Near-resonance approximation of rotating Navier–Stokes equations

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Received 6 October 2021; revised 17 January 2023
Accepted for publication 31 January 2023
Published 28 February 2023

Recommended by Dr Anna L Mazzucato

Abstract
We formalise the concept of near resonance for the rotating Navier–Stokes equations, based on which we propose a novel way to approximate the original partial differential equation (PDE). The spatial domain is a three-dimensional flat torus of arbitrary aspect ratios. We prove that the family of proposed PDEs are globally well-posed for any rotation rate and initial datum of any size in any $H^s$ space with $s \geq 0$. Such approximations retain many more 3-mode interactions, and are thus more accurate, than the conventional exact-resonance approach. Our approach is free from any limiting argument that requires physical parameters to tend to zero or infinity, and is free from any use of small divisors (so that all estimates depend smoothly on the torus’s aspect ratios). The key estimate hinges on the counting of integer solutions of Diophantine inequalities rather than Diophantine equations. Using a range of novel ideas, we handle rigorously and optimally challenges arising from the non-trivial irrational functions in these inequalities. The main results and ingredients of the proofs can form part of the mathematical foundation of a non-asymptotic approach to nonlinear, oscillatory dynamics in real-world applications.

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Keywords: near resonance, rotating Navier–Stokes equations, global well-posedness, restricted convolution, integer point counting, Diophantine inequalities, elliptic integrals
Mathematics Subject Classification numbers: Primary 35B25, 35B34, 35A01, 86A10, 42B37, secondary 35Q30

(Some figures may appear in colour only in the online journal)

1. Introduction

We investigate near-resonance (NR)-based approximations of the rotating Navier–Stokes (RNS) equations, a well-known model of geophysical fluid dynamics (GFD). We prove the global well-posedness of solutions under very mild conditions for this novel class of PDEs. These approximations are characterised by what we call ‘bandwidth’, a wavenumber-dependent parameter that allows many more three-mode interactions to be retained by the NR approach than by the conventional exact-resonance approach. We also prove explicit error bounds for such approximations. In a nutshell, we achieve in the proposed NR approximations two desirable properties: global solvability for arbitrary rotation rates and a large set of nonlinear interactions that dominate the dynamics for fast rotation rates.

The spatial domain, denoted by $\mathbb{T}^3$, is a three-dimensional flat torus with anisotropic periods $(2\pi L_1, 2\pi L_2, 2\pi)$ for any positive constants $L_1, L_2$. For a $\mathbb{C}^3$-valued, Lebesgue-integrable function $u$, the div-free condition $\nabla \cdot u = 0$ is understood in the weak sense, namely $\int_{\mathbb{T}^3} u \cdot \nabla h \, dx = 0$ for any $\mathbb{C}$-valued, smooth function $h$. Let $P_{\text{div}} u$ denote the Leray projection of $\mathbb{C}^3$-valued function $u \in L^p(\mathbb{T}^3)$ for $p \in (1, \infty)$ onto the subspace of div-free functions. More precisely,

$$P_{\text{div}} u := u - \nabla \Delta^{-1} \nabla \cdot u.$$ 

Then, we can define the bilinear form $B$ for $\mathbb{C}^3$-valued functions $u, v \in H^1(\mathbb{T}^3)$ as:

$$B(u, v) := P_{\text{div}}(u \cdot \nabla v).$$

Let $J$ denote the rotation matrix $J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and define the linear operator:

$$L := P_{\text{div}} J P_{\text{div}}.$$ 

Also, let $\mathbb{R}^+$ denote the set of positive real numbers and $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$. Let $t \in \mathbb{R}_0^+$ denote the time coordinate and $x \in \mathbb{T}^3$ the spatial coordinates.

The incompressible RNS equation is formulated for unknown $U: \mathbb{R}_0^+ \times \mathbb{T}^3 \mapsto \mathbb{R}^3$ as:

$$\partial_t U + B(U, U) = \Omega L U + \mu \Delta U, \quad \nabla \cdot U = 0, \quad (1.1)$$

here the scalar constant $\mu > 0$ denotes viscosity. The $\Omega L U$ term represents the Coriolis force due to the frame’s rotation, where the scalar constant $\Omega > 0$ is proportional to the rotation rate and takes the form of $1/(\text{Rossby number})$ upon nondimensionalisation. We omit an external forcing term since its treatment under various regularity assumptions is fairly well-documented. The initial datum $U(0, \cdot) \in L^2(\mathbb{T}^3)$ (or a more regular subspace of it) and div-free.
The linear term $\Omega \mathcal{L} \mathbf{U}$ with large $\Omega$ is responsible for inertial waves\(^1\). Early results of inertial waves by Poincaré [26] were discussed and extended in [18, section 2.7] in modern notation. This $\Omega \mathcal{L} \mathbf{U}$ term generates operator exponential $e^{\Omega \mathcal{L} t}$. This operator leads to a transformation of the RNS equations by way of Duhamel’s principle, which makes explicit the three-mode interactions in the eigen-expansion of the ‘transformed’ bilinear form. We then define an approximation of $B(\cdot, \cdot)$ via restricted convolution. Such a restriction retains or discards three-mode interactions based on the smallness of the linear combination of three frequencies, which will be called the ‘triplet value’, rather than on whether the triplet value vanishes as used for defining exact resonances. In detail, the triplet value $\omega_{\text{skew}}^\sigma$ is a function of $(n, k, m) \in (\mathbb{Z}^3 \setminus \{0\})^3$ for the three wavevectors and $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \{+, -, -\}^3$ for the three signs of temporal phases,

$$\omega_{\text{skew}}^\sigma := \sigma_1 \frac{\hat{n}_1}{|\hat{n}_1|} + \sigma_2 \frac{\hat{k}_3}{|\hat{k}_3|} + \sigma_3 \frac{\hat{m}_3}{|\hat{m}_3|},$$

(1.2)

where the accent ‘indicates a ‘domain-adjusted’ wavevector defined as:

$$\hat{n} := \left( \frac{n_1}{L_1}, \frac{n_2}{L_2}, \frac{n_3}{L_3} \right)^T \quad \text{for} \quad n = (n_1, n_2, n_3)^T \in \mathbb{Z}^3,$$

and the Euclidean length and dot product are applied as usual, e.g. $\hat{n} \cdot \hat{k} = \frac{n_1 k_1}{L_1} + \frac{n_2 k_2}{L_2} + n_3 k_3$.

Now we are ready to define the near-resonance (NR) set $\mathcal{N}[\delta_*]$ as:

$$\mathcal{N}[\delta_*] := \left\{ (n, k, m) \in (\mathbb{Z}^3 \setminus \{0\})^3 : \min_{\sigma \in \{+, -, -\}} |\omega_{\text{skew}}^\sigma| \leq \delta_* (n, k, m), \; n + k + m = 0 \right\},$$

(1.3)

where the bandwidth function:

$$\delta_* : (\mathbb{Z}^3 \setminus \{0\})^3 \to \mathbb{R}_0^+,$$

is related to but fundamentally different from the small divisor seen in literature (section 1.1). We should point out the $[\delta_*]$ notation is used to emphasize $\delta_*$ is regarded as a function here, not its value. In other words, $\mathcal{N}$ is a functional that maps a function to a subset of $(\mathbb{Z}^3)^3$. Also, the convolution condition $n + k + m = 0$ differs slightly from convention in order to make the symmetry of the indicator\(^2\) function $1_{\mathcal{N}[\delta_*]}(n, k, m)$ more user-friendly.

The Fourier series of function $g : \mathbb{T}^3 \mapsto \mathbb{C}$ is, as usual, $g(x) = \sum_{n \in \mathbb{Z}^3} \delta_n e^{i n \cdot x}$. In this notation, the bilinear form of RNS equations satisfies the convolution:

$$B(\mathbf{U}, \mathbf{V}) = \sum_{n \in \mathbb{Z}^3} \sum_{k \in \mathbb{Z}^3} B(e^{i k \cdot x} U_k, e^{-i (\hat{n} + \hat{k}) \cdot x} V_{-n-k}).$$

Then, postponing the detailed motivation to section 2.3, we introduce the restricted convolution based on the NR set $\mathcal{N}[\delta_*]$: $B_{\mathcal{N}[\delta_*]}(\mathbf{U}, \mathbf{V}) := \sum_{n \in \mathbb{Z}^3} \sum_{k \in \mathbb{Z}^3} B(e^{i k \cdot x} U_k, e^{-i (\hat{n} + \hat{k}) \cdot x} V_{-n-k}) 1_{\mathcal{N}[\delta_*]}(n, k, -n-k),$  

(1.4)

and arrive at the main equation:

$$\partial_t \tilde{\mathbf{U}} + B_{\mathcal{N}[\delta_*]}(\tilde{\mathbf{U}}, \tilde{\mathbf{U}}) = \Omega \mathcal{L} \tilde{\mathbf{U}} + \mu \Delta \tilde{\mathbf{U}}, \quad \nabla \cdot \tilde{\mathbf{U}} = 0,$$

(1.5)

with div-free initial datum $\mathbf{U}(0, \cdot) \in L^2(\mathbb{T}^3)$. We name this the near-resonance approximation of the RNS equations.

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\(^1\) Evident from dispersion relation (2.10), the Coriolis term does not generate (linear) oscillatory dynamics in horizontal modes. Such a large set of non-oscillatory modes is a common feature that separates many fluid models (even in the inviscid case) from classically known dispersive PDEs.

\(^2\) For set $S$, the associated indicator function is defined as $1_S(x) = 1$ if $x \in S$ and $1_S(x) = 0$ if $x \notin S$. 

\(2076\)
In this article, we always let bold italics symbols denote $\mathbb{C}^3$-valued functions in the $T^3$ spatial domain which may or may not be time-dependent as determined by the context.

For $s \in \mathbb{R}$, define the pseudo-differential operator $D^s := (\sqrt{-\Delta})^s$. For any function $g(x)$ with suitable regularity, define the homogeneous $H^s(T^3)$ norm, with shorthand notation $\| \cdot \|_s$, as:

$$\|g\|_s := \|D^s g\|_{L^2(T^3)}.$$  

We will use it only on zero-mean functions, in which case $\| \cdot \|_s$ and $\| \cdot \|_{H^s(T^3)}$ are equivalent.

The result of global well-posedness in theorem 1.3 shows the propagation of $H^s(T^3)$ regularity of the initial datum to all $t \geq 0$ where we note in particular that the bandwidth function $\delta_s$ is assumed to decay like $(\text{wavenumber})^{-1}$ with a minor logarithmic attenuation. When $s = 0$, the notion of weak solution we use coincides with the notion of weak solutions a la Leray [22]. Since various versions of weak solutions can be found in the literature, we specify the type used in this paper as follows. Firstly, recall the canonical inner-product pairing,

$$\langle f, g \rangle := \int_{T^3} f(x) \cdot \overline{g(x)} \, dx,$$

where the dot product is unrelated to conjugation, e.g. $\langle f_1, f_2, f_3 \rangle \cdot \langle g_1, g_2, g_3 \rangle^T = f_1 g_1 + f_2 g_2 + f_3 g_3$. The $\langle \cdot, \cdot \rangle$ pairing notation is used for scalar-, vector- and tensor-valued functions. Then we need to take sense of $B_{N[s]}(\cdot, \cdot)$ when both its arguments are only in $L^2(T^3)$. Let $D'(T^3)$ denote the space of distributions on $T^3$, namely the continuous dual space of $C^\infty_c(T^3)$ with the pairing denoted by $\langle \cdot, \cdot \rangle_{uk} : D'(T^3) \times C^\infty_c(T^3) \to \mathbb{C}$ as an extension to the $\langle \cdot, \cdot \rangle$ pairing.

**Definition 1.1.** Given $\mathbb{C}^3$-valued, zero-mean functions $u, v \in L^2(T^3)$ with $u$ div-free, define $B_{N[s]}(u, v)$ as an element of $D'(T^3)$ that satisfies, for any smooth function $h : T^3 \to \mathbb{C}$, that:

$$\langle B_{N[s]}(u, v), h \rangle_{uk} = \sum_{n \in \mathbb{Z}^3} \sum_{k \in \mathbb{Z}^3} -1_N[\delta_s](n, k, -n - k) \int_{T^3} e^{-in \cdot x} (u_k \otimes v_{n-k}) : \nabla P^{div} h(x) \, dx,$$

where $\otimes$ denotes tensor product and $:$ denotes the dot product between tensors.

**Definition 1.2.** We call $\tilde{U} \in D'([0, \infty) \cap L^2(\mathbb{R}^3)) \cap L^2([0, \infty); H^1(\mathbb{R}^3))$ a global weak solution of the NR approximation (1.5) if $\tilde{U}(t, \cdot)$ is div-free for all $t \geq 0$ and if, for any $\mathbb{C}^3$-valued, smooth function $\psi(t, x)$, the following weak formulation holds for any $T \geq 0$,

$$\begin{aligned}
\int_0^T \langle \tilde{U}(T, \cdot), \psi(t, \cdot) \rangle - \langle \tilde{U}(0, \cdot), \psi(0, \cdot) \rangle + \int_0^T \langle B_{N[\delta_s]}(\tilde{U}(t, \cdot)), \psi(t, \cdot) \rangle_{uk} \, dt
\end{aligned}$$

$$\begin{aligned}
= -\int_0^T \langle \tilde{U}, \partial_t \psi + \Omega \mathcal{L} \psi \rangle + \mu \langle \nabla \tilde{U}, \nabla \psi \rangle \, dt.
\end{aligned}$$

**Theorem 1.3 (Global well-posedness of NR approximation).** Consider (1.5), an NR approximation of the 3D RNS equations, where the bilinear from $B_{N[\delta_s]}$ is defined by (1.3), (1.4). Suppose the bandwidth function satisfies $\delta_s(n, k, m) \in [0, \frac{1}{2})$,

$$\delta_s(n, k, m) \text{ only depends on } \max \{|\hat{n}|, |\hat{k}|, |\hat{m}|\}, \quad (1.6)$$
and, for some constant $\hat{c}_1 > 0$,
\[
\delta_* (n, k, m) \log \frac{1}{\delta_* (n, k, m)} \leq \hat{c}_1 \min \{ |\hat{m}|^{-1}, |\hat{k}|^{-1}, |\hat{n}|^{-1} \},
\]
(1.7)
where $\delta_* (n, k, m) = 0$ is allowed under the convention $0 \log 0 = \lim_{\delta_* \to 0} \delta_* \log (1/\delta_*) = 0$.

Then, for any $\mathbb{R}^3$-valued, div-free, zero-mean$^3$ initial datum $\mathbf{U}(0, \cdot') = \mathbf{U}_0(\cdot') \in L^2 (\mathbb{T}^3)$, the NR approximation (1.5) admits a unique global weak solution. The solution satisfies the energy equality (rather than inequality),
\[
\| \mathbf{U}(T, \cdot') \|_{L^2}^2 + 2\mu \int_0^T \| \nabla \mathbf{U} \|_{L^2}^2 \, dt = \| \mathbf{U}_0 \|_{L^2}^2, \quad \forall \, T \geq 0.
\]
(1.8)
If furthermore $\mathbf{U}_0 \in H^s (\mathbb{T}^3)$ for some $s \in \mathbb{R}_+^*$, then the solution satisfies:
\[
\mathbf{U} \in \mathcal{C}^0 \left( [0, \infty); H^s (\mathbb{T}^3) \right) \cap L^2 \left( [0, \infty); H^{s+1} (\mathbb{T}^3) \right),
\]
and the global bounds:
\[
\max_{t \geq 0} \| \mathbf{U}(t, \cdot') \|_{L^2}^2 \leq e^{-2\sigma_0 C_E} \mathbf{E}_0 \quad \text{and} \quad \mu \int_0^\infty \| \mathbf{U} \|_{L^{s+1}}^2 \, dt \leq \mathbf{E}_0 + \mu^{-2} \bar{C} \mathbf{E}_0 e^{\mu^{-2} \bar{C} \mathbf{E}_0},
\]
(1.9)
for $\mathbf{E}_0 := \| \mathbf{U}_0 \|_{L^2}^2$ and constant $\bar{C} = \bar{C}(s, \hat{c}_1, \mathbf{T}_0)$.

The solution Lipschitz-continuously depends on the initial datum in the sense of lemma 6.3.

Remark 1.4. In this article, when we say a constant depends on the torus domain $\mathbb{T}^3$, it means the constant depends smoothly on its aspect ratios $L_1, L_2$. Though expected from physics, this is not achievable if a small-divisor argument is involved.

We also remark on the energy inequality (1.8) for initial data with only $L^2$ regularity. Firstly, with $= \cdot$ replaced by $\leq$, the energy inequality is satisfied by the celebrated Leray-Hopf solutions. Secondly, the energy inequality for weak solutions of the 2D RNS equations is well understood e.g. [7, theorem 3.2] (where the subtlety in Leray solution vs. weak solution is not essential). Thirdly, for 3D force-less Navier–Stokes equations, it is not known whether the energy equality holds for every weak or Leray solution in $L^\infty_t (\mathbb{R}; L^2 (\mathbb{T}^3)) \cap L^2_t (\mathbb{R}; H^s (\mathbb{T}^3))$. A recent result [3, theorem 1.1] shows the existence of a weak solution with its $L^2_t$ norm being any given non-negative smooth function of $t$, but the $L^2_t H^s_t$ regularity of the solution is not established.

The proof of theorem 1.3 is in section 6.1. It depends crucially on the following result that our proposed bilinear form $B_{\mathcal{X} / \mathcal{X}^*} (\mathbf{u}, \mathbf{v})$ satisfies estimates similar to those of the classic $\mathcal{P}^{\text{div}} (\mathbf{u} \cdot \nabla \mathbf{v})$ in 2D, even though the velocity fields are genuinely 3D, and there is no apparent decoupling of any kind. Then (1.5), with any rotation rate $\Omega$, enjoys most of the nice properties that 2D Navier–Stokes equations have [22], which leads to theorem 1.3. Further, such 2D-like feature can imply desirable numerical properties [31], which will be covered in future work.

Theorem 1.5 (2D-like estimates). Consider $B_{\mathcal{X} / \mathcal{X}^*}$ defined by (1.3)–(1.4). Suppose the bandwidth function $\delta_* (n, k, m) \in [0, 1/2]$ satisfies (1.6) and (1.7) for some constant $\hat{c}_1 > 0$. Then, for any $\mathbb{R}^3$-valued, div-free, zero-mean functions $\mathbf{u}, \mathbf{v}, \mathbf{w} \in H^{r+1} (\mathbb{T}^3)$ with $s \in \mathbb{R}_+^*$, we have:
\[
\left| \langle \mathbf{D} B_{\mathcal{X} / \mathcal{X}^*} (\mathbf{u}, \mathbf{v}), \mathbf{D} \mathbf{w} \rangle \right| \leq 2^r \left[ \| \mathbf{u} \|_1 \| \mathbf{v} \|_{s+1} + \| \mathbf{u} \|_s \| \mathbf{v} \|_1 \right] \| \mathbf{w} \|_{s+1}
+ 2^r \left[ \| \mathbf{u} \|_1 \| \mathbf{v} \|_{s+1} + \| \mathbf{u} \|_s \| \mathbf{v} \|_1 \right] \| \mathbf{w} \|_s.
\]
(1.10)

$^3$ The zero-mean assumption is imposed WLOG and this property propagates in time—see the start of section 2.
and a further estimate if \( v = w \),
\[
\left| \langle D' B_{N[\delta_s]}(u, w), D' w \rangle \right| \leq \begin{cases} 
   s \| u \|_1 \| w \|_s \| w \|_{s+1}, & \text{for } s \in (0, 1], \\
   2^s (\| u \|_1 \| w \|_s \| w \|_{s+1} + \| u \|_s \| w \|_1 \| w \|_{s+1} + \| u \|_{s+1} \| w \|_1 \| w \|_s), & \text{for } s \in (1, \infty).
\end{cases}
\]
(1.11)

The implied constants in the \( \lesssim \) notation only depend on \( c_1, T_3 \).

The proof of theorem 1.5 is at the start of section 6 and relies on results proven in the bulk of this article including section 3 (linking estimates of restricted convolution on torus domain to integer-point counting, a well-known technique in harmonic analysis), section 4 (linking point counting of sets in \( \mathbb{Z}^3 \) to volume of sublevel sets in \( \mathbb{R}^3 \)), section 5 (volume estimate) and appendix A (a general point-counting result concerning disjoint Jordan curves). These sections, especially the last three, contain the key technical novelties that we employ to resolve a range of challenges with rigour, optimality and a good level of generalisability.

Estimates (1.10), (1.11) are ‘2D-like’ in the following sense. The simple inequalities (6.14) for generic convolutions indicate the ‘cost’ of derivatives is usually \( \frac{1}{2} \), namely \( \frac{1}{2} \times \text{dimension} \), in view of the orders of derivatives on the left-hand and right-hand sides. In the above result, however, the cost of derivatives is reduced to 1 (note \( B_{N[\delta_s]} \) contains one derivative).

The next theorem on error estimates confirms that the NR approximation is close to RNS equations in the fast-rotation regime. It is noteworthy that the error bound does not grow out of bound as the viscosity \( \mu \to 0 \).

**Theorem 1.6 (O(\Omega^{-1}) error estimates, independent of \( \mu \)).** Let the bandwidth function \( \delta_*(n, k, m) \) in the NR set \( N[\delta_*] \) satisfy (1.6) and the lower bound:
\[
\delta_*(n, k, m) \max \left\{ 1, \log \frac{1}{\delta_*(n, k, m)} \right\} \geq c_2 \min \left\{ |\hat{n}|^{-1}, |\hat{k}|^{-1}, |\hat{n}|^{-1} \right\},
\]
(1.12)
for some constant \( c_2 > 0 \). Consider the RNS equations (1.1) and its NR approximation (1.5) with respective initial data \( \bar{U}_0 \) and \( \bar{U}_0^\prime \) that are \( \mathbb{R}^3 \)-valued, div-free, zero-mean, and satisfy:
\[
\| \bar{U}_0 \|_{2 \delta_*}^2 \leq E_0 \quad \text{and} \quad \| \bar{U}_0^\prime \|_{2 \delta_*}^2 \leq E_0, \quad \text{for real numbers } s > 3, s^* > \frac{5}{2}.
\]

Suppose real number \( s' = 0 \) or \( s' \in [1, s - 3) \) (if \( s > 4 \)) with \( s' \leq \delta^* \). Then there exist constants \( C(s, s^*, s', T_3, E_0) \) and \( c(s, s^*, s', T_3) \) independent of \( \mu, \Omega \) so that:
\[
\| \widetilde{U}(t, \cdot) - U(t, \cdot) \|_{2 s'}^2 \leq \| \widetilde{U}_0 - U_0 \|_{2 s'}^2 + Cc_2^{-2} \Omega^{-2} \quad \text{for } t \in \left[ 0, c/E_0^{\frac{1}{2}} \right].
\]
Its proof in section 6.2 essentially uses integration by parts to get estimates in the form of:
\[
\left\| \int_0^T h(t) \exp(i\omega_s \Omega t) \, dt \right\| \leq O\left( \frac{\| \partial h \| T + \| h \|}{|\omega_s| \Omega} \right), \quad \text{for } \omega_s \Omega \neq 0,
\]
(1.13)
(which is related to Riemann–Lebesgue lemma) where the norms are possibly different and \( \omega_s \) is just a placeholder for a triplet value that is outside the NR bandwidth. Based on that proof, it takes very little effort in remark 6.5 to show solvability of the original RNS equation for all \( t \geq 0 \) with sufficiently large \( \Omega \). Compared to error estimates of [5, theorem 6.3], [6, lemma 5.2], [16, theorem 2], [17, theorem 2] for the exact-resonance-based approximations, the above error estimate is explicit in terms of \( \Omega^{-1} \), independent of \( \mu \) and depends smoothly on the domain’s aspect ratios \( L_1, L_2 \) (see remark 1.4). The last property makes our error bound more physically relevant. Furthermore, our results reveal the ‘better’ nature of the RNS equations when
compared to what exact-resonance-based results can tell us, in that the ‘well behaved’ approximation we introduce contains many more three-mode interactions, and is thus much closer to the RNS solutions in the large $\Omega$ regime. This is for example reflected in a smaller derivative gap $s - s'$ we can prove. Due to the largeness requirement on $\Omega$, however, our approach sees limitation in the original Navier–Stokes equations. We also note that the choice of the lower-bound condition (1.12) on $\delta_*$ is solely for it to be relatable to the upper-bound condition (1.7). Removal of the $\log(1/\delta_*)$ term is almost trivial. Further, changing the $-1$ power in (1.12) will only require reconsidering the derivative gap $s - s'$ without essential modification of the proof. Finally we note that error estimates based on this type of proof are most relevant for $\delta_*$ values that are not too small or large because as argued in section 1.1, having $\delta_*$ too small amounts to selection based on exact resonances whereas having $\delta_* \geq 3$ clearly recovers the original bilinear form $B$.

Theorem 1.6 holds in the $\mu = 0$ case due to the result being independent of $\mu$, and as such, the estimates are local in time. On the other hand, global-in-time $O(\Omega^{-1})$ error estimates can be achieved via a fairly simple adaptation of the proof, but at the cost of the constants in the estimates depending on $\mu^{-1}$ and $t$, although the dependence on $t$ can be further avoided at the cost of lowering the regularity in the norm of the error.

The virtue of studying NR approximations is multi-fold, and our methodology and underlying philosophy bears promising potential in wider applications—see section 1.3 for more on adaptability.

For oscillatory dynamics, nonlinearity is the medium which allows modes of the solution to interact. One of the quantitative indicators of this nonlinear effect is the smallness of the triplet values as defined above. The exact-resonance approach is only concerned with zero triplet value which clearly exemplifies this effect. This is all that one can retain by ‘thinking infinitely’ and thus studying the limits, often known as singular limits—cf the seminal work of Schochet in [28]. In proposing the NR-based approach however, we are concerned with the reality that physical parameters, large or small, are always finite, and not $\infty$ or 0. In GFD applications [25], small parameters associated with oscillatory dynamics are not that small—for example the Rossby, Froude and Mach numbers typically range from 0.01 to 0.1. This means an asymptotic approach can miss a considerable amount of nonlinear coupling that plays a non-negligible part in the full dynamics. Therefore we propose to include contributions from small but finite triplet values, and hence introduce the notion of bandwidth for that smallness.

We note that one approach to complement the study of singular limits is to obtain explicit error bounds, namely convergence rates, in terms of the small/large parameters. Then, one can go even further to study next order corrections [16, 17].

The drastic increase of three-mode interactions included in the NR approximations is evidenced in appendix C. There, we show the lower bound of integer-point count arising from our proposed NR sets is far greater than the upper bound of integer-point count in [24, lemma 4.1] for exact-resonance sets.

Crucially, due to the widening of strict zero-ness to its fuzzy neighbourhood, we increase the dimension of the set of included modes when viewed in continuum. Specifically, we change from level sets to sublevel sets, which fundamentally requires novel ideas. For example, we need to study Diophantine inequalities. Then, number theoretical tools and small-divisor arguments, such as those applied to RNS equations in [5, 6, 17, 24] (based on exact resonance, and thus involving Diophantine equations) are often not applicable here.

This challenge is however countered by the benefit of robustness since the integer points in a neighbourhood of a curve/surface, compared to those on the curve/surface itself, form a set that is much more stable against perturbation. For example, the properties of the Diophantine equations and small divisors in the above works depend non-continuously on the domain’s
aspect ratios, but this rather unphysical sensitivity is no longer an issue when equations are relaxed to inequalities.

1.1. Literature

Recent decades have seen the power of PDE analysis in exact resonances. The list is so long that for brevity, we only mention a partial list [5–7, 14, 16, 17, 20] and references therein. In [28], Schochet proves for a large class of multi-time-scale PDEs the existence of a generic formula of singular limits—effectively exact-resonance-based approximations—for unprepared initial data. In many works, small-divisor condition/estimate is used to quantify how the lower bound of nonzero triplet values (in absolute value) depends on wavenumbers, so in our language, setting bandwidth $\delta$, less than such a lower bound will reduce the NR set to the exact-resonance set. More recently, results were proven in [8, 9, 12] for singular limit problems with physically relevant boundary conditions and domain geometry for which Fourier analysis is not applicable. Although exact resonance is considered therein, the convergence estimates (i.e. error bounds) are given explicitly in terms of the small parameter, which does not require any infinitesimal assumption. In recent years, there have been the first results in three-scale singular-limit problems [10, 11].

In [6], the authors prove global solvability of RNS equations in torus domains of any aspect ratios for sufficiently large rotation rate $\Omega$ with certain class of external forcing. For initial data in the critical $H^3(\mathbb{T}^2)$ space, global solvability is proven in [7, theorem 6.2] with the threshold for $\Omega$ depending on $U_0$ (not just its norm). For RNS equations with fractional Laplacian diffusion, global solvability is proven in [24]. All three proofs are based on the global existence of the exact resonance approximation in the $\Omega \to \infty$ limit, and rely on a certain decoupled structure of the limit equation between horizontal and vertical dimensions. Note, however, that there is no obvious decoupling in our NR approximations, and nor does the approach in remark 6.5 need the decoupling argument.

