SOME CHARACTERIZATIONS AND
A CONSTRUCTION OF
MIXED RENEWAL PROCESSES

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Abstract
Some characterizations of mixed renewal processes in terms of exchangeability and of different types of disintegrations are given, extending de Finetti’s Theorem. As a consequence, an existence result for mixed renewal processes, providing also a new construction for them, is obtained. As an application, some concrete examples of constructing such processes are presented and the corresponding disintegrating measures are explicitly computed.

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1 Introduction
Mixed renewal processes (MRPs for short) may serve as a source of challenging theoretical problems, since they are generalizations of mixed Poisson processes (MPPs for short) and closely connected with the concept of exchangeable stochastic processes (cf. e.g. [8]), as well as a useful tool for modelling real life situations, such as those emerging in actuarial practice (cf. e.g. [14], pages 164-165).

In Section 3 we introduce a new (to the best of our knowledge) definition of MRPs (see Definition 3.2) being in line with that of MPPs with parameter Θ. Such a definition seems to be a proper one as it involves explicitly the structural parameter Θ, which is usually essential in the study of risk-theoretical problems. Since conditioning is involved in this definition of MRPs, it seems to be natural to ask about the structural role of disintegrations in this field. For this reason, we recall the definitions of different types of disintegrations (see Definitions 3.3) and provide some characterizations of

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MRPs via disintegrations (see Proposition 3.9 and Corollary 3.10). By means of these disintegration results we can reduce MRPs to ordinary renewal processes for the disintegrating probability measures, showing in this way that Definition 3.2 is the natural one for MRPs. On this basis we obtain some new necessary and sufficient conditions for the exchangeability of the associated claim interarrival processes (see Theorem 4.9, (i) to (iii)). Furthermore, in Theorem 3.13 we extend a basic result of our previous work [10].

The second definition of MRPs investigated in this paper is due to Huang [8], see Definition 4.7. In Section 4, we first give some characterizations of exchangeability in terms of different types of disintegrations, extending de Finetti’s Theorem to families of measurable maps taking values in an arbitrary measurable space \((Y,T)\), without any topological assumption neither for the underlying probability space nor for the space \((Y,T)\), see Theorem 4.4. As a consequence, some further characterizations of MRPs in terms of exchangeability and of disintegrations are deduced (see Theorem 4.9, (iii) to (vi)). Theorem 4.9 provides amongst others a detailed discussion of the relation between the two definitions of MRPs and shows that in most cases appearing in applications both definitions coincide.

As another consequence of these characterizations, in Section 5 we deduce a new existence result for MRPs (see Theorem 5.1), extending a similar construction for MPPs, see [11], Theorem 3.1. As an application we construct concrete examples of MRPs and compute the corresponding disintegrating measures explicitly.

It is worth noticing that the standard construction of MPPs existing so far, due to Lundberg, requires the presence of birth processes (cf. e.g. [5], pages 61-63). This assumption is dropped in Theorem 5.1. Furthermore, Kolmogorov’s Consistency Theorem leads to an existence result for Markov processes (cf. e.g. [4], Theorem 455A). But since MRPs are not in general Markov (see [8], Theorem 3 together with Definitions 3.2 and 4.7), hence birth processes, the above methods cannot be applied for any MRP. On the contrary, Theorem 5.1 applies a new construction working for general MRPs. It completely differs from methods applied as yet and at the same time removes further restrictive assumptions.

## 2 Preliminaries

By \(\mathbb{N}\) is denoted the set of all natural numbers and \(\mathbb{N}_0 := \mathbb{N} \cup \{0\}\). The symbol \(\mathbb{R}\) stands for the set of all real numbers, while \(\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}\). If \(d \in \mathbb{N}\), then \(\mathbb{R}^d\) denotes the Euclidean space of dimension \(d\).

Given a probability space \((\Omega, \Sigma, P)\), a set \(N \in \Sigma\) with \(P(N) = 0\) is called a \textbf{\(P\)-null set} (or a null set for simplicity). The family of all \(P\)-null sets is denoted by \(\Sigma_0\). For random variables \(X, Y : \Omega \rightarrow \mathbb{R}\) we write \(X = Y \ P\text{-a.s.}\), if \(\{X \neq Y\} \in \Sigma_0\).

If \(A \subseteq \Omega\), then \(A^c := \Omega \setminus A\), while \(\chi_A\) denotes the indicator (or characteristic) function of the set \(A\). The identity map from \(\Omega\) onto itself is denoted by \(\text{id}_\Omega\). The \(\sigma\)-algebra generated by a family \(\mathcal{G}\) of subsets of \(\Omega\) is denoted by \(\sigma(\mathcal{G})\). A \(\sigma\)-algebra \(\mathcal{A}\) is \textbf{countably}
generated if there exists a countable family $\mathcal{G}$ of subsets of $\Omega$ such that $A = \sigma(\mathcal{G})$.

For any Hausdorff topology $\mathcal{T}$ on $\Omega$ by $\mathcal{B}(\Omega)$ is denoted the Borel $\sigma$-algebra on $\Omega$, i.e. the $\sigma$-algebra generated by $\mathcal{T}$. By $\mathcal{B} := \mathcal{B}(\mathbb{R})$, $\mathcal{B}_d := \mathcal{B}(\mathbb{R}^d)$ and $\mathcal{B}_N := \mathcal{B}(\mathbb{R}^N)$ is denoted the Borel $\sigma$-algebra of subsets of $\mathbb{R}$, $\mathbb{R}^d$ and $\mathbb{R}^N$, respectively, while $\mathcal{L}^1(P)$ stands for the family of all real-valued $P$-integrable functions on $\Omega$. Functions that are $P$-a.s. equal are not identified.

The probability measure $P$ is said to be perfect if for any random variable $X$ on $\Omega$ there exists a Borel set $B \subseteq X(\Omega) := \{X(\omega) : \omega \in \Omega\}$ such that $P(X^{-1}(B)) = 1$.

Given two probability spaces $(\Omega, \Sigma, P)$ and $(\Upsilon, \mathcal{T}, Q)$ as well as a $\Sigma$-$\mathcal{T}$-measureable map $X : \Omega \to \Upsilon$ we denote by $\sigma(X) := \{X^{-1}(B) : B \in \mathcal{T}\}$ the $\sigma$-algebra generated by $X$, while $\sigma(\{X_i\}_{i \in I}) := \sigma(\bigcup_{i \in I} \sigma(X_i))$ stands for the $\sigma$-algebra generated by a family $\{X_i\}_{i \in I}$ of $\Sigma$-$\mathcal{T}$-measurable maps from $\Omega$ into $\Upsilon$.

Setting $T_X := \{B \subseteq \Upsilon : X^{-1}(B) \in \Sigma\}$ for any given $\Sigma$-$\mathcal{T}$-measureable map $X$ from $\Omega$ into $\Upsilon$, we clearly get that $T \subseteq T_X$. Denote by $P_X : T_X \to \mathbb{R}$ the image measure of $P$ under $X$. The restriction of $P_X$ to $T$ is denoted again by $P_X$. By $K(\theta)$ is denoted an arbitrary probability distribution on $\Psi$ with parameter $\theta \in \Psi$. In particular, $P(\theta)$, $\text{Exp}(\theta)$ and $\text{Ga}(\gamma, \alpha)$, where $\theta, \gamma, \alpha$ are positive parameters, stand for the law of Poisson, exponential and gamma distribution, respectively (cf. e.g. [15]).

If $X \in \mathcal{L}^1(P)$ and $\mathcal{F}$ is a $\sigma$-subalgebra of $\Sigma$, then each function $Y \in \mathcal{L}^1(P \mid \mathcal{F})$ satisfying for each $A \in \mathcal{F}$ the equality $\int_A X \ dP = \int_A Y \ dP$ is said to be a version of the conditional expectation of $X$ with respect to (or given) $\mathcal{F}$ and it will be denoted by $\mathbb{E}_P[X \mid \mathcal{F}]$. For $X := \chi_B \in \mathcal{L}^1(P)$ with $B \in \Sigma$ we set $P(B \mid \mathcal{F}) := \mathbb{E}_P[\chi_B \mid \mathcal{F}]$.

By $(\Omega \times \Upsilon, \Sigma \otimes \mathcal{T}, P \otimes Q)$ is denoted the product probability space of $(\Omega, \Sigma, P)$ and $(\Upsilon, \mathcal{T}, Q)$, and by $\pi_\Omega$ and $\pi_\Upsilon$ the canonical projections from $\Omega \times \Upsilon$ onto $\Omega$ and $\Upsilon$, respectively.

Given two measurable spaces $(\Omega, \Sigma)$ and $(\Upsilon, T)$, a $T$-$\Sigma$-Markov kernel is a function $k$ from $T \times \Omega$ into $\mathbb{R}$ satisfying the following conditions:

(k1) The set-function $k(\cdot, \omega)$ is a probability measure on $T$ for any fixed $\omega \in \Omega$.

(k2) The function $\omega \mapsto k(B, \omega)$ is $\Sigma$-measurable for any fixed $B \in \mathcal{T}$.

Let be given a $\Sigma$-$\mathcal{T}$-measurable map $X$ from $\Omega$ into $\Upsilon$ and a $\sigma$-subalgebra $\mathcal{F}$ of $\Sigma$. A conditional distribution of $X$ over $\mathcal{F}$ is a $T$-$\mathcal{F}$-Markov kernel $k$ satisfying for each $B \in \mathcal{T}$ condition

$$k(B, \cdot) = P(X^{-1}(B) \mid \mathcal{F})(\cdot) \quad P \mid \mathcal{F} \ - \text{a.s.}$$

Such a Markov kernel $k$ will be denoted by $P_{X \mid \mathcal{F}}$. In particular, if $(\Psi, Z)$ is a measurable space, $\Theta$ is a $\Sigma$-$Z$-measurable map from $\Omega$ into $\Psi$ and $\mathcal{F} := \sigma(\Theta)$, then the function $P_{X \mid \Theta} := P_{X \mid \sigma(\Theta)}$ is called a conditional distribution of $X$ given $\Theta$. Note that if $\Upsilon$ is a Polish space (i.e. a topological space homeomorphic to a complete separable metric space) then a conditional distribution of $X$ over $\mathcal{F}$ always exists (cf. e.g. [2], Theorem 10.2.2).
Clearly, for every $T$-Z-Markov kernel $k$, the map $K(\Theta)$ from $T \times \Omega$ into $\mathbb{R}$ defined by means of

$$K(\Theta)(B, \omega) := (k(B, \cdot) \circ \Theta)(\omega)$$

for any $B \in T$ and $\omega \in \Omega$ is a $T$-$\sigma(\Theta)$-Markov kernel. In particular, for $(\mathcal{Y}, T) = (\mathbb{R}, \mathcal{B})$ its associated probability measures $k(\cdot, \theta)$ for $\theta = \Theta(\omega)$ with $\omega \in \Omega$ are distributions on $\mathcal{B}$ and so we may write $K(\theta)(\cdot)$ instead of $k(\cdot, \theta)$. Consequently, in this case $K(\Theta)$ will be denoted by $K(\Theta)$. For any $\sigma$-subalgebra $\mathcal{F}$ of $\Sigma$, we say that two $T$-$\mathcal{F}$-Markov kernels $k_i$, for $i \in \{1, 2\}$, are $P \mid \mathcal{F}$-equivalent and we write $k_1 = k_2$ $P \mid \mathcal{F}$-a.s., if there exists a $P$-null set $N \in \mathcal{F}$ such that for any $\omega \notin N$ and $B \in T$ the equality $k_1(B, \omega) = k_2(B, \omega)$ holds true.

From now on $(\Omega, \Sigma, P)$ is a probability space, while $(\mathcal{Y}, T)$ and $(\Psi, Z)$ are measurable spaces, all of them arbitrary but fixed.

