PLANARITY IN HIGHER-DIMENSIONAL CONTACT MANIFOLDS

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ABSTRACT. We obtain several results for (iterated) planar contact manifolds in higher dimensions: (1) Iterated planar contact manifolds are not weakly symplectically semi-fillable. This generalizes a 3-dimensional result of Etnyre [10] to a higher-dimensional setting. (2) They do not arise as nonseparating weak contact-type hypersurfaces in closed symplectic manifolds. This generalizes a result by Albers-Bramham-Wendl [4]. (3) They satisfy the Weinstein conjecture, i.e. every contact form admits a closed Reeb orbit. This is proved by an alternative approach as that of [2], and is a higher-dimensional generalization of a result of Abbas-Cieliebak-Hofer [1]. The results follow as applications from a suitable symplectic handle attachment, which bears some independent interest.

1. INTRODUCTION

Contact structures in every odd dimension can be understood via open book decompositions, a correspondence which has been established by important work of Giroux [16]. In dimension three, planar contact manifolds, those that correspond to an open book decomposition with genus zero pages, are in some sense the simplest contact 3-manifolds. For instance, their strong symplectic fillings, when they exist, carry Lefschetz fibrations inducing the given planar open book decomposition along their boundary [33]. This implies that the classification of strong symplectic fillings of a planar contact manifold boils down to studying Dehn twist factorizations in the mapping class of a genus zero surface.

While overtwisted contact structures are planar [10], there are known obstructions to planarity in dimension three. For instance, Etnyre [10] proves that if $W$ is a weak symplectic filling of a contact 3-manifold $M$ supported by a planar open book decomposition, then the boundary of $W$ is connected. Moreover, planar contact 3-manifolds do not admit nonseparating embeddings as weak contact-type hypersurfaces inside closed symplectic 4-manifolds [4, 29]. Etnyre’s result can actually be recovered from this result (in the strong filling case), as it follows easily from the existence of symplectic caps in dimension three [7, 11].

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In higher dimensions, there are natural generalizations of planarity. For instance, one can replace a fibration defining an open book decomposition by a fibration whose fibers are still genus zero surfaces with boundary, but such that the base is now a higher-dimensional closed contact manifold. One could also consider contact manifolds admitting a submanifold with the structure of a contact fibration over a Liouville domain of arbitrary dimension, whose fibers are closed planar contact 3-manifolds. Both of these lead naturally to (planar versions of) the notion of spinal open book decomposition or SOBD, as discussed in [28] based on the three dimensional notion defined in [25].

An alternative generalization is to consider standard open book decompositions in higher dimensions, keeping $S^1$ as the base of the symplectic fibration, but impose the condition that the fibers carry a suitable structure which is inductively built from a low-dimensional planar structure. In this vein, the following notion was introduced in [2].

Given a $(2n-2)$-dimensional Weinstein domain $(W^{2n-2}, \omega)$, we say that $W$ admits an *iterated planar Lefschetz fibration* if there exists a sequence $f_i : W^{2i} \to \mathbb{D}^2$ of exact symplectic Lefschetz fibrations, $i = 2, \ldots, n-1$, where the regular fiber of $f_{i+1}$ is the total space of $f_i$, and $f_2 : W^4 \to \mathbb{D}^2$ is a planar Lefschetz fibration. We denote its regular fiber by $W^{2n-2}$, which is simply a genus zero surface with boundary. Observe that when $n = 3$, an iterated planar Lefschetz fibration is a planar Lefschetz fibration.

An *iterated planar contact manifold* $(M, \xi)$ is a $(2n-1)$-dimensional contact manifold supported by an open book decomposition $\text{OB}(W^{2n-2}, \varphi)$ whose page $W^{2n-2}$ admits an iterated planar Lefschetz fibration. Note that when $n = 3$, $M$ is a contact 5-manifold supported by an open book decomposition whose page admits a planar Lefschetz fibration, inducing a planar open book decomposition in the binding. We refer the reader to Section 2 and also [3] for a more detailed discussion on iterated planarity.

**Conventions.** Throughout the paper, we will abbreviate iterated planar as $\text{IP}$, we will refer to an open book decomposition as in the above definition as an $\text{IP}$ open book decomposition, and we will denote $M^{2n-1} = \text{OB}(W^{2n-2}, \varphi)$. Lastly, when we talk about a weak symplectic semi-filling, we will mean a symplectic manifold with disconnected weakly dominated contact boundary. We will usually keep track of dimensions in the notation, in order to avoid confusion (while perhaps making the notation a bit cumbersome in some places).

**Statements of the results.** In [26], several examples of (exactly) symplectically semi-fillable higher-dimensional contact manifolds were constructed. It is then natural to wonder which contact manifolds can fit into a symplectic semi-filling.
The first result of this paper, generalizing Etnyre’s 3-dimensional result [10], is an obstruction to iterated planarity. The statement reads as follows:

**Theorem A.** IP contact manifolds are not weakly symplectically semi-fillable.

In other words, if a contact manifold $M$ is weakly symplectically semi-fillable, then it cannot be iterated planar. A further obstruction to iterated planarity, which generalizes a result by Albers-Bramham-Wendl [4], is the following:

**Theorem B.** IP contact manifolds do not embed as nonseparating weak contact-type hypersurfaces in closed symplectic manifolds.

In other words, if an IP contact manifold admits a weak contact-type embedding into a closed symplectic manifold, then it separates the latter into two disjoint pieces.

By the work of Lisca and Matić [24], we know that Stein fillings of contact manifolds can be embedded into closed symplectic manifolds. That is, Stein fillable contact manifolds admit symplectic caps. For general higher-dimensional contact manifolds, it was not known until very recently that they admit (strong) symplectic caps [6, 23]. This fact can be used to recover Theorem A (with the word “weakly” replaced by “strongly”) from Theorem B, but our proof is independent of the existence of caps.

Recall that given a contact form $\lambda$ for a contact manifold $(M, \xi)$, there exists a unique contact vector field called the **Reeb vector field** $R_\lambda$, defined by the equations $\iota_{R_\lambda} d\lambda = 0$ and $\lambda(R_\lambda) = 1$. The Weinstein conjecture [32] states that, on a compact contact manifold $(M, \xi)$ of any odd dimension, any Reeb vector field $R_\lambda$ for $\xi$ carries at least one closed periodic orbit. This is a central conjecture in contact and symplectic topology, which has been inspirational for many developments in these fields, and which has a rich history. In dimension three, it was established by Taubes [30] (based on Seiberg-Witten theory), which culminated a large body of work by several people extending over more than two decades. In higher dimensions, though there are several partial results [5, 14, 20, 31], it is still open.

Prior to Taubes’s work, Abbas-Cieliebak-Hofer [1] developed a program for proving the Weinstein conjecture based on open book decompositions, and proved the conjecture for planar contact 3-manifolds. In this paper, we prove the Weinstein conjecture for the case of IP contact manifolds, by an alternative approach as that of [2], which is then a generalization of the result by Abbas-Cieliebak-Hofer to higher dimensions. The proof of the following is a suitable adaptation of the proof in [5] for overtwisted contact manifolds:

**Theorem C.** IP contact manifolds satisfy the Weinstein conjecture.
Given the results in this paper, it is then natural to wonder whether there is any relationship between (a suitable version of) planarity in higher dimensions, and overtwistedness:

**Question.** Are higher-dimensional overtwisted contact structures iterated planar? Or rather, are overtwisted contact structures supported by a planar spinal open book decomposition, or a suitable higher-dimensional notion of planarity?

1.1. **Sketch of proofs.** The technical input for obtaining the results is a suitable symplectic handle attachment, which is directly inspired by the handle attachments in [7] and [25]. We fix an IP contact manifold $M_1^{2n-1} = \text{OB}(W_1^{2n-2}, \varphi)$ where $W_1^{2n-2}$ admits an IP Lefschetz fibration, yielding an IP open book decomposition in $M_1^{2n-1}$. The manifold $W_1^{2n-2}$ carries a natural homotopy class of Weinstein structures, which gives a supported contact structure in $M_1^{2n-1}$ via a construction due to Giroux. We will construct a symplectic cobordism $(C^{2n}, \omega_{C^{2n}})$ from $M_1^{2n-1}$ (a strongly concave boundary component), to a new manifold $M_2^{2n-1}$, which is a stable boundary component of $C^{2n}$, and so carries a stable Hamiltonian structure induced from the symplectic structure $\omega_{C^{2n}}$ in $C^{2n}$. We shall do this by induction in the dimension, constructing a “handle” at each step, using as input the handle constructed in the previous step of the induction. The base case corresponds to the first step of Eliashberg’s capping construction [7] (See also the spine removal surgery in [25]).

Topologically, the effect of this handle attachment is to replace the given IP open book decomposition in $M_1^{2n-1}$ by a topological open book decomposition in $M_2^{2n-1}$. Its pages $W_2^{2n-2} = W_1^{2n-2} \cup C^{2n-2}$ have been enlarged by a symplectic cobordism $C^{2n-2}$ (constructed in the previous step, where we replace $M_1^{2n-1}$ by $M_1^{2n-3} = \partial W_1^{2n-2}$), the monodromy $\varphi$ extends to $W_2^{2n-2}$ by the identity along $C^{2n-2}$, and the new binding $M_2^{2n-3} = \partial W_2^{2n-2}$ has an abundance of embedded 2-spheres. Here, we make a distinction between a topological open book, and a contact open book, since the pages of the former type of open book are no longer Liouville, and hence the manifold $M_2^{2n-1}$ does not carry any obvious supported contact structure. We shall denote this by $M_2^{2n-1} = \text{TOB}(W_2^{2n-2}, \varphi)$. For a suitable almost complex data on $C^{2n}$ compatible with the stable Hamiltonian structure at $M_2^{2n-1}$, and which makes $M_2^{2n-1}$ a weakly pseudoconvex boundary component, these spheres in the new binding $M_2^{2n-3}$ become holomorphic.

After adding a marked point, the virtual dimension of the resulting moduli space is $2n$, and we obtain an evaluation map. One can show that a local uniqueness lemma holds, implying that the only spheres in the moduli space that touch...
a suitable subregion of $M^{2n-1}_2$ are the ones that we constructed “by hand”. Moreover, the latter can be shown to be Fredholm regular, so that the moduli space is a manifold of dimension $2n$ near them.

The cobordism $C^{2n}$ can be further modified, so that it may be glued to any symplectic manifold having $M^{2n-1}_1$ as a weak boundary component. Whereas the stable Hamiltonian structure along $M^{2n-1}_2$ is slightly perturbed, the modification may be chosen so that it does not affect the holomorphic spheres and the Local Uniqueness Lemma. The regions along which the original stable Hamiltonian structure is actually contact, become weakly dominated after the perturbation, whereas the region where the spheres are defined, and the Local Uniqueness Lemma holds, remains stable.

