Bound States in Mildly Curved Layers

P. Exner$^{a,b}$ and D. Krejčířik$^{a,c,e}$

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Abstract

It has been shown recently that a nonrelativistic quantum particle constrained to a hard-wall layer of constant width built over a geodesically complete simply connected noncompact curved surface can have bound states provided the surface is not a plane. In this paper we study the weak-coupling asymptotics of these bound states, i.e., the situation when the surface is a mildly curved plane. Under suitable assumptions about regularity and decay of surface curvatures we derive the leading order in the ground-state eigenvalue expansion. The argument is based on Birman-Schwinger analysis of Schrödinger operators in a planar hard-wall layer.

1 Introduction

The investigation of quantum particles constrained to a spatial region $\Omega$ of a prescribed shape, in particular, relations between its spectral properties and the geometry of $\Omega$ became an attractive problem when the technology progress made it possible to fabricate various mesoscopic systems for which this is a reasonable model – see, e.g., [LCM]. Curvature induced bound states in hard-wall strips and tubes have been demonstrated more than a decade ago [ES] and studied subsequently in numerous papers – cf. [DE, LCM] and references therein.

Much less attention was paid to quantum mechanics in layers, apart of the trivial planar case. While for an experimenter it is easier to prepare a semiconductor film on a curved substrate than to fabricate a quantum wire, from the mathematical point of view the opposite is true, and solving
the Schrödinger equation in a nontrivial layer is more complicated than the corresponding problem in a tube. It was noticed long time ago that in the formal limit of zero width the layer curvatures give rise to an effective attractive potential \( K \) but the first results for curved finite-width layers, with the particle Hamiltonian being a multiple of the Dirichlet Laplacian \( -\Delta^D \), appeared only recently.

We restrict ourselves to the case when \( \Omega \) is non-compact and nontrivially curved. The first proof of existence of geometrically induced bound states was given in [DEK1] under the assumption that the planar layer is curved only locally. A more general case of curved layers which are \textit{asymptotically} planar in the sense that the curvatures of the generating surface vanish at large distances has been discussed in [DEK2]. We have derived there several sufficient conditions for the existence of bound states expressed in terms of geometric quantities characterizing the reference surface. While these conditions cover a wide class of layers, they do not represent an ultimate result: it is not clear, \textit{e.g.}, whether bound states exist in layers built over surfaces of positive total Gauss curvature unless the latter are thin enough or endowed with a cylindrical symmetry.

After demonstrating their existence one is naturally interested in properties of the curvature-induced bound states. A particular question which we are going to address in the present paper concerns the weak-coupling regime. Since in our case the binding comes from the curvature alone, the described situation is expected to occur in \textit{mildly curved} layers. We will show that the layer has then a unique eigenvalue and derive an asymptotic expansion for the gap between this eigenvalue and the threshold of the essential spectrum. The leading term in this formula will depend on the mean curvature of the reference surface.

Let us describe briefly the contents of the paper. In the next section we will give a precise formulation of the problem and describe the main result summarized in Theorem 2.1. The rest is devoted to the proof. The strategy is adopted from the “lower-dimensional” case of curved strips and tubes analyzed in [DE, Sec. 4]. First, in Section 3 we consider Schrödinger operators acting in a planar layer built over \( \mathbb{R}^2 \). We derive a necessary and sufficient condition under which such an operator has a bound state in the weak coupling limit and find the asymptotic expansion of the eigenvalue. The result contained in Theorem 3.4 is of an independent interest; we prove it in a greater generality for a layer in \( \mathbb{R}^{2+m} \), \( m \geq 1 \). Next, in Section 4, we apply it to the case of mildly curved quantum layers. We express the operator \( -\Delta^D_\Omega \) in the coordinates \( (x,u) \), where \( x \equiv (x^1,x^2) \in \mathbb{R}^2 \) and \( u \in I := (-a,a) \) with \( a > 0 \) parametrize the surface and its normal space, respectively, and pass to a unitarily equivalent operator with an effective potential, which can be consecutively estimated by operators to which Theorem 3.4 can be applied. We will also show how the leading term in the expansion looks like in the case of a thin layer.
2 The results

To give a precise statement of the main result we need first to specify what we mean by mildly curved quantum layers. Let a family of surfaces $\Sigma_{\varepsilon} := p(\mathbb{R}^2)$ be given by a Monge patch

$$p: \mathbb{R}^2 \to \mathbb{R}^3, \quad p(x^1, x^2; \varepsilon) := (x^1, x^2, \varepsilon f(x^1, x^2)), \quad (2.1)$$

where $f$ is supposed to be a $C^4$-smooth function. Here $\varepsilon > 0$ is the parameter which controls the deformation; it is supposed to be small so that $\Sigma_{\varepsilon}$ is a mildly curved plane. The cross-product of the tangent vectors $p_\mu := \partial p/\partial x^\mu, \mu = 1, 2$, defines a unit normal field $n$ on $\Sigma_{\varepsilon}$. We put $\Omega_0 := \mathbb{R}^2 \times I$ and define a layer $\Omega_{\varepsilon} := \mathcal{L}(\Omega_0)$ of width $d = 2a$ over the surface $\Sigma_{\varepsilon}$ by virtue of the mapping $\mathcal{L}: \Omega_0 \to \mathbb{R}^3$ which acts as

$$\mathcal{L}(x, u; \varepsilon) := p(x; \varepsilon) + u n(x; \varepsilon). \quad (2.2)$$

The layer in question is thus the spatial region closed between two parallel surfaces – cf. [Sp3, Prob. 12 of Chap. 3] – represented by $p \pm an$. In this sense, we will hereafter refer to the surface coordinates $(x^1, x^2)$ as to longitudinal variables, while $u$ will be the transverse variable.

We consider a nonrelativistic spinless particle confined to $\Omega_{\varepsilon}$ which is free within it, and suppose that the boundary of the layer is a hard wall, i.e., the wavefunctions satisfy the Dirichlet boundary condition there. For the sake of simplicity we set Planck’s constant $\hbar = 1$ and the mass of the particle $m = \frac{1}{2}$. Then the Hamiltonian of the system can be identified with the Dirichlet Laplacian $-\Delta_{\Omega_{\varepsilon}}^D$ on $L^2(\Omega_{\varepsilon})$, which is defined for an open set $\Omega_{\varepsilon} \subset \mathbb{R}^3$ as the Friedrichs extension of the Laplacian acting on $C_0^\infty(\Omega_{\varepsilon})$ – cf. [RS4, Sec. XIII.15] or [Dav, Chap. 6]. The domain of the closure of the corresponding quadratic form is the Sobolev space $W^{1,2}_{0}(\Omega_{\varepsilon})$.

Next we must introduce some notation and formulate assumptions about the function $f$, in addition to the smoothness requirement mentioned above. We consider asymptotically planar surfaces – cf. [DEK2] – which means that the Gauss $K$ and mean $M$ curvatures of $\Sigma_{\varepsilon}$ vanish at large distances from a fixed point. This will hold if we require

$$\langle d1 \rangle f_\mu \in L^\infty(\mathbb{R}^2)$$

together with

$$\langle d2 \rangle f_{\mu\nu} \to 0 \quad \text{as} \quad |x| \to \infty$$

for $\mu, \nu \in \{1, 2\}$. This allows us to localize the essential spectrum. Adapting from [DEK2, Thm. 4.1] a simple argument based on a Neumann bracketing one finds that $\inf \sigma_{\text{ess}}(-\Delta_{\Omega_{\varepsilon}}^D) \geq \kappa_1^2$. Here $\{\kappa_j^2 := (\kappa_1 j)^2\}_{j=1}^\infty$, with $\kappa_1 := \pi/d$, are the eigenvalues of $-\Delta_{\Omega}^D$; the corresponding eigenfunctions will be denoted by $\chi_j$. Assuming in addition

$$\langle d3 \rangle f_{\mu\nu} \to 0 \quad \text{as} \quad |x| \to \infty \quad \text{and} \quad \langle d4 \rangle f_{\mu\nu\rho\sigma} \in L^\infty(\mathbb{R}^2)$$

for $\mu, \nu, \rho, \sigma \in \{1, 2\}$.
for \( \mu, \nu, \rho, \sigma \in \{1, 2\} \) we will be able to prove in Section 4.3 that the bound is sharp,

\[
\inf \sigma_{\text{ess}}(-\Delta_D^{\Omega}) = \kappa_1^2.
\] (2.3)

We have also to require that the derivatives of \( f \) satisfy the following integrability hypotheses:

\[
\langle r_{1,2} \rangle f_{,\mu \nu}, f_{,\mu \nu \rho} \in L^2(\mathbb{R}^2, (1 + |x|^\delta) \, dx), \text{ respectively}
\]

and

\[
\langle r_3 \rangle f_{,\mu \nu \rho \sigma} \in L^1(\mathbb{R}^2, (1 + |x|^\delta) \, dx)
\]

for some \( \delta > 0 \) and \( \mu, \nu, \rho, \sigma \in \{1, 2\} \). The main result of the present work reads as follows:

**Theorem 2.1.** Let \( \Omega_\varepsilon \) be a family of layers generated by the surfaces \( \Sigma_\varepsilon \) given by (2.1). Suppose that the function \( f \in C^4(\mathbb{R}^2) \) satisfies the hypotheses \( \langle d_{-4} \rangle \) and \( \langle r_{-3} \rangle \). If \( \Sigma_1 \) is not planar, then for all \( \varepsilon \) small enough

\[-\Delta_D^{\Omega_\varepsilon} \text{ has exactly one isolated eigenvalue } E(\varepsilon) \text{ below the essential spectrum.}
\]

Moreover, it can be expressed as

\[E(\varepsilon) = \kappa_1^2 - e^{2w(\varepsilon)^{-1}},\]

where \( w(\varepsilon) \) has the following asymptotic expansion

\[w(\varepsilon) = -\varepsilon^2 \sum_{j=2}^{\infty} (\chi_1, u\chi_j)_I^2 \left( \kappa_j^2 - \kappa_1^2 \right)^2 \int_{\mathbb{R}^2} \frac{|\hat{m}_0(\omega)|^2}{|\omega|^2 + \kappa_j^2 - \kappa_1^2} \, d\omega + O(\varepsilon^{2+\gamma})\]

with \( \gamma := \min\{1, \delta/2\} \). Here \( m_0 \) is the lowest-order term in the expansion of the mean curvature of \( \Sigma_\varepsilon \) w.r.t. \( \varepsilon \) — cf. (4.3).

**Remarks.** (a) The subscript “\( I \)” indicates the norm and the inner product in the space \( L^2(I) \). The sum runs in fact over even \( n \) only because one integrates over the interval \( I = (-a, a) \) on which the function \( u \mapsto \chi_1(u)u\chi_j(u) \) is odd for odd \( j \).

(b) The expression for the leading-term coefficient \( w_1 \) in the expansion \( w(\varepsilon) =: \varepsilon^2 w_1 + O(\varepsilon^{2+\gamma}) \) does not have a very transparent structure, however, for thin layers it can be rewritten as

\[w_1 = -\frac{1}{2\pi} \|m_0\|^2 + \frac{\pi^2 - 6}{24\pi^3} \|\nabla m_0\|^2 d^2 + O(d^4).\] (2.4)

This formula is instructive because the first term comes from the surface attractive potential \( K - M^2 \) which dominates the picture in this case, while the \( O(d^4) \) error term expresses the contribution of higher transverse modes. We refer to Section 4.4 for more details.
3 Weakly Coupled Schrödinger Operators in a Planar Layer

Let $M$ be an open connected precompact set in $\mathbb{R}^m$, $m \geq 1$. The object of our interest in this section will be the operator

$$H_\lambda = -\Delta_D + \lambda V$$

with $\lambda > 0$ on $\mathcal{H} := L^2(\mathbb{R}^2) \otimes L^2(M)$, (3.5)

where $-\Delta_D$ is the Dirichlet Laplacian on $\mathbb{R}^2 \times M$ defined as the closure of $-\Delta \otimes I_m + I_2 \otimes -\Delta^M_D$. In the last expression the unindexed $-\Delta$ stays for the Laplace operator in $\mathbb{R}^2$. The potential $V$ is supposed to be $H^1_0$-bounded, in other words

$$\langle a1 \rangle \exists a, b \geq 0 \ \forall \psi \in \text{Dom } Q_0 : \|V\psi\| \leq a\|\psi\| + b\|H^1_0\psi\|$$

where $\text{Dom } Q_0 = W^{1,2}(\mathbb{R}^2 \times M)$ is the form domain of $H_\lambda$.

The Dirichlet Laplacian $-\Delta^M_D$ on $L^2(M)$ has a purely discrete spectrum consisting of eigenvalues $\kappa_j^2 < \kappa_{j+1}^2 \leq \cdots \leq \kappa_j^2 < \cdots$; the corresponding normalized eigenfunctions will be denoted as $\chi_j$, where $j = 1, 2, \ldots$. The lowest eigenvalue is, of course, simple and the eigenfunction $\chi_1$ can be chosen positive – cf. [RS4, Sec. XIII. 12]. Important for our purpose are the transverse projections of the potential,

$$V_{jj'} := \int_M \bar{\chi}_j(y) V(\cdot, y) \chi_j'(y) \, dy$$

and the analogous quantities for other functions on $\mathbb{R}^2 \times M$, in particular for $|V|$. We shall adopt the assumptions

$$\langle a2 \rangle |V|_{11} \in L^{1+\delta}(\mathbb{R}^2) \quad \text{and} \quad \langle a3 \rangle |V|_{11} \in L^1(\mathbb{R}^2, (1 + |x|^\delta) \, dx)$$

for some $\delta > 0$. We also suppose that the essential spectrum of $H_\lambda$ does not start below the lowest transverse-mode threshold, i.e.,

$$\langle a0 \rangle \inf \sigma_{\text{ess}}(H_\lambda) \geq \kappa_1^2$$

This is true, in particular, if $V$ vanishes at large distances from a fixed point.

The goal of this section is to show that for $\lambda$ small enough the discrete spectrum of $H_\lambda$ below $\kappa_j^2$ is not empty provided the projection of $V$ onto $\chi_1$ is not repulsive in the mean. This part of spectrum then contains only one eigenvalue $E(\lambda)$ for such a small $\lambda$ and it approaches $\kappa_j^2$ as $\lambda$ tends to zero. The last claim follows from the following elementary fact.

**Proposition 3.1.** Assume $\langle a1 \rangle$. Then there are positive constants $\lambda_0$ and $c$ such that for all $\lambda \in (0, \lambda_0)$ we have $H_\lambda \geq \kappa_1^2 - c\lambda$.  

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Proof: Using the Schwarz inequality and the assumption (a1) together with the inequality between the geometric and arithmetic means, we have for all $\psi \in \text{Dom } Q_0$ the bound

$$Q_\lambda[\psi] := \|H_0^{\frac{1}{2}} \psi\|^2 \geq \|H_0^{\frac{1}{2}} \psi\|^2 - \lambda \|\psi\| \|V \psi\| \geq (1 - \lambda b/2) \|H_0^{\frac{1}{2}} \psi\|^2 - \lambda (a + b/2) \|\psi\|^2.$$ 

Since $\|H_0^{\frac{1}{2}} \psi\| \geq \kappa_1 \|\psi\|$, we obtain the assertion by putting $\lambda_0 := 2/b$ and $c := a + (1 + \kappa_1^2) b/2$. \[\square\]

3.1 Birman-Schwinger Analysis

Using the orthonormal basis of $L^2(M)$ given by $\{\chi_j\}$, the free resolvent operator $R_0(\alpha) := (H_0 - \alpha^2)^{-1}$ is decoupled in the following way

$$R_0(\alpha) = \sum_{j=1}^{\infty} \chi_j (\Delta + k_j(\alpha)^2)^{-1} \bar{\chi}_j, \quad k_j(\alpha) := \sqrt{\kappa_j^2 - \alpha^2}.$$ 

If we are interested in eigenvalues of $H_\lambda$ below the lowest transverse mode, we have to consider $\alpha \in [0, \kappa_1)$ and the two-dimensional resolvent in the middle of the expansion is well defined for any $j = 1, 2, \ldots$. It can be expressed in terms of Hankel’s function – cf. [AGHH, Chap. I.5]. Passing to Macdonald’s functions $K_0$ by [AS, 9.6.4], we arrive at the following integral kernel formula

$$R_0(x, y, x', y'; \alpha) = \frac{1}{2\pi} \sum_{j=1}^{\infty} \chi_j(y) K_0 \left( k_j(\alpha)|x - x'| \right) \bar{\chi}_j(y').$$ \hspace{1cm} (3.6)$$

Define $K(\alpha) := |V|^{\frac{1}{2}} R_0(\alpha) V^{\frac{1}{2}}$, where we have employed the usual sign convention, $V^\frac{1}{2} := \sqrt{|V|} \text{ sgn } V$. According to Birman-Schwinger principle – cf. [Sim] – the function $\alpha(\lambda)^2 \equiv E(\lambda)$ is an eigenvalue of $H_\lambda$ if and only if the operator $\lambda K(\alpha)$ has the eigenvalue $-1$, i.e.,

$$\alpha^2 \in \sigma_{\text{disc}}(H_\lambda) \iff -1 \in \sigma_{\text{disc}}(\lambda K(\alpha)).$$ \hspace{1cm} (3.7)$$

The first term in the expansion (3.6) referring to $j = 1$ has a singularity at $\alpha = \kappa_1$, and as usual in such problems we have to single it out. For this purpose we first introduce the following decomposition.

**Lemma 3.2.** There are real-analytic functions $f$ and $g$ such that

$$\forall u \in (0, \infty) : \quad K_0(u) = f(u) \ln u + g(u)$$

and
(i) $f(u) = -1 + O(u^2)$, $g(u) = (\ln 2 - \gamma_E) + O(u^2)$ as $u \to 0^+$,

(ii) $\exists C_1 > 0 \forall u \in (0, \infty) : \max\{f(u), g(u)\} \leq C_1 e^{-u},$

where $\gamma_E$ denotes Euler’s constant.

