Intermittent direction reversals of moving spatially-localized turbulence observed in two-dimensional Kolmogorov flow

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(Dated: April 10, 2018)

We have found that in two-dimensional Kolmogorov flow a single spatially-localized turbulence (SLT) exists stably and travels with a constant speed on average switching the moving direction randomly and intermittently for moderate values of control parameters: Reynolds number and the flow rate. We define the coarse-grained position and velocity of an SLT and separate the motion of the SLT from its internal turbulent dynamics by introducing a co-moving frame. The switching process of an SLT represented by the coarse-grained velocity seems to be a random telegraph signal. Focusing on the asymmetry of the internal turbulence we introduce two coarse-grained variables characterizing the internal dynamics. These quantities follow the switching process reasonably. This suggests that the twin attracting invariant sets each of which corresponds to a one-way traveling SLT are embedded in the attractor of the moving SLT and the connection of the two sets is too complicated to be represented by a few degrees of freedom but the motion of an SLT is controlled by the internal turbulent dynamics.

I. INTRODUCTION

Spatially-localized turbulent states (SLT) embedded in laminar flows such as puff and stripe, are observed mainly in subcritical transient flows around nonlinear critical Reynolds number both experimentally and numerically [1][2]. These SLTs play a fundamental role in elucidation of generation, evolution and sustenance of turbulence as well as transition to turbulence.

Considering not globally-occupied but spatially-localized states, new aspects of turbulence are emerged such as motion of turbulent regions. Since turbulence states are localized, the position and velocity of a turbulent state can be defined. Furthermore, these facts may stimulate researchers in more general context such as dissipative soliton and self-propelled particle: the former is a moving solitary state in a dissipative system and the latter is a simple model of animate lives such as microorganism, bird, fish and their collective motion.

In these contexts, a spatially-localized turbulent state can be regarded as a moving element coupled with complex internal freedoms. These moving turbulent regions also are connected with phenomena interfering with our daily life. For example, typhoons, which cause severe disasters, are fully developed complex turbulence and needlessly to say, prediction of their paths is not still easy.

Collective behavior of SLTs plays also an essential role in subcritical transitions. In such transient flows, SLTs create their copies and annihilate stochastically [1]. Recently, experimental and numerical researches have uncovered that subcritical transitions in shear flows can be regarded as the absorbing phase transition and its scaling exponents accord with those of directed percolation [7][8].

To describe the dynamics of complex turbulent states, dynamical systems approaches have been widely applied nowadays. In these approaches, simple invariant solutions of governing equations such as periodic solutions are adopted as landmarks embedded in a phase space, and a certain realization is identified as a single trajectory visiting these unstable invariant solutions [9][10][11]. Numerical methods to find unstable solutions based on Newton method have been developed to obtain a good guess for dealing with complex flows even at relatively higher Reynolds number [12][13]. This approach has been extended to the results of laboratory experiments [14]. However, it is a hard task to research dynamical properties of spatially-localized states because we must treat a wide range of spatial modes from small ones representing turbulence to large ones isolating turbulence from laminar regions.

While dynamical and statistical properties of flows in relatively small systems at low or moderate Reynolds numbers have been well understood, those of turbulent flows in extended domains at higher Reynolds number are not still clarified. This is partially because inhomogeneity induced by walls plays a crucial role in developed wall-bounded flows. In fact, many ingredients of turbulent flows including near-wall dynamics and large scale structures in bulk spontaneously coexist and interact with each other [15]. In addition, dynamical description of systems with translational symmetries has been studied for a long time [16][17][18]. However, its extension to dynamical systems with huge degrees of freedom such as turbulent flows has just come to be considered recently and is still one of challenging issues [19].

As a tractable and simple model representing localized turbulence, we deal with a two-dimensional flow in doubly-periodic box forced by a single monochromatic external force called Kolmogorov flow. Kolmogorov flow has been widely examined for a longtime to understand mainly mathematical aspects of Navier-Stokes flow such as cascades of supercritical bifurcations to turbulence [18][19][20][21]. Recently, spatially-localized dynamical states and its dynamical properties have been

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reported\cite{19} \cite{20}. Solitary spatially-localized turbulent states can exist and even be isolated by introducing the flow rate as a control parameter in the direction in which the Galilean invariance is broken by the forcing\cite{30}.

