AUGMENTATIONS ARE SHEAVES FOR LEGENDRIAN GRAPHS

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Abstract. In this article, associated to a (bordered) Legendrian graph, we study and show the equivalence between two categorical Legendrian isotopy invariants: the augmentation category, a unital \( A_\infty \)-category, which lifts the set of augmentations of the associated Chekanov-Eliashberg DGA, and a DG category of constructible sheaves on the front plane, with micro-support at contact infinity controlled by the (bordered) Legendrian graph. In other words, generalizing [21], we prove “augmentations are sheaves” in the singular case.

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1. Introduction

Nowadays, it has been an increasingly rich subject, to study the microlocal sheaf theoretic aspects of symplectic topology, or vice versa. This, has already been of interest in the case of a cotangent bundle \( T^*M \). Along this direction, one fundamental result is the microlocalization equivalence [19, 20], between the DG category of constructible sheaves on the base \( M \) and the infinitesimally wrapped Fukaya category of the cotangent bundle \( T^*M \):

\[
\text{NZ} : \mathcal{Sh}(M; \mathbb{K}) \cong \mathcal{Fuk}^\epsilon(T^*M; \mathbb{K}).
\]

A more refined version of the correspondence involves introducing a Legendrian \( \Lambda \) contained in \( T^{\infty\infty}M \). On one side, replace \( \mathcal{Sh}(M; \mathbb{K}) \) with the co-sphere bundle (equivalently, the set of points at contact infinity) of \( M \) due to the

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by a full subcategory $\mathcal{S}_\Lambda(M; \mathbb{K})$ of constructible sheaves with micro-support at infinity contained in $\Lambda$. On the other side, replace $\mathcal{F}\text{uk}^\bullet(T^*M; \mathbb{K})$ by a full subcategory of Lagrangian branes asymptotic to $\Lambda$ at infinity. Then, the Nadler-Zaslow correspondence produces an equivalence:

$$\text{NZ} : \mathcal{S}_\Lambda(M; \mathbb{K}) \rightarrow \mathcal{F}\text{uk}^\bullet(T^*M; \mathbb{K})$$

Here, both sides are categorical Legendrian isotopy invariants for $\Lambda \subset T^\infty M$. More recently, a correspondence [12, 14, 13] has also been established, between the DG category $\mathcal{S}_\Lambda(M)\mathcal{C}$ of compact objects in the larger category of (weakly) constructible sheaves on $M$, with micro-support contained in $\Lambda$, and a wrapped Fukaya category $\text{Perf} \mathcal{W}(T^*M, \Lambda)^{\text{op}}$ of $T^*M$, whose Lagrangians are disjoint from $\Lambda$ at infinity:

$$\text{GPS} : \mathcal{S}_\Lambda(M)\mathcal{C} \approx \text{Perf} \mathcal{W}(T^*M, \Lambda)^{\text{op}}.$$ 

Notice that the latter is of central interest in Homological Mirror Symmetry.

On the other hand, there is another powerful modern Legendrian isotopy invariant for a Legendrian submanifold $\Lambda$ in any contact manifold $V$. It is called the Legendrian contact homology DGA $A^{\text{CE}}(V, \Lambda)$, obtained by counting of holomorphic disks. More specifically, the generators of this algebra are the Reeb chords of $\Lambda$, and the differential counts holomorphic disks with punctures on the boundary, in the symplectization $\mathbb{R} \times V$, with boundary living on the Lagrangian $\mathbb{R} \times \Lambda$ and approaching the Reeb chords at the punctures. The DGA $A^{\text{CE}}(V, \Lambda)$ is an extended isotopy invariant up to homotopy equivalence.

Motivated by the Nadler-Zaslow correspondence, it is expected that certain representation category of $A^{\text{CE}}(\Lambda)$ is equivalent to the sheaf category $\mathcal{S}_\Lambda(N \times \mathbb{R}_z; \mathbb{K})$, when $V = J^1 N \equiv T^\infty (N \times \mathbb{R}_z)$ is a one-jet bundle naturally identified with an open contact submanifold of $T^\infty (M = N \times \mathbb{R}_z)$.

In this article, the case of the major interest to us is when $N = \mathbb{R}_x$, in which there is a rich interaction with Legendrian knot theory. That is, $\Lambda \subset J^1 \mathbb{R}_x = \mathbb{R}^3_{\text{std}}$ with a Maslov potential $\mu$ is now a Legendrian knot in the standard contact three space. In this case, “augmentations are sheaves” holds.

**Theorem 1.1** ([21]). There is an $A_{\infty}$-equivalence:

$$\text{NRSSZ} : \text{Aug}_+(\Lambda, \mu; \mathbb{K}) \rightarrow C_1(\Lambda, \mu; \mathbb{K})$$

Here, $\text{Aug}_+(\Lambda, \mu; \mathbb{K})$ is a unital $A_{\infty}$-category whose objects are the set of augmentations of the DGA $A^{\text{CE}}(\Lambda, \mu)$, a representation category of rank 1 representations of $A^{\text{CE}}(\Lambda, \mu)$, and $C_1(\Lambda, \mu; \mathbb{K})$ is the full subcategory of $\mathcal{S}_\Lambda(\mathbb{R}^2_{x,z}; \mathbb{K})$ whose objects are microlocal rank 1 sheaves with acyclic stalks when $z \ll 0$. Concerning “representations are sheaves”, or the correspondence between higher rank representations and higher microlocal rank sheaves, an equivalence in the cohomological level [8] was shown for $(2, m)$-torus knots. In the case when $M$ is of higher dimension, the correspondence in question is widely open. However, see [7, 6, 23, 24] and [13, Sec.6.4] for related results along this direction.

Our input originates from a following speculation in the perspective of microlocal sheaf theory: given a constructible sheaf $\mathcal{F}$ on $M = \mathbb{R}^2_{x,z}$, the micro-support of $\mathcal{F}$ at infinity is in general a singular Legendrian, typically a Legendrian graph in $T^\infty M$. So it is natural to seek a contact-geometric interpretation of the sheaf category $\mathcal{S}_\Lambda(M; \mathbb{K})$ when $\Lambda$ is a Legendrian graph. More generally, it is natural to ask the same question with essentially no more difficulty, when $N = I_x \subset \mathbb{R}_x$ is an open interval, and $T \subset J^1 N$ is a bordered Legendrian graph. Our main result gives a positive answer to this question.

**Theorem 1.2** (Theorems 4.19, 4.21 and 6.26). Let $(T, \mu)$ be a bordered Legendrian graph $T := (T_L \rightarrow T \leftarrow T_R)$ in $J^1 I_x \equiv T^\infty (I_x \times \mathbb{R}_z)$ with a Maslov potential $\mu = (\mu_L, \mu_R)$. There is a well-defined diagram of unital $A_{\infty}$-categories:

$$\text{Aug}_+(T, \mu; \mathbb{K}) := (\text{Aug}_+(T_L, \mu_L; \mathbb{K}) \leftarrow \text{Aug}_+(T, \mu; \mathbb{K}) \rightarrow \text{Aug}_+(T_R, \mu_R; \mathbb{K}))$$

where the objects of $\text{Aug}_+(T, \mu; \mathbb{K})$ are the augmentations of the LCH DGA $A^{\text{CE}}(T, \mu)$ defined as in [1]. The diagram is invariant under Legendrian isotopy and basepoint moves up to $A_{\infty}$-equivalence. Moreover, there is an $A_{\infty}$-equivalence between two diagrams of unital $A_{\infty}$-categories:

$$\text{Aug}_+(T, \mu; \mathbb{K}) \rightarrow C_1(T, \mu; \mathbb{K}),$$

where

$$C_1(T, \mu; \mathbb{K}) := (C_1(T_L, \mu_L; \mathbb{K}) \leftarrow C_1(T, \mu; \mathbb{K}) \rightarrow C_1(T_R, \mu_R; \mathbb{K}))$$

is a diagram of DG categories of constructible sheaves.
The DG category $C_l(T, \mu; \mathbb{K})$ is the full subcategory of $\text{Sh}_T(I_x \times \mathbb{R}_z; \mathbb{K})$ whose objects are microlocal rank 1 sheaves with acyclic stalks for $z \ll 0$. In particular, when $T = \Lambda \subset T^\infty - \mathbb{R}^2_z$, is a Legendrian graph, we get an $A_{\infty}$-equivalence $\text{Aug}_{\ast}(\Lambda, \mu; \mathbb{K}) \to C_l(\Lambda, \mu; \mathbb{K})$, i.e. “augmentations are sheaves” holds in the singular case.

As a consequence of Theorem 1.2, up to a normalization, the point-counting of sheaves in $C_l(T, \mu; \mathbb{K})$ (with boundary conditions) over a finite field $\mathbb{K} = \mathbb{F}_q$, is equivalent to that of augmentations in $\text{Aug}_{\ast}(T, \mu; \mathbb{K})$ (with boundary conditions), called augmentation number and denoted by $\text{aug}(T, \mu; \rho_L, \rho_R; \mathbb{F}_q)$. Generalizing the results in [16, 26], our preceding paper [3] solves this counting problem. More precisely, augmentation numbers are computed by ruling polynomials of $T$, defined via the combinatorics of decompositions of the front projection $T$:

**Theorem 1.3.** [3] Let $(T, \mu)$ be as above. Let $\rho_L \in \text{NR}(T_L, \mu_L)$ and $\rho_R \in \text{NR}(T_R, \mu_R)$ be two boundary conditions (i.e. normal rulings). Then the following two Legendrian isotopy invariants are the same:

$$\text{aug}(T, \mu; \rho_L, \rho_R; \mathbb{F}_q) = q^{-\max deg} z^h \rho(L(T, \mu; q, z)|\rho_R)$$

Here, $\langle \rho_L | R(T, \mu; q, z)\rho_R \rangle \in \mathbb{Z}[z^{\pm 1}]$ is the ruling polynomial for $(T, \mu)$ with boundary conditions $(\rho_L, \rho_R)$, $d := \max \text{deg}_{z}(\rho_L | R(T, \mu; z^\pm, z)|\rho_R)$. In the formula, we take $z = q^\frac{1}{2} - q^{-\frac{1}{2}}$, and

$$B := B + \sum_{v \in V(T)} \frac{\text{val}(v)}{2}$$

counts the number of “generalized” basepoints in $T$.

Moreover, the ruling polynomials satisfy the gluing property: If $(T, \mu) = (T^1, \mu^1) \circ (T^2, \mu^2)$ is a composition of two bordered Legendrian graphs, that is, $(T^1_L, \mu^1_L) = (T^2_L, \mu^2_L)$, then

$$\langle \rho_L | R(T, \mu; q, z)\rho_R \rangle = \sum_{\rho_1 \in \text{NR}(T^1_L, \mu^1_L)} \langle \rho_L | R(T^1, \mu^1; q, z)\rho_1 \rangle \langle \rho_1 | R(T^2, \mu^2; q, z)\rho_R \rangle.$$

**Organization.** The article is organized as follows. In Section 2, we review the basic backgrounds from [3] on Legendrian graphs and bordered Legendrian graphs $(T, \mu)$ with a Maslov potential. The main ingredients include the Legendrian contact homology (LCH) DGAs $A^{CSE}(T, \mu) = (A^{CSE}(T_L, \mu_L) \to A^{CSE}(T_R, \mu_R))$ for bordered Legendrian graphs.

In Section 3, we give the algebraic preliminaries before defining augmentation categories. In particular, we introduce two categories of consistent sequences: consistent sequences of bordered Legendrian graphs, and consistent sequences of DGAs. In Section 4, we define augmentation categories for bordered Legendrian graphs, and then show their unitarity and invariance.

In Section 5, we firstly give the preliminaries on the microlocal theory of sheaves. Then we construct the necessary combinatorial tools, which we call legible models for $\text{Sh}(T; \mathbb{K}) = (\text{Sh}(T_L; \mathbb{K}) \leftarrow \text{Sh}(T; \mathbb{K}) \to \text{Sh}(T_R; \mathbb{K}))$, the diagram of sheaf categories for a bordered Legendrian graph $T$ in $T^\infty - (I_x \times \mathbb{R}_z)$. As an application, we prove the invariance of $\text{Sh}(T; \mathbb{K})$ via combinatorics, hence the invariance of $C_l(T, \mu; \mathbb{K})$, the diagram of full-subcategories of $\text{Sh}(T; \mathbb{K})$ whose objects are microlocal rank 1 objects with acyclic stalks for $z \ll 0$.

In Section 6, we prove our main result “augmentations are sheaves” for bordered Legendrian graphs. The basic idea is as follows: By the invariance results in Section 4 and Section 5, we can assume the vertices in the front projection $T$ are all of type $(0, r)$ for some $r$, all the left cusps and vertices are to the left of the crossings of $T$, and all the right cusps are to the right of the crossings of $T$. Moreover, we can assume all right cusps are marked. Then both of the two diagrams of $A_{\infty}$-categories satisfy the sheaf property over $I_x$. Hence, by decomposing the front diagram $T$ into the composition of elementary pieces, it suffices to show the theorem for each elementary piece. By the results for Legendrian knots in [21], it suffices to show the case of an elementary bordered Legendrian graph $(V, \mu)$ involving only a vertex. This is done by an explicit description for both of the two diagrams $\text{Aug}_{\ast}(V, \mu; \mathbb{K})$ and $C_l(V, \mu; \mathbb{K})$. The augmentation side is done in Lemma 6.14. The sheaf side is a direct application of the legible model in Section 5.3 for $C_l(V, \mu; \mathbb{K})$. Then we are done.
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2. Setup

Notation 2.1. For each $m \geq 0$, we denote the set $\{1, \cdots, m\}$ equipped with the natural order by $[m]$.

2.1. Bordered Legendrian graphs. In this section, we briefly review the definition of bordered Legendrian graphs defined in [3, §2]. A graph is a finite regular one dimensional CW complex, whose 0-cells and closed 1-cells are called vertices and edges. For each vertex $v$, a half-edge at $v$ is a small enough restriction of an edge adjacent to $v$. Then as usual, the valency of $v$ is the number of half-edges at $v$ and denoted by $\text{val}(v)$.

A (based) bordered graph $\Gamma = (V, V_L, V_R, B, E)$ of type $(n_L, n_R)$ consists of the following data:

- the pair $(V \sqcup V_L \sqcup V_R \sqcup B, E)$ defines a graph $|\Gamma|$, 
- each $b \in B$ of $|\Gamma|$ is bivalent, and 
- two disjoint subsets $V_L$ and $V_R$ consist of $n_L$ and $n_R$ univalent vertices of $|\Gamma|$.

Elements in $V, V_L, V_R, B$ and $E$ will be called vertices, left borders, right borders, basepoints and edges, respectively. The interior $\hat{\Gamma}$ of $\Gamma$ is defined to be the complement of $V_L \sqcup V_R$.

Notation 2.2. In order to emphasize the border structures, we will denote the bordered graph $\Gamma$ as

$$\Gamma = (\Gamma_L \rightarrow \Gamma \leftarrow \Gamma_R),$$

where $\Gamma_L$ and $\Gamma_R$ are defined by $V_L$ and $V_R$ by, respectively, and both arrows are inclusions.

From now on, we mean by a graph a bordered graph with empty borders ($\varnothing \rightarrow \Gamma \leftarrow \varnothing$), which will be denoted simply by $\Gamma$.

For a closed interval $U = [x_L, x_R] \subset \mathbb{R}$, let the bordered one-jet space $J^1U = (J^1U_L \rightarrow J^1U \leftarrow J^1U_R)$ be the one-jet space $J^1U := (U \times \mathbb{R}^{xyz}, dz - ydx) \subset J^1\mathbb{R}^x = (\mathbb{R}_x^{xyz}, dz - ydx)$ together with two contact submanifolds

$$J^1U_L := ((x_L) \times \mathbb{R}_x, dz) \quad \text{and} \quad J^1U_R := ((x_R) \times \mathbb{R}_x, dz).$$

Definition 2.3 (bordered Legendrian graphs). A bordered Legendrian graph $\mathcal{T} = (T_L \rightarrow T \leftarrow T_R)$ of a bordered graph $\Gamma$ of type $(n_L, n_R)$ in $J^1U$ is defined as an embedding $T : \Gamma \rightarrow J^1U$ such that

(1) $T$ is transverse to the boundary $\partial J^1U = \partial U \times \mathbb{R}^{xyz}$ and the restrictions on the interior $\hat{T}$ and both borders $\Gamma_L$ and $\Gamma_R$ are contained in $J^1\hat{U}, J^1U_L$ and $J^1U_R$, respectively.

$$T \cap \partial J^1U, \quad T_L := T(\Gamma_L) \subset J^1U_L, \quad T_R := T(\Gamma_R) \subset J^1U_R, \quad \hat{T} := T(\hat{\Gamma}) \subset J^1\hat{U}.$$

(2) $T$ on edges are smooth Legendrian with boundary and pairwise non-tangent to each other at all vertices and basepoints, in particular, two edges adjacent to each basepoint form a smooth arc.

By labeling borders in $T_L$ and $T_R$ in top-to-bottom ways with respect to $z$-coordinates, we identify the left and right border $T_L$ and $T_R$ with the set $\{n_L\} = \{1, \cdots, n_L\}$ and $\{n_R\} = \{1, \cdots, n_R\}$.

There are two projections for $J^1\mathbb{R}^x \cong \mathbb{R}^{xyz}_x$, called the front and Lagrangian projections $\pi_{fr} : \mathbb{R}^{xyz}_x \rightarrow \mathbb{R}^2_x$ and $\pi_{lag} : \mathbb{R}^{xyz}_x \rightarrow \mathbb{R}^2_{xy}$, respectively.

Definition 2.4 (Regular projections). For a bordered Legendrian graph $\mathcal{T} = (T_L \rightarrow T \leftarrow T_R)$, the front and Lagrangian projections $\mathcal{T} = (T_L \rightarrow T \leftarrow T_R) := \pi_{fr}(\mathcal{T})$ and $\mathcal{T}_{lag} = (T_{lag,L} \rightarrow T_{lag} \leftarrow T_{lag,R}) := \pi_{lag}(\mathcal{T})$ are said to be regular if in their interiors,

(1) there are only finitely many transverse double points, called crossings, and 
(2) no vertices, basepoints or x-extreme points are crossings, 
(3) each edge containing a x-maximal point must involve at least one vertex or a basepoint.
where a point in the interior \( \hat{t} \) is said to be \( x \)-**maximal** or \( x \)-**minimal** if it is maximal or minimal with respect to the \( x \)-coordinate, and \( x \)-**extreme** if it is either \( x \)-maximal or \( x \)-minimal.\(^1\)

A bordered Legendrian graph of type \((0,0)\) is called a **Legendrian graph** and we denote the sets of regular front and Lagrangian projections of non-bordered and bordered Legendrian graphs by \( \mathcal{L} \), \( \mathcal{L}_{\text{Lag}} \) and \( \mathcal{B} \mathcal{L} \mathcal{G}, \mathcal{B} \mathcal{L} \mathcal{G}_{\text{Lag}} \), respectively.

**Remark.** Due to the Legendrian property, there are no vertical tangencies and no non-transverse double points in the front projection. Instead, it contains **cusps**, which is obviously, \( x \)-extreme.

**Notation.** The front and Lagrangian projection of \( \Gamma = T(\Gamma) \) with \( \Gamma = (V, V_L, V_R, B, E) \) will be denoted by \( T = (V, V_L, V_R, B, E) \) and \( T_{\text{Lag}} = (V_{\text{Lag}}, V_{\text{Lag},L}, V_{\text{Lag},R}, B_{\text{Lag}}, E_{\text{Lag}}) \), respectively.

For examples of regular and non-regular projection, see Figure 1. To avoid any confusion, we denote vertices and basepoints by small dots and bars, respectively.

\[
\begin{array}{c}
\text{(a) Non-regular local front projections} \\
\text{(b) Non-regular local Lagrangian projections}
\end{array}
\]

\[
\begin{array}{c}
\text{(c) Regular front projections} \\
\text{(d) Non-regular front projections}
\end{array}
\]

**Figure 1.** Regular and non-regular projections of bordered Legendrian graphs

**Definition 2.7** (Types and orientations). For a vertex \( v \) or a basepoint \( b \) of a bordered Legendrian graph, we say that it is of type \((\ell, r)\) if there are \( \ell \) and \( r \) half-edges adjacent to \( v \) or \( b \) on the left and right, respectively. We label the set \( H_v := \{h_{v,1}, \ldots, h_{v,n}\} \) of (small enough) half-edges in front and Lagrangian projections as follows:

\[
\begin{align*}
&h_{v,1} \quad h_{v,2} \quad \cdots \quad h_{v,\ell} \\
&\vdots \quad \vdots \quad \ddots \quad \vdots \\
&h_{v,\ell} \quad h_{v,\ell+1} \quad \cdots \quad h_{v,r} \\
&v \quad h_{v,\ell+1} \quad \cdots \quad h_{v,r+1}
\end{align*}
\]

In particular, each basepoint \( b \in B \) is assumed to be **oriented** from the half-edge \( h_{b,1} \) to \( h_{b,2} \) in the above convention.

**Example/Definition 2.8** (The trivial and vertex bordered Legendrian graphs). Let \( n \geq 1 \). The front projections of the trivial bordered Legendrian graph \( T_n = (T_{n,L} \to T_n \leftarrow T_{n,R}) \) of type \((n,n)\) and the vertex bordered graphs \( 0_n = (\varnothing \to 0_n \leftarrow 0_n) \) and \( \infty_n = (\infty_{n,L} \to \infty_n \leftarrow \varnothing) \) of type \((0,n)\) and \((n,0)\) as depicted in Figure 1(c), whose left and right borders are points lying on the red lines at the left and right, respectively.

For convenience’s sake, we define \( \gamma_0 = \gamma_1 = (\varnothing \to \varnothing \leftarrow \varnothing) \).

**Definition 2.9** (Equivalences and isomorphisms). We say that two bordered Legendrian graphs \( \mathcal{T} \) and \( \mathcal{T}' \) are **equivalent** if they are isotopic, that is, there exists a family of bordered Legendrian graphs

\[
\mathcal{T}_t : \Gamma \times [0,1] \to J^1U_t \subset J^1\mathbb{R}_x, \quad \mathcal{T}_0 = \mathcal{T}, \quad \mathcal{T}_1 = \mathcal{T}'.
\]

Two regular front (or Lagrangian) projections \( \mathcal{T} \) (or \( \mathcal{T}_{\text{Lag}} \)) and \( \mathcal{T}' \) (or \( \mathcal{T}'_{\text{Lag}} \)) of bordered Legendrian graphs \( \mathcal{T} \) and \( \mathcal{T}' \) are said to be **isomorphic** if there is an isotopy Legendrian \( \gamma_T \) such that \( \mathcal{T} \) (or \( \mathcal{T}_{\text{Lag}} \)) and \( \mathcal{T}' \) (or \( \mathcal{T}'_{\text{Lag}} \)) are Lagrangian (or front) projections of \( \mathcal{T}_0 \) and \( \mathcal{T}_1 \) and Lagrangian (or front) projections \( \mathcal{T}_t := \pi_t(\mathcal{T}_t) \) (or \( \mathcal{T}_{\text{Lag},t} := \pi_{\text{Lag}}(\mathcal{T}_t) \)) are regular for all \( t \).

\(^1\)In the front projection, an \( x \)-extreme point is either a cusp or a vertex of type \((0,n)\) or \((n,0)\).
Remark 2.10. It is important to note that during the isotopy between two bordered Legendrian graphs, the ambient manifold $J^1U_t$ may changes. For example, any translation along the $x$-axis will give us an equivalence.

Lemma 2.11. Up to isomorphism, every pair of equivalent front or Lagrangian projections can be connected by a zig-zag sequence of front or Lagrangian Reidemeister moves depicted in Figures 2(a) or 2(b), respectively.

This is well-known and we omit the proof. One may refer [5, Proposition 4.2] or [2, Proposition 2.1]. Notice also that these moves are not optimal. Namely, the move (IV) is a special case of (VI).

Notation 2.12. The sets of Lagrangian and front Reidemeister moves will be denoted as follows:

$$ RM := \{(I), (II), (III), (V), (VI)\}, \quad RM_{Lag} := \{(0_a), (0_b), (0_c), (ii), (iii_a), (iii_b), (iv)\}. $$

Remark 2.13. All front and Lagrangian Reidemeister moves $(M) \in RM$ and $(m) \in RM_{Lag}$ never increase the number of crossings.

Figure 2. Front and Lagrangian Reidemeister moves: Reflections are possible, the valency of vertex is arbitrary and the vertex can be replaced with a basepoint if it is bivalent.

On the other hand, one can consider the weaker equivalence given by the Legendrian isotopy up to basepoints. In other words, two bordered Legendrian graphs are Legendrian isotopic after forgetting basepoints.

Lemma 2.14. Two regular front or Lagrangian projections of bordered Legendrian graphs are equivalent up to basepoints if and only if they can be connected by a zig-zag sequence of front or Lagrangian Reidemeister moves together with basepoint splittings depicted in Figure 3.

Proof. This is obvious. □

Remark 2.15. Note that the operations $(B_1)$ and $(b_1)$ that move a basepoint through a crossing below or above can be realized as sequences of front and Lagrangian Reidemeister moves, respectively.
Definition 2.16 (Categories of bordered Legendrian graphs). We regard $\mathcal{B}L\mathcal{G}$ and $\mathcal{B}L\mathcal{G}_\text{Lag}$ of regular front and Lagrangian projections of isomorphic classes of bordered Legendrian graphs as categories, whose morphisms are freely generated by front and Lagrangian Reidemeister moves.

Therefore, projections up to zig-zags of morphisms are the same as those up to Reidemeister moves. Or equivalently, the isomorphism classes in the localized category of $\mathcal{B}L\mathcal{G}$ or $\mathcal{B}L\mathcal{G}_\text{Lag}$ by all Reidemeister moves are the same as the usual equivalent classes of bordered Legendrian graphs.

Example/Definition 2.17. A bordered Legendrian graph $\mathcal{T} \in \mathcal{B}L\mathcal{G}$ is said to be in a normal form if

1. every vertex is of type $(\text{val}(v), 0)$ and on the right above position,
2. every non-vertex $x$-maximum is a basepoint and vice versa so that each $x$-maximum or a basepoint looks as follows:

![Normal Form](image)

Namely, all vertices are lying near $J^1 U_R$, moreover, have the larger $z$-coordinate than any point in the right border $T_R$ as depicted in Figure 4.

![Figure 4](image)

Figure 4. A bordered Legendrian graph in a normal form

Lemma 2.18. Let $\mathcal{T} \in \mathcal{B}L\mathcal{G}$ be a bordered Legendrian graph. Then there exists a sequence of front Reidemeister moves consisting of (V), (VI) together with $(B_i)$’s which transforms a (non-unique) bordered Legendrian graph in a normal form to $\mathcal{T}$.

Proof. We first use $(B_i)$ to make each $x$-maximum to be a basepoint and vice versa.

For each vertex $v \in V$ of type $(\ell, r)$ with $r > 0$, we apply (VI) several times to make $v$ of type $(\ell + r, 0)$ in the reverse direction and we move a small neighborhood $U_v$ of each vertex $v$ to the right upward position by applying only front Reidemeister moves $(V)$ in the reverse direction and we are done. $\square$

2.1.1. Maslov potentials for bordered Legendrian graphs. Let $\mathcal{T} \in \mathcal{B}L\mathcal{G}$ and $S = S(\mathcal{T}) \subset \mathcal{T}$ be the set of $x$-extreme points in its interior. An $\mathbb{Z}$-valued Maslov potential of $\mathcal{T}$ is a function $\mu : R \to \mathbb{Z}$ from the set $R := \pi_0(\mathcal{T} \setminus (V \cup S))$ of connected components of the complement of vertices and cusps such that for all $s \in S \setminus V$,

$$\mu(s^+) - \mu(s^-) = 1 \in \mathbb{Z},$$

(2.1)
where $s^+$ (resp. $s^-$) is the upper (resp. lower) strand near $s$.

For $\mathcal{T}_{\text{Lag}} \in \mathcal{BLG}_{\text{Lag}}$, let $S_{\text{Lag}} = S(\mathcal{T}_{\text{Lag}}) \subset \mathcal{T}_{\text{Lag}}$ be the set of $x$-extreme points. As before we define the set $R_{\text{Lag}} := \pi_0(\mathcal{T}_{\text{Lag}} \setminus (V_{\text{Lag}} \cup S_{\text{Lag}}))$ of connected components of the complement of vertices and $x$-extreme points. Then an $\mathbb{Z}$-valued Maslov potential of $\mathcal{T}_{\text{Lag}}$ is a function $\mu : R_{\text{Lag}} \to \mathbb{Z}$ satisfying the following condition: for each $s \in S_{\text{Lag}} \setminus V_{\text{Lag}}$,

$$\mu(s^+) - \mu(s^-) = \begin{cases} 1 & \text{if } s \text{ is } x\text{-minimal;} \\ -1 & \text{if } s \text{ is } x\text{-maximal.} \end{cases} \tag{2.2}$$

Diagrammatically, the above definition is depicted in Figures 5(a) and 5(b).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Diagrams.png}
\caption{Defining diagrams for Maslov potentials}
\end{figure}

Moreover, one can define $\mu_L := \mu|_{T_L} : [n_L] \to \mathbb{Z}$ and $\mu_R := \mu|_{T_R} : [n_R] \to \mathbb{Z}$ as the restrictions of $\mu$ to the connected components containing $T_L$ and $T_R$, respectively, under the canonical identifications $T_L \cong [n_L]$ and $T_R \cong [n_R]$.

**Definition 2.19** (Maslov potentials for bordered Legendrian graphs). A Maslov potential for a bordered Legendrian graph $\mathcal{T}$ is a triple $(\mu_L, \mu_R)$ denoted simply by $\mu$.

We denote Legendrian graphs with Maslov potentials by using the superscript ‘$\mu$’ such as $\mathcal{BLG}_\mu$.

**Example 2.20.** Recall the bordered Legendrian graphs $\mathcal{T}_n$, $\mathcal{0}_n$ and $\mathcal{\infty}_n$ defined in Example/Definition 2.8. Since they have no $x$-extreme points except for a vertex, there are no conditions for Maslov potentials. That is, any function $\mu : [n] = \{1, \ldots, n\} \to \mathbb{Z}$ can be realized as a Maslov potential for $\mathcal{T}_n$, $\mathcal{0}_n$ or $\mathcal{\infty}_n$.

Then each Lagrangian Reidemeister move induces a unique isotopy between bordered Legendrian graphs with Maslov potentials.

**Lemma 2.21.** Let $(\mathcal{M}) : \mathcal{T}' \to \mathcal{T}$ and $(\mathcal{M}') : \mathcal{T}'_{\text{Lag}} \to \mathcal{T}_{\text{Lag}}$ be front and Lagrangian Reidemeister moves. Then they lift uniquely to $(\mathcal{M}) : (\mathcal{T}', \mu') \to (\mathcal{T}, \mu)$ and $(\mathcal{M}') : (\mathcal{T}'_{\text{Lag}}, \mu') \to (\mathcal{T}_{\text{Lag}}, \mu)$. Namely, for given $\mu'$, there is a unique Maslov potential $\mu$ on each $\mathcal{T}$ or $\mathcal{T}_{\text{Lag}}$ such that $(\mathcal{M}), \mu' = \mu$ and $(\mathcal{M}'), \mu' = \mu$.

**Proof.** This is an extension of Theorem 2.21 in [1] for Lagrangian Reidemeister moves. The proof is straightforward and we omit the proof. \qed

**Definition 2.22** (Restrictions of Maslov potentials on vertices). For each vertex $v$ of type $(\ell, r)$ with $\ell + r = n$, the set $H_v$ of half-edges can be identified with $\mathbb{Z}/n\mathbb{Z}$ by Definition 2.7 and we denote the restriction $\mu|_{H_v} : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}$ of a Maslov potential to $H_v$ by $\mu_v$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Diagrams.png}
\caption{Defining diagrams for Maslov potentials}
\end{figure}

**1.1.2. Concatenations of bordered Legendrian graphs.** For $i = 1, 2$, let $\mathcal{T}^i \in \mathcal{BLG}$ be a bordered Legendrian graph of type $(n^i_L, n^i_R)$. Suppose that $n^i_L = n^i_R$. Then we can naturally define the concatenation $\mathcal{T} = \mathcal{T}^1 \cdot \mathcal{T}^2 = (T_L \to T \leftarrow T_R)$ such that $T$ is obtained simply by concatenating and identifying $T_R^1$ and $T_R^2$ up to small isotopy near borders if necessary, and two borders $T_L := T_L^1$ and $T_R := T_R^2$ come naturally from $\mathcal{T}^1$ and $\mathcal{T}^2$, respectively.

**Remark 2.23.** It is important to note that we will not regard the points of concatenations as vertices of $T$. Therefore $T$ has $n$-less edges than the disjoint union of $\mathcal{T}^1$ and $\mathcal{T}^2$.

**Definition 2.24** (Closure). For $\mathcal{T} \in \mathcal{BLG}$ of type $(n_L, n_R)$, the closure $\widehat{\mathcal{T}}$ is defined by the Legendrian graph obtained by the concatenation

$$\widehat{\mathcal{T}} := \mathcal{0}_n \cdot \mathcal{T} \cdot \mathcal{\infty}_n \in \mathcal{LG}$$

as depicted in Figure 6.
**Lemma 2.25.** The closure \( \hat{T} : \mathcal{BG}^\mu \rightarrow \mathcal{LG}^\mu \) is a functor.

*Proof.* It is obvious that there is a unique way to extend \( \mu \) on \( T \) to \( \hat{\mu} \) on \( \hat{T} \) by definition of the closure, which is well-defined since any function on \([n]\) can be realized as a Maslov potential of \( 0_n \) or \( \infty_n \) as seen in Example 2.20.

Each Reidemeister move \((M) : T' \rightarrow T\) between bordered Legendrian graphs induces the exactly same move \((M) : \hat{T}' \rightarrow \hat{T}\) between closures. Therefore it becomes a functor preserving homotopy. □

We introduce a combinatorial way, called the Ng’s resolution to obtain a regular Lagrangian projection \( T_{\text{Lag}} \in \mathcal{LG}_{\text{Lag}}^\mu \) for given front projection \( T \in \mathcal{LG}^\mu \) defining the equivalent Legendrian graphs.

**Remark 2.26.** This is an extension of the original Ng’s resolution for Legendrian knots to Legendrian graphs.

**Definition 2.27** (Ng’s resolution). [22, Definition 2.1] For \( T \in \mathcal{LG}^\mu \), the Ng’s resolution \( \text{Res}_{\text{Ng}}(T) \) is a Lagrangian projection obtained by (combinatorially) replacing the local pieces as follows:

\[
\begin{array}{c}
\left< \left< \left< \left< \left< \left< \left< \right< \right< \right< \right< \right< \right< \right< \right>
\end{array}
\]

and for a vertex \( v \) of type \((f, r)\) and a basepoint \( b \), we take a replacement as follows:

\[
\begin{array}{c}
h_v,1 & h_v,2 & \cdots & h_v,f & h_v,f+1 & \cdots & h_v,r + 1 & h_v,r + 2 & \cdots & h_v,r + r \\
\end{array}
\]

Notice that if we have two equivalent front projections, then the Ng’s resolution induces equivalences between resolutions. Indeed, for each front Reidemeister move \((M)\), we have a sequence of Lagrangian Reidemeister move(s) \( \text{Res}_{\text{Ng}}(M) \).

One candidate for the choice of \( \text{Res}_{\text{Ng}}(M) \) for each \((M) \in \mathcal{RM}\) is given in [3, Figure 6].

**Lemma 2.28.** The Ng’s resolution \( \text{Res}_{\text{Ng}} : \mathcal{LG}^\mu \rightarrow \mathcal{LG}_{\text{Lag}}^\mu \) is a functor.

*Proof.* The well-definedness is well-known and as observed above, each morphism corresponding to front Reidemeister move \((M)\) will be mapped to the morphism corresponding to the chosen sequence \( \text{Res}_{\text{Ng}}(M) \) of Lagrangian Reidemeister moves. Therefore it is functorial. □

2.2. **Bordered Chekanov-Eliashberg DGAs.** Throughout this paper, we mean by a *differential graded algebra* (DGA) a pair \( A = (A, \partial) \) of a unital associative graded algebra \( A \) over a field \( \mathbb{K} \) with the unit \( 1 \in \mathbb{K} \).

A DGA \( A = (A, \partial) \) is said to be *semi-free* and generated by \( S \) if its underlying algebra \( A \) is a tensor algebra of a graded vector space \( \mathbb{K}(S) \)

\[
A = T(\mathbb{K}(S)) = \bigoplus_{\ell \geq 0} (\mathbb{K}(S))^{\otimes \ell}.
\]

for a (possibly infinite) generating set \( S \) with a grading \( | \cdot | : S \rightarrow \mathbb{Z} \).

**Remark 2.29.** This is the usual notion of the semi-freeness while it is used differently in [21].
We denote the category of semi-free DGAs by $\mathcal{D}_{SA}$.

**Assumption 2.30.** From now on, we mean by a DGA a semi-free DGA unless mentioned otherwise.

**Example/Definition 2.31** (Border DGAs and internal DGAs). There are two important examples of DGAs, called the border DGA $A_0(\mu) = (A_0, \partial_0)$ and the internal DGA $I_n(\mu) = (I_n, \partial_n)$ which are edge-graded over $E_n := [n]$ and defined as follows:

- For a function $\mu : [n] \to \mathbb{Z}$, the algebras $A_n$ and $I_n$ are generated by two sets $K_n$ and $\tilde{V}_n$
  
  $K_n := \{k_{ab} \mid 1 \leq a < b \leq n\}$, $\tilde{V}_n := \{\xi_{a,i} \mid a \in \mathbb{Z}/n\mathbb{Z}, i \geq 1\}.$

- The edge-gradings on generator $k_{ab}$ and $\xi_{a,i}$ are given as $(a, b)$ and $(a, a + i)$, where all indices are modulo $n$.

- The homological gradings are $|k_{ab}| := \mu(a) - \mu(b) - 1$, $|\xi_{a,i}| := \mu(a) - \mu(a + i) + N(n, a, i) - 1$,

where $N(n, a, i) = \sum_{j=0}^{n} N(n, a + j, 1)$ and $N(n, a, 1) := \begin{cases} 0 & a \neq n; \\ 2 & a = n; \end{cases}$ (2.3)

- The differentials for $k_{ab}$ and $\xi_{a,i}$ are defined as follows:
  
  $\partial_n(k_{ab}) := \sum_{a < c < b} (-1)^{k_{ac} - 1} k_{ac} k_{cb}$, $\partial_n(\xi_{a,i}) := \delta_{tn}1_a + \sum_{i_1 + i_2 = i} (-1)^{|\xi_{a,i_1} - 1|} \xi_{a,i_1} \xi_{a+i_2, i_2}.$ (2.4)

Then it is obvious that there is a canonical inclusion $k_{ab} \mapsto \xi_{a,b-a}$ from $A_n(\mu) \to I_n(\mu)$.

**Remark 2.32.** We have $A_0(\mu) \cong I_0(\mu) \cong \mathbb{K}$.

Especially, when $n = 2$ and $\mu$ is constant, say $m$, then the internal DGA $I_2(m) = (I_2(m), \partial_2)$ is given as

$I_2(m) = \mathbb{K} \langle \xi_{1,i}, \xi_{2,i} \mid i \geq 1 \rangle,$

where $|\xi_{1,i}| = i - 1$. Hence, it is independent to the constant $m$ and denoted by $I_2$ and moreover, we have the DGA morphism

$I_2 \to (\mathbb{K}[t, t^{-1}], \partial \equiv 0)$ $|t| = |t^{-1}| = 0, \quad b_{a, \ell} \mapsto \begin{cases} t & a = 1, \ell = 1; \\ t^{-1} & a = 2, \ell = 1; \\ 0 & \ell > 1. \end{cases}$ (2.7)

**Definition 2.33** (Tame isomorphisms). We say that a DGA morphism $f : A' = (A' = T(\mathbb{K}(S)), \partial') \to A = (A = T(\mathbb{K}(S)), \partial)$ is called an elementary isomorphism if for some $g \in R_{ee'}$,

$f(h) = \begin{cases} g + u & \text{if } h = g \\ h & \text{if } h \neq g, \end{cases}$

where $u \in A(e, e')$ is a composable word of type $(e, e')$ not containing $g$, and called a tame isomorphism if $f$ is a composition of countably many (possibly finite) elementary isomorphisms.

We consider a stabilization in the sense of [10, Definition 2.16] and a generalized stabilization defined in [1] as follows:

**Definition 2.34** (Stabilizations). A stabilization of $A = (A = T(\mathbb{K}(S)), \partial) \in \mathcal{D}_{SA}$ is a DGA which is tame isomorphic to a DGA $SA = (SA, \partial)$ obtained from $A$ by adding a countably many (possibly finite) number of canceling pairs of edge-graded generators $\{\partial^i, e^i \mid i \in I\}$ for some index set $I$ so that $\partial^i$ and $e^i$ have the same edge-grading and

$SA = T \left( \mathbb{K} \langle S \amalg \{\partial^i, e^i \mid i \in I\} \rangle \right), \quad |\partial^i| = |e^i| + 1, \quad \bar{\partial}(\partial^i) = e^i, \quad \bar{\partial}(e^i) = 0.$
Then the canonical forgetful map \( \pi : SA \to A \) sending both \( \hat{e} \) and \( e' \) to zero induces the isomorphism on homology groups, whose homotopy inverse is precisely the canonical embedding \( i : A \to SA \).

**Definition 2.35** (Generalized stabilizations). For \( A = (A = T(\mathbb{K}(S)), \partial) \in \mathcal{D} \mathcal{S} A \) and \( \varphi : I_n(\mu) \to A \) for some \( \mu : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z} \), the \( b \)-th positive or negative stabilization of \( A \) with respect to \( \varphi \) is the DGA \( S^b_\varphi A = \left( S^b_\varphi A, \overline{\partial}^b \right) \), where for \( v_{a,i} := \varphi(\xi_{a,i}) \in A \),

- the graded algebra \( S^b_\varphi A \) is given as
  \[
  S^b_\varphi A := T\left( \mathbb{K}\left\{ e^1, \ldots, e^n \right\} \right), \quad \left| e^b \right| := b + \sum_{a < b} \left( |v_{a,i}| + 1 \right),
  \]
- the differentials \( \overline{\partial}^b \) for \( e^b \) are given as
  \[
  \overline{\partial}^b e^b := \sum_{a < b} (-1)^{|e^a| - 1} e^a v_{a,b-a}.
  \]

As observed in [1, Remark 3.8] and proved in [3, Appendix A], the generalized stabilization is a composition of a stabilization and a destabilization.

**Proposition 2.36.** [3, Appendix A] There exists a semi-free edge-graded DGA \( \hat{S}^b_\varphi A \) which is a common stabilization of both \( A \) and \( S^b_\varphi A \).

### 2.2.1. Bordered DGAs

We consider DGAs together with additional structures, called bordered DGAs.

**Definition 2.37** (Bordered DGAs). A bordered DGA \( \mathcal{A} = \left( A_L \xrightarrow{\phi_L} A \xleftarrow{\phi_R} A_R \right) \) of type \( (n_L, n_R) \) consists of DGAs \( A, A_L \) and \( A_R \), and two DGA morphisms \( \phi_L : A_L \to A \) and \( \phi_R : A_R \to A \) such that

1. \( \phi_L \) is injective, and
2. for some \( \mu_L : [n_L] \to \mathbb{Z} \) and \( \mu_R : [n_R] \to \mathbb{Z} \),
   \[
   A_L \cong A_n(\mu_L) \quad \text{and} \quad A_R \cong A_m(\mu_R).
   \]

A bordered morphism \( \hat{f} : \mathcal{A}' \to \mathcal{A} \) is a triple \((f_L, f, f_R)\) of DGA morphisms such that they fit into the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{A}' & \xrightarrow{f} & \mathcal{A} \\
\downarrow & & \downarrow \\
\mathcal{A}' & \xrightarrow{\phi_L} & A' \leftarrow \phi_R
\end{array}
\]

We say that \( \mathcal{A} \) is a cofibrant if \( \phi_R \) is also injective.

**Definition 2.38** (Stabilizations). We say that \( \mathcal{A}' = (A'_L \to A' \leftarrow A'_R) \) is a (weak) stabilization of \( \mathcal{A} = (A_L \to A \leftarrow A_R) \) if \( A' \) is a stabilization of \( A \) and there exists a canonical projection

\[
\pi = (\text{Id}_L, \pi, \text{Id}_R) : \mathcal{A}' \to \mathcal{A},
\]

and is a strong stabilization if it is a weak stabilization and there exists the canonical embedding \( i : \mathcal{A} \to \mathcal{A}' \) as well.

**Definition 2.39** (Category of bordered DGAs). The category of bordered DGAs will be denoted by \( \mathcal{B} \mathcal{D} \mathcal{S} A \), and the full subcategory of \( \mathcal{B} \mathcal{D} \mathcal{S} A \) consisting of cofibrants by \( \mathcal{B} \mathcal{D} \mathcal{S} A^c \). Then the category \( \mathcal{D} \mathcal{S} A \) is the full subcategory of \( \mathcal{B} \mathcal{D} \mathcal{S} A^c \) consisting of bordered DGAs of type \( (0, 0) \).

**Definition 2.40** (Cofibrant replacements of bordered DGAs). Let \( \mathcal{A} = \left( A_L \xrightarrow{\phi_L} A \xleftarrow{\phi_R} A_R \right) \in \mathcal{B} \mathcal{D} \mathcal{S} A \) be a bordered DGA. The cofibrant replacement of \( \mathcal{A} \)

\[
\hat{\mathcal{A}} := \left( A_L \xrightarrow{\hat{\phi}_L} \hat{A} \xleftarrow{\hat{\phi}_R} A_R \right) \in \mathcal{B} \mathcal{D} \mathcal{S} A^c
\]

is defined by the mapping cylinder construction as follows: if \( A = (A = T(\mathbb{K}(S)), \partial) \),
• the DGA \( \hat{A} = (\hat{A}, \hat{\partial}) \) and its graded algebra \( \hat{A} \) is defined as
  \[
  \hat{A} := T \left( \bigoplus \left( R \amalg K_R \amalg \hat{K_R} \right) \right), \quad \hat{K_R} := \left\{ \hat{k}_{ab} \mid k_{ab} \in K_R \right\}, \quad |\hat{k}_{ab}| := |k_{ab}| + 1.
  \]

• The differential \( \hat{\partial} \) for each \( k_{ab} \) is the same as \( \partial_R(k_{ab}) \) and for \( \hat{k}_{ab} \) it is defined as
  \[
  \hat{\partial}(\hat{k}_{ab}) := k_{ab} - \phi_R(k_{ab}) + \sum_{a < c < b} (-1)^{|k_{ac}|} \hat{k}_{ac} \Phi_R(k_{cb}) + k_{ac} \hat{k}_{cb}.
  \]  
  (2.8)

• The morphism \( \hat{\phi}_L \) is the composition of \( \phi_L \) and the canonical inclusion \( A \to \hat{A} \), and \( \hat{\phi}_R \) is defined by \( \hat{\phi}_R(k_{ab}) = k_{ab} \in \hat{A} \).

Let \( \hat{\pi} = (\text{Id}, \hat{\pi}, \text{Id}) : \hat{A} \to A \) be the bordered morphism which fixes both border DGAs and sends all \( \hat{k}_{ab} \) to zero and each \( k_{ab} \) to \( \phi_R(k_{ab}) \).

**Lemma 2.41.** The bordered DGA \( \hat{A} \) is a weak stabilization.

**Proof.** Notice that in each differential \( \hat{\partial}(\hat{k}_{ab}) \) there is one and only one generator \( k_{ab} \). Therefore one may find a tame automorphism \( \Phi \) on \( \hat{A} \) that sends \( \hat{\partial}(\hat{k}_{ab}) \) to \( k_{ab} \) so that the DGA \( (\hat{A}, \hat{\partial}_b) \) with the twisted differential \( \hat{\partial}_b := \Phi \circ \hat{\partial} \circ \Phi^{-1} \) becomes a stabilization of \( A \). \( \square \)

**Remark 2.42.** The cofibrant replacement \( \hat{A} \) is not a strong stabilization. Indeed, the canonical inclusion \( \text{i} \) may involve the DGA homotopy. However it still satisfies the good property, for example, induces an \( A_{w,\text{equ}} \)-quasi-equivalence between augmentation categories. We will see this in Section 4.1.

2.2.2. Bordered Chekanov-Eliashberg DGAs. Let \( (T, \mu) \in \mathcal{E} \mathcal{L} \mathcal{Y}^\mu \) be a front projection of a bordered Legendrian graph of type \( (n_L, n_R) \) with a Maslov potential. We consider the Ng’s resolution \( \hat{\text{Res}}^{\text{Ng}}(\hat{T}) \) of the closure of \( T \), which has two distinguished vertices 0 and \( \infty \) of valency \( n_L \) and \( n_R \), respectively. See Figure 7.

![Figure 7. The Ng’s resolution of the closure of \( T \)](image)

Then it is known that the Chekanov-Eliashberg DGA \( A^{\text{CE}}(\hat{\text{Res}}^{\text{Ng}}(\hat{T})) \) has the following generating sets:

1. Crossings \( C(\hat{\text{Res}}^{\text{Ng}}(\hat{T})) \);
2. Vertex generators \( \hat{V}(\hat{\text{Res}}^{\text{Ng}}(\hat{T})) = \{ v_{a,i} \mid v \in \hat{V}(\hat{\text{Res}}^{\text{Ng}}(\hat{T})), a \in \mathbb{Z}/\text{val}(v)\mathbb{Z}, i \geq 1 \} \);
3. Basepoint generators \( \hat{B}(\hat{\text{Res}}^{\text{Ng}}(\hat{T})) = \{ b_{a,i} \mid b \in \hat{B}(\hat{\text{Res}}^{\text{Ng}}(\hat{T})), a = 1, 2, i \geq 1 \} \).