Proof: Around the origin a good choice to approximate the Macdonald function $K_0$ would be $f := -I_0$, where $I_0$ is the other modified Bessel function [AS, 9.6] but it has a bad behaviour at large distances. Hence we use an interpolation, for instance,

$$f(u) := -e^{-u^2}I_0(u) - (1 - e^{-u^2})K_0(u),$$

$$g(u) := e^{-u^2}I_0(u) \ln u + [1 + (1 - e^{-u^2})\ln u]K_0(u).$$

Using the relation [AS, 9.6.13], we obtain the behaviour at the origin (i). On the other hand, it follows by [AS, 9.7.2] that $f$ and $g$ have a faster-than-exponential decay at infinity, which gives together with (i) that there is a positive $C_1$ such that (ii) holds. We will thus use the decomposition $K(\alpha) = L_\alpha + M_\alpha$, where

$$L_\alpha(x, y, x', y') := -\frac{1}{2\pi} |V(x, y)|^{\frac{1}{2}} \chi_1(y) \ln k_1(\alpha) \chi_1(y') V(x', y')^{\frac{1}{2}}$$

contains the singularity and the regular $M_\alpha$ splits into two parts again,

$$M_\alpha = A_\alpha + B_\alpha.$$ The operator $B_\alpha$ is given by the projection of the resolvent on higher transverse modes, i.e., $B_\alpha := |V|^{\frac{1}{2}} R_0^\perp(\alpha) V^{\frac{1}{2}}$ with

$$R_0^\perp(\alpha) := \sum_{j=2}^{\infty} \chi_j \left(-\Delta + k_j(\alpha)^2\right)^{-1} \bar{\chi}_j,$$

and the kernel of the remaining term is therefore

$$A_\alpha(x, y, x', y') := \frac{1}{2\pi} |V(x, y)|^{\frac{1}{2}} \chi_1(y) \left(K_0(k_1(\alpha)|x - x'|) + \ln k_1(\alpha)\right) \chi_1(y') V(x', y')^{\frac{1}{2}}.$$

We note that $M_\alpha$ is by definition well defined for $\alpha = \kappa_1$. In particular,

$$A_{\kappa_1}(x, y, x', y') = \frac{1}{2\pi} |V(x, y)|^{\frac{1}{2}} \chi_1(y) \left(\gamma_E + \ln \frac{|x - x'|}{2}\right) \chi_1(y') V(x', y')^{\frac{1}{2}}.$$

Furthermore, we have the following lemma. Since its proof is purely technical, we postpone it to Section 3.2 below.

**Lemma 3.3.** Assume (a1–3). Then there are positive $C_2, C_3$ and $C_4$ such that

(i) $\forall \alpha \in [0, \kappa_1] : \|M_\alpha\| < C_2,$
(ii) \( \|M_\alpha - M_{\kappa_1}\| \leq C_3 \lambda^\gamma \), with \( \gamma := \min\{1, \delta/2\} \).

(iii) \( \left\| \frac{dM_\alpha(w)}{dw} \right\| < C_4 |w|^{-1} \) for \( \lambda \) sufficiently small, where \( w := (\ln k_1(\alpha))^{-1} \).

We recall that Proposition 3.1 yields \( \alpha^2 \to \kappa_1^2 \) as \( \lambda \to 0^+ \), and consequently, \( k_1(\alpha) \to 0^+ \). Hence the auxiliary variable \( w \) is well defined and negative for \( \lambda \) small enough, and \( w \to 0^- \) as \( \lambda \to 0^+ \).

By the Birman-Schwinger principle (3.7) eigenvalues of \( \mathcal{H}_\lambda \) correspond to singularities of the operator \( (I + \lambda \mathcal{K}(\alpha))^{-1} \) which we can equivalently express as

\[
(I + \lambda \mathcal{K}(\alpha))^{-1} = [I + \lambda(I + \lambda M_\alpha)^{-1}L_\alpha]^{-1} (I + \lambda M_\alpha)^{-1}.
\]

Since \( M_\alpha \) is bounded independently of \( \alpha \) due to Lemma 3.3(i), we have \( \|\lambda M_\alpha\| < 1 \) for all sufficiently small \( \lambda \), and therefore the second term on the right hand side of the last relation is a bounded operator. On the other hand, \( \lambda(I + \lambda M_\alpha)^{-1}L_\alpha \) is a rank-one operator of the form \((\psi,\cdot)|\varphi\rangle\), where

\[
\psi(x,y) := -\frac{\lambda}{2\pi} \ln k_1(\alpha) V(x,y)^{\frac{1}{2}} \chi_1(y),
\]

\[
\varphi(x,y) := [(I + \lambda M_\alpha)^{-1}V]^\frac{1}{2}\chi_1(x,y),
\]

so it has just one eigenvalue which is \((\psi,\varphi)\). The requirement that the latter equals \(-1\) yields the implicit equation

\[
w = F(\lambda, w), \quad F(\lambda, w) := \frac{\lambda}{2\pi} \left( V^{\frac{1}{2}} \chi_1, (I + \lambda M_{\alpha(w)})^{-1} |V|^\frac{1}{2} \chi_1 \right),
\]

(3.9)

where we use the auxiliary variable \( w \) defined in Lemma 3.3 which determines the energy via \( \alpha^2 = \kappa_1^2 - e^{2w^{-1}} \). Solving (3.9), we arrive at the main result of this section.

**Theorem 3.4.** Assume \( (a0-3) \) and exclude the trivial case, \( V \equiv 0 \). Then \( \mathcal{H}_\lambda \) has for sufficiently small \( \lambda > 0 \) exactly one eigenvalue \( E(\lambda) \) if and only if

\[
\int_{\mathbb{R}^2} V_{11}(x) \, dx \leq 0,
\]

(3.10)

and in this case we can express it as \( E(\lambda) = \kappa_1^2 - e^{2w(\lambda)^{-1}} \), where \( w(\lambda) \) has the following asymptotic expansion:

\[
w(\lambda) = \frac{\lambda}{2\pi} \int_{\mathbb{R}^2} V_{11}(x) \, dx
\]

\[
+ \left\{ \frac{\lambda}{2\pi} \left( \int_{\mathbb{R}^2 \times \mathbb{R}^2} V_{11}(x) \left( \gamma_E + \ln \frac{|x - x'|}{2} \right) V_{11}(x') \, dx \, dx' \right. \right.
\]

\[
- \sum_{j=2}^{\infty} \int_{\mathbb{R}^2 \times \mathbb{R}^2} V_{1j}(x) K_0(k_j(\kappa_1)|x - x'|) V_{j1}(x') \, dx \, dx' \left. \right\}
\]

\[
+ O(\lambda^{2+\gamma})
\]

(3.11)
with \( \gamma := \min\{1, \delta/2\} \).

**Proof:** Inserting the identity

\[
(I + \lambda M_\alpha)^{-1} = I - \lambda M_{\kappa_1} - \lambda(M_\alpha - M_{\kappa_1}) + \lambda^2 M^2_\alpha (I + \lambda M_\alpha)^{-1},
\]

into (3.9) and employing Lemma 3.3(i-ii), we get the asymptotic expansion (3.11). Note that its coefficients are well defined owing to (3.3); in particular, for the second-order terms it is true since \( (2\pi)^j \mathcal{X}_1, M_{\kappa_1} |V|^{1/2} \mathcal{X}_1 \) estimated by the Schwarz inequality is finite because of (3.3) and Lemma 3.3(i). The sufficient and necessary condition (3.10) follows from the fact that \( \tilde{V}(\lambda) \) converges to \( \kappa_1^2 \) as \( \lambda \to 0^+ \) because of Proposition 3.1, which corresponds to the situation that \( w \) goes to zero assuming negative values. In view of (3.11) it is evident that this is the case if \( \int V_{11} \) is strictly negative. We want to show that \( w \) is negative for small \( \lambda \) also if the first term in the expansion vanishes. Suppose first that the potential projections \( V_{jj'} \) belong to \( L^2(\mathbb{R}^2) \). Then we have

\[
\int_{\mathbb{R}^2 \times \mathbb{R}^2} V_{jj'}(x) K_0(k_j(\kappa_1)|x - x'|) V_{jj'}(x') \, dx \, dx' = (V_{jj'}, K_0 \ast V_{jj'}) = (\tilde{V}_{jj}, \tilde{K}_0 \ast \tilde{V}_{jj}) = 2\pi (\tilde{V}_{jj}, \tilde{K}_0 \tilde{V}_{jj}) = (2\pi)^2 \int_{\mathbb{R}^2} |\tilde{V}_{jj}(\omega)|^2 \, d\omega > 0, \quad (3.12)
\]

because \( (2\pi)^{-1} K_0(k_j(\kappa_1)|x|) \) is the Green function of \( -\Delta + k_j(\kappa_1)^2 \). At the same time, by Lemma 3.2(i) and the fact that we deal with the case \( \int V_{00} = 0 \) now,

