Abstract

In this paper we derive strong linear inequalities for sets of the form
\[ \{(x, q) \in \mathbb{R}^d \times \mathbb{R} : q \geq Q(x), \ x \in \mathbb{R}^d - \text{int}(P)\}, \]
where \( Q(x) : \mathbb{R}^d \to \mathbb{R} \) is a quadratic function, \( P \subset \mathbb{R}^d \) and “int” denotes interior. Of particular but not exclusive interest is the case where \( P \) denotes a closed convex set. In this paper, we present several cases where it is possible to characterize the convex hull by efficiently separable linear inequalities.

1 The positive-definite case

We consider sets of the form
\[ S = \{(x, q) \in \mathbb{R}^d \times \mathbb{R} : q \geq Q(x), \ x \in \mathbb{R}^d - \text{int}(P)\}, \tag{1} \]
where \( Q(x) : \mathbb{R}^d \to \mathbb{R} \) is a positive-definite quadratic function, and each connected component of \( P \subset \mathbb{R}^d \) is a homeomorph of either a half-plane or a ball. Thus, each connected component of \( P \) is a closed set with nonempty interior.

Since \( Q(x) \) is positive definite, we may assume without loss of generality that \( Q(x) = \|x\|^2 \) (achieved via a linear transformation). For any \( y \in \mathbb{R}^d \), the linearization inequality
\[ q \geq 2y^T(x - y) + \|y\|^2 = 2y^Tx - \|y\|^2 \tag{2} \]
is valid for all \((x, q) \in \mathbb{R}^d \times \mathbb{R} \). We seek ways of making this inequality stronger.

Definition 1.1 Given \( \mu \in \mathbb{R}^d \) and \( R \geq 0 \), we write \( B(\mu, R) = \{x \in \mathbb{R}^d : \|x - \mu\| \leq R\} \).

1.1 Geometric characterization

Let \( x \in \mathbb{R}^d \). Then \( x \in \mathbb{R}^d - \text{int}(P) \) if and only if
\[ \|x - \mu\|^2 \geq \rho, \quad \text{for each ball } B(\mu, \sqrt{\rho}) \subseteq P. \tag{3} \]
In terms of our set \( S \), we can rewrite (3) as
\[ q \geq 2\mu^Tx - \|\mu\|^2 + \rho, \quad \text{for each ball } B(\mu, \sqrt{\rho}) \subseteq P. \tag{4} \]
On the other hand, suppose
\[ \delta q - 2\beta^Tx \geq \beta_0 \tag{5} \]
is valid for \( S \). Since \( \mathbb{R}^d - P \) contains points with arbitrarily large norm it follows \( \delta \geq 0 \). Suppose that \( \delta > 0 \); then without loss of generality \( \delta = 1 \). Further, given \( x \in \mathbb{R}^d \), (5) is satisfied by \((x, q) \) with \( q \geq \|x\|^2 \) if and only if it is satisfied by \((x, \|x\|^2) \), and so if and only if we have
\[ \|x - \beta\|^2 \geq \|\beta\|^2 + \beta_0. \tag{6} \]
Since (5) is valid for \( S \), we have that (6) holds for each \( x \in \mathbb{R}^d - \text{int}(P) \). Assuming further that (5) is not trivial, that is to say, it is violated by some \((z, \|z\|^2) \) with \( z \in \text{int}(P) \), we must therefore have that \( \|\beta\|^2 + \beta_0 > 0 \) and \( B(\beta, \sqrt{\|\beta\|^2 + \beta_0}) \subseteq P \), i.e. statement (6) is an example of (3). Below we discuss several ways of sharpening these observations.
1.2 Lifted first-order cuts

Let \( y \in \partial P \). Then we can always find a ball \( B(\mu, \sqrt{\rho}) \subseteq P \) such that \( ||\mu - y||^2 = \rho \), possibly by setting \( \mu = y \) and \( \rho = 0 \).

**Definition 1.2** Given \( y \in \partial P \), we say \( P \) is locally flat at \( y \) if there is a ball \( B(\mu, \sqrt{\rho}) \subseteq P \) with \( ||\mu - y||^2 = \rho \) and \( \rho > 0 \).