The unifying idea for all the results is inspired by the proof of Theorem 6.1 in [26]. Given a symplectic manifold $W^{2n}$ having $M^{2n-1}_1$ as a weak or strong–contact-type boundary component (either a hypothetical weak symplectic semi-filling, or one we construct), we attach the cobordism $(C^{2n}, \omega_{C^{2n}})$ to $W^{2n}$. We thus obtain a new symplectic manifold which we still call $W^{2n}$, having stable boundary $M^{2n-1}_2$. We then extend the moduli space of spheres to $W^{2n}$ together with the evaluation map. After choosing a generic and properly embedded path in $W^{2n}$ starting from the region where local uniqueness holds, we study the sub-moduli space of spheres which are constrained to intersect this path via the evaluation map, which has virtual dimension equal to 1. While, after taking the Gromov compactification, there could be multiply covered spheres in this moduli preventing transversality by standard methods, this is dealt with via the polyfold machinery of Hofer-Wysocki-Zehnder [21] (in the Gromov-Witten case). After introducing an abstract and generic multivalued perturbation to the Cauchy-Riemann equation, and appealing to the polyfold regularization of constrained moduli spaces as in [13], we obtain a 1-dimensional, oriented, weighted branched orbifold with non-empty boundary. In the case where $W^{2n}$ is compact (e.g. a semi-filling), this moduli space is also compact (and we call it a COWBOY). In the case where $W^{2n}$ has a noncompact end, this moduli space fails to be compact (and therefore is an OW-BOY), but its noncompactness is tractable, in the sense that it corresponds to spheres escaping down the noncompact end of $W^{2n}$. Moreover, we may choose the abstract perturbation away from the subregion where the spheres are known to be regular and local uniqueness holds.

For Theorem A, $W^{2n}$ is a hypothetical weak symplectic semi-filling. The boundary of the resulting 1-dimensional COWBOY corresponds to spheres in the constrained and perturbed moduli space which touch a boundary component of $W^{2n}$. The local uniqueness statement then yields that there is a unique curve in the
boundary of this moduli space which touches the boundary component $M^{2n-1}_{2n-1}$ (along the *weakly* pseudoconvex subregion), so that the curves in the other boundary components of the **COWBOY** necessarily touch other boundary components of $W^{2n}$ tangentially. But this is a contradiction, since we can take the almost complex structure so that the latter are *strictly* pseudoconvex. Observe that, while the spheres are not necessarily holomorphic (they are solutions to a perturbed Cauchy-Riemann equation), we can still appeal to pseudoconvexity for sufficiently small abstract perturbation, as follows easily from a Gromov compactness argument.

For Theorem B, we adapt the proof in [4]. Given a hypothetical weak contact-type embedding of $M^{2n-1}_{2n-1}$ into a closed symplectic manifold, we cut the latter along the former. The result is a weak symplectic cobordism $W^{2n}_{1}$ having $M^{2n-1}_{2n-1}$ both as positive and negative weakly dominated boundary components. As in [4], we glue infinite copies of $W^{2n}_{1}$ to itself at the negative end, by inductively identifying positive end with negative end. The result is a noncompact symplectic manifold $W^{2n}_{\infty}$ with weak boundary $M^{2n-1}_{1}$, to which we may apply the symplectic handle attachment construction described above. We get a 1-dimensional **OW-BOY**, with one boundary component. By construction, $W^{2n}_{\infty}$ is periodic, and has infinitely many copies of $M^{2n-1}_{1}$ sitting as weak contact-type hypersurfaces which can be made strictly pseudoconvex. Periodicity implies that the geometry of $W^{2n}_{\infty}$ is bounded, and so closed holomorphic curves with bounded energy have bounded diameter. But then the spheres in the **OW-BOY** cannot shoot all the way down the noncompact end, since otherwise this would contradict with pseudoconvexity. They also cannot stretch infinitely far down while staying at a bounded distance from the boundary of $W^{2n}_{\infty}$, because of the diameter bounds. This is a contradiction.

The arguments for Theorem C are a reformulation of those in [5]. We construct an exact symplectic cobordism from an arbitrary contact form to a specific Giroux form supported by the **IP** open book decomposition. We attach the symplectic handle, and study the resulting **OWL-BOY** of spheres. Then, Gromov compactness and the exactness away from the handle implies breaking at the negative end, resulting in the desired closed Reeb orbit.

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2. Background

In this section, we recall various facts about open book decompositions, Lefschetz fibrations, symplectic fillings, and almost complex manifolds as a preparation for setting up the language for iterated planarity of contact manifolds.
Unless otherwise indicated, throughout this paper, all contact structures are positive and co-oriented. Contact and symplectic manifolds will be oriented by their contact and symplectic structures, respectively.

**Open book decompositions.** Recall that an open book decomposition of an oriented \((2n+1)\)-dimensional manifold \(M\) is a pair \((B, \pi)\), where \(B\) is a codimension 2 submanifold of \(M\) with trivial normal bundle, and \(\pi : M - B \rightarrow S^1\) is a fiber bundle, such that \(\pi\) agrees with the angular coordinate \(\theta\) on the normal disk \(\mathbb{D}^2\) when restricted to a neighborhood \(B \times \mathbb{D}^2\) of \(B\). The closure of the fibers in the fibration are called pages and \(B\) is called the binding of the open book decomposition.

Alternatively, an abstract open book decomposition of a \((2n+1)\)-dimensional manifold \(M\) is a pair \((W, \varphi)\), where \(W\) is a compact \(2n\)-dimensional manifold with boundary, and \(\varphi : W \rightarrow W\) is a diffeomorphism which restricts to the identity on a neighborhood of \(\partial W\), such that

\[
M = W \cup (\mathbb{D}^2 \times \partial W)
\]

Here, we denote

\[
W_\varphi = [0, 1] \times W \big/ (0, z) \sim (1, \varphi(z)),
\]

the mapping torus of \(\varphi\). We glue the latter to \(\mathbb{D}^2 \times \partial W\) via the obvious identification:

\[
((\theta, p) \in \partial W_\varphi) \sim ((\theta, p) \in \partial \mathbb{D}^2 \times \partial W).
\]

The boundary of \(W_\varphi\) is given by

\[
\partial W_\varphi = [0, 1] \times \partial W \big/ (0, z) \sim (1, z),
\]

since \(\varphi(z) = z\) on \(\partial W\). We then call the map \(\varphi\) the monodromy and the submanifold \(W\) the page of the open book decomposition. We denote \(M = \text{OB}(W, \varphi)\).

Such an open book decomposition is said to support a contact structure \(\xi\) on \(M\) if it is the kernel of a contact form \(\lambda\) satisfying the following:

1. \(\lambda\) restricts to a contact form on the binding and
2. \(d\lambda\) is positively symplectic on the pages \(W\), and the orientation on \(B\) induced by the contact form agrees with the boundary orientation on \(B\) induced by the symplectic form on \(W\).

If these two conditions hold, then the open book decomposition \(\text{OB}(W, \varphi)\) is called a supporting open book for the contact manifold \((M, \xi)\) and the contact form \(\lambda\) (a Giroux form) is said to be adapted to the open book decomposition \(\text{OB}(W, \varphi)\).

In 1978, Lawson \cite{22} proved that every odd-dimensional manifold admits an open book decomposition. However, the question of whether there is an accompanying supported contact structure, and vice versa, remained open until early 2000s. In dimension three, the statement of this correspondence, also known as
Giroux correspondence [16], reads as follows: Let $M$ be a closed oriented 3-manifold. Then there is a one to one correspondence between oriented contact structures on $M$ up to isotopy and open book decompositions of $M$ up to positive stabilization.

Giroux correspondence plays a pivotal role in understanding symplectic filling properties and cobordisms of contact structures (e.g. [7, 10, 11, 15]), which lead to various topological applications.

There is also a version in higher-dimensions. Recall that a *Liouville domain* is a triple $(W, \omega, \lambda)$, where $(W, \omega = d\lambda)$ is a compact exact symplectic manifold with nonempty boundary, such that the *Liouville vector field* $V_{\lambda}$ on $W$, associated to the *Liouville form* $\lambda$ via the equation $\omega(V_{\lambda}, \cdot) = \lambda$, points transversely outwards along the boundary of $W$. Note that $\ker \lambda$ is a contact structure on the boundary of $W$. For this reason, we say that $(W, \omega)$ has *convex* boundary. Moreover, if $\varphi : W \to \mathbb{R}$ is a Morse function for which $V_{\lambda}$ is gradient-like and $\partial W$ is a regular level set of $\varphi$, then the quadruple $(W, \omega, \lambda, \varphi)$ is called a *Weinstein domain*.

The higher-dimensional correspondence due to Giroux [16] then reads as follows: Any exact symplectomorphism $\varphi$ of a Liouville domain $(W, \omega, \lambda)$ gives rise to a contact structure on the resulting open book decomposition with page $W$ and monodromy $\varphi$. Conversely, every closed contact manifold $(M, \xi)$ admits a supporting open book decomposition with Weinstein pages.

In fact, any symplectomorphism of a Liouville domain $W$ which is the identity near its boundary is isotopic to an exact symplectomorphism relative boundary (by a lemma of Giroux). The first statement of the correspondence then follows by an adaptation of Thurston-Winkelnkemper’s construction, whereas the second is much harder, and uses Donaldson’s almost holomorphic techniques. However, uniqueness up to positive stabilization is open in higher dimensions.

**Symplectic Lefschetz fibrations.** Let $E$ be a compact $2n$-dimensional manifold with corners, whose boundary is the union of a “horizontal boundary” $\partial_h E$ and a “vertical boundary” $\partial_v E$ meeting in a codimension 2 corner. Let $\omega = d\lambda$ be an exact symplectic form on $W$ such that both pieces of the boundary are convex. Now consider a proper smooth map $f : E \to \mathbb{D}^2$ with finitely many critical points $\text{Crit}(f)$, and denote a regular fiber by $F$. We then say that $f : E \to \mathbb{D}^2$ is an *exact symplectic Lefschetz fibration* if it satisfies the following properties:

1. (Lefschetz-type critical points)
   The critical points of $f$ are nondegenerate, isolated and belong to the interior of $W$. For any $p \in \text{Crit}(f)$, there are orientation preserving local complex coordinates $(z_1, \ldots, z_n)$ about $p$ on $E$ and $f(p)$ on $\mathbb{D}^2$ such that, with
respect to these coordinates, \( f \) is given by the complex map

\[
f(z_1, \ldots, z_n) = z_1^2 + \cdots + z_n^2.
\]

Moreover, there is at most one critical point in each fiber of \( f \).

(2) \textit{(Symplectic fibers)}

Denote by \( E_z \) the fiber \( f^{-1}(z) \) for any \( z \in \mathbb{D}^2 \). We require that the restriction of \( \omega|_{E_z \setminus \text{Crit}(f)} \) is symplectic, so that the boundary of each regular fiber is convex. That is, regular fibers of \( f \) carry the structure of an exact symplectic manifold with contact type boundary.

