ABSENCE OF LOCAL UNCONDITIONAL STRUCTURE IN SPACES OF SMOOTH FUNCTIONS ON TWO-DIMENSIONAL TORUS

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Consider a finite collection \( \{T_1, \ldots, T_J\} \) of differential operators with constant coefficients on \( \mathbb{T}^2 \) and the space of smooth functions generated by this collection, namely, the space of functions \( f \) such that \( T_j f \in C(\mathbb{T}^2) \). Under a certain natural condition, we prove that this space is not isomorphic to a quotient of a \( C(\mathbb{S}) \)-space and does not have a local unconditional structure. This fact generalizes the previously known result that such spaces are not isomorphic to a complemented subspace of \( C(\mathbb{S}) \). Bibliography: 19 titles.

1. Introduction

It is well known and easy to see that the space \( C^k(\mathbb{T}) \) of \( k \) times continuously differentiable functions on the unit circle is isomorphic to \( C(\mathbb{T}) \). Also, it has long been known that in higher dimensions the situation is different—already for two dimensions the space \( C^k(\mathbb{T}^2) \) is not isomorphic to \( C(\mathbb{T}^2) \).

This fact was first announced in [4] and later generalized in many directions (see [5, 7–10, 13–16, 18]). However, the most general and natural framework was introduced only in the quite recent paper [11] (see also preprint [12] for the two-dimensional case).

More specifically, suppose we have a collection \( T = \{T_1, T_2, \ldots, T_J\} \) of differential operators with constant coefficients on the torus \( \mathbb{T}^2 \). Hence, each \( T_j \) is a linear combination of operators \( \partial_1^\alpha \partial_2^\beta \). The number \( \alpha + \beta \) is called the order of such a differential monomial; the order of \( T_j \) is the maximal order among all monomials involved in it. We consider the following seminorm on trigonometric polynomials \( f \):

\[
\|f\|_T = \max_{1 \leq j \leq J} \|T_j f\|_{C(\mathbb{T}^2)}.
\]

We define the Banach space \( C^T(\mathbb{T}^2) \) by the above seminorm, that is, we factorize over the null space and consider the completion. For example, if \( T \) consists of all differential monomials of order at most \( k \), then we get the space \( C^k(\mathbb{T}^2) \).

In [9, 10], the following assertion is proved. Suppose that all differential monomials involved in any of \( T_j \) are of order not exceeding \( k \). Let us drop the junior part of each \( T_j \), that is, let us drop all monomials of order strictly smaller than \( k \). If among the remaining senior parts there are at least two linearly independent parts, then \( C^T(\mathbb{T}^2) \) is not isomorphic to a complemented subspace of \( C(\mathbb{S}) \). Here, \( \mathbb{S} \) denotes an arbitrary uncountable compact metric space. According to Milutin’s theorem, all the resulting \( C(\mathbb{S}) \) spaces are isomorphic. However, this result gives no answer if all senior parts are multiples of one of them.

Thus, in preprint [12], and in [11] for arbitrary dimensions, a refinement of the above assertion is proved. In order to state it, we need the concept of mixed homogeneity.

Fix a mixed homogeneity pattern, that is, a line \( \Lambda \) that intersects the positive semiaxes. The equation of such a line is \( \frac{a}{x} + \frac{b}{y} = 1 \), where \( a \) and \( b \) are positive numbers. We say that \( \Lambda \) is admissible if all multiindices \( (\alpha, \beta) \) with \( \partial_1^\alpha \partial_2^\beta \) involved in one of \( T_j \) are below \( \Lambda \) or on it. This

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means that all multiindices under consideration must satisfy the following inequality:
\[
\frac{\alpha}{a} + \frac{\beta}{b} \leq 1.
\]

The senior part of \( T_j \) is defined as the sum of all differential monomials involved in \( T_j \) whose multiindices are on the line \( \Lambda \); the junior part is defined as the sum of all other monomials of \( T_j \). The senior and junior parts are denoted by \( \sigma_j \) and \( \tau_j \), respectively.

Suppose that, for some choice of \( \Lambda \), there are at least two linearly independent operators among all senior parts \( \sigma_j \). Then, as proved in [11, 12], \( C^T(\mathbb{T}^2) \) is not isomorphic to a complemented subspace of a \( C(S) \) space.

However, in a less general setting, this is not the best known result. For example, it is proved in [7] that \( C^k(\mathbb{T}^2) \) is not isomorphic to any quotient space of \( C(S) \). The following theorem generalizes this statement in the described setting.

**Theorem 1.** If there are at least two linearly independent operators among \( \sigma_j \) for a collection \( \mathcal{T} \) (for some choice of an admissible line \( \Lambda \)), then \( C^T(\mathbb{T}^2) \) is not isomorphic to any quotient space of \( C(S) \).

This is the first result of the present paper.

Also, observe that there is another generalization of the theorem from [11, 12], again in a less general setting. In [13], it is proved that if all operators in \( \mathcal{T} \) are differential monomials and at least two senior monomials (with respect to some pattern) are linearly independent, then the space \( C^T(\mathbb{T}^2) \) does not have local unconditional structure.

The following definition is from [3]. A Banach space \( X \) is said to have local unconditional structure if there exists a constant \( C > 0 \) such that for any finite-dimensional subspace \( F \subset X \), there exists a Banach space \( E \) with 1-unconditional basis and two linear operators \( R : F \to E \) and \( S : E \to X \) such that \( SRx = x \) for all \( x \in F \) and \( \|S\| \cdot \|R\| \leq C \). A basis \( \{e_n\} \) is 1-unconditional if for any numbers \( \varepsilon_n \) with \( |\varepsilon_n| \leq 1 \) and any finitary sequence \( (\alpha_n) \), the following inequality holds: \( \|\sum \varepsilon_n \alpha_n x_n\| \leq \| \sum \alpha_n x_n \| \).

