Ladder climbing and autoresonant acceleration of the spherical plasma density wave

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Abstract

Ladder climbing (LC) and autoresonance (AR) of the spherical plasma density wave are studied for the first time. The governing equation of the perturbed spherical density wave in the energy level space based on a hydrodynamic model of the electron plasma is presented, and it is demonstrated that the quantum LC and classical AR transition can be achieved in the spherical plasma. The asymptotic thresholds of the LC and AR transition of the spherical plasma wave are obtained analytically and confirmed numerically. We find that the spherical wave energy is concentrated to the sphere center as the density wave climbs to the higher level, the spherical plasma behaves obvious compression character, and the perturbed density of the sphere center even can be amplified to 100 times larger of the initial perturbed density. Compared to the one-dimensional case, the energy spectrum of the spherical plasma wave shifts upward, and the energy level spacing of the spherical plasma wave is broadened. These result in the facts that the spherical plasma needs the larger driving strength to achieve the LC and AR, while the total perturbed density of the spherical plasma always is larger than that of the one-dimensional case.

1. Introduction

It is known that the quantum mechanics is closely related to the mechanics of the classical waves [1]. Therefore, the transition between the quantum and classical descriptions of dynamical system plays an important role in the foundation of quantum mechanics. Studying the subtleties of the quantum-classical crossover is still an active research field. An effective framework of this field is to achieve a cascading transition from the initial to the final state through a series of intermediate levels by using a chirped light pulse with a continuously varying frequency [2–6]. This method is usually named the autoresonance (AR) in the classical mechanics, while the ladder climbing (LC) is used generally in the quantum mechanics.

AR is a continuing phase-locking phenomenon, which can excite a classical nonlinear oscillatory system to high energy by a weak chirped driving perturbation and control the excited state by changing the driving frequency. This method is first mentioned in relativistic particle accelerators [7] and by now it has been applied widely in numerous fields of physics, such as optics [8], fluid dynamics [9], Josephson junctions [10], nonlinear waves [11] and even planetary dynamics [12]. LC is the quantum counterpart of the classical AR and it is a series of successive two-level Landau–Zener (LZ) transitions, where only two adjacent energy levels of the driven oscillator are coupled at any given time. Compared with AR, the discrete nature of LC is visible only when spectra of the systems is discrete enough, therefore, it is studied only in quantum systems including anharmonic oscillators [13–15], Rydberg atom [16] and bouncing neutrons [17]. Interestingly, it is demonstrated that classical systems, for example, Langmuir waves in bounded plasma, also can exhibit LC much like a quantum system [18, 19]. This is the first report about the LC phenomenon of the classical system and it has a significant impact on the acceleration and manipulation of the electrons.

However, the prediction of the LC and AR in the classical plasma system given by [18, 19] is based on a one-dimensional plasma model in terms of the energy of the electrons. Obviously, to describe the general and real
problems, considering the case of the higher dimension is urgent, particularly, for the spherical plasma wave. The spherical plasma is an interesting subject of studying and it is significant for the injection and acceleration of the electrons [20, 21]. In addition, the most of the laboratory facilities about the inertial confinement fusion are spherically symmetric [22, 23]. Clearly, the existence of the LC and AR in the spherically symmetric plasma has important impact on the electrons acceleration and inertial confinement fusion. However, this has never been studied and is still a pending question. The purpose of this paper is to analyse the LC and AR dynamics of the spherically symmetric plasma wave in terms of the perturbed density of the plasma.

Starting from the basic equations of the collisionless electron plasma described by a spherical hydrodynamic model, we first derive the governing equation of the perturbed spherical density wave in the energy level space, and demonstrate that the spherical plasma also exists LC and AR phenomena. The asymptotic thresholds for the quantum LC and classical AR transition are obtained analytically and confirmed numerically. We find the driving strength of the spherical plasma is approximately 1.66 times that of the one-dimensional plasma to achieve these phenomena due to its spherical geometric structure. Interestingly, the spherical wave energy is concentrated to the sphere center as the density wave climbs to the higher level, the spherical plasma behaves obvious compression character, and the perturbed density of the sphere center even can be amplified to 100 times larger of the initial perturbed density. In addition, we also find that the total perturbed density of the spherical plasma always is larger than that of the one-dimensional plasma at any time due to the dispersion relation and the distributions of the occupation numbers of the spherical wave. These can help us to deep understand the AR acceleration of the electrons and have important impact on the inertial confinement fusion.

2. The model and theoretical analysis

2.1. The model and the evolution equation of the spherical plasma wave

Considering the spherical symmetry, we use the spherical coordinate system and consider all the variables that describe the dynamics of the plasma only depend on the position \( r \) and time \( t \). For simplicity, we consider a collisionless electron plasma described by a hydrodynamic model. In general, the continuity and momentum equations for the electrons, as well as the Poisson equation are respectively

\[
\frac{\partial n_e}{\partial t} + \frac{1}{r^2} \frac{\partial (r^2 n_e u_e)}{\partial r} = 0,
\]

\[
\frac{\partial (n_e u_e)}{\partial t} + \frac{1}{r^2} \frac{\partial (r^2 n_e u_e^2)}{\partial r} = -\frac{e}{m_e} n_e E - \frac{1}{m_e} \frac{\partial P}{\partial r},
\]

\[
\frac{1}{r^2} \frac{\partial (r^2 E)}{\partial r} = -4\pi e(n_e - Z n_i),
\]

where \( n_e, u_e, E, -e, m_e, P, Z e \) and \( n_i \) denote the electron density, the electron flow velocity, the electric field, the electron charge, the electron mass, the electron pressure, the ion charge and the ion density, respectively.