Especially, \( C(\hat{\text{Res}}^{\text{Ng}}(\hat{T})) \) has the subset \( \hat{K_R} \) of \( \binom{n_R}{2} \) crossings coming from the right border by \( \hat{k}_{ab} \)

\[
\hat{K_R} := \left\{ \hat{k}_{ab} \mid 1 \leq a < b \leq n_R \right\},
\]

which are in the dashed box in Figure 7. Similarly, two distinguished vertices 0 and \( \infty \) have the subsets of vertex generators

\[
K_L := \{ k_{a'b'} = 0_{a',b'-a'} \mid 1 \leq a' < b' \leq n_L \} \quad \text{and} \quad K_R := \{ k_{ab} = \infty_{a,b-a} \mid 1 \leq a < b \leq n_R \}.
\]

Then it is known that the differential for \( \hat{k}_{ab} \) is given as

\[
\hat{\partial}(\hat{k}_{ab}) = (-1)^{|k_{ab}|} k_{ab} + g_{ab} + \sum_{a < c < b} (-1)^{|k_{ac}|} k_{ac} k_{cb} + g_{ac} k_{cb},
\]

which is essentially the same as the equation (2.8) by replacing \( k_{ab} \) and \( g_{ab} \) with \((-1)^{|k_{ab}|} k_{ab}\) and \((-1)^{|k_{ab}|} g_{ab}\), respectively. In particular, the assignment \( k_{ab} \mapsto (-1)^{|k_{ab}|} g_{ab} \) defines a DGA morphism.
We define a bordered DGA
\[
\hat{A}^{CE}(T, \mu) = \left( A^{CE}(T_L, \mu_L) \xrightarrow{\partial} \hat{A}^{CE}(T, \mu) \xrightarrow{\hat{\partial}} A^{CE}(T_R, \mu_R) \right)
\]
associated to \((T, \mu)\) as follows: two border DGAs are as before and the DGA \(\hat{A}^{CE}(T, \mu) = (\hat{A}, \hat{\partial})\) is the DG-subalgebra of \(A^{CE}(\hat{T}_{Lag}, \hat{\mu})\) generated by all crossings in \(\hat{T}_{Lag}\), vertex generators for all but 0 and \(\infty\) and two sets \(K_L\) and \(K_R\).

The bordered Chekanov-Eliashberg DGA (CE DGA) or Legendrian contact homology DGA (LCH DGA) for \(T^\mu\) is given as
\[
A^{CE}(T, \mu) := \left( A^{CE}(T_L, \mu_L) \xrightarrow{\partial} A^{CE}(T, \mu) \xrightarrow{\partial} A^{CE}(T_R, \mu_R) \right),
\]
where

1. two DGAs \(A^{CE}(T_L, \mu_L)\) and \(A^{CE}(T_R, \mu_R)\) are isomorphic to border DGAs \(A_{n_0}(\mu_L)\) and \(A_{n_0}(\mu_R)\), respectively, whose generating sets \(K_L\) and \(K_R\) are identified with
   \[K_L := \{ k_{a,b'} = 0_{a,b'-a} | 1 \leq a' < b' \leq n_L \}, \quad K_R := \{ k_{a,b} = 0_{a,b-a} | 1 \leq a < b \leq n_R \}.\]
2. the DGA \(A^{CE}(T, \mu)\) is the DG-subalgebra generated by crossings, vertex and basepoint generators contained only in \(\text{Res}^{\mathbb{Q}}(T)\) together with \(K_L\), and
3. the DGA morphism \(\phi_n\) is an obvious inclusion while \(\phi_R\) is defined as \(k_{ab} \mapsto (-1)^{|k_{ab}|} g_{ab} \) as above.

**Theorem 2.43.** [3, Theorem 3.2.11 and Proposition 3.2.13] Let \((T, \mu) \in \mathcal{BCL}^\mu\). The bordered DGA \(A^{CE}(T, \mu)\) is well-defined and invariant under Legendrian isotopy up to stabilizations.

Indeed, for each front Reidemeister move \((M) : (T', \mu') \to (T, \mu)\), there exists a zig-zag
\[
A^{CE}(T', \mu') \xleftarrow{\pi} \hat{A}^{CE}(T, \mu) = \mathcal{A}_0 \xleftarrow{\pi_1} \cdots \xleftarrow{\pi_{n-1}} \mathcal{A}_n = \hat{A}^{CE}(T, \mu) \xrightarrow{\pi} A^{CE}(T, \mu)
\]
of stabilizations of bordered DGAs between \(A^{CE}(T', \mu')\) and \(A^{CE}(T, \mu)\), where the bordered DGAs \(\hat{A}^{CE}(T', \mu')\) and \(\hat{A}^{CE}(T, \mu)\) are well-defined and the cofibrant replacements of \(A^{CE}(T', \mu')\) and \(A^{CE}(T, \mu)\), respectively, and all but the left- and rightmost are strong stabilizations.

We explain briefly the reason why the DG-subalgebra \(A^{CE}(\text{Res}^{\mathbb{Q}}(\hat{T}, \hat{\mu}))\) is well-defined. Since two vertex 0 and \(\infty\) face the unbounded region in \(\mathbb{R}^2\) especially between two half-edges \(h_0,n_0\) and \(h_0,1\), and \(h_{n_0},n_0\) and \(h_{n_0,1}\), there are no crossing generators involving vertex generators at 0 or \(\infty\) which pass these unbounded regions in their differentials.

In other words, the bordered DGA \(\hat{A}^{CE}(T, \mu)\) is the same as the LCH DGA of the Legendrian graph \((\hat{T}, \hat{\mu})\) together with two DG-subalgebras \(A^{CE}(T_L, \mu_L)\) and \(A^{CE}(T_R, \mu_R)\) of the distinguished internal DG-subalgebras \(I_0\) and \(I_{\infty}\). This data is obviously invariant under any front Reidemeister moves fixing two vertices 0 and \(\infty\).

**Remark 2.44.** The exactly same statement holds for Lagrangian projections. That is, for each \((m) : (\hat{T}_{Lag}, \mu') \to (\hat{T}_{Lag}, \mu)\), there is a zig-zag of stabilizations between two bordered LCH DGAs \(A^{CE}(\hat{T}_{Lag}, \mu')\) and \(A^{CE}(\hat{T}_{Lag}, \mu)\).

More precisely, \((\hat{\mu})\) is an isomorphism and \((\hat{M})\) is a canonical projection of a stabilization for \((M) \in \{(I), (II), (VI)\}\). However, we have a zig-zag of stabilization for \((VI)\)

\[
A^{CE}(T', \mu') \xleftarrow{\pi'} \hat{A}'^{CE}(T, \mu) \xrightarrow{\pi} \hat{A}^{CE}(T, \mu) \xrightarrow{\pi} A^{CE}(T, \mu).
\]

Similarly, for Lagrangian Reidemeister moves \((m)\), we have an isomorphism \((\hat{m})\), for \((m) \in \{(0), (III)\}\), a canonical projection of a stabilization for \((ii)\) and a zig-zag of stabilizations for \((iv)\).
Example 2.45 (DGAs for bordered Legendrian graphs in a normal form). Recall a bordered Legendrian graph \((T, \mu) \in BLG^R\) in a normal form in Definition 2.17. Let \((\bar{T}_{lag}, \mu)\) be the Ng’s resolution of \((T, \mu)\). Then the set of vertex generators at each \(v\) can be decomposed into two subsets as follows:

\[ V_{v,L} := \left\{ v_{a,j} \in \bar{V}_v \mid a + j < \text{val}(v) \right\}, \quad V_{v,O} := \left\{ v_{a,j} \in \bar{V}_v \mid a + j \geq \text{val}(v) \right\} \]

and we denote their unions by \(\bar{V}_L\) and \(\bar{V}_O\)

\[ \bar{V}_L := \bigsqcup_{v \in V} \bar{V}_{v,L}, \quad \bar{V}_O := \bigsqcup_{v \in V} \bar{V}_{v,O}. \]

Notice that the elements in \(\bar{V}_{v,O}\) correspond to the vertex generators which are lying on the left hand side of the vertex \(v\):

\[
\begin{align*}
\bar{V}_{v,O} = \{ v = e_3 \} & \quad \text{if } v = e_3 \\
\bar{V}_{v,O} = \{ v = e_3 \} & \quad \text{if } v = e_3
\end{align*}
\]

Since all vertices are in the upper right position, there are no immersed polygons for differentials involving generators in \(\bar{V}_O\) except for infinitesimal ones. That is, due to this geometric reason, no generators in \(\bar{V}_O\) appears in differentials for any crossing generator and in the image of the right border generators \(k_{ab}\)’s under \(\phi_R\).

Before closing this section, we remark the following. The LCH DGA \(A^{CE}(T, \mu)\) has infinitely many generators \(b_{a,\ell}\) for each basepoint \(b \in B\) while it usually contributes two generator inverses to each other such as \(t_b\) and \(t^{-1}_b\). However, as observed in [11, Proposition 14], the internal DGA for each basepoint \(I_b\) is quasi-isomorphic to the Laurent polynomial ring \(\mathbb{K}[t_b, t^{-1}_b]\). Therefore one can regard \(I_b\) as the free resolution of \(\mathbb{K}[t_b, t^{-1}_b]\).

Indeed, \(\mathbb{K}[t_b, t^{-1}_b]\) can be obtained by taking a quotient by all \(b_{a,\ell}\)’s with \(\ell > 1\), whose degrees are not zero. Due to this observation, we can get one another consequence such that any DGA morphism from \(A^{CE}(T, \mu)\) to a field \(K = (\mathbb{K}, \partial = 0)\) factors through the quotient DGA. Therefore there is an isomorphism between sets\(^2\) of DGA morphisms.

Now let us consider the effects of operations on basepoints to LCH DGAs. As seen earlier, there are three more moves on basepoints. As mentioned earlier, the operation \((b_1) : (\bar{T}_{lag'}, \mu') \rightarrow (\bar{T}_{lag}, \mu)\) can be viewed as a sequence of front Reidemeister moves. Hence we have a zig-zag of stabilizations which is not easy to describe directly.

\[ b_{1,1}c = \quad \overset{(b_1)}{\longrightarrow} \quad c = \]

On the other hand, two operations \((b_2)\) and \((b_3)\) can be regarded as tangle replacements so that they induce DGA morphisms \((b_2)_*, (b_3)_* : A^{CE}(T', \mu') \rightarrow A^{CE}(T, \mu)\). Indeed, for each generator \(b_{a,\ell}\) or \(v_{a,\ell}\) will be mapped to elements obtained by counting certain polygons in the support of \((b_2)\) or \((b_3)\) in \(T\). For example, \(b_{1,1}\) or \(v_{4,2}\) will be mapped to \(b_{1,1}'b_{1,1}''\) and \(b_{2,1}v_{4,2}\). See [1, §6.5] for detail.

\[ b_{1,1} = \quad \overset{(b_2)}{\longrightarrow} \quad b_{1,1}'b_{1,1}'' = \quad \overset{(b_3)}{\longrightarrow} \quad b_{2,1}v_{4,2} = \]

However, when we consider the quotient DGA by all generators \(b_{a,\ell}\)’s with \(\ell > 1\), then the move \((b_1)\) induces a DGA isomorphism \((b_1)_*\), which sends \(c\) to \(t^{-1}_b c\). The induced map \((b_2)_*\) sends \(t_b\) to \(t_b t_b^{-1}\) and \((b_3)_*\), sends \(v_{a,\ell}\) to either \(v_{a,\ell} t_b\) or \(t^{-1}_b v_{a,\ell}\) according to where the basepoint \(b\) is lying on.

\(^2\)Indeed, we have an isomorphism between augmentation varieties. See [3].
Conversely, one can find a DGA morphism \((b^{-1}_i)_* : A^{CE}(\mathcal{T}^\prime_{Lag}, \mu) \to A^{CE}(\mathcal{T}^\prime_{Lag}, \mu')\) for \(i = 2, 3\) which are left inverses of \((b_i)_*\), and defined as

\[
(b^{-1}_2)_*(b''_{a, \ell}) := \begin{cases} 1 & \ell = 1; \\ 0 & \ell > 1. 
\end{cases}
(b^{-1}_3)_*(b'_{a, \ell}) := \begin{cases} 1 & \ell = 1; \\ 0 & \ell > 1. 
\end{cases}
\]

In summary, we have the following:

**Lemma 2.46.** Let \((b_i)_* : (\mathcal{T}^\prime_{Lag}, \mu) \to (\mathcal{T}^\prime_{Lag}, \mu)\) be a basepoint move. Then either

1. for \(i = 1\), there exists a zig-zag of stabilizations between \(A^{CE}(\mathcal{T}^\prime_{Lag}, \mu')\) and \(A^{CE}(\mathcal{T}^\prime_{Lag}, \mu)\), or
2. for \(i = 2, 3\), there exists a pair of DGA morphisms

\[
A^{CE}(\mathcal{T}^\prime_{Lag}, \mu') \xrightarrow{(b_i)_*} A^{CE}(\mathcal{T}^\prime_{Lag}, \mu),
\]

where \((b_i^{-1})_*\) is a left inverse of the DGA morphism \((b_i)_*\), induced from the tangle replacement.

**Remark 2.47.** For \(i = 2, 3\), there is no issue on the commutativity of \((b_i^{-1})_*\), with the structure morphisms \(\phi_L\) and \(\phi_R\).

### 3. Consistent sequences

In this section, we first briefly review the definition of a consistent sequence of DGAs and the construction of the \(A_\infty\)-category associated to the consistent sequence of DGAs and we will basically follow the definition and construction described in [21, §3.3] but may use the different notations or conventions.

Let \(\mathcal{C}\) be a category. An \(m\)-component object \(C^{(m)}\) in \(\mathcal{C}\) is a functor from \(\mathcal{O}(\{m\})\) to \(\mathcal{C}\). That is, for any \(V \subset U \subset \{m\}\), there exist restrictions \(C^{(m)}|_{V}, C^{(m)}|_{U}\) and a restriction morphism \(C^{(m)}|_{U} \to C^{(m)}|_{V}\) in \(\mathcal{C}\). A morphism \(f^{(m)} : C^{(m)} \to C^{(m)}\) is a natural transformation between \(C^{(m)}\) and \(C^{(m)}\). We denote the category of \(m\)-component objects in \(\mathcal{C}\) by \(\mathcal{C}^{(m)}\).

Let \(\Delta_+\) be the category of all finite ordinals

\[
[1] = \{1\}, \hspace{1cm} [2] = \{1, 2\}, \hspace{1cm} [3] = \{1, 2, 3\}, \hspace{1cm} \cdots,
\]

whose morphisms are order-preserving inclusions.

A sequence \(C^{(*)}\) of \(m\)-component objects in \(\mathcal{C}\) is semi-simplicial if there is a morphism \(C^{h}|_{U} : C^{(n)}|_{h(U)} \to C^{(m)}|_{U}\) in \(\mathcal{C}\) for each \((h : [m] \to [n]) \in \Delta_+\) and \(U \subset [m]\), and is consistent if it is semi-simplicial and each morphism \(C^{h}|_{U}\) is an isomorphism.

For two consistent sequences \(C^{(*)}\) and \(C^{(*)}\), a consistent morphism \(f^{(*)} : C^{(*)} \to C^{(*)}\) is a collection of morphisms between \(m\)-component objects satisfying the consistency: for each \((h : [m] \to [n]) \in \Delta_+\) and \(U \subset [m]\), the following diagram is commutative

\[
\begin{array}{ccc}
C^{(n)}|_{h(U)} & \xrightarrow{C^{h}|_{U}} & C^{(m)}|_{U} \\
\downarrow f^{(n)}|_{h(U)} & & \downarrow f^{(m)}|_{U} \\
C^{(n)}|_{h(U)} & \xrightarrow{C^{h}|_{U}} & C^{(m)}|_{U}
\end{array}
\]

The composition of two consistent morphisms \(f^{(*)} : C^{(*)} \to C^{(*)}\) and \(g^{(*)} : C^{(*)} \to C^{(*)}\) is given as

\[
f^{(*)} \circ g^{(*)} := \left( f^{(m)} \circ g^{(m)} \right)_{m \geq 1} : C^{(*)} \to C^{(*)}.
\]

**Notation 3.1.** For each \(h : [m] \to [n], U \subset [m]\) and \(i \in [m]\), we may denote \(C^{h}|_{U}, C^{(m)}|_{U}\) and \(C^{(m)}|_{U}\) by \(C(h)|_{U}, C^{U}\) and \(C^{U}\), respectively. We further denote \(C^{(1)}\) and \(f^{(1)}\) simply by \(C\) and \(f\).

**Definition 3.2** (Category of consistent sequences). The category of all consistent sequences and morphisms will be denoted by \(\mathcal{C}^{(*)}\) and called a category of consistent sequences in \(\mathcal{C}\).

**Lemma 3.3.** Let \(C^{(*)} \in \mathcal{C}^{(*)}\) be a consistent sequence in \(\mathcal{C}\). Then for each \(m \geq 1\) and \(i \in [m]\), we have

\[
C^{i} = C^{(m)}|_{i} \cong C^{(1)} = C^{1}.
\]

**Proof.** This is obvious from the definition by considering \(h_i : [1] \to [m]\) with \(i = h_i(1)\).
3.1. **Consistent sequences of bordered Legendrian graphs.** We define categories of consistent sequences of bordered Legendrian graphs given in terms of both Lagrangian and front projections, which are related via Ng’s resolution as usual.

For a bordered graph $\Gamma = (\Gamma_L \rightarrow \Gamma \leftarrow \Gamma_R)$, an $m$-component bordered graph $\Gamma^{(m)}$ is a bordered graph defined as

$$\Gamma^{(m)} := (\Gamma_L^{(m)} \rightarrow \Gamma^{(m)} \leftarrow \Gamma_R^{(m)}),$$

where the inclusions preserve the label. Then for each $U \subset \{m\}$ and $i \in \{m\}$, let the restriction $\Gamma^{(m)} := (\Gamma_L \rightarrow \Gamma^{(m)} \leftarrow \Gamma_R)$ and $\Gamma^{(U)} := \bigsqcup_{i \in U} \Gamma^{(i)}$, where $\Gamma^{(i)} = \{i\} \times \Gamma$, for $* = L, R$ or empty.

For each $m$-component bordered graph $\Gamma^{(m)}$, its Legendrian embedding defines an $m$-component bordered Legendrian graph $\mathcal{G}^{(m)} = (\mathcal{G}_L^{(m)} \rightarrow \mathcal{G}^{(m)} \leftarrow \mathcal{G}_R^{(m)}) = (\mathcal{G}^{(m)} \rightarrow J^U)$, whose restriction $\mathcal{G}^{(U)}$ on $U \subset \{m\}$ is obviously defined as $\mathcal{G}^{(m)}|_{\mathcal{G}^{(U)}}$.

We can naturally consider the front and Lagrangian projections of $m$-component bordered Legendrian graphs which will be denoted by

$$\mathcal{G}^{(m)} = (\mathcal{G}_L^{(m)} \rightarrow \mathcal{G}^{(m)} \leftarrow \mathcal{G}_R^{(m)}) : \mathcal{L}=\pi_0(\mathcal{G}^{(m)}) \quad \text{and} \quad \mathcal{T}^{(m)} = (\mathcal{T}_{\mathcal{L}}^{(m)} \rightarrow \mathcal{T}^{(m)} \leftarrow \mathcal{T}_{\mathcal{R}}^{(m)}) : \mathcal{L} = \pi_{\mathcal{L}}(\mathcal{G}^{(m)})$$

and their restrictions on $U \subset \{m\}$ are denoted by $\mathcal{G}^{(U)}$ and $\mathcal{T}^{(U)}$ as before.

One can equip a Maslov potential $\mu^{(m)}$ on $\mathcal{G}^{(m)}$ or $\mathcal{T}^{(m)}$ so that the restriction $\mu^{(U)}$ is the restriction of $\mu^{(m)}$ on $\mathcal{T}^{(U)}$ or $\mathcal{T}^{(U)}$.

**Notation 3.4.** We denote the sets of all regular front and Lagrangian projections of $m$-component bordered Legendrian graphs with Maslov potentials by $\mathcal{B}\mathcal{L}\mathcal{G}^{\mu}_{\mathcal{L}}$ and $\mathcal{B}\mathcal{L}\mathcal{G}^{\mu}_{\mathcal{L}}$, respectively.

For a sequence $(\mathcal{G}^{(m)}, \mu^{(m)}), m \geq 1$ of $m$-component front projections $(\mathcal{G}^{(m)}, \mu^{(m)}) \in \mathcal{B}\mathcal{L}\mathcal{G}^{\mu}_{\mathcal{L}}$, the consistency is as follows: for each $(h : \{m\} \rightarrow \{n\}) \in \Delta_\alpha$, and $U \subset \{m\}$, there is an isomorphism between two front projections in the sense of Definition 2.9

$$(\mathcal{T}(h)^U, \mu^{(h)^U}) : (\mathcal{T}(h)^U, \mu^{(h)^U}) \cong (\mathcal{T}^{(U)}, \mu^{(U)}).$$

**Definition 3.5 (Consistent moves).** A consistent front (or Lagrangian) Reidemeister move or basepoint move $(M)^{(m)} : (\mathcal{T}^{(m)}, \mu^{(m)}) \rightarrow (\mathcal{T}^{(m)}, \mu^{(m)})$ between two consistent sequences of front (or Lagrangian) projections is a sequence of sequences of front (or Lagrangian) Reidemeister moves or basepoint moves satisfying the following conditions:

1. for each $m \geq 1$, $(M)^{(m)} : (\mathcal{T}^{(m)}, \mu^{(m)}) \rightarrow (\mathcal{T}^{(m)}, \mu^{(m)})$ is a sequence of front (or Lagrangian) Reidemeister moves or basepoint moves and in particular, $(M)^{(1)} = (M) \in \mathcal{RM}$ is a (possibly empty) front (or Lagrangian) Reidemeister move or a basepoint move;

2. it is compatible with restrictions, i.e., for each $U \subset \{m\}$, we have a sequence of front (or Lagrangian) Reidemeister moves or basepoint moves

$$(M)^{U} : (\mathcal{T}^{(U)}, \mu^{(U)}) \rightarrow (\mathcal{T}^{(U)}, \mu^{(U)});$$

3. it satisfies the consistency, i.e., for each $(h : \{m\} \rightarrow \{n\}) \in \Delta_\alpha$, and $U \subset \{m\}$, the following diagram is commutative:

$$(\mathcal{T}^{(h)}^{(U)}, \mu^{(h)^{(U)}}) \rightarrow (\mathcal{T}^{(h)}^{(U)}, \mu^{(h)^{(U)}}) \rightarrow (\mathcal{T}^{(U)}, \mu^{(U)}).$$

We furthermore say that a consistent move $(M)^{(m)}$ is elementary if it consists of Reidemeister moves or basepoint moves of only one type.

Two consistent sequences of front or Lagrangian projections are said to be (consistently) equivalent if and only if they are connected by a zig-zag sequence of consistent front or Lagrangian Reidemeister moves.
**Notation 3.6.** From now on, we will denote the elementary consistent Reidemeister move or basepoint move by the double arrow ‘⇒’ such as

\[(M)^{\bullet} : \mathcal{T} \longrightarrow \mathcal{T}.\]

**Example 3.7** (Elementary consistent Reidemeister moves). One example for elementary consistent Reidemeister move is a consistent Lagrangian Reidemeister move as follows: for each \(m \geq 1\),

\[
\begin{align*}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{example3.7.png}
\end{array}
\end{align*}
\]

**Definition 3.8** (Categories of consistent sequences of bordered Legendrian graphs). The categories of consistent sequences and Reidemeister moves of regular front and Lagrangian projections of bordered Legendrian graphs with Maslov potentials will be denoted by \(\mathcal{B}L\mathcal{G}_{\mu}^{\bullet}(\bullet)\) and \(\mathcal{B}L\mathcal{G}_{\text{Lag}}^{\mu}(\bullet)\), and two full subcategories consisting of consistent sequences of non-bordered Legendrian graphs will be denoted by \(\mathcal{L}\mathcal{G}_{\mu}^{\bullet}(\bullet)\) and \(\mathcal{L}\mathcal{G}_{\text{Lag}}^{\mu}(\bullet)\).

Recall the Ng’s resolution \(\text{Res}^{\text{Ng}} : \mathcal{B}L\mathcal{G}^{\mu} \rightarrow \mathcal{B}L\mathcal{G}_{\text{Lag}}^{\mu}\) which is a functor as seen earlier and therefore it preserves isomorphisms. In other words, each isomorphic pair of front projections will be mapped to an isomorphic pair of Lagrangian projections. It is obvious that \(\text{Res}^{\text{Ng}}\) sends each component to the corresponding component, it induces a functor between consistent sequences of front and Lagrangian projections.

**Lemma 3.9.** The Ng’s resolution induces a functor

\[
\text{Res}^{\text{Ng}}(\bullet) : \mathcal{B}L\mathcal{G}^{\mu}^{\bullet}(\bullet) \rightarrow \mathcal{B}L\mathcal{G}_{\text{Lag}}^{\mu}(\bullet)
\]

which preserves the homotopy relation.

**Proof.** This is nothing but the Lemma 2.28. \(\square\)

3.1.1. **Canonical front and Lagrangian parallel copies.** We introduce a canonical way to obtain a consistent sequence for a bordered Legendrian graph in terms of front projections or Lagrangian projections.

**Definition 3.10** (Canonical consistent sequences). Let \((\mathcal{T}, \mu) \in \mathcal{B}L\mathcal{G}^{\mu}\) and \((\mathcal{T}_{\text{Lag}}, \mu) \in \mathcal{B}L\mathcal{G}_{\text{Lag}}^{\mu}\) be regular front and Lagrangian projections. Then the **canonical consistent sequences** \((\mathcal{T}^{\bullet}(\bullet), \mu^{\bullet}(\bullet)) \in \mathcal{B}L\mathcal{G}^{\mu}(\bullet)\) and \((\mathcal{T}_{\text{Lag}}^{\bullet}(\bullet), \mu_{\text{Lag}}^{\bullet}(\bullet)) \in \mathcal{B}L\mathcal{G}_{\text{Lag}}^{\mu}(\bullet)\) are given by the \(z\)- and \(y\)-translations as depicted in Figures 8 and 9, respectively.

**Remark 3.11.** For Lagrangian projections, the \(y\)-translation comes from a strict contactomorphism

\[
(x, y, z) \rightarrow (x, y - \epsilon, z + x\epsilon),
\]

whose Lagrangian projection is the desired \(y\)-translation.

Notice that it is not obvious if \(\mathcal{T}^{(m)}\) or \(\mathcal{T}_{\text{Lag}}^{(m)}\) is regular again. Indeed, we need to check the regularity holds near each vertex.

There are many ways to achieve the regularity but in this paper, we fix the convention as follows: let \(U_1\) be a neighborhood of the vertex \(v\). If we take a parallel copy and make \(\mathcal{T}^{(2)}\) or \(\mathcal{T}_{\text{Lag}}^{(2)}\) by the small enough translation, then all additional crossings are contained in a small neighborhood \(U_2\). Now we take the third copy such that the newly appeared crossings avoid the region \(U_2\) and are contained in a larger neighborhood \(U_3\) and so on. See Figure 10. The shaded region represents neighborhoods \(U_i\). Then it is obvious that for each \(\mathcal{T}\) or \(\mathcal{T}_{\text{Lag}}\), the sequence \(\mathcal{T}^{(\bullet)}\) or \(\mathcal{T}_{\text{Lag}}^{(\bullet)}\) of canonical projections becomes a consistent sequence of \(m\)-component Lagrangian projections.
Remark 3.12. In the canonical $m$-copies near a vertex $v$ of type $(\ell, r)$, there are exactly \( \binom{m}{2} \binom{\ell}{2} + \binom{m}{2} \binom{r}{2} \) many additional crossings.
One can choose (zig-zag) sequences of consistent front and Lagrangian Reidemeister moves as follows:
Here, the numbers $N_i$ are given as
\[ N_1 = \frac{m(m - 1)(2m - 1)}{6}, \quad N_2 = \binom{m}{2}, \quad N_3 = \binom{m}{2}(\ell - 1), \quad N_4 = \binom{m}{3}, \quad N_5 = \binom{m}{2}(2\ell + 1), \]
where $\ell$ and $r$ are the numbers of half-edges on the left and right, respectively.

In addition, one can choose the consistent morphisms corresponding to the operations $(B_i)$ on basepoints as depicted in Figure 11.

The one important observation is that all parallel copies are consistent and all arrows are elementary. Therefore in summary, we have the following proposition.
Proposition 3.13. The canonical consistent sequences
\[ BLG^\mu \rightarrow BLG^{\mu(\bullet)} \quad \text{and} \quad BLG^\mu_{\text{Lag}} \rightarrow BLG^{\mu(\bullet)}_{\text{Lag}} \]
are well-defined and preserves the equivalences.

In particular, each consistent Reidemeister move and basepoint move can be mapped to a zig-zag of elementary consistent Reidemeister moves or basepoint moves.

Example 3.14. Due to the definition of the bordered Legendrian graph in a normal form, the canonical consistent sequence of a bordered Legendrian graph in a normal form is a sequence of bordered Legendrian graphs in a normal form.

On the other hand, as seen in Lemma 2.18, for each consistent sequence \((T^{(\bullet)}, \mu^{(\bullet)}) \in BLG^{\mu(\bullet)}\) of canonical front parallel copies, one can find a representative in a normal form up to consistent front Reidemeister moves. In other words, there are a consistent sequence \((T^{(\bullet)}', \mu^{(\bullet)}') \in BLG^{\mu(\bullet)}\) of canonical front parallel copies in a normal form and a sequence of consistent front Reidemeister moves between \((T^{(\bullet)}', \mu^{(\bullet)}') \rightarrow \cdots \rightarrow (T^{(\bullet)}, \mu^{(\bullet)})\).

Moreover, since the transformation from \(T^{(\bullet)}_1\) to \(T^{(\bullet)}_1\) only needs \((V), (V)\) and \((B)\), so does the sequence of consistent Reidemeister moves \((M_i)^{(\bullet)}\). That is, for each \(i\),
\[
(M_i)^{(\bullet)} \in \{(VI)^{(\bullet)}, (V)^{(\bullet)}, (B_1)^{(\bullet)}, (B_2)^{(\bullet)}, (B_3)^{(\bullet)}\}.
\]

Remark 3.15. It is easy to see that the Ng’s resolution does not map canonical parallel copies in front projections onto those in Lagrangian projections. Indeed, if \(T\) contains a right cusp or a vertex of type \((\ell, r)\) with \(\ell \geq 2\), then the resolution of the canonical front \(m\)-copies is not the same as the canonical Lagrangian \(m\)-copies of the resolution.

However, we can always find a consistent Lagrangian Reidemeister move consisting only of \((\text{ii})'\)’s and \((\text{iii}a)'\)’s between the canonical Lagrangian copy of Ng’s resolution and Ng’s resolutions of the canonical front copy. See Figure 12.

3.2. Consistent sequences of bordered Chekanov-Eliashberg DGAs. In this section, we will consider consistent sequences of bordered Chekanov-Eliashberg DGAs.

For each \(m \geq 1\), we define the ring \(\mathbb{K}^{(m)}\) as
\[
\mathbb{K}^{(m)} := \bigoplus_{i \in [m]} \mathbb{K}^i,
\]
where \(\mathbb{K}^i\) is the copy of \(\mathbb{K}\) with the unit \(1^i\). Then \(\mathbb{K}^{(m)}\) has the unit \(1^{(m)} := \sum_{i \in [m]} 1^i\) and for each subset \(U \subset [m]\), there exists a unique ring homomorphism
\[
\mathbb{K}^{(m)} \rightarrow \mathbb{K}^U := \bigoplus_{i \in U} \mathbb{K}^i.
\]
Therefore \(\mathbb{K}^{(m)}\) becomes an \(m\)-component object and furthermore it is obviously consistent.
3.2.1. Link-grading and composable DGAs. We first introduce an $m$-component link-grading on DGAs as described in [21, §3.2].

**Definition 3.16** (Link-graded and composable DGAs). Let $A^{(m)} = (A^{(m)}, \partial^{(m)})$ be a DGA which is not necessarily semi-free. We say that $A$ is $m$-component link-graded for some $m \geq 1$ if there exist functions $r, c : A^{(m)} \to [m]$ such that for each $g \in A^{(m)}$ and any word $g_1 \ldots g_k$ in $\delta(g)$ different from $ni$ for any $n \in \mathbb{Z}$ is composable in the sense that

$$
\begin{align*}
    r(g) &= r(g_1), & c(g) &= c(g_k), & r(g_i) &= c(g_{i+1})
\end{align*}
$$

for each $1 \leq i < k$. We say that a composable word $g_1 \ldots g_k$ is said to be of type $(i, j)$ if $r(g_1) = i$ and $c(g_k) = j$.

An $m$-component link-graded DGA $A^{(m)} = (A^{(m)}, \partial^{(m)})$ is composable over $[m]$ if $A^{(m)}$ is decomposed as

$$
A^{(m)} = \bigoplus_{(i, j) \in [m]^2} A^{(m)}(i, j)
$$

such that it satisfies the following:

- for each $i, j, k, l \in [m]$, the multiplication on $A^{(m)}(i, j) \otimes A^{(m)}(k, l)$ does not vanish only if $j = k$. In this case, the result is contained in $A^{(m)}(i, l)$.

- for each $(i, j)$, the restriction $A^{(m)}(i, j) = (A^{(m)}(i, j), \partial^{(m)})|_{A^{(m)}(i, j)}$ is a subchain complex. In particular, $A^{(m)}(i, i)$ becomes the DG-subalgebra.

One equivalent definition of composable DGAs is as follows: a DGA $A^{(m)} = (A^{(m)}, \partial^{(m)})$ is composable over $[m]$ if and only if it is a subalgebra of the matrix algebra so that

$$
A^{(m)} = \text{Mat}_m \left( A^{(m)}(i, j) \right) := \left\{ \left( a^{ij} \right) \mid a^{ij} \in A^{(m)}(i, j) \right\},
$$

and also it is a $\mathbb{K}^{(m)}$-module.

**Definition 3.17** (Semi-free composable DGAs). We say that a composable DGA $A^{(m)} = (A^{(m)}, \partial^{(m)})$ is semi-free if it is tensor-algebra-like as follows: there exists a graded set $S^{(m)}$

$$
S^{(m)} = \bigsqcup_{i, j \in [m]^2} S^{ij}
$$

and the underlying graded algebra $A^{(m)}$ is the direct sum

$$
A^{(m)} = T_{co} M^{(m)} := \bigoplus_{\ell \geq 0} \left( M^{(m)} \right)^{\ell},
$$

where $M^{(m)} = (M^{ij}) \subset A^{(m)}$ is a $\mathbb{K}^{(m)}$-module and each $M^{ij}$ is a free $(\mathbb{K}^i, \mathbb{K}^j)$-bimodule generated by $S^{ij}$

$$
M^{ij} := \mathbb{K}^i \otimes \mathbb{K}(S^{ij}) \otimes \mathbb{K}^j,
$$

and $\left( M^{(m)} \right)^{\ell}$ is the $\ell$-fold product of $M^{(m)}$ with respect to the matrix multiplication so that $\left( M^{(m)} \right)^{0} := \mathbb{K}^{(m)}$.

A DGA morphism $f : A^{(n)} \to A^{(m)}$ between semi-free composable DGAs is said to be an composable morphism if there exists $h : [m] \to [n] \in \Delta_+$ such that for each $i', j' \in [n]$, $f(A^{(n)}(i', j'))$ is either contained in $A(i, j)$ for $h(i, j) = (i', j')$ or 0 otherwise.

Let $i = (i_0, \ldots, i_k)$ be a sequence in $[m]$. We denote $M^{i}$ by the submodule defined as

$$
M^{i} := M^{i_0 i_1} M^{i_1 i_2} \ldots M^{i_{k-1} i_k}.
$$

Then it is easy to see that

$$
A^{(m)}(i, j) = \bigoplus_{i} M^{i},
$$

where $i$ ranges over all sequences starting and ending at $i$ and $j$, respectively.

**Notation 3.18.** For simplicity, we will use the notation $M^{(m)} = (\mathbb{K}^{(m)}(S^{(m)}))$ and so

$$
A^{(m)} = T_{co}(\mathbb{K}^{(m)}(S^{(m)})).
$$
**Remark 3.20.** It is obvious that each morphism \( f \in DSA_{co}^{(m)} \) preserves the link-grading. Especially, the subcategory \( DSA_{co}^{(1)} \) is equivalent to \( DSA \).

Let \( A^{(m)} \in DSA_{co}^{(m)} \). Then for each \( U \subset [m] \) with \( |U| = k \), there exists a \( k \)-component semi-free composable DGA \( A^U \in DSA_{co}^{(k)} \) which is the image of the quotient map

\[
res_U : A^{(m)} \to A^U
\]

defined by \( 1^i \mapsto 0 \) for each \( i \notin U \). Therefore each object in \( DSA_{co}^{(m)} \) satisfies the condition for \( m \)-components object.

**Example 3.21 (Composable DGAs from link-graded semi-free DGAs).** For each \( m \)-component link-graded semi-free DGA \( A^{(m)} = (A^{(m)} = (A^{(m)} = T(\mathbb{K}(S^m)), \delta^{(m)}) \), one can associate an \( m \)-component composable DGA \( A^{co}_{co}^{(m)} \) as described in [21, §3.2]. We will briefly review the construction as follows:

1. Add idempotent elements \( 1^i \) for \( i \in [m] \) and declare \( 1^i \partial_j = \delta_{ij} \).
2. For each element \( g \) of type \((i, j)\), declare \( 1^i g = g = g1^j \).
3. For each element \( g \) of type \((i, i)\) having \( n1 \) in \( \partial(g) \), replace \( n1 \) with \( n1^i \).
4. The unit 1 is replaced with \( \sum_{i \in [m]} 1^i \).

Then it is easy to check that

\[
A^{co}_{co}^{(m)} = (T_{co}(\mathbb{K}(S^m)), \delta^{(m)})
\]

and therefore it is semi-free composable.

**Definition 3.22 (Consistent sequences of composable DGAs).** The category of consistent sequences in \( DSA_{co} \) will be denoted by \( DSA_{co}^{(*)} \).

More precisely, a sequence \( A^{(*)} \) of \( m \)-component composable DGAs is consistent if and only if for each \((h : [m] \to [n]) \in \Delta_+ \) and \( U \subset [m] \), we have an isomorphism

\[
A^{h(U)} \to A^U.
\]

**Example/Definition 3.23 (Consistent sequences of composable border DGAs).** Let \( \mu : [n] \to \mathbb{Z} \) be a function. Recall the border DGA \( A_n(\mu) \in DSB = DSA_{co}^{(1)} \). For each \( m \geq 1 \), we will define the \( m \)-component border DGA \( A_n^{(m)}(\mu) \in DSB_{co}^{(m)} \) such that \( A_{n}^{(1)}(\mu) = A_n(\mu) \).

Let \( A_n^{(m)}(\mu) = (A_n^{(m)}, \delta_n^{(m)}) \) be an \( m \)-component link graded DGA defined as follows:

1. The algebra \( A_n^{(m)} \) is the algebra \( T_{co} M^{(m)} \) over the ring \( \mathbb{K}(m) \)

\[
A_n^{(m)} := T_{co}(\mathbb{K}(m)(K^{ij} \coprod Y^{ij})), \quad K^{ij}_n := \{k_{ab} \mid 1 \leq a < b \leq n\}, \quad Y^{ij}_n := \{(y_{a}^{ij} \mid a \in [n]) \mid i < j; \quad i \geq j.
\]

2. The link-grading for each \( k_{ab}^{ij} \) and \( y_{a}^{ij} \) is defined as \((i, j)\) and the (homological) grading of each is defined as

\[
|k_{ab}^{ij}| := |k_{ab}| \quad \text{and} \quad |y_{a}^{ij}| := -1.
\]

3. The differential \( \delta_n^{(m)} \) is defined as

\[
\delta_n^{(m)}(k_{ab}^{ij}) := \sum_{a \leq c \leq b; \quad i < k < j} (-1)^{|k_{ac}|}k_{ac}^{ij}k_{cb}^{kj} + \sum_{k < j} (-1)^{|k_{ab}| - 1}k_{ab}^{ij}y_{b}^{kj} + \sum_{i < k} y_{a}^{ik}k_{ab}^{kj}.
\]

\[
\delta_n^{(m)}(y_{a}^{ij}) := \sum_{i < k} y_{a}^{ik}k_{ab}^{kj}.
\]

Geometrically, generators \( k_{ab}^{ij} \) and \( y_{a}^{ij} \) correspond to the Reeb chord from the \( b \)-th edge in the \( j \)-th copy \( T_a^j \) to the \( a \)-th edge in the \( i \)-th copy \( T_a^i \) and the Reeb chord from the \( a \)-th edge of \( T_a^j \) to \( a \)-th edge of \( T_a^i \), respectively.
For each $U \subset [m]$, the restriction $A^{U}_{ij}(\mu)$ will be defined as the image of the quotient map sending all $k_{ab}^{ij}$'s and $y_{ij}'$'s to zero unless $i, j \in U$. This implies that $A^{(m)}_{ij}(\mu)$ satisfies all axioms for $m$-component objects.

Moreover, the sequence $(A^{(m)}_{ij}(\mu))_{m \geq 1}$ satisfies the consistency as follows: for each $h : [m] \to [m']$ and $U \subset [m]$, there is an obvious isomorphism $A^{(hU)}_{ij}(\mu) \to A^{(U)}_{ij}(\mu)$ sending each $k_{ab}^{ij}$ and $y_{ij}'$ to $k_{ab}^{ij}$ and $y_{ij}'$ if $i' = h(i)$ and $j' = h(j)$. Therefore we have a consistent sequence of composable DGAs $A^{(U)}_{ij}(\mu) \in \mathcal{D} \mathcal{A}^{(m)}_{co}$.

**Corollary 3.24.** The composable DGA $A^{(m)}_{ij}(\mu)$ can be obtained from the semi-free border DGA $A^{(m)}_{co}(\mu)$ via the standard recipe described in Example 3.21, where $\mu^{(m)} : [mn] \equiv [m] \times [n] \to \mathbb{Z}$ is defined to be

$$\mu^{(m)}(i) = \mu(a).$$

**Proof.** This follows directly from the definition and we are done. $\square$

Now we consider consistent sequences of bordered DGAs.

**Definition 3.25 (m-component bordered DGAs).** An $m$-component bordered DGA of type $(n_L, n_R)$

$$\mathcal{A}^{(m)} = \left( A^{(m)}_{L} \xrightarrow{\phi^{(m)}_{L}} A^{(m)} \xleftarrow{\phi^{(m)}_{R}} A^{(m)}_{R} \right)$$

is a bordered DGA satisfying the following conditions:

1. All DGAs $A^{(m)}_{L}$ for $* = L, R$ or empty are in $\mathcal{D} \mathcal{A}^{(m)}_{co}$,
2. two DGAs $A^{(m)}_{L}$ and $A^{(m)}_{R}$ are isomorphic to $A^{(m)}_{L} (\mu_L)$ and $A^{(m)}_{R} (\mu_R)$ for some $\mu_L$ and $\mu_R$,
3. two structure morphisms $\phi^{(m)}_{L}$ and $\phi^{(m)}_{R}$ are composable morphisms.

For each $U \subset [m]$, the restriction $\mathcal{A}^{U}$ for $U \subset [m]$ is defined by the image of the quotient map sending all elements in $A^{(m)}_{ij}(U)$ to zero unless $i, j \in U$

$$\mathcal{A}^{U} := \left( A^{U}_{L} \xrightarrow{\phi^{U}_{L}} A^{U} \xleftarrow{\phi^{U}_{R}} A^{U}_{R} \right),$$

where $\phi^{U}_{L}$ and $\phi^{U}_{R}$ are the induced maps by $\phi^{(m)}_{L}$ and $\phi^{(m)}_{R}$, respectively.

A bordered composable morphism $\mathcal{A}^{(n)} \to \mathcal{A}^{(m)}$ between $n$ and $m$-components bordered composable DGAs is a collection of bordered DGA morphisms $\{ \mathcal{A}^{(hU)} \to \mathcal{A}^{U} | U \subset [m] \}$ for a uniquely determined $(h : [m] \to [n]) \in \Delta_s$, which is compatible with the restriction map $\mathcal{A}^{(m)} \to \mathcal{A}^{U}$.

We denote the category of all bordered composable DGAs by $\mathcal{B} \mathcal{D} \mathcal{A}^{(m)}_{co}$ and its full subcategory of $m$-components by $\mathcal{B} \mathcal{D} \mathcal{A}^{(m)}_{co}^{(m)}$.

By definition, the categories $\mathcal{D} \mathcal{A}^{(m)}_{co}$ and $\mathcal{D} \mathcal{A}^{(m)}_{co}^{(m)}$ are full subcategory of $\mathcal{B} \mathcal{D} \mathcal{A}^{(m)}_{co}$ and $\mathcal{B} \mathcal{D} \mathcal{A}^{(m)}_{co}^{(m)}$ of type $(0, 0)$, respectively.

Let $\mathcal{A}^{(m)} \in \mathcal{B} \mathcal{D} \mathcal{A}^{(m)}_{co}$ be a sequence of $m$-components bordered DGAs. Then it is consistent if and only if for each $(h : [m] \to [n]) \in \Delta_s$ and $U \subset [m]$, we have an isomorphism $\mathcal{A}(h)^U : \mathcal{A}(h) \to \mathcal{A}^{U}$ between composable bordered DGAs.

Similarly, a sequence $\mathcal{F}^{(m)} : \mathcal{A}^{(*)} \to \mathcal{A}^{(*)}$ between consistent sequences of bordered DGAs is consistent if and only if for each $m \geq 1$, $\mathcal{F}^{(m)}$ is a bordered morphism between $m$-component bordered DGAs $\mathcal{A}^{(m)}$ and $\mathcal{A}^{(m)}$, and it satisfies the consistency, i.e., for each $h : [m] \to [n]$ and $U \subset [m]$, the following diagram is commutative:

$$\begin{array}{ccc}
\mathcal{A}^{(hU)} & \xrightarrow{\mathcal{F}^{(h)^U}} & \mathcal{A}^{U} \\
\downarrow \mathcal{A}(h)^U & & \downarrow \mathcal{F}^{U} \\
\mathcal{A}(h) & \xrightarrow{\mathcal{F}^{U}} & \mathcal{A}^{U}
\end{array}$$

**Definition 3.26 (Consistent sequences of composable bordered DGAs).** We denote the category of consistent sequences of composable bordered DGAs by $\mathcal{B} \mathcal{D} \mathcal{A}^{(*)}_{co}$. 
In terms of generators, we have the following observation. Let \( A^{(m)} = \left( A^{(m)} = T_{\text{co}}(\mathbb{E}^{(m)}(S^{(m)})), \hat{\partial}^{(m)} \right) \) be a composable \( m \)-component DGA. Then the restriction on \( U \) is now given by the image of the map \( I_k \mapsto 0 \) for all \( e \in E \) unless \( i \in U \). Moreover, the consistency implies that for each \( (h : [m] \to [n]) \in \Delta_{\ast} \), the DGA \( A^{(m)} \) is isomorphic to the DGA generated by \( S^{i j}_{\ast} \)'s for \( i', j' \in h([m]) \). Therefore we may identify

\[
S^{ij} \cong S^{11} \quad \text{and} \quad S^{ij} \cong S^{12}
\]

by considering \( h_i : [1] \to [m] \) and \( h_{ij} : [2] \to [m] \) with \( h_i(1) = i \) and \( h_{ij}(1) = i, h_{ij}(2) = j \).

Finally, the differential \( \hat{\partial}^{(m)} \) is the same as the push-forward of \( \hat{\partial}^{(n)} \) on \( A(h) : A^{(n)} \to A^{(m)} \). In other words, for each generator \( \hat{a}^{ij} \in S^U \),

\[
\hat{\partial}^{(m)}\left( a^{ij} \right) = (A(h) \circ \hat{\partial}^{(n)} \circ A(h)^{-1})(a^{ij}) = A(h)\left( \hat{\partial}^{(n)}(a^{i'j'}) \right),
\]

where \( i' = h(i), j' = h(j) \) and \( a^{i'j'} \) is regarded as an element of \( S^{i'j'} \cong S^U \). In other words, the differential in \( A^{(m)} \) can be obtained from the differential in \( A^{(n)} \) by removing generators not coming from \( A^{(m)} \).

The following is the direct consequence of Lemma 3.3 and summarizes the above discussion.

**Corollary 3.27.** [21, Lemma 3.15] Let \( \mathcal{A}^{(\ast)} \in \mathcal{BDGA}^{(\ast)}_{\text{co}} \) be a consistent sequence of bordered DGAs. Then for each \( h_i : [1] \to [m] \) with \( h_i(1) = i \) and \( h_{ij} : [2] \to [m] \) with \( h_{ij}(1) < j = h_{ij}(2) \), we have isomorphisms between DGAs

\[
\mathcal{A}^{ij} \cong \mathcal{A}^{11} \quad \text{and} \quad \mathcal{A}^{ij} \cong \mathcal{A}^{12}.
\]

**Definition 3.28** (Consistent stabilizations). A consistent sequence of bordered DGAs \( \mathcal{A}^{(\ast)} \in \mathcal{BDGA}^{(\ast)}_{\text{co}} \) is a (weak) consistent stabilization of \( \mathcal{A}^{(\ast)} \in \mathcal{BDGA}^{(\ast)}_{\text{co}} \) if there exists a consistent morphism \( \pi^{(\ast)} : \mathcal{A}^{(\ast)} \to \mathcal{A}^{(\ast)} \) of canonical projections of stabilization, and is a strong consistent stabilization if it is a weak stabilization and there exists a consistent morphism \( \hat{\pi}^{(\ast)} : \mathcal{A}^{(\ast)} \to \mathcal{A}^{(\ast)} \) of canonical embeddings.

Let \( A^{(\ast)} = (A^{(\ast)} = T_{\text{co}}(\mathbb{E}^{(\ast)}(S^{(\ast)})), \hat{\partial}^{(m)} \) be a consistent stabilization of \( A^{(\ast)} = (A^{(\ast)} = T_{\text{co}}(\mathbb{E}^{(\ast)}(S^{(\ast)})), \partial^{(m)} \).

Then in terms of generating sets, for each \( m \geq 1 \), there exists an index set \( I^{(m)} = (I_{\text{co}}^{(m)}) \) such that

\[
S^{ij} = S^U \sqcup \{ e^{ij}_k, e^{ij}_k | k \in I^{(m)} \}, \quad |e^{ij}_k| = |e^{ij}_k| + 1, \quad \hat{\partial}^{(m)}(e^{ij}_k) = e^{ij}_k, \quad \partial^{(m)}(e^{ij}_k) = 0.
\]

Then for each \( U \subset \Delta_{\ast} \), the restriction \( A^U \) is generated by

\[
S^U = S^U \sqcup \{ e^{ij}_k, e^{ij}_k | k \in I^{(m)} \}
\]

which is the image of the quotient map sending all \( e^{ij}_k \)'s and \( e^{ij}_k \)'s to zero unless both \( i, j \in U \). Finally, for each \( (h : [m] \to [n]) \in \Delta_{\ast} \), the consistency implies that we have an isomorphism

\[
A^{h(U)} \cong A^U
\]

sending each generator \( e^{i'j'}_k \) and \( e^{i'j'}_k \) to \( e^{ij}_k \) and \( e^{ij}_k \), respectively, for \( h(i, j) = (i', j') \). Therefore it is necessary and sufficient to have the following conditions

\[
I^{(m)} \cong I^{(m)} \quad \text{and} \quad |e^{ij}_k| = |e^{ij}_k|
\]

for \( (k, l) = (1, 1), (1, 2) \) or \( (2, 1) \) according to whether \( i = j, i < j \) or \( i > j \).

