Current statistics in the \( q \)-boson zero range process

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Abstract
We obtain exact formulas of the first two cumulants of particle current in the \( q \)-boson zero range process on a ring via exact perturbative solution of the TQ-equation. The result is represented as an infinite sum of double contour integrals. We perform the asymptotic analysis of the large system size limit \( N \to \infty \) of the expressions obtained. For \( |q| \neq 1 \) the leading terms of the second cumulant reproduce the \( N^{3/2} \) scaling expected for models in the Kardar–Parisi–Zhang universality class. The scaling \( q \approx \exp(-\alpha/\sqrt{N}) \to 1 \) corresponds to the crossover between the Kardar–Parisi–Zhang and Edwards–Wilkinson universality classes. Under this scaling the sum converges to an integral, resulting in the crossover scaling function derived previously for the asymmetric simple exclusion process and conjectured to be universal.

Keywords: \( q \)-boson zero range process, diffusion constant, T–Q equation, Kardar–Parisi–Zhang universality class

1. Introduction

Kardar–Parisi–Zhang (KPZ) and Edwards–Wilkinson (EW) universality classes unify plenty natural phenomena like interface random growth including the spread of fire [1, 2], the growth of bacterial colonies [3], solidification, wetting and instability fronts [4, 5]. They are also believed to capture universal features of the large scale behaviour of traffic flows [6] and polymers in random media [7]. For a review see [8] and references therein. Both classes were extensively studied starting from the 1980s [9]. The first analytic efforts were concerned with
corresponding stochastic PDEs for the interface growth, the continuous models, which were expected to give a universal description of the whole universality classes.

Description of the universal behaviour of an interface governed by the stochastic linear EW PDE [10] was relatively straightforward. In particular, it was shown to be characterized by two independent critical exponents, which can be chosen e.g. to be the roughness exponent $\zeta$ and dynamical exponent $z$. In $1 + 1$ dimension, we deal with in this paper, they are $\zeta = 1/2$ and $z = 2$. The first one shows that the stationary EW interface is like the Brownian motion. The second suggests that the propagation of fluctuations is purely diffusive.

The nonlinear KPZ equation resisted the exact analytic treatment for other 25 years. Luckily, the exact critical exponents in $(1 + 1)$ dimensions were guessed from heuristic arguments already in the seminal paper of Kardar et al [11]. The roughness exponent $\zeta = 1/2$ can be found from the fact that an addition of the non-linearity that transforms the EW equation to the KPZ one does not affect its stationary solution. However, off the stationary state the propagation of fluctuations becomes highly nontrivial being characterized by $z = 3/2$ exponent. The exponents $\zeta$ and $z$ can be translated to yet another pair of exponents $\alpha = \zeta/z$ and $\beta = 1/z$, responsible for fluctuation and correlation scales respectively. Correspondingly, their values are $\alpha = 1/4$, $\beta = 1/2$ for EW and $\alpha = 1/3$, $\beta = 2/3$ for KPZ class.

At that early stage the dimensional analysis, mode-coupling method and the renormalization group applied to the KPZ equation were useful for finding (at least heuristically) critical exponents and conjecturing scaling hypotheses about KPZ universality [9]. However, the exact scaling functions were off the scope of those approaches. This is when the integrable stochastic interacting particle systems came on stage. Thinking about interacting particles in $1 + 1$ dimensions we mean a stochastic diffusive or driven-diffusive particle system subject to an uncorrelated random force with local inter-particle interactions. The particle density field in such a system can be thought of as a gradient of the associated interface height. Conversely, the interface height is nothing but the time-integrated particle current in the particle system. Using this correspondence, the results obtained for a particle system can be translated to the interface language and vice versa. In addition, the integrability implies a special mathematical structure behind the stochastic process that allows one to obtain exact results, which produce the universal functions in the scaling limit.

The paradigmatic examples are the symmetric and asymmetric simple exclusion processes abbreviated as SSEP and ASEP [12], which together with associated ‘solid-on-solid’ interfaces give examples of systems belonging to the EW and KPZ universality classes respectively. In these models particles perform symmetric or asymmetric random walks on a 1D lattice subject to exclusion interaction, which prevents two particles from coming to the same site. Results on these models can roughly be divided into two groups.

The first group consists of results about the stationary state and the large time behaviour of finite systems like a periodic lattice or a segment with particle reservoirs attached to the ends. Among them are the stationary state density and current profiles [13] and correlation functions [14], the large deviation functions of the particle density and current on a ring [15, 16] and on a segment [17–22], etc. In particular the current cumulants were obtained, which we discuss below in detail. (For historical account and references see reviews [23–25].)

The second group describes the transient dynamics in the infinite system. These are the one-point current distributions for several special types of initial conditions [26–29]. In the case of TASEP, the totally asymmetric version of ASEP, it was also possible to find all the equal-time multipoint current distributions [30, 31] as well as those for unequal times [32, 33] and space–time points on space-like paths [34–36]. Some progress also was achieved for the time-like correlation functions [37].
Recently, the finite time results were also extended to systems with periodic boundary conditions [38–43], where one can study the transition between transient and stationary regimes. The latter set actually belongs to the intersection of the two mentioned groups.

Though the mentioned results give a vast picture of the EW and KPZ universality in 1+1 dimensions, we are still far from constructing the general theory. This is why testing the results obtained against other interacting particle models with richer dynamics is of interest. In this paper we address probably the next simplest interacting particle model, the zero range process (ZRP) [12]. This is a particle system on 1D lattice with totally asymmetric jumps, where the jump probability depends only on the occupation number of the site of departure. The peculiarity of this system, which reveals its simplicity already at the level of the stationary state, is the factorized form of the stationary measure in the infinite or periodic system [44]. Here we consider the continuous time $q$-boson ZRP, where the functional form of the jump rates ensures the Bethe ansatz integrability of the stochastic generator. The quantum integrable model based on the quantum deformation of the boson algebra was first proposed by Bogoliubov and Bullough [45]. It later was adapted by Sasamoto and Wadati [46] to be considered as the stochastic interacting particle model. One of the authors of this paper rediscovered it looking for the example of ZRP solvable by the coordinate Bethe ansatz [47]. Surprisingly, it was found to be dual under particle–hole transformation of the $q$-TASEP model that appeared much later as a degeneration of the Macdonald process [48].

The hopping rates of the model are parameterized by a real parameter $q \in \mathbb{R}$. The model is believed to belong to the KPZ universality class when $q \neq 1$ and to the EW universality class when $q = 1$. Several results were obtained on the $q$-boson ZRP. The mean group velocity and the diffusion coefficient for two particles on the infinite lattice were calculated [46]. The scaling form of the large deviation function of the particle current was obtained for the periodic lattice in the large system size limit and $q \neq 1$ [47].

More recently several results on the non-stationary dynamics on the infinite lattice were obtained. The spectral theory for the $q$-boson ZRP on the infinite lattice was constructed in [49]. In particular the Green functions of the evolution operator [50] and the exact expression of the leftmost particle position [51] were obtained using the Bethe ansatz diagonalization of the Markov generator [47]. Also many interesting results were obtained for $q$-TASEP, the dual partner of $q$-boson ZRP, using both the Markov duality [52] and the relation between $q$-TASEP and Macdonald process [48]. In particular the $q$-Laplace transform of the particle position was obtained [53], using which the scaling limit of the distribution of particle position was shown to converge to Tracy–Widom distribution [54]. Finally we mention that several generalizations of the $q$-boson ZRP were proposed like discrete time $q$-boson ZRP [55], $q$-Hahn ZRP [56, 57] and $q$-PushASEP [58]. Also the multi-species version of $q$-boson model was constructed [59, 60]. The stationary measure for the latter was obtained in the matrix product form and used for determining of the current and density profile [61].

In the present paper we undertake a further study of the large time regime of $q$-boson ZRP on the ring of size $N$ evaluating the exact diffusion coefficient for the particle position and the associated interface height. The problem of calculation of the current cumulants in exclusion processes has a long history. The diffusion coefficient was first obtained for TASEP on a ring [62] and on a segment [63] using the matrix product ansatz [64]. Later the matrix product technique was extended to ASEP [65]. The whole large deviation function of the distance traveled by a particle in TASEP, which in particular yielded all the exact scaled cumulants including the diffusion coefficient, was derived using the Bethe ansatz in [15]. This solution used significantly a special structure of the TASEP Bethe equations, which is not present in the more general ASEP case. The large deviation function for the ASEP was constructed in the large system size limit under a special KPZ scaling by the method of asymptotic
solution of the Bethe equations proposed in [66, 67]. The universal current cumulants in SSEP on the ring were obtained asymptotically both from the Bethe ansatz and from the fluctuating hydrodynamics in [16]. Technique based on asymptotic solutions of the Bethe equations was also applied to evaluate the current large deviation function in the ASEP on the segment with open ends [19]. Finally, the approach to finding the exact expressions of current cumulants based on the functional Bethe ansatz or T–Q Baxter equation was developed for ASEP on the finite ring by Prolhac and Mallick in [68]. The exact large deviation function for the ASEP on a segment was also found by adapting the matrix product [20, 21] and using the T–Q Baxter equation [22, 69].

Here we apply the method developed by Prolhac and Mallick to derive the first two cumulants of the particle current in the $q$-boson ZRP on the ring. Our interest is two-fold. First, it is the technical aspect of the perturbative solution of T–Q equation and as a result finding the exact form of the second current cumulant. Though both the ASEP and $q$-boson ZRP are solved in terms of the similar trigonometric Bethe ansatz, the concrete details are very different. While the former is related to the two-dimensional representation of the quantum affine algebra $U_q(\hat{sl}_2)$, the latter is constructed in terms of its infinite-dimensional $q$-boson representation. These facts reveal themselves in the structure of solution of the T–Q equation. In both cases the solutions are given in terms of polynomial truncation of the generating function of stationary weights. Complexity of this function seems to depend crucially on the dimension of underlying representation. In the ASEP case the single site weight is a simple binomial and the weight of $N$ sites is its $N$th power. In contrast, for the $q$-boson ZRP the single site function is an infinite sum representing the entire or meromorphic $q$-exponential function, of which the $N$th power is much harder to manipulate with. As a consequence the exact expression of the diffusion coefficient obtained in [65] is an explicit sum of quantities constructed of binomial coefficients. In our case we were able to represent the final result in the form of an infinite sum of double contour integrals, which being less explicit, however, is well suited for asymptotic analysis.

Second but not the least goal we pursue is to find the scaling limit of the formulas obtained and, thus, to check the scaling hypothesis made before on the basis of the analysis of EW and KPZ equations and ASEP. As due to the universality the results of this type are meaningful far beyond the specific exactly solvable model, we give here a brief account of the previous studies of the universal scaling behaviour of cumulants of particle current and interface height.

In general it is expected that in the infinite system of KPZ and EW universality classes a particle moves subdiffusively, with fluctuations growing with time $t$ as $t^\alpha$. So does the height of associated interface. However, in the finite periodic system of size $N$ at large time $t \gg N^z$ particles move diffusively, i.e. the variance of the fluctuations grows linearly with time with the proportionality factor, named diffusion coefficient, that vanishes in the infinite system size limit as $N^{2z-\zeta}$, i.e. $1/N$ for EW and as $1/\sqrt{N}$ for KPZ universality class (for details see review [9] and discussion in the text). The latter power law was one of the first demonstrations of KPZ scaling behaviour obtained from exact solution [62]. The universal power law form of cumulants of KPZ interface height of arbitrary order was conjectured in [70, 71] basing on the analysis of dimensions supplied with the scaling invariance arguments. Specifically the model dependent dimensionful factors were predicted, which come with the power laws, while the universal dimensionless constants are to be determined from exact solutions. The program of determining the universal constants was first realized within the exact calculation of the second current cumulant [62, 65] and of the cumulants of arbitrary order [15, 67] in TASEP and ASEP. The asymptotic scaling form of the cumulants was also verified at several
other models [47, 55, 72, 73]. The same program for the EW class was undertaken in the mentioned solution of SSEP in [16] and extended to ASEP under \(1/N\) weak asymmetry scaling [74].

