Application of Implicit Finite Difference Method to Determine the 2D Patterns of Unsteady State Thermal Spreading of Geothermal Systems

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Abstract. In this study, we built a thermal numerical modelling scheme using finite difference method based on the implicit Crank-Nicholson algorithm. The aim of the study is to describe the process of heat transfer, which is calculated using a thermal diffusion equation (2D vertical) at the unsteady-state conditions, in the geothermal area. The physical parameter used as the input is the thermal diffusivity of the rocks. This parameter was calculated from the other parameters (thermal conductivity, heat capacity, and density). The model was applied to the Wapsalit geothermal area whose geological structure has been known. The model successfully describes the heat flow that spread from the bottom layer to the surface by adjusting to the conditions of the rock.

1. Introduction

In a geothermal system, temperature distribution of the subsurface is formed by conduction process, convection process, or both [1]. The pattern of the temperature distribution is directly controlled by the variation of rock thermal conductivity in the area. The composition of the rock thermal conductivity will affect the pattern of heat distribution from the reservoir layer to the surface of the earth.

Geothermal system generally consists of four main parts: heat source, reservoir rock (permeable rock), cap rock and fluid circulation [2]. The most important part of this system is the reservoir. In this part processes of heat transfer and mass transfer take place. The processes that occur in this reservoir will affect the manifestations that appear at the surface.

Within the reservoir, the energy is stored at certain geological conditions. This energy is obtained from the system that has unique characteristics. The main characteristic of the system is the pattern of heat distribution. Therefore, the pattern of the heat distribution becomes a very important to be investigated.
In this study, we built a numerical modeling scheme using finite difference method based on the implicit Crank-Nicholson algorithm to describe the conduction process of heat transfer at the unsteady-state condition in the geothermal subsurface area. This modeling scheme was applied to some synthetic geological models (for testing the sensitivity of the models) and the subsurface structure of the Wapsalit geothermal area. The modeling result is used to interpret the subsurface thermal structure of the study area.

2. Method

2.1. The Equation Generator and Implicit Finite Difference Approach

Convection and conduction are the heat transfer processes that occur in the earth. In a geothermal system, the convection process occurs in the recharge area and in the permeable reservoir layer. On the other hand, conduction process occurs when heat migrates from the heat source to the surface through an impermeable rock layer. The 2D heat conduction equation in unsteady state condition is given by [3]:

\[ \frac{\partial T}{\partial t} = \kappa \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right) \quad (1) \]

With \( \kappa = \frac{k}{\rho c_p} \)

Where:
- \( \kappa \): Coefficient of thermal diffusivity (\( m^2/s \))
- \( k \): Thermal conductivity of the rock (\( W/mK \))
- \( \rho \): Density of the fluid and the rock (\( kg/m^3 \))
- \( c_p \): Specific heat capacity (\(kJ/kg\)).

To solve the equation, we applied a numerical approach using the finite-difference method based on the implicit Crank-Nicholson (CN) algorithm. The implicit Crank-Nicholson algorithm produces two sets of tridiagonal simultaneous equations which have to be solved in sequence. The first step is solving the equation on the \( x \)-axis, while the second step is solving the equation on the \( y \)-axis [4].

For \( x \)-axis, the numerical solution of the Equation (1) is:

\[
\frac{\kappa \Delta t}{4 \Delta x^2} T_{i,j}^{n+1} + \left( 1 + 2 \frac{\kappa \Delta t}{4 \Delta x^2} \right) T_{i,j}^{n+1} + \frac{\kappa \Delta t}{4 \Delta x^2} T_{i+1,j}^{n+1} = T_{i,j}^n + \frac{\kappa \Delta t}{4 \Delta x^2} (T_{i+1,j}^n - 2T_{i,j}^n + T_{i-1,j}^n) \quad (2)
\]