We consider an external driving created by the ponderomotive forces of the laser, which leads to a fluctuation of the electrons density, i.e. \( n_e = n_0 + n_d(r, t) + n(r, t) \), where \( n_0 \) is the unperturbed electron density, \( n_d \) is a slow modulation density caused by the driving, and \( n \ll n_0 \) is a small perturbed density, which is created by the smaller inertia of the electron. Note that because the mass of the electrons is much smaller than that of the ions, the characteristic time that the electrons can respond to the external driving is much shorter than that for the ions. So, we neglect high-frequency oscillations of \( n_i \) and consider \( n_i \) to be a slow function, \( Z n_i = n_0 + n_d(r, t) \). For simplicity, we assume the whole process is adiabatic, thus, the electron pressure \( P \) can be described by \( P(n_e) \approx P_0 + 3 n_e v_{th}^2 (n + n_d) \), where \( v_{th} = \sqrt{k_B T_0/m_e} \) is the electron thermal speed, \( k_B \) is the Boltzmann constant, and \( T_0 \) is the background temperature of the electrons.

We consider all of the perturbation quantities (the first-order quantity) are small, i.e. \( n \ll n_0 \). In this case, the higher order quantities \( O(n_e^2), O(nE) \) and \( O(n^2) \) can be neglected. In order to combine equations (1)–(3) into a single equation, we subtract the divergence of equation (2) from the temporal derivative of equation (1), and then substitute \( r^{-2} \partial_t (r^2 E) = -4\pi e n \) (which comes from equation (3)), and neglect the higher order terms, we can obtain a dimensionless equation as follow

\[
\frac{\partial^2 n_d}{\partial t^2} - 3 \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial n_d}{\partial r} \right) = -\frac{\partial^2 n}{\partial t^2} + \frac{3}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial n}{\partial r} \right) - n - mn_d + \frac{\partial n_d E}{\partial r},
\]

where the variables \( t, r, n_d(t, n) \) and \( E \) are normalized by \( \omega_p^{-1} \), \( \omega_p = \sqrt{4\pi e^2 n_0/m_e} \) is the electron plasma frequency, \( v_{th}/\omega_p n_0 \) and \( m_e \omega_p v_{th}/e \), respectively. Note that the electric field \( E \) in equation (4) satisfies \( r^{-2} \partial_t (r^2 E) = -n \), i.e. the electric field \( E \) depends on the perturbed density \( n \), thus, equation (4) actually has the two variables \( n_d \) and \( n \), equation (4) is not closed. However, we notice that the left hand side of equation (4) is just about the modulation density \( n_d \), therefore, in order to simplify and close equation (4), we choose the
is the zeroth order spherical Bessel function, here, 

\[ w_{j1, 2, 3, 2} \]

\[ \text{via} \]

\[ y_{mm m} \]

\[ we \introduce a new function \]

\[ nm \]

\[ where \]

\[ R \]

\[ A \]

\[ \text{note that equations (5)} \rightarrow (6) \] are a closed set of equations. We can obtain the driving modulation density \( n_d(r, t) \) as the form of the \( Nth \) spherical standing-wave mode by solving equation (5) with the given boundary conditions \( \partial_t n_{d|t=0} = \partial_t n_{d|t=R_0} = 0 \) (i.e. \( u_{d|t=0} = u_{d|t=R_0} = 0 \))

\[ n_d(r, t) = A j_0(k_N r) \cos \varphi_d(t), \]

where \( A \) is the amplitude of the driving modulation density, \( \varphi_d = \int \omega_d dt \) is the phase of the driving, \( \omega_d \) is the driving frequency. In particular, it is important to note that in order to study the LC and AR dynamics of the chirped system, we consider the case that the modulation is weak and slow, i.e. \( A \ll 1 \) and \( \omega_d \ll \omega_p \) (the response time of the perturbation is much shorter than the period of the driving). The concrete form of \( \omega_d \) and the details about \( \omega_d \) are presented in the later discussions. Here, the spherical plasma wave can be excited by the ponderomotive force of the driving chirped laser beams imposed on the surface of the spherical plasma. By controlling the chirped laser parameters, the driving ponderomotive force of the laser beam imposing on the surface of the spherical plasma changes with time, which can perturb the surface density changing with the time. Because we consider a very slow chirped laser pulse, i.e. \( \omega_d \ll \omega_p \), the response time of the disturbance is much shorter than the period of the driving, i.e. the formation of a Bessel function is much faster compared to the chirp. The perturbed density wave propagates towards the sphere center, and as time goes on, the waves bounce back and forth between the spherical center and the surface of the plasma sphere due to the hardwall boundary conditions. Then, the standing spherical plasma wave can be excited.

In equation (7), \( j_0(k_N r) = \sin(k_N r)/(k_N r) \) is the zeroth order spherical Bessel function, here, \( k_N \) is the wave number of the \( Nth \) standing-wave \((N = 1, \) generally), and it can be given in a universal form as follow

\[ k_N R_0 = \tan(k_N R_0), \]

where \( R_0 \) is the radius of the plasma sphere, and the subscript \( m \) is a positive integer, i.e. \( m = 1, 2, 3, \cdots \). Hence, the wave numbers \( k_m \) determined by equation (8) are discrete.

