Comparison between variational optimal mass transportation and Lie advection

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Optimal mass transportation plays a fundamental role in graphics, vision and machine learning. Conventional variational approach based on Brenier’s theorem gives accurate optimal transportation mapping and the Wasserstein distance but with high computational cost.

This work generalizes the Lie advection method to Riemannian manifolds with any dimensions, and compares the variational approach with Lie advection approach. Our experimental results show the efficiency and efficacy of the Lie advection method and demonstrate the Lie advection map can approximate the optimal transportation map with high accuracy.

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1. Introduction

Monge raised the classical Optimal Mass Transport Problem that concerns determining the optimal way, with minimal transportation cost, to move a pile of soil from one place to another [6]. Kantorovich [16] has proved the existence and uniqueness of the optimal transport plan based on linear programming. Monge-Kantorovich optimization has been used in numerous fields from physics, econometrics to computer science including data compression and image processing [23].

Intuitively, given two probability measures $\mu$ and $\nu$ in a Euclidean domain $\Omega$, an automorphism $\varphi : \Omega \to \Omega$ changes the volume-element, indicated by the Jacobian, therefore changes the probability densities. There are special automorphisms that map $\mu$ to $\nu$, which we call measure-preserving mappings. Each mapping $\varphi$ transports the mass $d\mu(p)$ at $p$ to another point $\varphi(p)$,

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the total transportation cost is the integration of the product of the transportation distance and the local mass. Among all the measure-preserving maps, there is a unique one with the least transportation cost, that is called the \textit{optimal mass transportation map}.

1.1. Direct applications

1.1.0.1. Parameterization  In computer graphics, parameterization refers to the process of mapping a 3D surface onto a planar domain with minimal distortions. The distortions can be classified into angle distortion and area distortion. Angle-preserving parameterization has been thoroughly studied, but area-preserving parameterization has not been fully explored yet. Optimal mass transportation map preserves measure, can be directly applied for area-preserving parameterization [27]. Furthermore, optimal mass transportation theory holds for arbitrary dimension, therefore it can be applied for measure-preserving mappings of volumetric data [25]. The target measures can be fully controlled by the users, the region of interests can be enlarged for better examination.

1.1.0.2. Image registration  In medical imaging, image registration aims at finding a good mapping between two images acquired using MRI or CT [11]. The matching process needs to consider both the geometric shapes in the images and the intensity of the pixels. One can treat the intensities as a probability measure, the optimal mass transportation map gives a reasonably good registration which matches both the shape and the intensity [15]. Furthermore, the method can be applied for both 2D images and 3D volumetric images with multi-modalities. The algorithm can be implemented in GPU to speed up the process [30].

1.1.0.3. Surface registration  In computer vision, dynamic surface registration with large deformations has fundamental importance. One approach is to flatten the surfaces onto planar domains using conformal mapping methods, and represent the area distortion factors as probability measures. Then one can find the unique optimal mass transportation map between two measured parameter domains. The composition of the conformal mapping and the OMT mapping gives the final registration [29]. If the surface deformation is isometric, the registration can fully recover the original mapping. If the surfaces are with complicated geometric features, the area-preserving property of the OMT mapping improves the robustness and the accuracy of the registration.
1.1.0.4. Shape classification  In engineering and medicine fields, shape classification plays a fundamental role. Optimal mass transportation offers a practical way to compute the distance between two probability measures, the so-called Wasserstein distance, which can be applied to measure the deviations among shapes. In [28], OMT map is applied for 3D facial expression classification. 3D human facial surfaces with different expressions are conformally mapped onto the unit disk using Riemann mapping. In this way, the Riemannian metric on the facial surface is converted into the area distortion factor (conformal factor) because of the conformality of the mapping. The conformal factor defines a probability measure on the disk. By computing the Wasserstein distance between two conformal factors, one can define a distance among the surfaces. Hence different expressions can be recognized using clustering techniques based on the Wasserstein distance.

1.1.0.5. Machine learning  In machine learning field, the Generative Adverser Networks (GAN) become very popular in recent years. In the GAN model, there are two neuron networks, the generator and the discriminator. The generator converts a Gaussian distribution in the latent space to a distribution learned from the real data, the discriminator verifies whether a sample is from the real data or generated by the generator. The generator and the discriminator compete with each other, and eventually reach the equilibrium, where the discriminator cannot differentiate the generated samples from the real sample. Optimal mass transportation theory has been applied for the GAN model, in fact the discriminator computes the Wasserstein distance between the real data distribution and the generated distribution, the generator finds the tranportation map from the Gaussian distribution to the read data distribution. This framework has been applied in Wasserstein GAN model, such as WGAN [2] and RWGAN [14] etc.

1.1.0.6. Shape interpolation  In graphics and geometric modeling, it is highly desirable to interpolate shapes. Optimal mass transportation theory has been broadly applied for shape interpolation [24], texture mixture [21]. Given a closed surface, one can define the characteristic function for it, which is positive inside the surface and zero outside. Then the characteristic functions are treated as probability measures. Given two surfaces, the Wasserstein geodesic between them can be easily constructed, the probability measures along the geodesic can be computed, whose support gives the interpolated shapes. For more shapes, one can compute the Wasserstein geodesic mass centers with different weights, which gives the interpolated shapes.
1.2. Motivation

The work mainly compares two computational approaches for optimal transportation, variational method and Lie advection method.

In practice, the variational optimal mass transportation map method based on Brenier’s theorem gives the accurate mapping and the Wasserstein distance, the number of unknowns equals to the number of sample points. On the other hand, the governing PDE Monge-Ampere is highly non-linear, the construction and maintenance of the geometric data structure is complicated, and with high spacial complexity with the increase of the dimension. Furthermore, this method requires the source domain to be convex, otherwise the resulting diffeomorphic map can not be extended to the boundary.

The Lie advection method leads to a measure-preserving mapping, which may not be the optimal. The number of unknowns equals to the number of sample points. The governing PDE is the linear Poisson equation. The method also requires geometric data structure to represent the Delaunay triangulation, but the maintenance is relatively easier. The method can handle non-convex domains.