It is worth noting that \( X \) has local unconditional structure if and only if its conjugate \( X^* \) is a direct factor of a Banach lattice (see [17]). Since the space \( C(S) \) has local unconditional structure, we also obtain the non-isomorphism of \( C^T(\mathbb{T}^2) \) to a complemented subspace of \( C(S) \) provided we prove that \( C^T(\mathbb{T}^2) \) does not have local unconditional structure. This is exactly the statement of the following theorem.

**Theorem 2.** If there are at least two linearly independent operators among \( \sigma_j \) for a collection \( \mathcal{T} \) (for some choice of an admissible line \( \Lambda \)), then \( C^T(\mathbb{T}^2) \) does not have local unconditional structure.

The main ingredients of our proofs are the same as in [11, 12]. We use the new embedding theorem established there together with certain facts about \( p \)-summing operators.

First, we introduce several definitions. A distribution \( f \) on the torus \( \mathbb{T}^2 \) is called proper if \( \hat{f}(s, t) = 0 \) whenever \( s = 0 \) or \( t = 0 \). Next, we need the following notion of Sobolev spaces with nonintegral smoothness:

\[
W^\alpha,\beta_2(\mathbb{T}^2) = \{ f \in C^\infty(\mathbb{T}^2)^\prime : \{(1 + m^2)^{\alpha/2}(1 + n^2)^{\beta/2} \hat{f}(m, n)\} \in \ell^2(\mathbb{Z}^2) \}.
\]

Clearly, the norm of \( f \) in \( W^\alpha,\beta_2(\mathbb{T}^2) \) is defined as \( \|\{(1 + m^2)^{\alpha/2}(1 + n^2)^{\beta/2} \hat{f}(m, n)\}\|_{\ell^2} \).

Now, we formulate the embedding theorem (see [11, Theorem 0.2 and Remark 1.6]), which we are going to use.

**Fact 1.** Suppose that proper distributions \( \phi_1, \ldots, \phi_N \) satisfy the following system of equations:
\[
-\partial_1^j \varphi_1 = \mu_0; \quad \partial_2^j \varphi_j - \partial_1^j \varphi_{j+1} = \mu_j, \quad j = 1, \ldots, N - 1; \quad \partial_2^N \phi_N = \mu_N,
\]
Therefore, the equation of $\Lambda$ rewrites in the following form:

$$\sum_{j=1}^{N} \|\varphi_j\|_{W^1_2(T^2)} \lesssim \sum_{j=0}^{N} \|\mu_j\|.$$ 

Here and in what follows, the notation $A \lesssim B$ means that there exists a constant $C > 0$ such that $A \leq CB$.

Several remarks are in order. First, Theorems 1 and 2 also hold for the torus $T^n$ of arbitrary dimension. But this fact cannot be derived from the 2-dimensional statements, or at least it is unclear how to do this; see [11] for certain details. The proofs in higher dimensions are somewhat similar, however, they are much more technically sophisticated, and even require a different embedding theorem; see [11] and Theorem 1.1 there. Thus, in the present article, we restrict ourselves to the two-dimensional case.

In this paper, we present proofs of Theorem 1 and Theorem 2. We start with the first theorem because its proof is easier and contains less technical details; in fact, the proofs of both theorems are quite similar and similar to the arguments from preprint [12].

Observe that it is also proved in [13] (again, in the case where all operators in $\mathcal{T}$ are differential monomials and there are at least two linearly independent operators among their senior parts) that if $C^T(T^2)^*$ is isomorphic to a subspace of a space $Y$ with local unconditional structure, then $Y$ contains the spaces $\ell_k^\infty$ uniformly; see [13] for definitions. The same statement also can be proved in our situation, but we do not present the details here, since our main goal is to show that, using the embedding theorem from [12], we can adapt various techniques to a more general context. While this result implies Theorem 1, we choose to sacrifice the generality for the sake of simplicity and transparency of presentation.

Also, a few words should be said about notation. As mentioned above, we write $A \lesssim B$ if $A \leq CB$ for a constant $C > 0$. It is always clear from the context which parameters $C$ can depend on and which not. Besides that, the notation $A \asymp B$ means that $A \lesssim B$ and $B \lesssim A$.

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2. Nonisomorphism to a Quotient of a $C(S)$-space

As in [12], we start our proof of Theorem 1 with several simple but helpful observations.

2.1. Several reductions. Let $C_0^T(T^2)$ denote the space of proper functions in $C^T(T^2)$. It is clear that this space is complemented in $C^T(T^2)$; indeed, a projection is given by convolution with a measure. Hence, it suffices to prove Theorem 1 for $C_0^T(T^2)$ in the place of $C^T(T^2)$.

Next, suppose that the admissible line $\Lambda$ is given by the equation $x/a + y/b = 1$. Let us show that, without loss of generality, we may assume that $a$ and $b$ are positive integers. Indeed, according to the assumptions of Theorem 1, there are at least two points $(r_1, r_2)$ and $(\rho_1, \rho_2)$ with nonnegative integral coordinates on $\Lambda$. We may assume that $r_1 > \rho_1$ and $r_2 < \rho_2$. Therefore, the equation of $\Lambda$ rewrites in the following form:

$$\frac{x}{r_1 - \rho_1} + \frac{y}{r_2 - \rho_2} = \frac{\rho_1}{r_1 - \rho_1} + \frac{\rho_2}{r_2 - \rho_2}.$$ 

Now, observe that we may shift the line $\Lambda$ and the whole construction by a vector with integral coordinates. This means that we may replace the collection $\mathcal{T}$ by $\{T_1 \partial_1^\rho \partial_2^\rho, \ldots, T_J \partial_1^\rho \partial_2^\rho\}$. The corresponding spaces

$$C_0^{\{T_1, \ldots, T_J\}}(T^2) \quad \text{and} \quad C_0^{\{T_1 \partial_1^\rho \partial_2^\rho, \ldots, T_J \partial_1^\rho \partial_2^\rho\}}(T^2)$$
are isomorphic—an isomorphism is given by the map \( f \mapsto \partial_1^m \partial_2^n f \). Hence, applying this shift, we may assume that \( \Lambda \) is defined by the following equation:

\[
\frac{x}{r_1 - r_2} + \frac{y}{\rho_2 - \rho_1} = \frac{\rho_1 + u}{r_1 - r_2} + \frac{\rho_2 + v}{\rho_2 - \rho_1}.
\]

