Nonlinear supersymmetric quantum mechanics: concepts and realizations

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Abstract

The nonlinear supersymmetric (SUSY) approach to spectral problems in quantum mechanics (QM) is reviewed. Its building from the chains (ladders) of linear SUSY systems is outlined and different one-dimensional and two-dimensional realizations are described. It is elaborated how the nonlinear SUSY approach provides two new methods of SUSY separation of variables for various two-dimensional models. In the framework of these methods, a partial and/or complete solution of some two-dimensional models becomes possible. The full classification of ladder-reducible and irreducible chains of SUSY algebras in one-dimensional QM is given. The emergence of hidden symmetries and spectrum generating algebras is elucidated in the context of the nonlinear SUSY in one-dimensional stationary and non-stationary, as well as in two-dimensional QM.

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1. Introduction

The concept of supersymmetric quantum mechanics (SUSY QM) embodies an algebraic form of transformations of a (complete or partial) spectral equivalence between different dynamical systems and in this sense it gives the algebraic tools for the spectral design of new quantum systems from a given set with controllable energy spectra. At present, there are a number of reviews [1–18] devoted to development and various applications, mostly of the linear SUSY QM. The very isospectral transformations realizing SUSY represent the Darboux (Darboux–Moutard) transformations [19–29], which, in the theory of ordinary differential equations, have been known about for a long time. According to [21] these transformations must be referred to as Euler–Imshenetsky–Darboux–Moutard ones having their origin in the paper by Leonard Euler [23]. We thank our referee for drawing our attention to this chain of papers.

In its original form, SUSY QM was formulated for one-dimensional space by Witten [30] about 30 years ago (see also [31–33]). This approach has two predecessors: the factorization method [34] in QM and Darboux’s transformation [19–28] for the Sturm–Liouville equation in mathematics.
The first one was proposed by Schrödinger in the 1940s [34] (see the excellent detailed review [35]); its basic relation is the representation of the Hamiltonian in a factorized form,

\[ h^{(0)}(x) = -\partial^2 + V^{(0)}(x) = q^+ q^-, \]

where operators \( q^\pm \) are,

\[ q^\pm = \mp \partial + (\partial W(x)); \quad q^+ = (q^-)^\dagger; \quad \partial \equiv \frac{d}{dx}. \]

The factorization (1) is in general possible for an arbitrary potential \( V(x) \) (see details below). The main idea of the factorization method is to consider together with the Hamiltonian (1) the partner Hamiltonian, also of Schrödinger form,

\[ h^{(1)}(x) = -\partial^2 + V^{(1)}(x) = q^- q^+, \]

with the same operators \( q^\pm \). By construction, these Hamiltonians satisfy the intertwining relations,

\[ h^{(0)} q^+ = q^+ h^{(1)}; \quad q^- h^{(0)} = h^{(1)} q^-, \]

which play a key role in the method under discussion. Just the intertwining relations (4) allow one to obtain the simple relations between wave functions \( \Psi^{(0)}_n \) and \( \Psi^{(1)}_n \) of \( h^{(0)} \) and \( h^{(1)} \), correspondingly. Indeed, according to (4) an arbitrary wave function \( \Psi^{(0)}_n(x) \) of \( h^{(0)} \) gives the wave function \( \Psi^{(1)}_n(x) \) of \( h^{(1)} \), and vice versa,

\[ q^- \Psi^{(0)}_n \propto \Psi^{(1)}_n; \quad q^+ \Psi^{(1)}_n \propto \Psi^{(0)}_n. \]

The energy eigenvalues for \( \Psi^{(0)}_n \) and \( \Psi^{(1)}_n \) are the same, \( E^{(0)}_n = E^{(1)}_n \). There are three options [36] for the spectra of \( h^{(0)} \) and \( h^{(1)} \):

1. Spectra coincide completely if neither \( q^+ \), nor \( q^- \) have normalizable zero modes;
2. Spectra almost coincide: the spectrum of \( h^{(0)} \) has one bound state less, if the normalizable zero mode of \( q^+ \) exists;
3. Spectra almost coincide: the spectrum of \( h^{(1)} \) has one bound state less, if the normalizable zero mode of \( q^- \) exists.

This is the essence of the spectral design method based on the super-Hamiltonian \( H \) with two non-singular real potentials. The supersymmetry relations between the super-Hamiltonian components \( h^{(0)} \) and \( h^{(1)} \), whereas the two non-singular real potentials are different. Each of the operators \( q^\pm \) may have one zero mode \( \exp(\pm W(x)) \); their normalizability properties depend on the behavior of function \( W(x) \): the Hamiltonians \( h^{(0)} \) and \( h^{(1)} \) are [36–43] either isospectral or almost isospectral (up to one bound state).

The connection between a given potential \( V^{(0)}(x) \) and a function \( W(x) \) provides a basis for the link of the factorization method with a Darboux transformation [19] and represents the analytic tool of spectral design. The substitution of (2) into (1), (3) results in:

\[ V^{(0)}(x) = (\partial W)^2(x) - (\partial^2 W(x)); \]

\[ V^{(1)}(x) = (\partial W)^2(x) + (\partial^2 W(x)). \]

In terms of the theory of ordinary second-order differential Sturm–Liouville equations (adapted to the Schrödinger form),

\[ - \Psi^{(0)''}(x) + V^{(0)}(x)\Psi^{(0)}(x) = E^{(0)}\Psi^{(0)}(x), \]

the Darboux transformation can be formulated as follows. If one knows its particular solution \( \psi^{(0)}(x) \equiv \exp(-W(x)) \) for the value \( E^{(0)} = 0 \), the equation (6) is satisfied; an arbitrary
solution $\Psi(0)(x; E(0))$ of (8) can be transformed (Darboux transformation) into a solution of the transformed equation of the same form and with the same value of parameter $E(0)$,

$$-\Psi^{(1)\prime\prime}(x) + V^{(1)}(x)\Psi^{(1)}(x) = E(0)\Psi^{(1)}(x).$$

(9)

The initial and transformed solutions are connected,

$$\Psi^{(1)}(x; E(0)) \propto (\partial + (\partial W)(x))\Psi(0)(x; E(0));$$

$$\Psi^{(0)}(x; E(0)) \propto (-\partial + (\partial W)(x))\Psi^{(1)}(x; E(0)).$$

(10)

It is clear that one obtains a one-to-one correspondence between solutions of (8) and (9), but up to zero modes of operators $(\mp \partial + (3W)(x))$.

The nonlinear first-order differential equation (6) (Riccati [44] equation) can be solved analytically for a restricted class of potentials $V^{(0)}(x)$ only. The class of analytically solvable Riccati equations corresponds to the so-called exactly solvable models in one-dimensional QM, latter ones are related to the so-called shape-invariant [45] potentials.

The above construction can be easily presented in a SUSY form. Indeed, the Hamiltonians $h^{(0)}$, $h^{(1)}$ and intertwining operators $q^\pm$ can be considered as elements of matrix $2 \times 2$ operators—super-Hamiltonian and supercharges,

$$H = \begin{pmatrix} h^{(0)} & 0 \\ 0 & h^{(1)} \end{pmatrix}; \quad Q^+ = \begin{pmatrix} 0 & 0 \\ q^- & 0 \end{pmatrix}; \quad Q^- = (Q^+)^\dagger = \begin{pmatrix} 0 & q^+ \\ 0 & 0 \end{pmatrix},$$

(11)

which obey the (anti)commutation relations—the simplest realization of SUSY algebra,

$$[H, Q^\pm] = 0; \quad \{q^\pm, Q^\mp\} = H; \quad (Q^\pm)^2 = 0.$$  

(12)

The relations (12) of the superalgebra express in compact form the factorization (1) (3) and the intertwining relations (4), while the nilpotency of supercharges $Q^\pm$ follows directly from their matrix structure. By an analogy with the SUSY quantum field theory, the component $h^{(0)}$ of $H$ is called the bosonic component and $h^{(1)}$ the fermionic one. Correspondingly, their eigenstates form bosonic and fermionic sectors, although this terminology has no physical sense in the present non-relativistic context. Nevertheless, the nilpotency of $Q^\pm$ can be associated with the fermion property—an analogue of the Pauli principle.

Thus in the simplest cases (linear SUSY algebras), an intertwining of two differential operators (for instance, Hamiltonians of one-dimensional quantum systems) by means of Darboux operators of first order in derivatives entails their factorization into differential multipliers which are formed by the same Darboux operators. However, in general, both the interrelation between pairs of dynamical operators (Hamiltonians) with (almost) equivalent spectra and the structure of operators which generate the spectral equivalence are not that simple [46–67]. In particular, the closing of a nonlinear SUSY algebra often produces a hidden symmetry generator which helps the partial solvability of spectral problem for Schrödinger Hamiltonians. This interrelation in the one- and two-dimensional QM is within the scope of this review. In particular, we survey well motivated solutions to the following problems:

- What is the correct self-consistent way to generalize the main relations of one-dimensional SUSY QM, including factorization and intertwining relations, to the case of multidimensional space?

- In what cases can higher-order Darboux–Crum transformations be constructed with the help of a sequence of intertwining transformations of lower order SUSY which relate a chain of (almost) isospectral intermediate Hamiltonians with real nonsingular potentials?

- What are the elementary blocks for a nonsingular factorization of intertwining operators?

- In what way does the irreducibility of elementary blocks of isospectral transformations reveal itself in the SUSY algebra and in the structure of the kernels of those transformations?
• Are there more complicated SUSY algebras in one dimensional QM which accept hidden symmetry generators (‘central charges’)?
• What is the relationship between shape-invariance and spectrum generating algebras for SUSY QM systems?
• How does SUSY for non-stationary Schrödinger operators produce hidden symmetries and spectrum generating algebras?
• Is it possible to find pairs of (almost) isospectral scalar two-dimensional Hamiltonians intertwined by higher order supercharges, i.e. to avoid the matrix components of the super-Hamiltonian by means of using Nonlinear SUSY algebra?
• Does the nonlinear two-dimensional SUSY QM give the constructive way to obtain new exactly solvable and/or quasi-exactly solvable models not amenable to the conventional separation of variables?

The structure of the paper is as follows. After a short reminder of the notation and basic definitions of the SUSY theory of isospectral transformations given above in the introduction, the generalizations of linear SUSY QM for arbitrary space dimensions [36, 68] are considered in section 2. All of them can be called direct generalizations since they are based on the supercharge operators of first order in derivatives. In particular, subsection 2.1 includes a detailed description of two-dimensional generalization [36] in a form similar to the factorization method of Schrödinger [34, 35]. Subsection 2.2 contains two illustrative examples: the Pauli equation for fermion motion on the plane with an external magnetic field [69–71] and the SUSY extension of a two-dimensional attractive Coulomb potential [36]. In subsection 2.3, the well developed formalism of SUSY quantum field theory is used to present the direct (first-order) generalization [68] of Witten’s one-dimensional SUSY QM onto the space with an arbitrary number \( d \) of dimensions. The main element of construction is the \( d \)-component’s real scalar superfield depending both on the time and the pair of Grassmann variables. This multidimensional generalization also allows the alternative interpretation as a model of \( d \) interacting particles on a line [18]. In subsection 2.4, the structure of the obtained super-Hamiltonian is analyzed [68] in the Hilbert space, which is a direct sum of the subspaces with fixed fermion numbers. On this basis, the super-Hamiltonian has the block-diagonal form with matrix blocks of different matrix dimensionalities. The SUSY intertwining relations between different matrix components of the super-Hamiltonian are derived providing their isospectrality. Subsection 2.5 gives a brief description of the illustrative example [72]: \( d = 3 \) SUSY QM with specific Yukawa-like superpotential provides relations between different sectors of nucleon-nucleon and nucleon–antinucleon interactions.

Section 3 is devoted to building polynomial SUSY algebra for \( d = 1 \) with the help of a dressing chain or ladder construction [46]. It is elucidated how they entail the SUSY algebra representation by differential intertwining operators of a higher-order in derivatives. For algebraic formulation, its key ingredient is given by SUSY Hamiltonian projections on the formal zero-mode space (the kernel) of supercharge operators, which are finite-dimensional constant matrices [65, 73]. However, the basic elements of spectral design in polynomial SUSY consist not only of intermediate isospectral transformations of first order in derivative (of linear SUSYs) but also of the transformations of second order in derivative [47, 54, 74]. Three irreducible classes of binary SUSY algebras are revealed and it is shown how they serve to complete the tool kit for spectral design. In particular, they may help to add or remove one or two adjacent levels for excited states. Thus the classification of irreducible (almost) isospectral transformations and related SUSY algebras is outlined. It was described first in [14], then systematically in [75, 76] and the exhaustive fine classification of irreducible operators was completed in [77]. In connection to this classification, the important problem of a possible
redundancy in supercharges, which can be eliminated without any changes in the super-Hamiltonian, is analyzed in subsection 3.4. With the help of a (redundant) multiplication of the supercharge by a polynomial of super-Hamiltonian, it is elucidated that certain irreducible second-order supercharges can be embedded into a higher-order SUSY, which in turn is decomposable in a chain of linear SUSYs. In section 4 we formulate basic theorems, which give a full description of the structure of polynomial one-dimensional SUSY algebras and on minimization of this algebra up to its essential part (proofs of these theorems can be found in [73, 75–77]). Apart from the above mentioned trivial ways to generate several SUSY algebras for the same super-Hamiltonian, there exist possibilities to obtain two non-minimizable SUSY algebras with supercharges of even and odd order in derivatives [73, 78]. Their combination forms a non-trivial hidden-symmetry operator which cannot be expressed as a polynomial of the super-Hamiltonian. Such systems and algebras are described in section 5. They are governed by (reflectionless) potentials of a special form arising also as soliton solutions of some nonlinear equations [28, 29]. The related SUSY algebra can be qualified as $\mathcal{N} = 2$ superalgebra with a central charge made of hidden symmetry operators.

In section 6 the concept of shape invariance is formulated for $d = 1$ polynomial SUSY algebras with the help of intertwining with shift [78]. The arising spectrum generating algebras represent intrinsic features of shape invariance; they can be considered as deformations of the hidden symmetries studied in section 5. It is elucidated on and exemplified for the second- and third-order SUSY shape invariance. The relations with deformed Heisenberg algebras [11, 58, 59] are also described. In the case of third-order SUSY its remarkable connection to the Painlevé-IV equations [79] is demonstrated (subsection 6.4) and the typical patterns of the energy spectra are listed. The latter connection has been the focus of recent investigations [80–86].

Section 7 is devoted to an extension of the spectral design methods on nonstationary Schrödinger equations [87]. We follow the scheme of SUSY intertwining relations and show that the addition of time dependence leads to certain restrictions on the potentials which are assembled in a super-Hamiltonian. In all cases, the closure of the SUSY algebra reveals a hidden symmetry, which becomes a typical rather than an exceptional feature for non-stationary Hamiltonians. For linear SUSY (subsection 7.1), this hidden symmetry entails simply a full separation of the variables. For the second-order SUSY (subsection 7.2), one discovers the spectrum generating algebra previously found for the third-order shape invariance; this algebra is completely encoded in time-dependent hidden symmetry operators. Thus hidden symmetries for nonstationary Schrödinger super-Hamiltonians give the universal framework for both stationary hidden symmetries and for spectrum generating algebras of shape invariant SUSY systems.

In section 8, the idea of one-dimensional polynomial SUSY QM is generalized onto the two-dimensional case providing isospectrality of scalar Schrödinger Hamiltonians without any intermediate matrix components [88–105]. Subsection 8.1 gives the basic formulas [88–91] of this approach for an arbitrary form of the highest (second-order) terms in the supercharge. The most promising case of the Lorentz (hyperbolic) form of second-order terms is studied in subsection 8.2 with special attention paid to reducible (with twist) and irreducible supercharges. The fundamental difference of two-dimensional polynomial SUSY QM is that both SUSY-partner Hamiltonians are completely integrable: they commute with symmetry operators which are of fourth order in momenta (subsection 8.3). Other examples of the relation between nonlinear supersymmetry and quasi-exact solvability can be found in [106]. Section 9 is devoted to two new methods [13, 16] of a solution of two-dimensional Schrödinger equations, which are not amenable to the conventional method of the separation of variables. Both new methods are based on the isospectrality of scalar Hamiltonians established in the previous
section and on the property of shape invariance (section 6). These methods can be considered as a SUSY separation of variables. The first of them (subsection 9.1) provides the quasi-exact solvability of several two-dimensional models, i.e. it allows one to find analytically a part of the spectrum and corresponding wave functions. The second one (subsection 9.2) gives the exact analytical solution of the models but it works only for particular values of parameters of the models. Both methods can be applied to several potentials not amenable to standard separation of variables; here they are illustrated by the solution of the two-dimensional generalization of the Morse model. In section 10, we list some of the related problems of SUSY QM which are outside the scope of this review and we mention some related open problems and perspectives for future study.

2. SUSY QM in arbitrary space dimension

2.1. Two-dimensional SUSY QM with first-order supercharges

The generalization of one-dimensional SUSY QM onto higher space dimensionality \( d \geq 2 \) in the Schrödinger equation is of great interest both from the conceptual and practical point of view. The forthright attempt to preserve the one-dimensional formulas given above but in the Schrödinger equation is of great interest both from the conceptual and practical point of view. The forthright attempt to preserve the one-dimensional formulas given above but in the Schrödinger equation is of great interest both from the conceptual and practical point of view. The forthright attempt to preserve the one-dimensional formulas given above but in the Schrödinger equation is of great interest both from the conceptual and practical point of view.

In terms of these operators, the initial Hamiltonian \( h^{(0)}(\vec{x}) \) can be quasifactorized,}

\[
q^\pm (\vec{x}) = \mp \partial_l + (\partial_l W)(\vec{x}); \quad \vec{x} = (x_1, x_2, \ldots, x_d); \quad \partial_l \equiv \frac{\partial}{\partial x_l}.
\] (13)

In terms of these operators, the initial Hamiltonian \( h^{(0)}(\vec{x}) \) can be quasifactorized,}

\[
h^{(0)}(\vec{x}) = -\Delta^{(2)} + V^{(0)}(\vec{x}) = q^+_l q^-_l; \quad \Delta^{(2)} \equiv \partial_l \partial_l,
\] (14)

where the sum over the repeated index \( l \) is implied. In the framework of this generalization, which can be named a ‘direct generalization’, the Schrödinger operator \( h^{(0)} \) is related to the chain \( (d) \) of other Hamiltonians of the Schrödinger form, but \( (d - 1) \) of them with matrix potentials (see subsection 2.5).

Now we shall describe this construction [36] in the simplest—two-dimensional case, where the chain of Hamiltonians includes three species, the original \( h^{(0)} \) and two others, \( h^{(1)} \) and \( h^{(2)} \), both of the Schrödinger form as well. The first of them has \( 2 \times 2 \) matrix potential \( V^{(1)}(\vec{x}) \), while the second one has a scalar potential \( V^{(2)}(\vec{x}) \). These partner Hamiltonians (superpartners) are expressible in a quasifactorized form as well,}

\[
h_{1k}^{(1)} = q^-_l q^-_k + p^-_l p^-_k = -\delta_{lk} \partial_l^2 + \delta_{lk} ((\partial_l W)^2(\vec{x}) - (\partial_l^2 W)(\vec{x})) + 2(\partial_l \partial_k W)(\vec{x});
\] (15)

\[
h^{(2)} = p^+_l p^-_l = -\partial_l^2 + V^{(2)}(\vec{x}) = -\partial_l^2 + (\partial_l W)^2(\vec{x}) + (\partial_l^2 W)(\vec{x}).
\] (16)

The operators \( q^\pm_1 \) were defined in (13), and the new vector operators \( p^\pm_1 \) are also of the first order in derivatives,}

\[
p^\pm_1 \equiv \epsilon_{1k} q^\mp_1,
\] (17)

where \( \epsilon_{1k} \) is a standard antisymmetric tensor with \( \epsilon_{12} = +1 \).
The chain of two-dimensional Hamiltonians $h^{(0)}$, $h^{(1)}_{ik}$, $h^{(2)}$ is analogous to the one-dimensional partnership $h^{(0)}$, $h^{(1)}$, since its components are also related by the intertwining relations with operators $q^+_k$, $p^+_k$ playing the role of intertwining operators,

$$h^{(0)} q^+_k = q^+_k h^{(1)}_{ik}, \quad h^{(1)}_{ik} q^+_k = q^+_k h^{(0)};$$  \hfill (18)

$$h^{(1)}_{ik} p^+_k = p^+_k h^{(2)}; \quad p^+_k h^{(1)}_{ik} = h^{(2)} p^+_k.$$  \hfill (19)

By definition, the intertwining operators $q^+_k$, $p^+_k$, and, correspondingly, two terms in the definition (15) of $h^{(1)}_{ik}$ are mutually orthogonal, $q^+_k p^+_k = 0$. These two terms in $h^{(1)}_{ik}$ were necessary to obtain the Hamiltonian of the Schrödinger form; the orthogonality is important to provide the intertwining relations (18), (19).

The energy spectra of $h^{(0)}$ and $h^{(2)}$ are, in general, non-overlapping, but the intertwining relations (18), (19) provide the equivalence of an energy spectra between a pair of two scalar Hamiltonians $h^{(0)}$, $h^{(2)}$ and $2 \times 2$ matrix Hamiltonian $h^{(1)}_{ik}$. The ‘equivalence’ means a coincidence of their spectra up to zero modes of operators $q^+_k$, $p^+_k$. Thus, the supersymmetry (SUSY transformation) allows one to reduce the solution of the spectral problem for the matrix Hamiltonian $h^{(1)}_{ik}$ to solutions of the couple of scalar spectral problems $h^{(0)}$, $h^{(2)}$ realizing the SUSY diagonalization method. Due to the intertwining relations, the vector wave functions of matrix Hamiltonian $h^{(1)}_{ik}$ are also connected (up to a normalization factor) with the scalar wave functions of scalar Hamiltonians $h^{(0)}$, $h^{(2)}$,.

$$\Psi^{(1)}_{i} (\vec{\chi}, E) = q^+_i \Psi^{(0)} (\vec{\chi}, E); \quad i = 1, 2; \quad \Psi^{(0)} (\vec{\chi}, E) = q^+_i \Psi^{(1)}_{i} (\vec{\chi}, E)$$

$$\Psi^{(1)}_{i} (\vec{\chi}, E) = p^+_i \Psi^{(2)} (\vec{\chi}, E); \quad \Psi^{(2)} (\vec{\chi}, E) = p^+_i \Psi^{(1)}_{i} (\vec{\chi}, E).$$  \hfill (20)

The two-dimensional generalization (14), (15), (16), (18), (19) of the Factorization Method can also be formulated compactly [68] in terms of exactly the same superalgebra (12). Indeed, three Hamiltonians $h^{(0)}$, $h^{(1)}_{ik}$, $h^{(2)}$ are combined in the super-Hamiltonian $H$, and intertwining operators $q^+_k$, $p^+_k$ - in the supercharges $Q^\pm$, correspondingly,

$$H = \begin{pmatrix} h^{(0)} & 0 & 0 \\ 0 & h^{(1)}_{ik} & 0 \\ 0 & 0 & h^{(2)} \end{pmatrix}; \quad Q^+ = (Q^-)^\dagger = \begin{pmatrix} 0 & 0 & 0 \\ q^+_1 & 0 & 0 \\ 0 & p^+_1 & p^+_2 \end{pmatrix},$$  \hfill (21)

all matrices are $4 \times 4$. The components $h^{(0)}$, $h^{(2)}$ are bosonic, and $h^{(1)}_{ik}$—fermionic component of the super-Hamiltonian.

