Abstract

We calculate thermal corrections to the non-linear QED effective action for low-energy photon interactions in a background electromagnetic field. The high-temperature expansion shows that at $T \gg m$ the vacuum contribution is exactly cancelled to all orders in the external field except for a non-trivial two-point function contribution. The high-temperature expansion derived reveals a remarkable cancellation of infrared sensitive contributions. As a result photon-splitting in the presence of a magnetic field is suppressed in the presence of an electron-positron QED-plasma at very high temperatures. In a cold and dense plasma a similar suppression takes place. At the same time Compton scattering dominates for weak fields and the suppression is rarely important in physical situations.
1 INTRODUCTION

Recently the non-linear effects induced by virtual electron-positron pairs in quantum electrodynamics (QED) have been discussed in great detail in the literature ([1, 2, 3] and references cited therein) confirming earlier calculations concerning in particular photon splitting in external magnetic fields [4, 5]. It is of great interest to notice that such non-linear QED effects may lead to observable physical effects in e.g. the gamma-ray burst spectra (for reviews see e.g. [6, 7]) in particular with regard to the so called soft gamma-ray repeaters [8, 9]. Recently, photon splitting processes in the magnetosphere of γ-ray pulsars has also been discussed [10], where it has been argued that such processes can be comparable to pair-production processes if the magnetic field \( B \) is sufficiently high, i.e. \( B \geq B_c \), where \( B_c \approx 4 \times 10^{13} \) Gauss is the critical field in QED (i.e. \( eB_c = m^2 \)).

In vacuum the presence of virtual electron-positron pairs leads to the well-known Euler-Kockel[11]–Heisenberg[12]–Weisskopf[13]–Schwinger[14] (EKHWS) effective action, which was used in the classical paper by Adler on the subject of computing photon-splitting amplitudes and absorption coefficients in magnetic fields [5]. Astrophysical models of neutron stars, which have been used in explanations for the attenuation of γ-rays, suggest that photon-splitting processes in the presence of strong magnetic fields do not necessarily take place in an environment which may be approximated by a vacuum. Instead finite temperature \( T \) and/or chemical potential \( µ \) may have important effects which motivates a study of photon splitting at finite temperature (see e.g. [15, 16, 17, 18]). We are considering the limit of weak fields in order to see how large the effects can be and it turns out that for an electrically neutral \( e^+e^- \)-plasma in a weak field the Compton scattering is dominating whenever the thermally induces splitting amplitude is appreciable. There is a possibility that the thermal splitting rate becomes larger than the Compton rate for finite chemical potential and very low temperature, but our calculational technique breaks down in that limit.

It has been noticed in the literature that if \( T \gg m \) all the non-linear terms in the effective EKHWS action are cancelled by thermal corrections and the remaining effective action becomes quadratic in the electro-magnetic field strengths with a non-trivial dependence in \( T, \mu \) [19, 20, 21, 22]. Below we reconsider this cancellation explicitly including \( O(m/T) \) corrections. As a function of the photon energy \( \omega \) the results obtained in the present paper strictly only apply to the situation where \( \omega/m \ll 1 \). In the vacuum sector Adler [5] has, however, calculated the exact \( \omega/m \) dependence and verified that this de-
pendence on the photon splitting processes is rather weak apart from phase-space factors. The $\omega/m$-dependence can therefore be extrapolated to $\omega/m \simeq 1$, at least for magnetic fields such that $B/B_c \leq 1$. We expect a similar weak $\omega/m$-dependence also in the presence of a thermal environment even though this has not yet been verified explicitly. Since we are mostly interested in the temperature and magnetic field dependence we will simply put $\omega = m$.

The paper is organised as follows. In Section 2 we recall the form of the QED effective action including one-loop thermal corrections due to the presence of a fermion-antifermion heat-bath. In Section 3 we perform a weak field expansion of the effective action and a high temperature expansion is considered in Section 4. In Section 5 a similar low temperature expansion is given. A non-covariant contribution to the effective action, fourth order in the external fields, together with sixth order vacuum and thermal corrections, are considered in Section 6 with regard to their effect on photon splitting processes. Final comments are given in Section 7.

