Fast Decentralized Federated Low Rank Matrix Recovery from Column-wise Linear Projections

Shana Moothedath and Namrata Vaswani*

Abstract—This work develops an alternating projected gradient descent and minimization algorithm for recovering a low rank (LR) matrix from mutually independent linear projections of each of its columns in a “decentralized federated” fashion. This means that the data is federated but there is no central coordinating node. To be precise, our goal is to recover an unknown \( n \times q \) rank-\( r \) matrix \( X^* = [x_1^\star, x_2^\star, \ldots, x_k^\star] \) from \( y_k := A_k x_k^\star, \) \( k = 1, 2, \ldots, q \), when each \( y_k \) is an \( m \)-length vector with \( m < n \); different subsets of \( y_k \)'s are acquired at different nodes (vertical federation); and there is no central coordinate node (each node can only exchange information with its neighbors). We obtain constructive provable guarantees that provide a lower bound on the required sample complexity and an upper bound on the iteration complexity (total number of iterations needed to achieve a certain error level) of our proposed algorithm. This latter bound allows us to bound the time and communication complexity of our algorithm and argue that it is fast and communication-efficient. We performed extensive simulations to validate the performance of our algorithm and compared it with benchmark algorithms.

I. INTRODUCTION

In this work we develop and analyze a gradient descent (GD) based algorithm for solving the low rank (LR) column-wise Compressive Sensing (LRcCS) problem [4], [5], [6] in a decentralized federated setting. The data is federated and there is no central coordinating node. LRcCS is an LR matrix recovery problem that involves recovering an LR matrix from mutually independent linear projections of each of its columns. We define it below. Two important applications where the LRcCS problem occurs are distributed/federated sketching [7], [5] and MRI (dynamic MRI or MRI of a set of similar images from multiple patients across multiple decentralized sites with each site being a node) [8]. Sketching refers to lossy data compression where the compression step needs to be computationally inexpensive, typically a linear projection, but the decompression can be more complex. Distributed/federated sketching means that the data to be compressed, e.g., images or videos, is acquired at geographically distributed nodes, e.g., mobile phones or Internet-of-Things (IoT) devices [5], [7]. For this application, a decentralized solution that works without the need for a central coordinating node is especially useful when nodes are geographically distant (if some or all nodes are far from others, a decentralized setting results in lower communication costs and is scalable). Moreover, in some distributed settings, it is impossible/impractical to have a central coordinating node. Another emerging widely popular application for the LRcCS problem is the multi-task representation learning, including multi-task supervised learning [9], [10] multi-task bandit learning [11], [12], [13], and multi-task reinforcement learning [14], [15]. In these learning problems, the unknown parameter between different yet related tasks have an LR structure and the goal is to jointly learn the tasks by leveraging the problem structure. Moreover, often there may not be a central server that all nodes can communicate with. This necessitates a decentralized solution such as the one we develop here. Our recent blind conference submission shows how our algorithm applies to this problem.

The centralized version of the LRcCS problem itself has only been studied starting in 2019. This is unlike other LR matrix recovery problems – LR matrix completion (LRMC) [16], [17] and LR matrix sensing (LRMS) [17] – that have been studied for more than a decade.

In the last decade, the design of decentralized GD algorithms for minimizing \( f(x) = \sum_{g=1}^G f_g(x) \), where \( x \) is a vector and \( f_g \) is known only to agent \( g \), has received a lot of attention [18], [19], [20], [21], [22], [23], [24], starting with the seminal work of Nedic et al. [18]. These works consider unconstrained convex optimization problems and study the following overall approach (which we henceforth refer to as DGD): at each iteration, replace the estimate \( x^{(t)} \) at node \( g \) by its weighted average over all its neighbors followed by a local GD step. A non-asymptotic guarantee for this approach was proved in later work by Yuan et al in [22]. There is also some existing work on projected GD for decentralized constrained optimization [25], [26] or for imposing the consensus constraint [27]. There is also a large amount of work that considers non-convex optimization problems and proves convergence to a stationary point under additional assumptions [28], [29], [30], [31], [32], [33], [34].

Decentralized PCA (computing top \( r \) singular vectors of a symmetric matrix) using the power method (PM) [35], also known as orthogonal iteration, has been studied in [36], [37], [38]. We explain the most related works in detail in Sec. III-D.

A. Problem setting, notation, and assumptions

Problem setting. The goal is to recover a set of \( q \) \( n \times \) \( q \) dimensional vectors/signals, \( x_1^\star, x_2^\star, \ldots, x_q^\star \) that are such that the \( n \times q \) matrix \( X^* := [x_1^\star, x_2^\star, \ldots, x_q^\star] \) has rank \( r \ll \min(n,q) \), from \( m \)-length vectors \( y_k \) satisfying

\[
y_k := A_k x_k^\star, \quad k = 1, 2, \ldots, q.
\]

The \( m \times n \) matrices \( A_k \) are known and mutually independent for different \( k \). The regime of interest is \( m < n \) and the goal is to have to use as few number of samples \( m \) as possible. We consider a decentralized federated setting; there are a total of \( L \leq q \) distributed nodes/sensors and each node measures/acquires/sketches a different subset of the \( y_k \)'s. We denote the set of indices of the \( y_k \)'s available at node \( g \) by...
The sets $S_g$ are mutually disjoint and $\cup_{g=1}^{q} S_g = \{g\}$. Here $\{g\} := \{1, 2, \ldots, q\}$. The $L$ nodes are connected by a communication network whose topology (connectivity) is specified by an undirected graph $G = (V, E)$, where $V$ is the set of $L$ nodes ($|V| = L$) and $E$ denotes the set of edges. The neighbor set of the $g$-th node is denoted by $N_g(G)$, i.e., $N_g(G) := \{j : (g, j) \in E\}$. There is no central coordinating node, each node can only exchange information with its neighbors. We note here that no one node can recover the entire matrix $X^\ast$. Node $g$ only recovers the columns $x^\ast_k$, for $k \in S_g$, i.e., the sub-matrix $X^\ast_{\{g\}} := [x^\ast_k, k \in S_g]$. Let

$$X^\ast \overset{\text{SVD}}{=} U^\ast \Sigma^\ast V^\ast := U^\ast B^\ast$$

(2)

denote its reduced (rank $r$) SVD, i.e., $U^\ast$ and $V^\ast$ are matrices with orthonormal columns (basis matrices), $U^\ast$ is $n \times r$, $V^\ast$ is $r \times q$, and $\Sigma^\ast$ is an $r \times r$ diagonal matrix with non-negative entries (singular values). We let $B^\ast := \Sigma^\ast V^\ast$. We use $\sigma^\ast_{\text{max}}$ and $\sigma^\ast_{\text{min}}$ to denote the maximum and minimum singular values of $\Sigma^\ast$ and we define its condition number as

$$\kappa := \sigma^\ast_{\text{max}}/\sigma^\ast_{\text{min}}.$$ 

**Assumptions.** Another way to understand (1) is as follows: each scalar measurement $y_{ki}$ (i-th entry of $y_k$)

$$y_{ki} := \langle a_{ki}, x^\ast_i \rangle, \quad i \in [m], \quad k \in [q]$$

(3)

with $a_{ki}$ being the i-th row of $A_k$. Notice from (3) that $y_{ki}$s are not global functions of $X^\ast$: no $y_{ki}$ is a function of the entire matrix $X^\ast$. We thus need an assumption that enables correct interpolation across the different columns. The following incoherence (w.r.t. the canonical basis) assumption on the right singular vectors suffices for this purpose [4], [5], [39].

Such an assumption on both left and right singular vectors was first introduced in [16] for making the LR matrix completion problem well posed.

**Assumption 1.1** (Incoherence of right singular vectors). Assume that $\|b^\ast_k\|^2 \leq \mu^2 \sigma^\ast_{\text{max}}^2 / q$ for a numerical constant $\mu$ that is greater than one. Since $\|x^\ast_i\|^2 = \|b^\ast_k\|^2$, this also implies that $\|x^\ast_i\|^2 \leq \mu^2 \sigma^\ast_{\text{max}}^2 / 2q$. Here, and below, $\|\cdot\|$ without a subscript denotes the $\ell_2$ norm.

In addition, we also make the following two simple and commonly used assumptions.

**Assumption 2.** The matrices $A_k$ are independent and identically distributed (i.i.d.) random Gaussian (each matrix entry is i.i.d. standard Gaussian).

**Assumption 3.** The graph, $G$, specifying the network topology is connected.

**Notation.** We use $\{g\} := \{1, 2, \ldots, q\}$. We denote the Frobenius norm as $||\cdot||_F$, and the induced $\ell_2$ norm (often called the operator norm or spectral norm) as $||\cdot||$ without any subscript. Further, $^\dagger$ denotes matrix or vector transpose. For a tall matrix $M$, $M^\dagger := (M^\top M)^{-1} M^\top$ (pseudo-inverse). We use $e_k$ to denote the $k^{th}$ canonical basis vector. We use $\mathbb{I}_{[\text{statement}]}$ to denote an indicator function that takes the value 1 if statement is true and zero otherwise. For a vector $w$ and a scalar $\alpha$, $\mathbb{I}_{\{w \leq \alpha\}}$ returns a vector of 1s and 0s of the same size with 1s where $(w(k) \leq \alpha)$ and zero everywhere else. Also, $|w|$ takes element-wise magnitude of the vector. We use $\circ$ to denote the Hadamard product. Thus $z := w \circ \mathbb{I}_{\{w \leq \alpha\}}$ zeroes out entries of $w$ with magnitude larger than $\alpha$. We use $W \in \mathbb{R}^{L \times L}$ to denote the weighted adjacency matrix of the network $G$ and $\lambda_k(W)$ to refer to the $k$-th largest eigenvalue of $W$. We use $g^\oplus$ for quantities for which each node obtains an estimate by consensus and we use $g$ for the sub-matrix of a larger matrix available at node $g$. For example, we use $U^\oplus$ to denote the consensus estimate of $U$ obtained at node-$g$ and $B_g$ is a submatrix of $B$ available at node $g$.

At various places in the paper, for tall $n \times r$ matrices, we are only interested in the orthonormal basis spanned by the matrix. We compute this basis using QR decomposition, i.e., $U^\ast \overset{\text{QR decomposition}}{=} QR$. We refer to the matrix $U$ as the basis matrix. For two $n \times r$ basis matrices $U_1, U_2$ (corresponding to two subspaces defined by their column spans), the measure of Subspace Distance (SD) used in this work is

$$\text{SD}_2(U_1, U_2) := \|\|I - U_1 U_1^\top\|\|_2$$

We reuse $c, C$ to denote different numerical constants in each use with $c < 1$ and $C \geq 1$ (these do not depend on the signal model parameters $n, q, r$). This is a standard approach in literature when only the order of the sample or time complexity is of interest (and not the exact constants). The notation $a \gtrsim b$ means that $a \geq Cb$. We will prove our results to hold with probability (w.p.) at least $1 - 1/n$ or $1 - C/n$ for a small constant $C$. We say a bound holds with high probability (whp) if it holds with at least this much probability.

The entire matrix $X$ (estimate of $X^\ast$) is not estimated at any one node. We use $X_\ast$ to denote its sub-matrix estimated at node $g$. Thus,

$$X = [X_1, X_2, \ldots, X_\ast], \quad \text{with} \quad X_k := [x^\ast_k, k \in S_g].$$

We use this type of notation at various places for $X, X^\ast$ as well as for other $q$ column matrices like $B$ and $B^\ast$. We form a $q \times q$ diagonal matrix $D$ with entries $\beta_k, k \in S_g, g \in [L]$, using the notation $D = \text{diagonal}(\beta_k, k \in S_g, g \in [L])$. Since $\{g\} = \cup_{g \in [L]} S_g$ and the sets $S_g$ are mutually disjoint, thus $\sum_{k \in [q]} \text{term}_k = \sum_{g \in [L]} \sum_{k \in S_g} \text{term}_k$.

**B. Contributions and Paper Organization**

We develop an alternating GD and minimization (AltGDmin) solution for solving the LRcCS problem in a decentralized federated fashion and provide a non-asymptotic theoretical guarantee for it. Simulation experiments are used to corroborate our results as well as compare with other approaches. Our theoretical guarantee provides an order-wise lower bound on the required sample complexity and an upper bound on the required number of iterations to achieve a certain accuracy level (iteration complexity). This upper bound enables us to also obtain time and communication complexity guarantees, and show that our approach is both fast and communication-efficient. The only assumptions in our work are right singular vectors’ incoherence and connected network.

To our best knowledge, there is no other work that provides a provably accurate algorithm for solving any LR matrix
recovery problem in a decentralized fashion. Hence we cannot borrow algorithmic or proof techniques from any other work. We also cannot borrow ideas from the existing works on decentralized optimization because most of these either study convex optimization problems or provide asymptotic performance. Instead we attempt to modify the best existing centralized solution, AltGDmin [6], by adapting the multi-consensus (gossip algorithms) idea from [40], [41]. The novelty of our proof techniques is explained in Sec. III-E and a detailed discussion that places our work in relation to existing work is provided in Sec. III-D.

We also develop, and experiment with, using the existing DGD idea [18], [22] for developing a decentralized version of AltGDmin and explain why it fails: its consensus step is averaging\( n \times r \) matrices with orthonormal columns; numerical averaging does not improve subspace estimation quality. As we show in Sec. VI, this fails to provide an accurate estimate even when a very large number of iterations are run. Lastly, we also experimentally compare with four other LR matrix recovery algorithms from the centralized LR recovery literature - AltGDmin [6], AltGD [42], Projected GD on \( X \) (ProjGD) [43], and AltMin [39], [4].

**Organization.** We develop the algorithm in Sec. II. We provide and discuss our theoretical guarantees in Sec. III. These are proved in Sec. IV and V and in the Appendix. Simulation experiments are described in Sec. VI. We conclude in Sec. VII.

II. DECENTRALIZED ALTERNATING GD AND MINIMIZATION (ALTGDMIN)

In this section, we present our proposed decentralized AltGDmin algorithm. We also discuss the existing works that are most closely related to ours.

A. AltGDmin main idea

The optimization problem that needs to be solved to solve our problem can be stated as

\[
\min_{X} f(X), \quad f(X) := \sum_{k=1}^{q} \| y_k - A_k x_k \|^2. \tag{4}
\]

Centralized AltGDmin proceeds as follows [6]. We rewrite the unknown matrix \( X \) as \( X = UB \), where \( U \) is \( n \times r \) and \( B \) is \( r \times q \), and consider the cost function

\[
f(U, B) := f(UB) = \sum_{k=1}^{q} \| y_k - A_k Ub_k \|^2. \tag{5}
\]

Starting with a careful initialization of \( U \), at each iteration, AltGDmin alternates between the following two steps [6]:

- **Minimization over \( B \):** For each new estimate of \( U \), solve for \( B \) by minimizing \( f(U, B) \) over it while keeping \( U \) fixed at its current value. Because of the LRcCS measurement model, this becomes a decoupled column-wise least squares (LS) step and the solution can be written in closed form as \( b_k = (A_k U)^{\dagger} y_k \) for each \( k \). In this step, the most expensive part is the computation of the matrices \( A_k U \) for all \( k \), this takes time of order \( mnr \cdot q \).

- **Projected-GD (ProjGD) for \( U \):** compute \( \tilde{U} \leftarrow U - \eta \nabla_U f(U, B) \) and orthonormalize it using QR decomposition: \( U \leftarrow QR(\tilde{U}) \). Here \( \nabla_U f(U, B) = \sum_{k=1}^{q} A_k (A_k Ub_k - y_k) b_k^{\dagger} \). The gradient computation needs time \( mqn \), QR only needs time \( nr^2 \).

The per-iteration time cost of AltGDmin is order \( mqn \).

B. Decentralized AltGDmin

Due to the non-convexity of our cost \( f(U, B) \), and of our constraint on \( U \), we are unable to borrow DGD and related ideas. We utilize the scalar average-consensus algorithm on each entry of the gradient at each node to approximate its sum.

**Average Consensus (AvgCons) algorithm.** We use the equal-neighbor consensus algorithm, summarized in Algorithm 1. In each of its iterations, nodes update their values by taking a weighted sum of their own and their neighbors’ partial sums. By doing this enough number of times, if the graph is connected, one can show that one finally computes an approximation of the true sum at each node [44], [41]. The weights are equal to \( 1/d_g \) where \( d_g = |N_g(G)| \) is the degree of node \( g \) (number of nodes that \( g \) is connected to). The weights’ matrix \( W \) is thus

\[
W_{gj} = \begin{cases} 
1/d_g, & \text{for } j \in N_g(G), \\
0, & \text{for } j \notin N_g(G). 
\end{cases} \tag{6}
\]

Observe that \( W \) is symmetric and doubly stochastic\(^1\); and it depends on the graph of the network as well. We use \( \lambda_k(W) \) to refer to the \( k \)-th largest eigenvalue of \( W \in \mathbb{R}^{n \times n} \). Then the second largest magnitude eigenvalue of \( W \) is defined as

\[
\gamma(W) := \max(\| \lambda_2(W) \|, |\lambda_L(W)|). \tag{7}
\]

The result below presents a convergence result on the number of \( T_{\text{con}} \) for convergence of Algorithm 1.

**Proposition 2.1** ([41], [44], [45]). Consider the average consensus algorithm (AvgCons) given in Algorithm 1 with a doubly stochastic weights’ matrix \( W \). Let \( \zeta_{\text{true}} := \sum_{g=1}^{L} \zeta_{g}^{(e)} \) be the true sum that we want to compute in a decentralized fashion. Pick an \( \epsilon_{\text{con}} < 1 \). If the graph of the network is connected, and if \( T_{\text{con}} \geq \frac{1}{\log(1/\gamma(W))} \frac{\log(L/\epsilon_{\text{con}})}{\log(1/\gamma(W))} \), then

\[
\max_{g} |z_{g}^{(e)} - \zeta_{\text{true}}| \leq \epsilon_{\text{con}} \max_{g} |z_{g}^{(e)} - \zeta_{\text{true}}|.
\]

We assumed the network (connectivity graph) is given. If the network topology can be designed, it is possible to design networks (and weighting matrices for networks) in such a way that the consensus algorithm converges at a faster rate [44].

**Decentralizing AltGDmin iterations.** Since the minimization step decouples for each column \( b_k \), this step is done locally at the node that has the corresponding \( y_k \). Thus, at node \( g \), we compute \( b_k = (A_k U)^{\dagger} y_k \) for all \( k \in S_g \). Each node thus updates a different \( r \times |S_g| \) sub-matrix \( B_g = [b_k, k \in S_g] \).

