Hybrid Optimal Tracking Control for Linear Continuous-Time Systems by Simultaneous Optimization Approach of System States and Mode Distributions*

Gou Nakura†

In this paper, we study optimal LQG tracking control problems for linear continuous-time hybrid systems. We adopt an approach to prepare some candidates of mode distributions and minimize value functions at each time for the systems and performance indices. By the approach, system states and mode distributions are simultaneously optimized so that the performance indices are minimized for available information of reference signals. We consider optimization problems for averaged systems and averaged performance indices throughout mode distributions. Finally we give numerical examples and verify that we can obtain both optimal tracking performance and optimal candidate of mode distributions by the algorithm presented in this paper.

1. Introduction

On design of hybrid systems composed of multi modes, it is very significant research issue to optimize both system states and modes. It is also very important to consider simultaneous estimation of both system states and inaccessible modes for hybrid systems with unknown modes[1,3]. This estimation is called hybrid estimation. By the hybrid estimation, we often want to know current mode at each time through information of observation. However there exist cases that we want to know distributions of modes on long run time interval rather than each estimate of the modes themselves at each time to grasp global performance over long time intervals, for example, distributions of active modes in solar systems[2,7], distributions of active agents on formation or consensus via hybrid systems representation and so on.

In[9], Q. Zhang has presented hybrid filtering algorithm by most probable trajectory (MPT) approach for linear continuous-time systems. In the paper, concerned systems are general hybrid systems which aren’t restricted to Markovian jump systems and where added noises aren’t restricted to be Gaussian. Throughout the paper, it is assumed that modes of the systems are not directly accessible, and the author has considered optimal estimation problems to find both estimated states of the systems and an optimal candidate of the distributions of the modes over the finite time interval. In the paper, extended most probable trajectory (MPT) approach to the hybrid systems has adopted to guarantee the optimality of estimation methods. On the approach, given information of observation, he has considered optimal control problems where we seek optimal control by which averaged noises energies are minimized for averaged systems. The approach has been modified for the hybrid systems with multi-modes to seek an optimal value of some candidates of value functions corresponding to the number of some candidates of mode distributions at each time. He has also considered cases that mode transitions follow Markovian jump processes and clarified relationship between estimators based on the Markovian mode transition probabilities and ones based on the limiting probabilities, i.e., he has shown near optimality of the limiting estimators.

G. Nakura has extended the filtering theory by Q. Zhang[9] to smoothing theory by two filters approach[5,6]. He has split performance indices over fixed time intervals into two parts, i.e., forward parts and backward parts at a current time, introduced backward filtering differential equations, and presented the smoothing algorithm by connecting the solutions of forward filtering equations and those of backward filtering equations. Also he has presented numerical examples and verified that better estimation performance by smoothing than filtering can be obtained. Note that, on these research, the concerned hybrid systems are transformed to the averaged systems throughout some candidates of mode distributions, and the optimal estimation problems are reduced to some optimal control problems. Therefore we must be able to consider optimal control problems to optimize con-
control inputs and mode distributions simultaneously for general hybrid systems.

In this paper, we study a linear quadratic Gaussian (LQG) optimal tracking control problems for linear continuous-time hybrid systems over the fixed time interval. The concerned systems are general hybrid systems given below which aren’t restricted to Markovian jump systems. It is assumed that modes of the systems are not decided in advance throughout this paper. We consider optimal tracking control problems to find both optimized states of the systems and an optimal candidate of the distributions of the modes for some reference signals over the finite time interval. We adopt simultaneous optimization approach of system states and mode distributions to guarantee the optimality of tracking control systems. In this paper, we transform the MPT approach for the hybrid estimation problems to control problems in order to solve the optimal hybrid tracking control problems. On this approach, given information of reference signals, we consider optimal control problems where we seek optimal control by which averaged energies are minimized for averaged systems. Finally we give numerical examples and verify that we can obtain both optimal tracking performance and optimal candidate of mode distributions by the algorithm presented in this paper.