The pure power laws for cumulants, which take place at the KPZ and EW fixed points, are expected to be connected by universal scaling functions. One example of such a function was derived in [65] from the exact formula for the diffusion coefficient in ASEP under weak \(1/\sqrt{N}\) asymmetry scaling. Another candidate for the scaling function describing the crossover of the third cumulant was also derived under the \(1/\sqrt{N}\) asymmetry scaling in [75].

Note that the crossover between KPZ and EW universality classes is exactly the regime of applicability of the KPZ equation, the two extreme cases being attained in the infinity and zero limits of the non-linearity parameter or equivalently at the late and early stages of the time evolution respectively. An alternative approach to the search for the universal cumulants and crossover scaling functions for EW and KPZ classes exploits the relation of KPZ equation with the problem of polymer in random media [7], which suggests that the corresponding interface height is distributed as the free energy of the latter. Treating the problem within the framework of replica method [87], reduces the calculation of \(n\)-replica polymer partition function to the quantum problem of \(n\) Lieb–Liniger bosons with attractive delta-interaction [76], solvable via the Bethe ansatz. On the other hand, the expectation of the \(n\)-polymer free energy per unit length plays the role of the rescaled generating function of the polymer free energy with \(n\) being the parameter of the generating function. Then, derivation of the rescaled cumulants proceeds via the heuristic trick of the expansion of the result around \(n = 0\). The way of analytical continuation of the solution of \(n\)-particle problem to the non-integer number of particles was proposed by Brunet and Derrida in [77]. In particular, the second order of the small \(n\) expansion of the \(n\)-polymer free energy reproduces the crossover function obtained in [74]. Also the third order term has been obtained, a correspondence of which with the result of [75] for the third cumulant has not been studied yet. In the joint limit \(n \to 0\) and boson interaction constant \(c \to \infty\) considered in [78] the \(n\)-polymer free energy attains the scaling form of the scaled cumulant generating function of KPZ class from [15].

To complete the review of studies of the universal cumulant behaviour we also mention the later results on the universal cumulants in a non-stationary setting. Specifically on the fluctuation scale the fluctuations of the interface in the EW universality class are purely Gaussian [10] while those in the KPZ class are described by universal distributions dependent on the global shape of initial conditions, some of which were previously known from the theory of random matrices [26–29]. The crossover between KPZ and EW universality classes is described by the solution of the KPZ equation, which was initially obtained for the narrow wedge initial conditions both from the weakly asymmetric limit of ASEP [79, 80] and from the polymer in random medium [81]. Finally we mention the finite time results [38–43] for TASEP in the ring geometry, which have the stationary fluctuations on the ring and the transient fluctuations in the infinite system as limiting cases.

In our paper we perform the asymptotic analysis of the exact formula obtained for the second stationary current cumulant in the \(q\)-boson ZRP both under in the KPZ and crossover scaling limits. The formula obtained in the KPZ regime confirms the earlier result [47]. Under the crossover scaling we obtain the scaling function interpolating between the EW and KPZ classes, which confirms the universality of the expression conjectured from the ASEP solution [65].

Our paper is organized as follows. In section 2 we define the model, sketch necessary information about its stationary state and current cumulants and state the first main result of the paper, exact formula of the diffusion coefficient. In section 3 we explain the mapping of the particle system to the interface on the cylinder and survey predictions based of
the scaling hypotheses for such an interface belonging to KPZ and EW classes. We conclude the section 3 formulating the results of the asymptotic analysis, which confirms these conjectures. In section 4 we outline the derivation of exact formulas of the two first scaled cumulants of particle current. Section 5 is devoted to asymptotic analysis. We summarize and conclude in section 6. The appendix contains details of calculations, which were moved there for the purpose of better readability of the main text.

2. \( q \)-boson zero range process: model, stationary state and results

2.1. The model and its observables

ZRP is a stochastic interacting particle system. We define it on a periodic one dimensional lattice with \( N \) sites (sites \( i \) and \( N + i \) are identical) and \( p \) particles. Each lattice site can be occupied by an integer number of particles \( n_i \geq 0 \). A particle configuration is specified by the set of occupation numbers \( n = \{ n_1, \ldots, n_N \} \). The total number of configurations is \( C_{N+p-1}^p \).

We consider a continuous time Markov process on the set of particle configurations. Each site has its own Poissonian alarm clock, which rings with rate \( u(n_i) \). When the clock rings a particle from site \( i \) jumps to the neighbouring site \( i + 1 \) (we imply that \( u(0) = 0 \)) (figure 1).

Let \( P_t(C) \) be the probability for the system to be in configuration \( C \) at time \( t \). The probability solves the master or forward Kolmogorov equation

\[
\frac{\partial}{\partial t} P_t(n) = \mathcal{L} P_t(n).
\]

Here \( \mathcal{L} \) is the operator, whose action on probability is defined by

\[
\mathcal{L} P_t(n) = \sum_{n'} (u(n' \rightarrow n) P_t(n') - u(n \rightarrow n') P_t(n)),
\]

where the rate \( u(n' \rightarrow n) \) of transition from configuration \( n' \) to \( n \) is equal to \( u(n_i) \) if the configuration \( n \) is obtained from \( n' \) by a single jump of a particle from site \( i \) to the site \( i + 1 \) and zero otherwise. In the following we will deal with the particular choice of the rates

\[
u(n) = [n]_q = \frac{1 - q^n}{1 - q},
\]

which was shown to be the one necessary for the Bethe ansatz integrability [47]. These rates are positive when \( q > -1 \). This is the range we consider below.

Having a solution of the master equation corresponding to particular initial conditions one can compute the expectation of any function of configuration at given time. Often one would
also like to study the statistics of additive functionals on trajectories of the process, by which we mean a quantity \( Y_t \) changing its value by a fixed amount \( \delta \) every time the system jumps from \( n' \) to \( n \). To this end, one considers the joint probability \( P_t(n, Y) \) for the configuration to be \( n \) and the value of \( Y_t \) to be \( Y \) at time \( t \). Its generating function

\[
G_t(n, \gamma) = \sum_{Y=0}^{\infty} P_t(n, Y) e^{\gamma Y}
\]

is a solution of the non-stochastic deformation of (1)

\[
\partial_t G_t(n, \gamma) = \mathcal{L}_\gamma G_t(n, \gamma),
\]

where the matrix of the deformed operator \( \mathcal{L}_\gamma \) is obtained from that of \( \mathcal{L} \) by multiplying every off-diagonal element corresponding to transition from \( n' \) to \( n \) by \( e^{\gamma \delta} \). We consider a particular example of \( Y_t \), the total distance traveled by all particles by time \( t \). In this case the increase of \( Y_t \) due to jump of a single particle is always \( \delta n \), so that the action of \( \mathcal{L}_\gamma \) is as follows.

\[
\mathcal{L}_\gamma G_t(n, \gamma) = \sum_{n'} (e^{\gamma u(n' \rightarrow n, \gamma)} G_t(n', \gamma) - u(n \rightarrow n') G_t(n, \gamma))
\]

(2)

The moment generating function of the random variable \( Y_t \) is given in terms of \( G_t(n, \gamma) \).

\[
\mathbb{E} e^{\gamma Y} = \sum_n G_t(n, \gamma)
\]

The utility of \( G_t(n, \gamma) \) reveals itself in an observation that in the long time limit its behaviour is dominated by the largest eigenvalue \( \lambda(\gamma) \) of matrix \( \mathcal{L}_\gamma \),

\[
\lambda(\gamma) = \lim_{t \to \infty} \frac{\ln \mathbb{E} e^{\gamma Y_t}}{t} = \sum_{n=1}^{\infty} c_n \frac{\gamma^n}{n!},
\]

i.e. the function \( \lambda(\gamma) \) plays the role of the generating function of scaled cumulants

\[
c_n = \lim_{t \to \infty} \frac{\langle Y^n_t \rangle_c}{t}
\]

of \( Y_t \), where we use notation \( \langle \xi^n \rangle_c \) for \( n \)th cumulant of the random variable \( \xi \). In particular, the first two scaled cumulants, which we deal with below, have a simple physical meaning. The first one

\[
J = J(N, p) := c_1 = \lambda'(0)
\]

is the expected number of particle jumps in the system per unit time, aka mean integral particle current, obtained by time-averaging of the expected total number of jumps made by all particles by time \( t \) growing to infinity. The second scaled cumulant is the group diffusion coefficient

\[
\Delta = \Delta(N, p) := c_2 = \lambda''(0)
\]

associated with the joint motion of all particles. Instead of these extensive quantities associated with the whole system we can consider intensive quantities associated with a single bond or a particle. In a sense the intensive quantities are more convenient to characterize the infinite system.
For example, one can consider the number of particles $y_i(t)$ that have passed through the bond $(i, i+1)$ by time $t$. Obviously the variation of the number of particles passing through different bonds on the ring is bounded by the total number of particles, i.e. $y_i(t) = Y_i/N + O(N)$, where $O(N)$ is time independent. Hence, the first two scaled cumulants of $y_i(t)$, the mean current and current diffusion coefficient, are independent of the bond being equal to

$$ j_N = \lim_{t\to\infty} \frac{\langle y_i(t) \rangle}{t} = \frac{J}{N}. $$

$$ \Delta_N^1 = \lim_{t\to\infty} \frac{\langle y_i^2(t) \rangle}{t} = \frac{\Delta}{N^2}. $$

Suppose also that particles on the ring always preserve their order. For example we may think that the particles in every site are arranged into a column. Only the top particle is allowed to jump, while the particle which jumps into a site takes the lowest position. Then, the same argument applies to the coordinate of a particle $x_i(t)$. Therefore, the first two scaled cumulants of $x_i(t)$, the mean particle velocity and the single particle diffusion coefficient, are

$$ v_N^p = j_N / \rho, \quad \Delta_N^p = \Delta_N / \rho^2. $$

As will be seen below the first intensive cumulants approach finite nonzero values in the thermodynamic limit, while the diffusion coefficients vanish indicating that the corresponding quantities evolve subdiffusively in the infinite system.

2.2. Stationary state and scaled current cumulants

Here, before stating the results we recap briefly the properties of the stationary state of the $q$-boson ZRP and fix necessary notations.

The peculiarity of ZRP is the factorized form of the stationary probability distribution [44], which makes the analysis of the stationary state particularly simple. This is to say that the probability of finding the system in a configuration $n$ is given by a product of one-site weights

$$ P_n = \frac{\prod_{i=1}^{N} f(n_i)}{Z(N, p)}, $$

(3)

where the one-site weight is given by

$$ f(m) = \begin{cases} 
\prod_{j=1}^{m} \frac{1}{u(j)}, & m > 0 \\
1, & m = 0 
\end{cases} $$

(4)

and

$$ Z(N, p) = \sum_{\{n_1, \ldots, n_N \mid \sum_{i=1}^{N} n_i = p\}} \prod_{i=1}^{N} f(n_i). $$

(5)

is the normalization factor referred to as the (canonical) partition function. The partition function can be given an integral representation with the use of the generating function of one-site weights

$$ F(z) = \sum_{n=0}^{\infty} f(n) z^n. $$
In the case of $q$-boson ZRP the series $F(z)$ is convergent for $z$ in the disk $|z| < 1/(1 - q)$ when $|q| \leq 1$ and for $z$ in the whole complex plane when $|q| > 1$ to infinite products, which give two different $q$-exponential functions.

$$
F(z) = \begin{cases} 
\exp_q(z(1 - q)), & |q| < 1, \\
\exp_q(z(q^{-1} - 1)), & |q| > 1,
\end{cases}
$$

(6)

where $(z; q)_\infty = \prod_{i=0}^{\infty} (1 - zq^i)$. From the statistical physics point of view the generating function of weights of particle configurations in $N$-site system, $F^N(z)$, considered as a function of fugacity $z$ can be thought of as the grand-canonical partition function. The canonical partition function $Z(N, p)$ has a contour integral representation

$$
Z(N, p) = \oint \frac{F^N(z)}{z^{p+1}} \frac{dz}{2\pi i},
$$

(7)

which is a standard relation between the canonical and grand-canonical partition functions.

