STABILITY OF AN ELECTROWEAK STRING WITH A FERMION CONDENSATE

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ABSTRACT

A solution of the standard electroweak theory with a single lepton family is constructed, consisting of a cosmic string and a fermion condensate within its core. The stability of this system to small perturbations is examined, and it is found that stability is not enhanced relative to the bare electroweak string. The presence of quark zero modes is shown to violate the existence criteria for embedded defects.
1. INTRODUCTION

The electroweak phase transition has attracted a great deal of attention in recent years as a promising epoch for the generation of the primordial baryon asymmetry\(^1\). One mechanism examined in detail in the literature is that of baryon generation at the walls of bubbles of false vacuum created during a first order phase transition\(^2\). Another proposal, which does not depend on the electroweak phase transition being first order is that of the shrinking and eventual collapse of strands of cosmic string\(^3\). A common thread linking both mechanisms is the requirement of a boundary between false and true vacua moving in a definite direction.

It is well known that the Nielsen-Olesen vortex solution\(^4\) of the abelian Higgs model may be embedded in the Weinberg-Salam electroweak theory; this new defect is a vortex of Z-particles, and is known as a Z-string\(^5,6\). To produce a sufficiently large baryon asymmetry from collapsing Z-strings, it is necessary to have a network of these strings formed during the electroweak phase transition. The detailed structure of such a network is difficult to analyse, but an important factor in determining the length of string formed is the stability or otherwise of such embedded strings. In contrast with vortices in the abelian Higgs model, Z-strings are not topologically stable, and the stability question becomes a dynamical one.

If one’s analysis is restricted to the bosonic sector of the electroweak theory, the Z-string is found to be unstable in the physical region of parameter space\(^7,8\). Adding a second Higgs doublet to the model does not extend the region of parameter space admitting stable strings sufficiently to reach the physical values\(^9\). On replacing the leptonic sector of the theory, it is seen that leptonic zero modes may be present along the string. A fermion whose mass arises from a Yukawa coupling will be massless within the string core and massive without. It has been suggested that, since the collapse of the string will cause any fermion condensate within to become massive, the existence of such a condensate will extend the region of parameter space admitting stable strings\(^10\). In the following work, this conjecture is examined in a concrete model: the standard electroweak theory with a single lepton family.
In section two, the structure of the Z-string and fermion condensate is explicitly described. In section three this configuration is perturbed, and the change in energy studied to second order in the perturbations. The case of quark zero modes is discussed in section four and the results are briefly summarised in section five.

2. THE Z-STRING AND FERMION ZERO MODE

The model under investigation is the Weinberg-Salam electroweak theory with a single lepton family, namely

\[
\mathcal{L} = -\frac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \Phi)^\dagger (D^\mu \Phi) - \lambda (\Phi^\dagger \Phi - \eta^2)^2 - i\bar{\Psi} \gamma^\mu D_\mu \Psi - i\bar{e}_R \gamma^\mu D_\mu e_R + h (\bar{e}_R \Phi^\dagger \Psi + \Psi \Phi e_R),
\]

where

\[
D_\mu \Psi = \left( \partial_\mu - ig^2/2 \tau^a W^a_\mu + ig'/2 B_\mu \right) \Psi,
\]

\[
D_\mu e_R = \left( \partial_\mu + ig' B_\mu \right) e_R,
\]

\[
D_\mu \Phi = \left( \partial_\mu - ig^2/2 \tau^a W^a_\mu - ig'/2 B_\mu \right) \Phi,
\]

\[
W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + ge^{abc} W^b_\mu W^c_\nu, F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \text{ and } \tau^a \text{ are the Pauli matrices.}
\]

