RECTANGULAR HEFFTER ARRAYS: A REDUCTION THEOREM

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Abstract. Let $m, n, s, k$ be four integers such that $3 \leq s \leq n$, $3 \leq k \leq m$ and $ms = nk$. Set $d = \gcd(s, k)$. In this paper we show how one can construct a Heffter array $H(m, n; s, k)$ starting from a square Heffter array $H(nk/d; d)$ whose elements belong to $d$ consecutive diagonals. As an example of application of this method, we prove that there exists an integer $H(m, n; s, k)$ in each of the following cases: (i) $d \equiv 0 \pmod{4}$; (ii) $5 \leq d \equiv 1 \pmod{4}$ and $nk \equiv 3 \pmod{4}$; (iii) $d \equiv 2 \pmod{4}$ and $nk \equiv 0 \pmod{4}$; (iv) $d \equiv 3 \pmod{4}$ and $nk \equiv 0, 3 \pmod{4}$. The same method can be applied also for signed magic arrays $SMA(m, n; s, k)$ and for magic rectangles $MR(m, n; s, k)$. In fact, we prove that there exists an SMA($m, n; s, k$) when $d \geq 2$, and there exists an MR($m, n; s, k$) when either $d \geq 2$ is even or $d \geq 3$ and $nk$ are odd. We also provide constructions of integer Heffter arrays and signed magic arrays when $k$ is odd and $s \equiv 0 \pmod{4}$.

1. Introduction

Heffter arrays are partially filled (pf, for short) arrays introduced by Archdeacon in \cite{A}. 

Definition 1.1. A Heffter array $H(m, n; s, k)$ is an $m \times n$ pf array with elements in $(\mathbb{Z}_{2nk+1}, +)$ such that

(a) each row contains $s$ filled cells and each column contains $k$ filled cells;
(b) for every $x \in \mathbb{Z}_{2nk+1} \setminus \{0\}$, either $x$ or $-x$ appears in the array;
(c) the elements in every row and column sum to 0 (in $\mathbb{Z}_{2nk+1}$).

Trivial necessary conditions for the existence of an $H(m, n; s, k)$ are $ms = nk$, $3 \leq s \leq n$ and $3 \leq k \leq m$. Instead of working with elements of a finite cyclic group, one can work with integers.

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Definition 1.2. An integer Heffter array $H(m, n; s, k)$ is an $m \times n$ pf array with elements in \{±1, ±2, . . . , ±nk\} ⊂ Z such that

(a) each row contains $s$ filled cells and each column contains $k$ filled cells;
(b) no two entries agree in absolute value;
(c) the elements in every row and column sum to 0.

As shown in [1], an additional necessary condition for the existence of an integer $H(m, n; s, k)$ is that $nk \equiv 0, 3 \pmod{4}$. The study of these objects began with the square case (i.e., when $m = n$, and so $s = k$), and the tight case (i.e., when $m = k$ and $n = s$). In particular, in [6] it was proved that a square Heffter array $H(n, n; k, k)$ (that we simply denote by $H(n; k)$) exists for all $n \geq k \geq 3$, while by [3, 11] an integer Heffter array $H(n; k)$ exists if and only if $n \geq k \geq 3$ and $nk \equiv 0, 3 \pmod{4}$. On the other hand, by [2] an $H(m; n, n)$ exists for all $m, n \geq 3$, while an integer $H(m; n; n, 3)$ exists if and only if the additional condition $mn \equiv 0, 3 \pmod{4}$ holds. These results confirm the validity of the following conjecture, originally proposed by Archdeacon himself.

Conjecture 1.3. [1] Conjecture 6.3] Given four integers $m, n, s, k$ such that $3 \leq s \leq n$, $3 \leq k \leq m$ and $ms = nk$, there exists a Heffter array $H(m, n; s, k)$. If the additional condition $nk \equiv 0, 3 \pmod{4}$ holds, there exists an integer Heffter array $H(m; n; s, k)$.

The first aim of this paper is to show how it is possible to reduce the problem of the existence of a rectangular Heffter array to the square case where the elements belong to consecutive diagonals (see Theorem 3.3). This process, combined with [22, Theorem 1.3], allows us to prove the following.

Theorem 1.4. Let $m, n, s, k$ be four integers such that $3 \leq s \leq n$, $3 \leq k \leq m$ and $ms = nk$. Set $d = \gcd(s, k)$. There exists an integer $H(m, n; s, k)$ in each of the following cases:

1. $d \equiv 0 \pmod{4}$;
2. $d \equiv 1 \pmod{4}$ with $d \geq 5$ and $nk \equiv 3 \pmod{4}$;
3. $d \equiv 2 \pmod{4}$ and $nk \equiv 0 \pmod{4}$;
4. $d \equiv 3 \pmod{4}$ and $nk \equiv 0, 3 \pmod{4}$.

Thanks to this result, Conjecture 1.3 for the integer case remains open only when $d = 1$ or $5 \leq d \equiv 1 \pmod{4}$ and $nk \equiv 0 \pmod{4}$. In particular, one can assume $k$ odd: in fact, the transpose of an (integer) $H(m, n; s, k)$ is an (integer) $H(n, m; k, s)$. A further result in this direction is given in Section 4 where we construct integer Heffter arrays when $s \equiv 0 \pmod{4}$.

Theorem 1.5. Let $m, n, s, k$ be four integers such that $3 \leq s \leq n$, $3 \leq k \leq m$ and $ms = nk$. If $s \equiv 0 \pmod{4}$ and $k \not\equiv 5$ is odd, then there exists an integer Heffter array $H(m, n; s, k)$.

Our reduction theorem will be proved in the more general context of $\lambda$-fold relative Heffter arrays, objects introduced in [11]. One of the main reasons that justifies the study of Heffter arrays and of their generalizations is that they allow, under suitable conditions (or assuming the validity of [8, Conjecture 3]), to produce biembeddings of pairs of cyclic decompositions (one decomposition consisting of $s$-cycles and the other one consisting of $k$-cycles) of the complete multipartite multigraph $\lambda K_{\ell \times t}$, with $\ell$ parts each of size $t$, onto orientable surfaces (see [5, 7, 9, 11, 12, 13]). Partial results about the existence of these generalizations have been obtained in [10, 11, 12, 22, 23].

In Section 5 we provide a similar reduction theorem for magic rectangles and signed magic arrays.
**Definition 1.6.** A magic rectangle $\text{MR}(m; n; s, k)$ is an $m \times n$ pf array with elements in $\Omega = \{0, 1, \ldots, nk - 1\} \subset \mathbb{Z}$ such that

(a) each row contains $s$ filled cells and each column contains $k$ filled cells;
(b) every $x \in \Omega$ appears exactly once in the array;
(c) the sum of the elements in each row is a constant value $c_1$ and the sum of the elements in each column is a constant value $c_2$.

**Definition 1.7.** A signed magic array $\text{SMA}(m; n; s, k)$ is an $m \times n$ pf array with elements in $\Omega \subset \mathbb{Z}$, where $\Omega = \{0, \pm 1, \pm 2, \ldots\}$ if $nk$ is odd and $\Omega = \{\pm 1, \pm 2, \ldots, \pm nk/2\}$ if $nk$ is even, such that

(a) each row contains $s$ filled cells and each column contains $k$ filled cells;
(b) every $x \in \Omega$ appears exactly once in the array;
(c) the elements in every row and column sum to 0.

The existence problem for diagonal magic squares (that is, magic squares whose elements belong to consecutive diagonals) has been already solved in [19]; then our reduction theorem gives the following.

**Theorem 1.8.** Let $m, n, s, k$ be four integers such that $3 \leq s \leq n$, $3 \leq k \leq m$ and $ms = nk$. Set $d = \gcd(s, k)$ and suppose $d \geq 2$. If $d$ is even or $nk$ is odd, then there exists an $\text{MR}(m; n; s, k)$.

In Section 5 we also consider the existence of a diagonal square $\text{SMA}(n; k)$. In fact, some of the signed magic squares obtained by Khodkar, Schulz, and Wagner in [21] are not diagonal. So, we firstly construct a diagonal $\text{SMA}(n; 3)$ for all $n \geq 3$ such that $n \not\equiv 0 \pmod{4}$, a diagonal $\text{SMA}(n; 5)$ for all odd $n \geq 5$, and a diagonal $\text{SMA}(n; 6)$ for all $n \geq 6$. This allows us to prove that a diagonal $\text{SMA}(n; k)$ exists for all $n \geq k \geq 3$. Then, applying our reduction theorem we obtain the following result.

**Theorem 1.9.** Let $m, n, s, k$ be four integers such that $3 \leq s \leq n$, $3 \leq k \leq m$ and $ms = nk$. There exists an $\text{SMA}(m; n; s, k)$ whenever $\gcd(s, k) \geq 2$.

To conclude our paper, we construct signed magic arrays when $s \equiv 0 \pmod{4}$.

**Theorem 1.10.** Let $m, n, s, k$ be four integers such that $3 \leq s \leq n$, $3 \leq k \leq m$ and $ms = nk$. If $s \equiv 0 \pmod{4}$, then there exists an $\text{SMA}(m; n; s, k)$.

2. Notation

In this paper, the arithmetic on the row (respectively, on the column) indices is performed modulo $m$ (respectively, modulo $n$), where the set of reduced residues is $\{1, 2, \ldots, m\}$ (respectively, $\{1, 2, \ldots, n\}$), while the entries of the arrays are taken in $\mathbb{Z}$. Given two integers $a \leq b$, we denote by $[a, b]$ the interval consisting of the integers $a, a + 1, \ldots, b$. If $a > b$, then $[a, b]$ is empty.

We denote by $(i, j)$ the cell in the $i$-th row and $j$-th column of a pf array $A$. The skeleton of $A$ is the set of its filled positions: it will be denoted by $\text{skel}(A)$. The support of $A$, denoted by $\text{supp}(A)$, is defined to be the set of the absolute values of the elements contained in $A$, while $\mathcal{E}(A)$ denotes the list of the entries of the filled cells of $A$. We also write $\mathcal{E}(i, j)$ to indicate the entry of the cell $(i, j)$ of $A$. Given a sequence $S = (B_1, B_2, \ldots, B_r)$ of pf arrays, we set $\text{supp}(S) = \cup_i \text{supp}(B_i)$ and $\mathcal{E}(S) = \cup_i \mathcal{E}(B_i)$. 

