We employ Riesz’s fractional derivative into the Wheeler–DeWitt equation for a closed de Sitter geometry and obtain the no-boundary and tunneling wavefunctions. From the corresponding probability distributions, the event horizon of the nucleated universe can be a fractal surface with dimensions between $2 \leq D < 3$. Concretely, the tunneling wavefunction favors fractal dimensions less than 2.5 and an accelerated power-law phase. Differently, the no-boundary proposal conveys fractal dimensions close to 3, with the universe instead entering a decelerated phase. Subsequently, we extend our discussion towards (non-trivial compact) flat and open scenarios. Results suggest that given the probability of creation of a closed inflationary universe in the tunneling proposal is exponentially suppressed, a flat or an open universe becomes favored within fractional inflationary quantum universe.

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I. INTRODUCTION

The entire visible universe appears to be highly unique in a way that is not suggested solely by known dynamical laws. Being more specific, from displaying a small large-scale curvature and approximate homogeneity and isotropy of the matter distribution on the largest current scales, the universe seems to demand judicious boundary conditions for its emergence. Namely, an exceptionally high degree of order in its very early stage. From such, it will then be evolving with increasing entropy, aligned with the second law of thermodynamics. The Hartle–Hawking ‘no-boundary’ proposal [1, 2] and the Linde–Vilenkin ‘tunneling’ proposal [3, 4] are two popular proposals for quantum states of the model universes that accommodate this perspective [5].

Herewith, we will consider the above context with the assistance of a new probe, not yet given widespread use. More concretely, a series of arguments and findings using fractional calculus in quantum physics have been recently presented [6–8]. The primary rationale is that if we limit the path integral description of quantum mechanics to Brownian paths exclusively, explaining concrete essential quantum phenomena would be complicated [12]. Due to these issues, extended variants of the Schrödinger equation (SE), have been considered, where derivatives of fractional order, e.g., $\partial^\alpha/\partial x^\alpha, 1 < \alpha \leq 2$, $\alpha$ being a rational number, are used. Such tool from fractional calculus assists fractional quantum mechanics (FQM), where space-fractional [13], time-fractional [14], and space-time-fractional [15] variants of the ordinary SE have been the subject of attention. Furthermore, in the last few years, FQM has been employed as a means to explore features within quantum field theory and gravity for fractional space-time [16, 17], and the fractional quantum field theory at positive temperature [18, 19]. It has pointed to exciting opportunities; please see [16–21], for a recent survey.

The purpose of this paper is therefore to employ a particular fractional WDW equation of the de Sitter (deS) space, embracing it as a probe model. We are aware of the restrictive scope but absorbing features can be retrieved. More concretely, for a closed de Sitter geometry, we obtain the fractional tunneling (Linde–Vilenkin) wavefunction and the fractional extension of the no-boundary wavefunction (Hartle–Hawking), followed by computing the corresponding probability distributions. Subsequently comparing those expressions with the corresponding usual nucleation rates of the Linde–Vilenkin and for the Hartle–Hawking deS wavefunction, our results suggest that a fractal behavior for the de Sitter horizon can be estimated: the tunneling (Linde–Vilenkin) wavefunction favors fractal dimensions less than 2.5; whereas in the no-boundary proposal (Hartle–Hawking) case, the corresponding wavefunction selects a fractal horizon with dimensions close to 3. Furthermore, after nucleating, the universe may either enter an accelerated power-law phase (Linde–Vilenkin wavefunction) or may instead proceed into a decelerated phase (Hartle–Hawking case). In addition, our analysis allows to discuss (non-trivial compact) flat and open scenarios. Since the probability of creation of a closed inflationary universe in the tunneling proposal is exponentially suppressed, a flat or an open universe emerges favored within fractional
inflationary quantum universe.

Let us start by briefly summarizing the canonical quantization of a deS universe.

