On the Undecidability of Fuzzy Description Logics with GCIs
with Łukasiewicz $t$-norm

Marco Cerami
IIIA - CSIC
Bellaterra, Catalunya
cerami@iiia.csic.es

Umberto Straccia
ISTI - CNR
Pisa, Italy
umberto.straccia@isti.cnr.it

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Abstract

Recently there have been some unexpected results concerning Fuzzy Description Logics (FDLs) with General Concept Inclusions (GCIs). They show that, unlike the classical case, the DL $\mathcal{ALC}$ with GCIs does not have the finite model property under Łukasiewicz Logic or Product Logic and, specifically, knowledge base satisfiability is an undecidable problem for Product Logic. We complete here the analysis by showing that knowledge base satisfiability is also an undecidable problem for Łukasiewicz Logic.

1 Introduction

Description Logics (DLs) [1] play a key role in the design of Ontologies. Indeed, DLs are important as they are essentially the theoretical counterpart of the Web Ontology Language OWL 2 [19], the standard language to represent ontologies.

It is very natural to extend DLs to the fuzzy case and several fuzzy extensions of DLs can be found in the literature. For a recent survey on the advances in the field of fuzzy DLs, we refer the reader to [18]. Besides the enrichment of DLs with fuzzy features, one of the challenges of the research in this community is the fact that different families of fuzzy operators (or fuzzy logics) lead to fuzzy DLs with different computational properties.

Decidability of fuzzy DLs is often shown by adapting crisp DL tableau-based algorithms to the fuzzy DL case [3, 21, 22, 23, 25, 26], or a reduction to classical DLs [5, 6, 7, 9, 24], or relying on some Mathematical Fuzzy Logic [13] based procedures [11, 12, 14, 15].

However, recently there have been some unexpected surprises [2, 3, 4]. Indeed, unlike the classical case, for the DL $\mathcal{ALC}$ with GCIs (i) [4] shows that it does not have the finite model property under Łukasiewicz Logic or Product Logic, illustrates that some algorithms are neither complete nor correct, and shows some interesting conditions under which decidability is still guaranteed; and (ii) [2, 4] show that knowledge base satisfiability
is an undecidable problem for it under Product Logic. Also worth mentioning is [10], which illustrates the undecidability of knowledge base satisfiability if one replaces the truth set \([0, 1]\) with complete De Morgan lattices equipped with a t-norm operator.

In this paper, we complete the analysis by showing that knowledge base satisfiability is an undecidable problem for the DL \(\mathcal{ALC}\) with GCIs under \([0, 1]-valued /\) suppress Lukasiewicz Logic as well. We prove our result following conceptually the methods devised in [2, 3, 10].

We next introduce briefly our fuzzy DL, then we illustrate the undecidability result.

2 The FDL /\ suppress \(\mathcal{L-ALC}\)

In this section we are going to introduce the general definitions of \(\mathcal{L-ALC}\) based on /\ suppress Lukasiewicz \(t\)-norm.

**Syntax.** Let \(A\) be a set of concept names, \(R\) be a set of role names. Concept names denote unary predicates, while role names denote binary predicates. The set of \(\mathcal{L-ALC}\) concepts are built from concept names \(A\) (also called atomic concepts) using connectives and quantification constructs over roles \(R\) as described by the following syntactic rules:

\[
C \rightarrow \top \mid \bot \mid A \mid C_1 \cap C_2 \mid C_1 \sqcup C_2 \mid \neg C \mid \exists R.C \mid \forall R.C.
\]

An assertion axiom is an expression of the form \(\langle a:C, n \rangle\) (concept assertion, \(a\) is an instance of concept \(C\) to degree at least \(n\)) or of the form \(\langle (a_1, a_2):R, n \rangle\) (role assertion, \((a_1, a_2)\) is an instance of role \(R\) to degree at least \(n\)), where \(a, a_1, a_2\) are individual names, \(C\) is a concept, \(R\) is a role name and \(n \in (0, 1]\) is a rational (a truth value). An ABox \(A\) consists of a finite set of assertion axioms.