In numerical analysis, the results of [19] showed the success of adapting singular limit theory to parareal algorithms. It employs a finite and discretised form of the infinite integral formula of [28], and its GFD version in [14]. Such adaptation always retains all exact resonances but due to its (numerically necessary) finiteness, near resonances are also retained in some form.

On the physical front, NR-based study has led to exciting results in GFD, e.g. [21, 23, 32] which show that NR is closely linked to some notable features in large-scale GFD such as zonal flows. The work of [33] quantifies exchanges of energies among various modes, which strongly suggests that one must keep the Rossby and Froude numbers finite in order to explain these phenomena. In another physical area, inertial waves are of particular significance in the study of Earth’s fluid interior: recall that the Coriolis term in the RNS equations is responsible for inertial waves. Notably, such waves are identified in gravimetric data [1] which inspired modelling work in [2, 27]. They also play an important role in turbulence theory [15] for GFD applications.

1.2. Outline

The rest of the article is organised as follows. First, section 1.3 is a discussion on potential adaption of our methodology to other oscillatory PDEs. The detailed motivation of our proposed NR approximation is given in section 2. Next in section 3, inspired by [6, lemma 3.1], we prove a result that gives an upper bound on restricted convolution (which is used in our proposed bilinear form) in terms of integer-point counting.
From section 4 onwards, we embark on a journey of bounding integer-point count in two-sided sublevel sets of a function of \( k \), namely \( |\varphi_{nk(-n-k)}^\delta| \leq \delta \) with given parameters \( n, \tilde{\sigma}, \delta \). Here, we see connections to group theory and topology. The essence of section 4, as suggested by the coined word ‘anti-discretise’ in its title, is to convert integer-point counting to volume estimate of sublevel sets. As discussed later in the last paragraph of the Introduction, the counting depends on meaningful lower bounds on the derivatives of the sublevel-set defining function, which can be challenging even in 1D. For the 2D case, the counting further requires consideration of the set perimeter. For example, the finite set:

\[
\{ (x_1, x_2) \in \mathbb{R}^2 : N < x_1 \leq 2N, \quad x_2^2 < x_1^{-4} \},
\]

for any large \( N \) contains \( N \) integer points but its area is as small as \( O(N^{-1}) \). This is indeed the essence of the elementary result of Jarnik and Steinhaus in [29] for when the 2D set is the interior region of a single Jordan curve. We prove in lemma A.2 an extended version allowing multiple disjoint Jordan curves—this means the set can be in either the interior or the exterior of an individual curve. Finally, 3D problems require even more information. For example,

\[
\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : N < x_1 \leq 2N, \quad x_2^2 + x_3^2 < x_1^{-4} \},
\]

contains \( N \) integer points but its volume is \( O(N^{-3}) \) and the area of its boundary is \( O(N^{-1}) \). Therefore it is impossible to have a 3D counterpart of the previous result in 2D, which is an issue that we will resolve with a straightforward treatment of the third dimension.

In section 5, we prove a sharp upper bound for the volume of the sublevel set \( |\varphi_{nk(-n-k)}^\delta| \leq \delta \) in the \( k \) space. The volume integral is first rewritten in spherical coordinates. Then, as the triplet value is monotone in the azimuthal angle for half of the \( 2\pi \)-period, we perform a crucial change of coordinates to replace the azimuthal angle with a new coordinate essentially equivalent to the triplet value. Since the aforementioned monotonicity must change due to periodicity, the Jacobian inevitably contains singular points. In fact, there are four singularities of \( \frac{1}{\sqrt{n}} \) nature (in the radial coordinate), which gives rise to elliptic integrals. Estimates of the integrals not only can depend on the ordering of the singular points, but can potentially degenerate if the singular points cross each other, i.e. when the order changes. Although there are \( 4! = 24 \) ways of ordering, we will show there are essentially two cases using simple group theory. In fact, elliptic integrals with four singularities of \( \frac{1}{\sqrt{n}} \) type are invariant under double transpositions (and identity permutation) of the singular points. They form the Klein 4-group \( K_4 \) which is a normal subgroup of the 4-permutation group \( S_4 \), so a coset of \( K_4 \) in \( S_4 \) is regarded as one equivalent case. Rather nicely, \( S_4/K_4 \equiv S_3 \) plays a role in proving lemma 5.3 since an affine transform of the singular points is used to make one of them fixed so that only the ordering of the other three matters.

We will then have all the lemmas needed for the proof of the 2D-like estimates given in the statement of theorem 1.5. This proof is at the start of section 6. After that, we prove theorems 1.3 and 1.6 using standard arguments.

In appendix A we prove lemma A.2 for counting integer points enclosed by multiple disjoint Jordan curves, and only use elementary mathematics except the Jordan-Schoenflies theorem from topology. In appendix B we prove elementary estimates for elliptic integrals. In appendix C, we construct examples of direct integer-point counting with lower bounds that evidence the optimality of the choices of bandwidth \( \delta_n \), including the logarithmic attenuation.

We remark that if the spatial domain is \( \mathbb{R}^3 \) instead, the concept of NR approximation and many of the above techniques still apply, in fact without the need to anti-discretise. But the dispersive nature of the inertial waves in the whole space is stronger than in a compact domain, and thus this case is usually treated very differently—see e.g. [7, chapter 5].
1.3. Adaptability

Many PDEs can be written in a form similar to (1.1) where linear operator $\mathcal{L}$ is responsible for generating fast waves and a dissipative term is optional. In our opinion, it is not entirely straightforward to determine whether a system would be amenable to this approach that allows us to do both of the following things: (1) to use as large as possible bandwidths in the definition of near resonance; (2) to rigorously prove, depending on the application, certain properties for the corresponding NR approximation such as improved estimates. A key ingredient in the adaptation requires meaningful bounds on the number of NR interactions, upon suitable localisation, whose triplet values are no more than the localised bandwidth (say $\delta$) away from zero. This number of NR interactions as a function of $\delta$ can be informally viewed as the cumulative distribution function (CDF) of near resonances. This CDF is bounded in our approach by the volume of the continuous set that tightly encloses the NR interactions being counted. At the end of this section, we discuss heuristically that such estimates as a function of $\delta$ should depend delicately on the degeneracy of certain stationary points.

In what follows, we describe some informal criteria for adaptation as well as some areas that may require modification, organised according to sections 2–5. The spatial domain is assumed to be a torus.

First, operator $\mathcal{L}$ needs to be skew-adjoint with respect to the $L^2$ inner product. The zero roots of the dispersion relation of $\mathcal{L}$ can play a special role, in part because zero-zero-zero frequency triplets are trivially near resonances. In section 2.2, we see such zero roots for the RNS system only occur at horizontal wavenumbers. In some other systems, however, zero roots occur at all wavenumbers. Then, any type of NR approximation must include interactions covering all wavenumber triplets, only subject to convolution condition. In the authors’ work [13] on the rotating Boussinesq system for example, this part of the nonlinearity is that of the 3D quasi-geostrophic system. Such full-dimensionality can affect every part of the dynamics via nonlinear coupling. To tackle this, some case-specific additional structure is usually needed—see [13] for example.

Furthermore, an assumption of lemma 3.1 is that the restricting set (of near resonances) should have full symmetry with respect to any permutations of the wavenumber triplet. If zero frequency exists for all wavenumbers, then similar estimates are needed for convolutions restricted on sets that only have partial symmetries due to frequency triplets of mixed zero and nonzero signs [13].

Next, for adapting the anti-discretisation idea in section 4, one needs to estimate the discretisation error $\sum_{k \in \mathbb{Z}^3} 1_v(k) - \int_{[-\delta,\delta]^3} 1_v(k) \, dV$ with $V$ being the sublevel set in which the triplet-value function takes values in $[-\delta,\delta]$ for a localised bandwidth $\delta$. See theorem 4.2. The dispersion relation needs to be in such form that one can estimate the number of monotonicity intervals of the triplet-value function with all but one coordinate fixed, since this controls the discretisation error in the 1D situation. Also see [13] for a variation of this approach.

Finally, the adaptation of volume estimates in section 5 is the least obvious to us. It is often technically necessary to perform a change of variables in the volume integral so that the triplet value becomes a coordinate. This clearly depends on the dispersion relation. Then, one needs to be able to track singularities caused by the stationary points of the variable-changing mapping and to control the associated Jacobian in places where it degenerates. To our knowledge, such degeneracy is often ‘controllable’ thanks to the specific dispersion relations in systems of hydrodynamic type, but there is great variation in how relevant estimates depend on the localised bandwidth $\delta$ and their proofs can be tricky. Below is some heuristic discussion
where we address the $\delta$-dependence in terms of integer-point counting, knowing that a similar heuristic discussion can also be done in terms of volume estimates. Recall definitions (1.2), (1.3) and consider the number of $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$ satisfying:

$$f(k_1, k_2, k_3) := C_n + \sigma_2 \frac{k_3}{|k|} - \sigma_3 \frac{i k_3 + k_2}{|a + k|} \in [-\delta, \delta], \quad \text{with} \quad \frac{1}{2} |a| \leq |k| \leq |a|,$$

(1.16)

for fixed $0 < \delta < 1$ and $n, \bar{a}$ as given parameters and with constant $C_n := \sigma_1 \frac{\bar{a}}{|n|}$. The intuitive $O(|\bar{a}|^3 \delta + |a|^2)$ upper bound for the integer point count of $k$ is not true due to the $O(|\bar{a}|^3 \delta \log \frac{1}{\delta} + |a|^2)$ lower bound for certain choices of $n$—see appendix C. This shows that there is no obvious choice for the function of $\delta$ that can serve as a sharp upper bound for this type of counting problem. In fact, take the simplest 1D case of $f_1 : \mathbb{R} \to \mathbb{R}$ and ask for an upper bound on the number of integers $k_1$ in the sublevel set given by $|f_1(k_1)| \leq \delta$ subject to $\frac{1}{2} N \leq k_1 \leq N$ for a large constant $N$. Then one should at least prove a lower bound of $|f_1'(x)|$ for $x$ near a zero of $f_1$, but if the zero of $f_1$ is too close to a stationary point then further information is needed. For example, the counting estimate for $f_1(k_1) = \frac{x}{N} - \frac{1}{4}$ with some $\alpha > 0$ is $O(N^{1/\alpha})$. Furthermore, back to example (1.16) in 3D, $f(k_1, k_2, k_3)$ combines square roots in such a fashion (e.g. without obvious convexity) that it is highly non-trivial to prove sharp lower bounds for its derivatives in the neighbourhood of its zero level set while also addressing degeneracy near stationary points where in fact higher derivatives of $f$ can possibly get too close to zero as well. Further rationalisation of (1.16) would result in an 8th degree polynomial in $(k_1, k_2, k_3)$ with $(\delta, n, \bar{a})$-dependent coefficients, and it is still highly non-trivial to optimally quantify those lower bounds of derivatives. There is also a lesser issue in topology: the level set of a function at a critical value is not necessarily manifold(s), for example the boundary of the 2D set defined by $0 < x^2 - y^2 < \delta$. But thanks to the inequality nature of sublevel sets, it can be approximated by a sequence of regular sets $\epsilon < x^2 - y^2 < \delta$ with $\epsilon \to 0$. Also see figure 1.

2. Motivation for near resonance approximations

First, a few basic details for clarity.

For a function $g(x) \in L^p(\mathbb{T}^3)$ for some $p \in (1, \infty)$, its Fourier series expansion is:

$$g(x) = \sum_{n \in \mathbb{Z}^3} g_n e^{i n \cdot x},$$

(2.1)

with coefficients:

$$g_n := \frac{\langle g, e^{i n \cdot x} \rangle}{\langle e^{i n \cdot x}, e^{i n \cdot x} \rangle} = \frac{1}{|\mathbb{T}^3|} \int_{\mathbb{T}^3} g(x) e^{-i n \cdot x} \, dx.$$

(2.2)

Combining these definitions gives this version of Parseval’s theorem/identity:

$$\langle f, g \rangle = |\mathbb{T}^3| \sum_{n \in \mathbb{Z}^3} f_n \cdot g_n.$$

(2.3)

For the Leray projection $\mathcal{P}^{\text{div}}$, classical theory of elliptic operators in Lebesgue spaces shows it is a bounded operator on $L^p(\mathbb{T}^3)$ for any $p \in (1, \infty)$. It is straightforward to show

4 Such study can appear in literature concerning the van der Corput lemma.
self-adjoint property $\langle \mathcal{P} \text{div} u, v \rangle = \langle u, \mathcal{P} \text{div} v \rangle$ for $\mathcal{C}^1$-valued functions $u \in L^p(T^3)$ and $v \in L^{2p}(T^3)$. Operator $\mathcal{L}$ is skew-self-adjoint w.r.t. the $L^2$ inner product in the sense that:

$$\int_{T^3} u \cdot \mathcal{L} u \, dx = 0, \quad \text{for } \mathbb{R}^3\text{-valued function } u \in L^2(T^3). \quad (2.4)$$

Also note the classic property $\langle B(u, v), v \rangle = 0$ for $\mathbb{R}^3$-valued functions $u, v \in H^1(T^3)$ with $u$ also being div-free.

In an equivalent form, we remove the mean drift from the velocity field as follows and assume zero-mean velocity throughout the article. By definition of $\mathcal{P} \text{div}$ and identity:

$$U \cdot \nabla V = \nabla \cdot (U \otimes V), \quad \text{for div-free } U, \quad (2.5)$$

where $H^1$ regularity of $U, V$ is assumed, we find $B(U, V)$ is of zero mean. (This still holds for $U, V \in L^2(T^3)$ in which case one regards $B(U, V)$ as a functional on smooth functions. See definition 1.1 where setting $\hbar \equiv 1$ gives the mean. Also see definition 1.2 with $\psi \equiv 1$.) Thus, for $U_0(t) = \frac{1}{|T^3|} \int_{T^3} U(t, x) \, dx$, integrate (1.1) to have $\frac{d}{dt} U_0 = \Omega J U_0$. Then upon substitution $U(t, x) = U^\prime (t, x - \int_0^t U_0(s) \, ds) + U_0(t)$ in (1.1) followed by dropping the primes, we obtain (1.1) again but with an additional invariant:

$$\int_{T^3} U(t, x) \, dx \equiv 0, \quad \text{for all } t \geq 0.$$

For operator $D^s = (\sqrt{-\Delta})^s$ with $s \in \mathbb{R}$, clearly $D^s g(x) = \sum_{n \in \mathbb{Z}^3} |\hat{n}|^s |g_n| e^{i\hat{n} \cdot x}$. Then,

$$\|g\|_s^2 = |T^3| \sum_{n \in \mathbb{Z}^3} |\hat{n}|^{2s} |g_n|^2, \quad \forall s \in \mathbb{R}, \quad (2.6)$$

by Parseval’s theorem (2.3).

We let $A \lesssim C_1, C_2, \ldots$ denote $A \leq CB$ for a nonnegative constant $C$ that depends on quantities $C_1, C_2, \ldots$. The specific relation between $C$ and $C_1, C_2, \ldots$ or the specific value of $C$ may change from inequality to inequality. For simplicity, we further omit the dependence on the domain $T^3$ in this notation—note remark 1.4.

2.1. Transformed bilinear form

Introduce the operator exponential $e^{\tau \mathcal{L}}$ for $\tau \in \mathbb{R}$ so that, for $\mathcal{C}^1$-valued, div-free function $V_m \in L^2(T^3)$, the function $V(\tau) = e^{\tau \mathcal{L}} V_m$ is the weak solution to:

$$\partial_\tau V = \mathcal{L} V \quad \text{with } \quad V(0) = V_m.$$

Existence and uniqueness of solution in $\mathcal{C}^\infty(\mathbb{R}; L^2(T^3))$ follows from proposition 2.1. For a solution $U(t, \cdot)$ to the RNS equation, perform transformation:

$$u(t, \cdot) = \left. \left( e^{-\tau \mathcal{L}} U(t, \cdot) \right) \right|_{\tau = \Omega t}, \quad (2.7)$$

(also cf (2.23)). Then by the usual chain rule,

$$\partial_t u(t) = \left. \left( e^{-\tau \mathcal{L}} \partial_\tau \left( e^{-\tau \mathcal{L}} U(t) \right) \right) \right|_{\tau = \Omega t} + \left. \left( e^{-\tau \mathcal{L}} \partial_\tau U(t) \right) \right|_{\tau = \Omega t}$$

$$= \left. \left( e^{-\tau \mathcal{L}} \partial_\tau U(t) - \Omega \mathcal{L} e^{-\tau \mathcal{L}} U(t) \right) \right|_{\tau = \Omega t}.$$
It is straightforward to show that $e^{r\mathcal{L}}u$ is div-free and $e^{r\mathcal{L}}$ commutes with $\mathcal{L}$ and $\Delta$, so if $U$ with sufficient regularity satisfies the RNS equation (1.1), then $u$ satisfies:

$$\partial_t u + B(\tau; u, u) \bigg|_{\tau=\Omega_0} = \mu \Delta u, \quad \text{with} \quad u(0, \cdot) = U(0, \cdot),$$

where the transformed bilinear form:

$$B(\tau; u, v) := e^{-r\mathcal{L}}B(e^{r\mathcal{L}}u, e^{r\mathcal{L}}v).$$

From now on, a symbol such as $B$ that denotes a bilinear form can also denote its transformed version, with the distinction that we explicitly write the dependence of the latter on the extra variable $\tau$ (which is sometimes referred to as the ‘fast time’).

### 2.2. Eigen-basis representation

For any $n \in \mathbb{Z} \setminus \{0\}$ and associated domain-adjusted wavevector $\hat{n}$, there exists (not uniquely) an orthonormal, right-hand oriented basis of $\mathbb{R}^3$, in the form $\frac{\hat{n}}{|\hat{n}|}, \tilde{r}_n, \hat{n}$ so that $\frac{\hat{n}}{|\hat{n}|} \times \tilde{r}_n = \hat{n}$ and $\frac{\hat{n}}{|\hat{n}|} \times \hat{n} = -\hat{n}$. Then for
denotes a bilinear form can also denote its trans-

$$r_n^+ := \frac{1}{\sqrt{2}} \left( \tilde{r}_n + i \hat{n} \right), \quad r_n^- := \overline{r_n^+},$$

the vectors $\frac{\hat{n}}{|\hat{n}|}, r_n^-, r_n^+$ form an orthonormal basis of $\mathbb{C}^3$ (so e.g. $r_n^- \cdot \overline{r_n^+} = 0$). Next, introduce:

$$e_n^\sigma := r_n^\sigma e^{i\sigma x}, \quad \sigma \in \{+, -\}, \ n \in \mathbb{Z} \setminus \{0\}.$$  

Since combining identity $v = \Delta^{-1} \Delta v$ for zero-mean $v$ with identity $\Delta = \nabla \nabla \cdot -\nabla \times (\nabla \times)$ shows $\mathcal{D} v = -\Delta^{-1} \nabla \times (\nabla \times v)$, we show that:

$$\mathcal{L} U = -\Delta^{-1} \nabla \times (\nabla \times (J U)) = \Delta^{-1} \nabla \times \partial_t U, \quad \text{for \ div-free, zero-mean function } U.$$  

Next, it is easy to see $\frac{\hat{n}}{|\hat{n}|} \times r_n^\sigma = \sigma i r_n^\sigma$ (so, in fact, $e_n^\sigma$ is also eigenfunction of $\nabla \times$). Note that orthogonality $\hat{n} \cdot r_n^\sigma = 0$ implies div-free condition $\nabla \cdot e_n^\sigma = 0$ and that $n \neq 0$ implies $e_n^\sigma$ is zero-mean, so by the above identity, we obtain the following eigen-pair relation for operator $\mathcal{L}$,

$$\mathcal{L} e_n^\sigma = i \omega_n^\sigma e_n^\sigma \quad \text{for dispersion relation } \omega_n^\sigma := \sigma \frac{\hat{n}}{|\hat{n}|}, \quad \forall \ \sigma \in \{+, -\}, \ n \in \mathbb{Z} \setminus \{0\}.$$  

Introduce eigen-projection, as an operator acting on $\mathbb{C}^3$-valued functions,

$$\mathcal{P}_n^\sigma u := \frac{\langle u, e_n^\sigma \rangle}{\langle e_n^\sigma, e_n^\sigma \rangle} \ e_n^\sigma = (u_n \cdot \overline{e_n^\sigma}) \ r_n^\sigma e^{i\sigma x}, \quad \forall \ n \in \mathbb{Z} \setminus \{0\},$$

with the $u_n$ notation used in the same fashion as Fourier series (2.1). Also define:

$$\mathcal{P}_0^\sigma u \equiv 0.$$  

Directly from definition, we have orthogonality property:

$$\mathcal{P}_n^\sigma \mathcal{P}_{n'}^\sigma' = \begin{cases} \mathcal{P}_n^\sigma, & \text{if } (n, \sigma) = (n', \sigma'), \\ 0, & \text{otherwise}, \end{cases}$$

and adjoint property:

$$\langle \mathcal{P}_n^\sigma u, v \rangle = \langle u, \mathcal{P}_n^\sigma v \rangle = \langle \mathcal{P}_n^\sigma u, \mathcal{P}_n^\sigma v \rangle.$$
Recall $\frac{\hat{n}}{|\hat{n}|}, r_n^+, r_n^-$ (for $n \neq \hat{0}$) form an orthonormal basis and recall $\nabla \cdot u = 0$ iff $\hat{n} \cdot u_n = 0$. Therefore,
\[ p_n^\sigma p_n^{\text{div}} = p_n^\sigma = p_n^{\text{div}} p_n^\sigma, \quad \forall \ n \in \mathbb{Z}^3 \setminus \{\hat{0}\}, \]
and
\[ p_n^\sigma u + p_n^\sigma u = p_n^{\text{div}} (u_n e^{i\hat{n} \cdot x}), \quad \forall \ n \in \mathbb{Z}^3 \setminus \{\hat{0}\}, \tag{2.14} \]
for any $\mathbb{C}^3$-valued function $u \in L^2(\mathbb{T}^3)$ that is not necessarily div-free. Thus by the completeness of the Fourier basis, we have expansion:
\[ p_n^{\text{div}} u = \sum_{n, \sigma} p_n^\sigma u, \quad \text{for } \mathbb{C}^3\text{-valued, zero-mean function } u \in L^2(\mathbb{T}^3), \tag{2.15} \]
here and below, $\sum_{n, \sigma}$ stands for $\sum_{n \in \mathbb{Z}^3} \sum_{\sigma \in \{+, -\}}$. Then, by (2.12), (2.13), we have:
\[ \|p_n^{\text{div}} u\|_{L^2}^2 = \sum_{n, \sigma} \left\| p_n^\sigma u \right\|_{L^2}^2, \quad \text{for zero-mean function } u \in L^2(\mathbb{T}^3). \tag{2.16} \]

**Proposition 2.1.** For any $\mathbb{C}^3$-valued, div-free, zero-mean functions $u, v \in L^2(\mathbb{T}^3)$, we have:
\[ L p_n^\sigma u = i \omega_n^\sigma p_n^\sigma u = p_n^\sigma L u, \tag{2.17} \]
the eigen-expansion of the operator exponential:
\[ e^{\tau L} u = \sum_{n, \sigma} e^{i \omega_n^\sigma \tau} p_n^\sigma u, \tag{2.18} \]
and the adjoint property:
\[ \langle e^{\tau L} u, v \rangle = \langle u, e^{-\tau L} v \rangle. \tag{2.19} \]

**Proof.** By (2.10), we show the first equality of (2.17). Combining it with (2.12), (2.15), we show the second equality of (2.17). Recalling the definition of $e^{\tau L}$, we use (2.15), (2.17) to prove (2.18). Combining the latter with (2.13), we see adjointness (2.19) naturally follows. \[ \square \]

Combining this with (2.11) shows operators $D^i$, $p_n^\sigma$ and $e^{\tau L}$ all commute. Then by (2.16),
\[ \|D^i g(x)\|_{L^2(\mathbb{T}^3)} = \|e^{\tau L} D^i g(x)\|_{L^2(\mathbb{T}^3)} = \|D^i e^{\tau L} g(x)\|_{L^2(\mathbb{T}^3)}, \quad \forall \ \tau \in \mathbb{R}. \tag{2.20} \]
Also, combining (2.19) with (2.9) shows:
\[ \langle B(\tau; u, v), w \rangle = \langle B(e^{\tau L} u, e^{\tau L} v), e^{\tau L} w \rangle, \quad \text{for } \mathbb{C}^3\text{-valued, div-free, zero-mean functions } u, v, w. \tag{2.21} \]

Finally, the convolution condition $n + k + m = \hat{0}$ is equivalent to the fact that:
\[ p_n^\sigma B(p_{k+}^\sigma u, p_{m-}^\sigma v) = 0, \quad \text{if } \ n + k + m \neq \hat{0}. \tag{2.22} \]
Thus, for a mapping $F$ from $(\mathbb{Z}^3)^3$ to a Banach space, introduce shorthand notation:
\[ \sum_{n, k, m; \text{conv}} F(n, k, m) \equiv \sum_{n \in \mathbb{Z}^3} \sum_{k \in \mathbb{Z}^3} F(n, k, -n - k). \]
Remark 2.2. The absolute convergence of the above sums in the Banach space that is the codomain of $F$ implies the commutability of the nested sums, and implies symmetry with respect the choice of running indices. For example,

$$
\sum_{n,k,m,\text{conv}} F(n,k,m) = \sum_{n \in \mathbb{Z}^d} \sum_{m \in \mathbb{Z}^d} F(n,-n-m,m) = \sum_{m \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} F(n,-n-m,m),
$$

and thus

$$
\sum_{n,k,m,\text{conv}} F(n,k,m) = \sum_{n,k,m,\text{conv}} F(n,m,k) = \sum_{n,k,m,\text{conv}} F(k,m,n),
$$

Without absolute convergence, even the simple sum $\sum_{n \in \mathbb{Z}} (\sum_{k=-1}^{1} k)$ is not commutable. In this article, all nested sums converge absolutely in a Banach space that is evident from context. Also note the completeness of Fourier basis in $H^s(\mathbb{T}^3)$ spaces.

For the reader’s reference, we also expand (2.7) for div-free, zero-mean functions $u, U$ as follows:

$$
u(t,x) = \sum_{n,\sigma} \exp(-i \omega_n^\sigma \Omega t) P_n^\sigma u(t,x) \iff U(t,x) = \sum_{n,\sigma} \exp(i \omega_n^\sigma \Omega t) P_n^\sigma u(t,x).$$

2.3. Motivating NR approximations

Introduce the following bilinear form via restricted convolution on a generic set $\mathcal{M} \subset (\mathbb{Z}^d)^3$ using a notation consistent with $B_{\mathcal{N}[s,\cdot]}$,

$$B_{\mathcal{M}}(U, V) := \sum_{n,k,m,\text{conv}} B(e^{i k \cdot x} U_k, e^{i m \cdot x} V_m) 1_{\mathcal{M}}(n,k,m).$$

We can easily prove the following identity for $\mathbb{C}^3$-valued, smooth functions $U, V, W$:

$$\langle B_{\mathcal{M}}(U, V), W \rangle = \langle [\mathbb{T}^3] \sum_{n,k,m,\text{conv}} (U_k \cdot \bar{m} i) V_m \cdot (\bar{W})_m 1_{\mathcal{M}}(n,k,m) \quad \text{with div-free } W,$$

where $(W)_m$ denotes the Fourier coefficients of $W$.

Analogously to (2.9), define

$$B_{\mathcal{M}}(\tau; u, v) := e^{-\tau \mathcal{L}} B_{\mathcal{M}}(e^{\tau \mathcal{L}} u, e^{\tau \mathcal{L}} v).$$

Apply expansion (2.18) on all three operator exponentials above and expand the result using the bilinearity of $B_{\mathcal{M}}(\cdot; \cdot)$. Then, noting identity $B_{\mathcal{M}}(P_k^{\sigma} u, P_m^{\sigma} v) = B(P_k^{\sigma} u, P_m^{\sigma} v) 1_{\mathcal{M}}(n,k,m)$ by definitions (2.11), (2.24), we find:

$$B_{\mathcal{M}}(\tau; u, v) = \sum_{n,k,m,\text{conv}} \sum_{\tilde{\sigma} \in \{+,-\}^3} P_n^{\tilde{\sigma}} B(P_k^{\sigma} u, P_m^{\sigma} v) \exp(i \omega_{nkm}^\sigma \tau) 1_{\mathcal{M}}(n,k,m),$$

for div-free, zero-mean functions $u, v$,

$$B(U, V) = B_{(\mathbb{Z}^d)^3}(U, V), \quad \text{and thus } B(\tau; u, v) = B_{(\mathbb{Z}^d)^3}(\tau; u, v).$$
Since the heuristic bound $u_1 \sim O(1)$ is independent of $\Omega$ as seen from the transformed RNS equation (2.8), we argue that (2.27), (2.28) suggest the applicability of (1.13) with $\omega^s = \omega^s_{\text{nr}}$ on the analysis of (2.8). The smaller the triplet value is, the closer it is to exact resonance and a more important role it plays in the dynamics. This prompts definition (1.3) of NR set $N[\delta_s]$ which selects all those triplet values $|\omega^s_{\text{nr}}|$ that fall below bandwidths given by the values of the function $\delta_s(n, k, m)$. Since the bilinear form $B_N[\delta_s](\tau; u, v)$ defined in (1.4), being the same as (2.24) with $M = N[\delta_s]$, can be expanded using (2.27), and thus is clearly the NR-selecting counterpart of the unrestricted $B(\tau; u, v) = B(\Omega)^3(\tau; u, v)$, we have arrived at the PDE (1.5) from the Introduction where it is named the NR approximation of the RNS equations.