**Notations:** Throughout this paper, the superscript “m” stands for the matrix transposition, \( \| \cdot \| \) denotes the Euclidean vector norm and \( \| v \|_2^2 \) also denotes the weighted norm \( v' R v \). \( O \) denotes the matrix with all zero components.

## 2. Problem Formulation

Let \( (\Omega, \mathcal{F}, \mathcal{P}) \) be a probability space, where \( \Omega \) is the sample space, \( \mathcal{F} \) is a \( \sigma \)-algebra of a subset of \( \Omega \) called events and \( \mathcal{P} \) is the probability measure on \( \mathcal{F} \). On this space, we consider the following system with mode transition and affected by Gaussian noises:

\[
\begin{align*}
    dx(t) &= A(t, \theta(t))x(t)dt + B_2(t, \theta(t))u(t, \theta(t))dt \\
         &+ B_3(t, \theta(t))r_c(t)dt + G(t)d\omega(t), \\
    x(0) &= x_0, \theta(0) = \theta_0 \\
    z_e(t) &= C_1(t, \theta(t))x(t) + D_{12}(t, \theta(t))u(t, \theta(t)) \\
         &+ D_{13}(t, \theta(t))r_c(t)
\end{align*}
\]

where \( x \in \mathbb{R}^n \) is the state, \( \omega(\cdot) \in \mathbb{R}^r \) is the random Wiener process satisfying the assumption \( A3 \) described below, \( u \in \mathbb{R}^m \) is the control input, \( z_e \in \mathbb{R}^k \) is the controlled output and \( r_c(\cdot) \in \mathbb{R}^{r_c} \) is a known or measurable reference signal. It is assumed that distributions of an initial state \( x_0 \) and an initial mode \( \theta_0 \) are given.

We assume that all these matrices are of compatible dimensions. For this system (1), we assume the following conditions:

**A1:** \( \text{rank } B_2(t, \theta(t)) = m \)

**A2:** \( D_{12}^T(t, \theta(t))D_{12}(t, \theta(t)) > O \)

**A3:**

\[
\begin{align*}
    \mathbb{E}[\omega(t)] &= 0, \\
    \mathbb{E}[\omega(t)\omega'(s)] &= W(t)\delta(t-s), \, t, s \in [0, T], \\
    \mathbb{E}[x(0)1_{\{\theta(0) = \theta_0\}}] &= \mu_0, \\
    \mathbb{E}[x(0)x'(0)1_{\{\theta(0) = \theta_0\}}] &= Q_0, \\
    \mathbb{E}[\omega(0)\omega'(0)] &= O, \\
    \mathbb{E}[\omega(t)\omega'(t)] &= O, \\
    \mathbb{E}[\omega(t)u'(t)] &= O, \\
    \mathbb{E}[\omega(t)r_c'(t)] &= O
\end{align*}
\]

where \( \mathbb{E} \) is the expectation with respect to \( \omega \). The indicator function \( 1_{\{\theta(t) = i\}} := 1 \) if \( \theta(t) = i \), and \( 1_{\{\theta(t) = i\}} := 0 \) if \( \theta(t) \neq i \). As the above assumption \( A3 \), \( x(t), r_c(t) \) and \( \omega(t) \) are mutually independent.

\( \theta(t) \) represents a mode sequence of the hybrid system (1). Let \( \mathcal{M} = \{1, 2, \ldots, M\} \) denote the state space of \( \theta(t) \). We define \( A(t, l) := A_1(t) \) and so on. In this paper, it is assumed that the probability distribution of \( \theta(\cdot) \) is undecided in advance, but among a finite number of candidate distributions. Let \( r \in \mathcal{N}_0 = \{1, 2, \ldots, n_0\} \), and let \( \mathcal{P} = \{\phi_1(\cdot), \ldots, \phi_{n_0}(\cdot)\} \) denote the set of such candidate distributions on \( \mathcal{M} \), i.e., for \( r \in \mathcal{N}_0 \) and \( t \in [0, T] \), \( \phi^{(r)}(t) = (\phi_1^{(r)}(t), \ldots, \phi_{n_0}^{(r)}(t)) \) with \( \phi_i^{(r)}(t) \geq 0 \) and \( \sum_{i=1}^{M} \phi_i^{(r)}(t) = 1 \).