2 Effective Action

The thermal one-loop non-linear QED effective Lagrangian $\mathcal{L}^{\beta,\mu}(E, B)$ for slowly varying electric and magnetic fields has been calculated \cite{20} with the result

$$
\mathcal{L}^{\beta,\mu}(E, B) = -\frac{1}{2\pi^{3/2}} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} f_F(p_0; A_0) \text{Im} \left\{ \int_{0}^{s_{\text{max}}} \frac{ds}{s} e^{2ab \cot(esa) \coth(esb)} \times (h(s) - i\epsilon)^{-1/2} \exp \left[ -i(m^2 - i\epsilon)s + i\frac{(p_0 - eA_0)^2}{h(s) - i\epsilon} + i\frac{\pi}{4} \right] \right\} . \quad (2.1)
$$

The field-independent contribution of Eq. (2.1) can be obtained by making use of dimensional regularisation techniques as described in \cite{21}. The function $h(s) = (eF \coth eFs)_{00}$, where $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$, was calculated in \cite{20} and is given by

$$
h(s) = \left( \frac{1}{2} - \frac{\mathcal{H}}{a^2 + b^2} \right) ea \cot[esa] + \left( \frac{1}{2} + \frac{\mathcal{H}}{a^2 + b^2} \right) eb \coth[esb] . \quad (2.2)
$$

In terms of the field-strength we use the notation

$$
\mathcal{H} = \frac{E^2 + B^2}{2} , \quad a = \left( \sqrt{\mathcal{F}^2 + \mathcal{G}^2 + \mathcal{F}} \right)^{1/2} , \quad b = \left( \sqrt{\mathcal{F}^2 + \mathcal{G}^2 - \mathcal{F}} \right)^{1/2} , \quad (2.3)
$$
where the Lorentz-invariant quantities $\mathcal{F}$ and $\mathcal{G}$ are given by
\[
\mathcal{F} = \frac{a^2 - b^2}{2} = \frac{B^2 - E^2}{2}, \quad \mathcal{G} = ab = \mathbf{E} \cdot \mathbf{B},
\]
in terms of the magnetic and electric fields $\mathbf{B}$ and $\mathbf{E}$ ($B = |\mathbf{B}|$, $E = |\mathbf{E}|$). In [20] it was argued that the fermion equilibrium distribution function, $f_F(p_0, eA_0)$, in the case of an external electromagnetic field, should be chosen in the following form
\[
f_F(p_0; eA_0) = \frac{\theta(p_0 - eA_0)}{e^{\beta(p_0 - \mu)} + 1} + \frac{\theta(-p_0 + eA_0)}{e^{\beta(-p_0 + \mu)} + 1},
\]
where $\beta = 1/T$ is the inverse temperature and $\mu$ the chemical potential related to the conserved charge of the system. In this notation the renormalised low energy EKHWS vacuum effective action for non-linear electromagnetic fields in QED has the well-known form (see e.g. [23]):
\[
L_{\text{vac}}(\mathbf{E}, \mathbf{B}) = -\frac{1}{8\pi^2} \int_{0}^{\infty} \frac{ds}{s^3} \left[ s^2 e^2 ab \coth(esa) \cot(esb) - 1 - \frac{s^2 e^2}{3} (a^2 - b^2) \right] e^{-m^2 s}.
\]

