The Formal Affine Demazure Algebra and Real Finite Reflection Groups

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Abstract

In this paper, we generalize the formal affine Demazure algebra of Hoffnung-Malagon-Lopez-Savage-Zainoulline to all real finite reflection groups. We begin by generalizing the formal group ring of Calmes-Petrov-Zainoulline to all real finite reflection groups. We then define and study the formal Demazure operators that act on the formal group ring. Using these results and constructions, we define and study the formal affine Demazure algebra for all real finite reflection groups. Finally, we compute several structure coefficients that appear in a braid relation among the formal Demazure elements, and we conclude this paper by computing all structure coefficients for the reflection groups $I_2(5)$, $I_2(7)$, $H_3$, and $H_4$.

Keywords

Demazure operator · Formal group law · Finite reflection group

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1 Introduction

Let $h^*$ be an algebraic oriented cohomology theory in the sense of Levine-Morel [22]. If $X$ is a smooth projective variety over a field $k$, then, given a vector bundle $E \to X$ over $X$ of rank $n$, there is a set of Chern classes $c_i(E) \in h^i(X)$, $i \in \{0, \ldots, n\}$. The first Chern classes of any two line bundles $L_1$ and $L_2$ over $X$ satisfy Quillen’s formula,

$$c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2)),$$

where $F$ is the formal group law over the ring $R = h^*(\text{Spec}(k))$ associated to $h^*$.

Assume for the rest of this section that $k$ has characteristic 0. Suppose $G$ is a split semisimple simply-connected linear algebraic group over the field $k$, and let $T$ be a maximal split torus sitting inside a Borel subgroup $B$ of $G$, so that $G/B$ is a complete flag variety. Let $\Lambda$ be the character lattice of $T$. In [6], Calmes-Petrov-Zainoulline constructed...
the formal group ring, which we will denote $\widehat{R[\Lambda]}_F$. In addition, the authors defined a ring homomorphism, called the characteristic map,

$$\epsilon_{G/B} : \widehat{R[\Lambda]}_F \rightarrow h^*(G/B).$$

Under certain conditions, the authors used the characteristic map to construct an algebraic model $\mathcal{H}(\Lambda)_F$ for the algebraic oriented cohomology ring $h^*(G/B)$ in terms of augmented Demazure operators. Specializing $h^*$ to the Chow theory, $\mathcal{H}(\Lambda)_F$ is the algebraic model for the Chow theory of the flag variety $G/B$ constructed by Demazure in [10]. We note that the formal group ring $\widehat{R[\Lambda]}_F$ is a purely algebraic object, and it can be defined with respect to any finitely generated free abelian group $\Lambda$, and any one-dimensional commutative formal group law $F$ over any commutative unital ring $R$.

In [18, 19], Kostant-Kumar described the equivariant cohomology and $K$-theory of flag varieties using the techniques of nil-Hecke and 0-Hecke algebras. These algebras are generated by Demazure operators, which satisfy a braid relation. Calmès, Hoffnung, Malagón-López, Savage, Zainoulline, and Zhong used the formal group ring $\widehat{R[\Lambda]}_F$ to generalize the nil-Hecke and 0-Hecke algebras of Kostant-Kumar to an arbitrary algebraic oriented cohomology theory [7–9, 16]. In particular, they constructed and studied the formal affine Demazure algebra, $D_F$, which is generated by formal Demazure elements that satisfy a twisted braid relation (see [16, Prop. 6.8]). After specializing $F$ to the additive and multiplicative formal group laws, $D_F$ equals completion of the nil-Hecke and 0-Hecke algebra, respectively.

Let $W$ be a real finite reflection group, and let $\Sigma$ be a root system of $W$ sitting in $\mathbb{R}^n$. The first goal of this paper is to generalize the formal group ring of [6] to all real finite reflection groups and to define and study divided difference operators that act on the generalized formal group ring. To begin, we construct a finitely generated free additive subgroup $\Lambda$ of $\mathbb{R}^n$ containing $\Sigma$, such that the natural action of $W$ on $\Sigma$ extends to an action of $W$ on $\Lambda$. Given a simple system $\Delta$ of $\Sigma$, let $\mathcal{R}$ be the unital subring of $\mathbb{R}$ generated by the elements $\alpha_i^\vee(\alpha_j)$ over all simple roots $\alpha_i, \alpha_j \in \Delta$ (here, we use the notation $\alpha_i^\vee$ to denote the coroot associated with $\alpha_i$). We choose $\Delta$ so that the ring $\mathcal{R}$ is a finitely generated free abelian group with a power basis $B$. In Section 2, we show that it is always possible to choose such a simple system $\Delta$. Now we define $\Lambda$ as the finitely generated free additive subgroup of $\mathbb{R}^n$ with basis $\{e_i \alpha_j\}$, where $e_i \in B$ and $\alpha_j \in \Delta$. There is a natural action of $W$ on $\Lambda$, defined on generators by $w(e_i \alpha_j) := e_i w(\alpha_j)$ for all $w \in W$, and we call $\Lambda$ the real root lattice. If $\Sigma$ is crystallographic, then the elements $\alpha_i^\vee(\alpha_j)$ are all integers for any choice of simple system $\Delta$. In this case, $B = \{1\}$ and $\Lambda$ is the usual crystallographic root lattice of $\Sigma$, as described in [8, §2].

Suppose $R$ is a subring of $\mathbb{C}$ that contains $\mathcal{R}$ and satisfies the conditions of Assumption 4.1. Let $F$ be a one-dimensional commutative formal group law over $R$. The conditions of Assumption 4.1 imply that there is an isomorphism of formal group laws $\log_F : F \rightarrow F_a$ over $R$, called the logarithm. Using the data of the real root lattice $\Lambda$ and the formal group law $F$ over $R$, we will now describe the construction of the formal group ring $\widehat{R[\Lambda]}_F$ in this paper.

For the remainder of this section, we will call $\widehat{R[\Lambda]}_F$ the classical formal group ring. There is a natural action of $W$ on $\widehat{R[\Lambda]}_F$ via the action of $W$ on $\Lambda$. If $\Sigma$ is crystallographic, then it is possible to define divided difference operators (i.e., formal Demazure operators) that act on $\widehat{R[\Lambda]}_F$. This was done in [6]. However, if $\Sigma$ is noncrystallographic, then there is
no clear way to define formal Demazure operators that act on \( R[\Lambda]_F \) (see Remark 3.14 for further details). To overcome this obstruction, we will attempt to define formal Demazure operators that act on the quotient ring \( R[\Lambda]_F / J_F^* \), where \( J_F^* \) is a certain closed \( W \)-invariant ideal of \( R[\Lambda]_F \) such that \( J_F^* = (0) \) when \( \Sigma \) is crystallographic (see the discussion immediately after Example 4.2 for the definition of \( J_F^* \)). As \( R[\Lambda]_F \) is a complete Hausdorff ring, we require \( J_F^* \) to be closed so that the quotient ring \( R[\Lambda]_F / J_F^* \) is also complete and Hausdorff; we require \( J_F^* \) to be \( W \)-invariant so that there is a well-defined action of \( W \) on \( R[\Lambda]_F / J_F^* \); and we require that \( J_F^* = (0) \) when \( \Sigma \) is crystallographic so that \( R[\Lambda]_F / J_F^* \) equals \( R[\Lambda]_F \) when \( \Sigma \) is crystallographic.

We will now describe how to construct the ideal \( J_F^* \). First assume \( F = F_a \) is the additive formal group law over \( R \). In this case, there is a clear candidate for the ideal \( J_F^* \). Namely, we can choose \( J_{F_a} \) so that \( R[\Lambda]_F / J_{F_a}^* \) is isomorphic to the completion of the symmetric algebra \( S_{R_a}(\Lambda) \) over \( R \) (see Example 4.19). It is possible to define Demazure operators that act on \( S_{R_a}(\Lambda) \) (see [15, Ch. IV]), and, hence, we are able to define Demazure operators that act on \( R[\Lambda]_F / J_{F_a}^* \). Now assume \( F \) is any formal group law over \( R \). The logarithm \( \log_F : F \to F_a \) induces a \( W \)-equivariant isomorphism of classical formal group rings \( \log_F^*: \ R[\Lambda]_F / J_{F_a}^* \to \ R[\Lambda]_F \). In particular, the image \( J_F^* := \log_F^* ( J_{F_a}^* ) \) is a closed \( W \)-invariant ideal in \( R[\Lambda]_F \), such that \( J_F^* = (0) \) when \( \Sigma \) is crystallographic. Note that the continuous \( W \)-equivariant ring isomorphism \( \log_F^*: \ R[\Lambda]_F / J_{F_a}^* \to \ R[\Lambda]_F \) induces an isomorphism \( R[\Lambda]_F / J_{F_a}^* \simeq \ R[\Lambda]_F / J_F^* \). In Lemma 5.1, we show that it is possible to define formal Demazure operators that act on \( R[\Lambda]_F / J_F^* \).

We are now ready to define the formal group ring \( R[\Lambda]_F \). Suppose \( R[x_\Lambda] \) is the polynomial ring over \( R \), with variables indexed by elements in \( \Lambda \), and let \( R[x_\Lambda] \) be the completion of \( R[x_\Lambda] \) at the ideal \( \ker(\epsilon) \), where \( \epsilon: R[x_\Lambda] \to R \) is the ring homomorphism that sends \( x_\lambda \mapsto 0 \) for all \( \lambda \in \Lambda \). We define a certain closed ideal \( J_F \) of \( R[x_\Lambda] \), and we show that \( R[x_\Lambda] / J_F \simeq R[\Lambda]_F / J_F^* \) as topological rings (see Lemma 4.14). Since we are able to define formal Demazure operators that act on \( R[\Lambda]_F / J_F^* \), it follows that we can define formal Demazure operators that act on \( R[x_\Lambda] / J_F \). We define the formal group ring as the quotient ring \( R[\Lambda]_F : = R[x_\Lambda] / J_F \) (see Definition 4.7). The formal Demazure operators that act on \( R[\Lambda]_F \) generalize the formal Demazure operators that were studied in [6] to all real finite reflection groups.

Next, we study the subalgebra \( D_{(R,F)}(\Lambda) \) of the endomorphism algebra of \( R[\Lambda]_F \) generated by formal Demazure operators and by multiplication by elements in \( R[\Lambda]_F \). In particular, we show that \( D_{(R,F)}(\Lambda) \) is a finitely generated free \( R[\Lambda]_F \)-module. When \( \Sigma \) is crystallographic, the \( R[\Lambda]_F \)-algebra \( D_{(R,F)}(\Lambda) \) is the endomorphism algebra studied in [6]. When \( F \) is the additive formal group law and \( R = \mathbb{C} \), the \( R[\Lambda]_F \)-algebra \( D_{(R,F)}(\Lambda) \) is the completion of the endomorphism algebra studied by Hiller in [15, Ch. IV]. When \( F \) is the additive formal group law, \( R = \mathbb{Z} \), and \( \Sigma \) is crystallographic, the \( R[\Lambda]_F \)-algebra \( D_{(R,F)}(\Lambda) \) is the completion of the endomorphism algebra studied by Demazure in [10].

The second goal of this paper is to generalize the formal affine Demazure algebra of [16] to all real finite reflection groups. To begin, we use the formal group ring \( R[\Lambda]_F \) to define the formal affine Demazure algebra \( D_F \) for all real finite reflection groups. Next, we show that several results in [8, 16] regarding the crystallographic formal affine Demazure
algebra extend to all real finite reflection groups. In particular, we describe $D_F$ in terms of generators and relations in Theorem 8.9, as was done in the crystallographic case in [8, Thm. 7.9]. We then compute several structure coefficients that appear in a braid relation among the formal Demazure elements (see Theorem 9.10), and we introduce a seemingly new formula involving the formal Demazure elements in the process (see Corollary 9.4). In addition, we compute all structure coefficients for the dihedral groups $I_2(5)$ and $I_2(7)$ in Examples 10.4 and 10.5, and the reflection groups $H_3$ and $H_4$ in Examples 10.11 and 10.12. The formulas for the structure coefficients that we obtain in this paper generalize several formulas that appear in the literature, such as the formulas in [16, Prop. 6.8] and [14, Section 8], to all real finite reflection groups.

This paper is organized as follows. In Section 2, we recall the definition and basic properties of real finite reflection groups and their associated root systems. We then define the real root lattice $\Lambda_1$, generalizing the crystallographic root lattice to noncrystallographic root systems. In Section 3, we recall the definition and basic properties of formal groups laws, and we discuss the formal group ring constructed in [6] with respect to the real root lattice. In Section 4, we construct the formal group ring for all real finite reflection groups. In Section 5, we define and study the formal Demazure operators that act on the formal group ring. In Section 6, we define the associated graded ring with respect to the formal group ring and use it to prove a basis theorem for a subalgebra of the endomorphism algebra of the formal group ring containing the formal Demazure operators. This section closely follows [6, §4]. In Section 7, we define the formal affine Demazure algebra for all real finite reflection groups, and we discuss several properties of this algebra. In Section 8, we give a presentation for the formal affine Demazure algebra in terms of the formal Demazure elements. This section closely follows [8, §6 and §7]. In Section 9, we compute several structure coefficients that appear in a braid relation among the formal Demazure elements. In Section 10, we analyze the structure coefficients derived in Section 9 after specializing to certain formal group laws, and we compute all structure coefficients for the groups $I_2(5), I_2(7), H_3,$ and $H_4$. In Appendix, we provide computations for products of up to seven formal Demazure elements for all real finite reflection groups.

### 2 Real Finite Reflection Groups and the Real Root Lattice

In this section, we recall the definition and basic properties of real finite reflection groups and their associated root systems. We then define the real root lattice $\Lambda$, generalizing the crystallographic root lattice to noncrystallographic root systems.

Let $V = \mathbb{R}^n$, and let $(\cdot, \cdot)$ be the standard inner product on $V$. For any $\alpha \in V \setminus \{0\}$, the reflection across $\alpha$ is the linear operator $s_\alpha$ defined by the formula

$$s_\alpha(v) = v - 2\frac{(\alpha, v)}{(\alpha, \alpha)}\alpha, \quad v \in V.$$ 

One can easily compute that $s_\alpha^2 = 1$.

**Definition 2.1** (see [17, pp. 6] or [15, Ch. I, §3, Def. 3.1]) A root system $\Sigma$ in $V$ is a finite set of nonzero vectors in $V$ satisfying the conditions:

1. $\Sigma \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Sigma$;
2. $s_\alpha(\Sigma) = \Sigma$ for all $\alpha \in \Sigma$;
3. The roots $\alpha \in \Sigma$ span $V$.

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The rank of a root system in $V$ is the dimension of $V$ as an $\mathbb{R}$-vector space. Fix any root system $\Sigma$ in $V$, and let $\alpha \in \Sigma$ be any root. Let $\alpha^\vee : V \to \mathbb{R}$ be the linear function defined by $\alpha^\vee(\beta) = \frac{2(\alpha, \beta)}{(\alpha, \alpha)}$, $\beta \in V$. We call $\alpha^\vee$ the coroot of $\alpha$. We adopt the following naming convention:

- If $\alpha^\vee(\beta) \in \mathbb{Z}$ for all $\alpha, \beta \in \Sigma$, we call $\Sigma$ a crystallographic root system. Otherwise, we call $\Sigma$ a noncrystallographic root system.

Let $W$ be the group generated by the reflections $s_{\alpha}$ over all roots $\alpha \in \Sigma$. We call $W$ the real finite reflection group of $\Sigma$. If $\Sigma$ is crystallographic, we call $W$ a Weyl group. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a subset of $\Sigma$. We call $\Delta$ a simple system of $\Sigma$ if it is an $\mathbb{R}$-basis of $V$, and if every root $\alpha \in \Sigma$ can be written as an $\mathbb{R}$-linear combination of elements in $\Delta$ with all coefficients nonpositive or all coefficients nonnegative. The elements in $\Delta$ are called simple roots and the reflection across a simple root is called a simple reflection. It is shown in [17, Ch. I, Thm. 1.3] that every root system contains a simple system. If a root $\alpha$ can be written as a linear combination of the simple roots with all coefficients nonpositive (resp. nonnegative), then $\alpha$ is called a negative root (resp. positive root). The positive system $\Sigma^+$ is the set of positive roots of $\Sigma$ with respect to $\Delta$, and the negative system $\Sigma^-$ is the set of negative roots of $\Sigma$ with respect to $\Delta$. If $\alpha_i$ is a simple root, we denote the reflection $s_{\alpha_i}$ by $s_i$, and we set $S = \{s_i \mid \alpha_i \in \Delta\}$.

**Lemma 2.2** (see [17, Ch. I, Cor. 1.5]) For any $\alpha \in \Sigma$, there is a reflection $w \in W$ and a simple root $\alpha_i \in \Delta$ such that $\alpha = w(\alpha_i)$.

**Theorem 2.3** (see [17, Ch. I, Thm. 1.9]) The group $W$ is generated by the set of simple reflections $S$, subject only to the relations:

$$(s_i s_j)^{m_{i,j}} = 1, \quad \alpha_i, \alpha_j \in \Delta,$$

where $m_{i,j}$ is the order of $s_i s_j$ in $W$.

**Remark 2.4** Lemma 2.2 and Theorem 2.3 tell us that, if $\Delta$ is a simple system of a root system $\Sigma$, then $\Sigma$ consists of the elements $w(\alpha_j)$, where $w = s_{i_1} \cdots s_{i_k}$ is a product of simple reflections and $\alpha_j \in \Delta$.

**Theorem 2.5** (see [17, Ch. I, §2]) Let $\Sigma$ be a noncrystallographic root system, and let $W$ be the real finite reflection group of $\Sigma$. Then $W$ is either the dihedral group $I_2(m)$, $m \geq 3$, of order $2m$, the full icosahedral group $H_3$, or the symmetry group $H_4$ of the hyperdodecahedron.

**Definition 2.6** (see [17, Ch. I, §1.6 and Ch. II, §5.9]) By Theorem 2.3, we can write $w \in W$ as a product $w = s_{i_1} \cdots s_{i_r}$ of simple reflections. The length $l(w)$ of $w$ is the minimal $r$ for which such an expression exists, and we call such an expression a reduced decomposition of $w$. The longest word in $W$ is the unique word in $W$ of maximal length. Write $w' \to w$ if $l(w) > l(w')$ and $w = w'u$ for some $u \in W$. We define the Bruhat order on $W$ to be the partial ordering $< on $W$ defined by $w' < w$ if and only if there is a sequence $w' = w_0 \to w_1 \to \cdots \to w_m = w$.

We will refer to the next few results frequently throughout this paper. We use the notation $s^k_{i,j,\ldots}$ to denote the product of $k$ simple reflections $s_i s_j s_i s_j \cdots$. 

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Proposition 2.7 (see [15, Ch. I, Prop. 3.6])  Let $v = s_i \cdots s_t$ be a reduced decomposition of $v \in W$. Then

$$v(\Sigma) \cap \Sigma^+ = \{ \theta_{ij} : 1 \leq j \leq t \},$$

where $\theta_{ij} = \alpha_{ij}$, and $\theta_{ij} = s_{ij} \cdots s_{i-1}(\alpha_{ij})$ if $j > 1$.

If $v$ is the longest word in $W$, then

$$\Sigma^+ = \{ \theta_{ij} : 1 \leq j \leq t \}.$$

Lemma 2.8 For $i \neq j$, we have

$$\alpha_j = \begin{cases} s_i^{-1}(\alpha_j), & m_{i,j} \text{ even;} \\ s_i^{-1}(\alpha_i), & m_{i,j} \text{ odd;} \end{cases}$$

where $m_{i,j}$ is the order of $s_is_j$ in $W$.

Proof Suppose $m_{i,j}$ is odd. Since $s_i^{-1}(\alpha_j) = s_i^{-1}(\alpha_j)$, it follows from Proposition 2.7 that

$$\Sigma_i := \{ \alpha_i, s_i(\alpha_j), s_is_j(\alpha_i), \ldots, s_i^{-(m_{i,j}-1)}(\alpha_i) \} = \{ \alpha_j, s_j(\alpha_i), s_j s_i(\alpha_j), \ldots, s_i^{-(m_{i,j}-1)}(\alpha_j) \}.$$ 

Thus,

$$s_i(\Sigma_i^+) = \{ s_j(\alpha_i), s_j s_i(\alpha_j), s_j s_i s_j(\alpha_i), \ldots, s_j^{-(m_{i,j}-2)}(\alpha_i) \} = \{ s_j(\alpha_j), s_i(\alpha_i), s_i(\alpha_j), \ldots, s_i^{-(m_{i,j}-2)}(\alpha_j) \}.$$ 

Since $s_j(\Sigma_i^+) \cap \Sigma^- = \{ s_j(\alpha_j) \}$, we must have that $s_i^{-(m_{i,j}-1)}(\alpha_i) = \alpha_j$. The proof for even $m_{i,j}$ is similar.

Proposition 2.9 (see [17, Ch. I, §1.7, Ex. 2]) Let $w = s_i \cdots s_t$ be a (not necessarily reduced) decomposition of $w \in W$ in terms of simple reflections. If $l(sw) < l(w)$ for some simple reflection $s$, then there is an index $ij$ for which $sw = s_is_i \cdots s_{ij} \cdots s_t$.

Theorem 2.10 (see [2, Ch. 2, §2.2, Thm. 2.2.2]) Let $w = s_i \cdots s_t$ be a reduced decomposition of $w \in W$. Then $u \leq w$ if and only if there is a reduced decomposition $u = s_{r_1} s_{r_2} \cdots s_{r_t}$ of $u$ such that $1 \leq r_1 < r_2 < \cdots < r_t \leq q$.

Proposition 2.11 (see [17, Ch. I, Prop. 1.2]) For any root $\beta \in \Sigma$ and reflection $w \in W$, we have $ws_\alpha w^{-1} = s_{w(\alpha)}$.

Definition 2.12 (cf. [15, pp. 22]) If $\Sigma$ is a root system in $V$ with finite reflection group $W$, and $\Delta$ is a simple system of $\Sigma$, then we call the pair $(\Sigma, \Delta)$ a geometric realization of $W$ in $V$.

Definition 2.13 Let $\alpha \in \Sigma$. The length $l(\alpha)$ of $\alpha$ is its length as a vector in $V = \mathbb{R}^n$. In other words, $l(\alpha) = \sqrt{\langle \alpha, \alpha \rangle}$.

Remark 2.14 As stated in [17, Ch. I, §1.1, Ex. 1], the reflections form a single conjugacy class in $I_2(m)$ when $m$ is odd, but form two conjugacy classes when $m$ is even. This tells us that, given a geometric realization of $I_2(m)$, all roots have the same length when $m$ is odd. However, only the lengths of the roots in each conjugacy class are guaranteed to be the equal when $m$ is even.
Remark 2.15 The rank 2 root systems are $A_1 \times A_1$, $A_2$, $B_2$, $G_2$, and the root systems of $I_2(m), m \geq 3$.

Let $W$ be a real finite reflection group, and let $(\Sigma, \Delta)$ be a geometric realization of $W$ in $V$, with simple system $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. Let $\alpha \in \Sigma$ be any root. By definition, there exist unique elements $c_{ij}^\alpha \in \mathbb{R}$ such that $\alpha = c_{ij}^\alpha \alpha_i + \cdots + c_{in}^\alpha \alpha_n$. We would like to determine whether the subring $\mathcal{R}$ of $\mathbb{R}$ generated by the elements $c_{ij}^\alpha$ over all $i = 1, \ldots, n$ and $\alpha \in \Sigma$ is a finitely generated free abelian group with a power basis, i.e., a basis of the form $\{1, \beta, \beta^2, \ldots, \beta^{l-1}\}$ for some $l \geq 1$, where $\beta \in \mathcal{R}$. As $c_{ij}^{\alpha_i} = 1$, the ring $\mathcal{R}$ must contain 1. If $\alpha$ is not a simple root, then Lemma 2.2 and Theorem 2.3 imply that $\alpha = s_{i_1} \cdots s_{i_k}(\alpha_j)$, where $\alpha_j \in \Delta$ and the $s_{i_k}$ are simple reflections. Thus, $\mathcal{R}$ is the unital subring of $\mathbb{R}$ generated by the elements $c_{ij}^\alpha(\alpha_j)$ over all $\alpha_i, \alpha_j \in \Delta$.

For the remainder of this paper, we will work under the following assumption:

Assumption 2.16 We assume $W$ is a real finite reflection group, and $(\Sigma, \Delta)$ is a geometric realization of $W$ in $V$, such that $\mathcal{R}$ is a finitely generated free abelian group with a power basis.

We will show the existence of a geometric realization for $W$ satisfying the conditions of Assumption 2.16 for every real finite reflection group $W$. If $W$ is a Weyl group, then $\mathcal{R} = \mathbb{Z}$, since the elements $\alpha_i^\vee(\alpha_j) \in \mathbb{Z}$ for all $\alpha_i, \alpha_j \in \Delta$.

The geometric realizations of $W = H_3$ and $W = H_4$ that follow are taken from [17, §2.13], and the geometric realizations of $W = I_2(m), m \geq 3$, that follow are taken from [17, Ch. §1.1]. First, we will consider the case $W = H_4$. In this case, $V = \mathbb{R}^4$. Set $a := \frac{1+\sqrt{5}}{4}$ and $b := -\frac{1+\sqrt{5}}{4}$. Let $(\Sigma, \Delta)$ be the geometric realization of $W = H_4$ in $V$ with simple roots:

$\alpha_1 = \left(a, -\frac{1}{2}, b, 0\right); \quad \alpha_2 = \left(-a, \frac{1}{2}, b, 0\right); \quad \alpha_3 = \left(\frac{1}{2}, b, -a, 0\right); \quad \alpha_4 = \left(-\frac{1}{2}, -a, 0, b\right).$

A computation gives the following result:

$\alpha_i^\vee(\alpha_j) = \begin{cases} 2, & \text{if } i = j; \\ -\tau, & \text{if } (i, j) \in \{(1, 2), (2, 1)\}; \\ -1, & \text{if } (i, j) \in \{(2, 3), (3, 2), (3, 4), (4, 3)\}; \\ 0, & \text{if } (i, j) \in \{(1, 3), (3, 1), (1, 4), (4, 1), (2, 4), (4, 2)\}, \end{cases}$

where $\tau = \frac{1+\sqrt{5}}{2}$ is the well-known golden section. The constant $\tau$ is a root of the characteristic equation $x^2 - x - 1 = 0$. Thus, we may take $\mathcal{R} = \mathbb{Z}[\tau]$, which has a power basis $\{1, \tau\}$ over $\mathbb{Z}$.

Now we will consider the case $W = H_3$. In this case, $V = \mathbb{R}^3$. Let $(\Sigma, \Delta)$ be the geometric realization of $W = H_3$ in $V$ with simple roots:

$\alpha_1 = \left(a, -\frac{1}{2}, b\right); \quad \alpha_2 = \left(-a, \frac{1}{2}, b\right); \quad \alpha_3 = \left(\frac{1}{2}, b, -a\right).$

By the earlier computation, we may take $\mathcal{R} = \mathbb{Z}[\tau]$ in this case as well.

