Tiling Generating Functions of Halved Hexagons and Quartered Hexagons

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Abstract. We prove exact product formulas for the tiling generating functions of various halved hexagons and quartered hexagons with defects on boundary. Our results generalize the previous work of the first author and the work of Ciucu.

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1. Introduction

Working on weighted enumerations of tilings often gives more insight than working on unweighted or ‘plain’ counting of tilings, and can be more challenging. It is a fact that most known results in the field are unweighted tiling enumerations; results about tiling generating functions are very rare. In this paper, we provide simple product formulas for weighted enumerations of tilings of different types of quartered hexagons and halved hexagons.

A ‘halved hexagon’ is half of a vertically symmetric hexagon divided by a vertical zigzag cut along the symmetry axis (see Fig. 4a, b for examples). The study of halved hexagons began with the work of Proctor on certain classes of staircase plane partitions [24, Corollary 4.1]. His result implies an elegant tiling formula for a hexagon with a maximal staircase cut-off, which can be viewed as a halved hexagon with a defect. We note that the tilings of a halved hexagon are in bijection with the transpose-complementary plane partitions, one of the ten symmetry classes of plane partitions [28]. We refer the reader to, e.g., [5, 6, 18–22, 26] for more discussion about tiling enumerations of halved hexagons.
A ‘quartered hexagon’ is half of a horizontally symmetric halved hexagon divided along its horizontal symmetric axis. These regions have been investigated in several different contexts, see, e.g., [1,11,13–15]. Some of these results show that tilings of quartered hexagons have fundamental connections to anti-symmetric monotone triangles and classic group characters.

As MacMahon’s classical theorem on boxed plain partitions [23] yields beautiful $q$-enumerations of lozenge tilings of a ‘quasi-regular hexagon’, one would expect the existence of nice $q$-enumerations for halved hexagons and quartered hexagons. Here a quasi-regular hexagon is a centrally symmetric hexagon with all 120° angles. However, all known enumerations of these regions are unweighted ones. In this paper, we provide nice $q$-enumerations for four different families of quartered hexagons and six families of halved hexagons.

Borodin et al. [3] provide a way to define weights for tilings as follows. They first assign an ‘elliptic weight’ to lozenges on the plane, based on some coordinate system. Then the weight of a tiling is the product of weights of its lozenges. We will adapt and specialize Borodin–Gorin–Rains’ elliptic weight in this paper.

The $i$-axis of our coordinate system runs along a horizontal lattice line. The length of one unit on the $i$-axis is precisely half the width of a vertical lozenge or, equivalently, half the length of an edge of a lozenge. The $j$-axis is perpendicular to the $i$-axis at a lattice vertex (this vertex is the origin of our coordinate system); the length of one unit on the $j$-axis is equal to half the height of a vertical lozenge or, equivalently, $\sqrt{3}/2$ times the length of an edge of a lozenge. Figure 1 shows a particular placement and tiling of a hexagon on our coordinate system.

Each vertical lozenge with center at the point $(i,j)$ has weight

$$\text{wt}_1(i,j) = \frac{q^i + q^{-i}}{2}.$$  (1.1)
(The weight does not depend on \(j\).) All other lozenges have weight 1. The weight of a tiling is the product of the weights of its lozenges. We call \(w_1\) the ‘symmetric weight.’ We also consider a variation \(w_2\) of the weight \(w_1\) by assigning to each vertical lozenge that intersects the \(j\)-axis a weight of \(1/2\) (these lozenges have weight 1 in \(w_1\)).

The remainder of this paper is organized as follows. In Sect. 2, we state in detail our main results. We provide exact formulas for the tiling generating functions (TGFs) of four families of quartered hexagons (see Theorems 2.1–2.4). Additionally, we investigate six families of halved hexagons with defects on the vertical (west) side. We also provide simple product formulas for their TGFs (see Theorems 2.7–2.12). Section 3 is devoted to several fundamental results in the enumeration of tilings. We also state two versions of Kuo condensation [12] that will be employed in our proofs. Section 4 contains a proof of Theorem 2.1 using Kuo condensation and induction. The proofs of Theorems 2.2–2.4 and Theorems 2.7–2.12 are also done in this manner and hence omitted.

2. Main Results

All regions considered in this paper are weighted regions. Strictly speaking, a ‘weighted region’ is a pair \((R, \text{wt})\), where \(R\) is an unweighted region on the triangular lattice, called the “shape” of the region, and \(\text{wt}\) is a weight assignment for the tilings of \(R\). Whenever the weight assignment is clearly given, we abuse the notation by viewing \(R\) as the weighted region. In the rest of the paper, we use the notation \(M(R)\) for the weighted sum of all tilings of \(R\). If \(R\) does not have any tilings, then \(M(R) = 0\). When \(R\) is a degenerate region (i.e., a region with empty interior), \(M(R) = 1\) by convention. We call \(M(R)\) the tiling generating function of \(R\).

2.1. Quartered Hexagons

While investigating a generalization of Jockusch–Propp’s quartered Aztec diamond [10], the first author proved simple product formulas for four families of the quartered hexagons [13,14] (see Fig. 2). The tilings of the quartered hexagons have interesting connections to antisymmetric monotone triangles [15] and characters of classical groups [1,11].

The four different types of quartered hexagons are as follows.

