The (2, 5) minimal model on genus two surfaces

Marianne Leitner
School of Mathematics, Trinity College Dublin, the University of Dublin,
College Green, Dublin 2, Ireland
& School of Theoretical Physics, Dublin Institute for Advanced Studies,
10 Burlington Road, Dublin 4, Ireland
leitner@stp.dias.ie

June 17, 2021

Abstract

In the (2, 5) minimal model, the partition function for genus \( g = 2 \) Riemann surfaces is given by a 5-tuple of functions with appropriate transformation under the mapping class group. These functions generalise the two Rogers-Ramanujan functions for the torus. Their expansions around a locus of surfaces with conical singularities in the interior of the \( g = 2 \) Siegel upper half plane are obtained in terms of standard modular forms. The dependence on the metric is controlled by a canonical choice of flat surface metrics. In the alternative case where a handle of the \( g = 2 \) surface is pinched, our method requires knowledge of the two-point function of the fundamental lowest-weight vector in the non-vacuum representation of the Virasoro algebra, for which we derive a 3rd order ODE. In order to make the paper more accessible to mathematicians, the exposition includes a short introduction to conformal field theory on Riemann surfaces, which may be of independent interest.

MSC 11F03 (Primary), 57M50, 81T40 (Secondary)

Contents

1 Motivation .......................... 2

2 Short introduction to CFT 3
   2.1 The partition function .................. 3
   2.2 Sewing Riemann surfaces ............... 4
   2.3 The vacuum module .................... 5
   2.4 The partition function on sewn Riemann surfaces ................. 6
   2.5 Example: The (2,5) minimal model in genus one ................. 7
   2.6 The Virasoro field, and more general fields ................ 8
   2.7 Segal’s formulae ........................ 10
1 Motivation

Two-dimensional conformal field theories (CFTs) are naturally defined on pairs (Σ, G), where Σ is a compact Riemann surface (of genus \( g \geq 0 \)) and G is a metric in the conformal class of Σ. Denote by \( X \) the space of pairs \( M = (\Sigma, G) \). A two-dimensional CFT is characterised by its partition function \( Z : X \rightarrow \mathbb{R} \), which is a smooth map. (This paper shows that the range of \( Z \) can in general not be restricted to \( \mathbb{R}^+ \).) The image of \( M = (\Sigma, G) \in X \) will be written \( Z_M \) or \( Z^g \).

Quantum field theories are supposed to yield many of the simplest smooth maps of this kind. What has been most studied is the restriction of \( Z \) to flat tori, where \( Z^\tau \) is modular on the full modular group. In the (2, 5) minimal model, also called Yang-Lee model, one has

\[
Z_{C(\mathbb{Z}^2)} = |q^{-1/60}G(q)|^2 + |q^{11/60}H(q)|^2,
\]

where

\[
G(q) = \sum_{n \geq 0} \frac{q^n}{(q; q)_n}, \quad H(q) = \sum_{n \geq 0} \frac{q^{n^2+n}}{(q; q)_n}
\]

is the pair of Rogers-Ramanujan functions. Here \( q = \exp(2\pi i) \) and \( (q; q)_n \) is the \( q \)-Pochhammer symbol. The so-called holomorphic blocks, the functions \( q^{-1/60}G \) and \( q^{11/60}H \), span a unitary representation of the mapping class group. Note that fivefold Dehn twists act by multiplication with a common twelfth root of unity. This fact should generalise to higher genus. The aim of the present paper is a calculation of the genus two partition function of this model and its holomorphic blocks. In particular, this requires the choice of suitable metrics.
Here we follow the approach by Graeme Segal [19] who correctly believed that it “is equivalent to that used by physicists”. The author learned the method from Werner Nahm, who had learned it from him. Segal’s concepts were further explored by physicists [23, 21, 15].

This paper provides a calculation of $\mathcal{Z}$ for genus two surfaces but no proof that it yields the required map on the full moduli space. Thus it does not aim at proving the existence of the corresponding CFT but provides the data for comparison with known functions, in particular Siegel modular forms.

2 Short introduction to CFT

2.1 The partition function

Let $M = (\Sigma, \mathcal{G}) \in \mathcal{X}$. Every point $P \in \Sigma$ has a neighbourhood $U \subset \Sigma$ together with a local coordinate $z$ on $U$ such that the metric on $U$ takes the form $\mathcal{G} = 2\mathcal{G}_{\bar{z}} dz d\bar{z}$, where in our conventions, $2\mathcal{G}_{\bar{z}} = e^\sigma(z)$ with $\sigma \in C^0(U, \mathbb{R})$. We require that the Gauss curvature two form, $K = -i \partial_\bar{z} \partial_z \log \mathcal{G}_{\bar{z}} dz \wedge d\bar{z}$, defines an $\mathbb{R}$-valued linear functional on $C^0(\Sigma, \mathbb{R})$.

By definition, $\mathcal{Z}_M$ is invariant under diffeomorphisms of $M$. For disjoint unions,

$$\mathcal{Z}_{M_1 \cup M_2} = \mathcal{Z}_{M_1} \mathcal{Z}_{M_2}.$$  

For a CFT of central charge $c \in \mathbb{R}$, it is postulated that [17]

$$\mathcal{G}_{\mu\nu} \frac{\delta \mathcal{Z}}{\delta \mathcal{G}_{\mu\nu}(x)} = \frac{c}{24\pi} K(x) \mathcal{Z}.$$  

(3)

Here $K$ is the Gauss curvature of the Levi-Civita connection, and the non-vanishing is a feature of the conformal anomaly.

As observed by W. Nahm [16], eq. (3) yields a simple relationship between the values of the partition function when $M_A = (\Sigma, \mathcal{G}_A)$ and $M_B = (\Sigma, \mathcal{G}_B)$ project to the same point in the moduli space of conformal structures. Under a finite Weyl transformation $\mathcal{G}_A \rightarrow \mathcal{G}_B = \varrho_{BA} \mathcal{G}_A$, the partition functions changes according to

$$\log \left( \frac{\mathcal{Z}_{M_B}}{\mathcal{Z}_{M_A}} \right) = \frac{c}{48\pi} \int_{\Sigma_g} \Delta \sigma \left( \mathcal{K}_A + \mathcal{K}_B \right),$$  

(4)

where $\Delta \sigma = \log \varrho_{BA}$ and $\mathcal{K}$ with index $A, B$ is the curvature two-form of the corresponding metric. In particular, for $A \in \mathbb{R}^+$,

$$\mathcal{Z}^x_{2g} = \lambda^{x(1-x)} \mathcal{Z}^x_{g}.$$  

(5)

Example 1. We consider $\mathbb{P}^1_C$ with the metric

$$\mathcal{G}(t) = \begin{cases} 
    e^t |dz|^2, & 0 \leq |z| \leq e^{-t/2} , \\
    e^{-t/2} |dz|^2, & e^{-t/2} \leq |z| \leq e^{t/2} , \\
    e^{t/2} |dz|^2, & e^{t/2} \leq |z| , 
\end{cases}$$

3
| $\Delta r$ | $|z| = e^{-\ell/2}$ | $|z| = 1$ | $|z| = e^{\ell/2}$ |
|---|---|---|---|
| $-\ell$ | 0 | 0 | $-\ell$ |
| $\int X_A$ | $2\pi$ | 0 | $2\pi$ |
| $\int X_B$ | 0 | $4\pi$ | 0 |

Table 1: The curvature of the metric $G_A = G(\ell)$ for $\ell > 0$ is evenly spread over two circles. The metric $G_B = G(0)$ has all curvature evenly distributed over the single circle $|z| = 1$.

for $\ell \geq 0$, $G(\ell)$ is continuous and its curvature is supported on the two circles $\{ |z| = e^{\pm\ell/2} \}$.

By eq. (4),

$$\gamma_{G(\ell)} = e^{\ell/12} \gamma_{G(0)} .$$

(6)

The individual contributions to formula (4) are listed in table 1.

We will only have to study the map on finite dimensional subspaces of $X$. The moduli spaces relevant here are all complex manifolds.

### 2.2 Sewing Riemann surfaces

The values of $\mathcal{Z}$ on higher genus Riemann surfaces can be constructed from data on lower genus Riemann surfaces by sewing along circles. Let $M = (\Sigma, G)$ be a connected Riemann surface with metric $G$. Let $(U, z)$ be a holomorphic coordinate chart, where $U \subset \Sigma$ is mapped to a neighbourhood of the unit circle $S^1$ in $\mathbb{C}$. We assign to $\mathbb{C}$ its standard metric and assume that the restriction of $z$ to modulus one yields an isometry between a curve $\gamma \subset U$ and $S^1$. Cut along $\gamma$ and let $M^0$ be the resulting metric manifold with two boundary curves with oriented isometries to $S^1$ and marked points corresponding to $1 \in S^1$. For any manifold $M$ with such a left boundary, we denote by $M^\theta$ the manifold obtained by rotating the marked point on that boundary by the angle $\theta$.

When $\gamma$ is a separating curve, we have

$$M^0 = M_L \cup M_R,$$

where $M_L$ and $M_R$ define manifolds with $S^1$ boundaries to the left and to the right, respectively, with the corresponding marked point. In the following, such surfaces $M_L$ and $M_R$ with parametrised boundary $\gamma$ will be referred to as left and right surfaces (for $\gamma$), respectively.

Conversely, given such a pair, we can construct $M$ as the connected sum $M(1)$, where

$$M(e^{i\theta}) = M_L \cup_\gamma M_R^\theta .$$

This is called $s$ formalism (where $s$ stands for “separating”). A special case is where $D_L$ and $D_R$ are that flat unit discs and the double disc

$$DD = D_L \cup_\gamma D_R$$

has genus $g = 0$. 

4
When $\gamma \subset \Sigma$ is a non-separating curve ($q$ formalism), then $M^\theta$ is connected. We write

$$M(e^\theta) = \sqcup_{\gamma} M^\theta$$

for the self-connected sum of $M^\theta$ along $\gamma$. Clearly the existence of a non-separating curve requires $\Sigma$ to have genus $g \geq 1$.

Our $s$ and $q$ formalisms were dubbed $s$ and $q$ formalisms in [14].

2.3 The vacuum module

Let $\mathcal{V} = \bigoplus_{h \in \mathbb{Z}} \mathcal{V}^h$ be the Verma module of the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n, 0},$$

with central charge $c \in \mathbb{R}$, lowest weight vector $v$, modulo the relations

$$L_0 v = 0, \quad L_1 v = 0,$$

in addition to $L_n v = 0$ for $n < 0$. (Note that our $L_n$ is $L_{-n}$ for many others; the convention here is chosen so that $L_1 = d/(dz)$). The degree $h$ is the eigenvalue of $L_0$. We consider $\mathcal{V}^h$ as vector spaces over $\mathbb{C}$, where the number 1 is identified with $v$.

Thus

$$\sum_{h=0}^\infty \dim(\mathcal{V}^h) q^h = \prod_{n=2}^\infty (1-q^n)^{-1} = 1 + q^2 + q^3 + 2q^4 + 2q^5 + 4q^6 + 4q^7 + 7q^8 + 8q^9 + O(q^{10}).$$

Let $\overline{\mathcal{V}}$ be the corresponding complex conjugate space. This means that there exists an antilinear involution denoted by an overbar that exchanges $\mathcal{V}$ and $\overline{\mathcal{V}}$. $L_0$ is defined accordingly, so that $\overline{\mathcal{V}}$ has a grading defined by $\overline{L}_0$ (with eigenvalues $\bar{h}$). Let $\bar{v} \in \overline{\mathcal{V}}$ be the lowest weight vector. The overbar operation extends to an antilinear involution on $\mathcal{V} \otimes \overline{\mathcal{V}}$. Let $R_2(\mathcal{V}) \subset \mathcal{V} \otimes \overline{\mathcal{V}}$ be the corresponding real subspace. We call $R_2(\mathcal{V})$ the vacuum sector and its element $v \otimes \bar{v}$ the vacuum vector.

The Shapovalov form [20]

$$\langle . | . \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$$

is the sesquilinear form defined by $L_n^* = L_{-n}$ (where $L_n^*$ denotes the Hermitian adjoint of $L_n$) and $\langle v | v \rangle = \| v \|^2 = 1$. A null vector is one for which the linear functional defined by the Shapovalov form vanishes.

For $c = -22/5$, the generic vacuum module $\mathcal{V}$ has a maximal invariant submodule generated by $(L_2L_2 - \frac{c}{2}L_4) v$. This is a null vector, so we can replace $(\mathcal{V}, \langle . | . \rangle)$ by the corresponding quotient space $\mathcal{B}$ with the induced Shapovalov form. We have

$$\sum_{h=0}^\infty \dim(\mathcal{B}^h) q^h = H(q),$$

cf. the second of eqs (2). The resulting theory is the $(2, 5)$ minimal model (with vacuum sector $R_2(\mathcal{B})$) mentioned above. It is the only model of interest here.
2.4 The partition function on sewn Riemann surfaces

We explore ways to compute the partition function for a pair \((\Sigma, G) \in \mathcal{X}\) using either sewing scheme, and for generic central charge \(c \in \mathbb{R}\). Let \(\mathcal{M}_R\) (resp. \(\mathcal{M}_L\)) be the space of all finite formal \(\mathbb{R}\)-linear combinations of right (resp. left) surfaces of genus zero. For the double disc given by the pair \(D_L, D_R\), put

\[
\mathcal{Z}_{DD} = 1.
\]  

Eqs (9) and (4) determine \(\mathcal{Z}_M\) for any \(M \in \mathcal{X}\) of genus \(g = 0\).

The following proposition is well motivated from physics and probably implied by the work of Graeme Segal though the author is not aware of a detailed proof.

**Proposition 2.** There exists a space \(\mathcal{R}_2(\mathcal{V})\) together with a pair of natural maps

\[
|\cdot\rangle : \mathcal{M}_R \to \mathcal{R}_2(\mathcal{V}),
\]

\[
\langle \cdot | : \mathcal{M}_L \to \mathcal{R}_2(\mathcal{V}),
\]

which are related by reversal of orientation and have the following properties:

1. \(\mathcal{R}_2(\mathcal{V})\) is dense in \(\mathcal{R}_2(\mathcal{V})\).
2. The Shapovalov form (8) on \(\mathcal{V}\) induces a bilinear form \(\langle \cdot, \cdot \rangle\) on \(\mathcal{R}_2(\mathcal{V})\).
3. \(|D_R\rangle = v \otimes \bar{v} \in \mathcal{R}_2(\mathcal{V})\).
4. For \((M_L, M_R) \in \mathcal{M}_L \times \mathcal{M}_R\),

\[
\langle M_L | M_R \rangle = \mathcal{Z}_{M_L \cup M_R}.
\] 

(In the product, angle brackets and vertical bars of inserted bra- and ket vectors will be omitted and we simply write \(\langle M_L | \rangle\) and \(\langle | M_R \rangle\).)

5. Let the cylinder \(\gamma \times [0, \ell]\) have the natural (flat) metric. Then

\[
|(\gamma \times [0, \ell])^g \cup_{\gamma \times \ell} M_R \rangle = e^{-(\ell_0 + \ell_0 - c/12)} e^{g(\ell_0 - \ell_0)} |M_R\rangle.
\] 

Note that eq. (6) is a special case of eq. (11).

A proof should follow from a Taylor expansion of \(\mathcal{Z}\) about \(DD\), as explained in Section 2.6.

**Corollary 3.** The Shapovalov form is positive on pairs \((M_L, M_R) \in \mathcal{M}_L \times \mathcal{M}_R\).

**Proof.** By eq. (9), \(\langle D_L | D_R \rangle = 1\). Positivity follows by conformal symmetry from eq. (10). \(\Box\)

Let \(c = -22/5\). The family of projections \(\mathfrak{P} \otimes \mathfrak{P} \to \mathfrak{P}^h \otimes \bar{\mathfrak{P}}^\bar{h}\) for \((h, \bar{h}) \in \mathbb{N}_0 \times \mathbb{N}_0\) gives rise to a set

\[
\{e_i\}_i = \cup_{h, \bar{h}} \{ \text{orthogonal basis in } \mathfrak{P}^h \otimes \bar{\mathfrak{P}}^\bar{h} \}
\] 

satisfying the following properties:
1. For \( i, j \geq 0 \),
\[
\langle e_i | e_j \rangle = \varepsilon_i \delta_{ij}, \quad e_i \in \{1, -1\}.
\] (13)

2. \( e_0 = 1 \) and
\[
e_0 = v \otimes \emptyset,
\]
where \( v \otimes \emptyset \) is the vacuum vector in \( \mathbb{R}^0 \otimes \mathbb{R}^0 \).

A set of vectors satisfying property (13) will be said to be standard orthogonal.

The set of \( \{e_i\} \) is useful for computing the partition function (10) for the \((2,5)\) minimal model, due to the identity
\[
\mathfrak{Z}_{M,\gamma} = \sum_i \langle M_L | e_i \rangle \varepsilon_i \langle e_i | M_R \rangle.
\] (14)

There is a variant for the self-connected sum of an oriented metric manifold \( M'' \) with boundaries \( \gamma_1, \gamma_2 \), which are isometrically isomorphic to \( S^1 \). For typographical reasons, will write \( \gamma \) instead of \( S^1 \). Denote by \( M_L^{(1)} \) and \( M_R^{(2)} \) two bounded manifolds which are a left and right surface for \( \gamma_1 \) and \( \gamma_2 \), respectively. Thus the connected sum \( M'' \equiv \gamma_1 M_R^{(2)} \) yields a manifold \( M_R^{(1)} \) with boundary \( \gamma_1 \). We set
\[
\langle M_L^{(1)} | M'' | M_R^{(2)} \rangle := \langle M_L^{(1)} | M'' \cup \gamma_2 M_R^{(2)} \rangle
\]
To simplify notations, the upper indices of \( M_L^{(1)} \) and \( M_R^{(2)} \) will be dropped. If \( M = M_L \cup M_R \), the corresponding vector is \( |M_L \rangle \langle M_R| \) and the partition function of the self-connected sum \( \gamma M = M_L \cup \gamma M_R \) is given by
\[
\mathfrak{Z}_{\gamma M} = \sum_i \varepsilon_i \langle e_i | M | e_i \rangle.
\] (15)

We define CFTs by the assumption that eqs (11), (14) and (15) extend to surfaces of higher genus but with an enlarged range of summation. The new \( e_i \) are orthogonal to \( \mathcal{M}_R \) so that they do not contribute for genus zero. They characterise the corresponding CFT. In general, the pairs \( (h_i, \bar{h}_i) \) will not be integral, but one needs \( (h_i - \bar{h}_i) \in \mathbb{Z} \) since \( \mathfrak{Z} \) is invariant under a \( 2\pi \) rotation along \( \gamma \). Moreover, the values of \( (h_i + \bar{h}_i) \) have to be bounded from below.

2.5 Example: The \((2,5)\) minimal model in genus one

Let \( M = \gamma \times [0, \ell] \) be the flat cylinder with boundary \( \gamma \times \{0\} \cup \gamma \times \{\ell\} \). By eqs (15) and (11),
\[
\mathfrak{Z}_{\gamma \times [0, \ell]} = \sum_{e_i} \varepsilon_i \langle e_i | (\gamma \times [0, \ell])^{\emptyset} | e_i \rangle = \sum_{e_i} \varepsilon_i \langle e_i | q^{La-c/24} \bar{q}^{La-c/24} | e_i \rangle,
\]
where \( q = \exp(i\theta - \ell) \). For \( c = -22/5 \) and for \( \tau = (\theta + i\ell)/2\pi \), the contribution of the vacuum sector is
\[
\sum_{e_i \in \mathbb{R}^0 \otimes \overline{\mathbb{R}}} \varepsilon_i \langle e_i | q^{La-c/24} \bar{q}^{La-c/24} | e_i \rangle = e^{c\ell/12} \sum_{(h_n, \bar{h}_n)} q^{h_n} \bar{q}^{\bar{h}_n} = |q^{1/60} H(q)|^2,
\]
where \((\hbar, \bar{\hbar})\) runs over the weights occurring in the vacuum sector. This function cannot be a complete partition function, since it is not invariant under the modular group. To get the partition function from eq. (1), the representation of the Virasoro algebra has to be extended to include one further sector.

Let \(W = \bigoplus_{h \in \mathbb{H}} W_h\) be the Verma module of the Virasoro algebra (7) with central charge \(c = -22/5\), and lowest weight vector \(w\) of holomorphic conformal weight \((L_0\text{eigenvalue})\) equal to \(-1/5\). Let \(\overline{W}\) be the corresponding complex conjugate space. \(\bar{L}_n\) is defined accordingly, so that \(\overline{W}\) has a grading defined by \(\bar{L}_0\) (with eigenvalues \(\bar{\hbar}\)). Let \(w \in \overline{W}\) be the lowest weight vector. The Shapovalov form on \(\overline{W}\) is non-degenerate only on the quotient \(\overline{W}/\mathbb{R}\) of \(\overline{W}\) by the invariant subspace generated by the nullvectors \(p L_1 L_1' 2 \bar{\hbar} w\) and \(p L_2 L_1' 1 \bar{\hbar} w\). Let \(R_2(\mathbb{W}) = \mathbb{W} \otimes \overline{\mathbb{W}}\) be the real subspace w.r.t. the overbar operation on \(\mathbb{W}\). We take

\[ R_2(\mathbb{W}) \oplus R_2(\overline{\mathbb{W}}) \quad (16) \]

as the space of (real) fields in the \((2, 5)\) minimal model. Complex fields are obtained by extending scalars. The fact that there is no pairing of fields between different sectors follows from \((h - \bar{\hbar}) \in \mathbb{Z}\), cf. the first of eqs (2), we recover the full partition function from eq. (1).

We call \(R_2(\mathbb{W})\) the non-vacuum sector of the \((2, 5)\) minimal model. Since there are no other representations of the Virasoro algebra that share the properties of \(R_2(\mathbb{W})\) and \(R_2(\overline{\mathbb{W}})\), one expects that (16) will do for any genus. This assumption characterises the model.

### 2.6 The Virasoro field, and more general fields

Let \(M\) be a manifold with Riemannian metric, and let \(U \subseteq M\) be an open set with coordinate \(x\). Based on Einstein’s work, Hilbert in classical field theory [8, p. 404] and Weinberg in quantum field theory [25, p. 360] identified the energy-stress tensor \(T^{\mu\nu}\) in the chart \((U, x)\) with the functional derivative w.r.t. the metric tensor \(g_{\mu\nu}(x)\),

\[ T^{\mu\nu}(x) = \frac{\delta}{\delta g_{\mu\nu}(x)} \]

Invariance of the partition function under change of coordinates implies

\[ \nabla_\mu T^{\mu\nu} = 0 , \quad (17) \]

where \(\nabla_\mu\) is the Levi-Civita derivative. Since for Weyl transformations, \(\delta g_{\mu\nu} \propto g_{\mu\nu}\), these transformations are described by the trace of \(T\). Eq. (3) now reads

\[ T_{\mu}^{\mu} = g^{\mu\nu} T_{\mu\nu} = \frac{c}{12 \pi} K \]

On a Riemann surface \(\Sigma\) with local complex coordinate \(z = x^1 + ix^2\) and metric \(g(z) = 2dz^1 dz\bar{z}\), volume preserving (i.e. tracefree) variations of the metric decompose into
\( \delta \mathcal{G}; \) and \( \delta \mathcal{G}_{\bar{z}} \). For locally flat metrics, the corresponding functional derivatives yield components of the complexified energy-momentum tensor, which are written \( T \) and \( T \), respectively. For general metrics, one defines \( T, \overline{T} \) so that they commute with Weyl transformations [5].