II. CANONICAL QUANTIZATION OF A DE SITTER UNIVERSE

The minisuperspace ADM action for a deS universe with a positive constant curvature geometry is given by \[ S_{\text{ADM}} = \frac{3\pi}{4G} \int \left( -\frac{a(t)}{N(t)} \dot{a}(t)^2 + a(t)N(t) - \frac{\Lambda}{3} N(t)a(t)^3 \right) dt, \] where \( G \) is the gravitational constant, \( \Lambda \) is the cosmological constant, \( N(t) \) is the lapse function, and \( a(t) \) represents the scale factor. The minisuperspace ADM Hamiltonian for the deS universe with a positive constant curvature geometry is then given by \[ H_{\text{ADM}} = -N \left\{ \frac{1}{3m_p^2 a(t)} \Pi^2 + \frac{3\pi m_p^2}{4} (a(t) \left( 1 - \frac{\Lambda}{3}a(t)^2 \right) \right\}, \] where \( m_p = 1/\sqrt{G} \) is the Planck mass, and \[ \Pi = -\frac{3\pi m_p^2 a(t) \dot{a}(t)}{2N(t)}, \] is the conjugate momentum of the scale factor, \( a(t) \). The WDW equation for the wavefunction of the deS universe is retrieved as \[ \left\{ -a^p \frac{d}{da} a^p \frac{d}{da} + \left( \frac{3\pi m_p^2}{2} \right)^2 a^2 \left( 1 - \frac{\Lambda}{3}a^2 \right) \right\} \Psi(a) = 0, \] where the parameter \( p \) represents the factor-ordering ambiguity. In this paper, we are interested in a semi-classical approximation, so we neglect the operator-ordering, and hence we set \( p = 0 \). The WKB Linde–Vilenkin wavefunction of the above one-dimensional WDW equation, disregarding the preexponential factor, is \[ \psi_{\text{LV}}(a) = \begin{cases} e^{\int_L^a \left| \Pi(a') \right| da'}, & a < L, \\ e^{-i \int_L^a \Pi(a') da' + i \frac{\pi}{4}}, & a \geq L, \end{cases} \] where \( L = \sqrt{3/\Lambda} \) is the deS horizon radius. The wavefunction decays exponentially toward \( a = L \) and after that, it oscillates, describing an expanding deS universe. On the other hand, the Hartle–Hawking wavefunction is \[ \psi_{\text{HH}}(x) = \begin{cases} e^{\int_L^a \left| \Pi(a') \right| da'}, & a < L, \\ \cos \left( \int_L^a \Pi(a') da' - \frac{\pi}{4} \right), & a \geq L. \end{cases} \] The nucleation rate for the Linde–Vilenkin wavefunction in this WKB approximation is given by \[ \mathcal{P}_\text{LV} = \frac{|\psi_{\text{LV}}(L)|^2}{|\psi_{\text{LV}}(0)|^2} \propto e^{-2\int_0^L |\Pi(a')| da'} = e^{-S_{\text{DS}}}, \] where \( S_{\text{DS}} \) is the entropy of deS space-time. The nucleation rate for the Hartle–Hawking wavefunction is \[ \mathcal{P}_{\text{HH}} = |\psi_{\text{HH}}(L)|^2 \propto e^{2\int_0^L |\Pi(a')| da'} = e^{S_{\text{DS}}}. \] It can be further added that our deS setting can be taken as a background upon quantum fluctuations may be considered. Albeit somewhat descriptive at this perspective, we can take \( \Lambda \) as an effective cosmological constant, associated as \( \Lambda_{\text{eff}} = 8\pi G V_0 = 8\pi V_0/m_p^2 \) to some potential of some field and in a maximum. Quantum fluctuations may then push such field away from the maximum; such vacuum energy remains essentially constant, \( V = V(0) := V_0 \), if the potential is suitably flat, and the universe then expands exponentially \( a(t) = \exp(t/L) \), where \[ L := \sqrt{\frac{3m_p^2}{8\pi V_0}}. \] From Eqs. 3 and 4, it is clear that the tunneling and the no-boundary states lead to different physical implications. According to the tunneling proposal, the highest probability is obtained for the smallest entropy values. Thus, the tunneling wavefunction ‘predicts’ that the universe is likely to nucleate with the smallest possible entropy. On the contrary, the Hartle–Hawking probability is peaked at the smallest values of the cosmological constant, and thus the corresponding wavefunction tends to predict initial conditions with maximum possible entropy. Or, in other words, the Linde–Vilenkin wavefunction ‘predicts’ that the universe will nucleate with a large vacuum energy. Due to the negative sign in 4, the no-boundary state instead increases the contribution of empty universes with \( V_0 = 0 \) in the entire quantum state, leading to the seemingly contradictory conclusion that indefinitely huge universes are infinitely more likely than finite-sized universes. The probability in equation 3, on the other hand, promotes large values of \( V_0 \) capable of causing deS inflationary scenarios. As a result, it appears that the Linde–Vilenkin prescription is physically more alluring than the no-boundary prescription, as far
as a deS inflation is remarked in this reduced scope of discussion.

