WEIGHTED INEQUALITIES RELATED TO A MUCKENHOUPT AND WHEEDEN PROBLEM FOR ONE–SIDE SINGULAR INTEGRALS

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Abstract. In this paper we obtain for $T^+$, a one-sided singular integral given by a Calderón-Zygmund kernel with support in $(-\infty, 0)$, a $L^p(w)$ bound when $w \in A^+_1$. In [A. K. Lerner, S. Ombrosi and C. Pérez, $A_1$ Bounds for Calderón-Zygmund operators related to a problem of Muckenhoupt and Wheeden, Math. Res. Lett. 16 (2009), no. 1, 149–156.], the authors proved that this bound is sharp with respect to $||w||_{A_1}$ and with respect to $p$. We also give a $L^{1,\infty}(w)$ estimate, for a related problem of Muckenhoupt and Wheeden for $w \in A^+_1$. We improve the classical results, for one-sided singular integrals, by putting in the inequalities a wider class of weights.

1. Introduction

Let $M$ be the classical Hardy-Littlewood maximal operator and $w$ a weight (i.e. $w \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $w > 0$). C. Fefferman and E. M. Stein in [5] proved an extension of the classical weak-type $(1, 1)$ estimate:

$$||Mf||_{L^{1,\infty}(w)} \leq C \int_{\mathbb{R}^n} |f(x)| Mw(x) \, dx,$$

(1.1)

where $C = C(n)$. This is a sort of duality for $M$. A consequence of this result, using an interpolation argument, is the following: if $1 < p < \infty$ and $p' = \frac{p}{p-1}$ then,

$$\int_{\mathbb{R}^n} (Mf(x))^p w(x) \, dx \leq C p' \int_{\mathbb{R}^n} |f(x)|^p Mw(x) \, dx,$$

where $C = C(n)$.

B. Muckenhoupt and R. Wheeden many years ago, in [18], conjectured that the analogue of (1.1) should hold for $T$, a singular integral operator, namely

$$\sup_{\lambda > 0} \lambda w(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda \}) \leq C \int_{\mathbb{R}^n} |f(x)| Mw(x) \, dx,$$

(1.2)

where $C = C(n,T)$.

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The best result along this line was given by C. Pérez in [20], where $M$ is replaced by the slightly larger operator $M_{L(\log L)^{\varepsilon}}$, $\varepsilon > 0$,

$$||Tf||_{L^{1,\infty}(w)} \leq C 2^{\frac{1}{\varepsilon}} \int_{\mathbb{R}^n} |f(x)| M_{L(\log L)^{\varepsilon}} w(x) \, dx,$$

where $C = C(n, T)$.

The one-sided version of this result was obtained in [12] by M. Lorente, J. M. Martell, C. Pérez and M. S. Riveros.

M. C. Reguera in [22] and M. C. Reguera and C. Thiele in [23] proved that the Muckenhoupt-Wheeden conjecture is false. In [22] the author gives a first approach by putting in the right hand side the dyadic maximal operator. In [23] they disproved (1.2) for $T$ the Hilbert transform.

On the other hand there is a variant of the conjecture (1.2) which has a lot of interest, namely the weak Muckenhoupt-Wheeden conjecture. The idea is to assume an a priori condition on the weight $w$. This condition can be read essentially from inequality (1.2): a weight $w \in A_1$ if there is a finite constant $C$ such that $Mw(x) \leq Cw(x)$ a.e. $x \in \mathbb{R}^n$. Denote $||w||_{A_1}$ the smallest of these $C$. The conjecture is the following:

Let $w \in A_1$, then

$$\sup_{\lambda > 0} \lambda w(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) \leq C ||w||_{A_1} \int_{\mathbb{R}^n} |f(x)| w(x) \, dx,$$

where $C = C(n, T)$.

In [11], A. K. Lerner, S. Ombrosy and C. Pérez, exhibit a logarithmic growth

$$||Tf||_{L^{1,\infty}(w)} \leq C ||w||_{A_1} \log(e + ||w||_{A_1}) ||f||_{L^{1}(w)},$$

(1.3)

where $C = C(n, T)$. It is an open problem if this result obtained by the authors in [11] is the best possible. Recently, F. Nazarov, A. Reznikov, V. Vasyunin, A. Volberg, proved that the weak Muckenhoupt-Wheeden conjecture is also false. See [19].

To prove this logarithmic growth result, they had to study first the corresponding weighted $L^p$ estimate for $1 < p < \infty$ and $w \in A_1$, $\varepsilon$

$$||Tf||_{L^p(w)} \leq C p p' ||w||_{A_1} ||f||_{L^p(w)},$$

(1.4)

where $C = C(n, T)$, being the last inequality, this time, fully sharp. See [11]. As a consequence of (1.3) and applying the Rubio de Francia’s algorithm they also get the following result, let $1 < p < \infty$, $w \in A_p$ and let $T$ be a Calderón-Zygmund operator then

$$||T||_{L^{p,\infty}(w)} \leq C ||w||_{A_p} \log(e + ||w||_{A_p}) ||f||_{L^p(w)},$$

(1.5)

where $C = C(n, p, T)$.

The first result of this kind obtaining the precise constant dependence on the $A_p$ norm of $w$ of the operator norms of singular integrals, maximal functions, and other operators in $L^p(w)$ was obtained by S. M. Buckley in [2]. There, he proves that

$$||M||_{L^p(w)} \leq C ||w||_{A_p}^{\frac{1}{p-1}},$$
where \( C = C(n, p) \).

Recently T. Hytönen, C. Pérez and E. Rela in [9] improved this result by giving a sharp weighted bound for the Hardy-Littlewood maximal operator involving the Fujii-Wilson \( A_{\infty} \)-constant. Also T. Hytönen and C. Pérez in [8] improved (1.3), (1.4) and several well known results, using for all these cases the Fujii-Wilson \( A_{\infty} \)-constant.

In this paper we obtain similar results as the ones in (1.3), (1.4) and (1.5) for one-sided weights and one-sided singular integrals.

Now we will state the results obtained in this work. The definitions of Sawyer’s weights and one-sided operators will appear in the next section.

**Theorem 1.1.** Let \( 1 < p < \infty \), \( w \in A_{1}^{+} \) and \( T^{+} \) be a one-sided singular integral, then,
\[
||T^{+}f||_{L^{p}(w)} \leq C p p' ||w||_{A_{1}^{+}} ||f||_{L^{p}(w)},
\]
(1.6)
where \( C = C(T^{+}) \).

**Theorem 1.2.** Let \( w \in A_{1}^{+} \) and \( T^{+} \) be a one-sided singular integral, then,
\[
||T^{+}f||_{L^{1, \infty}(w)} \leq C ||w||_{A_{1}^{+}} \log(e + ||w||_{A_{1}^{+}}) ||f||_{L^{1}(w)},
\]
(1.7)
where \( C = C(T^{+}) \).

**Corollary 1.3.** Let \( 1 < p < \infty \), \( w \in A_{p}^{+} \) and \( T^{+} \) be a one-sided singular integral, then
\[
||T^{+}f||_{L^{p, \infty}(w)} \leq C ||w||_{A_{p}^{+}} \log(e + ||w||_{A_{p}^{+}}) ||f||_{L^{p}(w)},
\]
(1.8)
where \( C = C(T^{+}) \).

By a duality argument, Corollary 1.3 implies the following:

**Corollary 1.4.** Let \( 1 < p < \infty \), \( w \in A_{p}^{-} \) and \( T^{-} \) be a one-sided singular integral, then for any measurable set \( E \)
\[
||T^{-}(\sigma \chi_{E})||_{L^{p}(w)} \leq C ||w||_{A_{p}^{-}} \frac{1}{A_{p}^{-}} \log(e + ||w||_{A_{p}^{-}}) \sigma(E) \frac{1}{p},
\]
(1.9)
where \( C = C(T^{-}) \) and \( \sigma = w^{\frac{1}{p} - 1} \).

In these results \( ||w||_{A_{1}^{+}} \) is the best constant of the weight \( w \in A_{1}^{+} \). Clearly, every theorem has a corresponding one, reversing the orientation of \( \mathbb{R} \).