A simple rearrangement of this representation gives an idea of SUSY diagonalization of matrix Hamiltonians in terms of the conventional SUSY algebra (12),

$$\tilde{H} = \begin{pmatrix} \tilde{h}^{(0)} & 0 \\ 0 & \tilde{h}^{(1)} \end{pmatrix}; \quad \tilde{h}^{(0)} = \begin{pmatrix} h^{(0)} & 0 \\ 0 & h^{(2)} \end{pmatrix}; \quad \tilde{h}^{(1)} = (h^{(1)}_{ik});$$

$$\tilde{Q}^+ = (\tilde{Q}^-)^\dagger = \begin{pmatrix} 0 & 0 \\ q^- & 0 \\ 0 & p^-_1 & p^-_2 \end{pmatrix}; \quad \tilde{q}^- = (\tilde{q}^+)\dagger = \begin{pmatrix} q^+_1 & q^+_2 \\ p^+_1 & p^+_2 \end{pmatrix}.$$  \hfill (22)

2.2. Examples

The Schrödinger operators with matrix potentials are not something very exotic in QM. In particular, the described two-dimensional generalization of SUSY QM was successfully used [69–71] to investigate the spectra of a stationary Pauli operator for fermion in external nonhomogeneous electromagnetic fields.
1. The Pauli Hamiltonian for a nonrelativistic spin 1/2 particle in three-dimensional space is written as follows,

$$H_p = (i\delta_a + eA_a)^2 - \mu \sigma_a B_a + U,$$

where \(a = 1, 2, 3\), \(e\) and \(\mu\) are the electric charge and magnetic momentum of the particle, \(A_a(x)\) is the vector part of the electromagnetic potential, and \(\sigma_a\) are the standard Pauli matrices. Restricting the problem to the case of external potentials which do not depend on one of the coordinates (on \(x_2\) below), the Pauli Hamiltonian (23), being a \(2 \times 2\) matrix operator, can be identified with the matrix component \(h^{(i)}_k\); \((i, k = 1, 2)\) of the super-Hamiltonian in the previous subsection. Indeed, the fermionic (two-component) wave function has to be factorized into the trivial plane wave along \(x_2\) with some momentum \(k\) and a nontrivial *two-dimensional* wave function,

$$\Psi_p(x) = \exp(i k x_2) \Psi(x_1, x_3).$$

(24)

In the subspace of such functions, the Pauli operator can be rewritten as,

$$H_p(x_1, x_3) = -(\delta_1 + eA_1)^2 + U + (k + eA_2)^2 - \mu \sigma_3 B_3; \quad l = 1, 3; \quad A_l = A_l(x_1, x_3).$$

(25)

Comparing (25) and (15), one concludes that their identification (up to replacing \((x_1, x_3) \leftrightarrow (x_1, x_2)\)) is possible if the external fields in (25) are expressed in terms of the only function \(W(x_1, x_3)\),

$$-\mu \vec{B} = (2(\delta_1 \delta_3 W)(x_1, x_3), 0, ((\delta_1^2 - \delta_3^2)W)(x_1, x_3)).$$

(26)

For simplicity, let the external sources of magnetic field be absent,

$$\text{rot} \ \vec{B} = \text{div} \ \vec{B} = 0$$

(27)

Then, the function \(W\) is a polynomial of the fourth order and the magnetic field is of the form,

$$-\mu \vec{B} = \{(2ax_1 x_3 + cx_1 + bx_3 + g), 0, (ax_1^2 - ax_3^2 + bx_1 - cx_3 + d)\}.$$ 

(28)

with arbitrary constant coefficients \(a, b, \ldots\). The scalar potential \(U(x_1, x_3)\) is a polynomial of the sixth order in coordinates (see details in [69–71]). Due to the results of the previous subsection, the spectral problem for a *matrix* \(2 \times 2\) Pauli Hamiltonian is reduced to a couple of spectral problems with scalar Hamiltonians \(h^{(0)}, h^{(1)}\), which is much simpler. This example demonstrates the opportunity of dynamical SUSY diagonalization of a class of matrix problems by means of the two-dimensional SUSY Quantum Mechanical approach. The identification of the Pauli operator \(H_p\) with a matrix component of a super-Hamiltonian is possible for a much wider class of external fields, if some unitary rotation is preliminary done (more details in [69–71]).

It is instructive to notice that this method works for a wide class of non-homogeneous magnetic fields, which can be directed at any angle with respect to the plane, including the field parallel to the plane. The magnetic dipole moment \(\mu\) of the particle is also arbitrary. It is necessary to mention the different SUSY approach to the solution of the Pauli equation for the fermion in the plane [110–113], which works only for a gyromagnetic ratio equal to 2 and only for (non-homogeneous) magnetic fields directed orthogonally to the plane. The latter condition in itself provides the diagonalizability of the problem, but SUSY allows one to study the symmetry properties and the spectrum of the Pauli operator. The method of [110–112] basically has the form of one-dimensional SUSY QM but with a matrix \(2 \times 2\) the supercharge operators \(Q^a\) (see also [114]). The initially diagonalizable Pauli equation for fermions with even integer gyromagnetic ratios in some specific magnetic fields orthogonal to the plane was also considered in [115] in the framework of nonlinear SUSY.
2. The second illustrative two-dimensional model [36] corresponds to the initial potential of a Coulomb form,

\[ V^{(0)}(\vec{x}) = -\frac{\alpha}{r}; \quad r^2 = x_1^2 + x_2^2; \quad \alpha > 0. \]  

(29)

In this case, the superpotential can be taken as

\[ W = 2\alpha r, \]

and therefore, the component \( V^{(2)} \) has no bound states at all,

\[ V^{(2)}(\vec{x}) = +\frac{\alpha}{r}. \]  

(30)

These specific circumstances lead to an equivalence of the spectra of the matrix Hamiltonian operator and the scalar component of the super-Hamiltonian \( h^{(0)}(\vec{x}) \). Since the discrete spectrum of the latter is well known, the spectrum of the matrix Hamiltonian with potential,

\[ V^{(1)}(\vec{x}) = -\alpha (\frac{x_i x_k}{r^2} - \delta_{ik}), \]  

(31)

is known as well. Its wave functions can be obtained from the wave functions of the scalar Coulomb problem \( V^{(0)} \) by means of intertwining the relations (18) with the operators \( q_{-l} \). The potential (31) can be physically interpreted in curvilinear (polar) coordinates [202]. It corresponds to an attractive Coulomb potential for a 1/2-spin particle with a positive spin projection \( s_z = 1/2 \) and to a repulsive Coulomb - for a particle with negative \( s_z = -1/2 \).

2.3. SUSY QM in arbitrary dimensionality of space as \((0 + 1)\)-dimensional quantum field theory

Both the one-dimensional factorization method and its two-dimensional generalization were shown above to realize the simplest superalgebra (12) of SUSY QM. This observation gives us an idea of how we could generalize this approach further for the arbitrary dimensionality \( d \) of space: it is necessary to build new realizations of the same SUSY algebra (12). For this purpose it is helpful [68] to exploit some methods and experience of well-developed SUSY quantum field theory [116]. Indeed, such new realizations of the SUSY algebra were obtained [68] by studying some simple models of SUSY quantum field theory. The simplest QFT, which can be interpreted as nonrelativistic QM, is the theory in \((0 + 1)\)-dimensional space, i.e. fields depend on one variable—the time \( t \). Keeping in mind that supersymmetry must be a property of the model, we consider the superfield, which depends both on one bosonic coordinate (time \( t \)) and a pair of mutually conjugate Grassmann variables \( \theta, \bar{\theta} \). The dimensionality \( d \) of SUSY QM on target is provided by choosing real fields, but with \( d \)-components.

Thus, the main ingredient of the model is [68] the \( d \)-dimensional real superfield,

\[ \varphi_l(t; \theta, \bar{\theta}); \quad l = 1, 2, \ldots, d; \quad \varphi_l = \bar{\varphi}_l; \]  

(32)

\[ \{\theta, \bar{\theta} = [\bar{\theta}, \theta] = [\bar{\theta}, \bar{\theta}] = [\theta, \bar{\theta}] = [\theta, \bar{\theta}] = 0. \]  

(33)

As it is typical for the genuine superfield theory, the infinitesimal super-transformation in the space \((t; \theta, \bar{\theta})\) is defined as,

\[ t \to t' = t - i(\bar{\theta} \epsilon - \theta \bar{\epsilon}); \quad \theta \to \theta' = \theta + \epsilon; \quad \bar{\theta} \to \bar{\theta}' = \bar{\theta} + \bar{\epsilon}, \]  

(34)

with a pair of Grassmann mutually conjugate parameters of transformation \( \epsilon, \bar{\epsilon} \). The ‘infinitesimal’ (linear in \( \epsilon, \bar{\epsilon} \)) transformations (34) can be generated by differential operators (super-generators) in the superspace \((t; \theta, \bar{\theta})\),

\[ Q = i \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial t}; \quad \bar{Q} = -i \frac{\partial}{\partial \bar{\theta}} + \bar{\theta} \frac{\partial}{\partial t}, \]  

(35)
which produce finite transformations by,
\[
U = \exp \{ i(\theta \vec{Q} + Q \vec{e}) \}.
\]
Operators \(Q, \vec{Q}\) satisfy the anticommutation relations,
\[
\{Q, \bar{Q}\} = \{\bar{Q}, \vec{Q}\} = 0; \quad \{Q, \vec{Q}\} = i \frac{\partial}{\partial t}.
\]
Due to nilpotency of \(\theta, \bar{\theta}\), the bosonic superfield can be expanded as,
\[
\psi_i(t; \theta, \bar{\theta}) = x_i(t) + i\theta \bar{\psi}_i(t) - i\bar{\psi}_i(t)\theta + \partial \partial D_i(t),
\]
where \(x_i(t), D_i(t)\) are bosonic real \(d\)-component fields, and \(\bar{\psi}_i(t), \psi_i(t)\) mutually conjugate \(d\)-component fermionic (anticommuting) fields. Restricting ourselves to super-scalar fields,
\[
\psi_i'(t; \theta', \bar{\theta}') = \psi_i(t; \theta, \bar{\theta}),
\]
one obtains the following rules of transformation,
\[
\begin{align*}
\delta x_i(t) &= i \epsilon \bar{\psi}_i(t) - i\bar{\psi}_i(t)\epsilon; \\
\delta D_i(t) &= \bar{\epsilon} \psi_i(t) + \psi_i(t)\bar{\epsilon}; \\
\delta \psi_i(t) &= -i \epsilon D_i(t) - \epsilon x_i(t),
\end{align*}
\]
where dot means derivative over \(t\).

As usual, for construction of the superinvariant action functional we need the operators of supercovariant derivatives,
\[
D = \frac{\partial}{\partial \theta} - i\bar{\theta} \frac{\partial}{\partial t}; \quad \bar{D} = \frac{\partial}{\partial \bar{\theta}} - i\bar{\theta} \frac{\partial}{\partial t},
\]
which allow us to build the super invariant action,
\[
S = \int dt \bar{L}(\bar{\varphi}) = \int dt \int d\theta d\bar{\theta} (\bar{D} \bar{\varphi} D \bar{\varphi} + W(\bar{\varphi}))
\]
with the arbitrary functional \(W(\bar{\varphi})\)—superpotential of the model. Expanding it around the point \(x_i(t)\), integrating the action (42) over both Grassmann variables and substituting nondynamical variables \(D_i, \bar{D}_i\) from the corresponding equations of motion, one obtains the action as functional of the component fields and superpotential,
\[
S = \int dt (\bar{\psi}_i \partial \psi_i) + (\bar{\psi}_i \partial \psi_i) - (\partial \bar{\psi}_i \partial \psi_i) - (\bar{\psi}_i \partial \partial W)(\bar{x})(\bar{x}) - (\partial \bar{\psi}_i \partial \partial W)(\bar{x})(\bar{x})[\bar{\psi}_i(t), \psi_m(t)]). \tag{43}
\]
The corresponding classical Hamiltonian is,
\[
H_{cl} = \bar{p}_1 p_1 + (\partial \bar{\psi}_i \partial \psi_i) + (\partial \bar{\psi}_i \partial \psi_i), \tag{44}
\]
where \(p_i\) is the momentum \(p_i = \delta L/\delta \dot{x}_i\). According to the Noether theorem (in the SUSY context), supercharges in the present field theory are \([117]\),
\[
Q^+ = \frac{\delta L}{\delta \epsilon} = (i p_i + (\partial \bar{\psi}_i \partial \psi_i)) \psi_i; \quad Q^- = \frac{\delta L}{\delta \bar{\epsilon}} = (-i p_i + (\partial \bar{\psi}_i \partial \psi_i)) \bar{\psi}_i = (Q^+)^+. \tag{45}
\]
To present this point, the model was formulated on the classical level, but now everything is prepared for its quantization (the prescriptions for canonical quantization of mixed systems both with bosonic and fermionic degrees of freedom, see \([118]\)). After quantization, coordinates \(x_i\), momenta \(p_i\) and fermionic variables \(\psi_i, \bar{\psi}_i\) become the operators \(\hat{x}_i, \hat{p}_i, \hat{b}_i^+, \hat{b}_i^-\), correspondingly,
\[
[\hat{p}_i, \hat{x}_m] = -i \delta_{im}; \quad [\hat{b}_i^+, \hat{b}_m^-] = \delta_{im}; \quad [\hat{b}_i^-, \hat{b}_m^+] = 0, \tag{46}
\]
and anticommuting operators \(\hat{b}_i^-\) can be interpreted as creation and annihilation operators for the system of \(d\) localized scalar particles, but with Fermi statistics. The quantized super-Hamiltonian has the form,
\[
H = \hat{p}_i \hat{p}_i + (\partial \bar{\psi}_i \partial \psi_i) + (\partial \bar{\psi}_i \partial \psi_i) \hat{b}_i^+ \hat{b}_i^-, \tag{47}
\]
and quantum supercharges are,

$$Q^\pm = (\pm i\hat{p}_i + (\partial_i W)\hat{x})b_i^\pm.$$  

(48)

These quantum operators satisfy all the relations of superalgebra (12) and they provide the required generalization of the one- and two-dimensional SUSY models of the introduction and subsection 2.1 to the case of arbitrary dimensionality $d$.

2.4. The structure of super-Hamiltonian in SUSY QM for arbitrary dimensionality of space

In order to understand the structure of $d$-dimensional quantum super-Hamiltonian (47) and supercharges (48), it is useful [68] to use the fermion number operator,

$$N_F = b_i^+ b_i^-,$$

(49)

which commutes with $H$, $Q^\pm$ as follows,

$$[H, N_F] = 0; \quad [N_F, Q^\pm] = \pm Q^\pm.$$  

(50)

If the basis in the Hilbert space $\mathcal{H}$ of states is chosen so that the space is a direct sum of the subspaces $\mathcal{H}_n$ with a fixed number $n$ of fermions ($n = 0, 1, 2, \ldots, d$), the conservation law (50) means that the super-Hamiltonian has the block-diagonal form in the fermionic Fock space,

$$H = \begin{pmatrix}
& & & \\
& h^{(d)} & & \\
& & & \\
0 & 0 & \cdots & 0 \\
& 0 & & h^{(1)} \\
& & & \\
0 & 0 & \cdots & 0 \\
& 0 & & 0 \\
& & & \\
0 & 0 & \cdots & 0 \\
& 0 & & 0 \\
\end{pmatrix}$$

(51)

with components $h^{(n)}$ acting in the subspace $\mathcal{H}_n$. The convenient basis in $\mathcal{H}_n$ is the basis of occupation numbers - $C_d^n$ vectors $|n_1, n_2, \ldots, n_d\rangle$ with $n_i = 0, 1$, and $\sum_{i=1}^{d} n_i = n$. In turn, the supercharge $Q^+$ has the block structure with a nonvanishing matrix $C_d^{n+1} \times C_d^n$ blocks $q_{n,n+1}^+$ below the main diagonal and $Q^-$-matrix $C_d^n \times C_d^{n+1}$ blocks $q_{n+1,n}^-$ above the main diagonal,

$$Q^+ = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
q_{0,1}^+ & 0 & 0 & 0 & 0 \\
0 & q_{1,2}^+ & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & q_{d-1,d}^+ & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}; \quad Q^- = \begin{pmatrix}
0 & q_{1,0}^- & 0 & 0 & 0 \\
0 & 0 & q_{2,1}^- & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & q_{d,d-1}^- \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.$$  

(52)

The algebra (12) and matrix representations (51), (52) provide the following relations,

$$h^{(n)} = q_{n-1,n}^- q_{n,n-1}^+ + q_{n+1,n}^+ q_{n,n+1}^- \equiv h^{(n)} h^{(n)} = \frac{h^{(n)} + h^{(n)}}{2}$$

(53)

(quasifactorization—compare with (15)),

$$q_{n+1,n+2}^+ q_{n,n+1}^- = q_{n-1,n-2}^- q_{n,n-1}^+ = 0; \quad h^{(n+1)} h^{(n)} = \frac{h^{(n)} + h^{(n+1)}}{2} = 0,$$

(54)

(orthogonality—compare with the orthogonality $p_i^+ q_i^- = p_i^- q_i^+ = 0$ in subsection 2.1),

$$h^{(n+1)} q_{n,n+1}^+ = q_{n+1,n}^- h^{(n)} = h^{(n)} q_{n+1,n}^+ = q_{n,n+1}^- h^{(n+1)} = 0; \quad q_{n,n+1}^- h^{(n)} = h^{(n)} q_{n+1,n}^+,$$

(55)

(intertwining relations analogous to (4)).

Thus, each matrix component $h^{(n)}$, $(n = 1, \ldots, d-1)$ of the super-Hamiltonian $H$ contains two mutually orthogonal terms $h^{(n)}$, $h^{(n)}$. These terms are intertwined with analogous terms of neighboring matrix components, providing their spectral equivalence (again, up to zero modes of supercharges). Due to orthogonality (54), each matrix component $h^{(n)}$, as a whole, is also intertwined with its neighbor $h^{(n-1)}$ and $h^{(n+1)}$. As for connection of the wave functions, those
of $h^{(n)}$ are connected, analogously to (5) and (20) with the wave functions of neighboring Hamiltonians $h^{(n+1)}$, $h^{(n-1)}$. This scheme can be illustrated by the following diagram,

$$[h^{(0)}] \supset [h^{(1)} + \frac{b}{2}] \supset [h^{(2)} + \frac{b}{2}] \supset \ldots [h^{(d-1)} + \frac{b}{2}] \supset h^{(d)}. \quad (56)$$

Thereby, the structure of elements of SUSY QM for arbitrary dimensionality of space is known. In order to obtain the explicit expressions for potentials, one has to choose any matrix representation for the fermionic creation and annihilation operators $b^\pm$. In particular, choosing

$$b^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \quad b^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (57)$$

the relations of the introduction for $d = 1$ are obtained. For the space dimensionality $d = 2$, the suitable representation for fermionic creation and annihilation operators $b^\pm_l$, $l = 1, 2$ is $4 \times 4$ matrix,

$$b^+_l = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} = (b^1_l)^\dagger; \quad b^-_l = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = (b^2_l)^\dagger. \quad (58)$$

leading [68] to the same $4 \times 4$ matrix dimensionality of the supercharges $Q^\pm$ and the super-Hamiltonian $H$ given by (21).

### 2.5. The particular case of $d = 3$: nucleon–nucleon interaction

For the physically interesting case of $d = 3$, three creation and annihilation operators $b^\pm_l$, $l = 1, 2, 3$, and the supercharges $Q^\pm$ are of matrix dimensionality $8 \times 8$. The super-Hamiltonian $H$ includes [68] four components: two scalars $h^{(0)}$, $h^{(3)}$ and two $3 \times 3$ matrices $h^{(1)}_d$, $h^{(2)}_d$. According to the general scheme of the previous subsection, there are intertwining relations in the pairs $h^{(0)} = h^{(1)}_0$, $h^{(1)}_1 = h^{(2)}_0$, $h^{(2)}_1 = h^{(3)}$, providing the relations between corresponding scalar and vector (three-component) wave functions. In this case, the same commutation and anticommutation relations of the superalgebra (12) are fulfilled.

The possibility of using this $8 \times 8$ realization of (12) can be illustrated by a derivation [72] of the simple connections between the different sectors of the nucleon-nucleon and nucleon–antinucleon interactions in the OPEP (one-pion-exchange-potential) approximation. The general form of the Hamiltonian of the interaction of two non-relativistic particles of spin 1/2 with scalar, spin-spin and tensor forces is:

$$V = \frac{1}{2} (\Delta^{(3)} + V_S(r) + \frac{1}{2} (1 + \vec{\sigma}_1 \vec{\sigma}_2) V_T(r) + (3(\vec{\sigma}_1 \vec{\sigma}_2) \Delta^{(3)} V_T(r)), \quad (59)$$

where $\vec{\sigma}_1$, $\vec{\sigma}_2$ are Pauli matrices describing the spin operators of two nucleons. Separating (59) onto singlet spin ($S = 0$) and triplet spin ($S = 1$) parts,

$$V^{S=0} = V_S(r) - V_T(r); \quad V^{S=1}_{k} = (V_S + V_T + 2\Delta^{(3)} V_T) \delta_{lk} - 6(\vec{\sigma}_l \vec{\sigma}_k V_T); \quad l, k = 1, 2, 3, \quad (60)$$

and suitably choosing the spherically symmetric superpotential $W(r)$, one may identify the first part $V^{S=0}$ with the scalar potential of the component $h^{(0)}$ of the super-Hamiltonian:

$$V^{S=0} = \frac{1}{2} \left[ (\vec{\sigma} W)^2(r) - (\Delta^{(3)} W)(r) \right] = \frac{1}{2} \left[ (\vec{\sigma} W)^2(r) - W^2 + \frac{2}{r} W \right]. \quad (61)$$

where prime means derivative over $r$. Simultaneously, it is possible [72] to identify the triplet spin part $V^{S=1}_k$ with the $3 \times 3$ component $h^{(2)}_k$ of the super-Hamiltonian, i.e. to put:

$$V_T(r) = \frac{1}{6} W(r); \quad V_S(r) = \frac{1}{3} (\Delta^{(3)} W)(r); \quad V_{\otimes_3}(r) = \frac{1}{2} (\vec{\sigma} W)^2(r) - \frac{1}{6} (\Delta^{(3)} W)(r). \quad (62)$$

13
In the OPEP-approximation, the explicit form of NN interaction potential is:

\[ V_{NN} = f^2 (\vec{\tau}_1 \cdot \vec{\sigma}_1) (\vec{\tau}_2 \cdot \vec{\sigma}_2) \frac{e^{-\mu r}}{r}, \]  

(63)

where \( \vec{\tau}_1, \vec{\tau}_2 \) are isospin operators of nucleons, \( f^2 \) is a coupling constant and \( \mu \) pion mass. The expression (63) corresponds to the specific forms of tensor, spin-spin and scalar terms:

\[ V_{NN}^T = \frac{1}{3} f^2 (\vec{\tau}_1 \cdot \vec{\tau}_2) e^{-\mu r}; \quad V_{NN}^S = \frac{2}{\Delta_1(n)} V_{NN}^T; \quad V_{NN}^S = -\frac{1}{\Delta_1(n)} V_{NN}^T. \]  

(64)

Choosing the superpotential \( W \) in the super-Hamiltonian \( H \) as

\[ W(r) = 2(\vec{\tau}_1 \cdot \vec{\tau}_2) f^2 e^{-\mu r}, \]  

(65)

one can identify [72] the Hamiltonian \( H^{NN} \) with \( h^{(0)} \oplus h^{(2)} \) with exponential accuracy \( (W')^2/W'' \) (see [119]).

It is important that in the OPEP-approximation, two other components \( h^{(1)} \) and \( h^{(3)} \) of the super-Hamiltonian \( H \) have also simple physical sense (see [72]): they describe the triplet spin \( S = 1 \) and singlet spin \( S = 0 \), correspondingly, of the nucleon–antinucleon interaction \( H^{NN} = h^{(1)} \oplus h^{(3)} \). Thus, just the parts of \( H^{NN} \) and \( H^{NN} \), with the definite total spin being the components of the same super-Hamiltonian, satisfy the SUSY intertwining relations of the form (55). In this case, if there were bound NN states, their spectra would (almost) coincide and we could write the relations between the wave functions \( \Psi^{NN}_n \) and \( \Psi^{\bar{N}N}_n \), similar to (20). In this case, the SUSY intertwining relations allow us to connect the asymptotic behavior of the wave functions of continuous spectra. The non-stationary and stationary non-relativistic scattering theory for a SUSY Hamiltonian was developed in [72], where the simple connections between scattering data—partial amplitudes, phase shifts and mixing parameters—of the components of the super-Hamiltonian were derived.

3. Polynomial SUSY in \( d = 1 \)

3.1. Supercharges of higher order in derivatives from a ladder construction

The SUSY methods of the previous section were based on superalgebra (12) which includes: supersymmetry, i.e. conservation of the supercharges \( Q^\pm \), (quasi)factorization of the super-Hamiltonian components and the nilpotency of supercharges, which is provided by their matrix form. Just the supersymmetry (i.e. the intertwining relations) guarantees the (almost exact) isospectrality between the components of the super-Hamiltonian with non-singular potentials, providing the most interesting practical results of the method. This is a reason to consider generalized SUSY with supercharges of a higher order in the derivatives and, therefore, without quasi-factorization of Hamiltonians in terms of the components of supercharges. In the context of the Darboux transformation [19], such a generalization of the intertwining relations (4) corresponds to the so-called Crum transformation [22]:

\[ h^{(0)} q_N^+ = q_N^+ h^{(1)}; \quad q_N^- h^{(0)} = h^{(1)} q_N^-, \]  

(66)

where \( N \)th order differential operators are:

\[ q_N^+ = \sum_{k=0}^{N} \omega_k^+(x) \partial^k, \quad q_N^- = \sum_{k=0}^{N} \omega_k^-(x) \partial^k, \]  

(67)

with differentiable coefficient functions. Herein let us take them as real and the pairs of intertwining operators to be mutually Hermitian conjugated \( q_N^- = (q_N^+)^{\dagger} \).
In the literature there are several synonyms for the higher-order SUSY algebra: originally it was named as a polynomial (or higher-derivative) one [46, 47], later a more general term of Nonlinear SUSY has been used [106, 120] in a certain relationship to a nonlinear SUSY algebra arising in the conformal QM [121]. The title of N-fold SUSY has been also suggested in [65].