A General Concept Inclusion (GCI) axiom is of the form \(\langle C_1 \sqsubseteq C_2, n \rangle\) (\(C_1\) is a subconcept of \(C_2\) to degree at least \(n\)), where \(C_1\) is a concept and \(n \in (0, 1]\) is a rational. A concept hierarchy \(T\), also called \(TBox\), is a finite set of GCIs. In what follows we will use the following shorthands:

- \(C_1 \sqsubseteq C_2\) for \(\langle C_1 \sqsubseteq C_2, 1 \rangle\) and \(a:C\) for \(\langle a:C, 1 \rangle\);
- \(C_1 \equiv C_2\) for the two axioms \(C_1 \sqsubseteq C_2\) and \(C_2 \sqsubseteq C_1\);
- \(C_1 \rightarrow C_2\) for \(\neg C_1 \sqcup C_2\);
- \(C_1 \leftrightarrow C_2\) for \((C_1 \rightarrow C_2) \cap (C_2 \rightarrow C_1)\);
- \(\min\{C_1, C_2\}\) for \(C_1 \sqcap (C_1 \rightarrow C_2)\), and \(\min\{C_1, \ldots, C_n\}\) for \(\min\{\ldots \min\{C_1, C_2\}, \ldots\}\);
- \(\max\{C_1, C_2\}\) for \((C_1 \rightarrow C_2) \rightarrow C_2\) and \(\max\{C_1, \ldots, C_n\}\) for \(\max\{\ldots \max\{C_1, C_2\}, \ldots\}\);

\footnote{1Each symbol may have super- and/or subscripts.}
Table 1: Semantics for L-\(\mathcal{ALC}\).

- \(n \cdot C\) for the \(n\)-ary disjunction \(C \sqcup \ldots \sqcup C\);

Finally, a knowledge base \(\mathcal{K} = \langle T, A \rangle\) consists of a TBox \(T\) and an ABox \(A\).

**Semantics.** From a semantics point of view, an axiom \(\langle \alpha, n \rangle\) constrains the truth degree of the expression \(\alpha\) to be at least \(n\). In the following, we use \(\otimes, \oplus, \ominus\) and \(\Rightarrow\) to denote Lukasiewicz \(t\)-norm, \(t\)-conorm, negation function, and implication function, respectively \([17]\). They are defined as operations in \([0, 1]\) by means of the following functions:

\[
\begin{align*}
    a \otimes b &= \max\{0, a + b - 1\} \\
    a \oplus b &= \min\{1, a + b\} \\
    \ominus a &= 1 - a \\
    a \Rightarrow b &= \min\{1, 1 - a + b\}
\end{align*}
\]

where \(a, b\) are arbitrary elements in \([0, 1]\). As in the classical framework, the implication can be defined in terms of disjunction (whose semantics is the \(t\)-conorm) and negation in the usual way: \(a \Rightarrow b = \ominus a \oplus b\). Note also that for any implication defined from a continuous \(t\)-norm \(\otimes\), it holds that: \(x \Rightarrow y = \max\{z \mid x \otimes z \leq y\}\), which is equivalent to the condition: \(y \geq x \otimes z\) iff \((x \Rightarrow y) \geq z\).

A fuzzy interpretation (or model) is a pair \(\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})\) consisting of a nonempty (crisp) set \(\Delta^\mathcal{I}\) (the domain) and of a fuzzy interpretation function \(\cdot^\mathcal{I}\) that assigns:

1. to each atomic concept \(A\) a function \(A^\mathcal{I} : \Delta^\mathcal{I} \rightarrow [0, 1]\),
2. to each role \(R\) a function \(R^\mathcal{I} : \Delta^\mathcal{I} \times \Delta^\mathcal{I} \rightarrow [0, 1]\),
3. to each individual \(a\) an element \(a^\mathcal{I} \in \Delta^\mathcal{I}\) such that \(a^\mathcal{I} \neq b^\mathcal{I}\) if \(a \neq b\) (Unique Name Assumption, different individuals denote different objects of the domain).

The fuzzy interpretation function is extended to complex concepts as specified in Table 1 (where \(x, y \in \Delta^\mathcal{I}\) are elements of the domain). Hence, for every complex concept \(C\) we get a function \(C^\mathcal{I} : \Delta^\mathcal{I} \rightarrow [0, 1]\). The satisfiability of axioms is then defined by the following conditions:
1. \( \mathcal{I} \) satisfies an axiom \( \langle a; C, \alpha \rangle \) if \( C^\mathcal{I}(a^\mathcal{I}) \geq \alpha \),

2. \( \mathcal{I} \) satisfies an axiom \( \langle (a, b); R, \alpha \rangle \) if \( R^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I}) \geq \alpha \),

3. \( \mathcal{I} \) satisfies an axiom \( \langle C \sqsubseteq D, \alpha \rangle \) if 
\[
(C \sqsubseteq D)^\mathcal{I} = \inf_{x \in \Delta^\mathcal{I}} \{C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x)\}. 
\]

It is interesting to point out that the satisfaction of a GCI of the form \( \langle C \sqsubseteq D, 1 \rangle \) is exactly the requirement that \( \forall x \in \Delta^\mathcal{I}, C^\mathcal{I}(x) \leq D^\mathcal{I}(x) \) (i.e., Zadeh’s set inclusion); hence, in this particular case for the satisfaction it only matters the partial order and not the exact value of the implication \( \Rightarrow \).