Under linear fractional coordinate change \( z \mapsto u \), \( T(z) \) transforms according to

\[
T(u) = \left( \frac{dz}{du} \right)^{h(T)} T(z),
\]

where \( h(T) = 2 \) is called the (holomorphic) conformal weight of \( T \). Analogous equations hold for \( \overline{T} \) in terms of the anti-holomorphic coordinate \( \bar{z} \) and weight \( \bar{h} = 2 \). Conformal invariance and eq. (17) imply that \( \partial_z T = 0 \) (and \( \partial_{\bar{z}} \overline{T} = 0 \)). One calls \( T \) a holomorphic field, which is reflected by the identity \( h(T) = 0 \). More specifically, \( T \) is called the Virasoro field. By holomorphicity of \( T \) (and antiholomorphicity of \( \overline{T} \)),

\[
T(z_1) \ldots T(z_k) \overline{T}(\bar{z}_{k+1}) \ldots \overline{T}(\bar{z}_{k+\ell}) \mathcal{Z}_{M_{k+\ell},D_R}
\]

is a holomorphic function in \( z_1, \ldots, z_k \in D_R \) (and antiholomorphic in \( \bar{z}_{k+1}, \ldots, \bar{z}_{k+\ell} \in D_R \)) for any \( M_L \) as above, provided \( |z_i| < |z_{i+1}| \) (and \( |\bar{z}_{k+i}| < |\bar{z}_{k+i+1}| \)). Its Laurent series coefficients are denoted by

\[
\langle M_L \rangle \langle L_{m_1} \ldots L_{n_1} v \rangle \otimes \langle L_{m_{k+\ell}} \ldots L_{n_{k+\ell}} \bar{v} \rangle
\]

where \( v \otimes \bar{v} \) is the vacuum vector. In this sense,

\[
T(z) = \sum_{n \in \mathbb{Z}} \bar{z}^{-n-2} L_n,
\]

(and the analogous equation for \( \overline{T} \) so that \( T \) corresponds to \( L_2 v \in \mathcal{V} \). By the Virasoro algebra (7),

\[
-L_2L_2v = \frac{c}{2} v, \quad L_{-1}L_2v = L_3v, \quad L_0L_2v = 2L_2v.
\]

For \( z \) close to \( z_0 = 0 \), this yields the Virasoro operator product expansion

\[
T(z) T(0) = \frac{c/2}{z^2} + \frac{2}{z^4} T(0) + \frac{1}{z} T'(0) + N_0(T,T)(0) + O(z).
\]

The field 1 on the r.h.s. of eq. (20) represents the identity mapping, \( T' = \partial_z T \) is the derivative field and \( N_0(T,T) \) is the so-called normal ordered product of \( T \), which corresponds to \( L_2 L_2 v \).

More generally, fields are commuting operators acting on smooth functions \( \mathcal{X} \to \mathbb{R} \).

Given \( M \in \mathcal{X} \) with coordinate chart \( (U, z) \), each \( e_i \) defines a field \( \psi_i \) on \( U \) as follows:

If \( M = M_L \cup Y M_R \) and if \( z \) defines an isometry between \( M_R \) and \( D_R(z_0) \), where \( D_R(z_0) \) denotes the flat unit disc centred at \( z_0 = 0 \), then

\[
\psi_i(z_0) \mathcal{Z}_M = \langle M_L \rangle \langle e_i \rangle.
\]

More generally, we let \( \psi_i \) commute with Weyl transformations. \( N \)-point (or correlation) functions \( \phi_1(z_1) \ldots \phi_N(z_N) \mathcal{Z}_M \) are defined analogously by simultaneous insertion of the corresponding vectors at \( z_1, \ldots, z_N \). This makes sense as long as the \( z_i \) are pairwise different. In physics, the usual notation is

\[
\langle \phi_1(z_1) \ldots \phi_N(z_N) \rangle_M = \phi_1(z_1) \ldots \phi_N(z_N) \mathcal{Z}_M.
\]
The quotient \( \langle \phi(z_1) \ldots \phi_N(z_N) \rangle_M / \mathcal{Z}_M \) is independent of the metric on \( M \) in a conformal class. Note that the insertion of a field requires a coordinate. By abuse of notation, one uses the arguments \( z_i \) to refer both to the coordinate function and its values.

Given a holomorphic field \( \phi \) and any field \( \psi \), the Laurent expansion of the product operator \( \phi(z) \psi(0) \) for \( z \) close to \( z_0 \),

\[
\phi(z) \psi(z_0) = \sum_{k \in \mathbb{Z}} (z - z_0)^k N_k(\phi, \psi)(z_0) ,
\]

defines a sequence of fields \( N_k(\phi, \psi) \). This is a special case of the operator product expansion, which will be introduced later. Eq. (11) implies scale invariance of the fields:

\[
\bar{z} = \lambda z \implies \psi_i(z) = \lambda^{h_i} \tilde{\psi}_i(\bar{z}) .
\]

Here \( \psi_i \) and \( \tilde{\psi}_i \) refer to the insertion of \( e_i \) using the coordinate \( z_i \) and \( \bar{z}_i \), respectively. For the conformal weights, this yields

\[
h(N_k(\phi, \psi)) = h(\phi) + h(\psi) + k, \quad \bar{h}(N_k(\phi, \psi)) = \bar{h}(\psi) .
\]

Thus the summation in eq. (22) has a lower bound. For \( \phi = T \), one finds \( N_{-1}(T, \psi) = L_0 \psi \).

As a generalisation of the OPE (20), one verifies (using contour integration) that \( L_1 \psi \) is the derivative of \( \psi \) w.r.t. \( z \). That is, \( N_{-1}(T, \psi) = d\psi / dz \).

Fields in the image of \( L_1 \) or \( \bar{L}_1 \) will be called derivative fields. The space of quasi-primary fields is the orthogonal complement of the derivative fields w.r.t. the Shapovalov form (8), thus the intersection of the kernels of \( L_{-1} = L_1^* \) and of \( \bar{L}_{-1} \).

Let \( \mathcal{M}_h \) be any irreducible representation of the Virasoro algebra, with lowest weight vector \( \omega_h \). \( \mathcal{M}_h \) is spanned by vectors of the form \( L_{n_1} \ldots L_{n_k} \omega_h \) with \( n_k \in \mathbb{Z} \), \( k \geq 0 \). The generating function of \( \mathcal{M}_h \) is the character

\[
\chi_h := \text{tr} q^L .
\]

where the trace is taken in \( \mathcal{M}_h \). Let \( \mathcal{M}_h^{\text{qp}} \) be the space of quasi-primary fields in the representation \( \mathcal{M}_h \). For \( h = 0 \) and \( \mathcal{M}_0^{\text{qp}} = \mathcal{M} \), the generating function of \( \mathcal{M}_0^{\text{qp}} \) is

\[
\chi_0^{\text{qp}} = (1 - q)(\chi_0 - 1) .
\]

For other weights \( h \), one has

\[
\chi_h^{\text{qp}} = (1 - q)\chi_h .
\]

### 2.7 Segal’s formulae

Segal’s formula (14) for the partition function of the connected sum \( M_L \sqcup_y M_R \) reads

\[
\mathcal{Z}_{M_L \sqcup_y M_R} = \sum_i \left( \frac{\psi_i(z_R = 0) \mathcal{Z}_{M_L \sqcup_y M_R}(\psi_i(z_L = 0) \mathcal{Z}_{D_L \sqcup_y M_R})}{\psi_i(z_L = 0) \psi_i(z_R = 0) \mathcal{Z}_{DD}} \right) .
\]

Here the summation is over the fields defined by eq. (21). When \( \Sigma_L \) has genus \( g_1 \) and \( \Sigma_R \) has genus \( g_2 \), then \( M_L \sqcup_y M_R \) has genus equal to \( g_1 + g_2 \). Eq. (24) is used in what we call the \( s \) formalism.
A special case of the $s$ formalism is the operator product expansion (OPE), since
\[
\langle \phi_1(z_1) \phi_2(0) \ldots \rangle_M = \sum_i \epsilon_i \langle \phi_1(z) \phi_2(0) \psi_i(z_R = 0) \rangle_{DB} \langle \psi_i(z_L = 0) \ldots \rangle_M.
\]
Scale invariance, eq. (23), implies
\[
\langle \phi_1(z) \phi_2(0) \psi_i(z_R = 0) \rangle = z^{-h_1-h_2+h} z^{-h_1-h_2+h} \langle \phi_1(1) \phi_2(0) \psi_i(z_R = 0) \rangle_{DB} \psi_i(0).
\]
Thus
\[
\phi_1(z_1) \phi_2(0) = \sum_i \epsilon_i z^{-h_1-h_2+h} z^{-h_1-h_2+h} \langle \phi_1(1) \phi_2(0) \psi_i(z_R = 0) \rangle_{DB} \psi_i(0).
\]
This generalises eq. (22).

When a single manifold $M^\theta$ with two boundary curves is self-sewn, a new handle is attached to it. Segal’s formula (15) for the so-called $q$ formalism is
\[
Z_{\gamma_M^\theta} = \sum_i \psi(x_L = 0) \psi(x_R = 0) Z_{D_L \gamma_{M^\theta} D_R} \psi(0) \psi_i(0) = \sum_i \psi(x_L = 0) \psi(x_R = 0) Z_{D_\mu \gamma_{M^\theta} D_\mu} \psi(0) \psi_i(0).
\]
In theories with superselection rules, the $q$ formalism requires additional terms, but this is irrelevant for the model we consider.

### 2.8 Notations and conventions

By a surface we mean a smooth manifold of real dimension two. By a (closed or open) disc we mean a simply connected orientable surface bounded by a circle. All occurring contour integrals are to be read counterclockwise. $\mathbb{N}_0$ and $\mathbb{N}$ denote the non-negative and positive integers, respectively, and $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. For a functional $F$ of the metric $\mathcal{G}$, we use the functional derivative
\[
\delta F := \int \frac{\delta F}{\delta \mathcal{G}^{\mu \nu}} \delta \mathcal{G}^{\mu \nu} d\text{vol}(\mathcal{G})
\]
where $d\text{vol}(\mathcal{G})$ is the volume form given by $\mathcal{G}$. We use Einstein’s convention, i.e. repeated (upper and lower) indices are summed over. The curvature two-form $K d\text{vol}(\mathcal{G}) = \mathcal{K}$ where $K$ is the Gauss curvature of the Levi-Civit`a connection.

We use the conventions from [27], in particular $G_2 = \pi^2 E_2/6$, $G_4 = \pi^4 E_4/90$ and $G_6 = \pi^6 E_6/945$, where $E_2$ resp. $E_4$ and $E_6$ are the quasi-modular resp. modular Eisenstein series of weight 2 resp. 4 and 6.
3 Correlation functions in the $(2, 5)$ minimal model

3.1 The fields

Now we specialise to the $(2, 5)$ minimal model, which is characterised by $c = -22/5$ and by the space of (real) fields (16). In this model, to the lowest weight vector $w \otimes \overline{w} \in R_2(\mathcal{W})$ is associated a real field $\Phi$, whose holomorphic and non-holomorphic conformal weight equal $h(\Phi) = \bar{h}(\Phi) = -1/5$. $\Phi$ is said to be primary, which means that $L_n \Phi = 0$ and $\bar{L}_n \Phi = 0$ for $n > 0$. To normalise the Shapovalov form on $R_2(\mathcal{W})$, we use

$$\langle w \otimes \overline{w} \vert w \otimes \overline{w} \rangle = \varepsilon_{\Phi},$$

in accordance with eq. (13). Later (in Section 4.1) we will see that $\varepsilon_{\Phi} = -1$. Arbitrarily but for definiteness, we normalise the Shapovalov forms on $\mathcal{W}$ and $\overline{\mathcal{W}}$ individually by setting

$$\langle w \vert w \rangle = 1,$$
$$\langle \overline{w} \vert \overline{w} \rangle = \varepsilon_{\Phi}.$$  

After extending, in eq. (15), the summation over $\lbrace e_i \in \mathcal{W} \otimes \overline{\mathcal{W}} \rbrace$ from eq. (12) to include a standard orthogonal set

$$\lbrace e_i \in \mathcal{W} \otimes \overline{\mathcal{W}} \rbrace = \mathcal{V}_{h, \bar{h}} \lbrace \text{orthogonal basis in } \mathcal{W}^h \otimes \overline{\mathcal{W}}^\bar{h} \rbrace$$

w.r.t. the Shapovalov form on $\mathcal{W}$, we recover the full partition function from eq. (1).

In any irreducible representation $\mathcal{W}_0$ of a minimal model, two fundamental linear relations hold. Recall that for the $(2, 5)$ minimal model and the vacuum representation $\mathcal{W}_0 = \mathcal{W}$ (with lowest weight vector $v$), these are

$$L_1 v = 0,$$
$$L_2 L_2 v = \frac{3}{5} L_4 v.$$ 

From the latter equation follows that

$$N_0(T, T) = \frac{3}{10} T^v,$$  

by the Virasoro OPE (20). The corresponding Fourier components satisfy

$$\sum_{m \geq 2} L_m L_{N-m} + \sum_{m < 2} L_{N-m} L_m = \frac{3}{10} (N-2)(N-3) L_N,$$  

by eq. (28) and Cauchy’s Theorem. For later use, we note that for $N \geq 3$ odd, eq. (29) is equivalent to

$$\sum_{m=-\infty}^{1} L_{N-m} L_m + \sum_{m=2}^{(N-1)/2} L_{N-m} L_m = \frac{1}{40} (N^2 - 9) L_N,$$

and for $N \geq 4$ even, to

$$\frac{1}{2} L_{N/2} L_{N/2} + \sum_{m=-\infty}^{N/2} L_{N-m} L_m + \sum_{m=2}^{N/2-1} L_{N-m} L_m = \frac{1}{40} (N^2 - 4) L_N.$$ 

12
Apart from \( \mathfrak{m}_0 = \mathfrak{g} \), the only lowest weight representation of the Virasoro algebra satisfying eq. (29) is \( \mathfrak{m}_{1/5} = \mathfrak{g} \) (with lowest weight vector \( \omega \)). For \( N = 2 \) and \( N = 3 \), eq. (29) yields the fundamental identities

\[
L_2 \omega = \frac{5}{2} L_1 L_1 \omega ,
\]
\[
L_3 \omega = \frac{25}{12} L_1 L_1 L_1 \omega .
\]

These two equations are equivalent to the OPE

\[
T(z) \Phi(0) = -\frac{1}{2z^2} \Phi(0) + \frac{1}{z} \Phi'(0) + \frac{5}{2} \Phi''(0) + \frac{25}{12} \Phi'''(0) + O(z^2) ,
\]  
(32)

where the dash denotes application of \( d/dz \). Note that eq. (32) is compatible with the Virasoro OPE (20) and eq. (28).

Let \( \mathfrak{m}_h \) be an irreducible representation with lowest weight vector \( \omega_h \), of weight \( h \). For \( h' \geq 1 \), the set of partitions of \( h' \) defines a generating set of the subvector space of vectors of conformal weight \( h + h' \): To every sequence \((n_1, \ldots, n_N) \in \mathbb{N}^N\) with \( 1 \leq n_1 \leq \ldots \leq n_N \) and \( \sum_{j=1}^N n_j = h' \) is associated the vector \( L_{n_1} \ldots L_{n_N} \omega_h \in \mathfrak{m}_h \). Eqs (30) and (31) show that when \( \mathfrak{m}_h \) is a representation of the (2, 5) minimal model, we may require \( n_i + 2 \leq n_{i+1} \) for \( i \geq 1 \). Thus the generating function for the number of holomorphic fields of a given weight in \( \mathfrak{g} \) and in \( \mathfrak{g} \) is the character

\[
\chi_\mathfrak{g} := \sum_{n \geq 0} \frac{q^{h+n}}{(q; q)_n} = 1 + q^2 + q^3 + q^4 + q^5 + 2q^6 + 2q^7 + 3q^8 + 3q^9 + 4q^{10} + 4q^{11} + 6q^{12} + \ldots ,
\]
\[
\chi_\mathfrak{g} := q^{-1/5} \sum_{n \geq 0} \frac{q^{h'_n}}{(q; q)_n} = q^{-1/5} \left( 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 4q^8 + 5q^9 + 6q^{10} + 7q^{11} + \ldots \right)
\]

respectively.

**Proposition 4.** In the (2, 5) minimal model, to every conformal weight \( 0 < h \leq 11 \), the space of quasi-primary fields of weight \( h \) in \( \mathfrak{g} \) is at most one-dimensional, while for \( h = 12 \) it is of dimension two. For \( h \leq 12 \), an orthogonal basis is given by

| \( h \) | quasi-primary vector in \( \mathfrak{g} \) | squared norm |
|---|---|---|
| 2 | \( L_2 \omega \) | \( c/2 \) |
| 4 | \( - \) | \( - \) |
| 6 | \( (7L_4L_2 - 2L_6) \omega \) | \( -217c/5 \) |
| 8 | \( \left( \frac{27}{4} L_4L_3 + \frac{12}{5} L_6L_2 - \frac{28}{7} L_8 \right) \omega \) | \( -12792c/5 \) |
| 10 | \( \left( 12L_6L_4 + 3L_7L_3 + 8L_8L_2 - \frac{756}{55} L_{10} \right) \omega \) | 105400 |
| 12 | \( v_{12,1} \) | \( -111587133c/13 \) |
| 12 | \( v_{12,2} \) | \( 4842771607546875c \) |

where

\[
v_{12,1} = \left( \frac{495}{4} L_7L_5 + \frac{55}{2} L_5L_4 + \frac{341}{4} L_6L_3 + 59L_{10}L_2 - \frac{2772}{13} L_{12} \right) \omega
\]
\[
v_{12,2} = \left( -7402950L_8L_2L_4L_5 - \frac{16923555}{4} L_7L_5 - \frac{7473735}{2} L_5L_4 - \frac{24424425}{4} L_9L_3 + 1260405L_{10}L_2 + 14784210L_{12} \right) \omega.
\]
Proof. The number of linearly independent quasi-primary fields of conformal weight \( h \) in \( \mathfrak{B} \) equals the coefficient of \( q^h \) in the series

\[
\chi^{\mathfrak{B}}_1 - 1 = (1 - q)(\chi_0 - 1) = q^2 + q^6 + q^8 + q^{10} + 2q^{12} + 2q^{14} + q^{15} + \ldots .
\]

The fields and their respective norm are obtained by direct computation.

The identity \( L_1 L_n v = (n - 1) L_{n+1} v \) shows that \( L_n v \) for any \( n \geq 3 \) correspond to iterated derivatives of \( T \). For the relatively prime integers \( m = 1275 627 234 375 \) and \( n = 2261 \), and for either choice of sign, the linear combination

\[
v_{12,1} \sqrt{m} \pm v_{12,2} \sqrt{n}
\]

is a null vector,

\[
n \parallel v_{12,1} \parallel^2 + m \parallel v_{12,2} \parallel^2 = 0.
\]

As such it is unique up to rescaling.

**Proposition 5.** In the \((2,5)\) minimal model, to every conformal weight \( h = h' - 1/5 \) with \( 0 < h' \leq 11 \), the space of quasi-primary fields in \( \mathfrak{B} \) of weight \( h \) is at most one-dimensional, with the first quasi-primary field occurring at weight \( 19/5 \). An orthogonal basis is given by

| \( h' \) | quasi-primary vector in \( \mathfrak{B} \) | squared norm |
|---|---|---|
| 4 | \((25L_3L_1 - 2L_4)\) \( w \) | \(5928 \parallel w \parallel^2 /5\) |
| 6 | \((21L_6L_2 + 7L_5L_4 - 43L_6)\) \( w \) | \(1634.817 \parallel w \parallel^2 /125\) |
| 8 | \((7L_3L_5 + 2L_6L_2 + 9L_6L_1 - 22L_6)\) \( w \) | \(-10 759 \parallel w \parallel^2 /5\) |
| 9 | \((7L_3L_3L_4 + \frac{7}{3}L_4L_3 + \frac{2}{5}L_7L_2 - \frac{2}{5}L_6L_1 - \frac{87}{5}L_6)\) \( w \) | \(-4 018 443 \parallel w \parallel^2 /3 125\) |
| 10 | \((12L_6L_2 + 3L_7L_3 + 8L_8L_2 + 6L_9L_1 - \frac{62}{5}L_10)\) \( w \) | \(356 414 461 \parallel w \parallel^2 /3 025\) |
| 11 | \((3L_7L_4 + 7L_9L_1 + 14L_9L_2 - 42L_10L_1 + 60L_6L_4L_1 + 15L_7L_3L_1 - \frac{72}{5}L_6L_1)\) \( w \) | \(-335 699 c^2 \parallel w \parallel^2 /10\) |
| 12 | \(w_{12,1}\) | \(\parallel w_{12,1} \parallel^2\) |
| | \(w_{12,2}\) | \(4 577 888 353 871 \parallel w \parallel^2 /3\) |

where \( \parallel w_{12,1} \parallel^2 = -401 304 741 575 703 261 976 819 992 213 522 \parallel w \parallel^2 /5 \) and

\[
w_{12,1} = \left(\frac{490 940 248 301 793}{2}L_7L_5 + 331 118 144 757 253L_6L_4 + \frac{750 814 671 380 107}{2}L_9L_3ight.
\]
\[-6 914 283 463 318L_10L_2 + \frac{1 810 769 754 094 025}{2}L_1L_1 + 448 490 640 452 880L_6L_4L_2
\]
\[-280 306 650 283 050L_7L_4L_1 - 541 926 190 547 230L_8L_3L_4 - 1 025 605 090 206 371L_{12}\) \( w \).

\[
w_{12,2} = \left(-\frac{132 275}{6}L_8L_4 - 16 445L_9L_3 + 9 880L_{10}L_2 - \frac{122 915}{2}L_{11}L_4 - 35 750L_6L_4L_2
\]
\[+ \frac{89 375}{4}L_7L_3L_1 + \frac{518 375}{12}L_9L_3L_1 + 45 959L_{12}\) \( w \).

Proof. The number of quasi-primary fields of conformal weight \( h = h' - 1/5 \) in \( \mathfrak{B} \) is given by the coefficient of \( q^h \) in the series

\[
\chi^{\mathfrak{B}}_1 = (1 - q)\chi^{\mathfrak{B}} = q^{-1/5}(1 + q^4 + q^6 + q^8 + q^{10} + q^{11} + 2q^{12} + \ldots ).
\]

The fields and their respective norm are obtained by direct computation, using in particular eqs (30) and (31). □
For the relatively prime integers \( m = 2775 \) and \( n = 144904525088234638887846 \), the linear combinations
\[
w_{12,1} \sqrt{m} \pm w_{12,2} \sqrt{n}
\]
are null vectors.