3 Weak Field Expansion

The total effective action for a slowly varying background field is given by
\[
L_{\text{eff}}(\mathbf{E}, \mathbf{B}) = -\mathcal{F} + L_{\text{vac}}(\mathbf{E}, \mathbf{B}) + L_{\beta,\mu}(\mathbf{E}, \mathbf{B}).
\]
The vacuum contribution $L_{\text{vac}}(\mathbf{E}, \mathbf{B})$ has a straightforward expansion in terms of the invariants $\mathcal{F}$ and $\mathcal{G}$. We are now interested in a similar weak-field expansion of the thermal contribution $L_{\beta,\mu}(\mathbf{E}, \mathbf{B})$. Care must then be taken since, as we will shortly see, one easily encounters severe infrared problems. The basic idea is to collect everything in the integrand in Eq. (2.1) which depends on the fields into one function multiplying the zero-field part and then expand it formally in $\mathbf{E}$ and $\mathbf{B}$. For a strictly degenerate plasma $L_{\beta,\mu}(\mathbf{E}, \mathbf{B})$ is not analytic in the fields but shows de Haas–van Alphen oscillations as discussed in [21]. However, at finite temperature and weak enough fields these oscillations average out over the Fermi surface and a weak field expansion becomes meaningful. In this situation we can neglect the $i\epsilon$ in the combination $h(s) - i\epsilon$ which is only needed for the non-analytic structure. In particular we need the following expansion
\[
\frac{esa \cot [esa]esb \coth [esb]}{(sh(s) - i\epsilon)^{1/2}} \exp \left[ i\omega^2 \frac{1 - sh(s)}{sh(s) - i\epsilon} \right] = \sum_{k=0}^{\infty} \alpha_{k-2}(\omega; \mathcal{H}; \mathcal{F}; \mathcal{G}) s^k,
\]
The lowest order terms are given through effective chemical potential \( \mu \omega \) where \( \Delta (2 \alpha) \) denoted by \( \Delta L \) of the external fields) which comes out in a standard way \([21]\). For the rest of Eq. (3.3), term since it only gives the field independent part (i.e. the free energy in the absence of the external fields) we in \( \alpha \) where we in \( \omega \) this point we would get serious artificial infrared problems with the \( \omega \)-integral. We avoid these divergences by first performing a number of partial integrations using

\[
\left( -i(\omega^2 - m^2) + \epsilon \right)^{-\frac{1}{2} - k} = \frac{\pi^{1/2}}{i^k \Gamma(k + \frac{1}{2})} D_{\omega^2}^k \left( -i(\omega^2 - m^2) + \epsilon \right)^{-1/2},
\]

where \( D_{\omega^2} \) is a derivative with respect to \( \omega^2 \). From now on we shall drop the \( k = -2 \) term since it only gives the field independent part (i.e. the free energy in the absence of the external fields) which comes out in a standard way \([21]\). For the rest of Eq. (3.3), denoted by \( \Delta \mathcal{L}^{\beta,\mu} \), we rewrite \( D_{\omega^2}^k \) as derivatives with respect to \( \omega \) and integrate them all by parts. This is not a problem since the thermal distribution function is zero at the boundaries. We obtain the result (notice that \( \alpha_{k=-1}(\omega) = 0 \)):

\[
\Delta \mathcal{L}^{\beta,\mu} = -\frac{1}{4\pi^2} \int_{-\infty}^{\infty} d\omega \frac{\theta(\omega^2 - m^2)}{\sqrt{\omega^2 - m^2}} \sum_{k=0}^{\infty} i^k \left( \frac{d}{d\omega 2\omega} \right)^k (\alpha_k(\omega) f_F(\omega)) ,
\]

where we have used \( \text{Im} \left[ (-i(\omega^2 - m^2) + \epsilon)^{-1/2} e^{i\frac{\pi}{4}} \right] \to \theta(\omega^2 - m^2)(\omega^2 - m^2)^{-1/2} \). The expansion in Eq. (3.3) is only valid if \( f_F(\omega) \) is smooth enough. Otherwise, a non-analytic structure like the de Haas–van Alphen oscillation can occur. We can then write

\[
\Delta \mathcal{L}^{\beta,\mu} = -\frac{1}{4\pi^2} \int_{-\infty}^{\infty} d\omega \frac{\theta(\omega^2 - m^2)}{\sqrt{\omega^2 - m^2}} \sum_{n=0}^{\infty} e^{2n} \Delta^{(2n)} \mathcal{L}^{\beta,\mu}(\omega) ,
\]

where \( \Delta^{(2n)} \mathcal{L}^{\beta,\mu}(\omega) \) is of the order \( 2n \) in the electromagnetic field-strengths \( E \) and/or \( B \). The lowest order terms are given through

