ON THE ANALYTICITY OF SOLUTIONS TO NON-LINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

SIMON BLATT

ABSTRACT. We present an elementary and easy proof of the fact that $C^\infty$ solutions to non-linear elliptic equations of second order

$$\phi(x, u, Du, D^2 u) = 0$$

are analytic. Following ideas of Kato [10], the proof uses an inductive estimate for weighted derivatives. We conclude the proof using Cauchy’s method of majorants [3].

CONTENTS

1. Introduction .......................... 1
2. Preliminaries ......................... 2
2.1. Characterisation of analytic functions 2
2.2. $L^2$-estimates ..................... 2
2.3. A Banach algebra ................. 3
2.4. Binomials .......................... 3
2.5. Higher order chain rule .......... 4
3. The proof ................................ 5
3.1. Reduction of the problem ......... 5
3.2. A recursive inequality .......... 6
3.3. Cauchy’s method of majorants .... 7
References ............................. 8

1. Introduction

We consider a solution $u \in C^\infty(\Omega, \mathbb{R})$ to the fully non-linear equation

$$(1.1) \quad \phi(x, u(x), Du(x), D^2 u(x)) \equiv 0 \text{ on } \Omega.$$ 

Here, $\Omega \subset \mathbb{R}^n$ is an open subset of the Euclidean space of dimension $n \geq 2$. Furthermore, we assume that $\phi : \Omega \times \mathbb{R} \times \mathbb{R}^n \times S(n) \to \mathbb{R}$ is a real analytic function that is elliptic in the sense that the functions $a_{ij}(x, y, p, q) := \frac{\partial^2 \phi}{\partial q_i \partial q_j}(x, y, p, q)$ satisfy

$$(1.2) \quad 0 < a_{ij}(x, y, p, q)\xi_i \xi_j \quad \text{for all } \xi \in \mathbb{R}^n.$$ 

We will give a short and elementary proof of the following well-known result:

**Theorem 1.1.** Let $u \in C^\infty(\Omega, \mathbb{R})$ be a solution to (1.1), $x_0 \in \Omega$, and $\phi$ be analytic in a neighborhood of $(x_0, u(x_0), Du(x_0), D^2 u(x_0))$. Then $u$ is real analytic near $x_0$. 

*Date: August 7, 2024.*

2010 Mathematics Subject Classification. 35A20, 35B65.

Key words and phrases. Non-linear elliptic equation, real analytic solutions, Faà di Bruno’s formula, method of majorants, Hilbert’s 19th problem.
Shortly after Hilbert asked whether solutions to regular variational problems are always analytic in his famous speech at the ICM in 1900 [9, Problem 19], Bernstein gave the first proof of the theorem above for the case $n = 2$ [2] under the assumption that $u \in C^3$. Gevrey [7] extend Bernstein’s result to parabolic equations inventing a completely different method. Petrowsky [14] generalised the result for a function in two independent variables to systems on a Euclidean space of arbitrary finite dimension. The most general results for elliptic systems can be found in [6] and [13, 12].

In this article we want to give a simple proof of analyticity of a solution $u$ as above. Our proof uses cut-off functions in a way inspired by the work of Kato [10] to derive a recursive estimate for the weighted differentials of $u$. Unfortunately, Kato only considered the case
\[ \Delta u = u^2 \]
in his paper and announced to extend his proof to arbitrary non-linear equations. In [8] Hashimoto tried to carry out this program. In this paper we clarify and complete the proof of Hashimoto.

Our proof combines basic $L^2$ estimates for the Laplacian with a higher order chain rule to derive in straight forward way a recursive estimate for weighted derivates. We use Cauchy method of majorants [3] to show that these recursive estimates imply the analyticity of $u$. That means that we compare the quantities to the derivatives of a solution to a carefully chosen analytic ordinary differential equation. Using that such solutions are known to be analytic, we can close the argument.

