A universality theorem for Voevodsky’s algebraic cobordism spectrum

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Abstract

An algebraic version of a theorem due to Quillen is proved. More precisely, for a ground field $k$ we consider the motivic stable homotopy category $\text{SH}(k)$ of $\mathbb{P}^1$-spectra, equipped with the symmetric monoidal structure described in [3]. The algebraic cobordism $\mathbb{P}^1$-spectrum $\text{MGL}$ is considered as a commutative monoid equipped with a canonical orientation $\text{th}^{\text{MGL}} \in \text{MGL}^{2,1}(\text{Th}(\mathcal{O}(-1)))$. For a commutative monoid $E$ in the category $\text{SH}(k)$ it is proved that assignment $\varphi \mapsto \varphi(\text{th}^{\text{MGL}})$ identifies the set of monoid homomorphisms $\varphi : \text{MGL} \rightarrow E$ in the motivic stable homotopy category $\text{SH}(k)$ with the set of all orientations of $E$. This result was stated originally in a slightly different form by G. Vezzosi in [8].

1 Introduction

Quillen proved in [7] that the formal group law associated to the complex cobordism spectrum $\text{MU}$ is the universal one on the Lazard ring. As a consequence, the set of orientations on a commutative ring spectrum $E$ in the stable homotopy category is in bijective correspondence with the set of homomorphisms of ring spectra from $\text{MU}$ to $E$ in the stable homotopy category. This result allowed a whole new approach to understanding the stable homotopy category, which is still actively pursued today.

On the algebraic side of things, there is a similar $\mathbb{P}^1$-ring spectrum $\text{MGL}$ in the motivic stable homotopy category of a field $k$. The formal group law associated to $\text{MGL}$ is not known to be the universal one, although unpublished work of Hopkins and Morel claims this if $k$ has characteristic zero. Nevertheless, the set of orientations on a $\mathbb{P}^1$-ring spectrum in the motivic stable homotopy category over $k$ can be identified in the same fashion.

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Theorem 1.0.1. Let $E$ be a commutative $\mathbf{P}^1$-ring spectrum over $k$. The set of orientations on $E$ is in bijection with the set of homomorphisms of $\mathbf{P}^1$-ring spectra from $\text{MGL}$ to $E$ in the motivic stable homotopy category over $k$.

For a more detailed formulation, see [2.3.1]. Our main motivation to write this paper was to prove the universality theorem 1.0.1 in a form convenient for its application in [4]. Theorem 1.0.1 was stated originally in a slightly different form by G. Vezzosi in [8], although he ignored certain aspects of the multiplicative structure on $\text{MGL}$.

1.1 Preliminaries

We refer to [3, Appendix] for the basic terminology, notation, constructions, definitions, results. For the convenience of the reader we recall the basic definitions. Let $S$ be a Noetherian scheme of finite Krull dimension. One may think of $S$ being the spectrum of a field or the integers. Let $S\text{m}/S$ be the category of smooth quasi-projective $S$-schemes, and let $\text{sSet}$ be the category of simplicial sets. A motivic space over $S$ is a functor $A: S\text{m}/S^{\text{op}} \to \text{sSet}$ (see [3, A.1.1]). The category of motivic spaces over $S$ is denoted $\text{M}(S)$. This definition of a motivic space is different from the one considered by Morel and Voevodsky in [2] – they consider only those simplicial presheaves which are sheaves in the Nisnevich topology on $S\text{m}/S$. With our definition the Thomason-Trobaugh $K$-theory functor obtained by using big vector bundles is a motivic space on the nose. It is not a simplicial Nisnevich sheaf. This is why we prefer to work with the above notion of “space”.

We write $H^\cm_\circ(S)$ for the pointed motivic homotopy category and $\text{SH}^\cm_\circ(S)$ for the stable motivic homotopy category over $S$ as constructed in [3, A.3.9, A.5.6]. By [3 A.3.11 resp. A.5.6] there are canonical equivalences to $H^\circ_\circ(S)$ of [2] resp. $\text{SH}(S)$ of [9]. Both $H^\cm_\circ(S)$ and $\text{SH}^\cm_\circ(S)$ are equipped with closed symmetric monoidal structures such that the $\mathbf{P}^1$-suspension spectrum functor is a strict symmetric monoidal functor

$$\Sigma_{\mathbf{P}^1}^\infty: H^\cm_\circ(S) \to \text{SH}^\cm_\circ(S).$$

Here $\mathbf{P}^1$ is considered as a motivic space pointed by $\infty \in \mathbf{P}^1$. The symmetric monoidal structure $(\wedge, I_S = \Sigma_{\mathbf{P}^1}^\infty S_+)$ on the homotopy category $\text{SH}^\cm(S)$ is constructed on the model category level by employing symmetric $\mathbf{P}^1$-spectra. It satisfies the properties required by Theorem 5.6 of Voevodsky congress talk [9]. From now on we will usually omit the superscript $^\cm$.

Every $\mathbf{P}^1$-spectrum $E = (E_0, E_1, \ldots)$ represents a cohomology theory on the category of pointed motivic spaces. Namely, for a pointed motivic space $(A, a)$ set

$$E^{p,q}(A, a) = \text{Hom}_{\text{SH}(S)}(\Sigma_{\mathbf{P}^1}^\infty(A, a), \Sigma^{p,q}(E))$$

and $E^{*,*}(A, a) = \bigoplus_{p,q} E^{p,q}(A, a)$. This definition extends to motivic spaces via the functor $A \mapsto A_+$ which adds a disjoint basepoint. That is, for a non-pointed motivic space $A$
set $E^{p,q}(A) = E^{p,q}(A_+, +)$ and $E^{*,*}(A) = \oplus_{p,q} E^{p,q}(A)$. Recall that there is a canonical element in $E^{2n,n}(E_n)$, denoted as $\Sigma^\infty_{P^n} E_n(-n) \to E$. It is represented by the canonical map $(\star, \ldots, \star, E_n, E_n \wedge \mathbb{P}^1, \ldots) \to (E_0, E_1, \ldots, E_n, \ldots)$ of $\mathbb{P}^1$-spectra. 

Every $X \in \mathbb{S}m/S$ defines a representable motivic space constant in the simplicial direction taking an $S$-smooth scheme $U$ to $\text{Hom}_{\mathbb{S}m/S}(U, X)$. It is not possible in general to choose a basepoint for representable motivic spaces. So we regard $S$-smooth varieties as motivic spaces (non-pointed) and set

$$E^{p,q}(X) = E^{p,q}(X_+, +).$$

Given a $\mathbb{P}^1$-spectrum $E$ we will reduce the double grading on the cohomology theory $E^{*,*}$ to a grading. Namely, set $E^m = \oplus_{m=p-2q} E^{p,q}$ and $E^* = \oplus_m E^m$. We often write $E^*(k)$ for $E^*(\text{Spec}(k))$ below.