In practice, many applications require optimal mass transporation maps, but the exact computation based on solving non-linear Monge-Ampere equation is expensive. Instead, the Lie advection method based on linear Poisson equation is more economic. Therefore, it is sensible to use Lie advection method to approximate the optimal transportation map. But many applications in machine learning and vision need to compute the Wasserstein distance between two probability measures. It is important to examine how well the Lie advection algorithm approximates the Wasserstein distance. In this work, we compare the Brenier’s approach and Lie advection method, measure the difference between the transportation costs obtained by the two methods. Our empirical results show that the measure-preserving mapping obtained by the Lie advection is a good approximation to the optimal transportation map, the Lie advection transportation cost is close to the Wasserstein distance.

1.3. Contributions

The major contributions of this work are as follows:

1. A theorem is proven, which generalizes the mathematical formulation of the Lie advection method based on Lie derivative and Cartan’s
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magic formula to Riemannian manifolds with arbitrary dimensions. The Lie advection algorithm based on dynamic meshes is designed, which is much more efficient than variational approach and robust to non-convex domains.

2. Comparison between Brenier’s approach and the Lie advection approach is conducted, which shows that the Lie advection mapping is very close to the optimal transportation mapping.

The manuscript is organized as follows: section 2 reviews three approaches for computing the OMT map: linear programming, the variational convex geometric method and the Lie advection; section 3 introduces the theoretic background, and gives the detailed proof for the Lie advection theorem; the computational algorithms are explained in details in section 4; the experimental results are reported in section 5; the conclusion and the future work are discussed in section 6.

2. Previous works

There are many methods to approximate optimal mass transportation map, in the following we only briefly review the most relevant approaches. For more thorough treatments, we refer readers to [31], [32] and [9].

Due to the vast potential applications, great research efforts have been spent for seeking efficient algorithms for computing the optimal mass transportation map recently. There are several approaches to approximate the OMT map using different computational techniques.

2.0.0.7. Kantarovich’s linear programming Kantarovich solved Monge’s problem by relaxing the optimal transportation map to transportation plans. Given \((X, \mu)\) and \((Y, \nu)\), one can define a joint distribution \(\gamma : X \times Y \rightarrow \mathbb{R}\), the marginal distribution of \(\gamma\) on \(X\) equals to \(\mu\), the marginal distribution of \(\gamma\) on \(Y\) equals to \(\nu\). The transportation cost is the expectation of the distance between a source in \(X\) and a target in \(Y\), under the joint distribution \(\gamma\). The optimal transportation plan \(\gamma\) can be obtained by minimizing the transportation cost with the marginal distribution constraints. By discretizing the distributions \(\mu, \nu\) and \(\gamma\), the problem is equivalent to a linear programming problem. The major merit of this approach is its flexibility for any types of cost functions. The drawback is the computational complexity. Suppose \(X\) and \(Y\) are discretized to \(n\) points respectively, then \(\mu\) and \(\nu\) are represented as Dirac measures, the joint measure is represented as \(n^2\) unknowns. This type of problem (referred to as an assignment problem) can
be solved by different methods of linear programming [8]. In the worst case, the popular simplex algorithm for linear programming has exponential complexity; the Khachiyan’s ellipsoid algorithm has the worst-case-polynomial complexity. By adding a regularization term, such as the entropy of the transport plan, the method can be accelerated [18]. This regularized version of optimal transport can be solved by highly efficient numerical algorithms [10].

2.0.0.8. Brenier’s approach

If the transformation cost is the square of Euclidean distance, then Brenier’s theorem 3.1 states that there exists a convex function \( u : X \rightarrow \mathbb{R} \), such that the gradient map \( \nabla u : X \rightarrow Y \) gives the optimal mass transportation map. By measure-preserving property, we obtain that \( u \) satisfies the Monge-Ampere equation,

\[
\det \left( \frac{\partial^2 u}{\partial x^i \partial x^j} \right)(x) = \frac{\mu(x)}{\nu \circ \nabla u(x)}.
\]

Monge-Ampere equation is closely related to the Alexandrov problem in convex geometry, which claims that given a convex polytope, the normal to each face is given \( n_i \), the projected volume of each face is given \( \nu_i \), then the polytope can be determined uniquely up to a vertical translation. The existence and uniqueness was first proven by Alexandrov [1] using a topological method; the existence was also proven by Aurenhammer in [3] and [4], the uniqueness and optimality was proven by Brenier [7]. Recently, Gu et al. [12] gave a novel proof for the existence and uniqueness based on the variational principle, which leads to the computational algorithm directly. In this approach, the target measure \( \nu \) is discretized to a Dirac measure with \( n \) points, the number of unknowns is \( n \), therefore much less than that in linear programming approach. In practice, the convex polytope is represented as its dual triangulation. During the optimization, the combinatorial structure of the polytope/triangulation is modified dynamically. It is difficult to construct and maintain the high dimensional simplicial complex (triangulation) data structure. This prevents algorithm from applying for solving high dimensional OMT problem. The variational approach is applied for measure-preserving parameterization in [27], [28], [25], [26], it is used for visualization in [35]. Similar methods are introduced in [19] and [17], where the target energy is the transportation cost directly, which is the Legendre dual to that in [12].
2.0.0.9. Lie advection approach  Suppose two probability measures $\mu$ and $\nu$ are given on the space $X$, we would like to find a measure-preserving automorphism $\varphi$ of $X$, which maps $\mu$ to $\nu$. We can find a flow, at each time $t$, the flow carries a particlal from $p$ to $\varphi_t(p)$, this induces a diffeomorphism $\varphi_t : X \to X$. $\varphi_t$ pulls back $\nu$ to $\mu_t$. We can design $\mu_t$ as the linear interpolation between $\mu$ and $\nu$. The velocity field of the flow can be represented as $V(p, t)$, then the differential relation $V(p, t)$ and the measure $\mu_t$ are related by Cartan’s magic formula. Hence, we can solve a special differential equation to obtain $V(p, t)$, then the one parameter family of diffeomorphisms $\varphi_t$. The measure-preserving mapping is given by $\varphi_1$. By using this approach, the measure-preserving mapping is given by

$$\log \frac{\mu}{\nu} u,$$

where the function $u$ is the solution to the Poisson equation with Neuman boundary condition,

$$\Delta u(p) = \nu - \mu, \quad p \notin \partial X$$
$$\frac{\partial u}{\partial n} = 0, \quad p \in \partial X$$

Zhou et al. introduced Lie advection on planar domain in [36]. The current work generalizes it to general Riemannian manifold setting. The Lie advection method gives one measure-preserving mapping, but may not be the optimal transportation mapping. It only involves solving linear PDEs, hence it is more efficient and simpler to implement.