Let \( A = (a_{i,j}) \) be a pf square array of size \( n \). We say that the element \( a_{i,j} \) belongs to the diagonal \( D_r \) if \( j - i \equiv r \pmod{n} \). Moreover, \( A \) is said to be cyclically \( b \)-diagonal if the nonempty cells of \( A \) are exactly those of \( b \) consecutive diagonals. For instance, the pf array \( A \) given in Figure 1 is an integer \( H(12;3) \) which is cyclically 3-diagonal. In fact, \( \text{skel}(A) = D_0 \cup D_1 \cup D_2 \).

**Definition 2.1.** A pf array with entries in \( \mathbb{Z} \) is said to be *shiftable* if every row and every column contains an equal number of positive and negative entries.

Let \( A \) be a shiftable pf array and \( x \) be a nonnegative integer. Let \( A \pm x \) be the (shiftable) pf array obtained by adding \( x \) to each positive entry of \( A \) and \(-x\) to each negative entry of \( A \). Observe that, since \( A \) is shiftable, the row and column sums of \( A \pm x \) are exactly the row and column sums of \( A \).

### 3. The reduction theorem

In this section we show how one can construct rectangular Heffter arrays starting from a square Heffter array having a particular shape. This result will be proved in the more general context of \( \lambda \)-fold relative Heffter arrays, see [11, 23].

**Definition 3.1.** Let \( m,n,s,k,t,\lambda \) be positive integers such that \( \lambda \) divides \( 2nk \) and \( t \) divides \( \frac{2nk}{\lambda} \). Let \( J \) be the subgroup of order \( t \) of \( \mathbb{Z}_v \), where \( v = \frac{2nk}{\lambda} + t \). A \( \lambda \)-fold Heffter array over \( \mathbb{Z}_v \) relative to \( J \), denoted by \( ^{\lambda}H_t(m,n; s,k) \), is an \( m \times n \) pf array \( A \) with elements in \( \Omega = \mathbb{Z}_v \setminus J \) such that:

(a) each row contains \( s \) filled cells and each column contains \( k \) filled cells;
(b) every element of \( \Omega \) appears exactly \( \lambda \) times in the list \( E(A) \cup -E(A) \);
(c) the elements in every row and column sum to 0.

Also for these arrays, there exists an ‘integer’ version.

**Definition 3.2.** Let \( m,n,s,k,t,\lambda \) be positive integers such that \( \lambda \) divides \( 2nk \) and \( t \) divides \( \frac{2nk}{\lambda} \). Let

\[
\Phi = \left\{ 1, 2, \ldots, \left\lfloor \frac{v}{2} \right\rfloor \right\} \setminus \left\{ \ell, 2\ell, \ldots, \left\lfloor \frac{t}{2} \right\rfloor \ell \right\} \subset \mathbb{Z}, \quad \text{where } v = \frac{2nk}{\lambda} + t \text{ and } \ell = \frac{v}{t}.
\]

**Figure 1.** An integer cyclically 3-diagonal \( H(12;3) \).
An integer $^λH_t(m,n;s,k)$ is an $m \times n$ pf array with elements in $\Phi$ such that:

(a) each row contains $s$ filled cells and each column contains $k$ filled cells;
(b) if $v$ is odd or if $t$ is even, every element of $\Phi$ appears, up to sign, exactly $λ$ times in the array; if $v$ is even and $t$ is odd, every element of $\Phi \setminus \{\frac{1}{2}\}$ appears, up to sign, exactly $λ$ times while $\frac{1}{2}$ appears, up to sign, exactly $\frac{1}{2}$ times;
(c) the elements in every row and column sum to 0.

Clearly, an (integer) $^1H_1(m,n;s,k)$ is nothing but an (integer) $H(m,n;s,k)$. In the following, we write $^λH_t(n;k)$ for $^λH_t(n,n;k,k)$. Also, we say that a $^λH_t(n;k)$ is diagonal if it is cyclically $k$-diagonal.

**Theorem 3.3.** Let $m,n,s,k$ be four integers such that $2 \leq s \leq n$, $2 \leq k \leq m$ and $ms = nk$. Let $λ$ be a divisor of $2nk$ and let $d$ be a divisor of $\frac{2nk}{λ}$. Set $d = \gcd(s,k)$. If there exists a (shift/integer) diagonal $^λH_t\left(\frac{2nk}{λ};d\right)$, then there exists a (shift/integer) $^λH_t(m,n;s,k)$.

**Proof.** Write $s = d\bar{s}$ and $k = d\bar{k}$. Since $\gcd(s,\bar{k}) = 1$, the equality $ms = nk$ implies that $\bar{k}$ divides $m$ and $\bar{s}$ divides $n$. Hence, we can write $m = \bar{c}\bar{k}$ and $n = \bar{s}\bar{c}$. Note that $c = \gcd(m,n)$. Let $A = (a_{i,j})$ be an (integer) diagonal $^λH_t(n;k;d)$ and suppose that the filled cells of $A$ belongs to the diagonals $D_0, D_1, \ldots, D_{d-1}$, where

$$D_i = \{(x,y) : 1 \leq x,y \leq nk, y-x \equiv i \pmod{\bar{c}\bar{k}}\}.$$

We define a function $ψ$ between $\skel(A)$ and the set of the cells of an empty array $B$ of size $m \times n$, as follows: for every $(i,j) \in \skel(A)$ with $i,j \in [1, nk]$, let $ψ((i,j)) = (u,v)$, where $u \in [1, m]$ and $v \in [1, n]$ are such that $u \equiv i \mod{m}$ and $v \equiv j \mod{n}$.

We prove that the function $ψ$ is injective. In fact, take two filled cells $(x_1,y_1) \in D_{i_1}$ and $(x_2,y_2) \in D_{i_2}$ of $A$, where $0 \leq i_1 \leq i_2 < d$. Suppose $ψ((x_1,y_1)) = ψ((x_2,y_2))$. Then $x_2 \equiv x_1 \mod{m}$ and $y_2 \equiv y_1 \mod{n}$ and hence $y_2 - x_2 \equiv y_1 - x_1 \mod{c}$. This implies $i_2 \equiv i_1 \mod{c}$ and so either $i_2 = i_1$ or $i_2 - i_1 \geq c = \frac{nc}{s} \geq d$, since $n \geq s$. As the second possibility is excluded by the hypothesis $i_2 - i_1 < d$, the two cells $(x_1,y_1), (x_2,y_2)$ belong to the same diagonal $D_{i_1}$. From $y_2 - x_2 \equiv y_1 - x_1 \mod{\bar{c}\bar{k}}$, we get $x_2 \equiv x_1 \mod{\bar{c}\bar{k}}$. Since $x_2 \equiv x_1 \mod{\bar{c}\bar{k}}$, we conclude that $x_2 \equiv x_1 \mod{\lcm(m,n)}$. Thus, from $\lcm(m,n) = \bar{s}\bar{c}\bar{k} = nk$ we get $(x_1,y_1) = (x_2,y_2)$.

Because of the injectivity of $ψ$, we can fill the cell $ψ((i,j))$ of $B$ with the entry $a_{i,j}$ of $A$, obtaining a bijection from $\skel(A)$ to $\skel(B)$. We now prove that the pf array $B$ so constructed is an (integer) $^λH_t(m,n;s,k)$. Clearly, $\mathcal{E}(A) = \mathcal{E}(B)$. Also, each row of $B$ contains $d \cdot \frac{ma}{m} = s$ filled cells and every column of $B$ contains $d \cdot \frac{mb}{m} = k$ filled cells. Finally, take the filled cells of a row of $A$: $(x_1,y_1), (x_2,y_1), \ldots, (x_d,y_1)$. Applying the function $ψ$, we obtain cells that belong to the same row $u$ of $B$. So, the elements of this row of $B$ are exactly the elements of $\bar{s}$ distinct rows of $A$. Since the elements of any row of $A$ sum to zero in $\mathbb{Z}_{\frac{nk}{λ}+t}$ (or in $\mathbb{Z}$), the corresponding elements of the row $u$ of $B$ sum to zero in $\mathbb{Z}_{\frac{nk}{λ}+t}$ (respectively, in $\mathbb{Z}$). The same for the columns.

This shows that $B$ is an (integer) $^λH_t(m,n;s,k)$. If we make the further assumption that $A$ is a shift/integer $^λH_t(n;k d)$, then $d$ is even and every row/column of $A$ has $\frac{k}{2}$ positive entries and $\frac{k}{2}$ negative entries. As remarked before, the elements of every row of $B$ are exactly the elements of $\bar{s}$ distinct rows of $A$. Hence, each row of $B$ has $\frac{s}{2}$ positive entries and $\frac{s}{2}$ negative entries. Similarly, each column of $B$ has $\frac{k}{2}$ positive entries and $\frac{k}{2}$ negative entries, proving that $B$ is a shift/integer $^λH_t(m,n;s,k)$.
For instance, we can take the integer diagonal $H(12; 3)$ of Figure 1, obtained by using Theorem 3.11. Following the proof of Theorem 3.3, we obtain the integer $H(6, 12; 6, 3)$ of Figure 2.

| -36 | 12 | 24 | 36 | -33 | 5 | 28 |
|-----|----|----|----|-----|--|----|
| -25 | -10 | 35 | 14 | -33 | -21 | 4 | 17 |
| -14 | -9 | 23 | 30 | -26 | -8 | 34 | -20 | 2 | 18 |
| -30 | -15 | -7 | 22 | -31 | -17 | -9 | 23 | -32 | 3 | 29 |
| 6 | 13 | -27 | 11 | 16 | -19 | 22 | -21 | -10 | 35 |

**Figure 2.** An integer $H(6, 12; 6, 3)$.

As an example of application of our reduction theorem, we get the following.

**Corollary 3.4.** Let $m, n, s, k$ be four integers such that $3 \leq s \leq n$, $3 \leq k \leq m$ and $ms = nk$. Suppose $d = \gcd(s, k) \geq 3$.