\[
-\int_{\mathbb{R}^2 \times \mathbb{R}^2} V_{11}(x) \left( \gamma_E + \ln \frac{|x - x'|}{2} \right) V_{11}(x') \, dx \, dx' = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^2 \times \mathbb{R}^2} V_{11}(x) \left( K_0(\varepsilon|x - x'|) + \ln \varepsilon \right) V_{11}(x') \, dx \, dx' = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^2 \times \mathbb{R}^2} V_{11}(x) K_0(\varepsilon|x - x'|) V_{11}(x') \, dx \, dx' \quad (3.13)
\]

is also positive, which follows by the consecutive use of the Fourier transform trick (3.12). For a general \( V_{jj'} \notin L^2(\mathbb{R}^2) \) we can approximate the potential by the cut-off functions \( V^N := \chi_{[-N,N]^2} \text{sgn } V \min\{|V|, N\} \), where \( \chi_\mathcal{A} \) denotes the characteristic function of a set \( \mathcal{A} \). Then the expressions in the first line of (3.12) and the last line of (3.13) are approximated by sequences whose elements are positive by the above argument. Using the dominated convergence and the absolute continuity of the Lebesgue integral we find that the limits exist, and of course they are positive again. Together we get that the second-order term in the expansion (3.11) is negative.
It remains to check that (3.11) is the only solution of (3.9) for $\lambda$ small. This will be true if we prove it for a non-positive $V$ since it represents a more attractive interaction; we thus replace $V$ by $-|V|$ in the expression for $F$. Using the Schwarz and triangle inequalities together with Lemma 3.3(i), (iii), we arrive at the estimate

$$
\left| \frac{\partial F}{\partial w} (\lambda, w) \right| = \frac{\lambda}{2\pi} \left| |V|^2 \chi_1, (I + \lambda M_{\alpha(w)})^{-1} \lambda \frac{dM_{\alpha(w)}}{dw} (I + \lambda M_{\alpha(w)})^{-1} |V|^2 \chi_1 \right| 
\leq \frac{\lambda^2}{2\pi} \left\| |V|^2 \chi_1 \right\|_2 \left\| (I + \lambda M_{\alpha(w)})^{-1} \right\|_2^2 \left\| \frac{dM_{\alpha(w)}}{dw} \right\| 
\leq \frac{\lambda^2}{2\pi} (1 - \lambda C_2)^{-2} C_4 \frac{|V|_{11}}{|w|} \left\| |V|_{11} \right\|^2_1,
$$

which holds for $\lambda$ sufficiently small and strictly less than $C_2^{-1}$. The norm of the potential is finite by assumption $\langle a3 \rangle$. Excluding the trivial case $V \equiv 0$ we note that there exists a $c' > 0$ such that the inequality $|w|^{-1} \leq c' \lambda^{-1}$ is valid for any solution $w$ of the implicit equation (3.9) and $\lambda$ small enough. So there is a $C_5 > 0$ such that the partial derivative of $F$ w.r.t. $w$ is bounded by $C_5 \lambda$ for all sufficiently small $\lambda$. Since any two solutions $w_1, w_2$ of the equation $w = F(\lambda, w)$ have to fulfill

$$
|w_2 - w_1| = \left| \int_{w_1}^{w_2} \frac{\partial F}{\partial w} dw \right| \leq \left| \int_{w_1}^{w_2} \left| \frac{\partial F}{\partial w} \right| dw \right| \leq C_5 \lambda |w_2 - w_1|,
$$

the uniqueness is ensured for $\lambda < C_5^{-1}$. □

### 3.2 Proof of Lemma 3.3

A reader not interested in the following technical analysis of the operator $M_{\alpha}$ may skip this subsection. We recall that $M_{\alpha}$ is given by the sum of $A_{\alpha}$ and $B_{\alpha}$, which are of a different nature. To prove the assertions of Lemma 3.3 we consider each of the operators separately.

#### 3.2.1 Analysis of $A_{\alpha}$

We show first that this operator is of the Hilbert-Schmidt class for $\alpha = \kappa_1$.

**Lemma 3.5.** Assume $\langle a2 \rangle$, $\langle a3 \rangle$, then $A_{\kappa_1}$ is a Hilbert-Schmidt operator.

**Proof:** Let us compute the HS-norm

$$
\|A_{\kappa_1}\|_{HS}^2 = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |V|_{11}(x) \left| \gamma_E + \ln \frac{|x - x'|}{\gamma_0} \right|^2 |V|_{11}(x') dx dx'.
$$
It can be estimated by a sum of two integrals. The first one will be finite if $|V|_{11} \in L^1(\mathbb{R}^2)$ which is true by (a3), so it remains to check that the integral

$$J := \int_{\mathbb{R}^2 \times \mathbb{R}^2} |V|_{11}(x) \ln^2 |x - x'| |V|_{11}(x') \, dx \, dx'$$

is finite. To estimate it, we divide the region of integration into two parts: if $|x - x'| \geq 1$, then for any $\delta > 0$ there is a $C_\delta > 0$ such that

$$\ln^2 |x - x'| \leq \ln^2(|x| + |x'|) \leq \ln^2(1 + |x|)(1 + |x'|) \leq C_\delta (1 + |x|^\delta)(1 + |x'|^\delta)$$

and the contribution to $J$ is thus finite because of (a3). On the other hand, for $|x - x'| < 1$ we use the Hölder and Young inequalities

$$J = \int_{\mathbb{R}^2} dx |V|_{11}(x) \int_{\mathbb{R}^2} dx' \chi_{[0,1]}(|x - x'|) \ln^2 |x - x'| |V|_{11}(x')$$

$$\leq \| |V|_{11} \|_1 + \delta \| \chi_{[0,1]} \|_1 \ln^2 \| |V|_{11} \|_1,$$

where $r := (1 + \delta^{-1})/2$, which yields a finite value owing to (a2). \qed

In a similar way we can estimate the HS-norm of $A_\alpha - A_{\kappa_1}$.

**Lemma 3.6.** Assume (a3), then $\|A_\alpha - A_{\kappa_1}\|_{HS} \leq C_6 k_1(\alpha)^\delta$ holds with a positive constant $C_6$ independent of $\alpha$.

**Proof:** Using the decomposition of Lemma 3.2,

$$(A_\alpha - A_{\kappa_1})(x, y, x', y')$$

$$= \ln^2 |x - x'| |V|_{11}(x) \left\{ (f(k_1(\alpha)|x - x'|) - f(0)) \ln k_1(\alpha)|x - x'|$$

$$+ g(k_1(\alpha)|x - x'|) - g(0) \right\} \chi_{1}(y') V(x', y')^{1/2},$$

which yields the estimate $\|A_\alpha - A_{\kappa_1}\|_{HS}^2 \leq 2(J_1^2 + J_2^2),$ where

$$J_\ell := \int_{\mathbb{R}^2 \times \mathbb{R}^2} |V|_{11}(x) h_\ell(k_1(\alpha)|x - x'|) |V|_{11}(x') \, dx \, dx', \quad \ell = 1, 2,$$

with $h_1(u) := [(f(u) - f(0)) \ln u]^2$ and $h_2(u) := [g(u) - g(0)]^2$. Since these functions are bounded, both integrals are finite under the assumption $|V|_{11} \in L^1(\mathbb{R}^2)$. Consequently, $\|A_\alpha - A_{\kappa_1}\|_{HS}$ has a bound independent of $\alpha$. However, we want to prove in addition that $A_\alpha$ converges in HS-norm to $A_{\kappa_1}$, i.e., the stated assertion. By virtue of Lemma 3.2, we have for any $\delta \in (0, 2)$ the rough bounds $h_\ell(u) \leq c_\ell u^\delta$ for some constants $c_\ell > 0$. This bound together with $|x - x'|^\delta \leq \max\{1, 2^{\delta-1}\} \max(|x|^{\delta} + |x'|^{\delta})$, and $|x|^\delta + |x'|^\delta \leq (1 + |x|^\delta)(1 + |x'|^\delta)$, yields the estimate

$$J_\ell \leq \tilde{C}_6 k_1(\alpha)^\delta \left( \int_{\mathbb{R}^2} |V|_{11}(x) (1 + |x|^\delta) \, dx \right)^2,$$
Lemma 3.7. Assume \( \langle a2 \rangle \) and \( \langle a3 \rangle \), then for sufficiently small \( w \),
\[
\left\| \frac{dA_\alpha(w)}{dw} \right\| < \frac{C_7}{|w|} \quad \text{with some} \quad C_7 > 0.
\]

Proof: Let us choose the contour \( \Gamma := \{ z = w + we^{it} : t \in [0, 2\pi) \} \) in the complex plane, where \( w \) is supposed to be negative and small enough so that \( A_{\alpha(w)} \) is analytic in the interior of the curve. We can use the Cauchy integral formula to estimate the complex derivative
\[
\left\| \frac{dA_\alpha(w)}{dw} \right\| = \left\| \frac{1}{2\pi i} \oint_{\Gamma} \frac{A_{\alpha(z)}}{(z-w)^2} \, dz \right\| \leq \sup_{z \in \Gamma} \left\| A_{\alpha(z)} \right\| \frac{1}{|w|}.
\]

It is straightforward to modify the proof of Lemma 3.6 (including the technical Lemma 3.2) in order to check the continuity of \( A_\alpha \) in \( \kappa_1 \) w.r.t. to the HS-norm also for complex values of \( \alpha \). This yields the desired result.