(3) \textit{(Conditions on the boundary)}

We require that

\[
\partial_v E = f^{-1}(\partial \mathbb{D}^2), \quad \partial_h E = \bigcup_{z \in \partial \mathbb{D}^2} \partial (f^{-1}(z)),
\]

and \( f|_{\partial_v E} : \partial_v E \to \partial \mathbb{D}^2 \) and \( f|_{\partial_h E} : \partial_h E \to \mathbb{D}^2 \) are surjective smooth fiber bundles. Moreover, there is a neighborhood \( N(\partial_h E) \) of \( \partial_h E \) such that the restriction map \( f|_{N(\partial_h E)} : N(\partial_h E) \to \mathbb{D}^2 \) is a product fibration \( \mathbb{D}^2 \times N(\partial F) \) where \( N(\partial F) \) denotes a neighborhood of \( \partial F \).

We note that the corners of \( E \) can be smoothed to make the resulting manifold into an honest Liouville domain \((W, \omega, \lambda)\).

\textbf{Iterated planar contact manifolds.} By a recent result of Giroux and Pardon \[17\], we know that any Weinstein domain admits a Weinstein Lefschetz fibration over \( \mathbb{D}^2 \) with Weinstein fibers. We can consider Weinstein Lefschetz fibrations as a special case of the symplectic Lefschetz fibrations on Liouville domains. Once we are given an exact symplectic Lefschetz fibration on a Weinstein domain, one can further construct an exact symplectic Lefschetz fibration on the codimension two Weinstein fiber, and iterate this process until the Liouville fiber is 4-dimensional. This idea leads us to the following definitions introduced in \[2\].

\textbf{Definition 2.1.} An \textit{iterated planar Lefschetz fibration} \( f : (W^{2n}, \omega) \to \mathbb{D}^2 \) on a \( 2n \)-dimensional Weinstein domain \((W^{2n}, \omega)\) is an exact symplectic Lefschetz fibration satisfying the following properties:

(1) There exists a sequence of exact symplectic Lefschetz fibrations \( f_i : (W^{2i}, \omega_i) \to \mathbb{D}^2 \) for \( i = 2, \ldots, n \) with \( f = f_n \).

(2) The total space \( (W^{2i}, \omega_i) \) of \( f_i \) is a regular fiber of \( f_{i+1} \), for \( i = 2, \ldots, n - 1 \).

(3) \( f_2 : (W^4, \omega_2) \to \mathbb{D}^2 \) is a planar Lefschetz fibration, i.e. the regular fiber of \( f_2 \) is a genus zero surface with nonempty boundary, which we denote by \( W^2 \).
For $n \geq 2$, the unit disk cotangent bundle $W^{2n} = \mathbb{D}^*S^n$ admits an iterated planar Lefschetz fibration where each regular fiber is $\mathbb{D}^*S^{n-1}$, and the Lefschetz fibration on $\mathbb{D}^*S^2$ is planar with fibers $\mathbb{D}^*S^1 = [0, 1] \times S^1$. Similarly, consider the $A_k$-singularity, which can be symplectically identified with 

$$\{(z_1, \ldots, z_n) \mid z_1^2 + \cdots + z_{n-1}^2 + z_{n+1} = 1\} \subset (\mathbb{C}^n, \omega_{std})$$

for $n \geq 3$ and $k \geq 2$. It is important to note that the $A_k$-singularity can be expressed as a plumbing of $k$ copies of $\mathbb{D}^*S^{n-1}$. Observe that each regular fiber of the Lefschetz fibration on the $A_k$-singularity, defined by the projection onto the last coordinate $z_n$, is $\mathbb{D}^*S^{n-1}$. This observation together with the existence of an iterated planar Lefschetz fibration on $\mathbb{D}^*S^{n-1}$ imply that the $A_k$-singularity admits an iterated planar Lefschetz fibration.

Here we remark that not every 4-manifold with nonempty boundary admits a planar Lefschetz fibration over $\mathbb{D}^2$. As a counterexample, consider $T^2 \times \mathbb{D}^2$. Assume to the contrary that $T^2 \times \mathbb{D}^2$ admits an iterated planar Lefschetz fibration. Then there must be a planar Stein fillable contact structure on the boundary of $T^2 \times \mathbb{D}^2$. Note that $\partial(T^2 \times \mathbb{D}^2) = T^3$, which admits a unique Stein fillable contact structure [9] and is known to be nonplanar [10]. Hence, $T^2 \times \mathbb{D}^2$ does not admit such a fibration.

If $f : W \to \mathbb{D}^2$ is an iterated planar Lefschetz fibration, then, after smoothing the corners, the boundary of $W$ inherits an open book decomposition whose pages are diffeomorphic to the regular fibers of $f$. The following definition is motivated by looking at the open book decomposition induced by the boundary restriction of an iterated planar Lefschetz fibration.

**Definition 2.2.** An iterated planar open book decomposition of a contact manifold $(M^{2n+1}, \xi)$ is an open book decomposition $\text{OB}(W, \varphi)$ whose page $W$ admits an iterated planar Lefschetz fibration.

Recall from the higher-dimensional correspondence [16] that every closed contact manifold admits a supporting open book decomposition with Weinstein pages. In what follows, we make use of this fact and iterated planar open book decompositions to define iterated planar contact manifolds.

**Definition 2.3.** For any $n > 1$, an iterated planar contact manifold $(M, \xi)$ is a $(2n+1)$-dimensional contact manifold supported by an open book decomposition whose Weinstein page admits an iterated planar Lefschetz fibration.

Consider the standard contact 5-sphere $(S^5, \xi_{st})$. We know that it is supported by an open book decomposition with planar binding $(S^3, \xi_{st})$ and exact symplectic page $(D^4, \omega_{st})$ whose monodromy is the identity. Hence, $(S^5, \xi_{st})$ is an iterated planar contact manifold.
Almost complex and stable Hamiltonian structures. Let \((M, \xi)\) be a \((2n + 1)\)-dimensional contact manifold and \(\lambda\) be a nondegenerate contact form on \(M\). Let \(W\) be a \(2n\)-dimensional manifold. An almost complex structure \(J\) on \(W\) is a linear isomorphism \(J: TW \to TW\) such that \(J^2 = -1\). \((W, J)\) is then called an almost complex manifold.

When \(W\) is equipped with a symplectic form, one has to examine the compatibility of symplectic form with almost complex structure. An almost complex structure \(J\) on a symplectic manifold \((W, \omega)\) is said to be compatible with \(\omega\) (or \(\omega\)-compatible) if \(\omega(Ju, Jv) = \omega(u, v)\), for any \(u, v \in TW\), and \(\omega(v, Jv) > 0\) for any nonzero vector \(v \in TW\).

It is well-known that the set of all almost complex structures on a symplectic manifold \((W, \omega)\) is nonempty and contractible. Hence, the tangent bundle \(TW\) can be considered as a complex vector bundle, uniquely up to homotopy.

Let \((F, j)\) be a Riemann surface and \((W, J)\) be an almost complex manifold. A \(J\)-holomorphic (or pseudoholomorphic) curve is then a smooth map \(u: (F, j) \to (W, J)\) such that it satisfies the following non-linear Cauchy-Riemann equation at every point:

\[
du \circ j = J \circ du
\]

or, equivalently, \(\bar{\partial}_J u = 0\).

We now define a special class of compatible almost complex structures on the symplectization \((\mathbb{R} \times M, d(e^s\lambda))\). Since \(\mathbb{R} \times M\) is noncompact, an almost complex structure \(J\) on \(\mathbb{R} \times M\) needs to satisfy certain conditions near infinity in order to control the behavior of the space of pseudoholomorphic curves far away. Note that the symplectization of \(M\) inherits a natural splitting of the tangent bundle

\[
T(\mathbb{R} \times M) = \mathbb{R} \langle \partial_s \rangle \oplus \mathbb{R} \langle R_\lambda \rangle \oplus \xi,
\]

where \(\partial_s\) is the unit vector in the \(\mathbb{R}\)-direction.

An almost complex structure \(J\) on \(\mathbb{R} \times M\) is then called \(\lambda\)-compatible if

1. \(J\) is \(\mathbb{R}\)-invariant,
2. \(J(\partial_s) = R_\lambda\) and \(J(-R_\lambda) = \partial_s\),
3. \(J(\xi) = \xi\),
4. \(J|_\xi\) is \(d\lambda\)-compatible.

Let \((M, \xi)\) be a contact type hypersurface in a Weinstein domain \((W, \omega = d\lambda)\). An \(\omega\)-compatible almost complex structure \(J\) on \(W\) is called \((M, \lambda)\)-compatible if \(J\)
restricts to a $\lambda|_M$-compatible almost complex structure on a collar neighborhood of $M$.

A stable Hamiltonian structure $\mathcal{H}$ on a $(2n+1)$-dimensional oriented manifold $M$ is a pair $(\Omega, \Lambda)$ consisting of a closed 2-form $\Omega$ and 1-form $\Lambda$ defined on $M$ with the following properties:

1. $\ker \Omega \subset \ker d\Lambda$
2. $\Lambda \wedge \Omega^n > 0$

In dimension three, the condition (1) can be equivalently written as $d\Lambda = g\Omega$ where $g : M \to \mathbb{R}$ is a smooth function. Observe also that the condition (2) is equivalent to $\Omega|_\xi$ is nondegenerate, where $\xi = \ker \Lambda$ is a co-oriented hyperplane distribution. In other words, $(\xi, \Omega|_\xi)$ is a symplectic vector bundle. We call condition (1) the stabilizing condition, whereas condition (2), the framing condition.

The Reeb vector field $R$ associated to $\mathcal{H}$ is defined by the equations $\Omega(R, \cdot) = 0$, $\Lambda(R) = 1$. Observe that if $\lambda$ is a contact form, then $(d\lambda, \lambda)$ is a stable Hamiltonian structure, which we say is a contact stable Hamiltonian structure. On a neighborhood $(-\epsilon, \epsilon) \times M$ of $M$, for $\epsilon > 0$ sufficiently small, $d(s\lambda) + \Omega$ is a symplectic form where $s \in (-\epsilon, \epsilon)$. One can generalize this further to $\mathbb{R} \times M$ by letting the symplectic form on $\mathbb{R} \times M$ be $\omega_\phi = d(\phi(s)\lambda) + \Omega$ where $\phi : \mathbb{R} \to (-\epsilon, \epsilon)$ is a strictly increasing function. The symplectization of $(M, \mathcal{H})$ is then symplectomorphic to $(\mathbb{R} \times M, \omega_\phi)$. Moreover, we have an analogous notion of almost complex structures compatible with a given stable Hamiltonian structure.

**Symplectic fillings and pseudoconvexity.** In this section, we recall some notions of symplectic fillings of contact manifolds.