One way to achieve the nonlinear SUSY is in combining several linear \( \mathcal{N} = 1 \) SUSY QM systems in the fermion number representation, which are implemented by nilpotent supercharges of the first order in derivatives (2) built from the real super-potentials \( W(x) \),

\[
q^\pm_1 \equiv \mp \partial + \chi(x); \quad \chi(x) \equiv (\partial W)(x).
\]

In the latter case, the SUSY algebra is completed (12) by the appropriate factorization (1), (3) of the super-Hamiltonian [34, 35].

Let us proceed by recursion: from a simple Darboux transformation to a ladder [37] or a dressing chain [122] made of several simple Darboux steps. To produce the required transformation operators, the two different linear SUSY systems may be overlapped. Namely, consider two super-Hamiltonians \( H_i \), \( i = 1, 2 \), equation (11), respectively, two sets of supercharges \( Q_i^\pm \) with coefficient functions \( \chi_i = (\partial W_i) \) and supercharge components \( r_i^\pm = \mp \partial + \chi_i \). Let us match two elements of the super-Hamiltonians with a constant shift,\(^{(69)}\)

\[
h_1^{(1)} = h_2^{(0)} + \lambda; \quad \chi_1^2 + \chi_1^1 = \chi_2^2 - \chi_2^1 + \lambda.
\]

Evidently, the ground state energies for the above Hamiltonians \( \bar{E}_1, \bar{E}_2 \) satisfy the inequalities \( \bar{E}_1 > \lambda > -\bar{E}_2 \). The constant shift of the super-Hamiltonian \( H_2 \rightarrow H_2 + \lambda \) does not break or change its supersymmetry but modifies the superalgebra.

After matching, the chain of intertwining relations (4) can be assembled to produce the final Hamiltonian \( h_2^{(1)} \) only from the initial one, \( h_1^{(0)} \) by means of a second-order Darboux transformation,

\[
h_1^{(0)} r_1^+ r_2^- = r_1^+ h_2^{(1)} r_2^- = r_1^+ (h_2^{(0)} + \lambda) r_2^- = r_1^+ r_2^- (h_2^{(1)} + \lambda); \quad r_2^- r_1^+ h_2^{(0)} = (h_2^{(1)} + \lambda) r_2^- r_1^+.
\]

Following the spectral design one concludes that the two-step ladder SUSY dynamics contains the redundant information, namely, about the intermediate Hamiltonians \( h_1^{(1)} = h_2^{(0)} + \lambda \). Let us remove it and define the two (almost) isospectral components,

\[
h^{(0)} = h_1^{(0)} + \lambda_1 = r_1^+ r_1^- + \lambda_1; \quad h^{(1)} = h_2^{(1)} + \lambda_2 = r_2^- r_2^+ + \lambda_2;
\]

for the super-Hamiltonian (11), for which we have employed an energy reference shifted by arbitrary \( \lambda_1, \lambda_2 ; \lambda = \lambda_2 - \lambda_1 \). Then the intertwining relations (66) are identical to equation (70) with \( q_2^+ = r_1^+ r_1^- = (Q_2^+)^\dagger \) and the supersymmetry \([H, Q_2^+] = [H, Q_2^-] = 0\) is generated by the conserved nilpotent supercharges,

\[
Q_2^+ = \begin{pmatrix} 0 & 0 \\ q_2^+ & 0 \end{pmatrix} = (Q_2^+)^\dagger, \quad (Q_2^+)^2 = (Q_2^-)^2 = 0.
\]

Now, in virtue of (71), the algebraic closure of the SUSY algebra is given by,

\[
\{Q_2^+, Q_2^-\} = \begin{pmatrix} r_1^+ r_2^- r_1^- r_2^+ & 0 \\ 0 & r_2^- r_1^+ r_1^- r_2^+ \end{pmatrix} = (H - \lambda_1)(H - \lambda_2),
\]

generalizing equation (12). Thus we have obtained the second-order polynomial SUSY algebra [46–51] as a concise form of isospectral deformation of a potential system realized by a ladder [33–43] or a dressing chain [122–125] of a couple of one-step Darboux transformations or,
equivalently, by a second-order Crum–Darboux intertwining [22] or by a blocking of two linear SUSY with a partial overlapping of super-Hamiltonians [46] (weak SUSY [126]), or by a tower of para-SUSY transformations [127–131] (see how to proceed in [14]) or by a reduction of spin-\(j\) parasupersymmetry [132].

The formal zero-modes\(^3\) of intertwining operators \(q_\pm^i\) form the basis of a two-dimensional representation of the super-Hamiltonian,

\[
h^{(0)} q_\pm^i \phi_\pm^i(x) = 0 = q_\pm^i h^{(1)} \phi_\pm^i(x); \quad i = 1, 2;
\]

\[
h^{(1)} \phi_\pm^i(x) = \sum_{j=1}^{2} S^-_{ij} \phi_\pm^j(x); \quad h^{(0)} \phi_\pm^i(x) = \sum_{j=1}^{2} S^+_{ij} \phi_\pm^j(x),
\]

(75)
due to intertwining relations (66), (70). In terms of these Hamiltonian projections – constant zero-mode \(S^\pm\), the SUSY algebra closure takes the polynomial form [73] (see also [65]),

\[
\{ Q_+^j, Q_-^j \} = \det[ E I - S^+]_{E = H} = \det[ E I - S^-]_{E = H} \equiv P_+(H).
\]

(76)
Thus both matrices have the same set of eigenvalues, which for the ladder construction (74) consists of \(\lambda_1, \lambda_2\). As the formal zero-mode set is not uniquely derived from (75) the matrices \(S^\pm\) are not necessarily diagonal. For instance, the equation \(r_1^+ q_1^- \phi_1^-(x) = 0\) has one formal zero-mode \(\phi_1^-(x)\) obeying \(r_1^+ \phi_1^-(x) = 0\) and another one obeying \(r_1^- \phi_1^-(x) = 0\); \(\phi_1^+ = r_1^+ \phi_1^-(x) \neq 0\). Evidently the zero-mode solution \(\phi_1^-(x)\) is determined up to an addition of \(\phi_1^+\). When multiplying these linear equations by \(r_2^-\) one proves with the help of equations (71), (72) that

\[
(h^{(1)} - \lambda_2) \phi_2^-(x) = 0; \quad (h^{(1)} - \lambda_1) \phi_2^+(x) = C \phi_2^+(x); \quad S^- = \begin{pmatrix} \lambda_1 & C \\ 0 & \lambda_2 \end{pmatrix},
\]

(77)
where \(C\) is an arbitrary real constant. If \(\lambda_1 \neq \lambda_2\) then by the redefinition \((\lambda_1 - \lambda_2) \phi_2^+ = (\lambda_1 - \lambda_2) \phi_2^+ + C \phi_2^+(x)\) one arrives at the diagonal matrix \(S^-\). However, in the confluent case, \(\lambda_1 = \lambda_2 \equiv \lambda\), \(C \neq 0\) it is impossible to diagonalize and by a proper normalization of the zero-mode \(\phi_1^+\) one gets the elementary Jordan cell,

\[
(h^{(1)} - \lambda) \phi_2^+(x) = 0; \quad (h^{(1)} - \lambda) \phi_2^+(x) = \phi_2^+(x); \quad S^- = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.
\]

(78)
Therefore, in the confluent case, the zero-mode \(\phi_1^+\) is no longer a solution of the Schrödinger equation but it is an adjoint solution, which can be derived by differentiation of the first relation in (78), \(\phi_1^+ = d\phi_2^+/d\lambda + c_1 \phi_2^+\) where \(c_1\) is an arbitrary constant. Still, the intermediate Hamiltonian \(\tilde{h} = r_1^- r_1^+ + \lambda = r_1^- r_2^+ + \lambda\) is well defined; therefore, the intermediate isospectral partner \(\phi_1^+(x)\) of the zero-mode \(\phi_1^+(x)\) is a solution of the Schrödinger equation with the above Hamiltonian. The analysis of the matrix \(S^+\) is similar. Thus we convince ourselves that in general the Hamiltonian projection onto the subspace of supercharge zero-modes is not diagonalizable but can always be transformed into a canonical Jordan form.

To complete the description of polynomial SUSY algebras generated by a second-order ladder, one should also take into consideration the degenerate case when \(\lambda_1 = \lambda_2 \equiv \lambda\), \(C = 0\). For this choice the matrix \(S^-\) is automatically diagonal and both zero modes \(\phi_{1,2}^+(x)\) are (independent) solutions of the Schrödinger equation with the Hamiltonian \(h\). Then it can be proved [73] that the intertwining operator \(q_2^+\) is just a linear function of this Hamiltonian, \(q_2^+ = \lambda - h^{(1)}\). Hence the intertwining is trivial \(h^{(1)} = h^{(0)}\) and such supercharges must be ruled out due to their triviality. However, for SUSY of higher order \(N \geq 3\) the removal of such blocks from the supercharges may lead to less-dimensional, ladder-irreducible SUSY algebras (see subsection 3.3).

\(^3\) We mark them as ‘formal’ because they are not supposed to be normalizable, just being solutions of the Schrödinger equation but not necessarily belonging to the set of spectrum eigenfunctions.
3.2. Polynomial SUSY QM from a ladder of linear SUSY systems

Let us give the general description of the $N$-step ladder of linear SUSY algebras, which entails the polynomial superalgebra of $N$th-order [46, 47]. We introduce a set of first-order differential operators for intermediate intertwining,

$$r^+_l = \mp \partial + \chi_l(x), \quad l = 1, \ldots, N,$$

and the relevant number of intermediate coefficient functions $\chi_l(x) = (\partial W_l)$ related to intermediate superpotentials $W_l(x)$. The set of the initial, $h^{(0)} \equiv h_0$, the final, $h^{(1)} \equiv h_N$ and intermediate Hamiltonians, $h_l = -\partial^2 + v_l(x)$ consists of Schrödinger operators, nonsingular and real ones. They obey the matching (ladder) relations,

$$h_l = r^+_l \cdot h_{l-1} = r^+_l \cdot h_l + \lambda_l, \quad l = 1, \ldots, N - 1,$$

$$h_0 = h^{(0)} = r^+_1 \cdot h_1, \quad h_N = h^{(1)} = r^+_N \cdot h_N + \lambda_N.$$  

(80)

These ladder relations correspond to the (dressing) chain equations on super-potentials, $W_l(x)$, $\chi_l(x) = (\partial W_l)(x)$,

$$v_l(x) = (\chi_l(x))^2 + (\chi_l(x))' + \lambda_l = (\chi_{l+1}(x))^2 - (\chi_{l+1}(x))' + \lambda_{l+1}.$$  

(81)

The corresponding intertwining (Darboux) transformations hold in each adjacent pair of Hamiltonians,

$$h_{l-1} \cdot r^+_l = r^+_l \cdot h_l, \quad r^+_l \cdot h_{l-1} = h_l \cdot r^-_l,$$

and therefore the chain of $N$ overlapping SUSY systems is properly built,

$$H_l = \begin{pmatrix} h_{l-1} & 0 \\ 0 & h_l \end{pmatrix}, \quad R^-_l = \begin{pmatrix} 0 & r^+_l \\ 0 & 0 \end{pmatrix} = (R^+_l)';$$

$$[H_l, R^+_l] = [H_l, R^-_l] = 0, \quad H_l - \lambda_l = [R^+_l, R^-_l].$$  

(83)

Now let us omit a chain of intermediate Hamiltonians between $h^{(0)}$ and $h^{(1)}$ and produce the higher-derivative $\simeq$ polynomial $\simeq$ nonlinear SUSY algebra for the super-Hamiltonian $H$ given in equation (11). The intertwining between $h^{(0)}$ and $h^{(1)}$ is performed by the Cruc–Darboux operators,

$$q^+_N = r^+_1 \cdots r^+_N, \quad q^-_N = r^-_1 \cdots r^-_N.$$  

(84)

The SUSY symmetry $[H, Q^-_N] = [H, Q^+_N] = 0$, is accomplished by the supercharges of the same matrix structure (73); the super-Hamiltonian is represented by finite-dimensional matrices on the subspaces of supercharge zero-modes,

$$q^+_i \phi^+_i(x) = 0; \quad i = 1, 2, \ldots, N;$$

$$h^{(0)} \phi^+_i(x) = \sum_{j=1}^N S^+_j \phi^+_j(x); \quad h^{(1)} \phi^+_i(x) = \sum_{j=1}^N S^-_j \phi^-_j(x)$$

(85)

due to intertwining relations (66). In terms of the constant matrices $S^\pm$, the algebraic closure is given by a nonlinear SUSY relation [73, 65].

$$\{Q^+_N, Q^-_N\} = \text{det}[E I - S^+ I_{N \times N}] = \text{det}[E I - S^- I_{N \times N} \equiv \mathcal{P}_N(H) = \prod_{l=1}^N x(H - \lambda_l).$$  

(86)

Again both matrices have the same set of eigenvalues, which for the ladder construction (80) consists of $\lambda_1, \ldots, \lambda_N$. If the degenerate roots appear, the normal (canonical) form of matrices $S^\pm$ may consist of nontrivial Jordan cells. If all intermediate $h_l$ are Hermitian and nonsingular, then $\lambda_l$ are real and each ladder step is well defined.

However, it turns out that not all higher-order intertwining operators can be built by a ladder algorithm based on linear SUSY elements with real and nonsingular intermediate Hamiltonians.
3.3. Generation of polynomial SUSY QM by ladder: irreducible SUSY blocks of types I, II and III

Let us elucidate the circumstances which may obstruct the SUSY ladder decomposition of polynomial SUSY algebra. The class of polynomial SUSY algebras can be extended admitting complex $\lambda_i$ and singular $h_i$. In fact, the full variety of building elements for nonlinear SUSY can be established within the class of intertwining (or transformation) operators $q^\pm_k$ of the second-order in derivatives. Eventually one has to find the basic set of Crum–Darboux operators (67) with nonsingular coefficient functions, which produce a nonsingular potential $V^{(1)}$ after intertwining (66) with the Hamiltonian possessing a smooth initial potential $V^{(0)}$.

When taking a supercharge of the second order in a derivative with real coefficient functions, one always can find formal (not necessarily normalizable) real zero-modes of the supercharges with real coefficients. We call this kind of irreducibility type I. Its elementary block corresponds to the polynomial SUSY algebra can be easily read off from its closure (76), as supercharges with real coefficients polynomials $\mathcal{P}_2(H)$ possess real coefficients. We call this kind of irreducibility type I. Its elementary block corresponds to the polynomial $\mathcal{P}_2^{(1)}(H) = (H + a)^2 + d$, $d > 0$; its analytical properties have been investigated in [47, 133]. Some examples of related isospectral potentials are described in [134].

Next, one has to ensure the positivity of the SUSY algebra relation (76) in a particular differential realization of a super-Hamiltonian $H$ with real non-singular potentials and the supercharges $Q^+_N, Q^-_N$ (with $N = 2$ in our case) made of differential operators with real coefficients. Let the energy spectrum $E_j$; $j = 0, 1, \ldots; E_j < E_{j+1}$ of $H$ be discrete, for simplicity. Then,

$$\mathcal{P}_N(E_j) = \langle Q^+_N \Psi\rvert Q^+_N \Psi \rangle + \langle Q^-_N \Psi\rvert Q^-_N \Psi \rangle \geq 0,$$

if the action of supercharges is well defined in the Hilbert space spanned on eigenfunctions of a super-Hamiltonian. It can be extended on a continuous energy spectrum with the use of wave packets.

Thus, for non-singular potentials, the acceptable disposition of polynomial roots related to zero-modes of a supercharge ensures non-negative values of $\mathcal{P}(E)$ for each energy level of a Hamiltonian. Accordingly, the following positions for polynomial real roots are acceptable (for a pair of complex, mutually conjugated roots the positivity is obvious).

A. $\lambda_1 \leq \lambda_2 \leq E_0$ or $\lambda_1 = E_0$; $\lambda_2 = E_1$.

The related SUSY algebra is well embedded into a chain or ladder realization. It is reducible because one can gradually add/remove $\lambda_1$ and then $\lambda_2$ without breaking the positivity of intermediate SUSY algebra. The coincidence of roots and energies correspond to an isospectral transformation with deleting/inserting energy levels. For instance, if $\lambda_1 = E_0$; $\lambda_2 = E_1$ then two pairs of zero-modes of $q^\pm_2$ can be chosen as solutions of Schrödinger equations with Hamiltonians $h^{(0)}$, $h^{(1)}$. One can implement [37] the energy levels $E_0, E_1$ to appear in any of the Hamiltonians $h^{(0)}$, $h^{(1)}$ but each level only once, either in $h^{(0)}$ or $h^{(1)}$.

B. $E_0 < \lambda_1 < \lambda_2 \leq E_1$ or $E_j < \lambda_1 < \lambda_2 \leq E_{j+1}$, $1 \leq j$.

A pair of real roots is inserted between adjacent energy levels. Such an algebra cannot be decomposed into a chain of two linear SUSY with non-singular intermediate potentials, as
the removal of any of roots $\lambda_{1,2}$ immediately breaks the positivity in (87). Then the intermediate Hamiltonian acquires inevitably a singular potential, whose usage would lead to the loss of isospectrality. The related Darboux transformations had been known in the 1950s [25]. We call this irreducibility type II. The examples and certain theorems are given in [54, 134] (see also [77] for an exhaustive analysis of this type of irreducible Darboux transformation).

C. $E_0 < \lambda_1 = \lambda_2 \leq E_1$ or $E_j < \lambda_1 = \lambda_2 \leq E_{j+1}$, $1 \leq j$.

This is a confluent case which seems to be obtained as a limit of the previous one. However, one-dimensional QM does not allow degenerate levels and, besides, the matrix projection for the corresponding super-Hamiltonian contains a non-trivial Jordan cell. Therefore we specify this case as a separate one. This kind of irreducibility is called type III. One may find more information on the analytical properties of the related potentials in [135].

One can use these second-order blocks to build an $N$th-order polynomial SUSY system. Their general form is again given by equation (86), if accepting the presence of complex conjugated roots $\lambda_i$.

Meanwhile, it seems that a pair of supercharge zero-modes or even a pair of new excited energy levels of the super-Hamiltonian can always be inserted by the successive application of an appropriate ladder construction, described in the previous section, using first-order intertwining transformations between regular Hamiltonians. But the order of a relevant ladder of first-order transformations and respectively of the final Polynomial SUSY, will certainly be higher than two. We come to the problem of a possible relationship between first-order reducible and irreducible SUSY algebras having the same super-Hamiltonian.

The related question concerns the degenerate roots. These roots are distributed between different Jordan cells in the matrices $S^\pm$. The problem concerns how many Jordan cells may coexist for the same eigenvalue and what their role in the supercharge structure is.

3.4. Minimization of SUSY algebra for a given Hamiltonian and the emergence of irreducible blocks of type II and III

Let us reveal a possible redundancy in supercharges which can be eliminated without any changes in the super-Hamiltonian. There exists a possibility when the intertwining operators $q_N^+$ and $p_N^+$ for $N > N_1$ are related by a polynomial factor $F(x)$ depending on the Hamiltonian,

$$ q_N^+ = F(h(0))p_{N_1}^+ = p_{N_1}^+ F(h(1)); \quad q_N^- = F(h(1))p_{N_1}^- = p_{N_1}^- F(h(0)). $$

Obviously in this case the reduction to the second supercharge does not result in any modifications of potentials.

Thus the problem consists of how to factorize a minimal essential part of the supercharge and avoid numerous SUSY algebras generated by means of ‘dressing’ (88).

Let us illustrate how it could work in the following example:

the matrix $S^-$ for the minimizable intertwining operator $k_3^+$ with Jordan cells of different sizes having the same eigenvalue. It is generated by the operators,

$$ p^+ = \bar{\phi} \partial + \chi, \quad h(1) = p^- p^+ + \lambda, \quad k_3^+ = -p^+ p^- p^+ = p^+(\lambda - h(1)). $$

Respectively, the basis of formal zero-modes (eigen- and associated functions) for the intertwining operator $k_3^+ \phi_i = 0$ generates the non-diagonal matrix $S^-$.

$$ \begin{align*}
\phi^+_1 & : \quad p^- p^+ \phi^+_1 = \phi^+_2 \quad \longrightarrow \quad h(1) \phi^+_1 = \lambda \phi^+_1 + \phi^+_2; \\
\phi^+_2 & : \quad p^- \phi^+_2 = 0 \quad \longrightarrow \quad h(1) \phi^+_2 = \lambda \phi^+_2; \\
\phi^+_3 & : \quad p^+ \phi^+_3 \neq 0, \quad p^- p^+ \phi^+_3 = 0 \quad \longrightarrow \quad h(1) \phi^+_3 = \lambda \phi^+_3.
\end{align*} $$

Thus the algebraic redundancy in the operator $k_3^+$ finds its unambiguous track in the presence of two Jordan cells in the characteristic matrix $S^-$ with the same eigenvalues.
The supercharge components cannot be factorized in the form (88) if the polynomial $P_N(x)$ in the SUSY algebra closure (76) does not reveal the degenerate zeros. Indeed the SUSY algebra closure contains the square of polynomial $F(x)$, for instance,

$$k_N k_N = F(h^{(1)}) P_N F(h^{(1)}) = F^2(h^{(1)}) P_N(h^{(1)}),$$  

(91)

where $P_N(x)$ is a polynomial of a lower order, $N_1 \leq N - 2$. Therefore each zero of the polynomial $F(x)$ will produce a double zero in the SUSY algebra provided by (91).

Thus the absence of double zeros is sufficient to deal with the SUSY charges, non-factorizable in the sense of equation (88). However, it is not necessary because the degenerate zeros may well appear in the ladder construction giving new pairs of isospectral potentials (see, for instance, [47] for the polynomial SUSY of the second order).

Suppose that (see, for instance, [47] for the polynomial SUSY of the second order).

The supercharge components cannot be factorized in the form (88) if the polynomial $F(x)$ in (92) is achieved by means of re-factorization $e^{N_1} e^{N_1} e^{N_1} e^{N_1}$ so that the generation of the ground state for $h^{(0)}$, $h^{(1)}$ the Hamiltonian projections on the $q_{\pm}^4$ zero-mode space – the matrices $S^\pm$ are, in general, not diagonalizable but have one rank-two Jordan cell each. Thus, for instance,

$$S^- = \begin{pmatrix} E_0 & 0 & 0 & C \\ 0 & E_2 & 0 & 0 \\ 0 & 0 & E_1 & 0 \\ 0 & 0 & 0 & E_0 \end{pmatrix} \quad \implies \quad \hat{S}^- = \begin{pmatrix} E_0 & C & 0 & 0 \\ 0 & E_0 & 0 & 0 \\ 0 & 0 & E_2 & 0 \\ 0 & 0 & 0 & E_1 \end{pmatrix},$$  

(92)

where a non-zero constant $C$ can be normalized to $C = 1$. The canonical Jordan form $\hat{S}^-$ in (92) is achieved by means of re-factorization $q^+ = r_1^4 r_2^4 r_3^4 r_4^4 = r_1^4 r_2^4 r_3^4 r_4^4$ so that the generation of the ground state for $h^{(0)}$ is associated now with $r_2^4$. Respectively, the polynomial SUSY algebra shows up one degenerate root,

$$P_4(H) = (H - E_0)^2 \{H - E_1 \}(H - E_2).$$  

(93)

Evidently, this fourth-order algebra cannot be optimized to a lower-order one because there is no replication of roots in different Jordan cells of matrices $S^\pm$ (see the ‘Strip-off’ theorem on minimization in the next section). However, one may perform fine-tuning of the Darboux transformation parameters to provide the constant $C = 0$ in (92). This peculiar choice provides two rank-1 cells in (92) with the same eigenvalue $E_0$. The SUSY algebra is still given by equation (93) but the intertwining operators reveal a redundancy,

$$q_{\pm}^4 = (h^{(0)} - E_0) q_{\pm}^4.$$  

(94)
By construction, the left-hand side of this relation is fully factorizable in elementary binomials \( r_j^+ \) with Hermitian nonsingular intermediate Hamiltonians. But in the right-hand side the operator \( q_2^+ = r_1^+ r_1^+ \) does not admit a further factorization with a nonsingular intermediate Hamiltonian because after removal of the redundant factor \((h^{(0)} - E_0)\) such a factorization is forbidden by the positivity of the SUSY algebra, equation (87).

One can easily extrapolate the previous argumentation onto the case of additional degeneracy of excited levels \( E_1 = E_2 \) to analyze the irreducible SUSY of type III. Thus we come to the conjectures that:

(a) the factorization (84) of intertwining operators \( q_N^\pm \) is not unique and there exist options to have more reducible ladders and less reducible ones with a larger number of singular intermediate Hamiltonians;

(b) irreducible algebras of type II and III can be identified with special cases of fully reducible higher-order algebras of ladder type when the Hamiltonian projections \( S^\pm \) have an appropriate number of pairs of Jordan cells with coinciding eigenvalues.