The partition function $Z(N, p)$ is a basic object in our consideration, which will frequently appear further. The generating function approach to its calculation is also useful to evaluation of various stationary state observables. For example the cumulants of the occupation number in the stationary state are given by

$$
\langle n_i^k \rangle_c = \frac{d^k}{d\chi^k} \ln \oint \frac{F^{N-1}(z)F(e^\chi z)}{z^{p+1}} \frac{dz}{2\pi i} \bigg|_{\chi=0},
$$

(8)

while for the mean current through the bond $(i, i+1)$, which is the mean number of particles jumping out of the site $i$, we have

$$
j_N = \mathbb{E}[n_i] = \frac{Z(N, p - 1)}{Z(N, p)}.
$$

(9)

These quantities manifestly do not depend on the site $i$. Thus for the integrated current we have

$$
J = \mathbb{E} \left( \sum_{i=1}^{N} u(n_i) \right) = NV \frac{Z(N, p - 1)}{Z(N, p)}.
$$

(10)

The second scaled cumulant, aka diffusion coefficient, can not be obtained from the simple stationary state analysis, being the simplest observable that implicitly contains unequal time correlations. To find this quantity we need to address first the full dynamical problem. This is what will be done below. Here we give the final expression, which is the main result of this article.

**Result 1.** The group diffusion coefficient $\Delta$ has the following representation

$$
\Delta = pJ + \frac{2N^2}{Z(N, p)^2} \oint \frac{dy}{2\pi i} \frac{F^N(y)}{y^p} \oint \frac{dt}{2\pi i} \frac{F^N(t) \phi(y)}{t-y}
$$

$$
+ \frac{2N^2}{Z(N, p)^2} \sum_{i=1}^{\infty} \oint \frac{dt}{2\pi i} \frac{F^N(t)}{t^p} \oint \frac{dy}{2\pi i} \frac{F^N(y) q^{\pm i} \phi(yq^{\pm i}) + \phi(y)}{t-yq^{\pm i}},
$$

(10)
where plus and minus signs in the powers of $q$ correspond to $|q| < 1$ and $q > 1$ respectively and the integration contours are two nested simple counterclockwise loops around the origin, which do not contain any other poles. Also we defined the function

$$\phi(z) = \frac{J}{p} (\ln F(z)')' - 1. \quad (11)$$

where we use notation $(\ln F(z)')' = \partial_z (\ln F(z))$ for derivative of function $\ln F(z)$.

The formula (10) is valid for $−1 < q \neq 1$. Taking the limit $q \to 1$ is a nontrivial exercise. It is, however, straightforward to show that at $q = 1$

$$J(N, p) = \Delta(N, p) = p. \quad (12)$$

Indeed, in this case a particle jumps from a site with the rate $u(n) = n$. If we forget about the order of particles, we may think that every particle tries to jump with unit rate independently of the others. Then the above result trivially follows from the properties of the sum of $p$ independent Poisson processes. We do not consider taking the limit of the exact formula here. Rather we will demonstrate that it is restored from the scaling limit below.

For $q \neq 1$ it is not difficult to use these formulas to see that for $p = 1$ particle on the lattice

$$J(N, 1) = 1, \quad \Delta(N, 1) = 1,$$

which, as expected, do not depend on the lattice size $N$. A little more work is needed to find these quantities in the two-particle case

$$J(N, 2) = \frac{2N}{N + (1 - q) 2^{-1} q^1},$$

$$\Delta(N, 2) = \frac{4N}{3(N + (1 - q) 2^{-1} q^1)^3} \frac{3Nq + (2N + 1)(N + 1)3q}{(1 + q)^2},$$

of which the limit $N \to \infty$ yields

$$J(\infty, 2) = 2, \quad \Delta(\infty, 2) = 2 + \frac{2 (1 - q)^2}{3 (1 + q)^2},$$

where $n_q = \frac{1}{1-q^2}$. The $q = 0$ limit $\Delta = 8/3$ agrees with the result obtained for TASEP [15], while the $q \to 1$ limit $\Delta = 2$ corresponds to the non-interacting particle picture discussed above. In principle it is possible to analyze also the larger values of $p$, though the complexity quickly increases.

Of course of physical interest is the behaviour of the cumulants in the thermodynamic limit, in which the notion of universality becomes relevant. Below we study the asymptotic limit of the exact formulas to demonstrate that they fit with existing conjectures on the universality in KPZ and EW universality class.

3. Interface growth and KPZ–EW universality

In this section exploiting the relation of $q$-boson ZRP with an interface growth model we discuss the asymptotic limit of the announced exact formulas in context of the KPZ–EW universality. The $q$-boson zero range process on the periodic lattice can be mapped onto a growing
interface on a cylinder $\mathbb{R} \times [0, N]$. For $x \in [0, N]$ and time $t$ we define a piece-wise constant height function $h(x, t)$, which experiences a jump

$$h(x + 0, t) - h(x - 0, t) = n_x(t)$$

at each integer coordinate $x = 1, \ldots, N$ and is constant otherwise. Periodicity of the particle system implies helicoidal boundary conditions for the height (see figure 2)

$$h(x + N, t) = \rho N + h(x, t),$$

where the particle density $\rho = p/N$ plays the role of the mean tilt of the interface. We are now interested in the late-time behaviour of the large system, implying that the limit $t \to \infty$ is taken first. The statistics of the interface height in this limit is dictated by the KPZ–EW universality. We first discuss the aspects of the universal behaviour, which can be extracted from the KPZ equation. Then we match this picture with the asymptotic results obtained for the interface associated with the $q$-boson ZRP.

### 3.1. KPZ equation and scaling hypotheses

Let us consider the KPZ equation for the interface height $h(x, t)$

$$\frac{\partial h}{\partial t} = \nu \frac{\partial^2 h}{\partial x^2} + \frac{\lambda}{2} \left( \frac{\partial h}{\partial x} \right)^2 + \eta(x, t),$$

where $\nu, \lambda$ and $D$ are three parameters of the model. The particular case $\lambda = 0$ is referred to as EW equation. We choose helicoidal boundary conditions $h(x + N, t) = h(x, t) + \rho N$ with the tilt $\rho$ for it to agree with the mapping (13). We are interested in the statistics of interface height function in the large time limit. In this limit the statistics should not depend on initial
conditions. Consider the quantity characterizing the fluctuations of the interface height on the cylinder, its dispersion,

$$W^2(N, t) = \langle h^2(x, t) \rangle_c,$$

(17)

with a particular case of the flat initial conditions, which ensure that this quantity do not depend on the coordinate $x$ at any time. The dimensional analysis shows that the general dependence of $W^2(N, t)$ on its arguments is given in terms of the function of $\mathcal{V}(x, y)$ of two dimensionless variables.

$$W^2(N, t) = \nu^2 \lambda^2 \mathcal{W} \left( \frac{\lambda^2 D N}{\nu^3}, \frac{\lambda^2 D \lambda}{\nu^5} t \right).$$

(18)

To make further predictions about its asymptotic limits one needs to draw additional scaling arguments. We expect that at large time under a specific mutual scaling this quantity interpolates between the size independent small time regime, where $W^2(N, t) \sim t^{2\alpha}$, and the diffusive late time regime, $W^2(N, t) \sim \Delta h^2(N)$ with the height diffusion coefficient $\Delta h^2(N)$ depending on $N$. The latter behaviour is due to the fact that at large time the motion of a point of the interface is dominated by the motion of the interface center of mass

$$\bar{h}_N(t) = \frac{1}{N} \int_0^N h(x, t) dx,$$

in view of which we could equally discuss the quantity

$$W^2_c(N, t) = \langle \bar{h}_N^2(t) \rangle_c$$

(19)

characterizing the fluctuations of the center mass of the interface. Though the quantities $W(N, t)$ and $W_c(N, t)$ defined by (17) and (19) respectively are manifestly different, they asymptotically coincide in the limit under consideration. In view of this, we do not distinguish between them in what follows. The small and large time regimes are expected to meet at the characteristic relaxation time $\tau(N) \approx N\zeta$. In particular, when the temporal and spacial scales are related by this scaling we require the functional form (18) to be consistent with the Family–Viseck-like ansatz [82]

$$W^2(N, t) = t^{2\alpha} \Phi(t/\tau(N)).$$

(20)

with the scaling function $\Phi(x)$, that behaves as $\Phi(x) \to \text{const}$ and $\Phi(x) \sim x^{1-2\alpha}$ as $x \to 0$ and $x \to \infty$ respectively. To connect this ansatz to (18) we introduce the function $\mathcal{H}(x, y)$

$$\mathcal{V}(x, y) = y^{2\alpha} \mathcal{H}(x, y).$$

(21)

When $y \propto x^2 \to \infty$, $\mathcal{H}(x, y)$ is supposed to take the scaling form

$$\lim_{L \to \infty} \mathcal{H}(Lx, L^2 y) = \Phi(y/x^2).$$

(22)

Substituting (21) and (22) into (18) we obtain for the dependence of $W^2(N, t)$ on the parameters

$$W^2(N, t) \sim \kappa_{KPZ} (D/2\nu)^2 |\lambda| N^{1/2} t, \quad t \gg N^{3/2}, \quad N \to \infty,$$

(23)

and

$$W^2(N, t) \sim \kappa_{EW} \frac{Dt}{N}, \quad t \gg N^2, \quad N \to \infty$$

(24)
for KPZ and EW classes respectively, where $\kappa_{\text{KPZ}}$ and $\kappa_{\text{EW}}$ are the universal dimensionless constants specific for given universality class. These constants can be obtained from exact solutions. The latter

$$\kappa_{\text{EW}} = 1$$  \hspace{1cm} (25)

follow directly from the EW equation, which suggests that $\bar{h}(t)$ moves as a Brownian motion with the dispersion $D$ at any time $t$. The former

$$\kappa_{\text{KPZ}} = \frac{\sqrt{\pi}}{4}$$  \hspace{1cm} (26)

was first obtained from the exact solution of the TASEP [15] and conjectured to be universal for the whole KPZ universality class.

Note that there are two different functions $\Phi(x)$ for EW and KPZ classes. The crossover between their late time asymptotics is given in terms of yet another scaling function $\mathcal{F}(x, y)$,

$$\mathcal{W}(x, y) = y\mathcal{F}(x, y/x^2)/x.$$  

The crossover describes the late time stage of the evolution in the diffusive scale $t/N^2 \to \infty$ under the scaling $\lambda \approx 1/\sqrt{N}$, $N \to \infty$. Then, the height dispersion

$$\lim_{t \to \infty} \frac{W^2(N, t)}{t} = \frac{D}{N^\nu} \mathcal{F}(g, \infty), \quad g = \frac{\lambda^2 DN}{\nu^3}.$$  \hspace{1cm} (27)

is given in terms of $\mathcal{F}(g, \infty)$, which is conjecturally the universal crossover function. The candidate for this function

$$\mathcal{F}(g, \infty) = \frac{\sqrt{g}}{2\sqrt{2}} \int_0^{\infty} \frac{y^2 e^{-y^2}}{\tanh((\sqrt{g}/\sqrt{32})y)} \, dy.$$  \hspace{1cm} (28)

was first was obtained in [65] as a scaling limit of the exact diffusion constant at the weak asymmetry.

Below we are going to test these conjectures against the asymptotic results obtained from the $q$-boson ZRP. To this end, we, however, still need a way of identification of the parameters of the model with those of the KPZ equation.

