Gravitational collapse of an isentropic perfect fluid with a linear equation of state

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We investigate here the gravitational collapse end states for a spherically symmetric perfect fluid with an equation of state \( p = k \rho \). It is shown that given a regular initial data in terms of the density and pressure profiles at the initial epoch from which the collapse develops, the black hole or naked singularity outcomes depend on the choice of rest of the free functions available, such as the velocities of the collapsing shells, and the dynamical evolutions as allowed by Einstein equations. This clarifies the role that equation of state and initial data play towards determining the final fate of gravitational collapse.

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I. INTRODUCTION

Considerable interest is seen in recent years to examine the final fate of gravitational collapse of a massive matter cloud within the framework of Einstein’s theory of gravity. This is due to the importance of this problem in black hole physics, and the related cosmic censorship conjecture which states the spacetime singularities of collapse must be hidden within black holes, and would not communicate with faraway observers in the spacetime (for some recent reviews, see e.g. [1-5]. The assumption that continual collapse of a matter cloud, such as a massive star which has exhausted its nuclear fuel, ends only in a black hole (BH) and not naked singularity (NS) is crucial to many of the considerations in the physics and astrophysics of black holes. Several dynamical collapse scenarios have been investigated extensively from such a perspective, which include the radiation collapse within the framework of a Vaidya metric (see e.g. [6], and references therein), collapse of a dust cloud [6-10], some considerations on perfect fluids (many of these being numerical) [11-19], massless scalar fields [20-26], and also more general matter fields [27-29].

Our purpose in this note is to study analytic models of spherical gravitational collapse of a perfect fluid with an equation of state \( p = k \rho \). This is of interest because this is a well-studied and extensively used case in astrophysics, which offers a physically interesting model. While modeling a realistic collapse, one has to consider the equation of state (EOS) of the collapsing matter as an additional constraint over the Einstein equations. This is a long standing interesting question in the arena of dynamical collapse theories as to how a physically realistic EOS affects the evolution of a collapse in terms of it’s final state. In the later stages of collapse one could not ignore the pressures, and hence the case when matter is an isentropic perfect fluid with a linear equation of state offers a useful scenario to examine collapse in ultra-relativistic limits.

We show here that a perfect fluid collapse could end in either of the BH/NS final states, depending on the nature of the initial data, and the allowed evolutions for the collapsing matter. Given a regular initial data for matter in terms of the regular density and pressure profiles, the exact outcome in terms of the above depends on the choice of rest of the free functions available, such as the velocities of the collapsing shells, and the allowed evolutions for the collapsing matter. Our results thus provide some insight on the role that equation of state and initial data play to determine the final fate of continual gravitational collapse in terms of the BH/NS end states.

The basic regularity conditions and the Einstein equations are given in Section 2. The collapse is examined in Section 3, and in Section 4 we study how the nature of the singularity is determined by the initial data and the evolution chosen. Some conclusions are summarized in Section 5.

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II. EINSTEIN EQUATIONS AND REGULARITY CONDITIONS

The spacetime geometry within the spherically symmetric collapsing cloud can be described by the metric in the comoving coordinates \((t, r, \theta, \phi)\) as given by,

\[
ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\psi(t,r)}dr^2 + R^2(t,r)d\Omega^2
\]

(1)

where \(d\Omega^2\) is the line element on a two-sphere. In this reference frame the energy-momentum tensor for a perfect fluid is given by,

\[
T^{ij} = (\rho + p)V^iV^j + pg^{ij}
\]

(2)

where \(V^i\) is unit timelike vector. We also take the matter fields to satisfy the weak energy condition, i.e. the energy density measured by any local observer is non-negative. Then we must have,

\[
T_{ik}V^iV^k \geq 0
\]

(3)

which amounts to,

\[
\rho \geq 0; \quad \rho + p \geq 0;
\]

(4)

Since we consider here an isentropic fluid, whose pressure is a linear function of the density only, the equation of state of the collapsing matter is given by,

\[
p(t, r) = k\rho(t, r)
\]

