| \mathcal{K}_{\lambda,\omega,\varnothing}(\lambda) \|_{L(L^{p}_{R+2}(\Omega))} \leq C_{R,\lambda,\varnothing,\omega} (1 + |\lambda|)^{-1/2+1/(2p)}, \quad \lambda \in C_{+}.
\] (4.9)

which together with Theorem 2.2 and Corollary 2.7 implies that \( K_{\lambda,\omega,\varnothing}(\lambda) \) is a compact operator from \( L^{p}_{R+2}(\Omega) \) to itself and satisfies (4.8).

If \( (I + T + K_{\lambda,\omega,\varnothing}(\lambda))^{-1} \in L(L^{p}_{R+2}(\Omega)) \), we construct the solution operators

\[
\begin{aligned}
\mathcal{R}_{\lambda,\omega,\varnothing}(\lambda) & = \Phi_{\lambda,\omega,\varnothing}(\lambda)(I + T + K_{\lambda,\omega,\varnothing}(\lambda))^{-1}, \\
\Pi_{\lambda,\omega,\varnothing}(\lambda) & = \Psi_{\lambda,\omega,\varnothing}(\lambda)(I + T + K_{\lambda,\omega,\varnothing}(\lambda))^{-1},
\end{aligned}
\] (4.10)

such that \( u = \mathcal{R}_{\lambda,\omega,\varnothing}(\lambda) f + P = \Pi_{\lambda,\omega,\varnothing}(\lambda) f \) satisfy (4.1) provided \( f \in L^{p}_{R+2}(\Omega) \).

Now we are in position to prove the invertibility of \( I + T + K_{\lambda,\omega,\varnothing}(\lambda) \) for all \( \lambda \in C_{+} \).

**Proposition 4.1.** Let \( p \in (1, \infty), \varepsilon \in (0, \frac{1}{2}], 0 < R_{*} \leq |R| \leq R^{*} \) and \( |\omega| \leq \omega^{*} \). Then, there exists a constant \( \eta = \eta_{p,R,\varnothing,\varnothing,\omega} > 0 \) such that if \( \| w \|_{L^{p}(\Omega)} \leq \eta \), then, \( (I + T + K_{\lambda,\omega,\varnothing}(\lambda))^{-1} \in \mathscr{A}(C_{+}, L(L^{p}_{R+2})) \cap C(C_{+}, L(L^{p}_{R+2}(\Omega))) \) satisfies

\[
\|(I + T + K_{\lambda,\omega,\varnothing}(\lambda))^{-1}\|_{L(L^{p}_{R+2}(\Omega))} \leq C_{R,\lambda,\varnothing,\varnothing,\omega}, \quad \lambda \in C_{+}.
\] (4.11)

We begin the proof with an known lemma.

**Lemma 4.2** \((23)\). Let \( p \in (1, \infty) \), and \( T \) be the operator defined in \((4.6)\). Then, \( (I + T)^{-1} \in L(L^{p}_{R+2}(\Omega)) \) satisfies \( \|(I + T)^{-1}\|_{L(L^{p}_{R+2}(\Omega))} \leq C_{R} \).

**Proof of Proposition 4.1.** Define \( C_{\ell} \triangleq \{ \lambda \in C_{+} | |\lambda| \geq \ell \} \). We will divide into two cases to prove this proposition.

**Case 1:** \( \lambda \in C_{0} \). From (4.8), there exists an \( \ell_{0} = C_{p,R,\varnothing,\varnothing,\omega}^{*} \), such that

\[
\|(I + T)^{-1} K_{\lambda,\omega,\varnothing}(\lambda)\|_{L(L^{p}_{R+2}(\Omega))} \leq 1/2, \quad \lambda \in C_{0}.
\] (4.12)

Hence, we have from the Neumann series expansion that

\[
(I + T + K_{\lambda,\omega,\varnothing}(\lambda))^{-1} = \sum_{j=0}^{\infty} (- (I + T)^{-1} K_{\lambda,\omega,\varnothing}(\lambda))^{j} (I + T)^{-1},
\]

satisfies

\[
\begin{aligned}
(I + T + K_{\lambda,\omega,\varnothing}(\lambda))^{-1} & \in \mathscr{A}(C_{\ell_{0}}, L(L^{p}_{R+2}(\Omega))) \cap C(C_{\ell_{0}}, L(L^{p}_{R+2}(\Omega))), \\
\|(I + T + K_{\lambda,\omega,\varnothing}(\lambda))^{-1}\|_{L(L^{p}_{R+2}(\Omega))} & \leq C_{R}, \quad \lambda \in C_{0}.
\end{aligned}
\]
Case 2: $\lambda \in \overline{C_+}$ and $|\lambda| \leq \ell_0$. If

$$(I + T + K_{\mathcal{G}, \omega, \mathcal{W}}(\lambda))^{-1} \in \mathcal{L}(L^p_{R+2}(\Omega)), $$

we have $(I + T + K_{\mathcal{G}, \omega, \mathcal{W}}(\lambda))^{-1} \in C(\overline{C_+}, \mathcal{L}(L^p_{R+2}(\Omega)))$ by (4.13). This implies that (4.11) holds for all $\lambda \in \overline{C_+}$ with $|\lambda| \leq \ell_0$.

Thus, in what follows, we will focus on the proof of (4.13). Since $T + K_{\mathcal{G}, \omega, \mathcal{W}}(\lambda) \in \mathcal{L}(L^p_{R+2}(\Omega))$ is a compact operator from $L^p_{R+2}(\Omega)$ to itself, it suffices to prove that $I + T + K_{\mathcal{G}, \omega, \mathcal{W}}(\lambda)$ is an injection by the Fredholm alternative theorem. For this purpose, we will prove that, given $\lambda \in \overline{C_+}$, if $f \in L^p_{R+2}(\Omega)$ satisfies

$$(I + T + K_{\mathcal{G}, \omega, \mathcal{W}}(\lambda)) f = 0,$$

then $f = 0$. For such $f$, we construct by (4.15)

$$(u, P) = (\Phi_{\mathcal{G}, \omega, \mathcal{W}}(\lambda) f, \Psi_{\mathcal{G}, \omega, \mathcal{W}}(\lambda) f)$$

such that

$$(\lambda - \Delta) u + \nabla P = -(L_{\mathcal{G}, \omega, \mathcal{W}} + \Delta) u, \quad \text{div} u = 0 \text{ in } \Omega, \quad u|_{\partial \Omega} = 0. \quad (4.14)$$

We claim that

$$\text{Re} \lambda \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)} = -\text{Re} \langle u \cdot \nabla \phi, u \rangle_{\Omega}. \quad (4.15)$$

In fact, from the local solvability theory of the classical Stokes equations with non-slip boundary condition and the bootstrap argument used in the proof of Theorem 5.1, we obtain that $u \in \mathcal{W}_{R+2}^{2,2}(\Omega)$ and $P \in W^{1,2}_{loc}(\Omega)$. Choose a bump function $\phi \in C^\infty_0(\Omega)$ satisfying $\phi = 1$ in $B_1$ and $\phi = 0$ outside $B_2$. Multiplying (4.14) by $\phi \bar{u}$ with $\phi(x) = \phi(|x|)$, we get

$$\lambda \langle u, \phi \bar{u} \rangle_{\Omega} + \langle \nabla u, \phi \nabla \bar{u} \rangle_{\Omega} = -\langle \nabla u, \nabla \phi \otimes u \rangle_{\Omega} - \langle (L_{\mathcal{G}, \omega, \mathcal{W}} + \Delta) u, \phi \bar{u} \rangle_{\Omega} + \langle \bar{P}, \nabla \phi \cdot u \rangle_{\Omega} \quad (4.16)$$

where

$$\bar{P} = P + \frac{1}{|\Omega_{R+3}|} \int_{\Omega_{R+3}} Q_{R^3} f_0 \, dx + \frac{1}{|\Omega_{R+3}|} \int_{\Omega_{R+3}} \Pi_{G, \omega, \mathcal{W}}(\lambda) f_0 \, dx.$$ 

Invoking that $\text{div} u = \text{div} w = 0$, we easily calculate

$$\text{Re} \langle (L_{\mathcal{G}, \omega, \mathcal{W}} + \Delta) u, \phi \bar{u} \rangle_{\Omega} = -\frac{1}{2} \int_\Omega |u|^2 (\mathcal{G} \partial_1 \phi \bar{u} + w \cdot \nabla \phi) \, dx + \text{Re} \langle u \cdot \nabla \phi \bar{u} \rangle_{\Omega} \quad \Delta J_1 + J_2.$$ 

Since $u = \mathcal{R}_{\mathcal{G}, \omega, \mathcal{W}}(\lambda) f_0$ in $B_{2(R+2)}$, we have by (2.60)

$$|\nabla^k u(x)| = O(|x|^{-1-k/2}(1 + s_\mathcal{G}(x))^{-1}), \quad k = 0, 1, \quad x \in B_{2(R+2)}.$$ 

This, combining with Lemma 8.3 yields for $\ell \geq 10(R+2)$,

$$|\langle \nabla u, \nabla \phi \otimes u \rangle_{\Omega}| \leq \ell^{-1} \int_{|x| \leq 2 \ell} |x|^{-\frac{\ell}{2}} \, dx = O(\ell^{-\frac{\ell}{2}}),$$

$$|J_1| \leq \ell^{-1} \int_{|x| \leq 2 \ell} \left( \frac{1}{|x|^2(1 + s_\mathcal{G}(x))} + \frac{\|w\|_{L^2(\Omega)}}{|x|^2} \right) \, dx = O(\ell^{-\frac{\ell}{2}}).$$
Moreover, since $f_0 = 0$ in $B_{R+2}^c$ and the kernel function of $Q_{R^3}$ is bounded by $|x|^{-2}$, we obtain for $x \in B_{9(R+2)}^c$

$$
\bigg| (Q_{R^3} f_0)(x) \bigg| \leq \int_{|y| \leq R+2} \frac{|f_0(y)|}{|x-y|^2} \, dy \leq \int_{|x-y| \geq \frac{\varepsilon}{|y|}} \frac{|f_0(y)|}{|x-y|^2} \, dy = O(|x|^{-2}).
$$

This, together with (2.136) and Lemma 8.4, implies

$$
|\bar{P}(x)| = O(|x|^{-3/2}), \quad x \in B_{10(R+2)}^c,
$$

which implies

$$
\left| \langle \bar{P}, u \cdot \nabla \phi \rangle \right| \leq \ell^{-1} \int_{|x| \leq 2\ell} |x|^{-5/2} \, dx = O(\ell^{-1/2}).
$$

Hence, letting $\ell \to \infty$ in (4.16), we prove (4.15) by the Lebesgue dominated convergence theorem.

Owing to $\text{div } u = \text{div } w = 0$ and $u_{\partial \Omega} = 0$, one gets

$$
\Re \langle u \cdot \nabla u, w \rangle = -\int_{\Omega} (\Re u \cdot \nabla) \Re u \cdot w + (\text{Im } u \cdot \nabla) \text{Im } u \cdot w \, dx.
$$

This identity, together with (4.15) and Lemma 8.4, implies

$$
\Re \lambda \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 \leq C \|u\|_{E, \Omega} \|\nabla u\|_{L^2(\Omega)}^2.
$$

Hence, we deduce $\nabla u = 0$ in $\Omega$ only if $C \|u\|_{E, \Omega} < 1$. This fact, combining with (4.14) and (4.17), we get $\tilde{P} = 0$ in $\Omega$. Thus, we have from (4.3)

$$
\begin{cases}
(1 - \varphi) R^G_{\partial \Omega, w, \Omega}(\lambda) f_0 + \varphi R^I_{\partial \Omega, w, \Omega}(\lambda) f_{\Omega_{R+3}} + B[\nabla \varphi \cdot \mathcal{D}(\lambda) f] = 0, \\
(1 - \varphi) (Q_{R^3} + \bar{P}_{\partial \Omega, w, \Omega}(\lambda)) f_0 + \varphi (Q_{\Omega_{R+3}} + \bar{P}_{\partial \Omega, w, \Omega}(\lambda)) f_{\Omega_{R+3}} = -\frac{1}{|\Omega_{R+3}|} \int_{\Omega_{R+3}} (Q_{R^3} + \bar{P}_{\partial \Omega, w, \Omega}(\lambda)) f_0 \, dx,
\end{cases}
$$

which implies

$$
\begin{cases}
R^G_{\partial \Omega, w, \Omega}(\lambda) f_0 = 0, \quad Q_{R^3} f_0 + \bar{P}_{\partial \Omega, w, \Omega}(\lambda) f_{\Omega_{R+3}} = 0 \quad \text{in } B_{R+2}, \\
R^I_{\partial \Omega, w, \Omega}(\lambda) f_{\Omega_{R+3}} = 0 \quad \text{in } B_{R+1}, \\
(Q_{\Omega_{R+3}} + \bar{P}_{\partial \Omega, w, \Omega}(\lambda)) f_{\Omega_{R+3}} = -\frac{1}{|\Omega_{R+3}|} \int_{\Omega_{R+3}} (Q_{R^3} + \bar{P}_{\partial \Omega, w, \Omega}(\lambda)) f_0 \, dx \quad \text{in } B_{R+1}.
\end{cases}
$$