As before, the one important example of a consistent stabilization is the cofibrant replacement of a consistent sequence of bordered DGAs defined as follows:

**Example/Definition 3.29** (Consistent cofibrant replacement). Let \( \mathcal{A}^{(\ast)} \in \mathcal{BDGA}^{(\ast)}_{\text{co}} \) be a consistent sequence of borded composable DGAs. For each \( m \geq 1 \), we consider the cofibrant replacement \( \mathcal{A}^{(m)} \) of \( \mathcal{A}^{(m)} \) defined in Definition 2.40.

For each \( U \subset [m] \), the restriction \( \mathcal{A}^{(m)}_U \) will be given as the cofibrant replacement of \( \mathcal{A}^U \) and therefore \( \mathcal{A}^{(m)}_U \) is an \( m \)-component bordered DGA as desired.

Finally, the consistency of \( \mathcal{A}^{(m)}_U \) comes easily from the consistency of \( \mathcal{A}^{(m)} \) as follows: for each \( h : [m] \to [n] \) and \( U \subset [m] \), the bordered DGA \( \mathcal{A}^{(m)}_U \) is the cofibrant replacement of \( \mathcal{A}^{h(U)} \) as above which is isomorphic to \( \mathcal{A}^U \) due to the consistency of \( \mathcal{A}^{(\ast)} \). Since the cofibrant replacement remains the same under isomorphism, we have a desired isomorphism

\[
\mathcal{A}^{h(U)} \cong \mathcal{A}^U.
\]
Lemma 3.30. For each $\mathcal{A}(\oslash) \in \mathcal{B}\mathcal{D}\mathcal{G}\mathcal{A}_{\mathcal{L}}$, its cofibrant replacement $\overline{\mathcal{A}}(\oslash) \in \mathcal{B}\mathcal{D}\mathcal{G}\mathcal{A}_{\mathcal{L}}$ is well-defined.

Proof. This is a summary of discussion above and we omit the proof. □

3.2.2. Consistent sequences of LCH DGAs. Let $(T(\oslash), \mu(\oslash)) \in \mathcal{B}\mathcal{L}\mathcal{C}(\oslash)$ be a consistent sequence of front projections of bordered Legendrian graphs with Maslov potentials. Then for each $m \geq 1$, we have the bordered LCH DGA $A^{CE}(T(m), \mu(m))$

$$A^{CE}(T(m), \mu(m)) = \left( A^{CE}(T(m), \mu(m)) \right)_{m \geq 1}$$

It is obvious that the DGA $A^{CE}(T(m), \mu(m))$ for each $\oslash = L, R$ or empty is $m$-component link-graded but is not composable. However, by using the standard recipe described in Example 3.21, we obtain an $m$-component composable bordered DGA $A^{CE}_{co}(T(m), \mu(m))$

$$A^{CE}_{co}(T(m), \mu(m)) = \left( A^{CE}_{co}(T(m), \mu(m)) \right)_{m \geq 1}$$

Then it is obvious that two induced morphisms $\phi_{L}^{(m)}$ and $\phi_{R}^{(m)}$ are composable morphisms but it is not yet known to satisfy the axiom for $m$-component objects.

Proposition 3.31. The bordered composable DGA $A^{CE}_{co}(T(m), \mu(m)) \in \mathcal{B}\mathcal{D}\mathcal{G}\mathcal{A}_{\mathcal{L}}$ is an $m$-component bordered composable DGA. Moreover, if $(T^{(\oslash)}, \mu^{(\oslash)})) \in \mathcal{B}\mathcal{L}\mathcal{C}(\oslash)$ is a consistent sequence, then the corresponding sequence

$$A^{CE}_{co}(T^{(\oslash)}, \mu^{(\oslash)}) := \left( A^{CE}_{co}(T^{(\oslash)}, \mu^{(\oslash)}) \right)_{m \geq 1}$$

of bordered DGAs is consistent.

Proof. We know that two border DGAs $A^{CE}_{co}(T^{(\oslash)}, \mu^{(\oslash)})$ and $A^{CE}_{co}(T^{(\oslash)}, \mu^{(\oslash)})$ are isomorphic to the $m$-component border DGAs

$$A^{CE}_{co}(T^{(\oslash)}, \mu^{(\oslash)}) \equiv A^{CE}_{co}(T^{(\oslash)}, \mu^{(\oslash)})$$

by Corollary 3.24.

For each $U \subset [m]$, we define the restriction $A^{CE}_{co}(T^{(\oslash)}, \mu^{(\oslash)})_{U}$ as the image of the quotient map sending all generators of type $(i, j)$ to zero unless $i, j \in U$. Hence $A^{CE}_{co}(T^{(\oslash)}, \mu^{(\oslash)})$ admits an $m$-component object structure.

Moreover, we have an isomorphism

$$A^{CE}_{co}(T^{(\oslash)}, \mu^{(\oslash)})_{U} \equiv A^{CE}_{co}(T^{(\oslash)}, \mu^{(\oslash)})$$

since we only consider generators and immersed disks lying on $T^{(\oslash)}$ in the left hand side.

Now assume that $(T^{(\oslash)}, \mu^{(\oslash)})$ is consistent and let $A^{CE}_{co}(T^{(\oslash)}, \mu^{(\oslash)})$ be the sequence of bordered LCH DGAs of $(T^{(\oslash)}, \mu^{(\oslash)}) \in \mathcal{B}\mathcal{L}\mathcal{C}(\oslash)$. Then for each $(h : [m] \rightarrow [n]) \in \Delta_{+}$ and $U \subset [m]$, we have isomorphisms between bordered composable DGAs

$$A^{CE}_{co}(T^{(\oslash)}, \mu^{(\oslash)})_{h(U)} \equiv A^{CE}_{co}(T^{(\oslash)}, \mu^{(\oslash)})_{h(U)} \quad \text{and} \quad A^{CE}_{co}(T^{(\oslash)}, \mu^{(\oslash)})_{U} \equiv A^{CE}_{co}(T^{(\oslash)}, \mu^{(\oslash)})_{U}$$

by the observation (3.2). Since two restrictions $(T^{(\oslash)}, \mu^{(\oslash)})$ and $(T^{(\oslash)}, \mu^{(\oslash)})$ are isomorphic due to the consistency of $(T^{(\oslash)}, \mu^{(\oslash)})$, we have a desired isomorphism

$$A^{CE}_{co}(T^{(\oslash)}, \mu^{(\oslash)})_{h(U)} \equiv A^{CE}_{co}(T^{(\oslash)}, \mu^{(\oslash)})_{U}$$

which means the consistency of $A^{CE}_{co}(T^{(\oslash)}, \mu^{(\oslash)})$.

Let $(M^{(\oslash)} : (T^{(\oslash)}, \mu^{(\oslash)}) \rightarrow (T^{(\oslash)}, \mu^{(\oslash)})$ be an elementary consistent front Reidemeister move and let us denote their bordered composable LCH DGAs by $\mathcal{A}(\oslash)$ and $\mathcal{A}(\oslash)$

$$\mathcal{A}(\oslash) = \left( A^{L}_{\mathcal{A}} \xrightarrow{\phi^{(\oslash)}} A^{\oslash} \xleftarrow{\phi^{(\oslash)}} A^{R}_{\mathcal{A}} \right) := A^{CE}_{co}(T^{(\oslash)}, \mu^{(\oslash)})$$

$$\mathcal{A}(\oslash) = \left( A^{L}_{\mathcal{A}} \xrightarrow{\phi^{(\oslash)}} A^{\oslash} \xleftarrow{\phi^{(\oslash)}} A^{R}_{\mathcal{A}} \right) := A^{CE}_{co}(T^{(\oslash)}, \mu^{(\oslash)})$$
Then as before, we will pass through the cofibrant replacement as follows. By Lemma 3.30, the cofibrant replacements

\[
\tilde{\mathcal{R}}^{(*)} = \left( A_L^{(*)} \xrightarrow{\hat{\varphi}_L^{(*)}} \tilde{A}^{(*)} \xleftarrow{\hat{\varphi}_R^{(*)}} A_R^{(*)} \right) \quad \text{and} \quad \hat{\mathcal{R}}^{(*)} = \left( A_L^{(*)} \xrightarrow{\hat{\varphi}_L^{(*)}} \hat{A}^{(*)} \xleftarrow{\hat{\varphi}_R^{(*)}} A_R^{(*)} \right)
\]

are also consistent and furthermore stabilizations of \( \mathcal{R}^{(*)} \) and \( \hat{\mathcal{R}}^{(*)} \), respectively. That is, there are consistent morphisms \( \tilde{\mathcal{R}}^{(*)} \) and \( \hat{\mathcal{R}}^{(*)} \) of canonical projections

\[
\tilde{\mathcal{R}}^{(*)} : \tilde{\mathcal{R}}^{(*)} \xrightarrow{\sim} \mathcal{R}^{(*)} \quad \text{and} \quad \hat{\mathcal{R}}^{(*)} : \hat{\mathcal{R}}^{(*)} \xrightarrow{\sim} \mathcal{R}^{(*)}.
\]

On the other hand, \((M)^{(*)}\) gives us an elementary consistent Reidemeister move

\[
(M)^{(*)} : (\tilde{T}^{(*)}, \tilde{\mu}^{(*)}) \rightarrow (\tilde{T}^{(*)}, \tilde{\mu}^{(*)}).
\]

Then by the condition (2) of consistent Reidemeister moves in Definition 3.5 and as observed in the discussion after Theorem 2.43, for each \( m \geq 1 \) it induces a zig-zag of stabilizations

\[
A_{co}^{CE}(\tilde{T}^{(m)}, \mu^{(m)}) \xleftarrow{\tilde{\mathcal{R}}^{(m)}} A_{co}^{CE}(\tilde{T}^{(m)}, \mu^{(m)}) \xrightarrow{\hat{\mathcal{R}}^{(m)}} A_{co}^{CE}(\tilde{T}^{(m)}, \mu^{(m)}),
\]

\[
A_{co}^{CE}(\tilde{T}^{(m)}, \mu^{(m)}) \xleftarrow{\tilde{\mathcal{R}}^{(m)}} A_{co}^{CE}(\tilde{T}^{(m)}, \mu^{(m)}) \xrightarrow{\hat{\mathcal{R}}^{(m)}} A_{co}^{CE}(\tilde{T}^{(m)}, \mu^{(m)})
\]

for \((M) \in \{ (I), (II), (VI) \} \), or

\[
A_{co}^{CE}(\tilde{T}^{(m)}, \mu^{(m)}) \xleftarrow{\tilde{\mathcal{R}}^{(m)}} A_{co}^{CE}(\tilde{T}^{(m)}, \mu^{(m)}) \xrightarrow{\hat{\mathcal{R}}^{(m)}} A_{co}^{CE}(\tilde{T}^{(m)}, \mu^{(m)})
\]

between \( m \)-component bordered composable LCH DGAs. Moreover, the condition (3) in Definition 3.5 implies the consistency of all arrows above and therefore we have a zig-zag of consistent stabilizations corresponding to \((M)^{(*)}\).

Similarly, one can observe the same property for elementary consistent Lagrangian Reidemeister moves as well and we omit the detail.

In summary, we have the following theorem.

**Theorem 3.32.** For \((\tilde{T}^{(*)}, \mu^{(*)}) \in \mathcal{B}L\mathcal{S}_{\text{Co}}^{(*)}\) and \((\tilde{T}^{(*)}_{\text{Lag}}, \mu^{(*)}) \in \mathcal{B}L\mathcal{S}_{\text{Lag}}^{(*)}\), the assignments

\[
(\tilde{T}^{(*)}, \mu^{(*)}) \mapsto A_{co}^{CE} \left( \tilde{T}^{(*)}, \mu^{(*)} \right) \in \mathcal{B}DG\mathcal{A}_{co}^{(*)} \quad \text{and} \quad (\tilde{T}^{(*)}_{\text{Lag}}, \mu^{(*)}) \mapsto A_{co}^{CE} \left( \tilde{T}^{(*)}_{\text{Lag}}, \mu^{(*)} \right) \in \mathcal{B}DG\mathcal{A}_{co}^{(*)}
\]

are well-defined and each elementary consistent front or Lagrangian Reidemeister move induces a zig-zag of consistent stabilizations.

Now let us consider the consistent basepoint move. For an elementary consistent basepoint move \((B_i)^{(*)}\) or \((b_i)^{(*)}\), we already know that it induces a zig-zag of consistent stabilizations since it is indeed a sequence of elementary consistent Reidemeister moves.

For the moves \((B_i)^{(*)}\) or \((b_i)^{(*)}\) with \( i = 2 \) or 3, it is obvious that there is a consistent morphisms

\[
A_{co}^{CE}(\tilde{T}^{(*)}, \mu^{(*)}) \xrightarrow{(B_i)^{(*)}} A_{co}^{CE}(\tilde{T}^{(*)}, \mu^{(*)}) \quad \text{and} \quad A_{co}^{CE}(\tilde{T}^{(*)}_{\text{Lag}}, \mu^{(*)}) \xrightarrow{(b_i)^{(*)}} A_{co}^{CE}(\tilde{T}^{(*)}_{\text{Lag}}, \mu^{(*)}).
\]

Then as mentioned in Remark 2.47, both bordered consistent morphisms \((B_i)^{(*)}\) and \((b_i)^{(*)}\) are well-defined and left inverses of consistent morphisms \((B_i)^{(*)}\) and \((b_i)^{(*)}\), respectively.
3.2.3. Legendrian graphs in a normal form. Recall the definition of Legendrian graphs or bordered Legendrian graphs in a normal form defined in Definition 2.17.

Let \((T, \mu) \in \mathcal{BCL}_\text{Lag}^\mu\) be a bordered Legendrian graph of type \((n_L, n_R)\) in a normal form and \((T_{\text{Lag}}, \mu) \in \mathcal{BCL}_\text{Lag}^\mu\) be its image of \(\text{Ng}\)'s resolution.

Let us consider the canonical Lagrangian \(m\)-copies \(T_{\text{Lag}}^{(m)}\). One of the benefit of being in a normal form is indeed that the DGA \(A^{CE}_{\text{Lag}}(T_{\text{Lag}}^{(m)} , \mu^{(m)})\) can be easily described in terms of the data of \(T_{\text{Lag}}\) and the DGA \(A^{CE}_{\text{Lag}}(T_{\text{Lag}}, \mu)\).

Recall that the \(x\)-maximal points in \(T\) are basepoints and their \(\text{Ng}\)’s resolutions look as depicted in Definition 2.27. Therefore the canonical Lagrangian \(m\)-copy near each basepoint \(b \in T_{\text{Lag}}\) (or equivalently, a \(x\)-maximum) looks like

\[
\xymatrix{ c \ar[r] & T \ar[r] & T_{\text{Lag}} \ar[r]^{(-)^{(m)}} & T_{\text{Lag}}^{(m)} \ar[r] & c }
\]

Let us consider the set of \(x\)-minimal points in \(T_{\text{Lag}}\). Then it has one-to-one correspondence with the set of connected components of the complement of \(V \sqcup B\), which is the set of edges.

The set of generators for \(A^{CE}(T_{\text{Lag}}, \mu)\) is the union of \(C, K_L, V\) and \(\bar{B}\)

\[
C = \{ c \mid c \text{ is a double point} \}, \quad K_L = \{ k_{ab} \mid 1 \leq a < b \leq n_L \}, \\
V = \{ v_{a,\ell} \mid v \in V, 1 \leq a \leq \text{val}(v), \ell \geq 1 \}, \\
\bar{B} = \{ b_{a,\ell} \mid b \in B, a = 1, 2, \ell \geq 1 \}.
\]

As seen in Definition 2.17 and Example 2.45, we define the sets \(\bar{V}_L\) and \(\bar{V}_C\). Then the generating set for \(A^{CE}(T_{\text{Lag}}^{(m)}, \mu^{(m)})\) is the union of the following sets:

\[
C^{(m)} \sqcup K_L^{(m)} \sqcup V^{(m)}_L \sqcup \bar{V}^{(m)}_C \sqcup \bar{B}^{(m)} \sqcup X^{(m)} \sqcup Y^{(m)},
\]

where

\[
C^{(m)} := \{ c_{ij} \mid c \in C, i, j \in [m] \}, \quad |c_{ij}| := |c|, \\
K_L^{(m)} := \{ k_{ij}^{ab} \mid 1 \leq a < b \leq n_L \}, \quad |k_{ij}^{ab}| := |k_{ab}|, \\
V^{(m)}_L := \{ v_{a,\ell}^{ij} \mid v_{a,\ell} \in \bar{V}_L, i \leq j \in [m] \}, \quad |v_{a,\ell}^{ij}| := |v_{a,\ell}|, \\
\bar{V}^{(m)}_C := \{ v_{a,\ell}^{ii} \mid v_{a,\ell} \in \bar{V}_C, i \in [m] \}, \quad |v_{a,\ell}^{ii}| := |v_{a,\ell}|, \\
\bar{B}^{(m)} := \{ b_{a,\ell}^{ij} \mid b_{a,\ell} \in \bar{B}, i \in [m] \}, \quad |b_{a,\ell}^{ij}| := |b_{a,\ell}|, \\
X^{(m)} := \{ x_{ij}^b \mid b \in B, i < j \in [m] \}, \quad |x_{ij}^b| := 0, \\
Y^{(m)} := \{ y_{ij}^e \mid y_e \in E, i < j \in [m] \}, \quad |y_{ij}^e| := -1.
\]

Geometrically, these generating sets corresponding to the crossings and vertex generators of \(T^{(m)}\) as follows:
• Crossing: for each $c \in C$, there are $m^2$-many copies $c_{ij}$ in $C^{(m)}$

• Vertex: for each $v \in V$, there are two sets of generators $\overline{v}_{V,L}^{(m)}$ and $\overline{v}_{V,C}^{(m)}$

• Left border: for each $k_{ab} \in K_L$ with $a < b$, there is the set $k_{l}^{(m)}$ of $m^2$ generators $k_{ab}^{ij}$, and for each $i < j$ and $1 \leq a \leq \ell$, there are $\binom{m}{2}$ generators $\gamma_{ab}^{ij}$

• Basepoint: for each $b_{a,\ell} \in B$, we have the set of $m$ generators $b_{a,\ell}^{ij}$

• $x$-maximum: for each $b \in B$, we have the set of $\binom{m}{2}$ generators $x_{b}^{ij}$ for $i < j$
• $x$-minimum: for each $e \in E$, we have the subset of $\binom{m}{2}$ generators $y^j_i$ for $i < j$

\begin{align}
\Phi(c) &:= \begin{pmatrix}
c^{11} & c^{12} & \cdots & c^{1m} \\
c^{21} & c^{22} & \cdots & c^{2m} \\
\vdots & \vdots & \ddots & \vdots \\
c^{m1} & c^{m2} & \cdots & c^{mm}
\end{pmatrix}, & \Phi(k_{ab}) &:= \begin{pmatrix}k^{11}_{ab} & k^{12}_{ab} & \cdots & k^{1m}_{ab} \\
k^{21}_{ab} & k^{22}_{ab} & \cdots & k^{2m}_{ab} \\
\vdots & \vdots & \ddots & \vdots \\
k^{m1}_{ab} & k^{m2}_{ab} & \cdots & k^{mm}_{ab}
\end{pmatrix}, \\
\Phi(v_{a,\ell}) &:= \begin{pmatrix}v^{11}_{a,\ell} & v^{12}_{a,\ell} & \cdots & v^{1m}_{a,\ell} \\
v^{21}_{a,\ell} & v^{22}_{a,\ell} & \cdots & v^{2m}_{a,\ell} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & v^{mm}_{a,\ell}
\end{pmatrix}, & \Phi(w_{a,\ell}) &:= \begin{pmatrix}w^{11}_{a,\ell} & w^{12}_{a,\ell} & \cdots & w^{1m}_{a,\ell} \\
w^{21}_{a,\ell} & w^{22}_{a,\ell} & \cdots & w^{2m}_{a,\ell} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & w^{mm}_{a,\ell}
\end{pmatrix}, \\
\Delta(b_{a,\ell}) &:= \begin{pmatrix}h^{11}_{a,\ell} & 0 & \cdots & 0 \\
0 & b^{22}_{a,\ell} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & b^{mm}_{a,\ell}
\end{pmatrix}, & X_b &:= \begin{pmatrix}1 & x^{12}_b & \cdots & x^{1m}_b \\
x^{21}_b & 1 & \cdots & x^{2m}_b \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}.
\end{align}

\[
Y_e := \begin{pmatrix}
y^{e1}_c & y^{e2}_c & \cdots & y^{em}_c \\
y^{e2}_c & y^{e3}_c & \cdots & y^{em}_c \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}
\]

We also define
\[
\Phi(b_{1,1}) := \Delta(b_{1,1})X_b \quad \text{and} \quad \Phi(b_{2,1}) := X_b^{-1}\Delta(b_{2,1}).
\]

The differential $\partial^{(m)}$ can be given as follows:

• for each $c^{ij} \in C^{(m)}$ with upper and lower edges $e, e' \in E$,

\[
\partial^{(m)}(c^{ij}) := \Phi(\partial(c))_{ij} + \sum_{k<l}(-1)^{|e'|}c^{ik}y^{kj}_e + \sum_{i<k}y^{ik}c^{kj}
\]

• for each $k_{ab}^{ij} \in K^{(m)}_L$ with two edges $e$ and $e'$ containing left borders corresponding $a$ and $b$,

\[
\partial^{(m)}(k_{ab}^{ij}) := \Phi(\partial(k_{ab}))_{ij} + \sum_{k<l}(-1)^{|e'|}k_{ab}^{ik}y^{kj}_{e'} + \sum_{i<k}y^{ik}k_{ab}^{kj}
\]

• for each $v_{a,\ell} \in \mathcal{V}_L$ with two half-edges $h_{v,a} \subset e \in E$ and $h_{v,a+\ell} \subset e' \in E$,

\[
\partial^{(m)}(v_{a,\ell}^{ij}) := \Phi(\partial(v_{a,\ell}))_{ij} + \sum_{i<k<j}(-1)^{|v_{a,\ell}'|}v_{a,\ell}^{ik}y^{kj}_{e'} + y^{ik}v_{a,\ell}^{kj}
\]
• for each $b \in B$ with two edges $(e, e')$ adjacent to $b$,
  \[
  \partial^{(m)}(x_{ij}^{b}) = \sum_{i < k < j} -x_{ik}^{b} y_{kj}^{e} + y_{ik}^{b} x_{kj}^{e}.
  \]

• for each $e \in E$, $b_{a,\ell} \in \tilde{B}$ and $w_{a,\ell} \in \widetilde{V}^{\mathcal{O}_{\partial}}$,
  \[
  \partial^{(m)}(x_{ii}^{b_{a,\ell}}) := \Phi(\partial(b_{a,\ell}))_{ii} = \delta_{\ell,2} + \sum_{\ell_{1}+\ell_{2} = \ell} (-1)^{|b_{a,\ell_{1}}-1|} |b_{a,\ell_{1}} b_{a+\ell_{1},\ell_{2}},
  \partial^{(m)}(w_{ii}^{b_{a,\ell}}) := \Phi(\partial(w_{a,\ell}))_{ii} = \delta_{\ell,\text{val}(w)} + \sum_{\ell_{1}+\ell_{2} = \ell} (-1)^{|w_{a,\ell_{1}}-1|} |w_{a,\ell_{1}} w_{a+\ell_{1},\ell_{2}},
  \partial^{(m)}(y_{i}^{e}) := (Y_{e})^{2}_{i} = \sum_{i < k < j} y_{ik}^{e} y_{kj}^{e}
  \]

Now consider the structure morphisms $\phi^{(m)}_{\mathcal{L}} : A^{(m)} \rightarrow A^{(m)}$ and $\phi^{(m)}_{\mathcal{R}} : A^{(m)}_{\mathcal{R}} \rightarrow A^{(m)}$. Since $\phi^{(m)}_{\mathcal{L}}$ is obvious, we need to consider $\phi^{(m)}_{\mathcal{R}}$. Then due to the geometric definition for $\phi^{(m)}_{\mathcal{R}}$ which counts immersed polygons with the positive end at each generator for $A^{(m)}_{\mathcal{R}}$, we have the similar description to the differential $\partial^{(m)}$ so that

\[
\phi^{(m)}_{\mathcal{R}}(k_{ij}) = \Phi(\partial(k_{ij}))_{ij}, \quad \phi^{(m)}_{\mathcal{R}}(y_{i}^{e}) = y_{i}^{e},
\]

where $1 \leq a' < b' \leq n_{\mathcal{R}}$, $e' \in \{n_{\mathcal{R}}\}$ and $e$ is the edge whose one end point is $e'$.

Here we are using implicitly the fact that any generator in $\overline{V}_{\mathcal{O}_{\partial}}$ never appear in a differential of other types of generators or an image of $\phi^{(m)}_{\mathcal{R}}$ as observed in Example 2.45.

### 4. Augmentation categories

In this section, we review the construction of the augmentation categories which are $A_{\text{coe}}$-categories obtained from consistent sequences of DGAs.

#### 4.1. Augmentation categories for consistent sequences of bordered DGAs

Let $K = (\mathbb{K}, \partial_{K} \equiv 0)$ be a DGA consisting of the base field $\mathbb{K}$ with the trivial differential. An augmentation for $A = (A = TM, \partial) \in \mathcal{D}\mathcal{G}\mathcal{A}$ is a DGA morphism

\[
\epsilon : A \rightarrow K.
\]

One can extend naturally the definition of augmentations to $m$-component DGAs. Let $K^{(m)} := \text{Mat}_{m}(K)$. Then it satisfies the axiom of $m$-component DGAs.

**Definition 4.1** (Augmentations for $m$-component DGAs). For $A^{(m)} = (A^{(m)} = TM^{(m)}, \partial^{(m)}) \in \mathcal{D}\mathcal{G}\mathcal{A}_{\text{coe}}^{(m)}$ with $M^{(m)} = (M^{ij})$, an augmentation of $A^{(m)}$ is a morphism

\[
\epsilon^{(m)} : A^{(m)} \rightarrow K^{(m)},
\]

where

\[
\epsilon^{(m)} = (\epsilon^{(m)}(i, j), \quad \epsilon^{(m)}(i, j) : A^{(m)}(i, j) \rightarrow K, \quad \epsilon^{(m)}(i, i)(1_{\mathbb{K}}^{j}) = 1 \in \mathbb{K}.
\]

We denote the restriction of $\epsilon^{(m)}$ to the graded submodule $M^{ij}$ by $\epsilon^{ij} := \epsilon^{(m)}(i, j)|_{M^{ij}} : M^{ij} \rightarrow \mathbb{K}$, and we say that $\epsilon^{(m)}$ is diagonal if $\epsilon^{ij} \equiv 0$ for $i \neq j$. Conversely, for a sequence $\epsilon = (\epsilon_{1}, \ldots, \epsilon_{m})$ of augmentations of $A$, the diagonal augmentation $\epsilon^{(m)}_{\epsilon} := (\epsilon^{ij}_{\epsilon})$ of $A^{(m)}$ is defined as

\[
\epsilon^{ii} : M^{ii} \xrightarrow{\epsilon_{i}} \mathbb{K}, \quad \epsilon^{ii}_{\epsilon} \equiv 0, \quad \forall i \neq j.
\]

The DGA $A^{(m)}_{\epsilon} = (A^{(m)}_{\epsilon}, \partial^{(m)}_{\epsilon})$ is defined as

\[
A^{(m)}_{\epsilon} := A^{(m)} \quad \text{and} \quad \partial^{(m)}_{\epsilon} := \phi^{(m)}_{\epsilon} \circ \partial^{(m)} \circ (\phi^{(m)}_{\epsilon})^{-1},
\]

where $\phi^{(m)}_{\epsilon} : A^{(m)} \rightarrow A^{(m)}_{\epsilon}$ is an algebra tame isomorphism such that for each generator $s \in R^{(m)}$,

\[
\phi^{(m)}_{\epsilon}(s) := s + \epsilon^{(m)}_{\epsilon}(s).
\]
Lemma 4.2. The 0-th differential \( \partial^{(m)}_{e_{\ell}} \) of the length filtration \( \partial^{(m)}_{e_{\ell}} \) vanishes.

Proof. The proof follows from the direct computation. □

Let \( A^{(i)} \) be a sequence of augmentations of \( \mathcal{A} \). Then by Lemma 3.3, it is obvious that for any \( i < j \in [m] \), there is an isomorphism between \( \left( \mathbb{K}_{\mathcal{E}}, \mathbb{K}_{\mathcal{E}}' \right) \)-modules

\[ M^{ij} = A^{ij} \cong A^{ij}_{(e_i, e_j)}. \]

Let \( M^{ij}_j := (M^{ij})^* [-1] \) be the dual space of \( M^{ij} \) with grading shift by \(-1\). That is, both \( M^{ij} \) and \( M^{ij}_j \) are decomposed into graded pieces

\[ M^{ij} = \bigoplus_{d \in \mathbb{Z}} M_{ij}^d, \quad M^{ij}_j = \bigoplus_{d \in \mathbb{Z}} M_{ij}^{jd}, \quad M_{ij}^{jd+1} := \text{Hom}_{\mathbb{K}} \left( \left( M^{ij} \right)^d, \mathbb{K} \right). \]

For each increasing sequence \( i := (i_1, \ldots, i_{k+1}) \) in \([m]\) and a sequence of integers \( d = (d_1, \ldots, d_k) \in \mathbb{Z}^k \), let us denote the spaces

\[ M^d := M^{i_1 i_2} \otimes M^{i_2 i_3} \otimes \cdots \otimes M^{i_k i_{k+1}}, \quad M^d_j := M^{i_1 i_2} \otimes M^{i_2 i_3} \otimes \cdots \otimes M^{i_k i_{k+1}} \]

whose elements will be denoted as

\[ a^d = a^{i_1 i_2} \otimes a^{i_2 i_3} \otimes \cdots \otimes a^{i_k i_{k+1}} \in M^d, \quad a^{ij} \in M^{ij}. \]

We equip a grading on \( M^d_j \) so that for homogeneous elements \( a^{ij} \in M^d_j \),

\[ |a^{ij}| := |a^{ij}| - |a^d|. \]

Consider the natural pairing \( \langle -, - \rangle : M^d_j \otimes M^d \to \mathbb{K} \) between \( M^d_j \) and \( M^d \) which is nonvanishing only on

\[ \bigoplus_{d \in \mathbb{Z}^k} \left( M^d_j \otimes \left( M^d \right)^d \right) \]

where \( d + 1 := (d_1 + 1, d_2 + 1, \ldots, d_k + 1) \). Since each evaluation \( \langle -, - \rangle_{ij} : M^{ij}_{d+1} \otimes (M^{ij})^d \to \mathbb{K} \) is of degree \(-1\), the pairing \( \langle -, - \rangle \) is of grading \(-k\).

Now let us define a composition map \( m_k : M^d_j \to M^d_{i_1 i_{k+1}} \) as follows: for each \( a^{ij} \in M^d_j \), we require that \( m_k \left( a^{ij} \right) \) satisfies that for each \( a^{i_1 i_{k+1}} \in M^{i_1 i_{k+1}} \),

\[ \langle m_k \left( a^{ij} \right), a^{i_1 i_{k+1}} \rangle_{i_1 i_{k+1}} = (-1)^{|i|} a^{ij} \sigma \left( a^{ij}, a^{i_1 i_{k+1}} \right), \]

where \( \sigma \) is the twisted differential on \( A^{(m)}. \)

Definition 4.3. For any \( k \geq 1 \), and any sequence \( (x_1, x_2, \ldots, x_k) \) with grading \( |x_i| \in \mathbb{Z} \), define

\[ \sigma_k \left( x_1, x_2, \ldots, x_k \right) := k(k - 1)/2 + \sum_{i < j} |x_i||x_j| + \sum_{1 \leq i \leq k} (j - 1)|x_j| \]

In particular, \( \sigma_1 = 0 \) and \( \sigma_2(x_1, x_2) = 1 + |x_1||x_2| + |x_2| \).

Then \( \sigma \) in (4.1) is given as

\[ \sigma = \sigma_k \left( a^{i_1 i_{k+1}}, \ldots, a^{i_1 i_{k+1}} \right) \]

and \( m_k \left( a^{ij} \right) \) is completely determined by this equation.

The degree of \( m_k \) can be computed as follows: since the degrees of pairings on the left and right are \(-1\) and \(-k\), respectively, we have

\[ |m_k \left( a^{ij} \right) \otimes a^{i_1 i_{k+1}}| + |\langle -, - \rangle_{i_1 i_{k+1}}| = 0 \]

and

\[ |m_k \left( a^{ij} \right) \otimes a^{i_1 i_{k+1}}| + |\langle -, - \rangle_{i_1 i_{k+1}}| = |\langle -, - \rangle_{i_1 i_{k+1}}|. \]
This is equivalent to
\[
|m_i (a_i^y) - |a_i^x| = 1 - k + |a^{i_k+i}| - |\partial_{k+1} (a^{i_k+i})| = 2 - k,
\]
which implies that the composition map \(m_i\) is of degree \(2 - k\).

**Definition 4.4.** The augmentation category \(\mathcal{Aug}_+ \left(A^\bullet; K\right)\) is an \(\mathcal{A}_{\infty}\)-category defined as follows:

- The objects are augmentations \(e : A = A^{11} \to K\);
- The morphisms are graded vector spaces
  \[\text{Hom}_{\mathcal{Aug}_+} (e_1, e_2) := \left( A_{(e_1, e_2)}^{12} \right)^\vee \cong \mathbb{S}^{12}_1 \; ;\]
- For \(k \geq 1\), the composition map
  \[m_k : \text{Hom}_{\mathcal{Aug}_+} (e_k, e_{k+1}) \otimes \cdots \otimes \text{Hom}_{\mathcal{Aug}_+} (e_1, e_2) \to \text{Hom}_{\mathcal{Aug}_+} (e_1, e_{k+1})\]
  is defined as \(m_i\) with the sequence \(i = (1, 2, \cdots, k + 1)\).

**Proposition 4.5** (Functoriality of Aug. [21]). The assignment \(A^\bullet \mapsto \mathcal{Aug}_+ \left(A^\bullet; K\right)\) defines a contravariant functor from the category of consistent sequences of DGAs onto the category \(\mathcal{Aug}_{\infty}\) of \(\mathcal{A}_{\infty}\)-categories.

**Proof.** This is just a combination of Propositions 3.17 and 3.20 in [21] and so we omit the proof. \(\Box\)

We briefly review the construction of the \(\mathcal{A}_{\infty}\)-functor
\[\mathcal{F}^\bullet = \left( \mathcal{F}^{(k)} \right)_{k \geq 1} : \mathcal{Aug}_+ \left(A^\bullet; K\right) \to \mathcal{Aug}_+ \left(A^\bullet; K\right)\]
induced from the consistent morphism \(f^\bullet : A^\bullet \to A^\bullet\)
\[\mathcal{F}^{(0)} : \text{Ob} (\mathcal{Aug}_+) \to \text{Ob} (\mathcal{Aug}_+),\]
\[\mathcal{F}^{(k)} : \text{Hom}_{\mathcal{Aug}_+} (e_k, e_{k+1}) \otimes \cdots \otimes \text{Hom}_{\mathcal{Aug}_+} (e_1, e_2) \to \text{Hom}_{\mathcal{Aug}_+} \left(\mathcal{F}^{(0)} (e_1), \mathcal{F}^{(0)} (e_{k+1})\right),\]
where \(e = (e_1, \cdots, e_{k+1})\) is a sequence of augmentations for \(A\), and
\[\mathcal{Aug}_+ := \mathcal{Aug}_+ \left(A^\bullet, K\right), \quad \mathcal{Aug}_+ := \mathcal{Aug}_+ \left(A^\bullet; K\right) .\]

Let \(\mathcal{F}^{(0)} (e_i) := e_i^f\) be the pull-back for each \(i\) by \(e_i\), which is an augmentation for \(A\).
\[e_i^f := (e_1^f, \cdots, e_{k+1}^f), \quad e_i^f (a) := e_i (f(a)).\]

Recall the DGAs \(A_{e}^{(k+1)} = \left( A_{e}^{(k+1)}, \partial_{e}^{(k+1)} \right)\) and \(A_{e}^{(k+1)} = \left( A_{e}^{(k+1)}, \partial_{e}^{(k+1)} \right)\) twisted by the algebra auto-morphisms \(\phi_{e}^{(k+1)}\) and \(\phi_{e}^{(k+1)}\) on \(A^{(k+1)}\) and \(A^{(k+1)}\), respectively. Then the composition
\[f_{e}^{(k+1)} := \phi_{e}^{(k+1)} \circ f^{(k+1)} \circ \left( \phi_{e}^{(k+1)} \right)^{-1} : A_{e}^{(k+1)} \to A_{e}^{(k+1)}\]
becomes a DGA morphism since
\[f_{e}^{(k+1)} \circ \partial_{e}^{(k+1)} = \left( \phi_{e}^{(k+1)} \circ f^{(k+1)} \circ \left( \phi_{e}^{(k+1)} \right)^{-1} \right) \circ \partial_{e}^{(k+1)} = \phi_{e}^{(k+1)} \circ f^{(k+1)} \circ \partial_{e}^{(k+1)} \circ \left( \phi_{e}^{(k+1)} \right)^{-1} = \phi_{e}^{(k+1)} \circ f^{(k+1)} \circ \partial_{e}^{(k+1)},\]
\[\phi_{e}^{(k+1)} \circ f^{(k+1)} \circ \partial_{e}^{(k+1)} = \left( \phi_{e}^{(k+1)} \circ f^{(k+1)} \circ \left( \phi_{e}^{(k+1)} \right)^{-1} \right) \circ \partial_{e}^{(k+1)} = \phi_{e}^{(k+1)} \circ f^{(k+1)} \circ \partial_{e}^{(k+1)} \circ \left( \phi_{e}^{(k+1)} \right)^{-1} = \partial_{e}^{(k+1)} \circ \phi_{e}^{(k+1)} \circ \partial_{e}^{(k+1)} \circ \left( \phi_{e}^{(k+1)} \right)^{-1} = \partial_{e}^{(k+1)} \circ \phi_{e}^{(k+1)} \circ \partial_{e}^{(k+1)} \circ \phi_{e}^{(k+1)} \circ \partial_{e}^{(k+1)} = \partial_{e}^{(k+1)} \circ \partial_{e}^{(k+1)} .\]

**Notation 4.6.** The \(\ell\)-th length filtration of \(f_{e}^{(k+1)}\) will be denoted by \(f_{e,\ell}^{(k+1)}\).
The $A_n$-functor $\mathcal{F}^{(\bullet)}$ will be defined by dualizing the composition $f^{(k+1)}$. More precisely, 

$$
\mathcal{F}^{(k)} : \text{Hom}_{\text{Aug}}(e_i, e_{k+1}) \otimes \cdots \otimes \text{Hom}_{\text{Aug}}(e_1, e_2) \rightarrow \text{Hom}_{\text{Aug}}(e^f_1, e^f_{k+1})
$$

is defined as follows: for each $a_i^\vee := a^\vee_{k,i+1} \otimes \cdots \otimes a^\vee_{1,2}$ with $a^\vee_{k,i+1} \in (A^1_{(e_i, e_{i+1})})^\vee$, 

$$
\left(\mathcal{F}^{(k)}(a^\vee), a^\ast\{^{L^k}\right\}_{1,k+1} = (-1)^\sigma \left(\mathcal{F}^{(k)}(a^\vee_{k,i+1}) \right)^\vee_{1,i+1}
$$

where $\sigma$ is the same as (4.2) in Definition 4.3.

Example 4.7 (Augmentation category for border DGAs). Let $\mu : [n] \rightarrow \mathbb{Z}$ be a function and $A^\ast_n(\mu)$ be the consistent sequence of border DGAs defined in Section 3.2. For simplicity, we denote $\text{Aug}_+(A^\ast_n(\mu); \mathbb{K})$ by $\text{Aug}_+$.

For each $m \geq 1$, the algebra $A^\ast_n$ has the generating sets

$$K_n := \left\{ k^*_{ij} \mid a < b, 1 \leq i, j \leq m \right\} \quad \text{and} \quad Y_n := \left\{ y^*_{ij} \mid 1 \leq a \leq n, 1 \leq i < j \leq m \right\},
$$

where the grading is given as

$$|k^*_{ij}| := \mu(a) - \mu(b) - 1, \quad |y^*_{ij}| := -1.
$$

The differential for each generator is given as follows:

$$
\partial^m_n(k^*_{ij}) := \sum_{a < c < b \atop i < k < j} (-1)^{|k_{ac}| - 1} k^*_{ac} \left( k^*_{ij} \right) + \sum_{k < j} (-1)^{|k_{ab}| - 1} k^*_{ij} \left( k^*_{ab} \right) + \sum_{i < k} y^*_{a} \left( k^*_{ij} \right) y^*_{ik}.
$$

$$
\partial^m_n(y^*_{ij}) := \sum_{i < k < j} y^*_{a} \left( k^*_{ij} \right) y^*_{a} \left( k^*_{ij} \right).
$$

The set of objects of $\text{Aug}_+$ is the augmentation variety for $A_n(\mu)$:

$$\text{Ob} (\text{Aug}_+) = \text{Aug}(A_n(\mu); \mathbb{K}) := \left\{ e : A_n(\mu) \rightarrow \mathbb{K} \right\},
$$

and for any $e_1, e_2 \in \text{Aug}_+$, the set $\text{Hom}_{\text{Aug}_+}(e_1, e_2)$ of morphisms is

$$\text{Hom}_{\text{Aug}_+}(e_1, e_2) = M^{12V} = \mathbb{K} \left\{ k_{ab}^{12V}, y_{c}^{12V} \mid 1 \leq a < b \leq n, c \in [n] \right\}.
$$

The composition map $m_k$ is given as follows:

- For $e_1, e_2 \in \text{Aug}_+$, the map $m_1 : \text{Hom}_{\text{Aug}_+}(e_1, e_2) \rightarrow \text{Hom}_{\text{Aug}_+}(e_1, e_2)$ is defined as

$$m_1 \left( k_{ab}^{12V} \right) = -\sum_{c < a} e_1(k_{ca}) k_{cb}^{12V} + \sum_{b < d} (-1)^{|k_{ab}| - 1} k_{cd}^{12V} e_2(k_{cd}),
$$

$$m_1 \left( y_{c}^{12V} \right) = -\sum_{a < c} e_1(k_{ac}) k_{ac}^{12V} + \sum_{c < d} k_{cd}^{12V} e_2(k_{cd}).
$$

- For $e_1, e_2, e_3 \in \text{Aug}_+$, the map $m_2 : \text{Hom}_{\text{Aug}_+}(e_2, e_3) \otimes \text{Hom}_{\text{Aug}_+}(e_1, e_2) \rightarrow \text{Hom}_{\text{Aug}_+}(e_1, e_3)$ is defined as

$$m_2 \left( k_{cd}^{12V} \otimes k_{ab}^{12V} \right) = \partial_{cd}(-1)^{|k_{ab}| + |k_{cd}|} k_{ab}^{12V} k_{cd}^{12V},
$$

$$m_2 \left( y_{c}^{12V} \otimes y_{a}^{12V} \right) = -\partial_{cd} k_{cd}^{12V},
$$

$$m_2 \left( k_{cd}^{12V} \otimes y_{a}^{12V} \right) = -\partial_{ac} k_{cd}^{12V},
$$

$$m_2 \left( y_{c}^{12V} \otimes k_{ab}^{12V} \right) = -\partial_{ac} y_{a}^{12V}.
$$

- For $m_k$ with $k \geq 3$, the higher composition $m_k$ vanishes.

In particular, the $A_{infty}$-category $\text{Aug}_+(A^\ast_n(\mu); \mathbb{K})$ is in fact a DG category.
Notice that for each $\epsilon \in \text{Aug}_+(A_n^{(*)} (\mu); K)$, the element defined as
\[ y := - \sum_{c \in [n]} y_{12}^{c} \epsilon \in \text{Hom}_{\text{Aug}_+}(\epsilon, \epsilon) \] (4.5)
becomes a cocycle since
\[ m_1(y) = \sum_{c \in [n]} \left( - \sum_{a < c} \epsilon(k_{ac})k^{12}_{ac} + \sum_{c < d} k^{12}_{cd} \epsilon(k_{cd}) \right) = 0. \]

**Corollary 4.8.** For each $\epsilon \in \text{Aug}_+(A_n^{(*)} (\mu); K)$, the element $-y \in \text{Hom}_{\text{Aug}_+}(\epsilon, \epsilon)$ defined in (4.5) becomes the unit of the augmentation category $\text{Aug}_+(A_n^{(*)} (\mu); K)$.

**Proof.** This is a direct consequence of the above computation. \qed

**Proposition 4.9.** The augmentation category $\text{Aug}_+\left( A^{(*)}; K \right)$ of a stabilization $A^{(*)}$ of $A^{(*)}$ is $A_{\infty}$-quasi-equivalent to the augmentation category $\text{Aug}_+(A^{(*)}; K)$ of $A^{(*)}$.

**Proof.** We first identify $A^{(*)}$ with a stabilization of $A^{(*)}$ for each $m \geq 1$ and denote generators for stabilizations by $e$'s and $\hat{c}$'s. For simplicity, we will denote augmentation categories of $A^{(*)}$ and $A^{(*)}$ by $\text{Aug}_+$ and $\text{Aug}_+$, respectively.

Let us consider two consistent morphisms $\iota^{(*)}$ and $\pi^{(*)}$
\[ \iota^{(*)}: A^{(*)} \to A^{(*)}, \quad \pi^{(*)}: A^{(*)} \to A^{(*)}, \]
where $\iota^{(*)}$ is the canonical inclusion of $A^{(*)} \to A^{(*)}$ and $\pi^{(*)}$ sends each $e$ and $\hat{c}$ in $A^{(*)}$ to zero. Then it is obvious that they are homotopy-inverses to each other so that
\[ \pi^{(*)} \circ \iota^{(*)} = \text{Id}_{A^{(*)}}, \quad \iota^{(*)} \circ \pi^{(*)} = \text{Id}_{A^{(*)}}, \]
where the sequence $H^{(*)}$ of homotopies is given by
\[ H^{(*)}(s) := \begin{cases} \hat{c} & s \in A \subset A^{(*)}; \\ 0 & s = e \in A^{(*)}; \\ s = \hat{c} & s \in A^{(*)}. \end{cases} \] (4.6)

It is well-known that an $A_{\infty}$-functor is an $A_{\infty}$-(quasi)-equivalence if it satisfies two condition, (i) essentially (quasi-)surjective, and (ii) (quasi-)fully faithful. In other words, we need to show the following: let $\gamma^{(*)}$ be the $A_{\infty}$-functor induced from $\iota^{(*)}$:

- for each $\epsilon \in \text{Aug}_+$, there exists $\epsilon' \in \text{Aug}_+$ such that $\gamma^{(0)}(\epsilon') := \iota^{(*)} \epsilon'$ and $\epsilon$ are isomorphic up to homotopy,
- for $\epsilon'_1, \epsilon'_2 \in \text{Aug}_+$, the induced chain map
  \[ \gamma^{(1)}: \text{Hom}_{\text{Aug}_+}(\epsilon'_1, \epsilon'_2) \to \text{Hom}_{\text{Aug}_+}\left( \gamma^{(0)}(\epsilon'_1), \gamma^{(0)}(\epsilon'_2) \right) \]
  is a quasi-isomorphism, i.e., an isomorphism between their cohomology groups.

The essential quasi-surjectivity is obvious. Indeed, for any $\epsilon \in \text{Aug}_+$, there is an augmentation $\epsilon' \in \text{Aug}_+$ extended from $\epsilon$ as
\[ \epsilon'(s) := \begin{cases} \epsilon(s) & s \in A \subset A^{(*)}; \\ 0 & \text{otherwise}, \end{cases} \]
such that the pull-back of $\epsilon'$ is precisely $\epsilon$
\[ \gamma^{(0)}(\epsilon') = \epsilon' \circ \iota = \epsilon, \]
and therefore it is indeed surjective.

Let $\epsilon'_1, \epsilon'_2 \in \text{Aug}_+$ be two augmentations. Then by the identification of $A^{(*)}$, we have
\[ \text{Hom}_{\text{Aug}_+}(\epsilon'_1, \epsilon'_2) \cong \text{Hom}_{\text{Aug}_+}\left( \gamma^{(0)}(\epsilon'_1), \gamma^{(0)}(\epsilon'_2) \right) \oplus \bigoplus_{i \in I_{12}} \mathbb{K} A \left( \epsilon'^{(1)}_{12}, \epsilon'^{(2)}_{12}, m_1 \right), \] (4.7)
for some index set \( I^{12} \), where
\[
m'_i(e'_{12}) = \hat{\partial'}_{12} \quad \text{and} \quad m'_i(\hat{\partial'}_{12}) = 0.
\] (4.8)

Then the induced chain map \( \mathcal{I}^{(1)} \) is the projection onto \( \text{Hom}_{\text{Aug}} \left( (\mathcal{I}^{(0)}(e'_1), \mathcal{I}^{(0)}(e'_2)) \right) \), which is surjective with the kernel
\[
\ker \left( \mathcal{I}^{(1)} \right) \cong \left( \bigoplus_{i \in I^{12}} \mathbb{K}_A \left( e'_{12}, \hat{\partial'}_{12} \right), m'_i \right).
\]

Since the kernel is acyclic as seen in (4.8), \( \mathcal{I}^{(1)} \) is a quasi-isomorphism, which implies the quasi-fully faithfulness, and we are done.

\[ \square \]

**Corollary 4.10.** If \( \text{Aug}^*_1 \) is homologically unital, then so is \( \text{Aug}^*_+ \).

**Proof.** Since the induced \( A_\infty \) functor \( \mathcal{I}^{(*)} \) is surjective, \( \text{Aug}^*_+ \) is homologically unital if so is \( \text{Aug}^*_1 \). \[ \square \]

**Remark 4.11.** One may expect that the \( A_\infty \) functor \( \Pi^{(*)} : \text{Aug}_+ \to \text{Aug}^*_+ \) induced from \( \pi^{(*)} \) is again an \( A_\infty \)-quasi-equivalence but the essential quasi-surjectivity is not obvious unless \( \text{Aug}^*_+ \) is unital.