Theorems 1.1, 1.2 and Corollaries 1.3, 1.4, for one-sided singular integrals, improve the ones obtained in [11] by putting in the inequalities a wider class of weights (the Sawyer classes).

The article is organized as follows: in Section 2 we introduce notation, definitions and well known results. In Section 3 we prove some previous lemmas that will be essential to obtain the proofs of Theorems and Corollaries given in Section 4. In Section 5 we give a weaker version and a simplest proof of Lemma 3.2 of Section 3.
2. Preliminaries

In this section we give some definitions and well known results.

2.1. One-side singular integral operators and Sawyer’s weights

**Definition 2.1.** Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. The one-sided maximal operators are defined as

$$M^+ f(x) = \sup_{h > 0} \frac{1}{h} \int_{x-h}^{x+h} |f(t)| \, dt, \quad M^- f(x) = \sup_{h > 0} \frac{1}{h} \int_{x-h}^{x} |f(t)| \, dt.$$  

The good weights for these operators are the Sawyer weights $A^+_p$ and $A^-_p$, see [25], [13], [14]. We recall the definition.

**Definition 2.2.** Let $w$ be a non-negative locally integrable function and $1 \leq p < \infty$. We say that $w \in A^+_p$ if there exists $C(p) < \infty$ such that for every $a < x < b$

$$\frac{1}{(b-a)^p} \left( \int_a^x w \right)^{p-1} \left( \int_x^b w^{p-1} \right)^{-1} \leq C(p), \quad (2.1)$$

when $1 < p < \infty$, and for $p = 1$,

$$M^- w(x) \leq C(1) w(x), \quad \text{for a.e. } x \in (a,b), \quad (2.2)$$

finally $A^+_\infty = \bigcup_{p \geq 1} A^+_p$, see [15].

The smallest possible $C(1)$ in (2.2) here is denoted by $\|w\|_{A^+_1}$ and the smallest possible $C(p)$ in (2.1) here is denoted by $\|w\|_{A^+_p}$.

The classes $A^-_p$ for $1 \leq p \leq \infty$ are defined in a similar way.

We also define

$$M^+_r f(x) = \sup_{h > 0} \left( \frac{1}{h} \int_{x-h}^{x+h} |f(t)|^r \, dt \right)^{\frac{1}{r}}, \quad M^-_r f(x) = \sup_{h > 0} \left( \frac{1}{h} \int_{x-h}^{x} |f(t)|^r \, dt \right)^{\frac{1}{r}},$$

where $r \geq 1$. Observe that $M^+_f \leq M^+_r f$ for all $r \geq 1$. Also, we will consider the following maximal operators introduced by F. J. Martín-Reyes, P. Ortega and A. de la Torre in [14],

$$M^+_g f(x) = \sup_{h > 0} \int_{x-h}^{x+h} |f(t)| g(t) \, dt \left( \int_{x-h}^{x+h} g(t) \, dt \right)^{-1},$$

$$M^-_g f(x) = \sup_{h > 0} \int_{x-h}^{x} |f(t)| g(t) \, dt \left( \int_{x-h}^{x} g(t) \, dt \right)^{-1},$$

where $g$ is a positive locally integrable function on $\mathbb{R}$. 

The classes $A^+_p(g)$, $1 \leq p \leq \infty$ are defined as follows, let $w$ be non-negative locally integrable functions and let $1 \leq p < \infty$. We say that $w \in A^+_p(g)$ if there exists $C(p) < \infty$ such that for every $a < x < b$

$$\left( \int_a^x w \right) \left( \int_x^b g^p' \sigma \right)^{p-1} \leq C(p) \left( \int_a^b g \right)^p,$$

(2.3)

where $\sigma = w^{\frac{1}{p}-1}$, $\frac{1}{p} + \frac{1}{p'} = 1$, when $1 < p < \infty$. For $p = 1$,

$$M_g^{-}(g^{-1}w)(x) \leq C(1)g^{-1}w(x), \quad \text{a.e.} \ x \in (a,b).$$

In [14] it was proved that $w \in A^+_p(g)$, if, and only if $M_g^+$ is bounded from $L^p(w)$ into $L^p(w)$, for $1 < p < \infty$, and $w \in A^+_1(g)$, if, and only if $M_g^+$ maps $L^1(w)$ into $L^{1,\infty}(w)$. Observe that if $g \equiv 1$ then $A^+_p(g) = A^+_p$, for $1 \leq p \leq \infty$.

**DEFINITION 2.3.** We shall say that a function $K$ in $L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ is a Calderón-Zygmund kernel if the following properties are satisfied:

- $||\hat{K}||_{\infty} < C_1$
- $|K(x)| < \frac{C_2}{|x|^n}$
- $|K(x) - K(x-y)| < \frac{C_3|y|}{|x|^n + |y|}$, where $|y| < \frac{|x|}{2}$.

The Calderón-Zygmund singular integral operator associated to $K$ is defined by

$$Tf(x) = p.v.(K * f)(x) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^n / B_\epsilon(x)} K(x-y)f(y) dy,$$

and the maximal operator associated with this kernel $K$ is

$$T^+f(x) = \sup_{\epsilon > 0} \int_{\mathbb{R}^n / B_\epsilon(x)} |K(x-y)||f(y)| dy.$$

A one-sided singular integral $T^+$ is a singular integral associated to a Calderón-Zygmund kernel with support in $(-\infty,0)$; therefore, in that case,

$$T^+f(x) = \lim_{\epsilon \to 0^+} \int_{x + \epsilon}^{\infty} K(x-y)f(y) dy.$$

Examples of such kernels are given by H. Aimar, L. Forzani and F. J. Martín-Reyes in [1]. The operator $T^-$ is defined similarly.

**REMARK 2.4.**

1. In [1], it is proved that the one-sided singular integral $T^+$ is controlled by the one-sided maximal functions $M^+$ in the $L^p(w)$ norm if $w \in A^+_\infty$. 

2. It is well known to that the classes $A_p$ are included in $A_p^+$ and $A_p^-$; namely $A_p = A_p^- \cap A_p^+$. See [25], [13], [14].

3. The one-sided classes of weights satisfy the following factorization, $w \in A_p^+$ if only if $w = w_1 w_2^{1-p}$ with $w_1 \in A_1^+$ and $w_2 \in A_1^-$, and $\|w\|_{A_p^+} \leq \|w_1\|_{A_1^+} \|w_2\|_{A_1^-}^{p-1}$. See [25], [13], [14].

4. It is easy to check that $(M^- f)^\delta \in A_1^+$ for all $0 < \delta < 1$ with $\|(M^- f)^\delta\|_{A_1^+} \leq C \frac{1}{1-\delta}$.

Finally, we recall some definition concerning Lorentz $L^{p,q}(\mu)$ spaces. Let $f$ be a measurable function on a measure space $(X, \mathcal{M}, \mu)$. The non-increasing rearrangement $f^*(t)$ of $f$ is defined as

$$f^*(t) = \inf\{\lambda > 0 : \mu(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}) \leq t\},$$

for all $0 < t < \infty$. The function $f$ is said to belong to the Lorentz space $L^{p,q}(\mu)$ if the quantities

$$\|f\|_{L^{p,q}(\mu)} = \left(\frac{q}{p} \int_0^\infty \left[t^{1/p} f^*(t)\right]^q \frac{dt}{t}\right)^{1/q},$$

whenever $0 < p < \infty$ and $0 < q < \infty$, and

$$\|f\|_{L^{p,\infty}(\mu)} = \sup_{t > 0} \left[t^{1/p} f^*(t)\right],$$

when $0 < p \leq \infty$, are finite. For more details see [26].

3. Previous Lemmas

To obtain Theorems 1.1 and 1.2 we need to prove some previous results: a sharp weak reverse Hölder’s inequality for one-sided weights, a particular case of the Coifman-type estimate for one-sided singular integrals and one-sided maximal operator, and also build $A_1^+$ weights from duality with special control on the constant, based in the Rubio de Francia algorithm, see [6].