Note that by the same argument as for the RNS equations, the NR approximation (1.5) is equivalent to its transformed version:

$$
\partial_\tau \tilde{u} + B_N[\delta_s](\tau; \tilde{u}, \tilde{u}) \bigg|_{\tau = \Omega} = \mu \Delta \tilde{u}, \quad \text{with} \quad \tilde{u}(0, \cdot) = \tilde{U}(0, \cdot),
$$

(2.29)

where $\tilde{u}(t, \cdot) = e^{-\Omega t} \tilde{U}(t, \cdot)$.

For brevity, the NR set $N[\delta_s]$ is defined without concerning the membership of the sign triplet $\bar{\sigma}$, as we omitted an alternative definition of the NR set in the form of:

$$
\left\{(n, k, m, \bar{\sigma}) \in (\mathbb{Z}^3 \setminus \{0\})^3 \times \{+, -\}^3 : |\omega^s_{\text{nr}}| \leq \delta_s(n, k, m), \; n + k + m = 0 \right\}.
$$

Results analogous to theorems 1.3, 1.5, 1.6 would still hold in this alternative setting. Firstly, the above NR set would retain less triplet interactions, thus making the 2D-like estimates still valid. Secondly, any discarded interaction would still satisfy $|\omega^s_{\text{nr}}| > \delta_s(n, k, m)$, thus making the key ingredients of the error estimates still valid. The changes in the proofs, however, would essentially require no more mathematics than lengthier expansions and use of the $P^s_n$ notation.

To conclude this section, for physical relevance, we need to know whether $B_N[\delta_s]$ still have these basic properties that $B$ has: mapping real-valued inputs to real-valued outputs and being skew-symmetric with respect to the $L^2$ inner product so that the approximate system still conserves kinetic energy in the absence of forcing.

**Proposition 2.3.** Consider bilinear form $B_M(\cdot, \cdot)$ defined in (2.24) with $M \subset (\mathbb{Z}^3)^3$. Then, for $\mathbb{R}^3$-valued, div-free, zero-mean smooth functions $U, V$,

symmetry $1_M(n, k, m) = 1_M(\bar{n}, \bar{k}, \bar{m})$ implies $B_M(U, V)$ is $\mathbb{R}^3$-valued.

(2.30)

We also have

symmetry $1_M(n, k, m) = 1_M(m, k, n)$ implies $\langle B_M(U, V) \rangle = 0$.

(2.31)

The same properties still hold with $B_M(U, V)$ replaced by $e^{\tau \mathcal{L}}B_M(e^{\tau \mathcal{L}}U, e^{\tau \mathcal{L}}V)$ for any $\tau \in \mathbb{R}$.

In particular, under assumption (1.6), the conclusions of (2.30), (2.31) hold for $B_N[\delta_s]$.

**Proof.** First, due to (2.26), (2.19) and the fact that $e^{\tau \mathcal{L}}$ maps the set of $\mathbb{R}^3$-valued, div-free, zero-mean functions onto the set itself, it suffices to just prove (2.30), (2.31).

By taking the complex conjugate of (2.1), we find that $g(x)$ is $\mathbb{R}^3$-valued iff $g^* = g$ for all $n$. Then take the complex conjugate of (2.24) with real-valued functions $U, V$ to prove (2.30).

By self-adjointness of $\mathcal{P}^\text{div}$, the assumed $V = \mathcal{P}^\text{div}V$ and expansion (2.25), noting $\mathcal{V} = V$, we show $\langle B_M(U, V) \rangle = \langle \mathcal{T}^3 \sum_{n, k, m} \text{comp} \left( U_k \cdot \bar{n} i \right) V_m \cdot V_n \rangle 1_M(n, k, m)$. Switching $m, n$ and adding it back, we prove (2.31) by incompressibility $U_k \cdot (\bar{n} + m) = 0$ and summation symmetry as in remark 2.2. □
3. Restricted convolution and integer-point counting

The restriction of 3-mode interactions to the NR set $\mathcal{N}[\delta_*]$ leads to the need for estimating restricted convolution. The lemma below is inspired by [6, lemma 3.1] and extends it to a version allowing different factors in the triple product. Also see [7, lemma 6.2].

**Lemma 3.1.** Let $1_{\mathcal{M}}(\cdot,\cdot,\cdot)$ denote the characteristic of set $\mathcal{M} \subset (\mathbb{Z}^3)^3$ and suppose $1_{\mathcal{M}}(\cdot,\cdot,\cdot)$ is symmetric with respect to all permutations in its three arguments. Define

$$\mathcal{M}_{\text{loc}} := \{(n,k,m) \in \mathcal{M} : |\hat{n}| \geq |\hat{k}| \geq |\hat{m}|\}.$$ 

Suppose there exist a constant $\beta \in [0,3]$ and a constant $C_0$ so that the counting condition:

$$\sum_{k \in \mathbb{Z}^3} 1_{\mathcal{M}_{\text{loc}}}(n,k,-n-k) \leq C_0 |\hat{n}|^\beta, \quad \forall \, n \in \mathbb{Z}^3 \setminus \{0\}, \tag{3.1}$$

holds. Then there exists a constant $C$ so that for zero-mean $u,v \in H^2(\mathbb{T}^3)$ and $w \in L^2(\mathbb{T}^3)$,

$$|\mathbb{T}^3|^{\gamma/2} \sum_{n,k,m,\text{conv}} |u_n| |v_k| |w_m| 1_{\mathcal{M}_{\text{loc}}}(n,k,m) \leq C \sqrt{C_0} \left( \|u\|_0 \|v\|_\frac{\gamma}{2} + \|u\|_\frac{\gamma}{2} \|v\|_0 \right) \|w\|_0. \tag{3.2}$$

The estimate still holds if we permute $u,v,w$ on the left-hand side.

When applied to standard product $\langle fg, h \rangle = \int_{\mathbb{T}^3} (fg) \overline{h} \, dx$, the counting condition (3.1) holds for $\beta = 3$, so the upper bound in (3.2) requires $\gamma$ derivatives on either $f$ or $g$. This is in fact a consequence of combining (6.14) (ii) with interpolation, and this is different from the common upper bound $\|fg\|_0 \lesssim \|f\|_{\frac{\gamma}{2} + \gamma} \|g\|_0$ for $\gamma > 0$ by (6.14) (i). In contrast, as $\beta = 2$ will later be used in the proof of 2D-like estimates theorem 1.5, restriction to $\mathcal{N}[\delta_*]$ ‘gains’ half a derivative.

**Proof.** It suffices to consider scalar-valued functions. Also note that, since the following proof only involves absolute values of the summands, all nested sums are commutable.

By the ordering encoded in $\mathcal{M}_{\text{loc}}$ and the triangle inequality $|\hat{n}| \leq |\hat{k}| + |\hat{n} - \hat{k}|$, we find:

$$1_{\mathcal{M}_{\text{loc}}}(n,k,-n-k) = 1 \implies |\hat{k}| \leq |\hat{n}| \leq 2|\hat{k}|. \tag{3.3}$$

This is to say, the $1_{\mathcal{M}_{\text{loc}}}(n,k,-n-k)$ term effectively localises the non-zero contributions to the sum in (3.1) to be around the ‘diagonal’ where $|\hat{k}|$ is of comparable size to $|\hat{n}|$.

Define

$$\eta(a,b,c) := \begin{cases} 1, & \text{if } a \geq b \geq c, \\ 0, & \text{otherwise}, \end{cases}$$

so that $1_{\mathcal{M}_{\text{loc}}}(n,k,m) = 1_{\mathcal{M}}(n,k,m) \eta(|\hat{n}|, |\hat{k}|, |\hat{m}|)$. For any wavevectors $n,k,m$, we have:

$$1 \leq \eta(|\hat{n}|, |\hat{k}|, |\hat{m}|) + \eta(|\hat{m}|, |\hat{n}|, |\hat{k}|) + \eta(|\hat{k}|, |\hat{m}|, |\hat{n}|) + \eta(|\hat{n}|, |\hat{k}|, |\hat{m}|).$$

Then the summand in (3.2) is bounded by the corresponding six terms $|u_nv_kw_m| 1_{\mathcal{M}_{\text{loc}}}(n,k,m) + |u_nv_kw_m| 1_{\mathcal{M}}(n,k,m) \eta(|\hat{n}|, |\hat{k}|, |\hat{m}|) + |u_nv_kw_m| 1_{\mathcal{M}_{\text{loc}}}(n,k,m) \eta(|\hat{m}|, |\hat{k}|, |\hat{n}|) + \ldots$ Next, permute $n,k,m$ in each term in such a way that the $\eta$ function appears as $\eta(|\hat{n}|, |\hat{k}|, |\hat{m}|)$. For example, switch $n,k$ in the second term, which results in $|u_nv_kw_m| 1_{\mathcal{M}}(k,n,m) \eta(|\hat{n}|, |\hat{k}|, |\hat{m}|)$. \[2090\]}
Also note the assumed symmetry of $I_{\mathcal{M}}(\cdot, \cdot, \cdot)$ and symmetry of convolution sum by remark 2.2. After these steps, all six terms can be rewritten so that $I_{\mathcal{M}_{\text{loc}}}(n, k, m)$ is a common factor:

$$
\sum_{n,k,m,\text{conv}} |u_n v_k w_m| I_{\mathcal{M}}(n, k, m) \leq \sum_{n,k,m,\text{conv}} \left( |u_n v_k w_m| + |u_k v_n w_m| + \text{four terms} \right) I_{\mathcal{M}_{\text{loc}}}(n, k, m).
$$

(3.4)

Next introduce annuli

$$
\mathcal{A}_i := \{ k \in \mathbb{Z}^3 : 2^{i-1} \leq |k| < 2^i \}, \quad \forall i \in \mathbb{Z}.
$$

Then the first term on the right-hand side of (3.4) satisfies:

$$
\sum_{n,k,m,\text{conv}} |u_n v_k w_m| I_{\mathcal{M}_{\text{loc}}}(n, k, m) = \sum_{n \in \mathbb{Z}^3} \left( \sum_{k \in \mathbb{Z}^3} |u_n v_k w_{-k-n}| I_{\mathcal{M}_{\text{loc}}}(n, k, -n-k) \right) = \sum_{i \in \mathbb{Z}} \sum_{n \in \mathcal{A}_i} \left( |u_n| \sum_{k \in \mathbb{Z}^3} |v_k w_{-k-n}| I_{\mathcal{M}_{\text{loc}}}. \right). \tag{3.5}
$$

Here and below, the shortened $I_{\mathcal{M}_{\text{loc}}}$ notation is always understood as $I_{\mathcal{M}_{\text{loc}}}(n, k, -k - n)$. Define

$$
a_i := \sum_{n \in \mathcal{A}_i} \left( \sum_{k \in \mathbb{Z}^3} |v_k w_{-k-n}| I_{\mathcal{M}_{\text{loc}}} \right)^2,
$$

and apply Cauchy-Schwarz inequality on the combined $(i, n)$-sum:

$$
(3.5) \leq \sqrt{\sum_{i \in \mathbb{Z}} \sum_{n \in \mathcal{A}_i} |u_n|^2} \sqrt{\sum_{i \in \mathbb{Z}} a_i}.
$$

(3.6)

To estimate $a_i$, apply Cauchy-Schwarz inequality on the $k$-sum to have:

$$
a_i \leq \sum_{n \in \mathcal{A}_i} \left[ \left( \sum_{k \in \mathbb{Z}^3} |v_k w_{-k-n}|^2 I_{\mathcal{M}_{\text{loc}}} \right) \left( \sum_{k \in \mathbb{Z}^3} I_{\mathcal{M}_{\text{loc}}} \right) \right] \leq C_0 \sum_{n \in \mathcal{A}_i} \left[ \sum_{k \in \mathcal{A}_{i-1} \cup \mathcal{A}_i} |\hat{\beta}| |v_k w_{-k-n}|^2 I_{\mathcal{M}_{\text{loc}}} \right] \leq C_0 \sum_{k \in \mathcal{A}_{i-1} \cup \mathcal{A}_i} \left[ |\hat{k}|^2 |v_k|^2 \sum_{n \in \mathcal{A}_i} |w_{-k-n}|^2 \right] \leq C_0 \sum_{k \in \mathcal{A}_{i-1} \cup \mathcal{A}_i} |\hat{k}|^2 |v_k|^2 |\mathbb{T}^3|^{-1} \|w\|_{L^2}^2,
$$

where in the last step we relaxed the range $n \in \mathcal{A}_i$ to all $n \in \mathbb{Z}^3$ and used (2.6).

Finally, combining the above and (3.6), we apply (2.6) again to prove:

$$
\sum_{n,k,m,\text{conv}} |u_n v_k w_m| I_{\mathcal{M}_{\text{loc}}}(n, k, m) \leq \sqrt{C_0} \|u\|_0 \|v\|_2 \|w\|_0 |\mathbb{T}^3|^{-1/2}.
$$

(3.7)

To generalise the above method to any of the six terms in (3.4), we will first establish a different estimate on (3.5). By (3.3), we have:

$$
(3.5) \leq \sum_{i \in \mathbb{Z}} \sum_{n \in \mathcal{A}_i} \left( |\hat{\beta}| |\beta|/2 u_n \sum_{k \in \mathbb{Z}^3} \left( |\hat{k}|^{-\beta/2} v_k \right) w_{-k-n} I_{\mathcal{M}_{\text{loc}}} \right).
$$
and immediately let \( |\hat{n}|^{1/2} u_n \) be denoted by \( u'_n \) and \( |\hat{k}|^{-1/2} v_k \) by \( v'_k \). Then we perform the same steps (except with the primes) afterwards until the right-hand side of (3.7); therefore we prove:

\[
\sum_{n,k,m: \text{conv}} |w_{n,k,m}| \mathbf{1}_{M_{\text{lsc}}} (n,k,m) \lesssim \sqrt{C_0} \| u \|_2 \| v \|_0 \| w \|_0 |T^2|^{-\frac{1}{2}}.
\]

The essence of this estimate and (3.7) is that every one of the six terms on the right-hand side of (3.4) can be bounded by the product of three norms where we have the freedom to apply the \( H^2 \) norm on either the factor with \( n \) index or the factor with \( k \) index. Since at least one of \( u \) and \( v \) is equipped with either \( n \) or \( k \) index on the right-hand side of (3.4), the proof of (3.2) is complete. The very last statement of the lemma follows from the assumed symmetry on \( \mathbf{1}_{M}(\cdot,\cdot,\cdot) \).

\( \square \)

4. 'Anti-discretise' from integer-point counting to integrals

With motivation already given in section 1.2, we start the mathematics right away.

Recall definition (1.3) of the NR set \( N[\delta] \) and the type of integer-point counting used in assumption (3.1) of lemma 3.1. They inspire us to define mapping \( F_{n,\sigma_1,\sigma_2} : \mathbb{R}^3 \setminus \{0,-n\} \to \mathbb{R} \) as follows (recall \( \hat{n}_3 = n_3 \), etc),

\[
F_{n,\sigma_1,\sigma_2}(k) := \sigma_1 \frac{n_3}{|\hat{n}|} + \sigma_2 \frac{k_3}{|\hat{k}|} + \frac{m_3}{|\hat{m}|}, \quad \text{for } m = -n - k,
\]

\[ \forall \ n \in \mathbb{Z}^3 \setminus \{\hat{0}\}, \ (\sigma_1,\sigma_2) \in \{+,-\}^2. \tag{4.1} \]

Define the following subset of \( \mathbb{R}^3 \) (not just \( \mathbb{Z}^3 \)),

\[
V_{n,\sigma_1,\sigma_2}^\delta := \{ k \in \mathbb{R}^3 \setminus \{0,-n\} : |F_{n,\sigma_1,\sigma_2}(k)| \leq \delta \text{ and } \frac{1}{2} |\hat{n}| \leq |\hat{k}| \leq |\hat{n}|, \forall \delta \in \mathbb{R}^+ \}
\]

\[ \forall \ n \in \mathbb{Z}^3 \setminus \{\hat{0}\}, \ (\sigma_1,\sigma_2) \in \{+,-\}^2. \tag{4.2} \]

Following the notation of lemma 3.1, define:

\[
N[\delta] := \{ (n,k,m) \in N[\delta] : |\hat{n}| \geq |\hat{k}| \geq |\hat{m}| \}.
\]

Then, the condition \( \mathbf{1}_{N[\delta]} (n,k,-n-k) = 1 \) implies \( k \not\in \{0,-n\} \) and implies there exists \( (\sigma_1,\sigma_2) \in \{+,-\}^2 \) so that \( |F_{n,\sigma_1,\sigma_2}(k)| \leq \delta \). Also, under (1.6), it implies \( \delta_\ast (n,k,-n-k) = \delta_\ast (n,n,n) \) and implies localisation (3.3). In short, under (1.6), we have \( \mathbf{1}_{N[\delta]} (n,k,-n-k) = 1 \) implies \( k \in \bigcup_{(\sigma_1,\sigma_2) \in \{+,-\}^2} V_{n,\sigma_1,\sigma_2}^\delta \). Therefore,

\[
\text{if } (1.6) \text{ then } \sum_{k \in \mathbb{Z}^3} \mathbf{1}_{N[\delta]} (n,k,-n-k) \leq \sum_{(\sigma_1,\sigma_2) \in \{+,-\}^2} \sum_{k \in \mathbb{Z}^3} \mathbf{1}_{V_{n,\sigma_1,\sigma_2}^\delta} (k), \forall n \in \mathbb{Z}^3 \setminus \{\hat{0}\}. \tag{4.4} \]

From here on until the last paragraph before theorem 4.2, we assume \( n,\sigma_1,\sigma_2 \) are fixed, let \( F = F_{n,\sigma_1,\sigma_2} \) and \( V^\delta = V_{n,\sigma_1,\sigma_2}^\delta \).

The symbol \( \delta \) can be regarded as a generic nonnegative value for the sake of generality. The case \( \delta = \delta_\ast (n,n,n) \) is our main interest and can be viewed as a localised bandwidth. Since \( V^\delta \) is Lebesgue measurable, the integral \( \int_{\mathbb{R}^3} V^\delta (k) \, dk \) is regarded as volume \( \text{vol}(V^\delta) \).

The heuristic is that integer-point counting resembles the Riemann sum as a discrete approximation of an integral of a suitable indicator function—note however, all integrals here are Lebesgue integrals. Thus, much of the work in section 4 is to bound \( \sum_{k \in \mathbb{Z}^3} V^\delta (k) = \int_{k \in \mathbb{R}^3} V^\delta (k) \). The estimate of the volume integral itself is given in section 5.
4.1. Reduction of 3D integer-point counting to 2D

We will use lemma A.2 for estimating number of planar integer points. Priori to that, we first ‘anti-discretise’ in the vertical dimension since it plays a different role in the dispersion relation. For \( k = (k_1, k_2, k_3) \), let:

\[
k_H = (k_1, k_2) \quad \text{and} \quad \hat{k}_H = \left( \frac{k_1}{L_1}, \frac{k_2}{L_2} \right).
\]

Since \( \frac{\partial F}{\partial k_3} = \sigma_2 \frac{|k_H|^2}{|k|^3} - \frac{|m_H|^2}{|m|^3} \), we have,

\[
\frac{\partial F}{\partial k_3} = 0 \implies |\hat{k}_H|^7 (k_3^2 + 2k_3n_3 + n_3^2 + |\bar{m}_H|^2) = |\bar{m}_H|^7 (k_3^2 + |k_H|^2).
\]

At any fixed \( n \in \mathbb{R}^3 \setminus \{0\} \) and \( k_H \in \mathbb{R}^2 \setminus \{0, -n_H\} \) (thus fixed \( m_H \in \mathbb{R}^2 \setminus \{0\} \)), \( F \) is defined for all \( k_3 \in \mathbb{R} \) and what is in above shows \( \frac{\partial F}{\partial k_3} \) either vanishes at most two \( k_3 \) lie in \( \mathbb{R} \) or vanishes everywhere. Then \( \mathbb{R} \) is the union of no more than three \( k_3 \)-monotonic intervals for \( F \), so \( \{ k_3 \mid |F(k)| \leq \delta \} \) is the union of at most three intervals (an isolated point is counted as an interval). In each interval, the number of integers \( k_3 \) and the integral of the indicator function in \( k_3 \) differ by at most 1. But even for \( k_H \in \{0, -n_H\} \), since \( F \) is defined in \( \mathbb{R}^3 \setminus \{0, -n\} \) and the form of \( \frac{\partial F}{\partial k_3} \) is simplified, we can show the previous statement still holds. Therefore,

\[
\sum_{k_3 \in \mathbb{Z}} 1_{V^\delta}(k_H, k_3) \leq 3 \quad \text{and} \quad \forall k_H \in \mathbb{Z}^2.
\]

(4.6)

Since any \( k \in V^\delta \) satisfies \( |\hat{k}_H| \leq |\bar{n}| \), we sum both sides of (4.6) for all \( k_H \in \mathbb{Z}^2 \) with \( |\hat{k}_H| \leq |\bar{n}| \) and apply lemma A.2 to estimate \( \sum_{k_3 \in \mathbb{Z}^2} 1_{V^\delta}(k_H, k_3) \). This shows:

\[
\sum_{k_3 \in \mathbb{Z}^2} 1_{V^\delta}(k) \leq C_{n,L_1} + \sum_{k_3 \in \mathbb{Z}^2} \int_{\mathbb{R}} 1_{V^\delta}(k_H, k_3) \, dk_3
\]

\[
\leq C_{n,L_1} + \int_{\mathbb{R}} \sum_{k_3 \in \mathbb{Z}^2} 1_{V^\delta}(k_H, k_3) \, dk_3, \quad \text{for constant } C_{n,L_1} \leq L_1L_2|\bar{n}|^2 + (L_1 + L_2)|\bar{n}|.
\]

(4.7)

Here, we switched the sum and the integral because the sum only contains finitely many terms.

4.2. Planar integer-point counting with topological singularity

We move to the 2D counting problem from (4.7), that is \( \sum_{k_3} 1_{V^\delta}(k_H, k_3) \) at given \( k_3 \). By an elementary result due to Jarnik and Steinhaus [29], given an open set \( \delta \) that is the interior of a closed, rectifiable Jordan curve \( \gamma^\delta \subset \mathbb{R}^2 \) (cf Jordan curve theorem), we have:

\[
\# \left( \mathbb{Z}^2 \cap (S \cup \gamma^\delta) \right) - \text{Area}(S) < \text{Len}(\gamma^\delta).
\]

For application to our problem, we prove lemma A.2 for multiple disjoint Jordan curves where the set of interest can be in either the interior or the exterior of a given curve.

Therefore we are interested in the continuous 2D set \( \{ k_H \in \mathbb{R}^2 : (k_H, k_3) \in V^\delta \} \) at given \( k_3, \delta \). A topological subtlety is that its boundary may contain tangentially contacting curves—see figure 1 for a 3D version of this. Another subtlety is that it is a closed set, so may contain isolated curve or point which is not covered by lemma A.2 according to the type of open sets it requires.
Lemma 4.1. Let \( n \in \mathbb{R}^3 \setminus \{0\}, \ (\sigma_1, \sigma_2) \in \{+, -\}^2 \) and \( \delta \in \mathbb{R}_0^+ \) be fixed. Also fix, \( k_3 \in (\pm |\bar{n}|, \mp |\bar{n}|) \setminus \{0, -m_3, -\frac{1}{2}m_3\} \). (4.8)

With definitions (4.1), (4.2) and notational convention (4.5), define 2D set:
\[
S(k_3) := \{k_H \in \mathbb{R}^2 : (k_H, k_3) \in V^3\},
\]
where the dependence of \( S(k_3) \) on \( \delta \) is implied.

Then there exist constants \( C, C' \) independent of \( n, \sigma_1, \sigma_2, k_3, \delta, L_1, L_2 \) so that:
\[
\#(\mathbb{Z}^2 \cap S(k_3)) \leq \text{Area}(S(k_3)) + C(L_1 + L_2)\sqrt{|\bar{n}|^2 - k_3^2 + C'},
\]
where if \( \delta = 0 \) then \( \text{Area}(S(k_3)) = 0 \).

First, introduce ‘oblong’ cylindrical coordinates \( (r_k, \phi_k, k_3) \in [0, \infty) \times \mathbb{T}_{2\pi} \times \mathbb{R} \) for \( k \) so that:
\[
k_H = (L_1 r_k \cos \phi_k, L_2 r_k \sin \phi_k, k_3).
\]

Then \( |k_H| = r_k \) and \( |\hat{k}| = \sqrt{r_k^2 + k_3^2} \). Similar notations apply to wavevector \( n \).

At fixed \( n \) and \( k_3 \), since assumption (4.8) ensures \( F(k_H, k_3) \) is defined for all \( k_H \in \mathbb{R}^2 \), we consider the cylindrical form of \( F \) which is the mapping \( G : [0, \infty) \times \mathbb{T}_{2\pi} \mapsto \mathbb{R} \) defined as:
\[
G(r_k, \phi_k) := F(L_1 r_k \cos \phi_k, L_2 r_k \sin \phi_k, k_3) = \sigma_1 \frac{n_3}{|\bar{n}|} + \sigma_2 \frac{k_3}{\sqrt{r_k^2 + k_3^2}} + m_3 \frac{|\bar{m}|}{r_k},
\]
where
\[
|\bar{m}|^2 = r_k^2 + 2r_k r_n \cos(\phi_k - \phi_n) + r_n^2 + m_3^2
\]
and
\[
m_3 = -(k_3 + m_3).
\]
(4.11)

with (4.11) due to \( \bar{m} = -\hat{k} - \bar{n} \) and \( \hat{k} \cdot \bar{n} = k_3 n_3 + \hat{k}_H \cdot \bar{n}_H = k_3 n_3 + r_k r_n \cos(\phi_k - \phi_n) \).

The azimuthal derivative will be of particular importance:
\[
\frac{\partial G}{\partial \phi_k} = \frac{m_3}{|\bar{m}|} r_n r_k \sin(\phi_k - \phi_n).
\]
(4.12)
Also, by the definition of $G$, we have:

\[ \frac{\partial}{\partial r_k} G(r_k, \phi_k) = 0 \]  
\[ \implies -\sigma^2 \frac{k_3 r_k}{(r_k^2 + k_3^2)^{3/2}} = \frac{m_3 (r_k + r_n \cos(\phi_k - \phi_n))}{(r_k^2 + 2 r_k r_n \cos(\phi_k - \phi_n) + r_n^2 + m_n^2)^{3/2}} \]  
\[ \implies r_k \text{ is a zero of a polynomial with leading term } (m_3^2 - k_3^2) r_k^4 \neq 0 \text{ under (4.8)}. \]

**Proof of lemma 4.1.** At fixed $n, k_3$, (4.8) ensures $F(k_H, k_3)$ is defined for all $k_H \in \mathbb{R}^2$. Let

\[ r_{in} := \max \{ 0, \frac{1}{4} |\eta|^2 - k_3^2 \} \quad \text{and} \quad r_{out} := \sqrt{|\eta|^2 - k_3^2}, \]

that correspond to the inner edge (if $r_{in} > 0$) and outer edge of the intersection of 3D annulus $\frac{1}{4} |\eta| \leq |\eta| \leq |\eta|$ and the plane $k_3 = constant$. If $r_{in} = 0$ then the inner edge does not exist.

If $r_{in} = 0$ then $G(r_k, \phi_k)$ is independent of $\phi_k$ by (4.12). In view of (4.13), the boundary $\partial S(k_3)$ consists of at most 11 concentric ellipses (on which either $|F| = 0$ or $|k| = r_{in}$ or $r_{out}$), hence proving (4.9) by virtue of lemma A.2—even if some of these ellipses are possibly not in the closure of the interior of $S(k_3)$. Note, for the case of $\delta = 0$, the set $S(k_3)$ is a subset of these ellipses, and thus $Area(S(k_3)) = 0$. Thus, for the rest of the proof, we assume

\[ r_{in} > 0. \]  
\[ (4.14) \]

Combining this with (4.8), (4.12), we find that $G$ at a fixed $r_k > 0$ is strictly monotonic for $\phi_k \in [\phi_n - \pi, \phi_n]$ and $\phi_k \in [\phi_n, \phi_n + \pi]$, changing monotonicity at $\phi_k \equiv \phi_n (\text{mod } \pi)$. We will refer to this property as **half-period monotonicity**. Immediately, for the case of $\delta = 0$, the set $S(k_3)$ intersects with any ellipse of a fixed $r_k$ coordinate at no more than 2 points, which proves $Area(S(k_3)) = 0$, i.e. zero Lebesgue measure in $\mathbb{R}^2$, as the last statement of the lemma.

We recognise that $S(k_3) = S_+ \cap S_-$ where

\[ S_+ := \{ k_H \in \mathbb{R}^2 : |k_H| \in [r_{in}, r_{out}] \text{ and } F(k_H, k_3) \leq \delta \}, \]

and $S_-$ is defined similarly but with $F$ replaced by $-F$.