To complete this section, note that for us a $\mathbb{P}^1$-ring spectrum is a monoid $(E, \mu, e)$ in $(\mathbb{S}h(S), \wedge, \mathbb{I}_S)$. A commutative $\mathbb{P}^1$-ring spectrum is a commutative monoid $(E, \mu, e)$ in $(\mathbb{S}h(S), \wedge, 1)$. The cohomology theory $E^*$ defined by a $\mathbb{P}^1$-ring spectrum is a ring cohomology theory. The cohomology theory $E^*$ defined by a commutative $\mathbb{P}^1$-ring spectrum is a ring cohomology theory, however it is not necessary graded commutative. The cohomology theory $E^*$ defined by an oriented commutative $\mathbb{P}^1$-ring spectrum is a graded commutative ring cohomology theory $[\mathbb{L}]$.

### 1.2 Oriented commutative ring spectra

Following Adams and Morel we define an orientation of a commutative $\mathbb{P}^1$-ring spectrum. However we prefer to use Thom classes instead of Chern classes. Consider the pointed motivic space $\mathbb{P}^\infty = \text{colim}_{n \geq 0} \mathbb{P}^n$ having base point $g_1 : S = \mathbb{P}^0 \hookrightarrow \mathbb{P}^\infty$.

The tautological “vector bundle” $\mathcal{J}(1) = \mathcal{O}_{\mathbb{P}^\infty}(-1)$ is also known as the Hopf bundle. It has zero section $z : \mathbb{P}^\infty \hookrightarrow \mathcal{J}(1)$. The fiber over the point $g_1 \in \mathbb{P}^\infty$ is $\mathbb{A}^1$. For a vector bundle $V$ over a smooth $S$-scheme $X$, with zero section $z : X \hookrightarrow V$, its Thom space $\text{Th}(V)$ is the Nisnevich sheaf associated to the presheaf $Y \mapsto V(Y)/(V \smallsetminus z(X))(Y)$ on the Nisnevich site $\mathbb{S}m/S$. In particular, $\text{Th}(V)$ is a pointed motivic space in the sense of $[\mathbb{L}]$ Defn. A.1.1. It coincides with Voevodsky’s Thom space $[\mathbb{V}]$ p. 422, since $\text{Th}(V)$ already is a Nisnevich sheaf. The Thom space of the Hopf bundle is then defined as the colimit $\text{Th}(\mathcal{J}(1)) = \text{colim}_{n \geq 0} \text{Th}((\mathcal{O}_{\mathbb{P}^n}(-1)))$. Abbreviate $T = \text{Th}(\mathbb{A}^1_{\mathbb{S}^2})$.

Let $E$ be a commutative $\mathbb{P}^1$-ring spectrum. The unit gives rise to an element $1 \in E^{0,0}(\text{Spec}(k)_+)$. Applying the $\mathbb{P}^1$-suspension isomorphism to that element we get an element $\Sigma_{\mathbb{P}^1}(1) \in E^{2,1}(\mathbb{P}^1, \infty)$. The canonical covering of $\mathbb{P}^1$ defines motivic weak equivalences

$$\mathbb{P}^1 \xrightarrow{\sim} \mathbb{P}^1/\mathbb{A}^1 \xleftarrow{\sim} \mathbb{A}^1/\mathbb{A}^1 \setminus \{0\} = T$$

of pointed motivic spaces inducing isomorphisms $E(\mathbb{P}^1, \infty) \leftrightarrow E(\mathbb{A}^1/\mathbb{A}^1 \setminus \{0\}) \to E(T)$. Let $\Sigma_T(1)$ be the image of $\Sigma_{\mathbb{P}^1}(1)$ in $E^{2,1}(T)$.

**Definition 1.2.1.** Let $E$ be a commutative $\mathbb{P}^1$-ring spectrum. A Thom orientation of $E$ is an element $th \in E^{2,1}(\text{Th}(\mathcal{J}(1)))$ such that its restriction to the Thom space of the
fibre over the distinguished point coincides with the element $\Sigma_T(1) \in E^{2,1}(T)$. A Chern orientation of $E$ is an element $c \in E^{2,1}(P^\infty)$ such that $c|_{P^1} = -\Sigma_{P^1}(1)$. An orientation of $E$ is either a Thom orientation or a Chern orientation. One says that a Thom orientation $th$ of $E$ coincides with a Chern orientation $c$ of $E$ provided that $c = z^*(th)$ or equivalently the element $th$ coincides with the one $th(0(-1))$ given by (2) below.

**Remark 1.2.2.** The element $th$ should be regarded as the Thom class of the tautological line bundle $T(1) = 0(-1)$ over $P^\infty$. The element $c$ should be regarded as the Chern class of the tautological line bundle $T(1) = 0(-1)$ over $P^\infty$.

**Example 1.2.3.** The following orientations given right below are relevant for our work. Here $MGL$ denotes the $P^1$-ring spectrum representing algebraic cobordism obtained below in Definition [2.1.1] and $BGL$ denotes the $P^1$-ring spectrum representing algebraic $K$-theory constructed in [3, Theorem 2.2.1].

- Let $u_1 : \Sigma_{P^1}(|T(1)|)(-1) \to MGL$ be the canonical map of $P^1$-spectra. Set $th_{MGL} = u_1 \in MGL^{2,1}(|T(1)|)$. Since $th_{MGL}|_{Th(1)} = \Sigma_{P^1}(1)$ in $MGL^{2,1}(|T(1)|)$, the class $th_{MGL}$ is an orientation of $MGL$.

- Set $c = (-\beta) \cup (\{0\} - \{0(1)\}) \in BGL^{2,1}(P^\infty)$. The relation (11) from [3] shows that the class $c$ is an orientation of $BGL$.

### 2 Oriented ring spectra and infinite Grassmannians

Let $(E, c)$ be an oriented commutative $P^1$-ring spectrum. In this section we compute the $E$-cohomology of infinite Grassmannians and their products. The results are the expected ones – see Theorems 2.0.7 and 2.0.8.