We notice that one may save time in calculations when, instead of a more lengthy binomial ladder construction, the isospectral irreducible transformations of type II or III are exploited to embed pairs of energy levels between two excited ones. In the next section, the results of a more rigorous investigation of the relationship between the reducible and irreducible intertwining are presented.

4. Rigorous results on structure and (ir)reducibility of polynomial SUSY

Let us summarize the organization of polynomial SUSY QM in a more systematic way. In this section the single supercharge components are analyzed. In the general situation when the unique supercharge exists, the minimization procedure is the way to factor out the redundant polynomials of the Hamiltonian to leave the part essential for an isospectral transformation of potentials after intertwining. In the consequent section 5 devoted to hidden symmetries, a more specific class of reflectionless potentials is involved and the two inequivalent supercharges are revealed. Then for lowering of their order as differential operators, an additional procedure of optimization can be employed [73], which exploits linear combinations of different intertwining operators for a given pair of Hamiltonians with coefficients polynomial in Hamiltonians. Nevertheless, eventually the two supercharges of minimal order in derivatives survive after optimization; this very procedure is substantially based on the following theorems. We notice that the two theorems of subsection 5.1 are applicable to both non-periodic and periodic potentials, whereas the two theorems of subsection 5.2 have been proven for the special class of potentials (the Sokolov class), which does not include periodic potentials.

4.1. Basic theorems on the structure of QM with a nonlinear SUSY

First the following theorem regulates the very structure of polynomial SUSY in terms of the Hamiltonian projection of the kernels of intertwining operators.

**Theorem 1.** Let \( \phi_n^\pm(x), n = 1, \ldots, N \) be a basis in \( \ker q_N^\pm \):

\[
q_0^\pm \phi_n^\pm = 0, \quad q_N^\pm = (q_N^\pm)^\dagger.
\]  

Then:
(1) the action of the Hamiltonians \( h^{(0)}, h^{(1)} \) on the functions \( \phi_n^\pm(x) \) is described by constant \( N \times N \) matrices,

\[
\begin{align*}
  h^{(0)} \phi_n^- &= \sum_{m=1}^{N} S_{mn}^+ \phi_m^-; \\
  h^{(1)} \phi_n^+ &= \sum_{m=1}^{N} S_{mn}^- \phi_m^+; \quad n = 1, \ldots, N;
\end{align*}
\]  

(96)

(2) the closure of the supersymmetry algebra takes a polynomial form,

\[
\{ Q^+, Q^- \} = \det[E I - S^+]_{E=H} = \det[E I - S^-]_{E=H} \equiv P_N(H),
\]  

where \( I \) is an identity matrix and \( S^\pm \) is the matrix with entries \( S_{mn}^\pm \).

**Corollary 1.** The spectra of the matrices \( S^+ \) and \( S^- \) are equal.

Now we describe the minimization (‘strip-off’) theorem.

A basis in the kernel of the intertwining operator, in which the matrix \( S \) of this operator has a Jordan form, is called canonical; elements of a canonical basis are called transformation functions.

The potentials \( V^{(0)}(x) \) and \( V^{(1)}(x) \) of the Hamiltonians \( h^{(0)} \) and \( h^{(1)} \) are interrelated by the equation [28]

\[
V^{(1)}(x) = V^{(0)}(x) - 2[\ln W(x)]'',
\]  

(98)

where \( W(x) \) is the Wronskian of elements of an arbitrary (a canonical as well) basis in \( \ker q_N^- \).

The intertwining operators \( q_N^\pm \) are called minimizable if these operators can be presented in the form

\[
\begin{align*}
  q_N^+ = P(h^{(0)}) p_M^+ = p_M^+ P(h^{(1)}), \\
  q_N^- = P(h^{(1)}) p_M^- = p_M^- P(h^{(0)}),
\end{align*}
\]  

(99)

where \( p_M^\pm \) are operators of order \( M \) which intertwine the same Hamiltonians as \( q_N^\pm \) and \( P(x) \) is a polynomial of degree \( (N - M) / 2 > 0 \). Otherwise the intertwining operators \( q_N^\pm \) are named as non-minimizable.

The following theorem contains the necessary and sufficient conditions under which an intertwining operator \( q_N^+ = (q_N^+)^\dagger \) is minimizable or not (a proof can be found in [73]).

**Theorem 2.** ‘Strip-off’ theorem on minimization of an intertwining operator \( q_N^+ \) (and equivalently \( q_N^- \)).

An intertwining operator \( q_N^+ \) can be presented in the form

\[
q_N^+ = p_M^+ \prod_{l=1}^{m} (\lambda_l - h^{(1)})^{\delta k_l},
\]  

(100)

where \( p_M^+ \) is a non-minimizable operator intertwining the same Hamiltonians as \( q_N^+ \) (so that \( p_M^+ h^{(1)} = h^{(0)} p_M^\dagger \)), if and only if a Jordan form of the matrix \( S^+ \) of the operator \( q_N^+ \) has \( m \) pairs (and no more) of Jordan cells with equal eigenvalues \( \lambda_l \) such that, for the \( l \)th pair, \( \delta k_l \) is an order of the smallest cell and \( k_l + 2 \delta k_l \) is an order of the largest cell. In this case, \( M = N - 2 \sum_{l=1}^{n} \delta k_l = \sum_{l=1}^{n} k_l \), where the \( k_l, m + 1 \leq l \leq n \) are orders of the remaining unpaired Jordan cells.

**Remark 1.** A Jordan form of the matrix \( S^+ \) of the intertwining operator \( q_N^+ \) cannot have more than two cells with the same eigenvalue \( \lambda \); otherwise \( \ker(\lambda - h^{(1)}) \) includes more than two linearly independent elements.

**Corollary 2.** Jordan forms of the matrices \( S \) of the operators \( q_N^+ \) and \( q_N^- \) coincide up to permutation of Jordan cells.
Remark 2. Theorems 1 and 2 are valid for a broad class of potentials including the periodic ones.

If a Jordan form of the matrix \( S \) of an intertwining operator has cells of an order higher than one, then the corresponding canonical bases contain not only formal solutions of the Schrödinger equation but also formal associated functions, which are defined as follows [136].

A function \( \psi_{n,i}(x) \) is called a formal associated function of \( i \)th order of the Hamiltonian \( h \) for a spectral value \( \lambda_n \) if

\[
(h - \lambda_n)^{i+1}\psi_{n,i} \equiv 0, \quad \text{and} \quad (h - \lambda_n)^i\psi_{n,i} \not\equiv 0. \tag{101}
\]

The term ‘formal’ emphasizes that this function is not necessarily normalizable (not necessarily belonging to \( L_2(\mathbb{R}) \)) and therefore it does not belong to the energy spectrum and, accordingly, is not included in the spectral resolution of identity. In particular, an associated function \( \psi_{n,0} \) of zero order is a formal eigenfunction of \( h \) (not necessarily a normalizable solution of the homogeneous Schrödinger equation). They appear in the subspace of zero-mode solutions for irreducible intertwining operators and play a role in building the S-matrix of the Hamiltonian projection.

4.2. Classification of really (ir)reducible SUSY transformations

Assume that the intertwining operators \( q_N^\pm \) are represented as products of the intertwining operators \( k_{N-M}^\pm \) and \( p_M^\pm \), \( 0 < M < N \) so that

\[
q_N^+ = p_{M}^+ k_{N-M}^+, \quad q_N^- = k_{N-M}^- p_M^-, \quad p_M^+ h_M = h(0) p_M^+, \quad p_M^- h(0) = h_M p_M^-; \nonumber
\]

\[
k_{N-M}^+ h = h_M k_{N-M}^+, \quad k_{N-M}^- h_M = h(1) k_{N-M}^-, \quad h_M = -\partial^2 + v_M(x), \tag{102}
\]

where the coefficients \( k_{N-M}^\pm \) and \( p_M^\pm \) as well as the potential \( v_M(x) \) may be complex and/or singular. The Hamiltonian \( h_M \) is called intermediate with respect to \( h(0) \) and \( h(1) \). In this case, by theorem 1, the spectrum of the matrix \( S \) of the operator \( q_N^+ \) is a union of the spectra of the matrices \( S \) for the operators \( k_{N-M}^\pm \) and \( p_M^\pm \).

The intertwining operators \( q_N^\pm \) are called reducible if these operators can be presented as products of two nonsingular intertwining operators (with real coefficients) \( k_{N-M}^\pm \) and \( p_M^\pm \), \( 0 < M < N \) so that equations (102) are valid and the intermediate Hamiltonian \( h_M \) has a real nonsingular potential. Otherwise \( q_N^\pm \) are called irreducible.

Irreducible, non-minimizable, intertwining operators of the second order with real coefficients are divided into three types: [14].

An irreducible intertwining operator of I type is a differential intertwining operator with real coefficients for which eigenvalues of the matrix \( S \) have nontrivial imaginary parts and are mutually complex conjugate.

An irreducible intertwining operator of the II type is a differential intertwining operator \( q_2^\pm \) (or equivalently \( q_2^\pm \)) of the second order with real coefficients such that:

1. eigenvalues of the matrix \( S^\pm \) of the operator \( q_2^\pm \) are real and different;
2. both elements \( \psi_1^\pm(x) \) and \( \psi_2^\pm(x) \) of a canonical basis of ker \( q_2^\pm \) have zeros.

An irreducible intertwining operator of the III type is a differential intertwining operator \( q_2^\pm \) (or equivalently \( q_2^\pm \)) of the second order with real coefficients such that:

1. the eigenvalues \( \lambda_{1,2} \) of the matrix \( S^\pm \) of the operator \( q_2^\pm \) are equal, \( \lambda_1 = \lambda_2 \);
2. a canonical basis in ker \( q_2^\pm \) consists of formal eigenfunctions, \( \psi_{10}^\pm(x) \), and associated functions, \( \psi_{11}^\pm(x) \), of the Hamiltonian \( h(1) \) which assemble into a Jordan cell,

\[
h(1) \psi_{10}^\pm = \lambda_{1} \psi_{10}^\pm, \quad (h(1) - \lambda_1) \psi_{11}^\pm = \psi_{10}^\pm;
\]

3. \( \psi_{10}(x) \) has at least one root.
Other types of irreducible non-minimizable intertwining operators of the second order do not exist.

Now let us state (in a concise form) two theorems which characterize the reducibility of intertwining operators of any order (proven in [75–77]):

**Theorem 3.** On reducibility of ‘dressed’ non-minimizable intertwining operators.

It asserts that for any non-minimizable intertwining operator with a real spectrum of the matrix $S$, one can ‘dress’ it, i.e. find and multiply by an appropriate polynomial of the Hamiltonian, so that the resulting (certainly minimizable) intertwining operator can be factorized into a product of intertwining operators solely of the first order with intermediate real non-singular Hamiltonians. Thus any polynomial SUSY algebra with real roots can be embedded into a higher order algebra corresponding to a ladder of linear SUSY.

**Theorem 4.** On complete reducibility of non-minimizable intertwining operators.

It asserts that any non-minimizable intertwining operator with an arbitrary spectrum of the matrix $S$ can be factorized into a product of intertwining operators of the first order and irreducible second-order intertwining operators of the I, II and III type with intermediate real non-singular Hamiltonians. Still this factorization is not necessarily unique and, in particular, it may contain more or less blocks of the first order.

These theorems have been formulated at full length and rigorously proven in [75–77] for the class $K$ (Sokolov class) of potentials $V(x)$ such that:

1. $V(x)$ is a real-valued function from $C_0^\infty$;
2. there exist numbers $R_0 > 0$, $C > 0$ (they depend on $V(x)$) such that the inequality $\inf_{|x| \geq R_0} |V(x)| + C > 0$ takes place for any $|x| \geq R_0$;
3. the functions
   \[
   \left( \int_{\pm R_0}^x \sqrt{|V(x_1)| + C} \, dx_1 \right)^2 \left( \frac{|V'(x)|^2}{|V(x)| + C^3} + \frac{|V''(x)|}{|V(x)| + C^2} \right)
   \]
   are bounded for $x \geq R_0$ and $x \leq -R_0$, respectively. It can be proven [75] that the set $K$ is closed under an intertwining of the Hamiltonians.

The last condition is not very restrictive: for example, it is fulfilled (for $x \geq R_0$), for potentials:

1. $V(x) = ax^\gamma [1 + o(1)]$, $a > 0$, $\gamma > 0$;
2. $V(x) = V_0 + ax^{-\gamma} [1 + o(1)]$, $V_0 > 0$, $a \in \mathbb{R}$, $\gamma > 0$;
3. $V(x) = ax^\alpha e^{bx^\beta} [1 + o(1)]$, $a > 0$, $b > 0$, $\alpha \in \mathbb{R}$, $\beta > 0$;
4. $V(x) = V_0 + ax^\alpha e^{-bx^\beta} [1 + o(1)]$, $V_0 > 0$, $a \in \mathbb{R}$, $b > 0$, $\alpha \in \mathbb{R}$, $\beta > 0$.

A similar statement also holds for $x \leq -R_0$. However, periodic potentials do not belong to this class $K$.

5. Hidden symmetry in one-dimensional SUSY QM

5.1. Simplest SUSY with hidden symmetry

The emergence of hidden symmetries happens to be an intrinsic feature of SUSY algebra being related to so-called central charges. Specifically in SUSY QM, there are classes of potentials in super-Hamiltonians whose structure is dictated by a hidden symmetry. In two (and more) dimensions, the variety of such isospectral QM systems is rather typical; their spectra and
eigenfunctions reveal partial or complete solvability accounted for by SUSY separation of variables (see sections 8 and 9).

However, hidden symmetries also exist in one-dimensional SUSY QM [78] if the same super-Hamiltonian conserves several non-trivial (non-minimizable and optimized [73]) supercharges. Following the algorithm of minimization on the variety of supercharges one can apply the optimization process [73] and prove that the maximal number of independent and non-minimizable (in the terminology of the previous section) supercharges is two. Thus the two pairs of conserved supercharges \( K^+, K^- \) and \( P^+, P^- \) may form two SUSY algebras for a Hermitian super-Hamiltonian \( H \).

Let us examine the algebraic structure of the simplest nonlinear SUSY with two non-minimizable supercharges,

\[
\begin{align*}
  k^\pm &\equiv \partial^2 \mp 2f(x)\partial x + \hat{b}(x) \mp f'(x); \\
  p^\pm &\equiv \mp \partial + \chi(x),
\end{align*}
\]

induced by a complex supercharge of the second order in derivatives (see the preprint version of [78] and [73]). The supersymmetries (121) generated by \( K^+, K^- \) and \( P^+, P^- \) prescribe that

\[
V^{(0),(1)} = \mp 2f' + f^2 + \frac{f''}{2f} - \left(\frac{f'}{2f}\right)^2 - \frac{d}{4f^2} - a, \\
\hat{b} = f^2 - \frac{f''}{2f} + \left(\frac{f'}{2f}\right)^2 + \frac{d}{4f^2},
\]

(105)

\[
V^{(0),(1)} = \chi^2 \mp \chi',
\]

(106)

where \( \chi, f \) are real functions and \( a, d \) are real constants. The related superalgebra closure for \( K^+, K^- \) and \( P^+, P^- \) takes the form,

\[
\{K^+, K^-\} = (H + a)^2 + d, \quad \{P^+, P^-\} = H.
\]

(107)

The compatibility of two supersymmetries is achieved by solutions of the following equations

\[
\chi = 2f, \quad f^2 + \frac{f''}{2f} - \left(\frac{f'}{2f}\right)^2 = \frac{d}{4f^2} - a = \chi^2 = 4f^2.
\]

(108)

Equation (108) represents a nonlinear second-order differential equation whose solutions are parameterized by two integration constants. Therefore the existence of two SUSY reduces substantially the class of potentials for which they may appear. Evidently equation (108) can be integrated into the first-order one,

\[
(f')^2 = 4f^4 + 4af^2 + 4G_0 f - d \equiv \Phi_4(f),
\]

(109)

where \( G_0 \) is a real constant.

The solutions of this equation are elliptic functions which can be found in the implicit form,

\[
\int_{f_0}^{f(x)} \frac{df}{\sqrt{\Phi_4(f)}} = \pm (x - x_0),
\]

(110)

where \( f_0 \) and \( x_0 \) are real constants.

They are nonsingular if:

(a) \( \Phi_4(f) \) has three different real roots and the double root \( \beta^2/2 \) is either the maximal one or a minimal one,

\[
\Phi_4(f) = 4 \left( f - \frac{\beta}{2} \right)^2 \left( f + \frac{\beta}{2} \right)^2 - \left( \frac{\beta^2}{2} - \epsilon \right), \quad 0 < \epsilon < \beta^2.
\]

(111)
Then there exists a relation between constants \( a, d, G_0 \) in terms of coefficients \( \beta, \epsilon \),
\[
a = \epsilon - \frac{3\beta^2}{2} < 0, \quad G_0 = \beta(\beta^2 - \epsilon), \quad d = \beta^2 \left( \frac{3\beta^2}{4} - \epsilon \right).
\] (112)

The constant \( f_0 \) is taken between the double root and a nearest simple root.
(b) \( \Phi_4(f) \) has two different real double roots which corresponds in (111), (112) to
\( G_0 = 0, \quad \beta^2 = \epsilon > 0, \quad a = -\epsilon/2, \quad d = -\epsilon^2/4. \) The constant \( f_0 \) is taken between the roots.

The corresponding potentials \( V^{(0)}, V^{(1)} \) are well known [137, 138] to be reflectionless, with one bound state at the energy \( \beta^2 - \epsilon \) and with the continuum spectrum starting from \( \beta^2 \). Respectively the scattering wave function is proportional to \( \exp(ikx) \) with \( k = \sqrt{E - \beta^2} \).

In the case (a) the potentials coincide in their form and differ only by a shift in the coordinate (Darboux displacement [139]),
\[
V^{(0),(1)} = \beta^2 - \frac{2\epsilon}{\text{ch}^2(\sqrt{\epsilon}(x - \chi^{(0),(1)}))}, \quad \chi^{(0),(1)} = x_0 \pm \frac{1}{4\epsilon} \ln \frac{\beta - \sqrt{\epsilon}}{\beta + \sqrt{\epsilon}},
\] (113)
and in the case (b) one of the potentials can be taken as constant,
\[
V^{(0)} = \beta^2, \quad V^{(1)} = \beta^2 \left( 1 - \frac{2}{\text{ch}^2(\beta(x - x_0))} \right).
\] (114)

For these potentials, one can elaborate the relations of extended SUSY algebra.

The initial algebra is given by the relations (107). It must be completed by the mixed anti-commutators containing symmetry operators,
\[
\{K^+, P^- \} = \{K^-, P^+ \}^\dagger = B(H) + \mathcal{E}(H),
\] (115)
where the first term is Hermitian and in general (see the next subsection) a polynomial of the super-Hamiltonian and the second one is anti-Hermitian and not a polynomial of \( H \). In the example under consideration, the first symmetry operator turns out to be constant, \( B(H) = G_0 \) after taking into account (104) and (109). Meanwhile the second operator reads,
\[
\mathcal{E}(H) = \left[ a^3 - (a + \frac{1}{2} V(x)) \partial - \frac{3}{2} V'(x) \right],
\] (116)
in the notations \( H = -\partial^2 I + V(x) \). By construction, the operator \( \mathcal{E}(H) \) realizes a new symmetry for the super-Hamiltonian. Directly from equation (116) one derives that,
\[
-\mathcal{E}^2(H) = H[H(a + d)]^2 + G_0^2 = (H - E_0)^2(H - \beta^2),
\] (117)
where \( E_0 = \beta^2 - \epsilon \) is the energy of a bound state. Thus (some) of the zero modes of \( \mathcal{E}(H) \) characterize either bound states or zero-energy states in the continuum. We remark that in the case (b) only the Hamiltonian \( H^{(1)} \) has a bound state. Accordingly, the physical zero modes of \( \mathcal{E}(H) \) may be either degenerate (case (a), broken SUSY) or non-degenerate (case (b), unbroken SUSY).

The square root in (117) can be established from the analysis of scattering (transmission) coefficients,
\[
\mathcal{E}(H) = i(E_0 - H)^{1/2}H^{1/2} - \beta^2.
\] (118)
An unambiguous determination of this square root needs to specify the space of asymptotic ‘incoming’ states on which this operator acts, i.e. the scattering condition which selects out the direction of scattering either from the left to the right or in the opposite direction. These two spaces do not overlap. In (118) the first type of scattering condition has been selected: from the left to the right with the asymptotic limit of an incoming state, \( x \to -\infty, \Psi(k, x) \to \exp(ikx), k > 0. \) For the alternative selection, one should change the
sign of square root. We notice that the symmetry operator (116), (118) is irreducible, i.e. the binomial \((E_b - H)\) cannot be removed. Indeed the elimination of this binomial would lead to an essentially nonlocal operator.

When taking equation (118) into account, one finds (on the same space of scattering states) the mixed anti-commutators of the extended SUSY algebra (115) in a non-polynomial form,

\[
\{ K^+ , P^- \} = \{ K^- , P^+ \}^\dagger = G_0 - i(H - E_b)\sqrt{H - \beta^2},
\]

Thus the ‘central charge’ of the extended SUSY is built of the elements (107) and (119) and cannot be diagonalized by a unitary rotation with elements polynomial in \(H\). Therefore, the SUSY algebra is extended in the class of differential operators of finite order. The existence of polynomial relations between commuting operators can in fact be established from the Burchnall–Chaundy theorem [140], as was noticed recently in [141].

5.2. General case: one-dim SUSY algebra with Hidden symmetry

Let us examine the very possibility of having several supercharges for the same super-Hamiltonian. We remind the reader here that a number of supercharges can be produced trivially with the help of multiplication on a polynomial of the Hamiltonian (see section 4). Certainly such supersymmetries are absolutely equivalent for the purposes of spectral design and one must get rid of them. It was proved in [73] that the infinite set of possible supercharges for a given super-Hamiltonian can be always optimized so that no more than two nontrivial supercharges remain and that an optimal set of two real supercharges \(K^\pm\) and \(P^\pm\) with components of minimal order in derivatives contains one operator of an odd-order in derivatives and another one of an even-order. They can be used to generate all possible conserved supercharges \(Q_i\) by ‘dressing’ their components with polynomials of the Hamiltonians,

\[
q_i^\pm = \alpha_i^\pm k^\pm + \beta_i^\pm p^\pm.
\]

Thus in one-dimensional (scalar) QM one may have the \(N = 1, 2\) SUSY only.

Correspondingly suppose that the super-Hamiltonian \(H\) admits two supersymmetries with supercharges \(K^\pm\) and \(P^\pm\) made of differential intertwining operators of order \(N\) and \(N_1\) respectively,

\[
[H, K^\pm] = [H, P^\pm] = 0.
\]

The second pair of supercharges \(P^\pm\) is assumed to be made of differential operators of a lower order \(N_1 < N\).

To close the algebra one has to include all anti-commutators between supercharges, i.e. the full algebra based on two pairs of supercharges \(K^\pm\) and \(P^\pm\) with real intertwining operators. Two supercharges generate two polynomial SUSY,

\[
\{ K^+, K^- \} = \tilde{P}_N(H), \quad \{ P^+, P^- \} = \tilde{P}_{N_1}(H).
\]

The closure of the extended, \(N = 2\) SUSY algebra is given by

\[
\{ P^-, K^+ \} \equiv \mathcal{R} = \begin{pmatrix} p^+_{N_1} k^-_N & 0 \\ 0 & k^+_N p^+_N \end{pmatrix},
\]

\[
\{ K^-, P^+ \} \equiv \mathcal{R}^\dagger = \begin{pmatrix} k^+_{N_1} p^-_N & 0 \\ 0 & p^-_N k^-_{N_1} \end{pmatrix}.
\]

Evidently the components of operators \(\mathcal{R}, \mathcal{R}^\dagger\) are differential operators of \(N + N_1\) order commuting with the Hamiltonians \(h^{(0)}, h^{(1)}\), hence they form symmetry operators \(\mathcal{R}, \mathcal{R}^\dagger\) for
the super-Hamiltonian. However, in general, they are not polynomials of the Hamiltonians \( h^\pm \) and these symmetries impose certain constraints on potentials.

All four operators \( \tilde{\mathcal{P}}_N(H), \tilde{\mathcal{P}}_{N_1}(H), \mathcal{R}, \mathcal{R}^\dagger \) commute each to the other. The Hermitian matrix characterizing this \( \mathcal{N} = 2 \) SUSY,

\[
\mathcal{Z}(H) = \begin{pmatrix} \tilde{\mathcal{P}}_N(H) & \mathcal{R} \\ \mathcal{R}^\dagger & \tilde{\mathcal{P}}_{N_1}(H) \end{pmatrix}, \quad \det [\mathcal{Z}(H)] = \tilde{\mathcal{P}}_N \tilde{\mathcal{P}}_{N_1} - \mathcal{R} \mathcal{R}^\dagger = 0, \tag{124}
\]

is degenerate. Therefore, it seems that the two supercharges are not independent and by their redefinition (unitary rotation) one might reduce the extended SUSY to an ordinary \( \mathcal{N} = 1 \) one. However, such rotations cannot be global (constant) and must use non-polynomial, pseudo-differential operators as 'parameters'. Indeed, the operator components of the 'central charge' matrix \( \mathcal{Z}(H) \) have a different order in the derivatives. Thus, globally the extended nonlinear SUSY deals with two sets of supercharges but when they act on a given eigenfunction of the super-Hamiltonian \( H \) one could, in principle, perform the energy-dependent rotation and eliminate a pair of supercharges. Nevertheless this reduction can be possible only after the constraints on the potentials have been resolved.