In this paper, we investigate novel translational motion of an SLT. In two-dimensional Kolmogorov flow at moderate values of Reynolds number and the flow rate, an SLT as shown in FIG.1 moves with a nearly constant speed sustaining its direction for a long time and suddenly and intermittently turns around as shown in FIG.2. Our motivation is to clarify the relationship between this translational motion and the internal dynamics of a single SLT.

The rest of paper is organized as follows: Sec.II is devoted to define and characterize the flow system. We introduce a co-moving frame to decompose an SLT into its spatial translation and internal dynamics. In sec.III, the coarse-grained motion of the center of an SLT is examined. In sec.IV we try to describe the motion of the center with representative variables of the internal dynamics of the SLT in the co-moving frame. Concluding remarks are presented in the final section.

II. GOVERNING EQUATION AND SETTING

We focus on two-dimensional (2D) Kolmogorov flow which is 2D flow sustained by a steady sinusoidal force. The velocity field \( \mathbf{u} = (u_x, u_y) \), where the subscripts \( x \) and \( y \) denote the directions parallel and perpendicular to the force, is governed by the following 2D Navier-Stokes equation in doubly periodic domain \( (x, y) \in [0, 2\pi/\alpha] \times [0, 2\pi] \):

\[
\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \frac{1}{\text{Re}} \nabla^2 \mathbf{u} + \sin(ny)\hat{x},
\]

\[
\nabla \cdot \mathbf{u} = 0.
\]

Here, the pressure \( p \) is doubly periodic and \( \alpha, \text{Re}, n \) and \( \hat{x} \) denote the aspect ratio of the rectangular domain, Reynolds number, the wave number of the external sinusoidal force and the unit vector in the \( x \)-direction, respectively.

The average flow rate in \( y \)-direction denoted by \( U_y \) is a conserved quantity and controls the nature of the flow\cite{30}:

\[
U_y = \frac{\alpha}{4\pi^2} \int_0^{2\pi/\alpha} dx \int_0^{2\pi} dy u_y = \langle u_y \rangle_{xy}.
\]

Direct numerical simulation (DNS) solves the following equation for the vorticity, \( \omega = \partial_x u_y - \partial_y u_x \), with the pseudo-spectral method for spatial discretization using the tow-thirds rule for dealiasing and the 2nd order Runge-Kutta (Heun) method for time evolution.

\[
\partial_t \omega + (\mathbf{u} \cdot \nabla) \omega = \frac{1}{\text{Re}} \nabla^2 \omega - n \cos(ny).
\]

The time and spatial resolutions used for DNSs are \( 2 \times 10^{-3} \) and 128 points per \( 2\pi \), respectively.

This system has the following fundamental symmetries for \( U_y \neq 0 \):

\[
T_l : \omega(x, y) \rightarrow \omega(x + l, y) \quad (0 \leq l < \frac{2\pi}{\alpha}),
\]

\[
S : \omega(x, y) \rightarrow -\omega(-x, y + \frac{\pi}{n}).
\]

Here, \( T_l \) is a continuous translational symmetry in \( x \)-direction, and \( S \) is a discrete shift-and-reflect symmetry which is represented by cyclic group of order \( 2n \). We also use these two symbols to denote actions on states of a flow as long as there is no misunderstanding.

There are two main control parameters in 2D Kolmogorov flow: \( \text{Re} \) and the flow rate \( U_y \). Note that for most researches on 2D Kolmogorov flow, \( U_y \) is fixed to 0. Since we are interested in the relationship between the motion and the internal turbulent dynamics of a single SLT, we limit \( \text{Re} \) to two values relatively close to the critical \( \text{Re} \) at which a moving SLT begins to switch its moving direction in \( x \): \( \text{Re} = 26.75 \) and 50 which are slightly and relatively higher than the critical \( \text{Re} \). For a single SLT to exist in the box, the mean flow rate \( U_y \) is set to 0.933 for \( \text{Re} = 26.75 \) and 1.46 for \( \text{Re} = 50 \), respectively. For the latter parameter set, the moving direction of an SLT contains quick fluctuations as well as relatively slow and intermittent switching. The other system parameters, \( n \) and \( \alpha \) are fixed to \( (n, \alpha) = (4, 0.25) \) in this paper.