If we write this equation in the form \( x/a_1 + y/b_1 = 1 \), then \( a_1 \) and \( b_1 \) are equal to

\[
\rho_1 + u + (\rho_2 + v)\frac{r_1 - r_2}{\rho_2 - \rho_1} \quad \text{and} \quad \rho_2 + v + (\rho_1 + u)\frac{\rho_2 - \rho_1}{r_1 - r_2},
\]

respectively. Clearly, there exist positive integers \( u \) and \( v \) such that the above expressions are integers.

So, we assume that the equation of \( \Lambda \) is \( x/a + y/b = 1 \), where \( a \) and \( b \) are positive integers. Let \( N \) denote the greatest common divisor of \( a \) and \( b \). Then all points on \( \Lambda \) with integral coordinates are of the form \((jm,(N-j)n)\) with \( 0 \leq j \leq N \); here \( m = a/N \) and \( n = b/N \), hence, \( m \) and \( n \) are coprime.

2.2. Main construction. Suppose that \( C_T^0(\mathbb{T}^2) \) is isomorphic to a quotient space of \( C(S) \). Let \( P : C(S) \to C_T^0(\mathbb{T}^2) \) be the quotient map.

Due to the above reductions, the senior part of every operator from \( \mathcal{T} \) has the following form:

\[
\sigma_s = \sum_{j=0}^{N} a_{sj} \partial_1^m \partial_2^{(N-j)n}.
\]

Observe that the space \( C_T^0(\mathbb{T}^2) \) depends only on the linear span of operators in \( \mathcal{T} \), thus, we may change our collection provided the corresponding changes do not affect its linear span.

Now, consider the matrix \( (a_{sj}) \). Suppose \( j_0 \) is the smallest index such that \( a_{sj_0} \neq 0 \) for at least one \( s \). Without loss of generality, we may assume that \( a_{1j_0} \neq 0 \). So, multiplying \( T_1 \) by a constant and subtracting a multiple of \( T_1 \) from other operators, we ensure that \( a_{1j_0} = -1 \) and \( a_{sj_0} = 0 \) for every \( s > 1 \). By the assumptions of the theorem, there exists \( j_1 \) such that \( a_{sj_1} \neq 0 \) for some \( s > 1 \). Without loss of generality, we assume that \( a_{2j_1} = 1 \) and \( a_{sj_1} = 0 \) for all \( s > 2 \).

Therefore, we have two operators, \( T_1 \) and \( T_2 \), whose senior parts are linearly independent. For simplicity, let \( a_j \) and \( b_j \) denote the coefficients of their senior parts, that is,

\[
\sigma_1 = \sum_{j=0}^{N} a_j \partial_1^m \partial_2^{(N-j)n};
\]

\[
\sigma_2 = \sum_{j=0}^{N} b_j \partial_1^m \partial_2^{(N-j)n}.
\]

Moreover, \( T_1 \) is the only operator in \( \mathcal{T} \) that involves the differential monomial \( \partial_1^m \partial_2^{(N-j_0)n} \), and \( T_2 \) is the only operator in \( \mathcal{T} \), besides probably \( T_1 \), that includes the differential monomial \( \partial_1^m \partial_2^{(N-j_1)n} \).

Let \( i \) denote the embedding of the space \( C_T^0(\mathbb{T}^2) \) into \( C_{T_1,T_2}^0(\mathbb{T}^2) \). The last space embeds into \( W_{T_1,T_2}(\mathbb{T}^2) \). Let \( g \) denote this embedding. Here, the spaces \( C_{T_1,T_2}^0(\mathbb{T}^2) \) and \( W_{T_1,T_2}(\mathbb{T}^2) \) are defined by the seminorms \( \max\{\|T_1f\|_{C(T^2)},\|T_2f\|_{C(T^2)}\} \) and \( \max\{\|T_1f\|_{L^1(T^2)},\|T_2f\|_{L^1(T^2)}\} \), respectively, and consist of proper functions only. Observe that the operator \( g \) is 1-summing; this property follows easily from the Pietsch factorization theorem. A good reference for the theory of \( p \)-summing operators is [19, Chap. III.F].

Next, we are going to construct an operator \( s \) from \( W_{T_1,T_2}(\mathbb{T}^2) \) into \( W_{2^{-\frac{m-1}{2}},2^{-\frac{n-1}{2}}}(\mathbb{T}^2) \). Again, the construction is very similar to that in [12], with certain simplifications.
First, we need the following simple fact.

**Fact 2.** System (1) with proper measures (or $L^1$ functions) $\mu_j$ is solvable if and only if the following relation holds true:

$$\sum_{j=0}^{N} \partial_1^k \partial_2^{(N-j)m} \mu_j = 0. \quad (2)$$

The proof is quite easily done by induction; see [12, Lemma 2.1].