\[
\Delta^{(2)} \mathcal{L}^{\beta,\mu}(\omega) = -\frac{2f_F(\omega)}{3} \mathcal{F} + \frac{\omega f_F^{(1)}(\omega) E^2}{6} ,
\]
\[ \Delta^{(4)} L^{\beta,\mu}(\omega) = \frac{f_F(\omega)}{60\omega^4} \left( 4F^2 + 7G^2 \right) \]
\[ - \frac{f_F^{(1)}(\omega)}{60\omega^3} \left( 4F^2 + 7G^2 \right) + \frac{f_F^{(2)}(\omega)}{360\omega^2} \left( 16F^2 - 8FH + 13G^2 \right) \]
\[ + \frac{f_F^{(3)}(\omega)}{360\omega} \left( 8FH - 8F^2 + G^2 \right) - \frac{f_F^{(4)}(\omega)E^4}{288}, \] (3.8)
where we have used \( H - F = E^2 \) and the notation \( f_F^{(n)}(\omega) = \frac{d^n f_F(\omega)}{d\omega^n} \).

## 4 High Temperature Expansion

In the very high temperature limit, i.e. \( T \gg m, |\mu_{\text{eff}}| \), we see that \( f_F(\omega) \to \frac{1}{2} \) and therefore only a few terms in the expansion of \( \Delta^{(2n)} L^{\beta,\mu}(\omega) \) survive. The quadratic terms go like \( \log(T/m) \) at high temperature and the higher order corrections go to constants. Then the \( \omega \)-integral can be performed for the higher corrections using
\[ \int_0^\infty d\omega \frac{\theta(\omega^2 - m^2)}{\sqrt{\omega^2 - m^2}} w^{-k} = \frac{\sqrt{\pi} \Gamma\left(\frac{1}{2}\right)}{2m^k \Gamma\left(\frac{k+1}{2}\right)}, \] (4.1)
where \( k > 0 \). In this limit the effective Lagrangian then becomes
\[ \Delta L^{\beta,\mu} = -\frac{e^2}{24\pi^2} (E^2 - B^2) \ln\left(\frac{T}{m}\right) + \frac{e^2 E^2}{24\pi^2} \]
\[ - \frac{e^4}{360\pi^2 m^4} [(E^2 - B^2)^2 + 7(E \cdot B)^2] \]
\[ - \frac{e^6}{1260\pi^2 m^8} [(E^2 - B^2)(2(E^2 - B^2)^2 + 13(E \cdot B)^2)] + \mathcal{O}(F^8), \] (4.2)
where \( F \) generically stands for an external field. It is interesting to note that the higher order corrections are exactly equal, but with opposite sign, to the vacuum corrections obtained from Eq. (2.6):
\[ \mathcal{L}^{\text{vac}}(E, B) = \frac{(a^2 - b^2)^2 + 7(ab)^2}{360m^4 \pi^2} + \frac{(b^2 - a^2)(2(b^2 - a^2)^2 + 13(ab)^2)}{1260m^8 \pi^2} + \mathcal{O}(F^8), \] (4.3)
As a matter of fact, by inspection of Eq. (2.1) and Eq. (2.6), this cancellation is true to all orders in the external fields except for quadratic terms exhibited in Eq. (4.2).
In passing, we also notice that at zero temperature and for a large chemical potential, $|\mu_{\text{eff}}| \gg m$, the thermal non-linear effective action $\mathcal{L}^{\beta,\mu}(E, B)$ also cancels the vacuum effective action $\mathcal{L}^{\text{vac}}(E, B)$.

At high temperature the vacuum and thermal effects thus cancel and we have to go to higher order in $m/T$ to find the leading contribution. If we first subtract the $f_F(\omega) = \frac{1}{2}$ part and the scale out a factor of $T^{-2n-2}$ in the remaining $\mathcal{O}(F^{2n})$ terms, we find that the integrand is infrared finite as $m/T \to 0$. This is far from obvious and comes from a remarkable cancellation of different infrared sensitive terms. To get the first sub-leading term we thus take the limit $m/T \to 0$ and then we are left with a dimensionless integral.