As it is expected we will say that a fuzzy interpretation \( \mathcal{I} \) satisfies a KB \( \mathcal{K} \) in case that it satisfies all axioms in \( \mathcal{K} \). And it is said that a fuzzy KB \( \mathcal{K} \) is satisfiable iff there exist a fuzzy interpretation \( \mathcal{I} \) satisfying every axiom in \( \mathcal{K} \).

In this paper, we mainly focus on witnessed models. This notion (see [14]) corresponds to the restriction to the DL language of the notion of witnessed model introduced, in the context of the first-order language, by Hájek in [16]. Specifically, a fuzzy interpretation \( \mathcal{I} \) is said to be witnessed iff it holds that for every complex concepts \( C, D \), every role \( R \), and every \( x \in \Delta^\mathcal{I} \) there is some \( y \in \Delta^\mathcal{I} \) such that

1. \( y \in \Delta^\mathcal{I} \) such that \( (\exists R.C)^\mathcal{I}(x) = R^\mathcal{I}(x, y) \otimes C^\mathcal{I}(y) \)

2. \( y \in \Delta^\mathcal{I} \) such that \( (\forall R.C)^\mathcal{I}(x) = R^\mathcal{I}(x, y) \Rightarrow C^\mathcal{I}(y) \)

If \( \mathcal{I} \) satisfies only condition 1. then \( \mathcal{I} \) is said to be weakly witnessed. Note that for Lukasiewicz logic, condition 1. and 2. are equivalent, so \( \mathcal{I} \) is weakly witnessed iff \( I \) is witnessed. Thorough the paper we will rely on the notion of witnessed interpretation only, but keep in mind that the results apply, thus, to weakly witnessed interpretations as well.

Note also that it is obvious that all finite fuzzy interpretations (this means that \( \Delta^\mathcal{I} \) is a finite set) are indeed strongly witnessed but the opposite is not true.

Sometimes (see, e.g., [3]), the notion of witnessed interpretation is strengthened to so-called strongly witnessed interpretations by imposing that additionally that for every complex concepts \( C, D \) and every \( x \in \Delta^\mathcal{I} \) there is some

\[ y \in \Delta^\mathcal{I} \text{ such that } (C \sqsubseteq D)^\mathcal{I} = C^\mathcal{I}(y) \Rightarrow D^\mathcal{I}(y) \]

has to hold. We do not deal with strongly witnessed interpretations here.

A fuzzy KB \( \mathcal{K} \) is said to be satisfiable iff there exist a fuzzy interpretation \( \mathcal{I} \) satisfying every axiom in \( \mathcal{K} \).
3 Undecidability of L-\(\text{ALC}\) with GCIs

Our proof consists in a reduction of the reverse of the Post Correspondence Problem (PCP) and follows conceptually the one in \[2, 3, 10\]. PCP is well-known to be undecidable \[20\], so is the reverse PCP, as shown next.

**Definition 1** (PCP). Let \(v_1, \ldots, v_p\) and \(w_1, \ldots, w_p\) be two finite lists of words over an alphabet \(\Sigma = \{1, \ldots, s\}\). The Post Correspondence Problem (PCP) asks whether there is a non-empty sequence \(i_1, i_2, \ldots, i_k\), with \(1 \leq i_j \leq p\) such that \(v_1 v_2 \ldots v_i k = w_1 w_2 \ldots w_i k\). Such a sequence, if it exists, is called a solution of the problem instance.

For the sake of our purpose, we will rely on a variant of the PCP, which we call Reverse PCP (RPCP). Essentially, words are concatenated from right to left rather than from left to right.

**Definition 2** (RPCP). Let \(v_1, \ldots, v_p\) and \(w_1, \ldots, w_p\) be two finite lists of words over an alphabet \(\Sigma = \{1, \ldots, s\}\). The Reverse Post Correspondence Problem (RPCP) asks whether there is a non-empty sequence \(i_1, i_2, \ldots, i_k\), with \(1 \leq i_j \leq p\) such that \(v_1 v_2 \ldots v_i k = w_1 w_2 \ldots w_i k\). Such a sequence, if it exists, is called a solution of the problem instance.