3. Theoretic background

This section briefly introduces the theoretic background of Optimal Mass Transport theory. We refer readers to the classical work [16] on optimal transport map with Kantorovich’s method, [7] and [32] for Brenier’s approach, [12] for more detailed proofs of the variational method.

3.1. Conformal mapping

Suppose $\varphi : (M, g) \to (N, h)$ is a diffeomorphism (smooth, bijective mapping) between two Riemannian surfaces. If $\varphi$ preserves angles, then we call it conformal.
Definition 3.1 (Conformal Mapping). A diffeomorphism \( \varphi : (M, g) \to (N, h) \) between two Riemannian surfaces is called conformal, if the pull back metric \( \varphi^*h \) induced by \( \varphi \) and the original metric \( g \) on the source surface differ by a scalar function, namely there exists a real function \( \lambda : M \to \mathbb{R} \), such that \( \varphi^*h = e^{2\lambda}g \).

For each point \( p \in M \) on the surface, one can find a neighborhood \( U(p) \), a special local parameter \((x, y)\) can be found within the neighborhood, the so-called isothermal parameters, such that the Riemannian metric can be written as \( g = e^{2\lambda(x,y)}(dx^2 + dy^2) \).

We can cover the surfaces \( M \) and \( N \) with isothermal coordinates, and get the local complex coordinates \( z = x + iy \) and \( w = u + iy \) respectively. The mapping \( \varphi : z \to w \) can be locally represented as a complex-valued function \( w = \varphi(z) \). The complex differential operators are defined as

\[
\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right).
\]

Then the mapping is conformal, if and only if \( \frac{\partial \varphi}{\partial \bar{z}}(p) = 0, \forall p \in M \). For general \( \varphi \in C^1(M, N) \), the mapping \( \varphi \) transforms infinitesimal circles on \( M \) to infinitesimal ellipses. The orientation and the eccentricity of the ellipses are given by the so-called Beltrami coefficient,

\[
(4) \quad \mu_\varphi := \frac{\partial \varphi}{\partial \bar{z}} / \frac{\partial \varphi}{\partial z}.
\]

If \( |\mu_\varphi| \) is less than 1 everywhere, then \( \varphi \) is a diffeomorphism; if \( \mu_\varphi \) is 0 everywhere, then \( \varphi \) is a conformal mapping. Under the normalization conditions, the norm of the Beltrami coefficient measure the distance from \( \varphi \) to the identity.

3.2. Optimal mass transportation map

Mongé [6] raised the optimal mass transportation problem in the 18th century.

Problem 3.1 (Optimal Mass Transport). Suppose \((X, \mu), (Y, \nu)\) are metric spaces with probabilities measures, which has the same total mass \( \int_X \mu dx = \int_Y \nu dy \). A map \( T : X \to Y \) is measure preserving, if for any measurable set \( B \subset Y \), \( \mu(T^{-1}(B)) = \nu(B) \). Given a transportation cost function \( c :\)
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\( X \times Y \to \mathbb{R} \), find the measure preserving map \( T : X \to Y \) that minimizes the total transportation cost

\[
C(T) := \int_X c(x, T(x))d\mu(x).
\]

In the 1940s, Kantorovich introduced the relaxation of Monge’s problem and solved it using linear programming method [16].

At the end of 1980s, Brenier [7] discovered the intrinsic connection between optimal mass transport map and convex geometry.

**Theorem 3.1** (Brenier). Suppose \( X \) and \( Y \) are the Euclidean space \( \mathbb{R}^n \), and the transportation cost is the quadratic Euclidean distance \( c(x, y) = |x - y|^2 \). If \( \mu \) is absolutely continuous and \( \mu \) and \( \nu \) have finite second order moments, then there exists a convex function \( f : X \to \mathbb{R} \), its gradient map \( \nabla f \) gives the solution to the Monge’s problem. Furthermore, the optimal mass transportation map is unique.

This theorem converts the Monge’s problem to solving the following Monge-Amperé partial differential equation:

\[
\det \left( \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \right) = \frac{\mu(x)}{\nu \circ \nabla f(x)}.
\]

Detailed proof can be found in [31].

### 3.3. Discrete optimal mass transport

We focus on the Brenier’s approach. Suppose \( \mu \) has compact support on \( X \), define \( \Omega = \text{supp} \mu = \{x \in X | \mu(x) > 0\} \), assume \( \Omega \) is a convex domain in \( X \). The space \( Y \) is discretized to \( Y = \{y_1, y_2, \cdots, y_k\} \) with Dirac measure \( \nu = \sum_{j=1}^{k} \nu_j \delta(y - y_j) \).

We define a height vector \( h = (h_1, h_2, \cdots, h_k) \in \mathbb{R}^k \), consisting of \( k \) real numbers. For each \( y_i \in Y \), we construct a hyperplane defined on \( X \),

\[
\pi_i(h) : \langle x, y_i \rangle + h_i = 0.
\]

Define a function

\[
u_h(x) = \max_{i=1}^{k} \{\langle x, y_i \rangle + h_i\},
\]
then \( u_h(x) \) is a convex function. We denote its graph by \( G(h) \), which is an infinite convex polyhedron with supporting planes \( \pi_i(h) \). The projection of \( G(h) \) induces a polygonal partition of \( \Omega \),

\[
\Omega = \bigcup_{i=1}^{k} W_i(h),
\]

where each cell \( W_i(h) \) is the projection of a facet of the convex polyhedron \( G(h) \) onto \( \Omega \),

\[
W_i(h) = \{ x \in X | u_h(x) = \langle x, y_i \rangle + h_i \} \cap \Omega.
\]

Note that, this partition is similar to the *power diagram* concept in computational geometry [5]. The area of \( W_i(h) \) is given by

\[
w_i(h) = \int_{W_i(h)} \mu(x) dx.
\]

The convex function \( u_h \) on each cell \( W_i(h) \) is a linear function \( \pi_i(h) \), therefore, the gradient map

\[
\text{grad } u_h : W_i(h) \rightarrow y_i, i = 1, 2, \ldots, k.
\]

maps each \( W_i(h) \) to a single point \( y_i \).