While the solution for \( y \)-axis is:

\[
\frac{\kappa \Delta t}{4 \Delta y^2} T_{i,j}^{n+1} + \left( 1 + 2 \frac{\kappa \Delta t}{4 \Delta y^2} \right) T_{i,j}^{n+1} + \frac{\kappa \Delta t}{4 \Delta y^2} T_{i,j+1}^{n+1} = T_{i,j}^n + \frac{\kappa \Delta t}{4 \Delta y^2} \left( T_{i,j+1}^n - 2T_{i,j}^n + T_{i,j-1}^n \right) \quad (3)
\]

2.2. Stability analysis of Crank Nicholson’s Method

The derivative approximation of Equation (1) using Crank-Nicholson’s method is:

\[
\frac{T_{i,j}^{n+1} - T_{i,j}^n}{\Delta t} = \frac{1}{2 \kappa} \left( \frac{T_{i+1,j}^{n+1} - 2T_{i,j}^{n+1} + T_{i-1,j}^{n+1}}{\Delta x^2} + \frac{T_{i,j+1}^{n+1} - 2T_{i,j}^{n+1} + T_{i,j-1}^{n+1}}{\Delta y^2} \right) \quad (4)
\]

By defining that \( d_1 = \frac{1}{2} \frac{\kappa \Delta t}{\Delta x^2} \) and \( d_2 = \frac{1}{2} \frac{\kappa \Delta t}{\Delta y^2} \) we obtain:
\[ T_{i,j}^{n+1} = T_{i,j}^n + d_1 \left( T_{i+1,j}^{n+1} + T_{i-1,j}^{n+1} + T_{i,j+1}^n + T_{i,j-1}^n \right) + d_2 \left( T_{i,j+1}^{n+1} + T_{i,j-1}^{n+1} + T_{i+1,j}^n + T_{i,j}^n \right) \]

The Fourier Component was defined as:

\[ T_{i,j}^n = T^n e^{i(\theta + \phi)} ; \]
\[ T_{i,j}^{n+1} = T^{n+1} e^{i(\theta + \phi)} ; \]
\[ T_{i\pm 1,j}^n = T^n e^{i(\theta \pm \phi)} ; \]
\[ T_{i,j\pm 1}^{n+1} = T^{n+1} e^{i(\theta \pm \phi)} ; \]
\[ T_{i,j\pm 2}^n = T^n e^{i(\theta \pm \phi(\pm 1))} \]

So, by substituting the Equations (6) into Equation (5), we obtain:

\[ T^{n+1} e^{i(\theta + \phi)} = T^n e^{i(\theta + \phi)} + d_1 \left( T^{n+1} e^{i(\theta + \phi)} - 2T^{n+1} e^{i(\theta + \phi)} + T^{n+1} e^{i(\theta(\pm 1) \phi)} \right) + d_2 \left( T^{n+1} e^{i(\theta + \phi)} - 2T^n e^{i(\theta + \phi)} T_{i,j}^n + T^n e^{i(\theta(\pm 1) + \phi)} \right) \]

After doing some eliminations, we obtain:

\[ T^{n+1} \left( 1 + d_1 + d_2 - d_1 \left( \frac{e^{i\theta} + e^{-i\theta}}{2} \right) - d_2 \left( \frac{e^{i\phi} + e^{-i\phi}}{2} \right) \right) = T^n \left( 1 - d_1 - d_2 + d_1 \left( \frac{e^{i\theta} + e^{-i\theta}}{2} \right) + d_2 \left( \frac{e^{i\phi} + e^{-i\phi}}{2} \right) \right) \]

\[ T^{n+1} = T^n \left( \frac{1 - d_1 - d_2 + d_1 \left( \frac{e^{i\theta} + e^{-i\theta}}{2} \right) + d_2 \left( \frac{e^{i\phi} + e^{-i\phi}}{2} \right)} {1 + d_1 + d_2 - d_1 \left( \frac{e^{i\theta} + e^{-i\theta}}{2} \right) - d_2 \left( \frac{e^{i\phi} + e^{-i\phi}}{2} \right)} \right) \]