The optimal tracking control problem on the finite time interval we address in this paper for the system (1) is to find an optimal input and the MPT (most probable trajectory) trajectory of \( \theta(t) \) and \( x(t) \), \( t \in [0, T] \), over the finite horizon \([0, T]\), using the information available on the known part of the information of the reference signal \( r_c(\cdot) \) for the given distributions of initial mode \( i_0 \) and initial state \( x_0 \). However we seek the optimal mode distribution from the prepared candidates of mode distributions not the optimal mode trajectory itself from the point of view of grasping the mode changes globally in this paper.

We define the following performance index for \( r \in \mathcal{N}_0 \) and \( t \in [0, T] \):

\[
J^{(r)}_{x, u}(x_0, u) := \mathbb{E}\left\{ \int_0^T \sum_{i=1}^{M} \phi_i^{(r)}(s)\|C_1(s, i)x(s) + D_12(s, i)u(s, i) + D_13(s, i)r_c(s)\|^2 dx + x'(T)Q_T x(T) \right\}
\]

where \( QT > O \) is a symmetric weighting matrix for the terminal state. Thus this performance index means the energies of controlled output and terminal state under some uncertainties averaged by the mode distributions for each \( r \in \mathcal{N}_0 \). Originally the control objective should be to minimize the tracking error between \( C_1(t, \theta(t))x(t) + D_{12}(t, \theta(t))u(t, \theta(t)) \) and the weighted reference signal \( -D_{13}(t, \theta(t))r_c(t) \). However, since \( \theta(t) \) is undecided in advance, we consider the averaged performance index (2) throughout the can-
didates $\phi_{(r)}(t)$ of mode distributions. We consider the optimization problems to decide $u(\cdot,i)$ and $r \in \mathbb{N}_0$ minimizing and $J^I_0(t)$ utilizing the known parts of the information of reference signal $R_T = \{r_e(t)|0 \leq l \leq T\}$. 

(Remark 1) While we may partially obtain any worse tracking performance over some time intervals within the whole time interval for the original hybrid system (1), since we consider the average value throughout the candidates of mode distributions by the performance index (2), the averaged tracking performance can be optimized over long run time interval. It is a significant point of a stochastic approach to design theory of hybrid systems, and also refer to[2] with regard to such motivations and examples of stochastic hybrid approach.

Since the mode $\theta(t)$ at each time is undecided in advance, we cannot directly design controllers for the system (1) including the unknown modes. Also, even if the modes are decided and accessible, the computational complexity can exponentially increase if we directly design the controllers for the system (1) including $\theta(t)$ explicitly. Hence we introduce the system averaged through the mode distributions for each $r \in \mathbb{N}_0$ as follows:

For notational simplicity, we adopt the following notation:

$$\mathcal{P}^{(r)}(t) = \sum_{i=1}^{M} \phi^{(r)}_i(t)F(t,i)$$

for a matrix function $F(t,i)$ and $r \in \mathbb{N}_0$. Similarly $F_0F_2^{(r)}(t) = \sum_{i=1}^{M} \phi^{(r)}_i(t)F_1(t,i)F_2(t,i)$ for matrix functions $F_1(t,i)$ and $F_2(t,i)$ and so on. Using these notations, we can shift the drift term in the system (1) to $\bar{A}^{(r)}(t)$ as follows:

$$dx(t) = \bar{A}^{(r)}(t)x(t)dt + u^{(r)}(t)dt + \mathcal{B}_3^{(r)}(t)r_c(t)dt + G(t)d\omega(t)$$

where

$$u^{(r)}(t) = (A(t,\theta(t)) - \bar{A}^{(r)}(t))x(t) + B_2(t,\theta(t))u(t,\theta(t)) + (B_3(t,\theta(t)) - \mathcal{B}_3^{(r)}(t))r_c(t)$$

