Rabi oscillations of matter wave solitons in optical lattices

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Inter-band Rabi oscillations of gap soliton matter waves induced by time dependent periodic forces in combined linear and nonlinear optical lattices are for the first time demonstrated. It is shown that under suitable conditions these oscillations can become long-lived. By switching off the external force at proper time it is possible to create either pure (stationary macroscopically populated) gap soliton states or linear combination of two gap solitons with appreciably long life-time.

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I. INTRODUCTION

In spite of its long history, the phenomenon of quantum tunneling still attracts a great deal of attention, particularly in connection with its macro-world manifestations. In condensed matter physics tunneling phenomena occur in connection with inter-band transitions of electrons (from the valence to the conduction band) under the action of external perturbations such as electric fields (Landau-Zener tunneling [1]) or radiation fields (photons) of proper energy (inter-band Rabi oscillations). The phenomenon is highly facilitated by time periodic perturbations with frequencies matching the energy gaps between the bands. In the context of cold atoms Rabi oscillations (RO) between Bloch bands have been considered both theoretically [2] and experimentally [3]. Possible extensions of these studies to ultracold atoms and to Bose-Einstein condensates (BEC) give an interesting perspective for inter-band transitions and ROs detection at the macroscopic level.

Macroscopic quantum tunneling phenomena are presently investigated in many physical systems, including superfluidity and superconductivity [4] and, more recently, in optical systems with ultra-cold atoms [5] and Bose-Einstein condensates (BECs) embedded in optical lattices (OLs) [6], where the Landau-Zener tunneling has been recently observed [7]. In contrast with solids, however, BEC systems are intrinsically nonlinear, being described by the mean field periodic Gross-Pitaevskii equation with the cubic nonlinearity arising from the interatomic two-body interactions. The correspondence with solid state physics occurs either at very low densities or when the interatomic interaction are “artificially” detuned to zero by means of external magnetic fields, using Feshbach resonances (see e.g. [8]).

The presence of nonlinearity in the periodic Gross-Pitaevskii equation introduces modulational (dynamical) instabilities of Bloch wavefunctions [9] with substantial effects on the inter-band tunneling [7, 10]. The appearance of spatially localized states inside gaps of the underlying linear band is associated to these modulational instabilities. These states, also known as gap-solitons (GSs) first discovered in nonlinear optics [11] and observed in fibers with Bragg gratings [12], in BECs were experimentally realized for the first time in [13]. Such states have energies which can scan the whole gap (or a large portion of it) giving rise to families of modes with different symmetry properties with respect to the OL (see e.g. [14]). The existence of GSs with chemical potentials arbitrarily close to band edges introduces new possibilities for a stimulated Landau-Zener tunneling and RO of matter waves across the gap.

The aim of the present paper is to show for the first time how the inter-band Rabi oscillations of matter wave solitons can be induced by means of combined linear and nonlinear OLs under the action of a time periodic linear potential. By switching off the external force at proper times we show that it is possible to put the system either in a stationary GS state or in a time dependent linear combination of two GS states which, in spite of the nonlinearity of the system, can persist in this superposition for a reasonable long time.

The paper is organized as follows. In section II we introduce the model equations while in section III we provide an analytical description of the long-lived Rabi oscillations of GSs in terms of a modified two-mode model valid for GS states close to gap edges. A small detuning of the force frequency from the gap-width will be used as parameter to achieve the absence of matter leakage (emission of radiation) from the soliton states. We show that under this condition the ROs become long lived and manageable. In Section IV we compare our analytical results with direct numerical simulations of the Gross-Pitaevskii equation. In this section we also discuss the possibility to create pure (stationary macroscopically populated) gap soliton states or linear combination of two of them which survive for appreciably long time, by switching off the external force sustaining the Rabi oscillation at a proper time. In the last section we design parameters for a possible experimental observation of the nonlinear ROs and briefly summarize the main results of the paper.
with quasi-momentum $U$ amplitude of the linear OL of the linear potential linear lattice can be created by optically [15] or magnet- 

cal systems modeled by the following periodic nonlinear gap coincide with the boundaries of the $E$-axis) for the parameters $V = 3$, $G = -0.5$. Lower panels show shapes of the localized modes at points $A$ – $D$ of the panel b.

