Repulsive Casimir-Polder forces from cosmic strings

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Abstract

We investigate the Casimir-Polder force acting on a polarizable microparticle in the geometry of a straight cosmic string. In order to develop this analysis we evaluate the electromagnetic field Green tensor on the imaginary frequency axis. The expression for the Casimir-Polder force is derived in the general case of anisotropic polarizability. In dependence of the eigenvalues for the polarizability tensor and of the orientation of its principal axes, the Casimir-Polder force can be either repulsive or attractive. Moreover, there are situations where the force changes the sign with separation. We show that for an isotropic polarizability tensor the force is always repulsive. At large separations between the microparticle and the string, the force varies inversely as the fifth power of the distance. In the non-retarded regime, corresponding to separations smaller than the relevant transition wavelengths, the force decays as the inverse fourth power of the distance. In the case of anisotropic polarizability, the dependence of the Casimir-Polder potential on the orientation of the polarizability tensor principal axes also leads to the moment of force acting on the particle.

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1 Introduction

As a result of symmetry breaking phase transitions during the evolution of the early universe various topological defects could arise [1]. In particular, cosmic strings have been of increasing interest due to the importance that they may have in cosmology. Though the recent observational data on the cosmic microwave background radiation have ruled out cosmic strings as the primary source for primordial density perturbations, they are still candidates for the generation of a number of interesting physical effects such as the generation of gravitational waves, high-energy cosmic rays, and gamma ray bursts. More recently it has been shown that cosmic stings form in brane inflation models as a by product of the annihilation of the branes (for a review see [2]).

In the simplest theoretical model the spacetime of a cosmic string is described by the flat metric with a deficit angle around the string. The corresponding non-trivial topology results in the distortion of the zero-point vacuum fluctuations of quantized fields and induces non-zero vacuum expectation values for physical observables. Another type of vacuum polarization arises when boundaries are present. The imposed boundary conditions on quantum fields alter the

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zero-point oscillations spectrum and lead to additional shifts in the vacuum expectation values. This is the well-known Casimir effect (for a review see [3]). Combined effects of topology and boundaries on the quantum vacuum in the geometry of a cosmic string have been investigated for scalar [4], vector [5, 6] and fermionic fields [7, 8], obeying boundary conditions on cylindrical surfaces (for the Casimir effect in the closely related geometry of a wedge with cylindrical boundary see Ref. [9]). The analysis of the Casimir force for massless scalar fields subject to Dirichlet, Neumann and hybrid boundary conditions in the setting of the conical piston has been recently developed in [10]. The vacuum polarization effects in a cosmic string spacetime induced by a scalar field obeying Dirichlet or Neumann boundary conditions on a surface orthogonal to the string are considered in [11].

The distortion of the vacuum fluctuations spectrum by the cosmic string also gives rise to Casimir-Polder forces acting on a polarizable microparticle. Nowadays, these forces have attracted a great deal of attention because of their important role in many areas of science [12, 13]. With advances in nanotechnology and successes in the production of ultracold atoms it became possible to measure the both Casimir and Casimir-Polder forces with increased precision (for a review see [14]). In particular, related to possible applications for micro and nano electromechanical systems, the search for geometries that would lead to repulsive Casimir-Polder forces is under active research (see, for instance, discussion in [15] and references therein). In Ref. [16] the Casimir-Polder force is considered in the static limit for a polarizable microparticle located near a cosmic string. This approximation is valid at large distances from the string where the dominant contribution to the force comes from low frequencies. In the present paper we derive the exact Casimir-Polder potential for the general case of frequency dependent anisotropic polarizability which allows us to consider the both non-retarded and retarded regimes. In particular, we show that in the special case of isotropic polarizability the force is always repulsive.

The paper is organized as follows. In the next section we present our calculation of the electromagnetic field Green tensor in the frequency domain for the geometry of a cosmic string. The Casimir-Polder potential and the force acting on a polarizable microparticle are investigated in sections 3 and 4 respectively. The main results are summarized in section 5. The evaluation of the functions appearing in the expressions of the components for the Green tensor is given in Appendix.

## 2 Electromagnetic field Green tensor

We consider the geometry of an infinitely long straight cosmic string. In the cylindrical coordinates \((x^1, x^2, x^3) = (r, \phi, z)\) the corresponding line element has the standard Minkowskian form

\[
d s^2 = d t^2 - d r^2 - r^2 d \phi^2 - d z^2,
\]

with the difference that now \(0 \leq \phi \leq \phi_0\) and the spatial points \((r, \phi, z)\) and \((r, \phi + \phi_0, z)\) are to be identified. The planar angle deficit is related to the mass \(\mu_0\) per unit length of the string by \(2\pi - \phi_0 = 8\pi G\mu_0\), where \(G\) is the gravitational constant. Effective metric with a planar angle deficit also arises in a number of condensed matter systems (see, for instance, [17]). The nontrivial topology due to the cosmic string changes the structure of the vacuum electromagnetic field. One of consequences is that a neutral polarizable microparticle placed close to the string experiences a dispersion force, the Casimir-Polder force. For a microparticle situated at a point \(r\), the corresponding interaction energy can be expressed as (see [13])

\[
U(r) = \frac{1}{2\pi} \int_0^\infty d \xi \, a_{jl}(i\xi)G^{(0)}_{jl}(r, r; i\xi),
\]

