From 1 to 6: A Finer Analysis of Perturbed Branching Brownian Motion

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Abstract

The logarithmic correction for the order of the maximum for two-speed branching Brownian motion changes discontinuously when approaching slopes \( \sigma_1^2 = \sigma_2^2 = 1 \), which corresponds to standard branching Brownian motion. In this article we study this transition more closely by choosing \( \sigma_1^2 = 1 \pm t^{-\alpha} \) and \( \sigma_2^2 = 1 \mp t^{-\alpha} \). We show that the logarithmic correction for the order of the maximum now smoothly interpolates between the correction in the i.i.d. case \( \frac{1}{2} \sqrt{2} \ln(t) \), \( \frac{3}{2} \sqrt{2} \ln(t) \), and \( \frac{6}{2} \sqrt{2} \ln(t) \) when \( 0 < \alpha < \frac{1}{2} \). This is due to the localization of extremal particles at the time of speed change, which depends on \( \alpha \) and differs from the one in standard branching Brownian motion. We also establish in all cases the asymptotic law of the maximum and characterize the extremal process, which turns out to coincide essentially with that of standard branching Brownian motion. © 2020 The Authors. Communications on Pure and Applied Mathematics published by Wiley Periodicals LLC

1 Introduction

So-called log-correlated (Gaussian) processes have received considerable attention over the last few years; see, e.g., [2,4,8,9,27]. One of the reasons for this is that they represent processes where the correlations are on the borderline of becoming relevant for the properties of the extremes of the process. A paradigmatic example for such processes is branching Brownian motion (BBM) [1,33]. This process has been intensely investigated from the point of view of extreme value theory over the last 40 years; see, e.g., [2,5,7,10,15,17,19,30]. To understand what we mean by BBM being borderline, it is useful to consider BBM as a special case of a class of Gaussian processes labeled by a function \( A: [0, 1] \rightarrow [0, 1] \) with
$A(0) = 0$ and $A(1) = 1$ that is increasing and right-continuous. Given such a function, so-called variable speed branching Brownian motion \cite{[11][12][20][21][31]} can then be constructed in two equivalent ways\footnote{Actually, it can be constructed in three different ways: instead of making a time change in the Brownian motions, one can alternatively make the branching rates explicitly time dependent.}

Fix a time horizon $t$ and let

\begin{equation}
\Sigma^2_t(s) = tA(s/t), \quad s \in [0,t].
\end{equation}

Define Brownian motion with speed function $\Sigma^2_t$ as a time change of ordinary Brownian motion on $[0,t]$ as

\begin{equation}
B^\Sigma_s = B_{\Sigma^2_t(s)}.
\end{equation}

Branching Brownian motion with speed function $\Sigma^2_t$ is constructed like ordinary branching Brownian motion except that, if a particle splits at some time $s < t$, then the offspring particles perform variable speed Brownian motion with speed function $\Sigma^2_t$; i.e., their laws are independent copies of $\{B^\Sigma_t - B^\Sigma_s\}_{t \geq s}$, all starting at the position of the parent particle at time $s$. We assume here and throughout this paper that particles in BBM branch after an exponential time of parameter one with probability $p_k$ into $k$ independent copies of themselves where the branching law $p_k$ satisfies $\sum_{k=1}^{\infty} k p_k = 1$, $\sum_{k=1}^{\infty} k p_k = 2$, and $K = \sum_{k=1}^{\infty} k(k-1) p_k < \infty$. This ensures, in particular, that the process cannot die out. It also normalizes that number of particles at time $t$, $n(t)$ to satisfy $\mathbb{E}[n(t)] = \mathcal{C}$.

Alternatively, variable speed BBM can be constructed as a Gaussian process indexed by a continuous-time Galton-Watson tree with mean zero and covariances

\begin{equation}
\mathbb{E}[x_k(s)x_\ell(r)] = \Sigma^2_t(d(x_k(t), x_\ell(t)) \wedge s \wedge r),
\end{equation}

where the $x_k$ label the $n(t)$ particles present at time $t$ and $d(x_k(t), x_\ell(t))$ is the time of the most recent common ancestor of the particles labeled $k$ and $\ell$ in the Galton-Watson tree.

The authors’ interest in this model was actually sparked by the second construction, which exhibits the connection to the generalized random energy models (GREM) introduced by Derrida \cite{23} (see also \cite{20}). These models were introduced as toy models for spin glasses for which the structure of extreme values is important. The major goal here is to understand the dependence of the structure of extremes on the covariance function. An analysis of the order of the maximum was carried out in \cite{13,14}. Already in this work, the phase transition happening at the identity function (which is described in more detail below) is visible. This is a main motivation for the study of arbitrary covariance functions and in particular this work, as it sheds light on how this transition exactly happens on a microscopic level.
After this small detour, let us now connect the two definitions of branching Brownian motion. The case $A(x) = x$ corresponds to standard branching Brownian motion. The behavior of the extremes of these processes are dramatically different according to whether $A$ stays below $x$ or whether it crosses this line:

(i) if $A(x) < x$ for all $x \in (0, 1)$, then, to first sub-leading order,

$$\max_{k \leq n(t)} x_k(t) \approx \sqrt{2} t - \frac{1}{2\sqrt{2}} \ln t;$$

(ii) if $A(x) = x$, then Bramson [15, 16] has shown that

$$\max_{k \leq n(t)} x_k(t) \approx \sqrt{2} t - \frac{3}{2\sqrt{2}} \ln t;$$

(iii) if, for some $x \in (0, 1)$, $A(x) > x$, then, to leading order

$$\max_{k \leq n(t)} x_k(t) \approx \sqrt{2} t \int_0^1 \sqrt{\overline{A}(y)} \, dy,$$

where $\overline{A}$ denotes the concave hull of the function $A$. The sub-leading corrections depend on the details of the function $\overline{A}$. For instance, if $A$ is piecewise linear with slopes $\sigma_1^2$ and $\sigma_2^2$ (and necessarily $\sigma_1^2 > \sigma_2^2$ to be in this sub-case) on $[0, 1/2]$, respectively, $[1/2, 1]$, then the correction is given by (see, e.g., [21])

$$- \frac{3}{2\sqrt{2}} (\sigma_1 + \sigma_2) \ln t.$$

Note that, as a functional of the function $A$, the linear term in $t$ is continuous, but the coefficient multiplying $\ln t$ is discontinuous at the function $A(x) = x$. For instance, in the example above with two speeds, the limit of this coefficient is

$$\left\{ \begin{array}{ll}
\frac{1}{2\sqrt{2}} & \text{if } \sigma_1^2 \uparrow 1, \\
\frac{3}{2\sqrt{2}} & \text{if } \sigma_1^2 = 1, \\
\frac{6}{2\sqrt{2}} & \text{if } \sigma_1^2 \downarrow 1.
\end{array} \right.$$  

(1.8)

If different sequences of functions $A$ that converge to $A(x) = x$ from above are considered, a huge variety of limiting values can be produced.

Branching Brownian motion has strong connections to the F-KPP equation, which is a well-known reaction-diffusion equation admitting traveling wave solutions,

$$\partial_t u = \frac{1}{2} \partial_x^2 u + F(u).$$  

(1.9)
where $F$ depends on the branching law. This connection can be extended to variable speed branching Brownian motion, in which case one obtains the time-inhomogeneous F-KPP equation,

\[(1.10) \quad \partial_s u_t = \frac{1}{2} \sigma^2(s/t) \partial_x^2 u_t + F(u_t),\]

where $\sigma^2(s/t) = \partial_s \Sigma^2(t)$ (for all $s$ where $\Sigma(t)$ is differentiable). Note that (1.10) is really a family of PDEs indexed by $t \in \mathbb{R}_+$, and $u_t : [0,t] \times \mathbb{R} \to \mathbb{R}$. Equation (1.10) was studied in [34]. While in the standard F-KPP case the issue is to find a scale function $m(s)$ such that, for suitable initial conditions $u_{st}, x + m(s)$ converges to a traveling wave, in the time inhomogeneous case where are strictly speaking no traveling waves. However, one can still analyze the “front” position by defining $X(t) = \sup_{x} u_t(t, x) = 1/2$ and show that $u_t(t, x + X(t))$ converges to some limiting profile. By (1.9), this then still gives the law of the maximum and other functionals related to variable speed BBM.

Further properties, in particular the laws of the rescaled maxima and the extremal processes, are fully understood in the cases when $A(x) \leq x$ for all $x \in [0, 1]$ and in the case when $A$ is a piecewise linear function [11, 12].

In this paper we have a closer look at the apparent discontinuities that happen when $A$ crosses the identity line (see (1.8)). To do so, we consider functions $A = A_t$ that depend explicitly on the time horizon $t$. Kistler and Schmidt [28] have considered the case when $A_t$ is a step function with step sizes $t^{\alpha - 1}$ and step heights $t^{\alpha - 1}$ that converges to $A(x) = x$ from below. They showed that in this case, the logarithmic correction is given by $\frac{3-2\alpha}{2\sqrt{2}} \ln t$, which interpolates nicely between cases (i) and (ii).

Here we consider piecewise linear functions that lie slightly above or below $A(x) = x$. More precisely, we restrict ourselves to the simplest example, where

\[(1.11) \quad A_t(x) = \begin{cases} \sigma^2_1(t)x & \text{if } x < 1/2, \\ \sigma^2_2(t) / 2 + \sigma^2_2(t)(x - 1/2) & \text{if } x \geq 1/2, \end{cases}\]

with $\sigma^2_1(t) = 1 \pm t^{-\alpha}$ and $\sigma^2_2(t) = 1 \mp t^{-\alpha}$. Different cases can be treated using essentially the same techniques, if necessary in an iterative way.

In this case, we will show that

(i) If $\sigma^2_1(t) = 1 - t^{-\alpha}$, the leading term is $\sqrt{2}t$ for all $\alpha > 0$, and the logarithmic corrections are

\[(1.12) \quad - \begin{cases} \frac{1+4\alpha}{2\sqrt{2}} \ln t & \text{if } \alpha \in (0, 1/2], \\ \frac{3}{2\sqrt{2}} \ln t & \text{if } \alpha \in [1/2, \infty), \end{cases}\]
Figure 1.1. Localization: If the speeds are decreasing (left), then an extremal particle is $O(t^\alpha)$ below the maximum at the time of the speed change. Until this time it has to stay below the barrier $s = \sqrt{2}r$, $s > r$. In the case of increasing speeds (right), an extremal particle is $\sqrt{2}r^{1-\alpha}/4 = O(\sqrt{t})$ below the maximum at the time of the speed change. Until then it has again to stay below the barrier.

(ii) If $\sigma_1^2(t) = 1 + t^{-\alpha}$, the leading term is $\sqrt{2}\frac{\sigma_1 + \sigma_2}{2}t^\alpha$ and the logarithmic correction is

$$
(1.13) \quad - \begin{cases} 
\frac{3}{2\sqrt{2}}(\sigma_1 + \sigma_2)(1 - 2\alpha) \ln t \approx \frac{3}{2\sqrt{2}}(2 - 2\alpha) \ln t & \text{if } \alpha \in [0, 1/2), \\
\frac{3}{2\sqrt{2}}\sigma_1 \ln t \approx \frac{3}{2\sqrt{2}} \ln t & \text{if } \alpha \geq 1/2.
\end{cases}
$$

Interpreting this result in the context of the F-KPP equation, this hints at a continuity result for the speed of the front positions.

Localization

The key observation that will be needed to prove this and more detailed facts is a localization result on the position of the ancestors of extremal particles at a time $t/2$. It is known that in the case when $\sigma_1^2 = 1 + O(1)$ the ancestors of extremal particles at time $t$ are also extremal at time $t/2$, and so are just a logarithm of $t$ below $\sqrt{2}t\sigma_1$. For standard BBM, these particles are $O(\sqrt{t})$ below $\sqrt{2}t/2$. In the case $\sigma_1^2 = 1 - O(1)$, these particles are even further below, namely by $\sqrt{2}(\sigma_1 - \sigma_1^2)\ln t/2$ [11]. We will show (in Chapters 3 and 4, resp.), that the ancestors of extremal particles at time $t$ are below $\sqrt{2}\sigma_1 t/2$ by $O(t^\alpha)$ in the case $\sigma_1^2 = 1 + t^{-\alpha}$, and by $\sqrt{2}t^{1-\alpha}/4 + O(\sqrt{t})$ in the case $\sigma_1^2 = 1 - t^{-\alpha}$, when $\alpha \in (0, 1/2]$ (see Figure 1.1). Aficionados of BBM will readily infer (1.12) and (1.13) from this information. To actually prove this is, however, a bit more delicate. The basic strategy is similar to that used in the case of two-speed BBM with $\sigma_1^2 < 1$ in [11], but there are some interesting twists.

$^2$Note that $\sqrt{2}\frac{\sigma_1 + \sigma_2}{2}t \approx \sqrt{2}(t - t^{1-2\alpha}/2)$, which is already different from the BBM case if $\alpha \leq 1/2$. 
Apart from the analysis of the log-correction to the value of the maximum, we also analyze the law of the maximum and the nature of the extremal process in these cases. Of course, in both cases the law of the maximum converges to a randomly shifted Gumbel distribution. Less obviously, whenever $\alpha \in (0, 1/2)$, the random shift is always given by the derivative martingale (see (1.16) below). The extremal process has the same structure as in BBM, i.e., a decorated Cox process, where the decoration process is independent of $\alpha$.