By replacing the control input $u(t,i)$ by

$$B_2^+(t,i)[(\bar{A}^{(r)}(t) - A(t,i))x(t) + u^{(r)}(t)]$$

where $B_2^+(t,i)$ is the pseudo inverse matrix of $B_2(t,i)$ in the performance index (2), we define

$$L^{(r)}(t,x,u^{(r)}) := \sum_{i=1}^{M} \phi^{(r)}_i(t)[C_1(t,i)x(t) + D_{12}(t,i)B_2^+(t,i)[(\bar{A}^{(r)}(t)) - A(t,i)]x(t) + u^{(r)}(t)]$$

and

$$(\mathcal{B}_3^{(r)}(t) - B_3(t,i))r_c(t)]^2.$$ 

Then we can define the following performance index:

$$J^I_1(t,x,u^{(r)}(\cdot)) := \mathbb{E}\left\{\int_t^T L^{(r)}(s,x(s),u^{(r)}(s))ds + x'(T)Q_Tx(T)\right\}$$

We consider the optimal control problems to minimize $J^I_1(t)$ for the given parts of $R_T$. Let $V^{(r)}(t,x)$ and $u^{(r)}(t)$ be respectively the value function of these control problems and the optimal control input for each $r \in \mathbb{N}_0$ as follows:

$$V^{(r)}(t,x) := \inf_{u(\cdot)} J^I_1(t,x,u^{(r)}(\cdot))$$

$$u^{(r)}(t) := \arg\min\{V^{(r)}(t,x^{(r)*}(t)): r \in \mathbb{N}_0\}$$

and then define

$$V^{(r)}(t) := V^{(r)}(t,x^{(r)*}(t)).$$

Then the most probable distribution is $\phi^{(\hat{r}(t))}(\cdot)$. Let $x^{*}(t) = x^{(r(t))}(t)$ and we have

$$V^{(r)}(t) = V^{(r(t))}(t,x^{*}(t))$$

under the state history $\{x^{(r)}(s):0 \leq s < t\}$ with the given $x_0$ and $i_0$ where

$$\hat{r}(t) := \arg\min\{V^{(r)}(t,x^{(r)}(t)): r \in \mathbb{N}_0\}.$$ 

Now we define the following optimal trajectory in the sense of modified most probable trajectory (MPT).

[Definition 1] Given the matrices $Q_T$, $(\hat{r}(t),x^{*}(t))$, $t \geq 0$, is called an optimal trajectory if it minimizes $V^{(r)}(t,x)$. 

Then we formulate the following optimal hybrid LQG tracking control problems for the performance index (4).

The Optimal Hybrid LQG Tracking Control Problem for Linear Continuous-Time Systems on the Finite Time Interval:

Consider the system (1) and the performance index (4) with $t = 0$. Find the control input $\{u^*(t)\}$ and the pair $\{(\hat{r}(t),x^{*}(t))\}$, $t \in [0,T]$ minimizing the performance index (4) where the control strategy $u^*(t)$, $0 \leq t \leq T$, is decided based on the state information $X_t = \{x(s)|0 \leq s \leq t\}$ and the causal information of the reference signal $R_T = \{r_c(t)|0 \leq l \leq T\}$ until the time $t$. 

That is, in this paper, we consider causal tracking problems. 

(Remark 2) Originally the MPT approach has been adopted to obtain some solutions of nonlinear filtering problems ([4]) where exogenous noises are optimized to minimize noise energies. Q. Zhang has extended
the approach for filtering problems with single mode to hybrid systems with multi modes \((8,9)\). In this paper, we transform the extended MPT approach for the hybrid estimation problems to optimal tracking control problems. In this approach, the optimal hybrid estimation problems to optimal hybrid tracking control problems.

(Remark 3) Since we cannot directly construct any optimization algorithms because the original system (1) and the original performance index (2) includes some undecided modes, we introduce the averaged system (3) by the transformation of inputs utilizing any available information of the candidates of mode distributions. For the averaged system (3), we introduce the transformed performance index (4) and construct the optimization algorithm by these (3) and (4). After the optimal state and mode distribution are decided, the inversely transformed unique \(u(t,i)\) under the assumption \(A1\) is utilized to drive the original hybrid systems.