The oriented $P^1$-ring spectrum $(E, c)$ defines an oriented cohomology theory on $Sm/S$ in the sense of [5, Defn. 3.1] as follows. The restriction of the functor $E^{*,*}$ to the category $Sm/S$ is a ring cohomology theory. By [5, Th. 3.35] it remains to construct a Chern structure on $E^{*,*}|_{Sm/S}$ in the sense of [5, Defn. 3.2]. Let $H_*(k)$ be the homotopy category of pointed motivic spaces over $k$. The functor isomorphism $\text{Hom}_{H_*(k)}(-, P^\infty) \to \text{Pic}(-)$ on the category $Sm/S$ provided by [2, Thm. 4.3.8] sends the class of the identity map $P^\infty \to P^\infty$ to the class of the tautological line bundle $0(-1)$ over $P^\infty$. For a line bundle $L$ over $X \in Sm/S$ let $[L]$ be the class of $L$ in the group $\text{Pic}(X)$. Let $f_L : X \to P^\infty$ be a morphism in $H(k)$ corresponding to the class $[L]$ under the functor isomorphism above. For a line bundle $L$ over $X \in Sm/S$ set $c(L) = f_L^*(c) \in E^{2,1}(X)$. Clearly, $c(0(-1)) = c$. The assignment $L/X \mapsto c(L)$ is a Chern structure on $E^{*,*}|_{Sm/S}$ since $c|_{P^1} = -\Sigma_{P^1}(1) \in E^{2,1}(P^1, \infty)$. With that Chern structure $E^{*,*}|_{Sm/S}$ is an oriented ring cohomology theory in the sense of [5]. In particular, $(BGL, c^K)$ defines an oriented ring cohomology theory on $Sm/S$.

Given this Chern structure, one obtains a theory of Thom classes $V/X \mapsto th(V) \in E^{2\text{rank}(V), \text{rank}(V)}(Th_X(V))$ on the cohomology theory $E^{*,*}|_{Sm/S}$ in the sense of [5, Defn. 3.32] as follows. There is a unique theory of Chern classes $V \mapsto c_i(V) \in E^{2i,i}(X)$ such that
for every line bundle $L$ on $X$ one has $c_1(L) = c(L)$. For a rank $r$ vector bundle $V$ over $X$ consider the vector bundle $W := 1 \oplus V$ and the associated projective vector bundle $P(W)$ of lines in $W$. Set

$$\bar{th}(V) = c_r(p^*(V) \otimes \mathcal{O}_{P(W)}(1)) \in E_{2r,r}^2 (P(W)).$$

(1)

It follows from [5, Cor. 3.18] that the support extension map

$$E_{2r,r}^2 (P(W)/(P(W) \setminus P(1))) \to E_{2r,r}^2 (P(W))$$

is injective and $\bar{th}(E) \in E_{2r,r}^2 (P(W)/(P(W) \setminus P(1)))$. Set

$$th(E) = j^* (\bar{th}(E)) \in E_{2r,r}^2 (Th_X(V)),$$

(2)

where $j : Th_X(V) \to P(W)/(P(W) \setminus P(1))$ is the canonical motivic weak equivalence of pointed motivic spaces induced by the open embedding $V \subset P(W)$. The assignment $V/X$ to $\bar{th}(E)$ is a theory of Thom classes on $E^*_{*,*} |_{Sm/S}$ (see the proof of [5, Thm. 3.35]). Hence the Thom classes are natural, multiplicative and satisfy the following Thom isomorphism property.

**Theorem 2.0.4.** For a rank $r$ vector bundle $p : V \to X$ on $X \in Sm/S$ with zero section $z : X \hookrightarrow V$, the map

$$- \cup th(V) : E^{*}_{*,*}(X) \to E^{*+2r,*+r}_{*,*} (V/(V \setminus z(X)))$$

is an isomorphism of two-sided $E^{*}_{*,*}(X)$-modules, where $- \cup th(V)$ is written for the composition map $(- \cup th(V)) \circ p^*$.

**Proof.** See [5, Defn. 3.32.(4)].

Analogous to [9, p. 422] one obtains for vector bundles $V \to X$ and $W \to Y$ in $Sm/S$ a canonical map of pointed motivic spaces $Th(V) \land Th(W) \to Th(V \times_S W)$ which is a motivic weak equivalence as defined in [3, Defn. 3.1.6]. In fact, the canonical map becomes an isomorphism after Nisnevich (even Zariski) sheafification. Taking $Y = S$ and $W = 1$ the trivial line bundle yields a motivic weak equivalence $Th(V) \land T \to Th(V \oplus 1)$. The canonical covering of $P^1$ defines motivic weak equivalences

$$T = A^1/A^1 \setminus \{0\} \xrightarrow{\sim} P^1/A^1 \leftarrow P^1$$

and the arrow $T = A^1/A^1 \setminus \{0\} \to P^1/P^1 \setminus \{0\}$ is an isomorphism. Hence one may switch between $T$ and $P^1$ as desired.

**Corollary 2.0.5.** For $W = V \oplus 1$ consider the motivic weak equivalences

$$c : Th(V) \land P^1 \to Th(V) \land P^1/A^1 \leftarrow Th(V) \land T \to Th(W)$$

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of pointed motivic spaces over $S$. The diagram

$$
\begin{array}{ccc}
E^{*+2r,*+r}(\Sigma_{\mathbb{P}^1}(Th(V))) & \xrightarrow{\Sigma_{\mathbb{P}^1}} & E^{*+2r+2,*+r+1}(Th(V) \wedge \mathbb{P}^1) \\
\downarrow{id} & & \downarrow{e^*} \\
E^{*+2r,*+r}(Th(V)) & \xrightarrow{\Sigma_{Th(V)}} & E^{*+2r+2,*+r+1}(Th(W)) \\
& \downarrow{-\cup th(V)} & \downarrow{-\cup th(W)} \\
E^{*,*}(X) & \xrightarrow{id} & E^{*,*}(X)
\end{array}
$$

commutes.

Let $Gr(n, n+m)$ be the Grassmann scheme of $n$-dimensional linear subspaces of $A_{S}^{n+m}$. The closed embedding $A^{n+m} = A^{n+m} \times \{0\} \hookrightarrow A^{n+m+1}$ defines a closed embedding

$$Gr(n, n+m) \hookrightarrow Gr(n, n+m+1).$$

(3)

The tautological vector bundle is denoted $\mathcal{T}(n, n+m) \rightarrow Gr(n, n+m)$. The closed embedding (3) is covered by a map of vector bundles $\mathcal{T}(n, n+m) \hookrightarrow \mathcal{T}(n, n+m+1)$. Let $Gr(n) = \text{colim}_{m \geq 0} Gr(n, n+m)$, $\mathcal{T}(n) = \text{colim}_{m \geq 0} \mathcal{T}(n, n+m)$ and $\text{Th}(\mathcal{T}(n)) = \text{colim}_{m \geq 0} \text{Th}(\mathcal{T}(n, n+m))$. These colimits are taken in the category of motivic spaces over $S$.