In the remainder of this paper, when we consider the case $\sigma_1 > \sigma_2$, we always set $\sigma_1 = 1 + t^{-\alpha}, \sigma_2^2 = 1 - t^{-\alpha}$, and

$$m(t) = m_{\sigma_1}(t) = \sqrt{2} \frac{\sigma_1 + \sigma_2}{2} t - \frac{3}{2\sqrt{2}} (2 - 2\alpha) \ln t$$

In the case $\sigma_1 < \sigma_2$, we will set

$$m(t) = m_{\sigma_2}(t) = \sqrt{2} t - \frac{1 + 4\alpha}{2\sqrt{2}} \ln t,$$

In both cases, this is correct for $0 < \alpha \leq 1/2$. If $\alpha > 1/2$, all is exactly as in standard BBM.

We will denote particles of two-speed BBM with variances $\sigma_1^2$ on $[0, t/2]$ and $\sigma_2^2$ on $[t/2, t]$ by $x_k(s)$ and those of standard BBM by $x_k(s)$.

Before stating the main result of this paper, let us recall the two key martingales that were introduced by Lalley and Sellke (30), the derivative martingale $Z(t)$ and (what we like to call) the McKean martingale $Y_\sigma(t)$. They are defined in terms of a standard BBM $x(t)$ via

$$Z(t) = \sum_{k=1}^{n(t)} (\sqrt{2} t - x_k(t)) e^{\sqrt{2}(x_k(t) - \sqrt{2} t)},$$

and

$$Y_\sigma(t) = \sum_{k=1}^{n(t)} e^{\sqrt{2}\alpha x_k(t) - (1 + \alpha^2)t},$$

Lalley and Sellke have shown that $Z(t)$ converges, as $t \uparrow \infty$, to an a.s. positive random variable $Z$, while for $\sigma \geq 1$, $Y_\sigma(t)$ converges a.s. to 0. On the other hand, if $\sigma < 1$, then $Y_\sigma(t)$ is uniformly integrable and converges to a random variable $Y_\sigma$ (see (11)).

We can now state the main results of this paper.
THEOREM 1.1. Let \( \tilde{x}(t) \) be two-speed BBM with \( \sigma_{1}^{2}(t) = 1 + t^{-\alpha} \) and \( \sigma_{2}^{2} = 1 + t^{-\alpha} \). Then, if \( \alpha \in (0, 1/2) \), for all \( y \in \mathbb{R} \),

\[
\lim_{t \uparrow \infty} \mathbb{P} \left( \max_{k \leq n(t)} \tilde{x}_{k}(t) - m_{\alpha}^{\pm}(t) \leq y \right) = \begin{cases} 
\mathbb{E}_{Z} \left[ e^{-\frac{2CZe^{-\sqrt{2}y}}{\sqrt{\pi}}} \right] & \text{in the } + \text{ case,} \\
\mathbb{E}_{Z} \left[ e^{-CZe^{-\sqrt{2}y}} \right] & \text{in the } - \text{ case,}
\end{cases}
\]

where \( Z \) is the limit of the derivative martingale (cf. (1.16)) and \( C \) is the positive constant

\[
C = \lim_{r \uparrow \infty} \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} u(r, y + \sqrt{2}r) e^{\sqrt{2}y} dy,
\]

where \( u \) is the solution of the F-KPP equation with Heaviside initial conditions.

Similarly, we get the convergence of Laplace functionals, that then imply the convergence of the extremal process.

THEOREM 1.2. Under the same hypotheses as in Theorem 1.1 for any bounded nonnegative function \( \phi \) with compact support, for all \( y \in \mathbb{R} \),

\[
\lim_{t \uparrow \infty} \mathbb{E} \left[ e^{-\sum_{k=1}^{n(t)} \phi(\tilde{x}_{k}(t) - m_{\alpha}^{\pm}(t) - y)} \right] = \begin{cases} 
\mathbb{E}_{Z} \left[ e^{-\frac{2C\phi(Z)e^{-\sqrt{2}y}}{\sqrt{\pi}}} \right] & \text{in the } + \text{ case,} \\
\mathbb{E}_{Z} \left[ e^{-C\phi(Z)e^{-\sqrt{2}y}} \right] & \text{in the } - \text{ case,}
\end{cases}
\]

where \( Z \) is the limit of the derivative martingale and \( C(\phi) \) is the positive constant

\[
C(\phi) = \lim_{r \uparrow \infty} \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} u(r, y + \sqrt{2}r) e^{\sqrt{2}y} dy,
\]

where \( u \) is the solution of the F-KPP equation with initial condition \( u(y, 0) = \exp(-\phi(-y)) \).

Remark 1.3. Theorem 1.2 implies that the extremal process is, up to a constant shift, always the same as that of standard BBM (see [7]), if \( \alpha > 0 \).

Outline of the Paper

The remainder of this paper is organized as follows. In Section 2 we recall some facts on the tail behavior of solutions of the F-KPP equation that form the crucial input in the analysis. Sections 3 and 4 contain the proof of Theorem 1.1. We deal separately with the cases \( \sigma_{1} > \frac{1}{2} \) and \( \sigma_{1} < \frac{1}{2} \). The structure of the proof is the same in both cases, but the details of the calculations are different and it appears easier to follow the arguments in each case rather then to jump back and forth.

The way both chapters are organized is as follows. First, we show where the extremal particles are localized at the change time \( t/2 \). Then we exploit the branching
Figure 1.2. Phase diagram of two-speed BBM. In the inner phase ($\alpha > \frac{1}{2}$), everything is as in standard BBM. In the northwest regime, the order of the maximum and the extremal process are a concatenation of two such processes for standard BBM. In the regime in between ($0 < \alpha < \frac{1}{2}$), the order of the maximum interpolates smoothly between the surrounding regimes. In the southeast regime, the order of the maximum coincides with the one in the i.i.d. case. The extremal process is similar to the one for BBM but the martingale appearing is different. In the regime, with $\sigma_1^2 = 1 - t^{-\alpha}$, $0 < \alpha < \frac{1}{2}$, the order of the maximum interpolates smoothly between the i.i.d. and the BBM order of the maximum. Observe that in the three middle regimes the extremal process coincides up to constant shift with the one of standard BBM and the martingale is always the derivative martingale.

Property at time $t/2$ to set up a recursion where the tail asymptotics of the law of the maximum of the BBMs after time $t/2$ are used. This results in a formula that is already somewhat reminiscent of the Lalley-Sellke representation [30] of the limiting distribution of the maximum of BBM. However, to prove convergence, we need to exhibit more independence by splitting paths at time $t^\beta$ for some suitable small $\beta$. This results in an expression that in all cases involves a slight modification of the derivative martingale that we then show to converge towards the limit of the usual derivative martingale.

In Section 5, we prove convergence of the the Laplace functionals and hence the extremal process. This is essentially identical to the proof of the law of the maximum and requires just a slight extension of the results on the asymptotics of
solutions of the F-KPP equation to the case of weakly \( t \)-dependent initial conditions.

2 Preliminaries about BBM

In this section we collect some known results about standard branching Brownian motion. A fundamental property of BBM is its relation to the Fisher-Kolmogorov-Petrovsky-Piscounov (F-KPP) equation \([22, 29]\) that was established by Ikeda, Nagasawa, and Watanabe \([24, 26]\) and McKean \([32]\). Namely, if we set, for some function \( f : [0, 1] \to [0, 1] \),

\[
  v(t, x) = \mathbb{E} \left[ \prod_{k=1}^{n(t)} f(x - x_k(t)) \right],
\]

then \( u(t, x) \equiv 1 - v(t, x) \) is the solution of the F-KPP equation \((1.9)\) with initial condition \( u(0, x) = 1 - f(x) \), and

\[
  F(u) = (1 - u) - \sum_{k=1}^{\infty} p_k (1 - u)^k.
\]

The following proposition is based on the deep analysis of the behavior of solutions to the F-KPP equation presented in Bramson’s monograph \([16]\).

**Proposition 2.1.** Let \( u \) be a solution to the F-KPP equation with initial data satisfying

(i) \( 0 \leq u(0, x) \leq 1 \);

(ii) \( \exists h > 0, \limsup_{t \to \infty} \frac{1}{t} \ln \int_0^t f^{(1+h)}(y) \, dy \leq -\sqrt{2} \);

(iii) \( \exists v > 0, M > 0, N > 0, \) it holds that \( \int_x^{x+N} u(0, y) \, dy > v, \forall x \leq -M \);

(iv) moreover, \( \int_0^{\infty} u(0, y) y e^{2y} \, dy < \infty \).

Then we have, for \( 0 < x = x(t) \) such that \( \lim_{t \to \infty} x(t)/t = 0 \),

\[
  \lim_{t \to \infty} e^{\sqrt{2}x} e^{x^2/2t} x^{-1} u \left( t, x + \sqrt{2} t - \frac{3}{2\sqrt{2}} \ln t \right) = C,
\]

where \( C \) is a strictly positive constant that depends only on the initial condition \( u(0, \cdot) \). More precisely,

\[
  C = \lim_{r \to \infty} \sqrt{\frac{2}{\pi}} \int_0^\infty u(r, y + \sqrt{2}r) e^{\sqrt{2}y} y \, dy.
\]

**Proof.** The proof of this proposition is a direct adaptation of the proofs of the corresponding propositions in \([7, 11]\) for the cases \( x \sim \sqrt{t} \) and \( x \sim t \).

**Remark 2.2.** Choosing for \( f \) the Heaviside function, this proposition implies in particular that, for \( x > 0 \),

\[
  \lim_{t \to \infty} e^{\sqrt{2}x} e^{x^2/2t} x^{-1} \mathbb{P} \left( \max_{k \leq n(t)} x_k(t) > x + \sqrt{2} t - \frac{3}{2\sqrt{2}} \ln t \right) = C.
\]
The rougher bound below follows by using the many-to-one lemma and standard Gaussian asymptotics:

**Lemma 2.3.** For any $x \in \mathbb{R}_+$,

\[
\mathbb{P} \left( \max_{k \leq n(t)} x_k(t) > x + \sqrt{2t} \right) \leq \frac{e^{-\sqrt{2}x - \frac{x^2}{2t}}}{\sqrt{2\pi} (\sqrt{2t} + x/\sqrt{t})}.
\]

### 3 The Law of the Maximum: The Case $\sigma_1^2 = 1 + \alpha$

The aim of this section is to prove Theorem 1.1 in the case when $\sigma_1^2 = 1 + \alpha$, i.e., to show that

\[
\lim_{t \to \infty} \mathbb{P} \left[ \max_{1 \leq k \leq n(t)} \tilde{X}_k(t) - \tilde{m}_\alpha(t) \leq y \right] = \mathbb{E} \left[ \exp \left( -\frac{\tilde{C}Z}{\sqrt{2}t} \frac{e^{-\sqrt{2}y}}{\sqrt{2\pi}} \right) \right],
\]

where we set $\tilde{C} = 2^{3/2}/C$. In this section we will always write $m(t) = m^+_{\alpha}(t)$. $Z$ is the limit of the derivative martingale and $C$ is the

#### 3.1 Localization of Paths

To prove (3.1), we need to control the position of particles until time $t/2$. To this end, we define three sets on the space of paths $X: \mathbb{R}_+ \to \mathbb{R}$. The first controls the position at time $s$. The second ensures that the path of the particle does not exceed a certain value, and the third controls the positions of particles at time $t/2$.

\[
\mathcal{G}_{s,A,B,y} = \{ X \mid X(s) - \sqrt{2s} \in [-A, y] \},
\]

\[
\mathcal{G}_{s_1,s_2} = \{ X \mid \forall s_1 \leq q \leq s_2 X(q) \leq 2y \},
\]

\[
\mathcal{H}_\beta = \{ X \mid X(t/2) \leq \sqrt{2\beta} \}.
\]

In the case of standard BBM, it was shown in Bramson [15] (see also the detailed analysis in [5]) that the positions of particles that are near the maximum at time $t$ are at time $t/2$ in a window of order $\sqrt{t}$ below $\sqrt{2t}/2$. In the case of 2-speed BBM with $\sigma_1 < \sigma_2$, it was shown in [11] that the corresponding window is of width $\sqrt{t}$ around $\sqrt{2\sigma_1^2}t/2$, which is a linear order in $t$ below the level of the maximal particles at time $t/2$ (which is near $\sqrt{2\sigma_1^2}t/2$). If $\sigma_1 > 1$, then extremal particles descend from the actual extremal particles at time $t/2$. So we expect that in our case, we see a transition from $\sqrt{t}$ to $O(1)$ as we vary $\alpha$.

**Proposition 3.1.** Let $\sigma_1^2 = 1 + \alpha$, $\sigma_2^2 = 1 - \alpha$. For any $\epsilon > 0$, there is $r_0 < \infty$ such that for all $r > r_0$ and for all $t$ large enough,

\[
\mathbb{P} \left[ \exists j \leq n(t) : \sigma_1^{-1} X_j \notin \mathcal{G}_{r,t/2} \right] \leq \epsilon.
\]

**Proof.** The event considered depends only on standard BBM up to time $t/2$. The well-known estimate for standard BBM follows from Bramson’s results in [15], see also [5].

□
The next proposition states that extremal particles stay by $t^\alpha$ below $\sqrt{2}t/2$ at time $t/2$ when the speed change happens.