3. Hybrid Optimal Tracking Algorithms

The Hamilton-Jacobi-Bellman (HJB) equations associated with the optimal control problem to minimize \(J_\text{tr}^r\) with regards to \(u(\cdot)\) are given as follows:

\[
\frac{\partial V(r)(t,x)}{\partial t} = \min_u \left\{ L(r)(t,x,u(r)) \right\}
\]

\[
= +\partial_x \left\{ G(t)W(t)G^r(t)\frac{\partial}{\partial x}(\frac{\partial V(r)(t,x)}{\partial x}) \right\},
\]

\[
V(r)(T,x) = x'(T)Q_T x(T), \ r \in N_0
\]

Then we obtain the following minimizing \(u(\cdot)\).

\[
u(r)(t,x) = -D(r)\left\{ \frac{D(r)}{D(r)} - \frac{D(r)}{D(r)} \right\} \cdot \left( \frac{D(r)}{D(r)} + \frac{D(r)}{D(r)} \right)
\]

\[
= -\partial_x D(r)^{-1}(t) \frac{D(r)}{D(r)} \left( \frac{D(r)}{D(r)} + \frac{D(r)}{D(r)} \right) \frac{1}{2} \frac{\partial V(r)(t,x)}{\partial x}
\]

where \(D(t) = D_{12}B_{2}^\dagger(t)\).

Let

\[
V(r)(t,x) = x'(r)(t)x(t) + \frac{1}{2} \frac{\partial V(r)(t,x)}{\partial x}
\]

for some functions \(X(r), a(r)\) and \(b(r)\) with appropriate dimensions.

Then we obtain the following differential matrix equations, vector equations and scalar equations with terminal conditions:

\[
\dot{X}(r)(t) + X(r)(t)\hat{A}(r)(t) + \hat{A}(r)(t)X(r)(t) - X(r)(t)\left( \frac{D(r)}{D(r)} \right)^{-1} X(r)(t)
\]

\[
- \frac{\partial V(r)(t)}{\partial x} \left( \frac{D(r)}{D(r)} \right)^{-1} X(r)(t)
\]

\[
- \hat{A}(r)(t) + \partial_x \left( \frac{D(r)}{D(r)} \right)^{-1} X(r)(t)
\]

\[
- \partial_x \left( \frac{D(r)}{D(r)} \right)^{-1} X(r)(t)
\]

\[
\hat{A}(r)(t) \left( \frac{D(r)}{D(r)} \right)^{-1} X(r)(t)
\]

\[
= -\hat{a}(r)(t) = \left( \frac{D(r)}{D(r)} \right)^{-1} \hat{a}(r)(t)
\]

\[
\hat{b}(T) = 0
\]

If the Optimal LQG Hybrid Tracking Problem for Linear Continuous-Time Systems on the Finite Time Interval is solvable by state feedback for arbitrary \(r_c(\cdot)\), the equations (7) and (8) hold for \(r_c(\cdot) \equiv 0\). Then we have

\[
\hat{a}(r)(t) = -\hat{a}(r)(t) = \left( \frac{D(r)}{D(r)} \right)^{-1} \hat{a}(r)(t)
\]

\[
\hat{b}(T) = 0
\]

\[
\hat{a}(r)(t) = \partial_x \left( \frac{D(r)}{D(r)} \right)^{-1} \hat{a}(r)(t)
\]

\[
\hat{b}(T) = 0
\]
Consequently the equality (5) is reduced to $V^{(r)}(t,x) = x'X^{(r)}(t)x + b^{(r)}(t)$, and its value is determined by solving (6) and (10).  

Now we have the following theorem which gives the necessary conditions for the solvability of the **Optimal LQG Hybrid Tracking Problem for Linear Continuous-Time Systems on the Finite Time Interval** by state feedback.