II. MODEL EQUATION

Although in the following we refer specifically to a BEC, the obtained results will be valid for all physical systems modeled by the following periodic nonlinear Schrödinger (NLS) equation with time dependent linear potential:

$$i\psi_t = -\psi_{xx} + U(x)\psi + \mathcal{F}(t)\psi + \mathcal{G}(x)|\psi|^2\psi,$$  

Here $U(x)$ and $\mathcal{G}(x)$ are $\pi$-periodic linear and nonlinear lattices, respectively, and $\mathcal{F}(t)$ is the time periodic amplitude of a linear potential. In BEC experiments the nonlinear lattice can be created by optically [15] or magnetically [16] induced Feshbach resonances (for the sake of brevity we speak about OL) while the external force can be induced simply by a time periodic acceleration of the linear OL. In absence of the nonlinearity, $\mathcal{G}(x) \equiv 0$ and of the linear potential $\mathcal{F}(t) \equiv 0$, Eq. (1) reduces to the familiar band theory framework, with the Schrödinger operator $\mathcal{H} = -d^2/dx^2 + U(x)$ and associated stationary eigenvalue problem $\mathcal{H}\varphi_{n}(x) = \mathcal{E}_{n}(x)\varphi_{n}(x)$, this providing energy bands $\mathcal{E}_{n}(x)$ and Bloch functions $\varphi_{n}(x)$ with quasi-momentum $q$ and band index $n$, normalized as $\int_{-\infty}^{\infty} |\varphi_{n}(x)|^2 dx = 1$. We will concentrate on the lowest gap extending from the top of the first band, denoted as $E_-$, to the bottom of the second band, denoted as $E_+$. The respective Bloch functions will be denoted by $\varphi_{\pm}(x)$.

The condition for the existence of GS states at both edges of a gap can be expressed in terms of effective masses $M_{\pm}(x) = (d^2\mathcal{E}_{\pm}/dx^2)^{-1}$ and effective nonlinearities $\chi_{\pm} = \int_{-\infty}^{\infty} \mathcal{G}(x)|\varphi_{\pm}(x)|^4 dx$, as: $M_{\pm}(x) > 0$ [17]. In analogy to what done in Refs. [17, 18], we satisfy this condition by considering OLs of the form: $U(x) = -V \cos(2x)$, $\mathcal{G}(x) = G + \cos(2x)$, with $V$ and $G$ related to the physical amplitude of the linear OL $V$ and to the scattering length $a_{s}(x) = a_{s0}(G + \cos(2x))$ by $V = VE_{R}$ and $G = \langle a_{s} \rangle / a_{s0}$ where $\langle a_{s} \rangle = \pi^{-1}\int_{-\infty}^{\infty} a_{s}(x) dx$ denotes the average of the scattering length over one period of the nonlinear OL. Here $E_{R} = \hbar^2 \pi^2 / (2m a^2)$ is the recoil energy, $d$ is the lattice constant, $a_{s0} > 0$ is the amplitude of the scattering length modulation, and $m$ the mass of an atom. Space and time variables in our formulas are measured in units $d/\pi$, and $h/E_{R}$, respectively, with the norm $N$ related to the physical number of atoms $N_0 = \frac{4a_{s0}d}{\pi}\sqrt{\langle a_{s} \rangle}$ and with $a_{s}$ the transverse oscillator length. From Fig. 1a we see that the OLs appropriate for our task are the ones with $G, V$ taken in the region between curves $G_{\pm}$. Notice that for fixed $G$ and $V$ a family of solutions, parameterized by the soliton norm $N = \int_{-\infty}^{\infty} |\psi|^2 dx$ (for a cigar-shaped configuration bifurcates from the opposite band edges (panel b). The bottom panels of Fig. 1 illustrate change of the symmetry of a GS as the different edges of the gap.

To stimulate RO between GSs we use a force $F = \nu \cos(\omega t)$ with amplitude $\nu \ll 1$ and frequency $\omega = E_g - 2\Delta$ slightly detuned from the gap width $E_g = E_+ - E_-$ (i.e. $\Delta \ll E_g$). The oscillation must involve a nontrivial rearrangement of the matter at each cycle, due to the change of symmetry of the state and the corresponding dynamics can be sustained for a long times only if the force is properly designed. In the following we use $\Delta$ as a free parameter to find optimal inter-band tunneling conditions.