## 3 Characterizations of mixed renewal processes via disintegrations

A family $\{N_t\}_{t \in \mathbb{R}_+}$ of random variables on $\Omega$ is a **counting** or a **claim number process** if there exists a null set $\Omega_N \in \Sigma_0$ such that for all $\omega \in \Omega \setminus \Omega_N$

(n1) $N_0(\omega) = 0$;

(n2) $N_t(\omega) \in \mathbb{N}_0 \cup \{\infty\}$ for each $t \in (0, \infty)$;

(n3) $N_t(\omega) = \inf_{u \in (t, \infty)} N_u(\omega)$ for each $t \in \mathbb{R}_+$;

(n4) $\sup_{u \in [0, t]} N_u(\omega) \leq N_t(\omega) \leq \sup_{u \in [0, t]} N_u(\omega) + 1$ for each $t \in (0, \infty)$;

(n5) $\sup_{t \in \mathbb{R}_+} N_t(\omega) = \infty$.

The null set $\Omega_N$ is called the **exceptional null set** of the counting process $\{N_t\}_{t \in \mathbb{R}_+}$.

A sequence $\{T_n\}_{n \in \mathbb{N}_0}$ of random variables on $\Omega$ is a **claim arrival process** if there exists a null set $\Omega_T \in \Sigma_0$ such that for all $\omega \in \Omega \setminus \Omega_T$ we have $T_0(\omega) = 0$ and $T_{n-1}(\omega) < T_n(\omega)$ for all $n \in \mathbb{N}$. The null set $\Omega_T$ is said to be the **exceptional null set** of the claim arrival process $\{T_n\}_{n \in \mathbb{N}_0}$. The sequence $\{W_n\}_{n \in \mathbb{N}}$, given by $W_n := T_n - T_{n-1}$ for each $n \in \mathbb{N}$, is then called the **claim interarrival process** induced by the claim arrival process $\{T_n\}_{n \in \mathbb{N}_0}$ (cf. e.g. [15], Section 1.1, page 6). Obviously, $\{W_n\}_{n \in \mathbb{N}}$ has the same exceptional null set with $\{T_n\}_{n \in \mathbb{N}_0}$, that is, $\Omega_W = \Omega_T$.

**Remark 3.1** If $\{N_t\}_{t \in \mathbb{R}_+}$ is a counting process with exceptional null set $\Omega_N$, then the sequence $\{T_n\}_{n \in \mathbb{N}_0}$ defined by

$$T_n := \inf\{t \in \mathbb{R}_+ : N_t = n\} \quad \text{for each} \quad n \in \mathbb{N}_0$$

...
is a claim arrival process with exceptional null set \( \Omega_T = \Omega_N \), while the sequence \( \{W_n\}_{n \in \mathbb{N}} \), where each \( W_n := T_n - T_{n-1} \), is obviously a claim interarrival process with exceptional null set \( \Omega_W = \Omega_T \).

The sequences \( \{T_n\}_{n \in \mathbb{N}_0} \) and \( \{W_n\}_{n \in \mathbb{N}} \) defined in the above remark are called the **claim arrival and interarrival process**, respectively, **induced by the counting process** \( \{N_t\}_{t \in \mathbb{R}_+} \). Conversely, given a claim interarrival process \( \{W_n\}_{n \in \mathbb{N}} \) we define the *induced claim arrival and counting processes* by setting \( T_n := \sum_{k=1}^n W_k \) for all \( n \in \mathbb{N}_0 \) and \( N_t := \sum_{n=1}^\infty \chi\{T_n \leq t\} \) for all \( t \in \mathbb{R}_+ \), respectively (cf. e.g. [15], Theorem 2.1.1).

Recall that a family \( \{\Sigma_i\}_{i \in I} \) of \( \sigma \)-subalgebras of \( \Sigma \) is **P-conditionally (stochastically) independent** over a \( \sigma \)-algebra \( \mathcal{F} \) of \( \Sigma \), if for each \( n \in \mathbb{N} \) with \( n \geq 2 \) we have

\[
P(E_1 \cap \cdots \cap E_n \mid \mathcal{F}) = \prod_{j=1}^n P(E_j \mid \mathcal{F}) \quad P \mid \mathcal{F} \text{-a.s.}
\]

whenever \( i_1, \ldots, i_n \) are distinct members of \( I \) and \( E_j \in \Sigma_{i_j} \) for every \( j \leq n \).

Let be given a family \( \{X_i\}_{i \in I} \) of \( \Sigma \)-\( T \)-measurable maps from \( \Omega \) into \( \Psi \). We say that \( \{X_i\}_{i \in I} \) is **P-conditionally (stochastically) independent** over a \( \sigma \)-algebra \( \mathcal{F} \subseteq \Sigma \), if the family \( \{\sigma(X_i)\}_{i \in I} \) of \( \sigma \)-algebras is P-conditionally independent over \( \mathcal{F} \).

The family \( \{X_i\}_{i \in I} \) is **P-conditionally identically distributed** over \( \mathcal{F} \), if

\[
P(F \cap X_i^{-1}(B)) = P(F \cap X_j^{-1}(B))
\]

whenever \( i, j \in I \), \( F \in \mathcal{F} \) and \( B \in \mathcal{T} \). For simplicity, we write P-conditionally i.i.d. instead of P-conditionally independent and identically distributed. Recall that “i.i.d.” is the abbreviation for “independent and identically distributed”.

Furthermore, if \( \Theta \) is a \( \Sigma \)-\( Z \)-measurable map from \( \Omega \) into \( \Psi \), we say that \( \{X_i\}_{i \in I} \) is **P-conditionally (stochastically) independent or identically distributed given \( \Theta \)**, if it is conditionally independent or identically distributed over the \( \sigma \)-algebra \( \sigma(\Theta) \).

***Throughout what follows, unless it is stated otherwise, \( \Theta \) is a \( \Sigma \)-\( Z \)-measurable map from \( \Omega \) into \( \Psi \), and we simply write “conditionally” in the place of “conditionally given \( \Theta \)” whenever conditioning refers to \( \Theta \). Moreover, \( \{N_t\}_{t \in \mathbb{R}_+} \) is a counting process and without loss of generality we may and do assume that \( \Omega_N = \emptyset \).***

The counting process \( \{N_t\}_{t \in \mathbb{R}_+} \) is said to be a **P-renewal process** with claim interarrival time distribution \( K(\theta_0) \), where \( \theta_0 \in \Psi \) is a parameter (or a \( (P, K(\theta_0)) \)-RP for short), if its associated claim interarrival times \( W_n \), \( n \in \mathbb{N} \), are independent and \( K(\theta_0) \)-distributed under the probability measure \( P \).

The following definition of an MRP being in line with the definition of a mixed Poisson process (MPP for short) with parameter \( \Theta \) (see [15], Section 4.2, page 87 or [10], Section 4) seems to be the natural one, since among others it involves explicitly the **structural parameter** \( \Theta \).
Definition 3.2 The counting process \( \{N_t\}_{t \in \mathbb{R}^+} \) is said to be a mixed renewal process on \((\Omega, \Sigma, P)\) with parameter the map \( \Theta \) and claim interarrival time conditional distribution \( K(\Theta) \) (or a \((P, K(\Theta))\)-MRP for short), if \( \{W_n\}_{n \in \mathbb{N}} \) is \( P \)-conditionally independent and
\[
P_{W_n|\Theta} = K(\Theta) \quad P \mid \sigma(\Theta) - \text{a.s.}
\]
for all \( n \in \mathbb{N} \).

In particular, for \((\Psi, Z) = (\mathbb{R}, \mathcal{B})\) and \( P_{\Theta}((0, \infty)) = 1 \) a claim number process \( \{N_t\}_{t \in \mathbb{R}^+} \) is a \( P \)-mixed Poisson process on \((\Omega, \Sigma, P)\) (or a \( P \)-MPP for short) with parameter the random variable \( \Theta \), if it has \( P \)-conditionally stationary independent increments (cf. e.g. [15], Section 4.1, page 86 for the definition), such that
\[
P_{N_t|\Theta} = P(t\Theta) \quad P \mid \sigma(\Theta) - \text{a.s.}
\]
holds true for each \( t \in (0, \infty) \).

Note that, for \((\Psi, Z) = (\mathbb{R}, \mathcal{B})\), \( P_{\Theta}((0, \infty)) = 1 \) and \( K(\Theta) = \text{Exp}(\Theta) \) \( P \mid \sigma(\Theta) \)-a.s. the \((P, K(\Theta))\)-MRP \( \{N_t\}_{t \in \mathbb{R}^+} \) becomes a \( P \)-MPP with parameter \( \Theta \) (see [10], Proposition 4.5).

Definitions 3.3 (a) Let \( Q \) be a probability measure on \( T \). A family \( \{P_y\}_{y \in \Upsilon} \) of probability measures on \( \Sigma \) is called a disintegration of \( P \) over \( Q \) if

\[ (d1) \text{ for each } D \in \Sigma \text{ the map } y \mapsto P_y(D) \text{ is } T\text{-measurable;} \]

\[ (d2) \int P_y(D)Q(dy) = P(D) \text{ for each } D \in \Sigma. \]

If \( f : \Omega \to T \) is an inverse-measure-preserving function (i.e. \( P(f^{-1}(B)) = Q(B) \) for each \( B \in T \)), a disintegration \( \{P_y\}_{y \in \Upsilon} \) of \( P \) over \( Q \) is called consistent with \( f \) if, for each \( B \in T \), the equality \( P_y(f^{-1}(B)) = 1 \) holds for \( Q \)-almost every \( y \in B \).

(b) Assume that \( M \) is a probability on the \( \sigma \)-algebra \( \Sigma \otimes T \) such that \( P \) and \( Q \) are the marginals of \( M \). Assume also that for each \( y \in \Upsilon \) there exists a probability \( P_y \) on \( \Sigma \), satisfying the following properties:

\[ (D1) \text{ For every } A \in \Sigma \text{ the map } y \mapsto P_y(A) \text{ is } T\text{-measurable;} \]

\[ (D2) M(A \times B) = \int_B P_y(A)Q(dy) \text{ for each } A \times B \in \Sigma \times T. \]

Then, \( \{P_y\}_{y \in \Upsilon} \) is said to be a product regular conditional probability (product r.c.p. for short) on \( \Sigma \) for \( M \) with respect to \( Q \) (see [3], Section 2 or [16], Definition 1.1).

(c) Let \( \mathcal{F} \) be a \( \sigma \)-subalgebra of \( \Sigma \) and \( R := P \mid \mathcal{F} \). A subfield r.c.p. for \( P \) over \( R \) (see [3], Section 2) is a family \( \{P_\omega\}_{\omega \in \Omega} \) of probability measures on \( \Sigma \) satisfying the following conditions:

\[ (sf1) \text{ for each } E \in \Sigma \text{ the map } \omega \mapsto P_\omega(E) \text{ is } \mathcal{F}\text{-measurable;} \]
\( (sf2) \int_F P_\omega(E) R(d\omega) = P(E \cap F) \) for all \( F \in \mathcal{F} \) and \( E \in \Sigma \).

**Remarks 3.4** (a) If \( \Sigma \) is countably generated and \( P \) is perfect, then there always exists a disintegration \( \{P_y\}_{y \in \mathcal{Y}} \) of \( P \) over \( Q \) consistent with any inverse-measure-preserving map \( f \) from \( \Omega \) into \( \mathcal{Y} \) providing that \( T \) is countably generated (see [3], Theorems 6 and 3), a product r.c.p. (see [3], Theorem 6) and a subfield r.c.p. (see [3], Theorems 6 and 2). So, in most cases appearing in applications (e.g. Polish spaces) all types of disintegrations always exist.

(b) Note also that the hypothesis “perfect” for the existence of a disintegration consistent with \( f \) is in fact necessary (see [3], Theorem 4').

(c) Let \( \{P_y\}_{y \in \mathcal{Y}} \) be a disintegration of \( P \) over \( Q \), and let \( f \) be an inverse-measure-preserving map from \( \Omega \) into \( \mathcal{Y} \). Then the following are equivalent:

\[ \{P_y\}_{y \in \mathcal{Y}} \text{ is consistent with } f \quad (1) \]

\[ P(A \cap f^{-1}(B)) = \int_B P_y(A) Q(dy) \quad \text{for each } A \in \Sigma \text{ and } B \in T \quad (2) \]

For each \( A \in \Sigma \) \( \mathbb{E}_P[\chi_A | \sigma(f)] = P_\bullet(A) \circ f \quad P | \sigma(f) \text{ - a.s.} \) \quad (3) \]

(d) Let \( X \) be a \( \Sigma \)-\( T \)-measurable map from \( \Omega \) into \( \mathcal{Y} \), let \( \{P_\theta\}_{\theta \in \Psi} \) be a disintegration of \( P \) over \( P_\theta \) consistent with \( \Theta \), and let \( k \) be a \( T \)-\( Z \)-Markov kernel. If \( k(\cdot, \cdot) \) is the distribution of \( X \) under \( P_\theta \) for \( \theta \in \Psi \), then the map \( K(\Theta) \) is a conditional distribution of \( X \) given \( \Theta \), since by condition \( (3) \) of \( (b) \) we get for \( A = X^{-1}(B) \) with \( B \in T \) that \( P_{X|\Theta}(B, \cdot) = K(\Theta)(B, \cdot) \quad P | \sigma(\Theta) \text{-a.s.} \).