To simplify equation (6), we expand the perturbed density \( n(r, t) \) with the unperturbed eigenmodes of equation (6). When the external drive disappears in equation (6), i.e. \( n_d(r, t) = 0 \), the unperturbed solution of equation (6) with the given boundary conditions \( \partial_t n_{|t=0} = \partial_t n_{|t=R_0} = 0 \) can be obtained \( n \sim e^{-i \omega_d t} j_0(k_m r) \)as follow

\[ n(r, t) = \Re \sum_{m=1}^{\infty} n_m(t) e^{-i \omega_d t} j_0(k_m r), \]

where \( n_m(t) \) are complex coefficients, \( m = 1, 2, 3, \cdots \) are the mode numbers, \( k_m \) are the wave numbers of the spherical plasma wave, and it is given by equation (8). In addition, there is a dimensionless discrete dispersion relation \( \omega_m^2 = 1 + 3 k_m^2 \) (here, \( \omega_m \) and \( k_m \) are normalized by \( \omega_p \) and \( 1/\lambda_D \), \( \lambda_D = \nu_D/\omega_p \) is the Debye length), which is coincident with the Bohm–Gross relation in the mathematical form. Note that the dispersion relation of the spherical plasma wave is identical to the one-dimensional case, but it is the description of the wave number \( k_m \), that is very different. The detailed discussions are presented in the section 2.2.

Then, we study the characteristic of the perturbed density wave in the energy level space. For convenience, we introduce a new function \( \psi_m \) via \( \rho_m \psi_m(t) = n_m(t) e^{-i \omega_m t} \), which can be viewed as the plasma density wave function and \( |\psi_m|^2 \) are the actions of individual modes. The coefficients \( \rho_m \) guarantee that the total action of the plasma wave \( \sum_m |\psi_m|^2 \) is conserved. It is shown that the wave action density \( |\psi_m|^2 \) should satisfy the relationship \( |\psi_m|^2 = \omega_m n_m^2 / k_m^2 \) [24]. Then, using the relationship \( \rho_m \psi_m = n_m e^{-i \omega_m t} \), we can obtain \( \rho_m = \frac{k_m}{\sqrt{\omega_m}} \). Substituting equation (7) and equation (9) into equation (6), making the Fourier-transform of equation (6), and only considering the correlations of modes \( m' = m - 1, \) \( m, \) \( m + 1 \) (note that \( m' > 1 \) must be met since \( m \geq 1 \)), we can obtain the evolution equation of the perturbed density wave in the energy level space

\[ i \frac{d\psi_m}{dt} = \phi_m \psi_m + [J_{m-1} \psi_{m-1} + J_{m+1} \psi_{m+1}] \cos \varphi_d(t), \]

where

\[ \phi_m = \omega_m + \frac{A \cos \varphi_d}{4 \omega_m k_m R_0} \times [2 \text{Si}(k_0 R_0) + \text{Si}((2k_m - k_0) R_0) - \text{Si}((2k_m + k_0) R_0)], \]

\[ \text{Si}(x) = \int_0^x \frac{\sin t}{t} dt, \]

\[ \text{Si}(x) = \frac{\sin x}{x}, \]
\[ J_m \pm 1 = \frac{A}{4k_l R_0} \left( \frac{k_m^2 m + k_{m+1}^2}{2k_m k_{m+1}} \right) \]

\[ \times \left\{ \sum_{i=0}^{n} (-1)^i \sin((k_{m+1} + (-1)^i k_m)R_0) \right\} \]

where \( \sin(\sigma) = \text{SinIntegral}(\sigma) = \int_0^\sigma \sin x \, dx \) is the sine integral function. Equation (10) can be viewed as the Schrödinger equation of the density representation, where the driving modulation density \( n_L \) serves as an effective potential. The first item on the right hand side of equation (10) is the potential energy, it is comprised of two parts: the harmonic oscillator potential (including harmonic and anharmonic potential) and the potential produced by the drive. The second term on the right hand side of equation (10) refers to the coupling between the mth and \((m-1)\)th energy level and the coupling between the mth and \((m+1)\)th energy level. It is worth noting that the coefficients of equation (10) change with the energy level \( m \), and the coupling strength between the mth and \((m-1)\)th energy level is different from that between the mth and \((m+1)\)th energy level due to the special geometric structure of the spherical plasma.

In addition, to research LC and AR dynamics of the chirped system, we assume that the resonance condition is \( \omega_d \approx \omega_{m,m+1} \), where \( \omega_{m,m+1} = \omega_{m+1} - \omega_m \) is the level spacing. For the modulation density (7), we consider the case that the dimensionless driving frequency \( \omega_d \) (which is normalized by \( \omega_p \)) changes linearly and slowly with the time, i.e. \( \omega_d = \omega_{1,2} + \alpha t \), here, \( \alpha \) is a dimensionless constant and describes the changing rate of the driving frequency over time. Because the driving frequency changes slowly with the time, we assume that \( 0 < \alpha \ll 1 \) (\( \alpha = 10^{-8} \) is used in our model). Therefore, the resonant transition between level \( m = 1 \) and \( m = 2 \) will be excited at \( t = 0 \), which we denote as \( 1 \rightarrow 2 \). Subsequently, transition between higher level, \( m \rightarrow m + 1 \), will occur when the resonance condition is satisfied.

To study more clearly the evolution of the driven system, we make a rotation transformation as follow. First, we introduce a new function \( C_m \) via \( C_m = e^{i \omega_d t} \psi_m \), neglect the nonresonant terms, we can obtain the equation for \( C_m \) from equation (10)

\[ i \frac{dC_m}{dt} = (\phi_m - \omega_m)C_m + \frac{J_{m-1} - J_{m+1}}{2} C_m + e^{i(\omega_{m-1} - \omega_m) t} C_{m+1} + e^{i(\omega_{m+1} - \omega_m) t} C_m \].