We now focus on the upper bound of $\#(\mathbb{Z}^2 \cap S_+)$. Since $\partial S_+$ is related to level set $F = \delta$, we have some topological concerns as described before the lemma. To this end, define:

\[ T := \bigcup_{\epsilon = 0, 1} \left\{ G(r_k, \phi_n + \epsilon \pi) : r_k \text{ is a stationary point of } G(r_k, \phi_n + \epsilon \pi), \text{ or } r_k \in \{0, r_{in}, r_{out}\} \right\}. \]

Next, we claim: there exist constants $C, C'$ independent of $n, k_3, \delta, L_1, L_2$ so that:

\[ \text{if } \delta \notin T \cup \{ \sigma_1 \frac{1}{|\alpha|} \}, \text{ then } \#(\mathbb{Z}^2 \cap S_+) \leq Area(S_+) + C(L_1 + L_2) r_{out} + C'. \]  
\[ (4.15) \]

The proof of this claim is as follows, where when we say (4.15) is proven ‘by virtue of lemma A.2’, it is understood that the role of $S$ is played by the interior of $S_+$ and a sub-lety (related to what was pointed out before) is addressed here: we must show $S_+$ equals the closure of its interior, namely, every point of $S_+$ is a limit point of its interior. In fact, recall $F(k)$ is continuous and a limit point of limit points is a limit point. Let $S'$ be the set of points in $S_+$ where $F < \delta$. Any point of $S'$ is either away from the edge(s) of the annulus, thus in the interior of $S_+$, or on an edge, thus a limit point of the previous type of points. Next, at every point of $S_+ \setminus S'$ we must have $F = G(r_k, \phi_k) = \delta$, so it is a limit point of $S'$ due to $\delta \notin T$ if $\phi_k \equiv \phi_n (\text{mod } \pi)$, and due to the half-period monotonicity otherwise.
Now, define mapping $H : [0, \infty) \mapsto \mathbb{R}$ as:

$$H(r_k) = \left( G(r_k, \phi_n) - \delta \right) \left( G(r_k, \phi_n + \pi) - \delta \right).$$

If $H(r_k) > 0$ for all $r_k \in [r_{in}, r_{out}]$, then by continuity, both $G(r_k, \phi_n)$ and $G(r_k, \phi_n + \pi)$ stay on the same side of $\delta$. By the half-period monotonicity, we find that $S_{r_k}$ is either empty or the entire annulus $r_{in} \leq r_k \leq r_{out}$, which then proves (4.15) by virtue of lemma A.2.

Now we prove (4.15) assuming additionally $H(\cdot)$ in $[r_{in}, r_{out}]$ takes non-positive value(s). If there is a zero $r_k^*$ of $H$ so that, e.g. $G(r_k^*, \phi_n) - \delta = 0$, then $G(r_k^*, \phi_n + \pi) - \delta \neq 0$ by the half-period monotonicity. Also, assumption $\delta \notin \mathcal{T}$ implies $\frac{\partial G}{\partial r_k}(r_k^*, \phi_n) \neq 0$. The argument so far shows $H$ is strictly monotonic at a neighbourhood of any of its zeros. Since $H$ has at most 18 zeros due to (4.13) and since $\delta \notin \mathcal{T}$ implies none of these zeros is in $\{0, r_{in}, r_{out}\}$ and $H(0) > 0$, there exist:

$$r_{in} \leq r_k^{(1)} < r_k^{(2)} < \ldots < r_k^{(2j)} \leq r_{out}, \quad \text{for some } J \in [1, 10],$$

that consist of all the zeros of $H$ located in $[r_{in}, r_{out}]$ and if necessary, also include $r_{in}$ and/or $r_{out}$, so that, at any $r_k \in [r_{in}, r_{out}]$,

$$H(r_k) \leq 0 \iff r_k \in I := \bigcup_{j=1}^{J} \left( r_k^{(2j-1)}, r_k^{(2j)} \right].$$

Then by half-period monotonicity, at given $r_k \in [r_{in}, r_{out}]$, equation $G(r_k, \phi_k) = \delta$ has

$$\begin{align*}
\text{exactly one root } \phi_k \in [\phi_n, \phi_n + \pi], & \quad \text{if } r_k \in I, \\
\text{no root } \phi_k \in \mathbb{T}_{2\pi}, & \quad \text{otherwise.}
\end{align*}$$

This allows us to define function $\Phi : I \mapsto [\phi_n, \phi_n + \pi]$ to be that one root, namely:

$$G(r_k, \Phi(r_k)) = \delta, \quad \forall r_k \in I.$$  \hfill (4.17)

We will only be concerned with $r_k \in I$ from now on. By definitions of $H, I$ and $\Phi$, we have:

$$\Phi(r_k^*) \equiv \phi_n \quad \text{mod } \pi \iff H(r_k^*) = 0 \iff r_k^* \in \left\{ r_k^{(1)}, r_k^{(2)}, \ldots, r_k^{(2J)} \right\} \setminus \{0, r_{in}, r_{out}\},$$  \hfill (4.18)

where the exclusion of the last small set is due to $\delta \notin \mathcal{T}$. Then, in view of (4.12), (4.14) and $m_1 \neq 0$ by assumption (4.8), we apply the implicit function theorem to find that $\Phi(r_k)$ is differentiable provided that $r_k$ is not from any of the $r_s$ satisfying (4.18). On the other hand, for any of the $r_s$ satisfying (4.18), we restrict the level set $G = \delta$ in a small neighbourhood of such point $(r_k^*, \Phi(r_k^*))$, and use assumption $\delta \notin \mathcal{T}$ and the implicit function theorem to find that the $r_k$ coordinate can be expressed as a differentiable function of $\phi_k$. In summary, the set:

$$\Gamma_\delta := \{ (L_1 r_k \cos \Phi(r_k), L_2 r_k \sin \Phi(r_k)) \in \mathbb{R}^2 : r_k \in I \},$$  \hfill (4.19)

consists of differentiable simple\(^5\) curve(s) on which $F(\cdot, \cdot, k_3) = \delta$.

The curves of $\Gamma_\delta$ are rectifiable as long as the following improper integral is finite:

$$\text{Len}(\Gamma_\delta) = \int_{I} \left| (L_1 (r_k \cos \Phi(r_k))^3, L_2 (r_k \sin \Phi(r_k))^3) \right| \, dr_k \lesssim (L_1 + L_2) \left( 1 + \int_{I} |\Phi'|^2 \, dr_k \right) r_{out}. \hfill (4.20)$$

\(^5\) A simple curve is a curve that does not cross itself.
Now \( \int_{r_k^{(2)}} \Phi \, dr \) is the total variation of \( \Phi(r_k) \) for \( r_k \in (r_k^{(2-1)}, r_k^{(2)}) \) and is bounded by 1 plus the number of its extremum points, up to constant factor. We are allowed to assume \( r_k \neq 0 \) as far as the integral (4.20) is concerned. By (4.11) we have:

\[
2r_k \cos (\Phi(r_k) - \phi_n) = -r_k + \frac{|m|^2 - r_n^2 - m_3^2}{r_k}.
\]

Also, note \( m_3 \neq 0 \) by (4.8). Since \( F(\cdot, \cdot, k_3) |_{\Gamma_\delta} = \delta \), we use (4.10) and the above to find:

\[
2r_n \cos (\Phi(r_k) - \phi_n) = -r_k + \frac{|k|^2m_2^2}{(C_1|k| - k_3)^2} - \frac{r_n^2 - m_3^2}{r_k} \quad \text{on } \Gamma_\delta \text{ with } C_1 := \sigma_2 \left( \delta - \sigma_1 \frac{n_3}{|m|} \right).
\]

Then, at an extremum point of \( \Phi(r_k) \), the \( r_k \) derivative of the above vanishes, i.e.

\[
0 = -1 + \frac{1}{r_k} \frac{d}{dr_k} \left( C_1|k| - k_3 \right) = \frac{|k|^2m_2^2}{(C_1|k| - k_3)^2} - \frac{r_n^2 - m_3^2}{r_k}.
\]

By identity \( \frac{1}{n} \frac{d}{dn} = \frac{1}{|k|} \frac{d}{dk} \) at fixed \( k \), we calculate the above derivative and substitute \( r_k^2 = |k|^2 - k_3^2 \) to find that \( |k| \) satisfies a polynomial equation with leading term \( C_1^2 |k|^2 \) with \( C_1 \neq 0 \) under (4.15). Thus, we have established an upper bound on the number of extremum points of \( \Phi(r_k) \). Since the number of intervals in \( \mathbf{I} \) is \( J \in [1, 10] \), we continue from (4.20) to find:

\[
\text{Len} (\Gamma_\delta) \lesssim (L_1 + L_2) r_{out}.
\]

We move on to \( \partial S_+ \). First, we consider the half annulus excluding the edge(s), namely \( (r, \phi_k) \in (r_m, r_{out}) \times [\phi_n, \phi_n + \pi] \). Any such point belongs to \( \partial S_+ \) if and only if it belongs to level set \( F = \delta \), which is due to half-period monotonicity and continuity. Then, by (4.16), (4.19), the closure of all such points equals \( \Gamma_\delta \) which of course is still part of \( \partial S_+ \).

Next, consider the half inner edge, namely \( r_k = r_m \) and \( \phi_k \in [\phi_n, \phi_n + \pi] \). Recall that if \( r_m = 0 \) then inner edge does not exist. So, let \( r_m > 0 \). If \( r_k^{(1)} = r_m > 0 \), then by (4.18) we have \( \Phi(r_m) \in (\phi_n, \phi_n + \pi) \). By half-period monotonicity, we extend \( \Gamma_\delta \) by part of the inner edge for either \( \phi_k \in [\phi_n, \Phi(r_m)] \) or \( \phi_k \in (\Phi(r_m), \phi_n + \pi] \) so that \( G(\phi_k, r_k) \leq \delta \), and the result of this extension is still a simple curve. If \( r_k^{(1)} > r_m > 0 \), then by (4.16), the sign of \( G - \delta \neq 0 \) is fixed on the half inner edge which therefore is either in \( \partial S_+ \) or in \( \mathbb{R}^2 \setminus \partial S_+ \). In the former case, we also add it to \( \Gamma_\delta \). Then, for such added curve (if any),

- its length is controlled by the right-hand side of (4.21);
- it equals the intersection of \( \partial S_+ \) and the half inner edge;
- its endpoints behave in exactly one of the two ways: either one endpoint satisfies \( \phi_k \equiv \phi_n \) (mod \( \pi \)) and the other endpoint on the pre-extended \( \Gamma_\delta \), or, both endpoints satisfy \( \phi_k \equiv \phi_n \) (mod \( \pi \)) and the pre-extended \( \Gamma_\delta \) does not intersect the half inner edge.

The same type of construction is used to extend \( \Gamma_\delta \) onto the half outer edge. We name the result of these extensions as \( \Gamma_\delta^\pm \). By construction, it is the intersection \( \partial S_\pm \), with the closed half annulus \( (r_m, r_{out}) \times [\phi_n, \phi_n + \pi] \). Further, every connected component of \( \Gamma_\delta^\pm \) is a simple curve with endpoints satisfying \( \phi_k \equiv \phi_n \) (mod \( \pi \)), and conversely, by (4.18) and the disjoint
nature of components of \( I \), every point on \( \Gamma_\delta^2 \) satisfying \( \phi_k \equiv \phi_n \pmod{\pi} \) is an endpoint of the aforementioned connected component. Then, by symmetry \( G(r_k, \phi_k) = G(r_k, 2\phi_n - \phi_k) \), we find that performing such a symmetry action on \( \Gamma_\delta^2 \) yields the intersection \( \partial S_+ \) with the closed half annulus \([r_{in}, r_{out}] \times [\phi_n - \pi, \phi_n]\). The above information on the overlapping of \( \Gamma_\delta^2 \) and \( \phi_k \equiv \phi_n \pmod{\pi} \) ensures that the two sets are joined together to form disjoint Jordan curves. In view of \((4.21)\), the proof of \((4.15)\) is complete by virtue of lemma \(A.2\).

But what about the exclude \( \delta \) values?

Since \((4.13)\) ensures \( T \) is a finite set\(^6\), any non-negative \( \delta \in T \cup \{\sigma_m \pmod{\pi}, 0\} \) can be approximated by \( \delta_1 > \delta \) from outside the previous set, so that \( \delta_1 \) satisfies \((4.15)\). In the upper bound in \((4.15)\), the excessive amount in the \(\text{Area}(S_+)\) term due to such approximation equals:

\[
\text{Area}(S_{\delta_1}) \quad \text{for} \quad S_{\delta_1} := \{(k\mu) \in \mathbb{R}^2 \mid \delta < f(k\mu, k_3) \leq \delta_1 \text{ and } r_{in} \leq |k\mu| \leq r_{out}\}
\]

Since the integrable function \( I_{S_{\delta_1}}(k_4) \) is monotonic in \( \delta_1 \) and \( \lim_{\delta_1 \to 0} I_{S_{\delta_1}}(k_4) = 0 \), we apply the monotone convergence theorem to show that \(\text{Area}(S_{\delta_1}) \to 0\) as \(\delta_1 \to \delta\). Noting the constants \(C, C'\) in \((4.15)\) are independent of \(\delta\), we prove the conclusion of \((4.15)\) for any \(\delta \geq 0\).

The counting of \# \((\mathbb{Z}^2 \cap S_-)\) follows the same steps except for \(F, G\) replaced by \(-F, -G\). Then, we apply inclusion-exclusion principle on \((\mathbb{Z}^2 \cap S(k_3)) = (\mathbb{Z}^2 \cap S_+) \cap (\mathbb{Z}^2 \cap S_-)\) to have:

\[
\#(\mathbb{Z}^2 \cap S(k_3)) \leq C(L_1 + L_2)r_{out} + C' + \text{Area}(S_+) + \text{Area}(S_-) - \#(\mathbb{Z}^2 \cap \{r_{in} \leq r_k \leq r_{out}\}).
\]

Apply inclusion-exclusion principle on \(S(k_3) = S_+ \cap S_-\) to have:

\[
\text{Area}(S_+) + \text{Area}(S_-) = \text{Area}(S(k_3)) + \text{Area}(\{r_{in} \leq r_k \leq r_{out}\}).
\]

Therefore it suffices to prove the following estimate:

\[
\text{Area}(\{r_{in} \leq r_k \leq r_{out}\}) - \#(\mathbb{Z}^2 \cap \{r_{in} \leq r_k \leq r_{out}\}) \leq C(L_1 + L_2)r_{out}.
\]

This is achieved by applying lemma \(A.2\) on the complement set of annulus \(\{r_{in} \leq r_k \leq r_{out}\}\) relative to a rectangle containing that annulus. The proof of \((4.9)\) is complete.

To link back to 3D volume integral, we first note that the exclusion of finite number of \(k_3\) values in lemma \(4.1\) does not affect the Lebesgue integral on the right-hand side of \((4.7)\). Then, recall \((4.5)\) to revert the \(V^0\) notation in \((4.7)\) and lemma \(4.1\) back to \(V_{n,\sigma_1,\sigma_2}^0\), and use lemma \(4.1\) to bound the right-hand side of \((4.7)\) to show:

\[
\sum_{k \in \mathbb{Z}^2} \begin{cases} 1_{V_{n,\sigma_1,\sigma_2}^0}(k) \leq C|\hat{n}|^2(L_1L_2 + L_1 + L_2) + \text{vol}(V_{n,\sigma_1,\sigma_2}^0). \end{cases}
\]

Combine this with \((4.4)\) to prove the following key result.

**Theorem 4.2.** For the NR set \(N[\delta_\ast] \) in \((1.3)\) subject to \((1.6)\), and for \(N[\delta_\ast]_{\text{loc}} \subset N[\delta_\ast] \) defined in \((4.3)\), consider the sets \(V_{n,\sigma_1,\sigma_2}^0\) given by \((4.1), (4.2)\). Fix \(n \) and \(\delta = \delta_\ast(n, n, n)\). Then there exists a constant \(C\) independent of \(n, \delta, L_1, L_2\) so that:

\[
\sum_{k \in \mathbb{Z}^2} 1_{N[\delta_\ast]_{\text{loc}}}(n, k, -n - k) \leq C|\hat{n}|^2(L_1L_2 + L_1 + L_2) + \sum_{(\sigma_1, \sigma_2) \in \{+,-\}^2} \text{vol}(V_{n,\sigma_1,\sigma_2}^0),
\]

(4.22)

where it is understood that \(\text{vol}(V_{n,\sigma_1,\sigma_2}^0) = 0\).

---

\(^6\) In fact, it suffices to know that \(T\) has zero Lebesgue measure in \(\mathbb{R}\) by Sard’s theorem.
5. Volume-integral estimates

Continuing from theorem 4.2, we focus our attention on upper bound of the volume \( \text{vol}(V^\text{d}_{n,\sigma_1,\sigma_2}) \), and it suffices to consider \( \delta > 0 \). The reader may refer to section 1.2 for a bigger picture of this multi-stage proof with technical highlights including the involvement of group theory.

Just like in the previous section, the symbol \( \delta \) can be regarded as a generic nonnegative value for the sake of generality. The case of localised bandwidth \( \delta = \delta_0(n,n,n) \) is our main interest.

**Theorem 5.1.** Consider \( V^\delta_{n,\sigma_1,\sigma_2} \) given in (4.1), (4.2) for any fixed bandwidth \( \delta \in (0,\frac{1}{2}) \), wavevector \( n \in \mathbb{Z}^3 \setminus \{0\} \) and sign choice \( (\sigma_1,\sigma_2) \in \{+,−\}^2 \). Then

\[
\frac{\text{vol}(V^\text{d}_{n,\sigma_1,\sigma_2})}{L_1L_2|n|^3} \lesssim \delta + \delta \log^+ \frac{1}{\delta} \frac{1}{|n|},
\]

where the implied constant factor in the \( \lesssim \) notation is independent of \( n,\ delta, \sigma_1, \sigma_2, L_1, L_2. \)

The proof is given in section 5.3.

Although the logarithmic factor in (5.1) is for volume estimate in continuum, it is actually optimal even for integer point counting in the sense detailed in appendix C.

To streamline the proof, we first do some housekeeping.

Similar to last section’s strategy of excluding some rare \( \delta \) values from the main argument, it suffices to prove (5.1) for \( n \) in a dense set of \( \mathbb{R}^3 \). Assume it indeed holds. For any \( n \neq 0 \) not in that dense set and any sufficiently small \( \epsilon > 0 \), there exists \( n^{\text{app}} \) in that dense set, so that:

\[
\left| \frac{n_3}{|n|} - n_3 \right| < \epsilon,
\]

so the corresponding \( F^{\text{app}}_{n,\delta,\sigma_1}(k) \) differs from \( F_{n,\sigma_1,\sigma_2}(k) \) by less than \( \epsilon \). Then, by the (temporary) assumption that the \( n^{\text{app}}, \delta + \epsilon \) version of (5.1) holds, we find that the \( n, \delta \) version of LHS (5.1) is controlled by the \( n^{\text{app}}, \delta + \epsilon \) version of RHS (5.1), hence proving the original form of (5.1).

Henceforth for the rest of section 5, we assume:

\[
n \text{ with } n_1n_2n_3 \neq 0 \quad \text{and} \quad \delta \in (0,\frac{1}{2}) \text{ are fixed.}
\]

We will use \( \mathcal{B}\{\cdot\} \) that takes a Boolean expression as the argument and gives numerical value 1 if the expression is true, and 0 otherwise. The composition of \( \mathcal{B}\{\cdot\} \) with its argument is then a mapping from the space where the variables of the Boolean expression reside to \( \{0,1\} \).

Then, when \( \mathcal{B}\{\cdot\} \) takes inequalities of simple enough nature, the corresponding mapping is Lebesgue measurable. For example, \( \mathcal{B}\{x \leq y\} \) for \( (x,y) \in \mathbb{R}^2 \) is Lebesgue measurable in \( \mathbb{R}^2 \).

Recall that the domain-adjusted wavevector \( \hat{k} \) is indicated with an accent ‘. So we rewrite the volume element in \( \int_{\mathbb{R}^3} 1_{V^\text{d}_{n,\sigma_1,\sigma_2}}(k) \, dk = L_1L_2 \, dk \) followed by substituting the definitions (4.1), (4.2), and dropping all accents. This result is:

\[
\frac{\text{vol}(V^\text{d}_{n,\sigma_1,\sigma_2})}{L_1L_2} = \int_{\mathbb{R}^3 \setminus \{0,−n\}} \mathcal{B}\left\{ \left| \frac{n_3}{|n|} + \frac{k_3}{|k|} \right| \leq \delta \right\} \mathcal{B}\{ \frac{1}{2}|n| \leq |k| \leq |n| \} \, dk,
\]

here the only abuse of notation is renaming \( \hat{n} \) as \( n \). The value of the integral does not depend on the choice of \( \hat{k} \) or \( k \) notation. Henceforth, we discontinue the use of accent ‘.
5.1. Change of coordinates, twice

For estimating the measure of a sublevel set of a function (here, it is the triplet value as function of $k$), we can swap the roles of dependent variable, i.e. the function itself, with one of the coordinates (here, it is the azimuthal angle of $k$). In what follows, the new coordinate is basically \( \frac{m}{n} \) which has a simple affine relation to the triplet value if we fix every quantity except the azimuthal angle of $k$.

The first step is therefore to represent wavevector $k$ in spherical coordinates but with rescaled radius $\lambda_k := \frac{m}{n}$ i.e. $(\lambda_k, \theta_k, \phi_k) \in [0, \infty) \times [0, \pi] \times T_2$ satisfying:

\[
(k_1, k_2, k_3) = |n| \lambda_k (\sin \theta_k \cos \phi_k, \sin \theta_k \sin \phi_k, c_k), \quad \text{with} \quad c_k := \cos \theta_k.
\]

Similarly notations are used for $n,m$. Note $\lambda_n = 1$.

We impose the following working assumptions until they are addressed above (5.10):

\[
\lambda_k \in \left[\frac{1}{2}, 1\right], \quad \theta_k \in (0, \frac{1}{2} \pi) \cup \left(\frac{1}{2} \pi, \pi\right) \quad \text{and} \quad \lambda_k c_k + c_n \neq 0. \quad (5.4)
\]

Naturally, the convolution condition $m = -k - n$ is always assumed.

By $k_n = k_n1 + k_n2 + n = |k| \cdot |n| (c_n1 \sin \theta_k \sin \theta_n, \cos(\phi_k - \phi_n))$, we expand $|m|^2 = |k + n|^2$, effectively expressing $\lambda_m$ in terms of $k, n$,

\[
\lambda_m = \sqrt{\lambda_k^2 + 1 + 2 \lambda_k c_n \sin \theta_k \sin \theta_n \cos(\phi_k - \phi_n)} \in (0, \lambda_k + 1], \quad (5.5)
\]

where $\lambda_m > 0$ is by the last part of (5.4), i.e. $m_3 \neq 0$, and $\lambda_m \leq \lambda_k + 1$ due to $|m| \leq |k| + |n|$.

Next, define mapping $M : (0, \infty) \times [0, \pi] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ as:

\[
M(\lambda_k, \theta_k, \phi_k) := -\frac{\lambda_k c_k + c_n}{\lambda_m} \left(1 - \frac{n_3 + k_3}{|n + k|}\right). \quad (5.6)
\]

Define mapping $q : (0, \infty) \times [0, \pi] \times [-1, 1] \rightarrow \mathbb{R}$ (that is ‘quartic’ in either $\lambda_k, c_k$ or $c_n$) as:

\[
q(\lambda_k, \theta_k, c_n) := \left[2 \lambda_k \sin \theta_k \sin \theta_n \sin(\phi_k - \phi_n) c_n^2 \right] - \left[\lambda_k^2 + 1 + 2 \lambda_k c_n c_k \right] c_n^2 - (\lambda_k c_k + c_n)^2 \right]^2. \quad (5.7)
\]

Then substitute (5.5) into (5.6) and rearrange using $\sin^2(\phi_k - \phi_n) + \cos^2(\phi_k - \phi_n) = 1$ to deduce:

\[
\left( q(\lambda_k, \theta_k, c_n) - \left[2 \lambda_k \sin \theta_k \sin \theta_n \sin(\phi_k - \phi_n) c_n^2 \right] \right) \bigg|_{c_n = M(\lambda_k, \theta_k, \phi_k)} = 0. \quad (5.8)
\]

By $\partial_{\phi_k} M(\lambda_k, \theta_k, \phi_k) = \lambda_k \sin \theta_k \sin \theta_n \sin(\phi_k - \phi_n) \lambda_m^{-2} (\lambda_k c_k + c_n)$ and under (5.2), (5.4) with fixed $\lambda_k, \theta_k$, we find the sign of $M \neq 0$ is fixed, and the sign of $\partial_{\phi_k} M \neq 0$ is fixed for $\phi_k \in (\phi_n - \pi, \phi_n)$ and for $\phi_k \in (\phi_n - \pi, \phi_n)$, so $M$ is monotonic in $\phi_k$ in each interval, prompting the next change of coordinates. The Jacobian will be the reciprocal of $\partial_{\phi_k} M$ as expressed above.

By (5.5), relax $\lambda_k^2$ to $(\lambda_k + 1)^2$ in above so that the only $\phi_k$ dependence is via $c_n$, the upcoming new coordinate. Then, at fixed $\lambda_k, \theta_k$ coordinates subject to (5.4), we change the $\phi_k$ coordinate to $c_n$ via $c_n = M(\lambda_k, \theta_k, \phi_k)$ in the following Lebesgue integral,
the following working assumptions that both have the same integrand and, when we switch back to un-scaled integrals:

\[
\int_{\phi_k \in \mathbb{T}^n \setminus \{\phi_k, \phi_k + \pi\}} \mathcal{B} \left\{ |\sigma_1 c_n + \sigma_2 c_k + M(\lambda_k, \theta_k, \phi_k) | \leq \delta \right\} d\phi_k \\
\leq 2 \int_{\min\{M(\lambda_k, c_k, \phi_k), M(\lambda_k, c_k, \phi_k + \pi)\}}^{\max\{M(\lambda_k, c_k, \phi_k), M(\lambda_k, c_k, \phi_k + \pi)\}} \mathcal{B} \left\{ |\sigma_1 c_n + \sigma_2 c_k + c_m| \leq \delta \right\} \frac{2(\lambda_k + 1)^2 |c_m|}{\sqrt{q(\lambda_k, \theta_k, c_m)}} \ dc_m.
\]

Now, the above information on the signs of \(\partial_\phi M\) and M ensures that any \(c_m\) in the right-hand side integral domain excluding endpoints corresponds to a \(\phi_k \in (\phi_n, \phi_n + \pi)\) so that \(c_m = M(\lambda_k, c_k, \phi_k) \neq 0\). This has two implications. First \(|c_m| = |M(\lambda_k, c_k, \phi_k)| \leq 1\) by (5.6) and simple geometry. Second, \(q(\lambda_k, \theta_k, c_m) > 0\) by (5.8), (5.2), (5.4). Therefore, we define:

\[
q^*(\lambda_k, \theta_k, c_m) = \begin{cases} 
\frac{1}{\sqrt{q(\lambda_k, \theta_k, c_m)}}, & \text{if } q(\lambda_k, \theta_k, c_m) > 0, \\
0, & \text{otherwise},
\end{cases}
\]

and continue from the previous estimate,

\[
\int_{\phi_k \in \mathbb{T}^n \setminus \{\phi_k, \phi_k + \pi\}} \mathcal{B} \left\{ |\sigma_1 c_n + \sigma_2 c_k + M(\lambda_k, \theta_k, \phi_k) | \leq \delta \right\} d\phi_k \\
\leq 4(\lambda_k + 1)^2 \int_{-1}^{1} \mathcal{B} \left\{ |\sigma_1 c_n + \sigma_2 c_k + c_m| \leq \delta \right\} q^*(\lambda_k, \theta_k, c_m) |c_m| \ dc_m.
\]

Recall this estimate holds under assumption (5.4), which allows us to apply double integrals:

\[
\int_{(0,\frac{1}{2} \pi) \cup (\frac{1}{2} \pi, \pi)} \int_{\left[\frac{1}{2} \pi, 1\right] \setminus (-\frac{\pi}{2})} \sin \theta_k \ d\lambda_k d\theta_k.
\]

Compare the LHS of the result (where M satisfies (5.6)) with the RHS of (5.3) to find that both have the same integrand and, when we switch back to un-scaled \(|k| = |n| \lambda_k\), the two integral domains differ by a zero-measure set in \(\mathbb{R}^3\). Therefore

\[
\frac{\text{vol}(V^0_{n,\sigma_1,\sigma_2})}{L_1 L_2 |n|^{\frac{3}{2}}} \leq 16 \int_{0}^{\pi} \int_{-1}^{1} \int_{-1}^{1} \mathcal{B} \left\{ |\sigma_1 c_n + \sigma_2 c_k + c_m| \leq \delta \right\} q^*(\lambda_k, \theta_k, c_m) |c_m| \sin \theta_k \ dc_m \ d\lambda_k \ d\theta_k,
\]

where we also filled back a zero-measure set in the RHS integral domain. Then by the virtue of Fubini-Tonelli theorem we have:

\[
\frac{\text{vol}(V^0_{n,\sigma_1,\sigma_2})}{L_1 L_2 |n|^{\frac{3}{2}}} \leq 16 \int_{-1}^{1} \int_{0}^{\pi} \mathcal{B} \left\{ |\sigma_1 c_n + \sigma_2 c_k + \theta_k| \leq \delta \right\} Q(\theta_k, c_m) \sin \theta_k \ dc_m \ d\theta_k, \quad (5.10)
\]

where

\[
Q(\theta_k, c_m) := |c_m| \int_{\frac{1}{2} \pi}^{\pi} q^*(\lambda_k, \theta_k, c_m) \ d\lambda_k.
\]

\[
5.2. \text{Elliptic integrals}
\]

Recall (5.2) so that \(c_k\) is fixed and \(0 < |c_n| < 1\). Noting the coordinates of (5.10) are \(\lambda_k, c_m, \theta_k\) (equivalently \(c_k\)), we impose for the rest of section 5.2 the following working assumptions (they are addressed below (5.32)):

\[
c_n c_k c_m \left(1 - c_n^2\right) \left(1 - c_k^2\right) \left(1 - c_m^2\right) \left(c_m^2 - c_k^2\right) \left(c_k^2 - c_n^2\right) \left(c_n^2 - c_m^2\right) \neq 0. \quad (5.12)
\]
To estimate $Q$ with its integrand $q^*$ defined in (5.9) having singularities caused by the $\lambda_k$ roots of $q$ defined in (5.7), we appeal to the tool of elliptic integrals. Start with factorisation:

$$q = - \left[ (c_m^2 - c_k^2) \lambda_k^2 + 2(\cos(\theta_k + \theta_n) c_m^2 - c_n c_k) \lambda_k + c_m^2 - c_n^2 \right] \times \left[ (c_m^2 - c_k^2) \lambda_k^2 + 2(\cos(\theta_k - \theta_n) c_m^2 - c_n c_k) \lambda_k + c_m^2 - c_n^2 \right],$$  \hspace{1cm} (5.13)

in which the discriminants of the two quadratic-in-$\lambda_k$ factors are, via trigonometry,

$$4 \sin^2(\theta_k \pm \theta_n)(1 - c_n^2)c_m^2 \geq 0.$$  

Thus, all possible $\lambda_k$ values that make $q(\lambda_k, \theta_k, c_m) = 0$ are:

$$\frac{c_n c_k - \cos(\theta_k + \tilde{\sigma} \theta_n) c_m^2 + \tilde{\sigma} \sin(\theta_k + \tilde{\sigma} \theta_n) \sin(\theta_m) c_m}{c_m^2 - c_k^2},$$

for $\tilde{\sigma} = \pm 1$, $\tilde{\sigma} = \pm 1$.

