Derivations of the
Lie algebras of differential operators *

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Abstract

This paper encloses a complete and explicit description of the derivations of the Lie algebra $D(M)$ of all linear differential operators of a smooth manifold $M$, of its Lie subalgebra $D^1(M)$ of all linear first-order differential operators of $M$, and of the Poisson algebra $S(M) = \text{Pol}(T^*M)$ of all polynomial functions on $T^*M$, the symbols of the operators in $D(M)$. It turns out that, in terms of the Chevalley cohomology, $H^1(D(M), D(M)) = H^1_{\text{DR}}(M)$, $H^1(D^1(M), D^1(M)) = H^1_{\text{DR}}(M) \oplus \mathbb{R}^2$, and $H^1(S(M), S(M)) = H^1_{\text{DR}}(M) \oplus \mathbb{R}$. The problem of distinguishing those derivations that generate one-parameter groups of automorphisms and describing these one-parameter groups is also solved.

1 Introduction

In [PS54], Pursell and Shanks proved the well-known result stating that the Lie algebra of all smooth compactly supported vector fields of a smooth manifold characterizes the differentiable structure of the variety. Similar upshots were obtained in numerous subsequent papers dealing with different Lie algebras of vector fields and related algebras (see e.g. [Abe82, Ame75, AG90, Gra78, Gra93, HM93, Omo76, Skr87]).

Derivations of certain infinite-dimensional Lie algebras arising in Geometry were also studied in different situations (note that in infinite dimension there is no such a clear correspondence between derivations and one-parameter groups of automorphisms as in the finite-dimensional case). Let us mention a result of L. S. Wollenberg [Wol69] who described all derivations of the Lie algebra of polynomial functions on the canonical symplectic space $\mathbb{R}^2$ with respect to the Poisson bracket. It turned out that there are outer derivations of this algebra in contrast to the corresponding Weyl algebra. This can be viewed as a variant of a "no-go" theorem (see [Jos70]) stating that the Dirac quantization problem [Dir58] cannot be solved satisfactorily because the classical and the corresponding quantum algebras are not isomorphic as Lie algebras. An algebraic generalization of the latter fact, known as the algebraic "no-go" theorem, has been proved in [GG01] by different methods. Derivations of the Poisson bracket of all smooth functions on a symplectic manifold have been determined in [ADML74] (for the real-analytic case, see [Gra86]). Another important result is the one by F. Takens [Tak73] stating that all derivations of the Lie algebra $\mathcal{X}(M)$ of smooth vector fields on a manifold $M$ are inner. The same turned out to be valid for analytic cases [Gra81]. Some cases of the Lie algebras of vector fields associated with different geometric structures were studied in a series of papers by Y. Kanke [Kan75, Kan81].

Our work [GP03] contains Shanks-Pursell type results for the Lie algebra $D(M)$ of all linear differential operators of a smooth manifold $M$, for its Lie subalgebra $D^1(M)$ of all linear first-order differential operators of $M$, and for the Poisson algebra $S(M) = \text{Pol}(T^*M)$ of all polynomial functions on $T^*M$, the symbols of the operators in $D(M)$. Furthermore, we computed all the automorphisms of these algebras and showed that $D(M)$ and $S(M)$ are not integrable. The current paper contains a description of their derivations, so it is a natural continuation of this previous work and can be considered as a generalization of the results of Wollenberg and Takens. It is also shown which derivations

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generate one-parameter groups of automorphisms and the explicit form of such one-parameter groups is provided.

### 2 Notations and definitions

Throughout this paper, $M$ is as usually assumed to be a smooth, Hausdorff, second countable, connected manifold of dimension $n$.

Recall that the space $\mathcal{D}(M)$ (or $\mathcal{D}$ for short) of linear differential operators on $C^\infty(M)$ (or $\mathcal{A}$ for short) is filtered by the order of differentiation, $\mathcal{D}^i$ being the space of at most $i$-th order operators (for $i \geq 0$; $\mathcal{D}^i = \{0\}$ for $i < 0$), and is equipped with an associative and so a Lie algebra structure, $\circ$ and $[.,.]$ respectively, such that $\mathcal{D}^i \circ \mathcal{D}^j \subset \mathcal{D}^{i+j}$ and $[\mathcal{D}^i, \mathcal{D}^j] \subset \mathcal{D}^{i+j-1}$. Obviously, $\mathcal{D}^0 = \mathcal{A}$ is an associative commutative subalgebra and $\mathcal{D}^1$ is a Lie subalgebra of $\mathcal{D}$. We denote by $\mathcal{D}_c$ (respectively $\mathcal{D}_c^i$) the algebra of differential operators (respectively the space of at most $i$-th order operators) that vanish on constants. For instance $\mathcal{D}_c^1$ is the Lie algebra $\mathcal{A}(M) = \text{Vect}(M)$ (or $\mathcal{X}$ for short) of vector fields on $M$, i.e. the Lie algebra $\text{Der} \mathcal{A}$ of derivations of the algebra of functions. Observe also that we have the canonical splittings $\mathcal{D} = \mathcal{A} \oplus \mathcal{D}_c$, $\mathcal{D}^i = \mathcal{A} \oplus \mathcal{D}_c^i$.

The classical counterpart of $\mathcal{D}$, the space $\mathcal{S}(M)$ (or $\mathcal{S}$ for short) of symmetric contravariant tensor fields on $M$, is of course naturally graded, $\mathcal{S}_i$ being the space of $i$-tensor fields (for $i \geq 0$; $\mathcal{S}_i = \{0\}$ for $i < 0$). This counterpart $\mathcal{S}$ is isomorphic— even as a $\mathcal{D}_c$-module—to the space $\text{Pol}(T^*M)$ of smooth functions on $T^*M$ that are polynomial on the fibers. Furthermore, it is a commutative associative and a Poisson algebra. These structures $\circ$ and $[,]$ verify $\mathcal{S}_i \circ \mathcal{S}_j \subset \mathcal{S}_{i+j}$ and $\mathcal{S}_i, \mathcal{S}_j \subset \mathcal{S}_{i+j}$ respectively. The Poisson bracket can be viewed as the symmetric Schouten bracket or the standard symplectic bracket. Note that $\mathcal{S}_0 = \mathcal{A}$ is an associative and Lie-commutative subalgebra of $\mathcal{S}$. Clearly, $\mathcal{S}$ is filtered by $\mathcal{S}^i = \oplus_{i \leq j} \mathcal{S}_j$ and $\mathcal{S}^i$ is a Lie subalgebra of $\mathcal{S}$ isomorphic to $\mathcal{D}^i$ and $\mathcal{A} \triangleleft \mathcal{X}$.

The algebras $\mathcal{D}$ and $\mathcal{S}$ are models of a quantum and a classical Poisson algebra in the sense of [GP03]. All the results of this paper apply to these algebras. It is well known that $\mathcal{D}^i$, $\mathcal{S}_i$, and $\mathcal{S}^i$ are algebraically characterized in the following way:

\begin{align*}
\{ D \in \mathcal{D} : [D, \mathcal{A}] \subset \mathcal{D}^i \} &= \mathcal{D}^{i+1} (i \geq -1), \\
\{ S \in \mathcal{S} : [S, \mathcal{A}] \subset S_i \} &= \mathcal{A} + \mathcal{S}_{i+1} (i \geq -1), \\
and\{ S \in \mathcal{S} : [S, \mathcal{A}] \subset S^i \} &= \mathcal{S}^{i+1} (i \geq -1).
\end{align*}