We start by recalling some notions from the world of almost complex manifolds. Let $(W, J)$ be an almost complex manifold with boundary $M = \partial W$. We then say that $(W, J)$ has strictly pseudoconvex (or sometimes called strictly $J$-convex) boundary $(M, \xi)$ if $M$ is a regular level set of a $J$-convex function whose gradient points outward at the boundary, i.e. the subbundle of complex tangencies $TM \cap J(TM) \subset TM$ is a positive contact structure whose conformal symplectic structure tames the restriction of $J$. Because $M$ is a codimension one submanifold of $W$, there exists a unique hyperplane field $\xi$ of complex tangencies in $TM$. As $\xi$ is oriented as a complex bundle, there is a contact form $\lambda$ on $M$. Equivalently, we call $M$ strictly pseudoconvex if $d\lambda(\cdot, J\cdot)|_\xi > 0$. Moreover, $M$ is called weakly pseudoconvex if $d\lambda(\cdot, J\cdot)|_\xi \geq 0$.

We will now recall the notions from the symplectic side.
A contact 3-manifold \((M^3, \xi)\) is \textit{weakly fillable} if it is the smooth boundary of a symplectic 4-manifold \((W^4, \omega)\) such that \(\omega|_\xi > 0\). We then call \((W^4, \omega)\) a \textit{weak-filling} of \((M^3, \xi)\). A contact manifold \((M^{2n-1}, \xi)\) is \textit{strongly fillable} if there exists a weak filling \((W^{2n}, \omega)\) such that one can find a Liouville vector field \(Z\) in a neighborhood of \(M\). We then call \(W\) \textit{strong symplectic filling} (or \textit{convex filling}) of \((M, \xi)\). One can also think of strong symplectic fillings as a strong symplectic cobordism from the empty manifold to \((M, \xi)\). Notice that the above definition of weak-filling is strictly for dimension three. One can generalize this idea to higher dimensions [26] by further requiring that \(\omega + \tau d\lambda|_\xi\) is symplectic for every \(\tau \geq 0\), for one (and hence every) choice of contact form \(\lambda\). One says that \(M\) is weakly dominated by \(\omega\). From [26], pseudoconvexity and weak domination are equivalent notions in all dimensions.

In dimension 3, there are several notable results concerning weak filling properties of contact manifolds. Eliashberg [7] and Etnyre [11], independently, show that any weak filling of a contact 3-manifold can be symplectically embedded into a closed symplectic 4-manifold. This implies that every contact 3-manifold can be symplectically capped.

Let \((W, \omega)\) be a symplectic manifold with more than one boundary component, one of which is \((M, \xi)\). If all the boundary components of \(W\) are weakly dominated by \(\omega\) then \((W, \omega)\) is called a \textit{weak semi-filling} of \((M, \xi)\), and \((M, \xi)\) is called \textit{weakly semi-fillable}. Semi-fillings of contact manifolds can always be turned into honest symplectic fillings if other boundary components can be symplectically capped off, which can always be done in dimension three.

### 3. A Handle Attachment

We consider an IP contact manifold \(M^3_{2n-1} = \text{OB}(W^2_{2n-2}, \varphi)\) with an IP open book decomposition. We will construct a symplectic cobordism \((C^{2n}, \omega_{C^{2n}})\) as described in the introduction. The pictorial reference for all of this construction is given in Figure 1, which contains most of the relevant information, and which the reader will be encouraged to consult in multiple instances as new objects are introduced.

**Base case** \(n = 2\). In the case where \(M^3_1\) is a planar contact 3-manifold, we make use of a construction originally due to Eliashberg [7], and the further adaptations in [25] used to define the notion of a \textit{spine removal surgery}.

By doing 0-surgery along the binding \(M^3_1\) of the planar open book decomposition in \(M^3_1 = \text{OB}(W^2, \varphi)\) (with respect to the page framing), we obtain the manifold \(M^3_2 := S^1 \times S^2\), the unique \(S^2\)-fibration over \(S^1\). Here, we have used that the pages of the original open book decomposition have genus zero. The resulting 4-dimensional cobordism \(C^4\) carries a symplectic form \(\omega_{C^4}\), which satisfies:
Figure 1. A diagrammatic picture of the cobordism $C^{2n}$. In order to obtain $C^{2n}$ from the above diagram, one needs to rotate the output manifold in the upper left around the $S^1$-direction of the disk $D^2$ in the base, and glue it to itself by the action of the monodromy $\varphi$ (as indicated by the symbol $\varphi$ in $\partial D^2$).

- $\omega_{C^3} = d(e^t \alpha)$ in a collar neighborhood $[0, \delta) \times M^3_1$, where $\alpha$ is a Giroux form for the open book decomposition in $M^3_1$.
- $\omega_{C^3} = d(e^s d\theta) + \omega_{S^2}$ in a collar neighborhood $(-\epsilon, 0] \times S^1 \times S^2$, where $\epsilon > 0$, $s \in (-\epsilon, 0]$, and $\omega_{S^2}$ is an area form in $S^2$.

In other words, the negative end $M^3_1$ is strongly concave, and the positive end $M^3_2$ is stable. The vector field $\partial_s$ is stabilizing, and $M^3_2$ carries the stable Hamiltonian structure

$$\mathcal{H}^3 := (i_{\partial_s} \omega|_{M^3_2}, \omega|_{M^3_2}) = (d\theta, \omega_{S^2})$$

Observe that its Reeb vector field is $\partial_\theta$, and its kernel $\ker d\theta = TS^2$ gives a foliation by spheres.

Given a Weinstein manifold $(W^4_1, \lambda)$ with contact-type boundary $(M^3_1, \alpha)$ (e.g. the page of the open book for an IP contact manifold $M^3_1$), we may attach $C^4$ on top of $W^4_1$ to obtain a symplectic manifold $(W^4_2, \omega_2) := (W^4_1, d\lambda) \cup_{M^3_1} (C^4, \omega_{C^4})$ with
stable boundary \((M^3_2, \mathcal{H}^3)\). We will use this 4-dimensional construction as input for our handle construction in dimension 6.

**Inductive step.** In arbitrary odd dimension \(2n - 1 \geq 5\), the inductive input is as follows:

- An IP contact manifold \(M^{2n-1} = \text{OB}(W^{2n-2}_1, \varphi)\), together with an IP open book decomposition.
- A symplectic cobordism \((C^{2n-2}, \omega_{C^{2n-2}})\) with concave contact-type boundary \((M^{2n-3}_1, \alpha) = \partial(W^{2n-2}_1, d\lambda)\), where \(\alpha\) is a Giroux form for the open book decomposition induced by the Lefschetz fibration structure in \(W^{2n-2}_1\); and stable positive boundary \((M^{2n-3}_2, \mathcal{H}^{2n-3} = (\Lambda', \Omega'))\).

Observe that \(M^{2n-3}_2 = \text{OB}(W^{2n-4}_1, \varphi_1)\), where \(W^{2n-4}_1\) is the regular fiber of the Lefschetz fibration in \(W^{2n-2}_1\), and \(\varphi_1\) is the product of positive Dehn twists along the vanishing cycles in \(W^{2n-4}_1\). This means that \(M^{2n-3}_2\) is again an IP contact manifold, and so we may assume that the symplectic cobordism \((C^{2n-2}, \omega_{C^{2n-2}})\) was constructed in the previous step.

We will denote \((W^{2n-2}_2, \omega_2) = (W^{2n-2}_1, d\lambda) \cup (C^{2n-2}, \omega_{C^{2n-2}})\), where the gluing takes place along a collar of the form \((-\delta, \delta) \times M^{2n-3}_1, d(e^t \alpha)\), for some \(\delta > 0\). The result is a symplectic manifold with stable boundary \((M^{2n-3}_2, \mathcal{H}^{2n-3})\) (See Figure 1). The monodromy \(\varphi\) extends to \(W^{2n-2}_2\) by the identity along \(C^{2n-2}\). We will also rename \((W^{2n-2}_1, d\lambda)\), replacing it with the slightly enlarged copy \((W^{2n-2}_1, d\lambda) \cup ([0, \delta] \times M^{2n-3}_1, d(e^t \alpha))\), without changing notation.

**Generalized mapping tori and Giroux forms.** Let \(\lambda\) be a Liouville form for a representative of the natural homotopy class of Weinstein structures associated to \(W^{2n-2}_1\), such that \(\varphi\) is a symplectomorphism with respect to \(d\lambda\). We ask also that \(\lambda = e^t \alpha\) in the collar neighborhood \((-\delta, 0] \times M^{2n-3}_1, \delta > 0\) as before, where \(t\) is the coordinate in the first factor, and \(\alpha\) is the given Giroux form in \(M^{2n-3}_1\). We denote by \(V_\lambda\) the Liouville vector field associated to \(d\lambda\), and we assume that \(\varphi\) is the identity in \((-\delta, 0] \times M^{2n-3}_1\). By a lemma of Giroux, up to isotopy we may assume that \(\varphi\) is an exact symplectomorphism, i.e. we have

\[
\varphi^* \lambda = \lambda - dh,
\]
for some positive smooth function \( h : W_1^{2n-2} \to \mathbb{R} \), which is constant equal to 1 in the \( \delta \)-collar neighborhood of \( M_1^{2n-3} \). We then write

\[
M_1^{2n-1} = M_1^{2n-3} \times \mathbb{D}^2 \cup MT(W_1^{2n-2}, \varphi),
\]

where

\[
MT(W_1^{2n-2}, \varphi) := W_1^{2n-2} \times \mathbb{R} / (x, 0) \sim (\varphi(x), h(x))
\]

Observe that, by construction, the “generalized” mapping torus \( MT(W_1^{2n-2}, \varphi) \) carries a well-defined contact form \( \alpha_1 := \lambda + d\phi \), where \( \phi \) is the coordinate in the \( \mathbb{R} \)-factor. The following modification will also be useful: For any \( r \in \mathbb{R}^+ \), we may define

\[
MT_r(W_1^{2n-2}, \varphi) = W_1^{2n-2} \times \mathbb{R} / (x, 0) \sim (\varphi(x), h(x)/r).
\]

By construction, \( MT_r(W_1^{2n-2}, \varphi) \) carries a well-defined contact form \( \alpha_r := \lambda + rd\phi \) (and its diffeomorphism type is clearly \( r \)-independent). This construction is originally due to Giroux, and this 1-form is the restriction of a Giroux form to the generalized mapping torus \( MT_r(W_1^{2n-2}, \varphi) \). We also have the following model for a Giroux form along \( M_1^{2n-3} \times \mathbb{D}^2 \): If \((r, \theta)\) are polar coordinates on \( \mathbb{D}^2 \), we may take \( \alpha_1 = \sigma + \alpha \). Here, \( \alpha \) is the Giroux form on \( M_1^{2n-3} \), and \( \sigma \) is a Liouville form on \( \mathbb{D}^2 \), chosen so that \( \sigma = rd\theta \) near \( r = 1 \). One needs to smoothen the two expressions of \( \alpha_1 \) so that they glue together smoothly. The complete construction of \( \alpha_1 \), including how to actually glue the two expressions together, will actually follow from our construction below. Our choice of smoothening of the corner \( (M_1^{2n-3} \times \mathbb{D}^2) \cap MT_1(W_1^{2n-2}, \varphi) \) will implicitly do the gluing. The result will be a Giroux form \( \alpha_1 \) on \( M_1^{2n-1} \) (so that the induced contact structure is supported by the \( \mathbb{I} \) open book decomposition in \( M_1^{2n-1} \)), which is modelled by the above expressions on each separate piece.