Next, using the identities \( \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \) and \( \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \), we get:

\[ T^{n+1} = T^n \left[ \frac{1 - d_1 - d_2 + d_1 \cos \theta + d_2 \cos \phi}{1 + d_2 - d_1 \cos \theta - d_2 \cos \phi} \right] \]

\[ G = \left[ \frac{1 - d_1 - d_2 \cos \theta + d_2 \cos \phi}{1 + d_1 + d_2 - d_1 \cos \theta - d_2 \cos \phi} \right] \]

A stable solution is fulfilled when \( |G| \leq 1 \) or:

\[ \left| \frac{(1 - d_1 - d_2 + d_1 \cos \theta + d_2 \cos \phi)}{(1 + d_1 + d_2 - d_1 \cos \theta - d_2 \cos \phi)} \right| \leq 1 \]
Since $-2 \leq \cos \theta - 1 \leq 0$ and $-2 \leq \cos \phi - 1 \leq 0$, then Equation (11) will be fulfilled for every \( d_1 \in R \) and \( d_2 \in R \). As the conclusion, we can define that stability of the 2D heat conduction equation in unsteady state condition using Crank Nicholson’s scheme is unconditionally stable.

2.3. Wapsalit Geological Models

Figure 1 is the geological model of Wapsalit geothermal area taken from Idral [5]. The Wapsalit geothermal area is located in the district of Apo Wae, Buru, Maluku. This area is one of the prospective zones of the geothermal energy in Indonesia.

The subsurface structure shown in Figure 1 is represented by the rock resistivity. Based on the rock resistivity, it can be known that the Wapsalit geothermal area is composed of several layers. The top layer consists of sandy soil that stretches from the southwest to the northeast. Marble (fresh metamorphic rock) can be found at some depth. Slate (weathered metamorphic rock) is found under the marble layer. Granite (as the altered rock) is found at the top of cap rock layers. The cap rock layers are composed of shale. Reservoir layer is composed of limestone. The heat source is located under the reservoir layer.

![Figure 1. Resistivity cross-section of Wapsalit geothermal area [5](image)](image)

In this study, we solved the thermal distributions for the Wapsalit geological models. The geological model was interpreted to obtain the diffusivity coefficient of rocks. The diffusivity coefficient was calculated based on the thermal conductivity, specific heat capacity, and density of rocks (see Table 1). The geological model was used as the input for the modeling of the thermal distributions.

Figure 2 depicts the geometric domain of Wapsalit geological models. The geometry consists of 12 rows \( \times \) 40 columns grid points. The grid spacing of the model is 100 meters, so the total dimension is 1200 \( \times \) 4000 meters. The initial value for this model is zero \( (T(x, y) = 0) \) at all points. The boundary condition at the left side, right side, and the top of the model is Neumann boundary condition. The heat source is located under the reservoir rock. The temperature of the heat source is assumed about 600\(^\circ\)C to 900\(^\circ\)C.

In this modelling, the iteration process will be stopped if the surface temperature of the model has reached about 60.7\(^\circ\)C. This criterion was chosen based on the real temperature at the study site [6].
Figure 2. Geometry of the modelling domain of Wapsalit geological model (numbers and colours within the cells correspond to the types of rocks shown in Table 1)

Table 1. Thermal diffusivity of the rock types used in Wapsalit geological model [7]

| No | Color Index | Rock types                          | Thermal diffusivity ($10^{-3} \text{ cm}^2/\text{s}$) |
|----|-------------|-------------------------------------|-------------------------------------------------------|
| 1  | Yellow      | Sandy Soil/top soil                 | 3                                                     |
| 2  | Green       | Marble/fresh metamorphic rock       | 10                                                   |
| 3  | Red         | Slate/weathered metamorphic rock    | 11                                                   |
| 4  | Orange      | Granite/altered rock                | 16                                                   |
| 5  | Blue        | Shale/cap rock                      | 8                                                    |
| 6  | Blue        | Limestone/reservoir rock            | 11                                                   |

3. Results and Discuss

3.1. Sensitivity Study

Before applying the model to the Wapsalit geothermal area, firstly we conducted a sensitivity test of the model to see the effect of the geometry, the coefficient of thermal diffusivity parameters (presented in Table 2) and the boundary condition using some simple synthetic models. The simple models for sensitivity study consist of a homogeneous earth model (shown in Figure 3a), an anomalous body embedded in a homogeneous earth model (shown in Figure 3b) and an anomalous body trapped within a layered earth model (shown in Figure 3c). Colors depicted within the cells correspond to the type of rocks listed in Table 2.
Figure 3. Simple Models for sensitivity study consist of homogeneous earth model (a); anomalous body embedded in homogeneous earth model (b); anomalous body within layered earth model (c). Colours within the cells correspond to the types of rocks listed in Table 2 [8].

