Non-diffusive large time behaviour for a degenerate viscous Hamilton-Jacobi equation

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Abstract

The convergence to non-diffusive self-similar solutions is investigated for non-negative solutions to the Cauchy problem \( \partial_t u = \Delta_p u + |\nabla u|^q \) when the initial data converge to zero at infinity. Sufficient conditions on the exponents \( p > 2 \) and \( q > 1 \) are given that guarantee that the diffusion becomes negligible for large times and the \( L^\infty \)-norm of \( u(t) \) converges to a positive value as \( t \to \infty \).

1 Introduction

The quasilinear degenerate parabolic equation

\[
\partial_t u = \Delta_p u + |\nabla u|^q , \quad (t, x) \in Q_\infty := (0, \infty) \times \mathbb{R}^N ,
\]

includes two competing mechanisms acting on the space variable \( x \), a degenerate diffusion \( \Delta_p u \) involving the \( p \)-Laplacian operator defined by

\[
\Delta_p u := \text{div} \left( |\nabla u|^{p-2} \nabla u \right) , \quad p > 2 ,
\]

and a source term \( |\nabla u|^q \), \( q > 1 \), depending solely on the gradient of \( u \). The aim of this work is to identify a range of the parameters \( p \) and \( q \) for which the large time behaviour of non-negative solutions to (1.1) is dominated by the source term. More precisely, we consider the Cauchy problem and supplement (1.1) with the initial condition

\[
u(0) = u_0 \geq 0 , \quad x \in \mathbb{R}^N .
\]

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Throughout the paper, the initial condition $u_0$ is assumed to fulfill
\[ u_0 \in C_0(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N), \quad u_0 \geq 0, \quad u_0 \not\equiv 0, \]
where
\[ C_0(\mathbb{R}^N) := \left\{ w \in BC(\mathbb{R}^N) : \lim_{R \to \infty} \sup_{\{x| \geq R\}} \{|w(x)|\} = 0 \right\}, \]
and $BC(\mathbb{R}^N) := C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

For such an initial condition, the Cauchy problem (1.1), (1.2) has a unique non-negative (viscosity) solution $u \in BC([0, \infty) \times \mathbb{R}^N)$ (see Proposition 2.1 below). Moreover, $t \mapsto \|u(t)\|_\infty$ is a non-increasing function and has a limit $M_\infty \in [0, \|u_0\|_\infty]$ as $t \to \infty$. Our main result is then the following:

**Theorem 1.1** Assume that $p > 2$ and $q \in (1, p)$. Consider a non-negative function $u_0$ satisfying (1.3) and let $u$ be the corresponding (viscosity) solution to (1.1), (1.2). Assume further that
\[ M_\infty := \lim_{t \to \infty} \|u(t)\|_\infty > 0. \]
Then
\[ \lim_{t \to \infty} \|u(t) - h_\infty(t)\|_\infty = 0, \]
where $h_\infty$ is given by
\[ h_\infty(t, x) := H_\infty \left( \frac{x}{t^{1/q}} \right) \quad \text{and} \quad H_\infty(x) := \left( M_\infty - \gamma_q |x|^q/(q-1) \right)_+ \]
for $(t, x) \in Q_\infty$ and $\gamma_q := (q - 1) q^{-q/(q-1)}$.

Here and below, $r_+ := \max \{r, 0\}$ denotes the positive part of the real number $r$.

The convergence (1.5) clearly indicates that the large time behaviour of non-negative solutions to (1.1), (1.2) fulfilling the condition (1.4) is governed by the gradient source term. Indeed, $h_\infty$ is actually a self-similar solution to the Hamilton-Jacobi equation
\[ \partial_t h = |\nabla h|^q, \quad (t, x) \in Q_\infty, \]
and an alternative formula for $h_\infty$ reads
\[ h_\infty(t, x) = \sup_{y \in \mathbb{R}^N} \left\{ M_\infty \mathbf{1}_{\{0\}}(y) - \gamma_q \frac{|x - y|^{q/(q-1)}}{t^{1/(q-1)}} \right\} \]
for $(t, x) \in [0, \infty) \times \mathbb{R}^N$, $\mathbf{1}_{\{0\}}$ denoting the indicator function of the singleton set $\{0\}$.

The formula (1.8) is the well-known Hopf-Lax-Oleinik representation formula for viscosity solutions to (1.7) (see, e.g., [10, Chapter 3]) and $h_\infty$ turns out to be the unique viscosity solution in $BUC(Q_\infty)$ to (1.7) with the bounded and upper semicontinuous initial condition $h_\infty(0, x) = \mathbf{1}_{\{0\}}(x)$ for $x \in \mathbb{R}^N$ [23].
Remark 1.2 The convergence (1.5) also holds true for the viscosity solution to the Hamilton-Jacobi equation (1.7) with a non-negative initial condition $u_0 \in C_0(\mathbb{R}^N)$ but with $\|u_0\|_{\infty}$ instead of $M_{\infty}$ in the formula (1.6) giving $H_{\infty}$. For (1.1), (1.2), the constant $M_{\infty}$ takes into account that, though negligible for large times, the diffusion erodes the supremum of $u$ during the time evolution.

For $p = 2$, Theorem 1.1 is also valid and is proved in [7], the proof relying on a rescaling technique: The crucial step is then to identify the possible limits of the rescaled sequence and this is done by an extensive use of the Hopf-Lax-Oleinik representation formula. The proof we perform here is of a completely different nature and relies on the relaxed half-limits method introduced in [3]. A similar approach has been used in [21] and [22] to investigate the large time behaviour of solutions to first-order Hamilton-Jacobi equations $\partial_t w + H(x, \nabla w) = 0$ in $Q_{\infty}$. It has also been used in [19] to study the convergence to non-diffusive localized self-similar patterns for non-negative and compactly supported solutions to $\partial_t w - \Delta_p w + |\nabla w|^q = 0$ in $Q_{\infty}$ when $p > 2$ and $q \in (1, p - 1)$.

In order to apply Theorem 1.1 one should check whether there are non-negative solutions to (1.1), (1.2) for which (1.4) holds true. The next result provides sufficient conditions for (1.4) to be fulfilled.

**Theorem 1.3** Assume that $p > 2$ and $q > 1$. Consider a non-negative function $u_0$ satisfying (1.3) and let $u$ be the corresponding solution to (1.1), (1.2). Introducing

\[(1.9) \quad q_\ast := p - \frac{N}{N + 1},\]

then $u$ fulfills (1.4) if

(a) either $q \in (1, q_\ast]$,

(b) or $q \in (q_\ast, p)$, $u_0 \in W^{2,\infty}(\mathbb{R}^N)$, and

\[(1.10) \quad \|u_0\|_{\infty} > \kappa_0 \inf_{y \in \mathbb{R}^N} \{\Delta_p u_0(y)\}^{(p-q)/q} \quad for\ some \ \kappa_0 > 0 \ which \ depends \ only \ on \ N, \ p, \ and \ q.\]

A similar result is already available for $p = 2$ and has been established in [7, 12]. The proof of Theorem 1.3 for $q \in (p - 1, p)$ and $p > 2$ borrows some steps from the case $p = 2$. However, it relies on semiconvexity estimates for solutions to (1.1), (1.2) which seem to be new for $p > 2$ and $q \in (1, p)$ and are stated now.

**Proposition 1.4** Assume that $p > 2$ and $q \in (1, p]$. Let $u$ be the viscosity solution to (1.1), (1.2) with initial condition $u_0 \in BUC(\mathbb{R}^N)$ (that is, $u_0 \in BC(\mathbb{R}^N)$ and is uniformly continuous
in \( \mathbb{R}^N \). Then \( \nabla u(t) \) belongs to \( L^\infty(\mathbb{R}^N) \) for each \( t > 0 \) and there is \( \kappa_1 > 0 \) depending only on \( N, p, \) and \( q \) such that

\[
\Delta_p u(t, x) \geq -\kappa_1 \|u(s)\|^{(p-q)/q} (t-s)^{-p/q}, \quad t > s \geq 0,
\]

in the sense of distributions. In addition, if \( u_0 \in W^{1,\infty}(\mathbb{R}^N) \), there holds

\[
\Delta_p u(t, x) \geq -\frac{N(p-1)}{q(q-1)} \frac{\|\nabla u_0\|^{p-q}}{t}, \quad t > 0
\]

for \( t > 0 \) in the sense of distributions.

The proof of Proposition 1.4 relies on the comparison principle combined with a gradient estimate established in [6].

Similar semiconvexity estimates for solutions to (1.1), (1.2) have already been obtained in [14] and [20, Lemma 5.1] for \( p = q = 2 \), in [7, Proposition 3.2] for \( p = 2 \) and \( q \in (1, 2) \), and in [7, Theorem 1] for \( p = q > 2 \). We extend these results to the range \( p > 2 \) and \( q \in (1, p] \). As we shall see below, the estimate (1.11) plays an important role in the proof of Theorem 1.3 and is also helpful to construct a subsolution in the proof of Theorem 1.1.

Let us finally emphasize that the validity of Proposition 1.4 is not restricted to non-negative solutions and that the solutions to the Hamilton-Jacobi equation (1.7) also enjoy the semiconvexity estimates (1.11) and (1.12). These two estimates thus stem from the reaction term \(|\nabla u|^q\) and not from the diffusion.

In the next section, we recall the well-posedness of (1.1), (1.2) in \( \text{BUC}(\mathbb{R}^N) \), as well as some properties of the solutions established in [6]. We also show the finite speed of propagation of the support for non-negative compactly supported initial data. Section 3 is devoted to the proof of the semiconvexity estimates (Proposition 1.4) and Section 4 to that of Theorem 1.1. Theorem 1.3 is shown in the last section, its proof combining arguments of [7, 12, 18] used to established analogous results when \( p = 2 \).