We start with a right trapezoidal region whose side-lengths are \(x, 2n - 1, x + n, 2n - 1\) in counter-clockwise order, starting from the north side\(^1\). The north, northeast, and south sides of the region follow lattice lines, and the west side follows a vertical zigzag. We also remove from the base of the region \(n\) up-pointing unit triangles at the positions \(s_1, s_2, \ldots, s_n\), from left to right (see the shape of the region in Fig. 2a; the black unit triangles indicate the removed ones). These removed triangles are called “dents”. Next, we assign weights to lozenges of the region (which may be used in tilings) using the

\(^1\)From now on, we always list the side-lengths of a region in this order.
Figure 2. Two different types of quartered hexagons

symmetric weight \( w_{t1} \) as in Fig. 3a. The \( j \)-axis is touching the vertical west side of the region, and the \( i \)-axis runs along the base. The vertical lozenge with center at the point \((i, j)\) has weight \( \frac{j + q^{-1}}{2} \), and the weight of a tiling is the product of the lozenge-weights, as usual. Denote by \( R^1_x(s_1, s_2, \ldots, s_n) \) the resulting weighted region.

The second family of weighted quartered hexagons is defined similarly. We start with a right trapezoidal region of side-lengths \( x, 2n, x + n, 2n \) and remove \( n \) up-pointing unit triangles along the base. Assume that the positions of the removed unit triangles are \( s_1, s_2, \ldots, s_n \). See Fig. 2b for the shape of the region. We again use the symmetric weight \( w_{t1} \) for lozenges in the new region, as shown in Fig. 3b. Denote by \( R^2_x(s_1, s_2, \ldots, s_n) \) this new weighted region. The TGFs of these two types of quartered hexagons are given by simple product formulas.

The \( q \)-integer \([n]_q\) is defined as \([n]_q = 1 + q + \cdots + q^{n-1}\). Then the \( q \)-factorial is defined to be the product of \( q \) integers: \([n]_q! = [1]_q[2]_q \cdots [n]_q\).

**Theorem 2.1.** Assume that \( x \) and \( n \) are non-negative integers and \((s_i)_{i=1}^n\) is a sequence of positive integers between 1 and \( n + x \). Then

\[
M(R^1_x(s_1, s_2, \ldots, s_n)) = 2^{n(n-1)}q^{-\sum_{i=1}^n(i-1)(4s_i-2i-1)} \frac{\prod_{i=1}^n[2(s_i + s_j - 1)]q^2}{\prod_{i=1}^n[2i-2]q^2!} \\
= 2^{n(n-1)}q^{-\sum_{i=1}^n(i-1)(4s_i-2i-1)} \prod_{1 \leq i < j \leq n} \frac{[2(s_i + s_j - 1)]q^2}{[i + j - 1]q^2} \prod_{1 \leq i < j \leq n} \frac{[2(s_j - s_i)]q^2}{[j - i]q^2}.
\]

**Theorem 2.2.** Assume that \( x \) and \( n \) are non-negative integers and \((s_i)_{i=1}^n\) is a sequence of positive integers between 1 and \( n + x \). Then

\[
M(R^2_x(s_1, s_2, \ldots, s_n)) = \frac{2^{n(n-1)}q^{-\sum_{i=1}^n(i-1)(4s_i-2i-1)}}{\prod_{i=1}^n[2i-2]q^2!} \\
= 2^{n(n-1)}q^{-\sum_{i=1}^n(i-1)(4s_i-2i-1)} \prod_{1 \leq i < j \leq n} \frac{[2(s_i + s_j - 1)]q^2}{[i + j - 1]q^2} \prod_{1 \leq i < j \leq n} \frac{[2(s_j - s_i)]q^2}{[j - i]q^2}.
\]
Figure 3. Four types of weighted quartered hexagons arising from two different shapes and two different weight assignments

\[
2^{-n^2} q^{-\sum_{i=1}^n (4i-2)s_i-2i^2+i} \prod_{i=1}^n [4s_i]q^2 \prod_{1 \leq i < j \leq n} [2(s_i+s_j)]q^2 [2(s_j-s_i)]q^2 \\
= 2^{-n^2} q^{-\sum_{i=1}^n (4i-2)s_i-2i^2+i} \prod_{1 \leq i \leq j \leq n} [2(s_i+s_j)]q^2 \prod_{1 \leq i < j \leq n} [2(s_j-s_i)]q^2.
\]

(2.2)

We are also interested in the siblings of the previous regions which use the variation \(\text{wt}_2\) of the symmetric weight \(\text{wt}_1\). The third family of weighted quartered hexagons has the same shape as the first one; the only difference is the weight assignment \(\text{wt}_2\) has been used. In particular, the \(j\)-axis now passes through the center of the vertical lozenges along the west side of the region. Each vertical lozenge with center at the point \((i, j)\) still has weight \(q^{i+j}+q^{-i-j}\), with one exception: the vertical lozenges intersected by the \(j\)-axis have weight 1/2 (see Fig. 3c). Denote by \(R^3_x(s_1, s_2, \ldots, s_n)\) this variant of an \(R^1\)-type region. Finally, the weighted quartered hexagon \(R^4_x(s_1, s_2, \ldots, s_n)\) has the same shape as the one in the second family; the only difference is the weight assignment.
We again use the weight $w_t$, instead of $w_1$. See Fig. 3d for the details. We also have nice formulas for the TGFs of these new quartered hexagons.