3.1. Sharp weak reverse Hölder’s inequality

F. J. Martín-Reyes proved a weak reverse Hölder’s inequality (see [13] Lemma 5).

**Lemma 3.1.** [13] (Sharp weak reverse Hölder’s inequality) Let $1 \leq p < \infty$ and $w \in A_p^+$. There exist positive numbers $\delta$ and $C$ such that

$$\int_a^b w^{1+\delta} \leq CM^- (w \chi_{(a,b)})(b)^\delta \int_a^b w,$$

(3.1)
for every bounded interval \((a,b)\), and therefore
\[
M_{1+\delta}^{-}(w\chi_{(a,b)})(b) \leq CM^{-}(w\chi_{(a,b)})(b),
\]
and
\[
M_{1+\delta}^{-}(w)(b) \leq CM^{-}(w)(b).
\]
The constant \(C\) depends only on \(\delta\) and the constant of the \(A^+_p\) condition.

Here we will need to be more precise in the constants. In the proof of Lemma 3.1 the constant \(C\) depends on a constant \(\beta\) and the constant \(\delta\). If we choose \(\beta = (4^p||w||_{A^+_p})^{-1}\) and \(\delta = \frac{1}{4^{p+2}e^2||w||_{A^+_p}}\), when \(1 < p < \infty\), and \(\beta = (||w||_{A^+_1})^{-1}\) and \(\delta = \frac{1}{16e^2||w||_{A^+_1}}\), when \(p = 1\), following the same steps of the proof, we obtain that \(C \leq 2\).

Then we can rewrite the equation (3.1) as
\[
\int_{a}^{b} w^r w \leq 2M^{-}(w\chi_{(a,b)})(b)r_{w}^{-1}\int_{a}^{b} w, \tag{3.2}
\]
for every bounded interval \((a,b)\), and therefore
\[
M_{r_{w}}^{-}(w\chi_{(a,b)})(b) \leq 2M^{-}(w\chi_{(a,b)})(b),
\]
and
\[
M_{r_{w}}^{-}(w)(b) \leq 2M^{-}(w)(b), \tag{3.3}
\]
where \(r_{w} = 1 + \frac{1}{4^{p+2}e^2||w||_{A^+_p}}\), when \(p > 1\), and \(r_{w} = 1 + \frac{1}{16e^2||w||_{A^+_1}}\) when \(p = 1\).

The following Lemma will be necessary to prove good-\(\lambda\) result in the next section.

**Lemma 3.2.** Let \(1 < p < \infty\), \(w \in A^{-}_p\), \(a < b < c\) and \(E \subseteq (b,c)\) a measurable set. For all \(\varepsilon > 0\), there exists \(C = C(\varepsilon, p)\) such that if \(|E| < \varepsilon^{-C||w||_{A^{-}_p}}(b-a)\) then \(w(E) < \varepsilon w(a,c)\).

**Proof.** Let \(w \in A^{-}_p\). Let apply the analogous to equation (3.2), i.e.
\[
\int_{b}^{c} w^r \leq 2M^{+}(w\chi_{(b,c)})(b)r_{w}^{-1}\int_{b}^{c} w.
\]
This last inequality implies
\[
(M_{r_{w}}^{+}(w^{r_{w}-1}\chi_{(b,c)})(b))^{\frac{1}{r_{w}}} \leq 2M^{+}(w\chi_{(b,c)})(b),
\]
where we take \(r = r_{w} = 1 + \frac{1}{4^{p+2}e^2||w||_{A^+_p}}\), when \(p > 1\) and \(r = r_{w} = 1 + \frac{1}{16e^2||w||_{A^+_1}}\), when \(p = 1\).
Using the definition of $M_g^+$, with $g = w$, we have that for all $x \in (a, b)$

$$
\left( \frac{1}{w(a, c)} \int_b^c w^{-1} w \right)^{\frac{1}{r'}} \leq (M_w^+(w^{-1} \chi_{(x,c)})(x))^{\frac{1}{r'}} 
\leq 2M^+(w\chi_{(x,c)})(x) 
\leq 2M^+(w\chi_{(a,c)})(x),
$$

then

$$(a, b) \subseteq \left\{ x : M^+(w\chi_{(a,c)})(x) > \frac{1}{4} \left( \frac{1}{w(a, c)} \int_b^c w \right)^{\frac{1}{r'}} w(a, c),
\right\}.$$

Recalling that $M^+$ is of weak type $(1, 1)$ with respect to the Lebesgue measure, we get

$$
b - a < C_1 w(a, c)^{\frac{1}{r'}} \left( \int_b^c w \right)^{\frac{1}{r'}} w(a, c),
$$

where $C_1$ does not depend on the weight $w$. This last inequality says that $1 \in A^+_w$ (where $r'$ is such that $\frac{1}{r'} + \frac{1}{p} = 1$), with constant $C_1$, see (2.3). Let $x \in (a, b)$, by hypothesis $E \subset (b, c)$ then

$$
M_w^+(\chi_E(x)) \geq \frac{1}{w(x, c)} \int_x^c \chi_E(t)w(t) dt \geq \frac{w(E)}{w(a, c)},
$$

obtaining that the interval $(a, b) \subset \{ x : M_w^+(\chi_E(x)) > \frac{w(E)}{2w(a, c)} \}$. Observe that $M_w^+$ is of weak type $(r', r')$ with respect to the Lebesgue measure with constant $||M_w^+||_{L^{r'} \rightarrow L^{r', \infty}} = C_2$ (see [14]), then

$$
b - a \leq \left| \left\{ x : M_w^+(\chi_E(x)) > \frac{w(E)}{2w(a, c)} \right\} \right| \leq C_2 \left( \frac{2w(a, c)}{w(E)} \right)^{r'} |E|.
$$

Taking into account that $1 < r < 2$ and $C_2^{\frac{1}{r'}} \leq C_3$, where $C_3$ does not depend on $p$ or $||w||_{A^p_\infty}$ (see [14]), then

$$
\frac{w(E)}{w(a, c)} < \left( C_2 \frac{|E|}{b - a} \right)^{\frac{1}{r'}} < C_3 e^{-\frac{C||w||_{A^p_\infty}}{r'}} < \epsilon,
$$

where the last inequality holds by choosing an appropriate $C$ depending only on $p$ and $\epsilon$. □

3.2. The Coifman-type estimate

Now we give a particular case of the Coifman-type estimate. In order to do this we need a kind of good-$\lambda$ inequality result. We will use the next result due to S. M. Buckley in [2].
Lemma 3.3. [2] Let \( g \in L^\infty(I) \) and \( T \) be an operator for which
\[
|\{x : T\phi(x) > \alpha\}| \leq \left( \frac{C p |\phi|_p}{\alpha} \right),
\]
for all \( \phi \in L^p(\mathbb{R}) \), sufficiently large \( p \) and \( \alpha \) and \( C \) being a constant independent of \( p \).

Then,
\[
|\{x : Tg(x) > \alpha\}| \leq C e^{\frac{c}{p}} |I|.
\]

Lemma 3.4. Let \( 1 \leq p < \infty \), \( w \in A_p^- \), \( T^- \) be a one-sided singular integral and \( T^* \) the maximal operator related to \( T^- \). Then, there exist positive constants \( c_1, c_2 \), \( \gamma_0 > 0 \) such that for every \( 0 < \gamma < \gamma_0 \)
\[
|\{x \in \mathbb{R} : T^* f(x) > 2\lambda, M^- f(x) < \gamma \lambda\}| < c_1 e^{-\frac{c_2}{p}} |\{T^* f(x) > \lambda\}|
\]
holds for \( f \in L^1(\mathbb{R}) \), \( \lambda > 0 \). Also, for all \( \epsilon > 0 \), there exists \( c' \) depending on \( \epsilon, \gamma_0 \) and \( p \) such that
\[
w \left( \left\{ x \in \mathbb{R} : T^* f(x) > 2\lambda, M^- f(x) < \frac{c' \lambda}{||w||_{A_p^-}} \right\} \right) < \epsilon w(\{T^* f(x) > \lambda\}).
\]