The gradient computation for the GD step over \( U \) requires a sum over all \( q \) terms. Recall from (5) that the centralized cost function can be expressed as

\[
f(U, B) = \sum_{g=1}^{L} f_g(U, B),
\]

\(^1\)Since \( W \) is doubly stochastic, its largest eigenvalue is 1 and all eigenvalues are between 1 and -1. The second largest magnitude eigenvalue is either \( |\lambda_2(W)| \) or \( |\lambda_L(W)| \).
Algorithm 1 AvgCons: Equal-neighbor average consensus

Input: $Z^{(i)}_{\text{in}}$, for all $g \in [L]$ ($Z^{(i)}_{\text{in}}$ is a matrix)
Parameters: $T_{\text{cons}}, G$ (graph connectivity)
1: Initialize $Z^{(i)}_{\text{0}} \leftarrow Z^{(i)}_{\text{in}}$, for all $g \in [L]$
2: for $t = 1$ to $T_{\text{cons}}$
3: \hspace{1em} $Z^{(i+1)}_{\text{t}} \leftarrow Z^{(i)}_{\text{t}} + \sum_{j \in N_G(g)} \frac{1}{d_g} (Z^{(i)}_{\text{t}} - Z^{(i)}_{j})$, for $g \in [L]$
4: end for
5: Output: $Z^{(i)}_{\text{out}} \leftarrow L \cdot Z^{(i)}_{T_{\text{cons}}}$

Algorithm 2 The complete Dec-AltProjGDmin algorithm. It calls the AvgCons function summarized in Algorithm 1.

1: Input: $A_k, y_k, k \in S_g, g \in [L]$ graph $G$
2: Output: $U^{(g)}, B_g$ and $X_g = U^{(g)} B_g$
3: Parameters: $\eta, T_{\text{cons}}, T$
4: Sample-split: Partition $A_k, y_k$ into $2T + 2$ equal-sized disjoint sets: $A_k = \{A_k^{(f)}, y_k^{(f)} \}_{f=0,1,2,\ldots,2T}$
5: Initialization: Run Algorithm 3 or 4 to get $U^{(g)}$
6: AltGDmin iterations:
7: \hspace{1em} for $t = 1$ to $T$
8: \hspace{2em} $U^{(g)} \leftarrow U^{(g)}_{t-1}$
9: \hspace{2em} Let $y_k = y_k^{(T+t)}, A_k = A_k^{(T+t)}$
10: \hspace{2em} $b_k \leftarrow (A_k U^{(g)})^\top y_k$ for all $k \in S_g$
11: \hspace{2em} $\bar{x}_k \leftarrow U^{(g)} b_k$ for all $k \in S_g$
12: \hspace{2em} $y^*_k = y_k - \beta k (A_k U^{(g)} b_k - y_k) b_k^\top$
13: \hspace{2em} $\nabla f_g \leftarrow \sum_{k \in S_g} \nabla f_g (A_k U^{(g)} b_k - y_k) b_k^\top$
14: \hspace{2em} $\text{gradU}^{(g)} \leftarrow \text{AvgCons}_{\lambda G^{(g)} \nabla f_g, g' \in [L], G, T_{\text{cons}}}$
15: \hspace{2em} $\tilde{U}^{(g)}_{+} \leftarrow U^{(g)} - \eta \text{gradU}^{(g)}$
16: \hspace{2em} QR decompose $\tilde{U}^{(g)}_{+} \equiv U^{(g)} R^{(g)} +$ to get $U^{(g)}_{+}$
17: \hspace{1em} $U^{(g)} \leftarrow U^{(g)}_{+}$
19: Output $U^{(g)}, B_g$ and $X_g = U^{(g)} B_g$

with $f_g(U, B) = \sum_{k \in S_g} \|y_k - A_k U b_k\|^2$. In a decentralized setting, the estimates of $\bar{U}$ are different at the different nodes $g$, i.e., at node $g$, the cost function is $f_g(U^{(g)}, B_g)$. Thus, node $g$ can only compute $\nabla f_g := \nabla U f_g(U^{(g)}, B_g)$. $\nabla f_g$ is the input to AvgCons given in Algorithm 1, at node $g$. It uses these inputs to compute an approximation to the sum, $\text{gradU} := \sum_g \nabla f_g(U^{(g)}, B_g)$. We denote the computed approximation (output of AvgCons) at node $g$ by $\text{gradU}^{(g)}$. Each node then uses this in a GD step
\[ \bar{U}^{(g)}_{+} \leftarrow U^{(g)} - \eta \text{gradU}^{(g)} \]
followed by using QR decomposition to get $U^{(g)}_{+}$
\[ U^{(g)}_{+} \equiv U^{(g)} R^{(g)} + \]
This algorithm is summarized in Algorithm 2. We provide a guarantee for the GDmin steps in Theorem 3.3. In the rest of this paper, for notational simplicity, we remove the subscript $U$ from the gradient notation. All gradients are always w.r.t. the first argument of $f(\cdot)$.

Decentralizing Initialization. Since $f(U, B)$ is not a convex function of the unknowns $\{U, B\}$, a careful initialization is needed. We initialize the algorithm by computing $U^{(g)}_{0,\text{central}}$ as the top $r$ left singular vectors of
\[ X^{(g)}_{0,\text{central}} := \frac{1}{m} (A_k^\top y_{1,\text{trunc}}(\alpha)), \ldots, \frac{1}{m} (A_k^\top y_{k,\text{trunc}}(\alpha)), \]
where $\alpha := \tilde{C} \sum_{k \in [m]} (y_k)^2$ and $y_{k,\text{trunc}}(\alpha) := y_k \frac{1}{\sqrt{\alpha}} 1_{\{y_k \leq \sqrt{\alpha} \}}$ and $\tilde{C} = 9k^2 \mu^2$. Using $e_k$'s, this can be expressed as $X^{(g)}_{0,\text{central}} = (1/m) \sum_{k \in [m]} (y_k) \sum_{j=1}^m a_k y_{k} e_k \frac{1}{\sqrt{\alpha}} 1_{\{y_k^2 \leq \alpha \}}$. In the above, observe that we are summing $a_k y_{k} e_k$ over only those values $i$, $k$ for which $y_{k,i}$ is not too large (is not much larger than its empirically computed average value). This truncation filters out the too large (outlier-like) measurements and sums over the rest. Theoretically, this converts the summands into sub-Gaussian r.v.'s which have lighter tails than the un-truncated ones which are sub-exponential [6]. The idea of “truncating” (zeroing out very large magnitude entries of the observation vector to make it lighter tailed) is borrowed from [46]; it was first used in the above fashion in [6]. We compute the threshold $\alpha$ using Algorithm 1. Next, each node $g$ truncates its $y_k$'s (zeros out entries with magnitude greater than $\sqrt{\alpha}$) to obtain $y_{k,\text{trunc}}$. This is then used to compute the $n \times \{S_g\}$ sub-matrix of $X_0$.
\[ (X_0)_g = \left[ \frac{1}{m} A_k^\top y_{k,\text{trunc}}(\alpha), k \in S_g \right] \]
The full matrix
\[ X_0 := [(X_0)_0, (X_0)_2, \ldots, (X_0)_g, \ldots, (X_0)_k, \ldots] \]
is not available at any one node. However, we do not need this full matrix, we only need its top $r$ singular vectors.

We use a consensus-based approximation of the power method (PM) [35, 47] to approximate the desired singular vectors. We proceed as follows. Using the same random seed, each node $g$ generates the same $U_{\text{init}}$ by generating an $n \times r$ matrix with i.i.d. standard Gaussian entries and orthonormalizing it. Because the same random seed is used, this matrix is the same at each node. This serves as the initialization for the PM, i.e., $U_{\text{0}}^g = U_{\text{init}}$. This same random seed assumption is made in all previous works that study decentralized PM, e.g., [36, 37]. Next, each node computes
\[ \tilde{U}^{(g)}_{+} = \text{AvgCons}_{\lambda G^{(g)} \nabla f_g, g' \in [L], G, T_{\text{cons}}} \]
This is the input to the AvgCons algorithm, which outputs at node $g$
\[ \tilde{U}^{(g)}_{+} = \text{AvgCons}_{\lambda G^{(g)} \nabla f_g, g' \in [L], G, T_{\text{cons}}} \]
This is an approximation to $\sum_g (X_0)(g \in S_g) \tilde{U}^{(g)}$. Finally, each node uses QR decomposition on $U_{\text{0}}^g$ to get $U_{\text{0}}^g$, i.e.,
\[ U_{\text{0}}^g \equiv U_{\text{0}}^g R_{\text{0}}^g \]
These steps are repeated $T_{\text{pm}}$ times. We summarize them in Algorithm 3. To argue that this approach provably works under the desired sample complexity, we need a $U_{\text{init}}$ that is such that the minimum singular value of $U_{\text{0}}^T U_{\text{init}}$ is lower bounded by a constant $c_1 < 1$; see Theorem 3.5. The above assumption is
Algorithm 3 Initialization - basic (one consensus loop).
1: Let \( y_k \equiv y^{(0)}_k, A_k \equiv A^{(0)}_k \) for all \( k \in [q] \).
2: \( \alpha^{(0)}_{\text{in}} \leftarrow 9n^2 \mu^2 \frac{1}{mq} \sum_{k \in S_g} \sum_{i=1}^{n} y_{k,i}^2 \) for all \( g \in [L] \).
3: \( \alpha^{(0)} \leftarrow \text{AvgCons}_{g}(\alpha^{(0)}_{\text{in}}, g' \in [L], G, T_{\text{con}}) \)
4: Let \( y_k \equiv y^{(0)}_k, A_k \equiv A^{(0)}_k \)
5: \( y_{k,\text{trnc}} := y_k \ominus I \{ |y_{k,i}| \leq \sqrt{\alpha^{(0)}_{\text{in}}} \} \) for all \( k \in S_g, g \in [L] \)
6: Compute \( (X_0)_{g,k} = \frac{1}{1 + A_k y_{k,\text{trnc}}, \ k \in S_g}, \) for all \( g \in [L] \)
7: Generate \( U_{\text{init}} \) with each entry i.i.d standard Gaussian (use the same random seed at all nodes) for all \( g \in [L] \)
8: Orthogonalize \( U_{\text{init}} \) using QR to get \( U_{\text{init}} \) for all \( g \in [L] \)
9: \( U^{(0)} \leftarrow U_{\text{init}} \)
10: for \( \tau = 1 \) to \( T_{\text{pm}} \) do
11: \( \hat{U}^{(g)}_{\text{in}} \leftarrow (X_0)_{g}^{\top} U^{(0)}, \) for all \( g \in [L] \)
12: \( \bar{U}^{(g)} \leftarrow \text{AvgCons}_{g}(\hat{U}^{(g)}_{\text{in}}, g' \in [L], G, T_{\text{con}}) \)
13: QR on \( \hat{U}^{(g)}_{\text{in}} \) to get \( U^{(g)} \) for all \( g \in [L] \)
14: end for
15: Output \( U_{0}^{(0)} \leftarrow U^{(0)} \)

needed to prove that consensus of \( \tilde{U}^{(g)} \)'s implies consensus of \( U^{(g)} \). Similar (in fact, stronger) assumptions are also made in [37] and in [36]. We can remove the need for this assumption by modifying our algorithm as follows: (i) Do the QR step only for node-1. (ii) Run a consensus loop to share \( U^{(1)} \) with all the nodes as follows: node-1 transmits \( U^{(1)} \), while all other nodes transmit a zeros' matrix. All nodes other than node-1 use the output of this consensus loop as their \( U^{(g)} \). At each iteration \( \tau \), this step guarantees consensus of \( U^{(g)} \) to error level \( \epsilon_{\text{con}} \). We summarize this modified algorithm in Algorithm 4. While this algorithm is more complicated (and requires all nodes to know which node is “node-1”, as considered in multi-agent controls [48]), obtaining a guarantee for it is simpler and holds without the above assumption; see Theorem 3.4.

The final output of either of the above initialization algorithms serves as the input to the AltGDmin iterations. We specify the complete Dec-AltGDmin algorithm in Algorithm 2. This uses sample-splitting which is a commonly used approach in the LR recovery literature [6], [17], [49], [43] to simplify the analysis. It helps ensure that the measurement matrices in each iteration for updating each of \( U \) and \( B \) are independent of all previous iterates. This allows one to use concentration bounds for sums of independent random variables to bound the error terms at each iteration.

III. MAIN RESULTS AND DISCUSSION
A. Main Result

Our main result, Theorem 3.1, stated next says the following. Consider Algorithm 2 with algorithm parameters (step size, \( \eta \), number of consensus iterations at each GDmin or PM iteration, \( T_{\text{con}}, \) total number of PM iterations for initializing AltGDmin, \( T_{\text{pm}}, \) and total number of GDmin iterations, \( T \) set as specified in it. Assume that the graph of the underlying communication network is connected, the right singular vectors of \( X^* \) satisfy the incoherence assumption (Assumption 1.1), and the \( A_k \)'s are i.i.d. random Gaussian \( m \times n \) matrices. If the number of samples per column, \( m \), satisfies the specified lower bounds, then, with high probability (whp), the subspace estimation error after \( T \) GDmin iterations, \( \text{SD}_2(U^{(g)}_{\text{in}}, U^*) \), is at most \( \epsilon_{\text{in}} \). The same is true for the normalized error in estimating \( x^*_k \'s \).

Below we present our main result. Recall, \( c, C \) denote different numerical constants in each use with \( c < 1 \) and \( C \geq 1 \). Also recall \( W \) is the weighted adjacency matrix of \( G \) defined in (6). From Eq. (7) \( \gamma(W) = \max(|\lambda_2(W)|, |\lambda_L(W)|) \).

Theorem 3.1. Assume that Assumptions 1.1, 1.2, 1.3 hold. Consider Algorithm 2 initialized using Algorithm 4. Let \( \eta = 0.4/m\sigma^2_{\text{max}} \), \( T_{\text{pm}} = Ck^2(\log n + \log \kappa, T = Ck^2 \log (1/\epsilon_{\text{fin}}), T_{\text{con}} = C(\log(1/\gamma(W)))^{(T + \log(1/\epsilon_{\text{fin}})) + \log n + \log L + \log \kappa). \)

Assume \( m \geq Ck^6 \mu^2(n + q)r (k^2r + \log(1/\epsilon_{\text{fin}})) \).

Then, w.p. at least \( 1 - 1/n, \)
\[ \text{SD}_2(U^{(g)}_{\text{in}}, U^*) \leq \epsilon_{\text{fin}}, \text{ and } \|x_k - x^*_k\| \leq \epsilon_{\text{fin}}\|x^*_k\| \]
for all \( k \in S_g, g \in [L] \).

The above guarantee applies also to Algorithm 2 initialized using Algorithm 3 if the following additional assumption holds: \( \sigma_{\text{min}}(U^* U_{\text{init}}^{-1}) \geq c_1 \) for a constant \( c_1 < 1 \).

The above result is what is often referred to as a non-asymptotic or constructive convergence guarantee: the specified values of \( T, T_{\text{con}} \) enable us to also provide a bound on the time and the communication complexity of Algorithm 2. We obtain these below in Sec. III-B.

Consider the sample complexity (the required lower bound on \( m \)). To understand it simply suppose that \( n \approx q \). Ignoring log factors, and treating \( \kappa, \mu \) as numerical constants, our result implies that roughly order \( r^2 \) samples suffice per column. If the low rank (LR) assumption and our algorithm were not used, we would have to invert each \( A_k \) to recover each \( x^*_k \) from \( y_k \); this would require \( m \geq n \) samples per column instead of just \( r^2 \). Under the LR assumption (\( r \ll n \), e.g., \( r = \log n \)), our required sample complexity is much lower than what inverting \( A_k \) would require.

B. Time and Communication Complexity

The time complexity is \( (T_{\text{pm}} \cdot T_{\text{con}}) \cdot \mathcal{O}_\text{init} + (T \cdot T_{\text{con}}) \cdot \mathcal{O}_\text{GD} \)
where \( \mathcal{O}_\text{init} \) and \( \mathcal{O}_\text{GD} \) are the time needed by one consensus iteration of the initialization or the GDmin steps. In each GDmin consensus iteration, we need to (i) compute \( A_k U \) for
all $k \in S_g$, $g \in [L]$, (ii) solve the LS problem for updating $b_k$ for all $k \in S_g$, $g \in [L]$, and (iii) compute the gradient w.r.t. $U$ of $f_k(U, B)$, i.e., compute $A_k'(A_kUb_k - y_k)b_k'$ for all $k \in S_g$, $g \in [L]$, and (iv) implement the GD step followed by a QR decomposition at all the nodes. The QR of an $n \times r$ matrix takes time of order $nr^2$ and this needs to be done at all the $L$ nodes. Recall that $\sum_g |S_g| = q$. Thus, order-wise, the time taken per iteration is $\omega_{GD} = \max(q \cdot mnq, q \cdot nr^2, q \cdot mnq, L \cdot nr^2) = mnqr$ since $L \leq q$. In the initialization step, the computation cost for computing the threshold $\alpha$ is negligible since it is a scalar. One PM iteration needs time of order $\omega_{init} = mnqr$. Thus, the time complexity is $T_{con} = (T + T_{pm})mnq = C\kappa^4 T \log(1/T) \max\{\log^2(1/\epsilon_m), \log^2 n, \log^2 \kappa, \log^2 L\}mnqr$ per iteration per node is $\omega_{GD} \cdot \omega_{init}$.

In all consensus iterations, the nodes are exchanging approximations to the gradient of the cost function w.r.t. $U$. This is a matrix of size $n \times r$. In one such iteration, each node receives $nr$ scalars from its neighbors. Thus the communication complexity per iteration per node is $nr \cdot (\max_x d_q)$ where $(\max_x d_q)$ is the maximum degree of any node. The same is true for the initialization too. Thus the cost per iteration for all the nodes is $nr \cdot (\max_x d_q) \cdot L$ and so the overall communication cost is $T_{con} (T + T_{pm})nr \cdot (\max_x d_q) \cdot L$.

**Corollary 3.2.** The time complexity of Algorithm 2 is $C\kappa^4 T \log(1/T) \max\{\log^2(1/\epsilon_m), \log^2 n, \log^2 \kappa, \log^2 L\}mnqr$. Its communication complexity is $C\kappa^4 T \log(1/T) \max\{\log^2(1/\epsilon_m), \log^2 n, \log^2 \kappa, \log^2 L\} \cdot nr \cdot (\max_x d_q)$.