**Corollary 4.12.** If \( \text{Aug}_+ \) is homologically unital, \( \Pi^{(*)} \) is an \( A_\infty \)-quasi-equivalence and so \( \text{Aug}^*_+ \) is homologically unital.

**Proof.** Let us consider the essential quasi-surjectivity first. For each augmentation \( e' \in \text{Aug}^*_+ \), one can find an augmentation \( e \in \text{Aug}_+ \) such that \( e' \) and \( e^\pi \) agree with each other on \( A \) but may have different values for some \( \bar{e} \).

Similar to (4.7), the chain complex \( \text{Hom}_{\text{Aug}^*_+} \left( e', e^\pi \right) \) is obviously a direct sum
\[
\text{Hom}_{\text{Aug}^*_+} \left( e', e^\pi \right) \cong \text{Hom}_{\text{Aug}_+} \left( e, e \right) \oplus \left( \bigoplus_{i \in I^{12}} \mathbb{K}_A \left( e'_{12}, \hat{\partial'}_{12} \right), m'_i \right).
\]

As before, the second summand is acyclic and therefore the existence of the unit in \( \text{Aug}_+ \) implies the existence of an isomorphism in \( \text{Hom}_{\text{Aug}^*_+} \left( e', e^\pi \right) \).

For two augmentations \( e_1, e_2 \in \text{Aug}^*_+ \), the induced chain map \( \Pi^{(1)} \) sends all \( a_{12}^\pi \) to themselves
\[
\Pi^{(1)} : \text{Hom}_{\text{Aug}^*_+} \left( e_1, e_2 \right) \to \text{Hom}_{\text{Aug}^*_+} \left( e_1^\pi, e_2^\pi \right), \quad \Pi^{(1)} \left( a_{12}^\pi \right) = a_{12}^\pi.
\]

Therefore it is injective and its cokernel \( \text{Coker} \left( \Pi^{(1)} \right) \) is isomorphic to the chain complex
\[
\text{Coker} \left( \Pi^{(1)} \right) \cong \left( \bigoplus_{i \in I^{12}} \mathbb{K}_A \left( e'_{12}, \hat{\partial'}_{12} \right), m'_i \right)
\]
which is acyclic and so it is quasi-fully faithful as desired. \[ \square \]

In summary, we have the following proposition:

**Proposition 4.13.** Suppose that there is a zig-zag of stabilizations
\[
A_0^{(*)} \xleftarrow{\pi_1} A_1^{(*)} \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_{n-1}} A_n^{(*)} \xrightarrow{\pi_n} A_0^{(*)}
\]
and \( \text{Aug}_+ \left( A_0^{(*)}; K \right) \) is homologically unital. Then \( \text{Aug}_+ \left( A_i^{(*)}; K \right) \) for each \( i \) is homologically unital and \( A_\infty \)-quasi-equivalent to \( \text{Aug}_+ \left( A_0^{(*)}; K \right) \).

Now we consider the bordered version of augmentation categories.

**Definition 4.14** (Augmentations for bordered DGAs). Let \( K = (\mathbb{K}, \partial_\mathbb{K} \equiv 0) \) be a DGA consisting of the base field \( \mathbb{K} \) with the trivial differential. An augmentation for \( \mathcal{A} = (A_L \rightarrow A \leftarrow A_R) \) is a bordered DGA morphism
\[
\varepsilon = (\varepsilon_L, \varepsilon, \varepsilon_R) : \mathcal{A} \to \mathcal{K} := (K = K = K).
\]
which makes the following diagram commutative:

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\varepsilon} & A_R \\
\downarrow & & \downarrow e_R \\
\mathcal{K} & \xrightarrow{K} & K
\end{array}
\]

**Definition 4.15.** A bordered \(A\)-category \((A_L \leftarrow \mathcal{A} \rightarrow A_R)\) of type \((n_L, n_R)\) consists of \(A\)-categories, \(A_L\), \(\mathcal{A}\) and \(A_R\) and two \(A\)-functors \(\mathcal{A} \rightarrow A_L\) and \(\mathcal{A} \rightarrow A_R\) such that both \(A_L\) and \(A_R\) are \(A\)-equivariant to augmentation categories

\[
A_L \cong \text{Aug}(A_L(n_L); K) \quad \text{and} \quad A_R \cong \text{Aug}(A_R(n_R); K)
\]

for some \(\mu_L : [n_L] \rightarrow \mathbb{Z}\) and \(\mu_R : [n_R] \rightarrow \mathbb{Z}\), respectively.

The morphism \(\mathcal{F}(\bullet)\) between two bordered \(A\)-categories is a triple \((\mathcal{F}_L(\bullet), \mathcal{F}(\bullet), \mathcal{F}_R(\bullet))\) of \(A\)-functors making the following diagram commutative:

\[
\begin{array}{ccc}
A_L & \xrightarrow{\mathcal{F}_L(\bullet)} & A_R \\
\downarrow & & \downarrow e_R \\
A' & \xrightarrow{\mathcal{F}(\bullet)} & A'' \\
\downarrow & & \downarrow e'' \\
A'_L & \xrightarrow{\mathcal{F}_R(\bullet)} & A''_R
\end{array}
\]

We say that \(\mathcal{F}^{(\bullet)}\) is an \(A\)-equivalence (or quasi-equivalence) if so are \(\mathcal{F}_L^{(\bullet)}\), \(\mathcal{F}^{(\bullet)}\) and \(\mathcal{F}_R^{(\bullet)}\).

We denote the category of bordered \(A\)-categories by \(\mathcal{B}\mathcal{A}_{\mathcal{C}}\).

Indeed, for each consistent sequence \(\mathcal{A}^{(\bullet)} = (A_L^{(\bullet)}, \mathcal{A}^{(\bullet)}, A_R^{(\bullet)})\) of bordered DGAs, we have an associated bordered augmentation \(A\)-category

\[
\text{Aug}_+(\mathcal{A}^{(\bullet)}; K) = \left( \begin{array}{ccc}
\text{Aug}_+(A_L^{(\bullet)}; K) & \xrightarrow{\text{Aug}_+(\phi^{(\bullet)}_L)} & \text{Aug}_+(A_R^{(\bullet)}; K)
\end{array} \right)
\]

**Corollary 4.16.** The contravariant functor \(\text{Aug}_+(-; K) : \mathcal{B}\mathcal{D}\mathcal{A}_{\mathcal{C}} \rightarrow \mathcal{B}\mathcal{A}_{\mathcal{C}}\) is well-defined.

**Proof.** This is a corollary of Proposition 4.5. \(\square\)

**Proposition 4.17.** Let \(\mathcal{A}^{(\bullet)}\) be a stabilization of \(\mathcal{A}^{(\bullet)}\). Then two bordered augmentation categories \(\text{Aug}_+(\mathcal{A}^{(\bullet)}; K)\) and \(\text{Aug}_+(\mathcal{A}'^{(\bullet)}; K)\) are \(A\)-equivariant.

**Proof.** Let \(\Pi^{(\bullet)} : \text{Aug}_+(\mathcal{A}^{(\bullet)}; K) \rightarrow \text{Aug}_+(\mathcal{A}'^{(\bullet)}; K)\) be the induced \(A\)-functor from the canonical projection \(\Pi : \mathcal{A}^{(\bullet)} \rightarrow \mathcal{A}'^{(\bullet)}\). Since \(\mathcal{A}^{(\bullet)}\) is a stabilization, two induced functors \(\text{Aug}_+(\pi_L)\) and \(\text{Aug}_+(\pi_R)\) on borders are equivalent. Hence it suffices to prove the \(A\)-equivariance of \(\text{Aug}_+(\pi^{(\bullet)})\).

Due to Proposition 4.13, it is obvious that the \(A\)-functor \(\mathcal{I}^{(\bullet)} : \text{Aug}_+(\mathcal{A}^{(\bullet)}; K) \rightarrow \text{Aug}_+(\mathcal{A}'^{(\bullet)}; K)\) induced from \(\mathcal{I}^{(\bullet)} : \mathcal{A}^{(\bullet)} \rightarrow \mathcal{A}'^{(\bullet)}\) is an \(A\)-equivariance whose quasi-inverse is precisely \(\text{Aug}_+(\pi^{(\bullet)})\) and is also an \(A\)-equivariance as desired. \(\square\)

### 4.2. Augmentation categories for bordered Legendrian graphs

Let \((\mathcal{T}^{(*)}, \mu^{(*)}) \in \mathcal{B}\mathcal{L}\mathcal{G}^{(*)}\) be a consistent sequence of bordered Legendrian graphs. Then by taking \(A_{\mathcal{C}}\), we have a consistent sequence of DGAs \(A_{\mathcal{CE}}^{(\mathcal{T}^{(*)}, \mu^{(*)})}\) as seen in Theorem 3.32.

**Definition 4.18 (Augmentation categories for consistent sequences of bordered Legendrian graphs).** Let \((\mathcal{T}^{(*)}, \mu^{(*)}) \in \mathcal{B}\mathcal{D}\mathcal{A}_{\mathcal{C}}^{(*)}\). The bordered augmentation category for \((\mathcal{T}^{(*)}, \mu^{(*)})\) is the composition

\[
\text{Aug}_+(\mathcal{T}^{(*)}, \mu^{(*)}; K) = \left( \begin{array}{ccc}
\text{Aug}_+(T_L^{(*)}, \mu_L^{(*)}; K) & \xrightarrow{\text{Aug}_+(T_R^{(*)}, \mu_R^{(*)}; K)} & \text{Aug}_+(T_R^{(*)}, \mu_R^{(*)}; K)
\end{array} \right)
\]

\[
\cong \text{Aug}_+(A_{\mathcal{CE}}^{(\mathcal{T}^{(*)}, \mu^{(*)})}; K),
\]
where for \( *= \mathbb{L}, \mathbb{R} \) or empty,
\[
\text{Aug}_* \left( T^{(*)}_i, \mu^{(*)}_i; K \right) := \text{Aug}_* \left( A_{co}^{CE}(T^{(*)}_i, \mu^{(*)}_i); K \right).
\]

In particular, if \( T^{(*)} \) is the consistent sequence of canonical front copies of \( T \), then we denote simply by
\[
\text{Aug}_* \left( T, \mu; K \right) := \text{Aug}_* \left( T_L, \mu_L; K \right) \leftarrow \text{Aug}_* \left( T, \mu; K \right) \rightarrow \text{Aug}_* \left( T_R, \mu_R; K \right).
\]

For an example of computing the augmentation category, see Section 6.4.

**Theorem 4.19** (Invariance theorem). The assignment \( (T, \mu) \mapsto \text{Aug}_* \left( T, \mu; K \right) \) is well-defined and invariant under Legendrian isotopy and basepoint moves up to zig-zags of \( A_{co}-\)quasi-equivalences.

**Proof.** The well-definedness is obvious. Indeed, the canonical front copy is well-defined and every morphism is a zig-zag of elementary morphisms by Proposition 3.13, it induces a well-defined consistent sequence of bordered LCH DGAs and elementary morphisms correspond to stabilizations by Theorem 3.32.

Now as always, we pass through the bordered augmentation categories of cofibrant replacements. Then we have a zig-zag of stabilizations of consistent bordered DGAs
\[
\mathcal{A}' \leftarrow \tilde{\mathcal{A}}' \leftrightarrow \tilde{\mathcal{A}}' \leftrightarrow \mathcal{A} \leftarrow \tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{A}}',
\]
where \( \mathcal{A}' = A_{co}^{CE}(T^{(*)}, \mu^{(*)}) \) and \( \mathcal{A} = A_{co}^{CE}(T^{(*)}, \mu^{(*)}) \) and \( \tilde{\mathcal{A}}' \) and \( \tilde{\mathcal{A}} \) are cofibrant replacements, respectively.

Due to Corollary 4.16, the bordered augmentation categories for these zig-zags are well-defined and by Proposition 4.17, every stabilization gives us an \( A_{co}-\)quasi-equivalences.

We assume that \( (T', \mu') \) and \( (T, \mu) \) are related by a basepoint move \( (B_i) \). For \( (B_i) \), we are done since \((B_i)^{(*)}\) induces a zig-zag of stabilizations as seen already in Section 3.2.2.

For \((B_i)\) with \( i \in \{2, 3\} \), the induced consistent basepoint move on the canonical front copies is a sequence of an elementary basepoint move and a Reidemeister move as seen in Figure 11. Now suppose that two consistent sequences \((T^{(*)}, \mu^{(*)})\) and \((T^{(*)}, \mu^{(*)})\) are related with an elementary consistent basepoint change move \((B_i)^{(*)}\).

Due to the discussion after Theorem 3.32, we have the induced consistent morphism \((B_i)^{(*)}\) and its left inverse \((B_i^{-1})^{(*)}\). Therefore we have a surjective \( A_{co}-\)functor
\[
\text{Aug}_* ((B_i)^{(*)}) : \text{Aug}_* (T^{(*)}, \mu^{(*)}; K) \rightarrow \text{Aug}_* (T'^{(*)}, \mu'^{*}; K).
\]

For any pair of augmentation \( \epsilon_1, \epsilon_2 \in \text{Aug}_* (T^{(*)}, \mu^{(*)}; K) \), we have a chain map
\[
\text{Aug}_* ((B_i)^{(2)}) : \text{Hom}_* (\epsilon_1, \epsilon_2) \rightarrow \text{Hom}_* (\epsilon'_1, \epsilon'_2),
\]
where \( \epsilon'_i \in \text{Aug}_* (T'^{(*)}, \mu'^{*}; K) \) is an induced augmentation via \((B_i)_i\). Notice that the basepoint move \((B_i)\) for \( i \in \{2, 3\} \) does not alter the crossings, especially crossings between the 1st and 2nd copies of \( T^{(*)} \).

Therefore the above chain map is an isomorphism between graded vector spaces which implies the fully faithfulness of \( \text{Aug}_* ((B_i)^{(*)}) \) and we are done. \qed

The following theorem is a generalization of Proposition 3.25 in [21].

**Theorem 4.20.** Let \( (T, \mu) \in \mathcal{BCL}^0 \) be a bordered Legendrian graph in a normal form. Then its augmentation category \( \text{Aug}_* (T, \mu; K) \) is strictly unital. That is, the \( A_{co} \) category \( \text{Aug}_* (T, \mu; K) \) for each \(* = \mathbb{L}, \mathbb{R}\) and empty is strictly unital and two \( A_{co}-\)functors are unit-preserving.

**Proof.** Let \( \epsilon \in \text{Aug}_* (T, \mu; K) \). We need to show that there exists an element in \( \text{Hom}_{\text{Aug}_*} (\epsilon, \epsilon) \) which plays the role of the identity.

Recall the generating sets for \( A_{co}^{CE}(T^{(*)}, \mu^{(*)}) \) described in Section 3.2.3. We define an element \( -\gamma^\vee_{12} \in \text{Hom}_{\text{Aug}_*} (\epsilon, \epsilon) \) as follows:
\[
-\gamma^\vee_{12} := \sum_{\epsilon \in E} -\gamma_{12}^\vee \in \text{Hom}_{\text{Aug}_*} (\epsilon, \epsilon),
\]
where \( \gamma_{12}^\vee \) is the dual of \( \gamma_{12} \).
Due to the formula of the differential $\partial_{1}^{(m)}$ given in Section 3.2.3, the only possibility for $s_{13}$ containing $y_{12}^{\ell}$ or $y_{c}^{\ell}$ in its differential is either

$$(-1)^{s_{13}^{1}-1}s_{12}^{12}, \text{ or } y_{c}^{12}s_{12}^{12},$$

respectively. Note that when $s_{13}^{13} = y_{c}^{13}$ for some $c$, then these two terms coincide. Moreover, the situation is essentially the same as the augmentation category for border DGAs as computed in Example 4.7.

Therefore, one and only one generator appears in both $m_{2}(-y_{12}^{\ell} \otimes s_{12}^{12})$ and $m_{2}(s_{12}^{12} \otimes -y_{12}^{\ell})$ which is precisely $s_{12}$ itself

$$m_{2}(-y_{12}^{\ell} \otimes s_{12}^{12}) = s_{12}^{12} = m_{2}(s_{12}^{12} \otimes -y_{12}^{\ell}).$$

Finally, the absence of terms of length at least 3 in any differential containing $y$ implies that the higher composition $m_{k}$ will vanish whenever it contains $-y_{12}^{\ell}$.

Now we prove that two $A_{\infty}$ functors $\text{Aug}_{*}(\phi^{(\bullet)}): \text{Aug}_{*}(T, \mu; K) \rightarrow \text{Aug}_{*}(T_{*, \mu}; K)$ for $* = L$ and $R$ are unit-preserving. Due to the definition of the induced $A_{\infty}$-functor briefly reviewed in Section 4 after Proposition 4.5, we have for each augmentation $e \in \text{Aug}_{*}(T, \mu; K)$

$$\text{Aug}_{*}(\phi^{(\bullet)}e)(-y_{12}^{\ell}) = \sum_{1, \ell \in R_{12}}(-1)^{r_{1}}s_{12}^{12} = \sum_{1, \ell \in R_{12}}(-y_{12}^{\ell}, \phi_{*, \ell}^{(s_{12}^{12})}),$$

where $r_{1} = 0$ as seen in Definition 4.3 and $\phi_{*, \ell}^{(s_{12}^{12})}$ is a twisted DGA morphism by using the diagonal augmentation $e = (e, e)$.

More precisely, since $\phi_{L}^{(s_{12}^{12})}$ identifies each generators in $A_{\infty,m_{1}} \cong A^{CE}_{\infty}(T_{L}, \mu_{L})$ with the corresponding generator in $A^{CE}_{\infty}(T_{R}, \mu_{R})$,

$$\phi_{L}(k_{12}^{12}) = k_{ab}^{12}, \quad \text{and} \quad \phi_{L}(c_{12}^{12}) = y_{c}^{12},$$

and therefore the image of $\phi_{L}^{(s_{12}^{12})}$ does not involve any $y_{c}^{12}$'s as observed before and therefore the pairing survives only for $y_{c}^{12}$'s. In other words,

$$\text{Aug}_{*}(\phi_{L}^{(\bullet)})(-y_{12}^{\ell}) = \sum_{c \in [m_{L}]}(-y_{12}^{\ell}, \phi_{L}^{(s_{12}^{12})}(y_{c}^{12})) \cdot y_{c}^{12} = \sum_{c \in [m_{L}]}(-y_{12}^{\ell}, \phi_{L}^{(s_{12}^{12})}(y_{c}^{12})) \cdot y_{c}^{12} \in \text{Hom}_{A_{\infty,m_{1}}}(\phi^{(e)}(e), \phi_{L}^{(e)}),$$

which is the unit as desired.

On the other hand, for the right border, we recall the map $\phi_{R}^{(m)}$ described in (3.9). That is, for $m = 2$, we have

$$\phi_{R}^{(2)}(k_{12}^{12}) = \Phi(\phi_{R}(k_{12}^{12}))),$$

Then $\phi_{R}^{(2)}(k_{12}^{12})$ does not involve any $y_{c}^{12}$'s as observed before and therefore the pairing survives only for $y_{c}^{12}$'s. In that case $\phi_{R}^{(2)}(y_{c}^{12}) = y_{c}^{12}$ and we have

$$\text{Aug}_{*}(\phi_{R}^{(\bullet)})(-y_{12}^{\ell}) = \sum_{c' \in [m_{R}]}(-y_{12}^{\ell}, \phi_{R}^{(s_{12}^{12})}(y_{c'}^{12})) \cdot y_{c'}^{12} = \sum_{c' \in [m_{R}]}(-y_{12}^{\ell}, \phi_{R}^{(s_{12}^{12})}(y_{c'}^{12})) \cdot y_{c'}^{12} \in \text{Hom}_{A_{\infty,m_{1}}}(\phi^{(e)}(e), \phi_{R}^{(e)}),$$

which is the unit as well. This completes the proof. \qed

**Theorem 4.21 (Unitarity).** For any bordered Legendrian graph $(T, \mu) \in \mathcal{B}_{\mathcal{L}}T^{\mu}$, the augmentation category $\text{Aug}_{*}(T, \mu; \mathcal{K})$ is homologically unital.

**Proof.** Due to Lemma 2.18, one can obtain $T$ from a Legendrian graph $T'$ in a normal form up to Legendrian isotopy and basepoint moves. Then by Theorem 4.20, the augmentation category $\text{Aug}_{*}(T', \mu'; \mathcal{K})$ is strictly unital.

As seen in the proof of Theorem 4.19, for each Legendrian isotopy, we have a zig-zag of stabilizations between consistent sequences of DGAs for $(T', \mu')$ and $(T, \mu)$. By Proposition 4.13, we have the homological unitality for $\text{Aug}_{*}(T, \mu; \mathcal{K})$ which implies that so is $\text{Aug}_{*}(T', \mu'; \mathcal{K})$ as desired.

For each basepoint move, we have either a zig-zag of stabilizations for $(B_{i})$ where the above argument is applicable, or an $A_{\infty}$-equivalence for $(B_{i})$ with $i = 2$ or 3 by the proof of Theorem 4.19 again, which preserves of course the (homological) unitality. \qed
Proposition 4.22. Let \((T, \mu) \in \mathcal{BCL} \) and \((T_{\text{Lag}}, \mu) := \text{Res}^{N_0}(T)\). Suppose that \((T(\cdot), \mu(\cdot)) \) and \((T(\cdot)_{\text{Lag}}, \mu(\cdot)) \) are consistent sequences of canonical front and Lagrangian copies, respectively. Then two augmentation categories \(\text{Aug}_*(T(\cdot), \mu(\cdot); K) \) and \(\text{Aug}_*(T(\cdot)_{\text{Lag}}, \mu(\cdot); K) \) are \(A_{\infty}\)-quasi-equivalent.

Proof. As seen in Section 3.1.1 and Figure 12, there exists a zig-zag of elementary Lagrangian Reidemeister moves between \(\text{Res}^{N_0}(T(\cdot)) \) and \(T(\cdot)_{\text{Lag}}\). By Theorem 3.32, we have a zig-zag of consistent stabilizations between

\[ A_{\text{CE}}(T(\cdot), \mu(\cdot)) := A_{\text{CE}}(\text{Res}^{N_0}(T(\cdot)), \mu(\cdot)) \quad \text{and} \quad A_{\text{CE}}(T(\cdot)_{\text{Lag}}, \mu(\cdot)). \]

Finally, Proposition 4.17 completes the proof. \(\square\)

5. Sheaf categories for Legendrian graphs

In this section we give the preliminaries on the microlocal theory of sheaves, the main reference is [17]. We also establish the necessary combinatorial tools for constructible sheaves, which will be used in the proof of the augmentation-sheaf correspondence in the next section.

5.1. Micro-support and constructible sheaves. For the moment, let \(M\) be any smooth manifold, and \(\mathbb{K}\) be a base field. We use the same notations as in [25]. Let \(\text{Sh}(M; \mathbb{K})\) to be the abelian category of sheaves of \(\mathbb{K}\)-modules. Let \(\text{Sh}_{\text{naive}}(M; \mathbb{K})\) to be the triangulated DG category of complexes of sheaves of \(\mathbb{K}\)-modules on \(M\) whose cohomology sheaves are constructible (that is, locally constant with perfect stalks on each stratum) with respect to some nice stratification (e.g. Whitney stratification). Let \(\text{Sh}(M; \mathbb{K})\) be the DG quotient of [9] of the DG category \(\text{Sh}_{\text{naive}}(M; \mathbb{K})\) with respect to acyclic complexes. Given a Whitney stratification \(S\) of \(M\), define \(\text{Sh}_{S}(M; \mathbb{K})\) to be the full subcategory of \(\text{Sh}(M; \mathbb{K})\) consisting of objects whose cohomology sheaves are constructible with respect to \(S\).

We firstly recall the notion of micro-support introduced by Kashiwara and Schapira:

Proposition/Definition 5.1 (Micro-support). [17, Prop.5.1.1, Def.5.1.2] Let \(\mathcal{F}\) be a sheaf \(^3\) on \(M\), and \(p = (x_0, \xi_0) \in T^*M\). We say \(p \notin \text{SS}(\mathcal{F})\) if one of the following equivalent conditions holds:

1. There exists a neighborhood \(U\) of \(p\), such that for any \(x_1 \in M\) and any \(C^1\)-function \(\varphi\) on a neighborhood of \(x_1\) satisfying \((x_1, d\varphi(x_1)) \in U\) and \(\varphi(x_1) = 0\), we have

\[ R\Gamma_{\{\varphi(x) > 0\}}(\mathcal{F}) = 0. \]

Equivalently, by the distinguished triangle \(R\Gamma_{\{\varphi(x) > 0\}}(\mathcal{F}) \to \mathcal{F} \to R\Gamma_{\{\varphi(x) < 0\}}(\mathcal{F}) \to R\Gamma_{\{\varphi(x) = 0\}}(\mathcal{F}) \to 1\), we get a quasi-isomorphism

\[ \mathcal{F}_{x_1} \xrightarrow{\sim} R\Gamma_{\{\varphi(x) < 0\}}(\mathcal{F})_{x_1}. \]

2. Up to taking an open chart near \(x_0\), we can assume \(M\) is an open subset in a vector space \(E\). Then there exists a neighborhood \(U\) of \(x_0\), an \(\varepsilon > 0\), and a proper closed convex cone \(\gamma\) in \(E\) with \(0 \in \gamma\), satisfying \(\gamma \setminus \{0\} \subset \{v \mid \langle v, \xi_0 \rangle < 0\}\), such that if we set

\[ H := \{x \mid \langle x - x_0, \xi_0 \rangle \geq -\varepsilon\} \quad \text{and} \quad L := \{x \mid \langle x - x_0, \xi_0 \rangle = -\varepsilon\}, \]

then \(H \cap (U + \gamma) \subset M\) and we have the natural isomorphism:

\[ R\Gamma(H \cap (x + \gamma); \mathcal{F}) \xrightarrow{\sim} R\Gamma(L \cap (x + \gamma); \mathcal{F}) \]

for all \(x \in U\). Recall that a cone is called proper if it contains no lines.

The set \(\text{SS}(\mathcal{F})\) is called the micro-support (or singular support) of \(\mathcal{F}\).

The micro-support satisfies the following properties:

1. \(\text{SS}(\mathcal{F})\cap 0_M = \text{Supp}(\mathcal{F})\) is the support of \(\mathcal{F}\), where \(0_M\) the zero section of \(T^*M\).

2. For any sheaf \(\mathcal{F}\), \(\text{SS}(\mathcal{F})\) is a conical (i.e. invariant under the action of \(R_+\) which scales the cotangent fibers) and closed co-isotropic subset of \(T^*M\).

3. If \(\mathcal{F}\) is a constructible sheaf with respect to a Whitney stratification \(S\), then \(\text{SS}(\mathcal{F})\) is a conical Lagrangian (i.e. Lagrangian wherever it is smooth) subset of \(T^*M\) with \(\text{SS}(\mathcal{F}) \subset \bigcup_{i \in S} T^*_i M\).

\(^3\)Whenever we say a sheaf, we mean a complex of sheaves of \(\mathbb{K}\)-modules.
Definition 5.2. The first proof is a direct consequence of the results of Guillermou-Kashiwara-Schapira [15]. For proof.

Theorem 5.3. The diagram of DG categories

\[ \text{Sh}(T'; \mathbb{K}) \xleftarrow{r_L} \text{Sh}(T; \mathbb{K}) \xrightarrow{r_R} \text{Sh}(T_R; \mathbb{K}) \]

is a Legendrian isotopy invariant, up to DG equivalence. That is, for any Legendrian isotopy between two bordered Legendrian graphs \( T, T' \), there is an equivalence between two diagrams of DG categories taking the form:

\[
\begin{array}{c}
\text{Sh}(T_L; \mathbb{K}) \\
\text{Sh}(T'; \mathbb{K})
\end{array}
\xleftarrow{r_L} \xrightarrow{r_R}
\begin{array}{c}
\text{Sh}(T; \mathbb{K}) \\
\text{Sh}(T_R; \mathbb{K})
\end{array}
\]

where the arrow \( \circ \) in the middle is an equivalence of DG categories which, under restrictions, commutes with the identity functor \( \text{Id} : \text{Sh}(T_L; \mathbb{K}) \rightarrow \text{Sh}(T'_L; \mathbb{K}) \) (resp. \( \text{Id} : \text{Sh}(T_R; \mathbb{K}) \rightarrow \text{Sh}(T'_R; \mathbb{K}) \)) up to specified natural isomorphism \( h_L : r_L \circ \text{Id} \) and \( h_R : r_R \circ \text{Id} \). Also, the same holds for \( \text{Sh}(T'; \mathbb{K}) \).

Proof. The first proof is a direct consequence of the results of Guillermou-Kashiwara-Schapira [15]. For the more related details, see [25, Thm.4.1, Rmk.4.2,4.3].

In the rest of this section, we will give a combinatorial description of the sheaf categories. In particular, we will use this description to give an alternative proof of the invariance theorem. The combinatorial description will also be needed in proving our main result “mutations are sheaves”.

5.2. Combinatorial description for constructible sheaves. As before, let \( M = I_x \times \mathbb{R}_c \) be the base manifold, and \( \mathcal{T} \) be a bordered Legendrian graph in \( J^1 I_x = T^{\infty,-} M \).

We can always assume the front projection \( T \) is regular so that the only singularities of \( T \) are crossings, cusps, and vertices. This induces a Whitney stratification \( \mathcal{S}_T \) of \( M \) whose 0-dimensional strata are the singularities, 1-dimensional strata are the arcs—the connected components of \( T \setminus \{ \text{singularities} \} \), and 2-dimensional strata are the regions—connected components of \( M \setminus T \). By definition, \( \text{Sh}(T; \mathbb{K}) \subset \text{Sh}(T^*; \mathbb{K}) \) are full subcategories of \( \text{Sh}(\mathcal{S}_T(M; \mathbb{K})) \).

Given a stratification \( \mathcal{S} \), the star of a stratum \( S \in \mathcal{S} \) is the union of strata whose closure contains \( S \), denoted by \( \text{Star}(S) \). Given 2 strata \( S \) and \( S' \), we denote by \( S \preceq S' \) and define an arrow \( S \rightarrow S' \) if \( S \subset S' \), or equivalently, \( \text{Star}(S) \supset \text{Star}(S') \). This defines \( \mathcal{S} \) as a poset category. We say that \( \mathcal{S} \) is a regular cell complex.
if every stratum is contractible as well as the star of each stratum is contractible. As in [25], we can choose a regular cell complex \( S \) refining the stratification \( S_T \) with the additional 1-dimensional strata away from the crossings of \( T \) if necessary.

**Assumption 5.4.** For the regular cell complex \( S \) refining the stratification \( S_T \) induced by \( T \), we assume that each additional 1-dimensional strata contains no vertical tangent and

1. it is tangent to the existing arcs at the singularity if it has an end at a cusp or a vertex, or
2. it is transverse to the boundary if it has an end at the boundary \( \partial M = \{(x, z) \mid x = x_L, x_R\} \).

The assumption can always be satisfied by choosing \( S \) appropriately.

**Definition 5.5.** For any regular cell complex \( S \) of an oriented manifold \( M \) and any stratum \( S \in S \), let us define a **co-standard object** \( \omega_S \) of \( \mathcal{S}h_S(M; \mathcal{K}) \) as

\[
\omega_S := \mathcal{K}_S[\dim S] = R(j_S)_!(\mathcal{K}[\dim S]) \in \mathcal{S}h_S(M; \mathcal{K}),
\]

where \( j_S : S \to M \) is the inclusion and \( \mathcal{K} \) is regarded as the constant sheaf on \( S \).

**Lemma 5.6 ([19, Lem.2.3.2]).** The triangulated DG category \( \mathcal{S}h_S(M; \mathcal{K}) \) is the triangulated envelope of the co-standard objects.

An immediate corollary is as follows:

**Corollary 5.7.** For any \( \mathcal{F} \in \mathcal{S}h_S(M; \mathcal{K}) \) and \( x \in w \in S \), we have natural quasi-isomorphisms

\[
\mathcal{F}_x \overset{\sim}{\longrightarrow} (\mathcal{R}\Gamma(\mathcal{F}))_x \quad \longleftrightarrow \quad \mathcal{R}\Gamma(\text{Star}(w); \mathcal{F}).
\]

**Proof.** A direct computation shows that this holds for all the co-standard objects. Moreover, the above property of \( \mathcal{F} \) is preserved under taking quasi-isomorphisms, shifts, and cones. Hence by the lemma above, we are done. \( \square \)

Now we come back to our setting where \( M = I_x \times \mathbb{R}_\varepsilon \). By definition, \( \mathcal{S}h_S(M; \mathcal{K}) \) contains \( \mathcal{S}h_{S_T}(M; \mathcal{K}) \), hence \( \mathcal{S}h(T^*; \mathcal{K}) \) and \( \mathcal{S}h(T; \mathcal{K}) \) as full subcategories as well. For an open subinterval \( I_x \) of \( I_x \), we denote \( M|_{I_x} := J_x \times \mathbb{R}_\varepsilon \) and define \( S|_{I_x} \) to be the stratification of \( M|_{I_x} \) induced by \( S \), whose strata are the connected components of \( S \cap (M|_{I_x}) \) for all \( S \in S \). Then \( S|_{I_x} \) is a Whitney stratification of \( M|_{I_x} \) refining the stratification induced by \( T|_{I_x} \) and we obtain a natural dg functor \( r : \mathcal{S}h_S(M; \mathcal{K}) \to \mathcal{S}h_{S|_{I_x}}(M|_{I_x}; \mathcal{K}) \) coming from the restriction.

Given a regular cell complex \( S \) of \( M \), there is a combinatorial description of \( \mathcal{S}h_S(M; \mathcal{K}) \) as follows: Denote the induced poset category by \( S \) again and denote the category of cochain complexes of \( \mathcal{K} \)-modules with cochain maps by \( \mathcal{C} \mathcal{h}(\mathcal{K}) \).

**Definition 5.8.** For any poset category \( S \), let \( \mathcal{F} \text{un}_{\text{naive}}(S, \mathcal{K}) \) be the DG category of functors from \( S \) to \( \mathcal{C} \mathcal{h}(\mathcal{K}) \), which are valued on perfect complexes, that is, complexes which are quasi-isomorphic to a bounded complex of finite projective \( \mathcal{K} \)-modules. We define \( \mathcal{F} \text{un}(S, \mathcal{K}) \) to be the DG quotient (see [9]) of \( \mathcal{F} \text{un}_{\text{naive}}(S, \mathcal{K}) \) with respect to the thick subcategory of objects taking values in acyclic complexes.

**Notation 5.9.** We denote the abelian category of functors from \( S \) to the abelian category \( \mathcal{K} \text{ - Mod} \) of \( \mathcal{K} \)-modules by \( \text{Fun}(S, \mathcal{K}) \).

We denote by \( \mathcal{C} \mathcal{h}_{\text{naive}}(S, \mathcal{K}) \) and \( \mathcal{C} \mathcal{h}_{\text{dg}}(S, \mathcal{K}) \) the DG category of cochain complexes of objects in the abelian category \( \text{Fun}(S, \mathcal{K}) \) with morphisms the usual complexes of maps between complexes and its DG quotient of \( \mathcal{C} \mathcal{h}_{\text{naive}}(S, \mathcal{K}) \) by the full subcategory of acyclic objects, respectively. Then \( \mathcal{F} \text{un}_{\text{naive}}(S, \mathcal{K}) \) and \( \mathcal{F} \text{un}(S, \mathcal{K}) \) are full subcategories of \( \mathcal{C} \mathcal{h}_{\text{naive}}(S, \mathcal{K}) \) and \( \mathcal{C} \mathcal{h}_{\text{dg}}(S, \mathcal{K}) \) respectively.

Observe that we obtain a functor of poset categories \( i : S|_{I_x} \to S \) by inclusion of strata, which then induces a natural DG functor \( i^* : \mathcal{F} \text{un}(S, \mathcal{K}) \to \mathcal{F} \text{un}(S|_{I_x}, \mathcal{K}) \).

**Proposition/Definition 5.10.** There is a functor \( i_s : \mathcal{F} \text{un}(S, \mathcal{K}) \to \mathcal{S}h(M; \mathcal{K}) \) defined as follows: let \( F \in \mathcal{F} \text{un}(S, \mathcal{K}) \) and \( w \in S \).

1. The stalk \( i_s(F)|_w \) is \( F(w) \) viewed as a constant sheaf. In particular, \( i_s(F) \) is constructible with respect to \( S \).
(2) Let $U_w$ be any contractible open subset of $\operatorname{Star}(w)$ such that $S_{|U_w}$ is a regular cell complex and the map $S_{|U_w} \to S_{|\operatorname{Star}(w)}$ of partially ordered sets of strata is a bijection. Then $\Gamma(U_w; i_S(F)) = F(w)$ and the restriction map $\Gamma(\operatorname{Star}(w); i_S(F)) \to \Gamma(U_w; i_S(F))$ is the identity. In particular, $(i_S(F))_x = F(w)$ for all $x \in w$ and it follows that $i_S$ is exact.

(3) The restriction map $\Gamma(\operatorname{Star}(w_1); i_S(F)) \to \Gamma(\operatorname{Star}(w_2); i_S(F))$ is $F(w_1 \to w_2)$ for all arrows $w_1 \to w_2$ in $S$.

Let $\gamma_S : \Sh(S; \mathbb{K}) \to \Fun(S; \mathbb{K})$ be a functor defined as
$$\mathcal{F} \mapsto [S \mapsto \Gamma(\operatorname{Star}(S); \mathcal{F})].$$

Then $\gamma_S \circ i_S = \operatorname{Id}$ and $(i_S, \gamma_S)$ is an adjoint pair and so $\gamma_S$ is left exact. As a consequence, we obtain an adjoint pair $(i_S, \Gamma_S)$ in the DG lifting:
$$i_S : \Fun(S; \mathbb{K}) \cong \Sh(S; \mathbb{K}) : \Gamma_S = R\gamma_S$$
and in fact the essential image of $i_S$ is contained in $\Sh(S; \mathbb{K})$. More explicitly, $\Gamma_S = R\gamma_S$ is given by $\mathcal{F} \mapsto [w \mapsto R\Gamma(\operatorname{Star}(w); \mathcal{F})]$, and for any $F \in \Fun(S; \mathbb{K})$ and $G \in \Sh(S; \mathbb{K})$, we have a natural quasi-isomorphism:
$$\RHom^*(i_S(F^*), G^*) \cong \RHom^*(\mathcal{F}^*, \Gamma_S(G^*))$$
(5.2)
Moreover, we get a natural isomorphism $\beta : \operatorname{Id} \Rightarrow \Gamma_S \circ i_S$.

Proof. Firstly, let us show that for any $F \in \Fun(S; \mathbb{K})$, $i_S(F)$ indeed defines a sheaf on $M$. By [27, Thm.2.7.1], it suffices to define $i_S(F)$ as a sheaf on a base of $M$.

We take a base $\mathcal{B} = \{B_i\}$ for the topology of $M$ such that each $B_i$ is of the form $U_w$ as in (2). Let $\mathcal{F}^\preceq$ be a presheaf on $\mathcal{B}$ defined as follows: for each $B_i = U_w$ with $w \in S$, $\mathcal{F}^\preceq(B_i) = F(w)$ and for each inclusion $B_i = V_{w_2} \hookrightarrow B_i = U_{w_1}$, we require that $\operatorname{Star}(w_2) \subseteq \operatorname{Star}(w_1)$. In other words, if $w_1 \to w_2$ is an arrow in $S$, then the restriction map $\Gamma(U_{w_1}; \mathcal{F}^\preceq) \to \Gamma(V_{w_2}; \mathcal{F}^\preceq)$ is defined to be $F(w_1 \to w_2)$. Clearly, this is a well-defined presheaf on $\mathcal{B}$. Then $i_S(F) := \mathcal{F}$ is defined to be the sheafification of $\mathcal{F}^\preceq$. Recall that for any open subset $U$ in $M$, we have
$$\Gamma(U; \mathcal{F}) = \{(f_p \in \mathcal{F}^\preceq_p)_{p \in \mathcal{U}} \mid \text{For any } p, \text{ there exists a pair } (V_S, s) \text{ such that } V_S \subseteq U \text{ is a neighborhood of } p \text{ and } V_S \in \mathcal{B} \text{ for some } s \in \mathcal{S}, \\ \text{and } s \in \mathcal{F}^\preceq(V_S) = F(s) \text{ with } f_q = f_{s_q} \text{ on } V_S. \}$$

Also, for all $S \in \mathcal{S}$ with $S \cap U \neq \emptyset$, each $p \in S \cap U$ has a system of neighborhoods in $\mathcal{B}$ of the form $U_S$. Then by definition, we have $\Gamma(U_S; \mathcal{F}^\preceq) = F(S) \cong \mathcal{F}^\preceq$. Hence, it follows that any section of $\Gamma(U; \mathcal{F})$ is locally constant on $S \cap U$ with values in $F(S)$. In particular, for any connected component $w$ in $S \cap U$, we have $\Gamma(w; \mathcal{F}) = F(S)$. Thus $\mathcal{F}|_S$ is $F(S)$ viewed as a (locally) constant sheaf. This shows (1), hence $\mathcal{F} \in \Sh(S; \mathbb{K})$.

Moreover, we can rewrite the definition as follows. Let us denote by $S|_U$ the stratification of $U$ consisting of the connected components of $S \cap U$ for all $S \in \mathcal{S}$ and let $\tau_U : S|_U \to S$ be the map of partially ordered sets induced by inclusion of strata. Then the previous definition can be translated into
$$\Gamma(U; \mathcal{F}) = \lim_{w \in S|_U} \left( F \circ \tau_U(w) \right).$$
(5.3)
In other words,
$$\Gamma(U; \mathcal{F}) = \left\{ (f(w))_w \in \prod_{w \in S|_U} (F \circ \tau_U(w)) \mid w_1 \leq w_2 \Rightarrow (F \circ \tau_U)(w_1 \to w_2)(f(w_1)) = f(w_2) \right\}.$$  

Besides, for any inclusion $V \hookrightarrow U$, there is an induced map of partially ordered sets $\tau_{U; V} : S|_V \to S|_U$ via inclusions of strata. Then $\tau_V = \tau_U \circ \tau_{U; V}$, and the restriction map is just $\tau^*_{U; V} : \Gamma(U; \mathcal{F}) \to \Gamma(V; \mathcal{F})$ via the pullback of functions. That is, for any $f \in \Gamma(U; \mathcal{F}) \subseteq \prod_{w \in S|_U} (F \circ \tau_U)(w)$, we have $(\tau^*_{U; V}f)(w) = f(\tau_U \circ \tau_V[w])$ for all $w \in S|_V$.

Now, let $U = U_w$ (including $\operatorname{Star}(w)$) as in (2). We obtain $\Gamma(U; \mathcal{F}) \cong F(\tau(w \cap U)) = F(w)$ as $w \cap U$ is the unique minimum in $S|_U$. For any inclusion $V_{w_2} \hookrightarrow U_{w_1}$ of two open subsets as in (2) with $w_1, w_2 \in S$,
we also know $w_2 \leq w_1$ and $\tau_{U_{w_1}, V_{w_2}}(w_2 \cap V_{w_2}) = w_2 \cap U_{w_1} \leq w_1 \cap U_{w_1}$ in $S|_{U_{w_1}}$. Then the restriction map $\Gamma(U_{w_1}; F) \cong F(w_1) \to \Gamma(V_{w_2}; F) \cong F(w_2)$ is identified with $F(w_1 \to w_2)$ as follows:

$$f \equiv f(w_1 \cap U) \mapsto \tau_{U_{w_1}, V_{w_2}} f \equiv (\tau_{U_{w_1}, V_{w_2}} f)(w_2 \cap V) = f(w_2 \cap U) = F(w_1 \to w_2)(f(w_1 \cap U)).$$

This shows (2) and (3) and in fact $F^\preceq \cong F$ is already a sheaf on $B$.

The functoriality of $i_S$ follows directly from the definition of $F$ above. By (1), the essential image of $i_S$ is contained in $\mathcal{H}_S(M, \underline{E})$ and it follows immediately from (2) that $\gamma_S \circ i_S = 1d$.

By [27, Ex.2.7.C], morphisms of sheaves correspond to morphisms of sheaves on a base. By a direct computation, we have an adjunction of $(i_S, \gamma_S)$ which yields the following: In (5.2), take an injective resolution of $G^*$, say $G^* \to I^*$, then by adjunction of $(i_S, \gamma_S)$, we have an injective object $\gamma_S(I^*)$ and it follows that

$$\text{RHom}^\bullet(i_S(F^*), G^*) \cong \text{Hom}^\bullet(i_S(F^*), I^*)$$

$$\cong \text{Hom}^\bullet(F^*, \gamma_S(I^*))$$

$$\cong \text{RHom}^\bullet(F^*, \gamma_S(G^*)).$$

where the last quasi-isomorphism follows from the fact that $\Gamma_S(G^*) \xrightarrow{\sim} \Gamma_S(I^*) = \gamma_S(I^*)$ is a quasi-isomorphism.

Finally, for any $F^* \in \mathcal{F}un(S, \underline{E})$ and $x \in w \in S$, by (2) and Corollary 5.7, we have a natural quasi-isomorphism

$$F^*(w) = \Gamma(\text{Star}(w); i_S(F^*)) \to R\Gamma(\text{Star}(w); i_S(F^*)) = R\gamma_S(i_S(F^*))(w) \cong (i_S(F^*))_{x}.$$

Hence, we get a natural isomorphism $\beta : 1d \xrightarrow{\sim} \Gamma_S \circ i_S$. This completes the proof. \hfill \square

We recall the following lemma:

Lemma 5.11. [25, Prop.3.9], [19, Lem.2.3.2] Let $S$ be a regular cell complex for $M$. Then the functor

$$\Gamma_S : \mathcal{H}_S(M, \underline{E}) \to \mathcal{F}un(S, \underline{E}), \quad F \mapsto [S \mapsto R\Gamma(\text{Star}(S); F)]$$

is a quasi-equivalence with a quasi-inverse $i_S$. Moreover, $i_S$ commutes with the functors induced by restriction to $J_x$.

Proof. The last statement follows directly from the definition of $i_S$ and it suffices to show the quasi-equivalence.

For any $S \in S$, we define a functor $\delta_S \in \mathcal{F}un(S, \underline{E})$ by $\delta_S(w) = 0$ if $w \not\in S$, and $\delta_S(S) := \underline{E}[[\text{dim } S]]$, and sends all arrows $w_1 \to w_2$ to zero. Clearly, $\mathcal{F}un(S, \underline{E})$ is the triangulated envelope of $\delta_S$. Moreover by Proposition/Definition 5.10, we have $i_S \circ \delta_S \cong \mathcal{H}_S[[\text{dim } S]] = \omega$. Hence, $i_S$ is essentially surjective by Lemma 5.6 and so is $\Gamma_S$ by the natural isomorphism $\beta : 1d \xrightarrow{\sim} \Gamma_S \circ i_S$ in Proposition/Definition 5.10.

It suffices to show that $i_S$ is fully faithful. In fact, for any $F, G \in \mathcal{F}un(S, \underline{E})$, by the adjunction $(i_S, \Gamma_S)$ in Proposition/Definition 5.10, we have

$$\text{RHom}^\bullet(i_S(F), i_S(G)) \cong \text{RHom}^\bullet(F, \Gamma_S(i_S(G))) \cong \text{RHom}^\bullet(F, G)$$

where the last quasi-isomorphism follows from the natural isomorphism $\beta : 1d \xrightarrow{\sim} \Gamma_S \circ i_S$, which then implies the quasi-isomorphism $G \cong \Gamma_S \circ i_S(G)$. This finishes the proof. \hfill \square

Remark 5.12. For the last part of the proof above, we can also finish the argument by showing that $\Gamma_S$ is fully faithful in a different way as follows: For all $S, W \in S$, a direct computation shows

$$\Gamma_S(\underline{E}_{[\text{dim } S]})(W) := R\Gamma(\text{Star}(W); \underline{E}_{[\text{dim } S]}) \cong (\underline{E}_{\text{dim } S})_p [\text{dim } S]$$

for any $p \in W$. That is, $\Gamma_S(\underline{E}_{[\text{dim } S]})(W)$ is acyclic unless $W = S$ when $\Gamma_S(\underline{E}_{[\text{dim } S]})(S) \cong \underline{E}_{[\text{dim } S]}$. In other words, $\Gamma_S(\underline{E}_{[\text{dim } S]}) \cong \delta_S$. Now, for any $S \in S$, we take $F^* = \Gamma_S(\underline{E}_{[\text{dim } S]}) \cong \delta_S$. Then $\delta_S(F^*) \cong \underline{E}_{[\text{dim } S]}$. We apply the adjunction (5.2) to obtain

$$\text{RHom}^\bullet(\underline{E}_{[\text{dim } S]}, G^*) \cong \text{RHom}^\bullet(\Gamma_S(\underline{E}_{[\text{dim } S]}), \Gamma_S(G^*))$$

By an argument of taking shifts, exact triangles, and quasi-isomorphisms in the place of $\underline{E}_{[\text{dim } S]}$, we then obtain

$$\text{RHom}^\bullet(F^*, G^*) \cong \text{RHom}^\bullet(\Gamma_S(F^*), \Gamma_S(G^*))$$
for any $F^*, G^* \in \Sh_{\mathcal{S}}(M; \mathbb{K})$. This shows the fully faithfulness of $\Gamma_{\mathcal{S}}$ by a different argument.

**Remark 5.13.** By [19, Lem.2.3.2], we also know that for all $S, W \in \mathcal{S}$, the morphism complex is given as

$$\text{RHom}_{\Sh_{\mathcal{S}}(M; \mathbb{K})}(\mathcal{K}_{\mathcal{S}}[\dim S], \mathbb{K}_{W}[\dim W]) \simeq \begin{cases} \mathbb{K} & S \leq W, \\ 0 & \text{otherwise}. \end{cases}$$

Hence so is $\text{RHom}_{\Fun(S, \mathbb{K})}(\delta_S, \delta_W)$.