Throughout the paper, \( C \) and \( C_i, \ i \geq 1 \), denote positive constants depending only on \( p, q, \) and \( N \). Dependence upon additional parameters will be indicated explicitly. Also, \( \mathcal{M}_N(\mathbb{R}) \) denotes the space of real-valued \( N \times N \) matrices and \( \delta_{ij} = 1 \) if \( i = j \) and \( \delta_{ij} = 0 \) if \( i \neq j \), \( 1 \leq i, j \leq N \). Given a matrix \( A = (a_{ij}) \in \mathcal{M}_N(\mathbb{R}) \), \( \text{tr}(A) \) denotes its trace and is given by \( \text{tr}(A) := \sum a_{ii} \).

2 Preliminary results

Let us first recall the well-posedness (in the framework of viscosity solutions) of (1.1), (1.2), together with some properties of the solutions established in [6].
Proposition 2.1 Consider a non-negative initial condition \( u_0 \in \mathcal{BUC}(\mathbb{R}^N) \). There is a unique non-negative viscosity solution \( u \in \mathcal{BC}([0, \infty) \times \mathbb{R}^N) \) to (1.1), (1.2) such that

\[
0 \leq u(t, x) \leq \|u_0\|_\infty, \quad (t, x) \in Q_\infty,
\]

(2.1)

\[
\|\nabla u(t)\|_\infty \leq \min \left\{ C_1 \|u(s)\|_\infty^{1/q} (t - s)^{-1/q}, \|\nabla u(s)\|_\infty \right\},
\]

(2.2)

and

\[
\int_{\mathbb{R}^N} (u(t, x) - u(s, x)) \vartheta(x) \, dx + \int_s^t \int_{\mathbb{R}^N} (|\nabla u|^{p-2} \nabla u \cdot \nabla \vartheta - |\nabla u|^q \vartheta) \, dx \, d\tau = 0
\]

(2.3)

for \( t > s \geq 0 \) and \( \vartheta \in \mathcal{C}_0^\infty(\mathbb{R}^N) \). In addition, \( t \mapsto \|u(t)\|_\infty \) is a non-increasing function.

Proof. We put \( \tilde{u}_0 := \|u_0\|_\infty - u_0 \). As \( \tilde{u}_0 \) is a non-negative function in \( \mathcal{BUC}(\mathbb{R}^N) \), it follows from [6, Theorem 1.1] that there is a unique non-negative viscosity solution \( \tilde{u} \) to

\[
\partial_t \tilde{u} - \Delta_p \tilde{u} + |\nabla \tilde{u}|^q = 0, \quad (t, x) \in Q_\infty := (0, \infty) \times \mathbb{R}^N,
\]

(2.4)

with initial condition \( \tilde{u}(0, x) = \tilde{u}_0(x) \) for \( x \in \mathbb{R}^N \). It also satisfies \( 0 \leq \tilde{u}(t, x) \leq \|u_0\|_\infty \) and

\[
\int_{\mathbb{R}^N} (\tilde{u}(t, x) - \tilde{u}(s, x)) \vartheta(x) \, dx + \int_s^t \int_{\mathbb{R}^N} (|\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \cdot \nabla \vartheta + |\nabla \tilde{u}|^q \vartheta) \, dx \, d\tau = 0
\]

(2.3)

for \( t > s \geq 0 \), \( x \in \mathbb{R}^N \), and \( \vartheta \in \mathcal{C}_0^\infty(\mathbb{R}^N) \). In addition, \( \nabla \tilde{u}(t) \) belongs to \( L^\infty(\mathbb{R}^N) \) for each \( t > 0 \) and

\[
\|\nabla \tilde{u}(t)\|_\infty \leq C_1 \|\tilde{u}_0\|_\infty^{1/q} \quad t^{-1/q}
\]

by [6, Lemma 4.1]. Setting \( u := \|u_0\|_\infty - \tilde{u} \), we readily deduce from the properties of \( \tilde{u} \) that \( u \) is a non-negative viscosity solution to (1.1), (1.2) satisfying (2.1) and (2.3). Also, \( \nabla u(t) \) belongs to \( L^\infty(\mathbb{R}^N) \) for each \( t > 0 \). The uniqueness and the time monotonicity of \( \|u\|_\infty \) then both follow from the comparison principle, see [8] or [11, Theorem 2.1]. Finally, given \( s \geq 0 \), \( (t, x) \mapsto \|u(s)\|_\infty - u(t + s, x) \) is the unique non-negative viscosity solution to the Cauchy problem (2.4) with initial condition \( x \mapsto \|u(s)\|_\infty - u(s, x) \) and we infer from [6, Lemma 4.1] that

\[
\|\nabla u(t + s)\|_\infty \leq C_1 \|\|u(s)\|_\infty - u(s)\|_{\infty}^{1/q} \quad t^{-1/q} \leq C_1 \|u(s)\|_\infty^{1/q} \quad t^{-1/q}
\]

for \( t > 0 \), whence (2.2).

We next turn to the propagation of the support of non-negative solutions to (1.1), (1.2) with non-negative compactly supported initial data.

Proposition 2.2 Consider a non-negative solution \( u \) to (1.1), (1.2) with an initial condition \( u_0 \) satisfying (1.3). Assume further that \( u_0 \) is compactly supported in a ball \( B(0, R_0) \) of \( \mathbb{R}^N \) for some \( R_0 > 0 \). Then \( u(t) \) is compactly supported for each \( t \geq 0 \).
Consequently, \( u \) supported solutions to the \( \max \) stated in [11, Theorem 2.1] to conclude that

\[
\partial_t w - \partial_p^2 (w^{p-1}) + \partial_1 (w^q) = 0, \quad (t, x_1) \in (0, \infty) \times \mathbb{R},
\]

with wave speed unity. It is given by \( w(t, x_1) = f(x_1 - t) \) for \( (t, x_1) \in (0, \infty) \times \mathbb{R} \), the function \( f \) being implicitly defined by

\[
(p - 1) \int_0^y \frac{z^{p-3}}{1 - z^{q-1}} \, dz = (-y)_+, \quad y \in \mathbb{R}.
\]

In particular, \( f \) satisfies \( f(y) = 0 \) if \( y > 0 \) and \( f(y) \to 1 \) as \( y \to -\infty \). Introducing

\[
F(y) := \int_y^\infty f(z) \, dz, \quad y \in \mathbb{R},
\]

the properties of \( f \) ensure that \( F \) is a decreasing function on \( (-\infty, 0) \) with \( F(y) = 0 \) if \( y > 0 \), \( F(y) \leq |y| \) if \( y < 0 \), and \( F(y) \to \infty \) as \( y \to -\infty \). There is therefore a unique \( \mu \in (-\infty, 0) \) such that \( F(R_0 + \mu) = \|u_0\|_\infty \). In addition, it readily follows from (2.5) and the invariance by translation of (1.1) that \( W_\mu(t, x) := F(x_1 + \mu - t) \) is a travelling wave solution to (1.1).

Now, \( u \) and \( W_\mu \) are both solutions to (1.1) in \( (0, \infty) \times H_+ \), the half-space \( H_+ \) being defined by \( H_+ := \{ x \in \mathbb{R}^N : x_1 > R_0 \} \). Owing to the monotonicity of \( F \), the bound \( 0 \leq f \leq 1 \), and (2.1), we have also

\[
u_0(x) - W_\mu(0, y) = 0 - W_\mu(0, y) \leq W_\mu(0, x) - W_\mu(0, y) \leq |x - y|
\]

for \( x \in H_+ \) and \( y \in H_+ \),

\[
u(t, x) - W_\mu(t, y) \leq \|u_0\|_\infty - W_\mu(t, y) \\
\leq F(R_0 + \mu - t) - W_\mu(t, y) = W_\mu(t, x) - W_\mu(t, y) \leq |x - y|
\]

for \( t > 0, x \in \partial H_+, y \in H_+ \), and

\[
u(t, x) - W_\mu(t, y) \leq \|u_0\|_\infty - F(R_0 + \mu - t) \leq 0
\]

for \( t > 0, x \in H_+, y \in \partial H_+ \). We are then in a position to use the comparison principle stated in [11, Theorem 2.1] to conclude that \( u(t, x) \leq W_\mu(t, x) \) for \( (t, x) \in (0, \infty) \times H_+ \). Consequently, \( u(t, x) \leq F(x_1 + \mu - t) = 0 \) if \( t \geq 0 \) and \( x_1 \geq \max \{ R_0, t - \mu \} \), and the rotational invariance of (1.1) allows us to conclude that \( u(t, x) = 0 \) for \( t \geq 0 \) and \( |x| > \max \{ R_0, t - \mu \} \).

We finally recall the convergence to self-similar solutions for non-negative and compactly supported solutions to the \( p \)-Laplacian equation [17]

\[
\partial_t \varphi = \Delta_p \varphi, \quad (t, x) \in Q_\infty.
\]
Proposition 2.3 Let $\varphi_0$ be a non-negative and compactly supported function in $L^1(\mathbb{R}^N)$ and $\varphi$ denote the unique weak solution to (2.6) with initial condition $\varphi_0$. Then

$$\lim_{t \to \infty} t^{(N(r-1))/(r(N(p-2)+p))} \| \varphi(t) - B_{\|\varphi_0\|_1}(t) \|_r = 0 \quad \text{for} \quad r \in [1, \infty),$$

where $B_L$ denotes the Barenblatt solution to (2.6) given by

$$B_L(t,x) := t^{-N/(N(p-2)+p)} b_L \left( xt^{-1/(N(p-2)+p)} \right),$$

$$b_L(x) := \left( C_2 L^{(p(p-2))/(p-1)(N(p-2)+p)} - C_3 |x|^{p/(p-1)} \right)^{(p-1)/(p-2)}$$

for $(t,x) \in (0, \infty) \times \mathbb{R}^N$ and $L > 0$.

