Lagrangian densities of hypergraph cycles

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Abstract

The Lagrangian density of an \( r \)-uniform hypergraph \( F \) is \( r! \) multiplying the supremum of the Lagrangians of all \( F \)-free \( r \)-uniform hypergraphs. For an \( r \)-graph \( H \) with \( t \) vertices, it is clear that \( \pi_\lambda(H) \geq r! \lambda(K^r_t-1) \). We say that an \( r \)-uniform hypergraph \( H \) with \( t \) vertices is perfect if \( \pi_\lambda(H) = r! \lambda(K^r_t-1) \). A theorem of Motzkin-Straus implies that all 2-uniform graphs are perfect. It is interesting to explore what kind of hypergraphs are perfect. A hypergraph is linear if any 2 edges have at most 1 vertex in common. We propose the following conjecture: (1) For \( r \geq 3 \), there exists \( n \) such that a linear \( r \)-uniform hypergraph with at least \( n \) vertices is perfect. (2) For \( r \geq 3 \), there exists \( n \) such that if \( G, H \) are perfect \( r \)-uniform hypergraphs with at least \( n \) vertices, then \( G \uplus H \) is perfect.

Regarding this conjecture, we obtain a partial result: Let \( S^2_{2,t} = \{ 123, 124, 125, 126, \ldots, 12(t+2) \} \). (An earlier result of Sidorenko states that \( S^2_{2,t} \) is perfect [20].) Let \( H \) be a perfect 3-graph with \( s \) vertices. Then \( F = S^2_{2,t} \uplus H \) is perfect if \( s \geq 3 \) and \( t \geq 3 \).

There was no known result on Lagrangian densities of hypergraph cycles and there were 3 unsolved cases for 3-uniform graphs spanned by 3 edges: a linear cycle of length 3: \( C^3_3 = \{ 123, 345, 561 \} \), the generalized triangle: \( F_5 = \{ 123, 124, 345 \} \) and \( K^3_4 = \{ 123, 124, 134 \} \). In this paper, we obtain the Lagrangian density of \( F_5 \) and this is the first example of non-perfect 3-uniform graph. We also obtain an extension of the above result to \( r \)-uniform hypergraphs. We show that \( C^3_3 \) is perfect, and among all \( C^3_3 \)-free 3-graphs \( G \), only those hypergraphs containing \( K^3_4 \) achieves the Lagrangian \( \lambda(K^3_4) \). An extension of this result to the 3-uniform linear cycle of length \( t \) is also given in the paper. The Turán densities of extensions of the above hypergraphs can be obtained by applying a transference technique of Pikhurko.

Key Words: Hypergraph Lagrangian

1 Introduction

1.1 Notations and definitions

For a set \( V \) and a positive integer \( r \), \( V^r \) denotes the family of all \( r \)-subsets of \( V \). An \( r \)-uniform graph or \( r \)-graph \( G \) consists of a set \( V(G) \) of vertices and a set \( E(G) \subseteq V(G)^r \) of edges. Let \( |G| \) denote the number of edges of \( G \). An edge \( e = \{ a_1, a_2, \ldots, a_r \} \) will be simply denoted by \( a_1a_2 \ldots a_r \). An

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In extremal problems, it is interesting in its own right to determine the maximum Lagrangian of a graph. The maximum number of edges in a graph $G$ is denoted by $\lambda(G)$, and it's maximum complete subgraphs. For a fixed positive integer $n$ and an $r$-graph $F$, the Turán number of $F$, denoted by $ex(n, F)$, is the maximum number of edges in an $F$-free $r$-graph on $n$ vertices. An averaging argument of Katona, Nemetz and Simonovits [12] shows that the sequence $\frac{ex(n, F)}{\binom{n}{r}}$ is a non-increasing sequence of real numbers in $[0, 1]$. Hence, $\lim_{n \to \infty} \frac{ex(n, F)}{\binom{n}{r}}$ exists. The Turán density of $F$ is defined as

$$\pi(F) = \lim_{n \to \infty} \frac{ex(n, F)}{\binom{n}{r}}.$$ 

For 2-graphs, Erdős-Stone-Simonovits determined the Turán numbers of all non-bipartite graphs asymptotically. Very few results are known for hypergraphs and a recent survey on this topic can be found in Keevash’s survey paper [13].

**Definition 1.1** Let $G$ be an $r$-graph on $[n]$ and let $\vec{x} = (x_1, \ldots, x_n) \in [0, \infty)^n$. For every subgraph $H \subseteq G$, define the Lagrangian function

$$\lambda(H, \vec{x}) = \sum_{e \in E(H)} \prod_{i \in e} x_i.$$ 

The Lagrangian of $G$, denoted by $\lambda(G)$, is defined as

$$\lambda(G) = \max\{\lambda(G, \vec{x}) : \vec{x} \in \Delta\},$$

where

$$\Delta = \{\vec{x} = (x_1, x_2, \ldots, x_n) \in [0, 1]^n : x_1 + x_2 + \cdots + x_n = 1\}.$$ 

The value $x_i$ is called the weight of the vertex $i$ and a vector $\vec{x} \in \Delta$ is called a feasible weight vector on $G$. A feasible weight vector $\vec{y} \in \Delta$ is called an optimum weight vector for $G$ if $\lambda(G, \vec{y}) = \lambda(G)$. In [13], Motzkin and Straus established a connection between the Lagrangian of any given 2-graph and its maximum complete subgraphs.

**Theorem 1.2** (13) If $G$ is a 2-graph in which a maximum complete subgraph has $t$ vertices, then $\lambda(G) = \lambda(K^2_t) = \frac{1}{2}(1 - \frac{1}{t}).$

They also applied this connection to give another proof of the theorem of Turán on the Turán density of complete graphs. Since then, the Lagrangian method has been a useful tool in hypergraph extremal problems. Earlier applications include that Frankl and Rödl [6] applied it in disproving the long standing jumping constant conjecture of Erdős. Sidorenko [20] and Frankl-Füredi [6] applied Lagrangians of hypergraphs in finding Turán densities of some hypergraphs. More recent developments of this method were obtained by Pikhurko [18] and in the papers [8] [16] [2] [17] [10]. In addition to its applications in extremal problems, it is interesting in its own right to determine the maximum Lagrangian of $r$-graphs with certain properties as remarked by Hefetz and Keevash [8]. For example, an interesting conjecture...
of Frankl-Füredi [5] considers the question of determining the maximum Lagrangian among all \( r \)-graphs with the fixed number of edges. Talbot [21] made a first breakthrough in confirming this conjecture for some cases. Subsequent progress in this conjecture were made in the papers of Tyomkyn [23], Lei-Lu-Peng [14] and Tang-Peng-Zhang-Zhao [22]. Recently, Gruslys-Letzter-Morrison [7] confirmed this conjecture for \( r = 3 \) and sufficiently large \( m \). We focus on the Lagrangian density of an \( r \)-graph \( F \) in this paper.

Given an \( r \)-graph \( F \), the Lagrangian density \( \pi_\lambda(F) \) of \( F \) is defined to be

\[
\pi_\lambda(F) = \sup \{ r! \lambda(G) : G \text{ is } F \text{-free} \}.
\]

The Lagrangian density is closely related to the Turán density. It’s easy to show the following fact.

**Fact 1.3** \( \pi(F) \leq \pi_\lambda(F) \).

**Proof.** Let \( \epsilon > 0 \) be arbitrary. Let \( n \) be large enough and let \( G_n \) be a maximum \( F \)-free \( r \)-graph on \( n \) vertices such that \( \pi(F) \leq \frac{|G_n|}{\binom{n}{r}} + \frac{\epsilon}{2} \). Then

\[
\pi(F) \leq \frac{|G_n|}{\binom{n}{r}} + \frac{\epsilon}{2} \leq r! \sum_{e \in E(G_n)} \frac{1}{n^r} + \epsilon = r! \lambda(G_n, \left( \frac{1}{n}, \ldots, \frac{1}{n} \right)) + \epsilon \leq r! \lambda(G_n) + \epsilon \leq \pi_\lambda(F) + \epsilon.
\]

A pair of vertices \( \{i, j\} \) is **covered** in a hypergraph \( F \) if there exists an edge \( e \) in \( F \) such that \( \{i, j\} \subseteq e \). We say that \( F \) covers pairs if every pair of vertices in \( F \) is covered. Let \( r \geq 3 \) and \( F \) be an \( r \)-graph. The extension of \( F \), denoted by \( H^F \) is obtained as follows: For each pair of vertices \( v_i, v_j \) not covered in \( F \), we add a set \( B_{ij} \) of \( r-2 \) new vertices and the edge \( \{v_i, v_j\} \cup B_{ij} \), where the \( B_{ij} \)'s are pairwise disjoint over all such pairs \( \{i, j\} \). A transference technique of Sidorenko [19] and Pikhurko [18] gave the following connection between the Lagrangian density of a hypergraph \( F \) and the Turán density of its extension.

**Proposition 1.4** [19, 18] \( \pi(H^F) = \pi_\lambda(F) \). In particular, if \( F \) covers pairs, then \( \pi(F) = \pi_\lambda(F) \).

For example, to determine the Turán density of \( K_4^3 \) (a long standing conjecture of Turán) is equivalent to determine the Lagrangian density of \( K_4^3 \).

The Lagrangian density of the enlargement of a tree satisfying Erdős-Sós’s conjecture is determined by Sidorenko [20] and Brandt-Irwin-Jiang [2]. Pikhurko [18] determined the Lagrangian density of a 4-uniform tight path of length 2 and this led to confirm the conjecture of Frankl-Füredi on the Turán number of its extension, the \( r \)-uniform genearlized triangle for the case \( r = 4 \). Norin and Yepremyan [17] determined for \( r = 5 \) or 6 by extending the earlier result of Frankl-Füredi in [5]. Jenssen [10] determined the Lagrangian density of a path of length 2 formed by two edges intersecting at \( r-2 \) vertices for \( r = 3, 4, 5, 6, 7 \). Hefetz and Keevash [8] determined the Lagrangian density of a 3-uniform matching of size 2. Jiang-Peng-Wu [11] obtained for any 3-uniform matching. The case for an \( r \)-uniform matching of size 2 was given in [11] (independently, in [24] for \( r = 4 \)). In [25] and [8], we obtained the Lagrangian densities of a 3-uniform linear path of length 3 or 4, the disjoint union of a 3-uniform linear path of length 2 or 3 and a 3-uniform matching, and the disjoint union of a 3-uniform tight path of length 2 and a 3-uniform matching. These were all the known results on Lagrangian densities.
1.2 Main results, open problems and Remarks

For an $r$-graph $H$ on $t$ vertices, it is clear that $\pi_\lambda(H) \geq r!\lambda(K_{r-1}^*)$. We say that an $r$-uniform hypergraph $H$ on $t$ vertices is perfect if $\pi_\lambda(H) = r!\lambda(K_{r-1}^*)$. Theorem 1.2 implies that all 2-graphs are perfect. Currently, all hypergraphs with known Lagrangian densities are perfect (mentioned in the previous paragraph) except an $r$-uniform matching of size 2 for $r \geq 4$. It is interesting to explore what kind of hypergraphs are perfect. An $r$-uniform hypergraph is linear if any two edges have at most 1 vertex in common. Let $G \cup H$ denote the disjoint union of $G$ and $H$. Let us propose the following conjecture.

