Abstract

We compute the $r$-matrix for the elliptic Euler-Calogero-Moser model. In the trigonometric limit we show that the model possesses an exact Yangian symmetry.
1 Introduction

The Euler-Calogero-Moser model was defined in [1, 2]. In [3] we considered the rational case and we derived the $r$-matrix. In this paper we are interested in its trigonometric and elliptic generalizations. In the elliptic case we compute the $r$-matrix and show that the usual elliptic Calogero-Moser model and its $r$-matrix are obtained by Hamiltonian reduction. In the trigonometric case we show that the current algebra symmetry exhibited by Gibbons and Hermsen [1] in the rational case, is deformed into a Yangian symmetry algebra.

We consider a system of $N$ particles on a line with pairwise interactions. The degrees of freedom consist of the positions and momenta $(p_i, q_i)_{i=1\ldots N}$ and of antisymmetric additional variables $(f_{ij} = -f_{ji})_{i,j=1\ldots N}$, with the Poisson brackets

$$\{ p_i, q_j \} = \delta_{ij} \quad (1)$$

$$\{ f_{ij}, f_{kl} \} = \frac{1}{2} (\delta_{il} f_{jk} + \delta_{ki} f_{lj} + \delta_{jk} f_{il} + \delta_{lj} f_{ki}). \quad (2)$$

The Poisson brackets of the $f_{ij}$ just reproduce the $O(N)$ Lie algebra. The Hamiltonian will be taken of the form

$$H = \frac{1}{2} \sum_{i=1}^N p_i^2 - \frac{1}{2} \sum_{i,j=1\atop i \neq j}^N f_{ij} f_{ji} V(q_{ij}), \quad q_{ij} = q_i - q_j \quad (3)$$

with an even potential $V(-x) = V(x)$.

The equations of motion are easily derived:

$$\dot{q}_i = p_i$$

$$\dot{p}_i = \sum_{j=1\atop j \neq i}^N f_{ij} f_{ji} V'(q_{ij})$$

$$\dot{f}_{ij} = \sum_{k=1\atop k \neq i,j}^N f_{ik} f_{jk} \left[ V(q_{ik}) - V(q_{jk}) \right].$$

Such a system admits a Lax representation only for specific potentials. Indeed writing the following ansatz for the Lax pair

$$L(\lambda) = \sum_{i=1}^N p_i e_{ii} + \sum_{i,j=1\atop i \neq j}^N l(q_{ij}, \lambda) f_{ij} e_{ij} \quad (4)$$

$$M(\lambda) = \sum_{i,j=1\atop i \neq j}^N m(q_{ij}, \lambda) f_{ij} e_{ij} \quad (5)$$

where $e_{ij}$ is the $N \times N$ matrix $(e_{ij})_{kl} = \delta_{ik} \delta_{jl}$ and $\lambda \in \mathbb{C}$ is the spectral parameter, we find that the equations of motion can be written in the Lax form

$$\dot{L}(\lambda) = [M(\lambda), L(\lambda)] \quad (6)$$

if and only if the following equalities are satisfied:

$$m(x, \lambda) = -\frac{\partial}{\partial x} l(x, \lambda) = -l'(x, \lambda) \quad (7)$$

$$l'(x, \lambda) l(y, \lambda) - l'(y, \lambda) l(x, \lambda) = l(x + y, \lambda) [V(x) - V(y)] \quad (8)$$

$$l(x) \sim -\frac{1}{x} \text{ when } x \to 0. \quad (9)$$
Eq.(8) is the famous functional equation of Calogero. Its general solution is:

$$ l(x, \lambda) = -\frac{\sigma(x+\lambda)}{\sigma(x) \sigma(\lambda)}, \quad V(x) = \varphi(x) $$

(10)

where $\sigma$ and $\varphi$ are Weierstrass elliptic functions, the relevant properties of which are recalled in the appendix. The elliptic $O(N)$ Euler-Calogero-Moser model is precisely defined by eq.(8) with $V(x) = \varphi(x)$ together with the Poisson brackets (12).