**Proposition 3.2.** Let $\sigma_1^2 = 1 + t^{-\alpha}, \sigma_2^2 = 1 - t^{-\alpha}$. For any $d \in \mathbb{R}$ and any $\epsilon > 0$, there exist constants $A > B > 0$ such that, for all $t$ large enough,

$$\mathbb{P}[\exists j \leq n(t) : \{\mathcal{F}_j(t) > m(t) - d\} \land \{\sigma_1^{-1} \mathcal{F}_j \notin \mathcal{G}_{t/2,A,B,\alpha}\}] \leq \epsilon.$$

**Proof.** Abbreviate $I = [\sqrt{2}t/2 - At^\alpha, \sqrt{2}t/2 - Bt^\alpha]$. The probability in question can be written in the form

$$\mathbb{P}(\exists k \leq n(t/2) : \{\sigma_1 \max_{\ell \leq n^k(t/2)} x_\ell^k(t/2) > m(t) - \sigma_2 \max_{\ell \leq n^k(t/2)} x_\ell^k(t/2) - d\}$$

$$\land \{x_k(t/2) \notin I\} \land \{x_k(s) \leq \sqrt{2}s, \forall s \in [t/2]\}.$$  

(3.5)

By the many-to-one lemma, this is bounded from above by

$$e^{t/2} \mathbb{E} \left[ 1_{X_1(t/2) \notin I} 1_{\frac{1}{2} \leq X_k(s) \leq \sqrt{2}s, \forall s \in [t/2]} \right]$$

$$\times 1_{\max_{\ell \leq n^k(t/2)} \sigma_2 x_\ell^k(t/2) > m(t) - \sigma_1 x_k(t/2) - d}$$

$$= e^{t/2} \int e^{-\frac{c^2}{2t}} \mathbb{P} \left( \mathcal{F}_{t/2}^{\text{c}} \left( \max_{\ell \leq n^k(t/2)} x_\ell^k(t/2) > m(t) - \sigma_1 x_k(t/2) - d \right) \right)$$

$$\times \mathbb{P} \left( \sigma_2 \max_{\ell \leq n^k(t/2)} x_\ell^k(t/2) > m(t) - \sigma_1 x_k(t/2) - d \right) \frac{dz}{d\mu_0}.$$  

(3.7)

where $\mathcal{F}_{t/2}^{\text{c}}$ denotes the Brownian bridge from 0 to $y$ in time $t/2$ and we wrote $I^c$ short for $I \cap (-\infty, \sqrt{2}t/2]$. The probability regarding the Brownian bridge satisfies

$$\mathbb{P} \left( \mathcal{F}_{t/2}^{\text{c}} \left( \frac{1}{2} \leq X_0, \sqrt{2}t/2 - z \right) \leq 0, \forall s \in [t/2] \right) \leq \sqrt{\frac{2}{\pi}} \frac{\sqrt{\mathcal{F}_{t/2}^{\text{c}}(\sqrt{2}t/2 - z)}}{t/2},$$

as long as $\sqrt{2}t/2 - z \ll \sqrt{t}$, and is bounded by 1 otherwise. We now split the integral into the parts where $z$ is above $\sqrt{2}t/2 - Bt^\alpha$ and where it is below $\sqrt{2}t/2 - At^\alpha$. The first part gives, with a change of variables,

$$e^{t/2} \int_0^{Bt^\alpha} e^{-\frac{c^2}{2t}} \mathbb{P} \left( \mathcal{F}_{t/2}^{\text{c}} \left( \frac{1}{2} \leq X_0, \sqrt{2}t/2 - z \right) \leq 0, \forall s \in [t/2] \right)$$

$$\times \mathbb{P} \left( \max_{\ell \leq n^k(t/2)} x_\ell^k(t/2) > m(t) - \sigma_1 \sqrt{2}t/2 + \sigma_1 y - d \right) dy.$$  

(3.9)
Recall that 
\[ m(t) - \sigma_1 \sqrt{2t/2} = \sigma_2 \sqrt{2t/2} - \frac{3}{2\sqrt{2}} (\sigma_1 + \sigma_2 (1 - 2\alpha)) \ln t. \] Hence, the probability involving the maximum in (3.9) reads

\[
\mathbb{P} \left( \max_{\ell \leq m^k(t/2)} \chi^k(t/2) > \sqrt{2}t/2 + \frac{3}{2\sqrt{2}} \left( \frac{\sigma_1}{\sigma_2} + (1 - 2\alpha) \right) \ln t + \frac{\sigma_1}{\sigma_2} \frac{y}{d} - d/\sigma_2 \right).
\]

(3.10)

Using Proposition 2.1, respectively (2.5), we see that this probability equals, asymptotically, as 
\[ t \uparrow \infty, \] for 
\[ y > \frac{3}{2\sqrt{2}} (1 - 2\alpha) \ln t + \frac{\sigma_1}{\sigma_2} \frac{y}{d} - d/\sigma_2, \]

(3.11)

For 
\[ y \leq \frac{3}{2\sqrt{2}} (1 - 2\alpha) \ln t + \frac{\sigma_1}{\sigma_2} \frac{y}{d} - d/\sigma_2, \] we simply bound the probability by 1. Inserting this and the bound (3.8) into (3.9), we see that this term is not larger than

\[
\int_0^B e^{-\frac{y}{\sqrt{2}} \sqrt{\frac{2}{\pi t}}} \left( \frac{\sigma_1}{\sigma_2} \frac{y}{d} - \frac{3}{2\sqrt{2}} (1 - 2\alpha) \ln t + \frac{\sigma_1}{\sigma_2} \frac{y}{d} - d/\sigma_2 \right) dy
\]

(3.12)

which is finite and tends to zero as 
\[ B \downarrow 0. \] Finally, for the remaining part of the integral in (3.9), we bound by

\[
\int_0^{\frac{3}{2\sqrt{2}} (1 - 2\alpha) \ln t + \frac{\sigma_1}{\sigma_2} \frac{y}{d} - d/\sigma_2} e^{y \sqrt{\frac{2}{\pi t}}} \sqrt{\frac{2}{\pi t}} C y^2 e^{\sqrt{\frac{2}{\pi t}} dy} \leq C (\ln t)^2 r t^{-3\alpha}.
\]

(3.13)
which tends to zero as \( t \uparrow \infty \). The part of the integral in (3.7) involving the terms below \( \sqrt{2}t/2 - At^\alpha \) can be written as

\[
(3.14) \quad e^{t/2} \int_0^\infty \frac{e^{-(\frac{\sqrt{2}}{2}t)} \, y}{\sqrt{\pi t}} \frac{\, dy}{\sqrt{2\pi}} \left( I_{\frac{1}{2}}(t) \, \mathbb{P} \left( \frac{t}{2} \leq 0, \forall s \in (t/2] \right) \right) \times \mathbb{P} \left( \sigma_2 \max_{\ell \leq n^{k}(t/2)} x_\ell(t/2) > m(t) - \sigma_1 \sqrt{2}t/2 + \sigma_1 y - d \right). \]

We have to distinguish the cases where \( z \leq K\sqrt{r} \) and the rest. In the former, we can proceed as in the case above and we get, up to vanishing terms, for any \( K > 0 \),

\[
(3.15) \quad \int_A^{Kt^{1/2-\epsilon}} \frac{\, dy}{\sqrt{\pi t}} \frac{\, C_y \, e^{\sqrt{2}d/\sigma_2}}{\sqrt{2\pi}} \int_0^\infty \frac{\, e^{-y\sqrt{t}} \, \sqrt{\pi} \frac{\, dy}{\sqrt{2\pi}} \frac{\, C_y \, e^{\sqrt{2}d/\sigma_2}}{\sqrt{2\pi}} \, dy}{\sqrt{2\pi}} \, dy \rightarrow \int_A^\infty \frac{\, e^{-y\sqrt{t}} \, \sqrt{\pi} \frac{\, dy}{\sqrt{2\pi}} \frac{\, C_y \, e^{\sqrt{2}d/\sigma_2}}{\sqrt{2\pi}} \, dy}{\sqrt{2\pi}},
\]

which in turn converges to 0 as \( A \uparrow \infty \). For the remaining term, it is enough to bound the probability involving the Brownian bridge by one and to use the bound (2.6). One then gets a bound

\[
(3.16) \quad \int_{t^\epsilon}^\infty \frac{\, e^{-y\sqrt{2t}} \, \sqrt{\pi} \frac{\, dy}{\sqrt{2\pi}} \frac{\, C_y \, e^{\sqrt{2}d/\sigma_2}}{\sqrt{2\pi}} \, dy}{\sqrt{2\pi}} \leq \int_{Kt^{\epsilon/2-\epsilon}}^\infty \frac{\, e^{-y\sqrt{2t}} \, \sqrt{\pi} \frac{\, dy}{\sqrt{2\pi}} \frac{\, C_y \, e^{\sqrt{2}d/\sigma_2}}{\sqrt{2\pi}} \, dy}{\sqrt{2\pi}},
\]

which tends to zero rapidly as \( t \uparrow \infty \). This concludes the proof. \( \square \)

The next proposition states that \( \mathcal{H}_\delta \) holds for all extremal particles for \( 0 < \delta < 1/2 \). This is a weaker form of the localization results shown in [5].

**Proposition 3.3.** Let \( \sigma_1^2 = 1 + t^{-\alpha}, \sigma_2^2 = 1 - t^{-\alpha} \). For any \( d \in \mathbb{R} \) and any \( \epsilon > 0 \), there exists \( 0 < \delta < 1/2 \) such that, for all \( t \) large enough,

\[
(3.17) \quad \mathbb{P} \left[ \exists j \leq n(t) : \{ \tilde{x}_j(t) > m(t) - d \} \land \{ \sigma_1^{-1}x_j \notin \mathcal{H}_\delta \} \right] \leq \epsilon.
\]

**Proof.** To prove this proposition, we may use Proposition 3.2 and the fact that any path starting at zero, ending at some \( \sqrt{2}t/2 - z \) with \( z \in [At^\alpha, Bt^\alpha] \), and staying below the line \( \sqrt{2}s \) will not be above \( \sqrt{2}t^\beta - t^\beta \delta \) at time \( t^\beta \) with high probability.

To do so, we decompose a bridge in time \( t/2 \) from 0 to \( z \) into two pieces, one from 0 to \( \sqrt{2}t^\beta - y \) in time \( t^\beta \) and one from \( \sqrt{2}t^\beta - y \) to \( \sqrt{2}t/2 - z \) in time \( t^* = t/2 - t^\beta \). Then the probability that the first bridge stays below \( \sqrt{2}s \) is, to leading order in \( t \), given by

\[
(3.18) \quad \frac{\sqrt{\frac{2}{\pi} t^\beta \, y}}{t^\beta - r},
\]

while the probability for the second bridge is \( 2yz/t^* \). These estimates follow from lemma 2.2 in [16]. Thus the probability that the bridge is above \( \sqrt{2}t^\beta - t^\beta \delta \) is
given by

\[
\int_0^{t^\delta} e^{-\frac{y^2}{2 \pi t^\delta}} \sqrt{\frac{\pi}{t^\delta - \gamma}} dy = \int_0^{t^{\delta(\delta-1/2)/2}} e^{-\frac{y^2}{2 \pi \gamma}} y^2 dy \leq t^{3\delta(\delta-1/2)}.
\]

The right-hand side tends to zero for any \( \delta < 1/2 \), which implies the assertion of the proposition.

The following simple lemma shows that if a condition holds for all paths that exceed some level, then this condition can also be imposed on the paths when computing the probability that the maximum stays below that level.

\begin{lemma}
Let \( x_k, k = 1, \ldots, n \) be path-valued random variables and \( \mathcal{G} \) be any event such that, for some \( \epsilon > 0 \),
\[
\mathbb{P} \left( \exists_{k \leq n} : \{x_k(t) > y\} \land \{x_k \in \mathcal{G}\} \right) \geq \mathbb{P} \left( \exists_{k \leq n} x_k(t) > y \right) - \epsilon.
\]
Then
\[
\left| \mathbb{P} \left( \max_{k \leq n} x_k(t) \leq y \right) - \mathbb{P} \left( \max_{k \leq n : x_k(t) \in \mathcal{G}} x_k(t) \leq y \right) \right| \leq \epsilon.
\]
\end{lemma}

\begin{proof}
Obviously,
\[
\mathbb{P} \left( \max_{k \leq n} x_k(t) \leq y \right) \leq \mathbb{P} \left( \max_{k \leq n : x_k(t) \in \mathcal{G}} x_k(t) \leq y \right)
\]
\[
= 1 - \mathbb{P} \left( \exists_{k \leq n} : \{x_k(t) > y\} \land \{x_k \in \mathcal{G}\} \right)
\]
\[
\leq 1 - \mathbb{P} \left( \exists_{k \leq n} : x_k(t) > y \right) + \epsilon
\]
\[
= \mathbb{P} \left( \max_{k \leq n} x_k(t) \leq y \right) + \epsilon,
\]
which proves the lemma.
\end{proof}

\subsection{3.2 Recursive Structure}

We want to control
\[
\mathbb{P} \left[ \max_{1 \leq k \leq n(t)} x_k(t) - m(t) > y \right]
\]
\[
= \mathbb{P} \left[ \max_{k \leq n(t) \land \ell \leq n^{k} (t/2)} \sigma_1 x_k(t/2) + \sigma_2 x_\ell(t/2) - m(t) > y \right],
\]
where, for each \( k \), \( (x^k_\ell(\cdot))_{\ell \leq n^k (t/2)} \) are particles of an independent standard branching Brownian motion. First, we introduce several localization conditions in (3.23).