Proof. Since \( \{x : T^* f(x) > \lambda\} \) is an open set and it has finite measure for \( f \in L^1(\mathbb{R}) \), it can be written as a disjoint countable union of open intervals. Let \( J = (a, b) \) be such an interval. It is enough to prove that there exist \( c_1, c_2, c' \) and \( \gamma_0 \) such that
\[
|\{x \in J : T^* f(x) > 2\lambda, M^- f(x) < \gamma \lambda\}| < c_1 e^{-\frac{c_2}{p}} |J|,
\]
and
\[
w \left( \left\{ x \in J : T^* f(x) > 2\lambda, M^- f(x) < \frac{c' \lambda}{||w||_{A_p^-}} \right\} \right) < \epsilon w(J),
\]
for every \( 0 < \gamma < \gamma_0 \) and every \( \lambda > 0 \). Let us take a sequence \( \{x_i\}_{i=0}^{\infty} \) in \( J = (a, b) \) in such a way that \( x_0 = b \) and \( x_{i-1} - x_i = x_i - a \) for every \( i > 0 \). Observe that we only
need to prove that
\[
|\{x \in (x_{i+1}, x_i) : T^* f(x) > 2\lambda, M^- f(x) < \gamma \lambda\}| < c_1 e^{-\frac{c_2}{p}} (x_{i+1} - x_{i+2}). \quad (3.5)
\]
By Lemma 3.2 there exists \( c' \) depending on \( \epsilon, \gamma_0, p, c_1, c_2 \), such that
\[
w \left( \left\{ x \in (x_{i+1}, x_i) : T^* f(x) > 2\lambda, M^- f(x) < \frac{c' \lambda}{||w||_{A_p^-}} \right\} \right) < \epsilon w(x_i, x_{i+2}). \quad (3.6)
\]
Let us show (3.5). Let \( i \in \mathbb{N} \), if \( \{ x \in (x_{i+1}, x_i) : T^*f(x) > 2\lambda, M^-f(x) < \gamma \lambda \} = \emptyset \) there is nothing to prove. We choose \( \bar{\sigma} < a \) such that \( x_i - a = a - \bar{\sigma} \) and

\[
\xi = \sup\{ x \in (x_{i+1}, x_i) : M^-f(x) \leq \gamma \lambda \}.
\]

Let us write \( f = f_1 + f_2 \) with \( f_1 = f \chi(\bar{\sigma}, \xi) \) then

\[
\{ x \in (x_{i+1}, x_i) : T^*f(x) > 2\lambda, M^-f(x) < \gamma \lambda \} \subset A \cup B,
\]

where

\[
A = \{ x \in (x_{i+1}, \xi) : T^*f_1(x) > \frac{1}{2} \lambda, M^-f(x) < \gamma \lambda \},
\]

\[
B = \{ x \in (x_{i+1}, \xi) : T^*f_2(x) > \frac{3}{2} \lambda, M^-f(x) < \gamma \lambda \}.
\]

The second set \( B \) is essentially empty for \( \gamma \) small enough. By standard estimation (see [1]), we get that for \( x \in (x_{i+1}, \xi) \), \( T^*f_2(x) \leq \frac{3}{2} \lambda \) then

\[
\left\{ x \in (x_{i+1}, \xi) : T^*f_2(x) > \frac{3}{2} \lambda, M^-f(x) < \gamma \lambda \right\} = \emptyset,
\]

for \( 0 < \gamma < \gamma_0 \) small enough.

Now we work with set \( A \). Let \( \Omega = \{ x \in (x_{i+1}, \xi) : M^-f_1(x) > 3 \gamma \lambda \} \), observe that

\[
\int_{\mathbb{R}} f_1(t) \, dt \leq 4 \gamma \lambda (x_i - x_{i+1}).
\]

The last inequality implies that \( \Omega \subset (\bar{\sigma}, \bar{a}) \) with \( \bar{a} - \xi = \frac{\gamma}{4}(x_i - x_{i+1}) \). Let us write \( \Omega = \bigcup I_j \) where \( I_j = (a_j, b_j) \) are disjoint maximal intervals. Then

\[
\frac{1}{|I_j|} \int_{I_j} f_1(t) \, dt = 3 \gamma \lambda.
\]

We define \( I_j^+ = (b_j, c_j) \), \( |I_j^+| = 2 |I_j| \), \( \bar{\Omega} = \bigcup (I_j^+ \cup I_j) = \bigcup I_j \) and \( f_1 = g + h \) with

\[
g = f_1 \chi_{\mathbb{R}/\Omega} + \sum_j 3 \gamma \lambda \chi_{I_j}, \quad h = \sum_j h_j = \sum_j (f_1 - 3 \gamma \lambda) \chi_{I_j}.
\]

Observe that \( g \leq 3 \gamma \lambda \) and \( g \) has support in \( (\bar{\sigma}, \bar{a}) \). Then using Lemma 3.3 we have

\[
\left\{ \left\{ x : T^*g(x) > \frac{\lambda}{4} \right\} \leq e^{\frac{\gamma}{\lambda}} (\bar{a} - \bar{\sigma}) \leq 32 e^{\frac{\gamma}{\lambda}} (x_{i+1} - x_{i+2}).\right.
\]
Now let us study $T^* h$ for $x \notin \tilde{\Omega}$,

$$|T^* h(x)| \leq \sum_j \int_{I_j} |h_j(y)(K(x - y) - K(x - b_j))| \, dy \leq C \sum_j \int_{I_j} |h_j(y)| \frac{y - b_j}{(x - b_j)^2} \, dy \leq \frac{3}{2} C \sum_j \frac{\delta_j}{\delta_j^2 + (x - b_j)^2} \int_{I_j} |h_j(y)| \, dy \leq 9 C \gamma \lambda \sum_j \frac{\delta_j}{\delta_j^2 + (x - b_j)^2} |I_j| \leq C \gamma \lambda \sum_j \frac{\delta_j^2}{\delta_j^2 + (x - b_j)^2},$$

where $\delta_j = c_j - a_j$. We write $\Delta(x) = \sum_j \frac{\delta_j^2}{\delta_j^2 + (x - b_j)^2}$.

Observe that if $x \in \tilde{\Omega}$ then $M^- f(x) \geq \gamma \lambda$. In fact, if $x \in I_j$, for some $j$, then by definition of $\Omega$ we have that $3 \gamma \lambda < M^- f_1(x) < M^- f(x)$. If $x \in I_{j+1}$ then

$$3 \gamma \lambda = \frac{1}{|I_j|} \int_{I_j} f_1(t) \, dt = \frac{x - a_j}{(x - a_j)|I_j|} \int_{a_j}^x f(t) \, dt \leq 3 M^- f(x).$$

By the exponential Carleson’s estimation, (see [3]), we have

$$\left| \left\{ x \in (x_{i+1}, \xi) : \Delta(x) > \frac{c}{f} \right\} \right| < C e^{-\frac{c}{f}} |(x_{i+1}, \xi)| \leq 2 C e^{-\frac{c}{f}} (x_{i+1} - x_{i+2}),$$

therefore

$$\left| \left\{ x \in (x_{i+1}, \xi) : T^* h(x) > \frac{1}{4} \lambda, M^- f(x) < \gamma \lambda \right\} \right| \leq C e^{-\frac{c}{f}} (x_{i+1} - x_{i+2}).$$

Putting together this last estimate with the ones for $f_2$ and $g$, we obtain the desired result. \(\square\)

**Lemma 3.5.** Let $p \geq 1$, $w \in A_p^-$ and let $T^-$ be a one-sided singular integral. Then there exists a constant $C = C(p, T^-)$, such that

$$||T^- f||_{L^1(w)} \leq C ||w||_{A_p^-} ||M^- f||_{L^1(w)}.$$

**Proof.** By Lemma 3.4, for $\varepsilon = \frac{1}{4}$ exists $c'$ such that

$$w\left( \left\{ x \in \mathbb{R} : T^* f(x) > 2 \lambda, M^- f(x) < \frac{c' \lambda}{||w||_{A_p^-}} \right\} \right) < \frac{1}{4} w(\{ T^* f(x) > \lambda \}).$$
Observe that
\[ \int_0^N w(\{ T^- f > \lambda \}) \, d\lambda \leq 2 \int_0^N w(\{ T^* f > 2\lambda \}) \, d\lambda \leq B_1 + B_2, \]
where
\[ B_1 = 2 \int_0^N w \left( \left\{ T^* f > 2\lambda, M^- f < \frac{c'r'}{|w|_{A_p}} \right\} \right) \, d\lambda, \]
\[ B_2 = 2 \int_0^N w \left( \left\{ M^- f \geq \frac{c'r'}{|w|_{A_p}} \right\} \right) \, d\lambda. \]

