A SPHERE HARD TO CUT

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Abstract. We show that for any \( \epsilon, M > 0 \) there is a Riemannian 3-sphere \( S \) of volume 1, such that any (not necessarily connected) surface separating \( S \) in two regions of volume greater than \( \epsilon \), has area greater than \( M \).

1. Introduction

Glynn-Adey and Zhu show in [4] that for any \( \epsilon > 0, M > 0 \) there is a Riemannian 3-ball \( B \) of volume 1 such that any smooth disk separating \( B \) in two regions of volume greater than \( \epsilon \) has area greater than \( M \). We prove the same result here both for the 3-ball and the 3-sphere for separations by arbitrary surfaces and not just disks. Glynn-Adey and Zhu assume further that the ball \( B \) has bounded diameter and boundary surface area but these are properties that are easy to arrange in general modifying slightly the ball \( B \).

These results contrast with the situation in dimension 2. Liokumovich, Nabutovsky and Rotman showed in [7] that if \( D \) is a Riemannian 2-disc there is a simple arc of length bounded by \( 2\sqrt{3} \sqrt{\text{area}(D)} + \delta \) which cuts the disc into two regions of area greater than \( \text{area}(D)/4 - \delta \) where \( \delta \) is any positive real. A similar result was shown in [9] for the sphere. The results in [7] were prompted by a question of Gromov [5] and Frankel-Katz [2] concerning bounding the length of contracting homotopies of a 2-disk.

Balacheff-Sabourau [1] showed that there is some \( c > 0 \) such that any Riemannian surface \( M \) of genus \( g \) can be separated in two domains of equal area by a 1-cycle of length bounded by \( c\sqrt{g + 1}\sqrt{\text{area}(M)} \). Liokumovich [6] on the other hand showed that given \( C > 0 \) and a closed surface \( M \) there is a Riemannian metric of diameter 1 on \( M \) such that any 1-cycle splitting it into two regions of equal area has length greater than \( C \).
2. A sphere hard to cut

**Definition.** Let $B$ be a Riemannian 3-ball. If $F \subset B$ is a smoothly embedded orientable surface with boundary we say that $F$ *separates* $B$ if $F \cap \partial B = \partial F$.

If $F$ is a surface separating a Riemannian 3-ball $B$ we say that $F$ *cuts an $\epsilon$-piece* of $B$ if $B - F$ can be written as a union of two disjoint open sets $U, V$ both of which have volume greater than $\epsilon$.

We define similarly what it means for a closed surface to cut an $\epsilon$-piece of a Riemannian 3-sphere.

Our construction relies on the existence of expander graphs. We recall now the definition of expanders. Let $\Gamma = (V, E)$ be a graph. For $S, T \subseteq V$ denote the set of all edges between $S$ and $T$ by $E(S, T) = \{(u, v) : u \in S, v \in T, (u, v) \in E\}$.

**Definition.** The *edge boundary* of a set $S \subseteq V$, denoted $\partial S$ is defined as $\partial S = E(S, S^c)$.

A $k$-regular graph $\Gamma = (V, E)$ is called a *$c$-expander graph* if for all $S \subset V$ with $|S| \leq |V|/2$, $|\partial S| \geq c|S|$.

Pinsker [8] has shown that there is a $c > 0$ such that for any $n$ large enough there is a 3-regular expander graph with $n^3$ vertices.

Consider a 3-regular $c$-expander graph $\Gamma_n$ with $n^3$ vertices. We give a way to ‘thicken’ this graph, i.e. replace it by a Riemannian handlebody. For each vertex we pick a Euclidean 3-ball $B_v$ of radius $1/n$. Recall that the volume of this ball is $4\pi/3 \cdot (1/n)^3$. Let $S_v$ be the boundary sphere of $B_v$. If $l$ is an equator of of $S_v$ we pick 3 equidistant points $e_1, e_2, e_3$ on $l$ and we consider 3 disjoint (spherical) discs on $S_v$ with centers $e_1, e_2, e_3$ and radii equal to $1/n$. Clearly these discs are disjoint. Now to each edge $E_i$ in $\Gamma$ leaving $v$ we associate the disc with center $e_i$. If an edge $e$ joins the vertices $v, w$ of $\Gamma$ we identify the discs of the balls $B_v, B_w$ corresponding to this edge.

In this way we obtain a handlebody $\Sigma_n$. Note that $\partial \Sigma_n \cap B_v$ is a pair of pants. We will refer to $B_v$ later on as a filled in pair of pants and we will call the discs with centers $e_1, e_2, e_3$ on $S_v$ the holes of this pair of pants. We note that the area of $S_v$ is $4\pi (1/n)^2$ and the area of of $S_v$ minus the 3 spherical discs is

$$4\pi (1/n)^2 - 6\pi (1/n)^2 (1 - \sin 0.5) = \pi (1/n)^2 (6 \sin 0.5 - 2).$$

By changing the metric of $\Sigma_n$ slightly we get a smooth handlebody, denoted still by $\Sigma_n$, of volume $4\pi/3$. Finally by gluing appropriately thickened discs to this handlebody we obtain a ball $B_n$. We may assume that this gluing operation changes the volume of $B_n$ and the area of
its boundary by a negligible amount. We may pick a simple curve \( \gamma \) on \( \partial B_n \) such that every point of \( \partial B_n \) is at distance at most \( 1/n \) from \( \gamma \). By gluing a thickened disk of diameter \( 1/n \) and negligible volume to \( \partial B_n \) along \( \gamma \) we obtain a new ball of arbitrarily small diameter. We still denote this 3-ball by \( B_n \). In fact it follows also directly by the properties of expander graphs that the diameter of \( B_n \) is bounded.