On the other hand, observing $R^I_{\partial \Omega, w, \Omega}(\lambda) f_{\Omega_{R+3}}|_{\partial \Omega_{R+3}} = 0$, $\chi_{\Omega_{R+3}} f_{\Omega_{R+3}} = f_0$ and the first inequality in (1.12), we easily verify that

$$
\begin{align*}
\nu_1 &= R^G_{\partial \Omega, w, \Omega}(\lambda) f_0, \quad \theta_1 = Q_{R^3} f_0 + \bar{P}_{\partial \Omega, w, \Omega}(\lambda) f_0, \\
\nu_2 &= \chi_{\Omega_{R+3}} R^I_{\partial \Omega, w, \Omega}(\lambda) f_{\Omega_{R+3}}, \quad \theta_2 = \chi_{\Omega_{R+3}} (Q_{\Omega_{R+3}} + \bar{P}_{\partial \Omega, w, \Omega}(\lambda)) f_{\Omega_{R+3}} + \frac{1}{|\Omega_{R+3}|} \int_{\Omega_{R+3}} (Q_{R^3} + \bar{P}_{\partial \Omega, w, \Omega}(\lambda)) f_0 \, dx,
\end{align*}
$$

all solve
\[(\lambda + L_{\Omega,w})v + \nabla \theta = f_0, \quad \text{div } v = 0 \quad \text{in } B_{R+3}, \quad v|_{\partial B_{R+3}} = 0.\]

By uniqueness, we have \(v_1 = v_2\) and \(\theta_1 - \theta_2 = c\) for some constant \(c\) in \(B_{R+3}\), which implies in \(\Omega_{R+3}\)
\[R_{\Omega,w}(\lambda)f_{\Omega_{R+3}} = R_{\Omega,w}(\lambda)f_0,\]
\[Q_{R^2}f_0 + \Pi_{R,\Omega,w}^G(\lambda)f_{\Omega_{R+3}} - \Pi_{R,\Omega,w}^I(\lambda)f_{\Omega_{R+3}} = c.\]

Plugging these equalities into (4.18), we get
\[R_{\Omega,w}(\lambda)f_0 = 0, \quad Q_{R^2}f_0 + \Pi_{R,\Omega,w}^G(\lambda)f_0 = c \quad \text{in } \Omega_{R+3}. \quad (4.20)\]

Since
\[c|_{\Omega_{R+3}} = \int_{\Omega_{R+3}} \left[ Q_{R^2}f_0 + \Pi_{R,\Omega,w}^Gf_0 - \Pi_{R,\Omega,w}^If_{\Omega_{R+3}} \right] dx = 0,\]
we deduce by (4.19)-(4.20)
\[f = (\lambda + L_{\Omega,w})R_{\Omega,w}(\lambda)f_0 + \nabla (Q_{R^2}f_0 + \Pi_{R,\Omega,w}^G(\lambda)f_0) = 0 \quad \text{in } \Omega,\]
and so complete the proof of Proposition 4.1. \(\square\)

Next, we will discuss the decay of \((I + T + K_{\Omega,w,\Omega}(\lambda))^{-1}\) with respect to \(\lambda\).

**Proposition 4.2.** Assume that \(p \in (1, \infty)\), \(\theta \in (0, \frac{\pi}{2})\), \(0 < |\mathcal{R}| \leq \mathcal{R}^*\) and \(|\omega| \leq \omega^*\). Let \(\varepsilon \in (0, \frac{1}{2})\) if \(p \geq \frac{6}{5}\) otherwise \(\varepsilon \in (0, \frac{3p-3}{4p})\). Then, there exist constants \(\eta = \eta_{p, R, \mathcal{R}^*, \omega^*} > 0\) and
\[\ell_3 = \ell_{p, R, \mathcal{R}^*, \omega^*} > \ell_2 \triangleq \max(\ell_0, \ell_1), \quad \ell_0, \ell_1 \quad \text{same as in Th.2.4 and Th.3.1}\]
such that if \(\|w\|_{L^p, \Omega} \leq \eta\), then
\[(I + T + K_{\Omega,w,\Omega}(\lambda))^{-1} = (I + T)^{-1} + S_{\Omega,w,\Omega}^1(\lambda) + S_{\Omega,w,\Omega}^2(\lambda), \quad \lambda \in C_{+\ell_3} \quad (4.21)\]
where
\[S_{\Omega,w,\Omega}^1(\lambda) \in \mathcal{A}(\Sigma_{\ell, \ell_3}, L(L^p_{R+2}(\Omega))), \quad S_{\Omega,w,\Omega}^2(\lambda) \in \mathcal{A}(C_{+\ell_3}, L(L^p_{R+2}(\Omega))).\]

satisfying
\[\|S_{\Omega,w,\Omega}^1(\lambda)\|_{L(L^p_{R+2}(\Omega)))} \leq C_{\theta, R, \mathcal{R}^*, \omega^*}|\lambda|^{-\frac{1}{2} + (1/2p)}, \quad \lambda \in \Sigma_{\theta, \ell_3}, \quad (4.22)\]
\[\|S_{\Omega,w,\Omega}^2(\lambda)\|_{L(L^p_{R+2}(\Omega)))} \leq C_{\theta, R, \delta, \mathcal{R}^*, \omega^*}|\lambda|^{-\frac{1}{2} + \delta}, \quad \lambda \in C_{+\ell_3}, \quad 0 < \delta < 1/2. \quad (4.23)\]

**Proof.** Let \(f \in L^p_{R+2}(\Omega)\), We split \(K_{\Omega,w,\Omega}(\lambda)f = K_{\Omega,w,\Omega}^1(\lambda)f + K_{\Omega,w,\Omega}^2(\lambda)f\) from (2.36) and (2.52), where
\[K_{\Omega,w,\Omega}^1(\lambda)f \triangleq (\Delta \varphi)D^1(\lambda)f + 2\varphi \cdot \nabla D^1(\lambda)f + \mathcal{R}(\partial_1 \varphi)D^1(\lambda)f + \mathcal{R}(\varphi \cdot \nabla D^1(\lambda)f + L_{\Omega,w,\Omega}[\nabla \varphi \cdot D^1(\lambda)f] + \mathcal{R}[\nabla \varphi \cdot (L_{\Omega,w,\Omega}D^1(\lambda)f)] - \mathcal{R}[\nabla \varphi \cdot \nabla \Xi^1(\lambda)f] - \nabla \varphi \cdot \Xi^1(\lambda)f,\]
\[K_{\Omega,w,\Omega}^2(\lambda)f \triangleq K_{\Omega,w,\Omega}(\lambda)f - K_{\Omega,w,\Omega}^1(\lambda)f.\]
with

$$D^1(\lambda) f = (\lambda I - \Delta - \mathcal{R}_0 \partial_1)^{-1} \mathcal{P}_R^3 + \mathcal{R}_{R_0}^{G,1}(\lambda) f_0 - \mathcal{R}_{R_0}^I(\lambda) f \in \Omega_{R_0^3},$$

$$\Xi^1(\lambda) f = \Pi_{R_0}^{G,1}(\lambda) f_0 - \Pi_{R_0}^I(\lambda) f \in \Omega_{R_0^3}.$$

By Theorem 2.4, Corollary 2.5, Theorem 3.1 and Lemma 4.1, we deduce

$$K_{R_0}^1(\lambda) \in \mathcal{S}(\Sigma_{\theta, \ell_2}, L(|R_0|^{3/2}(\Omega))), \quad K_{R_0}^2(\lambda) \in \mathcal{S}(\Sigma_{\theta, \ell_2}, L(|R_0|^{9/2}(\Omega))),$$

$$\|K_{R_0}^1(\lambda)\|_{L(|R_0|^{3/2}(\Omega))} \leq C_{\theta, R} R_0^{9/2} \lambda^{-1/2 + 1/(2p)}, \quad \lambda \in \Sigma_{\theta, \ell_2},$$

$$\|K_{R_0}^2(\lambda)\|_{L(|R_0|^{9/2}(\Omega))} \leq C_{\theta, R} R_0^{9/2} \lambda^{-1/2 + \delta}, \quad \lambda \in C_{\theta, \ell_2}, \quad 0 < \delta \ll 1/2. \quad (4.24)$$

Hence, we choose a integer $$N > 0$$ such that $$(1/2 - 1/2p) N \geq 2$$, and set

$$S_{R_0}^1(\lambda) = \sum_{j=1}^N (-(I + T)^{-1} K_{R_0}^1(\lambda))^j (I + T)^{-1},$$

$$S_{R_0}^2(\lambda) = \sum_{j=1}^\infty (-(I + T)^{-1} K_{R_0}^1(\lambda))^j (I + T)^{-1} - S_{R_0}^1(\lambda).$$

By (4.24)-(4.25), we deduce that there exists a $$\ell_3 = C_{\theta, R} R_0^{9/2} \lambda > \ell_2$$ such that (4.21)-(4.23) hold. So we complete the proof of this proposition.

**Proposition 4.3.** Let $$p \in (0, 1/2)$$. Under the assumption of Proposition 4.1, there exists a constant $$\eta = \eta_{p, R, \mathcal{G}, \omega} > 0$$ such that if $$\|w\|_{L^p(\Omega)} \leq \eta$$, then for $$\lambda, \lambda + h \in \mathbb{C}_+$$

$$\|\partial_\lambda (I + T + K_{R_0}^1(\lambda))^{-1}\|_{L(|R_0|^{3/2}(\Omega))} \leq C_{\theta, R} R_0^{9/2} \lambda^{-1/2 + \delta}, \quad 0 < \frac{\eta}{\ell_2} \ll 1/2,$$

$$\|\Delta_h \partial_\lambda (I + T + K_{R_0}^1(\lambda))^{-1}\|_{L(|R_0|^{3/2}(\Omega))} \leq C_{\theta, R} R_0^{9/2} \lambda^{-1/2 + \delta}. \quad (4.27)$$

In particular, for $$0 < |h| \leq h_0$$

$$\|\Delta_h \partial_\lambda (I + T + K_{R_0}^1(\lambda))^{-1}\|_{L(|R_0|^{3/2}(\Omega))} \leq C_{\theta, R} R_0^{9/2} \lambda^{-1/2 + \delta}, \quad 0 < \eta \ll 1/2.$$

**Proof.** By Theorem 2.6, Corollary 2.7, Theorem 3.1 and Lemma 4.1, we deduce for $$\lambda, \lambda + h \in \mathbb{C}_+, \rho \in (0, 1/2)$$ and $$0 < \eta \ll 1/2$$

$$\|\partial_\lambda K_{R_0}^1(\lambda)\|_{L(|R_0|^{3/2}(\Omega))} \leq C_{\theta, R} R_0^{9/2} \lambda^{-1/2 + \delta},$$

$$\|\partial\partial_\lambda K_{R_0}^1(\lambda)\|_{L(|R_0|^{3/2}(\Omega))} \leq C_{\rho, R} R_0^{9/2} \lambda^{-1/2 + \delta},$$

$$\|\partial_\lambda (I + T + K_{R_0}^1(\lambda))^{-1}\|_{L(|R_0|^{3/2}(\Omega))} \leq C_{\theta, R} R_0^{9/2} \lambda^{-1/2 + \delta}.$$ 

These estimates, combining with (4.11) and the fact that

$$\partial_\lambda (I + T + K_{R_0}^1(\lambda))^{-1} = (I + T + K_{R_0}^1(\lambda))^{-1} K_{R_0}^1(\lambda) (I + T + K_{R_0}^1(\lambda))^{-1},$$

yield Proposition 4.3. 

Finally, relying on the above analysis of $$(I + T + K_{R_0}^1(\lambda))^{-1}$$, we have the solution operators defined in (4.10) process the following properties.
Theorem 4.4. Under the assumption of Proposition 4.2, there exists a positive constant \( \eta = \eta_{p,R,\mathcal{R},\omega^*} \) such that if \( \| w \|_{L^q} \leq \eta \), then

\[
\begin{cases}
R_{\mathcal{R},R}^1(\omega) \in \mathcal{A}(C+\ell_3, \mathcal{L}(L^p_{R+2}(\Omega), \mathcal{W}^{2,p}(\Omega) \cap \mathcal{P}(\Omega))), \\
\Pi_{\mathcal{R},R}^1(\omega) \in \mathcal{A}(C+\ell_3, \mathcal{L}(L^{p}_{R+2}(\Omega), \hat{W}^{1,p}(\Omega))),
\end{cases}
\]

satisfying for \( f \in L^p_{R+2}(\Omega) \) and \( \lambda \in C+\ell_2 \)

\[
\begin{cases}
R_{\mathcal{R},R}^1(\omega) f = R_{\mathcal{R},R}^1(\omega) f + R_{\mathcal{R},R}^2(\omega) f, \\
\Pi_{\mathcal{R},R}^1(\omega) f = \Pi f + \Pi_{\mathcal{R},R}^1(\omega) f + \Pi_{\mathcal{R},R}^2(\omega) f, \\
\Pi f = (1 - \varphi) \hat{Q}_{\Omega}^1 [ (1 + T)^{-1} f ]_0 + \varphi \hat{Q}_{\Omega+3} [(1 + T)^{-1} f]_{\Omega+3},
\end{cases}
\]