To be more explicit let us take $\mu_{\text{eff}} = 0$ and write the subtracted $\mathcal{O}(F^4)$ Lagrangian as

$$\Delta \mathcal{L}^{(4)} = -\frac{1}{2\pi^2} \int_0^\infty \frac{dp}{\omega} \left[ \frac{C_0}{\omega^4} (f_F(\omega) - \frac{1}{2}) + \frac{C_1}{\omega^3} D_\omega f_F(\omega) + \frac{C_2}{\omega^2} D^2_\omega f_F(\omega) + \frac{C_3}{\omega} D^3_\omega f_F(\omega) + C_4 D^4_\omega f_F(\omega) \right] , \quad \text{(4.4)}$$

where now $f_F(\omega) = (e^{\beta \omega} + 1)^{-1}$ and $p = \sqrt{\omega^2 - m^2}$. The full expression is perfectly finite, due to cancellation mechanism alluded to above, but each term diverges at $\beta \omega = 0$. To be able to treat the terms separately we use a zeta-function regularisation technique by multiplying the integrand in Eq. (4.4) by $\omega^\nu$, considering $\nu$ large enough first, and then take the limit $\nu \to 0$ at the end of the calculation. Let us therefore define the function

$$I(z, n) \equiv \int_0^\infty \frac{dx}{(e^x + 1)^n} , \quad \text{(4.5)}$$

and notice that

$$I(z, n) = I(z, n - 1) + \frac{z}{n-1} I(z + 1, n - 1) ,$$

$$I(z, 1) = \frac{1 - 2^z \pi^{1-z} \zeta(z)}{2^z \sin(z \pi/2)} . \quad \text{(4.6)}$$

For positive $z$ the function $I(z, n)$ has been defined in terms of an analytical continuation using the Riemann $\zeta$-function. The integrals in Eq. (4.4) can then be performed by partial integrations, where the boundary terms vanish for large enough $\nu$, and thereby reducing the result to

$$\Delta \mathcal{L}^{(4)} = \frac{31 \zeta(5)}{3840 \pi^6 T^4} [B^4 + 6 B^2 E^2 - 12 E^2 + 9 (E \cdot B)^2] . \quad \text{(4.7)}$$
The $\mathcal{O}(F^6)$ correction can be treated in a similar way and we find

$$
\Delta \mathcal{L}^{(6)} = -\frac{511\zeta(9)}{147456\pi^{10}T^8} \left[ 2B^6 + 42B^4E^2 + 36B^2E^4 - 220E^6 - 7B^2(E \cdot B)^2 
+ 279E^2(E \cdot B)^2 \right].
$$

\(\text{(4.8)}\)

## 5 Low Temperature Expansion

In many applications, such as photon splitting (see Section 6), it is more interesting to have a low temperature expansion. At high temperature Compton scattering would anyway dominate over pure splitting (at least for low $eB/m^2$) and in several astrophysical situations the temperature is relatively low compared to the electron mass. On general grounds we expect low temperature effects to be exponentially small since they are Boltzmann suppressed, but as we shall find there is a region where the effects still are appreciable. The formula we need for the expansion is for $\mu_{\text{eff}} = 0$

$$
\int_0^\infty d\omega \frac{\theta(\omega^2 - m^2)}{\sqrt{\omega^2 - m^2}} \frac{1}{\omega^q} d\omega \rho_f(\omega) \simeq \begin{cases} (-1)^p e^{-m/T} \frac{\Gamma(p/2)}{2m^p m^q} \frac{\Gamma(q/2)}{\Gamma(q/2 + 1)} & \text{for } q \geq 1, \\
\frac{(-1)^p}{m^q} \left( \frac{\pi T}{2m} \right)^{1/2} e^{-m/T} & \text{for } q = -1, 0. \end{cases}
$$

\(\text{(5.1)}\)

We find that, although all terms go to zero exponentially, the ones with more derivatives with respect to $\omega$ have higher inverse powers of $T$ which makes them relatively large for $T/m \lesssim 1$, where also the approximations in Eq. (5.1) are numerically good. Using these terms the effective action to $\mathcal{O}(F^4)$ becomes

$$
\mathcal{L}^{3,\mu}(E, B) \simeq \frac{e^4}{4\pi^2} \left[ \frac{2}{3} \left( \frac{2\pi T}{m} \right)^{1/2} F + \frac{1}{6} \left( \frac{2\pi m}{T} \right)^{1/2} E^2 \right] e^{-m/T}
- \frac{e^4}{4\pi^2} \left[ \frac{14F^2 + 7G^2}{45m^4} + \frac{4F^2 + 7G^2}{120m^3T} + \frac{16F^2 - 8FH + 13G^2}{180m^2T^2} 
- \frac{2\pi}{360} \frac{8F^2 - 8F^2 + G^2}{mT^3} + \frac{E^4}{144T^4} \left( \frac{\pi T}{2m} \right)^{1/2} \right] e^{-m/T}.
$$