**Conjecture 1.5** (1) For $r \geq 3$, there exists $n$ such that a linear $r$-uniform hypergraph with at least $n$ vertices is perfect.

(2) For $r \geq 3$, there exists $n$ such that if $G, H$ are perfect $r$-graphs with at least $n$ vertices, then $G \cup H$ is perfect.

For $r \geq 4$, the condition that the number of vertices is large enough cannot be removed from the above conjecture as the $r$-uniform matching of size 2 is not perfect for $r \geq 4$.

Regarding this conjecture, we provide a partial result.

**Theorem 1.6** Let $S_{2,t} = \{123, 124, 125, 126, \ldots, 12(t + 2)\}$. Let $H$ be a perfect 3-graph with $s$ vertices. Then $F = S_{2,t} \cup H$ is perfect if $s \geq 3$ and $t \geq 3$.

An earlier result of Sidorenko states that $S_{2,t}$ is perfect [20]. Combining previous known results this Theorem generates many perfect hypergraphs.

In view of the known results, there was no result on hypergraph cycles and there were 3 unsolved cases for 3-uniform graphs spanned by 3 edges: a linear cycle of length 3 denoted by $C_3^3 = \{123, 345, 561\}$, the generalized triangle denoted by $F_5 = \{123, 124, 345\}$ (a Berge cycle) and $K_3^3 = \{123, 124, 134\}$ (a tight cycle). In this paper, we also obtain the Lagrangian density of $F_5$ and $C_3^3$. We prove the following result implying that $F_5$ is not perfect.

**Theorem 1.7** Let $G$ be an $F_5$-free 3-graph with $n$ vertices. Then $\lambda(G) \leq \frac{27}{32}$. Furthermore, equality holds if and only if $G$ contains a copy of $S_{n,1}^3(1)$ with $n' \leq n$ and $n' \to \infty$, where $S_{n,1}^3(1)$ is the 3-graph with vertex set $[n]$ and edge set $\{(i,j) \mid i, j \in [n] \setminus \{1\}\}$.

**Corollary 1.8** $\pi_\lambda(F_5) = \frac{4}{9}$.

**Proof of Corollary 1.8** Since $S_{n,1}^3(1)$ is $F_5$-free, then $\pi_\lambda(F_5) \geq \lim_{n \to \infty} 3!\lambda(S_{n,1}^3(1)) = \frac{4}{9}$. On the other hand, by Theorem 1.7, $\pi_\lambda(F_5) \leq \frac{4}{9}$. Therefore, $\pi_\lambda(F_5) = \frac{4}{9}$. \hfill \qedsymbol

We also consider an extension of the above result to $r$-graphs. Let edges $e_1 = \{1, 2, 3, \ldots, r-2, a_1, a_2\}$, $e_2 = \{1, 2, 3, \ldots, r-2, a_1, a_3\}, \ldots, e_{r-1} = \{1, 2, 3, \ldots, r-2, a_1, a_{r-1}\}$. Let $r$-uniform graphs $F_0^r = e_1 \cup e_2 \cup \{a_2, a_3, m_1, m_2, \ldots, m_{r-2}\}$, $F_i^r = e_1 \cup e_2 \cup \{a_2, a_3, 1, 2, \ldots, i, m_1, m_2, \ldots, m_{r-i-2}\}$ for $1 \leq i \leq r-3$. Let $\mathcal{F}^r = \{F_0^r, F_1^r, \ldots, F_{r-3}^r\}$. Note that $F_0^r$ is $F_5$. We show that

**Theorem 1.9** Let $s = \left\lceil \frac{r-1}{2(r-1)} \right\rceil$ and $G$ be an $\mathcal{F}^r$-free $r$-graph satisfying $e_1, e_2, \ldots, e_{s-1} \in E(G)$. Then $\lambda(G) \leq \frac{\alpha}{r_0}$.
As remarked in Remark 3.5, when \( r = 3 \), Theorem 1.9 implies the first part of Theorem 1.7.

An \( r \)-uniform linear cycle of length \( t \) denoted by \( C^3_1 \) is isomorphic to \( \{123, 345, \ldots, (2t - 1)(2t)1\} \). Since \( C_1^3 \) has \( 2t \) vertices, then \( K^3_{2t-1} \) is \( C^3_1 \)-free, consequently, \( \pi_\lambda(C^3_1) \geq 3!\lambda(K^3_{2t-1}) \). We show that \( C^3_3 \) is perfect, and among all \( C^3_3 \)-free 3-graphs \( G \), only those containing \( K^3_3 \) achieves the Lagrangian \( \lambda(K^3_3) \).

**Theorem 1.10** Let \( G \) be a \( C^3_3 \)-free 3-graph. Then \( \lambda(G) \leq \lambda(K^3_3) = \frac{4}{25} \), and equality holds if and only if \( G \) contains \( K^3_3 \) as a subgraph.

**Corollary 1.11** \( \pi_\lambda(C^3_3) = \frac{12}{25} \).

Proof of Corollary 1.11 Since \( K^3_3 \) is \( C^3_3 \)-free and \( \lambda(K^3_3) = \frac{4}{25} \), then \( \pi_\lambda(C^3_3) \geq 3!\lambda(K^3_3) = \frac{12}{25} \). On the other hand, by Theorem 1.10 \( \pi_\lambda(C^3_3) \leq \frac{12}{25} \). Therefore, \( \pi_\lambda(C^3_3) = \frac{12}{25} \). \( \square \)

An \( r \)-graph \( G \) is dense if \( \lambda(G') < \lambda(G) \) for every proper subgraph \( G' \) of \( G \). As remarked in Remark 3.7 if \( G \) is \( F \)-free \( r \)-graph, then \( G \) contains a dense subgraph with the same Lagrangian as \( G \). So to estimate an upper bound of the Lagrangians of all \( C^3_3 \)-free 3-graphs, we only need to consider all dense \( C^3_3 \)-free 3-graphs. The following result is an extension of Theorem 1.10 (this will be explained in Section 5).

**Theorem 1.12** Let \( G \) be a dense and \( C^3_3 \)-free 3-graph satisfying \( K^3_{2t-2} \subseteq G \) \( (t \geq 4) \), then \( \lambda(G) \leq \lambda(K^3_{2t-1}) \).

The condition that \( K^3_{2t-2} \subseteq G \) in Theorem 1.12 is very strong, with effort, this condition can be weakened (We omit the technical argument. Instead, only the proof of Theorem 1.12 will be given). We propose the following conjecture.

**Conjecture 1.13** \( C^3_3 \) is perfect for any \( t \geq 3 \). A \( C^3_3 \)-free 3-graph achieves the maximum Lagrangian density only on 3-graphs containing \( K^3_{2t-1} \).

The first part of the conjecture is included in Conjecture 1.5.

In Section 2, we give some preliminary results on the Lagrangian function. In Section 3, we give the proofs of Theorems 1.7 and 1.9. The proof of Theorem 1.6 will be given in Section 4, and the proof of Theorem 1.10 and 1.12 will be given in Section 5.

Remark. The Turán densities of extensions of the above hypergraphs can be obtained by applying Proposition 1.4. There is still one unsolved case for 3-uniform graphs spanned by 3 edges: \( K^3_{4-} = \{123, 124, 134\} \). Since \( K^3_{4-} \) covers pairs, the Lagrangian density of \( K^3_{4-} \) is the same as the Turán density of \( K^3_{4-} \). It would be very interesting if the Turán density of \( K^3_{4-} \) can be obtained by determining the Lagrangian density of \( K^3_{4-} \).

2 Preliminaries

The following fact follows immediately from the definition of the Lagrangian.

**Fact 2.1** Let \( G_1, G_2 \) be \( r \)-graphs and \( G_1 \subseteq G_2 \). Then \( \lambda(G_1) \leq \lambda(G_2) \).
Fact 2.2 (II) Let $G$ be an $r$-graph on $[n]$. Let $\vec{x} = (x_1, x_2, \ldots, x_n)$ be an optimum weight vector on $G$. Then
\[
\frac{\partial \lambda(G, \vec{x})}{\partial x_i} = r \lambda(G)
\]
for every $i \in [n]$ satisfying $x_i > 0$.

This can be generalized to the following result.

Fact 2.3 Let $E \subset [n]^r$ and $f(\vec{x})$ be a homogeneous function with degree $r$ in the form of
\[
a_{i_1i_2\ldots i_r}x_{i_1}x_{i_2}\ldots x_{i_r}.
\]
Let $\vec{x} = (x_1, x_2, \ldots, x_n)$ be an optimum weight vector for $\text{Max}\{f(\vec{x}), \vec{x} \in S\}$. Then
\[
\frac{\partial f(\vec{x})}{\partial x_i} = rf(\vec{x})
\]
for every $i \in [n]$ satisfying $x_i > 0$.

Given an $r$-graph $G$, and $i, j \in V(G)$, define
\[
L_G(j \setminus i) = \{e : i \notin e, e \cup \{j\} \in E(G) \text{ and } e \cup \{i\} \notin E(G)\}.
\]

Fact 2.4 Let $G$ be an $r$-graph on $[n]$. Let $\vec{x} = (x_1, x_2, \ldots, x_n)$ be a feasible weight vector on $G$. Let $i, j \in [n]$, $i \neq j$ satisfying $L_G(i \setminus j) = L_G(j \setminus i) = \emptyset$. Let $\vec{y} = (y_1, y_2, \ldots, y_n)$ be defined by letting $y_i = x_i$ for every $\ell \in [n] \setminus \{i, j\}$ and $y_i = y_j = \frac{1}{2}(x_i + x_j)$. Then $\lambda(G, \vec{y}) \geq \lambda(G, \vec{x})$. Furthermore, if the pair $\{i, j\}$ is covered by an edge of $G$, $x_i > 0$ for each $1 \leq i \leq n$, and $\lambda(G, \vec{y}) = \lambda(G, \vec{x})$, then $x_i = x_j$.