**Definition 5.14.** [25, Def.3.11] Let $\mathcal{T} = (T_\ell \rightarrow T \leftarrow T_\ell) \in \mathcal{BLG}$ be a bordered Legendrian graph in $J^1I_\mathcal{S} = T^{\infty, \cdot} M$ and $\mathcal{S}$ be a regular cell complex refining the stratification $\mathcal{S}_T$ induced by $T$. Let $\Fun_T^+(S; \mathbb{K})$ (resp. $\Fun_T^+(S; \mathbb{K})_0$) be the full subcategory of $\Fun(S, \mathbb{K})$ of objects $F$ satisfying (1) and (2) (resp. (1)–(3)) as follows:

1. Every arrow from a 0-dimensional stratum which is not a crossing, a cusp, or a vertex, or from a 1-dimensional stratum which is not contained in an arc of $T$, is sent to a quasi-isomorphism. In other words, every arrow from a zero- or one-dimensional stratum contained in $\mathcal{S}$ but not in $\mathcal{S}_T$, is sent to a quasi-isomorphism.
2. If $S, S' \in \mathcal{S}$, with $S'$ bounds $S$ from above, then $S' \rightarrow S$ is sent to a quasi-isomorphism. In other words, every downward arrow is sent to a quasi-isomorphism.
3. If $S \in \mathcal{S}$ is contained in the bottom region of $T$ (i.e. the region contains the points with $z \ll 0$), then $F(S)$ is acyclic.
4. (Crossing condition) At each crossing $c$ of $\mathcal{T}$, which is also a 0-dimensional stratum in $\mathcal{S}$, there is an induced subcategory of $\mathcal{S}$ as follows:

```
\begin{center}
\begin{tikzcd}
& N \arrow[rd] & \\
W \arrow[ru] \arrow[rd] & & E \\
& c \arrow[ru] \arrow[rd] & \\
S \arrow[ru] \arrow[ru] & & \end{tikzcd}
\end{center}
```

All triangles in the diagram are commutative and the total complex of the bicomplex $F(c) \rightarrow F(nw) \oplus F(ne) \rightarrow F(N)$ is acyclic.

We also define $\Fun_T^+(S; \mathbb{K})$ and $\Fun_T^+(S; \mathbb{K})_0$ to be the full subcategories of $\Fun_T^+(S; \mathbb{K})$ and $\Fun_T^+(S; \mathbb{K})_0$, respectively, consisting of functors $F$ satisfying the extra crossing condition (4).

Then we have the following result similar to [25, Thm.3.12].

**Lemma 5.15** (Combinatorial model). Let $\mathcal{T} \in \mathcal{BLG}$. For a regular cell complex $\mathcal{S}$ for $M = I_\mathcal{S} \times \mathbb{R}_c$, obtained by refining the stratification $\mathcal{S}_T$, the functor $\Gamma_{\mathcal{S}} : \Sh(T; \mathbb{K}) \rightarrow \Fun_T^+(S; \mathbb{K})$ with quasi-inverse $i_{\mathcal{S}}$, and $i_{\mathcal{S}}$ commutes with restriction to $I_\mathcal{S}$. So is $\Gamma_{\mathcal{S}} : \Sh(T; \mathbb{K})_0 \rightarrow \Fun_T^+(S; \mathbb{K})$. The results also hold when $T$ is replaced by $T^+$.

**Proof.** The proof is entirely similar to that of [25, Thm.3.12]. Both the micro-support condition of a sheaf and the properties in Definition 5.14 of functors in $\Fun(S, \mathbb{K})$ can be checked locally. This reduces the proof to match the micro-support condition with the corresponding property in Definition 5.14 near an arc, a cusp, a crossing, and a vertex. The first three cases have been already covered by [25, Thm.3.12] and the only new ingredient is what happens at a vertex.

**Local combinatorial model near a vertex.** Let $v \in T$ be a vertex of type $(\ell, r)$. At first we assume that there are no additional 1-dimensional strata in $\mathcal{S}$ ending at $v$ and that $y(v) = 0$ and $T^*_v M \cap (\mathbb{R}_{>0} \cdot T) = \mathbb{R}_{>0} \cdot (-dz)$.

Let $p = \alpha dx + \beta dz \in T^*_v M \setminus \mathbb{R}_{>0} \cdot T$. Then either $\alpha \neq 0$ or $\alpha = 0$ and $\beta > 0$. Near $v$, we label the region below the arc $i$ by $I_i$, for $1 \leq i \leq \ell + r$, and label the regions above $v$ and below $v$ by $N$ and $S$, respectively.
Then $I = S = I + r$ and the poset subcategory of $S$ near $v$ looks as

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```

Observe that the downward arrow $v \to S$ is the same as the compositions $v \to \ell + r \to S$ and $v \to \ell \to S$. Thus the lemma in this case is equivalent to the following claim.

**Claim 5.16.** Let $B_r(v)$ be an $r$-neighborhood of $v$ for small enough $r \ll 1$. Then for any $F \in Sh_S(M; \mathbb{I})$, the restriction $C_{B_r}(v)$ is contained in $Sh_T(B_r(v); \mathbb{I})$ if and only if the arrows

$$\{ i \to I_i \mid 1 \leq i \leq \ell + r \} \cup \{ v \to \ell, v \to \ell + r \}$$

which are the dotted arrows in the diagram above, are sent to quasi-isomorphisms under $\Gamma_S(F)$.

**Proof.** For simplicity, we may assume that $v$ is at the origin $(0,0)$ and up to a local $C^1$-diffeomorphism near $v$, we can assume the half-edges $1, 2, \ldots, \ell$ are modelled on the graphs of the decreasing functions $z = a_i x^2$ on $(-1,0]$ with $a_i = 1 - \frac{1}{i}$, and the half-edges $\ell + 1, \ldots, \ell + r$ are modelled on the graphs of the increasing functions $z = b_i x^2$ on $[0,1)$ with $b_i = 1 - \frac{\ell + r}{i}$.

If $C_{B_r}(v) \in Sh_T(B_r; \mathbb{I})$, then by the local model near an arc in the smooth case, we have already known that $\Gamma_S(F)(i \to I_i)$ is a quasi-isomorphism for all $1 \leq i \leq \ell + r$.

We take a linear Morse function $\varphi = ax + \beta z$ with $\alpha \neq 0$, and then $\varphi(v) = 0, d\varphi(v) = \alpha dx + \beta dz \notin SS(F)$. By definition, $\varphi \sim R \Gamma_{\varphi(x) < 0}(F)_v$ is a quasi-isomorphism, which is equivalent to

$$R \Gamma(B_r(v) \cap \varphi^{-1}(-\infty, \varepsilon); F) \sim R \Gamma(B_r(v) \cap \varphi^{-1}(-\infty, -\varepsilon); F)$$

for a small enough $0 < \varepsilon$. We set $Y_r := B_r(v) \cap \varphi^{-1}(-\infty, \pm \varepsilon)$.

Then $Y_r$ and $Y_\ell$ are the regions which are to the left of the red and blue lines or to the right of those lines according to the sign of $\alpha$.

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Hence, we have quasi-isomorphisms

$$R \Gamma(Y_r; F) \simeq \Gamma_S(F)(v) \quad \text{and} \quad R \Gamma(Y_\ell; F) \simeq \text{Cone} \left( \bigoplus_{i=1}^\ell \Gamma_S(F)(I_i) \xrightarrow{d} \bigoplus_{i=1}^{\ell-1} \Gamma_S(F)(I_i) \right) [-1].$$
where the $i$-th component of $d$ is the difference of the restriction maps induced by $i \to I_i$ and $i + 1 \to I_i$, for $1 \leq i \leq l - 1$. Under this identification, the restriction map $RT(Y; \mathcal{F}) \to RT(Y; \mathcal{F})$ is induced by the morphisms $\{v \to i \mid 1 \leq i \leq l\}$ and is a quasi-isomorphism if and only if the total complex of

$$\Gamma_S(\mathcal{F})(v) \to \bigoplus_{i=1}^{\ell} \Gamma_S(\mathcal{F})(i) \to \bigoplus_{i=1}^{\ell-1} \Gamma_S(\mathcal{F})$$

is acyclic. As $\Gamma_S(\mathcal{F})(i \to I_i)$ is a quasi-isomorphism for all $1 \leq i \leq \ell - 1$, this happens if and only if $\Gamma_S(\mathcal{F})(v \to \ell)$ is a quasi-isomorphism as well.

Conversely, let $\mathcal{F} \in \mathcal{S}h_S(M; \mathbb{K})$ be a sheaf such that $\Gamma_S(\mathcal{F})(i \to I_i)$ for $1 \leq i \leq \ell$ and $\Gamma_S(\mathcal{F})(v \to \ell)$ are all quasi-isomorphisms. In order to prove that $\mathcal{F}|_{B_r(v)} \in \mathcal{S}h_Y(B_r(v); \mathbb{K})$, it suffices to show that $SS(\mathcal{F}) \cap (T_v^* M - \{0\}) \subset \mathbb{R}_{>0} \cdot (-dz)$ similar to the smooth case.

For any $p = \alpha dx + \beta dz$ in $T_v^* M$ with $\alpha \neq 0$, the same argument as above applies to any $C^1$-function $\varphi$ with $d\varphi(v)$ sufficiently close to $p$, rather than $\alpha x + \beta z$. Hence, $p \notin SS(\mathcal{F})$.

Suppose that $p = (\varphi, \beta dz) \in T_v^* M$ with $\beta > 0$. We use (2) in Proposition/Definition 5.1 to show that $p \notin SS(\mathcal{F})$. As seen in the picture below

we can take a smaller open ball $U$ around $v$ and a small $\epsilon > 0$ defining a line $L := \{x \mid \langle x - v, \beta dz \rangle = -\epsilon\}$, and take a proper closed convex cone $\gamma$ in $E = \mathbb{R}^2_{xz}$ with $0 \in \gamma$ so that

$$\gamma \setminus \{0\} \subset \{w \mid \langle w, \beta dz \rangle < 0\},$$

which is just the lower half-plane. For example, let $v_1, v_2$ be the two downward unit vectors generating the two green rays as shown above then $\gamma = \{tv_1 + sv_2 : t, s \geq 0\}$. Let $H := \{x : \langle x - v, \beta dz \rangle \geq -\epsilon\}$ be the region above $L$ and then we clearly have $H \cap (U + \gamma) \subset B_r(v)$. It suffices to show that for all $x \in U$, we have the natural quasi-isomorphism

$$r : RT(H \cap (x + \gamma); \mathcal{F}) \simto RT(L \cap (x + \gamma); \mathcal{F})$$

(5.4)

Recall [17, Rmk.2.6.9] that for any compact subset $Z$ of $B_r(v)$, there exists a quasi-isomorphism

$$\lim_{U \supset Z} RT(U; \mathcal{F}) \simto RT(Z; \mathcal{F}),$$

where $U$ runs over the open neighborhoods of $Z$.

As illustrated above, $H \cap (x + \gamma)$ is the region bounded by a triangle $\Delta_x$ whose sides are parallel to 3 lines $\mathbb{R} \cdot v_1, \mathbb{R} \cdot v_2$ and $L$, where $L \cap (x + \gamma)$ is the bottom edge of $\Delta_x$ contained in the region $S$. Hence, $RT(L \cap (x + \gamma); \mathcal{F}) \simto \Gamma_S(\mathcal{F})(S) = RT(S; \mathcal{F})$. Referring the picture above and according to the choice of $x$, we have the following four cases:

1. If $x = x_1$ so that $H \cap (x_1 + \gamma)$ has empty intersection with $T$, then $RT(H \cap (x + \gamma); \mathcal{F}) \simto \Gamma_S(\mathcal{F})(S)$ and so we have the natural isomorphism $r$ in (5.4).
2. If $x = x_2$ so that $v \in H \cap (x_2 + \gamma)$, then $RT(H \cap (x + \gamma); \mathcal{F}) \simto \Gamma_S(\mathcal{F})(v)$ and the map $r$ is the composition $\Gamma_S(\mathcal{F})(\ell + r \to S) \circ \Gamma_S(\mathcal{F})(v \to \ell + r)$. By the hypothesis of $\mathcal{F}$, each component in the composition is a quasi-isomorphism, hence so is the composition.
(3) If \( x = x_3 \) so that \( H \cap (x_3 + \gamma) \) has non-empty intersection with \( T \) exactly along half-edges \( i, i+1, \ldots, \ell \) for some \( 1 \leq i \leq \ell \), then we have

\[
R\Gamma(H \cap (x + \gamma); \mathcal{F}) \cong \text{Cone} \left( \bigoplus_{k=i}^\ell \Gamma_S(\mathcal{F})(k) \xrightarrow{d} \bigoplus_{k=i}^\ell \Gamma_S(\mathcal{F})(I_k) [1] \right) [-1],
\]

where the \( k \)-th component of \( d \) is the difference of the restriction maps induced by \( k \to I_k \) and \( k+1 \to I_k \) as before. Under this identification, the map (5.4) is induced by the morphism \( \ell \to S \).

Then by the nine lemma (or the 3 x 3-lemma) for triangulated categories [18, Lem.2.6], we obtain a diagram in which all squares commute except for the non-displayed one on the bottom right that anti-commutes, and all rows and columns are exact triangles:

\[
\begin{array}{ccc}
R\Gamma(H \cap (x + \gamma); \mathcal{F}) & \xrightarrow{\text{Id}} & \bigoplus_{k=i}^\ell \Gamma_S(\mathcal{F})(k) \\
\downarrow & & \downarrow \\
\Gamma_S(\mathcal{F})(S) & \xrightarrow{\text{Id}} & \bigoplus_{k=i}^\ell \Gamma_S(\mathcal{F})(I_k) \\
\downarrow & & \downarrow \\
\text{Cone}(r) & \xrightarrow{\text{Id}} & \bigoplus_{k=i}^{\ell-1} \Gamma_S(\mathcal{F})(k)[1] \\
\downarrow & & \downarrow \\
& & \bigoplus_{k=i}^{\ell-1} \Gamma_S(\mathcal{F})(I_k)[1] +1
\end{array}
\]

where the map \( r' \) is induced by \( \ell \to S \) and the \( k \)-th component of \( d' \) is the difference of restriction maps induced by \( k \to I_k \) and \( k+1 \to I_k \) if \( i \leq k < \ell - 1 \) and is the restriction map induced by \( \ell - 1 \to I_{\ell-1} \) if \( k = \ell - 1 \). By the hypothesis of \( \mathcal{F} \), the map \( \Gamma_S(\mathcal{F})(k) \to I_k \) is a quasi-isomorphism for \( k = \ell - 1, \ell - 2, \ldots, i \), and so is \( d' \). The exactness of the third row implies that \( \text{Cone}(r) \) is acyclic and therefore is a quasi-isomorphism.

(4) If \( x = x_4 \) so that \( H \cap (x + \gamma) \) has non-empty intersection with \( T \) exactly along half-edges \( j, j + 1, \ldots, \ell + r \), for some \( \ell + 1 \leq j \leq \ell + r \), then the same argument as in the third case holds and we conclude that the map \( r \) in (5.4) is a quasi-isomorphism.

This completes the proof of the claim. \( \square \)

Finally, suppose that there are some additional 1-dimensional strata ending at \( v \). By our assumption on \( S \) at the beginning of Section 5.2, we can regard the extra 1-dimensional strata near \( v \) as additional half-edges at \( v \) and obtain a new vertex \( v' \) with \( \ell' \) left half-edges and \( r' \) right half-edges for some \( \ell' \geq \ell \) and \( r' \geq r \).

We label the half-edges and regions near \( v' \) as before. Then exactly the same argument as above holds and proves the lemma for this case. Equivalently, for all \( \mathcal{F} \in \mathcal{SH}(M; \mathbb{K}) \), we have \( \mathcal{F} \in \mathcal{SH}(B_{r}(v); \mathbb{K}) \) if and only if all arrows \( \{i \to I_i \mid 1 \leq i \leq \ell' + r' \} \) are sent to quasi-isomorphisms under \( \Gamma_S(\mathcal{F}) \).

\[5.3. \textbf{A legible model for constructible sheaves}. \] Let \( \mathcal{T} = (T_L \to T \leftarrow T_R) \in \mathcal{BL}_S \) be a bordered Legendrian graph in \( J^1 I_x = T^\infty M \) as before and \( S \) be a stratification refining \( S_T \). We can simplify the combinatorial model further under the following stronger assumption:

\[\textbf{Assumption 5.17}. \] The stratification \( S \) is the induced stratification \( S_{\mathcal{T}} \) of a bordered Legendrian graph \( \mathcal{T}' = (T_L \to \mathcal{T} \leftarrow T_R) \) which extends \( \mathcal{T} \) so that \( \mathcal{T}' \) contains no cusps and no vertices with only left or right half-edges.

In particular, this implies that \( S \) is a regular cell complex and satisfies Assumption 5.4.

For a stratification \( S \) satisfying the above assumption, we denote by \( G_S \) the finite set of functions \( f \) on \((x_L, x_R)\) such that \( f \) is either \( \pm \infty \) or a continuous function whose graph is contained in a union of zero- and one-dimensional strata in \( S \).

We observe the following: suppose that \( S \) satisfies Assumption 5.17. Then any region \( S \) in \( S \) is an open disk in \( M \) bounded by the graphs of 2 functions \( f_L \leq f_U \) in \( G_S \). That is, \( S = \{(x, z) \mid x \in (x_L, x_R), f_L(x) < \)}.
We call \((f_L, f_U)\) a bounding pair for \(S\). The choice of the pair \((f_L, f_U)\) might not be unique. However, since \(S\) is contractible, there exist \(x_L \leq x_{1,S} < x_{2,S} \leq x_R\) depending only on \(S\) such that \(f_L < f_U\) on \((x_1,S, x_2,S)\) but \(f_L = f_U\) on \((x_1,S, x_1,S) \cup (x_2,S, x_R)\). In other words, the graphs of \(f_L\) and \(f_U\) on \((x_1,S, x_2,S)\) are the lower and upper boundary of \(S\), which will be denoted by \(B_S\) and \(A_S\), respectively.

For any two bounding pairs \((f_L, f_U)\) and \((f'_L, f'_U)\) for \(S\), we denote by \((f_L, f_U) \preceq (f'_L, f'_U)\) if \(f_L \leq f'_L\) and \(f_U \leq f'_U\). In addition, the pair \((\max\{f_L, f'_L\}, \max\{f_U, f'_U\})\) becomes also a bounding pair for \(S\) which is a common upper bound. It follows that there is a unique maximal bounding pair for \(S\), which will be denoted by \((I_S, U_S)\).

**Definition 5.18.** Let \(S = S(\overline{T})\) be a regular cell complex as above. A poset category or a finite (acyclic) quiver with relations, denoted by \(\mathcal{R}(S)\), is defined as follows:

- The objects are the 2-dimensional cells of \(S\);
- For any two objects \(R\) and \(S\) separated by the 1-dimensional stratum \(s\) with \(S\) below and \(R\) above, we assign an arrow \(e_s : S \to R\) that impose \(S \leq R\) and generates the partial order on \(\mathcal{R}(S)\);
- For any crossing or vertex \(v\) of \(\overline{T}\), there are unique regions \(N\) and \(S\) immediately above and below \(v\) and exactly two directed paths \(\gamma_L(v) : S \to N\) and \(\gamma_R(v) : S \to N\) which are compositions of arrows, such that \(\gamma_L(v)\) and \(\gamma_R(v)\) pass through objects corresponding to the regions in \(\text{Star}_S(v)\) and go around \(v\) from the left and right hand side, respectively. Then we impose the relation \(\gamma_L(v) \sim \gamma_R(v)\).

Moreover, we say that an arrow \(e : S \to R\) is simple if there are no regions between them. We say that a directed path \(\gamma : S \to R\) is simple if it is a composition of simple arrows.

We denote by \(A(\mathcal{R}(S))\) the quotient of the path algebra \(\mathbb{K}\langle \mathcal{R}(S) \rangle\) of the acyclic quiver \(\mathcal{R}(S)\) by the ideal generated by \((\gamma_L(v) - \gamma_R(v))\)’s

\[
A(\mathcal{R}(S)) := \mathbb{K}\langle \mathcal{R}(S) \rangle / (\gamma_L(v) - \gamma_R(v) : v \in V(S)).
\]

Here, \(V(S)\) is the set of 0-dimensional strata of \(S\).

We remark that while the path algebra of an acyclic quiver is of global cohomological dimension 1, our algebra \(A(\mathcal{R}(S))\) is of global cohomological dimension 2 in general.

**Proposition 5.19.** The poset category \(\mathcal{R}(S)\) is well-defined.

**Proof.** To show that the definition indeed gives a partial order, we need to check the antisymmetric property, that is, for any two 2-dimensional cells \(R, S\) in \(S\), both \(R \leq S\) and \(S \leq R\) implies that \(R = S\). This follows immediately from the last statement in the lemma below.

**Lemma 5.20.** For any two 2-dimensional cells \(R, S\) in \(S\), the following are all equivalent:

1. \(u_R \leq l_S\).
2. \(S\) is above the graph of \(u_R\).
3. \(R < S\).

Moreover, \(R \leq S\) if and only if \(l_R \leq l_S\).

**Proof.** \((1) \Rightarrow (2)\) is clear.

\((2) \Rightarrow (1)\): If \(S\) is above the graph of \(u_R\), then \(l_S \geq u_R\) on \([x_1,S, x_2,S]\). Suppose that \(l_S(x) < u_R(x)\) for some \(x\), say \(x > x_2,S\). Then \(l_W := \{x \in (x_2,S, x_R) | l_S(x) < u_R(x)\}\) is non-empty. Let \(x_0 := \inf(l_W) \geq x_2,S\), then \(l_S \geq u_R\) on \([x_1,S, x_0]\), \(u_S(x_0) = l_S(x_0) = u_R(x_0)\), and the graph of \(u_R\) is strictly above that of \(l_S\) on

\[^4\text{A function } f_L \text{ or } f_U \text{ may be constant at } \pm \infty \text{ and in this case the region is unbounded.}\]
an small open interval \((x_0, x_0 + \epsilon)\). Define a pair \((\ell'_S, u'_S)\), which coincides with \((\ell_S, u_S)\) on \((x_1, x_0)\), and \(\ell'_S = u'_S = \max\{\ell_S = u_S, u_R\} \) on \([x_0, x_1)\). This defines a bounding pair for \(S\), and \((\ell_S, u_S) < (\ell'_S, u'_S)\), contradicting to the maximality of \((\ell_S, u_S)\). Hence, \(\ell_S(x) \geq u_R(x)\) for all \(x \in (x_1, x_0)\).

\((3) \Rightarrow (2)\) : It suffices to show: If \(R\) and \(S\) are separated by an 1-dimensional stratum \(s\), with \(R\) below and \(S\) above. Then \(S\) is above the graph of \(u_R\).

In fact, \(s\) is the graph of the function \(u_R = \ell_S\) on an open interval \((x_1, x_2)\). Then, \(S\) contains points which are above the graph of \(u_R\), hence, the whole disk \(S\) is above the graph of \(u_R\).

\((2) \Rightarrow (3)\) : \(S \neq R\) is clear. It suffices to show \(R \leq S\). If the lower boundary of \(S_0 = S\) is not contained in the graph of \(u_R\), then it contains a 1-dimensional stratum \(s_1\), which is strictly above the graph of \(u_R\).

Let \(S_1\) be the 2-dimensional cell below \(s_1\), then \(S_1\) is above the graph of \(u_R\) as well. Also by \((3) \Rightarrow (2)\), \(S_0\) is above the graph of \(u_S\). By \((2) \Rightarrow (1)\), we then have \(u_R \leq \ell_{S_1} \leq \ell_{S_0} \leq u_{S_0} \) in \(G_S\), with \(l_{S_0} \neq u_{S_0}, \ell_{S_0} \neq u_{S_0}\). If the lower boundary of \(S_1\) is not contained in the graph of \(u_R\), we repeat the procedure above to obtain \(S_2, S_3\). In particular, \(S_2\) is above the graph of \(u_R\), \(S_1\) is above the graph of \(u_S\), and \(u_R \leq \ell_{S_2} \leq \ell_{S_1} \leq \ell_{S_0} \leq u_{S_0} \) in \(G_S\), with \(l_{S_i} \neq u_{S_i}\) for \(i = 0, 1, 2\). Since \(G_S\) is finite, after repeat the previous procedure finitely many times, we obtain a finite sequence \(s_i, S_i\) for \(1 \leq i \leq N\) for some \(N \geq 0\), such that, \(s_i\) separates the 2-dimensional cells \(S_{i-1}, S_i\) with \(S_{i-1}\) above and \(S_i\) below, \(u_{S_i} \leq \ell_{S_{i+1}}\), \(u_R \leq \ell_{S_N}\), and the lower boundary of \(S_N\) is contained in the graph of \(u_R\). The last condition just says that \(l_{S_N} = u_R([x_1, S_N] \times [x_2, S_N])\).

Let us show that there is a 1-dimensional cell \(S_{N+1}\) separating \(R, S_N\) with \(R\) below and \(S_N\) above. Otherwise, the open intervals \((x_1, x_2, R)\) and \((x_1, S_N, x_2, S_N)\) have empty intersection. Say, \(x_2, R \neq x_1, S_N\). Then by definition \(x_2, R\), have \(I_R = u_R = l_{S_N}\), on \([x_1, x_2, S_N]\). Define a pair \((I_R, u'_R)\) such that it coincides with \((I_R, u_R)\) outside \([x_1, S_N, x_2, S_N]\), and \(I'_R = u'_R = u_{S_N}\) on \([x_1, S_N, x_2, S_N]\). This defines a new bounding pair for \(R\) with \((I_R, u_R) \leq (I'_R, u'_R)\), contradiction. Now, we have shown that \(S = S_0 \geq S_1 \geq \ldots \geq S_N \geq R\), hence \(S \geq R\).

The last statement of the lemma: ‘\(\Rightarrow\)’ follows from \((3) \Rightarrow (1)\).

\(\Leftarrow\) If \(R = S\), we are done. Otherwise, assume \(I_R \leq \ell_{R}\) and \(R \neq S\). In particular, \(S\) is the graph of \(u_R\).

Then \(S\) is above the graph of \(u_R\) as well, as \(R \neq S\) is the only 2-dimensional cell bounded by the graphs of \(l_{R} \leq u_{R}\). Now \((2) \Rightarrow (3)\) implies \(R < S\).

This finishes the proof of the lemma.

For a region \(R \in \mathcal{R}(S)\), we define an open subset of \(M\) to be the upper half-space of \(I_R(x)\)

\[M_R := \{(x, z) | x \leq x_R, z > l_R(x)\}\]

Then \(M_R \subset M_R\) if and only if \(R' \geq R\).

**Lemma 5.21.** Let \(R \in \mathcal{R}(S)\) be fixed. Then for any \(R' > R\) in \(\mathcal{R}(S)\) with lower boundary \(B_R\), the intersection \(B_{R'}(R) := B_{R'} \cap M_R\) is non-empty and contractible.

**Proof.** \(B_{R}(R)\) is non-empty: Since \(R' > R\), by definition, there is a 1-dimensional cell \(s\) separating \(R'\) and another region \(S\), with \(R'\) above and \(S\) below, such that \(S \geq R\). In particular, \(S\) is above the graph of \(I_R\). It follows that \(s\) is contained in \(M_R\), hence contained in \(B_{R'}(R) = B_{R'} \cap M_R\).

\(B_{R}(R)\) is contractible. We use the notation at the beginning of Section 5.3. Recall that \(B_{R}\) is simply the graph of \(l_{R}\) over \((x_1, x_2, R)\). Moreover, \(l_{R} < u_{R}\) on \((x_1, x_2, R)\) and \(l_{R} = u_{R}\) outside \((x_1, x_2, R)\). Also, by Lemma 5.20, we have \(u_{R} \leq l_{R}\).

If \(B_{R}(R)\) is not contractible, then \(B_{R}\) has a 1-dimensional cell \(s\) contained in the graph of \(I_{R}\), such that, \(B_{R}\) contains points both on the left and right of \(S\). As \(R'\) is above the graph of \(u_{R}\), \(s\) has no points in the lower boundary of \(R\). In other words, \(s\) is contained in the graph of \(l_{R} = u_{R}\) over \([x_0, x_1]\) or \([x_2, x_0]\), say, the latter. Assume \(s\) lives over the interval \((x_1, x_2)\), that is, \(s\) is the graph of \(I_{R} = u_{R} = l_{R}\) over \((x_1, x_2)\). Then \(x_2, R \leq x_1\). Moreover, \(l_{R} \neq I_{R}\) as functions on \([x_2, x_0]\), as \(B_{R}\) contains points living over \([x_2, x_0]\).

Now, can define a pair \((\tilde{l}_{R}, \tilde{u}_{R})\) of continuous functions in \(G_S\) such that, it coincides with \((I_{R}, u_{R})\) on \((x_1, x_2)\), and \(\tilde{l}_{R} = \tilde{u}_{R} \leq \max\{l_{R} = u_{R}\} = I_{R}\) on \([x_2, x_0]\). Then \((\tilde{l}_{R}, \tilde{u}_{R})\) is a new bounding pair for \(R\), with \((I_{R}, u_{R}) \leq (\tilde{l}_{R}, \tilde{u}_{R})\), contradicting to the maximality of \((I_{R}, u_{R})\). This finishes the proof.\[\]

Now there is an induced functor of poset categories \(\rho : S \rightarrow \mathcal{R}(S)\), which sends \(w\) to \(\rho(w)\), the unique 2-dimensional cell ‘below’ \(w\). More precisely, we define \(\rho(w)\) to be \(w\) if \(w\) is a 2-dimensional stratum, or
to be the 2-dimensional cell immediately below \( w \) otherwise. Then for each region \( R \in \mathcal{R}(S) \), the upper half-space \( M_R \) is the union of the strata
\[
M_R = \bigcup_{\rho(w) \geq R} w
\]
by Lemma 5.20.

As usual, we define the abelian category \( \text{Fun}(\mathcal{R}(S), \mathbb{K}) \), the DG category \( \mathcal{F}\text{un}(\mathcal{R}(S); \mathbb{K}) \), and the restriction functors \( \mathcal{F}\text{un}(\mathcal{R}; \mathbb{K}) \to \mathcal{F}\text{un}(\mathcal{R}(S|_{I_S}); \mathbb{K}) \) for an open sub-interval \( I_S \) of \( I_x \). By pre-composition, we get a functor
\[
\rho^* : \text{Fun}(\mathcal{R}(S); \mathbb{K}) \to \text{Fun}(S; \mathbb{K}),
\]
which is clearly exact. We can then use the same letter \( \rho^* \) for its DG lifting
\[
\rho^* : \mathcal{F}\text{un}(\mathcal{R}(S); \mathbb{K}) \to \mathcal{F}\text{un}(S; \mathbb{K}).
\]

**Proposition/Definition 5.22.** Let us define a functor between two abelian categories
\[
\rho_* : \text{Fun}(S, \mathbb{K}) \to \text{Fun}(\mathcal{R}(S), \mathbb{K}), \quad F \mapsto [R \mapsto \Gamma(M_R; i_{S}(F))].
\]
Then \( \rho_* \circ \rho^* = \text{Id} \) and \( (\rho^*, \rho_*) \) is an adjoint pair
\[
\rho^* : \text{Fun}(\mathcal{R}(S), \mathbb{K}) \rightleftarrows \text{Fun}(S, \mathbb{K}) : \rho_*.
\]
As a consequence, we obtain an adjoint pair \( (\rho^*, R\rho_*) \) in the DG liftings
\[
\rho^* : \mathcal{F}\text{un}(\mathcal{R}(S), \mathbb{K}) \rightleftarrows \mathcal{F}\text{un}(S, \mathbb{K}) : R\rho_*
\]
where \( R\rho_* \) is given by \( F \mapsto [R \mapsto R\Gamma(M_R; i_{S}(F))] \). Moreover, we have a natural isomorphism \( \beta : \text{Id} \Rightarrow R\rho_* \circ \rho^* \).

**Proof.** We firstly show that \( \rho_* \circ \rho^* = \text{Id} \). By definition and formula (5.3), for any \( F \in \text{Fun}(\mathcal{R}(S), \mathbb{K}) \) and \( R \in \mathcal{R}(S) \), we have
\[
(\rho_* \circ \rho^*)(F)(R) = \Gamma(M_R; i_{S}(\rho^*F)) = \lim_{w \in S|_{M_R}} (\rho^*F)(\tau_{M_R}(w)).
\]

Since \( S|_{M_R} = \{ w \in S | \rho(w) \geq R \} \) and \( \tau_{M_R} \) is just the inclusion map, \( (\rho^*F)(\tau_{M_R}(w)) = F(\rho(w)) \) for all \( w \in S|_{M_R} \) and it follows that
\[
(\rho_* \circ \rho^*)(F)(R) = \lim_{\rho(w) \geq R} F(\rho(w)) \cong F(R).
\]

Therefore \( \rho_* \circ \rho^* = \text{Id} \).

Next, we show that \( (\rho^*, \rho_*) \) is an adjoint pair. For any \( F \in \text{Fun}(\mathcal{R}(S), \mathbb{K}) \) and \( G \in \text{Fun}(S, \mathbb{K}) \), a morphism \( f \in \text{Hom}(\rho^*F, G) \) is a collection of maps \( \{ f_w | w \in S \} \) with \( f_w : (\rho^*F)(w) = F(\rho(w)) \to G(w) \) such that for any arrow \( w_1 \to w_2 \), we get a commutative diagram
\[
\begin{array}{ccc}
F(\rho(w_1)) & \xrightarrow{f_{w_1}} & G(w_1) \\
\downarrow & & \downarrow \\
F(\rho(w_2)) & \xrightarrow{f_{w_2}} & G(w_2)
\end{array}
\]

Equivalently, we get a collection of maps \( \{ f_{R,w} | \rho(w) \geq R \} \)
\[
f_{R,w} : F(R) \to F(\rho(w)) = (\rho^*F)(w) \xrightarrow{f_w} G(w)
\]
for any fixed region \( R \) in \( \mathcal{R}(S) \) and \( w \in S|_{M_R} \), i.e., \( \rho(w) \geq R \) such that we get a commutative diagram
\[
\begin{array}{ccc}
F(R_1) & \xrightarrow{f_{R_1,w_1}} & G(w_1) \\
\downarrow & & \downarrow \\
F(R_2) & \xrightarrow{f_{R_2,w_2}} & G(w_2)
\end{array}
\]
for any \( R_1 \leq R_2 \) in \( \mathcal{R}(S) \) and \( w_1 \leq w_2 \) in \( S \) with \( \rho(w_1) \geq R \). This is again equivalent to having a collection of maps \( \{ f_R \mid R \in \mathcal{R}(S) \} \) with
\[
\tilde{f}_R = \lim_{\rho(w) \geq R} f_{R,w} : F(R) \to \lim_{\rho(w) \geq R} G(w),
\]
such that for any arrow \( R_1 \to R_2 \) in \( \mathcal{R}(S) \), there is a commutative diagram
\[
\begin{array}{ccc}
F(R_1) & \xrightarrow{f_{R_1}} & \lim_{\rho(w_1) \geq R_1} G(w_1) \\
\downarrow & & \downarrow \\
F(R_2) & \xrightarrow{f_{R_2}} & \lim_{\rho(w_2) \geq R_2} G(w_2).
\end{array}
\]

Since \( (\rho, G)(R) = \Gamma(M_R; i_S(G)) = \lim_{\rho(w) \geq R} G(w) \) by formula (5.3), the above data is identical to a morphism \( \tilde{f} \) in \( \text{Hom}(F, \rho, G) \). That is, we have a natural isomorphism \( \text{Hom}(\rho, F, G) \cong \text{Hom}(F, \rho, G) \).

Finally, we show the natural isomorphism \( \beta : \text{Id} \Rightarrow R\rho_* \circ \rho^* \). Indeed, for any \( F^\bullet \in \mathcal{F}\text{un}(\mathcal{R}(S), \mathbb{K}) \) and any region \( R \) in \( \mathcal{R}(S) \), we have a natural morphism
\[
\beta_{F^\bullet}(R) : F^\bullet(R) \to (R\rho_* \circ \rho^*)(R)
\]
defined by
\[
F^\bullet(R) = (\rho_* \circ \rho^* F^\bullet)(R) = \Gamma(M_R; i_S \rho^* F^\bullet) \to \Gamma(M_R; i_S \rho^* F^\bullet).
\]
We want to show that \( \beta_{F^\bullet}(R) \) is a quasi-isomorphism.

For any region \( R' \in \mathcal{R}(S) \), let \( \delta_{R'} : \mathcal{F}\text{un}(\mathcal{R}(S), \mathbb{K}) \) be a functor defined by \( \delta_{R'}(W) = 0 \) if \( W \neq R \), and \( \delta_{R'} = \mathbb{K} \) such that \( \delta'_{R'} \) sends all the arrows in \( \mathcal{R}(S) \) to zero. As seen in the proof of Lemma 5.11, \( \mathcal{F}\text{un}(\mathcal{R}(S), \mathbb{K}) \) is the triangulated envelope of the objects of the form \( \delta'_{R'} \). Observe that for fixed \( R \), the property of \( \beta_{F^\bullet}(R) \) in (5.5) being a quasi-isomorphism is preserved under taking quasi-isomorphisms, shifts, and cones in the place of \( F^\bullet \). Thus, it suffices to show that \( \beta_{F^\bullet}(R) \) is a quasi-isomorphism only when \( F^\bullet = \delta'_{R'} \) for \( R' \in \mathcal{R}(S) \).

By definition, we see that \( i_S \rho^* \delta'_{R'} = \mathbb{K}_{E_{R'}} \), where \( E_{R'} \) is the union of \( R' \) and its upper boundary \( A_{R'} \). It suffices to show that
\[
\Gamma(M_R; \mathbb{K}_{E_{R'}}) \cong \begin{cases} \mathbb{K} & \text{if } R' = R, \\ 0 & \text{otherwise.} \end{cases}
\]

If \( R' \) is not contained in \( M_R \), then neither is \( E_{R'} \) ans we have \( \Gamma(M_R; \mathbb{K}_{E_{R'}}) = 0 \). Otherwise, if \( R' \subset M_R \), then so is \( E_{R'} \). If \( R' \neq R \), by Lemma 5.21, we have \( \overline{R}(R) := \overline{R} \cap M_R = E_{R'} \cup B_{E_{R'}}(R) \), where both of \( \overline{R}(R) \) and \( B_{E_{R'}}(R) \) are non-empty contractible closed subsets of \( M_R \). We then obtain the following exact triangle:
\[
\Gamma(M_R; \mathbb{K}_{E_{R'}}) \to \Gamma(M_R; \mathbb{K}_{\overline{R}(R)}) \to \Gamma(M_R; \mathbb{K}_{B_{E_{R'}}(R)}) \xrightarrow{\delta^1}.
\]

where the second arrow is the same as the quasi-isomorphism \( \Gamma(\overline{R}(R); \mathbb{K}) \cong \mathbb{K} \to \Gamma(B_{E_{R'}}(R); \mathbb{K}) \cong \mathbb{K} \). Therefore, \( \Gamma(M_R; \mathbb{K}_{E_{R'}}) \) is acyclic. Finally, if \( R' = R \), then \( E_R \) is a contractible closed subset of \( M_R \) and it follows that \( \Gamma(M_R; \mathbb{K}_{E_R}) \cong \Gamma(E_R; \mathbb{K}) \cong \mathbb{K} \) as desired.

**Definition 5.23.** Let \( \mathcal{F}\text{un}_f(\mathcal{R}(S), \mathbb{K}) \) and \( \mathcal{F}\text{un}_f(\mathcal{R}(S), \mathbb{K}_0) \) be full subcategories of \( \mathcal{F}\text{un}(\mathcal{R}(S), \mathbb{K}) \) consisting of functors \( F \) satisfying the conditions (1)–(2) and (1)–(3) described below:

1. For any additional 1-dimensional stratum \( s \) of \( S \) separating two 2-dimensional cells \( R_1 \) below and \( R_2 \) above, the morpgism \( F(e_s) : F(R_1) \to F(R_2) \) is a quasi-isomorphism.
(2) Around any crossing $c$ of $T$, we label the 4 regions by $N, S, W$ and $E$ as in Definition 5.14 and there exists the following commutative diagram

$$
\begin{array}{ccc}
F(N) & \rightarrow & F(W) \\
\downarrow & & \downarrow \\
F(S) & \rightarrow & F(E)
\end{array}
$$

such that the total complex of $F(S) \rightarrow F(W) \oplus F(E) \rightarrow F(N)$ is acyclic.

(3) If $R$ is the bottom region in $S$, then $F(R)$ is acyclic.

**Proposition 5.24 (Legible model).** Let $S$ be a regular cell complex refining the stratification induced by $T$ and satisfying Assumption 5.17. Then

$$\rho^* : \text{Fun}(R(S), \mathbb{K}) \overset{\sim}{\rightarrow} \text{Fun}_{T^*}(S, \mathbb{K})$$

(5.7)

is a quasi-equivalence with a quasi-inverse $R_{\rho*}$, and $\rho^*$ is compatible with restriction to an open subinterval. Moreover, the similar statement holds for $\rho^* : \text{Fun}_T(R(S), \mathbb{K}) \rightarrow \text{Fun}_T(S, \mathbb{K})$.

For the proof of the proposition, we need the following lemma.

**Lemma 5.25.** For any $F \in \mathbf{Sh}_{T^*}(M; \mathbb{K})$, there exists a filtration

$$0 = F_m \rightarrow F_{m-1} \rightarrow \cdots \rightarrow F_0 = F$$

such that each of the associated graded pieces $Gr_i F_m$ is contained in $\mathbf{Sh}_{T^*}(M; \mathbb{K})$ and supported on a single region $R_i$ of $S$.

More precisely, for each $E_i$ which is the union of $R_i$ and its upper boundary, we have $Gr_i F_m = (A_i)_{E_i}$ for some perfect complex of $\mathbb{K}$-modules $A_i$ regarded as a constant sheaf on $M$.

**Proof.** We use the notations at the beginning of Section 5.3. For each function $f$ in $G_S$, we define an open subset of $M$

$$M_f := \{(x, z) \in M \mid x \in (x_L, x_R), z > f(x)\}.$$ 

Consider the maximal bounding pair $(l_{R_1}, u_{R_1})$ for the bottom region $R_1$ in $S$. Then $f_0 := l_{R_1} \equiv -\infty$ and $f_1 := u_{R_1}$ is the unique minimum in $G_S \setminus \{f_0\}$. In particular, $M_{f_0} = M$.

If $M_{f_1}$ is non-empty, then $R(S|_{M_{f_1}})$ is a finite non-empty partially ordered subset of $R(S)$ which contains a minimum, say $R_2$. In other words, $R_2$ is a minimum among those regions of $S$ which are above the graph of $f_1$. Then the lower boundary of $R_2$ is contained in the graph of $f_1$. Otherwise, the lower boundary contains a 1-dimensional stratum $s$ which is strictly above the graph of $f_1$. The region $S = \rho(s)$ below $s$ is less than $R_2$ and this is a contradiction to the minimality of $R_2$.

Now we take $f_2 \in G_S$ so that $(f_1, f_2)$ is a bounding pair of $R_2$. Then by induction, we can repeat the procedure above to obtain a sequence $(f_i)$ in $G_S$ such that for each $i$, the graphs of $f_{i-1}$ and $f_i$ bound a single region $R_i$. Since $G_S$ is finite, there exists a $m \in \mathbb{N}$ such that $f_m = \infty$ is the unique maximum in $G_S$. Hence, $M_{f_m} = \emptyset$ and the procedure stops. In fact, $m$ is the number of regions in $S$.

Let $M_i := M_{f_i}$. We obtain a sequence of open inclusions

$$\emptyset = M_m \subset M_{m-1} \subset \cdots \subset M_0 = M$$

and $F_i := F_{M_i} = R(j_{M_i}):j_{M_i}^{-1}F$, where $j_{M_i} : M_i \rightarrow M$ is the open inclusion. By [17, Prop.5.4.8.(ii)], we have $F_i \in \mathbf{Sh}_{T^*}(S; \mathbb{K})$ and obtain a filtration

$$0 = F_m \rightarrow F_{m-1} \rightarrow \cdots \rightarrow F_0 = F,$$

whose associated $i$-th graded piece is $Gr_i F_m = F_{M_{m-1} \setminus M_i}$ and induced by the exact triple

$$F_{M_i} \rightarrow F_{M_{i-1}} \rightarrow F_{M_{i-1} \setminus M_i} \rightarrow F_{M_{i+1} \setminus M_i}^{+1}.$$
By definition, $\text{Gr}_i \mathcal{F}_s$ is supported in a single region $R_i$. More precisely, it has possibly non-zero stalks only at points in the region $R_i$ and its upper boundary. Moreover, by the triangular inequality for micro-supports, we have $\text{Gr}_i \mathcal{F}_s \in \text{Sh}_{\mathcal{F}_s}(M; \mathbb{K})$. In fact,
\[
\text{Gr}_i \mathcal{F}_s \simeq \mathcal{F}_{M_{i-1}\setminus M_i} \simeq (F(R_i))_{M_{i-1}\setminus M_i},
\]
where the complex $F(R_i) = R\Gamma(R_i; \mathcal{F})$ is regarded as a constant sheaf on $M$. \hfill \Box

**Proof of Proposition 5.24.** The proof is similar to that of [25, Prop.3.22], the only nontrivial part is to show the quasi-equivalence (5.7).

At first, let us show that $\rho^*$ is fully faithful. For any $F, G \in \mathcal{F}\text{un}(\mathcal{R}(S), \mathbb{K})$, by the adjunction $(\rho^*, R\rho_*)$ in Proposition/Definition 5.22, we have
\[
\text{RHom}^*(\rho^*F, \rho^*G) \simeq \text{RHom}^*(F, R\rho_*\rho^*G) \simeq \text{RHom}^*(F, G),
\]
where the last quasi-isomorphism follows from the natural isomorphism $\beta : \text{Id} \Rightarrow R\rho_* \circ \rho^*$ in Proposition/Definition 5.22, which then implies the quasi-isomorphism $G \simeq R\rho_* \circ \rho^*(G)$. Thus, $\rho^*$ is fully faithful. The natural isomorphism also shows that $R\rho_*$ is essentially surjective.

Now, it suffices to show that $\rho^*$ is essentially surjective. For any functor $F \in \mathcal{F}\text{un}(\mathcal{S}; \mathbb{K})$, let $\mathcal{F} := i_S(F)$. Then by Proposition/Definition 5.10 and Corollary 5.7, we have
\[
R\Gamma(\text{Star}(w); \mathcal{F}) \simeq F_x = F(w) = \Gamma(\text{Star}(w); \mathcal{F})
\]
for all $x \in w \in \mathcal{S}$. The adjunction $(\rho^*, R\rho_*)$ gives us an adjunction map $\epsilon_F : \rho^* R\rho_* F \to F$, where for any $w \in \mathcal{S}$,
\[
\rho^* R\rho_* F(w) = R\rho_* F(\rho(w)) = R\Gamma(M_{\rho(w)}; \mathcal{F}),
\]
and by definition of $M_{\rho(w)}$, we have $\text{Star}(w) \subset M_{\rho(w)}$ and therefore
\[
\epsilon_F(w) : \rho^* R\rho_* F(w) = R\Gamma(M_{\rho(w)}; \mathcal{F}) \to R\Gamma(\text{Star}(w); \mathcal{F}) \simeq F(w)
\]
defined by the restriction of sections.

Hence it suffices to show that $\epsilon_F : \rho^* R\rho_* F \to F$ is a quasi-isomorphism for all $F \in \mathcal{F}\text{un}_{\text{Tor}}(\mathcal{S}; \mathbb{K})$, or equivalently, the restriction
\[
R\Gamma(M_{\rho(w)}; \mathcal{F}) \simeq R\Gamma(\text{Star}(w); \mathcal{F})
\]
is a quasi-isomorphism for any $w \in \mathcal{S}$.

Now we apply Lemma 5.25 to $\mathcal{F}$ to reduce the proof of the quasi-isomorphism (5.8) to the case when $m = 1$, that is, when the sheaf $\mathcal{F}$ is supported on a single region $R'$ and $\mathcal{F} \simeq A_{E_{R'}}$ for some perfect complex of $\mathbb{K}$-modules $A$ with $E_{R'}$ the union of $R'$ and its upper boundary.

However, for $\mathcal{F} \simeq A_{E_{R'}}$, we see that $R\Gamma(\text{Star}(w); \mathcal{F}) \simeq \mathcal{F}_x \simeq \mathcal{F}_y$ for all $w \in \mathcal{S}$ and $x \in w, y \in \rho(w)$ by Corollary 5.7. Therefore
\[
R\Gamma(\text{Star}(w); A_{E_{R'}}) \simeq \begin{cases} A & R' = \rho(w), \\ 0 & \text{otherwise}. \end{cases}
\]

Finally, the same holds for $R\Gamma(M_{\rho(w)}; A_{E_{R'}})$ as seen in (5.6) after replacing $\mathbb{K}$ with $A$ in the same argument. This completes the proof of Proposition 5.24. \hfill \Box

We can further simplify the legible model for sheaf categories by taking resolutions from the knowledge of quiver representations [4]. Indeed, the category $\mathcal{F}\text{un}(\mathcal{R}(S), \mathbb{K})$ is nothing but a representation category of the quiver with relations $\mathcal{R}(S)$, i.e., the representation category of $A(\mathcal{R}(S))$ with values on perfect complexes (i.e. complexes which are quasi-isomorphic to a bounded complex of finite projective $\mathbb{K}$-modules). Now, for any $F \in \mathcal{F}\text{un}(\mathcal{R}(S), \mathbb{K})$, we can replace $F$ by either a projective or injective resolution.

Let us review a key fact about $A(\mathcal{R}(S))$ which we will use later on. Recall some terminology in quiver representation theory. Let us denote by $\mathcal{R}(S)_0$ and $\mathcal{R}(S)_1$ the sets of objects and arrows respectively. For any object $a \in \mathcal{R}(S)_0$, the path of length 0 at $a$ will be denoted by $A_a$. Then $A(\mathcal{R}(S))$ is a free $\mathbb{K}$-module with the basis the set of all paths (of lengths $\geq 0$) in $\mathcal{R}(S)$ modulo the relation in Definition 5.18.

**Proposition 5.26.** Let $S$ be a regular cell complex refining the stratification induced by $\mathcal{T}$ and satisfying Assumption 5.17. Then for $A := A(\mathcal{R}(S))$, we have the following characterizations for indecomposable (left) projective and injective modules:
(1) The indecomposable (left) projective modules of $A$ are $P_{a} := \mathcal{A} \lambda_{a}$ with $a \in \mathcal{R}(S)_{0}$. In particular, $P_{a}(s)$ is injective for all arrows $s$ in $\mathcal{R}(S)_{1}$.

(2) The indecomposable (left) injective modules of $A$ are $I_{a} := \text{Hom}_{k}(\lambda_{a} A, k)$ with $a \in \mathcal{R}(S)_{0}$. In particular, $I_{a}(s)$ is surjective for all arrows $s$ in $\mathcal{R}(S)_{1}$.

(3) The (left) simple modules of $A$ are $S_{a}$ with $a \in \mathcal{R}(S)_{0}$, where $S_{a}$ consists of $k$ at the object $a$ and 0 otherwise, and $S_{a}$ sends all the arrows in $\mathcal{R}(S)_{1}$ to zero.