Moreover, $\mathcal{S}$ is the classical algebra induced by the quantum algebra $\mathcal{D}$. Thus, $\mathcal{S}_i = \mathcal{D}^i / \mathcal{D}^{i-1}$. For any non-zero $D \in \mathcal{D}$, the degree $\deg(D)$ of $D$ is the lowest $i$, such that $D \in \mathcal{D}^i \setminus \mathcal{D}^{i-1}$. If $\text{cl}_j$ is the class in the quotient $\mathcal{S}_j$, the (principal) symbol $\sigma(D)$ of $D$ is

$$\sigma(D) = \text{cl}_{\deg(D)}(D)$$

and the symbol $\sigma_i(D)$ of order $i \geq \deg(D)$ is defined by

$$\sigma_i(D) = \begin{cases} 0, & \text{if } i > \deg(D), \\ \sigma(D), & \text{if } i = \deg(D). \end{cases}$$

Then, the commutative multiplication and the Poisson bracket of $\mathcal{S}$ verify

$$\sigma(D_1) \cdot \sigma(D_2) = \sigma_{\deg(D_1)+\deg(D_2)}(D_1 \circ D_2) \quad (D_1, D_2 \in \mathcal{D})$$

and

$$\{ \sigma(D_1), \sigma(D_2) \} = \sigma_{\deg(D_1)+\deg(D_2)-1}([D_1, D_2]) \quad (D_1, D_2 \in \mathcal{D}).$$

}\[\textit{Derivations of the Lie algebras of differential operators}\]
3 Locality and weight

The characterizations (1), (2), and (3) of the filters $D^{i+1}$ of $D$ and the terms $S_{i+1}$ and filters $S^{i+1}$ of $S$ ($i \geq -1$), can be "extended" in the following way:

Lemma 1 For any $i \geq -1$ and any $k \geq 1$, we have

$\{D \in D : [D, D^k] \subset D^i\} = R \cdot 1 + D^{i-k+1}$, \hspace{1cm} (6)

$\{S \in S : \{S, S_k\} \subset S_i\} = R \cdot 1 + S_{i-k+1}$, \hspace{1cm} (7)

and

$\{S \in S : \{S, S_k\} \subset S^i\} = R \cdot 1 + S^{i-k+1}$. \hspace{1cm} (8)

Proof. (i) Note first that $\{S \in S : \{S, S_k\} = 0\} = R \cdot 1$. Of course, we need only show that the commutation of $S$ with $S_k$ implies $S \in R \cdot 1$. But this is obvious: on a connected Darboux chart domain $U$, take for instance the polynomials $S_k \in S_k$ defined by $S_k(x; \xi) = (\xi)^k$ and $S_k(x; \xi) = x^i(\xi)^k$ ($x \in U$, $\xi \in (R^n)^*$, $\{\xi, x^i\} = \delta_j^i, i, j \in \{1, \ldots, n\}$).

More generally, we have $\{S \in S : \{S, S_k\} \subset S_i\} = R \cdot 1 + S_{i-k+1}$ for all $i \geq -1$. Take $i \geq 0$. Writing $S = S_{i-k+1} + S'$ with $S_{i-k+1} \in S_{i-k+1}$ and $S' \in S \cap S_{i-k+1}$, we get $\{S', S_k\} \subset S \cap S_{i-k+1}$, so $\{S', S_k\} = 0$ and $S' \in R \cdot 1$. Hence the conclusion.

(ii) In order to prove (3), observe that it is enough to consider the case $i \geq 0$. If $\{S, S_k\} \subset S^i$ and $S = \sum_j S_j$, $S_j \in S_j$, we have $\{S_j, S_k\} = 0$ for all $j > i - k + 1$. So $S_j \in R \cdot 1$ and $S \subset R \cdot 1 + S^{i-k+1}$.

(iii) Assume $[D, D^k] \subset D^i$ and $D \notin D^{i-k+1}$, so that $\deg(D) > i - k + 1$. Clearly, $\sigma_k : D^k \to S_k$ is surjective, so any $S_k \in S_k \setminus S^i$ reads $S_k = \sigma(\Delta)$, $\deg(\Delta) = k$. Thus,

$\{\sigma(D), S_k\} = \sigma(D, \sigma(\Delta)) = \sigma_{\deg(D)+k-1}([D, \Delta]) = 0$.

So $\sigma(D) \in R \cdot 1$ and $D \in R \cdot 1$. Eventually, $D \in R \cdot 1 + D^{i-k+1}$. \[\Box]\n
Let $(P, [\cdot, \cdot])$ be either the Lie algebra $(D, [\cdot, \cdot])$, its Lie subalgebra $(D^1, [\cdot, \cdot])$, or the Poisson algebra $(S, \{\cdot, \cdot\})$. The sign "\cdot" stands for the multiplication "\o" of differential operators and the multiplication "\cdot" of polynomials of $T^*M$. We denote by Der $P$ the Lie algebra of all derivations of the Lie algebra $(P, [\cdot, \cdot])$.

Proposition 1 Any derivation of the Lie algebra $P$ is a local operator.

Proof. If $P \in P^i$ vanishes on an open $U \subset M$ and if $x_0 \in U$, we have

$P = \sum_k [X_k, P_k]$\n
for certain $X_k \in X$, $P_k \in P^i$ with $X_k|_V = P_k|_V = 0$ for some neighborhood $V \subset U$ of $x_0$. In the quantum case, this follows for instance from [Pon02]. In the classical case, a straightforward adaptation of [DWLS] Ex. 12 shows that the set $\{L_X : \Gamma(S^{\leq i}TM) \to \Gamma(S^{\leq i}TM) \mid X \in X\}$, where $\Gamma(S^{\leq i}TM)$ is the space of smooth sections of the tensor bundle $\oplus_{j \leq i} S^jTM$, is locally transitive. Since it is obviously stable under locally finite sums, the announced result is a direct consequence of [DWLS] Prop. 3, Def. 2.

For any $C \in \text{Der } P$, the preceding decomposition of $P$ and the derivation property then imply that $(CP)(x_0) = 0$. \[\Box]\n
Lemma 2 There is a finite set $F = \{f_1, \ldots, f_m\} \subset C^\infty(M)$ ($m \leq 2n + 1$), such that

(j) the $C^\infty(M)$-module $\Omega^1(M)$ of differential 1-forms on $M$ is spanned by $dF = \{df_1, \ldots, df_m\}$,

(ii) if $P \in P$ verifies $[P, F] \subset P^i$ then $P \in P^{i+1}$, for any $i \geq -1$.\n
Proof. Assertion (j) is a consequence of Whitney’s embedding theorem (see [AG90, Prop. 2.6], [Whit36]). It suffices to prove (jj) for \( i = -1 \). Indeed, by induction, if (jj) is verified for \( i (i \geq -1) \) then it is for \( i + 1 \):

\[
[P, F] \subset P^{i+1} \Rightarrow [P, F], A] \subset P^i \\
\Rightarrow [P, A], F] \subset P^i \\
\Rightarrow [P, A] \subset P^{i+1} \\
\Rightarrow P \in P^{i+2}.
\]

As

\[
[D, F] = 0 \Rightarrow \{\sigma(D), F\} = 0, \forall D \in \mathcal{D},
\]

it is enough to consider the classical case, which is obvious in view of (jj). Indeed, if \( f \in A \) we have

\[
df = \sum_{s=1}^{m} g_s df_s \quad (g_s \in A)
\]

and \( S \in S = S(M) = \text{Pol}(T^* M) \) and \( \Lambda \) denotes the canonical Poisson tensor of \( T^* M \), then

\[
\{S, f\} = \Lambda(dS, df) = \sum_{s=1}^{m} g_s \{S, f_s\} = 0.
\]

Hence the result. \( \blacksquare \)

**Proposition 2** Any derivation \( C \) of the Lie algebra \( \mathcal{P} \) has a bounded weight, i.e. there is \( d \in \mathbb{N} \), such that

\[
C(\mathcal{P}^i) \subset \mathcal{P}^{i+d}, \forall i \in \mathbb{N}.
\]