### 3.2. Dimensionful invariants and asymptotic results

The way of identification of the model dependent constants in the KPZ universality class was proposed in [70, 71]. It is based on the observation that the parameters $A = D/2\nu$ and $\lambda$ are stable with respect to scale transformation

$$x \to bx, \quad t \to b^z t, \quad h \to b^\zeta h,$$  \hspace{1cm} (29)

with $z = 3/2$ and $\zeta = 1/2$, which together with the corresponding transformation of $D$, $\lambda$ and $\nu$ leaves the KPZ equation invariant. It was then conjectured that the dimensionful model-dependent constants within the universal functions must appear as a combination of these two parameters. It is indeed the case in (23), which suggests

$$W^2(N, t) \approx \kappa_{\text{KPZ}}A^\frac{1}{2} |\lambda|N^{-\frac{1}{2}}t,$$  \hspace{1cm} (30)

and is conjecturally true for all the systems of KPZ class.
The quantities $A$ and $\lambda$ can be expressed in terms of the characteristics of the stationary state of the process in the infinite system. Specifically the non-linearity coefficient $\lambda$ is the second derivative

$$\lambda = \frac{\partial^2 v^h}{\partial \rho^2},$$

of the infinite system limit $v^h = \lim_{N \to \infty} v^h_N$ of the mean interface velocity

$$v^h_N = \lim_{t \to \infty} \frac{h(x, t)}{t},$$

with respect to the tilt, while the constant $A$ can be defined as the amplitude of the two-point interface height covariance

$$\lim_{N \to \infty} \lim_{t \to \infty} \langle (h(x, t) - h(y, t))^2 \rangle_c = A |x - y|.$$ (32)

Yet another quantity, which can be expressed in terms of the dimensionful invariants in a universal way is the leading finite size correction to the interface velocity

$$\lim_{N \to \infty} N^{-1} (v^h_N - v^h_{\infty}) = -\frac{A \lambda}{2}.$$ (33)

To apply these formulas to the interface associated with the $q$-boson ZRP we remind that from the ZRP-interface mapping introduced in the beginning of the section the number of particles $y_i(t)$ passed through the bond $(i, i + 1)$ by time $t$ is the increase of the height $(h(x, t) - h(x, 0))$ for $x \in (i, i + 1)$. Hence the height dispersion is

$$W(N, t) = \langle y^2 \rangle_c \simeq \frac{\Delta t}{N^2}, \quad t \to \infty.$$

The interface velocity is nothing but the mean particle current

$$v^h_N = j_N$$

and the two-point height covariance is the variance of the number of particles between the reference points

$$\lim_{N \to \infty} \lim_{t \to \infty} \langle (h(x, t) - h(y, t))^2 \rangle_c = \left( \sum_{x < y} n_i \right)^2.$$ (34)

The two latter quantities can alternatively be obtained as averages over the stationary state. To this end we evaluate the contour integrals from the subsection 2.2 asymptotically in the thermodynamic limit

$$p \to \infty, \quad N \to \infty, \quad \frac{p}{N} = \rho.$$ (35)

using the saddle point approximation. The details of the method will be given in section 5. Here we place the results of the calculation. They are given in terms of the derivatives

$$h_k = (z \partial_z)^k h(z) \bigg|_{z=\zeta}$$

of the function

$$h(z) = \ln F(z) - \rho \ln (z)$$
at the critical point $z^*$ of the integrand, given by the smallest positive solution of the equation
\[ z^*(\ln F(z^*))' = \rho \] (36)
with the function $F(z)$ defined in (6). In the statistical physics language the latter equation is the standard relation between the density $\rho$ and fugacity $z^*$, while the function $\ln F(z)$ plays the role of the grand-canonical free energy.

The evaluation of the integral (7) asymptotically yields
\[ Z(N, \rho) = e^{Nh_0} \sqrt{2\pi Nh^2} \left[ 1 + \frac{1}{2N} \left( \frac{h_1}{4h_2^2} - \frac{5h_3^2}{12|h_2|^3} \right) + O(N^{-2}) \right]. \] (37)
For the current in the infinite system we obtain
\[ j_\infty(\rho) = z^* \] (38)
with the finite size correction
\[ \lim_{N \to \infty} N(j_N - j_\infty) = \frac{1}{2} \left( \frac{h_3}{h_2^2} - \frac{1}{|h_2|} \right) \] (39)
For the non-linearity coefficient we have
\[ \lambda = \frac{\partial^2 j_\infty}{\partial \rho^2} = \frac{z^*}{h_2} \left( \frac{1}{|h_2|} - \frac{h_3}{h_2^2} \right). \] (40)
where we used that $dz^*/d\rho = z^*/h_2$, which follows from (36).

The variance of the particle number between two given points is the sum of variances of single site occupancies in this interval due to decorrelation of the occupancies in the thermodynamic limit
\[ \left< \left( \sum_{x<i<y} n_i \right)^2 \right>_c = \sum_{x<i<y} \left< n_i^2 \right>_c \approx h_2 |x-y|, \quad |x-y| \to \infty, \] (41)
which yields
\[ A = h_2. \] (42)

We thus have expressed the dimensionful invariants $A$ and $\lambda$ in terms of characteristics of the stationary state of an infinite system. It also can be checked by a direct computation that these definitions are consistent with the formula (33) applied to the finite size corrections (39) of the particle current $j_N$.

The results of substitution of the formulas (40) and (42) into (30) are to be compared with the following result of the asymptotic analysis of exact formula (10) carried out in section 5.

**Result 2.**
\[ \lim_{N \to \infty} \frac{\Delta}{N^2} = \frac{\sqrt{\pi}}{8\sqrt{h_2}} \left( \frac{\phi_1 h_3}{|h_2|} - \phi_2 \right), \] (43)
where $\phi_k = (z\partial_z^k \phi(z)|_{z=z^*}$ with $\phi(z)$ defined in (11). Surprisingly, up to the error of order of $O(1/N)$ this formula can also be recast in terms of the stationary state observables of the original system and of the similar system with twice larger size and number of particles,
\[ \frac{\Delta}{N^2} = \frac{Z(2N, 2\rho)}{Z(N, \rho)^2} (j_N(\rho) - j_{2N}(\rho)) + O \left( \frac{1}{N} \right). \] (44)
Specifically, its leading asymptotics is given in terms of the universal finite size corrections to the current. By virtue of (37), (39) and (33), (34) this is the same as (30). It is an interesting question whether this formula is meaningful in a more general context.

The formulas (43) and (44) also agree with the universal scaling expression of the large deviation function obtained in [47], where the dimensionful constants were expressed in terms of the function

$$g_\nu(z) = \sum_{i=1}^\infty \frac{z}{(1-q^i)}$$

related to the quantities under consideration by

$$g_{q \pm 1}(\pm(1-q^{\pm 2})) = \pm(\ln F(z))'$$

for $q \leq 1$ respectively.

The second part of the asymptotic analysis is devoted to the crossover regime. It corresponds to the scaling, in which the dimensionless variable

$$g = \lambda^2 DN^{-3}$$

from (27) stays finite as $N \to \infty$. This can be realized by taking

$$q = e^{-\sqrt{N}}.$$ 

In this case

$$j_N = \rho - \frac{\alpha \rho^2}{2 \sqrt{N}} + O\left(\frac{1}{N}\right)$$

and hence

$$\lambda \approx -\frac{\alpha}{\sqrt{N}}.$$ 

Thus $\alpha$ varying from zero to infinity brings the system from EW to KPZ universality class. Correspondingly the parameters, $D$ and $\nu$ should be given the limiting values they take in the EW limit. Note that like the parameters $A$ and $\lambda$ are the dimensionful invariants of the KPZ class, the parameters $D$ and $\nu$ are invariant with respect to the scaling transformation (29) with exponents $\zeta = 1/2$ and $\z = 2$ that leave the EW equation invariant. Thus, conjecturally these quantities can be ascribed to any model in the EW universality class. In our case, when $q = 1$ we use (12), (24) and (25) to show that

$$D = \rho.$$ 

To find $\nu$ we note that $A = D/(2\nu) = h_2 = \rho$ in the same limit, form where we conclude that

$$\nu = \frac{1}{2}$$

and $g = 8\rho \alpha^2$.

With such a defined $g$ the result of the asymptotic analysis of the exact formula (10) of the diffusion coefficient in the crossover regime is as follows.

**Result 3.**

$$\lim_{N \to \infty} \Delta_{N} = \rho F(g, \infty),$$

which is exactly matches with the conjectured expression (27) with the universal scaling function $F(g, \infty)$ from (28).

4. **Exact formulas for scaled current cumulants**

4.1. **Bethe ansatz and T~Q equation**

The purpose of this section is to derive the integral representations for the first two derivatives of the largest eigenvalue of the operator $\mathcal{L}_\gamma$. The operator $\mathcal{L}_\gamma$ for $q$-boson zero range process

$\ldots$
can be diagonalized using the Bethe ansatz [46, 47]. The eigenvectors and eigenvalues are parameterized by \( p \) complex numbers \( x_1, \ldots, x_p \) given by the roots of Bethe ansatz equations (BAE)

\[
e^{\gamma N} (1 - x_i)^{-N} = (-1)^{p-1} \prod_{j=1}^{p} \frac{x_j - q x_i}{x_j - q x_i}
\]

(46)

A particular solution corresponds to a particular eigenvector, and the corresponding eigenvalue has the form

\[
\lambda(\gamma) = - \sum_{i=1}^{p} x_i.
\]

(47)

We would like to find a particular solution of BAE corresponding to the Perron–Frobenius eigenvalue satisfying \( \lambda(0) = 0 \). The corresponding eigenvector is known to be translationally invariant, that is to say that for the solution of BAE we have

\[
\prod_{i=1}^{p} (1 - x_i) = e^{\gamma p}.
\]

(48)

Let us introduce a polynomial \( Q(x) \) of degree \( p \), with roots from the solution of BAE

\[
Q(x) = \prod_{i=1}^{p} (x - x_i).
\]

Rewriting the system (46) in terms of \( Q(x) \) we notice that the Bethe roots are also zeroes of the polynomial

\[
e^{\gamma N} Q(qx) + q^p (1 - x)^N Q(x/q).
\]

This fact suggests that the latter polynomial is divisible by \( Q(x) \). Hence we have the polynomial TQ-relation

\[
T(x)Q(x) = e^{\gamma N} Q(qx) + q^p (1 - x)^N Q(x/q).
\]

(49)

between the polynomial \( Q(x) \) of degree \( p \) and yet another unknown polynomial \( T(x) \) of degree \( N \). Together with the condition

\[
Q(1) = e^{\gamma p}
\]

(50)

following from (48) it determines the functional equation for \( Q(x) \) corresponding to the largest eigenvalue.

Once we know \( Q(x) \), the eigenvalue is

\[
\lambda(\gamma) = \frac{1}{(p-1)!} \left. \frac{d^{p-1} Q(x)}{dx^{p-1}} \right|_{x=0}.
\]

(51)

We are going to solve relation (49) perturbatively in powers of \( \gamma \) in the vicinity of \( \gamma = 0 \). To this end, we assume the following expansions

\[
Q(x) = Q_0(x) + \gamma Q_1(x) + \gamma^2 Q_2(x) + \ldots,
\]

\[
T(x) = T_0(x) + \gamma T_1(x) + \gamma^2 T_2(x) + \ldots,
\]

\[
\lambda(\gamma) = \gamma \lambda_1 + \gamma^2 \lambda_2 + \ldots
\]
The T–Q relation is equivalent to the system of equations for the polynomials $T_k(x)$ and $Q_k(x)$ which can be solved order by order. The leading orders of $Q(x)$ and $T(x)$ follow directly from (49).

\[ Q_0(x) = x^p, \quad T_0(x) = q^p + (1 - x)^N. \] (52)

The first two coefficients of the eigenvalue are of our interest, being related to the scaled cumulants of integrated current.

\[ \lambda_1 = J, \quad \lambda_2 = \frac{\Delta}{2} \]

To find them we solve T–Q relation in the first and second orders.