(2)
where $\alpha_{jl}(i\xi)$ is the polarizability tensor of a particle evaluated at a purely imaginary frequency, 
\[
G^{(s)}_{jl}(r, r'; \omega) = \int_{-\infty}^{+\infty} d\tau G^{(s)}_{jl}(x, x')e^{i\omega\tau}.
\] (3)
with $x = (t, r)$, $x' = (t', r')$, $\tau = t - t'$, and summation is understood over the indices $j, l = 1, 2, 3$. The expression (3) of the Casimir-Polder potential is given in terms of the subtracted Green tensor
\[
G^{(s)}_{jl}(x, x') = G_{jl}(x, x') - G_{jl}^{(M)}(x, x'),
\] (4)
where $G_{jl}(x, x')$ is the retarded Green tensor for the electromagnetic field in the geometry of a cosmic string and $G_{jl}^{(M)}(x, x')$ is the corresponding tensor in the Minkowski spacetime. Note that in (2) and in what follows we write the relations in terms of the physical components of the tensors. As the geometry of a cosmic string is flat outside the string core, the renormalization procedure is reduced to the subtraction of the Minkowskian part.

The retarded Green tensor for the electromagnetic field is given by the expression
\[
G_{jl}(x, x') = -i\theta(\tau)(E_j(x)E_l(x') - E_l(x')E_j(x)),
\] (5)
where $\theta(x)$ is the unit-step function, $E_j(x)$ is the operator of the $j$ component of an electric field, and the angular brackets mean the vacuum expectation value. By expanding the electric field operator over a complete set of mode functions $\{E_\alpha(x), E^*_\alpha(x)\}$ and using the commutation relations, the following mode sum formula is obtained for the Green tensor:
\[
G_{jl}(x, x') = -i\theta(\tau)\sum_\alpha \left[E_{\alpha j}(x)E^*_{\alpha l}(x') - E_{\alpha l}(x')E^*_{\alpha j}(x)\right],
\] (6)
where asterisk stands for a complex conjugate and the collective index $\alpha$ specifies the mode functions.

For the geometry under consideration, we have two different classes of mode functions corresponding to the cylindrical waves of the transverse magnetic (TM) and transverse electric (TE) types. They are obtained from the corresponding mode functions for the vector potential given in Ref. [6] and have the form
\[
E^{(\lambda)}_\alpha(x) = \frac{q}{2\pi \gamma \omega}E^{(\lambda)}(r)e^{i\gamma m \phi + ikz - i\omega t},
\] (7)
where $m = 0, \pm 1, \pm 2, \ldots$, $-\infty < k < +\infty$, $0 \leq \gamma < \infty$,
\[
\omega^2 = \gamma^2 + k^2, \quad q = 2\pi/\phi_0,
\] (8)
and $\lambda = 0, 1$ correspond to the TM and TE waves, respectively. In (7), the radial functions for separate components of the electric field, $E^{(\lambda)}_l(r)$, are given by the expressions
\[
E^{(0)}_1(r) = i\gamma J_{q|l|}(\gamma r), \quad E^{(0)}_2(r) = -\frac{kq}{r} J_{q|l|}(\gamma r), \quad E^{(0)}_3(r) = \gamma^2 J_{q|l|}(\gamma r), \\
E^{(1)}_1(r) = -\frac{\omega q}{r} J_{q|l|}(\gamma r), \quad E^{(1)}_2(r) = -i\omega \gamma J_{q|l|}(\gamma r), \quad E^{(1)}_3(r) = 0,
\] (9)
where $J_\nu(x)$ is the Bessel function, the prime means the derivative with respect to the argument, and the indices $l = 1, 2, 3$ correspond to the coordinates $r, \phi, z$, respectively.
Substituting the mode functions (7) into the mode sum formula (6), the following representation is obtained for the Green tensor:

\[
G_{jl}(x, x') = -\frac{iq(r)}{2\pi} \sum_{m=-\infty}^{+\infty} \sum_{\lambda=0,1} \int_{-\infty}^{+\infty} dk \int_{0}^{\infty} d\gamma \frac{1}{\gamma \omega} \times \left[ e^{iqm\Delta \phi + ik\Delta z - i\omega r} E_{j}^{(\lambda)}(r) E_{l}^{(\lambda)*}(r') - e^{-iqm\Delta \phi - ik\Delta z + i\omega r} E_{j}^{(\lambda)}(r') E_{l}^{(\lambda)*}(r) \right],
\]

where \(\Delta \phi = \phi - \phi'\) and \(\Delta z = z - z'\). By using this representation, the spectral component of the Green tensor, appearing in (2), is presented in the form:

\[
G_{jl}(r, r'; i\xi) = -\frac{q}{2\pi} \sum_{m=-\infty}^{+\infty} \sum_{\lambda=0,1} \int_{-\infty}^{+\infty} dk \int_{0}^{\infty} d\gamma \frac{1}{\gamma \omega} \times \left[ E_{j}^{(\lambda)}(r) E_{l}^{(\lambda)*}(r') \frac{e^{iqm\Delta \phi + ik\Delta z}}{\omega - i\xi} + E_{j}^{(\lambda)}(r') E_{l}^{(\lambda)*}(r) \frac{e^{-iqm\Delta \phi - ik\Delta z}}{\omega + i\xi} \right].
\]