In the lower \( \text{Re} \) case, the initial condition assigned is an unstable relative periodic solution (URO) which is a continuation solution of the stable solitary relative periodic solution obtained in Ref.\cite{30}. By the symmetry \( S \), this URO can have both positive and negative velocities, \( c_{\text{URO}} = \pm 0.02 \), in \( x \). The period of the URO is \( \sim 60 \)
and characterizes the time scale of the internal turbulent fluctuation.

Because of numerical errors in the initial condition, this solution falls into an SLT in a few periods and gets to switch intermittently its moving direction. Furthermore, around at \( t \sim 10^5 \) it suddenly ceases to move with a constant speed even on average and begins to hang around changing its moving direction quickly. This suggests that there exist several different types of SLT states: A kind of transition from traveling to standing SLT. However, we focus on the first (traveling) SLT state observed before the second transition. For \( \text{Re} = 50 \) and \( U_y = 1.46 \), an SLT continues to travel with switching the direction for a long sustaining time at least \( t \sim O(10^6) \) which is not affected by initial conditions with a single SLT.

We introduce a frame system to separate the motion from the internal turbulent dynamics of each SLT. Here the motion of an SLT stands for the evolution in a coarse-grained time of a point representing the location of the SLT. We call this point the center of the SLT. An SLT travels both in \( x \) and \( y \) directions: the translation in \( y \) is mainly caused by the mean flow rate and the vortex pair constituting the SLT but the translation in \( x \) is derived from the internal turbulent dynamics. Therefore we will give our attention to the motion in \( x \) of an SLT.

This frame system is an extension of Galilean transformation and is defined by formally applying a time-dependent translational symmetry:

\[
\hat{\omega}(x, y, t) = \mathcal{T}_{l(t)} \omega(x, y, t) = \omega(x + l(t), y, t),
\]

where \( l(t) \) is a time dependent shift in \( x \)-direction. We call the case of \( l(t) = 0 \) the laboratory frame and the case of \( l(t) = -X(t) \) the co-moving frame where \( X(t) \) is an approximate or coarse-grained location of the SLT but its definition includes some ambiguity originated from the internal turbulent fluctuation. We define \( X(t) \) by setting the phase of the first Fourier mode of the vorticity field with \( k_x = \alpha \) and \( k_y = 0 \) to \( \pi/2 \). Hereafter, we call \( X(t) \) the center of the SLT. Note that if the velocity of the center of the SLT, i.e., \( V(t) = dX(t)/dt \) is not a constant, the dynamics of the SLT even in the co-moving frame is coupled with the motion of the SLT.

Moreover, we expect (not guaranteed) that the center of the following average vorticity \( \Omega(x, t) \) stays around the same positions in the co-moving frame:

\[
\Omega(x, t) = \frac{1}{2\pi} \int_0^{2\pi} d\omega \omega(x, t),
\]

This method has been adopted for one-dimensional PDE and three-dimensional turbulent pipe [24, 31]. Both in the laboratory and co-moving frames, the time evolution of \( \Omega(x, t) \) is shown in FIG. 2 and FIG. 3.

In the laboratory frame, sudden and intermittent changes of the moving direction of the SLT can be observed. In the co-moving frame, the SLT stands around the same position in time with small fluctuation. This enable us to make a decomposition into the motion of the SLT, i.e., \( X(t) \) and the internal turbulent dynamics, \( \hat{\omega}(x, t) \) defined in (7).

\[ \text{III. MOTION OF MOVING TURBULENCE} \]

We focus on the nature of the switching of the moving direction. Because even the coarse-grained center of an SLT, \( X(t) \), still fluctuates in a time scale of the order of the internal dynamics of the SLT, the intervals between adjacent reverses of the moving direction denoted by \( \Delta t \), i.e., the residence time are evaluated with a velocity averaged over an interval \( T \) defined as

\[
\bar{v}_T(t) = \frac{1}{T} \int_t^{t+T} dt' \frac{dX(t')}{dt'} = \frac{X(T + t) - X(t)}{T}.
\]