Nonetheless, suppose we extend the above as to include an explicit inflaton scalar field. In that case, the situation changes drastically. There is a general domain \( V_0 \leq 10^{-8}m_p^4 \) derived from the amplitude constraint on gravitational waves generated during inflation in all models where the effective potential does not change significantly during the final stages of inflation. Many inflationary models, including new inflation, hybrid inflation predict that inflation will occur at \( V_0 \ll 10^{-8}m_p^4 \). Subsequently, the minimal size of a closed inflationary universe is given by \( 1/H = 3m_p^4 \). The probability of quantum creation of an inflationary universe is suppressed by a factor of \( 10^{-10^{10}} \) in a specific example with \( V_0 = 10^{-12}m_p^4 \). As remarkably shown in Refs. [27–28], the tunneling proposal, allows the creation of a topologically compact nontrivial open or flat universe. Moreover, it seems more likely to occur than the emergence of a closed universe. Since such processes are unaffected by exponential factors, the universe can be created even if the energy density during inflation is much lower than the Planck density. After a long period of inflation, such universes become indistinguishable from isotropic flat universes.

III. DE SITTER FRACTIONAL QUANTUM COSMOLOGY

Recent quantum gravity results have given a significant boost to the growing use of fractional calculus in quantum cosmology. In [31–35], the dimension of spacetime changes with scale. Because fractional integrodifferential operators can describe these processes, using fractal processes in quantum physics is a precursor to incorporating fractional calculus into quantum theory.

Let us now convey our discussion towards a fractional quantum formulation of deS space-time. To obtain a fractional WDW equation, let us see how a fractional SE can be expressed. An analogy can be drawn from the diffusion equation, which is extended to the fractional diffusion equation to describe anomalous diffusion in the case of fractional generalization of quantum mechanics. Because the Schrödinger equation is identical to the diffusion equation up to a few constants, fractional generalization of the Schrödinger equation is also possible. Consider the Hamiltonian of a classical system, \( H = \frac{p^2}{2m} + V(r) \), where \( r \) and \( p \) denote the space coordinates and momentum associated with a particle of mass \( m \), respectively. As previously stated, the Brownian-like quantum mechanical trajectories used in Feynman’s framework are replaced by Lévy-like ones in the Laskin approach. Applying a natural generalization of the preceding classical Hamiltonian as \( H_\alpha(p,r) := D_\alpha|p|^\alpha + V(r) \), where \( 1 < \alpha \leq 2 \) is the Lévy’s fractional parameter and it is associated to the concept of Lévy path [14]. \( D_\alpha \) is a scale coefficient. Applying the standard canonical quantization procedure, \( (r,p) \rightarrow (r,-i\hbar \nabla) \), to this extended Hamiltonian gives the corresponding quantum Hamiltonian of the system \( H_\alpha = D_\alpha(-\hbar^2\Delta)^{\alpha/2} + V(r) \). Therefore, to obtain a fractional extension of the Schrödinger equation,

