THE QUASICONFORMAL SUBINVARIANCE PROPERTY OF JOHN DOMAINS IN $\mathbb{R}^n$ AND ITS APPLICATION

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ABSTRACT. The main aim of this paper is to give a complete solution to one of the open problems, raised by Heinonen from 1989, concerning the subinvariance of John domains under quasiconformal mappings in $\mathbb{R}^n$. As application, the quasisymmetry of quasiconformal mappings is discussed.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we study the subinvariance property of John domains under quasiconformal mappings in $\mathbb{R}^n$. The motivation for this study stems from one of the open problems raised by Heinonen in [15]. The aim of this paper is to give a complete solution to this open problem and its application. John domains were introduced by John in [23] in his study of rigidity of local isometries. The term “John domain” was coined by Martio and Sarvas seventeen years later [29]. Among various equivalent characterizations, we shall adopt the definition for John domains in the sense of carrot property, which is called the carrot definition for John domains in the following. We shall give the precise definition together with other necessary notions and notations in the next section.

Throughout the paper we assume that $D$ and $D'$ are subdomains in $\mathbb{R}^n$ and that $f : D \to D'$ is a $K$-quasiconformal mapping with $K \geq 1$. See [32, 43] for definitions and properties of $K$-quasiconformal mappings.

In [8], the authors showed the following.

Theorem A. ([8, p. 120-121]) Suppose $D'$ is QED. Then for each QED subdomain $D_1$ in $D$, $f(D_1)$ is also QED.

The reader is referred to [11] for the definition of QED domains. By Theorem A and [34, Theorem 5.6], the following is obvious.

Theorem B. If $D'$ is $c$-uniform, then for each $c_1$-uniform subdomain $D_1$ in $D$, $f(D_1)$ is still $\rho$-uniform, where $\rho = \rho(n, K, c, c_1)$, which means that the constant $\rho$ depends only on $n$, $K$, $c$ and $c_1$.

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In the next result, the case (1) is due to Väisälä from [36, Theorem 2.20], whereas the case (2) is obtained by Heinonen [15, Theorem 7.1].

**Theorem C.** Suppose $D'$ is broad.

1. Then for each John subdomain $D_1$ in $D$, $f(D_1)$ is a John domain;
2. If both $D$ and $D'$ are bounded, then for each broad subdomain $D_1$ in $D$, $f(D_1)$ is a broad domain.

We refer to [29] for an early discussion on this topic. However, a natural problem is that whether $f(D_1)$ is a John domain for each John subdomain $D_1$ of $D$ when $D'$ is a John domain. This is an open problem raised by Heinonen [15] (in fact, this open problem was put forward by Väisälä) in the following form.

**Open Problem 1.1.** Suppose that $f$ is a quasiconformal mapping of a domain $D$ in $\mathbb{R}^n$ onto a John domain $D'$ in $\mathbb{R}^n$. Is it then true that every John subdomain of $D$ is mapped onto a John subdomain of $D'$ by $f$?

That is, if $D'$ is a $c$-John domain, then for every $c_1$-John subdomain in $D$, is its image under $f$ a $c_2$-John domain in $D'$, where $c_2 = c_2(n, K, c, c_1)$ (see e.g. [9] or [8])?

If the answer is yes, in what follows, we shall say that $f$ has the subinvariance property of John domains. Heinonen himself discussed this problem and as consequence he obtained that for a quasiconformal mapping from the unit ball $B$ in $\mathbb{R}^n$ onto a John domain $D'$, the image of each Stolz cone $C_M(w)$ with vertex at $w \in \partial B$, the boundary of $B$, under $f$ is uniform, see [15, Theorem 7.3], where $C_M(w)$ is defined to be the interior of the closed convex hull of $w$ and the hyperbolic ball centered at $0$ with radius $M > 0$. It is known that every ball in $\mathbb{R}^n$ is a $\frac{1}{2}$-uniform domain.

In this paper, we consider Open Problem 1.1 further. The examples constructed in Section 3 show that the answer to Open problem 1.1 is negative when one of $D$ and $D'$ is unbounded. This observation shows that it suffices for us to consider the case where both $D$ and $D'$ are bounded. In this case, we get the following affirmative answer.

**Theorem 1.1.** Suppose that $D$ and $D'$ are bounded subdomains in $\mathbb{R}^n$ and that $f : D \rightarrow D'$ is a $K$-quasiconformal mapping. If $D'$ is an $a$-John domain with center $y_0'$, then for every $c$-John subdomain $D_1$ in $D$ with center $z_0$, its image $f(D_1)$ is a $\tau$-John subdomain in $D'$ with center $f(z_0)$, where $\tau = \tau \left( n, K, a, c, \frac{\text{diam}(D)}{d_{D}(f^{-1}(y_0'))} \right)$.

In [36], Väisälä considered the question: When is $G \times \mathbb{R}^1$ quasiconformally equivalent to the unit open ball $B^3$ in $\mathbb{R}^3$, where $G$ is a domain in the plane $\mathbb{R}^2$? He proved that there is a quasiconformal mapping $f$ from $G \times \mathbb{R}^1$ to $B^3$ if and only if $G$ satisfies the so-called internal chord-arc condition, see [36, Theorem 5.2]. In the proof of [36, Theorem 5.2], the following result plays a key role.

**Theorem D.** ([36, Theorem 2.20]) Suppose that $f : D \rightarrow D'$ is a $K$-quasiconformal mapping between domains $D$ and $D' \subset \mathbb{R}^n$, where $D$ is $\phi$-broad. Suppose also that $A \subset D$ is a pathwise connected set and that $A' = f(A)$ has the $c_1$-carrot property in $D'$ with center $y_0' \in \overline{D'}$. If $y_0' \neq \infty$ and hence $y_0' \in D'$, we assume that $\text{diam}(A) \leq \frac{\text{diam}(D)}{d_{D}(f^{-1}(y_0'))}$.
Theorem E. ([15, Theorem 6.1]) Suppose that $D$ and $D'$ are bounded, that $f : D \to D'$ is $K$-quasiconformal, and that $D$ is $\varphi$-broad. If $A \subset D$ is such that $f(A)$ is $b$-LLC$_2$ with respect to $\delta_{D'}$ in $D'$, then $f|_A : A \to f(A)$ is weakly $H$-quasisymmetric in the metrics $\delta_D$ and $\delta_{D'}$, where $H$ depends only on the data
\[
\kappa_2 = \kappa_2 \left( n, K, b, \varphi, \frac{\delta_D(A)}{d_D(x_0)}, \frac{\delta_{D'}(f(A))}{d_{D'}(f(x_0))} \right),
\]
x$_0$ is a fixed point in $A$ and $\delta_D(A)$ denotes the $\delta_D$-diameter of $A$.

As an application of Theorem 1.1, we consider Theorem D further. Our result is as follows.

Theorem 1.2. Suppose that $f : D \to D'$ is a $K$-quasiconformal mapping between bounded subdomains $D$ and $D'$ in $\mathbb{R}^n$, where $D$ is a $b$-uniform domain. Suppose also that $A \subset D$ is a pathwise connected set and that $A' = f(A)$ has the $c_1$-carrot property in $D'$ with center $y'_0$. Then the restriction $f|_A$ is $\eta$-quasisymmetric in the metrics $\delta_D$ and $\delta_{D'}$ with $\eta$ depending only on the data $\kappa = \kappa \left( n, K, b, c_1, \frac{\text{diam}(D')}{d_{D'}(f(x_0))} \right)$, where $x_0 \in D$ satisfies $d_D(x_0) = \sup \{d_D(x) : x \in D \}$.

Remark 1.1. (1) Although, in Theorem 1.2, we replace the condition “$D$ being broad” in Theorem D by the one “$D$ being uniform”, the dependence of the function $\eta$ on the center of $A'$ in Theorem D is removed.
(2) Obviously, it follows from the proof of Theorem 1.2 that Theorem 1.2 still holds when $D$ is assumed to be broad and inner uniform.
(3) By Lemma 7.1 in Section 7, we see that $D$ has the $4b^2$-carrot property with center $x_0$.

The arrangement of this paper is as follows. In Section 2, we shall introduce necessary notions and notations, and record some useful results. Section 3 contains three counterexamples to Heinonen’s open problem and their proofs. These examples show that if one of the domains is unbounded then the answer to Heinonen’s open problem is negative. In view of these examples it suffices to consider the case where both domains are bounded. Thus, to give a positive answer to Heinonen’s open problem in this case, we need some auxiliary results which we include in Sections 4 and 5. Based on carrot arcs, in Section 4, we obtain a method of construction of uniform subdomains in a domain in $\mathbb{R}^n$ and, in addition, we present four more basic
results that will be useful for our discussion in the sequel. Section 5 is devoted to a
property for a class of special arcs which satisfy the so-called “QH-condition”. Prior
to the proof of Theorem 1.1, in Section 6, we give a way to construct new uniform
subdomains and new carrot arcs, and obtain several other related properties. Finally,
we present a proof of the main result in this section, namely, Theorem 6.1 from which
we prove Theorem 1.1 easily. In Section 7, based on Theorem 1.1 and a combination
of some related results obtained in Section 4, we prove Theorem 1.2.

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2. Preliminaries

2.1. Notation. Throughout the paper, we always use $B(x_0, r)$ to denote the open
ball \{ $x \in \mathbb{R}^n : |x - x_0| < r$ \} centered at $x_0$ with radius $r > 0$. Similarly, for the
closed balls and spheres, we use the notations $\overline{B}(x_0, r)$ and $S(x_0, r)$, respectively. Obvi-
ously, $B = B(0, 1)$.

2.2. John domains, uniform domains and carrot property.

Definition 2.1. A domain $D$ in $\mathbb{R}^n$ is said to be $c$-uniform if there exists a constant
c with the property that each pair of points $z_1, z_2$ in $D$ can be joined by a rectifiable
arc $\gamma$ in $D$ satisfying (cf. [28])

$$\ell(\gamma) \leq c |z_1 - z_2|$$

and

$$\min_{j=1,2} \{ \ell(\gamma[z_j, z]) \} \leq c d_D(z)$$

for all $z \in \gamma$, where $\ell(\gamma)$ denotes the arclength of $\gamma$, $\gamma[z_j, z]$ the part of $\gamma$ between $z_j$
and $z$, and $d_D(z)$ the distance from $z$ to the boundary $\partial D$ of $D$. Also we say that $\gamma$
is a double $c$-cone arc.

In order to introduce the definition of John domains, we need the following con-
cept.

Definition 2.2. A set $A \subset D$ in $\mathbb{R}^n$ is said to have the $c$-carrot property with center
$x_0 \in \overline{D}$ if there exists a constant $c$ with the property that for each point $z_1$ in $A$, $z_1$
and $x_0$ can be joined by a rectifiable arc $\gamma$ in $D$ satisfying (cf. [30, 36])

$$\ell(\gamma[z_1, z]) \leq c d_D(z)$$

for all $z \in \gamma$. Also we say that $\gamma$ is a $c$-carrot arc with center $x_0$.

Now, we are ready to introduce the carrot definition for John domains.

Definition 2.3. A domain $D$ in $\mathbb{R}^n$ is said to be a $c$-John domain with center $x_0 \in \overline{D}$
if $D$ has the $c$-carrot property with center $x_0 \in \overline{D}$. Especially, $x_0$ is taken to be $\infty$
when $D$ is unbounded.
This is the carrot definition for John domains. Now, there are plenty of alternative characterizations for uniform and John domains, see [7, 10, 12, 26, 28, 37, 38, 39, 40, 41]. The importance of uniform and John domains together with some special domains throughout the function theory is well documented, see [10, 26, 30, 34]. Moreover, John domains and uniform domains in \( \mathbb{R}^n \) enjoy numerous geometric and function theoretic features in many areas of modern mathematical analysis, see [2, 7, 12, 14, 24, 25, 34, 40] (see also [1]).

Definitions 2.1 and 2.3 (resp. Definition 2.2) are often referred to as the “arc-length” definitions for uniform domains and John domains (resp. the carrot property). When the arclength in Definitions 2.1 and 2.3 (resp. Definition 2.2) is replaced by the diameter, then it is called the “diameter” definition for uniform domains and John domains (resp. the carrot property), and in the latter definition, the assumption on the rectifiability of the arc is not necessary.

The following result reveals the close relationship between these two definitions.

**Theorem F.** ([29, 30, 35]) The “arc-length” definition for uniform domains and John domains (resp. the carrot property) is quantitatively equivalent to the “diameter” one. In particular, for a John domain the center can be taken to be the same.

### 2.3. Quasihyperbolic metric and solid arcs

Let \( \gamma \) be a rectifiable arc or path in \( D \). Then the *quasihyperbolic length* of \( \gamma \) is defined to be the number \( \ell_{k_D}(\gamma) \) given by (cf. [13]):

\[
\ell_{k_D}(\gamma) = \int_\gamma |dz|/d_D(z).
\]

For \( z_1, z_2 \) in \( D \), the *quasihyperbolic distance* \( k_D(z_1, z_2) \) between \( z_1 \) and \( z_2 \) is defined in the usual way:

\[
k_D(z_1, z_2) = \inf_{\gamma} \ell_{k_D}(\gamma),
\]

where the infimum is taken over all rectifiable arcs \( \gamma \) joining \( z_1 \) to \( z_2 \) in \( D \). An arc \( \gamma \) from \( z_1 \) to \( z_2 \) is called a *quasihyperbolic geodesic* if \( \ell_{k_D}(\gamma) = k_D(z_1, z_2) \). Each subarc of a quasihyperbolic geodesic is obviously a quasihyperbolic geodesic. It is known that a quasihyperbolic geodesic between two points in \( D \) always exists (cf. [12, Lemma 1]). Moreover, for \( z_1, z_2 \) in \( D \), we have (cf. [37, 43])

\[
k_D(z_1, z_2) \geq \inf_{\gamma} \log \left( 1 + \frac{\ell(\gamma)}{\min\{d_D(z_1), d_D(z_2)\}} \right) \geq \log \left( 1 + \frac{|z_1 - z_2|}{\min\{d_D(z_1), d_D(z_2)\}} \right)
\]

\[
\geq \log \left| \log \frac{d_D(z_2)}{d_D(z_1)} \right|,
\]

where \( \gamma \) denote curves in \( D \) connecting \( z_1 \) and \( z_2 \). If \( \gamma \) is a quasihyperbolic geodesic in \( D \) joining \( z_1 \) to \( z_2 \), then we know that for \( x \in \gamma \),

\[
k_D(z_1, z_2) \geq \log \left( 1 + \frac{\ell(x)}{d_D(x)} \right).
\]

If \( |z_1 - z_2| \leq d_D(z_1) \), we have [43]

\[
k_D(z_1, z_2) < \log \left( 1 + \frac{|z_1 - z_2|}{d_D(z_1) - |z_1 - z_2|} \right).
\]
The following characterization of uniform domains in terms of quasihyperbolic metric is useful for our discussions.

**Theorem G.** ([42, 2.50 (2)]) A domain $D \subset \mathbb{R}^n$ is $c$-uniform if and only if there is a constant $\mu_1$ such that for all $x, y \in D$,

$$k_D(x, y) \leq \mu_1 \log \left( 1 + \frac{|x - y|}{\min\{d_D(x), d_D(y)\}} \right),$$

where $\mu_1 = \mu_1(c)$.

This form of the definition of uniform domains is due to Gehring and Osgood [12]. As a matter of fact, in [12, Theorem 1], there was an additive constant in the inequality of Theorem G, but it was shown by Vuorinen [42, 2.50 (2)] that the additive constant can be chosen to be zero.

Next, we recall a relationship between the quasihyperbolic distance of points in a domain $D$ and the one of their images in $D'$ under $f$.

**Theorem H.** ([12, Theorem 3]) Suppose that $D$ and $D'$ are domains in $\mathbb{R}^n$, and that $f : D \to D'$ is a $K$-quasiconformal mapping. Then for $z_1, z_2 \in D$,

$$k_{D'}(z_1', z_2') \leq \mu_2 \max\{k_D(z_1, z_2), (k_D(z_1, z_2))^{1/\mu_2}\},$$

where the constant $\mu_2 = \mu_2(n, K) \geq 1$.

As a generalization of quasihyperbolic geodesics, Väisälä introduced the concept of solids in [39].

Let $\alpha$ be an arc in $D$. The arc may be closed, open or half open. Let $\vec{x} = (x_0, \ldots, x_n)$, $n \geq 1$, be a finite sequence of successive points of $\alpha$. For $h \geq 0$, we say that $\vec{x}$ is $h$-coarse quasihyperbolic if $k_D(x_{j-1}, x_j) \geq h$ for all $1 \leq j \leq n$. Let $\Phi_{k_D}(\alpha, h)$ be the family of all $h$-coarse quasihyperbolic sequences of $\alpha$. Set

$$s_{k_D}(\vec{x}) = \sum_{j=1}^{n} k_D(x_{j-1}, x_j)$$

and

$$\ell_{k_D}(\alpha, h) = \sup\{s_{k_D}(\vec{x}) : \vec{x} \in \Phi_{k_D}(\alpha, h)\}$$

with the agreement that $\ell_{k_D}(\alpha, h) = 0$ if $\Phi_{k_D}(\alpha, h) = \emptyset$. Then the number $\ell_{k_D}(\alpha, h)$ is the $h$-coarse quasihyperbolic length of $\alpha$.

**Definition 2.4.** Let $D$ be a domain in $\mathbb{R}^n$. An arc $\alpha \subset D$ is $(\nu, h)$-solid with $\nu \geq 1$ and $h \geq 0$ if

$$\ell_{k_D}(\alpha[x, y], h) \leq \nu k_D(x, y)$$

for all $x, y \in \alpha$.

Obviously, each quasihyperbolic geodesic is $(1, 0)$-solid. The following result is due to Väisälä.

**Theorem I.** ([39, Theorem 6.22]) Suppose that $\gamma \subset D \neq \mathbb{R}^n$ is a $(\nu, h)$-solid arc with endpoints $a_0, a_1$ and that $D$ is a $c$-uniform domain. Then there is a constant $c_2$ such that
Definition 2.5. A domain $D$ in $\mathbb{R}^n$ is said to be $c$-inner uniform if there exists a constant $c$ with the property that each pair of points $z_1, z_2$ in $D$ can be joined by a rectifiable arc $\gamma$ in $D$ satisfying (2.1) and $\ell(\gamma) \leq c\delta_D(z_1, z_2)$ (cf. [40]).

Obviously, “uniformity” implies “inner uniformity”.

Definition 2.6. Let $\varphi : (0, \infty) \to (0, \infty)$ be a decreasing homeomorphism. We say that $D$ is $\varphi$-broad if for each $t > 0$ and each pair $(C_0, C_1)$ of continua in $D$ the condition $\delta_D(C_0, C_1) \leq t\min\{\text{diam}(C_0), \text{diam}(C_1)\}$ implies

$$\text{Mod}(C_0, C_1; D) \geq \varphi(t),$$

where $\delta_D(C_0, C_1)$ denotes the $\delta_D$-distance between $C_0$ and $C_1$ and see, for example, [32] for the definition of $\text{Mod}(C_0, C_1; D)$, the modulus of a family of the curves in $D$ connecting $C_0$ and $C_1$.

Broad domains were introduced in [36] and it was later proved that a simply connected planar domain is broad if and only if it is inner uniform [30, 40]. Further, the following is known.

Theorem J. ([11, Lemma 2.6]) If $D$ is a $c$-uniform domain, then $D$ is $\varphi_c$-broad, where $\varphi_c$ depends only on $c$.

It is important to recall that the notion of broad domains also goes under the term Löwner space. The notion of a Löwner space was introduced by Heinonen and Koskela [17] in their study of quasiconformal mappings of metric spaces; Heinonens recent monograph [16] renders an enlightening account of these ideas. See [3, 4, 19, 31] etc for more related discussions.

2.5. Quasisymmetric mappings.

Definition 2.7. Let $X_1$ and $X_2$ be two metric spaces with distance written as $|x - y|$, and let $\eta : [0, \infty) \to [0, \infty)$ be a homeomorphism. We say that an embedding $f : X_1 \to X_2$ is $\eta$-quasisymmetric if $|a - x| \leq t|a - y|$ implies

$$|f(a) - f(x)| \leq \eta(t)|f(a) - f(y)|$$

for all $a, x, y \in X_1$, and if there is a constant $\tau \geq 1$ such that $|a - x| \leq |a - y|$ implies

$$|f(a) - f(x)| \leq \tau|f(a) - f(y)|,$$

then $f$ is said to be weakly $\tau$-quasisymmetric.
Obviously, “quasisymmetry” implies “weak quasisymmetry”. There are many references in literature in this line, see, for example, [3, 5, 6, 33] et c. Among them, Väisälä proved the following local quasisymmetry of quasiconformal mappings in $\mathbb{R}^n$.

**Theorem K.** ([33, Theorem 2.4]) Suppose that $D \subset \mathbb{R}^n$, and that $f$ is a $K$-quasiconformal mapping of $D$ onto a domain $D' \subset \mathbb{R}^n$. Then for $x \in D$ and $0 < \lambda < 1$, the restriction $f|_{B(x, \lambda D(x))}$ is $\eta$-quasisymmetric, where $\eta = \eta_{n,K,\lambda}$.

### 2.6. Linearly locally connected sets.

**Definition 2.8.** Suppose that $A \subset D$ and $b \geq 1$ is a constant. We say that $A$ is $b$-$LLC_2$ (resp. $b$-$LLC_2$ with respect to $\delta_D$) in $D$ if for all $x \in A$ and $r > 0$, any two points in $A \setminus B(x, br)$ (resp. $A \setminus B_\delta(x, br)$) can be joined in $D \setminus B(x, r)$ (resp. $D \setminus B_\delta(x, r)$), where

$$B_\delta(x, r) = \{z \in \mathbb{R}^n : \delta_D(z, x) < r\}.$$

If $A = D$, then we say that $D$ is $b$-$LLC_2$ (resp. $b$-$LLC_2$ with respect to $\delta_D$).

In [15], Heinonen proved

**Theorem L.** ([15, Lemma 7.2]) Let $D \subset \mathbb{R}^n$, and let $A$ be a $\varphi$-broad subdomain of $D$. Then $A$ is $b$-$LLC_2$ with $b = b(n, \varphi)$. In particular, $A$ is $\mu_3$-$LLC_2$ with respect to $\delta_D$ in $D$, where $\mu_3 = \mu_3(n, \varphi)$.

For convenience, in what follows in this paper, we always assume that $x, y, z, \ldots$ denote points in a domain $D$ in $\mathbb{R}^n$ and $x', y', z', \ldots$ the images in $D'$ of $x, y, z, \ldots$ under $f$, respectively. Also we assume that $\alpha, \beta, \gamma, \ldots$ denote curves in $D$ and $\alpha', \beta', \gamma', \ldots$ the images in $D'$ of $\alpha, \beta, \gamma, \ldots$ under $f$, respectively, and for a set $A$ in $D$, $A'$ always denotes its image in $D'$ under $f$.

### 3. Counterexamples to Heinonen's open problem

In this section, we shall construct three examples to show that the answer to Open problem 1.1 is negative when one of $D$ and $D'$ is unbounded, which are as follows.

**Example 3.1.** Suppose that $\mathbb{H}^+$ denotes the upper half plane of $\mathbb{R}^2 \cong \mathbb{C}$ and $\mathbb{D} = \{z : |z| < 1\}$, the unit disk in $\mathbb{C}$, and suppose that $f_1(z) = i \frac{z}{z+1}$, where $i^2 = -1$. Then $f_1$ maps $\mathbb{H}^+$ onto $\mathbb{D}$ and does not have the subinvariance property of John domains.

**Example 3.2.** Suppose that $f_2(z) = i \frac{1}{z}$. Obviously, $f_2$ maps $\mathbb{D}$ onto $\mathbb{H}^+$. Then $f_2$ does not have the subinvariance property of John domains.

**Example 3.3.** Suppose that $f_3(z) = -\frac{i}{z}$. Obviously, $f_3$ maps $\mathbb{H}^+$ onto $\mathbb{H}^+$. Then $f_3$ does not have the subinvariance property of John domains.