The velocity \( \bar{v}_{T=+0} \) crosses zero even if an SLT moves in the same direction in the coarse-grained scale because
the average vorticity Ω stays around the same position but strongly fluctuates in the translational direction especially in the higher Re case as shown in FIG.2 and FIG.3. To detect the direction reversal in a coarse-grained time, we set T = 100, which is longer than a typical time scale of the internal turbulence dynamics. This typical time scale is ~ 60 and of the order of the period of the URO adopted as the initial condition. The subscript T is omitted hereafter for simplicity.

The evolutions of the center X(t) and the average velocity  of vorticity Ω are shown in FIG.4. The average velocity  takes roughly two values, i.e. ±|ωmax| and a direction reversal occurs when  crosses zero. In this sense, the average velocity  is an adequate variable to detect direction reversals. This also suggests that at least there are two (that is, twin) unstable invariant sets with ±|ωmax| about one of which the SLT wanders and the direction reversal corresponds to switching between the stays around these sets. We expect that these invariant sets are close to the twin URO one of which is adopted as the initial condition.

The histogram of the number of the residence time ∆t larger than t denoted by F(∆t > t) is shown in FIG.5. The exponential-decay tendency observed in F(∆t > t) reminds us of a random telegraph signal which is produced by the Poisson process\[32, 33\]. This suggests that the aforementioned twin invariant sets corresponding to SLTs with the positive and negative velocities in x have complicated structures different from simple spiral chaos such as Lorenz attractor. It also seems to support the simplified picture mentioned above of the phase space in which the motion of a single SLT is embedded. Therefore we should carefully select a well-acted projection to describe the trajectory in a phase space.

IV. RELATION TO INTERNAL TURBULENT DYNAMICS

In this section, we try to describe the motion of a single SLT in the phase space in relation to internal turbulent dynamics. To begin with, we introduce some coarse-grained variables that characterize the asymmetric nature in x of the internal turbulent dynamics observed in the co-moving frame based on S which allows an SLT to travel both to the negative and positive directions in x. The first one is the following quantity evaluated simply by the maximum and minimum values of vorticity:

\[ s(t) = \max \omega(x, t) - \left| \min \omega(x, t) \right|, \]  

(10)

Since the transformation S, x → -x and y → y + π/n, changes the sign of the vorticity Ω(x, t), the sign of s(t) also changes as follows:

\[ Ss(t) = -\left( \min \omega(x, t) \right) - \left| \max \omega(x, t) \right| = -s(t). \]  

(11)

The variable s(t) evaluates the degree of the asymmetry of the vorticity distribution of an SLT. This asymmetry is the origin of the antisymmetry of s(t) and thus closely related to the direction reversal.

Since the maximum and minimum of ω(x, t) fluctuate quickly in time like \( t \), the average or coarse-grained  should be also introduced:

\[ \bar{s}_T = \frac{1}{T} \int_t^{t+T} ds(t). \]  

(12)

FIG. 4. Time evolutions of the average velocity  (blue solid line) and the central position of SLT X (black dotted line). Large black dots denote direction reversals at Re=26.5 (top panel) and 50 (bottom panel).

FIG. 5. F(∆t) for (Re, U_y)=(26.75,0.933) (left panel) and (Re, U_y)=(50,1.46) (right panel).
The subscript $T$ of $\varpi_T$ is omitted hereafter for simplicity. As shown in FIG. 3, $\varpi$ correlates adequately with the moving direction of an SLT in the two Re cases. However, they fluctuate more strongly than $c_T(t)$ and this tendency is enhanced in the higher Re case. This suggests that the internal turbulence controls the motion of a single SLT. However, the correlation between $\varpi_T(t)$ and $\varpi_T(t)$ is not sufficient enough for $\varpi_T$ to be used for quantitative description of the motion of the SLT.