**Remark 2.0.6.** It is not difficult to prove that $E^{*,*}(Gr(n, n+m))$ is multiplicatively generated by the Chern classes $c_i(\mathcal{T}(n, n+m))$ of the vector bundle $\mathcal{T}(n, n+m)$. This proves the surjectivity of the pull-back maps $E^{*,*}(Gr(n, n+m+1)) \rightarrow E^{*,*}(Gr(n, n+m))$ and shows that the canonical map $E^{*,*}(Gr(n)) \rightarrow \prod_{m} E^{*,*}(Gr(n, n+m))$ is an isomorphism. Thus for each $i$ there exists a unique element $c_i = c_i(\mathcal{T}(n)) \in E^{2i,i}(Gr(n))$ which for each $m$ restricts to the element $c_i(\mathcal{T}(n, n+m))$ under the obvious pull-back map.

**Theorem 2.0.7.** Let $E$ be an oriented $\mathbb{P}^1$-ring spectrum. Then

$$E^{*,*}(Gr(n)) = E^{*,*}(k)[[c_1, c_2, \ldots, c_n]]$$

is the formal power series ring. The inclusion $\text{inc}_n : Gr(n) \hookrightarrow Gr(n+1)$ satisfies $\text{inc}_n^*(c_m) = c_m$ for $m < n+1$ and $\text{inc}_n^*(c_{n+1}) = 0$.

**Proof.** The case $n = 1$ is well-known (see for instance [4, Thm. 3.9]). For a finite dimensional vector space $W$ and a positive integer $m$ let $F(m, W)$ be the flag variety of flags $W_1 \subset W_2 \subset \cdots \subset W_m$ of linear subspaces of $W$ such that the dimension of $W_i$ is $i$. Let $\mathcal{T}^i(m, W)$ be the tautological rank $i$ vector bundle on $F(m, W)$.

Let $V = A^\infty$ be an infinite dimensional vector bundle over $S$ and set $e = (1, 0, \ldots)$. Then $V_n$ denotes the $n$-fold product of $V$, and $e_i^n \in V_n$ the vector $(0, \ldots, 0, e, 0, \ldots, 0)$ having $e$ precisely at the $i$th position. Let $F(m) = \text{colim}_W F(m, W)$ and let $\mathcal{T}^i(m) = \text{colim}_W \mathcal{T}^i(m, W)$, where $W$ runs over all finite-dimensional vector subspaces of $V_n$. Thus
we have a flag $\mathcal{T}^1(m) \subset \mathcal{T}^2(m) \subset \cdots \subset \mathcal{T}^m(m)$ of vector bundles over $F(m)$. Set $L^i(m) = \mathcal{T}^i(m)/\mathcal{T}^{i-1}(m)$. It is a line bundle over $F(m)$.

Consider the morphism $p_m : F(m) \to F(m-1)$ which takes a flag $W_1 \subset W_2 \subset \cdots \subset W_m$ to the flag $W_1 \subset W_2 \subset \cdots \subset W_{m-1}$. It is a projective vector bundle over $F(m-1)$ such that the line bundle $L^i(m)$ is its tautological line bundle. Thus there exists a tower of projective vector bundles $F(m) \to F(m-1) \to \cdots \to F(1) = P(V_n)$. The projective bundle theorem implies that

\[ E^{*,*}(F(n)) = E^{*,*}(k)[[t_1, t_2, \ldots, t_n]] \]

(the formal power series in $n$ variables), where $t_i = c(L^i(n))$ is the first Chern class of the line bundle $L^i(n)$ over $F(n)$.

Consider the morphism $q : F(n) \to \text{Gr}(n)$, which takes a flag $W_1 \subset W_2 \subset \cdots \subset W_n$ to the space $W_n$. It can be decomposed as a tower of projective vector bundles. In particular, the pull-back map $q^* : E^{*,*}(\text{Gr}(n)) \to E^{*,*}(F(n))$ is a monomorphism. It takes the class $c_i$ to the symmetric polynomial $\sigma_i = t_1 t_2 \cdots t_i + \cdots + t_{n-i+1} \cdots t_{n-1} t_n$. So the image of $q^*$ contains $E^{*,*}(k)[[\sigma_1, \sigma_2, \ldots, \sigma_n]]$. It remains to check that the image of $q^*$ is contained in $E^{*,*}(k)[[\sigma_1, \sigma_2, \ldots, \sigma_n]]$. To do that consider another variety.

Namely, let $V^0$ be the $n$-dimensional subspace of $V_n$ generated by the vectors $e_i^n$'s. Let $l_i^n$ be the line generated by the vector $e_i^n$. Let $V^0_i$ be a subspace of $V^0$ generated by all $e_j^n$'s with $j \leq i$. So one has a flag $V^0_1 \subset V^0_2 \subset \cdots \subset V^0_n$. We denote this flag $F^0$. For each vector subspace $W$ in $V_n$ containing $V^0$ consider three algebraic subgroups of the general linear group $\mathbb{GL}_W$. Namely, set

\[ P_W = \text{Stab}(V^0), \quad B_W = \text{Stab}(F^0), \quad T_W = \text{Stab}(l_1^n, l_2^n, \ldots, l_i^n). \]

The group $T_W$ stabilizes each line $l_i^n$. Clearly, $T_W \subset B_W \subset P_W$ and $\text{Gr}(n, W) = \mathbb{GL}_W/P_W$, $\mathbb{F}(n, W) = \mathbb{GL}_W/B_W$. Set $M(n, W) = \mathbb{GL}_W/T_W$. One has a tower of obvious morphisms

\[ M(n, W) \xrightarrow{r_W} \mathbb{F}(n, W) \xrightarrow{q_W} \text{Gr}(n, W). \]

Set $M(n) = \text{colim}_W M(n, W)$, where $W$ runs over all finite dimensional subspace $W$ of $V_n$ containing $V^0$. Now one has a tower of morphisms

\[ M(n) \xrightarrow{r} F(n) \xrightarrow{q} \text{Gr}(n). \]

The morphisms $r_W$ can be decomposed in a tower of affine bundles. Hence it induces an isomorphism on any cohomology theory. The same then holds for the morphism $r$ and

\[ E^{*,*}(M(n)) = E^{*,*}(k)[[t_1, t_2, \ldots, t_n]]. \]