Proof. Since $L_G(i \setminus j) = L_G(j \setminus i) = \emptyset$, then
\[
\lambda(G, \vec{y}) - \lambda(G, \vec{x}) = \sum_{\{i,j\} \subseteq E} \left( \frac{(x_i + x_j)^2}{4} - x_i x_j \right) \prod_{k \in e \setminus \{i,j\}} x_k \geq 0.
\]
If the pair $\{i, j\}$ is covered by an edge of $G$ and $x_i > 0$ for each $1 \leq i \leq n$, then the equality holds only if $x_i = x_j$. \qed

Fact 2.5 (II) Let $G = (V, E)$ be a dense $r$-graph. Then $G$ covers pairs.

Note that the converse of Fact 2.5 is not true. For example, the Fano plane covers pairs but it is not dense. Indeed, many counterexamples exist by Theorem 2.1 in the paper of Talbot [21].

Fact 2.6 (II) If $G$ is a dense 3-graph on $[n]$ ($n \geq 4$). Then $G$ contains a copy isomorphic to $\{123, 124\}$.

While considering the Lagrangian density of an $r$-graph $F$, we can always reduce to consider dense $F$-free $r$-graph.

Remark 2.7 Let $F$, $G$ be $r$-graphs and $G$ be $F$-free. Then there exists a dense subgraph $G'$ of $G$ such that $\lambda(G') = \lambda(G)$ and $G'$ is $F$-free.

Proof. Let $G$ be an $r$-graph on $n$ vertices. If $G$ is dense, then we are fine. If not, then we can find $G' \subset G$ such that $\lambda(G') = \lambda(G)$ and $|V(G')| < |V(G)|$. If $G'$ is dense, then we stop. Otherwise, we continue this process until we find a dense subgraph. \qed
3 Lagrangian density of generalized triangle

In an $r$-graph $G$, $N(a)$ denotes the link of $a$, i.e. $N(a) = \{ S|\{a\} \cup S \in E(G) \}$. Let $N(a,b)$ denote the link of $\{a,b\}$, i.e. all $(r-2)$-sets $S$ such that $\{a,b\} \cup S \in E(G)$. Let $N_A(a)$ denote $N(a) \cap A$.

3.1 Lagrangian density of $F_5$

In the section, the proof of Theorem 1.7 will be given. Let $G$ be an $F_5$-free 3-graph. By Remark 2.7, we may assume that $G$ is dense, $F_5$-free, and $|G| > 5$. We first show several crucial facts.

Fact 3.1 Let $G$ be a dense graph on $n$ vertices and let $\vec{x} = (x_1, x_2, \ldots, x_n)$ be an optimal weight vector satisfying $x_i \geq x_{i+1} > 0$. If there exist $x_i$ such that $x_i \geq \frac{1}{3}$, then

(i) $\lambda(G) \leq \frac{2}{27}$,

(ii) $\lambda(G) \rightharpoonup \frac{2}{27}$ if and only if $x_1 = \frac{1}{3}$ and $G \supseteq S_3^1$ satisfying $t \to \infty$.

Proof. Since $x_1 \geq x_i \geq \frac{1}{3}$, then $(1 - x_1)^2 \leq \frac{1}{9}$. By Theorem 1.7 and Fact 2.2,

$$3\lambda(G) = \frac{\partial \lambda(G)}{\partial x_1} \leq \left( \frac{1 - x_1}{t-1} \right)^2 \left( \frac{t-1}{2} \right) = \frac{(t-2)(1-x_1)^2}{2(t-1)} \leq \frac{2}{9},$$

where $t$ is the maximum complete subgraph in $N(1)$.

So $\lambda(G) \leq \frac{2}{27}$. And $\lambda(G) \rightharpoonup \frac{2}{27}$ if and only if $x_1 = \frac{1}{3}$, $G \supseteq S_3^1$ satisfying $t \to \infty$. \hfill \Box

Lemma 3.2 If $G$ is dense, $F_5$-free and contains a $K_4^3$, then $\lambda(G) \leq \frac{1}{10}$.

Proof. Assume that $v_1, v_2, v_3, v_4 \in V(G)$ form a $K_4^3$. We first prove

Claim 3.3 Let $v_i, v_j \in V(G) \setminus \{v_1, v_2, v_3, v_4\}$. Let $k, t \in [4]$, and $k \neq t$, then the following properties hold:

(i) if $v_i v_j v_k \in E(G)$, then $v_i v_j v_t \notin E(G)$,

(ii) $v_i v_k v_t \notin E(G)$.

Proof. (i) If $v_i v_j v_k, v_i v_j v_t \in E(G)$, let $r \in [4] \setminus \{k, t\}$, then $v_i v_j v_k, v_i v_j v_t, v_k v_t v_r$ form a $F_5$, a contradiction.

(ii) If $v_i v_k v_t \in E(G)$, let $p, r \in [4] \setminus \{k, t\}$, then $v_i v_k v_t, v_p v_t v_k, v_p v_t v_t$ form a $F_5$, a contradiction. \hfill \Box

Let us continue the proof of Lemma 3.1. Let $\vec{x}$ be an optimal weight vector for $G$, and $x_1, x_2, x_3, x_4$ be the weights of $v_1, v_2, v_3, v_4$ respectively. Let $a = x_1 + x_2 + x_3 + x_4$, then by Fact 2.2, $12\lambda(G) = \sum_{i=1}^{4} \frac{\partial \lambda(G)}{\partial x_i} = \sum_{i=1}^{4} \lambda(N(v_i), \vec{x})$.

Since $\sum_{i=1}^{4} \lambda(N(v_i), \vec{x})$ contains each $x_ix_j$ $(5 \leq i < j \leq n)$ at most once by Claim 3.3(i), $x_ix_j$ $(1 \leq i < j \leq 4)$ exactly twice, and no term in the form of $x_ix_j$ $(1 \leq i \leq 4, 5 \leq j \leq n)$ by Claim 3.3(ii). Then

$$12\lambda(G) \leq \left( \frac{1-a}{n-4} \right)^2 \left( \frac{n-4}{2} \right) + 2(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4).$$

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Since \( \sum_{1 \leq i < j \leq 4} x_i x_j \leq 6(\frac{a}{2})^2 = \frac{3}{8}a^2 \), then
\[
\lambda(G) \leq \frac{(1-a)^2}{24} + \frac{a^2}{16} = \frac{5a^2 - 4a + 2}{48}.
\]
The above quadratic function obtains the maximum \( \frac{1}{16} \) at \( a = 1 \) (recall that \( a \in [0, 1] \)). So \( \lambda(G) \leq \frac{1}{16} \).

**Proof of Theorem 1.7.** By Remark 2.7 and Lemma 3.2 we may assume that \( G \) is a dense \( K_4^3 \)-free 3-graph with \( n \geq 5 \). By Fact 2.6 we may assume that \( G \) contains \{012, 013\}. Note that \( N(2, 3) \subset \{0, 1 \} \) since otherwise we will get a \( F_5 \). Without loss of generality, let \( 023 \in E(G) \). Note that \{012, 013, 023\} form a \( K_4^3 \).

Let \( A \) a maximal set containing \{1, 2, 3\} such that \( 0ij \in E(G) \) for any pair \( \{i, j \} \) in \( A \). Clearly \( |A| \geq 3 \). We divide \( V(G) \) into 3 parts, \( V(G) = \{0\} \cup A \cup B \), where \( B = V(G) \setminus A \cup \{0\} \). Then we have the following Observation.

**Observation 3.4**
(i) For \( v \in B \) and \( i, j \in A \), \( vij \notin E(G) \).
(ii) Let \( v \in B \), if \( v \notin E(G) \) for some \( i \in A \), then \( v0j \in E(G) \) for each \( j \in A \).
(iii) For any \( v \in B \) and any \( i \in A \), \( \{v, i\} \) is not covered in \( G \).
(iv) For any \( v_1, v_2 \in B \) we have \(|\{i | iv_1v_2 \in E(G), i \in A \cup \{0\}\}| \leq 1 \).
(v) If \( i, j, k \in A \), then \( ijk \notin E(G) \).

**Proof.**
(i) If \( vij \in E(G) \) then \( vij, 0ki, 0kj \) form a \( F_5 \) for any \( k \in A \setminus \{i, j\} \), such a \( k \) exists since \( |A| \geq 3 \).
(ii) Since \( G \) is dense then \( N(v, j) \neq \emptyset \) for each \( j \in A \). If \( N(v, j) \cap B \neq \emptyset \), let \( u \in N(v, j) \cap B \), then \( uvj, 0ui, 0ij \) form a \( F_5 \), so \( N(v, j) \cap B = \emptyset \). By (i), \( N(v, j) \cap A = \emptyset \). So \( N(v, j) = \{0\} \).
(iii) This is due to (i), (ii) and the maximality of \( A \).
(iv) If there exist \( i, j \in A \) s.t. \( iv_1v_2, jv_1v_2 \in E(G) \), then \( iv_1v_2, jv_1v_2, 0ij \) form a \( F_5 \) (change \( 0ij \) to \( ijk \) when \( i = 0 \) or \( j = 0 \) for \( k \in A \setminus \{i, j\} \)).
(v) If \( ijk \in E(G) \) then \( 0ij, 0ik, 0jk \), \( ijk \) form a \( K_4^3 \), a contradiction.