The convergence (2.7) is proved in [17, Theorem 2] for $r = \infty$. As $\varphi_0$ is compactly supported, so is $\varphi(t)$ for each $t > 0$ and the support of $\varphi(t)$ is included in $B(0, C_4(\varphi_0)^{1/(N(p-2)+p)})$ for $t \geq 1$ [17, Proposition 2.2]. Combining this property with [17, Theorem 2] readily provide the convergence (2.7) for all $r \in [1, \infty)$.

3 Semiconvexity

In this section, we prove the semiconvexity estimates (1.11) and (1.12). To this end, we would like to derive an equation for $\Delta_p u$ to which we could apply the comparison principle. The poor regularity of $u$ however does not allow to perform directly such a computation and an approximation procedure is needed. As a first step, we report the following result:

Lemma 3.1 Let $a$ and $b$ be two non-negative function in $C^\infty([0, \infty))$ satisfying

$$a(r) > 0, \quad a'(r) > 0, \quad a'(r) b'(r) - a(r) b''(r) > 0,$$

$$c(r) := 2 \left( \frac{b'}{a} \right)(r) + \frac{4 r (a b'' - a' b') (r)}{a^2(r) + 2r a(r) a'(r)} \geq 0.$$

Consider a classical solution $v$ to

$$\partial_t v - \text{div} \left( a(|\nabla v|^2) \nabla v \right) = b(|\nabla v|^2), \quad (t,x) \in Q_\infty,$$

and put

$$w := \text{div} \left( a(|\nabla v|^2) \nabla v \right) \quad \text{and} \quad z_i := a(|\nabla v|^2) \partial_i v$$

for $i \in \{1, \ldots, N\}$. Then

$$\partial_t w - \mathcal{L} w - \nabla \cdot \nabla w - \frac{c(|\nabla v|^2)}{N} w^2 \geq 0 \quad \text{in} \quad Q_\infty,$$
where
\[
\mathcal{L}w := \sum_{i,j} \partial_i \left( a \left( |\nabla v|^2 \right) E_{ij} \partial_j w \right), \quad \mathcal{V} := 2 b' \left( |\nabla v|^2 \right) \nabla v,
\]
\[
E_{ij} := \delta_{ij} + 2 \frac{a'}{a} \left( |\nabla v|^2 \right) \partial_i v \partial_j v, \quad 1 \leq i, j \leq N.
\]

The proof of Lemma 3.1 borrows some steps from the proof of [9, Theorem 1] for \( p = q > 2 \) but requires additional arguments to handle the term coming from the fact that \( q \neq p \). In particular, we recall the following elementary result which will be helpful to estimate this term.

**Lemma 3.2** Let \( A \) and \( B \) be two symmetric matrices in \( \mathcal{M}_N(\mathbb{R}) \) and put \( M := ABA \). Then \( M \) is a symmetric matrix in \( \mathcal{M}_N(\mathbb{R}) \) and
\[
|MX|^2 \leq \text{tr} \left( M^2 \right) |X|^2 \quad \text{for} \quad X \in \mathbb{R}^N.
\]

**Proof of Lemma 3.1** We first note that
\[
\partial_j z_i = a \left( |\nabla v|^2 \right) \sum_k E_{ik} \partial_k \partial_j v,
\]
\[
\partial_i z_i = a \left( |\nabla v|^2 \right) \sum_k E_{ik} \partial_k \partial_i v,
\]
for \( 1 \leq i, j \leq N \). According to the definition of \( w \), we infer from (3.3), (3.6), and (3.7) that
\[
\partial_t w = \sum_{i,k} \partial_i \left( a \left( |\nabla v|^2 \right) E_{ik} \partial_k \partial_t v \right)
\]
\[
= \sum_{i,k} \partial_i \left( a \left( |\nabla v|^2 \right) E_{ik} \partial_k \left( w + b \left( |\nabla v|^2 \right) \right) \right)
\]
\[
= \mathcal{L}w + 2 \sum_{i,k} \partial_i \left( a \left( |\nabla v|^2 \right) E_{ik} \partial_j v \partial_j \partial_k v \right)
\]
\[
= \mathcal{L}w + 2 \sum_{i,j} \partial_i \left( \frac{b'}{a} \left( |\nabla v|^2 \right) z_j \partial_j z_i \right)
\]
\[
= \mathcal{L}w + 4 \sum_{i,j} \left( \frac{b'}{a} \right) \left( |\nabla v|^2 \right) \partial_i v \partial_k \partial_i v \partial_j \partial_j z_i
\]
\[
+ 2 \sum_{i,j} \left( \frac{b'}{a} \right) \left( |\nabla v|^2 \right) \partial_i z_j \partial_j z_i + 2 \sum_{i,j} \left( \frac{b'}{a} \right) \left( |\nabla v|^2 \right) \partial_i \partial_j z_i .
\]
Since \( w = \sum \partial_i z_i \), the last term of the right-hand side of the above inequality is equal to \( \mathcal{V} \cdot \nabla w \) and

\[
(3.8) \quad \partial_t w = \mathcal{L} w + \mathcal{V} \cdot \nabla w + 4 \left[ a \left( \frac{b'}{a} \right) \right] (|\nabla v|^2) \sum_{i,j,k} \partial_j v \partial_k v \partial_k \partial_i v \partial_j z_i \\
+ 2 \left( \frac{b'}{a} \right) (|\nabla v|^2) \sum_{i,j} \partial_i z_j \partial_j z_i .
\]

On the one hand, introducing the matrix \( \mathcal{E} := (E_{ij}) \) and the Hessian matrix \( D^2 v = (\partial_i \partial_j v) \) of \( v \), we infer from (3.6) that

\[
(3.9) \quad \sum_{i,j} \partial_i z_j \partial_j z_i = a^2 (|\nabla v|^2) \sum_{i,j,k,l} E_{ik} \partial_k \partial_j v E_{jl} \partial_i \partial_l v
\]

\[
= a^2 (|\nabla v|^2) \sum_{i,j} (\mathcal{E} D^2 v)_{ij} (\mathcal{E} D^2 v)_{ji} .
\]

On the other hand, using once more (3.6), we obtain

\[
(3.10) \quad \sum_{i,j,k} \partial_j v \partial_k v \partial_k \partial_i v \partial_j z_i = a (|\nabla v|^2) \left( \mathcal{E} D^2 v \nabla v, (\mathcal{E} D^2 v) \nabla v \right) .
\]

Inserting (3.9) and (3.10) in (3.8), we end up with

\[
(3.11) \quad \partial_t w = \mathcal{L} w + \mathcal{V} \cdot \nabla w + 2 (ab') (|\nabla v|^2) \left( (\mathcal{E} D^2 v)^2 \right) \\
+ 4 (a b' - a' b') (|\nabla v|^2) \left( D^2 v \nabla v, (\mathcal{E} D^2 v) \nabla v \right) .
\]

We next observe that

\[
(3.12) \quad \mathcal{E} \nabla v = \left( 1 + 2 |\nabla v|^2 \left( \frac{a'}{a} \right) (|\nabla v|^2) \right) \nabla v
\]

and that, for \( X \in \mathbb{R}^N \),

\[
\langle \mathcal{E} X, X \rangle = |X|^2 + 2 \left( \frac{a'}{a} \right) (|\nabla v|^2) \langle X, \nabla v \rangle^2 \geq |X|^2
\]
as $a$ and $a'$ are both positive by (3.1). Consequently, $\mathcal{E}$ is a positive definite symmetric matrix in $\mathcal{M}_N(\mathbb{R})$ and there exists a positive definite matrix $\mathcal{E}_{1/2}$ such that $\mathcal{E}^{1/2} = \mathcal{E}$. We then infer from the definition of $\mathcal{E}_{1/2}$, (3.12), and Lemma 3.2 (with $A = \mathcal{E}_{1/2}$, $B = D^2v$ and $X = \mathcal{E}_{1/2}^{-1} \nabla v$) that

$$\langle D^2v \nabla v, (\mathcal{E} D^2v) \nabla v \rangle = |(\mathcal{E}_{1/2} D^2v) \nabla v|^2 = \left| (\mathcal{E}_{1/2} D^2v \mathcal{E}_{1/2}^{-1} \mathcal{E}_{1/2}^{-1}) \nabla v \right|^2 \leq tr \left( (\mathcal{E}_{1/2} D^2v \mathcal{E}_{1/2}^{-1} \mathcal{E}_{1/2}^{-1}) \langle \nabla v, \nabla v \rangle \right) \leq tr \left( (\mathcal{E} D^2v) \langle \nabla v, \nabla v \rangle \right) \leq tr \left( (\mathcal{E} D^2v) |\nabla v|^2 \left( 1 + 2 |\nabla v|^2 \left( \frac{a'}{a} \right) \right) \right)^{-1}.$$