For derivative fields, one has

**Proposition 6.** For \( k \geq 0 \) and the respective \( k^{th} \) derivative in the holomorphic coordinate,
\[
\| T^{(k)} \|_c^2 = \frac{k!(k+3)!}{3!} \in \{ 1, 4, 40, 720, 20\,160, \ldots \},
\]
and
\[
\frac{\| \Phi^{(k)} \|_c^2}{\| \Phi \|_c^2} = k! \prod_{n=1}^{k} \left( k - n - \frac{2}{5} \right) \in \{ 1, \frac{2}{5}, \frac{12}{25}, \frac{288}{125}, \frac{20160}{625}, \ldots \}.
\]

**Proof.** Let \( \phi \) be a field with the properties \( L_{-1} \phi = 0 \) and \( L_0 \phi = h \phi \). For \( k \geq 1 \),
\[
\| \phi^{(k)} \|_c^2 = \langle L^{k-1}_1 \phi | L^k_1 \phi \rangle = \langle L^{k-1}_1 \phi | L_{-1} L^k_1 \phi \rangle = \sum_{m=1}^{k} \langle L^{k-m}_1 \phi | L^{m-1}_1 [L_{-1} L_1] L^{k-m}_1 \phi \rangle = 2 \sum_{m=1}^{k} (k-m+h) \langle L^{k-1}_1 \phi | L^{k-m}_1 \phi \rangle = k(k-1+2h) \| \phi \|_c^2.
\]

By induction,
\[
\| \phi^{(k)} \|_c^2 = k! \prod_{n=1}^{k} (k-n+2h) \| \phi \|_c^2.
\]

Now we specialise to genus \( g = 1 \). Thus \( q \) is identified with the nome \( e^{2\pi i \tau} \) for \( \tau \in \mathfrak{H} \), the upper complex half plane. In the character formulae above, we have \( \chi_{\mathbb{R}} = H(q) \) and \( \chi_{\mathbb{C}} = q^{-1/5} G(q) \) in terms of the Rogers-Ramanujan functions eq. (2). The zero-point functions differ from the corresponding characters by a factor of \( q^{-1/24} \).

Conventionally one numbers them in lexicographic order,
\[
\langle 1 \rangle_1 \tau_1^e = q^{-1/60} G(q), \quad \langle 1 \rangle_2 \tau_2^e = q^{1/60} H(q). \quad (33)
\]

In accordance with Section 2.5, the first and second correspond to the non-vacuum and vacuum sector, respectively, and the partition function is given by eq. (1). Together, the zero-point functions form a vector valued modular function \( \left( \langle 1 \rangle_1 \tau_1^e \langle 1 \rangle_2 \tau_2^e \right) \) for the full modular group w.r.t. a representation
\[
\varphi_{\mathbb{R}} : \ SL(2, \mathbb{Z}) \to M_2(\mathbb{C})
\]
that factors through $SL(2, \mathbb{Z}/5\mathbb{Z})$. In particular, the genus one partition function eq. (1) is invariant under the full modular group.

For the remainder of this section, we drop the label $g = 1$. For $\ell \in \mathbb{R}$, let $\mathcal{D}_\ell = d/d(2\pi i \tau) - (\ell/12)E_2(\tau)$ be the Serre-derivative operator. (When the weight $\ell$ of a modular form $f$ is clear, we will often omit the index and write simply $\mathcal{D}f$ for $\mathcal{D}_\ell f$, $\mathcal{D}^2 f$ for $\mathcal{D}_{\ell+2} \mathcal{D}_\ell f$, etc. For instance, both $\langle 1 \rangle_1$ and $\langle 1 \rangle_2$ are solutions of the Kaneko-Zagier differential equation

$$\mathcal{D}^2 f = \frac{\ell(\ell + 2)}{144} E_4(\tau)f$$

which was originally studied for $\ell = 0$ or $\ell \equiv 4 \pmod{6}$ [26, §8] but needs $\ell = 1/5$ here.)

For $N \geq 0$, the $N$-point function of the Virasoro field $T$ is the $N$-fold functional derivative of the zero-point function. In this sense the Virasoro field generates changes of the metric. For volume preserving metric changes, we have for $a = 1, 2$ [5]\footnote{or preprint arXiv:1705.08294 for a direct proof}

$$\frac{1}{2\pi i} \frac{d}{dt} \langle 1 \rangle_a = \frac{1}{(2\pi)} \langle T \rangle_a ,$$

where $\langle T \rangle_a$ is the one-point function of the field $T$. (By translational invariance of $N$-point functions on the torus, $\langle T(z) \rangle_a$ is constant in position.) As an aside, the OPE (20) yields in addition

$$\mathcal{D}_2 \langle T \rangle_a = \frac{11}{3600} (2\pi)^3 E_4(\tau) \langle 1 \rangle_a ,$$

confirming eq. (34).

For fixed $\tau \in \mathbb{H}_1$, let $\wp(z) = \wp(z|\tau)$ and $\zeta(z) = \zeta(z|\tau)$ the Weierstrass $\wp$ and $\zeta$ function, respectively. Denote their respective value at $z_{ij} := z_i - z_j$ (for $z_i, z_j \in \mathbb{C}$) by $\wp_{ij}$ and $\zeta_{ij}$. We always set $z_0 = 0$ and omit the index in this case, writing simply $\wp_i$ for $\wp(z_i)$.

Now we calculate the one-point function w.r.t. the metric $|dz|^2$ of the field $\Phi \in R_{2}([8)$ which corresponds to the lowest weight vector $w \in \mathfrak{h}$.

**Proposition 7.** The one-point function of $\Phi$ satisfies

$$\mathcal{D}_{-1/5} \langle \Phi \rangle = 0 .$$

**Proof.** By the OPE (32),

$$\frac{\langle T(z) \Phi(0) \rangle}{\langle \Phi(0) \rangle} = h \wp(z) .$$

(Note that $\langle \Phi''(0) \rangle = \frac{d^2}{dz^2} \langle \Phi(z) \rangle |_{z=0} = 0$ by translational invariance.) Moreover,

$$\int_0^1 \wp(z)dz = -2 \zeta(1/2) = -\frac{\pi^2}{3} E_2(\tau) ,$$

so when the contour integral is taken along the real period, and $\oint dz = 1$,

$$\frac{1}{2\pi i} \frac{d}{dt} \langle \Phi(0) \rangle_a = \oint \langle T(z) \Phi(0) \rangle_a \frac{dz}{(2\pi i)^2} = -\frac{1}{60} E_2(\tau) \langle \Phi(0) \rangle_a .$$

This shows that $\langle \Phi \rangle$ lies in the kernel of $\mathcal{D}_{-1/5}$. 

\hfill $\square$
Since the kernel of $O$ is spanned by $\eta(\tau)^{2l}$, where $\eta(\tau)$ is Dedekind $\eta$-function, we have
\[
\langle \Phi \rangle = \mu_\eta |\eta|^{-4/5}
\] (39)
for some $\mu_\eta \in \mathbb{R}^*$, which we compute in Proposition 15 (up to a sign that we choose positive). We will use the factorisation $\langle \Phi \rangle = \langle w \rangle \langle \bar{w} \rangle$, where $\langle w \rangle = \sqrt{\mu_\eta \, \eta^{-2/5}}$ and $\langle \bar{w} \rangle = \sqrt{\mu_\eta \, \eta^{-2/5}}$

**Corollary 8.** We have
\[
\frac{\langle T(z_1)T(z_2)\Phi(0) \rangle}{\langle \Phi(0) \rangle} = \frac{c}{2} \varphi_{12}^2 - \frac{1}{5} \varphi_{12} (\varphi_1 + \varphi_2) + \frac{6}{25} \varphi_1 \varphi_2 + 24 G_4 .
\] (40)

**Proof.** On the one hand, from the OPE (32) for $T(z)$ and $\Phi(0)$, and eq. (37),
\[
\langle \Phi \rangle^{-1} \langle T(z)T(w) \Phi(0) \rangle = \frac{h^2}{\bar{z}^2} \varphi(w) - \frac{h}{\bar{z}} \varphi'(w) + \text{terms that are regular for } z \to 0 ,
\]
where the occurring even and odd negative power of $z$ can be replaced with $\varphi(z)$ and $z \varphi(z)$, respectively. The latter expression is not elliptic. However, we may use
\[
-z \varphi'(w) = \varphi(z - w) - \varphi(w) + O(z^2) .
\]
Thus
\[
\langle \Phi \rangle^{-1} \langle T(z)T(w) \Phi(0) \rangle = h \varphi(z) \varphi(z - w) + (h^2 - h) \varphi(z) \varphi(w) + \text{terms that are regular for } z \to 0 .
\]

On the other hand, the OPE (20) for $T(z)$ and $T(w)$, and eq. (37) yield
\[
\langle \Phi \rangle^{-1} \langle T(z)T(w) \Phi(0) \rangle = \frac{c/2}{(z - w)^2} + \frac{h}{(z - w)^2} \{ \varphi(z) + \varphi(w) \} - \frac{1}{5} h \varphi''(w) + O(z - w)
\]
\[
= \frac{c}{2} \varphi^2(z - w) + h \varphi(z - w) \{ \varphi(z) + \varphi(w) \} - \frac{6}{5} h \varphi(z) \varphi(w) + \tilde{C}_{4,\ell} ,
\]
where $\tilde{C}_{4,\ell}$ is constant in $z$ and $w$. By comparison, we obtain
\[
h \left( h + \frac{1}{5} \right) = 0 , \quad \tilde{C}_{4,\ell} = -(c - 2h) 6 G_4 = 24 G_4 ,
\]
since $c = -22/5$.

**Proposition 9.** The two-point function of $\Phi$ satisfies the ODE
\[
\left( \frac{25}{12} \frac{d^3}{dz^3} - \varphi(z) \frac{d}{dz} + \frac{1}{5} \varphi'(z) \right) \langle \Phi(z) \Phi(0) \rangle_A = 0 .
\] (41)

**Proposition 10.** For $z_2 \neq 0$,
\[
\langle T(z_1) \Phi(z_2) \Phi(0) \rangle_A = -\frac{1}{5} \{ \varphi_{12} + \varphi_1 - \varphi_2 \} \langle \Phi(z_2) \Phi(0) \rangle_A
\]
\[
+ (\xi_{12} - \xi_1 + \xi_2) \langle \Phi'(z_2) \Phi(0) \rangle_A + \frac{5}{2} \langle \Phi''(z_2) \Phi(0) \rangle_A .
\]
Proof. By the OPE of $T(u)$ with $\Phi(z)$ and with $\Phi(0)$, respectively,
\[
\langle T(u) \Phi(z) \Phi(0) \rangle_A \\
= h \varphi(u-z) \langle \Phi(z) \Phi(0) \rangle_A + \zeta(u-z) \langle \Phi'(z) \Phi(0) \rangle_A + \text{regular for } u \to z \\
= h \varphi(u) \langle \Phi(z) \Phi(0) \rangle_A + \zeta(u) \langle \Phi'(z) \Phi(0) \rangle_A + \text{regular for } u \to 0
\]
By translational invariance,
\[
\langle \Phi(z) \Phi'(0) \rangle_A = -\langle \Phi'(z) \Phi(0) \rangle_A.
\]
Considering all poles at once yields
\[
\langle T(u) \Phi(z) \Phi(0) \rangle_A \\
= h(\varphi(z-u) + \varphi(u)) \langle \Phi(z) \Phi(0) \rangle_A + (\zeta(z-u) - \zeta(u)) \langle \Phi'(z) \Phi(0) \rangle_A \quad (42)
\]
+ terms that are constant in $u$,
by ellipticity of the three-point function. Comparison of the $u^0$ terms on the r.h.s. of the first line of eq. (42) with the OPE (32) for $T(u)$ and $\Phi(0)$ shows that the terms constant in $u$ are equal to
\[
-h \varphi(z) \langle \Phi(z) \Phi(0) \rangle_A + \zeta(z) \langle \Phi'(z) \Phi(0) \rangle_A + \frac{5}{2} \langle \Phi''(z) \Phi(0) \rangle_A
\]
since $\langle \Phi(z) \Phi''(0) \rangle_A = \langle \Phi''(z) \Phi(0) \rangle_A$, by invariance under both translation and reflection. \hfill \Box

Proof of Proposition 9. Eq. (41) follows by comparison of the terms in eq. (42) which are linear in $u$, with the OPE (32) for $T(u)$ and $\Phi(0)$, using that
\[
\langle \Phi^{(3)}(z) \Phi(0) \rangle_a = -\langle \Phi^{(3)}(0) \Phi(z) \rangle_a.
\]
\hfill \Box

3.2 Graphical representation of correlation functions on the torus

We now consider general rational conformal field theories on the torus with holomorphic differential $dz$ and periods $1, \tau$. Throughout this section, all occurring correlation functions refer to one and the same holomorphic block (we omit the lower index $A$ $\in \mathbb{H}$).

The following proposition is the reformulation of a known result [10, 11] in terms of elliptic functions. Since the present version is somewhat simpler and uses an argument required for proving the related Proposition 12, which is new, we present it here.

Proposition 11. Let $\langle \rangle$ be a holomorphic block in a sector of a CFT on the torus with central charge $c$. For $N \in \mathbb{N}_0$, let $S_N^{[1]} := S(z_1, \ldots, z_N)$ be the set of oriented graphs with vertices $z_1, \ldots, z_N$ (which may or may not be connected), subject to the condition that every vertex has at most one ingoing and at most one outgoing line, and none is a tadpole (with the line incoming to a vertex being identical to its outgoing line). For $n \in \mathbb{N}$, there exist functions
\[ C_{2n,c} : \mathbb{H} \to \mathbb{C} \]
which depend on \(c\), such that for the \(N\)-point function of the Virasoro field, we have

\[
\langle 1 \rangle^{-1} \langle T(z_1) \ldots T(z_N) \rangle = \sum_{\Gamma \in \mathcal{S}_N^{[1]}} \gamma(\Gamma), \tag{43}
\]

where for \(\Gamma \in \mathcal{S}_N^{[1]}\),

\[
\gamma(\Gamma) := \left( \frac{c}{2} \right)^{\text{loops}} C_{\Gamma; (N-\text{edges})c} \prod_{(z_i,z_j) \in \Gamma} \varphi_{ij}.
\]

Here \((z_i, z_j) \in \Gamma\) is an oriented edge. Moreover, for all \(n \in \mathbb{N}\), \(C_{2n,c}\) is a modular form of weight \(2n\).

Note that the result holds true in general, and not just in the \((2, 5)\) minimal model. For later use, here is the result for \(N = 3\) and for \(N = 4\), respectively:

\[
\langle 1 \rangle^{-1} \langle T(z_2) T(z_1) T(z_3) \rangle
\]

\[
= c \varphi_{12} \varphi_{13} \varphi_{23} C_{0,c}
\]

\[
+ \frac{c}{2} \left( \varphi_{12}^2 + \varphi_{13}^2 + \varphi_{23}^2 \right) C_{2,c}
\]

\[
+ 2 \left( \varphi_{12} \varphi_{13} + \varphi_{12} \varphi_{23} + \varphi_{13} \varphi_{23} \right) C_{4,c},
\]

\[
\langle 1 \rangle^{-1} \langle T(z_2) T(z_1) T(z_3) T(z_4) \rangle
\]

\[
= \frac{c^2}{4} \left( \varphi_{12}^3 + \varphi_{13}^3 + \varphi_{23}^3 \right) C_{0,c}
\]

\[
+ c \left( \varphi_{12} \varphi_{23} \varphi_{13} + \varphi_{13} \varphi_{23} \varphi_{12} + \varphi_{12} \varphi_{24} \varphi_{34} \varphi_{31} \right) C_{0,c}
\]

\[
+ c \left( \varphi_{12}^2 \varphi_{23} + \varphi_{13}^2 \varphi_{12} + \varphi_{12} \varphi_{23} \right) C_{2,c}
\]

\[
+ 2 \left( \varphi_{12} \varphi_{13} \varphi_{23} + \varphi_{12} \varphi_{24} \varphi_{34} \varphi_{31} + \varphi_{13} \varphi_{23} \varphi_{12} + \varphi_{12} \varphi_{23} \varphi_{24} \right) C_{4,c}
\]

\[
+ \frac{c}{2} \left( \varphi_{12}^2 + \varphi_{13}^2 + \varphi_{23}^2 \right) C_{4,c}
\]

\[
+ 2 \left( \varphi_{12} \varphi_{13} + \varphi_{12} \varphi_{23} + \varphi_{13} \varphi_{23} \right) C_{6,c}.
\]

A proof of Proposition 11 is given in Appendix A.

In order to compute higher order terms in the \(s\)-expansion of the fifth solution \(\langle 1 \rangle_{s}^{5}\), we use the following result.

**Proposition 12.** Let \(\Phi\) be a primary field of holomorphic conformal weight \(h\). Let \(\langle \Phi \rangle\) be the corresponding one-point function of a CFT on the torus with central charge \(c\). For \(N \geq 0\), let \(\mathcal{S}_N^{[1]} := \mathcal{S}(0, z_1, \ldots, z_N)\) be the set of oriented graphs with vertices \(z_0 = 0, z_1, \ldots, z_N\) subject to the conditions...
1. Every vertex different from 0 has at most one ingoing and at most one outgoing line, and none is a tadpole.

2. No edge emanates from 0.

Let \( \lambda : \tilde{S}^{[1]}_N \rightarrow \{0, 1, \ldots, N\} \) be the map that counts a graph’s respective number of edges ending in 0. For \( n \in \mathbb{N} \), there exist functions

\[
\tilde{C}_{2n,c} : \tilde{S} \rightarrow \mathbb{C}
\]

which depend on \( c \), such that for the \((N + 1)\)-point function of \( N \) copies of the Virasoro field and one copy of \( \Phi \), we have

\[
\langle \Phi \rangle^{-1} \langle T(z_1) \cdots T(z_N) \Phi(0) \rangle = \sum_{\Gamma \in \tilde{S}(0,z_1,\ldots,z_N)} \tilde{y}(\Gamma),
\]

where for \( \Gamma \in \tilde{S}(0,z_1,\ldots,z_N) \) with \( \lambda = \lambda(\Gamma) \) and with number \#edges \( \geq \lambda \),

\[
\tilde{y}(\Gamma) := \left( \frac{c}{2} \right)^{\#\text{loops}} \tilde{C}_{2, (N-\#\text{edges}), c} \prod_{j=0}^{\lambda-1} (h - j) \prod_{(z_i, z_j) \in \Gamma} \varphi_{ij}.
\]

Moreover, for all \( n \in \mathbb{N} \), \( \tilde{C}_{2n,c} \) is a modular form of weight \( 2n \).

Since holomorphic and anti-holomorphic variables do not give rise to mixed singularities and can be treated separately, we can write

\[
\langle T(z_1) \cdots T(z_N) \tilde{T}(z'_1) \cdots \tilde{T}(z'_N) \Phi(0) \rangle = \langle T(z_1) \cdots T(z_N) w \rangle \langle \tilde{T}(z'_1) \cdots \tilde{T}(z'_N) w \rangle,
\]

where according to eq. (39), \( \langle w \rangle = \sqrt{\mu \eta} \). For illustration, Proposition 12 yields for \( N = 3 \):

\[
\langle \Phi \rangle^{-1} \langle T(z_1) T(z_2) T(z_3) \Phi(0) \rangle = c \varphi_{12} \varphi_{23} \varphi_{31} \tilde{C}_{0,c} - \frac{c}{10} (\varphi_{12}^2 \varphi_3 + \varphi_{23}^2 \varphi_1 + \varphi_{31}^2 \varphi_2) \tilde{C}_{0,c} - \frac{66}{125} \varphi_{12} \varphi_2 \varphi_3 \tilde{C}_{0,c}
\]

\[
- \frac{1}{5} (\varphi_{12} \varphi_{23} (\varphi_1 + \varphi_1) + \varphi_{13} \varphi_{32} (\varphi_1 + \varphi_2) + \varphi_{31} \varphi_{12} (\varphi_2 + \varphi_2)) \tilde{C}_{0,c}
\]

\[
+ \frac{6}{25} (\varphi_{12} \varphi_2 (\varphi_1 + \varphi_3) + \varphi_{31} \varphi_2 (\varphi_1 + \varphi_3) + \varphi_{23} \varphi_1 (\varphi_2 + \varphi_3)) \tilde{C}_{c}
\]

\[
+ 2 (\varphi_{12} + \varphi_{31} + \varphi_{23}) \tilde{C}_{c} - \frac{1}{5} (\varphi_1 + \varphi_2 + \varphi_3) \tilde{C}_{c} + \tilde{C}_{c}
\]

Since \( \tilde{C}_{2,c} = 0 \), the contribution of all graphs with \#edges = 2,

\[
\frac{c}{2} (\varphi_{12}^2 + \varphi_{31}^2 + \varphi_{23}^2) + 2 (\varphi_{12} \varphi_{23} + \varphi_{13} \varphi_{32} + \varphi_{31} \varphi_{12})
\]

\[
- \frac{1}{5} (\varphi_{12} (\varphi_1 + \varphi_2 + 2 \varphi_3) + \varphi_{31} (\varphi_1 + 2 \varphi_2 + \varphi_3) + \varphi_{23} (2 \varphi_1 + \varphi_2 + \varphi_3))
\]

\[
+ \frac{6}{25} (\varphi_1 \varphi_2 + \varphi_3 \varphi_1 + \varphi_2 \varphi_3)
\]

drops out. A proof of Proposition 12 has been moved to Appendix B.
We provide a machinery for computing successively, for \( N \geq 1 \), the modular forms \( C_{2N,c} = \gamma(\Gamma_0^N) \) and \( \tilde{C}_{2N,c} = \tilde{\gamma}(\Gamma_0^N) \) from Proposition 11 and Proposition 12, respectively.

Our method relies on a formula in Weinberg’s book [25, p. 360]. For the zero-point and for the one-point function of a primary field \( \Phi \) of holomorphic weight \( h \) of a CFT sector on the torus, we have

\[
\frac{d^N(1)}{dx^N} = \int \ldots \int \langle T(z_1) \ldots T(z_N) \rangle \frac{dz_2 \ldots dz_N}{(2\pi i)^N}
\]

and

\[
\frac{d^N(\Phi)}{dx^N} = \int \ldots \int \langle T(z_1) \ldots T(z_N) \Phi(z_0) \rangle \frac{dz_1 \ldots dz_N}{(2\pi i)^N}
\]

respectively. In the present discussion, we integrate along the real period (using that the fields are holomorphic, and Cauchy’s Theorem). This is particularly convenient for our purpose since the period integral over \( \varphi \) is proportional to \( E_2 \) by eq. (38), and does not contribute to \( C_{2N,c} \) and \( \tilde{C}_{2N,c} \). For \( N = 1 \), we recover eq. (35) and (36), respectively.

Let \( f, g : \mathbb{C} \to \mathbb{C} \) be two functions with period 1. Suppose \( f \) is meromorphic on \( \mathbb{C} \), and \( g \) is meromorphic in a tubular neighbourhood of the real line. The convolution of \( f \) and \( g \) along the real period is defined by

\[
(f \ast g)(x) := \lim_{\sigma \to 0^+} \int_{-\pi i}^{\pi i} f(z)g(x-z)dz \quad x \in \mathbb{R}.
\]

One can show that \( f \ast g \) has a unique analytic continuation to complex arguments, which is regular on \( \mathbb{R} \), meromorphic on \( \mathbb{C} \), and which has period one. Thus iterated convolutions can be considered. The case of interest to us is when \( f \) and \( g \) equal the same function, or an iterated convolution of it. For \( k \geq 1 \), we also write \( f \ast \ldots \ast f \) (or \( k \)-factors) for the \( k \)-fold convolution of \( f \), and we set \( f^{*0} = 1 \). For example, \( (1 \ast \varphi) \) is given by eq. (38).