1. If $d \equiv 0 \pmod{4}$, then there exists a shiftable $H(m, n; s, k)$.
2. If $d \equiv 1 \pmod{4}$ and $nk \equiv 3 \pmod{4}$, then there exists an integer $H(m, n; s, k)$.
3. If $d \equiv 3 \pmod{4}$ and $nk \equiv 0, 3 \pmod{4}$, then there exists an integer $H(m, n; s, k)$.

**Proof.** We recall that there exists an integer diagonal $H(a; b)$, with $a \geq b \geq 3$, in each of the following cases:

(i) $b \equiv 0 \pmod{4}$ (shiftable), see [3] and [22];
(ii) $b \equiv 1 \pmod{4}$ and $a \equiv 3 \pmod{4}$, see [9, 14];
(iii) $b \equiv 3 \pmod{4}$ and $a \equiv 0, 1 \pmod{4}$, see [3].

Now, it suffices to apply Theorem 3.3 taking $b = d$ and $a = \frac{nk}{d}$ (note that $\frac{nk}{d} \geq d$ as $n \geq s \geq d$).

**Example 3.5.** We construct the following integer $H(8, 12; 9, 6)$ starting from an integer diagonal $H(24; 3)$:

| -72 | 24 | 48 | -64 | 7 | 57 | -29 | -15 | 44 |
|-----|----|----|-----|--|----|-----|-----|----|
| -49 | -22 | 71 | -40 | 6 | 34 | -53 | -14 | 67 |
| 43 | -26 | -21 | 47 | -63 | 5 | 58 | -30 | -13 |
| 23 | 31 | -50 | -20 | 70 | -39 | 4 | 35 | -54 |
| -66 | 11 | 55 | -27 | -19 | 46 | -62 | 3 | 59 |
| -42 | 10 | 32 | -51 | -18 | 69 | -38 | 2 | 36 |
| 60 | -65 | 9 | 56 | -28 | -17 | 45 | -61 | 1 |
| 12 | 25 | -41 | 8 | 33 | -52 | -16 | 68 | -37 |

**Proof of Theorem 1.4.** By [22, Theorem 1.3], we may assume that $d \geq 3$ is odd. So, the result follows from Corollary 3.4 items (2) and (3).

We now apply our reduction theorem to the known results on diagonal $\lambda$-fold relative Heffter arrays.

**Corollary 3.6.** Let $m, n, s, k$ be four integers such that $3 \leq s \leq n$, $3 \leq k \leq m$ and $ms = nk$. Suppose $d = \gcd(s, k) \geq 3$. 


(1) If \( d \equiv 1 \pmod{4} \) and \( nk \equiv 3 \pmod{4} \), then there exists an integer \( \text{H}_d(m; n; s, k) \).
(2) If \( d \equiv 3 \pmod{4} \) and \( nk \equiv 0, 1 \pmod{4} \), then there exists an integer \( \text{H}_d(m; n; s, k) \).
(3) If \( d = 3 \) and \( nk \) is odd, then there exists an integer \( \text{H}_s;k(m; n; s, k) \) and an integer \( \text{H}_s;k(m; n; s, k) \).
(4) If \( d = 3 \) and \( nk \equiv 3 \pmod{4} \), then there exists an integer \( \text{H}_d(m; n; s, k) \).
(5) If \( d = 3 \) and \( nk \equiv 1 \pmod{4} \), then there exists an integer \( \text{H}_d(m; n; s, k) \).
(6) If \( d = 3 \), \( nk \) is odd, and \( \lambda \) divides \( n \), then there exists an integer \( \lambda \text{H}_s;k(m; n; s, k) \).
(7) If \( d = 3 \), \( nk \) is odd, and \( \lambda \) divides \( 2n \), then there exists an integer \( \lambda \text{H}_s;k(m; n; s, k) \).
(8) If \( d = 5 \) and \( nk \equiv 3 \pmod{4} \), then there exists an integer \( \text{H}_d(m; n; s, k) \).

**Proof.** There exists an integer diagonal \( \text{H}_a(a; b) \), with \( a \geq b \geq 3 \), when \( b \equiv 1 \pmod{4} \) and \( a \equiv 3 \pmod{4} \), and when \( b \equiv 3 \pmod{4} \) and \( a \equiv 0, 3 \pmod{4} \), see \([10]\). By \([12]\), there exist an integer diagonal \( \text{H}_a(a; 3) \) and an integer diagonal \( \text{H}_a(a; 3) \) when \( a \geq 3 \) is odd. By \([11]\), there exist the following diagonal non-integer arrays:

(i) \( \text{H}_1(a; 3) \) when \( a \geq 5 \) is such that \( a \equiv 1 \pmod{4} \);
(ii) \( \text{H}_1(a; 3) \) when \( a \geq 3 \) is such that \( a \equiv 3 \pmod{4} \);
(iii) \( \lambda \text{H}_s;k(a; 3) \) when \( a \geq 3 \) is odd and \( \lambda \) divides \( n \);
(iv) \( \lambda \text{H}_s;k(a; 3) \) when \( a \geq 3 \) is odd and \( \lambda \) divides \( 2n \);
(v) \( \text{H}_1(a; 5) \) when \( a \geq 7 \) is such that \( a \equiv 3 \pmod{4} \).

Now, it suffices to apply Theorem 3.3 taking \( b = d \) and \( a = \frac{nk}{2} \). \( \square \)

### 4. Direct constructions when \( k \) is odd

In this section, we construct integer Heffter arrays \( \text{H}(m; n; s, k) \) when \( k \neq 5 \) is odd and \( s \equiv 0 \pmod{4} \). Note that from the necessary condition \( ms = nk \) we obtain \( n \equiv 0 \pmod{4} \). By Theorem 1.3 we could also assume \( \gcd(s, k) = 1 \), however this hypothesis is not necessary for our constructions.

We begin considering the case \( k = 3 \). In \([2]\) the authors proved the existence of an integer \( \text{H}(n; 3; 3, n) \) using blocks of size \( 3 \times 8 \) and blocks of size \( 3 \times 12 \). Our first step is to rearrange the elements of these blocks, obtained by working with Skolem sequences \([24]\), in order to produce \( 3 \times 4 \) blocks whose rows and columns sum to zero. So, for any fixed integer \( \mu \geq 0 \), we can define:

|    | \( A_1 \) | \( A_2 \) | \( B_a \) | \( C_a \) |
|----|---------|---------|---------|---------|
| 1 | \( 4\mu + 4 \) | \( 4\mu + 6 \) | \( 8\mu + 5 \) | \( 4\mu + 1 \) |
| 2 | \( -8\mu - 7 \) | \( 4\mu + 3 \) | \( -8\mu - 4a \) | \( -8\mu - 6 + 4a \) |
| 3 | \( -14\mu - 15 \) | \( -1 \) | \( -20\mu - 20 \) | \( -4\mu - 2 + 4a \) |
| 4 | \( -12\mu - 12 \) | \( -8\mu - 8 \) | \( -16\mu - 16 \) | \( -8\mu - 3 + 4a \) |
| 5 | \( 10\mu + 10 \) | \( 20\mu + 21 \) | \( 10\mu + 13 + 2a \) | \( 8\mu + 4 - 4a \) |
| 6 | \( -14\mu - 11 - 2a \) | \( 14\mu + 14 - 2a \) | \( -8\mu - 11 - 2a \) | \( -8\mu - 12 - 2a \) |
| 7 | \( -2\mu - 23 \) | \( -22\mu - 22 \) | \( -16\mu + 14 - 2a \) | \( 4\mu - 4a \) |
| 8 | \( -16\mu - 15 + 2a \) | \( -10\mu - 12 - 2a \) | \( 10\mu + 13 + 2a \) | \( 8\mu + 4 - 4a \) |
| 9 | \( 14\mu + 14 - 2a \) | \( -20\mu - 20 \) | \( 8\mu + 4 - 4a \) | \( 8\mu + 4 - 4a \) |
| 10 | \( 10\mu + 13 + 2a \) | \( -22\mu - 22 \) | \( 8\mu + 4 - 4a \) | \( 8\mu + 4 - 4a \) |
with $a \in [0, \mu - 1]$. The sequence

\begin{equation}
A_1(\mu) = \begin{cases} (A_1, A_2) & \text{if } \mu = 0, \\
(A_1, A_2, B_0, B_1, \ldots, B_{\mu - 1}, C_0, C_1, \ldots, C_{\mu - 1}) & \text{if } \mu \geq 1
\end{cases}
\end{equation}

has length $2\mu + 2$ and support equals to $\text{supp}(A_1(\mu)) = [1, 24\mu + 24]$. In fact,

\[
\bigcup_{a=1}^{2} \text{supp}(A_a) = [1, 2] \cup [4\mu + 3, 4\mu + 6], \cup [8\mu + 7, 8\mu + 9] \cup [10\mu + 10, 10\mu + 11] \\
\cup [12\mu + 12, 12\mu + 14] \cup [14\mu + 15] \cup [16\mu + 16] \cup [18\mu + 17, 18\mu + 19] \\
\cup [20\mu + 20, 20\mu + 21] \cup [22\mu + 22, 22\mu + 23] \cup [24\mu + 24];
\]

\[
\bigcup_{a=0}^{\mu - 1} \text{supp}(B_a) = \{4, 6, \ldots, 4\mu + 2\} \cup \{4\mu + 7, 4\mu + 9, \ldots, 8\mu + 5\} \cup [8\mu + 10, 10\mu + 9] \\
\cup [10\mu + 12, 12\mu + 11] \cup [12\mu + 15, 14\mu + 14] \cup [14\mu + 16, 16\mu + 15];
\]

\[
\bigcup_{a=0}^{\mu - 1} \text{supp}(C_a) = \{3, 5, \ldots, 4\mu + 1\} \cup \{4\mu + 8, 4\mu + 10, \ldots, 8\mu + 6\} \cup [16\mu + 17, 18\mu + 16] \\
\cup [18\mu + 20, 20\mu + 19] \cup [20\mu + 22, 22\mu + 21] \cup [22\mu + 24, 24\mu + 23].
\]