2 The $r$-matrix

From eq.(8) it follows that $trL^n(\lambda)$ is a set of conserved quantities. In particular

$$ trL(\lambda) = \sum_{i=1}^{N} p_i, \quad trL^2(\lambda) = 2H + \varphi(\lambda). $$

The involution property of these quantities $trL^n(\lambda)$ will follow from the existence of an $r$-matrix which we now calculate (13). Introducing the notations $L_1(\lambda) = L(\lambda) \otimes 1$ and $L_2(\lambda) = 1 \otimes L(\lambda)$ we show that the Poisson brackets of the Lax matrix elements can be recast as

$$ \{L_1(\lambda), L_2(\mu)\} = [r_{12}(\lambda, \mu), L_1(\lambda)] - [r_{21}(\mu, \lambda), L_2(\mu)]. $$

(11)

Following (13) we assume that $r$ is of the form

$$ r_{12}(\lambda, \mu) = a(\lambda, \mu) \sum_{i=1}^{N} c_{ii} \otimes e_{ii} + \sum_{i,j=1 \atop i \neq j}^{N} b_{ij}(\lambda, \mu) e_{ij} \otimes e_{ij} + \sum_{i,j=1 \atop i \neq j}^{N} c_{ij}(\lambda, \mu) e_{ij} \otimes e_{ij}. $$

Requiring that $r_{12}(\lambda, \mu)$ be independent of the $p_i$ variables we obtain

$$ b_{ij}(\lambda, \mu) = -b_{ji}(\mu, \lambda) $$

(12)

$$ c_{ij}(\lambda, \mu) = c_{ij}(\mu, \lambda). $$

(13)

Moreover assuming that $r_{12}(\lambda, \mu)$ is independent of the $f_{ij}$ variables yields the following system:

$$ a(\lambda, \mu) l(q_{ij}, \lambda) - b_{ij}(\lambda, \mu) l(q_{ij}, \mu) + c_{ij}(\lambda, \mu) l(q_{ji}, \mu) = -l'(q_{ij}, \lambda) $$

(14)

$$ b_{ij}(\lambda, \mu) l(q_{jk}, \lambda) - b_{ik}(\lambda, \mu) l(q_{jk}, \mu) = \frac{1}{2} l(q_{jk}, \lambda) l(q_{ji}, \mu) $$

(15)

$$ c_{ij}(\lambda, \mu) l(q_{jk}, \lambda) + c_{ik}(\lambda, \mu) l(q_{jk}, \mu) = \frac{1}{2} l(q_{jk}, \lambda) l(q_{ij}, \mu) $$

(16)

$$ c_{ij}(\lambda, \mu) l(q_{ki}, \lambda) + c_{kj}(\lambda, \mu) l(q_{ki}, \mu) = \frac{1}{2} l(q_{ki}, \lambda) l(q_{ij}, \mu). $$

(17)

A solution to these equations is

$$ a(\lambda, \mu) = -\frac{1}{2} [\zeta(\lambda + \mu) + \zeta(\lambda - \mu)] $$

$$ b_{ij}(\lambda, \mu) = \frac{1}{2} l(q_{ij}, \lambda - \mu) $$

$$ c_{ij}(\lambda, \mu) = \frac{1}{2} l(q_{ij}, \lambda + \mu). $$

Indeed substituting the preceding expressions in eq.(13,16,17) leads to the same relation:

$$ l(q_{ij}, \lambda - \mu) l(q_{jk}, \lambda) + l(q_{ki}, \mu - \lambda) l(q_{jk}, \mu) + l(q_{ik}, \lambda) l(q_{ji}, \mu) = 0 $$
which upon setting \( x = \frac{1}{2} (\lambda + q_{ij}) \), \( y = \frac{1}{2} (2\mu - \lambda + q_{ij}) \), \( z = \frac{1}{2} (\lambda + q_{ji}) \) and \( t = \frac{1}{2} (-\lambda - q_{ki} + q_{kj}) \) is a direct consequence of relation (53). The expression for \( a(\lambda, \mu) \) is then given by eq.(14), and is simplified using eq.(51) and (55). Finally the r-matrix reads

\[
    r_{12}(\lambda, \mu) = \frac{1}{2} \sum_{i,j=1}^{N} l(q_{ij}, \lambda - \mu) e_{ij} \otimes e_{ji} + \frac{1}{2} \sum_{i,j=1}^{N} l(q_{ij}, \lambda + \mu) e_{ij} \otimes e_{ij}
\]

\[
    - \frac{1}{2} \left[ \zeta(\lambda + \mu) + \zeta(\lambda - \mu) \right] \sum_{i=1}^{N} e_{ii} \otimes e_{ii}.
\]