4.2. First order calculation

The equation (49) in the first order in $\gamma$ looks as follows

\[ T_0(x)Q_1(x) + T_1(x)Q_0(x) = Q_1(qx) + NQ_0(qx) + q^p(1 - x)^N Q_1(x/q). \] (53)

As $\deg Q_1(x) \leq p - 1$, it is enough to solve this equation mod $x^p$. Due to (52) it is then reduced to

\[ (q^p + (1 - x)^N)Q_1(x) = Q_1(qx) + q^p(1 - x)^N Q_1(x/q) \mod x^p. \] (54)

Introducing

\[ B_1(x) = q^p Q_1(x/q) - Q_1(x) \] (55)

we obtain

\[ B_1(qx) = (1 - x)^N B_1(x) \mod x^p. \] (56)

The cases $|q| < 1$ and $|q| > 1$ should be treated separately. In both cases we first construct the solution $\tilde{B}_1(x)$ of the difference equation regular at $x = 0$, which is globally a meromorphic function in the first case and entire function in the second. For that purpose we iterate equation (56) obtaining for both $|q| < 1$ and $|q| > 1$ the solution

\[ \tilde{B}_1(x) = B_1(0) F \left( \frac{x}{(1 - q)} \right)^N, \]

where $F(z)$ was defined in (6). Having written the formula for $\tilde{B}_1(x)$ in the same way for all values of $q \neq 1$ we can proceed further without referring to the value of $q$.

The first $p$ terms of the expansion $\tilde{B}_1(x)$ at the origin give the desired polynomial $B_1(x) = \sum_{i=0}^{p-1} b_i x^i$. Its coefficients define the polynomial $Q_1(x) = \sum_{i=0}^{p-1} q^i x^i$ due to the relation

\[ b_i = (q^{p-i} - 1)q^i. \] (57)

which follows from (55). Then one can write down the relation between the polynomials $Q_1(x)$ and $B_1(x)$ in an integral form

\[ Q_1(x) = \sum_{i=0}^{p-1} \frac{b_i x^i}{q^{p-i} - 1} = \sum_{i=1}^{p} \frac{b_{p-i} x^i}{q^i - 1} \]

\[ = -x^p \oint \frac{B_1(z)}{z^{p+1}} \sum_{i=1}^{\infty} \frac{(z/x)^i}{1 - q^i} \frac{dz}{2\pi i} \] (58)
Here the integration contour is a simple anticlockwise loop around the origin $z = 0$, which must be the only singularity inside the contour.

Then, it will be convenient to express everything in terms of $F(x)$. It turns out that the results expressed in terms of this function look the same for all values of $q \neq 1$. This is in particular due to the fact that for the logarithmic derivative of $F(z)$ we have

$$
(\ln F(x))' = \sum_{k=0}^{\infty} \frac{x^k(1-q)^{k+1}}{1-q^{k+1}}
$$

(59)

irrespective of whether $|q| < 1$ or $|q| > 1$. This expression, however, is valid in the domain of the series convergence, $|(1-q)x| < 1$ in the former case and $|(1-q)x/q| < 1$ in the latter.

Substituting $\tilde{B}_1(x)$ to (58) instead of $B_1(x)$, which does not affect the result of integration as only the first $p$ terms of the expansion of these functions near the origin contribute to the integral, we obtain

$$
Q_1(x) = -\frac{\tilde{B}_1(0)x^{p-1}}{(1-q)^p} \int \frac{F(z)^N}{z^p} (\ln F \left( \frac{z}{x} \right))^1 \frac{dz}{2\pi i},
$$

(60)

where the integration variable was scaled by $(1-q)^{-1}$. The constant $B_1(0)$ is defined from (50) where we substitute the integral form of $Q_1(x)$.

For further purposes we list the resulting representations for $Q_1(x), B_1(x)$

$$
\tilde{B}_1(x) = -\frac{N(1-q)^pF(x/(1-q))^N}{Z(N, p)},
$$

(61)

$$
Q_1(x) = \frac{N x^{p-1}}{Z(N, p)} \int \frac{F(z)^N}{z^p} (\ln F \left( \frac{z}{x} \right))^1 \frac{dz}{2\pi i},
$$

(62)

The integration contours in the latter formula satisfy $|(1-q)z| < |x|$ and $|(1-q)z| < |qx|$ for $|q| < 1$ and $|q| > 1$, respectively.

We are in a position to write the expression for $\lambda_1$.

$$
\lambda_1 = \frac{1}{(p-1)!} \frac{d^{p-1} Q_1(x)}{dx^{p-1}} = \frac{N}{Z(N, p)} \int \frac{F(z)^N}{z^p} \frac{dz}{2\pi i} = N \frac{Z(N, p-1)}{Z(N, p)},
$$

(63)

which coincides with stationary state result.

4.3. Second order calculation

The second order equation is as follows

$$
\frac{N^2}{2} Q_0(qx) + N Q_1(qx) + Q_2(qx) + q^p(1-x)^N Q_2(x/q)
$$

$$
= Q_1(x) T_1(x) + Q_2(x) T_0(x) + Q_0(x) T_2(x)
$$

(64)

with initial condition

$$
\frac{p^2}{2} = Q_2(1)
$$

(65)

which comes from the second order of (50). Repeating the same reasoning as in case of the
first correction we write
\[ NQ_1(qx) + Q_2(qx) + q^p(1 - x)^N Q_2(x/q) = Q_1(x)T_1(x) + Q_2(x)(q^p + (1 - x)^N) \mod x^p. \] (66)

Introducing
\[ B_2(x) = q^pQ_2(x/q) - Q_2(x) \] (67)
we obtain
\[ -B_2(qx) + (1 - x)^N B_2(x) = Q_1(x)T_1(x) - NQ_1(qx) \mod x^p. \] (68)

Let us denote \( f(x) \) the right-hand side of this equation with \( T_1(x) \) found from (53),
\[ T_1(x) = Nq^p + x^{-p}[(1 - x)^N B_1(x) - B_1(qx)]. \] (69)

To solve the equation (68) we rewrite it in the standard form with \( q \)-difference operator defined by
\[ D_qa(x) = \frac{a(qx) - a(x)}{(q - 1)x}, \] (70)
which yields
\[ D_qB_2(x) = -\frac{f(x)}{x(q - 1)} + \frac{(1 - x)^N - 1}{x(q - 1)}B_2(x) \mod x^p \] (71)

This is the first order linear inhomogeneous \( q \)-difference equation with non-constant coefficients. Depending on whether \(|q| < 1 \) or \(|q| > 1 \) corresponding homogeneous equation is solved by iterations with decreasing of increasing powers of \( q \), so that the final expressions slightly differ. Below we show the calculation for \(|q| < 1 \) in detail. Calculations for the case \(|q| > 1 \) are analogous and will be omitted.

The solution of (71) can be obtained by the variation of constant method
\[ \tilde{B}_2(x) = S(F \left( x/(1 - q) \right))^N \left( \frac{f(qx)}{x(q - 1)} \right) \sum_{i=0}^{\infty} \frac{f(q^i)}{x(q^i)^N} \] (72)
where \( S \) is the constant of integration to be defined from (65). The coefficients of polynomials \( Q_2 \) and \( B_2 \) are related by (57). Then, one has the integral representation for \( Q_2(x) \) similar to (62)
\[ Q_2(x) = -\frac{x^{p-1}}{(1 - q)^p} \int F^N(z) \left( S + \sum_{i=0}^{\infty} \frac{f(q^i(1 - q)z)}{F(q^{-1}z)^N} \right) \ln F \left( \frac{z}{x} \right) \frac{dz}{2\pi i} \] (73)

The constant \( S \) is found from the initial condition
\[ S = -\frac{pN(1 - q)^p}{2Z(N, p)} \int F^N(z) \frac{dz}{2\pi i} \sum_{i=0}^{\infty} \frac{f(q^i(1 - q)z)}{F(q^{-1}z)^N} \ln F(z) \frac{dz}{2\pi i} \] (74)
Finally we have
To obtain the final formula we notice that the integral in (76) are those coming from \( \ln F(z) \) calculus. All the poles inside the contour contributing to the first term in square brackets of which is a half of the expression for \( J \). The second term with number followingsum of triple integrals variable change \( \tilde{y} \) nomials \( B \)

\[
Q_2(x) = \frac{pNZ(N, p - 1)}{2Z(N, p)} + \frac{x^{p-1}}{(1-q)^p} \int F^N(z) \sum_{i=0}^{\infty} \frac{f(q^i(1-q)z)}{F(q^{i+1}z)^N} \times \left( \frac{N}{pZ(N, p)} \right) (\ln F(z))^p \int F^N(z) (\ln F \left( \frac{z}{x} \right))^p \frac{dz}{2\pi i} \frac{dz}{2\pi i}.
\]

We are interested in the highest coefficient of \( Q_2(x) \). To this end, we divide the last expression by \( x^{p-1} \) and send \( x \) to infinity. Only the free term of \( (\ln F(z/x))^p \) equal to unity remains.

\[
\lambda_2 = \frac{pNZ(N, p - 1)}{2Z(N, p)} + \frac{1}{(1-q)^p} \sum_{i=0}^{\infty} \int \frac{F^N(z)}{\phi(z)} \frac{dz}{F(q^{i+1}z)^N} \frac{dz}{2\pi i}
\]

with \( \phi(x) \) defined in (11).

The next step is to simplify this result by substituting \( f(x) \) and the integral forms of polynomials \( B_i(x) \) and \( Q_i(x) \) that is done step by step in appendix A. As a result we arrive at the following sum of triple integrals

\[
\lambda_2 = \frac{pI}{2} + \frac{N^2}{Z(N, p)} \int \frac{dz}{2\pi i^2} \left[ \int \frac{dy}{y^p} \frac{F^N(t)}{F(q^{i+1}z)^N} \frac{dy}{2\pi i} + \sum_{i=0}^{\infty} \int \frac{dy}{y^p} \frac{F^N(t)q^{i+1}y^{p-1}}{F(q^{i+1}z)^N} \frac{dy}{2\pi i} \right],
\]

where \( J \) is the mean integrated current given by (9) and

\[
F_{0.i}(z) := \frac{F(z)}{F(q^i z)}.
\]

To obtain the final formula we notice that the integral in \( z \) can be evaluated by the residue calculus. All the poles inside the contour contributing to the first term in square brackets of (76) are those coming from \( (\ln F (y/(1-q)z))^p \), i.e. \( z = q^i y, i = 0, 1, \ldots \) A summand of the second term with number \( i \) receives the contribution from the only pole \( z = q^{i+1}y \). After variable change \( \tilde{y} = q^{-i-1}y \) we obtain

\[
\lambda_2 = \frac{pI}{2} + \frac{N^2}{Z(N, p)} \sum_{i=0}^{\infty} \int \frac{dy}{y^p} \frac{F^N(t)}{F(q^{i+1}z)^N} \frac{dy}{2\pi i} \frac{dt}{2\pi i} \frac{F^N(t)q^{i} \phi(yq^{i+1})}{F(q^{i+1}z)^N} \frac{dt}{t - yq^{i+1}}
\]

which is a half of the expression for \( \Delta \) announced in (10) in the case \( |q| < 1 \). The case \( q > 1 \) derived in a similar way is obtained from this formula by replacing \( q \to 1/q \).

