Variations of Ramanujan’s Euler Products

Shin-ya Koyama* & Nobushige Kurokawa†

April 13, 2021

Abstract

We study the meromorphy of various Euler products of degree two attached to holomorphic Hecke eigen cusp forms for the elliptic modular group, including Ramanujan’s $\Delta$-function.

Key Words: Euler products; zeta functions; Ramanujan’s $\Delta$-function; cusp forms; modular group

AMS Subject Classifications: 11M06, 11M41,

Introduction

In 1916 Ramanujan [12] studied the Euler product associated to Ramanujan’s $\tau$-function defined as

$$\Delta(z) = e^{2\pi iz} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})^{24} = \sum_{n=1}^{\infty} \tau(n)e^{2\pi inz}$$

for $\Im(z) > 0$. This $\Delta(z)$ is a Hecke-eigen cusp form of weight 12 for the elliptic modular group $SL(2,\mathbb{Z})$. Ramanujan’s Euler product is written in the normalized form as

$$L_{\Delta}(s) = \prod_{p: \text{prime}} (1 - a(p)p^{-s} + p^{-2s})^{-1},$$

where $a(n) = \tau(n)n^{-\frac{3}{2}}$.

*Department of Biomedical Engineering, Toyo University, 2100 Kujirai, Kawagoe, Saitama, 350-8585, Japan.
†Department of Mathematics, Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo, 152-8551, Japan.
Ramanujan conjectured that $|a(p)| \leq 2$ for all primes $p$ and it was proved by Deligne [4] in 1974. Hence there exists a unique $\theta(p) \in [0, \pi]$ satisfying $a(p) = 2\cos(\theta(p))$.

Thus

$$L_\Delta(s) = \prod_{p: \text{prime}} (1 - 2\cos(\theta(p))p^{-s} + p^{-2s})^{-1}$$

$$= \prod_{p: \text{prime}} \det (1 - \alpha_0^\Delta(p)p^{-s})^{-1},$$

where

$$\alpha_0^\Delta(p) = \begin{pmatrix} e^{i\theta(p)} & 0 \\ 0 & e^{-i\theta(p)} \end{pmatrix} \in \text{Conj}(SU(2)).$$

More general Euler products were constructed by Serre [13], Langlands [9] and Tate [17] as

$$L_\Delta(s, \text{Sym}^m) = \prod_{p: \text{prime}} \det (1 - \text{Sym}^m(\alpha_0^\Delta(p))p^{-s})^{-1}$$

for the irreducible representation $\text{Sym}^m : SU(2) \to U(m + 1)$ with $m = 0, 1, 2, 3, \ldots$. Quite recently, Newton-Thorne [10, 11] proved that there exists an automorphic representation $\pi_m$ of $GL(m + 1, \mathbb{A}_\mathbb{Q})$ satisfying

$$L(s, \pi_m) = L_\Delta(s, \text{Sym}^m)$$

for each $m = 0, 1, 2, 3, \ldots$ and that $\pi_m$ ($m \geq 1$) are cuspidal corresponding to the holomorphy of $L_\Delta(s, \text{Sym}^m)$.

The situation is quite similar for each holomorphic Hecke eigen cusp form $\varphi$ of general weight for the elliptic modular group.

In this paper we first study the Euler product

$$Z_m^\pm(s) = \prod_{p: \text{prime}} (1 \pm 2\cos(m\theta(p))p^{-s} + p^{-2s})^{-1}$$

for $m = 0, 1, 2, 3, \ldots$ and show the following result.

**Theorem A.** $Z_m^\pm(s)$ has an analytic continuation to all $s \in \mathbb{C}$ as a meromorphic function.

In fact, this is easily seen from Newton-Thorne [10, 11] since each $Z_m^\pm(s)$ can be written explicitly in terms of $L_{\varphi}(s, \text{Sym}^n)$. In particular $Z_m^-(s)$ has a simple expression as

$$Z_m^-(s) = \begin{cases} 
L_{\varphi}(s, \text{Sym}^m) & (m \geq 2) \\
L_{\varphi}(s) & (m = 1).
\end{cases}$$
Next, we show the converse in some sense. Actually we determine all (non-constant) monic polynomials $f(x) \in \mathbb{Z}[x]$ such that

$$Z^\pm(s, f) = \prod_{p : \text{prime}} (1 \pm f(a(p))p^{-s} + p^{-2s})^{-1}$$

are meromorphic on $\mathbb{C}$.