We denote

\[
M_{1,S}^{2n-1} := M_1^{2n-3} \times \mathbb{D}^2,
\]

and call it the spine of \( M_1^{2n-1} \), and

\[
M_{1,P}^{2n-1} := MT_r(W_1^{2n-2}, \varphi),
\]

and call it the \( r \)-paper.

We refer to the 1-paper \( M_{1,P}^{2n-1} \) simply as the paper of \( M_1^{2n-1} \) (the 2-paper will be contained in the paper of the resulting manifold \( M_2^{2n-1} \)).

**A “trivial” symplectic cobordism.** We will now define an open symplectic manifold \((E, \omega_E)\) having \( M_1^{2n-1} \) as a contact-type hypersurface. Symplectically, \((E, \omega_E)\) will be symplectomorphic to an open piece of the symplectization of the contact manifold \((M_1^{2n-1}, \alpha_1)\). What follows is an adaptation of a construction in [25] (see also [28]).
We enlarge the spine $M_{1,S}^{2n-1}$ to
\[ \hat{M}_{1,S}^{2n-1} := M_{1}^{2n-3} \times \mathbb{D}^2(2), \]
where $\mathbb{D}^2(2)$ denotes the 2-disk of radius 2 in $\mathbb{C}$, and we take polar coordinates $re^{i\phi} \in \mathbb{D}^2(2)$. We pick a Liouville form $\sigma$ in $\mathbb{D}^2(2)$, so that $\sigma = rd\phi$ for $r \in (1 - \delta, 2]$. We denote the associated Liouville vector field by $V$, so that $V = r \partial_r$ in the collar $(1 - \delta, 2] \times S^1 \subset \mathbb{D}^2(2)$.

Again for the same $\delta > 0$ chosen before, we define
\[ \mathcal{N}_P^r := (-\delta, \delta) \times M_{1}^{2n-3} \times \mathbb{R} / \mathbb{Z} \subset M_{r,P}^{2n-1}, \]
which is a collar neighborhood of $\partial M_{r,P}^{2n-1}$. We take coordinates $(t, b, \phi_r) \in \mathcal{N}_P^r$, and we define $\phi := r\phi_r \in S^1$. We also define
\[ \mathcal{N}_S := M_{1}^{2n-3} \times (1 - \delta, 2] \times S^1 \subset \hat{M}_{1,S}^{2n-1}, \]
with coordinates $(b, r, \phi) \in \mathcal{N}_S$, and
\[ M_P^{2n} := \{(r, w) : r \in (1 - \delta, 2], w \in M_{r,P}^{2n-1}\} = \bigcup_{r \in (1 - \delta, 2]} M_{r,P}^{2n-1}. \]
Moreover, we have a collar
\[ \mathcal{N}_P := \{(r, w) : r \in (1 - \delta, 2], w \in \mathcal{N}_P^r\} = \bigcup_{r \in (1 - \delta, 2]} \mathcal{N}_P^r \subset M_P^{2n-1}. \]

Let $E$ be the open manifold
\[ E = (-\delta, \delta] \times \hat{M}_{1,S}^{2n-1} \bigcup M_P^{2n} / \sim, \]
where we identify a tuple $(t, b, r, \phi) \in (-\delta, \delta] \times \mathcal{N}_S \subset (-\delta, \delta] \times \hat{M}_{1,S}^{2n-1}$, with $(r, t, b, \phi_r = \phi/r) \in \mathcal{N}_P \subset M_P^{2n}$ (see, again, Figure 1).

Observe that $E$ has boundary and corners, and contains a copy of $M_1^{2n-1}$ as a hypersurface (with corners), depicted in blue in Figure 1. Indeed, we have
\[ M_1^{2n-1} \cong \{0\} \times M_{1,S}^{2n-1} \bigcup M_{1,P}^{2n-1} \subset E, \]
and its corner is $(\{0\} \times M_{1,S}^{2n-1}) \cap M_{1,P}^{2n-1}$. We can always smoothen the latter, and we shall make this smoothening explicit below. Moreover, there is a global and well-defined coordinate $r$ on $E$, as well as a coordinate $t$ (which is not globally defined). We denote by $E(t)$ the region of $E$ along which $t$ is defined. We also have a symplectic “fibration” $\pi : E \to \mathbb{D}^2(2)$, whose fibers change topological type. For $r > 1 - \delta$, the fiber over $(r, \phi)$ is $(W_{1}^{2n-2}, d\lambda)$, and for $r \leq 1 - \delta$, it is a copy of the collar neighborhood $((\delta, \delta] \times M_1^{2n-3}, d(e^t\alpha))$. 


The boundary of $E$ can be written as

$$\partial E = \partial_h E \cup \partial_v E,$$

where

$$\partial_h E := \{ t = \delta \} = \{ \delta \} \times \tilde{M}_{1,S}^{2n-1}$$

$$\partial_v E := \{ r = 2 \} = M_{2,P}^{2n-1}.$$

The corner of $E$ is then $\partial_h E \cap \partial_v E = \{ t = \delta, r = 2 \}$.

We now construct a symplectic form $\omega_E$ in $E$. Observe that we have a well-defined 1-form $\lambda_E := \lambda + \sigma$ on $E$, and it is straightforward to check that it is actually Liouville. Then $\omega_E := d\lambda_E$ is symplectic. Denote by $V_E$ the associated Liouville vector field, which is nothing else than $\omega_E$. Observe that $V_E$ is manifestly positively transverse to the hypersurface $M_1^{2n-1} \subset E$, away from its corner. Along the region $\{ r > 1 - \delta \} \cap E(t)$, which contains the corner of $E$ and that of $M_1^{2n-1} \subset E$, we have $\lambda_E = e^t \alpha + r \partial \phi$ and so $V_E = \partial_t + r \partial_\phi$. This means that $V_E$ will be positively transverse to any reasonable smoothening inside $E$ of the corner of $M_1^{2n-1}$. For example, we may choose smoothening functions $F, G : (-\delta, \delta) \to (-\delta, 0]$ satisfying:

$$\begin{cases}
(F(\rho), G(\rho)) = (\rho, 0), & \text{for } \rho \leq -\delta/3 \\
G'(\rho) < 0, & \text{for } \rho > -\delta/3 \\
F'(\rho) > 0, & \text{for } \rho < \delta/3 \\
(F(\rho), G(\rho)) = (0, -\rho), & \text{for } \rho \geq \delta/3
\end{cases}$$

And then we may replace the region $M_1^{2n-1} \cap \{ t \in (-\delta, 0], r \in (1 - \delta, 1] \}$, containing the corner of $M_1^{2n-1}$, with the smoothened corner

$$M_{1,C}^{2n-1} := \{ (r = 1 + F(\rho), t = G(\rho), b, \phi_1 + F(\rho) = \phi/(1 + F(\rho))) : (\rho, b, \phi) \in (-\delta, \delta) \times M_1^{2n-3} \times S^1 \} \quad (3.1)$$

See Figure 1. We then obtain a hypersurface of the form

$$\tilde{M}_1^{2n-1} = \tilde{M}_{1,S}^{2n-1} \cup M_{1,C}^{2n-1} \cup \tilde{M}_{1,P}^{2n-1},$$

where

$$\tilde{M}_{1,S}^{2n-1} = M_{1,S}^{2n-1} \setminus N_S = M_1^{2n-3} \times \mathbb{D}^2(1 - \delta),$$

and

$$\tilde{M}_{1,P}^{2n-1} = M_{1,P}^{2n-1} \setminus N_P.$$

Clearly, we have that

$$\tilde{M}_1^{2n-1} \cong M_1^{2n-1},$$

and so we will drop the tilde from all the notation. Then, $V_E$ is positively transverse to $M_1^{2n-1}$. It follows that $M_1^{2n-1}$ is indeed a contact-type hypersurface, inheriting
the contact form $\alpha_1 \coloneqq \lambda_E |_{M_1^{2n-1}}$. Our construction actually implies that $\alpha_1$ is a Giroux form for $M_1^{2n-1}$.

The handle. We now construct a handle, which we will attach on top of the manifold $E$. Topologically, this handle is of the form $C^{2n-2} \times \mathbb{D}^2(2)$, and we attach it on top of $E$ by the obvious identification (see Figure 1). Symplectically, we endow it with the 2-form $\omega_{C^{2n-2}} + d\sigma$, which is indeed symplectic, and by construction this glues smoothly to $\omega_E$. So we get a symplectic cobordism

$$(C^{2n}, \omega_{C^{2n}}) \coloneqq (E, \omega_E) \cup (C^{2n-2} \times \mathbb{D}^2(2), \omega_{C^{2n-2}} + d\sigma).$$

This cobordism contains a collar

$$((\epsilon, 0] \times M_2^{2n-1} \times \mathbb{D}^2(2), d(e^s\Lambda') + \Omega' + d\sigma),$$

and again has a corner, corresponding to $\{s = 0\} \cap \{r = 2\}$ in this collar, where $s \in (-\epsilon, 0]$. We smoothen this corner similarly as before, where we replace $\delta$ by $\epsilon$, and we set $r = 2 + F(\tau), s = G(\tau)$. We rename $C^{2n}$, replacing it with its smoothened version, and where we also remove the region of $E$ which is identified with the negative symplectization of the hypersurface $M_1^{2n-2}$ (this version of $C^{2n}$ is depicted in red in Figure 1). Then $C^{2n}$ now has $M_1^{2n-2}$ as a concave contact-type boundary component, and its positive boundary component can be written as

$$M_2^{2n-1} = M_{2,S}^{2n-1} \cup M_{2,C}^{2n-1} \cup \tilde{M}_{2,P}^{2n-1}.$$

Here we have the following:

1. $M_{2,S}^{2n-1} = M_2^{2n-3} \times \mathbb{D}^2(2 - \epsilon)$ is the spine of $M_2^{2n-1}$,
2. $M_{2,C}^{2n-1} \cong (-\epsilon, \epsilon) \times M_2^{2n-3} \times S^1$ is its smoothened corner, and
3. if $W_{2,n-2}^{2n-2} := W_2^{2n-3} \setminus ((-\epsilon, 0] \times M_2^{2n-1})$ and $C_{2,n-2}^{2n-2} := C^{2n-2} \setminus ((-\epsilon, 0] \times M_2^{2n-1})$,
   then $\tilde{M}_{2,P}^{2n-1} = MT_2(W_{2,n-2}^{2n-2}, \varphi) = M_{2,P}^{2n-1} \cup (C_{2,n-2}^{2n-2} \times S^1)$ is its paper.

Topologically, this is just an open book decomposition with page $W_{2,n-2}$ and monodromy $\varphi$. But, since $W_{2,n-2}$ is not Liouville, $M_2^{2n-1}$ does not carry any obvious contact structure supported by this open book decomposition. However, it does carry a stable Hamiltonian structure, constructed below. We call this decomposition a topological open book decomposition, and denote it by $M_2^{2n-1} = \text{TOB}(W_2^{2n-2}, \varphi)$.