By taking into account Eq. (9), we find the expressions

\[
G_{11}(r, r'; i\xi) = \frac{2q}{\pi} \left[ \partial_{\Delta z}^{2} B(r, r', \Delta \phi, \Delta z) + \frac{2}{rr'} \partial_{\Delta \phi}^{2} A(r, r', \Delta \phi, \Delta z) \right],
\]

\[
G_{22}(r, r'; i\xi) = \frac{2q}{\pi} \left[ \partial_{\Delta z}^{2} B(r, r', \Delta \phi, \Delta z) - 2\partial_{\Delta \phi} A(r, r', \Delta \phi, \Delta z) \right],
\]

\[
G_{33}(r, r'; i\xi) = \frac{4q}{\pi} (-\partial_{\Delta z}^{2} + \xi^{2}) A(r, r', \Delta \phi, \Delta z),
\]

for diagonal components and the expressions

\[
G_{12}(r, r'; i\xi) = \frac{4q}{\pi} \left[ 2\partial_{\xi z} \partial_{\Delta z}^{2} \partial_{\Delta \phi} + r' \partial_{\xi} \partial_{\Delta \phi} \right] A(r, r', \Delta \phi, \Delta z),
\]

\[
G_{13}(r, r'; i\xi) = -\frac{4q}{\pi} \partial_{\Delta \phi} \partial_{\xi} A(r, r', \Delta \phi, \Delta z),
\]

\[
G_{23}(r, r'; i\xi) = -\frac{4q}{\pi r} \partial_{\Delta \phi} \partial_{\Delta z} A(r, r', \Delta \phi, \Delta z),
\]

for off-diagonal ones. Here we have defined the functions

\[
A(r, r', \Delta \phi, \Delta z) = \sum_{m=0}^{\infty} \cos(qm\Delta \phi) \int_{0}^{\infty} dk \cos(k\Delta z) \times \int_{0}^{\infty} d\gamma J_{qm}(\gamma r) J_{qm}(\gamma r'),
\]

\[
B(r, r', \Delta \phi, \Delta z) = \sum_{j=\pm 1} \sum_{m=0}^{\infty} \cos(qm\Delta \phi) \int_{0}^{\infty} dk \cos(k\Delta z) \times \int_{0}^{\infty} d\gamma J_{qm-j}(\gamma r) J_{qm-j}(\gamma r'),
\]

where the prime on the sign of the sum means that the term \(m = 0\) should be taken with the weight 1/2. The remained off-diagonal components of the Green tensor are obtained from those in (13) by using the relation

\[
G_{ij}(r', r; -i\xi) = G_{ji}(r, r'; i\xi).
\]
In accordance with Eqs. (12) and (13), the evaluation of the components for the Green tensor is reduced to that for the functions (14) and (15). The corresponding procedure is described in Appendix. The expressions which will be used below are given by Eqs. (67) and (72).

For the evaluation of the Casimir-Polder potential by using Eq. (2), we need the components of the subtracted Green tensor (4) in the coincidence limit: \( r' \to r \). In this limit the Green tensor \( G_{jl}(r, r'; i\xi) \) is divergent. In the problem under consideration, the renormalization is reduced to the subtraction of the part corresponding to the Green tensor in the Minkowski spacetime. The components of the Green tensor in a cosmic string geometry are given by (12) and (13) with the functions \( A(r, r', \Delta \phi, \Delta z) \) and \( B(r, r', \Delta \phi, \Delta z) \) from Eqs. (67) and (72). As it has been emphasized in Appendix, the \( k = 0 \) terms in these expressions coincide with the corresponding components in the Minkowski spacetime. Hence, the subtracted Green tensor is obtained omitting the \( k = 0 \) terms in the expressions for \( G_{jl}(r, r'; i\xi) \). It can be seen that in the coincidence limit the corresponding off-diagonal components vanish. Further, in the evaluation of the coincidence limit for \( \partial^2_{\Delta \phi} A(r, r', \Delta \phi, \Delta z) \), it is convenient to use the relation

\[
\lim_{r' \to r} \partial^2_{\Delta \phi} A(r, r', \Delta \phi, \Delta z) = -\lim_{r' \to r} \left[ r \partial_r (r \partial_r) - 4r^2 \partial(\Delta z)^2 (r \partial_r + 1) \right] A(r, r', \Delta \phi, \Delta z).
\]

This relation is obtained from the representation (63) for the function \( A(r, r', \Delta \phi, \Delta z) \) on using the properties of the modified Bessel function. After a straightforward but long calculation one then obtains the following closed expression for the nonzero components of the subtracted Green tensor in the coincidence limit (no summation over \( l \)):

\[
G^{(s)}_{ll}(r, r; i\xi) = 2\xi^3 \left[ \sum_{k=1}^{[q/2]} f_l(2\xi rs_k, s_k) - \frac{q}{\pi} \sin(q\pi) \int_0^\infty dy \frac{f_l(2\xi r \cosh(y), \cosh(y))}{\cosh(2qy) - \cos(q\pi)} \right],
\]

where \([q/2]\) means the integer part of \(q/2\) and we have introduced the notation

\[
s_k = \sin(\pi k/q).
\]

In (20), the function \( f_l(u, v) \) is defined as

\[
f_l(u, v) = e^{-u} \sum_{p=1}^3 b_{lp}(v)u^{p-4},
\]

with

\[
b_{lp}(v) = b_{lp}^{(0)} + b_{lp}^{(1)} v^2,
\]

and with the coefficients

\[
b_{lp}^{(0)} = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -2 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad b_{lp}^{(1)} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix},
\]

where the rows and columns are numbered by \( l \) and \( p \) respectively. For integer values of the parameter \( q \), formula (18) is reduced to

\[
G^{(s)}_{ll}(r, r; i\xi) = \xi^3 \sum_{k=1}^{q-1} f_l(2\xi rs_k, s_k).
\]