For this we need to define shifted versions of the event \( \mathcal{G} \) and \( \mathcal{T} \) as
\[
\mathcal{G}_{s,A,B,T,s,T,y} = \{X | X(s) - \sqrt{2s} + S \in [-A(s + T)^y, -B(s + T)^y]\},
\]
\[
\mathcal{T}_{s_1,s_2,S} = \{X | \forall s_1 \leq s \leq s_2; X(s) + S \leq \sqrt{2s}\}.
\]
By Proposition 3.2 we have that

\[
P_h \max_{k \leq n(t/2), \ell \leq n^k\ell(t/2)} \sigma_1 x_k(t/2) + \sigma_2 x_\ell^k(t/2) - m(t) > y
\]

(3.26)

\[
\leq P \left[ \exists k \leq n(t/2), \ell \leq n^k\ell(t/2) : \{ \sigma_1 x_k(t/2) + \sigma_2 x_\ell^k(t/2) - m(t) > y \} \wedge \{ x_k \in \mathcal{G}_{t/2, A, B, \alpha} \} \right] + \epsilon,
\]

for \( B \) sufficiently close to zero and \( A \) large enough. The probability on the right-hand side of (3.26) is also a lower bound for (3.23). Proceeding similarly, the probability in (3.26) is equal (up to error terms of size \( \epsilon \)) to

\[
P_h \max_{k \leq n(t/2), \ell \leq n^k\ell(t/2)} \sigma_1 x_k(t/2) + \sigma_2 x_\ell^k(t/2) - m(t) > y
\]

(3.27)

\[
\wedge \{ x_k \in \mathcal{G}_{t/2, A, B, \alpha} \cap \mathcal{H}_\delta \}
\]

for any \( \delta > 1/2 \). Moreover, we can also introduce a condition on the path between time \( t \) and time \( t = 2 \) and get that (3.27) is equal to (again up to an error of order \( \epsilon \))

\[
P_h \max_{k \leq n(t/2), \ell \leq n^k\ell(t/2)} \sigma_1 x_k(t/2) + \sigma_2 x_\ell^k(t/2) - m(t) > y
\]

(3.28)

\[
\wedge \{ x_k \in \mathcal{G}_{t/2, A, B, \alpha} \cap \mathcal{H}_\delta \cap \mathcal{J}_{t/2}\}
\]

Set

\[
\mathcal{L}_{t, A, B} = \mathcal{G}_{t/2, A, B, \alpha} \cap \mathcal{H}_\delta \cap \mathcal{J}_{t/2}.
\]

In view of Lemma 3.4, we only need to analyze

\[
P_h \max_{k \leq n(t/2), \ell \leq n^k\ell(t/2)} \sigma_1 x_k(t/2) + \sigma_2 x_\ell^k(t/2) - m(t) \leq y
\]

(3.29)

in order to prove Theorem 1.1. Using the branching property, we can rewrite (3.30) as

\[
\mathbb{E} \left[ \prod_{k \leq n(t/2), x_k \in \mathcal{L}_{t/2, A, B}} \mathbb{P} \left[ \max_{l \leq n^k l(t/2)} x_l^k(t/2) \leq \frac{m(t) + y - \sigma_1 x_k(t/2)}{\sigma_2} \bigg| \mathcal{F}_{t/2} \right] \right]
\]

(3.31)

\[
= \mathbb{E} \left[ \prod_{k \leq n(t/2), x_k \in \mathcal{L}_{t/2, A, B}} \left( 1 - \mathbb{P} \left[ \max_{l \leq n^k l(t/2)} x_l^k(t/2) > \frac{m(t) + y - \sigma_1 x_k(t/2)}{\sigma_2} \bigg| \mathcal{F}_{t/2} \right] \right) \right].
\]

where \( \mathcal{F}_s \) with \( s \leq t/2 \) denotes the \( \sigma \)-algebra generated by \( (x(u))_{u \leq s} \).
As $x_k \in \mathcal{G}_{t',A,B,\alpha}$, we can use the tail asymptotics given in Proposition 2.1 to control the conditional probability in (3.31). Namely,

$\mathbb{P} \left[ \max_{t \leq n(t/2)} x_k^k(t/2) > \frac{m(t) + y - \sigma_1 x_k(t/2)}{\sigma_2} \left| \mathcal{F}_{t/2} \right. \right] = \tilde{C} \Gamma_k(t)e^{-\sqrt{2} \Gamma_k(t)}(1 + o(1)),$

where the $o(1)$ error term is uniform in the range of possible values for $x_k(t/2)$ since $x_k \in \mathcal{G}_{t',A,B,\alpha}$ and

$\Gamma_k(t) = \frac{m(t) + y - \sigma_1 x_k(t/2)}{\sigma_2} - \left( \sqrt{\frac{t}{2}} - \frac{3}{2\sqrt{2}} \ln(t) \right)$

$= \frac{\sigma_1}{\sigma_2} \left( \sqrt{\frac{t}{2}} - x_k(t/2) \right) - \frac{3}{2\sqrt{2}}(1 - 2\alpha) \ln(t) + y/\sigma_2.$

Plugging (3.32) back into (3.31) we obtain that the expectation in (3.31) is equal to

$\mathbb{E} \left[ \prod_{k \leq n(t/2), x_k \in \mathcal{G}_{t',t/2,A,B}} \left( 1 - \tilde{C} \Gamma_k(t)e^{-\sqrt{2} \Gamma_k(t)} \right) \right] (1 + o(1))$

$= \mathbb{E} \left[ \prod_{k \leq n(t/2), x_k \in \mathcal{G}_{t',t/2,A,B}} \exp \left( -\tilde{C} \Gamma_k(t)e^{-\sqrt{2} \Gamma_k(t)} \right) \right] (1 + o(1)),$

since $\Gamma_k(t) > At^\alpha$ if $x_k \in \mathcal{G}_{t',A,B,\alpha}$. Next, we rewrite the expectation in (3.34) by conditioning on $\mathcal{F}_{t,\beta}$ as

$\mathbb{E} \left[ \prod_{k \leq n(t/2), x_k \in \mathcal{G}_{t,\beta}} \mathbb{E} \left[ \prod_{j \leq n(t/2-\beta), x_j \in \mathcal{G}_{t,\beta}} \exp \left( -\tilde{C} \Delta_k(t)e^{-\sqrt{2} \Delta_k(t)} \right) \left| \mathcal{F}_{t,\beta} \right. \right] \right]$

with $\mathcal{G}_{t',t/2-\beta,A,B,x_k(\beta) - \sqrt{2} \beta \Gamma_k(t,\beta,\alpha)} \cap \mathcal{T}_{t/2-\beta,A,B,x_k(\beta) - \sqrt{2} \beta \Gamma_k(t,\beta,\alpha)}$ as defined in (3.24), and

$\Delta_k(t) = \frac{\sigma_1}{\sigma_2} \left( \sqrt{\frac{t}{2}} - x_k(t/2) \right) - x_k^k(t/2 - t^{\beta})$

$- \frac{3}{2\sqrt{2}}(1 - 2\alpha) \ln(t) + y/\sigma_2.$

where, for each $k$, $(x_j^k(\cdot))_{t \leq n(t/2-\beta)}$ are particles of an independent standard branching Brownian motion. We set

$\mathcal{G}_{t',t/2-\beta,A,B,x_k(\beta) - \sqrt{2} \beta \Gamma_k(t,\beta,\alpha)} \cap \mathcal{T}_{t/2-\beta,A,B,x_k(\beta) - \sqrt{2} \beta \Gamma_k(t,\beta,\alpha)}$
We rewrite the inner expectation in (3.34) as
\begin{equation}
\mathbb{E} \left[ \exp \left( - \sum_{j \leq n^x \left( t/2 - t^\beta \right), \ x_j \in \mathbb{F}_{t^\beta \cdot t/2 - t^\beta, x_k \cdot t^\beta}^{\mathcal{F}_t} \left( \sqrt{2} t^\beta - x_j \left( t^\beta \right) \right) e^{-\sqrt{2} t^\beta - x_k \left( t^\beta \right)} \right) \right]_{\mathcal{F}_{t, t^\beta}} \end{equation}

(3.37)

\[= \mathbb{E} \left[ \exp \left( - \frac{\sigma_1}{\sigma_2} C \left( \sqrt{2} t^\beta - x_k \left( t^\beta \right) \right) e^{-\sqrt{2} t^\beta - x_k \left( t^\beta \right)} \right) \right]_{\mathcal{F}_{t, t^\beta}} \]

where
\begin{equation}
\mathcal{B}_k(t) = \sum_{j \leq n^x \left( t/2 - t^\beta \right), \ x_j \in \mathbb{F}_{t^\beta \cdot t/2 - t^\beta, x_k \cdot t^\beta}} e^{-\sqrt{2} \left( \frac{\alpha_1}{\alpha_2} \left( \sqrt{2} t^\beta - x_j \left( t^\beta \right) \right) - \frac{3}{2} \sqrt{2} \left( 1 - 2 \alpha \right) \ln(t) + y/\sigma_2 \right)} \end{equation}

(3.38)

and
\begin{equation}
\mathcal{Z}_k(t) = \sum_{j \leq n^x \left( t/2 - t^\beta \right), \ x_j \in \mathbb{F}_{t^\beta \cdot t/2 - t^\beta, x_k \cdot t^\beta}} \left( \frac{\alpha_1}{\alpha_2} \left( \sqrt{2} t^\beta - x_j \left( t^\beta \right) \right) - \frac{3}{2} \sqrt{2} \left( 1 - 2 \alpha \right) \ln(t) + y/\sigma_2 \right) \end{equation}

(3.39)

Next, we want upper and lower bounds on the expression in (3.37). To this end we use the basic inequality
\begin{equation}
1 - x \leq e^{-x} \leq 1 - x + \frac{1}{2} x^2, \quad x > 0,
\end{equation}

(3.40)

for
\begin{equation}
x = \frac{\alpha_1}{\alpha_2} C \left( \sqrt{2} t^\beta - x_k \left( t^\beta \right) \right) e^{-\sqrt{2} t^\beta - x_k \left( t^\beta \right)} \mathcal{B}_k(t)
\end{equation}

(3.41)

As the term $e^{-x}$ appears in the conditional expectation with respect to $\mathcal{F}_{t^\beta}$, we need to compute $\mathbb{E}(\mathcal{B}_k(t) | \mathcal{F}_{t^\beta})$, $\mathbb{E}(\mathcal{Z}_k(t) | \mathcal{F}_{t^\beta})$, $\mathbb{E}(\mathcal{B}_k(t)^2 | \mathcal{F}_{t^\beta})$, and $\mathbb{E}(\mathcal{Z}_k(t) \mathcal{B}_k(t) | \mathcal{F}_{t^\beta})$.

### 3.3 Computation of the Main Term

We begin with the computation of the averages of the McKeans and the derivative martingale terms.
Lemma 3.5. With the notation from the last subsection, the McKean martingale term is

\[ \mathbb{E}[\mathcal{Y}_k(t) | \mathcal{F}_t] = 2^{3/2} (\sqrt{2t} \beta - x_k(t^\beta)) \mu^{-\alpha} \frac{e^{-\sqrt{2y}}}{\sqrt{2\pi}} (1 + o(1)), \]

and the derivative martingale term is

\[ \mathbb{E}[\mathcal{Z}_k(t) | \mathcal{F}_t] = (\sqrt{2t} \beta - x_k(t^\beta)) \frac{e^{-\sqrt{2y}}}{\sqrt{2\pi}} (1 + o(1)), \]

where \( o(1) \) tends to 0 as first \( t \uparrow \infty \) and then \( B \downarrow 0 \) and \( A \uparrow \infty \).