\(\text{(5.2)}\)

## 6 Weak Field Photon Splitting

In the classic paper by Adler [3] the QED photon splitting process in a background magnetic field is treated in great detail. One important observation in that paper is that
the $O(F^4)$-term does not contribute to the splitting amplitude in vacuum. This result relies on the covariant form of the effective action, i.e. that only $F$ and $G$ occurs in the explicit expression, and that all the photons are collinear due to kinematics. Non-linear effects on the dispersion relation either make it possible to have a non-zero opening angle between the split photons or prohibits the splitting altogether, depending on the polarisation states. At finite temperature the situation is, however, different since now the manifest Lorentz invariance is broken in the effective action by the dependence on $H$. The consequence is that photon splitting is allowed already from the $O(F^4)$-term even for collinear photons.

Thermal corrections also alter the dispersion relation of the photons which changes the kinematic conditions for splitting, and then not all combinations of photon polarisation are possible. These effects were studied in \cite{3} on a general basis and it was found that the only allowed process is $|| \rightarrow \perp_1 + \perp_2$, provided the electron density is not too large ($n_e \lesssim 10^{19} \text{cm}^{-3}$). Thermal corrections to the dispersion relation have several origins. First, there is the plasma mass which is independent of the background field. The static version, the Deb\'ye mass, can found from Eq. (2.1) by noting that the $O(F^0)$ term depends on $A_0$ through the thermal distribution function. Expanding $O(A_0^2)$ gives the Deb\'ye mass \cite{20}. Secondly, higher order terms like the $FH$-term change the dispersion relation in a background field. Details of the dispersion relation are complicated by the fact that polarisation tensor is non-analytic in the energy and momentum. We shall concentrate on situations where the photon frequency $\omega$ is smaller than the electron mass so that our effective action for slowly varying fields is valid. At the same time we take the temperature and chemical potential to be low so that correction to the dispersion relation is small and the photon almost light like. In this way we do not have any large correction to phase-space integrals which would arise when the thermal mass is important. To be more concrete we first consider an $e^+e^-$-plasma with $\mu_{\text{eff}} = 0$ and $T/m$ sufficiently small. Then we require that

$$\omega_p^2 = \frac{2\pi\alpha}{m} \left( \frac{2mT}{\pi} \right)^{3/2} e^{-m/T} \ll \omega^2, k^2 \lesssim m^2,$$

which in actual numbers means $T/m \ll 4.5$. It may at first seem inconsistent to assume the effect on the dispersion relation to be small and still argue that plasma effects on the splitting amplitude can be important. However, corrections to the dispersion relation should be compared with the photon energy and momentum, while thermal splitting amplitudes should be compared with other splitting or scattering processes. Moreover,
higher order terms in the field strength are more IR-sensitive at low $T$ and therefore relatively larger.

The new ingredient in a background plasma is the $\mathcal{F} \mathcal{H}$-term and the question is whether there is any regime in which the thermal splitting is important if the temperature is not much higher than the electron rest mass. At low temperature the vacuum effects dominate and at high temperature Compton scattering is overwhelmingly large.

The various processes to be considered depend differently on the photon energy $\omega$ and the scattering angle $\theta$ between the direction of the propagation of the initial photon and the external magnetic field, but we shall choose for reasons of simplicity $\sin \theta = 1$ and $\omega = m$ in the final numerical estimates. This choice of $\omega$ is in the upper limit of the validity since our approximation assumes a low photon frequency (almost constant fields) and it should be remembered that the splitting probability goes like $(\omega/m)^5$ (see below), and thus decreases rapidly for smaller $\omega$.