Next we will consider the dihedral groups $W = I_2(m)$, where $m \geq 3$. In this case, $V = \mathbb{R}^2$. Let $(\Sigma, \Delta)$ be the geometric realization of $W = I_2(m)$ in $V$ with simple roots:

$\alpha_1 = (-1, 0); \quad \alpha_2 = \left(\cos\left(\frac{\pi}{m}\right), \sin\left(\frac{\pi}{m}\right)\right).$
A computation gives the following result:

\[
\alpha_i^\gamma(\alpha_j) = \begin{cases} 
2, & \text{if } i = j; \\
-2 \cos \left( \frac{\pi}{m} \right), & \text{if } (i, j) \in \{(1, 2), (2, 1)\}.
\end{cases}
\]

Therefore, \(R = \mathbb{Z}[2 \cos \left( \frac{\pi}{m} \right)]\). We claim that \(R = \mathbb{Z}[2 \cos \left( \frac{\pi}{m} \right)]\) is a finitely generated free abelian group with a power basis. In [20, Thm. 1], it is proven that the element \(2 \cos \left( \frac{\pi}{m} \right)\) is an algebraic integer of degree \(l = \frac{\phi(2m)}{2}\), where \(\phi\) is the Euler totient function. In [23], it is shown that

\[
B = \left\{ 1, 2 \cos \left( \frac{\pi}{m} \right), \left( 2 \cos \left( \frac{\pi}{m} \right) \right)^2, \ldots, \left( 2 \cos \left( \frac{\pi}{m} \right) \right)^{l-1} \right\}
\]

is an integral basis for the number field \(\mathbb{Q}(2 \cos \left( \frac{\pi}{m} \right)) = \mathbb{Q}(2 \cos \left( \frac{2\pi}{m} \right))\). This means that the set \(B\) is a \(\mathbb{Z}\)-basis for the ring of integers of \(\mathbb{Q}(2 \cos \left( \frac{\pi}{m} \right))\). Since \((\cos \left( \frac{\pi}{m} \right), \sin \left( \frac{\pi}{m} \right)) = \left( \cos \left( \frac{2\pi}{m} \right), \sin \left( \frac{2\pi}{m} \right) \right)\) is a primitive \(2m\)-th root of unity, it follows from [26] that the ring \(R = \mathbb{Z}[2 \cos \left( \frac{\pi}{m} \right)]\) is the ring of integers of \(\mathbb{Q}(2 \cos \left( \frac{\pi}{m} \right))\). Therefore, \(B\) is a power basis for \(R = \mathbb{Z}[2 \cos \left( \frac{\pi}{m} \right)]\).

We now go over four concrete examples of the geometric realization of dihedral groups constructed above. Some of the formulas we compute in these examples are used in Section 10.

**Example 2.17** Suppose \(m = 3\). Since \(2 \cos \left( \frac{\pi}{3} \right) = 1\), we have \(R = \mathbb{Z}\). Clearly, \(\{1\}\) is a \(\mathbb{Z}\)-basis of \(R\). The roots of \(\Sigma\) are:

\[
\alpha_1; \quad s_2(\alpha_1) = \alpha_1 + \alpha_2; \quad s_1s_2(\alpha_1) = \alpha_2;
\]

\[
s_2s_1s_2(\alpha_1) = -\alpha_2; \quad s_1s_2s_1s_2(\alpha_1) = -\alpha_1 - \alpha_2; \quad s_2s_1s_2s_1s_2(\alpha_1) = -\alpha_1.
\]

**Example 2.18** Suppose \(m = 4\). Since \(2 \cos \left( \frac{\pi}{4} \right) = \sqrt{2}\), we have \(R = \mathbb{Z}[\sqrt{2}]\). Moreover, \(\{1, \sqrt{2}\}\) is a \(\mathbb{Z}\)-basis of \(R\), since \(\sqrt{2}\) is a root of the irreducible quadratic polynomial \(x^2 - 2\) over \(\mathbb{Q}\). The roots of \(\Sigma\) are:

\[
\alpha_1; \quad s_2(\alpha_1) = \alpha_1 + \sqrt{2}\alpha_2; \quad s_1s_2s_1s_2(\alpha_1) = -\alpha_1 - \alpha_2; \quad s_2s_1s_2s_1s_2(\alpha_1) = -\sqrt{2}\alpha_1 - \alpha_2;
\]

\[
\alpha_2; \quad s_1(\alpha_2) = \sqrt{2}\alpha_1 + \alpha_2; \quad s_2s_1s_2s_1(\alpha_2) = -\alpha_2; \quad s_1s_2s_1s_2s_1(\alpha_2) = -\sqrt{2}\alpha_1 - \alpha_2.
\]

**Example 2.19** Suppose \(m = 5\). Since \(2 \cos \left( \frac{\pi}{5} \right) = \frac{1+\sqrt{5}}{2} = \tau\) (this is the golden section, which was mentioned earlier), we have \(R = \mathbb{Z}[\tau]\). Moreover, \(\{1, \tau\}\) is a \(\mathbb{Z}\)-basis of \(R\), since \(\tau\) is a root of the irreducible quadratic polynomial \(x^2 - x - 1\) over \(\mathbb{Q}\). The roots of \(\Sigma\) are:

\[
\alpha_1; \quad s_2(\alpha_1) = \alpha_1 + \tau\alpha_2; \quad s_1s_2s_1s_2(\alpha_1) = \tau\alpha_1 + \tau\alpha_2; \quad s_2s_1s_2s_1s_2(\alpha_1) = \tau\alpha_1 + \alpha_2;
\]

\[
\alpha_2; \quad s_1(\alpha_2) = s_1^{(4)}(\alpha_1) = s_1^{(5)}(\alpha_1) = s_1^{(6)}(\alpha_1) = -\alpha_2; \quad s_1^{(7)}(\alpha_1) = -\tau\alpha_1 - \alpha_2; \quad s_1^{(8)}(\alpha_1) = -\alpha_1 - \tau\alpha_2; \quad s_1^{(9)}(\alpha_1) = -\alpha_1.
\]

**Example 2.20** Suppose \(m = 7\). Set \(\zeta = 2 \cos \left( \frac{\pi}{7} \right)\). We have \(R = \mathbb{Z}[\zeta]\). Moreover, \(\{1, \zeta, \zeta^2\}\) is a \(\mathbb{Z}\)-basis of \(R\), since \(\zeta\) is a root of the irreducible cubic polynomial \(x^3 - x^2 - 2x + 1\) over \(\mathbb{Q}\). The roots of \(\Sigma\) are:

\[
\alpha_1; \quad s_2(\alpha_1) = \alpha_1 + \zeta\alpha_2; \quad s_1s_2(\alpha_1) = (\zeta^2 - 1)\alpha_1 + \zeta\alpha_2; \quad s_2s_1s_2(\alpha_1) = (\zeta^2 - 1)\alpha_1 + (\zeta^2 - 1)\alpha_2;
\]

\[
s_1^{(4)}(\alpha_1) = \zeta\alpha_1 + (\zeta^2 - 1)\alpha_2; \quad s_1^{(5)}(\alpha_1) = \zeta\alpha_1 + \alpha_2; \quad s_1^{(6)}(\alpha_1) = \alpha_2; \quad s_1^{(7)}(\alpha_1) = -\alpha_2.
\]
For the remainder of this paper, we will fix a power basis $B$ of $\mathcal{R}$. This is recorded in Assumption 2.21:

**Assumption 2.21** The set $B = \{e_1, \ldots, e_l\}$ is a power basis of $\mathcal{R}$, and $e_1 = 1$.

**Definition 2.22** Let $\Lambda$ be the $\mathcal{R}$-module generated by the roots $\alpha \in \Sigma$. We call $\Lambda$ a real root lattice. If $\Sigma$ is crystallographic, then we call $\Lambda$ a crystallographic root lattice.

By construction of $\mathcal{R}$, the roots $\alpha \in \Sigma$ are $\mathcal{R}$-linear combinations of the simple roots $\alpha_i \in \Delta$. Thus, $\Lambda$ is generated as an $\mathcal{R}$-module by $\Delta$. Assume towards a contradiction that there exist $t_1, \ldots, t_n \in \mathcal{R}$ such that $t_1 \alpha_1 + \cdots + t_n \alpha_n = 0$ in $V$, and such that $t_i \neq 0$ for some index $i$. Since $\mathcal{R} \subseteq \mathbb{R}$, this contradicts the fact that $\Delta$ forms a basis of the $\mathbb{R}$-vector space $V$ spanned by the roots $\Sigma$. Therefore, $\Delta$ is linearly independent over $\mathcal{R}$, so $\Delta$ is an $\mathcal{R}$-basis of $\Lambda$. Furthermore, we showed earlier that $B$ is a $\mathbb{Z}$-basis of $\mathcal{R}$. Thus, we can take $\{e_i \alpha_j\}$ as a $\mathbb{Z}$-basis of $\Lambda$. In particular, $\Lambda$ is a finitely generated free abelian group of rank $nl$. Moreover, since $e_1 = 1$, the basis $\{e_i \alpha_j\}$ contains $\Delta$.

As $W$ acts on $\Sigma$ by permuting the roots, there is a natural action of $W$ on $\Lambda$ given by $w \cdot (r \alpha) = r(w(\alpha))$ for all $w \in W$, $\alpha \in \Sigma$, and $r \in \mathcal{R}$. By linearity, the action of $w \in W$ on $\Lambda$ is a group homomorphism.

**Remark 2.23** The lattice $\Lambda$ depends on the choice of simple system $\Delta$ and on the choice of power basis $B$. Thus, the construction of the formal group ring $R[\Lambda]_F$ in Section 4 depends on the choice of $\Delta$ and on the choice of $B$. Further details are given in Remark 3.14.

### 3 Formal Group Laws and the Classical Formal Group Ring

In this section, we recall the definition of a one-dimensional commutative formal group law, and we discuss several of its properties. We then analyze the formal group ring $R[\Lambda]_F$ of [6] associated to the real root lattice $\Lambda$ and the formal group law $(R, F)$.

**Definition 3.1** (see [22, pp. 4]) A one-dimensional commutative formal group law $(R, F)$ over a commutative unital ring $R$ is a power series $F(u, v) \in R[u, v]$ satisfying the following axioms:

1. $F(u, 0) = F(0, u) = u \in R[u]$;
2. $F(u, v) = F(v, u)$;
3. $F(u, F(v, w)) = F(F(u, v), w) \in R[u, v, w]$.

Fix a commutative unital ring $R$. By [24, Ch. IV, Exercise 4.1], given a formal group law $(R, F)$, there is a unique power series $i(u) \in R[u]$ such that $F(u, i(u)) = F(i(u), u) = 0$. The series $i(u)$ is called the formal inverse of $u$ with respect to the formal group law $(R, F)$.

Following [24, Ch. IV, §2], a morphism $f : (R, F) \to (R, F')$ of formal group laws over $R$ is a power series $f(u) \in R[u]$ such that $f(F(u, v)) = F'(f(u), f(v))$ and $f(0) = 0$. The morphism $f$ is called an isomorphism if there is a morphism $g : (R, F') \to (R, F)$ of formal group laws over $R$, such that $f(g(u)) = g(f(u)) = u$. 
Let \((\mathbb{C}, F)\) be a one-dimensional commutative formal group law over \(\mathbb{C}\). Write \(F(u, v) = u + v + \sum_{i,j \geq 1} ti,j u^i v^j\), where \(t_{i,j} \in \mathbb{C}\), and suppose \((\mathbb{C}, F_a)\) is the additive formal group law over \(\mathbb{C}\), i.e., \(F_a(u, v) = u + v\). In [24, Ch. IV.5], the author constructs an isomorphism of formal group laws \(\log_F : (\mathbb{C}, F) \to (\mathbb{C}, F_a)\) called the logarithm of \((\mathbb{C}, F)\). The inverse of the logarithm is called the exponential of \((\mathbb{C}, F)\) and is denoted \(\exp_F\). In [24, Ch. IV.4], it is shown that \(\log_F(u) = \int \omega(u)\), where \(\omega(v) = F_a(0, v)dv\) is the normalized invariant differential of \((\mathbb{C}, F)\) and \(F_a(x, y)\) is the partial derivative of \(F(u, v)\) with respect to \(u\), evaluated at \((x, y)\). There exist \(a_i, b_i \in \mathbb{C}, i \geq 2\), such that

\[
\exp_F(u) = u + \sum_{i \geq 2} a_i u^i; \quad \log_F(u) = u + \sum_{i \geq 2} b_i u^i.
\]

Note that \(\exp_{F_a}(u) = u\) and \(\log_{F_a}(u) = u\). Set \(C_F := \{a_i\}_{i \geq 2} \cup \{b_i\}_{i \geq 2} \cup \{t_{i,j}\}_{i,j \geq 1}\).

**Definition 3.2** If \(S\) is any subring of \(\mathbb{C}\) containing \(C_F\), then we call \(S\) an ample ring with respect to the formal group law \((\mathbb{C}, F)\).

If \(R\) is an ample ring with respect to \((\mathbb{C}, F)\), then we may view \((\mathbb{C}, F)\) as a formal group law \((R, F)\) over \(R\). In particular, there is an isomorphism of formal group laws \(\log_F : (R, F) \to (R, F_a)\), with inverse \(\exp_F : (R, F_a) \to (R, F)\), induced by the logarithm and exponential of \((\mathbb{C}, F)\). We call \(\log_F : (R, F) \to (R, F_a)\) and \(\exp_F : (R, F_a) \to (R, F)\) the logarithm and exponential of \((R, F)\), respectively.

**Example 3.3** (see [22, Ex. 1.1.4]) The additive formal group law \((R, F_a)\) over \(R\) is given by \(F_a(x, y) = x + y\). The formal inverse of \(x\) under \((R, F_a)\) is \(i(x) = -x\).

If \(R\) is an ample ring with respect to \((\mathbb{C}, F_a)\), then the logarithm of \((R, F_a)\) is \(\log_{F_a}(x) = x\), and the exponential of \((R, F_a)\) is \(\exp_{F_a}(x) = x\).

**Example 3.4** (see [22, Ex. 1.1.5] and [24, Ch. IV, §9, Ex. 5.1]) The multiplicative formal group law \((R, F_m)\) over \(R\) is given by \(F_m(x, y) = x + y + xy\). The formal inverse of \(x\) under \((R, F_m)\) is \(i(x) = \frac{x}{1+x} := -x \sum_{i \geq 0} x^i\).

If \(R\) is an ample ring with respect to \((\mathbb{C}, F_m)\), then the logarithm and exponential of \((R, F_m)\) are given by the formulas

\[
\log_{F_m}(x) = \log(1 + x) = \sum_{i \geq 1} (-1)^{i-1} \frac{x^i}{i}; \quad \exp_{F_m}(x) = \exp(x) - 1 = \sum_{i \geq 1} \frac{x^i}{i!}.
\]

Observe that \(R\) contains \(\mathbb{Q}\).

**Example 3.5** (see [25, Ch. IV, §9, Ex. 3.4(3)]) The Lorentz formal group law \((R, F_l)\) over \(R\) is given by \(F_l(x, y) = \frac{x+y}{1+xy} := (x + y) \sum_{i \geq 2} (xy)^i\). The formal inverse of \(x\) under \((R, F_l)\) is \(i(x) = -x\).

If \(R\) is an ample ring with respect to \((\mathbb{C}, F_l)\), then the logarithm and exponential of \((R, F_l)\) are given by the formulas

\[
\log_{F_l}(x) = \tanh^{-1}(x) = \sum_{i \geq 1} \frac{x^{2i-1}}{2i - 1}; \quad \exp_{F_l}(x) = \tan(x) = \sum_{i \geq 1} \frac{2^i (2^{2i} - 1) B_{2i}}{(2i)!} x^{2i-1},
\]

where \(B_{2i}\) is the \(2i\)-th Bernoulli number.
Example 3.6 (see [22, §1.1]) Let \( \mathbb{L} \) be Lazard ring, i.e., the commutative unital ring generated by some elements \( r_{i,j}, i, j \geq 1 \), subject only to the relations imposed by the definition of the formal group law. The universal formal group law \((\mathbb{L}, F_u)\) over the Lazard ring \( \mathbb{L} \) is given by
\[
F_u(x, y) = x + y + \sum_{i,j \geq 1} r_{i,j} x^i y^j.
\]
The formal inverse of \( x \) under \((\mathbb{L}, F_u)\) is a power series \( i(x) = -x - c_2 x^2 - c_3 x^3 - \cdots \in \mathbb{L}[x] \), where the \( c_i \) can be computed explicitly in terms of the \( r_{i,j} \) from the relation \( F_u(x, i(x)) = 0 \).

Definition 3.7 Let \((R, F)\) be a formal group law of the form
\[
F(x, y) = (x + y)g(x, y), \quad g(x, y) \in R[[x, y]].
\]
We say that \((R, F)\) is of additive type. In particular, the formal inverse of \( x \) under \((R, F)\) is \( i(x) = -x \).

Example 3.8 The additive formal group law and the Lorentz formal group law are examples of formal group laws of additive type.

We now state and prove some standard facts about topological rings.

Lemma 3.9 Let \( S \) be a commutative unital topological ring, and let \( I \) be an ideal in \( S \). The topological closure \( \overline{I} \) of \( I \) in \( S \) is an ideal in \( S \).

Proof Let \( x, y \in \overline{I} \) and \( s \in S \). The difference \( x - y \in \overline{I} \) by the proof of [4, Ch. III, §2.1, Prop. 1]. Now all we need to show is that \( sx \in \overline{I} \). Let \( U \) be a neighbourhood of \( sx \), and consider the multiplication map \( m: S \times S \rightarrow S \), where \( m(a, b) = ab \) for all \( a, b \in S \). The map \( m \) is continuous, so \( m^{-1}(U) \) is open in \( S \times S \) and contains the point \((s, x)\). Write \( m^{-1}(U) = (V_s, V_x) \) for some open neighbourhoods \( V_s \) and \( V_x \) of \( s \) and \( x \), respectively. Since \( x \in \overline{I} \), we have \( V_x \cap I \neq \emptyset \). Choose \( y \in V_x \cap I \). Since \( y \in V_x \), we have \( s \cdot y \in m(s, V_x) \subseteq m(V_s, V_x) = U \), and since \( y \in I \), we have \( s \cdot y \in I \). Thus, \( s \cdot y \in U \cap I \), so \( U \cap I \neq \emptyset \). Therefore, \( sx \in \overline{I} \).

Theorem 3.10 Suppose \( S \) is a commutative unital metrizable complete Hausdorff topological ring, and \( I \) is a closed ideal in \( S \). Then the quotient \( S/I \) is a complete Hausdorff ring.

Proof Since \( S \) is a commutative metrizable complete Hausdorff topological group, and \( I \) is a closed normal subgroup of \( S \), the quotient group \( S/I \) is Hausdorff by [4, Ch. III, §2.6, Prop. 18a] and complete by [5, Ch. IX, §3, no. 1, Prop. 4]. Thus, the quotient \( S/I \) is a complete Hausdorff topological ring.

Remark 3.11 (see [1, §10]) Let \( S \) be a commutative unital ring, and let \( I \) be an ideal in \( S \). The \( I \)-adic topology on \( S \) is the topology generated by elements of the form \( x + I^n \), where \( x \in S \) and \( n \geq 1 \). In particular, \( S \) is first-countable with respect to this topology. The ring \( S \) is complete in the \( I \)-adic topology if and only if the canonical ring homomorphism \( S \rightarrow \lim_{\overline{\cdot}}(S/I^n) \) is a ring isomorphism, and \( S \) is Hausdorff in the \( I \)-adic topology if and only if \( \cap_{i \geq n} I^i = (0) \). If \( S \) is Hausdorff, then there is a metric on \( S \) that generates the \( I \)-adic topology. Given \( s \in S \setminus \{0\} \), set \( \text{ord}_I(s) = t \), where \( t \geq 0 \) is the largest integer such that \( s \in I^t \) (here, we take \( I^0 := S \), and \( \text{ord}_I(0) = \infty \). For any \( s_1, s_2 \in S \), we define the metric
The following construction of the formal group ring was introduced in [6, §2]. Let $R$ be a commutative unital ring and $(R, F)$ a one-dimensional commutative formal group law over $R$. Let $R[x_\Lambda]$ be the polynomial ring over $R$ with variables indexed by elements of the real root lattice $\Lambda$. Here, we view $\Lambda$ as a finitely generated free abelian group with basis $\{e_i\alpha_j\}$. The augmentation map $\epsilon : R[x_\Lambda] \to R$ sends $x_\lambda \mapsto 0$ for each $\lambda \in \Lambda$ and fixes $R$. Let $\tilde{\text{R}}[x_\Lambda]$ be the ker$(\epsilon)$-adic completion of the polynomial ring $R[x_\Lambda]$. Given $u, v \in R[x_\Lambda]$, $m \in \mathbb{Z}_{\geq 0}$, we define the following notation:

$$u +_F v = F(u, v); \quad m \cdot_F u = u +_F +_F \cdots +_F u; \quad (-m) \cdot_F u = -_F (m \cdot_F u),$$

where $-_F u$ is the formal inverse of $u$ under $(R, F)$.

Let $\tilde{J}_F$ be the closure of the ideal in $\tilde{\text{R}}[x_\Lambda]$ generated by the elements

$$x_0 \quad \text{and} \quad x_{\lambda_1 + \lambda_2} = (x_{\lambda_1} +_F x_{\lambda_2}),$$

over all $\lambda_1, \lambda_2 \in \Lambda$.

**Definition 3.12** The quotient

$$\tilde{\text{R}}[\Lambda]_F := \tilde{\text{R}}[x_\Lambda]/\tilde{J}_F$$

is the formal group ring of [6, Def. 2.4]. For the remainder of this paper, we call $\tilde{\text{R}}[\Lambda]_F$ the **classical formal group ring**.

**Remark 3.13** If $\Sigma$ is crystallographic, then $\Lambda$ is a crystallographic root lattice, and $\tilde{\text{R}}[\Lambda]_F$ is the formal group ring studied in [8].

We denote the image of $x_\lambda$ in $\tilde{\text{R}}[\Lambda]_F$ by the same symbol. We will now review several facts about the classical formal group ring.

The ring $R[x_\Lambda]$ is a complete Hausdorff ring with respect to the $\mathcal{I}$-adic topology, where $\mathcal{I}$ is the kernel of the augmentation map $R[x_\Lambda] \to R$ that sends $x_\lambda \mapsto 0$ for all $\lambda \in \Lambda$ and fixes $R$. Note that $\tilde{\mathcal{I}}_F$ is contained in $\mathcal{I}$. Thus, Lemma 3.9 and Theorem 3.10 imply that the classical formal group ring $\tilde{\text{R}}[\Lambda]_F$ is a complete Hausdorff ring with respect to the $\tilde{\mathcal{I}}_F$-adic topology, where $\tilde{\mathcal{I}}_F$ is the kernel of the induced augmentation map $\tilde{\text{R}}[\Lambda]_F \to R$. By [6, Cor. 2.13], there is a continuous ring isomorphism $\varrho : \tilde{\text{R}}[\Lambda]_F \to R[x_1, \ldots, x_n]$ such that

$$\varrho \left( \sum_{i=1}^n \sum_{j=1}^n m_{i,j} e_i \alpha_j \right) = (m_{1,1} \cdot_F x_1 +_F \cdots +_F m_{1,n} \cdot_F x_n) +_F \cdots \cdots +_F (m_{l,1} \cdot_F x_{n(l-1)+1} +_F \cdots +_F m_{l,n} \cdot_F x_{nl}). \quad (1)$$

In particular, the isomorphism sends $x_{e_i \alpha_j} \mapsto x_{(i-1)n+j}$. If $R$ is an integral domain, then so is $\tilde{\text{R}}[\Lambda]_F$.

Since $W$ acts on $\Lambda$, it follows from [6, Lem. 2.6] that $W$ also acts on the classical formal group ring by

$$w(x_{\lambda}) = x_{w(\lambda)}, \quad w \in W, \quad \lambda \in \Lambda. \quad (2)$$
In fact, since each \( w \in W \) sends \( \tilde{I}_F \) to itself, the action of \( w \in W \) on \( \hat{R}[\Lambda]_F \) is a continuous ring homomorphism.

Let \((R, F')\) be another formal group law over \( R \). By [6, Lem. 2.6], any morphism \( f : (R, F) \to (R, F') \) of formal group laws over \( R \) induces a continuous ring homomorphism \( f^* : \hat{R}[\Lambda]_{F'} \to \hat{R}[\Lambda]_F \) sending \( x_\lambda \) to \( f(x_\lambda) \). To see that \( f^* \) is well-defined, note that

\[
f^*(x_{\lambda+\mu}) = f(x_{\lambda+\mu}) = f(x_\lambda + F_x \mu) = f(x_\lambda) + F' f(x_\mu) = f^*(x_\lambda) + F' f^*(x_\mu),
\]

for all \( \lambda, \mu \in \Lambda \). In addition, \( f^* \) is \( W \)-equivariant: write \( f(u) = \sum_{i \geq 1} c_i u^i \) for some \( c_i \in R \). Given \( w \in W \) and \( \lambda \in \Lambda \), we have

\[
f^*(w \cdot x_\lambda) = f^*(x_{w(\lambda)}) = \sum_{i \geq 1} c_i x_{w(\lambda)}^{i} = \sum_{i \geq 1} w \cdot (c_i x_\lambda^{i}) = w \cdot f^*(x_\lambda).
\]

If \((C, F)\) is a formal group law over \( C \), and \( R \) is an ample ring with respect to \((C, F)\), then the exponential of the formal group law \((R, F)\) induces a \( W \)-equivariant continuous ring isomorphism

\[
\exp^*_F : \hat{R}[\Lambda]_F \to \hat{R}[\Lambda]_{F_a}, \quad \exp_F^*(x_\lambda) = \exp_F(x_\lambda), \quad \lambda \in \Lambda.
\]

This is summarized by the following commutative diagram, which commutes for any \( w \in W \):

\[
\begin{array}{ccc}
\hat{R}[\Lambda]_F & \xrightarrow{\exp_F^*} & \hat{R}[\Lambda]_{F_a} \\
\downarrow w & & \downarrow w \\
\hat{R}[\Lambda]_F & \xrightarrow{\exp_F^*} & \hat{R}[\Lambda]_{F_a}.
\end{array}
\]

**Remark 3.14** Assume \( R \) is an integral domain. When \( \Sigma \) is crystallographic, it is shown in [6, Cor. 3.4] that \( x_\alpha \) divides the element \( u - s_\alpha(u) \) for all \( u \in \hat{R}[\Lambda]_F \) and \( \alpha \in \Sigma \). Therefore, one can define a formal Demazure operator \( \Delta^{(R, F)}_\alpha \) on \( \hat{R}[\Lambda]_F \) by the formula

\[
\Delta^{(R, F)}_\alpha(u) = \frac{u - s_\alpha(u)}{x_\alpha}, \quad u \in \hat{R}[\Lambda]_F.
\]

However, when \( \Sigma \) is noncrystallographic, it is not necessarily true that \( x_\alpha \) divides the element \( u - s_\alpha(u) \) for all \( u \in \hat{R}[\Lambda]_F \) and \( \alpha \in \Sigma \). We provide an example to demonstrate this. Suppose \((R, F_a)\) is the additive formal group law over the integral domain \( R \), the reflection group \( W = I_2(5) \), and \((\Sigma, \Delta)\) is the geometric realization of \( I_2(5) \) given in Example 2.19.

Then, in \( \hat{R}[\Lambda]_{F_a} \), we have

\[
x_{\tau \alpha_2} (x_{\alpha_1}) = x_{\alpha_1} - x_{\alpha_1 + \tau \alpha_2} = x_{\alpha_1} - (x_{\alpha_1} + x_{\tau \alpha_2}) = -x_{\tau \alpha_2}.
\]

If \( x_{\tau \alpha_2} \) divides \( x_{\alpha_1} - s_{\alpha_2}(x_{\alpha_1}) = -x_{\tau \alpha_2} \) in \( \hat{R}[\Lambda]_{F_a} \), then the isomorphism \( \hat{R}[\Lambda]_{F_a} \simeq \hat{R}[x_1, \ldots, x_4] \) of Eq. 1 implies that \( x_2 \) divides \( -x_4 \) in \( \hat{R}[x_1, \ldots, x_4] \), which is a contradiction.