(11)

Then, we define \( B_m = C_m e^{-i \int \phi_m dt} \) and the dimensionless slow time \( \tau = \sqrt{\alpha} t \), substitute these definitions into equation (11) and let the exponential term equal to zero, i.e. choose \( \int \gamma_{m,m+1} dt = \omega_{m,m+1} t - \phi_m \), where \( \gamma_{m,m+1} = \gamma_{m+1} - \gamma_m \), we can rewrite equation (11) as follow

\[ i \frac{dB_m}{d\tau} = \tilde{\phi}_m B_m + (\tilde{J}_{m-1} B_{m-1} + \tilde{J}_{m+1} B_{m+1}) \],

(12)

where

\[ \tilde{\phi}_m = \Gamma_m + \frac{\phi_m - \omega_m}{\sqrt{\alpha}} \]

\[ \tilde{J}_m = \frac{J_{m+1} + J_{m-1}}{2 \sqrt{\alpha}}, \quad \Gamma_m = \frac{\gamma_m}{\sqrt{\alpha}} \]

Note that the introduction of \( \gamma_m \) is only to simplify equation (11), i.e. to eliminate the exponential term in equation (11). Thus, \( \gamma_m \) is determined by the relationship \( \gamma_{m,m+1} = \omega_{m,m+1} - \omega_m \). Obviously, the expression of \( \gamma_m \) depends on the form of \( \omega_m \), which is determined by equation (8). Hence, as we will see in the section 2.2, the concrete form of \( \gamma_m \) can be given after the analytic expression of equation (8) is obtained approximately.

Equation (12) and equation (8) govern the evolution of the spherical plasma wave.

If the sphere standing wave \( \psi_s(kr) \) in equation (7) and equation (9) is replaced by the plane standing wave \( \cos(kr) \), the slow evolution equation of the one-dimensional plasma wave [18] can be given by equation (12) with \( \tilde{\phi}_m = \Gamma_m, \tilde{J}_{m-1} = \tilde{J}_{m+1} = A/(8\sqrt{\alpha}) \), and the dispersion relation is reduced to \( \omega_m^2 = 1 + 3k^2 = 1 + 3\pi^2 m^2/R_0^2 \). Obviously, the one-dimensional plasma wave is described by a Schrödinger equation with constant coefficients. In this case, i.e. when our model is reduced to the one-dimensional plasma case, \( R_0 \) stands for the length of the one-dimensional plasma. We notice that the evolution equations of the spherical and one-dimensional plasma wave are clearly different: the evolution equation of the spherical plasma wave is a variable coefficient equation and the coupling strength between the mth and \((m-1)\)th energy level is different from that between the mth and \((m+1)\)th energy level (the former is weaker than the latter). Therefore, when the electrons are excited from the initial state (the ground state) to the higher energy level, maybe there is a part of the electron that are not excited, which causes the incomplete transition of the energy level. This is further revealed by the numerical simulations.
We note that, the effects of the Landau damping in the system should be considered due to the collisionless dissipation is not contained in our fluid equations, and which may cause the energy loss at high mode number [19]. If Maxwellian distribution is assumed, the local Landau damping rate for the mth mode is given by 
\[ \xi_m \approx k_m^2 \sqrt{3/\pi} \exp(-k_m^2/2 - 3/2) \] [25, 26]. Therefore, the energy will decrease as the form of \( e^{\exp(2\Xi_m)} \), where \( \Xi_m = \xi_m \Delta t \). \( \Delta t \) is the time interval between neighboring energy level (transition time), which is determined by the relation \( \omega_{m,m+1} - \omega_{m-1,m} \). Obviously, \( \Xi_m \) grows rapidly with \( m \), hence, the value of \( m \) must be restricted in order to guarantee our model is still valuable. Specifically, the maximum mode \( m_{\text{max}} \) should satisfy \( m_{\text{max}} \leq m_{\text{kin}} \) in the system, where \( m_{\text{kin}} \) is defined as \( \Xi_m \sim 1 \) [18, 19].

2.2. The asymptotic theoretical prediction for the quantum LC and the classical AR transitions

In this section, we aim to find the asymptotic analytical threshold of the quantum LC and classical AR transition in the system. Equation (12) is a variable coefficient Schrödinger equation and is hard to derive the analytical thresholds. Hence, equation (12) should be simplified to obtain the approximate analytical thresholds. An explicit solution of equation (8) can be well fitted by

\[ k_m R_0 = \left( m + \frac{1}{2} \right) \pi. \] (13)

Note that the discrete wave number of the spherical plasma wave given by equation (13) is different from the one-dimensional plasma wave, i.e. \( k_m R_0 = m \pi \). This means that the energy spectrum of the spherical plasma wave shifts upward relative to the one-dimensional case, therefore, the energy of the spherical plasma wave is larger than that of the one-dimensional case for any energy level, which can be seen further in the latter discussions.

Then, the dimensionless discrete dispersion relation of the spherical plasma wave can be expressed by

\[ \omega_m^2 = 1 + 3k_m^2 \approx 1 + \left( m + \frac{1}{2} \right) \beta, \] (14)

where \( \beta = 3 \pi^2 / R_0^2 \) is linked to the scale of the spherical plasma and it also can be viewed as a measure of the anharmonicity in the system, i.e. of how strongly the level spacing \( \omega_{m,m+1} = \omega_{m+1} - \omega_m \) depends on the mode numbers \( m \). The dispersion relation of the spherical plasma wave given by equation (14) is different from that of the one-dimensional case, i.e. \( \omega_m^2 = 1 + m^2 \beta \). The dispersion relation has a significant impact on the energy of the system. The total energy of the plasma wave is given by \( \sum_m \omega_m B_m^2 \), here, \( |B_m|^2 \) represents the occupation number of the energy level. Obviously, the deviation of \( \omega_m \) of the spherical plasma wave from that of the one-dimensional case depends on the mode number \( m \) and the parameter \( \beta (\omega_m^2(\text{sphere}) - \omega_m^2(\text{1D})) = (0.25 + m) \beta \).