(e) Conversely, in the special case where \( \Sigma \) is countably generated and \( (\mathcal{Y}, T) = (\mathbb{R}, \mathcal{B}) \), given \( \{P_\theta\}_{\theta \in \Psi} \) as in (d), we get that for each conditional distribution \( K(\Theta) \) of \( X \) given \( \Theta \), there exists an **essentially unique** probability distribution \( (P_\theta)_X \) of \( X \), for \( \theta \in \Psi \), such that for each \( B \in \mathcal{B} \) we have

\[ K(\Theta)(B, \cdot) = (P_\bullet)_X(B) \circ \Theta \quad P | \sigma(\Theta) \text{- a.s..} \]

In fact, by applying a monotone class argument, it can be easily seen that the disintegration is essentially unique in the sense that if \( \{P'_\theta\}_{\theta \in \Psi} \) is any other disintegration of \( P \) over \( P_\Theta \) which is consistent with \( \Theta \), then \( P_\theta = P'_\theta \) for \( P_\Theta \)-almost all \( (P_\Theta \text{-a.a. for short}) \theta \in \Psi \). But the consistency of \( \{P_\theta\}_{\theta \in \Psi} \) together with (c) yields that condition \( (3) \) holds true; hence setting \( A = X^{-1}(B) \) with \( B \in \mathcal{B} \) we deduce that \( K(\Theta)(B, \cdot) = (P_\bullet)_X(B) \circ \Theta \quad P | \sigma(\Theta) \text{-a.s..} \).

If no confusion arises, we denote \( (P_\theta)_X \) by \( K(\theta) \) for \( \theta \in \Psi \).

*Throughout what follows, the conditional distribution \( K(\Theta) \) involving in Remark 3.4, (e) will be considered together with the distributions \( K(\theta) \), for \( \theta \in \Psi \), associated with \( K(\Theta) \) as in the above remark, without any additional comments.*

*For the remainder of this section, \( \{P_\theta\}_{\theta \in \Psi} \) is a disintegration of \( P \) over \( P_\Theta \) consistent with \( \Theta \) and \( \{X_i\}_{i \in I} \) is a non empty family of \( \Sigma \)-\( T \)-measurable maps from \( \Omega \) into \( \mathcal{Y} \).*

The next result extends Lemma 4.3 from [10].

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Lemma 3.5 If \( \{k_i\}_{i \in I} \) is a non empty family of T-Z-Markov kernels, then for each \( i \in I \) and for any fixed \( B \in T \) the following conditions are equivalent:

(i) \( P_{X_i \mid \Theta}(B, \cdot) = K_i(\Theta)(B, \cdot) \) \( P \mid \sigma(\Theta) \) - a.s.;

(ii) \( P_\theta(X_i^{-1}(B)) = k_i(B, \theta) \) for \( P_\Theta \)-a.a. \( \theta \in \Psi \).

In particular, the same remains true if \( K_i(\Theta)(B, \cdot) \) and \( k_i(B, \theta) \) are independent of \( i \) for all \( B \in T \) and \( P_\Theta \)-a.a. \( \theta \in \Psi \).

Proof. Let us fix on arbitrary \( i \in I \). For all \( B \in T \) and \( D \in Z \) we obtain that

\[
\int_{\Theta^{-1}(D)} P_{X_i \mid \Theta}(B, \cdot) dP = \int_{\Theta^{-1}(D)} K_i(\Theta)(B, \cdot) dP
\]

\[
\iff \int_{\Theta^{-1}(D)} \mathbb{E}_{P}[X_{X_i^{-1}(B)} \mid \sigma(\Theta)] dP = \int_{\Theta^{-1}(D)} k_i(B, \cdot) \circ \Theta dP
\]

\[(3) \iff \int_{D} P_{\theta}(X_i^{-1}(B)) P_{\Theta}(d\theta) = \int_{D} k_i(B, \theta) P_{\Theta}(d\theta). \]

Consequently, the equivalence of assertions (i) and (ii) follows. \( \Box \)

Lemma 3.6 Let \( \{k_i\}_{i \in I} \) be as in Lemma 3.5. Suppose that \( I \) is countable and \( T \) is countably generated. Then the following are equivalent:

(i) Condition \( P_{X_i \mid \Theta} = K_i(\Theta) \) \( P \mid \sigma(\Theta) \) - a.s. holds true for each \( i \in I \);

(ii) for \( P_\Theta \)-a.a. \( \theta \in \Psi \) condition \( P_\theta \circ X_i^{-1} = k_i \) holds true for each \( i \in I \).

In particular, the same remains true if \( K_i(\Theta) \) and \( k_i \) are independent of \( i \).

Proof. If (i) holds true, we then get by Lemma 3.5 that for each \( i \in I \) and \( B \in T \) condition

\( P_{\theta}(X_i^{-1}(B)) = k_i(B, \theta) \) for \( P_\Theta \)-a.a. \( \theta \in \Psi \).

is satisfied, which is equivalent to the fact that

\[
\forall i \in I \ \forall B \in T \ \exists L_{1,i,B} \in Z_0 \ \forall \theta \notin L_{1,i,B} \ P_\theta(X_i^{-1}(B)) = k_i(B, \theta),
\]

where \( Z_0 := \{L \in Z : P_\Theta(L) = 0\} \). Since \( I \) is countable, we find for each \( B \in T \) a \( P_\Theta \)-null set \( L_{1,B} := \bigcup_{i \in I} L_{1,i,B} \) such that

\[
\forall \theta \notin L_{1,B} \ \forall i \in I \ P_{\theta}(X_i^{-1}(B)) = k_i(B, \theta). \quad (4)
\]

Denote by \( \mathcal{G}_T \) a countable generator of \( T \). Without loss of generality we may assume that \( \mathcal{G}_T \) is closed under finite intersections. It follows by (4) that

\[
\forall n \in \mathbb{N} \ \forall B_n \in \mathcal{G}_T \ \exists L_{1,n} := L_{1,B_n} \in Z_0 \ \forall \theta \notin L_{1,n} \ \forall i \in I \ P_{\theta}(X_i^{-1}(B_n)) = k_i(B_n, \theta).
\]
Set $\tilde{L}_I := \bigcup_{k \in \mathbb{N}} \tilde{L}_{I,k} \in \mathbb{Z}_0$ and let

$$D := \{B \in T : P_\theta(X_i^{-1}(B)) = k_i(B, \theta) \ \forall \theta \notin \tilde{L}_I \ \forall i \in I\}.$$  

Then $\mathcal{G}_T \subseteq D$, and by applying a monotone class argument it can be easily seen that $D = T$. So assertion $(ii)$ follows.

Applying a similar reasoning we obtain the converse implication. \hfill $\square$

The following result extends Lemma 4.1 from [10].

**Lemma 3.7** Let $I$ be countable and $T$ countably generated. Then the family $\{X_i\}_{i \in I}$ is $P$-conditionally independent if and only if for $P_\theta$-a.a. $\theta \in \Psi$ it is $P_\theta$-independent.

**Proof.** Assume that $\{X_i\}_{i \in I}$ is $P$-conditionally independent. Then according to Remark 3.4, $(c)$ and following the same reasoning as in the proof of [10], Lemma 4.1, we get that

$$\int_D P_\theta\left(\bigcap_{j=1}^m \{X_{ij} \in B_j\}\right) P_\theta(d\theta) = \int_D \prod_{j=1}^m P_\theta(\{X_{ij} \in B_j\}) P_\theta(d\theta)$$

whenever $D \in \mathbb{Z}$, $m \in \mathbb{N}$, $i_1, \ldots, i_m \in I$ are distinct, and $B_1, \ldots, B_m \in T$, equivalently that for each $m \in \mathbb{N}$, for all $i_1, \ldots, i_m \in I$ distinct and for all $B_1, \ldots, B_m \in T$ there exists a $P_\theta$-null set $L_{I,m,i_1,\ldots,i_m,B_1,\ldots,B_m} \in \mathbb{Z}$ such that for any $\theta \notin L_{I,m,i_1,\ldots,i_m,B_1,\ldots,B_m}$

$$P_\theta\left(\bigcap_{j=1}^m \{X_{ij} \in B_j\}\right) = \prod_{j=1}^m P_\theta(\{X_{ij} \in B_j\}) \quad (5)$$

holds true. Without loss of generality we may and do assume that $m = 2$. Since $T$ is countably generated, applying successively two monotone class arguments we get that there exists a $P_\theta$-null set $L_I \in \mathbb{Z}$ such that for any $\theta \notin L_I$ condition (5) holds true for $m = 2$, for each $i_1, i_2 \in I$ with $i_1 \neq i_2$ and for each $B_1, B_2 \in T$; hence $\{X_i\}_{i \in I}$ is $P_\theta$-independent for any $\theta \notin L_I$. Since the inverse implication is clear, this completes the proof. \hfill $\square$

**Lemma 3.8** Let $I$ be countable and $T$ countably generated. Then the following hold true:

(i) The family $\{X_i\}_{i \in I}$ is $P$-conditionally identically distributed if and only if for $P_\theta$-a.a. $\theta \in \Psi$ it is $P_\theta$-identically distributed.

(ii) The family $\{X_i\}_{i \in I}$ is $P$-conditionally i.i.d. if and only if for $P_\theta$-a.a. $\theta \in \Psi$ it is $P_\theta$-i.i.d.

**Proof.** Ad (i): If $\{X_i\}_{i \in I}$ is $P$-conditionally identically distributed then for any two $i, j \in I$ and for each $B \in T$ the equality $P_{X_i|\Theta}(B) = P_{X_j|\Theta}(B)$ holds true $P | \sigma(\Theta)$-a.s., which due to Remark 3.4, $(c)$ yields that there exists a $P_\theta$-null set $\tilde{L}_{i,j,B} \in \mathbb{Z}$ such
that for any $\theta \notin \tilde{L}_j$ we have $P_\theta(X^{-1}_i(B)) = P_\theta(X^{-1}_j(B))$. Since $I$ is countable and $T$ is countably generated, letting $\tilde{L}_I := \bigcup_{B \in \mathcal{G}_T} \bigcup_{i,j \in I} \tilde{L}_{i,j}$, where $\mathcal{G}_T$ is a countable generator of $T$, and applying a monotone class argument, we find a $P_\theta$-null set $\tilde{L}_I \subseteq Z$ such that for any $\theta \notin \tilde{L}_I$ the equality $P_\theta(X^{-1}_i(B)) = P_\theta(X^{-1}_j(B))$ holds true for all $i, j \in I$ and $B \in T$; hence $\{X_i\}_{i \in I}$ is $P_\theta$-identically distributed. The inverse implication is immediate by Remark 3.4, (c).

Ad (ii): Assume that $\{X_i\}_{i \in I}$ is $P$-conditionally i.i.d.. It then follows by assertion (i) and Lemma 3.7 that there exist two $P_\theta$-null sets $\tilde{L}_I$ and $L_I$ in $Z$ such that for any $\theta \notin \tilde{L}_I := \tilde{L}_I \cup L_I$ the family $\{X_i\}_{i \in I}$ is $P_\theta$-i.i.d.. Since the inverse implication is clear, this completes the proof. \[ \square \]

**Proposition 3.9** The counting process $\{N_t\}_{t \in \mathbb{R}^+}$ is a $(P, K(\Theta))$-MRP if and only if for $P_\Theta$-a.a. $\theta \in \Psi$ it is a $(P_\theta, K(\theta))$-RP.