5. Asymptotic analysis

Let us move to the asymptotic analysis of the exact formulas in the thermodynamic limit

\[
N \to \infty, \quad p \to \infty, \quad p/N \to \rho.
\]
The main ingredient of the analysis is the evaluation of integrals of the form

\[ \int \frac{dt}{2\pi i} e^{F_N(t)} g(z) = \int \frac{dz}{2\pi i} e^{Nh(z)} g(z), \] (78)

and its two-dimensional analogues in the saddle point approximation. Here \( h(z) = \ln F(z) - \rho \ln(z) \) was defined in (13) and \( g(z) \) is an arbitrary function analytic on the integration contour. The main contribution to this integral comes from a critical point \( z^* \) of \( h(z) \) given by the solution of

\[ h'(z^*) = 0. \]

There are many solutions of this equation, all being on the real axis (see figure 3). It can be seen from the figure that the maximal contribution comes from the minimal positive solution that is found in the ranges \( z^* \in (0,1/(1-q)) \) when \( |q| < 1 \) and \( z^* \in (0,\infty) \) when \( q > 1 \). The integration contour can always be deformed to pass through \( z^* \) without crossing any singularities. One can in principle transform the integration contour into the steepest descent (stationary phase) contour, which explicit form however is not easy to analyze. Instead, for the saddle point method to be applicable it is enough to construct the steep descent contour, where the real part of \( h(z) \),

\[ \Re h(z) = \begin{cases} \sum_{i=0}^{\infty} \ln |1 - z(1 - q)q^i| - \rho \ln |z|, & |q| < 1; \\ \sum_{i=0}^{\infty} \ln |1 - z(1 - q)q^{-i}| - \rho \ln |z|, & |q| > 1, \end{cases} \]

monotonously decreases away from \( z^* \). In the case \( q > 0 \) the simplest choice is the circle \( |z| = z^* \), which ensures the correct sign of the derivative of the real part,

\[ \frac{d}{d\phi} \Re h(z^* e^{i\phi}) = \begin{cases} \sum_{i=0}^{\infty} \frac{-z^* \sin \phi(1 - q)q^i}{|1 - z^* e^{i\phi}(1 - q)q^i|^2}, & |q| < 1; \\ \sum_{i=0}^{\infty} \frac{z^* \sin \phi(1 - q)q^{-i}}{|1 - z^* e^{i\phi}(1 - q)q^{-i}|^2}, & |q| > 1, \end{cases} \] (79)

with respect to the polar angle \( \phi \in (-\pi, \pi) \):

\[ \text{sgn} \frac{d}{d\phi} \Re h(z^* e^{i\phi}) = -\text{sgn} \ \phi. \]

We do not have such an estimate for the negative values of \( q \). However we expect that the results of the saddle point approximation can be analytically continued to this region.

Deforming the integration contour to the steep descent contour we apply the standard saddle point technique, which yields the desired asymptotic approximation

\[ \int \frac{dz}{2\pi i} e^{Nh(z)} g(z) = \frac{e^{Nh_0} \sqrt{2\pi Nh_2}}{\sqrt{2\pi Nh_2}} \left[ g_0 + \frac{1}{2N} \left( \frac{h_2 g_1}{h_2} - \frac{g_2}{|h_2|^2} - \frac{5g_0h_2^2}{12|h_2|^3} + \frac{g_0h_2}{4h_2^2} \right) + O(N^{-2}) \right], \] (80)

of (78) in terms \( g_1 = (z\partial_z)^2 g(z)|_{z=z^*} \) and \( h_k = (z\partial_z)^k h(z)|_{z=z^*} \). In particular, for the asymptotics of the partition function given by (78) with \( g(z) = 1 \) we obtain (37).
Figure 3. The graphs of $\Re h(z)$ (a) and of $h'(z)$ (b) for different values of $q$. The density is $\rho = 2$ in all cases. The bottom row (c) shows corresponding contours of steepest descent in the complex plane.

Most quantities we deal with, like the stationary state averages, are given by the normalized contour integrals, those from (78) multiplied by the inverse partition function. Below we write such integrals as $\oint D_{N,p}(t)g(t)$, where for brevity we introduce the following notation for normalized differential

$$D_{N,p}(t) := \frac{dt}{2\pi i} \frac{1}{Z(N, p)^{\rho+1}} F_N(t).$$

The expansion (80) of the original unnormalized integral starts from the exponential in $N$ factor. The normalization makes the generic normalized integral (with $N$-independent integrand $g(z)$) of order of $O(1)$. The lower corrections are $O(N^{-k})$ with $k = 1, 2, \ldots$. In particular in the first two orders we have

$$\oint D_{N,p}(t)g(t) = g_0 + \frac{1}{2N} \left( \frac{h_1 g_1}{h_2} - \frac{g_2}{|h_2|} \right) + O(N^{-2}). \quad (81)$$

An example of the normalized integral is the mean particle current given by

$$j_N = \oint D_{N,p}(z). \quad (82)$$

Applying (81) we obtain formulas (38) and (39). The other asymptotic approximations of the stationary state averages listed in the subsection 2.2 are obtained similarly. Our further aim is to obtain asymptotic estimates of the diffusion coefficient.
5.1. KPZ regime $|q| \neq 1$

We first note that the formula (10) can be rewritten in the form

$$
\Delta = N^2 \frac{Z(2N, 2p)}{Z(N, p)^2} \iint D_{2N, 2p}(t) t \phi(t)
+ N^2 \iiint D_{N, p}(t) D_{N, p}(y) y t \left[ \frac{\phi(t) - \phi(y)}{t - y} \right]
+ 2N^2 \sum_{i=1}^{\infty} \iiint D_{N, p}(t) D_{N, p}(y) y t \frac{q^{\pm i} \phi( yq^{\mp i}) + \phi(y)}{t - yq^{\pm i}}
$$

(83)

(We remind that the plus and minus signs are taken for $|q| < 1$ and $q > 1$.) The main advantage of this representation is the appearance of the single integral in the first line. To motivate this we note that the integrand of the first double integral in (10) has a pole at $t = y$. Though it is not a problem for the integral at finite $N$, since the integration contours are nested, it becomes an obstacle for an application of the saddle point method, because the variables take identical values at the critical point,

$$
y = t = z^*.
$$

(84)

To overcome this difficulty we separate the pole contribution as follows

$$
\iint_{|y| < |t|} D_{N, p}(t) D_{N, p}(y) \frac{y t \phi(y)}{t - y} = \frac{1}{2} \left( \iiint D_{N, p}(t) D_{N, p}(y) y t \frac{\phi(y) - \phi(t)}{t - y} \right)
+ \frac{Z(2N, 2p)}{Z(N, p)^2} \int D_{2N, 2p}(t) \phi(t).
$$

(85)

To obtain this identity we first divide the integral in lhs into two identical parts and then exchange the integration variables, $t \leftrightarrow y$, in one of the parts. In this way we obtain the sum of two double integrals, where the contours of integration in the variables $y$ and $t$ are circles nested in two opposite ways, $|y| < |t|$ in the first and $|t| < |y|$ in the second. Then, we deform the contours in the second integral to restore back the nesting of the first one. On the way we pick up the contribution from the pole $y = t$ in the form of the single integral, which yields (85). Note that the integrand of the double integral in rhs is now regular at $y = t$ and, hence, the nesting of the contours is irrelevant.

Returning to (83) we find the single integral of (85) in the first line and the double integral in the second line of (83). We also used the representation (82) of $J$, where the first term in the square brackets comes from.

Now, we are going to analyze the rhs (83) evaluating the integrals in the steepest descent approximation. In particular we will do it term by term under the infinite sum in the last line. This interchange of the limit $N \to \infty$ and the infinite summation requires, however, a proper justification, which we return to in the end.

Let us first argue that the leading contribution to the diffusion coefficient comes from the first line of (83), while the second and third lines are of smaller order. First we observe that the asymptotic expansions of the terms with double integrals contain only integer powers of $N$, while the one of the term with the single integral is in half-integer powers. Indeed, as we have already noted, the normalized integrals, where the integrand (outside of the normalized differential) is $N$-independent, is normally $O(1)$ containing only integer powers of $N$ in corrections.
In our case the function $\phi(z)$ does depend on $N$ via the current $J$, which, however, being the normalized integral itself is also expanded in integer powers on $N$. As $\phi(z)$ depends linearly on $J$ its $N$-dependence does not spoil the power integrality or half-integrality of our expansions. Taking into account that the ratio of partition functions in the first line is $O(\sqrt{N})$, we would expect the first line to be $O(N^{5/2})$ and the sum of the second and third lines to be $O(N^2)$, both with corrections obtained by decreasing the powers of $N$ by integers. However, this is not the case, since in both cases the leading terms vanish. According to (81), to obtain the leading term of the asymptotic expansion of a normalized integral we just substitute the critical values of arguments $y = t = z^*$ to the integrand together with taking the limit $N \to \infty$. Thus, the leading term of the first line vanishes, as

$$\lim_{N \to \infty} \phi(z^*) = 0.$$  

To show that the leading term of the sum of the second and third lines vanish we first note that the second ratio in the second line turns into the derivative when its two arguments coincide

$$\phi'(y) = \rho_j N (\phi(z) + 1)^2 - \frac{2}{2} \sum_{i=1}^{\infty} q_i^j \frac{(\phi(z) - \phi(\frac{z}{1-q_i}))}{1-q_i},$$

which is justified by a direct check. Evaluating the integrands at $z^*$ with $\phi'(z^*)$ reexpressed using (87), taking the limit $N \to \infty$ and using (38) and (86) we find that the $O(N^2)$ asymptotics of the second plus the third line in (83) vanishes.

The next corrections to the two contributions are of order of $O(N^{3/2})$ and $O(N)$ respectively. Therefore it is enough to evaluate the first order correction of the single integral term using the formulas (37) and (81), which yields

$$\Delta = \frac{N^{3/2}\sqrt{\pi}}{8\sqrt{h_2}} \left( \frac{\phi_1 h_2}{|h_2|} - \phi_2 \right) + O(N).$$

This ensures (43).

To obtain another form of the same formula, we note that

$$\int \int D_{2N,2p}(t)t\phi(t) = \int \int D_{2N,2p}(t)t \left( \frac{1}{p} \ln F(t) - 1 \right)$$

$$= \frac{1}{Z(2N,2p)} ( \phi_1 Z(2N,2p) - Z(2N,2p-1) ),$$

where we applied the integration by parts to the first term in the brackets under the integral. Substituting the result into the first line of (83) we arrive at the alternative representation (44) of the main asymptotics of $\Delta$.

Finally let us justify the interchange of the limit $N \to \infty$ and the infinite sum in

$$\sum_{i=1}^{\infty} \int \int D_{N,p}(t)D_{N,p}(y) \frac{q^{\pm i} \phi(yq^{\pm i}) + \phi(y)}{t - yq^{\pm i}}.$$  

(88)
To this end, we expand the denominator of the ratio into the geometric series and exchange the summation and integration,

$$
\oint \oint D_{N,p}(t) D_{N,p}(y) y t \frac{q^{\pm i} \phi(y q^{\pm i}) + \phi(y)}{t - y q^{\pm i}} = \sum_{k=0}^{\infty} q^{\pm ik} \oint \oint D_{N,p}(t) D_{N,p}(y) y q^{\pm i} \phi(y q^{\pm i}) + \phi(y) \left( \frac{y}{t} \right)^k.
$$

Here we used the fact that the sum in $k$ is in fact finite, as at most $p$ integrals in $y$ are nonzero due to the integrand having a pole at the origin, while the others being analytic inside the integration contour. Choosing the integration contours to be the circles $|z| = z^*$ centered at the origin, one can show that the double integral can be bounded by an $N$-independent constant $C > 0$ (We use the boundedness of $\phi(z)$ in the disk $|z| < z^*$ and asymptotic equality of steep descent integrals $\oint |D_{N,p}(t)|$ and $\oint D_{N,p}(t)$, while for the latter $\lim_{N \to \infty} \oint D_{N,p}(t) = 1$ due to (81)). Hence the summand in (88) is bounded by the summable function $C q^{\pm}/(1 - q^{\pm i})$, which allows us to apply the dominated convergence theorem to justify the change of the order of limit and summation.