For a word \(\mu = i_1 i_2 \ldots i_k \in \{1, \ldots, p\}^*\) we will use \(v_\mu\), \(w_\mu\) to denote the words \(v_{i_k} v_{i_{k-1}} \ldots v_{i_1}\) and \(w_{i_k} w_{i_{k-1}} \ldots w_{i_1}\). We denote the empty string as \(\epsilon\) and define \(v_1\) is \(\epsilon\). The alphabet \(\Sigma\) consists of the first \(s\) positive integers. We can thus view every word in \(\Sigma^*\) as a natural number represented in base \(s + 1\) in which \(0\) never occurs. Using this intuition, we will use the number \(0\) to encode the empty word.

Now we show that the reduction from PCP to RPCP is a very simple matter and it can be done through the transformation of the instance lists to the lists of their palindromes defined as follows: let \(\Sigma = \{1, \ldots, s\}\) be an alphabet and \(v = t_1 t_2 \ldots t_{|v|}\) a word over \(\Sigma\), with \(t_i \in \Sigma\), for \(1 \leq j \leq |v|\), then the palindrome of \(v\) is defined as \(\text{pal}(v) = t_{|v|} t_{|v|-1} \ldots t_1\).

**Lemma 3.** Let \(v_1, \ldots, v_p\) and \(w_1, \ldots, w_p\) be two finite lists of words over an alphabet \(\Sigma = \{1, \ldots, s\}\). For every non-empty sequence \(i_1, i_2, \ldots, i_k\), with \(1 \leq i_j \leq p\) it holds that

\[
\text{pal}(v_{i_k}) \text{pal}(v_{i_{k-1}}) \ldots \text{pal}(v_{i_1}) = \text{pal}(w_{i_k}) \text{pal}(w_{i_{k-1}}) \ldots \text{pal}(w_{i_1}).
\]

**(Proof)** First we prove by induction on \(k\), that, for every sequence \(v = v_{i_1} v_{i_2} \ldots v_{i_k}\) of words over \(\Sigma\), it holds that \(\text{pal}(v) = \text{pal}(v_{i_k}) \text{pal}(v_{i_{k-1}}) \ldots \text{pal}(v_{i_1})\).

- The case \(k = 1\) is straightforward.
- Let \(v = v_{i_1} v_{i_2} \ldots v_{i_k}\) and suppose, by inductive hypothesis, that \(\text{pal}(v_{i_1} v_{i_2} \ldots v_{i_{k-1}}) = \text{pal}(v_{i_{k-1}}) \text{pal}(v_{i_{k-2}}) \ldots \text{pal}(v_{i_1})\). It follows that \(\text{pal}(v) = \text{pal}(v_{i_k}) \text{pal}(v_{i_{k-1}}) \ldots \text{pal}(v_{i_1})\).
Since the palindrome of a word is unique, we have that, if \(v_{i_1} v_{i_2} \cdots v_{i_k} = w_{i_1} w_{i_2} \cdots w_{i_k}\), then \(\text{pal}(v_{i_1} v_{i_2} \cdots v_{i_k}) = \text{pal}(w_{i_1} w_{i_2} \cdots w_{i_k})\) and, thus, \(\text{pal}(v_{i_k}) \text{pal}(v_{i_{k-1}}) \cdots \text{pal}(v_{i_1}) = \text{pal}(w_{i_k}) \text{pal}(w_{i_{k-1}}) \cdots \text{pal}(w_{i_1})\).

**Corollary 4.** The RPCP is undecidable.

*(Proof)* The proof is based on the reduction of PCPs to RCPs. For every instance \(\varphi = (v_1, w_1), \ldots, (v_p, w_p)\) of PCP, let \(f\) be the function

\[
\phi = (\text{pal}(v_1), \text{pal}(w_1)), \ldots, (\text{pal}(v_p), \text{pal}(w_p))
\]

Clearly \(f\) is a computable function. Moreover, \(\varphi \in \text{PCP}\) if and only if there exists a non-empty sequence \(i_1, i_2, \ldots, i_k\), with \(1 \leq i_j \leq p\) such that \(v_{i_1} v_{i_2} \cdots v_{i_k} = w_{i_1} w_{i_2} \cdots w_{i_k}\), that is, by Lemma 3,

\[
\text{pal}(v_{i_k}) \text{pal}(v_{i_{k-1}}) \cdots \text{pal}(v_{i_1}) = \text{pal}(w_{i_k}) \text{pal}(w_{i_{k-1}}) \cdots \text{pal}(w_{i_1})
\]

i.e., \(f(\varphi) \in \text{RPCP}\). Therefore, \(\varphi \in \text{PCP}\) if and only if \(f(\varphi) \in \text{RPCP}\).