However, on the one hand, the value of \( \beta \) depends on \( R_0 \), on the other hand, the maximum mode number \( m_{\text{kin}} \) allowed by the system also depends on \( R_0 \) due to the Landau damping rate \( \xi_m \approx k_m^2 \sqrt{3/\pi} \exp(-k_m^2/2 - 3/2) \). Hence, for the different scales of the plasma \( R_0 \), the difference of the dispersion relation between the spherical and one-dimensional plasma wave is different: when the scale of the plasma \( R_0 \) is small, i.e. the value of \( \beta (P_c) \) is large and the system belongs to the LC regime, the difference of the dispersion relation between the spherical and one-dimensional plasma is significant, therefore, for the same distributions of the energy level, the energy of the spherical plasma wave is larger than that of the one-dimensional plasma wave. In contrast, with the increasing of the scale of the plasma, i.e. the system enters into the AR regime, the difference of the energy between the spherical and one-dimensional plasma wave will decrease and even can be ignored.

All of these phenomena can be demonstrated by figure 1. Figure 1 shows the dispersion relation of the spherical (shown by the blue symbol) and one-dimensional (shown by the red symbol) plasma wave with the different scales of the plasma \( R_0 \). Figure 1(a) shows the dispersion relation in the LC regime and figure 1(b) represents that in the intermediate regime (shown by the circles) and AR regime (shown by the rectangles). Clearly, the smaller the scale of the plasma is, the greater the difference of the dispersion relation between the spherical and one-dimensional plasma wave does. In other words, for the same distributions of the energy level, when the system belongs to LC regime, the energy of the spherical plasma wave is larger than that of the one-dimensional plasma wave. Otherwise, when the system belongs to the AR regime, the difference of the energy between the spherical and one-dimensional plasma wave is very small and even can be ignored. Furthermore, assuming \( 3m^2 \ll 1 \), we can obtain the level spacing \( \omega_{m,m+1} \approx (m + 1) \beta \), which is also different from that of the one-dimensional case, i.e. \( \omega_{m,m+1} \approx (m + 1/2) \beta \) [19]. Obviously, the level spacing of the spherical plasma wave is larger than that of the one-dimensional case, which means that it is more difficult for the spherical plasma wave to climb to the higher level, therefore, the spherical plasma needs the larger driving amplitude to achieve LC and AR phenomena. This is further confirmed in figure 2. In addition, because the difference of \( K_m \) between the spherical and one-dimensional plasma wave, the propagation speed of the density wave \( (\nabla_\parallel e \omega_\parallel / \partial K_m \approx 3k_m - 9k_m^2 / 2) \) also is different. Hence, there will be a delay of the time for the propagation of the spherical and one-dimensional plasma wave. This can be confirmed by the numerical results in the section 3.
According to equation (14) and \( \gamma_m = \omega_m - \omega_d \), we can choose \( \gamma_m = 0.5 \beta m^2 - (1.5 \beta + \alpha t) m - 0.125 \beta \) in equation (12). Furthermore, we can obtain the transition time \( \Delta t = \beta / \alpha \) from the relation \( \omega_m \omega_{m+1} = \omega_{m-1,m} = \beta \). Then, the coefficients in equation (12) can be approximated: the second term in the definition of \( \phi_m \) after equation (10) can be neglected since it is a small and not resonant term, thus, \( P_m \approx P_m + 0.5 P_2; \) for a slightly larger \( m \) such as \( m \geq 5 \), we have \( S_i[(k_m - k_i)R_0] \approx S_i[(k_m + k_i)R_0] \), and \( (k_m - k_i)R_0 = \pi \), thus, \( f_{m+1} \approx f_m = 0.5 P_1 \).

Then, equation (12) is reduced to a Schrödinger equation with constant coefficients

\[
\frac{dB_m}{d\tau} = \Gamma_m B_m + \frac{P_1}{2} (B_{m-1} + B_{m+1}),
\]

where

\[
P_1 = \frac{f_{m+1}}{\sqrt{\alpha}} = \frac{2.7A}{4k_0R_0\sqrt{\alpha}} \approx \frac{A}{6.65\sqrt{\alpha}},
\]

\[
P_2 = \frac{\beta}{\sqrt{\alpha}}.
\]

The parameter \( P_1 \) measures the strength of the driving, \( P_2 \) characterizes the nonlinearity in the chirped system as well as the scale of the plasma [13]. The dynamics of the driven system depends on the parameters \( P_1 \) and \( P_2 \). Note that equation (15) is only an asymptotic approximation of equation (12) in order to obtain the asymptotic analytical threshold, the actual evolution equation of the spherical plasma wave is equation (12).
According equation (15), the asymptotic thresholds of quantum LC and classical AR transitions can be obtained. The quantum LC is a series of successive two-level LZ transitions and the transition probability for the $m \rightarrow m + 1$ transition can be given by the LZ theory [27]

$$P_{m \rightarrow m+1} = 1 - e^{-2\left(\frac{m+1}{\Delta}\right)^2} \approx 1 - e^{-\frac{\alpha^2}{\Delta^2}}. \tag{18}$$

We define the total probability $P \approx 0.85$ for capturing into the LC as the probability of occupying a high energy level $m_{\text{kin}}$ after $m_{\text{kin}}$ successive LZ transitions, i.e.