We will collect some well known results in Section 2 to make the article accessible to a broader audience. We characterise analytic functions (Subsection 2.1), repeat basic $L^2$ estimates for solutions to Poisson’s equation (Subsection 2.2) and state a higher order chain rule (Subsection 2.5). Furthermore, we prove some identities that help us to deal with the estimates we obtain applying these two ingredients to the problem. In Section 3 we end this note with a complete proof of Theorem 1.1.

2. Preliminaries

2.1. Characterisation of analytic functions. Our proof relies on the following well known fact:

Lemma 2.1. A function $u : \Omega \rightarrow \mathbb{R}^m$, $\Omega \subset \mathbb{R}^n$ is analytic if and only if for every compact set $K \subset \Omega$ there exist constants $C = C_K, A = A_K < \infty$ such that for every multi-index $\alpha \in \mathbb{R}^n$ we have
\[ \|\partial^\alpha u\|_{L^\infty(K)} \leq CA^{\|\alpha\| |\alpha|!}. \]

A proof of this statement can be found in [11]. Combining this with Sobolev’s embedding theorems we immediately get

Corollary 2.2. A function $u : \Omega \rightarrow \mathbb{R}^m$, $\Omega \subset \mathbb{R}^n$ is analytic if and only if for every $x \in \Omega$ there is a radius $r > 0$ and constants $C, A < \infty$ such that for every multi-index $\alpha \in \mathbb{R}^n$ we have
\[ \|\partial^\alpha u\|_{L^2(B(x,r))} \leq CA^{\|\alpha\| |\alpha|!}. \]

It is our aim in the proof of Theorem 1.1 in Section 3, to derive such estimates for a solution of (1.1) around the point $x_0 \in \Omega$ and its derivatives.

2.2. $L^2$-estimates. We recapitulate the following well-known $L^2$-estimate for the Laplacian.

Proposition 2.3. For $u \in C^2(\mathbb{R}^n)$ with compact support we have
\[ \|D^2 u\|_{L^2} \leq \|\Delta u\|_{L^2}. \]
Proof. Applying Green’s first identity twice we get
\[
\int_{\mathbb{R}^n} \Delta u \Delta u \, dx = - \int_{\mathbb{R}^n} \Delta \partial_i u \partial_i u \, dx = \int_{\mathbb{R}^n} \partial_j \partial_j \partial_i u \partial_i u \, dx.
\]
\[\square\]

For \( m = \left\lfloor \frac{n}{2} \right\rfloor + 1 \) we set
\[
\|f\|_{H^m} = \sum_{|\alpha| = m} \| \partial^\alpha u \|_{L^2}.
\]

Then the following proposition holds:

**Proposition 2.4.** For \( u \in C^{2+m}(\mathbb{R}^n) \) with compact support we have
\[
\|D^2 u\|_{H^m} = \|\Delta u\|_{H^m}.
\]

**Proof.** From Proposition 2.3 we get
\[
\|\partial^\alpha D^2 u\|_{L^2} = \|D^2 \partial^\alpha u\|_{L^2} = \|\Delta \partial^\alpha u\|_{L^2} = \|D^\alpha \Delta u\|_{L^2}.
\]

Summing over all \( \alpha \in \mathbb{N}^n \) with \( |\alpha| = m \) yields the result. \( \square \)

2.3. A Banach algebra. We will use later on that \( H^m(\mathbb{R}^n) \) is a Banach algebra, if we chose\( m = \left\lfloor \frac{n}{2} \right\rfloor + 1: \)

**Proposition 2.5** ([1]). For \( f, g \in C^\infty(\mathbb{R}^n) \) with compact support and \( m = \left\lfloor \frac{n}{2} \right\rfloor + 1 \) we have
\[
\|fg\|_{H^m} \leq C \|f\|_{H^m} \|g\|_{H^m}.
\]