Assuming that the number of nodes $L$ is small (is a numerical constant, i.e., it does not grow with $n, q, r$), our time complexity is nearly linear in the problem size $mnq$, thus making the algorithm fast. The communication complexity per node per iteration is order $nr$ which is the size of $U$ that needs to be shared. Thus, this cannot be improved any further.

**C. Proof of Theorem 3.1: GDmin and initialization guarantees**

We can prove the following results for the GD and minimization (GDmin) steps, and the initialization respectively. Combining these proves Theorem 3.1. The GDmin result says the following. If the number of samples per column, $m$, satisfies the specified lower bounds, if the initialization error is small enough, and if the initial estimates of $U^*$ at the different nodes are also close entry-wise (and hence in Frobenius norm), then, with high probability (whp), the subspace estimation error $SD_2(U_t^{(g)}, U^*)$ decays exponentially. The same is true for the normalized error in estimating $x_k^{(g)}$.

The initialization result helps prove that the initial estimates do indeed satisfy the required bounds. Combining the two results then proves Theorem 3.1.

**Theorem 3.3 (GDmin iterations).** Assume that Assumptions 1.1, 1.2, 1.3 hold. Consider Algorithm 2. Pick a desired error $\epsilon_m < 1$. Let $\eta = c_n/\max_n^{\kappa^2} 2$ with $c_n = 0.4$, $T = C\kappa^2 T \log(1/\epsilon_m)$, and $T_{con} = C\kappa^2 T \log(1/T) \max\{\log L + T + \log(1/\epsilon_m) + \log \kappa\}$. Assume that, at each iteration $t$,

$$mq \geq \kappa^4 \mu^2 nr, \; m \geq \max\{\log n, \log q, r\}.$$  

If $SD_2(U_t^{(g)}, U^*) \leq \delta_t = \frac{c}{\sqrt{\kappa^2}}, \; \text{and} \; \max g \neq g \|U_t^{(g)} - U_t^{(g')}\|_F \leq \rho_0 = \frac{c_n T \epsilon_m}{\kappa^2},$ then, w.p. at least $1 - Tn^{-10}$, at any iteration $t \leq T$,

$$SD_2(U_t^{(g)}, U^*) \leq \delta_t := (1 - \frac{c_n}{\kappa^4})^t \delta_0, \; \|x_k^{(g)} - x_k^{(g')}\| \leq \delta_t \|x_k^{(g)}\|,$$

for all $k \in S_g, g \in [L]$.

**Proof.** See Sec. IV and V.

**Theorem 3.4 (Initialization).** Assume that Assumptions 1.1, 1.2, 1.3 hold. Pick a $\delta_0 < 1$ and a $\rho_0 < 1$. Consider Algorithm 4. If $T_{con} = C\kappa^2 T \log(1/T) \max\{\log L + T + \log(1/\delta_0) + \log(1/\rho_0)), T_{pm} = C\kappa^2 \log(1/\delta_0)$, and

$$mq \geq \kappa^4 \mu^2 (n + q) r \frac{1}{\delta_0},$$

then, w.p. at least $1 - 1/n$,

$$SD_2(U_0^{(g)}, U^*) \leq \delta_0, \; \max g \neq g \|U_0^{(g)} - U_0^{(g')}\|_F \leq \rho_0.$$  

**Proof.** See Appendix C.

Observe that our initialization step (Algorithm 4) requires two consensus loops in each power method (PM) iteration. The second loop requires all nodes to know which node is “node-1”. This is not a big limitation, this type of assumption is made in other works on multi-agent control too [48]. We can remove the need for this second consensus loop (and hence the need for knowing which node is “node-1”) if we assume that the matrix used to initialize the PM, $U_{init}$, is such that the cosine of the largest principal angle between its column span and that of $U^*$ (true subspace) is at least a constant. We should mention that a stronger assumption than this is used in previous works on decentralized PCA as well [36], [37].

**Theorem 3.5 (Initialization - basic (single consensus loop)).** Assume everything assumed in Theorem 3.4. Consider Algorithm 3. If $\sigma_{min}(U^* U_{init}) \geq c_1$ for a numerical constant $c_1 < 1$, and if $mq \geq \kappa^4 \mu^2 (n + q) r \max(r, 1/\delta_0)$ then w.p. at least $1 - 1/n$, the conclusions of Theorem 3.4 hold.

**Proof.** See Appendix D.

**Remark 3.6.** If $U_{init}$ is generated as a random subspace (e.g., by orthonormalizing an $n \times r$ random Gaussian matrix), we cannot prove that $\sigma_{min}(U^* U_{init}) \geq c$. All we can show is that this singular value is lower bounded by $c/\sqrt{nr}$ using [51, Theorem 1.1]. However, it may be possible to get an initialization that satisfies the above requirement in one of two ways. (i) If other prior information is available, e.g., in a subspace tracking setting, one could get a $U_{init}$ with the desired property. (ii) Alternatively it may be possible to design a rejection sampling scheme in which one generates multiple random initializations $U_{init}$ until one of them is such that projecting $X_0$ orthogonal to it results in a matrix with sufficiently smaller Frobenius norm than that of $X_0$, i.e. it satisfies $\| (I - U_{init}) X_0 \|_F^2 < (1 - c_1) \| X_0 \|_F^2$. The LHS and RHS can both be computed using a scalar consensus loop to approximate the sums $\sum_g \| (I - U_{init}) X_0 \|_F^2$ and $\sum_g \| X_0 \|_F^2$, respectively.
Proof of Theorem 3.1. To combine our guarantee for the GDmin iterations with that for initialization, we need to set
\[ \delta_0 = \frac{c_0}{\sqrt{\tau} \kappa^2}, \quad \rho_0 = \frac{c^T \varepsilon_{\text{in}}}{\kappa^2}. \] (8)

This means, for initialization, we need \( mq \gtrsim n^8 \mu^2 (n + q) \tau^2 := m_{\text{in}} q, T_{\text{con}} = C \log(1/L) \log(\log L + T + \log(1/\varepsilon_{\text{in}}) + \log n + \log \kappa), \) and \( T_{\text{pm}} = C \kappa^2 \log(n \sqrt{\tau} \kappa^2) = C \kappa^2 \log(n \sqrt{\tau} \kappa^2) = C \kappa^2 (\log n + \log \kappa) \) since \( r \leq n \). GDmin iterations need \( mq \gtrsim n^8 \mu^2 \tau r := m_{\text{GD}} q \). Thus, the total number of samples \( mq \) that is needed is \( mq \gtrsim m_{\text{GD}} q + T m_{\text{GD}} q \). By plugging in \( T = C \kappa^2 \log(1/\varepsilon_{\text{in}}) \), we get the sample complexity of Theorem 3.1. The expression for \( T_{\text{con}} \) is also as given above.

D. Related Work Discussion

Centralized LRcCS. An alternating minimization (AltMin) solution for solving its generalization, LR phase retrieval, and hence also LRcCS itself, was developed and studied in [4], [39]. A convex relaxation (mixed norm minimization) for solving LRcCS was proposed and studied in [5]. Solving a convex relaxation is very slow (both theoretically and empirically). Altmin is faster but not as fast as GD-based solutions. Other existing algorithms in LR literature that can solve LRcCS include projected GD (ProjGD) introduced in [43] and alternating GD (AltGD) introduced in [42]. ProjGD is not communication efficient, whereas AltGD is slow. In recent work [6], a GD-based solution, called AltGDmin, was developed. This was shown to be faster than the Altmin solution, and much faster than the convex relaxation.

Other works on decentralized LR matrix recovery. There has been some work on decentralized LRMC which is a related but different problem [52], [53], [54]. [52], [53] designs a decentralized Gauss-Seidel method, while [54] develops a decentralized approach to solve the convex relaxation of robust LRMC. Both these approaches are slower than GD based solutions; and neither comes with provable guarantees. In a different work [55], the global geometry of the cost function for decentralized LRMS (which is a very different problem than LRcCS, since it involves recovery from linear projections of entire matrix \( X^* \)) was characterized. Recently, LRcCS has been studied in the decentralized setting in three conference papers [1], [2], [3] in which we introduced the algorithm or provided a theoretical guarantee and its improvement (proof were not provided and very limited simulations were done).

Consensus-based (gossip) algorithms. There is a large amount of literature on using multiple consensus iterations to approximate the gradient (or related quantities such as the Hessian weighted gradient used in Newton Raphson) [40], [34] for both unconstrained and constrained optimization problems. However there are a few differences. Most of these works study a general convex optimization problem, and hence can only prove convergence to a minimizer; and consequently the claims that can be proved are weaker. When non-convex problems are studied, then only convergence to a stationary point can be proved. Also almost all guarantees provided are asymptotic and require many assumptions.

A second difference is that all works consider decentralized versions of optimization problems of the form
\[ \min_{\theta} \sum_{g=1}^{L} f_g(\theta) \]

On the other hand, our works consider a specific optimization problem that needs to be solved to solve the LRcCS problem:

\[
\min_{U, B : U^\top U = I} \sum_{g=1}^{L} \sum_{k \in S_g} ||y_k - A_k U b_k||_2^2
\]

Observe that, in our problem, \( U \) is a global variable, while \( B \) is not: \( B = [B_1, B_2, \ldots, B_g, \ldots, B_L] \) and its sub-matrices \( B_g \) only occur in \( f_g(.) \). Thus these can be updated locally at respective nodes \( g \). This simplifies our algorithm (only gradients w.r.t. \( U \) need to be shared), but this also means that the proof requires more work. Also, our goal is to recover the unknown LR matrix \( X^* \); consequently we need our developed algorithm to converge to this specific minimizer. For this, (i) our algorithm needs a careful decentralized initialization; and (ii) our guarantee needs to a lower bound on the required sample complexity.

Lastly, even though our optimization problem is an instance of nonlinear LS (NLLS), it is not easy for us to use the standard optimization approach for NLLS: this involves obtaining a closed form expression for \( B_k : b_k = (A_k U)^\dagger y_k \) and substituting this into the cost function to get a new cost function that depends only on \( U \). One then develops a GD or Newton Raphson algorithm to minimize the new cost which in our case will be
\[
\min_{U, B : U^\top U = I} \sum_{g=1}^{L} \sum_{k \in S_g} ||I - A_k U (A_k U)^\dagger y_k||_2^2
\]

over only \( U \). However, it is not clear how to compute the gradient w.r.t \( U \) of the above in closed form. Even if one could, analyzing the resulting algorithm would be extremely difficult. Consequently we cannot borrow the decentralizing approach developed for NLLS in [40].

Initialization and decentralized PM. Our initialization step involves solving a decentralized PCA problem (computing top \( r \) singular vectors of matrix). There has been some existing works on decentralized PCA [36], [37], [38]. We can use any of these results to provide a very short proof of our initialization guarantee. However each of these results have limitations that we describe next. The result of [36] (Theorem 1) requires an assumption on intermediate algorithm outputs: it requires the minimum singular value of \( \hat{U}_r \) to be lower bounded by \( c \) at each iteration \( \tau \). The result of [37] (Theorem 1) has a similar issue, its required number of consensus iterations depends on a parameter \( \beta = \min_{s} \sigma_{\min}(U_s) \). Recall that \( U_{\tau} = X_0 X_0^\top U_{\tau-1} \). Both these are stronger requirements than what our initialization guarantees need. The work of [38] studies a subspace tracking based improvement of decentralized PM and shows that the required \( T_{\text{con}} \) for each iteration of this algorithm does not grow with the desired final error, \( \delta_0 \). But, instead, their \( T_{\text{con}} \) grows with \( 1/\sigma_{r+1}(\Phi) \) where \( \Phi \) is the matrix whose singular vectors are being approximated; see Eq. (3.11) of [38]. This means, it requires \( \sigma_{r+1}(\Phi) \) to be lower bounded. This
requirement is hard to interpret, since a larger $\sigma_{r+1}$ will imply smaller singular value gap.

**Our GDmin guarantee vs that for DGD from [18], [22].**

Consider our GDmin iterations’ guarantee. It requires the number of consensus iterations for each gradient computation, $T_{\text{con}}$, to be proportional to $\log(1/\epsilon_{\text{fin}})$ where $\epsilon_{\text{fin}}$ is the desired final error. Thus, one needs to know the desired level of the final error before starting the algorithm. Said another way, the choice of $T_{\text{con}}$ governs how accurate the final estimates can be. The advantage of our approach is that the total number of GD iterations required, $T$, also grows only logarithmically with $1/\epsilon_{\text{fin}}$. This result needs the total number of GD iterations, $T \cdot T_{\text{con}}$, and this depends linearly on $\log(1/\epsilon_{\text{fin}})$. In comparison, consider the DGD approach of [18], [22] which uses $T_{\text{con}} = 1$. The guarantees provided in [18] are asymptotic. The later result provided by Yuan et al in [22] is non-asymptotic. This result assumes a convex cost function and $\epsilon_{\text{fin}}$ is the desired distance of the final estimate from a stationary point of the cost function (minimizer since it is convex). We study a specific problem and our cost $f(U, B)$ is not convex. But since there is no other existing result for our specific problem, we compare with this result. This result needs the total number of GD iterations $T$ to grow linearly with $\frac{\eta}{\eta T}$, and needs $\eta < 1/L_h$. Here $L_h$ is the maximum of the Lipschitz constants of the cost functions at the different nodes. Consequently, even though DGD uses $T_{\text{con}} = 1$ in each GD iteration, because $T$ needs to grow linearly with $1/\epsilon_{\text{fin}}$, thus $T \cdot T_{\text{con}}$, also grow linearly with $1/\epsilon_{\text{fin}}$. Order-wise, this is a much larger total number of iterations than $\log^2(1/\epsilon_{\text{fin}})$ which is what our approach needs. Our simulation experiments demonstrate this fact, see Sec. VI.

**Gradient and subspace tracking.** It may be possible to improve our algorithm and guarantees for it (reduce the required value of $T_{\text{con}}$) by borrowing the gradient tracking ideas from [24] to modify our GD step and by using the subspace tracking ideas from [38] to modify our initialization.

**Decentralized non-convex algorithms [56], [56].** [56] studied a randomized incremental gradient method for decentralized unconstrained minimization of a non-convex cost function that is smooth and that satisfies the Polyak Łojasiewicz (PL) condition. The PL condition is a generalization of strong convexity to non-convex functions (if a function satisfies PL condition, then all its stationary points are global minimizers). Almost sure and mean-squared convergence are proved. In addition, the number of iterations required to reach within an $\epsilon$-neighborhood of a stationary point is provided. This result too shows that this number grows linearly with $1/\epsilon$.

**E. Proof novelty**

Because we study a decentralized implementation of Alt-GDmin for the LRecCS problem, the gradient estimate, $\mathbf{gradU}$ that is used in each GD iteration, is a consensus-based approximation of $\sum_{\mathbf{g}} \nabla f(U^{(g)}, B_g)$; notice that $U$ is different at each node $g$. This simple change implies that our analysis approach needs multiple important changes from the centralized case that was studied in [6].

(1) We cannot use the fundamental theorem of calculus approach used in the centralized setting [6] because the Hessian w.r.t. $U^{(g)}$ of $\sum_{\mathbf{g}} f^{(g)}(U^{(g)}, B_g^*)$ cannot be computed. We instead need to use a different proof approach, which, in fact, provides an alternate easier proof even for the centralized setting. See Sec. V.

(2) In order for our proof to work, we need to ensure that the estimates $U^{(g)}$ at the different nodes are close enough entry-wise, i.e., we need $\max_{g \neq g'} \| U^{(g)} - U^{(g')} \|_F$ to be small, at each iteration. This we henceforth refer to this as “consensus for $U^{(g)}$s”. We can obtain this bound as follows: (i) Use the average consensus guarantee [41] (Proposition 2.1) to argue that the gradient estimates at the different nodes are close at each iteration. This, along with assuming that the initial estimates $U_0^{(g)}$ are close, then implies that the same holds for $U^{(g)}$s at the different nodes. (ii) We need to use a result for perturbed QR decomposition [57], [58] (Proposition B.3) to argue that this implies a similar bound on $\| U^{(g)} - U^{(g')} \|_F$.

(3) The initialization step needs to be different from the centralized setting because there is no central coordinating node and because we want to design an algorithm with small communication complexity in each iteration. (i) We need an approach in which the number of scalars to be shared per iteration is of order $nr^2$ per node. This means we cannot run a consensus algorithm for summing/averaging $(X_0)_x (X_0)_x^T$. We thus use a decentralized version of the power method (PM). We analyze this by interpreting it as an instance of the noisy PM meta algorithm of [47]. (ii) Also, in obtaining the “consensus for $U^{(g)}$s” bound for the GDmin steps, we assumed that the initial estimates $U_0^{(g)}$ at the different nodes are close. We need to prove this while analyzing the PM.

IV. PROOF: MINIMIZATION STEP

A. Definitions

Recall that

\[
B = [B_1, B_2, \ldots, B_L], \quad \text{with } B_g := [b_k, \ k \in S_g] \\
X = [X_1, X_2, \ldots, X_L], \quad \text{with } X_g := U^{(g)} B_g
\]

and $x_k = U^{(g)} b_k$ for all $k \in S_g$. Define $g_k := U^{(g)}^T x_k^*$, for all $k \in S_g$ and

\[
G = [G_1, G_2, \ldots, G_L], \quad G_g := [g_k, \ k \in S_g] = U^{(g)}^T X_g^*.
\]

B. Main Claim

In this section, we prove the following result for the minimization step. These results get used in the next section when analyzing the GD step.

**Theorem 4.1** (Min step for $B$). Assume that, for all $g$, $SD_2(U^{(g)}, U^*) \leq \delta_t$, for some $\delta_t \geq 0$. Then, $w_p \geq 1 - \exp(\log q + r - cm)$

1) $\|b_k - g_{k^*}\| \leq 0.4\delta_t \|b_{k^*}\|$

2) $\|b_k\| \leq \|g_{k^*}\| + 0.4 \cdot 0.02 \|b_{k^*}\| \leq 1.1 \|b_{k^*}\|$

3) $\|x_k - x_{k^*}\| \leq 1.4\delta_t \|b_{k^*}\|$

4) $\|B - G\|_F \leq 0.4\delta_t \|B^*\|_F \leq 0.4\sqrt{r} \delta_t \sigma_{\max}^*$

5) $\|X - X^*\|_F \leq 1.4\sqrt{r} \delta_t \sigma_{\max}^*$
If $\delta_t \leq 0.02/\sqrt{K^2}$ and if $\max_{g \neq g'} \|U^{(g)} - U^{(g')}\|_F \leq \rho_t$ with $\rho_t < 0.1\delta_t$, then the above implies that

a) $\sigma_{\min}(B) \geq 0.9\sigma^*_\min$ and

b) $\sigma_{\max}(B) \leq 1.1\sigma^*_\max$.