Note that the first term in the square brackets of (18) dominates for large values of \( q \).
3 Casimir-Polder potential

Given the subtracted Green tensor in the coincidence limit of the arguments, we can evaluate the Casimir-Polder potential by using the formula (2). By taking into account Eq. (18), the potential is expressed as

\[ U(r) = \frac{r^{-4}}{16\pi} \sum_{l,p=1}^{3} \left[ \sum_{k=1}^{[q/2]} \frac{b_{lp}(s_k) h_{lp}(2rs_k)}{s_k^4} - \frac{q}{\pi} \sin(q\pi) \int_{0}^{\infty} dy \frac{h_{lp}(2r \cosh y)}{\cosh(2qy) - \cos(q\pi)} \frac{b_{lp}(\cosh y)}{\cosh^4 y} \right], \]

where we have defined the function

\[ h_{lp}(y) = \int_{0}^{\infty} dx \ x^{p-1} e^{-x} \alpha_{U}(ix/y). \]

For integer values of \( q \) the expression for the potential takes the form

\[ U(r) = \frac{r^{-4}}{32\pi} \sum_{k=1}^{q-1} \sum_{l,p=1}^{3} \frac{b_{lp}(s_k) h_{lp}(2rs_k)}{s_k^4}. \]

The Casimir-Polder potential, given by Eq. (24), depends on the distance from the string an on the angles determining the orientation of the principal axes of the polarizability tensor with respect to the cosmic string. These angles enter in the formula through the components \( \alpha_{U}(i\xi) \) of the polarizability tensor in the cylindrical coordinates \((r, \phi, z)\) with the \( z \)-axis along the string.

In order to make the dependence on the orientation of the microparticle more explicit, we introduce the Cartesian coordinates \( x' = (x', y', z') \) with the axes directed along the principal axes of the polarizability tensor. For the polarizability tensor in this coordinates one has:

\[ \alpha_{lm}'(\omega) = \text{diag}(\alpha_1(\omega), \alpha_2(\omega), \alpha_3(\omega)), \]

where \( \alpha_{l}(\omega) \) are the corresponding eigenvalues. In addition, we introduce an intermediate Cartesian coordinates \( x'' = (x'', y'', z'') \) with the \( z'' \)-axis along the string and with the particle location at \((r, 0, 0)\). Introducing the coefficients \( \beta_{lm} \) in accordance with \( x''_l = r \delta_{l1} + \sum_{m=1}^{3} \beta_{lm} x'_m \), it can be seen that (no summation over \( l \))

\[ \alpha_{U}(\omega) = \sum_{m=1}^{3} \beta_{lm}^2 \alpha_{m}(\omega). \]

The coefficients \( \beta_{lm} \), with \( \sum_{m=1}^{3} \beta_{lm}^2 = 1 \), can be given in terms of the Euler angles determining the orientation of the coordinate system \( x'_l \) with respect to the system \( x''_l \). For an isotropic particle with \( \alpha_{m}(\omega) \equiv \alpha(\omega) \) one has \( \alpha_{U}(\omega) = \alpha(\omega) \). The general expression for the matrix \( \beta_{lm} \) can be found, for example, in [18].

Here we shall consider an important special case of (27) with

\[ \alpha_1(\omega) = \alpha_2(\omega). \]

In this case the Casimir-Polder potential in Eq. (24) has the form \( \sum_{l,m=1}^{3} a_{lm} \beta_{lm}^2 \) with \( a_{11} = a_{12} \). Now, it can be seen that

\[ \sum_{m=1}^{3} a_{lm} \beta_{lm}^2 = a_{11} - (a_{11} - a_{13}) \beta_{13}^2, \]

where

\[ \beta_{13} = \cos \alpha \sin \beta, \ \beta_{23} = \sin \alpha \sin \beta, \ \beta_{33} = \cos \beta. \]
In (31), $\beta$ is the angle between the principal axis $z'$ of the polarizability tensor and the axis of the cosmic string, $\alpha$ is the angle between the intersection line of the planes $(x', y')$ and $(x'', y'')$ and the axis $y''$. The potential does not depend on the third Euler angle. The latter is a consequence of the symmetry (29).

Let us consider the asymptotic of the general formula (24) at large distances from the string. By taking into account that for $y \gg 1$ one has

$$h_{lp}(y) \approx \alpha_{ll}(0) \Gamma(p),$$

with $\alpha_{ll}(0)$ being the static polarizability of a particle, the potential is presented in the form

$$U(r) \approx \frac{1}{16\pi r^4} \sum_{l,p=1}^{3} \Gamma(p) \alpha_{ll}(0) \left[b_{lp}^{(0)} g_4(q) + b_{lp}^{(1)} g_2(q)\right].$$

Here we have introduce the notation

$$g_n(q) = \sum_{k=1}^{[q/2]} s_k^{-n} - \frac{q}{\pi} \sin(q\pi) \int_{0}^{\infty} dy \frac{\cosh^{-n}(y)}{\cosh(2qy) - \cos(q\pi)}.$$  

For the functions $g_n(q)$ in (33) one has the following expressions

$$g_2(q) = \frac{q^2 - 1}{6}, \quad g_4(q) = \frac{(q^2 - 1)(q^2 + 11)}{90}.$$  

By taking into account the expressions (22) of the coefficients, for the potential at large distances one gets (note that $\sum_{p=1}^{3} \Gamma(p) b_{lp}^{(1)} = 0$):

$$U(r) \approx \frac{(q^2 - 1)(q^2 + 11)}{360\pi r^4} \left[\alpha_{11}(0) - \alpha_{22}(0) + \alpha_{33}(0)\right].$$

This expression coincides with the result given in [16] (with the coefficient corrected) for the static limit. In particular, for an isotropic polarizability the corresponding force is repulsive.