The focus of the next two sections is to resolve this issue. Let \( W \) be any real finite reflection group, and let \((C, F)\) be a formal group law over \( C \). Choose any ample ring \( R \) with respect to \((C, F)\), such that \( R \) contains \( \mathcal{R} \) (see Assumption 4.1). In Section 4, we define the formal group ring \( \hat{R}[\Lambda]_F \) as the quotient of \( \hat{R}[x_\Lambda] \) by a certain closed \( W \)-invariant ideal.
J_F (see Definition 4.7). Note that R[x_\Lambda] can also be realized as the quotient of R[\Lambda]_F by a certain closed W-invariant ideal J_F^* (see Remark 4.8 and Lemma 4.14). We will denote the image of x_\lambda in R[\Lambda]_F by the same symbol. When \Sigma is crystallographic, R[\Lambda]_F collapses to the classical formal group ring R[\Lambda]_F with respect to the crystallographic root lattice \Lambda. The following three facts are recorded in Lemma 4.15: there is a well-defined action of W on R[\Lambda]_F given by w(x_\lambda) = x_{w(\lambda)} for all w \in W and \lambda \in \Lambda; there is a continuous ring isomorphism R[\Lambda]_F \rightarrow R[x_1, \ldots, x_n] sending x_{e_i \alpha_j} \mapsto \exp_F(e_i x_j) for all e_i \in B and \alpha_j \in \Delta; and x_{\alpha_j} divides x_{e_i \alpha_j} in R[\Lambda]_F for all e_i \in B and \alpha_j \in \Delta. In Section 5, we show that x_{\alpha_j} divides u = s_\alpha(u) in R[\Lambda]_F for all u \in R[\Lambda]_F and \alpha \in \Sigma, using the fact that x_{\alpha_j} divides x_{e_i \alpha_j} for all e_i \in B and \alpha_j \in \Delta (see Lemma 5.1). This allows us to define the formal Demazure operators that act on R[\Lambda]_F (see Definition 5.2).

Note that the real root lattice \Lambda is determined by the choice of simple system \Delta and the choice of power basis B of \mathcal{R}. Therefore, the construction of the formal group ring also depends on the choice of \Delta and on the choice of B. With that said, suppose \Delta' is another simple system of \Sigma, and let \mathcal{R}' be the subring of \mathcal{R} generated by the elements \beta_j \vee (\beta_j) over all \beta_j, \beta_j \in \Delta'. Suppose \mathcal{R}' is a finitely generated free abelian group with a power basis \mathcal{B}' = \{f_1, \ldots, f_n\} such that f_1 = 1, and let \Lambda' be the real root lattice determined by \Delta' and \mathcal{B}'. If R is any ample ring with respect to (C, F) that contains both \mathcal{R} and \mathcal{R}', then (2) of Lemma 4.15 implies that R[\Lambda']_F \simeq R[x_1, \ldots, x_n] \cong R[\Lambda]_F.

4 Construction of the Formal Group Ring

The purpose of this section is to construct the formal group ring R[\Lambda]_F for all real finite reflection groups. This construction will allow us to define formal Demazure operators on R[\Lambda]_F in Section 5. See Remark 3.14 for a detailed motivation for this construction. The definition of the formal group ring is given in Definition 4.7.

Let (C, F) be a formal group law over C. We work under Assumption 4.1 for the remainder of this paper.

Assumption 4.1 We assume R is an ample ring with respect to (C, F), such that R contains \mathcal{R}.

Example 4.2 Suppose W = I_2(5) and (\Sigma, \Delta) is the geometric realization of W given in Example 2.19. Let (C, F) = (C, F_{\tau}) be the multiplicative formal group law over C. Then the ring R satisfies Assumption 4.1 if and only if R is a subring of C containing Q[\tau], where \tau is the golden section.

The exponential exp_F (resp. logarithm log_F) of (R, F) induces a W-equivariant continuous ring isomorphism of classical formal group rings exp_F^* : R[\Lambda]_F \rightarrow R[\Lambda]_F (resp. log_F^* : R[\Lambda]_F \rightarrow R[\Lambda]_F). The isomorphisms exp_F^* and log_F^* are inverse to each other. Since R contains \mathcal{R}, we let J_F^* be the closure of the ideal in R[\Lambda]_F generated by the elements

\[ e_i \log_F(x_{\alpha_j}) - \log_F(x_{e_i \alpha_j}) , \]

over all e_i \in B and \alpha_j \in \Delta.
The ideal $\mathcal{J}_F^*$ is contained in the kernel of the augmentation map $\widehat{R[\Lambda]}_F \to R$, and, hence, the map $\widehat{R[\Lambda]}_F \to R$ factors through the quotient $\widehat{R[\Lambda]}_F/\mathcal{J}_F^*$. Therefore, Lemma 3.9 and Theorem 3.10 imply that $\widehat{R[\Lambda]}_F/\mathcal{J}_F^*$ is a complete Hausdorff ring with respect to the $\mathcal{I}_F$-adic topology, where $\mathcal{I}_F$ is the kernel of the induced augmentation map $\widehat{R[\Lambda]}_F/\mathcal{J}_F^* \to R$. We will denote the image of $x_1$ in the quotient $\widehat{R[\Lambda]}_F/\mathcal{J}_F^*$ by the same symbol. Observe that, if $\Sigma$ is crystallographic, then $B = \{e_1 = 1\}$, so $\mathcal{J}_F^* = (0)$ and $\widehat{R[\Lambda]}_F/\mathcal{J}_F^* \simeq \widehat{R[\Lambda]}_F$ in this case.

**Lemma 4.3** The ideal $\mathcal{J}_F^*$ in $\widehat{R[\Lambda]}_F$ is $W$-invariant.

**Proof** Under the continuous $W$-equivariant ring isomorphism $\exp_F^*: \widehat{R[\Lambda]}_F \to \widehat{R[\Lambda]}_F$, we have $\exp_F^*(\mathcal{J}_F^*) = \mathcal{J}_F^*$. To see this, we first note that, by continuity, $\exp_F^*(\mathcal{J}_F^*)$ is contained in the closure $\mathcal{J}_F^*$. Since $\exp_F^*$ is an isomorphism of topological rings, $\exp_F^*(\mathcal{J}_F^*)$ is closed in $\widehat{R[\Lambda]}_F$, so it must equal $\mathcal{J}_F^*$.

Since $\exp_F^*$ is $W$-equivariant, we see that $\mathcal{J}_F^*$ is $W$-invariant in $\widehat{R[\Lambda]}_F$ if and only if $\mathcal{J}_F^*$ is $W$-invariant in $\widehat{R[\Lambda]}_F$. Thus, we will show that $\mathcal{J}_F^*$ is $W$-invariant in $\widehat{R[\Lambda]}_F$. Let $\mathcal{J}$ be the ideal in $\widehat{R[\Lambda]}_F$ generated by the elements $e_i x_{\alpha_j} - x_{e_i \alpha_j}$, over all $e_i \in B$ and $\alpha_j \in \Delta$. The closure of $\mathcal{J}$ in $\widehat{R[\Lambda]}_F$ is $\mathcal{J}_F^*$. If $\mathcal{J}$ is $W$-invariant, then so is the closure $\mathcal{J}_F^*$ by continuity of the action of $w \in W$ on $\widehat{R[\Lambda]}_F$. In other words, suppose $\mathcal{J}$ is $W$-invariant. If $x \in \mathcal{J}_F^*$ and $(x_i)_{i \geq 0}$ is a sequence of elements in $\mathcal{J}$ converging to $x$, then the sequence $(w(x_i))_{i \geq 0}$ in $\mathcal{J}$ converges to $w(x)$. So $w(x)$ lies in the closure $\mathcal{J}_F^*$ as well. Thus, it is enough to show that $\mathcal{J}$ is $W$-invariant in $\widehat{R[\Lambda]}_F$. To do this, we will show that the image of $w(e_i x_{\alpha_j} - x_{e_i \alpha_j})$ in the quotient $\widehat{R[\Lambda]}_F/\mathcal{J}$ is zero for all $e_i \in B$, $\alpha_j \in \Delta$, and $w \in W$.

Here, we are denoting the image of $x_\lambda$ in the quotient $\widehat{R[\Lambda]}_F/\mathcal{J}$ by the same symbol.

Fix $w \in W$. Since $w(\alpha_j) \in \Lambda$, we write $w(\alpha_j) = \sum_{k=1}^l \sum_{r=1}^n c_{k,r} e_k \alpha_r$ for some $c_{k,r} \in \mathbb{Z}$. Furthermore, we can write $e_i e_k = \sum_{t=1}^l d_t^{(i,k)} e_t$ for some $d_t^{(i,k)} \in \mathbb{Z}$, since $\{e_1, \ldots, e_l\}$ is a $\mathbb{Z}$-basis of $\mathcal{R}$. Thus, in the quotient $\widehat{R[\Lambda]}_F/\mathcal{J}$, we have

$$w(x_{e_i \alpha_j}) = x_{e_i w(\alpha_j)} = x_{e_i \sum_{k=1}^l \sum_{r=1}^n c_{k,r} e_k \alpha_r} = x_{\sum_{k=1}^l \sum_{r=1}^n c_{k,r} d_t^{(i,k)} e_t x_{e_i \alpha_r}}.$$

Since $(R, F_0)$ is the additive formal group law, we have

$$x_{\sum_{k=1}^l \sum_{r=1}^n c_{k,r} d_t^{(i,k)} e_t x_{e_i \alpha_r}} = \sum_{k=1}^l \sum_{r=1}^n c_{k,r} d_t^{(i,k)} x_{e_i x_{e_i \alpha_r}}.$$

Finally, since $e_i x_{\alpha_j} = x_{e_i \alpha_j}$ in $\widehat{R[\Lambda]}_F/\mathcal{J}$, we have

$$w(x_{e_i \alpha_j}) = \sum_{k=1}^l \sum_{r=1}^n c_{k,r} d_t^{(i,k)} e_t x_{e_i \alpha_r} = e_i \sum_{k=1}^l \sum_{r=1}^n c_{k,r} e_k x_{e_i \alpha_r}$$

and

$$= e_i x_{\sum_{k=1}^l \sum_{r=1}^n c_{k,r} e_k \alpha_r} = e_i x_{w(\alpha_j)} = w(e_i x_{\alpha_j}).$$
Therefore, \( \mathcal{J} \) is a \( W \)-invariant ideal in \( \widehat{R[\Lambda]}_{F_a} \).

**Remark 4.4** Lemma 4.3 implies that there is a well-defined \( W \)-action on the quotient \( \widehat{R[\Lambda]}_{F}/\mathcal{J}_{F}^{\ast} \), given by

\[
    w(x_{\lambda}) = x_{w(\lambda)}, \quad w \in W, \lambda \in \Lambda.
\]

In the proof of Lemma 4.3, we showed that the continuous \( W \)-equivariant ring isomorphism \( \exp_{F}^{\ast} : \widehat{R[\Lambda]}_{F} \to \widehat{R[\Lambda]}_{F_{a}} \) exchanges the \( W \)-invariant ideals \( \mathcal{J}_{F}^{\ast} \) and \( \mathcal{J}_{F_{a}}^{\ast} \). Thus, \( \exp_{F}^{\ast} \) induces a continuous \( W \)-equivariant ring isomorphism on the quotients,

\[
    \widehat{\exp}_{F}^{\ast} : \widehat{R[\Lambda]}_{F}/\mathcal{J}_{F}^{\ast} \to \widehat{R[\Lambda]}_{F_{a}}/\mathcal{J}_{F_{a}}^{\ast}, \quad \widehat{\exp}_{F}^{\ast}(x_{\lambda}) = \exp_{F}(x_{\lambda}), \quad \lambda \in \Lambda.
\]

This is summarized by the following diagram, which commutes for all \( w \in W \):

\[
\begin{array}{ccc}
\widehat{R[\Lambda]}_{F}/\mathcal{J}_{F}^{\ast} & \xrightarrow{\exp_{F}^{\ast}} & \widehat{R[\Lambda]}_{F_{a}}/\mathcal{J}_{F_{a}}^{\ast} \\
\downarrow{w} & & \downarrow{w} \\
\widehat{R[\Lambda]}_{F}/\mathcal{J}_{F}^{\ast} & \xrightarrow{\exp_{F}^{\ast}} & \widehat{R[\Lambda]}_{F_{a}}/\mathcal{J}_{F_{a}}^{\ast}.
\end{array}
\]

**Lemma 4.5** There is a continuous ring isomorphism

\[
\widehat{R[\Lambda]}_{F}/\mathcal{J}_{F}^{\ast} \to R[x_{1}, \ldots, x_{n}],
\]

sending \( x_{e_{i}a_{j}} \mapsto \exp_{F}(e_{i}x_{j}) \). In particular, \( \widehat{R[\Lambda]}_{F}/\mathcal{J}_{F}^{\ast} \) is an integral domain.

**Proof** There is a continuous ring isomorphism

\[
\widehat{\exp}_{F}^{\ast} : \widehat{R[\Lambda]}_{F}/\mathcal{J}_{F}^{\ast} \to \widehat{R[\Lambda]}_{F_{a}}/\mathcal{J}_{F_{a}}^{\ast}, \quad \widehat{\exp}_{F}^{\ast}(x_{e_{i}a_{j}}) = \exp_{F}(x_{e_{i}a_{j}}).
\]

Suppose there is a continuous ring isomorphism

\[
\gamma : \widehat{R[\Lambda]}_{F_{a}}/\mathcal{J}_{F_{a}}^{\ast} \to R[x_{1}, \ldots, x_{n}], \quad \gamma(x_{e_{i}a_{j}}) = e_{i}x_{j}.
\]

Then the composition

\[
\gamma \circ \widehat{\exp}_{F}^{\ast} : \widehat{R[\Lambda]}_{F}/\mathcal{J}_{F}^{\ast} \to R[x_{1}, \ldots, x_{n}]
\]

is a continuous ring isomorphism such that \( (\gamma \circ \widehat{\exp}_{F}^{\ast})(x_{e_{i}a_{j}}) = \exp_{F}(e_{i}x_{j}) \), which proves the lemma. Therefore, we will show the existence of the isomorphism \( \gamma \).

Recall that \( \mathcal{J}_{F_{a}}^{\ast} \) is the closure of the ideal in \( \widehat{R[\Lambda]}_{F_{a}} \) generated by the elements \( x_{e_{i}a_{j}} - e_{i}x_{a_{j}} \), over all \( e_{i} \in B \) and \( a_{j} \in \Delta \). In addition, recall that there is a continuous ring isomorphism

\[
\varrho : \widehat{R[\Lambda]}_{F_{a}} \to R[x_{1}, \ldots, x_{n}], \quad \varrho(x_{me_{i}a_{j}}) = mx_{(i-1)n+j},
\]

for all \( e_{i} \in B, a_{j} \in \Delta, \) and \( m \in \mathbb{Z} \) (see Eq. 1). The image \( \varrho(\mathcal{J}_{F_{a}}^{\ast}) \) is the closure of the ideal in \( R[x_{1}, \ldots, x_{n}] \) generated by the elements \( x_{(i-1)n+j} - e_{i}x_{j} \), over all \( i = 1, \ldots, l, \) and \( j = 1, \ldots, n \). By Lemma 3.9 and Theorem 3.10, the quotient \( R[x_{1}, \ldots, x_{n}]/\varrho(\mathcal{J}_{F_{a}}^{\ast}) \) is a complete Hausdorff ring with respect to the topology induced by the kernel of the augmentation map \( R[x_{1}, \ldots, x_{n}]/\varrho(\mathcal{J}_{F_{a}}^{\ast}) \to R \). Thus, there is a continuous ring isomorphism on
the quotients,
\[
\tilde{\varrho} : \widehat{\mathcal{R} \left[ \Lambda \right]}_{F_a} / \mathcal{J}_F^* \to R[x_1, \ldots, x_n] / \mathcal{Q}(\mathcal{J}_F^*), \quad \tilde{\varrho}(x_{me_i \alpha_j}) = mx_{(i-1)n+j}.
\]
Here we denote the image of \( x_r \) in the quotient \( R[x_1, \ldots, x_n] / \mathcal{Q}(\mathcal{J}_F^*) \) by the same symbol. 

Suppose there is a continuous ring isomorphism
\[
\kappa : R[x_1, \ldots, x_n] / \mathcal{Q}(\mathcal{J}_F^*) \to R[x_1, \ldots, x_n], \quad \kappa(x_{(i-1)n+j}) = e_ix_j.
\]
Then the composition
\[
\kappa \circ \tilde{\varrho} : \widehat{\mathcal{R} \left[ \Lambda \right]}_{F_a} / \mathcal{J}_F^* \to R[x_1, \ldots, x_n]
\]
is a continuous ring isomorphism such that \((\kappa \circ \tilde{\varrho})(x_{e_i \alpha_j}) = e_ix_j\). In particular, we can take \( \gamma := \kappa \circ \tilde{\varrho} \). Therefore, all we need to do is show the existence of the isomorphism \( \kappa \).

The ring homomorphism
\[
\tilde{\kappa} : R[x_1, \ldots, x_n] \to R[x_1, \ldots, x_n], \quad \tilde{\kappa}(x_{(i-1)n+j}) = e_ix_j
\]
extends to a continuous ring homomorphism
\[
\kappa : R[x_1, \ldots, x_n] / \mathcal{Q}(\mathcal{J}_F^*) \to R[x_1, \ldots, x_n], \quad \kappa(x_{(i-1)n+j}) = e_ix_j.
\]
Since \( e_1 = 1 \), we have
\[
\tilde{\kappa}(x_{(i-1)n+j} - e_i x_j) = \tilde{\kappa}(x_{(i-1)n+j} - e_i x_{(i-1)n+j}) = e_i x_j - e_i x_j = 0.
\]
Therefore, \( \tilde{\kappa} \) induces a continuous ring homomorphism
\[
\kappa : R[x_1, \ldots, x_n] / \mathcal{Q}(\mathcal{J}_F^*) \to R[x_1, \ldots, x_n], \quad \kappa(x_{(i-1)n+j}) = e_i x_j.
\]
In the other direction, the ring homomorphism
\[
\iota : R[x_1, \ldots, x_n] \to R[x_1, \ldots, x_n] / \mathcal{Q}(\mathcal{J}_F^*), \quad \iota(x_j) = x_j
\]
extends to a continuous ring homomorphism
\[
\iota : R[x_1, \ldots, x_n] \to R[x_1, \ldots, x_n] / \mathcal{Q}(\mathcal{J}_F^*), \quad \iota(x_j) = x_j.
\]
We see that \((\kappa \circ \iota)(x_j) = \kappa(x_j) = x_j \) and that \((\iota \circ \kappa)(x_{(i-1)n+j}) = \iota(e_i x_j) = e_i x_j = x_{(i-1)n+j} \). Since the \( x_j, j = 1, \ldots, n \), generate \( R[x_1, \ldots, x_n] \) as a ring, and the \( x_k, k = 1, \ldots, n_l \), generate \( R[x_1, \ldots, x_n] / \mathcal{Q}(\mathcal{J}_F^*) \) as a ring, it follows from the continuity of \( \iota \) and \( \kappa \) that \( \iota \) and \( \kappa \) are continuous ring isomorphisms that are inverse to each other. This proves the lemma.

**Remark 4.6** Let \( f : \widehat{\mathcal{R} \left[ \Lambda \right]}_{F} / \mathcal{J}_F^* \to R[x_1, \ldots, x_n] \) be the isomorphism given in the statement of Lemma 4.5. In the proof of Lemma 4.5, we showed that there is a factorization \( f = g \circ h \), where
\[
h : \widehat{\mathcal{R} \left[ \Lambda \right]}_{F} / \mathcal{J}_F^* \to \widehat{\mathcal{R} \left[ \Lambda \right]}_{F_a} / \mathcal{J}_F^*, \quad h(x_{e_i \alpha_j}) = \exp_F(x_{e_i \alpha_j}) = \exp_F(e_i x_{e_i \alpha_j});
\]
\[
g : \widehat{\mathcal{R} \left[ \Lambda \right]}_{F_a} / \mathcal{J}_F^* \to R[x_1, \ldots, x_n], \quad g(x_{e_i \alpha_j}) = e_i x_j.
\]
The maps \( g \) and \( h \) are both continuous ring isomorphisms.

Let \( \mathcal{J}_F \) be the closure of the ideal in \( R[\Lambda] \) generated by the elements 
\[
x_0 \quad \text{and} \quad \log_F(x_{\lambda_1+\lambda_2}) - (\log_F(x_{\lambda_1}) + \log_F(x_{\lambda_2})) \quad \text{and} \quad e_i \log_F(x_{\alpha_j}) - \log_F(x_{e_i \alpha_j}),
\]
over all \( e_i \in B, \alpha_j \in \Delta, \) and \( \lambda_1, \lambda_2 \in \Lambda \). By similar reasoning as earlier, the quotient \( R[\Lambda] / \mathcal{J}_F \) is a complete Hausdorff ring with respect to the \( \mathcal{I}_F \)-adic topology, where \( \mathcal{I}_F \)
is the kernel of the induced augmentation map $R[x_{\Lambda}]/J_F \to R$. We will now define the formal group ring.

**Definition 4.7** The quotient ring

$$R[\Lambda]_F := R[x_{\Lambda}]/J_F$$

is the *formal group ring* with respect to $(R, F)$ and $\Lambda$.

From now on, we will denote the image of $x_{\lambda}$ in the quotient $R[\Lambda]_F = R[x_{\Lambda}]/J_F$ by the same symbol.

**Remark 4.8** In Lemma 4.14, we will show that there is a continuous ring isomorphism $R[\Lambda]_F \to \widetilde{R[\Lambda]}_F/J_F^*$, sending $x_{\lambda} \mapsto x_{\lambda}$ for all $\lambda \in \Lambda$. Once Lemma 4.14 is proven, it will follow from Lemma 4.3 that there is a well-defined action of $W$ on $R[\Lambda]_F$ given by $w(x_{\lambda}) = x_{w(\lambda)}$ for all $w \in W$ and $\lambda \in \Lambda$, and it will follow from Lemma 4.5 that $R[\Lambda]_F \cong R[x_1, \ldots, x_n]$.

**Remark 4.9** If $\Sigma$ is *crystallographic*, then $B = \{e_1 = 1\}$, and, therefore, $J_F$ is the closure of the ideal in $R[x_{\Lambda}]$ generated by the elements

$$x_0 \quad \text{and} \quad x_{\lambda_1 + \lambda_2} - (x_{\lambda_1} + F x_{\lambda_2}),$$

over all $\lambda_1, \lambda_2 \in \Lambda$. In this case, $R[\Lambda]_F$ is the classical formal group ring with respect to the crystallographic root lattice $\Lambda$.

**Remark 4.10** In [6], the formal group ring is defined over any commutative unital ring $S$. However, if $\Sigma$ is noncrystallographic, then the definition of $J_F$ requires that $S$ contains $R$. In order to provide a simple unified exposition of the formal group ring for all finite root systems, we restrict $S$ to a subring of $\mathbb{C}$ in this paper (subject to Assumption 4.1).

**Remark 4.11** Let $J^\dagger_F$ be the closure of the ideal in $R[x_{\Lambda}]$ generated by the elements

$$x_0 \quad \text{and} \quad \log_F(x_{\lambda_1 + \lambda_2}) - (\log_F(x_{\lambda_1}) + \log_F(x_{\lambda_2})) \quad \text{and} \quad \exp_F(e_i \log_F(x_{\alpha_j})) - x_{e_i \alpha_j},$$

over all $e_i \in B, \alpha_j \in \Delta$, and $\lambda_1, \lambda_2 \in \Lambda$. It is straightforward to verify that $J^\dagger_F = J_F$ in the ring $R[x_{\Lambda}]$. Recall that there are $a_k, b_k \in R$ such that $\exp_F(u) = u + \sum_{k \geq 2} a_k u^k$ and $\log_F(u) = u + \sum_{k \geq 2} b_k u^k$ for all $u \in R[\Lambda]_F$. Thus, in $R[\Lambda]_F$, we have

$$x_{e_i \alpha_j} = \exp_F(e_i \log_F(x_{\alpha_j})) = \exp_F\left(e_i x_{\alpha_j} + e_i \sum_{k \geq 2} b_k x_{\alpha_j}^k \right) = \left(e_i x_{\alpha_j} + e_i \sum_{k \geq 2} b_k x_{\alpha_j}^k \right)^r,$$

for all $e_i \in B$ and $\alpha_j \in \Delta$. Therefore, the generators $\exp_F(e_i \log_F(x_{\alpha_j})) - x_{e_i \alpha_j}$ of the ideal $J_F = J^\dagger_F$ tell us how to express $x_{e_i \alpha_j}$ as an $R$-linear combination of positive powers of the element $x_{\alpha_j}$ in the quotient ring $R[\Lambda]_F$ for all $e_i \in B$ and $\alpha_j \in \Delta$. 

[Springer]
Example 4.12 Let \((R, F) = (R, F_m)\) be the multiplicative formal group law over \(R\). Let \(W = I_2(4)\), and suppose \((\Sigma, \Delta)\) is the geometric realization of \(W = I_2(4)\) given in Example 2.18, with real root lattice \(\Lambda\). Then \(\Delta = \{\alpha_1, \alpha_2\}\) consists of two simple roots, and \(B = \{1, \sqrt{2}\}\). In this case, Remark 4.11 tells us that, in the quotient ring \(R[\Lambda]_{F_m}\), we have
\[
x_{\sqrt{2}\alpha_1} = \exp_{F_m}(\sqrt{2} \log_{F_m}(x_{\alpha_1})) \quad \text{and} \quad x_{\sqrt{2}\alpha_2} = \exp_{F_m}(\sqrt{2} \log_{F_m}(x_{\alpha_2})).
\]
In other words, \(x_{\sqrt{2}\alpha_1}\) and \(x_{\sqrt{2}\alpha_2}\) can be expressed as \(R\)-linear combinations of positive powers of \(x_{\alpha_1}\) and positive powers of \(x_{\alpha_2}\), respectively, in the quotient \(R[\Lambda]_{F_m}\).

Remark 4.13 Let \(\tilde{J}_F^\dagger\) be the closure of the ideal in \(\widetilde{R[\Lambda]}_F\) generated by the elements
\[
\exp_F(e_i \log_F(x_{\alpha_j})) - x_{e_i \alpha_j},
\]
over all \(e_i \in B\) and \(\alpha_j \in \Delta\). It is straightforward to verify that \(\tilde{J}_F^\dagger = J_F^\ast\) in the ring \(\widetilde{R[\Lambda]}_F\).

In particular, \(R[\Lambda]_F/\tilde{J}_F^\dagger = R[\Lambda]_F/J_F^\ast\). The ideal \(J_F^\ast = \tilde{J}_F^\dagger\) tells us how we can express \(x_{e_i \alpha_j}\) as an \(R\)-linear combination of positive powers of the element \(x_{\alpha_j}\) in \(R[\Lambda]_F/J_F^\ast\) for all \(e_i \in B\) and \(\alpha_j \in \Delta\).

Lemma 4.14 There is a canonical continuous ring isomorphism
\[
R[x_\Lambda]/J_F \rightarrow \widetilde{R[\Lambda]}_F/J_F^\ast,
\]
sending \(x_\lambda + J_F \mapsto x_\lambda + J_F^\ast\) for all \(\lambda \in \Lambda\).