Now, take any $f \in W_{1}^{T_1, T_2}(\mathbb{T}^2)$ and consider the pair of functions $(f_1, f_2) = (T_1 f, T_2 f)$. Clearly, they satisfy the equation $T_2 f_1 - T_1 f_2 = 0$. This is a differential equation and now we rewrite it in a different form. Namely, observe that for $\alpha/\alpha + \beta/b < 1$, the differential monomial $\partial_1^2 \partial_2^\beta$ has the following expression in terms of $\partial_1^\alpha$ and $\partial_2^\beta$ with the help of Fourier multipliers:

$$\partial_1^\alpha \partial_2^\beta f = I_{\alpha\beta} \partial_1^\alpha f + J_{\alpha\beta} \partial_2^\beta f,$$

where $I_{\alpha\beta}$ and $J_{\alpha\beta}$ are Fourier multipliers with the following symbols:

$$\frac{(iu)^{\alpha+a}}{(iu)^{2a} \pm (iv)^{2b}} \quad \text{and} \quad \pm \frac{(iu)^{\alpha} (iv)^{\beta+b}}{(iu)^{2a} \pm (iv)^{2b}},$$

respectively. By this we mean that $I_{\alpha\beta}$ and $J_{\alpha\beta}$ act on a function $g \in L^1_0(\mathbb{T}^2)$ by multiplying its Fourier coefficients $\hat{g}(u, v)$ by the corresponding expressions. The choice of $\pm$ is determined by the condition $(-1)^n = \pm (-1)^b$, so that the denominators do not vanish when $u$ and $v$ are not equal to zero. It is proved in [12] that such multipliers are bounded on $L^1_0(\mathbb{T}^2)$.

**Fact 3.** The Fourier multipliers $I_{\alpha\beta}$ and $J_{\alpha\beta}$ defined as above are bounded on $L^1_0(\mathbb{T}^2)$.

Using these multipliers, we rewrite the junior parts of operators $T_1$ and $T_2$ in the following form:

$$\sum_{\alpha, \beta} c_{\alpha\beta} (I_{\alpha\beta} \partial_1^\alpha + J_{\alpha\beta} \partial_2^\beta).$$

Therefore, regrouping the terms in the expression $T_2 f_1 - T_1 f_2$, we rewrite it as

$$\sum_{j=0}^{N} \partial_1^j \partial_2^{(N-j)m} \mu_j = 0,$$

where the $\mu_j$ are precisely the functions $b_j f_1 - a_j f_2$ for $j \neq 0, N$; $\mu_0$ is equal to $b_0 f_1 - a_0 f_2$ plus a linear combination of the operators $J_{\alpha\beta}$ applied to $f_1$ and $f_2$, and $\mu_N$ is equal to $b_N f_1 - a_N f_2$ plus a linear combination of the operators $I_{\alpha\beta}$ applied to $f_1$ and $f_2$.

Now, using Fact 2, we find a solution of the following system of differential equations:

$$-\partial_1^m \varphi_1 = \mu_0; \quad \partial_2^\beta \varphi_j - \partial_1^m \varphi_{j+1} = \mu_j, \quad j = 1, \ldots, N - 1; \quad \partial_2^\beta \varphi_N = \mu_N. \quad (3)$$

By Fact 1, all functions $\varphi_j$ are in $W^{m-1, n-1}_{2} S^{1/2} \mathbb{T}^2$. We take the function

$$\varphi_{j_0+1} \in W^{m-1, n-1}_{2} S^{1/2} \mathbb{T}^2,$$

it depends linearly on the initial function $f$ and therefore, we get a bounded linear operator $s$ from $W_{1}^{T_1, T_2}(\mathbb{T}^2)$ into $W^{m-1, n-1}_{2} S^{1/2} \mathbb{T}^2$. Summing up, we have the following diagram:

$$C(S) \xrightarrow{P} C^T_0(\mathbb{T}^2) \xrightarrow{i} C^T_0(\mathbb{T}^2) \xrightarrow{g} W_{1}^{T_1, T_2}(\mathbb{T}^2) \xrightarrow{d} W^{m-1, n-1}_{2} S^{1/2} \mathbb{T}^2.$$
2.3. Contradiction. We pass to the final part of the proof. We construct an operator from a finite-dimensional subspace of $W_{2,\alpha}^{\frac{m}{N} \cdot \frac{m}{n}}(\mathbb{T}^2)$ to $C(S)$ and we use certain standard facts from Banach space theory, mostly related to the absolutely summing operators, to get a contradiction. Now we present the details.

Consider the function $v_{pq} := z_1^p z_2^q \in C^r(\mathbb{T}^2)$. We assume that the natural numbers $p$ and $q$ satisfy the following inequality:

$$\delta \leq q^n \leq p^m \leq \delta q^n,$$

where $\delta$ is a small fixed constant (it depends on $T$ but not on $p$ and $q$) which will be chosen later. Also, we consider only large values of $p$: $p > C$ for a large constant $C$. We always assume that the numbers $p$ and $q$ satisfy these conditions, and do not emphasize this further in the present section.

First of all, observe that $\|v_{pq}\|_{C^r(\mathbb{T}^2)} \asymp p^{mN}$.

Indeed, if we take any differential monomial $\partial_1^a \partial_2^b$ involved in the senior part of any operator from $T$, then we have $\partial_1^a \partial_2^b z_1^p z_2^q = (ip)^a (iq)^b z_1^p z_2^q$. Since this monomial is in the senior part of some operator, the following inequality holds: $\frac{\alpha}{N_a} + \frac{\beta}{N_b} < 1$. Therefore, if $\alpha = \alpha_0 m$, then $\beta = (N - \alpha_0 - c)n$ for some $c > 0$. Hence, the norm of $\partial_1^a \partial_2^b v_{pq}$ in $C(\mathbb{T}^2)$ is equal to $p^\beta q^\beta = p^{\alpha_0 m} q^{(N - \alpha_0 - c)n} \asymp p^{m(N - c)}$. Clearly, this quantity is arbitrarily smaller than $p^{mN}$ provided $p$ is sufficiently large.

On the other hand, if we apply any differential monomial involved in the senior part of one of the operators, which is of the form $\partial_1^a \partial_2^b (N-j)^n$, to $v_{pq}$, then we get a function with norm $p^\beta q^\beta = p^{\alpha_0 m} q^{(N - \alpha_0 - c)n} \asymp p^{m(N - c)}$. Moreover, if $j > j_0$, then $p^\beta q^\beta < p^\beta q^\beta < p^\beta q^\beta < p^\beta q^\beta$ for sufficiently small $\delta$. Therefore, we have $\|T_1 v_{pq}\|_{C(\mathbb{T}^2)} \asymp p^{mN}$. Combining these facts, we easily obtain $\|v_{pq}\|_{C^r(\mathbb{T}^2)} \asymp p^{mN}$, as required.