Owing to the non-positivity (3.1) of $b'' - a' b'$, we deduce from (3.11) and the above inequality that

$$\partial_t w \geq \mathcal{L} w + \mathcal{V} \cdot \nabla w + (a^2 c) \left( |\nabla v|^2 \right) tr \left( (\mathcal{E} D^2v) \right),$$

the function $c$ being defined in (3.2). We finally use the inequality

$$tr \left( A^2 \right) \geq \frac{1}{N} tr(A)^2, \quad A \in \mathcal{M}_N(\mathbb{R}),$$

the identity

$$w = \sum_i \partial_i z_i = a \left( |\nabla v|^2 \right) tr \left( (\mathcal{E} D^2v) \right),$$

and the non-negativity (3.2) of $c$ to conclude that

$$\partial_t w \geq \mathcal{L} w + \mathcal{V} \cdot \nabla w + \frac{1}{N} \left( a^2 c \right) \left( |\nabla v|^2 \right) tr \left( \mathcal{E} D^2v \right)^2 \geq \mathcal{L} w + \mathcal{V} \cdot \nabla w + c \left( \frac{|\nabla v|^2}{N} \right) w^2,$$

and complete the proof. □

**Proof of Proposition 1.4.** To be able to use Lemma 3.1 we shall first construct a suitable approximation of (1.1), (1.2). Such a construction has already been performed in [6] for similar purposes and we recall it now. Given $u_0$ satisfying (1.3), there is a sequence of functions $(u_{0,k})_{k \geq 1}$ such that, for each integer $k \geq 1$, $u_{0,k} \in \mathcal{BC}^\infty(\mathbb{R}^N)$, $u_0 \leq u_{0,k+1} \leq u_{0,k}$, and $(u_{0,k}, \nabla u_{0,k})$ converge towards $(u_0, \nabla u_0)$ uniformly on every compact subset of $\mathbb{R}^N$ as $k \to \infty$. Next, for $\varepsilon \in (0, 1)$ and $r \geq 0$, we set

$$a_\varepsilon(r) := (r + \varepsilon^2)^{(p-2)/2} \quad \text{and} \quad b_\varepsilon(r) := (r + \varepsilon^2)^{q/2} - \varepsilon^q.$$
Then the Cauchy problem
\begin{align}
\partial_t u_{k,\varepsilon} &= \text{div} \left( a_{\varepsilon} \left( |\nabla u_{k,\varepsilon}|^2 \right) \nabla u_{k,\varepsilon} \right) + b_{\varepsilon} \left( |\nabla u_{k,\varepsilon}|^2 \right), \quad (t, x) \in Q_{\infty}, \\
u_{k,\varepsilon}(0) &= u_{0,k} + \varepsilon^\nu, \quad x \in \mathbb{R}^N,
\end{align}
has a unique classical solution \( u_{k,\varepsilon} \), the parameter \( \nu > 0 \) depending on \( p, q, \) and \( N \) and being appropriately chosen. Furthermore,
\begin{align}
\|\nabla u_{k,\varepsilon}(t)\|_\infty &\leq \|\nabla u_{0,k}\|_\infty, \quad t \geq 0, \\
\lim_{k \to \infty} \lim_{\varepsilon \to 0} u_{k,\varepsilon}(t, x) &= u(t, x),
\end{align}
the latter convergence being uniform on every compact subset of \([0, \infty) \times \mathbb{R}^N\), see [6, Section 3] (after performing the same change of unknown function as in the proof of Proposition 2.1).

Introducing
\[ c_{\varepsilon}(r) = 2 \left( \frac{b'_{\varepsilon}}{a_{\varepsilon}} \right)(r) + 4r \left( \frac{a_{\varepsilon} b'_{\varepsilon} - a'_{\varepsilon} b_{\varepsilon}}{a_{\varepsilon}^2(r) + 2r a_{\varepsilon}(r) a'_{\varepsilon}(r)} \right), \quad r \geq 0, \]
let us check that \( a_{\varepsilon} \) and \( b_{\varepsilon} \) fulfill the conditions (3.1) and (3.2). Clearly, \( a_{\varepsilon} > 0 \) and \( a'_{\varepsilon} > 0 \) as \( p > 2 \). Next, since \( 1 < q \leq p \),
\[ (a'_{\varepsilon} b_{\varepsilon} - a_{\varepsilon} b''_{\varepsilon})(r) = \frac{q (p-q)}{4} (r + \varepsilon^2)^{(p+q-6)/2} \geq 0, \]
\[ c_{\varepsilon}(r) = q \frac{r(q-1) + \varepsilon^2}{r(p-1) + \varepsilon^2} (r + \varepsilon^2)^{(q-p)/2} \geq 0. \]
We may then apply Lemma 3.1 to deduce that \( w_{k,\varepsilon} := \text{div} \left( a_{\varepsilon} \left( |\nabla u_{k,\varepsilon}|^2 \right) \nabla u_{k,\varepsilon} \right) \) satisfies
\begin{equation}
\partial_t w_{k,\varepsilon} - \mathcal{L}_{k,\varepsilon} w_{k,\varepsilon} - \mathcal{V}_{k,\varepsilon} \cdot \nabla w_{k,\varepsilon} - \frac{c_{\varepsilon} \left( |\nabla u_{k,\varepsilon}|^2 \right)}{N} w_{k,\varepsilon}^2 \geq 0
\end{equation}
in \( Q_{\infty} \). Observe next that the condition \( 1 < q \leq p \) implies that \( c_{\varepsilon} \) is a non-increasing function. It then follows from (3.15) that
\[ c_{\varepsilon} \left( |\nabla u_{k,\varepsilon}|^2 \right) \geq c_{\varepsilon} \left( |\nabla u_{0,k}|^2 \right) \]
and we end up with
\begin{equation}
\partial_t w_{k,\varepsilon} - \mathcal{L}_{k,\varepsilon} w_{k,\varepsilon} - \mathcal{V}_{k,\varepsilon} \cdot \nabla w_{k,\varepsilon} - \frac{c_{\varepsilon} \left( |\nabla u_{0,k}|^2 \right)}{N} w_{k,\varepsilon}^2 \geq 0
\end{equation}
in \( Q_{\infty} \). Clearly, \( t \mapsto -N/ \left( c_{\varepsilon} \left( |\nabla u_{0,k}|^2 \right) \right) \) is a subsolution to (3.17) and the comparison principle warrants that
\begin{equation}
w_{k,\varepsilon}(t, x) \geq - \frac{N}{c_{\varepsilon} \left( |\nabla u_{0,k}|^2 \right)} t, \quad (t, x) \in Q_{\infty}.
\end{equation}
Letting \( \varepsilon \to 0 \) and \( k \to \infty \) in the previous inequality with the help of (3.16) gives (1.12).
Next, since \( (1.1) \) is autonomous, we infer from \( (2.2) \) (with \( s = 0 \)) and \( (1.12) \) that

\[
\Delta_p u(t, x) \geq -\frac{2N(p-1)}{q(q-1)} \frac{\|\nabla u(t/2)\|^{p-q}}{t} \geq -\frac{2p/q N(p-1)}{q(q-1)} C \|u_0\|^{(p-q)/q} t^{-p/q},
\]

whence \( (1.11) \) for \( s = 0 \). To prove the general case \( s \in (0, t) \), we use again the fact that \( (1.1) \) is autonomous. \( \square \)

We have a similar result when \( u_0 \) is more regular.

**Corollary 3.3** Assume that \( p > 2 \) and \( q \in (1, p] \). Let \( u \) be the solution to \( (1.1) \), \( (1.2) \) with an initial condition \( u_0 \) satisfying \( u_0 \in W^{2,\infty}(\mathbb{R}^N) \) in addition to \( (1.3) \). Then

\[
(3.19) \quad \Delta_p u(t, x) \geq -\left| \inf_{y \in \mathbb{R}^N} \Delta_p u_0(y) \right|
\]

in the sense of distributions.

**Proof.** Keeping the notations introduced in the proof of Proposition 1.4, we readily infer from \( (3.17) \) and the comparison principle that

\[
(3.20) \quad w_{k,\varepsilon}(t, x) \geq -\left| \inf_{y \in \mathbb{R}^N} \Delta_p u_{0,k}(y) \right|, \quad (t, x) \in Q_\infty.
\]

Owing to the regularity of \( u_0 \), it is possible to construct the sequence \( (u_{0,k})_k \) such that it satisfies

\[
\lim_{k \to \infty} \inf_{y \in \mathbb{R}^N} \Delta_p u_{0,k}(y) = \inf_{y \in \mathbb{R}^N} \Delta_p u_0(y).
\]

We may then pass to the limit first as \( \varepsilon \to 0 \) and then as \( k \to \infty \) in \( (3.20) \) and use \( (3.16) \) and the above convergence to complete the proof. \( \square \)

Another useful consequence of the semiconvexity estimates derived in Proposition 1.4 is that the solution \( u \) to \( (1.1) \), \( (1.2) \) is a supersolution to a first-order Hamilton-Jacobi equation.

**Corollary 3.4** Consider an initial condition \( u_0 \) satisfying \( (1.3) \). Setting \( F(t, \xi_0, \xi) := \xi_0 - |\xi|^q + \kappa_1 \|u_0\|^{(p-q)/q} t^{-p/q} \) for \( t \in (0, \infty) \), \( \xi_0 \in \mathbb{R} \), and \( \xi \in \mathbb{R}^N \) (recall that \( \kappa_1 \) is defined in \( (1.11) \)), the solution \( u \) to \( (1.1) \), \( (1.2) \) is a supersolution to \( F(t, \partial_t w, \nabla w) = 0 \) in \( Q_\infty \).

**Proof.** We still use the notations introduced in the proof of Proposition 1.4. As \( w_{k,\varepsilon} = \text{div} \left( a_\varepsilon \left( |\nabla u_{k,\varepsilon}|^2 \right) \nabla u_{k,\varepsilon} \right) \), we infer from \( (3.13) \) and \( (3.18) \) that

\[
\partial_t u_{k,\varepsilon} - b_\varepsilon \left( |\nabla u_{k,\varepsilon}|^2 \right) \geq -\frac{N}{c_\varepsilon \left( \|\nabla u_{0,k}\|_\infty^2 \right) t}
\]

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We then use (3.16) and the stability of viscosity solutions \[1,2,8\] to pass to the limit as \(\varepsilon \to 0\) and \(k \to \infty\) in the previous inequality and conclude that \(u\) is a supersolution to

\[
\partial_t w - |\nabla w|^q + \frac{N(p-1)}{q(q-1)} \frac{\|\nabla u_0\|_{\infty}^{p-q}}{t} = 0 \quad \text{in} \quad Q_\infty.
\]

Now, fix \(T \geq 0\). As (1.1) is an autonomous equation, the function \( (t,x) \mapsto -u(t+T,x) \) is the solution to (1.1) with initial condition \(u(T)\) and the above analysis allows us to conclude that \(u\) is a supersolution to

\[
\partial_t w - |\nabla w|^q + \frac{N(p-1)}{q(q-1)} \frac{\|\nabla u(T)\|_{\infty}^{p-q}}{t-T} = 0 \quad \text{in} \quad (T, \infty) \times \mathbb{R}^N.
\]