**Undecidability of general KB satisfiability.** We show the undecidability by a reduction of RPCPs to KB satisfiability problems. Specifically, given an instance \(\varphi\) of RPCP, we will construct a Knowledge Base \(\mathcal{O}_\varphi\) that is satisfiable iff \(\varphi\) has no solution.

In order to do this we will encode words \(v\) from the alphabet \(\Sigma\) as rational numbers \(0.v\) in \([0, 1]\) in base \(s + 1\); the empty word will be encoded by the number 0.

So, let us define the TBox

\[
\mathcal{T} := \{ V \equiv V_1 \sqcup V_2, W \equiv W_1 \sqcup W_2 \}
\]

and for \(1 \leq i \leq p\) the TBoxes
\[ T^i_\varphi := \{ \top \sqsubseteq \exists R_i \top, \]
\[ V \sqsubseteq (s + 1)^{|v_i|} \cdot \forall R_i.V_1, \]
\[ (s + 1)^{|v_i|} \cdot \exists R_i.V_1 \sqsubseteq V, \]
\[ W \sqsubseteq (s + 1)^{|w_i|} \cdot \forall R_i.W_1, \]
\[ (s + 1)^{|w_i|} \cdot \exists R_i.W_1 \sqsubseteq W \]
\[ \langle \top \sqsubseteq \forall R_i.V_2, 0.v_i \rangle, \]
\[ \langle \top \sqsubseteq \forall R_i.\neg V_2, 1 - 0.v_i \rangle, \]
\[ \langle \top \sqsubseteq \forall R_i.W_2, 0.w_i \rangle, \]
\[ \langle \top \sqsubseteq \forall R_i.\neg W_2, 1 - 0.w_i \rangle, \]
\[ A \sqsubseteq (s + 1)^{\max\{|v_i|,|w_i|\}} \cdot \forall R_i.A \]
\[ (s + 1)^{\max\{|v_i|,|w_i|\}} \cdot \exists R_i.A \sqsubseteq A \}
\]

Now, let
\[ T_\varphi = T \cup \bigcup_{i=1}^{p} T^i_\varphi. \]

Further we define the ABox \( \mathcal{A} \) as follows:
\[ \mathcal{A} := \{ a : \neg V, a : \neg W, \langle a : A, 0.01 \rangle, \langle a : \neg A, 0.99 \rangle \}. \]

Finally, we define
\[ O_\varphi := (T_\varphi, \mathcal{A}). \]

We now define the interpretation
\[ I_\varphi := (\Delta I_\varphi, I_\varphi) \]

as follows:
\[ \Delta I_\varphi = \{1, \ldots, p\}^* \]
\[ a I_\varphi = \epsilon \]
\[ V I_\varphi(\epsilon) = W I_\varphi(\epsilon) = 0, A I_\varphi(\epsilon) = 0.01, \text{ and for } 1 \leq i \leq 2, V_i I_\varphi(\epsilon) = W_i I_\varphi(\epsilon) = 0 \]
Lemma 5. Let $\mu$ be a mapping $\mu : I \rightarrow \mathcal{P}$ for every $i \in I$. However, $\mu_i$ is a model of $\varphi$ and satisfies axiom $\varphi$, it is possible to prove that, for every witnessed model $\mu$, $\mu_i$ holds for every $i \in I$. Moreover, as in [2] it is possible to prove that, for every witnessed model $\mathcal{I}$ of $\varphi$, there is a mapping $g$ from $\mathcal{I}$ to $\mathcal{I}$.

Lemma 5. Let $\mathcal{I}$ be a witnessed model of $\mathcal{O}_\varphi$. Then there exists a function $g : \Delta_\varphi \rightarrow \Delta_\mathcal{I}$ such that for every $\mu \in \Delta_\varphi$, $C_\mathcal{I}(\mu) = C_\mathcal{I}(g(\mu))$ holds for every concept name $C$ and $R_\mathcal{I}(\mu, \mu_i) = R_\mathcal{I}(g(\mu), g(\mu_i))$ holds for every $i$, with $1 \leq i \leq p$.

(Proof) Let $\mathcal{I}$ be a witnessed model of $\mathcal{O}_\varphi$. We will build the function $g$ inductively on the length of $\mu$.