In the special case (29), by taking into account Eqs. (28) and (31), we find the asymptotic expression

$$U(r) \approx \frac{(q^2 - 1)(q^2 + 11)}{360\pi r^4} \left[\alpha_{3}(0) + 2(\alpha_{1}(0) - \alpha_{3}(0))\sin^2 \alpha \sin^2 \beta\right].$$

If $\alpha_{3}(0) > \alpha_{1}(0)$, with dependence on the orientation of the principal axes of the polarizability tensor with respect to the cosmic string, the force corresponding to Eq. (37) can be either attractive or repulsive.

In the case of isotropic polarizability, $\alpha_{jl}(\omega) = \delta_{jl}\alpha(\omega)$, the function $h_{lp}(v)$ in general formula (24) does not depend on $l$ and we can explicitly sum over $l$. Introducing the notation

$$b_p(v) = \sum_{l=1}^{3} b_{lp}(v),$$

the expression for the Casimir-Polder potential in the isotropic case is obtained from (24) by the replacement

$$\sum_{l=1}^{3} b_{lp}(v) h_{lp}(2rv) \rightarrow b_p(v) h_p(2rv),$$
with \( h_p(y) = \int_0^\infty dx \, x^{p-1}e^{-x}\alpha(ix/y) \). For the functions \( b_p(v) \) one has

\[
b_1(v) = b_2(v) = 2v^2, \quad b_3(v) = 2 - 2v^2.
\]  

Now we return to the general formula (24) for the Casimir-Polder potential. For the further transformation of the corresponding expression the polarizability tensor should be specified. For the functions \( \alpha_m(\omega) \) in (27) we use the anisotropic oscillator model:

\[
\alpha_m(i\xi) = \sum_j g_j^{(m)} \frac{\omega_j^{(m)}}{\omega_j^{(m)} + \xi^2},
\]  

where \( \omega_j^{(m)} \) and \( g_j^{(m)} \) are the oscillator frequencies and strengths, respectively. This model for the dynamic polarizability works well over a wide range of separations. Now, by taking into account (28), the functions in (25) are presented in the form

\[
h_{lp}(y) = y^2 \sum_{m=1}^3 \sum_j g_j^{(m)} \beta_{lm}^2 B_p(y\omega_j^{(m)}).\]  

with the notation

\[
B_p(z) = \int_0^\infty dx \, x^{p-1}e^{-x}x^{-2} + z^2.
\]

The expression for the integral in (43) for general \( p \) can be found in [19]. In the cases needed in (42) one has

\[
B_1(z) = z^{-1} [\sin(z)\text{ci}(z) - \cos(z)\text{si}(z)],
\]

\[
B_2(z) = -\cos(z)\text{ci}(z) - \sin(z)\text{si}(z),
\]

\[
B_3(z) = -z [\sin(z)\text{ci}(z) - \cos(z)\text{si}(z)] + 1.
\]

The functions in the right-hand side of Eq. (41) are defined as

\[
\text{ci}(x) = -\int_x^\infty \frac{\cos t}{t}dt, \quad \text{si}(x) = \text{Si}(x) - \pi/2 = -\int_x^\infty \frac{\sin t}{t}dt.
\]

As a result, for the Casimir-Polder potential we find the following expressions

\[
U(r) = \frac{1}{4\pi r^2} \sum_{l,m,p=1}^3 \sum_j g_j^{(m)} \beta_{lm}^2 \left[ \sum_{k=1}^{[q/2]} \frac{b_{lp}(s_k)}{s_k^2} B_p(2r\omega_j^{(m)} s_k) \right.
\]

\[
- \frac{q}{\pi} \sin(q\pi) \int_0^\infty dy \frac{B_p(2r\omega_j^{(m)} \cosh y)}{\cosh(2qy) - \cos(q\pi)} \frac{b_{lp}(\cosh y)}{\cosh^2 y},
\]

where the functions \( b_{lp}(v) \) are defined by Eq. (21). For the isotropic case the corresponding formula is obtained from (46) by the replacement

\[
\sum_{l,m=1}^3 \beta_{lm}^2 b_{lp}(v) B_p(2rv\omega_j^{(m)}) \rightarrow b_p(v) B_p(2rv\omega_j),
\]

with the functions \( b_p(v) \) given by (40).