Let us continue the proof of Theorem 1.7. Let \( k = |A| \), the weight of 0 be \( c \), the weight of vertices in \( A \) be \( a_1, a_2, ..., a_k \) in the optimal weighting vector. Let \( a = a_1 + a_2 + ... + a_k \), consider \( \frac{\partial \lambda(G)}{\partial x_i}, i \in A \) and \( \frac{\partial \lambda(G)}{\partial x_0} \). By Fact 2.2
\[
(3k + 3)\lambda(G) = \sum_{i=1}^{k} \frac{\partial \lambda(G)}{\partial x_i} + \frac{\partial \lambda(G)}{\partial x_0}.
\]
By Observation 3.4(iv), the terms in the form of \( x_i x_j \) where \( i, j \in B \) appear at most once in \( \sum_{i=1}^{k} \frac{\partial \lambda(G)}{\partial x_i} + \frac{\partial \lambda(G)}{\partial x_0} \), the terms in the form of \( a_i a_j (1 \leq i < j \leq k) \) appear exactly once in \( \frac{\partial \lambda(G)}{\partial x_k} \) and not in \( \sum_{i=1}^{k} \frac{\partial \lambda(G)}{\partial x_i} \) by the definition of \( A \) and Observation 3.4(iii)(v), the terms in the form of \( a_i c (1 \leq i \leq k) \) appear exactly \( k - 1 \) times in \( \sum_{i=1}^{k} \frac{\partial \lambda(G)}{\partial x_i} \) by the definition of \( A \). By Observation 3.4(iii), no term in the form of \( cx_j \).
\[ (j \in B) \text{ or } a_i x_j \ (1 \leq i \leq k, j \in B) \text{ appear in } \sum_{i=1}^{k} \frac{\partial \lambda(G)}{\partial x_i} + \frac{\partial \lambda(G)}{\partial a_i}. \] So

\[ (3k + 3)\lambda(G) \leq \left( \frac{n - k - 1}{2} \right) \left( \frac{1 - c - a}{n - k - 1} \right)^2 + \sum_{1 \leq i < j \leq k} a_i a_j + (k - 1)ac \]

\[ \leq \frac{(1 - a - c)^2}{2} + \frac{1}{2}a^2 + (k - 1)ac. \]

So

\[ \lambda(G) \leq \frac{1}{6(k + 1)} \left( c^2 + a^2 + 1 + 2ac - 2c - 2a \right) + \frac{k - 1}{3(k + 1)}ac + \frac{1}{6(k + 1)}a^2 \]

\[ \leq \frac{c^2 + 2a^2 + 1 + 2kac - 2c - 2a}{6(k + 1)} \]

\[ = f(c) \quad (c \leq \frac{1}{3}). \]

Since \( f(c) \) is quadratic, opens up and \( 0 \leq c \leq \frac{1}{3} \), then

\[ \lambda(G) \leq \max\{ f(0); f(\frac{1}{3}) \} = \max\{ \frac{2a^2 - 2a + 1}{6(k + 1)}(a < 1); \frac{a^2 + (2k - 2)a + \frac{4}{3}}{6(k + 1)}(a \leq \frac{2}{3}) \}. \]

Thus \( \lambda(G) \leq \min\{ \frac{1}{4k + 1}, \frac{k}{2(k + 1)} \}. \) Since \( k \geq 3 \), then \( \lambda(G) \leq \frac{k}{2(k + 1)}. \) The proof implies that the equality holds if and only if \( c = \frac{1}{3} \) and \( a = \frac{2}{3} \). This implies that the weight of each vertex in \( B \) is 0. Then \( G \) is dense implies that \( B = \emptyset \) and \( G \) is isomorphic to \( S_0^1(1) \) with \( n \to \infty \).

\[ \square \]

### 3.2 Extension to \( r \)-uniform graphs

The proof of Theorem 1.9 will be given in this section.

**Remark 3.5** When \( r = 3 \), Theorem 1.9 is equivalence to the first part of Theorem 1.7.

**Proof.** For \( r = 3 \), \( s = \frac{3^2}{2 \times 21} = \frac{9}{7} \leq 3 \). By Fact 2.6 and Remark 2.7 \( G \) contains a copy of \( \{123, 124\} \). \( \square \)

**Proof of Theorem 1.9** By Remark 2.7 we may assume that \( G \) is dense. Apply induction on \( r \). By Theorem 1.7 the conclusion holds for \( r = 3 \). Assume that the conclusion holds for all the integers less than \( r(> 3) \). Since \( G \) is dense, then there exist an edge \( e' \in E(G) \) which covers \( \{a_2, a_3\} \). Note that \( e' \subset V(e_1 \cup e_2) \) since otherwise \( e_1 \cup e_2 \cup e' \in \mathcal{F}^r \). Without loss of generality, let \( e' = \{1, 2, 3, \ldots, r - 2, a_2, a_3\} \). Similarly there exist an edge \( e'' \) covering \( a_3, a_4 \). We note that \( e'' \subset V(e_2 \cup e_3) \) and \( e'' = \{1, 2, 3, \ldots, r - 2, a_3, a_4\} \) since otherwise \( e_1 \cup e' \cup e'' = \mathcal{F}^r \). So we can divide \( V(G) \) into 3 parts. Let \( O = \{1, 2, 3, \ldots, r - 2\}, \{a_1, a_2, \ldots, a_s\} \subset A \) and \( A = \{a_1, a_2, \ldots, a_k\} \) be a maximal set such that for any \( a_i, a_j \in A \), \( 123 \ldots (r - 2) a_i a_j \in E(G) \). Let \( B = V(G) \setminus (O \cup A) = \{b_1, b_2, \ldots\} \), then we have the following facts.

**Fact 3.6** \( O^i \times A^i \times B^i \cap E(G) = \emptyset \), where \( j \geq 2, l \geq 1, i + j + l = r \), and \( O^i \times A^i \times B^i \) denotes the set of all \( r \)-sets consisting of \( i \) elements from \( O \), \( j \) elements from \( A \) and \( l \) elements from \( B \).

**Proof.** If \( e = 123 \ldots a_1 a_2 b_1 \ldots b_l \in O^i \times A^i \times B^i \cap E(G) \), then \( \{1, 2, \ldots, r - 2, a_2, a_3\} \cup \{1, 2, \ldots, r - 2, a_1, a_3\} \cup e \in \mathcal{F}^r \). \( \square \)
Fact 3.7 $O^{r-2} \times A \times B \cap E(G) = \emptyset$.

Proof. Let $e \in O^{r-2} \times A \times B \cap E(G)$. For each $i$, consider $e$ and $e_{i-1}$, then there exist an edge $e'$ covering $\{b_1, a_i\}$, and $V(e') \subset V(e \cup e_i)$. If $e' \neq 12, \ldots, r - 2a_i b_1$, then $a_1 \in e'$, and $e' \cup \{1, 2, \ldots, a_i, a_j\} \cup \{1, 2, \ldots, r - 2, a_1, a_j\} \in \mathcal{F}'$ for $j \notin \{i\}$. So $123, (r - 2)a_i b_1 \in E(G)$ for each $i \in [k]$, then $b_1 \in A$ by the maximality of $A$. A contradiction.

Let $\{x_1, x_2, \ldots, x_{r-2}, x_{a_1}, \ldots, x_{a_k}, x_{b_1}, \ldots, x_{b_t}, \ldots\}$ be an optimal weight of $V(G) = O \cup A \cup B$. If there exist $i \in V(G)$ such that $x_i \geq \frac{1}{r}$, then $N(i)$ is an $\mathcal{F}^{r-1}$-free $(r - 1)$-graph. By Fact 2.2 and the induction hypothesis, we have

$$r \lambda(G) = \frac{\partial \lambda(G)}{\partial x_i} \leq \frac{2}{(r - 1)(r - 1)} \left( 1 - \frac{1}{r} \right)^{r - 1} = \frac{2}{r^{r-1}}.$$ 

Consequently $\lambda(G) \leq \frac{2}{r}$.

So we may assume that $0 < x_i < \frac{1}{r}$ for each $i \in V(G)$. By Fact 2.2, we have

$$kr \lambda(G) = \sum_{i=1}^{k} \frac{\partial \lambda(G)}{\partial x_{a_i}}.$$ 

Now let function $g(x) = \sum_{i=1}^{k} \frac{\partial \lambda(G)}{\partial x_{a_i}}$. Consider the optimization problem

$$\text{Max}\{g(x)| \sum_{i \in V(G)} x_i = 1, 0 \leq x_i \leq \frac{1}{r}\},$$ 

and let $\overrightarrow{x}$ be a solution to the optimization problem. Then we know that

$$kr \lambda(G) \leq g(\overrightarrow{x}). \quad (3.1)$$

If there exist $a_j \in A$ such that $x'_{a_j} > 0$, then by Fact 2.2

$$(r - 1) g(\overrightarrow{x}) = \left. \frac{\partial g(x)}{\partial x_{a_j}} \right|_{\overrightarrow{x}} = (k - 1)x'_{a_1}x'_{a_2} \ldots x'_{a_{r-2}}.$$ 

So

$$kr(r - 1) \lambda(G) \leq (k - 1)x'_{a_1}x'_{a_2} \ldots x'_{a_{r-2}} \leq (k - 1) \left( \frac{1}{r} \right)^{r - 2}.$$ 

Thus

$$\lambda(G) \leq \frac{2}{r^r}.$$ 

So we can assume that $x'_{a_1} = x'_{a_2} = \ldots = x'_{a_k} = 0$. Let us estimate $g(\overrightarrow{x})$. In this case, all terms containing $x'_{a_i}$ for some $a_i \in A$ are 0. We only need to look at terms corresponding to $(r - 1)$-tuples in $O \cup B$. Note that $N_O \cup B(a_i) \cap N_{O \cup B}(a_j) = \emptyset$ since if $12 \ldots b_1 b_2 \ldots b_{r - 1} a_i, 12 \ldots b_1 b_2 \ldots b_{r - 1} a_j \in E(G)$, then these 2 edges together with $12 \ldots (r - 2) a_i a_j$ form an $r$-graph in $\mathcal{F}'$. So the set of subscripts of all non-zero terms in $g(\overrightarrow{x})$ is a subset of the complete $(r - 1)$-graph on $O \cup B$. Therefore,

$$g(\overrightarrow{x}) \leq \left( \frac{1}{n - k} \right)^{r - 1} \left( \frac{n - k}{r - 1} \right) \leq \frac{1}{(r - 1)!}. \quad (3.2)$$

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Combining (3.1) and (3.2), we have

$$\lambda(G) \leq \frac{1}{kr} g(x) \leq \frac{1}{kr(r-1)!} \leq \frac{2}{r!}$$

since $k \geq s \geq \frac{r^{r-1}}{2(r-1)!}$.

\[\square\]

4 Generating perfect hypergraphs

The proof of Theorem 1.6 will be given in this section.

Proof of Theorem 1.6 Let $H$ be a perfect 3-graph with $s$ vertices. Let $F = S_{2,t} \cup H$. Let $G$ be an $F$-free 3-graph such that $\lambda(G) \to \text{Sup}\{\lambda(G') | G' \text{ is F-free}\}$. By Remark 2.7 we may assume that $G$ is dense. If $F$ is not perfect, then $\lambda(G) > \lambda(K_{s+t+1})$ since the number of vertices in $F$ is $s + t + 2$. By a result of Sidorenko (20), $S_{2,s+t}$ is perfect. So $S_{2,s+t} \subseteq G$. Assume that $\{12a_1, 12a_2, ... 12a_{s+t}\} \subseteq G$. We show the following claim.

Claim 4.1 $G \setminus \{1,2\}$ is $H$-free.

