CRITICAL VALUES OF RANDOM ANALYTIC FUNCTIONS ON COMPLEX MANIFOLDS

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Abstract. We study the asymptotic distribution of critical values of random holomorphic sections $s_n \in H^0(M^m, L^n)$ of powers of a positive line bundle $(L, h) \to (M, \omega)$ on a general Kähler manifold of dimension $m$. By critical value is meant the value of $|s(z)|_h$ at a critical point where $\nabla_h s_n(z) = 0$, where $\nabla_h$ is the Chern connection. The distribution of critical values of $s_n$ is its empirical measure. Two main ensembles are considered: (i) the normalized Gaussian ensembles so that $E||s_n||_2^2 = 1$ and (ii) the spherical ensemble defined by Haar measure on the unit sphere $SH^0(M, L^n) \subset H^0(M, L^n)$ with $||s_n||_2^2 = 1$. The main result is that the expected distributions of critical values in both the normalized Gaussian ensemble and the spherical ensemble tend to the same universal limit as $n \to \infty$, given explicitly as an integral over $m \times m$ symmetric matrices.

1. Introduction

The purpose of this article is to determine the asymptotic distribution of critical values of random holomorphic polynomials of large degree, and their generalizations (‘holomorphic sections’) to all compact Kähler manifolds $(M, \omega)$ of any dimension $m$. We work in the same general setting as the articles [DSZ1, DSZ2, DSZ3] on the distribution of critical points on Kähler manifolds and recall the definitions in §2.

We first consider the critical value distribution of Gaussian random ‘polynomials’ $s_n$ as the degree $n \to \infty$ and then consider the more difficult and interesting problem of critical values of $L^2$ normalized random polynomials with $||s_n||_{L^2} = 1$. We refer to the latter as the spherical critical value distribution since the ‘polynomials’ are drawn at random from the unit sphere in Hilbert space. We regard the spherical distribution as primary for the critical value distribution since a critical value is only counted once in a line $\{cs_n, c \in \mathbb{C}\}$ of sections and one can relate the heights of the critical values to the other threshold heights of $L^2$ normalized ‘polynomials’.

Theorem 1 shows that a special normalized Gaussian critical value distribution has a universal limit independent of the manifold and Kähler metric. The main result of this article, Theorem 2, shows that the spherical critical value distribution has the same universal limit. We also give a limit formula for the simpler spherical value distribution in Theorem 3. The spherical limit results may be viewed as a Poincaré-Borel theorems for critical values.

The spherical distribution of critical values is potentially useful in analyzing the Morse theory of the modulus $|s_n(z)|_{h^n}$ of random sections, since it is at critical values that the level sets change topology. They are also important in understanding the topography of random surfaces, i.e. the graphs of the modulus $y = |s_n(z)|_{h^n}$.
of ‘polynomials’ of degree $n$, which are often visualized as mountain landscapes above some given sea level. It is known that $\sup_{z \in M} ||s_n(z)||_{h^n} \leq C m^{n/2}$ when $||s_n||_{L^2} = 1$ and $\dim_{\mathbb{C}} M = m$ and it is proved in [SZ] (after a long history of similar results in other settings) that the expected sup norm of such normalized sections is bounded by a universal constant times $\sqrt{\log n}$. Thus in a measure sense, typical ‘polynomials’ of degree $n$ and norm one have global maxima $\leq C \sqrt{\log n}$, and conjecturally the median should be of this form for some $C$. Deterministically, critical values of all normalized polynomials lie in $[0, C m^{n/2}]$. It would be interesting to know the exact height (for $L^2$ normalized polynomials) at which the peak of the random mountain first occurs. At a certain threshold height, the mountain tops are sometimes conjectured to form a Poisson spatial process, and it would be interesting to know the connectivity properties of the landscape at lower sea levels.

The calculation of the spherical density of critical points is only a calculation but it is probably a necessary one for the more involved landscape questions. The Kähler setting is a model for other settings in which one studies normed random waves with a notion of degree or eigenvalue, such as random spherical harmonics or more general Riemannian waves on Riemannian manifolds, in which the principal modification is in the asymptotics of the relevant covariance functions.

Before stating the results, we introduce some notation and background. Compact complex manifolds have no non-constant holomorphic functions and the natural replacement for them are twisted holomorphic functions known as holomorphic sections of complex line bundles $\pi : L \to M$. Here, the fiber $L_z$ over $z \in M$ is a one-complex dimensional space and a holomorphic section is a map $s : M \to L$ satisfying $\bar{\partial}s = 0, \pi \circ s(z) = z$. Degree $n$ sections are sections of the $n$th tensor power $L^n$ of $L$, and the space of holomorphic sections is denoted $H^0(M, L^n)$. Its dimension is given asymptotically by

\begin{equation}
\label{dim_H0}
d_n = \dim_{\mathbb{C}} H^0(M, L^n) \simeq C n^m.
\end{equation}

When $\dim_{\mathbb{C}} M = 1$, i.e. when $M$ is a Riemann surface, then the natural examples are polynomials of degree $n$ ($g = 0$), theta functions of degree $b$ ($g = 1$) and holomorphic differentials of type $(dz)^n$ for $g \geq 2$. The techniques and results of this article, as in the predecessors [BSZ, DSZ1], hold in this general geometric setting.

The Kähler metric $\omega$ determines a Hermitian metric $h$ and connection $\nabla$ on $L$ and on its powers. The Hermitian metric satisfies $\bar{\partial}\bar{\partial} \log h = \omega$, and the connection $\nabla$ is known as the Chern connection and is compatible with the Hermitian metric $h$ on and complex structure on $L$. As recalled in [2] the Hermitian metric $h$ and Kähler form $\omega$ give rise to a definition of Gaussian random holomorphic section in $H^0(M, L^n)$.

By a critical point of a holomorphic section $s \in H^0(M, L^n)$ we mean a point $z \in M$ so that

\begin{equation}
\label{critical_point}
\nabla s(z) = 0.
\end{equation}

Thus the section is ‘parallel’ at $z$. Equivalently, critical points are points where the norm square is critical $d|s_n|_{L^2}^2 = 0$ and so we are studying the critical points and values of the real-valued function $|s(z)|_{h^n}^2$. More precisely,

\begin{equation}
\label{critical_equivalence}
d|s_n|_{h^n}^2 = 0 \iff \nabla s_n = 0 \text{ or } s_n = 0.
\end{equation}

\textsuperscript{1}The methods of this article give an explicit value of $C$, which we defer to a future article.
We note that \( x = 0 \) is surely a critical value of \(|s_n(z)|^2 \) since every section has zeros. However we omit the zero value in the definition of the empirical measure of critical values. When the Hermitian line bundle \((L, h)\) is positive, as we assume, the only local minima of \(|s|_h \) are its zeros. Therefore the critical points in \([13]\) are either saddle points or local maxima.

To define Gaussian random holomorphic sections, introduce a family of Gaussian measures adapted to the Hermitian metric and the associated inner product \([20]\) on sections. For any \( \alpha > 0 \) we put

\[
d_{\gamma_n}^\alpha(s_n) = \left( \frac{\alpha}{\pi} \right)^{d_n} e^{-\alpha|a|^2} da, \quad s_n = \sum_{j=1}^{d_n} a^n_j s^n_j, \tag{4}
\]

where \( \{s^n_1, \cdots , s^n_{d_n}\} \) is the orthonormal basis of \( H^0(M, L^n) \) with respect to the inner product \([20]\). Equivalently, the coefficients \( a^n_j \) are complex Gaussian random variables which satisfy the following normalization conditions,

\[
E^\alpha_n a^n_k a^n_k = 0, \quad E^\alpha_n a^n_k a^n_j = 1/\alpha \delta_{kj}, \quad E^\alpha_n a^n_k a^n_j = 0
\]

Here, we denote the expectation with respect to \( \gamma_n^\alpha \) by \( E^\alpha_n \). Under this normalization, we have the expected \( L^2 \) norm of \( s_n \),

\[
E^\alpha_n \|s_n\|^2_{h^n} = \frac{d_n}{\alpha}. \tag{6}
\]

Thus the covariance kernel of \( \gamma_n^\alpha \) is

\[
E^\alpha_n(s(z)\overline{s(w)}) = \frac{1}{\alpha} \Pi_n(z, w), \tag{7}
\]

where \( \Pi_n \) is the Szegö projector with respect to \([20]\).

When \( \alpha = d_n \) we call the ensemble the normalized Gaussian measure. It is given by

\[
d\mu^\alpha_n(s_n) = \left( \frac{d_n}{\pi} \right)^{d_n} e^{-d_n|a|^2} da, \tag{8}
\]

and from \([6]\) it follows that

\[
E^\alpha_n \|s_n\|^2_{h^n} = 1. \tag{9}
\]

As will be seen below, the density of critical values has a limit for this sequence of probability measures. We discuss the relations of the densities as \( \alpha \) varies in \([3]\).

The distribution of critical points of a section is defined by the un-normalized empirical measure

\[
C_s = \sum_{z: \nabla s(z) = 0} \delta_z. \tag{10}
\]

The Chern connection \( \nabla \) associated with \( h \) is not holomorphic, and the number of critical points depends on the section \( s \). The statistics of the number of critical points in the normalized Gaussian ensemble was determined in \([DSZ1, DSZ2]\). The critical point distribution is invariant under \( s \to cs \) and therefore it is equivalent to work with Gaussian or spherical distributions. It is proved in Corollary 5 of \([DSZ2]\) that the expected number \( N_{n,q,h}^{\text{crit}} \) of critical points of Morse index \( q \) for any positively curved Hermitian metric \( h \) on a Kähler manifold of dimension \( m \) satisfies

\[
N_{n,q,h}^{\text{crit}} \sim \left[ \frac{\pi^m h_0 m!}{m!} c_1(L)^m \right] n^m. \tag{11}
\]
Here, $b_0$ is a Betti number and $c_1(L)$ is the first Chern class, both of which are topological invariants. For instance, in dimension one there are roughly $\frac{2}{3}n$ saddles points and $\frac{1}{3}n$ local maximal for Gaussian random polynomials of degree $n$ with the $SU(2)$ (or Fubini-Study) inner product.

1.1. Statement of results. In this article we study the Gaussian, resp. spherical, distribution of critical values,

$$CV_s := \{|s(z)|_{h^n} : \nabla s(z) = 0, \ s \in H^0(M, L^n)\}$$

in the limit as the degree $n$ tends to $\infty$. Thus, the “value” of the section at a critical point is the Hermitian norm in $\mathbb{R}$ of the section; since $s(z) \in L_z$ it would not make sense to study the values themselves. In view of (11), we define the (normalized) empirical measure of nonvanishing critical values of $|s_n|_{h^n}$ by

$$CV_s = \frac{1}{n^{m}} \sum_{z : \nabla' s_n(z) = 0} \delta_{|s_n|_{h^n}}.$$  

Note that it is not necessarily a probability measure but from the results of [DSZ2] (such as (11)) it follows that for any $\epsilon > 0$ there exists a constant $C$ so that

$$\# \{z : \nabla s(z) = 0\} \leq Cn^m$$

except for a set of sections of measure $< \epsilon$. We define the Gaussian density of critical values $D_n^\alpha(x)$ as the expected density of $E_n^\alpha CV_s$ in the sense of distribution,

$$E_n^\alpha \left( \frac{1}{n^{m}} \sum_{z : \nabla' s_n = 0} \psi(|s_n|_{h^n}) \right) = \int_{\mathbb{R}^+} \psi(x) D_n^\alpha(x) dx \text{ for } \psi \in C_0(\mathbb{R}^+)$$

where $dx$ is the Lebesgue measure on $\mathbb{R}$. In [3] we will calculate the densities for all $\alpha$. The Kac-Rice are quite complicated for fixed $n$, but for the normalized Gaussian ensemble there are simple asymptotics.