Now, for any fixed integer $\nu \geq 0$ define:

\[
E_1 = \begin{pmatrix} 4\nu + 1 & -4\nu - 5 & -4\nu - 8 & 4\nu + 12 \\
18\nu + 30 & -18\nu - 28 & -18\nu - 26 & 18\nu + 24 \\
-22\nu - 31 & 22\nu + 33 & 22\nu + 34 & -22\nu - 36 \\
4\nu + 6 & -18\nu - 25 & 4\nu + 2 & 10\nu + 17 \\
-12\nu - 19 & 22\nu + 35 & -14\nu - 20 & 4\nu + 4 \\
4\nu + 7 & 10\nu + 14 & 8\nu + 11 & -22\nu - 32 \\
10\nu + 15 & 4\nu + 9 & -18\nu - 27 & 4\nu + 3 \\
-14\nu - 22 & -14\nu - 23 & 10\nu + 16 & 18\nu + 29 \\
4\nu - 3 - a & -4\nu + 2 + 4a & -4\nu + 1 + 4a & 4\nu - 4a \\
18\nu + 32 + 2a & -10\nu - 20 - 2a & -18\nu - 31 - 2a & 10\nu + 19 + 2a \\
-22\nu - 29 + 2a & 14\nu + 18 - 2a & 22\nu + 30 - 2a & -14\nu - 19 + 2a \\
8\nu + 7 - 4a & -8\nu - 9 + 4a & -8\nu - 10 + 4a & 8\nu + 12 - 4a \\
8\nu + 15 + 2a & -8\nu - 14 - 2a & -16\nu - 25 - 2a & 16\nu + 24 + 2a \\
-16\nu - 22 + 2a & 16\nu + 23 - 2a & 24\nu + 35 - 2a & -24\nu - 36 + 2a
\end{pmatrix},
\]

with $a \in [0, \nu - 1]$. The sequence

\begin{equation}
A_2(\nu) = \begin{cases} (E_1, E_2, E_3) & \text{if } \nu = 0, \\
(E_1, E_2, E_3, F_0, F_1, \ldots, F_{\nu - 1}, G_0, G_1, \ldots, G_{\nu - 1}) & \text{if } \nu \geq 1
\end{cases}
\end{equation}

has length $2\nu + 3$ and support equals to $\text{supp}(A_2(\nu)) = [1, 24\nu + 36]$. In fact,

\[
\bigcup_{a=1}^{2} \text{supp}(E_a) = [4\nu + 1, 4\nu + 10] \cup [4\nu + 12] \cup [8\nu + 11] \cup [8\nu + 13] \\
\cup [10\nu + 14, 10\nu + 18] \cup [12\nu + 19] \cup [14\nu + 20, 14\nu + 23] \\
\cup [18\nu + 24, 18\nu + 30] \cup [22\nu + 31, 22\nu + 36];
\]

\[
\bigcup_{a=0}^{\mu - 1} \text{supp}(F_a) = [1, 4\nu] \cup [10\nu + 19, 12\nu + 18] \cup [12\nu + 20, 14\nu + 19] \\
\cup [18\nu + 31, 20\nu + 30] \cup [20\nu + 31, 22\nu + 30];
\]
Proof of Theorem 1.5. Our first step is to construct a sequence \( Z = (Z_0, Z_1, \ldots, Z_{n-4}) \) of length \( \frac{n}{4} \) such that its elements \( Z_i \) are \( k \times 4 \) blocks whose rows and columns sum to zero, and such that \( \supp(Z) = [1, nk] \).

To this intent, we define a sequence \( B = (U_0, U_1, \ldots, U_{n-4}) \) of length \( \frac{n}{4} \) consisting of blocks of size \( 3 \times 4 \), as follows. If \( n \equiv 0 \) (mod 8), we take the sequence \( B = A_1 \left( \frac{n-4}{8} \right) \) as defined in (4.2). If \( n \geq 12 \) and \( n \equiv 4 \) (mod 8), we set \( B = A_2 \left( \frac{n-12}{8} \right) \) as defined in (4.2). Finally, if \( n = 4 \), the sequence \( B \) consists only of the block \( H(3, 4; 4, 3) \) = 
\[
\begin{array}{cccc}
1 & 2 & 3 & -6 \\
8 & -12 & -7 & 11 \\
-9 & 10 & 4 & -5 \\
\end{array}
\]

that, in each case, \( \supp(B) = [1, 3n] \).

If \( k = 3 \), set \( Z_i = U_i \) for all \( i = 0, \ldots, \frac{n-4}{8} \). It is clear that \( Z \) satisfies all our requirements. Assume \( k \geq 7 \). To obtain our block \( Z_i \) we use the block \( U_i \) for the first three rows, and we need a shiftable \( H(k-3, 4; 4, k-3) \) to construct the remaining \( k-3 \) rows. The existence of this Heffter array was proved in [21, 22]. But here, for the reader’s sake, we prefer to give an explicit construction. So, consider the following shiftable blocks:

\[
Q_4 = \begin{array}{cccc}
1 & -2 & -3 & 4 \\
-5 & 6 & 7 & -8 \\
-9 & 10 & 11 & -12 \\
13 & -14 & -15 & 16 \\
\end{array}, \quad Q_6 = \begin{array}{cccc}
1 & -14 & -3 & 16 \\
-2 & 13 & 4 & -15 \\
5 & -18 & -7 & 20 \\
-6 & 17 & 8 & -19 \\
-9 & 23 & 10 & -24 \\
11 & -21 & -12 & 22 \\
\end{array}
\]

Note that \( Q_4 \) is an integer \( H(4, 4; 4, 4) \) and \( Q_6 \) is an integer \( H(6, 4; 4, 6) \). For all fixed \( h \geq 0 \) we take the \( (4h+4) \times 4 \) block \( V_h \) and the \( (4h+6) \times 4 \) block \( W_h \), as follows:

\[
V_h = \begin{array}{c}
Q_4 \\
Q_4 \pm 16 \\
Q_4 \pm 32 \\
\vdots \\
Q_4 \pm 16h \\
\end{array}, \quad W_h = \begin{array}{c}
Q_6 \\
Q_4 \pm 24 \\
Q_4 \pm 40 \\
\vdots \\
Q_4 \pm (16h+8) \\
\end{array}
\]

So, the blocks \( V_h \) and \( W_h \) have rows and columns that sum to zero. Furthermore, \( \supp(V_h) = [1, 16h+16] \) and \( \supp(W_h) = [1, 16h+24] \) (clearly, \( V_0 = Q_4 \) and \( W_0 = Q_6 \)).

If \( k \equiv 3 \) (mod 4), write \( k = 4q + 7 \) and define \( Z_i = U_i \) \( V_q \pm (3n + (16q+16)i) \). Note that

\[
\bigcup_{i=0}^{n-1} \supp(V_q \pm (3n + (16q+16)i)) = \bigcup_{i=0}^{n-1} [3n + 1 + 16(q+1)i, 3n + 16(q+1)(i+1)]
\]
If \( k \equiv 1 \pmod{4} \), write \( k = 4q + 9 \) and define \( Z_i = \begin{bmatrix} W_q \\ U_i \\ W_q \pm (3n + (16q + 24)i) \end{bmatrix} \). We have

\[
\bigcup_{i=0}^{\frac{n-1}{4}} \text{supp} (W_q \pm (3n + (16q + 24)i)) = \bigcup_{i=0}^{\frac{n-1}{4}} [3n + 1 + 8(2q + 3)i, 3n + 8(2q + 3)(i + 1)].
\]

In both case, we conclude that \( \text{supp}(Z) = [1, 3n] \cup [3n + 1, kn] = [1, kn] \).

Hence, for all \( k \neq 5 \) odd, we were able to construct the required sequence \( Z \). To obtain an integer \( H(m, n; s, k) \), we fill the cell \((a + ki, b + 4i)\) of an array \( H \) of size \( m \times n \) with the element of the cell \((a, b)\) of \( Z_i \), for all \( i = 0, \ldots, \frac{n-1}{4} \). Clearly, every column of \( H \) contains exactly \( k \) filled cells (a column of a unique block \( Z_i \)). Also, every row of \( H \) contains \( s \) filled cells (a row from each \( s \) distinct blocks \( Z_j \)), since \( k \frac{n}{4} = m \frac{s}{4} \). Hence, \( \text{supp}(H) = \text{supp}(Z) = [1, nk] \) and, since the elements of each row/column of \( Z_i \) sum to 0, the same holds for \( H \).

Following the proof of Theorem 1.5, we can construct an integer \( H(14, 8; 4, 7) \). We start taking the following two \( 7 \times 4 \) blocks:

\[
Z_0 = \begin{bmatrix} 4 & -7 & 5 & -2 \\ 9 & 18 & -15 & -12 \\ -13 & -11 & 10 & 14 \\ 25 & -26 & -27 & 28 \\ -29 & 30 & 31 & -32 \\ -33 & 34 & 35 & -36 \\ 37 & -38 & -39 & 40 \end{bmatrix}, \quad Z_1 = \begin{bmatrix} 6 & 3 & -1 & -8 \\ 17 & 19 & -20 & -16 \\ -23 & -22 & 21 & 24 \\ 41 & -42 & -43 & 44 \\ -45 & 46 & 47 & -48 \\ -49 & 50 & 51 & -52 \\ 53 & -54 & -55 & 56 \end{bmatrix}.
\]

then,

\[
\begin{array}{cccc}
4 & -7 & 5 & -2 \\
9 & 18 & -15 & -12 \\
-13 & -11 & 10 & 14 \\
25 & -26 & -27 & 28 \\
-29 & 30 & 31 & -32 \\
-33 & 34 & 35 & -36 \\
37 & -38 & -39 & 40 \\
\hline
6 & 3 & -1 & -8 \\
17 & 19 & -20 & -16 \\
-23 & -22 & 21 & 24 \\
41 & -42 & -43 & 44 \\
-45 & 46 & 47 & -48 \\
-49 & 50 & 51 & -52 \\
53 & -54 & -55 & 56 \\
\end{array}
\]

is an integer \( H(14, 8; 4, 7) \). An integer \( H(28, 16; 12, 21) \) is shown in Figure 3 where we highlighted the blocks \( U_0, U_1, U_2, U_3 \).