Since the numerator equals $c_n c_k - \cos(\theta_k + \tilde{\sigma} \theta_n + \tilde{\sigma} \theta_m)c_m$, we denote all $\lambda_k$-zeros of $q$ as:

$$\Lambda^{\sigma_k, \sigma_n} := \frac{c_n c_k - e^{\sigma_k \sigma_n} c_m}{c_m^2 - c_k^2}$$

with $e^{\sigma_k \sigma_n} := \cos(\theta_m + \sigma_k \theta_k + \sigma_n \theta_n)$

for $\sigma_k = \pm 1$, $\sigma_n = \pm 1$.

(5.14)

Note $\tilde{\sigma}, \tilde{\sigma}, \sigma_k, \sigma_n$ are independent of $\sigma_1, \sigma_2$ in the NR conditions.

By elementary trigonometric identity:

$$\cos^2\beta - \cos^2\gamma = -\sin(\beta + \gamma) \sin(\beta - \gamma),$$  \hspace{1cm} (5.15)

we go through more elementary trigonometry\textsuperscript{7} to find:

$$\Lambda^{\sigma_k, \sigma_n} = -\frac{\sin(\theta_m + \sigma_k \theta_n)}{\sin(\theta_m - \sigma_k \theta_n)}.$$  \hspace{1cm} (5.16)

Interested reader can explore the geometry of the above right-hand side with the help of (5.8) which shows that $q = 0$ is linked to $\phi_k \equiv \phi_n \pmod{\pi}$.

In applying the tool of elliptic integral, we rely on the following notations. For $(a, b, c, d) \in \mathbb{R}^4$, define products of interlaced (‘IL’), enclosed (‘EC’) and separated (‘SP’) differences:

$$\Pi_{IL}(a, b, c, d) := (a - c)(b - d),$$
$$\Pi_{EC}(a, b, c, d) := (a - d)(b - c),$$
$$\Pi_{SP}(a, b, c, d) := (a - b)(c - d).$$

They satisfy identity:

$$\Pi_{IL} = \Pi_{EC} + \Pi_{SP}.$$  \hspace{1cm} (5.17)

Also, they are invariant under two-pair swapping permutations, known as ‘double transpositions’, which consist three members: interlaced $(a \leftrightarrow c, b \leftrightarrow d)$, enclosed $(a \leftrightarrow d, b \leftrightarrow c)$, and separated $(a \leftrightarrow b, c \leftrightarrow d)$. We call this Klein 4-group symmetry\textsuperscript{8} or ‘$K_4$-symmetry’ for short.

For ordered real numbers $a > b > c > d$, introduce parameters (e.g. [4, p 8, p 102])

$$E^2(a, b, c, d) := \frac{(a - b)(c - d)}{(a - c)(b - d)} = \frac{\Pi_{SP}(a, b, c, d)}{\Pi_{IL}(a, b, c, d)} < 1,$$  \hspace{1cm} (5.18)

\textsuperscript{7} Rewrite the numerator of (5.14) as: $\frac{1}{4} \cos(\sigma_k \theta_k + \sigma_n \theta_n) + \frac{1}{4} \cos(\sigma_k \theta_k - \sigma_n \theta_n) - \frac{1}{4} \cos(\theta_n + \sigma_k \theta_k + \sigma_n \theta_n) - \frac{1}{4} \cos(\sigma_k \theta_k + \sigma_n \theta_n) = \sin(\theta_n + \sigma_k \theta_k) \sin(\theta_n - \sigma_k \theta_k)$, and the denominator as $\sin(\theta_n + \sigma_k \theta_k) \sin(\theta_n - \sigma_k \theta_k)$.

\textsuperscript{8} Together with identity, they are the only self-inverse, parity-preserving members of the 4-permutation group, and form a subgroup. Then, necessarily, they are commutative and the three non-identity elements are cyclic under the group operation. This subgroup is isomorphic to the Klein 4-group and to $\mathbb{Z}_2 \times \mathbb{Z}_2$. 

\[ g(a, b, c, d) := \frac{2}{\sqrt{(a - c)(b - d)}} = \frac{2}{\sqrt{\Pi_{\ell}(a, b, c, d)}} \]  
(5.19)

with \( \ell^2 < 1 \) due to the rearrangement inequality (or (5.17)). The incomplete elliptic integral of the first kind for any \( 0 \leq \ell < 1 \) is defined as:

\[ \int_0^\Psi \frac{1}{\sqrt{1 - \ell^2 \sin^2 x}} \, dx = \int_0^{\sin \Psi} \frac{1}{\sqrt{(1 - r^2)(1 - \ell^2 r^2)}} \, dr, \quad 0 \leq \Psi \leq \frac{\pi}{2}. \]

Its relation to \( \mathcal{Q} \) is given in proposition B.1 with the quartet of singular points being \( \Lambda^{\pm, \pm} \) from (5.14) rearranged in descending order. This would pose a lengthy task of tracing \( 4! = 24 \) cases, not to mention the inconvenient form of \( \Lambda^{\pm, \pm} \). However, we can make great simplification thanks to affine relation in (5.14), and the following two elegant correspondences.

**Lemma 5.2 (First Correspondence).** For any 4-permutation \( \pi \), the following identity holds,

\[ \Pi(\pi(C^{+, -}, C^{+, -}, C^{+, -})) = 4 \Pi(\pi(1, c_2^2, c_3^2, c_4^2)) \], \quad \text{for } \Pi \in \{ \Pi_{\mathcal{Q}}, \Pi_{\mathcal{EC}}, \Pi_{\mathcal{SP}} \}. \]

Thus, under working assumptions (5.12), all \( \mathcal{C}^{\pm, \pm} \) are distinct, and all \( \Lambda^{\pm, \pm} \) are distinct.

**Proof.** Let us start with the \( \pi = id \) version. Define mapping \( \mathcal{D}^{\sigma^+, \sigma^+} : [0, \pi]^3 \rightarrow [-1, 1] \) as:

\[ \mathcal{D}^{\sigma^+, \sigma^+}(\theta_1, \theta_2, \theta_3) := \cos(\theta_1 + \sigma^+ \theta_2 + \sigma^+ \theta_3), \]

so that,

\[ \mathcal{D}^{\sigma^+, \sigma^+}(\theta_m, \theta_k, \theta_a) = C^{\sigma^+, \sigma^+}. \]

Then, the case for \( \Pi = \Pi_{\mathcal{SP}} \) and \( \pi = id \) is due to the elementary identity:

\[ \Pi_{\mathcal{SP}}((D^{+, +}, D^{+, -}, D^{-, -}, D^{-, -})(\theta_1, \theta_2, \theta_3)) = 4 \Pi_{\mathcal{SP}}(1, \cos^2 \theta_2, \cos^2 \theta_2, \cos^2 \theta_1), \]

for \((\theta_1, \theta_2, \theta_3) = (\theta_m, \theta_k, \theta_a)\). With \((\theta_1, \theta_2, \theta_3) = (\theta_m, \theta_a, \theta_k)\), we find \((D^{+, +}, D^{+, -}, D^{-, -}, D^{-, -})(\theta_m, \theta_a, \theta_k) = (C^{+, +}, C^{+, +}, C^{+, +}, C^{+, +}), \) so the above identity also proves the case for \( \Pi = \Pi_{\mathcal{IL}} \) and \( \pi = id \). The case for \( \Pi = \Pi_{\mathcal{EC}} \) and \( \pi = id \) is proven similarly by combining the above identity and \((D^{+, +}, D^{+, -}, D^{-, -}, D^{-, -})(\theta_1, \theta_2, \theta_3) = (C^{+, +}, C^{+, +}, C^{+, +}, C^{+, +}), \)

For cases with general permutations \( \pi \), we just combine the proven case for with \( \pi = id \) with the fact that the composition \( \Pi \circ \pi \) changes the type of \( \Pi \) amongst \( \Pi_{\mathcal{IL}}, \Pi_{\mathcal{EC}}, \Pi_{\mathcal{SP}} \) and/or changes the sign in the same fashion for both sides of the claimed identity.

Let \( \vec{C} \) denote the vector resulting from sorting \((C^{+, +}, C^{+, +}, C^{+, +}, C^{+, +})\) in descending order, and let \( \vec{\zeta} \) denote the vector resulting from sorting \((1, c_2^2, c_3^2, c_4^2)\) in descending order.

**Lemma 5.3 (Second Correspondence).** For any \( \Pi \in \{ \Pi_{\mathcal{IL}}, \Pi_{\mathcal{EC}}, \Pi_{\mathcal{SP}} \} \), we have:

\[ \Pi(\vec{C}) = 4 \Pi(\vec{\zeta}). \]

**Proof.** For \( \vec{v} = (a, b, c, d) \) with mutually distinct real numbers \( a, b, c, d \), define \( \pi_{\vec{v}} \) to be the unique permutation that sorts \( \vec{v} \) in descending order, namely:

\[ \vec{v}_s = \pi_{\vec{v}}(\vec{v}). \]  
(5.20)

Let \( K_4 \) denote the Klein 4-group and \( S_4 \) the 4-permutation group. When \( K_4 \) is viewed as a subgroup of \( S_4 \), it consists of identity and the double transpositions given below (5.17). By group theory, \( K_4 \) is a normal subgroup of \( S_4 \), so the quotient group \( S_4/K_4 \) exists.

Now, take any 4-permutations \( \pi_1, \pi_2 \) so that \( \pi_2 \in K_4 \pi_1 \) with coset \( K_4 \pi_1 \in S_4/K_4 \). By the \( K_4 \)-symmetry below (5.17), the three signs of \( \Pi_{\mathcal{EC}}(\cdot), \Pi_{\mathcal{SP}}(\cdot), \Pi_{\mathcal{IL}}(\cdot) \) acting on \( \pi_1 \vec{v} \) and
Consider by notation (5.22) and (5.23). In particular \( \pi \in K_4 \pi^* = \pi^* \) holds the first element \( (a) \) fixed. In other words, \( a \) is always the largest entry in \( \vec{v} \). Thus the signs of \( \Pi_{EC}(\vec{v}), \Pi_{SP}(\vec{v}), \Pi_{IL}(\vec{v}) \) are simply computed as the signs of \((b - c), (c - d), (b - d)\).

\[ \Pi_{EC}(\vec{v}) \quad \Pi_{SP}(\vec{v}) \quad \Pi_{IL}(\vec{v}) \quad \pi \pi^* \in K_4 \pi^* \text{ where } \pi^* = \ldots \]

| \( \Pi_{EC}(\vec{v}) \) | \( \Pi_{SP}(\vec{v}) \) | \( \Pi_{IL}(\vec{v}) \) | \( \pi \pi^* \) |
|---|---|---|---|
| + | + | + | \( (a), b, c, d \) |
| - | - | - | \( (a), d, c, b \) |
| + | + | - | \( (a), d, b, c \) |
| - | + | - | \( (a), b, c, d \) |
| + | - | + | impossible |
| - | - | + | impossible |

\( \pi \vec{v} \) are identical. It turns out the reverse is also true: for a given \( \vec{v} \), the three signs of \( \Pi_{EC}(\vec{v}), \Pi_{SP}(\vec{v}), \Pi_{IL}(\vec{v}) \) suffice to uniquely determine the element of \( S_4/K_4 \) that contains \( \pi \vec{v} \). In particular \( \pi \pi^* \in K_4 \pi^* \) with \( \pi^* \) being the ‘special representative’ of the coset that holds the first element fixed, known as a stabiliser. (In fact, choosing stabilisers as representatives for the cosets is possible due to the isomorphism between \( S_4/K_4 \) and \( S_2 \).) See table 1.

Let \( \vec{c} = (\vec{c}^+, \vec{c}^-, \vec{c}^, \vec{c}^-) \) and \( \vec{c} = (1, c_1^2, c_2^2, c_m^2) \) so that \( \vec{c}_s = \pi \vec{c} (\vec{c}) \) and \( \vec{c}_t = \pi \vec{c} (\vec{c}) \) by notation (5.20). Using the one-to-one relation proven above and the \( \pi = id \) version of lemma 5.2, we prove \( \pi \vec{c} \in K_4 \pi \vec{c} \). Combine it with the \( K_4 \)-symmetry of the \( \Pi \)'s and the \( \pi = \pi \vec{c} \) version of lemma 5.2 to complete the proof.

Let \( \vec{\Lambda} \) denote the vector resulting from sorting \((\Lambda^+, \Lambda^-, \Lambda^+, \Lambda^-)\) in descending order. Since \( \Lambda^\pm, \Lambda^\pm, \Lambda^\pm, \Lambda^\pm \) are linked by an affine transform as in (5.14), we recall definitions (5.18), (5.19) and apply lemma 5.3 to obtain:

\[ g(\vec{\Lambda}) = \frac{|c_m^2 - c_1^2|}{|c_m|} \frac{1}{\Pi_{IL}(\vec{c}_d)}, \quad (5.21) \]

and (note (5.17)):

\[ 1 - \vec{c}_d^2(\vec{\Lambda}) = \frac{\Pi_{EC}(\vec{c}_d)}{\Pi_{IL}(\vec{c}_d)}. \quad (5.22) \]

Then, noting the first entry of \( \vec{c}_t \) is always 1, we have reduced the bookkeeping task of \( 4! = 24 \) cases to \( 3! = 6 \) cases that account the ordering of \( c_1^2, c_2^2, c_m^2 \). In fact, we show in the argument below (5.32) that there are only 2 essentially distinct scenarios. But first, let us estimate \( Q \).

**Lemma 5.4.** Consider \( Q(\theta, c_m) \) from (5.11). Under working assumptions (5.12), we have:

\[ Q \lesssim 1 + \log \frac{\Pi_{IL}(\vec{c}_d)/\Pi_{EC}(\vec{c}_d)}{\sqrt{\Pi_{IL}(\vec{c}_d)}} \quad (5.23) \]

and a (conditional) sharper estimate:

\[ |c_m| < \min\{|c_2|, |c_m|\} \implies Q \lesssim \frac{1}{\Pi_{IL}(\vec{c}_d)}. \quad (5.24) \]

**Proof.** Denote elements of the descending-ordered vectors:

\[ \vec{c}_s = (1, c_2^2, c_3, c_4) \quad \text{and} \quad \vec{\Lambda} = (\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4). \]

---

| Table 1. All possible descending order permutations of \( \vec{v} = (a, b, c, d) \) are divided into 6 cosets, distinguishable by the three signs \( \vec{v} \) produces on the left columns. The ‘special representative’ \( \pi^* \) of each coset holds the first element \( (a) \) fixed. In other words, \( a \) is always the largest entry in \( \vec{v} \). Thus the signs of \( \Pi_{EC}(\vec{v}), \Pi_{SP}(\vec{v}), \Pi_{IL}(\vec{v}) \) are simply computed as the signs of \((b - c), (c - d), (b - d)\). |
Combine (5.7), (5.9) and (5.11), noting the coefficient \(-\left(c_m^2 - c_k^2\right)^2\) of \(\Lambda_k^4\) therein, to show:

\[
Q(\theta_k, c_m) = \frac{|c_m|}{|c_m^2 - c_k^2|} \int_{\left[\frac{1}{2} \Lambda_k, 1\right]} \frac{d \lambda_k}{\sqrt{-(\lambda_k - \Lambda_1)(\lambda_k - \Lambda_2)(\lambda_k - \Lambda_3)(\lambda_k - \Lambda_4)}}. \tag{5.25}
\]

(i) For proving the general bound (5.23), we simply drop the \(\left[\frac{1}{2}, 1\right]\) restriction of the above integral and combine elementary estimate (B2) with identities (5.21), (5.22).

(ii) For proving the sharper bound (5.24), by the general bound (5.23) we just proved, it suffices to assume \(\Pi_{\text{IL}}(\zeta')/\Pi_{\text{EC}}(\zeta') > 9\) for the rest of the proof. Then

\[
\varsigma_2 - \varsigma_4 > 9(\varsigma_2 - \varsigma_3). \tag{5.26}
\]

Next, by (5.15), (5.16), we have:

\[
\Lambda^+ \sigma_n^{A^-} \Lambda^- \sigma_n \left(c_m^2 - c_k^2\right) < 0, \tag{5.27}
\]

\[
\Lambda^{\sigma_n A^+} \Lambda^{- \sigma_n} \left(c_m^2 - c_k^2\right) < 0,
\]

the latter of which together with the ordering in (5.24) implies:

\[
\Lambda_1 > \Lambda_2 > 0 > \Lambda_3 > \Lambda_4. \tag{5.28}
\]

Then in view of the restriction \(\lambda_k \in \left[\frac{1}{2}, 1\right]\) in the integral (5.25), it suffices to consider:

\[
0 < \Lambda_2 < 1, \tag{5.29}
\]

and we can relax the integral domain of (5.25) to \([\Lambda_2, 1]\). Then, by (5.21) and proposition B.1, the sharper bound (5.24) would hold if (B.3) holds for \((a, b, c) = (\Lambda_1, \Lambda_2, \Lambda_3)\) and \(\gamma = 1\). Since \(1 - \Lambda_2 < 1\) by (5.29), it suffices to bound \(\Lambda_2 - \Lambda_3\) and \(\Lambda_1 - \Lambda_2\) from below by positive constants with the latter bound strictly greater than 1. The rest of the proof is for this purpose.

(a) For bounding \(\Lambda_2 - \Lambda_3\) from below, we use affine relation (5.14) and lemma 5.2 together with \(\Pi_{\text{EC}}(1, c_m^2, c_k^2, c_m^2) < 0\) and \(\Pi_{\text{IL}}(1, c_m^2, c_k^2, c_m^2) < 0\) thanks to the ordering in (5.24), to have:

\[
\Pi_{\text{EC}}\left(\Lambda^{++}, \Lambda^{+-}, \Lambda^{-+}, \Lambda^{--}\right) < 0 \quad \text{and} \quad \Pi_{\text{IL}}\left(\Lambda^{++}, \Lambda^{+-}, \Lambda^{-+}, \Lambda^{--}\right) < 0.
\]

Then, using simple logic\(^9\), we find that \(\{\Lambda_1, \Lambda_4\}\) can not equal to any of these sets: \(\{\Lambda^{++}, \Lambda^{+-}\}, \{\Lambda^{++}, \Lambda^{--}\}, \{\Lambda^{+-}, \Lambda^{--}\}\). Therefore there are only \(\binom{4}{2} = 2\) choices for the set \(\{\Lambda_1, \Lambda_4\}\) and equivalently only 2 choices for \(\{\Lambda_2, \Lambda_3\}\). Then by (5.16), we find:

\[
\Lambda_2 - \Lambda_3 = \min \left\{|\Lambda^{++} - \Lambda^{+-}|, |\Lambda^{+-} - \Lambda^{--}|\right\} = \frac{2|c_m| |\sin \theta_n|}{|c_m| |\sin \theta_k| + |c_k| |\sin \theta_m|} > \frac{2|c_m|}{|c_m| + |c_k|},
\]

where we used the ordering in (5.24) and the positivity of all \(\sin \theta_k's\). Now, relax \(\varsigma_4 < 0\) in (5.26) and use it together with the ordering in (5.24) to find either \(c_k^2 < c_m^2\) or \(c_k^2 > \frac{9}{8} c_m^2\). Then continue from the above to prove a positive lower bound for \(\Lambda_2 - \Lambda_3\).

\(^9\) if the largest and smallest elements of \(x, y, z, w\) form set \(\{x, w\}\) or set \(\{y, z\}\), then \((x - y)(z - w) > 0\).
(b) For bounding \( \Lambda_1 - \Lambda_2 \) from below by a constant strictly greater than 1, there are two cases.

1. If \( 1 > |c_m| > |c_k| > |c_n| \), we observe that each pair of roots in either of the two quadratic-in-\( \Lambda_k \) factors of (5.13) multiply to \( \frac{c_k^2}{c_m^2} \) which is greater than 9 by (5.26). Then, by the sign information of (5.28), we have \( \Lambda_1 \Lambda_2 > 9 \). This together with (5.29) shows \( \Lambda_1 - \Lambda_2 > 8 \).

2. If \( 1 > |c_k| > |c_m| > |c_n| \), then (5.27) and (5.28) together imply \( \{ \Lambda_1, \Lambda_2 \} = \{ \Lambda^+, \sigma, \Lambda^- \} \) for \( \sigma = 1 \) or \( -1 \), so we use (5.15), (5.16) to obtain:

\[
\Lambda_1 + \Lambda_2 = -\sin(\theta_m + \sigma \theta_n) - \sin(\theta_m - \theta_k) + \sin(\theta_m + \theta_k) + \sin(\theta_m - \theta_k)
\]

\[
\frac{c_m^2 - c_k^2}{\sin(\theta_m - \sigma \theta_n)} + \frac{2c_k \sin \theta_m}{\sin(\theta_m + \sigma \theta_n)} > \frac{2\sin \theta_m}{\sin \theta_n + \sin \theta_m} \cdot \frac{c_m^2 - c_k^2}{\sin(\theta_m - \theta_k)} + \frac{c_m^2 - c_k^2}{\sin(\theta_m + \theta_k)}
\]

(5.30)

where the last step was to due to \( \Lambda_1 + \Lambda_2 > 0 \) and \( |\sin(\theta_m - \sigma \theta_n)| < (\sin \theta_n + \sin \theta_m) |c_k| \) thanks to the ordering assumption. Then in view of (5.26), we find:

\[
\Lambda_1 + \Lambda_2 > \frac{16\sin \theta_m}{\sin \theta_n + \sin \theta_m}.
\]

By relaxing \( c_k^2 < 1 \) and rewriting \( c_m^2 - c_n^2 = \sin^2 \theta_n - \sin^2 \theta_m \) in (5.30), we also have:

\[
\Lambda_1 + \Lambda_2 > \frac{2(\sin \theta_n - \sin \theta_m)}{\sin \theta_m}.
\]

Since \( \sin \theta_n > \sin \theta_m > 0 \) and the above two lower bounds have obvious monotonicity in \( \sin \theta_n \), we can show \( \Lambda_1 + \Lambda_2 > 4 \) which, with (5.29), proves \( \Lambda_1 - \Lambda_2 > 2 \). \( \square \)

5.3. Proof of theorem 5.1

We claim every NR condition with \( \delta \in (0, \frac{1}{2}) \) implies:

\[
\sqrt{s_2} \leq \frac{1}{2}, \quad (5.31)
\]

regardless of the signs \( \sigma_1, \sigma_2 \). Suppose not, so \( 1 \geq \sqrt{s_2} \geq \sqrt{s_3} \geq \sqrt{s_4} \geq \frac{1}{2} \). Note that, up to a harmless sign change, a triple value is either \( \sqrt{s_2} \pm \sqrt{s_3} \pm \sqrt{s_4} \). In the former case, the \( \sqrt{s_3} > \frac{1}{2} \) term is too large to make the entire expression fall within \( [-\delta, \delta] \). In the latter case, the first two terms combine to a value in \( (0, \frac{1}{2}) \), but \( \sqrt{s_4} > \frac{1}{2} \) so \( \sqrt{s_2} - \sqrt{s_3} \pm \sqrt{s_4} \in [-\delta, \delta] \) is not possible. Contradiction has been reached.

Applying \( \log x \leq x - 1 \) to (5.23) shows \( \mathcal{Q} \leq \frac{1}{\sqrt{\text{det}(c)}} \). Combining this with (5.31) to find:

\[
\mathcal{Q} \leq \frac{1}{\sqrt{s_2} - s_3}. \quad (5.32)
\]

Now, inspired by the ordering concern in lemma 5.4, introduce shorthand notations:

\[
\chi_0 := \mathfrak{M} \{ c_k c_m (1 - c_k^2) (1 - c_m^2) (c_k^2 - c_n^2) (c_n^2 - c_m^2) = 0 \},
\]

\[
\chi_1 := (1 - \chi_0) \cdot \mathfrak{M} \{ |c_k| < \min \{ |c_n|, |c_m| \} \},
\]

\[
\chi_2 := (1 - \chi_0) \cdot \mathfrak{M} \{ |c_m| < \min \{ |c_n|, |c_k| \} \},
\]

\[
\chi_3 := (1 - \chi_0) \cdot \mathfrak{M} \{ |c_n| < \min \{ |c_k|, |c_m| \} \}.
\]
Since their sum is at least 1, we perform simple change $c_k = \cos \theta_k$ in (5.10) and have,
\[
\frac{\text{vol}([V_{n,\sigma_1,\sigma_2}])}{L_1[L_2|n|^3]} \leq \sum_{j=0}^{3} \int_{(-1,1)^3} \mathbb{B} \{|\sigma_1 c_n + \sigma_2 c_k + c_m| \leq \delta\} \cdot \chi_j \cdot Q(\arccos c_k, c_m) \, dc_m \, dc_k. \tag{5.33}
\]

Note that we can view $\varsigma_2, \varsigma_3, \varsigma_4$ and $\chi_0, \ldots, \chi_3$ as measurable functions\(^\text{10}\) of $c_n, c_k, c_m$.

The $j=0$ summand of (5.33) vanishes due to the zero measure of the support of $\chi_0$.

For the $j=1$ summand of (5.33), we use (5.32) and change of coordinates from $(c_m, c_k)$ to $(c_m, \delta')$ with $\delta' = \sigma_1 c_n + \sigma_2 c_k + c_m$ to bound it as:
\[
\lesssim \int_{-1}^{1} \int_{-3}^{3} \mathbb{B} \{|\delta'| \leq \delta\} \cdot \frac{\chi_1}{\sqrt{\varsigma_2 - \varsigma_3}} \, \delta' \, dc_m = \int_{-\delta}^{\delta} \left( \int_{-1}^{1} \frac{\chi_1}{\sqrt{c_m^2 - c_k^2}} \, dc_m \right) \, dc_k,
\]
where $\varsigma_2 - \varsigma_3 = |c_n^2 - c_k^2|$ was due to $\chi_1 = 1$. Also,
\[
\chi_1 = 1 \implies \langle -|c_n|, |c_k| \rangle \geq -\sigma_2 c_k = c_m - \delta' + \sigma_1 c_n.
\]

Then, the $\varsigma_1$ factor in the above right-hand side integrand allows us to restrict the $c_m$-integral to an interval of length $2|c_n|$. Therefore, by elementary Calculus, the above inner integral is uniformly bounded. Thus the double integral is $\lesssim \delta$, which is consistent with theorem 5.1.

The $j=2$ summand of (5.33) is treated similarly to $j=1$ since (5.32) depends on ordering, not labels of the $c$’s, and since $c_k, c_m$ play symmetric roles in $\chi_1, \chi_2$. Also $c_k, c_m$ are symmetric in the NR condition in the sense of $|\sigma_1 c_n + \sigma_2 c_k + c_m| = |\sigma_1 c_2 c_n + \sigma_2 c_m + c_1|$.