Proof Let \(J_F^\circ\) be the closure of the ideal in \(R[x_\Lambda]\) generated by the elements
\[
x_0 \quad \text{and} \quad x_{\lambda_1 + \lambda_2} - (x_{\lambda_1} + x_{\lambda_2}) \quad \text{and} \quad e_i \log_F(x_{\alpha_j}) - \log_F(x_{e_i \alpha_j}),
\]
over all \(e_i \in B\), \(\alpha_j \in \Delta\), and \(\lambda_1, \lambda_2 \in \Lambda\). It is straightforward to verify that \(J_F^\circ = J_F\) in \(R[x_\Lambda]\). Thus, the map \(\psi : R[x_\Lambda]/J_F \rightarrow \widetilde{R[\Lambda]}_F/J_F^\ast\) defined by \(\psi(x_\lambda + J_F) = x_\lambda + J_F^\ast\) is a well-defined ring homomorphism. It is straightforward to verify that \(\psi\) is bijective and continuous, with a continuous inverse. \(\Box\)

Lemma 4.15 The following properties hold in \(R[\Lambda]_F\):

1. There is a well-defined \(W\)-action on \(R[\Lambda]_F\) given by
\[
w(x_\lambda) = x_{w(\lambda)}, \quad w \in W, \quad \lambda \in \Lambda.
\]

2. There is a continuous ring isomorphism
\[
R[\Lambda]_F \rightarrow R[x_1, \ldots, x_n], \quad x_{e_i \alpha_j} \mapsto \exp_F(e_i x_j).
\]

In particular, \(R[\Lambda]_F\) is an integral domain.

3. The element \(x_{\alpha_j}\) divides \(x_{e_i \alpha_j}\) in \(R[\Lambda]_F\) for all \(e_i \in B\) and \(\alpha_j \in \Delta\).

Proof Property (1) follows from Remark 4.4 and Lemma 4.14. Property (2) follows from Lemma 4.5 and Lemma 4.14. Property (3) follows from Remark 4.11. \(\Box\)

Remark 4.16 Since \(e_1 = 1\), the isomorphism of Lemma 4.15 (2) sends \(x_{\alpha_j} \mapsto \exp_F(x_j)\) for each \(\alpha_j \in \Delta\).
For a finite sequence of simple roots \( I = (\alpha_{i_1}, \ldots, \alpha_{i_r}) \), we set \( x_I := x_{\alpha_{i_1}} \cdots x_{\alpha_{i_r}} \). We define the length of \( I \) to be \( l(I) = r \). We say a sequence \( I = (\alpha_{i_1}, \ldots, \alpha_{i_r}) \) of simple roots is ordered if \( i_1 \leq i_2 \leq \cdots \leq i_r \). Let \( \Upsilon \) be the set of all ordered sequences \( I \) of simple roots such that \( l(I) \geq 1 \), and let \( \Upsilon_r \) be the subset of \( \Upsilon \) consisting of sequences of simple roots of length \( r \).

**Remark 4.17** Let \( \lambda \in \Lambda \). Since \( \{e_i \alpha_j\} \) is a \( \mathbb{Z} \)-basis for \( \Lambda \), we can write \( \lambda = \sum_{j=1}^{n} \sum_{i=1}^{l} c_{i,j} e_i \alpha_j \) for some \( c_{i,j} \in \mathbb{Z} \). The relations in \( R[\Lambda]_{F_a} \) allow us to write \( x_{\lambda} \in R[\Lambda]_{F_a} \) in the form \( x_{\lambda} = \sum_{j=1}^{n} \left( \sum_{i=1}^{l} c_{i,j} e_i \right) x_{\alpha_j} \), where \( \sum_{i=1}^{l} c_{i,j} e_i \in \mathcal{R} \) for each fixed \( j \). In particular, \( x_{\lambda} \) can be written as an \( \mathcal{R} \)-linear combination of the elements \( x_{\alpha_j} \), where \( \alpha_j \in \Delta \). Since \( e_1 = 1 \), the isomorphism of Lemma 4.15 (2) implies that \( x_{\lambda} \) can be written uniquely as an \( \mathcal{R} \)-linear combination of the elements \( x_{\alpha_j} \), where \( \alpha_j \in \Delta \). Thus, any product of \( r \) \( x_{\lambda} \)'s, where \( r \geq 1 \), can be written uniquely as an \( \mathcal{R} \)-linear combination of elements of the form \( x_I \), where \( I \in \Upsilon_r \).

**Remark 4.18** Since \( e_1 = 1 \), the isomorphism of Lemma 4.15 (2) implies that any element in \( R[\Lambda]_{F_a} \) can be written uniquely as an \( \mathcal{R} \)-linear combination of 1 and the elements \( x_I \), where \( I \in \Upsilon \).

**Example 4.19** (cf. [6, Ex. 2.19]) Let \( S^i_R(\Lambda) \) be the \( i \)-th symmetric power of the \( \mathcal{R} \)-module \( R \otimes_{\mathcal{R}} \Lambda \), where we view \( \Lambda \) as a finitely generated free \( \mathcal{R} \)-module with basis \( \Delta \). For \( I = (\alpha_{i_1}, \ldots, \alpha_{i_r}) \in \Upsilon \), set \( \alpha_I := \alpha_{i_1} \cdots \alpha_{i_r} \in S^r_R(\Lambda) \). By definition, every element in \( S^r_R(\Lambda) \) can be written uniquely as an \( \mathcal{R} \)-linear combination of 1 and the elements \( \alpha_I \), where \( I \in \Upsilon \). The \( \mathcal{R} \)-algebra \( (S^*_{\mathcal{R}}(\Lambda))^\wedge := \prod_{i=0}^{\infty} S^i_R(\Lambda) \) is the completion of the symmetric algebra \( S^*_R(\Lambda) := \bigoplus_{i=0}^{\infty} S^i_R(\Lambda) \) at the kernel of the augmentation map \( \alpha_I \mapsto 0 \), \( I \in \Upsilon \).

By Remark 4.18, every element in \( R[\Lambda]_{F_a} \) can be written uniquely as an \( \mathcal{R} \)-linear combination of 1 and the elements \( x_I \), where \( I \in \Upsilon \). Thus, there is a continuous \( \mathcal{R} \)-linear ring isomorphism

\[
\phi: \quad R[\Lambda]_{F_a} \longrightarrow (S^*_{\mathcal{R}}(\Lambda))^\wedge,
\]

sending \( x_I \mapsto \alpha_I \) for all \( I \in \Upsilon \), and extended by \( \mathcal{R} \)-linearity.

**Example 4.20** (cf. [6, Ex. 2.20]) Consider the group ring

\[
R[\Lambda] := \left\{ \sum_j r_j e^{\lambda_j} \mid r_j \in R, \quad \lambda_j \in \Lambda \right\}.
\]

Let \( \text{tr} : R[\Lambda] \to R \) be the \( \mathcal{R} \)-linear trace map that sends \( e^\lambda \mapsto 1 \) for all \( \lambda \in \Lambda \). Let \( R[\Lambda]^\wedge \) be the ker(tr)-adic completion of \( R[\Lambda] \). There is a continuous ring isomorphism

\[
\hat{h}: \quad \widehat{R[\Lambda]}_{F_m} \to R[\Lambda]^\wedge,
\]

where \( \hat{h}(x_{\lambda}) = e^\lambda - 1 \) and \( \hat{h}^{-1}(e^\lambda) = 1 + x_{\lambda} = (1 + x_{-\lambda})^{-1} \) for all \( \lambda \in \Lambda \).

Suppose that \( \Sigma \) is noncrystallographic. Let \( \mathcal{E} \) be the closure of the ideal in \( R[\Lambda]^\wedge \) generated by the elements

\[
\exp_{F_m}(e_i \log_{F_m}(e^{\alpha_j} - 1)) - (e^{e_i \alpha_j} - 1),
\]

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over all \( e_i \in B \) and \( \alpha_j \in \Delta \). We have \( \hat{h}(\tilde{\mathcal{J}}_{F_m}^t) = \mathcal{E} \), where the ideal \( \tilde{\mathcal{J}}_{F_m}^t \) of \( R[\Lambda]_F \) was defined in Remark 4.13. To see this, note that, by continuity, \( \hat{h}(\tilde{\mathcal{J}}_{F_m}^t) \subseteq \mathcal{E} \). Since \( \hat{h} \) is an isomorphism of topological rings, \( \hat{h}(\tilde{\mathcal{J}}_{F_m}^t) \) is closed, so we get equality. As mentioned in Remark 4.13, we have \( \tilde{\mathcal{J}}_{F_m}^t = \mathcal{J}_{F_m}^t \). Through the continuous ring isomorphism \( R[\Lambda]_F \to \mathcal{E} \), we see that \( \hat{h} \) induces a continuous ring isomorphism \( h : R[\Lambda]_F \to (R[\Lambda])^\wedge /\mathcal{E} \).

such that \( h(x_\lambda) = e^{x_\lambda} - 1 \) and \( h^{-1}(e^{x_\lambda}) = 1 + x_\lambda \) for all \( \lambda \in \Lambda \).

As discussed in Remark 4.11, the ideal \( \mathcal{J}_{F_m} \) tells us how to express \( x_{e_\alpha j} \) as an \( R \)-linear combination of positive powers of the element \( x_{\alpha_j} \) in the quotient \( R[\Lambda]_F \). Namely, we can write \( x_{e_\alpha j} = \exp_{F_m}(e_i \log_{F_m}(x_{\alpha_j})) \) in \( R[\Lambda]_F \) for all \( e_i \in B \) and \( \alpha_j \in \Delta \). Similarly, the ideal \( \mathcal{E} \) tells us how we can express the element \( e^{x_{e_\alpha j}} - 1 \) in terms of positive powers of \( e^{x_{\alpha_j}} - 1 \) in the quotient \( (R[\Lambda])^\wedge /\mathcal{E} \). Namely, we can write

\[
e^{x_{e_\alpha j}} - 1 = \exp_{F_m}(e_i \log_{F_m}(e^{x_{\alpha_j}} - 1))
\]

in \( (R[\Lambda])^\wedge /\mathcal{E} \) for all \( e_i \in B \) and \( \alpha_j \in \Delta \).

Suppose \( W = I_2(4) \), and let \( (\Sigma, \Delta) \) be the geometric realization of \( W \) given in Example 2.18. Then \( \Delta = \{ \alpha_1, \alpha_2 \} \) consists of two simple roots, and \( B = \{ 1, \sqrt{2} \} \). Therefore, \( \mathcal{E} \) is the closure of the ideal in \( R[\Lambda]^\wedge \) generated by the elements

\[
\exp_{F_m}((e^{x_{\alpha_1}} - 1)) - (e^{\sqrt{2}x_{\alpha_1}} - 1) \quad \text{and} \quad \exp_{F_m}((e^{x_{\alpha_2}} - 1)) - (e^{\sqrt{2}x_{\alpha_2}} - 1).
\]

Compare this with Example 4.12.

## 5 Formal Demazure Operators

In this section, we show that the element \( u - s_\alpha(u) \) is divisible by \( x_\alpha \) in \( R[\Lambda]_F \) for all \( u \in R[\Lambda]_F \) and \( \alpha \in \Sigma \). Using this fact, we define the formal Demazure operator \( \Delta_\alpha^{(R, F)} \) in this section. When \( (R, F) = (R, F_a) \) is the additive formal group law over \( R \), the isomorphism \( \phi \) of Example 4.19 exchanges the formal Demazure operator \( \Delta_\alpha^{(R, F_a)} \) on \( R[\Lambda]_{F_a} \) with an operator \( (\Delta_\alpha^R)^\wedge \) on \( (S^*_R[\Lambda])^\wedge \). We conclude this section by studying the restriction of \( (\Delta_\alpha^R)^\wedge \) to the symmetric algebra \( S^*_R[\Lambda] \).

The following result will allow us to define the formal Demazure operator on \( R[\Lambda]_F \).

**Lemma 5.1 (cf. [6, Cor. 3.4])** For any \( u \in R[\Lambda]_F \) and root \( \alpha \in \Sigma \), the element \( u - s_\alpha(u) \) is divisible by \( x_\alpha \) in \( R[\Lambda]_F \).

**Proof** First we assume that \( \alpha = \alpha_j \) is a simple root. Since \( \alpha_j^\wedge(\lambda) \in \mathcal{R} \) for any \( \lambda \in \Lambda \), it follows that \( \alpha_j^\wedge(\lambda) = \sum_{i=1}^l c_i e_i \) for some \( c_i \in \mathbb{Z} \). We have

\[
s_j(x_\lambda) = x_\lambda - \alpha_j^\wedge(\lambda) x_\lambda = x_\lambda - \sum_{i=1}^l c_i e_i x_\lambda = x_\lambda + F (((-c_1) \cdot F x_{e_1 \alpha_j}) + F \cdots + F ((-c_l) \cdot F x_{e_l \alpha_j})).
\]

Set \( r = ((-c_1) \cdot F x_{e_1 \alpha_j}) + F \cdots + F ((-c_l) \cdot F x_{e_l \alpha_j}) \). By (3) of Lemma 4.15, \( x_{e_i \alpha_j} \) divides each \( x_{e_i \alpha_j} \). Hence, \( x_{e_i \alpha_j} \) divides each \( (-c_i) \cdot F x_{e_i \alpha_j} \). Thus, \( x_{e_i \alpha_j} \) divides \( r \). Write \( u + F v = u + v g(u, v) \), where \( g(u, v) \in R[\Lambda]_F \). Then

\[
x_\lambda - s_\alpha(x_\lambda) = x_\lambda - (x_\lambda + F r) = x_\lambda - (x_\lambda + r g(x_\lambda, r)) = -r g(x_\lambda, r).
\]
So \( x_{\alpha j} \) divides \( x_i - s_j(x_i) \). Then, by the formula
\[
xy - s_j(xy) = (x - s_j(x))y + x(y - s_j(y)) - (x - s_j(x))(y - s_j(y)),
\]
the element \( x_{\alpha j} \) divides \( u - s_j(u) \) for any monomial \( u \in R[x_\Lambda] \). Finally, \( x_{\alpha j} \) divides \( u - s_j(u) \) for any \( u \in R[\Lambda]_F \) by density of \( R[\Lambda]_F \).

Now let \( \alpha \in \Sigma \) be any root. By Lemma 2.2, the root \( \alpha \) can be written \( \alpha = w(\alpha_j) \) for some \( \alpha_j \in \Delta \) and \( w \in \mathcal{W} \). Furthermore, Proposition 2.11 says that \( ws_\beta w^{-1} = s_{w(\beta)} \) for any \( w \in \mathcal{W} \) and \( \beta \in \Sigma \). Thus, given \( u \in R[\Lambda]_F \), we see that
\[
\frac{u - s_\alpha(u)}{x_\alpha} = \frac{u - s_{w(\alpha_j)}(u)}{x_{w(\alpha_j)}} = \frac{(ww^{-1}(u) - (ww^{-1}(u)) - (ww^{-1}(u))}{(ww^{-1})(x_{w(\alpha_j)})} = w\left(\frac{w^{-1}(u) - s_j(w^{-1}(u))}{x_{\alpha_j}}\right) \in R[\Lambda]_F.
\]

**Definition 5.2** Following [16, Def. 4.1], for each root \( \alpha \in \Sigma \), we define an \( R \)-linear operator \( \Delta^{(R,F)}_\alpha \) on \( R[\Lambda]_F \) by the formula
\[
\Delta^{(R,F)}_\alpha(u) = \frac{u - s_\alpha(u)}{x_\alpha}, \quad u \in R[\Lambda]_F.
\]

We call the operator \( \Delta^{(R,F)}_\alpha \) a formal Demazure operator.

**Proposition 5.3** (see [6, Prop. 3.8] and [10, §3]) The following formulas hold for any \( u, v \in R[\Lambda]_F \), \( \alpha \in \Sigma \), and \( w \in \mathcal{W} \).

1. \( \Delta^{(R,F)}_\alpha(1) = 0 \), \( \Delta^{(R,F)}_\alpha(u) x_\alpha = u - s_\alpha(u) \);
2. \( \Delta^{(R,F)}_\alpha(\Delta^{(R,F)}_\alpha(u)) x_\alpha = \Delta^{(R,F)}_\alpha(u) + \Delta^{(R,F)}_{-\alpha}(u) \), \( \Delta^{(R,F)}_\alpha(u) x_\alpha = \Delta^{(R,F)}_{-\alpha}(u) x_{-\alpha} \);
3. \( s_\beta \Delta^{(R,F)}_\alpha(u) = -\Delta^{(R,F)}_{\alpha}(u) \), \( \Delta^{(R,F)}_\alpha(s_\beta(u)) = -\Delta^{(R,F)}_{\alpha}(u) \);
4. \( \Delta^{(R,F)}_\alpha(uv) = \Delta^{(R,F)}_\alpha(u) v + u \Delta^{(R,F)}_\alpha(v) - \Delta^{(R,F)}_\alpha(u) \Delta^{(R,F)}_\alpha(v) \Delta^{(R,F)}_\alpha(u) + s_\alpha(u) \Delta^{(R,F)}_\alpha(v) \);
5. \( w \Delta^{(R,F)}_\alpha(w^{-1}(u)) = \Delta^{(R,F)}_w(u) \).

**Proof** The formulas (1)–(4) are straightforward computations using the definition of the formal Demazure operator. Relation (5) follows from Proposition 2.11, which says that \( ws_\beta w^{-1} = s_{w(\beta)} \). \( \square \)

Let \( (\Delta^{R})^\wedge : (S^*_R(\Lambda))^\wedge \to (S^*_R(\Lambda))^\wedge \) be the \( R \)-linear operator corresponding to the Demazure operator \( \Delta^{(R,F)_\alpha} : R[\Lambda]_{F_{\alpha}} \to R[\Lambda]_{F_{\alpha}} \) with respect to the isomorphism \( \phi : R[\Lambda]_{F_{\alpha}} \to (S^*_R(\Lambda))^\wedge \) of Example 4.19. The operator \( (\Delta^{R})^\wedge \) restricts to an \( R \)-linear operator \( \Delta^R : S^*_R(\Lambda) \to S^*_R(\Lambda) \). If \( \Sigma \) is crystallographic and \( R = \mathbb{Z} \), then \( \Delta^R \) is the classical Demazure operator of [10], which acts on the symmetric algebra \( S^*_\Sigma(\Lambda) := \bigotimes_{i=1}^\infty S^i_{\Sigma}(\Lambda) \). If \( R = \mathbb{C} \), then \( \Delta^C \) is the Demazure operator of [15, Ch. IV, pp. 135], which acts on the symmetric algebra \( S(V_C) = \bigotimes_{i=1}^\infty S^i_{\Sigma}(\Lambda) \), where \( V_C = \mathbb{C} \otimes \mathbb{R} V \) is the complex vector space spanned by the roots \( \beta \in \Sigma \).

For \( \alpha_i \in \Delta \), we will use the notation \( \Delta^{(R,F)}_i := \Delta^{(R,F)}_{\alpha_i} \) and \( \Delta^R_i := \Delta^R_{\alpha_i} \). For a finite sequence of simple roots \( I = (\alpha_{i_1}, \ldots, \alpha_{i_r}) \), we set:
\[
\Delta^R_I := \Delta^R_{i_1} \circ \cdots \circ \Delta^R_{i_r}, \quad \Delta^{(R,F)}_I := \Delta^{(R,F)}_{i_1} \circ \cdots \circ \Delta^{(R,F)}_{i_r}.
\]
We also set \( \alpha_I := \alpha_{i_1} \cdots \alpha_{i_r} \in S_R^*(\Lambda) \). We say that \( I \) is reduced if \( s_I \) is reduced in \( W \). Fix a reduced decomposition of a reflection \( w = s_{i_1} \cdots s_{i_r} \), where \( i_j \in \{1, \ldots, n\} \). We call \( I_w := (\alpha_{i_1}, \ldots, \alpha_{i_r}) \) a reduced sequence of \( w \).

**Lemma 5.4** Let \( I = (\alpha_{i_1}, \ldots, \alpha_{i_k}) \in \Upsilon_k \). We can write

\[
\Delta_I^R(\alpha_{i_1} \cdots \alpha_{i_k}) = \sum_{I' \in \Upsilon_{k-1}} c_{i'}^R \alpha_{I'},
\]

where the \( c_{i'}^R \in \mathcal{R} \) are independent of \( R \).

**Proof** First note that, for any \( \lambda \in \Lambda \), we have \( \Delta_I^R(\lambda) = \alpha_I^\vee(\lambda) \in \mathcal{R} \). Thus, since \( \mathcal{R} \) is a ring, the result follows by induction on the degree of monomials using (4) of Proposition 5.3. \( \square \)

**Proposition 5.5** (see [10, §4, Thm. 1] and [15, Ch. IV, Prop. 1.7]) Let \( I_w \) and \( I_w' \) be reduced sequences for the same element \( w \in W \). Then

\[
\Delta_{I_w}^R = \Delta_{I_w'}^R.
\]

**Proof** By [15, Ch. IV, Prop. 1.7], this result holds when \( R = \mathbb{C} \). Now the proposition follows from Lemma 5.4, and from the fact that \( \{1\} \cup \{\alpha_I\}_{I \in \Upsilon} \) is an \( \mathcal{R} \)-basis of \( S_R^*(\Lambda) \). \( \square \)

**Remark 5.6** Proposition 5.5 implies that \( \Delta_{I_w}^R \) is independent of the reduced sequence \( I_w \) of \( w \in W \). By the density of \( (S_R^*(\Lambda))^\wedge \), the operator \( \Delta_{I_w}^R \) is independent of the reduced sequence \( I_w \) of \( w \in W \) as well. Thus, the operator \( \Delta_{I_w}^{(R,F_a)} \) is independent of the reduced sequence \( I_w \) of \( w \in W \). In general, if \( (R,F) \) is an arbitrary formal group law over \( R \), then \( \Delta_{I_w}^{(R,F)} \) depends on the reduced sequence \( I_w \) of \( w \in W \).

From now on, we will simply write \( \Delta_{I_w}^R \) to denote the operator \( \Delta_{I_w}^R \), where \( I_w \) is any reduced sequence of \( w \in W \).

**Proposition 5.7** (see [10, §4, Prop. 3(a)] and [15, Ch. IV, Lem. 2.2]) Let \( w, w' \in W \). We have

\[
\Delta_{I_w}^R \Delta_{I_{w'}}^R = \begin{cases} 
\Delta_{I_{ww'}}^R, & \text{if } l(ww') = l(w) + l(w'), \\
0, & \text{otherwise.}
\end{cases}
\]

**Proof** By [15, Ch. IV, Lem. 2.2], this result holds when \( R = \mathbb{C} \). Now the proposition follows from Lemma 5.4, and from the fact that \( \{1\} \cup \{\alpha_I\}_{I \in \Upsilon} \) is an \( \mathcal{R} \)-basis of \( S_R^*(\Lambda) \). \( \square \)

Since \( S_R^*(\Lambda) \) injects into the integral domain \( (S_R^*(\Lambda))^\wedge \simeq R[[\Lambda]]_{F_a} \), the symmetric algebra \( S_R^*(\Lambda) \) is an integral domain. We let \( K \) be the field of fractions of \( S_R^*(\Lambda) \).

**Lemma 5.8** (cf. [10, Lem. 3]) Set \( \Sigma_w := \Sigma^+ \cap w(\Sigma^-) \) and \( q_w = \left( \prod_{\alpha \in \Sigma_w} \alpha \right) \). We have

\[
q_w \Delta_{I_w}^R = \det(w)w + \sum_{w' < w} a(I_{w'})w',
\]
where the $a(I_w) \in K$, and the ordering $w' < w$ is with respect to the Bruhat order on $W$.

Proof This proof is the same as the proof of [10, Lem. 3]; however, the proof of [10, Lem. 3] uses the result [3, Ch. VI, §1, No 6, Cor. 2], which is a property of the Weyl group. Since we are working with real finite reflection groups, we provide an updated reference. The result [3, Ch. VI, §1, No 6, Cor. 2] can be replaced by Proposition 2.7. \hfill \Box

Definition 5.9 Let $D_R(\Lambda)$ be the $R$-algebra of endomorphisms of $S_R^*(\Lambda)$ generated by the Demazure operators $\Delta_i^R$ and by multiplication by elements in $S_R^*(\Lambda)$.

Corollary 5.10 (cf. [10, Cor. 1] and [15, Ch. IV, Prop. 1.8]) The algebra $D_R(\Lambda)$ is free as a left $S_R^*(\Lambda)$-module with basis $\{\Delta_i^R\}_{w \in W}$.

Proof Since Proposition 5.3 and Lemma 5.8 hold, this proof is the same as the proof of [10, Cor. 1]. \hfill \Box

Remark 5.11 If $\Sigma$ is crystallographic and $R = \mathbb{Z}$, then Corollary 5.10 was proven by Demazure in [10, Cor. 1]. If $R = \mathbb{C}$, then Corollary 5.10 was proven by Hiller in [15, Ch. IV, Prop. 1.8].

6 Endomorphisms of the Formal Group Ring

In this section, we define the associated graded ring $Gr^*_F(\Lambda)$, and we discuss the subalgebra $D_{(R,F)}(\Lambda)$ of the endomorphism algebra of $R[\Lambda]^F$ generated by the formal Demazure operators and by multiplication by elements in $R[\Lambda]^F$. This section closely follows [6, §4].

If $I = (\alpha_1, \ldots, \alpha_r)$ is a sequence of simple roots, we set $x_I := x_{\alpha_1} \cdots x_{\alpha_r}$. Denote by $I$ the kernel of the augmentation map $R[x_\Lambda] \to R$. Recall the kernel $I_F$ of the augmentation map $R[\Lambda]^F \to R$. By convention, we set $I_i^F = R[\Lambda]^F$ for $i \leq 0$. Recall that $J_F$ is the closure of the ideal in $R[x_\Lambda]$ generated by the elements $x_0$ and $\log_F(x_{\lambda_1+\lambda_2}) - (\log_F(x_{\lambda_1}) + \log_F(x_{\lambda_2}))$ and $e_i \log_F(x_{\alpha_j}) - \log_F(x_{e_i\alpha_j})$, over all $\lambda_1, \lambda_2 \in \Lambda$, $e_i \in B$, and $\alpha_j \in \Delta$. Observe that, in $R[x_\Lambda]$, we can write

$$e_i \log_F(x_{\alpha_j}) - \log_F(x_{e_i\alpha_j}) = e_i x_{\alpha_j} - x_{e_i\alpha_j} + g_{i,j};$$

and

$$\log_F(x_{\lambda_1+\lambda_2}) - (\log_F(x_{\lambda_1}) + \log_F(x_{\lambda_2})) = x_{\lambda_1+\lambda_2} - (x_{\lambda_1} + x_{\lambda_2}) + h_{\lambda_1,\lambda_2},$$

for some $g_{i,j}, h_{\lambda_1,\lambda_2} \in I_F^2$.

We define the associated graded ring,

$$Gr^*_F(\Lambda) = \bigoplus_{i=0}^{\infty} I_i^F/I_i^F.$$

We have

$$I_F = I/J_F \quad \text{and} \quad I_k^F/I_k^F = I^k/(J_F \cap I^k + I^{k+1}).$$

Set $I^k := J_F \cap I^k + I^{k+1}$. Then Eqs. 5 and 6 imply that $I_k/I^{k+1}$ is generated by elements of the form $\rho \cdot x_0 + I^{k+1}$ and $\rho \cdot (x_{\lambda_1+\lambda_2} - x_{\lambda_1} - x_{\lambda_2}) + I^{k+1}$ and $\rho \cdot (e_i x_{\alpha_j} - x_{e_i\alpha_j}) + I^{k+1}$.
where $\lambda_1, \lambda_2 \in \Lambda$, $e_i \in B, \alpha_j \in \Delta$, and $\rho$ is a monomial of degree $k - 1$.