**Proof.** Assume that $\{N_t\}_{t \in \mathbb{R}^+}$ is a $(P, K(\Theta))$-MRP, i.e. that the process $\{W_n\}_{n \in \mathbb{N}}$ is $P$-conditionally independently and that for all claim interarrival times $W_n$ condition $P_{W_n|\Theta} = K(\Theta)$ holds true $P \mid \sigma(\Theta)$-a.s.. Applying now Lemmas 3.7 and 3.6, we equivalently get that there exist two $P_\Theta$-null sets $H_N$ and $\tilde{H}_N$ in $Z$ such that for any $\theta \notin H_* := H_N \cup \tilde{H}_N$ the sequence $\{W_n\}_{n \in \mathbb{N}}$ is $P_\theta$-independent and $(P_\theta)\tilde{W}_n = K(\theta)$ for each $n \in \mathbb{N}$, respectively, i.e. such that $\{N_t\}_{t \in \mathbb{R}^+}$ is a $(P_\theta, K(\theta))$-RP for any $\theta \notin H_*$. \[ \square \]

**Corollary 3.10** Let be given a $\sigma$-subalgebra $\mathcal{F}$ of $\Sigma$ and a subfield r.c.p. $\{P_\omega\}_{\omega \in \Omega}$ for $P$ over $R := P \mid \mathcal{F}$. Then $\{W_n\}_{n \in \mathbb{N}}$ is $P$-conditionally i.i.d. over $\mathcal{F}$ with a conditional probability distribution $K(id_\Omega) = P_{W_n|\mathcal{F}} P \mid \mathcal{F}$-a.s. for each $n \in \mathbb{N}$, if and only if $\{N_t\}_{t \in \mathbb{R}^+}$ is a $(P, K(id_\Omega))$-MRP if and only if for $R$-a.a. $\omega \in \Omega$ the family $\{N_t\}_{t \in \mathbb{R}^+}$ is a $(P_\omega, K(\omega))$-RP.

**Proof.** Put $(\Psi, Z) := (\Omega, \mathcal{F})$ and $\Theta := id_\Omega$. Then $\{P_\omega\}_{\omega \in \Omega}$ is a disintegration of $P$ over $P_\Theta = R$ consistent with the map $\Theta$. So the result follows by Proposition 3.9. \[ \square \]

Finally, we extend Lemmas 3.7 and 3.8 for uncountable index set. To this aim, we need to recall some notions more.

Given a partially ordered set $I$, any increasing family $\{\Sigma_i\}_{i \in I}$ of $\sigma$-subalgebras of $\Sigma$ is said to be a filtration for $(\Omega, \Sigma)$. For any family $\{Z_i\}_{i \in I}$ of $\Sigma$-$\mathcal{F}$-measurable maps, the filtration $\{Z_i\}_{i \in I}$ with $Z_i := \sigma(\bigcup_{j \leq i} \sigma(Z_j))$ for each $i \in I$, is called the canonical filtration for $\{Z_i\}_{i \in I}$. In particular, for $I = \mathbb{R}_+$ the filtration $\{Z_t\}_{t \in \mathbb{R}_+}$ is said to be right-continuous if $Z_t = \bigcap_{s > t} Z_s$ for any $t \in \mathbb{R}_+$.

Let $I$ be an arbitrary subset of $\mathbb{R}_+$ and let $\mathcal{Y}$ be a metric space. We say that the family $\{X_i\}_{i \in I}$ of $\Sigma$-$\mathcal{B}(\mathcal{Y})$-measurable maps from $\Omega$ into $\mathcal{Y}$ is separable, if there exists a countable set $G \subseteq I$ such that for each $\omega \in \Omega$ the set $\{(u, X_u(\omega)) : u \in G\}$ is dense in $\{(i, X_i(\omega)) : i \in I\}$. Any such set $G$ is called a separator (or separating set) for $\{X_i\}_{i \in I}$.
Remarks 3.11 Let $I \subseteq \mathbb{R}_+$ and let $Q$ be a probability measure on $\Sigma$. Then the following can be easily proven:

(a) If $\{U_t\}_{t \in I}$ is a family of $\Sigma$-$\mathcal{B}(\Upsilon)$-measurable maps from $\Omega$ into $\Upsilon$, and $\{Z_t\}_{t \in I}$ is its canonical filtration, then $\{U_t\}_{t \in I}$ is $Q$-independent if and only if for every bounded $\mathcal{B}(\Upsilon)$-measurable real-valued function $f$ on $\Upsilon$ the equality

$$
\mathbb{E}_Q[\chi f(U_t)] = Q(A)\mathbb{E}_Q[f(U_t)]
$$

holds true for each $s, t \in I$ with $s < t$ and for each $A \in \mathcal{Z}_s$.

(b) If $U_1$ and $U_2$ are two $\Sigma$-$\mathcal{B}(\Upsilon)$-measurable maps from $\Omega$ into $\Upsilon$, then they are $Q$-identically distributed if and only if $\mathbb{E}_Q[f(U_1)] = \mathbb{E}_Q[f(U_2)]$ for every bounded $\mathcal{B}(\Upsilon)$-measurable real-valued function $f$ on $\Upsilon$.

Recall that the family $\{X_t\}_{t \in \mathbb{R}_+}$ has $P$-(conditionally) independent increments, if for each $m \in \mathbb{N}$ and for each $t_0, t_1, \ldots, t_m \in \mathbb{R}_+$, such that $0 = t_0 < t_1 < \cdots < t_m$ the increments $X_{t_j} - X_{t_{j-1}}$ ($j \in \mathbb{N}_m$) are $P$-(conditionally) independent.

Proposition 3.12 Let $\Upsilon$ be a Polish space, let $\{X_t\}_{t \in \mathbb{R}_+}$ be a family of $\Sigma$-$\mathcal{B}(\Upsilon)$-measurable maps from $\Omega$ into $\Upsilon$ and let $\{\mathcal{H}_t\}_{t \in \mathbb{R}_+}$ be its canonical filtration. If the family $\{X_t\}_{t \in \mathbb{R}_+}$ is separable with separator $\mathcal{Q}_+$, then the following hold true:

(i) If $\{\mathcal{H}_t\}_{t \in \mathbb{R}_+}$ is right-continuous, then $\{X_t\}_{t \in \mathbb{R}_+}$ is $P$-conditionally independent if and only if for $P_\theta$-a.a. $\theta \in \Psi$ it is $P_\theta$-independent.

(ii) The family $\{X_t\}_{t \in \mathbb{R}_+}$ is $P$-conditionally identically distributed if and only if for $P_\theta$-a.a. $\theta \in \Psi$ it is $P_\theta$-identically distributed.

(iii) If $\{\mathcal{H}_t\}_{t \in \mathbb{R}_+}$ is right-continuous, then $\{X_t\}_{t \in \mathbb{R}_+}$ is $P$-conditionally i.i.d. if and only if for $P_\theta$-a.a. $\theta \in \Psi$ it is $P_\theta$-i.i.d.

(iv) If $\{\mathcal{K}_t\}_{t \in \mathbb{R}_+}$ is a family of $\mathcal{B}(\Upsilon)$-$Z$-Markov kernels such that $\mathcal{K}_t(\theta)$ is for every $t \in \mathbb{R}_+$ a probability distribution on $\mathcal{B}(\Upsilon)$ with parameter $\theta \in \Psi$, and the function $t \mapsto \mathcal{K}_t(\theta)(B)$ is continuous for any fixed $B \in \mathcal{B}(\Upsilon)$ and $\theta \in \Psi$, then condition $P_{X_t|\Theta} = \mathcal{K}_t(\Theta) P | \mathcal{\sigma}(\Theta)$-a.s. holds true for each $t \in \mathbb{R}_+$ if and only if for $P_\theta$-a.a. $\theta \in \Psi$ condition $P_\theta \circ X^{-1} = \mathcal{K}_t$ holds true for each $t \in \mathbb{R}_+$.

Moreover, assertions (i) to (iii) remain true for the increments of $\{X_t\}_{t \in \mathbb{R}_+}$ in the place of $\{X_t\}_{t \in \mathbb{R}_+}$.

Proof. Ad (i): The “if” implication follows as in the proof of Lemma 4.1 from [10]. For the “only if” part, assume that $\{X_t\}_{t \in \mathbb{R}_+}$ is $P$-conditionally independent and note that our assumptions for $\{X_t\}_{t \in \mathbb{R}_+}$ and $\{\mathcal{H}_t\}_{t \in \mathbb{R}_+}$ imply $\mathcal{H}_s = \sigma(\{X_u\}_{u \in \mathcal{Q}_+, u \leq s}) = \bigcap_{s \in \mathcal{Q}_+, s' > s} \mathcal{H}_s'$ for each $s \in \mathbb{R}_+$.

(a) It follows by Lemma 3.7 that there exists a $P_\theta$-null set $O_{\mathcal{Q}_+} \in Z$ such that for any $\theta \notin O_{\mathcal{Q}_+}$ condition (5) holds true with $\mathcal{Q}_+$ and $\mathcal{B}(\Upsilon)$ in the place of $I$ and $T$, respectively.
Throughout this proof fix on an arbitrary \( \theta \notin O_{Q+} \). Then condition (5) together with Remark 3.11, (a) implies that for all \( s,t \in Q_+ \) with \( s < t \), for every bounded \( \mathcal{B}(T) \)-measurable real-valued function \( f \) on \( T \) and for each \( A \in \mathcal{Z}_s \) we have

\[
\mathbb{E}_{P_\theta}[\chi_A f(X_t)] = P_\theta(A) \mathbb{E}_{P_\theta}[f(X_t)].
\] (7)

If we take \( s,t \in \mathbb{R}_+ \) with \( s < t \) and if we write (7) for \( s',t' \in Q_+ \) with \( s' < t' \) and then let \( s' \downarrow s \) and \( t' \downarrow t \), the separability of \( \{X_t\}_{t \in \mathbb{R}_+} \) together with an application of Lebesgue’s Dominated Convergence Theorem yields that for all \( A \in \bigcap_{s' \in Q_+, s' > s} \mathcal{H}_{s'} = \mathcal{H}_s \) and for every bounded continuous real-valued function \( f \) on \( T \) condition (7) holds true.

(b) Let \( s,t \in \mathbb{R}_+ \) with \( s < t \) and let \( f \) be a function as in (6). Then for each \( n \in \mathbb{N} \) there exists a bounded continuous real-valued function \( g_n \) on \( T \) satisfying the inequality \( \int |g_n - f|d(P_\theta)X_t \leq \frac{1}{n} \) (cf. e.g. [4], Proposition 415P); hence there exists a sequence \( \{g_n\}_{n \in \mathbb{N}} \) of bounded continuous real-valued functions on \( T \) such that condition

\[
\lim_{n \to \infty} \int \chi_A(|g_n - f| \circ X_t)dP_\theta = 0 \quad \text{holds true for all} \quad A \in \mathcal{F}_s.
\]

(c) Let \( s,t \in \mathbb{R}_+ \) with \( s < t \), \( A \in \mathcal{H}_s \) and \( f \) be a function as in (6). It then follows by (b) that there exists a sequence \( \{g_n\}_{n \in \mathbb{N}} \) of bounded continuous real-valued functions on \( T \) such that

\[
\mathbb{E}_{P_\theta}[\chi_A f(X_t)] = \lim_{n \to \infty} \mathbb{E}_{P_\theta}[\chi_A g_n(X_t)] \overset{(a)}{=} \lim_{n \to \infty} P_\theta(A) \mathbb{E}_{P_\theta}[g_n(X_t)] = P_\theta(A) \mathbb{E}_{P_\theta}[f(X_t)];
\]

hence by Remark 3.11, (a) we get that \( \{X_t\}_{t \in \mathbb{R}_+} \) is \( P_\theta \)-independent, which proves (i).

Ad (ii): The “if” implication is immediate by Remark 3.4, (c). For the “only if” part, assume that \( \{X_t\}_{t \in \mathbb{R}_+} \) is \( P \)-conditionally identically distributed.

(d) Since \( \{X_t\}_{t \in \mathbb{R}_+} \) is \( P \)-conditionally identically distributed, we get that for any two \( s,t \in Q_+ \) and for each \( B \in \mathcal{B}(T) \) the equality \( P_{X_t|\Theta}(B) = P_{X_s|\Theta}(B) \) holds \( P \mid \sigma(\Theta) \)-a.s.