5. Signed magic arrays and magic rectangles

The existence of magic rectangles and signed magic arrays has been considered mainly in the square case, in the tight case, or when each column contains two or three filled cells, see [15, 16, 17, 18, 19, 20, 21]. We can provide a reduction theorem also for magic rectangles and signed
Figure 3. An integer $H(28, 16; 12, 21).$

| 8   | -15 | 9   | -2 | -129 | 143 | 130 | -144 | -237 | 238 | 239 | -240 |
| 17   | 36  | -29 | -24 | 131  | -141 | -132 | 142  | -241 | 242 | 243 | -244 |
| -25   | -21 | 20  | 26  | 145  | -146 | -147 | 148  | 245  | -246 | -247 | 248  |
| 49   | -62 | -51 | 64  | -149 | 150  | 151  | -152 | 249  | -250 | -251 | 252  |
| -50   | 61  | 52  | -63 | -153 | 154  | 155  | -156 | 253  | 254  | 255  | -256 |
| 53   | -66 | -55 | 68  | 157  | -158 | -159 | 160  | -257 | 258  | 259  | -260 |
| -54   | 65  | 56  | -67 | 161  | -162 | -163 | 164  | 261  | -262 | -263 | 264  |
| -57   | 71  | 58  | -72 | -165 | 166 | 167  | -168 | 5    | -14  | -3   | 12   |
| 59   | -69 | -60 | 70  | -169 | 170 | 171  | -172 | 38   | -33  | -39 | 34   |
| 73   | -74 | -75 | 76  | 173  | -174 | -175 | 176 | -43  | 47  | 42  | -46  |
| -77   | 78  | 79  | -80 | 177  | -178 | -179 | 180 | 265  | -278 | -267 | 280  |
| -81   | 82  | 83  | -84 | -181 | 182 | 183  | -184 | -266 | 277  | 268  | -279 |
| 85   | -86 | -87 | 88  | -185 | 186 | 187  | -188 | 269  | -282 | -271 | 284  |
| 89   | -90 | -91 | 92  | -189 | 190 | -191 | 192 | -270 | 284  | 272  | -283 |
| -93   | 94  | 95  | -96 | 18  | 28  | -19  | -27 | 273  | 287  | 274  | -288 |
| -97   | 98  | 99  | -100 | 13   | -6  | -11  | 4   | 275  | -285 | -276 | 286  |
| 101  | -102 | -103 | 104 | -31  | -22  | 30  | 23  | 289  | -290 | -291 | 292  |
| 105  | -106 | -107 | 108 | 193  | -206 | -195 | 208  | -293 | 294  | 295  | -296 |
| -109  | 110 | 111  | -112 | -194 | 205  | 196  | -207 | -297 | 288  | 299  | -300 |
| -113  | 114 | 115  | -116 | 197  | -210 | -199 | 212  | 301  | -302 | -303 | 304  |
| 117  | -118 | -119 | 120 | -198 | 209  | 200  | -211 | 305  | -306 | -307 | 308  |
| 10   | 7  | -1  | -16 | -201 | 215 | 202  | -216 | -309 | 310  | 311  | -312 |
| 35   | 37  | -40  | -32 | 203  | -213 | -204 | 214  | -315 | 314  | 315  | -316 |
| -45  | -44 | 41  | 48  | 217  | -218 | -219 | 220  | 317  | -318 | -319 | 320  |
| 121  | -134 | -123 | 136 | -221 | 222  | 223  | -224 | 321  | -322 | -323 | 324  |
| -122  | 133 | 124  | -135 | -225 | 226  | 227  | -228 | -325 | 326  | 327  | -328 |
| 125  | -138 | -127 | 140 | 229  | -230 | -231 | 232  | -329 | 330  | 331  | -332 |
| -126 | 137  | 128  | -139 | 233  | -234 | -235 | 236  | 333  | -334 | -335 | 336  |
magic arrays. In fact, easily adapting the proof of Theorem 5.3 we obtain the following result, where we denote a magic square \( MR(n, n; k, k) \) and a signed magic square \( SMA(n, n; k, k) \) by \( MR(n; k) \) and \( SMA(n; k) \), respectively. Furthermore, we say that an \( MR(n; k) \) (or an \( SMA(n; k) \)) is diagonal if it is cyclically \( k \)-diagonal.

**Theorem 5.1.** Let \( m, n, s, k \) be four integers such that \( 2 \leq s \leq n \), \( 2 \leq k \leq m \) and \( ms = nk \). Set \( d = \gcd(s, k) \). If there exists a (shiftable) diagonal SMA \( (\frac{nk}{d}; d) \), then there exists a (shiftable) SMA \( (m, n; s, k) \). If there exists a diagonal MR \( (\frac{nk}{d}; d) \), then there exists an MR \( (m, n; s, k) \).

Now, we consider the existence of diagonal signed magic arrays, starting with the following constructions.

**Lemma 5.2.** There exists a diagonal \( SMA(a; 3) \) for all odd \( a \geq 3 \).

**Proof.** A diagonal \( SMA(a; 3) \), when \( a = 2g + 1 \), can be obtained by taking a pf array \( A \) whose rows \( R_1, R_2, \ldots, R_a \) are as follows. For each \( \ell \in [0, g] \), we fill the row \( R_{1+2\ell} \) with the entries
\[
E(1+2\ell, 1+2\ell) = -(2g+1) + \ell, \quad E(1+2\ell, 2+2\ell) = g - 2\ell, \quad E(1+2\ell, 3+2\ell) = (g+1) + \ell,
\]
and, for each \( \ell \in [0, g-1] \), we fill the row \( R_{2+2\ell} \) with the entries
\[
E(2+2\ell, 2+2\ell) = -(3g+1) + \ell, \quad E(2+2\ell, 3+2\ell) = (g-1) - 2\ell, \quad E(2+2\ell, 4+2\ell) = (2g+2) + \ell.
\]
It is clear that we filled the diagonals \( D_0, D_1, D_2 \) in such a way that the elements of every row of \( A \) sum to zero. Furthermore, \( E(D_0) = \{-3g+1, -(g+1)\} \), \( E(D_1) = \{-g, g\} \) and \( E(D_2) = \{g+1, 3g+1\} \). We now describe the columns \( C_1, C_2, \ldots, C_a \) of \( A \). For each \( \ell \in [0, g] \), we filled the column \( C_{2+2\ell} \) with the entries
\[
E(2\ell, 2+2\ell) = (2g+1) + \ell, \quad E(1+2\ell, 2+2\ell) = g - 2\ell, \quad E(2+2\ell, 2+2\ell) = -(3g+1) + \ell.
\]
Note that the column \( C_1 \) corresponds to the value \( \ell = g \), since the indexes are taken modulo \( a \). Also, for each \( \ell \in [0, g-1] \), we filled the column \( C_{3+2\ell} \) with the entries
\[
E(1+2\ell, 3+2\ell) = (g+1) + \ell, \quad E(2+2\ell, 3+2\ell) = (g-1) - 2\ell, \quad E(3+2\ell, 3+2\ell) = -2g+\ell.
\]
Thus, the elements of every column of \( A \) sum to zero: we conclude that \( A \) is a diagonal \( SMA(a; 3) \). \( \square \)