**Proof.** This is a standard fact in quiver representation theory. See [4, II.2.Lem.2.4, III.2.Lem.2.1]. □

5.4. A combinatorial proof of invariance of sheaf categories. We can now give an alternative proof of Theorem 5.3 via the combinatorial descriptions of sheaf categories, i.e. Lemma 5.11, Lemma 5.15, Proposition 5.24, and Proposition 5.26.

**Proof of Theorem 5.3.** It suffices to show the invariance of $\mathcal{Sh}(T; \mathcal{K})$ under the 6 types of Legendrian Reidemeister moves in the front projection. First observe that the diagram $\mathcal{Sh}(T; \mathcal{K})$ is unchanged if we regard each cusp in $\pi_{xT}(T)$ as a 2-valency vertex. Then the Legendrian Reidemeister moves (I) and (IV) are special cases of move (VI), and move (II) is a special case of move (V). So it suffices to show the invariance under moves (III), (V), and (VI). Let $T, T'$ be any pair of bordered Legendrian graphs in $T^{\infty, \sim} M$, which differ by any one of the above three moves. We can take a pair of regular cell complexes $S, S'$ refining the stratifications of $M$ induced by $T, T'$ respectively, such that:

1. $S, S'$ satisfy Assumption 5.17 for $T, T'$ respectively.
2. $S, S'$ coincide outside the local bordered Legendrian graphs involving the Legendrian Reidemeister move.

Denote by $S_{L} = S'_{L}$, $S_{R} = S'_{R}$ the induced stratifications of $S$ (equivalently, $S'$) near the left and right boundary of $M = I_{L} \times \mathbb{R}_{x}$ respectively. In other words, say $I_{L}$ (resp. $I_{R}$) lives over in the interval $I_{L} = (x_{L}, x_{L} + \epsilon) \subset I_{L}$ (resp. $I_{R} = (x_{R}, x_{R} - \epsilon) \subset I_{R}$), then $S_{L} = S|_{I_{L}}$ (resp. $S_{R} = S|_{I_{R}}$) is the stratification with strata the connected components of $S \cap I_{L} \times \mathbb{R}_{x}$ (resp. $S \cap I_{R} \times \mathbb{R}_{x}$) for all $S \in S$, as defined at the beginning of Section 5.2.

Apply Lemma 5.11, Lemma 5.15 and Proposition 5.24, it suffices to show, for each of three moves above, the equivalence between the 2 diagrams of DG categories: $\mathcal{F}un_{T}(\mathcal{R}(S), \mathcal{K}) := (\mathcal{F}un_{T_{L}}(\mathcal{R}(S_{L}), \mathcal{K}) \leftrightarrow \mathcal{F}un_{T_{R}}(\mathcal{R}(S_{R}), \mathcal{K})) \leftrightarrow \mathcal{F}un_{T}(\mathcal{R}(S'), \mathcal{K})$, and $\mathcal{F}un_{T}(\mathcal{R}(S'), \mathcal{K}) = (\mathcal{F}un_{T_{L}}(\mathcal{R}(S'_{L}), \mathcal{K}) \leftrightarrow \mathcal{F}un_{T_{R}}(\mathcal{R}(S'_{R}), \mathcal{K})).$ This can essentially be shown by the diagrams in Figures 14, 15, 16.

5.4.1. Move (III). If $T, T'$ differ by a move (III), can take a pair of regular cell complexes $S, S'$ as above so that, the local bordered Legendrian graphs involving the move (III) are as in Figure 14 (1), (4) respectively. In the figure, denote by $T_{L}, S_{L}$ the induced bordered Legendrian graph and regular cell complexes refining $S_{L}$ in (i), for $1 \leq i \leq 4$. In particular, we have $T_{1} := T, T_{4} := T'$, $T_{2} = T_{3},$ and $S_{1} = S, S_{4} = S', S_{2} = S_{3},$ and all the $(T_{i}, S_{i})$ coincide outside the local bordered Legendrian graphs in the figure. The letters in each picture label the regions.

**Proposition/Definition 5.27.** There is an adjunction $(i, \pi)$ of functors between two abelian categories

$$i : \mathcal{F}un(\mathcal{R}(S_{1}), \mathcal{K}) \rightleftharpoons \mathcal{F}un(\mathcal{R}(S_{2}), \mathcal{K}) : \pi$$

defined as follows:

1. $(i(F_{1}))$ forgets $F_{1}(C)$ in (I);
2. $(\pi(F_{2}))$ keeps the same data as $F_{2}$ on other regions, and $\pi(F_{2})(C) := F_{2}(X_{1}) \times_{F_{2}(Z)} F_{2}(X_{2}) = \ker(F_{2}(X_{1}) \oplus F_{2}(X_{2}) \xrightarrow{(\cdot, -)} F_{2}(Z)).$ Then the map $\pi(F_{2})(A \to C)$ is uniquely determined by the universal property of $\pi(F_{2})(C)$ as a kernel.

Moreover, both of $i$ and $\pi$ are exact, and $i \circ \pi = \text{id}$. As a consequence, we obtain an adjunction $(i, \pi)$ of DG functors in the DG lifting:

$$i : \mathcal{F}un(\mathcal{R}(S_{1}), \mathcal{K}) \rightleftharpoons \mathcal{F}un(\mathcal{R}(S_{2}), \mathcal{K}) : \pi$$

Moreover, we get a natural isomorphism $\beta : i \circ \pi \Rightarrow \text{id}$, and $\pi : \mathcal{F}un(\mathcal{R}(S_{2}), \mathcal{K}) \to \mathcal{F}un(\mathcal{R}(S_{1}), \mathcal{K})$ is fully faithful.
For any $F, G \in \mathcal{F}(\mathcal{R}(S_2), \mathcal{K})$, by the adjunction $(i, \pi)$, we have

$$\text{RHom}(\pi(F), \pi(G)) \cong \text{RHom}(i \circ \pi(F), G) \cong \text{RHom}(F, G)$$

where the last quasi-isomorphism follows from the natural isomorphism $\beta : i \circ \pi \Rightarrow \text{Id}$, which gives the quasi-isomorphism $\beta_F : i \circ \pi(F) \Rightarrow F$.

**Lemma 5.28.** Let $\mathcal{D}_1 := \mathcal{F}(\mathcal{R}(S_1), \mathcal{K})$ be the DG category in (1), and $\mathcal{D}_2$ be the DG full subcategory of $\mathcal{F}(\mathcal{R}(S_2), \mathcal{K})$ whose objects are functors $F_2$ such that two additional crossing conditions induced by the two red squares in (2) hold, i.e. both of the total complexes $\text{Tot}(F_2(A) \rightarrow F_2(B_1) \oplus F_2(X_1) \xrightarrow{+,-} F_2(Z))$ and $\text{Tot}(F_2(A) \rightarrow F_2(B_2) \oplus F_2(X_2) \xrightarrow{+,-} F_2(Z))$ are acyclic. Then the adjunction $(i, \pi)$ of DG functors in Proposition/Definition 5.27 induces equivalences

$$i : \mathcal{D}_1 \tilde{\rightarrow} \mathcal{D}_2 : \pi$$

which are quasi-inverses to each other.

**Proof.** Firstly, the adjunction of DG functors $(i, \pi)$ in Proposition/Definition 5.27 induces an adjunction of DG functors $i : \mathcal{D}_1 \rightarrow \mathcal{D}_2 : \pi$.

To show this, it suffices to show that the essential image of $\mathcal{D}_1$ under $i$ is contained in $\mathcal{D}_2$, and the essential image of $\mathcal{D}_2$ under $\pi$ is contained in $\mathcal{D}_1$. The former is clear, as for example, for any $F_1 \in \mathcal{D}_1$, the crossing conditions for $F_1$ at the crossing above and the crossing to the left of $C$ in (1) implies the first crossing condition for $i(F_1)$, i.e. the total complex $\text{Tot}(F_2(A) \rightarrow F_2(B_1) \oplus F_2(X_1) \xrightarrow{+,-} F_2(Z))$ is acyclic. The latter essentially follows from Proposition 5.26. More precisely, for any $F_2 \in \mathcal{D}_2 \subset \mathcal{F}(\mathcal{R}(S), \mathcal{K})$, by the proposition, $F_2$ is quasi-isomorphic to an object in which all the arrows are sent to surjections. So, we can assume $F_2$ itself has this property. Now, the crossing condition for $\pi(F_2)$ at the crossing above $C$ is automatic, and the crossing conditions for $F_2$ is equivalent to the natural cochain maps $F_2(A) \rightarrow \ker(F_2(B_1) \oplus F_2(X_1) \xrightarrow{+,-} F_2(Z))$, and $F_2(A) \rightarrow \ker(F_2(B_2) \oplus F_2(X_2) \xrightarrow{+,-} F_2(Z))$ being...
quasi-isomorphisms, which are equivalent to the crossing conditions for \(\pi(F_2)\) at the crossings to the left and to the right of \(C\). Hence, \(\pi(F_2) \in \mathcal{D}_2\).

Now, the natural isomorphism \(\beta: i \circ \pi \Rightarrow \text{Id} : \mathcal{D}_2 \to \mathcal{D}_2\) implies that \(i : \mathcal{D}_1 \to \mathcal{D}_2\) is essentially surjective. By Proposition/Definition 5.27, \(\pi : \mathcal{D}_2 \to \mathcal{D}_1\) is fully faithful. It then suffices to show \(\pi\) is essentially surjective.

For any \(F_1 \in \mathcal{F}\text{un}(\mathcal{R}(S_1), \mathbb{K})\), by Proposition 5.26, \(F_1\) is quasi-isomorphic to an object in which all arrows are sent to surjections. We can then assume \(F_1\) itself has this property. Now, the crossing conditions for \(F_1\) in the local bordered Legendrian graph are equivalent to the natural quasi-isomorphisms

\[
\begin{align*}
F_1(C) \xrightarrow{\phi} \ker(F_1(X_1) \oplus F_1(X_2) \xrightarrow{\phi} F_1(Z)) , \quad F_1(A) \xrightarrow{\phi} \ker(F_1(B_1) \oplus F_1(C) \xrightarrow{\phi} F_1(X_2)) , \quad F_1(A) \xrightarrow{\phi} \ker(F_1(B_2) \oplus F_1(C) \xrightarrow{\phi} F_1(X_1)) ,
\end{align*}
\]
equivalently, \(F_1(C) \xrightarrow{\phi} \pi \circ i(F_1)(C)\), and the two crossing conditions induced by the two red squares in (2) defining \(i(F_1) \in \mathcal{D}_2\). Hence, we get a natural quasi-isomorphism \(\alpha_{F_1} : F_1 \xrightarrow{\sim} \pi \circ i(F_1)\). In particular, \(\pi : \mathcal{D}_2 \to \mathcal{D}_1\) is essentially surjective.

Similar to \(\mathcal{D}_2\), let \(\mathcal{D}_3\) be the corresponding DG category defined by (3). Then (2) and (3) are identical, that is, \(\mathcal{D}_3 = \mathcal{D}_2\). We just have changed the letters labelling the regions from ‘\(X\)’ to ‘\(X’\’, to make it convenient for us to compare with (4).

Let \(\mathcal{D}_4 := \mathcal{F}\text{un}_{T'}(\mathcal{R}(S_2), \mathbb{K})\) be the DG category in (4). By a dual argument to that in proving the equivalence between \(\mathcal{D}_1\) and \(\mathcal{D}_2\), we immediately obtain equivalences

\[p : \mathcal{D}_3 \xRightarrow{\sim} \mathcal{D}_4 : j\]

which are quasi-inverses to each other. Here \(j\) is induced from the forgetful functor \(i : \mathcal{F}\text{un}(\mathcal{R}(S_4), \mathbb{K}) \to \mathcal{F}\text{un}(\mathcal{R}(S_3), \mathbb{K})\) with \(i(F_4)\) forgets \(F_4(Y')\), and \(p\) is induced from the functor \(p : \mathcal{F}\text{un}(\mathcal{R}(S_3), \mathbb{K}) \to \mathcal{F}\text{un}(\mathcal{R}(S_4), \mathbb{K})\) with \(p(F_3(Y')) = \text{Coker}(F_3(A') \xrightarrow{\phi} F_3(B'_1) \oplus F_3(B'_2))\).

Now, by composition, we then get an equivalence \(\mathcal{F}\text{un}_{T'}(\mathcal{R}(S_4), \mathbb{K}) \cong \mathcal{F}\text{un}_{T}(\mathcal{R}(S), \mathbb{K})\). Notice that the composition sends the object \(F|_{T_1'}\), \(F\) restricted to \(T_1\), to \(F|_{T_1''}\), which is quasi-isomorphic to \(F|_{T_{1=3}}\) in \(\mathcal{D}_2\). Hence, the equivalence commutes with \(\text{Id} : \mathcal{F}\text{un}_{T}(\mathcal{R}(S_1), \mathbb{K}) \to \mathcal{F}\text{un}_{T}(\mathcal{R}(S_2'), \mathbb{K})\) up to a specified natural isomorphism. The same holds for \(T_{1=2} = T_{1''}\). Therefore, we get an equivalence of diagrams of DG categories:

\[\mathcal{F}\text{un}_{T}(\mathcal{R}(S), \mathbb{K}) \cong \mathcal{F}\text{un}_{T}(\mathcal{R}(S), \mathbb{K})\]

as desired.

5.4.2. Move (V). The proof proceeds similarly as above. If \(T, T'\) differ by a move (V), can take the pair of regular cell complexes \(S, S'\) so that the local bordered Legendrian graphs involving the move are as in Figure 15. (1), (4) respectively. In the picture, \(T\) and \(T'\) both have \(l\) left and \(r\) right half-edges at the vertex, labelled by 1, \(\ldots\), \(l\) and \(l+1, \ldots, r\) from top to bottom respectively. In addition, the bordered Legendrian graph \(T(S)\) (resp. \(T(S')\)) underlying \(S\) (resp. \(S'\)) is \(T\) (resp. \(T'\)) plus the additional dashed arcs. We have added the dashed arcs to ensure that \(S, S'\) satisfy Assumption 5.17. As before, let \(T_i, S_i\) be the induced bordered Legendrian graphs and corresponding regular cell complexes in (i), for \(1 \leq i \leq 4\). In particular, \((T_1, S_1) = (T, S), (T_2, S_2) = (T', S_2), (T_3, S_3) = (T_3, S_3)\).

Similar to Proposition/Definition 5.27, we have

**Proposition/Definition 5.29.** There is an adjunction \((i, \pi)\) of functors between two abelian categories

\[i : \mathcal{F}\text{un}(\mathcal{R}(S_1), \mathbb{K}) \rightleftarrows \mathcal{F}\text{un}(\mathcal{R}(S_2), \mathbb{K}) : \pi\]

defined as follows:

1. \(i(F_1)\) forgets \(F_1(J_k)'\)’s in (1) for \(1 \leq k \leq l\);
2. \(\pi(F_2)\) keeps the same data as \(F_2\) on other regions, and \(\pi(F_2)(J_k) := F_2(I_k) \times_{F_2(N_1)} F_2(N_2) = \ker(F_2(I_k) \oplus F_2(N_2) \xrightarrow{\phi} F_2(N_1))\) for \(1 \leq k \leq l\). Then all the additional maps for \(\pi(F_2)\) are uniquely determined by the universal properties of \(\pi(F_2)(J_k)\)’s as kernels.

Moreover, both of \(i\) and \(\pi\) are exact, and \(i \circ \pi = \text{Id}\). As a consequence, we obtain an adjunction \((i, \pi)\) of DG functors in the DG lifting:

\[i : \mathcal{F}\text{un}(\mathcal{R}(S_1), \mathbb{K}) \rightleftarrows \mathcal{F}\text{un}(\mathcal{R}(S_2), \mathbb{K}) : \pi\]

Moreover, we get a natural isomorphism \(\beta : i \circ \pi \Rightarrow \text{Id}, \) and \(\pi : \mathcal{F}\text{un}(\mathcal{R}(S_2), \mathbb{K}) \to \mathcal{F}\text{un}(\mathcal{R}(S_1), \mathbb{K})\) is fully faithful.
Figure 15. Invariance of sheaf categories: Legendrian Reidemeister move (V).

Proof. The proof is identical to that of Proposition/Definition 5.27. □

Similar to Lemma 5.28, we have

Lemma 5.30. Let $D_1 := \text{Fun}_T(R(S_1), \mathbb{K})$ be the DG category in (1), and $D_2$ be the DG full subcategory of $\text{Fun}_T(R(S_2), \mathbb{K})$ whose objects are functors $F_2$ such that the additional crossing condition induced by the red square in (2) holds, i.e. the total complex $\text{Tot}(F_2(S_1) \to F_2(S_2) \oplus F_2(N_2) \xrightarrow{(r, -)} F_2(N_1))$ is acyclic. Then the adjunction $(i, \pi)$ of DG functors in Proposition/Definition 5.29 induces equivalences

$$i : D_1 \xrightarrow{\sim} D_2 : \pi$$

which are quasi-inverses to each other.

Proof. Similar to that of Lemma 5.28. □

Similar to the case of move III, let $D_3$ be the DG category defined by (3). Then $D_3 = D_2$. Let $D_4 := \text{Fun}_T(R(S_4), \mathbb{K})$ be the DG category in (4). By a dual argument to that in proving the equivalence between $D_1$ and $D_2$, we immediately obtain equivalences

$$p : D_3 \xrightarrow{\sim} D_4 : j$$

which are quasi-inverses to each other. Here $j$ is induced from the forgetful functor $i : \text{Fun}(R(S_4), \mathbb{K}) \to \text{Fun}(R(S_3), \mathbb{K})$ with $i(F_4)$ forgets $F_4(J'_k)$’s for $l + 1 \leq k \leq l + r$, and $p$ is induced from the functor

$$p : \text{Fun}(R(S_3), \mathbb{K}) \to \text{Fun}(R(S_4), \mathbb{K})$$

with $p(F_3)(J'_k) = \text{Coker}(F_3(S'_1) \xrightarrow{(r, -)} F_3(J'_k) \oplus F_3(S'_2))$.

Again by composition, we get an equivalence of diagrams of DG categories: $\text{Fun}_T(R(S), \mathbb{K}) \simeq \text{Fun}_T(R(S), \mathbb{K})$, as desired.
5.4.3. Move (VI). If \( T, T' \) differ by a move (VI), can take the pair of regular cell complexes \( S, S' \) so that the local bordered Legendrian graphs involving the move are as in Figure 16. In the picture, \( T \) has \( l \) left and \( r \geq 1 \) right half-edges, labelled by \( 1, \ldots, l \) and \( l + 1, \ldots, l + r \) from top to bottom respectively. \( T' \) has \( l + 1 \) left and \( r - 1 \) right half-edges at the vertex, with the labelling inherited from that of \( T \). In addition, the bordered Legendrian graph \( T(S) \) (resp. \( T(S') \)) underlying \( S \) (resp. \( S' \)) is \( T \) (resp. \( T' \)) plus the additional dashed arcs. We have added the dashed arcs to ensure that \( S, S' \) satisfy Assumption 5.17.

![Figure 16. Invariance of sheaf categories: Legendrian Reidemeister move (VI).](image)

Exactly the same as before, we have

**Proposition/Definition 5.31.** There is an adjunction \((i, \pi)\) of functors between two abelian categories

\[
i : \text{Fun}(\mathcal{R}(S'), \mathbb{K}) \simeq \text{Fun}(\mathcal{R}(S), \mathbb{K}) : \pi
\]

defined as follows:

1. \( i(F') \) forgets \( F'(J_k) \)'s in (2) for \( 1 \leq k \leq l \);
2. \( \pi(F) \) keeps the same data as \( F \) on other regions, and \( \pi(F)(J_k) := F(I_k) \times_{F(N)} F(I_{k+1}) = \ker(F(I_k)) \oplus F(I_{k+1}) \overset{(\pi_F)}{\rightarrow} F(N) \) for \( 1 \leq k \leq l \). Then all the additional maps for \( \pi(F) \) are uniquely determined by the universal properties of \( (F(J_k))'s \) as kernels.

Moreover, both of \( i \) and \( \pi \) are exact, and \( i \circ \pi = \text{id} \). As a consequence, we obtain an adjunction \((i, \pi)\) of DG functors in the DG lifting:

\[
i : \text{Fun}(\mathcal{R}(S'), \mathbb{K}) \simeq \text{Fun}(\mathcal{R}(S), \mathbb{K}) : \pi
\]

Moreover, we get a natural isomorphism \( \beta : i \circ \pi \Rightarrow \text{id} \), and \( \pi : \text{Fun}(\mathcal{R}(S), \mathbb{K}) \rightarrow \text{Fun}(\mathcal{R}(S'), \mathbb{K}) \) is fully faithful.

**Proof.** The proof is identical to that of Proposition/Definition 5.27. \( \square \)

Also, we have

**Lemma 5.32.** Let \( \mathcal{D} := \text{Fun}_T(\mathcal{R}(S), \mathbb{K}) \) and \( \mathcal{D}' := \text{Fun}_{T'}(\mathcal{R}(S'), \mathbb{K}) \) be the DG categories in (1) and (2) respectively. Then the adjunction \((i, \pi)\) of DG functors in Proposition/Definition 5.29 induces equivalences

\[
i : \mathcal{D}' \sim \mathcal{D} \ : \pi
\]

which are quasi-inverses to each other.

**Proof.** Similar to that of Lemma 5.28. \( \square \)

Again as before, we get an equivalence of diagrams of DG categories: \( \text{Fun}_T(\mathcal{R}(S), \mathbb{K}) \simeq \text{Fun}_{T'}(\mathcal{R}(S), \mathbb{K}) \), as desired. Now, We’ve finished the proof of Theorem 5.3. \( \square \)
5.5. Microlocal monodromy. Given a bordered Legendrian graph $\mathcal{T} = (T_L \to T \leftarrow T_R)$, equipped with a $\mathbb{Z}$-valued Maslov potential $\mu$, let $S$ be a regular cell complex refining the stratification $S_T$ induced by $\mathcal{T}$. Denote by $\mathcal{L}oc(T \setminus V_T)$ the category of local systems of cochain complexes of $K$-modules on the complement $T \setminus V_T$ of vertices.

As in [25, Def.5.4], we define the microlocal monodromy.

**Definition 5.33** (Microlocal monodromy). There is a natural functor $\mu\text{hom} : \text{Sh}(T; K) \to \mathcal{L}oc(T \setminus V_T)$, called microlocal monodromy, such that for each arc $a$ of $\mathcal{T}$, the region above $a$, denote by $N$, and the arrow $a \to N$ in $S$, we define

$$\mu\text{hom}(\mathcal{F})(a) := \text{Cone}(\mathcal{F}(\text{Star}(a)) \to \mathcal{F}(\text{Star}(N)))[-\mu(a)].$$

**Proposition 5.34.** Then the microlocal monodromy $\mu\text{hom}$ is invariant under Legendrian isotopy of $\mathcal{T}$.

**Proof.** The proof is entirely the same to that in [25, §5.1]. In another perspective, for any sheaf $F \in \text{Sh}(T; K)$ and any point $p \in T \setminus V_T$ which is a smooth Legendrian point of $\mathcal{T}$, by Proposition 7.5.3 in [17], the microlocal monodromy $\mu\text{hom}(F)(p)$ is nothing but the microlocal stalk of $\mathcal{F}$ at $p$, which is intrinsic. \hfill \Box

**Definition 5.35** (Subcategory of microlocal rank 1). We define $C_1(\mathcal{T}, \mu; K)$ to be the full DG subcategory of $\text{Sh}(T; K)$ whose objects are $\mathcal{F}$ such that $\mu\text{hom}(\mathcal{F})$ is a local system of rank 1 $K$-modules in cohomological degree 0, and define the induced diagram of constructible sheaf categories

$$C_1(\mathcal{T}, \mu; \mathbb{K}) := (C_1(T_L, \mu_{1}; \mathbb{K}) \leftarrow C_1(T; \mu; \mathbb{K}) \to C_1(T_R, \mu_{1}; \mathbb{K})).$$

This DG category $C_1(\mathcal{T}, \mu; \mathbb{K})$ will be the sheaf side of the augmentation-sheaf correspondence in the next section. As a consequence of Theorem 5.3, we obtain

**Corollary 5.36.** The category $C_1(\mathcal{T}, \mu; \mathbb{K})$ is a Legendrian isotopy invariant up to DG equivalence.

**Proof.** This follows from Theorem 5.3 and the fact that the notion of the microlocal monodromy is intrinsic as seen in Proposition 5.34. \hfill \Box

Let $S$ be a regular cell complex which refines $S_T$ and satisfies Assumption 5.17. We use the notations in Definitions 5.18 and 5.23.

**Definition 5.37.** We define $\mathcal{F}\text{un}_{\mathcal{T}, \mu_{1}}(\mathcal{R}(S), \mathbb{K})$ to be the full DG subcategory of $\mathcal{F}\text{un}_{\mathcal{T}}(\mathcal{R}(S), \mathbb{K})$ consisting of functors $F$ such that $\text{Cone}(F(e_s))[-\mu(s)] \cong \mathbb{K}$ for all arcs $s$ contained in $\mathcal{T}$.

By restriction, we then obtain a diagram of DG categories

$$\mathcal{F}\text{un}_{\mathcal{T}, \mu_{1}}(\mathcal{R}(S), \mathbb{K}) := (\mathcal{F}\text{un}_{\mathcal{T}_{L}, \mu_{1}}(\mathcal{R}(S|_{T_L}), \mathbb{K}) \leftarrow \mathcal{F}\text{un}_{\mathcal{T}_{1}, \mu_{1}}(\mathcal{R}(S), \mathbb{K}) \to \mathcal{F}\text{un}_{\mathcal{T}_{R}, \mu_{1}}(\mathcal{R}(S|_{T_R}), \mathbb{K}))$$

**Corollary 5.38.** There is an $A_{\infty}$-equivalence:

$$\mathcal{F}\text{un}_{\mathcal{T}, \mu_{1}}(\mathcal{R}(S), \mathbb{K}) \cong C_1(\mathcal{T}, \mu; \mathbb{K}).$$

**Proof.** This is a direct corollary of Lemma 5.15 and Proposition 5.24. \hfill \Box

6. AUGMENTATIONS ARE SHEAVES FOR LEGENDRIAN GRAPHS

6.1. Local calculation for augmentation categories. In this section, we will compute the $A_{\infty}$-structures completely for the trivial bordered Legendrian graphs and the bordered Legendrian graphs containing a vertex of type $(0, n_R)$.

6.1.1. Augmentation category for a trivial bordered Legendrian graph. Let $(I_n, \mu)$ be a trivial bordered Legendrian graph of $n$ parallel strands, equipped with a $\mathbb{Z}$-valued Maslov potential $\mu$. We will describe the augmentation category $\text{Aug}_{\infty}(I_n; \mathbb{K})$, which has been already seen in Corollary 4.8 for the unitarity.

**Notation 6.1.** In this section, we denote $\text{Aug}_{\infty}(I_n; \mathbb{K})$ by $\text{Aug}_{\infty}$ for simplicity.

As seen in Example 4.7, the Chekanov-Eliashberg DGA $A^{CE}(I_n^{(m)})$ is generated by the set

$$\left\{ \frac{e^{ij}}{a b} \mid a < b, 1 \leq i, j \leq m \right\} \bigcup \left\{ \frac{y_a^{ij}}{a} \mid 1 \leq a \leq n, 1 \leq i < j \leq m \right\},$$

where the grading is given as

$$|a^{ij}| := \mu(a) - \mu(b) - 1, \quad |y_a^{ij}| := -1.$$
**Assumption 6.2.** From now on, we denote $y_{ij}^{ij}$ by $k_{ij}^{ij}$. We regard $k_{ij}^{jj}$ as zero unless it is well-defined.

Then under the above assumption, the differential $\partial^{(m)}$ is simply given as

$$
\partial^{(m)} k_{ij}^{ab} = \sum_{a \leq b \leq m} (-1)^{|k_{ij}^{ab}|} k_{ik}^{ij} k_{cj}^{bc}.
$$

Recall from Example 4.7. $\text{Aug}_+$ is a DG category such that:

1. The objects are the augmentations for $A_n(\mu)$:

$$
\text{Ob}(\text{Aug}_+) = \text{Aug}(A_n(\mu); \mathbb{K}).
$$

(6.1)

2. In $A^{(2)}(I_n)$, we have

$$
M^{12} := \mathbb{K}\langle k_{ab}^{12} \mid 1 \leq a \leq b \leq n \rangle.
$$

Then, for any two objects $\epsilon_1, \epsilon_2$, the set of morphisms is

$$
\text{Hom}_{\text{Aug}_+}(\epsilon_1, \epsilon_2) = M^{12^{2}} = \mathbb{K}\langle k_{ab}^{12} \mid 1 \leq a \leq b \leq n \rangle.
$$

(6.2)

3. For $\epsilon_1, \epsilon_2 \in \text{Aug}_+$, the map

$$
m_1 : \text{Hom}_{\text{Aug}_+}(\epsilon_1, \epsilon_2) \rightarrow \text{Hom}_{\text{Aug}_+}(\epsilon_1, \epsilon_2)
$$

is defined as

$$
m_1(k_{ab}^{12}) = -\sum_{c < a} \epsilon_1(k_{ca}^{12}) + \sum_{b < d} (-1)^{|k_{ab}^{12}|} k_{ad}^{12} \epsilon_2(k_{bd}^{12}).
$$

(6.3)

4. For $\epsilon_1, \epsilon_2, \epsilon_3 \in \text{Aug}_+$, the map

$$
m_2 : \text{Hom}_{\text{Aug}_+}(\epsilon_2, \epsilon_3) \otimes \text{Hom}_{\text{Aug}_+}(\epsilon_1, \epsilon_2) \rightarrow \text{Hom}_{\text{Aug}_+}(\epsilon_1, \epsilon_3)
$$

is defined as

$$
m_2(k_{cd}^{12} \otimes k_{ab}^{12}) = \delta_{bc}(-1)^{\sigma_2}(-1)^{|k_{ab}^{12}|} k_{ad}^{12}.
$$

(6.4)

where

$$
\sigma_2 = 1 + |k_{ab}^{12}| + |k_{cd}^{12}| + |k_{ab}^{12}|
$$

**Definition 6.3 (Morse complex).** Consider a free graded $\mathbb{K}$-module $C = C(I_n)$,

$$
C := \bigoplus_{1 \leq a \leq n} \mathbb{K} e_a, \quad |e_a| := -\mu(a).
$$

equipped with a decreasing filtration $F^\ast$ via

$$
F^i C := \bigoplus_{k > i} \mathbb{K} \cdot e_k,
$$

for $0 \leq i \leq n$.

- A Morse complex for $(C, F^\ast)$ is a complex $(C, d)$ with a $\mathbb{K}$-linear filtration-preserving differential $d$ of degree 1.
- Let $MC = MC(C; \mathbb{K})$ be the set of Morse complexes of $(C, F^\ast)$.

**Lemma 6.4 ([26, Def.4.4]).** There is a canonical identification

$$
\Xi : \text{Aug}(A_n(\mu); \mathbb{K}) \overset{\simeq}{\rightarrow} MC
$$

$$
e \mapsto d = d(e),
$$

where

$$
de_i := \sum_{j > i} (-1)^{\mu(i)} e(a_{ij}) e_j.
$$

The set of Morse complexes $MC$ can be lifted to an obvious DG category $MC = MC(I_n; \mathbb{K})$:

1. The set of objects is $MC$. 

(1) The set of objects is $MC$. 

(2) The set of morphisms is $MC (\epsilon_1, \epsilon_2)$.

(3) The set of homomorphisms is $MC (\epsilon_1, \epsilon_2)$. 

(4) The set of morphisms is $MC (\epsilon_1, \epsilon_2)$. 

(5) The set of homomorphisms is $MC (\epsilon_1, \epsilon_2)$. 

(6) The set of morphisms is $MC (\epsilon_1, \epsilon_2)$. 

(7) The set of homomorphisms is $MC (\epsilon_1, \epsilon_2)$. 

(8) The set of morphisms is $MC (\epsilon_1, \epsilon_2)$. 

(9) The set of homomorphisms is $MC (\epsilon_1, \epsilon_2)$.
Lemma 6.6

Proof of Lemma 6.6.

The morphism space $\text{Hom}_{MC}(d_1, d_2)$ is

$$\text{Hom}_{MC}(d_1, d_2) := \text{End}(C, F^*),$$

the complex of endomorphisms of $(C, F^*)$ whose differential is given by

$$Df := d_2 \circ f - (-1)^{|f|} f \circ d_1.$$

(3) The composition $\cdot$ is the usual composition of endomorphisms of $C$.

Notation 6.5. We use the identification $\text{End}(C) \cong C \otimes C^*$ and so for $(e \otimes f^*), (e' \otimes f''^*) \in \text{End}(C),

$$(e \otimes f^*)(g) := (f^*, g) e \in C,$$

$$(e \otimes f^*) \cdot (e' \otimes f''^*) := (f^*, e')(e \otimes f'') \in \text{End}(C).$$

Lemma 6.6 ([21, Theorem 7.25]). There is a (strict) isomorphism of DG categories

$$\mathfrak{h} : \text{Aug}_+(I_n; \mathbb{K}) \longrightarrow MC$$

which is given on objects by

$$e \mapsto d(e) := \sum_{a < b} \left( (-1)^{s(a)} e(k_{ab}) \right) (e_b \otimes e_a^*),$$

as in Lemma 6.4, and on morphisms $\text{Hom}_{\text{Aug}_+}(\epsilon_1, \epsilon_2) \to \text{Hom}_{MC}(d(\epsilon_1), d(\epsilon_2))$ by

$$k_{ab}^{12v} \mapsto (-1)^{s(a, b)} (e_b \otimes e_a^*),$$

where the sign $s(a, b)$ is given by:

$$s(a, b) := \mu(a)(\mu(b) + 1) + 1.$$

Remark 6.7. Notice that the sign convention for $s(a, b)$ here is different from the original one, due to the fact that our definition for $d(e)$ differs by a sign from $\mathfrak{h}(e)$ in [21, Theorem 7.25].

Proof of Lemma 6.6. For completeness, we give the proof. By Lemma 6.4, it suffices to show that $\mathfrak{h}$ commutes with the differential and composition, that is, $\mathfrak{h} \circ m_1 = D \circ \mathfrak{h}$, and $\mathfrak{h} \circ m_2 = \mathfrak{h} \cdot D$.

Firstly, let us show $\mathfrak{h} \circ m_1 = D \circ \mathfrak{h}$. For two objects $\epsilon_1, \epsilon_2 \in \text{Aug}_+$, let $x \in \text{Hom}_+(\epsilon_1, \epsilon_2)$ be a homogeneous element of the form

$$x = \sum_{1 \leq a \leq b \leq n} x_{ab} k_{ab}^{12v}. $$

In particular, $x_{ab} \neq 0$ implies that $|x| = |k_{ab}^{12v}| = \mu(a) - \mu(b)$ since $x$ is homogeneous.

By definition of $\mathfrak{h}$, we have

$$\mathfrak{h}(x) = \sum_{1 \leq a \leq b \leq n} (-1)^{s(a, b)} x_{ab} (e_b \otimes e_a^*).$$

On the other hand, apply the formula (6.3) for $m_1$, we have

$$\mathfrak{h} \circ m_1(x) = - \sum_{c < a \leq b} e_1(k_{ac}) x_{ab} h \left( k_{cb}^{12v} \right) + \sum_{a \leq b < d} (-1)^{|k_{ab}^{12v}|} x_{ab} e_2(k_{bd}) h \left( k_{ad}^{12v} \right)$$

$$- \sum_{c < a \leq b} (-1)^{s(c, b)} e_1(k_{ca}) x_{ab} (e_b \otimes e_c^*)$$

$$+ \sum_{a \leq b < d} (-1)^{\mu(a) - \mu(b) + s(a, d)} x_{ab} e_2(k_{bd}) (e_c \otimes e_a^*)$$

$$= - \sum_{c < a \leq b} (-1)^{|x|} \left( (-1)^{\mu(c)} e_1(k_{ca}) \right) \left( (-1)^{s(a, b)} x_{ab} \right) \left( e_b \otimes e_a^* \right) \left( e_c \otimes e_a^* \right)$$

$$+ \sum_{a \leq b < d} \left( (-1)^{s(a, b)} x_{ab} \right) \left( (-1)^{\mu(b)} e_2(k_{bd}) \right) \left( e_d \otimes e_b^* \right) \left( e_b \otimes e_a^* \right)$$

$$= -(-1)^{|x|} \mathfrak{h}(x) \circ d(\epsilon_1) + d(\epsilon_2) \circ \mathfrak{h}(x)$$

$$= D \circ \mathfrak{h}(x)$$

as desired. Here, the fourth equality follows directly from the formulas for $\mathfrak{h}(x)$ and $d(\epsilon_1)$ above.
The only nontrivial part is the third equality involving two sign manipulations, which we check as follows: for the first sign manipulation, it suffices to show that, if $\epsilon_1(k_{ca})x_{ab} \neq 0$, then
\[
s(c, b) \equiv |x| + \mu(c) + s(ab) \mod 2.
\]
In fact, the condition implies that
\[
|k_{ca}| = \mu(c) - \mu(a) - 1 = 0,
\]
and therefore
\[
s(c, b) - s(a, b) - |x| = \mu(b) + 1 - \mu(c) - (\mu(c) - 1 - \mu(b)) \equiv 0 \mod 2
\]
as desired.
Similarly, for the second one, it suffices to show that,
\[
x_{ab}e_2(k_{bd}) \neq 0 \implies \mu(a) - \mu(b) + s(a, d) \equiv s(a, b) + \mu(b) \mod 2,
\]
or equivalently,
\[
s(a, d) - s(a, b) + \mu(a) \equiv 0 \mod 2.
\]
In fact, the condition implies that
\[
|k_{bd}| = \mu(b) - \mu(d) - 1 = 0,
\]
and therefore we have the desired identity
\[
s(a, d) - s(a, b) + \mu(a) = -\mu(a) + \mu(a) \equiv 0 \mod 2.
\]
It remains to show $\mathfrak{b} \circ m_2 = \mathfrak{b} \cdot \mathfrak{b}$. For any objects $\epsilon_1, \epsilon_2, \epsilon_3$ in $\text{Aug}_2(T_\mathbb{K})$, let
\[
x := \sum_{a \preceq b} x_{ab}k_{ab}^{12v}\in \text{Hom}_+(\epsilon_2, \epsilon_3), \quad y := \sum_{a \preceq b} y_{ab}k_{ab}^{12v}\in \text{Hom}_+(\epsilon_1, \epsilon_2)
\]
be two homogeneous elements. We want to show $\mathfrak{b} \circ m_2(x, y) = \mathfrak{b}(x) \cdot \mathfrak{b}(y)$. By definition of $\mathfrak{b}$, we have
\[
\mathfrak{b}(x) \cdot \mathfrak{b}(y) = \sum_{a \preceq b} \left((-1)^{s(c, b)x_{cb}(e_q \otimes e^*_r)} \cdot (-1)^{s(a, c)y_{ac}(e_c \otimes e^*_a)}\right)
\]
\[
= \sum_{a \preceq b} (-1)^{s(c, b)+s(a, c)}y_{ac}x_{c,b}(e_b \otimes e^*_a).
\]
On the other hand, apply the formula (6.4), we have:
\[
\mathfrak{b} \circ m_2(x, y) = \mathfrak{b} \left(\sum_{a \preceq b} x_{cb}y_{ac}m_2\left(k_{cb}^{12v} \otimes k_{ac}^{12v}\right)\right)
\]
\[
= \sum_{a \preceq b} y_{ac}x_{cb}(-1)^{1+|k_{ac}^{12v}||k_{cb}^{12v}|}(-1)^{s(a, b)}(e_b \otimes e^*_a)
\]
\[
= \mathfrak{b}(x) \circ \mathfrak{b}(y)
\]
as desired. Here, the last equality follows from the previous formula for $\mathfrak{b}(x) \circ \mathfrak{b}(y)$, and the following sign manipulation:
\[
1 + |k_{ac}^{12v}||k_{cb}^{12v}| + s(a, b)
\]
\[
= 1 + (\mu(a) - \mu(c))(\mu(c) - \mu(b)) + \mu(a)(\mu(b) + 1) + 1
\]
\[
\equiv \mu(a)\mu(c) + \mu(c)\mu(b) + \mu(c) + \mu(a) + 2
\]
\[
\equiv s(a, c) + s(c, b) \mod 2
\]
This finishes the proof of the lemma. □
6.1.2. Augmentation category for a vertex. Let \( \mathcal{V}_r \in \mathcal{BLG}_r \) be the bordered Legendrian graph in \( J^1U \) of type \((n_L, n_R)\), whose front projection is as the left picture in Figure 17. In particular, \( \mathcal{V}_r \) contains a vertex \( v \) with \( r \) right half-edges, labelled from top to bottom by \( 1, 2, \ldots, r \). Then \( n_R = n_L + r \).

As usual, we label the left (resp. right) ends from top to bottom by \( 1, 2, \ldots, n_L \) (resp. \( 1, 2, \ldots, n_R \)). We assume the right half-edges of \( v \) connect the right ends \( k, k + 1, \ldots, k + r - 1 \) from top to bottom. In other words, the right half-edge \( p \) is connected to the right end \( k + p - 1 \). For simplicity, we denote \( a' := a \) for \( 1 \leq a < k \) and \( b' := b + r \) for \( k \leq b \leq n_L \). We would like to compute the augmentation category

\[
\text{Aug}_+ (\mathcal{V}_r; \mathcal{K}) = (\text{Aug}_+(\mathcal{V}_{h,R}; \mathcal{K}) \leftarrow \text{Aug}_+(\mathcal{V}_r; \mathcal{K}) \rightarrow \text{Aug}_+(\mathcal{V}_{h,R}; \mathcal{K})) .
\]

For \( m \geq 1 \), let \( \mathcal{V}^{(m)}_r = \left( V^{(m)}_{h,L} \rightarrow V^{(m)}_r \leftarrow V^{(m)}_{h,R} \right) \in \mathcal{BLG}^{(m)}_r \) be the canonical front projection \( m \)-copy of \( \mathcal{V}_r \). For examples, \( \mathcal{V}^{(m)}_r \)'s for \( m = 1, 2, 3 \) are shown as in Figure 17 from left to right.

**Remark 6.8.** Here we are using the convention of the canonical front parallel copies described in Section 3.1.1.

**Notation 6.9.** For simplicity, let us denote

\[
A^{(m)} := A_{CE} (V^{(m)}_r), \quad A^{(m)}_+ := A_{CE} (V^{(m)}_r), \quad \text{Aug}_+ := \text{Aug}_+(\mathcal{V}_r; \mathcal{K}), \quad \text{Aug}_+ := \text{Aug}_+(\mathcal{V}_r; \mathcal{K}),
\]

where \( * = L, R \).

As usual, for each copy \( V^{(m)}_r \) in \( \mathcal{V}^{(m)}_r \), label the left ends and right ends from top to bottom by \( 1, 2, \ldots, n_L \) and \( 1, 2, \ldots, n_R \) respectively, label the vertex by \( v^i \) and the right half-edges from top to bottom by \( 1, 2, \ldots, r \) in \( \mathbb{Z}/r \).

In the bordered Legendrian graph \( \mathcal{V}^{(m)}_r \), label the Reeb chords of \( V^{(m)}_{h,L} \) (resp. \( V^{(m)}_{h,R} \)) corresponding to the pairs of strands by \( k_{ij}^{ab} \) (resp. \( k_{ij}^{ab} \)). That is, for \( 1 \leq a < b \leq n_L, 1 \leq i, j \leq m \) or \( 1 \leq a = b \leq n_L, 1 \leq i < j \leq m \) (resp. \( 1 \leq a < b \leq n_R, 1 \leq i, j \leq m \) or \( 1 \leq a = b \leq n_R, 1 \leq i < j \leq m \)), we obtain a Reeb chord (or line segment) of \( V^{(m)}_{h,L} \) (resp. \( V^{(m)}_{h,R} \)) going from the \( b \)-th strand of \( V^{(m)}_{h,L} \) (resp. \( V^{(m)}_{h,R} \)) to the \( a \)-th strand of \( V^{(m)}_{h,L} \) (resp. \( V^{(m)}_{h,R} \)).

Label the Reeb chords of \( V^{(m)}_r \) at the vertex \( v^i \) by either \( c_{ab}^{ij} \) if \( 1 \leq a < b \leq r \) or \( v_{a,\ell}^{ij} \) otherwise. i.e. \( c_{ab}^{ij} \) or \( v_{a,\ell}^{ij} \) is the “Reeb chord” starting from the initial half-edge \( a + \ell \in \mathbb{Z}/r \mathbb{Z} \), traveling around the vertex \( v^i \) counterclockwise and covering \( \ell \) minimal sectors, hence ending at the half-edge \( a \in \mathbb{Z}/r \mathbb{Z} \) of \( v^i \).

**Remark 6.10.** This is the same as the construction for Legendrian graphs in a normal form described in Section 3.2.3 but the reflected manner. In other words, the generators \( c_{ab}^{ij} \) correspond to vertex generators.
lying on the right side of $v$.

There are additional Reeb chords of $V_{t_r}^{(m)}$ corresponding to the crossings of the right half-edges of the vertices. Label the Reeb chord going from the $b$-th right half-edge of $v_j$ in $V_{t_r}^j$ to the $a$-th right half-edge of $v_i$ in $V_{t_r}^i$ by $c_{ij}^{ab}$ for $1 \leq b < a \leq r$ and $1 \leq i < j \leq m$. See Figure 18 for an example when $r = 3$, $m = 4$.

![Diagram showing Reeb chords and crossings](image)

**Figure 18.** The crossings in the front projection $m$-copy near a vertex: $r = 3$, $m = 4$.

We denote the set of Reeb chords by $R_{ij}^{fs}$ starting at the $j$-th copy $V_{t_r}^j$ and ending at the $i$-th copy $V_{t_r}^i$

$$R_{ij}^{fs} = \left\{ v_{pq}^{ij}, v_{a,\ell}^{ij}, c_{ij}^{ab} \right\}.$$  

Then the LCH DGA $A^{(m)} = (A^{(m)}, \delta^{(m)}) := A^{CE}(V_{t_r}^{(m)})$ can be described as follows.

$$A^{(m)} := \mathbb{Z}\langle \{ R_{ij}^{fs} \mid 1 \leq i \leq j \leq m \} \rangle,$$

the free associative algebra generated by the Reeb chord in $V_{t_r}^{(m)}$.

The grading for each generator is defined as follows: let $\mu_{a,\ell} := \mu_{V, a, \ell}$.

$$|k_{ab}^{ij}| = \mu_{a}(a) - \mu_{b}(b) - 1,$$

$$|v_{a,\ell}^{ij}| = |v_{a,\ell}| = \mu(a) - \mu(a + \ell) + N(n, a, \ell) - 1,$$

$$|c_{ij}^{ab}| = \begin{cases} \mu(a) - \mu(b) - 1 & i = j; \\ \mu(a) - \mu(b) & i < j. \end{cases}$$

where $N(n, a, \ell)$ is the same as described in (2.3).

The differentials of $k_{ab}^{ij}$ and $v_{a,\ell}^{ij}$ are given by the differentials of border DGAs and internal DGAs defined in Example/Definition 2.31. Indeed, under Assumption 6.2, we have

$$\delta^{(m)} k_{ab}^{ij} = \sum_{a \leq \ell \leq b} (-1)^{k_{ab}^{ij} + 1} k_{ac}^{ij} k_{cb}^{ij}. \tag{6.7}$$

$$\delta^{(m)} v_{a,\ell}^{ij} = \delta_{\ell, \tau} + \sum_{\ell_1 + \ell_2 = \ell} (-1)^{v_{a,\ell_1}^{ij} + 1} v_{a,\ell_1}^{ij} v_{a,\ell_2}^{ij}. \tag{6.8}$$

The differential of $c_{ij}^{ab}$ is obtained by counting certain admissible disks of degree 1 in $(J U, V_{t_r}^{(m)})$.

For simplicity, we denote $\tilde{a} := (-1)^{\mu(a) - 1} a$ for any Reeb chord $a$. Then for $i < j$,

$$\delta^{(m)} c_{ij}^{ab} = \sum_{a, \ell} (-1)^{\tilde{a} + 1} e_{a,\ell}. \tag{6.9}$$
where for sequences \( a = (a_1, \ldots, a_{\ell+1}) \) and \( i = (i_1, \ldots, i_{\ell+1}) \) with \( \ell \geq 1 \)
\[
f^a_i := \varepsilon_{a_1 a_2}^{i_1 i_2} \varepsilon_{a_2 a_3}^{i_1 i_3} \cdots \varepsilon_{a_{\ell+1} a_{\ell+1}}^{i_1 i_{\ell+1}}
\]
and the summation is over all possible sequences \( a \) and \( i \) in \([r]\) and \([m]\) such that
\[
a_1 = a, \quad a_{\ell} = b, \quad i_1 = i, \quad i_{\ell} = j,
\]
and either
\[
a_1 < a_2 > \cdots > a_{\ell}, \quad i_1 = i_2 < \cdots < i_{\ell}, \quad \epsilon = 0, \quad (6.10)
\]
or
\[
a_1 > a_2 < \cdots < a_{\ell}, \quad i_1 < i_2 = i_3 < \cdots < i_{\ell}, \quad \epsilon = |e_{a_1 a_2}^i| - 1. \quad (6.11)
\]

In Figure 18, two shaded regions labelled by \( A \) and \( B \) correspond to differentials of two types in (6.10) and (6.11), respectively.

In fact, we have a bordered DGAs
\[
\mathcal{A}^{(m)} = \left( \begin{array}{c}
A^{(m)} \xrightarrow{\phi^{(m)}} A^{(m)} \xrightarrow{\phi^{(m)}} A^{(m)} \\
\end{array} \right),
\]
where \( \phi^{(m)} \) is the inclusion of the sub-DGA \( A^{(m)} \) generated by \( \{k_{ab}^{ij}\} \), and
\[
A^{(m)} = \mathbb{Z}(k_{ab}^{ij} : 1 \leq p < q \leq n_r, 1 \leq t, s \leq m \text{ or } 1 \leq p = q \leq n_r, 1 \leq t < s \leq m)
\]
with the differential similar to \( \partial^{(m)} \) as in (6.7), that is:
\[
\partial^{(m)} b_{pq}^{rs} = \sum_{p < o < q, u} (-1)^{|b_{pq}^{rs}|+1} b_{pq}^{mu} b_{ov}^{sw}
\]
where we sum over all \( p \leq o \leq q \leq 1 \leq u \leq m \) such that \( b_{pq}^{mu} \) and \( b_{ov}^{sw} \) are well-defined.