(18)

\section{The \( sl(N) \) model}

The above \( O(N) \) model can be obtained from the more general \( sl(N) \) model by a mean procedure \cite{8, 9, 10}. The \( sl(N) \) elliptic Euler-Calogero Moser model is defined by the Hamiltonian

\[
    H = \frac{1}{2} \sum_{i=1}^{N} p_{i}^2 - \frac{1}{2} \sum_{i,j=1}^{N} f_{ij} f_{ji} \varphi(q_{ij})
\]

and the Poisson brackets

\[
    \{ p_i, q_j \} = \delta_{ij}
\]

\[
    \{ f_{ij}, f_{kl} \} = \delta_{jk} f_{il} - \delta_{il} f_{kj}.
\]

(20)\hspace{1cm} (21)

For this model we define a Lax matrix as

\[
    L(\lambda) = \sum_{i=1}^{N} (p_i - \zeta(\lambda) f_{ii}) e_{ii} + \sum_{i,j=1}^{N} l(q_{ij}, \lambda) f_{ij} e_{ij}.
\]

(22)

The Hamiltonian is given by \( H = \frac{1}{2} \int \frac{d\lambda}{2\pi iN} \text{tr} L^2(\lambda) \). A direct calculation gives

\[
    \{ L_1(\lambda), L_2(\mu) \} = \left[ r_{12}(\lambda, \mu), L_1(\lambda) \right] - \left[ r_{21}(\mu, \lambda), L_2(\mu) \right] - \sum_{i,j=1}^{N} l'(q_{ij}, \lambda - \mu) (f_{ii} - f_{jj}) e_{ij} \otimes e_{ji}
\]

(23)

with the beautifully simple r-matrix

\[
    r_{12}(\lambda, \mu) = -\zeta(\lambda - \mu) \sum_{i=1}^{N} e_{ii} \otimes e_{ii} + \sum_{i,j=1}^{N} l(q_{ij}, \lambda - \mu) e_{ij} \otimes e_{ji}.
\]

(24)

At this point let us make two remarks:

- Because of the third term in the right member of eq.(23) the integrals of motion \( \text{tr} L^n(\lambda) \) are not in involution. However we can restrict ourselves to the manifolds \( (f_{ii} = \text{constant})_{i=1...N} \) since \( \text{tr} L^n(\lambda) \) Poisson-commute with \( f_{ii} \). On these manifolds \( \text{tr} L^n(\lambda) \) are in involution.

- The r-matrix for the \( O(N) \) model eq.(24) is immediately seen to be of the form

\[
    r_{12}^{O(N)} = \frac{1}{2} (1 + \sigma \otimes 1) r_{12}^{sl(N)}
\]

where \( \sigma \) is the involutive automorphism

\[
    \sigma : \lambda e_{ij} \longrightarrow -(-\lambda)^n e_{ij}.
\]

This is typical of a mean construction.
In the following we will restrict the $f_{ij}$ to a symplectic leaf of the Poisson manifold (21). Introducing vectors
\[
(\xi_i)_{i=1\ldots N} \quad \text{with} \quad \xi_i = (\xi_i^a)_{a=1\ldots r}
\]
\[
(\eta_i)_{i=1\ldots N} \quad \text{with} \quad \eta_i = (\eta_i^b)_{a=1\ldots r}
\]
with the Poisson brackets
\[
\{\xi_i^a, \xi_j^b\} = 0, \quad \{\eta_i^a, \eta_j^b\} = 0, \quad \{\xi_i^a, \eta_j^b\} = -\delta_{ij} \delta_{ab},
\]
we parametrize the $f_{ij}$ as follows:
\[
f_{ij} = (\xi_i|\eta_j) = \sum_{a=1}^r \xi_i^a \eta_j^a.
\]

The phase space now becomes a true symplectic manifold.