To state the result we need some notations. We denote by $S(\mathbb{C}^m) \cong \mathbb{C}^{m^2 \times m}$ the space of complex symmetric matrices. We also denote by $d\xi$ the Legesgue measure on $S(\mathbb{C}^m)$. We also define the special matrix $P := (\delta_{jj'}\delta_{qq'} + \delta_{jj'}\delta_{qq'}) \frac{m^2 + m}{m^2} \times \frac{m^2 + m}{m^2}$.

**Theorem 1.** Let $(M, \omega)$ be an $m$-dimensional compact Kähler manifold. Let $(L, h) \to M$ be a polarized positive holomorphic line bundle. Let $D_n^\alpha$ be the expected density of critical values defined in (14) with $\alpha = d_n$, i.e. the normalized Gaussian density. Then we have,

$$D_n^d(x) = D_\infty(x) + O \left( \frac{1}{n} (x(1 + x^4)e^{-x^2}) \right) \quad \text{on} \quad (0, \infty),$$

where

$$D_\infty(x) = f_m(|x|) |x| e^{-|x|^2}, \text{ with } f_m(|x|) = c_m \int_{S(\mathbb{C}^m)} e^{-|\xi|^2} \left( |\sqrt{P}\xi|^2 - |x|^2 \right) d\xi.$$  

Here, $c_m = \frac{2^n}{m^2 \pi^{m^2 + m}} V$, where $V$ is the volume of $(M, \omega)$. The asymptotics can be differentiated any number of times (with appropriate changes in the polynomial growth in the remainder estimate.)

**Remark 1.** Henceforth we generally set $V = 1$ for notational simplicity.
In the case of Riemann surfaces when \( m = 1 \), we have \( P = 2 \). Assuming the volume of \( M \) is \( \pi \), then

\[
D_n^\infty(x) = x \left( 2x^2 - 4 + 8e^{-\frac{x^2}{2}} \right)e^{-x^2} + O\left( \frac{e^{-x^2}(1 + x^5)}{n} \right).
\]

Below is the computer graphic of the leading term,

Graph of \( D_\infty(x) \) in dimension one

The critical point densities for \( D_n^\alpha \) have the scaling relation \( D_n^\alpha(x) = \alpha^{\frac{1}{2}} D_n^{1\alpha}(\alpha^{\frac{1}{2}}x) \) (Lemma[2]), and from this one can determine the asymptotics for the other Gaussian ensembles. We also give a similar formula for the simpler expected distribution function of random sections in \( \mathfrak{9} \).

The proof of Theorem [1] is based on the Kac-Rice formula in Lemmas[3,4], which give exact formulae for all of the \( D_n^\alpha(x) \). The asymptotics then follow from the complete asymptotic expansion of the covariance kernel \( \frac{7}{4} \) in \( \mathfrak{7.3} \). In the case of \( SU(m + 1) \) polynomials on \( \text{CP}^m \) we give an exact formula for all \( n \) in \( \mathfrak{6} \).

Remark 2. As in [DSZ2] Theorem 1.2 (see also [Bau]), we can give similar formulae for the distribution of critical values when the critical point is constrained to have a specified Morse index. The formula only changes in that we integrate over the subset \( S^q(\mathbb{C}^m) \) of matrices of index \( q \).

1.2. Density of critical values in the spherical ensemble. As mentioned above, the critical point distribution is homogeneous, i.e the same for \( s \) and \( cs \). The critical value distribution is however not homogeneous, since the critical values are multiplied by \( |c| \). Since the mass of the normalized Gaussian measure is asymptotically concentrated near the unit sphere as \( d_n \to \infty \), the critical values of a line of sections \( \{cs\} \) are weighted most for values of \( |c| \) close to 1. This weighting is a re-scaled version of the one in the classical Poincaré Borel theorem, which states that the spherical probability measure \( \nu_d \) on the sphere \( S^d(\sqrt{d}) \) tends to the Gaussian measure as \( d \to \infty \). More precisely, if \( P_d : \mathbb{R}^d \to \mathbb{R}^k \) is \( P_d(x) = \sqrt{d}(x_1, \ldots, x_k) \), then for all \( k, P_d \nu_d \to \gamma_k = (2\pi)^{-d/2}e^{-|x|^2/2}dx \). Moreover,

\[
\gamma_d \{ x \in \mathbb{R}^d : ||x||^2 \geq \frac{d}{1 - \epsilon} \} \leq e^{-\epsilon^2 d/4}, \quad \gamma_d \{ x \in \mathbb{R}^d : ||x||^2 \leq (1 - \epsilon)d \} \leq e^{-\epsilon^2 d/4}.
\]
Our spherical probability measure is normalized Haar measure $d\nu_n$ on

$$SH^0(M,L^n) = \{ s \in H^0(M,L^n) : ||s||_{L^2} = 1 \}. \quad (17)$$

We refer to the corresponding probability space as the spherical ensemble. What we are calling the normalized Gaussian measure $\mathbb{E}_\nu$ concentrates exponentially on this unit sphere. We denote the expectation with respect to $d\nu_n$ by $\mathbb{E}_{\nu_n}$ and define the (normalized) spherical density of critical points $D_n^S(x)$ by

$$\mathbb{E}_{\nu_n} CV_s = D_n^S(x) dx. \quad (18)$$

In fact it makes more sense to pass to the quotient Fubini-Study probability measure on the projective space $\mathbb{P}H^0(M,L^n)$ of sections since the critical value distribution is invariant under multiplication by $e^{i\theta}$.

We view the spherical density (18) as primary because fixing $||s_n||_{L^2} = 1$ sets a scale against which one can calibrate the heights at which interesting features of the landscapes occur. For instance, as mentioned above, the spherical critical value distribution $D_n^S$ is supported in $[0,C\sqrt{d_n}]$, and its median should occur at a constant times $\sqrt{\log n}$. In the sequel we plan to study such distinguished levels in more detail. The main result of this article is:

**Theorem 2.** The density of critical values in the spherical ensemble $SH^0(M,L^n)$ on any compact Kähler manifold has the universal limit,

$$\lim_{n \to \infty} D_n^S(u) = D_\infty(u).$$

Thus, the spherical critical point distribution tends to the same universal limit as the normalized Gaussian measure of Theorem 1. As the asymptotics (and Graph) indicate, the most probable critical value and the median of the critical value distribution $D_n^S$ is around 1 in dimension one. An upper bound for the median may be derived from the exact formula for the spherical critical point density. In a subsequent article we will apply Theorem 2 or more precisely its proof to obtain a formula for the median and for the asymptotics of the critical point density with fixed Morse index in special $n$-dependent intervals.

As discussed above, Theorem 2 is a Poincaré-Borel type theorem for critical values. Intuitively it is based on the concentration of normalized Gaussian measure around the unit sphere, but in its details it uses the special scalings of the critical value distribution and the asymptotics of the covariance kernels and therefore does not seem to follow directly from the classical Poincaré-Borel theorem. The proof is based on a Laplace transform relation between the spherical and normalized Gaussian critical value distributions.

The Poincaré-Borel relation between the normalized Gaussian and spherical expectations of the critical value distribution holds also for the full value distribution. We denote the density of values in the spherical ensemble by $f_n^S(u) du$. In §9 we prove:

**Theorem 3.** The density of values in the spherical ensemble $SH^0(M,L^n)$ of any Kähler manifold has the limit,

$$\lim_{n \to \infty} f_n^S(u) = 2ue^{-u^2}.$$
1.3. **Related results and problems.** Other articles over the last ten years devoted to the statistics of critical points of Gaussian random fields include [DSZ1, DSZ2, DSZ3, Bau, Mac, NSV, B, GW, ABA, N1, N2, Z]. The Kac-Rice formula for the Gaussian critical value distribution of holomorphic sections were originally obtained in [SZ3] but the asymptotics were not determined as explicitly as in this article. As mentioned above, we view the Gaussian formalism mainly as a method for computing the spherical distribution. In the real domain, a Kac-Rice formula and an asymptotic analysis of the critical point distribution are given in [N2].

The expected value of the empirical measure (13) is only the first and simplest of the many probabilistic problems on critical values. As in the case of zeros or critical points, one may ask for the variance of linear statistics (pairings of smooth test functions with the empirical measure), the asymptotic normality of linear statistics, large deviations properties and so on. As mentioned above, the precise structure of the landscapes defined by \( y = |s_n(z)|_{h_n} \) is unknown in many respects.

In the case of polynomials of one complex variable \( p(z) \), one might instead use the standard complex derivative \( \frac{dp}{dz} \) to define critical points and critical values, but it is in fact a meromorphic connection with a pole at infinity and leads to quite a different theory. To our knowledge, statistics of critical points in the latter sense have only been studied in [H, FW]. In [FW], the authors studied the expected density of critical values of Gaussian \( SU(2) \) random polynomials \( p_n \) defined on \( \mathbb{C} \) with respect to the meromorphic connection \( \frac{d}{dz} \). Also, in [H], B. Hanin studies the correlation between zeros and critical points for this classical connection.

The main result of [FW] is that the (un-normalized) expected density of non-vanishing critical values of the modulus of \( |p_n| \) satisfies

\[
E \left( \sum_{z : |p_n|' = 0} \delta_{|p_n|} \right) \sim \frac{1}{x}
\]

on \( \mathbb{R}_+ \) as \( x \) away from 0 where \( |p_n|' = \frac{dp_n}{dz} \). This result is quite different from Theorem [1] due to the fact that the connection \( d/dz \) is a flat meromorphic one rather than the smooth but non-holomorphic connection of this article.

2. **Background on Gaussian measures in the Kähler setting**

2.1. **Kähler manifolds.** Let \((M, \omega)\) be a compact Kähler manifold of complex dimension \( m \) with the local Kähler potential \( \omega = \partial \overline{\partial} \phi \). Let \((L, h) \to M\) be a positive holomorphic line bundle such that the curvature of the Hermitian metric \( h \),

\[
\Theta_h = -\partial \overline{\partial} \log |e|_h^2
\]

is a positive (1, 1) form [GH]. Here, \( e \) is a local non-vanishing holomorphic section of \( L \) over an open set \( U \subset M \) such that locally \( L|_U \cong U \times \mathbb{C} \) and \( |e|_h = h(e, e)^{1/2} \) is the pointwise \( h \)-norm of \( e \).

We denote by \( H^0(M, L^n) \) the space of global holomorphic sections of \( L^n = L \otimes \cdots \otimes L \). Under the local coordinate, we can write the global holomorphic section as \( s_n = f_n e^{\phi_n} \) where \( f_n \) is a holomorphic function on \( U \). We denote the dimension of \( H^0(M, L^n) \) by \( d_n \).
The Hermitian metric $h$ induces a Hermitian metric $h^n$ on $L^n$ given by $|e^\otimes n|_{h^n} = |e|_h^n$. Throughout the article, we assume the polarized condition $\Theta_h = \omega$ such that in the local coordinate, we have the $h$-norm $|e|_h = e^{-\frac{\omega}{2}}$ and hence $|s_n|_{h^n} = |f_n|e^{-\frac{\omega}{2}}$.

We decompose the Chern connection $\nabla = \nabla' + \nabla''$ of the Hermitian line bundle $(L^n, h^n)$ into holomorphic and antiholomorphic parts where in the local coordinate $\nabla' = dz + n\partial\log h$ and $\nabla'' = d\bar{z}$ [GH].