We double \( B_n \) along its boundary to obtain a 3-sphere. By changing the metric slightly along the doubling locus we may ensure that we obtain a smooth sphere \( S_n \) of volume \( 8\pi/3 \).

**Theorem 2.1.** Given \( \epsilon, M > 0 \) there is some \( n \) such that any surface that cuts an \( \epsilon \)-piece of \( B_n \) (or \( S_n \)) has area greater than \( M \).

**Proof.** We may (and will) assume that \( \epsilon < 1/100 \). Let \( F \) be a (not necessarily connected) surface cutting an \( \epsilon \)-piece of \( B_n \). So \( B_n - F = U_1 \cup U_2 \) with \( U_1, U_2 \) open of volume greater than \( \epsilon \). We denote by \( Q_1, Q_2 \) the closures of \( U_1, U_2 \) respectively. Without loss of generality we assume that \( \text{vol}(U_2) \geq \text{vol}(U_1) \).

We note that \( B_n \) contains a handlebody \( \Sigma_n \) which is a union of filled in pairs of pants \( B_v \)-one for each vertex of the graph \( \Gamma_n \). Clearly \( S_v \cap \partial \Sigma_n \) is a pair of pants with 3 holes.

Let \( B_v \) be one such (filled in) pair of pants. Its volume is \( 4\pi/3n^3 \). By the solution of the isoperimetric problem for a ball \([10]\) if a surface cuts an \( \epsilon 4\pi/3n^3 \) piece of \( B_v \) then its area is greater than \( (4\pi\epsilon/3n^3)^{2/3} \geq \epsilon/n^2 \).

Let’s say that for \( n_1 \) filled in pairs of pants \( F \) cuts an \( \epsilon/n^3 \) piece and that for \( n_2 \) filled in pairs of pants more than \( 4\pi(1-\epsilon)/3n^3 \) of their volume is contained in \( U_1 \). Since \( \text{vol}(U_1) \leq \text{vol}(U_2) \)

\[
n_2 \leq 2\epsilon n^3 \leq n^3/2
\]

We distinguish two cases.

**Case 1.** \( n_1 \geq \epsilon n^3/2 \). Since the area of intersection of \( F \) with each one of these \( n_1 \) filled in pairs of pants is greater than \( \epsilon/n^2 \) the area of \( F \) is greater than \( \epsilon^2 n/2 \) which clearly tends to infinity as \( n \to \infty \).

**Case 2.** \( n_1 < \epsilon n^3/2 \). Since \( \text{vol}(U_1) > \epsilon \) we have that \( n_2 \geq \epsilon n^3/2 \). Let’s denote this set of \( n_2 \)-filled pairs of pants by \( A \). Let \( B_v \) be in \( A \), and let \( U_v = B_v \cap U_1 \). Since

\[
\text{vol}(U_v) \geq \frac{4\pi(1-\epsilon)}{3n^3}
\]
by the Euclidean isoperimetric inequality the boundary of $U_v$ has area at least
$$\frac{4\pi(1-\epsilon)^{2/3}}{n^2}.$$ 

Since $\epsilon < 1/100$ it follows that if the area of $F \cap B_v$ is less than $\epsilon/2n^2$ then $U_1$ intersects non-trivially all 3 holes of the filled-in pair of pants. In fact since the area of a spherical cap is given by $2\pi rh$ where $r$ is the radius and $h$ the height, the area of the intersection of $U_1$ with a hole is greater than
$$\frac{2\pi}{4n^2} > \frac{1}{n^2} \quad \text{(**).}$$

Let’s denote by $A_1$ the set of filled-in pair of pants in $A$ for which the area of intersection of $F \cap B_v$ is more than $\epsilon/2n^2$ and let $A_2 = A - A_1$. We set $k_1 = |A_1|$, $k_2 = |A_2|$ and note that
$$k_1 + k_2 = n^2 \geq \epsilon n^3 \frac{3}{2}.$$ 