The following theorem plays a fundamental role for discrete optimal mass transport theory,

**Theorem 3.2** (Gu-Luo-Sun-Yau 2013 [12]). For any given measure \( \nu \), such that \( \sum_{j=1}^{n} \nu_j = \int_{\Omega} \mu, \nu_j > 0 \), there must exist a height vector \( h \) unique up to adding a constant vector \( (c, c, \ldots, c) \), the convex function Eqn. 7 induces the cell decomposition of \( \Omega \), Eqn. 8, such that the following area-preserving constraints are satisfied for all cells,

\[
\int_{W_i(h)} \mu(x) dx = \nu_i, i = 1, 2, \ldots, n.
\]

Furthermore, the gradient map \( \text{grad } u_h \) optimizes the following transportation cost

\[
C(T) := \int_{\Omega} |x - T(x)|^2 \mu(x) dx.
\]

The existence and uniqueness was first proven by Alexandrov [1] using a topological method; the existence was also proven by Aurenhammer [3],
the uniqueness and optimality was proven by Brenier [7]. Recently, Gu et al. [12] gave a novel proof for the existence and uniqueness based on the variational principle, which leads to the computational algorithm directly.

Define the admissible space of height vectors $H_0 := \{ h | \sum_{j=1}^k h_j = 0 \text{ and } \int_{W_i(h)} \mu > 0, \forall i = 1, \cdots, k, \}$. Then define the energy $E(h)$,

$$E(h) = \int_{\Omega} u_h(x)\mu(x)dx - \sum_{i=1}^k \nu_i h_i,$$

or equivalently

$$E(h) = \int_0^h \sum_{i=1}^k w_i(\eta)d\eta_i - \sum_{i=1}^k \nu_i h_i + C,$$

where $C$ is a constant. Consider the shape bounded by the graph $G(h)$, the horizontal plane $\{ x_{n+1} = 0 \}$ and the cylinder consisting of vertical lines through $\partial \Omega$, the volume of the shape is given by the first term.

The gradient of the energy is given by

$$\nabla E(h) = (w_1(h) - \nu_1, \cdots, w_k(h) - \nu_k)^T,$$

Suppose the cells $W_i(h)$ and $W_j(h)$ intersect at an edge $e_{ij} = W_i(h) \cap W_j(h) \cap \Omega$, then the Hessian of $E(h)$ is given by

$$\frac{\partial^2 E(h)}{\partial h_i \partial h_j} = \begin{cases} \frac{\int_{e_{ij}} \mu(x)dx}{|y_j - y_i|} & W_i(h) \cap W_j(h) \cap \Omega \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

The following theorem lays down the theoretic foundation of our OMT map algorithm.

**Theorem 3.3 (Discrete Optimal Mass Transport [12]).** If $\Omega$ is convex, then the admissible space $H_0$ is convex, the energy (Eqn. 14) is convex. The unique global minimum $h_0$ is an interior point of $H_0$. Furthermore, the gradient map (Eqn. 11) induced by the minimum $h_0$ is the unique optimal mass transport map, which minimizes the total transportation cost (Eqn. 13).

The proof of Theorem 3.3 is reported in [12]. Due to the convexity of the volume energy Eqn. 14, With this theory, the global minimum can be obtained efficiently using Newton’s method. Comparing to Kantorovich's approach, where there are $O(n^2)$ unknowns, this approach has only $O(n)$ unknowns.
3.4. Lie advection

Measure preserving mapping can be achieved using Lie advection method, which simplifies the whole computation to solving a linear Poisson equation. In the following, we generalize the Lie advection method to Riemannian manifolds of any dimension. We state our result as the following theorem:

**Theorem 3.4** (Lie Advection on Riemannian manifolds). Suppose \((\Omega, g)\) is an \(n\) dimensional manifold with the Riemannian metric \(g\), \(\mu\) and \(\nu\) are two probability measures with the equal total mass

\[
\int_{\Omega} d\mu = \int_{\Omega} d\nu.
\]

There exists a one parameter family of diffeomorphisms \(\phi_t: \Omega \to \Omega,\ 0 \leq t \leq 1\), such that

\[
\phi_1^*(\nu) = \mu.
\]

Namely, \(\phi_1\) is measure-preserving.

**Proof.** Assume \((\Omega, g)\) is a Riemannian domain, with local coordinates \((x^1, x^2, \ldots, x^n)\). Then basis for the tangent vector fields are

\[
\left\{ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n} \right\},
\]

the the basis for the differential 1-forms are

\[
\{ dx^1, dx^2, \ldots, dx^n \}.
\]

Let \(T\Omega\) be the tangent bundle of \(\Omega\), \(\Gamma(T\Omega)\) the sections of the tangent bundle. One section \(X \in \Gamma(T\Omega)\) is a smooth vector field. \(\gamma(p, t)\) is the integral curve of \(X\) with initial point \(p \in \Omega\), namely

\[
\frac{d\gamma(p, t)}{dt} = X \circ \gamma(p, t), \quad \gamma(p, 0) = p
\]

Let \(\varphi_t\) be the one parameter family of diffeomorphisms generated by the vector field \(X\),

\[
\varphi_t(p) := \gamma(p, t).
\]

The probability measures are represented as \(n\)-differential forms,

\[
\omega_\mu(x) = \mu(x) dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n,
\]

\[
\omega_\nu(x) = \nu(x) dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.
\]
we linearly interpolate the two probability measures, for any \( t \in [0, 1] \)

\[
\omega_t = (1 - t)\omega_\nu + t\omega_\mu = ((1 - t)\nu + t\mu)dx_1 \wedge \cdots \wedge dx_n.
\]

Then we let the pull-back measure induced by \( \varphi_t \) as \( \omega_t \),

\[
\omega_t = \varphi_t^* (\omega_\nu),
\]

in turn

\[
\frac{d\varphi_t^* (\omega_\nu)}{dt} = \frac{d\omega_t}{dt} = \omega_\mu - \omega_\nu.
\]