In summary, we have

$$\partial C^{2n} = -\text{OB}(W_1^{2n-2}, \varphi) \sqcup \text{TOB}(W_2^{2n-2}, \varphi).$$
The stable Hamiltonian structure. We now make the boundary component \( M_2^{2n-1} \subset \partial C^{2n} \) a stable one. We write down a vector field \( X \), which is Liouville near \( M_1^{2n-1} \), and stabilizing near \( M_2^{2n-1} \), as follows.

Choose bump functions

\[
\beta_1 : (-\epsilon, 0] \to [0, 1], \\
\beta_0 : [0, \delta) \to [0, 1]
\]
satisfying:

- \( \beta_1 \equiv 0 \) near \( s = -\epsilon \), \( \beta_1 \equiv 1 \) near \( s = 0 \).
- \( \beta_0 \equiv 0 \) near \( t = \delta \), \( \beta_0 \equiv 1 \) near \( t = 0 \).

Denote by \( C^{2n}(s) \) the region of \( C^{2n} \) in which the coordinate \( s \) is defined, and similarly denote \( C^{2n}(t) \).

We then define the vector field \( X \) on \( C^{2n} \) by

\[
X = \begin{cases}
\beta_1(s) \partial_s + V_\sigma, & \text{along } C^{2n}(s) \\
V_\sigma, & \text{along } C^{2n}(E \cup C^{2n}(s)) \\
\beta_0(t) \partial_t + V_\sigma, & \text{along } C^{2n}(t) \\
V_\lambda + r \partial_r, & \text{along } M_{2,P}^{2n} \setminus C^{2n}(t)
\end{cases}
\tag{3.2}
\]

Then \( X \) is transverse to \( \partial C^{2n} \), negatively at \( -M_2^{2n-1} \), along which is Liouville, and positively at \( M_2^{2n-1} \). It is straightforward to check that \( X \) actually stabilizes \( M_2^{2n-1} \), and it is actually Liouville along \( M_{2,P}^{2n-1} \subset \hat{M}_{2,P}^{2n-1} \subset M_2^{2n-1} \). We obtain a stable Hamiltonian structure

\[
\mathcal{H}^{2n-1} = (\Lambda, \Omega) := (i_X \omega_{C^{2n}}|_{M_2^{2n-1}}, \omega_{C^{2n}}|_{M_2^{2n-1}}),
\]

which is contact along \( M_{2,P}^{2n-1} \).

This finishes the inductive construction.

Remark 3.1. Let us write explicit expressions for \( \mathcal{H}^{2n-1} \).

Along \( M_{2,P}^{2n-1} \), we have

\[
\mathcal{H}^{2n-1}|_{M_{2,P}^{2n-1}} = (\lambda + 2d\phi, d\lambda), \tag{3.3}
\]

and we clearly see in this expression that it is contact along this region.

Along the collar \((-\delta, \delta) \times M_1^{2n-1} \times S^1 \subset \hat{M}_{2,P}^{2n-1} \), we obtain

\[
\mathcal{H}^{2n-1}|_{(-\delta, \delta) \times M_1^{2n-1} \times S^1} = (\beta_0(t)e^t\alpha + 2d\phi, d(e^t\alpha)). \tag{3.4}
\]
Along $C_ε^{2n-2} \times S^1 \subset \hat{M}_{2,P}^{2n-1}$, we get

$$\mathcal{H}^{2n-1}|_{C_ε^{2n-2} \times S^1} = (2d\phi, \omega_{C^{2n-2}}).$$

(3.5)

Along the smoothened corner $M_{2,C}^{2n-1}$, we have

$$\mathcal{H}^{2n-1}|_{M_{2,C}^{2n-1}} = (\epsilon^{G(\tau)} \beta_1(G(\tau)) \Lambda' + (2 + F(\tau))d\phi, d(\epsilon^{G(\tau)} \Lambda') + \Omega' + F'(\tau) d\tau \wedge d\phi).$$

(3.6)

Finally, along $M_{2,S}^{2n-1}$, we see that

$$\mathcal{H}^{2n-1}|_{M_{2,S}^{2n-1}} = (\Lambda' + \sigma, \Omega' + d\sigma).$$

(3.7)

Observe that the expression (3.7) is contact along the regions where $\mathcal{H}^{2n-3} = (\Lambda', \Omega')$ is contact. For example, assuming $\mathcal{H}^{2n-3}$ is the stable Hamiltonian structure arising from this construction in the previous step, this is the case for $M_{2,P}^{2n-3} \times \mathbb{D}^2(2 - \epsilon) \subset M_{2,S}^{2n-1}$.

4. Modification for weak fillings

In this section, we modify the symplectic cobordism $C^{2n}$ so that we may glue it on top of an arbitrary symplectic manifold $(W^{2n}, \omega)$ with a weak boundary component $M_{1}^{2n-1} \subset \partial W^{2n}$, an IP contact manifold. What follows is an adaptation of a construction in [25].

We assume that we are given a closed 2-form $\omega$ in $M_{1}^{2n-1}$, such that $\omega|_\xi > 0$, where $\xi = \ker \alpha_1$ is the contact structure in $M_{1}^{2n-1}$ associated to the Giroux form $\alpha_1$, and supported by the IP open book decomposition. The 2-form $\omega$ takes the role of $\omega|_{M_{1}^{2n-1}}$ in the presence of $(W^{2n}, \omega)$.

By induction, we can arrange a closed 2-form $\eta$ in $M_{1}^{2n-1}$ such that:

1. $[\eta] = [\omega] \in H^2_{dR}(M_{1}^{2n-1})$.
2. $\eta$ is a pullback of a closed 2-form $\eta'$ in the binding $M_{1}^{2n-3}$ along $M_{1,S}^{2n-1} \cup M_{1,C}^{2n-1} \cong M_{1}^{2n-3} \times \mathbb{D}^2$, such that, recursively, $\eta'$ also satisfies conditions (1) and (2) for dimension $2n - 3$ (replacing $\omega$ by $\omega|_{M_{1}^{2n-3}}$ in (1)).

The second condition follows from contractibility of $\mathbb{D}^2$, and yields, in particular, that $\eta$ is independent on $\phi \in S^1$ along $M_{1,S}^{2n-1} \cup M_{1,C}^{2n-1}$. Observe that, because of dimensional reasons, $\eta' \equiv 0$ in the base case $n = 2$, and (2) is automatic.

4.1. Cohomological extension. We now wish to extend $\eta$ to a closed 2-form to $C^{2n}$. We do this, again, by induction. We will show that we can always extend closed 2-forms in $M_{1}^{2n-1}$ satisfying condition (2) above, to suitable closed 2-forms in $C^{2n}$.
Base case $n = 2$. The 4-dimensional cobordism $C^4$ is a result of attaching a collection of 2-handles to the planar contact 3-manifold $M_3$. In particular, we have $H_3(C^4, M_3; \mathbb{R}) = 0$, since there are no 3-handles, and it follows from the long exact sequence of the pair $(C^4, M_3)$ that the map $H_2(M_3; \mathbb{R}) \to H_2(C^4; \mathbb{R})$ is injective. By duality (over the field $\mathbb{R}$), the map $H_2(C^4; \mathbb{R}) \to H_2(M_3; \mathbb{R})$ is surjective. In terms of de Rham cohomology, this means that we can always extend closed 2-forms in $M_3$ to closed 2-forms in $C^4$. Moreover, by topological reasons, i.e. contractibility of the interval, we can choose the extension so that it is independent on the coordinates $s$ and $t$ along the collars where each are defined (meaning independence on both coefficients and differentials). This implies that $\eta$ is a 2-form in $S^1 \times S^2$, and, since $H_2^d(S^1 \times S^2) = H_2^d(S^2) \cong \mathbb{R}$, $\eta$ can be taken to be a constant multiple of $\omega_{S^2}$ near $M_3 = S^1 \times S^2$.

Inductive step. We assume by induction that we can always extend closed 2-forms on $M_1$ satisfying condition (2) (in dimension $2n-3$) to closed 2-forms on $C^{2n-2}$. Now, take $\eta$ a closed 2-form in $M_1$ satisfying (2). First, we observe that we can always extend it to the portion $M_P = \bigcup_{r \in [1,2]} M_P \cong M_1 \times [1,2] \subset C^{2n}$ in the obvious way, i.e. via pull-back, so that it is $r$-independent. The rest of $C^{2n}$ is diffeomorphic to $C^{2n-2} \times D^2(2)$. Identify $\eta$ with a closed 2-form $\eta'$ in $M_1$ along $M_1 \times D^2(2) \subset C^{2n}$ which also satisfies (2), and, by the induction hypothesis, choose an extension of $\eta'$ to $C^{2n-2}$ (also called $\eta'$). Again, by topological reasons, we can assume that $\eta'$ is $t$-independent in the collar neighborhood $(-\delta, \delta) \times M_1 \subset C^{2n-2}$, and $s$-independent in $(-\epsilon, 0] \times M_2 \subset C^{2n-2}$.

We then extend $\eta$ to $C^{2n-2} \times D^2(2)$ via pullback, so that it is $D^2(2)$-independent. Observe that the two extensions glue together smoothly, and that our assumptions on $\eta'$ of $s$- and $t$-independence are compatible with the construction of $\eta$ (when this one takes over the role of $\eta'$ in the next step). This finishes the induction argument.

Remark 4.1. Observe that it follows from the construction that $\eta$ is $r$-, $s$-, and $t$-independent wherever these variables are defined, and $\phi$-independent along the region $C^{2n} \setminus M_P$.
4.2. \(\eta\)-perturbation. Let us now fix a closed 2-form \(\eta\) in \(C^{2n}\), constructed inductively as above. We modify the symplectic form \(\omega_{C^{2n}}\) to the 2-form

\[\omega^\eta_{C^{2n}} := C\omega_{C^{2n}} + \eta,\]

for a large constant \(C \gg 1\). If \(C\) is chosen large enough, this 2-form is indeed symplectic, since the first summand then dominates, and nondegeneracy is an open condition. Since \([\omega] = [\eta]\) and \(\omega|_{\xi} > 0\) along \(M_1^{2n-1}\), by Lemma 1.10 in [26], we get a symplectic form on \([0, 1] \times M_1^{2n-1}\) which coincides with \(\omega\) on \(\{0\} \times M_1^{2n-1}\) and with \(C\alpha_1 + \eta\) on \(\{1\} \times M_1^{2n-1}\), for any \(C\) sufficiently large. Therefore, after attaching this symplectic cobordism \([0, 1] \times M_2^{2n-1}\), we may glue the symplectic manifold \((C^{2n}, \omega^\eta_{C^{2n}})\) to any symplectic manifold \((W^{2n}, \omega)\) having \(M_1^{2n-1}\) as a weak boundary component. We shall denote

\[H^{2n-1}_\eta := (i_X\omega^\eta_{C^{2n}}|_{M_2^{2n-1}}, \omega^\eta_{C^{2n}}|_{M_2^{2n-1}}) = CH^{2n-1} + (i_X\eta|_{M_2^{2n-1}}, \eta|_{M_2^{2n-1}}),\]

where \(X\) is the stabilizing vector field defined in (3.2). Observe that since \(H^{2n-1}_\eta\) is a perturbation of the SHS \(H^{2n-1}\), it is a framed Hamiltonian structure, i.e. the framing condition (which is open) still holds. Whereas the stability condition is not open, we will show in next section, nevertheless, that \(H^{2n-1}_\eta\) is indeed still a stable Hamiltonian structure along a specified region of \(M_2^{2n-1}\), which is full of 2-spheres. Observe also that the regions along which \(H^{2n-1}_\eta\) is contact, become weakly dominated under the \(\eta\)-perturbation (since weak domination is an open condition).