We can define an inner product on $H^0(M, L^n)$ as,

\[
\langle s^n_1, s^n_2 \rangle_{h^n} = \int_M h^n(s^n_1, s^n_2)dV
\]

where $dV = \frac{\omega^n}{m!}$ is the volume form. We recall that throughout the article we assume the volume is normalized as $\int_M dV = 1$.

We write this in the local coordinates,

\[
\langle s^n_1, s^n_2 \rangle_{h^n} = \int_M f^n_1\bar{f}^n_2e^{-n\frac{\omega}{2}}dV
\]

where $s^n_1 = f^n_1e^\otimes n$ and $s^n_2 = f^n_2e^\otimes n$.

2.2. Gaussian measures. Recall that a Gaussian measure on $\mathbb{R}^n$ is a measure of the form

\[
\gamma_\Delta = \frac{e^{-\frac{1}{2} \langle \Delta^{-1}x, x \rangle}}{(2\pi)^{n/2} \sqrt{\det \Delta}}dx_1 \cdots dx_n,
\]

where $\Delta$ is a positive definite symmetric $n \times n$ matrix. The matrix $\Delta$ gives the second moments of $\gamma_\Delta$:

\[
\langle x_j x_k \rangle_{\gamma_\Delta} = \Delta_{jk}.
\]

This Gaussian measure is also characterized by its Fourier transform

\[
\hat{\gamma}_\Delta(t_1, \ldots, t_n) = e^{-\frac{1}{2} \sum \Delta_{jk} t_j t_k}.
\]

If we let $\Delta$ be the $n \times n$ identity matrix, we obtain the standard Gaussian measure on $\mathbb{R}^n$,

\[
\gamma_n := \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2} |x|^2}dx_1 \cdots dx_n,
\]

with the property that the $x_j$ are independent Gaussian variables with mean 0 and variance 1.

A complex Gaussian measure on $\mathbb{C}^k$ is a measure of the form

\[
\gamma_\Delta = \frac{e^{-\langle \Delta^{-1}z, \bar{z} \rangle}}{\pi^k \det \Delta}dz,
\]

where $dz$ denotes Lebesgue measure on $\mathbb{C}^k$, and $\Delta$ is a positive definite Hermitian $k \times k$ matrix. The matrix $\Delta = (\Delta_{\alpha\beta})$ is the covariance matrix of $\gamma_\Delta$:

\[
\langle z_\alpha \bar{z}_\beta \rangle_{\gamma_\Delta} = \Delta_{\alpha\beta}, \quad 1 \leq \alpha, \beta \leq k.
\]
3. A one-parameter family of Gaussian measures and their critical point distributions

The one-parameter family of complex Gaussian measures \((4)\) on \(H^0(M, L^n)\) may be written formally as

\[
d\gamma^n_\alpha = \left(\frac{\alpha}{\pi}\right)^{d_n} e^{-\alpha ||s||^2} DS
\]

where \(DS\) is Lebesgue measure.

The normalization comes from the calculation

\[
1 = \frac{\alpha^{dn}}{\pi^{dn}} \int_{C^d} e^{-\alpha ||z||^2} dz = \frac{1}{\pi^{dn}} \int_{S^d} e^{-||z||^2} d\omega = \frac{\omega_{k}}{2\pi^{k/2}} \Gamma(d_n).
\]

where \(\omega_k = \frac{2\pi^{k/2}}{\Gamma(k/2)}\) is the surface measure of the unit sphere \(S^{k-1} \subset \mathbb{R}^k\).

As in (14) we denote the normalized density of critical values with respect to \(\gamma^n_\alpha\) by

\[
E^n_\alpha CV_s = D^n_\alpha(x) dx.
\]

We note that \(D^n_\alpha dx\) is not a probability measure on \(\mathbb{R}_+\) since \(CV_s\) is not in general a probability measure. The mass of the measure can be determined from the eventual Kac-Rice formulae of §3.

3.1. Normalized Hemitian Gaussian measure. As mentioned in the introduction, when \(\alpha = d_n\) it’s the normalized Gaussian measure \([8]\) which is characterized by

\[
E a^a_k = 0, \quad E a^a_k \bar{a}^a_j = \frac{1}{d_n} \delta_{kj}, \quad E a^a_k a^a_j = 0
\]

Under this normalization, the expected \(L^2\) norm of \(s_n\) is 1 \([1]\).

3.2. Relations between Gaussian critical value densities. Next we compare densities \((27)\) as \(\alpha\) changes. The first step is

**Lemma 1.** For any \(s \in H^0(M, L^n)\) with non-degenerate critical points, and for any \(r > 0\),

\[
CV_{rs} = CV_s(r^{-1}x).
\]

**Proof.** For any \(f \in C(\mathbb{R})\) we have,

\[
\langle CV_{rs}, f \rangle = \sum_{z: \nabla s(z) = 0} f(r|s(z)|_{H^n}) = \langle CV_s(x), f(rx) \rangle.
\]

Changing variables to \(y = rx\) completes the proof. \(\square\)

Since \(D^n_\alpha dx\) transforms by the inverse dilation, the density has the transformation law,

**Lemma 2.** \(D^n_\alpha(x) = \alpha^{\frac{1}{2}} D^n_{\alpha^{\frac{1}{2}}}(\alpha^{\frac{1}{2}} x)\).
Proof. We have,
\[ D_{\alpha}^n(x)dx = \mathbb{E}_\alpha^n CVs = (\frac{2}{\pi})^d \int_{H^0(M \times \mathbb{L}^n)} CVs(x)e^{-\alpha ||s||^2} Ds \]
\[ = (\frac{2}{\pi})^d \int_{H^0(M \times \mathbb{L}^n)} CVs(x)e^{-\|s\|^2} Ds \]
\[ = \frac{1}{\pi^n} \int_{H^0(M \times \mathbb{L}^n)} CVs(\alpha^{-\frac{1}{2}}s)(x)e^{-\|s\|^2} Ds' \]
\[ = \frac{1}{\pi^n} \int_{H^0(M \times \mathbb{L}^n)} CVs'(\alpha^\frac{1}{2}x)e^{-\|s\|^2} Ds' = \alpha^{\frac{1}{2}} D_1 n(\alpha^\frac{1}{2}x)dx, \]
where we change variables \( s \to s' = \alpha^{-\frac{1}{2}}s \) and apply Lemma 1.

Of course this implies that \( D_{\alpha}^n(x)dx \) all have the same mass. Since \( \alpha = d_n \) in the normalized Gaussian measure, we have:

**Corollary 1.** \( D_{d_n}^n(x) = d_n^\frac{1}{2} D_1 n(d_n^\frac{1}{2}x) \)

4. **Kac-Rice formula for the Gaussian critical point density**

The main result of this section (Lemma 3) gives a generic Kac-Rice formula for the expected density \( D_{d_n}^n \) of critical points with respect to the normalized Gaussian measure. Very general Kac-Rice formulae applicable to the critical point density in our setting are given in [BSZ]. Other presentations can be found in [AT, N1]. The historical references are [K, R].

The Kac-Rice formula is as follows, let \( f(t) \) be a real valued smooth stochastic process on the finite interval \( I \subset \mathbb{R} \). Then the expected number zeros,

\[ \mathbb{E} \# \{ t \in I : f(t) = 0 \} = \int_I \int_\mathbb{R} |y| p_t(0, y)dxdt \]

where \( p_t(0, y) \) is the joint density of \( (f, f') \) evaluated at \( (0, y) \), \( dy \) and \( dt \) are Lebesgue measures on \( \mathbb{R} \). If \( f \) is a Gaussian process, then the joint density \( p_t(x, y) \) is uniquely determined by the covariance matrix of \( (f, f') \) [BSZ, AT].

4.1. **Kac-Rice formula for the critical point density.** In this subsection, we will derive the Kac-Rice formula for the expected density \( D_{\gamma}^d_n \) of critical values of \( s_n \) with respect to \( \gamma d_n \). As we will show this particular Gaussian density has a limit as \( n \to \infty \). The formula may be derived from [BSZ, DSZ1] but we take advantage of some simplifications to speed up the proof. To simplify notation, we write

\[ D_n := D_{d_n}^d. \]

In the local coordinate \( U \cong \mathbb{C}^m \) and a local trivialization of \( L \), we write the normalized Gaussian random sections as,

\[ s_n = f_n e^{\otimes n}. \]

where

\[ f_n = \sum_{j=1}^{d_n} a_j f_j^n \]

where \( \{a_j\} \) are normalized Gaussian random variable [28] and locally \( \{s_j^n = f_j^n e^{\otimes n}, j = 1, \ldots, d_n\} \) is an orthogonal basis of \( H^0(M, L^n) \) with respect to the inner product [20].
The smooth Chern connection then has the form \([\text{GH}].\)

\[
\nabla' s_n = (f'_n - n\partial \varphi f_n)e^\varphi.
\]

Here, \(d = \partial + \bar{\partial}\) is the decomposition into terms of type \((1,0),\) resp. \((0,1).\) Here, \(f'_n = \partial f = \sum_j \frac{\partial f}{\partial z_j} dz_j.\) Thus, in the local coordinate, the empirical measure \([13]\) has the form

\[
CV = \frac{1}{n^m} \sum_{z \in \Omega} \delta_{|f_n|e^{-\varphi}}.
\]

where

\[
\Omega = \{z \in \mathbb{C}^m : (f'_n - n\partial \varphi f_n)e^{-\varphi} = 0\}.
\]

We also introduce the locally defined empirical measure of complex critical values

\[
\hat{CV} = \frac{1}{n^m} \sum_{z \in \Omega} \delta_{f_n e^{-\varphi}}.
\]

We will determine the expected density \(\hat{D}_n\) of \(\hat{CV}\) on \(\mathbb{C}\) and then integrate out the angle variable \(\int_0^{2\pi} \hat{D}_n(|x|, \theta)|x|d\theta\) to obtain the (global) expected density of \(CV.\)

In other words, we use that \(\langle \psi, CV \rangle = \langle \psi, \hat{CV} \rangle\) for radial functions \(\psi\) and thus, \(CV = \pi \hat{CV}\) where \(\pi : \mathbb{C}^* \to \mathbb{R}_+\) the map \(z \to |z|\).

The result is:

**Lemma 3.** The expected distribution of critical values \(CV\) is given by the formula,

\[
\hat{D}_n(x) = \frac{1}{n^m} \int_0^{2\pi} \int_{S(\mathbb{C}^m)} p^0_x(x, \theta, 0, \xi) |\det (\xi x^* - n^2 I x^2)| x d\xi dV d\theta
\]

where \(\xi \in S(\mathbb{C}^m) \cong \mathbb{C}^{m(m+1)/2}\) is the space of \(m \times m\) complex symmetric matrices and \(d\xi\) is the Lebesgue measure on \(S(\mathbb{C}^m),\) \(p^0_x(y, \theta, 0, \xi)\) is the joint density of \(p^0_x(y, 0, \xi)\) of normalized Gaussian random variables \((f_n, f'_n, f''_n)\) evaluate at \(f'_n = 0,\) here we substitute \(y := xe^{i\theta}\) by the map \(\pi.\) The formula of \(p^0_x(y, 0, \xi)\) is given explicitly in Lemma \([7].\)

**Proof.** We first introduce some notations:

\[
p_n = f_n e^{-\varphi} \in \mathbb{C}, \quad q_n = (f'_n - n\partial \varphi f_n)e^{-\varphi} \in \mathbb{C}^m, \quad r_n = f''_n e^{-\varphi} \in S(\mathbb{C})
\]

then \(p_n, q_n\) and \(r_n\) are all complex Gaussian random variables.