**Remark 2.6.** In order to get rid of the constant \( C \) in the last inequality, we work with the norms
\[
(2.1) \quad \|f\|_{\tilde{H}^m} := C \|f\|_{H^m}
\]
in the following where. Then
\[
\|fg\|_{\tilde{H}^m} = C \|fg\|_{H^m} \leq C^2 \|f\|_{H^m} \|g\|_{H^m} \leq \|f\|_{\tilde{H}^m} \|g\|_{\tilde{H}^m}.
\]

2.4. Binomials. For two multiindices \( \alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n \) the binomial coefficient is defined through
\[
\binom{\alpha}{\beta} = \prod_{j=1}^n \binom{\alpha_j}{\beta_j}.
\]

We will use the following result to deal with terms coming from the generalised product rule.

**Proposition 2.7** ([10, Proposition 2.1]). Let \( \alpha \) be a multiindex and \( k \leq |\alpha| \). Then
\[
\sum_{\beta \geq \alpha, \|\beta\| = k} \binom{\alpha}{\beta} = \binom{|\alpha|}{k}.
\]

**Proof.** This can be seen comparing the coefficients in the Taylor series expansions of the identity
\[
(1 + t)^{\alpha_1} \cdots (1 + t)^{\alpha_n} = (1 + t)^{|\alpha|}.
\]
\[\square\]
2.5. Higher order chain rule. The following fact for higher derivatives will be important in our proof:

**Proposition 2.8.** Let \( g : \mathbb{R}^{m_1} \to \mathbb{R}^{m_2} \) and \( f : \mathbb{R}^{m_2} \to \mathbb{R} \) be two \( C^k \)-functions. Then for any multiindex \( \alpha \in \mathbb{N}^{m_2} \) of length \( |\alpha| \leq k \) and \( x \in \Omega \) the identity

\[
\partial^\alpha (f \circ g)(x) = P^\alpha_{m_1,m_2}(\partial^\alpha f(g(x)), \partial^\alpha g)_{|\gamma| \leq |\alpha|, |\gamma| \leq |\alpha|}
\]

holds, where \( P^\alpha_{m_1,m_2} \) is a fixed linear combination with positive coefficients of terms of the form

\[
\partial^\gamma_{\gamma_1} g(x) \partial^{\alpha_1} g_1 \cdots \partial^{\alpha_l} g_l
\]

with \( 1 \leq l \leq |\alpha| \) and \( |\gamma_1| + \cdots + |\gamma_l| = |\alpha|. \)

For \( m_1 = m_2 = 1 \) we will use the notation \( P^\alpha \) instead of \( P^\alpha_{m_1,m_2} \). We leave the easy inductive proof of this statement to the reader. This proposition contains all the information about the higher order chain rule we need. Precise and explicit formulas for the higher order chain rule were given by Faa di Bruno for the univariate case \([5]\) and by for example Constanini and Savits in \([4]\) for the multivariate case.

Let us derive two easy consequences of Proposition 2.8. The first one will allow us to reduce the multivariate case to the univariate one.

**Lemma 2.9.** For constants \( a_\gamma = a_{|\gamma|}, b_\gamma = b_{|\gamma|} \in \mathbb{R} \), depending only on the length of the multiindex \( \gamma \), we have

\[
P^\alpha_{m_1,m_2}([a_\gamma], [b_\gamma]) = P^{\alpha_1}([a_{|\gamma|}], [b_{|\gamma|}]).
\]

**Proof.** Plugging functions \( g \) and \( f \) of the form

\[
g(x_1, \ldots, x_{m_1}) = \tilde{g}(x_1 + \cdots + x_{m_1}) \cdot (1, \ldots, 1)^T
\]

and

\[
f(y_1, \ldots, y_{m_2}) = f \left( \frac{y_1 + \cdots + y_{m_2}}{m_2} \right)
\]

into the higher order chain rule (Proposition 2.8), we get from

\[
f \circ g = (\tilde{f} \circ \tilde{g})(x_1 + \cdots + x_{m_1})
\]

that

\[
P^\alpha_{m_1,m_2}((\partial^\gamma f(g(x))), \partial^\gamma g)_{|\gamma| \leq |\alpha|, |\gamma| \leq |\alpha|} = P^{\alpha_1}((\partial^\gamma \tilde{f}(\tilde{g}(x))), \partial^\gamma \tilde{g})_{|\gamma| \leq |\alpha|, |\gamma| \leq |\alpha|}.
\]