**Theorem 2.3.** Assume that $x$ and $n$ are non-negative integers and $(s_i)^n_{i=1}$ is a sequence of positive integers between 1 and $n + x$. Then

$$M(R_x^3(s_1, s_2, \ldots, s_n)) = 2^{-n(n-1)} q^{-\sum_{i=1}^n ((4i-4)s_i - 2i^2 - i + 3)} \prod_{1 \leq i < j \leq n} [2(s_i + s_j - 2)]q^2 \frac{[2(s_j - s_i)]q^2}{\prod_{i=1}^n [2i - 2]q^{2i}}$$

$$= 2^{-n(n-1)} q^{-\sum_{i=1}^n ((4i-4)s_i - 2i^2 - i + 3)} \prod_{1 \leq i < j \leq n} \frac{[2(s_i + s_j - 2)]q^2}{[i + j - 1]q^2} \prod_{1 \leq i < j \leq n} \frac{[2(s_j - s_i)]q^2}{[j - i]q^2}. \quad (2.3)$$

**Theorem 2.4.** Assume that $x$ and $n$ are non-negative integers and $(s_i)^n_{i=1}$ is a sequence of positive integers between 1 and $n + x$. Then

$$M(R_x^4(s_1, s_2, \ldots, s_n)) = 2^{-n^2} q^{-\sum_{i=1}^n ((4i-2)s_i - 2i^2 - i + 1)} \prod_{1 \leq i < j \leq n} [2(s_i + s_j - 1)]q^2 \frac{[2(s_j - s_i)]q^2}{\prod_{i=1}^n [2i - 1]q^{2i}}$$

$$= 2^{-n^2} q^{-\sum_{i=1}^n ((4i-2)s_i - 2i^2 - i + 1)} \prod_{1 \leq i < j \leq n} \frac{[2(s_i + s_j - 1)]q^2}{[i + j - 1]q^2} \prod_{1 \leq i < j \leq n} \frac{[2(s_j - s_i)]q^2}{[j - i]q^2}. \quad (2.4)$$

**Remark 2.5.** (Combinatorial reciprocity phenomenon) From the four formulas (2.1)–(2.4), we realize that the TGF of $R_x^3(s_1, s_2, \ldots, s_n)$ is obtained from the TGF of $R_x^3(s_1, s_2, \ldots, s_n)$ by replacing $s_i$ with $s_i - 1/2$, for $i = 1, 2, \ldots, n$ and similarly, the TGF of $R_x^4(s_1, s_2, \ldots, s_n)$ is obtained from the TGF of $R_x^4(s_1, s_2, \ldots, s_n)$ by making the same replacement. These observations remind us of the “combinatorial reciprocity phenomenon”: even though the region $R_x^1(s_1, s_2, \ldots, s_n)$ (resp. $R_x^2(s_1, s_2, \ldots, s_n)$) is not defined when the $s_i$’s are half-integers, its tiling formula gives the number of combinatorial objects of a different sort (here, the tilings of $R_x^2(s_1, s_2, \ldots, s_n)$ (resp. $R_x^4(s_1, s_2, \ldots, s_n)$) when evaluated at half-integers. We refer the reader to, e.g., [2, 25, 27] for more discussion about the phenomenon. It would be very interesting to have a direct explanation for this which does not require calculating the TGFs.

**Remark 2.6.** As shown in [1], symplectic and orthogonal characters give certain weighted tiling enumerations of the quartered hexagon. In particular, by evaluating the principal specialization of the classical group characters, one could obtain nice $q$- enumerations of the quartered hexagon. However, these $q$- enumerations are different from those in our Theorems 2.1–2.4, as the weight assignments for the lozenges in the two cases are different.
2.2. Halved Hexagons with Dents on Vertical Side

We consider next four families of halved hexagons with dents on the vertical side.

The first family is illustrated in Fig. 4a. We start with a halved hexagon whose north, northeast, southeast, and south sides have side-lengths $x, 2d - m, m, x + d - m$ ($m \leq d$). The west side runs along a vertical zigzag with $2d$ steps. Along the west side, we remove $d - m$ up-pointing unit triangles (the black ones). We label the ‘bumps’ on the west side $1, 2, \ldots, d$ (from bottom to the top). Assume that the positions of the bumps which do not contain a removed unit triangle are $1 \leq l_1 < l_2 \cdots < l_m \leq d$. We now assign weights to the lozenges of the region using the weight assignment $w_{1, \text{asshow}}$, as shown by example in Fig. 5a. Denote by $A_{x,d}(l_1, l_2, \ldots, l_m)$ the resulting weighted region.
We next consider three more variations of $A_{x,d}(l_1, \ldots, l_m)$ as follows. The second family of weighted regions uses the same weight assignment $w_1$ as the $A$-type regions; the differences are the length of the northeast side (which now has side-length $2d - m + 1$) and the length of the bottom-most step of the zigzag on the west side (which is now 2). Denote by $B_{x,d}(l_1, \ldots, l_m)$, the resulting region. See Fig. 4b for the shape of the $B$-type region and Fig. 5b for the weight assignment of the region.

The third family of regions has exactly the same shape as the $A$-type region; however, we now use the weight assignment $w_2$. Recall that the vertical lozenges intersected by the $j$-axis now have weight $1/2$. We denote by $C_{x,u}(h_1, \ldots, h_n)$ these regions (shown in Figs. 4c and 5c). Similarly, the regions

**Figure 5.** Weight assignments for the four families of halved hexagons
in the fourth family are identical in shape to the $B$-type regions; the only difference is that we now apply weight assignment $w_{t_2}$ to their lozenges. These regions are denoted by $D_{x,u}(h_1, \ldots, h_n)$ (illustrated in Figs. 4d and 5d).

The TGFs of all four regions above are given by simple product formulas.