For \( B_1 \), we obtain
\[ B_1 = 2 \int_0^N w \left( \left\{ T^* f > 2\lambda, M^- f < \frac{c'r'}{|w|_{A_p}} \right\} \right) \, d\lambda \leq \frac{1}{2} \int_0^N w(\{ T^* f > \lambda \}) \, d\lambda. \]

It is easy to see that
\[ \frac{1}{2} \int_0^N w(\{ T^* f > \lambda \}) \, d\lambda \leq \frac{2||w||_{A_p}}{c'} \int_0^{\frac{N}{2||w||_{A_p}}} \int_0^N w(\{ T^* f > \lambda \}) \, d\lambda, \]
then
\[ ||T^- f||_{L^1(w)} \leq \frac{4||w||_{A_p}}{c'} ||M^- f||_{L^1(w)}, \]

obtaining the desired result. \( \square \)

For the next result we shall need the following Lemma due to A. K. Lerner, S. Ombrosi and C. Pérez in [11].

**Lemma 3.6.** [11] Let \( 1 < s < \infty \) and \( v \) be a weight. There exists an operator \( R \) in \( L^s(v) \) such that

- \( h \leq R(h) \)
- \( ||R(h)||_{L^s(v)} \leq 2||h||_{L^s(v)} \)
- \( R(h)(v)^{\frac{1}{s}} \in A_1 \) with \( ||R(h)(v)^{\frac{1}{s}}||_{A_1} \leq cs' \).

It is known that the weight \( (M_r^- w)^{1-p'} \) belongs to the \( A^\infty \) class with the corresponding constants independent of \( w \). Hence the next Lemma is a particular case of the Coifman-type estimate.

**Lemma 3.7.** Let \( T^- \) be a one-sided singular integral, \( p, r \geq 1 \). Then there exists \( C = C(T^-) \) such that
\[ \left| \frac{T^- f}{M_r^- w} \right|_{L^p(M_r^- w)} \leq C p' \left| \frac{M^- f}{M_r^- w} \right|_{L^{p'}(M_r^- w)}. \] (3.7)
Proof. By duality we have

\[ \left| \frac{T^-f}{M^-_r w} \right|_{L^p'(M^-_r w)} = \sup_{\|h\|_{L^p(M^-_r w)}} \int_{\mathbb{R}} |T^-f| h \, dx. \]

Choosing \( s = p \) and \( v = M^-_r w \), by Lemma 3.6, there exists an operator \( R \) such that 
\[ R(h)(M^-_r w) \frac{1}{p} \in A_1 \] with \[ \|R(h)(M^-_r w)\|_{A_1} \leq cp' \], then by Remark 2.4, item (2), 
\[ R(h)(M^-_r w) \frac{1}{p} \in A_1^- \] with \[ \|R(h)(M^-_r w)\|_{A_1^-} \leq cp'. \]

Now using the Remark 2.4, item (3) and item (4), we have 
\[ \|R(h)\|_{A_3^-} = \|R(h)(M^-_r w)\|_{A_3^-}^\frac{1}{p} (M^-_r w)^{\frac{1}{2p} - 2} \]
\[ \leq \|R(h)(M^-_r w)\|_{A_1^-}^\frac{1}{p} (M^-_r w)^{\frac{1}{2p}} \|A_1^-\|_{A_1^-}^2 \]
\[ \leq cp' \left( \frac{c}{1 - \frac{1}{2pr}} \right)^2 \leq Cp'. \]

Finally by Lemma 3.5,
\[ \int_{\mathbb{R}} |T^-f| h \, dx \leq \int_{\mathbb{R}} |T^-f| R(h) \, dx \leq C \|R(h)\|_{A_3^-} \int_{\mathbb{R}} M^-_r (f) R(h) \, dx \]
\[ \leq Cp' \int_{\mathbb{R}} \frac{M^-_r f}{M^-_r w} R(h) M^-_r w \, dx \leq Cp' \left\| \frac{M^-_r f}{M^-_r w} \right\|_{L^p'(M^-_r w)} \|R(h)\|_{L^p(M^-_r w)}. \]

As \( \|h\|_{L^p(M^-_r w)} = 1 \) we have
\[ \left| \frac{T^-f}{M^-_r w} \right|_{L^p'(M^-_r w)} \leq Cp' \left| \frac{M^-_r f}{M^-_r w} \right|_{L^p'(M^-_r w)}. \quad \square \]

4. Proof of the results

4.1. Proof of the Theorems

In order to prove Theorem 1.1 we first need to show the following result:

THEOREM 4.1. Let \( 1 < p < \infty, \ 1 < r < 2, \ w \) a weight and \( T^+ \) be a one-sided singular integral. Then
\[ \|T^+f\|_{L^p(w)} \leq Cpp'(r')^\frac{1}{p'} \|f\|_{L^p(M^-_r w)}, \] (4.1)

where \( C = C(T^+) \).
Proof. Observe that $T^−$ is the adjoint operator of $T^+$, with kernel supported in $(0, \infty)$. Also observe that as $(M^−_r w) ∈ A^+_1 \subset A^+_p$, then $(M^−_r w)^{1−\rho'} ∈ A^−_p \subset A^∞_w$. Therefore (4.1) is equivalent to prove

$$\left\| \frac{T^− f}{M^−_r w} \right\|_{L^p'(M^−_r w)} ≤ Cpp'(r') \left\| \frac{f}{w} \right\|_{L^p'(w)}.$$

By Hölder’s inequality

$$\frac{1}{b-a} \int_a^b f w^{−\frac{1}{p} \frac{1}{w}} ≤ \left( \frac{1}{b-a} \int_a^b w^r \right)^{\frac{1}{pr}} \left( \frac{1}{b-a} \int_a^b \left( f w^{−\frac{1}{p}} \right)^{(pr)'} \right)^{\frac{1}{pr'}},$$

and taking supremum we get

$$(M^− f(b))^{p'} ≤ (M^−_r w(b))^{p'−1}(M^−_{(pr)'_w}(f w^{−\frac{1}{p}})(b))^{p'},$$

then

$$\left\| \frac{M^− f}{M^−_r w} \right\|_{L^p'(M^−_r w)} ≤ \left\| M^−_{(pr)'} \left( f w^{−\frac{1}{p}} \right) \right\|_{L^p'}.$$

Now using that $||M^−_k g||_{L^s} ≤ C \left( \frac{s}{k} \right)^{\frac{1}{p'}} ||g||_{L^r}$, for $g = f w^{\frac{1}{p}}$, $k = (pr)'$ and $s = p'$ we get

$$\left\| \frac{M^− f}{M^−_r w} \right\|_{L^p'(M^−_r w)} ≤ C \left( \frac{rp−1}{r−1} \right)^{1−\frac{1}{pr}} \left\| \frac{f}{w} \right\|_{L^p'(w)} ≤ Cp \left( \frac{1}{r−1} \right)^{1−\frac{1}{pr}} \left\| \frac{f}{w} \right\|_{L^p'(w)}.$$

Observe that $t^{\frac{1}{r}} ≤ 2$ for $t ≥ 1$, then

$$\left( \frac{1}{r−1} \right)^{1−\frac{1}{pr}} ≤ (r')^{1−\frac{1}{r−1} + \frac{1}{pr}} ≤ 2(r')^{\frac{1}{pr'}}.$$

