When a charged heterotic string is placed in a constant magnetic field $B$, we show that this system can be solved exactly by using the cyclotron frequency. We then calculate anomalies of the super Virasoro algebra, and give the corresponding spectrum-generating algebra for this system. They differ from the free case by the cyclotron frequency. It is remarkable that our system is equivalent to the completely free system when $B$ takes integral values.

PACS numbers:

I. INTRODUCTION

The heterotic string\cite{1} is regarded as one of the most promising model for realistic particles. It is a closed string, which is composed of right-moving fermionic and bosonic strings with ten dimensions and left-moving bosonic string with 26 dimensions. As a result of the compactification of the 26-10=16-dimensional internal space, the heterotic string is associated with the phenomenologically promising gauge group $SO(32)$ or $E_8 \times E_8\cite{2,3}$.

The electromagnetic interaction of the heterotic string has been also considered. The electromagnetic field has so far been introduced as a vector field $g_{\mu I}$ of the Kaluza-Klein type, where $g_{\mu I}$ is a space-time metric with external components $\mu = 0, \cdots , 9$ and internal components $I = 10, \cdots , 25\cite{4}$. However, this type of electromagnetic interaction cannot be solved exactly, but we can treat it only by a perturbative method.

In this paper we propose an exactly solvable model of the heterotic string placed in a constant magnetic field. The electric charge was assumed to distribute uniformly along the closed string. The electromagnetic field is introduced in such a way that the interaction is invariant under the superconformal transformation and also under the gauge transformation of the electromagnetic field. When the charged heterotic string is placed in the constant magnetic field $B$, we show that the Lagrangian of the interacting heterotic string can be translated into the free type of Lagrangian. However, this free type of closed string is not periodic at the boundary $\sigma = 2\pi$, but yields the phase factors $\exp(\pm 2\pi i \omega)$, where $\omega$ is the cyclotron frequency. This causes the fact that the cyclotron frequency $\omega$ is included in all orders in mode expansions of the string and also in quantization conditions for mode operators. This differs from the completely free case. Therefore, our next task is to calculate the superconformal algebra together with anomalies, and also to give the spectrum-generating algebra\cite{5}, which is necessary to construct physical states satisfying the Virasoro conditions. Finally we point out that our system is equivalent to the completely free system when $B$ takes integral values.

In Sec\[II] we propose a new type of interaction of the electromagnetic field with the heterotic string. In Sec\[III] and Sec\[VII] we calculate anomalies associated with the super Virasoro algebra. In Sec\[V] the spectrum-generating algebra is constructed. In Sec\[VII] we consider the algebra isomorphic...
to the spectrum-generating algebra. From the equation of isomorphisms we derive the number of space-time dimensions together with constraint constants of the 0-mode Virasoro operators. Finally, Sec. VII is devoted to concluding remarks.

II. A HETEROSTATIC STRING IN THE EXTERNAL FIELD

The free Lagrangian is given by

$$L_0 = 2\partial_+ \hat{x}_a D \hat{x}^a.$$  (II.1)

The notation is summarized as follows:

$$(\tau, \sigma) = (s^0, s^1), \quad s^\pm = s^0 \pm s^1, \quad \partial_\pm = \partial / \partial s^\pm = (1/2)(\partial / \partial \tau \pm \partial / \partial \sigma),$$

$$D = i\partial_\theta + \theta \partial_-$$

and

$$\hat{x}^a(s^+, s^-, \theta) = x^a(s^+, s^-) + i\chi^a(s^+, s^-) \theta,$$

$$a = (\mu, I), \quad \mu = 0, 1, \cdots, 9 \quad I = 10, \cdots, 25$$  (II.3)

The free action is

$$S_0 = 2 \int d^2 s d\theta \partial_+ \hat{x}_a D \hat{x}^a = 2 \int d^2 s (\partial_+ x_a \partial_- x^a + i\chi_a \partial_+ \chi^a).$$  (II.4)

Under the superconformal transformation (SCT)

$$\delta s^+, = f^+(s^+),$$

$$\delta s^-, = f^-(s^-) + i\eta(s^-) \theta,$$

$$\delta \theta = \eta(s^-) + 1/2 \theta \partial_- f^-(s^-),$$

the action is invariant, if $\hat{x}^a$ is a superconformal scalar. The conserved superconformal charge is given by

$$Q = \int d\sigma (f^+ L_+ + f^- L_- + i\eta G_-),$$  (II.5)

where

$$L_+ = J_a J^a, \quad J^a = \partial_+ x^a,$$

$$L_- = I_\mu I^\mu + i\partial_\eta \chi_\mu \chi^\mu, \quad I^\mu = \partial_- x^\mu,$$

$$G_- = 2\chi_\mu I^\mu.$$  (II.6)