By definition of the delta function, we have for any test functions \(\psi \in C_0^\infty(\mathbb{C}),\)

\[
\frac{1}{n^m} \sum_{z \in \Omega} \delta_{f_n e^{-\varphi}}(\psi) = \frac{1}{n^m} \sum_{z : q_n(z) = 0} \psi(p_n)
\]

\[
= \frac{1}{n^m} \int_{\mathbb{C}^m} \delta_0(q_n) \psi(p_n) dq_n \wedge d\bar{q}_n
\]

\[
= \frac{1}{n^m} \int_{\mathbb{C}^m} \delta_0(q_n) \psi(p_n) \left| \det \left( \frac{|\partial q_n|^2}{|\partial z|^2} - \frac{|\partial q_n|}{|\partial z|^2} \right) \right| dz
\]
where $dz$ is the Lebesgue measure on $\mathbb{C}$. By direct computations, we have,
\[
\frac{\partial q_n}{\partial z} = (f''_n - n\partial \varphi f'_n - n\partial^2 \varphi f_n)e^{-\frac{n\varphi}{2}} - \frac{n}{2} \partial \varphi q_n
\]
\[
= f'_n e^{-\frac{n\varphi}{2}} - n\partial \varphi q_n + n^2 \partial^2 \varphi \partial \varphi^* p_n - n\partial^2 \varphi p_n - \frac{n}{2} \partial \varphi q_n^*
\]
and
\[
\frac{\partial q_n}{\partial \bar{z}} = -n\partial \bar{\varphi} p_n - \frac{n}{2} \partial \varphi q_n^*
\]
where $\frac{\partial q_n}{\partial z}$ and $\frac{\partial q_n}{\partial \bar{z}}$ are $m \times m$ symmetric matrices; for simplicity, for any matrix $A$ we denote,
\[
|A|^2 := AA^*
\]
By taking expectation on both sides, we have,
\[
\mathbb{E} \left( \frac{1}{n^m} \sum_{z \in \Omega} \delta_{f_n e^{-\frac{n\varphi}{2}}} \cdot \psi \right)
\]
\[
= \frac{1}{n^m} \mathbb{E} \int_{\mathbb{C}^m} \delta_0(q_n) \psi(p_n) \left| \det \left( \frac{\partial q_n}{\partial z} - \frac{\partial q_n}{\partial \bar{z}} \right) \right| dz
\]
\[
\overset{=} {\rightarrow} \frac{1}{n^m} \int_{\mathbb{C}^m} \int_{\mathbb{C}^m} \psi(y)p_z(y,0,\xi) \left| \det \left( \xi + n^2 \partial \varphi \partial \varphi^* y - n \partial^2 \varphi y \right) - n^2 \left| \partial \bar{\varphi} \right|^2 \right| dy dz
\]
where $p_z(y,s,\xi)$ is the joint probability of the Gaussian random field $(p_n, q_n, r_n)$ and $dx$ is the Lebesgue measure on $\mathbb{C}$.

Thus, the expected density $\mathbb{E} \hat{C} \hat{V}_s$ is,
\[
(37) \quad \hat{D}_n(y) = \frac{1}{n^m} \int_{\mathbb{C}^m} \int_{\mathbb{C}^m} p_z(y,0,\xi) \left| \det \left( \xi + n^2 \partial \varphi \partial \varphi^* y - n \partial^2 \varphi y \right) - n^2 \left| \partial \bar{\varphi} \right|^2 \right| dy dz
\]
We rewrite $y$ in polar coordinate $(x, \theta)$. Then the expected density of $CV_s$ is,
\[
(38) \quad D_n(x) = \int_0^{2\pi} \hat{D}_n(x, \theta) x d\theta
\]

The next step is to get an explicit geometric formula for $D_n(x)$. The empirical measure $CV_s$ and its average $D_n$ are independent of coordinates and frames on the Kähler manifold $M$ and line bundle $L^n$. We may choose Kähler normal coordinate to simplify the above integral.

We freeze at a point $z_0$ as the origin of the coordinate patch to simplify the integrand at $z_0$. It is well known in terms of Kähler normal coordinates $\{z_j\}$, the Kähler potential $\varphi$ has the expansion in the neighborhood of $z_0$:
\[
\varphi(z, \bar{z}) = ||z||^2 - \frac{1}{4} \sum R_{j kpq}(z_0) z_j \bar{z}_k z_p \bar{z}_q + O(||z||^5).
\]
In general, $\varphi$ contains a pluriharmonic term $f(z) + \overline{f(z)}$, but a change of frame for $L$ eliminates that term up to fourth order. Thus
\[
\partial \varphi(z_0) = 0, \quad \partial^2 \varphi(z_0) = 0, \quad \partial \bar{\varphi}(z_0) = 1, \quad dV(z_0) = dz.
\]
Such frames are called adapted in [BSZ, DSZ2]. Hence, the joint density of $(p_n, q_n, r_n)$ at $z_0$ is the same as the joint density of Gaussian process $(f_n, f'_n, f''_n)$.
Thus by (37)(38)(40), we obtain the global expression,

\[ D_n(x) = \frac{1}{n^m} \int_0^{2\pi} \int_M \int_{S(\mathbb{C}^m)} p_{\bar{z}}(x, \theta, 0, \xi) |\det \left( |\xi|^2 - n^2 I x^2 \right)| x d\xi dV d\theta \]

which completes the proof. \( \square \)

5. Calculation of the joint probability density

In this section we calculate the joint probability distribution \( p_{\bar{z}}(y, s, \xi) \) of Lemma 3 with respect to the normalized Gaussian measure \( \gamma_{d_n} \).

5.1. Density \( p_{\bar{z}}(y, s, \xi) \). In this subsection, we will derive the formula for \( p_{\bar{z}}(y, s, \xi) \) of the joint density of the Gaussian process \( (f_n, f'_n, f''_n) \). It is given by the formula \( [BSZ, AT] \),

\[
p_{\bar{z}}(y, s, \xi) = \frac{1}{\pi d_n} \frac{1}{\det \Delta_{\bar{z}}} \exp \left\{ - \left\langle \begin{pmatrix} y \\ s \\ \xi \end{pmatrix}, \left( \Delta_{\bar{z}} \right)^{-1} \begin{pmatrix} \bar{y} \\ \bar{s} \\ \bar{\xi} \end{pmatrix} \right\rangle \right\}
\]

where \( \frac{(m+1)(m+2)}{2} \) is the dimension of the Gaussian process \( (f_n, f'_n, f''_n) \) and \( \Delta_{\bar{z}} \) is the covariance matrix of this process.

We rearrange the order the Gaussian process and write \( \tilde{\Delta}_{\bar{z}} \) as the covariance matrix of \( (f'_n, f''_n, f_n) \), then we rewrite,

\[
p_{\bar{z}}(y, s, \xi) = \frac{1}{\pi d_n} \frac{1}{\det \tilde{\Delta}_{\bar{z}}} \exp \left\{ - \left\langle \begin{pmatrix} s \\ \xi \\ y \end{pmatrix}, \left( \tilde{\Delta}_{\bar{z}} \right)^{-1} \begin{pmatrix} \bar{s} \\ \bar{\xi} \\ \bar{y} \end{pmatrix} \right\rangle \right\}
\]

The covariance kernel is defined by

\[
\Pi_n(z, w) := \sum_{j=1}^{d_n} f^n_j(z) \bar{f}^n_j(w).
\]

where \( \{f^n_j\} \) is defined in (30).

Then, we have

\[
\mathbf{E} \left( f_n(z) \bar{f}_n(w) \right) = \frac{1}{d_n} \Pi_n(z, w)
\]

The notation \( \Pi_n(z, w) \) usually refers to the Szegö kernel but in fact \( [44] \) is the Bergman kernel, which has the pointwise TYZ expansion \( [I, T, Z] \).

\[
\Pi_n(z, z) = n^m e^{n\varphi(z)} \left[ 1 + a_1(z)n^{-1} + a_2(z)n^{-2} + \cdots \right],
\]

where \( a_1 \) is the scalar curvature. Integrating over \( M \) with respect to \( e^{-n\varphi} dV \) gives the well-known dimension polynomial,

\[
d_n = n^m (1 + n^{-1} \int_M a_1 dV + n^{-2} \int_M a_2 dV + \cdots )
\]

The covariance matrix is then given by,

\[
\tilde{\Delta}_{\bar{z}} = \begin{pmatrix} A_n & B_n \\ B_n^* & C_n \end{pmatrix},
\]

where

\[
A_n = \frac{1}{d_n} \frac{\partial^2 \Pi_n(z, w)}{\partial z \partial \bar{w}} \bigg|_{z=w}
\]
is an $m \times m$ matrix;

$$B_n = \frac{1}{d_n} \left( \frac{\partial^3 \Pi_n(z, w)}{\partial z^3 \partial w} \bigg|_{z=w} \frac{\partial^3 \Pi_n(z, w)}{\partial z^2} \bigg|_{z=w} \right)$$

is an $m \times \frac{m^2+m+2}{2}$ matrix;

$$C_n = \frac{1}{d_n} \left( \frac{\partial^3 \Pi_n(z, w)}{\partial z^3 \partial w} \bigg|_{z=w} \frac{\partial \Pi_n(z, w)}{\partial z} \bigg|_{z=w} \right)$$

is a $\frac{m^2+m+2}{2} \times \frac{m^2+m+2}{2}$ matrix.

Thus, we have

**Lemma 4.** With the above notations,

$$\rho_n^w(y, 0, \xi) = \frac{1}{\pi d_m} \frac{1}{\det A_n \det \Lambda_n} \exp \left\{ - \left( \left( \frac{\xi}{y} \right), \Lambda_n^{-1} \left( \frac{\xi}{y} \right) \right) \right\},$$

where

$$\Lambda_n = \frac{1}{d_n} (C_n - B_n^* A_n^{-1} B_n).$$

6. Calculation in the Fubini-Study case

In this section, we give explicit formulae for the Kac-Rice density of critical points for $SU(m + 1)$ polynomials. This is the case where $M = \mathbb{CP}^m$, where $L$ is the line bundle $\mathcal{O}(1)$ whose sections are linear functions on $\mathbb{CP}^m$, and so sections $L^n = \mathcal{O}(n)$ are homogeneous polynomials of degree $n$. We equip $\mathcal{O}(1)$ with its Fubini-Study metric $h_{FS}$ given by

$$\| s \|_{h_{FS}}(w) = \left| \frac{(s, w)}{|w|} \right|, \quad w = (w_0, \ldots, w_m) \in \mathbb{CP}^m,$$

for $s \in \mathbb{CP}^{m+1} \equiv H^0(\mathbb{CP}^m, \mathcal{O}(1))$, where $|w|^2 = \sum_{j=0}^m |w_j|^2$ and $|w| \in \mathbb{CP}^m$ denotes the complex line through $w$. The Kähler form on $\mathbb{CP}^m$ is the Fubini-Study form

$$\omega_{FS} = \frac{\sqrt{-1}}{2} \Theta_{h_{FS}} = \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log |w|^2.$$

We denote by $D_{n}^{SU(m+1)}$ the density of critical points for $\gamma_n^{d_n}$. Our result is:

**Proposition 1.**

$$D_{n}^{SU(m+1)}(x) = c_m e^{-x^2} \int_{S(\mathbb{C}^m)} e^{-|\xi|^2} \left| \det \left( \frac{n-1}{n} \sqrt{\xi} \right) \right| \left| \det \left( \frac{n-1}{n} \sqrt{\xi} \right) \right|^2 \right| d\xi$$

where $c_m$ is as in Theorem 7.