The characteristic length scale in the problem at hand is given by the wavelengths corresponding to oscillator frequencies \( \omega_j^{(m)} \). At small distances from the string, \( r \ll 1/\omega_j^{(m)} \), the
dominant contribution in \((46)\) comes from the term with \(p = 1\). By taking into account that for \(z \ll 1\) one has \(B_1(z) \approx \pi/(2z)\), in the leading order, for the potential one finds

\[
U(r) \approx \frac{1}{16r^3} \sum_{l,m=1}^{3} \sum_{j} \eta_j^{(m)} \beta_{lm}^2 \left[ b_{l1}^{(0)} g_3(q) + b_{l1}^{(1)} g_1(q) \right],
\]

(48)

where the function \(g_n(q)\) is defined in Eq. (34),

\[
b_{l1}^{(0)} = (1, -2, 1), \quad b_{l1}^{(1)} = (1, 1, 0),
\]

(49)

and we have introduced the notation

\[
\eta_j^{(m)} = g_j^{(m)}/\omega_j^{(m)}.
\]

(50)

Note that for odd values of \(n\) we have no closed expressions for \(g_n(q)\). The dependence of the asymptotic expression \((48)\) on the orientation of the polarizability tensor principal axes is simplified in the special case \((29)\):

\[
U(r) \approx \frac{1}{16r^3} \sum_j \left\{ 2\eta_j^{(1)} g_1(q) + (\eta_j^{(3)} - \eta_j^{(1)}) \left[ g_1(q) \sin^2 \beta + g_3(q)(1 - 3 \sin^2 \alpha \sin^2 \beta) \right] \right\}.
\]

(51)

With dependence of the orientation and on the values of \(\eta_j^{(m)}\), the corresponding force can be either repulsive or attractive. Moreover, we can have a situation where the Casimir-Polder force in retarded and non-retarded regimes has opposite signs. In the isotropic case the asymptotic expression \((48)\) reduces to

\[
U(r) \approx \frac{g_1(q)}{8r^3} \sum_j \frac{g_j}{\omega_j},
\]

(52)

and the corresponding force is repulsive.

### 4 Casimir-Polder force

Now we consider the Casimir-Polder force, \(\mathbf{F} = -\nabla U\). This force is perpendicular to the string and is directed along the radial direction with the unit vector \(\mathbf{n}_r\): \(\mathbf{F} = F_r \mathbf{n}_r\). In the general case of the polarizability tensor, by taking into account the expression \((24)\) for the potential, one gets

\[
F_r = \frac{r^{-5}}{16\pi} \sum_{p=1}^{4} \sum_{k=1}^{[q/2]} \frac{c_p(s_k)}{s_k^4} h_{lp}(2rs_k)
\]

\[
-\frac{q}{\pi} \sin(q\pi) \int_0^\infty dy \frac{h_{lp}(2r \cosh y)}{\cosh(2qy) - \cos(q\pi)} \frac{c_p(cosh y)}{\cosh^4 y}.
\]

(53)

In this formula we have defined

\[
c_p(v) = c_p^{(0)} + c_p^{(1)} v^2,
\]

(54)

with the matrices for the coefficients

\[
c_p^{(0)} = \begin{pmatrix}
-3 & -3 & -2 & -1 \\
6 & 6 & 2 & 0 \\
-3 & -3 & -2 & -1
\end{pmatrix},
\]

\[
c_p^{(1)} = \begin{pmatrix}
-3 & -3 & 0 & 1 \\
-3 & -3 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

(55)
The asymptotic expression for the force at large distances is directly obtained from (56) and the Casimir-Polder force behaves as $1/r^5$ power law (retarded regime). In dependence of the eigenvalues for the polarizability tensor and of the orientation of the principal axes, the Casimir-Polder force can be either repulsive or attractive. In the case of isotropic polarizability, the corresponding formula is obtained from [53] by the replacement $\sum_{l=1}^{3} c_p(v) h_p(2rv) \to c_p(v) h_p(2rv)$, where $c_p(v) = \sum_{l=1}^{3} c_{lp}(v)$. The functions $c_p(v)$ are given by

$$c_1(v) = c_2(v) = -6v^2, \quad c_3(v) = -2, \quad c_4(v) = -2 + 2v^2. \quad (56)$$

Note that in the closely related geometry of a perfectly conducting wedge the Casimir-Polder force has both radial and azimuthal components. In the static limit and for an isotropic polarizability tensor, this force has been considered in [20]. In particular, it has been shown that the radial component corresponds to an attractive force towards the cusp of the wedge. The most favorable case for Casimir-Polder repulsion, where the atom is only polarizable in the $z$ direction, has been recently investigated in [15]. In the geometry of a wedge, the force on the atom for both retarded and non-retarded regimes is discussed in [21] for integer values of $\pi/\phi_0$ with $\phi_0$ being the opening angle of the wedge.

In the oscillator model, on the base of (46), for the force we find the following expression

$$F_r = \frac{-1}{4\pi r^3} \sum_{l,m=1}^{3} \sum_{p=1}^{4} g_j^{(m)} / \beta_{lm} \left[ \sum_{k=1}^{[q/2]} \frac{c_{lp}(s_k)}{s_k^2} B_p(2r\omega_j^{(m)} s_k) \right. $$

$$\left. - \frac{q}{\pi} \sin(q\pi) \int_0^\infty dy \frac{B_p(2r\omega_j^{(m)} \cosh y)}{\cosh(2qy) - \cos(q\pi)} \frac{c_{lp}(\cosh y)}{\cosh^2 y} \right], \quad (57)$$

where the functions $B_p(z)$, $p = 1, 2, 3$, are given by Eq. (44) and

$$B_4(z) = -z^2 B_2(z) + 1. \quad (58)$$

At distances from the string smaller than the relevant transition wavelengths, the corresponding asymptotic is easily obtained from [52] and the force scales as $1/r^4$. For integer values of the parameter $q$, the formulas for the potential and force are obtained from (46), (53), and (57) omitting the integral terms and making the replacement (see also (26))