Now, the electromagnetic field $A_\mu(\hat{x})$ is introduced in such a way as

$$\hat{L} = 2[\partial_+ \hat{x}_a + 2A_\mu(\hat{x})] D \hat{x}^a.$$  (II.7)

If the action for the interaction term is integrated over $\theta$, we have

$$S_{int} = 4 \int d^2 s d\theta A_\mu(\hat{x}) D \hat{x}^a$$

$$= 4 \int d^2 s [A_\mu(x) \partial_- x^a - i\partial_\eta A_\mu(x) \chi^a \chi^b].$$  (II.11)
The interaction is clear to be invariant under the gauge transformation, $\delta A_a(x) = \partial_a \Lambda(x)$, or equivalently,

$$\delta A_a(x) = \partial_a \Lambda(x) \quad (\text{II.12})$$

We then choose the symmetric gauge

$$A_\mu(\hat{x}) = -\frac{1}{2} F_{\mu\nu} \hat{x}^\nu, \quad F_{\mu\nu} = \text{constant}, \quad (\text{II.13})$$

to obtain

$$\hat{L} = 2[\partial \hat{x} - F \cdot \hat{x}] \cdot D \hat{x}. \quad (\text{II.14})$$

If we use a new variable

$$\hat{X} = \exp(-F s^+ \hat{x}), \quad (\text{II.15})$$

we find that the Lagrangian (II.14) reduces to

$$\hat{L} = 2\partial_x \hat{X} \cdot D \hat{X}. \quad (\text{II.16})$$

This is the free type of Lagrangian with respect to $\hat{X}$. The action for Eq. (II.16) is invariant under SCT, if is a superconformal scalar. In this case the original variable $\hat{x}$ behaves as

$$\delta_{\text{SCT}} \hat{x} = \delta s^+ F \cdot \hat{x} \quad (\text{II.17})$$

under SCT.

We concentrate on one $2 \times 2$ block of $F_{\mu\nu}$ with $B$ real

$$F_{\mu\nu} = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix}, \quad \mu, \nu = 1, 2. \quad (\text{II.18})$$

Introducing

$$\hat{X}^{(\pm)} = (\hat{X}^1 \pm i \hat{X}^2)/\sqrt{2}. \quad (\text{II.19})$$

Eqs. (II.15) turns out to be

$$\hat{X}^{(\pm)} = \exp(\pm i B s^+) \hat{x}^{(\pm)}. \quad (\text{II.20})$$

Corresponding to Eq. (II.3), we expand $\hat{X}^{(\pm)}(\tau, \sigma, \theta)$ in $\theta$

$$\hat{X}^{(\pm)}(\tau, \sigma, \theta) = X^{(\pm)}(\tau, \sigma) + i \psi^{(\pm)}(\tau, \sigma) \theta. \quad (\text{II.21})$$

From Eq. (II.20) we find

$$X^{(\pm)}(\tau, \sigma) = \exp \left[ \pm i B(\tau + \sigma) \right] x^{(\pm)}(\tau, \sigma), \quad (\text{II.22})$$

$$\psi^{(\pm)}(\tau, \sigma) = \exp \left[ \pm i B(\tau + \sigma) \right] \chi^{(\pm)}(\tau, \sigma). \quad (\text{II.23})$$

Considering the periodicity of $x^{(\pm)}(\tau, \sigma)$ and $\chi^{(\pm)}(\tau, \sigma)$, we find the quasi-periodicity for $X^{(\pm)}(\tau, \sigma)$ and $\psi^{(\pm)}(\tau, \sigma)$ as

$$X^{(\pm)}(\tau, \sigma + 2\pi) = \exp \left( \pm 2\pi i B \right) X^{(\pm)}(\tau, \sigma), \quad (\text{II.24})$$

$$\psi^{(\pm)}(\tau, \sigma + 2\pi) = -\exp \left( \pm 2\pi i \omega \right) \psi^{(\pm)}(\tau, \sigma), \quad \text{for NS sector} \quad (\text{II.25})$$

$$\psi^{(\pm)}(\tau, \sigma + 2\pi) = \exp \left( \pm 2\pi i \omega \right) \psi^{(\pm)}(\tau, \sigma), \quad \text{for Ramond sector} \quad (\text{II.26})$$
where $B = q + \omega$, $q \in \mathbb{Z}$, $0 \leq \omega < 1$. In the following we call $\omega$ simply the cyclotron frequency, and consider only the range $0 < \omega < 1$, since our system is equivalent to the free system when $\omega = 0$. However, it is remarkable that when $B$ takes integral values, $B = q \in \mathbb{Z}$, our system is equivalent to the completely free system.