Observe that

$$\mathcal{T}_F^k / \mathcal{T}_F^{k+1} = \mathcal{T}_k / \mathcal{T}_k \cong (\mathcal{I}_k / \mathcal{I}_k) / (\mathcal{T}_k / \mathcal{T}_k^*).$$

(7)

Let $x_{\lambda_1} \cdots x_{\lambda_k} + \mathcal{T}_F^{k+1} \in \mathcal{T}_F^k / \mathcal{T}_F^{k+1}$ for some $\lambda_r \in \Lambda$. Through the identification of rings given in Eq. 7, the element $x_{\lambda_1} \cdots x_{\lambda_k} + \mathcal{T}_F^{k+1} \in \mathcal{T}_F^k / \mathcal{T}_F^{k+1}$ corresponds to the element $(x_{\lambda_1} \cdots x_{\lambda_k} + \mathcal{T}_k^{k+1}) + \mathcal{T}_k / \mathcal{T}_k^{k+1} \in (\mathcal{I}_k / \mathcal{I}_k^{k+1}) / (\mathcal{T}_k / \mathcal{T}_k^{k+1})$. Since $\{\epsilon_i \alpha_j\}$ is a $\mathbb{Z}$-basis for $\Lambda$, there are $c(r)_{i,j} \in \mathbb{Z}$ such that $\lambda_r = \sum_{j=1}^n \sum_{i=1}^l c(r)_{i,j} \epsilon_i \alpha_j$. Thus, the generators of $\mathcal{T}_k / \mathcal{T}_k^{k+1}$ allow us to write

$$(x_{\lambda_r} + \mathcal{T}_k^{k+1}) + \mathcal{T}_k / \mathcal{T}_k^{k+1} = \left( \sum_{j=1}^n \sum_{i=1}^l c(r)_{i,j} \epsilon_i \alpha_j + \mathcal{T}_k^{k+1} \right) + \mathcal{T}_k / \mathcal{T}_k^{k+1}.$$

Therefore, we can write $(x_{\lambda_1} \cdots x_{\lambda_k} + \mathcal{T}_F^{k+1}) + \mathcal{T}_k / \mathcal{T}_k^{k+1}$ as an $R$-linear combination of elements of the form $(x_{I} + \mathcal{T}_F^{k+1}) + \mathcal{T}_k / \mathcal{T}_k^{k+1}$, where $I \in \Upsilon_k$. In particular, $x_{\lambda_1} \cdots x_{\lambda_k} + \mathcal{T}_F^{k+1}$ can be written as an $R$-linear combination of elements of the form $(x_{I} + \mathcal{T}_F^{k+1})$, where $I \in \Upsilon_k$.

**Lemma 6.1** (cf. [6, Lem. 4.2]) The morphism of graded $R$-algebras

$$\psi : \mathcal{S}_R^k(\Lambda) \to \mathcal{G}_r^* (\mathcal{R}(\mathcal{F}),\Lambda)$$

defined by sending $\alpha_I$ to $x_I + \mathcal{T}_F^{k+1}$, $I \in \Upsilon_k$, and extended by $R$-linearity, is an isomorphism.

**Proof** Every element in $\mathcal{S}_R^k(\Lambda), k > 0$, can be written uniquely as an $R$-linear combination of monomials of the form $\alpha_I$, where $I \in \Upsilon_k$. The map $\psi$ sends the monomial $\alpha_I$ to the element $x_I + \mathcal{T}_F^{k+1} \in \mathcal{G}_r^* (\mathcal{R}(\mathcal{F}),\Lambda)$, and, therefore, $\psi$ is well-defined. Since $\mathcal{G}_r^* (\mathcal{R}(\mathcal{F}),\Lambda)$ is generated as a graded unital $R$-algebra by the elements $x_{\alpha_I} + \mathcal{T}_F^2$, where $\alpha_I \in \Delta$, the map $\psi$ is surjective. We define a map in the other direction.

Define the map of $R$-modules

$$\tilde{\theta}_k : \mathcal{T}_F^k / \mathcal{T}_F^{k+1} \to \mathcal{S}_R^k(\Lambda),$$

by sending a monomial of degree $k$ in some $x_{\lambda}$’s to the product of $\lambda$’s involved. The map $\tilde{\theta}_k$ factors through the quotient $\mathcal{T}_F^k / \mathcal{T}_F^{k+1} / (\mathcal{T}_k / \mathcal{T}_k^{k+1})$, hence giving a well-defined map $\theta_k : \mathcal{T}_F^k / \mathcal{T}_k \to \mathcal{S}_R^k(\Lambda)$. Equivalently, $\tilde{\theta}_k$ gives a well-defined map $\theta_k : \mathcal{T}_F^k / \mathcal{T}_F^{k+1} \to \mathcal{S}_R^k(\Lambda)$, since $\mathcal{T}_F^k / \mathcal{T}_k = \mathcal{T}_F^k / \mathcal{T}_F^{k+1}$. The sum $\theta := \oplus \theta_k$ is also well-defined. Since every element in $\mathcal{T}_F^k / \mathcal{T}_F^{k+1}$ is an $R$-linear combination of elements of the form $x_{I} + \mathcal{T}_F^{k+1}$, where $I \in \Upsilon_k$, we see that $\theta$ is the desired inverse map.

**Remark 6.2** Since every element in $\mathcal{S}_R^k(\Lambda), k > 0$, can be written uniquely as an $R$-linear combination of monomials of the form $\alpha_I$, where the $I \in \Upsilon_k$, it follows from the isomorphism $\psi$ of Lemma 6.1 that every element in $\mathcal{T}_F^k / \mathcal{T}_F^{k+1}$ can be written uniquely as an $R$-linear combination of elements of the form $x_{I} + \mathcal{T}_F^{k+1}$, where $I \in \Upsilon_k$.

**Lemma 6.3** (cf. [6, Prop. 4.6 (1)]) For any root $\alpha \in \Sigma$, the operator $\Delta_{\alpha}^{(R,F)}$ sends $\mathcal{T}_F^k$ to $\mathcal{T}_F^{k-1}$.

**Proof** This follows directly from (4) of Proposition 5.3.
By Lemma 6.3, the operator $\Delta^{(R,F)}_\alpha$ induces an $R$-linear operator of degree $-1$ on the graded ring $\mathcal{G}r^*_R(\Lambda)$, denoted $\mathcal{G}r \Delta^{(R,F)}_\alpha$, for all $\alpha \in \Sigma$.

**Proposition 6.4** (see [6, Prop. 4.4]) For each root $\alpha \in \Sigma$, the isomorphism of Lemma 6.1 exchanges the operator $\mathcal{G}r \Delta^{(R,F)}_\alpha$ on $\mathcal{G}r^*_R(\Lambda)$ with $\Delta^R_\alpha$ on the symmetric algebra $S^*_R(\Lambda)$.

**Proof** By Remark 6.2, every element in $I^k / I^k + 1$, $k > 0$, can be written uniquely as an $R$-linear combination of elements of the form $x_\lambda + I^k + 1$, where $\lambda \in \Upsilon_k$. Now the result follows by induction on the degree of monomials using (4) of Proposition 5.3.

**Definition 6.5** (see [6, Def. 4.5]) Let $\mathcal{D}(R,F)(\Lambda)$ be the algebra of $R$-linear endomorphisms of $R[\Lambda]_F$ generated by the formal Demazure operators $\Delta^{(R,F)}_\alpha$ for all roots $\alpha$, and by multiplication by elements in $R[\Lambda]_F$.

For each element $w \in W$, fix a reduced sequence of simple roots $I_w$.

**Theorem 6.6** (see [6, Lem. 4.11 (3)]) The operators $\Delta^{(R,F)}_{I_w}$ form a basis of $\mathcal{D}(R,F)(\Lambda)$ as a left $R[\Lambda]_F$-module.

**Proof** By Proposition 5.5 and Corollary 5.10, the operators $\Delta^R_{I_w}$ form a basis of $\mathcal{D}R(\Lambda)$ over $S^*_R(\Lambda)$. Therefore, the rest of this proof is the same as the proof of [6, Lem. 4.11].

**Remark 6.7** We can view $\mathcal{D}R(\Lambda)$ as the subalgebra of the endomorphism algebra of $(S^*_R(\Lambda))^{\wedge}$ generated by the Demazure operators $\Delta^R_\alpha$ and by multiplication by elements in $S^*_R(\Lambda)$. If $(R, F) = (R, F_a)$ is the additive formal group law over $R$, then, through the isomorphism $\phi : R[\Lambda]_{F_a} \to (S^*_R(\Lambda))^{\wedge}$ of Example 4.19, we can view $\mathcal{D}(R,F_a)(\Lambda)$ as the completion of $\mathcal{D}R(\Lambda)$ at the kernel of the map $\mathcal{D}R(\Lambda) \to R$, sending $\alpha^* \mapsto 0$ for all $\alpha \in \Sigma$. Here, we have used $\alpha^*$ to denote multiplication by $\alpha$.

## 7 The Formal Affine Demazure Algebra

In this section, we define the formal affine Demazure algebra $\mathcal{D}F$ and discuss several of its properties. All results discussed in this section are extensions of results in [8] and [16] to all real finite reflection groups.

By (2) of Lemma 4.15, the formal group ring $R[\Lambda]_F$ is an integral domain. Thus, we let $Q^{(R,F)}_\Lambda$ be the localization of $R[\Lambda]_F$ at the multiplicative subset generated by the elements $x_\alpha$ for all $\alpha \in \Sigma$. The ring $Q^{(R,F)}_\Lambda$ is an integral domain since $R[\Lambda]_F$ is an integral domain, and we call $Q^{(R,F)}_\Lambda$ the localized formal group ring. As $W$ acts on $R[\Lambda]_F$ by permuting the roots $\alpha \in \Sigma$, it also acts on $Q^{(R,F)}_\Lambda$.

**Lemma 7.1** (see [8, Lem. 3.2]) The localization map $R[\Lambda]_F \to Q^{(R,F)}_\Lambda$ is injective.

**Proof** This follows from the fact that $R[\Lambda]_F$ is an integral domain.
Following [16, Def. 6.1] and [18, §4.1], we define the twisted formal group ring to be the $R$-module $Q^{(R,F)}_{\Lambda,W} := Q^{(R,F)}_{\Lambda} \otimes_R R[W]$ with multiplication given by

\[
(q \delta_w)(q' \delta_{w'}) = qw(q')\delta_{ww'}, \quad w, w' \in W, \quad q, q' \in Q^{(R,F)}_{\Lambda},
\]

and extended by $R$-linearity. Here, $\delta_w$ is the element in $R[W]$ corresponding to $w \in W$ (so that $\delta_w \delta_{w'} = \delta_{ww'}$ for all $w, w' \in W$). We will denote the identity $1 = \delta_1$. The set $\{\delta_w\}_{w \in W}$ is the canonical basis of the group ring $R[W]$, and, hence, it is a basis of $Q^{(R,F)}_{\Lambda,W}$ as a left $Q^{(R,F)}_{\Lambda}$-module.

Consider the $R$-subalgebra $D_{R[\Lambda]}_F$ of $Q^{(R,F)}_{\Lambda,W}$ generated by $R[\Lambda]_F$ and the elements

\[
X^{(R,F)}_{\alpha} = \frac{1}{x_\alpha} (1 - \delta_\alpha),
\]

for all roots $\alpha \in \Sigma$, where $\delta_\alpha = \delta_{s_\alpha}$ for all $\alpha \in \Sigma$. Following [16, Def. 6.2 and Def. 6.3], we call the elements $X^{(R,F)}_{\alpha}$, $\alpha \in \Sigma$, formal Demazure elements and we call $D_{R[\Lambda]}_F$ the formal affine Demazure algebra. Often, we denote the formal Demazure elements by $X_\alpha$, omitting the superscript, and we denote the formal affine Demazure algebra by $D_F$. For any simple root $\alpha_i \in \Delta$, we define $X_i := X_{\alpha_i}$ and $\delta_i := \delta_{s_{\alpha_i}}$. For any sequence of simple roots $I = (\alpha_{i_1}, \ldots, \alpha_{i_n})$, we will use the notation $\delta_I := \delta_{i_1} \cdots \delta_{i_n}$ and $X_I := X_{i_1} \cdots X_{i_n}$.

**Lemma 7.2** (see [8, Lem. 5.8]) *The formal affine Demazure algebra $D_F$ coincides with the $R$-subalgebra of $Q^{(R,F)}_W$ generated by the elements of $R[\Lambda]_F$ and the formal Demazure elements $X_i$, $i \in \{1, \ldots, n\}$.***

**Proof** By Lemma 2.2, any root $\alpha \in \Sigma$ can be written $\alpha = w(\alpha_i)$ for some $\alpha_i \in \Delta$ and $w \in W$. Now this result follows by the proof of [8, Lem. 5.8].

**Lemma 7.3** (see [8, Lem. 5.4]) *Given a reduced sequence $I_v$ of $v \in W$ of length $l(v)$, let

\[
X_{I_v} = \sum_{w \in W} a_{v,w} \delta_w
\]

for some $a_{v,w} \in Q^{(R,F)}_{\Lambda}$. Then

(a) $a_{v,w} = 0$ unless $w \leq v$ with respect to the Bruhat order on $W$;

(b) $a_{v,w} = (-1)^{l(v)} \prod_{\alpha \in \Gamma(v)[\Sigma^-] \cap \Sigma^+} x_\alpha^{-1}$.

**Proof** This proof is the same as the proof of [8, Lem. 5.4]; however, the proof of [8, Lem. 5.4] uses the results [12, Th. 1.1, III, (ii)] and [3, Ch. VI, §1, No 6, Cor. 2], which are properties of the Weyl group. Since we are working with real finite reflection groups, we provide updated references. The result [12, Th. 1.1, III, (ii)] can be replaced by Theorem 2.10, and instead of [3, Ch. VI, §1, No 6, Cor. 2], we use Proposition 2.7.

**Remark 7.4** By definition, we have $X_\alpha = \frac{1}{x_\alpha} (1 - \delta_\alpha)$. Therefore, since $W$ permutes the roots, the $a_{v,w}$ of Lemma 7.3 are sums of products of elements of the form $\frac{1}{x_\alpha}$, where $\alpha \in \Sigma$.

**Corollary 7.5** (see [8, Cor. 5.6]) *Choose a reduced sequence $I_w$ for each $w \in W$. The elements $\{X_{I_w}\}_{w \in W}$ form a basis of $Q^{(R,F)}_{\Lambda,W}$ as a left $Q^{(R,F)}_{\Lambda}$-module, and the element $\delta_v$
decomposes as $\sum_{w \leq v} b_{w,v} X_{I_w}$ with $b_{w,v} \in \mathbb{Q}^{(R,F)}$. Furthermore,

$$b_{v,v} = (-1)^{(v)} \prod_{\alpha \in v(\Sigma^-) \cap \Sigma^+} x_\alpha.$$ 

**Proof** Since Lemma 7.3 holds, this proof is the same as the proof of [8, Cor. 5.6].

The proofs of Lemmas 7.6 and 7.7 are direct computations:

**Lemma 7.6** (see [16, Lemma 6.5]) For all $\alpha \in \Sigma$, we have the following commuting relation in $\mathbb{Q}^{(R,F)}$,

$$X_\alpha q = s_\alpha(q) X_\alpha + \Delta_\alpha^{(R,F)}(q), \quad q \in \mathbb{Q}^{(R,F)}.$$

Here, $\Delta_\alpha^{(R,F)}(q)$ denotes the element $\frac{q - s_\alpha(q)}{x_\alpha} \in \mathbb{Q}^{(R,F)}$. By Lemma 5.1, if $q \in R[\Lambda]_F$, then $\Delta_\alpha^{(R,F)}(q) \in R[\Lambda]_F$. Hence, the formula above gives a relation in $D_F$.

**Lemma 7.7** (see [16, Eq. (6.1)]) For any $\alpha \in \Sigma$ we have

$$X_{\alpha}^2 = \kappa_\alpha X_\alpha, \quad \text{where} \quad \kappa_\alpha = \frac{1}{x_\alpha} + \frac{1}{x_{-\alpha}}.$$

Here, $\kappa_\alpha \in R[\Lambda]_F$ by the argument in [6, Def. 3.12].

**Remark 7.8** If $(R, F)$ is a formal group law of additive type, then the $\kappa_\alpha$ of Lemma 7.7 is 0 for each $\alpha \in \Sigma$. If $(R, F)$ is the multiplicative formal group over $R$, then $\kappa_\alpha = -1$ for all $\alpha \in \Sigma$.

The following proposition says that all formal affine Demazure algebras are isomorphic. Its proof is the same as the proof of [16, Thm. 7.4]. We include the proof.

**Proposition 7.9** (see [16, Thm. 7.4]) There is an $R$-algebra isomorphism $D_{R[\Lambda]_F} \cong D_{R[\Lambda]_{Fa}}$.

**Proof** The $W$-equivariant continuous ring isomorphism $\widetilde{\exp}_F^* : R[\Lambda]_F \rightarrow R[\Lambda]_{Fa}$ of formal group rings sends $x_\alpha \mapsto \exp_F(x_\alpha)$. Thus, $\exp_F^*$ induces an isomorphism of twisted formal group rings $\widetilde{\exp}_F^* : \mathbb{Q}_{A}^{(R,F)} \rightarrow \mathbb{Q}_{A}^{(R,Fa)}$, and $D_{R[\Lambda]_{Fa}}$ is isomorphic to its image $D' := \exp_F^*(D_{R[\Lambda]_{Fa}})$ under this map. Now, $D'$ is generated over $R[\Lambda]_{Fa}$ by the elements

$$\exp_F^*\left(\Delta_i^{(R,F)}\right) = \frac{1}{\exp_F(x_{a_i}) (1 - \delta_i)} = \frac{x_{a_i}}{\exp_F(x_{a_i})} \Delta_i^{(R,Fa)},$$

where $\alpha_i \in \Delta$. Recall that there are $a_k \in R$ such that $\exp_F(x_{a_i}) = x_{a_i} + \sum_{k \geq 2} a_k x_{a_i}^k = x_{a_i} g(x_{a_i})$, where $g(x_{a_i}) = 1 + \sum_{k \geq 2} a_k x_{a_i}^{k-1} \in R[\Lambda]_F$. Observe that $g(x_{a_i})$ is invertible in $R[\Lambda]_F$, since its constant term is 1. Thus, $\frac{1}{\exp_F(x_{a_i})} = g(x_{a_i}) \in R[\Lambda]_{Fa}$ is invertible in $R[\Lambda]_{Fa}$. Therefore, $D'$ is generated over $R[\Lambda]_{Fa}$ by the elements $\Delta_i^{(R,Fa)}, \alpha_i \in \Delta$, and is, thus, isomorphic to $D_{R[\Lambda]_{Fa}}$. 

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8 Presentation in Terms of Generators and Relations

In this section, we provide a presentation for the formal affine Demazure algebra in terms of formal Demazure elements. This section closely follows [6, §5.2], [8, §6 and §7], and [16, §6].

The arguments used from here up to and including the proof of Lemma 8.1 closely follow the arguments used in [6, §5.2]. Recall that we can view $S^*_R(\Lambda)$ as the symmetric algebra of the complex vector space $V_C = \mathbb{C} \otimes_R V$ spanned by the roots $\alpha \in \Sigma$. Let $w_0$ be the longest element of $W$, and let $N = l(w_0)$. By [15, Ch. IV, Prop. 1.6 and Prop. 1.7] (see also the proof of [15, Ch. IV, Thm. 1.10]), the element $d_0 := \prod_{\beta \in \Sigma^+} \beta$ satisfies $\Delta^{C}_{I_{w_0}}(d_0) = |W|$ for any reduced sequence $I_{w_0}$ of $w_0$.

Since each root $\beta$ is an $R$-linear combination of the simple roots, it follows from Lemma 5.4 that $\Delta^{R}_{I_{w_0}}(d_0) = \Delta^{C}_{I_{w_0}}(d_0) = |W|$ for any reduced sequence $I_{w_0}$ of $w_0$. Through the isomorphism $\psi : S^*_R(\Lambda) \to \mathcal{G}r^*_R(R,F)(\Lambda)$ of Lemma Lemma 6.1, we choose an element $u_0 \in \mathcal{I}^N_F / \mathcal{I}^{N+1}_F$ with $u_0 = \psi(d)$. Therefore, by Proposition 6.4, for any reduced sequence $I_{u_0}$ of $w_0$, we have that $\Delta^{(R,F)}_{I_{u_0}}(u_0) = \Delta^{R}_{I_{u_0}}(d) + \mathcal{I}_F$. Hence, $\epsilon \Delta^{(R,F)}_{I_{w_0}}(u_0) = \Delta^{R}_{I_{w_0}}(d) = |W|$ for any reduced sequence $I_{u_0}$ of $w_0$.

The proof of Lemma 8.1 is the same as the proof of [6, Lemma 6.3 (3)]. We include the proof of Lemma 8.1 because it refers to several facts in [6] that we have shown hold for all real finite reflection groups as well. The element $u_0$ has the following property:

**Lemma 8.1** (cf. [6, Lemma 5.3 (3)]) If $I$ is any sequence of simple roots, and $1 \leq l(I) \leq N$, then

$$\epsilon \Delta^{(R,F)}_{I}(u_0) = \begin{cases} |W|, & \text{if } I \text{ is reduced and } l(I) = N; \\ 0, & \text{otherwise.} \end{cases}$$

**Proof** For $k > 0$, Lemma 6.3 implies that the operator $\Delta^{(R,F)}_{I}$ sends $\mathcal{I}^k_F$ to $\mathcal{I}^{k-l(I)}_F$. Thus, if $l(I) < N$, then $\epsilon \Delta^{(R,F)}_{I}(u_0) \in \epsilon \left( \mathcal{I}^{N-l(I)}_F \right) = \{0\}$.

If $l(I) = N$ and $I$ is not reduced, then $\epsilon \Delta^{(R,F)}_{I}(u_0) = 0$. To see this, it suffices to show that, in this case, $\Delta^{(R,F)}_{I}$ sends $\mathcal{I}^k_F$ to $\mathcal{I}^{k-l(I)+1}_F$. Thus, it is enough to show that $\mathcal{G}r\Delta^{(R,F)}_{I}$ is invertible. The fact that $\Delta^{R}_{I} = 0$ when $I$ is not reduced follows from Proposition 5.7.

If $l(I) = N$ and $I$ is reduced, then $\epsilon \Delta^{(R,F)}_{I}(u_0) = |W|$ by definition. $\square$

In [6, 8], the torsion index $t$ of [10] serves the same role as the element $|W|$ does in this paper. For each $w \in W$, let $I_w$ be a reduced sequence of $w$. With the torsion index $t$ replaced by the element $|W|$, the following result goes through:

**Lemma 8.2** (see [8, Lem. 6.1]) Suppose $|W|$ is invertible in $R$. Then the matrix

$$\left( \Delta^{(R,F)}_{I_v} \Delta^{(R,F)}_{I_w}(u_0) \right)_{(v,w) \in W \times W}$$

with coefficients in $R[\Lambda]_F$ is invertible.
Proof By hypothesis, the element \(|W|\) is invertible in \(R\). In addition, Lemma 8.1 holds. Thus, this result follows by the proof of [6, Prop. 6.6].

The arguments used from here up to and including the proof of Theorem 8.9 closely follow the arguments used in [8, §6 and §7] and [16, §6]. Let \(\phi : Q^{(R,F)}_{\Lambda,W} \to \text{End}_R(Q^{(R,F)}_{\Lambda})\) be the \(R\)-algebra homomorphism induced by the left action of \(Q^{(R,F)}_{\Lambda}\) on \(Q^{(R,F)}_{\Lambda,W}\). By definition, \(R[\Lambda]_F\) acts on the left on both algebras, \(\phi \) is \(R[\Lambda]_F\)-linear, and \(\phi(X_\alpha) = \Delta^i(\alpha,R,F)\).

Let \(\phi_{DF} : DF \to D(R,F)\) be the \(R\)-algebra homomorphism induced by the left action of \(Q^{(R,F)}_{\Lambda,W}\) on \(Q^{(R,F)}_{\Lambda}\). By definition, \(R/\llbracket \Lambda \rrbracket F\) acts on the left on both algebras, \(\phi\) is \(R/\llbracket \Lambda \rrbracket F\)-linear, and \(\phi(X_\alpha) = \Delta^i(\alpha,R,F)\).

**Theorem 8.3** (see [8, Thm. 7.10]) The map \(\phi_{DF} : DF \to D(R,F)\) is both an \(R\)-algebra isomorphism and a left \(R/\llbracket \Lambda \rrbracket F\)-module isomorphism.

**Proof** Since Corollary 7.5 and Lemma 8.2 hold, this proof is the same as the proof of [8, Thm. 7.10].

**Corollary 8.4** The \(R\)-algebra \(DF\) is free as a left \(R/\llbracket \Lambda \rrbracket F\)-submodule of \(Q^{(R,F)}_{\Lambda,W}\) with basis \(\{X_{I_w}\}_{w \in W}\).

**Proof** This follows from Theorems 6.6 and 8.3.

**Lemma 8.5** (see [8, Lem. 7.1]) Let \(I\) and \(I'\) be reduced sequences of the same element \(w \in W\). Then in the formal affine Demazure algebra \(DF\), we have

\[ X_I - X_{I'} = \sum_{v < w} c_v X_{I_v} \quad \text{for some } c_v \in R[\Lambda]_F, \]

where the ordering \(v < w\) is with respect to the Bruhat order on \(W\).

**Proof** By Corollary 8.4, the difference \(X_I - X_{I'} = \sum_{v \in W} c_v X_{I_v}\) for some \(c_v \in R[\Lambda]_F\). Since Lemmas 7.1, 7.3, and Corollary 7.5 hold, the rest of this proof is the same as the proof of [8, Lem. 7.1].

**Remark 8.6** Let \(I\) and \(I'\) be reduced sequences of the same element \(u \in W\). Then Proposition 5.5 and Theorem 8.3 imply that, in \(D(R,F)\), we have \(X_I^{(R,F,u)} = X_{I'}^{(R,F,u)}\).

**Remark 8.7** Let \(I\) and \(I'\) be reduced sequences of the same element \(u \in W\). Suppose \(\Sigma\) is crystallographic. By [6, Thm. 3.10], in \(D(R,F)\), we have \(X_I^{(R,F_m)} = X_{I'}^{(R,F_m)}\). Note that [6, Thm. 3.10] refers to the proof of [11, Thm. 2, pp. 86], which relies on the geometry of flag varieties. If \(\Sigma\) is noncrystallographic, we do not know whether \(X_I^{(R,F_m)} = X_{I'}^{(R,F_m)}\) in \(D(R,F)\).

**Remark 8.8** (cf. [8, Ex. 7.3]) Suppose \(\alpha_i\) and \(\alpha_j\) are simple roots, and \(m_{i,j}\) is the order of \(s_is_j\) in \(W\). Set \(w_0^{i,j} = s_is_j\cdot\cdot\cdot = s_js_i\cdot\cdot\cdot\) \(m_{i,j}\) times \(m_{i,j}\) times. If \(\Sigma\) is a root system of rank 2, then \(w_0^{1,2}\) is the unique element in \(W\) with more than one reduced decomposition; \(w_0^{1,2}\) has exactly two
reduced sequences \((\alpha_1, \alpha_2, \alpha_1, \ldots)\) and \((\alpha_2, \alpha_1, \alpha_2, \ldots)\), both of length \(m_{1,2}\); and \(w_0^{1,2}\) is the longest word in \(W\).