4 The $r$-matrix of the elliptic Calogero model

We show here that the $r$-matrix for the elliptic Calogero model \cite{8, 12} can be obtained from eq. (23) by a Hamiltonian reduction procedure \cite{8, 9, 10}.

We choose $r = 1$ in eq. (26). On the manifold $f_{ij} = \xi_i \eta_j$ acts an Abelian Lie group
\[
\xi_i \rightarrow \lambda_i \xi_i, \quad \eta_i \rightarrow \lambda_i^{-1} \eta_i.
\]
Remark that the group acts on $L$ and therefore all the Hamiltonians $trL^n(\lambda)$ are invariant. Thus one can apply the method of Hamiltonian reduction. The infinitesimal generator of this action is
\[
H_\epsilon = \sum_{i=1}^N \epsilon_i f_{ii}, \quad \lambda_i = 1 + \epsilon_i.
\]

We fix the momentum by choosing
\[
f_{ii} = \alpha.
\]
To compute the reduced Poisson brackets of the Lax matrix, we remark that the matrix
\[
L^{Cal}(\lambda) = g^{-1}L(\lambda) g \quad \text{with} \quad g = \text{diag}(\xi_i)_{i=1\ldots N}
\]
\[
= \sum_{i=1}^N [g_i - \alpha \zeta(\lambda)] e_{ii} + \alpha \sum_{i,j=1 \atop i \neq j}^N l(q_{ij}, \lambda) e_{ij}
\]
is invariant under the isotropy group $G_\alpha$ of $\alpha$ (which is the whole group itself since it is Abelian) and we can compute the Poisson brackets of its matrix elements directly. We find
\[
\{L^{Cal}_1(\lambda), L^{Cal}_2(\mu)\} = \{L^{Cal}_2(\lambda, \mu), L^{Cal}_1(\lambda)\} - \{L^{Cal}_2(\mu, \lambda), L^{Cal}_2(\mu)\}
\]
with
\[
\sum_{i,j=1 \atop i \neq j}^N l(q_{ij}, \lambda - \mu) e_{ij} \otimes e_{ji} + \frac{1}{2} \sum_{i=1}^N e_{ii} \otimes e_{ii} 
\]
where $u_{12} = \{g_1, g_2\} g_1^{-1} g_2^{-1}$ is here equal to zero. Redefining
\[
r^{Cal}_{12}(\lambda, \mu) \rightarrow r^{Cal}_{12}(\lambda, \mu) + \left[ \frac{1}{2} \alpha \sum_{i=1}^N e_{ii} \otimes e_{ii}, L(\mu) \right]
\]
does not change eq. (23) and yields exactly the $r$-matrix found in \cite{11, 13}
\[
r^{Cal}_{12}(\lambda, \mu) = \sum_{i,j=1 \atop i \neq j}^N l(q_{ij}, \lambda - \mu) e_{ij} \otimes e_{ji} + \frac{1}{2} \sum_{i,j=1 \atop i \neq j}^N l(q_{ij}, \mu) (e_{ii} + e_{jj}) \otimes e_{ij}
\]
\[-[\zeta(\lambda - \mu) + \zeta(\mu)] \sum_{i=1}^N e_{ii} \otimes e_{ii}.
\]
5 Yangian symmetry in the trigonometric case

The parametrization (26) of $f_{ij}$ introduces a $sl(r)$ symmetry into the theory. The transformation

$$\eta^a_i \rightarrow \sum_{b=1}^r u^{ab} \eta^b_i,$$

$$\xi^a_i \rightarrow \sum_{b=1}^r (u^{-1})^{ab} \xi^b_i,$$

leaves the $f_{ij}$ invariant and therefore also the Hamiltonians. This symmetry is generated by a set of conserved currents

$$J^{ab}_0 = \sum_{i=1}^N \xi^b_i \eta^a_i.$$

(31)

It is remarkable that this current was shown, in the rational case [1], to be the first of a hierarchy building a current algebra commuting with the Hamiltonian — and more generally with a subset of the commuting Hamiltonians.