III. TWO MODE MODEL FOR NONLINEAR RABI OSCILLATIONS

An analytical description of the phenomenon can be made in terms of a two-level model. To this end we consider $\sqrt{\nu/\omega} \ll 1$ as a small parameter and recall the semi-classical equation of motion $dq/dt = -\nu \cos(\omega t)$, leading to the change of the phase of the wavefunction as follows $e^{i(q(t) - q(0))} = e^{-i\nu x(\omega) \sin(\omega t)}$, and take into account that in the leading order the solution is a superposition of the states bordering the gap edges:

$$\psi \approx A_+ \varphi_+ e^{-i(E_++\Delta)t} + A_- \varphi_- e^{-i(E_+-\Delta)t}. \tag{2}$$

Here $A_\pm \equiv A_\pm(x,t) \sim \nu/\omega$ are slow varying amplitudes $(\partial A_\pm / \partial t) \sim \partial^2 A_\pm / \partial x^2 \sim (\nu/\omega)A_\pm$ of the Bloch functions $\varphi_\pm$ bordering the edges with energies $\mu_\pm = E_\pm + \Delta$, providing $\mu_+ - \mu_- = \omega$ and satisfying the above criteria $\Delta \sim \nu \ll \omega$ being small detuning towards the gap. Dropping details (see e.g. [10, 14]), we give the final system for the evolution of the amplitudes

$$\frac{\partial A_\pm}{\partial t} = -\frac{1}{2M_\pm} \frac{\partial^2 A_\pm}{\partial x^2} + \Delta A_\pm + \gamma A_\mp \pm (x_\pm |A_\mp|^2 + 2x_\pm |A_\pm|^2)A_\pm, \tag{3}$$

where $x = \int_{-\infty}^{\infty} \mathcal{G}(x)|\varphi_\pm|^2 dx$ and $\gamma = \frac{\nu}{\omega} \int_{-\infty}^{\infty} \frac{d^2 \varphi_\pm}{dx^2} \varphi_\pm dx$. In absence of nonlinearity and for spatially homogeneous amplitudes this system reduces to the model of
to estimate \( T \) accurately. Since \( \Omega \) imply conservation of \( N \) and \( \Omega \) between the two energy levels of a gap soliton, this firms the interpretation of the phenomenon as RO of two-level atoms in a light field. This confirms the interpretation of the phenomenon as Rabi oscillations of a gap soliton between the two energy levels \( \mu_\pm \). Neglecting the second derivative and nonlinearity in Eq.(3), the inter-band tunneling time can be estimated as half of the Rabi period, i.e. \( T_{\text{tun}} = \pi/\Omega \), with \( \Omega \) the usual linear Rabi frequency \( \Omega = 2\sqrt{\gamma^2 + \Delta^2} \). The results are depicted in Fig.2 where, in panels a,b, \( \nu = 0.1 \) (solid lines) and \( \nu = 0.2 \) (dashed lines); (c,d) dynamics of the populations, \( p_{1,2}(t) \), of the two bands for \( \nu = 0.1 \). Parameters are: \( V = 3 \), \( G = −0.644 \) (panels a,c) and \( E = 2.152 \) (panels b,d). Lower panels show the solitonic shapes corresponding to points A–F. In panels B and E soliton profiles are compared to stationary solutions (dashed lines) with the same \( N \) of the initial conditions \( (N = 0.61) \), but with energies in the vicinity of the opposite gap edge \( (E = 2.152 \) in panel B and \( E = −0.73 \) in panel E).