In the low photon frequency limit the amplitude for the $\parallel \rightarrow \perp_1 + \perp_2$ photon splitting process is given by

$$M[\parallel \rightarrow \perp_1 + \perp_2] = \omega_1 \omega_2 \left\{ \left( \hat{k} \times \hat{e}^\parallel \right)_i \frac{\partial}{\partial B_i} + \hat{e}^\parallel_i \frac{\partial}{\partial B_i} \right\}$$

$$\times \left( \left( \hat{k} \times \hat{e}^\perp \right)_j \frac{\partial}{\partial B_j} + \hat{e}^\perp_j \frac{\partial}{\partial E_j} \right)^2 \mathcal{L}_{\text{eff}}(\mathbf{E}, \mathbf{B}) \bigg|_{E = 0}.$$  \hspace{1cm} (6.2)

The photon absorption coefficient is then given by

$$\kappa(\parallel \rightarrow \perp_1 + \perp_2) = \frac{1}{32 \pi \omega^2} \int_{0}^{\omega} d\omega_1 \int_{0}^{\omega} d\omega_2 \delta(\omega - \omega_1 - \omega_2) |M[\parallel \rightarrow \perp_1 + \perp_2]|^2.$$  \hspace{1cm} (6.3)

The appropriate vacuum $\mathcal{O}(F^6)$ splitting probability for the process $\parallel \rightarrow \perp_1 + \perp_2$ can be obtained from Eq. (4.3):

$$\kappa_6^{\text{vac}} = \left( \frac{e B}{m^2} \right)^6 \left( \frac{\omega}{m} \right) \sin^6 \theta \left( \frac{169 \alpha^3}{1488375 \pi^2} \right).$$  \hspace{1cm} (6.4)

The only difference from the well-known result of [5] is that we use units where $\alpha = e^2/4\pi$ and $F_{\text{Adler}}^{\mu \nu} = \sqrt{4\pi} F^{\mu \nu}$.

Next, we need the splitting probability from the thermal $\mathcal{O}(F^4)$ term. The non-covariant part of $\mathcal{L}^{\beta,\mu}$ to that order, which is the only one contributing to photon splitting, is obtained from Eq. (3.8) and is given by

$$\Delta \mathcal{L}^{\beta,\mu} = -\frac{1}{2 \pi^2} \frac{B^4}{360} \int_{-\infty}^{\infty} d\omega \frac{\theta(\omega^2 - m^2)}{\sqrt{\omega^2 - m^2}} \frac{d}{d\omega} \left( \frac{f^{(2)}(\omega)}{\omega} \right).$$  \hspace{1cm} (6.5)
As in the case of photon-splitting processes in the vacuum, we derive the splitting rate for the process $\parallel \rightarrow \perp + \perp$ using Eq. (6.5)

$$\frac{\kappa_6}{m} = \left(\frac{eB}{m^2}\right)^2 \left(\frac{\omega}{m}\right)^5 \sin^2 \theta \frac{\alpha^3}{30375 \pi^2} \left(m^4 \int_0^\infty dk \frac{d}{d\omega} \left(\frac{f^{(2)}(\omega)}{\omega}\right)\right)^2 .$$  \hspace{1cm} (6.6)

To be consistent we also compute the thermal $\mathcal{O}(F^6)$ splitting rate $\kappa_6$ in a similar way. Finally it should be noticed that the amplitudes for all these processes should be added coherently since the final states are the same and the external field is constant and thus coherent. The total splitting rate $\kappa_{\text{tot}}$ shows the characteristic decrease for high $T$ when the thermal contribution interfere destructively with the vacuum part. The three contributions and the coherent sum are compared in Fig. 1. (The physical attenuation length in a QED plasma can be obtained using $m_\gamma^{-1} = 3.86 \times 10^{-11}$ cm.)

In addition to splitting the photon can scatter directly with the plasma which turns out to be the dominant process for a large parameter range. We estimate the scattering rate using the total Compton cross-section which for unpolarised photons is $\sigma_C = 8\pi \alpha^2 f(\omega/m)/3m^2$, where $f(\omega/m)$ is a slowly varying function such that $f(\omega/m) \simeq 1/2$ if $\omega \simeq m$ (see e.g. [23]). In order to get the absorption coefficient $\kappa_C$ we multiply $\sigma_C$ with the appropriate density of electrons and positrons

$$\frac{\kappa_C}{m} = \frac{\sigma_C}{m^2 \pi^2} \int_0^\infty dp \frac{p^2}{(e^{\beta p} + 1)} .$$  \hspace{1cm} (6.7)

It is clear from Fig. 1 that the only region where the thermal splitting can dominate over the vacuum one it is already far below the Compton scattering rate and will thus not be very important.