(5)

where \(k \in (-1, 1]\) is a constant. We note that the cases of negative \(k\) are the dark energy fluids with negative pressures. Now for the metric \(1\) the Einstein equations take the form (in the units \(8\pi G = c = 1\))

\[
\rho = \frac{F'}{R^2 R'} = -\frac{1}{k} \frac{\dot{F}}{R^2 R}
\]

(6)

\[
\nu' = -\frac{k}{k+1} [\ln(\rho)]'
\]

(7)

\[
R'\dot{G} - 2\dot{R}\nu' G = 0
\]

(8)

\[
G - H = 1 - \frac{F}{R}
\]

(9)

where,

\[
G(t, r) = e^{-2\psi}(R')^2; \quad H(t, r) = e^{-2\nu}(\dot{R})^2
\]

(10)

The arbitrary function \(F = F(t, r)\) here has an interpretation of the mass function for the cloud, and it gives the total mass in a shell of comoving radius \(r\) on any spacelike slice \(t = \text{const}\). We have \(F \geq 0\) from the energy conditions. In order to preserve the regularity at the initial epoch, we have \(F(t_i, 0) = 0\), that is, the mass function should vanish at the center of the cloud. Since we are considering collapse, we have \(\dot{R} < 0\), i.e. the physical radius \(R\) of the cloud keeps decreasing in time and ultimately reaches \(R = 0\), which denotes the singularity where all matter shells collapse to a zero physical radius. We can use the scaling freedom available for the radial co-ordinate \(r\) to write \(R = r\) at the initial epoch \(t = t_i\). Let us introduce a function \(v(t, r)\) as defined by,

\[
v(t, r) \equiv R/r
\]

(11)

we then have \(R(t, r) = rv(t, r)\), and

\[
v(t_i, r) = 1; \quad v(t_s(r), r) = 0; \quad \dot{v} < 0
\]

(12)
The time \( t = t_i(r) \) corresponds to the shell-focusing singularity at \( R = 0 \), where all the matter shells collapse to a vanishing physical radius.

The description of the singularity in terms of \( v(t, r) \) has the following advantage. The physical radius goes to the value zero at the shell-focusing singularity, but we also have \( R = 0 \) at the regular center of the cloud at \( r = 0 \). This is to be distinguished from the genuine singularity by the fact, for example, that the density and other physical quantities including the curvature scalars are all finite at a regular center \( r = 0 \), even though \( R = 0 \) holds there. This is achieved, as we point out below, by a suitable behaviour of the mass function, which should go to a vanishing value sufficiently fast in the limit of approach to the regular center where even though \( R \) goes to zero the density must remain finite. On the other hand, we note that at \( t = t_i \) we have \( v = 1 \), and then as the collapse evolves \( v \) continuously decreases to become zero only at the singularity, i.e. \( v = 0 \) uniquely corresponds to the genuine spacetime singularity at \( R = 0 \).

We thus see that there are now five total field equations with five unknowns, which are \( \rho, \psi, \nu, R, \) and \( F \). Solutions of these equations, subject to the weak energy condition and the given regular initial data for collapse at the initial spacelike surface \( t = t_i \), determine the matter distribution and metric of the space-time. We then have specific time evolutions of the initial data which define the collapse final states. It turns out that there exist classes of solutions, which give either a black hole or a naked singularity as the end state of the collapse, depending on the nature of the initial data, and the class of evolutions chosen.

### III. COLLAPSING MATTER CLOUDS

It is now possible to consider the gravitational collapse of a perfect fluid within this framework as we discuss below. The regularity conditions as defined above set up the initial data at the initial surface \( t = t_i \) from which the collapse develops. We now consider the collapse equations which allow us to see when the singularity occurs, and how the initial data and the classes of evolutions as governed by the Einstein equations lead to the formation of the spacetime singularity.