**Proof.** Set \( d = \max\{\text{deg}(Cf_s), s \in \{1, \ldots, m\}\} \), where \( \text{deg} \) is the degree in the filtered algebra \( \mathcal{P} \) and where the set \( \mathcal{F} = \{f_1, \ldots, f_m\} \) is that of Lemma 2. Then \( C \) maps all functions into \( \mathcal{P}^d \). Indeed, if \( f \in A \) we have for any \( f_s \in \mathcal{F} \),

\[
0 = C[f, f_s] = [Cf, f_s] + [f, Cf_s]
\]

and \( [Cf, f_s] \in \mathcal{P}^{d-1} \), so that \( Cf \in \mathcal{P}^d \). The announced result can then once more be obtained by induction. Take \( P \in \mathcal{P}^{i+1} (i \geq 0) \) and apply again the derivation property:

\[
C[P, f_s] = [CP, f_s] + [P, Cf_s], \forall s \in \{1, \ldots, m\}.
\]

Hence the conclusion. \( \blacksquare \)

**Remark:** Evidently, for \( C \in \text{Der} \mathcal{D}^1 \), we have \( C(\mathcal{D}^0) \subset \mathcal{D}^1 \) and \( C(\mathcal{D}^1) \subset \mathcal{D}^1 \subset \mathcal{D}^2 \).

4 Corrections by inner derivations

**Proposition 3** Let \( C \in \text{Der} \mathcal{P} \). There is (a non-unique) \( P \in \mathcal{P} \), such that \( C - \text{ad} P \in \text{Der} \mathcal{P} \) respects the filtration. The set of all elements of \( \mathcal{P} \) that have this property is then \( P + \mathcal{P}^1 \).

**Proof.** Take an arbitrary derivation \( C \) of the Lie algebra \( \mathcal{P} \). Let \( (U_i, \varphi_i), i \in I \) be an atlas of \( M \) and \( \mathcal{U} = (U, \varphi = (x^1, \ldots, x^n)) \) any chart of this atlas. The restriction \( C_{|U} \) of the local operator \( C \) to the domain \( U \) is of course a derivation of the Lie algebra \( \mathcal{P}_U \), similar to \( \mathcal{P} \) but defined on \( U \) instead of \( M \).

Set now

\[
P^{d,i}_{C} = C_{|U} (x^i) \in \mathcal{P}^d_U.
\]

This element \( P^{d,i}_{C} \) is equal to or symbolically represented by a polynomial of \( T^* U \) of type

\[
P^{d,i}_{C}(x; \xi) = \sum_{|\alpha| \leq d} \gamma^i_\alpha (x) \xi^\alpha,
\]

where we used standard notations, \( \gamma^i_\alpha \in \mathcal{C}^\infty(U) \) and \( \xi \in (\mathbb{R}^n)^* \).

Since it follows from \( C_{|U} [x^i, x^j] = 0 \), that \( [P^{d,i}_{C}, x^i] = [P^{d,j}_{C}, x^i] \), we get

\[
\partial_{\xi^j} P^{d,i}_{C}(x; \xi) = \partial_{\xi^j} P^{d,j}_{C}(x; \xi).
\]
Thus, there is a polynomial of \( T^* U \),
\[
P_C^d(x; \xi) = \sum_{|\alpha| \leq d+1} \gamma_\alpha(x) \xi^\alpha
\]
(polynomial character in \( \xi \) and smooth dependence on \( x \) easily checked), such that
\[
\partial_\xi P_C^d(x; \xi) = P_C^{d+1}(x; \xi), \forall i \in \{1, \ldots, n\}.
\]
Finally, \( P_C^d \in \mathcal{P}_U^{d+1} \) (interpret—if necessary—the polynomial as differential operator) and
\[
C|_U (x^i) = [P_C^d, x^i], \forall i.
\]
For any function \( f \in \mathcal{A} \) and any \( i \in \{1, \ldots, n\} \), we then have
\[
0 = C|_U [f|_U, x^i] = [(Cf)|_U, x^i] + [f|_U, [P_C^d, x^i]]
\]
\[
= [(Cf)|_U - [P_C^d, f|_U], x^i].
\]
In view of Lemma 2, this entails that
\[
(Cf)|_U - [P_C^d, f|_U] \in C^\infty(U).
\]
Now we will glue together the elements \( P_C^d \in \mathcal{P}_U^{d+1} \). Let \((U_i, \phi_i, \psi_i)_{i \in I}\) be a partition of unity subordinated to the considered atlas and set
\[
P_C = \sum_i \psi_i P_C^d.
\]
Clearly \( P_C \in \mathcal{P}^{d+1} \). Furthermore, \( C - \text{ad} \ P_C \) is a derivation of \( \mathcal{P} = \mathcal{D} \) and of \( \mathcal{P} = \mathcal{S} \). Let us emphasize that for \( \mathcal{P} = \mathcal{D}^1 \), this map \( C - \text{ad} \ P_C \) verifies the derivation property in \( \mathcal{D}^1 \), but is a priori only linear from \( \mathcal{D}^1 \) into \( \mathcal{D} \). For any \( \mathcal{P} \), it respects the filtration. Indeed, for any \( f \in \mathcal{A} \) and any open \( V \subset M \), we have
\[
(Cf)|_V - [P_C^d, f|_V] = \sum_i \psi_i|_V \left( (Cf)|_V - [P_C^d, f|_V] \right) \in C^\infty(V),
\]
in view of (9). We can now proceed by induction. So assume that \( CP - [P_C, P] \in \mathcal{P}^i, \forall P \in \mathcal{P}^i \). Then, if \( P \in \mathcal{P}^{i+1} \) and \( f \in \mathcal{A} \),
\[
[CP - [P_C, P], f] = C(Pf) - [P, Pf] - [P, [P_C, f]] = (Cf - [P_C, Pf]) - [P, [P_C, f]] \in \mathcal{P}^i.
\]
Hence the result for \( \mathcal{P} = \mathcal{D} \) and \( \mathcal{P} = \mathcal{S} \). For \( \mathcal{P} = \mathcal{D}^1 \), Equation (11) shows that \( [P_C, \mathcal{D}^1] \subset \mathcal{D}^1 \). In view of Lemma 11, this means that \( P_C \in \mathcal{D}^1 \). Finally, Equation (10) allows to see that \( C(\mathcal{D}^0) \subset \mathcal{D}^0 \), so that \( C \) respects the filtration.

**Remark:** Thus the inner derivation of Proposition 3 can be taken equal to 0, in the case \( \mathcal{P} = \mathcal{D}^1 \).

**Proposition 4** If \( C \subset \text{Der} \, \mathcal{P} \) respects the filtration, there is a unique vector field \( Y \subset \text{Der} \, \mathcal{A} \subset \mathcal{P} \) such that \( C - \text{ad} \, Y \subset \text{Der} \, \mathcal{P} \) respects the filtration and
\[
(C - \text{ad} \, Y)|_\mathcal{A} = \kappa \, \text{id},
\]
where \( \kappa \in \mathbb{R} \) is uniquely determined by \( C \).

**Proof.** Consider a derivation \( C \) of \( \mathcal{P} \) that respects the filtration and denote by \( C(\mathcal{P}) \) the centralizer of \( \text{ad} \, \mathcal{A} \) in the Lie algebra \( \mathcal{E} = \text{End} \, \mathcal{P} \) of endomorphisms of \( \mathcal{P} \), i.e. the Lie subalgebra \( C(\mathcal{P}) = \{ \psi \in \mathcal{E} : [\psi, \text{ad} \, \mathcal{A}] = 0 \} \), where \([\cdot, \cdot]\) is the commutator of endomorphisms of \( \mathcal{P} \). The derivation \( C \subset \text{Der} \, \mathcal{P} \) induces a derivation \( \text{ad}_C \subset \text{Der} \, \mathcal{E} \), which respects the centralizer. Indeed, for any \( \psi \subset C(\mathcal{P}) \), we have
\[
[\text{ad}_C \psi, \text{ad} \, \mathcal{A}] = [[C, \psi], \text{ad} \, \mathcal{A}] = -[[\psi, \text{ad} \, \mathcal{A}], C] = [\psi, [C, \text{ad} \, \mathcal{A}]] = 0,
\]
as \([C, \text{ad } f]_E = \text{ad}(C f)\) in \(\text{ad}\mathbb{A}\) for each \(f \in \mathbb{A}\), since \(C\) is a derivation that respects the filtration.