Similarly, $\|v_{pq}\|_{C^{r_1}, r_2(\mathbb{T}^2)} \asymp p^{mN}$ and $\|v_{pq}\|_{W^{r_1}, r_2(\mathbb{T}^2)} \asymp p^{mN}$. Therefore, consider the following functions:

$$w_{pq} := \frac{v_{pq}}{p^{mN}}.$$

The above arguments guarantee that $\|w_{pq}\|_{C^r(\mathbb{T}^2)} \asymp 1$. Hence, there exist functions $f_{pq}$ in $C(S)$ such that $P(f_{pq}) = w_{pq}$ and $\|f_{pq}\|_{C(S)} \leq C$. Besides that, we have $T_1 w_{pq} = c_{pq} v_{pq}$ and $T_2 w_{pq} = d_{pq} v_{pq}$, where $|c_{pq}|, |d_{pq}| \asymp 1$.

Next, we need to solve the system of differential equations (3). Recall that $\mu_0$ is equal to $b_0 c_{pq} v_{pq} - a_0 d_{pq} v_{pq}$ plus a linear combination of the operators $I_{a\beta}$ and $J_{a\beta}$ applied to $T_1 w_{pq}$ and $T_2 w_{pq}$. Therefore,

$$\mu_0 = \lambda_{pq} c_{pq} v_{pq} + \eta_{pq} d_{pq} v_{pq} + (b_0 c_{pq} v_{pq} - a_0 d_{pq} v_{pq}).$$

It is easy to see that $\xi_{pq} \eta_{pq} = O(p^{-\varepsilon})$ for a small fixed $\varepsilon > 0$. Indeed, recall that the symbol of any Fourier multiplier $I_{a\beta}$ has the following form:

$$\frac{(ip)^{\alpha + a} (iq)^{\beta}}{(ip)^{2a} \pm (iq)^{2b}}.$$

The absolute value of this expression can be estimated by

$$\left| \frac{(ip)^{\alpha + a} (iq)^{\beta}}{(ip)^{2N} \pm (iq)^{2N}} \right| \asymp \frac{p^\alpha q^\beta}{p^{N} q^{N}} \times \frac{p^{m} q^{m}}{p^{N} q^{N}}.$$

We have $\alpha/m + \beta/n < N$; therefore, the expression under consideration is $O(p^{-\varepsilon})$. The same is true for all operators $I_{a\beta}$ and $J_{a\beta}$.
Now we find a solution of the system of differential equations (3). Specifically, we are interested in the function \( \varphi_{j_0+1} \) used to define the operator \( s \).

If \( j_0 = 0 \), then we need only the first differential equation to find \( \varphi_1 \). By construction, \( j_0 = 0 \) means that \( a_0 = -1 \) and \( b_0 = 0 \). Therefore, we clearly have \( \varphi_1 = k_{pq} \frac{v_{pq}}{p^m} \), where \( |k_{pq}| \asymp 1 \).

If \( j_0 > 0 \), then, again by construction, \( a_0 = b_0 = 0 \) and we use the first equation from system (3) to conclude that
\[
\varphi_1 = \xi_{pq}^{(1)} \cdot \frac{v_{pq}}{p^m}, \quad \text{where} \quad \xi_{pq}^{(1)} = O(p^{-\varepsilon}).
\]

Observe that \( |\partial_2^n \varphi_1| = |\xi_{pq}^{(1)} \frac{v_{pq}}{p^m} | \asymp |\xi_{pq}^{(1)} v_{pq}| \) in this case. Now, if \( j_0 = 1 \), then we use the second equation to conclude that \( \varphi_2 = k_{pq} \frac{v_{pq}}{p^m} \) with \( |k_{pq}| \asymp 1 \). Indeed, in this case, \( \mu_1 = b_1 c_{pq} v_{pq} - a_1 d_{pq} v_{pq} \), and \( j_0 = -1 \) guarantee that \( a_1 = -1, b_1 = 0 \). If \( j_0 > 1 \), then we conclude from the second equation that \( \varphi_2 = \xi_{pq}^{(2)} v_{pq} \) with \( |\xi_{pq}^{(2)}| = O(p^{-\varepsilon}) \), etc.

In any case, the following relation holds for \( \varphi_{j_0+1} \):
\[
\varphi_{j_0+1} = k_{pq} \frac{v_{pq}}{p^m}, \quad \text{where} \quad |k_{pq}| \asymp 1.
\]

Now, to emphasize the dependence of \( \varphi_{j_0+1} \) on \( p \) and \( q \), we put \( \varphi^{(p,q)} := \varphi_{j_0} \). The system \( \{\varphi^{(p,q)}\} \) is orthogonal in \( W_{2}^{m-1, \frac{n-1}{2}} \) and we have
\[
\|\varphi^{(p,q)}\|_{W_{2}^{m-1, \frac{n-1}{2}}} \asymp p^{-\frac{m}{2}} q^{\frac{n-1}{2}} p^{-1/2} q^{-1/2}.
\]

Finally, consider the finite-dimensional operator
\[
A: W_{2}^{m-1, \frac{n-1}{2}} C(S),
\]
which takes \( p^{1/2} q^{1/2} \varphi^{(p,q)} \) to \( \alpha_{pq} f_{pq} \), where \( \{\alpha_{pq}\} \) is an arbitrary sequence of numbers such that \( \sum |\alpha_{pq}|^2 = 1 \). Here, we assume that \( p \) and \( q \) satisfy the previous conditions and \( p \leq M \) for a large number \( M \). More precisely, the operator \( A \) is the composition of the orthogonal projection onto \( \text{span}\{f_{p,q}\}_{p < M} \) and the described operator.