**Theorem 8.9** (see [8, Thm. 7.9] and [16, Thm. 6.14]) Let \(\Sigma\) be a root system of a real finite reflection group \(W\). Let \(Q_{\Lambda}^{(R,F)}\) denote the localization of \(R[\Lambda]_F\) at the elements \(x_\alpha, \alpha \in \Sigma\). Given a set of simple roots \(\{\alpha_1, \ldots, \alpha_n\}\) with associated simple reflections \(\{s_1, \ldots, s_n\}\), let \(m_{i,j}\) denote the order of the product \(s_is_j\) in \(W\). The elements \(q \in Q_{\Lambda}^{(R,F)}\) (resp. \(q \in R[\Lambda]_F\)) and the formal Demazure elements \(X_i\) satisfy the following relations for all \(i, j = 1, \ldots, n\):

1. \(X_iq = \Delta_i^{(R,F)}(q) + s_i(q)X_i\);
2. \(X_i^2 = \kappa_i X_i\), where \(\kappa_i = \frac{1}{x_{a_i}} + \frac{1}{x_{-a_i}}\);
3. \(X_i X_j X_i \cdots - X_j X_i X_j \cdots = \sum_{w < w_0^{i,j}} \eta_{i,j}^w X_{I_w}\), \(\eta_{i,j}^w \in R[\Lambda]_F\).

Here, \(w_0^{i,j}\) is defined in Remark 8.8, and the ordering \(w < w_0^{i,j}\) is with respect to the Bruhat order on \(W\). These relations, together with the ring law in \(R[\Lambda]_F\) and the fact that the \(X_i\) are \(R\)-linear, form a complete set of relations in the localized twisted formal group ring \(Q_{\Lambda,W}^{(R,F)}\) (resp. the formal affine Demazure algebra \(D_F\)).

**Proof** The three relations hold by Lemmas 7.6, 7.7, and 8.5. Since Corollaries 7.5 and 8.4 hold and \(W\) is a Coxeter group (see Theorem 2.3), the proofs of the presentations of \(Q_{\Lambda,W}^{(R,F)}\) and \(D_F\) are similar to the proof of [16, Thm. 6.14].

### 9 Computations of Structure Coefficients

In the present section, we study the braid relation (3) of Theorem 8.9. In particular, we give general formulas for several structure coefficients \(\eta_{i,j}^w\) that appear in this braid relation, thus generalizing several formulas obtained in [16, Prop. 6.8] to all real finite reflection groups.

Before we proceed, we will define some notation that will be used in this section and in Section 10.

**Notation 9.1** For any root \(\gamma \in \Sigma\), set \(y_\gamma = \frac{1}{s_\gamma}\). Let \(\{\alpha, \beta\}\) be distinct simple roots, and let \(m\) be the order of \(s_\alpha s_\beta\) in \(W\). Write

\[
w_{0}^{\alpha,\beta} = s_\alpha s_\beta s_\alpha \cdots = s_\beta s_\alpha s_\beta \cdots.
\]

As in the proof of Lemma 2.8, we set

\[
\Sigma^+_{\alpha,\beta} : = \{\alpha, s_\alpha(\beta), s_\alpha s_\beta(\alpha), \ldots, s_{(m-1)}(\alpha)\} = \{\beta, s_\beta(\alpha), s_\beta s_\alpha(\beta), \ldots, s_{(m-1)}(\beta)\},
\]

and we will use the notation

\[
y_{\Sigma^+_{\alpha,\beta}} = \prod_{\gamma \in \Sigma^+_{\alpha,\beta}} y_\gamma.
\]
We define the following notation for products of \( i \) formal Demazure elements, \( \delta \)'s, and simple reflections:

\[
X_{\alpha,\beta,\ldots}^{(i)} = X_{\alpha}X_{\beta}X_{\alpha}^{\ldots}
\]
\[
X_{\beta,\alpha,\ldots}^{(i)} = X_{\beta}X_{\alpha}X_{\beta}^{\ldots}
\]
\[
\delta_{\alpha,\beta,\ldots}^{(i)} = \delta_{\alpha}\delta_{\beta}\delta_{\alpha}^{\ldots}
\]
\[
\delta_{\beta,\alpha,\ldots}^{(i)} = \delta_{\beta}\delta_{\alpha}\delta_{\beta}^{\ldots}
\]
\[
s_{\alpha,\beta,\ldots}^{(i)} = s_{\alpha}s_{\beta}s_{\alpha}^{\ldots}
\]
\[
s_{\beta,\alpha,\ldots}^{(i)} = s_{\beta}s_{\alpha}s_{\beta}^{\ldots}
\]

So, for example,

\[
X_{\alpha,\beta,\ldots}^{(3)} = X_{\alpha}X_{\beta}X_{\alpha}
\]
\[
s_{\beta,\alpha,\ldots}^{(7)} = s_{\beta}s_{\alpha}s_{\beta}s_{\alpha}s_{\beta}s_{\alpha}s_{\beta}
\]

We define the following:

\[
\omega_i = \begin{cases} 
\alpha, & i \text{ even}, \\
\beta, & i \text{ odd}.
\end{cases}
\]

Let \( i = 1, \ldots, m - 2 \). We define:

\[
u_{\alpha,\beta}^{(i)} = \prod_{j=0}^{m-i-1} s_{\alpha,\beta,\ldots}(y_{\omega_j}), \quad 
u_{\beta,\alpha}^{(i)} = \prod_{j=0}^{m-i-1} s_{\beta,\alpha,\ldots}(y_{\omega_j+1}).
\]

By Proposition 2.7, \( \nu_{\alpha,\beta}^{(0)} = \nu_{\beta,\alpha}^{(0)} = y_{\Sigma^+} \).

For \( j > i \geq 0 \), we define the operators \( S_{\beta,\alpha}^{(i,j)} := \sum_{k=i}^{j} s_{\beta,\alpha,\ldots}^{(k)} \) and \( S_{\alpha,\beta}^{(i,j)} := \sum_{k=i}^{j} s_{\alpha,\beta,\ldots}^{(k)} \),

which act on a root \( \gamma \in \Sigma \) as follows:

\[
S_{\beta,\alpha}^{(i,j)}(\gamma) = s_{\beta,\alpha,\ldots}^{(i)}(\gamma) + s_{\beta,\alpha,\ldots}^{(i+1)}(\gamma) + \cdots + s_{\beta,\alpha,\ldots}^{(j)}(\gamma),
\]
\[
S_{\alpha,\beta}^{(i,j)}(\gamma) = s_{\alpha,\beta,\ldots}^{(i)}(\gamma) + s_{\alpha,\beta,\ldots}^{(i+1)}(\gamma) + \cdots + s_{\alpha,\beta,\ldots}^{(j)}(\gamma).
\]

We have the following extension of [16, Prop. 6.8] to all real finite reflection groups. The proof is very similar to the proof of [16, Prop. 6.8]. We include it because we will refer to the proof several times in this section.

**Lemma 9.2** (cf. [16, Prop. 6.8]) *The difference* \( X_{\alpha,\beta,\ldots}^{(m)} - X_{\beta,\alpha,\ldots}^{(m)} \) *can be written as a linear combination*

\[
X_{\alpha,\beta,\ldots}^{(m)} - X_{\beta,\alpha,\ldots}^{(m)} = \sum_{i=1}^{m-2} (\kappa_{\alpha,\beta}^{(i)} X_{\alpha,\beta,\ldots}^{(i)} - \kappa_{\beta,\alpha}^{(i)} X_{\beta,\alpha,\ldots}^{(i)}),
\]

*where* \( \kappa_{\alpha,\beta}^{(i)}, \kappa_{\beta,\alpha}^{(i)} \in \mathbb{R}[\Lambda]_F, i = 1, \ldots, m - 2.\)

**Proof** We let \( \kappa_{\alpha,\beta}^{(m-1)} \) and \( \kappa_{\beta,\alpha}^{(m-1)} \) denote the coefficients of \( X_{\alpha,\beta,\ldots}^{(m-1)} \) and \( X_{\beta,\alpha,\ldots}^{(m-1)} \) on the right side of Eq. 8. Similarly, we let \( \kappa_{\alpha,\beta}^{(m)} \) and \( \kappa_{\beta,\alpha}^{(m)} \) denote the coefficients of \( X_{\alpha,\beta,\ldots}^{(m)} \) and \( X_{\beta,\alpha,\ldots}^{(m)} \) on the right side of Eq. 8. We will show by direct computation that all four of these coefficients equal 0.
The fact that \( \kappa_{\alpha,\beta}^{(m)} = \kappa_{\beta,\alpha}^{(m)} = 0 \) follows by the same reasoning as in the proof of [16, Prop. 6.8]. We will include the proof because we use it in Corollary 9.4 and Theorem 9.10. Alternatively, \( \kappa_{\alpha,\beta}^{(m)} = \kappa_{\beta,\alpha}^{(m)} = 0 \) by Lemma 8.5.

In the expansion of the product

\[
X_{\alpha,\beta}^{(m)} = \sum_{\alpha,\beta,\ldots} X_{\alpha} X_{\beta} X_{\alpha} \cdots X_{\omega_{m+1}} = (y_{\alpha} - y_{\alpha} \delta_{\alpha})(y_{\beta} - y_{\beta} \delta_{\beta})(y_{\alpha} - y_{\alpha} \delta_{\alpha}) \cdots (y_{\omega_{m+1}} - y_{\omega_{m+1}} \delta_{\omega_{m+1}}),
\]

and in the expansion of the product \( X_{\beta,\alpha}^{(m)} \), the coefficient of \( \delta_{w_{0,\beta}} = \delta_{(m)}_{\alpha,\beta,\ldots} = \delta_{(m)}_{\beta,\alpha,\ldots} \) is

\[
\pm y_{\alpha} s_{\alpha}(y_{\beta}) \cdots s_{(m-1)}(y_{\omega_{m+1}}) = \pm y_{\beta} s_{\beta}(y_{\alpha}) \cdots s_{(m-1)}(y_{\omega_{m}}) = \pm y_{\Sigma_{a,\beta}^{+}},
\]

where we use ‘+’ when \( m \) is even and we use ‘−’ when \( m \) is odd. Hence, in the difference \( X_{\alpha,\beta,\ldots} - X_{\beta,\alpha,\ldots} \), the \( \delta_{w_{0,\beta}} \)-term is

\[
\pm y_{\Sigma_{a,\beta}^{+}} \delta_{(m)}_{a,\beta,\ldots} - (\pm y_{\Sigma_{a,\beta}^{+}} \delta_{a,\beta,\ldots}) = 0.
\]

So \( \kappa_{\alpha,\beta}^{(m)} = \kappa_{\beta,\alpha}^{(m)} = 0 \). Next we will show that \( \kappa_{\alpha,\beta}^{(m-1)} = \kappa_{\beta,\alpha}^{(m-1)} = 0 \).

Suppose \( m \) is odd. To obtain a \( \delta_{w_{0,\beta}}^{(m-1)} \)-term in the expansion of the product \( X_{\alpha,\beta}^{(m)} \), except for the last \( X_{\alpha} \), we must choose the \( \delta \)-term in each factor, except for the last factor, where we use ‘+’ if \( m \) is even and ‘−’ if \( m \) is odd. Hence, the \( \delta_{w_{0,\beta}}^{(m-1)} \)-term is

\[
(y_{\alpha} \delta_{\alpha})(y_{\beta} \delta_{\beta}) \cdots (y_{\beta} \delta_{\beta}) y_{\alpha} = y_{\alpha} s_{\alpha}(y_{\beta}) \cdots s_{(m-1)}(y_{\omega_{m+1}}) = y_{\Sigma_{a,\beta}^{+}} \delta_{(m-1)}^{a,\beta}.
\]

By a similar argument, the \( \delta_{w_{0,\beta}}^{(m-1)} \)-term in \( X_{\beta,\alpha}^{(m)} \) is

\[
y_{\alpha} (y_{\beta} \delta_{\beta})(y_{\alpha} \delta_{\alpha}) \cdots (y_{\beta} \delta_{\beta}) y_{\alpha} = y_{\alpha} y_{\beta} s_{\beta}(y_{\alpha}) \cdots s_{(m-2)}(y_{\omega_{m+1}}) = c \delta_{(m-1)}^{a,\beta}.
\]

By Lemma 2.8, \( \alpha = s_{(m-1)}^{a,\beta} \). Thus, Proposition 2.7 implies that \( c = y_{\Sigma_{a,\beta}^{+}} \). Therefore, in the difference \( X_{\alpha,\beta}^{(m)} - X_{\beta,\alpha}^{(m)} \), the coefficients of \( \delta_{w_{0,\beta}}^{(m-1)} \) and \( \delta_{w_{0,\beta}}^{(m-1)} \) are zero, which implies that \( \kappa_{\alpha,\beta}^{(m-1)} = \kappa_{\beta,\alpha}^{(m-1)} = 0 \).

Now suppose \( m \) is even. One can show by a similar argument that the \( \delta_{w_{0,\beta}}^{(m-1)} \)-term and the \( \delta_{w_{0,\beta}}^{(m-1)} \)-term in \( X_{\alpha,\beta}^{(m)} \) are both \( -y_{\Sigma_{a,\beta}^{+}} \). Thus, \( \kappa_{\alpha,\beta}^{(m-1)} = \kappa_{\beta,\alpha}^{(m-1)} = 0 \) when \( m \) is even as well.

Now we consider the constant terms. By the discussion just before the statement of Theorem 8.3, there is a natural action of \( Q_{a,\beta}^{(R,F)} \) on \( Q_{\Lambda}^{(R,F)} \), where \( X_{\alpha}(r) = \Delta_{a}^{(R,F)}(r) = 0 \) and \( X_{\beta}(r) = \Delta_{\beta}^{(R,F)}(r) = 0 \) for all \( r \in R \). Therefore, the constant term on the right side of Eq. 8 is zero.

Finally, it follows from Lemma 8.5 that all coefficients on the right hand side of Eq. 8 are in \( R[\Lambda]_{F} \).

**Remark 9.3** If \( \Sigma \) is crystallographic, then the structure coefficients \( \kappa_{a,\beta}^{(i)} \) and \( \kappa_{\beta,\alpha}^{(i)} \) can be explicitly computed from the formulas in [16, Prop. 6.8] by applying Lemmas 7.6 and 7.7.
We assume that $m \geq 3$ is odd for the rest of this section. We believe that results similar to those proven in the rest of this section hold when $m$ is even. However, for simplicity of our exposition, we will only consider the case when $m$ is odd.

We will use the formulas given in the following Corollary to prove Theorem 9.10.

**Corollary 9.4** Assume $m \geq 3$ is odd. The $\kappa^{(1)}_{\beta,\alpha}, \ldots, \kappa^{(m-2)}_{\beta,\alpha}, \kappa^{(1)}_{\alpha,\beta}, \ldots, \kappa^{(m-2)}_{\alpha,\beta}$ of Lemma 9.2 satisfy

\[
X^{(m)}_{\alpha,\beta,...} = \sum_{i=1}^{m-2} (-1)^{m-i} \kappa^{(i)}_{\alpha_{i+1},\alpha_{i}} X^{(i)}_{\alpha_{i+1},\alpha_{i}} - \gamma_{\alpha,\beta} \left( \delta^{(m)}_{\alpha,\beta,...} + \sum_{i=1}^{m-1} (-1)^i (\delta^{(m-i)}_{\alpha,\beta,...} + \delta^{(m-i)}_{\beta,\alpha,...}) - I \right);
\]

\[
X^{(m)}_{\beta,\alpha,...} = \sum_{i=1}^{m-2} (-1)^{m-i} \kappa^{(i)}_{\alpha_{i+1},\alpha_{i}} X^{(i)}_{\alpha_{i+1},\alpha_{i}} - \gamma_{\beta,\alpha} \left( \delta^{(m)}_{\beta,\alpha,...} + \sum_{i=1}^{m-1} (-1)^i (\delta^{(m-i)}_{\alpha,\beta,...} + \delta^{(m-i)}_{\beta,\alpha,...}) - I \right).
\]

**Proof** Theorem 2.10 and Lemma 7.3 imply that the elements $X^{(m)}_{\alpha,\beta,...}$ and $X^{(m)}_{\beta,\alpha,...}$ are $Q^{(R,F)}_\Lambda$-linear combinations of the elements in

\[
\{1, \delta_{\alpha}, \delta_{\beta}, \delta_{\alpha,\beta}, \ldots, \delta_{\alpha,\beta,...}, \delta^{(m-1)}_{\alpha,\beta,...}, \delta^{(m-1)}_{\beta,\alpha,...}, \delta^{(m)}_{\alpha,\beta,...}, \delta^{(m)}_{\beta,\alpha,...}\}.
\]

So there must exist coefficients $p_{1,j}, p_{\alpha,j}, p_{\beta,j} \in Q^{(R,F)}_\Lambda; i = 1, \ldots, m; j = 1, 2,$ that satisfy

\[
X^{(m)}_{\alpha,\beta,...} = \sum_{i=1}^{m-2} (-1)^{m-i} \kappa^{(i)}_{\alpha_{i+1},\alpha_{i}} X^{(i)}_{\alpha_{i+1},\alpha_{i}} + p^{(m)}_{\alpha,1} \delta^{(m)}_{\alpha,\beta,...} + p^{(m-1)}_{\alpha,1} \delta^{(m-1)}_{\alpha,\beta,...} + p^{(m-1)}_{\beta,1} \delta^{(m-1)}_{\beta,\alpha,...} + \cdots + p^{(1)}_{\alpha,1} \delta_{\alpha} + p^{(1)}_{\beta,1} \delta_{\beta} + p_{1,1} 1,
\]

(9)

\[
X^{(m)}_{\beta,\alpha,...} = \sum_{i=1}^{m-2} (-1)^{m-i} \kappa^{(i)}_{\alpha_{i+1},\alpha_{i}} X^{(i)}_{\alpha_{i+1},\alpha_{i}} + p^{(m)}_{\beta,1} \delta^{(m)}_{\beta,\alpha,...} + p^{(m-1)}_{\alpha,2} \delta^{(m-1)}_{\alpha,\beta,...} + p^{(m-1)}_{\beta,2} \delta^{(m-1)}_{\beta,\alpha,...} + \cdots + p^{(1)}_{\alpha,2} \delta_{\alpha} + p^{(1)}_{\beta,2} \delta_{\beta} + p_{1,2} 1.
\]

(10)

By Lemma 9.2, together with the fact that $\delta^{(m)}_{\alpha,\beta,...} = \delta^{(m)}_{\beta,\alpha,...}$ and the fact that the $\delta_{w}, w \in W$, are $Q^{(R,F)}_\Lambda$-linearly independent, we find that

\[
p_{1,1} = p_{1,2}, \quad p^{(i)}_{\beta,1} = p^{(i)}_{\beta,2}, \quad p^{(i)}_{\alpha,1} = p^{(i)}_{\alpha,2}, \quad i = 1, \ldots, m - 1.
\]

(11)

Now, in the proof of Lemma 9.2, we showed that, for $j = 1, 2$, the coefficients

\[
p^{(m)}_{\alpha,1} = p^{(m)}_{\alpha,2} = -y_{\Sigma^+}^+, \quad p^{(m-1)}_{\beta,1} = p^{(m-1)}_{\beta,2} = y_{\Sigma^+}^+.
\]

Furthermore, since $m$ is odd, in the expansion of the product $X^{(m)}_{\alpha,\beta,...}$, the coefficient of $\delta^{(i)}_{\beta,\alpha,...}$ equals the coefficient of $-\delta^{(i-1)}_{\beta,\alpha,...}$ when $i$ is odd, and the coefficient of $\delta^{(i)}_{\alpha,\beta,...}$ equals the coefficient of $-\delta^{(i-1)}_{\alpha,\beta,...}$ when $i$ is even. This follows from the definition of the formal Demazure element, $X_{\alpha} = y_{\alpha}(1 - \delta_{\alpha})$, and from the fact that $\delta_{\alpha}^2 = \delta_{\beta}^2 = 1$ and $\delta_{\alpha,\beta,...} = \delta_{\beta,\alpha,...}$. Switching $\alpha$ with $\beta$ gives a similar result for $X^{(m)}_{\beta,\alpha,...}$. See the computations in Calculation 11.1 for concrete examples that display this property. This tells us that

\[
p^{(m-2)}_{\beta,1} = p^{(m-1)}_{\beta,1} = -y_{\Sigma^+}^+ \quad \text{and} \quad p^{(m-2)}_{\alpha,2} = p^{(m-1)}_{\alpha,2} = -y_{\Sigma^+}^+.
\]

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Combining this with Eq. 11 gives us

\[ p_{\alpha,1}^{(m-3)} = -p_{\alpha,1}^{(m-2)} = -p_{\alpha,2}^{(m-2)} = y_{\Sigma^+_{\alpha,\beta}}, \text{ and} \]

\[ p_{\beta,2}^{(m-3)} = -p_{\beta,2}^{(m-2)} = -p_{\beta,1}^{(m-2)} = y_{\Sigma^+_{\alpha,\beta}}. \]

Now we continue recursively to obtain the formulas for the coefficients that appear in the statement of this corollary.

Remark 9.5 Assume \( m \geq 3 \) is odd. Suppose \((R, F)\) is the additive formal group law over \( R \); or \((R, F)\) is the multiplicative formal group law over \( R \) and \( \Sigma \) is crystallographic. In the first case, Remark 8.6 implies that all the \( \kappa_{ij}^{(i)} = 0 \). In the second case, Remark 8.7 implies that all the \( \kappa_{ij}^{(i)} = 0 \). Thus, in either case, Corollary 9.4 implies that

\[ X_{\alpha,\beta,...}^{(m)} = X_{\beta,\alpha,...}^{(m)} = y_{\Sigma^+_{\alpha,\beta}} \left( \delta_{\alpha,\beta,...}^{(m)} + \sum_{i=1}^{m-1} (-1)^i (\delta_{\alpha,\beta,...}^{(m-i)} + \delta_{\beta,\alpha,...}^{(m-i)}) - 1 \right). \]

We will now motivate Lemma 9.7 with the following example.

Example 9.6 Suppose \( m = 5 \). The coefficient of \( \delta_{3}^{(3)} \alpha,\beta,... \) in the expansion of the product \( X_{\alpha,\beta,...}^{(5)} \) is

\[ c_{\alpha,\beta}^{(3)} = -\nu_{\alpha,\beta}^{(2)} \delta_{\alpha,\beta}^{(0,3)} (y_\alpha y_\beta). \]

To see this, note that in the expansion of

\[ X_{\alpha,\beta,...}^{(5)} = (y_\alpha - y_\alpha \delta_\alpha)(y_\beta - y_\beta \delta_\beta)(y_\alpha - y_\alpha \delta_\alpha)(y_\beta - y_\beta \delta_\beta)(y_\alpha - y_\alpha \delta_\alpha), \]

all \( \delta_{3}^{(3)} \alpha,\beta,... \) summands are obtained by choosing a \( \delta \)-term in each factor, except for two adjacent factors. In the adjacent factors, one should instead choose the constant term. This way, one obtains 4 summands:

\[ c_{\alpha,\beta}^{(3)} \delta_{\alpha,\beta,\beta,...}^{(3)} = -(y_\alpha)(y_\beta)(y_\alpha \delta_\alpha)(y_\beta \delta_\beta)(y_\alpha \delta_\alpha) - (y_\alpha \delta_\alpha)(y_\beta)(y_\alpha)(y_\beta \delta_\beta)(y_\alpha \delta_\alpha) \]
\[ - (y_\alpha \delta_\alpha)(y_\beta \delta_\beta)(y_\alpha) - (y_\alpha \delta_\alpha)(y_\beta \delta_\beta)(y_\alpha \delta_\alpha)(y_\beta)(y_\alpha). \]

The first summand is

\[ -(y_\alpha)(y_\beta)(y_\alpha \delta_\alpha)(y_\beta \delta_\beta)(y_\alpha \delta_\alpha) = -y_\alpha y_\beta \nu_{\alpha,\beta}^{(2)} \delta_{\alpha,\beta,\beta,...}^{(3)}. \]
The second summand is
\[-(y_\alpha \delta_\alpha)(y_\beta)(y_\alpha)(y_\beta \delta_\beta)(y_\alpha \delta_\alpha) = -y_\alpha s_\alpha (y_\beta y_\alpha) \delta_\alpha (y_\beta \delta_\beta) (y_\alpha \delta_\alpha) \]
\[= -s_\alpha (y_\alpha y_\beta) (y_\alpha \delta_\alpha) (y_\beta \delta_\beta) (y_\alpha \delta_\alpha) \]
\[= -s_\alpha (y_\alpha y_\beta) y_\alpha s_\alpha (y_\beta) s_\beta (y_\alpha) \delta_\alpha \delta_\beta \delta_\alpha \]
\[= -s_\alpha (y_\alpha y_\beta) v_{\alpha, \beta}^{(3)} \delta(3). \]

The third summand is
\[-(y_\alpha \delta_\alpha)(y_\beta \delta_\beta)(y_\alpha)(y_\beta)(y_\alpha \delta_\alpha) \]
\[= -y_\alpha s_\alpha (y_\beta) s_\alpha (y_\alpha y_\beta) \delta_\alpha \delta_\beta (y_\alpha \delta_\alpha) \]
\[= -s_\alpha s_\beta (y_\alpha y_\beta) y_\alpha (y_\delta \beta) (y_\alpha \delta_\alpha) \]
\[= -s_\alpha s_\beta (y_\alpha y_\beta) y_\alpha s_\alpha (y_\beta) s_\beta (y_\alpha) \delta_\alpha \delta_\beta \delta_\alpha \]
\[= -s_\alpha s_\beta (y_\alpha y_\beta) v_{\alpha, \beta}^{(3)} \delta(3). \]

The fourth summand is
\[-(y_\alpha \delta_\alpha)(y_\beta \delta_\beta)(y_\alpha \delta_\alpha)(y_\beta)(y_\alpha) \]
\[= -y_\alpha s_\alpha (y_\beta) s_\alpha s_\beta (y_\alpha) s_\alpha s_\beta s_\alpha (y_\alpha y_\beta) \delta_\alpha \delta_\beta \delta_\alpha \]
\[= -s_\alpha s_\beta s_\alpha (y_\alpha y_\beta) v_{\alpha, \beta}^{(3)} \delta(3). \]

Combining these summands, we see that
\[c_{\alpha, \beta}^{(3)} = -(y_\alpha y_\beta + s_\alpha (y_\alpha y_\beta) + s_\alpha s_\beta (y_\alpha y_\beta) + s_\alpha s_\beta s_\alpha (y_\alpha y_\beta)) v_{\alpha, \beta}^{(2)} \]
\[= -v_{\alpha, \beta}^{(2)} s_{\alpha, \beta}^{(0, 3)} (y_\alpha y_\beta). \]

**Lemma 9.7** Assume $m \geq 3$ is odd. The coefficient of $\delta^{(m-2)}_{\alpha, \beta, \ldots}$ in the expansion of the product $X^{(m)}_{\alpha, \beta, \ldots}$ is
\[c_{\alpha, \beta}^{(m-2)} = -v_{\alpha, \beta}^{(2)} s_{\alpha, \beta}^{(0, m-2)} (y_\alpha y_\beta). \]

By symmetry, the coefficient of $\delta^{(m-2)}_{\beta, \alpha, \ldots}$ in the expansion of $X^{(m)}_{\beta, \alpha, \ldots}$ is $c_{\beta, \alpha}^{(m-2)} = -v_{\beta, \alpha}^{(2)} s_{\beta, \alpha}^{(0, m-2)} (y_\alpha y_\beta)$.