5. Moduli space of spheres

In this section, we construct an almost complex structure in the symplectic cobordism \(C^{2n}\) constructed in Section 3, together with a moduli space of holomorphic spheres.

5.1. Almost complex structure on a local model. After running the inductive step of the construction of \(C^{2n}\), with the particular base case we chose, from Equation (3.7), we obtain the following local model for the stable Hamiltonian manifold \((M_2^{2n-1}, H^{2n-1})\) around the binding \(M_2^3 = S^1 \times S^2\) of \(M_2^3 \subset M_2^4 \subset \cdots \subset M_2^{2n-1}\):

\[Y := S^1 \times S^2 \times \mathbb{D}^2_1 \times \cdots \times \mathbb{D}^2_{n-2}, (d\theta + \sum_{i=1}^{n-2} \sigma_i, \omega_{S^2} + \sum_{i=1}^{n-2} d\sigma_i)\]

Here, \(\mathbb{D}^2\) is a copy of the 2-disk \(\mathbb{D}^2\), \(\omega_{S^2}\) is an area form on \(S^2\), and \(\sigma_i\) is a Liouville form in the \(i\)-th disk \(\mathbb{D}^2_i\). In particular, its kernel is

\[\xi_Y := TS^2 \oplus \xi_0,\]
where $\xi_0 = \ker (d\theta + \sum_{i=1}^{n-2} \sigma_i)$ is a contactization contact structure associated to the Liouville domain $(\mathbb{D} := \prod_{i=1}^{n-2} \mathbb{D}_i^2, \sigma := \sum_{i=1}^{n-2} \sigma_i)$, and so it is a “confoliation”. Observe also that its Reeb vector field is $\partial_\theta$.

Let us then choose almost complex structures $j_{S^2}$ and $J_{\xi_0}$ which are respectively compatible with $\omega_{S^2}$ and $d\sigma|_{\xi_0}$, and define an almost complex structure $J$ on $\xi_Y$ by

$$J = j_{S^2} \oplus J_{\xi_0}$$

We extend $J$ to $(-\epsilon, 0] \times Y$ so that it maps $\partial_s$ to $\partial_\theta$. Then, $J$ is adapted to the symplectization of the stable Hamiltonian structure $\mathcal{H}_{2n-1}|_Y$ along $Y$, which is just the collar $(-\epsilon, 0] \times Y$ inside $C^{2n}$. Observe that, whereas $Y$ is weakly pseudoconvex for this choice of $J$, the region $S^1 \times \mathbb{D} \subset (-\epsilon, 0] \times Y$ becomes a strictly pseudoconvex portion of the boundary.

By our specific choice of extension $\eta$, the region $Y \subset M_{2n-1}$ remains stable under the $\eta$-perturbation. The analogous local model for the $\eta$-perturbed stable Hamiltonian structure along $Y$, for inductively constructed $\eta$, is

$$\left( Y = S^1 \times S^2 \times \mathbb{D}_1^2 \times \cdots \times \mathbb{D}_{n-2}^2, C \left( d\theta + \sum_{i=1}^{n-2} \sigma_i, \omega_{S^2} + \sum_{i=1}^{n-2} d\sigma_i \right) + (0, \eta) \right),$$

where $\eta = K \omega_{S^2}$ for some $K \in \mathbb{R}$, and $C$ needs to be chosen sufficiently large so that $C + K > 0$. In particular, the same $J$ as for the unperturbed stable Hamiltonian structure $\mathcal{H}_{2n-1}$ is adapted to the symplectization of $\mathcal{H}_{\eta}^{2n-1}$ along $Y$.

5.2. Extension to $C^{2n}$. Choose any extension of $J$ to an $\omega_{C^{2n}}$-compatible almost complex structure on $C^{2n}$, so that $J$ is compatible with the stable Hamiltonian structure $\mathcal{H}_{2n-1}^{2n-1}$ near $M_{2n-1}^{2n-1}$, and makes the latter a weakly pseudoconvex boundary component, which is strictly pseudoconvex along the regions where it is weakly dominated. In the non-perturbed cobordism $(C^{2n}, \omega_{C^{2n}})$, we take $J$ to be cylindrical near the concave contact-type boundary component $-M_{2}^{2n-1} \subset \partial C^{2n}$, and also cylindrical near the convex contact-type portion

$$M_{2,p}^{2n-1} \subset M_{2}^{2n-1} \subset \partial C^{2n}$$

so that in particular the latter region is actually strictly pseudoconvex. In the $\eta$-perturbed one $(C^{2n}, \omega_{C^{2n}}^\eta)$, we take $J = J_{\eta}$ to be a perturbation of the $J$ for the unperturbed data, which is still compatible with the perturbed stable Hamiltonian structure in the regions which remain stable under the perturbation, and so that the weak boundary piece $M_{2,p}^{2n-1}$ is still strictly pseudoconvex.
As in previous sections, a more detailed inductive construction is also possible. However, it will not be too relevant for our purposes, for which the above local model will suffice, so we will not include further details. For example, we may also assume inductively that, along \( C_{\epsilon} \times S^1 \subset M^{2n-1}_2 \), where \( \mathcal{H}^{2n-1} = (2d\phi, \omega_{C^{2n-2}}) \), the almost complex structure \( J \) coincides with the almost complex structure \( J' \) corresponding to dimension \( 2n-2 \) on \( TC^{2n-2} = \ker d\phi \) (and maps \( \partial_r \) to \( \partial_\theta \) in a collar).

Similarly, along \( M_{2,P} \), we have \( \ker \Lambda = \ker \Lambda' \oplus \langle v - \sigma(v)R' : v \in T\mathbb{D}^2(2-\delta) \rangle \), where \( R' \) is the Reeb vector field of \( \mathcal{H}^{2n-3} = (\Lambda', \Omega') \). So, we may choose \( J = J' \oplus \tilde{j} \), where \( j \) is any \( \sigma \)-compatible almost complex structure on \( D^2_{2n-2} \), and \( \tilde{j} \) its lift to \( \langle v - \sigma(v)R' : v \in T\mathbb{D}^2(2-\delta) \rangle \cong T\mathbb{D}^2(2-\delta) \). This is compatible with the local model above, where \( J_{\xi_0} \) becomes \( J_{\xi_0} = \bigoplus_{i=1}^{n-2} j_i \) for \( j_i \) a \( \sigma_i \)-compatible almost complex structure on \( \mathbb{D}^2_i \). All local expressions can be suitably glued together along the smoothened corners.

5.3. **A local moduli space.** For this choice of almost complex structure \( J \), the spheres \( u(s,\theta,z) = \{s\} \times \{\theta\} \times S^2 \times \{z\} \subset (-\epsilon,0] \times S^1 \times S^2 \times \mathbb{D} = (-\epsilon,0] \times Y \) are clearly \( J \)-holomorphic in \( C^{2n} \). Since their normal bundles are trivial, by the Riemann-Roch formula, their index is \( 2n-2 \). Moreover, they are Fredholm regular. Indeed, since \( J \) splits, so does the associated normal linearized Cauchy-Riemann operator, which is the direct sum of trivial \( \overline{\partial} \) operators acting on trivial line bundles, each having index 2. These are surjective by automatic transversality, and therefore so is the normal linearized Cauchy-Riemann operator.

We will denote by \( \mathcal{M} \) the moduli space of \( J \)-holomorphic spheres in \( C^{2n} \) containing the spheres \( u(s,\theta,z) \). Observe that we have shown that \( \mathcal{M} \) is a manifold of dimension \( (2n-2) \) around the latter.

5.4. **Local uniqueness.** In this section, we prove the following:

**Lemma 5.1** (Local Uniqueness Lemma). By shrinking \( \epsilon \) if necessary, any holomorphic map \( u : (S^2, j_{S^2}) \to (C^{2n}, J) \) in the moduli space \( \mathcal{M} \), which intersects \( (-\epsilon,0] \times Y \), is (a reparametrization of) one of the spheres \( u(s,\theta,z) \).

**Proof.** First, we assume that \( u \) intersects \( \{0\} \times Y \). In this case, we define the open set \( U := u^{-1}(\{0\} \times Y) \).
Then \( u|_\mathcal{U} = (u_1, u_2) \), where \( u_1 : \mathcal{U} \to S^2 \), and \( u_2 : \mathcal{U} \to (-\epsilon, 0] \times S^1 \times \mathbb{D} \) are holomorphic maps. Since \( u_2 \) touches the strictly pseudoconvex boundary of \( (-\epsilon, 0] \times S^1 \times \mathbb{D} \) tangentially, it follows that it is constant. This implies that \( \mathcal{U} = S^2 \), and \( u \) is a reparametrization of a sphere \( u(0,\theta,z) \).

In the general case, we will assume to the contrary that we have a sequence \( u_k \) of holomorphic spheres intersecting \( \{ -\epsilon_k \} \times Y \), for \( \epsilon_k \to 0 \), and that are not reparametrizations of any of the \( u_k(\theta,z) \). Then, a subsequence converges to a nodal configuration in \( \mathcal{M} \) intersecting \( \{ 0 \} \times Y \). This configuration is then a reparametrization of a sphere \( u(0,\theta,z) \). But the implicit function theorem implies that every sphere in \( \mathcal{M} \) near \( u(0,\theta,z) \) is of the form \( u(s',\theta',z') \) for some \( (s',\theta',z') \), which is a contradiction. This proves the lemma. \( \square \)

6. Weak Semi-fillability

We now proceed to the proof of the Theorem A.

**Proof.** Assume that \((W^{2n}, \omega)\) is a symplectic semi-filling, having the IP contact manifold \((M^{2n-1}, \xi)\) as a boundary component, where \( \xi \) is supported by an IP open book decomposition, and other nonempty boundary components. Given any Giroux form \( \alpha \) for \( \xi \), in the strong case, we may assume that \( \omega|_{(-\delta,0] \times M} = d(e^t\alpha) \) in some \( \delta \)-collar neighborhood of \( M \). In the weak case, we may take \( \omega|_{(-\delta,0] \times M} = Cd(e^t\alpha) + \omega|_{\xi} \), for some \( C > 0 \) constant.