For any \( g \in W_{2}^{m-1, \frac{n-1}{2}} \) with
\[
g = \sum_{p < M} \chi_{pq} p^{1/2} q^{1/2} \varphi^{(p,q)},
\]
we have
\[
Ag = \sum_{p < M} \alpha_{pq} \chi_{pq} f_{pq}.
\]

Hence,
\[
\|Ag\|_{C(S)} \lesssim \sum_{p < M} \left| \chi_{pq} \alpha_{pq} \right| \leq \left( \sum_{p < M} |\chi_{pq}|^2 \right)^{1/2} \left( \sum_{p < M} |\alpha_{pq}|^2 \right)^{1/2} \leq \left( \sum_{p < M} |\chi_{pq}|^2 \right)^{1/2} \lesssim \|g\|_{W_{2}^{m-1, \frac{n-1}{2}}},
\]
and we conclude that \( \|A\| \lesssim 1 \).

Besides that, recall that the operator \( g \) (see the diagram at the end of Sec. 2.2) is 1-summing, therefore, \( AsgP \) is also 1-summing. Since this operator acts on a \( C(S) \)-space, it is also 1-integral; therefore, its restriction to a finite-dimensional subspace, namely, to \( \text{span}_{p < M}\{f_{pq}\} \), is 1-nuclear. The last property simply means that it has a finite trace and its trace can be estimated by its norm; see, for example, \([19, pp. 218–219]\). We have \( \text{tr}(AsgP) \lesssim \|AsgP\| \lesssim 1 \).
Now, we are going to prove that the last property is false. Recall that, by construction, the operator $sgiP$ takes $f_{pq}$ to $\psi^{(p,q)}$, and $A$ takes $\psi^{(p,q)}$ to $p^{-1/2}q^{-1/2}\alpha_{pq}f_{pq}$. Hence, the operator $AsgiP$ has diagonal form in the basis $\{f_{pq}\}$:

$$AsgiP(f_{pq}) = p^{-1/2}q^{-1/2}\alpha_{pq}f_{pq}.$$  

Since the trace of the operator under consideration is bounded, we infer that

$$\left|\sum p^{-1/2}q^{-1/2}\alpha_{pq}\right| \lesssim 1.$$  

The above inequality holds for an arbitrary sequence $(\alpha_{pq})$ with $\sum |\alpha_{pq}|^2 = 1$, hence,

$$\sum p^{-1}q^{-1} \lesssim 1.$$  

However, this is clearly false. Indeed, recall that the number of admissible $q$ is of order $p^{m/n}$, and we have $q \asymp p^{m/n}$ for every such $q$. Therefore, our sum is estimated from below simply by

$$\sum_{C<p<M} p^{-1}.$$  

Since $\sum p^{-1}$ is a divergent series, we arrive at a contradiction. The proof of Theorem 1 is finished.

### 3. Absence of local unconditional structure

Now, we start the proof of Theorem 2, the second main result of this paper. This proof mostly involves methods from [13] and uses the embedding theorem from [11, Fact 1]. Exactly as in the proof of Theorem 1, we suppose that the space $C^T(\mathbb{T}^2)$ has local unconditional structure.

#### 3.1. Main constructions

First, we repeat the same reduction procedures as in the previous section. We may consider the space $C^0_0(\mathbb{T}^2)$ in the place of $C^T(\mathbb{T}^2)$, since the passage to a complemented subspace preserves local unconditional structure. Besides that, we assume that all the additional assumptions from Sec. 2.1 are fulfilled. Next, we define the operators $i$, $g$, $s$ in the same way as in Sec. 2.2. We denote the senior and junior parts of $T_j$ by $\sigma_j$ and $\tau_j$, respectively.

Let $H$ denote the collection of differential operators corresponding to all points with integral coordinates on the line $\Lambda$. Then the embedding $j : C^0_0(\mathbb{T}^2) \to C^0(\mathbb{T}^2)$ is a continuous operator (see [1, Theorem 9.5]). Therefore, we have the following diagram:

$$C^H_0(\mathbb{T}^2) \overset{j}{\rightarrow} C^T_0(\mathbb{T}^2) \overset{i}{\rightarrow} C^{T_1,T_2}_0(\mathbb{T}^2) \overset{g}{\rightarrow} W_1^{T_1,T_2}(\mathbb{T}^2) \overset{s}{\rightarrow} W_2^{m-1,n-1}(\mathbb{T}^2).$$

Recall that the operator $g$ is 1-summing. Also, we need the following fact (see [3] or [17, Sec. 23]):

**Fact 4.** Let $X$ be a Banach space having local unconditional structure. Then every 1-summing operator $T$ from $X$ to an arbitrary Banach space $Y$ can be factored through the space $L^1$, i.e., there is a measure $\mu$ and operators $V : X \to L^1(\mu)$ and $U : L^1(\mu) \to Y^*$ such that $UV = \kappa T$, where $\kappa : Y \to Y^*$ is the canonical embedding.

Using the above fact, we obtain the following commutative diagram:

$$C^H_0(\mathbb{T}^2) \overset{j}{\rightarrow} C^T_0(\mathbb{T}^2) \overset{i}{\rightarrow} C^{T_1,T_2}_0(\mathbb{T}^2) \overset{g}{\rightarrow} W_1^{T_1,T_2}(\mathbb{T}^2) \overset{s}{\rightarrow} W_2^{m-1,n-1}(\mathbb{T}^2) \overset{V}{\rightarrow} L^1(\mu) \overset{U}{\rightarrow}$$
Now, we consider the following dual diagram:

\[
\begin{align*}
C_0^H(\mathbb{T}^2)^* & \xrightarrow{\lambda^*} C_0^0(\mathbb{T}^2)^* \xrightarrow{\eta^*} C_0^{T_1,T_2}(\mathbb{T}^2)^* \xrightarrow{\kappa^*} W_1^{T_1,T_2}(\mathbb{T}^2)^* \xrightarrow{s^*} W_2^{\frac{m-1}{n-1}}(\mathbb{T}^2) \\
& \downarrow L^\infty(\mu) \quad \downarrow V^* \quad \downarrow U^*
\end{align*}
\]

The next step of the proof is to construct a specific operator which takes elements of \(C_0^H(\mathbb{T}^2)^*\) to elements of the space \(W_1^H(\mathbb{T}^2)\). The last space is quasi-Banach; the corresponding definition is given below. This construction is the same as in [13]; we repeat it for the sake of completeness.