Proof. If $H \subset G \setminus \{1,2\}$, we know that $S_{2,s+t} \subseteq G$, so there exist at least $t$ vertices $\{a_{i_1}, a_{i_2}, ..., a_{i_t}\}$ in $G \setminus (\{1,2\} \cup H)$ such that $12a_{i_1}, 12a_{i_2}, ..., a_{i_t} \in E(G)$ since $|V(H)| = s$. So $G$ is not $F$-free, a contradiction. \[\square\]

Let us continue the proof of Theorem 1.6. Since $G \setminus \{1,2\}$ is $H$-free and $H$ is perfect, then $\lambda(G \setminus \{1,2\}) \leq \lambda(K^{3}_{s-1})$. Let $G'$ be the 3-graph with edge set $E(G) \cup \{12i : i \in V(G) \setminus \{1,2\}\} \cup \{1jk, 2jk : j,k \in V(G) \setminus \{1,2\}\}$. Clearly $G \subseteq G'$. By Fact 2.1 we know that $\lambda(G) \leq \lambda(G')$. By Fact 2.4 we can assume that, the vertices 1, 2 have the same weights $a$ in an optimal weight vector of $G'$. Then we have,

$$\lambda(G) \leq \lambda(G') \leq a^2(1 - 2a) + 2a \left( \frac{1 - 2a}{n - 2} \right)^2 \binom{n - 2}{2} + \lambda(K^{3}_{s-1})(1 - 2a)^3$$

$$\leq 2a^3 - 3a^2 + a + \frac{(1 - 2a)^3}{6}m = f(a) \quad (m = \frac{(s - 2)(s - 3)}{(s - 1)^2}).$$

By direct calculation $f'(a) = (6 - 4m)a^2 + (4m - 6)a + 1 - m$ and $f''(a) = 2(6 - 4m)(a - \frac{1}{2}) \leq 0$ when $a \leq \frac{1}{2}$. So $f'(a)$ is monotone decreasing in $(0,\frac{1}{2})$. Since $f'(\frac{1}{2} - \frac{\sqrt{3} - 2m}{6 - 4m}) = 0$, then $f(a)$ is monotone increasing in $(0,\frac{1}{2} - \frac{\sqrt{3} - 2m}{6 - 4m})$, monotone decreasing in $(\frac{1}{2} - \frac{\sqrt{3} - 2m}{6 - 4m}, \frac{1}{2})$. Since $a \in (0,\frac{1}{2})$, then

$$f(a) \leq f\left(\frac{1}{2} - \frac{\sqrt{3} - 2m}{6 - 4m}\right) = \frac{1}{6\sqrt{3} - 2m} = \frac{s - 1}{6\sqrt{s^2 + 4s - 9}}.$$

So

$$\lambda(G) \leq \frac{s - 1}{6\sqrt{s^2 + 4s - 9}}.$$

Our goal is to show that $\lambda(G) < \lambda(K^{3}_{s+t+1})$. Since for $t \geq 3$

$$\frac{(s + 3)(s + 2)}{6(s + 4)^2} = \lambda(K^{3}_{s+4}) \leq \lambda(K^{3}_{s+t+1}).$$
So it’s sufficient to show that \( \frac{s-1}{6\sqrt{s^2+4s-9}} \leq \frac{(s+3)(s+2)}{6(s+4)^2} \). This is equivalent to show that

\[
(s-1)(s+4) < (s+3)(s+2)\sqrt{s^2+4s-9}.
\]
i.e.

\[
(s^2 - 2s + 1)(s^2 + 8s + 16) < (s^2 + 5s + 6)^2(s^2 + 4s - 9).
\]
Subtracting the left hand side in both sides, we got

\[0 < 3s^4 + 38s^3 + 103s^2 - 140s - 580 = g(s)\).

So our goal is to show the above inequality \( g(s) > 0 \). Since \( g'(s) = 12s^3 + 114s^2 + 206s - 140 \), and \( g''(s) = 36s^2 + 228s + 206 \) \( (s > 0) \), then \( g''(s) > 0 \) for all \( s > 0 \). So \( g'(s) \) is monotone increasing. By direct calculation \( g'(0) < 0, g'(1) > 0 \), so \( g(s) \) is monotone increasing in \( [1, +\infty) \). Since \( g(3) > 0 \), then \( g(s) > 0 \) holds for all \( s \geq 3 \). This completes the proof. \( \square \)

5 Lagrangian density of a 3-uniform linear cycle

Let \( P^3_t \) denote the 3-uniform linear path with length \( t \). Let \( G \) be a \( C^3_t \)-free 3-graph and \( A \subset V(G) \). By Remark 2.7 we may assume that \( G \) is dense. We will give some structure analysis on \( G \) and show that \( G \) is a ‘good’ 3-graph with some nice structure.

**Definition 5.1** For two vertices \( a_i, b_i \in V(G) \setminus A \), we say that \( \{a_i, b_i\} \) is a good pair to \( A \) if \( N(a_i, k) = b_i \) and \( N(b_i, k) = a_i \) for all \( k \in A \).

**Definition 5.2** We say that a dense 3-graph \( G \) is a good graph to \( A \) if \( V(G) = \bigcup_{i=1}^n\{a_i, b_i\} \cup A \) and \( \{a_i, b_i\} \) is a good pair to \( A \) for each \( 1 \leq i \leq s \), where \( s = \frac{|V(G) \setminus A|}{2} \).

**Fact 5.3** Let \( G \) be a dense and \( C^3_t \)-free graph. Let \( \{2t-2\} \setminus \{123\} \) be a \( \overline{K}^3_{2t-2} \) in \( G \). Let \( a, b, c \in V(G) \setminus \{2t-2\} \) and \( i, j, k, l \in \{t-2\} \) be four different integers. Then the following properties hold.

(i) If \( t \geq 4 \), then there is at most one of \( aij \) and \( bkl \) in \( E(G) \).

(ii) There is at most one of \( abi \) and \( acj \) in \( E(G) \).

(iii) There is at most one of \( aij, abk \) in \( E(G) \).

**Proof.**

(i) If there exist \( i, j, k, l \in \{t-2\} \) such that \( aij \in E(G) \) and \( bkl \in E(G) \), then there exist \( u \in \{2t-2\} \) such that \( iku \in E(G) \) and \( \{i, k, u\} \cap \{1, 2, 3\} \neq \emptyset \). Then there exists a linear path \( P_{t-3} \) in \( \{2t-2\} \setminus \{i, k, u\} \) with endpoints \( j, l \) for \( t \geq 4 \). Then \( P_{t-3}, aij, iku, bkl \) form a \( C^3_t \), a contradiction.

(ii) If both \( abi, acj \in E(G) \), then \( abi, acj \) and a \( P_{t-2} \) in \( \{2t-2\} \setminus \{k\} \) with endpoints \( \{i, j\} \), where \( k \in \{3\} \setminus \{i, j\} \), form a \( C^3_t \).

(iii) If both \( aij, abk \in E(G) \), then \( aij, abk \) and a \( P_{t-2} \) in \( \{2t-2\} \) with endpoints \( \{i, k\} \) or endpoints \( \{j, k\} \) form a \( C^3_t \). \( \square \)

**Fact 5.4** Let \( G \) be a dense and \( C^3_t \)-free 3-graph, and let \( \{2t-2\} \setminus \{123\} \subseteq G[A] \). Then each \( a \in V(G) \setminus A \) belongs to at most 1 good pair to \( A \).
Proof. If \( a \in V(G) \setminus A \) belongs to 2 good pairs \( \{a, b_1\}, \{a, b_2\} \) to \( A \), then \( ab_1, ab_2 \in E(G) \). This contradicts to Fact 5.3(ii). \( \Box \)

**Fact 5.5** Let \( G \) be a good graph to \( A \) with \( V(G) = \bigcup_{i=1}^{t} \{a_i, b_i\} \cup A \), where \( s = \frac{|V(G)\setminus A|}{2} \) and let \( K_{2t-2}^3 \subseteq G[A] \). Then for each pair \( \{i, j\} \), \( N(a_i, a_j) \subseteq \{b_i, b_j\} \) and \( N(a_i, b_j) \subseteq \{b_i, a_j\} \).

Proof. If \( k \in N(a_i, a_j) \cap A \), then \( a_i a_j k, a_i b_j' \in E(G) \) for \( k' \in A \setminus \{k\} \), a contradiction to Fact 5.3(ii). So \( N(a_i, a_j) \cap A = \emptyset \). Suppose that \( x \in N(a_i, a_j) \subseteq V(G) \setminus (A \cup \{b_i, b_j\}) \). If \( t \geq 4 \), then \( a_i a_j x, a_i b_j' \in E(G) \). But \( a_i b_j, a_i a_j x, a_i b_j' \) and a path \( P_{t-3} \) connecting 1 and 2 form a \( C_3^3 \), a contradiction. If \( t=3 \), then \( a_i a_j x, a_i b_j' \) and \( a_i b_j' \) form a \( C_3^3 \), a contradiction. So \( N(a_i, a_j) \subseteq \{b_i, b_j\} \). Similarly, \( N(a_i, b_j) \subseteq \{b_i, a_j\} \). \( \Box \)

**Definition 5.6** For good pairs \( \{a_1, b_1\}, \{a_2, b_2\} \) to \( A \), we say \( \{a_2, b_2\} \geq \{a_1, b_1\} \) if \( a_2 b_2 a_1, a_2 b_2 b_1 \in E(G) \).

**Fact 5.7** Let \( G \) be a good graph to \( A \) with \( V(G) = \bigcup_{i=1}^{t} \{a_i, b_i\} \cup A \), where \( s = \frac{|V(G)\setminus A|}{2} \), and let \( K_{2t-2}^3 \subseteq G[A] \). Then either \( \{a_i, b_i\} \geq \{a_j, b_j\} \) or \( \{a_j, b_j\} \geq \{a_i, b_i\} \).

Proof. By Fact 5.5 \( N(a_i, a_j) \subseteq \{b_i, b_j\} \). Without loss of generality, let \( a_i b_i a_j \in E(G) \). By Fact 5.5 \( N(a_i, b_j) \subseteq \{b_i, a_j\} \). If \( a_i b_j b_i \in E(G) \), then \( \{a_i, b_i\} \geq \{a_j, b_j\} \). Else \( a_i b_j a_j \in E(G) \), by Fact 5.5 \( N(b_i, b_j) \subseteq \{a_i, a_j\} \). If \( a_i b_i b_j \in E(G) \), then \( \{a_i, b_i\} \geq \{a_j, b_j\} \). Otherwise \( \{a_i, b_i\} \leq \{a_j, b_j\} \). \( \Box \)

**Fact 5.8** Let \( G \) be a good graph to \( A \) with \( V(G) = \bigcup_{i=1}^{t} \{a_i, b_i\} \cup A \), where \( s = \frac{|V(G)\setminus A|}{2} \). If \( \lambda(G[A]) \leq \lambda(K_{2t-1}^3) \), then \( \lambda(G) \leq \lambda(K_{2t-1}^3) \), equality holds if and only if \( G = G[A] \) and \( \lambda(G[A]) = \lambda(K_{2t-1}^3) \).