**The proof of Example 3.1.** It is known that both $\mathbb{H}^+$ and $\mathbb{D}$ are $\frac{\pi}{2}$-John disks. Suppose on the contrary that there is a constant $c \geq 1$ such that for every 4-John
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subdomain in $\mathbb{H}^+$, its image under the conformal mapping $f_1$ is $c$-John in $\mathbb{D}$, where $f_1(z) = \frac{z - i}{z + i}$ and $i^2 = -1$. We let

$$G_1 = \mathbb{H}^+ \setminus (0, (8c - 1)i].$$

Elementary calculations show that $G_1$ is a 4-John domain and

$$G_1' = \mathbb{D} \setminus (-i, (1 - \frac{1}{4c})i].$$

We take

$$z'_1 = \frac{1}{2} - \frac{3}{4}i \text{ and } z'_2 = \frac{1}{2} - \frac{3}{4}i.$$

Then there must exist a $c$-cone arc $\beta'$ in $G_1'$ joining $z'_1$ and $z'_2$. The fact that $G_1'$ is a bounded $c$-John domain guarantees the existence of such an arc $\beta'$. Let $z'_0$ be one of the intersection points of $\beta'$ with the interval $[(1 - \frac{1}{4c})i, i)$ in the imaginary axis (see Figure 1). Obviously, $d_{G_1'}(z'_0) \leq \frac{1}{8c}$, and thus

$$1 < \min\{|z'_1 - z'_0|, |z'_2 - z'_0|\} \leq \min\{\ell(\beta'[z'_1, z'_0]), \ell(\beta'[z'_0, z'_2])\} \leq c d_{D_1'}(z'_0) \leq \frac{1}{8}.$$

This obvious contradiction completes the proof of this example. □

![Figure 1](image1.png)

**Figure 1.** The domains $G_1$, $G_1'$, the points $z'_0$, $z'_1$, $z'_2$ and the arc $\beta'$ ($w = (8c - 1)i$ and $w' = (1 - \frac{1}{4c})i$)

The proof of Example 3.2. Let $G_2 = \mathbb{D} \setminus [0, 1)$. Since $f_2(z) = \frac{1 + z}{1 - z}$, we see that the image of $G_2$ under $f_2$ is $G'_2 = \mathbb{H}^+ \setminus [i, +\infty)$ (see Figure 2). It easily follows from [30, Section 2.4] that $G_2$ is a John disk, but obviously, $G'_2$ is not a John domain. □

![Figure 2](image2.png)

**Figure 2.** The domains $G_2$ and $G'_2
The proof of Example 3.3. Let $G_3 = \mathbb{H}^+ \setminus (0, i]$. Then under the mapping $f_3(z) = -\frac{1}{z}$, the image $G'_3 = \mathbb{H}^+ \setminus [i, +\infty)$ (see Figure 3). Elementary calculations show that $G_3$ is a 4-John domain, but obviously, $G'_3$ is not a John domain. \qed

4. The properties of carrot arcs

In this section, we shall prove some auxiliary results which are useful for the proofs of Theorems 1.1 and 1.2 given in the last two sections. Our first results concern the construction of uniform subdomains in a domain based on carrot arcs.

Suppose $G$ is a domain in $\mathbb{R}^n$. For $z_1$ and $z_0$ in $G$, if $\gamma$ is a $c$-carrot arc joining $z_1$ and $z_0$ in $G$ with center $z_0$, then we have the following lemma.

**Lemma 4.1.** There exists a simply connected domain $G_{1,0} = \bigcup_{i=1}^{k_0} B_i \subset G$ such that

1. $z_1, z_0 \in G_{1,0}$;
2. for each $i \in \{1, \ldots, k_0\}$,

$$\frac{1}{3\mu_4} d_G(x_i) \leq r_i \leq \frac{1}{\mu_4} d_G(x_i);$$

3. if $k_0 \geq 3$, then for all $i, j \in \{1, \ldots, k_0\}$ with $|i - j| \geq 2$,

$$\text{dist}(B_i, B_j) \geq \frac{1}{2^5 \mu_5} \max\{r_i, r_j\};$$
(4) if \( k_0 \geq 2 \), then for each \( i \in \{1, \ldots, k_0 - 1\} \)
\[ r_i + r_{i+1} - |x_i - x_{i+1}| \geq \frac{1}{2^9 \mu_5} \max\{r_i, r_{i+1}\}; \]

(5) \( d_G(z_0) \leq 2^8 \mu_4 \mu_5 d_{G_{1,0}}(z_0) \),
where \( B_i = B(x_i, r_i), x_i \in \gamma, x_i \not\in B_{i-1} \) for each \( i \in \{2, \ldots, k_0\} \).

Remark 4.1. It easily follows from the construction in Lemma 4.1 that the intersection \( \partial G_{1,0} \cap \partial G \) is an empty set.

Lemma 4.2. The domain \( G_{1,0} \) constructed in Lemma 4.1 is a \( 2^{12} c^2 \mu_4 \mu_5 \)-uniform domain.

Proof. By definition, we only need to show that for \( u_1 \) and \( u_2 \in G_{1,0} \), there is a double \( 2^{12} c^2 \mu_4 \mu_5 \)-cone arc \( \lambda \) in \( G_{1,0} \) connecting \( u_1 \) and \( u_2 \).

We first consider the case where there is an \( i \in \{1, \ldots, k_0 - 1\} \) such that \( u_1, u_2 \in B_i \cup B_{i+1} \). Under this assumption, clearly, the existence of \( \lambda \) easily follows from [27, Theorem 1.2].

Next, we consider the case where there are \( i, j \in \{1, \ldots, k_0\} \) such that \( j - i \geq 2, u_1 \in B_i, u_2 \in B_j \) and \( \{u_1, u_2\} \) is not contained in \( B_i \cup B_{t+1} \) for all \( t \in \{i, \ldots, j-1\} \).

It suffices to discuss the case: \( u_1 \not\in [x_i, x_{i+1}] \) and \( u_2 \not\in [x_{j-1}, x_j] \) since the discussions for other cases are similar, where \( [x_i, x_{i+1}] \) denotes the line segment with endpoints \( x_i \) and \( x_{i+1} \). In this case, we let
\[ \lambda = [u_1, x_i] \cup [x_i, x_{i+1}] \cup \cdots \cup [x_{j-1}, x_j] \cup [x_j, u_2]. \]

For each \( u \in \lambda \), if \( u \in [u_1, x_i] \), then
\[ \ell(\lambda[u_1, u]) = |u_1 - u| < d_{G_{1,0}}(u). \]

Also if \( u \in [u_2, x_j] \), then
\[ \ell(\lambda[u_2, u]) = |u_2 - u| < d_{G_{1,0}}(u). \]

In the following, we consider the case \( u \in \lambda[x_i, x_j] \). Clearly, there exists a \( t \in \{i, \ldots, j\} \) such that \( u \in B_t \). If \( u \in B_j \), then by Lemma 4.1,
\[ \ell(\lambda[u_2, u]) \leq 2r_j \leq 2^7 \mu_5 d_{G_{1,0}}(u). \]

Similarly, if \( u \in B_i \), then
\[ \ell(\lambda[u_1, u]) \leq 2r_i \leq 2^7 \mu_5 d_{G_{1,0}}(u). \]

So it remains to consider the case: \( u \in \lambda \setminus (B_i \cup B_j) \). Then Lemma 4.1 implies
\[ \ell(\lambda[u_1, u]) \leq \ell(\lambda[u_1, x_i]) + r_i \leq 3c_4 d_{G_{1,0}}(x_t) + r_t \leq 2^6 (3c_4 + 1) \mu_5 d_{G_{1,0}}(u). \]
Hence $\lambda$ is a $2^6(3c\mu_4 + 1)\mu_5$-cone arc. It follows from Lemma 4.1 that
\[
\max\{r_i, r_j\} \leq 2^5\mu_5|u_1 - u_2|,
\]
whence
\[
\ell(\lambda[u_1, u_2]) \leq r_i + \ell(\lambda[x_i, x_j]) + r_j \leq 2^6\mu_5|u_1 - u_2| + cd_G(x_j) \leq 2^7c\mu_4\mu_5|u_1 - u_2|.
\]

The proof of the existence of $\lambda$ is finished. \qed

Our next result is an estimate on the distance from the points in carrot arcs to the boundary of $G$, which plays a key role in the discussions in Section 5. For the proof of the estimate, the following result is necessary.

Lemma 4.3. For $x, y \in G$, if $|x - y| \leq \mu_6 d_G(y)$, then $d_G(y) \geq \frac{1}{2\mu_6} d_G(x)$, where $\mu_6 \geq 1$ is a constant.

Proof. We divide the proof into two cases: $|x - y| < \frac{1}{2} d_G(x)$ and $|x - y| \geq \frac{1}{2} d_G(x)$. For the first case, we see that
\[
d_G(y) \geq d_G(x) - |x - y| > \frac{1}{2} d_G(x).
\]
For the second case,
\[
d_G(y) \geq \frac{1}{2\mu_6} |x - y| \geq \frac{1}{2\mu_6} d_G(x).
\]
The proof is complete. \qed

Suppose $\chi \subset G$ is a $\mu_7$-carrot arc from $x_1$ to $x_2$ with the center $x_2$ in $G$, where $\mu_7 \geq 1$ is a constant. Then we have

Lemma 4.4. For every $u \in \chi[x_1, x_2]$,
\[
d_G(u) \geq \frac{2\ell(\chi[x_1, u]) + d_G(x_1)}{4\mu_7}.
\]

Proof. For every $u \in \chi[x_1, x_2]$,
\[
d_G(u) \geq \frac{\ell(\chi[x_1, u])}{\mu_7}.
\]
Then by Lemma 4.3, we have
\[
d_G(u) \geq \frac{1}{2\mu_7} d_G(x_1).
\]
Hence
\[
d_G(u) \geq \frac{2\ell(\chi[x_1, u]) + d_G(x_1)}{4\mu_7}
\]
as required. \qed

Our last two results are on the estimate on the quasihyperbolic distance between two points in a ball in $G$ and the comparison of the distances from the images of the points in a ball in $G$ to the boundary of $G'$, which will be crucial for the discussions in Sections 5 and 6.
Lemma 4.5. Suppose \( G \) is a domain in \( \mathbb{R}^n \) and \( x \in G \). Then for all \( y \in \overline{B}(x, \frac{1}{b}d_G(x)) \) with \( b > 1 \), we have

\[
k_G(x, y) < \frac{1}{b-1}.
\]

Proof. Since \( d_G(z) \geq d_G(x) - |x - z| \geq (1 - \frac{1}{b})d_G(x) \) for each \( z \in [x, y] \), we get

\[
k_G(x, y) \leq \int_{[x,y]} \frac{|dz|}{d_G(z)} \leq \frac{|x - y|}{(1 - \frac{1}{b})d_G(x)} < \frac{1}{b-1}.
\]

The proof is complete. \( \square \)

Lemma 4.6. Suppose that \( G, G' \subset \mathbb{R}^n \), that \( f : G \rightarrow G' \) is a \( K \)-quasiconformal mapping, and that \( x \in G \). Then for each \( y \in \overline{B}(x, \frac{24}{\rho_1}d_G(x)) \), we have

\[
|x' - y'| < \frac{1}{5} \min\{d_{G'}(x'), d_{G'}(y')\} \quad \text{and} \quad \frac{4}{5}d_{G'}(y') < d_{G'}(x') < \frac{5}{4}d_{G'}(y'),
\]

where \( \rho_1 = (6\mu_2)^{4\mu_2} \) and \( \mu_2 \) is from Theorem H.

Proof. For \( x \in G \) and \( y \in \overline{B}(x, \frac{24}{\rho_1}d_G(x)) \), by Lemma 4.5, we have

\[
k_G(x, y) \leq \frac{24}{\rho_1 - 24} < 1,
\]

and so Theorem H implies

\[
\log \left( 1 + \frac{|x' - y'|}{\min\{d_{G'}(x'), d_{G'}(y')\}} \right) \leq k_{G'}(x', y') \leq \mu_2(k_G(x, y))^{\frac{1}{\mu_2}} < \frac{1}{6}.
\]

It follows that

\[
|y' - x'| < \frac{1}{5} \min\{d_{G'}(x'), d_{G'}(y')\},
\]

whence

\[
d_{G'}(y') \geq d_{G'}(x') - |x' - y'| > \frac{4}{5}d_{G'}(x')
\]

and

\[
d_{G'}(x') \geq d_{G'}(y') - |x' - y'| > \frac{4}{5}d_{G'}(y'),
\]

from which the proof follows. \( \square \)
In this section, we introduce a class of special arcs and prove a related lemma. This lemma is useful for the proof of Theorem 6.1 given in the next section.

Suppose that \( G \) is a domain in \( \mathbb{R}^n \). For \( w_1, w_2 \in G \), we suppose \( \beta \subset G \) is an arc with endpoints \( w_1 \) and \( w_2 \). Then we introduce the following concept.

**QH condition.** We say that \( \beta \) satisfies \( QH(s_0, s_1, s_2) \)-condition in \( G \) if there exist positive constants \( s_0, s_1 \) and \( s_2 \), where \( s_1 \) is an integer and \( s_2 \geq 1 \), satisfying the following (see Figure 5):

1. \( k_G(w_1, w_2) \geq s_0 \);
2. There exist successive points \( \eta_0 (= w_1), \eta_1, \ldots, \eta_{s_1} (= w_2) \) in \( \beta \) such that for all \( i, j \in \{0, \ldots, s_1 - 1\} \),
   \[
   \frac{1}{s_2} k_G(\eta_j, \eta_{j+1}) \leq k_G(\eta_i, \eta_{i+1}) \leq s_2 k_G(\eta_j, \eta_{j+1});
   \]
3. For each \( i \in \{0, \ldots, s_1 - 2\} \) and \( \eta \in \beta[\eta_{i+1}, w_2] \),
   \[
   k_G(\eta_i, \eta_{i+1}) \leq s_2 k_G(\eta_i, \eta).
   \]

Roughly speaking, the \( QH \) condition says that \( \beta \) can be partitioned into a number of pieces, such that the \( k_G \)-distance between the endpoints of the pieces are comparable, and so that the \( k_G \)-distance between \( \eta_i \) and \( \eta_{i+1} \) of the \( i \)th piece is comparable to the \( k_G \)-distance from the left endpoint \( \eta_i \) to any point in the subcurve of \( \beta \) from the right endpoint \( \eta_{i+1} \) to the final endpoint \( w_2 \). By the definition of \( QH \) condition, the following is obvious.

**Corollary 5.1.** Suppose \( \beta \) is a quasihyperbolic geodesic in \( G \) and \( 0 < s_0 \leq \ell_{k_G}(\beta) < \infty \). Then it satisfies \( QH(s_0, s_1, s_2) \)-condition in \( G \), where \( s_1 = 2 \) and \( s_2 = 1 \).

Let \( w_0 \in \beta \) be such that (see Figure 5)

\[
(5.1) \quad d_G(w_0) = \inf_{z \in \beta} d_G(z).
\]
For each $i \in \{0, \ldots, s_1 - 1\}$, we let $u_i \in \beta[\eta_i, \eta_{i+1}]$ (see Figure 5) satisfy
\[
k_G(u_i, \eta_{i+1}) = \frac{1}{4s_2}k_G(\eta_i, \eta_{i+1}).
\]

Suppose that $G$ is an $a$-John domain with center $x_0$, and that $s_1 \geq [(4a)^2n]$ is an integer, $s_2 \geq 1$, $s_0 \geq 8s_1s_2^2$. Then we have the following result.

**Theorem 5.1.** Suppose $\beta$ denotes an arc in $G$ which satisfies $QH(s_0, s_1, s_2)$-condition in $G$. Then

1. for every $i \in \{0, 1, \ldots, s_1 - 1\}$,
   \[
   \min\{d_G(u_i), d_G(\eta_{i+1})\} < \text{diam}(\beta);
   \]
2. $k_G(w_1, w_2) \leq 48a^2s_1s_2^2 \log \left(1 + \frac{\text{diam}(\beta)}{d_G(w_0)}\right)$.

**Proof.** We prove the conclusion (1) in Theorem 5.1 by a method of contradiction. Suppose on the contrary that there exists some $t \in \{0, 1, \ldots, s_1 - 1\}$ such that
\[
\min\{d_G(u_t), d_G(\eta_{t+1})\} \geq \text{diam}(\beta).
\]

Then
\[
[u_t, \eta_{t+1}] \subset \overline{B}(u_t, \frac{1}{2}d_G(u_t)) \cup \overline{B}(\eta_{t+1}, \frac{1}{2}d_G(\eta_{t+1}))
\]
which implies
\[
k_G(u_t, \eta_{t+1}) < 2.
\]

Then by the definition of $QH$ condition, we see that for each $i \in \{0, 1, \ldots, s_1 - 1\}$,
\[
k_G(\eta_i, \eta_{i+1}) \leq s_2k_G(\eta_i, \eta_{i+1}) = 4s_2^2k_G(u_t, \eta_{t+1}),
\]
and so
\[
k_G(w_1, w_2) \leq \sum_{i=0}^{s_1-1} k_G(\eta_i, \eta_{i+1}) \leq 4s_2^2 \sum_{i=0}^{s_1-1} k_G(u_t, \eta_{t+1}) < 8s_1s_2^2,
\]
which contradicts the assumption that “$k_G(w_1, w_2) \geq s_0$”, and thus the proof of (1) in Theorem 5.1 is complete.

To prove (2) of this lemma, we let $\beta_{1i}$ be an $a$-carrot arc joining $u_i$ and $x_0$ in $G$, and let $\beta_{2i}$ be an $a$-carrot arc joining $\eta_{i+1}$ and $x_0$ in $G$ for each $i \in \{0, \ldots, s_1 - 1\}$ (see Figure 6). This can be done because $G$ is an $a$-John domain with center $x_0$. Let
\[
\beta_i = \beta_{1i} \cup \beta_{2i}.
\]

If there exists an $s \in \{0, \ldots, s_1 - 1\}$ such that
\[
\ell(\beta_s) \leq \text{diam}(\beta),
\]
then we see that $x_0 \in G$. Thus from Lemma 4.4, we deduce that
\[
k_G(u_s, \eta_{s+1}) \leq \int_{\beta_s} \frac{|dx|}{d_G(x)} \leq \int_{\beta_{1s}} \frac{|dx|}{d_G(x)} + \int_{\beta_{2s}} \frac{|dx|}{d_G(x)}
\]
\[
< 8a \log \left(1 + \frac{\text{diam}(\beta)}{d_G(w_0)}\right),
\]
In the remaining case, that is, for each $i \in \{0, \ldots, s_1-1\}$, 

$$\ell(\beta_i) > \text{diam}(\beta).$$

It is possible that $x_0 = \infty$. To apply Lemma 4.4 to continue the proof, we choose a point from each $\beta_i$ ($i \in \{0, \ldots, s_1-1\}$) to replace $x_0$ in the following way: It follows from (1) in this lemma that for each $i \in \{0,1,\ldots,s_1-1\}$, there exists $v_i \in \beta_i$ satisfying

$$\frac{1}{2a} \text{diam}(\beta) \leq d_G(v_i) < \text{diam}(\beta).$$

Then we easily obtain that for all $i \in \{0,\ldots,s_1-1\}$,

$$\mathbb{B}(v_i, \frac{1}{2}d_G(v_i)) \subset \mathbb{B}(\eta_0, (a + \frac{3}{2})\text{diam}(\beta)),$$

since $|v_i - \eta_0| < (1+a)\text{diam}(\beta)$. We see that there exist $p \neq q \in \{0,\ldots,s_1-1\}$ such that

$$(5.3) \quad \mathbb{B}(v_p, \frac{1}{2}d_G(v_p)) \cap \mathbb{B}(v_q, \frac{1}{2}d_G(v_q)) \neq \emptyset,$$

because otherwise,

$$(a + \frac{3}{2})^n \text{Vol} \left( \mathbb{B}(\eta_0, \text{diam}(\beta)) \right) = \text{Vol} \left( \mathbb{B}(\eta_0, (a + \frac{3}{2})\text{diam}(\beta)) \right) > \left( \frac{1}{4a} \right)^n s_1 \text{Vol} \left( \mathbb{B}(\eta_0, \text{diam}(\beta)) \right),$$

where “Vol” denotes the volume. This is clearly a contradiction since $s_1 \geq [(4a)^2]$. We divide the rest of the arguments into four cases:

1. $v_p \in \beta_{2p}$ and $v_q \in \beta_{1q}$ (see Figure 7);
2. $v_p \in \beta_{2p}$ and $v_q \in \beta_{2q}$;
3. $v_p \in \beta_{1p}$ and $v_q \in \beta_{1q}$;
4. $v_p \in \beta_{1p}$ and $v_q \in \beta_{2q}$,
The quasiconformal subinvariance property of John domains in $\mathbb{R}^n$ and its application

where $p < q$.

It suffices to discuss the first case since the discussions for the remaining three cases are similar. First, we exploit Lemma 4.4 to estimate $k_G(\eta_{p+1}, u_q)$ in terms of $\frac{\text{diam}(\beta)}{d_G(w_0)}$. It follows from Lemma 4.4 that

$$k_G(v_p, \eta_{p+1}) \leq \ell k_G(\beta_p[v_p, \eta_{p+1}]) \leq 4a^2 \log \left(1 + \frac{\text{diam}(\beta)}{d_G(w_0)}\right),$$

and by (5.3), obviously,

$$k_G(v_p, v_q) < 2.$$

Thus

(5.4) $$k_G(\eta_{p+1}, u_q) \leq k_G(\eta_{p+1}, v_p) + k_G(v_p, v_q) + k_G(v_q, u_q) \leq 8a^2 \log \left(1 + \frac{\text{diam}(\beta)}{d_G(w_0)}\right) + 2.$$

Since $k_G(w_1, w_2) > s_1^2$, we see that $\text{diam}(\beta) > \frac{2}{3}d_G(w_0)$. For otherwise,

$$|w_1 - w_0| \leq \frac{2}{3}d_G(w_0) \quad \text{and} \quad |w_2 - w_0| \leq \frac{2}{3}d_G(w_0),$$

which implies that $k_G(w_1, w_2) < 3$. It is impossible. Therefore, an elementary computation shows that

$$2 \leq 4a^2 \log \left(1 + \frac{\text{diam}(\beta)}{d_G(w_0)}\right),$$

whence (5.4) yields

(5.5) $$k_G(\eta_{p+1}, u_q) \leq 12a^2 \log \left(1 + \frac{\text{diam}(\beta)}{d_G(w_0)}\right).$$

Next, we establish a relationship between $k_G(\eta_i, \eta_{i+1})$ and $k_G(\eta_{p+1}, u_q)$ for $i \in \{0, \ldots, s_1 - 1\}$. We know from the $QH$ condition and the choice of $u_q$ that for each
The combination of (5.2) and (5.7) completes the proof of (2) in Theorem 5.1.

and if \( q = p + 1 \), then

\[
k_G(\eta_i, \eta_{i+1}) \leq s_2 k_G(\eta_{p+1}, \eta_{p+2}) \leq \frac{4}{4s_2 - 1} s_2^2 k_G(\eta_{p+1}, u_q),
\]

since \( k_G(\eta_{p+1}, u_q) \geq k_G(\eta_{p+1}, \eta_{p+2}) - k_G(\eta_{p+2}, u_q) \). Hence we have proved that for each \( i \in \{0, \ldots, s_1 - 1\} \),

\[
(5.6) \quad k_G(\eta_i, \eta_{i+1}) \leq s_2 k_G(\eta_{p+1}, \eta_{p+2}) \leq 4s_2^2 k_G(\eta_{p+1}, u_q).
\]

Now, we are ready to finish the proof for this case. By (5.5) and (5.6), we have

\[
(5.7) \quad k_G(w_1, w_2) \leq \sum_{i=0}^{s_1-1} k_G(\eta_i, \eta_{i+1}) \leq 48a_2 s_1 s_2^2 \log \left( 1 + \frac{\text{diam}(\beta)}{d_G(w_0)} \right).
\]

The combination of (5.2) and (5.7) completes the proof of (2) in Theorem 5.1. \( \square \)

6. Quasiconformal subinvariance property of John domains

In this section, we assume that \( D \) and \( D' \) are bounded subdomains in \( \mathbb{R}^n \), that \( D' \) is an \( a \)-John domain with center \( y'_0 \in D' \), that \( f : D \to D' \) is a \( K \)-quasiconformal mapping, and that \( D_1 \subset D \) is a \( c \)-John domain with center \( z_0 \). For any \( z_1 \in D_1 \), we shall construct a carrot arc in \( D'_1 \) to join \( z'_1 \) and \( z'_0 \) in the sense of “diameter”, which is stated as Theorem 6.1. Clearly, this approach shows that \( D'_1 \) is a John domain, from which our main result in this paper, Theorem 1.1, will be easily proved. In order to get such a carrot arc in \( D' \), we first construct a carrot arc in \( D \), which is included in Lemma 6.3 or Lemma 6.4, and then we shall prove that the image of the obtained carrot arc under \( f \) is our desired carrot arc in \( D' \). This will be reached through a series of lemmas which are divided into two groups. To this end, we let \( \gamma_1 \subset D_1 \) denote a rectifiable arc with endpoints \( z_1 \) and \( z_0 \) satisfying

\[
\ell(\gamma_1[z_1, z]) \leq c d_{D_1}(z)
\]

for all \( z \in \gamma_1 \), i.e. \( \gamma_1 \) is a \( c \)-carrot arc with center \( z_0 \).