We consider a general mass function \( F(t, r) \) for the collapsing cloud, which can be written as

\[
F(t, r) = r^3 M(r, v)
\]  

(13)

where \( M \) is a regular and suitably differentiable function, and \( M > 0 \). As seen from equations below, the regularity and finiteness of the density profile at the initial epoch \( t = t_i \), and at all other regular epochs before the singularity at \( R = 0 \) develops, requires that \( F \) goes as \( r^3 \) close to the center. Hence we note that since \( M \) is a general (at least \( C^2 \)) function, the equation equation (13) is not really any ansatz, or a specific choice, but quite a generic form of the mass profile for the collapsing cloud. We note that [19] considered mass profiles which are analytic (with \( F(t, r) = F(r, R) \)), and perfect fluids could form naked singularity, but we do not impose such an assumption on mass function here.

Then equation (13) gives,

\[
\rho = \frac{3M + r[M_r + M_v v']}{v^2(v + rv')} = \frac{1}{k} \frac{M_r}{v^2}
\]

(14)

The regular density distribution at the initial epoch is given by,

\[
\rho_0(r) = 3M(r, 1) + rM(r, 1)_r
\]

(15)

It is evident that, in general, as \( v \to 0, \rho \to \infty \). That is, both the density and pressure blow up at the singularity. We note from equation (14) that \( \rho = \rho(r, v) \) and hence \( v' = f(r, v) \). Now rewriting equation (14) we get,

\[
3kM + krM_r + Q(r, v)M_v = 0
\]

(16)

where,

\[
Q(r, v) = (k + 1)rv' + v
\]

(17)

Now the above equation (16) has a general solution of the form (18),

\[
F(X, Y) = 0
\]

(18)

where \( X(r, v, M) \) and \( Y(r, v, M) \) are the solutions of the system of equations,

\[
\frac{dM}{3k} = \frac{dr}{kr} = \frac{dv}{Q}
\]

(19)
Amongst all the classes of solutions of $M(r,v)$ as given by equation (18), only those are to be considered which obey the energy condition, which are regular, and which in the limit of $v \to 0$ give $\rho \to \infty$. In other words, the equation of state as given by the perfect fluid condition $p = k\rho$, and the energy condition isolate the class of the mass functions to be considered.

We can now directly integrate equation (7) to get,

$$\nu(r,v) = -\frac{k}{k+1} \ln(\rho)$$

(20)

Let us now define a suitably differentiable function $A(r,v)$ in the following way,

$$\nu'(r,v) = A(r,v),\frac{R'}{R}$$

(21)

That is $A(r,v),\nu' = \nu'/R'$. We note that equation (14) can in principle give solutions which are not regular, or such that the function $A(r,v)$ as defined above is not regular. However, regularity of $M$ and $A$ is a necessary assumption we make here, which will be used later also (see e.g. remarks after equation (30)). Hence, only the class of regular solutions is to be considered.

Our main interest here is in studying the shell-focusing singularity at $R = 0$ which is physical singularity where all the shells collapse to zero radius. Hence we assume that there are no shell-crossing singularities in the spacetime, where $R' = 0$ and so the function $A(r,v)$ is well-defined.

Some comments are in order here on our assumption that $R' > 0$, that is, we consider the situation with no shell-crossing singularities. This is because, it is generally believed (see e.g. [31] that such singularities can be possibly removed from the spacetime as they are typically gravitationally weak, and also because spacetime extensions have been constructed through the same in certain cases. Under the situation, we are interested only in examining the nature of the shell-focusing singularities at $R = 0$, which are genuine curvature singularities, where the physical radii for all collapsing shells vanish, and the spacetime necessarily terminates without extension.