Further, if $m \geq \max(\log q, \log n, r)$, then the above conditions hold with probability $1 - n^{-10}$.

**Proof.** Proof of Theorem 3.3 is given in Appendix A.

V. PROOF OF THEOREM 3.3: GD step

A. Definitions

Define

$$\nabla f_g := \nabla f_g(U^{(g)}, B_g) := \sum_{k \in S_g} \nabla U f_k(U^{(g)}, b_k)$$

$$\nabla \nabla g := \text{AvgCons}_g(\nabla f_g(U^{(g)}, B_g), \; g' \in [L])$$

$$\text{gradU} := \sum_{g'} \nabla f_{g'}(U^{(g')}, B_{g'})$$

$\nabla f_g$ is the input to the AvgCons algorithm at node $g$ and $\text{gradU}$ is an estimate of $\text{gradU}$ obtained by running $T_{\text{con}}$ iterations of the consensus algorithm AvgCons at node $g$.

Define the error terms

$$\text{ConsErr}^{(g)} := \text{gradU} - \text{gradU}^{(g)}$$

$$\text{Err} := \mathbb{E}[\text{gradU}] - \text{gradU}$$

$$\text{InpErr}^{(g)} := \sum_{g'} \max_{U^{(g')} \neq U^{(g)}} \|U^{(g')} - U^{(g')}\|_F$$

and define $P_{U^\tau} := I - U^\tau U^\tau$.

B. Main Claims

In this section, we prove the following result. This uses Theorem 4.1 from the previous section in its proof.

**Theorem 5.1** (GD step and consensus for $U$). Assume that $\eta = c_\eta/m\sigma^*_{\min}^2$ with $c_\eta \leq 0.5$. Assume that $\text{SD}_2(U^{(g)}, U^*) \leq \delta_t$, $\|U^{(g)} - U^*\|_F \leq \rho_t$, $\|\text{ConsErr}^{(g)}\|_F \leq m\epsilon_{\text{con}}\sigma^*_{\min}$.

If $\delta_t \leq 0.02/\sqrt{K^2}$, $\rho_t < 0.1\delta_t\epsilon_{\text{con}} < 0.1\delta_t$, and if $mq \geq C\epsilon_{\min}^2 nr$ and $m \geq \max(\log n, \log q, r)$, then, w.p. at least $1 - n^{-10}$

$$\text{SD}_2(U^{(g)}, U^*) \leq \delta_{t+1} := \delta_t(1 - 0.6 c_\eta K^2)$$

and

$$\max_{g \neq g'} \|U^{(g')} - U^{(g')}\| \leq \rho_{t+1} := 1.7(\rho_t + 3K^2 c_\eta)$$

**Proof.** See Appendix B-A in supplementary material.

The next lemma bounds the consensus error.

**Lemma 5.2.** Assume that, for all $g$, $\text{SD}_2(U^{(g)}, U^*) \leq \delta_t$ and $\max_{g \neq g'} \|U^{(g')} - U^{(g')}\| \leq \rho_t$. Also, $mq \geq C\epsilon_{\min}^2 nr$. If $T_{\text{con}} = \frac{\log(1/\gamma(W))}{2\log(L/\epsilon_{\text{con}})}$, then, w.p., $\|\text{ConsErr}^{(g)}\|_F \leq \epsilon_{\text{con}}\sigma^*_{\min}$

**Proof.** See Sec. V-C.

**Proof of Theorem 3.3.** By simplifying the recursion of Theorem 5.1, observing that $\rho_t$ increases with $t$ and $\delta_t$ decreases with $t$, and using Lemma 5.2, we obtain Theorem 3.3. See Sec. V-D for the complete proof.

C. **Proof of Lemma 5.2:** Bounding ConsErr

**Proposition 5.3.** If

$$T_{\text{con}} \geq \frac{3}{2} \frac{1}{\log(1/\gamma(W))} \log(L/\epsilon_{\text{con}}) = C \frac{1}{\log(1/\gamma(W))} \log(L/\epsilon_{\text{con}}),$$

then

$$\|\text{ConsErr}^{(g)}\|_F \leq \epsilon_{\text{con}} \max_g \|\text{InpErr}^{(g)}\|_F$$

Here $\text{InpErr}^{(g)}$ is the error between input of node $g$ to the consensus algorithm and the desired sum to be computed and $\text{ConsErr}^{(g)}$ is the error between the final output at node $g$ and $T_{\text{con}}$ iterations and the desired sum.

**Proof.** Recall that $\text{ConsErr}^{(g)} := \text{gradU}^{(g)} - \text{gradU}$ and $\text{gradU}$ is the output of our AvgCons algorithm applied to each entry of $\nabla f_g$. Define

$$\text{InpErr}^{(g)} := \nabla f_g - \text{gradU} = \sum_{g' \neq g} \nabla f_{g'}$$

with $\nabla f_g$ being the partial gradient sum at node $g$. This is the input to the AvgCons algorithm.

We bound $\text{ConsErr}$ in terms of $\text{InpErr}$ by using Proposition 2.1 for scalar consensus. We apply this result to each of the $nr$ scalar entries of $\nabla f_g$. Thus, for the $(j, j')$-th entry, $z_{\text{true}}^{(g)}_{j, j'} = (\text{gradU})_{j, j'}$, $z_{\text{in}}^{(g)}_{j, j'} = (\nabla f_g)_{j, j'}$, and $z_{\text{out}}^{(g)}_{j, j'} = (\text{gradU})_{j, j'}$. Assume the graph of the network is connected. Using this result, if $T_{\text{con}} \geq \frac{1}{\log(1/\gamma(W))} \log(L/\epsilon_{\text{con}})$, then

$$\max_g \|\text{ConsErr}^{(g)}\|_F \leq \epsilon_{\text{con}} \max_g \|\text{InpErr}^{(g)}\|_F$$

Using $\sum_{j, j'} max_g \|\text{InpErr}^{(g)}_{j, j'}\|^2 \leq \sum_{j, j'} \sum_g \|\text{InpErr}^{(g)}_{j, j'}\|^2 = \sum_g \|\text{InpErr}^{(g)}\|^2_F \leq L m max_g \|\text{InpErr}^{(g)}\|^2_F$

$$\max_g \|\text{ConsErr}^{(g)}\|_F \leq \epsilon_{\text{con}} \sqrt{L} \max_g \|\text{InpErr}^{(g)}\|_F$$

By setting $\epsilon_{\text{con}} = \epsilon_{\text{con}} / \sqrt{L}$, we prove the result.

The next step is to obtain a bound on $\text{InpErr}^{(g)}$ and we use the sub-exponential Bernstein inequality followed by a standard epsilon-net argument. To bound $\mathbb{E}[\|\text{InpErr}^{(g)}\|_F]$, we also need the following simple claim.

**Claim 5.4.** If $M'$ is $M$ with some columns removed, then $\|M'\| \leq \|M\|$.

**Proof of Lemma 5.2.** For a matrix $Z$, let $Z_{ij}$ denotes a submatrix of $Z$ obtained after eliminating the columns $k \in S_g$. Using Claim 5.4 and bounds on $\|B\|$ and $\|X^* - X\|_F$ from Theorem 4.1, if $\delta_t < \frac{c}{\sqrt{K^2}}$

$$\|\mathbb{E}[\text{InpErr}^{(g)}]\|_F = \| \sum_{g' \neq g} \sum_{k \in S_g} m(x_k - x_k^*) b_k^T \|_F$$

$$= m \| (X_{ij} - X_{ij}^*) B_{ij}^T \|_F \leq m \| (X_{ij} - X_{ij}^*) F \cdot \|B_{ij}^T \| \leq m \| X - X^* \|_F \cdot \|B\| \leq 1.1 m \delta_t \sqrt{K^2} \sigma^*_{\max}.$$

w.p. $1 - \exp(\log q + r - cm)$. The last step used Theorem 4.1.
Proceeding exactly as in the proof of Lemma B.1, we can show that, if \( \delta_t \leq c/\sqrt{\kappa} \), \( \rho_t \leq 0.1\delta_t/\kappa^2 \), and \( m \) satisfies the bounds stated in that lemma, then,

\[
\max_g \| \text{InpErr}^{(t)} - \mathbb{E}[\text{InpErr}^{(g)}] \| \leq 1.1c_1 \delta_t m \sigma_{\min}^* \cdot \kappa^2
\]

w.p. at least \( 1 - \exp(C(n + r) - \alpha c_t^2 m \sigma^*/\kappa^4 \mu^2 r) - \exp(\log q + r - cm) \).

Thus, with the above probability, \( \| \text{InpErr}^{(t)} - \mathbb{E}[\text{InpErr}^{(g)}] \|_F \leq \sqrt{r} \cdot 1.1c_1 \delta_t m \sigma_{\min}^* \cdot \kappa^2 \). Setting \( c_1 = 0.1 \), we conclude that w.p. at least \( 1 - L \exp(C(n + r) - c_{\min}^* m \sigma_{\max}^* \cdot \kappa^2) - \exp(\log q + r - cm) \),

\[
\max_g \| \text{InpErr}^{(t)} \|_F \leq (1.1+0.11)m(\delta_t \sqrt{r}) \sigma_{\max}^* \cdot \kappa^2 \leq 0.1 m \sigma_{\min}^* \cdot \kappa^2
\]

if \( \delta_t \) and \( \rho_t \) are bounded as stated above. Combining this with Proposition 5.3, we have proved the lemma. \( \square \)

D. Proof of Theorem 3.3: combine Theorem 5.1 and Lemma 5.2

Simplifying the recursion of Theorem 5.1,

\[
\rho_{t+1} = 1.7t^3 + \rho_0 + 3.3t^3 \cdot c_t \epsilon_{\text{con}} \quad \text{and} \quad \delta_{t+1} = (1 - 0.6c_t^2) \cdot \delta_t
\]

Since \( \delta_t \) decreases with \( t \) and \( \rho_t \) increases with \( t \), thus, the bounds of Theorem 5.1 hold if \( \delta_0 \leq c/\sqrt{\kappa} \), \( \rho_T \leq 0.1 \delta_T \cdot \kappa^2 \), and \( \epsilon_{\text{con}} < 0.1 \delta_T \cdot \kappa^2 \). Using the expressions above, in order to ensure \( \delta_T = \epsilon_{\text{in}} \), we need

1. \( T \geq C \sigma_{\min}^2 \log(1/\epsilon_{\text{in}}) \),
2. \( \delta_0 \leq c/\sqrt{\kappa} \),
3. \( \epsilon_{\text{con}} < 0.1 \epsilon_{\text{in}} \cdot \kappa^2 \),
4. \( \rho_T = 1.7t^3 \rho_0 + 3.3t \cdot T \cdot \delta_0 \cdot \epsilon_{\text{con}} \leq 0.1 \epsilon_{\text{in}} \cdot \kappa^2 \).

This holds if

a) \( \rho_0 \leq c^2 \epsilon_{\text{in}} / \kappa^2 \), with \( c = 1/1.6 \), and
b) \( \epsilon_{\text{con}} \leq c^2 \epsilon_{\text{in}} / (T \kappa^2) \).

Setting \( c_t = 0.4 \) (or any constant \( \leq 0.5 \)), we thus need (1) \( T = C \sigma_{\min}^2 \log(1/\epsilon_{\text{in}}) \) and (2) \( \epsilon_{\text{con}} = c^2 \epsilon_{\text{in}} / (T \kappa^2) \).

By Lemma 5.2, if \( T_{\text{con}} = \frac{1}{\log(1/\gamma(W))} \log(L/\epsilon_{\text{con}}) \), then \( \| \text{ConsErr}^{(g)}(W) \|_F \leq \epsilon_{\text{con}} m \sigma_{\min}^* \cdot \kappa^2 \). Thus, to get \( \epsilon_{\text{con}} = c^2 \epsilon_{\text{in}} / (T \kappa^2) \), we need to set

\[
T_{\text{con}} = C \frac{1}{\log(1/\gamma(W))} \log(L + T + \log(1/\epsilon_{\text{in}}) + \log \kappa)
\]

By using this expression and the above and Theorem 5.1, we have proved Theorem 3.3.

Remark 5.5. We get the \( t_{\text{con}} \epsilon_{\text{con}} \) type factor in the expression for \( \rho_{t+1} \) given above because of the projected GD step. If there was no projection (simple GD), we would get a factor of only \( t_{\text{con}} \). This would eliminate the need for \( T_{\text{con}} \). To depend on \( T \), it would instead only depend on \( \log T \).
Additionally, we compare all cases studied in this experiment although error does go down a little but saturates at around 0.1.

In all cases $n = 600$, $r = 4$, $q = 600$, $p = 0.5$, and $L = 20$. We compared performance of our proposed algorithm (Dec-AltGDmin) with the centralized counterpart from [6] and with the LR matrix recovery algorithms, Altmin [4], [39], ProjGD [43], and AltGD [42] algorithms. The network was generated as an ER graph with $p = 0.5$ probability. In Figures 1a, 1b, we present the plots for error vs. execution time for $m = 50, 30$, respectively. For improved visualization, we have set the $x$-limit to 10 seconds. This choice aligns with the runtime of all algorithms except Altmin, which exceeds 1000 seconds. From Figures 1a, 1b, only Altmin, AltGDmin, and Dec-AltGDmin converges in both setting, however, the execution time for Altmin is too large as compared to the other two algorithms, thereby validating the effectiveness of the proposed decentralized approach and its centralized counter part.

![Fig. 1: In all cases $n = 600$, $r = 4$, $q = 600$, $p = 0.5$, and $L = 20$. We compared performance of our proposed algorithm (Dec-AltGDmin) with the centralized counterpart from [6] and with the LR matrix recovery algorithms, Altmin [4], [39], ProjGD [43], and AltGD [42] algorithms.](image1)

In Figure 2 we set $n = 600$, $m = 50$, $r = 2$, $q = 600$ and $p = 0.5$. In Figure 2a we set $L = 20$, and we compared Dec-AltGDmin with the centralized algorithm in [6], AltGDmin, and the three algorithms obtained by modifying the GD step in AltGDmin using approaches in [18], [22], as discussed in 2) and 3) above. In Figure 2a DGD-AltGDmin-rand refers to the setting described in 2-a), DGD-AltGDmin-zero refers to the setting described in 2-b), and DGD-AltGDmin-spect refers to the setting described in 2-c). From the plots we observe that our algorithm and AltGDmin converges, although our algorithm converges slower than AltGDmin. This is expected since Dec-AltGDmin requires $T_{con}$ iterations of AvgCons in every GD step, while in a centralized setting a central server performs the data aggregation. On the other hand, the errors for the three DGD algorithms are high. The key reason why the approach of [18], [22] and the follow-up works designed for standard GD cannot be used for updating $U$ is because it involves averaging the partial estimates $U^{(g)}$, $g \in [L]$, obtained locally at the different nodes. However, since $U^{(g)}$s are subspace basis matrices, their numerical average will not provide a valid “subspace mean” (see below). Moreover, the cost functions considered in [18], [22] are convex while LRcCS problem is non-convex.

In all cases $n = 600$, $m = 50$, $r = 2$, $q = 600$, and $L = 20$. We compared performance of our proposed algorithm (Dec-AltGDmin) with the centralized counterpart from [6] and with the LR matrix recovery algorithms, Altmin [4], [39], ProjGD [43], and AltGD [42] algorithms. The network was generated as an ER graph with $p = 0.5$ probability. In Figures 1a, 1b, we present the plots for error vs. execution time for $m = 50, 30$, respectively. For improved visualization, we have set the $x$-limit to 10 seconds. This choice aligns with the runtime of all algorithms except Altmin, which exceeds 1000 seconds. From Figures 1a, 1b, only Altmin, AltGDmin, and Dec-AltGDmin converges in both setting, however, the execution time for Altmin is too large as compared to the other two algorithms, thereby validating the effectiveness of the proposed decentralized approach and its centralized counter part.

![Fig. 2: In all cases $n = 600$, $r = 2$, $q = 600$, $m = 50$, and $p = 0.5$. We compare performance of our proposed algorithm (Dec-AltGDmin) with the centralized one from [6], with the three versions of AltGDmin modified using the DGD approach of [18], [22], and with AltGDmin run for a single node. In all figures, the network was generated as an ER graph with $p = 0.5$ probability of nodes being connected. For Figure 2a, we considered a large-sized problem and simulated until $T = 400$ total GDmin iterations. In this case all versions of DGD-based AltGDmin modification fail. Small network: $L = 2, 20$, $n = 600$, $r = 2$, $q = 600$, $m = 50$. Large network: $n = 100$, $r = 2$, $q = 100$, $m = 40$. For Figure 2b, we considered a smaller network with two agents and simulated until $T = 600$ total GDmin iterations. In this case all three algorithms converge. In order to test the performance of the algorithms for a larger network we considered a setting with 20 agents and tried the experiment for $T = 3000$ iterations as shown in Figure 2c: DGD [18], [22] and One-node-AltGDmin fail although error does go down a little but saturates at around 0.1.

![Fig. 2: In all cases $n = 600$, $r = 2$, $q = 600$, $m = 50$, and $p = 0.5$. We compare performance of our proposed algorithm (Dec-AltGDmin) with the centralized one from [6], with the three versions of AltGDmin modified using the DGD approach of [18], [22], and with AltGDmin run for a single node. In all figures, the network was generated as an ER graph with $p = 0.5$ probability of nodes being connected. For Figure 2a, we considered a large-sized problem and simulated until $T = 400$ total GDmin iterations. In this case all versions of DGD-based AltGDmin modification fail. Small network: $L = 2, 20$, $n = 600$, $r = 2$, $q = 600$, $m = 50$. Large network: $n = 100$, $r = 2$, $q = 100$, $m = 40$. For Figure 2b, we considered a smaller network with two agents and simulated until $T = 600$ total GDmin iterations. In this case all three algorithms converge. In order to test the performance of the algorithms for a larger network we considered a setting with 20 agents and tried the experiment for $T = 3000$ iterations as shown in Figure 2c: DGD [18], [22] and One-node-AltGDmin fail although error does go down a little but saturates at around 0.1.

no existing algorithms for solving LRcCS in a decentralized federated setting. Potential benchmarks to compare with are

1) DGD-basic: A naive application of the DGD idea is to use DGD to minimize $f(X)$ given in (4) without using the LR assumption at all (run a separate DGD for estimating each column $x_k^*$). This cannot correctly recover $x_k^*$ because we have $m < n$ and thus the least squares problem is under-determined. We do not show experiments for this case because it is clear it will fail to correctly recover $X^*$.