6.1. The constructions of a uniform domain and an arc in \( D_1 \). It follows from Lemma 4.1 that the following result is obvious.

**Lemma 6.1.** There exists a simply connected domain \( D_{1,0} = \bigcup_{i=1}^{k_1} B_{i,j} \subset D_1 \) such that

1. \( z_1, z_0 \in D_{1,0} \);
2. for each \( i \in \{1, \ldots, k_1\} \),

\[
\frac{1}{3} \rho_1 d_{D_1}(x_{1,i}) \leq r_{1,i} \leq \frac{1}{\rho_1} d_{D_1}(x_{1,i});
\]
(3) if $k_1 \geq 3$, then for all $i, j \in \{1, \ldots, k_1\}$ with $|i - j| \geq 2$,
\[
\text{dist}(B_{1,i}, B_{1,j}) \geq \frac{1}{2^5 \rho_2} \max\{r_{1,i}, r_{1,j}\};
\]
(4) if $k_1 \geq 2$, then for each $i \in \{1, \ldots, k_1 - 1\}$,
\[
r_{1,i} + r_{1,i+1} - |x_{1,i} - x_{1,i+1}| \geq \frac{1}{2^5 \rho_2} \max\{r_{1,i}, r_{1,i+1}\},
\]
where $B_{1,i} = B(x_{1,i}, r_{1,i})$, $x_{1,i} \in \gamma_1$, $x_{1,i} \not\in B_{1,i-1}$ for each $i \in \{2, \ldots, k_1\}$, $\rho_1$ is from Lemma 4.6 and $\rho_2 = 2^{32+c^2} \rho_1^2$.

Moreover, Lemma 4.2 gives

**Lemma 6.2.** The domain $D_{1,0}$ constructed in Lemma 6.1 is a $\rho_3$-uniform domain, where $\rho_3 = 2^{12} c^2 \rho_1 \rho_2$.

Now we set
\[
\gamma_{1,0} = [z_1, x_{1,2}] \cup \cdots \cup [x_{1,k_1-1}, x_{1,k_1}] \cup [x_{1,k_1}, z_0] \quad \text{(see Figure 8)}.
\]

Obviously, $\gamma_{1,0} \subset D_{1,0}$: In the rest of this section, our main aim is to prove that the image $\gamma'_{1,0}$ of $\gamma_{1,0}$ under $f$ is the desired carrot arc in $D'$. First, we need properties on $\gamma_{1,0}$, $D_{1,0}$ and $D'_{1,0}$, which are included in the next two subsections.

![Figure 8](image_url)  
**Figure 8.** The arc $\gamma_{1,0}$ joining $z_0$ and $z_1$

### 6.2. The properties of $\gamma_{1,0}$.

In this subsection, we show several lemmas on $\gamma_{1,0}$ which will be used later on.

The following lemma shows that $\gamma_{1,0}$ is a carrot arc in $D_1$ with center $z_0$.

**Lemma 6.3.** For each $z \in \gamma_{1,0}$, we have $\ell(\gamma_{1,0}[z_1, z]) \leq \rho_5 d_{D_1}(z)$, where $\rho_5 = \frac{8}{7} c$.
Lemma 6.5. Clearly, for every \( z \in \gamma_{1,0} \), there exists an \( i \in \{1, \ldots, k_1\} \) such that \( z \in B_{1,i} = \mathbb{B}(x_{1,i}, r_{1,i}) \). It follows from Lemma 6.1 that

\[
\ell(\gamma_{1,0}[z_1, z]) \leq \ell(\gamma_{1,0}[z_1, x_{1,i}]) + r_{1,i} \leq c d_{D_i}(x_{1,i}) + r_{1,i} \leq \rho_5 d_{D_i}(z),
\]

since \( d_{D_i}(x_{1,i}) \leq |x_{1,i} - z| + d_{D_i}(z) \) and \( r_{1,i} = \frac{1}{\rho_i} d_{D_i}(x_{1,i}) \). Hence the lemma holds. \( \square \)

Further, from Lemma 6.1 and the similar reasoning as in the proof of [22, Theorem 1.8], we have the following.

**Lemma 6.4.** Suppose that \( x_1 \in \gamma_{1,0} \). Then for \( x_2 \in \gamma_{1,0}[x_1, z_0] \), the part \( \gamma_{1,0}[x_1, x_2] \) of \( \gamma_{1,0} \) is a double \((2^3c \rho_5)^2(2^6 \rho_2 + 1)\)-cone arc in \( D_{1,0} \).

The next two results show that every part of \( \gamma_{1,0} \) is solid in \( D \) and in \( D_1 \), respectively.

**Lemma 6.5.** For \( x_1, x_2 \in \gamma_{1,0} \), \( \ell_{k_D}(\gamma_{1,0}[x_1, x_2]) \leq 2^{16} c^3 \rho_1 \rho_2 k_D(x_1, x_2) \), where \( \rho_1 \) and \( \rho_2 \) are from Lemma 6.1.

**Proof.** In view of Lemmas 4.4, 6.4 and 6.3, we have

\[
\ell_{k_D}(\gamma_{1,0}[x_1, x_2]) = \int_{\gamma_{1,0}[x_1, x_2]} \frac{|dx|}{d_D(x)} \leq 2 \rho_5 \log \left( 1 + \frac{2(\ell(\gamma_{1,0}[x_1, x_2]))}{d_D(x_1)} \right)
\]

\[
< 2^{16} c^3 \rho_1 \rho_2 k_D(x_1, x_2)
\]
as required. \( \square \)

Similarly, we have

**Lemma 6.6.** For \( u_1, u_2 \in \gamma_{1,0} \), \( \ell_{k_D}(\gamma_{1,0}[u_1, u_2]) \leq 2^{16} c^3 \rho_1 \rho_2 k_{D_1}(u_1, u_2) \).

We easily infer from Lemmas 4.4 and 6.3 the following which will be used later on.

**Lemma 6.7.** Suppose \( x_1 \) and \( x_2 \in \gamma_{1,0} \). Then for every \( u \in \gamma_{1,0}[x_1, x_2] \), we have

\[
(1) \ d_{D_i}(u) \geq \frac{2(\ell(\gamma_{1,0}[x_1, u])) + d_{D_i}(x_1)}{4 \rho_5}, \quad \text{and}
\]

\[
(2) \ d_D(u) \geq \frac{2(\ell(\gamma_{1,0}[x_1, u])) + d_D(x_1)}{4 \rho_5}.
\]

### 6.3. The properties of the domain \( D_{1,0} \) and its image \( D'_{1,0} \).

In the proof of the main result in this section, Theorem 6.1, the following two results on \( D_{1,0} \) and \( D'_{1,0} \) are useful.

**Lemma 6.8.** For each \( z \in D_{1,0} \), we have

\[
d_{D_i}(z) \geq (\rho_1 - 1) d_{D_{1,0}}(z),
\]

and for each \( z \in \gamma_{1,0}[z_1, x_{1,k_1}] \),

\[
(1) \ d_{D_i}(z) \leq 2^6(3 \rho_1 + 1) \rho_2 d_{D_{1,0}}(z);
\]

\[
(2) \ d_{D'_i}(z') \leq (2^7(3 \rho_1 + 1) \rho_2 \mu_2)^\rho_2 d_{D'_{1,0}}(z'),
\]

where \( \mu_2 \) is from Theorem H.
Proof. For $z \in D_{1,0}$, by Lemma 6.1, there exists a $j \in \{1, \ldots, k_1\}$ such that $z \in B_{1,j}$. Further, we assume that $B_{1,i}$ is the last ball from $B_{1,1}$ to $B_{1,k_1}$ such that $z \in B_{1,i}$. Then

$$d_{D_1}(z) \geq d_{D_1}(x_{1,i}) - |z - x_{1,i}| \geq (\rho_1 - 1)d_{D_{1,0}}(x_{1,i}) \geq (\rho_1 - 1)d_{D_{1,0}}(z),$$

from which the first inequality follows.

For the inequality (1), let $z \in \gamma_{1,0}[z_1, x_{1,k_1}]$. We still assume that $B_{1,i}$ is the last ball from $B_{1,1}$ to $B_{1,k_1}$ such that $z \in B_{1,i}$. Again, Lemma 6.1 implies

$$d_{D_{1,0}}(z) \geq \frac{1}{2^6} \rho_1 r_{1,i}$$

and

$$d_{D_1}(z) \leq d_{D_1}(x_{1,i}) + r_{1,i} \leq (3\rho_1 + 1)r_{1,i},$$

whence

$$d_{D_1}(z) \leq 2^6(3\rho_1 + 1)\rho_2 d_{D_{1,0}}(z),$$

which shows that (1) holds.

For the remaining inequality, let $u \in S(z, d_{D_{1,0}}(z))$. Then we have

$$\log \left(1 + \frac{1}{2^6(3\rho_1 + 1)\rho_2}\right) \leq \log \left(1 + \frac{|u - z|}{d_{D_1}(z)}\right) \leq k_{D_1}(z, u) \leq \int_{[z,u]} \frac{|dw|}{d_{D_1}(w)} \leq \frac{1}{\rho_1 - 2},$$

since for $w \in [z,u]$, $d_{D_1}(w) \geq d_{D_1}(z) - |w - z|$ and $d_{D_1}(z) \geq (\rho_1 - 1)d_{D_{1,0}}(z)$, whence we see from Theorem H that

$$k_{D_1}(z', u') \geq \left(\frac{1}{\mu_2} \log \left(1 + \frac{1}{2^6(3\rho_1 + 1)\rho_2}\right)\right)^{\mu_2} > \frac{1}{((2^7(3\rho_1 + 1)\rho_2)\mu_2)^{\mu_2} - 1}.$$ 

Necessarily, we have

$$u' \in \mathbb{R}^n \setminus \overline{B}(z', (2^7(3\rho_1 + 1)\rho_2\mu_2)^{\mu_2} d_{D_1}(z')), \quad \text{then the proof of (2) easily follows from the fact that } f(\overline{B}(z, d_{D_{1,0}}(z))) \subset D_{1,0}'$. \quad \Box$$

It follows from [12, Theorem 3] and [37, Theorem 4.15] that solid arcs have the quasiconformal invariance property, i.e., the image of each solid arc under a conformal mapping is still a solid arc. Since each quasihyperbolic geodesic is a $(0, 1)$-solid arc, we easily see that the following corollary is a consequence of Lemma 6.2, [21, Lemma 2.1], Theorems G and I.

**Corollary 6.1.** There exists a constant $\mu_8$ such that for $w_1$ and $w_2 \in D_{1,0}$, the following hold:

1. $k_{D_{1,0}}(w_1, w_2) \leq \mu_8 \log \left(1 + \frac{|w_1 - w_2|}{\min\{d_{D_{1,0}}(w_1), d_{D_{1,0}}(w_2)\}}\right).$

Suppose that $\zeta'$ is a quasihyperbolic geodesic joining $w_1'$ and $w_2'$ in $D_{1,0}'$. Then

2. $\zeta$ is a $(\nu, h_1)$-solid arc and

$$\text{diam}(\zeta) \leq \mu_8 \max\{|w_1 - w_2|, \min\{d_{D_{1,0}}(w_1), d_{D_{1,0}}(w_2)\}\},$$

where $(\nu, h_1)$ depends only on $(n, K)$, and
(3) for all \( w \in \zeta \), \( \min \{ \text{diam}(\zeta[w_1, w]), \text{diam}(\zeta[w, w_2]) \} \leq \mu_8 d_{D_{1,0}}(w) \);

(4) Let \( v_{0,0} \in \zeta \) be such that
\[
d_{D_{1,0}}(v_{0,0}) = \sup_{p \in \zeta} d_{D_{1,0}}(p).
\]

Then for \( w \in \zeta[w_j, v_{0,0}] \) (\( j = 1, 2 \)),
\[|w_j - w| \leq \mu_8 d_{D_{1,0}}(w),\]
where \( \mu_8 = \mu_8(n, K, \rho_3, \nu, h_1) = \mu_8(n, K, \rho_3) \) and \( \rho_3 \) is from Lemma 6.2.

Now, we are in a position to state and prove our main result in this section.

6.4. The statement of Theorem 6.1. The following is the main result in this section which shows that \( \gamma'_{1,0} \) is a carrot arc with center \( z'_0 \).

**Theorem 6.1.** There is a constant \( \rho \) such that for any \( z' \in \gamma'_{1,0} \),
\[
\text{diam}(\gamma'_{1,0}[z'_1, z'_2]) < \rho d_{D_{t}}(z'),
\]
where \( \rho = \rho \left( n, K, a, \frac{1}{\gamma}, \frac{\text{diam}(D)}{d_D(f^{-1}(y_0))} \right) \).

6.5. The proof of Theorem 6.1. In this subsection, by a method of contradiction, we shall prove that the constant \( \rho \) in Theorem 6.1 can be taken to be \( 2\rho_{10} \). Here

(1) \( \rho_{10} = \rho_{10}^{\rho_0} \)
(2) \( \rho_9 = \rho_9^{\Gamma(\lambda T_{1+3})} \)
(3) \( \rho_8 = 48a^2 \mu_2 \rho_7 \)
(4) \( \rho_7 = 2^{12} \frac{a^2}{c^2} (\mu_8 \rho_2 \rho_6) \frac{2^{\mu_2}}{\mu_8} \)
(5) \( \rho_6 = \frac{2^{12} \mu_2 (3 \rho_1 + 1) \rho_2 \rho_5}{\eta_1^{-1}(\frac{1}{\rho_5})} \)

where we need to remember the following:

(i) \( \mu_2 \) (resp. \( \mu_8 \)) is from Theorem H (resp. Corollary 6.1); \( \rho_1 \) and \( \rho_2 \) are from Lemma 6.1 and \( \rho_3 \) (resp. \( \rho_5 \)) is from Lemma 6.2 (resp. Lemma 6.3);

(ii) in \( \rho_6, \eta_1 = \eta_{n, K, \lambda}, \lambda_1 = \frac{1}{5} \) and \( \eta \) is from Theorem K;

(iii) in \( \rho_6, \Lambda_0 = \rho_8^2 \) and for each \( i \in \{1, \ldots, T_i + 3\} \) (\( T_i = [16a]^{2n} \)), \( \Lambda_i = \Gamma(\Lambda_{i-1}) \), here
\[
\Gamma(t) = \left( (1 + 216 \rho_{17} t) \rho(t) H(t) \right)^{2\mu_2 \rho_{15} \mu(t)}.
\]

(iv) in \( \Gamma(t), \rho(t) = \rho(n, K, \rho, 2^{47 + (3a\rho t)^2} a^2 \rho t^2) \) and \( \rho \) is from theorem B;

(v) in \( \Gamma(t), \mu(t) = \mu_1(c_1(t)), c_1(t) = 2^{47 + (3a\rho t)^2} a^2 \rho t^2 \) and \( \mu_1 \) is from Theorem G;

(vi) in \( \Gamma(t), H(t) \) is defined as follows:
\[
H(t) = \kappa_2 \left( n, K, \mu_0 t, \varphi(t), 6a(\rho_{17} t + 1), \rho_{17}^{2\mu_2} \frac{\text{diam}(D)}{d_D(y_0)} \right);
\]

(vii) in \( H(t), \varphi(t) = \varphi_{c_2(t)}, c_2(t) = 2^{47 + (3a\rho t)^2} a^2 \rho t^2 \) and \( \varphi \) is from Theorem J;

(viii) in \( H(t), \mu_0 t = \mu_3(n, \varphi_{\rho(t)}) \) and \( \mu_3 \) is from Theorem L;

(ix) in \( H(t), \kappa_2 \) is from Theorem E.
Suppose on the contrary that there exists a point \( w' \in \gamma_{1,0}' \) satisfying
\[
\text{diam}(\gamma_{1,0}'[z_1', w']) \geq 2\rho_{10}d_{D_1'}(w').
\]
(6.1)

Based on this assumption, we shall prove a series of lemmas. Through these lemmas, a contradiction will be reached, from which the proof of Theorem 6.1 is complete. In the following, we divide these lemmas into two groups.

6.5.1. Lemmas: Part I.

The sketch for this part: In this part, under the assumption of (6.1), we will pick up two special points \( z_1', z_0' \) from the image \( \gamma_{1,0}' \) of \( \gamma_{1,0} \) constructed in Section 6. Then these two points determine a quasihyperbolic geodesic \( \gamma_{2,0}' \) in \( D_{1,0}' \). We shall apply the obtained results, especially Theorem 5.1, to analyze \( \gamma_{2,0} \) and \( \gamma_{2,0}' \) together with some related points in these two arcs, and seven Lemmas will be proved.

We let \( z_1' \) be the first point in \( \gamma_{1,0}' \) along the direction from \( z_1' \) to \( z_0' \) such that
\[
\text{diam}(\gamma_{1,0}'[z_1', z_0']) = \rho_{10}d_{D_1'}(z_1').
\]
(6.2)

The existence of \( z_1' \) follows from (6.1). And let
\[
z_0', \in \gamma_{1,0}'[z_1', z_1'] \cup \mathcal{S}(z_1', \frac{1}{4}\text{diam}(\gamma_{1,0}'[z_1', z_1']))
\]
be such that \( \gamma_{1,0}'[z_0', z_1'] \subset \overline{B}(z_1', \frac{1}{4}\text{diam}(\gamma_{1,0}'[z_1', z_1'])) \). Obviously, \( z_0' \) satisfies the following (see Figure 9):
\[
2\text{diam}(\gamma_{1,0}'[z_0', z_1']) \leq \text{diam}(\gamma_{1,0}'[z_1', z_1']) \leq 2\text{diam}(\gamma_{1,0}'[z_1', z_0'])
\]
(6.3)

and
\[
|z_1' - z_0'| = \frac{1}{4}\text{diam}(\gamma_{1,0}'[z_1', z_1']).
\]
(6.4)

Next, we determine the positions of points \( z_0 \) and \( z_1 \) in \( \gamma_{1,0} \).
Lemma 6.9. The points \( z_{0,0} \) and \( z_{1,0} \) are not contained in the part \( \gamma_{1,0}[x_{1,k_1}, z_0] \) of \( \gamma_{1,0} \), where the point \( x_{1,k_1} \) is from Lemma 6.1 (see Figure 8).

Now, we prove this Lemma. For each \( x \in \gamma_{1,0}[x_{1,k_1}, z_0] \), by Lemma 6.1, we see that \( x \in B_{1,k_1} = \mathbb{B}(x_{1,k_1}, \frac{1}{\rho_1}d_{D_1}(x_{1,k_1})) \), and thus, Lemma 4.6 implies

\[
|x' - x'_{1,k_1}| < \frac{1}{5}d_{D_1}(x'_{1,k_1}) \quad \text{and} \quad d_{D_1}(x'_{1,k_1}) \leq \frac{5}{4}d_{D_1}(x').
\]

Suppose \( z_{0,0} \in \gamma_{1,0}[x_{1,k_1}, z_0] \) or \( z_{1,0} \in \gamma_{1,0}[x_{1,k_1}, z_0] \). Then for each \( w' \in \gamma_{1,0}[z'_{1,0}, z'_0] \), we deduce from the choice of \( z'_{1,0} \) and (6.5) that

\[
\text{diam}(\gamma_{1,0}[z'_{1,0}, w']) \leq \text{diam}(\gamma_{1,0}[z'_{1,0}, x'_{1,k_1}]) + \text{diam}(\gamma_{1,0}[x'_{1,k_1}, z'_0]) \leq \rho_{10}d_{D'_1}(x'_{1,k_1}) + \frac{2}{5}d_{D'_1}(x'_{1,k_1}) \leq \frac{2 + 5\rho_{10}}{4}d_{D'_1}(w') \leq 2\rho_{10}d_{D'_1}(w'),
\]

which, together with the choice of \( z'_{1,0} \), shows that for all \( w' \in \gamma'_{1,0} \), \( \text{diam}(\gamma'_{1,0}[z'_{1,0}, w']) < 2\rho_{10}d_{D'_1}(w') \). It follows from (6.1) that this is impossible. Hence Lemma 6.9 is true.

Further, we have

Lemma 6.10. (1) For \( z \in \gamma_{1,0}[z_{0,0}, z_{1,0}] \),

\[
d_D(z) < \rho_1\ell(\gamma_{1,0}[z_1, z]) \quad \text{and} \quad d_D(z) < \rho_1\rho_5d_D(z);
\]

(2) For \( x_1, x_2 \in \gamma_{1,0}[z_{0,0}, z_{1,0}] \),

\[
k_{D_1}(x_1, x_2) \leq 2^6\rho_2k_D(x_1, x_2).
\]

First, we prove (1). Suppose on the contrary that there exists some point \( z \in \gamma_{1,0}[z_{0,0}, z_{1,0}] \) such that

\[
d_D(z) \geq \rho_1\ell(\gamma_{1,0}[z_1, z]).
\]

We shall use the local quasisymmetry of \( f \) to get a contradiction. We first do some preparation.

Let \( x_{1,0} \) be the first point of \( \gamma_{1,0}[z_{0,0}, z_{1,0}] \) from \( z_{0,0} \) to \( z_{1,0} \) satisfying (see Figure 9)

\[
d_D(x_{1,0}) \geq \rho_1\ell(\gamma_{1,0}[z_1, x_{1,0}]).
\]

It is possible that \( x_{1,0} = z_{0,0} \) or \( z_{1,0} \). For every \( y \in \gamma_{1,0}[z_{1,0}, x_{1,0}] \), since \( y \in \mathbb{B}(x_{1,0}, \frac{1}{\rho_1}d_D(x_{1,0})) \), by Lemma 4.6, we get

\[
|y' - x'_{1,0}| < \frac{1}{5}d_D(x'_{1,0}),
\]

which yields

\[
\gamma'_{1,0}[z'_{1,0}, x'_{1,0}] \subset \mathbb{B}(x'_{1,0}, \frac{1}{5}d_D(x'_{1,0})),
\]

and so by (6.3),

\[
diam(\gamma'_{1,0}[x'_{1,0}, z'_{1,0}]) \leq diam(\gamma'_{1,0}[z'_{1,0}, x'_{1,0}]) \leq \frac{2}{5}d_D(x'_{1,0}).
\]

Let

\[
B'_1 = \mathbb{B}(x'_{1,0}, \frac{3}{5}d_D(x'_{1,0})),
\]
and let $v'_1$ be a point in the boundary of $D'_1$ such that
\begin{equation}
|z'_{1,0} - v'_1| = d_{D'_1}(z'_{1,0}) \quad \text{(see Figure 10).}
\end{equation}

**Figure 10.** The points $z'_{1,0}$ in $\gamma'_{1,0}[z'_{1,0}, z'_{1,0}]$ and $v'_1$ in $\partial D'_1$

It follows from (6.2) and (6.6) that
\[
d_{D'_1}(z'_{1,0}) = \frac{1}{\rho_{10}} \text{diam}(\gamma'_{1,0}[z'_{1,0}, z'_{1,0}]) \leq \frac{3}{5\rho_{10}} d_{D'}(x'_{1,0}).
\]

Then we easily know that $v'_1, z'_{0,0}, z'_{1,0} \in B'_1$. We are ready to get a contradiction by using the local quasisymmetry of $f^{-1}$. By Theorem K, the restriction $f^{-1}|_{B'_1}$ is $\eta_1$-quasisymmetric, where $\eta_1 = \eta_{m,K,\lambda_1}$ and $\lambda_1 = \frac{4}{5}$. It follows from Lemma 6.3, (6.2), (6.4) and (6.7) that
\[
\frac{1}{\rho_5} \leq \frac{|v_1 - z_{1,0}|}{|z_{0,0} - z_{1,0}|} \leq \eta_1\left(\frac{|v'_1 - z'_{1,0}|}{|z'_{0,0} - z'_{1,0}|}\right) \leq \eta_1\left(\frac{4}{\rho_{10}}\right).
\]

This is the desired contradiction. Hence the first inequality in Lemma 6.10(1) is true. The second inequality in Lemma 6.10(1) easily follows from the first one and Lemma 6.3.