So for constants \( a_\gamma = a_{|\gamma|}, b_\gamma = b_{|\gamma|} \in \mathbb{R} \) depending only on the length of the multiindex \( \gamma \), we obtain

\[
P^\alpha_{m_1,m_2}([a_\gamma], [b_\gamma]) = P^{\alpha_1}([a_{|\gamma|}], [b_{|\gamma|}]).
\]

\( \Box \)

We will use Lemma 2.9 to estimate derivatives of analytic function in \( x, u(x), D u(x), \) and \( D^2 u(x) \). Let \( u \in C^\infty(B_1(0), \mathbb{R}) \) and \( f : \Omega \times \mathbb{R} \times \mathbb{R}^n \times S(n) \to \mathbb{R} \) be a function that is infinitely often differentiable on a neighborhood of the image

\[
K = \{(x, u(x), D u(x), D^2 u(x) : x \in B_1(0))\}.
\]

For a fixed cutoff function \( \rho \in C^\infty(\mathbb{R}, [0, 1]) \) with support contained in \( B_1(0) \) and \( \rho = 1 \) on \( B_{\frac{1}{2}}(0) \) we will consider the quantities

\[
\begin{align*}
M_N := & \sup_{|\alpha| \leq N} |\partial^\alpha u|_{((B_1(0))} & \text{ for } N = 0, 1, 2 \\
\tilde{M}_N := & \sup_{|\alpha| \leq N} |\rho^{N-1} \partial^\alpha u|_{\bar{\Omega}} & \text{ for } N \geq 3
\end{align*}
\]

and

\[
M_N := \tilde{M}_{N+2} + \tilde{M}_{N+1} + \tilde{M}_N + 1
\]

for all \( N \geq 0 \). We prove the following basic higher order estimate for the right hand side of our equation using the chain rule.
Lemma 2.10. We have
\[
|\rho^{|\alpha|} \partial^\alpha f(x, u, Du, D^2u)|_{\|p\|_{L^\infty(0)}} \leq |\rho^{|\alpha|} (|E^{(0)}| |D^{(1)} f|_{C^0(K)}), \{M_k\}_{k=0,...,|\alpha|})
\]
where \(E = (1 + |p|)^3\).

Proof. Applying Faas di Brunos formula (Proposition 2.8) to \(f \circ g\) where
\[
g(x) = (x, u(x), Du(x), D^2u(x))
\]
we get
\[
\rho^{|\alpha|} \partial^\alpha f(x, u, Du, D^2u) = \rho^{|\alpha|} p^{|\alpha|}_{n+2n+1+1} \{\partial^\gamma f, \{\partial^\gamma g\}
\]
where \(p^{|\alpha|}_{n+2n+1} \{\partial^\gamma f, \{\partial^\gamma g\}\} \) is a linear combination with positive coefficients of terms of the form
\[
\partial^\gamma_{x_{i_1}...x_{i_l}} f(g(x)) \partial^\gamma g_{i_1} \cdots \partial^\gamma g_{i_l}
\]
with \(1 \leq l \leq |\alpha|\) and \(\gamma_1 + \cdots + \gamma_l = \alpha\). Due to the special structure of \(g\), we have \(\partial^\gamma g_{i_l} = 1\) for \(i = 1, \ldots, n\) and \(|\gamma| = 1\) and \(\partial^\gamma g_{i_l} = 0\) for \(i = 1, \ldots, n\) and \(|\gamma| \geq 2\). For such terms we get using that \(H^m\) is a Banach algebra \((2.1)\)
\[
|\rho^{|\alpha|} \partial^\gamma_{x_{i_1}...x_{i_l}} f(g(x)) \partial^\gamma g_{i_1} \cdots \partial^\gamma g_{i_l}|_{\|p\|_{L^\infty(0)}} \leq E|\rho^{|\alpha|} \partial^\gamma_{x_{i_1}...x_{i_l}} f(g(x))|_{\|p\|_{L^\infty(0)}} M_{|\gamma|} \cdots M_{|\gamma|}
\]
where \(E = (|p| + 1)^3\). We hence deduce using Lemma 2.9 that
\[
|\rho^{|\alpha|} \partial^\gamma f(x, u, Du, D^2u)|_{\|p\|_{L^\infty(0)}} \leq C|\rho^{|\alpha|} (|E^{(0)}| |D^{(1)} f|_{C^0(K)}), \{M_k\}_{k=0,...,|\alpha|})
\]
\[
\square
\]