Permuting vectors $e_i^n$'s yields an inclusion $\Sigma_n \subset GL(V^0)$ of the symmetric group $\Sigma_n$ in $\mathbb{GL}(V^0)$. The action of $\Sigma_n$ by the conjugation on $\mathbb{GL}_W$ normalizes the subgroups $T_W$ and $P_W$. Thus $\Sigma_n$ acts as on $M(n)$ so on $\text{Gr}(n)$ and the morphism $q \circ r : M(n) \to \text{Gr}(n)$ respects this action. Note that the action of $\Sigma_n$ on $\text{Gr}(n)$ is trivial and the action of $\Sigma_n$ on $E^{*,*}(M(n))$ permutes the variable $t_1, t_2, \ldots, t_n$. Thus the image of $(q \circ r)^*$ is contained in $E^{*,*}(k)[[\sigma_1, \sigma_2, \ldots, \sigma_n]]$. Whence the same holds for the image of $q^*$. The Theorem is proven.

\[ \square \]
The projection from the product $\text{Gr}(m) \times \text{Gr}(n)$, to the $j$-th factor is called $p_j$. For every integer $i \geq 0$ set $c_i^j = p_i^* (c_i(\mathcal{J}(m)))$ and $c_i^j = p_j^* (c_i(\mathcal{J}(n)))$

**Theorem 2.0.8.** Suppose $E$ is an oriented commutative $\mathbf{P}^1$-ring spectrum. There is an isomorphism

$$E^{*, *}(\text{Gr}(m) \times \text{Gr}(n)) = E^{*, *}(k)[[c_1^1, c_2^1, \ldots, c_m^1, c_1^2, c_2^2, \ldots, c_n^2]]$$

is the formal power series on the $c_i^j$'s and $c_i^j$'s. The inclusion $i_{m,n} : G(m) \times \text{Gr}(n) \hookrightarrow G(m+1) \times G(n+1)$ satisfies $i_{m,n}^*(c_i^j) = c_r^j$ for $r < m+1$, $i_{m,n}^*(c_{m+1}^j) = 0$, and $i_{m,n}^*(c_{n+1}^j) = c_r^j$ for $r < n+1$, $i_{m,n}^*(c_{n+1}^j) = 0$.

**Proof.** Follows as in the proof of Theorem 2.0.7.

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### 2.1 The symmetric ring spectrum representing algebraic cobordism

To give a construction of the symmetric $\mathbf{P}^1$-ring spectrum $\text{MGL}$, recall the external product of Thom spaces described in [3, p. 422]. For vector bundles $V \rightarrow X$ and $W \rightarrow Y$ in $\mathcal{S}m/S$ one obtains a canonical map of pointed motivic spaces $\text{Th}(V) \wedge \text{Th}(W) \rightarrow \text{Th}(V \times_S W)$ which is a motivic weak equivalence as defined in [3, Defn. 3.1.6]. In fact, the canonical map becomes an isomorphism after Nisnevich (even Zariski) sheafification.

The algebraic cobordism spectrum appears naturally as a $T$-spectrum, not as a $\mathbf{P}^1$-spectrum. Hence we describe it as a symmetric $T$-ring spectrum and obtain a symmetric $\mathbf{P}^1$-ring spectrum (and in particular a $\mathbf{P}^1$-ring spectrum) by switching the suspension coordinate (see [3, A.6.9]). For $m, n \geq 0$ let $\mathcal{J}(n, mn) \rightarrow \text{Gr}(n, mn)$ denote the tautological vector bundle over the Grassmann scheme of $n$-dimensional linear subspaces of $\mathbf{A}^m_S = \mathbf{A}^m_S \times_S \cdots \times_S \mathbf{A}^m_S$. Permuting the copies of $\mathbf{A}^m_S$ induces a $\Sigma_n$-action on $\mathcal{J}(n, mn)$ and $\text{Gr}(n, mn)$ such that the bundle projection is equivariant. The closed embedding $\mathbf{A}^m_S = \mathbf{A}^m_S \times \{0\} \hookrightarrow \mathbf{A}^{m+1}_S$ defines a closed $\Sigma_n$-equivariant embedding $\text{Gr}(n, mn) \hookrightarrow \text{Gr}(n, (m+1)n)$. In particular, $\text{Gr}(n, mn)$ is pointed by $g_n : S = \text{Gr}(n, n) \hookrightarrow \text{Gr}(n, mn)$. The fiber of $\text{Gr}(n, mn)$ over $g_n$ is $\mathbf{A}^m_S$. Let $\text{Gr}(n)$ be the colimit of the sequence

$$\text{Gr}(n, n) \hookrightarrow \text{Gr}(n, 2n) \hookrightarrow \cdots \hookrightarrow \text{Gr}(n, mn) \hookrightarrow \cdots$$

in the category of pointed motivic spaces over $S$. The pullback diagram

$$\begin{array}{ccc}
\mathcal{J}(n, mn) & \longrightarrow & \mathcal{J}(n, (m+1)n) \\
\downarrow & & \downarrow \\
\text{Gr}(n, mn) & \longrightarrow & \text{Gr}(n, (m+1)n)
\end{array}$$

induces a $\Sigma_n$-equivariant inclusion of Thom spaces

$$\text{Th}(\mathcal{J}(n, mn)) \hookrightarrow \text{Th}(\mathcal{J}(n, (m+1)n)).$$
Let $\text{MGL}_n$ denote the colimit of the resulting sequence

$$\text{MGL}_n = \text{colim}_{m \geq n} \text{Th}(\mathcal{F}(n, mn))$$

(4)

with the induced $\Sigma_n$-action. There is a closed embedding

$$\text{Gr}(n, mn) \times \text{Gr}(p, mp) \hookrightarrow \text{Gr}(n + p, m(n + p))$$

(5)

which sends the linear subspaces $V \hookrightarrow A^{mn}$ and $W \hookrightarrow A^{mp}$ to the product subspace $V \times W \hookrightarrow A^{mn} \times A^{mp} = A^{m(n+p)}$. In particular $(g_n, g_p)$ maps to $g_{n+p}$. The inclusion (5) is covered by a map of tautological vector bundles and thus gives a canonical map of Thom spaces

$$\text{Th}(\mathcal{F}(n, mn)) \wedge \text{Th}(\mathcal{F}(p, mp)) \to \text{Th}(\mathcal{F}(n + p, m(n + p)))$$