We note that the only difference between $D_{n}^{SU(m+1)}$ and the limit $D_{\infty}$ of Theorem 1 is in the limit $\frac{n-1}{n} \rightarrow 1$. In dimension $m = 1$,

$$D_{n}^{SU(2)}(x) = \frac{4}{\pi} xe^{-x^2} \int_0^{\infty} e^{-r} \left| \frac{n-1}{n} - 2r - Ix^2 \right| dr.$$
Proof. The Szegő kernel for $O(n)$ is given in an affine chart $Z_0 = 1$ with $z_j = \frac{z_j}{Z_0}$ by

$$\Pi^{O_0(\mathbb{C}P^1, O(n))}(z, w) = \frac{n + 1}{\pi} (1 + z\bar{w})^n e^{\bar{\phi}(z)} \otimes e^{\bar{\phi}(w)}.$$ 

Since our formula is invariant when the Szegő kernel is multiplied by a constant, we can replace the above by the normalized Szegő kernel

$$\tilde{\Pi}_n(z, w) := (1 + z\bar{w})^n$$

in our computation.

Since

$$\varphi(z) = -\log |e(z)|^2_{FS} = \log(1 + |z|^2),$$

we have

$$\varphi(0) = \frac{\partial^2 \varphi}{\partial z}(0) = \frac{\partial^2 \varphi}{\partial \bar{z}}(0) = 0;$$

i.e., $e_n$ is an adapted frame at $z = 0$. Hence when computing the (normalized) matrices $B_n$, $C_n$ for $H^0(\mathbb{C}P^1, O(n))$, we can take the usual derivatives of $\tilde{\Pi}_N$. Indeed, we have

$$\frac{\partial \tilde{\Pi}_n}{\partial z} = n(1 + z\bar{w})^{n-1}\bar{w},$$

$$\frac{\partial^2 \tilde{\Pi}_n}{\partial z^2} = n(n - 1)(1 + z\bar{w})^{n-2}(\bar{w})^2,$$

$$\frac{\partial^2 \tilde{\Pi}_n}{\partial \bar{z}^2} = n(n - 1)(1 + z\bar{w})^{n-2}z^2,$$

$$\frac{\partial^2 \tilde{\Pi}_n}{\partial z \partial \bar{w}} = n(1 + z\bar{w})^{n-1} + n(n - 1)(1 + z\bar{w})^{n-2}z\bar{w},$$

$$\frac{\partial^3 \tilde{\Pi}_n}{\partial z^2 \partial \bar{w}} = 2n(n - 1)(1 + z\bar{w})^{n-2}z + n(n - 1)(n - 2)(1 + z\bar{w})^{n-3}z^2\bar{w}$$

$$\frac{\partial^3 \tilde{\Pi}_n}{\partial z \partial \bar{w}^2} = 2n(n - 1)(1 + z\bar{w})^{n-2} + 4n(n - 1)(n - 2)(1 + z\bar{w})^{n-3}z\bar{w}$$

$$+ n(n - 1)(n - 2)(n - 3)(1 + z\bar{w})^{n-4}z^2\bar{w}^2$$

It follows that

$$A_n = \frac{1}{d_n} \frac{\partial^2 \Pi_n(z, w)}{\partial z \partial \bar{w}}|_{z=w=0} = (n\delta_{ij})^m_{i,j=1}$$

$$B_n = \frac{1}{d_n} \begin{pmatrix} \frac{\partial^3 \Pi_n(z, w)}{\partial z^2 \partial \bar{w}}|_{z=w} & \frac{\partial^2 \Pi_n(z, w)}{\partial z \partial \bar{w}}|_{z=w} \\ \frac{\partial^2 \Pi_n(z, w)}{\partial \bar{w}^2}|_{z=w} & \frac{\partial \Pi_n(z, w)}{\partial \bar{w}}|_{z=w} \end{pmatrix} = \begin{pmatrix} 2n(n - 1)I & 0 \\ 0 & 1 \end{pmatrix}$$

(an $m \times \frac{m^2 + m + 2}{2}$ matrix);

$$C_n = \frac{1}{d_n} \begin{pmatrix} \frac{\partial^3 \Pi_n(z, w)}{\partial z \partial \bar{w}^2}|_{z=w} & \frac{\partial^2 \Pi_n(z, w)}{\partial z \partial \bar{w}}|_{z=w} \\ \frac{\partial^2 \Pi_n(z, w)}{\partial \bar{w}^2}|_{z=w} & \frac{\partial \Pi_n(z, w)}{\partial \bar{w}}|_{z=w} \end{pmatrix} = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$$

(an $\frac{m^2 + m + 2}{2} \times \frac{m^2 + m + 2}{2}$ matrix.)
Since $B = 0$,

$$\Lambda_n = \frac{n^m}{d_n} (C_n - B_n A_n^{-1} B_n^*) = \frac{n^m}{d_n} C_n$$

\[= \frac{n^m}{d_n} \begin{pmatrix} n(n-1) & 0 \\ 0 & 1 \end{pmatrix} \]

where

$$P := (\delta_{jj'} \delta_{qq'} + \delta_{jj'} \delta_{qq'}) \frac{m^2 + n}{m^2 - n}$$

Thus

$$\det \Lambda_n = (n^2)^{\frac{m^2 + n}{m^2 - n}} \det P,$$

and

$$p_z(y, 0, \xi) = \frac{1}{\pi^{d_m}} \frac{1}{n^m n^{m^2 + m}} e^{-(n(n-1))^{-1} P^{-1} |\xi|^2 - |y|^2}$$

Write $y$ in the polar coordinate $(x, \theta)$ and combine Lemmas 3 and 4, we have

$$\mathbf{D}_n^{SU(m+1)}(x) = \frac{1}{\pi^{d_m}} \frac{x e^{-x^2}}{n^{m^2 + 3m}} \int_0^{2\pi} \int_M \int_{S(\mathbb{C}^m)} \frac{1}{\det P} e^{-(n(n-1))^{-1} P^{-1} |\xi|^2}$$

$$\times \left| \det \left( |\xi|^2 - n^2 x^2 \right) \right| d\xi dV d\theta$$

Now we change variables $\xi \to \sqrt{n(n-1)} \xi$, apply the assumption $\int_M dV = 1$ and integrate $\theta$ variable to get,

$$\mathbf{D}_n^{SU(m+1)}(x) = \frac{2 \pi^2 x e^{-x^2}}{\pi^{d_m}} \int_{S(\mathbb{C}^m)} \frac{1}{\det P} e^{-P^{-1} |\xi|^2} \left| \det \left( \frac{n-1}{n} |\xi|^2 - x^2 \right) \right| d\xi$$

Now change variables $\xi \to \sqrt{P} \xi$ to obtain the stated result.

$$\square$$

For $m = 1$ we get

$$\mathbf{D}_n^{SU(2)}(x) = \frac{4}{\pi} x e^{-x^2} \int_0^\infty e^{-r} \left| 2 \frac{n-1}{n} r - x^2 \right| \, dr$$

\[= \frac{n}{n-1} \frac{4}{\pi} x e^{-x^2} \int_0^\infty e^{-r} \left| 2r - \frac{n-1}{n} x^2 \right| \, dr \]

\[= x \left( 2x^2 - 4 \frac{n}{n-1} + 8 \exp \left( -\frac{n-1}{n} x^2 \right) \right) e^{-x^2} \]

7. Proof of Theorem 1

In this section, we will complete the proof of Theorem 1. The main additional ingredient is the asymptotic expansion of the Szegő kernel. Otherwise the computation is similar to the case of $\mathbb{C}^m$.

7.1. Covariance matrix. We now calculate the leading terms in $A_n$ and $\Lambda_n$. The key point is to calculate the mixed derivatives of $\Pi_n$ on the diagonal. It is convenient to do the calculation in Kähler normal coordinates about a point $z_0$ in $M$.

Now take two derivatives on both sides of (45),

$$\partial_j \bar{\partial}_{j'} \Pi_n(z, z) = n^m (n \partial_j \bar{\partial}_{j'} \varphi e^{n \varphi} [1 + n^{-1} a_1 + \cdots] + e^{n \varphi} [n^{-1} \partial_j \bar{\partial}_{j'} a_1 + \cdots])$$

\[+ 2 \Re \partial_j \varphi e^{n \varphi} \bar{\partial}_{j'} a_1 + \cdots + n^2 \partial_j \bar{\partial}_{j'} \varphi e^{n \varphi} [1 + n^{-1} a_1 + \cdots]) \]
We apply (39) at the origin $z_0$ to get,
\[(68)\quad A_n(z_0) = \partial_j \partial_j \Pi_n(z_0, z_0) = \frac{n^n}{d_n} (nI + a_1 I + n^{-1}(a_2 I + \partial_j \partial_j a_1 + \cdots)).\]

By the same arguments, we compute $B(z_0)$ up to the leading order term,
\[(69)\quad B_n(z_0) = \partial_j \partial_j \partial_j \Pi_n(z_0, z_0) = \frac{n^n}{d_n} [\partial_j \partial_j \partial_j a_1 + \partial_j \partial_j a_1 + \cdots, n^{-1} \partial_j a_1 + \cdots];\]

and the leading terms in each entry of $C_n(z_0)$,
\[(70)\quad C_n(z_0) = \frac{n^n}{d_n} \left( \frac{n^2(\partial_j \partial_j \partial_j \partial_j + \partial_j \partial_j \partial_j \partial_j)(1 + n^{-1}a_1 + \cdots)}{n^{-1} \partial_j \partial_j a_1 + n^{-2} \partial_j \partial_j a_2 + \cdots} \frac{n^{-1} \partial_j \partial_j a_1 + n^{-2} \partial_j \partial_j a_2 + \cdots}{1 + n^{-1}a_1 + \cdots} \right)\]

We refer to [DSZ2] for more details of those computations.
Together with Lemmas 3 and 4, the formulae for $A_n, B_n, C_n$ give an explicit formula for $D_n$ on a general Kähler manifold.

7.2. Proof of Theorem 1. In this section we calculate the leading order term as $n \to \infty$ of the expected density $D_n(x)$. We have,
\[
\Lambda_n = \frac{n^n}{d_n} (C_n - B_n A_n B_n^*)
\]
\[
= \frac{n^n}{d_n} \left( \frac{n^2 P + O(n)}{n^{-1} \partial_j \partial_j a_1 + O(n^{-2})} \frac{n^{-1} \partial_j \partial_j a_1 + O(n^{-2})}{1 + n^{-1}a_1 + O(n^{-2})} \right)
\]
\[
= \left( \frac{n^2 P + O(n)}{n^{-1} \partial_j \partial_j a_1 + O(n^{-2})} \frac{n^{-1} \partial_j \partial_j a_1 + O(n^{-2})}{1 + n^{-1}a_1 + O(n^{-2})} \right)
\]
in the last step, we apply the full expansion of $d_n$ [40] and where $P$ is given in (61).
Thus
\[(71)\quad \det \Lambda_n \sim (n^2)^{m^2-m} \det P.\]

Applying (46) to (48) again, we obtain the asymptotic analogue of (62),
\[(72)\quad \det A_n \sim n^m.\]

We then have,
\[(73)\quad A_n^{-1} \sim \left( \frac{n^{-2} P^{-1}}{n^{-3} \partial_j \partial_j a_1} \frac{n^{-3} \partial_j \partial_j a_1}{1} \right) + \text{lower order terms}\]

Thus,
\[(74)\quad p_z(y, 0, \xi) = \frac{1}{\pi^{d+m} \det P} \frac{1}{n^{m(n^2+m)}} e^{-n^{-2} P^{-1} |\xi|^2 - |y|^2} + \text{lower order terms},\]

hence by Lemma 3
\[
(75)\quad D_n(x) = \frac{1}{\pi^{d+m} n^{m^2+3m}} \int_0^{2\pi} \int_M \int_S(\xi) \frac{1}{\det P} e^{-n^{-2} P^{-1} |\xi|^2} |\det (|\xi|^2 - n^2 x^2)| d\xi dVd\theta + \text{lower order terms}\]

here we substitute $y = xe^{i\theta}$ in the expression of (74).