The latter together with Lemma 3.8, (i) yields the existence of a \( P_\Theta \)-null set \( \tilde{O}_{Q+} \in Z \) such that for any \( \theta \notin \tilde{O}_{Q+} \) and for all \( s,t \in Q_+ \) condition \( P_\theta(X_t) = (P_\theta)_{X_s} \) holds true, which by Remark 3.11, (b) equivalently yields that for any \( \theta \notin \tilde{O}_{Q+} \), for every function \( f \) as in the above remark, and for all \( s,t \in Q_+ \) we have \( \mathbb{E}_{P_\theta}[f(X_t)] = \mathbb{E}_{P_\theta}[f(X_s)] \).

Till the end of the proof of (i), fix on an arbitrary \( \theta \notin \tilde{O}_{Q+} \).

(e) If we take \( s,t \in \mathbb{R}_+ \) and if we write the last equality for \( s',t' \in Q_+ \) and then let \( s' \downarrow s \) and \( t' \downarrow t \), the separability of \( \{X_t\}_{t \in \mathbb{R}_+} \) together with an application of Lebesgue’s Dominated Convergence Theorem yields that for every bounded continuous real-valued function \( f \) on \( T \) condition \( \mathbb{E}_{P_\theta}[f(X_t)] = \mathbb{E}_{P_\theta}[f(X_s)] \) holds true.

Following now the same reasoning with that of steps (b) and (c), we obtain that the last equality is satisfied by all functions \( f \) as in Remark 3.11, and all \( s,t \in \mathbb{R}_+ \), which is equivalent to the fact that condition \( P_\theta(X_t) = (P_\theta)_{X_s} \) holds true for all \( s,t \in \mathbb{R}_+ \); hence the family \( \{X_t\}_{t \in \mathbb{R}_+} \) is \( P_\theta \)-identically distributed, which proves (ii).

Ad (iii): Assume that \( \{X_t\}_{t \in \mathbb{R}_+} \) is \( P \)-conditionally i.i.d. and that its canonical filtration is right-continuous. It then follows by assertions (i) and (ii) there exist two \( P_\Theta \)-null sets \( \tilde{O}_{Q+} \) and \( O_{Q+} \) in \( Z \) such that for any \( \theta \notin \tilde{O}_{Q+} := \tilde{O}_{Q+} \cup O_{Q+} \) the family \( \{X_t\}_{t \in \mathbb{R}_+} \) is \( P_\theta \)-i.i.d.. Since the inverse implication is clear, assertion (iii) follows.
Ad (iv): Assume that for any \( t \in \mathbb{R}_+ \) condition \( P_{X_t|\Theta} = \tilde{K}_t(\Theta) \) holds \( P | \sigma(\Theta) \)-a.s.. It then follows by Lemma 3.6 that there exists a \( P_\Theta \)-null set \( \tilde{O}'_{Q_+} \in Z \) such that for any \( \theta \notin \tilde{O}'_{Q_+} \), for each \( B \in \mathfrak{B}(\mathcal{Y}) \) and for any \( t \in Q_+ \) the following condition holds true:

\[
P_{\theta}(X_t^{-1}(B)) = \tilde{K}_t(\theta)(B).
\] (8)

Fix on arbitrary \( \theta \notin \tilde{O}'_{Q_+} \) and \( t \in \mathbb{R}_+ \). Then the separability of \( \{X_t\}_{t \in \mathbb{R}_+} \) implies that there exists a monotone sequence \( \{X_s\}_{s \in Q_+} \) such that \( s \to t \) and \( X_t = \lim_{s \to t} X_s \), which together with (8) and the Monotone Convergence Theorem yields that

\[
(P_{\theta})_{X_t} = \lim_{s \to t} (P_{\theta})_{X_s} = \lim_{s \to t} \tilde{K}_s(\theta) = \tilde{K}_t(\theta).
\]

Since the inverse implication follows by applying similar arguments, we obtain (iv).

Moreover, the proofs of assertions (i) to (iii) for the increments of \( \{X_t\}_{t \in \mathbb{R}_+} \) run in the same way as for \( \{X_t\}_{t \in \mathbb{R}_+} \).

It is immediate from the corresponding definitions that if \( \{X_t\}_{t \in \mathbb{R}_+} \) satisfies condition \( X_0(\omega) = 0 \) for each \( \omega \in \Omega \) and has \( P \)-conditionally independent increments, then it will have \( P \)-conditionally stationary increments, if and only if for each \( t, h \in \mathbb{R}_+ \) the equality \( P_{X_{t+h} - X_t|\Theta} = P_{X_h|\Theta} \) holds \( P | \sigma(\Theta) \)-a.s. true (cf. e.g. [10], page 68 for the definition of conditionally stationary increments).

The following result extends a basic one from [10], that is Proposition 4.4.

**Theorem 3.13** Let \( \{N_t\}_{t \in \mathbb{R}_+} \) be a counting process and let \( \{\tilde{K}_t\}_{t \in \mathbb{R}_+} \) be as in Proposition 3.12 but with \( \mathcal{Y} = [0, \infty] \). Then \( \{N_t\}_{t \in \mathbb{R}_+} \) has \( P \)-conditionally stationary independent increments such that condition

\[
P_{N_t|\Theta} = \tilde{K}_t(\Theta) \quad P | \sigma(\Theta) - a.s.
\]

holds true for each \( t \in \mathbb{R}_+ \) if and only if for \( P_{\Theta} \)-a.a. \( \theta \in \Psi \) it has \( P_{\theta} \)-stationary independent increments such that \( (P_{\theta})_{N_t} = \tilde{K}_t(\theta) \) for each \( t \in \mathbb{R}_+ \).

**Proof.** Since \( \{N_t\}_{t \in \mathbb{R}_+} \) is a counting process it has right-continuous paths; hence it is separable with separator \( Q_+ \). Note also that the canonical filtration of \( \{N_t\}_{t \in \mathbb{R}_+} \) is right-continuous (see [13], Theorem 25, where the proof works for any probability space not necessarily complete). Thus, all assumptions of Proposition 3.12 are fulfilled, and so we may apply it to deduce the thesis of the theorem.

**Corollary 3.14 ([10], Proposition 4.4)** The family \( \{N_t\}_{t \in \mathbb{R}_+} \) is a \( P \)-MPP with parameter \( \Theta \) if and only if it is a \( P_{\Theta} \)-Poisson process with parameter \( \theta \) for \( P_{\Theta} \)-a.a. \( \theta \in \mathbb{R} \).
4 Mixed renewal processes and exchangeability

An infinite family \( \{X_i\}_{i \in I} \) of \( \Sigma\)-\(T\)-measurable maps from \( \Omega \) into \( \mathcal{Y} \) is said to be exchangeable under \( P \) or \( P\text{-exchangeable} \), if for each \( r \in \mathbb{N} \) we have

\[
P\left( \bigcap_{k=1}^{r} X_{i_k}^{-1}(B_k) \right) = P\left( \bigcap_{k=1}^{r} X_{j_k}^{-1}(B_k) \right)
\]

whenever \( i_1, \ldots, i_r \in I \) are distinct, \( j_1, \ldots, j_r \in I \) are distinct, and \( B_k \in T \) for each \( k \leq r \) (cf. e.g. [4], 459C).

For the purposes of this section we recall the notion of infinite products of measure spaces. Let \( I \) be an arbitrary non-empty index set. If \( \{(\Omega_i, \Sigma_i, P_i)\}_{i \in I} \) is a family of probability spaces then, for each \( \emptyset \neq J \subseteq I \) we denote by \((\Omega_J, \Sigma_J, P_J)\) the product probability space \( \otimes_{i \in J} (\Omega_i, \Sigma_i, P_i) := (\prod_{i \in J} \Omega_i, \otimes_{i \in J} \Sigma_i, \otimes_{i \in J} P_i) \). If \((\Omega, \Sigma, P)\) is a probability space, we write \( P_i \) for the product measure on \( \Omega^I \) and \( \Sigma_I \) for its domain.

**Lemma 4.1** Let \( \mathcal{F} \) be a \( \sigma \)-subalgebra of \( \Sigma \) and let \( \{X_i\}_{i \in I} \) be a non-empty family of \( \Sigma\)-\(T\)-measurable maps from \( \Omega \) into \( \mathcal{Y} \) such that \( \{X_i\}_{i \in I} \) is \( \mathcal{P} \)-conditionally i.i.d. over \( \mathcal{F} \). Suppose that \( T \) is countably generated and that \( P_{X_i} \) is perfect for each \( i \in I \). Then there exists a probability measure \( M \) on \( T \otimes \mathcal{F} \) with marginal \( R := P \mid \mathcal{F} \) on \( \mathcal{F} \) such that \( M := P \circ (X_i \times \text{id}_\Omega)^{-1} \) for every \( i \in I \), and a product r.c.p. \( \{Q_\omega\}_{\omega \in \Omega} \) on \( T \) for \( M \) with respect to \( R \), such that

(i) for any fixed \( B \in T \) and \( i \in I \) the map \( Q_\cdot(B) : \Omega \rightarrow [0,1] \) is \( R \)-a.s. equal to \( P(X_i^{-1}(B) \mid \mathcal{F})(\cdot) \);

(ii) \( \int_{\mathcal{F}} Q^I_\omega(H) R(d\omega) = P(F \cap X^{-1}(H)) \) for every \( F \in \mathcal{F} \) and \( H \in T_I \), where \( Q^I_\omega \) denotes the \( I \)-fold product probability \( \otimes_{i \in I} P_i \) of copies \( P_i := Q_\omega \) of \( Q_\omega \) for \( i \in I \), and \( X : \Omega \rightarrow \mathcal{Y}^I \) is defined by \( X(\omega) = (X_i(\omega))_{i \in I} \) for each \( \omega \in \Omega \).

**Proof.** First fix on an arbitrary \( i \in I \).

(a) The function \( X_i \times \text{id}_\Omega \) from \( \Omega \) into \( \mathcal{Y} \times \Omega \) defined by means of

\[
(X_i \times \text{id}_\Omega)(\omega) := (X_i(\omega), \omega) \quad \text{for each} \quad \omega \in \Omega
\]

is \( \Sigma\)-\(T \otimes \mathcal{F} \)-measurable. So, we have a probability measure \( M_i := P \circ (X_i \times \text{id}_\Omega)^{-1} \) on \( T \otimes \mathcal{F} \). Since all \( X_i \) are \( \mathcal{P} \)-conditionally identically distributed over \( \mathcal{F} \), it follows that \( M_i \) is independent of \( i \), so we may write \( M := M_i \) for any fixed \( i^* \in I \).

(b) There exists a product r.c.p. \( \{Q_\omega\}_{\omega \in \Omega} \) on \( T \) for \( M \) with respect to \( R = P \mid \mathcal{F} \) such that for any fixed \( B \in T \)

\[
Q_\cdot(B) = P(X_i^{-1}(B) \mid \mathcal{F})(\cdot) \quad R \text{-a.s.}
\]

In fact, by assumption each marginal measure \( P_{X_i} \) of \( M \) on \( T \) is perfect and \( T \) is countably generated; hence by Remark 3.4, (a), there exists a product r.c.p. \( \{Q_\omega\}_{\omega \in \Omega} \) on \( T \) for \( M \) with respect to \( R \).
Since \( \{Q_\omega\}_{\omega \in \Omega} \) satisfies (D2), we get that
\[
\int_F Q_\omega(F)\,d\omega = P(F) = \int_F P(F \cap X^{-1}(F))(\omega)\,d\omega
\]
for every \( F \in T \) and \( F \in \mathcal{F} \), which proves (b); hence (i) follows.

(c) Using (i) and a monotone class argument we get that (ii) holds true. \( \square \)

The next result extends a corresponding one due to Olshen (see [12], Theorem (3)) concerning a generalization of de Finetti’s Theorem.

**Proposition 4.2** Let \( \{X_i\}_{i \in I} \) be a \( \Sigma\)-\( T \)-measurable family of \( \Delta \)-measurable maps from \( \Omega \) into \( \mathcal{Y} \). Suppose that \( T \) is countably generated and \( P_{X_i} \) is perfect for each \( i \in I \). Then there exists a \( \Sigma\)-\( g\)-measurable map \( \Theta \) from \( \Omega \) into \( \mathbb{R}^d \) such that \( \{X_i\}_{i \in I} \) is \( P \)-conditionally i.i.d. given \( \Theta \).