where \( \varphi \) is the same function as in (4.4) such that

\[
\begin{cases}
R_{\mathcal{R},R}^1(\omega) \in \mathcal{A}(\Sigma_{\theta,\ell_3}, \mathcal{L}(L^p_{R+2}(\Omega), \mathcal{W}^{2,p}(\Omega))), \\
R_{\mathcal{R},R}^2(\omega) \in \mathcal{A}(\Sigma_{\theta,\ell_3}, \mathcal{L}(L^p_{R+2}(\Omega), \mathcal{W}^{2,p}(\Omega))), \\
\Pi_{\mathcal{R},R}^1(\omega) \in \mathcal{A}(\Sigma_{\theta,\ell_3}, \mathcal{L}(L^p_{R+2}(\Omega), \hat{W}^{1,p}(\Omega))), \\
\Pi_{\mathcal{R},R}^2(\omega) \in \mathcal{A}(\Sigma_{\theta,\ell_3}, \mathcal{L}(L^p_{R+2}(\Omega), \hat{W}^{1,p}(\Omega))),
\end{cases}
\]

satisfying for every \( |\beta| \leq 2 \)

\[
\| \partial^2_{\beta} R_{\mathcal{R},R}^1(\omega) \|_{L^p(\Omega)} \leq C_{\theta,R,\mathcal{R},\omega^*} |\lambda|^{-1+\frac{|\beta|}{2}}, \quad \lambda \in \Sigma_{\theta,\ell_2},
\]

\[
\| \partial^2_{\beta} R_{\mathcal{R},R}^2(\omega) \|_{L^p(\Omega)} \leq C_{\theta,R,\mathcal{R},\omega^*} |\lambda|^{-1+\frac{|\beta|}{2}}, \quad \lambda \in \Sigma_{\theta,\ell_2},
\]

\[
\| \nabla \Pi_{\mathcal{R},R}^1(\omega) \|_{L^p(\Omega)} \leq C_{\theta,R,\mathcal{R},\omega^*}, \quad \lambda \in \Sigma_{\theta,\ell_2},
\]

\[
\| \nabla \Pi_{\mathcal{R},R}^2(\omega) \|_{L^p(\Omega)} \leq C_{\theta,R,\mathcal{R},\omega^*}, \quad \lambda \in \Sigma_{\theta,\ell_2},
\]

\[
\| \Pi_{\mathcal{R},R}^1(\omega) \|_{L^p(\Omega)} \leq C_{\theta,R,\mathcal{R},\omega^*}, \quad \lambda \in \Sigma_{\theta,\ell_2},
\]

Proof. In view of (2.33), (2.59) and (1.21), we have the decomposition (1.26) with

\[
\begin{align*}
R_{\mathcal{R},R}^1(\omega) f &= (1 - \varphi)(\lambda - \Delta - \mathcal{R}\partial_1)^{-1} \mathcal{P}_{\mathcal{R}}^1 [(I + T)^{-1} f + S_{\mathcal{R},\omega}^1(\omega) f]_0 + (1 - \varphi)^{\mathcal{R}}_{\mathcal{R},\omega} f + S_{\mathcal{R},\omega}^1(\omega) f, \\
&\quad + \varphi \mathcal{R}_{\mathcal{R},\omega}^1(\omega) [(I + T)^{-1} f + S_{\mathcal{R},\omega}^1(\omega) f]_{\Omega+3} + \mathcal{B} \left[ \nabla \varphi \cdot D^1(\lambda) [(I + T)^{-1} f + S_{\mathcal{R},\omega}^1(\omega) f]_{\Omega+3} \right],
\end{align*}
\]

\[
\begin{align*}
R_{\mathcal{R},R}^2(\omega) f &= R_{\mathcal{R},R}^2(\omega) f - R_{\mathcal{R},R}^1(\omega) f, \\
\Pi_{\mathcal{R},R}^1(\omega) f &= (1 - \varphi) \hat{Q}_{\Omega}^1 [ S_{\mathcal{R},\omega}^1(\omega) f ]_0 + \varphi \hat{Q}_{\Omega+3} [ S_{\mathcal{R},\omega}^1(\omega) f ]_{\Omega+3} + (1 - \varphi)^{\mathcal{R}}_{\mathcal{R},\omega} f + S_{\mathcal{R},\omega}^1(\omega) f, \\
&\quad + \varphi \mathcal{R}_{\mathcal{R},\omega}^1(\omega) f + (I + T)^{-1} f + S_{\mathcal{R},\omega}^1(\omega) f, \\
\Pi_{\mathcal{R},R}^2(\omega) f &= \Pi_{\mathcal{R},R}^2(\omega) f - \Pi f - \Pi E_{\mathcal{R},R}^1(\omega) f.
\end{align*}
\]

By Theorem 2.1, Corollary 2.5, Theorem 3.1, Lemma 4.2 and Proposition 4.2, we deduce (1.27) + (1.33), and so finish the proof of Theorem 4.4. □
Theorem 4.5. Let $\rho \in (0, \frac{1}{2})$. Under the assumption of Proposition 4.4, there exists a constant $\eta = \eta_{p,R, R^*, \omega^*} > 0$ such that if $\|w\|_{\varepsilon, \Omega} \leq \eta$, then

$$R_{\varepsilon,\omega,w}(\lambda) \in C(\overline{C_+}, L^p(\mathbb{R}^d; \mathbb{R}^d), \mathbb{W}^2p(\mathbb{R}^{d+1}))$$

satisfying for every $\lambda, \lambda + h \in \overline{C_+}$, $0 < \rho < \frac{1}{2}$ and $j \leq 2$,

\begin{align}
\|\nabla^j R_{\varepsilon,\omega,w}(\lambda)\|_{L^p(\mathbb{R}^{d+1}; L^p(\mathbb{R}^{d+1}))} &\leq C_{\varepsilon,\omega,w}(1 + |\lambda|)^{-1+(j/2)}, \\
\|\nabla^j (\partial_\lambda R_{\varepsilon,\omega,w}(\lambda))\|_{L^p(\mathbb{R}^{d+1}; L^p(\mathbb{R}^{d+1}))} &\leq C_{\varepsilon,\omega,w}(1 + |\lambda|)^{-2+(j/2)+\rho}, \\
\|\partial^h (\Delta_h \partial_\lambda R_{\varepsilon,\omega,w}(\lambda))\|_{L^p(\mathbb{R}^{d+1}; L^p(\mathbb{R}^{d+1}))} &\leq C_{\varepsilon,\omega,w}(1 + |\lambda|)^{\rho/2}.
\end{align}

In particular, for $0 < |h| \leq h_0$

$$\|\nabla^2 (\Delta_h \partial_\lambda R_{\varepsilon,\omega,w}(\lambda))\|_{L^p(\mathbb{R}^{d+1}; L^p(\mathbb{R}^{d+1}))} \leq C_{\varepsilon,\omega,w}(1 + |\lambda|)^{\rho/2}.$$

Since Theorem 4.5 is a direct consequence of Theorem 2.6, Theorem 3.1, Lemma 4.1 and Proposition 4.3, here we omit its proof.

Remark 4.6. Following the proof of Theorem 4.4 and Theorem 4.5, we can deduce that there exist operators $R_{\varepsilon,\omega,w}^*, \Pi_{\varepsilon,\omega,w}^*$ satisfying all estimates in Theorem 4.4 and Theorem 4.5 such that $(\mathbf{v}, \Theta) = (R_{\varepsilon,\omega,w}^*(\lambda) f, \Pi_{\varepsilon,\omega,w}^*(\lambda) f)$ solves

$$\begin{align}
(\lambda I + L_{\varepsilon,\omega,w}^*) v + \nabla \Theta &= f \in L^p(\mathbb{R}_+^2; \Omega), \\
\text{div} \mathbf{v} &= 0 \quad \text{in} \, \Omega, \\
\mathbf{v}\big|_{\partial \Omega} &= 0.
\end{align}$$

5. Behavior of $T_{\varepsilon,\omega,w}(t)P_\Omega$ and $T_{\varepsilon,\omega,w}^*(t)P_\Omega$ acting on $L^p_{\mathbb{R}_+^2}(\Omega)$.

In this section, we consider the behavior with respect to $t$ of the solution to the linear problem:

$$\begin{align}
\partial_t \mathbf{u} + L_{\varepsilon,\omega,w} \mathbf{u} + \nabla P &= \mathbf{0}, \\
\text{div} \mathbf{u} &= 0 \quad \text{in} \, \Omega \times (0, \infty), \\
\mathbf{u}\big|_{\partial \Omega} &= 0, \\
\mathbf{u}(x, 0) &= P_\Omega f, \\
f &\in L^p_{\mathbb{R}_+^2}(\Omega).
\end{align}$$

5.1. Behavior in a short time.

Theorem 5.1. Let $p \in (1, \infty)$, $0 < \varepsilon \leq |\varepsilon| \leq R^* \leq |\omega| \leq \omega^*$. Assume that $\varepsilon \in (0, \frac{1}{2})$ if $p \geq \frac{6}{5}$ otherwise $\varepsilon \in (0, \frac{2p-3}{p})$. Then there exists a constant $\eta = \eta_{p,R, R^*, \omega^*} > 0$ such that if $\|w\|_{\varepsilon, \Omega} \leq \eta$, then problem (5.1) admits a solution $(\mathbf{u}, P)$ with $\mathbf{u}$ represented by

$$\mathbf{u}(t) = \lim_{\ell \to \infty} \frac{1}{2\pi i} \int_{\gamma - i\ell}^{\gamma + i\ell} e^{\lambda t} R_{\varepsilon,\omega,w}(\lambda) f \, d\lambda, \quad \gamma > \ell_3,$$

satisfying

$$\mathbf{u} \in C(\overline{R_+}; \mathbb{W}^2(\Omega)) \cap C(\overline{R_+}; \mathbb{W}^2p(\Omega)) \cap C^1(\overline{R_+}; \mathbb{L}^p(\Omega)), \quad P \in C(\overline{R_+}; \mathbb{W}^1p(\Omega)), \quad P \in C(\overline{R_+}; \mathbb{W}^1p(\Omega)),$$

where $\ell_3$ is the same constant as in Theorem 4.4.
Proof. For $\gamma > \ell_3 > 0$, there exists $\theta_0 \in (\pi/2, \pi)$ such that
\[ \Gamma_{\theta_0, \gamma} \triangleq \{ \gamma + re^{\pm i\theta_0} | r \geq 0 \} \subset \Sigma_{\theta, \ell_3}. \]

We set from Theorem \[ \text{[4.4]} \]
\[ u^{(k)}_{\ell}(t) = \frac{1}{2\pi i} \int_{\Gamma_{\theta_0, \gamma}} e^{\lambda t} R^k_{\gamma, \omega, \omega} (\lambda) f \, d\lambda, \quad P^{(k)}_{\ell}(t) = \frac{1}{2\pi i} \int_{\Gamma_{\theta_0, \gamma}} e^{\lambda t} \Pi^k_{\gamma, \omega, \omega} (\lambda) f \, d\lambda, \]
\[ u^{(1)}(t) = \frac{1}{2\pi i} \int_{\Gamma_{\theta_0, \gamma}} e^{\lambda t} R^1_{\gamma, \omega, \omega} (\lambda) f \, d\lambda, \quad u^{(2)}(t) = \frac{1}{2\pi i} \int_{\gamma - i\ell}^{\gamma + i\ell} e^{\lambda t} R^2_{\gamma, \omega, \omega} (\lambda) f \, d\lambda, \]
\[ P^{(1)}_{\ell}(t) = \frac{1}{2\pi i} \int_{\Gamma_{\theta_0, \gamma}} e^{\lambda t} \Pi^1_{\gamma, \omega, \omega} (\lambda) f \, d\lambda, \quad P^{(2)}_{\ell}(t) = \frac{1}{2\pi i} \int_{\gamma - i\ell}^{\gamma + i\ell} e^{\lambda t} \Pi^2_{\gamma, \omega, \omega} (\lambda) f \, d\lambda, \]
\[ P^0_{\ell}(t) = \frac{1}{2\pi i} \int_{\gamma - i\ell}^{\gamma + i\ell} e^{\lambda t} f \, d\lambda. \]

with $k = 0, 1$. By Theorem \[ \text{[4.4]} \] and Lemma 5.1 in \[ \text{[23]} \], we deduce that
\[ u^{(1)}, u^{(2)} \in C^1(\mathbb{R}_+; L^p(\Omega)) \cap C(\mathbb{R}_+; W^{1,p}(\Omega)), \]
\[ P^{(1)}, P^{(2)} \in C(\mathbb{R}_+; W^{1,p}(\Omega)), \]

satisfying
\[
\begin{aligned}
&\lim_{t \to \infty} \sup_{0 < t_1 \leq t \leq t_2} \| u^{(1)}_\ell(t) - u^{(1)}(t) \|_{W^{1,p}(\Omega)} = 0, \\
&\lim_{t \to \infty} \sup_{0 \leq t \leq T} \| u^{(2)}_\ell(t) - u^{(2)}(t) \|_{W^{2,p}(\Omega)} + \| \partial_t u^{(2)}_\ell(t) - \partial_t u^{(2)}(t) \|_{L^p(\Omega)} = 0, \\
&\lim_{t \to \infty} \sup_{0 < t_1 \leq t \leq t_2} \| \nabla P^{(1)}_\ell(t) - \nabla P^{(1)}(t) \|_{L^p(\Omega)} + \| P^{(1)}_\ell(t) - P^{(1)}(t) \|_{L^p(\Omega_{\ell+3})} = 0, \\
&\lim_{t \to \infty} \sup_{0 \leq t \leq T} \| \nabla P^{(2)}_\ell(t) - \nabla P^{(2)}(t) \|_{L^p(\Omega)} + \| P^{(2)}_\ell(t) - P^{(2)}(t) \|_{L^p(\Omega_{\ell+3})} = 0,
\end{aligned}
\]

and
\[
\begin{aligned}
&t^{1/2} \| \nabla^j u^{(1)}_\ell(t) \|_{L^p(\Omega)} + t \| \partial_t u^{(1)}_\ell(t) \|_{L^p(\Omega)} \leq C_{R, \gamma, \mathcal{A}^*, \omega, \mathcal{E}^*} e^{\gamma t} \| f \|_{L^p(\Omega)}, \quad j \leq 2, \\
&t \| \nabla u^{(2)}_\ell(t) \|_{L^p(\Omega)} + t \| \partial_t u^{(2)}_\ell(t) \|_{L^p(\Omega)} \leq C_{R, \gamma, \mathcal{A}^*, \omega, \mathcal{E}^*} e^{\gamma t} \| f \|_{L^p(\Omega)}, \\
&t \| \nabla P^{(1)}_\ell(t) \|_{L^p(\Omega)} + t \ell^{(1/2) + (1/2p)} \| P^{(1)}_\ell(t) \|_{L^p(\Omega_{\ell+3})} \leq C_{R, \gamma, \mathcal{A}^*, \omega, \mathcal{E}^*} e^{\gamma t} \| f \|_{L^p(\Omega)}, \\
&t \| \nabla P^{(2)}_\ell(t) \|_{L^p(\Omega)} + \| P^{(2)}_\ell(t) \|_{L^p(\Omega_{\ell+3})} \leq C_{R, \gamma, \mathcal{A}^*, \omega, \mathcal{E}^*} e^{\gamma t} \| f \|_{L^p(\Omega)}.
\end{aligned}
\]