The relevant parts in Lagrangian (II.16) can be written as

$$\hat{\mathcal{L}} = 2[\partial_+ \hat{X}^+(s) \hat{D}\hat{X}^-(s) + \partial_+ \hat{X}^-(s) \hat{D}\hat{X}^+(s)].$$

(II.27)

Integrating over $\theta$, the Lagrangian reduces to

$$\mathcal{L} = 2[\partial_+ \hat{X}^+(s) \partial_- \hat{X}^-(s) + \partial_+ \hat{X}^-(s) \partial_- \hat{X}^+(s) + i\psi^+(s) \partial_+ \psi^-(s) + i\psi^-(s) \partial_+ \psi^+(s)].$$

(II.28)

In spite of the quasi-periodicity of $X$ and $\psi$, the periodic boundary condition, which is necessary in the variational principle, is guaranteed, because the aperiodic phase factors $\exp(\pm 2\pi i \omega)$ are always cancelled out between the $(+)$ and $(-)$ components in the Lagrangian.

Equations of motion are all of free type:

$$\partial_+ \partial_- X^\pm = 0,$$

$$\partial_+ \psi^\pm = 0.$$  

(II.29)  

(II.30)

Their solutions with boundary conditions (II.24), (II.25) and (II.26) are given by

$$X^\pm(\tau, \sigma) = X^\pm_+ (s^+) + X^\pm_- (s^-),$$

(II.31)

$$X^\pm_+ (s^+) = \frac{i}{\pi} \sum_n \frac{1}{n \pm \omega} \exp[-i(n \mp \omega)s^+] \alpha_n^\pm,$$

$$X^\pm_- (s^-) = \frac{i}{\pi} \sum_n \frac{1}{n \pm \omega} \exp[-i(n \pm \omega)s^-] \beta_n^\pm,$$

and

$$\psi^\pm (s^-) = \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_r^\pm \exp[-i(r \pm \omega)s^-], \quad \text{for NS sector}$$

(II.32)

$$\psi^\pm (s^-) = \frac{1}{2} \sum_{n \in \mathbb{Z}} d_n^\pm \exp[-i(n \pm \omega)s^-], \quad \text{for Ramond sector}.$$

The conjugate momenta to $X^\pm(\tau, \sigma)$ are

$$P^{(\mp)} = \frac{\partial \mathcal{L}}{\partial (\partial_+ X^{(\mp)})} = \partial_- X^{(\mp)} + \partial_+ X^{(\mp)} = \hat{\dot{X}}^{(\mp)}.$$  

(II.33)

The quantization is accomplished by setting the commutation rules

$$[X^{(\pm)}(\tau, \sigma), \ P^{(\mp)}(\tau, \sigma')] = i \pi \delta_{\pm \omega}(\sigma - \sigma'),$$

(II.34)

and other combinations are zero. Here $\delta_{\pm \omega}(\sigma - \sigma')$ is the delta function with the same quasi-periodicity as $X^{(\pm)}(\tau, \sigma)$ with respect to its argument. From these we have commutation relations:

$$[\alpha_n^{(\pm)}, \alpha_m^{(-)}] = (m - \omega)\delta_{m+n,0}, \quad 0 < \omega < 1,$$

(II.35)

$$[\beta_m^{(\pm)}, \beta_n^{(-)}] = (m + \omega)\delta_{m+n,0},$$

(II.36)
and other combinations are zero. Since \(0 < \omega < 1\), \(\alpha_m^{(+)}\), \(\beta_m^{(-)}\) are annihilation operators for \(m > 0\), and creation operators for \(m < 0\), while \(\alpha_m^{(-)}\), \(\beta_m^{(+)}\) are creation operators for \(m < 0\), and annihilation operators for \(m \geq 0\). As for the fermionic parts, we get, after the Dirac quantization,

\[
\{ b_\tau^{(+)} , b_r^{(-)} \} = \delta_{\tau+r,0} , \quad \text{others} = 0 , \tag{II.37}
\]

\[
\{ d_m^{(+)} , d_n^{(-)} \} = \delta_{m+n,0} , \quad \text{others} = 0 . \tag{II.38}
\]

\(b_r^{(\pm)}\) are annihilation operators for \(r > 0\), and creation operators for \(r < 0\).