In order to give a proof of (2) in Lemma 6.10, we divide the discussions into two cases: $|x_2 - x_1| \leq \frac{1}{2}d_{D_1}(x_1)$ and $|x_2 - x_1| > \frac{1}{2}d_{D_1}(x_1)$.

For the first case, Lemma 6.10(1) implies that
\begin{equation}
k_{D_1}(x_1, x_2) \leq \int_{[x_1, x_2]} \frac{|dx|}{d_{D_1}(x)} \leq \frac{2|x_2 - x_1|}{d_{D_1}(x_1)} < 3\rho_1\rho_5 k_D(x_1, x_2).
\end{equation}
For the second case, that's, \( |x_2 - x_1| > \frac{1}{2} d_{D_1}(x_1) \), it follows from Lemmas 6.4, 6.7 and 6.10(1) that

\[
(6.9) \quad k_{D_1}(x_1, x_2) \leq \int_{\gamma_{1,0}[x_1,x_2]} \frac{|dx|}{d_{D_1}(x)} \leq 2\rho_5 \log \left( 1 + \frac{2\ell(1,0|x_1,x_2|)}{d_{D_1}(x_1)} \right) \\
\leq 2\rho_5 \log \left( 1 + 2^{14}c^2\rho_1^2\rho_5 \frac{|x_2 - x_1|}{d_D(x_1)} \right) \\
< 2^6 \rho_2 k_{D}(x_1, x_2),
\]

where, in the third inequality, the inequalities

\[
\frac{|x_2 - x_1|}{d_D(x_1)} \geq \frac{|x_2 - x_1|}{\rho_1 \rho_5 d_{D_1}(x_1)} > \frac{1}{2\rho_1 \rho_5}
\]

are used. The combination of (6.8) and (6.9) completes the proof of the inequality (2) in Lemma 6.10. Hence the proof of Lemma 6.10 is complete.

In order to apply Theorem 5.1 to continue the proof, the following lower bound for the quasihyperbolic distance in \( D_1' \) between \( z_0', z_1' \) is useful.

**Lemma 6.11.** \( k_{D_1'}(z_0', z_1') \geq \log \frac{\rho_1}{4} \) and \( \min\{|z_0' - z_1'|, d_{D_1}(z_1, 0)\} \geq \rho_9 d_{D_1}(z_0, 0) \).

The proof of the first inequality is obvious, since it follows from (6.2) and (6.4) that

\[
k_{D_1'}(z_0', z_1') \geq \log \frac{|z_0' - z_1'|}{d_{D_1}(z_1, 0)} = \log \frac{\rho_1}{4}.
\]

For the second inequality, we infer from Theorem H and the proved inequality that \( k_{D_1}(z_0, z_1) > 1 \), and thus

\[
k_{D_1}(z_0, z_1) \geq \frac{1}{\mu_2} k_{D_1}(z_0', z_1') \geq \frac{1}{\mu_2} \log \frac{\rho_1}{4}.
\]

Consequently, Corollary 6.1, Lemmas 6.3, 6.8 and 6.9 yield

\[
\frac{1}{\mu_2} \log \frac{\rho_1}{4} \leq k_{D_1}(z_0, z_1) \leq \mu_8 \log \left( 1 + \frac{|z_0 - z_1|}{\min\{d_{D_1}(z_0, 0), d_{D_1}(z_1, 0)\}} \right) \\
\leq \mu_8 \log \left( 1 + \frac{2^6(3\rho_1 + 1)\rho_2 \rho_5 d_{D_1}(z_1, 0)}{\min\{d_{D_1}(z_0, 0), d_{D_1}(z_1, 0)\}} \right),
\]

and necessarily, we see

\[
\min\{d_{D_1}(z_0, 0), d_{D_1}(z_1, 0)\} = d_{D_1}(z_0, 0),
\]

whence

\[
\min\{|z_0 - z_1|, d_{D_1}(z_1, 0)\} \geq \rho_9 d_{D_1}(z_0, 0).
\]

The proof of Lemma 6.11 is complete.

Let \( \gamma_{2,0}' \) be a quasihyperbolic geodesic joining \( z_0' \) and \( z_1' \) in \( D_1' \) (see Figure 11). By Proposition 5.1, we see that \( \gamma_{2,0}' \) satisfies the \( QH (\ell_{k_{D_1}}(\gamma_{2,0}'), 2, 1) \)-condition. Further, we shall prove that if the quasihyperbolic distance between two points in \( \gamma_{2,0}' \) is big enough, then the part of \( \gamma_{2,0}' \) with these two points as its endpoints satisfies some much “stronger” \( QH \) condition. This is included in Lemma 6.14. Before the
statement and the proof of Lemma 6.14, we need some inequalities, which are stated
in Lemmas 6.12 and 6.13. We recall that $D_{1,0}$ is a $\rho_3$-uniform domain (see Lemma
6.2).

![Figure 11. The arc $\gamma'_{2,0}$ in $D'_{1,0}$](image)

**Lemma 6.12.** For $z \in \gamma_{2,0}$, we have

1. $(\rho_1 - 1)d_{D_{1,0}}(z) \leq d_{D_1}(z) \leq 2^5 \rho_2 \rho_6 d_{D_{1,0}}(z)$;
2. $d_D(z) \leq 2^5 \rho_1 \rho_2 \rho_6 d_{D_{1,0}}(z)$;
3. $d_{D_1}(z_{o,0}) < 2^9 \mu_8 \rho_2 d_{D_1}(z)$;
4. $d_{D_1}(z') \leq (2^6 \rho_2 \rho_6 \mu_2)^{\mu_2} d_{D_{1,0}}(z')$;
5. $d_{D'}(z') \leq (2^6 \rho_1 \rho_2 \rho_6 \mu_2)^{\mu_2} d_{D_{1,0}}(z')$.

For a proof of this Lemma, it follows from the fact “$\gamma_{2,0} \subset D_{1,0}$” and Lemma 6.8
that for $z \in \gamma_{2,0}$,

\[(6.10) \quad d_{D_1}(z) \geq (\rho_1 - 1)d_{D_{1,0}}(z),\]

and we obtain from Corollary 6.1 that for $z \in \gamma_{2,0}$,

\[(6.11) \quad \min\{|z_{0,0} - z|, |z_{1,0} - z|\} \leq \mu_8 d_{D_{1,0}}(z),\]

where $\mu_8 = \mu_8(n, K, \rho_3)$ is from Corollary 6.1.

Let $z$ be a point in $\gamma_{2,0}$. To prove that $z$ satisfies the inequalities (1) and (2) in
Lemma 6.12, we only need to consider the case $\min\{|z_{0,0} - z|, |z_{1,0} - z|\} = |z_{0,0} - z|$ since the proof for the other case $\min\{|z_{0,0} - z|, |z_{1,0} - z|\} = |z_{1,0} - z|$ is similar.

Further, we distinguish two possibilities: $|z_{0,0} - z| \leq \frac{1}{2}d_{D_{1,0}}(z_{0,0})$ and $|z_{0,0} - z| > \frac{1}{2}d_{D_{1,0}}(z_{0,0})$. For the first case, we have

\[
d_{D_1}(z) \leq d_{D_1}(z_{0,0}) + |z_{0,0} - z| \leq \frac{3}{2}d_{D_1}(z_{0,0}),
\]

\[
d_{D_{1,0}}(z) \geq d_{D_{1,0}}(z_{0,0}) - |z_{0,0} - z| \geq \frac{1}{2}d_{D_{1,0}}(z_{0,0})
\]

and

\[
d_D(z) \leq d_D(z_{0,0}) + |z_{0,0} - z| \leq \frac{3}{2}d_D(z_{0,0}).
\]
Since Lemma 6.9 implies that \( z_{0,0} \in \gamma_{1,0}[z_1, x_{1,k_1}] \), we see from Lemma 6.8 that

\[
(6.12) \quad d_{D_{1,0}}(z) \geq \frac{1}{2} d_{D_{1,0}}(z_{0,0}) \geq \frac{1}{2^7(3\rho_1 + 1)\rho_2} d_{D_1}(z_{0,0}) \geq \frac{1}{2^6(9\rho_1 + 3)\rho_2} d_{D_1}(z),
\]

and Lemma 6.10 leads to

\[
(6.13) \quad d_{D_{1,0}}(z) \geq \frac{1}{2^7(3\rho_1 + 1)\rho_2} d_{D_1}(z_{0,0}) \geq \frac{1}{2^7(3\rho_1 + 1)\rho_1\rho_2\rho_5} d_D(z_{0,0}) \geq \frac{1}{2^6\rho_1(9\rho_1 + 3)\rho_2\rho_5} d_D(z).
\]

For the second case, that’s, \( |z_{0,0} - z| > \frac{1}{2} d_{D_{1,0}}(z_{0,0}) \), again, by Lemmas 6.8, 6.9 and 6.10, we see that

\[
d_{D_1}(z) \leq d_{D_1}(z_{0,0}) + |z_{0,0} - z| \leq 2^6(3\rho_1 + 1)\rho_2 d_{D_{1,0}}(z_{0,0}) + |z_{0,0} - z| \leq (2^7(3\rho_1 + 1)\rho_2 + 1)|z_{0,0} - z|
\]

and

\[
d_D(z) \leq d_D(z_{0,0}) + |z_{0,0} - z| \leq 2^6\rho_1(3\rho_1 + 1)\rho_2\rho_5 d_{D_{1,0}}(z_{0,0}) + |z_{0,0} - z| \leq (2^7\rho_1(3\rho_1 + 1)\rho_2\rho_5 + 1)|z_{0,0} - z|,
\]

which, together with (6.11), implies

\[
(6.14) \quad d_{D_{1,0}}(z) \geq \frac{1}{\mu_8}|z_{0,0} - z| \geq \frac{1}{\mu_8(2^7(3\rho_1 + 1)\rho_2 + 1)} d_{D_1}(z)
\]

and

\[
(6.15) \quad d_{D_{1,0}}(z) \geq \frac{1}{\mu_8}|z_{0,0} - z| \geq \frac{1}{\mu_8(2^7\rho_1(3\rho_1 + 1)\rho_2\rho_5 + 1)} d_D(z).
\]

We conclude from (6.10), (6.12), (6.13), (6.14) and (6.15) that the inequalities (1) and (2) in Lemma 6.12 are true.

It follows directly from (6.11), Lemmas 4.3 and 6.11 that

\[
d_{D_{1,0}}(z) \geq \frac{1}{2\mu_8} \min\{d_{D_{1,0}}(z_{0,0}), d_{D_{1,0}}(z_{1,0})\} \geq \frac{1}{2\mu_8} d_{D_{1,0}}(z_{0,0}).
\]

Hence, by Lemmas 6.8 and 6.9, we have

\[
d_{D_1}(z) \geq (\rho_1 - 1)d_{D_{1,0}}(z) \geq \frac{\rho_1 - 1}{2\mu_8} d_{D_{1,0}}(z_{0,0}) > \frac{1}{2^6\mu_8\rho_2} d_{D_1}(z_{0,0}),
\]

which shows that Lemma 6.12(3) holds.

Based on the inequalities (1) and (2) in Lemma 6.12, the proofs of (4) and (5) in Lemma 6.12 easily follow from a similar argument as in that of Lemma 6.8(2). Hence the proof of Lemma 6.12 is complete.

**Lemma 6.13.** For \( w_1', w_2' \in \gamma'_{2,0} \), if \( k_{D_{1,0}}(w_1', w_2') \geq \mu_2^2\rho_7 \), then

\[
k_{D'}(w_1', w_2') \geq \frac{1}{\mu_3^2\rho_7} k_{D_{1,0}}(w_1', w_2').
\]
We shall apply Theorem H to obtain a proof of Lemma 6.13. Indeed, since the assumption \( k_{D_1,0}(w'_1, w'_2) \geq \mu_2^2 \rho_7 \) implies \( k_{D_1,0}(w_1, w_2) \geq 1 \), by Theorem H, we deduce that

\[
(6.16) \quad k_{D_1,0}(w_1, w_2) \geq \mu_2 k_{D_1,0}(w'_1, w'_2) \geq \mu_2 \rho_7.
\]

Further, we obtain from Corollary 6.1 and Lemma 6.12 that

\[
(6.17) \quad k_{D_1,0}(w_1, w_2) \leq \mu_8 \log \left(1 + \frac{|w_1 - w_2|}{\min\{d_D(w_1), d_D(w_2)\}}\right)
\]

\[
\leq \mu_8 \log \left(1 + 2^5 \rho_1 \rho_2 \rho_8 \frac{|w_1 - w_2|}{\min\{d_D(w_1), d_D(w_2)\}}\right)
\]

\[
< \rho_7 \log \left(1 + \frac{|w_1 - w_2|}{\min\{d_D(w_1), d_D(w_2)\}}\right)
\]

\[
\leq \rho_7 k_D(w_1, w_2),
\]

whence the combination with (6.16) shows \( k_D(w_1, w_2) \geq \mu_2 \), which implies \( k_{D'}(w'_1, w'_2) \geq 1 \). Therefore, (6.16), (6.17) and Theorem H yield

\[
k_{D'}(w'_1, w'_2) \geq \frac{1}{\mu_2} k_D(w_1, w_2) \geq \frac{1}{\mu_2^2 \rho_7} k_{D_1,0}(w'_1, w'_2)
\]

as required.

Our next lemma is as follows.

**Lemma 6.14.** Suppose \( s_1 = \lfloor \rho_7 \rfloor \) and \( s_2 = \mu_2^2 \rho_7 \). For \( w'_1, w'_2 \in \gamma_{2,0}' \), if \( k_{D_1,0}(w'_1, w'_2) \geq \mu_2^2 \rho_7^2 \) and \( k_{D'}(w'_1, w'_2) \geq \rho_7 \), then

1. \( \gamma_{2,0}'[w'_1, w'_2] \) satisfies QH \((s_0, s_1, s_2)\)-condition in \( D' \);
2. \( k_{D'_1,0}(w'_1, w'_2) \leq 48a^2 \mu_2^2 \rho_7 \log \left(1 + \frac{\ell_{\gamma_{2,0}[w'_1, w'_2]}}{d_{D'}(y_0')}\right) \) provided \( s_0 \geq 8s_1 s_2^2 \), where

\[
d_{D'}(y_0') = \inf_{z \in \gamma_{2,0}[w'_1, w'_2]} d_{D'}(z').
\]

To prove Lemma 6.14(1), since \( w'_1 \neq w'_2 \), we only need to check Conditions (2) and (3) in the definition of QH condition.

First, we partition the part \( \gamma_{2,0}'[w'_1, w'_2] \) of \( \gamma_{2,0}' \). Let \( x'_0 = w'_1 \), and let \( x'_1, \ldots, x'_{[\rho_7]} \in \gamma_{2,0}'[w'_1, w'_2] \) be successive points such that for each \( i \in \{1, \ldots, [\rho_7]\} \),

\[
k_{D'_1,0}(x'_{i-1}, x'_i) = \frac{1}{\rho_7} k_{D_1,0}(w'_1, w'_2),
\]

and thus \( k_{D'_1,0}(x'_{i-1}, x'_i) \geq \mu_2^2 \rho_7 \). Then we see from Lemma 6.13 that for all \( i \neq j \in \{1, \ldots, [\rho_7]\} \),

\[
k_{D'}(x'_{i-1}, x'_i) \leq k_{D_1,0}(x'_{i-1}, x'_i) = k_{D'_1,0}(x'_{i-1}, x'_i) \leq \mu_2^2 \rho_7 k_{D'}(x'_{j-1}, x'_j).
\]

Similarly, for each \( x' \in \gamma_{2,0}'[x'_i, x'_j] \), again, we get from Lemma 6.13 that

\[
k_{D'}(x'_{i-1}, x'_i) \leq \ell_{k_{D'_1,0} \gamma_{2,0}'[x'_{i-1}, x'_i]} \leq \ell_{k_{D'_1,0} \gamma_{2,0}'[x'_{i-1}, x'_i]} = k_{D'_1,0}(x'_{i-1}, x'_i) \leq \mu_2^2 \rho_7 k_{D'}(x'_{i-1}, x'_i).
\]
We complete the proof of (1) in Lemma 6.14. Since $s_0 \geq 8s_1s_2^2$, again, it follows from Lemma 6.13 together with Theorem 5.1 and (1) in Lemma 6.14 that

$$k_{D'_1,0}(w'_1, w'_2) \leq \mu_2^2 \rho_7 k_{D'}(w'_1, w'_2) \leq 48a^2 \mu_2^6 \rho_7^4 \log \left(1 + \frac{\text{diam}(\gamma'_2,0, w'_1, w'_2)}{d_{D'}(y'_0)}\right).$$

The proof of Lemma 6.14 is complete.

**Lemma 6.15.**

1. For each $z' \in \gamma'_2,0$,

$$\text{diam}(\gamma'_1,0[z'_0,0, z'_1,0]) \leq \frac{5}{4} \rho_{10} e^{2\mu_2 \rho_8 (\varepsilon \rho_1 \rho_5)^2} d_{D'_1}(z').$$

2. $\text{diam}(\gamma'_2,0) \leq \frac{8}{5} \left(6 + \frac{1}{\rho_{10}}\right) \text{diam}(\gamma'_1,0[z'_0,0, z'_1,0]);$

3. $k_{D'_1,0}(z'_1,0, z'_0,0) \leq \rho_8 \rho_9.$

For a proof of Lemma 6.15, we first need some preparation. From Lemma 6.11, we know that

$$(6.18) \quad \max\{|z_{0,0} - z_{1,0}|, \min\{d_{D_1,0}(z_{0,0}), d_{D_1,0}(z_{1,0})\}\} = |z_{0,0} - z_{1,0}|.$$

Then Corollary 6.1 yields

$$\text{diam}(\gamma'_{2,0}) \leq \mu_8 |z_{0,0} - z_{1,0}|,$$

and so by Lemma 6.3, we see that for each $z \in \gamma'_{2,0},$

$$(6.19) \quad d_{D_1}(z) \leq \text{diam}(\gamma'_{2,0}) + d_{D_1}(z_{1,0}) \leq (\mu_8 \rho_5 + 1)d_{D_1}(z_{1,0}).$$

Since $z \in \gamma'_{2,0}$, it follows from Lemma 6.1 that there must exist some $t_0 \in \{1, \ldots, k_1\}$ such that $z \in B_{1,t_0} = B(x_{1,t_0}, r_{1,t_0})$. Then Lemma 4.6 shows that

$$|(z' - x'_{1,t_0})| < \frac{1}{5} d_{D'_1}(x'_{1,t_0}) \quad \text{and} \quad d_{D'_1}(x'_{1,t_0}) < \frac{5}{4} d_{D'_1}(z').$$

Now, it is possible to prove Lemma 6.15. For a proof of (1) and (2) in the lemma, we need upper bounds for the quotients

$$\frac{|z'_{0,0} - z'|}{\text{diam}(\gamma'_{1,0}[z'_{0,0}, z'_{1,0}])} \quad \text{and} \quad \frac{\text{diam}(\gamma'_{1,0}[z'_{0,0}, z'_{1,0}])}{d_{D'_1}(z')}.$$

To get these upper bounds, we divide the proof into three cases based on the positions of $x_{1,t_0}$ in $\gamma'_{1,0}.$
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For the first case when $x_{1,t_0} \in \gamma_{1,0}[z_{0,0}, z_{1,0}]$ (see Figure 12), we know from (6.2), (6.4) and (6.20) that

\begin{align}
|z'_{0,0} - z'| &\leq |z' - x'_{1,t_0}| + |z'_{0,0} - x'_{1,t_0}| \\
&\leq \frac{1}{5}d'_{D_1}(x'_{1,t_0}) + |z'_{0,0} - x'_{1,t_0}| \\
&\leq \frac{1}{5}(|x'_{1,t_0} - z'_{1,0}| + d'_{D_1}(z'_{1,0})) + \text{diam}(\gamma'_{1,0}[z'_{0,0}, z'_{1,0}]) \\
&\leq \frac{6}{5}\text{diam}(\gamma'_{1,0}[z'_{0,0}, z'_{1,0}]) + \frac{1}{5\rho_{10}}\text{diam}(\gamma'_{1,0}[z'_{1,0}, z'_{1,0}]) \\
&= \frac{6}{5}\text{diam}(\gamma'_{1,0}[z'_{0,0}, z'_{1,0}]) + \frac{4}{5\rho_{10}}|z'_{1,0} - z'_{0,0}| \\
&\leq \frac{2}{5}(3 + \frac{2}{\rho_{10}})\text{diam}(\gamma'_{1,0}[z'_{0,0}, z'_{1,0}]),
\end{align}

and the choice of $z'_{1,0}$, (6.3) and (6.20) lead to

\begin{align}
d'_{D_1}(z') &> \frac{4}{5}d'_{D_1}(x'_{1,t_0}) \geq \frac{4}{5\rho_{10}}\text{diam}(\gamma'_{1,0}[z'_{1,0}, x'_{1,t_0}]) \\
&\geq \frac{4}{5\rho_{10}}\text{diam}(\gamma'_{1,0}[z'_{0,0}, z'_{0,0}]) \geq \frac{2}{5\rho_{10}}\text{diam}(\gamma'_{1,0}[z'_{1,0}, z'_{1,0}]) \\
&\geq \frac{4}{5\rho_{10}}\text{diam}(\gamma'_{1,0}[z'_{0,0}, z'_{1,0}]).
\end{align}

\textbf{Figure 12. The case } x_{1,t_0} \in \gamma_{1,0}[z_{0,0}, z_{1,0}]
For the second case when \(x_{1,t_0} \in \gamma_{1,0}[z_1, z_{0,0}]\), it follows from Lemmas 6.3, 6.7 and 6.12 that

\[
k_{D_1}(z_{0,0}, x_{1,t_0}) \leq \int_{\gamma_{1,0}[x_{1,t_0}, z_{0,0}]} \frac{|dx|}{d_{D_1}(x)} \leq 2\rho_5 \log \left(1 + \frac{2\ell(\gamma_{1,0}[x_{1,t_0}, z_{0,0}])}{d_{D_1}(x_{1,t_0})}\right)
\]

\[
\leq 2\rho_5 \log \left(1 + \frac{2\rho_5 d_{D_1}(z_{0,0})}{d_{D_1}(x_{1,t_0})}\right) \leq 2\rho_5 \log \left(1 + \frac{2^9 \mu_8 \rho_2 \rho_5}{\rho_1}\right)
\]

\[
< 2^7 \mu_8 (c\rho_1 \rho_5)^2,
\]
since \(d_{D_1}(z) \leq (1 + \frac{1}{\rho_1})d_{D_1}(x_{1,t_0})\). Hence we see from Theorem H that

\[
\max \left\{ \log \left(1 + \frac{|z_{0,0}' - x_{1,t_0}'|}{\rho_{D_1}'(z_{0,0}'')}\right), \log \frac{d_{D_1}'(x_{1,t_0}')}{d_{D_1}'(z_{0,0}'')} \right\} \leq k_{D_1}'(z_{0,0}', x_{1,t_0}') < 2^7 \mu_2 \mu_8 (c\rho_1 \rho_5)^2
\]

which implies

\[
(6.23) \quad \max \{|z_{0,0}' - x_{1,t_0}'|, d_{D_1}'(x_{1,t_0}')\} \leq e^{2^7 \mu_2 \mu_8 (c\rho_1 \rho_5)^2} d_{D_1}'(z_{0,0}')
\]

and

\[
(6.24) \quad d_{D_1}'(z_{0,0}') \leq e^{2^7 \mu_2 \mu_8 (c\rho_1 \rho_5)^2} d_{D_1}'(x_{1,t_0}').
\]