(6)

which is compatible with the colimit (4). Furthermore, the map (6) is $\Sigma_n \times \Sigma_p$-equivariant, where the product acts on the target via the standard inclusion $\Sigma_n \times \Sigma_p \subseteq \Sigma_{n+p}$. After taking colimits, the result is a $\Sigma_n \times \Sigma_p$-equivariant map

$$\mu_{n,p} : \text{MGL}_n \wedge \text{MGL}_p \to \text{MGL}_{n+p}$$

(7)

of pointed motivic spaces (see [9, p. 422]). The inclusion of the fiber $A^p$ over $g_p$ in $\mathcal{F}(p)$ induces an inclusion $\text{Th}(A^p) \subset \text{Th}(\mathcal{F}(p)) = \text{MGL}_p$. Precomposing it with the canonical $\Sigma_p$-equivariant map of pointed motivic spaces

$$\text{Th}(\mathcal{A}^1) \wedge \text{Th}(\mathcal{A}^1) \wedge \cdots \wedge \text{Th}(\mathcal{A}^1) \to \text{Th}(\mathcal{A}^p)$$

defines a family of maps $e_p : (\Sigma_+^\infty S_+)_p = T^p \to \text{MGL}_p$. Inserting it in the inclusion (7) yields $\Sigma_n \times \Sigma_p$-equivariant structure maps

$$\text{MGL}_n \wedge \text{Th}(\mathcal{A}^1) \wedge \text{Th}(\mathcal{A}^1) \wedge \cdots \wedge \text{Th}(\mathcal{A}^1) \to \text{MGL}_{n+p}$$

(8)

of the symmetric $T$-spectrum $\text{MGL}$. The family of $\Sigma_n \times \Sigma_p$-equivariant maps (7) form a commutative, associative and unital multiplication on the symmetric $T$-spectrum $\text{MGL}$ (see [1] Sect. 4.3). Regarded as a $T$-spectrum it coincides with Voevodsky’s spectrum $\text{MGL}$ described in [2, 6.3].

Let $\overline{T}$ be the Nisnevich sheaf associated to the presheaf $X \mapsto \mathbf{P}^1(X)/(\mathbf{P}^1 - \{0\})(X)$ on the Nisnevich site $Sm/S$. The canonical covering of $\mathbf{P}^1$ supplies an isomorphism

$$T = \text{Th}(\mathbf{A}^1_{\overline{\mathbb{F}}}) \xrightarrow{\cong} \overline{T}$$

of pointed motivic spaces. This isomorphism induces an isomorphism $\text{MSS}_T(S) \cong \text{MSS}_{\overline{T}}(S)$ of the categories of symmetric $T$-spectra and symmetric $\overline{T}$-spectra. In particular, $\text{MGL}$ may be regarded as a symmetric $\overline{T}$-spectrum by just changing the structure maps up to an isomorphism. Note that the isomorphism of categories respects both the symmetric monoidal structure and the model structure. The canonical projection
$p: \mathbb{P}^1 \to \overline{T}$ is a motivic weak equivalence, because $\mathbb{A}^1$ is contractible. It induces a Quillen equivalence

$$\text{MSS}(S) = \text{MSS}_{\mathbb{P}^1}(S) \xrightarrow{p_*} \text{MSS}_{\overline{T}}(S)$$

when equipped with model structures as described in [1] (see [3, A.6.9]). The right adjoint $p^*$ is very simple: it sends a symmetric $\overline{T}$-spectrum $E$ to the symmetric $\mathbb{P}^1$-spectrum having terms $(p^*(E))_n = E_n$ and structure maps

$$E_n \wedge \mathbb{P}^1 \xrightarrow{E_n \wedge p} E \wedge \overline{T} \xrightarrow{\text{structure map}} E_{n+1}.$$

In particular $\text{MGL} := p^*\text{MGL}$ is a symmetric $\mathbb{P}^1$-spectrum by just changing the structure maps. Since $p^*$ is a lax symmetric monoidal functor, $\text{MGL}$ is a commutative monoid in a canonical way. Finally, the identity is a left Quillen equivalence from the model category $\text{MSS}^{\text{cm}}(S)$ used in [3] to Jardine’s model structure by the proof of [3, A.6.4]. Let $\gamma: \text{Ho}(\text{MSS}^{\text{cm}}(S)) \to \text{SH}(S)$ denote the equivalence obtained by regarding a symmetric $\mathbb{P}^1$-spectrum just as a $\mathbb{P}^1$-spectrum.

**Definition 2.1.1.** Let $(\text{MGL}, \mu_{\text{MGL}}, e_{\text{MGL}})$ denote the commutative $\mathbb{P}^1$-ring spectrum which is the image $\gamma(\text{MGL})$ of the commutative symmetric $\mathbb{P}^1$-ring spectrum $\text{MGL}$ in the motivic stable homotopy category $\text{SH}(S)$.

### 2.2 Cohomology of the algebraic cobordism spectrum

Let $E$ be an oriented commutative $\mathbb{P}^1$-ring spectrum and let $S = \text{Spec}(k)$ for a field $k$. We will compute $E^{\ast,\ast}(\text{MGL})$ and $E^{\ast,\ast}(\text{MGL} \wedge \text{MGL})$ in this short section.

By [3, Cor. 2.1.4], the group $E^{\ast,\ast}(\text{MGL})$ fits into the short exact sequence

$$0 \to \lim^1 E^{\ast+2i-1,\ast+i}(\text{Th}(\mathcal{J}(i))) \to E^{\ast,\ast}(\text{MGL}) \to \lim E^{\ast+2i,\ast+i}(\text{Th}(\mathcal{J}(i))) \to 0$$

where the connecting maps in the tower are given by the top line of the commutative diagram

$$
\begin{array}{c}
E^{\ast+2i-1,\ast+i}(\text{Th}(i)) \leftarrow_{\Sigma_{\mathcal{J}(i)}}^{\text{inc}} E^{\ast+2i+1,\ast+i+1}(\text{Th}(i) \wedge \mathbb{P}^1) \leftarrow E^{\ast+2i+1,\ast+i+1}(\text{Th}(i+1)) \\
\text{id} \uparrow \quad \epsilon^\ast((\text{Th}(\mathcal{J}(i))\oplus 1)) \uparrow \quad \text{inc}^\ast \\
E^{\ast,\ast}(\text{Gr}(i)) \leftarrow E^{\ast,\ast}(\text{Gr}(i)) \leftarrow E^{\ast,\ast}(\text{Gr}(i+1))
\end{array}
$$

Here $\epsilon: \text{Th}(V) \wedge \mathbb{P}^1 \to \text{Th}(V \oplus 1)$ is the canonical map described in Corollary 2.0.5. The pull-backs $\text{inc}^\ast$ are all surjective by Theorem 2.0.4. So we proved the following