\[
\frac{i\hbar}{\partial t} \frac{\partial \psi(r,t)}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi(r,t) + V(r,t)\psi(r,t),
\]

we may replace the ordinary Laplacian, \( \Delta \), with the fractional Riesz, \( (-\hbar^2\Delta)^{\alpha/2} \), namely

\[
-\frac{\hbar^2}{2m} \Delta \rightarrow D_\alpha(-\hbar^2\Delta)^{\alpha/2}.
\]

The fractional Laplacian in the Riesz form, \( (-\hbar^2\Delta)^{\alpha/2} \), for 3-dimensional Euclidean space is defined in terms of the Fourier transformation [37]

\[
(-\hbar^2\Delta)^{\alpha/2} \psi(r,t) = \mathcal{F}^{-1}[|p|^{\alpha} \mathcal{F} \psi(r,t)]
\]

\[= \frac{1}{(2\pi\hbar)^3} \int d^3p |p|^{\alpha} \mathcal{F} \psi(r',t) d^3r',
\]

in which \( p = \sqrt{p_1^2 + p_2^2 + p_3^2} \). The result is the fractional SE

\[
\frac{i\hbar}{\partial t} \frac{\partial \psi(r,t)}{\partial t} = D_\alpha(-\hbar^2\Delta)^{\alpha/2} \psi(r,t) + V(r,t)\psi(r,t).
\]

Returning to our case study, the supermomentum constraint \( H_\text{L} \) is identically zero in homogeneous cosmological models, and the shift function \( N_\text{L} \) can be adjusted to zero without neglecting any of Einstein’s equations. As a result, the superspace WDW equation is simplified to the minisuperspace WDW equation:

\[
\left\{ \frac{1}{2} \Box + U(q^\nu) \right\} \Psi(q^\nu) = 0, \quad \nu = 0, \ldots, N-1,
\]

where \( q^\nu \) are coordinates of \( N \)-dimensional minisuperspace, \( \Box = \sum_{\alpha=1}^{N} \partial_{\alpha} \left( \sqrt{-g} f^{\alpha\beta} \partial_{\beta} \right) \) is the d’Alembert operator, \( f_{\alpha\beta} \) denotes the corresponding minisuperspace metric with signature \((-++\ldots+)\) and \( U(q^\nu) \) is the potential. To obtain a fractional extension of the above WDW equation we may replace the ordinary d’Alembertian operator in [15] as [37–40], namely

\[
\left\{ \Box \right\}^{\alpha/2} \Psi(q^\nu) = \mathcal{F}^{-1}[|p|^{\alpha} \mathcal{F} \Psi(p)],
\]

where \( |p| = \sqrt{-p_1^2 + p_2^2 + p_3^2} \), \( i = 1, 2, \ldots, N \), and \( \mathcal{F} \) is a Fourier transformation. Let us be more clear and specific. The WDW equation [41] is a one-dimensional ES with zero energy. Thus, to construct a particular fractional WDW equation, we may, similarly to the SE case, replace the ordinary derivative for the fractional Riesz derivative, namely [20–21] [14]

\[
-\frac{d^2}{da^2} \rightarrow m_p^{2-\alpha} \left( -\frac{d^2}{da^2} \right)^{\alpha/2}, \quad 1 < \alpha \leq 2.
\]
We remind that the fractional Riesz derivative is further defined in terms of the Fourier transformation \[ \mathcal{F}(\frac{d^2}{dx^2})^{\alpha/2} \Psi(a) = \frac{1}{2\pi} \int d\Pi e^{-i\Pi a} \mathcal{F} \int e^{-i\Pi a} \Psi(a') da'. \]