(e) Since $\mathcal{I}$ is a model of $\mathcal{O}_\varphi$, then there is an element $\delta \in \Delta_\mathcal{I}$ such that $a_\mathcal{I} = \delta$. Since $\mathcal{I}$ is a model of $\mathcal{A}_\varphi$, setting $g(\epsilon) = \delta$, we have that $V_\mathcal{I}(\epsilon) = 0 = V_\mathcal{I}(g(\epsilon))$ and the same holds for concept $W$. Moreover, since $\mathcal{I}$ is a model of $\mathcal{T}_\varphi$, we have that $\forall V_1 \sqcup V_2 \subseteq I, \mathcal{I}(\delta)$ and, therefore, $V_1^{\mathcal{T}_\varphi}(\epsilon) = 0 = V_1^{\mathcal{I}}(g(\epsilon))$ and the same holds for $V_2, W_1$ and $W_2$. On the other hand, we have that $A_\mathcal{I}(\epsilon) = 0.01 = A_\mathcal{I}(g(\epsilon))$, as well. So, $g(\epsilon) = \delta$ satisfies the condition of the lemma.

(\mu_i) Let now $\mu$ be such that $g(\mu)$ has already been defined. Now, since $\mathcal{I}$ is a witnessed model and satisfies axiom $\exists R_1, R_1 \subseteq \exists R_1, \forall R_1, V_1$, then for all $i$, with $1 \leq i \leq p$, there exists a $\gamma \in \Delta_\mathcal{I}$ such that $R_i^{\mathcal{I}}(g(\mu), \gamma) = 1$. So, setting $g(\mu_i) = \gamma$ we get $1 = R_i^{\mathcal{I}}(\mu, \mu_i) = R_i^{\mathcal{I}}(g(\mu), g(\mu_i))$.

Furthermore, by inductive hypothesis, we can assume that $V_\mathcal{I}(g(\mu)) = 0.01$ and $W_\mathcal{I}(g(\mu)) = 0.01$.

Since $\mathcal{I}$ satisfies axiom $V \subseteq (s + 1)^{|v|} \cdot \forall R_1, V_1$, then $0.01 = V_\mathcal{I}(g(\mu)) \leq (s + 1)^{|v|} \cdot \forall R_1, V_1 g(\mu) = (s + 1)^{|v|} \cdot \forall R_1, V_1 g(\mu) = (s + 1)^{|v|} \cdot \forall R_1, V_1 g(\mu) = (s + 1)^{|v|} \cdot \forall R_1, V_1 g(\mu) = (s + 1)^{|v|} \cdot \forall R_1, V_1 g(\mu)$.

However, $\mathcal{I}$ is not a strongly witnessed model of $\mathcal{O}_\varphi$. 2
Since $I$ satisfies axiom $(s + 1)^{|v_1|} \cdot \exists R_i, V_1 \subseteq V$, then $0.v_\mu = V^I(g(\mu)) \geq (s + 1)^{|v_1|}.(\exists R_i, V_1)^I(g(\mu)) = (s + 1)^{|v_1|} \cdot \sup_{\gamma \in \Delta^I} \{R_1^I(g(\mu), \gamma) \otimes V_1^I(\gamma)\} \geq (s + 1)^{|v_1|}.R_1^I(g(\mu), \mu_i) \otimes V_1^I(\mu_i) = (s + 1)^{|v_1|}.V_1^I(g(\mu_i)).$ Therefore, $(s + 1)^{|v_1|}.V_1^I(g(\mu_i)) = 0.v_\mu$ and $V_1^I(g(\mu_i)) = 0.v_\mu \cdot (s + 1)^{-|v_1|} = V_1^I(\mu_i).

Similarly, it can be shown that $W_1^I(g(\mu_i)) = 0.w_1 \cdot (s + 1)^{-|w_1|} = W_1^I(\mu_i)$.

Since $I$ satisfies axioms $\langle \top \sqsubseteq \forall R_i, V_2, 0.v_i \rangle$ and $\langle \top \sqsubseteq \forall R_i, V_2, 1 - 0.v_i \rangle$, it follows that $(\forall R_i, V_2)^I(g(\mu)) \geq 0.v_i$ and $\langle \forall R_i, V_2^I(g(\mu)) \geq 1 - 0.v_i \rangle$. Therefore, for $R_2^I(g(\mu), g(\mu_i)) = 1$ we have $V_2^I(g(\mu_i)) = 0.v_i = V_2^I(\mu_i)$. Similarly, it can be shown that $W_2^I(\mu_i) = 0.w_i = W_2^I(\mu_i)$.

Now, since $I$ satisfies axiom $V \equiv V_1 \sqcup V_2$, then, $V^I(g(\mu_i)) = V_1^I(g(\mu_i)) + V_2^I(g(\mu_i)) = 0.v_\mu \cdot (s + 1)^{-|v_1|} + 0.v_i = 0.v_\mu = V^I(\mu_i)$.