The field equations arising from this Lagrangian are

\[
\partial_\nu F^{\mu\nu} = \frac{i g'}{2} \left( \Phi^\dagger (D^\mu \Phi) - (D^\mu \Phi)^\dagger \Phi \right) + \frac{g'}{2} \left( \bar{\Psi} \gamma^\mu \Psi + 2\bar{e}_R \gamma^\mu e_R \right),
\]

\[
(\delta^{ab} \partial_\nu + ge^{abc} W^c_\nu) W^{a\mu\nu} = \frac{i g}{2} \left( \Phi^\dagger \tau^b (D^\mu \Phi) - (D^\mu \Phi)^\dagger \tau^b \Phi \right) - \frac{g}{2} \bar{\Psi} \gamma^\mu \tau^b \Psi,
\]

\[
D^\mu D_\mu \Phi + 2\lambda (\Phi^\dagger \Phi - \eta^2) \Phi = h\bar{e}_R \Psi,
\]

\[
i\gamma^\mu D_\mu \Psi = h\Phi e_R,
\]

\[
i\gamma^\mu D_\mu e_R = h\Phi^\dagger \Psi.
\]

A Z-string ansatz of the form \( W^1_\mu = W^2_\mu = 0, \Phi = (0_\phi), \Psi = (0_{e_L}) \) satisfies the field equations for \( W^1_\mu \) and \( W^2_\mu \) and the remaining equations become

\[
-\Box A_\mu + \partial_\mu (\partial A) = q \sin 2\Theta_W \bar{e}_e \gamma_\mu e,
\]

\[
-\Box Z_\mu + \partial_\mu (\partial Z) = -iq \left( \phi^\dagger (D_\mu \phi) - (D_\mu \phi)^\dagger \phi \right) + q \cos 2\Theta_W \bar{e}_e \gamma_\mu e e_L - 2q \sin^2 \Theta_W \bar{e}_e \gamma_\mu e e_R,
\]
\[ D^\mu D_\mu \phi + 2\lambda \left( \phi^\dagger \phi - \eta^2 \right) \phi = h\tilde{e}_R e_L, \]

where \( q = \frac{1}{2} \alpha, \ Z_\mu = \cos \Theta_W W^3_\mu - \sin \Theta_W B_\mu, \ A_\mu = \sin \Theta_W W^3_\mu + \cos \Theta_W B_\mu, \ g' = \alpha \sin \Theta_W, \ g = \alpha \cos \Theta_W \) and \( D_\mu \phi = (\partial_\mu - iqZ_\mu)\phi \). Working in cylindrical polar coordinates \((r, \theta, z)\) and writing \( A^\mu = 0, \)

\[ qZ^\mu = \left( 0, -\frac{v(r)}{r}\bar{e}_\theta \right), \quad \phi = \eta f(r)e^{i\theta}, \]

leads to the slightly modified vortex profile equations

\[ -v'' + \frac{1}{r}v' + 2q^2\eta^2 f^2(v - 1) = rq^2(\cos 2\Theta_W \bar{e}_L \gamma^0 e_L - 2\sin^2 \Theta_W \bar{e}_R \gamma^0 e_R), \]

\[ -f'' - \frac{1}{r}f' + \frac{1}{r^2}f(1 - v)^2 + 2\lambda\eta^2 f(f^2 - 1) = \frac{g}{\eta}e^{-i\theta} \bar{e}_R e_L, \]

where the Dirac matrices are

\[ \gamma^r = \begin{pmatrix} 0 & e^{-i\theta} & 0 & 0 \\ -e^{i\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & -e^{-i\theta} \\ 0 & 0 & e^{i\theta} & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & -ie^{-i\theta} & 0 & 0 \\ -ie^{i\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & ie^{-i\theta} \\ 0 & 0 & ie^{i\theta} & 0 \end{pmatrix}, \]

\[ \gamma^0 = \begin{pmatrix} \tau^3 & 0 \\ 0 & -\tau^3 \end{pmatrix}, \quad \gamma^\pm = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

The boundary conditions on \( f \) and \( v \) are

\[ f(0) = v(0) = 0, \quad f(\infty) = v(\infty) = 1. \]  