Suppose \( P \) is locally flat at \( y \) and let \( B(\mu, \sqrt{\rho}) \) be as in the definition. Let \( a^T x \geq a_0 \) be a supporting hyperplane for \( B(\mu, \sqrt{\rho}) \) at \( y \), i.e., \( a^T y = a_0 \) and \( a^T x \geq a_0 \) for all \( x \in B(\mu, \sqrt{\rho}) \). We claim that

\[
q \geq 2y^T x - ||y||^2 + 2\alpha(a^T x - a_0)
\]

is valid for \( S \) if \( \alpha \geq 0 \) is small enough. To see this, note that since \( a^T x \geq a_0 \) supports \( B(\mu, \sqrt{\rho}) \) at \( y \), it follows that \( \mu - y = \hat{\alpha} a \) for small enough, but positive \( \hat{\alpha} \), i.e.,

\[
B(y + \hat{\alpha} a, \sqrt{\hat{\alpha}^2||a||^2}) = B(\mu, \sqrt{\rho}).
\]

Now, assume \( \alpha \leq \hat{\alpha} \). Then \((v, ||v||^2)\) violates (7) iff

\[
||v||^2 < 2y^T v - ||y||^2 + 2\alpha(a^T v - a_0)
\]

\[
= 2(y + \alpha a)^T v - ||y + \alpha a||^2 + \alpha^2||a||^2 + 2\alpha(y^T a - a_0)
\]

\[
= 2(y + \alpha a)^T v - ||y + \alpha a||^2 + \alpha^2||a||^2,
\]

which implies

\[
v \in B(y + \alpha a, \sqrt{\alpha^2||a||^2}) \subset B(\mu, \sqrt{\rho})
\]

since \( \alpha \leq \hat{\alpha} \). In other words, for small enough, but positive \( \alpha \), (7) is valid for \( S \).

In fact, the above derivation implies a stronger statement: since \( a^T x \geq a_0 \) supports \( B(y + \alpha a, \sqrt{\alpha^2||a||^2}) \) at \( y \), for any \( \alpha > 0 \), it follows (7) is valid for \( S \) iff \( B(y + \alpha a, \sqrt{\alpha^2||a||^2}) \subseteq P \). Define

\[
\hat{\alpha} = \sup \{ \alpha : \text{(7) is valid} \}.
\]

If there exists \( v \notin P \) such that \( a^T v > a_0 \) then the assumptions on \( P \) imply that \( \hat{\alpha} < +\infty \) and the 'sup' is a 'max'. If on the other hand \( a^T v \leq a_0 \) for all \( v \notin P \) then \( \hat{\alpha} = +\infty \) (and, of course, \( a^T x \leq a_0 \) is valid for \( S \)).

In the former case, we call

\[
q \geq 2y^T x - ||y||^2 + 2\hat{\alpha}(a^T x - a_0)
\]

a lifted first-order inequality.

**Theorem 1.3** Any linear inequality

\[
\delta q - \beta^T x \geq \beta_0
\]

valid for \( S \) either has \( \delta = 0 \) (in which case the inequality is valid for \( \mathbb{R}^d - P \)), or \( \delta > 0 \) and (14) is dominated by a lifted first-order inequality or by a linearization inequality (2).

**Proof.** Consider a valid inequality (14). As above we either have \( \delta = 0 \), in which case we are done, or without loss of generality \( \delta = 1 \), and by increasing \( \beta_0 \) if necessary we have that (14) is tight at some point \((y, ||y||^2) \in \mathbb{R}^d \times \mathbb{R} \).

Write

\[
\beta^T x + \beta_0 = 2y^T x - ||y||^2 + 2\gamma^T x + \gamma_0,
\]