We then use (2.2) (with \(T = t/2\)) to complete the proof. \(\square\)

### 4 Convergence to self-similarity

We change the variables and the unknown function so that the convergence (1.5) is transformed to the convergence towards a steady state. More precisely, we introduce the self-similar (or scaling) variables

\[
\tau = \frac{1}{q} \log (1+t), \quad y = \frac{x}{(1+t)^{1/q}},
\]

and the new unknown function \(v\) defined by

\[
(4.1) \quad u(t,x) = v \left( \frac{\log (1+t)}{q}, \frac{x}{(1+t)^{1/q}} \right), \quad (t,x) \in [0, \infty) \times \mathbb{R}^N.
\]

Equivalently, \(v(\tau,y) = u(e^{\tau t} - 1, ye^{\tau})\) for \((\tau,y) \in [0, \infty) \times \mathbb{R}^N\) and it follows from (1.1), (1.2) that \(v\) solves

\[
(4.2) \quad \partial_\tau v = y \cdot \nabla v + q |\nabla v|^q + q e^{-(p-q)\tau} \Delta_p v, \quad (\tau,y) \in (0, \infty) \times \mathbb{R}^N,
\]

\[
(4.3) \quad v(0) = u_0, \quad y \in \mathbb{R}^N.
\]

We also infer from (2.1) and (2.2) that there is a positive constant \(C_5(u_0)\) depending only on \(N, p, q,\) and \(u_0\) such that

\[
(4.4) \quad \|v(\tau)\|_\infty + \|\nabla v(\tau)\|_\infty \leq C_5(u_0), \quad \tau \geq 0,
\]

while (1.4) reads

\[
(4.5) \quad \lim_{\tau \to \infty} \|v(\tau)\|_\infty = M_\infty > 0.
\]
Formally, since \( p > q \), the diffusion term vanishes in the large time limit and we expect the large time behaviour of the solution \( v \) to (4.2), (4.3) to look like that of the solutions to the first-order Hamilton-Jacobi equation

\[
\partial_\tau w - y \cdot \nabla w - q |\nabla w|^q = 0 \quad \text{in} \quad Q_\infty.
\]

Now, to investigate the large time behaviour of first-order Hamilton-Jacobi equations, an efficient approach has been developed in \([21, 22]\) which relies on the relaxed half-limits method introduced in \([3]\). More precisely, for \((\tau, y) \in (0, \infty) \times \mathbb{R}^N\), we define the relaxed half-limits \( v^* \) and \( v^* \) by

\[
v^*_y(\cdot) := \liminf_{(\sigma, z, \lambda) \to (\tau, y, \infty)} v(\sigma + \lambda, z) \quad \text{and} \quad v^*_y(\cdot) := \limsup_{(\sigma, z, \lambda) \to (\tau, y, \infty)} v(\sigma + \lambda, z).
\]

These relaxed half-limits are well-defined thanks to (4.4) and we first note that the right-hand sides of the above definitions indeed do not depend on \( \tau > 0 \). In addition,

\[
0 \leq v_*(x) \leq v^*(x) \leq M_\infty \quad \text{for} \quad y \in \mathbb{R}^N
\]

by (4.5), while (4.4) and the Rademacher theorem ensure that \( v_* \) and \( v^* \) both belong to \( W^{1, \infty}(\mathbb{R}^N) \). Finally, by \([2, \text{Théorème 4.1}]\) applied to equation (4.2), \( v^* \) and \( v_* \) are viscosity subsolution and supersolution, respectively, to the Hamilton-Jacobi equation

\[
\mathcal{H}(y, \nabla w) := -y \cdot \nabla w - q |\nabla w|^q = 0 \quad \text{in} \quad \mathbb{R}^N.
\]

We now aim at showing that \( v^* \) and \( v_* \) coincide. However, the equation (4.9) has infinitely many solutions as \( y \mapsto -c - \gamma_q |y|^{q/(q-1)} \) solves (4.9) for any \( c > 0 \).

The information obtained so far on \( v_* \) and \( v^* \) are thus not sufficient and are supplemented by the next two results.

**Lemma 4.1** Given \( \varepsilon \in (0, 1) \), there is \( R_\varepsilon > 1/\varepsilon \) such that

\[
v(\tau, y) \leq \varepsilon \quad \text{for} \quad \tau \geq 0 \quad \text{and} \quad y \in \mathbb{R}^N \setminus B(0, R_\varepsilon),
\]

and \( 0 \leq v_*(y) \leq v^*(y) \leq \varepsilon \) for \( y \in \mathbb{R}^N \setminus B(0, R_\varepsilon) \).

In other words, \( \v(\tau) \) belongs to \( C_0(\mathbb{R}^N) \) for each \( \tau \geq 0 \) in a way which is uniform with respect to \( \tau \geq 0 \).

**Proof.** We first construct a supersolution to (4.2) in \((0, \infty) \times \mathbb{R}^N \setminus B(0, R)\) for \( R \) large enough. To this end, consider \( R \geq R_\varepsilon := 1 + (q \|u_0\|_\infty)^{q-1} + 3pq (2 \|u_0\|_\infty)^{p-2})^{1/q} \) and put \( \Sigma_R(y) = \|u_0\|_\infty R^2 |y|^{-2} \) for \( y \in \mathbb{R}^N \setminus B(0, R) \). Let \( \mathcal{L} \) be the parabolic operator defined by

\[
\mathcal{L} w(\tau, y) := \partial_\tau w(\tau, y) - y \cdot \nabla w(\tau, y) - q |\nabla w(\tau, y)|^q - q e^{-(p-q)\tau} \Delta_p w(\tau, y)
\]
for \((\tau, y) \in Q_\infty\) (so that \(\mathcal{L}v = 0\) by (1.2)). Then, if \(y \in \mathbb{R}^N \setminus B(0, R)\), we have

\[
\mathcal{L}\Sigma_R(y) = 2\Sigma_R(y) - q \frac{2^q}{|y|^q} \Sigma_R(y)^q + q^2 - 3p - \left(\Sigma_R(y)^p - e^{-(p-q)\tau}\right)
\]

\[
\geq 2\Sigma_R(y) \left\{ 1 - q (2\|u_0\|_\infty)^q - \frac{R^{2(p-1)}}{|y|^{3q-2}} - 3pq e^{-(p-q)\tau} (2\|u_0\|_\infty)^{p-2} R \right\}
\]

\[
\geq 0
\]

by the choice of \(R\). Consequently, \(\Sigma_R\) is a supersolution to (1.2) in \((0, \infty) \times \mathbb{R}^N \setminus B(0, R)\) for \(R \geq R_\epsilon\).

Now, fix \(\varepsilon \in (0, 1)\). Since \(u_0 \in C_0(\mathbb{R}^N)\), there is \(\rho_\varepsilon \geq \max \{R_\epsilon, \varepsilon^{-1}\}\) such that \(u_0(y) \leq \varepsilon/2\) if \(|y| \geq \rho_\varepsilon\). We then infer from the monotonicity of \(\Sigma_R\) and (2.1) that

\[
u_0(y) - \frac{\varepsilon}{2} - \Sigma_{\rho_\varepsilon}(z) \leq -\Sigma_{\rho_\varepsilon}(z) \leq 0
\]

if \(|y| \geq \rho_\varepsilon\) and \(|z| \geq \rho_\varepsilon\),

\[
v(\tau, y) - \frac{\varepsilon}{2} - \Sigma_{\rho_\varepsilon}(z) \leq \|u_0\|_\infty - \Sigma_{\rho_\varepsilon}(z) = \Sigma_{\rho_\varepsilon}(y) - \Sigma_{\rho_\varepsilon}(z) \leq \frac{2\|u_0\|_\infty}{\rho_\varepsilon} |y - z|
\]

if \(|y| = \rho_\varepsilon\), \(|z| \geq \rho_\varepsilon\), and \(\tau \geq 0\), and

\[
v(\tau, y) - \frac{\varepsilon}{2} - \Sigma_{\rho_\varepsilon}(z) \leq \|u_0\|_\infty - \|u_0\|_\infty \leq 0
\]

if \(|y| \geq \rho_\varepsilon\), \(|z| = \rho_\varepsilon\), and \(\tau \geq 0\). As \(v - \varepsilon/2\) and \(\Sigma_{\rho_\varepsilon}\) are subsolution and supersolution, respectively, to (4.2), the comparison principle [11] Theorem 4.1 warrants that \(v(\tau, y) - \varepsilon/2 \leq \Sigma_{\rho_\varepsilon}(y)\) for \(\tau \geq 0\) and \(|y| \geq \rho_\varepsilon\). It remains to choose \(R_\varepsilon \geq \rho_\varepsilon\) such that \(\Sigma_{\rho_\varepsilon}(y) \leq \varepsilon/2\) for \(|y| \geq R_\varepsilon\) to complete the proof of (4.10). The last assertion of Lemma 4.1 is then a straightforward consequence of the definition (4.7) and (4.10).

We next use the semiconvexity estimate (1.11) (and more precisely its consequence stated in Corollary 3.4) to show that \(v_\ast\) lies above the profile \(H_\infty\) defined in (1.6).