Therefore, our fractional WDW equation of a deS spacetime is

\[
\left( -\frac{d^2}{da^2} \right)^{\alpha/2} \Psi(x) + \frac{9\pi^2 m_p^{2+\alpha}}{4} \left( a^2 - \frac{\Lambda}{3} a^4 \right) \Psi(a) = 0.
\]

To obtain a WKB approximation, we rewrite the wavefunction as the exponential of another function, \( f \), namely

\[
\Psi(x) = e^{f(x)}, \quad f(x) = \mathbb{C}.
\]

Using (20) and (18) in (19) and assuming the wavefunction is square-integrable, we find

\[
\frac{d^2 f}{dx^2} + \left( \frac{df}{dx} \right)^2 + |\Pi|^2 = 0,
\]

where \( \Pi \) satisfies the following classical fractional super-Hamiltonian constraint

\[
|\Pi|^\alpha + \frac{9\pi^2}{4} m_p^{2+\alpha} a^2 \left( 1 - \frac{a^2}{L^2} \right) = 0.
\]

Within our chosen WKB approximation procedure, we can show that the corresponding WKB wavefunctions in the fractional tunneling or no-boundary case still bear the formal structure as in (15) or (16). Of course, the explicit presence of \( \alpha \) as induced from the use of the Riesz derivative, conveys significant modifications.

The WKB Linde–Vilenkin wavefunction of the above one-dimensional fractional WDW equation, disregarding the preexponential factor, is

\[
\psi_{LV}(a) \simeq \begin{cases} e^{Ca/D} F(\frac{D}{2} , \frac{1-D}{2} I + \frac{D}{2} (\frac{a}{L})^2), & a < L, \\ e^{-iCa/D} F(\frac{D}{2} , \frac{1-D}{2} I + \frac{D}{2} (\frac{a}{L})^2) + i \frac{1}{\Gamma(D/2)} C^{-1}, & a \geq L, \end{cases}
\]

where \( C := (\frac{3\pi}{2})^{D-1} m_p^D \), and \( F(a, b; c; z) \) is the hypergeometric function and \( D = 1 + 2/\alpha \). It is likewise possible to write the Hartle–Hawking wavefunction in terms of hypergeometric functions.

Consider now the Linde–Vilenkin and Hartle–Hawking probability distributions. These are subsequently written as

\[
\mathcal{P}_{LV} = e^{-\left( \frac{a}{L} \right)^{D-1} \Gamma(D/2) \left( \frac{3\pi}{2} \right)^{D/2} \left( \frac{a}{2 \pi} \right)^{3-D}}, \quad \mathcal{P}_{HH} = \mathcal{P}_{LV}^{-1}.
\]

Comparing Eqs. (7) and (8) with the above results, it suggest us to define a fractional entropy as

\[
S_{Frac} := \frac{A_{Frac}}{4G},
\]

where

\[
A_{Frac} := 4\sqrt{\pi} \left( \frac{3\pi}{4} \right)^{D-1} \frac{\Gamma(D)}{\Gamma(D+1/2)} \left( \frac{L}{L_p} \right)^{D},
\]

would be the fractal area of the deS horizon, \( D \) is the fractal dimension of the horizon and \( L_p = 1/m_p \) is the Planck length. Specifically, in the above expression the horizon surface is a fractal surface whose dimension, \( D \), is dependent in the Lévy’s fractional parameter \( \alpha \)

\[
D = \frac{2 + \alpha}{\alpha}, \quad 2 < D < 3.
\]

Note that for \( \alpha = 2 \) (or equivalently, \( D = 2 \)) the fractal area of the horizon will reduce the original smooth area of the deS space. Eqs. (26) and (27) show that increasing the Lévy’s fractional parameter, \( \alpha \), also increases the fractality of the horizon area, which in turns causes the change of the effective Gibbons–Hawking entropy, making it larger than in the smooth standard case.