We now extend this result to the trigonometric case, and we will show that the hierarchy of currents form a Yangian symmetry in this case. Taking the trigonometric limit ($\omega_1 = \infty$ and $\omega_2 = i\pi$) in the above formulas, we see that the Lax matrix can be taken of the form

$$L(\lambda) = L_0 - \coth(\lambda)F$$

(32)

with

$$L_0 = \sum_{i=1}^N p_i e_{ii} - \sum_{i,j=1 \atop i \neq j}^N \coth(q_{ij}) f_{ij} e_{ij}, \quad F = \sum_{i,j=1}^N f_{ij} e_{ij}.$$

(33)

By a straightforward calculation, or taking the limit of the elliptic case, we find

$$\{L_1(\lambda), L_2(\mu)\} = \frac{1}{2} (1 - \coth(\lambda) \coth(\mu)) ([C, F_1] - [C, F_2])$$

$$- \sum_{i,j=1 \atop i \neq j}^N (f_{ii} - f_{jj}) \frac{1}{\sinh^2(q_{ij})} e_{ij} \otimes e_{ji}$$

(34)

where

$$r^{0}_{12} = - \sum_{i,j=1 \atop i \neq j}^N \coth(q_{ij}) e_{ij} \otimes e_{ji}$$

(35)

and $C$ is the Casimir element of $sl(N)$

$$C = \sum_{i,j=1}^N e_{ij} \otimes e_{ji}.$$

(36)

In spite of the unusual second term in eq. (34), the quantities $trL^n(\lambda)$ are still in involution on the manifolds $\Sigma_\alpha : (f_{ii} = \alpha)_{i=1...N}$. Indeed,

$$\{trL^n(\lambda), trL^m(\mu)\} = \frac{n m}{2} \sum_{i,j=1 \atop i \neq j}^N \frac{f_{ii} - f_{jj}}{\sinh^2(q_{ij})} [L^{n-1}(\lambda)]_{ij} \frac{1}{\sinh^2(q_{ij})} [L^{m-1}(\mu)]_{ji}$$

$$- \frac{n m}{2} (1 - \coth(\lambda) \coth(\mu)) tr_{12} (L_1^{n-1}(\lambda)L_2^{m-1}(\mu)[C, F_1 - F_2])$$

5
and since $\text{tr}_2 ((1 \otimes A) C) = A$, we obtain

$$\{ \text{tr} L^n(\lambda), \text{tr} L^m(\mu) \} = n\ m \sum_{i,j=1}^{N} f_{ii} - f_{jj} \left[ L^{n-1}(\lambda) \right]_{ij} \left[ L^{m-1}(\mu) \right]_{ji}$$

$$- \frac{n\ m}{2} (1 - \coth(\lambda) \coth(\mu)) \ \text{tr} \{ L^{n-1}(\lambda) [L^{m-1}(\mu), F] - L^{m-1}(\mu)[L^{n-1}(\lambda), F] \}. $$

If we notice that

$$F = -\frac{L(\lambda) - L(\mu)}{\coth(\lambda) - \coth(\mu)}$$

we immediately get the involution property.

We consider now the subset $\text{tr}(L^n) = \text{tr}(L_0 + F)^n$ of commuting Hamiltonians; notice that $H$ belongs to this subset, since $H = \frac{1}{2} \text{tr} L^2 - \alpha \text{tr} L + \frac{1}{2} N \alpha^2$.

We introduce the following quantities:

$$J_{ab}^n = \text{tr}(L^n F_{ab}), \quad a, b = 1, \ldots, r \quad n = 0, 1, \ldots, \infty$$

where $F_{ab}$ is the $N \times N$ matrix of elements

$$(F_{ab})_{ij} = f_{ij}^a = \xi_i^a \eta_j^a.$$ (38)

We define the generating functional of the currents $J_{ab}^n$. It is the $r \times r$ matrix $T(z)$ of elements

$$T_{ab}(z) = -\frac{1}{2} \delta_{ab} - \sum_{n \in \mathbb{N}} z^{n+1} J_{ab}^n = -\frac{1}{2} \delta_{ab} + \text{tr} \left( \frac{1}{L - z} F_{ab} \right).$$ (39)