**Theorem B.** Let $f(x) \in \mathbb{Z}[x]$ be a monic polynomial of degree $m \geq 1$. Then the following properties are equivalent.

1. $Z^\pm(s, f)$ is meromorphic on $\mathbb{C}$.
2. $|f(x)| \leq 2$ for $-2 \leq x \leq 2$.
3. $f(x) = 2T_m(x)$. Here, $T_m(x)$ is the Chebyshev polynomial defined by $T_m(\cos \theta) = \cos(m \theta)$.
4. $Z^\pm(s, f) = Z^\pm_m(s)$.

**Example 1.** Let

$$Z^m_m(s) = \prod_{p : \text{prime}} (1 - (a(p)^2 - m)p^{-s} + p^{-2s})^{-1}$$

for $m \in \mathbb{Z}$. Then,

$$Z^m_m(s) \text{ is meromorphic on } \mathbb{C} \iff m = 1.$$

Moreover,

$$L_\varphi(s, \text{Sym}^2) = \zeta(s) \prod_{p : \text{prime}} (1 - (a(p)^2 - 1)p^{-s} + p^{-2s})^{-1}$$

$$= \prod_{p : \text{prime}} (1 - a(p^2)p^{-s} + a(p^2)p^{-2s} - p^{-3s})^{-1}.$$

This expression was used by Shimura [16] for the proof of the holomorphy of $L_\varphi(s, \text{Sym}^2)$.

**Example 2.** For $m \in \mathbb{Z}$,

$$\prod_{p : \text{prime}} (1 - (a(p) - m)p^{-s} + p^{-2s})^{-1} \text{ is meromorphic on } \mathbb{C} \iff m = 0.$$

Otherwise, each $Z^\pm(s, f)$ has the natural boundary $\text{Re}(s) = 0$ as in the following result.
**Theorem C.** Let \( f(x) \in \mathbb{Z}[x] \) be a non-constant monic polynomial. Then the following properties are equivalent.

(1) \( Z^\pm(s, f) \) is not meromorphic on \( \mathbb{C} \).

(2) \( Z^\pm(s, f) \) has an analytic continuation in \( \text{Re}(s) > 0 \) with the natural boundary \( \text{Re}(s) = 0 \).

Our results come from a general meromorphy theorem for Euler products shown in [6, 7] assuming [10, 11]. We notice that the proof is similar to the case of meromorphy of Dirichlet series \( \sum_{n=1}^{\infty} a(n)m^{-s} \) and \( \sum_{n=1}^{\infty} a(n)m^{-s} \) treated in [6,7]. For \( m = 1, 2 \) they are meromorphic on \( \mathbb{C} \) and for \( m \geq 3 \) they are meromorphic in \( \text{Re}(s) > 0 \) with the natural boundary \( \text{Re}(s) = 0 \). We refer to [8] for an application to rigidity of Euler products.

1 Proof of Theorem A

(1) **The case** \( m = 0 \)

From
\[
Z_0^\pm(s) = \prod_{p: \text{prime}} (1 \pm 2p^{-s} + p^{-2s})^{-1} = \prod_{p: \text{prime}} (1 \pm p^{-s})^{-2}
\]
we see that
\[
Z_0^-(s) = \prod_{p: \text{prime}} (1 - p^{-s})^{-2} = \zeta(s)^2
\]
and
\[
Z_0^+(s) = \prod_{p: \text{prime}} (1 + p^{-s})^{-2} = \prod_{p: \text{prime}} \left( \frac{1 - p^{-2s}}{1 - p^{-s}} \right)^{-2} = \frac{\zeta(2s)^2}{\zeta(s)^2}
\]
are meromorphic on \( \mathbb{C} \).