The remainder estimate is discussed in §7.3 below.
Now we change variables $\xi \to n\xi$, apply the assumption $\int_M dV = 1$ and integrate in the $\theta$ variable to get,

\begin{equation}
D_n(x) = \frac{2\pi xe^{-x^2}}{\pi^m} \int_{S(C^n)} \frac{1}{\det P} e^{-n^{-1}|\xi|^2} |\det (|\xi|^2 - Ix^2)| \, d\xi + O(\frac{1}{n}) \tag{76}
\end{equation}

Now change variable $\xi \to \sqrt{P}\xi$ to get,

\begin{equation}
D_n(x) = c_m xe^{-x^2} \int_{S(C^n)} e^{-|\xi|^2} |\det (|\sqrt{P}\xi|^2 - Ix^2)| \, d\xi + O(\frac{1}{n}) \tag{77}
\end{equation}

\begin{equation}
= f_m(x) + O(\frac{1}{n}).
\end{equation}

7.3. **Remainder estimate.** To prove the remainder estimate we use more terms in the Bergman kernel expansion and Taylor expansions of the various functions in the Kac-Rice integrand. We sketch the proof as follows. First we have,

\begin{equation}
\Lambda_n = \frac{n^m}{d_n} (C_n - B_n A_n^{-1} B_n^*)
= \frac{n^m}{d_n} \left( n^2 P(1 + n^{-1}a_1 + \ldots) - n^{-1} \partial_j \partial_q a_1 + O(n^{-2}) \right)

= \frac{1}{1 + n^{-1}a_1 + \ldots} \left( n^2 P(1 + n^{-1}a_1 + \ldots) - n^{-1} \partial_j \partial_q a_1 + O(n^{-2}) \right)
\end{equation}

Hence there exists a complete asymptotic expansion,

\begin{equation}
\det \Lambda_n = (n^2)^{\frac{n^2}{d_n} - 1} P(1 + n^{-1}b_1 + n^{-2}b_2 + \ldots)
\end{equation}

Same apply to

\begin{equation}
\det A_n = n^m (1 + n^{-1}c_1 + n^{-2}c_2 + \ldots)
\end{equation}

where $b_j$ and $c_j$ are polynomials of curvature and uniformly bounded.

This two identities imply the full expansion,

\begin{equation}
\det \Delta = \det A_n \det \Lambda_n = n^{m+2m}(1 + n^{-1}d_1 + n^{-2}d_2 + \ldots) \det P
\end{equation}

We also have,

\begin{equation}
\Lambda_n^{-1} = \left( \begin{array}{cccc}
n^{-2}P^{-1}(1 - n^{-1}a_1 + \ldots) & n^{-3}(\partial_j \partial_q a_1 + \ldots) \\
n^{-3}(\partial_j \partial_q a_1 + \ldots) & 1 - n^{-1}a_1 + \ldots.
\end{array} \right)
\end{equation}

It follows that

\begin{equation}
p_2^2(y, 0, \xi) = \frac{1}{\pi^m (1 + n^{-1}d_1 + n^{-2}d_2 + \ldots) \det P n^{m+n^2}} e^{-n^2(1-n^{-1}a_1 + \ldots)\xi P^{-1} \xi - (1-n^{-1}a_1 + \ldots)|y|^2 - 2n^{-3} \Re \langle \partial_j \partial_q a_1 + \ldots, \xi, y \rangle},
\end{equation}

where $\langle , \rangle$ is the nature inner product on $\mathbb{C}$.

We substitute this formula into the following integration in Lemma [3]

\begin{equation}
D_n(x) = \frac{1}{n^m \pi^d_m} \int_0^{2\pi} \int_M \int_{S(C^n)} p_2^2(x, \theta, 0, \xi) |\det (\xi \xi^* - n^2Ix^2)| \, x \, d\xi \, dV \, d\theta,
\end{equation}

then we change variables $\xi \to n\xi$ and $y = xe^{i\theta}$ to obtain the exact formula,

\begin{equation}
D_n(x) = \frac{x}{\pi^d_m} \int_0^{2\pi} \int_M \int_{S(C^n)} e^{-n^{-1}a_1 + \ldots} \xi P^{-1} \xi - (1-n^{-1}a_1 + \ldots)x^2 - 2n^{-3} \Re \langle \partial_j \partial_q a_1 + \ldots, \xi, xe^{i\theta} \rangle |\det (\xi \xi^* - Ix^2)| \, d\xi \, d\theta \, dV
\end{equation}

|\det (\xi \xi^* - Ix^2)|d\xi d\theta dV
Put $h = \frac{1}{n}$, then we rewrite the above formula $D_n(h) := D_n(x)$. $D_n(h)$ is analytic with respect to $h$ for fixed $x$. We Taylor expand $D_n(h)$ at $h = 0$ to obtain,

$$D_n(x) = D_{\infty}(x) + O\left(\frac{1}{n}(x(1 + x^4)e^{-x^4})\right).$$

7.4. Riemann surface. On a Riemann surface of area 1, we have

$$D_n(x) = \frac{2}{\pi^2} xe^{-x^2} \int_{\mathcal{C}} e^{-|\xi|^2} \left|2|\xi|^2 - x^2\right| d\xi + O\left(\frac{1}{n}\right),$$

or equivalently

$$D_n(x) = \frac{4}{\pi} xe^{-x^2} \int_0^{\infty} e^{-r} \left|2r - x^2\right| dr + O\left(\frac{1}{n}\right)$$

8. Spherical ensemble: Proof of Theorem 2

In this section, we relate the expected density of critical points of an $L^2$ normalized random $s_n \in SH^0(M, L^n)$ with the spherical Haar measure to the expected density $D_n = D_n^{s_n}$ in the normalized Gaussian ensemble. We begin by relating Gaussian and spherical averages of density-valued random variables in Lemma 5. This relation is valid both for the critical point distribution and the value distribution of $\xi_0$

8.1. Relation between spherical and normalized Gaussian averages of density-valued random variables.

Lemma 5. The spherical density $D_n^S$ and the $\gamma_n$ densities of critical values are related by,

$$D_n^S(x) = \frac{1}{\pi^2} \omega_{2d_n} \int_0^{\infty} D_n^S(\rho^{-\frac{1}{2}} x) e^{-\alpha \rho} \rho^{d_n - \frac{3}{2}} d\rho$$

$$= \frac{1}{\pi^2} \omega_{2d_n} \int_0^{\infty} D_n^S(\rho^{-\frac{1}{2}}) e^{-\alpha \rho^2} \rho^{d_n - \frac{3}{2}} d\rho.$$

Proof. By (20) and Lemma 1 we have

$$E_n^{CV} = \frac{a_n}{\pi^2} \int_{S^0(M, L^n)} \mathcal{CV}_s e^{-\alpha \|s\|^2} Ds$$

$$= \frac{a_n}{\pi^2} \omega_{2d_n} \int_0^{\infty} \int_{S^0(M, L^n)} \mathcal{CV}_s e^{-\alpha r^2} r^{2d_n - 1} dr d\nu_n$$

by definition of the normalization of the Haar spherical form on $SH^0(M, L^n)$. We then change variables $\rho = r^2$.

When $\alpha = d_n$, it follows from Lemma 5 that

$$D_n^S(x) = \frac{1}{\pi^2} \omega_{2d_n} \int_0^{\infty} D_n^S(\rho^{-\frac{1}{2}}) e^{-\rho} \rho^{d_n - \frac{3}{2}} d\rho.$$

Changing variables to $y = x^2$, (81) is equivalent to

$$K_n y^{-d_n - \frac{1}{2}} d_n^* D_n^S(y^2) = \mathcal{L}(\rho^{d_n - \frac{3}{2}} d_n^*(\rho^2)) (d_n y),$$

where $\mathcal{L}$ denotes the Laplace transform and we put

$$K_n = 2\pi^d \omega_{2d_n}^{-\frac{1}{2}}, \quad d_n^* = D_n^S(\rho^{-\frac{1}{2}}).$$
If we change variables \(d_n \rho \to \rho\) in (81) we also get,

\[
K_n y^{-d_n+\frac{3}{2}} d_n^{\frac{1}{2}} D_n^m(y^{\frac{1}{2}}) = \mathcal{L}(\rho^{d_n-\frac{3}{2}} D_{d_n}^m d_S^n(y)).
\]

In the last line, \(D_r f(x) = f(r^{-1} x)\) denotes the dilation operator.

8.2. **Spherical density for SU(2) polynomials.** Before proving Theorem 2 it is helpful to do the calculations first for the case of SU(m + 1) polynomials, where one has explicit formulae for \(D_{m+1} \text{SU}(m+1)\). The formulae are simplest when \(m = 1\) and so we start with this case. We then follow the outline for the general case.

For SU(2) polynomials, where \(d_n = n + 1\), (66) and (81)-(84) imply that

\[
K_n d_n^{-\frac{3}{2}} y^{-n} \left(2y - \frac{n}{n - 1} + 8 \exp\left(-\frac{n - 1}{2n} y\right)\right) e^{-y} = \mathcal{L}(\rho^{n-\frac{3}{2}} D_{d_n}^m d_S^n(y))
\]

We recall that

\[
\rho^{-\nu} e^{-\alpha \rho} = \mathcal{L}_1[a,\infty] \frac{(x - a)^{\nu-1}}{\Gamma(\nu)}.
\]

Hence the left side of (85) is \(K_n d_n^{\frac{3}{2}}\) times

\[
\mathcal{L} \left(21_{[1,\infty]}(\rho) \left(\frac{\rho - 1}{n - 1} - 4 \frac{n}{n - 1} \frac{(\rho - 1)^{n-1}}{\Gamma(n)}\right) + 81_{[2n-1,\infty]}(\rho) \left(\frac{3n-1}{2n} \frac{(\rho - 1)^{n-1}}{\Gamma(n)}\right)\right).
\]

It follows that \(D_{2n}^m \text{SU}(2)(\sqrt{d_n} \rho^{-\frac{3}{2}}) = D_{d_n}^m d_S^n\) equals

\[
K_n \rho^{-n+\frac{3}{2}} d_n^{\frac{3}{2}} \left(21_{[1,\infty]}(\rho) \left(\frac{\rho - 1}{n - 1} - 4 \frac{n}{n - 1} \frac{(\rho - 1)^{n-1}}{\Gamma(n)}\right) + 81_{[2n-1,\infty]}(\rho) \left(\frac{3n-1}{2n} \frac{(\rho - 1)^{n-1}}{\Gamma(n)}\right)\right)
\]

We then put \(u = \sqrt{d_n} \rho^{-\frac{3}{2}}\) to get

\[
D_{2n}^m \text{SU}(2)(u) = K_n d_n^{-1} u \left(21_{[0,\infty]}(u) \left[u^{2n-2} \frac{n-1}{2n-1} (\frac{1}{n-1})^{n-1} - 4 \frac{n}{n - 1} (\frac{1}{n})^{n-1}\right] + 81_{[0,\infty]}(u) \left(\frac{1}{n} \frac{1}{n-1}\right)\right),
\]

where \(b_n = \frac{2n(n+1)}{3n-1}\). Recalling that \(d_n = n + 1\), we conclude that

\[
D_{2n}^m \text{SU}(2)(u) = K_n d_n^{-1} u \left(21_{[0,\infty]}(u) \left[u^{2n} \frac{n-1}{2n} \frac{n-1}{2n} (\frac{1}{n-1})^{n-1} - 4 \frac{n}{n - 1} (\frac{1}{n})^{n-1}\right] + 81_{[0,\infty]}(u) \left(\frac{1}{n} \frac{1}{n-1}\right)\right).
\]

We further recall from (26) that \(\omega_{2(n+1)} = \frac{n+1}{2(n+1)}\) and combine \((n + 1)^{-2} \frac{\Gamma(n - 1)^{-1}}{\Gamma(n + 1)^{-1}}\) (resp. \((n + 1)^{-2} \frac{\Gamma(n - 1)}{\Gamma(n + 1)}\) to reach the final:

**Corollary 2.** The spherical density for SU(2) polynomials of degree \(n\) is given exactly by

\[
D_{2n}^m \text{SU}(2)(u) = u \left(2u^{2} \frac{n-1}{2n} \frac{n-1}{2n} (\frac{1}{n-1})^{n-1} - 4 \frac{n}{n - 1} (\frac{1}{n})^{n-1}\right) + 81_{[0,\infty]}(u) \left(\frac{1}{n} \frac{1}{n-1}\right).
\]

It follows that

\[
\lim_{n \to \infty} D_{2n}^m \text{SU}(2)(u) = u \left(2u^{2} - 4 + 8e^{-\frac{3}{2}u^{2}}\right) e^{-u^{2}},
\]

proving Theorem 2 in the SU(2) case.
8.3. Spherical density for $SU(m+1)$ polynomials. We now extend the proof to $SU(m+1)$ polynomials for general $m$. We do not simplify the integral for $f_m$ in that case but still may use the explicit $n$-dependence to confirm the main result.