Consider the space \(W_2^H(\mathbb{T}^2)\) containing proper functions only and defined by the following seminorm:

\[
\|f\|_{W_2^H(\mathbb{T}^2)} = \max_{T \in H} \|Tf\|_{L^2(\mathbb{T}^2)}.
\]

Clearly, this is a Hilbert space. Recall that \(H = \{\partial_1^{pm_1} \partial_2^{(N-j)n}\}_{j=0}^N\). The space \(W_2^H(\mathbb{T}^2)\) is identifiable with a subspace of \(L^2(\mathbb{T}^2) \oplus \ldots \oplus L^2(\mathbb{T}^2)\) (there are \(N + 1\) copies of \(L^2(\mathbb{T}^2)\)); the identification is given by the following map:

\[
f \mapsto (\partial_1^{pm_1} \partial_2^{(N-j)n} f)^{(j)}_{j=0}.
\]

Therefore, consider the orthogonal projection \(P\) from the direct sum of \(N + 1\) copies of \(L^2(\mathbb{T}^2)\) onto \(W_2^H(\mathbb{T}^2)\). Below we reproduce from [13] certain properties of the operators under consideration.

First, it is easy to see how \(P\) acts on a natural basis of \(L^2(\mathbb{T}^2) \oplus \ldots \oplus L^2(\mathbb{T}^2)\). Suppose that \(k = (k_1, k_2)\) is a pair of integers and denote by \(\phi^l_k\) the following element of the space \(L^2(\mathbb{T}^2) \oplus \ldots \oplus L^2(\mathbb{T}^2)\):

\[
\phi^l_k = (0, 0, \ldots, 0, z_{1}^{k_1} z_{2}^{k_2}, 0, 0, \ldots, 0),
\]

where \(z_{1}^{k_1} z_{2}^{k_1}\) is at the \(l\)-th position, \(0 \leq l \leq N\). The following fact is obtained by simple computations.

**Fact 5.** If either \(k_1\) or \(k_2\) equals 0, then \(P(\phi^l_k) = 0\). Otherwise,

\[
P(\phi^l_k) = \lambda_l \left( \sum_{j=0}^{N} |\lambda_j|^2 \right)^{-1} \lambda_0 z_{1}^{k_1} z_{2}^{k_2}, \ldots, \lambda_N z_{1}^{k_1} z_{2}^{k_2},
\]

where \(\lambda_j = (ik_1)^{pm_1} (ik_2)^{(N-j)n}\).

Next, we need to understand how \(P\) acts on the space \(C_0^H(\mathbb{T}^2)^*\). The space \(C_0^H(\mathbb{T}^2)^*\) is identifiable with a subspace of \(C(\mathbb{T}^2) \oplus \ldots \oplus C(\mathbb{T}^2)\), in the same way as \(W_2^H(\mathbb{T}^2)\) is identified with a subspace of \(L^2(\mathbb{T}^2) \oplus \ldots \oplus L^2(\mathbb{T}^2)\). Therefore, we have

\[
C_0^H(\mathbb{T}^2)^* = (\mathcal{M}(\mathbb{T}^2) \oplus \ldots \oplus \mathcal{M}(\mathbb{T}^2))/\mathcal{X}.
\]

where \(\mathcal{X}\) is the annihilator of \(C_0^H(\mathbb{T}^2)\) in \((C(\mathbb{T}^2) \oplus \ldots \oplus C(\mathbb{T}^2))^*\), that is,

\[
\mathcal{X} = \left\{ (\mu_0, \mu_1, \ldots, \mu_N) : \sum_{j=0}^{N} \int \partial_1^{pm_1} \partial_2^{(N-j)n} g d\mu_j = 0 \ \forall \ g \in C_0^H(\mathbb{T}^2) \right\}.
\]

Formally speaking, at this point, we should consider \(\Phi_M\), the convolution with \(M\)-th Fejér kernel in both variables, and the operators \(P_M\) such that

\[
P_M(F) = P(\Phi_M \mu_0, \Phi_M \mu_1, \ldots, \Phi_M \mu_N), \quad F \in C_0^H(\mathbb{T}^2)^*.
\]

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where \((\mu_0, \ldots, \mu_N)\) is any representative of the functional \(F\). This formula is meaningful because \(P\) is an orthogonal projection and if \((\nu_0, \ldots, \nu_N)\) is in \(X\), then \((\Phi_M \nu_0, \ldots, \Phi_M \nu_N)\) is in \(X \cap (L^2(T^2) \oplus \ldots \oplus L^2(T^2))\), that is, in the kernel of the projection \(P\). Now, we formulate the following fact from [13].

**Fact 6.** The operators \(P_M : C^H_0(T^2) \to W^H_{1/2}(T^2)\) are uniformly bounded in \(M\).

The definition of the space \(W^H_{1/2}(T^2)\) is now clear from the context.

The proof, modulo certain technical details, follows from the theory of singular integrals and Fourier multipliers with mixed homogeneity developed in [2] (the formula in Fact 5 guarantees that the components of \(P\) are Fourier multipliers with a certain homogeneity), and it is even true that these operators are uniformly of weak type \((1,1)\). Of course, there are certain technical differences; for example, in [2], everything is done for the space \(\mathbb{R}^n\) instead of \(T^n\). Nevertheless, these differences can be overcome quite easily; see [13] for details.