For $P^0_\ell$, we get by Lemma 5.2 and its remark in \[ \text{[23]} \]
\[ P^0_\ell \to 0 \quad \text{in} \quad \mathcal{D}'(\Omega \times \mathbb{R}_+). \]

Set
\[ u_\ell = u^{(1)}_\ell + u^{(2)}_\ell, \quad P_\ell = P^{(0)}_\ell + P^{(1)}_\ell + P^{(2)}_\ell, \quad u = u^{(1)} + u^{(2)}, \quad P = P^{(1)} + P^{(2)}. \]

Obviously,
\[ \partial_t u_\ell + L_{\mathcal{A}^*, \omega} u_\ell + \nabla P_\ell = \frac{1}{2\pi i} \int_{\gamma - i\ell}^{\gamma + i\ell} e^{\lambda t} f \, d\lambda, \quad \text{div} \ u_\ell = 0 \text{ in } \Omega \times \mathbb{R}_+, \quad u_\ell|_{\partial \Omega} = 0. \]
Letting $\ell \to \infty$, we have by \eqref{5.6} and \eqref{5.8} that $\text{div } u = 0$ in $\Omega \times \mathbb{R}_+$, $u|_{\partial \Omega} = 0$ and
\[
\partial_t u + L_{\Omega,\omega,\theta} u + \nabla P = 0 \quad \text{in } D'(\Omega \times \mathbb{R}_+). \tag{5.9}
\]

Next, we prove
\[
\lim_{t \to 0^+} \| u(t) - P_{\Omega} f \|_{L^p(\Omega)} = 0. \tag{5.10}
\]

From \eqref{4.29} and Lemma 2.1 in \cite{23}, we know
\[
\lim_{t \to 0^+} \| u^2(t) \|_{L^p(\Omega)} = 0.
\]

So it suffices to prove
\[
\lim_{t \to 0^+} \| u^{(1)}(t) - P_{\Omega} f \|_{L^p(\Omega)} = 0. \tag{5.11}
\]

For this propose, we decompose $\mathcal{R}_{\Omega,\omega,\theta}^{1,k}(\lambda)$ defined in \eqref{4.34} as
\[
\mathcal{R}_{\Omega,\omega,\theta}^{1,k}(\lambda) f \triangleq (1 - \varphi)(\lambda - \Delta - \mathcal{R} \partial_1)^{-1} \mathcal{P}_{\mathbb{R}^3}[\varphi R_{\Omega,\omega,\theta}^I(\lambda)] \left( (I + T)^{-1} f \right)_{\Omega_R^3} + \mathbb{B} \left[ \nabla \varphi \cdot \left( (\lambda - \Delta - \mathcal{R} \partial_1)^{-1} \mathcal{P}_{\mathbb{R}^3}[\varphi R_{\Omega,\omega,\theta}^I(\lambda)] (I + T)^{-1} f \right)_{\Omega_R^3} ight],
\]
\[
\mathcal{R}_{\Omega,\omega,\theta}^{1,2}(\lambda) f \triangleq \mathcal{P}_{\Omega,\omega,\theta}^I(\lambda) f - \mathcal{R}_{\Omega,\omega,\theta}^{1,k}(\lambda) f.
\]

which induces us to rewrite $u^{(1)}(t) = u^{(1,1)}(t) + u^{(1,2)}(t)$ with
\[
u^{(1,k)}(t) \triangleq \frac{1}{2\pi i} \int_{\Gamma_{0,\gamma}} e^{\lambda t} \mathcal{R}_{\Omega,\omega,\theta}^{1,k}(\lambda) f \, d\lambda, \quad k = 1, 2.
\]

By \eqref{4.37}, \eqref{4.22}, Lemma 4.1 and Lemma 4.2, we have
\[
\mathcal{R}_{\Omega,\omega,\theta}^{1,2}(\lambda) \in \mathcal{A}(\Sigma_{\theta,\ell}, \mathcal{L}(L^p_{\Omega_R^3}(\Omega), L^2(\Omega))), \quad \| \mathcal{R}_{\Omega,\omega,\theta}^{1,2}(\lambda) f \|_{L^p(\Omega)} \leq C_{R,\theta,\Omega_R^3} |\lambda|^{-\frac{1}{2} + \frac{3}{2}} \| f \|_{L^p(\Omega)}, \quad \lambda \in \Sigma_{\theta,\ell}.
\]

This, combining with Lemma 2.1 in \cite{23}, yields
\[
\lim_{t \to 0^+} \| u^{(1,2)}(t) \|_{L^p(\Omega)} = 0
\]

For $u^{(1,1)}(t)$, we know that
\[
\lim_{t \to 0^+} \left\| \frac{1}{2\pi i} \int_{\Gamma_{0,\gamma}} e^{\lambda t} (\lambda - \Delta - \mathcal{R} \partial_1)^{-1} \mathcal{P}_{\mathbb{R}^3} g \, d\lambda - \mathcal{P}_{\mathbb{R}^3} g \right\|_{L^p(\mathbb{R}^3)} = 0, \quad g \in L^p(\mathbb{R}^3),
\]
\[
\lim_{t \to 0^+} \left\| \frac{1}{2\pi i} \int_{\Gamma_{0,\gamma}} e^{\lambda t} \mathcal{R}_{\Omega,\omega,\theta}^I(\lambda) g \, d\lambda - \mathcal{P}_{\Omega_{R+3}} g \right\|_{L^p(\Omega_{R+3})} = 0, \quad g \in L^p(\Omega_{R+3}),
\]

which imply
\[
\lim_{t \to 0^+} \| u^{(1,1)}(t) - W f \|_{L^p(\Omega)} = 0
\]

with
\[
W f \triangleq (1 - \varphi) \mathcal{P}_{\mathbb{R}^3}(I + T)^{-1} f_0 + \varphi \mathcal{P}_{\Omega_{R+3}}[(I + T)^{-1} f]_{\Omega_{R+3}} + \mathbb{B} \left[ \nabla \varphi \cdot \left( \mathcal{P}_{\mathbb{R}^3}(I + T)^{-1} f \right)_0 - \mathcal{P}_{\Omega_{R+3}}[(I + T)^{-1} f]_{\Omega_{R+3}} \right].
\]
This, together with $W f = \mathcal{P}_\Omega f$ in Lemma 5.3 of [23], yields

$$\lim_{t \to 0^+} \|u^{(1)}(t) - \mathcal{P}_\Omega f\|_{L^p(\Omega)} = 0.$$  

This finishes the proof of (5.11). Finally, we prove

$$\|\partial_t u(t)\|_{W^{-1,p}(\Omega_{R+3})} \leq C_{\gamma, R, R^*}, e^{\gamma t L^{(1/2)-(1/2p)}} \|f\|_{L^p(\Omega)}.  \tag{5.12}$$

Since

$$\partial_t u_\ell(t) = \frac{1}{2\pi i} \int_{\gamma - i\ell}^{\gamma + i\ell} e^{\lambda t} R_{\beta, \omega, w}(\lambda) f \, d\lambda,$$

and

$$\left\{ \begin{array}{ll}
(\lambda I + L_{\beta, \omega, w}) R_{\beta, \omega, w}(\lambda) f + \nabla \Pi_{\beta, \omega, w}(\lambda) f = f & \text{in } \Omega, \\
\text{div } R_{\beta, \omega, w}(\lambda) f = 0 & \text{in } \Omega, \quad R_{\beta, \omega, w}(\lambda) |_{\partial \Omega} = 0, \end{array} \right.  \tag{5.13}$$

we decompose $\partial_t u_\ell(t) = v_\ell^{(0)}(t) + v_\ell^{(1)}(t) + v_\ell^{(2)}(t) + v_\ell^{(3)}(t)$ with

$$v_\ell^{(0)}(t) \triangleq \frac{1}{2\pi i} \int_{\gamma - i\ell}^{\gamma + i\ell} e^{\lambda t} (f - \nabla \Pi f) \, d\lambda,$$

$$v_\ell^{(1)}(t) \triangleq -\frac{1}{2\pi i} \int_{\gamma - i\ell}^{\gamma + i\ell} e^{\lambda t} (\Delta R_{\beta, \omega, w}(\lambda) f + \nabla \Pi_{\beta, \omega, w}(\lambda) f) \, d\lambda,$$

$$v_\ell^{(2)}(t) \triangleq -\frac{1}{2\pi i} \int_{\gamma - i\ell}^{\gamma + i\ell} e^{\lambda t} (L_{\beta, \omega, w} - \Delta) R_{\beta, \omega, w}(\lambda) f \, d\lambda,$$

$$v_\ell^{(3)}(t) \triangleq -\frac{1}{2\pi i} \int_{\gamma - i\ell}^{\gamma + i\ell} e^{\lambda t} (L_{\beta, \omega, w} R_{\beta, \omega, w}(\lambda) f + \nabla \Pi_{\beta, \omega, w}(\lambda) f) \, d\lambda.$$

Set

$$v^{(1)}(t) \triangleq -\frac{1}{2\pi i} \int_{\Gamma_{\theta, \gamma}} e^{\lambda t} (\Delta R_{\beta, \omega, w}(\lambda) f + \nabla \Pi_{\beta, \omega, w}(\lambda) f) \, d\lambda,$$

$$v^{(2)}(t) \triangleq -\frac{1}{2\pi i} \int_{\Gamma_{\theta, \gamma}} e^{\lambda t} (L_{\beta, \omega, w} - \Delta) R_{\beta, \omega, w}(\lambda) f \, d\lambda,$$

$$v^{(3)}(t) \triangleq -\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} (L_{\beta, \omega, w} R_{\beta, \omega, w}(\lambda) f + \nabla \Pi_{\beta, \omega, w}(\lambda) f) \, d\lambda.$$

Then, by (1.12), Theorem 4.3 and Lemma 5.1-Lemma 5.2 in [23], we obtain

$$\|v^{(1)}\|_{W^{-1,p}(\Omega_{R+3})} \leq C_{\gamma, R, R^*}, e^{\gamma t L^{(1/2)-(1/2p)}} \|f\|_{L^p(\Omega)},$$

$$\|v^{(2)}\|_{L^p(\Omega_{R+3})} \leq C_{\gamma, R, R^*}, e^{\gamma t L^{(1/2)}} \|f\|_{L^p(\Omega)},$$

$$\|v^{(3)}\|_{L^p(\Omega_{R+3})} \leq C_{\gamma, R, R^*}, e^{\gamma t} \|f\|_{L^p(\Omega)},$$

$$\lim_{\ell \to \infty} \sup_{0 < T_1 < T_2} \left\| v_\ell^{(1)}(t) - v^{(1)}(t) \right\|_{W^{-1,p}(\Omega_{R+3})} = 0,$$

$$\lim_{\ell \to \infty} \sup_{0 < T_1 < T_2} \left\| v_\ell^{(k)}(t) - v^{(k)}(t) \right\|_{L^p(\Omega_{R+3})} = 0, \quad k = 1, 2,$$

$$\lim_{\ell \to \infty} v_\ell^{(0)} = 0 \text{ in } \mathcal{D}'(\Omega \times R_+).$$
Since $u(t) \to u(t)$ in $V'((0,\infty) \times \mathbb{R}_+)$, we conclude $\partial_t u(t) = v^{(1)}(t) + v^{(2)}(t) + v^{(3)}(t)$ by uniqueness. This proves (5.12) and so completes the proof of Theorem 5.1. \qed

To show the uniqueness of the solution $u$ obtained in Theorem 5.1, we consider the linear nonstationary problem associated to $L^*_{2l,\omega,w}$

\begin{equation}
\begin{cases}
\partial_t v + L^*_{2l,\omega,w} v + \nabla \Theta = 0, & \text{div } v = 0 \quad \text{in } \Omega \times \mathbb{R}_+, \\
v|_{\partial \Omega} = 0, & v|_{t=0} = P_\Omega f, \quad f \in L^p_{R+2}(\Omega).
\end{cases}
\end{equation}