$$\sum_{k=1}^{[q/2]} \rightarrow \frac{1}{2} \sum_{k=1}^{q-1}. \quad (59)$$

In the case of isotropic polarizability with oscillator frequencies $\omega_j$ and oscillator strengths $g_j$, the formula for the Casimir-Polder force takes the form

$$F_r = \frac{-1}{4\pi r^3} \sum_{p=1}^{4} \sum_{j} g_j \left[ \sum_{k=1}^{[q/2]} \frac{c_p(s_k)}{s_k^2} B_p(2r\omega_j s_k) \right. $$

$$\left. - \frac{q}{\pi} \sin(q\pi) \int_0^\infty dy \frac{B_p(2r\omega_j \cosh y)}{\cosh(2qy) - \cos(q\pi)} \frac{c_p(\cosh y)}{\cosh^2 y} \right], \quad (60)$$

with the functions $c_p(v)$ defined by Eq. (56). Note that in accordance with (56) one has $c_p(s_k) < 0$ and, hence, the first term in the square brackets of (60) is always negative. This term dominates and the Casimir-Polder force in the isotropic case is always repulsive. In figure [1] the Casimir-Polder force (60) is plotted as a function of the distance for separate values of $q$. The single oscillator model is used with isotropic polarizability and with the parameters $g_j = g$,
Figure 1: The Casimir-Polder force as a function of the distance from the string for separate values of $q$ (numbers near the curves).

Figure 2: The Casimir-Polder force as a function of the parameter $q$ for a fixed separation from the string corresponding to $\omega_0 r = 1$. 
\[ \omega_j = \omega_0. \] Figure 2 shows the Casimir-Polder force in terms of the parameter \( q \) for fixed value of the distance from the string, corresponding to \( \omega_0 r = 1 \).

Note that the dependence of the Casimir-Polder potential on the orientation of the polarizability tensor principal axes with respect to the string will also lead to the moment of force acting on the particle. This moment is obtained differentiating the potential with respect to the respective angles. As a consequence, the influence of the cosmic string on the system of particles with anisotropic polarizability results in the macroscopic polarization.

### 5 Conclusion

In the present paper we have investigated the Casimir-Polder force acting on a polarizable microparticle placed near an infinitely thin straight cosmic string. As the first step in the evaluation of this force for the general case of anisotropic polarizability tensor, we have considered the retarded Green tensor for the electromagnetic field. By using the mode sum formula we have evaluated all components of this tensor. They are expressed in terms of two functions (14) and (15). Employing the summation formula (64), the representations of these functions are obtained in which the parts corresponding to the geometry of the Minkowski spacetime are explicitly extracted. The renormalization of the Green tensor in the coincidence limit of the spatial arguments is reduced to the omitting of these parts and the renormalized Green tensor is given by expression (18).

The investigation of the Casimir-Polder interaction between the string and a polarizable microparticle, based on the expression for the Green tensor, is given in sections 3 and 4. For the general case of the polarizability tensor, the Casimir-Polder potential and force are given by the expressions (24) and (53), respectively. The force depends on the distance from the string and on the angles determining the orientation of the principal axes of the polarizability tensor with respect to the cosmic string. These angles enter in the general formulas through the coefficients \( \beta_{lm} \) in Eq. (28). In dependence of the eigenvalues for the polarizability tensor and of the orientation of the principal axes, the Casimir-Polder force can be either repulsive or attractive. Moreover, there are situations where the force changes the sign with separation. For an isotropic polarizability tensor the force is always repulsive. At large distances from the string (retarded regime), the dominant contribution to the Casimir-Polder potential comes from low frequencies and in the leading order the potential is given by Eq. (36). The Casimir-Polder force behaves as \( 1/r^5 \) power law. When the two eigenvalues of the polarizability tensor are equal, the explicit dependence on the orientation of the particle is given by formula (37).

Generalizing the results of Ref. [22], we can study the Casimir-Polder interaction for the geometry of a metallic cylindrical shell with the cosmic string along its axis. The corresponding mode functions are given in [6]. The results of this investigation will be reported elsewhere. As it already has been discussed in [6], from the point of view of the physics in the region outside the string core, the geometry with a cylindrical boundary can be viewed as a simplified model for the superconducting string, in which the string core in what concerns its superconducting effects is taken to be an ideal conductor.
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\section*{A Evaluation of the functions}
In this section, the representations are derived for the functions, appearing in the expressions (12) and (13) for the components of the Green tensor. First we consider the function \(A(r, r', \Delta \phi, \Delta z)\), defined by (14). As the first step we use the relation

\[
\frac{1}{\omega^2 + \xi^2} = \int_0^\infty dx e^{-(\omega^2+\xi^2)x}. \tag{61}
\]

Substituting this into (14), the integral over \(\gamma\) is performed with the help of the formula [19]

\[
\int_0^\infty d\gamma \gamma J_{qm}(\gamma r) J_{qm}(\gamma r') e^{-\gamma^2 x} = \frac{1}{2x} \exp \left( -\frac{r^2 + r'^2}{4x} \right) I_{qm} \left( \frac{rr'}{2x} \right). \tag{62}
\]

After integrating over \(k\) and changing the integration variable to \(y = 1/(2x)\), we find the following representation

\[
A(r, r', \Delta \phi, \Delta z) = \sqrt{\pi} \frac{q}{8} \int_0^\infty dy \frac{y^{1/2}}{y^{1/2}} \exp \left( -\frac{y^2 + r'^2 + (\Delta z)^2}{2y} - \frac{\xi^2}{2y} \right) \cos(qm\Delta \phi) I_{qm} \left( \frac{yr}{2y} \right). \tag{63}
\]