Specifically, $R' > 0$ implies that we must have $v + rv' > 0$. Since $v$ is necessarily positive, it follow that this will be satisfied whenever $v'$ is greater or equal to zero, or even when it is negative the magnitude of $rv'$ should be less then that of $v$. Later in this section we shall derive an expression for the quantity $v'$, in terms of the initial data and the other free evolutions as allowed by the Einstein equations. Hence it follows that we can specifically state the condition for avoidance of shell-crossings in terms of the behaviour of these functions. In particular, it turns out that whenever the singularity curve $t_s(r)$ (which corresponds to $R = 0$) is increasing (or when it decreases at a sufficiently slow rate) with a slope greater or equal to zero at the origin, the shell-crossing singularities are avoided at least in the vicinity of the regular center $r = 0$. We then have a ball of finite radius around the regular center which contains no shell-crossings till the final singularity formation at $R = 0$. We shall, however, not go into further details here.

At the initial epoch we have,

$$A(r,v)\bigg|_{v=1} = -\frac{k}{k+1} \left[ \frac{\rho_0'(r)}{\rho_0(r)} \right]$$

(22)

In fact, for all epochs, the relation between the function $M$ and $A$ is given by equation (21) as $A,v,R' = -\frac{k}{k+1} \ln \left[ -\frac{M}{k\rho v^2} \right]'$. If we consider a smooth initial profile, i.e. the gradient of the initial density vanishes at the center, then we must have $A(r,v) = rg(r,v)$, where $g(r,v)$ is another suitably differentiable function.

Now using equation (21) we can integrate (22) to get,

$$G(r,v) = b(r)e^{2rA}$$

(23)

Here $b(r)$ is another arbitrary function of the comoving coordinate $r$. A comparison with dust collapse models interprets $b(r)$ as the velocity function for the collapsing shells. Following this parallel, we can write,

$$b(r) = 1 + v^2b_0(r)$$

(24)

Finally, using equations (22), (20) and (24) in (21) we have,

$$\sqrt{vv} = -\rho^{\frac{k}{k+1}} \sqrt{e^{2rA}v_0(r) + vh(r,v) + M(r,v)}$$

(25)

where,

$$h(r,v) = \frac{e^{2rA} - 1}{r^2}$$

(26)
Integrating the above equation we have,

\[ t(v, r) = \int_{v}^{1} \frac{\sqrt{v} dv}{\rho - k + 1} \sqrt{e^{2rA} vB_0 + vh + M} \]  

(27)

Note that the variable \( r \) is treated as a constant in the above equation. Close to the center we can write \( t(v, r) \) as,

\[ t(v, r) = t(v, 0) + rX(v) + O(r^2) \]  

(28)

Here the function \( X(v) \) is given by,

\[ X(v) = -\frac{1}{2} \int_{v}^{1} \frac{\sqrt{v} B_1(0, v)}{B(0, v)} \]  

(29)

where,

\[ B(r, v) = \rho - \frac{k}{1 + k} \sqrt{e^{2rA} vB_0 + vh + M}; \quad B_1 = B_{v, r} \]  

(30)

In order to obtain equation (28), we note that we require the integral (27) could be differentiated. This is possible because it is finite by definition, and then we need all the functions, namely \( A(r, v), b_0(r) \) and \( M(r, v) \) to be suitably differentiable. In our case we require them to be at least \( C^2 \) for \( r \) not equal to zero, and \( C^1 \) for \( r = 0 \).

Thus we see that the time taken for the central shell to reach the singularity is given as

\[ t_{s_0} = \int_{0}^{1} \frac{\sqrt{v} dv}{B(0, v)} \]  

(31)

The time for other shells to reach the singularity is given by the following, which defines the singularity curve developing in the spacetime as end result of collapse,

\[ t_s(r) = t_{s_0} + rX(0) + O(r^2) \]  

(32)

It is now clear that the value of the quantity \( X(0) \), which represents the tangent to the singularity curve, depends on the functions \( b_0, M \) and \( h \), which have the initial values as dictated by the initial data at \( t = t_i \), and which are functions of \( r \) and \( v \) as the case may be. Hence, a given set of density and velocity profiles, together with the evolutions chosen, completely determines the tangent at the center to the singularity curve. Further, from equation (25), we get,

\[ \sqrt{vv'} = X(v)B(0, v) + O(r) \]  

(33)

We note that as seen above, \( X(0) \) involves functions depending on the initial data, and also the evolutions \( A \) and \( M \). The relation between the functions \( A \) and \( M \) is given by the perfect fluid equation of state, for which we have shown that solutions exist, and the Einstein equations as noted earlier. Among different classes of solutions only those are to be considered which ensure the density and pressures to blow up at the singularity.