For the $j=3$ summand of (5.33), estimate (5.23) or (5.32) is not sharp enough, especially since we will argue optimality in appendix C. Instead (5.24) is used for this case. Then, by different but simple considerations for the cases of $\varsigma_2 < \frac{1}{\pi}$ and $\varsigma_2 \geq \frac{1}{\pi}$ (also note (5.31)), we have:

the $j=3$ summand of (5.33)
\[
\lesssim \int_{-1}^{1} \int_{-3}^{3} \mathbb{B} \{|\sigma_1 c_n + \sigma_2 c_k + c_m| \leq \delta\} \cdot \max \left\{ \frac{\chi_3}{\sqrt{\varsigma_2 - \varsigma_4}}, \frac{\chi_3}{\sqrt{1 - \varsigma_3}} \right\} \, dc_m \, dc_k. \tag{5.34}
\]

We prove the following estimates on the contributions of the two arguments of max:

• By definition of $\chi_3$, we have $\frac{\chi_3}{\sqrt{1 - \varsigma_3}} \leq \frac{1}{\sqrt{1 - \varsigma_2}}$ so the contribution due to $\frac{\chi_3}{\sqrt{1 - \varsigma_3}}$ is:
\[
\lesssim \int_{-1}^{1} \int_{-1}^{1} \mathbb{B} \{|\sigma_1 c_n + \sigma_2 c_k + c_m| \leq \delta\} \cdot \frac{1}{\sqrt{1 - c_k^2}} \, dc_m \, dc_k \leq \int_{-1}^{1} 2\delta \cdot \frac{1}{\sqrt{1 - c_k^2}} \, dc_k \lesssim \delta,
\]
which is consistent with theorem 5.1.

• For the contribution to (5.34) due to $\frac{\chi_3}{\sqrt{\varsigma_2 - \varsigma_4}}$, we divide it into two sub-cases using:
\[
\chi_3 = \chi_3 \cdot \mathbb{B} \{|c_m| < |c_n|\} + \chi_3 \cdot \mathbb{B} \{|c_k| > |c_m|\} \overset{\text{def}}{=} \chi_{3,1} + \chi_{3,2}.
\]

Thanks again to the $c_k, c_m$ symmetry, it suffices to treat the case $\chi_{3,1} = 1$. Then by change of coordinates from $(c_m, c_k)$ to $(c_m, \delta')$ with $\delta' = \sigma_1 c_n + \sigma_2 c_k + c_m$, we bound the contribution to (5.34) due to the double integral of $\mathbb{B} \{ \cdot \} \cdot \frac{\chi_3}{\sqrt{\varsigma_2 - \varsigma_4}} \, dc_m \, dc_k$ as:

\(^{10}\) e.g. $\min\{a, b\} = \frac{1}{2}(a + b) - \frac{1}{2}|a - b|$ and $\varsigma_3 = c_n^2 + c_k^2 + c_m^2 - \min\{c_n^2, c_k^2, c_m^2\} + \min\{-c_n^2, -c_k^2, -c_m^2\}$. 

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\[ \lesssim \int_{-1}^{1} \int_{-3}^{3} B \{ |\delta'| \leq \delta \} \cdot \frac{\chi_{3,1}}{\sqrt{\xi_2 - \xi_4}} \, d\xi' \, dc_m = \int_{-\delta}^{\delta} \left( \int_{-1}^{1} \frac{\chi_{3,1}}{\sqrt{c_{m}^2 - c_{n}^2}} \, dc_m \right) \, d\xi', \]  

with \( \xi_2 - \xi_4 = c_m^2 - c_n^2 \) by \( \chi_{3,1} = 1 \). Also \( \chi_{3,1} = 1 \) implies:

\[ (\pm c_m |c_m|) \cup \left( (\pm 1, |c_n|) \cup (|c_n|, 1) \right) \ni -\sigma_2 c_k = c_m - \delta^*, \quad \text{for} \quad \delta^* := \delta - \sigma_1 c_n. \]

Now, for brevity, it suffices to integrate over \( c_m \in (0,1) \) as the case of \( c_m \in (-1,0) \) is treated similarly. Thus, we have (with the convention that \( (b, a) = 0 \) if \( b \geq a \)):

\[ \chi_{3,1} = 1 \text{ and } c_m \in (0,1) \implies \delta^* - c_m < \delta^* + c_m \text{ and } (0, \delta^* - |c_n|) \cup (\delta^* + |c_n|, 1) \ni c_m \implies \delta^* > 0 \text{ and } (\frac{1}{2} \delta^*, \delta^* - |c_n|) \cup (\delta^* + |c_n|, 1) \ni c_m. \]

Combining them with the simple estimates:

\[ \int_{b}^{a} \frac{dx}{\sqrt{x^2 - c_n^2}} = \log \left( x + \sqrt{x^2 - c_n^2} \right) \bigg|_{b}^{a} < \log \frac{2a}{b}, \quad \text{for } a > b > |c_n|, \]

and

\[ \int_{b}^{a} \log \frac{1}{x} \, dx = (x - \log x) \bigg|_{b}^{a} < (a - b) \left( 1 + \log \frac{1}{a} \right), \quad \text{for } a > b \geq 0, \]

we find

\[ (5.35) \lesssim \delta + \max_{\sigma_1, \sigma_2} \int_{-\delta}^{\delta} B \{ \delta' - \sigma_1 c_n > 0 \} \cdot \log^+ \frac{1}{\delta' - \sigma_1 c_n + |c_n|} \, d\xi' \lesssim \begin{cases} \delta + \max_{\sigma_1, \sigma_2} \left( \log^+ \frac{1}{\delta + 2|c_n|}, \log^+ \frac{1}{\delta + 2|c_n|}, \log \frac{1}{\delta} \right), & \text{if } \delta < |c_n|, \\ \delta + \max \left\{ (\delta + |c_n|) \log^+ \frac{1}{\delta + 2|c_n|}, (\delta - |c_n|) \log \frac{1}{\delta} \right\}, & \text{if } \delta > |c_n|. \end{cases} \]

The latter bound can be relaxed using \( \delta + |c_n| < 2\delta \) and \( \log \frac{1}{\delta} < \log 3 + \log \frac{1}{\delta + 2|c_n|} \), hence merged into the former estimate.

We have completed the proof of theorem 5.1 (also see below (5.3) regarding the notational change between \( \tilde{n} \) and \( n \)).

6. Proofs of the main theorems

We are ready to prove the 2D-like estimates for bilinear form \( B_{\mathcal{N}[\delta_s]} \) under our proposed conditions on the bandwidth function \( \delta_s \).

Listed below are elementary inequalities for later use.

\[ |\hat{k} + \hat{m}|^s \leq (|\hat{k}| + |\hat{m}|)^s \leq 2^{(s-1)^s} (|\hat{k}|^s + |\hat{m}|^s), \quad \forall s \in \mathbb{R}^+, \quad (6.1) \]

where the latter inequality is shown by minimising \( 1/(\alpha^s + (1 - \alpha)^s) \) for \( \alpha = |\hat{k}|/(|\hat{k}| + |\hat{m}|) \).

By the mean value theorem and \( ||\hat{n}|| - |\hat{m}| \leq |\hat{n} + \hat{m}| \leq |\hat{n} + \hat{m}| \), \( \forall s \in \mathbb{R}^+ \), we have:

\[ ||\hat{n}|| - |\hat{m}|^{s-1} \leq \max \{ ||\hat{n}||^{s-1}, |\hat{m}|^{s-1} \} \cdot |\hat{n} + \hat{m}|, \quad \forall s \in \mathbb{R}_0^+. \]
Also,

\[ u_k \cdot \hat{k} = 0 \implies |u_k \cdot \hat{m}| \leq |u_k| \min \{ |\hat{m}|, |\hat{k} + \hat{m}| \}. \tag{6.3} \]

**Proof of theorem 1.5.** Recall assumptions (1.6) and \( \delta_s(n, k, m) \in [0, \frac{1}{2}] \), so that we can apply counting estimate theorem 4.2 and volume estimate theorem 5.1 with \( \delta = \delta_s(n, n, n) \) to show:

\[
\sum_{k \in \mathbb{Z}^3} I_{N[\delta_s]}(n, k, -n - k) \lesssim_{L_1, L_2} |\hat{n}|^2 + |\hat{n}|^3 \delta_s(n, n, n) \log \frac{1}{\delta_s(n, n, n)} , \quad \forall n \in \mathbb{Z}^3 \setminus \{0\}.
\]

Then bound the above right-hand side using assumption (1.7). Also note that assumption (1.6) implies the symmetry of \( I_{N[\delta_s]}(\cdot, \cdot, \cdot) \) with respect to argument permutations. Therefore, we have validated all the assumptions of lemma 3.1 with \( \mathcal{M} = N[\delta_s] \) and crucially \( \beta = 2 \).

For proving (1.10), we use (2.25) to expand its left-hand side as,

\[
\langle D'B_{N[\delta_s]}(u, v), D'w \rangle = |T^3| \sum_{n, k, m, \text{conv}} |\hat{n}|^s (u_k \cdot \hat{m} i) \cdot (v_m \cdot w_n) |\hat{n}|^s \cdot |\hat{m}|^t \cdot |\hat{n}|^t + |\hat{m}|^t \cdot 1_{N[\delta_s]}(n, k, m),
\]

and relax one \( |\hat{n}|^s = |\hat{k} + \hat{m}|^s \) factor using (6.1). Apply lemma 3.1 to prove (1.10).

For proving (1.11), use (2.25) to expand its left-hand side, and add the expansion to an equivalent version of itself with \( n, m \) switched. Then, by incompressibility \( u_k \cdot \hat{n} = -u_k \cdot \hat{m} \), symmetry \( I_{N[\delta_s]}(n, k, m) = I_{N[\delta_s]}(m, k, n) \) and the symmetry of convolution sum from remark 2.2, we show:

\[
2|\langle D'B_{N[\delta_s]}(u, w), D'w \rangle| \leq |T^3| \sum_{n, k, m, \text{conv}} |u_k \cdot \hat{m}| \cdot |w_m \cdot w_n| \cdot |\hat{n}|^s - |\hat{m}|^t \cdot |\hat{n}|^t + |\hat{m}|^t \cdot 1_{N[\delta_s]}(n, k, m). \tag{6.4}
\]

Then the first case of (1.11) for \( s \in (0, 1] \) is due to (6.2), (6.3), \( |\hat{n}|^s + |\hat{m}|^t \leq 2 \max \{ |\hat{n}|^s, |\hat{m}|^t \} \) and, again, lemma 3.1 with \( \mathcal{M} = N[\delta_s] \) and \( \beta = 2 \). The second case for \( s > 1 \) is done similarly with the additional estimate (by (6.1)):

\[
\max \{ |\hat{n}|^{s-1}, |\hat{m}|^{t-1} \} \leq 2^{s-1} \min \{ |\hat{n}|^{s-1}, |\hat{m}|^{t-1} \} + 2^{t-1} |\hat{k}|^{t-1}.
\]

The proof of theorem 1.5 on 2D-like estimates is complete. \( \square \)

We move onto proving the ingredients need for the proof of theorem 1.3 on global well-posedness of the NR approximation. For simplicity, from here on, we let:

\[ \tilde{B} := B_{N[\delta_s]}. \]

Some preliminary remarks are in order.

First, we recall definitions 1.1 and 1.2 regarding weak solutions. The sum in definition 1.1 converges absolutely, because \( h(x) \) is smooth (making \( |h_n| \lesssim |\hat{n}|^{-10}|h|_{L_2} \), and by Cauchy-Schwarz inequality, we have:

\[
|T^3| \sum_{k \in \mathbb{Z}^3} |u_k| |v_{-n-k}| \leq ||u||_0 ||v||_0 , \quad \forall n \in \mathbb{Z}^3 \tag{6.5}.
\]

By a similar method, the canonical inner product on the right-hand side below can be Fourier-expanded into an absolutely convergent sum, so it is well defined and satisfies:

\[
\langle \tilde{B}(u, v), D'h \rangle_{w_k} = \langle D'\tilde{B}(u, v), D'h \rangle, \quad \forall u, v \in H^{r+1}(T^3), \ s \in \mathbb{R}_+ \text{ and smooth } h(x). \tag{6.6}
\]
In definition 1.2, the continuity in time for $\tilde{U}(t, \cdot)$ with values in $L^2_T(L^2)$ is stronger than the commonly seen $L^2_tL^2_x$ regularity. Also, as theorem 1.3 covers solutions in higher regularity spaces, we note that weak solutions in $H^0([0, \infty); H^s(\mathbb{T}^3)) \cap L^2([0, \infty); H^{s+1}(\mathbb{T}^3))$ coincide with other notions of solutions as $s$ increases. For $s \geq 1$, both $\Omega\partial_t \tilde{U}$ and $\mu \partial_t \tilde{U}$ belong to $L^2([0, \infty); H^{s-1}(\mathbb{T}^3))$. One can show $\tilde{B}(\tilde{U}, \tilde{U})$ also belongs to this space using (6.1) and (6.14) (ii) with $h$ being a test function. Then, the result of setting $\psi = \psi(x)$ in definition 1.2:

$$
\tilde{U}(T, \cdot) - \tilde{U}(0, \cdot) = \int_0^T -\tilde{B}(\tilde{U}, \tilde{U}) + \Omega \partial_t \tilde{U} + \mu \Delta \tilde{U} \mathrm{d}t,
$$

is a Bochner integral with integrand in $L^2([0, \infty); H^{s-1}(\mathbb{T}^3))$. Therefore, $\partial_t \tilde{U}$ is also in this space and importantly, the two sides of PDE (1.5) are regarded as an identical element in this space. In short, for $s \geq 1$, the weak formulation and such identity in the PDE form are equivalent, so one can manipulate the latter without involving the former. Similar argument shows, with $s > \frac{7}{2}$, the PDE is satisfied in the $C^0([0, \infty); H^{s-\frac{7}{2}}(\mathbb{T}^3))$ space which, if $s > \frac{7}{2}$, is embedded in $C^0([0, \infty) \times \mathbb{T}^3)$, the last case called classical solutions.

We also introduce projection $P_{<R} w := \sum_{n \in \mathbb{Z}^2, |\omega| < R} e^{i\omega \cdot x} u_n$, known as low-pass filter.

**Remark 6.1.** $P_{<R}$ preserves the realness of its argument, is self-adjoint with respect to the $\langle \cdot, \cdot \rangle$ inner product, and commutes with $P_{\text{div}}, L, D, \text{ differentiation and integration.}$

**Remark 6.2.** For a weak solution from definition 1.2, using $\psi = e^{i\omega \cdot x}$ and (6.5), we find Fourier coefficients of $\tilde{U}(t, \cdot)$ are smooth functions of $t$, hence $P_{<R} \tilde{U} \in C^0([0, T] \times \mathbb{T}^3)$.

The first ingredient is the next lemma on stability.

**Lemma 6.3 (Stability of NR approximate dynamics).** For the NR approximation (1.5) under the same assumptions on $\delta_t$ and $N(\delta_t)$ as in theorem 1.3, suppose it admits two weak solutions with regularity $\tilde{U}, \tilde{U}' \in C^0([0, T]; H^s(\mathbb{T}^3)) \cap L^2([0, T]; H^{s+1}(\mathbb{T}^3))$ for some $s \in \mathbb{R}^+$ and $T \in \mathbb{R}^+$. Then, for $E = \int_0^T \|\tilde{U}, \tilde{U}'\|^2_{s+1} dt$ and constant $C = C(s, T^3)$, we have:

$$
\|\tilde{U} - \tilde{U}'\|^2_{s+1}(t) \leq e^{CtE}\|\tilde{U} - \tilde{U}'\|^2_{s+1}(0), \quad \forall \ t \in [0, T],
$$

and

$$
\int_0^T \|\tilde{U} - \tilde{U}'\|^2_{s+1}(t) \mathrm{d}t \leq e^{-t}\|\tilde{U} - \tilde{U}'\|^2_{s+1}(0).
$$

**Proof.** Let $w = \tilde{U} - \tilde{U}'$ and $w_R = P_{<R} w$. By taking the difference of the weak formulation in definition 1.2 for $\tilde{U}$ and $\tilde{U}'$, setting $\psi = D^2 w_R$ (smooth by remark 6.2) and noting the term involving $L$ vanishes, we have, for any $t \in [0, T],$

$$
\|w_R\|^2_{s+1}(t_1) - \|w_R\|^2_{s+1}(0) + I_R = \int_{0}^{t_1} \frac{1}{2} \|w_R\|^2_{s+1} - \mu \|\nabla w_R\|^2_{s+1} dt,
$$

$$
\text{with } I_R = \int_0^{t_1} \left( \tilde{B}(\tilde{U}, w_R) + \tilde{B}(\tilde{U}', w - w_R) + \tilde{B}(w, \tilde{U}') , D^2 w_R \right)_{w_R} dt.
$$

Since $\tilde{U}(t, \cdot), \tilde{U}'(t, \cdot) \in H^{s+1}(\mathbb{T}^3)$ for almost every $t \in [0, T]$, and since $I_R$ is an integral in $t$, we can transform its integrand using (6.6). Then, by theorem 1.5, we find:

$$
\|w_R\|^2_{s+1}(t) - \|w_R\|^2_{s+1}(0) + I_R = \int_{0}^{t_1} \frac{1}{2} \|w_R\|^2_{s+1} - \mu \|\nabla w_R\|^2_{s+1} dt,
$$

$$
\|w_R\|^2_{s+1}(0) \leq e^{-t}\|w_R\|^2_{s+1}(0).
$$

$$
\int_0^T \|\tilde{U} - \tilde{U}'\|^2_{s+1}(t) \mathrm{d}t \leq e^{-t}\|\tilde{U} - \tilde{U}'\|^2_{s+1}(0).
$$
Then, take the limit of (6.7) as \( R \to \infty \). The assumed regularity on \( \bar{U}, \bar{U}' \) ensures Lebesgue dominated convergence theorem can be applied to the above right-hand side and also ensures \( \|w - w_R\|_{x+1}(t) \to 0 \) as \( R \to \infty \) for almost every \( t \in [0, T] \). So,

\[
\|w\|_2^2(t_1) - \|w\|_2^2(0) + 2 \int_0^{t_1} \mu \|\nabla w\|^2_2 \, dt \lesssim \int_0^{t_1} \|(\bar{U}, \bar{U}')\|_{x+1} \|w\|_2 \, \|w\|_{x+1} \, dt.
\]

Using the assumed regularity and the dominated convergence argument again, we can replace \((\bar{U}, \bar{U}')\) by \( \mathcal{P}_{<R}(\bar{U}, \bar{U}') \) up to arbitrary error. Then, further applying Young’s inequality shows:

\[
\|w\|_2^2(t_1) - \|w\|_2^2(0) + \int_0^{t_1} \mu \|\nabla w\|^2_2 \, dt \lesssim \epsilon_R + \int_0^{t_1} \|\mathcal{P}_{<R}(\bar{U}, \bar{U}')\|_{x+1}^2(t) \|w\|_2^2(t) \, dt, \quad \forall t_1 \in [0, T],
\]

with \( \epsilon_R \to 0 \) as \( R \to \infty \). Dropping the left-hand side integral gives a Grönwall’s inequality for \( \|w\|_2^2(t) \) with \( \|\mathcal{P}_{<R}(\bar{U}, \bar{U}')\|_{x+1}^2(t) \) being smooth due to remark 6.2. This then proves the first part of the conclusion. Substituting it into the above proves the second part. \( \square \)

6.1. **Proof of the global well-posedness result theorem 1.3**

The stability statement of the theorem is self-explanatory. Together with estimate (1.9), it implies uniqueness. Let us prove global existence and (1.8), (1.9).

First, fix large \( R > 0 \) (to appear in (6.8)). For any \( R > \rho \), define bilinear form:

\[
\tilde{B}^\rho(U, V) := \mathcal{P}_{<R} \tilde{B}(\mathcal{P}_{<R} U, \mathcal{P}_{<R} V).
\]

Its eigen-basis expansion is similar to (1.4) but with an additional \( 1_{\{\max(\|n\|, \|\bar{n}\|) < R\}}(n, k, m) \) factor in the summand. Consider the following equation:

\[
\partial_t V^\rho + \tilde{B}^\rho(V^\rho, V^\rho) = \Omega \mathcal{L} V^\rho + \mu \Delta V^\rho, \quad \text{with} \quad V^\rho(0, \cdot) = \mathcal{P}_{<\rho} \bar{U}_0. \tag{6.8}
\]

Define also \( E^\rho_j := \|V^\rho_j(0, \cdot)\|_2^2 \) for any \( j \in \mathbb{R}_+^+ \). Apparently \( E^\rho_j \) is finite and independent of \( R \).

The collection of Fourier modes \( V^\rho_n(t) \) with \( |n| < R \) satisfy a self-contained, finite dimensional ODE system. (A side note: nonlinearity \( \tilde{B}^\rho \) can cause \( V^\rho_n(t) \neq 0 \) for modes \( \rho \leq |n| < R \) at \( t > 0 \), even for initial data as in (6.8).) We refer to it as the ‘\( R \)-low-pass ODE’ and prove it is globally soluble as follows. Apparently \( V^\rho_n(t, \cdot) \) is div-free. By remark 6.1 and proposition 2.3, we find \( \langle \tilde{B}^\rho(V^\rho_n, V^\rho_n), \mathcal{P}_{<R} V^\rho_n \rangle = 0 \) and also \( \langle \mathcal{P}_{<R}(\Omega \mathcal{L} V^\rho_n + \mu \Delta V^\rho_n), \mathcal{P}_{<R} V^\rho_n \rangle \leq 0 \). Therefore, applying \( L^2 \) energy method on (6.8) shows, if the ODE is soluble for \( t \in [0, T] \), then:

\[
\sum_{n \in \mathbb{T}^2, |n| < R} |V^\rho_n|^2(t) \leq \sum_{n \in \mathbb{T}^2, |n| < R} |V^\rho_n|^2(0) = |\mathbb{T}^2|^{-1} E^\rho_0, \quad \forall t \in [0, T]. \tag{6.9}
\]

Now, the \( R \)-low-pass ODE is autonomous. Its right-hand side function is Lipschitz continuous in variables \( \{V^\rho_n\}_{|n| < R} \) with respect to the max norm, and the Lipschitz constant is controlled by max \( \left\{ |\mathcal{L} V^\rho_n| \right\}_{|n| < R} \). Then apply Picard iteration on the \( R \)-low-pass ODE in an interval \( t \in [0, t_1] \), for \( t_1 \) small enough so that the result of each iteration has its max norm bounded by \( 2 \sqrt{|\mathbb{T}^2|^{-1} E^\rho_0} \). Note that this bound holds for the base step of the iteration because we know (6.9) holds at \( t = 0 \). The aforementioned Lipschitz continuity implies that we can decrease \( t_1 \) (if need be) to make the iteration mapping a contraction with respect to the max norm, so by the Banach fixed-point theorem, the solution exists for \( t \in [0, t_1] \). We can choose \( t_1 \) to depend only on \( R \) and \( |\mathbb{T}^2|^{-1} E^\rho_0 \), the proof of which is elementary, and thus omitted. The solvability proven so far implies (6.9) holds at \( t = t_1 \), then by the same Picard
iteration and fixed-point argument, we prove the $R$-low-pass ODE is solvable for $t \in [t_1, 2t_1]$, for $t \in [2t_1, 3t_1]$, and so on.

On the other hand, the choice of initial data in (6.8) and the definition of $\tilde{B}^R$ with $R > \rho$ implies $V^R(t) \equiv 0$ if $|n| \geq R$. Then, the information on $V^R(t)$ we have shown so far together with (6.8) implies $V^R(t, x)$ is smooth in $[0, \infty) \times \mathbb{T}^3$. So we take the inner product $D^j (6.8)$ with $D/V^R$, knowing Calculus rules for smooth functions apply, to obtain:

$$
\frac{1}{2} \frac{d}{dt} \|V^R\|_2^2 + \mu \|V^R\|_p^2 = -\langle D\tilde{B}^R(V^R, V^R), D/V^R \rangle, \quad \forall j \in \mathbb{R}^d.
$$

The proof of 2D-like estimate (1.11) can be repeated verbatim for $\tilde{B}^R$ except that we apply the $n, m$ symmetry of $1 \chi_{\{n, k, m\}}(n, k, m) I_{[\max(\{|n|, |k|, |m|) < R}\}}(n, k, m)$ to obtain a version of (6.4) where $\tilde{B}^R$ replaces $B$ and there is an additional harmless $1_{[\max(\{|n|, |k|, |m|) < R\}}(n, k, m)$ factor in the upper bound. Therefore, for any $j \in \mathbb{R}^d$, we have:

$$
\frac{1}{2} \frac{d}{dt} \|V^R\|_2^2 + \mu \|V^R\|_p^2 \leq \sqrt{2C_j} \|V^R\|_1 \|V^R\|_2 \|V^R\|_{p+1},
$$

for $0 \leq C_j \leq 4|\beta| \mathbb{T}^3|^{-1}$. (6.10)

Since setting $j = 0$ gives zero value after the first $\leq$, we obtain the analogue of equality (1.8),

$$
\|V^R(T, \cdot)\|_2^2 + 2 \mu \int_0^T \|V^R\|_2^2 \, dt = \|V^R(0, \cdot)\|_2^2, \quad \forall T \geq 0.
$$

Solving Grönwall’s inequality (6.10) for general $j$ gives:

$$
\|V^R(T, \cdot)\|_j^2 \leq \exp \left( \frac{2C_j}{\mu} \int_0^T \|V^R(t, \cdot)\|_j^2 \, dt \right) E^V_0, \quad \forall T \geq 0 \text{ and } j \geq 0.
$$

Then, integrate (6.10) in time and relax every $\|V^R\|_2^2$ to using the above estimate, followed by applying $\int_0^\infty \|V^R(t, \cdot)\|^2_2 \, dt \leq \frac{1}{2} E^V_0$ (by (6.11)), to obtain an upper bound on $\int_0^\infty \|V^R\|^2_2 \, dt$. In summary, we prove (1.9) with every $\tilde{U}$ replaced by $V^R$, $s$ replaced by $j$, $E_0$ replaced by $E^V_0$ and $E_{0b}$ replaced by $E^{V_0}_{0b}$. As any order of time derivatives of $V^R$ can be estimated using (6.8), (6.1), (6.14)(ii), it is easy to show that all $\{V^R\}_{R > \rho}$ have their $H^3([0, \infty) \times \mathbb{T}^3)$ bounds finite and independent of $R$. Obviously $j$ can be arbitrarily large. Therefore by compact embedding, for any positive $j, T$, there exist $V_j, T \in H^3([0, T] \times \mathbb{T}^3)$ and a sequence of $R$ values (indices omitted for brevity) tending to $\infty$ so that:

$$
\lim_{R \to \infty} \|V^R - V_j, T\|_{H^3([0, T] \times \mathbb{T}^3)} = 0.
$$

Using the standard technique of repeatedly extracting subsequences of $R$ values (again, indices omitted for brevity), we show there exists a smooth function $V(t, x)$ so that:

$$
\lim_{R \to \infty} \|V^R - V\|_{H^3([0, T] \times \mathbb{T}^3)} = 0,
$$

for any positive $j, T$. By Sobolev embedding, this implies convergence in $W^{4-3}(0, T] \times \mathbb{T}^3)$. Then for very positive $j$, every linear term of (6.8) converges to its $V$ analogue in $H^3([0, T] \times \mathbb{T}^3)$ and the initial condition part also converges in $H^3(\mathbb{T}^3)$. For the bilinear term of (6.8), split the following difference into three parts,

$$
\tilde{B}^R(V^R, V^R) - \tilde{B}(V, V) = (\tilde{B}^R(V^R, V^R) - \tilde{B}(V, V)) + (\tilde{B}(V, V) - \tilde{B}(P_{<R/2} V, P_{<R/2} V)) + (\tilde{B}(P_{<R/2} V, P_{<R/2} V) - \tilde{B}(V, V)),
$$
where the two terms involving $\mathcal{P}_{< R/2}$ cancel due to the definition of $\tilde{B}^R$. Each difference is concerned with the same type of bilinear form and can be estimated similarly. For example, by definition 1.1, (6.5) and choosing any smooth $h(x)$ (making $|h_n| \lesssim |\hat{h}|^{10}_1 |h|_2$), we have:

$$\langle \tilde{B}(u, u) - \tilde{B}(u', u'), h \rangle_{wk} \lesssim \left( \|u\|_0 + \|u'\|_0 \right) \|u - u'\|_0 \|h\|_2. \quad (6.12)$$

At any fixed $t \in [0, \infty)$, let $h = D^2(\tilde{B}(\mathcal{P}_{< R/2} V, \mathcal{P}_{< R/2} V) - \tilde{B}(V, V), u = \mathcal{P}_{< R/2} V, u' = V,$ and use the smoothness of $V$ as proven above, noting (6.6), to find $\|\tilde{B}(\mathcal{P}_{< R/2} V, \mathcal{P}_{< R/2} V) - \tilde{B}(V, V)\|_t$ to be essentially controlled by $\|\mathcal{P}_{< R/2} V - V\|_0$, hence tending to 0 as $R \to \infty$.