(2) **The case** \( m = 1 \)

By identifying
\[
Z_1^-(s) = \prod_{p: \text{prime}} (1 - 2\cos(\theta(p))p^{-s} + p^{-2s})^{-1} = L_\varphi(s)
\]
and

\[ Z_1^+(s) = \prod_{p: \text{prime}} \frac{(1 + 2 \cos(\theta(p)) p^{-s} + p^{-2s})^{-1}}{(1 + e^{i\theta(p)} p^{-s})(1 + e^{-i\theta(p)} p^{-s})} \]

\[ = \prod_{p: \text{prime}} \frac{[(1 + e^{i\theta(p)} p^{-s})(1 + e^{-2i\theta(p)} p^{-2s})]^{-1}}{(1 + e^{i\theta(p)} p^{-s})(1 + e^{-i\theta(p)} p^{-s})} \]

\[ = \frac{L_\varphi(2s, \text{Sym}^2)}{L_\varphi(s, \text{Sym}) L_\varphi(2s, \text{Sym}^0)}, \]

where

\[ L_\varphi(s, \text{Sym}^1) = L_\varphi(s), \]

\[ L_\varphi(s, \text{Sym}^0) = \zeta(s), \]

we know that both are meromorphic on \( \mathbb{C} \).

**3** The case \( m \geq 2 \)

First

\[ Z_m^-(s) = \prod_{p: \text{prime}} \frac{[(1 - e^{im\theta(p)} p^{-s})(1 - e^{-im\theta(p)} p^{-s})]^{-1}}{(1 - e^{i(m-2)\theta(p)} p^{-s}) \cdots (1 - e^{-i(m-2)\theta(p)} p^{-s})} \]

\[ = \frac{L_\varphi(s, \text{Sym}^m)}{L_\varphi(s, \text{Sym}^{m-2})} \]

is a meromorphic function on \( \mathbb{C} \) by means of \([1, 10, 11]\).

Secondly

\[ Z_m^+(s) = \prod_{p: \text{prime}} \frac{[(1 + e^{im\theta(p)} p^{-s})(1 + e^{-im\theta(p)} p^{-s})]^{-1}}{(1 - e^{-2i\theta(p)} p^{-2s})(1 - e^{-i\theta(p)} p^{-s})} \]

\[ = \frac{L_\varphi(2s, \text{Sym}^{2m})}{L_\varphi(2s, \text{Sym}^{m-2})} \]

is meromorphic on \( \mathbb{C} \) using \([1, 10, 11]\). \( \square \)
2 Meromorphy of Euler products

We recall the meromorphy result of [7, §3 Theorem 7] needed in this paper. Take the triple

\[ E = (P(\mathbb{Z}), SU(2), \alpha_0^\phi), \]

where \( P(\mathbb{Z}) \) is the set of prime numbers, and the map

\[ \alpha_0^\phi : P(\mathbb{Z}) \to \text{Conj}(SU(2)) \]

defined as

\[ \alpha_0^\phi(p) = \left[ \begin{array}{cc} e^{i\theta(p)} & 0 \\ 0 & e^{-i\theta(p)} \end{array} \right]. \]

We identify \([0, \pi]\) and \( \text{Conj}(SU(2)) \) via the following map

\[ \begin{array}{ccc} [0, \pi] & \longrightarrow & \text{Conj}(SU(2)) \\ \psi & \mapsto & \left[ \begin{array}{cc} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{array} \right] \end{array} \]

Notice that the normalized measure on \( \text{Conj}(SU(2)) = [0, \pi] \) coming from the normalized Haar measure of \( SU(2) \) is given by \( \frac{2}{\pi} \sin^2 \theta d\theta \) used in the Sato-Tate conjecture proved in [1].

Main results of [6, 7] give the criterion of the meromorphy of Euler product

\[ L(s, E, H) = \prod_{p: \text{prime}} H_{\alpha_0^\phi(p)}(p^{-s})^{-1} \]

for each polynomial

\[ H(T) \in 1 + TR(SU(2))[T] \]

when specialized to the present situation, where \( R(SU(2)) \) is the ring of virtual characters of \( SU(2) \) and

\[ H_{\alpha_0^\phi(p)}(T) \in 1 + TC[T] \]

denotes the polynomial obtained by taking values of coefficients.