Following the same argument as in the proof of Theorem 5.1, we have

**Corollary 5.2.** Under the assumption of Theorem 5.1, there exists a positive constant $\eta = \eta_{p,\Omega,\omega}^*$ such that if $\|u\|_{\theta,T} \leq \eta$, then problem (5.14) admits a solution $(v,\Theta)$ with

\begin{equation}
v(t) = \lim_{\ell \to \infty} \frac{1}{2\pi i} \int_{\gamma_{-\ell}}^{\gamma_{+\ell}} e^{\lambda t} R^*_{2l,\omega,w}(\lambda) f \, d\lambda, \quad \gamma > \ell_3,
\end{equation}

satisfying

\begin{equation}
v \in C([0,\infty) ; \mathbb{R}^p(\Omega)) \cap C([0,\infty) ; \mathbb{W}^{2,p}(\Omega)) \cap C^1([0,\infty) ; L^p(\Omega)), \quad \Theta \in C([0,\infty) ; \mathbb{W}^1,p(\Omega)).
\end{equation}

Moreover, $v$ and $\Theta$ possess

\begin{equation}
\left\| \left( v(t), t \partial_t v(t), t^{1/2} \nabla v(t), t^2 \nabla v(t), t \nabla \Theta(t) \right) \right\|_{L^p(\Omega)} \leq C_{\gamma,\Omega,\omega} e^{\gamma t} \|f\|_{L^p(\Omega)},
\end{equation}

\begin{equation}
t^{(1/2) + (1/p)} \left( \|\partial_t v(t)\|_{W^{-1,p}(\Omega)} + \|\Theta(t)\|_{L^p(\Omega)} \right) \leq C_{\gamma,\Omega,\omega} e^{\gamma t} \|f\|_{L^p(\Omega)}.
\end{equation}

Invoking Corollary 5.2, we obtain the following proposition, which gives the uniqueness of $u$ in Theorem 5.1

**Proposition 5.3.** Let $p \in (1,\infty)$. Assume that $u \in C([0,\infty) ; \mathbb{R}^p(\Omega)) \cap C([0,\infty) ; \mathbb{W}^{2,p}(\Omega)) \cap C^1([0,\infty) ; L^p(\Omega))$ and $P \in C([0,\infty) ; \mathbb{W}^1,p(\Omega))$ such that

\begin{equation}
\begin{cases}
\partial_t u + L_{2l,\omega,w} u + \nabla P = 0, & \text{div } u = 0 \quad \text{in } \Omega \times \mathbb{R}_+, \\
u|_{\partial \Omega} = 0, & u|_{t=0} = 0.
\end{cases}
\end{equation}

Then $u = 0$ in $\Omega \times \mathbb{R}_+$.

**Proof.** For any $\phi \in C_{0,\omega}^\infty$, problem (5.14) with $f = \phi$ admits a solution $(v,\Theta)$ satisfying (5.15)-(5.18) by Corollary 5.2. For every $T > 0$, one easily verifies that $\bar{v}(t) \triangleq v(T-t)$ and $\bar{\Theta}(t) \triangleq \Theta(T-t)$ solve

\begin{equation}
\begin{cases}
- \partial_t \bar{v} + L^*_{2l,\omega,w} \bar{v} + \nabla \bar{\Theta} = 0, & \text{div } \bar{v} = 0 \quad \text{in } \Omega \times (-\infty, T), \\
\bar{v}|_{\partial \Omega} = 0, & \bar{v}|_{t=T} = \phi.
\end{cases}
\end{equation}

We have by integration by parts

\begin{align*}
0 &= \int_0^T \int_\Omega \left( \partial_t u(t) + L_{2l,\omega,w} u(t) + \nabla P \right) \cdot \bar{v}(t) \, dx \, dt \\
&= \int_\Omega u(T) \cdot \phi \, dx + \int_0^T \int_\Omega u(t) \cdot \left( - \partial_t \bar{v}(t) + L^*_{2l,\omega,w} \bar{v}(t) \right) \, dx \, dt \\
&= \int_\Omega u(T) \cdot \phi \, dx - \int_0^T \int_\Omega u(t) \cdot \nabla \bar{\Theta} \, dx \, dt = \int_\Omega u(T) \cdot \phi \, dx.
\end{align*}
Remark 5.4. By duality, it is obvious that $v$ defined in (5.15) is unique.

5.2. Behavior in Large time. From Proposition 5.2, Theorem 5.1 and Corollary 5.3 we have the following representation of $T$:

With it, we will follow the idea in [24, 25] to study the behavior of $\phi$. Hence, in the light of the arbitrariness of $\eta$, $\gamma$, $\gamma = \ell_3$. Letting $\gamma = \ell_3$, we deduce by (4.38) $\parallel T^r(\gamma)_{|\Omega} \parallel_{L^p(\gamma)} < t^{-1-p}$, $t \geq 1$

$$\parallel T^r(\gamma)_{|\Omega} \parallel_{L^p(\gamma)} = \int_{\gamma_0}^{+\infty} e^{\lambda t} R_{\gamma, \omega, w}(\gamma + i\ell) f - e^{(\gamma - i\ell) t} R_{\gamma, \omega, w}(\gamma - i\ell) f$$

In addition, by (4.35), (4.36), we obtain

$$\lim_{t \to \infty} \frac{1}{2\pi i} \int_{\gamma_0}^{+\infty} e^{\lambda t} \partial_{\gamma} R_{\gamma, \omega, w}(\gamma + i\ell) f = 0, \text{ in } L^p(\gamma_{R+3}),$$

$$\frac{1}{2\pi i} \int_{\gamma_0}^{+\infty} \|e^{\lambda t} \partial_{\gamma} R_{\gamma, \omega, w}(\gamma + i\ell) f \|_{L^p(\gamma_{R+3})} d\lambda < \infty.$$
Hence, by Lemma 6.1 in [23], we conclude from (5.26)
\[
\|T_{\Omega}(t)\|_{W^{1,p}(\Omega_{R+3})} \leq R_{p,\Omega,t} 1^{-\rho} \|f\|_{L^p(\Omega)}, \quad \rho \in (0, 1/2). \tag{5.27}
\]

For \(\partial_t T_{\Omega}(t)\), we calculate
\[
\partial_t T_{\Omega}(t) = \frac{1}{2\pi t^2} \int_{-\infty}^{\infty} e^{ist} (\partial_t \mathcal{R}_{\Omega}(is)) f \, ds - \frac{1}{2\pi t} \int_{-\infty}^{\infty} is e^{ist} (\partial_t \mathcal{R}_{\Omega}(is)) f \, ds
\]
\[
= \frac{1 + i}{2\pi t^2} \int_{-\infty}^{\infty} e^{ist} (\partial_t \mathcal{R}_{\Omega}(is)) f \, ds - \frac{1}{2\pi t} \int_{-\infty}^{\infty} e^{ist} (\partial_t (\mathcal{R}_{\Omega}(is))) f \, ds
\]
\[
\triangle v_1(t) + v_1(t).
\]
Since \(v_1 = \frac{1 + i}{t} T_{\Omega}(t)\), we have from (5.27)
\[
\|v_1(t)\|_{W^{1,p}(\Omega_{R+3})} \leq C R_{p,\Omega,t} 1^{-\rho} \|f\|_{L^p(\Omega)}, \quad \rho \in (0, 1/2). \tag{5.28}
\]

For \(v_2(t)\), we have by (5.15)
\[
\partial_t (\mathcal{R}_{\Omega}(is)) f = \Delta (\partial_t \mathcal{R}_{\Omega}(is)) f + R \partial_1 (\partial_t \mathcal{R}_{\Omega}(is)) f
\]
\[
+ \omega((e_1 \times x) \cdot \nabla (\partial_t \mathcal{R}_{\Omega}(is)) f - e_1 \cdot (\partial_t \mathcal{R}_{\Omega}(is)) f
\]
\[
- \mathbf{w} \cdot \nabla (\partial_t \mathcal{R}_{\Omega}(is)) f - (\partial_t \mathcal{R}_{\Omega}(is)) f \cdot \nabla w - \nabla (\partial_t \Pi_{\Omega}(is)) f.
\]
Thus, we deduce by (4.36) and (4.38) that \((\partial_t (\mathcal{R}_{\Omega}(is))) f \in L^1(\mathbb{R}; W^{-1,p}(\Omega_{R+3}))\) and
\[
\sup_{0 < |\Delta| \leq 1} \|\partial_t (\mathcal{R}_{\Omega}(is)) f - (\partial_t (\mathcal{R}_{\Omega}(is))) f \|_{W^{-1,p}(\Omega_{R+3})} \, ds
\]
\[
\leq C \|f\|_{L^p(\Omega)}, \quad \rho \in (0, 1/2),
\]
which implies
\[
\|v_2(t)\|_{W^{-1,p}(\Omega_{R+3})} \leq t^{-\rho} \|f\|_{L^p(\Omega)}, \quad \rho \in (0, 1/2). \tag{5.29}
\]
Collecting (5.27)–(5.29), we prove (5.22). In the same way as deriving (5.23), we obtain (5.23), and so complete the proof of Theorem 5.5.

6. \(L^p-L^q\) estimates of \(T_{\Omega,w}(t)\) and \(T_{\Omega,w}(t)\)

We first study the \(L^p-L^q\) estimates of \(T_{\Omega,w}(t)\), that is, the \(L^p-L^q\) estimates of the solution map \(f \mapsto u(t)\) of the following Cauchy problem:
\[
\begin{cases}
\partial_t u + L_{\Omega,w}u + \nabla P = 0, & \text{div } u = 0 \quad \text{in } \Omega \times (0, \infty), \\
u|_{\partial \Omega} = 0, & u|_{t=0} = f \in \mathbb{H}(\Omega).
\end{cases} \tag{6.1}
\]
Without of generality, we assume the pressure \(P(x,t)\) such that
\[
\int_{\Omega_{R+3}} P(x,t) \, dx = 0 \quad \text{for every } t > 0. \tag{6.2}
\]
We start from the regularity estimates of the solution of (6.1)
Proposition 6.1. Let \( p \in (1, 3) \), \( 0 < \Re \leq |\Re| \leq \Re^* \) and \( |\omega| \leq \omega^* \). Assume that \( \varepsilon \in (0, \frac{1}{2}) \) if \( p \geq \frac{6}{5} \) otherwise \( \varepsilon \in (0, \frac{3p-3}{p}) \). Then, there exists a constant \( \eta = \eta_{p, R, \Re, \Re^*, \omega^*} > 0 \) such that if \( \|w\|_{L^p(\Omega)} \leq \eta \), then for \( f \in L^p(\Omega) \)

\[
\|u(t)\|_{L^p(\Omega)} + \frac{1}{t^{1/2}} \|\nabla u(t)\|_{L^p(\Omega)} + t^{\frac{1}{2} + \frac{3-p}{p}} \left( |\partial_t u(t)|_{W^{-1,p}(\Omega_{R+3})} + \|P(t)\|_{L^p(\Omega_{R+3})} \right) \\
\leq C_{R, \Re, \Re^*, \omega^*} \|f\|_{L^p(\Omega)}, \quad 0 < t \leq 2,
\]

\[
\|u(t)\|_{W^1,p(\Omega_{R+3})} + \||\partial_t u(t)|_{W^{-1,p}(\Omega_{R+3})} + \|P(t)\|_{L^p(\Omega_{R+3})} \\
\leq C_{\varepsilon, R, \Re, \Re^*, \omega^*} t^{-\frac{3}{2}} \|f\|_{L^p(\Omega)}, \quad t > 2.
\]

Proof. We first estimate \( u(t) \). Let \( \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^3) \) be the bump function in \([1,4]\). Set

\[
v_0 = (1 - \varphi)f + \mathbb{B} [\nabla \varphi \cdot f],
\]

where \( \mathbb{B} \) is a Bogovskii’s operator. By Lemma 4.1, we have

\[
\|v_0\|_{L^p(\mathbb{R}^3)} \lesssim \|f\|_{L^p(\Omega)}, \quad \text{div } v_0 = 0.
\]

Set \((v(t), \Theta(t)) = (T_{\theta, \omega, \varphi}(t)v_0, \tilde{Q}_t(\Theta \cdot \nabla v(t) + v(t) \cdot \nabla \Theta))\). Then \((v(t), \Theta(t))\) solves

\[
\begin{aligned}
\partial_t v + L_{\theta, \omega, \varphi} v + \nabla \Theta &= 0, & \text{div } v &= 0 \text{ in } \mathbb{R}^3 \times (0, \infty), \\
v_{t=0} &= v_0, & \int_{\Omega_{R+3}} \Theta \, dx &= 0,
\end{aligned}
\]

and satisfies

\[
\begin{aligned}
\|v(t)\|_{L^p(\mathbb{R}^3)} &\leq C t^{-\frac{3}{2} + \frac{3-p}{p}} \|f\|_{L^p(\Omega)}, \quad p \leq q \leq \infty, \\
\|\nabla v(t)\|_{L^p(\mathbb{R}^3)} &\leq C t^{-\frac{1}{2} + \frac{3-p}{p}} \|f\|_{L^p(\Omega)}, \quad p \leq q \leq 3, \\
\|\nabla \Theta(t)\|_{L^p(\mathbb{R}^3)} &+ \|\Theta(t)\|_{L^q(\Omega_{R+3})} \leq C t^{-\frac{1}{2} + \frac{3-p}{p}} \|f\|_{L^p(\Omega)}, \quad p \leq q \leq 3,
\end{aligned}
\]

by making use of \((2.4), (2.31)\) and Lemma 8.4.