Let us further elaborate on the implications from (20)–(21), or instead, just (22), and establish how the situation in the fractional cases become quite different. Eq. (22) allows to extract and write that the effective fractional ADM Hamiltonian is given by

\[
H_{ADM}(\alpha) = -N \left( \frac{1}{3\pi m_p^2} \left| \Pi \right|^{\alpha} + U(a) \right),
\]

where

\[
U(a) := \frac{3\pi m_p^2}{4} a(t) \left( 1 - \frac{8\pi V_0}{3m_p} a(t)^2 \right).
\]

Then, the Hamilton’s equations, \( \dot{\alpha} = \partial H_{ADM}(\alpha) / \partial \Pi, \Pi = -\partial H_{ADM}(\alpha) / \partial \alpha \) together with the Hamiltonian constraint \( H(\alpha) = H_{ADM}(\alpha)/N = 0 \) lead us to the fractional Friedmann equations in comoving frame \( (N = 1) \):

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{2a^2(1-D)}{(D-1)^2} \left( \frac{2\sqrt{G}}{3\pi} \right)^{2(D-2)} \left( \frac{a^2}{L^2} - 1 \right)^{3-D},
\]

\[
\frac{\ddot{a}}{a} = \frac{(5-2D)(-U)^{2-D}}{(D-1)^2(3\pi)^{D-2}a^D} \left( \frac{a^2}{L^2} + \frac{D-2}{5-2D} \right),
\]

where \( L \) is defined by (16).

Note as well that for \( D = 2 \) the above field equations reduce the original usual Friedmann equations. Furthermore, as the above field equations show, for \( D \neq 2 \) (or equivalently \( \alpha \neq 2 \), see Fig. (11)), our deS model universe does not expand exponentially. The universe nucleated from nothing and is accelerated only for \( D < 2.5 \), increasing as a power-law expansion:

\[
a(t) \propto t^{\frac{1}{\alpha - 1}}.
\]

The number of e-foldings before the \( t_{end} \) of a suitable inflation period and from a time \( t_* \) becomes
In \((t_{	ext{end}}/t_*)/(2(D-1))\). For example, for \(D = 2.001\), we find \(N_* = 50 \ln(t_{	ext{end}}/t_*) > 50\); Nb. the scale we observe from the cosmic microwave background data conforms to the e-folding in the range \(50 < N_* < 60\) [13]. Usually, this kind of power-law expansion may be realized if a scalar field with an exponential potential dominates the universe’s energy density at an early stage [44]. As we saw, in the fractional extension of quantum cosmology, for \(D \neq 2\), the probability of having a power-law expansion is real and without such exponential potential. This allows to raise the possibility that inflation, an early accelerated stage of expansion, may happen from fractional quantum mechanical perspective. This is of interest and the usual form of the deS inflation will take place only for \(D = 2\).

Let us add that due to the negative sign in (24), the no-boundary state increases not only the contribution of empty universes with \(V_0 = 0\) in the entire quantum state, but also highly fractal horizons with dimension close to three, \(D \to 3\). This two effects lead to the seemingly contradictory conclusion that quite huge and decelerating universes are infinitely more likely than finite-sized universes. The tunneling wavefunction probability as in equation (24), on the other hand, promotes large values of \(V_0\) simultaneously with fractal horizon with dimension close to two, \(D \to 2\), and capable of inducing power law accelerated expansion scenarios. As a result, it appears that the tunneling prescription becomes more alluring within the fractional quantum cosmology herewith, when contrasting with the no-boundary prescription.