**Proposition.** On the manifolds $\Sigma_\alpha$ we have the following two properties:

1. The currents $J_{ab}^n$ Poisson commute with all the quantities of the form $\text{tr}(L^n)$.

2. The generating functional $T(z)$ satisfies the defining relation of a (classical) Yangian algebra:

$$\{ T(y) \otimes T(z) \} = [R(y, z), T(y) \otimes T(z)]$$

with

$$R(y, z) = -2 \frac{\Pi}{y - z} , \quad \Pi = \sum_{a,b=1}^{r} e_{ab} \otimes e_{ba}. $$ (41)

**Proof.** To prove this proposition we need the Poisson brackets

$$\{ L_1, L_2 \} = [r_{12}^0, L_1] - [r_{21}^0, L_2] + \sum_{i,j=1}^{N} (f_{ii} - f_{jj}) \frac{1}{\sinh^2(q_{ij})} e_{ij} \otimes e_{ji} $$

$$\{ L_1, F_{2}^{ab} \} = [-r_{21}^0 + C, F_{2}^{ab}]$$

$$\{ F_{1}^{ab}, F_{2}^{cd} \} = (\delta_{ad} F_{1}^{cb} - \delta_{bc} F_{2}^{ad}) C.$$ (44)

Remark that the currents $J_{ab}^n$ and the Hamiltonians $\text{tr}(L^n)$ are invariant under the symmetry

$$\xi_i^a \longrightarrow \lambda_i \xi_i^a , \quad \eta_i^a \longrightarrow \lambda_i^{-1} \eta_i^a.$$  

Therefore we can compute their Poisson brackets on the reduced phase space straightforwardly: restricting ourselves to the manifolds $f_{ii} = \alpha$, the last term in eq.(42) vanishes, and we will systematically drop its contribution in intermediate calculations.