**Proposition 13.** Let \( (1) \) be a holomorphic block in a sector of a CFT on the torus with central charge \( c \). Let \( (\Phi) \) be the one-point function of a primary field \( \Phi \) of holomorphic weight \( h \) in the same sector. For \( N \geq 1 \), let \( S_{2N} \subset \mathcal{S}_N^{[1]} \) and \( S_{2N} \subset \mathcal{S}_N^{[1]} \), respectively, be the set of graphs whose connected components are all either isolated points, or loops. We have

\[
(2\pi i)^N \frac{d^N}{dx^N}(1) = \int \ldots \int \sum_{\Gamma \in S_{2N} \cap \Gamma_0^N} \gamma(\Gamma) \frac{dz_2 \ldots dz_N}{(2\pi i)^N} = C_{2N,c} + O(E_2)
\]

and

\[
(2\pi i)^N \frac{d^N}{dx^N}(\Phi) = \int \ldots \int \sum_{\Gamma \in S_{2N} \cap \Gamma_0^N} \tilde{\gamma}(\Gamma) \frac{dz_1 \ldots dz_N}{(2\pi i)^N} = \tilde{C}_{2N,c} + O(E_2),
\]

respectively, where integration is performed along the real period.

Note that the meaning of “\( O(E_2) \)” here is that the l.h.s. in each case is a quasi-modular form (of weight 2N) which, when it is expressed as a polynomial in \( E_2 \) with modular coefficients, has the constant term \( C_{2N,c} \) or \( \tilde{C}_{2N,c} \), respectively. Note that this determines the modular forms \( C_{2N,c} \) and \( \tilde{C}_{2N,c} \) uniquely.
Proof. In the graphical representation of $\langle T(z_1) \ldots T(z_N) \rangle$ and of $\langle T(z_1) \ldots T(z_N) \Phi(z_0) \rangle$, for $0 \leq k \leq N$, the iterated period integral over a connected component of $k$ edges in a graph is proportional to the period integral over $\varphi^{\ast k}(z)$. For all graphs other than $\Gamma_0^\ast$, we have $k \geq 1$, and for $z \neq 0$,
\[
\int \varphi^{\ast k}(z) \, dz = (1 \ast \varphi)^k,
\]
where $(1 \ast \varphi)$ is proportional to $E_2(\tau)$ by eq. (38). Thus the only relevant convolutions are those at $z = 0$. These correspond precisely to loops. \hfill \Box

It turns out that it is useful to work with Eisenstein’s zeta function [4, 24]. In modern terminology, it is defined as the modified Weierstrass zeta function
\[
\mathcal{Z}(z) := \zeta(z) + z(1 \ast \varphi).
\]
For $m, n \in \mathbb{Z}$, the zeta function satisfies [1]
\[
\zeta(z + m + n\tau) - \zeta(z) = 2m\zeta(1/2) + 2n\zeta(\tau/2),
\]
and Legendre’s relation (together with eq. (38)) yields
\[
\mathcal{Z}(z + m + n\tau) - \mathcal{Z}(z) = -2\pi ni.
\]
Thus $1 \ast \mathcal{Z} = -\pi i$. Under differentiation, the convolution behaves according to $(f \ast g)' = f' \ast g = f \ast g'$, and we have
\[
\varphi^{\ast k}(z) = (-1)^k \frac{d^k}{dz^k} \mathcal{Z}^{\ast k}(z) + (1 \ast \varphi)^k. \tag{46}
\]
For convenience of the reader, we list the first few relevant terms:
\[
\mathcal{Z} \ast \mathcal{Z} = -\frac{1}{2} \mathcal{Z}^2 + \frac{1}{2} \varphi + \frac{2\pi^2}{3},
\]
\[
\mathcal{Z} \ast \mathcal{Z} \ast \mathcal{Z} = \frac{1}{6} \mathcal{Z}^3 - \frac{\pi i}{2} \mathcal{Z}^2 - \frac{1}{2} \mathcal{Z} \varphi - \frac{1}{6} \varphi' + \frac{\pi i}{2} \varphi.
\]
Eq. (46) keeps being true if we replace the operation $\ast$ by a closely related commutative convolution $\otimes$, for which iterated convolutions of $\mathcal{Z}$ are machine computable. $\varphi^{\ast k}$ differs from $\varphi^{\otimes k}$ by a residue [13, cf. Proposition 5].

Example 14. We address the modular forms $C_{2N_e}$ and $\tilde{C}_{2N_e}$ from Proposition 13. We have $C_{0_e} = 1$ and $C_{2_e} = t$, where $t := \langle 1 \rangle^{-1} \langle T \rangle = d \log(1) / d(2\pi i r)$. For $N \leq 3$, the computation of at most 3-fold convolutions of $\varphi$ are required. In the $(2, 5)$ minimal model, earlier results [11, and references therein] include
\[
\langle T(z) T(0) \rangle = \frac{c}{2} \varphi(z)^2 \langle 1 \rangle + 2 \varphi(z) \langle T \rangle - 6c G_4 \langle 1 \rangle. \tag{47}
\]
Thus $C_{4_e} = -6c G_4$, in accordance with Proposition 13. Moreover, we have
\[
C_{6_e} = -\frac{84}{5} \left( 5c G_6 - 3 G_4 \right),
\]
\[
C_{8_e} = -\frac{16}{5} \left( 367c G_4^2 - 336 G_6 \right).
\]
In Theorem 12, $\hat{C}_{0_e} = 1$, $\hat{C}_{2_e} = 0$, and
\[
\hat{C}_{4_e} = 24 G_4, \quad \hat{C}_{6_e} = 336 G_6.
\]
Note that this recovers the modular form $\tilde{C}_{4_e}$ from Corollary 8.
4 The genus 2 partition function

4.1 The holomorphic blocks

The (2, 5) minimal model has the two sectors $R_2(3)$ and $R_2(8)$. In order to calculate, for \( g \geq 1 \), the partition function $\gamma_g$ for $(\Sigma, \mathcal{F}) \in \mathcal{X}$, one proceeds as follows:

1. Cut $\Sigma$ along $g - 1$ homologically trivial cycles $\gamma_i$, $i = 1, \ldots, g - 1$, so that $\Sigma$ decomposes into $g$ tori with boundary $\gamma_{i-1} \cup \gamma_i$, where $\gamma_0$ and $\gamma_g$ are empty.

2. Cut the $i$th component ($i = 1, \ldots, g$) along a homologically non-trivial cycle $\gamma_i$. This reduces its genus to zero.

3. For $i = 2, \ldots, g - 1$, cut once more along a separating cycle $\gamma_i$ so that the connected components (pants) all have three boundary circles.

Altogether we have cut along $3(g - 1)$ cycles and obtained $2(g - 1)$ pants (if $g \geq 2$).

We encode the cutting scheme in the vector

\[
(\gamma'_1, \ldots, \gamma'_{g-1}; \gamma'_1, \ldots, \gamma'_1, \ldots, \gamma'_{g-2})
\]  

(48)

whose components will be referred to as cutting cycles. The calculations are performed by using the corresponding combination of $s$ and $q$ expansions.

Thus an $N$-point function on $M \in \mathcal{X}$ can be defined as a sum of products of three-point functions on a sphere. The result does not depend on the cutting scheme provided that the four-point functions on a sphere and the one-point functions on a torus are globally defined [21, 15]. It suffices to check this for the primary fields, and thus, in the (2, 5) minimal model, for the field $\Phi$. For $\langle \Phi \rangle_{q}^{s=1}$, eq. (39) is sufficient. Moreover, $\langle \Phi \Phi \Phi \rangle_{q}^{s=0}$ can be explicitly calculated: Here is a short derivation. A comprehensive discussion is contained in [3].

\[
\langle T(z) \Phi(z_1) \ldots \Phi(z_N) \rangle_{q}^{s=0}
\]

is a meromorphic function of $z$ whose poles are prescribed by the OPE (32) for each of copy of $\Phi$. By the Liouville proposition, for $N \geq 0$, we can write

\[
\langle T(z) \Phi(z_1) \ldots \Phi(z_N) \rangle_{q}^{s=0} = \langle \Phi(z_1) \ldots \Phi(z_N) \rangle_{q}^{s=0} \sum_{i=1}^{N} \frac{-1/5}{(z - z_i)^2} + \sum_{i=1}^{N} \frac{1}{z - z_i} \frac{\partial}{\partial z_i} \langle \Phi(z_1) \ldots \Phi(z_N) \rangle_{q}^{s=0}.
\]

(49)

According to eq. (18), for $z \to \infty$, $\langle T(z) \Phi(z_1) \ldots \Phi(z_N) \rangle_{q}^{s=0} = O(1/z^4)$. Thus there are no additional regular terms in eq. (49). Moreover, the coefficient of $1/z^k$ for $k = 1, 2, 3$ in eq. (49) must vanish,

\[
\sum_{i=1}^{N} \left( \frac{k-1}{z_i} \frac{\partial}{\partial z_i} - \frac{1}{5} z_i^{k-2} (k - 1) \right) \langle \Phi(z_1) \ldots \Phi(z_N) \rangle_{q}^{s=0} = 0, \quad k = 1, 2, 3.
\]

Equivalently, $\langle \Phi(z_1) \ldots \Phi(z_4) \rangle_{q}^{s=0}$ is invariant under translations, dilatations, and special conformal transformations. Up to overall normalisation, this fully determines the
Two fundamental solutions are for the normalisation from eq. (26). For Eqs (49) and (52) together yield a \( N \) for some function \( \lambda_{2p} \in \mathbb{R}^+ \), for \( N = 4 \),

\[
\langle \Phi(z_1) \Phi(z_2) \Phi(z_3) \Phi(z_4) \rangle_{\epsilon} = \left( \frac{z_{12} z_{34}}{z_{14} z_{23}} \right)^{2/5} \epsilon \left( \frac{z_{12} z_{34}}{z_{14} z_{23}} \right)^{2/5} 3_{\epsilon}^{\epsilon-0},
\]

for some function \( f \), which determine as follows: By the OPE (32) for \( T \) and \( \Phi \) about \( z = z_1 \),

\[
\langle \Phi(z_1) \Phi(0) \Phi(1) \Phi(0) \rangle_{\epsilon} = f(z) 3_{\epsilon}^{\epsilon-0}.
\]

(A corresponding ODE holds for the anti-holomorphic coordinate.)

For \( N = 4 \), \((z_1, z_2, z_3) = (z, 0, 1) \) and \( z_4 \to \infty \),

\[
\langle \Phi(z) \Phi(0) \Phi(1) \Phi(0) \rangle_{\epsilon} = f(z) 3_{\epsilon}^{\epsilon-0}.
\]

where \( \Phi \) denotes the field in the inverse coordinate. Eq. (53) is equivalent to a hypergeometric ODE for the function \( f \). Using the Frobenius ansatz \( f(z) = |z(1 - z)|^{4/5} \), the ODE takes the standard form

\[
z(1 - z) \mathcal{F}''(z) + \frac{6}{5} (1 - 2z) \mathcal{F}'(z) - \frac{12}{25} \mathcal{F}(z) = 0.
\]

Two fundamental solutions are \( \mathcal{F}_1(z) = z F_1 \left( \frac{2}{5}, \frac{3}{5}; \frac{4}{5}; z \right) \) and \( \mathcal{F}_2(z) = z^{-1/5} \tilde{z} F_1 \left( \frac{1}{5}, \frac{2}{5}; \frac{6}{5}; \tilde{z} \right) \).

Since \( f \) is real, the full solution reads

\[
\mathcal{F}(z) = m_1 |\mathcal{F}_1(z)|^2 + m_2 |\mathcal{F}_2(z)|^2
\]

for \( m_1, m_2 \in \mathbb{R}^+ \). The condition \( \mathcal{F}(z) = \mathcal{F}(1 - z) \) yields \( m_2/m_1 = \mathcal{C} \), where

\[
\mathcal{C} = -\frac{\Gamma(6/5) \Gamma(1/5) \Gamma(2/5)}{\Gamma(4/5)^3} \approx -3.653116237
\]

is Cardy’s constant [2]. In order to fix the constant \( m_1 \), we apply the \( s \) formalism to \((P^4, \mathcal{G}) = M_L \sqcup \mathcal{M}_R \). Thus by eqs (24) and (54),

\[
f(z) 3_{\epsilon}^{\epsilon-0} = \langle \Phi(z) \Phi(0) \rangle_{M_L \sqcup \mathcal{M}_R} \langle \Phi(1) \Phi(0) \rangle_{D_L \sqcup \mathcal{M}_R} + \ldots
\]

This result provides the first rigorous upper bound on the rate at which LO loops decay in the critical Ising model.
where the summand in the first line corresponds to \( 1 \in \mathcal{R}_2(\mathbb{B}) \) and that in the second to \( \Phi \in \mathcal{R}_2(\mathbb{B}) \). By eq. (50) and eq. (51), the lowest order term in \( z \), for \( z \) close to zero, equals \( \xi^2_2|z|^{4/5} \) in the first and \( \varepsilon_\Phi \lambda_{\Phi\Phi}^2 |z|^{2/5} \) in the second line, respectively. By comparison with

\[
 f(z) = |z(1-z)|^{4/5} m_1 \left( 2 F_1 \left( \frac{4}{5}, \frac{3}{5}, \frac{6}{5}; \frac{2}{5}; z \right) \right)^2 + |z|^{-2/5} \mathcal{C} \left( 2 F_1 \left( \frac{3}{5}, \frac{2}{5}, \frac{4}{5}; \frac{2}{5}; z \right) \right)^2,
\]

the first and the second summand correspond to the fields in \( \mathcal{R}_2(\mathbb{B}) \) and in \( \mathcal{R}_2(\mathbb{B}) \), respectively. In particular, \( m_1 = 1 \) and \( \mathcal{C} = \varepsilon_\Phi \lambda_{\Phi\Phi}^2 \). This implies

\[
 \varepsilon_\Phi = -1.
\] (56)

Note that in Cardy’s paper, \( \Phi \) is an imaginary field and its squared norm is positive.

**Proposition 15.** The positive normalisation factor of \( \langle \Phi(z) \rangle_{|z|}^{-1} \) in eq. (39) equals

\[
 \mu_\Phi = \sqrt{|\mathcal{C}|},
\]

where \( \mathcal{C} \) is Cardy’s constant (55). Moreover,

\[
 \langle \Phi(z_L = 0) \Phi(z = 1) \Phi(z_R = 0) \rangle_{D\bar{D}} = -\sqrt{|\mathcal{C}|}.
\]

**Proof.** By translational symmetry, we may set \( z = 1 \). In the \( q \) formalism we let the flat torus of modulus \( \tau = \log q/(2\pi i) \) for \( 0 < |q| < 1 \) degenerate into the cylinder \((\gamma \times [0, \ell])^\nu\) for \( \ell = -\log |q| \) and \( \theta = \arg(q) \). According to eq. (74) for the single field \( \Phi \) of weight \((h, \bar{h}) = (-1/5, -1/5)\),

\[
 \langle \Phi(z = 1) \rangle_{|z|}^{-1} = -|q|^{-c/12} |g|^{-2/5} \langle \Phi(z_L = 0) \Phi(z_R = 0) \rangle_{D\bar{D}} + \ldots
\]

Note that all contributions from the vacuum sector vanish since \( \langle \Phi(z) \rangle_{|z|}^{-1} = 0 \). In the three-point function, we perform the coordinate change \( z_3 = 1/z_R \). Thus

\[
 \langle \Phi(z_L = 0) \Phi(z = 1) \Phi(z_R = 0) \rangle_{D\bar{D}} = \lim_{z_3 \to \infty} |z_3|^{-4/5} \langle \Phi(z_L = 0) \Phi(z = 1) \Phi(z_3) \rangle_{D\bar{D}} = \lambda_{\Phi\Phi\Phi},
\]

by eqs (51) and (9). Thus for \( c = -22/5 \),

\[
 \langle \Phi(z = 1) \rangle_{|z|}^{-1} = -\lambda_{\Phi\Phi\Phi} |q|^{-1/30} + \ldots
\]

Comparison with eq. (39) yields \( \mu_\Phi = -\lambda_{\Phi\Phi\Phi} \), as required. \( \Box \)

Because of Proposition 15, the partition function \( Z_M \) is computable for \( M \in \mathcal{X} \) of arbitrary genus, in terms of an iterated infinite series.

Let \( \mathcal{A} \) the set of all maps from the cutting scheme (48) into the set \( \{ v, w \} \). Thus an element \( A \in \mathcal{A} \) is an assignment of one of \( v \) and \( w \) to each component in (48). Thus \( A \) specifies for each individual cutting cycle whether a pair of discs endowed with fields \( e_i \in \mathcal{W} \otimes \mathcal{R} \) or \( e_j \in \mathcal{W} \otimes \mathcal{R} \) is inserted. Write

\[
 Z_M = \sum_{A \in \mathcal{A}} Z_{A,M}
\]
for the corresponding decomposition. The tensor product structure of the two sectors implies that
\[ \mathcal{Z}_{A,M} = e_A \mathcal{R}_{A,M} \overline{\mathcal{R}_{A,M}} \]  \tag{57} \]
where \( \mathcal{R}_{A,M} \) is a holomorphic function and \( \overline{\mathcal{R}_{A,M}} \) is its complex conjugate. The factor \( e_A \) is a sign, which is due to \( \langle w | w \rangle = e \Phi \langle w | w \rangle \).

For \( A \in \mathcal{A} \) and \( 0 \leq j \leq 3 \), let \( p_{a_j}(A) \) denote the number of pants with exactly \( j \) boundary cycles to which \( w \) is assigned. If \( p_{a_0}(A) > 0 \) then \( \mathcal{R}_{A,M} = 0 \), since \( \langle \Phi \rangle = 0 \), for every metric. (Also the correlation functions of all other fields in the non-vacuum sector are zero.) Thus we exclude the corresponding assignment from \( \mathcal{A} \).

**Proposition 16.** In eq. (57), for \( A \in \mathcal{A} \),
\[ e_a = e \Phi^{p_{a_0}(A) - cc(A)} \]
where \( cc(A) := \text{number of cutting cycles that are assigned } w \text{ by } A \).

**Proof.** According to Segal’s formulae for the partition function in the \( s \) and the \( q \) formalism, the (nonzero) factor \( \mathcal{R}_{A,M} \) is a product of three-point functions \( \langle \psi, \phi, \phi \rangle \) and inverses of \( \langle \psi, \phi \rangle_{DD} \) for fields \( \psi_i, \phi \in R_2(\mathbb{H}) \). More precisely, for \( m = 0, 2, 3 \), \( p_{a_m}(A) \) three-point functions have \( m \) fields in \( R_2(\mathbb{H}) \), and \( 3 - m \) fields in \( R_2(\mathbb{H}) \). \( cc(A) \) factors \( \langle \psi, \phi \rangle_{DD} \) correspond to \( \psi_i \in R_2(\mathbb{H}) \). The remaining \((3g - 3 - cc(A))\) factors correspond to \( \psi_i \in R_2(\mathbb{H}) \). Thus we have a decomposition. Since \( p_{a_1}(A) = 0 \) and \( p_{a_0}(A) \) is even by construction. Thus the numerator contributes \( e_{p_{a_0}(A)} \), while the denominator contributes \( e^{-cc(A)} \), as required.

Different choices of cutting cycles induce unitary transformations of \((\mathcal{R}_{A,M})_{A \in \mathcal{A}}\). A convenient notation for \( \mathcal{R}_{A,M} \) is \((\mathcal{R}_{A,M})_{A \in \mathcal{A}}(\Phi) \) (in accordance with \( \mathcal{Z}_M \) and \( \Phi^\phi \)), and it is called a holomorphic (or conformal) block (associated to \( A \in \mathcal{A} \) on \( M \). We also use the word zero-point function.

The cardinality \( |\mathcal{A}| \) of \( \mathcal{A} \) equals one for \( g = 0 \), two for \( g = 1 \), and \( |\mathcal{A}| = 2^{3g-3} \) for \( g \geq 2 \). For hyperelliptic \( M = X \), \( |\mathcal{A}| \) is a Fibonacci number \([12]\). Here is the example we will use later:

**Proposition 17.** In the \((2, 5)\) minimal model for genus two, \( |\mathcal{A}| = 5 \).

**Proof.** The cutting scheme for \( g = 2 \) results in two pants. When \( A \in \mathcal{A} \) assigns to \( \gamma^0 \) the value \( w \), then \( p_{a_0}(A), p_{a_2}(A) \in \{0, 1, 2\} \) and \( p_{a_0}(A) + p_{a_2}(A) = 2 \). In the case where \( p_{a_0}(A) = 1 = p_{a_2}(A) \), there is a choice for which of the pants has no \( w \) assignment, while the other has two. When the value \( w \) is assigned to \( \gamma^0 \), only \( p_{a_3}(A) \) does not vanish, so \( p_{a_3}(A) = 2 \). (The nonzero elements are listed in Table 2.)

\( N \)-point functions can be treated in the same way. One can use additional circles with \( s \) expansions to locate each field insertion in a separate component with two boundary circles only.

Since \( T \) commutes with anti-holomorphic derivatives, application of \( T(z) \) to the partition function in eq. (57) yields an expression of the form
\[ T(z) \mathcal{Z}_M = \sum_{A \in \mathcal{A}} \langle T(z) \rangle_{A,M} e_A \mathcal{R}_{A,M} \].

26
For $A \in \mathfrak{g}$, this defines the corresponding 1-point function $\langle T(z) \rangle_{A,M}$ on $M$. Successive application of $T(z_1) \ldots T(z_N)$ to $\mathfrak{g}_M$ gives rise to Virasoro $N$-point (or correlation) functions $\langle T(z_1) \ldots T(z_N) \rangle_{A,M}$ provided that $z_i \neq z_j$ for $1 \leq i < j \leq N$. The index $A$ will be omitted whenever possible.

4.2 Degenerating hyperelliptic Riemann surfaces

For $k \geq 1$ and for $0 \leq g \leq k - 1$, let $(\mathbb{P}^1_0, P_1, P_2, \ldots, P_{2g}, P_{2k})$ be the Riemann sphere marked with $2g + 2$ distinct points $P_1, P_2, \ldots, P_{2g}, P_{2k}$, and let

$$\Sigma_g = \Sigma(P_1, P_2, \ldots, P_{2g}, P_{2k}) : \quad \gamma^2 = p(s)$$

be its double cover, which is ramified precisely at these $2g + 2$ marked points, namely the roots of the polynomial $p(s)$. A metric $\mathcal{G}_g$ on $\Sigma_g$ will be specified later. For the purpose of this paper, $k = 3$ and $g \in \{0, 1, 2\}$. We shall give a perturbative expansion around the complex structure at which $\Sigma(P_1, \ldots, P_k)$ degenerates. This is achieved by pinching a suitably chosen cycle $\gamma \subset \Sigma_2$, or by letting $m$ ramification points run together, where $m = 2$ or $m = 3$. The parameter describing the degeneration can be introduced through the following procedure:

1. Cut along the cycle $\gamma \subset \Sigma_2$ such that $\gamma$ is the inverse image of the curve with the equation $|x| = \text{const.}$ on $\mathbb{P}^1_0$ that encloses precisely $m$ ramification points $P_1, \ldots, P_m$.

2. Replace $x(P_i)$ for $i \leq m$ by $s x(P_i)$ for $0 < |s| < 1$. This is equivalent to cutting along $\gamma$ and inserting a cylinder with parameter $\log s = i \theta - \ell$.

The perturbative expansion is a power series in $s$.