This is just the Lie derivative of \( \omega_t \) with respect to \( X \),

\[
\mathcal{L}_X \omega_t := \frac{d\varphi_t^* (\omega_\nu)}{dt}.
\]

According to Cartan’s magic formula, we have

\[
\mathcal{L}_X \omega_t = d \circ \iota_X \omega_t + \iota_X \circ d\omega_t.
\]

Because \( \Omega \) is of \( n \) dimensional, the exterior differential of \( \omega_t \) is 0, \( d\omega_t = 0 \). The above equation yields to

\[
d \circ \iota_X \omega_t = \omega_\mu - \omega_\nu
\]

Without loss of generality, we assume there exists a function \( u : \Omega \to \mathbb{R} \) and the vector field has the form

\[
X(t) = \frac{1}{(1 - t)\nu + t\mu} \nabla u = \frac{1}{(1 - t)\nu + t\mu} \sum_{i=1}^n \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_i}.
\]

By definition of \( \iota_X(t)\omega_t \) and Eqn. 20 and Eqn. 18, we have

\[
\iota_X(t)\omega_t(X_1, \ldots, X_{n-1}) = \omega_t(X(t), X_1, \ldots, X_{n-1})
\]

\[
= dx^1 \wedge \cdots \wedge dx^n \left( \sum_{i=1}^n \frac{\partial u}{\partial x^i}, X_1, \ldots, X_{n-1} \right)
\]
Using this formula, we obtain
\[ t_X(t) \omega_\nu \left( \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n} \right) \]
\[ = d x^1 \wedge \cdots \wedge d x^n \left( \sum_{i=1}^{n} \frac{\partial u}{\partial x^i}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\hat{\partial}}{\partial x^i}, \ldots, \frac{\partial}{\partial x^n} \right) \]
\[ = d x^1 \wedge \cdots \wedge d x^n \left( \frac{\partial u}{\partial x^1}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\hat{\partial}}{\partial x^i}, \ldots, \frac{\partial}{\partial x^n} \right) \]
\[ = (-1)^{i+1} d x^1 \wedge \cdots \wedge d x^n \left( \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial u}{\partial x^i}, \ldots, \frac{\partial}{\partial x^n} \right) \]
\[ = (-1)^{i+1} \frac{\partial u}{\partial x^i} \]

Hence, we get
\[ t_X(t) \omega_\nu = \sum_{i=1}^{n} (-1)^{i+1} d x^1 \wedge d x^2 \wedge \cdots \wedge \hat{d} x^i \wedge \cdots \wedge d x^n. \]

It yields to
\[ d \circ t_X(t) \omega_\nu = \Delta u d x^1 \wedge d x^2 \wedge \cdots \wedge d x^i \wedge \cdots \wedge d x^n. \]

From Eqn. 19, we obtain the Poisson equation
\[ (21) \quad \Delta u = \mu - \nu, \]

with Neumann boundary condition. The solution to the Poisson equation exists and is unique, hence the vector field \( X(t) \) exists, the one parameter family of diffeomorphisms \( \varphi_t \) exist. By construction, \( \varphi_1^*(\omega_\nu) = \omega_\mu \), \( \varphi_1 \) is the desired measure-preserving mapping. \( \varphi_1 \) can be obtained by integrating the vector field \( X(t) \) from 0 to 1,
\[ \varphi_1 = \int_0^1 X(t) dt = \int_0^1 \frac{1}{(1-t)\nu + t\mu} \nabla u dt = \frac{\log \mu - \log \nu}{\mu - \nu} \nabla u. \]

4. Computational algorithms

This section focuses on the algorithms. The theoretic deduction for spherical harmonic map can be found in [13], for volumetric harmonic map in [33], for
optimal mass transportation map in [28]. In order to be complete, we give all details of these algorithms.

4.1. Discrete optimal mass transportation map

Given a discrete point set \( P \subset \mathbb{R}^2 \), \( P = \{(p_u, A_u), u \in \mathcal{L}\} \), where \( \mathcal{L} \) is a index set, such that \( \sum_{u \in \mathcal{L}} A_u = \pi \). Our goal is to find a discrete optimal mass transportation from the unit disk to the measured point set \( P \), denoted as \( \varphi : \mathbb{D}^2 \to P \). The algorithm pipeline is summarized in Alg. 1.

4.1.0.10. Brenier potential

According to Brenier’s theorem, there should be a convex function, the so-called Brenier potential \( f : \mathbb{D}^2 \to \mathbb{R} \), the optimal mass transportation map is given by the gradient map of the Brenier potential.

In the discrete setting, the Brenier potential is a piecewise linear convex function, constructed as follows. For each vertex \( u \in M \), suppose \( f(u) = (a_u, b_u) \), we construct a plane \( \pi_u \) in the four dimensional Euclidean space \( \mathbb{R}^3 \), namely a linear function \( \pi_u(x, y) = a_u x + b_u y + h_u \). The Brenier potential is defined as \( f(x, y) := \max_{u \in \mathcal{L}} \pi_u(x, y) \), its graph is the upper envelope of family of planes \( \{ \pi_u, u \in M \} \) in \( \mathbb{R}^3 \).

We use \( \Omega(h) \) to represent the (open) convex polyhedron of the upper envelope, where \( h = (h_u) \) is the height vector. \( \pi_u \)'s are the supporting planes of the upper envelope \( \Omega(h) \), the face \( F_u \) is the intersection between \( \pi_u \) and \( \Omega(h) \), the projection of \( F_u \) into \( \mathbb{R}^2 \) is defined as a cell \( W_u \) in \( \mathbb{R}^2 \), therefore, the projection of the upper envelope \( \Omega(h) \) to \( \mathbb{R}^2 \) induces a cell decomposition of the unit solid ball, denoted as \( D(h) \). we call this cell decomposition as the power diagram of \( \mathbb{D}^2 \). The area of each cell is defined as \( w_u(h) = \text{vol}(\mathbb{D}^2 \cap W_u(h)) \).

4.1.0.11. Legendre dual

The computation of the upper envelop of a family of planes \( \{ \pi_u, u \in M \} \) is converted to convex construction of its Legendre dual.