**Lemma 4.2** For \(y \in \mathbb{R}^N\), we have

\[
H_\infty(y) \leq v_\ast(y) \leq v_\ast(y).
\]

**Proof.** For \(\tau \geq 0\), \(y \in \mathbb{R}^N\), \(\xi_0 \in \mathbb{R}\) and \(\xi \in \mathbb{R}^N\), we set \(\mathcal{F}(\tau, y, \xi_0, \xi) := \xi_0 - y \cdot \xi - q |\xi|^q + \kappa_2 e^{-(p-q)\tau}\) with \(\kappa_2 := q \kappa_1 e^{q}/(e^q - 1)\), the constant \(\kappa_1\) being defined in (1.11). It then readily follows from Corollary 3.4 that

\[
v\text{ is a supersolution to } \mathcal{F}(\tau, y, \partial_\tau w, \nabla w) = 0 \text{ in } (1, \infty) \times \mathbb{R}^N.
\]
We next fix $\tau_0 > 1$ and denote by $\tilde{V}$ the (viscosity) solution to
\[
\partial \tau V - y \cdot \nabla V - q |\nabla V|^q = 0, \quad (\tau, y) \in (\tau_0, \infty) \times \mathbb{R}^N,
\]
\[
V(\tau_0) = v(\tau_0), \quad y \in \mathbb{R}^N.
\]
On the one hand, a straightforward computation shows that the function $\tilde{V}$ defined by
\[
\tilde{V}(\tau, y) := V(\tau, y) - \kappa_2 \int_{\tau_0}^{\tau} e^{-(p-q)s} ds, \quad (\tau, y) \in (\tau_0, \infty) \times \mathbb{R}^N,
\]
is the (viscosity) solution to $F(\tau, y, \partial \tau \tilde{V}, \nabla \tilde{V}) = 0$ in $(\tau_0, \infty) \times \mathbb{R}^N$ with initial condition $\tilde{V}(\tau_0) = v(\tau_0)$. Recalling (4.12), we infer from the comparison principle that
\[
(4.13) \quad \tilde{V}(\tau, y) \leq v(\tau, y) \quad \text{for} \quad (\tau, y) \in (\tau_0, \infty) \times \mathbb{R}^N.
\]
On the other hand, it follows from Proposition [A.1] that $(\text{Lemma 4.1 ensures that})$ there is \(v^* \in \mathbb{R}^N\) such that
\[
\lim_{\tau \to \infty} \sup_{y \in \mathbb{R}^N} \left| V(\tau, y) - (\|v(\tau_0)\|_\infty - \gamma_q |y|^{q/(q-1)})_+ \right| = 0.
\]
We may then pass to the limit as $\tau \to \infty$ in (4.13) and use the definition (4.7) to conclude that
\[
(\|v(\tau_0)\|_\infty - \gamma_q |y|^{q/(q-1)})_+ - \kappa_2 \int_{\tau_0}^{\infty} e^{-(p-q)s} ds \leq v_*(y) \leq v^*(y)
\]
for $y \in \mathbb{R}^N$. Letting $\tau_0 \to \infty$ in the above inequality with the help of (4.5) completes the proof of the lemma.

We are now in a position to complete the proof of Theorem 1.1. To this end, fix $\varepsilon \in (0, 1)$. Lemma 4.1 ensures that $v^*(y) \leq \varepsilon$ for $|y| \geq R_\varepsilon \geq 1/\varepsilon$ while the continuity of $H_\infty$ implies that there is $r_\varepsilon \in (0, \varepsilon)$ such that $H_\infty(y) \geq M_\infty - \varepsilon$ for $|y| \leq r_\varepsilon$. Recalling (4.8), we realize that
\[
(4.14) \quad \begin{cases} v^*(y) - \varepsilon \leq 0 \leq H_\infty(y) & \text{if} \quad |y| = R_\varepsilon, \\
v^*(y) - \varepsilon \leq M_\infty - \varepsilon \leq H_\infty(y) & \text{if} \quad |y| = r_\varepsilon. 
\end{cases}
\]
Moreover, introducing $\psi(y) = -\gamma_q |y|^{q/(q-1)}/2$, we have
\[
(4.15) \quad H(y, \nabla \psi(y)) = \frac{q \gamma_q}{2(q-1)} |y|^{q/(q-1)} \left(1 - \frac{1}{2q-1}\right) > 0 \quad \text{if} \quad r_\varepsilon < |y| < R_\varepsilon,
\]
the Hamiltonian $H$ being defined in (4.9). Summarizing, we have shown that $H_\infty$ and $v^*-\varepsilon$ are supersolution and subsolution, respectively, to (4.9) in $\Omega_\varepsilon := \{y \in \mathbb{R}^N : r_\varepsilon < |y| < R_\varepsilon\}$ with $v^*-\varepsilon \leq H_\infty$ on $\partial \Omega_\varepsilon$ by (4.14). Owing to (4.15) and the concavity of $H$ with respect to its second variable, we may apply [15, Theorem 1] to conclude that $v^*-\varepsilon \leq H_\infty$ in $\Omega_\varepsilon$. 

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This property being valid for each \( \varepsilon \in (0, 1) \), we actually have \( v^* \leq H_\infty \) in \( \mathbb{R}^N \) by passing to the limit as \( \varepsilon \to 0 \) thanks to the properties of \( r_\varepsilon \) and \( R_\varepsilon \). Recalling (4.11), we have thus established that \( v^* = v_* = H_\infty \) in \( \mathbb{R}^N \). In particular, the property \( v^* = v_* \) and the definition (4.7) provide the uniform convergence of \( \{v(\tau)\}_{\tau \geq 0} \) towards \( v^* = H_\infty \) on every compact subset of \( \mathbb{R}^N \) as \( \tau \to \infty \), see [2, Lemme 4.1] or [1, Lemma V.1.9]. Combining this local convergence with Lemma 4.1 actually gives

\[
\lim_{\tau \to \infty} \|v(\tau) - H_\infty\|_\infty = 0.
\]

Theorem 1.1 then readily follows after writing the convergence (4.16) in the original variables (4.16) and noticing that \( \|h_\infty(1 + t) - h_\infty(t)\|_\infty \to 0 \) as \( t \to \infty \).

## 5 Limit value of \( \|u(t)\|_\infty \)

This section is devoted to the proof of Proposition 1.4 for which three cases are to be distinguished and handled differently: \( q \in (1, p - 1] \), \( q \in (p - 1, q_*] \), and \( q \in (q_*, p) \).

**Proof of Proposition 1.4** \( q \in (1, p - 1) \). We proceed as in [18, Proposition 1] (where a similar result is proved for \( p = 2 \) and \( q = 1 \)). For \( \alpha > N/2 \), \( \delta > 0 \), and \( x \in \mathbb{R}^N \), we set \( \varrho_\delta(x) := (1 + \delta |x|^2)^{-\alpha} \). Clearly, \( \varrho_\delta \in L^1(\mathbb{R}^N) \) and it follows from (2.3) that

\[
\frac{d}{dt} \int_{\mathbb{R}^N} \varrho_\delta(x) u(t, x) \, dx = \int_{\mathbb{R}^N} \left\{ \varrho_\delta(x) |\nabla u(t, x)|^q - |\nabla u(t, x)|^{p-2} \nabla u(t, x) \cdot \nabla \varrho_\delta(x) \right\} \, dx \\
\geq \int_{\mathbb{R}^N} \varrho_\delta(x) |\nabla u(t, x)|^q \left( 1 - |\nabla u(t, x)|^{p-2} \frac{|\nabla \varrho_\delta(x)|}{\varrho_\delta(x)} \right) \, dx.
\]

Recalling that \( \|\nabla u(t)\|_\infty \leq \|\nabla u_0\|_\infty \) by (1.3) and (2.2) and noticing that \( |\nabla \varrho_\delta| \leq \alpha \delta^1/2 \varrho_\delta \), we further obtain

\[
\frac{d}{dt} \int_{\mathbb{R}^N} \varrho_\delta(x) u(t, x) \, dx \geq \int_{\mathbb{R}^N} \varrho_\delta(x) |\nabla u(t, x)|^q \left( 1 - \alpha \delta^1/2 \|\nabla u_0\|_\infty^{p-1-q} \right) \, dx.
\]

Choosing \( \delta = \|\nabla u_0\|_\infty^{2(q+1-p)/\alpha^2} \) and integrating with respect to time give

\[
\|u(t)\|_\infty \|\varrho_\delta\|_1 \geq \int_{\mathbb{R}^N} \varrho_\delta(x) u(t, x) \, dx \geq \int_{\mathbb{R}^N} \varrho_\delta(x) u_0(x) \, dx > 0.
\]

We then pass to the limit as \( t \to \infty \) to conclude that \( M_\infty > 0 \).

We next turn to the case \( q \in (p-1, q_*) \) which turns out to be more complicated and requires two preparatory results.
Lemma 5.1 Assume that \( q \in (1, q_*) \) and let \( u \) be a non-negative solution to (1.1), (1.2) with a compactly supported initial condition \( u_0 \) satisfying (1.3). Then \( u(t) \in L^1(\mathbb{R}^N) \) for each \( t \geq 0 \), the function \( t \mapsto \|u(t)\|_1 \) is non-decreasing and

\[
\lim_{t \to \infty} \|u(t)\|_1 = \infty.
\]

Proof. For every \( t \geq 0 \), \( u(t) \) is bounded and compactly supported by (2.1) and Proposition 2.2, and is thus in \( L^1(\mathbb{R}^N) \). The time monotonicity of the \( L^1 \)-norm of \( u \) then readily follows from (2.3) with \( \vartheta = 1 \), a valid choice in this particular case as \( u(t) \) is compactly supported. It further follows from (2.3) with \( \vartheta = 1 \) that

\[
\|u(t)\|_1 \geq \|u(T)\|_1 + \int_T^t \|\nabla u(s)\|_q^q \, ds \quad \text{for } t > T \geq 0.
\]

Consider next \( T > 0 \) and \( t > T \). Recalling the Gagliardo-Nirenberg inequality

\[
w_q \leq C_6 \|\nabla w\|_{L^q(N(q-1)/(N(q-1)+q)} \|w\|_{L^1(N(q-1)+q)}, \quad w \in W^{1,q}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N),
\]

we infer from (5.2), (5.3), and the time monotonicity of the \( L^1 \)-norm of \( u \) that

\[
\|u(t)\|_1^{1+(q^2/N(q-1))} \geq \|u(t)\|_1^{2/(N(q-1))} \left( \|u(T)\|_1 + \int_T^t \|\nabla u(s)\|_q^q \, ds \right)
\]
\[
\geq \int_T^t \|u(s)\|_1^{2/(N(q-1))} \|\nabla u(s)\|_q^q \, ds
\]
\[
\geq C_7 \int_T^t (\|u(s)\|_q^q)^{(N(q-1)+q)/(N(q-1))} \, ds.
\]