3. The proof

After the above preliminaries, we can now prove Theorem 1.1.

3.1. Reduction of the problem. Let \(x_0 \in \Omega\). We will show that \(u\) is analytic in a neighborhood of \(x_0\) in \(\Omega\). We can assume that \(x_0 = 0, B_1(0) \subset \Omega, a_j(0) = \delta_j,\) and that the \(a_{ij}\) have an arbitrarily small oscillation on \(B(0),\) say smaller than \(\varepsilon\) for an \(\varepsilon > 0\) to be chosen below. This can be achieved by a standard change of coordinates combined with a scaling of the solution.

Differentiating Equation (1.1), we see that the function \(v_k = \partial_j u\) satisfies the equation
\[
a_{ij}(x, u, Du, D^2u) \partial_j v_k(x) = -(b_{ij}(x, u, Du, D^2u) \partial_j v_k(x) + c(x, u, \Delta u, D^2u) v_k + d_k(x, u, Du, D^2u))
\]
\[
f_k(x, u, Du, D^2u),
\]
where
\[
a_{ij}(x, y, p, q) = \frac{\partial \phi}{\partial q_j}(x, y, p, q), \quad b_{ij}(x, y, p, q) = \frac{\partial \phi}{\partial p_j}(x, y, p, q),
\]
\[
c(x, y, p, q) = \frac{\partial \phi}{\partial y}(x, y, p, q), \quad d_k(x, y, p, q) = \frac{\partial \phi}{\partial x_k}(x, y, p, q).
\]
We can read off from its definition that \(f_k\) is an analytic function around 0.

We now freeze the coefficients, i.e. we set \(a_{ij}^0 := a_{ij}(0, u(0), Du(0), D^2u(0))\) and observe that \(v_k\) solves the equation
\[
a_{ij}^0 \partial_j v_k = \tilde{a}_{ij}(x, u, Du, D^2u) \partial_j v_k + f_k(x, u, Du, D^2u)
\]
where \(\tilde{a}_{ij} = a_{ij}^0 - a_{ij}\). Due to the bound on the oscillation of the \(a_{ij}\), we have
\[
|\tilde{a}_{ij}| \leq \varepsilon
\]
on \(B_1(0)\).

As \(a_{ij}^0 = \delta_{ij}\), the function \(v_k\) solves
\[
\Delta v_k = \tilde{a}_{ij}(x, u, Du, D^2u) \partial_j v_k + f_k(x, u, Du, D^2u) = R_k.
\]
3.2. A recursive inequality. Since $\partial_i j$ and $f_k$ are analytic, there are constants $C, A < \infty$ such that for every multi-index $\alpha \in \mathbb{N}^{2\omega+1}$ we have

$$\|\partial^\alpha \partial_i j(x, u, Du, D^2 u)\|_{C^\alpha(B_1(0))}, \|\partial^\alpha f_k(x, u, Du, D^2 u)\|_{C^\alpha(B_1(0))} \leq CA^{\|\alpha\|}!$$