Similarly to the ending paragraph in the previous section, let us like re-address the points therein introduced. However, we will discuss them extending the context towards fractional quantum cosmology. More precisely, remind the general range \(V_0 \leq 10^{-12}m_P^4\) as derived from the amplitude constraint on gravitational waves generated during inflation. From \(V_0 = 10^{-12}m_P^4\), the probability distribution [24], yields \(10^{-10^7} < \mathcal{P}_{LV} \leq 10^{-10^{16}}\), where the lower limit is obtained for \(D \simeq 3\) and the upper limit in calculated for \(D = 2\). Thus, the herewith closed fractional universe may not be favored. In fact, if we consider instead creating a compact topologically nontrivial open or flat universes (see Refs. [27, 30]), then the fractional WDW equation (19) becomes

\[
\left( -\frac{d^2}{da^2} \right)^{\frac{D}{2}} \Psi(x) + \frac{3V_k m_P^4}{16\pi^2} \left( k\alpha^2 - \frac{\Lambda}{3} \right) \Psi(a) = 0,
\]

where \(k = -1, 0\), and \(V_k\) denotes the finite volume of the non-trivial compact spatial hypersurfaces [45]. Besides, the effective fractional ADM Hamiltonian [28] will generalize to

\[
H_{\text{ADM}}(\alpha) = -\mathcal{N} \left\{ \frac{2\pi}{3V_k m_P a} \left[ \left| \Pi \right|^2 + \frac{2V_k m_P^2}{8\pi} \left( k - \frac{\Lambda}{3} \alpha^2 \right) \right] \right\}.
\]

The fractional Friedmann equation obtained for such universe will be

\[
\left( \frac{a'}{a} \right)^2 = \frac{a^{2(D-1)} \left( 3V_k m_P \right)^2 (D-2)}{4\pi \left( 1 + \frac{D}{2} \right)} \left( \frac{\alpha^2}{L^2} - k \right)^{3-D}.
\]

The semiclassical solution of (32) for open universe, \((k = -1)\), disregarding the pre-exponential factor is

\[
\Psi(a) \approx \exp \left\{ \pm iC \alpha^D \left( \frac{D}{2} - 1 \left| \frac{1 - D}{2} + 1 + \frac{D}{2} \right| \alpha^D \right) \right\},
\]

where \(C = \frac{3V_k}{8\pi} m_P^D\), and a positive sign corresponds to an expanding universe. It is easy to show that for flat fractional quantum cosmology, \((k = 0)\), the semiclassical wavefunction is given by

\[
\Psi(a) \approx a^{-\gamma} \exp \left\{ \pm i m_P \frac{D}{2D-1} \left( \frac{3V_k^2 V_0}{2\pi} \right)^{\frac{D}{2D-1}} a^{2D-1} \right\},
\]

where a positive sign corresponds to an expanding universe. Note that for \(D = 2\) the above expression reduce to the semiclassical wavefunction obtained by Linde in [27]. This approximation breaks down at \(a \leq (m_P V_0^{(D-1)/2})^{1/(1-2D)}\). Similar to the closed universe, for scale factors much larger than this value, the classical scale factor satisfies (31). This shows that the discussion as proposed in Refs. [4, 28], can also be of value within our appraisal of de Sitter fractional quantum cosmology. For the flat universe is also correct for our model.

**IV. CONCLUSIONS**

Our simple model brings about a new perspective, whereby FQM features could have played an influence
at the early stages of inflation of the universe. The properties of an inflaton field are restricted by observations of fluctuations in the CMB and the universe’s matter distribution. Despite the fact that the mass of such inflaton and its interaction with matter fields are not determined, well-known arguments favor a heavy (scalar) field with a mass of $10^{13}$ GeV, near to the GUT scale, which is frequently used as evidence for the presence of new physics at the junction of the electroweak and Planck scales. Our findings suggest that inflation within fractional quantum cosmology may emerge through a different perspective. Namely, taking into consideration the event horizon of the nucleated universe as a fractal surface with dimension $D$. Our analysis is broad when we consider that inflation occurs at energy densities lower than the Planck density. As the tunneling proposal indicates an exponentially suppression of the probability of quantum creation of a closed inflationary universe, results suggest that a fractional inflationary universe may favour instead a flat or open universes.

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