2) DGD-AltGDmin: modify the centralized AltGDmin algorithm as follows: replace the centralized projected GD step for $U$ by the DGD approach: $U_t^{(g)} \leftarrow Q R( \sum_{g' \in \mathcal{G}} U_t^{(g')} - \eta \nabla U_t f_g(U^{(g)}, B^g))$. To initialize this we can use one of three methods given below.

   a) use random initialization as suggested in [18]

   b) use zero initialization as suggested in [22]

   c) or use our spectral initialization approach, Algorithm 3, which guarantees a good initialization.

Additionally, we compare all cases studied in this experiment with the centralized AltGDmin algorithm from [6] and AltGDmin running for a single node (One-node-AltGDmin).
In Figure 2b, we compare the performance of our proposed algorithm with DGD-AltGDmin-spect and One node-AltGDmin. We considered a very small network with only two agents, i.e., $L = 2$. It is evident from the figure that when $L = 2$, all three algorithms converge. Notably, our algorithm achieves a faster convergence compared to the others. This outcome is expected, as when $L = 2$, the One node-AltGDmin algorithm, which has access to 50% of the data, is generally adequate to derive a reasonable estimate of the 2-dimensional subspace. Later in Figure 2c we considered $L = 20$ case after increasing the iteration count to $T = 3000$. In Figure 2c, with each agent having access to only 5% of the data, both DGD-AltGDmin and One node-AltGDmin fail to accurately estimate the subspace. In contrast, our proposed algorithm demonstrates accurate subspace estimation.

**Why DGD idea does not work.** From figures, even after using the AltGDmin initialization, the DGD modified approach fails. The reason this does not work even experimentally is because DGD is designed for standard GD, where as we are using it here for projected GD and the projection is onto the set of $n \times r$ matrices with orthonormal columns (basis matrices for the subspaces spanned by their columns). In this case, averaging, i.e. computing $\sum_{g \in N_i(g)} U_g^{(t)} / d_g$ does not improve the subspace distance because $SD(U, U^*) = SD((1/d_g) U, (1/d_g) U^*)$. This is why the DGD estimate is roughly only as good as local GD (without any communication with any other nodes) would be. If each node has the same number of $y_k$s, then this is AltGDmin with only $q/L y_k$s instead of $q$, thus making it a lot less accurate.

**Experiment 3: Comparison by varying $T_{con}$ and $p$.** In this experiment, we tested the performance of our Dec-AltGDmin algorithm for different values of consensus iterations, $T_{con}$, and edge probability of the communication network $G$. The plots of these experiments are given in Figure 3a. We set the parameters as $n = q = 600, m = 50, r = 2, L = 20$ and $T = 400$. We plot the matrix estimation error (at the end of the iteration) $SD(U^*, U_{\hat{t}}^{(1)})$ and the execution time-taken (until the end of that iteration) on the y-axis and x-axis, respectively. To compare with we also implemented the centralized algorithm in [6]. In Figure 3a, we set $p = 0.5$ and varied the consensus iterations as, $T_{con} = 5, 10, 25, 100$. The experimental results show that subspace error decreases while increasing the $T_{con}$ and for $T_{con} = 100$ the error values of the Dec-AltGDmin algorithm and AltGDmin algorithms are the same at the end of the GD iterations. In Fig. 3b, we set $T_{con} = 30$ and we varied the values of the edge probability as, $p = 0.4, 0.5, 0.6, 0.7$. From Fig. 3b we observe that as expected the subspace error $SD(U^*, U_{\hat{t}}^{(1)})$ decreases when the connectivity of the network improves. From experiment 2 (Figure 3a) we observe that when $p = 0.5$, the number of consensus iterations required for the error in Dec-AltGDmin algorithm to reach the same order as that of the AltGDmin algorithm is $T_{con} = 100$. However, when the graph is more connected, e.g., $p = 0.7$, the errors in Dec-AltGDmin and AltGDmin algorithms coincides for $T_{con} = 30$. For $p = 0.6$ or less, to bring the error to a value close to that of the AltGDmin algorithm, $T_{con}$ must be increased to at least 100.

**VII. Conclusions**

In this work, we developed and analyzed an alternating projected gradient descent and minimization algorithm for recovering a low rank (LR) matrix from mutually independent linear projections of each of its columns in a “decentralized federated” fashion. This means that the data is federated but there is no central coordinating node. Some works refer to this setting as “fully-decentralized”. Open questions for future work include (i) how to obtain a guarantee for which the number of consensus iterations needed in each GD step iteration does not depend on the final desired accuracy level, and (ii) how to improve the initialization guarantee.

**References**

[1] S. Moothedath and N. Vaswani, “Fully decentralized and federated low rank compressive sensing,” 2021.
[2] ——, “Dec-AltProjGD: Fully-decentralized alternating projected gradient descent for low rank column-wise compressive sensing,” Conference on Decision and Control (CDC), 2022.
[3] ——, “Comparing decentralized gradient descent approaches and guarantees,” in IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), 2023, pp. 1–5.
[4] S. Nayer and N. Vaswani, “Sample-efficient low rank phase retrieval,” IEEE Trans. Info. Th., Dec. 2021.
[5] R. S. Srinivasas, K. Lee, M. Junge, and J. Romberg, “Decentralized sketching of low rank matrices,” in Neur. Info. Proc. Sys. (NeurIPS), 2019, pp. 10101–10110.
[6] S. Nayer and N. Vaswani, “Fast and sample-efficient federated low rank matrix recovery from column-wise linear and quadratic projections,” IEEE Trans. Info. Th., Feb. 2023.
[7] F. P. Anaraki and S. Hughes, “Memory and computation efficient pca via very sparse random projections,” in Intl. Conf. Machine Learning (ICML), 2014, pp. 1341–1349.
[8] S. Babu, S. G. Lingala, and N. Vaswani, “Fast low rank compressive sensing for accelerated dynamic MRI,” IEEE Trans. Comput. Imag., 2023.
[9] S. S. Du, W. Hu, S. M. Kakade, J. D. Lee, and Q. Lei, “Few-shot learning via learning the representation, provably,” in Intl. Conf. Learning Representations (ICLR), 2021.
[10] N. Tripuraneni, C. Jin, and M. Jordan, “Provably meta-learning of linear representations,” in International Conference on Machine Learning, 2021, pp. 10434–10443.
[11] J. Hu, X. Chen, C. Jin, L. Li, and L. Wang, “Near-optimal representation learning for linear bandits and linear rl,” in International Conference on Machine Learning, 2021, pp. 4349–4358.
[12] L. Yang, W. Hu, J. D. Lee, and S. S. Du, “Impact of representation learning in linear bandits,” arXiv:2010.06531, 2020.
[13] L. Cella, K. Lounici, G. Pacreau, and M. Pontil, “Multi-task representation learning with stochastic linear bandits,” in International Conference on Artificial Intelligence and Statistics, 2023, pp. 4822–4847.
[14] C. D’Erramo, D. Tateo, A. Bonarini, M. Restelli, and J. Peters, “Sharing knowledge in multi-task deep reinforcement learning,” arXiv:2401.09561.
[15] S. Arora, S. Du, S. Kakade, Y. Luo, and N. Saurshi, “Provable representation learning for imitation learning via bi-level optimization,” in International Conference on Machine Learning, 2020, pp. 367–376.
[16] E. J. Candes and B. Recht, “Exact matrix completion using alternating minimization,” in Annual ACM Symp. on Th. of Comp. (STOC), 2013.
[17] A. Nedic and A. Ozdaglar, “Distributed subgradient methods for multi-agent optimization,” IEEE Transactions on Automatic Control, vol. 54, no. 1, pp. 48–61, 2009.
[18] S. Lee and A. Nedić, “Distributed random projection algorithm for convex optimization,” IEEE Journal of Selected Topics in Signal Processing, vol. 7, no. 2, pp. 221–229, 2013.
[19] A. Nedic, “Convergence rate of distributed averaging dynamics and optimization in networks,” Foundations and Trends® in Systems and Control, vol. 2, no. 1, pp. 1–100, 2015.
[20] I. Lobel and A. Ozdaglar, “Distributed subgradient methods for convex optimization over random networks,” IEEE Transactions on Automatic Control, vol. 56, no. 6, pp. 1291–1306, 2010.
Fig. 3: Error versus execution time plot with time in seconds. We compare the performance of our algorithm by varying the consensus iterations, $T_{con}$, and by varying the edge probability $p$ of the network. In Figure 3a, we vary $T_{con}$ as 5, 10, 25, and 100. In Figure 3b, we vary the probability of edge in the communication network $G$ as 0.4, 0.5, 0.6 and 0.7. The parameters used are: $n = q = 600$, $r = 2$, $m = 50$, and $L = 20$.

[22] K. Yuan, Q. Ling, and W. Yin, “On the convergence of decentralized gradient descent,” SIAM Journal on Optimization, vol. 26, no. 3, pp. 1835–1854, 2016.

[23] A. Reisizadeh, A. Mokhtari, H. Hassani, and R. Pedarsani, “An exact quantized decentralized gradient descent algorithm,” IEEE Transactions on Signal Processing, vol. 67, no. 19, pp. 4934–4947, 2019.

[24] W. Shi, Q. Ling, G. Wu, and W. Yin, “Extra: An exact first-order algorithm for decentralized consensus optimization,” SIAM Journal on Optimization, vol. 25, no. 2, pp. 944–966, 2015.

[25] A. Nedić, A. Ozdaglar, and P. A. Parrilo, “Constrained consensus and optimization in multi-agent networks,” IEEE Transactions on Automatic Control, vol. 55, no. 4, pp. 922–938, 2010.

[26] F. Shahriari-Mehr, D. Bosch, and A. Panahi, “Decentralized constrained optimization: Double averaging and gradient projection,” arXiv:2106.11408, 2021.

[27] A. Rogozin and A. Gasnikov, “Projected gradient method for decentralized optimization over time-varying networks,” arXiv:1911.08527, 2019.

[28] P. Bianchi and J. Jakubowicz, “Convergence of a multi-agent projected stochastic gradient algorithm for non-convex optimization,” IEEE transactions on automatic control, vol. 58, no. 2, pp. 391–405, 2012.

[29] P. Bianchi, G. Fort, and W. Hachem, “Performance of a distributed stochastic approximation algorithm,” IEEE Transactions on Information Theory, vol. 59, no. 11, pp. 7405–7418, 2013.

[30] J. Lafond, H.-T. Wai, and E. Moulines, “D-Fw: Communication efficient distributed algorithms for high-dimensional sparse optimization,” in 2016 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), 2016, pp. 4144–4148.

[31] P. Di Lorenzo and G. Scutari, “Next-In-network nonconvex optimization,” IEEE Transactions on Signal and Information Processing over Networks, vol. 2, no. 2, pp. 120–136, 2016.

[32] T. Tatarenko and B. Touri, “Nonconvex distributed optimization,” IEEE Transactions on Automatic Control, vol. 62, no. 8, pp. 3744–3757, 2017.

[33] J. Zeng and W. Yin, “On nonconvex decentralized gradient descent,” IEEE Transactions on Signal processing, vol. 66, no. 11, pp. 2834–2848, 2018.

[34] H.-T. Wai, J. Lafond, A. Scaglione, and E. Moulines, “Decentralized frank–wolfe algorithm for convex and nonconvex problems,” Transactions on Automatic Control, vol. 62, no. 11, pp. 5522–5537, 2017.

[35] G. H. Golub and C. F. Van Loan, Matrix computations. John Hopkins University Press, 2013.

[36] D. Kempe and F. McSherry, “A decentralized algorithm for spectral analysis,” Journal of Computer and System Sciences, vol. 74, no. 1, pp. 70–83, 2008.

[37] A. Gang, B. Xiang, and W. U. Bajwa, “Directed principal subspace analysis for partitioned big data: Algorithms, analysis, and implementation,” IEEE Transactions on Signal and Information Processing over Networks, vol. 7, pp. 699–715, 2021.

[38] H. Ye and T. Zhang, “DeEPCA: Decentralized exact PCA with linear convergence rate,” J. Mach. Learn. Res., vol. 22, no. 238, pp. 1–27, 2021.

[39] S. Nayer, P. Narayananamurthy, and N. Vaswani, “Provable low rank phase retrieval,” IEEE Trans. Info. Th., March 2020.

[40] X. Li and A. Scaglione, “Convergence and applications of a gossip-based gauss-Newton algorithm,” IEEE transactions on signal processing, vol. 61, no. 21, pp. 5231–5246, 2013.

[41] A. Olshovsky and J. N. Tsitsiklis, “Convergence speed in distributed consensus and averaging,” SIAM journal on control and optimization, vol. 48, no. 1, pp. 33–55, 2009.

[42] X. Yi, D. Park, Y. Chen, and C. Caramanis, “Fast algorithms for robust pca via gradient descent,” in Neur. Info. Proc. Sys. (NeurIPS), 2016.

[43] P. Jain and N. Netrapalli, “Fast exact matrix completion with finite samples,” in Conf. on Learning Theory, 2015, pp. 1007–1034.

[44] S. Boyd, P. Diaconis, and L. Xiao, “Fastest mixing markov chain on a graph,” SIAM review, vol. 46, no. 4, pp. 667–689, 2004.

[45] L. Lovász, “Random walks on graphs,” Combinatorics, Paul erdos is eighty, vol. 2, no. 1–46, p. 4, 1993.

[46] Y. Chen and E. Candès, “Solving random quadratic systems of equations is nearly as easy as solving linear systems,” in Neur. Info. Proc. Sys. (NeurIPS), 2015, pp. 739–747.

[47] M. Hardt and E. Price, “The noisy power method: A meta algorithm with applications,” Advances in neural information processing systems (NeurIPS), 2014.

[48] A. Olshovsky, “Linear time average consensus on fixed graphs and implications for decentralized optimization and multi-agent control,” arXiv:1411.4186, 2014.

[49] Y. Cherapanamjeri, K. Gupta, and P. Jain, “Nearly-optimal robust matrix completion,” ICML, 2016.

[50] S. Moothedath and N. Vaswani, “Fast, communication-efficient, and provable decentralized low rank matrix recovery,” Arxiv: 2204.08117, 2023.

[51] M. Rudelson and R. Vershynin, “Smallest singular value of a random rectangular matrix,” Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences, vol. 62, no. 12, pp. 1707–1739, 2009.

[52] Q. Ling, Y. Xu, W. Yin, and Z. Wen, “Decentralized low-rank matrix completion,” in 2012 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), 2012, pp. 2925–2928.

[53] A.-Y. Lin and Q. Ling, “Decentralized and privacy-preserving low-rank matrix completion,” Journal of the Operations Research Society of China, vol. 3, no. 2, pp. 189–205, 2015.

[54] M. Mardani, G. Mateos, and G. Giannakis, “Decentralized sparsity-regularized rank minimization: Algorithms and applications,” IEEE Trans. Sig. Proc., 2013.

[55] S. Li, Q. Li, Z. Zhu, G. Tang, and M. B. Wakin, “The global geometry of centralized and distributed low-rank matrix recovery without regularization,” IEEE Signal Processing Letters, vol. 27, pp. 1400–1404, 2020.

[56] R. Xin, U. A. Khan, and S. Kar, “A fast randomized incremental gradient method for decentralized non-convex optimization,” IEEE Transactions on Automatic Control, 2021.

[57] G. Stewart, “Perturbation bounds for the QR factorization of a matrix,” SIAM Journal on Numerical Analysis, vol. 14, no. 3, pp. 509–518, 1977.

[58] J.-g. Sun, “On perturbation bounds for the qr factorization,” Linear algebra and its applications, vol. 215, pp. 95–111, 1995.

[59] X. Yi, D. Park, Y. Chen, and C. Caramanis, “Fast algorithms for robust pca via gradient descent,” in Advances in neural information processing systems, 2016, pp. 4152–4160.
[60] Y. Chen, Y. Chi, J. Fan, and C. Ma, “Spectral methods for data science: A statistical perspective,” arXiv preprint arXiv:2012.08496, 2020.

[61] R. Vershynin, High-dimensional probability: An introduction with applications in data science. Cambridge University Press, 2018, vol. 47.
Recall that
\[ B = [B_1, B_2, \ldots, B_k], \text{ with } B_g := [b_k, \ k \in S_g] \]
\[ X = [X_1, X_2, \ldots, X_k], \text{ with } X_g := U^{(g)} B_g \]
and \( x_k = U^{(g)T} b_k \) for all \( k \in S_g \).

Define \( g_k := U^{(g)T} x_k^* \), for all \( k \in S_g \) and
\[ G = [G_1, G_2, \ldots, G_k], \ G_g := [g_k, \ k \in S_g] = U^{(g)T} X_g^* \]
\[ F = [F_1, F_2, \ldots, F_k], \ F_g := U^{(g)T} X_g^*. \]
Thus, \( F = U^{(g)T} X^* \). Notice the difference between \( G \) and \( F \). Both will be used in our proof.

We use the following lemma from [6].

**Lemma A.1** ([6], Lemma 3.3, first part). Let \( g_k := U^{(g)T} x_k^* \).
Then, w.p. \( \geq 1 - q \exp(r - cm) \),
\[ \| g_k - b_k \| \leq 0.4 \| (I_n - U^{(g)T} U^{(g)T}) U^* b_k^\prime \|, \text{ for all } k \in S_g. \]

**Proof of Theorem 4.1:** By Lemma A.1, we have w.p. \( \geq 1 - \exp(\log q + r - cm) \)
\[ \| b_k - g_k \| \leq 0.4 \| (I_n - U^{(g)T} U^{(g)T}) U^* b_k^\prime \|. \]
Under the assumption that \( SD_2(U^{(g)}, U^*) \leq \delta_t \) with \( \delta_t < 0.02 \), this directly implies the following
1) \( \| b_k - g_k \| \leq 0.4 \| b_k^\prime \| \)
2) \( \| b_k \| \leq \| g_k \| + 0.4 \cdot 0.02 \| b_k^\prime \| \leq 1.1 \| b_k^\prime \| \)
3) \( \| x_k - x_k^\prime \| \leq 1.4 \delta_t \| b_k^\prime \| \)
( follow by adding/subtracting \( U^{(g)} g_k \) and using triangle inequality).