Finally, by inductive hypothesis, assume that $A^I(g(\mu)) = A^I(\mu) = 0.01 \cdot (s + 1)^{-\sum_{j \in \{i_1, i_2, \ldots, i_k\}} \max\{|v_j|, |w_j|\}}$, where $\mu = i_1 i_2 \ldots i_k$.

Since $I$ satisfies axioms $A \subseteq (s + 1)^{\max\{|v_1|, |w_1|\}} \cdot \forall R_i.A$, we have that $A^I(g(\mu)) \leq (s + 1)^{\max\{|v_1|, |w_1|\}} \cdot (\forall R_i, A)^I(g(\mu)) \leq (s + 1)^{\max\{|v_1|, |w_1|\}} \cdot A^I(g(\mu_i))$.

Likewise, since $I$ satisfies axioms $(s + 1)^{\max\{|v_1|, |w_1|\}} \cdot \exists R_i.A \subseteq A$, we have that $A^I(g(\mu)) \geq (s + 1)^{\max\{|v_1|, |w_1|\}} \cdot (\exists R_i, A)^I(g(\mu)) \geq (s + 1)^{\max\{|v_1|, |w_1|\}} \cdot A^I(g(\mu_i))$

and, thus,

$$A^I(g(\mu)) = (s + 1)^{\max\{|v_1|, |w_1|\}} \cdot A^I(g(\mu_i)).$$

Therefore,

$$A^I(g(\mu_i)) = (s + 1)^{-\max\{|v_1|, |w_1|\}} \cdot A^I(g(\mu))$$

$$= (s + 1)^{-\max\{|v_1|, |w_1|\}} \cdot A^I(\mu)$$

$$= (s + 1)^{-\max\{|v_1|, |w_1|\}} \cdot 0.01 \cdot (s + 1)^{-\sum_{j \in \{i_1, i_2, \ldots, i_k\}} \max\{|v_j|, |w_j|\}}$$

$$= 0.01 \cdot (s + 1)^{-\max\{|v_1|, |w_1|\} + \sum_{j \in \{i_1, i_2, \ldots, i_k\}} \max\{|v_j|, |w_j|\}}$$

$$= 0.01 \cdot (s + 1)^{-\sum_{j \in \{i_1, i_2, \ldots, i_k\}} \max\{|v_j|, |w_j|\}}$$

$$= A^I(\mu_i),$$

which completes the proof. ∎

From the last Lemma it follows that if the RPCP instance $\phi$ has a solution $\mu$, for some $\mu \in \{1, \ldots, p\}^+$, then $v_\mu = w_\mu$ and, thus, $0.v_\mu = 0.w_\mu$. Therefore, every witnessed model $I$ of $O_\phi$ contains an element $\delta = g(\mu)$ such that $V^I(\delta) = V^I(g(\mu)) = 0.v_\mu = 0.w_\mu = W^I(\phi) = W^I(\mu)$. Conversely, from the definition of $I_\phi$, if $\phi$ has no solution, then there is no $\mu$ such that $0.v_\mu = 0.w_\mu$, i.e., there is no $\mu$ such that $V^I(g(\mu)) = W^I(g(\mu))$. 9
However, as $O_\varphi$ is always satisfiable, it does not yet help us to decide the RPCP. We next extend $O_\varphi$ to $O'_\varphi$ in such a way that an instance $\varphi$ of the RPCP has a solution iff the ontology $O'_\varphi$ is not witnessed satisfiable and, thus, establish that the KB satisfiability problem is undecidable. To this end, consider

$$O'_\varphi := (T'_\varphi, A),$$

where

$$T'_\varphi := T_\varphi \cup \bigcup_{1 \leq i \leq p} \{\top \sqsubseteq \forall \cdot (\lnot (V \leftrightarrow W) \sqcup \lnot A)\}.$$

The intuition here is the following. If there is a solution for RPCP then, by the observation before, there is a point $\delta$ in which the value of $V$ and $W$ coincide under $I$. That is, the value of $\lnot (V \leftrightarrow W)$ is 0 and, thus, the one of $\lnot (V \leftrightarrow W) \sqcup \lnot A$ is less than 1. So, $I$ cannot satisfy the new GCI in $T'_\varphi$ and, thus, $O'_\varphi$ is not satisfiable. On the other hand, if there is no solution to the RPCP then in $I$ there is no point in which $V$ and $W$ coincide and, thus, $\lnot (V \leftrightarrow W) > 0$. However, we will show that the value of $\lnot (V \leftrightarrow W)$ in all points is strictly greater than $A$ and, as $A \sqcup \lnot A$ is 1, so also $\lnot (V \leftrightarrow W) \sqcup \lnot A$ will be 1 in any point. Hence, $I$ is a model of the aditional axiom in $T'_\varphi$, i.e., $O'_\varphi$ is satisfiable.