We adopt the following notation for $0 \leq a \leq b$:

$$\langle b \rangle := a + (a + 1) + (a + 2) + \cdots + b.$$  \hfill (2.5)

We define two polynomials as follows:

$$P(x, u, d, (l_i)_{i=1}^m, (h_j)_{j=1}^n) = 2^{-E} q^{-F} \frac{\prod_{1 \leq i < j \leq m}[2(l_j - l_i)]q^2 \prod_{1 \leq i < j \leq n}[2(h_j - h_i)]q^2}{\prod_{i=1}^m[2(l_i - 1)]q^2 \prod_{j=1}^n[2(h_j - 1)]q^2!} \prod_{i=1}^m \prod_{j=1}^n [2\bar{x} + 2i + j - 1]q^2 \prod_{i=1}^m [2(\bar{x} + m + i)]q^2 \prod_{i=1}^m \prod_{j=1}^n [2\bar{x} + m + i + j]q^2$$

$$\times \prod_{i=1}^m \prod_{j=1}^n [2\bar{x} + i + j - 1]q^2 \prod_{i=1}^m \prod_{j=1}^n [2\bar{x} - i - j + m + 1]q^2 \prod_{i=1}^m \prod_{j=1}^n [2\bar{x} + i + j - 1]q^2 \prod_{i=1}^m \prod_{j=1}^n [2\bar{x} - i - j + n + 1]q^2,$$  \hfill (2.6)

where $\bar{x} = x + u - n + d - m - e = x + \max(u - n, d - m)$ and $E$ and $F$ are defined below.

$$Q(x, u, d, (l_i)_{i=1}^m, (h_j)_{j=1}^n) = 2^{-E'} q^{-F'} \frac{\prod_{1 \leq i < j \leq m}[2(l_j - l_i)]q^2 \prod_{1 \leq i < j \leq n}[2(h_j - h_i)]q^2}{\prod_{i=1}^m[2(l_i - 1)]q^2 \prod_{j=1}^n[2(h_j - 1)]q^2!}$$

$$\times \prod_{i=1}^m \prod_{j=1}^n [2\bar{x} + 2i + j]q^2 \prod_{i=1}^m [2(\bar{x} + n + i)]q^2 \prod_{i=1}^m \prod_{j=1}^n [2\bar{x} + n + i + j + 1]q^2$$

$$\times \prod_{i=1}^m \prod_{j=1}^n [2\bar{x} + n + i + j + 1]q^2$$

$$\times \prod_{i=1}^m \prod_{j=1}^n [2\bar{x} + i + j - 1]q^2 \prod_{i=1}^m \prod_{j=1}^n [2\bar{x} + i + j - 1 + 1]q^2$$

$$\times \prod_{i=1}^m \prod_{j=1}^n [2\bar{x} + i + j + n + 1]q^2 [2\bar{x} - i - j + m + 1]q^2$$
\begin{equation}
\times \prod_{i=1}^{n} \prod_{j=1}^{h_i} [2(\bar{x} + i + j + m)]_q^2 [2(\bar{x} - i + j + n + 2)]_q^2.
\end{equation}

where \( \bar{x} = x + u - n + d - m - e' = x + \max(u - n, d - m + 1) - 1 \) and \( E' \) and \( F' \) are defined below.

We define the exponents \( E, F, E', F' \) as follows:

\begin{align*}
E &= E(x, u, d, (l_i)^{m}_{i=1}, (h_j)^{n}_{j=1}) = f(t_1 + 2d - m, n, m + 1) \\
& \quad + \sum_{i=1}^{m} \left( \frac{t_1 + 2d - m - (m - i + 1)}{t_1 + 2(d - l_i) - 2(m - i)} \right) + \sum_{j=1}^{n} \left( \frac{t_2 + 2u - n - (n - j + 1)}{t_2 + 2(u - h_j) - 2(n - j)} \right),
\end{align*}

\begin{align*}
F &= F((l_i)^{m}_{i=1}, (h_j)^{n}_{j=1}) = n(m + 1) + \sum_{i=1}^{m} (2l_i - i) + \sum_{j=1}^{n} (2h_j - j),
\end{align*}

\begin{align*}
E' &= E'(x, u, d, (l_i)^{m}_{i=1}, (h_j)^{n}_{j=1}) = f(t'_1 + 2d - m, n + 1, m) \\
& \quad + \sum_{i=1}^{m} \left( \frac{t'_1 + 2d - m - (m - i + 1)}{t'_1 + 2(d - l_i) - 2(m - i)} \right) + \sum_{j=1}^{n} \left( \frac{t'_2 + 2u - n - (n - j + 1)}{t'_2 + 2(u - h_j) - 2(n - j)} \right),
\end{align*}

\begin{align*}
F' &= F'((l_i)^{m}_{i=1}, (h_j)^{n}_{j=1}) = (n + 1)m + \sum_{i=1}^{m} (2l_i - i) + \sum_{j=1}^{n} (2h_j - j),
\end{align*}

where \( f(t, x, y) = \frac{xy(2t + x - y)}{2} \), \( t_1 = 2(x + u - n - e) + 1, t_2 = 2(x + d - m - e) \),
\( t'_1 = 2(x + u - n - e') + 1, t'_2 = 2(x + d - m + 1 - e') \), and \( l_0 = h_0 = 0 \) by convention.