Let us find the formal relation between the symmetry operators \( \mathcal{R}, \mathcal{R}^\dagger \) and the super-Hamiltonian. These operators can be decomposed into Hermitian and anti-Hermitian parts,

\[
\mathcal{B} \equiv \frac{1}{2} (\mathcal{R} + \mathcal{R}^\dagger) \equiv \begin{pmatrix} b^+ & 0 \\ 0 & b^- \end{pmatrix}, \quad \mathcal{E} \equiv \frac{1}{2} (\mathcal{R}^\dagger - \mathcal{R}) \equiv \begin{pmatrix} e^+ & 0 \\ 0 & e^- \end{pmatrix}. \tag{125}
\]

The operator \( \mathcal{B} \) is a differential operator of an even order and therefore it may be a polynomial of the super-Hamiltonian \( H \). But if the operator \( \mathcal{E} \) does not vanish identically, it is a differential operator of an odd order and cannot be realized by a polynomial of \( H \).

It can be proven [73] that the Hermitian operator \( \mathcal{B} \) is indeed a polynomial of the super-Hamiltonian of the order \( N_0 \leq N - 1 \). Let us use it to unravel the super-Hamiltonian content of the operator \( \mathcal{E} \),

\[
\mathcal{E}^2 (H) = \tilde{\mathcal{P}}_N(H) \tilde{\mathcal{P}}_{N_1}(H) - \mathcal{B}^2 (H), \tag{126}
\]

which follows directly from (124) and (125). As the (nontrivial) operator \( \mathcal{E}(H) \) is a differential operator of an odd order \( N_e \), it may have only a realization non-polynomial in \( H \) being a square root of (126) in an operator sense. An unambiguous determination of this square root needs to specify the space of asymptotic ‘incoming’ states on which this operator acts, i.e. the scattering condition (see the previous subsection). The symmetry operator is certainly non-trivial if the sum of the orders \( N + N_1 \) of the operators \( k^\pm_N \) and \( p^\pm_{N_1} \) is odd and therefore \( N_e = N + N_1 \).

The existence of a nontrivial symmetry operator \( \mathcal{E} \) commuting with the super-Hamiltonian results in common eigenstates which, however, are not necessarily physical wave functions. In general they may be combinations of two solutions of the Schrödinger equation with a given energy, the physical and unphysical ones. But if the symmetry operator \( \mathcal{E} \) is anti-Hermitian in the Hilbert space spanned on the eigenfunctions of the super-Hamiltonian \( H \), then both operators have a common set of physical wave functions. This fact imposes quite rigid conditions on partner potentials \( V^{(0)}, (1) \). It can be proven [73] that they inevitably belong to the class of transparent or reflectionless ones [137, 138]. Such a symmetry has relations to the Lax method in soliton theory [28, 29].

As the symmetry operator \( \mathcal{E} \) is anti-Hermitian its eigenvalues are imaginary but, by construction, its coefficients are real. Since the wave functions of bound states of the system \( H \) can always be chosen to be real, we conclude that they must be zero-modes of the operator \( \mathcal{E}(H) \),

\[
\mathcal{E}(H) \psi_i = \mathcal{E}(E_i) \psi_i = 0, \quad \tilde{\mathcal{P}}_N(E_i) \tilde{\mathcal{P}}_{N_1}(E_i) - \mathcal{B}^2 (E_i) = 0, \tag{127}
\]
which represents the algebraic equation on bound state energies of a system possessing two supersymmetries. Among solutions of (127) one reveals also a zero-energy state at the bottom of the continuum spectrum.

6. Shape invariance for SUSY related potentials

6.1. Shape invariance in linear SUSY

One of the goals of SUSY QM design consists of the search for exactly solvable models of QM. For a one-dimensional Schrödinger equation, a list of exactly solvable potentials is well known: harmonic oscillator, Coulomb, Morse, Pöschl–Teller, Scarf, Eckart potentials etc [142]. Each of these potentials has its own history of solutions, but all of them were reproduced in the framework of a single algebraic procedure of the factorization method [35] in the middle of the last century. The related method was formulated in the context of modern SUSY QM with the help of a new notion—shape invariance—introduced by Gendenshtein in [45].

The basic steps of standard shape invariance in absence of spontaneous SUSY breaking [45, 143–145, 4] are the following. Let us consider a parametric family of one-dimensional superpartners \( h^{(0)}(a) \), \( h^{(1)}(a) \) and first order supercharges \( q^\pm(a) \) depending on a parameter \( a \). We say that the Hamiltonians are shape invariant when, in addition to SUSYal intertwining relations (4), they have the property (analogous to ‘cyclic’ in [123, 125]),

\[
\tag{128} h^{(1)}(a) = h^{(0)}(\bar{a}) + \mathcal{R}(a),
\]

where \( \bar{a} = \tilde{a}(a) \) is a modified value of the parameter depending on \( a \) and \( \mathcal{R}(a) \) is a \((c-)\) function of \( a \) taken as positive for definiteness. Then the absence of spontaneous breaking of supersymmetry for all values of \( a \) implies that just the lowest eigenvalue \( E_0(a) \) of \( h^{(0)}(a) \) vanishes, and the corresponding eigenfunctions \( \Psi^{(0)}_0(a) \) are normalizable zero modes of \( q^-(a) \). This property allows one to solve the models (i.e. to find the spectrum and all bound state wave functions) algebraically.

To perform it we start from

\[
\tag{129} h^{(0)}(\bar{a})\Psi^{(0)}_0(\bar{a}) = E_0(\bar{a})\Psi^{(0)}_0(\bar{a}) = 0.
\]

Consider the relation (128) to obtain,

\[
\tag{130} h^{(1)}(a)\Psi^{(0)}_0(\bar{a}) = \mathcal{R}(a)\Psi^{(0)}_0(\bar{a}).
\]

We notice that \( \Psi^{(0)}_0(\bar{a}) \equiv \Psi^{(1)}_0(a) \) has no nodes and therefore is the ground state wave function of \( h^{(1)}(a) \). The combination of (4) and (130) yields,

\[
\tag{131} h^{(0)}(a)q^+(a)\Psi^{(0)}_0(\bar{a}) = \mathcal{R}(a)q^+(a)\Psi^{(0)}_0(\bar{a}).
\]

Provided \( q^+(a)\Psi^{(0)}_0(\bar{a}) \) is normalizable, we have generated an excited state of \( h^{(0)}(a) \). These steps can be repeated up to the last step, where the resulting wave function \( \Psi \) will not be normalizable anymore. There are notorious cases (oscillator-like potentials) where the spectrum is not bounded from above. The simplest and famous case is the harmonic oscillator with \( q^- = \pm \partial + \lambda \) and \( \mathcal{R}(a) = 2\lambda \). By the cyclic algorithm, one easily reconstructs the energy spectrum \( E^{(0)}_n = 2n\lambda \).

It is clear that the isospectrality of \( h^{(0)}(a) \) and \( h^{(1)}(a) \) (up to the only zero mode \( \Psi^{(0)}_0(a) \) ) implies that there is no eigenvalue of \( h^{(0)}(a) \) between zero and the ground state energy \( E^{(1)}_0(a) \) of \( h^{(1)} \). This observation can be used to prove that after a number of iterations one gets the entire spectrum of \( h^{(0)}(a) \). Thus the shape invariance or cyclic method leads to a spectrum generating algebras which allow one to find algebraically all energy eigenvalues and eigenfunctions of shape invariant Hamiltonians in one-dimensional SUSY QM—complete solvability.
We notice that the shape invariance relation (128) combined with SUSY factorization of Hamiltonians (1), (3) entails the conventional Heisenberg algebra,

\[ [q^-, q^+] = \mathcal{R}(a), \quad (132) \]

to supply \( q^\pm \) with the meaning of creation and annihilation operators.

### 6.2. Intertwining with shift and higher order shape-invariance

The particular construction of higher order shape-invariance can be realized [78] by breaking the hidden symmetry (121) for two polynomial SUSY algebras (122),

\[
\begin{align*}
    h^{(0)}k_N^+ &= k_N^+h^{(1)}; \quad k_N^+k_N^- = \tilde{\mathcal{P}}_N(h^{(0)}); \quad k_{N}^-k_{N}^+ = \tilde{\mathcal{P}}_N(h^{(1)}); \\
    h^{(0)}p_{N}^+ &= p_{N}^+(h^{(1)} + 2\lambda); \quad p_{N}^+p_{N}^- = \tilde{\mathcal{P}}_N(h^{(0)}); \quad p_{N}^-p_{N}^+ = \tilde{\mathcal{P}}_N(h^{(1)} + 2\lambda);
\end{align*}
\]

(133)

with the help of the shift on a positive constant \( \mathcal{R}(a) \equiv 2\lambda \). In the sector of one of the partner Hamiltonians (for example, \( h^{(0)} \)) we obtain,

\[
    h^{(0)}a^+ = a^+(h^{(0)} + 2\lambda), \quad a^-h^{(0)} = (h^{(0)} + 2\lambda)a^-
\]

(135)

if introducing the product operators \( a^+ \equiv p_{N}^+k_{N}^- = (a^-)^\dagger \). We will call such a Hamiltonian ‘higher order shape-invariant’. One can also work the other way around and start from (135) with a differential operator \( a^+ \) of \( N + N \) order in derivatives, factorize the latter as a product \( p_{N}^+k_{N}^- \) with nonsingular operator multipliers and find an auxiliary super-partner Hamiltonian \( h^{(1)} \) to obtain (133).

Equation (135) is a spectrum generating, ladder (dressing chain [123, 125]) equation where \( a^+ \) plays the role of the generalized creation operator, which provides an excitation energy of \( 2\lambda \). In order to study the spectrum, it is essential to determine normalizable zero modes of \( a^- \) and \( a^+ \). The former ones describe the lowest lying levels of the system for \( \lambda \) and one has to apply recursively the operator \( a^+ \) to them in order to generate the excitation spectrum. The energies of the zero modes can be obtained from vanishing the average of the operator product \( a^+a^- \). In nonlinear (polynomial) SUSY, this product can be evaluated algebraically because [47]

\[
    a^+a^- = p_{N}^+k_{N}^-p_{N}^- = p_{N}^+\tilde{\mathcal{P}}_N(h^{(1)})p_{N}^- = \tilde{\mathcal{P}}_N(h^{(0)} - 2\lambda)\tilde{\mathcal{P}}_N(h^{(0)}),
\]

(136)

see (122). In contrast to the simple harmonic oscillator, one also has the possibility of the zero modes of the operator \( a^+ \). Accordingly in this case, the relevant operator product reads

\[
    a^-a^+ = \tilde{\mathcal{P}}_N(h^{(0)})\tilde{\mathcal{P}}_N(h^{(0)} + 2\lambda),
\]

(137)

where the polynomials contain zeros, some of which correspond to the zero modes of the operator \( a^+ \). Thus (135) provides a connection between the different levels of \( h^{(0)} \) by a given shift, which is the simplest realization of the notion of shape-invariance [45]. Although some properties of the spectrum, like the normalizable zero modes of \( a^\pm \), will depend on the explicit product structure of \( a^\pm \), the equidistant excitation spectrum and the corresponding wave functions can be mainly obtained algebraically.

The generalized (or deformed) Heisenberg algebra is finally built by the following closure,

\[
    [a^+, a^-] = \tilde{\mathcal{P}}_N(h^{(0)} - 2\lambda)\tilde{\mathcal{P}}_N(h^{(0)}) - \tilde{\mathcal{P}}_N(h^{(0)})\tilde{\mathcal{P}}_N(h^{(0)} + 2\lambda) = 2\lambda\mathcal{P}_{N+1}^{-1}(h^{(0)}).
\]

(138)

It is possible to study certain consequences of (135) algebraically, using the deformed Heisenberg algebra (138) without taking into account the specific definition of the operators \( a^\pm \), although the physical spectrum is essentially based on normalizable zero modes of \( a^\pm \).

An important comment is due. Taking into account the minimization recipe from subsection 3.4 and the theorem 2 from subsection 4.1, one can find, in certain cases, that
the order of operators $p_{N_l}^\pm$ and $k_{N_l}^\pm$ can be lowered without changing the potential in the Hamiltonian $h^{(0)}$. However, the appropriate factorization of the redundant polynomials of the Hamiltonian can be typically realized only at the level of individual SUSY algebras and, in general, cannot be extended to the deformed Heisenberg algebra (138). Thus, at the formal level, there may be several algebras (138) with the same spectrum pattern.

6.3. Intertwining with shift and second-order shape-invariance

Let us elucidate the particular case when $a^\pm$ is of the second order, in which we can obtain a generalized singular harmonic oscillator that is also shape-invariant. Correspondingly, we consider in (133), (134) $k^\pm, p^\pm$ of the first order. We can thereby explore that they can lead to nontrivial consequences. The superpotentials which solve (133), (134) are growing linearly for large $k$.$\psi$

Similar results hold for $h^{(0)}$ and $h^{(1)}$ as follows,

$$V^{(0)}(x) = \frac{\rho(\rho - 1)}{x^2} + \frac{\lambda^2 x^2}{4} + \lambda \left(\rho + \frac{1}{2}\right); \quad V^{(1)}(x) = \frac{\rho(\rho + 1)}{x^2} + \frac{\lambda^2 x^2}{4} + \lambda \left(\rho - \frac{1}{2}\right).$$

(139)

These potentials are shape-invariant in the standard sense [45] and belong to the class of algebraically solvable models, because $V^{(1)}(x; \rho, \lambda) = V^{(0)}(x; \rho + 1, \lambda - 2\lambda)$. On the half-line with suitable boundary conditions for wave functions [47] at the origin, the above potentials can be interpreted as radial harmonic oscillators for integer values of $\rho$. However, this system also makes sense for arbitrary real $\rho$.

The algebraic properties of these systems are based on the following factorization,

$$h^{(0)} = k^+ k^- = p^+ p^- + \lambda(2\rho + 1); \quad h^{(1)} = k^+ k^- = p^+ p^- + \lambda(2\rho - 1),$$

(140)
satisfying the relations (133), (134). The intertwining operators are given by,

$$k^+ = (k^-)^+ = -\partial - \frac{\rho}{x} - \frac{\lambda x}{2}; \quad p^+ = -\partial - \frac{\rho}{x} + \frac{\lambda x}{2}.$$ (141)

In terms of the product operators $a^+ \equiv p^+ k^-$ and $a^- \equiv k^+ p^-$ one obtains an algebra suggestive of a generalization of the standard harmonic oscillator algebra,

$$[h^{(0)}, a^\pm] = \pm 2a^\pm,$$ (142)

where the raising/lowering (creation/annihilation) operators $a^\pm$ can be presented in a form convenient for further evaluations,

$$a^+ = (a^-)^+ = \exp\left\{\frac{\lambda x^2}{4}\right\} \left(-\partial^2 + \frac{\rho(\rho - 1)}{x^2}\right) \exp\left\{-\frac{\lambda x^2}{4}\right\}. (143)$$

Similar results hold for $h^{(1)}$.

Meanwhile the crucial polynomial algebra describes the individual products,

$$a^+ a^- = (h^{(0)} - 2\lambda)(h^{(0)} - \lambda(2\rho + 1)); \quad a^- a^+ = h^{(0)}(h^{(0)} - \lambda(2\rho - 1)).$$ (144)

The zeros of polynomials in (144) may indicate the zero modes (‘vacuum states’) of the annihilation, $a^-$ or creation, $a^+$ operators. In the case under discussion, it happens that the appropriate normalizable zero modes appear for the operators $k^+$ and $p^-$,

$$k^+ \psi_{0,1} = 0; \quad \psi_{0,1} = x^{-\rho} \exp\left\{-\frac{\lambda x^2}{4}\right\}; \quad p^- \tilde{\psi}_{0,2} = 0; \quad \tilde{\psi}_{0,2} = x^\rho \exp\left\{-\frac{\lambda x^2}{4}\right\}, (145)$$

which entails the two zero modes for the annihilation operator,

$$a^- \psi_{0,1} = 0; \quad \psi_{0,1} = x^{-\rho} \exp\left\{-\frac{\lambda x^2}{4}\right\}; \quad \psi_{0,2} = x^{1-\rho} \exp\left\{-\frac{\lambda x^2}{4}\right\}. (146)$$
Accordingly they are associated to two zeros of the first polynomial in (144),
\[(\hbar^{(0)} - 2\lambda)\psi_{0,1} = 0 = (\hbar^{(0)} - \lambda(2\rho + 1))\psi_{0,2}.
\] (147)

Another couple of zeros for the second polynomial in (144) are not related to any normalizable solution.

Eventually the oscillator algebra (142) generates one ladder of equidistant levels for \(\rho \geq \frac{3}{2}\) or, equivalently for \(\rho \leq \frac{1}{2}\) and two independent ladders of equidistant levels for \(-\frac{1}{2} < \rho < \frac{1}{2}\) or, equivalently, \(\frac{1}{2} < \rho < \frac{3}{2}\). For \(\rho = \frac{1}{2}\) these two ladders coincide. The corresponding spectra of \(\hbar^{(0)}\) are: \(E_{n,1} = 2\lambda(n + 1)\) or \(E_{n,2} = \lambda(2\rho + 1 + 2n)\) for \(n = 0, 1, \ldots\).

The generalized Heisenberg algebra for this sort of oscillator reads,
\[ [a^+, a^-] = -4\lambda\hbar^{(0)} + 2\lambda^2(2\rho + 1). \] (148)

There might be a (superficial) impression of the algebra (142) as to be a broken hidden symmetry of the \(\mathcal{N} = 2\) SUSY (122), (123). However, the contraction \(\lambda \to 0\) results in the system with \(a^+ \sim \hbar^{(0)}\) and continuum spectrum and therefore no nontrivial hidden symmetry emerges. One can find the deformed Heisenberg algebra (148) arising in a more complicated setting of superconformal mechanics [146].

A nontrivial, shape-invariant breaking of the \(\mathcal{N} = 2\) SUSY arises for the third-order spectrum-generating algebra.

### 6.4. Intertwining with shift and third-order shape-invariance

Let us now consider the intertwining operator \(k^\pm\) to be a reducible or irreducible operator of the second order but \(p^\pm\) still of the first order like in (104) [78, 80]. Accordingly, they are parameterized by
\[ k^\pm \equiv \delta^2 \mp 2f(x)\partial + b(x) \mp f'(x); \] (149)
\[ p^\pm = \mp \partial + \chi(x). \] (150)

The solutions of (133) are unchanged in respect to equations (105) of subsection 5.1, but equation (134) implies a shift for the potential \(V^{(1)}(x)\) in (106). Thus the consistency equations for \(\chi(x)\) and \(f(x)\) are modified. The first equation (108) becomes:
\[ \chi(x) = 2f(x) + \lambda x, \] (151)
where a possible integration constant can be ignored because of a shift of \(x\), which fixes the origin of the coordinate. Then the potential \(V^{(0)}(x)\) can be written from (105) as:
\[ V^{(0)}(x) = -2f''(x) + 4f^2(x) + 4\lambda x f(x) + \lambda^2 x^2 - \lambda. \] (152)

The coefficient in the intertwining operator \(k^\pm\) reads,
\[ b = -(2f'^2 + 4\lambda xf + 4\lambda^2 x^2 + a). \] (153)

In the second equation (108), one has to replace \(\chi^2\) in accordance with (151), i.e. \(f(x)\) satisfies the following equation,
\[ f'' = \frac{f'^2(x)}{2f(x)} + 6f^3(x) + 8\lambda xf^2(x) + 2(\lambda^2 x^2 - \lambda + a)f(x) + \frac{d}{2f(x)}. \] (154)

The equation (154) can be transformed by the substitution \(f(x) \equiv 1/2\sqrt{x}, g(y); y = \sqrt{x}\) to the Painlevé-IV equation [79],
\[ g'' = \frac{g^2(y)}{2g(y)} + \frac{3}{2} g^3(y) + 4yg^2(y) + 2(y^2 - a)g(y) - \frac{b}{2g(y)} \] (155)
where
\[ a \equiv 1 - \frac{a}{\lambda} \quad \text{and} \quad b \equiv -\frac{4d}{\lambda^2}. \]  

(156)

This equation has been studied intensively in the last few years [81, 82]. In what follows we will focus mainly on the asymptotic properties of its solutions, which will determine the asymptotics of the potentials (152) and the normalizability of the eigenfunctions.

The spectrum generating operators \( a^+ = p^+ k^+ = (a^-)^\dagger \) form the polynomial algebra,
\[ a^+ a^- = h(0) [ (h(0) + a - 2\lambda)^2 + d] ; \quad a^- a^+ = [ h(0) + 2\lambda ] [ (h(0) + a)^2 + d]. \]  

(157)

We show how one can derive the spectrum from (135) and (160) if normalizable zero modes of the annihilation operator \( a^- \) exist. We stress that this algebraic method is very powerful now since the explicit form of the potential is known only in terms of Painlevé transcendents.

The equation for zero modes of \( a^- \) reads
\[ a^- \Psi_k^{(0)} = k^+ p^- \Psi_k^{(0)} = 0 \]  

(158)

where \( k \) labels the normalizable solutions.

In accordance to the nonlinear SUSY algebra (157), the equation for eigenvalues \( E_k^{(0)} \) is,
\[ E^{(0)} \cdot [ (E^{(0)} + a - 2\lambda)^2 + d] = 0, \]  

(159)

and has at most three real solutions.

Concerning the generalized Heisenberg algebra [83], the modification in respect to (142) in the previous subsection is given by,
\[ [a^+, a^-] = -2\lambda (3(h(0))^2 + (4a - 2\lambda)h(0) + a^2 + d) \]  

(160)

and similarly for \( h(1) \).

6.5. Spectrum patterns for third order shape-invariance

We give now a short description of typical spectrum patterns (the details can be found in [78]), [80]. The case of equal asymptotics of \( f(x) \) at \( \pm \infty \) is discussed below.

- Three normalizable zero modes of \( a^- \) and, respectively, three equidistant sequences of levels may arise only for \( \lambda > \sqrt{-d} \) if \( -\sqrt{-d} > a > \sqrt{-d} - \sqrt{2}\lambda \) or \( 2\lambda - \sqrt{-d} > a > \sqrt{-d} \). The corresponding solutions of the Painlevé-IV equation must have the leading asymptotics \( f(x) \sim -\lambda x/3 \) with subleading oscillations \( \sim \sqrt{\lambda^2/3} + o(x) \).

- Two normalizable zero modes of \( a^- \) (and two sequences of levels) may exist for different solutions of the Painlevé-IV equation.
  - Namely, for \( \lambda > \sqrt{-d} \) the solution with asymptotics \( f(x) \sim -\lambda x/3 \) provides the fall off of \( \Psi^{(0)}_0(x) \), \( \Psi^{(0)}_1(x) \) if \( a < \sqrt{-d} - \sqrt{2}\lambda \) and of \( \Psi^{(0)}_0(x) \), \( \Psi^{(0)}_1(x) \) if \( 2\lambda - \sqrt{-d} > a > \sqrt{-d} \).
  - For \( \lambda > \sqrt{-d} \) the solution with asymptotics \( f(x) \sim -\sqrt{-d}/2\lambda x \) generates two sequences of levels starting from \( \Psi^{(0)}_0(x) \), \( \Psi^{(0)}_1(x) \) if \( 2\lambda n + \sqrt{-d} < -a < 2\lambda (n+1) - \sqrt{-d} ; n = 0, 1, 2,... \).
  - For \( \lambda < \sqrt{-d} \) two sequences (from \( \Psi^{(0)}_0(x) \) and \( \Psi^{(0)}_1(x) \)) are generated for two possible asymptotics \( f(x) \sim -\sqrt{-d}/2\lambda x \) and \( f(x) \sim -\lambda x/3 \) if \( 2\lambda + \sqrt{-d} > a > \sqrt{-d} \).

- For \( a \geq 2\lambda + \sqrt{-d} \) and arbitrary positive \( \lambda \) one sequence of equidistant levels will be realized with ground state \( \Psi^{(0)}_0(x) \) and one of three asymptotics: \( f(x) \sim -\lambda x/3, \sim \pm \sqrt{-d}/2\lambda x \).
  - For \( f(x) \sim -\lambda x/3 \) only one sequence of levels will be realized also for \( \lambda = \sqrt{-d}, \ a = \pm \sqrt{-d} \) (starting from \( \Psi^{(0)}_0(x) \)), for \( \lambda = \sqrt{-d}, \ a < \sqrt{-d} \) (starting from \( \Psi^{(0)}_1(x) \)) and for \( \lambda < \sqrt{-d}, \ a \leq 2\lambda + \sqrt{-d} \) (starting from \( \Psi^{(0)}_0(x) \)).
7. Nonlinear SUSY for non-stationary Schrödinger equations

7.1. Linear SUSY and hidden symmetry

For asymptotic behavior $f(x) \sim -\sqrt{-d}/2\lambda x$ and $\lambda > \sqrt{-d}$ the spectrum consists of one sequence of levels if $|a + 2\lambda n| < \sqrt{-d}$, $n = 0, 1, 2, ...$ (starts from $\Psi_0^{(0)}(x)$) or if $a = -\sqrt{-d}$ (starts from $\Psi_0^{(0)}$).