Finally applying Lemma 3.7 we get,

$$\left\| \frac{T^− f}{M^−_r w} \right\|_{L^p'(M^−_r w)} ≤ Cpp' \left\| \frac{M^− f}{M^−_r w} \right\|_{L^p'(M^−_r w)} ≤ Cpp'(r') \left\| \frac{f}{w} \right\|_{L^p'(w)}.$$

Proof of Theorem 1.1. This result is a consequence of Theorem 4.1. Using the equation (3.3), we observe that $r'_w \lesssim ||w||_{A^+_1}$ and $M^−_{r'_w}(w)(x) ≤ 2M^−(w)(x) ≤ 2||w||_{A^+_1} w(x)$ a.e. $x$. Then,

$$||T^+ f||_{L^p(w)} ≤ Cpp'(r'_w) \left\{ \int_R |f|^p(x) M^−_{r'_w}(w)(x) dx \right\}^{\frac{1}{p}} \lesssim Cpp'(||w||_{A^+_1}) \left\| \frac{1}{w} \right\|_{L^p(w)}.$$

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$$||T^+ f||_{L^p(w)} ≤ Cpp'(r'_w) \left\{ \int_R |f|^p(x) M^−_{r'_w}(w)(x) dx \right\}^{\frac{1}{p}} \lesssim Cpp'(||w||_{A^+_1}) \left\| \frac{1}{w} \right\|_{L^p(w)}.$$

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**Proof of Theorem 1.2.** Without loss of generality we assume that \(0 \leq f \in L^\infty_c(\mathbb{R})\).

Let

\[
\Omega = \{x \in \mathbb{R} : M^+f(x) > \lambda\} = \bigcup_j I_j = \bigcup_j (a_j, b_j),
\]

where \(I_j = (a_j, b_j)\) are the connected component of \(\Omega\) and they satisfy

\[
\frac{1}{|I_j|} \int_{I_j} f(y) \, dy = \lambda.
\]

Note that if \(x \notin \Omega\), then for all \(h \geq 0\)

\[
\frac{1}{h} \int_x^{x+h} f(y) \, dy \leq \lambda.
\]

Therefore \(f(x) \leq \lambda\) for a.e \(x \in \mathbb{R} \setminus \Omega\). Let \(I^-_j = (c_j, a_j)\) with \(c_j\) chosen so that \(|I^-_j| = 2|I_j|\) and set

\[
\tilde{\Omega} = \bigcup_j (I^-_j \cup I_j) = \bigcup_j I_j.
\]

We write \(f = g + h\) where

\[
g = f \chi_{\mathbb{R} \setminus \Omega} + \sum_{j=1}^{\infty} \lambda \chi_{I_j}, \quad h = \sum_{j=1}^{\infty} h_j = \sum_{j=1}^{\infty} (f - \lambda) \chi_{I_j}.
\]

Observe that \(0 \leq g(x) \leq \lambda\) for a.e. \(x\) and also that \(h_j\) has vanishing integral. Then

\[
w(\{x : |T^+f(x)| > \lambda\}) \leq w(\tilde{\Omega}) + w \left( \left\{ x \in \mathbb{R} \setminus \tilde{\Omega} : |T^+h(x)| > \frac{\lambda}{2} \right\} \right)
\]

\[
+ w \left( \left\{ x \in \mathbb{R} \setminus \tilde{\Omega} : |T^+g(x)| > \frac{\lambda}{2} \right\} \right) = \text{I} + \text{II} + \text{III}.
\]

We estimate \(\text{I}\):

\[
\text{I} = w(\tilde{\Omega}) \leq \sum_j (w(I^-_j) + w(I_j)),
\]

for each \(j\)

\[
w(I^-_j) = \frac{w(I^-_j)}{|I^-_j|} |I_j| = \frac{w(I^-_j)}{|I^-_j|} \frac{1}{\lambda} \int_{I_j} f(x) \, dx
\]

\[
= \frac{1}{\lambda} \int_{I^-_j} \int_{I_j} w(t) \, dt \, f(x) \, dx \leq \frac{3}{\lambda} \int_{I_j} \frac{1}{(x-c_j)^2} \int_{c_j}^x w(t) \, dt \, f(x) \, dx
\]

\[
\leq \frac{3}{\lambda} \int_{I_j} f(x) M^- w(x) \, dx.
\]
On the other hand, $(w,M^- w) \in A^+_1$ then $M^+$ is weak type $(1,1)$ with respect to this pair of weights, then
\[ \sum_j w(I_j) = w(\{ x : M^+ f(x) > \lambda \}) < \frac{4}{\lambda} \int_{\mathbb{R}} f(t)M^- w(t) dt, \]
therefore
\[ I = w(\tilde{\Omega}) \leq \frac{7}{\lambda} \int_{\mathbb{R}} f(t)M^- w(t) dt \leq \frac{7}{\lambda} ||w||_{A^+_1} \int_{\mathbb{R}} f(t)w(t) dt. \]

To estimate $II$, let $r_j = |I_j|/|I^- j|/2$. Now we use that $h_j$ is supported in $I_j$, $\int_{I_j} h_j = 0$, and that $K$ is supported in $(-\infty,0)$:
\[ II = w\left( \left\{ x \in \mathbb{R} \setminus \tilde{\Omega} : |T^+ h(x)| > \frac{\lambda}{2} \right\} \right) \leq \frac{2}{\lambda} \int_{\mathbb{R} \setminus \tilde{\Omega}} |T^+ h(t)| w(t) dt \]
\[ \leq \frac{2}{\lambda} \sum_j \int_{I_j} |h_j(y)| \int_{\mathbb{R} \setminus \tilde{\Omega}} |K(t-y) - K(t-a_j)| w(t) dt dy \]
\[ = \frac{2}{\lambda} \sum_j \int_{I_j} |h_j(y)| \int_{-\infty}^{c_j} |K(t-y) - K(t-a_j)| w(t) dt dy. \]

Observe that it is suffice to obtain that for all $y \in I_j$,
\[ \int_{-\infty}^{c_j} |K(t-y) - K(t-a_j)| w(t) dt \leq C \essinf_{\bar{I}_j} M^- (w\chi_{\mathbb{R} \setminus \bar{I}_j}). \]
To see this we use the condition of the kernel $K$,
\[ \int_{-\infty}^{c_j} |K(t-y) - K(t-a_j)| w(t) dt = \sum_{k=1}^{\infty} \int_{a_j - 2^{-k} r_j}^{a_j - 2^{k+1} r_j} |K(t-y) - K(t-a_j)| w(t) dt \]
\[ \leq C \sum_{k=1}^{\infty} \int_{a_j - 2^{k+1} r_j}^{a_j - 2^k r_j} \left| \frac{y-a_j}{(t-a_j)^2} \right| w(t) dt \]
\[ \leq C \sum_{k=1}^{\infty} \frac{y-a_j}{(2^k r_j)^2} \int_{a_j - 2^k r_j}^{a_j - 2^{k+1} r_j} w(t) \chi_{(a_j - 2^k r_j, a_j - 2^{k+1} r_j)} dt \]
\[ \leq C \sum_{k=1}^{\infty} \frac{1}{2^k (2^k r_j)^2} \int_{a_j - 2^k r_j}^{a_j - 2^{k+1} r_j} w(t) \chi_{(a_j - 2^k r_j, a_j - 2^{k+1} r_j)} dt, \]
where $C = C(T^+)$. If $x \in I_j$
\[ \int_{-\infty}^{c_j} |K(t-y) - K(t-a_j)| w(t) dt \]
\[ \leq C \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{x-a_j+2^{k+1} r_j}{2^k r_j} \frac{1}{x-a_j+2^{k+1} r_j} \int_{a_j - 2^{k+1} r_j}^{x} w(t) \chi_{(a_j - 2^k r_j, a_j - 2^{k+1} r_j)} dt \]
\[ \leq CM^- w\chi_{\mathbb{R} \setminus \bar{I}_j}(x) \sum_{k=1}^{\infty} \frac{1}{2^k} \left( \frac{x-a_j}{2^k r_j} + \frac{2^{k+1} r_j}{2^k r_j} \right) \leq CM^- (w\chi_{\mathbb{R} \setminus \bar{I}_j}(x), \]

therefore
\[
II \leq \frac{C}{\lambda} \sum_j \text{ess inf}_{I_j} M^- (w \chi_{R \setminus I_j}) \int_{I_j} |h_j(y)| \, dy \\
\leq \frac{C}{\lambda} \sum_j \int_{I_j} |h_j(y)| M^- (w \chi_{R \setminus I_j})(y) \, dy \\
\leq \frac{C}{\lambda} \left[ \sum_j \int_{I_j} f(y) M^- (w \chi_{R \setminus I_j})(y) \, dy + \sum_j \int_{I_j} |g(y)| M^- (w \chi_{R \setminus I_j})(y) \, dy \right] \\
= \frac{C}{\lambda} (A + B).
\]