Using above,
\[ \| B - G \|_F \leq 0.4 \delta_t \sqrt{\sum_k \| b_k^\prime \|^2 \leq 0.4 \delta_t \sqrt{r} \sigma_{\text{max}}^*} \]
and similarly for \( \| X - X^\prime \|_F \). Thus,
1) \( \| B - G \|_F \leq 0.4 \delta_t |B^*| \| F \leq 0.4 \delta_t \sqrt{r} \sigma_{\text{max}}^* \)
2) \( \| X - X^\prime \|_F \leq 1.4 \sqrt{r} \delta_t \sigma_{\text{max}}^* \)

To bound the singular values of \( B \), we show next that \( B \) is close to \( F \) and bound the singular values of \( F \). To do this, we use the fact that every \( U^{(g)} \) is close to \( U^{(g)} \) to show that \( G \) is close to \( F \). Since we have already bounded \( B - G \) this implies a bound on \( B - F \).

Recall from the theorem statement that \( \rho_i \) is the upper bound on \( \max_{g \neq g'} \| U^{(g)} - U^{(g')} \|_F \). We have
\[ \| G - F \|_F^2 \leq \max_g \| U^{(g)} - U^{(g')} \|_F^2 \sum_g \| X^{(g)} \|_F^2 \leq \rho_t^2 \sqrt{r} \sigma_{\text{max}}^* \]
Thus, using the above and the bound on \( B - G \),
\[ \| B - F \|_F \leq (0.4 \delta_t + \rho_t) \sqrt{r} \sigma_{\text{max}}^* \]
Next, we have
\[ \sigma_{\text{min}}(F) \geq \sigma_{\text{min}}(U^{(g)T} U^*) \sigma_{\text{min}}(B^*) = \sigma_{\text{min}}(U^{(g)T} U^*) \sigma_{\text{min}}(U^{(g)T} U^*) \]
and
\[ \sigma_{\text{min}}^2(U^{(g)T} U^*) = \lambda_{\text{min}}(U^{(g)T} U^{(g)T} U^{(g)T} U^*) \]
\[ = \lambda_{\text{min}}(I - U^{(g)T} U^* + U^{(g)T} U^{(g)T} U^{(g)T} U^*) \]
\[ = \lambda_{\text{min}}(I - U^* U^* (I - U^{(g)T} U^{(g)T}) U^*) \]
\[ = 1 \lambda_{\text{max}}(U^{(g)T} I - U^{(g)T} U^{(g)T}) U^* \]
\[ = 1 \lambda_{\text{max}}(U^{(g)T} I - U^{(g)T} U^{(g)T}) U^* \]
\[ = 1 \| (I - U^{(g)T} U^{(g)T}) U^* \|_F^2 \leq 1 - \delta_t^2. \]
We used \( U^{(g)T} U^* = I \) and \( (I - U^{(g)T} U^{(g)T}) = (I - U^{(g)T} U^{(g)T})^2 \).

Also,
\[ \sigma_{\text{min}}(B) \geq \sigma_{\text{min}}(F) - \| B - F \| \geq \sigma_{\text{min}}(F) - \| B - F \|_F. \]
Combining the above three bounds, if \( \delta_t < 0.02/\sqrt{r} \sigma_{\text{max}}^* \), then
\[ \sigma_{\text{min}}(B) \geq \sqrt{1 - \delta_t^2 \sigma_{\text{min}}^* - (0.4 \delta_t + \rho_t) \sqrt{r} \sigma_{\text{max}}^*} \geq 0.9 \sigma_{\text{max}}^*, \]
if \( \rho_t \leq 0.1 \delta_t \) and \( \delta_t \leq 0.1 / \sqrt{r} \).
Finally, if \( \| F \| \leq \| B^* \| = \sigma_{\text{max}}^*, \) under the above bounds on \( \delta_t, \rho_t \),
\[ \sigma_{\text{max}}(B) \leq \sigma_{\text{max}}^* + (0.4 \delta_t + \rho_t) \sqrt{r} \sigma_{\text{max}}^* \leq 1.1 \sigma_{\text{max}}^*. \]
Now if \( m \geq \max(\log q, \log n, r) \), then the probability is \( \geq 1 - n^{-10} \). We have thus proved Theorem 4.1. \( \square \)

**Appendix B**

**Proof of Theorem 5.1**

### A. Proof of Theorem 5.1, first part

Assume that, for all \( g \),
- \( \| \text{ConsErr}^{(g)} \|_F \leq m \epsilon_{\text{con}} \sigma_{\text{min}}^* \) for all \( g \)
- \( SD_2(U^{(g)}, U^*) \leq \delta_t \)
- \( \| U^{(g)} - U^{(g')} \|_F \leq \rho_t \)

We will use the following lemma in our proof. Since its proof uses Theorem 4.1, this lemma also needs all the assumptions of that result (it also needs the bound on \( \max_{g \neq g'} \| U^{(g)} - U^{(g')} \| \)).

**Lemma B.1** (Improved version of gradient deviation lemma from [6]). Consider Theorem 4.1. Assume that, for all \( g \), \( SD_2(U^{(g)}, U^*) \leq \delta_t \) and \( \max_{g \neq g'} \| U^{(g)} - U^{(g')} \| \leq \rho_t \). The following hold:

1) \( \mathbb{E}[\text{gradU}] = m(X - X^*) B^T \)
2) \( \| \mathbb{E}[\text{gradU}] \|_F \leq 1.1 m \delta_t \sqrt{r} \sigma_{\text{max}}^* \)
3) If \( \delta_t < \frac{1}{\sqrt{r}} \) and \( \rho_t < 0.1 \delta_t \) then,

\[ \text{w.p. at least } 1 - \exp(C(n + r) - c \sqrt{mq} - \exp(\log q + r - cm)) \]
\[ \| \text{gradU} - \mathbb{E}[\text{gradU}] \| \leq \epsilon_1 \delta_t m \sigma_{\text{min}}^* \]

**Proof.** See Appendix F. This lemma follows in a fashion similar to that in the centralized case. The key difference is that, in the gradU expression, different matrices \( U^{(g)} \) are used at the different nodes. The centralized case bound itself is a strengthening of what was proved in [6] (Lemma 3.5, part 1). Differences from [6] (Lemma 3.5, part 1): (i) We use \( SD_2 \) instead of \( SD_F \) as our subspace distance measure, and hence we use Theorem 4.1 instead of the corresponding result used in [6]. (ii) Because of this, there is one key change in how we bound the probability while applying the sub-exponential
Bernstein inequality. These two changes in fact allow us to get a bound with a better sample complexity than the one proved in [6].

We have the following lemma.

**Lemma B.2.** If \( \delta_t < c/\sqrt{\tau k^2} \), \( \rho_t < 0.1 \delta_t / \sqrt{k} \), \( \varepsilon_{\text{con}} < 0.1 \delta_t / \sqrt{k} \), if \( \eta = c_n / m \sigma_{\max}^2 \) with \( c_n < 0.5 \), and if \( mq \geq \kappa^4 \mu^2 nr \) and \( m \geq \max(\log n, \log q, m) \), w.p. at least \( 1 - n^{-10} \)

\[
SD_2(U^{(g)}_+, U^*) \leq \delta_{t+1} := (1 - 0.6 \frac{\tau}{\kappa^2}) \delta_t
\]

**Proof.** Recall that the GD step of the algorithm is

\[
\tilde{U}^{(g)}_+ = U^{(g)}_+ - \eta \nabla U^{(g)}_+ \\
\tilde{U}^{(g)}_{QR} = U^{(g)}_{QR}
\]

Since \( U_+ = \tilde{U}_+(R_+)^{-1} \) and since \( \|R_+^{-1}\| = 1/\sigma_{\min}(R_+) = 1/\sigma_{\min}(\tilde{U}_+) \), thus,

\[
SD_2(U^{(g)}_+, U^*) = \|P_{U_+} \cdot \tilde{U}^{(g)}_+\| \leq \frac{\|P_{U_+} \cdot \tilde{U}^{(g)}_+\|}{\sigma_{\min}(U^{(g)}_+)}.
\]

Consider the numerator. Adding/subtracting \( \nabla U^{(g)} \) and then also adding/subtracting \( \mathbb{E}[\nabla U^{(g)}] \), we get

\[
\tilde{U}^{(g)}_+ = U^{(g)}_+ - \eta \mathbb{E}[\nabla U^{(g)}] + \eta \text{Err} + \eta \text{ConsErr}^{(g)}.
\]

Using the first claim of Lemma B.1, and projecting orthogonal to \( U^* \),

\[
P_{U_{\perp}} \cdot \tilde{U}^{(g)}_+ = P_{U_{\perp}} \cdot U^{(g)}_+ - \eta \text{Err} + \eta \text{ConsErr}^{(g)}
\]

since \( P_{U_{\perp}} X^* = 0 \). Now

\[
XB^T = \sum \nabla U^{(g)}_+ B_+^T B_+ B_+^T.
\]

Adding/subtracting \( U^{(g)}_+ \sum \nabla U^{(g)}_+ B_+^T B_+ B_+^T \) gives

\[
XB^T = U^{(g)}_+ \sum \nabla U^{(g)}_+ B_+^T B_+ B_+^T - U_{\text{ConsErr}^{(g)}}
\]

Using above and the fact that \( \sum \nabla U^{(g)}_+ B_+^T B_+ B_+^T = BB^T \),

\[
P_{U_{\perp}} \cdot \tilde{U}^{(g)}_+ = P_{U_{\perp}} \cdot U^{(g)}_+ (I - \eta \mu B B^T) + \eta \text{Err} + \eta \text{ConsErr}^{(g)}
\]

By our assumption in the beginning of this section (also the assumption made in the Theorem),

\[
\|\text{ConsErr}^{(g)}\| \leq \varepsilon_{\text{con}} m \sigma_{\min}^2, \\
\|\text{ConsErr}^{(g)}\| \leq \rho_t \|B\|^2 \leq \rho_t \sigma_{\max}^2
\]

where the last inequality used Theorem 4.1.

By Lemma B.1 with \( \varepsilon_1 = 0.1 \), if \( \delta_t < 0.1/\sqrt{\tau k}, \rho_t < 0.1 \delta_t \), then

\[
\|\text{Err}\| \leq 0.1 m \delta_t \sigma_{\max}^2.
\]

Consider the first term of (12). Using Theorem 4.1,

\[
\lambda_{\min} (I - \eta \mu B B^T) = 1 - \eta \|B\|^2 \geq 1 - 1.2 \eta m \sigma_{\max}^2.
\]

Thus, if \( \eta < 0.5 / m \sigma_{\max}^2 \), then the above matrix is p.s.d. This along with Theorem 4.1 then implies that

\[
\|I - \eta \mu B B^T\| = \lambda_{\max} (I - \eta \mu B B^T) \leq 1 - 0.9 m \sigma_{\max}^2
\]

Thus, if \( \eta = c_n / m \sigma_{\max}^2 \) with \( c_n < 0.5 \), \( \delta_t < 0.02 / \sqrt{\tau k^2} \), \( \rho_t \kappa^2 < 0.1 \delta_t \), \( \varepsilon_{\text{con}} < 0.1 \delta_t \), and if \( mq \geq \kappa^4 \mu^2 nr \) and \( m \geq \max(\log n, \log q, m) \), then,

\[
\|P_{U_{\perp}} \cdot \tilde{U}^{(g)}_+\| \leq \|P_{U_{\perp}} \cdot U^{(g)}_+\| (1 - 0.9 \frac{c_n}{\kappa^2}) + \eta (\rho_t \kappa^2 + 0.1 m \delta_t + \epsilon_{\text{con}}) \sigma_{\max}^2 \]

\[
\leq \delta_t (1 - 0.9 \frac{c_n}{\kappa^2}) + \frac{c_n}{\kappa^2} (\rho_t \kappa^2 + 0.1 \delta_t + \epsilon_{\text{con}}) \sigma_{\max}^2 \]

\[
\leq \delta_t (1 - 0.9 \frac{c_n}{\kappa^2}) + \frac{c_n}{\kappa^2} 0.3 \delta_t = \delta_t (1 - 0.6 \frac{c_n}{\kappa^2}).
\]

The above used \( \|P_{U_{\perp}} \cdot U^{(g)}_+\| = \delta_t \). Next, we obtain a lower bound on \( \sigma_{\min}(U^{(g)}_+) \). If \( \delta_t < 0.02 / \sqrt{\tau k^2} \), \( \rho_t < 0.1 \delta_t \) (needed for using the bound on \( \text{Err} \) from Lemma B.1), \( \epsilon_{\text{con}} \kappa^2 < 0.1 \delta_t \), and the lower bounds on \( m \) from above hold, then

\[
\sigma_{\min}(U^{(g)}_+) \geq \sigma_{\min}(\tilde{U}^{(g)}_+) - \eta \|\mathbb{E}[\nabla U^{(g)}] - \eta \|\text{Err}\| - \eta \|\text{ConsErr}^{(g)}\| \]

\[
\geq 1 - c_n \delta_t \sqrt{\tau k^2} (1.4 + \frac{0.1}{\kappa^2} \sqrt{\tau^2}) \frac{c_n}{\kappa^2} \delta_t
\]

\[
\geq 1 - 1.7c_n \delta_t \sqrt{\tau k^2}
\]

The above used the bound on \( \|\mathbb{E}[\nabla U^{(g)}]\| \) from Lemma B.1 and \( \kappa^2 \sqrt{\tau} > 1 \) and \( \epsilon_{\text{con}} < 0.1 \delta_t \).

Using the last two bounds above and (11),

\[
SD_2(U^{(g)}_+, U^*) \leq \delta_t (1 - 0.6 \frac{c_n}{\kappa^2}) \frac{1}{1 - 1.7c_n \delta_t \sqrt{\tau k^2}} \]

\[
\leq \delta_t (1 - 0.6 \frac{c_n}{\kappa^2}) (1 + 1.7c_n \delta_t \sqrt{\tau k^2}) \]

\[
\leq \delta_t (1 - 0.6 \frac{c_n}{\kappa^2}) (0.6 \delta_t \sqrt{\tau k^2}) \leq \delta_t (1 - 0.6 \frac{c_n}{\kappa^2}).
\]

We used \( (1 - x)^{-1} < (1 + x) \) if \( |x| < 1 \), and \( \delta_t < 0.01 / \sqrt{\tau k^2} \) (last row). Now if \( mq \geq \kappa^4 \mu^2 nr \) and \( m \geq \max(\log n, \log q, m) \), the result holds w.p. at least \( 1 - n^{-10} \). □

**B. Proof of Theorem 5.1, second part (recursion for \( \rho_t \))**

We need the following [57, Theorem 3.1], [58, Corollary 4.2].

**Proposition B.3** (Perturbed QR factorization, Corollary 4.2 of [58]). Let \( \tilde{Z}_1 \overset{QR}{=} Z_1 R_1 \) and \( \tilde{Z}_2 \overset{QR}{=} Z_2 R_2 \). Then,

\[
\|Z_2 - Z_1\|_F \leq \sqrt{\frac{\|Z_2 - \tilde{Z}_1\|_F}{\sigma_{\min}(Z_1)}}
\]

if the RHS is less than \( \sqrt{2} \cdot 4 / \sqrt{10} \).

Assume everything from the previous subsection above. Then we have the following result.

**Lemma B.4.** Assume everything from Lemma B.2. Then,

\[
\|U^{(g)}_+ - U^{(g)}_+^{(t+1)}\| \leq \rho_{t+1} := 1.7 (\rho_t + 2 \kappa^2 c_n \epsilon_{\text{con}})
\]
Using Proposition B.3 and (13), under the assumptions made for it,
\[
\|U_{+}^{(0)} - U_{+}^{(g)}\|_F \leq 1.5\left(\rho_t + \eta \cdot \|\text{ConsErr}\|_F\right) \leq 1.5\left(\rho_t + \eta \cdot \|\text{ConsErr}\|_F\right) = \rho_{t+1}.
\]

We have thus proved the lemma. \qed

Combining the last two lemmas proves Theorem 5.1.

APPENDIX C

PROOF OF THEOREM 3.4

A. Results from [6]

The following is an easy corollary of (i) the lemmas proved in [6] and (ii) scalar consensus guarantee Proposition 2.1 applied to show that each \(\alpha^{(g)}\) is a close approximation of \(\alpha\) (truncation threshold). We provide a proof in Appendix E.

Corollary C.1. Let \(E[\cdot]\) be \(E[\cdot|\alpha^{(g)}], \text{for all } g \in [L]\). The following holds

1) \(E[X_0] = X^*D = U^*\Sigma^*V^*D\), where
\[
D := \text{diagonal}(\beta_k(\alpha^{(g)}), k \in S_g, g \in [L]),
\]
\[
\beta_k(s) := E[\zeta^2 I_{\{\|x\|_2^2 \leq s\}}].
\]
with \(\zeta\) being a standard normal Gaussian r.v. Thus, \(E[X_0]\)

is a rank \(r\) matrix and
\[
E[X_0] \overset{\text{SVD}}{=} (U^*Q)\Sigma^*V^*\]

where \(Q\) is an \(r \times r\) unitary matrix, \(\Sigma^*\) is an \(r \times r\) diagonal matrix of its singular values and \(V^*\) is an \(r \times q\) matrix with orthonormal rows.

2) Fix \(0 < \epsilon_0 \leq 1\). Then, w.p. at least \(1 - \exp\left(\frac{\epsilon_0^2 nq}{c q^2} - \exp\left(-c q^2 \epsilon_0^2\right)\right)\)
\[
\|X_0 - E[X_0]\| \leq \epsilon_0 \sigma_{\max}^*\]

3) Also, with the above probability,
\[
0.92 \leq \min_k \beta_k(\alpha^{(g)}) \leq \max_k \beta_k(\alpha^{(g)}) \leq 1
\]
and so \(\sigma_{\min}(D) \geq 0.92\) and \(\|D\| \leq 1\).