The four-component Dirac spinor \( e \) is split into the two parts \( e_L \) and \( e_R \). Writing \( e_L^T = (a, b, -a, -b), \ e_R^T = (c, d, c, d) \) leads to Dirac equations which may be solved using the ansatz

\[ a = \psi_1(r)e^{ikz - i\omega t + in\theta}, \]

\[ b = i\psi_2(r)e^{ikz - i\omega t + (n+1)\theta}, \]

\[ c = \psi_3(r)e^{ikz - i\omega t + (n-1)\theta}, \]

\[ d = i\psi_4(r)e^{ikz - i\omega t + in\theta}, \]

where \( \psi_1, \psi_2, \psi_3, \psi_4 \) are real. This ansatz results in the coupled ordinary differential equations

\[ (\omega + k)\psi_1 - \psi_2' - \frac{1}{r}(n + 1 + \cos 2\Theta_W v)\psi_2 = hnf\psi_3, \]

\[ (k - \omega)\psi_2 - \psi_1' + \frac{1}{r}(n + \cos 2\Theta_W v)\psi_1 = hnf\psi_4, \]

\[ (\omega - k)\psi_3 - \psi_4' + \frac{1}{r}(n - 2\sin^2 \Theta_W v)\psi_4 = hnf\psi_1, \]

\[ -(\omega + k)\psi_4 - \psi_3' + \frac{1}{r}(n - 1 - 2\sin^2 \Theta_W v)\psi_3 = hnf\psi_2. \]
We are interested in finding zero energy condensates, that is solutions to equations (15) with \( \omega = k = 0 \) such that all the spinor components decay exponentially at large \( r \) and remain bounded in the string core. On setting \( \omega = k = 0 \), the equations decouple into two pairs, namely

\[
\begin{align*}
\psi_1' - \frac{1}{r}(n + \cos 2\Theta_W v)\psi_1 &= -h\eta f \psi_4, \\
\psi_4' + \frac{1}{r}(n - 2\sin^2 \Theta_W v)\psi_4 &= -h\eta f \psi_1,
\end{align*}
\]

and

\[
\begin{align*}
\psi_2' + \frac{1}{r}(n + 1 + \cos 2\Theta_W v)\psi_2 &= -h\eta f \psi_3, \\
\psi_3' - \frac{1}{r}(n - 1 - 2\sin^2 \Theta_W v)\psi_3 &= -h\eta f \psi_2.
\end{align*}
\]

To match the exponentially decaying large \( r \) solution in a given pair with solution at the origin it is required that both solutions at small \( r \) are non-singular\(^{[11]} \). If the right hand sides of equations (11) and (12) are both zero, then \( f \) and \( v \) are the familiar Nielsen-Olesen profiles, with the limiting behaviour \( f \propto r \) and \( v \propto r^2 \) for small \( r \). Assuming this limiting behaviour remains the same, it is found that, whatever value of \( n \) is chosen, the pair of equations (17) always has at least one singular solution for small \( r \), hence we must set \( \psi_2 = \psi_3 = 0 \) to avoid exponentially large profiles at large \( r \) or singular behaviour at the origin. For the pair of equations (16), both solutions for small \( r \) are non-singular iff \( n = 0 \), yielding a single leptonic zero energy condensate of the form

\[
e_L = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \psi_1(r), \quad e_R = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} i\psi_4(r),
\]

where

\[
\begin{align*}
\psi_1' - \frac{1}{r} \cos 2\Theta_W v \psi_1 &= -h\eta f \psi_4, \\
\psi_4' - 2\frac{2}{r} \sin^2 \Theta_W v \psi_4 &= -h\eta f \psi_1.
\end{align*}
\]

With this form of the condensate, equations (11) and (12) reduce to

\[
\begin{align*}
-v'' + \frac{1}{r}v' + 2q^2\eta^2 f^2(v - 1) &= 0, \\
-f'' - \frac{1}{r}f' + \frac{1}{r^2} f(1 - v)^2 + 2\lambda\eta^2 f(f^2 - 1) &= 0.
\end{align*}
\]
thus the Higgs field and gauge field profiles are identical to those for the usual Z-string, and the earlier assumptions about their limiting behaviour for small $r$ are valid.