For a fixed cutoff function $\rho \in C_0^\infty(\mathbb{R}^n), [0, 1])$ with support contained in $B_1(0)$ and $\rho = 1$ on $B_2(0)$ we now consider as above the quantities

$$\tilde{M}_N := \sup_{|\alpha|=N} \|\partial^\alpha u\|_{l^p} \text{ for } N = 0, 1, 2$$
$$\tilde{M}_N := \sup_{|\alpha|=N} \|\partial^{N+3} \partial^\alpha u\|_{l^p} \text{ for } N \geq 3$$

and

$$M_N := \tilde{M}_{N+2} + \tilde{M}_{N+1} + \tilde{M}_N + 1$$

for all $N \geq 0$. We prove the following recursive estimate:

**Proposition 3.1.** For all $N \geq 1$

$$M_{N+1} \leq C\varepsilon M_{N+1} + C \sum_{0 \leq \gamma \leq N} \binom{N}{\gamma} P^\gamma((E^k A^k)!, \{M_k\})M_{N(\gamma+1)}$$

+ CP^N((E^k A^k)!, \{M_k\}) + CNM_N + CN(N-1)M_{N-1}.$$

**Proof.** We consider the weighted function $\rho^\|\alpha\| \partial^\alpha \rho \partial^\beta \partial_i j(x)$ for a multiindices $\alpha$ and $\beta$ with $|\beta| = 2$ and $N = |\alpha|$. Proposition 2.4 and Equation (3.1) yield

$$\|\partial^\alpha \rho^\|\alpha\| \partial^\beta \partial_i j(x)\|_{l^p} \leq \|\partial^\alpha \rho^\|\alpha\| \partial^\beta \partial_i j(x)\|_{l^p} + \|\rho^N, \partial^\beta \partial^\alpha j(x)\|_{l^p}$$

$$\leq C\|\Delta^N \partial^\alpha \partial_i j(x)\|_{l^p} + C[\|\Delta^N, \partial^\beta \partial^\alpha j(x)\|_{l^p} + \|\rho^N, \partial^\beta \partial^\alpha j(x)\|_{l^p}$$

$$= C\|\rho^N \partial^\beta \partial_i j(x)\|_{l^p} + C[\|\Delta^N, \rho^N \partial^\beta \partial_i j(x)\|_{l^p} + \|\rho^N, \partial^\beta \partial^\alpha j(x)\|_{l^p}$$

$$= I_1 + I_2 + I_3.$$}

Here, $[A, B] = AB - BA$ denotes the commutator. Using the identity

$$\partial_i j(\rho^N h) = \{N(N-1)\rho^{N-2} \partial_i \partial_j \rho + N\rho^{N-1} \partial_i^2 h\} h + N^2 \partial_i \partial_j \rho + \rho^N \partial_i \partial_j h$$

together with the fact that $H^m$ is a Banach algebra, we get

$$I_1 + I_2 \leq C(NM_N + N(N-1)M_{N-1}).$$

To estimate the term $I_3$ we use

$$\|\rho^N \partial^\alpha \partial_i j(\rho^N \partial^\beta \partial_i j(x))\|_{l^p} \leq \|\rho^N \partial^\alpha \partial_i j(\rho^N \partial^\beta \partial_i j(x))\|_{l^p} + \|\partial_i j\|_{l^p} = \|\rho^N \partial^\alpha \partial_i j(\rho^N \partial^\beta \partial_i j(x))\|_{l^p} \leq \varepsilon M_{N+1} + CM_N. $$