When $\gamma$ is homologous trivial (though nontrivial in the fundamental group), which is the case discussed in Section 4.3, $m$ equals 3, and the cutting results in two separate tori with a disc centred at $\infty$ and 0, respectively, removed. When $\gamma$ is non-homologous to zero, which is the case addressed in Section 4.4, $m = 2$, and the cutting results in a single torus with two discs removed and a cylinder. To distinguish the two cases we shall refer to the first and second case as the $q$ and the $s$ formalism, respectively, though both refer to the same geometric quantities $\ell, \theta$. In the case of the $q$-formalism, the perturbation parameter will be named accordingly.

4.3 The $s$ formalism in genus $g = 2$

Let $(\Sigma_1, P)$ and $(\hat{\Sigma}_1, \hat{P})$ be two flat tori with a single puncture and with holomorphic one-forms $dz$ and $d\hat{z}$, normalised by period one along some cycle. Let $\mathcal{D}_L$ and $\mathcal{D}_R$ be the flat unit disc centred at $P$ and $\hat{P}$, respectively, and containing no other ramification point. Let $z$ and $\hat{z}$ be a corresponding pair of coordinates with $z(P) = 0$ and $\hat{z}(\hat{P}) = 0$. By eq. (5), we may w.l.o.g. assume that the two discs lie inside the respective fundamental cell of the two tori. If $M_L^{(1)} \subset \Sigma_1$ and $\hat{M}_R^{(2)} \subset \hat{\Sigma}_1$ are the left and right manifolds for $\gamma_1$ and $\gamma_2$, respectively, the connected sum

$$M_2(s) := M_L^{(1)} \cup_{\gamma_1} (\gamma \times [0, \ell])^0 \cup_{\gamma_2} \hat{M}_R^{(2)}$$ (58)
along $\gamma_1$ and $\gamma_2$ defines an element of genus two in $\mathcal{X}$ (with the induced metric). Here $s = \exp(i\theta - \ell)$. We continue the coordinates $z$ and $\hat{z}$ to the cylinder so that

$$z\hat{z} = s.$$  \hfill (59)

Since the sewing is compatible with the involutions $z \mapsto -z$, $\hat{z} \mapsto -\hat{z}$, the corresponding $\mathbb{P}^1_C$ projections of the two tori are sewn and form a new $\mathbb{P}^1_C$. On this space, we obtain a pair of almost global coordinates:

**Proposition 18.** Let $\Sigma_1, \hat{\Sigma}_1$ be flat tori of modulus $\tau, \hat{\tau} \in \mathfrak{S}_1$ with a single puncture, and let $z$ and $\hat{z}$ be a corresponding pair of local analytic coordinates vanishing at the respective punctures and satisfying the sewing condition (59) as stated above. There exists a triple $(X, \hat{X}, \xi)$ satisfying the following properties:

$X$ and $\hat{X}$ define a pair of coordinates on $\mathbb{P}^1_C$, such that $X$ is defined on $\mathbb{P}^1_C \setminus \{\infty\}$, $\hat{X}$ is defined on $\mathbb{P}^1_C \setminus \{0\}$ and

$$X\hat{X} = \xi/s^2.$$  \hfill (60)

holds on $\mathbb{P}^1_C \setminus \{\infty, 0\}$. Here $\xi \in \mathbb{Q}[G_4, \hat{G}_4, G_6, \hat{G}_6][[s^2]]$ has an expansion for $0 < |s| < 1$ which starts as follows:

$$\frac{\xi}{s^2} = \frac{1}{s^2} - \frac{1}{144} s^2 \hat{G}_4 \hat{G}_4 - \frac{320}{12} s^4 G_6 \hat{G}_6 + 18720 G_6^2 \hat{G}_6^2 s^6 + \frac{1000240}{121} G_4 G_6 \hat{G}_4 \hat{G}_6 s^8 + \mathcal{O}(s^{10}).$$

Moreover, for $x = \varphi(z \mid \tau)$ and $\hat{x} = \varphi(\hat{z} \mid \hat{\tau})$, the terms of low order in $s$ read

$$X(x) = x \left(1 - \frac{6}{7} \hat{G}_4 s^3 x^2 - \frac{10}{9} \hat{G}_6 s^6 x (x^2 - 30 G_4) + \frac{24}{11} G_4^2 s^8 x (x^3 + 9 G_4 x - 25 G_6) + \frac{60}{11} \hat{G}_4 \hat{G}_6 s^{10} x (19 x^4 - 402 G_4 x^2 - 200 G_6 x - 3060 G_4^2) - \frac{216}{143} \hat{G}_4^3 s^{12} x (55 s^3 - 66 G_4 x^3 - 4950 G_6 x^2 + 33308 G_6^2 x + 71100 G_4 G_6) - \frac{600}{143} \hat{G}_6^2 s^{12} x (-22 x^5 + 913 G_4 x^3 - 165 G_6 x^2 - 9834 G_6^2 x + 19790 G_4 G_6) + O(s^{14}) \right),$$

resp. $\hat{X}(\hat{x})$ is obtained by replacing, in the above formula, $x$ by $\hat{x}$ and by exchanging $G_{2k} \leftrightarrow \hat{G}_{2k}$ at every occurrence. Here and in the following, $G_{2k} = G_{2k}(\tau)$ and $\hat{G}_{2k} = G_{2k}(\hat{\tau})$ and likewise for other modular forms.

**Proof.** In a tubular neighbourhood of $\{|z| = |\hat{z}|\}$, which in this proof we refer to as the annulus, define a function $f_s$ by

$$f_s(x) := \log x + \log \hat{x}.$$  

By eq. (59) for $0 < s < 1$,

$$f_s(x) = \log \left( \frac{z\hat{z}}{s} \right)^2 x \hat{x} = -2 \log s + \log z^2 x + \log \hat{z}^2 \hat{x}.$$  

28
\[
\log z^2 x \text{ and } \log \hat{z}^2 \hat{x} \text{ are individually holomorphic in } s. \text{ Indeed, we have an expansion }
\]

\[
\log z^2 x = a_1 z^4 + a_2 z^6 + \left(- \frac{a_1^2}{2} + a_3\right) z^8 + (-a_1 a_2 + a_4) z^{10} + O(z^{12}),
\]

where for \( m \geq 1, a_m \in \mathbb{Q}[G_{2m+2}] \), more specifically,

\[
a_m(x) = 2(2m + 1) G_{2m+2}(x).
\]

According to eq. (59), for \( n \geq 1 \), we have \( z^{2n} = x^{2n} / z^{2n} \), where

\[
\frac{1}{z^{2n}} = \frac{1}{(2n - 1)!} \left( \frac{d^{2n-2}}{dz^{2n-2}} z^{2k} \right).
\]

Here \( \hat{a}_m = a_m(\hat{x}) \), and \( \frac{d^{2n-2}}{dz^{2n-2}} z^{2k} \) denotes the \( (2n - 2) \text{nd} \) derivative w.r.t. \( z \). For \( n \geq 1 \), it is a polynomial of degree \( n \) in \( x \). For \( n = 1 \), it equals \( \hat{x} \), and the polynomials for higher values of \( n \) are listed e.g. in [1, p. 640] for \( n \leq 8 \). As a result, \( \log x = O(s^4) \), where the coefficient of \( s^4 \) equals

\[
a_1 \hat{x}^2 + a_1 \hat{a}_1 / 3 - 10 a_1 \hat{G}_4 + O(\hat{x}^2).
\]

Using the corresponding expansion for \( \hat{z}^{2m} = \hat{x}^{2m} / \hat{z}^{2m} \) clears the coefficient of its \( \hat{x} \) dependence. Every such replacement introduces another positive power of \( s \) and the alternating \( k \geq 2 \) fold application yields an expansion such that the coefficient of \( x^{2n} \) for \( n \leq k \) is a linear combination of a polynomial in \( x \) and one in \( \hat{x} \), respectively. Thus we have a unique splitting

\[
\log (\hat{z}^2 x) + \log (\hat{z}^2 \ hat{x}) = A(x) + B(\hat{x}) + C \tag{61}
\]

where \( A, B \) are expansions of the form

\[
A(x) = \sum_{n=1}^{\infty} A_n x^n, \quad B(\hat{x}) = \sum_{n=1}^{\infty} B_n \hat{x}^n,
\]

respectively, and \( C / s^4 = 2 a_1 \hat{a}_1 / 3 - 10 (\hat{a}_1 \hat{G}_4 + a_1 \hat{G}_4) + O(s^2) \). For \( n \geq 1, A_n \) and \( B_n \) are at least of order \( O(s^{3n}) \), by construction. \( \log (\hat{z}^2 x) + \log (\hat{z}^2 \hat{x}) \) is holomorphic in both \( x \) and \( \hat{x} \) on the annulus, so the series converge. Moreover, \( A, B \) have an analytic continuation to the outside of the annulus. Indeed, \( A \) can be extended to small values of \( |x| \) and thus to some open neighbourhood \( U \) of \( |x| = 0 \) containing the annulus, while \( B \) can be extended to some open neighbourhood \( \hat{U} \) of \( |\hat{x}| = 0 \). In the notations of eq. (61), we define for \( x, \hat{x} \in \mathbb{C}, \)

\[
\log X := \log x - A(x), \quad \log \hat{X} := \log \hat{x} - B(\hat{x}).
\]

\( X \) and \( \hat{X} \) define coordinates on \( \mathbb{P}^1 \setminus \{\infty\} \) and on \( \mathbb{P}^1 \setminus \{0\} \), respectively, which for \( \xi = e^C \) satisfy

\[
X \hat{X} = x \hat{x} e^{-(A + B)} = x \hat{x} e^{-f(x) - 2 \log x} = \xi / x^2,
\]

as required. \( \square \)

The triple \((r, \hat{r}, s)\) of the moduli of the individual tori \( \Sigma \) and \( \hat{\Sigma} \) and the local sewing parameter is related to the period matrix as follows:

29
Proposition 19. The period matrix of $M_2(s)$ for the sewing condition (59) equals

$$\Omega(\tau, \hat{\tau}, s) = \begin{pmatrix} \tau & 0 \\ 0 & \hat{\tau} \end{pmatrix} + 2\pi is \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 4\pi is^2 \begin{pmatrix} \hat{G}_2 & 0 \\ 0 & \hat{G}_2 \end{pmatrix} + O(s^3).$$

Here $G_2$ is the quasi-modular form of weight 2.

The result is consistent with [14].

Proof. For $j = 2, 3, 4$, let $\theta_j = \theta_j(0; \tau)$ and $\hat{\theta}_j = \theta_j(0; \hat{\tau})$ be the Jacobi theta constants. Set

$$x_1 = \frac{\pi^2}{3} (\theta_2^4 + \theta_4^4), \quad x_2 = \frac{\pi^2}{3} (\theta_2^4 - \theta_4^4), \quad x_3 = -\frac{\pi^2}{3} (\theta_2^4 + \theta_4^4),$$

and define $\hat{x}_k$ for $k = 1, 2, 3$ as a function of $\hat{\tau}$ accordingly. Thus $x_1, x_2, x_3$ and $\hat{x}_1, \hat{x}_2, \hat{x}_3$ are the ramification points of $\Sigma$ and of $\hat{\Sigma}$, respectively [1]. For the pair of coordinates $X, \hat{X}$ from Proposition 18, set

$$X_k = X(x_k) \quad \hat{X}_k = \hat{X}(\hat{x}_k), \quad k = 1, 2, 3.$$ 

The linear fractional transformation $x \mapsto \frac{x - x_k}{1 - \frac{x_k}{x_{k+1}} (\frac{\theta_k}{\theta_2})^4}$ maps $(x_1, x_2, x_3)$ to $(\infty, 0, 1)$, and it induces for $k = 4, 5, 6$ the association

$$X_k \mapsto \left( \frac{\theta_3}{\theta_2}(\tau) \right)^4 \left( 1 + \pi^2 s^2 \theta_4^4 \hat{X}_k + O(s^4) \right),$$

by eq. (60) and since $X_1 - X_2 = \pi^2 \theta_4^4 + O(s^2)$. On the other hand, set $\Omega(\tau, \hat{\tau}, s) = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}$. Let $X_{1, \Omega} = \infty, X_{2, \Omega} = 0, X_{3, \Omega} = 1$ and

$$X_{4, \Omega} = \Theta_{3,2}(\Omega), \quad X_{5, \Omega} = \Theta_{3,4}(\Omega), \quad X_{6, \Omega} = \Theta_{3,6}(\Omega),$$

where for $k, \ell = 2, 3, 4$,

$$\Theta_{k,\ell}(\Omega) = \left( \frac{\theta_k}{\theta_2}(\Omega_{11}) \right)^4 \left( 1 + 4\pi^2 d \log \frac{\theta_1}{\theta_2}(\Omega_{11}) d \log \left( \theta_k \theta_\ell \right) (\Omega_{22}) + O(v^4) \right).$$

is the Riemann theta function with rational characteristics. We use the identities

$$\left( \log \frac{\theta_1}{\theta_2} \right)' = -\frac{\pi i}{4} \theta_4^4,$$

where the dash indicates differentiation w.r.t. the modulus,

$$\left( \theta_k \theta_\ell \right)' = 2\pi i \left( \mathcal{D} \log (\theta_k \theta_\ell) + \frac{G_2}{2\pi^2} \right)$$

where $\mathcal{D} \log (\theta_k \theta_\ell) = \mathcal{D} \log \theta_k + \mathcal{D} \log \theta_\ell$ with

$$\mathcal{D} \log \theta_k = \frac{1}{24} (\theta_4^4 + \theta_2^4), \quad \mathcal{D} \log \theta_1 = \frac{1}{24} (\theta_4^4 - \theta_2^4), \quad \mathcal{D} \log \theta_3 = \frac{1}{24} (\theta_4^4 + \theta_2^4).$$

By eqs (62), this yields for $k = 1, 2, 3$,

$$X_{7-k, \Omega} = \left( \frac{\theta_3}{\theta_2}(\Omega_{11}) \right)^4 \left( 1 + \nu^2 \theta_4^4 (\Omega_{11}) (G_2(\Omega_{22}) - \frac{1}{4} \hat{x}_k(\Omega_{22})) + O(v^4) \right).$$

30
where $\hat{\gamma} (\Omega_{22})$ is defined as a function of $\Omega_{22}$ by eqs (62). Since the $X_{\Omega, \Omega}$ are the ramification points of $\Sigma_2$ [18, p. 56, eq. 91], the set $\{X_{\Omega, \Omega}\}_{\Omega = 4, 5, 6}$ given by eq. (65) must equal the image of the association (63). By the fact that $\sum_k \hat{\gamma}_k = 0$, it follows that

$$
\left( \frac{\theta_1}{\theta_2} (\tau) \right)^4 = \left( \frac{\theta_1}{\theta_2} (\Omega_{11}) \right)^4 \left( 1 + v^2 \theta_1^2 (\Omega_{11}) G_2 (\Omega_{22}) + O(v^3) \right).
$$

(66)

Since $\theta_1/\theta_2$ is invertible, we find in particular that $\Omega_{11} - \tau = O(v^2)$. Subtracting eq. (66) from eq. (65) yields

$$
X_{7-L, \Omega} - \left( \frac{\theta_1}{\theta_2} (\tau) \right)^4 = \left( \frac{\theta_1}{\theta_2} (\Omega_{11}) \right)^3 \left( 1 + v^2 \theta_1^2 \hat{\gamma}_k + O(v^3) \right),
$$

and we conclude that

$$
v^2 = (2\pi i)^2 \bar{s}^2 + O(s^3).
$$

(67)

By eqs (64), (66) and (67),

$$
\Omega_{11} = \tau + 4\pi i s \hat{\gamma}_2 + O(s^3).
$$

This yields the expansion of the period matrix to the required order. □

By eq. (58), $M_2(s)$ carries an induced metric given by

$$
\mathcal{g}_{LN} (s) = \begin{cases}
\frac{|dz|^2}{1 - |z|}, & 1 \leq |z|,
\frac{|d \log z|^2}{\sqrt{|z|}} & |z| \leq 1,
\frac{|d \log \bar{z}|^2}{\sqrt{|z|}} & |\bar{z}| \leq 1,
\frac{|d \bar{z}|^2}{1 - |\bar{z}|} & |\bar{z}|.
\end{cases}
$$

(68)

where $z, \bar{z}$ lie in the fundamental cell of the respective torus and the pair satisfies eq. (59).

This metric is continuous and the curvature is supported on the two circles $\gamma_1$ and $\gamma_2$. Mark the latter by one point each and let

$$
M_1 = M_1^{(1)} \cup_{\gamma_1} D_R^{(1)},
$$

$$
\hat{M}_1 (s) = D_R^{(1)} \cup_{\gamma_1} (\gamma \times [0, \ell])^s \cup_{\gamma_2} \hat{M}_R^{(2)}
$$

(69)

be the corresponding elements in $\mathcal{X}$ of genus one. For $s = 1$, we drop the argument and write $\hat{M}_1$.

By Proposition 17, in the (2, 5) minimal model on $M_2(s), |\mathfrak{M}| = 5$. As illustrated by Table 2, it is convenient to use the description

$$
\mathfrak{M} = \{(a, b), \Phi | a, b \in \{1, 2\}\}
$$

(70)

where $a = 1$ and $a = 2$ stand for the non-vacuum and the vacuum sector, respectively, in accordance with eqs (33).

Segal’s formula (24) together with eq. (11) yield the zero-point function associated to $(a, b) \in \mathfrak{M}$

$$
\langle 1 \rangle_{(a, b), M_2(s)} = \sum_n \frac{\langle \psi_n (z_L = 0) \rangle_{a, M_1, \hat{M}_1(s)} \langle \psi_n (z_L = 0) \rangle_{b, \hat{M}_1(s)}}{\langle \psi_n (z_L = 0) \rangle_{DDO}},
$$

(71)
Table 2: Holomorphic blocks in the (2,5) minimal model on \(M_2(s)\): We write \(q = \exp(2\pi ir)\) and \(\bar{q} = \exp(2\pi i\bar{r})\). Using the \(q\) formalism, \(<1(\tau)\rangle_{1}^{s=1} = <\Phi \Phi>^{s=0} + \ldots\) in the non-vacuum resp. \(<1(\tau)\rangle_{2}^{s=1} = <1>^{s=0} + \ldots\) in the vacuum sector on the flat torus. An additional holomorphic block is given by \(<\Phi(\tau)>_{\Phi}^{s=1} = <\Phi \Phi \Phi>^{s=0} + \ldots\)

| \(A(y', \quad y', \quad \hat{y}')\) | resp. lowest weight terms in \(g = 0\) | \(<1(\tau, \hat{\tau})>_{A}^{s=2}\) | \(\epsilon_{A}\) |
|---|---|---|---|
| \(u, \quad u, \quad v\) | \(<1>^{s=0}, <1>^{s=0}\rangle_{1}\rangle_{2}\) | \(<1(\tau, \hat{\tau})>_{1}^{s=2}\rangle_{2}\rangle_{2}\) | 1 |
| \(u, \quad u, \quad w\) | \(<1>^{s=0}, <\Phi \Phi>^{s=0}\rangle_{1}\rangle_{2}\) | \(<1(\tau, \hat{\tau})>_{1}^{s=2}\rangle_{2}\rangle_{2}\) | 1 |
| \(u, \quad w, \quad v\) | \(<1>^{s=0}, <\Phi \Phi>^{s=0}\rangle_{1}\rangle_{2}\) | \(<1(\tau, \hat{\tau})>_{1}^{s=2}\rangle_{1}\rangle_{2}\) | 1 |
| \(u, \quad w, \quad w\) | \(<\Phi \Phi \Phi>^{s=0}, <\Phi \Phi>^{s=0}\rangle_{1}\rangle_{2}\) | \(<1(\tau, \hat{\tau})>_{1}^{s=2}\rangle_{1}\rangle_{2}\) | 1 |
| \(w, \quad w, \quad w\) | \(<\Phi \Phi \Phi>^{s=0}, <\Phi \Phi \Phi>^{s=0}\rangle_{1}\rangle_{2}\) | \(<1(\tau, \hat{\tau})>_{1}^{s=2}\rangle_{1}\rangle_{1}\) | 1 |

where \(\{\psi_{a}\}\) be a standard orthogonal basis of \(R_{2}(\mathfrak{B})\). (We indiscriminately write \(\psi\) for the field in either coordinate.) By eqs (21) and (11),

\[
<\psi(\zeta_{L} = 0)\rangle_{a, M_{1}(s)} = s^{h_{a} - 1/24} \langle e_{i} | \hat{\Phi}_{R}^{(2)} \rangle_{a} , \tag{72}
\]

where \(\langle e_{i} | \hat{\Phi}_{R}^{(2)} \rangle_{a} = \hat{\psi}(\zeta = 0)\rangle_{a, M_{1}}\). Since all one-point functions in the analytic coordinate on \(M_{1}\) and \(\hat{M}_{1}\) are constant in position, we write (for \(z_{h} = z\))

\[
<\phi(z)\rangle_{a, M_{1}} = <\psi_{a} \rangle , \quad <\hat{\psi}(\zeta)\rangle_{a, M_{1}} = <\hat{\psi}_{a} \rangle .
\]

By considering \(s = -1\), we see that \(<\hat{\psi}_{a} \rangle = 0\) for \(\psi \in \mathfrak{B}\) for which \(h\) is odd. Moreover, only the quasi-primary fields contribute to the sum in eq. (71).

**Proposition 20.** Let \(M_{1}\) and \(\hat{M}_{1}\) be the genus one Riemann surfaces defined by eqs (69). Let \(M_{2}(s)\) be the genus two Riemann surface from eq. (58). Let \(<1>_{a} = <1>_{a, M_{1}}\) and \(<1>_{b} = <1>_{b, M_{b}}\) be the holomorphic blocks of the (2,5) minimal model from eqs (33). Set \(t_{a} := d \log <1>_{a} / d(2\pi ir)\) and \(t_{b} := d \log <1>_{b} / d(2\pi i\bar{r})\) accordingly. Associated to the set \(\mathfrak{P}\) from eq. (70) are the five holomorphic blocks

\[
<1(\tau, \hat{\tau})>_{a,b}^{s=2} = s^{1/60} \langle 1 \rangle_{a,b}^{s=2} \Xi_{a,b}^{s=2} , \quad a, b = 1, 2,
\]

and

\[
<1(\tau, \hat{\tau})>_{\Phi}^{s=2} = s^{-1/60} \epsilon_{\Phi} \mu_{\Phi} (\bar{\eta} \eta)^{-2/5} \Xi_{\Phi}^{s=2} , \tag{73}
\]

where \(\Xi_{a,b}^{s=2}\) has an expansion in modular forms as follows:

\[
\Xi_{a,b}^{s=2} = \sum_{n \geq 0} \Xi_{a,b,n}^{s=2} s^{2n} ,
\]

where for \(n \geq 0\), \(\Xi_{a,b,n}^{s=2} \in Q[G_{4}, G_{6}, \hat{G}_{4}, \hat{G}_{6}, t_{a}, t_{b}]\) is a modular form of weight \(2n\) both in \(\tau\) and in \(\hat{\tau}\). It is linear in \(t_{a}\) and in \(t_{b}\) for \(A = (a, b)\), and of degree zero for \(A = \Phi\). Exchanging \(\tau \leftrightarrow \hat{\tau}\), maps the matrix \(\Xi_{(a,b),n}^{s=2}\) to its transpose, while \(\Xi_{\Phi,n}^{s=2}\) is left invariant.
Proof. The product structure and related symmetry between $\tau$ and $\hat{\tau}$ follow from Segal’s formula (71). We address the coefficients $\Xi_{(a,b)}^{g=2}$. For $a = 1, 2$, the correct Laurent coefficient of the Virasoro $N$-point function $\langle T(z_1) \ldots T(z_N) \rangle_z$, or of one of its derivatives, is of the form $\langle L_{n_1} \ldots L_{n_k} \Phi(z) \rangle_z$, thus, by Proposition 11, it defines an element in the polynomial ring $\mathbb{Q}[G_4, G_6, \bar{t}_0]$, which is linear in $\bar{t}_0$. Moreover, as mentioned previously, only one-point functions associated to even values of $h$ can be nonzero. We address $\Phi$. By Proposition 12, any $\langle L_{n_1} \ldots L_{n_k} \rangle$ is modular or a weakly holomorphic modular form. Only powers of $s$ of the form $h' - 1/5$ occur with $h'$ even. For $N \geq 0$, while being constant in position,

$$\langle L_{n_1} \ldots L_{n_k} \Phi(-z) \rangle = (-1)^{h-h'} \langle L_{n_1} \ldots L_{n_k} \Phi(z) \rangle,$$

since $i(L_0 - \bar{L}_0)$ generates rotations. Here $h = \sum_{i=1}^{N} n_i - 1/5$ and $h' = -1/5$. Thus

$$\langle L_{n_1} \ldots L_{n_k} \Phi(z) \rangle = 0 \quad \text{for } h' = h + 1/5 \text{ odd}$$

Restricting to the holomorphic part $\Phi$ shows that, apart from a factor of $s^{-1/5}$, only even powers of $s$ occur.