It follows from the description of the centralizer, see [GP03, Theo. 3], that if \(\psi \in \mathcal{C}(\mathcal{P})\), then \(\psi\) respects the filtration and there is \(\psi_1 \in \mathcal{C}(\mathcal{P})\), such that \(\psi_1(\mathcal{P}) \subset \mathcal{P}^{i-1}\) and \(\psi = \ell_{\psi(1)} + \psi_1\). Obviously, the left multiplication \(\ell_{f} : \mathcal{P} \ni P \to f \cdot P \in \mathcal{P}\) by an arbitrary \(f \in \mathbb{A}\) belongs to the centralizer \(\mathcal{C}(\mathcal{P})\). So \((\text{ad}_C \ell_{f})(\ell_{g}) \in \mathcal{C}(\mathcal{P})\) and for any \(g \in \mathcal{A}\), \([C, \ell_{f}](g) = [C, \ell_{f}](1) \cdot g\), i.e. \((C - C(1))(f \cdot g) = (C - C(1))(f) \cdot g + f \cdot (C - C(1))(g)\). As constants are the only central elements and as derivations map central elements to central elements, \(C(1) = \kappa (\kappa \in \mathbb{R})\), and the preceding result means that \((C - \kappa \text{id})|_{\mathcal{A}}\) is a vector field \(Y\). Finally, \(C - \text{ad } Y\) \(\in \text{Der} \mathcal{P}\) respects the filtration and \((C - \text{ad } Y)|_{\mathcal{A}} = \kappa \text{id}|_{\mathcal{A}}\). Uniqueness of \(Y\) is readily obtained. Indeed, if \(Y\) is a suitable vector field, then the corresponding constant \(\kappa\) is necessarily \(\kappa = C(1)\) and \(Y\) is unique. \(\blacksquare\)

5 Characterization of the derivations for the Lie algebra \(\mathcal{D}^1(M)\)

Let \(|\eta|\) be a fixed smooth nowhere zero 1-density. The associated divergence \(\text{div}|_{\eta}\) (or simply \(\text{div}\)) is defined for any vector field \(X\) as the unique function \(\text{div}|_{\eta}(X)\) that verifies \(L_X|\eta| = (\text{div}|_{\eta}(X))|\eta|\), where \(L_X|\eta|\) is the Lie derivative of \(|\eta|\) in the direction of \(X\). In any local coordinate system in which \(|\eta|\) is a constant multiple of the standard density, this divergence reads \(\text{div}|_{\eta}(X) = \sum_{i} \partial_{x_i} X^i\), with self-explaining notations. For details regarding the origin of the class of the divergence, we refer the reader to [Lec02].

**Theorem 1** A map \(C : \mathcal{D}^1(M) \to \mathcal{D}^1(M)\) is a derivation of the Lie algebra \(\mathcal{D}^1(M) = \mathcal{C}^\infty(M) \oplus \text{Vect}(M)\) of first order differential operators on \(\mathcal{C}^\infty(M)\), if and only if it can be written in the form

\[
C_Y,\kappa,\lambda,\omega(X + f) = [Y, X + f] + \kappa f + \lambda \text{ div } X + \omega(X),
\]

where \(Y \in \text{Vect}(M)\), \(\kappa, \lambda \in \mathbb{R}\), and \(\omega \in \Omega^1(M) \cap \text{ker } d\). All these objects \(Y, \kappa, \lambda, \omega\) are uniquely determined by \(C\).

**Corollary 1** The first group of the Chevalley-Eilenberg cohomology of the Lie algebra \(\mathcal{D}^1(M)\) of first order differential operators on \(\mathcal{C}^\infty(M)\) with coefficients in the adjoint representation, is given by

\[
H^1(\mathcal{D}^1(M), \mathcal{D}^1(M)) \simeq \mathbb{R}^2 \oplus H^1_{\text{DR}}(M),
\]

where \(H^1_{\text{DR}}(M)\) stands for the first space of the de Rham cohomology of \(M\).

**Proof.** Let \(C_1\) be a derivation of \(\mathcal{P}\) that respects the filtration and reduces to \(\kappa \text{id}\) \((\kappa \in \mathbb{R})\) on functions. The derivation property, written for \(f \in \mathbb{A}\) and \(X \in \mathcal{X}\), shows that \(C_1(\mathcal{X}) \subset \mathcal{A}\), and written for \(X, Y \in \mathcal{X}\), it means that \(C_1|_{\mathcal{X}}\) is a 1-cocycle of the Lie algebra \(\mathcal{X}\) canonically represented upon \(\mathbb{A}\). The cohomology \(H(\mathcal{X}, \mathcal{A})\) is known (see e.g. [Fuc87] or [DWL83]). Having fixed a divergence on \(\mathcal{X}\), we get

\[
C_1|_{\mathcal{X}} = \lambda \text{ div } + \omega,
\]

with \(\lambda \in \mathbb{R}\) and \(\omega \in \Omega^1(M) \cap \text{ker } d\). Finally,

\[
C_1(X + f) = \kappa f + \lambda \text{ div } X + \omega(X), \forall X \in \mathcal{X}, \forall f \in \mathbb{A}.
\]

To prove uniqueness of \(Y, \omega, \kappa, \lambda\), it suffices to write Equation (12) successively for \(1 \in \mathbb{A}\), \(f \in \mathbb{A}\), and \(X \in \mathcal{X}\). Hence Theorem 1, Corollary 1 is clear. Indeed, if \(C\) and \(C' = C + \text{ad } (Z + h)\) \((Z \in \mathcal{X}, h \in \mathcal{A})\) are two cohomologous 1-cocycles and if we denote by \((Y, \omega, \kappa, \lambda)\) and \((Y', \omega', \kappa', \lambda')\) the respective unique quadruples, then necessarily \(\kappa' = \kappa\), \(X' = \lambda\), and \(\omega' = \omega - dh\), so that the map

\[
H^1(\mathcal{D}^1, \mathcal{D}^1) \ni [C] \to \kappa + \lambda + [\omega] \in \mathbb{R}^2 \oplus H^1_{\text{DR}}(M)
\]

is a well-defined vector space isomorphism. \(\blacksquare\)

**Remarks:** 1. Observe that the preceding proof is valid for the generic algebra \(\mathcal{P}\), so not only for \(\mathcal{D}^1\) but also for \(\mathcal{S}\) and \(\mathcal{D}\).
2. Note that, for \( \lambda \neq 0 \) and \( \omega = df \, (f \in \mathcal{A}) \), \( \lambda \div X + \omega(X) = \lambda \div' X \), where \( \div' X = \div X + \lambda^{-1} \omega(X) \) is another divergence.