for appropriate \( \gamma \) and \( \gamma_0 \). Suppose first that \( y \in \text{int}(\mathbb{R}^d - P) \). Then \((\gamma, \gamma_0) = (0, 0) \), or else (14) would not be valid in a neighborhood of \( y \). Thus, (14) is a linearization inequality.
Suppose next that \( y \in \partial P \), and that \((\text{14})\) is not a linearization inequality, i.e. \((\gamma, \gamma_0) \neq (0, 0)\). We can write \((\text{14})\) as
\[
q \geq 2y^T x - \|y\|^2 + 2\gamma x + \gamma_0 = 2(y + \gamma)^T x - \|y + \gamma\|^2 - 2\gamma^T y - \|\gamma\|^2 + \gamma_0.
\] (16)
Since \((\text{14})\) is not a linearization inequality, and is satisfied at \((y, \|y\|^2)\) there exist points \((v, \|v\|^2)\) (with \(v\) near \(y\)) which do not satisfy it. Necessarily, any such \(v\) must not lie in \(\mathbb{R}^d - P\) (since \((\text{14})\) is valid for \(S\)). Using \((\text{16})\) this happens iff
\[
\|v\|^2 < 2(y + \gamma)^T v - \|y + \gamma\|^2 - 2\gamma^T y - \|\gamma\|^2 + \gamma_0, \quad \text{that is,} \quad v \in \text{int} \left( \mathcal{B} \left( y + \gamma, \sqrt{2\gamma^T y - \|\gamma\|^2 + \gamma_0} \right) \right).
\] (17)
In other words, the set of points that violate \((\text{14})\) is the interior of some ball \(\mathcal{B}\) with positive radius, which necessarily must be contained in \(P\). Since \((y, \|y\|^2)\) satisfies \((\text{14})\) with inequality, \(y\) is in the boundary of \(\mathcal{B}\). Thus, \(P\) is locally flat at \(y\); writing \(a^T x = a_0\) to denote the hyperplane orthogonal to \(\gamma\) through \(y\), we have that \((\text{14})\) is dominated by the resulting lifted first-order inequality.

### 1.3 The polyhedral case

Here we will discuss an efficient separation procedure for lifted first-order inequalities in the case that \(P\) is a polyhedron. Further properties of these inequalities are discussed in \((\text{11})\).

Suppose that \(P = \{ x \in \mathbb{R}^d : a_i^T x \geq b_i, \ 1 \leq i \leq m \}\) is a full-dimensional polyhedron, where each inequality is facet-defining and the representation of \(P\) is minimal. For \(1 \leq i \leq m\) let \(H_i = \{ x \in \mathbb{R}^d : a_i^T x = b_i \}\). For \(i \neq j\) let \(H_{i,j} = \{ x \in \mathbb{R}^d : a_i^T x = b_i, a_j^T x = b_j \}\). \(H_{i,j}\) is \((d-2)\)-dimensional; we denote by \(\omega_{ij}\) the unique unit norm vector orthogonal to both \(H_i\) and \(a_i\) (unique up to reversal).

Consider a fixed pair of indices \(i \neq j\), and let \(\mu \in \text{int}(P)\). Let \(\Omega_{ij}\) be the 2-dimensional hyperplane through \(\mu\) generated by \(a_i\) and \(\omega_{ij}\). By construction, therefore, \(\Omega_{ij}\) is orthogonal to \(H_{i,j}\) and is thus the orthogonal complement to \(H_{i,j}\) through \(\mu\). It follows that \(\Omega_{ij} = \Omega_{ji}\) and that this hyperplane contains the orthogonal projection of \(\mu\) onto \(H_i\) (which we denote by \(\pi_i(\mu)\)) and the orthogonal projection of \(\mu\) onto \(H_j\) (\(\pi_j(\mu)\), respectively). Further, \(\Omega_{ij} \cap H_{i,j}\) consists of a single point \(k_{i,j}(\mu)\) satisfying
\[
\|\mu - k_{i,j}(\mu)\|^2 = \|\mu - \pi_i(\mu)\|^2 + \|\pi_i(\mu) - k_{i,j}(\mu)\|^2 = \|\mu - \pi_j(\mu)\|^2 + \|\pi_j(\mu) - k_{i,j}(\mu)\|^2.
\] (19)
Now we return to the question of separating lifted first-order inequalities. Note that \(P\) is locally flat at a point \(y\) if and only if \(y\) is in the relative interior of one of the facets. Suppose that \(y\) is in the relative interior of the \(i^{th}\) facet. Then the lifting coefficient corresponding to the lifted first-order inequality at \(y\) is tight at some other point \(\hat{y}\) in a different facet, facet \(j\), say. Thus, there is a ball \(\mathcal{B}(\mu, \sqrt{\rho})\) contained in \(P\) which is tangent to \(H_i\) at \(y\) and tangent to \(H_j\) at \(\hat{y}\), that is to say,
\[
y = \pi_i(\mu) \quad \text{and} \quad \hat{y} = \pi_j(\mu),
\] (20)
\[
y - k_{i,j}(\mu) \quad \text{is parallel to} \quad \omega_{ij} \quad \text{and} \quad \hat{y} - k_{i,j}(\mu) \quad \text{is parallel to} \quad \omega_{ji},
\] (21)
\[
\|\mu - y\|^2 = \|\mu - \hat{y}\|^2 = \rho, \quad \text{and by (19)},
\] (22)
\[
\|y - k_{i,j}(\mu)\| = \|\hat{y} - k_{i,j}(\mu)\|, \quad \text{and}
\] (23)
\[
\|\mu - y\| = \tan \phi \|y - k_{i,j}(\mu)\|,
\] (24)
where \(2\phi\) is the angle formed by \(\omega_{ij}\) and \(\omega_{ji}\). By the preceding discussion, \(\rho = \hat{a}^2\|a_i\|^2\); using (22) and (23) we will next argue that the lifting coefficient, \(\hat{a}\), is an affine function of \(y\).