Proof. Let \( O_s \) be the 3-graph whose vertex set is \( \{a_1, b_1, a_2, b_2, \ldots, a_s, b_s\} \), and edge set is \( \{a_i b_i a_j, a_i b_j b_i | i \neq j, 1 \leq i, j \leq s\} \). By Fact 5.7 and the definition of a good 3-graph, we have \( E(G) \subset E(G[A]) \cup E(O_s) \cup \{a_i b_I a | a \in A, 1 \leq i, j \leq s\} \).

Let the sums of weights of \( \bigcup_{i=1}^{t} \{a_i, b_i\} \) in an optimal weighting is \( a \). Then

\[
\lambda(G) \leq \lambda(O_s) a^3 + \frac{a^2}{4}(1-a) + \lambda(K_{2t-1}^3)(1-a)^3.
\]

Let’s prove the following fact.

**Fact 5.9** \( \lambda(O_s) = \frac{1}{10} \).

Proof. By Fact 2.4 we may assume that \( a_i, b_i \) have the same weighting (say \( w_i \)) in an optimal weighting. Then

\[
\sum_{i=1}^{s} w_i = \frac{1}{2},
\]

and

\[
\lambda(O_s) = \sum_{i=1}^{s} w_i^2 (1-2w_i).
\]

By Fact 2.2 we know that if \( w_i, w_j > 0 \), then

\[
3\lambda(O_s) = w_i(1-2w_i) = w_j(1-2w_j),
\]

13
i.e. 
\[(w_i - w_j)(1 - 2(w_i + w_j)) = 0.\]

So either \(w_i = w_j\) or \(w_i + w_j = \frac{1}{2}\). If \(w_i + w_j = \frac{1}{2}\), then all other \(w_k = 0\) and this is \(O_2\) which is isomorphic to \(K_4^3\). It’s easy to check that \(\lambda(O_2) = \frac{1}{16}\). So we only need to verify the case that \(w_1 = w_2 = \cdots = w_k = \frac{1}{2k}\). In this case, \(\lambda(O_k, \overrightarrow{F}) = k(\frac{1}{16k})^2(1 - \frac{1}{k}) = \frac{k^2}{16k^2} \leq \frac{1}{16}\) for \(k \geq 2\). \(\square\)

Let us continue the proof of Fact 5.8. Applying Fact 5.9 to (1), we have
\[\lambda(G) \leq \frac{1}{16}a^3 + \frac{a^2}{4}(1 - a) + m(1 - a)^3 \quad (m = \lambda(K_{2t-1}^3) = \frac{(2t-2)(2t-3)}{6(2t-1)^2}). \quad (2)\]

**Case 1.** \(t \geq 4\).

By (2),
\[\lambda(G) \leq \frac{1}{12}a^3 + \frac{a^2}{4}(1 - a) + m(1 - a)^3 \]
\[\quad = -(\frac{1}{6} + m)a^3 + (3m + \frac{1}{4})a^2 - 3ma + m = f(a).\]

Then \(f'(a) = -(\frac{1}{2} + 3m)a^2 + (6m + \frac{3}{2})a - 3m\), and the zeros of \(f'(a)\) are \(\frac{6m}{6m+1}\) and 1. Since \(f'(a)\) opens down, then \(f(a)\) decrease in \([0, \frac{6m}{6m+1}]\) and increase in \([\frac{6m}{6m+1}, 1]\). So
\[f(a) \leq Max\{f(0), f(1)\} = \{m, \frac{1}{12}\} = m\]
since \(m > \frac{12}{25}\). So \(\lambda(G) \leq m = \lambda(K_{2t-1}^3)\). Equality holds if and only if \(a = 0\) and \(\lambda(G[A]) = \lambda(K_{2t-1}^3)\).

**Case 2.** \(t = 3\).

In this case, \(m = \frac{2}{25}\). In view of (2),
\[\lambda(G) \leq \frac{1}{16}a^3 + \frac{a^2}{4}(1 - a) + \frac{2}{25}(1 - a)^3 \]
\[\quad = \frac{1}{400}(-107a^3 + 196a^2 - 96a + 32) = g(a).\]

Then \(g'(a) = \frac{1}{400}(-321a^2 + 392a - 96)\) and the zeros of \(g'(a)\) are \(x_1 = (96 - \sqrt{7600})/321\) and \(x_2 = (96 + \sqrt{7600})/321\). Consequently, \(g(a)\) decreases in \([0, x_1]\), increases in \([x_1, x_2]\) and decreases in \([x_2, 1]\). So
\[g(a) \leq Max\{g(0), g(x_2)\} = \frac{2}{25}\]
and consequently \(\lambda(G) \leq \frac{2}{25}\). Equality holds if and only if \(a = 0\) and \(\lambda(G[A]) = \lambda(K_3^3)\). \(\square\)

### 5.1 Lagrangian density of \(C_3^3\)

We give the proof of Theorem 1.10 in this section. Let \(G\) be a dense and \(C_3^3\)-free 3-graph, we show that \(G\) is a good graph satisfying Fact 5.8 and obtain the conclusion by applying Fact 5.8.

**Proof of Theorem 1.10.** By Remark 2.7, we may assume that \(G\) is dense. By a result of de Caen in [4], \(\pi(K_3^-) \leq \frac{1}{3}\). Since \(K_3^-\) cover pairs, by Proposition 1.14 \(\pi_0(K_3^-) \leq \frac{1}{3}\). If \(\lambda(G) > \frac{2}{25}\), then \(G\) contains \(K_3^-\). Let us assume that \(\{123, 124, 134\} \subseteq G\). We will show that \(G\) is a good graph and apply Fact 5.8.
Case 1 Assume $G$ doesn’t contain an isomorphic copy of $\{123, 124, 134, x12\}$.

Case 1.1 $G$ contains $\{123, 124, 134, x23\}$.

Then $xy1 \in E(G)$ for $y \in V(G) \setminus \{1, 2, 3, y\}$, otherwise $xy1, x23, 124$ form a $C^3_3$. So, $N(x, 1) \subseteq \{2, 3, y\}$, however this contradicts to the assumption of Case 1.

Case 1.2 $G$ doesn’t contain a copy isomorphic to $\{123, 124, 134, x23\}$.

In this case, we have $x23, x24, x34, x12, x13, x14 \not\in E(G)$ for any $x$. Then there exists $y \not\in \{2, 3, y\}$ such that $xy1 \in E(G)$. We claim that $N(x, 2) = N(x, 3) = N(x, 4) = \{y\}$. Otherwise, let $y_1 \in N(x, 2)$ and $y \neq y_1$, clearly $y_1 \not\in \{1, 3, y\}$. Then $xy1, xy1, 123$ form a $C^3_3$, a contradiction. Similarly, the same holds for $N(x, 3)$ and $N(x, 4)$. Let $A = \{1, 2, 3, y\}$, then what we have obtained is that for each $a$ there exists $y$ such that $\{x, y\}$ is a good pair to $A$, so $G$ is a good graph to $A$. Since $\lambda(G[\{1, 2, 3, y\}]) < \frac{2}{25}$, then $\lambda(G) < \frac{2}{25}$ by Fact 5.8.

Case 2 Assume that $G$ contains $\{x12, 123, 124, 134\}$ for some $x \in V(G)$.

Case 2.1 $x23 \in E(G)$.

In this case we have the following observation.

Observation 5.10 For any $a \in V(G) \setminus \{1, 2, 3, y\}$, there exists $b \in V(G) \setminus \{1, 2, 3, y\}$ such that $ab3 \in E(G)$.

Proof. Note that $a13 \not\in E(G)$ since otherwise $a13, x23, 124$ form a $C^3_3$, $a23 \not\in E(G)$ since otherwise $a23, x12, 134$ form a $C^3_3, a34 \not\in E(G)$ since otherwise $a34, x23, 124$ form a $C^3_3$, and $ax3 \not\in E(G)$ since otherwise $ax3, x12, 134$ form a $C^3_3$. So there exists $b \in V(G) \setminus \{1, 2, 3, y\}$ such that $b \in N(a, 3)$ since $G$ covers pair.

Now we fix such a $b$.

Observation 5.11 $N(a, 1) = N(a, 2) = N(a, 3) = N(a, 4) = N(a, x) = \{b\}$.

Proof. Note that $a12 \not\in E(G)$ since otherwise $a12, ab3, 134$ form a $C^3_3$, $a13 \not\in E(G)$ since otherwise $a13, x23, 124$ form a $C^3_3, a14 \not\in E(G)$ since otherwise $a14, ab3, 123$ form a $C^3_3, ax1 \not\in E(G)$ since otherwise $ax1, ab3, 134$ form a $C^3_3$, and $ay1 \not\in E(G)$ for $y \in V(G) \setminus \{1, 2, 3, y\}$ since otherwise $ay1, ab3, 134$ form a $C^3_3$. So $N(a, 1) = \{b\}$.

We have shown that $a12, a23 \not\in E(G)$. Note that $a24 \not\in E(G)$ since otherwise $a24, ab3, 123$ form a $C^3_3, ax2 \not\in E(G)$ since otherwise $ax2, ab3, 123$ form a $C^3_3$, and $ay2 \not\in E(G)$ for $y \in V(G) \setminus \{1, 2, 3, y\}$ since otherwise $ay2, ab3, 123$ form a $C^3_3$. So $N(a, 2) = \{b\}$.

We have shown that $a14, a24, a34 \not\in E(G)$. Note that $ax4 \not\in E(G)$ since otherwise $ax4, ab3, 134$ form a $C^3_3$, and $ay4 \not\in E(G)$ for $y \in V(G) \setminus \{1, 2, 3, y\}$ since otherwise $ay4, ab3, 134$ form a $C^3_3$. So $N(a, 4) = \{b\}$.

We have shown that $\{ax1, ax2, ax3, ax4\} \cap E(G) = \emptyset$. Note that $axy \not\in E(G)$ for $y \in V(G) \setminus \{1, 2, 3, y\}$ since otherwise $axy, ab3, x23$ form a $C^3_3$. So $N(a, x) = \{b\}$.

By Observation 5.11 we know that $b \in N(a, 3)$. Assume that there exist $b' \neq b$ such that $ab13 \in E(G)$.

Then $ab13, ab1, 123$ form a $C^3_3$. So $N(a, 3) = \{b\}$.

Case 2.2 $G$ doesn’t contain a copy isomorphic to $\{x23, x12, 123, 134, 124\}$.  