3. Denote \( C_{Y,0,0,0} = C_Y \), \( C_{0,1,0,0} = C_A \), \( C_{0,0,1,0} = C_{\div} \), \( C_{0,0,0,\omega} = C_\omega \). The Lie algebra structure of \( \text{Der} \, D^1 \) is determined by the following commutation relations (the commutators we miss are just 0):

\[
[C_Y, C_{Y'}] = C_{[Y,Y']}, \quad [C_Y, C_{\div}] = C_{\div(Y)}, \quad [C_Y, C_\omega] = C_{\omega(Y)}, \quad [C_A, C_{\div}] = C_{\div}, \quad [C_A, C_\omega] = C_\omega.
\]

(13)

6 Characterization of the derivations for the Lie algebra \( S(M) \)

Remark: Let us recall that we mentioned in [GP03] Sect. 4] two specific types of derivations: the canonical derivation \( \text{Deg} \in \text{Der} \, S \), \( \text{Deg} : S_i \ni S \rightarrow (i - 1)S \in S_i \) and the derivation \( \overline{\omega} \in \text{Der} \, P \) implemented by a closed 1-form \( \omega \) of \( M \). If \( U \) is an arbitrary open subset of \( M \) and if \( \omega|_U = df_U \), \( f_U \in C^\infty(U) \), this cocycle \( \overline{\omega} \) is defined by

\[
\overline{\omega}(P)|_U = [P|_U, f_U], \forall P \in P.
\]

Remark that, if \( P = D^1 \), \( \overline{\omega} \) coincides with the derivation \( \omega \). In the case \( P = S \) the lowering derivation \( \overline{\omega} \) can be interpreted as the action of the vertical vector field \( \omega^v \) (the vertical lift of the section \( \omega \) of \( T^* M \)) on polynomial functions on \( T^* M \).

**Theorem 2** A map \( C : S(M) \rightarrow S(M) \) is a derivation of the Lie algebra \( S(M) \) of all infinitely differentiable functions of \( T^* M \) that are polynomial along the fibers, if and only if it is of the form

\[
C_{P,\kappa,\omega}(S) = \{P, S\} + \kappa \text{ Deg}(S) + \omega^v(S),
\]

where \( P \in S(M) \), \( \kappa \in \mathbb{R} \), and \( \omega \in \Omega^1(M) \cap \ker d \). Here \( \kappa \) is uniquely determined by \( C \), but \( P \) and \( \omega \) are not. The set of all fitting pairs is \( \{(P + h, \omega + dh) : h \in C^\infty(M)\} \), so we get uniqueness if we impose that the polynomial function \( P \) vanishes on the 0-section of \( T^* M \).

**Corollary 2** The first group of the Chevalley-Eilenberg cohomology of the Lie algebra \( S(M) \) of all polynomial functions on \( T^* M \) with coefficients in the adjoint representation, is given by

\[
H^1(S(M), S(M)) \simeq \mathbb{R} \oplus H^1_{\text{DR}}(M).
\]
For \( f \in A, X \in \mathcal{X} \), and \( S \in \mathcal{S}_{k+1} \), we have \( 0 = C(\{S, f\}) = \{C(S), f\} \) and \( C(\{X, S\}) = \{X, C(S)\} \), so \( C \) maps \( \mathcal{S}_{k+1} \) into \( A \) and intertwines the adjoint action of \( \mathcal{X} \). Hence \( C(\{fX, X^{k+1}\}) = fX(C(X^{k+1})) \) and the map \( D_X : A \ni f \to C(\{fX, X^{k+1}\}) \in A \) is a differential operator of order 0. On the other hand, \( D_X(f) = -(k + 1)C(X(f)X^{k+1}) \) is of order 0 only if it is just 0. Thus, \( C(X(f)X^{k+1}) = 0 \), for all \( f \in A \) and all \( X \in \mathcal{X} \). Let us now work locally. Homogeneous polynomials of degree \( k + 1 \) on \( (\mathbb{R}^n)^* \), \( n = \dim M \) are spanned by \((k + 1)\)-th powers \( X^{k+1}, X \in \mathbb{R}^n \), and any function reads \( X(f) \) for any non-vanishing \( X \in \mathbb{R}^n \). So polynomials of the form \( X(f)X^{k+1} \) locally span \( \mathcal{S}_{k+1} \) and \( C = 0 \). This completes the proof of Theorem 2, except for uniqueness of \( \kappa \) and the convenient pairs \((P, \omega)\).

Equation (14), written for \( S \in \mathcal{S} \), determines by the following commutation relations (the missing commutators are just 0):

\[
\{C_P, C_{P'}\} = C_{\{P, P'\}}, [\text{Deg}, C_P] = C_{\text{Deg}(P)}, [\omega^n, C_P] = C_{\omega^n(P)}, [\omega^n, \text{Deg}] = \omega^n.
\]

### 7 Characterization of the derivations for the Lie algebra \( \mathcal{D}(M) \)

#### Lemma 3
Any derivation \( C \in \text{Der} \mathcal{D}(M) \) that respects the filtration induces a derivation \( \tilde{C} \in \text{Der} \mathcal{S}(M) \), which respects the graduation:

\[
\tilde{C} : \mathcal{S}_i(M) \ni S \to \sigma_i(C(\sigma_i^{-1}(S))) \in \mathcal{S}_i(M),
\]

for all \( i \in \mathbb{N} \).

#### Theorem 3
A map \( C : \mathcal{D}(M) \to \mathcal{D}(M) \) is a derivation of the Lie algebra \( \mathcal{D}(M) \) of all linear differential operators on \( \mathcal{C}^{\infty}(M) \) if and only if it can be written in the form

\[
C_{P, \omega}(D) = [P, D] + \omega(D),
\]

where \( P \in \mathcal{D}(M) \) and \( \omega \in \Omega^1(M) \cap \ker d \) are not unique. Again the appropriate pairs are \((P + h, \omega + dh)\), \( h \in \mathcal{C}^{\infty}(M) \), so we get uniqueness if we impose that \( P \) is vanishing on constants.

#### Corollary 3
The first cohomology group of the Lie algebra \( \mathcal{D}(M) \) of all linear differential operators on \( \mathcal{C}^{\infty}(M) \) with coefficients in the adjoint representation is isomorphic to the first space of the de Rham cohomology of \( M \):

\[
H^1(\mathcal{D}(M), \mathcal{D}(M)) \simeq H^1_{\text{DR}}(M).
\]

**Proof.** Lemma 3 is a consequence of the surjective character of \( \sigma_i : \mathcal{D}^i \to \mathcal{S}_i \) and Equation (16) that links the Poisson and the Lie brackets.

Propositions 3 and Theorem 1 show that in order to establish Theorem 3 we can start with \( C_1 \in \text{Der} \mathcal{D} \), such that \( C_1(\mathcal{D}^i) \subset \mathcal{D}^i, \forall i \in \mathbb{N} \) and \( C_1(X + f) = \kappa f + \lambda \text{ div } X + \omega(X) \), with the usual notations. When correcting by \( \omega \), we get a filtration-respecting derivation \( C_2 = C_1 - \omega \), which maps \( X + f \) to \( C_2(X + f) = \kappa f + \lambda \text{ div } X \). The derivation \( C_2 \) induced on the classical level then verifies \( \tilde{C}_2(X + f) = \kappa f \). Theorem 2 now implies that

\[
\tilde{C}_2(S) = -\kappa \text{ Deg}(S),
\]

for all \( S \in \mathcal{S} \).

Let us emphasize that Theorem 3 is based upon Theorem 1 and Theorem 2 itself built upon Theorem 1. The point is that the degree-derivation is not generated by any canonical quantum derivation. Therefore the proof of Theorem 3 is a little bit more complicated than that of Theorem 2.
Observe first that Equation (17) means that, for each $i \in \mathbb{N}$,
\begin{equation}
C_2|_{D^i} = \kappa (1 - i) \mathrm{id} + \chi_i,
\end{equation}
where $\chi_i \in \text{Hom}_R(D^i, D^{i-1})$. Indeed, it entails that, if $D \in D^i$, $i \in \mathbb{N}$, the operator $C_2 D - \kappa (1 - i) D$
has a vanishing $i$-th order symbol. As easily checked, $\chi_0 = 0$, $\chi_1 (X + f) = \kappa f + \lambda \text{div } X$, and $\chi_i f = i \kappa f$
($X \in \mathcal{X}, f \in \mathcal{A}, i \in \mathbb{N}$). Injecting now this structure into the derivation property, written for $D^i \in D^i$
and $\Delta^j \in D^j$, $i, j \in \mathbb{N}$, we obtain
\begin{equation}
\chi_{i+j-1}[D^i, \Delta^j] = [\chi_i D^i, \Delta^j] + [D^i, \chi_j \Delta^j].
\end{equation}

When using the decomposition $D = \mathcal{A} \oplus D_c$, we denote by $\pi_0$ and $\pi_\kappa$ the projections onto $\mathcal{A}$ and
$D_c$ respectively. Furthermore, if $D \in \mathcal{D}$, we set $D_0 = \pi_0 D = D(1)$ and $D_c = \pi_\kappa D = D - D(1)$, and if
$C \in \text{End } \mathcal{D}$, we set $C_0 = \pi_0 \circ C \in \text{Hom}_R(\mathcal{D}, \mathcal{A})$ and $C_c = \pi_\kappa \circ C \in \text{Hom}_R(\mathcal{D}, D_c)$.