Let \(h_{i,j}^q (1 \leq q \leq d - 2)\) be a basis for \(\{ x \in \mathbb{R}^d : a_i^T x = a_j^T x = 0 \}\). Then \(a_i\), together with \(\omega_{ij}\) and the \(h_{i,j}^q\) form a basis for \(\mathbb{R}^d\). Let
- \(O_i\) be the projection of the origin onto \(H_i\) - hence \(O_i\) is a multiple of \(a_i\),
• \( N_i \) be the projection of \( O_i \) onto \( H_{ij} \).

We have
\[
y = O_i + (N_i - O_i) + (k_{ij}(\mu) - N_i) + (y - k_{ij}(\mu)),
\]
and thus, since \( N_i - O_i \) and \( y - k_{ij}(\mu) \) are parallel to \( \omega_{ij} \), and \( k_{ij}(\mu) - N_i \) and \( O_i \) are orthogonal to \( \omega_{ij} \),
\[
\omega_{ij}^T y = \omega_{ij}^T (N_i - O_i) + \omega_{ij}^T (y - k_{ij}(\mu)) = \omega_{ij}^T (N_i - O_i) + \| \omega_{ij} \| \| y - k_{ij}(\mu) \|,
\]
or
\[
\| y - k_{ij}(\mu) \| = \| \omega_{ij} \|^{-1} \omega_{ij}^T (y - N_i + O_i).
\]
Consequently,
\[
\hat{\alpha} = \frac{\rho}{\| a_i \|} = \frac{\tan \phi}{\| a_i \|} \| y - k_{ij}(\mu) \|
\]
\[
= \frac{\tan \phi}{\| a_i \|} \| \omega_{ij} \|^{-1} \omega_{ij}^T (y - N_i + O_i).
\]
We will abbreviate this expression as \( p_{ij} y + q_{ij} \). Let \( x^* \in \mathbb{R}^d \). The problem of finding the strongest possible lifted first-order inequality at \( x^* \) chosen from among those obtained by starting from a point on face \( i \), can be written as follows:
\[
\begin{align*}
\min & \quad -2y^T x^* + \| y \|^2 - 2\alpha(a^T x^* - a_0) \\
\text{s.t.} & \quad y \in P \\
& \quad a_i^T y = b_i \\
& \quad 0 \leq \alpha \leq p_{ij} y + q_{ij} \quad \forall \ j \neq i.
\end{align*}
\]
This is a linearly constrained, convex quadratic program with \( d + 1 \) variables and \( 2m - 1 \) constraints. By solving this problem for each choice of \( 1 \leq i \leq m \) we obtain the the strongest inequality overall.