3.2.2 Analysis of \( B_\alpha \)

Mimicking [BGRS, Lemma 2.2] and [BEGK, Lemma 2.3] we denote as \( \mathcal{H}_1 \subset \mathcal{H} \) the subspace \( L^2(\mathbb{R}^2) \otimes \{ \chi_1 \} \). Let \( P_1 \) be the corresponding projection and \( P_1^\perp := I - P_1 \), then we can write \( R_0^\perp(\alpha) \) defined in (3.8) as \( P_1^\perp R_0(\alpha) P_1^\perp \).

Lemma 3.8. Assume \( \langle a1 \rangle \), then \( \alpha \mapsto B_\alpha \) is uniformly bounded in the operator-norm topology on the interval \( [0, \kappa_2 - \varepsilon] \) for any \( \varepsilon \in (0, \kappa_2] \).

Proof: Since the lowest point in the spectrum of \( H_0 P_1^\perp \downarrow P_1^\perp \mathcal{H} \) is \( \kappa_2^2 \), the operator-valued function \( R_0^\perp(\cdot) \) has an analytic continuation into the region \( \{ \alpha \in \mathbb{C} | \alpha^2 \in \mathbb{C} \setminus [\kappa_2^2, \infty) \} \). In particular, this region includes the interval \( [0, \kappa_1] \) actually considered, where one has the following estimate on the norm
\[
\left\| R_0^\perp(\alpha) \right\| = \sup_{j=2,3,...} (\kappa_j^2 - \alpha^2)^{-1} = (\kappa_2^2 - \alpha^2)^{-1} < (\kappa_2^2 - \kappa_1^2)^{-1}.
\]
If $V$ was essentially bounded then this result would be sufficient for the boundedness of $B_\alpha$ because $\|B_\alpha\| \leq \|V\frac{1}{2}\|_\infty \|R_0^\perp(\alpha)\| \|V\frac{1}{2}\|_\infty$. In order to accommodate the extra factors $|V\frac{1}{2}|, V\frac{1}{2}$ under our weaker assumption $(a1)$, we introduce the quadratic form

$$b_\alpha(\phi, \psi) := (\phi, B_\alpha \psi) = \left(R_0^\perp(\alpha)^\frac{1}{2} \, P_1^\perp \left|V\right|^\frac{1}{2} \phi, R_0^\perp(\alpha)^\frac{1}{2} \, P_1^\perp \left|V\right|^\frac{1}{2} \psi \right)$$

with $\alpha$ supposed to be a real number from an interval $[0, \kappa_2 - \varepsilon]$. To check boundedness of this form, it is sufficient to verify that $R_0^\perp(\alpha)^\frac{1}{2} \, P_1^\perp |V|^\frac{1}{2}$ is a bounded operator, i.e., that $|V|^\frac{1}{2}$ is $(R_0^\perp(\alpha)^\frac{1}{2} \, P_1^\perp)$-bounded, which is equivalent to the statement that there exist $c_1, c_2 \geq 0$ such that

$$\forall \psi \in \text{Dom} \, Q_0 : \left\|V\frac{1}{2} \, P_1^\perp \psi\right\|^2 \leq c_1 \left\|\psi\right\|^2 + c_2 \left\|R_0^\perp(\alpha)^\frac{1}{2} \, P_1^\perp \psi\right\|^2.$$

However,

$$\left\|V\frac{1}{2} \, P_1^\perp \psi\right\|^2 = \left(P_1^\perp \psi, |V| P_1^\perp \psi \right) \leq \left\|P_1^\perp \psi\right\| \left\|V \, P_1^\perp \psi\right\|$$

$$\leq \left\|P_1^\perp \psi\right\| \left(a \left\|P_1^\perp \psi\right\| + b \left\|H_0^\frac{1}{2} \, P_1^\perp \psi\right\| \right)$$

$$\leq \left(a + \frac{b}{2}\right) \left\|\psi\right\|^2 + \frac{b}{2} \left\|H_0^\frac{1}{2} \, P_1^\perp \psi\right\|^2.$$

In the first step we have used the Schwarz inequality, in the second our hypothesis $(a1)$, and finally the inequality between the geometric and arithmetic means together with $\|P_1^\perp \psi\| \leq \|\psi\|$. At the same time,

$$\left\|R_0^\perp(\alpha)^\frac{1}{2} \, P_1^\perp \psi\right\|^2 = \left(P_1^\perp \psi, (H_0 - \alpha^2) \, P_1^\perp \psi \right) \geq \left\|H_0^\frac{1}{2} \, P_1^\perp \psi\right\|^2,$$

so we can identify $c_1 := a + b/2$ and $c_2 := b/2$ which completes our proof. \qed

This establishes Lemma 3.3(i) for the operator $B_\alpha$. Since $k_1(\alpha)^2 \leq c\lambda$ owing to Proposition 3.1, the property (ii) is included in the following result.

**Lemma 3.9.** Assume $(a1)$, then there exists $C_7 > 0$ such that

$$\|B_\alpha - B_{\kappa_1}\| \leq C_7 k_1(\alpha)^2.$$

**Proof:** Using the first resolvent identity [Wei, Thm. 5.13], we infer

$$\|B_\alpha - B_{\kappa_1}\| = |\alpha - \kappa_1| \left\|V\frac{1}{2} \, R_0^\perp(\alpha) R_0^\perp(\kappa_1) V\frac{1}{2}\right\|$$

$$\leq |\alpha - \kappa_1| \left\|V\frac{1}{2} R_0^\perp(\alpha)\right\| \left\|R_0^\perp(\alpha)\right\| \left\|R_0^\perp(\kappa_1)\right\| \left\|R_0^\perp(\kappa_1)\right\| \left\|V\frac{1}{2}\right\|.$$

However, $|\kappa_1 - \alpha| \leq k_1(\alpha)^2$ and it is clear from the proof of Lemma 3.8 that the remaining factors at the r.h.s. of the above estimate are finite. \qed
Similarly to the operator $A_\alpha$, we need also a bound on the derivative of $B_\alpha(w)$ w.r.t. $w$. However, the situation in the present case is simpler because $R_0^\perp(\alpha)$ itself is analytic in an open complex set containing $\alpha = \kappa_1$.

**Lemma 3.10.** Assume $\langle a1 \rangle$, then for sufficiently small $w$,

\[
\left\| \frac{dB_\alpha(w)}{dw} \right\| \leq C_8 \quad \text{with some} \quad C_8 > 0.
\]

**Proof:** Since

\[
\frac{dB_\alpha(w)}{dw} = \frac{dB_\alpha}{d\alpha} \frac{d\alpha}{dw} = 2\frac{\varepsilon^{2w-1}}{w^2} |V|^{\frac{1}{2}} R_0^\perp(\alpha(w))^2 V^{\frac{1}{2}}
\]

and the prefactor function $w^{-2}\varepsilon^{2w-1}$ is uniformly bounded in $(-\infty, 0)$, we can employ the Schwarz inequality to get the estimate

\[
\left\| |V|^{\frac{1}{2}} R_0^\perp(\alpha)^2 V^{\frac{1}{2}} \right\| \leq \left\| |V|^{\frac{1}{2}} R_0^\perp(\alpha)^2 \right\| \left\| R_0^\perp(\alpha) V^{\frac{1}{2}} \right\|.
\]

It is clear from the proof of Lemma 3.8 that the right hand side of this inequality is uniformly bounded in $[0, \kappa_2 - \varepsilon]$ for any $\varepsilon \in (0, \kappa_2]$. The former interval contains a neighbourhood of $\kappa_1$ in which $w$ is well defined.

This yields the remaining assertion (iii) of Lemma 3.3 for $|w| < 1$.

## 4 Mildly Curved Layers

Our strategy in proving the ground-state asymptotic expansion in mildly curved layers $\Omega_\varepsilon$ will be to estimate the corresponding Hamiltonian $-\Delta^\Omega_{\varepsilon}$ by an operator of the form $-\Delta - \Delta^D + \varepsilon V$ and to apply Theorem 3.4 to the latter. Here $-\Delta$ is the Laplacian in the plane, $-\Delta^D$ is the transverse operator which is the particular case of $-\Delta^M_D$ discussed in the preceding section, and $V$ is an effective potential given by curvatures of $\Sigma_\varepsilon$.