We first report Theorems 1 and 2. The important assumption made in [6, 7] for \( E = (P(\mathbb{Z}), SU(2), \alpha_0^\phi) \) on the analytic properties of the symmetric power \( L \)-functions follow from recently proved automorphy due to Newton-Thorne [10, Theorem A]. Especially the boundedness in vertical strips of the symmetric power \( L \)-functions needed in [6, 7] follow from Gelbart-Shahidi [5] (see Shahidi [15] and Cogdell-Piatetskii-Shapiro [3]).

**Theorem 1.** Let \( H(T) \in 1 + TR(SU(2))[T] \) be a polynomial of degree \( n \). Then the following properties are equivalent.
(1) $H(T)$ is unitary in the sense that there exist functions $\varphi_j : \mathbb{R} \to \mathbb{R}$ ($j = 1, 2, ..., n$) such that

$$H_\theta(T) = \prod_{j=1}^{n} (1 - e^{i\varphi_j(\theta)}T)$$

for $\theta \in [0, \pi] = \text{Conj}(SU(2))$.

(2) $L(s, E, H)$ is meromorphic on $\mathbb{C}$.

**Theorem 2.** Let $H(T) \in 1 + TR(SU(2))[T]$. Then the following properties are equivalent.

(1) $H(T)$ is not unitary.

(2) $L(s, E, H)$ is meromorphic in $\text{Re}(s) > 0$ with the natural boundary.

Here we prove a concrete example of degree two for applications in this paper.

**Theorem 3.** Let $H(T) = 1 \pm hT + T^2 \in 1 + TR(SU(2))[T]$. Then the following properties are equivalent.

(1) $H(T)$ is unitary.

(2) $|h(\theta)| \leq 2$ for all $\theta \in [0, \pi]$.

(3) $L(s, E, H)$ is meromorphic on $\mathbb{C}$.

**Proof.** Since (1) $\iff$ (3) is implied in Theorem 1, it suffices to show (1) $\iff$ (2).

(1) $\implies$ (2): From

$$H_\theta(T) = (1 - e^{i\varphi(\theta)}T)(1 - e^{-i\varphi(\theta)}T)$$

for a function $\varphi : \mathbb{R} \to \mathbb{R}$, we have

$$|h(\theta)| = |2\cos(\varphi(\theta))| \leq 2$$

for $\theta \in [0, \pi]$.

(2) $\implies$ (1): From $|h(\theta)| \leq 2$ we have

$$H_\theta(T) = \left(1 \pm \frac{h(\theta) + i\sqrt{4 - h(\theta)^2}}{2}T\right)\left(1 \pm \frac{h(\theta) - i\sqrt{4 - h(\theta)^2}}{2}T\right)$$

and

$$\left|\frac{h(\theta) \pm i\sqrt{4 - h(\theta)^2}}{2}\right| = 1.$$
3 Proofs of Theorems B and C

From Theorems 1, 2 and 3 we see that Theorems B and C are proved by the following result.

**Theorem 4.** For a monic polynomial \( f(x) \in \mathbb{Z}[x] \) of degree \( m \geq 1 \), the following conditions are equivalent.

1. \( |f(x)| \leq 2 \) for \(-2 \leq x \leq 2\).
2. \( f(x) = 2T_m(\frac{x}{2}) \).

Actually, let

\[ H(T) = 1 \pm f(2 \cos \theta)T + T^2 \in 1 + TR(SU(2))[T], \]

then we see the following equivalence:

\[ L(s, E, H) \text{ is meromorphic on } \mathbb{C} \]

\[ \overset{\text{Th } 1}{\iff} \quad H(T) \text{ is unitary} \]

\[ \overset{\text{Th } 3}{\iff} \quad |f(2 \cos \theta)| \leq 2 \text{ for all } \theta \in [0, \pi] \]

\[ \iff \quad |f(x)| \leq 2 \text{ for } -1 \leq x \leq 1 \]

\[ \overset{\text{Th } 4}{\iff} \quad f(x) = 2T_m(\frac{x}{2}) \]

\[ \iff \quad L(s, E, H) = \prod_p (1 \pm 2 \cos(m \theta(p))p^{-s} + p^{-2s})^{-1}. \]

We remark that this also gives a second proof of Theorem A.