For the summation of the series in this expression we use the formula (see also [23])

\[
\sum_{m=0}^\infty \cos(qm\Delta \phi) I_{qm}(w) = \frac{1}{2q} \sum_k e^{w \cos(2k\pi/q - \Delta \phi)} - \frac{1}{4\pi} \sum_{j=\pm 1} \int_0^\infty dx \frac{\sin(q\pi + jq\Delta \phi)e^{-w \cosh x}}{\cosh(qx) - \cos(q\pi + jq\Delta \phi)} \tag{64}
\]

where in the first term on the right-hand side the summation goes under the condition

\[-q/2 + q\Delta \phi/(2\pi) \leq k \leq q/2 + q\Delta \phi/(2\pi). \tag{65}
\]

The formula (64) is obtained on the base of the integral representation 9.6.20 from [24] for the modified Bessel function \(I_{qm}(w)\). For integer values of \(q\), formula (64) reduces to the known result [19, 25]

\[
\sum_{m=0}^\infty \cos(qm\Delta \phi) I_{qm}(w) = \frac{1}{2q} \sum_{k=0}^{q-1} e^{w \cos(2k\pi/q - \Delta \phi)}. \tag{66}
\]

The application of formula (64) to the series in (63), after the integration over \(y\), leads to the following final expression:

\[
A(r, r', \Delta \phi, \Delta z) = \frac{\pi}{4q} \left[ \sum_k e^{-\xi u_k}u_k - \frac{q}{2\pi} \sum_{j=\pm 1} \int_0^\infty dx \frac{\sin(q\pi + jq\Delta \phi)e^{-\xi v(x)}}{\cosh(qx) - \cos(q\pi + jq\Delta \phi)} \right]. \tag{67}
\]

where we have defined

\[
u_k = \sqrt{r^2 + r'^2 + (\Delta z)^2 - 2rr' \cos(2\pi k/q - \Delta \phi)},
\]

\[
v(x) = \sqrt{r^2 + r'^2 + (\Delta z)^2 + 2rr' \cosh x}. \tag{68}
\]
Now we turn to the evaluation of the function $B(r, r', \Delta \phi, \Delta z)$, defined by (15). Using (61), it is presented in the form

$$B(r, r', \Delta \phi, \Delta z) = \sqrt{\frac{\pi}{8}} \int_0^\infty dy \, y^{1/2} \exp \left( -\frac{r^2 + r'^2 + (\Delta z)^2}{2} y - \frac{\xi^2}{2y} \right) \times \sum_{m=0}^\infty \cos(qm \Delta \phi) \sum_{j=\pm 1} I_{qm-j} (yr'r') .$$

(69)

By taking into account that

$$\sum_{j=\pm 1} I_{qm-j} (yr'r') = \frac{2}{yr'} \partial_r I_{qm} (yr'r') ,$$

(70)

we can rewrite this expression in the form

$$B(r, r', \Delta \phi, \Delta z) = 2 \frac{r}{r'} A(r, r', \Delta \phi, \Delta z) + \sqrt{\frac{\pi}{2}} \frac{1}{r'} \partial_r \int_0^\infty dy \, y^{3/2} \exp \left( -\frac{r^2 + r'^2 + (\Delta z)^2}{2} y - \frac{\xi^2}{2y} \right) \times \sum_{m=0}^\infty \cos(qm \Delta \phi) I_{qm} (yr'r') .$$

(71)

The second term in the right-hand side is evaluated by using the summation formula (64), in a way similar to that we have used for the function $A(r, r', \Delta \phi, \Delta z)$. As a result, we find the following representation:

$$B(r, r', \Delta \phi, \Delta z) = 2 \frac{r}{r'} A(r, r', \Delta \phi, \Delta z) + \sqrt{\frac{\pi}{2}} \frac{1}{r'} \partial_r \left[ \sum_{k} e^{-\xi u_k} \frac{q}{2\pi} \sum_{j=\pm 1} \int_0^\infty dx \frac{\sin(q\pi + jq\Delta \phi) e^{-\xi v(x)}}{\cosh(qx) - \cos(q\pi + jq\Delta \phi)} \right] .$$

(72)

The evaluation of the Green tensor components in section 2 is based on formulas (67) and (72). The corresponding expressions in the Minkowski spacetime are obtained taking $q = 1$ in (67) and (72). In this case the integral terms vanish and we have

$$A^{(M)}(r, r', \Delta \phi, \Delta z) = \frac{\pi}{4} \frac{e^{-\xi u_0}}{u_0}, \quad B^{(M)}(r, r', \Delta \phi, \Delta z) = \frac{\pi}{2} \frac{e^{-\xi u_0}}{u_0} \cos(\Delta \phi) ,$$

(73)

where $u_0$ is given by expression (68) with $k = 0$. It is important to note that the $k = 0$ term in the expressions for $qA(r, r', \Delta \phi, \Delta z)$ and $qB(r, r', \Delta \phi, \Delta z)$ coincide with $A^{(M)}(r, r', \Delta \phi, \Delta z)$ and $B^{(M)}(r, r', \Delta \phi, \Delta z)$, respectively. Hence, in the representations (67) and (72) the Minkowskian parts are explicitly extracted. As a result, the renormalization procedure is simply reduced to the omitting of the $k = 0$ terms in the corresponding expressions of the Green tensor components.

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