Last, for $f(x) \sim -\sqrt{-d}/2\lambda x$ one obtains one sequence with ground state $\Psi_0^{(0)}$ if $\lambda \leq \sqrt{-d}$ and $a < -\sqrt{-d}$ or if $\lambda > -\sqrt{-d}$ and $[a + 2\lambda n] < \sqrt{-d}$, $n = 0, 1, 2, ...$.

For specific values of parameter $a$, the normalizable zero mode of $a^+$ can cause a truncation of one of two sequences (with ground state $\Psi_0^{(0)}(x)$).

Let us also remark that the particular value $a = -2\lambda - \sqrt{-d}$, i.e. $E_+^{(0)} = 0$, $E_-^{(0)} < 0$, just corresponds to the case when the Painlevé-IV equation (155) has a class of particular solutions, which coincide with solutions of the Riccati equation $g'(y) = g^2(y) + 2yg(y) + \sqrt{d}$. Substituting this Riccati equation into equation (152), one finds that the potential is the pure harmonic oscillator: $V_1(x) = \lambda^2 x^2 - \lambda$.

One additional singlet state satisfies the equation

$$(161) \quad a^+ \Psi_0^{(0)}(x) = a^- \Psi_0^{(0)}(x) = 0.$$ 

For $\lambda > -\sqrt{-d}$ ($\lambda < -\sqrt{-d}$) it occurs when $a = \pm \sqrt{-d}$ and $\Psi_0^{(0)}(x) = \Psi_{2,1}(x)$ which entails the equation:

$$f'(x) = -2f^2(x) - 2\lambda xf(x) \mp \sqrt{-d}$$

with asymptotic behavior $f(x) \sim \mp \sqrt{-d}/2\lambda x$. The spacing between the two ground states $\Psi_0^{(0)}(x)$ and $\Psi_{0}^{(0)}(x)$ is: $\Delta E = 2\lambda \mp 2\sqrt{-d}$.

The doublet representation $(\Psi_0^{(0)}(x), a^+ \Psi_0^{(0)}(x))$ of the spectrum generating algebra (135), (160) is built on the solutions of the equation:

$$(a^+)^2 \Psi_0^{(0)}(x) = a^- \Psi_0^{(0)}(x) = 0.$$ 

It may hold when $a = -2\lambda - \sqrt{-d}$ for an arbitrary positive value of $\lambda$ and when $a = -2\lambda + \sqrt{-d}$ for $\lambda > -\sqrt{-d}$. It is equivalent to $a^+ \Psi_0^{(0)}(x) = \Psi_{1,2}(x)$, which is satisfied when $f(x)$ obeys the following equation:

$$8\lambda f^2(x)(f'(x) + 2f^2(x) + 2\lambda xf(x) - 2\lambda \mp \sqrt{-d})$$
$$= (f'(x) + 2f^2(x) + 2\lambda xf(x) \mp \sqrt{-d}) \cdot [(f'(x) + 2f^2(x) + 2\lambda xf(x) - 2\lambda \pm \sqrt{-d}]$$

One can show that all the solutions of this equation fulfill the Painlevé-IV equation (154). These solutions have the asymptotics $f(x) \sim \pm \sqrt{-d}/2\lambda x$ and cannot have any (pole) singularity. The spectrum consists of a doublet $(0, 2\lambda)$ and infinite sequence $E_n = \pm 2\sqrt{-d} + 2(n + 2)\lambda$, $n = 0, 1, 2, ...$.

Recently some higher-order generalizations of shape-invariance have been elaborated leading to the Painlevé-V etc equations [84, 85]. As well, new solutions of Painlevé-IV with complex parameters based on the third-order shape invariance have been found [86].

7. Nonlinear SUSY for non-stationary Schrödinger equations

7.1. Linear SUSY and hidden symmetry

In this section, our aim is to elucidate that many of the nonlinear SUSY constructions illustrated before can also be implemented in the Schrödinger time dependent framework [147, 87, 6, 148].
The first- and higher-order intertwining operators and the corresponding relations between non-stationary one-dimensional Schrödinger operators can be introduced straightforwardly. But, as compared to the stationary case, already the first-order intertwining relations imply some hidden symmetry\(^4\), which leads to a specific quantum dynamics when the evolution is described by quantum orbits and results in the R-separation of variables [87]. In turn, second-order intertwining operators [87, 149] and the corresponding nonlinear SUSY give rise to the quantum motion governed by the spectrum generating algebras. We start with the non-stationary Schrödinger operator

\[
\mathcal{S}[V] = i\hbar \partial_t + \partial_x^2 - V(x,t).
\] (163)

Here, \(\partial_t = \partial / \partial t\) and \(\partial_x = \partial / \partial x\) denote the partial derivatives with respect to time and position. When they are applied to some function \(f\), the following notations for these derivatives are used: \(\dot{f}(x,t) = (\partial_t f)(x,t)\) and \(\ddot{f}(x,t) = (\partial_x f)(x,t)\).

The intertwining operator of the first order [87] is given by

\[
\xi^+(x,t)\partial_t + \xi_1(x,t)\partial_x + \xi_2(x,t).
\] (164)

with, in general, complex-valued functions \(\xi_0, \xi_1, \xi_2\). The possibility of a complexification of the intertwining (Darboux) (first and also higher order) operator was emphasized by [149].

For the Schrödinger operator (163), the intertwining relation reads

\[
\mathcal{S}[V^{(0)}]q^+_i = q^+_i \mathcal{S}[V^{(1)}],
\] (165)

where the functions \(\xi_i (i = 0, 1, 2)\) and \(V^{(0),(1)}\) are not independent of each other. It can also be represented in the SUSY form equations (11), (12) when the stationary Hamiltonians \(h^{(0),(1)}\) are extended to the Schrödinger operators \(\mathcal{S}[V^{(0),(1)}]\), then

\[
[\mathcal{S}_t, Q_i^+] = 0, \quad \mathcal{S}_t = \begin{pmatrix} \mathcal{S}[V^{(0)}] & 0 \\ 0 & \mathcal{S}[V^{(1)}] \end{pmatrix}, \quad Q_i^- = \begin{pmatrix} 0 & q_i^+ \\ 0 & 0 \end{pmatrix}.
\] (166)

After inserting (163) and the intertwining operator (164) into relation (165) it can be derived that \(\xi_0\) and \(\xi_1\) may depend only on time, i.e. \(\xi_0 = 0 = \xi_1\). Then the assumption that \(\xi_0\) does not vanish identically would entail that the potential difference \(V^{(0)} - V^{(1)}\) does depend on time only. This is a rather trivial case and therefore let us take \(\xi_0 \equiv 0\). Making the appropriate choice of variables \(\xi_1(t) = e^{i\beta(t)}\rho(t)\) and \(\xi_2(x,t) = e^{i\beta(t)}\rho(t)\omega(x,t)\) with real \(\beta\), positive \(\rho\) and complex \(\omega\) functions, one finds

\[
V^{(0)}(x,t) = \omega^2(x,t) + \omega'(x,t) - i\omega(x,t) + \alpha(t) - \beta(t) + i\dot{\beta}(t)/\rho(t),
\]

\[
V^{(1)}(x,t) = \omega^2(x,t) - \omega'(x,t) + i\omega(x,t) + \alpha(t),
\] (167)

where \(\alpha\) is a time-dependent complex-valued integration constant. One may set [87] \(\beta \equiv 0\) without loss of generality. Furthermore, one may also remove \(\alpha \to 0\) as being absorbed in \(\omega\) by the shift \(\omega \to \omega - i\int \mathrm{d}x\alpha\). Finally, we are left with

\[
V^{(0)}(x,t) = \omega^2(x,t) + \omega'(x,t) - i\omega(x,t) + i\dot{\beta}(t)/\rho(t),
\]

\[
V^{(1)}(x,t) = \omega^2(x,t) - \omega'(x,t) - i\omega(x,t). \quad \text{(168)}
\]

Here the super-potential \(\omega\) is not arbitrary as the potentials are assumed to be real. It could be achieved by taking a stationary real super-potential. However, this choice leads to the stationary SUSY QM discussed previously. Therefore, we consider a complex super-potential

\[
\omega(x,t) = \omega_R(x,t) + i\omega_I(x,t)
\] (169)

---

\(^4\) Symmetry properties of the time dependent Schrödinger equation were studied in general in [150, 151] (and references therein).
with real functions $\omega_R$ and $\omega_I$. The reality condition $\text{Im} \, V^{(0)} = \text{Im} \, V^{(1)} = 0$ is fulfilled if

$$2(\omega_R)^2 + \dot{\rho}/\rho = 0, \quad 2(\omega_R)(\omega_I) - (\omega_I)^2 - \omega_R = 0,$$

which can be integrated to

$$\omega_I(x, t) = -\frac{1}{4} \frac{\dot{\rho}(t)}{\rho(t)} x^2 + \frac{1}{2} \rho(t) \dot{\mu}(t)x + \gamma(t),$$

$$\omega_R(x, t) = \frac{1}{2} \ln \rho(t) + K(x/\rho(t) + \mu(t)),$$

where $\mu$ and $\gamma$ are arbitrary real functions of time and $K$ is an arbitrary real function of the variable $y = x/\rho + \mu$. In terms of these functions, the final form of the two partner potentials is

$$V^{(0), (1)}(x, t) = \frac{1}{\rho^2(t)} [K''(y) \pm K''(y)] - \frac{\ddot{\rho}(t)}{4\rho(t)} x^2 + \left(\frac{\rho(t)\dot{\mu}(t) + \dot{\rho}(t)\dot{\mu}(t)}{2}\right)x$$

and the intertwining operator reads

$$q^+_I(x, t) = \rho(t)\partial_t + K'(x/\rho(t) + \mu(t)) - i\frac{1}{2}(\dot{\rho}(t)x - \rho^2(t)\dot{\mu}(t)).$$

Let us demonstrate that the non-stationary Schrödinger equation $S[V^{(0), (1)}] \psi^{(0), (1)} = 0$ with the potentials given in equation (172) (which is equivalent to the intertwining (165)) admits a separation of variables. Indeed, after the transformation

$$y = x/\rho(t) + \mu(t), \quad \psi^{(0), (1)}(x, t) = \frac{1}{\sqrt{\rho(t)}} e^{-i\omega t} \phi^{(0), (1)}(y, t) \equiv \Omega(y, t) \phi^{(0), (1)}(y, t),$$

this Schrödinger equation becomes quasi-stationary [152]

$$i\rho^2(t)\partial_t \phi^{(0), (1)}(y, t) = \left[-\partial_y^2 + K''(y) \pm K''(y)\right] \phi^{(0), (1)}(y, t),$$

which is obviously separable in $y$ and $t$. Hence, the solutions of the original Schrödinger equations have the general form $\psi(x, t) = \Omega(y, t)Y(y)T(t)$, which is known as the R-separation of variables [153]. In other words, for any pair of Schrödinger operators $S[V^{(0), (1)}]$ that admits a first-order intertwining relation (165), there exists a transformation (174) to some new coordinate in which the potentials become stationary (see also [148]).

This $R$-separation of variables is certainly related to the existence of a symmetry operator for the super-Hamiltonian. First, one can directly verify the adjoint intertwining relation for real potentials,

$$q^-_I S[V^{(0)}] = S[V^{(1)}]q^-_I,$$

where

$$q^-_I \equiv (q_I^+)^\dagger = -\rho(t)\partial_t + K'(x/\rho(t) + \mu(t)) + i\frac{1}{2}(\dot{\rho}(t)x - \rho^2(t)\dot{\mu}(t)).$$

Then from (165), (166) and (176) we obtain the closure of the SUSY algebra,

$$[Q, \tilde{Q}] = \mathcal{R}_\tau, \quad [\hat{S}_I, \mathcal{R}_\tau] = 0, \quad \hat{Q}_I = (Q_I)^\dagger,$$

where the symmetry operator $\mathcal{R}_\tau$ has the following components

$$R^\pm_I = q^\pm_I q^-_I = -\rho^3(t)\partial^3_x + \frac{i}{2}(\dot{\rho}(t)[x, \partial_x] - 2\rho^3(t)\dot{\mu}(t)\partial_x) + [K'(x/\rho(t) + \mu(t))]^2 \pm K''(x/\rho(t) + \mu(t)) + \frac{1}{4}(\dot{\rho}(t)x - \rho^2(t)\dot{\mu}(t))^2$$

$$= \exp[-i\omega_I(x, t)][-\partial^2_y + K''(y) \pm K''(y)]\exp[i\omega_I(x, t)].$$

(179)
Thus the quasi-stationary Hamiltonians in equation (175) are just a unitary equivalent to the symmetry operators $R^\pm$. It means that the supersymmetry entails the separation of variables because it provides a new symmetry for the super-Hamiltonian. As consequence, the quantum dynamics splits in orbits with a given eigenvalue of the symmetry operator.

### 7.2. Second-order intertwining for stationary potentials: spectrum generating algebra

Let us relate a pair of Schrödinger operators $\hat{S}[V^{(0)}]$ and $\hat{S}[V^{(1)}]$ by second-order (intertwining) operators of the form,

$$ q^+_t (x, t) = G(x, t) \partial^2_t - 2F(x, t) \partial_x + B(x, t). \quad (180) $$

We will explore the existence of the time-dependent SUSY charges with the appearance of the spectrum generating (oscillator like) algebras for corresponding Hamiltonians.

As in the first-order, case it can be shown [87] that the inclusion of an additional term of the first order in $\partial_t$ leads to the trivial solutions when the difference $V^{(0)} - V^{(1)}$ depends on the time $t$ only. Furthermore, from the intertwining relation (165) with the above $q^+_t$, one can conclude that the function $G$ cannot depend on $x$ and it is even possible to exclude a phase.

In other words, without loss of generality $G(x, t) \equiv g(t)$ and accordingly, from now on, an intertwining operator can be reduced to,

$$ q^+_t (x, t) = g(t) \partial^2_t - 2F(x, t) \partial_x + B(x, t). \quad (181) $$

In [87], particular solutions of the intertwining relation (165) were constructed with $q^+_t$ as given above. In this section we shall analyze the solutions of the intertwining relation (165) for the case where both potentials $V^{(0)}$ and $V^{(1)}$ are stationary, i.e. do not depend on $t$.

One class of such solutions is known from [47]. Assuming a supercharge $q^+_t$ with real coefficient functions independent on $t$, one finds that the corresponding solutions of (165) coincide with those of the stationary intertwining relations $(-\partial^2_t + V^{(0)}(x))q^+_t(x) = q^+_t(x)(-\partial^2_t + V^{(1)}(x))$.

Here we are interested in more general solutions of (165) when operators $q^+_t$ depend on $t$,

$$(i\partial_t - h^{(0)})q^+_t(x, t) = q^+_t(x, t)(i\partial_t - h^{(1)}), \quad (182)$$

with stationary Hamiltonians $h^{(0),(1)} = -\partial^2_t + V^{(0),(1)}(x)$ but time-dependent intertwining operators.

Let us employ the suitable ansatz with simple $t$-dependence in (181),

$$ q^+_t (x, t) = k^+_t(x) + A(t)p^+_t(x), \quad (183) $$

where

$$ k^+_t(x) \equiv \partial^2_t - 2f(x)\partial_x + \hat{b}(x) - f'(x), \quad p^+_t(x) \equiv -\partial_x + \chi(x). \quad (184) $$

Here all functions besides $A$ are considered to be real. We also assume $A \neq 0$. With this ansatz the intertwining relation (182) can be shown [87] to yield,

\[
\begin{align*}
\text{iA} &= 2\hat{m} + 2mA, \\
\dot{k}^+_t h^{(0)} - h^{(1)}k^+_t &= 2\hat{m}p^+_t, \\
\dot{h}^{(0)}p^+_t - p^+_th^{(1)} &= 2m p^+_t,
\end{align*}
\]

with real constants $\hat{m}$ and $m$. 

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We find it interesting to focus on the case $\mu \neq 0$ to explore certain spectrum generating algebras. The first equation in (185) immediately leads to
\[ A(t) = m_0 e^{-2i\mu t} - \tilde{m}/m \]
with a real $m_0$ and,
\[ q_i^r(x,t) = \partial_t^2 - \left( 2f(x) + \frac{\tilde{m}}{m} \right) \partial_x + b(x) - f'(x) + \frac{\tilde{m}}{m} \chi(x) - m_0 e^{-2i\mu t} p^+(x). \]
Without loss of generality we may set $\tilde{m} = 0$, because a non-vanishing $\tilde{m}$ may always be absorbed via a proper redefinition of $f$ and $b$, i.e. of the operator $K$.

As a consequence, the second relation in (185) leads to a second-order intertwining between $h^{(0)}$ and $h^{(1)}$. This has already been considered in [47] and it was found that the potentials $V^{(0)}$, $V^{(1)}$ and the function $\tilde{b}$ can be expressed in terms of $f$ and two arbitrary real constants $a$ and $d$ as in subsection 5.1, equation(105),
\[ V^{(0),(1)}(x) = \pm 2f'(x) + f''(x) + \frac{f'(x)}{2f(x)} - \frac{f''(x)}{4f^2(x)} - \frac{d}{4f^2(x)} - a, \]
\[ \tilde{b}(x) = f''(x) - \frac{f'(x)}{2f(x)} + \frac{f''(x)}{4f^2(x)} + \frac{d}{4f^2(x)}. \]
The corresponding second-order SUSY algebra generated by the supercharge $K$ is similar to (107).

One may find some similarities between the present intertwining algebra (185) and the extended SUSY relations discussed in section 5. But we emphasize that now for $\mu \neq 0$ the last relation in (185) does not generate a second SUSY. Rather it creates the equivalence of relatively shifted spectra of two Hamiltonians $h^{(0)}$ and $h^{(1)}$, which is typical for spectrum generating algebras emerging for the shape-invariant Hamiltonians (section 6). Specifically,
\[ p^+ p^- = h^{(0)} - m + c; \quad p^+ p^+ = h^{(1)} + m + c, \]
where $c$ is a real constant. Therefore, the reflectionless potentials found in section 5 are produced only in the limit of $m = 0$.

The genuine spectrum generating algebra for stationary Hamiltonians $h^{(0),(1)}$ can be derived from equation (185)
\[ [h^{(0)}, a^{(0)}_m] = -2ma^{(0)}_m, \quad a^{(0)}_m \equiv k^+ p^-, \]
\[ [h^{(0)}, a^{(0)}_m] = 2ma^{(0)}_m, \quad a^{(0)}_m \equiv p^+ k^-, \]
\[ [h^{(1)}, a^{(1)}_m] = 2ma^{(1)}_m, \quad a^{(1)}_m \equiv k^- p^+, \]
\[ [h^{(1)}, a^{(1)}_m] = -2ma^{(1)}_m, \quad a^{(1)}_m \equiv p^- k^+, \]
where $p^+ = (p^+)^\dagger$, $k^- = (k^+)^\dagger$ and $a^{(0),(1)}_m = (a^{(0),(1)}_m)^\dagger$.

The closure of this spectrum generating algebra is a polynomial deformation of Heisenberg algebra [83] (see subsection 6.4),
\[ [a^{(0),(1)}_m, a^{(0),(1)}_n] = P^{(0),(1)}(h^{(0),(1)}). \]
The explicit form of the polynomials $P^{(0),(1)}$ can be obtained with the help of equations (185) and (189) for $m = 0$. For instance,
\[ a^{(0)}_m a^{(0)}_n = (h^{(0)} - m + c)(h^{(0)} - 2m + a)^2 + d; \]
\[ a^{(0)}_m a^{(0)}_n = (h^{(0)} + m + c)(h^{(0)} + a)^2 + d, \]
where the notations from equations (107) and (189) are employed. The polynomials $P^{(0),(1)}$ turn out to be different for the isospectral partners $h^+$,
\[ P^{(0)}(h^{(0)}) = -6m(h^{(0)})^2 + 4m(2m - 2a - c)h^{(0)} - 2m[a^2 + d + 2(a - m)(c - m)]; \]
\[ P^{(1)}(h^{(1)}) = -6m(h^{(1)})^2 - 4m(2m + 2a + c)h^{(1)} - 2m[a^2 + d + 2(a + m)(c + m)]. \]
Hence the two spectrum generating algebras are, in general, different; this is essentially due to the shift in intertwining relations (189). When comparing to the equations in subsection 6.4, we find that the generating algebras (192), (193) are in fact originating from the third-order shape invariance.

There is a formal discrete symmetry between their constants and Hamiltonians $h^{(0)}$, $a$, $c \mapsto -h^{(1)}$, $-a$, $-c$.

The intertwining relation (182) and its adjoint give rise to the symmetry operators $q^+_t$, $q^-_t$ and $q^+_q$, $q^-_q$ for $(i\hbar - h^{(0)})$ and $(i\hbar - h^{(1)})$, respectively. Using equations (107), (186), (189) and after elimination of the polynomials of the Hamiltonians $h^{(0,1)}$, these symmetry operators may be reduced to,

$$
R^{(0)}(x, t) = m_0[e^{2ia_1}a_{(0)} + e^{-2ia_0}a_{(0)}],
$$

$$
R^{(1)}(x, t) = m_0[e^{2ia_1}a_{(1)} + e^{-2ia_0}a_{(1)}].
$$

As our potentials do not depend on time, the time derivatives $\dot{R}^{(0,1)}(x, t)$ of the Hermitian symmetry operators $R^{(0,1)}(x, t)$ form the independent set of Hermitian symmetry operators which do not commute between themselves. Similar results have also been obtained in [151] using a different approach. We see that the non-stationary SUSY delivers the time-dependent hidden symmetry operators [14],

$$
e^{2ia_0}a_{(1)} = \frac{1}{2m_0}R^{(0,1)}(x, t) - \frac{i}{4m_0}R^{(0,1)}(x, t),$$

which give the entire set of spectrum generating algebras previously found for the third-order shape invariance. Thus hidden symmetries for nonstationary Schrödinger super-Hamiltonians give the universal framework for both stationary hidden symmetries and for spectrum generating algebras of shape invariant SUSY systems.

8. Polynomial SUSY QM in $d = 2$

8.1. Supercharges of second order in derivatives: generalities

It was shown in section 2 that a multidimensional generalization of SUSY QM includes both scalar and matrix Hamiltonians; some physical examples of matrix problems, incorporated into SUSY QM, were given in subsections 2.2 and 2.5. In general, the construction does not provide any intertwining and therefore any direct relations between spectra and wave functions of two scalar components of the super-Hamiltonian, if we are not interested in models with a separation of variables. However, such an opportunity can be opened by applying a development of the ideas of polynomial SUSY of section 3 for higher dimensionality of space. In particular, this approach turned out [88–92, 94, 95, 97–100, 102–105] to be very fruitful for the study of nontrivial spectral problems in the case of $d = 2$ (see also [13, 16]).

Thus, in order to get rid of the matrix component of a two-dimensional super-Hamiltonian, we shall explore the supercharges of the second order in partial derivatives. The simplest variant of second order supercharges—of reducible form $q^\pm = (q^+_q)(q^\mp_q)$—is not very promising: the intertwined partner Hamiltonians differ by a trivial constant only and both of them admit the separation of variables (see details in [88, 89]). For this reason, here we are interested in general irreducible second order components of supercharges,

$$q^+ = g_{a}(\vec{x})\partial_x\partial_t + C_{t}(\vec{x})\partial_t + B(\vec{x}); \quad q^- = (q^+)^\dagger,$$

with real ‘metrics’ $g_{a}(\vec{x})$ and coefficient functions $C_{t}(\vec{x}), B(\vec{x})$. The familiar intertwining relations

$$h^{(0)}q^+ = q^+h^{(1)}; \quad q^-h^{(0)} = h^{(1)}q^-.$$

(196)
for two scalar two-dimensional Hamiltonians of Schrödinger form,

$$h^{(0),(1)} = -\Delta^{(2)} + V^{(0),(1)}(\vec{x})$$

(198)
can be rewritten as a system of nonlinear partial differential equations for \( g_{\delta \lambda}(\vec{x}) \), the coefficient functions \( C_{i}(\vec{x}), B(\vec{x}) \) and potentials \( V^{(0),(1)}(\vec{x}) \). The general solution for the ‘metrics’ \( g_{\delta \lambda}(\vec{x}) \) is [88, 89],

\[
g_{11} = ax_{1}^{2} + a_{1}x_{2} + b_{1}; \quad g_{22} = ax_{2}^{2} + a_{2}x_{1} + b_{2}; \quad g_{12} = -\frac{1}{2}(2ax_{1}x_{2} + a_{1}x_{1} + a_{2}x_{2}) + b_{1},
\]

(199)

where \((a, a_{1}, b_{1} = \text{const})\). Thus, the senior in the derivative part of supercharges belongs to the \( E(2) \)-universal enveloping algebra [153]. We distinguish four different (inequivalent) classes in second derivatives [88, 89],

\[
q^{(1)+} = \gamma(P_{1}^{2} + P_{2}^{2}) + C_{i}\partial_{i} + B;
\]

(200)

\[
q^{(2)+} = aP_{1}^{2} + (\alpha + \gamma)P_{2}^{2} + C_{i}\partial_{i} + B;
\]

(201)

\[
q^{(3)+} = \gamma[J_{3}, P_{1}] + \alpha(P_{1}^{2} + P_{2}^{2}) + C_{i}\partial_{i} + B;
\]

(202)

\[
q^{(4)+} = \gamma P_{3}^{2} + \beta P_{2}^{2} + aP_{1}^{2} + C_{i}\partial_{i} + B
\]

(203)

where \( J_{3} \) and \( P_{1}, P_{2} \) are generators of rotations and translations, correspondingly, and \( \gamma \neq 0 \).