Definition 4.1 (Legendre Dual). Suppose \( f : \mathbb{R}^2 \to \mathbb{R} \), its Legendre dual is a function \( f^* : \mathbb{R}^2 \to \mathbb{R} \), defined as

\[
f^*(x^*, y^*) := \sup_{(x, y) \in \mathbb{R}^2} \{ xx^* + yy^* - f(x, y) \}.
\]

Each plane \( \pi_u(x, y, z) = a_u x + b_u y + c_u z + h_u \) is dual to a point

\[
\pi^*_u := (a_u, b_u, c_u, -h_u) \in \mathbb{R}^4.
\]
The upper envelope of the supporting planes
\[ \Omega(h) = \text{Env}(\{\pi_u, u \in M\}) \]
is the graph of the Brenier potential \( f \); the graph of the Legendre dual of the Brenier potential \( f^* \) is the lower convex hull of the dual points of the supporting planes
\[ \Omega^*(h) := \text{Conv}(\{\pi_u^*, u \in M\}). \]
The projection of the lower convex hull \( \Omega^*(h) \) to \( \mathbb{R}^2 \) forms a triangulation, which we call as the power Delaunay triangulation, denoted as \( T(h) \).

4.1.0.12. Optimization

The key to find the optimal mass transportation map is to find the appropriate height vector \( h \), then construct the upper envelop \( \Omega(h) \), and the Brenier potential function, whose gradient map is the desired map. The height vector is the unique minimizer of the following convex energy
\[
E(h) = \sum_{u \in M} A_u h_u - \int h \sum_{u \in M} w_u(\eta) d\eta_u,
\]
with the constraint \( \sum_{u \in M} A_u h_u = 0. \)

Due to the convexity of the energy, it can be optimized using Newton’s method. The gradient of the energy has the form
\[
\nabla E(h) = (A_u - w_u(h)), u \in M
\]
The Hessian matrix of the energy is constructed as follows. If the cell \( W_u(h) \) and \( W_v(h) \) are adjacent in the power diagram \( D(h) \), then their intersection is a 2-cell \( W_u(h) \cap W_v(h) \), the dual of this 1-cell in the power Delaunay triangulation \( T(h) \) is an edge \( \{f(u), f(v)\} \), then we define
\[
k_{u,v} := \frac{|W_u(h) \cap W_v(h)|}{|f(u) - f(v)|}.
\]
If \( W_u(h) \cap W_v(h) = \emptyset \), then the corresponding \( k_{u,v} \) is 0. The Hessian matrix is given by
\[
\frac{\partial^2 E(h)}{\partial h_u \partial h_v} := \begin{cases} -k_{u,v} & u \neq v \\ \sum_w k_{u,w} & u = v \end{cases}
\]
At the first step, we initialize the height vector as
\[ h_u = -\frac{1}{2} (f(u), f(u)) = \frac{1}{2} (a_u^2 + b_u^2). \]

At each step, we solve the linear system
\[ \left( \frac{\partial^2 E(h)}{\partial h_u \partial h_v} \right) \delta h = \nabla E(h) \]
then update the height vector \( h \leftarrow h - \delta h \), until the norm of the gradient \( \nabla E(h) \) is less than a predefined threshold.

**Algorithm 1** Optimal Mass Transport Map (OMT-Map)

**Input:** A discrete point set in \( \mathbb{R}^2 \) with measure \( P = \{(p_u, A_u)\}, A_u > 0, \sum_u A_u = \pi \); a threshold \( \epsilon \).

**Output:** The unique discrete OMT-Map \( \varphi : \mathbb{D}^2 \rightarrow P \).

Scale and translate \( P \), such that \( P \subset \mathbb{D}^2 \).
Initialize \( h_u \leftarrow -\frac{1}{2} (p_u, p_u) \).

repeat

- Compute the upper envelope \( \text{Env}(\{\pi_u\}) \), where the plane \( \pi_u(p) = \langle p_u, p \rangle + h_u \),
- Project the upper envelope to obtain the power diagram \( D(h) \),
- Compute the dual power Delaunay triangulation \( T(h) \),
- Compute the cell areas \( w(h) = (w_u(h)) \).
- Compute \( \nabla E(h) \) using Eqn. 22.
- Compute the Hessian matrix using Eqn. 23.
- Update the height vector \( h \leftarrow h - \delta H^{-1} \nabla E(h) \).

until \( \|\nabla E\| < \epsilon \).

return \( \varphi : \Omega \rightarrow P, W_u(h) \rightarrow p_u \).

### 4.2. Area-preserving parameterization

Let \( M \) be the simply connected tetrahedral mesh with a single boundary surface, we would like to compute a area-preserving mapping from \( M \) to the unit disk \( \mathbb{D}^2 \). The algorithm pipeline is described in Alg. 2.

We first scale the whole mesh \( M \), such that the total area equals to \( \pi \), then compute the initial parameterization \( f : M \rightarrow \mathbb{D}^2 \). For each vertex \( u \in M \), we define the measure associated with it as one third the total area
of all faces adjacent to it,

\[ A_u := \frac{1}{3} \sum_{\{u,v,w\} \in M} \text{Area}(\{u,v,w\}), \]  

therefore, the total measure equals to the area of the unit disk; \( \sum_{u \in M} A_u = \pi \), then we define the discrete point set with measures \( P = \bigcup_{u \in M} \{(f(u), A_u)\} \). 

We compute the discrete optimal mass transportation map \( \varphi : \mathbb{D}^2 \rightarrow P \), the inverse map \( \varphi^{-1} \) maps each point \( p_u \in P \) to the mass center of \( W_u \). The composition \( \varphi^{-1} \circ f : M \rightarrow \mathbb{D}^2 \) is the desired area-preserving parameterization.

**Algorithm 2** Area-preserving Parameterization

**Input:** A simply connected mesh \( M \) with single boundary.

**Output:** An area-preserving parameterization.

Compute an initial parameterization \( f : M \rightarrow \mathbb{D}^2 \).

Compute the discrete measures using Eqn. 24, construct the measured point set using Eqn. 25.