For the Ramond sector, we need a special care on the 0-modes, so the detail will be discussed in Sec.4.

### III. CALCULATION OF ANOMALY

The Lagrangian (II.27) happens to appear as if it is a free Lagrangian. However, the dynamical variables \(X^{(\pm)}(\tau, \sigma)\) and \(\psi^{(\pm)}(\tau, \sigma)\) are subject to the quasi-periodicity (II.24)-(II.26), and this causes the inclusion of the cyclotron frequency \(\omega\) in the commutators (II.35) and (II.36) for mode operators, different from the completely free case. Considering this fact, we should examine the validity of the super Virasoro algebra together with calculation of anomaly.

Let us define the current operators by

\[
J^{(\pm)}(z) = i\partial X^{(\pm)} = z^{\pm\omega} \sum_n z^{-n-1} \alpha_n^{(\pm)} = z^{\pm\omega} \tilde{j}^{(\pm)}(z) , \quad z = \exp(is^+) , \tag{III.1}
\]

\[
J^{(\pm)}(z) = i\partial X^{(\pm)} = z^{\mp\omega} \sum_n z^{-n-1} \beta_n^{(\pm)} = z^{\mp\omega} \tilde{\bar{j}}^{(\pm)}(z) , \quad z = \exp(is^-) . \tag{III.2}
\]

Here, the exponent -1 on \(z\) is only for convenience. The operator product expansions for them are given by

\[
\tilde{j}^{(+)}(z) \tilde{j}^{(-)}(z') = \frac{1}{(z - z')^2} - \frac{\omega}{z(z' - z)} , \tag{III.3}
\]

\[
\tilde{j}^{(-)}(z) \tilde{j}^{(+)}(z') = \frac{1}{(z - z')^2} + \frac{\omega}{z'(z - z')} , \tag{III.4}
\]

\[
\tilde{\bar{j}}^{(+)}(z) \tilde{\bar{j}}^{(-)}(z') = \frac{1}{(z - z')^2} + \frac{\omega}{z'(z - z')} , \tag{III.5}
\]

\[
\tilde{\bar{j}}^{(-)}(z) \tilde{\bar{j}}^{(+)}(z') = \frac{1}{(z - z')^2} - \frac{\omega}{z(z' - z)} . \tag{III.6}
\]

Here, we have used the following contractions:

\[
\langle \alpha_m^{(\pm)} \alpha_n^{(\mp)} \rangle = \begin{cases} (m \mp \omega)\delta_{m+n,0} , & (m \mp \omega > 0) \\ 0 , & (m \mp \omega < 0) \end{cases} \tag{III.7}
\]

\[
\langle \beta_m^{(\pm)} \beta_n^{(\mp)} \rangle = \begin{cases} (m \pm \omega)\delta_{m+n,0} , & (m \pm \omega > 0) \\ 0 , & (m \pm \omega < 0) \end{cases} \tag{III.8}
\]

For the fermionic fields we confine ourselves to the NS sector,

\[
\psi^{(\pm)}(z) = \frac{1}{2} z^{\mp\omega} \sum_r z^{-r-1/2} b_r^{(\pm)} = z^{\mp\omega} \tilde{\psi}^{(\pm)}(z) , \tag{III.9}
\]

\[
\tilde{\bar{\psi}}^{(\pm)}(z) \tilde{\bar{\psi}}^{(\mp)}(z') = \frac{1}{4} \frac{1}{z - z'} , \tag{III.10}
\]
Finally, the algebra is supplemented by the + mode operator $T_+(z) := \tilde{J}^{(+)}(z) \tilde{J}^{(-)}(z)$;,

$$T_+(z)T_+(z') = \frac{1}{(z-z')^2} + \frac{\omega - \omega^2}{z'z(z-z')^2} + \frac{2T_+(z')}{(z-z')^2} + \frac{\partial' T_+(z')}{z - z'} .$$