In this case, we know that \(x_23, x_24 \notin E(G)\). Then \(N(x, 3) \subseteq \{1, 4\}\) since otherwise let \(y \in N(x, 3)\) and \(y \in V(G) \setminus \{1, 2, 3, 4, x\}\), then \(xy_3, x_23, 134\) form a \(C_3^1\).

**Case 2.2.1** \(x_3 \in E(G)\).

**Case 2.2.1.1** There exists \(a \notin \{1, 2, 3, 4, x\}\) such that \(ax_1 \in E(G)\).

In this case, we know that \(ax_2 \notin E(G)\) since otherwise \(\{x_12, 123, x_13, ax_1, ax_2\}\) is isomorphic to \(\{123, 124, 134, x_12, x_23\}\). And \(a_12, a_13, a_14 \in E(G)\) since otherwise if \(1 \in N(a, 2)\), then there exist \(y \notin \{1, 2, x\}\) such that \(ay_2 \in E(G)\), however \(ax_1, ay_2, 123\) form a \(C_3^1\) (change 123 to 124 if \(y = 3\)). Similarly the same holds for \(N(a, 3)\) and \(N(a, 4)\). We claim that \(x_14 \in E(G)\) since otherwise there exists \(z \in N(x, 4) \setminus \{1\}\), then \(zx_4, ax_1, 134\) form a \(C_3^1\) (change 134 to 124 if \(z = 3\)). Let \(A\) be a maximal set containing \(\{2, 3, 4, x, a\}\) such that \(1A^2 \subseteq E(G)\) i.e. \(1ij \in E(G)\) for all \(i, j \in A\) and \(i \neq j\). We have the following claim.

**Claim 5.12** For any \(a_i \in V(G) \setminus (A \cup \{1\})\), there exists exactly one \(b_i\) such that \(\{a_i, b_i\}\) is a good pair to \(A \cup \{1\}\) of \(G\).

**Proof.** We claim that \(a_1k \notin E(G)\) for any \(k \in A\) and \(a_i \in V(G) \setminus (A \cup \{1\})\). If the conclusion does not hold i.e. there exist \(a_i \in V(G) \setminus (A \cup \{1\})\) such that \(a_1l_j \in E(G)\) for some \(j \in A\). Without loss of generality, let \(a_1, 12 \in E(G)\). Consider \(N(a_i, 3)\), note that \(t \notin N(a_i, 3)\) for \(t \in A\) since otherwise \(a_t, 1a, 132\) form a \(C_3^1\) (change \(a\) to \(2\) if \(t = a\) or change \(x\) to \(2\) if \(t = x\)). And \(N(a_i, 3) \cap (V(G) \setminus (A \cup \{1\})) = \emptyset\) since otherwise if \(y \in N(a_i, 3) \cap (V(G) \setminus (A \cup \{1\}))\) then \(a_3y_3, a_1, 134\) form a \(C_3^1\). So \(N(a_i, 3) = \{1\}\). Similarly \(N(a_i, k) = \{1\}\) for all \(k \in A\), a contradiction to the maximality of \(A\). So for any \(a_i \in V(G) \setminus (A \cup \{1\})\) there exist \(b_i \in V(G) \setminus (A \cup \{1\})\) such that \(b_i \in N(a_i, 1)\). And \(b_i \in N(a_i, k)\) for \(k \in A\) since otherwise if \(b_i \notin N(a_i, k)\) for some \(k\), then there exists \(b_i' \in N(a_i, k)\), consequently \(a_i b_i^3, a_i b_i 1, 134\) form a \(C_3^1\). So \(N(a_i, k) = \{b_i\}\). If there exist \(b_i'' \neq b_i\) such that \(b_i'' \in N(a_i, 1)\), then \(a_i b_i'' 1, a_i b_i 2, 123\) form a \(C_3^1\), a contradiction. So \(N(a_i, 1) = b_i\) as well and we have shown that \(\{a_i, b_i\}\) is a good pair to \(A \cup \{1\}\). \(\square\)

By the definition of \(A\), \(S_k^1(1) \subseteq G[A \cup \{1\}]\), where \(k = |A| + 1\). And any edge in \(A\) would lead to a \(C_3^1\), so \(G[A \cup \{1\}] = S_k^1(1)\) and it’s easy to get that \(\lambda(G[A \cup \{1\}]) < \frac{2}{k} \). By Fact 5.8, \(\lambda(G) < \frac{2}{k}\).

**Case 2.2.1.2** For any \(a \in V(G) \setminus \{1, 2, 3, 4, x\}\), \(ax_1 \notin E(G)\).

In this case we claim that

**Claim 5.13** There exists \(b \in V(G) \setminus \{1, 2, 3, 4, x\}\) such that \(\{a, b\}\) is a good pair to \(\{1, 2, 3, 4, x\}\).

**Proof.** Let’s consider \(N(a, x)\). Note that \(2 \notin N(a, 1)\) since otherwise \(ax_2, x_12, 124\) form a \(C_3^1\), \(3 \notin N(a, x)\) since otherwise \(ax_3, x_12, 134\) form a \(C_3^1\), and \(4 \notin N(a, x)\) since otherwise \(ax_4, x_12, 134\) form a \(C_3^1\). So there exists \(b \in V(G) \setminus \{1, 2, 3, 4, x\}\) such that \(abx \in E(G)\).

Consider \(N(a, 1)\). Note that \(2 \notin N(a, 1)\) since otherwise \(abx, a_12, x_12, 134\) form a \(C_3^1\), \(3 \notin N(a, 1)\) since otherwise \(abx, ax_12, x_12, 134\) form a \(C_3^1\), \(4 \notin N(a, 1)\) since otherwise \(abx, a_14, x_12, 134\) form a \(C_3^1\), and \(N(a, 1) \cap (V(G) \setminus \{1, 2, 3, 4, x, b\}) = \emptyset\) since otherwise \(abx, ay_1, x_12, 134\) form a \(C_3^1\) for some \(y \in V(G) \setminus \{1, 2, 3, 4, x, b\}\). So \(N(a, 1) = \{b\}\).

If \(N(a, 2) \neq \{b\}\), then there exists \(b' \in N(a, 2)\). Note that \(b' \neq 1, x\) since we have shown that \(a_12, ax_2 \notin E(G)\), then \(ab_1'' \notin E(G)\). So \(N(a, 2) = \{b\}\).

If \(N(a, 3) \neq \{b\}\), then there exists \(b'' \notin N(a, 3)\). Note that \(b'' \neq 1, x\) since we have shown that \(a_13, ax_3 \notin E(G)\), then \(ab'' \notin E(G)\). So \(N(a, 3) = \{b\}\).

If \(N(a, 4) \neq \{b\}\), then there exists \(b''' \notin N(a, 4)\). Note that \(b''' \neq 1, 3\) since we have shown that \(a_14, a_34 \notin E(G)\), then \(ab''' \notin E(G)\). So \(N(a, 4) = \{b\}\).
Therefore \( \{a,b\} \) is a good pair to \( \{1,2,3,4,x\} \).

By Claim 5.13 \( G \) is a good graph to \( \{1,2,3,4,x\} \). Since \( \lambda(G[1,2,3,4,x]) \leq \frac{2}{25} \), then \( \lambda(G) \leq \frac{2}{25} \) by Fact 5.8. Equality holds if and only if \( G[1,2,3,4,x] = K_3^3 \).

Case 2.2.2 \( x3 \notin E(G) \).

Recall that \( N(3,x) \subseteq \{1,4\} \), then \( x34 \in E(G) \).

**Observation 5.14** For every \( a \in V(G) \setminus \{1,2,3,4\} \) there exists \( b \in V(G) \setminus \{1,2,3,4,a\} \) such that \( N(a,b) \supseteq \{b\} \).

**Proof.** Note that \( ax1 \notin E(G) \) since otherwise \( ax1,x34,123 \) form a \( C_3^3 \), \( ax2 \notin E(G) \) since otherwise \( ax2,x34,123 \) form a \( C_3^3 \), \( ax3 \notin E(G) \) since otherwise \( ax3,x12,134 \) form a \( C_3^3 \), and \( ax4 \notin E(G) \) since otherwise \( ax4,x12,134 \) form a \( C_3^3 \). So there exists \( b \in V(G) \setminus \{1,2,3,4,a\} \) such that \( abx \in E(G) \). \( \square \)

**Claim 5.15** Let \( A = \{a : a12, a34 \in E(G)\} \), then \( G \) is a good graph to \( A \cup \{1,2,3,4\} \).

**Proof.** Note that \( A \neq \emptyset \) since \( x \in A \). We will prove an observation first.

**Observation 5.16** For any \( y \in V(G) \setminus (A \cup \{1,2,3,4\}) \), \( y12 \notin E(G) \).

**Proof.** If \( y12 \in E(G) \), then consider \( N(y,3) \). Take \( a \in A \). If \( 1 \in N(y,3) \), then \( a12, y13, a34 \) form a \( C_3^3 \).

If \( 2 \in N(y,3) \), then \( y3, a12, a34 \) form a \( C_3^3 \). If \( a \in N(y,3) \), then \( ay3, a12, a34 \) form a \( C_3^3 \). If \( x' \in N(y,3) \), where \( x' \notin A \cup \{1,2,3,4\} \), then \( x'y3, y12, a34 \) form a \( C_3^3 \). So \( N(y,3) = \{4\} \), then \( y \in A \), a contradiction to the maximality of \( A \). \( \square \)

For any \( z \in V(G) \setminus (A \cup \{1,2,3,4\}) \), we consider \( N(z,1) \), note that \( 2 \notin N(z,1) \) by Observation 5.16. Take \( a \in A \). Note that \( 3 \notin N(z,1) \) since otherwise \( z13, a34, a12 \) form a \( C_3^3 \). If \( 4 \notin N(z,1) \) since otherwise \( z14, a34, a12 \) form a \( C_3^3 \). If \( a \notin N(z,1) \) since otherwise \( a12, a34, 123 \) form a \( C_3^3 \). So there exists \( u \in V(G) \setminus A \cup \{1,2,3,4\} \) such that \( uz1 \in E(G) \).