### 1.3.1 The Disjunctive Approach

For \( 1 \leq i \leq m \) let \( \bar{P}^i = \{ x \in \mathbb{R}^d : a_i^T x \leq b_i \} \); thus \( \mathbb{R}^d - P = \bigcup_{i} \bar{P}^i \). Further, for \( 1 \leq i \leq m \) write:
\[
\bar{Q}^i = \{ (x, q) \in \mathbb{R}^d \times R : a_i^T x \leq b_i, \ q \geq \| x \|^2 \}.
\]
Thus, \( (x^*, q^*) \in \text{conv}(S) \) if and only if \( (x^*, q^*) \) can be written as a convex combination of points in the sets \( \bar{Q}^i \). This is the approach pioneered in Ceria and Soares [6] (also see [13]). The resulting separation problem is carried out by solving a second-order cone program with \( m \) conic constraints and \( md \) variables, and then using second-order cone duality in order to obtain a linear inequality (details in [10]).

Thus, the derivation we presented above amounts to a possibly simpler alternative to the Ceria-Soares approach, which also makes explicit the geometric nature of the resulting cuts.

### 1.4 The ellipsoidal case

In this section we will discuss an efficient separation procedure for lifted first-order inequalities in the case that \( P \) is a convex ellipsoid, in other words,
\[
P = \{ x \in \mathbb{R}^d : x^T Ax - 2c^T x + b \leq 0 \},
\]
for appropriate \( A \geq 0 \), \( c \) and \( b \). The separation problem to solve can be written as follows: given \((x^*, q^*) \in \mathbb{R}^{d+1}\),
\[
\max \{ \Theta(\rho) : \rho \geq 0 \} \text{ where, for fixed } \rho \geq 0,
\]
\[
(34)
\]
\[ \Theta(\rho) \doteq \max \rho - (q^* - 2\mu^T x^* + \mu^T \mu) = \rho - \|x^* - \mu\|^2 - q^* + \|x^*\|^2 \quad (35) \]
\[ \text{s.t. } \mu \in \mathbb{R}^d, \rho \geq 0 \text{ and } B(\mu, \sqrt{\rho}) \subseteq P \quad (36) \]

Consider a fixed value \( \rho > 0 \). We will first show that with this proviso the condition
\[ B(\mu, \sqrt{\rho}) \subseteq P \quad (37) \]
is SOCP-representable. We note that \[ 11 \] considers the problem of finding a minimum-radius ball containing a family of ellipsoids; our separation problem addresses, in a sense, the opposite situation, which leads to a somewhat different analysis. Our equations \[ 10 ] - \[ 11 \] are related to formulae found in \[ 11 \] (also see \[ 7 \]) but again reflecting the opposite nature of the problem. Also see \[ 4 \]. Some of the earliest studies in this direction are found in \[ 5 \].

Returning to \[ 37 \], notice that this condition is equivalent to stating
\[ \|x\|^2 - \rho \geq 0, \quad \forall x \text{ s.t. } (x + \mu)^T A(x + \mu) - 2c^T (x + \mu) + b \geq 0. \quad (38) \]
Using the S-Lemma \[ 14 \], \[ 11 \], \[ 2 \], \[ 38 \] holds if and only if there exists a quantity \( \theta \geq 0 \) such that, for all \( x \in \mathbb{R}^d \)
\[ x^T (I - \theta A)x + 2\theta(c^T - \mu^T A)x + \theta(-\mu^T A\mu + 2c^T \mu - b) - \rho \geq 0. \]
Clearly we must have \( \theta > 0 \); writing \( \tau = \theta^{-1} \) we have that \[ 38 \] holds if and only if there exists \( \tau > 0 \) such that
\[ x^T \left( I - \frac{1}{\tau} A \right)x + \frac{2}{\tau}(c^T - \mu^T A)x + \frac{1}{\tau}(-\mu^T A\mu + 2c^T \mu - b) - \rho \geq 0 \quad \forall x \in \mathbb{R}^d. \quad (39) \]
Let the eigenspace decomposition of \( A \) be \( A = U\Lambda U^T \) and write
\[ y = U^T x, \quad v = v(\mu) = U^T (c - A\mu). \]
Then we have that \[ 39 \] holds if for all \( y \in \mathbb{R}^d \),
\[ y^T \left( I - \frac{1}{\tau} A \right) y + \frac{2}{\tau}v^Ty + \frac{1}{\tau}(-\mu^T A\mu + 2c^T \mu - b) - \rho \geq 0, \]
or, equivalently,
\[ \begin{pmatrix} I - \frac{1}{\tau} A & \frac{1}{\tau}v \\ \frac{1}{\tau}v^T & \frac{1}{\tau}(-\mu^T A\mu + 2c^T \mu - b) - \rho \end{pmatrix} \geq 0. \quad (40) \]
Let \( \lambda_{\max} \) denote the largest eigenvalue of \( A \). Then \[ 40 \] holds if \( \tau \geq \lambda_{\max} \), and
\[ -\frac{1}{\tau^2} \sum_{j=1}^d \frac{v_j^2}{1 - \lambda_j/\tau} + \frac{1}{\tau}(-\mu^T A\mu + 2c^T \mu - b) - \rho \geq 0, \]
or, equivalently
\[ -\sum_{j=1}^d \frac{v_j^2}{\tau - \lambda_j} - \mu^T A\mu + 2c^T \mu - b - \rho \tau \geq 0 \quad (41) \]
which is SOCP-representable. Formally this is done as follows: \[ 11 \] holds if there exist quantities \( y_j, 1 \leq j \leq d \) such that
\[ y_j(\tau - \lambda_j) \geq v_j^2, \quad 1 \leq j \leq d, \quad \text{and} \quad -\sum_{j=1}^d y_j - \mu^T A\mu + 2c^T \mu - b - \rho \tau \geq 0. \quad (42) \]
In summary, then, for fixed $\rho$ the problem of finding the most violated lifted first-order inequality can be formulated as the following SOCP, with variables $\mu$, $\tau$, $v$ and $y$:

$$\begin{align*}
\min & \quad 2\mu^T x^* + \mu^T \mu + q^* - \rho \\
\text{s.t.} & \quad v = U^T (e - A\mu) & \quad (43) \\
& \quad \tau \geq \lambda_{\max} & \quad (44) \\
& \quad y_j (\tau - \lambda_j) \geq v_j^2, \quad 1 \leq j \leq d, & \quad (45) \\
& \quad - \sum_{j=1}^{d} y_j + 2c^T \mu - b - \rho \tau \geq \mu^T A\mu. & \quad (46)
\end{align*}$$

(47)

Here, constraints (46) and (47) are conic (in (47), it is critical that $\rho$ is a fixed value, since $\tau$ is a variable).

**Lemma 1.4** Let $K$ be an arbitrary convex set and $v \in K$. For $\rho > 0$ the function

$$N(\rho) = \min \|v - \mu\|^2$$

s.t.

$$B(\mu, \sqrt{\rho}) \subseteq K,$$

is convex.

Pending the proof of this result, we note that as per eq. (35), if $A \succeq 0$ then $\Theta(\rho)$ is a concave function of $\rho$. Thus the separation problem can be solved to arbitrary tolerance using e.g. golden ratio search, with the SOCP (43)-(47) as a subroutine.

**Proof of Lemma 1.4** To prove convexity of $N$, it suffices to show that for any pair of values $\rho_1 \neq \rho_2$ there exists a function $g(\rho)$ such that

(a) $g(\rho_i) = N(\rho_i)$, $i = 1, 2,$

(b) $g(\rho) \geq N(\rho_1)$ for every $\rho$ between $\rho_1$ and $\rho_2$,

(c) $g(\rho)$ is convex between $\rho_1$ and $\rho_2$.

Thus, let $\rho_1, \rho_2$ be given. For $i = 1, 2$ let $\mu_i = \arg\min N(\rho_i)$ and $R_i = \sqrt{\rho_i}$. Assume without loss of generality that $R_1 < R_2$. Let $0 < \lambda \leq 1$. Since $K$ is convex,

$$B( (1 - \lambda)\mu_1 + \lambda \mu_2, \sqrt{((1 - \lambda)R_1 + \lambda R_2)^2}) \subseteq K,$$

(49)

in other words, for any point $\mu$ in the segment $[\mu_1, \mu_2]$, there is a ball with center $\mu$, contained in $K$ and with radius

$$R_1 + \frac{R_2 - R_1}{\|\mu_2 - \mu_1\|} \|\mu - \mu_1\|,$$

(50)

or, to put it even more explicitly, as a point $\mu$ moves from $\mu_1$ to $\mu_2$ there is a ball with center $\mu$ contained in $K$, whose radius is obtained by linearly interpolating between $R_1$ and $R_2$. Let $\mu^*$ be the nearest point to $v$ on the line defined by $\mu_1$ and $\mu_2$ (possibly $\mu^* \notin K$). For $i = 1, 2$, let $t_i = \|\mu^* - \mu_i\|$.