**Lemma 5.3.** There exists a diagonal \( SMA(a; 3) \) for all \( a \geq 6 \) such that \( a \equiv 2 \pmod{4} \).

**Proof.** A diagonal \( SMA(a; 3) \), when \( a = 4g + 2 \), can be obtained by taking a pf array \( A \) whose rows \( R_1, R_2, \ldots, R_a \) are as follows. We fill the row \( R_1 \) with the entries
\[
E(1, 1) = -(3g+2), \quad E(1, 2) = 1, \quad E(1, 3) = 3g+1.
\]
For each \( \ell \in [0, g-1] \), we fill the row \( R_{2+2\ell} \) with the entries
\[
E(2+2\ell, 2+2\ell) = -(5g+4) - \ell, \quad E(2+2\ell, 3+2\ell) = 3+2\ell, \quad E(2+2\ell, 4+2\ell) = (5g+1) - \ell,
\]
and, for each \( \ell \in [0, g-2] \), we fill the row \( R_{3+2\ell} \) with the entries
\[
E(3+2\ell, 3+2\ell) = -(3g+4) - \ell, \quad E(3+2\ell, 4+2\ell) = 4+2\ell, \quad E(3+2\ell, 5+2\ell) = 3g - \ell.
\]
Next, we fill the row \( R_{2g+1} \) with the entries
\[
E(2g+1, 2g+1) = -(4g+3), \quad E(2g+1, 2g+2) = 2, \quad E(2g+1, 2g+3) = 4g+1.
\]
For each \( \ell \in [0, g - 1] \), we fill the row \( R_{2g + 2 + 2\ell} \) with the entries
\[
E(2g + 2 + 2\ell, 2g + 2 + 2\ell) = -(4g + 4) - \ell,
\]
\[
E(2g + 2 + 2\ell, 2g + 3 + 2\ell) = -(2g - 1) + 2\ell, \quad E(2g + 2 + 2\ell, 2g + 4 + 2\ell) = (6g + 3) - \ell,
\]
and, for each \( \ell \in [0, g - 2] \), we fill the row \( R_{2g + 3 + 2\ell} \) with the entries
\[
E(2g + 3 + 2\ell, 2g + 3 + 2\ell) = -(2g + 2) - \ell,
\]
\[
E(2g + 3 + 2\ell, 2g + 4 + 2\ell) = -(2g - 2) + 2\ell, \quad E(2g + 3 + 2\ell, 2g + 5 + 2\ell) = 4g - \ell.
\]
Finally, we fill the rows \( R_{g-1}, R_a \) with the entries
\[
E(a - 1, a - 1) = -(3g + 1), \quad E(a - 1, a) = -(2g + 1), \quad E(a - 1, 1) = 5g + 2,
\]
\[
E(a, a) = -(3g + 3), \quad E(a, 1) = -2g, \quad E(a, 2) = 5g + 3.
\]
It is clear that we filled the diagonals \( D_0, D_1, D_2 \) in such a way that the elements of every row of \( A \) sum to zero. Furthermore, \( E(D_0) = [--(6g + 3), -(2g + 2)], E(D_1) = [--(2g + 1), -1 union [1, 2g + 1] \) and \( E(D_2) = [2g + 2, 6g + 3] \). We now describe the columns \( C_1, C_2, \ldots, C_a \) of \( A \). We filled the columns \( C_1, C_2 \) with the entries
\[
E(a - 1, 1) = 5g + 2, \quad E(a, 1) = -2g, \quad E(1, 1) = -(3g + 2),
\]
\[
E(a, 2) = 5g + 3, \quad E(1, 2) = 1, \quad E(2, 2) = -(5g + 4).
\]
For each \( \ell \in [0, g - 1] \), we filled the column \( C_{3+2\ell} \) with the entries
\[
E(1 + 2\ell, 3 + 2\ell) = (3g + 1) - \ell,
\]
\[
E(2 + 2\ell, 3 + 2\ell) = 3 + 2\ell, \quad E(3 + 2\ell, 3 + 2\ell) = -(3g + 4) - \ell,
\]
and, for each \( \ell \in [0, g - 2] \), we filled the column \( C_{4+2\ell} \) with the entries
\[
E(2 + 2\ell, 4 + 2\ell) = (5g + 1) - \ell,
\]
\[
E(3 + 2\ell, 4 + 2\ell) = 4 + 2\ell, \quad E(4 + 2\ell, 4 + 2\ell) = -(5g + 5) - \ell.
\]
Next, we filled the column \( C_{2g+2} \) with the entries
\[
E(2g, 2g + 2) = 4g + 2, \quad E(2g + 1, 2g + 2) = 2, \quad E(2g + 2, 2g + 2) = -(4g + 4).
\]
Then, for each \( \ell \in [0, g - 1] \), we filled the column \( C_{2g+3+2\ell} \) with the entries
\[
E(2g + 1 + 2\ell, 2g + 3 + 2\ell) = (4g + 1) - \ell
\]
\[
E(2g + 2 + 2\ell, 2g + 3 + 2\ell) = -(2g - 1) + 2\ell, \quad E(2g + 3 + 2\ell, 2g + 3 + 2\ell) = -(2g + 2) - \ell,
\]
and, for each \( \ell \in [0, g - 2] \), we filled the column \( C_{2g+4+2\ell} \) with the entries
\[
E(2g + 2 + 2\ell, 2g + 4 + 2\ell) = (6g + 3) - \ell,
\]
\[
E(2g + 3 + 2\ell, 2g + 4 + 2\ell) = -(2g - 2) + 2\ell, \quad E(2g + 4 + 2\ell, 2g + 4 + 2\ell) = -(4g + 5) - \ell.
\]
Finally, we filled the column \( C_a \) with the entries
\[
E(a - 2, a) = 5g + 4, \quad E(a - 1, a) = -(2g + 1), \quad E(a, a) = -(3g + 3).
\]
Thus, the elements of every column of \( A \) sum to zero: we conclude that \( A \) is a diagonal SMA\((a; 3)\).

In Figure 4 the reader can find a diagonal SMA\((14; 3)\), obtained by following the proof of Lemma 5.3.

**Lemma 5.4.** There exists a diagonal SMA\((a; 5)\) for all odd \( a \geq 5 \).
We now describe the columns $C_1, C_2, \ldots, C_a$ of $A$. For each $\ell \in [0, g]$, we filled the column $C_{3+2\ell}$ with the entries

$E(1, 1) = -(5g + 2), \quad E(1, 2) = -(g + 1), \quad E(1, 3) = g, \quad E(1, 4) = g + 2, \quad E(1, 5) = 4g + 1.$

For each $\ell \in [0, g - 1]$, we filled the row $R_{2+2\ell}$ with the entries

$E(2 + 2\ell, 0) = -(4g + 2) - \ell, \quad E(2 + 2\ell, 1) = -(3g + 1) + 2\ell,$

$E(2 + 2\ell, 4 + 2\ell) = (g - 1) - 2\ell, \quad E(2 + 2\ell, 5 + 2\ell) = (g - 3) + 2\ell, \quad E(2 + 2\ell, 6 + 2\ell) = (5g + 1) - \ell,$

and, for each $\ell \in [0, g - 2]$, we filled the row $R_{3+2\ell}$ with the entries

$E(3 + 2\ell, 0) = -(3g + 2) - \ell, \quad E(3 + 2\ell, 1) = -3g + 2\ell,$

$E(3 + 2\ell, 5 + 2\ell) = -(g - 2) - 2\ell, \quad E(3 + 2\ell, 6 + 2\ell) = (g + 4) + 2\ell, \quad E(3 + 2\ell, 7 + 2\ell) = 4g - \ell.$

Finally, we filled the row $R_a$ with the entries

$E(a, a) = -(4g + 1), \quad E(a, 1) = -(g + 2), \quad E(a, 2) = -g,$

$E(a, 3) = g + 1, \quad E(a, 4) = 5g + 2.$

It is clear that we filled the diagonals $D_0, D_1, D_2, D_3, D_4$ in such a way that the elements of every row of $A$ sum to zero. Furthermore, $E(D_0) = [-5g + 2, -3g + 2]$, $E(D_1) = [-3g + 1, -(g + 1)]$, $E(D_2) = [-g, g]$, $E(D_3) = [g + 1, 3g + 1]$ and $E(D_4) = [3g + 2, 5g + 2].$

We now describe the columns $C_1, C_2, \ldots, C_a$ of $A$. For each $\ell \in [0, g]$, we filled the column $C_{3+2\ell}$ with the entries

$E(-1 + 2\ell, 1) = (4g + 2) - \ell, \quad E(2\ell, 2) = (g + 1) + 2\ell,$

$E(1 + 2\ell, 3 + 2\ell) = g - 2\ell, \quad E(2 + 2\ell, 3 + 2\ell) = -(3g + 1) + 2\ell,$

$E(3 + 2\ell, 3 + 2\ell) = -(3g + 2) - \ell.$

Note that the column $C_2$ corresponds to the value $\ell = g$. Also, for each $\ell \in [0, g - 1]$, we filled the column $C_{4+2\ell}$ with the entries

$E(2, 4 + 2\ell) = (5g + 2) - \ell, \quad E(1 + 2\ell, 4 + 2\ell) = (g + 2) + 2\ell,$

$E(2 + 2\ell, 4 + 2\ell) = (g - 1) - 2\ell, \quad E(3 + 2\ell, 4 + 2\ell) = -3g + 2\ell, \quad E(4 + 2\ell, 4 + 2\ell) = -(4g + 3) - \ell.$

Figure 4. A diagonal SMA(14; 3).
Finally, we fill the row $R$. Note that the column $C_1$ corresponds to the value $\ell = g - 1$. Thus, the elements of every column of $A$ sum to zero: we conclude that $A$ is a diagonal $\text{SMA}(a;5)$. \hfill $\square$

**Lemma 5.5.** There exists a shiftable diagonal $\text{SMA}(a;6)$ for all $a \geq 6$.

**Proof.** For any $a \geq 6$, a shiftable diagonal $\text{SMA}(a;6)$ can be obtained by taking a pf array $A$ whose rows $R_1, R_2, \ldots, R_a$ are as follows. For each $\ell \in [1, a - 3]$, we fill the row $R_\ell$ with the entries

$$E(\ell, \ell) = \ell, \quad E(\ell, \ell + 1) = a + \ell, \quad E(\ell, \ell + 2) = -a - 2\ell,$$

$$E(\ell, \ell + 3) = -3 - \ell, \quad E(\ell, \ell + 4) = (3a - 2) - \ell, \quad E(\ell, \ell + 5) = -(3a - 5) + 2\ell.$$ 

We fill the rows $R_{a-2}, R_{a-1}$ as follows:

$$E(a - 2, a - 2) = a - 2, \quad E(a - 2, a - 1) = 2a - 2, \quad E(a - 2, a) = -(3a - 4),$$

$$E(a - 2, 1) = -1, \quad E(a - 2, 2) = 3a, \quad E(a - 2, 3) = -(3a - 1);$$

$$E(a - 1, a - 1) = a - 1, \quad E(a - 1, a) = 2a - 1, \quad E(a - 1, 1) = -(3a - 2),$$

$$E(a - 1, 2) = -2, \quad E(a - 1, 3) = 3a - 1, \quad E(a - 1, 4) = -(3a - 3).$$

Finally, we fill the row $R_a$ with the entries

$$E(a, a) = a, \quad E(a, 1) = 2a, \quad E(a, 2) = -3a,$$

$$E(a, 3) = -3, \quad E(a, 4) = 3a - 2, \quad E(a, 5) = -(3a - 5).$$

It is clear that we filled the diagonals $D_0, D_1, \ldots, D_5$ in such a way that the elements of every row of $A$ sum to zero. Note that each row contains three positive entries and three negative entries. Also, $E(D_0) = [1, a], E(D_1) = [a + 1, 2a], E(D_2) = \{-(a^2 - (3a - 2), \ldots, -(a + 2)\}, E(D_3) = [-a, -1], E(D_4) = [2a + 1, 3a] and E(D_5) = \{-(3a - 1), -(3a - 3), \ldots, -(a + 1)\}$. We now describe the columns $C_1, C_2, \ldots, C_a$ of $A$. We filled the columns $C_1, C_2$ with the entries

$$E(a - 4, 1) = -(a + 3), \quad E(a - 3, 1) = 2a + 1, \quad E(a - 2, 1) = -1,$$

$$E(a - 1, 1) = -(3a - 2), \quad E(a, 1) = 2a, \quad E(1, 1) = 1;$$

$$E(a - 3, 2) = -(a + 1), \quad E(a - 2, 2) = 3a, \quad E(a - 1, 2) = -2,$$

$$E(a, 2) = -3a, \quad E(1, 2) = a + 1, \quad E(2, 2) = 2.$$ 

For each $\ell \in [3, a]$, we filled the column $C_\ell$ with the entries

$$E(\ell - 5, \ell) = -(3a + 5) + 2\ell, \quad E(\ell - 4, \ell) = (3a + 2) - \ell, \quad E(\ell - 3, \ell) = -\ell,$$

$$E(\ell - 2, \ell) = -(a - 4) - 2\ell, \quad E(\ell - 1, \ell) = (a - 1) + \ell, \quad E(\ell, \ell) = \ell.$$ 

Thus, the elements of every column of $A$ sum to zero, and every column contains three positive entries and three negative entries. We conclude that $A$ is a shiftable diagonal $\text{SMA}(a;6)$. \hfill $\square$

Note that, because of Lemmas 5.2, 5.3, 5.4 and 5.5 many of the constructions given in [21] can be shortened and simplified.