We next introduce another variable representing a kind of the distance from these invariant sets more quantitatively. The vorticity field in the co-moving frame $\hat{\omega}(x, t)$ is projected onto the two fields defined by the average under the condition that the traveling direction is positive or negative, respectively. The negative mean state $\phi_n$ and the positive mean state $\phi_p$ are defined numerically as follows:

$$\phi_n(x, t) = \langle \hat{\omega}(x, t) \rangle_{\varpi(\omega)<0}, \quad \phi_p(x, t) = \langle \hat{\omega}(x, t) \rangle_{\varpi(\omega)>0},$$

where the bracket $\langle \rangle$ and its subscript denote an ensemble average in the co-moving frame and the condition under which the average is calculated, respectively. These two fields are expected to approximate the twin invariant sets.

The projections onto $\phi_n$ and $\phi_p$ are carried out with the internal product $\langle \phi | \omega \rangle$ between real functions as follows:

$$a_n(\varpi) = \frac{\langle \phi_n(\omega) \rangle}{||\omega||^2_2},$$

$$a_p(\varpi) = \frac{\langle \phi_p(\omega) \rangle}{||\omega||^2_2},$$

$$\langle f | g \rangle = \alpha \frac{4\pi^2}{\varpi} \int dxdy f(x, y)g(x, y),$$

$$||f|| = \sqrt{\langle f | f \rangle}.$$

The difference between the coefficients denoted by $a(t) = a_p(t) - a_n(t)$ also evaluates the asymmetry of an SLT and is expected to indicate the direction of the motion, because the moving direction of the URO is determined by the asymmetry of the vorticity field and an SLT seems to stay around one of the invariant sets close to the corresponding URO. As shown in FIG. 5, $a(t)$ reproduces roughly the switching process better than $\varpi(t)$. However, in the higher Re case, both the two quantities, $\varpi(t)$ and $a(t)$, which are coarse-grained representatives of the internal turbulent dynamics, tend to be less able to follow the average velocity $\bar{\varpi}(t)$, although the two moving states with the velocities $\pm c_{\text{max}}$ and the direction reversal are still clearly identified. This suggests that even though the twin or multiple invariant sets are still discriminated clearly by the moving direction, the number of variables or the dimension of the phase space required to describe the internal turbulent dynamics which is directly related to the coarse-grained motion of an SLT increases with Re.

This difficulty is inherited by their higher order moments. To see this, we define the dispersion, the second order fluctuation of the vorticity field under the condition $\varpi < 0$ as follows:

$$\delta\Omega^2_n(x) = \int_0^{2\pi} dy \delta\omega^2_n(x, y),$$

$$\delta\omega^2_n(x, y) = \langle \varpi^2(x, y) \rangle_{\varpi < 0} - \langle \varpi(x, y) \rangle^2_{\varpi < 0}.$$
constituted by several twin unstable invariant sets each of which might correspond to a one-way traveling SLT. The asymmetry of the vorticity fluctuation is observed in other traveling localized states or invasion-fronts of turbulence. In the higher Re case, the asymmetry of vorticity fluctuation is weaker than that in the lower Re case. Therefore we need variables more susceptible to geometrical or temporal characteristics of the SLT to resolve the internal turbulent dynamics.