Suppose first that $\mu^*$ is in the line segment between $\mu_1$ and $\mu_2$ and $\mu^* \neq \mu_1$. By (49) there is a ball centered at $\mu$ and contained in $K$ with radius strictly larger than $R_1$, a contradiction by definition of $\mu_1$. The same contradiction would arise if $\mu_2$ separates $\mu^*$ and $\mu_1$.

Thus $\mu_1$ separates $\mu^*$ and $\mu_2$. Defining

$$s = \frac{R_2 - R_1}{t_2 - t_1} > 0,$$

(51)

we have that for $-t_1 \leq t \leq t_2 - 2t_1$ the point

$$\mu(t) = \mu_1 + \frac{\mu_2 - \mu_1}{t_2 - t_1} (t + t_1)$$

(52)
lies in the segment \([\mu_1, \mu_2]\) and is the center of a ball of radius

\[ R(t) = R_1 + s(t_1 + t); \quad (53) \]

further \(\mu(-2t_1) = \mu^*\). Since \(K\) is convex, the segment between \(v\) and \(\mu_2\) is contained in \(K\); let \(w\) be the point in that segment with \(\|v - w\| = \|v - \mu_1\|\); by the triangle inequality

\[ \|\mu_1 - \mu_2\| \geq \|w - \mu_2\|. \quad (54) \]

Let \(\pi\) be the slope of the linear interpolant, between values \(R^*\) and \(R_1\), along the segment \([v, \mu_2]\), i.e. \(R^* + \pi\|v - \mu_2\| = R_2\). Then, as previously, \(B(w, \sqrt{R_w}) \subseteq K\) where \(R_w = R^* + \pi\|v - w\|\). But then it follows by definition of \(\mu_1\) that

\[ R_w \leq R_1. \quad (55) \]

By \(54\) and \(51\), we have \(\pi \geq s\), and therefore, by \(55\),

\[ R_1 \geq R^* + \pi\|v - w\| = R^* + \pi\|v - \mu_1\| \geq R^* + \pi\|\mu^* - \mu_1\| \geq st_1, \quad (56) \]

Now, for any \(t\), since \(\mu(-2t_1) = \mu^*\),

\[ \|\mu(t) - \mu^*\| = \frac{\|\mu_2 - \mu_1\|}{t_2 - t_1}|t + 2t_1|. \quad (57) \]

Define \(\gamma = (\mu_2 - \mu_1)/(t_2 - t_1)\), and

\[ g(\rho) = \gamma^2\left(\frac{\sqrt{\rho} - R_1}{s} + t_1\right)^2 + \|\mu^* - v\|^2. \]

We will prove that \(g\) satisfies properties (i)-(iii) listed above. For \(\rho\) between \(\rho_1\) and \(\rho_2\), writing \(R = \sqrt{\rho}\) and

\[ t = (R - R_1)/s - t_1, \]

it follows that \(\mu(t)\) is the center of a ball of radius \(R\) contained in \(K\). Further, since \(\|\mu(t) - \mu^*\| = \gamma|t + 2t_1|\),

\[ g(\rho) = \|\mu(t) - \mu^*\|^2 + \|\mu^* - v\|^2, \]

and so \(g\) satisfies (i) and (ii). Finally, to see that \(g\) is convex, note that the coefficient of \(\sqrt{\rho}\) in the expansion of \(g(\rho)\) in \(58\) equals

\[ 2\gamma^2\left(\frac{t_1}{s} - \frac{R_1}{s^2}\right) \leq 0, \quad (59) \]

by \(57\). ■

**Note:** We speculate that \(A \geq 0\) (i.e., convexity of \(P\)) is not required for Lemma 1.4, and that further the overall separation algorithm can be improved to avoid dealing with the fixed \(\rho\) case.