**Theorem 1** Consider the system (1) and the performance index (4). Suppose the conditions A1, A2 and A3. Then, if the Optimal LQG Hybrid Tracking Problem for Linear Continuous-Time Systems on the Finite Time Interval is solvable by state feedback for (1) and (4), there exist positive semi-definite matrices $X^{(r)}(t)$, $r \in N_0$, and scalar functions $b^{(r)}(t)$, $r \in N_0$, satisfying the conditions $X^{(r)}(T) = Q_T$ and $b^{(r)}(T) = 0$ such that the matrix differential equations (6) and the scalar differential equations (10) hold over $[0,T]$. Moreover an optimal control strategy for the **Optimal LQG Hybrid Tracking Problem for Linear Continuous-Time Systems on the Finite Time Interval** by state feedback for (1) and (4) is given by

$$u^{(r)}(t,x^*) = -\left[DF^{(r)}(t)\right]^{-1}\left(DC_1^{(r)}(t)x^*(t) + DDA^{(r)}(t)x^*(t) + DDA^{(r)}(t)x^*(t) + DDA^{(r)}(t)x^*(t)\right).$$

Now we have the following optimal LQG tracking algorithm, which gives the solution of the **Optimal LQG Hybrid Tracking Problem for Linear Continuous-Time Systems on the Finite Time Interval** by state feedback.

***Optimal hybrid tracking control algorithm***

Step 1) Obtain $X^{(r)}(t)$ and $b^{(r)}(t)$ for $r \in N_0$ and $t \in [0,T]$ by solving (6) and (10) with terminal conditions.

Step 2) Using the above $X^{(r)}(t)$, design the candidate of control inputs

$$u^{(r)}(t,x) = -\left[DF^{(r)}(t)\right]^{-1}\left(DC_1^{(r)}(t)x(t) + DDA^{(r)}(t)x(t) + DDA^{(r)}(t)x(t) + DDA^{(r)}(t)x(t)\right)$$

and obtain the solution $x^{(r)*}(t)$ of the differential equation

$$dx(t) = A^{(r)}(t)x(t)dt + u(t)dt + B_3^{(r)}(t)r_c(t)dt + G(t)d\omega, \quad x(t - \delta t) = x_0$$

for small sampling interval $\delta t > 0$.

Step 3) Choose $\dot{r}(t)$ that minimizes $V^{(r)}(t,x) = x'X^{(r)}(t)x + b^{(r)}(t)$ for $X^{(r)}(t)$ and $b^{(r)}(t)$ obtained in the Step 1) at the current time $t$.

Step 4) For $t + \delta t$, repeat Step 2) and Step 3), and obtain $\dot{r}, u^{(r)}$ and $x^{(r)*}$.

Step 5) Repeat from Step 2) to Step 4) until $T$.

Then the most probable distribution is $\phi^{(r)}(\cdot)$ and the optimal trajectory is given by

$$(\hat{r}(t), x^*(t)) = (\hat{r}(t), x^{(r)*}(t)).$$

(Remark 4) Notice that the optimization algorithm presented in this paper includes a remarkably different point from state estimation algorithms in [5,6,9] and so on, while the hybrid estimation algorithms in [5,6,9] and so on have been constructed by reducing the estimation problems to some optimal control problems. Because the modes change continuously as time passes, in the cases of the optimal control problems, we need to discretize and update the optimal control input considering what values of the system states and value functions based on them can be obtained just before the current time while we can design the hybrid estimators at each time without considering the changes of the system states, modes and value functions based on them just before the current time. Actually, in the case of the estimation problems, because the optimal inputs are originally exogenous noises, we can design the optimal inputs and estimators independently both theoretically and on the constructions of optimization algorithms. However, in the case of optimal control problems by state feedback presented in this paper, the state feedback control input at the current time always depends on the system states and the optimal candidate of mode distributions just before the time. Therefore also notice that the appropriateness of the algorithm presented in this paper depends on the sampling interval seriously.

4. **Numerical Examples**

In this section, we study numerical examples to demonstrate the effectiveness of the presented design algorithm.