Notice that in the case when \( m = 1 \), we have the identification \( i^{(1)}_L = i : \mathcal{A}^{(1)} \rightarrow \mathcal{A} = \mathcal{A}(V) \leftarrow \mathcal{A}(V) = \mathcal{A}^{(1)} : i = i^{(1)}_L \), with \( a_{pq} := a_{pq}^{11}, v_{pq} := v_{pq}^{11} \) and \( b_{pq} := b_{pq}^{11} \).

In addition, the DGA map \( i^{(m)}_L \) is defined via counting certain admissible disks of index 0 in \((I_s \times \mathbb{R}, V^{(m)})).

More precisely, we have

1. If \( 1 \leq p \leq q \leq r \) and \( 1 \leq t, s \leq m \) such that \( b_{k+p-1,k+q-1}^{ts} \) is well-defined, then:
\[
i^{(m)}_L(b_{k+p-1,k+q-1}^{ts}) = \sum v_{p,a,p}^{tu} v_{ab}^{12} \cdots v_{cq}^{12} + \sum c_{pq}^{tu} v_{a,b}^{12} \cdots v_{dq}^{12}
\]
where the summation is over all possible such \textit{composable} words so that the summand is well-defined. In particular, this implies that \( i^{(m)}_L(b_{k+p-1,k+q-1}^{ts}) = 0 \) if \( t > s \), and if \( t = s \), then
\[
i^{(m)}_L(k_{k+p-1,k+q-1}^{ts}) = v_{p,q}^{tu} \text{ with } p < q.
\]
2. If \( 1 \leq p \leq q \leq n_r, 1 \leq t, s \leq m \) such that \( b_{k+p,q}^{ts} \) is well-defined, in particular, \( 1 \leq p' \leq q' \leq n_r \) and \( p', q' \notin \{k, k+1, \ldots, k+r-1\} \), then
\[
i^{(m)}_L(b_{k+p,q}^{ts}) = \frac{v_{pq}^{tu}}{v_{pq}^{tu}}
\]
3. Otherwise, that is, if \( 1 \leq p \leq q \leq n_r, 1 \leq t, s \leq m \) such that, exactly one of \( p, q \)'s belongs to \( \{k, k+1, \ldots, k+r-1\} \) and \( b_{k+p}^{ts} \) is well-defined, then
\[
i^{(m)}_L(b_{k+p,q}^{ts}) = 0
\]
For example, in the case \( r = 3, m = 4 \) as in Figure 18, we have
\[
i^{(4)}_L(b_{k+p-1,k+q-1}^{ts}) = v_{p,q}^{12} \quad \text{if } 1 \leq p < q \leq r;
\]
\[
i^{(4)}_L(b_{p,q}^{ts}) = a_{pq}^{12} \quad \text{if } 1 \leq p < q \leq n_r;
\]
\[
i^{(4)}_L(b_{pq}^{ts}) = 0 \quad \text{else.}
\]

Notice that when \( m = 1 \), we have
\[
i_L(b_{k+p-1,k+q-1}^{ts}) = v_{p,q}^{tu} \quad \text{if } 1 \leq p < q \leq r;
\]
i_L(b_{pq}^{ts}) = a_{pq}^{12} \quad \text{if } 1 \leq p < q \leq n_r;
\]
i_L(b_{pq}^{ts}) = 0 \quad \text{else.}
To describe the augmentation category \(\text{Aug}_v(V; \mathbb{K})\), firstly observe that \(\mathcal{A}(V^{(m)})\) is the push-out of \(\mathcal{A}(V^{(m)}_v)\) and \(\mathcal{I}^{(m)}_v\), where \(\mathcal{I}^{(m)}_v\) is the sub-DGA of \(\mathcal{A}(m)\) generated by \(\{v_{pq}^{ij}, c_{pq}^{ij}\}\). That is, for \(m \geq 1\), we have the following push-out diagram of DGAs:

\[
\begin{array}{ccc}
\mathbb{Z} & \longrightarrow & \mathcal{I}^{(m)}_v \\
\downarrow & & \downarrow \\& \\
\mathcal{A}(V^{(m)}) \ & \longrightarrow \ & \mathcal{A}(m)
\end{array}
\]

Also, \(\mathcal{I}^{(m)}_v = \mathcal{A}(v^{(m)})\) is nothing but the LCH DGA for the front projection \(m\)-copy of \(v\), viewed as a bordered Legendrian graph, obtained from \(V\) by removing the extra parallel strands. It follows that

\[
\text{Aug}_v(V; \mathbb{K}) \equiv \text{Aug}_v(V;\mathbb{K}) \times \text{Aug}_v(v;\mathbb{K})
\]

is a strict product of two \(A_{\infty}\)-categories. By Section 6.1.1, we’ve already seen the identification \(b_h : \text{Aug}_v(V; \mathbb{K}) \equiv \text{MC}(V; \mathbb{K})\). It suffices to describe \(\text{Aug}_v(v; \mathbb{K})\).

Firstly, we introduce a Morse complex for a vertex as follows:

**Definition 6.11.** [Def.4.3.1] Let \(\mathbb{K}[Z]\) be the graded polynomial ring in one variable \(Z\) with \(|Z| = 1\). We define a free graded left \(\mathbb{K}[Z]\)-module

\[
C(v) := \mathbb{K}[Z]\langle e_1, \ldots, e_r \rangle, \quad |e_a| := -\mu(a)
\]

and a decreasing filtration

\[
C(v) \supseteq F^1C(v) \supseteq \cdots \supseteq F^rC(v) \supseteq Z^2 \cdot F^{r+1}C(v) = Z^2 \cdot F^1C(v)
\]

of \(C(v)\) by free graded left \(\mathbb{K}\)-submodules such that for each \(a \in \mathbb{Z}/r\mathbb{Z}\),

\[
F^aC(v) := \mathbb{K}\langle e_a \rangle \oplus \bigoplus_{i > 0} \mathbb{K}\langle Z^{\mu(v,a,i)} e_{a+i} \rangle.
\]

We define \(\text{MC}(v; \mathbb{K})\) to be the set of all \(\mathbb{K}[Z]\)-superlinear endomorphisms \(d\) of \(C(v)\) of degree 1 which preserves \(F^*\) and satisfies \(d^2 + Z^2 = 0\).

**Lemma 6.12.** [Lem.4.3.3] There is a canonical identification

\[
\Xi_v : \text{Aug}(v; \mathbb{K}) \cong \text{MC}(v; \mathbb{K}); \quad \epsilon \mapsto d = d(\epsilon),
\]

where

\[
(-1)^{\mu(i)}d_i = \sum_{j > 0} \epsilon(v_{ij})Z^{\mu(v_{ij})} e_{i+j}.
\]

From now on, we will always use the identification above.

**Lemma 6.13.** [Lem. 4.3.6] Let \(v\) be a vertex as above. There are decompositions of \(\text{Aug}(v; \mathbb{K})\) and \(\text{MC}(v; \mathbb{K})\) over the finite set \(\text{NR}(v)\)

\[
\text{MC}(v; \mathbb{K}) = \bigcup_{\rho \in \text{NR}(v)} B(v; \mathbb{K}) \cdot d_\rho
\]

In particular, \(\text{MC}(v; \mathbb{K}) = \emptyset\) if \(\operatorname{val}(v)\) is odd.

Now we can describe \(\text{Aug}_v(v; \mathbb{K})\).

**Objects.** Notice that \(\mathcal{A}(v^{(1)}) \cong \mathcal{I}_v\) with \(v_{pq}^{11} = \hat{v}_{pq}\), where \(\mathcal{I}_v\) is the internal DGA of \(v\) defined in Example/Definition 2.31. Hence, the set of objects of \(\text{Aug}_v(v; \mathbb{K})\) is:

\[
\text{Ob } \text{Aug}_v(v; \mathbb{K}) = \text{Aug}(v; \mathbb{K}),
\]

the variety of augmentations for \(\mathcal{I}_v\). Notice also that, by Lemma 6.13, \(\text{Aug}(v; \mathbb{K}) = \emptyset\) if \(r\) is odd. So, from now on, we will assume \(r\) is even.

**Morphisms.** As in Section 6.1.1, define \(C_{12}\) to be the free \(\mathbb{K}\)-module generated by \(\{c_{pq}^{12}\}\) in \(\mathcal{I}_v^{(2)}\). Then, for any two objects \(e_1, e_2\), the set of morphisms in \(\text{Aug}_v(v; \mathbb{K})\) is

\[
\text{Hom}_+(e_1, e_2) = C_{12}^+ = \mathbb{K}\langle c_{pq}^+, \ 1 \leq q < p \leq r \rangle
\]
as a free $K$-module, where $C^i_{pq} := C^i_{12}[-1]$, and $x^+ := (x^{12})^v = (x^{12})^r[-1]$. In particular, $|c^+_{pq}| = |c^+_{pq}| + 1$.

**Compositions $m_K$.** For any $K \geq 1$, and objects $e_1, e_2, \ldots, e_{K+1}$, the composition map

$$m_K : \text{Hom}_+(e_K, e_{K+1}) \otimes \ldots \otimes \text{Hom}_+(e_2, e_3) \otimes \text{Hom}_+(e_1, e_2) \to \text{Hom}_+(e_1, e_{K+1})$$

is defined as before.

$m_1$. For $K = 1$ and $\epsilon = (e_1, e_2)$, as in Section 6.1.1, we compute

$$\partial_-^2|c^+_{pq}| = \sum_{A \geq p} \phi(v^{11}_{p, A-p})c^+_{pq} + \sum_{B < q} c^+_{pq} \partial_-^2\phi(v^{12}_{R, B-q})$$

where $1 \leq q < p \leq r$. It follows that

$$m_1(c^+_{AB}) = \sum_{B < q < A} \epsilon(v_{p, A-p})c^+_{AB} + (-1)^{|c^+_{AB}|} \sum_{B < q < A} \epsilon(v_{B, B-q})c^+_{AB}$$

(6.16)

where $1 \leq B < A \leq r$.

**$m_K$ for $K \geq 2$.** In general, for $K \geq 2$ and given $\epsilon = (e_1, e_2, \ldots, e_{K+1})$ defining a diagonal augmentation for $I^+_V(K+1)$, we need to compute $m_K(c^+_{e_1}, \ldots, c^+_{e_{K+1}})$, where $c^+_{e_i} \in (c^+_{pq})$ is a generator for $\text{Hom}_+(e_i, e_{i+1})$. To do that, we need to calculate $\partial_-^1(c^+_{K+1})$ and look at the $K$-coefficient of $c^+_{12} \ldots c^+_{K+1}$ in the result.

By the formula (6.9) for $\partial_-^1(c^+_{K+1})$, with $1 \leq q < p \leq r$, we have

$$\partial_-^1(c^+_{K+1}) = \sum \phi(v^{11}_{p, 0})c^+_{12} \ldots c^+_{K+1}$$

$$+ \sum c^+_{12} \phi(v^{12}_{1, 0})c^+_{23} \ldots c^+_{K+1} + \text{else}$$

in which only the first two terms contribute to $m_K$. In addition, the first summation is over all $1 \leq a_1, \ldots, a_k \leq r$ such that $p < a_1 < a_2 > \ldots > a_K > q$; the second summation is over all $1 \leq a_1, \ldots, a_k \leq r$ such that $p > a_1 < a_2 > a_3 > \ldots > a_K > q$.

We can then divide the computation of $m_K(c^+_{e_1}, \ldots, c^+_{e_{K+1}})$ into 4 cases:

1. If $K > r + 1$, then both of the two summations above are empty. Hence, we have

$$m_K = 0, \text{ for all } K \geq r + 1.$$  

(6.17)

2. For $2 \leq K \leq r$, and any $1 \leq p = a_0, a_1, a_2, \ldots, a_k = a_{K+1} \leq r$ such that $a_0 > a_1 < a_2 > a_3 > \ldots > a_K > a_{K+1}$ and $a_0 > a_{K+1}$, have

$$m_K(c^+_{e_{a_0}, a_{a_1}}, \ldots, c^+_{a_{a_2}, a_{a_3}}, c^+_{a_{a_3}}) = (-1)^{|c^+_{a_{a_1}}|} \epsilon(v_{a_0, a_2})c^+_{a_{a_3}}$$

(6.18)

where $\sigma_K = \sigma_K(c^+_{a_{a_0}, a_{a_1}}, \ldots, c^+_{a_{a_2}, a_{a_3}})$ as in Definition 4.3.

3. For $2 \leq K \leq r - 1$, and any $1 \leq a_1, a_2, \ldots, q = a_{K+1} \leq r$ such that $a_1 > a_2 > \ldots > a_{K+1}$, have

$$m_K(c^+_{e_{a_1}, a_{a_2}}, \ldots, c^+_{a_{a_3}, a_{a_3}}, c^+_{a_{a_3}}) = (-1)^{|c^+_{a_{a_3}}|} \sum_{a_{K+1} < p < a_1} \epsilon(v_{p, a_1-p})c^+_{a_{a_3}}$$

(6.19)

where $\sigma_K = \sigma_K(c^+_{a_{a_1}, a_{a_2}}, \ldots, c^+_{a_{a_3}, a_{a_3}})$ as in Definition 4.3.

4. Else, $2 \leq K \leq r$, and $(c^+_{e_1}, c^+_{i_2}, \ldots, c^+_{e_k})$ is not of the form $(c^+_{a_{a_1}, a_{a_2}}, \ldots, c^+_{a_{a_3}})$ in (2) or $(c^+_{a_{a_1}, a_{a_2}}, \ldots, c^+_{a_{a_3}})$ in (3) above, then

$$m_K(c^+_{i_1}, \ldots, c^+_{i_2}, c^+_{i_1}) = 0.$$  

(6.20)

This gives the description of $\text{Aug}_+(V; K)$.

Now we want to compute the whole diagram of augmentation categories:

$$\text{Aug}_+(V; K) := (\text{Aug}_+(V; K) \xrightarrow{\epsilon} \text{Aug}_+(V; K) \xrightarrow{\epsilon} \text{Aug}_+(V; K))$$

(6.21)

The functor $\mu_K$ is clear, under the identification $\text{Aug}_+(V; K) \cong \text{Aug}_+(V; K) \times \text{Aug}_+(V; K)$, it is the obvious restriction functor $\text{Aug}_+(V; K) \times \text{Aug}_+(V; K) \to \text{Aug}_+(V; K)$. Recall that by Lemma 6.6, we have the identification $\mu_K : \text{Aug}_+(V; K) \cong \mathcal{M}(V; K)$. It suffices to describe $\mu_K$.

**On objects.** Given any object $e$ in $\text{Aug}_+(V; K)$, denote $e_l := e \circ i_l$, $e_r := e \circ i_r$, and $e_r := e \circ i_r$, where $i_l : I \to \mathcal{A}(V)$ is the natural inclusion of DGAs. Then by definition, $\mu_K(e) = e_r$. We will use the identification $\text{Aug}_+(V; K) \cong \text{Aug}_+(V; K) \times \text{Aug}_+(V; K)$ and $e_r := \text{Aug}_+(V; K) \cong \mathcal{M}(V; K)$. Then $e = (d_l, d_r)$, where $(C(V_l), d_l := d(e_l))$ and $(C(V_r), d_r := d(e_r))$ are Morse complexes defined as in Lemma 6.4 and Lemma 6.12 respectively. Also, we have identification $e_r = d_r$, i.e. $(C(V_r), d(e_r))$. 

\[ \text{Augustination are sheaves for Legendrian graphs} \]
In fact, \( \tau_R \) is a quotient of cochain complexes \( \tau_R : (C(V_\lambda), d_\lambda) \oplus (C(v), d_\lambda) \rightarrow (C(V_\lambda), d_\lambda) \oplus (C(V_R), d_\lambda) \) as follows. By definition, \( \varphi_\lambda = e_\lambda^n \otimes e_\lambda^\ell \) with \( e_\lambda^\ell \) the basis element corresponding to the strand \( \lambda \) of \( V_\lambda \). \( C(v) = \prod_{p \leq R} \mathbb{Z}[\delta] e_\lambda^\ell \) with \( e_\lambda^\ell \) corresponding to the right half-edge \( p \) of \( v \). \( C(V_R) = \prod_{p \leq R} \mathbb{Z}[\delta] e_\lambda^\ell \) with \( e_\lambda^\ell \) corresponding to the strand \( \lambda \) of \( V_R \). Now, define \( \varphi_\lambda : C(T_\lambda) \oplus C(v) \rightarrow C(V_R) \) by \( \varphi_\lambda(e_\lambda^\ell) = e_\lambda^\ell \), \( \varphi_\lambda(Z_n e_\lambda^\ell) = 0 \) for \( n > 0 \). Then by the formula (6.15) for \( \tau_R \), \( \tau_R = \varphi_\lambda : (C(V_\lambda), d_\lambda) \oplus (C(v), d_\lambda) \oplus (C(V_R), d_\lambda) \oplus (C(V_R), d_\lambda) = \text{im} \cap \ker \) is indeed a quotient of complexes, where \( (C(V_\lambda), d_\lambda) \) is the subcomplex of \( (C(V_R), d_\lambda) \) with \( C(V_\lambda) = \oplus_{1 \leq p \leq R} \mathbb{Z}[\delta] e_\lambda^\ell \). Observe that \( \varphi_\lambda \) in fact defines naturally a DG functor \( MC(V_\lambda; \mathbb{K}) \times MC(v; \mathbb{K}) \rightarrow MC(V_\lambda; \mathbb{K}) \times MC(v_\lambda; \mathbb{K}) \implies MC(V_R; \mathbb{K}) \) of DG categories.

On morphisms. By definition, the \( A_\infty \)-functor \( \tau_R \) on morphisms is given by a collection of assignments \( \{\tau_{R,K} = (\tau_{R,K})_{K \geq 1}\} \). That is, for any \( K \geq 1 \) and \( K + 1 \) objects \( e_1, e_2, \ldots, e_{K+1} \) in \( \text{Aug}_*(V; \mathbb{K}) \), we have a map

\[
\tau_{R,K} : \text{Hom}_*(e_K, e_{K+1}) \otimes \ldots \otimes \text{Hom}_*(e_2, e_1) \rightarrow \text{Hom}_*(e_1, e_2) \otimes \text{Hom}_*(e_{K+1}, e_K)
\]

of degree \( 1 - K \), where \( e_{i,R} := \tau_{R}(e_i) \). It suffices to determine \( \tau_{R,K}(c_{i,K}^+, \ldots, c_{i,K}^+) \) for any collection of generators, where \( c_{i,K}^+ \in \{a_{i,K}^{p,q}, c_{i,K}^{p,q}\} \) is a generator for \( \text{Hom}_*(e_j, e_{j+1}) = \hat{\mathbb{K}}(a_{i,K}^{p,q}, c_{i,K}^{p,q}) \) as a free \( \mathbb{K} \)-module.

As in Section 6.1.1, \( \varepsilon := (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_K) \) (resp. \( \varepsilon_n := (\varepsilon_{1,K}, \varepsilon_{2,K}, \ldots, \varepsilon_{K+1,K}) \)) defines a diagonal augmentation for \( \mathcal{A}^*(K+1) \) (resp. \( \mathcal{A}^*(K+1) \)). Define \( \iota_{R,K} : \mathcal{A}^*(K+1) \otimes \mathbb{K} \rightarrow \mathcal{A}^*(K+1) \otimes \mathbb{K} \). Recall that we have

\[
\iota_{R,K}(c_{i,K}^+, \ldots, c_{i,K}^+) = (-1)^{v_{K+1}} \sum_{i,j} b_{i,j}^+ (i_{R,K}^{(K+1)})_{i,j} \cdot c_{i,K}^+ \cdot c_{j,K}^+
\]

(6.22)

where \( (i_{R,K}^{(K+1)})_{i,j} \) denotes the \( \mathbb{K} \)-coefficient of \( c_{i,K}^+ \cdot c_{j,K}^+ \) in \( i_{R,K}^{(K+1)} b_{i,j}^+ \), and \( \sigma_K := \sigma_K(c_{i,K}^+, \ldots, c_{i,K}^+) \) as in Definition 4.3. Observe that, by the formula for \( \iota_{R,K} \), the \( A_\infty \)-functor \( \tau_R \) is simply the composition

\[
\tau_R : \text{Aug}_*(V_\lambda; \mathbb{K}) \times \text{Aug}_*(v; \mathbb{K}) \xrightarrow{\text{Id} \times \tau_{V,R}} \text{Aug}_*(V_\lambda; \mathbb{K}) \times \text{Aug}_*(v_\lambda; \mathbb{K}) \xrightarrow{\text{Id} \times \tau_{V,R}} \text{Aug}_*(V_\lambda; \mathbb{K})
\]

Here, \( \tau_{V,R} : \text{Aug}_*(v; \mathbb{K}) \rightarrow \text{Aug}_*(v; \mathbb{K}) \) is the right restriction functor for the bordered Legendrian graph \( v \), by removing the extra parallel strands of \( V \). And, the second arrow in the composition is the natural inclusion of DG categories \( MC(V_\lambda; \mathbb{K}) \times MC(v; \mathbb{K}) \hookrightarrow MC(V_\lambda; \mathbb{K}) \), under the obvious identification. It suffices to describe \( \tau_{V,R} = \tau_{V,R} \mid \text{Aug}_*(v; \mathbb{K}) \).

(\( \tau_{V,R} \)). Let \( K = 1 \) and \( \varepsilon = (\varepsilon_1, \varepsilon_2) \). Apply formula (6.12), for \( 1 \leq p \leq q \leq r \), we compute:

\[
i_{R,K}^{(2)}(i_{R,K}^{(2)})_{p,q} = \sum_{L \leq q} \chi_L(v_{p,L}^+, p_{L,q}^+) \chi_{L} + \sum_{L < p} \chi_{L,q}^+(p_{L,q}^+)_{v_{p,L}^+, p_{L,q}^+}
\]

Then by (6.22), for \( 1 \leq B < A \leq r \), we have

\[
(\tau_{V,R})(c_{i,R}^+) = \sum_{p \leq B} \varepsilon_1(v_{p,A}^-, p_{B,q}^+)_{k+1,B-1} + \sum_{q > A} \varepsilon_2(v_{B,q}^-, p_{B,q}^+)_{k+1,k+q-1}
\]

(6.23)

(\( \tau_{V,R} \)) for \( K \geq 2 \). In general, let \( K \geq 2 \) and \( \varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{K+1}) \). Apply formula (6.12), for \( 1 \leq p \leq q \leq r \), we have

\[
i_{R,K}^{(K+1)}(i_{R,K}^{(K+1)})_{p,q} = \sum_{L \leq q} \chi_L(v_{p,L}^+, p_{L,q}^+) \chi_{L} + \sum_{L < p} \chi_{L,q}^+(p_{L,q}^+)_{v_{p,L}^+, p_{L,q}^+}
\]

in which only the first two terms contribute to \( (\tau_{V,R})_K \). In addition, the first summation is over all \( 1 \leq a_1, \ldots, a_k \leq r \) such that \( p < a_1 > a_2 > \ldots > a_k > q \); the second summation is over all \( 1 \leq a_1, \ldots, a_k \leq r \) such that \( p > a_1 < a_2 < a_3 \ldots > a_k > q \).

Similar to \( m_K \), we can divide the computation of \( (\tau_{V,R})_K(c_{i,K}^+, \ldots, c_{i,K}^+) \) into 4 cases:

1. If \( K \geq r + 1 \), then both of the two summations above are empty. Hence, we have

\[
(\tau_{V,R})_K = 0, \text{ for all } K \geq r + 1.
\]

(6.24)
(2) For $2 \leq K \leq r$, and any $1 \leq p = a_0, a_1, a_2, \ldots, a_K, q = a_{K+1} \leq r$ such that $a_0 > a_1 < a_2 > a_3 > \ldots > a_K > a_{K+1}$ and $a_0 \leq a_{K+1}$, have

$$\tau_{v,R}(c_{a_0 a_1}^+, \ldots, c_{a_2 a_1}^+, \ldots, c_{a_{K+1}}^+) = (1)^{r_{K+1} - r} b_{a_0 a_1 - 1, k + a_{K+1} - 1}^+(v_{a_1, a_2 - a_1})$$

where $r_K = r_{K+1} - r$ and $c_{a_0 a_1}^+, \ldots, c_{a_{K+1}}^+$ as in Definition 4.3.

(3) For $2 \leq K \leq r - 1$, and any $1 \leq a_1, a_2, \ldots, q = a_{K+1} \leq r$ such that $a_1 > a_2 > \ldots > a_K$, have

$$\tau_{v,R}(c_{a_1 a_2}^+, \ldots, c_{a_2 a_1}^+, \ldots, c_{a_{K+1}}^+) = (1)^{r_{K}} \sum_{p \leq a_{K+1}} \varepsilon_1(v_{p, a_1 - p}) b_{p + a_{K+1} - 1}^+(v_{a_1, a_2 - a_1})$$

where $r_K = r_{K+1} - r$ and $c_{a_1 a_2}^+, \ldots, c_{a_{K+1}}^+$ as in Definition 4.3.

(4) Else, $2 \leq K \leq r$, and $(c_{a_1}^+, c_{a_2}^+, \ldots, c_{a_k}^+)$ is not of the form $(c_{a_0 a_1}^+, \ldots, c_{a_{K+1}}^+)$ in (2) or $(c_{a_1 a_2}^+, \ldots, c_{a_{K+1}}^+)$ in (3) above, then

$$\tau_{v,R}(c_{a_1}^+, \ldots, c_{a_k}^+) = 0.$$

This finishes the description of $\tau_{v,R} : \text{Aug}_+(V; K) \rightarrow \text{Aug}_+(V_R; K)$, hence that of $\tau_R$.

Furthermore, the following gives a simpler description of $\text{Aug}_+(V; K)$:

**Lemma 6.14.** The $A_{\infty}$-functor

$$\tau_{v,R} : \text{Aug}_+(V; K) \rightarrow \text{Aug}_+(V_R; K).$$

is an $A_{\infty}$-equivalence. As a consequence, we obtain a commutative diagram of $A_{\infty}$-categories:

$$\begin{array}{ccc}
\text{Aug}_+(V; K) & \xrightarrow{\tau_{v,R}} & \text{Aug}_+(V_R; K) \\
\downarrow \cong & & \downarrow \cong \\
\text{Aug}_+(V_1; K) & \xrightarrow{\tau_{v,R}} & \text{Aug}_+(V_1; V_R; K) \\
\downarrow \cong & & \downarrow \cong \\
\text{MC}(V_1; K) & \xrightarrow{\tau_{v,R}} & \text{MC}(V_1; V_R; K)
\end{array}$$

which induces equivalences between all the three rows, with each row viewed as a diagram of $A_{\infty}$-categories. Here, $\tau_1$'s are the projections to the first factors, and $\epsilon$'s are the natural inclusions, and the functor $\text{Id} \times \tau_R$ is defined via

$$\text{Aug}_+(V; K) \equiv \text{Aug}_+(V_1; K) \times \text{Aug}_+(V_R; K) \xrightarrow{\text{Id} \times \tau_{v,R}} \text{Aug}_+(V_1; V_R; K) \equiv \text{MC}(V_R; K).$$

In particular, the result implies that $\text{Aug}_+(V; K)$ is equivalent to the third row of Morse complex categories, which involves only DG categories and DG functors. This will be one key step in showing our main result “augmentations are sheaves”.

**An algebraic proof of Lemma 6.14.** Notice that the equivalence between the second and third row is just a direct consequence of Lemma 6.6. The only nontrivial part is to show $\tau_{v,R}$ is an $A_{\infty}$-equivalence. For simplicity, we can now assume $V = v$, that is, $n_k = 0, k = 1$, and $\tau_{v,R} = \tau_R$.

Remember that on objects $\tau_R$ is just the map of augmentation varieties $\tau_R : \text{Aug}(V; K) \rightarrow \text{Aug}(V_R; K)$, which is surjective. In particular, $\tau_R$ is essentially surjective. It suffices to show that $\tau_R$ is fully faithful, i.e. for any two objects $\epsilon_1, \epsilon_2$ in $\text{Aug}_+(V; K)$, the co-chain map

$$\tau_{R,1} = (\tau_R)_! : \text{Hom}_V(\epsilon_1, \epsilon_2) \rightarrow \text{Hom}_{V_R}(\epsilon_{1,R}, \epsilon_{2,R})$$

is a quasi-isomorphism. Here $\text{Hom}_T(-, -) := \text{Hom}_{\text{Aug}_+(T; K)}(-, -)$ for any bordered Legendrian graph $T$. Equivalently, it suffices to show that the cone of $-\tau_{R,1}$ is acyclic. By definition, the cone of $-\tau_{R,1}$ is the cochain complex

$$\text{Cone}(-\tau_{R,1}) := \left( \text{Hom}_{V_R}(\epsilon_{1,R}, \epsilon_{2,R}) \oplus \text{Hom}_V(\epsilon_1, \epsilon_2)[1], \delta \right)$$

with differential

$$\delta := \begin{pmatrix} m_{1,R} & -1 & 0 \\ 0 & -m_1 & \end{pmatrix}$$

Let $\text{End}^*(C(V_R))$ be the free $\mathbb{Z}$-graded $K$-module of $K$-linear endomorphisms of $C(V_R) = \oplus_{p \geq 0} \mathbb{K} \cdot e_p$ without filtration. For $i = 1, 2$, remember that $d(\epsilon_{i,R})$ is the image of $\epsilon_{i,R}$ under the DG equivalence.
$b_R: \text{Aug}_1(V_R; \mathbb{K}) \cong MC(V_R; \mathbb{K})$ in Lemma 6.6 for $V_R$. Then define a cochain complex $(E, D) \coloneqq (\text{End}^*(C(V_R)), D)$ with differential:

$$D(f) := d(e_2, R) \circ f - (−1)^{|f|} f \circ d(e_1, R).$$

Notice that $\text{Hom}_V(\epsilon_1, \epsilon_2)[1] = \oplus_{1 \leq q \leq p \leq r} \mathbb{K} \cdot c_{p, q}^+[1]$, and $\text{Hom}_V(\epsilon_1, \epsilon_2, R) = \oplus_{1 \leq p \leq q \leq r} \mathbb{K} \cdot b_{p, q}^+$. In particular, $|c_{p, q}^+[1]| = |c_{p, q}^+| - 1 = \mu(p) - \mu(q)$. Define a $\mathbb{K}$-linear map $\alpha: \text{Cone}(−\tau_{R, 1}) \to E$ with:

$$\alpha(b_{p, q}^+) := (−1)^{x(p,q)+|q|} \langle p \rangle, \quad 1 \leq p \leq q \leq r;$$

$$\alpha(c_{p, q}^+[1]) := (−1)^{x(p,q)+|q|} \langle p \rangle, \quad 1 \leq q < p \leq r.$$

where $s(p, q) = \mu(p)(\mu(q) + 1) + 1$ and $|q \rangle \langle p|$ are defined as in Lemma 6.6.

Claim 1: The map $\alpha: \text{Cone}(−\tau_{R, 1}) \to E$ is an isomorphism of cochain complexes.

Proof of Claim 1. Clearly, $\alpha$ is a $\mathbb{K}$-linear isomorphism of degree 0. It suffices to show that $\alpha$ is a cochain map. Firstly observe that, $\alpha|_{\text{Hom}_V(\epsilon_1, \epsilon_2, R)}$ is just the composition

$$\text{Hom}_V(\epsilon_1, \epsilon_2, R) \xrightarrow{(b_R)} \text{Hom}_{MC(V_R, \mathbb{K})}(d(e_1, R), d(e_2, R)) \xrightarrow{\iota} E,$$

where $(b_R)_1$ is induced from the DG equivalence $b_R: \text{Aug}_1(V_R; \mathbb{K}) \cong MC(V_R; \mathbb{K})$, and the second arrow is the inclusion of complexes. Hence, it follows by Lemma 6.6 that $\alpha|_{\text{Hom}_V(\epsilon_1, \epsilon_2, R)}$ is a cochain map. It remains to show that, for any homogeneous element $x \in \text{Hom}_V(\epsilon_1, \epsilon_2)[1]$, we have $\alpha \circ \delta(x) = D \circ \alpha(x)$.

Say, $x = \sum_{1 \leq B < A \leq r} x_{AB} c_{AB}^+[1]$, and denote $x^+ := x[-1]$. In particular, $x_{AB} \neq 0$ implies that $|x| = |c_{AB}^+[1]| = \mu(A) - \mu(B)$. Firstly, we have

$$\alpha(x) = \sum_{B < A} (−1)^{x(AB)} x_{AB} |B \rangle \langle A|$$

$$d(e_1, R) = \sum_{p < A} (−1)^{\mu(p)} e_1, R(b_{p, A}) |A \rangle \langle p|$$

$$d(e_2, R) = \sum_{B < q} (−1)^{\mu(B)} e_2, R(b_{B, q}) |q \rangle \langle B|$$

On the other hand, by definition, $\delta(x) = −\tau_{R, 1}(x^+) − m_1(x^+) [1]$. Apply formula (6.16), we obtain:

$$\alpha(m_1(x^+) [1]) = \sum_{B < P < A} \epsilon_1(v_p, \alpha, −p) x_{AB} \alpha(c_{AB}^+[1]) + \sum_{B < q < A} (−1)^{|c_{AB}^+|} x_{AB} \alpha(c_{AB}^+[1])$$

$$= \sum_{q < p < A} \epsilon_1, R(b_{p, A}) x_{AQ} \alpha(c_{p, q}^+[1]) + \sum_{B < q < p} (−1)^{|c_{p, q}^+|} x_{p, B, q} \alpha(c_{p, q}^+[1])$$

$$= \sum_{q < p < A} (−1)^{s(p, q)} \epsilon_1, R(b_{p, A}) x_{AQ} \langle q \rangle \langle p| + \sum_{B < q < p} (−1)^{|c_{p, q}^+| + 1} x_{p, B, q} \alpha(c_{p, q}^+[1]) \langle q \rangle \langle p|$$

Similarly, by formula (6.23), we have:

$$\alpha(\tau_{R, 1}(x^+)) = \sum_{p \leq B < A} \epsilon_1(v_p, \alpha, −p) x_{AB} \alpha(b_{p, B}^+[1]) + \sum_{B \leq q < A} (−1)^{|c_{AB}^+|} x_{AB} \alpha(b_{AB}^+[1])$$

$$= \sum_{p \leq q < A} \epsilon_1, R(b_{p, A}) x_{AQ} \alpha(b_{p, q}^+[1]) + \sum_{B \leq q < p} (−1)^{|c_{p, q}^+|} x_{p, B, q} \alpha(b_{p, q}^+[1])$$

$$= \sum_{p \leq q < A} (−1)^{s(p, q)} \epsilon_1, R(b_{p, A}) x_{AQ} \langle q \rangle \langle p| + \sum_{B \leq q < p} (−1)^{|c_{p, q}^+| + 1} x_{p, B, q} \alpha(b_{p, q}^+[1]) \langle q \rangle \langle p|$$
Combine the two computations above, we then obtain:

\[ \alpha \circ \delta(x) = - \sum_{p,q \leq A} (-1)^{p+q} \epsilon_1(b_{p,A}) \chi_{A,q} \langle q \rangle \langle p \rangle - \sum_{B < p,q} (-1)^{k_{p,B} + q_{p,q}} \chi_{p,B} \epsilon_2(b_{B,q}) \langle q \rangle \langle p \rangle \]

\[ = - \sum_{p,q \leq A} (-1)^{p+q} \epsilon_1(b_{p,A}) \chi_{A,q} \langle q \rangle \langle p \rangle + \sum_{B < p,q} (-1)^{\mu(B)+q_{p,q}} \chi_{p,B} \epsilon_2(b_{B,q}) \langle q \rangle \langle p \rangle \]

\[ = - \sum_{p,q \leq A} (-1)^{|x|} (-1)^{|p|} \epsilon_1(b_{p,A})((-1)^{|A,q|} \chi_{A,q}) \langle A \circ |A \rangle \langle p \rangle + \sum_{B < p,q} ((-1)^{|p,B|} \chi_{p,B})((-1)^{|B|} \epsilon_2(b_{B,q})) \langle B \circ |B \rangle \langle p \rangle \]

\[ = - (-1)^{|x|} \alpha(x) \circ d(\epsilon_1,R) + d(\epsilon_2,R) \circ \alpha(x) \]

\[ = D \circ \alpha(x) \]

as desired. In the computation above, the third equality follows directly from the formulas for \( \alpha(x) \) and \( d(\epsilon_i,R) \) above. The only nontrivial part is the second equality involving two sign manipulations, which are of the same form as those in (6.5), hence follow by exactly the same argument. This finishes the proof of Claim 1.

Now, by Claim 1, it suffices to show that \((\mathcal{E}, D)\) is acyclic, which is purely a homological algebra problem. In fact, we can prove a stronger statement:

**Claim 2:** For any free \( \mathbb{Z} \)-graded \( \mathbb{K} \)-module \( C := \oplus_{p \leq 1} \mathbb{K} \cdot \epsilon_p \), let \((C, D_1), (C, D_2) \in \text{MC}(C; \mathbb{K})\) be two Morse complexes as in Definition 6.3, such that either \( D_1 \) or \( D_2 \) is acyclic. Let \((\mathcal{E}, D) := (\text{End}^\bullet(C), D)\) be the complex of \( \mathbb{K} \)-linear endomorphisms of \( C \), with differential given by \( D(f) := D_2 \circ f - (-1)^{|f|} f \circ D_1 \). Then \((\mathcal{E}, D)\) is chain homotopic to zero.

To prove the claim, we firstly make some preparation.

**Definition 6.15.** Define \( \text{GNR}(C) \) to be the set of all involutions \( \rho \) possibly with fixed points on \( I(C) := \{1, 2, \ldots, n\} \) such that

\[ |e_i| = |e_j| - 1, \quad \text{if } \rho(i) = j \text{ and } i < j. \]

Let \( B(C; \mathbb{K}) \) be the automorphism group of \((C, F^\bullet)\) in Definition 6.3. Then \( B(C; \mathbb{K}) \) acts on \( \text{MC}(C; \mathbb{K}) \) via conjugation: For any \( g \in B(C; \mathbb{K}) \) and \( d \in \text{MC}(C; \mathbb{K}) \),

\[ g \cdot d = g \circ d \circ g^{-1}. \]

**Definition 6.16** (Canonical differentials). For each involution \( \rho \in \text{GNR}(C) \), define a canonical differential \( d_\rho \in \text{MC}(C; \mathbb{K}) \) as follows:

\[ (-1)^{\alpha(i)} d_\rho e_i = \begin{cases} e_j & \text{if } \rho(i) = j \text{ and } i < j; \\ 0 & \text{otherwise}. \end{cases} \]

Recall that, we have

**Lemma 6.17** ([26, Lem.4.5]). The group action of \( B(C; \mathbb{K}) \) induces a decomposition of \( \text{MC}(C; \mathbb{K}) \) into finitely many orbits

\[ \text{MC}(C; \mathbb{K}) = \bigsqcup_{\rho \in \text{GNR}(C)} B(C; \mathbb{K}) \cdot d_\rho. \]

Now, we can prove Claim 2.

**Proof of Claim 2.** By Lemma 6.17, if \( d_1 \) is acyclic, then \( d_1 = g_1 \circ d_{k_1} = g_1 \circ d_k \circ g_1^{-1} \). Then clearly we have \( d_2 = 0 \), and \( d_{k_i} \circ \alpha_{k_i} = 0 \). Define \( \delta_i \in \mathcal{E}^{-1} \) by \( \delta_i \cdot e_q = e_p \) if \( p < q = \rho_1(p) \), and \( \delta_i \cdot e_q = 0 \). Then clearly we have \( \delta_1 = 0 \), and \( d_{k_1} \circ \alpha_{k_1} = 0 \). Define \( \delta_i \in \mathcal{E}^{-1} \) as \( \delta_i := g_i \circ \delta_{k_i} = g_i \circ \delta_{k_i} \circ g_i^{-1} \). It follows immediately that \( \delta_1 = 0 \) and \( d_1 \circ \delta_1 = \delta_1 \circ d_1 = id \).
Now, if $d_2$ is acyclic, *define* a map $H_2 : E \to E$ of degree $-1$ by $H_2(f) := \delta_2 \circ f$. We then compute:

$$[D, H_2](f) = D \circ H_2(f) + H_2 \circ D(f)$$

$$= D(\delta_2 \circ f) + \delta_2 \circ D(f)$$

$$= d_2 \circ (\delta_2 \circ f) - (-1)^{|f-1|}(\delta_2 \circ f) \circ d_1 + \delta_2 \circ (d_2 \circ f - (-1)^{|f|}f \circ d_1)$$

$$= f$$

That is, $\text{Id} = [D, H_2] : E \to E$ is chain homotopic to zero, as desired. If $d_1$ is acyclic, by a similar argument, one can show that, $\text{Id} = [D, H_1]$ with $H_1(f) = (-1)^{|f|}f \circ \delta_1$, as desired. $\square$

This finishes the proof of the lemma. $\square$

Now we give an alternative geometric proof of Lemma 6.14. 

A geometric proof of Lemma 6.14. Let us start by introducing an alternative parallel copies, let us say $v^{(\ast)}$, of the vertex $v$ of type $(0, r)$ as follows: Clearly $v^{(\ast)}$ is in $\mathcal{B}[\mathcal{L}^{2^{(\ast)}}]_{\text{Aug}}$, and let us consider its consistent DGA $A^{(\ast)}(v')$. Let us use the same labeling convention of the crossing as in Figure 18, then $A^{(\ast)}(v')$ is generated by the following sets of generators:

$$C_1 := \{ e_{a,a}^{i,j} \mid 1 \leq i < j \leq m, 1 \leq a \leq r \};$$

$$C_2 := \{ e_{a,b}^{i,j} \mid 1 \leq i < j \leq m, 1 \leq a < b \leq r \};$$

$$\tilde{V} := \{ v_{a,\ell}^{i} \mid 1 \leq i \leq m, 1 \leq a \leq r, \ell \in \mathbb{N} \}.$$}

Note that the generators in $C_1$ are the crossings near the right border while the ones in $C_2$ are crossings near the vertices $v^i$ in Figure 19. The grading is given by

$$|e_{a,a}^{i,j}| = \mu(a) - \mu(b) - 1, \quad |v_{a,\ell}^{i}| = \mu(a) - \mu(a + \ell) + (N - 1),$$

where $N = N(n, a, \ell)$ is the same as in (2.3). The differential for the vertex generators $\tilde{V}$ is the same as before, see (2.5) in Example/Definition 2.31. For the generators in $C_1$ and $C_2$, the differential is given by

$$\partial^m e_{a,a}^{i,j} = \sum_{i < k < j} e_{a,a}^{i,k} e_{a,a}^{k,j},$$

$$\partial^m e_{a,b}^{i,j} = (-1)^{|v'_{a,b-a}^{i,j}|+1} v_{a,b-a}^{i,j} + \sum_{a \leq c < b} (-1)^{|v_{a,c}^{i,j}|+1} e_{a,c}^{i,j} v_{c,b-c}^{i,j}$$

$$+ \sum_{i < k < j} (-1)^{|e_{a,c}^{i,k}|+1} e_{a,c}^{i,k} e_{c,b}^{k,j}$$

Now describe the corresponding augmentation category $\mathcal{A}_{\text{Aug}}(v')$. The objects of $\mathcal{A}_{\text{Aug}}(v')$ is the augmentations of $A(v; \mathbb{K})$:

$$\text{Ob}(\mathcal{A}_{\text{Aug}}(v')) = \text{Aug}(v; \mathbb{K}).$$

For any two objects $\epsilon_1, \epsilon_2$, the set of morphism becomes

$$\text{Hom}_{\mathcal{A}_{\text{Aug}}(v')}(\epsilon_1, \epsilon_2) = \mathbb{K} \left\{ e_{a,b}^{12} \mid 1 \leq a \leq b \leq r \right\}.$$
By dualizing the differential $d^{(m)}$, we have the following $A_\infty$ structure $\{m^k\}_{k \geq 1}$ as follows:

- For $\epsilon_1, \epsilon_2 \in \text{Aug}(v; \mathbb{K})$, the map
  
  $m^1_1 : \text{Hom}_{\mathcal{A}_{\text{Aug}}(v)}(\epsilon_1, \epsilon_2) \to \text{Hom}_{\mathcal{A}_{\text{Aug}}(v)}(\epsilon_1, \epsilon_2)$

  is defined as
  
  $m^1_1(\epsilon_{ab}) = -\sum_{c < a} \epsilon_1(v_{c,a-c})\epsilon_{cb} + \sum_{b < d} (-1)^{|\epsilon_{ab}|} \epsilon_{ab} \epsilon_2(v_{b,d-b})$  \hspace{1cm} (6.29)

- For $\epsilon_1, \epsilon_2, \epsilon_3 \in \text{Aug}(v; \mathbb{K})$, the map
  
  $m^2_2 : \text{Hom}_{\mathcal{A}_{\text{Aug}}(v)}(\epsilon_2, \epsilon_3) \otimes \text{Hom}_{\mathcal{A}_{\text{Aug}}(v)}(\epsilon_1, \epsilon_2) \to \text{Hom}_{\mathcal{A}_{\text{Aug}}(v)}(\epsilon_1, \epsilon_3)$

  is defined as
  
  $m^2_2(\epsilon_{cd} \otimes \epsilon_{ab}) = \delta_{bc}(-1)^{|\epsilon_{ab}|} \epsilon_{cd}$

- For $m^k_k$ with $k \geq 3$, the higher composition $m^k_k$ vanishes.

Note that

$1' := \sum_{1 \leq a \leq r} -c^{12}_a \in \text{Hom}_{\mathcal{A}_{\text{Aug}}(v')}(\epsilon, \epsilon)$

is the unit for all $\epsilon \in \text{Aug}(v')$, and hence $\mathcal{A}_{\text{Aug}}(v')$ is a (strictly) unital $A_\infty$-category.

We show that $\tau_{R'}^i : \text{Aug}_+(v') \to \mathcal{A}_{\text{Aug}}(v'_R)$ is an $A_\infty$-equivalence. Similar as before, it suffices to check that the $A_\infty$-functor $\tau_{R'}^i$ is essentially surjective and fully-faithful. Indeed, $\tau_{R'}^i$ is surjective (and hence essentially surjective). The main issue again is to show the following

Lemma 6.18. As in the above setup, 

$\tau_{R,1}' = (\tau_{R}^i)_1 : \text{Hom}_{\mathcal{A}_{\text{Aug}}(v')}(\epsilon_1, \epsilon_2) \to \text{Hom}_{\mathcal{A}_{\text{Aug}}(v'_R)}(\epsilon_1, R; \epsilon_2, R)$

is a quasi-isomorphism.

Proof of Lemma 6.18. The DGA $A^{(m)}(v'_R)$ is nothing but a trivial bordered Legendrian described in Section 6.1.1. The generators are

\[
\left\{ k^{ij}_{ab} \mid 1 \leq a \leq r, 1 \leq i < j \leq m \right\} \bigcup \left\{ k_{ab}^{1j} \mid 1 \leq a < b \leq r, 1 \leq i, j \leq m \right\},
\]

with grading $|k^{ij}_{ab}| := \mu(a) - \mu(b) - 1$, and the differential

$k_{ab}^{(m)} (k^{ij}_{ab}) = \sum_{a \leq c \leq b} (-1)^{|k_{ac}^{ij}|} k^{ij}_{ac} k^{kj}_{cb}.$

Here we use the convention on generators in Assumption 6.2.

Now consider the DGA map

$\phi^{(m)}_{R} : A^{(m)}(v'_R) \to A^{(m)}(v').$

By a direct counting of polygons in Figure 19, we have

$\phi^{(m)}_{R}(k^{ij}_{ab}) = v_{a,b-a}^{i,j}, \quad \phi^{(m)}_{R}(k^{ij}_{ab}) = c^{i,j}_{a,b}.$

Then by the construction below Proposition 4.5, we obtain

$\tau_{R,1}' = (\tau_{R}^i)_1 : \text{Hom}_{\mathcal{A}_{\text{Aug}}(v')}(\epsilon_1, \epsilon_2) \to \text{Hom}_{\mathcal{A}_{\text{Aug}}(v'_R)}(\epsilon_1, R; \epsilon_2, R);$

$c^{12v}_{a,b} \mapsto k^{12v}_{a,b}$

for $1 \leq a \leq b \leq r$. By comparing (6.3) and (6.29), we conclude that $\tau_{R,1}'$ is a (strict) isomorphism between two chain complexes, in particular, a quasi-isomorphism. \hfill \Box

As a second step, let us show

Lemma 6.19. There is a $A_\infty$ quasi equivalence between $\mathcal{A}_{\text{Aug}}(v; \mathbb{K})$ and $\mathcal{A}_{\text{Aug}}(v'; \mathbb{K})$. 
Proof of Lemma 6.19. Let us denote the intermediate Lagrangian projections between $\nu^m$ and $\nu^n$ as in Figure 20. Even though the illustration is given when $r = 3$ and $m = 4$, the similar works for arbitrary $r$ and $m$. Note that each of them is in $BLG^{(m)}_{\text{Log}}$, so it makes sense to consider the corresponding consistent DGAs.

Each pair of consecutive consistent Lagrangian projections is related by a sequence of consistent moves as depicted in Figure 20. This implies that there is a zig-zag of stabilizations between $A^\bullet(v)$ and $A^\bullet(v')$. Moreover, we already argued that $\text{Aug}_+(v')$ is strictly unital. Then by Proposition 4.13, we conclude that $\text{Aug}_+(v; \mathbb{R})$ and $\text{Aug}_+(v'; \mathbb{R})$ are $A_\infty$ quasi equivalent to each other which proves Lemma 6.19.

Note that the consistent Reidemeister moves in Figure 20 fix the right border. So Lemma 6.19 and the fact that $\tau_R : \text{Aug}_+(v') \rightarrow \text{Aug}_+(v'_R)$ is an $A_\infty$ equivalence imply that

$$\tau_R : \text{Aug}_+(v) \rightarrow \text{Aug}_+(v_R)$$

is also a $A_\infty$ quasi equivalence. This gives a geometric proof of Lemma 6.14.

6.2. Sheaf property of augmentation categories. Similar to [21], for the purpose of proving “augmentations are sheaves” for Legendrian graphs, or more generally, bordered Legendrian graphs, we will need a sheaf property of augmentation categories, which we now explain.