Proof. We start with the conditional expectation of \( \mathcal{Y}_k(t) \). Using the many-to-one lemma, we get

\[ \mathbb{E}[\mathcal{Y}_k(t) | \mathcal{F}_t] = e^{\star} \mathbb{E} \left[ e^{-\sqrt{2} \left( \frac{\sigma_1}{\sigma_2} (\sqrt{2t} \ast - x(t^\ast)) - \frac{3}{2} \frac{1}{\sqrt{2t}} (1 - 2\alpha) \ln(t) + y/\sigma_2 \right)} \right. \]

\[ \times \mathbb{1}_{\{0 \leq s \leq t^\ast \ x(s) + x_k(t^\beta) - \sqrt{2t} \beta \leq \sqrt{2}s \}} \times \mathbb{1}_{\{x(t^\ast) - \sqrt{2t} \ast + x_k(t^\beta) - \sqrt{2t} \beta \in [-A(t/2)^\alpha, -B(t/2)^\alpha] \}}. \]

The two conditions in the indicator functions can be expressed in terms of a Brownian bridge from \( x_k(t^\beta) \) to its endpoint \( x(t^\ast) \) that must stay below \( \sqrt{2}s \) all the time. This condition produces a factor \( 2 (\sqrt{2t} \beta - x_k(t^\beta)(\sqrt{2t}/2 - x_k(t^\ast))) \). Using the independence of the bridge from its endpoint, this allows us to write

\[ \mathbb{E}[\mathcal{Y}_k(t) | \mathcal{F}_t] = 2t^{3/2 - 3\alpha} \ e^{\star} \]

\[ \times \int_{\sqrt{2t} \ast - A t^\alpha}^{\sqrt{2t} \ast - B t^\alpha} \ (\sqrt{2t} \beta - x_k(t^\beta)) (\sqrt{2t}/2 - z) \]

\[ \times \frac{e^{-\frac{z^2}{2t}} - \sqrt{2y}/\sigma_2 + \sqrt{2y}/\sigma_2 (z - \sqrt{2t} \ast)}{\sqrt{2\pi} t^\ast} \ dz \]

\[ = 2t^{3/2 - 3\alpha} \times \int_{-B t^\alpha}^{-A t^\alpha} (-z) \frac{e^{-\frac{(z+t^\ast)^2}{2t^\ast}}}{\sqrt{2\pi} t^\ast} \ e^{\sqrt{2y}/\sigma_2} \ dz \ e^{-\sqrt{2y}} (1 + o(1)) \]

\[ = 2^{5/2} (\sqrt{2t} \beta - x_k(t^\beta)) \mu^{-3\alpha} \]

\[ \times \int_{-B t^\alpha}^{-A t^\alpha} (-z) \frac{e^{\sqrt{2y}/\sigma_2 - 1} \ e^{-\frac{z^2}{2t^\ast}}}{\sqrt{2\pi}} \ e^{\sqrt{2y}/\sigma_2} \ dz \ e^{-\sqrt{2y}} (1 + o(1)). \]
The second inequality uses that we have chosen $\beta \ll z$ in the domain of integration so that we can replace $t/2$ by $t^{*}$ without making a significant error. In the range of integration, the term $\frac{1}{2\pi} z^2$ vanishes as $t \uparrow \infty$. The integral in the last line thus becomes

$$
\int_{-At^{*}}^{-Bt^{*}} \left( -z \right) \frac{e^{\sqrt{2} z t^{*}}}{\sqrt{2\pi}} \, dz = \frac{t^{2\alpha}}{\sqrt{2\pi}} \int_{B}^{A} z \, e^{-\sqrt{2} z} \, dz.
$$

As $A \uparrow \infty$ and $B \downarrow 0$, the last integral converges to $1/2$. This yields (3.42).

Next, we treat the conditional expectation of $\mathcal{A}_{k}(t)$. It is evident from the previous calculations that the terms in front of the exponential with the logarithm and the $y$ in (3.39) will tend to $0$. What is left of the conditional expectation of $\mathcal{A}^{*}$ is

$$
2^{5/2} \left( \sqrt{2} t^{\beta} - x_k(t^{\beta}) \right) y^{-3\alpha} \int_{-At^{*}}^{-Bt^{*}} z^2 \, e^{-\sqrt{2} t^{*} z - \frac{z^2}{2\pi}} \, dz \, e^{-\sqrt{2} y} \left( 1 + o(1) \right)
$$

(3.47)

$$
= 2^{5/2} \left( \sqrt{2} t^{\beta} - x_k(t^{\beta}) \right) y^{-\frac{3\alpha}{2}} \int_{B}^{A} z^2 \, e^{-\sqrt{2} z} \, dz \left( 1 + o(1) \right).
$$

The last integral converges to $2^{-5/2}$ as $A \uparrow \infty$ and $B \downarrow 0$. Thus we get (3.43). This concludes the proof of the lemma.

**Remark 3.6.** It is curious to see that the terms $\sqrt{2} t^{\beta} - x_k(t^{\beta})$ appear and recreate the derivative martingale as a factor of $\mathbb{E} \mathcal{A}_{k}(t)$. If we had been a bit more sloppy and used as the probability for the bridge just $t^{\beta/2 \alpha} t^{*}$, we would instead have gotten just a factor $t^{1/2}$ multiplying the McKean martingale. But nothing would have changed, since by a result of Aïdékon and Shi [3], this would converge in probability to a limit that has the same law as the limit of the derivative martingale.

### 3.4 Controlling the Second Moment

We now show that the expectations of the quadratic terms are bounded by a polynomial term in $t^{\alpha}$; see (3.55) below. For this it is enough to show that $\mathbb{E}[\mathcal{A}^{*}(t^{*})^2] \leq P(t^{\alpha})$ for $P$ some polynomial. Dropping all irrelevant terms that are controlled by some power of $t$, we are left with computing

$$
\mathbb{E} \left( \sum_{j=1}^{n(t^{*})} \frac{1}{x_j \in \mathcal{Z}_{t^{*}} \beta_{t^{*}} - \beta_{x_{k}(t^{*})}} e^{\sqrt{2} \pi \frac{1}{\sqrt{2}} (x_j(t^{*}) - \sqrt{2} t^{*})^2} \right)^2.
$$

(3.48)

Using the many-to-two lemma, this is bounded by

$$
\int_{0}^{t^{*}} ds \, e^{s} + \int_{-\infty}^{\sqrt{2} t^{*} - s} \frac{d w}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} \int_{-\infty}^{\sqrt{2} t^{*} - s} \frac{d z}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \int_{B(t^{*})}^{A(t^{*})} e^{-\sqrt{2} y} \left( 1 + o(1) \right)
$$

(3.49)

$$
\times \left( \int_{\sqrt{2} t^{*} - w - Bt^{*}}^{\sqrt{2} t^{*} - w - At^{*}} \frac{d z}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \right)^2.
$$
Shifting the $w$-integral, this equals
\[
\int_0^t ds \ e^{\ast + s} \int_{-\infty}^0 dw \ e^{\frac{(w + \sqrt{2}(r^* - s))^2}{2(t^* - s)}}
\times \left( \int_{2s - w - A_t^\alpha}^{\sqrt{2}s - w - B_t^\alpha} dw \ e^{-\frac{w^2}{2s}} e^{\sqrt{2\alpha_1}(w + z - \sqrt{2}s)} \right)^2
\]
(3.50)
\[
= \int_0^t ds \ e^{2s} \int_{-\infty}^0 dw \ e^{-\frac{w^2}{2s}} e^{\frac{\alpha_1(\alpha_2)w^2}{2}} e^{\sqrt{2\alpha_1}(w + z - \sqrt{2}s)}
\times \left( \int_{2s - w - A_t^\alpha}^{\sqrt{2}s - w - B_t^\alpha} dw \ e^{-\frac{w^2}{2s}} e^{\sqrt{2\alpha_1}(w + z - \sqrt{2}s)} \right)^2.
\]
Now,
\[
\int_{2s - w - A_t^\alpha}^{\sqrt{2}s - w - B_t^\alpha} dw \ e^{-\frac{w^2}{2s}} e^{\sqrt{2\alpha_1}(w + z - \sqrt{2}s)}
\]
\[
= \int_{-B_t^\alpha}^{-A_t^\alpha} dz \ e^{-\frac{(z + w + \sqrt{2}s)^2}{2s}} e^{\sqrt{2\alpha_1}z}
\]
\[
= e^{-s} e^{\sqrt{2}w - \frac{w^2}{2s}} \int_{-B_t^\alpha}^{-A_t^\alpha} dz \ e^{-z w/s + \sqrt{2}t^{-\alpha}z - \frac{z^2}{2s}}
\]
(3.51)
Inserting this into (3.50), we arrive at
\[
\int_0^t ds \int_{-\infty}^0 dw \ e^{\sqrt{2}w - \frac{w^2(2r^* - s)}{2s(t^* - s)}} \left( \int_{-A_t^\alpha}^{-B_t^\alpha} dz \ e^{-z w/s + \sqrt{2}t^{-\alpha}z - \frac{z^2}{2s}} \right)^2.
\]
(3.52)
To the level of precision we care about, the integral in the square can be bounded by the maximum of its integrand, i.e.,
\[
\int_{-A_t^\alpha}^{-B_t^\alpha} dw \ e^{-z w/s + \sqrt{2}t^{-\alpha}z - \frac{z^2}{2s}} \leq e^{\sqrt{2}A}.
\]
(3.53)
The remaining integral over $w$ is trivially bounded by $\sqrt{s/(2t^* - s)}$, which is smaller than 1, and we are done.

### 3.5 Towards the Derivative Martingale

We have seen that
\[
\mathbb{E}(\mathcal{Y}_k(t) | \mathcal{F}_t^\beta) = \text{const}(\sqrt{2} t^\beta - x_k(t^\beta)) t^{-\alpha} (1 + o(1)) \quad \text{and}
\]
\[
\mathbb{E}(\mathcal{X}_k(t) | \mathcal{F}_t^\beta) = \text{const}(\sqrt{2} t^\beta - x_k(t^\beta)) (1 + o(1)).
\]
(3.54)
Also, \( \mathbb{E}(\beta_k(t)^2 | \mathcal{F}_t) \), \( \mathbb{E}(2\beta_k(t)^2 | \mathcal{F}_t) \), and \( \mathbb{E}(\beta_k(t) \mathcal{Z}_k(t) | \mathcal{F}_t) \) grow at most polynomially in \( t^\alpha \). Then it follows, with \( \chi \) as in (3.41) and since \( x_k \in \mathcal{H}_t \), that

\[
\frac{\mathbb{E}[x^2 | \mathcal{F}_t]}{\mathbb{E}[x | \mathcal{F}_t]} \leq e^{\frac{\sigma_1}{\sigma_2} (\sqrt{2} \alpha - x_k (t^\beta) \mathcal{Z}_k(t^\beta - x_k (t^\beta) \mathcal{Z}_k(t | \mathcal{F}_t))} \leq C e^{t^{\beta \delta} \mathcal{P}(t^\alpha)}.
\]  

The right-hand side of (3.55) converges to zero as \( t \uparrow \infty \). Using (3.40) together with (3.55), we get that the expected value in (3.37) is equal to

\[
\left( 1 - \frac{\sigma_1}{\sigma_2} \mathcal{C} (\sqrt{2} t^\beta - x_k (t^\beta)) e^{-\frac{\sigma_1}{\sigma_2} (\sqrt{2} t^\beta - x_k (t^\beta)) \mathbb{E}[\beta_k(t) | \mathcal{F}_t]} \right) \frac{\mathbb{E} \left[ \mathcal{X}_k(t) | \mathcal{F}_t \right]}{\mathbb{E} \left[ \mathcal{Z}_k(t) | \mathcal{F}_t \right]} (1 + o(1)).
\]  

Plugging (3.56) back into (3.34), we get that (3.34) is equal to

\[
\mathbb{E} \left[ \prod_{k \leq n(t^\rho), x_k \in \mathcal{H}_t} \left( 1 - \frac{\sigma_1}{\sigma_2} \mathcal{C} (\sqrt{2} t^\beta - x_k (t^\beta)) e^{-\frac{\sigma_1}{\sigma_2} (\sqrt{2} t^\beta - x_k (t^\beta)) \mathbb{E}[\beta_k(t) | \mathcal{F}_t]} \right) \right]
\]

\[
\mathbb{E} \left[ \exp \left( - \sum_{k \leq n(t^\rho), x_k \in \mathcal{H}_t} \left( \frac{\sigma_1}{\sigma_2} \mathcal{C} (\sqrt{2} t^\beta - x_k (t^\beta)) e^{-\frac{\sigma_1}{\sigma_2} (\sqrt{2} t^\beta - x_k (t^\beta)) \mathbb{E}[\beta_k(t) | \mathcal{F}_t]} \right) \right) \right] (1 + o(1)),
\]

since \( \mathbb{E}(x) \leq Q(t^\alpha) e^{-At^\alpha} \), uniformly in \( x_k \), for some polynomial \( Q \) as \( x_k \in \mathcal{H}_t, A, B, \gamma \).

Next, we observe that, by Lemma 3.5,

\[
\sum_{k \leq n(t^\rho), x_k \in \mathcal{H}_t} \frac{\sigma_1}{\sigma_2} \mathcal{C} (\sqrt{2} t^\beta - x_k (t^\beta)) e^{-\frac{\sigma_1}{\sigma_2} (\sqrt{2} t^\beta - x_k (t^\beta)) \mathbb{E}[\beta_k(t) | \mathcal{F}_t]} \mathbb{E}[\beta_k(t) | \mathcal{F}_t]
\]

\[
= \sum_{k \leq n(t^\rho), x_k \in \mathcal{H}_t} \frac{\sigma_1}{\sigma_2} \mathcal{C} (\sqrt{2} t^\beta - x_k (t^\beta)) e^{-\frac{\sigma_1}{\sigma_2} (\sqrt{2} t^\beta - x_k (t^\beta)) \mathbb{E}[\beta_k(t) | \mathcal{F}_t]} \frac{e^{-\sqrt{2} y} + o(1)}{\sqrt{2} \pi}
\]

\[
\leq \text{const} t^{\beta \delta - \alpha} \sum_{k \leq n(t^\rho), x_k \in \mathcal{H}_t} (\sqrt{2} t^\beta - x_k (t^\beta)) e^{-\frac{\sigma_1}{\sigma_2} (\sqrt{2} t^\beta - x_k (t^\beta))},
\]

which converges to 0 in probability since \( \beta \delta < \alpha \) and the sum in the last line converges to the limit of the derivative martingale in probability, as will be shown in Lemma 3.7 below. On the other hand, using (3.43) to estimate the remaining
term in (3.57), we obtain
\[
\sum_{k \leq n(t^\beta), x_k \in \mathcal{N}} \frac{\sigma_1}{\sigma_2} C e^{-\sqrt{2} \sigma_1 (\sqrt{2} t^\beta - x_k(t^\beta))} \mathbb{E}[\mathcal{F}_k(t)|\mathcal{F}_t] \]
\[
= \sum_{k \leq n(t^\beta), x_k \in \mathcal{N}} \frac{\sigma_1}{\sigma_2} C (\sqrt{2} t^\beta - x_k(t^\beta)) e^{-\sqrt{2} \sigma_1 (\sqrt{2} t^\beta - x_k(t^\beta))} \left( e^{-\frac{y}{2 \pi}} + o(1) \right). \tag{3.59}
\]

The last expression almost looks like (recall (1.16))
\[
\tilde{C} Z(t^\beta) \cdot \frac{e^{-\sqrt{2} y}}{\sqrt{2 \pi}}. \tag{3.60}
\]

The next lemma asserts that this is indeed the case.