We have so far considered only the $(1, 2)$ plane, where the constant magnetic field is placed. The
factors, \( \beta \) the Virasoro condition \( F \) sector of the presence of magnetic fields, therefore,

\[
T_+(z)T_+(z') = \frac{d+2}{(z-z')^2} + \frac{\omega - \omega^2}{zz'(z-z')^2} + \frac{2T+(z')}{z-z'} + \partial T+(z'),
\]

\[
T_-(z)T_-(z') = \frac{3d_+}{4(z-z')^4} + \frac{\omega}{zz'(z-z')^2} + \frac{2T-(z')}{z-z'} + \partial T-(z'),
\]

\[
T_-(z)G(z') = \frac{3}{2(z-z')^2}G(z') + \frac{1}{z-z'} \partial G(z').
\]

\[
G(z)G(z') = \frac{d_-}{(z-z')^2} + \frac{\omega}{zz'(z-z')^2} + \frac{2T_-(z')}{z-z'}.
\]

These are equivalent to the super Virasoro algebra

\[
\begin{align*}
[L^+_m, L^+_n] &= (m-n)L^+_{m+n} + \delta_{m+n,0}A^+_m, \\
[L^-_m, L^-_n] &= (m-n)L^-_{m+n} + \delta_{m+n,0}A^-_m, \\
[L^-_m, G_r] &= \left(\frac{m}{2} - r\right)G_{m+r}, \\
\{G_r, G_s\} &= 2L^-_{r+s} + \delta_{r+s,0}B^-_r,
\end{align*}
\]

where the anomaly terms are given by

\[
\begin{align*}
A^+_m &= \frac{d+2}{12}m(m^2 - 1) + m(\omega - \omega^2), \\
A^-_m &= \frac{d-2}{8}m(m^2 - 1) + m\omega, \\
B^-_r &= \frac{d-2}{2}(r^2 - \frac{1}{4}) + \omega, \\
B &= q + \omega, \quad q \in \mathbb{Z}, \quad 0 < \omega < 1.
\end{align*}
\]

IV. ANOMALY IN THE RAMOND SECTOR

As for the Ramond sector, we should be careful for the 0-mode. The mode expansions of fermionic fields are given by

\[
\psi^\pm_R(z) = \frac{1}{2}z^{-\omega} \sum_n z^{-n}d_n^{(-)} = z^{-\omega}\widetilde{\psi}^{(\pm)}(z).
\]

where \( n \) runs over the integral-numbers. The mode operators obey the commutation relation, after the Dirac quantization,

\[
\{d_m^{(-)}, d_n^{(-)}\} = \delta_{m+n,0}.
\]

Usually the 0-mode \( d_0^{(-)} \) is regarded as the Dirac \( \gamma \)-matrix. However, in the presence of the magnetic field, it is not the case. The reason is as follows: Note that the super Virasoro operator \( F_0 \) contains factors, \( \beta_0^{(+)}d_0^{(-)} + \beta_0^{(-)}d_0^{(+)} \). Since \( \beta_0^{(-)} \) is the creation operator, the second term contradicts with the Virasoro condition \( F_0 | \text{ground state } \rangle = 0 \), if \( d_0^{(\pm)} \) is regarded as the Dirac \( \gamma \) matrix. In the sector of the presence of magnetic fields, therefore, \( d_0^{(+)} \) should be regarded as the annihilation operator, whereas other components \( d_0^{(-)} \) without magnetic fields behave as \( \gamma \) matrices.
In this reason we consider that \( d_m^+ \) is annihilation operator for \( m \geq 0 \), and creation operator for \( m < 0 \), while \( d_m^- \) is annihilation operator for \( m > 0 \), and creation operator for \( m \leq 0 \). The contractions are, therefore, defined as

\[
\begin{align*}
\langle d_m^+ d_n^- \rangle &= \begin{cases} 
\delta_{m+n,0}, & (m \geq 0) \\
0, & (m < 0) 
\end{cases} \\
\langle d_m^- d_n^+ \rangle &= \begin{cases} 
\delta_{m+n,0}, & (m > 0) \\
0, & (m \leq 0) 
\end{cases}
\end{align*}
\]

The operator product expansions for fermionic fields are, then, given by

\[
\begin{align*}
\hat{\psi}_R^+(z) \hat{\psi}_R^-(z') &= \frac{z}{4(z-z')^4}, \\
\hat{\psi}_R^-(z) \hat{\psi}_R^+(z') &= \frac{z'}{4(z-z')^4}.
\end{align*}
\]

For the super operator, \( F(z) = 2[\hat{\psi}_R^+(z) \hat{\psi}_R^-(z) + \hat{\psi}_R^-(z) \hat{\psi}_R^+(z)] \), we have

anomaly terms of \( F(z)F(z') = \frac{z + z'}{(z-z')^3} \).

