Extended axial model

R. Amorim* and J. Barcelos-Neto†
Instituto de Física
Universidade Federal do Rio de Janeiro
RJ 21945-970 - Caixa Postal 68528 - Brasil

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Abstract

We study an extension of the axial model where local gauge symmetries are taken into account. The anomaly of the axial current is calculated by the Fujikawa formalism and the model is also solved. Besides the well known features of the particular models (axial and Schwinger) it was obtained an effective interaction of scalar and gauge fields via a topological current.

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*Electronic mail: amorim@if.ufrj
†Electronic mails: ift03001@ufrj and barcelos@vms1.nce.ufrj.br
1 Introduction

The axial model corresponds to a theory where a real scalar field interacts with a fermionic axial current via a derivative coupling. It was introduced by Rothe and Stamatescu almost twenty years ago \cite{1} and has been studied since then on its more diverse aspects \cite{2,3,4,5}. Its Lagrangian density reads \cite{1}

\[
\mathcal{L} = i \bar{\psi} \gamma^\mu (\partial_\mu + ig_0 \gamma_5 \partial_\mu \phi) \psi + \mathcal{L}_\phi,
\]

where

\[
\mathcal{L}_\phi = \frac{1}{2} \left( \partial_\mu \phi \partial^\mu \phi - m_0^2 \phi^2 \right).
\]

We easily observe that it exhibits global gauge and chiral-gauge symmetries. In consequence, from the Noether’s theorem, both vector and axial currents are conserved. The two main features of this model are

(i) The divergence of the axial current is anomalous:

\[
\partial_\mu j_5^\mu = \frac{i g_0}{\pi} \Box \phi,
\]

where \( j_5^\mu \) is defined as

\[
j_5^\mu = \bar{\psi} \gamma^\mu \gamma_5 \psi.
\]

(ii) The mass \( m_0 \) is renormalized to

\[
m^2 = \frac{m_0^2}{1 - \frac{g_0^2}{\pi}}.
\]

This model does not have the corresponding local gauge symmetries. Apparently, one possibility of having them would be to consider the coupling of the axial current with a gauge field \( A_\mu \), namely, \( g_0 \bar{\psi} \gamma^\mu \psi A_\mu \), and taking \( A_\mu = \partial_\mu \phi \) as a particular gauge choice. However, this cannot be true. A simple argument against this procedure is that the coupling term above does not have the correct dimension.

The consistent way of implementing local gauge transformations in the axial model is actually by means of a gauge field, but taken independently of \( \partial_\mu \phi \). The general Lagrangian density then reads

\footnote{We adopt throughout this paper the following convention and notation: \( \eta_{\mu\nu} = \text{diag.}(+1, -1) \), \( \epsilon^{01} = 1 \), \( \{\gamma^\mu, \gamma^\nu\} = 2 \eta^{\mu\nu} \), \( \gamma^\mu = \gamma^0 \gamma^\mu \gamma^0 \), \( \gamma^{01} = \gamma^0 \gamma^1 = -\gamma^1 \), \( \gamma_5 = \gamma^0 \gamma^1 \), \( \gamma^\mu \gamma^\nu = \eta^{\mu\nu} + \epsilon^{\mu\nu} \gamma_5 \), \( \gamma_5 \gamma^\mu = \epsilon^{\mu\nu} \gamma_\nu \) (these last two are only true in two dimensions).}
\[ \mathcal{L} = i \bar{\psi} \gamma^\mu \partial_\mu \psi - g_0 \bar{\psi} \gamma^\mu \gamma_5 \psi \partial_\mu \phi + \mathcal{L}_\phi + \mathcal{L}_A, \]  

(6)

where \( \mathcal{L}_A \) is the well known gauge field term

\[ \mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \]  

(7)

and the covariant derivative reads

\[ D_\mu = \partial_\mu - ie A_\mu. \]  

(8)

We list below the mass dimension of the quantities that appear in the Lagrangian (6):

\[ [\psi] = \frac{1}{2}, \quad [\phi] = 0, \quad [A_\mu] = 0, \quad [e] = 1, \quad [g_0] = 0, \quad [m_0] = 1. \]  

(9)

The model described by (6) is in fact a mixing of the axial model, described by expressions (1) and (2), and the Schwinger model (3). The purpose of this work is to study the features of such extended model, which will be called “extended axial model” (EAM). It might be opportune to first make some comments about the EAM:

(i) The derivative which is acting on the scalar field cannot be replaced by a covariant one. This is so because the scalar field is real and consequently does not couple to the electromagnetic one (the gauge transformation of \( \phi \) is zero).