**Claim 2.2.1.** The canonical map

$$E^{\ast,\ast}(\text{MGL}) \to \lim E^{\ast+2i,\ast+i}(\text{Th}(\mathcal{J}(i))) = E^{\ast,\ast}(k)[c_1, c_2, c_3, \ldots]$$

is an isomorphism of two-sided $E^{\ast,\ast}(k)$-modules.
Now compute $E^{*,*}(\text{MGL} \wedge \text{MGL})$. By [3, Cor. 2.1.5] the group $E^{*,*}(\text{MGL} \wedge \text{MGL})$ fits into the short exact sequence

$$0 \to \lim_1 E^{*+4i-1,*+2i}(\text{Th}(\mathcal{J}(i)) \wedge \text{Th}(\mathcal{J}(i))) \to E^{*,*}(\text{MGL} \wedge \text{MGL})$$

$$\to \lim E^{*+4i,*+2i}(\text{Th}(\mathcal{J}(i)) \wedge \text{Th}(\mathcal{J}(i))) \to 0.$$ 

Note that since $\text{Th}(\mathcal{J}(i)) \wedge \text{Th}(\mathcal{J}(i)) \cong \text{Th}(\mathcal{J}(i) \times \mathcal{J}(i))$, there is a Thom isomorphism $E^{*+4i-1,*+2i}(\text{Th}(\mathcal{J}(i) \times \mathcal{J}(i))) \cong E^{*+1,*}(\text{Gr}(i) \times \text{Gr}(i))$ by Theorem 2.0.4. The $\lim_1$-group is trivial because the connecting maps coincide with the pull-back maps

$$E^{*-1,*}(\text{Gr}(i + 1) \times \text{Gr}(i + 1)) \to E^{*-1,*}(\text{Gr}(i) \times \text{Gr}(i))$$

and these are surjective by Theorem 2.0.8. This implies the following

**Claim 2.2.2.** The canonical map

$$E^{*,*}(\text{MGL} \wedge \text{MGL}) \to \lim E^{*+2i,*+i}(\text{Th}(\mathcal{J}(i)) \wedge \text{Th}(\mathcal{J}(i))) = E^{*,*}(k)[[c'_1, c''_1, c'_2, c''_2, \ldots]]$$

is an isomorphism of two-sided $E^{*,*}(k)$-modules. Here $c'_i$ is the $i$-th Chern class coming from the first factor of $\text{Gr} \times \text{Gr}$ and $c''_i$ is the $i$-th Chern class coming from the second factor.

### 2.3 A universality theorem for the algebraic cobordism spectrum

The complex cobordism spectrum, equipped with its natural orientation, is a universal oriented ring cohomology theory by Quillen’s universality theorem [7]. In this section we prove a motivic version of Quillen’s universality theorem. The statement is contained already in [8]. Recall that the $\mathbb{P}^1$-ring spectrum $\text{MGL}$ carries a canonical orientation $\text{th}^{\text{MGL}}$ as defined in 1.2.3. It is the canonical map $\text{th}^{\text{MGL}}: \Sigma^\infty_\mathbb{P} (\text{Th}(\mathcal{O}(-1)))(-1) \to \text{MGL}$ of $\mathbb{P}^1$-spectra.

**Theorem 2.3.1** (Universality Theorem). Let $E$ be a commutative $\mathbb{P}^1$-ring spectrum and let $S = \text{Spec}(k)$ for a field $k$. The assignment $\varphi \mapsto \varphi(\text{th}^{\text{MGL}}) \in E^{2,1}(\text{Th}(\mathcal{J}(1)))$ identifies the set of monoid homomorphisms

$$\varphi: \text{MGL} \to E$$

in the motivic stable homotopy category $\text{SH}^{\text{mot}}(S)$ with the set of orientations of $E$. The inverse bijection sends an orientation $\text{th} \in E^{2,1}(\text{Th}(\mathcal{J}(1)))$ to the unique morphism

$$\varphi \in E^{0,0}(\text{MGL}) = \text{Hom}_{\text{SH}(S)}(\text{MGL}, E)$$

such that $u_1^*(\varphi) = \text{th}(\mathcal{J}(i)) \in E^{2i,1}(\text{Th}(\mathcal{J}(i)))$, where $\text{th}(\mathcal{J}(i))$ is given by (2) and $u_i: \Sigma^\infty_\mathbb{P} (\text{Th}(\mathcal{J}(i)))(-i) \to \text{MGL}$ is the canonical map of $\mathbb{P}^1$-spectra.
Proof. Let \( \varphi : \text{MGL} \rightarrow E \) be a homomorphism of monoids in \( \text{SH}(S) \). The class \( \text{th} := \varphi(\text{th}^{\text{MGL}}) \) is an orientation of \( E \), because

\[
\varphi(\text{th}|_{\text{Th}(1)}) = \varphi(\text{th}|_{\text{Th}(1)}) = \varphi(\Sigma_{P^i}(1)) = \Sigma_{P^i}(\varphi(1)) = \Sigma_{P^i}(1).
\]

Now suppose \( \text{th}^E \in E^{2i}(\text{Th}(\emptyset(-1))) \) is an orientation of \( E \). We will construct a monoid homomorphism \( \varphi : \text{MGL} \rightarrow E \) in \( \text{SH}(S) \) such that \( u^\varphi_i(\varphi) = \text{th}(\mathcal{I}(i)) \) and prove its uniqueness. To do so recall that by Claim [2.2.1] the canonical map \( E^{*,*}(\text{MGL}) \rightarrow \lim_i E^{*,*,+2i}(\text{Th}(\mathcal{I}(i))) \) is an isomorphism. The family of elements \( \text{th}(\mathcal{I}(i)) \) is an element in the \( \lim \)-group, thus there is a unique element \( \varphi \in E^{0,0}(\text{MGL}) \) with \( u^\varphi_i(\varphi) = \text{th}(\mathcal{I}(i)) \).