**Theorem 2.7** (Formula for a region of type A). For non-negative integers \( x, d, m \) (\( d \geq m \)) and a sequence \( (l_i)^{m}_{i=1} \) of positive integers between 1 and \( d \), we have

\begin{equation}
M(A_{x,d}(l_1, l_2, \ldots, l_m)) = P(x, 0, d, (l_i)^{m}_{i=1}, \emptyset).
\end{equation}

**Theorem 2.8** (Formula for a region of type B). For non-negative integers \( x, d, m \) (\( d \geq m \)) and a sequence \( (l_i)^{m}_{i=1} \) of positive integers between 1 and \( d \), we have

\begin{equation}
M(B_{x,d}(l_1, l_2, \ldots, l_m)) = Q(x, 0, d, (l_i)^{m}_{i=1}, \emptyset).
\end{equation}

**Theorem 2.9** (Formula for a region of type C). For non-negative integers \( x, u, n \) (\( u \geq n \)) and a sequence \( (h_j)^{n}_{j=1} \) of positive integers between 1 and \( u \), we have

\begin{equation}
M(C_{x,u}(h_1, h_2, \ldots, h_n)) = Q(x, u, 0, \emptyset, (h_j)^{n}_{j=1}).
\end{equation}

**Theorem 2.10** (Formula for a region of type D). For non-negative integers \( x, u, n \) (\( u \geq n \)) and a sequence \( (h_j)^{n}_{j=1} \) of positive integers between 1 and \( u \), we have

\begin{equation}
M(D_{x,u}(h_1, h_2, \ldots, h_n)) = P(x, u, 0, \emptyset, (h_j)^{n}_{j=1}).
\end{equation}
We consider next a hybrid of the $A$-type and $D$-type regions. Our new region can roughly be viewed as a $D$-type region and (the vertical reflection of) an $A$-type region glued together along their bases. Let $x, u, d, m,$ and $n$ be non-negative integers, with $m \leq d$ and $n \leq u$. We consider the halved hexagons whose north, northeast, southeast and south sides have the side-lengths $x + d - m - e, 2u + m - n + 1, 2d - m + n, x + u - n + e$, respectively, where $e = \min(u - n, d - m)$. The west side is now the concatenation of two vertical zigzag paths with $2d$ and $2u$ steps as in Fig. 6a. Strictly speaking, there is an additional unit step between the two zigzag paths; this brings the length of the west side to $2u + 2d + 1$. We remove $u - n$ up-pointing unit triangles from the upper part of the west side and $d - m$ down-pointing unit triangles from the lower part. Next, we assign the weights to the vertical lozenges of the region using the weight assignment $\text{wt}_2$ as in Fig. 7a. (The $j$-axis passes the upper-left vertex of the region, and the $i$-axis runs along the base.) Denote by $S_{x,u,d}(\{(l_i)_{i=1}^m\}; \{(h_j)_{j=1}^n\})$ the resulting weighted region.

We are also interested in a counterpart of the $S$-type region as follows. Our base halved hexagon has side-lengths $x + d - m + 1 - e', 2u + m - n, 2d - m + n + 1, x + u - n - e', 2u + 2d + 1$, where $e' = \min(u - n, d - m + 1)$. We still remove $u - n$ up-pointing and $d - m$ down-pointing unit triangles from the west side,
as in the case of the $S$-type regions. However, we remove an additional down-pointing triangle in the “middle” of the west side (in particular, we remove the only down-pointing unit triangle on the west side between the first upper and lower bumps). We use the weight assignment $w_2$ just as in the $S$-type regions. We denote by $T_{x,u,d}(\ell_i^{m_i=1}, h_j^{n_j=1})$ the new weighted region. See Figs. 6b and 7b for examples. One can also view a $T$-type region as a hybrid version of a $C$-type region and (the vertical reflection of) a $B$-type region.

The TGFs of the $S$- and $T$-type regions are also given by simple product formulas. Moreover, their tiling generating function are generalizations of those of the $A$- and $D$-type regions and the $B$- and $C$-regions, respectively.

**Theorem 2.11** (Formula for region of type $S$). Assume that $x, u, d, m,$ and $n$ are non-negative integers ($d \geq m$, $u \geq n$), and $(\ell_i^{m_i=1}, (h_j^{n_j=1})$ are two sequences of positive integers between 1 and $d$ and between 1 and $u$, respectively. Then

$$M(S_{x,u,d}(\ell_i^{m_i=1}, h_j^{n_j=1}) = P(x,u,d,\ell_i^{m_i=1}, h_j^{n_j=1}). \quad (2.16)$$

**Theorem 2.12** (Formula for region of type $T$). Assume that $x, u, d, m,$ and $n$ are non-negative integers ($d \geq m$, $u \geq n$), and $(\ell_i^{m_i=1}, (h_j^{n_j=1})$ are two sequences of positive integers between 1 and $d$ and between 1 and $u$, respectively. Then

$$M(T_{x,u,d}(\ell_i^{m_i=1}, h_j^{n_j=1}) = Q(x,u,d,\ell_i^{m_i=1}, h_j^{n_j=1}). \quad (2.17)$$

**Remark 2.13.** We note that the unweighted versions of Theorems 2.7–2.12 were treated by Ciucu in [5]. Moreover, following the line of work in Ciucu’s paper,
one can obtain the tiling generating function of the symmetric hexagon with two families of holes on the symmetry axis, say using our Theorems 2.11 and 2.12 along with Ciucu’s Factorization Theorem [4, Theorem 1.2]. We leave it as an exercise for the reader.

3. Preliminaries

A perfect matching of a graph is a collection of disjoint edges that covers all vertices of the graph. There is a bijection between tilings of a region $R$ on the triangular lattice and perfect matchings of its (planar) dual graph $G$ (i.e., the graph whose vertices are the unit triangles in $R$ and whose edges connect precisely two unit triangles sharing an edge). The weight of an edge in the dual graph is equal to that of the corresponding lozenge in the region. We use the notation $M(G)$ for the weighted sum of the perfect matchings of the graph $G$, where the weight of a perfect matching is the product of its edge-weights. We often call $M(G)$ the matching generating function of $G$.

