Characterization of the Fibonacci Cobweb Poset as oDAG

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Abstract

The characterization of Fibonacci Cobweb poset $P$ as DAG and oDAG is given. The $dim$ 2 poset such that its Hasse diagram coincide with digraf of $P$ is constructed.

1 Fibonacci cobweb poset

The Fibonacci cobweb poset $P$ has been invented by A.K.Kwaśniewski in [1, 2, 3] for the purpose of finding combinatorial interpretation of fibonomial coefficients and eventually their recurrance relation.

In [1] A. K. Kwaśniewski defined cobweb poset $P$ as infinite labeled digraph oriented upwards as follows: Let us label vertices of $P$ by pairs of coordinates: $\langle i, j \rangle \in \mathbb{N}_0 \times \mathbb{N}_0$, where the second coordinate is the number of level in which the element of $P$ lies (here it is the $j$-th level) and the first one is the number of this element in his level (from left to the right), here $i$.

Following [1] we shall refer to $\Phi_s$ as to the set of vertices (elements) of the $s$-th level, i.e.:

$$\Phi_s = \{\langle j, s \rangle, \ 1 \leq j \leq F_s \}, \ s \in \mathbb{N} \cup \{0\},$$

where $\{F_n\}_{n \geq 0}$ stands for Fibonacci sequence.

Then $P$ is a labeled graph $P = (V, E)$ where

$$V = \bigcup_{p \geq 0} \Phi_p, \quad E = \{\langle \langle j, p \rangle, \langle q, p + 1 \rangle \rangle \}, \ 1 \leq j \leq F_p, \ 1 \leq q \leq F_{p+1}.$$
We can now define the partial order relation on $P$ as follows: let $x = (s, t), y = (u, v)$ be elements of cobweb poset $P$. Then

$$(x \leq_P y) \iff [(t < v) \lor (t = v \land s = u)].$$

Fig. 1. The picture of the Fibonacci ”cobweb” poset

2 DAG $\rightarrow$ oDAG problem

In [5] A. D. Plotnikov considered the so called ”DAG $\rightarrow$ oDAG problem”. He determined condition when a digraph $G$ may may be presented by the corresponding $dim \ 2$ poset $R$ and he established the algorithm for finding it.

Before citing Plotnikov’s results let us recall (following [5]) some indispensable definitions.

If $P$ and $Q$ are partial orders on the same set $A$, $Q$ is said to be an extension of $P$ if $a \leq_P b$ implies $a \leq_Q b$, for all $a, b \in A$. A poset $L$ is a chain, or a linear order if we have either $a \leq_L b$ or $b \leq_L a$ for any $a, b \in A$. If $Q$ is a linear order then it is a linear extension of $P$. 
The **dimension** $\dim R$ of $R$ being a partial order is the least positive integer $s$ for which there exists a family $F = (L_1, L_2, \ldots, L_s)$ of linear extensions of $R$ such that $R = \bigcap_{i=1}^{s} L_i$. A family $F = (L_1, L_2, \ldots, L_s)$ of linear orders on $A$ is called a **realizer** of $R$ on $A$ if

$$R = \bigcap_{i=1}^{s} L_i.$$ 

We denote by $D_n$ the set of all acyclic directed $n$-vertex graphs without loops and multiple edges. Each digraph $\vec{G} = (V, \vec{E}) \in D_n$ will be called **DAG**.

A digraph $\vec{G} \in D_n$ will be called **orderable** (oDAG) if there exists are $\dim 2$ poset such that its Hasse diagram coincide with the digraph $\vec{G}$.

Let $\vec{G} \in D_n$ be a digraph, which does not contain the arc $(v_i, v_j)$ if there exists the directed path $p(v_i, v_j)$ from the vertex $v_i$ into the vertex $v_j$ for any $v_i, v_j \in V$. Such digraph is called **regular**. Let $D \subset D_n$ is the set of all regular graphs.

Let there is a some regular digraph $\vec{G} = (V, \vec{E}) \in D_n$, and let the chain $\vec{X}$ has three elements $x_{i_1}, x_{i_2}, x_{i_3} \in X$ such that $i_1 < i_2 < i_3$, and, in the digraph $\vec{G}$, there are not paths $p(v_{i_1}, v_{i_2})$, $p(v_{i_2}, v_{i_3})$ and there exists a path $p(v_{i_1}, v_{i_3})$. Such representation of graph vertices by elements of the chain $\vec{X}$ is called the representation in **inadmissible form**. Otherwise, the chain $\vec{X}$ presents the graph vertices in **admissible form**.