If \( \varphi \) denotes the solution to the \( p \)-Laplacian equation \( \partial_t \varphi - \Delta_p \varphi = 0 \) in \( Q_\infty \) with initial condition \( \varphi(0) = u_0 \), the comparison principle readily implies that

\[
\varphi(t, x) \leq u(t, x), \quad (t, x) \in Q_\infty.
\]

Inserting this estimate in the previous lower bound for \( \|u(t)\|_1 \), we end up with

\[
\|u(t)\|_1^{1+(q^2/N(q-1))} \geq C_7 \int_T^t (\|\varphi(s)\|_q^q)^{(N(q-1)+q)/(N(q-1))} \, ds.
\]

Now, by Proposition 2.3 we have

\[
\lim_{s \to \infty} s^{N(q-1)/(N(p-2)+p)} \|\varphi(s) - B_{\|u_0\|_1}(s)\|_q^q = 0
\]

and

\[
\|B_{\|u_0\|_1}(s)\|_q^q = C_8 \ s^{-N(q-1)/(N(p-2)+p)},
\]

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so that
\[
\| \varphi(s) \|_q^q \geq \left( \| B_{\| u_0 \|_1} (s) \|_q - \| \varphi(s) - B_{\| u_0 \|_1} (s) \|_q \right)^q \\
\geq s^{-N(q-1)/(N(p-2)+p)} \left( C_8 - \| \varphi(s) - B_{\| u_0 \|_1} (s) \|_q \right)^q \\
\geq \left( \frac{C_8}{2} \right)^q s^{-N(q-1)/(N(p-2)+p)}
\]
for \( s \geq T \), provided \( T \) is chosen sufficiently large. Inserting this estimate in (5.5) gives
\[
\| u(t) \|_1^{1+(q^2/N(q-1))} \geq C_9 \int_T^t s^{-N(q-1)/(N(p-2)+p)} \, ds \\
\geq C_{10} \begin{cases} \\
\quad t^{(N+1)(q_*-q)/(N(p-2)+p)} - T^{(N+1)(q_*-q)/(N(p-2)+p)} & \text{if } q \in (1, q_*], \\
\quad \log(t/T) & \text{if } q = q_*.
\end{cases}
\]
We then let \( t \rightarrow \infty \) to obtain the claimed result. \( \square \)

We next argue as in [12, Lemma 14] (for \( p = 2 \)) to show that, if \( q \in (p-1, p) \) and \( M_{\infty} = 0 \), then the \( L^\infty \)-norm of \( u(t) \) decays faster than an explicit rate.

**Lemma 5.2** Assume that \( q \in (p-1, p) \) and let \( u \) be a non-negative solution to (1.1), (1.2) with an initial condition \( u_0 \) satisfying (1.3). If \( M_{\infty} = 0 \) in (1.4), then
\[
\| u(t) \|_\infty \leq C_{11} \, t^{-(p-q)/(2q-p)} \quad \text{for} \quad t > 0.
\]
Observe that the assumptions \( p > 2 \) and \( q \in (p-1, p) \) imply that \( 2q > p \) and \( (p-q)/(2q-p) > 0 \).

**Proof.** Consider a non-negative function \( \eta \in C^\infty (\mathbb{R}^N) \) with compact support in \( B(0, 1) \) and \( \| \eta \|_1 = 1 \). We then define a sequence of mollifiers \( \eta_\delta \) by \( \eta_\delta (x) := \eta(x/\delta)/\delta^N \) for \( x \in \mathbb{R}^N \) and \( \delta \in (0, 1) \). For \( (t,x) \in Q_\infty \) and \( T > t \), we take \( \vartheta(x) = \eta_\delta (x - x_0) \) in (2.3) and infer from (1.11) (with \( s = t/2 \)) that
\[
\| u(T) \|_\infty \geq \int_{\mathbb{R}^N} u(T, x) \, \eta_\delta (x - x_0) \, dx \\
\geq \int_{\mathbb{R}^N} u(t, x) \, \eta_\delta (x - x_0) \, dx - \int_t^T \int_{\mathbb{R}^N} |\nabla u(s, x)|^{p-2} \, \nabla u(s, x) \cdot \nabla \eta_\delta(t, x - x_0) \, dx \, ds \\
\geq \int_{\mathbb{R}^N} u(t, x) \, \eta_\delta (x - x_0) \, dx - 2^{p/q} \, \kappa_1 \| u \|_\infty \left( t^{(p-q)/q} \int_t^T (2s-t)^{-p/q} \, ds \right) \\
\geq \int_{\mathbb{R}^N} u(t, x) \, \eta_\delta (x - x_0) \, dx - C_{12} \left\| u \left( \frac{t}{2} \right) \right\|_\infty \left( t^{(p-q)/p} - T^{(q-p)/p} \right).
\]
Owing to the continuity of \( u \), we may pass to the limit as \( \delta \to 0 \) in the above inequality and deduce that
\[
\|u(T)\|_\infty \geq u(t, x_0) - C_{12} \left\| u \left( \frac{t}{2} \right) \right\|_\infty^{(p-q)/q} \left( t^{(q-p)/p} - T^{(q-p)/p} \right).
\]

But the above inequality is valid for all \( x_0 \in \mathbb{R}^N \) and we thus end up with
\[
\|u(T)\|_\infty \geq \|u(t)\|_\infty - C_{12} \left\| u \left( \frac{t}{2} \right) \right\|_\infty^{(p-q)/q} \left( t^{(q-p)/p} - T^{(q-p)/p} \right).
\]

Finally, as \( q < p \), we may let \( T \to \infty \) in the previous inequality and use the assumption \( M_\infty = 0 \) to conclude that
\[
\|u(t)\|_\infty \leq C_{13} \left\| u \left( \frac{t}{2} \right) \right\|_\infty^{(p-q)/q} \left( t^{(q-p)/p} \right),
\]
or, equivalently, as \( 2q > p \),
\[
t^{(p-q)/(2q-p)} \|u(t)\|_\infty \leq C_{13} \left\{ \left( \frac{t}{2} \right)^{(p-q)/(2q-p)} \left\| u \left( \frac{t}{2} \right) \right\|_\infty \right\}^{(p-q)/q}
\]
for \( t \geq 0 \). Introducing
\[
A(t) := \sup_{s \in (0, t)} \left\{ s^{(p-q)/(2q-p)} \|u(s)\|_\infty \right\} \in [0, \infty), \ t \geq 0,
\]
we deduce from the previous inequality that \( A(t) \leq C_{13} A(t)^{(p-q)/q} \), whence \( A(t) \leq C_{13}^{q/(2q-p)} \) for \( t \geq 0 \). This bound being valid for each \( t > 0 \), the proof of (5.6) is complete. \( \square \)

**Proof of Proposition 1.4:** \( q \in (p-1, q_*) \).

**Step 1:** We first consider a compactly supported initial condition \( u_0 \) satisfying (1.3) and assume for contradiction that \( M_\infty = 0 \). On the one hand, according to Lemma 5.2 and the assumption \( q \leq q_* \), there holds
\[
\limsup_{t \to \infty} t^{N/(N(p-2)+p)} \|u(t)\|_\infty \leq C_{11} t^{(N+1)(q-q_*)/((2q-p)(N-2)+p))} \leq C_{11}.
\]

On the other hand, fix \( t_0 > 0 \) and let \( \varphi \) be the solution to the \( p \)-Laplacian equation \( \partial_t \varphi - \Delta_p \varphi = 0 \) in \( Q_\infty \) with initial condition \( \varphi(0) = u(t_0) \). As \( u_0 \) is compactly supported, so is \( u(t_0) \) by Proposition 2.2 and \( u(t_0) \) thus belongs to \( L^1(\mathbb{R}^N) \). Moreover, the comparison principle warrants that \( u(t, x) \geq \varphi(t - t_0, x) \) for \( (t, x) \in [t_0, \infty) \times \mathbb{R}^N \). We then infer from the above
Using once more Proposition 2.3, we may pass to the limit as \( t \to \infty \) in the previous inequality to obtain

\[ \liminf_{t \to \infty} t^{\frac{N}{(p-2) + p}} \|u(t)\|_\infty \geq C_{14} \|u(t_0)\|_1^{\frac{N}{(p-2) + p}}. \]

Combining (5.7) and (5.8) yields \( \|u(t_0)\|_1 \leq C_{15} \) for all \( t_0 > 0 \) which contradicts Lemma 5.1. Therefore, \( M_\infty > 0 \).

**Step 2:** Now, if \( u_0 \) is an arbitrary initial condition satisfying (1.3), there clearly exists a compactly supported initial condition \( \tilde{u}_0 \) satisfying (1.3) and such that \( u_0 \geq \tilde{u}_0 \) in \( \mathbb{R}^N \). Introducing the solution \( \tilde{u} \) to (1.1) with initial condition \( \tilde{u}_0 \), the comparison principle entails that \( u \geq \tilde{u} \) in \( Q_\infty \), hence

\[ M_\infty \geq \lim_{t \to \infty} \|\tilde{u}(t)\|_\infty. \]

The first step of the proof ensures that the right-hand side of the above inequality is positive which completes the proof.

It remains to investigate the case \( q \in (q_*, p) \), for which we adapt the proof of [7, Theorem 2.4(b)].