We consider the following two mode systems and assume that the system parameters are as follows:

$$dx(t) = A(\theta(t))x(t)dt + B_2u(t,\theta(t))dt + B_3r_c(t)dt + G(t)d\omega, \quad x(0) = x_0, \theta(0) = 0$$

$$z_c(t) = C_1x(t) + D_{12}u(t) + D_{13}r_c(t)$$

where

\begin{align*}
\cdot \text{ Mode 1:} & \quad A_1 = \begin{bmatrix} 0 & 1 \\ -1 & -0.4 \end{bmatrix}, \\
\cdot \text{ Mode 2:} & \quad A_2 = \begin{bmatrix} 0.5 & 1 \\ 0.8 & -0.2 \end{bmatrix},
\end{align*}
\[B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\]

\[C_1 = \begin{bmatrix} -0.5 & 0.1 \\ 0 & 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad D_{13} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.\]

We set \(x_0 \sim \mathcal{N}(0,0.5)\), the distribution of the initial mode \(t_0\) as \((1/2,1/2)\) and the covariance \(W(t) = 0.3\). The candidates of mode distributions are given as follows:

**Case I:**
\[\phi^{(1)} = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}, \phi^{(2)} = \begin{bmatrix} 1/10 \\ 9/10 \end{bmatrix}, \phi^{(3)} = \begin{bmatrix} 7/10 \\ 3/10 \end{bmatrix}\]

**Case II:**
\[\phi^{(1)} = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}, \phi^{(2)} = \begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}, \phi^{(3)} = \begin{bmatrix} 7/10 \\ 3/10 \end{bmatrix}\]

The paths of \(\theta(t)\) are generated randomly, and the performances are compared under the same circumstance, that is, the same set of the paths so that the performances can be easily compared.

We verify the effectiveness of the presented hybrid tracking control algorithm for reference signals \(r_c(t) = \sin(t)\) and \(r_c(t) = \sin(\pi t/30)\). In order to carry out the algorithms, we solve the backward pair of the differential equations (6) and (10) with the terminal conditions for the given \(r_c(t)\) and each candidate \(r = 1,2,3\) of given distributions, and obtain the pair \((\hat{r}(t), x^*(t))\) minimizing \(V^{(r)}(t)\) for \(t \in [0,T]\). We calculate sample trajectories for the original plant (11) with the conversely transformed original input \(u(t,\theta(t))\) through the same paths of the mode \(\theta(t)\).

Fig. 1 and Fig. 3 show the values of \(\|C_1 x(t) + \hat{D}_{12} u(t,\theta(t)) + \hat{D}_{13} r_c(t)\|^2\) for \(r_c(t) = \sin(t)\) in the cases I and II respectively. Fig. 2 and Fig. 4 show the optimal mode distributions for \(r_c(t) = \sin(t)\) in the cases I and II respectively. Notice that these figures do not show mode changes themselves. By these figures, it is shown that we can obtain better tracking performance in the Case II than the Case I for \(r_c(t) = \sin(t)\). Fig. 5 and Fig. 7 show the values of \(\|C_1 x(t) + D_{12} u(t,\theta(t)) + D_{13} r_c(t)\|^2\) for \(r_c(t) = \sin(\pi t/30)\) in the cases I and II respectively. Fig. 6 and Fig. 8 show the optimal mode distributions for \(r_c(t) = \sin(\pi t/30)\) in the cases I and II respectively. By these figures, it is shown that we obtain the different tracking performances at each time but almost the same values of tracking errors averaged over \([0,30]\) in the Case I and II for \(r_c(t) = \sin(\pi t/30)\). These results show that the optimized tracking performances and mode distributions depend on the cases of the prepared candidates of the mode distributions and the types of the reference signals numerically while we may prepare the candidates of mode distribution, the performance index and the reference signals independently.
5. Concluding Remarks

In this paper, we have studied the optimal LQG tracking control problems for linear continuous-time hybrid systems over the fixed time interval. The systems consist of several vector fields which transit between them, and aren’t restricted to the Markovian jump systems. We have considered the causal cases of reference signals as the tracking problems. We have adopted the modified MPT approach to solve the optimal hybrid tracking control problems. The modified MPT approach means that the optimal value functions are chosen at each time so that the corresponding candidates of mode distributions and optimal state feedback laws give the solutions of the optimal tracking control problems. The optimal tracking approach adopted in this paper guarantees the optimality of tracking performance in the meaning of the modified MPT.