## 2 Indefinite Quadratics

The general case of a set \(\{ (x, q) \in \mathbb{R}^d \times \mathbb{R} : q \geq Q(x), \ x \in \mathbb{R}^d - \text{int}(P) \}\), where \(Q(x)\) is a semidefinite quadratic can be approached in much the same way as that employed above, but with some important differences.

We first consider the case where \(P\) is a polyhedron. Let \(P = \{ (x, w) \in \mathbb{R}^{d+1} : a_i^T x - w \leq b_i, \ 1 \leq i \leq m \}\) (here, \(w\) is a scalar). Consider a set of the form

\[ S = \{ (x, w, q) \in \mathbb{R}^{d+2} : q \geq \|x\|^2, \ (x, w) \in \mathbb{R}^{d+1} - P \}. \quad (60) \]
Many examples can be brought into this form, or similar, by an appropriate affine transformation. Consider a point \((x^*, w^*)\) in the relative interior of the \(i\)th facet of \(P\). We seek a lifted first-order inequality of the form
\[
(2x^* - \alpha a_i)^T x + \alpha w + \alpha b_i - \|x^*\|^2 \leq q,
\]
for appropriate \(\alpha \geq 0\). If we are lifting to the \(j\)th facet, then we must have \(v_{ij} = \alpha b_i - \|x^*\|^2\), where
\[
v_{ij} = \min \|x\|^2 - (2x^* - \alpha a_i)^T x - \alpha w \\
s.t. \quad a_j^T x - w = b_j.
\]
(61) (62)

To solve this optimization problem, consider its Lagrangian:
\[
L(x, w, \nu) = \|x\|^2 - (2x^* - \alpha a_i)^T x - \alpha w - \nu(a_j^T x - w - b_j)
\]
Taking the gradient in \(x\) and setting it to 0:
\[
\nabla_x L = 0 \iff 2x - 2x^* + \alpha a_i - \nu a_j = 0 \\
\iff x = x^* - \frac{\alpha}{2} a_i + \frac{\nu}{2} a_j.
\]
Now doing the same for \(w\):
\[
\nabla_w L = 0 \iff -\alpha + \nu = 0 \\
\iff \nu = \alpha.
\]
Combining these two gives
\[
x = x^* - \frac{\alpha}{2} a_i + \frac{\alpha}{2} a_j
\]
then using the constraint \(a_j^T x - w = b_j\) gives
\[
w = a_j^T x^* - b_j - \frac{\alpha}{2} a_j^T a_i + \frac{\alpha}{2} a_j^T a_j
\]
Next we expand out the objective value using the expressions we have derived for \(x\) and \(w\), and set the result equal to \(\alpha b_i - \|x^*\|^2\). Omitting the intermediate algebra, the result is the quadratic equation
\[
\alpha(a_i^T x^* - b_i - (a_j^T x^* - b_j)) - \frac{1}{4}\alpha^2(a_i^T a_i - 2a_i^T a_j + a_j^T a_j) = 0
\]
One root of this equation is \(\alpha = 0\). The other root is
\[
\hat{\alpha} = \frac{4(a_i^T x^* - b_i - (a_j^T x^* - b_j))}{a_i^T a_i - 2a_i^T a_j + a_j^T a_j}.
\]
(63)
Since \(a_i^T x^* - w^* = b_i\), and \(a_j^T x^* - w^* \leq b_j\), we have
\[
a_j^T x^* - b_i - (a_j^T x^* - b_j) > 0
\]
so \(\hat{\alpha} > 0\) (the denominator is a squared distance between some two vectors so it is non-negative). Moreover, the expression for \(\hat{\alpha}\) is an affine function of \(x^*\). Thus, as in Section 1.3, the computation of a maximally violated lifted first-order inequality is a convex optimization problem.

In this case there is an additional detail of interest: note that the points cut-off by the inequality are precisely those of the form \((x, w, \|x\|^2)\) such that
\[
(2x^* - \hat{\alpha} a_i)^T x + \alpha w + \alpha b_i - \|x^*\|^2 > \|x\|^2.
\]
(64)
This condition defines the interior of a paraboloid; this is the proper generalization of condition 3 in the indefinite case.
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