**Corollary 5.6.** There exists a diagonal $\text{SMA}(a;b)$ for any $a \geq b \geq 3$. Also, if $b$ is even, there exists a shiftable diagonal $\text{SMA}(a;b)$ for any $a \geq b \geq 3$. 
Proof. The statement follows from [21] Theorems 7 and 9 when (i) \( b \) is even and \( a \) is odd; or (ii) \( b \geq 5 \) is odd and \( a \) is even; or (iii) \( b = 3 \) and \( a \equiv 0 \pmod{4} \). The case \( b = 3 \) and \( a \equiv 2 \pmod{4} \) follows from Lemma 5.3 while the case \( b \equiv 0 \pmod{4} \) and \( a \) even follows from [21] Lemmas 10 and 11.

We are left with the following two cases: (1) \( a, b \) are both odd; (2) \( a \) is even and \( b \equiv 2 \pmod{4} \). So, write \( b = 4h + r \) with \( r \in \{3, 5, 6\} \). Let \( A \) be the diagonal SMA\((a; r)\) constructed in Lemmas 5.2, 5.3 and 5.5 and let \( B \) be the shiftable diagonal SMA\((a; 4)\) of [21] Lemma 10. If \( h = 0 \), set \( C = A \). If \( h > 0 \), let \( C \) be the pf array obtained by filling the 4\( h \) diagonals \( D_r, D_{r+1}, \ldots, D_{r+4h-1} \) of \( A \) with the elements of the filled diagonals of \( B \pm \frac{a-1}{2}, B \pm \frac{a(r+4)-1}{2}, \ldots, B \pm \frac{a(c+4h-4)-1}{2} \). Then, \( C \) is a diagonal SMA\((a; b)\). □

Proposition 5.7. Let \( m, n, s, k \) be four integers such that \( 3 \leq s \leq n \), \( 3 \leq k \leq m \) and \( ms = nk \). If \( s \) and \( k \) are both even, there exists a shiftable SMA\((m, n; s, k)\).

Proof. This result was already proved in [23], except when \( s, k \equiv 2 \pmod{4} \) and \( m, n \) are both odd. In this case, without loss of generality, we may assume \( m \geq n \) (and so \( s \leq k \)). Let \( A_1 \) be a shiftable SMA\((n; s)\) obtained in Corollary 5.6. Clearly if \( m = n \) we have nothing to prove. So, suppose \( m > n \). Since \( m - n \geq 2 \) is even and \( k - s \equiv 0 \pmod{4} \) with \( k - s \geq 4 \), by [23] Proposition 4.14 there exists a shiftable SMA\((m - n, n; s, k - s)\), say \( A_2 \). Let \( A \) be the pf array of size \( m \times n \) obtained by taking

\[
A = \begin{bmatrix}
A_1 \\
A_2 \pm ns/2
\end{bmatrix}
\]

Each row of \( A \) contains \( s \) filled cells and each of its columns contains \( s + (k - s) = k \) filled cells. Also, note that \( \mathcal{E}(A_1) = \{\pm 1, \pm 2, \ldots, \pm ns/2\} \) and \( \mathcal{E}(A_2 \pm ns/2) = \{\pm(1 + ns/2), \pm(2 + ns/2), \ldots, \pm ms/2\} \). Since \( \mathcal{E}(A) = \mathcal{E}(A_1) \cup \mathcal{E}(A_2) = \{\pm 1, \pm 2, \ldots, \pm ms/2\} \), \( A \) is a shiftable SMA\((m, n; s, k)\). □

Proof of Theorem 1.8 Set \( d = \gcd(s, k) \). If \( d = 2 \), then \( s \) and \( k \) are both even, and the result follows from Proposition 5.7. Assume \( d \geq 3 \). By Corollary 5.6 there exists a diagonal SMA\((\frac{a}{d}; d)\): the result follows from Theorem 5.1. □

We now consider the existence of magic rectangles.

Lemma 5.8. Let \( m, n, s, k \) be four integers such that \( 2 \leq s \leq n \), \( 2 \leq k \leq m \) and \( ms = nk \). If there exists a shiftable SMA\((m, n; s, k)\), then there exists an MR\((m, n; s, k)\).

Proof. See the proof of [23] Theorem 1.12. □

Thanks to the classification of the diagonal MR\((a; b)\) given in [19], we get Theorem 1.8.

Proof of Theorem 1.8 If \( d \) is even, the result follows from Proposition 5.7 and Lemma 5.8. In [19] it was proved that a diagonal MR\((a; b)\) exists for all \( a \geq b \geq 3 \) such that \( a \) is odd or \( b \) is even. So, if \( d \geq 3 \) and \( nk \) is odd, apply Theorem 5.1 taking \( b = d \) and \( a = \frac{nk}{d} \). □

In Figure 5 we give an MR\((9, 18; 12, 6)\) obtained by applying our reduction theorem to a diagonal MR\((18; 6)\), whose construction is described in the proof of [19] Theorem 6.

To conclude our paper, we show how the ideas of Section 4 can be adapted to construct signed magic arrays. Also in this case, we prefer to work in the more general context of integer \( \lambda \)-fold relative Heffter arrays.
Proposition 5.9. Let \( m, n, s, k \) be four integers such that \( 2 \leq s \leq n \), \( 2 \leq k \leq m \) and \( ms = nk \). Let \( \lambda \) be a divisor of \( n \) such that \( n \geq 3 \lambda \). Suppose that \( n, s, \lambda \) are even and such that \( n \lambda k \equiv 0, 3 \pmod{4} \). Then, there exists an integer \( \lambda H_1(m, n; s, k) \).

Proof. Let \( H \) be an integer \( H(k, \frac{n}{\lambda}, \frac{n}{\lambda}; k) \), and call \( C_1, C_2, \ldots, C_\frac{n}{\lambda} \) its columns. By [2], \( H \) exists since \( n \lambda \geq 3 \) and \( n \lambda k \equiv 0, 3 \pmod{4} \). For any \( i = 1, \ldots, \frac{n}{\lambda} \), define the \( k \times 2 \) block \( F_i = \begin{bmatrix} C_i & -C_i \end{bmatrix} \). Hence, each block \( F_i \) has rows and columns that sum to zero. Now, take the sequence \( B \) obtained by concatenating \( \frac{n}{\lambda} \) copies of the sequence \((F_1, F_2, \ldots, F_\frac{n}{\lambda})\). The sequence \( B \) has length \( \frac{n}{\lambda} \) and is such that each of the elements of \([1, k\frac{n}{\lambda}]\) appears in \( E(B) \) exactly \( \frac{k}{2} \) times with sign + and \( \frac{k}{2} \) times with sign −. Write \( B = (B_0, B_1, \ldots, B_{\frac{n}{\lambda}}) \). To obtain an integer \( \lambda H_1(m, n; s, k) \), we fill the cell \((a + ki, b + 2i)\) of an array \( A \) of size \( m \times n \) with the element of the cell \((a, b)\) of block \( B_i \), for all \( i = 0, \ldots, \frac{n}{\lambda} \). Also in this case, every column of \( A \) contains exactly \( k \) filled cells (a column of a unique block \( B_i \)), and every row of \( A \) contains \( s \) filled cells (a row from each \( s \frac{n}{\lambda} \) distinct blocks \( B_j \)), since \( \frac{n}{\lambda} k = m \frac{s}{\lambda} \). \( \square \)

In general, a \( \lambda^2 H_1(m, n; s, k) \) is not necessarily a signed magic array. However, by the particular shape of the pf array constructed in the proof of the previous proposition, for \( \lambda = 2 \) we get the following.

Corollary 5.10. Let \( m, n, s, k \) be four integers such that \( 3 \leq s \leq n \), \( 3 \leq k \leq m \) and \( ms = nk \). If \( n \) and \( s \) are even and \( nk \equiv 0, 6 \pmod{8} \), then there exists an SMA(m, n; s, k).

In Figure 5 we give an SMA(20, 8; 6, 15) obtained by following the previous construction.

Now, we show how to construct an SMA(m, n; s, k) when \( k \) is odd and \( s \equiv 0 \pmod{4} \). It will be useful to introduce the following notation: if \( 0 \leq a < b \), then \( \pm[a, b] \) denotes the set \([−b, −a] \cup [a, b] \). As done in Section 4, we begin considering the case \( k = 3 \). Working with signed Skolem sequences [4, Theorem 3.1], we produce \( 3 \times 4 \) blocks whose rows and columns
sum to zero. In fact, for any fixed integer \( \mu \geq 0 \), we can define:

\[
A = \begin{pmatrix}
1 & 2 & -2 & -1 \\
4 \mu + 3 & -(4 \mu + 6) & -(12 \mu + 3) & 12 \mu + 6 \\
-4 \mu + 4 & 4 \mu + 4 & 12 \mu + 5 & -(12 \mu + 5)
\end{pmatrix},
\]

\[
B_a = \begin{pmatrix}
4 a + 3 & 4 a + 5 & -(4 a + 5) & -(4 a + 3) \\
4 \mu + 4 a + 5 & -(4 \mu + 8 a + 12) & -(12 \mu - 8 a - 3) & 12 \mu - 4 a + 4 \\
-(4 \mu + 8 a + 8) & 4 \mu + 4 a + 7 & 12 \mu - 4 a + 2 & -(12 \mu - 8 a + 1)
\end{pmatrix},
\]

\[
C_a = \begin{pmatrix}
4 a + 4 & 4 a + 6 & -(4 a + 6) & -(4 a + 4) \\
4 \mu + 4 a + 6 & -(4 \mu + 8 a + 14) & -(12 \mu - 8 a - 5) & 12 \mu - 4 a + 3 \\
-(4 \mu + 8 a + 10) & 4 \mu + 4 a + 8 & 12 \mu - 4 a + 1 & -(12 \mu - 8 a - 1)
\end{pmatrix},
\]