Using the modification of Section 4, if necessary, we can then attach the symplectic cobordism \((C, \omega_C)\) to \((W, \omega)\) along \((M, \xi = \ker \alpha)\), where \( \alpha \) is the explicit Giroux form arising from the construction of \((C, \omega_C)\). We obtain a new boundary component \( M' \), which is stable in the strong case, and, in the weak case, has portions which are stable, and others which are weakly dominated (arising as a small perturbation of a contact-type data). Near \( M' \), we have a moduli space of \( J \)-holomorphic spheres for the almost complex structure \( J \) from Section 5 defined on \( C \), stemming from a subregion of the form \((-\epsilon, 0] \times Y \subset C \).

We now extend the almost complex structure \( J \) on \( C \) to \( W \), so that it makes all the boundary components different from \( M' \) strictly pseudoconvex. We then obtain a moduli space of holomorphic spheres \( \mathcal{M} \) in \( \bar{W} \), containing the spheres in \( C \). Its virtual dimension is \( 2n-2 \), and it is a manifold near the spheres in \((-\epsilon, 0] \times Y \). We add a marked point to the domain of each curve in \( \mathcal{M} \), obtaining a moduli space \( \mathcal{M}_* \) of virtual dimension \( 2n \), and an evaluation map \( ev : \mathcal{M}_* \to \bar{W} \).

Observe that, while the spheres in \( \mathcal{M}_* \) are somewhere injective if they lie in \((-\epsilon, 0] \times Y \), there could be multiple covers somewhere else, or in components of nodal configurations in the compactification \( \overline{\mathcal{M}}_* \). We then appeal to the polyfold technology of [21], together the regularization of constrained moduli spaces of [13]...
(cf. the proof of Theorem 6.1 in [26]). We view the Gromov compactification \( \overline{\mathcal{M}} \) of \( \mathcal{M} \) as sitting inside a Gromov-Witten polyfold \( B \) containing stable nodal configurations of spheres with one marked point and possibly multiple components. We view the nonlinear Cauchy-Riemann operator \( \overline{\partial} J \) as a section of a strong polyfold bundle \( E \to B \) with zero set \( \overline{\partial} J^{-1}(0) = \overline{\mathcal{M}} \). We then introduce an abstract perturbation \( p \), which is a multivalued section of \( E \) and in general can be chosen arbitrarily small and supported near \( \overline{\mathcal{M}} \), and which is generic in the sense that \( \overline{\partial} J + p \) is transverse to the zero section. Moreover, we are free to choose it in such a way that it is supported away from an open region of the moduli space \( \mathcal{M} \subset \overline{\mathcal{M}} \) containing the spheres in \( (-\epsilon, 0] \times Y \), since we already know these are regular. We obtain a perturbed moduli space

\[
\overline{\mathcal{M}}' = (\overline{\partial} J + p)^{-1}(0),
\]

which is a 2n-dimensional compact, oriented, weighted branched orbifold, which we will denote by COWBOY, with boundary and corners, and it comes equipped with an evaluation map

\[
ev : \overline{\mathcal{M}}' \to W
\]
at the marked point.

Observe that our choice of perturbation implies that the elements of \( \overline{\mathcal{M}}' \) approaching \( (-\epsilon, 0] \times Y \) are still actually \( J \)-holomorphic curves, and so the Local Uniqueness Lemma 5.1 continues to hold. Also, no element in the unperturbed moduli space \( \overline{\mathcal{M}} \) is allowed to tangentially touch the portions of the boundary of \( W \) along which \( J \) is strictly pseudoconvex. By a Gromov compactness argument, the same is true for the perturbed moduli \( \overline{\mathcal{M}}' \) (for sufficiently small perturbation).

Pick a generic smooth path \( l : [0, 1] \to W \) with \( l(0) = (0, \theta, p, z) \in \{0\} \times Y \), for some \( p \in S^2 \), and with \( l(1) \) lying in a boundary component \( M'' \) of \( W \) different from \( M' \) meeting both transversely inside \( W \). Consider the constrained moduli space \( \overline{\mathcal{M}}_{*,l} := ev^{-1}(l) \). By the results in [13], \( \overline{\mathcal{M}}_{*,l} \) is, for generic choices, a 1-dimensional COWBOY with boundary \( ev^{-1}(\partial l) \). By strict pseudoconvexity, no curve in \( \overline{\mathcal{M}}_{*,l} \) intersects \( M'' \) tangentially, and hence

\[
\partial \overline{\mathcal{M}}_{*,l} = ev^{-1}(l(0)) = \{u(0, \theta, z)\}
\]

by the Local Uniqueness Lemma 5.1. But there exist no 1-dimensional COWBOY with connected boundary, and hence we get a contradiction. \( \square \)
7. NONSEPARATING WEAK CONTACT-TYPE HYPERSURFACES

We now prove the Theorem B.

Proof. Assume to the contrary that the IP contact manifold \((M^{2n-1}, \xi)\) embeds into a closed symplectic manifold \((W^{2n}, \omega)\) as a nonseparating weak contact-type hypersurface. We now adapt the proof in [4], to which we refer the reader for more details.

We cut \(W\) open along \(M\), obtaining a symplectic cobordism \((W_1, \omega_1)\) having two weakly dominated boundary components \(M_\pm\), one positive and one negative, and both diffeomorphic to \(M\). We “get rid” of the negative boundary component \(M^-\), by attaching infinitely many copies of \(W_1\) at \(M^-\), at each step identifying \(M^-\) with \(M^+\). That is, we inductively define

\[
(W_n, \omega_n) = (W_{n-1}, \omega_{n-1}) \bigcup_{M_- \sim M_+} (W_1, \omega_1).
\]

After the induction, the result is a noncompact symplectic manifold \((W_\infty, \omega_\infty)\) with boundary \(M_+ \cong M\), which is positively weakly dominated. By construction, \(W_\infty\) contains infinitely many copies of \(M\), as weak contact-type hypersurfaces. We attach the perturbed version of the symplectic cobordism \((C_{2n}, \omega_{C_{2n}})\) (of Section 4) to \((W_\infty, \omega_\infty)\) along its positive boundary \(M_+\), obtaining a new symplectic manifold \((W'_\infty, \omega'_\infty)\) with stable boundary \(M'_+\).

Observe that, by construction, the symplectic form \(\omega_\infty\) is periodic. Therefore, we may choose any \(\omega'_\infty\)-compatible almost complex structure \(J\) in \(W'_\infty\), which extends the almost complex structure along \(C_{2n}\) of Section 5, which is also periodic along \(W_\infty \subset W'_\infty\), and which makes every copy of \(M\) inside \(W_\infty\) strictly pseudoconvex. Periodicity then implies that \(W'_\infty\) has bounded geometry, which means in particular that holomorphic curves with bounded energy have bounded diameter, where the bound on the diameter depends only on the energy bound.

From Section 5, we obtain a moduli space \(M'_\bullet\) of spheres with a marked point in \(W'_\infty\), of virtual dimension \(2n\), stemming from a region \((-\epsilon, 0] \times Y \subset C_{2n}\). We may now use similar arguments as in the proof of Theorem A. Namely, we choose a properly embedded generic path \(l: [0, +\infty) \to W'_\infty\), with \(l(0) \in (-\epsilon, 0] \times Y\), such that \(l\) is transverse to \(M'\). As in the proof of Theorem A, by abstractly perturbing the Cauchy-Riemann equation, we obtain a 1-dimensional noncompact, oriented, weighted branched orbifold (an OW-BOY) \(M'_{\bullet, l}\), consisting of spheres constrained to intersect \(l\). Observe that no sphere in \(M'_{\bullet, l}\) is allowed to shoot off down the noncompact piece of \(W'_\infty\) in such a way that its distance to \(M'\) goes to infinity, since otherwise there would exist a sphere which tangentially intersects (from below) one of the infinitely many strictly pseudoconvex hypersurfaces. This means that
there is a sequence of spheres in $M_*, l$ which “stretches out” down the noncompact piece, but always staying at a bounded distance from $M$. But this cannot happen, because of the diameter bounds. □

8. THE WEINSTEIN CONJECTURE

We now prove the Theorem C.

Proof. Let $(M^{2n-1}, \xi)$ be an IP contact manifold and $\alpha_0$ be any contact form inducing $\xi$. Let $\alpha_1$ be the Giroux form adapted to the IP open book decomposition with $\ker \alpha_1 = \xi$, which we explicitly constructed in Section 3. Then, as in e.g. [5], we may construct an exact symplectic cobordism $((-\infty, 0] \times M, d\lambda)$, where $\lambda = e^t \alpha_0$ for $t \leq -1$, and $\lambda = e^t \alpha_1$ for $t \in [-\epsilon, 0]$ for any given small $\epsilon > 0$.

We attach the (non-perturbed) symplectic cobordism $(C^{2n}, \omega_{C^{2n}})$ constructed in Section 3 to $((-\infty, 0] \times M, d\lambda)$ along $\{0\} \times M$, obtaining a (non-exact and noncompact) symplectic cobordism $$(W, \omega) = ((-\infty, 0] \times M, d\lambda) \bigcup_M (C^{2n}, \omega_{C^{2n}})$$ with stable boundary $M'$. We extend the almost complex structure $J$ constructed in Section 5 to $W$, so that it is cylindrical in the cylindrical ends of $W$. We thus obtain a moduli space $\mathcal{M}$ of spheres in $W$ stemming from $M'$, for which the Local Uniqueness Lemma holds in a subregion of the form $(-\epsilon, 0] \times Y \subset (-\epsilon, 0] \times M' \subset W$.

Choose a generic path $l : [0, +\infty) \to W$ with $l(0) \in \{0\} \times Y \subset \partial W$, which is transverse to $M'$, and properly embedded in $W$. Add a marked point to the domain of the spheres, obtaining a moduli space $\mathcal{M}_\bullet$ of virtual dimension $2n$. As in the proof of Theorem B, by introducing an abstract perturbation which vanishes near the spheres in $(-\epsilon, 0] \times Y$, we obtain a $2n$-dimensional OW-BOY with boundary and corners $\mathcal{M}_\bullet$, together with an evaluation map $ev : \mathcal{M}_\bullet \to W$. By the exactness of $\omega$ in $(-\infty, 0] \times M \subset W$, the noncompactness of $\mathcal{M}_\bullet$ corresponds only to spheres escaping down the negative end.

Consider the constrained moduli space $\mathcal{M}_{*, l} = ev^{-1}(l)$, which is, for generic choices, a 1-dimensional OW-BOY with boundary $ev^{-1}(l(0))$ (by the Local Uniqueness Lemma). Observe that, by the exactness of $\omega$ along $(-\infty, 0] \times M$, no sphere in $\mathcal{M}_{*, l}$ can completely lie in $(-\infty, 0] \times M \subset W$. It follows that, as we transverse $\mathcal{M}_{*, l}$ towards its (possibly multiple) noncompact ends, we obtain a sequence of spheres which are “stretching out” towards the negative end of $W$. By Gromov compactness, this sequence breaks at the negative end, and we obtain a finite energy holomorphic curve in the negative symplectization of $(M, \alpha_0)$. This proves Theorem C. □
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