Table 2. Thermal diffusivity of rocks used in the sensitivity study [7]

| No | Color Index | Parameter | Rock types | Thermal diffusivity ($10^{-3}$ cm$^2$/s) |
|----|-------------|-----------|------------|------------------------------------------|
| 1  |             |           | Quartzite  | 26                                       |
| 2  |             |           | Peridotite | 17                                       |
| 3  |             |           | Serpentine | 13                                       |
| 4  |             |           | Slate      | 11                                       |
| 5  |             |           | Basalt     | 9                                        |
| 6  |             |           | Sandy Soil | 3                                        |
| 7  |             |           | Steel      | 173                                      |

The geometry models shown in Figure 3 have a size of 4000 x 6000 meters, divided into 20 rows x 30 columns. The initial temperature is zero ($T(x, y) = 0$) for all points in the grid. The boundary condition at the left, the right, and the top of the model are Neumann boundary condition. The temperature of the heat source is $500^{\circ}$C and it is located at the bottom of the model.
Figure 4. Thermal distribution of a homogeneous earth model (a); an anomalous body embedded in a homogeneous earth model (b); an anomalous body embedded within a layered earth model (c).

Modeling result for the first model (Figure 4a) shows that the temperature is decreased (evenly) when it gets closer to the surface. This shows that there is no difference in conductivity at each depth (homogeneous layers). From the second model (shown in Figure 4b), we can see a change in temperature in the anomalous rock. The rock causes the changes of the temperature distribution to be faster. This is because the thermal diffusivity of the anomalous rock has greater value than the surrounding rocks.

From the third model (shown in Figure 4c) we can see that the distribution of temperature flow is disturbed by rock anomalies. The model shows that the temperature flows from the bottom layer to the surface slowly. This is because each layer in this model has a different thermal diffusivity. However, because there is an anomalous rock in the middle of the model, which has a high thermal diffusivity, then there is a heat flow disorder in that layer. Yet, the contours of the thermal distribution models still show a good distribution pattern.

The results of sensitivity studies of three models of the earth in Figure 4 show that the pattern of heat distribution corresponds to the synthetic geological models. The sensitivity study of the model also shows a good performance. The variation of the model structure can be successfully represented by the heat distribution. Within this perspective, the Crank-Nicholson algorithm can be used to determine the distribution of heat in the Wapsalit geothermal area.

3.2. Thermal Numerical Modelling using Wapsalit Geological Model

Figure 5a shows the results of numerical modeling of 2D unsteady state thermal diffusivity at the Wapsalit geothermal area. The heat distribution pattern of the model shows that heat flow spreads by adjusting the conditions of the rock layers. The temperature of the reservoir layer is higher than surrounding due to direct contact with the heat source (probably derived from magma intrusion).
Figure 5a shows the result of the thermal spreading model in the Wapsalit geothermal area. The heat distribution pattern shows that the heat flow adjusts to the rock layers in its path. On the other hand, Figure 5b depicts the profile of surface temperature in the Wapsalit geothermal area. It can be seen that for this geothermal area the temperature tends to be higher in zone that contain a reservoir in the subsurface. Based on this pattern, we have an opportunity to build an inversion method to investigate the subsurface in the future.

4. Conclusions
The finite difference method based on the Implicit Crank-Nicholson algorithm has successfully applied to describe the distribution of 2D heat diffusion in the unsteady state condition of geothermal systems. The sensitivity study shows that the modeling scheme exhibits acceptable performance. The model of the Wapsalit geothermal area successfully exhibits the pattern of heat diffusion, which is in accordance with the conditions of rock layers.

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