From the formula

\[
\{ F_m, F_n \} = \oint dzd\bar{z'} (z^m \bar{z}^n) F(z)F(z'),
\]

it follows that

\[
\begin{align*}
\{ F_m, F_n \} &= 2L_{-m+n} + \delta_{m+n,0}B_m, \\
B_m(\text{Ramond}) &= \frac{d_2}{2}m^2.
\end{align*}
\]

For the fermionic part \( T^F = 2 : \partial \psi_R \cdot \psi_R : \) with

\[
\frac{1}{2}T^F =: \partial \hat{\psi}_R^+(\hat{\psi}_R^- + \partial \hat{\psi}_R^- \hat{\psi}_R^+) = \partial \hat{\psi}_R^+(\hat{\psi}_R^- + \partial \hat{\psi}_R^- \hat{\psi}_R^+): \]

we have

\[
\text{Anomaly parts of } T^F(z)T^F(z') = \frac{1}{(z-z')^4} \left( \frac{z^2 + z'^2}{4} + \frac{\omega^2 - \omega(z-z')^2}{(z-z')^2} \right).
\]

For the bosonic part \( T^B =: \hat{f}(+ \hat{f}(-) : \), we already had the product \( T^B(z)T^B(z') \) before as

\[
\text{Anomaly parts of } T^B(z)T^B(z') = \frac{zz'}{(z-z')^4} + \frac{\omega - \omega^2}{(z-z')^2}.
\]

Here the equation has been multiplied by the factor \( zz' \), in order to make it the same power as the fermionic one. Then the total sum of the anomaly is given by

\[
\text{Anomaly of } [T^B(z) + T^F(z)][T^B(z') + T^F(z')] = \frac{1}{(z-z')^4} \left( zz' + \frac{z^2 + z'^2}{4} \right).
\]
where \( \omega \) terms are cancelled out from Eqs.(IV.12) and (IV.12). This gives the anomaly term without the cyclotron frequency
\[
A_m^-(\text{Ramond}) = \frac{d}{8} m^3 ,
\]
(IV.14)

together with
\[
B_m^-(\text{Ramond}) = \frac{d}{2} m^2 .
\]
(IV.15)

V. SPECTRUM-GENERATING ALGEBRA

Our SGA is characterized by the cyclotron frequency \( \omega \). We summarize it for the right-moving NS sector:
\[
\begin{align*}
[A_m^{\pm}, A_n^{\pm}] &= (m + \omega)\delta_{m+n,0} , \quad \{ B_r^{\pm}, B_s^{\pm} \} = \delta_{r+s,0} , \quad [A_m^i, A_n^j] = 0 , \\
[A_m^{\pm}, A_n^{\pm}] &= (m \pm \omega)A_{m+n}^{\pm} , \quad \{ B_r^{\pm}, A_n^+ \} = \left( \frac{n}{2} + r \pm \omega \right) B_{r+n}^{\pm} , \\
[A_m^{\pm}, B_r^{\pm}] &= (m \pm \omega)B_{m+r}^{\pm} , \quad \{ B_r^{\pm}, B_s^{\pm} \} = A_{r+s}^{\pm} ,
\end{align*}
\]
(V.1)

Any operator in Eqs.(V.1) and (V.2) is commutable with the super Virasoro operator \( G_r \). Each definition of the operators in Eq.(V.1) is as follows:
\[
\begin{align*}
A_m^{\pm} &= \frac{1}{2\pi i} \oint dz A_m^{\pm}(z) , \\
A_m^{\pm}(z) &= [r^{\pm} - (m \pm \omega)\psi^{\pm}(\psi_-)]V^{m\pm\omega} , \\
V &= \exp[iX_-(z)] , \\
X_-(z) &= x_- - ip_- \ln z + i \sum_{n=0}^{\infty} \frac{\alpha_n}{n} z^{-n} , \quad p_- = 1 ,
\end{align*}
\]
(V.3)

\[
\begin{align*}
B_r^{\pm} &= \frac{1}{2\pi i} \oint dz B_r^{\pm}(z) , \\
B_r^{\pm}(z) &= [\psi^{\pm}(1 + \frac{1}{2} \psi_- \partial \psi_- J_-^2)J_-^{1/2} - \psi_- J_-^{(\pm)/2}]V^{r\pm\omega} .
\end{align*}
\]
(V.4)