**Theorem 6.** The instance $\varphi$ of the RPCP has a solution iff the ontology $O'_\varphi$ is not witnessed satisfiable.

*(Proof)* Assume first that $\varphi$ has a solution $\mu = i_1 \ldots i_k$ and let $I$ be a witnessed model of $O_\varphi$. Let $\bar{\mu} = i_1 i_2 \ldots i_{k-1}$ (last index $i_k$ is dropped from $\mu$). Then by Lemma 5 it follows that there are nodes $\delta, \delta' \in \Delta^I$ such that $\delta = g(\mu), \delta' = g(\bar{\mu})$, with $V^I(\delta) = V^{I_\varphi}(\mu) = W^{I_\varphi}(\mu) = W^I(\delta)$ and $R^{I_\varphi}_{i_k}(\delta', \delta) = 1$. Then $(V \leftrightarrow W)^I(\delta) = 1$. Since $(\lnot A)^I(\delta) < 1$, then $(\lnot (V \leftrightarrow W) \sqcup \lnot A)^I(\delta) < 1$. Hence there is $i$, with $1 \leq i \leq p$, such that $(\forall R_i.(\lnot (V \leftrightarrow W) \sqcup \lnot A))^I(\delta) < 1$. So, axiom $\top \sqsubseteq \forall R_i. (\lnot (V \leftrightarrow W) \sqcup \lnot A)$ is not satisfied and, therefore, $O_\varphi$ is not satisfiable.

For the converse, assume that $\varphi$ has no solution. On the one hand we know that $I_\varphi$ is a model of $O_\varphi$. On the other hand, since $\varphi$ has no solution, then there is no $\mu = i_1 \ldots i_k$ such that $v_\mu = w_\mu$ (i.e., $0.v_\mu = 0.w_\mu$) and, therefore, there is no $\mu \in \Delta^{I_\varphi}$ such that $V^{I_\varphi}(\mu) = W^{I_\varphi}(\mu)$. Consider $\mu \in \Delta^{I_\varphi}$ and, with $1 \leq i \leq p$ and assume,
without loss of generality, that $V^I_{\varphi}(\mu_i) < W^I_{\varphi}(\mu_i)$. Then

$$(V \leftrightarrow W)^I_{\varphi}(\mu_i) = (V^I_{\varphi}(\mu_i) \Rightarrow W^I_{\varphi}(\mu_i)) \otimes (W^I_{\varphi}(\mu_i) \Rightarrow V^I_{\varphi}(\mu_i))$$

$$= 1 \otimes (W^I_{\varphi}(\mu_i) \Rightarrow V^I_{\varphi}(\mu_i))$$

$$= W^I_{\varphi}(\mu_i) \Rightarrow V^I_{\varphi}(\mu_i)$$

$$= 1 - W^I_{\varphi}(\mu_i) + V^I_{\varphi}(\mu_i)$$

$$= 1 - (W^I_{\varphi}(\mu_i) - V^I_{\varphi}(\mu_i))$$

$$= 1 - (0.\mu_{j\mu} - 0.\nu_{\mu})$$

$$\leq 1 - 0.01 \cdot (s + 1)^{-\max\{|\nu_{\mu}|,|\mu_{\mu}|\}}$$

$$\leq 1 - 0.01 \cdot (s + 1)^{-\sum_{j \in \{1, 2, \ldots, i_1, i_2, \ldots, i_k, \ldots, i\}} \max\{|\nu_j|,|\mu_j|\}}$$

$$= (\neg A)^I_{\varphi}(\mu_i).$$

Therefore, $(\neg (V \leftrightarrow W))^I_{\varphi}(\mu_i) \geq A^I_{\varphi}(\mu_i)$. As $A^I_{\varphi}(\mu_i) \oplus (\neg A)^I_{\varphi}(\mu_i) = 1$, it follows that for every $\mu \in \Delta^I_{\varphi}$ and $i$, with $1 \leq i \leq p$, it holds that $(\forall R_i.(\neg (V \leftrightarrow W) \sqcup \neg A))^I_{\varphi}(\mu) = 1$ and, therefore, $I_{\varphi}$ is a witnessed model of $O'_{\varphi}$.

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