V. CONCLUDING REMARKS

We have found that a single spatially-localized turbulence (SLT) exists stably and travels in \( x \) switching the moving direction randomly and intermittently. By introducing the coarse-grained center \( X(t) \) and traveling velocity \( \tilde{v}_T(t) \) of the SLT, we have characterized the traveling motion and these switching events in the coarse-grained time scale. Even in the relatively high Re case, \( \tilde{v}_T(t) \) takes roughly two values \( \pm |c_{\text{max}}| \) and the residence time \( \Delta t \) is thought to obey the exponential distribution, which suggests that \( \tilde{v}_T(t) \) can be approximated by a random telegraph signal. At least (several) twin attracting invariant sets, each of which corresponds to a one-way traveling SLT and may be close to unstable relative periodic solutions (URO), are embedded in the attractor of the moving turbulence. Since we expect that like a self-propelled particle the motion of a single SLT is controlled by the characteristics of the internal turbulence, the time evolution of the flow is decomposed into the coarse-grained motion of the center of the SLT and the accompanying turbulent field by defining the co-moving frame with \( X(t) \). We have introduced two coarse-grained variables \( \tilde{v}_T(t) \) and \( a(t) \) characterizing the asymmetry in \( x \) of the internal turbulence: \( \tilde{v}_T(t) \) simply estimates the asymmetry of the vorticity distribution of the SLT and \( a(t) \) represents the difference between the approximated distances from the twin unstable sets one of which is projected onto the other by the discrete shift-and-reflect symmetry \( S \). In the lower Re case both the variables follow \( \tilde{v}_T(t) \) sufficiently. However, in the higher Re case, though the switching events are detected well and the latter seems to work relatively better than the former, they fluctuate more significantly than \( \tilde{v}_T(t) \). This implies that with Re the structure in the phase space of each one-way SLT gets complicated and thus more variables or dimensions are required to resolve it. The difficulty in the higher Re case might partly come from an arbitrary way of decomposition into position and internal dynamics. In other words, the fluctuations of quantities in co-moving frame are affected by the definition of the center of position, which might be solved by introducing a proper co-moving frame. However, such a frame could not be found a priori in general.

Since the switching processes are detected sharply and can be well approximated by a random telegraph signal even in the higher Re case, the twins must be still separated clearly in the attractor of a single SLT. In this paper, we have not dealt with the mechanism of the reversal of the moving direction and its relationship with the internal turbulence but focused on the characterization of the coarse-grained motion of a SLT and its internal turbulent. It is easy to make a Langevin model which can reproduce the stochastic nature of the switching events. However, we are now rather trying to study a deterministic model of the switching process in relation to the internal turbulence focusing on the structure of the attractor.

Invariant solutions should play a key role in more reliable description of states at higher Re. It is expected that a fixed point like URO embedded in each of the twin, i.e., a pair of chaotic attracting sets mimics the average quantity for each directions in FIG. The exponential-like decay of the residence time \( \Delta t \) suggests that the switching between the twin occurs randomly like a random telegraph signal and thus the way to connect the twin is complicated but expected to tightly relate to the internal turbulent dynamics. These invariant solutions also will help us to attain more quantitative and precise understanding of SLTs.

This type of intermittent switching can be observed in other flows: reversal of Large Scale Circulation in a steady forced flow and thermal driven flow. The approach based on a low dimensional model derived by Galerkin method is useful to study such a transition. We expect that by this approach with the invariant solutions mathematical models representing a moving SLT can be developed.

Since as mentioned above the residence time is suggested to obey an exponential distribution, the switching process seems to be Poisson process like interval statis-

![Figure 7](image-url)
tics. However, the total number of events obtained are too little to decide a class of direction reversals. Therefore we should perform longer DNS repeatedly to obtain precise statistical aspect of direction reversals.

It is interesting and important to study states consisting of a number of SLTs. Such a multiple SLT state in Kolmogorov flow also can contribute as one of the most simple and tractable examples in the elucidation of turbulence transitions observed in wall-bounded flows. However, our approach introduced in this paper needs some improvements in the definitions of the coarse-grained quantities such as positions, velocities and ones representing internal turbulent dynamics.

Concerning subcritical turbulence transition as non-equilibrium phase transition, ”moving” SLTs that are observed there may affect the determination of critical exponents and/or even a class of the transition itself. In fact, to do so spatial and temporal intervals of laminar regions have been utilized, but fast moving SLTs might modify the distribution of such intervals which blurs the estimation of critical exponents. At least, the correlation length between SLTs can be much longer than the length scale of the support of an SLT. In statistical physics, Mermin-Wagner theorem shows that no long-range order exists for systems at thermal equilibrium in two spatial dimension, e.g. XY model. However, XY model consisting of moving elements such as Viscek model can have a non-zero order parameter and discontinuous phase transition occurs even in two spatial dimensions though Viscek model is a non-equilibrium system. This might suggest that subcritical turbulence transition does not necessarily belong to the universal class of directed percolation if fast moving elements exist there.

ACKNOWLEDGMENTS

The authors thank Dr. Teramura for useful discussions and comments on our work.

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