\[ \chi_\pm \left( \frac{x}{2\gamma} \right) = \frac{\text{sech}[\sqrt{-2M\Omega}\:x\:\pi]}{\sqrt{\pi}} \] where \( M \), \( \Omega \) and \( \Omega \) were fixed by \( N = \text{sech}\|\pm\Omega_{\parallel}\|/\pi|\pm\Omega_{\perp}| \). In the following we use this solution to find an optimal design for long-lived ROs. To this regard, we remark that long-lived ROs can be achieved only if a minimal loss of matter (ideally zero) from the solitons occurs during each oscillation cycle. For this one must impose the conservation of the energy and the conservation of \( N \) (complete exchange of particles between components at the turning points). Moreover, since the states must change symmetry every half cycles (being states at opposite gap edges) one must also require that the spatial localization of the two GSs at the turning points is approximately the same (this facilitates the symmetry exchange). The last condition is satisfied if \( M_{\pm}\chi_\pm = M_{-}\chi_- \), while the conservation of \( N \) and the conservation of the energy imply \( M_{\pm}\Omega_{\pm} = M_{-}\Omega_{+} \), and \( \Delta = (\Omega_{-} - \Omega_{+})/6 \), respectively. Since \( \Omega_{\pm} \) depends on \( N \), \( M_{\pm} \), \( \chi_{\pm} \), one can adjust these parameters to have all the above relations satisfied with the driving frequency \( \omega_0 = \Omega_g - 2\Delta \) chosen as \( \omega_0 = \Omega_g - (\Omega_+ - \Omega_-)/3 \) for an optimal design (see Fig.3). Notice that for \( \omega \neq \omega_0 \), while \( N \) will be conserved, the soliton energy will not be perfectly conserved during the tunneling process and that for fixed \( M_{\pm} \), \( \chi_{\pm} \), a change of \( N \) requires the adjustment of the optimal frequency. Contrary to the linear case, where the optimal transfer between levels is achieved for zero detuning, here the optimal transfer is obtained for \( \Delta \neq 0 \). This is a direct consequence of the nonlinearity of the system which replaces linear atomic wavefunctions with energies exactly at gap edges with nonspreading GSs wavepackets with energies detuned from band edges inside the gap. In the limit of zero nonlinearity (\( N \to 0 \)) GSs tend to extended Bloch states at gap edges and the linear optimal condition (\( \Delta = 0 \)) for RO is recovered.

IV. NUMERICAL RESULTS

In order to check these predictions we have performed direct numerical integrations of Eq.(1) with initial condition in a form of a stationary GS close to the bottom edge of the gap. The frequency detuning, the parameter values of the OLs, and the initial conditions were chosen to met the requirements of conservation of energy, soliton norm and widths in the solitonic states during the oscillation, as described above.

A. Nonlinear Rabi oscillations and tunneling time

The results are depicted in Fig.2 where, in panels a,b, we show that minima and maxima of the periodically varying energy of the system well match the GS levels, what represents the most clear evidence of the inter-band tunneling. Notice that the frequency of the energy oscillations increases with the force strength \( \nu \) and corre-
lates with similar oscillations of the band populations (Fig. 2d) computed as \( \rho_{1,2} = \int_{-\infty}^{\infty} |c_{1,2}(q)|^2 dq/N \), where \( c_n(q) = \int_{-\infty}^{\infty} \psi_{n,q}(x) dx \). One also observes change of the wave-packet symmetry: initially symmetric (with respect to \( x = 0 \)) soliton excited in the vicinity the lower gap edge (Fig. 2 A), in the vicinity of the upper gap edge transforms into an odd solution (Fig. 2 B). Similarly, initial odd soliton (Fig. 2 D) is transformed into the even one (Fig. 2 E). After one cycle of oscillations (points C and F) in the upper panels of Fig. 2, the soliton approximately restores its initial shape (c.f. panels A and C, and panels D and F).

To compare RO under optimal and arbitrary conditions, we show in Fig. 3a,b, the time evolution of the GS states projected on the second band (i.e. the population \( |c_2(q)|^2 \)) as obtained from numerical integrations of the NLS in the case of an optimally chosen nonlinearity \( G \) (Fig. 3h) and for a non optimal value of \( G \) for which small-amplitude GSs do not exist close to the upper band edge (Fig. 3j). We see that while in the first case the ROs are long-lived, in the latter case the soliton is destroyed after few periods of oscillation. From Fig. 3 we also see that the numerical frequency for the maximal transfer of atoms from one band to another is in very good agreement with the analytical value \( \omega_0 \) estimated from the two mode model.

In Fig. 4 we present the comparison of the tunneling time \( T_{tun} \) obtained from the direct numerical simulations and in the framework of the two-mode model. One observes that \( T_{tun} \) decreases with the increasing the force strength \( \nu \), and increases with the amplitude of the linear OL. The discrepancies between the exact and approximated models are negligible for moderate amplitudes of the linear OL (namely \( V \lesssim 2 \) in Fig. 4h), and become appreciable for a deep OL and small external force strength (\( V = 3 \) and \( \nu \lesssim 0.01 \) in Fig. 4a and \( V = 3 \) in Fig. 4b).

### B. Superposition of two gap soliton states

It is worth to note that by switching off the linear potential at a time \( t_s = (2n + 1)T_{tun} \) (\( n = 0, 1, \ldots \)) the state of the system will coincide with one of the station-
soliton states or linear combination of two gap solitons with appreciably long life-time.

We finally remark that similar phenomena are expected to occur also in arrays of optical waveguides periodically modulated along the propagation direction.

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