(ii) Inadvertently, we could think to make \( g_0 = e \) to eliminate the second term of (6) by a gauge transformation of \( A_\mu \). As it was previously said, this cannot be done since \( g_0 \) and \( e \) do not have the same mass dimensions.

(iii) Another wrong reasoning would be the trial to obtain the scalar term \( -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi \) from the gauge field one \( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \) and vice-versa by considering that in two-dimensions the gauge field can always be written as

\[ A_\mu = \partial_\mu \sigma + \epsilon_{\mu\nu} \partial^\nu \rho. \]  

(10)

Also here, this would not be succeeded. \( \sigma \) and \( \rho \) have mass-dimensions \(-1\) and, consequently, cannot be related to \( \phi \). In fact, in case of replacing \( A_\mu \) given by (10) into (3), we would obtain terms in \( \Box \sigma \Box \sigma, \Box \rho \Box \rho \) etc.

So we conclude that Lagrangian (3) cannot be simplified. It has to be used in the way it appears. We shall see that the anomaly of the axial current of the EAM is more general and contains crossed terms of both sectors. However, the renormalized mass of the scalar field and the mass acquired by the photon field are precisely the
same of the usual models when taken separately. A point to be emphasized is that after solving the EAM, the scalar field effectively interacts with the gauge one via a topological current.

Our paper is organized as follows. In Sec. 2 we calculate the anomaly of the axial current of the EAM by means of the Fujikawa path integral technique \cite{7}. The model is solved by integrating on the fermionic fields in Sec. 3. Some final remarks are left for Sec. 4.

## 2 Axial current anomaly

To calculate the anomaly of the axial current we make use of the Fujikawa technique \cite{5}. We begin by writing down the general expression of the vacuum functional

$$Z = N \int [d\bar{\psi}] [d\psi] [d\phi] [dA] \exp \{iS\},$$

where $S$ is the action corresponding to the Lagrangian density \cite{3} plus some gauge-fixing term \cite{4}. The chiral anomaly arises in the path integral formalism from the fact that the measure $[d\bar{\psi}] [d\psi]$ is not invariant under chiral gauge transformations. In the infinitesimal case, these transformations read

$$\psi'(x) = [1 + i \epsilon(x) \gamma_5] \psi(x),$$
$$\bar{\psi}'(x) = \bar{\psi}(x) [1 + i \epsilon(x) \gamma_5],$$

and we obtain \cite{5}

$$[d\bar{\psi}] [d\psi] = [d\bar{\psi}'] [d\psi'] \exp \left\{ 2i \int d^2 x \epsilon(x) I(x) \right\},$$

where

$$I(x) = \sum_n \phi_n^\dagger(x) \gamma_5 \phi_n(x).$$

The quantities $\phi_n(x)$ form a complete and orthonormal set of eigenfunctions of some Hermitian operator $O_H$. To determine the anomaly, it is then necessary to perform the calculation of the sum above. However, it is well known that sums like $\sum_n \phi_n^\dagger(x) \Gamma_{\mu\nu...} \phi(y)$, where $\Gamma_{\mu\nu...}$ are product of gamma matrices, are divergent when $x = y$. The way found by Fujikawa to regularize these sums is to introduce an exponential factor in order to avoid contributions of big eigenvalues. Concentrating on our particular case, this is done in the following way

\footnote{It is understood that we have functionally integrated on ghost fields that come from the Faddeev-Popov method.}
\[ I(x) = \lim_{M \to \infty} \sum_n \phi_n^\dagger(x) \gamma_5 \phi_n(x) e^{-\lambda_n^2/M^2}, \]
\[ = \lim_{M \to \infty} \sum_n \phi_n^\dagger(x) \gamma_5 e^{-O_2^2/M^2} \phi_n(x), \]
\[ = \lim_{M \to \infty} \left( \gamma_5 e^{-O_2^2/M^2} \right)_{ij} \sum_n \phi_{nJ}(x) \phi_{nI}(y), \]
\[ = \lim_{M \to \infty} \left( \gamma_5 e^{-O_2^2/M^2} \right)_{ij} \delta_{ij} \delta^{(2)}(x - y), \]
\[ = \lim_{M \to \infty} \text{tr} \gamma_5 e^{-O_2^2/M^2} \delta^{(2)}(x - y). \quad (15) \]