Compute the discrete optimal mass transportation map \( \varphi : \mathbb{D}^2 \rightarrow P \),

**return** \( \varphi^{-1} \circ f : M \rightarrow \mathbb{D}^2 \).

4.3. Lie advection algorithm

In this subsection, we explain the Lie advection algorithm in details.

4.3.0.13. Solving Poisson equation  

The key step of the algorithm is to solve the Poisson equation, 

\[ \Delta u = \nu - \mu. \]

For each vertex \( v \in M \), the measure \( \mu \) is given by the Eqn. 24 using the initial vertex positions in \( \mathbb{R}^3 \), the measure \( \nu \) is defined similarly using the parameter vertex position. The Poisson equation can be solved using Finite element method, the function is defined on the vertex set, \( u : V \rightarrow \mathbb{R} \).

\[ \Delta u(v) = \sum_{w \sim v} k_{v,w} (u(w) - u(v)) = \nu(v) - \mu(v), \]
Comparison between variational OMT and Lie advection

Figure 1: Lie advection process. The top row shows the flow process from the conformal initial parametrization to authalic parametrization at different time; the bottom row depicts the vector field, the norms of the vectors are color encoded, the orientations are illustrated as arrows.

Figure 2: Area-preserving mapping by Lie advection.

Figure 3: Area-preserving mapping by Lie advection.

where $k_{v,w}$ is the cotangent edge weight: suppose the edge $\{v, w\}$ are against two corner angles $\alpha$ and $\beta$, then $k_{v,w} = \cot \alpha + \cot \beta$. The discrete Poisson equation can be solved using conjugate gradient algorithm.

4.3.0.14. Vector field The solution to the Poisson equation $u : V \rightarrow \mathbb{R}$ can be linear extended using the barycentric coordinates. Then the function $u$
is a piecewise linear one, its gradient is a piecewise constant vector field. Suppose \{v, w, l\} is a face of the mesh, then the gradient on this face can be represented as

\[
\nabla u|_{\{v, w, l\}} = u(l)(w - v) + u(v)(l - w) + u(w)(v - l).
\]

At each vertex \(v\), the vector field is defined as the blending of the adjacent face vectors weighted by the face area,

\[
\nabla u(v) = \sum_{w, l} \lambda_{w, l}^v \nabla u|_{\{v, w, l\}}.
\]

where

\[
\lambda_{w, l}^v := \frac{\text{Area}(\{v, w, l\})}{\sum_{w, l} \text{Area}(\{v, w, l\})}.
\]

According to theorem 3.4, the tangential vector field is given by

\[
X(v, t) = \frac{1}{(1 - t)\nu(v) + t\mu(v)} \nabla u(v).
\]

For a non-vertex point \(p\), we use barycentric coordinates to linearly extend the vector field \(X(p, t)\). The mapping is given by

\[
\varphi_1(p) = \int_0^1 X(p, t) = \frac{\ln \mu(p) - \ln \nu(p)}{\mu(p) - \nu(p)} \nabla u(p).
\]

4.3.0.15. Dynamic remeshing Although in smooth case, the solution to the Lie advection has close form, in practice, this formula leads to degenerated triangulations. Instead of using the one-step integrated formula in theorem 3.4, we compute the vector field and the mapping \(\varphi_t\) by several steps. In the beginning, we modify the triangulation of \(\Omega\) to be Delaunay by edge swaps. At each time \(t_k\), we compute the vector field \(X(t_k)\), then move each vertex \(v\) along \(X(v, t_k)\) with a step length \(\delta_k\). The step length \(\delta_k\) is maximized until one triangle is degenerated after the deformation. Then we update the triangulation to be Delaunay by edge swaps, update \(t_{k+1} \leftarrow t_k + \delta_k\), and calculate \(\mu_{k+1}\) and \(\nu\). Repeat this procedure, until the time exceeds 1 or \(\varphi_t^*\nu\) is close to \(\mu\) in \(L^2\) norm.

The algorithm is summarized in Alg. 3. Fig. 1 illustrates the computational process for the area-preserving parameterization of the Lion Vase model; Fig. 2 and Fig. 3 show the Lie advection for area-preserving mapping for the Buddha and Torso model, the initial mappings are the Riemann mappings.
Algorithm 3 Lie Advection Algorithm

**Input:** A Riemannian domain $\Omega$; Two probability measures $\mu$ and $\nu$.
**Output:** An area-preserving mapping $\phi : \Omega \rightarrow \Omega$, such that $\phi^* \nu = \mu$.

Set the initial mapping $\phi$ as the identity $id : \Omega \rightarrow \Omega$, $t = 0$.

repeat

- Update the triangulation of $\Omega$ to be Delaunay by edge swaps;
- Compute the current pull back measure $\phi^* \nu$;
- Solve the Possion equation $\Delta u = \phi^* \nu - \mu$ with Neumann boundary condition;
- Compute the gradient field $\nabla u$ and the vector field $X$;
- Compute the optimal step length $\delta$, $t \leftarrow t + \delta$;
- Update the mapping $\phi \leftarrow \phi + \delta X$

until $t$ exceeds 1 or $\phi^* \nu$ is close to $\mu$ in $L^2$ norm.

Return $\phi$.

5. Experimental results

We have implemented our algorithms using generic C++ on a Windows 10 platform, with 4 core 3.2GHz CPU i5-650, 8GB memory. All the geometric data are from public resources, either manually modeled, acquired from real life by laser scanning or reconstructed from geological data. All the geometric data are represented as triangle meshes.

5.1. Initial parametrization

There are several kinds of well-known surface flattening, such as harmonic map [33], discrete Ricci flow [34], least square conformal parameterization [20], scalable least injective mapping (SLIM) [22] and constant mean curvature flow (CMC) and so on.

5.2. Flexible boundary condition

One of the major draw-backs of the variational OMT algorithm is that if the parameter domain $\Omega$ is non-convex, then the optimal transportation mapping in the interior of $\Omega$ is homeomorphic, but the homeomorphism can not be extended to the boundary. The Lie advection algorithm overcomes this shortcoming, as long as the domain boundary is regular, the mapping is globally homeomorphic, up to the boundary. As shown in Fig. 4, the buddha surface is mapped on the planar domain using various parameterization algorithms, the planar domain is with piecewise smooth boundary, but may not be convex. The Lie advection algorithm achieves area-preserving homeomorphisms for all cases.
Figure 4: Lie advection algorithm can handle non-convex domains. The top row shows the initial mapping, the bottom row demonstrates the area-preserving mapping. Each column are initialized by CMC (a), SLIM (b), harmonic maps (c) and (d) algorithms.

