SIMPLE EXPRESSIONS FOR THE HOLED TORUS RELATIONS

NORIYUKI HAMADA

Abstract. In the mapping class group of a $k$-holed torus with $0 \leq k \leq 9$, one can factorize the boundary multi-twist (or the identity when $k = 0$) as the product of twelve right-handed Dehn twists. Such factorizations were explicitly given by Korkmaz and Ozbagci for each $k \leq 9$ and an alternative one for $k = 8$ by Tanaka. In this note, we simplify their expressions for the $k$-holed torus relations.

Let $\Sigma_k^\ell$ denote a torus with $k$ boundary components and $\text{Mod}(\Sigma_k^\ell)$ be its mapping class group where maps are assumed to be identity on the boundary. In $[3]$, Korkmaz and Ozbagci gave a factorization of the multi-boundary twist $t_{\delta_1} \cdots t_{\delta_k}$ as the product of twelve right-handed Dehn twists in $\text{Mod}(\Sigma_k^\ell)$ for each $k \leq 9$. Here $t_{\delta_i}$ stands for the right-handed Dehn twist about a curve parallel to the $i$-th boundary component. We call this type of factorization a $k$-holed torus relation in general. Moreover, Tanaka $[6]$ also gave such a factorization for $k = 8$, which turned out to be Hurwitz inequivalent to Korkmaz-Ozbagci’s 8-holed torus relation. Those factorizations have two major geometrical backgrounds, one as Lefschetz pencils and the other as Stein fillings, which demonstrate the fundamental importance of such $k$-holed torus relations.

It is a standard fact in the literature that the isomorphism classes of Lefschetz pencils are one-to-one correspondent to Dehn twist factorizations of the boundary multi-twists in the mapping class groups of holed surfaces, up to Hurwitz equivalence; a genus-$g$ Lefschetz pencil with $n$ critical points and $k$ base points gives a monodromy factorization $t_{a_n} \cdots t_{a_1} = t_{\delta_1} \cdots t_{\delta_k}$ in the mapping class group of a $k$-holed surface of genus $g$, and conversely such a factorization determines a Lefschetz pencil. One can show that Korkmaz-Ozbagci’s 9-holed torus relation yields a minimal elliptic Lefschetz pencil on $\mathbb{CP}^2$ (cf. $[2]$). For the other $k \leq 8$, Korkmaz-Ozbagci’s $k$-holed torus relations simply correspond to Lefschetz pencils obtained by blowing-up some of the base points of the above pencil on $\mathbb{CP}^2$. On the other hand, it turns out that Tanaka’s 8-holed torus relation yields a minimal elliptic Lefschetz pencil on $S^2 \times S^2$ (cf. $[4]$). Furthermore, it can be shown that a genus-1 Lefschetz pencil cannot have $k \geq 10$ base points (in fact, we might expect that the above pencils exhaust all possibilities). This is why Korkmaz and Ozbagci stopped at $k = 9$.

Dehn twist factorizations (with homologically nontrivial curves) of elements in mapping class groups of holed surfaces also provide positive allowable Lefschetz fibrations over $D^2$, which in turn represent Stein fillings of contact 3-manifolds. As summarized in $[5]$, there is another elegant interpretation of the $k$-holed torus relations in this point of view. As the monodromy of an open book, the boundary multi-twist $t_{\delta_1} \cdots t_{\delta_k}$ in $\text{Mod}(\Sigma_k^\ell)$ yields the contact 3-manifold $(Y_k, \xi_k)$ that is given as the boundary of the symplectic $D^2$-bundle over $T^2$ with Euler number $-k$. While the symplectic $D^2$-bundle naturally gives a Stein filling of $(Y_k, \xi_k)$, the positive allowable Lefschetz fibration over $D^2$ associated with the obvious Dehn

\footnote{The author also found 8-holed torus relations in $[1, 2]$ in different contexts, which in fact can be shown to be Hurwitz equivalent to either Korkmaz-Ozbagci’s or Tanaka’s.}
twist factorization $t_{δ_1} \cdots t_{δ_k}$ also gives the same Stein filling. If the boundary multi-twist $t_{δ_1} \cdots t_{δ_k}$ has another factorization (i.e. a $k$-holed torus relation) it also gives a Stein filling of $(Y_k, ξ_k)$. Actually, Stein fillings of $(Y_k, ξ_k)$ are already classified by Ohta and Ono [4]; besides the symplectic $S^2$-bundle there is (i) no more Stein filling when $k \geq 10$, (ii) one more Stein filling when $k \leq 9$ and $k \neq 8$, and (iii) two more Stein fillings when $k = 8$. Those Stein fillings can be realized by the positive allowable Lefschetz fibrations; the Stein filling in (ii) by Korkmaz-Ozbagci’s $k$-holed torus relation, the two Stein fillings in (iii) by Korkmaz-Ozbagci’s and Tanaka’s $8$-holed torus relations. The fact that $(Y_k, ξ_k)$ has another factorization (i.e. a $k$-holed torus relation) it also gives satisfactory simple expressions for Korkmaz-Ozbagci’s and Tanaka’s $8$-holed torus relation, the two Stein fillings in (iii) by Korkmaz-Ozbagci’s and Tanaka’s $8$-holed torus relations. The fact that $(Y_k, ξ_k)$ has a unique Stein filling for $k \geq 10$ is the second reason that there is no $k(\geq 10)$-holed torus relation.