**Proof** A $\delta^{(m-2)}_{\alpha, \beta, \ldots}$ summand in the expansion of $X^{(m)}_{\alpha, \beta, \ldots}$ is obtained by choosing $\delta$-terms in all factors except of two adjacent factors, $X_\alpha X_\beta$ or $X_\beta X_\alpha$, where one chooses constant terms (if one doesn’t choose adjacent factors, then there is cancellation of $\delta$’s, resulting in a word of length less than $m - 2$). So we obtain $(m - 1)$ summands,
\[c_{\alpha, \beta}^{(m-2)} \delta^{(m-2)}_{\alpha, \beta, \ldots} = -(y_\alpha)(y_\beta)(y_\alpha \delta_\alpha)(y_\beta \delta_\beta) \cdots (y_\beta \delta_\beta)(y_\alpha \delta_\alpha) \]
\[-(y_\alpha \delta_\alpha)(y_\beta)(y_\alpha \delta_\alpha)(y_\beta \delta_\beta) \cdots (y_\beta \delta_\beta)(y_\alpha \delta_\alpha) \]
\[-(y_\alpha \delta_\alpha)(y_\beta \delta_\beta)(y_\alpha)(y_\beta) \cdots (y_\beta \delta_\beta)(y_\alpha \delta_\alpha) \]
\[\vdots \]
\[-(y_\alpha \delta_\alpha)(y_\beta \delta_\beta)(y_\alpha \delta_\alpha)(y_\beta \delta_\beta) \cdots (y_\beta)(y_\alpha). \]
We use the multiplication in \(Q_{\Lambda, W}(R, F)\) to obtain a formula for \(c_{\alpha, \beta}^{(m-2)}\). Let \(i = 0, \ldots, m - 2\). Then the \((i + 1)^{st}\) summand above is

\[
- \sum_{\delta_{\alpha}} (y_\alpha s_\alpha) (y_\beta s_\beta) (y_\alpha s_\alpha) \cdots (y_{\omega_{i+1}} s_{\omega_{i+1}}) (y_{\alpha} s_{\alpha}) (y_{\omega_{i}} s_{\omega_{i}}) \cdots (y_{\beta} s_{\beta}) (y_{\alpha} s_{\alpha})
\]

\[
= -y_\alpha s_\alpha (y_\beta s_\beta) (y_\alpha s_\alpha) \cdots 
\]

\[
\cdots (i-1, i, i+1) s_{i-1, i} s_{i, i} \cdots (y_{\omega_{i+1}} s_{\omega_{i+1}}) (y_{\alpha} s_{\alpha}) (y_{\omega_{i}} s_{\omega_{i}}) \cdots (y_{\beta} s_{\beta}) (y_{\alpha} s_{\alpha})
\]

\[
= -s_{i, i} (i, i+1) s_{i, i} (y_{\alpha} s_{\alpha}) (y_{\beta} s_{\beta}) (y_{\alpha} s_{\alpha}) \cdots (y_{\beta} s_{\beta}) (y_{\alpha} s_{\alpha})
\]

\[
= -s_{i, i} (y_{\omega_{i+1}} s_{\omega_{i+1}}) (y_\alpha s_\alpha) (y_\beta s_\beta) (y_\alpha s_\alpha) \cdots (y_\beta s_\beta) (y_\alpha s_\alpha)
\]

\[
= -s_{i, i} (y_{\omega_{i+1}} s_{\omega_{i+1}}) (y_\alpha s_\alpha) (y_\beta s_\beta) (y_\alpha s_\alpha) \cdots (y_\beta s_\beta) (y_\alpha s_\alpha)
\]

\[
= -s_{i, i} (y_{\omega_{i+1}} y_{\omega_{i+1}}) s_{i, i} (y_\alpha s_\alpha) (y_\beta s_\beta) (y_\alpha s_\alpha) \cdots (y_\beta s_\beta) (y_\alpha s_\alpha)
\]

\[
= -s_{i, i} (y_{\omega_{i+1}} y_{\omega_{i+1}}) s_{i, i} (y_\alpha s_\alpha) (y_\beta s_\beta) (y_\alpha s_\alpha) \cdots (y_\beta s_\beta) (y_\alpha s_\alpha)
\]

Note that we can replace \(s_{i, i} (y_{\omega_{i+1}} y_{\omega_{i+1}}) s_{i, i} (y_\alpha s_\alpha) (y_\beta s_\beta) (y_\alpha s_\alpha) \cdots (y_\beta s_\beta) (y_\alpha s_\alpha)\) by \(s_{i, i} (y_\alpha s_\alpha) (y_\beta s_\beta) (y_\alpha s_\alpha) \cdots (y_\beta s_\beta) (y_\alpha s_\alpha)\). Combining these summands, we obtain the formula for the coefficient \(c_{\alpha, \beta}^{(m-2)}\) that appears in the statement of the lemma.

We will now motivate Lemma 9.9 with the following example.

**Example 9.8** Suppose \(m = 5\). The coefficient of \(\delta_{\beta, \alpha}^{(2)}\) in the expansion of the product \(X_{\alpha, \beta, \ldots}^{(5)}\) is

\[
c_{\beta, \alpha}^{(2)} = y_\alpha \upsilon_{\beta, \alpha}^{(3)} (s_{\beta, \alpha}^{(0, 2)} (y_\alpha y_\beta) + s_{\alpha} (y_\alpha y_\beta)).
\]

To see this, note that in the expansion of

\[
X_{\alpha, \beta, \ldots}^{(5)} = (y_\alpha - y_\alpha s_\alpha) (y_\beta - y_\beta s_\beta) (y_\alpha - y_\alpha s_\alpha) (y_\beta - y_\beta s_\beta) (y_\alpha - y_\alpha s_\alpha),
\]

all \(\delta_{\beta, \alpha}^{(2)}\) summands are obtained in one of two ways. In the first way: one chooses the constant term in the first factor, followed by the \(\delta\)-term in all remaining factors, except for two adjacent factors. In the two adjacent factors, one should instead choose the constant term. Here, one obtains 3 summands. In the second way: one chooses the constant term in the second factor, and a \(\delta\)-term in the remaining factors. Here, the \(\delta\)'s in the first and third factors will cancel each other, giving a summand of \(\delta_{\beta, \alpha}^{(2)}\). So we get 4 summands:

\[
c_{\beta, \alpha}^{(2)} \delta_{\beta, \alpha}^{(2)} = (y_\alpha s_\alpha) (y_\beta) (y_\alpha s_\alpha) (y_\beta s_\beta) (y_\alpha) + (y_\alpha) (y_\beta) (y_\alpha s_\alpha) (y_\beta s_\beta) (y_\alpha)
\]

\[
+ (y_\alpha) (y_\beta s_\beta) (y_\alpha) (y_\beta) (y_\alpha s_\alpha) + (y_\alpha) (y_\beta s_\beta) (y_\alpha) (y_\beta s_\beta) (y_\alpha).
\]

The first summand is

\[
(y_\alpha s_\alpha) (y_\beta) (y_\alpha s_\alpha) (y_\beta s_\beta) (y_\alpha) = y_\alpha s_\alpha (y_\beta) (y_\alpha s_\alpha) (y_\beta s_\beta) (y_\alpha s_\alpha)
\]

\[
= y_\alpha s_\alpha (y_\beta) (y_\alpha) (y_\beta s_\beta) (y_\alpha s_\alpha)
\]

\[
= y_\alpha \upsilon_{\beta, \alpha}^{(3)} s_{\alpha} (y_\alpha y_\beta) s_{\beta}^{(2)}.
\]

The second summand is

\[
(y_\alpha) (y_\beta) (y_\alpha) (y_\beta s_\beta) (y_\alpha s_\alpha) = y_\alpha y_\beta y_\alpha (y_\beta s_\beta) (y_\alpha s_\alpha)
\]

\[
= y_\alpha \upsilon_{\beta, \alpha}^{(3)} y_\alpha y_\beta s_{\beta}^{(2)}.
\]
The third summand is
\[
(y_\alpha)(y_\beta)(y_\alpha)(y_\beta)(y_\alpha) = y_\alpha y_\beta s_\beta (y_\alpha y_\beta) \delta_\beta (y_\alpha \delta_\alpha)
\]
\[
= y_\alpha s_\beta (y_\alpha y_\beta) (y_\beta \delta_\beta) (y_\alpha \delta_\alpha)
\]
\[
= y_\alpha s_\beta (y_\alpha y_\beta) y_\beta \delta_\beta (y_\alpha) \delta_\beta \delta_\alpha
\]
\[
= y_\alpha u^{(2)}_{\beta, \alpha} s_\beta (y_\alpha y_\beta) \delta^{(2)}_{\beta, \alpha}.
\]

The fourth summand is
\[
(y_\alpha)(y_\beta)(y_\alpha)(y_\beta)(y_\alpha) = y_\alpha y_\beta s_\beta (y_\alpha y_\beta) s_\alpha (y_\beta y_\alpha) \delta_\beta \delta_\alpha
\]
\[
= y_\alpha u^{(3)}_{\beta, \alpha} s_\beta s_\alpha (y_\alpha y_\beta) \delta^{(2)}_{\beta, \alpha}.
\]

Combining these summands, we see that
\[
c^{(2)}_{\beta, \alpha} = y_\alpha u^{(3)}_{\beta, \alpha} \{ s_\beta (y_\alpha y_\beta) + s_\beta s_\alpha (y_\alpha y_\beta) + s_\alpha (y_\alpha y_\beta) \}
\]
\[
= y_\alpha u^{(3)}_{\beta, \alpha} \{ s^{(0,2)}_{\beta, \alpha} (y_\alpha y_\beta) + s_\alpha (y_\alpha y_\beta) \}.
\]

**Lemma 9.9** Assume \( m \geq 5 \) is odd. Then the coefficient of \( \delta^{(m-3)}_{\beta, \alpha} \) in the expansion of the product \( X^{(m)}_{\alpha, \beta, \ldots} \) is
\[
c^{(m-3)}_{\beta, \alpha} = y_\alpha u^{(3)}_{\beta, \alpha} \{ s^{(0,m-3)}_{\beta, \alpha} (y_\alpha y_\beta) + s_\alpha (y_\alpha y_\beta) \}.
\]

By symmetry, the coefficient of \( \delta^{(m-3)}_{\alpha, \beta} \) in the expansion of \( X^{(m)}_{\alpha, \beta, \ldots} \) is
\[
c^{(m-3)}_{\alpha, \beta} = y_\beta u^{(3)}_{\alpha, \beta} \{ s^{(0,m-3)}_{\alpha, \beta} (y_\alpha y_\beta) + s_\beta (y_\alpha y_\beta) \}.
\]

**Proof** A \( \delta^{(m-3)}_{\beta, \alpha} \) summand in the expansion of \( X^{(m)}_{\alpha, \beta, \ldots} \) is obtained in one of two ways. In the first way, one chooses the constant term in the first \( X_\alpha \), followed by \( \delta \)-terms in the remaining factors, except for two adjacent factors. In the two adjacent factors, one should instead choose the constant terms (if one doesn’t choose adjacent factors, then there is a cancellation of \( \delta \)’s, resulting in a word of length less than \( m - 3 \); if one chooses the \( \delta \)-term in the first factor, the product of \( \delta \)’s will begin with \( \delta_\alpha \)). This gives \( m - 2 \) summands. In the second way, we obtain one additional summand by choosing the \( \delta \)-term in the first factor, followed by the constant term in the second factor, followed by the \( \delta \)-term in each of the remaining factors (so that the first and third \( \delta_\alpha \)’s will cancel). So we have \( m - 1 \) summands in total:
\[
c^{(m-3)}_{\beta, \alpha} \delta^{(m-3)}_{\beta, \alpha, \ldots} = (y_\alpha \delta_\alpha)(y_\beta)(y_\alpha \delta_\alpha)(y_\beta \delta_\beta) \cdots (y_\beta \delta_\beta)(y_\alpha \delta_\alpha) \]
\[
+ (y_\alpha)(y_\beta)(y_\alpha)(y_\beta \delta_\beta) \cdots (y_\beta \delta_\beta)(y_\alpha \delta_\alpha) \]
\[
+ (y_\alpha)(y_\beta \delta_\beta)(y_\alpha)(y_\beta) \cdots (y_\beta \delta_\beta)(y_\alpha \delta_\alpha) \]
\[
+ \cdots \]
\[
+ (y_\alpha)(y_\beta \delta_\beta)(y_\alpha \delta_\alpha)(y_\beta)(y_\alpha).
\]
We use the multiplication in $Q^{(R,F)}_{\Lambda, W}$ to obtain a formula for $c_{\beta,\alpha}^{(m-3)}$. The formula for the first summand above is

$$
(y_\alpha \delta_\alpha)(y_\beta)(y_\alpha \delta_\alpha) \cdots (y_\alpha \delta_\alpha)(y_\beta \delta_\beta)(y_\alpha \delta_\alpha) = y_\alpha s_{i_0}(y_\beta)(y_\alpha \delta_\alpha)
$$

$$
\cdots \cdots (y_\alpha \delta_\alpha)(y_\beta \delta_\beta)(y_\alpha \delta_\alpha)
$$

$$
= y_\alpha s_{i_0}(y_\beta)(y_\alpha \delta_\alpha)
$$

$$
\cdots \cdots (y_\alpha \delta_\alpha)(y_\beta \delta_\beta)(y_\alpha \delta_\alpha)
$$

Now, let $i = 0, \ldots, m - 3$. The $(i + 2)^{nd}$ summand above is

$$
(y_\alpha)(y_\beta \delta_\beta)(y_\alpha \delta_\alpha) \cdots (y_{i_0} \delta_{i_0})(y_{i_0+1})(y_{i_0})(y_{i_0+1} \delta_{i_0+1}) \cdots (y_\beta \delta_\beta)(y_\alpha \delta_\alpha)
$$

$$
= y_\alpha y_\beta s_{i_0}(y_\alpha \delta_\alpha)(y_\beta \delta_\beta)(y_\alpha \delta_\alpha)
$$

$$
\cdots \cdots (y_\alpha \delta_\alpha)(y_\beta \delta_\beta)(y_\alpha \delta_\alpha)
$$

$$
= y_\alpha y_\beta s_{i_0}(y_\alpha \delta_\alpha)(y_\beta \delta_\beta)(y_\alpha \delta_\alpha)
$$

$$
\cdots \cdots (y_\alpha \delta_\alpha)(y_\beta \delta_\beta)(y_\alpha \delta_\alpha)
$$

Note that we can replace $s_{i_0}(y_\alpha \delta_\alpha)(y_\beta \delta_\beta)(y_\alpha \delta_\alpha)$ by $y_\alpha \delta_\alpha$. Combining these summands, we obtain the general formula for the coefficient $c_{\beta,\alpha}^{(m-3)}$ that appears in the statement of the lemma.

**Theorem 9.10** Suppose $m \geq 3$ is odd. Below are explicit formulas for the structure coefficients $\kappa_{\alpha,\beta}^{(m-2)}$ and $\kappa_{\beta,\alpha}^{(m-2)}$:

$$
\kappa_{\alpha,\beta}^{(m-2)} = S_{\alpha,\beta}^{(0,m-2)}(y_\alpha y_\beta) - y_\beta s_{\alpha,\beta}^{(m-2)}(y_\beta);
$$

$$
\kappa_{\beta,\alpha}^{(m-2)} = S_{\beta,\alpha}^{(0,m-2)}(y_\alpha y_\beta) - y_\alpha s_{\beta,\alpha}^{(m-2)}(y_\alpha).
$$

Suppose $m \geq 5$ is odd. Below are explicit formulas for the structure coefficients $\kappa_{\alpha,\beta}^{(m-3)}$ and $\kappa_{\beta,\alpha}^{(m-3)}$:

$$
\kappa_{\alpha,\beta}^{(m-3)} = -y_\beta \{s_\alpha(y_\alpha y_\beta) + [S_{\alpha,\beta}^{(m-2)} - S_{\beta,\alpha}^{(m-2)}](y_\alpha y_\beta) - s_{\alpha,\beta}^{(m-2)}(y_\alpha) - s_{\beta,\alpha}^{(m-2)}(y_\beta) - s_{\alpha,\beta}^{(m-2)}(y_\beta) - s_{\beta,\alpha}^{(m-2)}(y_\alpha)\};
$$

$$
\kappa_{\beta,\alpha}^{(m-3)} = -y_\alpha \{s_\beta(y_\alpha y_\beta) + [S_{\beta,\alpha}^{(m-2)} - S_{\alpha,\beta}^{(m-2)}](y_\alpha y_\beta) - s_{\alpha,\beta}^{(m-2)}(y_\beta) - s_{\beta,\alpha}^{(m-2)}(y_\beta) - s_{\alpha,\beta}^{(m-2)}(y_\alpha) - s_{\beta,\alpha}^{(m-2)}(y_\alpha)\}.
$$

**Proof** We will prove that the explicit formulas for $\kappa_{\alpha,\beta}^{(m-2)}$ and $\kappa_{\alpha,\beta}^{(m-3)}$ hold. The proofs that the formulas for $\kappa_{\alpha,\beta}^{(m-2)}$ and $\kappa_{\alpha,\beta}^{(m-3)}$ hold are similar, and we will omit them.

By Lemma 2.8, we have $s_{\alpha,\beta}^{(m-1)}(\alpha) = \beta$ and $s_{\beta,\alpha}^{(m-1)}(\beta) = \alpha$ when $m$ is odd. We use this property implicitly in this proof.
First, we will determine the coefficient $\kappa^{(m-2)}_{\beta,\alpha}$. Replacing $m$ with $m-2$ in Lemma 9.2 and using the same method of proof, we deduce that the coefficient of $\delta^{(m-2)}_{\beta,\alpha}$ in the expansion of the product $X^{(m-2)}_{\beta,\alpha,...}$ is

$$b_{\beta,\alpha} = -\nu^{(2)}_{\beta,\alpha}. \quad (\star)$$

In Lemma 9.7, we found the coefficient of $\delta^{(m-2)}_{\beta,\alpha,...}$ in the expansion of $X^{(m)}_{\beta,\alpha,...}$, which we denoted $c^{(m-2)}_{\beta,\alpha}$. Therefore, by Corollary 9.4, we have the following formula:

$$X^{(m)}_{\beta,\alpha,...} = \left( \kappa^{(m-2)}_{\beta,\alpha} X^{(m-2)}_{\beta,\alpha,...} - \cdots \right) + y \Sigma_{\psi}^{+} \left( -\delta^{(m-2)}_{\beta,\alpha,...} + \cdots \right),$$

where

$$\kappa^{(m-2)}_{\beta,\alpha} = \frac{1}{b_{\beta,\alpha}} \left( c^{(m-2)}_{\beta,\alpha} + y \Sigma_{\psi}^{+} \right).$$

One can check that, after cancellations and simplifications, this equation is the same as the equation given in the statement of this theorem.

Now we will determine the coefficient $\kappa^{(m-3)}_{\alpha,\beta}$. Replacing $m$ with $m-2$ in Lemma 9.2 and using the same method of proof, we deduce that the coefficient of $\delta^{(m-3)}_{\alpha,\beta}$ in the expansion of the product $X^{(m-2)}_{\beta,\alpha,...}$ is

$$d_{\alpha,\beta} = y \nu^{(3)}_{\alpha,\beta}.$$

Replacing $m$ with $m-3$ instead, we deduce that the coefficient of $\delta^{(m-3)}_{\alpha,\beta}$ in the expansion of the product $X^{(m-3)}_{\alpha,\beta,...}$ is

$$e_{\alpha,\beta} = \nu^{(3)}_{\alpha,\beta}.$$

In Lemma 9.9, we found the coefficient of $\delta^{(m-3)}_{\alpha,\beta,...}$ in the expansion of $X^{(m)}_{\beta,\alpha,...}$, which we denoted $c^{(m-3)}_{\alpha,\beta}$. Therefore, by Corollary 9.4, we have the following formula:

$$X^{(m)}_{\beta,\alpha,...} = \left( \kappa^{(m-3)}_{\beta,\alpha} X^{(m-3)}_{\beta,\alpha,...} - \kappa^{(m-2)}_{\alpha,\beta} X^{(m-3)}_{\alpha,\beta,...} + \cdots \right) + y \Sigma_{\psi}^{+} \left( \delta^{(m-3)}_{\alpha,\beta,...} + \cdots \right),$$

where

$$\kappa^{(m-3)}_{\alpha,\beta} = \frac{1}{e_{\alpha,\beta}} \left( c^{(m-3)}_{\alpha,\beta} - \kappa^{(m-2)}_{\beta,\alpha} d_{\alpha,\beta} - y \Sigma_{\psi}^{+} \right).$$

One can check that, after cancellations and simplifications, this equation is the same as the equation given in the statement of this theorem.

**Remark 9.11** We do not have general formulas for the remaining structure coefficients of Lemma 9.2. In Section 10, we compute all structure coefficients for the groups $I_2(5)$, $I_2(7)$, $H_3$, and $H_4$.

### 10 Applications

In this section, we specialize the structure coefficients derived in Section 9 to certain formal group laws. We also compute all structure coefficients in Eq. 8 for the dihedral groups $I_2(5)$ and $I_2(7)$, and the reflection groups $H_3$ and $H_4$.  

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We will use the following lemma in Example 10.2.

**Lemma 10.1** Suppose \( m \geq 3 \) is odd. Let \( i = 1, \ldots, m-1 \). If \( i \) is odd, we have \( s_{\alpha, \beta, \ldots, (\beta)}^{(i)}(\alpha) = s_{\beta, \alpha, \ldots, (\alpha)}^{(m-i-1)}(\beta) \), and if \( i \) is even, we have \( s_{\alpha, \beta, \ldots, (\beta)}^{(i)}(\alpha) = s_{\beta, \alpha, \ldots, (\alpha)}^{(m-i-1)}(\beta) \).

**Proof** It follows from Lemma 2.8 that \( s_{\alpha, \beta, \ldots, (\beta)}^{(m)}(\alpha) = -\alpha \) when \( m \) is odd. Applying compositions of reflections \( s_{\alpha} \) and \( s_{\beta} \) to both sides of this equation to reduce the length of \( s_{\alpha, \beta, \ldots, (\beta)}^{(m)} \) gives the desired equations. \( \square \)

**Example 10.2** Suppose that \( m \geq 3 \) is odd. Let \((R, F)\) be any formal group law such that \( \kappa_{\gamma} = \frac{1}{x_\gamma} + \frac{1}{x_{-\gamma}} \) is constant over all roots \( \gamma \in \Sigma \). This includes the multiplicative formal group law over \( R \), and any formal group law over \( R \) of additive type. It also includes the hyperbolic formal group law over \( R \), which is described in [21, §2.3]. Observe that, in \( Q(\Lambda R, F) \), the condition that \( q := \kappa_{\gamma} \) is constant over all \( \gamma \in \Sigma \) is equivalent to the condition \( y_{-\gamma} = q - y_{\gamma} \) for all \( \gamma \in \Sigma \). We show that, under this condition, the structure coefficients of Theorem 9.10 satisfy

\[
\begin{align*}
\kappa_{\alpha, \beta}^{(m-2)} &= \kappa_{\beta, \alpha}^{(m-2)}, & \text{and} \\
\kappa_{\alpha, \beta}^{(m-3)} &= \kappa_{\beta, \alpha}^{(m-3)} = 0.
\end{align*}
\]

(a) Using the formulas for \( \kappa_{\alpha, \beta}^{(m-2)} \) and \( \kappa_{\beta, \alpha}^{(m-2)} \) obtained in Theorem 9.10 and making the substitution \( y_{-\alpha} = (q - y_{\alpha}) \), we can write

\[
\begin{align*}
\kappa_{\alpha, \beta}^{(m-2)} &= y_{\alpha} y_{\beta} + \{[q - y_{\beta}] s_{\beta}(y_{\alpha}) + [q - s_{\beta}(y_{\alpha})] s_{\beta} s_{\alpha}(y_{\beta}) + \cdots \\
&\quad + \{[q - s_{\beta, \alpha, \ldots, (\beta)}^{(m-3)}(y_{\alpha})] s_{\beta, \alpha, \ldots, (\beta)}^{(m-3)}(y_{\beta}) + [q - s_{\beta, \alpha, \ldots, (\beta)}^{(m-3)}(y_{\beta})] s_{\beta, \alpha, \ldots, (\beta)}^{(m-2)}(y_{\alpha}) \} \\
&\quad - y_{\alpha} s_{\beta, \alpha, \ldots, (\beta)}^{(m-2)}(y_{\alpha}) \text{, and}
\end{align*}
\]

\[
\begin{align*}
\kappa_{\alpha, \beta}^{(m-2)} &= y_{\alpha} y_{\beta} + \{[q - y_{\alpha}] s_{\alpha}(y_{\beta}) + [q - s_{\alpha}(y_{\beta})] s_{\alpha} s_{\beta}(y_{\alpha}) + \cdots \\
&\quad + \{[q - s_{\alpha, \beta, \ldots, (\alpha)}^{(m-4)}(y_{\beta})] s_{\alpha, \beta, \ldots, (\alpha)}^{(m-4)}(y_{\alpha}) + [q - s_{\alpha, \beta, \ldots, (\alpha)}^{(m-3)}(y_{\alpha})] s_{\alpha, \beta, \ldots, (\alpha)}^{(m-3)}(y_{\beta}) \} \\
&\quad - y_{\beta} s_{\alpha, \beta, \ldots, (\alpha)}^{(m-2)}(y_{\beta}).
\end{align*}
\]

By Lemma 10.1, we have the following relations:

\[
\begin{align*}
y_{\alpha} s_{\alpha}(y_{\beta}) &= y_{\alpha} s_{\beta, \alpha, \ldots, (\beta)}^{(m-2)}(y_{\alpha}), & y_{\beta} s_{\beta}(y_{\alpha}) &= y_{\beta} s_{\beta, \alpha, \ldots, (\beta)}^{(m-2)}(y_{\beta}), & (12) \\
q_{\beta} s_{\beta, \alpha, \ldots, (\beta)}^{(i)}(y_{\beta}) &= q_{\beta} s_{\beta, \alpha, \ldots, (\beta)}^{(m-i-1)}(y_{\alpha}), & i = 1, 3, 5, \ldots, m - 2, & (13) \\
q_{\beta} s_{\beta, \alpha, \ldots, (\beta)}^{(i)}(y_{\alpha}) &= q_{\beta} s_{\beta, \alpha, \ldots, (\beta)}^{(m-i-1)}(y_{\beta}), & i = 2, 4, \ldots, m - 3, & (14) \\
(q_{\beta} s_{\beta, \alpha, \ldots, (\beta)}^{(i+1)}(y_{\beta}) &= s_{\beta, \alpha, \ldots, (\beta)}^{(m-i-1)}(y_{\beta}) s_{\beta, \alpha, \ldots, (\beta)}^{(m-i-2)}(y_{\alpha}), & i = 2, 4, \ldots, m - 3, & (15) \\
(q_{\beta} s_{\beta, \alpha, \ldots, (\beta)}^{(i+1)}(y_{\alpha}) &= s_{\beta, \alpha, \ldots, (\beta)}^{(m-i-1)}(y_{\alpha}) s_{\beta, \alpha, \ldots, (\beta)}^{(m-i-2)}(y_{\beta}), & i = 1, 3, 5, \ldots, m - 4.
\end{align*}
\]

Using these relations and comparing \( \kappa_{\beta, \alpha}^{(m-2)} \) and \( \kappa_{\alpha, \beta}^{(m-2)} \), it is straightforward to verify that \( \kappa_{\alpha, \beta}^{(m-2)} = \kappa_{\beta, \alpha}^{(m-2)} \).
(b) Using the formulas for $\kappa_{\alpha,\beta}^{(m-3)}$ and $\kappa_{\beta,\alpha}^{(m-3)}$ obtained in Theorem 9.10 and making the substitution $y - \alpha = (q - y_\alpha)$, we can write

$$-\frac{\kappa_{\alpha,\beta}^{(m-3)}}{y_\beta} = [q - y_\alpha] s_\alpha(y_\beta) + [(q - s_\alpha(y_\beta)]s_\alpha s_\beta(y_\alpha) - [q - s_\beta(y_\alpha)] s_\beta s_\alpha(y_\beta)] + \cdots$$

$$+[q - s_{\alpha,\beta,\ldots}^{(m-4)}(y_\beta)] s_{\alpha,\beta,\ldots}^{(m-3)}(y_\alpha) - [q - s_{\beta,\alpha,\ldots}^{(m-4)}(y_\alpha)] s_{\beta,\alpha,\ldots}^{(m-3)}(y_\beta)]$$

$$-[q - s_{\beta,\alpha,\ldots}^{(m-3)}(y_\beta)] s_{\beta,\alpha,\ldots}^{(m-2)}(y_\alpha) + y_\alpha s_{\beta,\alpha,\ldots}^{(m-2)}(y_\alpha) - s_{\beta,\alpha,\ldots}^{(m-2)}(y_\alpha) s_{\alpha,\beta,\ldots}^{(m-2)}(y_\beta).$$

By Lemma 10.1, we have the following relations:

$$y_\alpha s_\alpha(y_\beta) = y_\alpha s_{\beta,\alpha}^{(m-2)}(y_\beta), \quad \text{and} \quad q s_{\beta,\alpha}^{(m-2)}(y_\alpha) = q s_\alpha(y_\beta).$$

These relations, together with the relations Eqs. 12 to 15, allow us to deduce that the terms of $\kappa_{\alpha,\beta}^{(m-3)}$ cancel in pairs. We conclude that $\kappa_{\alpha,\beta}^{(m-3)} = \kappa_{\beta,\alpha}^{(m-3)} = 0$.