**Proof of Proposition 1.4:** \( q \in (q_*, p) \). We put

\[ m_0 := \inf_{y \in \mathbb{R}^N} \Delta_p u_0(y). \]

As in the proof of Lemma 5.1 let \( \eta \in C^\infty(\mathbb{R}^N) \) be a non-negative function with compact support in \( B(0, 1) \) and \( \|\eta\|_1 = 1 \), and define a sequence of mollifiers \( (\eta_\delta) \) by \( \eta_\delta(x) := \eta(x/\delta)/\delta^N \) for \( x \in \mathbb{R}^N \) and \( \delta \in (0, 1) \). For \( (t, x_0) \in Q_\infty \) and \( T \in (0, t) \), we take \( \vartheta(x) = \eta_\delta(x - x_0) \) in (2.3) and infer from (1.11) (with \( s = 0 \)) and Corollary 3.3 that

\[ \|u(t)\|_\infty \geq \int_{\mathbb{R}^N} u(t, x) \eta_\delta(x - x_0) \, dx \]

\[ \geq \int_{\mathbb{R}^N} u_0(x) \eta_\delta(x - x_0) \, dx - \int_0^T \int_{\mathbb{R}^N} \sqrt{u(s, x)} |\nabla u(s, x)|^{p-2} \nabla u(s, x) \cdot \nabla \eta_\delta(t, x - x_0) \, dx \, ds \]

\[ \geq \int_{\mathbb{R}^N} u_0(x) \eta_\delta(x - x_0) \, dx - \int_0^T m_0 \, ds - \kappa_1 \|u_0\|_{\infty}^{(p-q)/q} \int_T^t s^{-(p/q)} \, ds \]

\[ \geq \int_{\mathbb{R}^N} u_0(x) \eta_\delta(x - x_0) \, dx - T \, m_0 - C_{16} \|u_0\|_{\infty}^{(p-q)/q} \left( T^{(q-p)/p} - t^{(q-p)/p} \right). \]
Owing to the continuity of $u_0$, we may pass to the limit as $\delta \to 0$ in the above inequality and deduce that
\[
\|u(t)\|_\infty \geq u_0(x_0) - T m_0 - C_{16} \|u_0\|_\infty^{(p-q)/q} \left(T^{(q-p)/p} - t^{(q-p)/p}\right).
\]
Since $q < p$, we may let $t \to \infty$ in the above inequality and take the supremum with respect to $x_0$ to conclude that
\[
M_\infty \geq \|u_0\|_\infty - T m_0 - C_{16} \|u_0\|_\infty^{(p-q)/q} T^{(q-p)/p}.
\]
Next, for $\beta \in (0, 1)$, the choice $T = \|u_0\|_\infty^{(p-q)/q} (\beta + m_0)^{-q/p}$ in the previous inequality yields
\[
M_\infty \geq \|u_0\|_\infty^{(p-q)/q} \left(\|u_0\|_\infty^{p/q} - (1 + C_{16}) (\beta + m_0)^{(p-q)/p}\right).
\]
This inequality being valid for every $\beta \in (0, 1)$, we conclude that
\[
M_\infty \geq \|u_0\|_\infty^{(p-q)/q} \left(\|u_0\|_\infty^{p/q} - (1 + C_{16}) m_0^{(p-q)/p}\right) > 0
\]
as soon as (1.10) is fulfilled with $\kappa_0 = (1 + C_{16})^{p/q}$.

\[\Box\]

A Convergence for the Hamilton-Jacobi equation (4.6)

In this section, we study the large behaviour of non-negative solutions to the Hamilton-Jacobi equation (4.6) with initial data in $C_0(\mathbb{R}^N)$ and show their convergence to a steady state uniquely determined by the $L^\infty$-norm of the initial data. Though the large time behaviour of solutions to first-order Hamilton-Jacobi equations has received considerable attention in recent years (see [4, 5, 16, 21, 22] and the references therein), the particular case of (4.6) does not seem to have been investigated in the literature. We thus provide a simple proof relying on the Hopf-Lax-Oleinik formula.

**Proposition A.1** Let $q > 1$ and consider a non-negative function $h_0 \in C_0(\mathbb{R}^N)$. Let $h$ be the unique viscosity solution to the Cauchy problem

\[
\begin{align*}
\partial_\tau h - y \cdot \nabla h - q \left|\nabla h\right|^q &= 0, \quad (\tau, y) \in (0, \infty) \times \mathbb{R}^N, \\
h(0) &= h_0, \quad y \in \mathbb{R}^N.
\end{align*}
\]

Then

\[
\lim_{\tau \to \infty} \|h(\tau) - h_s\|_\infty = 0
\]

with
\[
h_s(y) := \left(\|h_0\|_\infty - \gamma_q \left|y\right|^{q/(q-1)}\right)_+, \quad y \in \mathbb{R}^N,
\]
the constant $\gamma_q = (q - 1) q^{-q/(q-1)}$ being defined in Theorem [1.1].
Thanks to the concavity of the Hamiltonian \( H(y, \xi) = -y \cdot \xi - q \| \xi \|^q, (y, \xi) \in \mathbb{R}^N \times \mathbb{R}^N \), with respect to its second variable, the Hopf-Lax-Oleinik formula provides a representation formula for the solution \( h \) to (A.1), (A.2) which can be used to prove (A.3).

**Proof.** We first recall that \( h \) is given by the Hopf-Lax-Oleinik formula

\[
h(\tau, y) = \sup_{z \in \mathbb{R}^N} \left\{ h_0(z) - \gamma_q \, |y - z - e^{-\tau} e^{q/(q-1)} (1 - e^{-\tau})^{-1/(q-1)}| \right\}
\]

for \((\tau, y) \in [0, \infty) \times \mathbb{R}^N\), see, e.g., [10, Chapter 3]. Since \( h(\tau, y) \geq h_0(ye^\tau) \geq 0 \), we have in fact

\[
h(\tau, y) = \sup_{z \in \mathbb{R}^N} \left\{ (h_0(z) - \gamma_q \, |y - z - e^{-\tau} e^{q/(q-1)} (1 - e^{-\tau})^{-1/(q-1)}|) + \right\}
\]

for \((\tau, y) \in [0, \infty) \times \mathbb{R}^N\).

Consider now \( \beta \in (0, 1) \). As \( h_0 \in C_0(\mathbb{R}^N) \), there is \( R_\beta > (\|h_0\|_\infty / \gamma_q)^{(q-1)/q} \) such that

\[
(A.4) \quad h_0(z) \leq \beta \quad \text{for} \quad |z| \geq R_\beta.
\]

On the one hand, if \((\tau, y) \in [\log R_\beta, \infty) \times \mathbb{R}^N \) and \( z \in \mathbb{R}^N \), we have either \(|z| \geq R_\beta \) and

\[
\left| \left( h_0(z) - \gamma_q \, |y - z - e^{-\tau} e^{q/(q-1)} (1 - e^{-\tau})^{-1/(q-1)}| \right)_+ + \left( h_0(z) - \gamma_q \, |y|^{q/(q-1)} \right)_+ \right|
\]

\[
\leq \gamma_q \, |y - z - e^{-\tau} e^{q/(q-1)} \left\{ (1 - e^{-\tau})^{-1/(q-1)} - 1 \right\} + \gamma_q \, |y - z - e^{-\tau} e^{q/(q-1)} - |y|^{q/(q-1)}| \right.
\]

\[
\leq \gamma_q \, (|y| + R_\beta e^{-\tau})^{q/(q-1)} \left\{ (1 - e^{-\tau})^{-1/(q-1)} - 1 \right\} + \frac{q \gamma_q}{q - 1} \, (|y| + |z|)^{1/(q-1)} \, |z| \, e^{-\tau}
\]

as \( \tau \geq \log R_\beta \). Combining the above two estimates give

\[
\left| h(\tau, y) - \sup_{z \in \mathbb{R}^N} \left\{ (h_0(z) - \gamma_q \, |y|^{q/(q-1)} \right\}_+ \right|
\]

\[
\leq C(q) \, (|y| + 1)^{q/(q-1)} \left\{ (1 - e^{-\tau})^{-1/(q-1)} - 1 + R_\beta e^{-\tau} \right\} + 2 \beta,
\]

whence

\[
(A.5) \quad |h(\tau, y) - h_*(y)| \leq C(q) \, (|y| + 1)^{q/(q-1)} \left\{ (1 - e^{-\tau})^{-1/(q-1)} - 1 + R_\beta e^{-\tau} \right\} + 2 \beta
\]
for \((\tau, y) \in [\log R_\beta, \infty) \times \mathbb{R}^N\). On the other hand, if \(\tau \geq \log(R_\beta)\), \(|y| \geq Y := 1 + (\|h_0\|_\infty/\gamma_q)^{(q-1)/q}\) and \(z \in \mathbb{R}^N\), we have either \(|y - z e^{-\tau}| \geq Y - 1\) and

\[
\begin{align*}
  h_0(z) - \gamma_q |y - z e^{-\tau}|^{q/(q-1)} (1 - e^{-q\tau})^{-1/(q-1)} \\
  \leq (1 - e^{-q\tau})^{-1/(q-1)} \left\{ \|h_0\|_\infty (1 - e^{-q\tau})^{1/(q-1)} - \gamma_q |y - z e^{-\tau}|^{q/(q-1)} \right\} \\
  \leq (1 - e^{-q\tau})^{-1/(q-1)} \left\{ \|h_0\|_\infty - \gamma_q (Y - 1)^{q/(q-1)} \right\} \\
  \leq 0,
\end{align*}
\]

or \(|y - z e^{-\tau}| < Y - 1\) and

\[
|z| \geq |y e^\tau| - |z - y e^\tau| \geq Y e^\tau - (Y - 1) e^\tau = e^\tau \geq R_\beta,
\]

so that

\[
\begin{align*}
  h_0(z) - \gamma_q |y - z e^{-\tau}|^{q/(q-1)} (1 - e^{-q\tau})^{-1/(q-1)} \leq \beta
\end{align*}
\]

by \((\ref{eq:A.4})\). Therefore,

\[
\begin{align*}
  h(\tau, y) \leq \beta \quad \text{for} \quad (\tau, y) \in [\log R_\beta, \infty) \times \mathbb{R}^N \setminus B(0, Y).
\end{align*}
\]

The claim \((\ref{eq:A.3})\) then easily follows from \((\ref{eq:A.5})\) and \((\ref{eq:A.6})\). \(\Box\)

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