If $k_1 \geq \epsilon n^3/4$ then we see that the area of $F$ is greater than $\epsilon^2 n/8$ which clearly tends to infinity as $n \to \infty$. Otherwise $k_2 \geq \epsilon n^3/4$. By the expander property (and since $k_2 \leq n^3/2$) the (not necessarily connected) union of filled in pairs of pants in $A_2$, $\Sigma$, has a boundary that consists of at least
$$ck_2 \geq \frac{\epsilon n^3}{4}$$
holes. Let $B_v$ be a filled-in pair of pants adjacent to one of these holes. Clearly $B_v$ intersects $U_1$. We claim that the area of $B_v \cap F$ is at least $\epsilon/n^2$. This is clear if $F$ cuts an $4\pi\epsilon/3n^3$ piece of $B_v$ or if $B_v$ lies in $A_1$. If this is not the case then more than $(1 - \epsilon)4\pi/n^3$ of the volume of $B_v$ is contained in $U_2$. Let $O_v$ be the center of $B_v$. Let’s denote by $C_r$ the sphere with radius $r$ and center $O_v$. Let $l_r$ be the length of the intersection of $F$ with $C_r$. If $l_r > 1/10n$ for all $r$ with $1/n > r > 9/10n$ then by the co-area formula ([3], 3.2.22) the area of $F \cap B_v$ is greater than $1/100n^2 > \epsilon/n^2$. Otherwise we consider an $r_0 \in (9/10n, 1/n)$ for which $l_{r_0}$ is smaller than $1/10n$. We consider the portion $F_1$ of $F$ between $C_{r_0}$ and the boundary of $B_v$ and we fill the holes of $F_1$ lying on $C_{r_0}$ by minimal area discs. The total area of these discs is smaller than $\frac{\pi}{100n^2}$.

Let’s call the surface obtained this way by $F_2$. Note that $F_2$ separates $U_1 \cap B_v$ from $O_v$. Let $f$ be the radial projection from $O_v$ to $C_1 = S_v$. Clearly $f(F_2)$ contains $S_v \cap U_1$ and by inequality (***) the area of $S_v \cap U_1$ is greater than $\frac{1}{n^2}$. Also $f$ is Lipschitz with Lipschitz
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constant less than 2. So the area of \( f(F_2 - F_1) \) is less than \( \frac{\pi}{50n^2} \). It follows that the area of \( F_1 \) is greater than 
\[
\frac{1}{4n^2}
\]
so the area of \( F \cap B_\nu \) is greater than \( \epsilon/n^2 \) in this case too.

It follows as before that the area of \( F \) is at least
\[
\frac{c\epsilon n^3 \cdot \epsilon}{4} = \frac{c\epsilon^2 n}{4}
\]
which clearly tends to infinity as \( n \to \infty \).

The result for the 3-sphere \( S_n \) follows immediately from \( B_n \) as \( S_n \) is a union of two copies of \( B_n \) and if a surface cuts an \( \epsilon \)-piece of \( S_n \) is cuts an \( \epsilon/2 \) piece in one of these two copies of \( B_n \). Finally clearly we may normalize the volume of \( S_n, B_n \) to 1.

□

Remark 1. In [4] it is assumed additionally that the surface area and the diameter of the ball \( B_n \) is bounded. However both these properties are easy to arrange. As for the surface area one may excise a small ball from the 3-sphere \( S_n \) in the proof above and obtain a ball \( B \) such that the area of \( \partial B \) is arbitrarily small. By construction \( B_n, S_n \) have diameter less than 1. In fact given any ball (in any dimension \( \geq 3 \)) one can easily decrease its diameter by surgery: one may cut out a thickened simple curve and glue back in a ball with small diameter. This has no effect on the volume- or separation properties of the ball. Even though we stated our result only for dimension 3 the same construction applies for spheres (balls) of any dimension \( n \geq 3 \).

References

[1] F. Balacheff, S. Sabourau, Diastolic and isoperimetric inequalities on surfaces, Ann. Sci. Ecole Norm. Sup. 43 (2010) 579-605.
[2] S. Frankel, M. Katz, The Morse landscape of a Riemannian disc , Annales de l’ Inst. Fourier 43 (1993), no. 2, 503-507.
[3] H. Federer, Geometric Measure Theory, Springer Verlag, New York, 1969.
[4] P. Glynn-Adey, Z. Zhu, Subdividing 3-dimensional Riemannian disks, arXiv preprint, arXiv:1508.03746, 2015.
[5] M. Gromov, Asymptotic invariants of infinite groups, in Geometric Group Theory, v. 2, 1-295, London Math. Soc. Lecture Note Ser., vol. 182, Cambridge Univ. Press, Cambridge, 1993.
[6] Y. Liokumovich. Surfaces of small diameter with large width J. Topol. Anal., 06, 383 (2014). Surfaces of small diameter with large width
[7] Y. Liokumovich, A. Nabutovsky, R. Rotman Contracting the boundary of a Riemannian 2-disc, arXiv preprint arXiv:1205.5474 2012.
[8] M. Pinsker, *On the complexity of a concentrator*, 7th International Teletraffic Conference, Stockholm, June 1973, 318/1–318/4.

[9] P. Papasoglu, *Cheeger constants of surfaces and isoperimetric inequalities*, Trans. Amer. Math. Soc 361 (2009), no. 10, 5139-5162.

[10] A. Ros, *The isoperimetric problem* in Global theory of minimal surfaces, volume 2 of Clay Math. Proc., pages 175-209. Amer. Math. Soc., Providence, RI, 2005.

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