The actual expansions of $\Xi_{(a,b)}^{g=2}$ and of $\Xi_{(a,b)}^{g=2}$ are obtained from eq. (71) by direct computation and use of the list of quasi-primary fields from Proposition 4. For $a, b \in \{1, 2\}$,

$$\langle 1 \rangle^{(s,\tau,\hat{\tau})}_{(a,b)} = s^{11/60} \langle 1 \rangle^{(s,\tau,\hat{\tau})}_{(1,1)} + s^2 \langle T \rangle_a \langle \hat{T} \rangle_b + s^3 \frac{49}{2} \langle L_4 \rangle_{a} \langle L_4 \rangle_{b} + \ldots.$$  

Recall from Section 3.1 that $L_m \Phi$ for $m \geq 3$ is a derivative field. In the following, we drop the index $a, b \in \{1, 2\}$. By eq. (19),

$$\langle T(z) T^{(k)}(0) \rangle = \sum_{n \geq 2} e^{n-2} \langle L_n L_{k+2} \rangle.$$  

Comparison with eq. (47) yields: For $h = 6$,

$$\langle L_4 \rangle = 12 G_4 \langle T \rangle - 44 G_6 \langle 1 \rangle$$

for $h = 8$,

$$\langle L_5 \rangle = -4 \langle L_6 \rangle$$

$$\langle L_6 \rangle = 20 G_6 \langle T \rangle - 132 G_4 \langle 1 \rangle,$$

for $h = 10$,

$$\langle L_6 \rangle = 15 \langle L_8 \rangle$$

$$\langle L_7 \rangle = -6 \langle L_9 \rangle$$

$$\langle L_8 \rangle = 24 G_4 \langle T \rangle - 336 G_6 \langle 1 \rangle.$$

33
In order to compute terms of order $h \geq 12$, $N$-point functions for $N \geq 3$ must be taken into account. Firstly, we have for $h = 12$,

\[
\langle L_2 L_4 \rangle = -2 \langle L_9 L_4 \rangle \\
\langle L_4 L_4 \rangle = -\frac{133056}{13} G_4^2(1) + \frac{10080}{11} G_4 G_6(T) - \frac{92400}{13} G_6^2 \\
\langle L_9 L_3 \rangle = -8 \langle L_{10} L_2 \rangle \\
\langle L_{10} L_2 \rangle = -\frac{4752}{13} G_4^3 + \frac{360}{11} G_4 G_6(T) - \frac{3300}{13} G_6^2(1).
\]

Moreover, by eq. (19),

\[
\langle T(z_2)T(z_1) T(0) \rangle = \sum_{n_2 \geq 2, n_1 \geq 2} z_2^{n_2} z_1^{n_1} - 2 \langle L_{n_1} L_{n_1} L_{2} \rangle.
\]

Sorting out the coefficient of $z_2^{n_2} z_1^{n_1} z_0^0$ in eq. (44) with the correct modular coefficients from Example 14 yields

\[
\langle L_4 L_4 L_2 \rangle = -\frac{95040}{13} G_4^3 - \frac{7200}{11} G_4 G_6(T) - \frac{100320}{13} G_6^2(1).
\]

This yields an expansion of $\Xi^g=2_{(a,b)}$ up to terms of order $O(s^{14})$.

We address the fifth holomorphic block. In analogy to eq. (71), using Proposition 5,

\[
\langle 1 \rangle(s, r, t)^{g-2}_{\Phi} = s^{-1/60} \left( \langle w \rangle^{1/2}_{\Phi} + s^2 \frac{\langle 25L_3L_1 - 2L_4 \rangle w \langle (25L_3L_1 - 2L_4) w \rangle}{\| 25L_3L_1 - 2L_4 \| w^2} + \ldots \right).
\]

We have used eq. (39) for the one-point function of $\Phi$ and the factorisation given there. Put $h = h' - 1/5$ with $h' \in \mathbb{N}_0$. We have $\langle T(z) \Phi(0) \rangle = \sum_n z^{n-2} \langle L_n \Phi(0) \rangle$, so by eq. (37),

\[
\begin{align*}
\langle L_4 w \rangle &= -\frac{6}{5} G_4 \langle w \rangle, \\
\langle L_6 w \rangle &= -2 G_6 \langle w \rangle, \\
\langle L_8 w \rangle &= -\frac{12}{5} G_4^2 \langle w \rangle, \\
\langle L_{10} w \rangle &= -\frac{36}{11} G_4 G_6 \langle w \rangle, \\
\langle L_{12} w \rangle &= -\frac{4}{65} (36 G_4^3 + 25 G_6^2) \langle w \rangle.
\end{align*}
\]

We have $\langle T(z_2)T(z_1) \Phi(0) \rangle = \sum_{n} z_2^{n_2} z_1^{n_1} - 2 \langle L_{n_1} L_{n_1} \Phi(0) \rangle$, so by eq. (40), for $h' = 4$,

\[
\langle L_3 L_1 w \rangle = \frac{12}{5} G_4 \langle w \rangle,
\]

for $h' = 6$,

\[
\begin{align*}
\langle L_4 L_2 w \rangle &= -60 G_6 \langle w \rangle, \\
\langle L_5 L_1 w \rangle &= 8 G_6 \langle w \rangle.
\end{align*}
\]

34
for $h' = 8$,

$$\langle L_5 L_1 \rangle = 600 G_3^2 \langle \omega \rangle,$$
$$\langle L_6 L_2 \rangle = -180 G_3^2 \langle \omega \rangle,$$
$$\langle L_7 L_1 \rangle = \frac{72}{5} G_3^2 \langle \omega \rangle.$$

For $h' = 10$,

$$\langle L_6 L_4 \rangle = -\frac{299868}{55} G_4 G_6 \langle \omega \rangle,$$
$$\langle L_7 L_3 \rangle = \frac{25200}{11} G_4 G_6 \langle \omega \rangle,$$
$$\langle L_6 L_2 \rangle = -\frac{5040}{11} G_4 G_6 \langle \omega \rangle,$$
$$\langle L_9 L_1 \rangle = \frac{288}{11} G_4 G_6 \langle \omega \rangle.$$

In order to compute terms of order $h' \geq 12$ in the non-vacuum sector, correlation functions with $N \geq 3$ copies of $T$ and one copy of $\Phi$ must be taken into account. These are provided by Proposition 12. Thus for $h' = 12$,

$$\langle L_7 L_4 \rangle = \frac{1407888}{65} G_4^2 \langle \omega \rangle + \frac{193200}{13} G_4^2 \langle \omega \rangle,$$
$$\langle L_8 L_4 \rangle = -\frac{3586104}{325} G_4^2 \langle \omega \rangle - \frac{99120}{13} G_4^2 \langle \omega \rangle,$$
$$\langle L_8 L_3 \rangle = \frac{43200}{13} G_4^2 \langle \omega \rangle + \frac{30000}{13} G_4^2 \langle \omega \rangle,$$
$$\langle L_{10} L_2 \rangle = -\frac{6480}{13} G_4^2 \langle \omega \rangle - \frac{4500}{13} G_4^2 \langle \omega \rangle,$$
$$\langle L_{11} L_1 \rangle = \frac{288}{13} G_4^2 \langle \omega \rangle + \frac{200}{13} G_4^2 \langle \omega \rangle,$$

and

$$\langle L_6 L_4 \rangle = -\frac{18551736}{325} G_4^2 \langle \omega \rangle - \frac{541040}{13} G_4^2 \langle \omega \rangle,$$
$$\langle L_7 L_4 \rangle = -\frac{844560}{13} G_4^2 \langle \omega \rangle - \frac{579480}{13} G_4^2 \langle \omega \rangle,$$
$$\langle L_9 L_3 \rangle = \frac{7177968}{325} G_4^2 \langle \omega \rangle + \frac{198400}{13} G_4^2 \langle \omega \rangle.$$

Together with the first of eqs (27), this yields the expansion for $\Xi^{g-2}_0$ up to terms of
order $O(s^4)$. More specifically, we have $\Xi^{g=2}_{(a,b)} = \sum_{n \geq 0} \Xi^{g=2}_{(a,b),n} s^{2n}$, where

$$
\begin{align*}
\Xi^{g=2}_{(a,b),0} &= 1 \\
\Xi^{g=2}_{(a,b),1} &= -\frac{5}{11} t_a \hat{t}_b \\
\Xi^{g=2}_{(a,b),2} &= 0 \\
\Xi^{g=2}_{(a,b),3} &= \frac{280}{31} \left( \frac{9}{11} G_4 \hat{G}_4 t_a \hat{t}_b - 3 \left( G_4 \hat{G}_4 t_a + G_4 \hat{G}_4 t_b \right) + 11 G_6 \hat{G}_6 \right), \\
\Xi^{g=2}_{(a,b),4} &= \frac{624}{41} \left( G_6 \hat{G}_6 t_a \hat{t}_b - 15 \left( G_2 G_4 \hat{t}_b + G_6 \hat{G}_4^2 t_a \right) + 99 G_2 G_4^2 \right), \\
\Xi^{g=2}_{(a,b),5} &= \frac{4896}{31} \left( G_2^2 \hat{G}_4^2 t_a \hat{t}_b - 14 \left( G_2^2 \hat{G}_4 G_4 t_a + G_2 G_4 \hat{G}_4 G_6 \hat{t}_b \right) + 196 G_4 \hat{G}_4 G_6 \right), \\
\Xi^{g=2}_{(a,b),6} &= \frac{48}{61} \left( \frac{8817040260}{1691701} G_4 \hat{G}_4 \hat{G}_4 \hat{G}_6 t_a \hat{t}_b + \frac{4994136}{451} \left( G_4^2 \hat{G}_4 G_4 \hat{G}_6 \hat{t}_b + G_4 \hat{G}_6 \hat{G}_4 \hat{G}_4 t_a \right) \\
&\quad + \frac{688210}{341} \left( G_4 G_6 \hat{G}_6^2 t_a + G_6 \hat{G}_4 \hat{G}_6 \hat{t}_b \right) - \frac{529767216}{2665} G_4^3 \right) \\
&\quad - \frac{1970892}{13} \left( G_2^2 \hat{G}_4^2 + G_2 \hat{G}_4^3 \right) - \frac{37043545}{403} G_2 \hat{G}_4^2 \right).
\end{align*}
$$

Moreover, $\Xi^{g=2}_{\Phi} = \sum_{n \geq 0} \Xi^{g=2}_{\Phi,n} s^{2n}$, where

$$
\begin{align*}
\Xi^{g=2}_{\Phi,0} &= 1, \\
\Xi^{g=2}_{\Phi,1} &= 0, \\
\Xi^{g=2}_{\Phi,2} &= \frac{312}{95} G_4 \hat{G}_4, \\
\Xi^{g=2}_{\Phi,3} &= \frac{59340}{551} G_6 \hat{G}_6, \\
\Xi^{g=2}_{\Phi,4} &= \frac{1175328}{725} G_4 \hat{G}_4^2, \\
\Xi^{g=2}_{\Phi,5} &= \frac{75034656}{2299} G_4 \hat{G}_4 \hat{G}_6, \\
\Xi^{g=2}_{\Phi,6} &= \frac{32}{799833908590151400983568329597} \times \\
&\quad \left( \frac{8336522269110502535732151408980286372}{5} G_4 \hat{G}_4^3 \right) \\
&\quad + \frac{49126884084081132256334667534510924}{3} \left( G_2^2 \hat{G}_4^2 + G_2 \hat{G}_4^3 \right) \\
&\quad + \frac{108570072190577226179433424864396525}{3} G_2 \hat{G}_4^2 \right).
\end{align*}
$$

Several terms in the $q$-expansions had been found previously by T. Gilroy and M. Tuite [22] but these functions had not been given explicitly as modular forms.

### 4.4 The $q$ formalism in genus $g \geq 1$

Let $M = (\Sigma, \mathcal{A}) \in \mathcal{X}$, and let $M^f \subset M$ be the metric manifold obtained by cutting along a non-separating curve $\gamma \subset \Sigma$. Let $\gamma_1, \gamma_2$ be the boundary curves of $M^f$. Let
$D^{(1)}_L$ and $D^{(2)}_R$ be the corresponding flat left and right unit discs centred at $P_0$ and $P_x$, respectively, and containing no other ramification point. We consider the case where

$$D^{(1)}_L \sqsubset \gamma_1, M^\theta \sqsubset \gamma_2 D^{(2)}_R.$$ 

has genus zero or one. Let $M^\theta = (S^1)^\theta$. Let $z_L$ and $z_R$ be a pair of coordinates on $D^{(1)}_L$ and $D^{(2)}_R$ vanishing at the respective disc centre ($z = w$ and $z = -w$, respectively, for $w \neq 0$) and satisfying $z_L z_R = 1$ on the boundary. Now remove the discs (which leaves us with $S^1$ and an angle $\theta$) and insert the twisted cylinder bounded by $\gamma_1 = \gamma \times \{0\}$ and $\gamma_2 = \gamma \times \{\ell\}$. Thus $z(P) \sim \hat{q} \hat{z}(P)$ and

$$z_L(P) z_R(P') = \hat{q}$$

for $\gamma$, for $\hat{q} = \exp(i \theta - \ell)$. We conclude that the self-sewn surface

$$\sqsubset_{\gamma \times [0,\ell]} (S^1)^\theta = \sqsubset_{\gamma} (\gamma \times [0,\ell])^\theta$$

is a torus $M_1(q)$ of modulus $\tau = \log \hat{q}/(2\pi i)$. The latter carries the flat metric induced by that on the cylinder. We write $\sqsubset_{\gamma \times [0,\ell]} M^\theta$ when self-sewing is performed by inserting the flat cylinder of length $\ell$ (with the twist prescribed by the angle $\theta$).

For later use, we note that for $N \geq 0$, Segal’s formula (25) together with eq. (11) yields for the $N$-point function of $\phi_1, \ldots, \phi_N \in R_\zeta(\mathfrak{H}) \otimes R_2(\mathfrak{H})$ associated to $A \in \mathfrak{H}$:

$$\langle \phi_1(z_1) \ldots \phi_N(z_N) \rangle_{A, \gamma \times [0,\tau]} M^\theta = |\hat{q}|^{-c/12} \sum_n \hat{q}^{h_n} \hat{q}^{\tau h_n} \frac{\langle \psi_n(z_L = 0) \phi_1(z_1) \ldots \phi_N(z_N) \psi_n(z_R = 0) \rangle_{A, \gamma \times [0,\tau]} M^\theta D^{(1)}_L D^{(2)}_R}{\langle \psi_n(z_L = 0) \psi_n(z_R = 0) \rangle_{D^{(1)}_L D^{(2)}_R}},$$

(74)

provided $|\hat{q}| \ll 1$ and $\arg(\hat{q}) = \theta$. Here $\{\psi_n\}_n$ is a standard orthogonal basis of $R_2(\mathfrak{H}) \otimes R_2(\mathfrak{H})$. (We indiscriminately write $\psi$ for the field in either coordinate.)

Taking $N = 0$ and restricting the summation in eq. (74) to a basis $\{\psi_n\}_{n \geq 0}$ of $\mathfrak{H}$, yields in particular

$$\langle 1 \rangle(\hat{q}, w, \tau)_{A}^{g-2} = \hat{q}^{11/60} \sum_{n \geq 0} \hat{q}^{h_n} \langle \psi_n(z_L = z - w) \psi_n(z_R = z + w) \rangle_{A}^{\tau \tau - \tau - \tau - \tau}. \quad (75)$$

Note that $\langle 1 \rangle$ depends on $w$ rather than $z - w$ and $z + w$, by translational symmetry.

The period matrix can be computed in a way analogous to that in the $s$-formalism and explicit formulae are provided by [14].

For the $(2,5)$ minimal model, $|\mathfrak{H}| = 2$, corresponding to the pair of Rogers-Ramanujan functions (for $n = 0$). For higher weight terms, derivative fields have to be taken into account as well. In addition to two-point function for the quasi-primary fields listed in Proposition 4 for $h \geq 2$, there is one term corresponding to the field $T^{(h-2)}$ for each of $h = 3, 4, 5$, whose squared Shapovalov norm equals $\frac{1}{h-2} (h - 2)! (h + 1)!$ by Proposition 6. Note that for $h = 4$, the normal ordered square of $T$ does not yield an additional field by eq. (28).

The first interesting term in the series of eq. (75) occurs for weight $h = 6$, which we provide by Proposition 21. For the remainder of this section, all occurring correlation functions refer to the torus with holomorphic differential $dz$ and periods $1, \tau$. We use the notation introduced subsequent to eq. (35).
Table 3: List of the terms for \( c = -22/5 \) and conformal holomorphic weight \( 0 \leq h \leq 6 \) in eq. (75). (The label \( g = 1 \) is dropped.)

**Proposition 21.** For \( 0 \leq h \leq 6 \) and \( A \in \{1, 2\} \), the summands in eq. (75) are the terms listed in Table 3.

**Proof:** It suffices to show that the term for \( h = 6 \) is correct, and thus that

\[
\langle (7L_6L_2 - 2L_6)(z_1 \rangle (7L_4L_2 - 2L_6)(z_2) \rangle_A \n
= \left( -1 062 817 \psi_{12}^6 + 38 261 412 G_4 \psi_{12}^4 + 63 465 500 G_6 \psi_{12}^3 - 668 595 460 G_3 \psi_{12}^2 \right. \\
\left. \quad - 659 501 640 G_4 G_6 \psi_{12} + 1 512 \left( 46 062 G_4^2 - 176 645 G_6^2 \right) \right) c \langle 1 \rangle_A \\
+ \left( 1 21 065 \psi_{12}^5 - 3 723 006 G_4 \psi_{12}^3 - 6 180 846 G_6 \psi_{12}^2 + 15 845 760 G_3 \psi_{12} \right. \\
\left. \quad + 29 310 876 G_4 G_6 \right) \langle T \rangle_A / 4.
\]

We list the individual contributions: \( \langle L_6(z_1) L_6(z_2) \rangle_A \) is the coefficient of \( (z - z_1)^4(w - z_2)^4 \).
in $\langle T(z)T(w)\rangle_A$ for $|z_1 - z_2|$ and $|w - z_2|$ small.

$$\langle L_6(z_1)L_6(z_2)\rangle_A = \left( 5775 \varphi_{12}^6 - 207900 G_4 \varphi_{12}^4 - 346500 G_6 \varphi_{12}^3 + 1455300 G_4^2 \varphi_{12}^2 \\
+ 3591000 G_4 G_6 \varphi_{12} - 378000 G_4^3 + 1470000 G_6^2 \right) c\langle 1 \rangle_A \\
\left( 1260 \varphi_{12}^5 - (63000 G_6 + 37800 G_4) \varphi_{12}^3 + 151200 G_4^2 \varphi_{12} \\
+ 277200 G_4 G_6 \right) \langle T \rangle_A.$$

$\langle L_4 L_2(z_1)L_6(z_2)\rangle_A$ is the coefficient of $(z - z_1)^2$ in $\langle T(z)T(z_1)T(z_2)\rangle_A$ for $|z - z_2|$ small.

$$\langle L_4 L_2(z_1)L_6(z_2)\rangle_A = \left( 9887400 G_6^2 - 2592000 G_4^3 + 2449400 G_4 G_6 \varphi_{12} + 9979200 G_4^2 \varphi_{12} \\
- 2368000 G_6 \varphi_{12}^3 - 1425600 G_4 \varphi_{12}^4 + 39600 \varphi_{12}^5 \right) c\langle 1 \rangle_A \\
\left( 4233600 G_4 G_6 + 2332800 G_4^2 \varphi_{12} - 907200 G_6 \varphi_{12}^2 - 552960 G_4 \varphi_{12}^3 \\
+ 18144 \varphi_{12}^5 \right) \langle T \rangle_A.$$

$\langle L_4 L_2(z_1)L_4 L_2(z_2)\rangle_A$ is the coefficient of $(z - z_1)^2(w - z_2)^2$ in $\langle T(z)T(z_1)T(w)T(z_2)\rangle_A$ for $|z - z_1|$ and $|w - z_2|$ small.

$$\langle L_4 L_2(z_1)L_4 L_2(z_2)\rangle_A = \left( 74040 G_6^2 - 28944 G_4^3 + 244440 G_4 G_6 \varphi_{12} \\
+ 102060 G_4^2 \varphi_{12}^2 - 30100 G_6 \varphi_{12}^3 - 16812 G_4 \varphi_{12}^4 + 467 \varphi_{12}^5 \right) c\langle 1 \rangle_A \\
\left( 49104 G_4 G_6 + 51840 G_4^2 \varphi_{12} + 18984 G_6 \varphi_{12}^2 + 15144 G_4 \varphi_{12}^3 \\
- 588 \varphi_{12}^5 \right) \langle T \rangle_A.$$

This completes the proof. \[\square\]

Since the overall number of holomorphic blocks in genus two is five and in the $q$ formalism, only two of them are obtained from inserting the identity field (corresponding to the pair of the Rogers-Ramanujan functions), the remaining three solutions must be given by the two-point function of $\Phi$. Indeed, as we have shown in Proposition 9, $\langle \Phi(z) \Phi(0) \rangle$ satisfies a 3rd order ODE.