According to the Fujikawa method, the operator that has to be used to regularize the sum is the one that appears in the theory. In our case, we have

\[ \tilde{D} = \gamma^\mu \left( \partial_\mu - ieA_\mu - ig_0 \gamma_5 \partial_\mu \right). \quad (16) \]

The point is that this operator is not Hermitian. An usual procedure used in literature is to go to the Euclidean space. We do it by letting \( x^0 \to -ix^4, \partial_0 \to i\partial_4, \gamma^0 \to i\gamma_4, \gamma^\mu \partial_\mu \to -\gamma_\mu \partial_\mu. \) In this space, all gamma matrices are anti-Hermitians. So, the terms \( \gamma_\mu \partial_\mu \) and \( ie\gamma_\mu A_\mu \) are Hermitians, but \( ig_0 \gamma_\mu \gamma_5 \partial_\mu \phi \) is not. Consequently, the operator

\[ \tilde{D}_E = \gamma_\mu \left( \partial_\mu - ieA_\mu - ig_0 \gamma_5 \partial_\mu \right), \quad (17) \]

obtained from

\[ \tilde{D} \to - \tilde{D}_E \quad (18) \]

is still not Hermitian.

There is a simple way of circumventing this problem. It consists in taking a kind of analytical extension of the field \( \phi \) when we go to the Euclidean space \([4, 8]\). So, instead of the operator \((17)\) we use the Hermitian one

\[ \tilde{D}_H = \gamma_\mu \left( \partial_\mu - ieA_\mu + g_0 \gamma_5 \partial_\mu \right), \quad (19) \]

where it was taken

\[ \phi \to i\tilde{\phi}. \quad (20) \]

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\(^3\)In the Euclidean space, our initial convention and notation have to be changed to \( \{ \gamma_\mu, \gamma_\nu \} = -2\delta_{\mu \nu}, \gamma_\mu \gamma_\nu = -\delta_{\mu \nu} - ie_{\mu \nu} \gamma_5, \gamma_5 \gamma_\mu = ie_{\mu \nu} \gamma_5, \gamma_5 = i\gamma_1 \gamma_4, e_{14} = +1 \) etc.
We shall use $D_H$ as the regulating operator of the Fujikawa technique. It is then just a matter of algebraic calculation to obtain

$$I(x) = \frac{1}{2\pi} \left( g_0 \Box \tilde{\phi} - 2ieg_0 A_\mu \partial_\mu \tilde{\phi} + e\epsilon_{\mu\nu} \partial_\mu A_\nu \right). \quad (21)$$

Rotating back to the Minkowski space and not forgetting to make $i\tilde{\phi} \to \phi$, we have

$$I(x) = \frac{1}{2\pi} \left( ig_0 \Box \phi + 2eg_0 A_\mu \partial_\mu \phi + e\epsilon_{\mu\nu} \partial_\mu A_\nu \right). \quad (22)$$

With this result the measure changes to

$$[d\bar{\psi}] [d\psi] = [d\bar{\psi}'] [d\psi'] \exp \left\{ \frac{i}{\pi} \int d^2x \epsilon(x) \left( ig_0 \Box \phi + 2eg_0 A_\mu \partial_\mu \phi + e\epsilon_{\mu\nu} \partial_\mu A_\nu \right) \right\}, \quad (23)$$

which implies through

$$\lim_{\epsilon \to 0} \frac{\delta Z}{\delta \epsilon(x)} = 0 \quad (24)$$

that

$$\partial_\mu j^\mu_5 = -\frac{1}{2\pi} \left( e\epsilon_{\mu\nu} F^{\mu\nu} + 2ig_0 \Box \phi + 4eg_0 A_\mu \partial_\mu \phi \right). \quad (25)$$

We notice that when we take $e = 0$, we obtain the anomaly of the axial current of the Rothe and Stamatescu’s model. The corresponding anomaly of the Schwinger model is obtained by taking $g_0 = 0$. We emphasize the presence of a mixing term with $eg_0$ coupling. This leads to an effective interaction between $A_\mu$ and $\phi$. We are going to discuss the consequences of this fact with more details in the next section.