The projections on $\mathcal{A}$ of Equation (19), written for $D^i_c \in D^i_c$ ($i \geq 2$) and $f \in \mathcal{A}$, then for $D^i_c \in D^i_c$
and $\Delta^j_c \in D^j_c$ ($i + j \geq 3$), read
\begin{equation}
(\chi_{i,i} D^i_c)(f) = (i - 1) \kappa D^i_c(f) + \chi_{i-1,0} [D^i_c, f]_c
\end{equation}
and
\begin{equation}
\chi_{i+j-1,0}[D^i_c, \Delta^j_c] = D^i_c(\chi_j \Delta^j_c) - \Delta^j_c(\chi_i D^i_c)
\end{equation}
respectively. Exploiting first Equation (20) with $i = 2$ and $D^i_c = Y^2, Y \in \mathcal{X}$, we get the upshot
\begin{equation}
\chi_{2,c} Y^2 = (2\lambda + \kappa) Y^2 + 2 \lambda (\text{div } Y) Y,
\end{equation}
for all $Y \in \mathcal{X}$. If we apply the second order symbol $\sigma_2$ to both sides of this equation, we see that
\begin{equation}
2\lambda + \kappa = 0.
\end{equation}

There is an atlas of $M$ such that in each chart $(U, x^1, \ldots, x^n)$ the divergence takes the classical form, $\text{div } (\sum_i X^i \partial_{x^i}) = \sum_i \partial_{x^i} X^i$. We work in such a chart and write $\partial$ (resp. $f'$ and $B(f)$, $f \in C^\infty(U)$)
instead of $\partial_{x^i}$ (resp. $\partial f$ and $\chi_{2,0}(f \partial^2)$). For $i = 1$, $D^i_c = g \partial, j = 2, \Delta^j_c = f \partial^2, f, g \in C^\infty(U)$, Equation (21) yields
\begin{equation}
B(gf' - 2fg') = \lambda f' g'' + g (B(f))'.
\end{equation}
In particular, $B(f') = (B(f))'$ and $B(g') = 0$. But then $B = 0$, $\lambda = 0$ (see Equation (21)), $\kappa = 0$ (see Equation (20)), and $C_2|_{D^i_c} = 0$.

We now proceed by induction and show that $C_2|_{D^{k+1}} = 0$, if $C_2|_{D^k} = 0$, $k \geq 1$. As $C_2|_{D^{k+1}}$ only
depends on the $(k + 1)$-th order symbol, as $[C_2(D), f] = 0$, and $C_2([X, D]) = [X, C_2(D)]$, for all
$D \in D^{k+1}, X \in \mathcal{X}$, and $f \in \mathcal{A}$, $C_2$ defines a map $C_2 : S_{k+1} \rightarrow \mathcal{A}$ that intertwines the adjoint action of $\mathcal{X}$. We have shown in the proof of Theorem 4 that such a map necessarily vanishes. Hence $C_2 = 0$
This completes the proof of Theorem 4. Indeed, the statement regarding the appropriate pairs $(P, \omega)$ is obvious. The same is true for Corollary 5.

Remarks: 1. Denote $C_{P, \omega} = C_P, C_{\omega, \omega} = \overline{\omega}$. The Lie algebra structure of Der $\mathcal{D}$ is determined by
the following commutation relations (the missing commutator is 0):
\begin{equation}
[C_P, C_{P'}] = C_{[P, P']}, \quad [\overline{\omega}, C_P] = C_{\overline{\omega}(P)}.
\end{equation}
2. Corollary 1, Corollary 2, and Corollary 3 imply that the first adjoint cohomology spaces of the
Lie algebras $D^1, \mathcal{S}$, and $D$ are independent of the smooth structure of $M$, provided that the topology of $M$
remains unchanged.

3. It is worth comparing our cohomological results with those obtained in other recent papers. Let
$D_M = (\text{End } \mathcal{A})_{\text{loc}, c}$ be the Lie algebra of local endomorphisms of $\mathcal{A}$ that vanish on constants. A well-known
theorem of Peetre, [Pee60], guarantees that these operators are locally differential. The main
theorem of [Pon99] asserts that the first three local cohomology groups $H^p(D_M, \mathcal{A})_{\text{loc}}$ ($p \in \{1, 2, 3\}$)
of $D_M$ canonically represented upon $A$ are isomorphic to the corresponding groups $H^p_{DR}(M)$ of the de Rham cohomology of $M$. In particular,

$$H^1(D_M, C^\infty(M)) \simeq H^1_{DR}(M).$$

Let us quote from [AAL02] the outcome

$$H^1(Vect(M), D(M)) \simeq H^1(Vect(M), D^1(M)) \simeq \mathbb{R} \oplus H^1_{DR}(M),$$

for all $i \in \mathbb{N}$.

## 8 Integrability of derivations

In this section we distinguish those derivations that generate (smooth) one-parameter groups of automorphisms of the Lie algebra $\mathcal{P}$ (we will call such derivations integrable) and we find explicit forms of these one-parameter groups of automorphisms. The smoothness of a curve in $Aut \mathcal{P}$ is defined in the obvious way with relation to the smooth structure on $M$. For instance, $\Phi_t$ is smooth in $Aut D$ if for any $D \in \mathcal{D}$ and any $f \in C^\infty(M)$ the induced map $(t, x) \mapsto \Phi_t(D)(f)(x)$ is a smooth function on $\mathbb{R} \times M$, a curve $\Phi_t$ in $Aut S$ is smooth if for any $S \in S$ the induced map $(t, y) \mapsto \Phi_t(S)(y)$ is a smooth function on $\mathbb{R} \times T^*M$, etc. In the following all one-parameter groups will be assumed to be smooth.

Since the group $Diff(M)$ of smooth diffeomorphisms of $M$ is embedded in $Aut \mathcal{P}$ (see [GP03]), a partial problem is the determination of one-parameter groups of diffeomorphisms. This, however, is well known and the one-parameter groups of diffeomorphisms are just flows $\text{Exp}(tY)$ of complete vector fields $Y$. Note that in general it is hard to decide if a given diffeomorphism is implemented by a vector field, since neighbourhoods of identity in the connected component of the group $Diff(M)$ are far from being filled up by flows (even in the case when $M$ is compact and all vector fields are complete (see [Gra88, Kop70, Pal73])); that differs $Diff(M)$ from finite-dimensional Lie groups.