Then, by (6.2), (6.4) and (6.20), we have

\[
(6.25) \quad |z_{0,0}' - z'| \leq |z' - x_{1,t_0}'| + |z_{0,0}' - x_{1,t_0}'|
\]

\[
< \frac{1}{5} d_{D_1}'(x_{1,t_0}') + \text{diam}(\gamma_{1,0}'[z_1', z_{1,0}'])
\]

\[
\leq \frac{1}{5}(|x_{1,t_0}' - z_{1,0}'| + d_{D_1}'(z_{1,0}')) + \text{diam}(\gamma_{1,0}'[z_1', z_{1,0}'])
\]

\[
\leq \frac{1}{5} \left(6 + \frac{1}{\rho_{10}}\right) \text{diam}(\gamma_{1,0}'[z_1', z_{1,0}'])
\]

\[
\leq \frac{4}{5} \left(6 + \frac{1}{\rho_{10}}\right) \text{diam}(\gamma_{1,0}'[z_{0,0}', z_{1,0}'])
\]

and (6.3), (6.20) and the choice of \(z_{1,0}'\) lead to

\[
(6.26) \quad d_{D_1}'(z') \geq 4 \frac{d_{D_1}'(x_{1,t_0}')}{5} \geq 4 \frac{4}{5 e^{2^7 \mu_2 \mu_8 (c\rho_1 \rho_5)^2}} \text{diam}(\gamma_{1,0}'[z_1', z_{1,0}'])
\]

\[
\geq \frac{4}{5 \rho_{10} e^{2^7 \mu_2 \mu_8 (c\rho_1 \rho_5)^2}} \text{diam}(\gamma_{1,0}'[z_1', z_{1,0}'])
\]

\[
\geq \frac{2}{5 \rho_{10} e^{2^7 \mu_2 \mu_8 (c\rho_1 \rho_5)^2}} \text{diam}(\gamma_{1,0}'[z_1', z_{1,0}'])
\]

\[
\geq \frac{4}{5 \rho_{10} e^{2^7 \mu_2 \mu_8 (c\rho_1 \rho_5)^2}} \text{diam}(\gamma_{1,0}'[z_{0,0}', z_{1,0}']).
\]
For the last case, that's, \( x_{1,t_0} \in \gamma_{1,0}[z_{1,0}, z_0] \), we see from Lemmas 6.3 and 6.7, together with (6.19), that

\[
\begin{align*}
k_{D_1}(z_{1,0}, x_{1,t_0}) & \leq \int_{\gamma_{1,0}[z_{1,0}, x_{1,t_0}]} \frac{|dx|}{d_{D_1}(x)} \leq 2\rho_5 \log \left( 1 + \frac{2\ell(\gamma_{1,0}[z_{1,0}, x_{1,t_0}])}{d_{D_1}(z_{1,0})} \right) \\
& \leq 2\rho_5 \log \left( 1 + \frac{2\rho_5 d_{D_1}(x_{1,t_0})}{d_{D_1}(z_{1,0})} \right) \leq 2\rho_5 \log \left( 1 + \frac{2\rho_5}{\rho_1 - 1} \cdot \frac{d_{D_1}(z)}{d_{D_1}(z_{1,0})} \right) \\
& \leq 2\rho_5 \log \left( 1 + \frac{2\rho_5(\mu_8\rho_5 + 1)}{\rho_1 - 1} \right) < 5c\mu_8\rho_5,
\end{align*}
\]

since \( d_{D_1}(x_{1,t_0}) \leq \frac{\rho_1}{\rho_1 - 1} d_{D_1}(z) \) for \( z \in B_{1,t_0} \). By replacing \( 27\mu_8(c\rho_1\rho_5)^2 \) with \( 5c\mu_8\rho_5 \), a similar reasoning as in the proofs of (6.23) and (6.24) shows that

\[
\max \{|z'_{1,0} - x'_{1,t_0}|, d_{D_1}(x'_{1,t_0})\} \leq e^{5c\mu_8\rho_5} d_{D_1}(z'_{1,0})
\]

and

\[
d_{D_1}(z'_{1,0}) \leq e^{5c\mu_8\rho_5} d_{D_1}(x'_{1,t_0}).
\]

Hence, by (6.2), (6.4) and (6.20), we have

\[
|z'_{0,0} - z'| \leq (|z'_{0,0} - z'_{1,0}| + |z'_{1,0} - x'_{1,t_0}| + |x'_{1,t_0} - z'|)
\]

\[
\leq \text{diam}(\gamma'_{1,0}[z'_{0,0}, z'_{1,0}]) + \frac{6}{5} e^{5c\mu_8\rho_5} d_{D_1}(z'_{1,0})
\]

\[
\leq \left( 1 + \frac{24}{5\rho_{10}} e^{5c\mu_8\rho_5} \right) \text{diam}(\gamma'_{1,0}[z'_{0,0}, z'_{1,0}]),
\]

and (6.2), (6.3) and (6.20) lead to

\[
\begin{align*}
d_{D_1}'(z') & > \frac{4}{5} d_{D_1}'(x'_{1,t_0}) \geq \frac{4}{5 e^{5c\mu_8\rho_5}} d_{D_1}'(z'_{1,0}) \\
& \geq \frac{8}{5\rho_{10} e^{5c\mu_8\rho_5}} \text{diam}(\gamma'_{1,0}[z'_{0,0}, z'_{1,0}]).
\end{align*}
\]

The inequalities (6.21), (6.22), (6.25), (6.26), (6.27) and (6.28) are our requirements. Also, (6.22), (6.26) and (6.28) show that Lemma 6.15(1) is true.

Further, we get from (6.21), (6.25) and (6.27) that for \( z' \in \gamma'_{2,0} \),

\[
|z'_{0,0} - z'| \leq \frac{4}{5} \left( 6 + \frac{1}{\rho_{10}} \right) \text{diam}(\gamma'_{1,0}[z'_{0,0}, z'_{1,0}]).
\]

Let \( x'_{1} \in \gamma'_{2,0} \) be such that

\[
|z'_{0,0} - x'_{1}| = \frac{1}{2} \text{diam}(\gamma'_{2,0}).
\]

Then we know from (6.29) that

\[
\text{diam}(\gamma'_{2,0}) = 2|z'_{0,0} - x'_{1}| \leq \frac{8}{5} \left( 6 + \frac{1}{\rho_{10}} \right) \text{diam}(\gamma'_{1,0}[z'_{0,0}, z'_{1,0}]),
\]

whence Lemma 6.15(2) is also true.
For a proof of (3) in Lemma 6.15, we let \( s_1 = [\rho_7], s_2 = \mu^2 \rho_7 \) and \( s_0 = 8 s_1 s_2^2 \). Since Lemma 6.11 implies
\[
k_{D'_{1,0}}(z'_{0,0}, z'_{0,0}) \geq k_{D'_{1,0}}(z'_{0,0}, z'_{1,0}) \geq \log \frac{\rho_{10}}{4} > \mu^2 \rho_7^2,
\]
we see from Lemma 6.13 that
\[
k_D(z'_{1,0}, z'_{0,0}) \geq \frac{1}{\mu^2 \rho_7} k_{D'_{1,0}}(z'_{1,0}, z'_{0,0}) \geq \frac{1}{\mu^2 \rho_7} \log \frac{\rho_{10}}{4} > s_0.
\]
Hence Lemma 6.14 shows that \( \gamma'_{2,0} \) satisfies \((s_0, s_1, s_2)\)-\(QH\) condition. It follows from Lemma 6.14 together with (1) and (2) in this Lemma that
\[
k_{D'_{1,0}}(z'_{1,0}, z'_{0,0}) \leq 48 a^2 \mu^6 \rho_7^4 \log \left(1 + \frac{\text{diam}(\gamma'_{2,0})}{d_{D'}(w'_0)}\right)
\leq 48 a^2 \mu^6 \rho_7^4 \log \left(1 + \frac{\frac{8}{5} (6 + \frac{1}{\rho_9}) \text{diam}(\gamma'_{1,0}[z'_{0,0}, z'_{1,0}])}{d_{D'}(w'_0)}\right)
< 48 a^2 \mu^6 \rho_7^4 \log \left(1 + \frac{13 \rho_{10} e^{2 \pi \mu} (c \rho_{10} \rho_7^4) d_{D'}(w'_0)}{d_{D'}(w'_0)}\right)
< \rho_8 \rho_9,
\]
where \( w'_0 \in \gamma'_{2,0} \) satisfies
\[
d_{D'}(w'_0) = \inf \{d_{D'}(z') : z' \in \gamma'_{2,0}\}.
\]
Hence the inequality Lemma 6.15(3) holds, and thus the proof of Lemma 6.15 is complete.

6.5.2. Lemmas: Part II.

The sketch for this part: In this part, first, we choose a proper point \( \omega'_0 \) in \( \gamma'_{2,0} \) and partition the part \( \gamma'_{2,0}[\omega'_0, z'_{1,0}] \) of \( \gamma'_{2,0} \) with the aid of some special points \( \{y'_i\}_{i=1}^{m+1} \). After that, we shall pick up a needed point \( \{y'_i\} \) from \( \{y'_i\}_{i=1}^{m+1} \) and another special point \( v'_3 \) from \( \gamma'_{2,0} \), and then partition the part \( \gamma'_{2,0}[y'_i, v'_3] \) of \( \gamma'_{2,0} \) by using the points \( \{x'_i\}_{i=1}^{\lceil |A_{2,1} + 3| + 1 \rceil} \). Based on the obtained points, we shall construct the corresponding carrot arcs, balls etc. With the aid of the related points, carrot arcs, balls etc, seven Lemmas will be proved.

We begin to let \( \omega'_0 \) to be the first point in \( \gamma'_{2,0} \) along the direction from \( z'_{1,0} \) to \( z'_{0,0} \) such that
\[
d_{D'_{1,0}}(\omega'_0) = \sup_{p' \in \gamma'_{2,0}} d_{D'_{1,0}}(p').
\]
It is possible that \( \omega'_0 = z'_{1,0} \) or \( z'_{0,0} \). Clearly, there exists a nonnegative integer \( m \) such that
\[
2^m d_{D'_{1,0}}(z'_{1,0}) \leq d_{D'_{1,0}}(\omega'_0) < 2^{m+1} d_{D'_{1,0}}(z'_{1,0}).
\]
Let \( v'_0 \) be the first point in \( \gamma'_{2,0}[z'_{1,0}, \omega'_0] \) from \( \omega'_0 \) to \( z'_{1,0} \) satisfying (see Figure 13)
\[
d_{D'_{1,0}}(\omega'_0) = 2^m d_{D'_{1,0}}(v'_0).
\]
Now, we give a partition to $\gamma_{2,0}[\omega_0, v_0']$. Let $y_1' = \omega_0'$. If $v_0' = y_1'$, we let $y_2' = v_0'$. If $v_0' \neq y_1'$, then we let $y_2, \ldots, y_{m+1}' \in \gamma_{2,0}[\omega_0', v_0']$ be the points such that for each $i \in \{2, \ldots, m+1\}$, $y_i'$ denotes the first point from $\omega_0'$ to $v_0'$ with

$$d_{D_1'}(y_i') = \frac{1}{2^{i-1}} d_{D_1'}(y_1').$$

Obviously, $y_{m+1}' = v_0'$. If $v_0' \neq z_{1,0}'$, then we use $y_{m+2}'$ to denote $z_{1,0}'$ (see Figure 13).

![Figure 13. The arc $\gamma_{2,0}'$ and the related point $\omega_0'$](image)

**Lemma 6.16.** Suppose that $w_1' \in \gamma_{2,0}'[y_{m+1}', \omega_0']$ and $w_2' \in \gamma_{2,0}'[w_1', \omega_0']$ which satisfy

$$d_{D_1'}(w_2') \geq \Gamma(t) d_{D_1'}(w_1')$$

for some $t > (\mu_2 \rho_t)^2$. Then there exists some $w' \in \gamma_{2,0}'[w_1', w_2']$ such that

$$\ell(\gamma_{2,0}'[w_1', w']) \geq t d_{D_1'}(w'),$$

where $\Gamma(t)$ is defined as in Section 6.

We prove Lemma 6.16 by a method of contradiction. Suppose that for each $w' \in \gamma_{2,0}'[w_1', w_2']$, we have

$$\ell(\gamma_{2,0}'[w_1', w']) < t d_{D_1'}(w').$$

In the following, based on a new carrot arc $\beta_{2,0}'$, we shall construct a uniform domain $D_{3,0}'$ in $D'$ together with its uniform subdomain $D_{3,1}'$ and then show that the restriction $f^{-1}|_{D_{3,1}'}$ is weakly $H(t)$-quasisymmetric in $\delta_{D_{3,0}'}$ and $\delta_{D_{3,0}'}$ (see Subsection 2.4 for the definition of internal metrics). This weak quasisymmetry of $f^{-1}$ will help us to get a contradiction.

At first, let us construct a new arc in $D'$. Since $D'$ is a bounded $a$-John domain with center $y_0'$, there must exist an $a$-carrot arc $\beta_0'$ in $D'$ joining $w_2'$ and $y_0'$ with center $y_0'$. We set (see Figure 14)

$$\beta_{2,0}' = \gamma_{2,0}'[w_1', w_2'] \cup \beta_0'.$$

If $y_0' = w_2'$, we assume that $\beta_0' = \{w_2'\}$.

By using (6.33), we shall prove that $\beta_{2,0}'$ is a carrot arc. For $z' \in \beta_{2,0}'$, if $z' \in \gamma_{2,0}'[w_1', w_2']$, then (6.33) implies that

$$\ell(\gamma_{2,0}'[w_1', z']) < t d_{D_1'}(z');$$
On the other hand, if \( z' \in \beta_0' \), since \(|w'_2 - z'| \leq a d_{D'}(z')\), Lemma 4.3 yields
\[
d_{D'}(w'_2) \leq 2ad_{D'}(z'),
\]
and so (6.33) leads to
\[
\ell(\beta_{2,0}[w'_1, z']) = \ell(\gamma_{2,0}[w'_1, w'_2]) + \ell(\beta'_0[w'_2, z']) \leq t d_{D'_{1,0}}(w'_2) + a d_{D'}(z') \\
\leq 2at d_{D'}(z') + a d_{D'_{1,0}}(z') \\
< 3at d_{D'_{1,0}}(z').
\]
Hence we have proved that \( \beta'_{2,0} \) is a \( 3at \)-carrot arc from \( w'_1 \) to \( y'_0 \) with center \( y'_0 \), and thus Lemmas 4.1 and 4.2 guarantee the following.

**Proposition 6.1.** There exists a simply connected \( 2^{47+(3a\rho t)^2}a^2\rho t^2 \)-uniform domain 
\[
D'_{3,0} = \bigcup_{i=1}^{k_{3,0}} B_{3,i} \subset D' \text{ such that }
\]
\[
(1) \ w'_1, y'_0 \in D'_{3,0}; \\
(2) \text{For each } i \in \{1, \ldots, k_{3,0}\}, \\
\frac{1}{3\rho t} d_{D'}(x'_{3,i}) \leq r_{3,i} \leq \frac{1}{\rho t} d_{D'}(x'_{3,i}),
\]
where \( B_{3,i} = \mathbb{B}(x'_{3,i}, r_{3,i}) \), \( x'_{3,i} \in \beta'_{2,0}, x'_{3,i} \notin B_{3,i-1} \) for each \( i \in \{2, \ldots, k_{3,0}\} \) and \( x'_{3,1} = w'_1 \) (see Figure 14).
Next, we construct a subdomain of $D'_{3,1}$. Let

$$t_2 = \max \{ t : x'_{3,i} \in \gamma'_{2,0} \text{ for each } i \in \{1, \ldots, t\} \}. $$

Then we take (see Figure 14)

$$D'_{3,1} = \bigcup_{i=1}^{t_2} B_{3,i}. $$

We easily know that $D'_{3,1}$ is a $2^{47+(3\rho t^2)}a^2\rho_t t^2$-uniform domain. Since for each $i \in \{1, \ldots, t_2\}$, Lemma 6.12 and Proposition 6.1 imply

$$r_{3,i} \leq \frac{d_{D'}(x'_{3,i})}{\rho_7} \leq \frac{(2^6 \rho_1 \rho_2 \rho_6 \mu_2)^{\mu_2} d_{D'_{1,0}}(x'_{3,i})}{\rho_7} < d_{D'_{1,0}}(x'_{3,i}), $$

we see that

$$D'_{3,1} \subset D'_{1,0}, $$

whence it follows from Theorem B and the fact “$D_{1,0}$ being $\rho_0$-uniform” that $D_{3,1}$ is a $\rho(t)$-uniform domain, where $\rho(t) = \rho(n, K, \rho_3, 2^{47+(3\rho t^2)}a^2\rho_t t^2)$ and $\rho$ is from Theorem B. Hence, Theorem J implies that $D_{3,1}$ is $\varphi(\rho(t))$-broad, and then Theorem L shows that $D_{3,1}$ is $\mu_{0t}$-LLC$_2$ with respect to $\delta_{D_{3,1}}$ in $D_{3,0}$, where $\mu_{0t} = \mu_3(n, \varphi(\rho(t)))$ and $\mu_3$ is from Theorem L. Since Proposition 6.1 and Theorem J guarantee that $D'_{3,0}$ is a $\varphi(t)$-broad domain, where $\varphi(t) = \varphi_{c_2}(t)$, $c_2(t) = 2^{47+(3\rho t^2)}a^2\rho_t t^2$ and $\varphi$ is from Theorem J, we see from Theorem E that the restriction $f^{-1}|_{D'_{3,1}}$ is weakly $H(t)$-quasisymmetric in the metrics $\delta_{D'_{3,0}}$ and $\delta_{D_{3,0}}$, where

$$H(t) = \kappa_2 \left( n, K, \mu_{0t}, \varphi(t), \frac{\delta_{D'_{3,0}}(D'_{3,1})}{d_{D'_{3,0}}(x'_{3,k_{0,3}})}, \frac{\delta_{D_{3,0}}(D_{3,1})}{d_{D_{3,0}}(x_{3,k_{0,3}})} \right). $$

Now, we need to estimate the ratios $\frac{\delta_{D'_{3,0}}(D'_{3,1})}{d_{D'_{3,0}}(x'_{3,k_{0,3}})}$ and $\frac{\delta_{D_{3,0}}(D_{3,1})}{d_{D_{3,0}}(x_{3,k_{0,3}})}$ to improve the constant $H(t)$. We first get an estimate for the second quantity.

For each $z' \in S(x'_{3,k_{0,3}}, r_{3,k_{0,3}})$, we see from Lemma 4.6 that

$$|x_{3,k_{0,3}} - z| < \frac{1}{5} d_{D}(x_{3,k_{0,3}}), $$

and so (2.3) implies

$$k_{D}(x_{3,k_{0,3}}, z) \leq \log \left( 1 + \frac{|x_{3,k_{0,3}} - z|}{d_{D}(x_{3,k_{0,3}}) - |x_{3,k_{0,3}} - z|} \right) < 1. $$

Moreover, by Lemma 4.5 and Proposition 6.1, we have

$$\frac{1}{\rho_7 - 1} \geq k_{D'}(x'_{3,k_{0,3}}, z') \geq \log \left( 1 + \frac{|x'_{3,k_{0,3}} - z'|}{d_{D'}(x'_{3,k_{0,3}})} \right) \geq \log \left( 1 + \frac{1}{3 \rho_7} \right), $$

The quasiconformal subinvariance property of John domains in $\mathbb{R}^n$ and its application.
whence it follows from (6.34) and Theorem H that
\[
\left( \frac{1}{6 \mu_2 \rho_7} \right)^{\mu_2} < \left( \frac{1}{\mu_2} \log(1 + \frac{1}{3 \rho_7}) \right)^{\mu_2} \leq \left( \frac{1}{\mu_2} k_D'(x_{3,k_3,0}', z') \right)^{\mu_2} \leq k_D(x_{3,k_3,0}, z)
\]
and so
\[
\left| x_{3,k_3,0} - z \right| \geq \left( 1 - e^{-\left( \frac{1}{6 \mu_2 \rho_7} \right)^{\mu_2}} \right) d_D(x_{3,k_3,0}),
\]
which shows that
\[
d_{f^{-1}(B_{3,k_3,0})}(x_{3,k_3,0}) > \left( 1 - e^{-\left( \frac{1}{6 \mu_2 \rho_7} \right)^{\mu_2}} \right) d_D(y_0),
\]
Since \( y_0 \in B_{3,k_3,0} \), it follows from Lemma 4.6 that
\[
d_{D_{3,0}}(x_{3,k_3,0}) = d_{f^{-1}(B_{3,k_3,0})}(x_{3,k_3,0}) > \frac{4}{5} \left( 1 - e^{-\left( \frac{1}{6 \mu_2 \rho_7} \right)^{\mu_2}} \right) d_D(y_0),
\]
whence we have
\[
\frac{\delta_{D_{3,0}}(D_{3,1})}{d_{D_{3,0}}(x_{3,k_3,0})} \leq \frac{\delta_{D'_{3,0}}(D'_{3,1})}{d_{D'_{3,0}}(x_{3,k_3,0})} \leq \frac{5e^{\left( \frac{1}{6 \mu_2 \rho_7} \right)^{\mu_2}}}{4(1 + \rho_7)} \frac{\text{diam}(D)}{d_D(y_0)} < \rho_7^{\mu_2} \frac{\text{diam}(D)}{d_D(y_0)}.
\]
In order to get an estimate for \( \frac{\delta_{D'_{3,0}}(D'_{3,1})}{d_{D'_{3,0}}(x_{3,k_3,0})} \), we let \( v_1' \in \partial D'_{3,1} \) be such that
\[
|v_1' - y_0'| \geq \frac{1}{2} \delta_{D'_{3,0}}(D'_{3,1}) \quad \text{(see Figure 14)}.
\]
Since \( y_0 \in B_{3,k_3,0} \), we know from Proposition 6.1 that
\[
d_{D'}(y_0) \leq d_{D'}(x'_{3,k_3,0}) + |x'_{3,k_3,0} - y_0'| \leq (1 + \frac{1}{\rho_7})d_{D'}(x'_{3,k_3,0}),
\]
and, since \( D' \) has the \( a \)-carrot property with center \( y_0 \), we see that there is an \( a \)-carrot arc \( \tau' \) in \( D' \) connecting \( v_1' \) and \( y_0 \) with center \( y_0 \). Hence,
\[
\delta_{D'_{3,0}}(D'_{3,1}) \leq 2|v_1' - y_0'| \leq 2\ell(\tau') \leq 2a d_{D'}(y_0)
\]
\[
\leq 2a(1 + \frac{1}{\rho_7})d_{D'}(x'_{3,k_3,0}) \leq 6a(1 + \rho_7)\delta_{3,3,0}
\]
\[
= 6a(1 + \rho_7)d_{D'_{3,0}}(x'_{3,k_3,0})
\]
so that
\[
\frac{\delta_{D'_{3,0}}(D'_{3,1})}{d_{D'_{3,0}}(x'_{3,k_3,0})} \leq 6a(1 + \rho_7).
\]
Hence we infer from (6.35) that we can take \( H(t) \) to be as follows:
\[
H(t) = \kappa_2 \left( n, K, \mu_0, \varphi(t), 6a(\rho_7 + 1), \rho_7^{2\mu_2} \frac{\text{diam}(D)}{d_D(y_0)} \right).
\]
For each \( i \in \{1, \ldots, k_3,0 - 1\} \), we get from (2) in Proposition 6.1 that
\[
\frac{\rho_7}{\rho_7 + 1} d_{D'}(x'_{3,i}) \leq d_{D'}(x'_{3,i+1}) \leq \frac{\rho_7 + 1}{\rho_7 - 1} d_{D'}(x'_{3,i}),
\]
Lemma 4.3 implies and so by Lemma 6.12 and Proposition 6.1,

\[ \frac{\rho_T - 1}{3(\rho_T + 1)} r_{3,i} \leq r_{3,i+1} \leq \frac{3(\rho_T + 1)}{\rho_T - 1} r_{3,i}, \]

which guarantees that there is \( t_3 \in \{1, \ldots, t_2\} \) such that \( x'_{3,t_3} \in \gamma'_{2,0}(w_1', x'_{3,t_2}) \) and

\[ \frac{1}{48} \delta_{D'_{3,0}}(x'_{3,t_2}, w_1') \leq \delta_{D'_{3,0}}(x'_{3,t_2}, w_1') \leq \frac{1}{2} \delta_{D'_{3,0}}(x'_{3,t_2}, w_1'). \]

Obviously,

\[ \frac{1}{2} \delta_{D'_{3,0}}(x'_{3,t_2}, w_1') \leq \delta_{D'_{3,0}}(x'_{3,t_2}, x'_{3,t_3}) \leq \frac{3}{2} \delta_{D'_{3,0}}(x'_{3,t_2}, w_1'), \]

whence

\[ \frac{\delta_{D'_{3,0}}(x'_{3,t_3}, w_1')}{\delta_{D'_{3,0}}(x'_{3,t_2}, x'_{3,t_3})} \leq 1. \]

(6.36)

In order to apply the weak quasisymmetry of \( f^{-1} \) and then to arrive at a contradiction, we still need to estimate \( \frac{\delta_{D'_{3,0}}(x'_{3,t_3}, w_1')}{\delta_{D'_{3,0}}(x'_{3,t_2}, x'_{3,t_3})} \).