**Proof.** (a) Since \( \{X_i\}_{i \in I} \) is \( P \)-exchangeable, it follows by [4], Theorem 459B, that there exist a \( \sigma \)-subalgebra \( \mathcal{F} \) of \( \Sigma \) such that \( \{X_i\}_{i \in I} \) is \( P \)-conditionally i.i.d. over \( \mathcal{F} \).

So, applying Lemma 4.1, we deduce that there exists a family \( \{Q_\omega\}_{\omega \in \Omega} \) of \( T \)-\( \mathcal{F} \)-Markov kernels such that
\[
\int_F Q_\omega^i(H)\,d\omega = P(F \cap X^{-1}(H))
\]
for every \( H \in T_i \) and \( F \in \mathcal{F} \), where \( R := P \mid \mathcal{F} \). Then there exists a countably generated \( \sigma \)-subalgebra \( \mathcal{A} \) of \( \mathcal{F} \) such that \( Q_\omega(B) \) is \( \mathcal{A} \)-measurable for arbitrary but fixed \( B \in T \) (take e.g. \( \mathcal{A}_B := \sigma(\{[Q_\omega(B)]^{-1}(E) : E \in \mathcal{G}_\mathcal{B}\}) \) for \( B \in T \), and \( \mathcal{A} := \sigma(\bigcup_{B \in \mathcal{G}_\mathcal{B}} \mathcal{A}_B) \), where \( \mathcal{G}_\mathcal{B} \) and \( \mathcal{G}_T \) is a countable generator of \( \mathcal{B} \) and \( T \), respectively).

Since \( \mathcal{A} \) is countably generated, there exists a map \( \tilde{\Theta} : \Omega \rightarrow \mathbb{R} \) such that \( \mathcal{A} = \sigma(\tilde{\Theta}) \) (take e.g. \( \tilde{\Theta} \) to be the Marczewski functional on \( \Omega \), cf. e.g. [4], 343E for the definition).

But since \( \{X_i\}_{i \in I} \) is \( P \)-conditionally i.i.d. over \( \mathcal{F} \) and \( \mathcal{A} \subseteq \mathcal{F} \), it follows that \( \{X_i\}_{i \in I} \) is so over \( \mathcal{A} = \sigma(\tilde{\Theta}) \).

(b) There exists a \( \Sigma\)-\( \mathcal{B}_d \)-measurable map \( \Theta \) from \( \Omega \) into \( \mathbb{R}^d \) such that \( \{X_i\}_{i \in I} \) is \( P \)-conditionally i.i.d. given \( \Theta \).

In fact, since \( \mathbb{R} \) and \( \mathbb{R}^d \) are standard Borel spaces of the same cardinality, there exists a Borel isomorphism \( g \) from \( \mathbb{R} \) into \( \mathbb{R}^d \) (cf. e.g. [4], Corollary 424D(a)). Put \( \Theta := g \circ \tilde{\Theta} \). Then \( \Theta \) is a \( \Sigma\)-\( \mathcal{B}_d \)-measurable map from \( \Omega \) into \( \mathbb{R}^d \) such that \( \sigma(\Theta) = \sigma(\tilde{\Theta}) \). So, (b) follows by (a). This completes the proof. \( \square \)

**Corollary 4.3** (see Olshen, R. [12], Theorem (3)) If \( \{X_n\}_{n \in \mathbb{N}} \) is a \( P \)-exchangeable sequence of measurable maps from \( \Omega \) into a complete, separable metric space, then there exists a real-valued random variable \( \Theta \) on \( \Omega \) such that \( \{X_n\}_{n \in \mathbb{N}} \) is \( P \)-conditionally i.i.d. given \( \Theta \).

**Theorem 4.4** Let \( \{X_i\}_{i \in I} \) be an infinite family of \( \Sigma\)-\( T \)-measurable maps from \( \Omega \) into \( \mathcal{Y} \). Consider the following assertions:
(i) \{X_i\}_{i \in I} is \(P\)-exchangeable.

(ii) There exists a \(\sigma\)-subalgebra \(\mathcal{F}\) of \(\Sigma\) such that \{\(X_i\)\}_{i \in I} is \(P\)-conditionally i.i.d. over \(\mathcal{F}\).

(iii) There exists a \(\sigma\)-subalgebra \(\mathcal{F}\) of \(\Sigma\) and a family \(\{Q_\omega\}_{\omega \in \Omega}\) of \(T\)-\(\mathcal{F}\)-Markov kernels such that

\[
\int_{\mathcal{F}} Q_\omega (H) R(d\omega) = P(F \cap \mathcal{X}^{-1}(H))
\]

for every \(H \in \mathcal{T}_I\) and \(F \in \mathcal{F}\), where \(R := P | \mathcal{F}\) and \(Q_\omega^I, X\) are as in Lemma 4.1.

(iv) There exists a \(\Sigma\)-\(\mathcal{B}_d\)-measurable map \(\Theta\) from \(\Omega\) into \(\mathbb{R}^d\) such that \{\(X_i\)\}_{i \in I} is \(P\)-conditionally i.i.d. given \(\Theta\).

Then (i) \iff (ii), (iii) \implies (i) and (iv) \implies (i). If any one of conditions (i) to (iv) is satisfied, then all image measures \(P_{X_i}\) are equal. Moreover, if \(P_{X_i}\) is perfect for any \(i \in I\) and \(T\) is countably generated, then assertions (i) to (iv) are equivalent.

**Proof.** First note that if \(P\) is perfect then each \(P_{X_i}\) is perfect (cf. e.g. [4], Proposition 451E(a)). The equivalence (i) \iff (ii) follows by [4], Theorem 459B, while the implications (iii) \implies (i) and (iv) \implies (i) are evident.

Clearly, if assertion (i) or equivalently (ii) is satisfied then all \(P_{X_i}\) are equal and the same applies if (iii) or (iv) holds true.

Moreover, if every measure \(P_{X_i}\) is perfect and \(T\) is countably generated, then implications (ii) \implies (iii) and (i) \implies (iv) follow from Lemma 4.1 and Proposition 4.2, respectively. So we get that assertions (i) to (iv) are equivalent.

**Corollary 4.5** Let \{\(X_t\)\}_{t \in \mathbb{R}^+} be a family of \(\Sigma\)-\(T\)-measurable maps from \(\Omega\) into \(\Upsilon\). Suppose that \(\Sigma\) is countably generated, \(P\) is perfect, \(\Upsilon\) is a Polish space, \{\(X_t\)\}_{t \in \mathbb{R}^+} is separable with separator \(\mathbb{Q}_+\) and that its canonical filtration is right-continuous. Then each of the items (i) to (iv) of Theorem 4.4 is equivalent to condition

(v) there exist a \(\Sigma\)-\(\mathcal{B}_d\)-measurable map \(\Theta\) from \(\Omega\) into \(\mathbb{R}^d\) and a disintegration \(\{P_\theta\}_{\theta \in \mathbb{R}^d}\) of \(P\) over \(P_\Theta\) consistent with \(\Theta\) such that \{\(X_t\)\}_{t \in \mathbb{R}^+} is \(P_\theta\)-i.i.d. for \(P_\Theta\)-a.a. \(\theta \in \mathbb{R}^d\).

**Proof.** It follows by Remark 3.4, (a), that given a \(\Sigma\)-\(\mathcal{B}_d\)-measurable map \(\Theta\) from \(\Omega\) into \(\mathbb{R}^d\) there exists a disintegration \(\{P_\theta\}_{\theta \in \mathbb{R}^d}\) of \(P\) over \(P_\Theta\) consistent with \(\Theta\). Thus we may apply Proposition 3.12 to obtain that condition (v) is equivalent to (iv) of Theorem 4.4. The equivalence of all items (i) to (v) is immediate by Theorem 4.4. \(\square\)
Remarks 4.6 (a) The assumption “$P_{X_i}$ perfect” made in the last theorem is easy verified in the usual applications, since this is covered by the following facts: (α) If $\mathcal{Y}$ is a Polish space then each $P_{X_i}$ is Radon (cf. e.g. [4], Proposition 434K(b) and [4], Definition 411H(b) for the definition of a Radon measure); hence perfect (cf. e.g. [4], Proposition 416W(a)). (β) If $P$ is perfect then each $P_{X_i}$ is so (cf. e.g. [4], Proposition 451E(a)). (γ) If $\Omega = \mathcal{Y}^I$, $X_i (i \in I)$ are the canonical projections from $\Omega$ onto $\mathcal{Y}$, and $P$ is any probability measure on $\Sigma := T_I$ then each $P_{X_i}$ is perfect if and only if $P$ is perfect (cf. e.g. [4], Theorem 454A(b)(iii)).

(b) To the best of our knowledge, the most general result concerning the equivalence of assertions (i) and (iii) of Theorem 4.4 is Theorem 1.1 from [9] (which extends de Finetti’s Theorem), saying that for each infinite sequence $\{X_n\}_{n \in \mathbb{N}}$ of random variables taking values in a standard Borel space $\mathcal{Y}$ (i.e. $\mathcal{Y}$ is isomorphic to some Borel-measurable subset of $\mathbb{R}$) assertions (i) and (iii) of Theorem 4.4 with $\{X_n\}_{n \in \mathbb{N}}$ in the place of $\{X_i\}_{i \in I}$ are equivalent. It is well-known that any Polish space is standard Borel; in particular, $\mathbb{R}^d$ and $\mathbb{R}^N$ are such spaces.

(c) There are measurable spaces $(\mathcal{Y}, T)$ satisfying the assumptions of Theorem 4.4, i.e. that $T$ is countably generated and each $P_{X_i}$ is perfect, which are not standard Borel spaces; hence Theorem 4.4 extends Theorem 1.1 from [9]. In fact, it is known that each uncountable analytic Hausdorff space (i.e. a non-empty topological Hausdorff space being a continuous image of the space $\mathbb{N}^N$, cf. e.g. [4], Definition 423A) has a non-Borel analytic subset (cf. e.g. [4], Proposition 423L). It is also known that for each analytic Hausdorff space $\mathcal{Y}$ the Borel $\sigma$-algebra $\mathcal{B}(\mathcal{Y})$ is countably generated (cf. e.g. [4], 423X(d)), and that any Borel probability measure on $\mathcal{B}(\mathcal{Y})$ is always inner regular with respect to compact sets (see [7], Chapter IV, Theorem 1, page 195); hence it is perfect (cf. e.g. [4], Proposition 451C). Consequently, each uncountable analytic Hausdorff space has a subset satisfying the assumptions of Theorem 4.4, but not being a standard Borel space.

(d) Restricting attention to measurable spaces $(\mathcal{Y}, T)$ satisfying the countability assumption of $T$ as in Theorem 4.4, costs us some generality; for instance, the general compact Hausdorff space does not satisfy the countability assumption for $T$, and it is known that the equivalence of assertions (i) to (iii) of Theorem 4.4 is true for countable products of compact Hausdorff spaces (see [6] or [1]). More general, the equivalence of assertions (i) to (iii) of Theorem 4.4 is proven in [4], Theorem 459G for uncountable products of general Hausdorff spaces. But all the above are specialized in the product situation of topological spaces, while assertions (i) to (iii) of Theorem 4.4 have the advantage of being free from any topological assumption as well as from any product situation.

The following definition of an MRP traces back to Huang [8], Section 1, Definition 3.

**Definition 4.7** The counting process $\{N_t\}_{t \in \mathbb{R}^+}$ is said to be a $\nu$-mixed renewal
Definition 1 for the definition) implying that 

\[ P\left(\bigcap_{k=1}^{r}\{W_k \leq w_k\}\right) = \int \prod_{k=1}^{r} P_y(W_k \leq w_k) \nu(dy), \]

holds true, where \( \{P_y\}_{y \in \tilde{Y}} \) is a family of probability measures on \( \Sigma \) and \( \nu \) is a probability measure on \( B(\tilde{Y}) := \sigma(\{P_x(E) : E \in \Sigma\}) \) such that for \( \nu \)-a.a. \( \tilde{y} \in \tilde{Y} \) the process \( \{W_n\}_{n \in \mathbb{N}} \) is \( P_{\tilde{y}} \)-identically distributed.