Let $C$ be a constructible sheaf of (homotopically) unital $A_\infty$-categories on an interval $(x_L, x_R)$, with respect to a stratification $X$ consisting of zero dimensional strata $x_i$ and one-dimensional strata $u_{i:i+1} = (x_i, x_{i+1})$. Equivalently, the generization maps induce, for each $i$, a diagram of unital $A_\infty$-categories and (homotopically) unital $A_\infty$-functors:

$$C(u_{i-1:i}) \xleftarrow{\theta_L} C_{x_i} \xrightarrow{\theta_R} C(u_{i:i+1}).$$

It then follows from the sheaf axiom that, if $x_i < x_{i+1} < \ldots < x_j$ are the zero dimensional strata in the interval $(a, b) \subset (x_L, x_R)$, then

$$C((a, b)) \cong C_{x_i} \times_{C(u_{i:i+1})} C_{x_{i+1}} \times \ldots \times C_{x_j},$$

is a fiber product. Here, the objects are tuples $(\xi_i, \xi_{i+1}, \ldots, \xi_j; f_{i:i+1}, \ldots, f_{j-j-1})$ where $\xi_k \in C_{x_k}$ and $f_{k:k+1} : \eta_k(\xi_k) \rightarrow \eta_k(\xi_{k+1})$ is an isomorphism in $C$, i.e. a closed degree 0 morphism in $\text{Hom}_{C(u_{k:k+1})}(\eta_k(\xi_k), \eta_k(\xi_{k+1}))$, whose cohomology is invertible by passing to the cohomological category of $C(u_{k:k+1})$. 

![Figure 20. A sequence of consistent Reidemeister moves between $\nu^m$ and $\nu^n$.](image-url)
On the other hand, there is a full subcategory of this fiber product, called *strict fiber product*:

\[(C_x \times_{C(u_{i,j})} C_{x_{i+1}} \times \ldots \times C_{x_j})_{\text{strict}},\]

in which all the \(f_{k,k+1}\)'s are identity morphisms, that is, \(\varrho_R(\xi_k) = \varrho_R(\xi_{k+1})\). Recall that:

**Lemma 6.20** ([21, Lem.7.4]). If all \(\varrho_k\)'s satisfy the isomorphism lifting property: any isomorphism \(\phi : \varrho_k(\xi) \sim \eta\) is the image under \(\varrho_k\) of some isomorphism \(\psi : \xi \sim \xi'\). Then the inclusion of the strict fiber product in the actual fiber product is an \(A_{\infty}\)-equivalence.

In our case for augmentation categories, we have

**Lemma 6.21.** In the front projection picture, let \((T, \mu)\) be any of the following elementary bordered Legendrian graph: \(n\) parallel strands, a single crossing, a single left cusp, a single marked right cusp, or a single vertex as in Example 6.1.2. Then \(\text{Aug}_+(T_1; \mathbb{K}) \xrightarrow{\text{Lem.7.4}} \text{Aug}_+(T; \mathbb{K})\) satisfies the isomorphism lifting property as in Lemma 6.20.

**Proof.** The smooth cases are covered in [21, §7.2]. The only remaining case is when \(T = V\) is a single vertex as in Example 6.1.2. By Lemma 6.14, the restriction functor \(\nu_l\) is just the obvious projection \(\nu_l : \text{Aug}_+(V_1; \mathbb{K}) \times \text{Aug}_+(V_1; \mathbb{K}) \to \text{Aug}_+(V_1; \mathbb{K})\), which clearly satisfies the isomorphism lifting property. □

As a consequence, we obtain:

**Proposition 6.22.** Let \((T, \mu)\) be any bordered Legendrian graph in \(I^1 I_x\) with \(I_x = (x_L, x_R)\), such that the \(x\)-coordinates of the singularities in its front projection are all distinct, denoted by \(x_1 < x_2 < \ldots < x_N\). Assume each right cusp is marked, and each vertex has no left half-edges. Denote \(x_0 := x_L\) and \(x_{N+1} := x_R\), and \(u_{k,k+1} = (x_k, x_{k+1})\). Define a constructible sheaf \(C\) of unital \(A_{\infty}\)-categories on \(I_x\) via:

\[
(C(u_{x_{k-1,k}}) \leftarrow \varrho_l C_{x_k} \xrightarrow{\varrho_R} C(u_{k,k+1}))
\]

Then we have the following equivalences between the rows of diagrams of unital \(A_{\infty}\)-categories:

\[
\text{Aug}_+(T_1; \mathbb{K}) \xleftarrow{\nu_l} \text{Aug}_+(T; \mathbb{K}) \xrightarrow{\nu_R} \text{Aug}_+(T_1; \mathbb{K})
\]

\[
C(u_0) \xleftarrow{\varrho_l} (C_{x_1} \times_{C(u_1x_2)} C_{x_2} \times \ldots \times C_{x_N})_{\text{strict}} \xrightarrow{\varrho_R} C(u_N, N+1)
\]

In particular, the augmentation categories \{\text{Aug}_+(T_{[a,b]}; \mathbb{K})\}_{(a,b) \subset I_x} form a sheaf.

**Proof.** Similar to [21, §7.2], the identification between the first two rows follows directly from the definition of diagrams of augmentation categories, and the co-sheaf property of LCH DGAs associated to bordered Legendrian graphs. It remains to show the middle-lower inclusion is an equivalence, but this is done by Lemma 6.21 and Lemma 6.20 above. □

6.3. **Augmentations are sheaves.** Let us start by recalling the result and strategy from [21].

**Theorem 6.23.** [21] Let \(A \subset \mathbb{R}^3\) be a Legendrian knot, and let \(\mathbb{K}\) be a field. Then there is an \(A_{\infty}\) equivalence between two categories

\[
\text{Aug}_+(A; \mathbb{K}) \xrightarrow{\sim} C_1(A; \mathbb{K}).
\]

The first step is to cut the front \(\pi_0(A) \subset \mathbb{R}^2\) diagram into elementary pieces, bordered Legendrians in \((0, 1) \times \mathbb{R}\), only having a cusp, a crossing, or a basepoint. Next show the equivalence for each pieces, and then argue the sheaf property in both sides to conclude the global equivalence.

Among many ingredients in the proof, let us recall the equivalence when a bordered Legendrian is \(I_n\), the trivial \(n\)-strands.
Proposition 6.24. [21] Let \( I_n \subset (0, 1) \times \mathbb{R}_+ \) be a trivial tangle with \( n \)-strands, and let \( \mathbb{K} \) be a field. Then there are DG equivalences:

\[
\text{Aug}_*(I_n; \mathbb{K}) \xrightarrow{\sim} \text{MC}(I_n; \mathbb{K}), \quad \text{MC}(I_n; \mathbb{K}) \xrightarrow{\sim} C_1(I_n; \mathbb{K}).
\]

The first equivalence has been described in Lemma 6.6. The second equivalence can be described as follows:

Recall that the strands of \( I_n \) are labelled from top to bottom by \( 1, 2, \ldots, n \), and \( C = C(I_n) := \oplus_{i=1}^n \mathbb{K} \cdot e_i \), with \( |e_i| = -\mu(i) \). \( C \) is equipped with a filtration \( F^* : F^i = F^i C := \oplus_{j>i} \mathbb{K} \cdot e_j \). In particular, \( F^0 = C \) and \( F^n = 0 \). In addition, \( S_n \) is the regular cell complex induced by \( I_n \), and by Definition 5.18, \( \mathcal{R}(S) \) is the \( A_{n+1} \)-quiver:

\[
n \rightarrow n - 1 \rightarrow \ldots \rightarrow 0
\]

where the node \( i \) corresponds to the region immediately below the strand \( i \) for \( 1 \leq i \leq n \), and \( 0 \) corresponds to the upper region. Recall that the indecomposable projectives of the quiver \( A_{n+1} \) are:

\[
P_i := (0 \rightarrow \ldots \rightarrow \mathbb{K} \xrightarrow{id} \mathbb{K} \xrightarrow{id} \ldots \xrightarrow{id} \mathbb{K} \rightarrow 0 \rightarrow \ldots \rightarrow 0),
\]

i.e. \( P_i \) consists of a copy of \( \mathbb{K} \) at all nodes \( k \leq i \). The indecomposables of \( A_{n+1} \) are:

\[
S_{i,j} := P_j/P_{j-1} = (0 \rightarrow \ldots \rightarrow 0 \rightarrow \mathbb{K} \xrightarrow{id} \mathbb{K} \xrightarrow{id} \ldots \xrightarrow{id} \mathbb{K} \rightarrow 0 \rightarrow \ldots \rightarrow 0),
\]

i.e. \( S_{i,j} \) consists of a copy of \( \mathbb{K} \) at all nodes \( k \) for \( i \leq k \leq j \). In particular, \( S_{i,j}' := (P_{i-1} \hookrightarrow P_j) \twoheadrightarrow S_{i,j} \) is a projective resolution of \( S_{i,j} \). Moreover, the path algebra \( \mathbb{K}\langle A_{n+1} \rangle \) of \( A_{n+1} \) is of global cohomological dimension one, hence every sub-module of a projective for \( \mathbb{K}\langle A_{n+1} \rangle \) is still projective. It then follows that objects in the DG category \( \mathcal{F}un(A_{n+1}, \mathbb{K}) \) (Definition 5.8) split, hence quasi-isomorphic to a direct sum of \( S_{i,j}' \)'s. As an immediate consequence, we obtain:

Lemma 6.25. [21, Lem.7.38] Every representation \( R \) in \( \mathcal{F}un(A_{n+1}, \mathbb{K}) \) is quasi-isomorphic to a representation \( R' \):

\[
R'_n \hookrightarrow R'_{n-1} \hookrightarrow \ldots \hookrightarrow R'_0
\]

such that, by denoting \( R'_{-1} := 0 \), we have:

1. The maps \( R'_{i+1} \hookrightarrow R'_i \) are injective;
2. The quiver representation \( R' \) in each cohomological degree is projective;
3. The differential on each \( R'_i/R'_{i+1} \) is zero.

There is a natural DG equivalence \( \mathcal{R} : \text{MC}(I_n; \mathbb{K}) \xrightarrow{\sim} \mathcal{F}un_{I_n}(\mathcal{R}(S) = A_{n+1}, \mathbb{K}) \), such that, on objects, we have:

\[
\mathcal{R}(C, d) := ((F^n, d|_{F^n}) \hookrightarrow (F^{n-1}, d|_{F^{n-1}}) \hookrightarrow \ldots \hookrightarrow (F^0, d)),
\]

and \( \mathcal{R} \) sends Hom complexes in \( \text{MC}(I_n; \mathbb{K}) \) literally to the corresponding identical Hom complexes in \( \mathcal{F}un_{I_n}(A_{n+1}, \mathbb{K}) \). It follows immediately that \( \mathcal{R} \) is fully faithful. The essential surjectivity of \( \mathcal{R} \) is a direct corollary of Lemma 6.25 above. Now, the second equivalence in Proposition 6.24 is just the composition of the equivalence in Corollary 5.38 with \( \mathcal{R} \).

Finally, we come to our main theorem:

Theorem 6.26 (Augmentations are sheaves). Let \( \mathcal{T} \) be a bordered Legendrian graph, and let \( \mathbb{K} \) be a field. Then there is an \( A_{\infty} \)-equivalence:

\[
\text{Aug}_*(\mathcal{T}; \mathbb{K}) \xrightarrow{\sim} C_1(\mathcal{T}; \mathbb{K}).
\]

Proof. Let us recall the invariance property Theorem 4.19, Corollary 5.36 of the both sides, and sheaf property Proposition 6.22 of \( \text{Aug}_*(\mathcal{T} : \mathbb{K}) \). Thus it suffices to check the equivalence for a new type of elementary pieces \( \mathcal{V} = (V_t \xrightarrow{u_t} V \xleftarrow{m} V_b) \) in \( T^{\infty,-}(I_s \times \mathbb{R}_+^2) \), a bordered Legendrian with one vertex \( v \) of type \((0, m)\) and trivial \( k \)-strands as follows:
As in Section 5.3, let $S_V$ be the Whitney stratification of $I_x \times \mathbb{R}_z$ induced by $V$. Let $S$ be the regular cell complex for $I_x \times \mathbb{R}_z$ refining $S_V$, by adding one left half-edge at $v$. Clearly, $S$ satisfies Assumption 5.17. Then by Lemma 6.14 and Corollary 5.38, it suffices to show the equivalence between the diagram of Morse complex categories

$$\mathcal{MC} := (\mathcal{MC}(V_i; \mathbb{K}) \overset{p_i}{\leftarrow} \mathcal{MC}(V_i; \mathbb{K}) \times \mathcal{MC}(V_R; \mathbb{K}) \overset{\iota}{\rightarrow} \mathcal{MC}(V_R; \mathbb{K}))$$

and $\mathcal{Fun}_{V,1}(\mathcal{R}(S), \mathbb{K})$.

Notice that, $\mathcal{R}(S|_{V_i}) = A_{n+1}$ with $n = k + m$. And,

$$\mathcal{Fun}_{V,1}(\mathcal{R}(S), \mathbb{K}) \hookrightarrow \mathcal{Fun}_{V,1}(\mathcal{R}(S|_{V_i}), \mathbb{K})$$

is just the fully faithful embedding into the full DG subcategory of $\mathcal{Fun}_{V,1}(A_{n+1}, \mathbb{K})$, consisting of functors $F$ such that $F(k + m \rightarrow k)$ is a quasi-isomorphism.

On the other hand, by the discussion above, we have the DG equivalence $\mathcal{R} : \mathcal{MC}(V_R; \mathbb{K}) \rightarrow \mathcal{Fun}_{V,1}(A_{n+1}, \mathbb{K})$. Notice that the image of $i$ of $\mathcal{MC}(V_i; \mathbb{K}) \times \mathcal{MC}(V_R; \mathbb{K})$ consists of Morse complexes $(C(V_R), d) = (C(V_i), d_L) \oplus (C(v_R), d_n)$, such that each summand is acyclic. Then, $\mathcal{R} \circ i$ induces a fully faithful DG functor

$$\mathcal{R}_V : \mathcal{MC}(V_i; \mathbb{K}) \times \mathcal{MC}(V_R; \mathbb{K}) \hookrightarrow \mathcal{Fun}_{V,1}(\mathcal{R}(S), \mathbb{K}).$$

$\mathcal{R}_V$ is also essentially surjective. In fact, $\mathcal{R}$ is essentially surjective implies that, any functor $F \in \mathcal{Fun}_{V,1}(\mathcal{R}(S), \mathbb{K})$, is quasi-isomorphic to $\mathcal{R}(C(V_R), d)$, for some Morse complex $(C(V_R), d)$, which we can assume to be canonical (or Barannikov’s normal form). In addition, the subquotient $(C(v_R), d_{v_R})$ of $(C(V_R), d)$ is acyclic, since $F(k + m \rightarrow k)$ is a quasi-isomorphism. It follows that $(C(V_R), d) = (C(V_i), d_L) \oplus (C(v_R), d_n)$ for some canonical Morse complexes $(C(V_i), d_L)$ and $(C(v_R), d_n)$. We are done.

Finally, it is direct to check that $\mathcal{R}_V$ commutes with the DG equivalence $\mathcal{R}_L : \mathcal{MC}(V_i; \mathbb{K}) \rightarrow \mathcal{Fun}_{V,1}(\mathcal{R}(S|_{V_i}), \mathbb{K})$ defined as in the discussion above, up to a specified natural isomorphism. Thus, we obtain a DG equivalence $\mathcal{R}_V : \mathcal{MC} \rightarrow \mathcal{Fun}_{V,1}(\mathcal{R}(S), \mathbb{K})$, as desired. This finishes the proof of Theorem 6.26. 

6.4 Example. Let us consider a Legendrian graph $\Lambda$ having the following front with Maslov potential and Lagrangian projection. Note that $\Lambda$ has one vertex $v$ and two (oriented) edges $e_1, e_2$:

![Figure 21. The front and Lagrangian projection of $\Lambda$](image)
6.4.1. Augmentation category. Fix the base field \( \mathbb{K} = \mathbb{Z}/2\mathbb{Z} \). Firstly, let us consider the augmentation category \( \text{Aug}_* = \text{Aug}_*(\Lambda; \mathbb{K}) \). We consider CE DGA \( \Lambda(\Lambda) \) with \( \mathbb{Z}/2\mathbb{Z}[t_1^{2*}, t_2^{2*}] \)-coefficient. Here, \( t_i \) corresponds to the basepoint on the edge \( e_i \) for \( i = 1, 2 \) in Figure 21. Then the DGA \((\Lambda, \cdot, \partial)\) is
\[
A = \mathbb{Z}/2\mathbb{Z}[t_1^{2*}, t_2^{2*}]\langle G \rangle;
\]
\[
G = \{a, b, c \mid i \in \mathbb{Z}/4\mathbb{Z}, \ell \in \mathbb{N}\},
\]
and the grading is given by
\[
|a| = |b| = 1, \quad |c| = 0, \quad |v_{i, \ell}| = \mu(i) - \mu(i + \ell) + N(v; i, \ell) - 1.
\]
The differential on the crossing generators are as follows:
\[
\partial a = t_1 + v_{1, 2} + c v_{2, 1};
\]
\[
\partial b = t_2 + v_{2, 2} + v_{1, 3}^{-1}(v_{1, 3} + c v_{2, 2} + a v_{3, 1});
\]
\[
\partial c = v_{1, 1}.
\]
The following are the list of degree zero generators:
\[
\{a, b, \chi, v_{1, 2}, v_{1, 3}, v_{2, 1}, v_{2, 2}, v_{3, 2}, v_{3, 3}, v_{4, 1}, v_{4, 2} \}.
\]
Among infinitely many equation of differential on vertex generators, let us list relations coming from the differential of degree one generators that augmentation \( \tilde{g} \) of a generator \( g \) should satisfy:
\[
\begin{align*}
\frac{v_{1, 2} v_{1, 3}}{v_{1, 2}} + \frac{v_{1, 3} v_{4, 1}}{1} &= 1, \\
\frac{v_{2, 1} v_{1, 3}}{v_{2, 2}} + \frac{v_{2, 2} v_{4, 2}}{1} &= 1, \\
\frac{v_{3, 2} v_{1, 3}}{v_{3, 3}} + \frac{v_{3, 3} v_{2, 2}}{1} &= 1, \\
\frac{v_{4, 1} v_{1, 3}}{v_{4, 2}} + \frac{v_{4, 2} v_{2, 2}}{1} &= 1;
\end{align*}
\]
Then by direct computation, we can check that there are eight possible \((\mathbb{Z}/2\mathbb{Z}\text{-valued})\) augmentations:
\[
\begin{array}{cccccccc}
i & e_i(c) & e_i(v_{1, 2}) & e_i(v_{1, 3}) & e_i(v_{2, 1}) & e_i(v_{2, 2}) & e_i(v_{3, 2}) & e_i(v_{3, 3}) & e_i(v_{4, 1}) & e_i(v_{4, 2}) \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
2 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
3 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
4 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
5 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
6 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
7 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
8 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{array}
\]
Obviously \( e(t_1) = e(t_2) = 1 \).

Remark 6.27. Note that there are only two equivalence classes of augmentations up to isomorphisms. One can check that
\[
\epsilon_1 \sim \epsilon_3 \sim \epsilon_4 \sim \epsilon_8, \quad \epsilon_2 \sim \epsilon_5 \sim \epsilon_6 \sim \epsilon_7.
\]
In the rest of example, we mainly consider two non-equivalent augmentations \( \epsilon_1 \) and \( \epsilon_2 \). Moreover, two (equivalence classes of) augmentations corresponds to two possible resolutions at vertex \( v \)
\[
\epsilon_1 \in \text{Aug}_{d_1}(\Lambda), \quad \epsilon_2 \in \text{Aug}_{d_2}(\Lambda),
\]
where \( \rho^1_v = \{\{1, 4\}, \{2, 3\}\} \), and \( \rho^2_v = \{\{1, 3\}, \{2, 4\}\} \). See [3, §5] for the details.

Figure 22. Two copy of \( \Lambda \) in the Lagrangian projection
Now consider two copies $\Lambda^{(2)}$. The labeling of the crossing is given as in Section 3.2.3. Then the corresponding algebra $A^{(2)}$ is generated by vertex generators

$$\{v_{k,\ell}^m \mid m = 1, 2, k \in \mathbb{Z}/4\mathbb{Z}, \ell \in \mathbb{N}\},$$

pure Reeb chords

$$\{a^{mm}, b^{mm}, c^{mm} \mid m = 1, 2\},$$

and mixed Reeb chords

$$\{a^{12}, b^{12}, c^{12}, v_{1,1}^{12}, v_{4,1}^{12}, x_1^{12}, x_2^{12}, y_1^{12}, y_2^{12}\} \cup \{a^{21}, b^{21}, c^{21}\}.$$

Here $v_{1,1}^{12}, v_{4,1}^{12}$ are the crossings near two-copies of vertex, see (3.4) for the label convention near $m$-copy of the vertex, and the crossings $x_i^{12}, y_i^{12}$ arise from the right- and left cusps, respectively. The grading for the generator is straight forward from the DGA $A(\Lambda)$ except the following

$$|x_i^{12}| = 0, \quad |y_i^{12}| = -1 \quad \text{for } i = a, b,$$

$$|v_{1,1}^{12}| = -1, \quad |v_{4,1}^{12}| = 0.$$

Now let us consider the mixed chords in $\Lambda^{(2)}$, especially from the second to the first copy. Note that there are such Reeb chords $a^{12}, b^{12}$ of degree 1, $c^{12}, v_{4,1}^{12}, x_1^{12}, x_2^{12}$ of degree 0, and $y_1^{12}, y_2^{12}, v_{1,1}^{12}$ of degree $-1$.

For convenience of the computation, let us restrict the differential to the following submodule

$$M_0^{(2)} := (M_1^{(1)})^\otimes \cong M^{12} \otimes (M_0^{(2)})^\otimes,$$

where $n_i$ is a nonnegative integer, and $M_0^{(i)}$ denotes the submodule spanned by degree 0 elements for $i = 1, 2$.

Then the restriction of the differential becomes

$$\partial a^{12}_{M_0^{(2)}} = x_1^{12} + c^{12} + v_{4,1}^{12} + c^{12} v_{2,1}^{12} + v_{1,1}^{12} v_{2,1}^{12};$$

$$\partial b^{12}_{M_0^{(2)}} = x_2^{12} + v_{2,1}^{12} (t_1^{1})^{-1} c^{12} v_{2,2}^{12} + (v_{2,1}^{12} v_{2,2}^{12} + v_{2,2}^{12} v_{4,1}^{12} (t_1^{2})^{-1}) (c^{22} v_{2,2}^{22} + v_{1,3}^{22});$$

$$\partial c^{12}_{M_0^{(2)}} = y_1^{12} + c^{12} v_{1,1}^{12} + v_{1,1}^{12} v_{2,1}^{12};$$

$$\partial v_{4,1}^{12} = v_{4,1}^{12} + v_{4,2}^{12} + v_{4,1}^{12} v_{2,2}^{12} + v_{4,1}^{12} v_{1,2}^{12};$$

$$\partial x_1^{12}_{M_0^{(2)}} = t_1^{1} (v_{3,2}^{12} v_{2,1}^{12} + v_{3,2}^{12} v_{4,1}^{12} v_{2,1}^{12} + v_{3,3}^{12} v_{2,2}^{12} + v_{3,3}^{12} v_{1,2}^{12} v_{2,2}^{12} + v_{3,4}^{12} v_{1,2}^{12} v_{1,3}^{12}) (t_1^{2})^{-1} + y_1^{12};$$

$$\partial x_2^{12}_{M_0^{(2)}} = t_2^{1} (v_{4,1}^{12} v_{1,1}^{12} v_{2,2}^{12} + v_{4,2}^{12} v_{2,2}^{12} + v_{4,1}^{12} v_{1,2}^{12} v_{1,3}^{12}) (t_2^{2})^{-1} + y_2^{12} + y_1^{12};$$

$$\partial y_{1}^{12}_{M_0^{(2)}} = 0;$$

$$\partial y_{2}^{12}_{M_0^{(2)}} = 0;$$

Note that the above restriction only contribute the bi-linearized differentials. For example, we have the following bi-linearized differential with respect to a pair $(\epsilon_1, \epsilon_1)$:

$$\partial_{\epsilon_1, \epsilon_1} a^{12} = x_1^{12} + c^{12} + v_{4,1}^{12};$$

$$\partial_{\epsilon_1, \epsilon_1} b^{12} = x_2^{12} + x_2^{12};$$

$$\partial_{\epsilon_1, \epsilon_1} c^{12} = y_1^{12} + y_2^{12} + v_{1,1}^{12};$$

$$\partial_{\epsilon_1, \epsilon_1} v_{4,1}^{12} = v_{1,1}^{12};$$

$$\partial_{\epsilon_1, \epsilon_1} y_{1}^{12} = 0;$$

For a pair $(\epsilon_1, \epsilon_2)$, we have

$$\partial_{\epsilon_1, \epsilon_2} a^{12} = x_1^{12} + v_{4,1}^{12};$$

$$\partial_{\epsilon_1, \epsilon_2} b^{12} = x_2^{12} + c^{12};$$

$$\partial_{\epsilon_1, \epsilon_2} c^{12} = y_2^{12} + v_{1,1}^{12};$$

$$\partial_{\epsilon_1, \epsilon_2} v_{4,1}^{12} = y_1^{12};$$

$$\partial_{\epsilon_1, \epsilon_2} y_{1}^{12} = 0;$$

$$\partial_{\epsilon_1, \epsilon_2} y_{2}^{12} = 0.$$
In a case of \((\epsilon_2, \epsilon_1)\), we deduce
\[
\begin{align*}
\partial_{\epsilon_2, \epsilon_1} a_1^{12} &= x_1^{12} + c_1^{12}, \\
\partial_{\epsilon_2, \epsilon_1} b_1^{12} &= x_2^{12} + v_{4,1}^{12}, \\
\partial_{\epsilon_2, \epsilon_1} y_1^{12} &= 0, \\
\partial_{\epsilon_2, \epsilon_1} y_2^{12} &= 0, \\
\partial_{\epsilon_2, \epsilon_1} x_1^{12} &= v_{1,1}^{12} + y_1^{12}, \\
\partial_{\epsilon_2, \epsilon_1} x_2^{12} &= v_2^{12}.
\end{align*}
\]

When a pair is \((\epsilon_2, \epsilon_2)\), then
\[
\begin{align*}
\partial_{\epsilon_2, \epsilon_2} a_1^{12} &= x_1^{12}, \\
\partial_{\epsilon_2, \epsilon_2} b_1^{12} &= x_2^{12}, \\
\partial_{\epsilon_2, \epsilon_2} c_1^{12} &= v_{1,1}^{12},
\end{align*}
\]
with all other differentials are trivial.

For any pair of augmentation \((\epsilon_i, \epsilon_j)\), we have
\[
\begin{align*}
\text{Hom}_{A_{\text{Aug}, i}}(\epsilon_i, \epsilon_j) &= \text{Hom}_{A_{\text{Aug}, i}}^0(\epsilon_i, \epsilon_j) \oplus \text{Hom}_{A_{\text{Aug}, i}}^1(\epsilon_i, \epsilon_j) \oplus \text{Hom}_{A_{\text{Aug}, i}}^2(\epsilon_i, \epsilon_j); \\
\text{Hom}_{A_{\text{Aug}, i}}^0(\epsilon_i, \epsilon_j) &= \mathbb{Z}/2\mathbb{Z}(y_1^{12}, y_2^{12}, v_{1,1}^{12})^\vee; \\
\text{Hom}_{A_{\text{Aug}, i}}^1(\epsilon_i, \epsilon_j) &= \mathbb{Z}/2\mathbb{Z}(c_1^{12}, v_{4,1}^{12}, x_1^{12}, x_2^{12})^\vee; \\
\text{Hom}_{A_{\text{Aug}, i}}^2(\epsilon_i, \epsilon_j) &= \mathbb{Z}/2\mathbb{Z}(a_1^{12}, b_1^{12})^\vee.
\end{align*}
\]

By dualizing the bi-linearized differential \(\partial_{\epsilon_i, \epsilon_i}\), we have
\[
\begin{align*}
m_1^{\epsilon_i, \epsilon_i}(y_1^{12})^\vee &= (c_1^{12})^\vee + (x_1^{12})^\vee + (x_2^{12})^\vee, & m_1^{\epsilon_i, \epsilon_i}(x_1^{12})^\vee &= (a_1^{12})^\vee + (b_1^{12})^\vee, & m_1^{\epsilon_i, \epsilon_i}(a_1^{12})^\vee &= 0; \\
m_1^{\epsilon_i, \epsilon_i}(y_2^{12})^\vee &= (c_2^{12})^\vee + (x_2^{12})^\vee + (x_3^{12})^\vee, & m_1^{\epsilon_i, \epsilon_i}(x_2^{12})^\vee &= (b_2^{12})^\vee, & m_1^{\epsilon_i, \epsilon_i}(b_2^{12})^\vee &= 0; \\
m_1^{\epsilon_i, \epsilon_i}(v_{4,1}^{12})^\vee &= (c_2^{12})^\vee + (v_{4,1}^{12})^\vee, & m_1^{\epsilon_i, \epsilon_i}(v_{4,1}^{12})^\vee &= (a_2^{12})^\vee, & m_1^{\epsilon_i, \epsilon_i}(a_2^{12})^\vee &= 0.
\end{align*}
\]
The only non-trivial cohomology class with respect to \(m_1^{\epsilon_i, \epsilon_i}\) is \(\alpha := [(y_1^{12} + y_2^{12})^\vee] \in H^0\text{Hom}_{A_{\text{Aug}, i}}(\epsilon_i, \epsilon_i)\).

By the similar computation for \((\epsilon_1, \epsilon_2), (\epsilon_2, \epsilon_1),\) and \((\epsilon_2, \epsilon_2)\), we have the following non-trivial cohomology classes
\[
\begin{align*}
\beta := [(y_2^{12} + v_{1,1}^{12})^\vee] \in H^0\text{Hom}_{A_{\text{Aug}, i}}(\epsilon_2, \epsilon_1), & \quad \gamma := [(y_1^{12} + v_{1,1}^{12})^\vee] \in H^0\text{Hom}_{A_{\text{Aug}, i}}(\epsilon_2, \epsilon_1); \\
\delta_1 := [(y_2^{12})^\vee], & \quad \delta_2 := [(y_2^{12})^\vee] \in H^0\text{Hom}_{A_{\text{Aug}, i}}(\epsilon_2, \epsilon_2), & \quad \delta_3 := [(v_{4,1}^{12})^\vee] \in H^1\text{Hom}_{A_{\text{Aug}, i}}(\epsilon_2, \epsilon_2).
\end{align*}
\]

respectively.

For the \(A_{\infty}\) structure, especially \(m_2\), we consider three copy of \(\Lambda\) as in Figure 6.4.1. Again the labeling

\[\text{Figure 23. Three copy of } \Lambda\]

and grading convention for generators are as in Section 3.2.3. Since the DGA \(A(\Lambda^{(3)})\) is complicated, for simplicity, let us list the terms in the differential which are relevant to our construction of \(A_{\infty}\) structure. Especially, we only list term of type \(M_0^{(k)}\), where
\[
M_0^{(k+1)} := M_0^{(k)} \otimes M_{k+1}^{k+1} \otimes (M_{k+1}^{k+1} \otimes M_{n_k}^{n_k}).
\]

Here \(n_{k+1}\) is a nonnegative integer, and \(M_{k+1}^{k+1}\) denotes the submodule spanned by degree 0 elements.
Then the restriction of the differential becomes
\[
\partial a_{M^0}^{13} = c^{12}v^{22,23}_{2,2}x_{1,1}^{1,1};
\]
\[
\partial b_{M^0}^{13} = \left(v^{11}_{1,2}v^{12}_{1,1}(t_1^{1})^{-1} + v^{11}_{2,1}(t_1^{1})^{-1}x_{1,1}^{1,1}\right)\left(x_1^{23}c^{33}v^{22}_{1,2} + c^{23}v^{33}_{2,2} + x_1^{23}v^{1,1}_{1,1}\right);
\]
\[
\partial c_{M^0}^{13} = y_1^{1,1}c^{23} + c^{12}y_1^{1,1};
\]
\[
\partial v_{4,-1}^{13} = v^{12}_{4,2}v^{12,23}_{2,2}v^{1,1}x_{1,1}^{1,1} + v^{11}_{1,2}v^{12}_{1,1}v^{12}_{2,2}v^{23}_{4,-1}x_{1,1}^{1,1}\]
\[
+ v^{12}_{4,-1}\left(v^{22}_3v^{23}_{1,1} + v^{22}_3v^{23}_{1,1}\right)v^{33}_{2,1};
\]
\[
\partial x_1^{13} = y_1^{1,1}x_{1,1}^{1,1} + x_1^{1,1}x_{1,1}^{1,1}\left(v^{22}_3v^{23}_{1,1} + v^{22}_3v^{23}_{1,1}v^{23,23}_{1,1} + v^{22}_3v^{23,23}_{1,1}\right)(t_1^{1})^{-1};
\]
\[
+ x_1^{1,1}x_{1,1}^{1,1}\left(v^{22}_3v^{23}_{1,1} + v^{22}_3v^{23}_{1,1}v^{23,23}_{1,1} + v^{22}_3v^{23,23}_{1,1}\right)(t_1^{1})^{-1}v^{12}_{4,-1}\]
\[
\partial x_2^{13} = y_1^{1,1}x_{1,1}^{1,1} + x_1^{1,1}x_{1,1}^{1,1}\left(v^{22}_3v^{23}_{1,1} + v^{22}_3v^{23}_{1,1}v^{23,23}_{1,1} + v^{22}_3v^{23,23}_{1,1}\right)(t_1^{1})^{-1};
\]
\[
\partial v_{1,1}^{13} = v^{12}_{1,1}v^{12}_{2,2} + y_1^{1,1}x_{1,1}^{1,1};
\]
\[
\partial y_1^{13} = y_1^{1,1}x_{1,1}^{1,1};
\]
\[
\partial v_{2,2}^{13} = y_1^{1,1}v^{12}_{1,1}x_{1,1}^{1,1};
\]

If we choose the triple \((\epsilon_1, \epsilon_1, \epsilon_1)\), then \(\partial(\epsilon_1, \epsilon_1, \epsilon_1)\) induces a multiplication on cohomology classes
\[
\bar{\omega}_{\epsilon_1, \epsilon_1, \epsilon_1} : H^* \text{Hom}_{\text{Aug}}(\epsilon_1, \epsilon_1) \otimes H^* \text{Hom}_{\text{Aug}}(\epsilon_1, \epsilon_1) \to H^* \text{Hom}_{\text{Aug}}(\epsilon_1, \epsilon_1)
\]
is given by
\[
\bar{\omega}_{\epsilon_1, \epsilon_1, \epsilon_1}(\alpha \otimes \alpha) = \alpha.
\]

By the similar computation,
\[
\bar{\omega}_{\epsilon_1, \epsilon_1, \epsilon_1}(\beta \otimes \alpha) = \beta, \quad \bar{\omega}_{\epsilon_1, \epsilon_1, \epsilon_1}(\gamma \otimes \alpha) = \gamma, \quad \bar{\omega}_{\epsilon_1, \epsilon_1, \epsilon_1}(\alpha \otimes \gamma) = \gamma,
\]
and
\[
\bar{\omega}_{\epsilon_1, \epsilon_1, \epsilon_1}(\delta_1 \otimes \beta) = 0, \quad \bar{\omega}_{\epsilon_1, \epsilon_1, \epsilon_1}(\delta_2 \otimes \beta) = \beta, \quad \bar{\omega}_{\epsilon_1, \epsilon_1, \epsilon_1}(\delta_3 \otimes \beta) = 0;
\]
\[
\bar{\omega}_{\epsilon_1, \epsilon_1, \epsilon_1}(\gamma \otimes \delta_1) = \gamma, \quad \bar{\omega}_{\epsilon_1, \epsilon_1, \epsilon_1}(\gamma \otimes \delta_2) = 0, \quad \bar{\omega}_{\epsilon_1, \epsilon_1, \epsilon_1}(\gamma \otimes \delta_3) = 0.
\]

For the triple \((\epsilon_2, \epsilon_2, \epsilon_2)\), we deduce the following table of multiplication:

| \(\bar{\omega}_{\epsilon_2, \epsilon_2, \epsilon_2}\) | \(\delta_1\) | \(\delta_2\) | \(\delta_3\) |
|----------------|--------|--------|--------|
| \(\delta_1\)   |  \(\delta_1\) |  0     |  \(\delta_3\) |
| \(\delta_2\)   |  0     |  \(\delta_2\) |  0     |
| \(\delta_3\)   |  0     |  0     |  \(\delta_3\) |

For the higher multiplication \(m_k\), \(k \geq 3\), let us consider \((k + 1)\)-copy of \(\Lambda\), the induced differential, and its restriction to \(M^{0(k+1)}\). The followings are possible terms which may contribute to \(m_k\), \(k \geq 3\):
\[
\partial b^{14}_{M^0} = \left(v^{11,12}_{1,2}v^{12}_{1,1}(t_1^{1})^{-1} + v^{11}_{2,1}(t_1^{1})^{-1}x_{1,1}^{1,1}\right)\left(x_1^{23}c^{34,44}_{1,2} + x_1^{34,44} + x_1^{34,44}ight);
\]
\[
\partial v_{4,-1}^{14}_{M^0} = v^{12}_{4,2}v^{12,23}_{2,2}v^{1,1}x_{1,1}^{1,1} + v^{11}_{1,2}v^{12}_{1,1}v^{12}_{2,2}v^{23}_{4,-1}x_{1,1}^{1,1}\]
\[
+ v^{12}_{4,-1}\left(v^{23}_3v^{23}_{1,1} + v^{23}_3v^{23}_{1,1}\right)v^{34}_{2,1};
\]
\[
\partial x_1^{14}_{M^0} = x_1^{12,2}v^{12,23}_{2,1}x_{1,1}^{1,1} + v^{12}_{1,1}v^{12}_{2,2}v^{23}_{4,-1}x_{1,1}^{1,1}\]
\[
+ v^{12}_{4,-1}\left(v^{23}_3v^{23}_{1,1} + v^{23}_3v^{23}_{1,1}\right)v^{34}_{2,1};
\]
\[
\partial b^{1m}_{M^0} = \left(v^{11,12}_{1,2}v^{12}_{1,1}(t_1^{1})^{-1} + v^{11}_{2,1}(t_1^{1})^{-1}x_{1,1}^{1,1}\right)\left(x_1^{23}c^{34,44}_{1,2} + x_1^{34,44} + x_1^{34,44}\right);
\]
\[
\left(x_1^{23}c^{34,44}_{1,2} + x_1^{34,44} + x_1^{34,44}\right).
\]
For example, a term in $\partial b^{14}_{14\mid M^0}(4)$ has a chain level contribution
\[
\langle m_{1}^{12 \otimes \chi_{1}^{12 \otimes v_{12}^{12}}, b^{12}} \rangle = 1.
\]

6.4.2. Sheaf category. On the other hand, let us consider the sheaf category $C_{1}(\Lambda; \mathbb{K})$. By the discussion in Section 5.2 and 5.3, a sheaf $\mathcal{F} \in C_{1}(\Lambda; \mathbb{K}) \subseteq Sh_{\Lambda}(\mathbb{R}^{2}; \mathbb{K})$ of micro-local rank 1 whose singular support lies in $\Lambda$ can be identified with a representation of the following quiver diagram:

\begin{figure}[h]
\centering
\includegraphics[width=\linewidth]{figure24.png}
\caption{The front diagram of $\Lambda$, and the induced legible model.}
\end{figure}

More precisely, the representation of the above quiver diagram is determined by three lines

\[
\begin{array}{c}
\mathbb{K} \\
\uparrow \pi \\
\ell_{1} \\
\ell_{2} \\
\mathbb{K} \\
\end{array}
\hspace{2cm}
\begin{array}{c}
\mathbb{K} \\
\ell_{1} \\
\mathbb{K} \\
\ell_{2} \\
\mathbb{K} \\
\end{array}
\hspace{2cm}
\begin{array}{c}
\mathbb{K} \\
\ker \pi \\
\mathbb{K} \\
\end{array}
\]

\[\ell_{1} := \text{Image } \ell_{1} \hookrightarrow \mathbb{K}^{2}, \quad \ell_{2} := \text{Image } \ell_{2} \hookrightarrow \mathbb{K}^{2}, \quad \ell_{0} := \ker \pi \hookrightarrow \mathbb{K}^{2},\]

which are isomorphic to $\mathbb{P}^{1}(\mathbb{K})$ satisfying that $\pi \circ \ell_{i}$ is quasi-isomorphism for $i = 1, 2$, in other words,

\[\ell_{0} \neq [\ell_{1}], \quad \ell_{0} \neq [\ell_{2}].\]

There are two possible cases, $[\ell_{1}] \neq [\ell_{2}]$ and $[\ell_{1}] = [\ell_{2}]$, and let us denote the corresponding representation of quiver by $Q_{1}$ and $Q_{2}$, respectively. Note that $Q_{2} \cong Q_{2}^{1} \oplus Q_{2}^{2}$, where

\[Q_{2}^{1} := \begin{array}{ccc}
\mathbb{K} \\
\uparrow \\
\mathbb{K} \\
\mathbb{K} \\
\end{array}
\hspace{2cm}
\begin{array}{ccc}
\mathbb{K} \\
\uparrow \\
\mathbb{K} \\
\mathbb{K} \\
0 \\
\end{array}
\hspace{2cm}
\begin{array}{ccc}
\mathbb{K} \\
\uparrow \\
\mathbb{K} \\
\mathbb{K} \\
0 \\
\end{array}
\]

Firstly, let us list all indecomposable representations of the above quiver of type $D_{4}$ as follows:

\[P_{1} := \begin{array}{ccc}
\mathbb{K} \\
0 \\
\mathbb{K} \\
0 \\
\mathbb{K} \\
\end{array}
\hspace{2cm}
\begin{array}{ccc}
\mathbb{K} \\
0 \\
\mathbb{K} \\
0 \\
\mathbb{K} \\
\end{array}
\hspace{2cm}
\begin{array}{ccc}
\mathbb{K} \\
0 \\
\mathbb{K} \\
0 \\
\mathbb{K} \\
\end{array}
\]

6.4.3. Computation of differential. In the rest of example, we assume that our base field is $\mathbb{Z}/2\mathbb{Z}$. Recall for a vector space $V$ that $V[i]$ means the degree shift by $-i$.

Note that

\[\text{Hom}(P_{i}[k], P_{j}[\ell]) \cong \begin{cases} 
\mathbb{K}[\ell - k] & \text{if } (i, j) = (1, 2), (1, 3), (1, 4), (2, 4); \\
0 & \text{otherwise}.
\end{cases}\]

\[\text{We omit the degree shift notation } [\ell] \text{ when it is clear.}\]
We sometimes use $\mathbb{K}_{P, Q}$ instead of $\text{Hom}(P, Q)$ when it is non-trivial.

The quivers $Q_1, Q_1^1$ and $Q_2^1$ admit the following projective resolutions which will be denoted by $\widetilde{Q}_1, \widetilde{Q}_2^1$ and $\widetilde{Q}_2^2$, respectively:

\[
\begin{align*}
0 & \longrightarrow P_1 \overset{f}{\longrightarrow} P_3 \oplus P_4 \longrightarrow Q_1 \longrightarrow 0; \\
0 & \longrightarrow P_2 \overset{g}{\longrightarrow} P_3 \oplus P_4 \longrightarrow Q_2^1 \longrightarrow 0; \\
0 & \longrightarrow P_1 \overset{h}{\longrightarrow} P_2 \longrightarrow Q_2^2 \longrightarrow 0.
\end{align*}
\]

Here a matrix form of $f, g$ and $h$ are given by $\left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right)$, and $(1)$ with respect to a canonical choice of basis of each ordered summand. Then we compute that

\[
R \text{Hom}(Q_1, Q_1) = \text{Hom}(\widetilde{Q}_1, Q_1) \cong \left( \mathbb{K}_{P, Q_1} \oplus \mathbb{K}_{P, Q_1} \overset{(1, 1)}{\longrightarrow} \mathbb{K}_{P, Q_1}[-1] \right) \cong \mathbb{K}.
\]

Let $b_1$ and $b_2$ be the non-zero element of $\mathbb{K}_{P, Q_1}$ and $\mathbb{K}_{P, Q_1}$, then the only non-trivial homology class in $H^i \text{Hom}(\widetilde{Q}_1, Q_1)$ is $[b_1 + b_2]$.

Similarly we verify that

\[
\begin{align*}
\text{Hom}(\widetilde{Q}_2^1, Q_2^1) & \cong \left( \mathbb{K}^2 \to \mathbb{K}[-1] \right) \cong \mathbb{K}, \\
\text{Hom}(\widetilde{Q}_2^2, Q_2^1) & \cong \left( \mathbb{K}^2 \to \mathbb{K}[-1] \right) \cong 0,
\end{align*}
\]

which conclude

\[
R \text{Hom}(Q_2, Q_2) = \text{Hom}(\widetilde{Q}_2, Q_2) \cong \bigoplus_{i,j=1,2} \text{Hom}(\widetilde{Q}_2^i, Q_2^j) \cong \mathbb{K}^2 \oplus \mathbb{K}[-1].
\]

Also compute

\[
\begin{align*}
\text{Hom}(\widetilde{Q}_1, Q_1^1) & \cong \left( \mathbb{K}^2 \to \mathbb{K}[-1] \right) \cong \mathbb{K}, \\
\text{Hom}(\widetilde{Q}_1, Q_1^2) & \cong \left( \mathbb{K} \to 0[-1] \right) \cong 0;
\end{align*}
\]

which imply

\[
\begin{align*}
R \text{Hom}(Q_1, Q_1) & \cong \text{Hom}(\widetilde{Q}_1, Q_1^1) \oplus \text{Hom}(\widetilde{Q}_1, Q_1^2) \cong \mathbb{K}; \\
R \text{Hom}(Q_2, Q_1) & \cong \text{Hom}(\widetilde{Q}_2, Q_1) \oplus \text{Hom}(\widetilde{Q}_2, Q_1^1) \cong \mathbb{K}.
\end{align*}
\]

Note that the above computation coincide with the one in $\text{Aug}_*(\Lambda)$, see (6.30).

6.4.4. Computation of multiplication. Now consider the compositions between Hom spaces. Especially look at

\[
m : R \text{Hom}(Q_1, Q_1) \otimes R \text{Hom}(Q_2, Q_1) \to R \text{Hom}(Q_1, Q_1).
\]

The domain is isomorphic to $\text{Hom}(\widetilde{Q}_1, Q_1) \otimes \text{Hom}(\widetilde{Q}_2, \widetilde{Q}_1)$ and hence becomes the following:

\[
\left( \mathbb{K}_{P, Q_1} \oplus \mathbb{K}_{P, Q_1} \overset{(1, 1)}{\longrightarrow} \mathbb{K}_{P, Q_1}[-1] \right) \otimes \left( \mathbb{K}_{P, P_1} \otimes \mathbb{K}_{P, P_1} \otimes \mathbb{K}_{P, P_2} \overset{(1, 0)}{\longrightarrow} \mathbb{K}_{P, P_2}[-1] \oplus \mathbb{K}_{P, P_1}[-1] \right).
\]

Let $c_1, c_2$, and $c_3$ be the non-zero element in $\mathbb{K}_{P, P_1}, \mathbb{K}_{P, P_1}$, and $\mathbb{K}_{P, P_1}$, respectively, then $[c_1 + c_2 + c_3]$ be the unique non-trivial cohomology class in $H^i \text{Hom}(\widetilde{Q}_1, \widetilde{Q}_1)$.

Let us denote the domain of $m_2$ by $D^*$, then the degree $i$-th part $D^i$ becomes as follows for $i = 0, 1, 2$:

\[
\begin{align*}
D^0 & \cong (\mathbb{K}_{P, Q_1} \otimes \mathbb{K}_{P, P_1}) \oplus (\mathbb{K}_{P, Q_1} \otimes \mathbb{K}_{P, P_2}) \oplus (\mathbb{K}_{P, Q_1} \otimes \mathbb{K}_{P, P_1}) \oplus (\mathbb{K}_{P, Q_1} \otimes \mathbb{K}_{P, P_1}) ; \\
D^1 & \cong (\mathbb{K}_{P, Q_1} \otimes \mathbb{K}_{P, P_1}) \oplus (\mathbb{K}_{P, Q_1} \otimes \mathbb{K}_{P, P_2}) \oplus (\mathbb{K}_{P, Q_1} \otimes \mathbb{K}_{P, P_1}) \oplus (\mathbb{K}_{P, Q_1} \otimes \mathbb{K}_{P, P_1}) ; \\
D^2 & \cong (\mathbb{K}_{P, Q_1} \otimes \mathbb{K}_{P, P_1}) \oplus (\mathbb{K}_{P, Q_1} \otimes \mathbb{K}_{P, P_1})
\end{align*}
\]
The differential $d_i : D^i \to D^{i+1}$ for $i = 0, 1$ can be expressed by the following matrix form

$$d_0 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad d_1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{pmatrix},$$

with respect to basis $\{e^0_i\}_{i=1,\ldots,6}$, $\{e^1_i\}_{i=1,\ldots,7}$, $\{e^2_k\}_{k=1,2}$ for $D^0$, $D^1$, $D^2$. These are chosen by a canonical element in each ordered summand. Note that the only non-trivial cohomology class in the domain $D^*$ is $[e^1_1 + \cdots + e^1_6] \cong [b_1 + b_2] \otimes [c_1 + c_2 + c_3]$.

In the codomain of $m$, $\mathbb{K}_{P,Q_1} \otimes \mathbb{K}_{P,Q_1} \xrightarrow{(1,1)} \mathbb{K}_{P,Q_1}[-1]$, let $f_1$ and $f_2$ be the non-zero element in $\mathbb{K}_{P,Q_1}$ and $\mathbb{K}_{P,Q_1}$, respectively. Then $[f_1 + f_2]$ be the only non-trivial cohomology class.

The multiplication $m$ for each summand is defined by

$$m(\mathbb{K}_W \otimes \mathbb{K}_T) = \begin{cases} \mathbb{K}_S & \text{if } T = W; \\ 0 & \text{otherwise.} \end{cases}$$

and the induced multiplication

$$\overline{m} : H^* \text{Hom}(\overline{Q}, Q_1) \otimes H^* \text{Hom}(\overline{Q}, \overline{Q}_1) \to H^* \text{Hom}(\overline{Q}, Q_1)$$

sends $[e^1_1 + \cdots + e^1_6]$ to $[f_1 + f_2]$ which coincides with

$$\overline{m}_2 : H^* \text{Hom}(\epsilon_1, \epsilon_1) \otimes H^* \text{Hom}(\epsilon_1, \epsilon_1) \to H^* \text{Hom}(\epsilon_1, \epsilon_1).$$

One can also verify the other cases coincide with the computation of multiplication in $\text{Aug}_*(\Lambda)$.

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