Let \( v_2' \) be an intersection point of \( \beta_0' \) with the ball \( B_{3,t_2} \) (see Figure 14). Since

\[ |w_2 - v_2'| \leq \ell(\beta_0'[w_2', v_2']) \leq ad'_{D'}(v_2'), \]

Lemma 4.3 implies

\[ d'_{D'}(v_2') \geq \frac{1}{2a} d'_{D'}(w_2'), \]

and so by Lemma 6.12 and Proposition 6.1,

\[ d'_{D'_{1,0}}(x'_{3,t_2}) \geq \frac{1}{(2^6 \rho_1 \rho_2 \rho_6 \mu_2) \mu_2} d'_{D'}(x'_{3,t_2}) \]

\[ \geq \frac{\rho_T}{(\rho_T + 1)(2^6 \rho_1 \rho_2 \rho_6 \mu_2) \mu_2} d'_{D'}(v_2') \]

\[ > \frac{1}{3a(2^6 \rho_1 \rho_2 \rho_6 \mu_2) \mu_2} d'_{D'}(w_2') \]

\[ > \frac{1}{3a(2^6 \rho_1 \rho_2 \rho_6 \mu_2) \mu_2} d'_{D'_{1,0}}(w_2'), \]

which, together with the assumption in the lemma, yields that

(6.37)

\[ k_{D'_{1,0}}(x'_{3,t_2}, w_1') \geq \log \frac{d'_{D'_{1,0}}(x'_{3,t_2})}{d'_{D'_{1,0}}(w_1')} \]

\[ \geq \log \frac{d'_{D'_{1,0}}(w_2')}{d'_{D'_{1,0}}(w_1')} - \mu_2 \log (2^8 a \rho_1 \rho_2 \rho_6 \mu_2) \]

\[ > \frac{1}{2} \log \Gamma(t), \]

which, together with Theorem H, shows that

(6.38)

\[ k_{D_{1,0}}(x_{3,t_2}, w_1) > 1. \]
Let us leave the proof of the lemma for a moment and prove the following inequality:

\begin{equation}
\text{diam}(\gamma_{2,0}[z_{0,0}, w_1]) \leq 2^7 \mu_8^2(3\rho_1 + 1)\rho_2\rho_5 d_{D_1,0}(w_1).
\end{equation}

For a proof of this inequality, it follows from Corollary 6.1 that if

\[ \min\{\text{diam}(\gamma_{2,0}[z_{0,0}, w_1]), \text{diam}(\gamma_{2,0}[w_1, z_{1,0}])\} = \text{diam}(\gamma_{2,0}[z_{0,0}, w_1]), \]

then

\[ \text{diam}(\gamma_{2,0}[z_{0,0}, w_1]) \leq \mu_8 d_{D_1,0}(w_1). \]

For the remaining case

\[ \min\{\text{diam}(\gamma_{2,0}[z_{0,0}, w_1]), \text{diam}(\gamma_{2,0}[w_1, z_{1,0}])\} = \text{diam}(\gamma_{2,0}[w_1, z_{1,0}]), \]

Corollary 6.1 shows that

\[ \text{diam}(\gamma_{2,0}[w_1, z_{1,0}]) \leq \mu_8 d_{D_1,0}(w_1), \]

and by Lemma 4.3,

\[ d_{D_1,0}(z_{1,0}) \leq 2\mu_8 d_{D_1,0}(w_1). \]

It follows from Corollary 6.1 and (6.18) that \( \text{diam}(\gamma_{2,0}) \leq \mu_8|z_{0,0} - z_{1,0}| \), and thus by Lemmas 6.3 and 6.8, we have

\[ \text{diam}(\gamma_{2,0}[z_{0,0}, w_1]) \leq \text{diam}(\gamma_{2,0}) \leq \mu_8|z_{0,0} - z_{1,0}| \]

\[ \leq \mu_8\rho_5 d_{D_1}(z_{1,0}) \leq 2^6 \mu_8(3\rho_1 + 1)\rho_2\rho_5 d_{D_1,0}(z_{1,0}) \]

\[ \leq 2^7 \mu_8^2(3\rho_1 + 1)\rho_2\rho_5 d_{D_1,0}(w_1). \]

Hence the inequality (6.39) has been proved.

Let us continue the discussion with the aid of (6.39). Obviously, it follows from (6.39) that

\begin{equation}
|x_{3,t_2} - w_1| \leq \text{diam}(\gamma_{2,0}[z_{0,0}, w_1]) \leq 2^7 \mu_8^2(3\rho_1 + 1)\rho_2\rho_5 d_{D_1,0}(w_1).
\end{equation}

Also we see from (6.37), (6.38), Theorem H and Corollary 6.1 that

\[ \log \Gamma(t) \leq 2k_{D_{1,0}}(x_{3,t_2}, w_1') \leq 2\mu_2 k_{D_1,0}(x_{3,t_2}, w_1) \]

\[ \leq 2\mu_2\mu_8 \log \left(1 + \frac{|x_{3,t_2} - w_1|}{\min\{d_{D_{1,0}}(x_{3,t_2}), d_{D_{1,0}}'(w_1)\}}\right). \]

Necessarily, (6.40) leads to

\[ \min\{d_{D_{1,0}}(x_{3,t_2}), d_{D_{1,0}}'(w_1)\} = d_{D_{1,0}}(x_{3,t_2}), \]

and so, we have

\begin{equation}
|x_{3,t_2} - w_1| \geq \left(\Gamma(t) \frac{1}{2\mu_2\mu_8} - 1\right)d_{D_{1,0}}(x_{3,t_2}).
\end{equation}

It follows from (6.33) and Proposition 6.1 that

\[ \delta_{D_{3,0}}'(x_{3,t_2}', w_1') \leq t \min\{d_{D_{3,0}}'(x_{3,t_2}), d_{D_{3,0}}'(x_{3,t_2})\} \]

\[ \leq 3\rho_7 t \min\{d_{D_{3,0}}'(x_{3,t_2}), d_{D_{3,0}}'(x_{3,t_2})\}, \]
and thus we obtain from Theorem G and (6.36) that

\[ k_{D_{3,0}}(x_{3,t_2}, x_{3,t_3}') \leq \mu(t) \log \left(1 + \frac{|x_{3,t_2}' - x_{3,t_3}'|}{\min\{d_{D_{3,0}}(x_{3,t_2}'), d_{D_{3,0}}(x_{3,t_3}')\}}\right) < \mu(t) \log(1 + 147 \rho_t t), \]

since

\[ |x_{3,t_2}' - x_{3,t_3}'| \leq |x_{3,t_2}' - w_1'| + |w_1' - x_{3,t_3}'| \leq 49 \delta_{D_{3,0}}(x_{3,t_3}', w_1'), \]

where \( \mu(t) = \mu_1(2^{47+(3\alpha \rho t)^2} a^2 \rho t^2) \) and \( \mu_1 \) is from Theorem G. Then, Theorem H shows that

\[ \log \left(1 + \frac{|x_{3,t_2} - x_{3,t_3}|}{d_{D_{3,0}}(x_{3,t_2})}\right) \leq k_{D_{3,0}}(x_{3,t_2}, x_{3,t_3}) \leq \mu_2 \mu(t)(1 + 147 \rho t), \]

and obviously,

\[ |x_{3,t_2} - x_{3,t_3}| < (1 + 147 \rho t)^{\mu_2 \mu(t)} d_{D_{3,0}}(x_{3,t_2}). \]

Hence it follows from the fact “\( D_{3,1} \) being a \( \rho(t) \)-uniform domain” that

\[ \delta_{D_{3,0}}(x_{3,t_2}, x_{3,t_3}) \leq \delta_{D_{3,1}}(x_{3,t_2}, x_{3,t_3}) \leq \rho(t) |x_{3,t_2} - x_{3,t_3}| \leq \rho(t)(1 + 147 \rho t)^{\mu_2 \mu(t)} d_{D_{3,0}}(x_{3,t_2}), \]

which, together with (6.41), shows that

\[ \delta_{D_{3,0}}(x_{3,t_3}, w_1) \geq |x_{3,t_2} - w_1| - \delta_{D_{3,0}}(x_{3,t_3}, x_{3,t_2}) \geq \left( \Gamma(t)^{\frac{1}{2\nu_8}} - \rho(t)(1 + 147 \rho t)^{\mu_2 \mu(t)} - 1 \right) d_{D_{3,0}}(x_{3,t_2}), \]

whence

\[ \frac{\delta_{D_{3,0}}(x_{3,t_3}, w_1)}{\delta_{D_{3,0}}(x_{3,t_2}, x_{3,t_3})} \geq \frac{\Gamma(t)^{\frac{1}{2\nu_8}} - \rho(t)(1 + 147 \rho t)^{\mu_2 \mu(t)} - 1}{\rho(t)(1 + 147 \rho t)^{\mu_2 \mu(t)}} > H(t). \]

But the weak quasisymmetry of \( f^{-1} \) together with (6.36) shows that

\[ \frac{\delta_{D_{3,0}}(x_{3,t_3}, w_1)}{\delta_{D_{3,0}}(x_{3,t_2}, x_{3,t_3})} \leq H(t). \]

This obvious contradiction completes the proof of Lemma 6.16.

As an application of Lemma 6.16, we get the following result.

**Lemma 6.17.** There must exist some \( i \in \{m_1, \ldots, m\} \) such that

\[ k_{D_{1,0}}(y_{i+1}, y_i') \geq \frac{1}{2} \log \Lambda_{T_{1+3}}, \]

where \( m_1 = m - \left\lfloor \frac{\Gamma(\Lambda_{T_{1+3}})}{2} \right\rfloor \), \( \Gamma(\Lambda_{T_{1+3}}) \) and \( \Lambda_{T_{1+3}} \) are defined as in Section 6. Here \( m \) is defined by (6.31) and we refer to Figure 13 for the points \( y_i' \) in \( \gamma_{2,0}' \).

Suppose on the contrary that

\[ k_{D_{1,0}}(y_{i+1}, y_i') < \frac{1}{2} \log \Lambda_{T_{1+3}} \]

for each \( i \in \{m_1, \ldots, m\} \).
With the aid of Lemma 6.16, we shall gain a contradiction by considering the arc length of the part \( \gamma'_{2,0}[y'_{m+1}, w'] \) for any \( w' \) in \( \gamma'_{2,0}[y'_{m+1}, y'_m] \). It follows from the obvious fact

\[
\log \left( 1 + \frac{\ell(\gamma'_{2,0}[y'_{i+1}, y'_i])}{d_{D'_1,o}(y'_{i+1})} \right) \leq k_{D'_1,o}(y'_{i+1}, y'_i)
\]

that

\[
\ell(\gamma'_{2,0}[y'_{i+1}, y'_i]) < \Lambda_{T_{1}+3}^{\frac{1}{2}} d_{D'_1,o}(y'_{i+1}),
\]

and further, for each \( y' \in \gamma'_{2,0}[y'_{i+1}, y'_i] \),

\[
(6.42) \quad \ell(\gamma'_{2,0}[y'_{i+1}, y'_i]) < \Lambda_{T_{1}+3}^{\frac{1}{2}} d_{D'_1,o}(y').
\]

Since for every \( w' \in \gamma'_{2,0}[y'_{m+1}, y'_m] \), there exists some \( i \in \{m, \ldots, m\} \) such that \( w' \in \gamma'_{2,0}[y'_i, y'_i] \), we see from (6.42) that

\[
(6.43) \quad \ell(\gamma'_{2,0}[y'_{m+1}, w']) = \ell(\gamma'_{2,0}[y'_{m+1}, y'_m]) + \cdots + \ell(\gamma'_{2,0}[y'_{i+1}, y'_i]) + \ell(\gamma'_{2,0}[y'_{i}, w'])
\]

\[
\leq \Lambda_{T_{1}+3}^{\frac{1}{2}} d_{D'_1,o}(y'_{m+1}) + \cdots + d_{D'_1,o}(y'_{i+1})
\]

\[
< 2\Lambda_{T_{1}+3}^{\frac{1}{2}} d_{D'_1,o}(y'_{i+1})
\]

\[
\leq 2\Lambda_{T_{1}+3}^{\frac{1}{2}} d_{D'_1,o}(w'),
\]

and since

\[
d_{D'_1,o}(y'_{m+1}) = \frac{d_{D'_1,o}(y'_1)}{2^{m-1}} = 2^{\left[\frac{\Gamma(T_{1}+3)}{2}\right]} d_{D'_1,o}(y'_{m+1}) > \Gamma(T_{1}+3) d_{D'_1,o}(y'_{m+1}),
\]

by taking \( w'_1 = y'_{m+1}, \ w'_2 = y'_m \) and \( t = \Lambda_{T_{1}+3} \), we see from Lemma 6.16 that there exists \( y' \in \gamma'_{2,0}[y'_{m+1}, y'_m] \) such that

\[
\ell(\gamma'_{2,0}[y'_{m+1}, y'_m]) \geq t d_{D'_1,o}(y').
\]

A contradiction can be reached by taking \( w' = y' \) in (6.43) since, obviously, \( \Lambda_{T_{1}+3} > 2\Lambda_{T_{1}+3}^{\frac{1}{2}} \). The proof of the lemma is complete.

It follows from Lemma 6.17 that there exists \( i_1 \in \{m - \left[\frac{\Gamma(T_{1}+3)}{2}\right], \ldots, m\} \) such that

\[
(6.44) \quad k_{D'_1,o}(y'_{i_1}, y'_{i_1+1}) \geq \frac{1}{2} \log \Lambda_{T_{1}+3}.
\]

Now, we prove a lower bound for the number \( m \) below. Since Lemma 6.15 implies

\[
\frac{\ell(\gamma'_{2,0})}{d_{D'_1,o}(w'_0)} \leq \int_{\gamma'_{2,0}} \frac{|dw'|}{d_{D'_1,o}(w')} = k_{D'_1,o}(z'_{1,0}, z'_{0,0}) \leq \rho_8 \rho_9.
\]
we see from (6.2) and (6.4) that
\[
d_{D_{1,0}}(\omega'_0) \geq \frac{1}{\rho_8 \rho_9} \ell(\gamma'_{2,0}) \geq \frac{1}{\rho_8 \rho_9} |z'_{1,0} - z'_{0,0}|
\geq 2^{\frac{1}{2} \rho_9 \log \rho_8} d_{D_{1,0}}(z'_{1,0}),
\]
and so (6.31) leads to
\[
(6.45) \quad m > \frac{1}{2} \rho_9 \log \rho_8 - 1.
\]
Since \(d_{D_{1,0}}(y'_{i_1}) = \frac{1}{2^{i_1}} d_{D_{1,0}}(y'_{i_1})\), we know from (6.45) that
\[
(6.46) \quad d_{D_{1,0}}(y'_{i_1}) \leq \frac{1}{2^{m - \lfloor \frac{1}{2} (\gamma_{T_1 + 3}) \rfloor}} d_{D_{1,0}}(y'_{i_1}) < \frac{1}{2^{5 \Gamma(\Lambda_{T_1 + 3})}} d_{D_{1,0}}(y'_{i_1}),
\]
which implies \(|y'_{i_1} - y_{i_1}| > \frac{1}{2} d_{D_{1,0}}(y'_{i_1})\), and so
\[
|y'_{1} - y_{i_1}| > \frac{1}{2} d_{D_{1,0}}(y'_{i_1}).
\]
Let \(v'_3\) be the first point in \(\gamma'_{2,0}[y'_{i_1}, y'_1]\) from \(y'_{i_1}\) to \(y'_1\) such that (see Figure 15)
\[
(6.47) \quad \ell(\gamma'_{2,0}[y'_{i_1}, v'_3]) = 2^{4 \Gamma(\Lambda_{T_1 + 3})} d_{D_{1,0}}(y'_{i_1}).
\]
Now, we partition the part \(\gamma'_{2,0}[y'_{i_1}, v'_3]\) of \(\gamma'_{2,0}\) as follows. We set \(x'_1 = y'_{i_1}\) and let \(x'_2, \ldots, x'_{4 \Gamma(\Lambda_{T_1 + 3}) + 1} \in \gamma'_{2,0}[y'_{i_1}, v'_3]\) be the points such that for each \(i \in \{2, \ldots, 4 \Gamma(\Lambda_{T_1 + 3}) + 1\}\), \(x'_i\) denotes the first point from \(x'_1\) to \(v'_3\) with (see Figure 15)
\[
(6.48) \quad \ell(\gamma'_{2,0}[x'_1, x'_i]) = 2^{i - 1} d_{D_{1,0}}(x'_1).
\]
Obviously, \(x'_{4 \Gamma(\Lambda_{T_1 + 3}) + 1} = v'_3\). Then we have

\[\text{Figure 15. A partition to } \gamma'_{2,0}[y'_{i_1}, v'_3]\]

**Lemma 6.18.** For each pair \(h, j \in \{1, \ldots, [\Gamma(\Lambda_{T_1 + 3}) + 1]\}, \) if \(j \geq h + [\Gamma(\Lambda_i)]\)
for some \(i \in \{1, \ldots, T_1 + 3\}\), then there exists some \(s \in \{h + 1, \ldots, j\}\) such that
\(k_{D_{1,0}}(x'_s - 1, x'_s) \geq \frac{1}{2} \log \Lambda_i\), where \(\Lambda_i\) and \(\Gamma(\Lambda_i)\) are defined as in Section 6.
Suppose on the contrary that
\[ k_{D_{1,0}}(x_{s-1}', x_s') < \frac{1}{2} \log \Lambda_i \]
for each \( s \in \{ h + 1, \ldots, j \} \).

By considering the arclength of the part \( \gamma_{2,0}'[x_h', w'] \) for any \( w' \) in \( \gamma_{2,0}[x_h', x_j'] \) and by using Lemma 6.16, we shall get a contradiction. Since \( \gamma_{2,0}' \) is a quasihyperbolic geodesic, we see from (2.2) that for \( x' \in \gamma_{2,0}[x_{s-1}', x_s'] \),
\[
\log \left( 1 + \frac{\ell(\gamma_{2,0}'[x_{s-1}', x_s'])}{d_{D_{1,0}}(x')} \right) \leq k_{D_{1,0}}(x_{s-1}', x_s') < \frac{1}{2} \log \Lambda_i,
\]
and so
\[
(6.49) \quad \ell(\gamma_{2,0}'[x_{s-1}', x_s']) < \Lambda_i^\frac{1}{2} d_{D_{1,0}}(x').
\]

For \( w' \in \gamma_{2,0}'[x_h', x_j'] \), obviously, there exists some \( s_0 \in \{ h + 1, \ldots, j \} \) such that \( w' \in \gamma_{2,0}'[x_{s_0-1}', x_{s_0}'] \). Then we see from (6.49) that
\[
(6.50) \quad \ell(\gamma_{2,0}'[x_i', w']) = \ell(\gamma_{2,0}'[x_i', x_{i+1}']) + \cdots + \ell(\gamma_{2,0}'[x_{s_0-2}', x_{s_0-1}']) + \ell(\gamma_{2,0}'[x_{s_0-1}', w']) \leq \frac{1}{2} \ell(\gamma_{2,0}'[x_i', x_{i+1}']) + \cdots + \ell(\gamma_{2,0}'[x_{s_0-1}', x_{s_0}']) = 2 \ell(\gamma_{2,0}'[x_{s_0-1}', x_{s_0}']) < 2 \Lambda_i^\frac{1}{2} d_{D_{1,0}}(w'),
\]
and consequently,
\[
(6.51) \quad \ell(\gamma_{2,0}'[x_i', x_j']) < 2 \Lambda_i^\frac{1}{2} d_{D_{1,0}}(x_j').
\]

Since
\[
\ell(\gamma_{2,0}'[x_i', x_j']) = \ell(\gamma_{2,0}'[x_1', x_j']) - \ell(\gamma_{2,0}'[x_1', x_h']) = (2^j - 1 - 2^{h-1}) d_{D_{1,0}}(x_1') = 2^{h-1}(2^{j-h} - 1) d_{D_{1,0}}(x_1'),
\]
and
\[
d_{D_{1,0}}(x_i') \leq \ell(\gamma_{2,0}'[x_1', x_h']) + d_{D_{1,0}}(x_1') \leq (2^{h-1} + 1) d_{D_{1,0}}(x_1'),
\]
we know from (6.51) that
\[
\frac{d_{D_{1,0}}(x_j')}{d_{D_{1,0}}(x_h')} \geq \frac{\ell(\gamma_{2,0}'[x_h', x_j'])}{2 \Lambda_i^\frac{1}{2} d_{D_{1,0}}(x_h')} \geq \frac{2^{h-1}(2^{j-h} - 1) d_{D_{1,0}}(x_1')}{2 \Lambda_i^\frac{1}{2}(2^{h-1} + 1) d_{D_{1,0}}(x_1')} > \frac{2^{r(\Lambda_i)}}{2} > \Gamma(\Lambda_i).
\]

By taking \( w_1' = x_h', \ w_2' = x_j' \) and \( t = \Lambda_i \), Lemma 6.16 implies that there exists \( y' \in \gamma_{2,0}'[x_{h}', x_{j}'] \) such that
\[
\ell(\gamma_{2,0}'[x_h', y']) \geq t d_{D_{1,0}}(y').
\]
A contradiction can be reached by taking \( w' = y' \) in (6.50) since, obviously, \( \Lambda_i > 2 \Lambda_i^\frac{1}{2} \). We complete the proof of Lemma 6.18.
By using Lemma 6.18, we first choose some points from \( \{ x'_1, x'_2, \ldots, x'_{2[\Lambda_{T_1+1}]} \} \).