In short, we have just proven that every term in (6.8) converges strongly to its $V$ analogue in $\mathcal{C}^0([0, T]; H^s(\mathbb{T}^3))$ as $R \to \infty$ for any $j$, and in particular, the strong limit of $\tilde{B}^R(V^j, V^R)$ is $\tilde{B}(V, V)$ with the bilinear form also converging. To finish the proof, we rename $U^{(\rho)} := V$ to highlight its dependence on $\rho$. Then

$$\partial_t U^{(\rho)} + \tilde{B}(U^{(\rho)}, U^{(\rho)}) = \Omega \mathcal{L} U^{(\rho)} + \mu \Delta U^{(\rho)}, \quad \text{with } U^{(\rho)}(0, x) = \mathcal{P}_{< \rho} \tilde{U}_0. \quad (6.13)$$

Recall we have also proven (1.8), (1.9) with $\tilde{U}$ is replaced by $V^R$, and by taking the limit, we obtain their $U^{(\rho)}$ analogue. Then, by the stability lemma 6.3, we have $\{U^{(\rho)}\}_{\rho \in \mathbb{Z}^+}$ form a Cauchy sequence in $\mathcal{C}^0([0, \infty); H^s(\mathbb{T}^3))$ and $\{\mathcal{V} U^{(\rho)}\}_{\rho \in \mathbb{Z}^+}$ form a Cauchy sequence in $L^2((0, \infty); H^s(\mathbb{T}^3))$. Both are Banach spaces, so each sequence converge strongly in the respective space to, say, $\tilde{U}$ and $W$. Therefore, for any $3 \times 3$ tensor-valued smooth function $h(t, x)$, we have:

$$\int_0^T \int_{\mathbb{T}^3} \tilde{U} : (\nabla \cdot h) - W : h \, dx \, dt = \lim_{\rho \to \infty} \int_0^T \int_{\mathbb{T}^3} U^{(\rho)} : (\nabla \cdot h) - \mathcal{V} U^{(\rho)} : h \, dx \, dt = 0.\$$

By choosing the smooth function $h = \mathcal{P}_{< \rho} (\mathcal{V} \tilde{U} - W)$ for any $R > 0$, we show $\mathcal{V} \tilde{U} = W$.

The strong convergence of $U^{(\rho)} \to \tilde{U}$ in the above sense implies every term in the weak formulation (a la definition 1.2) of (6.13) converges to the desired limit. Note that convergence of the term with $\langle \cdot \rangle_{wk}$ is due to (6.12). The global existence part of the Theorem has been proven. Also, (1.8) and (1.9) follow from the strong convergence of $U^{(\rho)} \to \tilde{U}$ together with argument just below (6.13) and the fact that $\|U^{(\rho)}(0, x)\|_0^2 = \|\mathcal{P}_{< \rho} \tilde{U}_0\|_0^2 \to E_{00}$ as $\rho \to \infty$.

The proof of theorem 1.3 is complete.

6.2. Proof of the error estimate theorem 1.6

First, we show nonlinear estimates in a generic setting. For functions $f, g, h$ in $\mathbb{T}^3$, define:

$$f_{abs} : = \sum_n |f_n| e^{ik \cdot x}, \text{ and similarly define } g_{abs}, h_{abs}.\$$

As the coefficient of $e^{ik \cdot x}$ in the Fourier series of $f_{abs}$ is $|f_{-n}|$, we use Parseval’s theorem (2.3) to have:

$$\sum_{n,k,m,\text{conv}} |f_n| |g_k| |h_m| = \sum_n \left( f_n \sum_k |g_k| |h_{-n-k}| \right) = \sum_n \left( f_{-n} \sum_k |g_k| |h_{-k}| \right) = \sum_n \left( f_{-n} \sum_k |g_k| |h_{-n-k}| \right) = |n^3|^{-1} \left\langle g_{abs} h_{abs}, f_{abs} \right\rangle. \$$

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Then we apply Cauchy–Schwarz inequality on the right-hand side and apply Hölder’s inequality on $\|g_{ab}h_{ab}\|_{L^2}$ followed by applying Sobolev inequalities to show, for generic set $\mathcal{M} \subset \{Z \setminus \{0\}\}^3$ which surely satisfies $\mathbf{1}_\mathcal{M}(n,k,m) \leq 1$, that:

$$
\sum_{n,k,m,\text{conv}} |f_n| |g_k| |h_m| \mathbf{1}_\mathcal{M}(n,k,m) \lesssim \begin{cases} 
C_\gamma \|f\|_{L^{1+\gamma}} \|g\|_0 \|h\|_0, & \text{for real } \gamma > 0, \\
C_\beta \|f\|_{L^{\frac{2}{1-\beta}}} \|g\|_\beta \|h\|_0, & \text{for real } \beta \in (0, \frac{1}{2}).
\end{cases}
(6.14)
$$

By the same approach together with (2.31), (6.1), (6.2), we obtain the following lemma:

**Lemma 6.4.** Suppose set $\mathcal{M} \subset \{Z \setminus \{0\}\}^3$ satisfy symmetry $\mathbf{1}_\mathcal{M}(n,k,m) = \mathbf{1}_\mathcal{M}(m,k,n)$. Then for any $\mathbb{R}^3$-valued, div-free, zero-mean functions $u,v \in H^{1+1}(\mathbb{T}^3)$, we have:

$$
\left| \langle \mathbf{D}'B_\mathcal{M}(u,v), \mathbf{D}'v \rangle \right| = \left| \langle \mathbf{D}'B_\mathcal{M}(u,v) - B_\mathcal{M}(u,\mathbf{D}'v), \mathbf{D}'v \rangle \right|
$$

$$
\lesssim \begin{cases} 
C_{s,\gamma} \|u\|_{\text{max}(s;\frac{1+\gamma}{2})} \|v\|_{2}, & \text{for real } s \geq 1 \text{ and } \gamma > 0, \\
C_s (\|u\|_{s+1} \|v\|_{1} + \|u\|_{s+1} \|v\|_{s+1}) \|v\|_{s}, & \text{for real } s > 0.
\end{cases}
$$

Note that, in proving the case of $s \in (0, 1)$, we further use (6.3). Also, see e.g. [20, lemma A.1] for $B_\mathcal{M} = B$ and integer $s$.

Following from the above, we remark on local-in-time existence and estimates of solutions. For this purpose, the formalisms of the original RNS equations (1.1) and our NR approximation (1.5) are regarded in the same way, due to the fact that lemma 6.4 can be applied to both $B$ and $B_{\mathcal{M}(\delta, \epsilon)}$. Then, with the initial data for $\mathbf{U}$ and $\mathbf{u}$ given in theorem 1.6, one can mimic the local-in-time existence part of the proof of theorem 1.3—except for using 3D-like estimates (6.14), lemma 6.4 to replace 2D-like estimates of theorem 1.5—to show local-in-time existence and estimates for both solutions as:

$$
\max_{t \in [0,T]} \left( \|\mathbf{U}\|_{s}^2(t) + \|\mathbf{u}\|_{s}^2(t) + \mu \int_0^{\epsilon/E_0^2} \|\mathbf{U}\|_{s+1}^2(t) + \|\mathbf{u}\|_{s+1}^2(t) \, dt \lesssim \epsilon, E_0,
(6.15)
$$

with constants $c$, $C$, $s$, $s^p$ described in theorem 1.6. Note in particular $s, s^p > \frac{5}{2}$. Also note that, in proving the above, one should change the counterpart of the first inequality in (6.10) by raising the first $H^1$ norm to $H^{1+\gamma}$ for $\gamma > 0$ and lowering the second $H^{s+1}$ norm to $H^0$. The second inequality of (6.10) is irrelevant here. The resulting Grönwall’s inequality is controlled by a Ricatti-type ODE for which the solution remains bounded in time interval $[0, \epsilon/E_0^2]$ as claimed in (6.15). (Note: for Navier–Stokes equations, see standard results such as [30, chapter 17, theorem 4.1, (4.16), propositions 4.2, 4.3].)

**Proof of theorem 1.6.** We will use the norm-preserving property (2.20) of $e^{t\mathcal{L}}$ without reference. Then $\|\hat{\mathbf{U}}\|_s = \|\hat{\mathbf{u}}\|_s$. Also, property (2.21) implies, for any $s \in \mathbb{R}^+$, that:

$$
\langle \mathbf{D}'B(\tau; \mathbf{u}, \mathbf{v}), \mathbf{D}'\mathbf{w} \rangle = \langle \mathbf{D}'B(e^{t\mathcal{L}} \mathbf{u}, e^{t\mathcal{L}} \mathbf{v}), \mathbf{D}'e^{t\mathcal{L}} \mathbf{w} \rangle.
(6.16)
$$

So, tri-linear estimates (6.14), lemma 6.4 will be used in conjunction with the above as needed.
Subtract the two transformed systems (2.8), (2.29) to find:

\[-\partial_t (u - \bar{u}) + \mu \Delta (u - \bar{u}) = B(\Omega; \mathbf{u} - \bar{u}) + B(\Omega; u - \bar{u}) + B(\Omega; \bar{u}, \bar{u}) - \tilde{B}(\Omega; \bar{u}, \bar{u}).\]

(6.17)

Define \( \mathcal{N}[\delta_\ast] := \{(n, k, m) \in (\mathbb{Z}^3 \setminus \{0\})^3 : n + k + m = 0\} \setminus \mathcal{N}[\delta_\ast] \) and rewrite the ‘residual’:

\[B(\Omega; \bar{u}, \bar{u}) - \tilde{B}(\Omega; \bar{u}, \bar{u}) = \sum_{(n, k, m) \in \mathcal{N}[\delta_\ast]} \sum_{\delta \neq \ast} \mathcal{P}_n^{\delta} B(\mathcal{P}_n^{\delta} \bar{u}, \mathcal{P}_m^{\delta} \bar{u}) \exp(i\omega_{\text{dim}}(\Omega)) = \Omega^{\ast} \partial \delta_\ast - \Omega^{\ast} r_1,\]

where \( r_\ast := \sum_{(n, k, m) \in \mathcal{N}[\delta_\ast]} \sum_{\delta \neq \ast} \mathcal{P}_n^{\delta} B(\mathcal{P}_n^{\delta} \bar{u}, \mathcal{P}_m^{\delta} \bar{u}) \exp(i\omega_{\text{dim}}(\Omega))(i\omega_{\text{dim}}(\Omega))^{-1},\]

and \( r_1 := \sum_{(n, k, m) \in \mathcal{N}[\delta_\ast]} \sum_{\delta \neq \ast} \mathcal{P}_n^{\delta} \partial_{\Omega} (B(\mathcal{P}_n^{\delta} \bar{u}, \mathcal{P}_m^{\delta} \bar{u}) \exp(i\omega_{\text{dim}}(\Omega))(i\omega_{\text{dim}}(\Omega))^{-1}.\]

Let \( w := u - \bar{u} + \Omega^{\ast} r_\ast \), and recast (6.17) into:

\[-\partial_t w + \mu \Delta w = B(\Omega; u, \bar{u}) + B(\Omega; u, \bar{u}) - \Omega^{\ast} (r_1 + B(\Omega; u, r_\ast) + B(\Omega; r_\ast, \bar{u}) - \mu \Delta r_\ast).\]

(6.18)

By definition (2.11) where constant vector \( r_\ast^n \) is of unit length, we have, for vector fields \( \mathbf{u}, \mathbf{v} \),

\[|\mathcal{P}_n^{\delta} \mathbf{u}| \leq |u_n| \quad \text{and} \quad |B(\mathcal{P}_n^{\delta} \mathbf{u}, \mathcal{P}_m^{\delta} \mathbf{v})| \leq |u_n| |v_m| |\bar{m}|.\]

Recall definition (1.3) of \( \mathcal{N}[\delta_\ast] \) so that on the complement set \( \mathcal{N}[\delta_\ast]^c \), we have \( |\omega_{\text{dim}}| > \delta_\ast (n, k, m) \). Then by the lower bound (1.12) on \( \delta_\ast (n, k, m) \), we find, for any \( \gamma \in \mathbb{R}^+ \),

\[\left| \frac{1}{\omega_{\text{dim}}} \right| \lesssim \gamma (|\bar{k}| + |\bar{m}|)^{1+\gamma}, \quad \forall (n, k, m) \in \mathcal{N}[\delta_\ast]^c.\]

Now, for any \( \mathbb{C}^1 \)-valued function \( \mathbf{h} \in L^2(\mathbb{T}^3) \) and any \( j \in \mathbb{R}^+ \), we expand \( \langle \mathcal{D} \mathbf{r}_\ast, \mathbf{h} \rangle \) using the adjoint property (2.13); then, using the above inequalities, we find:

\[\langle \mathcal{D} \mathbf{r}_\ast, \mathbf{h} \rangle \lesssim_{j, \gamma} c_{\epsilon_1} \sum_{n, k, m, \text{conv}} |\bar{u}_k| |\bar{u}_m| |\bar{m}|^{1+\gamma} |h_n|.\]

Next, switch \( k, m \) and add the result to the above, applying (6.1) and relaxing the result into two sums that are equal thanks to \( k, m \) symmetry, to find:

\[\langle \mathcal{D} \mathbf{r}_\ast, \mathbf{h} \rangle \lesssim_{j, \gamma} c_{\epsilon_1} \sum_{n, k, m, \text{conv}} |\bar{u}_k| |\bar{u}_m| |\bar{m}|^{1+\gamma} |h_n|.\]

Similarly, we use respectively the definition of \( r_1 \) and the right-hand side of (2.29) for \( \partial \bar{u} \), and simplify using \( k, m \) symmetry where possible, to find, respectively, for \( j \geq 0 \):

\[\langle \mathcal{D} \mathbf{r}_1, \mathbf{h} \rangle \lesssim_{j, \gamma} c_{\epsilon_1} \sum_{n, k, m, \text{conv}} \left((|\bar{u}_k| |\bar{u}_m| |\bar{m}|^{1+\gamma} + |\bar{u}_k| |\bar{l}_m| |\bar{m}|^{1+\gamma}) |h_n|,\right)\]

\[\langle \mathcal{D} \partial \bar{u}, \mathbf{h} \rangle \lesssim_{j, \gamma} \sum_{n, k, m, \text{conv}} |\bar{u}_k| |\bar{u}_m| |\bar{m}|^{1+\gamma} |h_n| + \mu \|\bar{u}\|_{j+2} \|\mathbf{h}\|_0.\]

By the above three inequalities, (6.14) and treating \( \mathbf{h} \) as test function, we obtain, for \( s' \in [0, s - 3) \),
\[\|r_{\delta},\|_{s'+1} \lesssim \hat{c}_2^{-1}\|\hat{u}\|_{s'}\|\hat{u}\|_{s'}, \quad (6.19a)\]

\[\|r_1\|_{s'} \lesssim \hat{c}_2^{-1}\||\partial_t\hat{u}\|_{s'}\|\hat{u}\|_{s'-1} + ||\partial_t\hat{u}\||_{s'-1}\|\hat{u}\|_{s'}\]

\[\lesssim \hat{c}_2^{-1}\|\hat{u}\|_{s'} + \hat{c}_2^{-1}\|\hat{u}\|_{s'-1} + \hat{c}_2^{-1}\|\hat{u}\|_{s} + \hat{c}_2^{-1}\|\hat{u}\|_{s+1}\|\hat{u}\|_{s'},\quad (6.19b)\]

Here and below the implied constants in the \(\lesssim\) notation depend on \(s',s\).

For the last three bilinear terms in (6.18), by the same combination of testing function, (6.1) (in the spirit of “endpoint” estimates) and (6.14), we obtain, for real \(s' \in [0,s-3)\) and \(\gamma > 0,\)

\[\|B(\Omega; w, \hat{u})\|_{s'} \lesssim s'(\|w\|_{s'} + \|\hat{u}\|_{s'})\|w\|_{\max\{s'+1,\frac{s'}{2}+\gamma\}}, \quad (6.20a)\]

\[\|B(\Omega; u, r_{\delta})\|_{s'} \lesssim s'(\|u\|_{s'} + \|r_{\delta}\|_{s'})\|u\|_{\max\{s'+1,\frac{s'}{2}+\gamma\}}, \quad (6.20b)\]

\[\|B(\Omega; r_{\delta}, \hat{u})\|_{s'} \lesssim s'(\|r_{\delta}\|_{s'} + \|\hat{u}\|_{s'})\|r_{\delta}\|_{\max\{s'+1,\frac{s'}{2}+\gamma\}}. \quad (6.20c)\]

(Note: in proving them, one can consider the cases \(\frac{s'}{2} < \frac{1}{2}, \frac{s'}{2} = \frac{1}{2}, \frac{s'}{2} > \frac{1}{2}.\)

Take the \(L^2\) inner product of \(D^{s'}w\) with \(D^{s'}w\) for \(s'\) in the range prescribed in theorem 1.6. In the resulting right-hand side, the first term vanishes if \(s' = 0\) and is bounded using lemma 6.4 if \(s' \geq 1\). Also, bound the rest using (6.19), (6.20). Then we arrive at:

\[
\frac{d}{dt}\|w\|_{s'}^2 + 2\mu\|w\|_{s'+1}^2 \lesssim (\|u\|_{s'} + \|u\|_{s'})\|w\|_{s'}^2 + \hat{c}_2^{-1}\Omega^{-1}\|\hat{u}\|_{s'}^2 + \|u\|_{s'}^2 \|\hat{u}\|_{s'}^2 + \|u\|_{s'}^2 \|\hat{u}\|_{s'}^2.
\]

Relax both \(O(\Omega^{-1})\) terms using Young’s inequality so that part of the result cancels the \(O(1/\Omega)\) term on the left of the result is absorbed into the first term on right,

\[
\frac{d}{dt}\|w\|_{s'}^2 \lesssim (\|\hat{u}\|_{s'} + \|u\|_{s'})\|w\|_{s'}^2 + \hat{c}_2^{-2}\Omega^{-2}\|\hat{u}\|_{s'}^2 + \|u\|_{s'}^2 \|\hat{u}\|_{s'}^2 \|\hat{u}\|_{s'}^2. \quad (6.21)\]

Recall \(w = u - \hat{u} + \Omega^{-1}r_{\delta}\), so by (6.19a) we have \(\|w(0,\cdot)\|_{s'}^2 - \|U_0 - \hat{U}_0\|_{s'}^2 \lesssim \Omega^{-2}.\)

Then, integrate the above in time and apply local estimates (6.15) to show:

\[
\|w(t,\cdot)\|_{s'}^2 - \|U_0 - \hat{U}_0\|_{s'}^2 \lesssim \int_0^t \|w(t,\cdot)\|_{s'}^2 dt + \hat{c}_2^{-2}\Omega^{-2}, \quad \text{for} \quad t \in [0,c/\Omega^{-\frac{1}{2}}].
\]

Solve this Gronwall’s inequality and apply (6.19a) again to prove the required bound on \(u - \hat{u} = w - \Omega^{-1}r_{\delta}\), which of course holds also for \(U - \hat{U}\).

\(\square\)

**Remark 6.5.** It then requires minimal effort to prove the global-in-time solvability of RNS equations for any rotation rate \(\Omega\) above a threshold value that only depends on viscosity \(\mu\) and the size and type of the norm of the initial datum \(U_0\). First, for \(U_0\) from a sufficiently regular function space, we combine assumptions of theorems 1.3, 1.6 so that the bandwidth \(\delta_\ast(n,k,m)\) now satisfies two-sided bounds (1.7) and (1.12). Further assume,

\(s' = s'\) and \(U_0 = \hat{U}_0 \in H^{s'+1}(\mathbb{T}^3)\),

so by theorem 1.3, \(\|\hat{u}(t,\cdot)\|_{s'+1}\) is uniformly bounded for all \(t \geq 0\). Now, for as long as:

\[
\|u\|_{s'} \leq 100\|\hat{u}\|_{s'},
\]

(6.22)
we deduce from (6.21) that:
\[
\frac{d}{dt} \|\mathbf{w}\|^2 \leq 101\|\mathbf{u}\|^2 + \Omega^{-2}\left[101\|\mathbf{u}\|^2 + \mu\|\mathbf{u}\|^2\right].
\]

By (1.8) and \(\|\mathbf{u}\|^2 \leq \|\nabla\mathbf{u}\|^2\) due to \(\mathbf{u}\) being zero-mean, we have \(\|\mathbf{u}(t, \cdot)\|^2\) decays exponentially. Interpolating this and the above uniform \(\mathcal{H}^1(\mathbb{T}^3)\) bound, we find the factor \(101\|\mathbf{u}\|^2\) of the above differential inequality can be integrated for \(t \in [0, \infty)\) when used as an integrating factor. Therefore, in view of (6.19), the assumed \(\mathbf{U}_0 = \mathbf{U}_0\) and \(\mathbf{w} = \mathbf{u} - \mathbf{u} + \mathbf{u}^{-1}\mathbf{r}_0\), we can find a small enough threshold for \(\Omega^{-1}\) below which the upper bound on \(\mathbf{w}\) stays low enough to ensure (6.22) stays valid for all \(t \geq 0\), hence making the RNS equations solvable in the \(\mathcal{H}^s(\mathbb{T}^3)\) space for all \(t \geq 0\). Secondly, if \(\mathbf{U}_0\) is not regular enough to be covered by the first case, we use the local-in-time smoothing property to bootstrap the solution’s regularity. For instance, if \(\mathbf{U}_0 \in \mathcal{H}^s(\mathbb{T}^3)\) for \(s > \frac{1}{2}\), then by the usual \(\mathcal{H}^s(\mathbb{T}^3)\) energy method followed by applying lemma 6.4 on the nonlinear term and interpolating the \(\mathcal{H}^2\) norm in the result, we have:
\[
\frac{d}{dt} \|\mathbf{U}\|^2 + 2\mu\|\mathbf{U}\|^{2+1}_2 \leq C_s\|\mathbf{U}\|^{2+\beta}_2\|\mathbf{U}\|^{1-\beta}_2,
\]
for some \(\beta = \beta(s) \in (0, 1)\). Apply Young’s inequality to relax the right-hand side to the sum of \(\mu\|\mathbf{U}\|^{2+1}_2\) and some finite power of the rest, noting \(\|\mathbf{U}\|_0\) is non-increasing in time, to obtain a finite bound on sup\(\mathbf{U}_{\|\mathbf{U}\|_2<1}\) for some \(t_1 > 0\). This then implies a finite bound on \(\mu\int_0^{t_1}\|\mathbf{U}\|^{2+1}_2\) and thus a finite bound on \(\|\mathbf{U}\|^{2+1}_2(t'\) for some \(t' \in [0, t_1]\). All these bounds only depend on \(\mu, s\) and \(\|\mathbf{U}_0\|_2\). Repeat such process to achieve a sufficiently regular solution \(\mathbf{U}(t, \cdot)\) at some \(t > 0\), after which time the proof is the same as the above first case.

Acknowledgments

Cheng and Sakellaris are supported by the Leverhulme Trust (Award No. RPG-2017-098). Cheng is supported by the EPSRC (Grant No. EP/R029628/1). We thank Beth Wingate, Colin Cotter, David Fisher, Paul Skerritt and Jon Bevan for insightful discussion and valuable feedback. We are grateful to the anonymous referees and the editors of Nonlinearity for their careful reading of the manuscript and constructive comments for improvement.

Appendix A. Counting integer points bounded by Jordan curves

Inspired by [29] on estimating integer points inside a planar Jordan curve, we provide the following self-contained proof of lemma A.2 on such counting problem generalised to a set separated by multiple disjoint Jordan curves. Then, the set in question may overlap either the interior or exterior of an individual Jordan curve.

Let \(c(\cdot)\) denote the set closure operator. A curve is simple if it does not cross itself. A curve is rectifiable if it has finite length. Recall the area of a bounded open set in \(\mathbb{R}^2\), i.e. its Lebesgue measure, is well defined because it is Borel measurable and hence Lebesgue measurable.

A Jordan curve is a simple closed curve in the plane \(\mathbb{R}^2\), namely, it is the image of an injective continuous map of a unit circle into the plane. The Jordan curve theorem, seemingly intuitive but requiring nontrivial proofs, states that for any planar Jordan curve \(\gamma^o\), its complement \(\mathbb{R}^2 \setminus \gamma^o\) consists of two path-connected open subsets called ‘exterior’ and ‘interior’ regions so that the exterior region consists of all points path-connected to points that are arbitrarily and sufficiently far away from \(\gamma^o\), and also, \(\gamma^o\) is the boundary of each region. Note that the ‘topological exterior’ of a general set is a different notion which we do not use here. An
even more nontrivial Jordan-Schoenflies theorem further asserts that for a Jordan curve as a continuous injection from unit circle to \( \mathbb{R}^2 \), the domain of that injection can be extended so that it is a homeomorphism of \( \mathbb{R}^2 \).

Note: we will use notation \( \gamma^o \) whenever the curve is explicitly known as a Jordan curve.

We call \( \gamma \) an ‘arc’ of a Jordan curve \( \gamma^o \) if it is the image of a closed, positive-lengthed, sub-interval of the unit circle under the same mapping that defines \( \gamma^o \).

Now, consider

\[
\text{bounded open set } \emptyset \neq S \subset \mathbb{R}^2 \text{ so that } \partial S = \bigcup_{\gamma^o \in \mathcal{J}} \gamma^o, \\
\text{where finite set } \mathcal{J} \text{ consists of disjoint, rectifiable Jordan curves.} \tag{A1}
\]

For an integer-point \( p = (x, y) \in \mathbb{Z}^2 \), define the open square box

\[
B_p := (x - \frac{1}{2}, y + \frac{1}{2}) \times (x - \frac{1}{2}, y + \frac{1}{2}).
\]

For any Jordan curve \( \gamma^o \in \mathcal{J} \), an arc \( \gamma \subset \gamma^o \cap \text{cl}(B_p) \) is called ‘maximal’ relative to \( B_p \) if:

\[
\gamma \cap B_p \text{ is non-empty, path-connected and } \gamma \cap \partial B_p = \{q_1, q_2\}, \tag{A2}
\]

for points \( q_1, q_2 \) which we shall call the endpoints of \( \gamma \). It is possible \( q_1 = q_2 \). Note that a given box \( B_p \) can break up a Jordan curve \( \gamma^o \) into several maximal arcs, but our proof below will be localised entirely within \( B_p \) assuming no knowledge of such break-ups. See figure A1.

Any point of \( B_p \cap \partial S \) that is not on a maximal arc must be on an ‘interior’ Jordan curve, i.e. one that is entirely in \( B_p \). In other words, there exists a set \( \Gamma_p \) of curves so that:

\[
B_p \cap \partial S = \bigcup_{\gamma \in \Gamma_p} \left( \gamma \setminus \{ \text{its endpoint(s)} \} \right) \tag{A3}
\]

where each \( \gamma \in \Gamma_p \) is either a maximal arc or an interior Jordan curve.

There are countably many\(^{11}\) such \( \gamma \)'s, and their total length \( \text{Len}(B_p \cap \partial S) \) is defined and finite. An element of \( \Gamma_p \) can only intersect \( \partial B_p \) at 0, 1 or 2 points. This then excludes any positive-length part of \( \partial B_p \cap \partial S \) from our proofs, which is OK as far as the upper bound in lemma A.2 is concerned. The finiteness of \( \mathcal{J} \) however may be lost. For example, consider \( p = (0, 0) \) and this modified version of ‘topologist’s sine curve’ \( y = |\sin(1/x)| \times x^2 - \frac{1}{2} \) in the \( x-y \) plane.

Figure A1 helps visualisation of the following notions. For points \( p, q \), let \( \overline{pq} \) denote the straight line segment bounded by and including \( p, q \). For any two points \( q_1, q_2 \in \partial B_p \), let \( \overline{q_1q_2} \), denote the piecewise linear segment of the box’s edge \( \partial B_p \) that is bounded by and including \( q_1, q_2 \) so that its length is less than or equal to that of the rest of \( \partial B_p \). For a maximal arc \( \gamma \) relative to \( B_p \) with endpoints \( q_1, q_2 \), let \( \Omega_\gamma \) denote the interior region of the concatenated Jordan curve \( \gamma \cup \overline{q_1q_2} \). For an interior Jordan curve \( \gamma^o \) in \( B_p \), naturally let \( \Omega_{\gamma^o} \) denote the interior region of \( \gamma^o \).

**Proposition A.1.** For any \( \gamma \in \Gamma_p \), the following two statements hold.

\[
\text{Len}(\gamma) < 1 \quad \text{implies} \quad \text{Area}(\Omega_\gamma) < \text{Len}(\gamma). \tag{A4}
\]

\[
p \in \text{cl}(\Omega_\gamma) \text{ for a maximal arc } \gamma \quad \text{implies} \quad \text{Len}(\gamma) \geq 1. \tag{A5}
\]

Intuitively, (A4) helps the proof of lemma A.2 in the sense that if we do not already have \( \text{Len}(\gamma) \geq 1 \) to cover the 1 integer count of \( p \), then for every \( \text{Area}(\Omega_\gamma) \) removed from

\(^{11}\) Consider summability and \( \mathbb{R}^+ = \bigcup_{j \in \mathbb{Z}} [2^j, 2^{j+1}) \).
Area\((B_p) = 1\) (thus not contributing to Area\((S)\)), compensation is provided via the contribution of Len\((\gamma)\) to Len\((\partial S)\). Also, \((A5)\) highlights the challenging case of \(p \notin cl(\Omega_\gamma)\).