Let $\beta_m = \dim S(C^m)$, and let $S_1(C^m)$ denote the unit sphere in the space of complex symmetric matrices with respect to the usual inner product $\text{Tr} A^* A$.

By Proposition 1 and since the determinant is homogeneous of order $2\beta_m = \dim S(C^m)$, we have

\begin{align*}
D_n^{SU(m+1)}(y^{\frac{1}{2}}) &= c_m y^{\frac{1}{2}} e^{-y} \int_{S(C^n)} e^{-|\xi|^2} \det \left( \frac{n-1}{n} |\sqrt{F} \xi|^2 - y \right) d\xi \\
&= c_m y^{\frac{1}{2}} e^{-y} \int_0^\infty \int_{S_1} e^{-r^2} \det \left( \frac{n-1}{n} |\sqrt{F} r_\omega|^2 - y \right) r^{2\beta_m-1} dr d\omega \\
&= c_n y^{\frac{1}{2}} y^{3\beta_m} e^{-y} \int_0^\infty \int_{S_1} e^{-\rho y} \det \left( \frac{n-1}{n} |\sqrt{F} \omega|^2 - \rho^{-1} \right) \rho^{3\beta_m-1} d\rho d\omega \\
&= c_n y^{\frac{1}{2}} y^{3\beta_m} e^{-y} \int_0^\infty \int_{S_1} e^{-\rho y} \det \left( \frac{n-1}{n} |\sqrt{F} \omega|^2 - \rho^{-1} \right) \rho^{3\beta_m-1} d\rho d\omega
\end{align*}

where $F_n^{\text{CP}^m}(\rho) : = \int_{S_1} \det \left( \frac{n-1}{n} |\sqrt{F} \omega|^2 - \rho^{-1} \right) d\omega$

We further set $a_m = 3\beta_m$ and recall (83). It follows from (84) that (with $d_3^m(\rho) := D_n^{SU(m+1)}(\rho^{\frac{1}{2}})$)

\begin{equation}
K_n \frac{c_m}{2} d_n^{-\frac{1}{2}} y^{-d_n+1} y^{a_m} e^{-y} \mathcal{L}(\rho^{a_m-1} F_n^{\text{CP}^m})(y) = \mathcal{L}(\rho^{d_n-\frac{1}{2}} D_{d_n^{-1}} d_3^m(y))
\end{equation}

We now simplify the left side using further identities for the Laplace transform. Denote the translation operator by $\tau_a f(t) = f(t-a)$ and the Heaviside step function by $H(t) = I^0_+$. Also, denote the $\nu$-fold primitive (fractional integral) by

\begin{equation}
I^\nu f(x) = \int_0^x \frac{(x-t)^{\nu-1}}{\Gamma(\nu)} f(t) dt.
\end{equation}

We have (see e.g. [W] Theorem 8.1),

\begin{equation}
\begin{cases}
\mathcal{L}(H(t-a)f(t-a)) = e^{-as} \mathcal{L} f(s) \\
\mathcal{L} I^\nu f = s^{-\nu} \mathcal{L} f,
\end{cases}
\end{equation}

We use the identities to simplify the left side of (88):

\begin{equation}
y^{-d_n+a_m+1} \mathcal{L}(\rho^{a_m-1} F_n^{\text{CP}^m})(y) e^{-y} = y^{-d_n+a_m+1} \mathcal{L} \tau_1(H(\rho)\rho^{a_m-1} F_n^{\text{CP}^m})(y) \\
= \mathcal{L}(I^{d_n-1-a_m} \tau_1(H(\rho)\rho^{a_m-1} F_n^{\text{CP}^m})(y).
\end{equation}

Combining (88) and (90) and uniqueness of the Laplace transform gives

\begin{equation}
K_n \frac{c_m}{2} d_n^{-2-a_m} d_3^m \tau_1(H(\rho)\rho^{a_m-1} F_n^{\text{CP}^m})(\rho) = \rho^{d_n-\frac{1}{2}} D_{d_n^{-1}} d_3^m(\rho)
\end{equation}
We have,
\[
I^{d_n - 1 - a_m} \tau_1 (H(\rho) \rho^{a_m - 1} F^{CP^n}_m (\rho))(\rho) = \int_0^\rho \frac{(\rho - t)^{d_n - a_m - 1}}{\Gamma(d_n - a_m)} (H(t - 1)(t - 1)^{a_m - 1} F^{CP^n}_m (t - 1)) dt
\]
\[
= \int_0^1 \frac{(\rho - t)^{d_n - a_m - 2}}{\Gamma(d_n - a_m)} ((t - 1)^{a_m - 1} F^{CP^n}_m (t - 1)) dt
\]
\[
= \int_0^\rho \frac{(\rho - t)^{d_n - a_m - 2}}{\Gamma(d_n - a_m)} (t^{a_m - 1} F^{CP^n}_m (t)) dt.
\]

Therefore,
\[
D^{SU(m+1)}_n \left( d_n^2 \rho^{-\frac{1}{2}} \right) = K_n \frac{\epsilon_m}{\Gamma(d_n - 1 - a_m)} \rho^{-d_n + \frac{1}{2}} d_n^{-\frac{1}{2}} \int_0^{\rho - 1} \frac{(\rho - t)^{d_n - a_m - 2}}{\Gamma(d_n - a_m)} (t^{a_m - 1} F^{CP^n}_m (t)) dt
\]
or equivalently with \( u = d_n^2 \rho^{-\frac{1}{2}}, \rho = u^{-2} d_n \),
\[
D^{SU(m+1)}_n (u) = K_n \frac{\epsilon_m}{\Gamma(d_n - 1 - a_m)} d_n^{-\frac{1}{2}} (d_n u^{-2})^{-a_m - \frac{1}{2}} \int_0^{d_n u^{-2} - 1} (1 - \frac{(t+1)u^2}{d_n}) d_n^{-a_m - 2} (t^{a_m - 1} F^{CP^n}_m (t)) dt
\]

Substituting the definition of \( F^{CP^n}_m (t) \), we observe that
\[
\lim_{n \to \infty} \int_{S^1} \int_0^1 \frac{(\rho - t)^{d_n - a_m - 2}}{\Gamma(d_n - a_m)} \left| \det \left( \frac{a_n - 2}{\rho} \sqrt{P \omega} \right) \right| d\omega = e^{-u^2} \int_0^\infty e^{-tu^2} t^{a_m - 1} \left( \int_{S^1} \left| \det \left( \sqrt{P \omega} \right) \right| d\omega \right) dt.
\]

Also,
\[
\Gamma(d_n - 1 - a_m) d_n^{a_m + 1} = \frac{d_n^{a_m + 1}}{(d_n - 1) \cdots (d_n - 1 - a_m)} \Gamma(d_n) \approx \Gamma(d_n),
\]
hence
\[
\frac{K_n}{\Gamma(d_n - 1 - a_m)} d_n^{a_m - 1} \approx 1
\]
Reversing the steps in (87) and comparing with (77), it follows that as \( n \to \infty \),
\[
D^{SU(m+1)}_n (u) \approx \frac{c_m}{2} u^{2a_m + 1} e^{-u^2} \int_0^\infty e^{-t^2} \rho^{a_m - 1} \left( \int_{S^1} \left| \det \left( \sqrt{P \omega} \right) \right| d\omega \right) d\rho
\]
\[
\approx D_\infty (u).
\]

8.4. **Proof of Theorem 2** We now go through the general case, closely following the calculations in the special case of \( CP^m \).

By Theorem 1,
\[
D_n (y^{\frac{1}{2}}) = D_\infty (y^{\frac{1}{2}}) + O\left( \frac{y^k e^{-y}}{n} \right), \text{ with } D_\infty (y^{\frac{1}{2}}) = f_m (y^{\frac{1}{2}}) y^{\frac{1}{2}} e^{-y}.
\]
In determining the limit of \( D_n^2 \) it follows from (84) and (78) and the remainder estimate that we may drop the remainder term and then the calculation becomes almost identical to that of \( SU(m+1) \) polynomials, with \( F^{CP^n}_m (\rho) \) replaced by
\[
F_m (\rho) = \int_{S^1 (C^m)} \left| \sqrt{P \omega} \right|^2 - \rho^{-1} \right| d\omega.
\]
As in [87], we have

\[ f_m(\sqrt{y}) = \frac{c_m}{2} y^{a_m+\frac{1}{2}} e^{-y} \mathcal{L}(\rho^{a_m-1} F_m)(y). \]  

By [83] [94] (with \( d^S_n(\rho) = D^S_n(\rho^{-\frac{1}{2}}) \)) we have

\[ K_n \frac{c_m}{2} d_{n+1} y^{1+a_m-d_n} (\mathcal{L} \rho^{a_m-1} F_m)(y) e^{-y} = \mathcal{L}(\rho^{d_n-\frac{3}{2}} D_{d_n} d^S_n)(y). \]

Hence

\[ \frac{c_m}{2} K_n I_{d_n-2-a_m} H(\rho) \rho^{a_m} F_m(\rho) = \rho^{d_n-\frac{3}{2}} D_{d_n} d^S_n(\rho). \]

Repeating the calculation in the SU\((m)\) case and setting \( u = \sqrt{d_n} \rho^{-\frac{1}{2}} \) gives

\[ D^S_n(u) = (d_n u^{-2})^{\frac{1}{2}-d_n} \left( \frac{c_m}{2} K_n \right) \int_0^{d_n u^{-2}-1} (d_n u^{-2}-1+t)^{d_n-a_m-2} e^{t a_m-1} F_m(t) dt \]

\[ = \frac{c_m}{2} K_n \Gamma(d_n-1-a_m) u^{2a_m+2} \int_0^{d_n u^{-2}-1} (1-\frac{(1+t)u^2}{d_n})^{d_n-a_m-2} e^{t a_m-1} F_m(t) dt. \]

We note that

\[ 1_{[0,d_n u^{-2}-1]}(1-\frac{(1+t)u^2}{d_n})^{d_n-a_m-2} e^{t a_m-1} F_m(t) \rightarrow \exp(-(1+t)u^2) e^{t a_m-1} F_m(t) \]

monotonically as \( n \rightarrow \infty \) and so

\[ \lim_{n \rightarrow \infty} \int_0^{d_n u^{-2}-1} (1-\frac{(1+t)u^2}{d_n})^{d_n-a_m-2} e^{t a_m-1} F_m(t) dt = e^{-u^2} \int_0^{\infty} \exp(-\rho u^2) \rho^{a_m-1} F_m(\rho) dt. \]