In addition to those theoretical importance, the $k$-holed torus relations are also practically useful in combinatorial constructions of new relations in the mapping class groups. In the first place, Tanaka [3] constructed his 8-holed torus relation in order to find new relations (which locate $(-1)$-sections of a well-known Lefschetz fibration) in higher genus mapping class groups. Similarly, the author [11] also constructed an 8-holed torus relation in the search of new relations (which give much small Lefschetz fibrations over tori).

Our aim in this note is to present simplified expressions for Korkmaz-Ozbagci’s and Tanaka’s $k$-holed torus relations. Although Korkmaz-Ozbagci gave satisfactorily simple expressions for $k \leq 4$, their curves become more involved as $k$ increases. For the practical use of the $k$-holed torus relations, simpler expressions should be convenient for one who tries to use them.

In the rest of this paper, we basically follow the notation in [3] as follows. We will denote a right-handed Dehn twist along a curve $α$ also by $α$. A left-handed Dehn twist along $α$ will be denoted by $\bar{α}$. We use the functional notation for multiplication; $βα$ means we first apply $α$ and then $β$. In addition, we denote the conjugation $αβ\bar{α}$ by $α(β)$, which is the Dehn twist along the curve $t_{a}(β)$.

1. Simplification

We will simplify Korkmaz-Ozbagci’s $8$- and $9$-holed torus relations, and Tanaka’s $8$-holed torus relation. The other Korkmaz-Ozbagci’s $k(\leq 7)$-holed torus relations can be obtained from the $9$-holed torus relation by capping off some of the boundary components, therefore we do not give them explicitly.

With the curves shown in Figure 1(a) Korkmaz and Ozbagci [3] gave the 8-holed torus relation

$$α_4α_5β_1σ_3σ_6α_2β_6σ_4σ_7σ_4σ_5 = δ_1δ_2δ_3δ_4δ_5δ_6δ_7δ_8,$$

where $β_1 = a_1(β)$, $β_6 = a_6(β)$ and $β_4 = a_4(β)$. We modify this relation as follows:

$$δ_1δ_2δ_3δ_4δ_5δ_6δ_7δ_8 = α_4α_5β_1σ_3σ_6α_2β_6σ_4σ_7σ_4σ_5 = δ_1δ_2δ_3δ_4δ_5δ_6δ_7δ_8$$

By putting $a_1 = α_5$, $b_1 = α_1(β_4)$, $a_2 = α_4$, $b_2 = β_4(σ_3)$, $b_3 = β_4(σ_6)$, $a_4 = α_2$, $b_4 = a_2(β_1)$, $b_5 = β_6(σ_4)$, $b_6 = β_6(σ_7)$, $a_7 = α_7$, $b_7 = a_7(β_6)$ and $b_8 = β_6(σ_5)$, we have just obtained

$$a_1b_1a_2b_2b_3a_4b_5b_6a_7b_7b_8 = δ_1δ_2δ_3δ_4δ_5δ_6δ_7δ_8,$$

where the resulting curves are depicted in Figure 3(a). We refer to this relation as $A_8$. 
(a) Korkmaz-Ozbagci’s relation.
(b) Tanaka’s relation.

Figure 1. The curves for the two 8-holed torus relations.

Figure 2. The curves for Korkmaz-Ozbagci’s 9-holed torus relation.

With the curves shown in Figure 1(b), Tanaka [6] gave another 8-holed torus relation

$$
\alpha_5 \alpha_7 \beta_6 \beta_2 \sigma_2 \sigma_1 \alpha_1 \alpha_3 \beta_3 \beta_6 \sigma_4 \sigma_7 = \delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 \delta_8,
$$

where $\beta_6 = \alpha_6(\beta), \beta_2 = \alpha_2(\beta), \beta_3 = \alpha_3(\beta)$ and $\beta_4 = \alpha_4(\beta)$. We modify this relation as follows:

$$
\delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 \delta_8 = \alpha_5 \alpha_7 \beta_6 \beta_2 \sigma_2 \sigma_1 \alpha_1 \alpha_3 \beta_3 \beta_6 \sigma_4 \sigma_7
$$

where $\beta_6 = \alpha_6(\beta), \beta_2 = \alpha_2(\beta), \beta_3 = \alpha_3(\beta)$ and $\beta_4 = \alpha_4(\beta)$.
(a) The relation $A_8$.