One has now to understand the structure of this singularity, and to examine when it will be visible, and when covered within an event horizon of gravity, i.e. hidden within a black hole.

**IV. NATURE OF THE SINGULARITY**

It is now possible to see for a perfect fluid collapse with a linear equation of state, how the initial data and the allowed evolutions determine the final fate of collapse in terms of either a black hole or a naked singularity. The apparent horizon within the collapsing cloud is given by \( R = F \). If the neighborhood of the center gets trapped earlier than the singularity, then it is covered and a black hole results, otherwise it is visible with non-spacelike future directed trajectories escaping from it. In other words, we examine below when there will be families of null geodesics existing, which will be future directed and outgoing, and which terminate in the past at the singularity, thus making the communication from the singularity to an outside observer possible, as opposed to a black hole situation where this will not be the case.

In order to consider the possible existence of such trajectories and to examine the nature of the central singularity at \( R = 0, r = 0 \), let us consider the equation for outgoing radial null geodesics which is given by,

\[ \frac{dt}{dr} = e^{\psi - \nu} \]  

(34)
The singularity occurs at \( v(t, r) = 0 \), i.e. \( R(t, r) = 0 \). Therefore, if there are any future directed null geodesics existing, which terminate in the past at the singularity, we must have \( R \to 0 \) as \( t \to t_s \) along these curves. Now writing equation (39) in terms of variables \((u = r^\alpha, R)\), we have,

\[
\frac{dR}{du} = \frac{1}{\alpha} r^{-(\alpha - 1)} R' \left[ 1 + \frac{\dot{R}}{R} e^{\psi - \nu} \right]
\]  

(35)

Choosing \( \alpha = \frac{2}{3} \) and using equation (34) we get,

\[
\frac{dR}{du} = \frac{3}{5} \left( \frac{R}{u} + \frac{\sqrt{v'}}{\sqrt{R}} \right) \left( 1 - \frac{F}{R^2} \frac{\dot{R} \sqrt{G} + \sqrt{\dot{R}}}{\sqrt{G}} \right)
\]  

(36)

If there are null geodesics which terminate at the singularity in the past with a definite tangent, then at the singularity we have \( \frac{dR}{du} > 0 \), in the \((u, R)\) plane with a finite value. Hence it follows that all points \( r > 0 \) on the singularity curve are covered necessarily, because \( F/R \to \infty \) with \( \frac{dR}{du} \to -\infty \) for any of these, and hence no outgoing null geodesics can terminate at these points in the past. The central singularity at \( r = 0 \) could however be naked. Define the tangent to the outgoing null geodesic from the singularity as,

\[
x_0 = \lim_{t \to t_s, r \to 0} \frac{R}{u} = \frac{dR}{du} \bigg|_{t \to t_s, r \to 0}
\]  

(37)

Using equation (30) and (34), we then get,

\[
x_0 = \frac{3}{2} \sqrt{B(0, 0)} \mathcal{X}(0)
\]  

(38)

Let us now deduce the necessary and sufficient conditions for a naked singularity to exist, that is, for null geodesics with a well-defined tangent to come out from the central singularity. Suppose we have \( \mathcal{X}(0) > 0 \), then we always have (from equation (35)), \( x_0 > 0 \) and then in the \((R, u)\) plane, the equation for the null geodesic that comes out from the singularity is given by

\[
R = x_0 u
\]  

(39)

In other words, equation (39) is a solution of the null geodesic equation in the limit of the central singularity. Thus given \( \mathcal{X}(0) > 0 \), we can always construct a solution of radially outgoing null geodesics emerging from the singularity. This makes the central singularity visible. In the \((t, r)\) plane, the null geodesics outgoing from the singularity will be given as,

\[
t - t_s(0) = x_0 r^{\frac{2}{3}}
\]  

(40)

It follows that \( \mathcal{X}(0) > 0 \) implies \( x_0 > 0 \) and we get radially outgoing null geodesics emerging from the singularity, giving rise to the central naked singularity.