### 3 Path integral solution of the model

Both in Schwinger and axial models there are effective theories that can be obtained by integrating on the fermionic fields. Let us see what kind of effective theory can be obtained here. Considering the fermionic part of the Lagrangian (6) and taking the $A_\mu$ field just as

$$A_\mu = \epsilon_{\mu\nu} \partial^\nu \rho, \quad (26)$$

that corresponds to take the gauge condition where $\sigma = 0$ (see expression (2)), we have
\[ \mathcal{L}_F = i \bar{\psi} \gamma^\mu (\partial_\mu - ie \epsilon_{\mu\nu} \partial^\nu \rho) \psi + g_0 \bar{\psi} \gamma_5 \gamma^\mu \psi \partial_\mu \phi , \]

\[ = i \bar{\psi} \gamma^\mu (\partial_\mu - i \gamma_5 \partial_\mu \xi) \psi , \quad (27) \]

where

\[ \xi = e \rho - g_0 \phi . \quad (28) \]

We notice that if we choose the gauge transformation

\[ \psi = e^{i \gamma_5 \xi} \chi , \quad (29) \]

we will obtain that the coupling of the fermionic current with \( \partial_\mu \rho \) disappears. Doing this iteratively by means of infinitesimal gauge transformations we get for the measure \[ [d \bar{\psi}] [d \psi] = [d \bar{\chi}] [d \chi] \exp \left( - \frac{i}{2\pi} \int d^2 x \partial_\mu \xi \partial^\mu \xi \right) . \quad (30) \]

Replacing the expression of \( \xi \) given by (28), we get

\[ [d \bar{\psi}] [d \psi] = [d \bar{\chi}] [d \chi] \exp \left( - \frac{i}{2\pi} \int d^2 x \left( e^2 \partial_\mu \rho \partial^\mu \rho - 2eg_0 \partial_\mu \rho \partial^\mu \phi \right. \right. \]

\[ + \left. g_0^2 \partial_\mu \phi \partial^\mu \phi \right) , \quad (31) \]

The vacuum functional then turns to be

\[ Z = N \int [d \bar{\chi}] [d \chi] [d A] [d \phi] \exp \left\{ i \int d^2 x \left[ i \chi \not{\partial} \chi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{e^2}{2\pi} A_\mu A^\mu \right. \right. \]

\[ + \left. \frac{1}{2} \left( 1 - \frac{g_0^2}{\pi} \right) \partial_\mu \phi \partial^\mu \phi - m_0^2 \phi^2 \right. \]

\[ - \left. \frac{eg_0}{\pi} e^{\mu\nu} \partial_\nu \phi A_\mu \right] \} . \quad (32) \]

As one observes, the resulting fermionic field does not interact with \( A_\mu \) and \( \phi \) anymore. It may disappear from the theory by integrating over it and absorbing the result into the renormalization factor \( N \). The so obtained effective Lagrangian contains the well known results of the Schwinger and the axial model, that is to say,
the photon field acquires a mass given by $e/\sqrt{\pi}$ and the mass $m_0$ of the scalar field is renormalized to $m_0 (1 - g_0^2/\pi)^{-1/2}$. The new point here is that the scalar field, that initially did not interact with the gauge field, now does via the current

$$J^\mu = \frac{e g_0}{\pi} \epsilon^{\mu \nu \rho \sigma} \partial_\nu \phi. \quad (33)$$

As can be trivially verified, $J^\mu$ is conserved independently of the equation of motion and it is not associated to any symmetry of the action. These facts give a topological character to $J^\mu$. Due to the boundary conditions we are using in the evaluation of most the quantities throughout the work (fields vanishing at the spatial infinity), there is only a null topological charge associated to $J^\mu$.

4 Conclusion

In this brief report we have considered the quantization of the extended axial model, which contains the Schwinger and the axial models as convenient limits. By using the Fujikawa prescription, it was possible to show that the axial current is actually anomalous. This anomaly has the usual contributions from the axial model sector as well as from the Schwinger model, but it contains also a crossed term which is a new feature of the considered model, expressing the fact that at the quantum level there is an interference of both sectors.

This fact also comes true when the model is solved by functionally integrating on the fermionic sector. The effective Lagrangian which survives presents a coupling between the EM field and the scalar one by means of a topological current.

We note that as the coupling between the gauge field and the scalar one is done by a bilinear term, further integrations on $A_\mu$ or $\phi$ could be done, leading to effectively pure scalar or pure vectorial theories with non local kinetical terms.

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