To compensate the zero boundary condition of \( v \) on \( \partial \Omega \), we set

\[
\tilde{v}(t) = (1 - \varphi)v(t) + \mathbb{B} [\nabla \varphi \cdot v(t)], \quad \tilde{\Theta}(t) = (1 - \varphi)\Theta(t).
\]

Obviously, \((\tilde{v}(t), \tilde{\Theta}(t))\) solves

\[
\begin{aligned}
\partial_t \tilde{v} + L_{\theta, \omega, \varphi} \tilde{v} + \nabla \tilde{\Theta} &= F(t), & \text{div } \tilde{v} &= 0 \text{ in } \Omega \times (0, \infty), \\
\tilde{v}|_{\partial \Omega} &= 0, & \tilde{v}_{t=0} &= \tilde{v}_0 \triangleq (1 - \varphi)v_0 + \mathbb{B} [\nabla \varphi \cdot v_0]
\end{aligned}
\]

with

\[
F(t) = -\Theta \nabla \varphi + (\Delta \varphi) v + 2(\nabla \varphi) \cdot \nabla v + \Re (\partial_1 \varphi) v + \omega (\langle e_1 \times v \rangle \cdot \nabla \varphi) v \\
+ (w \cdot \nabla \varphi) v - \mathbb{B} [\nabla \varphi \cdot (L_{\theta, \omega, \varphi} v + \nabla \Theta)] + L_{\theta, \omega, \varphi} \mathbb{B} [\nabla \varphi \cdot v].
\]

When \( t \in (0, 2) \), we easily get from Lemma 4.1 and \((6.7)\)

\[
\|\tilde{v}(t)\|_{L^p(\mathbb{R}^3)} + t^{\frac{1}{2}} \|\tilde{\Theta}\|_{L^p(\Omega_{R+3})} \leq C_R \|f\|_{L^p(\Omega)}.
\]

Meanwhile, we can deduce for \( t > 2 \)

\[
\|\tilde{v}\|_{W^{1,p}(\Omega_{R+3})} + \|\tilde{\Theta}\|_{L^p(\Omega_{R+3})} + \|F(t)\|_{L^p(\mathbb{R}^3)}
\]
\[
\begin{align*}
\lesssim & \|\tilde{\mathbf{v}}\|_{L^\infty(\Omega_{R,t})} + \|\nabla \tilde{\mathbf{v}}\|_{L^3(\Omega_{R,t})} + \|	ilde{\mathbf{\Theta}}\|_{L^3(\Omega_{R,t})} \lesssim t^{-\frac{3}{p}} \|f\|_{L^p(\Omega)}. 
\end{align*}
\] (6.11)

Hence, we have from (6.8)
\[
\|\partial_t \tilde{\mathbf{u}}\|_{W^{-1,p}(\Omega_{R,t})} \leq \begin{cases} C_R t^{-1/2} \|f\|_{L^p(\Omega)} & \text{if } 0 < t < 2, \\
C_R t^{-\frac{3}{p}} \|f\|_{L^p(\Omega)} & \text{if } t > 2.
\end{cases}
\]

Now, we set \(\tilde{\mathbf{u}}(t) = \mathbf{u}(t) - \tilde{\mathbf{v}}(t)\) and \(\tilde{P} = P - \tilde{\mathbf{\Theta}}\). Obviously,
\[
\begin{cases}
\partial_t \tilde{\mathbf{u}} + L_{\mathbf{\eta}, \omega, \mathbf{u}} \tilde{\mathbf{u}} + \nabla \tilde{P} = -\mathbf{F}(t), & \text{div } \tilde{\mathbf{u}} = 0 \text{ in } \Omega \times (0, \infty), \\
\tilde{\mathbf{u}}|_{\partial \Omega} = 0, & \tilde{\mathbf{u}}|_{t=0} = \tilde{\mathbf{u}}_0 \triangleq f - \tilde{\mathbf{v}}_0.
\end{cases}
\] (6.13)

By Lemma 3.1, we have \(\tilde{\mathbf{u}}_0 \in L^p(\Omega)\) satisfying
\[
\text{div } \tilde{\mathbf{u}}_0 = 0 \text{ in } \Omega \quad \text{and} \quad \|\tilde{\mathbf{u}}_0\|_{L^p(\Omega)} \leq C \|f\|_{L^p(\Omega)}.
\] (6.14)

Noting that \(\mathbf{F}(t) \in L^p(\Omega)\), we write
\[
\tilde{\mathbf{u}}(t) = T_{\mathbf{\eta}, \omega, \mathbf{u}}(t) \tilde{\mathbf{u}}_0 - \int_0^t T_{\mathbf{\eta}, \omega, \mathbf{u}}(t - \tau) \mathcal{P}_\Omega \mathbf{F}(\tau) \, d\tau \triangleq \tilde{\mathbf{u}}_1 + \tilde{\mathbf{u}}_2.
\]

By Theorem 5.1 and Theorem 5.5 with \(\rho = \max(\frac{3}{2p} - 1, \frac{1}{4})\), we obtain
\[
\begin{align*}
\|\tilde{\mathbf{u}}_1(t), t^{\frac{1}{2}} \nabla \tilde{\mathbf{u}}_1(t)\|_{L^p(\Omega)} + t^{\frac{1}{2} + \frac{1}{p}} \|\partial_t \tilde{\mathbf{u}}_1\|_{W^{-1,p}(\Omega_{R,t})} & \leq C \|f\|_{L^p(\Omega)}, \quad 0 < t \leq 2, \\
\|\tilde{\mathbf{u}}_1(t)\|_{W^{1,p}(\Omega_{R,t})} + \|\partial_t \tilde{\mathbf{u}}_1\|_{W^{-1,p}(\Omega_{R,t})} & \leq C t^{-\frac{3}{p}} \|f\|_{L^p(\Omega)}, \quad t > 2.
\end{align*}
\] (6.15)

For \(\tilde{\mathbf{u}}_2(t)\), when \(t \in (0, 2]\) we deduce by Theorem 5.1 that
\[
\|\tilde{\mathbf{u}}_2(t)\|_{W^{1,p}(\Omega)} \lesssim \|f\|_{L^p(\Omega)} \int_0^t \left(1 + (t - \tau)^{-\frac{3}{4}}\tau^{-\frac{1}{2}}\right) d\tau \lesssim \|f\|_{L^p(\Omega)},
\]
which implies that
\[
\|\tilde{\mathbf{u}}_2(t), t^{\frac{1}{2}} \nabla \tilde{\mathbf{u}}_2(t)\|_{L^p(\Omega)} \leq C \|f\|_{L^p(\Omega)}, \quad 0 < t \leq 2.
\] (6.16)

When \(t > 2\), we decompose \(\tilde{\mathbf{u}}_2(t)\) as
\[
\tilde{\mathbf{u}}_2(t) = -\left[ \int_0^1 + \int_1^{t-1} + \int_{t-1}^t \right] T_{\mathbf{\eta}, \omega, \mathbf{u}}(t - \tau) \mathcal{P}_\Omega \mathbf{F}(\tau) \, d\tau.
\]

Then, by Theorem 5.1 and Theorem 5.5 with \(\rho = \max(\frac{3}{2p} - 1, \frac{1}{4}) + \frac{1}{2}\), we get
\[
\|\tilde{\mathbf{u}}_2(t)\|_{W^{1,p}(\Omega_{R,t})} \lesssim_{\rho} \|f\|_{L^p(\Omega)} \left( \int_0^1 (t - \tau)^{-1-\rho} \tau^{-\frac{1}{2}} \, d\tau \\
+ \int_1^{t-1} (t - \tau)^{-1-\rho} \tau^{-\frac{3}{4}} \, d\tau + \int_{t-1}^t (t - \tau)^{-\frac{1}{2} - \rho} \tau^{-\frac{3}{4}} \, d\tau \right)
\lesssim_{\rho} \|f\|_{L^p(\Omega)} \left( t^{-\frac{3}{4}} + \int_1^{t-1} (t - \tau)^{-1-\rho} \tau^{-\frac{3}{4}} \, d\tau \right).
\]

We observe that
\[
\int_1^{t-1} (t - \tau)^{-1-\rho} \tau^{-\frac{3}{4}} \, d\tau \leq \left[ \int_0^{t/2} + \int_{t/2}^t \right] (t - \tau + 1)^{-1-\rho} (\tau + 1)^{-\frac{3}{4}} \, d\tau
\]
Hence, we obtain from (6.10)-(6.12) and (6.15)-(6.18) that
\[ \| \tilde{u}_2(t) \|_{W^{1,p}(\Omega_{R+3})} \leq t^{-\frac{3}{p}} \| u_0 \|_{L^p(\Omega)}, \quad t > 2. \] (6.17)

On the other hand, since
\[ \theta > 0, \quad \text{ln } s \leq C_\theta s^0 \text{ for all } s \geq 1. \]
Thus, we have from (6.2)
\[ \text{Proof.} \]
\[ \text{Let } \eta \text{ a constant } \]

Proposition 6.2.

\[ \text{□} \]

Now we estimate \( P(t) \). Set \( \tilde{\phi} = \phi - \frac{1}{|\Omega_{R+3}|} \int_{\Omega_{R+3}} \phi \, dx, \phi \in C_0^\infty (\Omega_{R+3}). \) Noting that
\[ \| B[\tilde{\phi}] \|_{W^{1,p'}(R^3)} \leq C \| \tilde{\phi} \|_{L^p(\Omega_{R+3})}, \quad \text{div } B[\tilde{\phi}] = \tilde{\phi} \text{ in } \Omega_{R+3}, \quad B[\tilde{\phi}] = 0 \text{ in } \Omega_{R+3}, \]
we have from (6.2)
\[ \langle P(t), \phi \rangle_{\Omega_{R+3}} = \langle P(t), \tilde{\phi} \rangle_{\Omega_{R+3}} = \langle P(t), \text{div } B[\tilde{\phi}] \rangle_{\Omega_{R+3}} = -\langle \nabla P(t), B[\tilde{\phi}] \rangle_{\Omega_{R+3}} \]
\[ = \langle \partial_t u, B[\tilde{\phi}] \rangle_{\Omega_{R+3}} + \langle \nabla u, \nabla B[\tilde{\phi}] \rangle_{\Omega_{R+3}} + \langle (L_{\eta,\omega,w} - \Delta) u, B[\tilde{\phi}] \rangle_{\Omega_{R+3}}. \]

This equality implies
\[ | \langle P(t), \phi \rangle_{\Omega_{R+3}} | \leq (\| \partial_t u \|_{W^{1,p}(\Omega_{R+3})} + \| u \|_{W^{1,p}(\Omega_{R+3})}) \| \phi \|_{L^p(\Omega_{R+3})}. \]

Hence, we obtain from (6.10)-(6.12) and (6.15)-(6.18) that
\[ \| P(t) \|_{L^p(\Omega_{R+3})} \leq \begin{cases} Ct^{-\frac{1}{2} - \frac{1}{2p}} \| f \|_{L^p(\Omega)}, & 0 < t \leq 2, \\ C_\theta t^{-\frac{1}{p} + \epsilon} \| f \|_{L^p(\Omega)}, & t > 2. \end{cases} \]
(6.19)

This, together with (6.11) finishes the proof of Proposition 6.1 \( \square \)

Now we study the \( L^p-L^q \) estimates of \( T_{\eta,\omega,w}(t) \).