Solving eq. (41) will allow to compute the coefficients of $q^{k-1/5} / \| \Phi \|^2$ for $k \geq 4$ in the continuation of eq. (75) to $A = 3, 4, 5$. For example, $\langle L_4 \Phi(z_1) L_3 \Phi(z_2) \rangle$ sorts out (in particular) the coefficient proportional to $(z - z_1)^2 (w - z_2)^{-1} (v - z_2)$ in the 5-point function $\langle T(z)T(u)T(v) \Phi(z_1) \Phi(z_2) \rangle$. A graphical proposition for correlation functions of $T$ and two copies of $\Phi$ is desirable.
4.5 The genus two partition function in the (2,5) minimal model

Our methods provide the means to compute the genus two partition function to any order in the sewing parameter $s$:

**Theorem 1.** For $0 < |s| < 1$, let $M_2(s)$ be the genus two Riemann surface defined by eq. (58) with metric $\mathcal{G}_{2\times N}(s)$ from eq. (68). In the (2,5) minimal model, the partition function on $M_2(s)$ equals

$$3_{M_2(s)} = |s|^{15/2} \left( \sum_{a,b=1,2} \langle 1 \rangle \langle \tau, \hat{\tau} \rangle_{(a,b)}^{g-2} \right)^2 - |s|^{-2/5} \langle 1 \rangle \langle \tau, \hat{\tau} \rangle_{\Phi}^{g-2} |s|^{11/30},$$

where the holomorphic blocks are those from Proposition 20.

In particular, for small values of $|s|$, the partition function is negative.

**Proof.** For $g = 2$, we have $p_2(A) = 0$ and $cc(A) = 3$, so $\varepsilon_A = -1$ according to Proposition 16, since $\varepsilon_\Phi = -1$ by eq. (56). So for $M_2 = M_2(s = 1)$, Segal’s formula (24) gives

$$3_{M_2} = \frac{3^{g-1}}{|dz|^2} - \langle \Phi(z) \rangle_{|dz|^2}^{g-1} \langle \Phi(z) \rangle_{|dz|^2}^{g-1} + \ldots.$$

Inserting the cylinder $(\gamma \times [0, -\log |s|])^{2g-2}$ yields the claimed identity, by eq. (72). $\square$

When dealing with general genus two surfaces, the metric (68) is a rather unnatural choice. We conclude the chapter by discussing an alternative approach, which uses the description of genus $g$ Riemann surfaces as branched coverings of the Riemann sphere. The latter carries a metric that is induced by the natural distance function on $\mathbb{C}$. This metric lifts to a metric on the genus $g$ surface which is singular at the branch points and flat outside. Thus the metric surface resembles a polyhedron. To regularise the metric, every ramification point $P$ on the surface is replaced by a flat disc centred at $P$, with the amount of curvature spread evenly over the bounding circle. The radius $\rho > 0$ of the circle is chosen in such a way that for any pair of ramification points $P, P'$ on the surface, the corresponding pair of closed discs has nonempty intersection iff $P = P'$.

The partition function associated with the regularised polyhedral metric will be called regularised partition function and denoted by $3^g_{\text{polyh}}$ (in genus $g$). We compute the renormalised partition function in genus two, which is obtained from $3^{g=2}_{\text{polyh}}$ by omitting the dependence on the radii of the regularisation. (In practice, after application of a Weyl transformation if necessary, we set all radii equal to one.)

Let $\Sigma$ be a hyperelliptic surface of genus $g \geq 0$, which is unramified over the point at infinity. Thus all $2g + 2$ ramification points lie at finite distance from its opposite point, the curvature distribution derives from proposition of Gauss-Bonnet according to

$$4\pi (1 - g) = \int_{2g} \mathcal{K} = \alpha (2g + 2) + \beta,$$

where $\alpha, \beta \in \mathbb{R}$. Thus $\alpha g = -2\pi g$ and $2\alpha + \beta = -4\pi$, where $\beta$ is the total amount of curvature that corresponds (in the way mentioned above) to the point at infinity, while $\alpha$ is the curvature corresponding to a single ramification point. In genus $g \geq 1$, $\alpha = -2\pi$, and $\beta = 8\pi$. 

40
Let $\Sigma_1$ and $\tilde{\Sigma}_1$ be flat tori of modulus $\tau$ and $\tilde{\tau}$, respectively. For $k = 1, 2, 3$, let $x_k$ and $\tilde{x}_k$ be the value of $\varphi(z|\tau)$ and of $\varphi(\tilde{z}|\tilde{\tau})$, respectively, at the half periods. For the pair of coordinates $X, \tilde{X}$ from Proposition 18, let

$$X_k = X(x_k), \quad \tilde{X}_k = \tilde{X}(\tilde{x}_k), \quad X_{k+3} := \xi/(s^2\tilde{X}_k), \quad k = 1, 2, 3.$$  

Let $\Sigma_2$ be the genus $g = 2$ surface with ramification points $X_1, \ldots, X_6$. Let

$$Y_k := (X - X_k)^{1/2}, \quad k = 1, \ldots, 6,$$

be local coordinates on mutually disjoint discs in $\mathbb{C}$ of radius $\varrho_k > 0$ about $X_k$, and let $\tilde{X} := X^{-1}$ be the coordinate (on both sheets) close to the point at infinity. The regularised polyhedral metric on $\Sigma_2$ (associated to the numbers $\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5$, and $\varrho_6$) is defined by

$$\varrho_{\text{polyh}} = \begin{cases} 
4\varrho_k |dY_k|^2 & \text{for } |X - X_k| \leq \varrho_k \text{ for } k = 1, \ldots, 6, \\
|dX|^2 & \text{for } \varrho_k \leq |X - X_k| < \varrho_\infty, \text{ for } k = 1, \ldots, 6, \\
\varrho_\infty^4 |d\tilde{X}|^2 & \text{for } \varrho_\infty \leq |X|.
\end{cases} \quad (76)$$

Note that the metric is everywhere continuous.

**Theorem 2.** Let $\varrho = -2\pi/5$. Let $x_1, x_2, x_3$ and $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ be the functions of the modulus $\tau$ of $\Sigma_1$ and $\tilde{\tau}$ of $\tilde{\Sigma}_1$, respectively, defined by eq. (62). Let $X, \tilde{X}$ be the pair of almost global coordinates on $\mathbb{P}^1_\mathbb{C}$ from Proposition 18. Let $\mathcal{J}_{\Sigma}(\pi)$ be the genus two partition function from Theorem 1, where $0 < |\pi| < 1$. The renormalised genus two partition function w.r.t. the polyhedral metric (76) is given by

$$\left(\mathcal{J}^{\Sigma=2}_{\text{polyh}}\right)^{\text{renorm}} = 2^{-c/3}|\xi|^{3/2}\left|\frac{2^{11/6}}{\xi}\right|\left|\Delta_{1,2,3}^{\tilde{\Delta}_{1,2,3}}\right|^{-c/24}|X'(x_1)X'(x_2)X'(x_3)|\tilde{X}'(\tilde{x}_1)\tilde{X}'(\tilde{x}_2)\tilde{X}'(\tilde{x}_3)|^{-c/24}$$

$$\left|\tilde{X}_1\tilde{X}_2\tilde{X}_3\right|^{1/2}|\tilde{x}_1\tilde{x}_2\tilde{x}_3|^{c/6}\exp\left(\frac{c}{6}\text{Res}_{\xi\to 0}\left[\frac{1}{\xi}\log|X'(\xi)|\right]\right)\mathcal{J}_{\Sigma}(\pi),$$

where $X'(x) = \frac{dX}{ds}$ etc. and

$$\Delta_{1,2,3} = \prod_{j \neq k} |x_j - x_k|^2, \quad \tilde{\Delta}_{1,2,3} = \prod_{j \neq k} |\tilde{x}_j - \tilde{x}_k|^2.$$

We prove proof makes use of the following

**Lemma 3.** Let $P \in \mathbb{C}$, and let $U_P \subseteq \mathbb{C}$ be an open set containing $P$. Let $\mathcal{H}_P$ be a two-form with support on $U_P$. Suppose $U_P$ and $\mathcal{H}_P$ are invariant under rotation around $P$. Let $f$ be a harmonic function on $U_P$ (i.e. $\partial_z \tilde{\partial}_{\tilde{z}} f = 0$ on $U_P$). The following is true:

$$\iint_{U_P} f(z) \mathcal{H}_P = f(z(P)) \iint_{U_P} \mathcal{H}_P.$$

In particular, for $n \in \mathbb{Z}$,

$$\frac{\iint_{U_P} z^n \mathcal{H}_P}{\iint_{U_P} \mathcal{H}_P} = \begin{cases} 
(\text{degree} z(P)) n & \text{in case } z(P) \neq 0, \\
\delta_{n,0} & \text{in case } z(P) = 0. 
\end{cases}$$
Proof. By assumption on $f$, there exists a holomorphic function $F$ on $U_P$ such that $f = F + \overline{F}$. Close to $z(P)$, $F$ admits a Taylor expansion in powers of $(z - z(P))$. A rotation $z \mapsto z'$ around $P$ leaves $\mathcal{K}_F$ invariant, but the only term $(z' - z(P))^n$ invariant under the coordinate change is the one for $n = 0$. □

Proof of Theorem 2. Let $dz$ and $d\xi$ be the holomorphic one-form on the flat tori $\Sigma_1$ and $\Sigma_1$, respectively. The proof of eq. (6) shows that the partition function for the metric

$$p_F = \left\{ \begin{array}{ll} \{dz\}^2 & \sqrt{|z|} \leq |z|, \\
\{d\xi\}^2 & \sqrt{|\xi|} \leq |\xi|, \end{array} \right. \quad (77)$$

on $\Sigma_2$ equals

$$\mathcal{M}_{\Sigma} = |s|^{c/12} \mathcal{H}_{\Sigma}(z) \cdot \mathcal{H}_{\Sigma}(\xi). \quad (78)$$

The metric (77) is continuous by eq. (59). Let $p_A$ and $p_B$ be the metric (77) and (76), respectively. The curvature of the metric $p_A$ from eq. (77) is supported on the cycle

$$\{|z| = |s|^{1/2} \} = \{|\xi| = |s|^{1/2} \} \quad (79)$$

and has integral $-4\pi$ by Gauss-Bonnet. On the other hand, all curvature of the regularised polyhedral metric $p_B$ from eq. (76) is spread over the circles bounding the discs centred at $X_1, \ldots, X_6$ and the point at infinity. Let $\alpha = -2\pi$ and $\beta = 8\pi$ be the integrated curvature on the circle

$$\{|X - X_k| = \varrho_k \}, \quad (80)$$

for $k = 1, \ldots, 6$, and on

$$\{|X| = \varrho_{\infty} \}, \quad (81)$$

respectively. Let $\mathcal{K}_k$ for $k = 1, \ldots, 6$ and $\mathcal{K}_\infty$ be the corresponding curvature two-form.

Thus we have to compute the Weyl-factor on eight circles carrying curvature for one of the metrics $p_A$ and $p_B$. On each of them the almost global coordinate $X$ from Proposition 18 is defined. Namely, the respective Weyl factor is given as follows:

$$|dz|^2 \mapsto |dX|^2 = \left\{ \begin{array}{ll} \exp(\Delta \sigma_A) |dz|^2 & \text{on the cycle (79)}, \\
\exp(\Delta \sigma_B) |dz|^2 & \text{on the cycle (80) for } k = 1, 2, 3, \\
\exp(\Delta \sigma_A) |d\xi|^2 & \text{on the cycle (80) for } k = 4, 5, 6, \\
\exp(\Delta \sigma_{\infty}) |d\xi|^2 & \text{on the circle (81)}, \end{array} \right.$$

where

$$\Delta \sigma_A = \Delta \sigma_B = \log |dX|^2 + \log \left| \frac{dX}{dz} \right|^2,$$

$$\Delta \sigma_B = \Delta \sigma_{\infty} = \log |dX|^2 + \log \left| \frac{dX}{d\xi} \right|^2 + \log \left| \frac{d\xi}{dz} \right|^2.$$

Note that by eq. (60),

$$\log \left| \frac{dX}{dX} \right| = \log |X| - \log |\tilde{X}| = 2 \log |X| + \log \left| \frac{z^2}{\xi} \right|.$$
Now we have the following list of integrals:
\[
\int\int \mathcal{K}_k \log \left| \frac{dx}{dz} \right|^2 = -4\pi \left( 2 \log 2 - 3 \log |s| \right), \tag{82}
\]
\[
\int\int \mathcal{K}_k \log \left| \frac{dX}{dx} \right| = -4\pi \text{Res}_{z=0} \left[ \frac{1}{z} \log \left| \frac{dX}{dx} \right| \right]. \tag{83}
\]

For \( k = 1, 2, 3, \)
\[
\int\int \mathcal{K}_k \log \left| \frac{dx}{dz} \right|^2 = -2\pi \left( 2 \log 2 + \log \hat{p}_k + \sum_{j \neq k} \log |x_k - x_j| - \log |X'(x_k)| \right), \tag{84}
\]
\[
\int\int \mathcal{K}_k \log \left| \frac{dX}{dx} \right|^2 = -2\pi \log |X'(x_k)|^2, \tag{85}
\]
\[
\int\int \mathcal{K}_{k+3} \log |X| = -2\pi \log |X_{k+3}|. \tag{86}
\]

and
\[
\int\int \mathcal{K}_{k+3} \log \left| \frac{dx}{dz} \right|^2 = -2\pi \left( 2 \log 2 + \log \hat{p}_{k+3} + \sum_{j \neq k} \log |\hat{x}_k - \hat{x}_j| - \log |\hat{X}'(\hat{x}_k)| \right), \tag{87}
\]
\[
\int\int \mathcal{K}_{k+3} \log \left| \frac{dX}{dx} \right|^2 = -2\pi \log |\hat{X}'(\hat{x}_k)|^2, \tag{88}
\]

where for \( k = 1, 2, 3, \)
\[
\log \hat{p}_k = \log |\hat{X} - \hat{X}_k| = \log \hat{p}_{k+3} - \log |X_{k+3}| + \log |\hat{X}_k|. \]

Moreover,
\[
\int\int \mathcal{K}_2 \log |X| = 8\pi \log \hat{p}_2, \tag{89}
\]
\[
\int\int \mathcal{K}_3 \log \left| \frac{dX}{dz} \right| = 0, \tag{90}
\]
\[
\int\int \mathcal{K}_3 \log \left| \frac{dX}{dx} \right|^2 = 8\pi \left( 2 \log 2 + \log |\hat{x}_1, \hat{x}_2, \hat{x}_3| \right). \tag{91}
\]

We conclude that
\[
\frac{3^{g_{\text{p}}^2} / 3^{g_{\text{MT}}^2}}{} = \exp \left\{ \frac{c}{48\pi} \int \Delta \sigma_A \mathcal{K}_A + \sum_{k=1}^{3} \left( \Delta \sigma_B \mathcal{K}_B + \Delta \tilde{\sigma} B \mathcal{K}_{B+3} + \Delta \tilde{\sigma} A \mathcal{K}_A \right) \right\},
\]
where
\[
\exp \left\{ \frac{c}{48\pi} \int \Delta \sigma_A \mathcal{K}_A \right\} = 2^{-c/6} |s|^{c/4} \exp \left( \frac{c}{6} \text{Res}_{z=0} \left[ \frac{1}{z} \log X'(x) \right] \right),
\]
and
\[
\exp \left\{ \frac{c}{48\pi} \int \sum_{k=1}^{3} \left( \Delta \sigma_B \mathcal{K}_B + \Delta \tilde{\sigma} B \mathcal{K}_{B+3} + \Delta \tilde{\sigma} A \mathcal{K}_A \right) \right\}
\]
\[
= 2^{-c/6} \left( \hat{q}_1 \ldots \hat{q}_6 \right)^{-c/24} \hat{g}_B^{2c/3} \left| \frac{1}{s} \right|^{11c/24} \left| \Delta_{1,2,3} \right|^{-c/24} \left| \hat{\Delta}_{1,2,3} \right|^{-c/24} |X'(x_1)X'(x_2)X'(x_3)|^{-c/24} \left| \hat{X}'(\hat{x}_1) \hat{X}'(\hat{x}_2) \hat{X}'(\hat{x}_3) \right|^{-c/24} |\hat{X}_1, \hat{X}_2, \hat{X}_3|^{-c/24} |\hat{x}_1, \hat{x}_2, \hat{x}_3|^{-c/6}. \]
The renormalised partition function is obtained by setting \( q_1 = \ldots = q_6 = q_\infty = 1 \) and the claimed follows by taking eq. (78) into account.

Due to the choice of coordinate in the definition of the metric (76), the renormalised partition function in Theorem 2 is not symmetric under exchange \( \tau \leftrightarrow \hat{\tau} \). Indeed, for \( x_1, x_2, x_3 \) from eq. (62),

\[
\sum_{i=1}^{3} x_i = 0, \quad \sum_{i=1}^{3} x_i^2 = 60 G_4, \quad \sum_{i=1}^{3} x_i^3 = 210 G_6,
\]

and the corresponding set of equations for \( \sum_{i=1}^{3} x_i^4 \) with \( 4 \leq k \leq 6 \), together with Proposition 18 yields

\[
\text{Res}_{z=0} \left[ \frac{1}{z} \log \frac{dX}{dx} \right] = -216 s^4 G_4 \overrightarrow{G}_4 - 1200 s^6 G_6 \overrightarrow{G}_6 - 3312 s^8 G_2^2 \overrightarrow{G}_4^2 - \frac{12614400}{121} s^{10} G_4 G_6 \overrightarrow{G}_4 \overrightarrow{G}_6 - \frac{384}{169} s^{12} \left( 437157 G_2^3 \overrightarrow{G}_4^3 + 177075 G_2^2 \overrightarrow{G}_4^2 + 139050 G_2 \overrightarrow{G}_4 + 68125 G_2^2 \overrightarrow{G}_4^2 \right) + \frac{9845452800}{121} G_2^2 G_6 \overrightarrow{G}_4 \overrightarrow{G}_6 s^{14} + O(s^{16}),
\]

and

\[
\log | \hat{X}_1 \hat{X}_2 \hat{X}_3 | = -360 G_4 \overrightarrow{G}_4 s^4 - 2100 G_6 \overrightarrow{G}_6 s^6 + 23760 G_2^2 \overrightarrow{G}_4^2 s^8 + \frac{3412800}{11} G_4 G_6 \overrightarrow{G}_4 \overrightarrow{G}_6 s^{10} + \frac{120}{13} s^{12} \left( 119664 G_2^3 \overrightarrow{G}_4^3 - 70560 G_2^2 \overrightarrow{G}_4^2 - 61200 G_2 \overrightarrow{G}_4 + 83125 G_2^2 \overrightarrow{G}_4^2 \right) - \frac{1364904000}{11} G_2^2 G_6 \overrightarrow{G}_4 \overrightarrow{G}_6 s^{14} + O(s^{16}).
\]

Thus the coefficient of \( s^{12} \) in the expression \( \log | \hat{X}_1 \hat{X}_2 \hat{X}_3 | + \text{Res}_{z=0} \left[ \frac{1}{z} \log \frac{dX}{dx} \right] \) reads

\[
-\frac{354544}{169} G_2^2 \overrightarrow{G}_4^3 + \frac{42076800}{169} G_2^2 \overrightarrow{G}_6^2 + \frac{42076800}{169} G_2 \overrightarrow{G}_4 + \frac{103515000}{169} G_2^2 \overrightarrow{G}_4^2,
\]

so that

\[
(3_{\text{polyh}}^2)_{\text{renorm}} = | \hat{X}_1 \hat{X}_2 \hat{X}_3 |^{c/4} \times (\text{symmetric expression} + O(s^{16})).
\]

We explain the factor of \(| \hat{X}_1 \hat{X}_2 \hat{X}_3 |^{c/4} \). W. Nahm suggests the following:

**Definition 22.** For \( N \in \mathbb{N} \) and for \( i = 1, \ldots, N+1 \), let \( X_i \in \mathbb{C} \cup \{ \infty \} \) with \( X_i \neq X_j \) for \( i \neq j \). For \( A \in \text{SL}(2, \mathbb{C}) \), set

\[
\prod_{i<j} (A(X_i) - A(X_j))^2 = \prod_{i=1}^{N+1} \Gamma_A(X_i)^{-N} \prod_{i<j} (X_i - X_j)^2.
\]

Here the dash means that infinite factors are omitted from the product.
It follows immediately that the $\Gamma_A$ for $A \in SL(2, \mathbb{C})$ satisfy the cocycle condition, namely for $A, A' \in SL(2, \mathbb{C})$, we have
\[
\Gamma_{AA'}(X) = \Gamma_A(A'(X)) \Gamma_A(X).
\]
Explicitly, $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ acts on $X \in \mathbb{C} \cup \{\infty\}$ by $AX = \frac{\alpha X + \beta}{\gamma X + \delta}$, so
\[
\Gamma_A(X) = \begin{cases} 
(\gamma X + \delta)^2 & \text{for } X \in \mathbb{C}, \text{ provided } \gamma X + \delta \neq 0, \\
\gamma^{-2} & \text{for } X \in \mathbb{C} \text{ such that } \gamma X + \delta = 0, \\
\gamma^2 & \text{for } X = \infty \text{ and } \gamma \neq 0, \\
\delta^{-2} & \text{for } X = \infty \text{ and } \gamma = 0.
\end{cases}
\]

Eq. (4) implies

**Proposition 23.** Let $X = \{X_i\}_{i=1}^{2g+2} \subset \mathbb{C} \cup \{\infty\}$ be the set of ramification points of a hyperelliptic genus $g$ Riemann surface with metric $|dX|^2$. Let $(\mathcal{Z}(\mathcal{X}^+))^{\text{renorm}}$ be the corresponding renormalised partition function. For $A \in SL(2, \mathbb{C})$, we have
\[
(\mathcal{Z}(A(\mathcal{X}^+)))^{\text{renorm}} = \prod_{i=1}^{2g+2} |\Gamma_A(X_i)|^{c/4} \left(\mathcal{Z}(\mathcal{X}^+)\right)^{\text{renorm}}.
\]

**Example 24.** Let $A : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ be defined by $A(X) = \xi/(s^2 X) = \hat{X}$. Proposition 23 yields
\[
(\mathcal{Z}(\{X_i\}_{i=1}^6))^{\text{renorm}} = \frac{s^2}{\xi} \prod_{i=1}^6 |X_i|^{c/4} \left(\mathcal{Z}(\{X_i\}_{i=1}^6)\right)^{\text{renorm}}.
\]

These observations can be used to define a universal partition function, which does not depend on the specific set of ramification points of the genus $g$ Riemann surface, but rather on the conformal class they define, namely
\[
\prod_{i,j=1}^{2g+2} |X_i - X_j|^{-c/4} \left(\mathcal{Z}(\mathcal{X}^+)\right)^{\text{renorm}}.
\]

**4.6 Discussion and outlook**

We have obtained a power series description of an object with good automorphic properties under the mapping class group in genus two. This is provided by the $(2,5)$ minimal model studied above, apart from ramifications into questions of metric dependence. We suggest using the polyhedral metric $G_{\text{polyh}}$, which defines an almost natural metric on Riemann surfaces of arbitrary genus. A constant curvature metric seems preferable, and the corresponding partition function might be computable. We would like to make contact with Siegel modular forms, but this requires the consideration of Dehn twists.

Dehn twists about simple closed curves $\gamma$, the type considered both in the $s$ and the $q$ sewing scheme, define elements in the mapping class group $\Gamma_2$ (for genus two).
Those on separating simple closed (SSC) curves generate the Torelli group $\mathcal{F}_2$, which is related to the Siegel modular group $Sp(4, \mathbb{Z})$ by the short exact sequence

$$1 \rightarrow \mathcal{F}_2 \rightarrow \Gamma_2 \rightarrow Sp(4, \mathbb{Z}) \rightarrow 1.$$ 

Due to a fifth root singularity in $s$, the holomorphic block (73) transforms trivially only under a quotient of $\Gamma_2$ by fifth powers of such Dehn twists. This quotient is infinite for $g \geq 2$ \cite{9}.