Consider \( N(z,2) \). If \( u \notin N(z,2) \), then there exists \( v \in N(z,2) \) and \( vz2, uz1,123 \) form a \( C_3^3 \) (change 123 to 124 if \( v = 3 \)). So \( N(z,2) = \{u\} \). Similarly, we have \( N(z,4) = \{u\} \), and \( N(z,a) = \{u\} \) for \( a \in A \). So \( \{z,u\} \) is a good pair to \( A \cup \{1,2,3,4\} \). Hence \( G \) is a good graph to \( A \cup \{1,2,3,4\} \). \( \square \)

Since \( G \) is a good graph to \( A \cup \{1,2,3,4\} \), then for \( a \in A \), \( N(a,1) \subseteq \{2,3,4\} \). If \( a13 \in E(G) \) or \( a14 \in E(G) \), then it is Case 2.2.1 since \( a12 \in E(G) \). So we may assume that \( N(a,1) = \{2\} \). Similarly, we can show that \( N(a,2) = \{1\} \). Consequently, \( N(a,3) = \{4\} \) and \( N(a,4) = \{3\} \). Hence \( G \) is a good graph to \( A \). Applying the same procedure to \( A \). If \( \lambda(G[A]) < \frac{2}{25} \), then by Fact 5.8 we know that Theorem 1.10 holds. Else if \( \lambda(G[A]) \geq \frac{2}{25} \), then applying the same procedure to \( G[A] \) implies that there exist 4 vertices \( \{5,6,7,8\} \) such that \( \{5,6\} \) and \( \{7,8\} \) are good pairs to \( A^{(i)} = A \setminus \{5,6,7,8\} \). Continue this procedure, until we obtain that \( \lambda(G[A]) < \frac{2}{25} \) or \( |A^{(i)}| \leq 5 \) and \( G \) is a good graph to \( A^{(i)} \). By Fact 5.8, \( \lambda(G) \leq \frac{2}{25} \), and equality holds if and only if \( G[\{1,2,3,4,x\}] = K_5^3 \). \( \square \)

### 5.2 Extension to \( C_3^t \)

In this section we prove Theorem 1.12. Let \( G \) be a dense and \( C_3^3 \)-free 3-graph with \( K_3^3 \setminus 2 \in G \) (\( t \geq 4 \)). Without loss of generality, let \( \{2t-2\} \setminus \{123\} \) be a \( K_3^3 \setminus 2 \) in \( G \). We will show that \( G \) is a good graph satisfying Fact 5.8 and obtain the conclusion of Theorem 1.12 by applying Fact 5.8.
**Fact 5.17** Let $G$ be a dense and $C^3_2$-free 3-graph. Let $|V(G)| \leq t-1$, then $\lambda(G) \leq \lambda(K^3_{2t-1})$, equality holds if and only if $G = K^3_{2t-1}$. Without loss of generality, assume that $|V(G)| \geq 2t$. If $V(G) \varsubsetneq \{123\} \cup \{45\}$, then $\lambda(G) < \lambda(K^3_{2t-1})$.

**Proof.** Let $G$ be a dense and $C^3_2$-free 3-graph. Then $|V(G)| \leq t-1$, then $\lambda(G) \leq \lambda(K^3_{2t-1})$, equality holds if and only if $G = K^3_{2t-1}$. Without loss of generality, assume that $|V(G)| \geq 2t$. If $V(G) \varsubsetneq \{123\} \cup \{45\}$, then $\lambda(G) < \lambda(K^3_{2t-1})$.

**Case 1** $G$ doesn't contain a copy isomorphic to $\{x14, \{2t-2\} \setminus \{123\} \cup \{45\}$, where $x \in V(G) \setminus \{2t-2\}$.

In this case, we know that $xk1, xk2, xk3 \notin E(G)$ for $4 \leq k \leq 2t-2$ and $x \in V(G) \setminus \{2t-2\}$.

**Case 1.1** $G$ contains a copy isomorphic to $\{x45, \{2t-2\} \setminus \{123\}\}$.

**Case 1.1.1** $G$ doesn't contain a copy isomorphic to $\{x45, x46, \{2t-2\} \setminus \{123\}\}$.

Assume that $x45 \in E(G)$, then $x46 \notin E(G)$. If $y \in N(x, 6) \cap (V(G) \setminus \{2t-2\})$, then $x46, x45, 612$ together with a $P_{l-3}$ in $\{2t-2\} \setminus \{2, 5, 6\}$ connecting 1 and 4 form a $C^3_2$. Without loss of generality, let $x67 \in E(G)$, then similarly we have $x89, x(10)(11), \ldots, x(2t-4)(2t-3) \in E(G)$, however $N(x, 2t-2) = \emptyset$, a contradiction.

**Case 1.1.2** $G$ contains a copy isomorphic to $\{x45, x46, \{2t-2\} \setminus \{123\}\}$.

Without loss of generality, assume that $x45, x46 \in E(G)$. Then for any $a \in V(G) \setminus \{2t-2\}$, $x \notin N(a, 5)$ since otherwise $x05, x46$ together with a $P_{l-2}$ on $\{2t-2\} \setminus \{4\}$ connecting 5 and 6 form a $C^3_2$. Moreover $k \notin N(a, 5)$ for $k \in \{2t-2\} \setminus \{5\}$, otherwise $xk5, x45$ together with a $P_{l-2}$ on $\{2t-2\} \setminus \{5\}$ connecting 4 and $k$ form a $C^3_2$ (change $x45$ to $x46$ if $k = 4$). Then there exist $b \in V(G) \setminus V(C^3_{2t-2})$ such that $ab5 \in E(G)$. By Fact 5.17, $\{a, b\}$ is good pair to $\{2t-2\}$ such that $N(a, k) \notin \{5\}$ for $k \in \{2t-2\} \setminus \{i\}$. Then By Fact 5.8 $\lambda(G) < \lambda(K^3_{2t-1})$.

**Case 1.2** $G$ doesn't contain a copy isomorphic to $\{x45, 2t-2 \setminus \{123\}\}$.

In this case, for any $a \in V(G) \setminus \{2t-2\}$, there exists $b \in V(G) \setminus \{2t-2\}$ such that $abi \in E(G)$ and $N(a, k) \notin \{i\}$ for any $k \in \{2t-2\} \setminus \{i\}$. By Fact 5.17, $\{a, b\}$ is a good pair to $\{2t-2\}$. By Fact 5.8 $\lambda(G) < \lambda(K^3_{2t-1})$.

**Case 2** $G$ contains a copy isomorphic to $\{x41, \{2t-2\} \setminus \{123\}\}$.

Without loss of generality, assume that $x41, x45 \in E(G)$. Then for any $a \in V(G) \setminus \{2t-2\}$, we know that $x \notin N(a, 5)$ since otherwise $x41, x05$ together with a $P_{l-2}$ on $\{2t-2\} \setminus \{1\}$ connecting 4 and 5 form a $C^3_2$. Moreover $k \notin N(a, 5)$ for $k \in \{2t-2\} \setminus \{5\}$, since otherwise $xk5, x45$ together with a $P_{l-2}$ on $\{2t-2\} \setminus \{5\}$ connecting 4 and $k$ form a $C^3_2$ (change $x45$ to $x41$ if $k = 4$). Then there exist $b \in V(G) \setminus V(C^3_{2t-2})$ such that $ab5 \in E(G)$. Since we have already known that $N(a, k) \notin \{5\}$, then by Fact 5.17 $\{a, b\}$ is a good pair to $\{2t-2\}$. Applying Fact 5.8 we have $\lambda(G) < \lambda(K^3_{2t-1})$.
Case 2.2 $G$ contains a copy isomorphic to $\{x_{41}, x_{15}, [2t-2]^3 \setminus \{123\}\}$.
Without loss of generality, we can assume that $x_{15}, x_{41} \in E(G)$. Then for any $a \in V(G) \setminus [2t-2]$, we know that $x \notin N(a,5)$ since otherwise $x_{41}, xa5$ together with a $P_{1-2}$ on $[2t-2] \setminus \{1\}$ connecting 4 and 5 form a $C^3_1$. Moreover $k \notin N(a,5)$ for $k \in [2t-2] \setminus \{5\}$, since otherwise $ak5, xa15$ together with a $P_{1-2}$ on $[2t-2] \setminus \{5\}$ connecting 1 and $k$ form a $C^3_2$. Hence $x \notin 2\{1\}$ since otherwise $xy5, x141$ together with a $P_{1-2}$ on $[2t-2] \setminus \{1,4\}$ such that $xk5 \in E(G)$. Then for any $a \in V(G) \setminus [2t-2]$, we have $x, 1, 4 \notin N(a,5)$ since otherwise $x_{41}, ai5$ is a good pair to $[2t-2]$ and a copy of $C^3_2$. Note that $k \neq 1$, then $2, 3, k \notin N(a,5)$ since $x_{41}, ai5 \in E(G)$ is a good pair to $[2t-2]$. Moreover $y \notin N(a,5)$ for $y \in [2t-2] \setminus \{1,2,3,4,5,k\}$ since otherwise $ay5, x5k$ together with a $P_{1-2}$ in $[2t-2] \setminus \{5\}$ form a $C^3_3$. So there exists $b \in V(G) \setminus [2t-2]$ such that $ab5 \in E(G)$. Since $5 \notin N(a,k)$, then by Fact $5.17$ $\{a,b\}$ is a good pair to $[2t-2]$. By Fact $5.8$ $\lambda(G) < \lambda(K^3_{2t-1})$.

Case 2.3 $G$ does not contain a copy isomorphic to $\{x_{41}, x_{45}, [2t-2]^3 \setminus \{123\}\}$.
Without loss of generality, we assume that $x_{41} \in E(G)$, $x_{15}, x_{45} \notin E(G)$. Note that $N(x,5) \cap (V(G) \setminus [2t-2]) = \emptyset$. Otherwise let $y \in N(x,5) \cap (V(G) \setminus [2t-2])$, then $xy5, x41$ together with a $P_{1-2}$ on $[2t-2] \setminus \{1\}$ connecting 4 and 5 form a $C^3_1$. So there exist $k \in [2t-2] \setminus \{1,4\}$ such that $xk5 \in E(G)$. Then for any $a \in V(G) \setminus [2t-2]$, we have $x, 1, 4 \notin N(a,5)$ since otherwise $x_{41}, ai5$ is a good pair to $[2t-2]$ and a copy of $C^3_2$. Note that $k \neq 1$, then $2, 3, k \notin N(a,5)$ since $x_{41}, ai5 \in E(G)$ is a good pair to $[2t-2]$. Moreover $y \notin N(a,5)$ for $y \in [2t-2] \setminus \{1,2,3,4,5,k\}$ since otherwise $ay5, x5k$ together with a $P_{1-2}$ in $[2t-2] \setminus \{5\}$ form a $C^3_3$. So there exists $b \in V(G) \setminus [2t-2]$ such that $ab5 \in E(G)$. Since $5 \notin N(a,k)$, then by Fact $5.17$ $\{a,b\}$ is a good pair to $[2t-2]$. By Fact $5.8$ $\lambda(G) < \lambda(K^3_{2t-1})$.

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