As a second example we shall consider the, for astrophysics, more realistic case of an electron-proton plasma. Since the electron is much lighter than the proton it dominates the correction to the splitting amplitude so we shall only take into account the electron part. For a non-degenerate plasma at relatively low temperature ($T \ll m_e$) the effective chemical potential is close to the electron mass (larger $\mu_{\text{eff}}$ gives a degenerate plasma and smaller $\mu_{\text{eff}}$ corresponds to exponentially small electron density). In order to see the main features we can therefore put $\mu_{\text{eff}} = m_e$. In Fig. 2 we can see that now the Compton damping rate rapidly decreases for low $T$ while the splitting rates, which are more IR sensitive, actually increase. The fact that the thermal splitting rate becomes large shall in this case be taken as an indication of that we approach the limit where
Figure 1: The vacuum and thermal photon splitting absorption coefficients $\kappa^\text{vac}_6$, $\kappa^\beta_4$, $\kappa^\beta_6$, and the Compton absorption $\kappa_C$ (upper solid curve) are shown as a function of temperature in units of $m$ for $eB = 0.2 m^2$, $\omega = m$ and $\sin \theta = 1$. The lower solid curve corresponds to the coherent sum of amplitudes up to $O(F^6)$. By including thermal corrections of $O(F^6)$, the splitting amplitude goes to zero at sufficiently large $T/m$ as discussed in Section [4].

the weak field expansion is no longer valid. We know that for $T = 0$ and $\mu_{\text{eff}} > m_e$ we have de-Haas–van-Alphen oscillations which are non-analytic in $B$ and that is reflected in divergent coefficients in the weak field expansion. Also, $\kappa^\beta_6$ diverges more rapidly than $\kappa^\beta_4$ which shows the breakdown of the perturbative expansion [21]. On the other hand Fig. 2 also tells us that there might be a region for realistic plasmas where the thermal splitting actually is the dominant process, but to settle the question it would be necessary to perform the calculation using the exact Landau levels without expansion in powers of $B$. 
A detailed analysis of the astrophysical consequences of thermal photon splitting is outside the scope of this paper but we have found that in many generic cases this process is subdominant and not likely to be important. We have shown in Section 6 that in a QED plasma thermal corrections induce a non-Lorentz invariant $O(F^4)$ term which contributes to the photon splitting process. Compared to the $O(F^6)$ vacuum contribution this thermal correction can actually be quite large already in an $e^+e^-$-plasma for $T/m \lesssim 0.2$, as indicated in Fig. 1, at least if $eB/m^2 \simeq 0.2$. For an $e^-p^+$-plasma with chemical potential $\mu_{\text{eff}} = m_e$ there is a possibility of relatively large thermal correction at very low temperature (see Fig. 2) but the perturbative calculation in powers of $eB$ breaks down in this limit. In Fig. 2 the Compton and photon splitting absorption coefficients become comparable for physical parameters corresponding to an electron density of the order of $10^{28}\text{cm}^{-3}$. In a neutron star such an electron density $n_e$ would occur only below

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Figure 2: The same symbols and values of parameters as in Fig. 1 except that here the chemical potential is $\mu_{\text{eff}} = m$. 

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the surface of the star (the surface electron density can be the order of $10^{27}\text{cm}^{-3}$), where photon splitting processes are not likely to be of any importance anyhow [10]. In studies of one-temperature accretion disks around black holes temperatures like $T \simeq m$ can be achieved [18]. In such models the magnetic field is, however, very small ($B \simeq 10^5\text{G}$) and photon splitting is therefore most likely an unimportant physical process. We have not taken dispersive effects into account. For soft gamma-ray repeaters, dispersive effects can be of importance for large electron (and/or positron) densities in that e.g. the rate for Compton scattering is modified [17].

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