On the other hand, if \( \mathcal{X}(0) < 0 \), then we see that the singularity curve is a decreasing function of \( r \). Hence the region around the center gets singular before the central shell, and the spacetime then terminates there. In this case, if there were any outgoing null geodesic from the central singularity, it must then go to a singular region, or outside the spacetime which is impossible. Hence when \( \mathcal{X}(0) < 0 \), we always have a black hole solution.

If \( \mathcal{X}(0) = 0 \) then we will have to take into account the next higher order non-zero term in the singularity curve equation, and do a similar analysis by choosing a different value of \( \alpha \) in equation (35).

We have thus shown above that \( \mathcal{X}(0) > 0 \) is the necessary and sufficient condition for null geodesics to come out from the central singularity with a definite positive tangent. It is thus seen how the initial data, together with the evolutions chosen in terms of the free functions such as \( b_0, \mathcal{M} \) and \( h \), fully determine the final end product of collapse in terms of either a black hole or a naked singularity. This is as determined by the \( \mathcal{X}(0) \) values above, or more generally in terms of the behaviour of the singularity curve in the vicinity of the central singularity. This is because \( \mathcal{X}(0) \) is determined by these initial profiles and the evolutions chosen as given by equation (29), which in turn determine the end states. Therefore, given any initial regular density and pressure profiles for the matter cloud from which the collapse develops, there always exist velocity profiles for collapsing matter shells, and evolutions as determined by the Einstein equations, so that the end state of the collapse would be BH or NS, depending on the choice made.

As seen above, the different outcomes are characterized by the positive or negative values of \( \mathcal{X}(0) \). This in turn depends on the initial values of the functions such as the initial mass and velocity profiles, and the evolutions chosen, as permitted by the Einstein equations. Thus the measures of outcomes leading to BH/NS phases are accordingly decided (e.g. the entire set of initial data and evolutions giving \( \mathcal{X}(0) > 0 \) leads to NS).
V. CONCLUSIONS

While some of the numerical models for perfect fluids indicated that naked singularities could arise for only a ‘soft’ equation of state, our results point out that within a generic perfect fluid collapse scenario with equation of state $p = kp$, as such the value of $k$ chosen does not appear to have any special significance. What matters is the initial data, and the chosen evolutions (i.e. the classes of allowed solutions to the Einstein equations), which then take this given initial data to a specific outcome, depending on the choice made. Also, if for a given chosen evolution, if the value $\mathcal{X}(0)$ was negative (or positive), then such will be the case by continuity for all neighbouring or close by evolutions where ‘nearness’ is defined in some suitable sense. Hence these outcomes in terms of BH or NS may be considered stable in a certain sense (within spherical symmetry) as characterized above. We thus see that there are classes of solutions to Einstein equations for perfect fluid models where given the matter initial data at the initial surface $t = t_i$, these evolutions take the collapse to end up either as a black hole or the naked singularity, depending on the choice of the class. This also means that the total space of evolutions can be divided into distinct subspaces, those that evolve a given initial data into black holes, and others that go to a naked singularity. The results on dust collapse are of course contained here as a special case with $p = 0$.

We should mention here that though the above give some information on dynamical evolution of collapse while pressures are included in the analysis, non-spherical perturbations will have important inputs to decide on the issues such as genericity and stability. It is possible that methods such as those developed in [32-33] could be useful in that direction.

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