Multiplying the spinors given in equation (18) by $e^{-i\omega(t+z)}$ for any value of $\omega$ gives a zero mode moving along the string which is still a solution of equations (15).

It should be noted that all the field equations are satisfied, except those for $A_0$, $A_z$, $Z_0$ and $Z_z$, so the simple ansatz above only yields an approximate solution. A reduction of the problem to two space dimensions would remove two of these inconsistencies, still leaving problems with $A_0$ and $Z_0$, but would spoil the decomposition of the electron field into left and right handed components, which relies on there being exactly three spatial dimensions.

3. STABILITY OF THE Z-STRING WITH A LEPTON CONDENSATE

Since the vacuum manifold of the bosonic electroweak theory is $S^3$, cosmic string configurations are not topologically stable, but if the string solution is meta-stable, that is a local minimum of energy in configuration space, then such strings may have significant cosmological consequences. It has been shown that the Z-string is meta-stable in a region of parameter space near $\Theta_W = \pi/2$, but not at physical values of the parameters [7]. In this section, the configuration described in section two is perturbed infinitesimally, and the quadratic change in the energy of the system is examined to discover whether or not the fermion condensate enhances the stability properties.

The energy of a static configuration in the model under investigation is

$$E = \int d^3x \left\{ \frac{1}{4} W_{ij} W_{ij}^a + \frac{1}{4} F_{ij} F_{ij} + (D^i \Phi)\dagger (D^i \Phi) + \lambda (\Phi\dagger \Phi - \eta^2)^2 ight\} - i \bar{\Psi} \gamma^i D_i \Psi - i \bar{e}_R \gamma^i D^i e_R - h(\bar{e}_R \Phi\dagger \Psi + \bar{\Psi} \Phi e_R) \right\}.$$  \hspace{1cm} (22)

For simplicity, perturbations will be restricted to be independent of $z$, and the $z$-components of the gauge field perturbations will be set to zero, thus allowing a study of energy per unit length. The main conclusion concerning stability of the system is unaffected by this restriction, as will be explained later. Writing

$$\Phi = \begin{pmatrix} \chi \\ \phi + \delta \phi \end{pmatrix}, \quad \Psi = \begin{pmatrix} \delta \nu_L \\ e_L + \delta e_L \end{pmatrix},$$  \hspace{1cm} (23)
and replacing \( \vec{Z} \) by \( \vec{Z} + \delta \vec{Z}, \vec{W}^{1.2} \) by \( \delta \vec{W}^{1.2}, \vec{A} \) by \( \delta \vec{A} \), and \( e_R \) by \( e_R + \delta e_R \), where \( \vec{Z} \), \( \phi \), \( e_L \), and \( e_R \) are given by expressions (10) and (18), the change in energy per unit length to quadratic order in the perturbations is found to be of the form

\[
\delta E = \delta E_1(\delta \vec{W}^{1},\delta \vec{W}^{2},\chi,\nu_L) + \delta E_2(\delta \phi,\delta \vec{Z},\delta \vec{A},\delta e_L,\delta e_R)
\]

where

\[
\delta E_1 = \delta E_1^B(\delta \vec{W}^{1},\delta \vec{W}^{2},\chi) + \delta E_1^F(\delta \vec{W}^{1},\delta \vec{W}^{2},\chi,\nu_L),
\]

\[
\delta E_2 = \delta E_2^B(\delta \phi,\delta \vec{Z},\delta \vec{A}) + \delta E_2^F(\delta \phi,\delta \vec{Z},\delta \vec{A},\delta e_L,\delta e_R).
\]