4) With above probability,
\[
\sigma_r(E[X_0]) \geq 0.92 \sigma_{\min}^*\]

B. Proof: overall idea

We obtain bounds for node 1. We show that all other nodes’ subspace basis estimates are within an \(\epsilon_{\text{con}}\) distance of node 1, i.e. \(\|U_{+}^{(g)} - U_{+}^{(0)}\|_F \leq \epsilon_{\text{con}}\) at each PM iteration \(\tau\) including the final one. This implies a similar bound on SD\(U_{+}^{(g)},U_{+}^{(1)}\). This follows since SD\(U_{+}^{(0)},U_{+}^{(1)}\) ≤ 2\(U_{+}^{(1)} - U_{+}^{(2)}\|_F\).

Let \(U_{\text{true}}\) be the matrix of top \(r\) singular vectors of \(X_0\). By triangle inequality,
\[
SD(U_{+}^{(g)},U_{+}^{(0)}) \leq SD(U_{+}^{(g)},U_{+}^{(1)}) + SD(U_{+}^{(1)},U_{+}^{(0)}) \leq 2\|U_{+}^{(1)} - U_{+}^{(0)}\|_F.
\]

Using Corollary C.1 and Wedin’s sin \(\theta\) theorem [60, Theorem 2.9], we can show the following.

Lemma C.2. Pick a \(\delta_0 < 0.1\). If \(mq \geq C\kappa^2 \mu_2^2(n + q)r/\delta_0^2\) and the event \(E\) holds, then w.p. at least \(1 - \exp(-c(n + q))\),
\[
SD(U_{+}^{(g)},U_{+}^{(0)}) \leq 2\delta_0.
\]

Proof. We apply Wedin’s sin \(\theta\) theorem [60, Theorem 2.9] to \(X_0\) and \(E[X_0]\) and use the fact that the span of top \(r\) left singular vectors of \(E[X_0]\) equals that of \(U_{+}\) (see Corollary C.1 above). Applying Wedin and then using \(\|U_{+}\| = \|V_{+}\| = 1\), the bound on \(\|X_0 - E[X_0]\|\) with \(c_0 = 0.2\delta_0/\kappa\), and using the lower bound on \(\sigma_r(E[X_0])\), we get
\[
SD(U_{+}^{(g)},U_{+}^{(0)}) \leq 2\frac{0.2 \delta_0^*}{\sigma_{\min}} \leq 2\frac{0.2 \delta_0^*}{0.92 \sigma_{\min}^* - 0.2 \cdot 0.1 \sigma_{\min}^*} \leq \delta_0
\]

if \(mq \geq C\kappa^2 \mu_2^2(n + q)r/\delta_0^2\). We used \(\delta_0 < 0.1\) to simplify the denominator. \qed

To bound the second term of (14), we use the following noisy PM result from [47].

Proposition C.3 (Noisy PM [47], Corollary 1.1 with \(p = r^2\).

Let \(Z_{\text{true}}\) denote the matrix of top \(r\) singular vectors of a symmetric matrix \(\Phi\) and let \(\sigma_r\) denote it’s \(r\)-th singular value. Consider the following approach

• For an \(r^2 \geq r\), let \(Z_{0}\) be an \(n \times r^2\) matrix with i.i.d. standard Gaussian entries. For \(\tau = 1.0\) to \(T_{\text{pm}}\), do,

\[
Z_{\tau} \leftarrow \text{QR}(Z_{\tau})\text{ where } Z_{\tau} \leftarrow \Phi Z_{\tau - 1} + G_{T_{\text{pm}}}.
\]

Assume that for all \(\tau\), \(5\sigma_{r}G_{T_{\text{pm}}} \leq \frac{\varepsilon_{\text{pm}}(\sigma_{r} - \sigma_{r+1})}{\sqrt{r^2 - r^2 + 1}}\)

for some fixed parameter \(\varepsilon_{\text{pm}} \leq 1/2\). For any \(T_{\text{pm}} > C\sigma_{r}^{-\log(n\gamma/\varepsilon_{\text{pm}})}\) for a constant \(C\), with probability (w.p.) at least \(1 - \gamma^{-c_1(r^2 - r^2 + 1)} - \varepsilon^{-c_2 n}\),
\[
SD(Z_{T_{\text{pm}}}, Z_{\text{true}}) \leq \varepsilon_{\text{pm}}.
\]

We apply Proposition C.3 with \(\gamma = n^{-1/c_1}, r^2 \geq r\),
\[
\Phi \equiv X_0X_0^T = \sum_{g}(X_0)_{g}(X_0)_{g}^T,
\]
\[
Z_{\text{true}} = U_{\text{true}}, Z_{\tau} = \tilde{U}_{\tau}^{(1)}, \text{ and } Z_{\tau} \equiv \tilde{U}_{\tau}^{(1)} + G_{T_{\text{pm}}}.
\]

\[
G_{T_{\text{pm}}}^{(1)} - U_{T_{\text{pm}}}^{(1)} = U_{T_{\text{pm}}}^{(1)} - U_{T_{\text{pm}}}^{(0)}.
\]

Recall that \(U_{T_{\text{pm}}}^{(1)} = \text{AvgCons}_{i}((X_0)_{g}(X_0)_{g}^T)\), \(g \in [L]\).
Remark C.4. With picking \( \gamma = n^{1/c_1} \) and \( r' = r \), the result holds w.p. at least \( 1 - 1/n \). If we pick \( r' = r + 10 \), the result will w.p. probability \( 1 - n^{-10} \) and so on.

Since \( \|Z_{\text{true}}\| = 1 \), \( \|Z_{\text{true}}^T G_H\| \leq \|G_r\| \). Also, \( \sqrt{r} - \sqrt{r-1} \geq c / \sqrt{r} \) [51, Theorem 1.1]. Using this, and the above choices of \( \gamma, r' \), a sufficient condition to apply Prop C.3 is

\[
5\|G_r^{(1)}\| \leq (\sigma_r - \sigma_{r+1}) \min \left( \epsilon_{pm}, \frac{c}{n^{1/c_1} \sqrt{r}} \right)
\]

We first lower bound \( \sigma_r(X_0^\top X_0^\top - \sigma_{r+1}(X_0^T X_0) \) and upper bound \( \sigma_r(X_0^T X_0) \). Using Corollary C.1 with \( \epsilon_0 = 0.1/k \), if \( m q \geq n^4(n + q)r \), then, whp,

\[
\|X_0 - E[X_0]\| \leq 0.1\sigma_{\text{min}}.
\]

Using this and Weyl's inequality,

\[
\sigma_r - \sigma_{r+1} \geq 0.8\sigma_{\text{min}}^2,
\]

\[
\sigma_r \leq \|X_0\| \leq 1.1\sigma_{\text{max}}^2.
\]

This follows using the above bound on \( \|X_0 - E[X_0]\| \), Weyl's inequality, and the facts that \( \sigma_r(X_0X_0^\top) = \sigma_r(X_0)^2 \), \( \sigma_r(E[X_0]) \geq 0.92\sigma_{\text{min}}^2 \), \( \sigma_{r+1}(E[X_0]) = 0 \), and \( \sigma_{r+1}(E[X_0^2]) \leq \|E[X_0]\| \leq \sigma_{\text{min}}^2 \). Using these, \( \sigma_r - \sigma_{r+1} \geq (\sigma_r(E[X_0]) - \|X_0 - E[X_0]\|^2) \geq (0.92\sigma_{\text{min}}^2 - \epsilon_{\text{pm}}^2) \geq 0.92\sigma_{\text{min}}^2(0.82\sigma_{\text{min}}^2) - 0.01\sigma_{\text{min}}^2 \geq 0.6\sigma_{\text{min}}^2 \).

Next we bound \( \|G_r^{(1)}\| \). Notice that it can be split as

\[
G_r^{(1)} = \text{ConsErr}^{(1)} + \text{UnconsErr}^{(1)}
\]

where

\[
\text{ConsErr}^{(1)} := U_r^{(1)} - \sum_g (X_0^T (X_0^T U_r^{(1)})^{-1} - X_0^T U_r^{(1)})
\]

\[
\text{UnconsErr}^{(1)} := \sum_g (X_0^T (X_0^T U_r^{(1)})^{-1} - X_0^T U_r^{(1)})
\]

C. Bounding ConsErr

To bound ConsErr\(^{(1)}\), we need to define and bound the error of the input to AvgCons w.r.t. the desired sum, InpErr\(^{(g)}\). Let

\[
\text{InpErr}^{(g)} := \sum_{g'} (X_0^T (X_0^T U_r^{(1)})^{-1} - X_0^T U_r^{(1)})
\]

Using Claim 5.4,

\[
\|\text{InpErr}^{(g)}\|_F = \sum_{g' < g} \|\sum_{g'} (X_0^T (X_0^T U_r^{(1)})^{-1} \cdot \|F \leq \sum_{g'} \|U_r^{(1)}\|_F \sum_{g'} \| (X_0^T (X_0^T U_r^{(1)})^{-1} \cdot \| \leq \sqrt{r} \cdot L \cdot \|X_0\|^2 \leq 1.1L \sqrt{r} \sigma_{\text{max}}^2
\]

if \( m q \geq n^4 \mu^2(n + q)r \). The last inequality used (16). Thus, using Proposition 5.3 (consensus result Frob norm version), if \( T_{\text{con}} \geq C \log(1/\gamma) \log(\log(1/\epsilon_{\text{con}})) \) and \( m q \geq n^4 \mu^2(n + q)r \), then

\[
\|\text{ConsErr}^{(1)}\|_F \leq 1.1 \sqrt{r} \epsilon_{\text{con}} \sigma_{\text{max}}^2.
\]

D. Bounding UnconsErr

Recall from our algorithm that, at each iteration \( \tau \), we run a second consensus loop in which node-1 sends \( U_r^{(1)} \) while all other nodes send an \( n \times r \) matrix of zeros. Thus, their true sum equals \( U_r^{(1)} \). Using Proposition 5.3 (consensus result Frob norm version), we can show that this loop ensures that all nodes' estimates converge to within an \( \epsilon_{\text{con}} \cdot \|U_r^{(1)}\|_F \) Frobenius norm distance of \( U_r^{(1)} \). To be precise, the input error (error between input of node to the consensus algorithm and the desired sum to be computed) for this consensus loop satisfies

\[
\text{InpErr}^{(1)} = 0, \quad \text{InpErr}^{(g)} = U_r^{(1)} - U_r^{(1)}, \quad \text{for all } g \neq 1.
\]

Applying Proposition 5.3 if \( T_{\text{con}} \geq C \log(1/\gamma) \log(\log(1/\epsilon_{\text{con}})) \), then

\[
\max_g \|U_r^{(1)} - U_r^{(1)}\|_F \leq \epsilon_{\text{con}} \| U_{\text{true}} \|_F
\]

Thus, using above bound and (16),

\[
\|U_r^{(1)}\|_F \leq \epsilon_{\text{con}} \| U_{\text{true}} \|_F
\]

Thus, using above bound and (16),

\[
\|U_r^{(1)}\|_F \leq \epsilon_{\text{con}} \| U_{\text{true}} \|_F
\]

By (16), \( \sigma_r(X_0X_0^\top) - \sigma_{r+1}(X_0X_0^\top) \geq 0.8\sigma_{\text{min}}^2 \). Thus, in order to apply the noisy PM result to show that \( \|U_r^{(1)}\|_F \leq \epsilon_{\text{con}} \), we need

\[
\|U_r^{(1)}\|_F \leq \min \left( \frac{0.5\delta_0}{n^{1/c_1} \sqrt{r}}, \frac{1}{0.8} \right)
\]

Using (20), and \( \sigma_{\text{min}}^2 = 1/(k^2 \sigma_{\text{max}}^2) \), the above bound holds if

\[
\epsilon_{\text{con}} \leq \frac{1}{2.2k^2 \sqrt{r}} \cdot \min \left( \frac{0.5\delta_0, \sqrt{r}}{n} \right) 0.8
\]

To obtain the second bound of our initialization theorem, we need \( \epsilon_{\text{con}} \sqrt{r} \leq \rho_0 \). Both these requirements are satisfied if

\[
\epsilon_{\text{con}} = 0.8 \min \left( \frac{\delta_0}{2k^2 \sqrt{r}}, \frac{1}{\kappa^2 n^{1/c_1} \sqrt{r}} \right)
\]

This means that we need to set

\[
T_{\text{con}} = C \log(1/\gamma) \log(\log(1/\epsilon_{\text{con}})) \log(1/\rho_0)
\]

To get the \( T_{\text{con}} \), expression, using (16), \( \sigma_r(X_0X_0^\top) - \sigma_{r+1}(X_0X_0^\top) \geq 0.8\sigma_{\text{min}}^2 \) and \( \sigma_r(X_0X_0^\top) \leq 1.1\sigma_{\text{max}}^2 \). Thus, \( \sigma_r/(\sigma_r - \sigma_{r+1}) < Ck^2 \). Thus, if

\[
T_{\text{con}} = Ck^2 \log(n/\delta_0)
\]

\( T_{\text{con}} \) is set as above, and \( m q \geq \kappa^2 \mu^2(n + q)r \), then, by Proposition C.3 (noisy PM result), \( \|U_r^{(1)}\|_F < 0.5\delta_0 \).

By Lemma C.2, \( \|U_r^{(1)}\|_F < 0.5\delta_0 \) whp if \( m q \geq \kappa^2 \mu^2(n + q)r/\delta_0^2 \). Thus, we have proved Theorem 3.4.
APPENDIX D
PROOF OF THEOREM 3.5: INITIALIZATION WITH ONE CONSENSUS LOOP

The first part of this proof remains the same as in Appendix C; see Sec. C-A, C-B and C-C. The step to bound $\text{UconsErr}^{(1)}$ is different and much longer. We give this next.

A. Definitions

Recall that $\text{UconsErr}^{(1)} = \sum_g \langle X_0 \rangle_g \langle X_0 \rangle_g^\top U_{\tau-1}^{(g)} - X_0 \langle X_0 \rangle_g^\top U_{\tau-1}^{(g)}$. To bound it, we need to define and bound $\rho_\tau := \max_g \|U_{\tau-1}^{(g)} - U_{\tau-1}^{(g)}\|_F$ \hfill (21)

B. Bounding $\text{UconsErr}^{(1)}$, step 1

**Assumption D.1.** Assume that $\sigma_{\min}(U_{\tau-1}^{\top} U_{\init}) \geq c_1 = 0.1$. This is assumed in our Theorem statement.

Consider UconsErr. We have $\|\text{UconsErr}^{(1)}\|_F \leq \max_g \|U_{\tau-1}^{(g)} - U_{\tau-1}^{(g)}\|_F \sum_g \|\langle X_0 \rangle_g \langle X_0 \rangle_g^\top\| \leq \rho_{\tau-1} \cdot L \cdot \|X_0\|^2 \leq \rho_{\tau-1} \cdot L \cdot 1.1 \sigma_{\max}^2$.

Thus, using (17) and (18),

$$\|G_0^{\top}\|_F \leq (\rho_{\tau-1} + \epsilon_{\text{con}} \sqrt{r}) \cdot 1.1 L \sigma_{\max}^2 \tag{22}$$

C. Bounding $\text{UconsErr}^{(1)}$, step 2: bound $\rho_\tau$ using perturbed QR result

Recall that $\rho_\tau = \max_g \|U_{\tau-1}^{(g)} - U_{\tau-1}^{(g)}\|_F$. We bound this using Proposition B.3. For applying this, we need a lower bound $\sigma_{\min}(U_{\tau-1}^{(1)})$ and an upper bound on $\|\bar{U}_{\tau}^{(0)} - \bar{U}_{\tau}^{(1)}\|_F$. Using (17) and Weyl’s inequality, and using $\|U_{\tau-1}^{(1)}\|_F = 1$,

$$\sigma_{\min}(\bar{U}_{\tau}^{(1)}) = \sigma_{\tau}(\bar{U}_{\tau}^{(1)}) \geq \sigma_{\min}(X_0 \bar{X}_0^\top U_{\tau-1}^{(1)} - \|G_0^{\top}\| \geq \sigma_{\min}(E[X_0] E[X_0]^{\top}) U_{\tau-1}^{(1)} - \|\text{ErrX0}\| \cdot 1 - \|G_0^{\top}\| \tag{23}$$

with $\text{ErrX0} := \langle X_0 \rangle_g \langle X_0 \rangle_g^\top - E[X_0] E[X_0]^{\top}$

We first lower bound $\sigma_{\min}(E[X_0] E[X_0]^{\top}) U_{\tau-1}^{(1)}$. For simplicity we remove the superscript in some of the writing below. Using Corollary C.1 with the diagonal matrix $D$ defined there, and Claim E.1 from Appendix E,

$$\sigma_{\tau}(E[X_0] E[X_0]^{\top} U_{\tau-1}^{(1)}) = \sigma_{\min}(\Sigma^{\top} \Sigma D^2 V^{\top} \Sigma U_{\tau-1}^{\top}) \\geq \sigma_{\min}(\Sigma^{\top} \Sigma) \sigma_{\min}(V^2 D^2 V^{\top}) \sigma_{\min}(U_{\tau-1}^{\top}) \\geq \sigma_{\min}^2 \sigma_{\min}(\Sigma^{\top} \Sigma D^2 V^{\top} \Sigma) \sigma_{\min}(U_{\tau-1}^{\top}) \\geq \sigma_{\min}^2 \sigma_{\min}(D^2) \sigma_{\min}(U_{\tau-1}^{\top}) \\geq 0.8 \sigma_{\min}^2 \cdot 1 \cdot \sigma_{\min}(U_{\tau-1}^{\top}) \tag{24}$$

since $\|V^{\top} z\|^2 = \|z\|^2 = 1$. By the second equation (fourth line) of the proof of Theorem 2.4 of [47] and the fact that, for subspace basis matrices $U_1$, $U_2$, the cosine of the $r$-th principal angle between the corresponding subspaces, denoted $\cos \theta_r(U_1, U_2)$, is equal to the $r$-th singular value of $U_1^\top U_2$ [60, eq. 2.5],

$$\sigma_{\min}(U_{\tau-1}^\top U^*) = \sigma_r(U_{\tau-1}^\top U^*) \\geq \cos \theta_r(U_{\tau-1}, U^*) \\geq \cos \theta_r(U_{\init}, U^*) \\geq \sigma_r(U_{\init}^\top U^*) \geq c_1$$

by Assumption D.1. Thus, combining (24) and above,

$$\sigma_{\min}(E[X_0] E[X_0]^{\top} U_{\tau-1}^{(1)}) \geq 0.8 c_1 \sigma_{\max}^2 \tag{25}$$