**Proof of Theorem 4.** Let

\[ g(x) = \frac{1}{2^m} f(2x) \in \mathbb{R}[x]. \]

This is a monic polynomial of degree \( m \). Then, (1) is equivalent to the following condition:

\[ |g(x)| \leq \frac{1}{2^{m-1}} \text{ for } -1 \leq x \leq 1. \]

Such a monic polynomial \( g(x) \) is uniquely determined as \( \frac{1}{2^{m-1}} T_m(x) \) by the famous theorem of Chebyshev [2]; see Serre [14]. Hence \( f(x) = 2T_m \left( \frac{x}{2} \right) \).

**References**

[1] T. Barnet-Lamb, D. Geraghty, M. Harris, and R. Taylor. *A family of Calabi-Yau varieties and potential automorphy (II)*, Publ. Res. Inst. Math. Sci 47 (2011), 29-98.
[2] P.L. Chebyshev, *Théorie des mécanismes connus sous le nom de parallélogrammes*, Mém. Acad. Sci. Pétrograd. 7 (1854), 539-568. (= Oe I, 111-143).

[3] J. W. Cogdell and I. I. Piatetski-Shapiro, *Converse theorems, functoriality, and applications to number theory*, Proceedings of the International Congress of Mathematicians, Volume II (2002), 119-128.

[4] P. Deligne, *La conjecture de Weil (I)*, Publ. Math. IHES 43 (1974), 273-307.

[5] S. Gelbart and F. Shahidi, *Boundedness of automorphic L-functions in vertical strips*, J. Am. Math. Soc. 14 (2001), 79-107.

[6] N. Kurokawa, *On the meromorphy of Euler products (I)*, Proc. London Math. Soc. (3) 53 (1986), 1-47.

[7] ———, *On the meromorphy of Euler products (II)*, Proc. London Math. Soc. (3) 53 (1986), 209-236.

[8] S. Koyama and N. Kurokawa, *Rigidity of Euler products*, arXiv: 2013.06464 v1 [math. NT] 11 Mar (2021).

[9] R. P. Langlands, *Problems in the theory of automorphic forms*, Springer Lecture Notes in Math. 170 (1970), 18-61.

[10] J. Newton and J. A. Thorne, *Symmetric power functoriality for holomorphic modular forms*, arXiv: 1912.11261 v2 [math. NT] 17 July (2020).

[11] ———, *Symmetric power functoriality for holomorphic modular forms (II)*, arXiv: 2009.07180 v1 [math. NT] 15 Sep (2020).

[12] S. Ramanujan, *On certain arithmetical functions*, Trans. Cambridge Philosophical Society 22 (1916), 159-184.

[13] J. P. Serre, *Une interprétation des congruences relatives à la fonction τ de Ramanujan*, Séminaire Delange-Pisot-Poitou: 1967/68, Théorie des Nombres, Fasc. 1, Exp. 14 (1969), 17 pp.

[14] ———, *Distribution asymptotique des Valeurs Propres des Endomorphismes de Frobenius [d’après Abel, Chebyshev, Robinson,...]*, Astérisque 414 (2019), 1136-1150. Séminaire Bourbaki. 2017/2018.

[15] F. Shahidi, *Automorphic L-functions and functoriality*, Proceedings of the International Congress of Mathematicians, Volume II (2002), 655-666.

[16] G. Shimura, *On the holomorphy of certain Dirichlet series*, Proc. London Math. Soc. (3) 31 (1975), 79-98.

[17] J. Tate, *Algebraic cycles and poles of zeta functions*, Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963) (1965), 93-110.