Kuo [12] introduced a graphical version of the well-known Dodgson condensation [7]. He used his condensation to provide alternative proof for the well-known Aztec diamond theorem [8,9] and the classical MacMahon’s theorem on plane partitions fitting in a given box [23]. Kuo condensation is especially useful in proving a conjectured formula for a TGF. In many cases, one can use Kuo condensation to obtain recurrences for TGFs that yield an inductive proof for the tiling formula. In particular, we will use the following two versions of the Kuo condensation in our proofs.

**Lemma 3.1** (Theorem 5.1 in [12]). Let $G = (V_1, V_2, E)$ be a weighted plane bipartite graph in which $|V_1| = |V_2|$. Let vertices $u, v, w, s$ appear on a face of $G$, in that order. If $u, w \in V_1$ and $v, s \in V_2$, then

$$M(G) M(G - \{u, v, w, s\}) = M(G - \{u, v\}) \times M(G - \{w, s\}) + M(G - \{u, s\}) M(G - \{v, w\}). \tag{3.1}$$

**Lemma 3.2** (Theorem 5.3 in [12]). Let $G = (V_1, V_2, E)$ be a weighted plane bipartite graph in which $|V_1| = |V_2| + 1$. Let vertices $u, v, w, s$ appear on a face of $G$, in that order. If $u, v, w \in V_1$ and $s \in V_2$, then

$$M(G - \{v\}) M(G - \{u, w, s\}) = M(G - \{u\}) M(G - \{v, w, s\}) + M(G - \{w\}) M(G - \{u, v, s\}). \tag{3.2}$$

A forced lozenge of a region $R$ is a lozenge that must be contained in any tiling of $R$. Assume that we remove $k$ forced lozenges $l_1, l_2, \ldots, l_k$ from $R$ to obtain the region $R'$. Then we have

$$M(R') = \left( \prod_{i=1}^{k} \text{wt}(l_i) \right)^{-1} \cdot M(R), \tag{3.3}$$

where $\text{wt}(l_i)$ is the weight of the lozenge $l_i$.

If the region $R$ admits a tiling, then $R$ must have the same number of up-pointing and down-pointing unit triangles. We call such a region balanced.
Figure 8. Two ways to assign weights to the lozenges of a halved hexagon. The shaded lozenges intersected by the $j$-axis have weight $\frac{1}{2}$ while the ones with label $n$ have weight $q^n + q^{-n}$. The following simple lemma is especially useful in the enumeration of tilings as it allows us to decompose a large region into smaller ones.

**Lemma 3.3** (Region-splitting Lemma [16,17]). Assume $R$ is a balanced region and $Q$ is a subregion of $R$ satisfying two conditions:

1. The unit triangles in $Q$ that have an edge on the boundary between $Q$ and $R \setminus Q$ have the same orientation (all are up-pointing or all are down-pointing);
2. $Q$ is balanced.

Then we have $M(R) = M(Q) \cdot M(R \setminus Q)$.

### 4. Proof of Theorem 2.1

This section is devoted to the proof of Theorem 2.1. We use Kuo condensation to obtain a recurrence for the TGF and provide an inductive proof for the theorem.

Before we can prove Theorem 2.1 by induction, we prove a result on a weighted halved hexagon that serves as a base case. The halved hexagon is obtained by a symmetric hexagon of side-lengths $2x + 1, n, n, 2x + 1, n, n$ (in counter-clockwise order, starting from the top side) by diving along a vertical zigzag cut. We assign weights to lozenges of the halved hexagon as in Fig. 8a; this weight assignment is exactly $w_1$ defined earlier for quartered hexagons. Denote by $P_{n,x}$ the resulting weighted region.
Figure 9. Several special halved hexagons and quartered hexagons

Lemma 4.1. Assume that $x$ and $n$ are non-negative integers. Then we have

$$M(P_{n,x}) = 2^{-n^2} q^{-\sum_{i=1}^{n} (2i-1)(2x+i)} \prod_{i=1}^{n} [2(x+i)]_{q^2} \times \prod_{1 \leq i < j \leq n} [2(2x+i+j)]_{q^2} [2(j-i)]_{q^2}. \quad (4.1)$$

As in the case of the quartered hexagon, we consider a variant $P'_{n,x}$ of $P_{n,x}$ by re-assigning the lozenge-weights as in Fig. 8b. In particular, the weights of lozenges are the same, except for the ones intersected by the $j$-axis, which now have weight $1/2$ (this weighting is exactly that of $\text{wt}_2$ defined earlier for quartered hexagons).

Lemma 4.2. Assume that $x$ and $n$ are non-negative integers. Then we have

$$M(P'_{n,x}) = 2^{-n^2} q^{-\sum_{i=1}^{n} (2i-1)(2x+i-1)} \prod_{i=1}^{n} [2(x+i) - 1]_{q^2} \times \prod_{1 \leq i < j \leq n} [2(2x+i+j-1)]_{q^2} [2(j-i)]_{q^2}. \quad (4.2)$$

Next, we prove Lemma 4.1 (Lemma 4.2 can be proven in a similar manner).

Proof of Lemma 4.1. We prove this lemma by induction on $n + x$. The base cases are the situations in which $n = 0$, $n = 1$, or $x = 0$.

When $x = 0$, the region has only one tiling (see Fig. 9a). It is easy to verify our identity in this case. When $n = 0$, then the region is degenerate and our identity becomes “1=1”. If $n = 1$, then our region becomes a hexagon of side-lengths $x, 1, 1, x, 1, 1$. It is easy to see that the hexagon has exactly $x$ tilings, each consists of one vertical lozenge, $x$ left-tilted lozenges, and $x$ right-tilted lozenges (illustrated in Fig. 9b). It is easy to calculate the TGF in this case, and then verify the identity.