It follows from Lemma 6.18 that there exists \( j \in \{ \left[ \Gamma(\Lambda_{T_1+1})+1, \ldots, 2[\Gamma(\Lambda_{T_1+1})] \right] \} \) such that

\[
(6.52) \quad k_{D'_{1,0}}(x'_{j-1}, x'_j) \geq \frac{1}{2} \log \Lambda_{T_1+1}.
\]

Let \( x'_{j_1} \) denote the first point along the direction from \( x'_{[\Gamma(\Lambda_{T_1+1})]+1} \) to \( x'_{2[\Gamma(\Lambda_{T_1+1})]} \), which satisfies (6.52). Easily, we have

\[
\ell(\gamma'_{2,0}[x'_1, x'_{j_1}]) \leq \frac{2^{[\Gamma(\Lambda_{T_1+1})]+1}d_{D'_{1,0}}(x'_1)}{4}.
\]

Again, it follows from Lemma 6.18 that there exists \( j_2 = j_1 + 2[\Gamma(\Lambda_{T_1-1})] + 1 \) such that \( x'_{j_2} \) is the first point along the direction from \( x'_{j_1+2} \) to \( x'_{j_1+2[\Gamma(\Lambda_{T_1-1})]+1} \) satisfying

\[
k_{D'_{1,0}}(x'_{j_2-1}, x'_{j_2}) \geq \frac{1}{2} \log \Lambda_{T_1-1}.
\]

Also, we get

\[
\ell(\gamma'_{2,0}[x'_1, x'_{j_2}]) \leq \frac{2^{[\Gamma(\Lambda_{T_1+1})]+1}d_{D'_{1,0}}(x'_1)}{4}\]

Once again, we see that there exists some \( j_3 = j_2 + 2[\Gamma(\Lambda_{T_1-3})] + 1 \) such that \( x'_{j_3} \) is the first point along the direction from \( x'_{j_2+2} \) to \( x'_{j_2+2[\Gamma(\Lambda_{T_1-3})]+1} \) satisfying

\[
k_{D'_{1,0}}(x'_{j_3-1}, x'_{j_3}) \geq \frac{1}{2} \log \Lambda_{T_1-3}.
\]

Moreover, we obtain

\[
\ell(\gamma'_{2,0}[x'_1, x'_{j_3}]) \leq \frac{2^{[\Gamma(\Lambda_{T_1+1})]+1}d_{D'_{1,0}}(x'_1)}{4}\]

By repeating this procedure as above \( m \ (m \geq 3) \) times, we shall determine a point \( x'_{j_m} \) in \( \gamma'_{2,0} \) such that

\[
(6.53) \quad \ell(\gamma'_{2,0}[x'_1, x'_{j_m}]) \leq \frac{2^{[\Gamma(\Lambda_{T_1+1})]+1}d_{D'_{1,0}}(x'_1)}{4}\]

and (6.47) shows

\[
(6.54) \quad \ell(\gamma'_{2,0}[x'_1, x'_{j_m}]) \leq \frac{2^{[\Gamma(\Lambda_{T_1+1})]+1}d_{D'_{1,0}}(x'_1)}{4}\]

\[
< \frac{2\Gamma(\Lambda_{T_1+1})+2\Gamma(\Lambda_{T_1+2})+2\Gamma(\Lambda_{T_1+3})}{4}d_{D'_{1,0}}(x'_1)\]

\[
< \ell(\gamma'_{2,0}[y'_1, y'_3]),
\]
whence we can find a sequence \( \{x_{j_r'}\}_{r=1}^{\left\lfloor \frac{T_j}{2} \right\rfloor} \) such that for each \( r \in \{2, \ldots, \left\lfloor \frac{T_j}{2} \right\rfloor \} \), \( x_{j_r'} \) denotes the first point along the direction from \( x_{j_{r-1}} \) to \( x_{j_{r-1}+2} \) satisfying

\[
\tag{6.55} k_{D_{1,0}}'(x_{j_{r-1}}, x'_{j_r}) \geq \frac{1}{2} \log \Lambda_{T_1+3-2r} \quad \text{(see Figure 16)}.
\]

![Diagram](image)

**Figure 16.** Points \( x'_{j_r} \) (\( r \in \{1, \ldots, \left\lfloor \frac{T_j}{2} \right\rfloor \}) \) in \( \gamma_{2,0}'[y'_{r_1}, v'_{r_2}] \), \( z'_{j_r} \) and \( u'_{j_r} \) in \( \gamma_{2,0}'[x'_{j_{r-1}}, x'_{j_r}] \), the carrot arcs \( \xi'_r \) and the balls \( B_{j_r} \), with center \( v'_{j_r} \) in \( \xi'_r \).

In order to determine some finite sequences of points in \( \gamma_{2,0}' \), we fix \( r \in \{1, \ldots, \left\lfloor \frac{T_j}{2} \right\rfloor \} \) and emphasize our discussions on the part \( \gamma_{2,0}'[x'_{j_{r-1}}, x'_{j_r}] \). It follows from (6.53) that

\[
\tag{6.56} \ell(\gamma_{2,0}'[x'_1, x'_{j_r}]) < 2^{j_r + \sum_{j_{r-1}}^{\left\lfloor \frac{T_j}{2} \right\rfloor} (2\Gamma(\Lambda_{T_1+3-2j_r})+1)} d_{D_{1,0}}'(x'_1) < 2^{j_r + T_1 \Gamma(\Lambda_{T_1+1-r})} d_{D_{1,0}}'(x'_1) = 2^{1+T_1 \Gamma(\Lambda_{T_1+1-r})} \ell(\gamma_{2,0}'[x'_1, x'_{j_r}]).
\]

By letting (see Figure 16)

\[
d_{D_{1,0}}'(z'_{j_r}) = \min_{z' \in \gamma_{2,0}'[x'_{j_{r-1}}, x'_{j_r}]} d_{D_{1,0}}'(z'),
\]

we see that \( z'_{j_r} \) divides \( \gamma_{2,0}'[x'_{j_{r-1}}, x'_{j_r}] \) into two parts. Without loss of generality, we may assume that

\[
\max\{\ell(\gamma_{2,0}'[x'_{j_{r-1}}, z'_{j_r}]), \ell(\gamma_{2,0}'[z'_{j_r}, x'_{j_r}])\} = \ell(\gamma_{2,0}'[z'_{j_r}, x'_{j_r}]).
\]

Let \( u'_{j_r} \in \gamma_{2,0}'[z'_{j_r}, x'_{j_r}] \) be the last point along the direction from \( z'_{j_r} \) to \( x'_{j_r} \) such that (see Figure 16)

\[
\tag{6.57} \ell(\gamma_{2,0}'[z'_{j_r}, u'_{j_r}]) \leq \Lambda_{T_1-2r} d_{D_{1,0}}'(u'_{j_r}).
\]
Then for each $z' \in \gamma_{2,0}'[u_{j_r}, x_{j_r}']$, we have
\[
\ell(\gamma_{2,0}'[z_{j_r}', z']) \geq \Lambda_{T_1-2r} d_{D_{1,0}'}(z'),
\]
which, together with (6.57), implies
\[
d_{D_{1,0}'}(z') \leq \frac{1}{\Lambda_{T_1-2r}} \ell(\gamma_{2,0}'[z_{j_r}', z']) = \frac{1}{\Lambda_{T_1-2r}} \left( \ell(\gamma_{2,0}'[z_{j_r}', u_{j_r}']) + \ell(\gamma_{2,0}'[u_{j_r}', z']) \right)
\leq d_{D_{1,0}'}(u_{j_r}') + \frac{1}{\Lambda_{T_1-2r}} \ell(\gamma_{2,0}'[u_{j_r}', z'])
\]
whence
\[
(6.58) \quad \ell_{k_{D_{1,0}'}(\gamma_{2,0}'[u_{j_r}', x_{j_r}'])} = \int_{\gamma_{2,0}'[u_{j_r}', x_{j_r}']} \frac{|dz'|}{d_{D_{1,0}'}(z')}
\geq \Lambda_{T_1-2r} \log \left( 1 + \frac{\ell(\gamma_{2,0}'[u_{j_r}', x_{j_r}'])}{\Lambda_{T_1-2r} d_{D_{1,0}'}(u_{j_r}')} \right).
\]

We take $\xi_r'$ to be an a-carrot arc joining $u_{j_r}'$ and $y_0'$ in $D'$ with center $y_0'$ (see Figure 16). The existence of $\xi_r'$ comes from the assumption "$D'$ being an a-John domain with center $y_0'$".

Let $\lambda = j_r - 2$. The continuation of the discussions needs a lower bound for arclengt of $\xi_r'$ which is as follows:
\[
(6.59) \quad \ell(\xi_r') \geq 2^{\lambda+1} d_{D_{1,0}'}(x_1').
\]

Since $u_{j_r}' \in \overline{B}(x_1', 2^{\lambda-1} d_{D_{1,0}'}(x_1'))$, to get this lower bound, it suffices to show
\[
y_0' \notin \overline{B}(x_1', 2^{\lambda+1} d_{D_{1,0}'}(x_1')).
\]
Suppose $|y_0' - x_1'| \leq 2^{\lambda+1} d_{D_{1,0}'}(x_1')$. Then Lemma 6.12 and (6.54) imply
\[
d_{D'}(y_0') \leq |y_0' - x_1'| + d_{D'}(x_1') \leq (2^{\lambda+1} + (2^{\overline{j_r} - 1} + (\overline{j_r}^\rho_1 \rho_2 \mu_2^2) d_{D_{1,0}'}(x_1')) < 2^{\Gamma (\Lambda_{T_1+2})} d_{D_{1,0}'}(x_1').
\]
But it follows from the assumption "$D'$ being an a-John domain with center $y_0'$" and (6.46) that
\[
d_{D'}(y_0') \geq \frac{1}{3a} \text{diam}(D') \geq \frac{2}{3a} d_{D_{1,0}'}(y_0') \geq \frac{2^{\Gamma (\Lambda_{T_1+2})} + 1}{3a} d_{D_{1,0}'}(y_0') > 2^{\Gamma (\Lambda_{T_1+2})} d_{D_{1,0}'}(x_1').
\]
This obvious contradiction shows that (6.59) holds.

It follows from (6.59) that there is a point, denoted by $v_{j_r}'$ (see Figure 16), in the intersection $\xi_r' \cap \overline{B}(x_1', 2^{\lambda+1} d_{D_{1,0}'}(x_1'))$ such that
\[
(6.60) \quad \ell(\xi_r'[u_{j_r}', v_{j_r}']) = 2^{\lambda} d_{D_{1,0}'}(x_1').
\]
Then
\[
(6.61) \quad d_{D'}(v_{j_r}') \geq \frac{2^{\lambda}}{a} d_{D_{1,0}'}(x_1').
\]
Let \( B_0 = \mathbb{B}(x'_1, 2^{\lambda+2}d_{D'_1,0}(x'_1)) \), and for each \( r \in \{1, \ldots, \left\lfloor \frac{T_1}{2} \right\rfloor \} \), we let (see Figure 16)

\[
B_{jr} = \mathbb{B}(v'_r, \frac{2^{\lambda-1}}{\alpha}d_{D'_1,0}(x'_1)).
\]

We easily know that

\[ B_{jr} \subset B_0. \]

Hence we have determined finite sequences of points \( \{z'_j\}_{j=1}^{T_1} \) and \( \{u'_j\}_{j=1}^{T_1} \) in \( \gamma_{2,0} \) together with a finite sequence of balls \( \{B_{jr}\}_{r=1}^{\left\lfloor \frac{T_1}{2} \right\rfloor} \) in \( B_0 \). Similarly, we shall determine another finite sequence of points \( \{\zeta'_{jt_p}\}_{p=1}^{p_0} \) in \( \gamma'_{2,0} \) and another finite sequence of balls \( \{B_{1p}\}_{p=1}^{p_0} \) in \( B_0 \). The procedure is as follows:

Let

\[
J = \left\{ r : r \in \{2, \ldots, \left\lfloor \frac{T_1}{2} \right\rfloor \} \text{ and } j_r > j_{r-1} + 2 \right\}.
\]

It is possible that \( J = \emptyset \). If \( J \neq \emptyset \), we let \( J = \{t_1, \ldots, t_{p_0}\} \). In particular, in what follows, we always assume that \( p_0 = 0 \) provided \( J = \emptyset \). Then, obviously, \( p_0 \leq \left\lfloor \frac{T_1}{2} \right\rfloor \).

For each \( t_p \in J \), we let

\[
d_{D'_1,0}(\zeta'_{jt_p}) = \max_{z' \in \gamma'_{j-2}[x'_j, x'_j-2, x'_{jt_p-1}]} d_{D'_1,0}(z').
\]

In this way, we get a finite sequence of points

\[ \{\zeta'_{jt_1}, \ldots, \zeta'_{jt_{p_0}}\} \]

in \( \gamma'_{2,0} \) (see Figure 17).

**Figure 17.** Point \( w'_{1p} \) in \( \gamma'_{2,0}[x'_{jt_{p-2}}, x'_{jt_{p-1}}] \), the carrot arcs \( \xi'_{1p} \) and the balls \( B_{1p} \) with center \( \eta'_{1p} \) in \( \zeta'_{1p} \). For convenience, for each \( p \in \{1, \ldots, p_0\} \), we denote the point \( \zeta'_{jt_p} \) by \( w'_{1p} \), and set

\[
k_{D'_1,0}(x'_{jt_p-2}, x'_{jt_p-1}) = \vartheta_{1t_p}.
\]
Obviously, the choice of $x'_{jr}$ implies

$$\vartheta_{1p} < \frac{1}{2} \log \Lambda_{T_{1}+3-2r}. \quad (6.64)$$

For each $p \in \{1, \ldots, p_0\}$, let $\xi'_{ip}$ be an $a$-carrot arc joining $w'_{ip}$ and $y'_0$ in $D'$ with center $y'_0$ (see Figure 17). Denote by $\eta'_{ip}$ the point in the intersection $\xi'_{ip} \cap B(x'_1, 2^{\lambda+1} d'_{D_{1,0}}(x'_1))$ such that (see Figure 17)

$$\ell(\xi'_{ip}[[w'_{ip}, \eta'_{ip}]] = 2^{\lambda} d'_{D_{1,0}}(x'_1),$$

and then

$$d_{D'}(\eta'_{ip}) \geq \frac{2^{\lambda} - 1}{a} d_{D_{1,0}}(x'_1). \quad (6.65)$$

Let

$$B_{1p} = B(\eta'_{ip}, \frac{2^{\lambda-1}}{a} d_{D_{1,0}}(x'_1)) \quad (\text{see Figure 17})$$

for each $p \in \{1, \ldots, p_0\}$. Obviously,

$$B_{1p} \subset B_0. \quad (6.66)$$

Then we have

**Lemma 6.19.** (1) Suppose that there exist integers $m < q \in \{1, \ldots, \lfloor T/2 \rfloor\}$ such that $B_{jm} \cap B_{jq} \neq \emptyset$ (see Figure 18). Then

$$k_{D'}(u'_{jm}, u'_{jq}) \leq 2a \left( \log \left( 1 + \frac{2^{\lambda+1} d_{D_{1,0}}(x'_1)}{d'(u'_{jm})} \right) + \log \left( 1 + \frac{2^{\lambda+1} d_{D_{1,0}}(x'_1)}{d'(u'_{jq})} \right) + 1 \right);$$
(2) Suppose that there exist integers \( h < s \in \{1, \ldots, p_0\} \) such that \( B_{1h} \cap B_{1s} \neq \emptyset \). Then
\[
 k_D'(u_{1h}', w_{1s}') \leq 2a \left( \log \left( 1 + \frac{2^{\lambda+1}d_{D_{1,0}}(z_1')}{d_{D'}(u_{1h}')} \right) + \log \left( 1 + \frac{2^{\lambda+1}d_{D_{1,0}}(z_1')}{d_{D'}(u_{1s}')} \right) \right).
\]

To prove Lemma 6.19, we only need to prove (1) since the proof of (2) is similar. Since Lemma 4.4 and (6.60) imply
\[
k_D'(u_{j,m}', v_{j,m}') \leq 2 \log \left( 1 + \frac{2^{\lambda+1}d_{D_{1,0}}(z_1')}{{D'}'(u_{j,m}')} \right),
\]
and by (6.61), obviously,
\[
k_D'(v_{j,m}', v_{j,q}') < 2,
\]
we see that
\[
k_D'(u_{j,m}', u_{j,q}') \leq k_D'(u_{j,m}', v_{j,m}') + k_D'(v_{j,m}', v_{j,q}') + k_D'(u_{j,q}', v_{j,q}') \leq 2a \left( \log \left( 1 + \frac{2^{\lambda+1}d_{D_{1,0}}(z_1')}{{D'}'(u_{j,m}')} \right) + \log \left( 1 + \frac{2^{\lambda+1}d_{D_{1,0}}(z_1')}{{D'}'(u_{j,q}')} \right) \right)
\]
as required.

Our next Lemma below is on the upper bound for \( p_0 \). Its proof needs Propositions 6.2 and 6.3 below.

**Proposition 6.2.** For all \( x', y' \in \gamma_{2,0} \), if \( k_{D_{1,0}}(x', y') \leq \vartheta \), then there exists some point \( z' \in \gamma_{2,0}[x', y'] \) such that \( \ell(\gamma_{2,0}[x', y']) \leq \vartheta d_{D_{1,0}}(z') \).

For a proof of this result, we let \( z' \in \gamma_{2,0}[x', y'] \) be such that
\[
d_{D_{1,0}}(z') = \sup_{w' \in D_{1,0}} d_{D_{1,0}}(w').
\]
Then
\[
\frac{\ell(\gamma_{2,0}[x', y'])}{d_{D_{1,0}}(z')} \leq \int_{\gamma_{2,0}[x', y']} \frac{|dw'|}{d(w')} = k_{D_{1,0}}(x', y') \leq \vartheta,
\]
which implies that
\[
\ell(\gamma_{2,0}[x', y']) \leq \vartheta d_{D_{1,0}}(z')
\]
as required.

**Proposition 6.3.** For positive constants \( r, \lambda_1 \) and \( \lambda_3 \) with \( \lambda_1 > \lambda_3 \), there exists a positive integer \( \lambda_2 < \left( \frac{\lambda_1}{\lambda_3} \right)^n \) such that the ball \( B(x, \lambda_1r) \) contains at most \( \lambda_2 \) disjoint balls \( B_i \), each of which has radius \( \lambda_3 r \).
The proof easily follows from the following estimate:

\[
\frac{\sum_{i=1}^{\lambda_2} \text{Vol} \left( B_i \right)}{\text{Vol} \left( B(x, \lambda_1 r) \right)} = \lambda_2 \left( \frac{\lambda_3}{\lambda_1} \right)^n < 1,
\]

where "Vol" denotes the volume.

Now, we are in a position to state and prove our next Lemma.

**Lemma 6.20.** \( p_0 \leq [8a]^n + 2 \).

Suppose on the contrary that \( p_0 > [8a]^n + 2 \). Under this assumption, (6.66) and Proposition 6.3 show that there exist \( p_1 < p_2 \in \{1, \ldots, p_0\} \) such that

\[
B_{1p_1} \cap B_{1p_2} \neq \emptyset.
\]

By the construction of \( w_{1p} \), we know that there exist \( t_{p_1} < t_{p_2} \in \{1, \ldots, \left\lfloor \frac{p_0}{2} \right\rfloor \} \) such that

\[
\zeta'_{jp_{p_1}} = w_{1p_{p_1}}' \quad \text{and} \quad \zeta'_{jp_{p_2}} = w_{1p_{p_2}}'.
\]

By estimating \( k_{D_{1,0}}(\zeta'_{jp_{p_1}}, \zeta'_{jp_{p_2}}) \), we shall get a contradiction. First, we see from Proposition 6.2 and (6.63) that

\[
\ell(\gamma_2,0[x_{j_{p_1}p_1}, x'_{j_{p_1}p_1}] - 1) \leq \varphi_{1t_{p_1}} d_{D_{1,0}}(\zeta'_{jp_{p_1}})
\]

and

\[
\ell(\gamma_2,0[x'_{j_{p_2}p_2}, x'_{j_{p_2}p_2}] - 1) \leq \varphi_{1t_{p_2}} d_{D_{1,0}}(\zeta'_{jp_{p_2}}).
\]

Further, by (6.56), we get

\[
\ell(\gamma_2,0[x'_{j_{p_2}p_2}, x'_{j_{p_2}p_2}] - 1) = \frac{1}{2} \ell(\gamma_2,0[x', x'] - 1) = \frac{1}{4} \ell(\gamma_2,0[x', x'] - 1)
\]

\[
\geq 2^{3 + T_1(\lambda_{T_1 + 1} - 2r_{p_1})} \ell(\gamma_2,0[x', x'] - 1)
\]

\[
= 2^{\lambda - 4 - T_1(\lambda_{T_1 + 1} - 2r_{p_1})} d_{D_{1,0}}(x_1'),
\]

and similarly, we gain

\[
\ell(\gamma_2,0[x'_{j_{p_2}p_2}, x'_{j_{p_2}p_2}] - 1) \geq 2^{\lambda - 4 - T_1(\lambda_{T_1 + 1} - 2r_{p_2})} d_{D_{1,0}}(x_1').
\]

Then (6.55), Lemma 6.13 and Lemma 6.19(2) lead to

\[
\mu_2^2 \rho_7 < \frac{1}{2} \log \Lambda_{T_1 + 3 - 2r_{p_1}}
\]

\[
\leq k_{D_{1,0}}(x'_{j_{p_1}}, x'_{j_{p_1}})
\]

\[
< k_{D_{1,0}}(\zeta'_{jp_{p_1}}, \zeta'_{jp_{p_2}}) \leq \mu_2^2 \rho_7 k_{D'}(\zeta'_{jp_{p_1}}, \zeta'_{jp_{p_2}})
\]

\[
\leq 2a \mu_2^2 \rho_7 \left( \log \left( 1 + \frac{2^{\lambda + 1} d_{D_{1,0}}(x'_{j_{p_1}})}{d_{D'}(\zeta'_{jp_{p_1}})} \right) + \log \left( 1 + \frac{2^{\lambda + 1} d_{D_{1,0}}(x'_{j_{p_2}})}{d_{D'}(\zeta'_{jp_{p_2}})} \right) + 1 \right)
\]
But it follows from (6.64) that

\[
\max \{ \vartheta_{1t_{p_2}}, \vartheta_{1t_{p_1}} \} < \frac{1}{2} \log \Lambda_{T_1+3-2t_{p_1}},
\]

and so

\[
4a\mu_2^2 \rho T \left( 5 + T_1 \Gamma(\Lambda_{T_1+1-2t_{p_1}}) + \log(1 + \max \{ \vartheta_{1t_{p_1}}, \vartheta_{1t_{p_2}} \} \right) + 2a\mu_2^2 \rho T < \frac{1}{2} \log \Lambda_{T_1+3-2t_{p_1}}.
\]

This obvious contradiction completes the proof of Lemma 6.20.