The rest of the calculation is the same as for \( \mathbb{C}P^m \) and we get

\[ \lim_{n \rightarrow \infty} D^S_n(u) = D_\infty(u). \]

9. **Value distribution**

As a check on the results for the critical value distribution, we give the analogous calculation in this section of the much simpler expected value distribution, which is a probability measure on \( \mathbb{R}_+ \). By the value distribution of (the modulus of) a section \( s_n \in H^0(M, L^n) \) we mean the probability measure \( \mu_{s_n} \) on \( \mathbb{R}_+ \) defined on a test function \( \psi \in C(\mathbb{R}_+) \) by

\[ \mu_{s_n} = (|s_n| h^n)_* dV, \quad \langle \psi, (|s_n| h^n)_* dV \rangle = \int_M \psi(|s_n| h^n) dV. \]

As is well-known, the distribution is minus the derivative of the volume function,

\[ V_{s_n}(\lambda) := Vol\{ z \in M : |s_n(z)| h^n > \lambda \}. \]

Fix \( z \in M \) and define the random variable \( \rho_n^z(x) = |f_n(z)| e^{-n\varphi(z)/2} = |s_n(z)| h^n \). For a test function \( \psi \in C(\mathbb{R}_+) \) we have (for any probability measure on sections)

\[ E_n \psi(\rho_n^z) = \int_0^\infty \psi(x) f_n^z(x) dx \]

where \( f_n^z(x) \) is the pointwise density of the distribution of \( \rho_n^z \).

By the expected value distribution in the Gaussian ensemble we mean the measures

\[ \mu_n := E_n \mu_{s_n} \]
which is defined in the sense of distribution,

\[(102) \quad \langle \psi, E_n(|s_n| h_n) \rangle dV = \int_M E_n \psi(|s_n(z)| h_n) dV = \int_M E_n \psi(\rho_n^z) dV.\]

To determine the expected value distribution it thus suffices determine \(E_n \psi(|s_n(z)| h_n)\) or equivalently \(E_n V_{s_n}(\lambda)\). We note that \(\mu_n\) is always an absolutely continuous measure on \(\mathbb{R}^+\) since \(s_n\) is analytic. Hence we may write it as \(f_n(x) dx\).

We also define the expected value distribution in the spherical ensemble,

\[(103) \quad \mu_n^S = E_n^S \mu_n, \quad s_n \in SH^0(M, L^n).\]

It is also a continuous measure and we write \(\mu_n^S = f_n^S(x) dx\).

The random variable \(\rho_n^z\) is the modulus of the Gaussian random variable \(p_n^z = f_n^z(z)e^{-n\psi(z)/2}\) which is only defined with a choice of local coordinates. But if \(\psi\) is a radial test function on \(\mathbb{C}\), then

\[E_n \psi(p_n^z) = E_n \psi(\rho_n^z),\]

and the right side is globally well defined. We denote the expected density of \(p_n^z\) by \(\hat{f}_n^z(y)\), which is a density defined on \(\mathbb{C}\). We write it as \(\hat{f}_n^z(x, \theta)\) in polar coordinate \(y = xe^{i\theta}\).

**Lemma 6.** For any probability measure on sections, the expected density of \(f_n^z\) is given by the formula,

\[(104) \quad f_n^z(x) = \int_0^{2\pi} \hat{f}_n^z(x, \theta) x d\theta.\]

**9.1. Kac-Rice formula for the Gaussian value density.** First we compare Gaussian value densities as \(\alpha\) changes. We denote \(f_n^\alpha\) is the expected density under the Gaussian ensemble \([4]\). The result is analogous to that of Lemma [2].

**Lemma 7.** \(f_n^\alpha(x) = \alpha^{1/2} f_n^1(\alpha^{1/2} x)\).

Hence it suffices to fix \(\alpha = 1\), and in the next Proposition we determine the Gaussian value distribution when \(\alpha = 1\) in \([4]\) \([5]\). If we define the expected density of values,

\[(105) \quad E_n^1 \left( \int_M \psi(|s_n| h_n) dV \right) = \int_0^{\infty} \psi(x) f_n^1(x) dx.\]

Then we can prove,

**Proposition 2.** The expected value density in the \(\alpha = 1\) ensemble is given by the formula,

\[(106) \quad f_n^1(x) = \frac{1}{\pi} \int_M \frac{1}{\Pi_n(z, z)} e^{-\Pi_n(z, z)^{-1} x^2} dV(z), \quad x \in \mathbb{R}^+\]

**Proof.** It follows from the Kac-Rice formula that

\[(107) \quad \hat{f}_n^1(y) = \frac{1}{\pi} \frac{1}{\Pi_n(z, z)} \exp \left\{ -\langle y, \Pi_n(z, z)^{-1} y \rangle \right\},\]

since the covariance matrix of the random variable \(p_n^z\) is

\[E_n^1(f_n^z \hat{f}_n^z) = E_n^1 |s(z)|^2 h_n = \Pi_n(z, z).\]
To obtain the density of values, we apply Lemma [6] and the polar coordinate $y = xe^{i\theta}$ to (107), then integrate the pointwise densities over $M$. □

The density is especially simple for $SU(m+1)$ polynomials where $\Pi_n(z, z)$ is the constant $d_n$ where $V_m$ is the Fubini-Study volume of $\mathbb{CP}^m$.

**Corollary 3.** For $SU(m + 1)$ polynomials with $\alpha = 1$ resp. $d_n$ and with volume $V_m$, the expected density of critical values is given by

\begin{align*}
(108) \quad f_{1, \mathbb{CP}^m}^1(x) = \frac{2V_m}{d_n} xe^{-V_m d_n^{-1} x^2}, \quad f_{d_n, \mathbb{CP}^m}^d = 2V_m xe^{-V_m x^2}.
\end{align*}

As mentioned in the introduction, we usually set $V_m = 1$ for simplicity of notation.

9.2. Relation between spherical and Gaussian densities. The Gaussian value distribution has the same problem as the critical point distribution, namely the weighted repetition of sections. If we multiply $s \in H^0(M, L^n)$ by $r > 0$ then the volume function changes by $V_{rs} = V_s(r^{-1} - 1)$. Consequently,

\begin{align*}
(109) \quad f_{rs} = r^{-1} f_s(r^{-1} s).
\end{align*}

The discussion is a word-for-word repetition of §8.1. Indeed, in the calculations we only used the relation (109) and the fact that the random variables take values in the densities on $\mathbb{R}$. We therefore omit the proofs, but for clarity do state the analogous Lemmas.

Exactly as in Lemma [5] we have

**Lemma 8.** The spherical density $f_n^S$ and the $\gamma_n^\alpha$ densities of values are related by,

\begin{align*}
f_n^S(x) = K_n^{-1} \alpha^d_n \int_0^\infty f_n^S(\rho^{-\frac{1}{2}} x) e^{-\alpha \rho \rho^{-\frac{1}{2}} \rho^{d_n-1}} d\rho.
\end{align*}

We combine Lemmas [7] and [8] to obtain

**Corollary 4.**

\begin{align*}
f_n^1(\alpha^\frac{1}{2} x) &= \alpha^{-\frac{1}{2}} \alpha^d_n K_n^{-1} \int_0^\infty f_n^S(\rho^{-\frac{1}{2}} x) e^{-\alpha \rho \rho^{-\frac{1}{2}} \rho^{d_n-1}} d\rho \\
&= \alpha^{d_n - \frac{1}{2}} K_n^{-1} x^{2d_n - 1} \int_0^\infty f_n^S(\rho^{-\frac{1}{2}}) e^{-\alpha x^2 \rho \rho^{d_n-\frac{3}{2}}} d\rho.
\end{align*}

9.3. Spherical density of values: Proof of Theorem [3] In this section we prove Theorem [3]. In this section we prove Theorem [3]. First we prove the statement for $SU(m + 1)$ polynomials.

**Lemma 9.** Let $f_n^{S, \mathbb{CP}^m}(u)$ be the density of values of $SU(m + 1)$ polynomials in the spherical ensemble with volume $V_m = 1$. Then

\begin{align*}
\lim_{n \to \infty} f_n^{S, \mathbb{CP}^m}(u) = 2ue^{-u^2}.
\end{align*}

**Proof.** The spherical density of values is determined from the Corollaries [3] and [4] (with $V_m = 1$) by changing variable $y = ax^2$,

\begin{align*}
2K_n \frac{1}{d_n} y^{-d_n+1} e^{-d_n^{-1} y} = \mathcal{L}(\rho^{d_n^{-\frac{3}{2}}} F_n^S(y)).
\end{align*}

where we put

\begin{align*}
F_n^S(\rho) = f_n^{S, \mathbb{CP}^m}(\rho^{-\frac{1}{2}}).
\end{align*}
By \((86)\) with \(a = d_n^{-1}\) and \(\nu = d_n - 1\), we obtain
\[
L(\rho^{d_n - \frac{3}{2}} F_n^S)(y) = 2K_n \frac{1}{d_n} \mathcal{L}1_{[d_n^{-1}, \infty]} \left( \frac{\rho - d_n^{-1} d_n^{-2}}{\Gamma(d_n - 1)} \right).
\]
Thus,
\[
f_n^{S,CP^m}(\rho^{-\frac{1}{2}}) = 2K_n \rho^{-d_n + \frac{3}{2}} \frac{1}{d_n} \mathcal{L}1_{[d_n^{-1}, \infty]} \left( u^{-2} - d_n^{-1} d_n^{-2} \right).
\]
With \(u = \rho^{-\frac{1}{2}}\) we get
\[
f_n^{S,CP^m}(u) = u^{2d_n - 3} \frac{2K_n}{\Gamma(d_n - 1)} \frac{1}{d_n} \mathcal{L}1_{[d_n^{-1}, \infty]} \left( u^{-2} - d_n^{-1} d_n^{-2} \right) = 2K_n \frac{u}{\Gamma(d_n)} u \mathcal{L}1_{[0, \sqrt{d_n}]} \left( 1 - \frac{u^2}{d_n} \right) d_n^{-2}.
\]
(110)
The rest of the calculation is then identical to the case of \(SU(m+1)\) polynomials.

The Lemma follows.

\[\square\]

9.4. General Kähler manifolds. In the general case, we also have:

**Lemma 10.** Let \(f_n^S(u)\) be the density of values of random \(s \in \text{SH}^0(M, L^n)\) in the spherical ensemble. Then
\[
\lim_{n \to \infty} f_n^S(u) = 2ue^{-u^2}.
\]

**Proof.** We combine Proposition\[2\] and Corollary\[4\]. The exact formula of the Proposition shows that the general spherical density is asymptotic to the \(SU(m+1)\) case in dimension \(m\) with a remainder term of order \(n^{-1}(1 + |x|^4)e^{-|x|^2}\). The remainder estimate is similar to but simpler than that for the critical point density in \[7.3\] and is omitted. It follows that
\[
f_n^S(\rho^{-\frac{1}{2}}) \simeq \rho^{-d_n + \frac{3}{2}} \frac{2K_n}{\Gamma(d_n - 1)} \frac{1}{d_n} \mathcal{L}1_{[d_n^{-1}, \infty]} \left( \frac{\rho - d_n^{-1} d_n^{-2}}{\Gamma(d_n - 1)} \right).
\]
The rest of the calculation is then identical to the case of \(SU(m+1)\) polynomials.

\[\square\]

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