Plotnikov showed that:

**Lemma 1.** \cite{5} A digraph $\vec{G} \in D_n$ may be represented by a $\dim 2$ poset if:

1. there exist two chains $\vec{X}$ and $\vec{Y}$, each of which is a linear extension of $\vec{G}$;

2. the chain $\vec{Y}$ is a modification of $\vec{X}$ with inversions, which remove the ordered pairs of $\vec{X}$ that there do not exist in $\vec{G}$.

Above lemma results in the algorithm for finding $\dim 2$ representation of a given DAG (i.e. corresponding oDAG) while the following theorem establishes the conditions for constructing it.

**Theorem 1.** \cite{5} A digraph $\vec{G} = (V, \vec{E}) \in D_n$ can be represented by $\dim 2$ poset iff it is regular and its vertices can be presented by the chain $\vec{X}$ in admissible form.
3 Fibonacci cobweb poset as DAG and oDAG

In this section we show that Fibonacci cobweb poset is a DAG and it is orderable (oDAG).

Obviously, cobweb poset $P = (V, E)$ defined above is a DAG (it is directed acyclic graph without loops and multiple edges). One can also verify that it is regular. For two elements $\langle i, n \rangle, \langle j, m \rangle \in V$ a directed path $p(\langle i, n \rangle, \langle j, m \rangle) \not\in E$ will exist iff $n < m + 1$ but then $\langle (i, n), (j, m) \rangle \not\in E$ i.e. $P$ does not contain the edge $\langle (i, n), (j, m) \rangle$.

It is also possible to verify that vertices of cobweb poset $P$ can be presented in admissible form by the chain $\vec{X}$ being a linear extension of cobweb $P$ as follows:

$$\vec{X} = (\langle 1, 0 \rangle, \langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 1, 3 \rangle, \langle 2, 3 \rangle, \langle 1, 4 \rangle, \langle 2, 4 \rangle, \langle 3, 4 \rangle, \langle 1, 5 \rangle, \langle 2, 5 \rangle, \langle 3, 5 \rangle, \langle 4, 5 \rangle, \langle 5, 5 \rangle, ...),$$

where

$$(s, t) \leq_{\vec{X}} (u, v) \iff [(s \leq u) \land (t \leq v)]$$

for $1 \leq s \leq F_t$, $1 \leq u \leq F_v$, $t, v \in \mathbb{N} \cup \{0\}$.

Fibonacci cobweb poset $P$ satisfies the conditions of Theorem 1 so it is oDAG. To find the chain $\vec{Y}$ being a linear extension of cobweb $P$ one uses Lemma 1 and arrives at:

$$\vec{Y} = (\langle 1, 0 \rangle, \langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 3 \rangle, \langle 1, 3 \rangle, \langle 3, 4 \rangle, \langle 2, 4 \rangle, \langle 1, 4 \rangle, \langle 5, 5 \rangle, \langle 4, 5 \rangle, \langle 3, 5 \rangle, \langle 2, 5 \rangle, \langle 1, 5 \rangle, ...),$$

where

$$(s, t) \leq_{\vec{Y}} (u, v) \iff [(t < v) \lor (t = v \land s \geq u)]$$

for $1 \leq s \leq F_t$, $1 \leq u \leq F_v$, $t, v \in \mathbb{N} \cup \{0\}$ and finally

$$(P, \leq_P) = \vec{X} \cap \vec{Y}.$$

4 Closing remark

For any sequence $\{a_n\}$ of natural numbers one can define corresponding cobweb poset as follows:

$$\Phi_s = \{\langle j, s \rangle, 1 \leq j \leq a_s\}, \ s \in \mathbb{N} \cup \{0\},$$
and \( \text{cob}P = (V, E) \) where

\[
V = \bigcup_{p \geq 0} \Phi_p, \quad E = \{\langle j, p \rangle, \langle q, p + 1 \rangle \}, \quad 1 \leq j \leq a_p, \quad 1 \leq q \leq a_{p+1}
\]

with the partial order relation on \( \text{cob}P \):

\[
(x \leq_P y) \iff [(t < v) \lor (t = v \land s = u)]
\]

for \( x = \langle s, t \rangle, y = \langle u, v \rangle \) being elements of cobweb poset \( \text{cob}P \). Similarly as above one can show that the family of cobweb posets consist of DAGs representable by corresponding \( \text{dim} \ 2 \) posets (i.e. of oDAGs).

References

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