$$(1 - e^{-\frac{\alpha^2}{\Delta^2}})^{m_{\text{kin}}} = 0.85, \tag{19}$$

obviously, for the fixed total probability, the theoretical threshold $P_1$ of the LC transition depends on $m_{\text{kin}}$. However, according to the Landau damping, the value of $m_{\text{kin}}$ is determined by $P_2$. Thus, for the different values of $P_2$, we can obtain the corresponding values of $m_{\text{kin}}$, and furthermore, we can calculate analytically the corresponding values of $P_1$ via equation (19), the results are shown in table 1. As we can see from table 1, the values of $P_1$ only have little change when the values of $P_2$ range from 25 to 10 (The numerical results in figure 2 shown $P_2 \sim [10 \sim 25]$ when the system belongs to LC regime), therefore, we can obtain the analytical threshold of the LC by averaging the values of $P_1$

$$P_1 \approx 1.5. \tag{20}$$

In addition, on the one hand, the transition time is $\Delta \tau = \frac{\beta}{\sqrt{\Delta^2}} = P_2$, which is equivalent to $\Delta \tau = \frac{\beta}{\alpha}$. On the other hand, the typical duration $\Delta \tau_{LZ}$ of each LZ transition exists the two different limits [28]. In the nonadiabatic (sudden) limit ($P_1 \ll 1$), $\Delta \tau_{LZ}$ is of the order of unity, while in the opposite (adiabatic) limit, $\Delta \tau_{LZ} \sim P_1$, Therefore, we can obtain the separatrix of the classical AR and the quantum LC regimes in the $(P_1, P_2)$ parameter space by comparing $\Delta \tau$ and $\Delta \tau_{LZ}$ [14]

$$P_2 \approx 1 + P_1. \tag{21}$$

Equation (21) includes the adiabatic and nonadiabatic limits. The quantum LC can be achieved if $P_2 > 1 + P_1$ is satisfied, in contrast, the classical AR transition requires $P_2 < 1 + P_1$, the latter also can be got when $P_1 \gg 1$ by requiring that the classical resonance width would include more than two quantum levels [13]. Note that this only is a necessary condition to achieve the LC or AR phenomenon in the system.

It is known that to achieve the classical AR transition for the driven nonlinear oscillator, the amplitude of the driving, the nonlinearity of the system and the chirped rate exist a relation [13, 29] $A \approx 25\beta^{-1/2}/\alpha^{1/2}$, where the scale coefficient is determined by the system. For our system (15), when it is expressed in terms of the parameters $P_1$ and $P_2$, we can obtain the theoretical threshold of the classical AR transition

$$P_2 \approx \frac{14.44}{P_1^1}. \tag{22}$$

Equations (20)–(22) describe the approximate analytical thresholds for the different regimes of the phase-locking transition for the spherical plasma wave. Note that the analytical thresholds of the one-dimensional plasma are similar to equations (20)–(22) but the driving parameter $P_1$ should be replaced with 1.66$P_1$. So, the coupling strength in the spherical plasma is weaker than that in the one-dimensional plasma (the former is 0.5/0.83 $\approx$ 0.6 times that of the latter). Thus, compared with the one-dimensional plasma, the spherical plasma needs the larger driving amplitude to achieve LC and AR phenomena due to its special geometric structure (the driving amplitude of the spherical plasma is 1.66 times as large as that of the one-dimensional plasma).

The theoretical predictions given by equations (20)–(22) can be confirmed by numerical simulation of the slow evolution equation (12) with the initial condition $B_m(\tau = \tau_0) = \delta_{m,1}$. For a given set of $(P_1, P_2)$, we define the resonant capture probability

| $P_2$ | $m_{\text{kin}}$ | $P_1$ |
|------|--------|------|
| 25   | 4      | 1.43 |
| 20   | 5      | 1.48 |
| 15   | 6      | 1.52 |
| 13   | 7      | 1.55 |
| 10   | 8      | 1.58 |

Table 1. The values of $P_1$ for the different $P_2$. The values of $m_{\text{kin}}$ after $m_{\text{kin}}$ successive LZ transitions.
where \( m_{\text{r}} \) is defined as \( 0.8 m_{\text{init}} \), \( m_{\text{init}} \) is the maximum mode number allowed by the system, which is related to \( P_2 \). For a series of the random values of \((P_1, P_2)\), we define the threshold of transition as the values of \((P_1, P_2)\) satisfying 0.85 capture probability, i.e. \( P(P_1, P_2) = 0.85 \). When we numerically solve equation (12), the range of the parameters is \( P_1 \in [1, 8.5], P_2 \in [0.1, 25] \). Only when the values of \((P_1, P_2)\) satisfy the capture probability \( P = 0.85 \), the LC and AR of the chirped system can be achieved. Therefore, we only present the points of \((P_1, P_2)\) that satisfying \( P = 0.85 \) as the form of the blue full circles in figure 2. In addition, figure 2 also shows the different regimes of phase-locking transition of the system obtained by theoretical results of equations (20)–(22). The blue solid and red dash lines are approximated analytic thresholds of the spherical and one-dimensional plasma, respectively. As we can see that the numerical simulation results do not agree well with the asymptotic analytical thresholds. The reasons are: the numerical simulation results come from equation (12), which is a variable coefficient equation. However, the asymptotic analytical thresholds (i.e. equations (20)–(22)) are obtained according to the simplified constant coefficient equation (15) (it is only an approximate result of equation (12)), thus, obviously, there will be the deviation between the numerical simulation results and the asymptotic analytical thresholds. Furthermore, because the system belongs to the mixing of LC and AR when \( 1 < P_2 < 10 \) and the classical AR also can be achieved when a stronger inequality, \( P_2 < 1 \), is satisfied for \( P_1 \gg 1 \) [14]. Therefore, in the region of \( 1 < P_2 < 10 \) and \( P_2 \ll 1 \), the deviation between the numerical results and the analytical thresholds is more larger.