In Huang’s definition it is assumed that \( \{N_t\}_{t \in \mathbb{R}^+} \) takes values only in \( \mathbb{N}_0 \), which is equivalent to the mild assumption that \( \{N_t\}_{t \in \mathbb{R}^+} \) has zero probability of explosion, that is \( P(\bigcup_{t \in (0,\infty)}\{N_t = \infty\}) = 0 \) (cf. e.g. [15], Lemma 2.1.4).

Remark 4.8 Note that in Huang’s [8] definition the assumption that \( \{W_n\}_{n \in \mathbb{N}} \) is \( P_{\tilde{y}} \)-identically distributed for \( \nu \)-a.a. \( \tilde{y} \in \tilde{Y} \) is not written explicitly. But this assumption must be included there, since it is necessary for the validity of the Corollary on page 20 of [8], as it follows from Example 5.4 below.

In fact, consider the process \( \{N_t\}_{t \in \mathbb{R}^+} \) of Example 5.4, where the above assumption does not hold true, as well as Huang’s definition of a \( \nu \)-MRP without the above assumption. Also note that \( q := P(Z < \infty) = 0 < 1 \), where \( Z \) is the almost sure limit of \( \{N_t\}_{t \in \mathbb{R}^+} \) as \( t \to \infty \). Assume, if possible, that Corollary in [8], page 20, holds true. Then conditional on the event \( \{Z = \infty\} \) the process \( \{N_t\}_{t \in \mathbb{R}^+} \) has the exchangeable property \( (E) \) (see [8], Definition 1 for the definition) implying that \( \{W_n\}_{n \in \mathbb{N}} \) is exchangeable, a contradiction to Example 5.4.

Theorem 4.9 Consider the following assertions:

(i) There exists a \( \Sigma \)-\( Z \)-measurable map \( \Theta \) from \( \Omega \) into \( \Psi \) such that \( \{N_t\}_{t \in \mathbb{R}^+} \) is a \((P, K(\Theta))\)-MRP.

(ii) There exist a \( \Sigma \)-\( Z \)-measurable map \( \Theta \) from \( \Omega \) into \( \Psi \), a disintegration \( \{P_\theta\}_{\theta \in \Psi} \) of \( P \) over \( P_\Theta \) consistent with \( \Theta \), and a family \( \{K(\theta)\}_{\theta \in \Psi} \) of \( \mathfrak{B} \)-\( Z \)-Markov kernels such that for \( P_\Theta \)-a.a. \( \theta \in \Psi \) the family \( \{N_t\}_{t \in \mathbb{R}^+} \) is a \((P_\theta, K(\theta))\)-RP.

(iii) The process \( \{W_n\}_{n \in \mathbb{N}} \) is \( P \)-exchangeable.

(iv) There exist a \( \sigma \)-subalgebra \( \mathcal{F} \) of \( \Sigma \) and a family \( \{Q_\omega\}_{\omega \in \Omega} \) of \( \mathfrak{B} \)-\( \mathcal{F} \)-Markov kernels such that 

\[ \int_{\mathcal{F}} Q_\omega^N(H) R(d\omega) = P(F \cap W^{-1}(H)) \]

for every \( H \in \mathfrak{B}_\mathbb{N} \) and \( F \in \mathcal{F} \), where \( R := P \mid \mathcal{F} \), \( W := (W_1, \ldots, W_n, \ldots) \) and \( Q_\omega^n \) denotes the \( \mathbb{N} \)-fold product probability \( \otimes_{n \in \mathbb{N}} P_n \) of copies \( P_n := Q_\omega \) of \( Q_\omega \) for \( n \in \mathbb{N} \).
Then the following implications hold true:

$$(i) \iff (ii) \implies (v) \iff (vi)$$

Moreover, if $\Sigma$ is countably generated and $P$ is perfect then $(iv) \iff (v)$; if in addition, $Z$ is countably generated then $(i) \iff (ii)$. If $(\Psi, Z) = (\mathbb{R}^d, \mathcal{B}_d)$ for $d \in \mathbb{N}$ then $(i) \iff (iii)$. If $\Sigma$ is countably generated, $P$ is perfect and $(\Psi, Z) = (\mathbb{R}^d, \mathcal{B}_d)$ then items $(i)$ to $(vi)$ are all equivalent.

**Proof.** First note that the implications $(i) \implies (iii)$ and $(vi) \implies (iii)$ are obvious. The implication $(ii) \implies (i)$ is immediate by Proposition 3.9, while the equivalence of $(iii)$ and $(iv)$ follows directly by Theorem 4.4, since for $(\mathcal{T}, \mathcal{F}) = \langle \mathbb{R}, \mathcal{B} \rangle$ every measure $P_{W_n}$ on $\mathcal{B}$ is perfect and $\mathcal{B}$ is countably generated. The latter, together with the implication $(vi) \implies (iii)$ yields $(vi) \implies (iv)$.

Ad $(ii) \implies (v)$: If $(ii)$ holds true, then there exists a $P$-null set $H \subseteq Z$ such that for any $\theta \notin H$ the process $(\{W_n\}_{n \in \mathbb{N}})$ is $P$-exchangeable, implying together with property (d2) its $P$-exchangeability as well; hence $(iii)$ or equivalently $(iv)$ follows.

Ad $(ii) \implies (v)$: Assume that $(ii)$ is true. Putting $S_\omega(E) := P_\theta(E)$ for any $\omega \in \Omega$, $E \in \Sigma$ and $\theta = \Theta(\omega)$, we clearly get that $\{S_\omega\}_{\omega \in \Omega}$ is a subfield r.c.p. for $P$ over $R := P \mid \sigma(\Theta)$ such that $\{N_t\}_{t \in \mathbb{R}^+}$ is an $(S_\omega, K(\omega))$-RP for any $\omega \notin H^{\ast} := \Theta^{-1}(H)$, where $K(\omega) := K(\theta)$ for each $\omega \in \Omega$ and $\Theta(\omega) = \theta \notin H \subseteq Z$. Since clearly $H^{\ast}$ is an $R$-null set, it follows that $\{S_\omega\}_{\omega \in \Omega}, \mathcal{F} := \sigma(\Theta)$ and $\{K(\omega)\}_{\omega \in \Omega}$ satisfy assertion $(v)$.

Ad $(v) \implies (vi)$: Assume that $(v)$ holds true and let $\mathcal{F}, \{S_\omega\}_{\omega \in \Omega}$ and $R$ be as in $(v)$. Put $\bar{Y} := \Omega, \{S_{\bar{y}}\}_{\bar{y} \in \bar{Y}} := \{S_\omega\}_{\omega \in \Omega}$ and $B(\bar{Y}) := \sigma(\{S_\omega(E) : E \in \Sigma\})$. Then $B(\bar{Y}) \subseteq \mathcal{F}$ and so we may define the probability measure $\nu := R \mid B(\bar{Y})$. Since by $(v)$ the process $\{W_n\}_{n \in \mathbb{N}}$ is $S_\omega$-i.i.d. for $R$-a.a. $\omega \in \Omega$, we get that it is $S_{\bar{y}}$-i.i.d. for $\nu$-a.a. $\bar{y} \in \bar{Y}$. The latter together with an application of (sf2) yields that $\{N_t\}_{t \in \mathbb{R}^+}$ is a $\nu$-MRP associated with $\{S_{\bar{y}}\}_{\bar{y} \in \bar{Y}}$; hence assertion $(vi)$ follows.

Moreover, if $\Sigma$ is countably generated and $P$ is perfect, the implication $(iv) \implies (v)$ holds true. In fact, by Theorem 4.4 we obtain that assertion $(iv)$ is equivalent with the fact that $\{W_n\}_{n \in \mathbb{N}}$ is $P$-conditionally i.i.d. over $\mathcal{F}$. But note that according to Remark 3.4, (a) there exists a subfield r.c.p. $\{S_\omega\}_{\omega \in \Omega}$ for $P$ over $R := P \mid \mathcal{F}$. Thus, we may apply Corollary 3.10 to get $(v)$. 

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If in addition $Z$ is countably generated and (i) holds true, then by Remark 3.4, (a) there exists a disintegration $\{P_\theta\}_{\theta \in \Psi}$ of $P$ over $P_\Theta$ consistent with $\Theta$. So according to Proposition 3.9 we get that for $P_\Theta$-a.a. $\theta \in \Psi$ the family $\{N_t\}_{t \in \mathbb{R}_+}$ is a $(P_\theta, K(\theta))$-RP; hence (i) implies (ii).

If $(\Psi, Z) = (\mathbb{R}^d, \mathcal{B}_d)$ and (iii) holds true, then by Proposition 4.2 there exists a $\Sigma$-$\mathcal{B}_d$-measurable map $\Theta$ from $\Omega$ into $\mathbb{R}^d$ such that $\{W_n\}_{n \in \mathbb{N}}$ is $P$-conditionally i.i.d. given $\Theta$; hence (i) follows. As a consequence, assuming that $\Sigma$ is countably generated, $P$ is perfect and $(\Psi, Z) = (\mathbb{R}^d, \mathcal{B}_d)$ we obtain that items (i) to (vi) are all equivalent. This completes the whole proof.

Note that the most important applications in Probability Theory are still rooted in the case of standard Borel spaces; hence of spaces satisfying always the assumptions of the above theorem concerning $P$, $\Sigma$ and $(\Psi, Z)$.

**Question 4.10** However, it remains an open question, whether Theorems 4.4 and 4.9 can be extended by avoiding the countability assumptions for $T$ and $\Sigma$, respectively?

## 5 A Construction

In this section, the existence of $(P, K(\Theta))$-MRPs with prescribed distributions for their claim interarrival processes as well as for the parameter $\Theta$ is proven by an application of Proposition 3.9. As a consequence, a method of constructing concrete examples of MRPs is provided.

Throughout what follows, we put $\tilde{\Omega} := \mathbb{R}^\mathbb{N}$, $\Omega := \tilde{\Omega} \times \Psi$, $\tilde{\Sigma} := \mathcal{B}(\tilde{\Omega})$ and $\Sigma := \tilde{\Sigma} \otimes Z$ for simplicity. The next result extends Theorem 3.1 from [11], which provides a construction for MPPs.

**Theorem 5.1** Let $\mu$ be a probability measure on $Z$ and for any fixed $\theta \in \Psi$ let $Q_n(\theta)$ be probability measures on $\mathcal{B}$ with $Q_n(\theta) = K(\theta)$ for all $n \in \mathbb{N}$, where for any fixed $B \in \mathcal{B}$ the function $K(\cdot)(B) : \Psi \to \mathbb{R}$ is $Z$-measurable and $K(\theta)((0, \infty)) = 1$. Then there exist a map $\Theta$ from $\Omega$ into $\Psi$, a family of probability measures $\{P_\theta\}_{\theta \in \Psi}$ on $\Sigma$, a unique probability measure $P$ on $\Sigma$ such that $P_\Theta = \mu$ and $\{P_\theta\}_{\theta \in \Psi}$ is a disintegration of $P$ over $\mu$ consistent with $\Theta$, and a $(P, K(\Theta))$-MRP $\{N_t\}_{t \in \mathbb{R}_+}$, the claim interarrival process $\{W_n\}_{n \in \mathbb{N}}$ of which satisfies condition

$$(P_\theta)_{W_n} = Q_n(\theta) \text{ for all } n \in \mathbb{N},$$

if $\theta \in \Psi$ is fixed.