We emphasize that in eq.(43) the same $r$-matrix appears. Moreover it is the term $[C, F_{2}^{ab}]$ in eq.(43) which is responsible for the quadratic form of eq.(4), as we shall see in what follows.
Introducing the generating functional $H(z) = \text{tr}(\frac{1}{L-z})$ of the Hamiltonians $\text{tr}(L^n)$ we compute
\[
\left\{ \frac{1}{L_1 - y_1} F_{ab}^{cd}, \frac{1}{L_2 - z} \right\} = - \left[ \frac{1}{L_2 - z} r_{12}^{0}, \frac{1}{L_1 - y_1} F_{ab}^{cd} \right] + \left[ \frac{1}{L_1 - y_1} r_{21}^{0}, \frac{1}{L_2 - z} \right]
\]
\[
+ \frac{1}{L_1 - y_1 \ L_2 - z} \left[ \mathcal{C}, F_{ab}^{cd} \right] \frac{1}{L_2 - z}.
\]
Taking the trace we obtain
\[
\left\{ T_{ab}^{cd}(y), H(z) \right\} = \text{tr} \left( F_{ab}^{cd} \right) \left( \frac{1}{L - y} \right) (L-z)^2 = 0.
\]
This proves the first part of the proposition. To prove the second part we evaluate
\[
\left\{ \frac{1}{L_1 - y_1} F_{ab}^{cd}, \frac{1}{L_2 - z} F_{cd}^{ef} \right\} = - \left[ \frac{1}{L_2 - z} r_{12}^{0}, \frac{1}{L_1 - y_1} F_{ab}^{cd} \right] + \left[ \frac{1}{L_1 - y_1} r_{21}^{0}, \frac{1}{L_2 - z} F_{cd}^{ef} \right]
\]
\[
+ \frac{1}{L_1 - y_1 \ L_2 - z} \left( \delta_{ad} F_{bc}^{ef} - \delta_{eb} F_{ad}^{cd} \right) C
\]
\[
+ \frac{1}{L_1 - y_1 \ L_2 - z} \left[ \mathcal{C}, F_{ab}^{cd} \right] \frac{1}{L_2 - z} F_{cd}^{ef} - \left[ \mathcal{C}, F_{cd}^{ef} \right] \frac{1}{L_1 - y_1} F_{ab}^{cd} \right\}.
\]
Hence taking the trace we get
\[
\left\{ T_{ab}^{cd}(y), T_{cd}^{ef}(z) \right\} = \text{tr} \left( \frac{1}{L - y} \frac{1}{L - z} \left( \delta_{ad} F_{bc}^{ef} - \delta_{eb} F_{ad}^{cd} \right) \right)
\]
\[
+ \text{tr} \left( \frac{1}{L - y} \left[ \frac{1}{L - z} F_{cd}^{ef}, \frac{1}{L - y} F_{ab}^{cd} \right] \right) - \text{tr} \left( \frac{1}{L - z} \left[ \frac{1}{L - y} F_{ab}^{cd}, \frac{1}{L - y} F_{cd}^{ef} \right] \right).
\]
Using the cyclicity of the trace and
\[
\frac{1}{L - y} \frac{1}{L - z} = \frac{1}{y - z} \left( \frac{1}{L - y} - \frac{1}{L - z} \right)
\]
this becomes
\[
\left\{ T_{ab}^{cd}(y), T_{cd}^{ef}(z) \right\} = \frac{1}{y - z} \left( \delta_{ad} (T_{bc}^{eh}(y) - T_{ch}^{eb}(z)) - \delta_{eb} (T_{ad}^{ef}(y) - T_{ad}^{ef}(z)) \right)
\]
\[
+ \frac{2}{y - z} \text{tr} \left( \frac{1}{L - y} \frac{1}{L - z} F_{cd}^{ef}, \frac{1}{L - y} F_{ab}^{cd}, \frac{1}{L - y} F_{cd}^{ef} \right).
\]
Remarking that
\[
\text{tr} \left( \frac{1}{L - y} \frac{1}{L - z} F_{cd}^{ef}, \frac{1}{L - y} F_{ab}^{cd} \right) = \sum_{ijkl=1}^{N} \left( \frac{1}{L - y} \right)_{ij} \xi_{ij}^{k} \eta_{ij}^{l} \left( \frac{1}{L - z} \right)_{kl} \xi_{kl}^{j} \eta_{kl}^{a}
\]
\[
= \left( \sum_{ij=1}^{N} \left( \frac{1}{L - y} \right)_{ij} \xi_{ij}^{k} \eta_{ij}^{l} \right) \left( \sum_{kl=1}^{N} \left( \frac{1}{L - z} \right)_{kl} \xi_{kl}^{j} \eta_{kl}^{a} \right)
\]
\[
= \left( T_{ad}^{ef}(y) + \frac{1}{2} \delta_{ad} \right) \left( T_{ch}^{eb}(z) + \frac{1}{2} \delta_{eb} \right)
\]
we prove the result \( \square \).
The rational limit is obtained by applying the canonical transformation

\[
p_i \rightarrow \frac{1}{\epsilon} p_i \\
q_i \rightarrow \epsilon q_i
\]

and sending \(\epsilon\) to zero. In this limit

\[
L_0 \rightarrow \frac{1}{\epsilon} L_{\text{rational}} \\
r_{12}^0 \rightarrow \frac{1}{\epsilon} r_{12}^{\text{rational}}.
\]

The Casimir term drops therefore from eq.(43), leaving us with a linear Poisson algebra

\[
\{T(y) \otimes T(z)\} = -\frac{1}{2} [R(y, z), T(y) \otimes 1 + 1 \otimes T(z)] \tag{45}
\]

which is the result found by Gibbons and Hermsen.