We claim that \( \varphi \) is a monoid homomorphism. To check that it respects the multiplicative structure, consider the diagram

\[
\begin{array}{ccc}
\Sigma_{P^i}(\text{Th}(\mathcal{I}(i)))(-i) \& \Sigma_{P^j}(\text{Th}(\mathcal{I}(j)))(-j) & \xrightarrow{\Sigma_{P^i}((\mu_{i,j})^{-1}(-i-j))} & \Sigma_{P^i}(\text{Th}(\mathcal{I}(i+j)))(-i-j) \\
MGL \& MGL & \xrightarrow{\mu^{\text{MGL}}} & MGL \\
E \& E & \xrightarrow{\mu^E} & E.
\end{array}
\]

Its enveloping square commutes in \( \text{SH}(S) \) by the chain of relations

\[
\varphi \circ u_{i+j} \circ \Sigma_{P^i}((\mu_{i,j})(-i-j)) = \mu_{i,j}^\varphi(\text{th}(\mathcal{I}(i+j))) = \text{th}(\mu_{i,j}^\varphi(\mathcal{I}(i+j))) = \text{th}(\mathcal{I}(i) \times \mathcal{I}(j)) = \mu^E((\varphi \circ u_i) \land (\varphi \circ u_j)).
\]

The canonical map \( E^{*,*}(\text{MGL} \& \text{MGL}) \rightarrow \lim_i E^{*,*,+2i}(\text{Th}(\mathcal{I}(i)) \land \text{Th}(\mathcal{I}(i))) \) is an isomorphism by Claim [2.2.2] Now the equality

\[
\varphi \circ u_{i+i} \circ \Sigma_{P^i}((\mu_{i,i})(-2i)) = \mu^E((\varphi \circ u_i) \land (\varphi \circ u_i))
\]

shows that \( \mu^E((\varphi \land \varphi) = \varphi \circ \mu^{\text{MGL}} \) in \( \text{SH}(k) \).

To prove the Theorem it remains to check that the two assignments described in the Theorem are inverse to each other. An orientation \( \text{th} \in E^{2,1}(\text{Th}(\emptyset(-1))) \) induces a morphism \( \varphi \) such that for each \( i \) one has \( \varphi \circ u_i = \text{th}(\mathcal{I}_i) \). And the new orientation \( \text{th}' := \varphi(\text{th}^{\text{MGL}}) \) coincides with the original one, due to the chain of relations

\[
\text{th}' = \varphi(\text{th}^{\text{MGL}}) = \varphi(u_1) = \varphi \circ u_1 = \text{th}(\mathcal{I}(1)) = \text{th}(\emptyset(-1)) = \text{th}.
\]

On the other hand a monoid homomorphism \( \varphi \) defines an orientation \( \text{th} := \varphi(\text{th}^{\text{MGL}}) \) of \( E \). The monoid homomorphism \( \varphi' \) we obtain then satisfies \( u^\varphi_i(\varphi') = \text{th}(\mathcal{I}(i)) \) for every \( i \geq 0 \). To check that \( \varphi' = \varphi \), recall that MGL is oriented, so we may use Claim [2.2.1] with \( E = \text{MGL} \) to deduce an isomorphism

\[
\text{MGL}^{*,*}(\text{MGL}) \xrightarrow{\text{th}} \lim_i \text{MGL}^{*,*,+2i}(\text{Th}(\mathcal{I}(i))).
\]
This isomorphism shows that the identity $\varphi' = \varphi$ will follow from the identities $u^*_i(\varphi') = u^*_i(\varphi)$ for every $i \geq 0$. Since $u^*_i(\varphi') = th(\mathcal{I}_i)$ it remains to check the relation $u^*_i(\varphi) = th(\mathcal{I}(i))$. It follows from the

**Claim 2.3.2.** There is an equality $u_i = th^{MGL}(\mathcal{I}(i)) \in MGL^{2i}(Th(\mathcal{I}(i)))$.

In fact, $u^*_i(\varphi) = \varphi \circ u_i = \varphi(u_i) = \varphi(th^{MGL}(\mathcal{I}(i))) = th(\mathcal{I}(i))$. The last equality in this chain of relations holds, because $\varphi$ is a monoid homomorphism sending $th^{MGL}$ to $th$. It remains to prove the Claim. We will do this in the case $i = 2$. The general case can be proved similarly. The commutative diagram

$$
\begin{array}{cccc}
\Sigma_\infty^\infty Th(\mathcal{I}(1))(−1) \wedge \Sigma_\infty^\infty Th(\mathcal{I}(1))(−1) & \xrightarrow{\Sigma_\infty^\infty (\mu_1,1)(−2)} & \Sigma_\infty^\infty Th(\mathcal{I}(2))(−2) \\
\downarrow u_1 \wedge u_1 & & \downarrow u_2 \\
MGL \wedge MGL & \xrightarrow{\mu_{MGL}} & MGL
\end{array}
$$

in $SH(k)$ implies that

$$
\mu^*_{1,1}(u_2) = u_1 \times u_1 \in MGL^{4,2}(Th(\mathcal{I}(1)) \wedge Th(\mathcal{I}(1))) = MGL^{4,2}(Th(\mathcal{I}(1) \times \mathcal{I}(1))).
$$

The equalities

$$
\mu^*_1(th^{MGL}(\mathcal{I}(2))) = th^{MGL}(\mu^*_1,1(\mathcal{I}(2))) = th^{MGL}(\mathcal{I}(1) \times \mathcal{I}(1))
$$

$$
= th^{MGL}((\mathcal{I}(1)) \times th^{MGL}(\mathcal{I}(1))
$$

imply that it remains to prove the injectivity of the map $\mu^*_{1,1}$. Consider the commutative diagram

$$
\begin{array}{ccc}
MGL^\bullet\bullet(Th(\mathcal{I}(1) \times \mathcal{I}(1))) & \xleftarrow{\mu^*_{1,1}} & MGL^\bullet\bullet(Th(\mathcal{I}(2))) \\
\uparrow \cong & & \uparrow \cong \\
MGL^\bullet\bullet(Gr(1) \times Gr(1)) & \xleftarrow{\nu^*_{1,1}} & MGL^\bullet\bullet(Gr(2))
\end{array}
$$

where the vertical arrows are the Thom isomorphisms from Theorem 2.0.4 and $\nu_{1,1} : Gr(1) \times Gr(1) \hookrightarrow Gr(2)$ is the embedding described by equation (5). For an oriented commutative $P^1$-ring spectrum $(E, th)$ one has $E^\bullet\bullet(Gr(2)) = E^\bullet\bullet(k)[[c_1, c_2]]$ (the formal power series on $c_1, c_2$) by Theorem 2.0.7. From the other hand

$$
E^\bullet\bullet(Gr(1) \times Gr(1)) = E^\bullet\bullet(k)[[t_1, t_2]]
$$

(the formal power series on $t_1, t_2$) by Theorem 2.0.8 and the map $\nu^*_{1,1}$ takes $c_1$ to $t_1 + t_2$ and $c_2$ to $t_1 t_2$. Whence $\nu^*_{1,1}$ is injective. The proofs of the Claim and of the Theorem are completed. \qed
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