Since all the estimates are uniform in \(M\), we omit the letter \(M\) in our notation. We have the following commutative diagram:

\[
\begin{array}{c}
W^H_{1/2}(T^2) \xleftarrow{P} C^H_0(T^2) \xleftarrow{j^*} C^T_0(T^2) \xleftarrow{\bar{v}^*} C^T_{1/2}(T^2) \xleftarrow{\bar{v}} W^1_{T_1, T_2}(T^2) \xleftarrow{\bar{v}^*} W^2_{2^{m-1} \cdot 2^{n-1}}(T^2) \\
\end{array}
\]

Now, we use several facts from the theory of \(p\)-summing operators. A good reference for this topic is [6]. The space \(W^H_{1/2}(T^2)\) is a quasi-Banach space of cotype 2 and \(L^\infty(\mu)\) is a space of type \(C(K)\). Therefore, \(Pj^* V^*\) is a 2-summing operator; this is a generalization of Grothendieck’s theorem, see [6] for details. Hence, \(Pj^* i^* g^* s^*\) is also 2-summing. In the next subsection, we show that this is not the case.

### 3.2. Final computations and a contradiction

As in the proof of Theorem 1, let \(v_{pq}\) denote the function \(z_1^{p} z_2^{q}\). Again, we consider only sufficiently large values of \(p\) and assume that the pair \((p, q)\) in question satisfies the following condition:

\[
\frac{\delta}{2} q^n \leq p^m \leq \delta q^n.
\]

We have

\[
\|v_{pq}\|_{W^2} \leq p^{m-1} \cdot 2^{-\frac{n-1}{2}} \leq p^{m-1} \cdot q^{\frac{n-1}{2}} \leq p^m \cdot p^{-1/2} \cdot q^{-1/2}.
\]

Let \(w_{pq}\) denote the function \(\frac{v_{pq}}{\|v_{pq}\|}\). The corresponding system is orthonormal in the space \(W^2_{2^{m-1} \cdot 2^{n-1}}\), hence, it is weakly 2-sumnable. Since \(Pj^* i^* g^* s^*\) is a 2-summing operator (by definition, this means that the operator takes weakly 2-sumnable sequences to 2-sumnable sequences), we conclude that

\[
\sum \|Pj^* i^* g^* s^* w_{pq}\|_{W^2}^2 < \infty.
\]

First, let us find the image of the function \(w_{pq}\) under \(j^* i^* g^* s^*\). Choose a function \(v_{\bar{p} \bar{q}} = z_1^{\bar{p}} z_2^{\bar{q}} \in C_0(T^2)\); the linear combinations of such functions are dense in \(C_0(T^2)\). We have

\[
\langle v_{\bar{p} \bar{q}}, (j^* i^* g^* s^*) w_{pq}\rangle = \langle (sg^i j) v_{\bar{p} \bar{q}}, w_{pq}\rangle = \langle sv_{\bar{p} \bar{q}}, w_{pq}\rangle = \frac{m-1}{2} \cdot \frac{n-1}{2}.
\]

Now, recall how \(s\) acts on the function \(v_{\bar{p} \bar{q}}\). We have to solve the system of equations (3); also, by definition, all functions \(\mu_j\) are multiples of \(z_1^{\bar{p}} z_2^{\bar{q}}\). Therefore, the solution in question is also a multiple of \(z_1^{\bar{p}} z_2^{\bar{q}}\); thus, \(\langle sv_{\bar{p} \bar{q}}, w_{pq}\rangle = \frac{m-1}{2} \cdot \frac{n-1}{2} \neq 0\) only for \(p = \bar{p}\) and \(q = \bar{q}\).
Now, we find the function $s(v_{pq})$. As shown in the previous section, $s$ takes $\frac{v_{pq}}{p}m$ to $k_{pq}^m p^{-m}$, where $|k_{pq}| \approx 1$. Thus,

$$s(v_{pq}) = k_{pq}p^{mN} p^{-m} v_{pq}.$$  

Hence, we have the following identity:

$$\langle sv_{pq}, w_{pq} \rangle = k_{pq}p^{mN} p^{-m} \|v_{pq}\|_{w_{pq}}^{\frac{m-1}{n+1}} \approx k_{pq}p^{mN} p^{-m} p^{-1/2} q^{-1/2} = k_{pq}p^{mN} p^{-1/2} q^{-1/2}.$$  

Therefore, we arrive at the following formula for the right-hand side of (4):

$$\langle sv_{pq}, w_{pq} \rangle_{w_{pq}}^{m-1} \approx \begin{cases} 0, & (p,q) \neq (\tilde{p}, \tilde{q}), \\ k_{pq}p^{mN} p^{-1/2} q^{-1/2}, & (p,q) = (\tilde{p}, \tilde{q}). \end{cases}$$

Recall that the following element of $C(T^2) \oplus \ldots \oplus C(T^2)$ corresponds to $v_{pq} \in C^0(T^2)$:

$$(\hat{\partial}_1^{jm} \hat{\partial}_2^{(N-j)m} v_{pq})_{j=0} = ((ip)^j (iq)^{(N-j)n} v_{pq})_{j=0}.$$  

Since $p^{mN} \approx q^{nN}$, we can take the following representative from the equivalence class corresponding to $(j^* s^* g^* s^*)w_{pq}$:

$$(l_{pq}^{-1/2} q^{-1/2} v_{pq}, 0, 0, \ldots, 0), \quad \text{where} \quad |l_{pq}| \approx 1.$$  

Finally, we apply the projection $P$, using the formula from Fact 5; in our case, $|\lambda_j| = p^{jm} q^{(N-j)n} \approx p^{mN}$, hence, $\lambda_l \lambda_k (\sum |\lambda_j|^2)^{-1} \approx 1$. We have

$$\sum \|l_{pq}^{-1/2} q^{-1/2} v_{pq}\|_{L^1}^{1/2} \approx \sum p^{-1} q^{-1},$$

and we already know that this sum is divergent. Therefore, we arrive at a contradiction and the theorem is proved.

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