Since $s = 0$ is a regular interior point on the Siegel upper half plane $\mathbb{H}_2$, our holomorphic blocks do not form a Siegel modular form. Taking fifth powers doesn’t lift the singularity since the holomorphic blocks transform into non-trivial linear combinations among themselves. One should be able to construct a scalar representation of $\Gamma_2$, however, which can then be related to Siegel modular functions.

As discussed in Section 4.4, an alternatively way to compute $\mathcal{Z}_{M_2}$ is by use of the $q$ formalism though this requires knowledge of $\langle \Phi \Phi \rangle^{x=1}$. The singular Riemann surface corresponding to $q = 0$ defines a boundary point of the Siegel upper half plane $\mathbb{H}_2$. It is desirable to understand better the matching between the $s$ and the $q$ formalism, since the partition function they deliver is the same. A main task is to show that the solutions obtained through perturbative expansion extend as a one-valued object to the full moduli space. This requires an understanding of the monodromy group.

The occurrence of linear differential equations of the Kaneko-Zagier type \cite{26} in genus one, and the 3rd order ODE from Corollary 9 for $g = 2$ is a general feature of holomorphic blocks in any genus. Since the use of algebraic coordinates allows to deal with (hyperelliptic) Riemann surfaces of all genera at once, we provide here a reformulation of the ODE for $x \Phi y$ in the description $\Sigma_1$:

$$y^2 = p^4 \left( x \right) \left( x^3 - 30 G_4 x - 70 G_6 \right).$$

Corollary 24. $\Upsilon(x)$ defined by $y^{-1/5} \Upsilon(x) := \langle \Phi(z) \Phi(0) \rangle$ satisfies the ODE

$$\left( p(x) \frac{d^3}{dx^3} + f(x) \frac{d^2}{dx^2} + g(x) \frac{d}{dx} + h(x) \right) \Upsilon(x) = 0 , \quad (89)$$

where

$$f = \frac{6}{5} p' ,$$

$$g = \frac{3}{100} \frac{[p']^2}{p} + \frac{9}{50} p'' ,$$

$$h = - \frac{33}{50} \frac{[p']^3}{p^3} + \frac{33}{250} \frac{p'p''}{p} - \frac{288}{125} .$$

In particular, the ODE has simple poles at the four ramification points.

Proof. Set

$$L = \frac{d^3}{dz^3} - \frac{12}{25} \varphi(z) \frac{d}{dz} + \frac{12}{125} \varphi'(z) .$$

Since $d/dz = y d/dx$ and

$$\frac{d^3}{dz^3} = y \left( p(x) \frac{d^3}{dx^3} + \frac{3}{2} p'(x) \frac{d^2}{dx^2} + \frac{1}{2} p''(x) \frac{d}{dx} \right) ,$$
we have in algebraic coordinates

\[ L = y \left( p(x) \frac{d^3}{dx^3} + \frac{3}{2} p'(x) \frac{d^2}{dx^2} + \frac{12}{25} p''(x) \frac{d}{dx} + \frac{12}{125} \right). \]

Thus

\[ L\left(y^{-1/5} \tau(x)\right) = y^{4/5} \left( p(x) \frac{d^3}{dx^3} + f(x) \frac{d^2}{dx^2} + g(x) \frac{d}{dx} + h(x) \right) \tau(x), \]

where

\[
\begin{align*}
f &= y^{1/5} \left( 3p \frac{d}{dx} + \frac{3}{2} p' \right) y^{-1/5}, \\
g &= y^{1/5} \left( 3p \frac{d^2}{dx^2} + 3 p' \frac{d}{dx} + \frac{12}{25} p'' \right) y^{-1/5}, \\
h &= y^{-4/5} L y^{-1/5}.
\end{align*}
\]

Computation of \( d^k y^{-1/5} / dx^k \) for \( k = 1, 2, 3 \) leads to the expressions for the functions \( f, g, h \). We conclude that the ODE (89) is equivalent to that of eq. (41). \( \square \)

By virtue of the Frobenius Ansatz \( \langle \Phi(z) \Phi(0) \rangle \sim z^n \) times a function of \( \tau \), the differential equation (41) imposes the condition

\[ \frac{25}{12} \alpha(\alpha - 1)(\alpha - 2) = \frac{2}{5} + \alpha. \]

on \( \alpha \), which produces the values 1/5, 2/5 and 12/5. On the other hand, using what is called power counting, and eqs (50) and (51) yield the OPE

\[ \Phi(z) \Phi(0) = \varepsilon_\Phi |z|^{4/5} 1 + \varepsilon_\Phi \lambda_{\Phi \Phi} |z|^{2/5} \Phi(0) + \ldots \]

The obvious solutions to the ODE are, to leading order,

\[ z^{2/5} \langle 1 \rangle, \quad z^{1/5} \langle \omega \rangle, \quad z^{12/5} \langle T \rangle. \]

The first two terms obviously correspond to the leading terms in the OPE, but the third terms needs a better understanding.

The author is grateful to W. Nahm for introducing her to the subject and for commenting on the manuscript. Explicit computations in Propos. 18, 20 and 21 were performed using Mathematica. The author found [7] helpful for the calculations in Propos. 4 and 5, she thanks A. Honecker for mentioning the resource to her. This research was supported in part by the National Science Foundation under Grant No. NSF PHY-1748958. The author acknowledges receipt of Higher Education Authority Support for a Covid-19 related Research Costed Extension of her IRC Government of Ireland Postdoctoral Award 2018/583 through Trinity College Dublin.

A Proof of Proposition 11

We use induction on \( N \). For \( N \geq 0 \), let \( \Gamma_0^N \in S_N^{(1)} \) be the graph whose vertices are all isolated.
For \( N = 0, \langle 1 \rangle = C_{0e} \) (corresponding to the empty graph). For \( N = 1, \Gamma_0^1(z) \) is the only graph, and

\[
\langle 1 \rangle^{-1} \langle T \rangle = \langle 1 \rangle^{-1} \langle T(z) \rangle = \gamma(\Gamma_0^1(z)) = C_{2e}
\]

is a modular form of weight 2. For \( N = 2 \), the admissible graphs form a closed loop, a single line segment (with two possible orientations), and two isolated points. Thus by eq. (43),

\[
C_{4e} = \langle 1 \rangle^{-1} \langle T(z_1)T(z_2) \rangle - \frac{c}{2} \phi_{12}^2 C_{0e} - 2 \phi_{12} C_{2e}.
\]

According to the OPE (20), \( C_{4e} \) is regular, thus constant. Suppose \( C_{2k,e} \), for \( k \leq N \) has the required properties for \( k < N \). Let

\[
E_N := \{ 1 \leq j \leq N \mid \exists i \text{ such that } \langle z_i, z_j \rangle \in \Gamma \},
\]

and let \( E_N^c \) denote its complement in \( \{1, \ldots N\} \). We define

\[
\langle T(z_1) \ldots T(z_N) \rangle,
\]

by

\[
\gamma(\Gamma) = \left( \frac{c}{2} \right)^{2 \text{loops}} \langle 1 \rangle^{-1} \langle \bigotimes_{k \in E_N} T(z_k) \rangle \prod_{(z_i, z_j) \in \Gamma} \phi_{ij},
\]

and show first that \( \langle T(z_1) \ldots T(z_N) \rangle \), is regular on any partial diagonal, thus constant. In other words, \( \sum_{\Gamma \in S_N^2} \gamma(\Gamma) \) reproduces the correct singular part of the Virasoro field as prescribed by the OPE (20). By symmetry, it suffices to verify this for \( \langle T(z_1) \ldots T(z_N) \rangle \) as a function of \( z_1 \).

Since the \( z \), dependence is trivial, we write \( \langle T(z_1) \ldots T(z_N) \rangle \rangle = \langle 1 \rangle C_{2n,e} \).

Suppose the graphical representation for the \( k \)-point function of the Virasoro field is correct for \( 2 \leq k \leq N - 1 \). For \( 1 \leq i \leq N \), set \( S^{[i]} := S(z_i, \ldots, z_N) \). For \( 1 \leq i, j \leq N \), \( i \neq j \), define

\[
S_{(ij)} := \{ \Gamma \in S^{[i]} \mid \langle z_i, z_j \rangle, \langle z_j, z_i \rangle \in \Gamma \},
\]

\[
S_{(ij)} := \{ \Gamma \in S^{[i]} \mid \langle z_i, z_j \rangle, \langle z_j, z_i \rangle \notin \Gamma \},
\]

\[
S_{(ij)} := \{ \Gamma \in S^{[i]} \mid \langle z_i, z_j \rangle, \langle z_j, z_i \rangle \notin \Gamma \}.
\]

\( S^{[i]} \) decomposes as

\[
S^{[i]} = S_{(12)} \cup S_{(1,2)} \cup S_{(2,1)} \cup S_{(1)(2)}.
\]

Since \( S_{(12)} \cong S^{[1]} \), we have

\[
\sum_{\Gamma \in S_{(12)}} \gamma(\Gamma) = \frac{c}{2} \phi_{12}^2 \sum_{\Gamma \in S^{[i]}} \gamma(\Gamma) = \frac{c}{2} \phi_{12}^2 \langle 1 \rangle^{-1} \langle T(z_3) \ldots T(z_N) \rangle
\]

by the induction hypothesis. Moreover,

\[
\sum_{\Gamma \in S^{[1]} \setminus S_{(12)}} \gamma(\Gamma) = 2 \phi_{12} \langle 1 \rangle^{-1} \langle T(z_2)T(z_3) \ldots T(z_N) \rangle + O((z_1 - z_2)^{-1}),
\]

48
since by induction hypothesis on $S^{[2]}$, for $\Gamma \in S^{[2]}$, 
\[
\gamma(\varphi_{12}^{-1}(\Gamma)) + \gamma(\varphi_{21}^{-1}(\Gamma)) = 2 \varphi_{12} \, \gamma(\Gamma) + O((z_1 - z_2)^{-1}) .
\]
Here $\varphi_{12}$ and $\varphi_{21}$ are the isomorphisms 
\[
\varphi_{12} : S_{(1,2)} \to S^{[2]},
\]
\[
\varphi_{21} : S_{(2,1)} \to S^{[2]}
\]
given by contracting the link $(z_1, z_2)$ resp. $(z_2, z_1)$ into the point $z_2$ and leaving the graph unchanged otherwise: Let $\Gamma \in S_{(1,2)}$. For $j \neq 2$, we have $(z_i, z_j) \in \Gamma$ for $j \neq 1$ iff $(z_i, z_j) \in \varphi_{12}(\Gamma)$, and we have $(z_i, z_j) \in \Gamma$ iff $(z_i, z_j) \in \varphi_{12}(\Gamma)$. The case $\Gamma \in S_{(2,1)}$ works analogously by changing the orientation. One checks easily that either map defines an isomorphism.

We address the modularity property of $C_{2n,c}$. From the transformation behaviour of $T$ under conformal coordinate transformations $z \mapsto \lambda z$ (with $\lambda > 0$) follows that 
\[
\langle T(z_1) \ldots T(z_n) \rangle_{\lambda} = \lambda^{2N} \langle T(\lambda z_1) \ldots T(\lambda z_n) \rangle_{\lambda^N} .
\]
On the other hand, we have in eq. (43), for every edge of a graph, $\varphi(z|\tau) = \lambda^2 \varphi(\lambda z|\lambda \tau)$. It follows that for $1 \leq n \leq N$, $C_{2n,c}$ transforms as a modular form of weight $2n$, 
\[
C_{2n,c}(\Lambda) = \lambda^{2n} C_{2n,c}(\Lambda) .
\]
The zero-point function is differentiable and generates the $N$-point functions of $T$ for $N > 0$. Thus the latter are also non-singular for finite $\tau$, except possibly at the cusps. Since $3^{x-1}$ is modular on the full modular group, it suffices to verify regularity at the cusp at infinity. Let $E_0$ be the lowest energy eigenvalue of $L_0$ in the specific sector. Let $y = \text{Im} \, \tau$. For $y \to \infty$, we have 
\[
\frac{\langle T(z_1) \ldots T(z_n) \rangle}{\langle 1 \rangle} \sim \frac{O(e^{-2\pi E_0 y})}{e^{-2\pi E_0 \gamma}} = O(1) .
\]
This completes the proof.

## B Proof of Proposition 12

We show that for $\Gamma \in \tilde{S} \left( 0, z_1, \ldots, z_N \right)$ and some polynomial $\varrho_I(h)$,
\[
\tilde{\gamma}(\Gamma) := \left( \frac{3}{2} \right)^{\text{loops}} \tilde{C}_{2,\Gamma(\text{edges})} \varrho_I(h) \prod_{(z_i, z_j) \in \Gamma} \varrho_{ij} ,
\]
and we specify $\varrho_I$.

For $N \geq 0$, let $\Gamma_0^N \in \tilde{S}_N$ be the graph whose vertices are all isolated.

For $N = 0$, the only graph is $\Gamma_0^0$, and we have $\tilde{\gamma}(\Gamma_0^0) = \tilde{C}_{0,c} = 1$.

For $N = 1$, by lack of a modular form of weight 2, the only graph that contributes has one edge, 
\[
\langle \Phi \rangle^{-1} \langle T(z) \Phi(0) \rangle = h \varphi(z) \tilde{C}_{0,c} .
\]
This is consistent with eq. (37).

For \(1 \leq i \leq N\), set \(\tilde{S}[i] := \tilde{S}(z_0 = 0, z_i, \ldots, z_N)\). We define \(\tilde{S}(i,j)\) and \(\tilde{S}_k(i,j)\) for \(1 \leq i, j \leq N\) in the same way as we defined \(S(i,j)\), \(S_k(i,j)\) and \(S_k(i,j)\) (proof of Proposition 11) but with \(\tilde{S}[1]\) in place of \(S[1]\). Moreover, for \(1 \leq i \leq N\) we set
\[
\tilde{S}(i,0) := \{\Gamma \in \tilde{S}[1] \mid (z_i, 0) \in \Gamma\}.
\]

We show that
\[
\langle \Phi \rangle^{-1}\langle T(z_1) \ldots T(z_N) \Phi(0) \rangle \sim \sum_{\Gamma \in \tilde{S}[1]\backslash \tilde{S}[0]} \tilde{\gamma}(\Gamma) = \tilde{\gamma}(\Gamma_N^0).
\]

The l.h.s. is well-defined: Every \(\Gamma \in \tilde{S}[1]\backslash \{\Gamma_N^0\}\) has an edge \((z_i, z_j)\) \(\in \Gamma\) with \(1 \leq i \leq N\) and \(0 \leq j \leq N\). So \(\tilde{\gamma}(\Gamma)\) is proportional to \(\phi_{ij}\), according to eq. (45), and the induction hypothesis on \(\tilde{S}[2]\) applies.

The arguments employed previously in the proof of Proposition 11 show that
\[
\sum_{\Gamma \in \tilde{S}(1,2)} \tilde{\gamma}(\Gamma) = \frac{c}{2} \phi_{12}^2 \langle T(z_2) \ldots T(z_N) \Phi(0) \rangle \langle \Phi \rangle^{-1}
\]
(recall our convention \(\phi_{12} = \phi(z_1 - z_2)\)) and
\[
\sum_{\Gamma \in \tilde{S}(1,1)} \tilde{\gamma}(\Gamma) = 2\phi_{12} \langle T(z_2) \ldots T(z_N) \Phi(0) \rangle \langle \Phi \rangle^{-1} + O((z_1 - z_2)^{-1}).
\]

(Recall that \(z_0 = 0, \phi_1 = \phi_{10}\).) Now we address graphs in \(\tilde{S}(1,0)\). By the OPE (32) and the induction hypothesis on \(\tilde{S}[2]\), we have
\[
\langle T(z_1) \ldots T(z_N) \Phi(0) \rangle \langle \Phi \rangle^{-1} = h \phi_1 \sum_{\Gamma \in \tilde{S}[2]} \tilde{\gamma}(\Gamma') + O(z_1^{-1}).
\]

We show that
\[
h \phi_1 \sum_{\Gamma \in \tilde{S}[2]} \tilde{\gamma}(\Gamma') - \sum_{\Gamma \in \tilde{S}(1,0)} \tilde{\gamma}(\Gamma) = O(z_1^{-1}).
\]

Let
\[
\kappa : \tilde{S}(1,0) \to \tilde{S}[2]
\]
be the map that contracts the edge \((z_1, 0)\) into 0. We have the decomposition
\[
\sum_{\Gamma \in \tilde{S}(1,0)} \tilde{\gamma}(\Gamma) = \sum_{\Gamma \in \tilde{S}[2]} \sum_{\Gamma \in \tilde{S}(1,0)} \tilde{\gamma}(\Gamma).
\]

Thus it remains to show that for every \(\Gamma' \in \tilde{S}[2]\), we have
\[
h \phi_1 \tilde{\gamma}(\Gamma') - \sum_{\Gamma \in \tilde{S}(1,0), \kappa(\Gamma) = \Gamma'} \tilde{\gamma}(\Gamma) = O(z_1^{-1}). \tag{90}
\]

Let \(\Gamma' \in \tilde{S}[2]\) be such that \((z_2, 0) \in \Gamma'\) for all values of \(k\), where \(2 \leq k \leq N\). For each such value of \(k\) we obtain one graph \(\Gamma \in \tilde{S}(1,0)\) with \(\kappa(\Gamma) = \Gamma'\) by replacing, according to eq. (45),
\[
\phi_k \mapsto \phi_{k1} \phi_1.
\]
Different values of $k$ lead to different graphs in $\tilde{S}_{1,0}$. One more graph $\Gamma \in \tilde{S}_{1,0}$ is obtained by replacing

$$\prod_{(i,j) \in \Gamma'} \varphi_{ij} \mapsto \varphi_1 \prod_{(i,j) \in \Gamma'} \varphi_{ij}.$$  

This proves eq. (90) provided that for the polynomial $\varrho_{\lambda}(h)$,

$$h \varrho_{\lambda} - (h_{\lambda+1} + \varrho_{\lambda+1}) = 0$$

with $h_0(h) = 1$.

An argument similar to that seen in the proof of Proposition 11 also shows that $\tilde{C}_{2n,e}$ is a modular form of weight $2n$.

References

[1] Abramowitz, M., and Stegun, I.: Handbook of Mathematical Functions, Dover Publications Inc., New York (1965);

[2] Cardy, J.L.: Conformal Invariance and the Yang-Lee Edge Singularity in Two Dimensions, Phys. Rev. Lett. 54.13, 1354–1356 (1985);

[3] Dotsenko, Vl.S.: Lectures on Conformal Field Theory, Advanced Studies in Pure Mathematics 16 (1988), Conformal Field Theory and Solvable Lattice Models, pp. 123–170;

[4] Eisenstein, G: Beiträge zur Theorie der elliptischen Funktionen, Crelle, 35 (1847), part VI, 153–274, reprinted in Mathematische Abhandlungen besonders aus dem Gebiete der höheren Arithmetik und der elliptischen Funktionen. Mit einer Vorrede von C.F. Gauss, Georg Olms Verlagsbuchhandlung (1967), 213–334;

[5] Eguchi, T. and Ooguri, H.: Conformal and current algebras on a general Riemann surface, Nucl. Phys. B282, 308–328 (1987);

[6] Friedan, D. and Shenker, S.: The analytic geometry of two-dimensional conformal field theory, Nucl. Phys. B281, (1987), 509–545;

[7] Headrick, M.: Virasoro.nb (Mathematica notebook), available at http://people.brandeis.edu/headrick/Mathematica/;

[8] Hilbert, D.: Die Grundlagen der Physik, Erste Mitteilung, vorgelegt in der Sitzung vom 20. November 1915, Nachrichten von der Koeniglichen Gesellschaft der Wissenschaften zu Goettingen, Math-physik. Klasse, 1915, 395–407 (1915);

[9] Humphries, S.: Normal closures of powers of Dehn twists in mapping class groups, Glasgow Mathematical Journal, 34(3), (1992), 314–317;

[10] Leitner, M.: Virasoro correlation functions on hyperelliptic Riemann surfaces, Lett. Math. Phys. 103.7, 701–728 (2013);

[11] Leitner, M.: CFTs on Riemann Surfaces of genus $g \geq 1$, PhD thesis, TCD (2014);
[12] Leitner, M., Nahm, W.: Rational CFTs on Riemann surfaces, in preparation, preprint arXiv:1705.07627;

[13] Leitner, M.: Convolutions on the complex torus, Int. J. Number Theory doi.org/10.1142/S1793042121500391;

[14] Mason, G., Tuite, M.: On genus two Riemann surfaces formed from sewn tori, Commun. Math. Phys. 270, 587–634 (2007);

[15] Moore, G.W., Seiberg, N.: Classical and quantum conformal field theory, Commun. Math. Phys. 123 (1989), 177;

[16] Nahm, W.: Automorphic forms for $g \geq 1$, Mathematisches Forschungsinstitut Oberwolfach, Report 38:2388, (2017), Oberwolfach reports; Automorphic forms and quantum field theory, talk given at the conference on Integrable Systems and Automorphic Forms, February 2020, Sirius Mathematics Center, Sochi;

[17] Polyakov, A.M.: Quantum Geometry of bosonic strings; Phys. Lett. B103, 207–210 (1981);

[18] Rosenhain, G.: Sur les fonctions de deux variables à quatre périodes, qui sont les inverses des intégrales ultra-elliptiques de la première classe, in Mémoires présentés par divers savants, 2nd ser., 11 (1851), 361–468; also translated into German as Abhandlung über die Functionen zweier Variabler mit vier Perioden, Ostwalds Klassiker der Exakten Wissenschaften, no. 65 (Leipzig, 1895), Académie des sciences (France). The citations in the text refer to the German translation;

[19] Segal, G.B.: The Definition of Conformal Field Theory, in: Bleuler K., Werner M. (eds) Differential Geometrical Methods in Theoretical Physics. NATO ASI Series (Series C: Mathematical and Physical Sciences), vol 250. Springer (1988), 165–171;

[20] Shapovalov, N.N.: On a bilinear form on the universal enveloping algebra of a complex semisimple Lie algebra, Funktsional. Anal. i Prilozhen., 6:4 (1972), 65–70; Funct. Anal. Appl., 6:4 (1972), 307–312;

[21] Sonoda, H.: Sewing Conformal Field Theories I, Nucl.Phys. B311, 401–416 (1988), Sewing Conformal Field Theories II, ibid., 417–432;

[22] Tuite, M., private communication (2015);

[23] Vafa, C.: Conformal Theories and Punctured Surfaces, Phys. Lett. B199, 195–202 (1987);

[24] Weil, A.: Elliptic functions according to Eisenstein and Kronecker, Springer Verlag, New-York (1976);

[25] Weinberg, S.: Gravitation and Cosmology, John Wiley & Sons, New York-Chichester-Brisbane-Toronto-Singapore (1972);

[26] Zagier, D. and Kaneko, M.: Supersingular $j$-invariants, hypergeometric series, and Atkin’s orthogonal polynomials in Proceedings of the Conference on Computational Aspects of Number Theory, AMS/IP Studies in Advanced Math. 7, International Press, Cambridge (1997), 97–126;
[27] Zagier, D.: *Elliptic modular forms and their applications*, in The 1-2-3 of Modular Forms: Lectures at a Summer School in Nordfjordeid, Norway, Universitext, Springer-Verlag, Berlin-Heidelberg-New York (2008), pp. 1–103.