**Proposition 6.2.** Let \( q \in [p, \infty) \). Under the assumption in Proposition 6.1, there exists a constant \( \eta = \eta_{\eta,\omega,w}^* \) such that if \( \| w \|_{L^q} \leq \eta \), then for \( f \in \mathcal{P}(\Omega) \)
\[ \| T_{\eta,\omega,w}(t) f \|_{L^p(\Omega)} \leq C_{\eta,\omega,w} t^{-\frac{1}{2} \left( \frac{1}{p} - \frac{1}{q} \right)} \| f \|_{L^p(\Omega)}, \quad \frac{1}{p} - \frac{1}{q} < \frac{1}{2}, \] (6.20)
\[ \| \nabla T_{\eta,\omega,w}(t) f \|_{L^p(\Omega)} \leq C_{\eta,\omega,w} t^{-\frac{1}{2}} \| f \|_{L^p(\Omega)}. \] (6.21)

**Proof.** Set \( u(t) = T_{\eta,\omega,w}(t) f \) and \( P(t) = \dot{Q}_{R^3} u(t) \). Obviously, \( (u, P(t)) \) satisfies (6.11)-(6.2). By Proposition 6.1 and the Gagliardo-Nirenberg inequality, we have
\[ \begin{cases} \| u(t) \|_{L^q(\Omega_{R+3})} \leq C t^{-\frac{1}{2} \left( \frac{1}{p} - \frac{1}{q} \right)} \| f \|_{L^p(\Omega)}, & 0 \leq \frac{1}{p} - \frac{1}{q} < \frac{1}{2}, \quad t > 0, \\ \| \nabla u(t) \|_{L^p(\Omega_{R+3})} \leq C t^{-\frac{1}{2}} \| f \|_{L^p(\Omega)}, & t > 0. \end{cases} \] (6.22)
Now we are in position to estimate \( u(t) \) in \( B^s_{p+2} \). Let \( \phi \in C^\infty_0(\mathbb{R}^3) \) satisfy \( \phi = 0 \) in \( B_{R+2} \) and \( \phi = 1 \) in \( B^s_{R+3} \), and set \( z(t) = \phi u(t) - E\{\nabla \phi \cdot u\}. \) Obviously,

\[
\begin{aligned}
\partial_t z(t) + L_{\partial_1, \omega, \pi} z(t) + \nabla (\phi P(t)) &= H(t), \quad \text{div} z(t) = 0 \text{ in } \mathbb{R}^3 \times (0, \infty),
\end{aligned}
\]

with

\[
H(t) = P(t) \nabla \phi + E\{\nabla \phi \cdot (L_{\partial_1, \omega, \pi} u(t) + \nabla P(t))\} - L_{\partial_1, \omega, \pi} E\{\nabla \phi \cdot u(t)\} - (\Delta \phi) u(t) - 2(\nabla \phi \cdot \nabla) u(t) - Q(\partial_1 \phi) u(t) - \omega((e_1 \times x) \cdot \nabla \phi) u(t) + (w \cdot \nabla \phi) u(t).
\]

Since \( \text{supp} \ H(t) \subset B_{R+1, R+2} \), by Lemma 4.1 and Proposition 6.1, we rewrite

\[
\|H(t)\|_{{L^r(\mathbb{R}^3)}} \leq \begin{cases} C t^{-\frac{1}{r}} \|f\|_{L^p(\Omega)}, & 0 < t \leq 2, \\ C t^{-\frac{1}{r}} \|f\|_{L^p(\Omega)}, & t > 2, \end{cases}, \quad \forall r \in [1, p].
\]

Hence, we have

\[
z(t) = T_{\partial_1, \omega, \pi}^G(t) z_0 + \int_0^t T_{\partial_1, \omega, \pi}^G(t-s) \mathcal{P}_{R_3} H(s) \, ds \triangleq z_1(t) + z_2(t).
\]

For \( z_1(t) \), we easily get from \((2.31)\) and Lemma 4.1 that

\[
\begin{aligned}
\|z_1(t)\|_{L^q(\mathbb{R}^3)} &\leq C t^{-\frac{1}{r}} \|f\|_{L^p(\Omega)}, \quad \forall q \in [p, \infty], \\
\|\nabla z_1(t)\|_{L^q(\mathbb{R}^3)} &\leq C t^{-\frac{1}{r}} \|f\|_{L^p(\Omega)}, \quad \forall q \in [p, 3].
\end{aligned}
\]

For \( z_2(t) \), when \( t \in (0, 2] \), using \((2.31)\) and \((6.24)\), we obtain for \( 0 < \frac{1}{p} - \frac{1}{q} < \frac{1}{3} \)

\[
\|z_2(t)\|_{L^q(\mathbb{R}^3)} \leq C \|f\|_{L^p(\Omega)} \int_0^t (t-s)^{-\frac{1}{r}(\frac{1}{p} - \frac{1}{q})} s^{-\frac{1}{r} - \frac{1}{r'}} \, ds \leq C t^{-\frac{1}{r}(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\Omega)}.
\]

When \( t \geq 2 \), we decompose \( z_2(t) \)

\[
z_2(t) = \left[ \int_0^1 + \int_1^{t-1} + \int_{t-1}^t \right] T_{\partial_1, \omega, \pi}^G(t-s) \mathcal{P}_{R_3} H(s) \, ds \triangleq J_1(t) + J_2(t) + J_3(t).
\]

Then we calculate by \((2.4)\), \((2.31)\) and \((6.24)\) that

\[
\|z_2(t)\|_{L^q(\mathbb{R}^3)} \leq C \|f\|_{L^p(\Omega)} \left( \int_0^1 (t-s)^{-\frac{1}{r}(\frac{1}{p} - \frac{1}{q})} s^{-\frac{1}{r} - \frac{1}{r'}} \, ds + \int_1^{t-1} (t-s)^{-\frac{1}{r}(\frac{1}{p} - \frac{1}{q})} s^{-\frac{1}{r} - \frac{1}{r'}} \, ds \right)
\]

\[
+ \int_{t-1}^t (t-s)^{-\frac{1}{r}(\frac{1}{p} - \frac{1}{q})} s^{-\frac{1}{r} - \frac{1}{r'}} \, ds \right) \leq C \|f\|_{L^p(\Omega)} \left( t^{-\frac{1}{r}(\frac{1}{p} - \frac{1}{q})} + t^{-\frac{1}{r} - \frac{1}{r'}} + \int_0^1 (t-s + 1)^{-\frac{1}{r}(\frac{1}{p} - \frac{1}{q})} (s + 1)^{-\frac{1}{r} - \frac{1}{r'}} \, ds \right)
\]

\[
\leq t^{-\frac{1}{r}(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\Omega)}, \quad 0 \leq \frac{1}{p} - \frac{1}{q} < \frac{1}{3},
\]

where we choose \( 1 < r < \max\left(\frac{3}{2}, p\right) \). This, together with \((6.27)\), gives that

\[
\|z_2(t)\|_{L^q(\mathbb{R}^3)} \leq C t^{-\frac{1}{r}(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\Omega)}, \quad 0 \leq \frac{1}{p} - \frac{1}{q} < \frac{1}{3}, \quad t > 0.
\]
In same way as deducing (6.28), we obtain by (2.32)
\[ \| \nabla z_2(t) \|_{L^p(\mathbb{R}^3)} \leq Ct^{-\frac{2}{3}} \| f \|_{L^p(\Omega)}, \quad t > 0. \] (6.29)

Since \( u(t) = z(t) \) in \( B^c_{R+3} \), collecting (6.22), (6.26) and (6.28), we deduce (6.20) and (6.21), and so finish the proof of Proposition 6.2.

Following the argument used in the proof of Theorem 1.5, we obtain the same results for the operator semigroup \( \{T^*_t u(t)\}_{t \geq 0} \).

**Corollary 6.3.** Under the assumption in Proposition 6.2, there exists a constant \( \eta = \eta_{\mathbb{R}^*, \mathfrak{M}^*, \omega^*} > 0 \) such that if \( \| u \|_{L^\infty(\mathbb{R}^3 \times \Omega)} \leq \eta \), then for \( f \in \mathfrak{F}(\Omega) \)
\[ \| T^*_t u(t) f \|_{L^p(\Omega)} \leq C t^{-\frac{3}{2} \left( \frac{3}{4} - \frac{1}{q} \right)} \| f \|_{L^p(\Omega)}, \quad \frac{1}{p} - \frac{1}{q} < \frac{1}{3}, \] (6.30)
\[ \| \nabla T^*_t u(t) f \|_{L^p(\Omega)} \leq C t^{-\frac{3}{2} \left( \frac{3}{4} - \frac{1}{3q} \right)} \| f \|_{L^p(\Omega)}. \] (6.31)

By Proposition 6.2 and Corollary 6.3 we can complete the proof of Theorem 1.5.

**Proof of Theorem 1.5.** By duality, we have from Theorem 6.2 and Corollary 6.3 that for every \( 1 < p \leq q < \infty \) satisfying \( 0 \leq \frac{1}{p} - \frac{1}{q} < \frac{1}{3} \),
\[ \| T^*_t u(t) f \|_{L^p(\Omega)} + \| T^*_t u(t) f \|_{L^q(\Omega)} \leq C t^{-\frac{3}{2} \left( \frac{3}{4} - \frac{1}{q} \right)} \| f \|_{L^p(\Omega)}. \] (6.32)
To prove the valid of (6.32) for all \( 1 < p \leq q < \infty \) satisfying \( \frac{1}{p} - \frac{1}{q} \geq \frac{1}{3} \), we choose \( q_1, q_2, q_3 \in [p, \infty) \) satisfying
\[ q_0 = q, \quad q_4 = p, \quad 0 \leq \frac{1}{q_0} - \frac{1}{q_{j-1}} < \frac{1}{3}, \quad j = 1, 2, 3, 4, \]
and then deduce
\[ \| T^*_t u(t) f \|_{L^{q_j}(\Omega)} \leq C t^{-\frac{3}{2} \left( \frac{3}{4} - \frac{1}{q_j} \right)} \| T^*_t u(t) f \|_{L^{q_{j-1}}(\Omega)} \]
\[ \leq C t^{-\frac{3}{2} \left( \frac{3}{4} - \frac{1}{q} \right)} \ldots t^{-\frac{3}{2} \left( \frac{3}{4} - \frac{1}{q_4} \right)} \| f \|_{L^p(\Omega)} \leq C t^{-\frac{3}{2} \left( \frac{3}{4} - \frac{1}{q} \right)} \| f \|_{L^p(\Omega)}. \] (6.33)
Similarly, for every \( 1 < p \leq q < \infty \) satisfying \( \frac{1}{p} - \frac{1}{q} \geq \frac{1}{3} \), we have
\[ \| T^*_t u(t) f \|_{L^{q_j}(\Omega)} \leq C t^{-\frac{3}{2} \left( \frac{3}{4} - \frac{1}{q} \right)} \| f \|_{L^p(\Omega)} \] (6.34)
Summing up (6.32)-(6.34), we prove (1.17), which together with (6.21) and (6.31) yields (1.18). Thus we complete the proof of Theorem 1.5.

**7. Stability in \( \mathbb{L}^3(\Omega) \)**

In this section, we will use the operator semigroup theory, given in Proposition 1.2 and Theorem 1.5 to show the stability of stationary flows satisfying (1.10) and (1.12).

**Proof of Theorem 1.6.** Let \( (X, \| \cdot \|_X) \) be a Banach space defined as follows:
\[ X = \{ u \in C_b([0, \infty); \mathbb{L}^3(\Omega)) \mid t^{3/2} \nabla u(t) \in C_b([0, \infty); \mathbb{L}^3(\Omega)) \}, \]
\[ \| u \|_X = \| u \|_{L^\infty([0, \infty); \mathbb{L}^3(\Omega))} + \sup_{t \geq 0} t^{3/2} \| \nabla u(t) \|_{\mathbb{L}^3(\Omega)}. \]
We write
\[ u = T_{\theta, \omega, w}(t)u_0 + B(u, u) \triangleq T_{\theta, \omega, w}(t)u_0 + \int_0^t T_{\theta, \omega, w}(t - \tau)p_\Omega((u(\tau) \cdot \nabla)u(\tau)) d\tau. \]
By Theorem 1.5 we have
\[
t^\frac{1}{2} \| \nabla B(u, v)(t) \|_{L^2(\Omega)} \lesssim t^\frac{1}{2} \int_0^t (t - \tau)^{-\frac{3}{2}} \| u(\tau) \|_{L^2(\Omega)} \| \nabla v(\tau) \|_{L^2(\Omega)} d\tau
\]
\[
\lesssim \sup_{t \geq 0} t^\frac{1}{2} \| u(t) \|_{L^6(\Omega)} \sup_{t \geq 0} t^\frac{1}{2} \| \nabla u \|_{L^2(\Omega)}
\]
\[
\lesssim (\| u \|_{L^\infty([0, \infty); L^2(\Omega)} + \| \nabla u \|_{L^2(\Omega)}) \sup_{t \geq 0} t^\frac{1}{2} \| \nabla v(t) \|_{L^2(\Omega)} \quad (7.1)
\]
and for every 1 < p ≤ q < ∞ satisfying \( \frac{1}{p} - \frac{1}{q} < \frac{1}{3} \)
\[
t^\frac{1}{2}(\frac{1}{p} - \frac{1}{q}) \| B(u, v)(t) \|_{L^q(\Omega)} \lesssim \int_0^t (t - \tau)^{-\frac{3}{2}} \| u(\tau) \|_{L^2(\Omega)} \| \nabla v(\tau) \|_{L^2(\Omega)} d\tau
\]
\[
\lesssim \sup_{t \geq 0} t^\frac{1}{2}(\frac{1}{p} - \frac{1}{q}) \| u(t) \|_{L^q(\Omega)} \sup_{t \geq 0} t^\frac{1}{2} \| \nabla v(t) \|_{L^2(\Omega)}. \quad (7.2)
\]
Collecting (7.1) and (7.2) with \( p = q = 3 \), we have
\[
\| B(u, v)(t) \|_X \leq C_1 \| u \|_X \| v \|_X, \quad u, v \in X. \quad (7.3)
\]
This, combining with the fact that \( T_{\theta, \omega, w}(t)u_0 \in X_q \) satisfies
\[
\| T_{\theta, \omega, w}(t)u_0 \|_{X_q} \leq C_2 \| u_0 \|_{L^2(\Omega)}. \quad (7.4)
\]
as follows from Theorem 1.2 and Theorem 1.5, gives by Banach fixed point theorem that problem (1.19) admits a unique solution \( u \) in \( X \) satisfying \( \| u \|_X \leq 2C_2\eta \) if \( \| u_0 \|_{L^2(\Omega)} < \eta \) with \( \eta \) satisfying \( 4C_1C_2\eta < 1 \). This proves (1.22) by the interpolation inequality.