### 5.2. KPZ–EW crossover

Here we consider the asymptotic behaviour of the diffusion coefficient in the scaling limit, where the limit $q \to 1$ is taken together with $N \to \infty$.

$$
N \to \infty, \quad p \to \infty, \quad \frac{p}{N} = \rho, \quad q = e^{-\alpha \sqrt{N}}.
$$

We start with the following representation of the formula (10)

$$
\Delta = N^2 \left[ 2 \rho \sum_{i=0}^{\infty} \oint \oint D_{N,p}(t) D_{N,p}(y) y t \frac{\phi(y)}{t - y q^i} + 2 \sum_{i=1}^{\infty} \oint \oint D_{N,p}(t) D_{N,p}(y) y t \frac{q^i \phi(y q^i)}{t - y q^i} \right],
$$

written for the case $|q| < 1$. Below, for the sake of economy we consider the case $|q| < 1$, while the case $|q| > 1$ is obtained by the change $q \to 1/q$ in the series coefficients.

We want to transform this expression to the form of well convergent series, which could be integrated term by term. To this end, we use the representation of the function $\phi(z)$,

$$
\phi(z) = \frac{1}{p} \sum_{i=0}^{\infty} \frac{(1 - q)^{i+1}}{1 - q^{i+1}} z^i = 1,
$$

obtained from (11) by expanding the denominator as a geometric series and then changing the summation order. Also we make the similar expansion of the denominators under the integrals.
in (90). Performing the summation in $i$ we obtain

$$
\Delta = N^2 \rho J_N + 2N^2 \frac{J_N}{\rho} \left[ \sum_{l=2}^{\infty} \frac{(1-q)^l}{1-q^l} \sum_{k=1}^{l-1} \frac{1}{1-q^k} \int_{D} D_{N,p}(t) D_{N,p}(y) (y^{-k} q^{k+l}) \right]
$$

$$
+ \sum_{l=2}^{\infty} \frac{(1-q)^l}{1-q^l} \sum_{k=1}^{\infty} \frac{1}{1-q^k} \int_{D} D_{N,p}(t) D_{N,p}(y) \left( \frac{y}{t} \right)^k \left( y + t q^{k} \right),
$$

where we separated the sum in the first line from that in the second line, to collect the $l$th powers of $y$ and $t$ together. Remembering that $(1-q) = O\left(\frac{1}{\sqrt{N}}\right)$ we notice that the external summations in $i$ in the two first lines are already asymptotic in nature due to the coefficients $(1-q)^l / (1-q^l) = O\left(N^{-(l-1)/2}\right)$, each term being $O\left(N^{-1/2}\right)$ times the previous one. On the other hand, we will see that there are $O\left(\sqrt{N}\right)$ terms of the same order inside the sums in $k$, which bring the dominant contribution to these sums. Therefore, it is enough to limit our consideration by a few first values of $l$. Specifically, we leave the terms with $l = 2, 3, 4$ in the first line and the terms with $l = 2, 3, 4$ in the second line to find the expansion up to the order $O(N)$. Then, only finitely many terms remains in the first line and five infinite sums in the second and the third lines. We also separate finitely many terms from the latter, to unify the infinite sums into a single infinite sum starting from the index value $k = 5$. Thus, asymptotically up to the terms of order of $O(N)$ the diffusion coefficient can be represented as consisting of two parts,

$$
\Delta = FS + IS + O\left(\sqrt{N}\right),
$$

(91)
a finite sum

$$
FS = N^2 \left[ \rho J_N + 2 \frac{J_N}{\rho} \sum_{l=2}^{4} \frac{(1-q)^l}{1-q^l} \left( \sum_{k=1}^{l-1} \frac{J_k}{1-q^k} + \sum_{k=l}^{4} \frac{I_k}{1-q^k} \right) + 2 \left( \frac{J}{p} - 1 \right) \sum_{k=1}^{4} \frac{I_k}{1-q^k} \right]
$$

(92)
and an infinite sum

$$
IS = 2N^2 \sum_{k=5}^{\infty} \frac{1}{1-q^k} \left[ \left( \frac{J_N}{\rho} - 1 \right) I_k + \frac{J_N}{\rho} \frac{(1-q)^2}{1-q^2} I_k^2 + \cdots + \frac{(1-q)^5}{1-q^5} I_k^5 \right],
$$

(93)
where for brevity we introduce the following notations for two products of contour integrals appearing repeatedly in the calculations

$$
J_k = \int_{D} D_{N,p}(t) D_{N,p}(y) \left( \frac{y}{t} \right)^k y^k,
$$

$$
I_k = \int_{D} D_{N,p}(t) D_{N,p}(y) \left( \frac{y}{t} \right)^k \left( y + t q^k \right) = J_k + q^k J_{k-1}.
$$

(94)
(95)
The finite sum is easy to calculate asymptotically using the formula (81) for the normalized integrals, which yields

$$
J_k = (z^*)^l \left( 1 + \frac{1}{2N} \left( \frac{h_3 l}{h_2} - \frac{(k+l)^2 + k^2}{|h_2|} \right) \right).
$$

(96)
The ingredients of this formula are expressed in terms of the saddle point $z^*$, which can asymptotically be found in the limit (89) from perturbative solution of the saddle point equation as the expansion in powers of $N^{-1/2}$. The solution in the four first orders yields

$$z^* = \rho - \frac{\alpha \rho^2}{2\sqrt{N}} + \frac{\alpha^2 \rho^3}{6N} - \frac{\alpha^3 \rho^3 (\rho^2 - 1)}{24N^{3/2}} + O(N^{-2}).$$  \hspace{1cm} (97)

Then, the expansion of the derivatives of function $h(z)$ follows

$$h_2 = \rho + \frac{\alpha \rho^2}{2\sqrt{N}} + \frac{\alpha^2 \rho^3}{6N} + \frac{-\alpha^3 \rho^3 + \alpha^3 \rho^4}{24N^{3/2}} + O(N^{-2}),$$  \hspace{1cm} (98)

$$h_3 = \rho + \frac{3\alpha \rho^2}{2\sqrt{N}} + \frac{7\alpha^2 \rho^3}{6N} + \frac{-3\alpha^3 \rho^3 + 5\alpha^3 \rho^4}{8N^{3/2}} + O(N^{-2}).$$  \hspace{1cm} (99)

The expansions of $J_l^k$ and hence $I_k^l$ for finite values of $k$ and $l$ appearing in (92) can be obtained by substitution of (97)–(99) into (96). In particular for of the particle current we have

$$J_N = J_0 = z^* \left( 1 + \frac{1}{2N} \left( \frac{h_2}{h_2^*} - \frac{1}{|h_2|} \right) \right)$$  \hspace{1cm} (100)

$$= \rho - \frac{\alpha \rho^2}{2\sqrt{N}} + \frac{\alpha^2 \rho^3}{6N} + \frac{1}{N^{3/2}} \left( \frac{\alpha \rho}{2} - \frac{1}{24} \alpha^3 \rho^2 (\rho^2 - 1) \right) + O(N^{-2}).$$  \hspace{1cm} (101)

Substituting them into (92) we obtain the approximation of the finite sum

$$FS = \frac{\alpha \rho^2}{2} N^{3/2} - \frac{\alpha^2 \rho^3}{12N} + O \left( \sqrt{N} \right),$$  \hspace{1cm} (102)

where the terms of order of $O(N^{2})$ have canceled.

It is more tricky to analyze the infinite sum. First we note that under a closer look one finds that the infinite sum is in fact finite, since $I_k^l$ vanishes when $k > p + l - 1$. However, the summation index $k$ becomes unbounded in the limit $p \rightarrow \infty$, and its value affects the location of the saddle points in the integrals. To account for this effect let us write $I_k^l$ in the form.

$$I_k^l = \frac{Z(N, p - k)Z(N, p + k)}{Z(N, p)^2} \iiint D_{N,p+k}(t)D_{N,p-k}(y) (y^l + t^k q^k).$$  \hspace{1cm} (103)

We now claim that the coefficient before the integral is responsible for the effective range of the index $k$, where the main contribution to the infinite sum (93) comes from. To this end, we introduce the free energy

$$\hat{f}(\rho) = -\lim_{N \rightarrow \infty} N^{-1} \ln Z(N, [\rho N]) = -h(z^*).$$

Then the ratio of partition functions of interest is approximated by

$$\frac{Z(N, p - k)Z(N, p + k)}{Z(N, p)^2} \simeq \exp \left[ -N \left( \hat{f} \left( \rho + \frac{k}{N} \right) + \hat{f} \left( \rho - \frac{k}{N} \right) - 2\hat{f}(\rho) \right) \right]$$  \hspace{1cm} (104)

$$\simeq \exp \left( -\frac{\hat{f}'(\rho) k^2}{N} \right) = \exp \left( -\frac{k^2}{Nh_2} \right).$$
where we used the fact that
\[ f'(\rho) = -\frac{d}{d\rho} \left( \frac{dz^*}{d\rho} \frac{\partial h(z^*)}{\partial z^*} + \frac{\partial h(z^*)}{\partial \rho} \right) = \frac{dz^*}{d\rho} \frac{\partial \ln z^*}{\partial z^*} = \frac{1}{h_2}. \]

The estimate (104) suggests that the main contribution to the infinite sum (93) comes from the scale, in which \( k \) is of order of \( \sqrt{N} \). It is not difficult also to find corrections to the exponent. There are two sources of corrections. One source is from approximation of the \( \ln Z(N, \rho N) \) by the free energy. The other is from approximation of the expressions at shifted densities \( \rho \pm k/N \) with those at density \( \rho \). Their orders of magnitude are \( O \left( k^2 N^{-3/2} \right) \) and \( O \left( k^4 N^{-3} \right) \) respectively. Note that due to the symmetry of the exact expression with respect to the transformation \( k \leftrightarrow -k \) the corrections contain only even powers of \( k \). If we also replace \( h_2 \) in the denominator of the exponent by its limiting value \( \rho \), we also acquire the correction of order of \( (k^2 N^{-3/2}) \), which is larger of the two former. Since the latter correction is relevant for our further calculations, we leave its explicit value extracted from (98) in the final expression, which therefore looks as follows

\[
\frac{Z(N, p - k)Z(N, p + k)}{Z(N, p)^2} = \exp \left( \frac{k^2}{N\rho} - \frac{\alpha k^2}{2N^{3/2}} + O \left( \frac{k^2}{N^2} \right) + O \left( \frac{k^4}{N^3} \right) \right)
\]

Apparently this approximation works well for not too large values of \( k \). We will use it in the range \( 0 < k < \epsilon N \), where \( \epsilon \) is a constant that can be chosen arbitrarily small.