### 4.1 The Geometry

The family of metric tensors for the surfaces given by (2.1) has the form

\[
g_{\mu\nu}(\varepsilon) = \delta_{\mu\nu} + \varepsilon^2 \eta_{\mu\nu} \quad \text{with} \quad (\eta_{\mu\nu}) := \left( \begin{array}{cc} f_{1,1}^2 & f_{1,1}f_{2,2} \\ f_{1,1}f_{2,2} & f_{2,2}^2 \end{array} \right),
\]

where the symbol $\delta_{\mu\nu}$ (as well as $\delta^{\mu\nu}$) has to be understood as the identity matrix. Since $\det(\eta_{\mu\nu}) = 0$ we get immediately

\[
g(\varepsilon) := \det(g_{\mu\nu}) = 1 + \varepsilon^2 \text{tr}(\eta_{\mu\nu}) = 1 + \varepsilon^2(f_{1,1}^2 + f_{2,2}^2)
\]

(4.1)
At the same time, the vector $n$ second eigenvalue is a bounded function in $\mathbb{R}$ (as a manifold with a boundary in $\mathbb{R}^2$), i.e., if we adopt the assumption (d1). Denote the bound of $\eta_{\mu\nu}$ by $\eta^\infty$. Then it follows that $g_{\mu\nu}(\varepsilon)$ is uniformly elliptic for sufficiently small $\varepsilon$ because

$$c_- \delta_{\mu\nu} \leq g_{\mu\nu}(\varepsilon) \leq c_+ \delta_{\mu\nu} \quad \text{with} \quad c_\pm := 1 \pm \varepsilon^2 \eta^\infty. \quad (4.2)$$

At the same time, the vector $n(\varepsilon) = g(\varepsilon)^{-\frac{1}{2}}(-\varepsilon f_{11}, -\varepsilon f_{12}, 1)$ represents the surface normal and therefore the second fundamental form is given by

$$h_{\mu\nu}(\varepsilon) = \varepsilon g(\varepsilon)^{-\frac{1}{2}} \theta_{\mu\nu} \quad \text{with} \quad (\theta_{\mu\nu}) := \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}.$$ 

We can construct now the Weingarten tensor $[K]$, Def. 3.3.4 & Prop. 3.5.5]

$$h_{\mu\nu}(\varepsilon) := h_{\mu\rho} g^{\rho\nu} = \varepsilon g(\varepsilon)^{-\frac{1}{2}} \left( \theta_{\mu\rho} \delta^{\rho\nu} + \varepsilon^2 \theta_{\mu\rho} \tilde{\eta}^{\rho\nu} \right),$$

which determines respectively the Gauss curvature $K$ and the mean curvature $M$ of the surface $\Sigma_\varepsilon$:

$$K(\varepsilon) = \varepsilon^2 g(\varepsilon)^{-2} k_0 \quad \text{with} \quad k_0 := \det(\theta_{\mu\nu}) = f_{11} f_{22} - f_{12}^2,$$

$$M(\varepsilon) = \varepsilon g(\varepsilon)^{-\frac{3}{2}} \left( m_0 + \varepsilon^2 m_1 \right) \quad \text{with} \quad m_0 := \frac{1}{2} \tr(\theta_{\mu\nu}) = \frac{1}{2} (f_{11} + f_{22})$$

and

$$m_1 := \frac{1}{2} \tr(\theta_{\mu\rho} \tilde{\eta}^{\rho\nu}) = \frac{1}{2} \left( f_{11} f_{22} + f_{12}^2 f_{11} - 2 f_{11} f_{12} f_{22} \right). \quad (4.3)$$

Since we are interested in the case when $\Sigma_\varepsilon$ is asymptotically planar [DEK2], i.e., when $K, M \to 0$ as $|x| \to \infty$, we assume (d2).

It is clear from the definition (2.2) that the metric tensor of the layer $\Omega_\varepsilon$ (as a manifold with a boundary in $\mathbb{R}^3$) has the block form

$$\begin{pmatrix} G_{ij} \end{pmatrix} = \begin{pmatrix} (G_{\mu\nu}) & 0 \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad G_{\mu\nu} = (\delta^\rho_{\nu} - uh^\rho_{\nu})(\delta^\sigma_{\mu} - uh^\sigma_{\mu}) g_{\rho\mu}, \quad (4.4)$$

and thus $G := \det(G_{ij}) = g(1 - 2Mu + Ku^2)^2$. Following [DEK2], one should make sure that the layer mapping $\mathcal{L}$ is a diffeomorphism. However, this is automatically fulfilled for an arbitrary $a$ provided $\varepsilon$ is small enough and the layer is built over surfaces of the special form $\Sigma_\varepsilon$. At the same time, $G_{\mu\nu}$ can be estimated by the surface metric,

$$C_- g_{\mu\nu} \leq G_{\mu\nu} \leq C_+ g_{\mu\nu} \quad \text{with} \quad C_\pm := (1 \pm a \rho_m^{-1})^2, \quad (4.5)$$

where $\rho_m^{-1} := \max \{ \|k_1\|_\infty, \|k_2\|_\infty \} = O(\varepsilon)$ and $k_1, k_2$ are the principal curvatures of $\Sigma_\varepsilon$, i.e., the eigenvalues of $h_{\mu\nu}$. 

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4.2 The Hamiltonian

In the beginning we have identified the particle Hamiltonian with \(-\Delta^\Omega_D\). We want to replace it by a Schrödinger-type operator which would allow us to employ the result of the previous section. This is achieved by means of the unitary transformation

\[ U : L^2(\Omega_e) \to L^2(\Omega_0, g^{\frac{1}{2}} dx du) : \{ \psi \mapsto U \psi := (1 - 2Mu + Ku^2)^{\frac{1}{2}} \psi \circ \mathcal{L} \}, \]

which leads to the unitarily equivalent operator

\[ H := U(-\Delta^\Omega_D)U^{-1} = -g^{-\frac{1}{2}} \partial_\mu g^\frac{1}{2} G^{ij} \partial_\nu + V \]

\[ V = g^{-\frac{1}{2}} (g^{\frac{1}{2}} G^{ij} J_{ij}, i) + J_{ij} G^{ij} J_{ij} \quad \text{with} \quad J := \ln \sqrt{1 - 2Mu + Ku^2}. \]

It makes sense since we suppose that the surface is \(C^4\)-smooth. Using the block form (4.4) of \(G_{ij}\), we can split \(H\) into a sum, \(H = H_1 + H_2\). For the first operator which is given by the part of \(H\) where one sums over the Greek indices (referring to the longitudinal coordinates) we have owing to (4.5) the estimate

\[ (C_-/C_+) (-\Delta_g + v_1) \leq H_1 \leq (C_+/C^2) (-\Delta_g + v_1) \quad (4.6) \]

which holds in \(L^2(\mathbb{R}^2 \times I, g^{\frac{1}{2}} dx du)\) in the form sense. Here \(-\Delta_g\) denotes the surface Laplace-Beltrami operator which can be written in the component form as \(-g^{-\frac{1}{2}} \partial_\mu g^\frac{1}{2} g^{\mu\nu} \partial_\nu\), and

\[ v_1 := -\frac{|u^2 \nabla_g K - 2u \nabla_g M|_g^2}{4(1 - 2Mu + Ku^2)^2} + \frac{u^2 \Delta_g K - 2u \Delta_g M}{2(1 - 2Mu + Ku^2)}, \quad (4.7) \]

where \(| \cdot |_g\) and \(\nabla_g\) mean respectively the norm and the gradient operator induced by the metric \(g_{\mu\nu}\). On the other hand,

\[ H_2 = -\partial_2^2 + V_2, \quad V_2 = \frac{K - M^2}{(1 - 2Mu + Ku^2)^2}, \quad (4.8) \]

where the potential \(V_2\) is attractive because \(K - M^2\) can be rewritten by means of the principal curvatures as \(-\frac{1}{4}(k_1 - k_2)^2\).

In the second step one uses the inequalities (4.3), which are a consequence of (d1), in order to get the bounds

\[ (c_-^2/c_+^2)(-\Delta) \leq -\Delta_g \leq (c_+^2/c_-^2)(-\Delta) \quad \text{in} \quad L^2(\mathbb{R}^2, g^{\frac{1}{2}} dx).\]

It remains to realize that – as another consequence of the uniform ellipticity of the metric – one can identify \(L^2(\mathbb{R}^2, g^{\frac{1}{2}} dx) = L^2(\mathbb{R}^2)\), and also \(L^2(\mathbb{R}^2 \times I, g^{\frac{1}{2}} dx du) = L^2(\mathbb{R}^2 \times I)\) as sets. If we rescale the longitudinal variable by means of \(x \mapsto \sigma_\pm x\) with \(\sigma_\pm^2 := c_\pm^2 C^2_\pm/(c_\pm^2 C_\pm)\), we obtain finally the above indicated bounds

\[ H_- \leq H \leq H_+ \quad \text{with} \quad H_\pm := -\Delta - \partial_2^2 + \varepsilon V_\pm, \quad (4.9) \]

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where
\[ V_\pm(x,u) := \frac{1}{\varepsilon} \left( \frac{C_\pm}{C_2^\pm} v_1 + V_2 \right) (x/\sigma_\pm, u). \]

We notice that since \( v_1 \) and \( V_2 \) are \( \varepsilon \)-dependent – cf. (4.3) and the explicit formulae for the potentials (4.7) and (4.8) – the potentials \( V_\pm \) are well defined for \( \varepsilon = 0 \).

### 4.3 The Ground State Asymptotics

Suppose that each of the operators \( H_\pm \) has for all sufficiently small \( \varepsilon \) just one eigenvalue \( E_\pm(\varepsilon) \). Since \( H_- \) is below bounded, the minimax principle tells us that the same is true for \( H_\pm \) and \( E_-(\varepsilon) \leq E(\varepsilon) \leq E_+(\varepsilon). \) (4.10)

If we identify \( \varepsilon \) with \( \lambda \) of the previous section, we see that under our assumptions the operators \( H_\pm \) are well suited for application of Theorem 3.4.