Detailed expressions for \( \delta E_1^B \), \( \delta E_1^F \), \( \delta E_2^B \) and \( \delta E_2^F \) are

\[
\delta E_1^B = \int d^2x \left\{ \frac{1}{2} |\nabla \times \delta \vec{W} + g \cos \Theta_W \vec{Z} \times \delta \vec{W}^2|^2 + \frac{1}{2} |\nabla \times \delta \vec{W}^2 - g \cos \Theta_W \vec{Z} \times \delta \vec{W}^{1.2} |^2 
\right.
\]

\[
- g \cos \Theta_W \delta \vec{W}^{1} \times \delta \vec{W}^{2} \cdot \nabla \times \vec{Z} + \frac{1}{4} g^2 \phi^\dagger \phi \delta \vec{W}^{\dagger} \cdot \delta \vec{W} 
\]

\[
+ |\nabla \chi|^2 + 2 \lambda (\phi^\dagger \phi - \eta^2) \chi \chi^\dagger + \frac{1}{2} \alpha^2 \cos^2 2 \Theta_W \vec{Z}^2 \chi \chi^\dagger 
\]

\[
- \frac{1}{4} g \alpha \sin^2 \Theta_W \vec{Z} (\delta \vec{W} \phi \phi^\dagger + \delta \vec{W}^\dagger \phi^\dagger \phi) + \frac{i}{2} \alpha \cos 2 \Theta_W \vec{Z} (\chi \nabla \chi^\dagger - \chi^\dagger \nabla \chi) 
\]

\[
+ \frac{i}{2} g \left( \delta \vec{W} (\phi \nabla \chi^\dagger - \chi^\dagger \nabla \phi) + \delta \vec{W}^\dagger (\chi \nabla \phi^\dagger - \phi^\dagger \nabla \chi) \right) \right\},
\]

\[
\delta E_1^F = \int d^2x \left\{ i \dot{\nu}_L \gamma \nabla \nu_L + \frac{1}{2} \alpha \dot{\nu}_L \gamma \vec{Z} \delta \nu_L 
\right.
\]

\[
- \frac{1}{2} g (\dot{\nu}_L \gamma \dot{\vec{W}} e_L + \dot{e}_L \gamma \delta \vec{W} e_L) - h(\dot{\nu}_L \chi e_R + \dot{e}_R \chi^\dagger \delta \nu_L) \right\}
\]

and

\[
\delta E_2^B = \int d^2x \left\{ \frac{1}{2} |\nabla \delta \vec{Z}|^2 + \frac{1}{4} \alpha^2 \phi^\dagger \phi \delta \vec{Z}^2 + \frac{1}{2} |\nabla \delta \vec{A}|^2 
\right.
\]

\[
+ |\nabla \delta \phi|^2 + 2 \lambda (\phi^\dagger \phi - \eta^2) \delta \phi \phi + \lambda (\phi^\dagger \delta \phi + \delta \phi \phi^\dagger)^2 
\]

\[
+ \frac{1}{4} \alpha^2 (\vec{Z}^2 |\delta \phi|^2 + 2 \vec{Z} \cdot \delta \vec{Z} (\phi^\dagger \delta \phi + \delta \phi \phi^\dagger)) 
\]

\[
+ \frac{i}{2} \alpha \delta \vec{Z} \cdot (\delta \phi^\dagger \nabla \phi - \delta \phi \nabla \phi^\dagger + \phi^\dagger \nabla \delta \phi - \phi \nabla \delta \phi^\dagger) 
\]

\[
+ \frac{i}{2} \alpha \vec{Z} \cdot (\phi^\dagger \nabla \delta \phi - \delta \phi \nabla \delta \phi^\dagger) \right\},
\]

\[
\delta E_2^F = \int d^2x \left\{ i \dot{\nu}_L \gamma \nabla \nu_L + i \dot{e}_R \gamma \nabla \nu_R + \frac{1}{2} \alpha \cos 2 \Theta_W \delta \nu_L \vec{Z} \delta e_L - \alpha \sin^2 \Theta_W \delta \nu_R \vec{Z} \delta e_R 
\right.
\]

\[
+ \frac{1}{2} \alpha (\cos 2 \Theta_W \delta \vec{Z} + \sin 2 \Theta_W \delta \vec{A}).(\dot{\nu}_L \gamma \delta e_L + \dot{e}_L \gamma \vec{Z} e_L) 
\]