The solutions (199) must be inserted into six other equations which followed from the intertwining relations (197),

\[
\partial_{i}C_{i}(\vec{x}) + \partial_{\lambda}C_{\lambda}(\vec{x}) + \Delta^{(2)}g_{\delta \lambda}(\vec{x}) - (V^{(0)}(\vec{x}) - V^{(1)}(\vec{x}))g_{\delta \lambda}(\vec{x}) = 0;
\]

(204)

\[
\Delta^{(2)}C_{i}(\vec{x}) + 2\partial_{i}B(\vec{x}) + 2g_{\delta \lambda}(\vec{x})\partial_{\lambda}V^{(1)}(\vec{x}) - (V^{(0)}(\vec{x}) - V^{(1)}(\vec{x}))C_{i}(\vec{x}) = 0;
\]

(205)

\[
\Delta^{(2)}B(\vec{x}) + g_{\delta \lambda}(\vec{x})\partial_{\lambda}V^{(1)}(\vec{x}) + C_{i}(\vec{x})\partial_{i}V^{(1)}(\vec{x}) - (V^{(0)}(\vec{x}) - V^{(1)}(\vec{x}))B(\vec{x}) = 0.
\]

(206)

It is clear that we have no chance to solve these equations with a general form (199) of \( g_{\delta \lambda}(\vec{x}) \). Therefore, one is forced to consider the simplest cases of constant metrics \( g_{\delta \lambda} \) looking for particular solutions for functions \( C_{i}(\vec{x}), B(\vec{x}), V^{(0)}(\vec{x}), V^{(1)}(\vec{x}) \).

The first sample of constant metrics is the elliptic (Laplace) one corresponding to (200), \( g_{\delta \lambda} = \delta_{\delta \lambda} \). In this case, the system (204)–(206) is essentially simplified, so that all coefficient functions can be found analytically [88, 89]. In particular, the combination \( C \equiv C_{1} + iC_{2} \) depends only on \( z = x_{1} + ix_{2} \), and it has the specific form,

\[
C^{2}(z) = az^{2} + bx + c,
\]

(207)

where \( a \) is real and \( b, c \) are complex constants. All other functions can also be found explicitly, but depending on the values of constants in (207), both potentials allow \( R \)-separation [153] of variables—in elliptic (for \( b = 0, a \neq 0 \)), parabolic (for \( a = 0, b \neq 0 \)) or polar (\( b = c = 0, a \neq 0 \)) coordinates. The situation with both \( a \neq 0, b \neq 0 \) is reduced to the first variant by a suitable constant shift of the coordinates. The case of polar coordinates just corresponds to reducible supercharges \( q_{i}^{+} \bar{q}_{i}^{+} \). Thus, for all possible values of the coefficients, the class of two-dimensional problems with elliptic metrics of supercharges can be reduced to two one-dimensional models; it will not be considered below.
8.2. Hyperbolic (Lorentz) metrics

8.2.1. Supercharges with twisted reducibility. The constant metrics of hyperbolic (Lorentz) type, $g_{ab} = \text{diag}(+1, -1)$ gives much more interesting models \cite{89, 91, 90}. One specific class of such solutions of intertwining relations (197) with the so-called supercharges with twisted reducibility,

$$ q^- = (q^+)^\dagger = q_1^+(\sigma_3)x_kq^- = (-\partial_l + \partial_k W(x))\sigma_3x_k(\partial_l + \partial_k W(x)), $$

was studied in papers \cite{94, 97, 99}. In (208), $W(x), \tilde{W}(x)$ are two different functions (superpotentials), $\sigma_3$ is the Pauli matrix, and summation over $k, l = 1, 2$ is implied. This form of supercharges can be considered as a generalization of the simplest reducibility for elliptic metrics, mentioned in the beginning of previous subsection. In contrast to that case, the form (208) leads to very nontrivial models, which do not allow the standard separation of variables. This construction of supercharges can be interpreted as obtained after gluing of two systems of two-dimensional SUSY QM of subsection 2.1 with first order supercharges with different superpotentials $W, \tilde{W}$, correspondingly. These systems are glued by their matrix components of super-Hamiltonian: these components are taken coinciding but up to an unitary constant matrix rotation by $\sigma_3$. In such a case, both Hamiltonians $h^{(0)} = \tilde{q}_1^+ \tilde{q}_1$ and $h^{(1)} = q_1^+ q_1^-$ are quasifactorized in terms of $\tilde{q}_1^+ \tilde{q}_1^-$ and $q_1^+ q_1^-$ correspondingly.

It was shown in \cite{94, 97} in a general form that equations (197) and (208) lead to the following representation for superpotentials $W, \tilde{W}$ in terms of four functions $\mu_{1,2,\pm}$,

$$ \chi = \mu_1(x_1) + \mu_2(x_2) + \mu_+(x_+) + \mu_-(x_-), $$

$$ \tilde{\chi} = \mu_1(x_1) + \mu_2(x_2) - \mu_+(x_+) - \mu_-(x_-), $$

where the light cone variables $x_{\pm} = x_1 \pm x_2$ were introduced. These functions $\mu_{1,2,\pm}$ have to satisfy the following equation,

$$ \mu'_1(x_1)[\mu'_+(x_+) + \mu'_-(x_-)] + \mu'_2(x_2)[\mu'_+(x_+) - \mu'_-(x_-)] = 0, $$

where $\mu'(x)$ means derivative over the argument. By substitutions $\phi \equiv \mu'$, it is evident that we deal with a purely functional equation with no derivatives,

$$ \phi_1(x_1)[\phi_+(x_+) + \phi_-(x_-)] = -\phi_2(x_2)[\phi_+(x_+) - \phi_-(x_-)]. $$

Each solution of equation (210) gives the corresponding solution of the intertwining relations (197) for the potentials $V, \tilde{V}$ and for the supercharges $q^{\pm}$,

$$ V^{(0)}(x) = (\phi_1^2(x_1) - \phi_1'(x_1)) + (\phi_2^2(x_2) - \phi_2'(x_2)) + (\phi_1^2(x_+) + \sqrt{2}\phi_1'(x_+)) $$

$$ + (\phi_2^2(x_-) + \sqrt{2}\phi_2'(x_-)), $$

$$ V^{(1)}(x) = (\phi_1^2(x_1) - \phi_1'(x_1)) + (\phi_2^2(x_2) - \phi_2'(x_2)) + (\phi_1^2(x_+) - \sqrt{2}\phi_1'(x_+)) $$

$$ + (\phi_2^2(x_-) - \sqrt{2}\phi_2'(x_-)), $$

$$ Q^{\pm} = \partial_1^2 - \partial_2^2 \pm \sqrt{2}(\phi_+(x_+) + \phi_-(x_-))\partial_1 \mp \sqrt{2}(\phi_+(x_+) - \phi_-(x_-))\partial_2 $$

$$ - (\phi_1'(x_1) - \phi_1'(x_1)) + (\phi_2'(x_2) - \phi_2'(x_2)) + 2\phi_+(x_+)\phi_-(x_-). $$

Thus, in order to find the systems with intertwining (197) by supercharges of the form (208), it is necessary to solve (210). This equation seems to be rather complicated, but it appeared to be solvable in a general form (see details in \cite{99}). In particular, it was shown that $\phi_{1,2}(x)$ are defined from solutions of the first-order nonlinear differential equation,

$$ \phi_{1,2}'' = a\phi_{1,2}^4 + b\phi_{1,2}^2 + c, $$

(213)
where \(a, b, c\) are arbitrary real constants. All solutions of this equation can be expressed in terms of elliptic functions; they are described for different ranges of parameters, for example, in appendix B of [154]. In turn, the functions \(\phi_{\pm}\) are defined uniquely by the solution of (213): they also satisfy the equation of the form (213), but with different constant coefficients. Depending on the values of \(a, b, c\), solutions \(\phi\) can be functions of their argument with a finite or infinite period. Corresponding models were studied in detail in [99] (periodic potentials) and in [94, 97] (potentials with an infinite period).

8.2.2. Irreducible supercharges. Let us consider now the most general supercharges with hyperbolic (Lorentz) metrics without any kind of reducibility [88, 89, 91, 13]. The light cone variables \(x_{\pm} \equiv x_{1} \pm x_{2}\) are again the most convenient to look for the solution of intertwining relations. In particular, it follows from the system (204)–(206) that the combinations \(C_{\pm} \equiv C_{1} \mp C_{2}\) depend only on one argument:

\[
C_{+} = C_{+}(x_{+}), \quad C_{-} = C_{-}(x_{-}).
\]

(214)

Then, the whole system (204)–(206) can be rewritten in a very compact form of two equations for functions \(C_{\pm}(x_{\pm})\) and an auxiliary function \(F(\vec{x})\),

\[
\partial_{-}(C_{-}F) = -\partial_{+}(C_{+}F);
\]

(215)

\[
\partial_{\pm}^{2} F = \partial_{\mp}^{2} F.
\]

(216)

It is easy to write the general solution \(F(\vec{x})\) of equation (216): it is expressed in terms of two arbitrary functions,

\[
F = F_{1}(x_{+} + x_{-}) + F_{2}(x_{+} - x_{-}),
\]

(217)

both are defined up to an arbitrary real constant, \(F_{1} \to F_{1} + \gamma, F_{2} \to F_{2} - \gamma\). The potentials \(V^{(1),(2)}(\vec{x})\) and the function \(B(\vec{x})\) are expressed in terms of four functions \(F_{1}(2x_{1}), F_{2}(2x_{2})\) and \(C_{\pm}(x_{\pm})\), which must satisfy the only equation (215),

\[
V^{(0),(1)} = \pm \frac{1}{2}(C_{+}^{2} + C_{-}^{2}) + \frac{1}{8}(C_{+}^{2} + C_{-}^{2}) + \frac{1}{4}(F_{2}(x_{+} - x_{-}) - F_{1}(x_{+} + x_{-}));
\]

(218)

\[B = \pm \frac{1}{4}(C_{+}C_{-} + F_{1}(x_{+} + x_{-}) + F_{2}(x_{+} - x_{-})),\]

where \(C'\) means the derivative in its argument. Although equation (215) has a compact form, it is very nontrivial being the functional-differential equation: the functions each depend on a specific argument. As a rule, equations of this kind have no regular recipe for a solution; this is the case for equation (215). Usually, one may try to solve such an equation starting from different suitable ansatzes for functions \(C_{\pm}(x_{\pm}), F(\vec{x})\).

(1) Let us choose \(C_{-} = 0\), then from (215) one obtains \(F = \phi_{-}(x_{-})/C_{+}(x_{+})\). After inserting into equation (216) the separation of variables is possible, and the particular solution reads,

\[
C_{+}(x_{+}) = \frac{1}{\delta_{1} \exp(\sqrt{\lambda} \cdot x_{+}) + \delta_{2} \exp(-\sqrt{\lambda} \cdot x_{+})};
\]

\[
F_{1,2}(2x) = \delta_{1}\sigma_{1,2} \exp(2\sqrt{\lambda}x) + \delta_{2}\sigma_{2,1} \exp(-2\sqrt{\lambda}x),
\]

where the Greek letters are arbitrary constants: depending on the sign of \(\lambda\) they may be real/complex.

(2) Let us consider the ansatz with factorized function \(F(\vec{x}), F = F_{+}(x_{+}) \cdot F_{-}(x_{-})\). Then from equation (215),

\[
C_{\pm} = \frac{v_{\pm}}{F_{\pm}} \pm \frac{\gamma}{F_{\pm}} \int v_{\pm} d\vec{x}_{\pm},
\]

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there are two options to fulfill the condition (216), i.e. \( F(\tilde{x}) = F_1(2x_1) + F_2(2x_2) \),

\[(a) \quad F_\pm(x_\pm) = \epsilon_\pm x_\pm, \quad (b) \quad F_\pm = \sigma_\pm \exp(\sqrt{\lambda} \cdot x_\pm) + \delta_\pm \exp(-\sqrt{\lambda} \cdot x_\pm).\]

Corresponding potentials can be found according to equation (218), being similar to ones obtained in [155] in quite a different approach.

(3) Let us start now from the general solution of (215),

\[ F = L \left( \int \frac{dx_+}{C_+} - \int \frac{dx_-}{C_-} \right) / (C_+ C_-). \] (219)

Then equation (216) gives the functional-differential equation for the functional \( L(A_+ - A_-) \) with \( A_\pm \equiv 1/C_\pm(x_\pm) \),

\[ \left( \frac{A''_+}{A'_+} - \frac{A''_-}{A'_-} \right) L(A_+ - A_-) + 3(A''_+ + A''_-) L'(A_+ - A_-) + (A'^2_+ - A'^2_-) L''(A_+ - A_-) = 0, \] (220)

where \( L' \) denotes the derivative of \( L \) with respect to its argument. If we take functions \( A_\pm \) such that \( A''_\pm = \lambda^2 A_\pm, \lambda = \text{const.} \), equation (220) will become an ordinary differential equation for \( L \) with an independent variable \( (A_+ - A_-) \). It can be solved easily,

\[ L(A_+ - A_-) = \alpha (A_+ - A_-)^{-2} + \beta, \]

where \( A_\pm = \sigma_\pm \exp(\lambda x_\pm) + \delta_\pm \exp(-\lambda x_\pm) \) with \( \sigma_+ \cdot \delta_+ = \sigma_- \cdot \delta_- \) and \( \alpha, \beta - \text{real constants.} \)

For \( \lambda^2 > 0 \), choosing \( \sigma_\pm = -\delta_\pm = k/2 \) or \( \sigma_\pm = +\delta_\pm = k/2 \), we obtain (up to an arbitrary shift in \( x_\pm \)) two particular solutions,

\[(3a) \quad A_\pm = k \sinh(\lambda x_\pm), \quad (3b) \quad A_\pm = k \cosh(\lambda x_\pm).\]

Then (219) leads to,

\[(3a) \quad F_1(2x) = \frac{k_1}{\cosh^2(\lambda x) + k_2 \cosh(2\lambda x)}; \quad F_2(2x) = \frac{k_1}{\sinh^2(\lambda x)} + k_2 \cosh(2\lambda x); \quad C_\pm = \frac{k}{\cosh(\lambda x_\pm)}, \quad k \neq 0, \] (221)

\[(3b) \quad F_1(2x) = -F_2(2x) = \frac{k_1}{\sinh^2(\lambda x)} + k_2 \sinh(2\lambda x); \quad C_\pm = \frac{k}{\sinh(\lambda x_\pm)}, \quad k \neq 0. \] (222)

For \( \lambda^2 < 0 \) hyperbolic functions must be substituted by trigonometric ones.

We have to remark that the case \( \lambda^2 = 0 \), i.e. \( A''_\pm = 0 \), is not of interest, leading to trivial superpartners. However, choosing in (222) \( \lambda \to 0, k, k_1, k_2 \to 0 \) simultaneously, so that \( \lambda^2 \sim k_1 \sim k^{-1}_2 \sim k^2 \), we obtain the solution,

\[ F_1(2x) = -F_2(2x) = k_1 x^{-2} + k_2 x^2; \quad C_\pm = \frac{k}{x_\pm}. \] (223)

One can check that (215) is also satisfied by

\[ F_1(2x) = -F_2(2x) = k_1 x^2 + k_2 x^4; \quad C_\pm = \pm \frac{k}{x_\pm}. \] (224)
(4) Starting again from (219), it is convenient to pass on to new variable functions $C_{\pm} \equiv \pm f_{\pm}/f_{\pm}$. Then $F$ in (219) is represented in the form $F = U(f_{+}f_{-})f_{+}f_{-}$ with an arbitrary\(^5\) function $U$. After substitution in (216) one obtains the functional-differential equation,

$$\left(f_{+}^{2}f_{-}^{2} - f_{+}^{2}f_{-}^{2}\right)U''(f) + 3f\left(\frac{f_{+}'}{f_{+}} - \frac{f_{-}'}{f_{-}}\right)U'(f) + \left(\frac{f_{+}''}{f_{+}} - \frac{f_{-}''}{f_{-}}\right)U(f) = 0, \quad f = f_{+}f_{-}. $$

For particular form of functions $f_{\pm} = \alpha_{\pm}\exp(\lambda x_{\pm}) + \beta_{\pm}\exp(-\lambda x_{\pm})$, this equation becomes an ordinary differential equation for $U$ with independent variable $f$. Its solution is $U = a + 4bf_{+}f_{-}$ (a, b—real constants). Then functions

$$F_{1}(x) = k_{1}(\alpha_{+}\alpha_{-}\exp(\lambda x) + \beta_{+}\beta_{-}\exp(-\lambda x)) + k_{2}(\alpha_{+}^{2}\alpha_{-}^{2}\exp(2\lambda x) + \beta_{+}^{2}\beta_{-}^{2}\exp(-2\lambda x)),$$

$$-F_{2}(x) = k_{1}(\alpha_{+}\beta_{-}\exp(\lambda x) + \beta_{+}\alpha_{-}\exp(-\lambda x)) + k_{2}(\alpha_{+}^{2}\beta_{-}^{2}\exp(2\lambda x) + \beta_{+}^{2}\alpha_{-}^{2}\exp(-2\lambda x)).$$

$$C_{\pm} = \pm \frac{\alpha_{\pm}\exp(\lambda x_{\pm}) + \beta_{\pm}\exp(-\lambda x_{\pm})}{\lambda(\alpha_{\pm}\exp(\lambda x_{\pm}) - \beta_{\pm}\exp(-\lambda x_{\pm}))} \quad (225)$$

(with $k_{1} \equiv a\lambda^{2}$, $k_{2} \equiv 4b\lambda^{2}$) are real solutions of (215), (216), if $\alpha_{\pm}, \beta_{\pm}$ are real for $\lambda^{2} > 0$, and $\alpha_{\pm} = \beta_{\pm}$ for $\lambda^{2} < 0$.

(5) To find the next class of solutions, it is useful to rewrite (215) in terms of $x_{1,2}$,

$$(F_{1}(2x_{1}) + F_{2}(2x_{1}))\partial_{1}(C_{+} + C_{-}) + F_{1}(2x_{1})(C_{+} + C_{-}) + F_{2}(2x_{2})(C_{+} - C_{-}) = 0. $$

Among the known particular solutions the most compact one is,

$$C_{+}(x) = C_{-}(x) = ax^{2} + c, \quad F_{1}(2x_{1}) = 0, \quad F_{2}(2x_{2}) = \frac{b^{2}}{x_{2}^{2}}. \quad (226)$$

After inserting these solutions (221)–(226) into the general formulas (218), one obtains the analytical expressions for potentials. Their explicit form can be found in [91].

Two additional classes of particular solutions of the system (204)–(206) were obtained analogously by means of suitable ansatzes for the case of degenerate metrics $g_{\hat{a}} = \text{diag}(1,0)$ (see [91]) and the case of deformed hyperbolic metrics $g_{\hat{a}} = \text{diag}(1, -a^{2})$ with $a \neq 0$ (see [98]).

8.3. Integrability

Similarly to the case of one-dimensional polynomial SUSY QM (section 3), the partner two-dimensional Hamiltonians $h^{(0)}$ and $h^{(1)}$ can be considered as components of the diagonal super-Hamiltonian $H$, which are intertwined by components $q^{\pm}$ of the off-diagonal supercharges $Q^{\pm}$. Thus, the operators $H$, $Q^{\pm}$ constitute the deformed SUSY algebra: both supercharges are nilpotent operators and they commute with the super-Hamiltonian. The difference is in the third relation of SUSY algebra: while in the one-dimensional case, the anticommutator of $Q^{\pm}$ produced the second-order polynomial of $H$, in the two-dimensional situation it gives some fourth-order diagonal operator [88–90],

$$\{Q^{+}, Q^{-}\} = \hat{R}; \quad \hat{R} = \begin{pmatrix} R^{(0)} & 0 \\ 0 & R^{(1)} \end{pmatrix} = \begin{pmatrix} q^{+}q^{-} & 0 \\ 0 & q^{+}q^{-} \end{pmatrix}. \quad (227)$$

\(^5\) Due to equation (216), the function $F$ should be additionally representable in the form $F = F_{1}(2x_{1}) + F_{2}(2x_{2})$. 
In general, $R^{(0)}$, $R^{(1)}$ are not expressed in terms of $h^{(0)}$, $h^{(1)}$ in this case, but nevertheless they are related. Indeed, as it follows directly from the intertwining relations (197), they are the fourth order symmetry operators of corresponding systems,

$$[h^{(0)}, R^{(0)}] = [h^{(1)}, R^{(1)}] = 0. \quad (228)$$

In general, these symmetry operators might be expressed as functions of $h^{(i)}$, $i = 0, 1$, similarly to the one-dimensional case. But investigation [89, 91, 90] shows that this is not the case: two opportunities can be realized depending on the metrics $g_{ik}$ in the supercharges.

1. For the case of elliptic (Laplacian) metrics $g_{ik} = \delta_{ik}$ (and only for this one), the fourth order operators $R^{(i)}$, $i = 0, 1$, are expressed in terms of Hamiltonians $h^{(i)}$ and the second order operators $\bar{R}^{(i)}$, which also commute with $h^{(i)}$,

$$R^{(i)} = (h^{(i)})^2 + \eta h^{(i)} + \bar{R}^{(i)}, \quad [h^{(i)}, \bar{R}^{(i)}] = 0 \quad (229)$$

with $\eta = \text{const.}$ The explicit forms of differential operators $R^{(i)}$ and of corresponding Hamiltonians $h^{(i)}$ depend on the values of constants in (207); they are given in [89, 90]. It was shown that these Hamiltonians allow the standard procedure of $R$-separation of variables [153] in polar, elliptic and parabolic coordinates.

2. For all other allowed metrics (199), the fourth-order operators $R^{(i)}$, $i = 0, 1$ cannot be reduced to any symmetry operators of a lower order. This was shown in the general form in [89]; it follows also from the results of Eisenhart [156], where the exhaustive list of systems, which are amenable to a separation of variables, was given (just in Cartesian, polar, elliptic and parabolic coordinates).

Thus, for all possible metrics (199) each member of the variety of two-dimensional Hamiltonians $h^{(i)}$, which satisfy the intertwining relations (197), has the symmetry operator $R^{(i)}$, and therefore is completely integrable. In general, this fact does not provide the solvability of the models, but some of them are partially or completely solvable (see the next section).

In papers [90, 91], the classical limit of polynomial two-dimensional SUSY QM was considered. In particular, the prescription how to look for the integrals of motion (classical analogues of symmetry operators) of the fourth order in momenta for a given classical Hamiltonian was formulated. Namely, it is necessary to find such complex classical functions $q_{cl}^+(\vec{x}, \vec{p}) = (q_{cl}^-)(\vec{x}, \vec{p})^\dagger$ in the phase space, that they satisfy the following relations:

$$\{q_{cl}^+, h_{cl}\} = i f(\vec{x}) q_{cl}^{\dagger}, \quad \{q_{cl}^-, h_{cl}\} = -i f(\vec{x}) q_{cl}^\dagger, \quad (230)$$

where $\{\cdot, \cdot\}$ are Poisson brackets, and $f(\vec{x})$ is an arbitrary real function of $\vec{x}$. If such a real function exists, the classical Hamiltonian $h_{cl}$ obeys the factorizable integral of motion $I_{cl} = q_{cl}^+ q_{cl}^\dagger$ of the fourth order in momenta,

$$[h_{cl}, I_{cl}] = 0. \quad (231)$$

We see that the identification of a classical integral of motion from the quantum SUSY algebra is unambiguous. However, the opposite is not always true: a quantum anomaly may arise [106].