(b) The relation $B_8$.

Figure 3. The curves for the simplified 8-holed torus relations.

Figure 4. The curves for the simplified 9-holed torus relation $A_9$.

By putting $a_1 = \alpha_5$, $b_1 = \beta_2(\sigma_2)$, $b_2 = \beta_2(\sigma_1)$, $a_3 = \alpha_3$, $b_3 = \bar{\alpha}_3(\beta_2)$, $b_4 = \alpha_1(\beta_2)$, $a_5 = \alpha_1$, $b_5 = \beta_4(\sigma_4)$, $b_6 = \beta_4(\sigma_7)$, $a_7 = \alpha_7$, $b_7 = \alpha_7(\beta_6)$ and $b_8 = \alpha_5(\beta_6)$, we obtained

$$a_1 b_1 b_2 a_3 b_3 b_4 a_5 b_5 b_6 a_7 b_7 b_8 = \delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 \delta_8,$$

where the resulting curves are depicted in Figure 3(b). We refer to this relation as $B_8$.

Remark 1. Let $X_{A_8} \to D^2$ and $X_{B_8} \to D^2$ be the positive allowable Lefschetz fibrations associated with $A_8$ and $B_8$, respectively. As observed in [5], we have $H_1(X_{A_8}; \mathbb{Z}) = 0$ and $H_1(X_{B_8}; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, hence $X_{A_8}$ and $X_{B_8}$ give the two distinct Stein fillings (other than the $D^2$-bundle) of the contact 3-manifold $(Y_8, \xi_8)$. 
With the curves shown in Figure 4, Korkmaz and Ozbagci gave the 9-holed torus relation

\[ \beta_4 \sigma_3 \delta_6 \delta_5 \alpha_7 \delta_6 \delta_8 \delta_9 = \delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 \delta_8 \delta_9, \]

where \( \beta_4 = \alpha_4(\beta) \), \( \beta_1 = \alpha_1(\beta) \) and \( \beta_7 = \alpha_7(\beta) \). We modify this relation as follows:

\[ \delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 \delta_8 \delta_9 = \beta_4 \sigma_3 \delta_6 \delta_5 \alpha_7 \delta_6 \delta_8 \delta_9 \]

\[ = \beta_4 \sigma_3 \delta_6 \delta_5 \alpha_7 \delta_6 \delta_8 \delta_9 \]

By putting \( a_1 = \alpha_5 \), \( b_1 = \alpha_5(\beta) \), \( b_2 = \beta_2(\beta) \), \( b_3 = \beta_3(\beta) \), \( a_4 = \alpha_2 \), \( b_4 = \alpha_2(\beta) \), \( b_5 = \beta_5(\beta) \), \( b_6 = \beta_6(\beta) \), \( a_7 = \alpha_7 \), \( b_7 = \alpha_7(\beta) \), \( b_8 = \beta_8(\beta) \), we obtained

\[ a_1 b_1 b_2 a_3 b_3 b_5 b_6 a_7 b_7 b_8 = \delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 \delta_8 \delta_9, \]

where the resulting curves are depicted in Figure 4. We refer to this relation as the \textit{schetz fibration} \( E \).

Remark 2. As we mentioned earlier, the relations \( B_8 \) and \( A_9 \) correspond to minimal Lefschetz pencils on \( S^2 \times S^2 \) and \( \mathbb{CP}^2 \), respectively (while the others, including \( A_8 \), are just blow-up of the latter). In Figure 5, we draw two handle decompositions of the elliptic Lefschetz fibration \( E(1) = \mathbb{CP}^2 \# \mathbb{CP}^2 \to S^2 \) and locate the \((-1)\)-sections corresponding to \( B_8 \) and \( A_9 \). Blowing down those sections must yield the 4-manifolds \( S^2 \times S^2 \) and \( \mathbb{CP}^2 \), respectively, and the exceptional spheres become the base points of the Lefschetz pencils.

**Figure 5.** Handle decompositions of the elliptic Lefschetz fibration \( E(1) = \mathbb{CP}^2 \# \mathbb{CP}^2 \to S^2 \) with configurations of \((-1)\)-sections.

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Graduate School of Mathematical Sciences, The University of Tokyo, Komaba, Meguro-ku, Tokyo 153-8914, Japan

E-mail address: nhamada@ms.u-tokyo.ac.jp