6 Conclusion

The Euler-Calogero-Moser model is becoming more and more interesting. On the one hand the computation of the classical \(r\)-matrix is made considerably easier by the existence of the extra variables \(f_{ij}\), the more complicated \(r\)-matrix of the Calogero-Moser model following naturally from a Hamiltonian reduction procedure. On the other hand, this model exhibits an exact infinite symmetry which is just a current algebra symmetry in the rational case and becomes an exact Yangian symmetry in the trigonometric case. This structure is very much reminiscent of the one discovered in [13]. Actually the two currents \(J_0\) and \(J_1\) (which generate the full algebra) are identical in the two cases. Indeed, in our case we have

\[
J_{1}^{ab} = \sum_{i=1}^{N} p_i f_{ii}^{ab} - \sum_{i,j}^{N} \frac{e^{-q_{ij}}}{\sinh(q_{ij})} f_{ij} f_{ji}^{ab}.
\]

Setting \((X_i)^{ab} = f_{ii}^{ab}\), \(\Theta_{ij} = 2 \frac{e^{2q_{ij}}}{e^{2q_{ij}} - e^{-2q_{ij}}}\), and using eq.(28) we can rewrite

\[
J_{1}^{ab} = \sum_{i=1}^{N} p_i X_i^{ab} - \sum_{i,j}^{N} \Theta_{ij} (X_i X_j)^{ab}.
\]

This is exactly the current found in [13]. In fact the model considered in [14, 15, 13] is a quantum version of our model for a particular choice of orbit.

At this point two interesting problems arise. One is the understanding of the role of the \(r\)-matrix in the quantization of these models. The other is the hypothetical extension of these results to the elliptic case, which still remains quite mysterious.

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Appendix

The Weierstrass \(\sigma\) function of periods \(2\omega_1, 2\omega_2\) is the entire function defined by

\[
\sigma(z) = z \prod_{m,n \neq 0} \left( 1 - \frac{z}{\omega_{mn}} \right) \exp \left[ \frac{z}{\omega_{mn}} + \frac{1}{2} \left( \frac{z}{\omega_{mn}} \right)^2 \right] \tag{46}
\]
with $\omega_{mn} = 2m\omega_1 + 2n\omega_2$. The functions $\zeta$ and $\varphi$ are

$$\zeta(z) = \frac{\sigma'(z)}{\sigma(z)}, \quad \varphi(z) = -\zeta'(z),$$

(47)

these functions having the symmetries

$$\sigma(-z) = -\sigma(z), \quad \zeta(-z) = -\zeta(z), \quad \varphi(-z) = \varphi(z).$$

(48)

Their behaviour at the neighbourhood of zero is

$$\sigma(z) = z + O(z^5), \quad \zeta(z) = z^{-1} + O(z^3), \quad \varphi(z) = z^{-2} + O(z^2).$$

(49)

Setting

$$l(q, \lambda) = -\frac{\sigma(q + \lambda)}{\sigma(q) \sigma(\lambda)}$$

(50)

it is easy to check that

$$l(-q, \lambda) = -l(q, -\lambda), \quad l'(q, \lambda) = l(q, \lambda) [\zeta(\lambda + q) - \zeta(q)].$$

(51)

We need several non trivial relations:

$$-\frac{\sigma(\lambda - \mu) \sigma(\lambda + \mu)}{\sigma^2(\lambda) \sigma^2(\mu)} = \varphi(\lambda) - \varphi(\mu),$$

(52)

$$\sigma(x-y)\sigma(x+y)\sigma(z-t)\sigma(z+t)+\sigma(y-z)\sigma(y+z)\sigma(x-t)\sigma(x+t)+\sigma(z-x)\sigma(z+x)\sigma(y-t)\sigma(y+t) = 0,$$

(53)

this last equation becoming, in terms of the $l(q, \lambda)$ function,

$$\frac{l(q, \lambda)}{l(-q, \lambda - \mu)} = \zeta(\lambda) + \zeta(\mu - \lambda) + \zeta(q) - \zeta(\mu + q).$$

(55)

Choosing the periods $\omega_1 = \infty$ and $\omega_2 = i\frac{\pi}{2}$ we obtain the hyperbolic case

$$\sigma(z) = \sinh(z) \exp\left(-\frac{z^2}{6}\right), \quad \zeta(z) = \coth(z) - \frac{z}{3}, \quad \varphi(z) = \frac{1}{\sinh^2(z)} + \frac{1}{3}$$

(56)

and

$$l(q, \lambda) = -\frac{\sinh(\lambda + q)}{\sinh(\lambda) \sinh(q)} \exp\left(-\frac{\lambda q}{3}\right).$$

(57)

All these formulas were collected in [11].

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