In Eqs.(V.3)-(V.4) and Eqs.(V.5)-(V.6) below, for brevity, fermionic fields are normalized in such a way that the contraction is given by \( \langle \psi^\mu(z) \psi^\nu(z') \rangle = (z - z')^{-1} \eta^{\mu\nu} \). Here \( \psi_- , X_- \) are light-cone variables defined by \( X_\pm = \kappa^{\pm1}(X^0 \pm X^{d-1})/\sqrt{2} \) with a real parameter \( \kappa \). The superscripts \((\pm)\) of \( X^{\pm} = (X^1 \pm iX^2)/\sqrt{2} \) should be distinguished from the light-cone subscripts \( \mp \). Note that the vertex operator \( V^{m\pm\omega} = \exp[i(m \pm \omega)X_-](z) \) behaves like \( z^{m\pm\omega} \) at \( z = 0 \). The new definition for \( A_m^{(\pm)} \) and \( B_r^{(\pm)} \) reduce to the original ones proposed by Brower and Friedmann[5], if the cyclotron frequency \( \omega \) is set to be zero.
The sub-algebra (V.4) is the same as that in Ref. [5]. It is composed only of the light-cone variables. They are free operators and this algebra is well-known. The light-cone operators are defined by

\[ A_m^+ = \frac{1}{2\pi i} \oint dz A_m^+(z) , \quad (V.5) \]

\[ A_m^+(z) = [(J_+ - n\psi_+\psi_-) - \frac{1}{2} n(\partial J_+ J_+^{-1} - n\psi_- \partial J_+ J_+^{-1})] V^n , \]

\[ B_r^+ = \frac{1}{2\pi i} \oint dz B_r^+(z) , \quad (V.6) \]

\[ B_r^+(z) = [\psi_+ (1 + \frac{1}{2} \psi_- \partial J_+^{-2}) J_+^{1/2} - \psi_- J_+ J_+^{-1/2}] V^r + \text{(irrelevant term)} . \]

The proof of our SGA (V.1) is given by the same method as in Ref. [6].

VI. ISOMORPHISMS

The algebra (V.2) is similar to the super Virasoro algebra for transverse operators

\[ [L^T_m, L^T_n] = (m - n)L^T_{m+n} + A^T(m)\delta_{m+n,0} , \]

\[ [L^T_m, G^T_r] = \left(\frac{m}{2} - r\right)G^T_{m+r} , \quad (VI.1) \]

\[ [G^T_r, G^T_s] = 2L^T_{r+s} + B^T(r)\delta_{r+s,0} , \]

where

\[ A^T(m) = \frac{d_+ - 2}{8} m(m^2 - 1) + 2m a_- + m \omega , \quad (VI.2) \]

\[ B^T(r) = \frac{d_+ - 2}{2} (r^2 - \frac{1}{4}) + 2a_- + \omega , \quad (VI.3) \]

\[ B = q + \omega , \quad q \in \mathbb{Z} , \quad 0 < \omega < 1 . \quad (VI.4) \]

Here the superscript \( T \) means that the operators are constructed from \( L^-m, G^-r \), leaving oscillators with spacial components \( \mu = 1, 2, \cdots, d - 2 \). The constant \( a_- \) is included in \( L^T_m \) as \(-a_- \delta_{m,0}\).

The isomorphisms

\[ A^+_m \sim L^T_m , \quad B^+_r \sim G^T_r \quad (VI.5) \]

are completed, if there hold equations

\[ A^T(m) = \frac{d_+ - 2}{8} m(m^2 - 1) + 2m a_- + m \omega = m_+ \quad (VI.6) \]

\[ B^T(r) = \frac{d_+ - 2}{2} (r^2 - \frac{1}{4}) + 2a_- + \omega = 4r^2 , \quad (VI.7) \]

These two equations are consistent to give the solution,

\[ d_+ = 10 , \quad (VI.8) \]

\[ a_- = \frac{1}{2} (1 - \omega) . \quad (VI.9) \]
As for the Ramond sector, we have \( d^R = 10 \) and \( a^R = 0 \).

The isomorphisms (VI.5) are also extended to other components interacting with the magnetic field. The algebra (VI.5) is similar to

\[
\begin{align*}
[\beta_m^{(\pm)}, \beta_n^{(-)}] &= (m + \omega)\delta_{m+n,0}, \quad \{ b_r^{(\pm)}, b_s^{(-)} \} = \delta_{r+s,0}, \quad [\beta_m^{(\pm)}, b_r^{(-)}] = 0, \\
[\beta_m^{(\pm)}, L_n^T] &= (m \pm \omega)\beta_{m+n}^{(\pm)}, \quad [b_r^{(\pm)}, L_n^T] = \left(\frac{n}{2} + r \pm \omega\right)b_{r+n}^{(\pm)}, \\
[\beta_m^{(\pm)}, G_r^T] &= (m \pm \omega)b_{m+r}^{(\pm)}, \quad \{ b_r^{(\pm)}, G_s^T \} = \beta_{r+s}^{(\pm)}.
\end{align*}
\]