For \( k > \epsilon N \) the ratio is at least exponentially small in \( N \). This follows from the monotonicity of the exponent in (93) as the function of the ratio \( (k/N) \). Indeed for \( 0 < x < \rho \) we have

\[
\frac{d}{dx} \left( f(\rho + x) + f(\rho - x) - 2f(\rho) \right) = \frac{\partial h(z^*)}{\partial \rho} \bigg|_{\rho - \rho + x} - \frac{\partial h(z^*)}{\partial \rho} \bigg|_{\rho - \rho - x} = \ln z^* \bigg|_{\rho - \rho + x} - \ln z^* \bigg|_{\rho - \rho - x}
\]

\[
= 2\ln z^* \bigg|_{\rho - \rho'} = 2\ln z^* \bigg|_{\rho - \rho'} \frac{z^*}{h_2} \bigg|_{\rho - \rho'} > 0,
\]

where the subscripts show that the corresponding expressions are evaluated at shifted values of the density. We also used the mean value theorem, to approximate the function \( \ln z^* \) of the density in the interval \( [\rho - x, \rho + x] \) by the linear function with the slope equal to the derivative of \( \ln z^* \) in \( \rho \) at some intermediate point \( \rho' \in [\rho - x, \rho + x] \). From the monotonicity of the exponent we conclude that for any small \( \epsilon > 0 \) there exists a constant \( c \), such that

\[
\frac{Z(N, p - k)Z(N, p + k)}{Z(N, p)^2} = O \left( e^{-cN} \right), \quad k > \epsilon N. \tag{105}
\]

It remains to evaluate the double integral in (103) asymptotically. To this end, we must take into account that the dominating critical points in the integrals in \( y \) and \( t \) are now shifted due to the density shifts \( \rho \to \rho \pm k/N \).

\[
\int y^t q^k \mid_{\rho \to (\rho - k/N)} = [J_{0^+}]_{\rho \to (\rho - k/N)} + q^k [J_{0^+}]_{\rho \to (\rho + k/N)}
\]

Then, we use the asymptotic formulas (96)–(99) for \( J_{0^+} \) evaluated at shifted densities. The expression obtained is rather cumbersome and we omit its explicit form here. Instead, we
show the asymptotic form of the summand of (93), which simplifies significantly due to many cancellations,

\[
[IS]_k = 2N^2 \exp \left( -\frac{k^2}{N^\epsilon} - \frac{\alpha k^2}{2N^{3/2}} + O \left( \frac{k^4}{N} \right) \right) \\
\times \left[ -\frac{\alpha \rho}{2} \left( \frac{k}{\sqrt{N}} \right) \frac{1}{N} + \left( \frac{\alpha^2 \rho^2}{3} \left( \frac{k}{\sqrt{N}} \right) + \frac{\alpha (1 + q^2)}{2} \left( \frac{k}{\sqrt{N}} \right)^2 \right) \frac{1}{N^{3/2}} \right. \\
+ O \left( \frac{1}{N^2} \right) + O \left( \frac{k^2}{N^2} \right) + O \left( \frac{k^3}{N^{5/2}} \right) \right].
\]

(106)

Here we implied that \( k \) is of order of \( \sqrt{N} \), and thus the exact value of \( q^k \) is kept.

Now we are in a position to evaluate the infinite sum asymptotically. To this end, we divide it into two parts

\[
IS = \sum_{k=5}^{\lceil \sqrt{N} \rceil} [IS]_k + O(e^{-c_1 N}).
\]

The second part is exponentially small in \( N \) with some constant \( c_1 > 0 \) due to (105) and the fact that there is at most \( \rho \) nonzero summands in the whole sum. The first sum can be approximated by an integral

\[
\sum_{k=5}^{\lceil \sqrt{N} \rceil} [IS]_k = 2N^2 \sqrt{N\rho} \int_0^\infty e^{-x^2} \left[ -\frac{\alpha x \rho^{3/2}}{2} \frac{1}{N} \right. \left. + \left( \frac{\alpha^2 \rho^5/2}{3} x + \frac{\alpha \rho}{2} \frac{1 + e^{-\alpha x \sqrt{\rho}}}{1 - e^{-\alpha x \sqrt{\rho}}} x^2 - x^3 \frac{\alpha^2 \rho^{5/2}}{4} \right) \frac{1}{N^{3/2}} \right] \, dx + O \left( \sqrt{N} \right) \\
= -\frac{\alpha^2 \rho^3}{2} + \left( \frac{\alpha^2 \rho^3}{12} + \alpha^3/2 \int_0^\infty e^{-x^2} \frac{1 + e^{-\alpha x \sqrt{\rho}}}{1 - e^{-\alpha x \sqrt{\rho}}} x^2 \, dx \right) N + O \left( \sqrt{N} \right). 
\]

(107)

Going from (106) to (107) we left only the first term in the exponential factor. The exponential of the other three terms are replaced by their expansion to the second order. As a result we obtain an extra term comparing to the sum in the square brackets in (107) and several more correction terms under the sum. Finally, we replace the variables in the integral to \( x = k^2/(N\rho) \).

All the integrals except one can be evaluated explicitly, while one remains in the integral form.

To estimate the overall corrections we need to evaluate the difference of the exact expression (with corrections) and the limiting expression (without corrections) and make sure that the resulting sum, which we extend back to infinity converges uniformly in \( N \), so that the summation and the limit \( N \to \infty \) can be interchanged. The uniform convergence is ensured by the Gaussian prefactor, whose exponent with corrections can always be kept negative by the choice of small enough \( \epsilon \). The explicit calculation is standard, and we omit the details. In addition one has to take into account the Euler–Maclaurin corrections coming from the integral approximation of the sum. The resulting corrections have an order \( O \left( \sqrt{N} \right) \).

Finally, when we add the finite and infinite sums all the terms except one integral surprisingly cancel, yielding

\[
\Delta = N\alpha \rho^{3/2} \int_0^\infty e^{-x^2} \frac{1 + e^{-\alpha x \sqrt{\rho}}}{1 - e^{-\alpha x \sqrt{\rho}}} x^2 \, dx + O \left( \sqrt{N} \right),
\]

\[ \]
which agrees with (28) and (45). Following the same way in the case $|q| > 1$ we observe that the change $q \leftrightarrow 1/q$ corresponds to the change $\alpha \leftrightarrow -\alpha$ in the final expression, which leaves it invariant. Thus, the formula obtained is valid for approaching the limit $q \to 1$ from both sides.

6. Conclusion

To summarize, we have obtained the exact integral representation for the diffusion constant of the $q$-boson ZRP. Asymptotic analysis of the result obtained yielded the expressions of the diffusion coefficient in the KPZ regime as well as KPZ–EW crossover scaling function. Both asymptotic results agree with the scaling hypothesis about KPZ universality and earlier solutions of other models. Our results in principle should follow from the recent finite-time results on $q$-boson ZRP [83]. However, the connection is highly nontrivial.

This work can be considered a first step in the studies of the large time behaviour of $q$-boson type models in confined geometry. The next goal could be a construction of the higher cumulants of the model. In particular, it would be interesting to study KPZ–EW crossover not only for particular current cumulants, but also for the whole large deviation function.

Also richer structure of the $q$-boson-like models with more parameters, e.g. of the chippping model with factorized steady state also called $q$-Hahn or $q, \mu, \nu$-ZRP, opens perspectives for studies of transitions between KPZ and other than EW types of behaviour. We mention the examples of two particular limits of that model with such a transitions. First, the jamming transition in the TASEP with generalized update was recently studied in [73]. This model is of the same type as TASEP corresponding to the quantum group parameter $q = 0$. The Bethe ansatz solution simplifies greatly in that case, allowing obtaining the whole scaled cumulant generating function by the Derrida–Lebowitz method. It crosses over from the KPZ to Gaussian deterministic aggregation regime. We expect that similar transition must also take place in the chippping model [56] with $q \neq 0$. In the latter case however the method of Derrida–Lebowitz is inapplicable. The appropriate tool would then be the T–Q relation approach developed in [68] adopted for the $q$-boson models in the present paper.

Also, the asymmetric avalanche process [72, 84] gives an example of the transition from dispersive to continuous avalanche flow. The latter model is the stochastic particle model with particle flow maintained by the non-local avalanche events. The avalanches are finite in the infinite system at low particle densities, while the mean avalanche size diverges as the density approaches the critical value. In the context of dynamics of the associated interface the system suffers the transition from the KPZ to the tilted interface universality class [85, 86]. How the current diffusion coefficient describes the crossover between these two regimes is still an open problem.

Finally, we mention the similar problem for $q$-boson ZRP on the open segment. While the current-density diagram was obtained for this model using the matrix product representation of the stationary state, neither the further current cumulants nor the large deviation function was yet obtained. Development of the T–Q approach to this problem is a challenging problem. Similarly to the case of periodic system we expect that in the universal scaling limits the results will match with those obtained from the solution of ASEP with open boundaries [21, 63]. This is a matter of further studies.

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Collecting the terms with the same index from two successive summands of the sum in index

We simplify the result (75) by substituting

\[ f(x) = NB_1(qx) + \frac{Q_1(x)}{x^p}[(1 - x)^N B_1(x) - B_1(qx)] \quad \text{(A.1)} \]

to the integral

\[ \sum_{i=0}^{\infty} \oint_{|z|<1} \frac{F(z)^N}{z^p} \phi(z) f(q^i(1 - q)z) \frac{dz}{F(q^{i+1}z)^N} = \frac{2\pi i}{2\pi i}. \]

The first summand of \( f(x) \) does not contribute to the \( \lambda_2 \). Indeed, for any \( i \) we have

\[ \oint_{|z|<1} \frac{F(z)^N}{z^p} \phi(z) \frac{NB_1(q^{i+1}(1 - q)z)}{F(q^{i+1}z)^N} \frac{dz}{2\pi i} \]

\[ = -\frac{N(1 - q)^p}{Z(N, p)} \oint_{|z|<1} \frac{NF(z)^N}{z^p} \left( \frac{N}{Z(N, p)} \frac{F(z)}{F(1)} - 1 \right) \frac{dz}{2\pi i} = 0 \]

where we first replace the degree \((p - 1)\) polynomial \( B_1(q^{i+1}(1 - q)z) \), which is the truncation of the function \( N(1 - q)^p F(q^{i+1}z)^N / Z(N, p) \), by the whole function that cancels the denominator. Apparently, this replacement does not affect the value of the integral. Indeed, integrating the series representation of the function term by term we find that the integrals with the terms of higher than \((p - 1)\) order vanish for the integrands being analytic inside the integration contour. Substituting the explicit form of \( \phi(z) \) and integrating by parts the first term in the brackets in the second line we see that the whole integral vanishes.

Substitution of the second part of \( f(x) \) into the integral under the sum in (75) gives

\[ \frac{1}{(1 - q)^p} \oint_{|z|<1} \frac{dz}{2\pi i} \sum_{i=0}^{\infty} Q_1((1 - q)q^i z) \frac{F_{0i}(z)B_1((1 - q)q^i z)}{(1 - q)^p q^i} \left( F_{0i}(z)B_1((1 - q)q^i z) - F_{0i+1}(z)B_1((1 - q)q^{i+1} z) \right) \]

\[ = \frac{1}{(1 - q)^p} \oint_{|z|<1} \frac{dz}{2\pi i} \sum_{i=0}^{\infty} Q_1((1 - q)q^i z) \frac{F_{0i+1}(z)B_1^{p}(1 - q)q^{i+1} z)}{(1 - q)^p q^i} \]

\[ - F_{0i}(z)B_1^{p}(1 - q)q^i z)) \quad \text{(A.2)} \]

where we introduce

\[ B_1^{p}(1 - q)x = -\frac{N(1 - q)^p}{Z(N, p)} F_{0}(x) - B_1((1 - q)x) \]

\[ = -\frac{N(1 - q)^p}{Z(N, p)} x^p \oint_{|z|<1} \frac{dz}{2\pi i} \frac{F(z)}{z^p(z - x)} \quad \text{(A.3)} \]

The expression the square brackets in (A.2) is the sum of two functions with successive indices. Collecting the terms with the same index from two successive summands of the sum in index
we obtain

\[ i = -\frac{1}{(1-q)^2\varphi(z)} \int \frac{dz}{2\pi i \varphi(z)} Q_1((1-q)z) B_1^p((1-q)^z) + \frac{1}{(1-q)^2\varphi(z)} \int \frac{dz}{2\pi i \varphi(z)} \sum_{i=0}^{\infty} B_1^p((1-q)^{q^i+1}z) F_{\omega+1}(z) \]

\times \left[ \frac{Q_1((1-q)(q^i+z))}{q^i} - \frac{Q_1((1-q)q^{i+1})}{q^{i+1}} \right].

The content of the square brackets in the latter formula can be further modified to

\[ \frac{Q_1((1-q)(q^i+z))}{q^i} - \frac{Q_1((1-q)q^{i+1})}{q^{i+1}} = \frac{N(1-q)^{q^{-1}p} - \sum_{k=0}^{\infty} z - yq^{k-1}}{Z(N, p)} \]

\[ \int \frac{F_N(y)}{2\pi i y^p} \frac{dy}{y^p} - \sum_{k=0}^{\infty} \frac{q^{k-1}}{z - yq^{k-1}}, \]

where we used the integral form (62) of \( Q_1(x) \). Substituting also the integral of \( B_1^p((1-q)x) \) from ((A.3)) and inserting the resulting integral into (75) we arrive at (76).

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