Next, we show (1.21). For every 0 < \( \delta < 1 \), we choose a \( u_{0, \delta} \in \mathcal{J}^3(\Omega) \cap \mathcal{J}^3(\Omega) \) such that
\[
\| u_0 - u_{0, \delta} \|_{L^2(\Omega)} < \delta \| u_0 \|_{L^2(\Omega)} \leq \delta \eta.
\]
According to the above analysis, we know that problem (1.19) admits a unique solution \( u_{\delta} \) in \( X \) with \( u_{\delta}|_{t=0} = u_{0, \delta} \), satisfying \( \| u_{\delta} \|_X \leq 2C_2\eta \). So, \( u - u_{\delta} \in X \) satisfies
\[
u - u_{\delta} = T_{\theta, \omega, w}(t)(u_0 - v_0) + \int_0^t T_{\theta, \omega, w}(t - \tau)p_\Omega((u - v) \cdot \nabla u + v \cdot \nabla (u - v)) (\tau) d\tau.
\]
By Theorem 1.5 (7.1) and (7.2) with \( p = 2, q = 3 \) or \( p = q = 3 \), we have
\[
\sup_{0 \leq t < \infty} t^\frac{1}{2} \| u_{\delta}(t) \|_{L^2(\Omega)} \leq C_3(\| u_{0, \delta} \|_{L^2(\Omega)} + \| u_{\delta}(t) \|_{L^2(\Omega)} \sup_{0 \leq t < \infty} t^\frac{1}{2} \| \nabla u_{\delta}(t) \|_{L^2(\Omega)}
\]
\[
\| u - v \|_X \leq C_4 \| u_0 - u_{\delta} \|_{L^2(\Omega)} + C_4(\| u \|_X \| u_{\delta} \|_X) \| u - u_{\delta} \|_X.
\]
These implies that if \( 2C_2(C_3 + 2C_4) \| u_0 \|_{L^2(\Omega)} < 1/2 \), then for every \( t > 0 \)
\[
\| u_{\delta}(t) \|_{L^2(\Omega)} \leq 4C_3\eta t^{-1/4}, \quad \| u(t) - u_{\delta}(t) \|_{L^2(\Omega)} \leq 2\delta C_4\eta.
\]
Hence, for every 0 < \( \delta < 1 \), \( \lim_{t \to \infty} \| u(t) \|_{L^2(\Omega)} \leq 2\delta C_4\eta \). This yields (1.21) since \( \delta \) is chosen arbitrarily. So we complete the proof of Theorem 1.6. \( \square \)
Lemma 8.1. Let $f \in L^p(\mathbb{R}^3)$, $p \in (1, \infty)$, $\varepsilon \in (0, \frac{1}{2})$, and $0 < |\mathcal{R}| \leq \mathcal{R}^\ast$. Assume that $\tilde{w}(t)$ is a period function on $t \geq 0$ with period $T > 0$ satisfying

- There exist a constant $M > 0$ independent of $t$ such that $\|\tilde{w}(t)\|_{L^\infty} \leq M$;
- $\tilde{w}(t)$ is Hölder continuous with respect to $t$.

Then, $-\mathcal{P}_{\mathcal{R}^3}(-\Delta - \mathcal{R}\partial_1 + B_{\tilde{w}(t)})$ and its dual operator generate unique evolution operators $G(t, s)$ and $G^*(t, s)$ in $L^p(\mathbb{R}^3)$, respectively, such that for $j \leq 2$

$$\|\nabla^j G(t, s)f\|_{L^p(\mathbb{R}^3)}, \|\nabla^j G^*(t, s)f\|_{L^p(\mathbb{R}^3)} \leq C_T(t - s)^{-\frac{j}{2}} e^{C_T(t-s)} \|f\|_{L^p(\mathbb{R}^3)}.$$

Proof. Since

$$\|\nabla^j (\lambda I - \mathcal{P}_{\mathcal{R}^3} \Delta)^{-1}f\|_{L^p(\mathbb{R}^3)} \leq C_\theta \lambda^{-1+\frac{j}{2}} \|f\|_{L^p(\mathbb{R}^3)}, \quad j \leq 2, \quad \lambda \in \Sigma_\theta,$$

we have

$$\|\mathcal{P}_{\mathcal{R}^3}(\mathcal{R}\partial_1 + B_{\tilde{w}(t)})(\lambda I - \mathcal{P}_{\mathcal{R}^3} \Delta)^{-1}f\|_{L^p(\mathbb{R}^3)} \leq C_{\theta, \mathcal{R}^\ast, M}(|\lambda|^{-1} + |\lambda|^{-\frac{1}{2}}) \|f\|_{L^p(\mathbb{R}^3)}.$$

This, together with the Neumann series expansion

$$(\lambda I + \mathcal{P}_{\mathcal{R}^3}(-\Delta - \mathcal{R}\partial_1 + B_{\tilde{w}(t)}))^{-1} = (\lambda I - \mathcal{P}_{\mathcal{R}^3}(-\mathcal{R}\partial_1 + B_{\tilde{w}(t)})(\lambda I - \mathcal{P}_{\mathcal{R}^3} \Delta)^{-1})^{-1},$$

gives

$$\|\nabla^j (\lambda I + \mathcal{P}_{\mathcal{R}^3}(-\Delta - \mathcal{R}\partial_1 + B_{\tilde{w}(t)}))^{-1}f\|_{L^p(\mathbb{R}^3)} \leq C_\theta \lambda^{-1+\frac{j}{2}} \|f\|_{L^p(\mathbb{R}^3)},$$

for every $\lambda \in \Sigma_{\theta, \ell}$ where $\ell$ satisfies $C_{\theta, \mathcal{R}^\ast, M}(\ell^{-1} + \ell^{-1/2}) < 1$. Similarly, we have

$$\|\nabla^j (\mathcal{P}_{\mathcal{R}^3}(-\Delta - \mathcal{R}\partial_1 + B_{\tilde{w}(t)}))^{-1}f\|_{L^p(\mathbb{R}^3)} \leq C_{\theta} \lambda^{-1+\frac{j}{2}} \|f\|_{L^p(\mathbb{R}^3)}, \quad \lambda \in \Sigma_{\theta, \ell}.$$

Thus from the holomorphic semigroups theory in [29, 30], for every fixed $s \geq 0$, both $-\mathcal{P}_{\mathcal{R}^3}(-\Delta - \mathcal{R}\partial_1 + B_{\tilde{w}(s)})$ and its dual operator generate analytic semigroups in $L^p(\mathbb{R}^3)$. So we prove this lemma from the theory of parabolic evolution systems in [1, 2].

Lemma 8.2 ([35]). Let $X$, $Y$ be two measurable spaces, and $T$ be an integral operator

$$Tf(x) = \int_Y K(x, y)f(y) \, dy, \quad x \in X$$

with

$$\sup_{x \in X} \left( \int_Y |K(x, y)|^r \, dy \right)^{\frac{1}{r}} + \sup_{y \in Y} \left( \int_X |K(x, y)|^r \, dx \right)^{\frac{1}{r}} < C, \quad r \geq 1.$$

Then for every $1 \leq p \leq q \leq \infty$ satisfying $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$

$$\|Tf\|_{L^q(X)} \leq C \|f\|_{L^p(Y)}.$$

Lemma 8.3. Let $\ell \geq 0$ and $0 \leq \delta < 1$ satisfy $\ell + \delta > 3$. Then, for every $R > 0$

$$\int_{|x| \geq R} |x|^{-\ell}(1 + s_R(x))^{-\delta} \, dx \leq C_{\delta, \ell} R^{-\ell - \delta + 3}.$$

(8.1)
Proof. Using a change of variable:

\[ y = Sx, \quad S = \begin{pmatrix} \Re/|\Re| & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

and the polar coordinate:

\[ y_1 = r \cos \theta, \quad y_2 = r \sin \theta \cos \varphi, \quad y_3 = r \sin \theta \sin \varphi, \quad r \in [0, \infty), \quad \theta \in [0, \pi], \quad \varphi \in [0, 2\pi], \]

we have

\[
\int_{|x| \geq R} |x|^{-\ell}(1 + s_R(x))^{-\delta} \, dx = \int_{R}^{\infty} \int_{0}^{2\pi} \int_{\theta}^{\pi} \frac{r^{-\ell+2} \sin \theta}{(1 + r(1 - \cos \theta))^\delta} \, d\theta d\varphi dr
\]

\[
= 2\pi(1 - \delta)^{-1} \int_{R}^{\infty} r^{-\ell+1}((1 + 2r)^{-\delta+1} - 1) \, dr
\]

where we have used

\[
\int_{0}^{\pi} \frac{\sin \theta}{(1 + r(1 - \cos \theta))^\delta} \, d\theta = \int_{0}^{\pi} \frac{\partial_p(1 + r(1 - \cos \theta))^{-\delta+1}}{(1 - \delta)r} \, d\theta = \frac{(1 + 2r)^{-\delta+1} + 1}{(1 - \delta)r}.
\]

Hence, we deduce (8.1) for \( \ell \) and complete the proof of Lemma 8.3.

Lemma 8.4. Let \( \alpha \in [0, 1/2] \) and \( p \in (1, 3) \) satisfying \( 1 - \alpha p \neq 0 \). Then there exists a constant \( C_\alpha \) such that

\[
\| |x|^{-1-a s_R(x)} f \|_{L^p(\mathbb{R}^3)} \leq C_\alpha \| \nabla f \|_{L^p(\mathbb{R}^3)}.
\]

Proof. It is well known that

\[
\| |x|^{-1} f \|_{L^p(\mathbb{R}^3)} \lesssim \| \nabla f \|_{L^p(\mathbb{R}^3)}.
\]

Similar to the proof of Lemma 8.3, we have

\[
\| |x|^{-1-a s_R(x)} f \|_{L^p(\mathbb{R}^3)}^p = \int_{0}^{\infty} \int_{0}^{2\pi} \int_{\theta}^{\pi} r^{2-p} |f(S^T y)|^3 \sin \theta \frac{1}{(1 - \cos \theta)^p} \, d\theta d\varphi dr.
\]

Let \( \theta_0 \in (0, \pi/4) \) and \( \rho(\theta) \in C_0^\infty(\mathbb{R}) \) satisfying \( \rho(\theta) = 1 \) if \( |\theta| \leq \theta_0 \) and \( \rho(\theta) = 0 \) if \( |\theta| \geq 2\theta_0 \), we deduce

\[
\| |x|^{-1-a s_R(x)} f \|_{L^p(\mathbb{R}^3)}^p \leq \frac{1}{(1 - \cos \theta_0)^p} \int_{0}^{\infty} \int_{0}^{2\pi} \int_{\theta}^{\pi} r^{2-p} |f(S^T y)|^p \sin \theta \, d\theta d\varphi dr
\]

\[
+ \int_{0}^{\infty} \int_{\theta}^{\pi} I(r, \varphi) \, d\varphi dr
\]

where

\[
I(r, \varphi) = \int_{0}^{\pi} r^{2-p} \rho(\theta) |f(S^T y)|^p \sin \theta \frac{1}{(1 - \cos \theta)^p} \, d\theta.
\]

By integration by part, we have

\[
r^{p-2}(1 - \rho a)I(r, \varphi) - \int_{0}^{\pi} \rho'(\theta)(1 - \cos \theta)^{1-pa} |f(S^T y)|^p \, d\theta
\]

\[
= \int_{0}^{\pi} \rho(\theta)(1 - \cos \theta)^{1-pa} \partial_\theta |f(S^T y)|^p \, d\theta
\]
\[ r \int_0^\pi \rho(\theta)(1 - \cos \theta)^{1-\rho\alpha} |f(S^T y)|^{p-1} |(\nabla f)(S^T y)| \, d\theta \]
\[ \lesssim r \left( \int_0^\pi \frac{\rho(\theta)(1 - \cos \theta)^{\rho(1-\rho\alpha)/(p-1)}}{(\sin \theta)^{1/2}} |f(S^T y)|^p \, d\theta \right)^{\frac{1}{p}} \left( \int_0^\pi \rho(\theta) \sin \theta |(\nabla f)(S^T y)|^p \, d\theta \right)^{\frac{1}{2}}. \]

Since
\[ \frac{\rho(\theta)(1 - \cos \theta)^{\rho(1-\rho\alpha)/(p-1)}}{(\sin \theta)^{1/2}} = \frac{\rho(\theta) \sin \theta}{(1 - \cos \theta)^{\rho\alpha}} \left( \frac{(1 - \cos \theta)^{1-\alpha}}{\sin \theta} \right)^{\frac{\rho}{p-1}} \lesssim \frac{\rho(\theta) \sin \theta}{(1 - \cos \theta)^{\rho\alpha}} \]
which follows from that \( \frac{(1 - \cos \theta)^{1-\alpha}}{\sin \theta} \leq 1 \) for every \( \theta \in [0, \frac{\pi}{2}] \) if \( 1 - \alpha \geq \frac{1}{2} \), we get
\[ I(r, \varphi) \lesssim_\alpha (I(r, \varphi))^{\frac{1}{p}} \left( r^2 \int_0^\pi \rho(\theta) \sin \theta |\nabla f(S^T y)|^p \, d\theta \right)^{\frac{1}{p}} + \int_0^\pi r^{2-p} \sin \theta |f(S^T y)|^p \, d\theta. \]

Hence we conclude
\[ \left\| \frac{f}{|x|^{1+\alpha} \sigma(x)^p} \right\|_{L^p(R^3)}^p \leq C_\alpha \left( \left\| \frac{1}{|x|} f \right\|_{L^p(R^3)}^p + \left\| \nabla f \right\|_{L^p(R^3)}^p \right). \]

This, together with (8.3), gives (8.2) and so finishes the proof of Lemma 8.4. \( \square \)

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