As we can see from figure 2, the equations (20)–(22) divide the \((P_1, P_2)\) parameter space into the several regions, the region above the separatrix given by equation (21) belongs to the quantum LC regime of the plasma wave, however, the quantum LC phenomenon can be achieved only when equation (20) is satisfied approximately due to the transition probability. In contrast, the region below the separatrix belongs to the classical AR regime, and the classical AR phenomenon can take place when equation (22) is satisfied approximately.

3. Numerical results

We solve numerically the slow evolution equation (12) for given dimensionless parameters \( P_1 \) and \( P_2 \). The initial conditions are \( B_m(\tau = \tau_0) = \delta_{m,1} \) (the linear resonance corresponds to \( \tau = 0 \)). Figures 3–5 show the quantum LC regime, the intermediate regime and the classical AR regime of the plasma wave, respectively.

In figure 3, we present the LC dynamics of the spherical plasma wave (shown by the blue color) with \( P_1 = 1.5, P_2 = 10 \) and the one-dimensional plasma wave [18] (shown by the red color) with \( P_1 = 0.9, P_2 = 10 \). The dimensionless scale of the plasma is \( R_0 \approx 170 \). The distributions of the occupation numbers \( |B_m|^2 \) with the

\[
P = \sum_{m=m_{\text{r}}}^{m_{\text{max}}} |B_m|^2, \tag{23}
\]
mode numbers \( m \) are shown in the left column (figures 3(a), (c), (e), (g)) for the four different times \( \tau = -5, 15, 35, 55 \), while the right column (figures 3(b), (d), (f), (h)) are the perturbed density \( n \) as the function of the space coordinate \( r \). The inset in figure 3(a) shows the time evolution of the total perturbed density \( \sum_{n} \omega_{n}|B_{m}^{n}|^{2} \) as the form of the ladder.

From the left column in figure 3, we can easily find that the evolutions of the spherical plasma wave are similar to the one-dimensional plasma wave in the energy level space: only a single energy level is highly occupied for the any given time, which illustrates the only two adjacent energy levels of the driven system are coupled, the system yields the LC dynamic. We should notice that, as discussed in the section 2.2 of the section 2, to achieve the similar LC phenomenon, the driving strength \( (P_{1}) \) of the spherical plasma is 1.66 times that of the one-dimensional plasma. However, as we can see, the level climbing of the spherical plasma wave is not complete compared to the one-dimensional case due to the variable coefficients of equation (12), which is

Figure 4. The evolutions of plasma wave in the intermediate regime at the four different times \( \tau = \pm 0.75 \) [(a), (b)], 3.25 [(c), (f)] and 32.25 [(g), (h)]. The spherical plasma corresponds to the parameters \( P_{1} = 3.1, P_{2} = 1.5 \) and the one-dimensional plasma corresponds to \( P_{1} = 1.87, P_{2} = 1.5 \). The dimensionless scale of the plasma is \( R_{0} \approx 444 \). The left column: the occupation numbers \( |B_{m}|^{2} \) versus the level numbers \( m \). The right column: the perturbed density \( n \) versus the position of the plasma \( r \), \( n_{0\text{max}} \) is the maximum value of the perturbed density initially. Several levels are excited simultaneously for the any given time in the intermediate regime. The inset in (a) shows the time evolution of the total perturbed density \( \sum_{n} \omega_{n}|B_{m}^{n}|^{2} \).

Figure 5. The evolutions of the plasma wave in the classical AR regime at the four different times \( \tau = \pm 0.1 \) [(a), (b)], 5.7 [(c), (d)], 11.7 [(e), (f)] and 16.5 [(g), (h)]. The spherical plasma corresponds to the parameters \( P_{1} = 8.5, P_{2} = 0.2 \) and the one-dimensional plasma corresponds to \( P_{1} = 5.1, P_{2} = 0.2 \). The dimensionless scale of the plasma is \( R_{0} \approx 1216 \). The left column: the occupation numbers \( |B_{m}|^{2} \) versus the level numbers \( m \). The right column: the perturbed density \( n \) versus the position of the plasma \( r \), \( n_{0\text{max}} \) is the maximum value of the perturbed density initially. Many levels are excited simultaneously for the any given time in the AR regime. The inset in (a) shows the total perturbed density \( \sum_{n} \omega_{n}|B_{m}^{n}|^{2} \) increases monotonically with the time.
induced by the spherical geometry and decreases the energy of the spherical plasma wave. Therefore, as shown in the inset in figure 3(a), the total perturbed density of the spherical plasma wave is only slightly larger than the one-dimensional case at any time, although the difference of the dispersion relation is obvious (see figure 1(a)). The moments of $m \to m + 1$ transitions occur at the theoretically predicted times $\tau_m = mP_2$.

From the right column in figure 3, we can see that for the one-dimensional plasma, the perturbed density wave changes uniformly in space and behaves as a sinusoidal character. The wave number as a whole is corresponding to the mode numbers occupied highly in the energy level space (see the left column of figure 3). However, the change of the spherical plasma wave is nonuniform and the wave energy is concentrated to the sphere center as the density wave climbs to the higher level. That is, the spherical plasma wave behaves obvious compression character during the LC dynamics. The perturbed density of the sphere center at $\tau = 55$ can be amplified to 12 times larger of the initial perturbed density.