### 3.6 Control of the Almost Martingale

**Lemma 3.7.** With the notation above,
\[
\sum_{k \geq n(t^\beta)} \frac{\sigma_1}{\sigma_2} (\sqrt{2} t^\beta - x_k(t^\beta)) e^{-\sqrt{2} \sigma_1 (\sqrt{2} t^\beta - x_k(t^\beta))} \rightarrow Z, \tag{3.61}
\]
in probability, as \( t \uparrow \infty \), where \( Z \) is the limit of the derivative martingale. Moreover,
\[
\sum_{k \geq n(t^\beta)} 1_{\{x_k \in \mathcal{N}\}} \frac{\sigma_1}{\sigma_2} (\sqrt{2} t^\beta - x_k(t^\beta)) e^{-\sqrt{2} \sigma_1 (\sqrt{2} t^\beta - x_k(t^\beta))} \rightarrow Z, \tag{3.62}
\]
as \( t \uparrow \infty \), in probability.

**Proof.** As \( \sigma_2^2 = 1 + t^{-\alpha} \) and \( \sigma_2^2 = 1 - t^{-\alpha} \), we have \( \frac{\sigma_1}{\sigma_2} = 1 + t^{-\alpha} + o(t^{-\alpha}) \), and to prove (3.61), it is enough to show that
\[
\sum_{k \geq n(t^\beta)} (\sqrt{2} t^\beta - x_k(t^\beta)) e^{-\sqrt{2} \sigma_1 (\sqrt{2} t^\beta - x_k(t^\beta))} \rightarrow Z, \quad t \uparrow \infty. \tag{3.63}
\]

Next, we introduce, for some \( 1 > \gamma > 1/2 \),
\[
1 = 1_{x_k(t^\beta) > \sqrt{2} t^\beta - t^{\beta_r}} + 1_{x_k(t^\beta) \leq \sqrt{2} t^\beta - t^{\beta_r}}. \tag{3.64}
\]

We control the two resulting terms separately and start with
\[
\sum_{k \geq n(t^\beta)} 1_{x_k(t^\beta) \leq \sqrt{2} t^\beta - t^{\beta_r}} (\sqrt{2} t^\beta - x_k(t^\beta)) e^{-\sqrt{2} \sigma_1 (\sqrt{2} t^\beta - x_k(t^\beta))} \tag{3.65}
\]
We want to show that the term in (3.65) converges to 0 in probability. By the Markov inequality, it is enough to show that the expectation of (3.65) converges to 0, as \( t \to 1 \). by the many-to-one lemma, we have

\[
\begin{align*}
\mathbb{E} \left[ \sum_{k=1}^{n(t^\beta)} 1_{x_k(t^\beta) \leq \sqrt{2} t^\beta - t^\gamma (\sqrt{2} t^\beta - x_k(t^\beta))} \right] &\leq e^{t^\beta} \int_{-\infty}^{\sqrt{2} t^\beta - t^\gamma} dx \frac{\exp \left( - \frac{x^2}{2 t^\beta} \right)}{\sqrt{2 \pi t^\beta}} \left( \frac{\sqrt{2} t^\beta - x}{1 + o(1)} \right) \\
&= e^{t^\beta - 2 \alpha} \int_{-\infty}^{\sqrt{2} t^\beta - t^\gamma - \alpha} dx \frac{\exp \left( - \frac{x^2}{2 t^\beta} \right)}{\sqrt{2 \pi t^\beta}} (x + \sqrt{2} t^\beta - \alpha) (1 + o(1)) \\
&\leq \text{const} \exp \left( - \frac{(t^\gamma + \sqrt{2} t^\beta - \alpha)^2}{2 t^\beta} \right),
\end{align*}
\]

(3.66)

where we computed the integral explicitly for the first summand and used Gaussian tail asymptotics for the second. The term in (3.66) converges to 0 as \( t \to 1 \).

Next, we turn to

\[
\begin{align*}
\sum_{k=1}^{n(t^\beta)} 1_{x_k(t^\beta) > \sqrt{2} t^\beta - t^\gamma (\sqrt{2} t^\beta - x_k(t^\beta))} e^{-\sqrt{2} \sigma_1 \sigma_2 (\sqrt{2} t^\beta - x_k(t^\beta))} \\
&\leq e^{-2(t^\gamma \sigma_1 \sigma_2)} \sum_{k=1}^{n(t^\beta)} 1_{x_k(t^\beta) > \sqrt{2} t^\beta - t^\gamma (\sqrt{2} t^\beta - x_k(t^\beta))} e^{-\sqrt{2} \sigma_1 \sigma_2 (\sqrt{2} t^\beta - x_k(t^\beta))}.
\end{align*}
\]

(3.67)

Note that the prefactor in (3.67) converges to 1, as \( t^\gamma \sigma_1 \sigma_2 < t^\alpha \). Moreover, as in [30], since \( x_k(t^\beta) \leq \sqrt{2} t^\beta \) a.s., using just that \( \sigma_1 / \sigma_2 \geq 1 \), the first line in (3.67) is also bounded from above by

\[
\begin{align*}
\sum_{k=1}^{n(t^\beta)} 1_{x_k(t^\beta) > \sqrt{2} t^\beta - t^\gamma (\sqrt{2} t^\beta - x_k(t^\beta))} e^{-\sqrt{2} (\sqrt{2} t^\beta - x_k(t^\beta))} \quad \text{a.s.}
\end{align*}
\]

(3.68)
Since \( \sum_{k=1}^{n(t^\beta)} (\sqrt{2} t^\beta - x_k(t^\beta)) e^{-\sqrt{2} (\sqrt{2} t^\beta - x_k(t^\beta))} \) converges to \( Z \) almost surely (see [30]), to prove (3.62) it is enough to show that

\[
(3.69) \quad \sum_{k=1}^{n(t^\beta)} \mathbb{1}_{x_k(t^\beta) \leq \sqrt{2} t^\beta \in [A_k \delta, B_k r]} (\sqrt{2} t^\beta - x_k(t^\beta)) e^{-\sqrt{2} (\sqrt{2} t^\beta - x_k(t^\beta))} \rightarrow 0,
\]

in probability as \( t \uparrow \infty \). Putting this together with (3.67), the convergence claimed in (3.61) follows.

To show (3.61) we note that, by (3.66), it is enough to show that

\[
(3.70) \quad \sum_{k=1}^{n(t^\beta)} \mathbb{1}_{x_k(t^\beta) \leq \sqrt{2} t^\beta \in [A_k \delta, B_k r]} (\sqrt{2} t^\beta - x_k(t^\beta)) e^{-\sqrt{2} (\sqrt{2} t^\beta - x_k(t^\beta))} \rightarrow 0,
\]

in probability, as \( t \uparrow \infty \). Note that, by the same upper and lower bounds as in (3.67) and (3.68), to prove (3.70) and (3.69), we need to show that

\[
(3.71) \quad \sum_{k=1}^{n(t^\beta)} \left( \mathbb{1}_{x_k(t^\beta) \leq \sqrt{2} t^\beta < -\gamma} + \mathbb{1}_{x_k(t^\beta) > \sqrt{2} t^\beta < -\delta} \right) e^{-\sqrt{2} (\sqrt{2} t^\beta - x_k(t^\beta))}
\]

converges to 0 in probability as \( t \uparrow \infty \). Following the computation in (3.66), we get, using the many-to-lemma, that the expectation of (I) in (3.71) is equal to

\[
(3.72) \quad e^{\beta} \int_{-\infty}^{\sqrt{2} t^\beta - \gamma} \frac{dx}{2 \pi t^\beta} e^{-\frac{x^2}{2 t^\beta}} e^{-\sqrt{2} (\sqrt{2} t^\beta - x)} = \int_{t^{\beta/2}}^\infty \frac{dy}{2 \pi t^\beta} y e^{-\frac{y^2}{2 t^\beta}} = \frac{t^{\beta/2}}{\sqrt{2 \pi}} e^{-t^{\beta(2\gamma-1)/2}}.
\]

As \( \gamma > 1/2 \) the (3.72) converges to 0 as \( t \uparrow \infty \). For (II) in (3.71), we have that, for \( r \) large enough,

\[
(3.73) \quad \mathbb{P}( (\text{II}) > \epsilon ) \leq \mathbb{P}( \{ (\text{II}) > \epsilon \} \cap \{ \forall 1 \leq (t^\beta) x_k \in J_r(t^\beta) \} ) + \epsilon.
\]
Using again the Markov inequality and the many-to-one lemma, we obtain the bound

\[ P \left( \{ f > \epsilon \} \cap \{ \forall_{k \leq n(t^\beta)} x_k \in A_{r,t^\beta} \} \right) \]

\[ \leq \mathbb{E} \left[ \sum_{k=1}^{n(t^\beta)} \mathbb{1}_{x_k(t^\beta) - \sqrt{2}t^\beta > -\delta \mathbb{1}_{s \leq t^\beta} x_k(s)} \right] \times (\sqrt{2}t^\beta - x_k(t^\beta)) e^{-\sqrt{2}(\sqrt{2}t^\beta - x_k(t^\beta))} \]

(3.74)

\[ = e^\beta \int_{\sqrt{2}t^\beta}^{\sqrt{2}t^\beta-1/2} \frac{dx}{\sqrt{2\pi t^\beta}} e^{-\frac{x^2}{2t^\beta}} (\sqrt{2}t^\beta - x) e^{-\sqrt{2}(\sqrt{2}t^\beta - x)} \times \sqrt{\frac{2}{\pi}} \frac{t^\beta}{\sqrt{t^\beta}} \]

using that the Brownian bridge is independent from its endpoint and (3.8), with \( t / 2 \) replaced by \( t^\beta \). The integral in (3.74) is computed as in (3.72), and we see that (3.74) is equal to

(3.75)

\[ \sqrt{\frac{2}{\pi}} \int_0^{t^\beta(1/2)} \frac{dy}{\sqrt{2\pi}} y^2 e^{-y^2} . \]

which converges, for any \( r \) fixed, to \( 0 \) as \( t \uparrow \infty \), since \( \delta < 1/2 \). Putting the estimates in (3.72) and (3.75) together, we obtain that (3.71) converges to 0 in probability as \( t \uparrow \infty \). This concludes the proof of Lemma 3.7. \( \square \)

### 3.7 Conclusion of the Proof

Using Lemma 3.7 we see that indeed the right-hand side of (3.59) converges, as first \( t \uparrow \infty \) and then \( A \downarrow 0 \) and \( B \uparrow \infty \), in probability to

(3.76)

\[ \tilde{C} \frac{e^{-\sqrt{2}y}}{\sqrt{2\pi}} . \]

Together with the fact that the term in (3.58) converges to 0, we get that (3.34) converges to

(3.77)

\[ \mathbb{E} \left[ e^{-\frac{y^2}{2\pi}} \right] C \frac{e^{-\sqrt{2}y}}{\sqrt{2\pi}} \]

which implies (3.1) and proves Theorem 1.1 in the case when \( \sigma_1^2 = 1 + t^{-\alpha} \).

### 4 The Law of the Maximum: The Case \( \sigma_1^2 = 1 - t^{-\alpha} \)

We now consider the case when \( \sigma_1^2 = 1 - t^{-\alpha} \) and \( \sigma_2^2 = 1 + t^{-\alpha} \). In this case, we have

(4.1)

\[ m(t) = \sqrt{2} t - \frac{3 - \gamma}{2\sqrt{2}} \ln t . \]
where $\gamma = 2 - 4\alpha$ as long as $\alpha \leq 1/2$. The aim of this section is to prove the following:

**THEOREM 4.1.** Let $m(t)$ be as in (4.1). Then

$$
\lim_{t \to \infty} \mathbb{P} \left[ \max_{1 \leq k \leq n(t)} \bar{X}_k(t) - m(t) \leq y \right] = \mathbb{E}[\exp(-CZ e^{-\sqrt{2}y})].
$$

$Z$ is the limit of the derivative martingale and $C$ is a positive constant.

The structure of the proof is identical to that in the previous section.

