Node differentiation dynamics along the route to synchronization in complex networks

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Synchronization has been the subject of intense research during decades mainly focused on determining the structural and dynamical conditions driving a set of interacting units to a coherent state globally stable. However, little attention has been paid to the description of the dynamical development of each individual networked unit in the process towards the synchronization of the whole ensemble. In this paper, we show how in a network of identical dynamical systems, nodes belonging to the same degree class differentiate in the same manner visiting a sequence of states of diverse complexity along the route to synchronization independently on the global network structure. In particular, we observe, just after interaction starts pulling orbits from the initially uncoupled attractor, a general reduction of the complexity of the dynamics of all units being more pronounced in those with higher connectivity. In the weak coupling regime, when synchronization starts to build up, there is an increase in the dynamical complexity whose maximum is achieved, in general, first in the hubs due to their earlier synchronization with the mean field. For very strong coupling, just before complete synchronization, we found a hierarchical dynamical differentiation with lower degree nodes being the ones exhibiting the largest complexity departure. We unveil how this differentiation route holds for several models of nonlinear dynamics including toroidal chaos and how it depends on the coupling function. This study provides new insights to understand better strategies for network identification and control or to devise effective methods for network inference.

Keywords: Complex networks — Dynamical complexity — Route to synchronization — Chaos

I. INTRODUCTION

The description of networks can be structural, based on the characterization and modelling of their topological properties,1–3 or dynamical, usually referring to the collective behavior emerging from the interaction between node dynamics and the network architecture. Synchronization is the most thoroughly investigated ensemble dynamics,2–5 including a rich variety of related behaviors such as chimera states,6–8 cluster synchronization,9–11 or explosive synchronization.12

The route to the synchronous state has been approached from different points of view. Microscopically, it has been analyzed the different ways local synchronization grows as the coupling increases depending on the connectivity structure.4,12–14 Globally, the synchronizability of a population of identical networked oscillators, that is, the stability of the coherent state, has been tackled through the master stability function and foresees the emergence of patterns when the stability of the network uniform state is lost.15–17

While these approaches help to understand how synchronization clusters grow and eventually merge into a macroscopic coherent state or predict its stability for a particular network structure and coupling function, few works pay attention to the description of the nodal dynamics along the route to synchronization. Recently, it has been shown that the degree of complexity of the node dynamics is strongly dependent on the connectivity; in particular, the local connectedness appears to confer considerable node differentiation hallmarks, albeit following mechanisms that are intricate and delineate a nonmonotonic effect.18,19 These ensemble fingerprints in the node dynamics can be used to infer the network statistical description, or even the detailed connectivity in certain conditions.20 By node differentiation it is here meant as the dynamical departure of an isolated node due to the interaction with its neighborhood, being more pronounced in networks with nodes having degree heterogeneity. In particular, a relatively large range of coupling strengths over which nodes with a higher degree have a less complex dynamics has been reported.19 ex-
plaining the low complexity observed in hubs of functional brain networks. However, the reverse situation, the existence of an even lower range of coupling strengths where high degree nodes are more complex than lower degree ones, remains under study. The purpose of the present work is to elucidate the route of the node differentiation dynamics in a complex network before reaching a synchronous state, and, in particular, how it depends on the dynamical system, the coupling function and on the topological properties of the network structure.

In order to characterize and quantify this route to node differentiation dynamics as the coupling strength increases, we used the complexity measure recently introduced by Letellier et al., able to distinguish organized from disorganized chaotic behavior together with maps of the dynamical patterns associated to relevant degree classes. Section II is devoted to introducing terminology and providing a brief description of the dynamical complexity measure. We then investigated in Section III the dynamical differences among Rössler oscillators in a star network according to their degree, and how those differences dependent upon the coupling function, the nominal dynamics, and the size of the star. In Section IV we extended the analysis to star networks of dynamical systems featuring higher nominal complexity, namely the symmetric Lorenz system, the high-dimensional Mackey-Glass equation, and the toroidal Saito system, to determine whether a general scenario can be outlined. In Section V, we considered Rössler systems in larger networks to explore the node differentiation induced by embedding the oscillators in a much more complex topological environment. Finally, Section VI offers general conclusions.

II. DYNAMICAL COMPLEXITY MEASURE

Let us start by considering the general description of a network comprising $N$ diffusively-coupled $m$-dimensional identical dynamical systems, whose state vector $\mathbf{x}_i$ evolves as

$$\dot{x}_i = f(x_i) - d \sum_{j=1}^{N} L_{ij} h(x_j)$$

where $f: \mathbb{R}^m \rightarrow \mathbb{R}^m$ and $h: \mathbb{R}^m \rightarrow \mathbb{R}^m$ are the nominal dynamics — here, nominal refers to the vector field of the $i$-th nodal dynamics when isolated — and the coupling function, respectively, and $d$ is the coupling strength. $L_{ij}$ are the elements of the Laplacian matrix encoding the network’s connectivity, with $L_{ii} = k_i$ the node degree, $L_{ij} = -1$ if nodes $i$ and $j$ are connected, and $L_{ij} = 0$ otherwise.