For the induction step, we assume that $x > 0$, $n > 1$ and that the theorem holds for any halved hexagon in which the sum of the $x$- and $n$-parameters is strictly less than $x + n$. Apply Kuo condensation as in Lemma 3.1 to the dual
graph $G$ of $P_{n,x}$ with the four vertices $u, v, w, \text{ and } s$ chosen as in Fig. 10b. More precisely, the vertices $u, v, w, s$ of $G$ are indicated by the corresponding unit triangles in the halved hexagon. The $u$- and $v$-triangles are the shaded ones in the upper-right corner of the region, and the $w$- and $s$-triangles are the shaded ones in the lower-right corner. Let us consider the region corresponding to the
graph $G - \{u, v, w, s\}$. The removal of the four shaded unit triangles produces some forced lozenges as in Fig. 10b. By removing these forced lozenges, we get a new and smaller halved hexagon, namely $P_{n-2,x}$ (see the region restricted by the bold contour). This implies that

$$M(G - \{u, v, w, s\}) = W_1 \cdot M(P_{n-2,x}), \quad (4.3)$$

where $W_1$ is the product of the weights of the forced lozenges. Considering the removal of forced lozenges as in Fig. 10c–f, we get

$$M(G - \{u, v\}) = W_2 \cdot M(P_{n-1,x}), \quad (4.4)$$

$$M(G - \{w, s\}) = W_3 \cdot M(P_{n-1,x}), \quad (4.5)$$

$$M(G - \{u, s\}) = W_4 \cdot M(P_{n-2,x+1}), \quad (4.6)$$

$$M(G - \{v, w\}) = W_5 \cdot M(P_{n,x-1}), \quad (4.7)$$

where the $W_2, W_3, W_4$, and $W_5$ are the products of the weights of the forced lozenges in their respective regions. Plugging the above five equations into the recurrence in Lemma 3.1, we get a recurrence for the TGFs of halved hexagons:

$$W_1 \cdot M(P_{n,x}) M(P_{n-2,x}) = W_2 W_3 M(P_{n-1,x}) M(P_{n-1,x})$$

$$+ W_4 W_5 \cdot M(P_{n-2,x+1}) M(P_{n,x-1}). \quad (4.8)$$

We now carefully investigate the forced lozenges. We can see that only the vertical ones contribute to the factors $W_i$, for $i = 1, \ldots, 5$ (as the others have weight 1). First observe that $W_4 = 1$. Moreover, each weight of a vertical forced lozenge (except for the rightmost one) appears exactly one in each of $W_1, W_2 \cdot W_3$, and $W_5$. The rightmost vertical forced lozenge, which we denote by $l_0$, appears exactly once in $W_1$ and $W_5$, but twice in the product $W_2 \cdot W_3$. Therefore $\frac{W_2 \cdot W_3}{W_1} = \text{wt}(l_0)$ and $\frac{W_4 \cdot W_5}{W_1} = \frac{W_5}{W_1} = 1$, where $\text{wt}(l_0)$ is the weight of the lozenge $l_0$. By definition, we have $\text{wt}(l_0) = q^{2x+n} + q^{-2x-n}/2$. The recurrence (4.8) now reduces to

$$M(P_{n,x}) M(P_{n-2,x}) = q^{2x+n} + q^{-2x-n}/2 M(P_{n-1,x})$$

$$\times M(P_{n-1,x}) + M(P_{n-2,x+1}) M(P_{n,x-1}). \quad (4.9)$$

It is easy to see that the statistic $n + x$ of the last five regions in the above recurrence are all less than that of the first one. It is routine to check that the expression on the right-hand side of (4.1) satisfies this recurrence, and our theorem follows from the induction principle. \qed

Proof of Theorem 2.1. Assume that $l$ is the largest index such that there is not a dent at position $s_l - 1$ on the base of the quartered hexagon $R_1^x(s_1, \ldots, s_n)$, where $s_{n+1} = n + x + 1$ by convention. Then $n - l + 1$ is the size of the maximal cluster of dents attached to the lower-right corner of the region. We now prove
by induction on the statistic \( p := x + n + l \). The base cases are the situations in which \( n = 0, x = 0, \) or \( l = 1 \).

When \( n = 0 \), our region is a degenerate one. By convention, the TGF is taken to be 1, and our identity becomes “\( 1=1 \)”. When \( x = 0 \), the region only has one tiling consisting of all vertical lozenges (see Fig. 9c). The weight of this tiling is exactly the TGF of the region. It is easy to verify that (2.1) holds in this case.

When \( l = 1 \), all dents are contiguous and form a large cluster of size \( n \) attached to the lower-right corner of the region. By removing forced lozenges, we obtained the halved hexagon \( P_{x,n-1} \) (see Fig. 9d). Then (2.1) follows from Lemma 4.1.

For the induction step we assume that \( x, n > 0 \) and \( l > 1 \) and that (2.1) holds for any region with the \( p \)-statistic strictly less than \( x + n + l \).

If \( l = n + 1 \), then we can remove forced lozenges with weight 1 along the right side of the region to obtain a “smaller” region. Here, we say that the \( R^1 \)-type region \( A \) is “smaller” than a region \( B \) of the same type if the \( p \)-statistic in \( A \) is less than that in \( B \). Then (2.1) follows from the induction hypothesis.