### 4.1 Localization of Paths

To prove Theorem 4.1, we need to control the position of particles until time $t/2$. Only the position of the path at time $t/2$ needs to be modified from the previous section. Therefore we redefine

$$
G_{s,A} = \{X \mid X(s) - \sqrt{2}s \sigma_1 \in [-A \sqrt{s}, A \sqrt{s}]\}.
$$

**PROPOSITION 4.2.** Let $\sigma_1^2 = 1 - t^{-\alpha}$ and $\sigma_2^2 = 1 + t^{-\alpha}$ with $\alpha \in (0, 1/2)$. For any $d \in \mathbb{R}$ and any $\epsilon > 0$, there exists a constant $A > 0$ such that, for all $t$ large enough,

$$
\mathbb{P} \left[ \exists j \leq n(t) : \{\bar{X}_j(t) > m(t) - d\} \wedge \{\sigma_1^{-1} \bar{X}_j(t/2) \notin G_{t/2,A}\} \right] \leq \epsilon.
$$

**PROOF.** Note that $\sigma_1 \sqrt{2}t/2 = \sqrt{2}t/2 - \sqrt{2}t^{1-\alpha}/4 + O(t^{1-2\alpha})$. Abbreviate

$$
I = [\sqrt{2}t/2 - \tau - A\sqrt{t}, \sqrt{2}t/2 - \tau + A\sqrt{t}],
$$

with $\tau = \sqrt{2}(1 - \sigma_1)t/2$. Note that $\tau = \sqrt{2}t^{1-\alpha}/4 + O(t^{1-2\alpha}) \gg \sqrt{t}$. The probability in question can be written in the form

$$
\mathbb{P} \left( \exists k \leq \frac{n(t)}{2} : \left\{ \sigma_1 x_k(t/2) > m(t) - \sigma_2 \max_{\ell \leq n(t/2)} x_\ell^{k} - d \right\} \wedge \{x_k(t/2) \notin I\} \right).
$$

We can also insert the condition that particles stay below the line $\sqrt{2}s$ for all time at no cost. Then the expression in (4.6) becomes

$$
\mathbb{P} \left( \exists k \leq n(t/2) : \left\{ \sigma_2 \max_{\ell \leq n(t/2)} x_\ell^{k} > m(t) - \sigma_1 x_k(t/2) - d \right\} \wedge \{x_k(t/2) \notin I\} \wedge \{x_k(s) \leq \sqrt{2}s, \forall s \in [r,t/2]\} \right).
$$
(where it is understood that $r \uparrow \infty$ after $t \uparrow \infty$). By the many-to-one lemma, this is bounded from above by

\[
\mathcal{E}'/2 \mathbb{E}\left[ 1_{\sigma_1 x_1(t/2) \neq t} \mathbb{1}_{x_k(s) \leq \sqrt{2} s, \forall s \in [r,t/2]} \times 1_{\max_{1 \leq n k \leq t/2} \sigma_2 x_k^2(t/2) > m(t) - \sigma_1 x_k(t/2) - d} \right]
\]

(4.8)

\[
= \mathcal{E}'/2 \int_{I^c} \frac{e^{-\frac{t^2}{2}}}{\sqrt{\pi t}} \mathbb{P}\left( \delta_{0,\sqrt{2}t/2}^2(s) \leq 0, \forall s \in (r,t/2) \right) \times \mathbb{P}\left( \sigma_2 \max_{\ell \leq n k} x_\ell^2(t/2) > m(t) - \sigma_1 z - d \right) dz,
\]

where $\delta_{0,y}$ denotes the Brownian bridge from 0 to $y$ in time $t/2$, and we wrote just $I^c$ for $I^c \cap (-\infty, \sqrt{2} t/2]$. The probability regarding the Brownian bridge satisfies, since $\tau \gg \sqrt{t}$,

\[
\mathbb{P}\left( \delta_{0,\sqrt{2}t/2}^2(s) \leq 0, \forall s \in (r,t/2) \right) \leq \frac{\sqrt{t}}{\sqrt{\pi t} \alpha}.
\]

(4.9)

We now write the integral as (set $J_A = (-A \sqrt{t}, A \sqrt{t})$),

\[
\mathcal{E}'/2 \int_{J_A^c} \frac{e^{-\frac{(\sqrt{2} \sigma_1 t/2 - y)^2}{2}}}{\sqrt{\pi t}} \mathbb{P}\left( \delta_{0,\sqrt{2}t/2}^2(s) \leq 0, \forall s \in (r,t/2) \right) \times \mathbb{P}\left( \sigma_2 \max_{\ell \leq n k} x_\ell^2(t/2) > m(t) - \sigma_1^2 \sqrt{2} t/2 + \sigma_1 y - d \right) dy.
\]

(4.10)

A simple calculation shows that

\[
\frac{m(t) - \sigma_1^2 \sqrt{2} t/2}{\sigma_2} = \frac{\sqrt{2} t}{2 \sigma_2} + \frac{\sqrt{2} (\sigma_2 - \sigma_2^{-1}) t}{2} - \frac{3 - \gamma}{2 \sqrt{2} \sigma_2} \ln t
\]

(4.11)

\[
= \frac{\sqrt{2} t}{2} + \frac{\sqrt{2} t (\sigma_2 - 1)}{2} - \frac{3 - \gamma}{2 \sqrt{2} \sigma_2} \ln t.
\]

Hence the probability involving the maximum in (4.10) reads

\[
\mathbb{P}\left( \max_{\ell \leq n k} x_\ell^2(t/2) > \frac{\sqrt{2} t (\sigma_2 - 1)}{2} - \frac{3 - \gamma}{2 \sqrt{2} \sigma_2} \ln t + \frac{\sigma_1}{\sigma_2} y - d \right) dy.
\]

(4.12)
Using the bound (2.5), we see that this probability equals, asymptotically, as 
\( t \uparrow \infty \), to
\[
C t^{-3/2} \left( \frac{\sqrt{2}t (\sigma_2 - 1)}{2} - \frac{3 - \gamma}{2\sqrt{2}\sigma_2} \ln t + \frac{\sigma_1}{\sigma_2} y - d \right) 
\times e^{-\frac{3 - \gamma}{2\sqrt{2}\sigma_2} \ln t + \frac{\sigma_1}{\sigma_2} y - d}
\times e^{-\left( \frac{3 - \gamma}{2\sqrt{2}\sigma_2} \ln t + \frac{\sigma_1}{\sigma_2} y - d \right)^2 / t}.
\]

The terms in the last exponential can be written as
\[
(\sigma_2 - 1)^2 t / 2 + \frac{\sigma_1}{\sigma_2} (\sigma_2 - 1) y + \frac{\sigma_2^2}{\sigma_2} y^2 / t + o(1).
\]

Inserting this and the bound (4.9) into (4.10), we see that this term is not larger
than
\[
\int_{|y|>A} \frac{e^{-(1+\sigma_1^2/\sigma_2^2)y^2 / t}}{\sqrt{\pi t}} C t^{-\gamma/2} e^{\sqrt{\alpha} d} dy.
\]
Recalling that \( \gamma = 2 - 4\alpha \), this becomes
\[
\int_{|y|>A} \frac{e^{-(1+\sigma_1^2/\sigma_2^2)y^2 / t}}{\sqrt{\pi}} C \sqrt{\alpha} e^{\sqrt{\alpha} d} dy.
\]
For any finite \( r \), this tends to 0 as \( A \uparrow \infty \). This concludes the proof of the proposition. \( \square \)

### 4.2 Recursive Structure

As in the previous section, and with the same notation, we write
\[
\mathbb{P} \left[ \max_{1 \leq k \leq n(t)} \tilde{X}_k(t) - m(t) > y \right]
\]
\[
= \mathbb{P} \left[ \max_{k \leq n(t) / 2, i \leq n^k(t) / 2} \sigma_1 x_k(t / 2) + \sigma_2 x^{k}_{l} (t / 2) - m(t) > y \right].
\]

We again need to define shifted versions of the events \( \mathcal{G} \) and \( \mathcal{T} \) by
\[
\mathcal{G}_{s,A,S,T} = \{ X(s) - \sqrt{2s}\sigma_1 + S \in [-A(s + T)^{1/2}, A(s + T)^{1/2}] \}
\]
By Propositions 4.4, 3.1, and 3.3, we have that
\[
\mathbb{P} \left[ \max_{1 \leq k \leq n(t)} \tilde{X}_k(t) - m(t) > y \right]
\]
\[
= \mathbb{P} \left[ \exists k \leq n(t) / 2, l \leq n^k(t) / 2 : \right.
\]
\[
\left. \sigma_1 x_k(t / 2) + \sigma_2 x^{k}_{l} (t / 2) - m(t) > y \right] \wedge \{ x_k \in \mathcal{L}_{t / 2, A}, x_l \in \mathcal{H}_{t / 2, A} \} + O(\epsilon).
\]

where
\[
\mathcal{L}_{t / 2, A} = \mathcal{G}_{t / 2, A} \cap \mathcal{H}_{t / 2} \cap \mathcal{T}_{t / 2, A}.
\]
In view of Lemma 3.4, it is enough to analyze the probability in the second line of (4.19), which, as in equation (3.31), can be written as

$$1 - \mathbb{E} \left[ \prod_{k \leq n(t/2), \ x_k \in \mathcal{G}_{A,t/2}} \left( 1 - \mathbb{P} \left[ \max_{l \leq n(t/2)} X^k_l(t/2) \right. \right. \right. \right.$$

$$\left. \left. \left. > \frac{m(t) + y - \sigma_1 x_k(t/2)}{\sigma_2} \right| \mathcal{F}_{t/2} \right) \right] \right].$$

(4.21)

Since $x_k \in \mathcal{G}_{t/2,A}$, we can use the tail asymptotics given in Proposition 2.1 to control the conditional probability in (4.21). Namely,

$$\mathbb{P} \left[ \max_{l \leq n(t/2)} X^k_l(t/2) > \frac{m(t) + y - \sigma_1 x_k(t/2)}{\sigma_2} \left| \mathcal{F}_{t/2} \right. \right] = 2^{3/2} C \Gamma_k(t) e^{-\sqrt{2} \Gamma_k(t) - \Gamma_k(t)^2/t} (1 + o(1)),$$

where the $o(1)$ error term is uniform in the range of possible values for $x_k(t/2)$ as $x_k \in \mathcal{G}_{t/2,A}$ and

$$\Gamma_k(t) = \frac{m(t) + y - \sigma_1 x_k(t/2)}{\sigma_2} - \left( \frac{t}{\sqrt{2}} - \frac{3}{2 \sqrt{2}} \ln(t) \right).$$

(4.22)

Plugging (4.22) back into (4.21) we obtain that the expectation in (4.21) is equal to

$$\mathbb{E} \left[ \prod_{k \leq n(t/2), \ x_k \in \mathcal{G}_{A,t/2}} \exp \left( -2^{3/2} C \Gamma_k(t) e^{-\sqrt{2} \Gamma_k(t) - \Gamma_k(t)^2/t} \right) \right] (1 + o(1)),$$

(4.23)

since $\Gamma_k(t) > A t^{1/2}$ for $x_k \in \mathcal{G}_{t/2,A}$.

Next, we rewrite the expectation in (4.23) by conditioning on $\mathcal{F}_{t,\beta}$ as

$$\mathbb{E} \left[ \prod_{k \leq n(t/2), \ x_k \in \mathcal{G}_{\beta}} \mathbb{E} \left[ \prod_{j \leq n^{1/2} \theta / 2} \exp \left( -C \Delta_k(t) e^{-\sqrt{2} \Delta_k(t) - \Delta_k(t)^2/t} \right) \left| \mathcal{F}_{t,\beta} \right. \right. \right.$$

$$\left. \left. \right] \right].$$

(4.24)

Note that this is true if $\alpha > 0$. Otherwise, we cannot use the tail asymptotics and thus in the case $\sigma_1^2 - 1 = O(1)$, the behavior changes completely, see [12].
with $\mathcal{G}_{t/2-\beta, A, x_k(t^\beta)} - \sqrt{2t} \mathcal{I}_t^{\beta}$ as defined in (4.18) and

$$\Delta_k(t) = \frac{m(t) + y - \sigma_1(x_k(t^\beta) + x_j^k(t^*))}{\sigma_2} - \left( \frac{t}{\sqrt{2}} - \frac{3}{2\sqrt{2}} \ln(t) \right)$$

$$= \sqrt{2}t(\sigma_2 - 1)/2 + \frac{2 - 4\alpha}{2\sqrt{2}} \ln(t)$$

$$+ \frac{\sigma_1}{\sigma_2} \left( \sqrt{2}t^* \sigma_1 - x_j^k(t^*) + \sqrt{2}t^\beta \sigma_2 - x_k(t^\beta) \right)$$

$$+ \frac{y}{\sigma_2} + O(t^{-\alpha} \ln(t)),$$

(4.25)

where, for each $k$, $(x_j^k(t))_{t \leq \eta_k(t/2-\beta)}$ are particles of an independent standard branching Brownian motion. Note also that, taking into account the localization,

$$\Delta_k(t) = \sqrt{2}t(\sigma_2 - 1)/2 + O(1) = \sqrt{2}t^{1-\alpha}/4 + O(1).$$

Thus, in the prefactor of the exponential, we can replace $\Delta_k(t)$ by $\sqrt{2}t^{1-\alpha}/4$.