Example 10.3 follows from Lemma 9.2 and Theorem 9.10:

Example 10.3 1. Suppose $m = 2$. In this case,

$$X_\alpha X_\beta = X_\beta X_\alpha.$$

2. Suppose $m = 3$. In this case,

$$X_\alpha X_\beta X_\alpha - X_\beta X_\alpha X_\beta = \kappa_{\alpha,\beta}^{(1)} X_\alpha - \kappa_{\beta,\alpha}^{(1)} X_\beta,$$

where $\kappa_{\alpha,\beta}^{(1)} = y_\alpha y_\beta + s_\beta(y_\alpha y_\beta) - y_\alpha s_\beta(y_\alpha)$ and $\kappa_{\beta,\alpha}^{(1)} = y_\alpha y_\beta + s_\alpha(y_\alpha y_\beta) - y_\beta s_\alpha(y_\beta)$.

Example 10.4 Suppose $m = 5$. We provide explicit formulas for $\kappa_{\alpha,\beta}^{(1)}, \kappa_{\alpha,\beta}^{(2)},$ and $\kappa_{\beta,\alpha}^{(3)}$:

$$\kappa_{\beta,\alpha}^{(3)} = s_{\beta,\alpha}^{(0,3)}(y_\alpha y_\beta) - y_\alpha s_\beta s_\alpha s_\beta(y_\alpha),$$

$$\kappa_{\alpha,\beta}^{(2)} = -y_\beta (s_\alpha(y_\alpha y_\beta) + s_\alpha s_\beta(y_\alpha y_\beta) - s_\beta s_\alpha s_\beta(y_\alpha y_\beta) + y_\alpha s_\alpha s_\beta(y_\alpha) - s_\alpha s_\beta(y_\alpha)s_\alpha s_\beta s_\alpha(y_\beta)), $$

$$\kappa_{\beta,\alpha}^{(1)} = -s_\beta(y_\alpha y_\beta) s_\beta s_\alpha s_\beta(y_\alpha) s_\beta s_\alpha s_\beta(y_\beta) - y_\alpha - y_\alpha y_\beta s_\alpha s_\beta(y_\alpha) s_\alpha s_\beta(y_\beta) - s_\alpha s_\beta s_\alpha(y_\beta) - y_\alpha s_\beta(y_\alpha) s_\beta s_\alpha s_\beta s_\alpha(y_\beta).$$

We will use the results proven in Section 9 to justify the explicit formulas for $\kappa_{\alpha,\beta}^{(1)}, \kappa_{\beta,\alpha}^{(2)},$ and $\kappa_{\beta,\alpha}^{(3)}$ above. Explicit formulas for the coefficients $\kappa_{\alpha,\beta}^{(1)}, \kappa_{\alpha,\beta}^{(2)},$ and $\kappa_{\beta,\alpha}^{(3)}$ can also be computed by using a similar argument. By symmetry of the argument, the formulas for $\kappa_{\alpha,\beta}^{(1)}, \kappa_{\beta,\alpha}^{(2)},$ and $\kappa_{\alpha,\beta}^{(3)}$ can be obtained from the above formulas for $\kappa_{\beta,\alpha}^{(1)}, \kappa_{\alpha,\beta}^{(2)},$ and $\kappa_{\beta,\alpha}^{(3)}$, respectively, by swapping $\alpha$ and $\beta$. For example, $\kappa_{\beta,\alpha}^{(3)} = s_{\beta,\alpha}^{(0,3)}(y_\alpha y_\beta) - y_\alpha s_\beta s_\alpha s_\beta(y_\beta)$.

The formulas for $\kappa_{\beta,\alpha}^{(3)}$ and $\kappa_{\alpha,\beta}^{(2)}$ are obtained from Theorem 9.10. Now we will find $\kappa_{\beta,\alpha}^{(1)}$. In Calculation A.1, we provide formulas for products of up to seven formal Demazure elements. We use these formulas, together with the formulas for the coefficients $\kappa_{\beta,\alpha}^{(3)}$ and $\kappa_{\alpha,\beta}^{(2)}$ to determine $\kappa_{\beta,\alpha}^{(1)}$. We do this by subtracting $\kappa_{\beta,\alpha}^{(3)} X_\beta X_\alpha X_\beta - \kappa_{\alpha,\beta}^{(2)} X_\alpha X_\beta + y_\alpha s_{\beta,\alpha}^{(1)}(1-\delta_\beta)$ from $X_{\beta,\alpha}^{(5)}$, and then using Corollary 9.4.
The coefficient of \((1 - \delta \beta)\) in the expansion of the product \(X_{\beta,\alpha,\ldots}^{(5)}\) is

\[
+y_{\beta} \{ s_{\beta} (y_{\alpha} y_{\beta}) \}^{2} + s_{\beta} (y_{\alpha} y_{\beta}) s_{\beta} s_{\alpha} (y_{\alpha} y_{\beta}) \\
+y_{\alpha} y_{\beta}^{2} s_{\alpha} (y_{\alpha} y_{\beta}).
\]

The coefficient of \((1 - \delta \beta)\) in the expansion of the product \(\kappa_{\beta,\alpha,\ldots}^{(3)} X_{\beta} X_{\alpha} X_{\beta}\) is

\[
+y_{\beta} \{ [S_{\beta,\alpha}^{(0,1)} (y_{\alpha} y_{\beta})]^{2} + s_{\beta} (y_{\alpha} y_{\beta}) s_{\beta} s_{\alpha} (y_{\alpha} y_{\beta}) \\
+y_{\alpha} y_{\beta}^{2} (S_{\beta,\alpha}^{(2,3)} (y_{\alpha} y_{\beta}) - y_{\alpha} s_{\beta} s_{\alpha} s_{\beta} (y_{\alpha}) \\
+y_{\beta} s_{\beta} (y_{\alpha} y_{\beta}) (s_{\beta} s_{\alpha} s_{\beta} (y_{\alpha} y_{\beta}) - y_{\alpha} s_{\beta} s_{\alpha} s_{\beta} (y_{\alpha})).
\]

The coefficient of \((1 - \delta \beta)\) in the expansion of the product \(-\kappa_{\alpha,\beta,\ldots}^{(2)} X_{\alpha} X_{\beta}\) is

\[
+y_{\alpha} y_{\beta}^{2} s_{\alpha} (y_{\alpha} y_{\beta}) \\
-y_{\alpha} y_{\beta}^{2} \{ S_{\beta,\alpha}^{(2,3)} (y_{\alpha} y_{\beta}) - y_{\alpha} s_{\beta} s_{\alpha} s_{\beta} (y_{\alpha}) \\
+y_{\alpha} y_{\beta}^{2} \{ s_{\alpha} s_{\beta} (y_{\alpha} y_{\beta}) - s_{\alpha} s_{\beta} (y_{\alpha}) s_{\beta} s_{\alpha} (y_{\beta}).
\]

After cancellations, the coefficient of \((1 - \delta \beta)\) in the term \(X_{\beta,\alpha,\ldots}^{(5)} - \kappa_{\beta,\alpha,\ldots}^{(3)} X_{\beta} X_{\alpha} X_{\beta} + \kappa_{\alpha,\beta,\ldots}^{(2)} X_{\alpha} X_{\beta}\) is

\[
C = -y_{\beta} \{ s_{\beta} (y_{\alpha} y_{\beta}) s_{\beta} s_{\alpha} s_{\beta} (y_{\alpha} y_{\beta}) - y_{\alpha} s_{\beta} s_{\alpha} s_{\beta} (y_{\alpha}) \} + y_{\alpha} y_{\beta} \{ s_{\alpha} s_{\beta} (y_{\alpha} y_{\beta}) - s_{\alpha} s_{\beta} (y_{\alpha}) s_{\beta} s_{\alpha} (y_{\beta}) \}.
\]

The coefficient of \((1 - \delta \beta)\) in \(X_{\beta}\) is \(y_{\beta}\). Therefore, by Corollary 9.4, the coefficient at \(X_{\beta}\) in the braid relation is \(\kappa_{\beta,\alpha,\ldots}^{(1)} = \frac{C - y_{\alpha} y_{\beta}^{2}}{y_{\beta}^{2}}\). Finally, by Lemma 10.1, we have that \(y_{\alpha} = s_{\beta,\alpha,\ldots}^{(4)} (y_{\beta})\), and that \(y_{\beta} = y_{\alpha} s_{\beta,\alpha,\ldots}^{(4)} (y_{\beta}) s_{\beta} s_{\alpha} s_{\beta} (y_{\alpha}) s_{\beta,\alpha,\ldots}^{(4)} (y_{\beta})\).

**Example 10.5** Suppose \(m = 7\). We provide explicit formulas for \(\kappa_{\beta,\alpha,\ldots}^{(1)}, \kappa_{\beta,\alpha,\ldots}^{(2)}, \kappa_{\beta,\alpha,\ldots}^{(3)}, \kappa_{\beta,\alpha,\ldots}^{(4)}\), and \(\kappa_{\beta,\alpha,\ldots}^{(5)}\):

\[
k_{\beta,\alpha,\ldots}^{(5)} = S_{\beta,\alpha}^{(0,5)} (y_{\alpha} y_{\beta}) - y_{\alpha} s_{\beta,\alpha,\ldots}^{(5)} (y_{\alpha}),
\]

\[
k_{\beta,\alpha,\ldots}^{(4)} = -y_{\beta} \{ s_{\beta} (y_{\alpha} y_{\beta}) \}^{2} + s_{\beta} (y_{\alpha} y_{\beta}) s_{\beta} s_{\alpha} (y_{\alpha} y_{\beta}) + y_{\alpha} y_{\beta}^{2} s_{\alpha} (y_{\alpha} y_{\beta}) \\
-y_{\alpha} y_{\beta}^{2} \{ s_{\alpha} s_{\beta} (y_{\alpha} y_{\beta}) - s_{\alpha} s_{\beta} (y_{\alpha}) s_{\beta} s_{\alpha} (y_{\beta}).
\]

\[
k_{\beta,\alpha,\ldots}^{(3)} = -y_{\beta} \{ s_{\beta} (y_{\alpha} y_{\beta}) \}^{2} + s_{\beta} (y_{\alpha} y_{\beta}) s_{\beta} s_{\alpha} (y_{\alpha} y_{\beta}) - y_{\alpha} y_{\beta} \{ S_{\beta,\alpha}^{(2,4)} (y_{\alpha} y_{\beta}) - s_{\beta} s_{\alpha} s_{\beta} (y_{\alpha}) \} \\
-y_{\alpha} y_{\beta} \{ s_{\alpha} s_{\beta} s_{\beta} (y_{\alpha} y_{\beta}) - y_{\alpha} s_{\beta} s_{\alpha} (y_{\beta}) \\
+y_{\alpha} y_{\beta} \{ s_{\beta} s_{\alpha} s_{\beta} (y_{\alpha} y_{\beta}) - y_{\alpha} s_{\beta} s_{\alpha} s_{\beta} (y_{\beta}).
\]

\[
k_{\beta,\alpha,\ldots}^{(2)} = -y_{\beta} \{ s_{\beta} (y_{\alpha} y_{\beta}) \}^{2} + s_{\beta} (y_{\alpha} y_{\beta}) s_{\beta} s_{\alpha} (y_{\alpha} y_{\beta}) - y_{\alpha} y_{\beta} \{ s_{\alpha} s_{\beta} s_{\beta} (y_{\alpha} y_{\beta}) - y_{\alpha} s_{\beta} s_{\alpha} (y_{\beta}) \\
+y_{\alpha} y_{\beta} \{ s_{\beta} s_{\alpha} s_{\beta} s_{\beta} (y_{\alpha} y_{\beta}) - y_{\alpha} s_{\beta} s_{\alpha} s_{\beta} (y_{\beta}) \\
-y_{\alpha} s_{\beta} s_{\alpha} s_{\beta} (y_{\alpha} y_{\beta}) - y_{\alpha} s_{\beta} s_{\alpha} s_{\beta} (y_{\alpha}) s_{\beta,\alpha,\ldots}^{(4)} (y_{\beta}).
\]
\[ k_{\beta, \alpha}^{(1)} = s_{\beta}(y_\alpha y_\beta)s_{\beta} s_{\alpha} s_{\beta}(y_\alpha) s_{\beta, \alpha}^{(5)}(y_\beta) + s_{\alpha} s_{\beta}(y_\alpha y_\beta)s_{\beta, \alpha}^{(4)} - y_\alpha s_{\beta} s_{\alpha} s_{\beta}(y_\beta) \]

The explicit formulas for \( k_{\beta, \alpha}^{(1)}, k_{\alpha, \beta}^{(2)}, k_{\alpha, \beta}^{(3)}, k_{\beta, \alpha}^{(4)}, \) and \( k_{\beta, \alpha}^{(5)} \) are obtained using the method of Example 10.4. Explicit formulas for the coefficients \( k_{\alpha, \beta}^{(1)}, k_{\beta, \alpha}^{(2)}, k_{\alpha, \beta}^{(3)}, k_{\beta, \alpha}^{(4)}, \) and \( k_{\alpha, \beta}^{(5)} \) can also be computed by using a similar argument. By symmetry of the argument, the formulas for \( k_{\alpha, \beta}^{(1)}, k_{\beta, \alpha}^{(2)}, k_{\alpha, \beta}^{(3)}, k_{\beta, \alpha}^{(4)}, \) and \( k_{\alpha, \beta}^{(5)} \) can be obtained from the above formulas for \( k_{\beta, \alpha}^{(1)}, k_{\alpha, \beta}^{(2)}, k_{\beta, \alpha}^{(3)}, k_{\beta, \alpha}^{(4)}, \) and \( k_{\beta, \alpha}^{(5)} \), respectively, by swapping \( \alpha \) and \( \beta \). For example, \( k_{\beta, \alpha}^{(5)} = s_{\alpha, \beta}^{(0, 5)}(y_\alpha y_\beta) - y_\beta s_{\alpha, \beta}^{(5)}(y_\beta) \).

**Remark 10.6** Let \((R, F)\) be a formal group law such that \( k_\gamma \) is constant over all \( \gamma \in \Sigma \). By direct computation and using the method of Example 10.2, we compute \( k_{\alpha, \beta}^{(1)} = k_{\beta, \alpha}^{(1)} \) when \( m = 5 \). We also compute \( k_{\alpha, \beta}^{(3)} = k_{\beta, \alpha}^{(3)} \), \( k_{\alpha, \beta}^{(2)} = k_{\beta, \alpha}^{(2)} = 0 \), and \( k_{\alpha, \beta}^{(1)} = k_{\beta, \alpha}^{(1)} \) when \( m = 7 \).

**Remark 10.7** Suppose \( m \geq 3 \) is odd. Let \((R, F)\) be a formal group law such that \( k_\gamma \) is constant over all \( \gamma \in \Sigma \). In light of Example 10.2 and Remark 10.6, we conjecture that \( k_{\alpha, \beta}^{(i)} = k_{\beta, \alpha}^{(i)} \) when \( i \) is odd, and \( k_{\alpha, \beta}^{(i)} = k_{\beta, \alpha}^{(i)} = 0 \) when \( i \) is even.

**Remark 10.8** Let \((\Sigma, \Delta)\) be the geometric realization of \( W = I_2(3) \) given in Example 2.17, and let \((R, F_a)\) be the additive formal group law over \( R \). In this case, \( \Delta = \{ \alpha, \beta \} \). Using the formulas for the roots given in Example 2.17 and the formulas for \( k_{\alpha, \beta}^{(1)} \) and \( k_{\beta, \alpha}^{(1)} \) given in Example 10.3 (b), we compute directly that \( k_{\alpha, \beta}^{(1)} = k_{\beta, \alpha}^{(1)} = 0 \). Thus, we have given a combinatorial proof of Remark 8.6 for this geometric realization of \( I_2(3) \).

**Remark 10.9** Let \((\Sigma, \Delta)\) be the geometric realization of \( W = I_2(5) \) given in Example 2.19, and let \((R, F_a)\) be the additive formal group law over \( R \). In this case, \( \Delta = \{ \alpha, \beta \} \). We already know that \( k_{\beta, \alpha}^{(2)} = k_{\alpha, \beta}^{(2)} = 0 \) by Example 10.2. Moreover, using the formulas for the roots given in Example 2.19, the formulas for the structure coefficients given in Example 10.4, and the relation \( \tau^2 = \tau + 1 \), we compute directly that \( k_{\alpha, \beta}^{(1)} = k_{\beta, \alpha}^{(1)} = k_{\alpha, \beta}^{(3)} = k_{\beta, \alpha}^{(3)} = 0 \) as well. Thus, we have given a combinatorial proof of Remark 8.6 for this geometric realization of \( I_2(5) \).

**Remark 10.10** Let \((\Sigma, \Delta)\) be the geometric realization of \( W = I_2(7) \) given in Example 2.20, and let \((R, F_a)\) be the additive formal group law over \( R \). In this case, \( \Delta = \{ \alpha, \beta \} \). We already know that \( k_{\beta, \alpha}^{(4)} = k_{\alpha, \beta}^{(4)} = 0 \) by Example 10.2, and that \( k_{\beta, \alpha}^{(2)} = k_{\alpha, \beta}^{(2)} = 0 \) by Remark 10.6. Moreover, using the formulas for the roots given in Example 2.20, the formulas for the structure coefficients given in Example 10.5, and the relation \( \zeta^3 = \zeta^2 + 2 \zeta - 1 \), we compute directly that \( k_{\alpha, \beta}^{(1)} = k_{\beta, \alpha}^{(1)} = k_{\alpha, \beta}^{(3)} = k_{\beta, \alpha}^{(3)} = k_{\alpha, \beta}^{(5)} = k_{\beta, \alpha}^{(5)} = 0 \) as well. Thus, we have given a combinatorial proof of Remark 8.6 for this geometric realization of \( I_2(7) \).

We will now consider the reflection groups \( H_3 \) and \( H_4 \).
Example 10.11 Let \((\Sigma, \Delta)\) be the geometric realization of \(W = H_3\) given in Section 2. This geometric realization has simple roots \(\Delta = \{\alpha_1, \alpha_2, \alpha_3\}\), where

\[
m_{1,2} = 5, \quad m_{1,3} = 2, \quad m_{2,3} = 3
\]

are the orders of \(s_i s_j\) in \(W\). The coefficients \(\kappa^{(1)}_{\alpha_2, \alpha_3}\) and \(\kappa^{(1)}_{\alpha_3, \alpha_2}\) were computed in Example 10.3 (b). In addition, \(X_1 X_3 = X_3 X_1\) by Example 10.3 (a). Moreover, the coefficients \(\kappa^{(1)}_{\alpha_1, \alpha_2}, \kappa^{(1)}_{\alpha_2, \alpha_1}, \kappa^{(2)}_{\alpha_1, \alpha_2}, \kappa^{(2)}_{\alpha_2, \alpha_1}, \kappa^{(3)}_{\alpha_1, \alpha_2}, \) and \(\kappa^{(3)}_{\alpha_2, \alpha_1}\) were computed in Example 10.4.

When \((R, F_a)\) is the additive formal group law over \(R\), one can verify directly, as was done in Remark 10.8, that the coefficient \(\kappa^{(1)}_{\alpha_2, \alpha_3} = \kappa^{(1)}_{\alpha_3, \alpha_2} = 0\). One can also verify directly, as was done in Remark 10.9, that the coefficients \(\kappa^{(1)}_{\alpha_1, \alpha_2}, \kappa^{(2)}_{\alpha_2, \alpha_1}, \kappa^{(1)}_{\alpha_2, \alpha_1}, \kappa^{(2)}_{\alpha_1, \alpha_2}, \kappa^{(3)}_{\alpha_2, \alpha_1}\) and \(\kappa^{(3)}_{\alpha_2, \alpha_1}\) all equal 0 in this case.

Example 10.12 Let \((\Sigma, \Delta)\) be the geometric realization of \(W = H_4\) given in Section 2. This geometric realization has simple roots \(\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}\), where

\[
m_{1,2} = 5, \quad m_{1,3} = 2, \quad m_{1,4} = 2, \quad m_{2,3} = 3, \quad m_{2,4} = 2, \quad m_{3,4} = 3
\]

are the orders of \(s_i s_j\) in \(W\). The coefficients \(\kappa^{(1)}_{\alpha_2, \alpha_3}, \kappa^{(1)}_{\alpha_3, \alpha_2}, \kappa^{(1)}_{\alpha_2, \alpha_4}, \) and \(\kappa^{(1)}_{\alpha_3, \alpha_4}\) were computed in Example 10.3 (b). In addition, \(X_1 X_3 = X_3 X_1, X_1 X_4 = X_4 X_1, \) and \(X_2 X_4 = X_4 X_2\) by Example 10.3 (a). Moreover, the coefficients \(\kappa^{(1)}_{\alpha_1, \alpha_2}, \kappa^{(1)}_{\alpha_2, \alpha_1}, \kappa^{(2)}_{\alpha_1, \alpha_2}, \kappa^{(2)}_{\alpha_2, \alpha_1}, \kappa^{(3)}_{\alpha_1, \alpha_2}, \) and \(\kappa^{(3)}_{\alpha_2, \alpha_1}\) were computed in Example 10.4.

When \((R, F_a)\) is the additive formal group law over \(R\), one can verify directly, as was done in Remark 10.8, that the coefficients \(\kappa^{(1)}_{\alpha_2, \alpha_3}, \kappa^{(1)}_{\alpha_3, \alpha_2}, \kappa^{(1)}_{\alpha_2, \alpha_4}, \) and \(\kappa^{(1)}_{\alpha_3, \alpha_4}\) equal 0. One can also verify directly, as was done in Remark 10.9, that the coefficients \(\kappa^{(1)}_{\alpha_1, \alpha_2}, \kappa^{(1)}_{\alpha_2, \alpha_1}, \kappa^{(2)}_{\alpha_1, \alpha_2}, \kappa^{(2)}_{\alpha_2, \alpha_1}\), and \(\kappa^{(3)}_{\alpha_2, \alpha_1}\) all equal 0 in this case.

Appendix

In the present section, we provide computations for products of up to seven formal Demazure elements for all root systems.

Calculation A.1 Let \(W\) be a real finite reflection group, let \(\Sigma\) be a root system of \(W\), and let \(\alpha, \beta \in \Sigma\). Below are explicit formulas for the products of \(X_\alpha\) and \(X_\beta\) up to seven elements. The formulas are written so that the coefficient to the left of a \(\delta\)-term in the expansion of the product appears after the colon.

\[
\begin{align*}
X_\beta \\
(1 - \delta_\beta) : y_\beta \\
X_\alpha X_\beta \\
(1 - \delta_\beta) : y_\alpha y_\beta \\
(\delta_\alpha\beta - \delta_\alpha) : y_\alpha s_\alpha(y_\beta) \\
X_\beta X_\alpha X_\beta
\end{align*}
\]
\[ X_{\alpha,\beta,...} \]

\[ (1 - \delta\beta) : y_\beta \{ S_{\beta,\alpha}^{(0,1)} (y_\alpha y_\beta) \} \]

\[ (\delta\alpha\beta - \delta\alpha) : y_\alpha y_\beta s_\alpha (y_\beta) \]

\[ (\delta\beta\alpha - \delta\beta\alpha\beta) : y_\beta s_\beta (y_\alpha s_\beta s_\alpha (y_\beta)) \]

\[ X_{(4)}^{(4)} \]

\[ (1 - \delta\beta) : y_\alpha y_\beta \{ S_{\beta,\alpha}^{(0,2)} (y_\alpha y_\beta) \} \]

\[ (\delta\alpha\beta - \delta\alpha) : y_\alpha s_\beta s_\alpha (y_\beta) \]

\[ (\delta\beta\alpha - \delta\beta\alpha\beta) : y_\beta s_\beta s_\alpha s_\beta (y_\alpha s_\beta s_\alpha (y_\beta)) \]

\[ X_{(5)}^{(5)} \]

\[ (1 - \delta\beta) : y_\beta \{ S_{\beta,\alpha}^{(0,3)} (y_\alpha y_\beta) \}^2 + y_\alpha y_\beta s_\alpha s_\beta s_\alpha (y_\alpha y_\beta) \]

\[ (\delta\alpha\beta - \delta\alpha) : y_\alpha y_\beta s_\alpha s_\beta (y_\beta) \]

\[ (\delta\beta\alpha - \delta\beta\alpha\beta) : y_\beta s_\beta s_\alpha s_\beta s_\alpha (y_\alpha s_\beta s_\alpha (y_\beta)) \]

\[ X_{(6)}^{(6)} \]

\[ (1 - \delta\beta) : y_\beta \{ S_{\beta,\alpha}^{(0,4)} (y_\alpha y_\beta) \}^2 + 2 y_\alpha y_\beta s_\beta s_\alpha s_\beta s_\alpha (y_\alpha y_\beta) \]

\[ + 2 y_\alpha y_\beta s_\alpha s_\beta s_\alpha (y_\beta) \]

\[ (\delta\alpha\beta - \delta\alpha) : y_\alpha y_\beta s_\alpha s_\beta (y_\beta) \]

\[ (\delta\beta\alpha - \delta\beta\alpha\beta) : y_\beta s_\beta s_\alpha s_\beta s_\alpha (y_\alpha s_\beta s_\alpha (y_\beta)) \]

\[ X_{(7)}^{(7)} \]
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Data Availability  Data sharing is not applicable to this article as no datasets were generated or analysed during the current study. The computations performed in Examples 10.4, 10.5, and Remark 10.10 were verified using the Maple and Python scripts available here: [13].

Declarations

Competing interests  The author has no relevant non-financial interests to disclose.

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