5.3. Measure-controllable mapping & region of interests

Both OMT and Lie advection algorithms can achieve measure-controllable mappings, that allow users to choose region of interests and enlarge or shrink these regions. As shown in Fig. 5, the mountain region is selected as the region of interests (ROI). Fig. 6 shows another example. The owl model is mapped on the the parameter rectangle, different regions are chosen as ROIs. The ROIs are enlarged or shrunk by different factors.

5.4. Riemannian manifold

The Lie advection algorithm can be directly generalized from Euclidean domain to Riemannian manifolds with arbitrary dimensions. In this work, we illustrate the generality of the method by computing spherical area-preserving parameterizations. Fig. 7, Fig. 8 and Fig. 9 show the topological sphere cases, Fig. 11 and Fig. 12 show topological disk cases.
5.5. Accuracy and efficiency

In order to verify the accuracy of the computational results, we compute the area element ratio (the determinant of the Jacobian matrix) of the mapping from 3D meshes to the 2D parameter domain. For each triangle, we compute the area ratio between the initial position in the mesh and the target position on the parameter domain, then the histograms are illustrated in Fig. 14. The area-ratio histograms of the initial parameterization are colored in orange, those of the final results are colored in blue. The orange
histograms are almost uniform distributed, the blue ones are highly concentrated near the origin. This shows our computational results approximate the area-preserving mapping with high accuracy.

The running times for each model are reported in Table 1, it is obvious that the Lie advection method is much faster than that of the variational OMT method.
Comparison between variational OMT and Lie advection

Figure 11: Area-preserving texture mapping using Lie advection for the Granite model.

Figure 12: Area-preserving texture mapping using Lie advection for the Hercorando model.

Table 1: Timing comparison (in sec.) between OMT and Lie Advection

| Model    | V  | T  | OMT  | Lie  |
|----------|----|----|------|------|
| Buddha   | 252| 500| 41:39.3 | 02:22.0 |
| Buste    | 417| 830| 01:06.3 | 00:12.7 |
| Duck     | 1425| 0.10| 00:06.9 | 00:01.4 |
| foot     | 3818| 0.79| 00:13.2 | 00:08.2 |
| maxplanck| 5278| 111| 00:03.6 | 00:00.8 |
| Moai     | 8774| 17544| 00:08.8 | 00:01.9 |
| nicola   | 14584| 29164| 03:21.5 | 00:08.1 |
| owl      | 252| 500| 01:32.4 | 02:24.3 |
| pegaso   | 417| 830| 00:09.9 | 00:03.9 |
| torso    | 8774| 17544| 01:02 | 00:15.8 |
| totoro   | 14584| 29164| 00:21.3 | 00:06.5 |

5.6. Comparison between Lie advection and OMT

In theory, Lie advection algorithm is capable of producing measure-preserving maps, but in general not the optimal mass transportation map. We design experiments to measure the difference between the area-preserving mappings produced by the Lie advection algorithm and the variational OMT algorithm.
Specifically, let $(S, g)$ be a topological disk with a Riemannian metric, $\varphi_0, \varphi_1 : (S, g) \to \mathbb{D}^2$, where $\varphi_0$ is obtained by the variational OMT method, $\varphi_1$ by the Lie advection method. Visually, the two measure-preserving mappings are very similar as shown in Fig. 13.

We would like to precisely measure the difference between $\varphi_0$ and $\varphi_1$. The composition

$$\psi = \varphi_1 \circ \varphi_0^{-1} : (\mathbb{D}^2, dx \wedge dy) \to (\mathbb{D}^2, dx \wedge dy),$$

preserves the Euclidean area element, namely $\psi$ belongs to the special diffeomorphism group of the disk. As shown in Fig. 15, we use checkerboard
Figure 15: Comparison between the Lie advection and the variational OMT methods. The 1st row shows the models, the 2nd row shows the checkerboard texture mapping of the map $\psi$. The 3rd row illustrates the histograms of the norm of the Beltrami coefficients of $\mu_\psi$ and the mean.

texture mapping to visualize $\psi$ in the middle row, it is obvious that the mapping $\psi$ is very close to the identity.

We can measure the distance between $\psi$ and the identity quantitatively. We compute the Beltrami coefficient $\mu_\psi$ of the mapping $\psi$ using the formula 4. If $\psi$ is close to the identity, then $\mu_\psi$ is close to the constant 0 everywhere. We calculate the norm of the Beltrami coefficient $\mu_\psi$, and depicts the histogram of the norms in the 3rd row in Fig. 15. It is obvious that the norms are concentrated at the origin, hence the quantitatively measurement shows that the deviation of $\psi$ from the identity is very small. Table 2 compares the transportation costs obtained by the Lie advection method and the variational OMT method, the relative difference is less than 1%, this shows Lie advection transportation cost is a good estimate to the Wasserstein distance.
Table 2: Comparison between the transportation costs obtained by Lie advection and variational OMT methods

| Model | Lie       | OMT       | Difference |
|-------|-----------|-----------|------------|
| Buddha| 7748.534087 | 7750.490673 | 0.025251%  |
| Nicolo| 2348.137983 | 2355.678252 | 0.3211169% |
| Owl   | 10436.088994 | 10706.2624 | 0.026362686% |
| Terrain | 1210.911749 | 1211.831155 | 0.0759268% |

6. Conclusion

This work proposes the Lie advection approach to approximate the optimal mass transportation map. The Lie advection method is generalized to general Riemannian manifold setting with arbitrary dimension, and applied for multiple engineering applications. The Lie advection method has less computational complexity, is capable of handling non-convex domains, is general to Riemannian manifolds. The experimental results also shows the Lie advection mapping can approximate the optimal transportation mapping with high accuracy.

In the future, we will rigorously prove the approximation error bound between the Lie advection mapping and the optimal mass transportation mapping and conduct more experiments for high dimensional data sets.

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