Next, by using again localization we obtain

$$\frac{\Delta_k(t)^2}{t} = \frac{1}{2}t(\sigma_2 - 1)^2 + \left( \frac{\sigma_1}{\sigma_2} \left( \sqrt{2}t^* \sigma_1 - x_j^k(t^*) \right) \right)^2 / t$$

$$+ \frac{\sigma_1}{\sigma_2} \left( \sqrt{2}t^* \sigma_1 - x_j^k(t^*) \right) \sqrt{2}(\sigma_2 - 1) + o(1).$$

(4.27)

Putting both terms together, we get for the terms in the exponent,

$$- \frac{t}{2} \left( \frac{\sigma_2 - 1}{2} - \frac{4\alpha - 2}{2} \ln(t) \right)$$

$$+ \frac{\sigma_1}{\sigma_2} \left( \sqrt{2}t^* \sigma_1 - x_j^k(t^*) \right)$$

$$- \left( \frac{\sigma_1}{\sigma_2} \left( \sqrt{2}t^* \sigma_1 - x_j^k(t^*) \right) \right)^2 / t$$

$$- \frac{\sigma_1}{\sigma_2} \left( \sqrt{2}t^* \sigma_1 - x_j^k(t^*) \right) \sqrt{2}(\sigma_2 - 1) - \sqrt{2}y + o(1).$$

(4.28)

We set

$$\tilde{\mathcal{G}}_{t/2-\beta, x_k(t^\beta)} \equiv \mathcal{G}_{t/2-\beta, A, x_k(t^\beta)} - \sqrt{2t} \mathcal{I}_t^{\beta} \cap \mathcal{G}_{0,t/2-\beta, x_k(t^\beta)} - \sqrt{2t} \mathcal{I}_t^{\beta}.$$

(4.29)

We rewrite the inner expectation in (4.24) as

$$\mathbb{E} \left[ \exp \left( - \sum_{j \leq \eta_k(t/2-\beta)} \frac{2}{\sigma_2} C \Delta_k(t)e^{\sqrt{2} \Delta_k(t) - \Delta_k(t)^2/t} \right) \mid \mathcal{F}_t^{\beta} \right]$$

$$= \mathbb{E} \left[ \exp \left( - C \alpha e^{-\frac{1}{2}(\sigma^2_2 - 1) - \sqrt{2}y} e^{-\frac{\sigma_1}{\sigma_2} \left( \sqrt{2}t^* \sigma_1 - x_k(t^\beta) \right) \beta_k(t)} \right) \mid \mathcal{F}_t^{\beta} \right].$$

(4.30)
where

\[ \mathcal{B}_k(t) = \sum_{j \leq n_k(t/2-\beta)} e^{-\sqrt{\beta} \sigma_1 t_2 - x_j^k(t^*)} \]

(4.31)

Here we used that \( t^{1-\alpha} t^{\gamma/2} = t^\alpha \). Note that this time there is no term involving \( \mathcal{B}_k \)!

As in the case \( \sigma_1 > \sigma_2 \), we can effectively replace in (4.30) \( \exp() \) by \( 1 + () \), compute the conditional expectation, and return then to \( \exp() \). This gives

\[ E \left[ \exp \left( -C t \alpha e^{-\frac{\sigma_1}{2} \left( \sqrt{\beta} \sigma_1 - x_j^k(t^*) \right) \mathcal{F}_k(t) } \right) \right] \]

(4.32)

The proof of (4.32) is completely analogous to the corresponding result in the case \( \sigma_1 > 1 \) and will be skipped.

### 4.3 Computation of the Main Term

We now come to the computation of the averages of \( \mathcal{B}_k(t) \).

**Lemma 4.3.** With the notation from the last subsection,

\[ E[\mathcal{B}_k(t) | \mathcal{F}_t] = \frac{\sigma_2 \left( \sqrt{\beta} - x_k(t^*) \right)^2 \alpha}{\sqrt{2}(1 + o(1))} \]

(4.33)

where \( o(1) \) tends to 0 as first \( t \uparrow \infty \) and then \( A \uparrow \infty \).

**Proof.** Since the \( x_k(t^*) \) must be of order \( t^{\beta/2} \) below \( \sqrt{2} t^* \), the bridges involved must go from \( x_k(t^*) \) to its endpoint \( x_k(t) \) and stay below \( \sqrt{2} s \) all the time.

This condition produces a factor

\[ 2 \left( \sqrt{\beta} - x_k(t^*) \right) \left( \sqrt{2}/2 - x_k(t^*) \right) \]

(4.34)

Note that the constraint on the endpoint of \( x_j^k(t^*) \) is that \( x_j^k(t^*) + x_j^k(t^*) - \sqrt{2} t \sigma_1/2 \in (A \sqrt{t}, A \sqrt{t}) \), but since \( |x_j^k(t^*)| \) is at most of order \( t^\beta \ll \sqrt{t} \), this constraint is equivalent to \( x_j^k(t^*) - \sqrt{2} t \sigma_1/2 \in (A \sqrt{t}, A \sqrt{t}) \). Thus

\[ E[\mathcal{B}_k(t) | \mathcal{F}_t] = \sqrt{2} e e^{e^*} \left( \sqrt{\beta} - x_k(t^*) \right)^2 \]

(4.35)

\[ \times \int_{\mathcal{F}_t} e^{-\frac{\sigma_1}{2} \left( \sqrt{\beta} \sigma_1 - x_j^k(t^*) \right)^2} e^{-\frac{\sigma_1}{2} \left( \sqrt{\beta} \sigma_1 - x_j^k(t^*) \right)^2} \frac{dx_j^k(t^*)}{\sqrt{2\pi e}} \frac{\sigma_1}{2} \left( \sqrt{\beta} \sigma_1 - x_j^k(t^*) \right)^2 \]

Note that this holds only if \( a > 0 \). As soon as \( 1 - \sigma_1^2 = O(1) \), the bridge condition disappears completely. This is why in that case the McKean martingale appears instead of the derivative martingale.
Shifting the integration variable, the integral in the last expression becomes

\[ \int_{A}^{A+\sqrt{t}} e^{-(z + \sqrt{2}t\sigma_1)^2} z \left( \sigma_2 - 1 \right) \frac{e^{\frac{\sigma_1^2}{\sigma_2^2} z^2 / t}}{\sqrt{2\pi t}} d\bar{z} \]

This implies (4.33) and concludes the proof of the lemma.

4.4 Towards the Derivative Martingale

Inserting (4.33) into (4.32), we see that this now becomes

\[ \exp \left( -C e^{-\sqrt{2}y} (\sqrt{2}t^{\beta} - x_k(t^{\beta})) e^{-\frac{\sigma_1^2}{\sigma_2^2} (\sqrt{2}t^{\beta} - x_k(t^{\beta}))} \right) (1 + o(1)) \]

Plugging this into (4.24), this becomes

\[ \mathbb{E} \left[ \exp \left( - \sum_{k \leq n(t^{\beta}), \tilde{x} \neq x_k} \frac{\sigma_1}{\sigma_2} C (\sqrt{2}t^{\beta} - x_k(t^{\beta})) e^{-\sqrt{2}\frac{\sigma_1}{\sigma_2} (\sqrt{2}t^{\beta} - x_k(t^{\beta}))} \right) \right] (1 + o(1)). \]

It remains to show that the sum in the exponential converges to the limit of the derivative martingale:

**Lemma 4.4.** With the notation above,

\[ \sum_{k \leq n(t^{\beta}), \tilde{x} \neq x_k} \frac{\sigma_1}{\sigma_2} (\sqrt{2}t^{\beta} - x_k(t^{\beta})) e^{-\sqrt{2}\frac{\sigma_1}{\sigma_2} (\sqrt{2}t^{\beta} - x_k(t^{\beta}))} \rightarrow Z, \]

in probability, as \( t \uparrow \infty \), where \( Z \) is the limit of the derivative martingale.

**Proof.** The proof of this lemma is completely analogous to that of Lemma 3.7 and will be skipped.

From this the proof of Theorem 1.1 follows in the case \( \sigma_1 < 1 \).

5 The Laplace Functional. Proof of Theorem 1.2

To control the extremal processes, we need to analyze the Laplace functionals. It will in fact be enough to consider functions \( \phi: \mathbb{R} \rightarrow \mathbb{R}_+ \) of the form

\[ \phi(x) = \sum_{\ell=1}^{L} c_{\ell} \mathbb{1}_{x \geq M_{\ell}}, \]
with $L \in \mathbb{N}, c_\ell > 0$, and $u_\ell \in \mathbb{R}$ (see [10, 11]). We need to compute
\[
\Psi_t(\phi) = \mathbb{E}[e^{-\int \phi(x) \, \eta_t(\text{d}x)}]
= \mathbb{E}[e^{-\sum_{k=1}^{n(t)} \phi(\bar{x}_k(t)) \mu(t))}]
= \mathbb{E}[e^{-\sum_{k=1}^{n(t)/2} \sum_{j=1}^{n(t)/2} \phi(\sigma_1 x_k(t/2) + \sigma_2 x_j(t/2) \mu(t))}]
= \mathbb{E}\left[\prod_{k=1}^{n(t)/2} \mathbb{E}\left[e^{-\sum_{j=1}^{n(t)/2} \phi(\sigma_1 x_k(t/2) + \sigma_2 x_j(t/2) \mu(t))} \, \mathcal{F}_{t/2}\right]\right].
\]

As in the previous chapters, we would like to interpret the conditional expectation in the product as a solution of the F-KPP equation and use the asymptotics of these solutions. However, there is a small problem due to the fact that the $\sigma_2$ that multiplies $x_k^j(t/2)$ depends on $t$. We will see that this problem can be solved rather easily with the help of the maximum principle.

To see this, consider, for fixed $t \in \mathbb{R}$ and $f: \mathbb{R} \to \mathbb{R}_+$,
\[
\mathbb{E}\left[\prod_{j=1}^{n(s)} f(\sigma_1(t) x(t) - \sigma_2(t) x_j(s))\right]
= \mathbb{E}\left[\prod_{j=1}^{n(s)} f^t(\sigma_1(t) x(t) - \sigma_2(t) x_j(s))\right]
= v^t(s, x(t)).
\]

where $f^t(x) = f(x \sigma_2(t))$. Then, for fixed $t$, $1 - v^t$ is a solution of the F-KPP equation with initial condition $1 - v^t(0, x) = 1 - f^t(x)$. Provided that $f$ (and $f^t$) satisfies the assumptions of Bramson’s theorem, we can derive the large-$s$ asymptotics for $v^t$. However, we want to look at the asymptotics when $s = t/2$ and $t \uparrow \infty$. Since in our cases, $f^t(x) \to f(x)$ as $t \uparrow \infty$, the initial conditions satisfy Bramson’s conditions uniformly in $t$ and bounds on $v^t(s, x)$ for large $s$ hold uniformly in $t$.

Fortunately, the maximum principle allows us to overcome this difficulty.

**Lemma 5.1.** Assume that $f^t$ is such that, for all $t > t_0$ and all $x \geq 0$,
\[
f(x) \leq f^t(x) \leq f^{t_0}(x).
\]
Then, for all $x \geq 0$ and all $t > t_0$,
\[
v(s, x) \leq v^t(s, x) \leq v^{t_0}(s, x).
\]
In particular,
\[
v(t, x(t)) \leq v^t(t, x(t)) \leq v^{t_0}(t, x(t)).
\]
The same holds if all inequalities are reversed.
PROOF. The proof is straightforward from the maximum principle; see proposition 3.1 in [16] or proposition 6.4 in [10]. □

With this information in mind we get the following slight generalization of Proposition 2.1.

Proposition 5.2. Let $u_t$ be a family of solutions to the F-KPP equation with initial data satisfying

$$(5.6) \quad u_t(0, x) \to u(0, x),$$

pointwise and monotone, for $x \geq 0$ as $t \to \infty$, where $u(0, x)$ satisfies

(i) $0 \leq u(0, x) \leq 1$;

(ii) for some $h > 0$, $\lim \sup_{t \to \infty} \frac{1}{t} \ln \int_t^{(1+h)} u(0, y) dy \leq -\sqrt{2}$;

(iii) for some $v > 0$, $M > 0$, and $N > 0$, it holds that $\int_x^{x+N} u(0, y) dy > v$, for all $x \leq -M$;

(iv) moreover, $\int_0^\infty u(0, y) y e^{2y} dy < \infty$.

Then we have, for $0 < x = x(t)$ such that $\lim_{t \to \infty} x(t) / t = 0$,

$$(5.7) \quad \lim_{t \to \infty} e^{\sqrt{2}t} e^{x^2/2t} x^{-1} u \left( t, x + \sqrt{2}t - \frac{3}{2\sqrt{2}} \ln t \right),$$

where $C$ is a strictly positive constant that depends only on the initial condition $u(0, \cdot)$. More precisely,

$$(5.8) \quad C = \lim_{r \to \infty} \sqrt{\frac{2}{\pi}} \int_0^\infty u(r, y + \sqrt{2}r) e^{\sqrt{2}y} y dy,$$

where $u$ is the solution of the F-KPP equation with initial condition $u(0, \cdot)$.

PROOF. The proof is essentially a rerun of the proofs in the case of fixed initial condition (see, e.g., the proofs of propositions 7.1 and 9.8 in [10]). The main point is to control the limit of expressions of the type

$$(5.9) \quad \int_0^\infty u^t(r, y, \sqrt{2}r) e^{\sqrt{2}y - \frac{(x y - x y)^2}{2(1 - e^{-2y r})}} (1 - e^{-2y r}) dy.$$
Strategy - GZ 2047/1, Projekt-ID 390685813, and the Collaborative Research Center 1060 “The Mathematics of Emergent Effects.” This work was done during visits by A.B. at the Courant Institute at New York University and L.H. at the IAM at Bonn University while L.H. was a Courant Instructor. We thank both institutions for their hospitality.

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Received August 2018.