\[
+ g' (\cos \Theta_W \delta \vec{A} - \sin \Theta_W \delta \vec{Z} \cdot (\dot{e}_R \gamma \delta e_R + \dot{\nu}_L \gamma \delta e_L) 
\]

\[
- h(\phi^\dagger \delta \nu_R \delta e_L + \phi \delta e_L \delta e_R + \delta \phi^\dagger \delta \nu_R \delta e_R + \phi \delta e_R \delta e_L + \delta \phi^\dagger \delta e_R \delta e_L + \phi \delta e_L \delta e_R) \right\},
\]
where $\delta \vec{W} = \delta \vec{W}^1 - i \delta \vec{W}^2$.

It should be noted that a similar perturbative expansion in the pure bosonic theory yields the same functional form as $\delta E_1^B + \delta E_2^B$. Since the $\tilde{Z}$ and $\phi$ profiles for the Z-string are unchanged on addition of the fermion condensate constructed in section two, it thus follows that

$$\delta E^B = \delta E_1^B + \delta E_2^B.$$  \hspace{1cm} (31)

The pure bosonic theory has been studied in detail by James et al.\cite{7}, who determined the region of $(\lambda, \Theta_W)$ space in which the strings were unstable by explicitly constructing perturbations yielding a negative energy change to quadratic order. With the fermion condensate present, setting

$$\delta \nu_L = \delta e_L = \delta e_R = 0$$ \hspace{1cm} (32)

causes $\delta E_1^F$ and $\delta E_2^F$ to become zero, hence

$$\delta E = \delta E^B$$

for perturbations of this special type.

Suppose the parameters $\lambda$ and $\Theta_W$ are chosen such that the Z-string in the pure bosonic theory is unstable, and the point in parameter space is not on the boundary between stability and instability. Then one can construct a perturbation of the form

$$\chi = \chi(x, y), \quad \delta \vec{W} = (W_x(x, y), W_y(x, y), 0), \quad \delta \phi = 0, \quad \delta \tilde{Z} = \delta \tilde{A} = 0$$

with negative $\delta E^B$. Using this perturbation, along with the zero fermionic perturbations given in equation (32), $\delta E$ is found to be negative too, hence the region of $(\lambda, \Theta_W)$ parameter space yielding stable Z-strings with a fermion condensate is contained in that yielding stable Z-strings.

The full problem with $z$ dependence contains the sub-problem of energy change per unit length discussed above as a special case, and the negative energy perturbation constructed above can be embedded in the full perturbation expression yielding the same change in energy per unit length. Thus, the region of stability in the complete problem is contained within that of the Z-string with no condensate.
4. QUARK ZERO MODES

In the previous sections we have seen that the appearance of electron zero modes on the electroweak string does not enhance the stability of the vortex solution. The lack of backreaction on the vortex fields is due to the special form of the zero mode spinors (18). The nature of the spinors is determined by the form of the coupling of the fermion to the scalar field. This question is now considered in relation to the appearance of quark zero modes.

Consider a generalisation of the string background fields (10) to allow the string to have winding number $l$; the scalar field then has an angular dependence of the form $e^{il\theta}$. To remove the angular variation from the Dirac equations for $\Psi$ and $e^R$ we must modify our ansatz for the spinor components (14) by including an extra factor of $e^{-il\theta}$ in the definitions of $c$ and $d$ (the ansatz (14) is given for the case $l = 1$). The equations still decouple into two pairs on setting $w = k = 0$; equations (16) and (17) become

$$\psi'_1 - \frac{1}{r}(n + \cos 2\Theta_W v)\psi_1 = -h\eta f \psi_4,$$

$$\psi'_4 + \frac{1}{r}(n + 1 - l - 2\sin^2 \Theta_W v)\psi_4 = -h\eta f \psi_1,$$