Before we start the investigation into one-parameter subgroups in $Aut \mathcal{P}$ we have to define the group-analogue of the divergence, which is important for the case $\mathcal{P} = D^1(M)$. Let us stress that in this paper the divergence is not an arbitrary 1-cocycle of vector fields with coefficients into functions, but a cocycle obtained as described in [GP03] from a nowhere vanishing 1-density or as depicted in [GMM03] from an odd volume form. These cocycles or divergences form some privileged cohomology class. We will integrate any such divergence $\text{div} : \mathcal{X}(M) \to C^\infty(M)$ to a group 1-cocycle $J : Diff(M) \to C^\infty(M)$, which is a sort of Jacobian. Indeed, if $|\eta|$ is the odd volume form inducing the divergence and if $\phi \in Diff(M)$, we have $\phi^*|\eta| = J(\phi)|\eta|$ for a unique positive smooth function $J(\phi)$. It is easily verified that if $\phi$ is a diffeomorphism between two domains of local coordinates and if $f$ and $g$ are the component functions of $|\eta|$ in the corresponding bases, then locally

$$J(\phi)(x) = \frac{g(\phi(x))}{f(x)} \det \partial_x \Phi,$$

where $\Phi$ is the local form of $\phi$. For any $\phi, \psi \in Diff(M)$, we clearly have

$$J(\phi \circ \psi) = \psi^*(J(\phi)) \cdot J(\psi). \quad (25)$$

In particular,

$$J(\phi^{-1}) = \frac{1}{J(\phi) \circ \phi^{-1}}.$$

A similar concept may be found under the name of Jacobi determinant in [AMR88] Def. 6.5.12]. Let us put $\text{Div}(\phi) = \ln J(\phi)$.

**Proposition 5** For any $X \in \mathcal{X}(M)$ and $\phi \in Diff(M)$, we have

(a)  

$$\phi^*(\text{div} \phi_*(X)) = \text{div} X + X(\text{Div}(\phi)) \quad (26)$$

and, if $X$ is complete,
Derivations of the Lie algebras of differential operators

(b) \[ \text{Div}(\text{Exp}(tX)) = \int_0^t (\text{div } \circ \text{Exp}(sX))ds. \] (27)

Proof. (a) By definition of the action of \( \phi \) on vector fields and differential forms, \( \phi^*(i_{\phi_*(X)}|\eta|) = i_X(\phi^*|\eta|) \), so that
\[ \phi^*(\text{div}|\eta\phi(X)) = \text{div}\phi^*|\eta|(X). \]
Since \( \phi^*|\eta| = J(\phi)|\eta| \), (26) follows.

(b) Let us put \( \tilde{F}_t = \text{Div}(\text{Exp}(tX)) \). It is easy to see that \( F_{t+s} = F_t + F_s \circ \text{Exp}(tX) \), which implies the differential equation
\[ \tilde{F}_t = X(F_1) + \tilde{F}_0. \] (28)
Additionally, we have the initial conditions \( F_0 = 0 \) and, due to
\[ \tilde{F}_0|\eta| = \frac{d}{dt}|_{t=0}(\text{Exp}(tX))^*|\eta| = (\text{div } X)|\eta|, \]
\( \tilde{F}_0 = \text{div } X \). Applying formally the variation of constant method, we find
\[ F_t = (\text{Exp}(tX))^*(\int_0^t \tilde{F}_0 \circ \text{Exp}(-sX)ds) = \int_0^t (\text{div } X) \circ \text{Exp}(sX)ds. \]
It is easily verified that this integral is really a solution. Equation (28) is in fact a PDE of first order, which can be written in the form
\[ L_X F = \tilde{F}_0, \]
with \( \tilde{X} = \partial_t - X \in \mathcal{X}(\mathbf{R} \times M) \). A well-known consequence of the theorem of Frobenius allows to see that this equation, completed by the boundary condition \( F|_M = 0 \), has locally one unique solution. Hence,
\[ F_t = \int_0^t (\text{div } X) \circ \text{Exp}(sX)ds. \]

8.1 The case \( \mathcal{D}^1(M) \)

Theorem 8 of [CP03] states that an endomorphism \( \Phi \) of \( \mathcal{D}^1 \) is an automorphism of the Lie algebra \( \mathcal{D}^1 \) if and only if it reads
\[ \Phi_{\phi,K,\Lambda,\Omega}(X + f) = \phi_*(X) + (K f + \Lambda \text{ div } X + \Omega(X)) \circ \phi^{-1}, \] (29)
where \( \phi \) is a diffeomorphism of \( M \), \( K, \Lambda \) are constants, \( K \neq 0 \), \( \Omega \) is a closed 1-form on \( M \), and \( \phi_* \) is the push-forward
\[ (\phi_*(X))(f) = (X(f \circ \phi)) \circ \phi^{-1}, \] (30)
all the objects \( \phi, K, \Lambda, \Omega \) being uniquely determined by \( \Phi \). The one-parameter group condition
\[ \Phi_{\phi_*,K_*,\Lambda_*,\Omega_*} \circ \Phi_{\phi_*,K_*,\Lambda_*,\Omega_*} = \Phi_{\phi_*,K_*,\Lambda_*,\Omega_*}, \]
gives immediately
\[ \phi_{t+s} = \phi_t \circ \phi_s, \quad K_{t+s} = K_t \cdot K_s, \quad \Lambda_{t+s} = \Lambda_t + K_t \cdot \Lambda_s, \]
and, in view of (26),
\[ \Omega_{t+s} = K_t \Omega_s + \phi_*(\Lambda_t d(\text{Div}(\phi_t))) + \phi^*_t \omega, \] (31)
with the initial conditions \( \phi_0 = \text{id}_M, \quad K_0 = 1, \quad \Lambda_0 = 0, \quad \Omega_0 = 0 \). One solves easily: \( \phi_t = \text{Exp}(tY), \quad K_t = e^{\kappa t}, \quad \Lambda_t = \frac{\lambda - e^{-\kappa t}}{\kappa} \) (with \( \frac{\lambda - e^{-\kappa t}}{\kappa} = t \) if \( \kappa = 0 \)), for some unique complete vector field \( Y \) and some unique real numbers \( \kappa, \lambda \). To solve (31) we derive the differential equation
\[ \dot{\Omega}_t = \kappa \Omega_t + \lambda d(\text{Div}(\phi_t)) + \phi^*_t \omega, \]
where $\omega = \dot{\Omega}_0$. This is an inhomogeneous linear equation, which can be solved by the method of variation of the constant. We get

$$\Omega_t = e^{\kappa t} \int_0^t e^{-\kappa s} \left( \lambda d(\text{Div}(\phi_s)) + \phi_s^* \omega \right) ds$$

and, in view of (27),

$$\Omega_t = \int_0^t \int_0^s \lambda d \left( \text{Div}(\phi_s) \circ \text{Exp}(u_Y) du \right) + \left( \text{Exp}(s_Y) \circ \phi_s^* \omega \right) ds.$$

Since $\dot{\phi}_0 = Y$, $\dot{K}_0 = \kappa$, $\dot{\lambda}_0 = \lambda$, $\dot{\Omega}_0 = \omega$, we get the following:

**Theorem 4** A derivation

$$C_{\gamma, \kappa, \omega}(X + f) = [Y, X + f] + \kappa f + \lambda \text{Div}(X) + \omega(X)$$

of $\mathcal{D}^1(M)$ induces a one-parameter group $\Phi_t$ of automorphisms of $\mathcal{D}^1(M)$ if and only if the vector field $Y$ is complete. In this case the group is of the form

$$\Phi_t(x + f) = (\text{Exp}(tY))_* (x) + \left( e^{\kappa t} f + \lambda \frac{e^{\kappa t} - 1}{\kappa} \text{Div}(x) \right) \circ \text{Exp}(-tY) + \left( \int_0^t e^{\kappa (t-s)} \left( \lambda \int_0^s \text{Div}(x \circ \text{Exp}(u_Y)) du + \left( \text{Exp}(s_Y) \circ \phi_s^* \omega \right)(x) \right) ds \right) \circ \text{Exp}(-tY).$$

**8.2 The case $\mathcal{S}(M)$**

We know from [GP03, Theorem 9] that an endomorphism $\Phi$ of $\mathcal{S}$ is an automorphism of the Lie algebra $\mathcal{S}$ if and only if it has the form

$$\Phi = \overline{\phi} \circ \mathcal{U}_K \circ e^\Omega.$$

Here $\phi \in \text{Diff}(M)$ and if $\mathcal{S}$ is interpreted as the algebra $\text{Pol}(T^*M)$ of polynomial functions on $T^*M$, the automorphism $\overline{\phi}$ is implemented by the phase lift $\phi^*$ of $\phi$ to the cotangent bundle $T^*M$, a symplectomorphism of $T^*M$. If, on the other hand, $\mathcal{S}$ is viewed as the algebra $\Gamma(\mathcal{S}TM)$ of symmetric contravariant tensor fields on $M$, the automorphism $\overline{\phi}$ is the standard action of $\phi$ on such tensor fields.