**Proof.** The proof runs in a similar way with that of Theorem 3.1 from [11] but with $\Psi$, $\mathcal{B}$ and $Q_n(\theta) = K(\theta)$ in the place of $\Upsilon := (0, \infty)$, $\mathcal{B}(\Upsilon)$ and $Q_n(\theta) = \text{Exp}(\theta)$, respectively. Next we sketch the proof for completeness.
Fix on arbitrary \( \theta \in \Psi \). If \( Q_n(\theta) = K(\theta) \) for each \( n \in \mathbb{N} \), it follows that there exist a unique probability measure \( \bar{P}_n := \otimes_{\theta \in \Psi} Q_n(\theta) \) on \( \Sigma \), and a sequence \( \{\bar{W}_n\}_{n \in \mathbb{N}} \) of \( \bar{P}_n \)-independent random variables on \((\Omega, \Sigma)\) such that

\[
\bar{W}_n(\omega) = \omega_n = \bar{\pi}_n(\omega) \quad \text{for each} \quad \omega \in \bar{\Omega} \quad \text{and} \quad n \in \mathbb{N},
\]

where \( \bar{\pi}_n \) is the canonical projection from \( \mathbb{R}^N \) onto \( \mathbb{R} \), satisfying

\[
(\bar{P}_n)_{\bar{W}_n} = Q_n(\theta) \quad \text{for all} \quad n \in \mathbb{N}.
\]

Since by assumption, for any fixed \( B \in \mathfrak{B} \) each function \( Q_n(\cdot)(B) \) is \( Z \)-measurable, it follows by a monotone class argument that the same holds true for the function \( \bar{P}_n(\cdot) \) for fixed \( E \in \Sigma \).

For each \( \theta \in \Psi \) put \( P_\theta := \bar{P}_n \otimes \delta_\theta \), where \( \delta_\theta \) is the Dirac probability measure on \( Z \), and for each \( n \in \mathbb{N} \) set \( W_n := \bar{W}_n \circ \pi_{\bar{\theta}} = \pi_n \), where \( \pi_{\bar{\theta}} \) and \( \pi_n \) are the canonical projections from \( \Omega \) onto \( \bar{\Omega} \) and from \( \Omega \) onto \( \mathbb{R} \), respectively. Put now

\[
P(E) := \int P_\theta(E^\theta) \mu(d\theta) \quad \text{for each} \quad E \in \Sigma,
\]

where \( E^\theta := \{\omega \in \Omega : (\omega, \theta) \in E\} \). Then \( P \) is a probability measure on \( \Sigma \) such that \( \{P_\theta\}_{\theta \in \Psi} \) is a disintegration of \( P \) over \( \mu \) consistent with \( \pi_\Psi \), where \( \pi_\Psi \) is the canonical projection from \( \Omega \) onto \( \Psi \) (see [11], proof of Theorem 3.1). Furthermore, it can be proven that for all \( \theta \in \Psi \) the process \( \{W_n\}_{n \in \mathbb{N}} \) is \( P_\theta \)-independent and \( (P_\theta)_{\bar{W}_n} = K(\theta) \) for each \( n \in \mathbb{N} \). Clearly, putting \( \Theta := \pi_\Psi \) we get \( P_\Theta = \mu \).

It then follows that the claim number process \( \{N_t\}_{t \in \mathbb{R}^+} \) induced by \( \{W_n\}_{n \in \mathbb{N}} \) is a \((P_\theta, K(\theta))\)-RP for all \( \theta \in \Psi \); hence by Proposition 3.9 it is a \((P, K(\theta))\)-MRP.

Applying now Theorem 5.1, we compute the corresponding disintegrating probability measures \( P_\theta \) (\( \theta \in \mathbb{R}^d \)) as well as the probability measure \( P \) for some MRPs of special interest which are not MPPs. To this aim recall that by \( \lambda_d \) is denoted the restriction of the Lebesgue measure \( \lambda_d \) to \( \mathfrak{B}(\mathbb{R}^d) \), while any restriction of \( \lambda_d \) to \( \mathfrak{B}(A) \), where \( A \) is any Borel subset of \( \mathbb{R}^d \), will be denoted again by \( \lambda_d \). In particular, if \( d = 1 \) then \( \lambda := \lambda_1 = \lambda \mid \mathfrak{B} \), where \( \lambda \) is the Lebesgue measure on \( \mathbb{R} \).

In the following two examples, some concrete \((P, K(\theta))\)-MRPs together with their associated families of disintegrating measures are constructed for one of the most common choices that can be made for a claim interarrival time distribution, i.e. \( \text{Ga}(\theta_1, \theta_2) \) with \( \theta_1 > 0 \) and \( 0 < \theta_2 < 1 \) (cf. e.g. [5], page 95). In particular, in the next example where \( \theta_2 = 1/2 \) this class of distributions is of special interest, since none of its members satisfy Assumption 5.1 from [5], proposed by Huang in [8], Theorem 3, which is essential in Grandell’s study for MRPs (see [5], Section 5.3). Moreover, in the the same example it is shown that there are counting processes \( \{N_t\}_{t \in \mathbb{R}^+} \) being both \((P, K(\theta))\)-MRPs and \( P_\theta \mid B(\Psi) \)-ones, which are not, though, MRPs according to Grandell [5], Definition 5.3.
Example 5.2 Let \( Q_n(\theta) = \text{Ga}(\theta, 1/2) \) for each \( n \in \mathbb{N} \) and for any fixed \( \theta > 0 \), and let \( \mu = \text{Ga}(\gamma, \alpha) \). Then the assumptions of Theorem 5.1 are satisfied for \( (\Psi, Z) = ((0, \infty), \mathcal{B}((0, \infty))) \).

Hence the probability measures \( \tilde{P} \) and \( P \) on \( \tilde{\Sigma} = \mathcal{B}(\tilde{\Omega}) \) and \( \Sigma = \mathcal{B}(\Omega) \), where \( \tilde{\Omega} = \mathbb{R}^n \) and \( \Omega = \mathbb{R}^n \times (0, \infty) \), respectively, as well as the disintegrations \( \{\tilde{P}_\theta\}_{\theta \in \Psi} \) and \( \{P_\theta\}_{\theta \in \Psi} \) can be computed. Moreover, there exists a random variable \( \Theta \) on \( \Omega \) such that \( P_\Theta = \text{Ga}(\gamma, \alpha) \).

Next we construct an MRP with parameter a two-dimensional random variable \( \Theta \).

Example 5.3 Let \( Q_n(\theta) = Q_n(\theta_1, \theta_2) = \text{Ga}(\theta_1, \theta_2) \) for each \( n \in \mathbb{N} \) and for any fixed \( \theta = (\theta_1, \theta_2) \in (0, \infty) \times (0, 1) \), and let \( \mu \) be a bivariate probability distribution on \( \mathcal{B}_2 \) such that \( \mu((0, \infty) \times (0, 1)) = 1 \).

Then the assumptions of Theorem 5.1 are satisfied for \( (\Psi, Z) = (\mathbb{R}^2, \mathcal{B}_2) \); hence \( \tilde{\Omega} = \mathbb{R}^n \), \( \Omega = \mathbb{R}^n \times \mathbb{R}^2 \) and \( \Theta = \pi_{\mathbb{R}^2} \).

Following now the arguments of Example 5.2, we get that \( P_\Theta = \mu \). Again according to Example 5.2, it suffices to compute the probability measures of interest on measurable cylinders. Then we get

\[
\tilde{P}_\theta(\tilde{B}) = (\otimes_{n \in \mathbb{N}} Q_n(\theta))(\tilde{B}) = \prod_{k \in L} Q_k(\theta)(\tilde{C}_k) = \frac{\theta_1^{2k}}{\Gamma(\theta_1)} \prod_{k \in L} \int_{\tilde{C}_k} \omega_k^{\theta_1 - 1} e^{-\theta_1 \omega_k} \lambda(d\omega_k); \quad (11)
\]
for each \( \theta = (\theta_1, \theta_2) \in (0, \infty) \times (0, 1) \). Consider now a measurable cylinder \( \tilde{B} \times E \in \tilde{\mathcal{C}} \times \mathfrak{B}(\mathbb{R}^2) \). Applying (11), we get

\[
P_\theta(\tilde{B} \times E) = \chi_E(\theta_1, \theta_2) \frac{\theta^\theta_2}{\Gamma(\theta_2)} \prod_{k \in L} \int_{\tilde{C}_k} \omega_k^{\theta_2-1} e^{-\theta_1 \omega_k} \lambda(d\omega_k);
\]

for each \( \theta = (\theta_1, \theta_2) \in (0, \infty) \times (0, 1) \); hence

\[
P(\tilde{B} \times E) = \int_{E} \frac{\theta^\theta_2}{\Gamma(\theta_2)} \prod_{k \in L} \int_{\tilde{C}_k} \omega_k^{\theta_2-1} e^{-\theta_1 \omega_k} \lambda(d\omega_k) \] \[P_\Theta(d\theta_1, d\theta_2).
\]

Finally, it follows an example to show that we cannot avoid including in Huang’s definition of an MRP the assumption that \( \{W_n\}_{n \in \mathbb{N}} \) is \( \tilde{P}_\tilde{Y} \)-identically distributed for \( \nu \)-a.a. \( \tilde{y} \in \tilde{Y} \) (see also Remark 4.8).

**Example 5.4** Let \((\Psi, Z) = \emptyset,(\mathfrak{B}(\mathbb{R}^2))\). If \(Q_n(\theta) = \text{Exp}(n\theta)\) for each \(n \in \mathbb{N}\) and for any fixed \(\theta > 0\), and if \(\mu = \text{Ga}(2, 1)\) then all the assumptions of Theorem 5.1 except for

\[Q_n(\theta) = \textbf{K}(\theta)\quad \text{for all } n \in \mathbb{N}\text{ and for any fixed } \theta \in \Psi\]

are satisfied. In fact, in this case \(\textbf{K}(\theta)\) is substituted by \(\textbf{K}(n\theta) := \text{Exp}(n\theta)\).

So the probability measures \(\tilde{P}\) and \(P\) on \(\tilde{\Sigma} = \mathfrak{B}(\tilde{\Omega})\) and \(\Sigma = \mathfrak{B}(\Omega)\), where \(\tilde{\Omega} = \mathbb{R}^N\) and \(\Omega = \mathbb{R}^N \times (0, \infty)\), respectively, as well as the disintegrations \(\{\tilde{P}_\theta\}_{\theta \in \Psi}\) and \(\{P_\Theta\}_{\theta \in \Psi}\) can be computed. Moreover, there exists a random variable \(\Theta\) on \(\Omega\) such that \(P_\Theta = \text{Ga}(2, 1)\). Following the same reasoning as in the proof of Theorem 5.1, we also obtain an interarrival process \(\{W_n\}_{n \in \mathbb{N}}\) which is \(P_\Theta\)-independent for all \(\theta > 0\) and satisfies \(W_n = \pi_n\) as well as \((P_\Theta)W_n = \textbf{K}(n\theta)\) for all \(n \in \mathbb{N}\) and for any fixed \(\theta > 0\). But the \(P_\Theta\)-independence of \(\{W_n\}_{n \in \mathbb{N}}\), for all \(\theta > 0\), implies for every \(r \in \mathbb{N}\) and for all \(w_1, \ldots, w_r \in \mathbb{R}_+\) that

\[
P\left(\bigcap_{k=1}^{r} \{W_k \leq w_k\}\right) = \int \prod_{k=1}^{r} P_\theta(W_k \leq w_k) \nu(d\theta), \tag{12}
\]

where \(\nu = P_\Theta | B((0, \infty)) = \mu | B((0, \infty))\) and \(B((0, \infty)) = \sigma(\{P_\Theta(E) : E \in \Sigma\})\). So, \(\{W_n\}_{n \in \mathbb{N}}\) is a claim interarrival process which is not \(P_\theta\)-identically distributed for any fixed \(\theta > 0\) but which satisfies (12). As a consequence, the claim number process \(\{N_t\}_{t \in \mathbb{R}_+}\) induced by the sequence of canonical projections \(\{\pi_n\}_{n \in \mathbb{N}} = \{W_n\}_{n \in \mathbb{N}}\) is not a \(\nu\)-MRP associated with \(\{P_\Theta\}_{\theta \in \Psi}\). Furthermore, for every \(w_1, w_2 \in \mathbb{R}_+\) we have

\[
P(W_1 \leq w_1, W_2 \leq w_2) = 2 \int_0^\infty (1 - e^{-\theta w_1})(1 - e^{-2\theta w_2}) e^{-2\theta} d\theta
\]

\[= w_2(2w_2 + 1)^{-1} - 2[(w_1 + 2)^{-1} - (w_1 + 2w_2 + 2)^{-1}],
\]

implying that \(P(W_1 \leq 2, W_2 \leq 1) = \frac{1}{3} \neq \frac{2}{7} = P(W_1 \leq 1, W_2 \leq 2)\); hence \(\{W_n\}_{n \in \mathbb{N}}\) is not \(P\)-exchangeable.
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