For $A$ there is nothing to prove. To work with $B$ we need to prove the following inequality
\[
M^- (w \chi_{R \setminus I_j})(y) \leq \frac{3}{2} \text{ess inf}_{z \in I_j} M^- (w \chi_{R \setminus I_j})(z),
\]
(4.2)
for all $y \in I_j$. In fact for $y, z \in I_j$,
\[
M^- (w \chi_{R \setminus I_j})(y) = \sup_{t < y} \frac{1}{y - t} \int_t^y w(s) \chi_{R \setminus I_j}(s) \, ds = \sup_{t < c_j} \frac{1}{y - t} \int_t^{c_j} w(s) \chi_{R \setminus I_j}(s) \, ds \\
\leq \sup_{t < c_j} \frac{3}{2} \frac{1}{y - t} \int_t^{c_j} w(s) \chi_{R \setminus I_j}(s) \, ds \leq \frac{3}{2} M^- (w \chi_{R \setminus I_j})(z).
\]

Then
\[
B = \sum_j \int_{I_j} |g(y)| M^- (w \chi_{R \setminus I_j})(y) \, dy = \sum_j \int_{I_j} \lambda M^- (w \chi_{R \setminus I_j})(y) \, dy \\
\leq \sum_j \int_{I_j} f(t) \, dt \frac{1}{|I_j|} \int_{I_j} M^- (w \chi_{R \setminus I_j})(y) \, dy \\
\leq \frac{3}{2} \sum_j \int_{I_j} f(t) \, dt \text{ ess inf}_{I_j} M^- (w \chi_{R \setminus I_j}) \leq \frac{3}{2} \sum_j \int_{I_j} f(t) M^- (w \chi_{R \setminus I_j})(t) \, dt.
\]

So
\[
II \leq \frac{C}{\lambda} \sum_j \int_{I_j} f(t) M^- (w \chi_{R \setminus I_j})(t) \, dt \leq \frac{C}{\lambda} \|w\|_{A^+_1} \int_{\mathbb{R}} f(t) w(t) \, dt.
\]

Finally we estimate $III$. First observe that doing the same proof that in (4.2) we obtain
\[
M^- (w \chi_{R \setminus \Omega})(y) \leq \frac{3}{2} \text{ess inf}_{z \in I_j} M^- (w \chi_{R \setminus \Omega})(z),
\]
(4.3)
for all $y \in I_j$. 
By Chebichef’s inequality, using the fact that $g \leq \lambda$ and choosing $r = r_w = 1 + \frac{1}{16\pi ||w||_{A_1^+}}$ in order to apply Theorem 4.1 and equation (3.3), we get that

$$III = w \left( \left\{ x \in \mathbb{R} \setminus \tilde{\Omega} : |T^+ g| (x) > \frac{\lambda}{2} \right\} \right)$$

$$\leq \frac{2^p}{\lambda^p} \int_{\mathbb{R}} (|T^+ g| (x))^p w(x) \chi_{(\mathbb{R} \setminus \tilde{\Omega})} (x) \, dx$$

$$\leq \frac{2^p}{\lambda^p} (Cpp'((r')^{\frac{1}{p}}))^{\frac{1}{p}} \int_{\mathbb{R}} (|g| (x))^p M^-(w \chi_{(\mathbb{R} \setminus \tilde{\Omega})}) (x) \, dx$$

By Chebichef’s inequality, using the fact that $f$ in $\tilde{\Omega}$, we have

$$\lim_{x \to r} f(x) f'(x) \leq C||w||_{A_1^+}$$

implies

$$III \leq \frac{2^{p+1}}{\lambda} (Cpp'((r')^{\frac{1}{p}}))^{\frac{1}{p}} \int_{\mathbb{R}} f(x) M^- w(x) \, dx$$

$$\leq \frac{2^{p+1}}{\lambda} (Cpp'(|w||_{A_1^+})^{\frac{1}{p}})^{\frac{1}{p}} ||w||_{A_1^+} \int_{\mathbb{R}} f(x) w(x) \, dx$$

$$\leq \frac{Cp^{p+1}}{\lambda} [pp' ||w||_{A_1^+}]^{\frac{1}{p}} \int_{\mathbb{R}} f(x) w(x) \, dx.$$}

We take $p = 1 + \frac{1}{\log(e+||w||_{A_1^+})}$ and observing that $t^{(\log(e+t))^{-1}}$ and $t^{-1}$ are bounded for $t > 1$ we have

$$[pp' ||w||_{A_1^+}]^{\frac{1}{p}} \leq C \log(e+||w||_{A_1^+}) ||w||_{A_1^+},$$

and as $1 < p < 2$ we obtain

$$III \leq \frac{C}{\lambda} \log(e+||w||_{A_1^+}) ||w||_{A_1^+} \int_{\mathbb{R}} f(x) w(x) \, dx.$$}

Combining this estimate with $I$ and $II$ completes the proof. □

**Remark 4.2.** The choice of $p$, in proof of Theorem 1.2, is similar to the one given in [11]. The conjecture or goal was to find a linear dependence of the constant $||w||_{A_1}$. In [19] the authors proved that this is not possible. The nearest one is a $t \log t$.
dependence of the constant, by the kind of steps followed to approach to the result. Maybe it can be shown that a \( t \log^\varepsilon t \), dependence is possible, for some \( 0 < \varepsilon < 1 \). To prove this result, a better estimate in Theorem 4.1 it should be obtain. C. Pérez in [21] conjectures that a \( t \log^\varepsilon t \) dependence is not possible.

4.2. Proof of the Corollaries

To prove the Corollary 1.3, we need to build \( A_1^+ \) weights with special control on the constant, based in the Rubio de Francia algorithm. In order to do this we need the one-sided version of the Buckley result sharp estimate in norm \( L^p \) of the Hardy-Littlewood maximal operator \( M \) respect to weights \( w \in A_p \). This Theorem was proved by F. J. Martín-Reyes and A. de la Torre in [17].

**THEOREM 4.3.** [17] If \( w \in A_p^- \) then

\[
||M^-||_{L^p(w)} \leq Cp'2^{p'}||w||_{A_p}^{\frac{1}{p'}}.
\] (4.4)

The equivalent of the following Lemma for weights in \( A_p^+ \), is proven in [4].

**LEMMA 4.4.** Let \( 1 < q < \infty \) and let \( w \in A_q^+ \). Then there exists a nonnegative sublinear operator \( D \) bounded on \( L^q \) such that for any nonnegative \( h \in L^q(w) \):

1. \( h \leq D(h) \);
2. \( ||D(h)||_{L^q(w)} \leq 2||h||_{L^q(w)} \);
3. \( D(h).w \in A_1^+ \) with \( ||D(h).w||_{A_1^+} \leq Cq2^q||w||_{A_q^+} \),

where the constant \( C \) not depend on \( ||w||_{A_1^+} \) and \( q \).