Further, $K \in \mathbb{R}^*$, $\Omega$ is a closed 1-form on $M$ and the automorphism $\mathcal{U}_K \in \text{Aut} \mathcal{S}$, $\mathcal{U}_K : \mathcal{S}_i \ni S \mapsto K_i^{-1} S \in \mathcal{S}_i$, is, for $K > 0$, the exponential of the derivation $\ln K$ $\text{Deg}$, whereas the automorphism $e^\Omega$ induced by the lowering derivation $\Omega$, i.e. the action of the vertical vector field $\Omega$, is the composition with the translation $\mathcal{T}_t$ by $\Omega$ in $T^*M$. Note that since the homothety $h_K$ of $T^*M$ by $K$ acts on homogeneous polynomials of degree $i$ by multiplication by $K^i$, the automorphism $\mathcal{U}_K$ can be written also in the form $\mathcal{U}_K(S) = K^{-1} S \circ h_K$. Hence, every one-parameter group of automorphisms of the Lie algebra $\mathcal{S}$ has the form

$$\Phi_{\phi_t, K_i, \Omega_i}(S) = K_i^{-1} S \circ \mathcal{T}_{\Omega_i} \circ h_{K_i} \circ (\phi_t^{-1})^*.$$

It is easy to prove the following commutation relations.

**Proposition 6**

$$h_K \circ \phi^* = \phi^* \circ h_K,$$

$$\mathcal{T}_t \circ h_K = h_K \circ \mathcal{T}_t \circ \Omega,$$

$$h_K = \mathcal{T}_t \circ \phi_t^{-1} \circ \Omega^*.$$

These relations together with the one-parameter group property yield

$$\phi_{t+s} = \phi_t \circ \phi_s,$$

$$K_{t+s} = K_t \cdot K_s,$$

$$\Omega_{t+s} = \Omega_t + K_t \cdot \phi_t^* \Omega_s.$$
with the initial conditions $\phi_0 = \text{id}_M$, $K_0 = 1$, $\Omega_0 = 0$. The obvious unique solutions are $\phi_t = \text{Exp}(tY)$ for a certain complete vector field $Y$, $K_t = e^{\kappa t}$ for a certain $\kappa \in \mathbb{R}$, and

$$\Omega_t = \int_0^t e^{\kappa s} (\text{Exp}(sY))^* \omega ds$$

for a certain closed 1-form $\omega$ on $M$.

Let us systematically characterize derivations by the unique triplet with first member vanishing on the 0-section. As well-known there is a Lie algebra isomorphism between $S_1(M) = \text{Pol}^1(T^*M)$ and $\mathfrak{X}(M)$. We denote a homogeneous first order polynomial $P$ on $T^*M$ by $P_s$ when it is viewed as vector field of $M$. Note that the hamiltonian vector field $X_P$ of $P$ is nothing but the phase lift $(P_s)^*$ of $P_s$.

We then have the following theorem.

**Theorem 5** A derivation

$$C_{P,\kappa,\omega}(S) = \{P, S\} + \kappa \text{ Deg}(S) + \omega^v(S)$$

of the Lie algebra $S(M)$ of all infinitely differentiable functions of $T^*M$ that are polynomial along the fibers, where $P$ is vanishing on the 0-section, is integrable if and only if the polynomial function $P$ belongs to $S_1(M)$ and is complete, i.e. the hamiltonian vector field $X_P$ of $P$ is complete, i.e. the basis vector field $P_s$ is complete. In this case the one-parameter group of automorphisms $\Phi_t$ generated by $C_{P,\kappa,\omega}$ reads

$$\Phi_t(S) = e^{-\kappa t} S \circ T_{\int_0^t e^{\kappa s} (\text{Exp}(sP_s))^* \omega ds} \circ h_{e^{\kappa t}} \circ \text{Exp}(-tX_P).$$

### 8.3 The case $D(M)$

Let us eventually recall that Theorem 10 of [GP03] asserts that automorphisms of the Lie algebra $D$ have the form

$$\Phi = \phi_\ast \circ C^a \circ e^{\Omega},$$

where $e^{\Omega}$ ($\Omega \in \Omega^1(M) \cap \ker d$) is the formerly mentioned automorphism of $D$ and where $\phi_\ast$ ($\phi \in \text{Diff}(M)$) is the automorphism of $D$ defined by $\phi_\ast(D) = \phi \circ D \circ \phi^{-1}$, $\phi(f)$ being of course $f \circ \phi^{-1}$ ($D \in D, f \in A$). Moreover, superscript $a$ is 0 or 1, so that $C^0 = \text{id}$ or $C^1 = C$, $C$ being the opposite of the conjugation operator $\ast$. Remember that for an oriented manifold $M$ with volume form $\eta$, the conjugate $D^\ast \in D$ of a differential operator $D \in D$ is defined by

$$\int_M D(f) \cdot g \ | \ \eta | = \int_M f : D^\ast(g) \ | \ \eta |,$$

for any compactly supported $f, g \in A$. Since $(D \circ \Delta)^\ast = \Delta^\ast \circ D^\ast$ ($D, \Delta \in D$), the operator $C := -\ast$ verifies $C(D \circ \Delta) = -\Delta^\ast \circ C(D)$ and is thus an automorphism of $D$. Formal calculus allows to show that this automorphism exists for any manifold, orientable or not. Clearly, the automorphism $C$ is not implemented by a derivation and $\Omega$ belongs to the connected component of identity only if $a = 0$. Thus we can consider one-parameter groups of automorphisms of the form

$$\Phi_{\phi_\ast, \Omega_t} = (\phi_t)_\ast \circ e^{\Omega_t}.$$ 

It is easy to prove that

$$e^{\Omega_t} \circ \phi_\ast = \phi_\ast \circ e^{\phi_\ast \Omega_t},$$

so that the one-parameter group property yields

$$\phi_{t+s} = \phi_t \circ \phi_s,$$

$$\Omega_{t+s} = \Omega_t + \phi_\ast^t \Omega_s,$$

with initial conditions $\phi_0 = \text{id}_M$, $\Omega_0 = 0$. The obvious general solutions are $\phi_t = \text{Exp}(tY)$ for a complete vector field $Y$, and $\Omega_t = \int_0^t (\text{Exp}(sY))^* \omega ds$ for a certain closed 1-form $\omega$. Thus we get the following.
Theorem 6 A derivation

$$C_{P,\omega}(D) = [P, D] + \omega(D)$$

(34)

of the Lie algebra $\mathcal{D}(M)$ of all differential operators is integrable if and only if $P \in \mathcal{X}(M)$ and $P$ is complete. In this case the one-parameter group of automorphisms $\Phi_t$ generated by $C_{P,\omega}$ reads

$$\Phi_t = (\text{Exp}(tP))_* \circ e^{\int_0^t (\text{Exp}(sP))^* \omega ds}.$$

Remark: The results of this section describing commutations rules for automorphisms easily imply that $\text{Aut} \mathcal{P}$ is an infinite-dimensional regular Lie group in the sense of A. Kriegl and P. Michor (see [KM97a] or [KM97b, Ch. 8]). The integrable derivations (in fact, those with compact supports) form the Lie algebra of $\text{Aut} \mathcal{P}$.

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