Indeed, the decay requirements \( \langle d1-4 \rangle \) yield that \( K, M, |\nabla_g K|_g, |\nabla_g M|_g \) and \( \Delta_g K \) tend to zero at infinity, and consequently, \( (V_\pm)_{11} \rightarrow 0 \) as \( |x| \rightarrow \infty \). The reason is that the term \( u \Delta_g M \) itself in the potential (4.7) vanishes when projected onto the first transverse mode \( \chi_1 \).

One can thus take \( \psi_n(x,u) := \varphi_n(x)\chi_1(u) \), where \( \varphi_n \) with \( \|\varphi_n\| = 1 \) is a sequence of functions with increasing supports which move towards infinity as \( n \rightarrow \infty \). If \( \varphi_n \) is properly chosen such \( \psi_n \) form a Weyl sequence for \( H_- \) and any \( \kappa_1^2 + \alpha \) with \( \alpha \in \mathbb{R}^+ \). Using then a Neumann bracketing argument, as mentioned in Section 3, in order to show that no \( \kappa_1^2 + \alpha \) with \( \alpha < 0 \) can belong to the essential spectrum, we conclude that \( \sigma_{\text{ess}}(H_\pm) = [\kappa_1^2, \infty) \). This verifies \( \langle a0 \rangle \) of Theorem 3.4. Furthermore, using this result together with \( \langle 1.9 \rangle \) and the minimax principle, we infer that \( \inf \sigma_{\text{ess}}(H) = \kappa_1^2 \).

The requirement \( \langle a1 \rangle \) holds trivially true because \( V_\pm \) are essentially bounded due to the surface regularity assumption and \( \langle d1-4 \rangle \). Under \( \langle r1-3 \rangle \), explicit expressions for the covariant derivatives of \( K \) and \( M \) in terms of the partial derivatives of the function \( f \) show that even \( \sup \{V_\pm(\cdot,u) | u \in I \} \) belongs to \( L^1(\mathbb{R}^2, (1 + |x|^{\delta}) dx) \) which gives \( \langle a3 \rangle \). Let us remark that we require the \( L^1 \)-integrability in \( \langle r3 \rangle \), which is a stronger condition than the \( L^2 \)-integrability required for the second and third derivatives of \( f \). The assumption \( \langle a2 \rangle \) is a consequence of \( \langle a3 \rangle \) and the boundedness of \( V_\pm \).

On the other hand, there is a difference between the Hamiltonian (3.5) and our operators \( H_\pm \) because the potentials \( V_\pm \) depend themselves on \( \varepsilon \). Inspecting the proof of Theorem 3.4 we see that one can modify it to get the expansion (3.11) using \( \varepsilon \mapsto V_\pm(\varepsilon) \) as well. However, expanding the latter in terms of \( \varepsilon \), the final expansion resulting from (3.11) changes and to ensure that it represents an eigenvalue it is necessary that the lowest-order term in
this expansion is not only non-positive but rather strictly negative. Recall that one can find (physically interesting) examples of situations when the potential depends on the coupling parameter in a nonlinear way and a bound state in the critical zero-mean case may or may not exist [BCEZ]. Hence one has to examine carefully the behaviour with respect to powers of \( \varepsilon \) in the expansion obtained from (3.11) by an appropriate change of the integration variables,

\[
\begin{align*}
    w_\pm (\varepsilon) &= \frac{\varepsilon}{2\pi} \sigma_\pm^2 \int_{\mathbb{R}^2} (V_\pm)_{11}(\sigma_{\pm} x; \varepsilon) \, dx \\
    &+ \left( \frac{\varepsilon}{2\pi} \right)^2 \sigma_\pm^4 \int_{\mathbb{R}^2 \times \mathbb{R}^2} (V_\pm)_{11}(\sigma_{\pm} x; \varepsilon) \left( \gamma_E + \ln \frac{\sigma_{\pm} |x - x'|}{2} \right) \\
    &\times (V_\pm)_{11}(\sigma_{\pm} x'; \varepsilon) \, dx \, dx' \\
    &- \sum_{j=2}^{\infty} \int_{\mathbb{R}^2 \times \mathbb{R}^2} (V_\pm)_{1j}(\sigma_{\pm} x; \varepsilon) K_0 \left( k_j (\kappa_1) \sigma_{\pm} |x - x'| \right) \\
    &\times (V_\pm)_{j1}(\sigma_{\pm} x'; \varepsilon) \, dx \, dx' + \mathcal{O}(\varepsilon^{2+\gamma}),
\end{align*}
\]

which determines the possible eigenvalues of \( H_\pm \) via \( E_\pm = \kappa_1^2 - \exp(2w_\pm^{-1}) \). By virtue of (1.3) we can write

\[
\varepsilon^{-1} V_2(x, u; \varepsilon) = \varepsilon \left( k_0(x) - m_0(x)^2 \right) + r_2(x, u; \varepsilon),
\]

where \( r_2 \) is an integrable function collecting the terms of order \( \varepsilon^2 \) and higher. Hence

\[
\forall j \in \mathbb{N} : \quad \varepsilon^{-1} (V_2)_{1j}(x; \varepsilon) = \left[ \varepsilon \left( k_0(x) - m_0(x)^2 \right) + \mathcal{O}(\varepsilon^2) \right] \delta_{1j},
\]

where we abuse the notation a little because the error term depends on \( x \) as well. The expansion of \( \varepsilon^{-1} v_1 \) requires more attention because \( \varepsilon^{-1} \Delta_g M \) in the second term of (1.7) is of order one. First of all, we can write

\[
\begin{align*}
    v_1(x, u; \varepsilon) &= -u(\Delta_g M)(x; \varepsilon) \\
    &+ u^2 \left( \frac{1}{2} \Delta_g K - |\nabla_g M|^2 - 2M\Delta_g M \right)(x; \varepsilon) + r_1(x, u; \varepsilon),
\end{align*}
\]

where \( r_1 \) is an integrable function of order \( \mathcal{O}(\varepsilon^3) \). This does not mean, of course, that the first two terms constitute a quadratic polynomial in \( \varepsilon \), because \( | \cdot |_g, \nabla_g \) and \( \Delta_g \) expand as well. Since the first term in the above expansion of \( v_1 \) is an odd function of \( u \), it does not contribute to \((V_\pm)_{11}\). On the other hand, it plays an important role in the higher modes:

\[
\begin{align*}
    (v_1)_{11}(x; \varepsilon) &= \| u \chi_1 \|^2 \left( \frac{1}{2} \Delta_g K - |\nabla_g M|^2 - 2M\Delta_g M \right)(x; \varepsilon) + \mathcal{O}(\varepsilon^3), \\
    (v_1)_{1j}(x; \varepsilon) &= -\langle \chi_1, u\chi_j \rangle (\Delta_g M)(x; \varepsilon) + \mathcal{O}(\varepsilon^2) \quad \text{if} \quad j \in \mathbb{N} \setminus \{1\}.
\end{align*}
\]
Notice that the inner product in the second line is non-zero for even \( j \) only. Now we use the relations

\[
|\nabla g_M|^2 = |\nabla M|^2 + \varepsilon^2 M_{\mu} \tilde{\eta}^{\mu\nu} M_{\nu} \quad \text{and} \quad -\Delta_g = -\Delta + \varepsilon^2 L(\varepsilon),
\]

where \( L(\varepsilon) \) is a second-order differential operator with coefficients which expand as \( O(1) \). This together with (4.3) yields

\[
\varepsilon^{-1}(v_1)_{11}(x; \varepsilon) = \varepsilon \| u\chi_1 \|^2 (\frac{1}{2} \Delta k_0 - |\nabla m_0|^2 - 2m_0 \Delta m_0)(x) + O(\varepsilon^2)
\]

\[
\varepsilon^{-1}(v_1)_{1j}(x; \varepsilon) = -(\chi_1, u\chi_j)_{\mu} (\Delta m_0)(x) + O(\varepsilon).
\]

Since \( C_\pm/C^2_\pm \) and \( \sigma_\pm \) expand like \( 1 + O(\varepsilon) \), we arrive at

\[
\varepsilon^{-1} w(\varepsilon) = \frac{\varepsilon^2}{2\pi} \left\{ \int_{\mathbb{R}^2} (k_0 - m_0^2)(x) \, dx
\right. \\
+ \left. \| u\chi_1 \|^2 \int_{\mathbb{R}^2} (\frac{1}{2} \Delta k_0 - |\nabla m_0|^2 - 2m_0 \Delta m_0)(x) \, dx 
\right.
\]

\[
- \frac{1}{2\pi} \sum_{j=2}^{\infty} (\chi_1, u\chi_j)_{\mu} \int_{\mathbb{R}^2 \times \mathbb{R}^2} (\Delta m_0)(x) K_0(k_j(\kappa_1)\sigma_\pm|x-x'|)
\]

\[
\times (\Delta m_0)(x') \, dx \, dx' \right\} + O(\varepsilon^{2+\gamma}), \tag{4.11}
\]

where the sum runs in fact over even \( j \) only.