**Lemma 6.21.** \( k_{D_{1,0}'}(u_{j_r}', x_{j_r}') \leq 2\rho_8 k_{D_{1,0}'}(z_{j_r}', u_{j_r}') \).

Proof of this Lemma easily follows from the upper bounds for

\[
k_{D_{1,0}'}(x_{j_r,-1}', x_{j_r}') \quad \text{and} \quad \frac{d_{D_{1,0}'}(z_{j_r}')}{d_{D_{1,0}'}(u_{j_r}')}.\]

First, we get an upper bound for \( k_{D_{1,0}'}(x_{j_r,-1}', x_{j_r}') \). Since

\[
k_{D_{1,0}'}(x_{j_r,-1}', x_{j_r}') \geq \frac{1}{2} \log \Lambda_{T_1+3-2r} > 8\mu_2^4 \rho^4,
\]

we see from Lemma 6.13 that

\[
k_{D'}(x_{j_r,-1}', x_{j_r}') \geq \frac{1}{\mu_2^2 \rho^2} k_{D_{1,0}'}(x_{j_r,-1}', x_{j_r}') > 8\mu_2^4 \rho^4.
\]

Then by Lemma 6.14, we have

\[
(6.67) \quad \frac{1}{2} \log \Lambda_{T_1+3-2r} \leq k_{D_{1,0}'}(x_{j_r,-1}', x_{j_r}') \leq 48 a^2 \mu_2^6 \rho^2 \log \left( 1 + \frac{\ell(\gamma_{2,0}[x_{j_r,-1}', x_{j_r}'])}{d_{D_{1,0}'}(z_{j_r}')} \right),
\]

whence

\[
(6.68) \quad \ell(\gamma_{2,0}[x_{j_r,-1}', x_{j_r}']) > \Lambda_{T_1+2-2r} d_{D_{1,0}'}(z_{j_r}'),
\]

which shows that there exists an integer \( m_2 \) such that

\[
(6.69) \quad 2^{m_2} d_{D_{1,0}'}(z_{j_r}') \leq \ell(\gamma_{2,0}[x_{j_r,-1}', x_{j_r}']) < 2^{m_2+1} d_{D_{1,0}'}(z_{j_r}').
\]
Hence it follows from (6.68) that

\[(6.70) \quad m_2 > \log_2 \Lambda_{T_1+2-2r} - 1 > \max\{\Lambda_{T_1+1-2r}, 3 \log_2 \Lambda_{T_1-2r}, \Lambda_1, 2\rho_s\}.
\]

By (6.67) and (6.69), we have

\[(6.71) \quad k_{D_{1,0}}(z_{j_r}') \leq 48a^2 \mu_2^6 \rho_7^4 \log \left(1 + \frac{\ell(\gamma_{2,0}[x_{j_r}-1, x_{j_r}'])}{d_{\Lambda_{D_{1,0}}}(z_{j_r}')}\right)
\]

\[\leq 48a^2 \mu_2^6 \rho_7^4 \log \left(1 + \frac{2m_2+1}{d_{\Lambda_{D_{1,0}}}(z_{j_r}')}\right)
\]

\[= 48a^2 \mu_2^6 \rho_7^4 (1 + 2^{m_2+1})
\]

\[< 48a^2 \mu_2^6 \rho_7^4 (m_2 + 1)
\]

\[< \frac{2\rho_8 m_2}{3}.
\]

An upper bound for \(\frac{d_{D_{1,0}}(z_{j_r}')}{d_{D_{1,0}}(u_{j_r}')}\) is as follows:

\[(6.72) \quad d_{D_{1,0}}(u_{j_r}') \geq 2^{\frac{m_2}{4}+1} d_{D_{1,0}}(z_{j_r}').
\]

Otherwise, (6.57) and (6.69) imply that

\[\ell(\gamma_{2,0}[u_{j_r}', x_{j_r}']) = \ell(\gamma_{2,0}[z_{j_r}', x_{j_r}']) - \ell(\gamma_{2,0}[z_{j_r}', u_{j_r}'])
\]

\[\geq \frac{1}{2} \ell(\gamma_{2,0}[x_{j_r}-1, x_{j_r}']) - \Lambda_{T_1-2r} d_{D_{1,0}}(u_{j_r}')
\]

\[\geq \frac{1}{2} \ell(\gamma_{2,0}[x_{j_r}-1, x_{j_r}']) - 2^{\frac{m_2}{4}+1} \Lambda_{T_1-2r} d_{D_{1,0}}(z_{j_r}'),
\]

\[\geq \left(1 - \frac{\Lambda_{T_1-2r}}{2^{\frac{m_2}{4}+1}}\right) \ell(\gamma_{2,0}[x_{j_r}-1, x_{j_r}'])
\]

\[\geq \frac{1}{4} \ell(\gamma_{2,0}[x_{j_r}-1, x_{j_r}']),
\]

since (6.70) implies that \(\frac{\Lambda_{T_1-2r}}{2^{\frac{m_2}{4}+1}} < \frac{1}{4}\). Then it follows from (6.58) and (6.69) that

\[k_{D_{1,0}}(u_{j_r}', x_{j_r}') \geq \Lambda_{T_1-2r} \log \left(1 + \frac{\ell(\gamma_{2,0}[u_{j_r}', x_{j_r}'])}{\Lambda_{T_1-2r} d_{D_{1,0}}(u_{j_r}')}\right)
\]

\[\geq \Lambda_{T_1-2r} \log \left(1 + \frac{\ell(\gamma_{2,0}[x_{j_r}-1, x_{j_r}'])}{4\Lambda_{T_1-2r} d_{D_{1,0}}(u_{j_r}')}\right)
\]

\[\geq \Lambda_{T_1-2r} \log \left(1 + \frac{2^{m_2} d_{D_{1,0}}(z_{j_r}')}{2^{\frac{m_2}{4}+3} \Lambda_{T_1-2r} d_{D_{1,0}}(z_{j_r}')}\right)
\]

\[\geq \Lambda_{T_1-2r} \log \left(1 + \frac{2^{m_2}}{4\Lambda_{T_1-2r}}\right),
\]
whence (6.70) implies that
\[
\Lambda_{T_1-2r} \log \left( 1 + \frac{2^{m_2-1}}{4\Lambda_{T_1-2r}} \right) \geq 2\Lambda_{T_1-2r}^\frac{1}{2} \log (1 + 2^{m_2-1}) > 48a^2\mu_2\rho_7^2(m_2 + 1),
\]
which, apparently, contradicts (6.71) since \( k_{D_1,0}(u'_j, x'_j) \leq k_{D_1,0}(x'_{j-1}, x'_j) \). Thus (6.72) holds.

Now, we obtain from (6.71) and (6.72) that
\[
k_{D_1,0}^\prime(z'_j, u'_j) \geq \log \frac{d_{D_1,0}^\prime(u'_j)}{d_{D_1,0}^\prime(z'_j)} > \frac{m_2}{3} > \frac{1}{2\rho_8} k_{D_1,0}^\prime(x'_{j-1}, x'_j)
\]
as required.

6.5.3. **Completion of the proof of Theorem 6.1.** To complete the proof, we first do some preparation. Since
\[
\frac{[\frac{T_1}{2}]}{2[16a]^n} \geq 2[16a]^n > 2(8a^n + 2),
\]
Lemma 6.20 implies that there are \( s_0, s_1 \in \{1, \ldots, [\frac{T_1}{2}]\} \) with \( s_1 - s_0 \geq 2[16a]^n \) such that \( \gamma_{0,2}[x'_{j_0}, x'_{j_1}] \) doesn’t contain any \( u'_{j_0} \).

Since \( s_1 - s_0 > 2([8a]^n + 2) \), by Proposition 6.3, we know that there exist integers \( p < q \in \{s_0, \ldots, s_1\} \) such that \( B_{j_p} \cap B_{j_q} \neq \emptyset \), \( q - p \geq 2 \) and \( q \leq s_0 + [16a]^n \). It follows from (6.58) and Lemma 6.21 that
\[
(6.73) \quad k_{D_1,0}^\prime(u'_j, u'_{j'}) > k_{D_1,0}^\prime(u'_{j_p}, x'_{j_p}) + k_{D_1,0}^\prime(x'_{j_q}, u'_{j'})
\]
\[
> \frac{1}{2\rho_8} k_{D_1,0}^\prime(u'_{j_p}, x'_{j_p}) + k_{D_1,0}^\prime(u'_{j_q}, x'_{j_q})
\]
\[
\geq \frac{\Lambda_{T_1-2p}}{2\rho_8} \log \left( 1 + \frac{\ell(\gamma_{0,2}[u'_j, x'_j])}{\Lambda_{T_1-2p}d_{D_1,0}^\prime(u'_j)} \right)
\]
\[
+ \frac{\Lambda_{T_1-2q}}{2\rho_8} \log \left( 1 + \frac{\ell(\gamma_{0,2}[u'_j, x'_j])}{\Lambda_{T_1-2q}d_{D_1,0}^\prime(u'_j)} \right).
\]

By (6.55), we get
\[
(6.74) \quad k_{D_1,0}^\prime(u'_j, u'_{j'}) > k_{D_1,0}^\prime(x'_{j_p+1}, x'_{j_q+1}) \geq \frac{1}{2} \log \Lambda_{T_1+1-2p}
\]
\[
> \Lambda_{T_1-2p} > \mu_2^2\rho_7,
\]
and then it follows from Lemmas 6.13 and 6.19 that
\[
k_{D_1,0}^\prime(u'_j, u'_{j'}) \leq \mu_2^2\rho_7 k_{D'}(u'_j, u'_{j'})
\]
\[
\leq 2a\mu_2^2\rho_7 \left( \log \left( 1 + \frac{2^{\lambda+1}d_{D_1,0}^\prime(x'_1)}{d_{D'}(u'_j)} \right) + \log \left( 1 + \frac{2^{\lambda+1}d_{D_1,0}^\prime(x'_1)}{d_{D'}(u'_j)} \right) + 1 \right).
\]
Without loss of generality, we may assume that \( d_{D'}(u'_{j_p}) \leq d_{D'}(u'_{j_q}) \), and thus

\[
(6.75) \quad k_{D',0}(u'_{j_p}, u'_{j_q}) \leq 4a \mu_2^2 \rho_7 \log \left( 1 + \frac{2^{\lambda+1} d_{D',0}(x'_1)}{d_{D'}(u'_{j_p})} \right) + 2a \mu_2^2 \rho_7.
\]

Then we have

\[
(6.76) \quad 2^{\lambda+1} d_{D',0}(x'_1) \geq \Lambda_{T_1}^2 d_{D'}(u'_{j_p}),
\]

because otherwise, (6.75) implies that

\[
k_{D',0}(u'_{j_p}, u'_{j_q}) < 4a \mu_2^2 \rho_7 \log(1 + \Lambda_{T_1}^2) + 2a \mu_2^2 \rho_7 < \frac{1}{2} \log \Lambda_{T_1+1-2p},
\]

which contradicts (6.74). Hence (6.76) holds.

Next, since

\[
j_p+[16a]^n - j_p = 2[16a]^n \quad \text{and} \quad \lambda - j_p+[16a]^n \leq \sum_{i=p+[16a]^n}^{[\frac{T}{2}]} (2\Gamma(\Lambda_{T_1+3-2i}) + 1),
\]

we have

\[
\lambda \leq j_p + 2[16a]^n + \sum_{i=p+[16a]^n}^{[\frac{T}{2}]} (2\Gamma(\Lambda_{T_1+3-2i}) + 1).
\]

Then elementary computations show that

\[
(6.77) \quad \ell(\gamma'_2,0[x'_1, x'_\lambda]) = 2^{\lambda-1} d_{D',0}(x'_1)
\]

\[
\leq 2^{\lambda_p-1} + 2[16a]^n + \sum_{i=p+[16a]^n}^{[\frac{T}{2}]} (2\Gamma(\Lambda_{T_1+3-2i}) + 1) d_{D',0}(x'_1)
\]

\[
= 2^{\lambda_p-1} + 2[16a]^n + 2\sum_{i=p+[16a]^n}^{[\frac{T}{2}]} \Gamma(\Lambda_{T_1+3-2i}) \ell(\gamma'_2,0[x'_1, x'_j])
\]

\[
< 2^{\lambda_p-1} + [\frac{T}{2}] + 2[16a]^n + \Lambda_{T_1} \Gamma(\Lambda_{T_1+3-2[16a]^n-2p}) \ell(\gamma'_2,0[x'_{j_p-1}, x'_{j_p}])
\]

\[
< \Lambda_{T_1-2p} \ell(\gamma'_2,0[x'_{j_p-1}, x'_{j_p}]).
\]

Now, we are in a position to finish the proof. To this end, we separate the discussions into two cases. For the first case where

\[
\ell(\gamma'_2,0[u'_{j_p}, x'_{j_p}]) \leq \frac{1}{4} \ell(\gamma'_2,0[x'_{j_p-1}, x'_{j_p}]),
\]

we have

\[
\ell(\gamma'_2,0[z'_{j_p}, u'_{j_p}]) = \ell(\gamma'_2,0[x'_j, x'_{j_p}]) - \ell(\gamma'_2,0[u'_{j_p}, x'_{j_p}])
\]

\[
\geq \frac{1}{2} \ell(\gamma'_2,0[x'_{j_p-1}, x'_{j_p}]) - \frac{1}{4} \ell(\gamma'_2,0[x'_{j_p-1}, x'_{j_p}])
\]

\[
\geq \frac{1}{4} \ell(\gamma'_2,0[x'_{j_p-1}, x'_{j_p}])),
\]
which, together with (6.57) and (6.77), shows that

\[ d_{D',o}(u'_{j_p}) \geq \ell(\gamma'_{2,0}[z_{j_p}, u'_{j_p}]) \geq \ell(\gamma'_{2,0}[x'_{j_p-1}, x'_{j_p}]) \geq \frac{\ell(\gamma'_{2,0}[x'_{1}, x'_{1}])}{4A_{T_1-2p}} \]

whence (6.75) implies

\[ k_{D',o}(u'_p, u'_{j_p}) < 4a\mu_2^2\rho_7 \log(1 + 16A_{T_1-2p}^2) + 2a\mu_2^2\rho_7 < \frac{1}{2} \log A_{T_1+1-2p}, \]

which contradicts (6.74).

For the other case, namely,

\[ \ell(\gamma'_{2,0}[u'_p, x'_{j_p}]) > \frac{1}{4} \ell(\gamma'_{2,0}[x'_{j_p-1}, x'_{j_p}]), \]

we infer from (6.73) and (6.77) that

\[ k_{D',o}(u'_p, u'_{j_p}) > \frac{\Lambda_{T_1-2p}}{2\rho_8} \log \left(1 + \frac{\ell(\gamma'_{2,0}[u'_p, x'_{j_p}])}{\Lambda_{T_1-2p}d_{D',o}(u'_{j_p})}\right) \]

\[ \geq \frac{\Lambda_{T_1-2p}}{2\rho_8} \log \left(1 + \frac{\ell(\gamma'_{2,0}[x'_{j_p-1}, x'_{j_p}])}{4A_{T_1-2p}d_{D',o}(u'_{j_p})}\right) \]

\[ \geq \frac{\Lambda_{T_1-2p}}{2\rho_8} \log \left(1 + \frac{2^{\lambda-1}d_{D',o}(x'_{1})}{4A_{T_1-2p}d_{D'}(u'_{j_p})}\right) \]

\[ \geq \frac{\Lambda_{T_1-2p}}{2\rho_8} \log \left(1 + \frac{2^{\lambda-1}d_{D',o}(x'_{1})}{4A_{T_1-2p}d_{D'}(u'_{j_p})}\right), \]

whence (6.75) implies

\[ \frac{\Lambda_{T_1-2p}}{2\rho_8} \log \left(1 + \frac{2^{\lambda-1}d_{D',o}(x'_{1})}{4A_{T_1-2p}d_{D'}(u'_{j_p})}\right) \leq 4a\mu_2^2\rho_7 \log \left(1 + \frac{2^{\lambda+1}d_{D',o}(x'_{1})}{d_{D'}(u'_{j_p})}\right) + 2a\mu_2^2\rho_7. \]

Further, (6.76) leads to

\[ \log \left(1 + \frac{t}{16A_{T_1-2p}^2}\right) < \frac{\rho_8^2}{\Lambda_{T_1-2p}} \log(1 + t). \]

where

\[ t = \frac{2^{\lambda+1}d_{D',o}(x'_{1})}{d_{D'}(u'_{j_p})}. \]

Then (6.76) implies that \( t \geq A_{T_1-2p}^2 \). Finally, set

\[ f(t) = \log \left(1 + \frac{t}{16A_{T_1-2p}^2}\right) - \frac{\rho_8^2}{\Lambda_{T_1-2p}} \log(1 + t). \]
Elementary computations show that $f$ is increasing in $(\Lambda_{T_1-2p}^2, +\infty)$ and $f(\Lambda_{T_1-2p}^2) > 0$, which implies that (6.78) is impossible. This completes the proof of Theorem 6.1. □

6.6. **The proof of Theorem 1.1.** It follows from Basic assumptions in Section 6, Theorem 6.1 and the arbitrariness of $z_1$ in $D_1$ that $D'_1$ is a $2\rho_{10}$-John domain with center $z'_0 \in D'_1$ in the diameter metric. By Theorem F, we see that Theorem 1.1 is true. □

7. **Quasisymmetries**

The aim of this section is to prove Theorem 1.2. The proof easily follows from Lemma 7.2 below. But the proof of Lemma 7.2 depends on Theorem 1.1 together with a relationship between inner uniformity and carrot property of the related domains whose precise statement is as follows.

**Lemma 7.1.** Suppose that $D$ is a bounded $b$-inner uniform domain. Then there exists some $x_0 \in D$ such that $D$ has the $4b^2$-carrot property with center $x_0$.

**Proof.** Let $x_0 \in D$ be such that

$$d_D(x_0) = \sup_{x \in D} d_D(x).$$

Since $D$ is $b$-inner uniform, we see that for $y \in D$, there exists a curve $\alpha$ in $D$ connecting $y$ and $x'_0$ such that for every $x \in \alpha$,

$$\min\{\ell(\alpha[y, x]), \ell(\alpha[x, x_0])\} \leq bd_D(x).$$

We only need to prove that

$$\ell(\alpha[y, x]) \leq 4b^2d_D(x).$$

For this, we consider two cases. For the first case where

$$\min\{\ell(\alpha[y, x]), \ell(\alpha[x, x_0])\} = \ell(\alpha[y, x]),$$

apparently,

$$\ell(\alpha[y, x]) \leq bd_D(x).$$

For the remaining case where

$$\min\{\ell(\alpha[y, x]), \ell(\alpha[x, x_0])\} = \ell(\alpha[x, x_0]),$$

we get

$$|x - x_0| \leq \ell(\alpha[x, x_0]) \leq bd_D(x),$$

whence Lemma 4.3 implies

$$d_D(x_0) \leq 2bd_D(x),$$

and so

$$\ell(\alpha[y, x]) \leq \ell(\alpha) \leq 2bd_D(x_0) \leq 4b^2d_D(x).$$

Hence the proof of Lemma 7.1 is complete. □

Now, we are ready to state and prove the key lemma in this section.
Lemma 7.2. Suppose that $f : D \to D'$ is a $K$-quasiconformal mapping between bounded domains $D, D' \subset \mathbb{R}^n$, where $D$ is a $b$-inner uniform domain. Suppose also that $A \subset D$ is a pathwise connected set and that $A' = f(A)$ has the $c_1$-carrot property in $D'$ with center $y'_0 \in D'$. Then $\text{diam}(A) \leq c_2 d_D(y_0)$ with $c_2 = c_2(n, K, b, c_1, \frac{\text{diam}(D')}{d_{D'}(x'_0)})$, where $x_0 \in D$ is determined in Lemma 7.1.

Proof. Since $y'_0 \in D'$, we see that $y_0 \neq \infty$ and $y'_0 \neq \infty$. Let $y_1 \in A$ be such that

$$(7.1) \quad |y_1 - y_0| > \frac{1}{3} \text{diam}(A),$$

and let $\alpha'_0$ be a $c_1$-carrot arc joining $y'_1$ and $y'_0$ in $D'$ with center $y'_0$, i.e.,

$$(7.2) \quad \ell(\alpha'_0[y'_1, y'_0]) \leq c_1 d_{D'}(y')$$

for all $y' \in \alpha'_0$. Then from Lemmas 4.1 and 4.2, we obtain

Claim 7.1. There exists a simply connected $b_0$-uniform domain $A'_1 = \bigcup_{i=1}^{t_0} B_i \subset D'$ such that

(i) $y'_0, y'_1 \in A'_1$;

(ii) for each $i \in \{1, \ldots, t_0\}$,

$$\frac{1}{3c_1} d_{D'}(x'_i) \leq r_i \leq \frac{1}{c_1} d_{D'}(x'_i);$$

(iii) if $t_0 \geq 3$, then for all $i, j \in \{1, \ldots, t_0\}$ with $|i - j| \geq 2$,

$$\text{dist}(B_i, B_j) \geq \frac{1}{2^{37+c_1^2}} \max\{r_i, r_j\};$$

(iv) if $t_0 \geq 2$, then

$$r_i + r_{i+1} - |x'_i - x'_{i+1}| \geq \frac{1}{2^{37+c_1^2}} \max\{r_i, r_{i+1}\}$$

for each $i \in \{1, \ldots, k_0 - 1\}$;

(v) $d_{D'}(y'_0) \leq 2^{40+c_1^2} c_1 d_{A'_1}(y'_0)$;

where $B_i = \mathbb{B}(x'_i, r_i)$, $x'_i \in \alpha'_0$, $x'_i \not\subset B_{i-1}$ for each $i \in \{2, \ldots, t_0\}$, $x'_1 = y'_1$ and $b_0 = 2^{41+c_1^2} c_1^3$.

First, we prove an estimate on the diameter of $A'_1$ in terms of the distance from $y'_0$ to the boundary of $A'_1$. If $t_0 > 1$, then we know from (7.2) and Claim 7.1(v) that

$$\text{diam}(A'_1) \leq 2 \sum_{i=1}^{t_0} r_i \leq 4\ell(\alpha'_0) \leq 4c_1 d_{D'}(y'_0) \leq 2^{42+c_1^2} c_1^2 d_{A'_1}(y'_0).$$

If $t_0 = 1$, then by (ii) and (v) in Claim 7.1,

$$\text{diam}(A'_1) = 2r_1 \leq \frac{2}{c_1} d_{D'}(y'_1) \leq \frac{2}{c_1 - 1} d_{D'}(y'_0) < 2^{42+c_1^2} c_1^2 d_{A'_1}(y'_0),$$

since $d_{D'}(y'_0) \geq (1 - \frac{1}{c_1}) d_{D'}(y'_1)$. Hence we have proved

$$(7.3) \quad \text{diam}(A'_1) \leq 2^{42+c_1^2} c_1^2 d_{A'_1}(y'_0).$$
Next, we prove that $A'_1$ has the carrot property with center $y'_0$. For each $y' \in A'_1$, let $\alpha'$ be a curve in $A'_1$ joining $y'$ and $y'_0$ such that
\[
\min\{\ell(\alpha'[y', x']), \alpha'[y'_0, x'])\} \leq b_0 d_{A'_1}(x')
\]
for all $x' \in \alpha'$, since $A'_1$ is $b_0$-uniform.

For the case $\min\{\ell(\alpha'[y', x']), \alpha'[y'_0, x'])\} = \ell(\alpha'[y', x'])$, we obviously have
\[
\ell(\alpha'[y', x']) \leq b_0 d_{A'_1}(x') .
\]
For the other case $\min\{\ell(\alpha'[y', x']), \alpha'[y'_0, x'])\} = \ell(\alpha'[x', y'_0])$, by Lemma 4.3, we have
\[
\ell(\alpha'[y', x']) \leq b_0 d_{A'_1}(x') .
\]
and so (7.3) implies
\[
\ell(\alpha'[y', x']) \leq b_0 \text{diam}(A'_1) \leq 2^{42+c1}b_1c_1^2d_{A'_1}(y'_0) \leq 2^{43+c1}(b_0c_1)^2d_{A'_1}(x') .
\]
Hence $A'_1$ has the $2^{43+c1}(b_0c_1)^2$-carrot property with center $y'_0$.

Now, we are ready to finish the proof of the lemma. Since both $D$ and $D'$ are bounded, it follows from the assumption “$D$ being $b$-inner uniform” and Lemma 7.1 that $D$ is a $4b^2$-John domain with center $x_0$, where $x_0$ satisfies
\[
d_D(x_0) = \sup_{x \in D} d_D(x) .
\]
Then Theorem 1.1 implies that $A_1$ is $c_0$-John with center $y_0$, where
\[
c_0 = \tau\left(n, K, b, c_1, \frac{\text{diam}(D')}{d_D'(x'_0)}\right) .
\]
By (7.1), we have
\[
\text{diam}(A) \leq 3|y_1 - y_0| \leq 3c_0 d_{A_1}(y_0) \leq 3c_0 d_D(y_0) .
\]
Hence the proof of Lemma 7.2 is complete. \hfill \square

7.1. The proof of Theorem 1.2. The proof of Theorem 1.2 easily follows from Theorems D, J and Lemma 7.2. \hfill \square

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