The product rule and the higher order chain rule together with the fact that $H^m$ is a Banach algebra yields

$$\|\rho^N \partial^\alpha (\partial_i j \partial^\gamma \partial_i v_k)\|_{l^p} \leq \|\rho^N \partial^\alpha (\partial_i j \partial^\gamma \partial_i v_k)\|_{l^p} + \sum_{0 \leq \gamma \leq N} \binom{N}{\gamma} \|\rho^\gamma \partial^\alpha j(\partial_i j)\|_{l^p} \leq \|\rho^N \partial^\alpha (\partial_i j \partial^\gamma \partial_i v_k)\|_{l^p}$$

$$\leq C\varepsilon M_{N+1} + CM_N + C \sum_{0 \leq \gamma \leq N} \binom{N}{\gamma} P^\gamma((E^k A^k)!, \{M_k\})M_{N(\gamma+1)+1}$$

where we used Lemma 2.10 and (3.2) in the last step. Using Proposition 2.7 we deduce

$$\|\rho^N \partial^\alpha (\partial_i j \partial^\gamma \partial_i v_k)\|_{l^p} \leq C\varepsilon M_{N+1} + CM_N + C \sum_{0 \leq \gamma \leq N} \binom{N}{\gamma} P^\gamma((E^k A^k)!, \{M_k\})M_{N(\gamma+1)+1}. $$

Estimating the term $\|f_k\|_{l^p}$ in a similar way yields

$$I_1 \leq C\varepsilon M_{N+1} + CM_N + C \sum_{0 \leq \gamma \leq N} \binom{N}{\gamma} P^\gamma((E^k A^k)!, \{M_k\})M_{N(\gamma+1)+1} + CP^N((E^k A^k)!, \{M_k\})$$
and hence finally
\[
M_{N+1} \leq C e M_{N+1} + C \sum_{0 \leq i \leq N} \binom{N}{i} P^i([E^k A^i k!], [M_k]) M_{N+i-1}
\]
\[+ CP^N([E^k A^i k!], M_k) + CNM_N + CN(N - 1)M_{N-1}.\]

\[
\Box
\]

3.3. Cauchy’s method of majorants. From Proposition 3.1 we get with \( A_1 = E \cdot A \)
\[
M_{N+1} \leq C e M_{N+1} + C \sum_{0 \leq i \leq N} \binom{N}{i} P^i([A^i k!], [M_k]) M_{N+i-1}
\]
\[+ CP^N([A^i k!], M_k) + CNM_N + CN(N - 1)M_{N-1}.
\]

for all \( N \in \mathbb{N} \). We assume from now on that \( C \geq 1 \) and \( C e \leq \frac{1}{2} \), so that
\[
N \quad M_{N+1} \leq \frac{1}{2} M_{N+1} + C \sum_{0 \leq i \leq N} \binom{N}{i} P^i([A^i k!], [M_k]) M_{N+i-1}
\]
\[+ CP^N([A^i k!], M_k) + CNM_N + CN(N - 1)M_{N-1}.
\]

We now look for a suitable majorant for the right-hand side. For \( A_1 = E \cdot A \) we set
\[
G_1(z) := \frac{1}{2} + C \sum_{n \in \mathbb{N}} A^n_1 (z - M_0)^n
\]
and
\[
G_2(z) := C \sum_{n \in \mathbb{N}} A^n_1 (z - M_0)^n.
\]