with \( a \in [0, \mu - 1] \). The sequence

\[
(5.1) \quad A(\mu) = \begin{cases} 
(A, B_0, B_1, \ldots, B_{\mu-1}, C_0, C_1, \ldots, C_{\mu-1}) & \text{if } \mu = 0, \\
(A, B_0, B_1, \ldots) & \text{if } \mu \geq 1
\end{cases}
\]

has length \( 2 \mu + 1 \) and \( \mathcal{E}(A(\mu)) = \pm [1, 12 \mu + 6] \). In fact,

\[
\mathcal{E}(A) = \{-12 \mu + 5, -(12 \mu + 3)\} \cup \{-4 \mu + 6, -(4 \mu + 4)\} \cup \{-2, -1\} \cup [1, 2] \\
\cup [4 \mu + 3, 4 \mu + 4] \cup [12 \mu + 5, 12 \mu + 6];
\]

\[
\bigcup_{a=0}^{\mu-1} \mathcal{E}(B_a) = \{-12 \mu + 4, -12 \mu, \ldots, -(4 \mu + 8)\} \cup \{-12 \mu + 1, -(12 \mu - 3), \ldots, -(4 \mu + 5)\} \cup \{-4 \mu + 1, -(4 \mu - 1), \ldots, -3\} \cup \{3, 5, \ldots, 4 \mu + 1\} \\
\cup \{4 \mu + 5, 4 \mu + 7, \ldots, 8 \mu + 3\} \cup \{8 \mu + 6, 8 \mu + 8, \ldots, 12 \mu + 4\};
\]
\[
\bigcup_{a=0}^{\mu-1} \mathcal{E}(C_a) = \{-(12\mu + 6), -(12\mu + 2), \ldots, -(4\mu + 10)\} \cup \{-(12\mu - 1), -(12\mu - 5), \ldots, -(4\mu + 3)\} \cup \{-(4\mu + 2), -4\mu, \ldots, -4\} \cup \{4, 6, \ldots, 4\mu + 2\} \cup \{4\mu + 6, 4\mu + 8, \ldots, 8\mu + 4\} \cup \{8\mu + 5, 8\mu + 7, \ldots, 12\mu + 3\}.
\]

**Proof of Theorem 1.10.** If \(k\) is even, the statement follows from Theorem 1.9. So, assume \(k\) odd: from the hypothesis \(ms = nk\) we obtain that \(n \equiv 0\ (mod\ 4)\). Furthermore, if \(n \equiv 0\ (mod\ 8)\), the statement follows from Corollary 5.10. So, we may also assume \(n \equiv 4\ (mod\ 8)\).

Our first step is to construct a sequence \(Z = (Z_0, Z_1, \ldots, Z_{n-1})\) of length \(\frac{n}{4}\), such that its elements \(Z_i\) are \(k \times 4\) blocks whose rows and columns sum to zero, and such that \(\mathcal{E}(Z) = \pm[1, \frac{n}{2}]\). To this purpose, take the sequence \(B = A(\frac{n}{8}+1)\) and write \(B = (U_0, U_1, \ldots, U_{n-4})\).

Note that \(\mathcal{E}(B) = \pm[1, \frac{3n}{4}]\). If \(k = 3\), set \(Z_i = U_i\) for all \(i = 0, \ldots, \frac{n-4}{4}\). It is clear that \(Z\) satisfies all our requirements. Assume \(k \geq 5\). To obtain our block \(Z_i\) we use the block \(U_i\) for the first three rows, and we need a shiftable SMA\((k-3, 4; 4, k-3)\) to construct the remaining \(k-3\) rows. So, consider the following shiftable block:

\[
Q = \begin{bmatrix}
1 & -2 & -3 & 4 \\
-1 & 2 & 3 & -4
\end{bmatrix}
\]

For all fixed \(h \geq 0\) we take the \((2h + 2) \times 4\) block \(V_h\):

\[
V_h = \begin{bmatrix}
Q \\
Q + 4 \\
Q + 8 \\
\vdots \\
Q + 4h
\end{bmatrix}
\]

The block \(V_h\) has rows and columns that sum to zero; furthermore, \(\mathcal{E}(V_h) = \pm[1, 4h + 4]\).

Write \(k = 2q + 5\) and define \(Z_i = \begin{bmatrix}
U_i \\
V_q \pm (\frac{3n}{4} + (4q + 4)i)
\end{bmatrix}\)

Note that

\[
\bigcup_{i=0}^{\frac{n-4}{4}} \mathcal{E} \left( V_q \pm \left( \frac{3n}{4} + (4q + 4)i \right) \right) = \bigcup_{i=0}^{\frac{n-4}{4}} \pm\left[ \frac{3n}{4} + 1 + 4(q + 1)i, \frac{3n}{4} + 4(q + 1)(i+1) \right] = \pm\left[ \frac{3n}{4} + 1, \frac{n}{4} \right].
\]

We conclude that \(\mathcal{E}(Z) = \pm[1, \frac{n}{2}]\).

Hence, for all \(k\) odd, we were able to construct the required sequence \(Z\). To obtain an SMA\((m, n; s, k)\), once again, we fill the cell \((a + ki, b + 4i)\) of an array \(H\) of size \(m \times n\) with the element of the cell \((a, b)\) of \(Z_i\), for all \(i = 0, \ldots, \frac{n-4}{4}\). Clearly, every column of \(H\) contains exactly \(k\) filled cells (a column of a unique block \(Z_i\)), and every row of \(H\) contains \(s\) filled cells (a row from each \(\frac{4}{3}\) distinct blocks \(Z_j\)), as \(k\frac{n}{4} = m\frac{4}{3}\). Hence, \(\mathcal{E}(H) = \mathcal{E}(Z) = \pm[1, \frac{n}{2}]\) and, since the elements of each row/column of \(Z_i\) sum to 0, the same holds for \(H\). \(\square\)

**References**

[1] D.S. Archdeacon, Heffter arrays and biembedding graphs on surfaces, Electron. J. Combin. 22 (2015), #P1.74.

[2] D.S. Archdeacon, T. Boothby, J.H. Dinitz, Tight Heffter arrays exist for all possible values, J. Combin. Des. 25 (2017), 5–35.
[3] D.S. Archdeacon, J.H. Dinitz, D.M. Donovan, E.S. Yazici, Square integer Heffter arrays with empty cells, *Des. Codes Cryptogr.* **77** (2015), 409–426.

[4] N. Brown, H. Jordon, Signed Langford sequences and directed cyclic cycle systems, *Australas. J. Combin.* **79** (2021), 234–249.

[5] K. Burrage, D.M. Donovan, N.J. Cavenagh, E.Ş. Yazıcı, Globally simple Heffter arrays $H(n;k)$ when $k \equiv 0,3 \pmod{4}$, *Discrete Math.* **343** (2020), #111787.

[6] N.J. Cavenagh, J.H. Dinitz, D.M. Donovan, E.Ş. Yazıcı, The existence of square non-integer Heffter arrays, *Ars Math. Contemp.* **17** (2019), 369–395.

[7] N.J. Cavenagh, D.M. Donovan, E.Ş. Yazıcı, Biembeddings of cycle systems using integer Heffter arrays, *J. Combin. Des.* **28** (2020), 900–922.

[8] S. Costa, F. Morini, A. Pasotti, M.A. Pellegrini, A problem on partial sums in abelian groups, *Discrete Math.* **341** (2018), 705–712.

[9] S. Costa, F. Morini, A. Pasotti, M.A. Pellegrini, Globally simple Heffter arrays and orthogonal cyclic cycle decompositions, *Australas. J. Combin.* **72** (2018), 549–593.

[10] S. Costa, F. Morini, A. Pasotti, M.A. Pellegrini, A generalization of Heffter arrays, *J. Combin. Des.* **28** (2020), 171–206.

[11] S. Costa, A. Pasotti, On $\lambda$-fold relative Heffter arrays and biembedding multigraphs on surfaces, *European J. Combin.*, **97** (2021), #103370.

[12] S. Costa, A. Pasotti, M.A. Pellegrini, Relative Heffter arrays and biembeddings, *Ars Math. Contemp.* **18** (2020), 241–271.

[13] J.H. Dinitz, A.R.W. Mattern, Biembedding Steiner triple systems and $n$-cycle systems on orientable surfaces, *Australas. J. Combin.* **67** (2017), 327–344.

[14] J.H. Dinitz, I.M. Wanless, The existence of square integer Heffter arrays, *Ars Math. Contemp.* **13** (2017), 81–93.

[15] T. Harmuth, Über magische Quadrate und ähnliche Zahlenfiguren, *Arch. Math. Phys.* **66** (1881), 286–313.

[16] T. Harmuth, Über magische Rechtecke mit ungeraden Seitenzahlen, *Arch. Math. Phys.* **66** (1881), 413–447.

[17] A. Khodkar, B. Ellis, Signed magic rectangles with two filled cells in each column, preprint available at https://arxiv.org/abs/1901.05502.

[18] A. Khodkar, D. Leach, Magic rectangles with empty cells, *Util. Math.* **116** (2020), 45–56.

[19] A. Khodkar, D. Leach, Magic squares with empty cells, to appear in *Ars Combinatoria*, preprint available at https://arxiv.org/abs/1804.11189.

[20] A. Khodkar, D. Leach, B. Ellis, Signed magic rectangles with three filled cells in each column, *Bull. Inst. Combin. Appl.* **90** (2020), 87–106.

[21] A. Khodkar, C. Schulz, N. Wagner, Existence of some signed magic arrays, *Discrete Math.* **340** (2017), 906–926.

[22] F. Morini, M.A. Pellegrini, On the existence of integer relative Heffter arrays, *Discrete Math.* **343** (2020), #112088.

[23] F. Morini, M.A. Pellegrini, Magic rectangles, signed magic arrays and integer $\lambda$-fold relative Heffter arrays, *Australas. J. Combin.* **80** (2021), 249–280.

[24] T. Skolem, On certain distributions of integers in pairs with given differences, *Math. Scand.* **5** (1957), 57–68.