and

$$\psi'_2 + \frac{1}{r}(n + 1 + \cos 2\Theta_W v)\psi_2 = -h\eta f \psi_3,$$

$$\psi'_3 - \frac{1}{r}(n - l - 2\sin^2 \Theta_W v)\psi_3 = -h\eta f \psi_2.$$  

At large distances we have the usual exponentially growing and exponentially decaying solutions for massive fields. If we are to have zero modes the small distance solutions must all be regular so that we can construct a square integrable solution. Thus we need only look at the case of small $r$. Close to the string core, $h\eta f = sr^{|l|}$ for some constant $s$ and $Z_\theta \propto r$, thus we can drop the gauge field terms and eliminate either spinor component of either pair to obtain a second order equation for the other.

Eliminating $\psi_4$ from (33), we find

$$-\frac{|l|}{r}(-\psi'_1 + \frac{n}{r}\psi_1) - \psi''_1 - \frac{1-l}{r}\psi'_1 + \frac{n(n-l)}{r}\psi_1 = 0.$$  

(35)

Substituting $\psi_1 = r^t$, yields the condition

$$-|l|(n - t) - t^2 + lt + n(n - l) = 0.$$
If $|l| = -l$ the solutions are $t = \pm n$ and we have one regular and one irregular solution. However, if $|l| = l$ the solutions are $t = n$ and $t = -n + 2l$ which can both be regular if $l$ is nonzero. Similarly we can eliminate $\psi_1$ and on substituting $\psi_4 = r^t$, we obtain

$$|l|(t + n + 1 - l) - t^2 + lt + (n - l)(n + 1 - l) = 0.$$  

If $|l| = -l$ the solutions are $t = \pm(n + 1 - l)$ and we again have one regular and one irregular solution. However, if $|l| = l$ the solutions are $t = l \pm (n + 1)$ which can also both be regular if $l$ is nonzero.

Repeating this procedure for the other pair of fields we find that zero modes with nonvanishing $\psi_2$ and $\psi_3$ only occur for $l$ negative. Thus if $l$ is positive we have a zero mode of the form given in (18) and for $l$ negative we have modes of the form

$$e_L = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \psi_2(r), \quad e_R = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} i\psi_3(r),$$  

(36)

For $l$ positive the modes persist if we allow $w$ and $k$ to be nonzero but set $w + k = 0$. The corresponding condition for $l$ negative is $w = -k$. The sign of $l$ thus determines the direction of motion of the zero modes along the string and the form of the spinor. This will be important when studying quark zero modes.

Now consider introducing the up and down quarks to complete the first family of fermions. We need to introduce mass terms for both right handed quarks and so have a quark Lagrangian of the form

$$\mathcal{L}_{\text{quark}} = (\bar{u}, \bar{d}) L \gamma^\mu(-i\partial_\mu - \frac{g}{2} \tau^a W^a_\mu + \frac{g'}{6} B_\mu) \begin{pmatrix} u \\ d \end{pmatrix}_L + \bar{u}_R \gamma^\mu(-i\partial_\mu - \frac{2}{3} g' B_\mu) u_R$$

$$+ \bar{d}_R \gamma^\mu(-i\partial_\mu + \frac{1}{3} g' B_\mu) d_R - G_d \left[ (\bar{u}, \bar{d})_L \left( \begin{array}{c} \phi^+ \\ \phi \end{array} \right) d_R + \bar{d}_R (\phi^-, \bar{\phi}) \begin{pmatrix} u \\ d \end{pmatrix}_L \right]$$

$$- G_u \left[ (\bar{u}, \bar{d})_L \left( \begin{array}{c} -\bar{\phi} \\ \phi^- \end{array} \right) u_R + \bar{u}_R (\phi, \phi^+) \begin{pmatrix} u \\ d \end{pmatrix}_L \right],$$