Next we upper bound $\|\text{ErrX0}\|$. By adding/subtracting $X_0 E[X_0]^{\top}$ and using Corollary C.1 with an $\epsilon_0 < 1$, if $mq \gtrsim \kappa^2 \mu^2 (n + q) r / \epsilon_0^2$,

$$\|\text{ErrX0}\| \leq (2 \|E[X_0]\| + \epsilon_0 \sigma_{\max}) \epsilon_0 \sigma_{\max} \leq 3 \epsilon_0 \sigma_{\max}^2 \tag{26}$$

Using (23), (25), (26) and (22), using $\|U_{\tau-1}\| = 1$, and setting $\epsilon_0 = 0.1 \cdot 1.1 \kappa^2 / \kappa^2$, if $mq \gtrsim \kappa^2 \mu^2 (n + q) r$, then $\rho_\tau$,

$$\rho_\tau \leq \frac{3 \epsilon_0 \sigma_{\max}^2 \cdot \sqrt{r}}{0.7 - (\rho_{\tau-1} + \epsilon_{\text{con}}) \cdot \sqrt{r}} = \frac{3 L \epsilon_0 \sigma_{\max}^2}{0.7 - (\rho_{\tau-1} + \epsilon_{\text{con}})} \cdot \sqrt{r} \tag{27}$$

Thus, if we set $\epsilon_{\text{con}} = \frac{0.005}{1.1L^{0.75}} \sqrt{r}$, then we can show that if $\rho_{\tau-1} < \frac{0.005}{1.1L^{0.75}} \sqrt{r}$, then $\rho_{\tau} \leq 4 L \sqrt{r} \epsilon_{\text{con}} < \frac{0.005}{1.1L^{0.75}}$, i.e., the same bound holds for $\rho_{\tau}$ as well. At $\tau = 0$, $\rho_0 = 0$. Thus, we have proved the following

**Lemma D.2.** If $\epsilon_{\text{con}} < \frac{0.005}{4L\kappa^{0.75}}$, then, for all $\tau$,

$$\rho_\tau := \max_g \|U_{\tau-1}^{(g)} - U_{\tau-1}^{(g)}\|_F \leq 4 L \kappa^{1.5} \sqrt{r} \epsilon_{\text{con}} \leq \frac{0.005}{1.1L^{0.75}} \sqrt{r}$$
E. Bounding $\ucons$, step 4: Combining bound on $G_\tau$ with bound on $\rho_{\tau-1}$

Combining Lemma D.2 and (22),

$$\|G_\tau^{(1)}\| \leq (\rho_{\tau-1} + \epsilon_{\text{con}} \sqrt{\tau}) 1.1 L \sigma_{\text{max}}^2$$

$$\leq (4L^2 \sqrt{\tau + \tau}) 1.1 L \epsilon_{\text{con}} \sigma_{\text{max}}^2 \leq 5L^2 n^4 \sqrt{r} \epsilon_{\text{con}} \sigma_{\text{min}}^2 \tag{27}$$

F. Applying the noisy PM result and completing the proof

By (16), $\sigma_r(X_0 X_0^\top) - \sigma_{r+1}(X_0 X_0^\top) > 0.8 \sigma_{\text{min}}^2$. Thus, in order to apply the noisy PM result to show that $\SD_2(U_{pm}^{(1)}, U_{\text{true}}) \leq \epsilon_{\text{pm}} = \delta_0/2$, we need

$$\|G_\tau^{(1)}\| \leq \min \left( \delta_0/2, \sqrt{\frac{\tau}{n}} - \frac{\tau - 1}{n} \right) 0.8 \sigma_{\text{min}}^2$$

The above used $\|Z_{\text{true}} G_\tau^{(1)}\| \leq \|G_\tau^{(1)}\|$. Using (27), the above bound holds if

$$\epsilon_{\text{con}} \leq \frac{1}{5L^2 K^2 \sqrt{\tau}} \min \left( \frac{\delta_0}{2}, \sqrt{\frac{\tau}{n}} - \frac{\tau - 1}{n} \right) 0.8 \sigma_{\text{min}}$$

For Lemma D.2 to hold, we need another bound on $\epsilon_{\text{con}}$. To obtain the second bound of our Initialization theorem, we need $\rho_{\tau_{\text{pm}}} \leq \rho_0$. Using Lemma D.2, this means we need $4L^2 \sqrt{\tau} \epsilon_{\text{con}} \leq \rho_0$ or $\epsilon_{\text{con}} \leq \frac{1}{4L^2 \sqrt{\tau}} \rho_0$. All these requirements are satisfied if

$$\epsilon_{\text{con}} \leq \frac{1}{5L^2 K^2 \sqrt{\tau}} \min \left( \frac{\delta_0}{2}, \sqrt{\frac{\tau}{n}} - \frac{\tau - 1}{n} \right) 0.005 \frac{\rho_0}{L^2 K^2 \sqrt{\tau}}$$

This means that we need to set $T_{\text{con}} = C \frac{1}{\log(1/\gamma(W))} \log \left( \frac{L^2 \kappa^4 \nu}{\delta_0 - \frac{\tau - 1}{n}} \right)$. This simplifies to $T_{\text{con}} \geq C \kappa^2 \log\left(\frac{n}{\delta_0}\right)$. $T_{\text{con}}$ is set as above, and $mq \geq \kappa^4 \mu^2 (n + q) r^2$, then, by Proposition C.3 (noisy PM result),

$$\SD_2(U_{pm}^{(1)}, U_{\text{true}}) < 0.5 \delta_0$$

By Lemma C.2, $\SD_2(U_{\text{true}}, U^*) < 0.5 \delta_0$ whp if $mq \geq \kappa^2 \mu^2 (n + q) r^2$.

Combining these two bounds, if $mq \geq \kappa^4 \mu^2 (n + q) r^2$ max($r, \frac{1}{\delta_0}$), then, $\SD_2(U_{pm}^{(1)}, U^*) < \delta_0$ whp.

APPENDIX E

RESULTS FOR INITIALIZATION TAKEN FROM [6]

The following linear algebra results are used in the proofs below and also in the main paper.

**Claim E.1.** If $X = UM$ with $U$ being $n \times r$ and orthonormal, and $M$ being $r \times q$,

$$\sigma_r(X) = \min(M)$$

2(use max(a, b, c) < abc if a, b, c $\geq$ 1, use $r \leq n$ and use order notation so that $C \log L^3 = C \log L$ for a different $C$ and similarly for $n, \kappa$).

$$T_{\text{con}} = C \frac{1}{\log(1/\gamma(W))} \log \left( \frac{L^2 \log n + \log \log \kappa + \log(1/\delta_0) + \log(1/\rho_0)}{\log(1/\gamma(W))} \right)$$

To get the $T_{\text{pm}}$ expression, using (16), $\sigma_r(X_0 X_0^\top) - \sigma_{r+1}(X_0 X_0^\top) > 0.8 \sigma_{\text{min}}^2$ and $\sigma_r(X_0 X_0^\top) \leq 1.1 \sigma_{\text{max}}^2$. Thus, $\sigma_r/(\sigma_r - 1) < C \kappa^2$.

If $A = BC$ and $A$ is tall (or square) and both $B, C$ are tall (or square), then

$$\sigma_{\text{min}}(A) = \min_{\|z\|_2 \leq 1} \frac{\|Az\|}{\|z\|} \geq \sigma_{\text{min}}(B) \sigma_{\text{min}}(C)$$

**Lemma E.2** ([6]). Conditioned on $\alpha^{(g)}$, we have the following conclusions.

1) $\mathbb{E}[X_0] = X^* D$ where

$$D := \text{diagonal}(\beta_k(\alpha^{(g)}), k \in S_q, g \in [L])$$

with $\beta_k(.)$ is as defined earlier in Corollary C.1. Recall from notation that $X^* = U^* \Sigma^* V^*$. 2) Thus, $\mathbb{E}[X_0]$ is a rank $r$ matrix and

$$\mathbb{E}[X_0] = (U^* Q)^\top \Sigma^* V^*$$

where $Q$ is an $r \times r$ unitary matrix, $\Sigma^*$ is an $r \times r$ diagonal matrix of its singular values and $V^*$ is an $r \times q$ matrix with orthonormal rows. This means that the span of top $r$ left singular vectors of $\mathbb{E}[X_0]$ equals the column span of $U^*$.

3) Also, $\sigma_r(\mathbb{E}[X_0]) = \sigma_{\text{min}}(U^* \Sigma^* V^*) \geq \sigma_{\text{min}}(D \Sigma^* V^*) \geq \sigma_{\text{min}}(D) \sigma_{\text{min}}(V^* \Sigma^*) \geq \sigma_{\text{min}}(D) \sigma_{\text{min}}(V^* \Sigma^*) = \min_j |D_{jj}| \cdot 1 \cdot \sigma_{\text{min}}$

The following lemma is an easy consequence of the scalar consensus result and the sub-exponential Bernstein inequality.

**Lemma E.3** ($\alpha^{(g)}$ consensus). Let $T_{\text{con}} = C \frac{1}{\log(1/\gamma(W))} \log(1/\epsilon_{\text{con}})$. Recall that $\alpha = 9n^2 \mu^2 \sum g \sum \|y_k\|_2/(mq)$ and $\alpha^{(g)}$ is its consensus estimate after $T_{\text{con}}$ iterations. The following hold:

1) w.p. at least $1 - \exp(-c \epsilon_1 n \mu^2)$,

$$|\alpha - 9n^2 \mu^2 \|X^*\|_F^2/q| \leq \epsilon_1 9n^2 \mu^2 \|X^*\|_F^2/q$$

2) $\max_{g \in [L]} |\alpha^{(g)} - \alpha| \leq \epsilon_{\text{con}} \alpha$

3) Thus, setting $\epsilon_{\text{con}} = 0.01$ and $\epsilon_1 = 0.01$, w.p. at least $1 - \exp(-c \epsilon_1 n \mu^2)$, for all $g \in [L]$

$$\mathbb{E}^{(g)} : = \left\{ 8.98 \mu^2 \frac{\|X^*\|_F^2}{q} \leq \alpha^{(g)} \leq 9.02 \mu^2 \frac{\|X^*\|_F^2}{q} \right\}$$

holds.

**Proof.** The first claim is the same as Fact 3.7 of [6]. It is a direct consequence of the sub-exponential Bernstein inequality [61]. The second claim follows by applying Proposition 5.3 as follows. Let $C_1 = 9n^2 \mu^2 / mq$. The desired sum is $C_1 \sum g \sum k \in S_q \|y_k\|_2^2$, and the input at node $g$ is $C_1 \sum k \in S_q \|y_k\|_2^2$. Thus the input error at node $g$, $\text{InpErr}^{(g)} = C_1 \sum g' \sum \sum k \in S_q \|y_k\|_2^2 \leq C_1 \sum g' \sum \|y_k\|_2^2 = \alpha$. The inequality follows since all summands are non-negative. Applying Proposition 5.3, then $\text{ConsErr}^{(g)} = |\alpha - \alpha^{(g)}| \leq \epsilon_{\text{con}} \max_g \text{InpErr}^{(g)} \leq \epsilon_{\text{con}} \alpha$. The third claim is an easy consequence of the first two. □

**Fact E.4** ([6]). min$_g \mathbb{E} \left[ \sigma^2 \mathbb{I} \left\{ |g| \leq \sqrt{N \log n} \|X^*\|_F \sqrt{\|P\|} \right\} \right] \geq 0.9
Combining the above two results, w.p. at least $1 - \exp(-cmq/\kappa^2\mu^2)$,

$$\min_g \min_{k \in S_k} \beta_k(\alpha^{(g)}) \geq \min_k \mathbb{E} \left[ \zeta^2 1_{\{\zeta \leq \sqrt{r} \text{max}_{\alpha^{(g)}}^{\text{diam}}(X^* \| F) \}} \right] \geq 0.9$$  \hspace{1cm} (28)

The following lemma can be proved exactly as done in [6]. It uses the last item of Lemma E.3 in its proof. The only difference in the proof now is that the thresholds $\alpha^{(g)}$ are different for subsets of columns. But, since we condition on $\alpha^{(g)}$s with $\alpha^{(g)} \in \mathcal{E}^{(g)}$, nothing changes.

**Lemma E.5 ([6]).** Fix $0 < \epsilon_1 < 1$. Then, w.p. at least $1 - \exp((n+q) - \epsilon_1^4mq/\mu^2\kappa^2)s$, conditioned on $\alpha^{(g)}$s, for $\alpha^{(g)} \in \mathcal{E}^{(g)}$,

$$\|X_0 - \mathbb{E}[X_0]\| \leq 1.1\epsilon_1\|X^*\|_F$$

By setting $\epsilon_1 = \epsilon_0/1.1\sqrt{r}\kappa$ in the above lemma, and using Lemma E.3 (last claim) to remove the conditioning on $\alpha^{(g)}$s, we can conclude that

$$\|X_0 - \mathbb{E}[X_0]\| \leq \epsilon_0\sigma_{\min}^2 w.p. \text{ at least } 1 - \exp((n+q) - \epsilon_1^4mq/\mu^2\kappa^2) - \exp(\log L - cmq/\kappa^2\mu^2).$$

This fact, Lemma E.2, and (28) constitute Corollary C.1 in the main paper.

**APPENDIX F**

**PROOF OF LEMMA B.1**

Assume everything given in Theorem 4.1 holds. Using bounds on $\|B\|$ and $\|X^* - X\|_F$ from Theorem 4.1, if $\delta_t < \frac{\epsilon}{\sqrt{r}\kappa}$,

$$\|\mathbb{E}[\nabla U]\| = \| \sum_k m(x_k - x_k^*)b_k^\top \|$$

$$\leq m\|X^* - X\|_F \cdot \|B\|$$

$$\leq m\|X - X^*\|_F \cdot \|B\| \leq 1.1m\delta_t\sqrt{r}\sigma_{\max}^2.$$  \hspace{1cm}

Next, we bound $\|\nabla U - \mathbb{E}[\nabla U]\| = \max_{w,|z|=1} w^\top (\sum_k a_k \delta_k^t b_k^\top(x_k - x_k^*) - \mathbb{E}[\cdot])z$ We bound the above for fixed unit norm $w, z$ using sub-exponential Bernstein inequality (Theorem 2.8.1 of [61]), followed by a standard epsilon-net argument. Consider the following for fixed unit norm $w, z$,

$$\sum_k \sum_i (w^\top a_k^i)(b_k^\top z) a_k^i(x_k - x_k^*) - \mathbb{E}[\cdot])$$

Observe that the summands are independent, zero mean, sub-exponential r.v.s with sub-exponential norm $K_{ki} \leq C\|w\|\|z\|b_k^\top(x_k - x_k^*) = C\|z\|b_k\|x_k - x_k^*.\|$ We apply the sub-exponential Bernstein inequality, Theorem 2.8.1 of [61], with $t = \epsilon_1\delta_t\sigma_{\min}^2$. We have

$$\sum_{k_i} K_{ki}^2 \geq \frac{\epsilon_1^4\delta_t^2m^2\sigma_{\min}^4}{4} \geq \frac{m\max_k \|x_k - x_k^*\|^2 \sum_k (z^\top b_k)^2}{\epsilon_1^4\delta_t^2m^2\sigma_{\min}^4} \geq \frac{m\max_k \|x_k - x_k^*\|^2 \|z\|B\|^2}{\epsilon_1^4\delta_t^2m^2\sigma_{\min}^4} \geq \frac{1.44m\delta_t^2\sigma_{\min}^2}{\epsilon_1^2\delta_t^2m^2\sigma_{\min}^4} \frac{1.1\sigma_{\max}^2}{\epsilon_1^2\delta_t^2m^2\sigma_{\min}^4} = \frac{c_1^2mq}{\kappa^4\mu^2\sigma_{\max}^2}$$

In the above, we used (i) $\sum_k (z^\top b_k)^2 \leq \|z^\top B\|^2 \leq \|B\|^2$ since $z$ is unit norm, (ii) Theorem 4.1 to bound $\|B\| \leq 1.1\sigma_{\max}$, and (iii) Theorem 4.1 followed by Assumption 1.1 (right incoherence) to bound $\|x_k - x_k^*\| \leq \delta_t \cdot \mu\sigma_{\max} \sqrt{r}/q$ and $|z^\top b_k| \leq \|b_k\| \leq 1.1\|b_k\| \leq 1.1\|b_k\| \leq 1.1\mu\sigma_{\max} \sqrt{r}/q$. For $\epsilon_1 < 1$, the first term above is smaller (since $1/\kappa^4 \leq 1/\kappa^2$), i.e., $\min(\frac{1.1\sigma_{\max}^2}{\epsilon_1^2\delta_t^2m^2\sigma_{\min}^4}, \frac{c_1^2mq}{\kappa^4\mu^2\sigma_{\max}^2})$. Thus, by sub-exponential Bernstein, w.p. at least $1 - \exp(-c_1^2mq/\kappa^4\mu^2\sigma_{\max}^2)$

$$w^\top (\nabla U - \mathbb{E}[\nabla U])z \leq \epsilon_1\delta_t\sigma_{\min}^2$$

Using a standard epsilon-net argument to bound the maximum of the above over all unit norm $w, z$ (similar to that given in the proof of [6, Lemma 3.5, part 1]), we can conclude that

$$\|\mathbb{E}[\nabla U]\| \leq 1.1\epsilon_1\delta_t\sigma_{\min}^2$$

w.p. at least $1 - \exp(C(n+r) - c_1^2mq/\kappa^4\mu^2\sigma_{\max}^2)$). The factor of $\exp(n+r)$ is due to the epsilon-net over $w$ which is an $n$-length unit norm vector (and thus the epsilon net covering it is of size $(1 + 2/\epsilon_{\text{net}})^n = C^n$ with $\epsilon_{\text{net}} = \epsilon$) and the epsilon-net over $z$ (and thus the epsilon net covering it is of size $C^n$). Union bound over both thus gives a factor of $C^n$. Since Theorem 4.1 holds w.p. $1 - \exp(\log q + r - cm)$, by union bound all the above claims hold w.p. $1 - \exp(C(n+r) - c_1^2mq/\kappa^4\mu^2\sigma_{\max}^2) - \exp(\log q + r - cm)$.

By replacing $\epsilon_1$ by $\epsilon_1/1.1$, our bound becomes simpler (and $1/1.1^2$ gets incorporated into the factor $c$ in the exponent). We have thus proved Lemma B.1