**Proof.** We define the operator \( S(h) = w^{-1}M^- (|h|w) \), then \( S \) is bounded in \( L^q(w) \), moreover, \( ||S||_{L^q(w)} \leq Cq2^q||w||_{A_q^+} \), indeed using equation (4.4) we get,

\[
||S(h)||_{L^q(w)} = \left( \int_{\mathbb{R}} (w^{-1}M^- (|h|w))^{\frac{1}{q'}} w^{1-q'} \, dx \right)^{\frac{1}{q'}} = \left( \int_{\mathbb{R}} (M^- (|h|w))^{\frac{1}{q'}} w^{1-q'} \, dx \right)^{\frac{1}{q'}} \leq ||M^-||_{L^q(w)^{1-q'}} ||h||_{L^q(w)^{1-q'}} \leq Cq2^q ||w^{1-q'}||_{A_q^{1-q'}} ||h||_{L^q(w)}.
\]

Recalling that \( w \in A_q^+ \) implies \( w^{1-q'} \in A_{q'}^- \) and that \( ||w^{1-q'}||_{A_q^-} = ||w||_{A_q^{-\frac{1}{1-q'}}} \), we get \( ||S||_{L^q(w)} \leq Cq2^q||w||_{A_q^+} \), as claimed.

Now we define the operator \( D \) via the following convergent Neumann series:

\[
D(h) = \sum_{k=0}^{\infty} \frac{S^k(h)}{2^k||S||^k}, \quad \text{where} \quad ||S|| = ||S||_{L^q(w)}.
\]
Then (1) and (2) are clearly satisfied.
(3) It follows from the definition of $D$ and the sublinearity of $S$ that

$$S(D(h)) \leq 2\|S\|(D(h) - h) \leq 2\|S\|D(h),$$

therefore

$$M^-(D(h)w) = M^-(D(h)w)w^{-1}w = S(D(h))w \leq 2\|S\|D(h)w \leq cq2^q\|w\|_{A_q^+}D(h)w. \qed$$

Proof of Corollary 1.3. For $\alpha > 0$ we set $\Omega_\alpha = \{x \in \mathbb{R} : |T^+f(x)| > \alpha\}$ and let $\phi(t) = t \log(e + t)$. Applying Lemma 4.4 with $q = p$, we get a sublinear operator $D$ bounded on $L^p$ satisfying properties (1), (2), and (3). Using these properties and Theorem 1.2, we obtain

$$\int_{\Omega_\alpha} h\,wdx \leq \int_{\Omega_\alpha} D(h)\,wdx \leq \frac{C}{\alpha} \phi(\|D(h).w\|_{A_1^+})\|f\|_{L^1(D(h).w)}$$

$$\leq \frac{C}{\alpha} \phi(Cp2p\|w\|_{A_p^+}) \int_{\mathbb{R}} |f|D(h)\,wdx$$

$$\leq \frac{C}{\alpha} 2\phi(Cp2p)\phi(\|w\|_{A_p^+}) \left( \int_{\mathbb{R}} |f|^p\,wdx \right)^\frac{1}{p} \left( \int_{\mathbb{R}} D(h)^p wdx \right)^\frac{1}{p}$$

$$\leq \frac{C}{\alpha} \phi(\|w\|_{A_p^+})\|f\|_{L^p(w)}\|h\|_{L^p(w)}.$$

The proof is completed by taking the supremum over all $h$ with $\|h\|_{L^p(w)} = 1$. \qed

Proof of Corollary 1.4. Given a one-sided singular operator $T^-$, its adjoint operator is $T^+$. Let $w \in A_p^-$ then $\sigma = w^{-\frac{1}{p'}} \in A_{p'}^+ \text{ with } \|\sigma\|_{A_p^+} = \|w\|_{A_p}^{-\frac{1}{p-1}}$. Applying Corollary 1.3 to the one-sided singular operator $T^+$ and the weight $\sigma$, we get

$$\|T^+\|_{L^{p'}(\sigma)} \leq C\|w\|_{A_p}^{-\frac{1}{p-1}} \log \left( e + \|w\|_{A_p}^{-\frac{1}{p-1}} \right) \|f\|_{L^{p'}(\sigma)}$$

$$\leq C\|w\|_{A_p}^{-\frac{1}{p-1}} \log(e + \|w\|_{A_p})\|f\|_{L^{p'}(\sigma)}.$$

From this, by duality we obtain

$$\|T^-\|_{L^p(w)} \leq C\|w\|_{A_p}^{-\frac{1}{p-1}} \log\left(e + \|w\|_{A_p}\right) \|f\|_{\sigma} \|_{L^{p,1}(\sigma)},$$

where $L^{p,1}(\sigma)$ is the standard weighted Lorentz space. Setting here $f = \sigma \chi_E$, where $E$ is any measurable set, completes the proof. \qed
5. Appendix

We will give an easier proof of a slight weak version of Lemma 3.2.

In [24] M. S. Riveros and A. de la Torre, using Lemma 5 in [13], obtained another version of weak reverse Hölder’s inequality. If we use the equation (3.2), following the same steps of the proof in [24], we obtain a new result with special control on the constant.

**Lemma 5.1.** [24] (One-sided RHI) Let $1 \leq p < \infty$, $w \in A^+_p$ and $a < b < c$ with $b - a = 2(c - b)$. If $r = 1 + \frac{1}{4^{p+2}e^{e^p}\|w\|_{A^+_p}}$, for $p > 1$ and $r = 1 + \frac{1}{16e^{e\|w\|_{A^+_1}}}$, for $p = 1$ then

$$\frac{1}{b-a} \int_a^b w^r \leq C \left( \frac{1}{c-a} \int_a^c w \right)^r,$$

where $C$ does not depend on the weight $w$.

The following Lemma is a slight weak version of Lemma 3.2.

**Lemma 5.2.** Let $p \geq 1$, $w \in A^-_p$, $a < b < c$ such that $2(b-a) = (c-b)$ and $E \subseteq (b,c)$ a measurable set. Then for every $\varepsilon > 0$ there exists $C = C(\varepsilon, p)$ such that if $|E| < e^{-C\|w\|_{A^-_p}}(b-a)$ then $w(E) < \varepsilon w(a,c)$.

**Proof.** We will use the analogous to Lemma 5.1 for $A^-_p$ weights.

$$w(E) = \frac{1}{c-b} \int_b^c w \chi_E (c-b) \leq (c-b) \left( \frac{1}{c-b} \int_b^c w^r \right)^{\frac{1}{r}} \left( \frac{1}{c-b} \int_b^c \chi_E \right)^{\frac{1}{r'}}$$

$$= \left( \frac{|E|}{c-b} \right)^{\frac{1}{r'}} (c-b) C \frac{1}{c-a} \int_a^c w \leq \left( \frac{|E|}{b-a} \right)^{\frac{1}{r'}} C \int_a^c w \leq \varepsilon w(a,c),$$

where the last inequality is obtained by following the same steps as in (3.4). □

As a Corollary of Lemma 5.1 we obtain another proof of Proposition 3 in [13], this is

**Corollary 5.3.** Let $1 < p < \infty$ and $w \in A^+_p$. Then $w \in A^+_{p-\varepsilon}$, with $p - \varepsilon = \frac{p-1}{r(\sigma)} + 1$ where $\sigma = w^{1-p'}$ and $r(\sigma)$ is the one obtained in the analogous version of Lemma 5.1 for a weight in $A^-_p$.

**Proof.** In [24] it is proved that $w \in A^+_p$ if, and only if there exists $C > 0$ such that

$$\sup_{a,b,c,d} \frac{1}{(b-a)^p} \left( \int_a^b w \right)^p \left( \int_c^d w_p^{-1} \right)^{p-1} < C. \quad (5.1)$$
where the supremum is taken over all \( a, b, c, d \) such that \( a < b < c < d \) and \( 2(b - a) = 2(d - c) = c - b \).

Let \( r = r(\sigma) \) be the one of Lemma 5.1 and \( a, b, c, d \) as in the previous line, then

\[
\left( \frac{1}{b - a} \int_a^b w \right) \left( \frac{1}{d - c} \int_c^d w^{p-\varepsilon-1} \right) \leq \left( \frac{1}{b - a} \int_a^b w \right) \left( \frac{1}{d - c} \int_c^d \sigma^r \right)^{\frac{p-1}{r}} \leq \left( \frac{1}{b - a} \int_a^b w \right) \left( \frac{1}{d - b} \int_b^d \sigma \right)^{p-1} \leq (C)^{p-1} ||w||_{A_p^r},
\]

where \( C \) does not depend on \( p \) nor \( w \). \( \square \)

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