(37)

where $\Phi = (\phi^+, \phi^-)$. In the background fields of the electroweak string this reduces to

$$\mathcal{L}_{\text{quark}} = \bar{u}_L \gamma^\mu(-i\partial_\mu - \frac{1}{2G}(g^2 + g'^2/3)Z_\mu) u_L + \bar{d}_L \gamma^\mu(-i\partial_\mu + \frac{1}{2G}(g^2 + g'^2/3)Z_\mu) d_L$$

$$+ \bar{u}_R \gamma^\mu(-i\partial_\mu + \frac{2}{3} g'^2 Z_\mu) u_R + \bar{d}_R \gamma^\mu(-i\partial_\mu - \frac{1}{3} g'^2 Z_\mu) d_R - G_d (\bar{d}_L \phi d_R + \bar{d}_R \phi^* d_L)$$

$$+ G_u (\bar{u}_L \phi^* u_R + \bar{u}_R \phi u_L).$$

(38)
The generic form of the Lagrangian for each pair of particles is thus

\[ L_f = \bar{f}_L \gamma^\mu (-i\partial_\mu - aZ_\mu) f_L + \bar{f}_R \gamma^\mu (-i\partial_\mu - bZ_\mu) f_R + m(\bar{f}_L \phi^* f_R + \bar{f}_R \phi f_L), \]  

(39)

where \( \phi \) should be replaced by \( \phi^* \) in the case of the d quark and electron.

The coupling of the up quark to \( \phi^* \) rather than \( \phi \) leads to the up quark zero modes moving in the opposite direction along the string to the electron and down quark modes\(^{[12]}\) with the form of the up quark mode being given by (36) rather than (18).

Now, the quark Lagrangian (37) contains terms that are linear in the upper component of the Higgs field: \(-\phi^+ (G_d \bar{u}_L d_R + G_u \bar{d}_R u_L) + (h.c.)\). As the u and d quarks couple to the Higgs field in different ways, one is a left-mover and the other a right-mover. The terms that can act as sources for the upper component of the Higgs doublet thus contain both left and right handed and left and right moving fermions. In the gamma basis we are using, \( \gamma^0 = \text{diag}(1,-1,-1,1) \), thus using \( \uparrow \) and \( \downarrow \) to denote left and right movers, the source terms are

\[ \bar{f}_L \uparrow f_R \downarrow = (0, \psi^*_2, 0, -\psi^*_2) \gamma^0 \begin{pmatrix} 0 \\ \psi_4 \\ 0 \\ \psi_4 \end{pmatrix} = 2\psi^*_2 \psi_4 \]  

(40)

and

\[ \bar{f}_R \uparrow f_L \downarrow = (\psi^*_3, 0, \psi^*_3, 0) \gamma^0 \begin{pmatrix} \psi_1 \\ 0 \\ -\psi_1 \\ 0 \end{pmatrix} = 2\psi^*_3 \psi_1. \]  

(41)

Thus the combination of u and d quarks will act as a source for the upper component of the Higgs field, violating the Vachaspati existence criteria\(^{[5]}\). It is only possible to write down the electroweak string as a solution of the equations of motion if there are no terms in the Lagrangian that are linear in any of the fields that are assumed to vanish. The quark zero modes violate this condition by generating a non-vanishing term that is linear in the upper component of the Higgs doublet.
5. SUMMARY

A zero energy leptonic bound state has been constructed, and it is found that the profile of the Z-string is unaffected by its presence, in the approximation of neglecting source terms in the z-direction and the zero-direction. Negative energy perturbations for the bare Z-string remain so for the configuration of Z-string and condensate, and so the region of stability in parameter space is not extended – in particular, the configuration is unstable for physical values of the electroweak parameters.

The presence of a fermionic zero mode implies that the electroweak Z-string is superconducting. For this to have cosmological significance, it is necessary to form such a string and then allow the fermionic current to build up. As the string is unstable, even in the presence of a time-independent fermion condensate, such a configuration will not have been able to form at the electroweak phase transition, and so the attractive scenario of a network of stabilised loops of current-carrying electroweak string persisting to the present day is unrealistic in this model.

The appearance of quark zero modes on the electroweak string leads to a violation of the Vachaspati existence criteria for embedded defects as the quark modes provide a source for the upper component of the Higgs doublet.

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