(VI.10)

The isomorphisms are now completed by

\[
A^{(\pm)}_m \sim \beta^{(\pm)}_m, \quad B^{(\pm)}_r \sim b^{(\pm)}_r.
\]

(VI.11)

As for the + (left-moving) mode, it contains only the bosonic string. The algebra is obtained from the superstring by neglecting all fermionic parts. SGA is given by

\[
\begin{align*}
[A^{(\pm)}_m, A^{(\pm)}_n] &= (m - \omega)\delta_{m+n,0}, \\
[A^{(\pm)}_m, A^+_n] &= (m \pm \omega)A^{(\pm)}_{m+n}, \\
[A^+_m, A^+_n] &= (m - n)A^+_{m+n} + 2m^3\delta_{m+n,0}.
\end{align*}
\]

(VI.12)

The isomorphisms, \( A^{(\pm)}_m \sim A^{(\pm)}_m \) and \( A^+_m \sim L^T_m \), are completed if there holds the equation

\[
A^{T}_{\text{mode}}(m) = -\frac{d_+ - 2}{12}(m^3 - m) + 2ma_+ + m(\omega - \omega^2) = 2m^3.
\]

(VI.13)

From this we have

\[
d_+ = 26,
\]

(VI.14)

\[
a_+ = 1 - \frac{\omega - \omega^2}{2}.
\]

Any physical state should be satisfied by the BRST charge condition \( Q_{\text{BRST}} | \text{phys.} \rangle = 0 \), or equivalently by the super Virasoro conditions, \( G_{r>0} | \text{phys.} \rangle = 0 \), \( (L_{n\geq 0} - \delta_{n,0}a_-) | \text{phys.} \rangle = 0 \), \( (L^+_{n\geq 0} - \delta_{n,0}a_+) | \text{phys.} \rangle = 0 \), for the NS sector, and \( F_{n\geq 0} | \text{phys.} \rangle = 0 \), \( L_{n\geq 0} | \text{phys.} \rangle = 0 \), \( (L^+_{n\geq 0} - \delta_{n,0}a_+) | \text{phys.} \rangle = 0 \) for the Ramond sector. It is well known that such physical states can be constructed from the SGA operators.

VII. CONCLUDING REMARKS

We have proposed a new type of interaction of the electromagnetic field with the heterotic string. When the charged heterotic string is placed in the constant magnetic field \( B \), we have shown that the system can be solved exactly, so as to be translated into the free type of heterotic string. However, this free type of closed string is not periodic at the boundary \( \sigma = 2\pi \), but yields the phase factors \( \exp(\pm 2\pi i\omega) \), where \( \omega \) is the cyclotron frequency. This causes the fact that the cyclotron frequency \( \omega \) is included in all orders in mode expansions of the string and also in quantization conditions for mode operators. This differs from the completely free case. Therefore, our next task has been to calculate the superconformal algebra together with anomalies, and also to give the spectrum-generating algebra, which is necessary to construct physical states satisfying the Virasoro conditions. Finally, we point out that our system is equivalent to the completely free system when \( B \) takes integral values.
Any gauge symmetry derived from the internal space $I = 10, \cdots, 25$ is broken by the cyclotron frequency $\omega$. This comes from the constraint condition for the internal momentum

$$\sum_I p_I^2 = 1 + \omega^2 + 2(R^- - R^+) ,$$

(VII.1)

where $R^\pm$ are number operators. The internal momentum is expressed as

$$p_I = n_I R_I , \quad (n_I \in \mathbb{Z})$$

(VII.2)

where $R_I$ is the $I$th-radius of the torus. Since $2(R^- - R^+)$ takes integral values plus integral times of $2\omega$, we have

$$\sum_I p_I^2 = \sum_I n_I^2 R_I^2 = N + 2N'\omega + \omega^2 . \quad (N, N' \in \mathbb{Z})$$

(VII.3)

This can be regarded as the constraint for $R_I$. However, the Kac-Moody algebra is related only with $\sum_I p_I^2 =$ integral value, so that any internal gauge symmetry is violated by $\omega$.

When $\omega = 0$, the external magnetic field $B$ takes an integral value. In this case, as already noted before, our system is equivalent to completely free system, and we have the well-known internal gauge symmetries.

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