9. SUSY separation of variables for $d = 2$

The idea of shape invariance was very productive in one-dimensional SUSY QM (see section 6 and references therein). In this section, the two-dimensional analogue of that shape invariance will be exploited effectively [92, 94, 13, 95, 100, 16, 102–105] for two-dimensional Hamiltonians which satisfy the SUSY intertwining relations (197). In contrast to the one-dimensional case, there are essential differences in $d = 2$: in general, the ground state of
$h^{(1)}(a)$ does not coincide with zero modes of the second order supercharge $q^+(a)$, the value $E_0(a) \neq 0$, also many zero modes of $q^+$ exist. Up to these crucial differences, all other actions can be repeated. It was shown [92, 94, 13, 105] that such a direct generalization of the procedure above may provide only a partial (quasi-exact) solvability of the model (see the subsection 5.2); but for some models [100, 102, 16, 104, 106, 115] it will help to solve the model completely (see subsection 5.3).

The idea to consider the quasi-exactly-solvable models—the intermediate class between exactly solvable and unsolvable analytically models—was introduced in the 1980s in [157–163]. In particular, the series of papers by Turbiner, Ushveridze and Shifman was devoted to the elegant algebraic method of construction of one-dimensional quasi-exactly-solvable (and sometimes, of exactly solvable) quantum models. In general, this method works beyond the supersymmetry, but both approaches can be combined in the one-dimensional case as in [164]. This approach is applicable to two-dimensional problems as well, but only in curved spaces with nontrivial metrics [161].

The realization of such opportunities to solve non-trivial two-dimensional models partially, or even completely, would be very important since only one regular method of analytically solving the Schrödinger equation for two-dimensional models is known: a reduction to a pair of one-dimensional problems by means of the procedure of the separation of variables [153]. This method can be used for a very restrictive class of models; full classification of the models which allowed a separation of variables was given by Eisenhart [156]: four possibilities exist—Cartesian, polar, elliptic and parabolic coordinates. The general form of potentials amenable to a separation of variables is also known explicitly up to arbitrary functions of one variable. An analytical solution is possible only if these functions belong to the list of exactly solvable one-dimensional potentials. All these Hamiltonians $H$ are integrable: the symmetry operator $R$ of the second order in derivatives (in momenta) exists, $[H, R] = 0$. Besides models amenable to the separation of variables, the class of so-called Calogero-like models [165, 18] is known as well. They describe the specific forms of the pairwise interaction of $N$ particles on a line; they are solvable by means of a special transformation of variables which leads to a separation of variables. Recently, new classes of solvable two-dimensional models were built in [166–168], but all these models are superintegrable and amenable to a separation of variables.

In the two following subsections, we shall present two special procedures of SUSY-separation of variables in two-dimensional models, which do not allow the use of a conventional separation of variables. Subsection 9.2 presents the first procedure of SUSY separation of variables, where variables are separated in the supercharge. It leads to QES models; the specific model of two-dimensional Morse potential illustrates this method. In subsection 9.3, the second procedure of SUSY separation of variables is given where variables are separated in one of the partner Hamiltonians. In the case of the same Morse model, but with particular values of the parameter, it allows one to solve the model completely, i.e. to find analytically the whole spectrum and all wave functions.

### 9.1. SUSY-separation of variables I: QES models

The first variant of SUSY-separation of variables is realized when the Hamiltonians $h^{(0)}, h^{(1)}$ do not allow a standard separation of variables, but the supercharge $q^+$ does allow [92, 13, 94, 97, 99, 16, 105]. The general scheme is the following. Let us suppose that we know the zero modes of $q^+$,

$$q^+\Omega_n(x) = 0; \quad n = 0, 1, ..., N; \quad q^+\Phi(x) = 0.$$
Due to intertwining relations (197), the Hamiltonian $h^{(1)}$ obeys the important property: the space of zero modes $\Omega_1$ is closed under the action of $h^{(1)}$,

$$h^{(1)}\tilde{\Omega}(\vec{x}) = \tilde{\Omega}(\vec{x})$$

with constant matrix $\tilde{\mathcal{C}}$. If this matrix is known, and if it can be diagonalized,

$$\tilde{C}\tilde{\Omega}(\vec{x}) = \Lambda \tilde{\Omega}(\vec{x}),$$

the eigenvalues of $h^{(1)}$ can be found algebraically,

$$h^{(1)}(\tilde{B}\tilde{\Omega}(\vec{x})) = \Lambda(\tilde{B}\tilde{\Omega}(\vec{x})).$$

Thus, to realize this scheme of construction of some number of energy values and the corresponding wave functions, we need:

1. to find zero modes $\Omega_1(\vec{x})$;
2. to find constant matrix $B$, such that $\tilde{B}\tilde{C} = \Lambda \tilde{B}$.

As for the first step, the zero modes can be obtained by using the special similarity transformation (not unitary!), which removes the terms linear in derivatives from $q^+$,

$$\tilde{q}^+ = e^{-\chi(\vec{C})}q^+e^{\chi(\vec{C})} = \partial_1^2 - \partial_2^2 + \frac{1}{4}(F_1(2x_1) + F_2(2x_2));$$

$$\chi(\vec{x}) = \frac{1}{4} \left( \int C_+(x_+)dx_+ + \int C_-(x_-)dx_- \right).$$

Now, $\tilde{q}^+$ allows a separation of variables for an arbitrary solution of intertwining relations, and we obtain the first variant of the new procedure—SUSY-separation of variables. Similarly to the conventional separation of variables, separation of variables in the operator $\tilde{q}^+$ itself does not guarantee solvability of the problem.

The next task is to solve two one-dimensional problems,

$$\begin{align*}
(-\partial_1^2 - \frac{1}{4}F_1(2x_1))\eta_n(x_1) &= \epsilon_n \eta_n(x_1); \\
(-\partial_2^2 + \frac{1}{4}F_2(2x_2))\rho_n(x_2) &= \epsilon_n \rho_n(x_2).
\end{align*}$$

Three remarks are appropriate now.

**Remark 1.** The same similarity transformation of $h^{(1)}$ does not lead to an operator amenable to the separation of variables.

**Remark 2.** The normalizability of $\Omega_n$ has to be studied attentively due to the non-unitarity of the similarity transformation.

**Remark 3.** We have no reason to expect exact solvability of the model, but quasi-exact-solvability can be predicted.

As for the matrix $\tilde{B}$, it must be found by some specific procedure. Such a procedure was used in the example presented below.

In principle, the first scheme of the SUSY-separation of variables can be used for arbitrary models satisfying intertwining relations by supercharges with Lorentz metrics. The list of solutions of intertwining relations is already rather long (see subsection 8.2.2), and it may increase in future. The main obstacle is the analytical solvability of one-dimensional equations, obtained after a separation of the variables in the operator $\tilde{q}^+$. 
Below we describe briefly such a model, which can be considered as the generalized two-dimensional Morse potential,
\[ C_+ = 4a \alpha; \quad C_- = 4a \alpha \cdot \text{coth} \frac{\alpha x}{2}; \]
\[ f_i(x_i) = \frac{1}{4} F_i(2x_i) = -A(e^{-2ax_i} - 2e^{-\alpha x_i}); \quad i = 1, 2, \]
\[ V^{(0),(1)} = \alpha^2 a (2a \mp 1) \sinh^{-1} \left( \frac{\alpha x}{2} \right) + 4a^2 \alpha^2 + A[e^{-2ax_1} - 2e^{-\alpha x_1} + e^{-2ax_2} - 2e^{-\alpha x_2}]. \]  
(232)

where \( \Lambda > 0, \alpha > 0, a \)—real.

To explain the name, we present the potentials in the form,
\[ V(\vec{x}) = V_{\text{Morse}}(x_1) + V_{\text{Morse}}(x_2) + v(x_1, x_2), \]
where the first two terms are just one-dimensional Morse potentials and the last term mixes variables \( x_1, x_2 \).

The solutions of one-dimensional Schrödinger equations with Morse potentials \( V_{\text{Morse}}(x) \) are well known [169]; the zero modes can be written [92, 13] as,
\[ \Omega_n(\vec{x}) = \left( \frac{\alpha}{\sqrt{A}} \right)^n \exp \left( -\frac{\xi_1 + \xi_2}{2} \right) (\xi_1, \xi_2)^n \cdot F(-n, 2s_n + 1; \xi_1, \xi_2); \]
\[ \xi_i = 2\sqrt{\alpha} \exp(-\alpha x_i); \quad s_n = \frac{\sqrt{A}}{\alpha} - n - \frac{1}{2} > 0. \]

The conditions of normalizability and of an absence of the ‘fall to the center’ are,
\[ a \in (-\infty, -\frac{1}{4} - \frac{1}{4\sqrt{2}}); \quad s_n = \frac{\sqrt{A}}{\alpha} - n - \frac{1}{2} > -2a > 0 \]

To obtain the matrix \( \hat{C} \) explicitly, one must act by \( h^{(1)} \) on \( \Omega_n \). The matrix turns out to be triangular, and therefore, the energy eigenvalues coincide with its diagonal elements,
\[ E_k = c_{kk} = -2(2a^2 s_k - \epsilon_k). \]

To find a variety of wave functions is a more difficult task. For that it is necessary to find all elements of \( \hat{C} \) and all elements of matrix \( \hat{B} \). The recurrent procedure for the case of a two-dimensional Morse potential was given in [92, 13]. This variety can be enlarged by means of the shape invariance property (128) of the model, which can be easily checked. Similarly to one-dimensional shape invariance, each wave function constructed by a SUSY-separation of variables leads to a set of additional wave functions, which can be written as,
\[ h^{(0)}(a) = \left[ q^{-}(a)q^{-}(a - \frac{1}{2}) \cdots q^{-}(a - \frac{M - 1}{2}) \Psi \left( a - \frac{M}{2} \right) \right] \]
\[ = \left( E_0 \left( a - \frac{M}{2} \right) + \mathcal{R} \left( a - \frac{M - 1}{2} \right) + \cdots + \mathcal{R}(a) \right) \]
\[ \times \left[ q^{-}(a)q^{-}(a - \frac{1}{2}) \cdots q^{-}(a - \frac{M - 1}{2}) \Psi \left( a - \frac{M}{2} \right) \right] \]
with integer \( M \). An analogous approach works for the two-dimensional generalization of the Pöschl–Teller model [94], for some two-dimensional periodic potentials [99] and for the two-dimensional generalization of the Scarf II potential [105].
9.2. SUSY-separation of variables II: exact solvability

Among all known solutions of two-dimensional intertwining relations with second order supercharges a subclass exists \([100, 102, 96, 16, 104]\), where one of the intertwined Hamiltonians \(h^{(1)}\) is amenable to standard separation of the variables due to the specific choice of the parameters of the model. Its superpartner \(h^{(0)}\) still does not allow a conventional separation of variables.

The scheme will be described below for the same specific model—a two-dimensional generalization of the Morse potential,

\[
V^{(0),(1)} = \alpha^2 a(2a \mp 1) \sinh^{-2} \left(\frac{\alpha x}{2}\right) + 4\alpha^2 a^2 + A[e^{-2a x_1} - 2e^{-ax_1} + e^{-2ax_2} - 2e^{-ax_2}]
\]

Let us choose \(a_0 = -1/2\) in order to vanish the mixed term in \(V^{(1)}\), and therefore, the Hamiltonian \(h^{(1)}\) allows the conventional separation of variables. Moreover, after the separation of variables each of the obtained one-dimensional problems is again exactly solvable: they coincide with one-dimensional problems of the previous subsection, where they occurred in a different context (separation of variables in \(\hat{q}\)).

The discrete spectrum of this one-dimensional model is,

\[
\epsilon_n = -\alpha^2 x_n^2; \quad s_n \equiv \frac{\sqrt{A}}{\alpha} - n - \frac{1}{2} > 0; \quad n = 0, 1, 2, \ldots
\]

Wave functions are expressed in terms of confluent hypergeometric functions,

\[
\eta_n(x_i) = \exp\left(-\frac{\xi_i}{2}\right)(\xi_i)^s F(-n, 2s_n + 1; \xi_i); \quad \xi_i \equiv \frac{2\sqrt{A}}{\alpha}\exp(-\alpha x_i).
\]

Due to the separation of variables, the two-dimensional problem with \(h^{(1)}(\hat{x})\) is exactly solvable. Its energy eigenvalues are,

\[
E_{n,m} = E_{m,n} = \epsilon_n + \epsilon_m,
\]

being twofold degenerate for \(n \neq m\). The corresponding eigenfunctions can be chosen as symmetric or (for \(n \neq m\)) antisymmetric combinations,

\[
\Psi^{(1)SA}_{E_{n,m}}(\hat{x}) = \eta_n(x_1) \eta_m(x_2) \pm \eta_n(x_1) \eta_m(x_2).
\]

Our initial aim here is to completely solve the problem for \(h^{(0)}(\hat{x})\) with \(a_0 = -1/2\). The main tool is again the SUSY intertwining relations, i.e. the isospectrality of \(h^{(0)}\) and \(h^{(1)}\) but up to zero modes and singular properties of \(q^+\). In general, we may expect three kinds of levels of \(h^{(0)}(\hat{x})\):

(i) The levels, which coincide with \(E_{n,m}\). Their wave functions can be obtained from \(\Psi^{(1)}\) by means of \(q^+\).

(ii) The levels, which were absent in the spectrum of \(h^{(1)}(\hat{x})\), if some wave functions of \(h^{(0)}(\hat{x})\) are simultaneously the zero modes of \(q^-\). Then the second intertwining relation would not give any partner state among the bound states of \(h^{(1)}(\hat{x})\).

(iii) The levels, which were also absent in the spectrum of \(h^{(1)}(\hat{x})\), if some wave functions of \(h^{(0)}(\hat{x})\) become non-normalizable after the action of operator \(q^-\).

We have to analyze these three classes of possible bound states of \(h^{(0)}\) one after another.

(i) The first SUSY intertwining relation gives the twofold degenerate wave functions of \(h^{(0)}\) with energies \(E_{n,m}\). \(\Psi^{(0)}_{E_{n,m}} = q^+ \Psi^{(1)}_{E_{n,m}}\). But \(q^+\) includes singularity on the line \(x_1 = x_2\), therefore the normalizability of \(\Psi^{(0)}_{E_{n,m}}\) depends crucially on the behavior of \(\Psi^{(1)}_{E_{n,m}}\) on the line \(\xi_1 = \xi_2\). One can check that only antisymmetric functions \(\Psi^{(1)}\) survive, i.e. only
symmetric $\Psi^{(0)}$ survive. This fact can be demonstrated [100] both by direct calculation and by an indirect method - by means of the symmetry operator $R^{(0)}$.

The indirect method explores that the symmetry operator $R^{(0)} = q^{-}q^{+}$ for $a_{0} = -1/2$ can be written in terms of one-dimensional Morse parts $h_{1}(x_{1})$, $h_{2}(x_{2})$ of the Hamiltonian $h^{(1)} = h_{1}(x_{1}) + h_{2}(x_{2})$,

$$R^{(0)} = (h_{1}(x_{1}) - h_{2}(x_{2}))^{2} + 2a^{2}(h_{1}(x_{1}) + h_{2}(x_{2})) + a^{4}.$$ 

Therefore,

$$R^{(0)}\Psi_{E_{nm}}^{(0)A}(\vec{x}) = r_{n,m}\Psi_{E_{nm}}^{(A)}(\vec{x}); r_{n,m} = \alpha^{2}[(n-m)^{2} - 1][s_{n} + s_{m}]^{2} - 1,$$

and

$$\|\Psi_{E_{nm}}^{(1)S}\|^{2} = (\Psi_{E_{nm}}^{(0)A}|q^{-}q^{+}|\Psi_{E_{nm}}^{(0)A}) = r_{n,m}\|\Psi_{E_{nm}}^{(0)A}\|^{2}.$$ 

For $n = m$, wave functions $\Psi_{E_{nm}}^{(0)S}$ vanish identically by trivial reasons. It is clear now that wave functions $\Psi_{E_{nm+1}}^{(0)S}$ also vanish. For all other $n, m$, functions $\Psi_{E_{nm}}^{(0)S}$ have positive and finite norm and there is no degeneracy of these levels.

(ii) These possible bound states of $h^{(0)}$ are the normalizable zero modes of $q^{-}$. The variety of such zero modes is known from [92]: they exist only for positive values of $a$

$$a \in \left(\frac{1}{4} + \frac{1}{\sqrt{2}A}, +\infty\right),$$

which does not contain the value $a_{0} = -1/2$. Thus, no normalizable bound states of this class exist for $h^{(0)}$.

(iii) We have to study an opportunity that $q^{-}$ destroys the normalizability of some eigenfunctions of $h^{(0)}$. It could occur due to the singular character of $q^{-}$ at $x_{1} = x_{2}$. The analysis was performed [100] in suitable coordinates. It shows that $q^{-}$ is not able to transform the normalizable wave function to non-normalizable. Therefore, the third class of possible wave functions $h^{(0)}$ does not exist too.

Summing up, the spectrum of $h^{(0)}$ with $a_{0} = -1/2$ consists only of the bound states with energies $E_{nm}$ for $|n - m| > 1$. This spectrum is bounded from above by the condition of the positivity of $s_{n}, s_{m} : n, m < \sqrt{A}/\alpha - 1/2$. The corresponding wave functions are obtained analytically [100].

The results above can be expanded to the whole hierarchy of Morse potentials with $a_{k} = -(k + 1)/2$ with $k = 0, 1, ...$ by means of the shape invariance property. Let us denote elements of the hierarchy as $h^{(0)}(\vec{x}; a_{k})$, $h^{(1)}(\vec{x}; a_{k})$. All these Hamiltonians are also exactly solvable due to the shape invariance of the model,

$$h^{(0)}(\vec{x}; a_{k-1}) = h^{(1)}(\vec{x}; a_{k}); k = 1, 2, \ldots.$$ 

This means that the following chain (hierarchy) of Hamiltonians can be built,

$$h^{(1)}(\vec{x}; a_{0}) \div h^{(0)}(\vec{x}; a_{0}) = h^{(1)}(\vec{x}; a_{1}) \div h^{(0)}(\vec{x}; a_{1}) = \ldots \div h^{(1)}(\vec{x}; a_{k-1}) = h^{(0)}(\vec{x}; a_{k}) \div h^{(0)}(\vec{x}; a_{k}),$$

where the sign $\div$ denotes intertwining by $q^{+}(a_{i})$.

In the general case, the functions

$$\Psi_{E_{nm}}^{(k)}(\vec{x}; a_{k}) = q^{+}(a_{k})\Psi_{E_{nm}}^{(k-1)}(\vec{x}; a_{k}) = q^{+}(a_{k})q^{+}(a_{k-1}) \ldots q^{+}(a_{0})\Psi_{E_{nm}}^{(1)A}(\vec{x}; a_{0})$$

(if normalizable) are the wave functions of $h^{(0)}(\vec{x}; a_{k})$ with energies $E_{nm} = -\alpha^{2}(s_{n}^{2} + s_{m}^{2})$. The symmetries of the wave functions alternate and depend on the length of the chain. This is true, but up to zero modes of operators $q^{+}$. 

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It is necessary to keep under control the normalizability of $\Psi_1$ and zero modes of $q^+$. This control is performed algebraically by means of identity, which must be fulfilled up to a function of $H$,

$$R^{(1)}(a_k) = R^{(0)}(a_{k-1}).$$

Actually, the following equation holds,

$$q^- (a_k) q^+ (a_k) = q^+ (a_{k-1}) q^- (a_{k-1}) + a^2 (2k + 1) [2h^{(0)} (\vec{x}; a_k) + a^2 (2k^2 + 2k + 1)].$$

These relations allowed us to evaluate the norms of wave functions. The result is the following. The spectra of Hamiltonians $h^{(0)} (\vec{x}; a_k)$ are not degenerate. They consist of the bound states with energies $E_{n,m}$, with indices $|n - m| > k + 2$, and their wave functions $\Psi^{(0)}_{E_{n,m}} (\vec{x}; a_k)$ were given analytically above.

The procedure of this subsection was reproduced fully for two other two-dimensional models which are not amenable to standard separation: generalized Pöschl–Teller [102] and Scarf II [104] potentials. The full energy spectra and corresponding wave functions were also built analytically by the second variant of SUSY-separation of variables.

### 10. Perspectives, some applications and missing points

The purpose of this review has been to elucidate the recent progress in nonlinear SUSY realization by intertwining Darboux transformations for a community working on methods of isospectral design and their applications. We wanted also to draw attention to new (quasi-) exactly solvable potential systems in two-dimensions, non-separable by a conventional choice of coordinates. Summarizing our experience, we describe the general SUSY QM as governed by the extended nonlinear SUSY algebra with $N$ pairs of nilpotent supercharges $Q_j, \bar{Q}_j = Q_j^\dagger = \bar{Q}_j$, and a number of Hermitian hidden-symmetry differential operators $R_\alpha = R_\alpha^\dagger$, $[R_\alpha, R_\beta] = 0; \quad 0 \leq \alpha, \beta \leq M$.

Not all the research areas that are linked to the subject of this review have been covered in the main text. A number of extensions and applications of nonlinear SUSY QM have been found and yet there are still plenty of open questions and challenges that remain to be solved. Below we list some of these directions and mention a few missing topics.

- The method of multidimensional SUSY intertwining relations and shape invariance turned out to be useful for studying different variants of Calogero-like models of $N$ particles on a line, including the models with internal degrees of freedom and with pairwise interactions based on any root systems [108, 170–177] (see the recent review [18]).

- One-dimensional SUSY models with matrix superpotentials and supercharges [178–184] have been used either for a description of the motion of spin particles in external fields [69–71, 110, 114, 185–190] or to study scattering of particles with strong coupling of channels [185, 186, 191–194] with practical application of the results to specific processes or to investigate the spectral problem for the Bogoliubov–de Gennes system [196]. Such systems are not fully investigated in searching for extended SUSY systems with hidden symmetries. It is clear that a comprehensive understanding of the irreducible building blocks for spectral design analogous to the scalar case is not yet achieved. Also, matrix models with a higher dimensionality of space have to be studied. To start with, two-dimensional models with a matrix superpotential were studied in [195].

- A natural question appears in respect to the classical formulation of nonlinear SUSY whether a canonical quantization scheme similar to that of subsection 2.3 exists in this case. A corresponding pseudoclassical Lagrangian for the particular one-dimensional model was observed in [132] and investigated in a more general setting in [106, 253]. It was found
that quantization of one-dimensional nonlinear supersymmetry faces a problem of the quantum anomaly. Thus the problem arises to identify the quantum anomaly in the two- and higher dimensional case. On the other hand, the anomaly mentioned above originated from the noncommutativity of momenta and coordinates (ordering problem) which is unambiguously resolved in our recipe of supercharge identification and therefore does not affect any of the results presented in the review.

- Coherent states in the framework of SUSY QM models were built in [197–200], but only for one-dimensional systems [201].
- Formulation of multidimensional SUSY QM in arbitrary curvilinear coordinates was developed in [202] (see also [203–205]). This formalism could be useful for the investigation of SUSY design of different cosmological and brain-world models.
- Generalization of polynomial SUSY QM for higher (first of all, \( d = 3 \)) space dimensions seems to be very interesting (see [101]).
- Effective mass SUSY QM is a rather popular generalization of conventional SUSY QM for the Schrödinger equation with coordinate-dependent mass. This sort of model may have various applications in physics of solid state and nano devices as well as in cosmology. The well developed SUSY approach to effective mass QM is rather promising for the solution of different problems in physics. Some of suitable reference works are [206–212].
- The periodic potentials were studied actively in the framework of SUSY QM as well, see [213–222, 62] on one-dimensional models and [99] for two-dimensional quasi-exactly-solvable models with periodic potentials. Certainly, spectral design of periodic potential systems is promising, especially, in two-dimensional QM related to new nanoscale materials like graphene.
- The embedding of shape invariant potentials and their eigenfunctions into so-called discrete QM has been analyzed recently [223, 224].
- The generalizations of conventional SUSY QM to the models with complex potentials [93, 96, 225–227] provide an effective tool in the study of non-Hermitian [228–233] and, especially, PT symmetric [234–239] quantum mechanics with real energy spectrum (see reviews [240, 241]). Spectral design of non-Hermitian systems with non-diagonalizable Hamiltonians was investigated both in one-dimensional [242–244] and two-dimensional [245] cases. PT-symmetric periodic potentials with SUSY have been investigated in [246].
- The polynomial SUSY in two dimensions has already brought a number of examples of a new type of irreducible SUSY with hidden symmetries of a higher-order in derivatives [88, 89]. An interesting sample of nonlinear supersymmetry with holomorphic supercharges for a two-dimensional fermion in an external magnetic field was studied in [115]. One may expect a variety of new types of irreducible SUSY for third-order (and higher-order) supercharges as well as new discoveries in two and three dimensions.
- The perspective direction for investigations is the interrelations between the inverse scattering problem and methods of SUSY QM [247–252].
- Hidden and exotic nonlinear supersymmetries have been found in some models as, for instance, in pure parabosonic systems [132] and kink–antikink crystal [254]. There are also some connections of nonlinear SUSY with conformal SUSY mechanics [255].
The SUSY QM approach happens to be applicable and useful for building different brain-world models with extra space dimensions [256–259].

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