We consider the solution \( w \) to ordinary the differential equation
\[
w'(t) = G_1(w)w' + G_2(w) + C(t + t^2))w' + M_1
\]
with \( w(0) = M_0 \). As \( |G_1(w) + C(t + t^2)| < 1 \) near to \( t = 0 \) and \( w = M_0 \) this differential
equations is equivalent to
\[
w'(t) = \frac{G_2(w)}{1 - (G_1(w) + Ct + Ct^2)}
\]
near to \( t_0 \) and \( w = M_0 \). Hence, this initial value problem has a unique analytic solution on
an open time interval containing \( 0 \). If we assume that \( CA_1 \geq M_1/2 \), we get that
\[
w'(0) \geq M_1.
\]
Using the product and chain rule and using that \( G'_1(M_0) = 0 \), one sees that the derivatives
\( \hat{M}_N = w^{(N)}(0) \) satisfy \( M_1 \leq \hat{M}_1 \) and the equality
\[
\hat{M}_{N+1} = \frac{1}{2} \hat{M}_{N+1} + C \sum_{0 \leq i \leq N} \binom{N}{i} P^i([A^i k!], [M_k]) \hat{M}_{N+i-1}
\]
\[+ CP^N([CA^i k!], [M_k]) + CN\hat{M}_N + CN(N - 1)\hat{M}_{N-1}
\]
for \( N \geq 1 \). Comparing the last equality with inequality (3.4) we inductively deduce that
\[
M_N \leq \hat{M}_N
\]
for all \( N \in \mathbb{N}_0 \). Since \( w \) is analytic around \( 0 \), Lemma 2.1 gives us constants \( 0 < \hat{C}, B \) such
that \( \hat{M}_N = w^{(N)}(0) \leq CB^N N! \) for all \( N \in \mathbb{N}_0 \). Hence,
\[
M_N \leq \hat{M}_N \leq \hat{C}B^N N!
\]
and Lemma 2.1 tells us that \( u \) is analytic on \( B_1(0) \).
REFERENCES

[1] Robert A. Adams. *Sobolev spaces*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[2] S. Bernstein. Sur la nature analytique des solutions des équations aux dérivées partielles du second ordre. *Mathematische Annalen*, 59(1-2):20–76, 1904.

[3] Augustin Louis Cauchy. *Mémoire sur l'emploi du calcul des limites dans l'intégration des équations aux dérivées partielles*. 1842.

[4] G. M. Constantine and T. H. Savits. A multivariate Faà di Bruno formula with applications. *Transactions of the American Mathematical Society*, 348(2):503–520, 1996.

[5] F Faà Di Bruno. Note sur une nouvelle formule de calcul différentiel. *Quarterly J. Pure Appl. Math*, 1(359-360):12, 1857.

[6] Avner Friedman. On the regularity of the solutions of nonlinear elliptic and parabolic systems of partial differential equations. *J. Math. Mech.*, 7:43–59, 1958.

[7] Maurice Gevrey. Sur la nature analytique des solutions des équations aux dérivées partielles. Premier mémoire. *Annales Scientifiques de l'École Normale Supérieure. Troisième Série*, 35:129–190, 1918.

[8] Yoshiaki Hashimoto. A remark on the analyticity of the solutions for non-linear elliptic partial differential equations. *Tokyo Journal of Mathematics*, 29(2):271–281, 2006.

[9] David Hilbert. Mathematical problems. *Bull. Amer. Math. Soc.*, 8(10):437–479, 07 1902.

[10] Keiichi Kato. New idea for proof of analyticity of solutions to analytic nonlinear elliptic equations. *SUT Journal of Mathematics*, 32(2):157–161, 1996.

[11] Steven G. Krantz. A primer of real analytic functions, 1992.

[12] Charles B. Morrey, Jr. On the analyticity of the solutions of analytic non-linear elliptic systems of partial differential equations. I. Analyticity in the interior. *American Journal of Mathematics*, 80:198–218, 1958.

[13] Charles B. Morrey, Jr. On the analyticity of the solutions of analytic non-linear elliptic systems of partial differential equations. II. Analyticity at the boundary. *American Journal of Mathematics*, 80:219–237, 1958.

[14] I. G. Petrowsky. Sur l’analyticité des solutions des systèmes d’équations différentielles. *Rec. Math. N. S. [Mat. Sbornik]*, 5(47):3–70, 1939.

(Simon Blatt) DEPARTMENT OF MATHEMATICS, PARIS LODRON UNIVERSITÄT SALZBURG, HELLBRUNNER STRASSE 34, 5020 SALZBURG, AUSTRIA

Email address, Simon Blatt: simon.blatt@sbg.ac.at