In figure 4, we show the evolutions of the plasma wave in the intermediate regime at the four different times $[\tau = -0.75, 11.25, 23.25, 32.25]$. The spherical plasma shown by the blue color corresponds to the parameters $P_1 = 3.1, P_2 = 1.5$ and the one-dimensional plasma shown by the red color corresponds to $P_1 = 1.87, P_2 = 1.5$. The dimensionless scale of the plasma is $R_0 \approx 444$. From the left column in figure 4, we can easily find that the evolutions of the spherical plasma wave are similar to the one-dimensional plasma wave in the energy level space: several levels are excited simultaneously for the any given time but the occupation numbers of the spherical plasma wave are small, which illustrates the driven system belongs to the mixing of the quantum LC and the classical AR, in other words, the system yields the intermediate regime. The inset in figure 4(a) indicates that the evolution of density wave for the spherical and one-dimensional plasma is similar: compared with the LC regime, the total perturbed density still changes approximately as the form of the ladder in the intermediate regime but the width of the ladder becomes narrower, due to the transition time becomes short with the decrease of the $P_2$. However, the total perturbed density of the spherical plasma is larger than that of the one-dimensional plasma at any time due to the dispersion relation and the distributions of the occupation numbers, and this difference becomes more obvious with the increasing of the time. The right column in figure 4 illustrates that, in the intermediate regime, the spatial distribution of the perturbed density wave for the spherical and one-dimensional plasma is different significantly: for the one-dimensional plasma, the perturbed density wave seems to form an envelope and the envelope is reflected when it propagates to the boundaries (i.e. the propagation of the perturbed density wave is reversible). However, for the spherical plasma, the perturbed density changes most violently near the center of the sphere, i.e. the wave energy is concentrated to the sphere center, which is especially obvious in the AR regime.

In figure 5, we show the AR dynamics of the plasma wave at the four different times $[\tau = -0.1, 5.7, 11.7, 16.5]$. The spherical plasma shown by the blue color corresponds to the parameters $P_1 = 8.5, P_2 = 0.2$ and the one-dimensional plasma shown by the red color corresponds to $P_1 = 5.1, P_2 = 0.2$. The dimensionless scale of the plasma is $R_0 \approx 1216$. From the left column in figure 5, we can easily find that the evolutions of the spherical plasma wave are similar to the one-dimensional plasma wave in the energy level space: in contrast with the LC regime, many levels are excited simultaneously for the any given time, this illustrates that many levels of the system are coupled simultaneously, the system yields the AR regime. In addition, as mentioned in the section 2.2 of the section 2, the propagation speed of the spherical plasma wave is faster than that of the one-dimensional case. Thus, from the left column in figure 5, we can see that there is a delay of the time between the spherical and one-dimensional plasma wave. The inset in figure 5(a) indicates that the evolution of density wave for the spherical and one-dimensional plasma is similar: compared with the intermediate regime, the total perturbed density increases monotonically with the time in the AR regime, this can be interpreted as the fact that the transitions time becomes small enough due to the very weak nonlinearity ($P_2$). However, we also note that the total perturbed density of the spherical plasma wave is larger than that of the one-dimensional case, which is analogous to the phenomenon illustrated by the inset in figure 4(a). Especially, from the right column in figure 5, we can see that, in the AR regime, compared with the one-dimensional plasma, ‘the effect of the sphere center’ of the spherical plasma is more obvious, i.e. the oscillation of the spherical plasma wave is more severely concentrated to the centre of the sphere. The spherical plasma shows the extremely obvious compression character, the perturbed density of the sphere center at $\tau = 16.5$ can be amplified to 100 times larger of the initial perturbed density. This phenomenon can help us to better understand the AR acceleration of the electrons. For the one-dimensional case, however, an obvious envelope is formed and the propagation of the envelope is reversible in a bounded plasma [18]. Note that if the nonlinear effects (such as particle trapping) are considered, the propagation of the envelope will be non-reversible and the amplitude of the perturbed density wave will decrease to some extent [19].
4. Summary

We have discussed the evolutions of the collisionless spherical plasma wave described by a hydrodynamic model. Starting from the basic equations of the electrons plasma, we derive the governing equation of the perturbed density wave of the spherical plasma and find the spherical plasma also exists LC and AR phenomena. Then, we get analytically the asymptotic thresholds for the quantum LC and classical AR transition, and which also are further confirmed numerically. We find that in order to achieve LC and AR phenomenon, in terms of the driving strength of needling, the spherical plasma is approximately 1.66 times that of the one-dimensional plasma due to its spherical geometric structure. In addition, we also notice that, compared with the one-dimensional plasma case, the total perturbed density of the spherical plasma wave always is larger at any time, which is caused by the dispersion relation and the distributions of the occupation numbers of the spherical wave. In particular, we find that ‘the effect of the sphere center’ of the spherical plasma is extremely obvious, i.e. the spherical wave energy is concentrated largely to the sphere center as the density wave climbs to the higher energy level, the spherical plasma behaves obvious compression character, and the perturbed density of the center even can be amplified to 100 times larger of the initial perturbed density. On the one hand, this can help us to deeply understand the AR acceleration of the electrons. On the other hand, it is also closely relative to the dynamics of the inertial confinement fusion. In the actual spherical plasma experiment, such as the inertial confinement fusion, under the driving of the ponderomotive force of the laser beams, the massive electrons in the plasma are compressed toward the center of the sphere, which makes the electrons density near the sphere center increase dramatically [30]. Therefore, our results provide a possibility to improve the compression efficiency in the realistic spherical plasma experiment by designing appropriate LC and AR of the spherical plasma density wave.

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