In this paper, we have considered the problems that both the system states and modes are optimized. However, we have considered the problems that the distributions of the modes over the fixed time interval not the modes themselves are optimized to grasp the global behavior of the hybrid systems over the long time intervals. In order to optimize both the system states and distributions of the modes for the given reference signals, we have introduced the av-
eraged performance index with respect to the candidates of the mode distributions for the averaged systems. For this performance index, we have formulated the optimal tracking control problem for the information of the reference signals. We have derived the backward Riccati type differential equations and the backward scalar differential equations, which give the necessary conditions for the solvability of the optimal control problems. Then we have presented the optimal hybrid tracking control algorithm. Finally we have studied the numerical examples to compare the tracking performances.

References

[1] O. L. V. Costa: Linear minimum mean square error estimation for discrete-time Markovian jump linear systems; *IEEE Trans. Automat. Contr.* Vol. 39, No. 8, pp. 1685–1689 (1994)

[2] O. L. V. Costa, M. D. Fragoso and R. P. Marques: *Discrete-Time Markov Jump Linear Systems*, Springer (2005)

[3] M. D. Fragoso, O. L. V. Costa, J. Baczynski and N. Rocha: Optimal linear mean square filter for continuous-time jump linear systems; *IEEE Trans. Automat. Contr.* Vol. 50, No. 9, pp. 1364–1369 (2005)

[4] R. E. Mortensen: Maximum-likelihood recursive nonlinear filtering; *J. Optim. Theory Appl.* Vol. 2, pp. 386–394 (1968)

[5] G. Nakura: An approach to noncausal hybrid estimation for linear continuous-time systems with non-Gaussian noises; *Proceedings of SICE Annual Conference 2011*, WeC11-02, pp. 979–984 (CD-ROM) (2011)

[6] G. Nakura: An approach to hybrid smoothing for linear continuous-time systems with non-Gaussian noises; *Proceedings of the 43rd ISCIE International Symposium on Stochastic Systems Theory and Its Applications (SSS2011)*, pp. 63–72, FB2-2 (CD-ROM) (2011)

[7] D. D. Sworder and R. O. Rogers: An LQG solution to a control problem with solar thermal receiver; *IEEE Trans. Automat. Contr.* Vol. 28, pp. 971–978 (1983)

[8] Q. Zhang: Optimal filtering of discrete-time hybrid systems; *J. Optim. Theory Appl.* Vol. 100, No. 1, pp. 123-144 (1999)

[9] Q. Zhang: Hybrid filtering for linear systems with non-Gaussian disturbances; *IEEE Trans. Automat. Contr.* Vol. 45, pp. 50–61 (2000)

Author

Gou Nakura (Member)

Gou Nakura received the B. E. degree from Kyoto University, Kyoto, Japan, in 1996, and the M. E. and Ph. D. degrees in engineering from Shizuoka University, Hamamatsu, Japan, in 1999 and 2003, respectively. From April, 2003 to March, 2005, he was a Research Associate at the Department of Electrical and Electronic Engineering at Aoyama Gakuin University. From April, 2005 to September, 2005 and from April 2006 to September, 2006, he was a researcher at the Graduate School of Informatics, Kyoto University. From October, 2005 to March, 2006 and from November, 2006 to March, 2009, he was with the Graduate School of Engineering, Osaka University as a Specially Appointed Research Associate or Research Fellow at the Frontier Research Center (FRC) and the Department of Mechanical Engineering. His current research interests include robust control, preview control, estimation, stochastic systems and hybrid systems. He is a member of SICE, ISCIE and IEEJ.