This expression can be further simplified. By a double integration by parts where one uses the fact that \( f_{\mu} f_{\nu \rho} \to 0 \) as \( |x| \to \infty \) due to \( (d1) \) and \( (d2) \), we get that the integral of \( k_0 \) is equal to zero. We note in this connection that it follows then from \( (d1) \) and \( (d3) \) that the total Gauss curvature, \( K := \int_{\Sigma} K d\Sigma_x \), behaves as \( O(\varepsilon^4) \). Using the divergence theorem and the assumptions \( (d1) \), \( (d2) \) and \( (d3) \) by which \( \nabla K \to 0 \) as \( |x| \to \infty \), we find that the integral from \( \Delta k_0 \) does not contribute as well. A similar argument employing the Green formula gives

\[
\int_{\mathbb{R}^2} m_0 \Delta m_0(x) \, dx = -\int_{\mathbb{R}^2} |\nabla m_0|^2(x) \, dx. \tag{4.12}
\]

It is convenient to rewrite the integral part of the last term in (4.11) by means of a convolution

\[
\frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} (\Delta m_0)(x) K_0(k_j(\kappa_1)\sigma_\pm|x-x'|)(\Delta m_0)(x') \, dx \, dx' \\
= (\Delta m_0, G_k * \Delta m_0),
\]
where $G_k(\cdot) := (2\pi)^{-1} K_0(k \cdot \cdot)$ and $k$ abbreviates $k_j(k_1)\sigma_{\pm}$. Since $G_k$ is the fundamental solution of the distributional equation $(-\Delta + k^2)G_k = \delta$, we get

$$(\Delta m_0, G_k * \Delta m_0) = (\Delta m_0, \Delta G_k * m_0) = (\Delta m_0, (k^2 G_k - \delta) * m_0)$$

$$= k^2 (\Delta m_0, G_k * m_0) - (\Delta m_0, m_0) = k^2 (G_k * \Delta m_0, m_0) + \|\nabla m_0\|^2$$

$$= k^2 (\Delta G_k * m_0, m_0) + \|\nabla m_0\|^2 = k^2 ((k^2 G_k - \delta) * m_0, m_0) + \|\nabla m_0\|^2$$

$$= k^4 (m_0, G_k * m_0) - k^2 \|m_0\|^2 + \|\nabla m_0\|^2. \quad (4.13)$$

In the second line we have employed the identity $\|\nabla \parallel m_0\|$ term. If we insert the obtained expression into the expansion of $m_0$ and use, in addition, the Parseval identity,

$$\|u\chi_1\|^2 = \sum_{j=1}^{\infty} (\chi_1, u\chi_j)^2, \quad (4.14)$$

we find that the terms containing $\|\nabla m_0\|$ cancel, hence we arrive at

$$w_{\pm}(\varepsilon) = -\frac{\varepsilon^2}{2\pi} \left\{ \|m_0\|^2 + \sum_{j=2}^{\infty} (\chi_1, u\chi_j)^2 k^2 \left[ k^2 (m_0, G_k * m_0) - \|m_0\|^2 \right] \right\}$$

$$+ O(\varepsilon^{2+\gamma}),$$

where, of course, $k$ expands as $k_j(k_1) + O(\varepsilon)$. Since

$$\sum_{j=2}^{\infty} (\chi_1, u\chi_j)^2 k_j(k_1)^2 = \|u \partial_3 \chi_1\|_I^2 - \kappa_1^2 \|\chi_1\|_I^2 = 1,$$

we see that also the terms containing $\|m_0\|$ cancel. Using finally the Fourier transformation as in $\|\nabla m_0\|$, we obtain

$$w_{\pm}(\varepsilon) = -\varepsilon^2 \sum_{j=2}^{\infty} (\chi_1, u\chi_j)^2 k_j(k_1)^4 \int_{\mathbb{R}^2} \frac{|\tilde{m}_0(\omega)|^2}{|\omega|^2 + \kappa_2^2 - \kappa_1^2} d\omega + O(\varepsilon^{2+\gamma}),$$

where we have also expanded the remaining $k$ from $G_k$ w.r.t. $\varepsilon$ and included the higher orders into the error term.

We note that $m_0$ (and therefore $\tilde{m}_0$) cannot be identically zero once $\Sigma_1$ is not a plane. It can be seen from the formulae (4.3): suppose that $m_0 \equiv 0$. Then $f_{11} \equiv -f_{22}$, which yields $k_0 = -(f_{11}^2 + f_{12}^2)$ and consequently $k_0 \equiv 0$ because we have shown that $\int_{\mathbb{R}^2} k_0(x) dx = 0$. Hence $K \equiv 0$. At the same time, $f_{\mu\nu} \equiv 0$, which gives $m_1 \equiv 0$ and therefore $M \equiv 0$. If we thus exclude the trivial planar case, the lowest-order term in the expansion is strictly negative. Since it is identical for both $w_+$ and $w_-$, it follows by (1.10) that it is the same also for the expansion of $w(\varepsilon)$ in the ground-state energy $E = \kappa_1^2 - e^{2w-1}$ of the Hamiltonian $H$. This concludes the proof of Theorem 2.3.
Remark. The surface \( \Sigma_1 \) is not planar if the matrix \( \theta_{\mu\nu} \) is not identically zero, i.e., if \( f_{\mu\nu} \neq 0 \). This can be seen as follows. Suppose that \( f_{\mu\nu} = 0 \) identically. Solving this as a system of differential equations, we get \( f(x) = c_\mu x^\mu \), where \( c_\mu \) is a constant vector, which is exactly a plane parametrization.

4.4 The Thin Layer Limit

We say that the layer \( \Omega_\varepsilon \) is thin if \( a \ll \rho_m \). Since the minimum curvature radius introduced in (4.5) explodes as \( \varepsilon \to 0 \) this condition is eventually always satisfied. It is useful, however, to consider also a “true” thin-layer limit in which we start with a mildly curved layer with a fixed \( \varepsilon \) which is already in the asymptotic regime, and make its width \( d \) small. To this aim we rewrite the formula for \( w_1 \) in the expansion \( w_1(\varepsilon) = \varepsilon^2 w_1 + \mathcal{O}(\varepsilon^{2+\gamma}) \) of Theorem 2.1 in a different way. For this we go back to the intermediate expansion (4.11). We simplify it as above, however, do not use (4.13) and (4.14). Instead we apply the transformation (3.12) directly to \( (\Delta m_0, G_k \ast \Delta m_0) \), expand \( k \) in terms of \( \varepsilon \) obtaining thus

\[
w_1 = -\frac{1}{2\pi} \|m_0\|^2 + \frac{\|u\chi_1\|^2}{2\pi} \|\nabla m_0\|^2 - \sum_{j=2}^{\infty} (\chi_1, u\chi_j)^2 \int_{\mathbb{R}^2} \frac{\|\hat{\Delta} m_0(\omega)\|^2}{|\omega|^2 + k_j(\kappa_1)^2} \, d\omega.
\]

An explicit calculation yields \( \|u\chi_1\|^2 = (\pi^2 - 6)/(12\kappa_1^2) \), so the second term in the expression for \( w_1 \) is proportional to \( d^2 \) by the definition of \( \kappa_1 \). On the other hand, using the estimate \( |\omega|^2 + k_j(\kappa_1)^2 \geq k_j(\kappa_1)^2 \geq \kappa_2^2 - \kappa_1^2 = 3\kappa_1^2 \) together with (4.14) and the above explicit result, we see that the third term is \( \mathcal{O}(d^4) \). Note that this makes sense because \( \|\hat{\Delta} m_0\| = \|\Delta m_0\| \) is finite due to (r3). Summing up the argument, we obtain the expansion (2.4).

This formula has a transparent structure. By virtue of (4.3), (1.1), and the fact that \( \int_{\mathbb{R}^2} k_0(x) \, dx = 0 \), we get the behaviour

\[
\int_{\Sigma_\varepsilon} (K - M^2) \, d\Sigma_\varepsilon = -\varepsilon \left( \|m_0\|^2 + \mathcal{O}(\varepsilon^2) \right).
\]

Hence the first term of (2.4) comes from the surface attractive potential \( K - M^2 \) which dominates the picture for thin layers. The last claim follows from (1.7) and (1.8), because \( v_1 \to 0 \) and the leading term of \( V_2 \) is \( K - M^2 \) in the limit \( u \to 0 \), which one can do as soon as \( d \to 0 \). At the same time, \( C_{\pm} \to 1 \), which in view of (1.4) leads to the “limiting” Hamiltonian \( -\Delta g - \partial_3^2 + K - M^2 \). Such a limit is, of course, only formal since the transverse spectrum explodes, which classically corresponds to increasingly rapid oscillations in the transverse direction. Nevertheless, a similar limit with the confinement realized by a harmonic potential transverse to a compact curved surface was discussed recently in a rigorous way in [FH] and led to the same tangential operator \( -\Delta_g + K - M^2 \).
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