**Proof.** Statement \((A4)\) for an interior Jordan curve \(\gamma\) follows from the isoperimetric inequality. Consider maximal arc \(\gamma\) relative to \(B_p\) with endpoints \(q_1, q_2\). By simple geometry, we have:

\[
\text{Len}(q_1q_2) \leq \sqrt{2} \text{Len}(q_1q_2) \leq \sqrt{2} \text{Len}(\gamma).
\]

Then by the isoperimetric inequality and \(\text{Len}(\partial \Omega_\gamma) = \text{Len}(\gamma) + \text{Len}(q_1q_2)\), the above leads to:

\[
\text{Area}(\Omega_\gamma) < \left(1 + \frac{\sqrt{2}}{4}\right) (\text{Len}(\gamma))^2,
\]

which proves \((A4)\).

For the proof of \((A5)\), assume \(p \in cl(\Omega_\gamma)\) but suppose instead \(\text{Len}(\gamma) < 1\). Then \(q_1q_2\) overlaps with at most 2 sides of \(\partial B_p\), with the understanding that overlapping with a corner of \(B\) is on 2 sides. Let \(q_{opp}\) be the point that is symmetric to \(q_1\) about the centre \(p\). See figure A1. The overlapping argument we just made then implies \(q_{opp} \in \partial B_p \setminus q_1q_2\), so \(q_{opp}\) is in the exterior region of the concatenated Jordan curve \(\partial \Omega_\gamma\) since one can path-connect \(q_{opp}\) to any point outside the box without touching \(\partial \Omega_\gamma\). Then, by \(p \in cl(\Omega_\gamma)\) and the Jordan curve theorem, the closed line segment \([pq_{opp}]\) includes a point \(q_{arc} \in [pq_{opp}] \cap \gamma \cap B_p\) (may not be unique). See figure A1. Then,

\[
\text{Len}(q_1q_{arc}) = \text{Len}(q_1p) + \text{Len}(pq_{arc}).
\]

Helped by the geodesic nature of straight lines and \(\text{dist}(p, \partial B_p) = \frac{1}{2}\), this implies:

\[
\text{Len}(\gamma) \geq \text{Len}(q_1q_{arc}) + \text{Len}(q_{arc}q_2) \geq \text{Len}(q_1p) + \text{Len}(pq_2) \geq 1.
\]

Contradiction to \(\text{Len}(\gamma) < 1\) supposed before!

In the following main result on planar integer-point counting, we have to address exceptional cases where an integer point is surrounded by a possibly very small Jordan curve.
Lemma A.2. For open set $S$ introduced in (A1), we have:

$$
\# \{ Z^2 \cap cl(S) \} \leq Area(S) + \text{Len}(\partial S) + |E|,
$$

with the exceptional set:

$$
E := \{ p \in Z^2 \cap cl(S) : \text{there exists an interior Jordan curve } \gamma^o \in \mathcal{F} \text{ so that } p \in cl(\Omega_{\gamma^o}) \text{ and } Area(\gamma^o) + \text{Len}(\gamma^o) < 1 \}.
$$

Proof. It suffices to show, for any $p \in (Z^2 \cap cl(S)) \setminus E$ and $B := B_p$, we have:

$$
B \cap \partial S \neq \emptyset \quad \text{implies} \quad Area(B \cap S) + \text{Len}(B \cap \partial S) \geq 1.
$$

Recall (A3) and the text leading to it. Let $\Gamma := \Gamma_p$. The above then amounts to:

$$
B \cap \partial S \neq \emptyset \quad \text{implies} \quad Area(B \cap S) + \sum_{\gamma \in \Gamma} \text{Len}(\gamma) \geq 1. \quad (A6)
$$

If $p \in cl(\Omega_{\gamma^o})$ for an interior Jordan curve $\gamma^o \in \Gamma$, then $Area(\gamma^o) + \text{Len}(\gamma^o) \geq 1$ due to $p \notin E$, which proves (A6). If $p \in cl(\Omega_{\gamma})$ for maximal arc $\gamma \in \Gamma$, then (A6) follows from (A5).

Therefore from now on, for such $p \in Z^2 \cap cl(S)$, we assume:

$$
p \notin cl(\Omega_{\gamma}) \quad \text{for any } \gamma \in \Gamma, \text{ so } p \in S. \quad (A7)
$$

Define $S_{path} = \{ p_n \in B \setminus \partial S : \exists \text{a path, disjoint from } \partial S \cup \partial B, \text{ that connects } p \text{ to } p_n \} \cup \{ p \}$, which is open since any closed path in $\mathbb{R}^2 \setminus (\partial S \cup \partial B)$ is positively distanced from $\partial S \cup \partial B$. For any point $p_{out} \notin S$, we consider any path that connects $p$ to $p_{out}$ and is parametrised by $\psi : [0, 1] \to \mathbb{R}^2$. Then $\psi(0) = p \in S$ and the openness of $S$ implies that $\{ t \in (0, 1) \mid \psi(t) \subset S \} \neq \emptyset$, so its supremum corresponds to a point on $\partial S$. By the arbitrariness of $\psi$, we find $p_{out} \notin S_{path}$, and thus:

$$
S_{path} \subset B \cap S. \quad (A8)
$$

Next, by the definition of $S_{path}$, (A7) and Jordan curve theorem, we have:

$$
S_{path} \subset B \setminus \left( \bigcup_{\gamma \in \Gamma} \text{cl}(\Omega_{\gamma}) \right). \quad (A9)
$$

On the other hand, it is an intuitive but non-trivial statement to assert that the two sides above are equal, as suggested by the existence of ‘lakes of Wada’, three non-overlapping, connected open sets that share the same boundary! To this end, take any point:

$$
p_1 \in B \setminus \left( \bigcup_{\gamma \in \Gamma} \text{cl}(\Omega_{\gamma}) \right), \quad (A10)
$$

and we prove $p_1 \in S_{path}$ as follows. In fact, parametrise the line segment $pp_1$ as:

$$
\psi(t) = (1 - t)p + p_1 \quad \text{for } t \in [0, 1].
$$

Since $\psi(0) = p \in S_{path}$ which is open, the following is defined:

$$
\tau^- := \sup \{ t \in (0, 1) \mid \psi([0, t]) \cap \partial S = \emptyset \} > 0.
$$

Then $\psi(\tau^-)$ satisfies one and only one of the following.
(i) $\psi(t^-) \notin \partial S$. Then the maximality of $t^-$ and the closedness of $\partial S$ implies $t^- = 1$. Thus, $p$ and $p_1$ are connected by line segment $\psi([0,1])$ which does not intersect $\partial S \cup \partial B$. This proves $p_1 \in S_{\text{path}}$.

(ii) $\psi(t^-) \in \gamma^\psi$ for some $\gamma^\psi \in \Gamma$. Then by definition of $S_{\text{path}}$ and (A9), we have

$$\psi([0,t^-)) \subset S_{\text{path}}, \text{ and thus it is in the exterior of Jordan curve } \partial \Omega_{\gamma^\psi}. \quad \text{(A11)}$$

By (A10), we have $\psi(1) = p_1 \in \mathbb{R}^2 \setminus \text{cl}(\Omega_{\gamma^\psi})$ which is open, so:

$$t^+ := \inf \{ t \in (0,1) \mid \psi([t,1]) \cap \text{cl}(\Omega_{\gamma^\psi}) = \emptyset \} < 1.$$

Then $\psi(t^\pm) \in \gamma^\psi$ and $t^- \leq t^+$. Let $\gamma_{\text{sub}}^\psi \subset \gamma^\psi$ denote the closed sub-curve (possibly degenerate to a point) of $\gamma^\psi$ enclosed between $\psi(t^\pm)$ inclusive. By the definition of interior Jordan curve and definition (A2) of maximal arc, and the fact that $\psi(t^-), \psi(t^+) \subset B$, we have the closed sets $\gamma_{\text{sub}}^\psi$ and $\partial B$ are disjoint, hence

$$d := \text{dist}(\gamma_{\text{sub}}^\psi, \partial B) > 0.$$

Let $B_1$ denote the set of all points in $B$ that are more than $\frac{1}{2}d$ away from $\partial B$. Clearly any point not in $B_1$ is at least $\frac{1}{2}d$ away from $\gamma_{\text{sub}}^\psi$. Any point in $B_1 \cap (\partial S \backslash \gamma^\psi)$ must belong to some $\gamma \in \Gamma$ satisfying (A3) and distinct from $\gamma^\psi$, which implies the closed set $\gamma \cap B_1$ is disjoint from $\gamma_{\text{sub}}^\psi$. Also, since $\gamma$ is either an interior Jordan curve or a maximal arc of length no less that $d$ (due to $\gamma \cap B_1 \neq \emptyset$), the total number of such $\gamma$’s is finite. Then,

$$\text{dist}(\gamma_{\text{sub}}^\psi, \partial B \cup (\partial S \backslash \gamma^\psi)) > 0.$$

Now, by the Jordan-Schoenflies theorem, let $h$ be a homomorphism of $\mathbb{R}^2$ that maps the Jordan curve $\partial \Omega_{\gamma^\psi}$ to the unit circle while preserving the respective exterior (interior) region. By the local uniform continuity of $h^{-1}$ and the above positivity, we have:

$$\text{dist}(h(\gamma_{\text{sub}}^\psi), h(\partial B \cup (\partial S \backslash \gamma^\psi))) > 0.$$

Recall (A11) and $\psi(t^\pm) \in \gamma_{\text{sub}}^\psi$. Also recall that the definition of $t^+$ implies $\psi(t^+)$ is in the exterior of $\partial \Omega_{\gamma^\psi}$. Since topological properties remain valid under $h$, we use continuity of $h \circ \psi(t)$ to find sufficiently small $\epsilon > 0$ so that $h \circ \psi(t^- - \epsilon)$ and $h \circ \psi(t^+ + \epsilon)$ are in the exterior of $h(\partial \Omega_{\gamma^\psi})$ and are sufficiently close to $h(\gamma_{\text{sub}}^\psi)$ which is an arc of the circle $h(\partial \Omega_{\gamma^\psi})$. Thus, the above positive distance and the simple geometry of circle ensures $h \circ \psi(t^\pm \pm \epsilon)$ can be connected by a path disjoint from $h(\partial S \cup \partial B)$. This path can possibly intersect the part of $\psi(t)$ where $t \in [0, t_+ - \epsilon] \cup [t_+ + \epsilon, 1]$, which is OK as far as path-connectedness is concerned. Then, in view of (A11), we show $\psi(t^+ + \epsilon) \in S_{\text{path}}$.

If need be, iterate the above starting from below (A10), but focusing on the line segment between $\psi(t^+ + \epsilon)$ and $p_1$ so that the first step is the definition of a new $\psi(\cdot)$ with the new $\psi(0)$ being $\psi(t^+ + \epsilon) \in S_{\text{path}}$ from the previous iteration and $\psi(1)$ always being $p_1$. Since all $\gamma^\psi$ involved in the iteration form a finite set (because it is either an interior Jordan curve or a maximal curve of length no less than $2 \text{dist}(p_1, \partial B)$), and since the definition of $t^+$ prevents the iteration from intersecting the same $\gamma^\psi \in \Gamma$ twice, we must have the iteration of item (ii) halt in finite steps, after which we must reach item (i), hence showing $p_1 \in S_{\text{path}}$. 

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Therefore we have shown the two sides of (A9) are indeed identical, which implies:

$$\text{Area}(S_{\text{path}}) \geq 1 - \sum_{\gamma \in \Gamma} \text{Area}(\Omega_{\gamma}).$$

Finally, either $\sum_{\gamma \in \Gamma} \text{Len}(\gamma) \geq 1$ which trivially proves (A6) or otherwise, every $\gamma \in \Gamma$ satisfies $\text{Len}(\gamma) < 1$, thus proving (A4) of proposition A.1. Substitute the result into the above inequality and apply (A8) to prove (A6).

\[\square\]

**Appendix B. Elliptic integrals**

The following result on elliptic integrals is elementary but convenient.

**Proposition B.1.** Consider $a > y > b > c > d$ and parameters $\ell^2(a,b,c,d), \ g(a,b,c,d)$ defined in (5.18), (5.19). Then,

$$\int_{(b,c)} \frac{d\lambda}{\sqrt{(\lambda-a)(\lambda-b)(\lambda-c)(\lambda-d)}} = g \int_0^{\lambda_0} \frac{d\lambda_0}{\sqrt{(1-\lambda_0^2)(1-\ell^2\lambda_0^2)}},$$

for $\lambda_0 = \sqrt{\frac{(a-c)(y-b)}{(a-b)(y-c)}} < 1$. (B1)

Also, there exists positive constant $C_1$ independent of $a,b,c,d$ so that:

$$\int_{(d,c) \cup (b,a)} \frac{d\lambda}{\sqrt{(\lambda-a)(\lambda-b)(\lambda-c)(\lambda-d)}} \leq C_1 \ g \cdot (1 - \log \sqrt{1 - \ell^2}).$$

If, in addition, there exist positive constants $C_2, C_3$ so that:

$$\frac{y-b}{a-b} \leq C_2 < 1 \text{ and } \frac{y-b}{b-c} \leq C_3,$$

then a refined estimate:

$$\text{(B1)} \leq C_1 \ g,$$  

(B4)

holds for a positive constant $C_4 = C_4(C_2,C_3)$ that is otherwise independent of $a,b,c,d,y$.

**Proof.** By change of coordinates:

$$\lambda_0 = \sqrt{\frac{(a-c)(\lambda-b)}{(a-b)(\lambda-c)}} = \sqrt{\frac{\Pi_{\Sigma}(a,\ell,\lambda,b,c)}{\Pi_{\Sigma}(a,\lambda,b,c)}} \text{ for any } \lambda \in (b,a),$$

we use identity (5.17) to have:

$$(1 - \lambda_0^2)(1 - \ell^2\lambda_0^2) = \frac{\Pi_{\Sigma}(a,\ell,\lambda,b,c)}{\Pi_{\Sigma}(a,\lambda,b,c)} \left(1 - \frac{\Pi_{\Sigma}(\lambda,b,c,d)}{\Pi_{\Sigma}(\lambda,b,c,d)}\right) = \left(\frac{a-\lambda}{a-b}\right)\left(\frac{b-c}{\lambda-c}\right)^2.$$

Also, clearly

$$d\lambda_0 = \frac{1}{2} \sqrt{\frac{a-c}{a-b}} \sqrt{\frac{\lambda-c}{\lambda-b}} \frac{b-c}{(\lambda-c)^2}.$$

Therefore we prove (B1). The $y_0 < 1$ part is due to the rearrangement inequality (or (5.17)).

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Next, in (B1), we let \( y \nearrow a \) or equivalently \( y_0 \nearrow 1 \), relax the factor \((1 + \delta_0)(1 + \ell \delta_0) \geq 1\) on the right-hand side and then treat the cases of \( 0 < \ell^2 < \frac{1}{2} \) and \( \frac{1}{2} \leq \ell^2 < 1 \) differently to prove:

\[
\int_{(b,a)} \frac{d\lambda}{-(\lambda-a)(\lambda-b)(\lambda-c)(\lambda-d)} \leq \text{right-hand side of (B2)}.
\]

Apply this result to \(-d > -c > -b > -a\), noting \( \ell^2(-d,-c-a,-b) = \ell^2(b,a,c,d) \) and \( g(-d,-c,a,b) = g(a,b,c,d) \), to show the entire (B2).

B4 follows from relaxing \( 1 - \ell^2 \lambda_0^3 > 1 - \lambda_0^2 \) in (B1), and then using identity and estimate:

\[
1 - y_0^2 = \frac{1 - \frac{y-b}{a-b}}{b-a} + 1 \geq \frac{1 - C_2}{C_3 + 1} > 0.
\]

\( \square \)

Appendix C. Optimalty of choice of bandwidth \( \delta \) and integer count

We establish lower bounds for the number of \( k \) associated with \( N[\delta_{\cdot\cdot}]_{\text{loc}} \) (defined in (4.3)) at a fixed \( n \) and under assumption (1.6) on \( \delta_\cdot \). This will be done via direct counting, and not via volume estimate. If one accepts that estimate (3.1) with \( \beta = 1 \) and \( M_{\text{loc}} = N[\delta_{\cdot\cdot}]_{\text{loc}} \) is a condition that is not only sufficient but also technically necessary for the proof of theorem 1.5 to work, then the ‘optimality of (1.7)’ means that (1.7) being valid asymptotically for arbitrarily large wavenumbers is necessary for (3.1) with \( \beta = 1 \) and \( M_{\text{loc}} = N[\delta_{\cdot\cdot}]_{\text{loc}} \) to hold.

Set \( L_1 = L_2 = L_3 = 1 \) for simplicity. By Theorems 4.2 and 5.1, the count of integer points \( k \) as in \( \sum_{k \in \mathbb{Z}} 1_{N[\delta_{\cdot\cdot}]_{\text{loc}}} (n,k,-n-k) \) is shown to have an upper bound at the order of:

\[
|n|^2 + |n|^3 \delta + |n|^3 \delta \log + \frac{1}{\delta + \frac{2\pi n}{|n|}}, \quad \text{for any large } |n|\text{ and } \delta = \delta_\cdot(|n|,|n|,|n|).
\]

(C1)

In the following main results of this appendix, we provide examples to show that the same type of integer-point count has lower bounds of the same order for \( n_3 = 0 \) and \( n_3 \neq 0 \) (with a change in the latter case that is justified in the remarks), hence showing that if \( \delta \) violates (1.7) then (3.1) with \( \beta = 1 \) cannot hold.

**Proposition C.1.** Fix \( \delta \in (0,\frac{1}{2}) \) and \( n \in \mathbb{Z}^3 \). The number of all \( k \in \mathbb{Z}^3 \) subject to constraints:

\[
|n| \geq |k| \geq |n+k| \geq \frac{1}{3} |n|,
\]

(C2)

(so all 3 wavenumbers are ‘localised’), and

\[
\left| \frac{n_3}{|n|} + \frac{k_3}{|k|} - \frac{n_3 + k_3}{|n+k|} \right| \leq \delta,
\]

(C3)

for large enough \( |n| \) is at least of order:

(i) \( |n|^2 + |n|^3 \delta \log \frac{1}{\delta} \) under additional constraints \( n_3 = 0 \) and \( k_3 \neq 0 \);

(ii) \( |n|^2 \min \left\{ 1, (|n|^2)^{\frac{3}{2}} \right\} + |n|^3 \delta \min \left\{ \log \frac{1}{\delta}, \delta |n| \log |n| \right\} \) under additional constraints \( n_3k_3(n_3 + k_3) \neq 0 \) and \( 1 \lesssim |n|^2 \).

The first lower bound also holds for \( \delta = 0 \) if \( 0 \log 0 = 0 \) is understood.

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We make some remarks. By dispersion relation \((2.10)\) for the large operator \(\Omega L\) that is responsible for fast waves, any term in eigen-expansion \((2.18)\) with purely horizontal wavenumber, i.e. \(n_3 = 0\), is independent of the fast time \(\tau\), so it is called a slow mode in the dichotomic framework used in most literature whereas any mode with \(n_3 \neq 0\) is regarded fast. In the case of three slow/horizontal modes, the \(n_3 = k_3 = m_3 = 0\) condition makes the counting elementary, thus we omit the detail. There does not exist any triplet of one fast and two slow modes due to \(n_3 + k_3 + m_3 = 0\). Then proposition C.1 addresses both triplets of slow-fast interaction and fast-fast interaction—we made an effort on this separate treatment as it appears frequently in literature. The estimate for either case alone can justify the use of logarithmic factor in \((1.7)\). Also note the result for former case attests to the optimality of upper bound \((C1)\) for every \(\delta \in \left[0, \frac{1}{2}\right)\) and the result for the latter case deviates from \((C1)\) for \(|n|^{-2} \leq \delta \ll |n|^{-1}\). Such degeneracy is however expected. In fact, it can be rigorously argued in the limiting case \(\delta = 0\) for which it was shown in \([24, \text{lemma 4.1}]\) that the number of integer solution to the Diophantine equation associated with purely fast interaction has an \(O(|n|^{1+\epsilon})\) upper bound for any \(\epsilon > 0\), therefore it is not possible to close the gap with \((C1)\). We should note that this result does not apply to all aspect ratios of the domain \((|24, \text{remark 4.2})\).

**Proof.** For case (i), let \(N\) be any large positive integer and \(n = (-2N, 0, 0)\). Consider set

\[
\left\{ (k_1, k_2, k_3) \in \mathbb{Z}^3 : k_1 \in \left[ N, N + \frac{N}{3} \min \left\{ \frac{\delta |n|}{2k_3}, 1 \right\} \right] \text{ and } \max \{|k_2|, |k_3|, |k_1| \} \leq N, k_3 \neq 0 \right\}. \tag{C4}
\]

It is elementary to show any \(k\) from this set satisfies \((C2)\) and also, for \(m := -n - k\), we have:

\[
|k| |m| \geq k_1 m_1 = k_1(2N - k_1) \geq \frac{8}{9} N^2, \quad \text{and} \quad |k| + |m| \geq |n|. \tag{C5}
\]

Then any such \(k\) satisfies \((C3)\) since

\[
-\frac{n_3}{|n|} \frac{k_3}{|k|} \frac{m}{|m|} = \frac{k_3 (|k| - |m|)}{|k| |m|} = \frac{4k_3 N (k_1 - N)}{|k| |m| (|k| + |m|)} \leq \left[ -\frac{3}{4} \delta, \frac{3}{4} \delta \right]. \tag{C6}
\]

At a fixed \(k_3\), the count of integer pairs \((k_1, k_2)\) admissible by \((C4)\) is:

\[
(2N + 1) \left( 1 + \frac{N}{3} \min \left\{ \frac{\delta N}{k_3}, 1 \right\} \right).
\]

With \(1 + |x| > \frac{1}{2} (1 + x)\) for \(x \geq 0\), its \(k_3\)-sum for \(|k_3| \in [1, N]\) can be bounded from below by:

\[
(2N + 1) N + \frac{(2N + 1) N}{3} \int_1^{N+1} \min \left\{ \frac{\delta N}{k_3}, 1 \right\} \, dk_3.
\]

The integral equals \(\delta N \log(N + 1)\) if \(0 < \delta N < 1\), and equals \(\delta N \log \frac{N + 1}{N} + (\delta N - 1)\) if \(\delta N \geq 1\). By using \(\log x \geq -e^{-1}\), we lower the integral value in the \(0 < \delta N < 1\) case as:

\[
\delta N \log(N + 1) > \delta N \log \frac{1}{\delta} + \delta N \cdot \log(\delta N) \geq \delta N \log \frac{1}{\delta} - e^{-1},
\]

which clearly also works as lower bound for the \(\delta N \geq 1\) case. Noting \(|n| = 2N\), we prove the first case of the proposition which also holds for \(\delta = 0\) if \(0 \log 0 = 0\) is understood.

For case (ii), one may consider a proof by treating it as perturbation of case (i). In a nutshell, let \(n' = (-2N, 0, \alpha)\) and \(k' = (k_1, k_2, k_3 - \alpha)\) for any \((k_1, k_2, k_3)\) from \((C4)\). With \(m' = -n' - k'\), estimate the difference between \((C6)\) and

\[
-\frac{n}{|n'|} - \frac{k'}{|k'|} - \frac{m'}{|m'|} \at \alpha = 1 \text{ by considering the}
\]
\( \alpha \) derivative and showing that the difference is \( O(|n|^{-1}) \). Then, using the first case, one can prove the second case for as long as \( 1 \leq \delta |n| \). To cover a wider parameter range as claimed in the proposition, we adopt the following more refined approach.

For large integer \( N \), let \( n = (-2N, 0, -1) \) and consider set:

\[
\left\{ (k_1, k_2, k_3) \in \mathbb{Z}^3 : \frac{|n|}{1 + \delta |n|} \leq |k| \leq |n|, k_1 \in \left[ N, N + \frac{N}{3} \min \left\{ \frac{\delta |n|}{2(k_3 - 1)}, 1 \right\} \right] \text{ and } k_3 \in \left[ 2, \frac{N}{3} \min \left\{ \sqrt{\delta |n|}, 1 \right\} \right] \right\} \tag{C7}
\]

It is elementary to show any \( k \) from this set satisfies (C2). Next, with \( m := -n - k \), consider:

\[
-\frac{n_3}{|n|} - k_3 \frac{m_3}{|m|} = f^{\delta}(k) + f^{\bar{\delta}}(k) := \left( \frac{1}{|n|} - \frac{1}{|k|} \right) + \left( \frac{k_3 - 1}{|m|} - \frac{k_1 - 1}{|k|} \right).
\]

By (C2), we find \( f^\delta < 0 \) and \( f^{\bar{\delta}} > 0 \). By the first inequality of (C7), we have \( f^\delta > -\delta \). For the \( f^{\bar{\delta}}(k) \) term, we use the \( \delta \)-dependent upper bounds of \( k_1, k_3 \) in (C7) to have:

\[
f^{\bar{\delta}}(k) = (k_3 - 1) \cdot \frac{|k| - |m|}{|k||m|} = (k_3 - 1) \cdot \frac{4N(k_1 - N) + 2k_3 - 1}{|k||m||k + m|} \leq \frac{1}{6} N^2 \delta |n| + \frac{1}{2} N^2 \delta |n| |k||k + m| \leq \frac{\delta |n|^2}{1 + \delta |n|^2} |k||k + m|.
\]

Since it is easy to show (C5) still holds, we find \( f^\delta \leq \delta \), hence proving (C3).

We move on to counting of (C7) where the constraints on \( k_1, k_3 \) are independent of \( k_2 \). Then, at any such admissible \( k_1, k_3 \), the range of \( |k_3| \) is of length:

\[
\sqrt{|n|^2 - k_1^2} - \sqrt{|n|^2 - k_3^2} = \left( \frac{|n|^2}{|n|^2 - k_1^2} - k_1^2 - k_3^2 \right)^+.\]

By the first case, we have \( \delta |n|^2 \leq k_1^2 + k_3^2 \), thus:

\[
\sqrt{|n|^2 - k_1^2} - \sqrt{|n|^2 - k_3^2} \leq \delta |n|^2 \cdot \left( \frac{1}{1 + \delta |n|^2} - k_1^2 - k_3^2 \right)^+.
\]

where we used \( \frac{1}{1 + \delta |n|^2} < k_1^2 + k_3^2 + \frac{1}{2} |n|^2 \). This last bound also implies that the assumption for the second case above is only possible when \( \delta |n| < 1 \), a condition that can be used to show that the second lower bound is \( O(\delta |n|^2) \). Note there are at least \( (b - a) \) integers in \( [a, b] \). In summary, at any \( (k_1, k_3) \) admissible by (C7), the number of integer \( k_2 \) has a lower bound:

\[
\min \left\{ 1, \delta |n| \right\} O(|n|), \quad \text{provided } 1 \leq \delta |n|^2.
\]

Independently, we estimate \( \sum k_1 \sum k_3 1 \), the count of integer pairs \( (k_1, k_3) \) admissible by (C7). When \( \delta |n| \geq 1 \), we use \( 1 + |x| > \frac{1}{2}(1 + x) \) for \( x \geq 0 \) to find a lower bound as:

\[
\sum_{k_3 = 2}^{\lfloor \delta |n|/2 \rfloor} \left( 1 + \frac{N}{3} \right) + \sum_{k_3 = \lfloor \delta |n|/2 \rfloor + 1}^{\lfloor N/3 \rfloor} \left( 1 + \frac{N}{3} \frac{\delta |n|}{2(k_3 - 1)} \right) > \frac{1}{2} \left( \frac{N}{3} + \frac{N}{6} \right) \left( \frac{1}{2} \delta |n| - 1 \right)^+ + \frac{N}{12} \left( \int_{\delta |n|/2}^{\lfloor N/3 \rfloor} \frac{\delta |n|}{k_3 - 1} \, dk_3 \right)^+.
\]
At large $N$, the right-hand side is of order $N + N^2\delta + N^2\delta \log^+ \frac{1}{\delta N}$ which has a lower bound of order $|n| + |n|^3 \log^+ \frac{1}{|n|}$ since $|n|^2 = 4N^2 + 1$ and $\delta \in (0, \frac{1}{2})$. The case when $\delta |n| < 1$ is estimated similarly with a lower bound on $(\sum_{k_3=1}^{\infty} \frac{1}{k_3})$ to be:

$$\sum_{k_3=2}^{\infty} \left( 1 + \frac{N}{2} \log \frac{|n|}{\delta |n|} \right) > \frac{N}{2} \log \frac{|n|}{\delta |n|} + \frac{N}{12} \int_2^{\infty} \frac{\log \frac{|n|}{\delta |n|}}{k_3-1} dk_3.$$  

Using $\sqrt{\delta |n|} \log \sqrt{\delta |n|} \geq -e^{-1}$, we can show the right-hand side has an asymptotic lower bound of order $N\sqrt{\delta |n|} + N\delta |n| \log^+ \frac{1}{\delta N}$. The proof of case (ii) is complete. \hfill \Box

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