In the rest of the proof, we assume that \( l \leq n \). We now apply Kuo condensation to obtain a recurrence for the TGF of the \( R^1 \)-type region. We apply Lemma 3.2 to the dual graph \( G \) of the region \( R \) obtained from \( R^1_x(s_1, \ldots, s_n) \) by filling the leftmost dent \((s_1)\) with a unit triangle. The vertices \( u, v, w, s \) of \( G \) are indicated by the corresponding unit triangles of \( R \) in Fig. 11a. In particular, the \( u \)-triangle is the up-pointing unit triangle at position \( \alpha = s_1 - 1 \), the \( v \)-triangle is at position \( s_1 \), and the \( w \)- and \( s \)-triangles are at the upper-right corner of the region.

The removal of the \( u \)-, \( v \)-, \( w \)- and \( s \)-triangles yields forced lozenges as indicated in Fig. 12. By considering the weights of the forced lozenges as in the proof of Lemma 4.1, we have the following recurrence:

\[
M(R^1_x(\{s_i\}_{i=1}^n)) M(R^1_x(\{s_i\}_{i=2}^{n-1})) = M(R^1_x(\{s_i\}_{i=1}^n)) M(R^1_x(\{s_i\}_{i=1}^{n-1})) + M(R^1_x(\{s_i\}_{i=2}^{n})) M(R^1_x(\{s_i\}_{i=1}^{n-1})).
\]

(4.10)

Here we use the notation \( \alpha S \) for the ordered set obtained by including \( \alpha \) and reordering the elements.

To finish the proof we verify that the expression on the right-hand side of (2.1) satisfies the same recurrence. This verification is straightforward, but, due to the complexity of the formula, it is not trivial. For completeness, we briefly show the verification. Denote by \( f(\{s_1, s_2, \ldots, s_n\}) \) the expression on the right-hand side of (2.1). We would like to show that

\[
\frac{f(\{s_i\}_{i=2}^n) f(\{s_i\}_{i=2}^{n-1})}{f(\{s_i\}_{i=1}^n) f(\alpha s_{i=2}^{n-1})} + \frac{f(\{s_i\}_{i=2}^n) f(\alpha s_{i=1}^{n-1})}{f(\{s_i\}_{i=1}^n) f(\alpha s_{i=2}^{n-1})} = 1.
\]

(4.11)

By definition, the powers of 2 and \( q \) in the first term cancel out. Now we can write the first term as

\[
\frac{\Delta(\{s_i\}_{i=2}^n) \cdot \Box(\alpha s_{i=2}^{n-1}) \cdot \Delta(\{s_i\}_{i=1}^{n-1}) \cdot \Box(\alpha s_{i=1}^{n-1})}{\Delta(\{s_i\}_{i=1}^n) \cdot \Box(\{s_i\}_{i=1}^{n-1}) \cdot \Delta(\alpha s_{i=2}^{n-1}) \cdot \Box(\alpha s_{i=1}^{n-1})}.
\]

(4.12)
where $\Delta(S) := \prod_{1 \leq i < j \leq n} [2(s_j - s_i)]_{q^2}$ and $\square(S) := \prod_{1 \leq i < j \leq n} [2(s_j + s_i - 1)]_{q^2}$ for an ordered set $S = \{s_1 < s_2 < \cdots < s_n\}$. By definition, we can simplify

$$\frac{\Delta(\{s_i\}_{i=2}^n)\Delta(\{s_i\}_{i=1}^{n-1})}{\Delta(\{s_i\}_{i=1}^n)\Delta(\{s_i\}_{i=2}^{n-1})} = \frac{[2(s_n - \alpha)]_{q^2}}{[2(s_n - s_1)]_{q^2}} \quad \text{and} \quad \frac{\square(\{s_i\}_{i=2}^n)\square(\{s_i\}_{i=1}^{n-1})}{\square(\{s_i\}_{i=1}^n)\square(\{s_i\}_{i=2}^{n-1})} = \frac{[2(s_n + \alpha - 1)]_{q^2}}{[2(s_n + s_1 - 1)]_{q^2}}. \quad (4.13)$$

The first term now reduces to

$$\frac{[2(s_n - \alpha)]_{q^2}}{[2(s_n - s_1)]_{q^2}} \cdot \frac{[2(s_n + \alpha - 1)]_{q^2}}{[2(s_n + s_1 - 1)]_{q^2}}.$$
Figure 12. Obtaining a recurrence for the TGF of a quartered hexagon
Similarly, the second term is equal to

\[ q^{4(s_n - \alpha)} \frac{[2(\alpha - s_1)]_{q^2}}{[2(s_n - s_1)]_{q^2}} \frac{[2(\alpha + s_1)]_{q^2}}{[2(s_n + s_1 - 1)]_{q^2}}. \]

The identity (2.1) is now equivalent to

\[
\frac{[2(s_n - \alpha)]_{q^2}}{[2(s_n - s_1)]_{q^2}} \frac{[2(s_n + \alpha - 1)]_{q^2}}{[2(s_n + s_1 - 1)]_{q^2}} + q^{4(s_n - \alpha)} \frac{[2(\alpha - s_1)]_{q^2}}{[2(s_n - s_1)]_{q^2}} \frac{[2(\alpha + s_1)]_{q^2}}{[2(s_n + s_1 - 1)]_{q^2}} = 1,
\]

which is a true identity. This finishes the proof. \( \square \)

The proofs of Theorem 2.2–2.4 are essentially the same as that of Theorem 2.1. We apply Kuo condensation as in Fig. 11b–d and appeal to Lemmas 4.1 and 4.2 to verify the base cases.

The proofs of Theorems 2.7–2.11 are done using Kuo condensation and induction on \( h + x \), where \( h \) is the “height” of a region of type \( A, B, C, D, S, \) or \( T \) and is given by the sum of the side-lengths of the northeast and southeast sides of the region. We omit these proofs here.

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