To compute the nodal dynamics in a more reliable way, we computed a Poincaré section of the attractor when the dynamics has a toroidal structure, or a first-return map built with one of the coordinates of the Poincaré section in case it is non-toroidal. Then, each oscillator’s map is used to obtain a complexity coefficient $C_D$ introduced in Ref. and defined as

$$C_D = S_p + \Delta,$$

being $S_p$ a permutation entropy and $\Delta$ a structurality marker. We chose this complexity definition because of its ability to distinguish organized unpredictable behavior (dissipative chaos) from disorganized unpredictable phenomena (noise or conservative chaos). While the entropy quantifies the unpredictability of the dynamics, the structurality accounts for its undescribability, two very different aspects contributing to the complexity of a dynamics.

The permutation entropy $S_p$ is defined as in Ref.

$$S_p = -\frac{1}{\log(N_q)} \sum_{\pi=1}^{N_q!} p_{\pi} \log p_{\pi} \in [0,1].$$

It is based on the probability distribution $p_{\pi}$ of the $N_q!$ possible ordinal patterns constructed from the order relations of $N_q$ successive data-points in the Poincaré section. The structurality is computed by dividing the first-return map (or the two-dimensional projection of the Poincaré section) into a $N_q \times N_q$ boxes and

$$\Delta = \frac{1}{N_q} \sum_{i,j=1}^{N_q} q_{ij} \in [0,1],$$

where $q_{ij} = 1$ if the box $(i,j)$ is visited at least once, and $q_{ij} = 0$ otherwise. The structurality $\Delta$ is typically low ($\Delta < 0.2$) for organized dynamics, and large ($\Delta > 0.80$) for disorganized dynamics. We thus have $C_D \approx 0$ for a limit cycle ($S_p \approx 0$ and $\Delta \approx 0$), $0.5 < C_D < 1.0$ for dissipative chaotic behavior (organized but unpredictable), and $C_D > 1.5$ for weakly dissipative chaos or stochastic processes (both are unpredictable and disorganized).

In all our computations we used $N_q = 6$ as the length of the ordinal patterns. The number $N_d$ of data points in the Poincaré section to compute $S_p$ and $\Delta$ was $N_d = 10,000$, largely above the requirement $N_d > 5N_q!$ recommended by Riedl et al. The number of boxes was $N_q = 5 \log(N_q)$ chosen such that $N_d \gg N_q$, as prescribed in Ref.

The box width $\delta_p$ is determined by the largest range visited in our simulations for a given system as $\delta_p = (x_{\max} - x_{\min})/N_q$, where $x_{\max}$ and $x_{\min}$ are the maximal and minimal values recorded along one axis of the first-return map. To avoid excessively promoting noise contamination or small fluctuations around period-1 limit cycle, we introduced a “noise filter” to interpret the ordered patterns of the $N_q$ data points to allow permutation only if $|x_i - x_j| > \delta_p$.

Let us consider the paradigmatic Rössler system to exemplify the ability of Eq. to discriminate between different dynamics. The vector flow in Eq. is

$$f(x) = [-y - z, x + ay, b + z(x - c)]$$
taking \( b = 0.2 \) and \( c = 5.7 \) as fixed parameters and \( a \in [0.08, 0.38] \) as the bifurcation parameter. Equation (1) was integrated using a fourth-order Runge-Kutta scheme\(^{23}\) with a time step \( \delta t = 0.01 \). Initial conditions are randomly selected within a small neighborhood of radius 0.5 centered at the origin of the state space. Figure 1(a) shows the bifurcation diagram as a function of the parameter \( a \) for a single Rössler system (\( d = 0 \)) computed from the \( y_n \) coordinate of the Poincaré section
\[
\mathcal{P}_x = \{(y_n, z_n) \in \mathbb{R}^2 \mid x_n = x_-, \dot{x}_n > 0\},
\]
where \( x_- = (c - \sqrt{c^2 - 4ab})/2 \) is the coordinate of the inner singular point.\(^{22}\) The classical period-doubling cascade as a route to chaos is quantitatively characterized by the dynamical complexity \( C_D \) in Fig. 1(b). It perfectly captures the periodic windows (\( C_D < 0.2 \)) intermingled with chaotic behavior. Panels (c)-(e) show the first-return maps for the three \( a \) values marked in (a)-(b) with vertical lines. The first value, \( a = 0.26 \) is located between a period-3 and a period-2 window and its first-return map is bimidal [Fig. 1(c)] but with a slightly developed third branch appearing just after the period-3 window. For \( a = 0.31 \), Fig. 1(d), the bimodal map has the three branches. A more developed chaos, characterized with three monotone branches, is observed for \( a = 0.38 \) in Fig. 1(e). In particular, the right end of the third branch approaches the vicinity of the bisecting line, indicating that a new period-1 orbit is about to be created in the population of unstable periodic orbits. This is the most developed chaos that can be observed along this line of the parameter space, as indicated by the dynamical complexity \((C_D = 0.89)\), in agreement with the most developed first-return map.

**III. A STAR NETWORK OF RÖSSLER SYSTEMS**

Let us now consider a star network of \( N = 16 \) coupled Rössler oscillators in the \( d-a \) parameter plane. Actual complexity change of the hub \( C_{\text{hub}} - C_0 \) (a)-(b) and the difference \( C_{\text{hub}} - (C_{\text{leaves}}) \) (c)-(d) when oscillators are coupled through the \( y \) (a,c), and \( x \) (b,d) variables, respectively. Bottom panels (e)-(f) show the synchronization error \( E \). The color scales are provided on the right sides. In (a)-(f) panels, white curves are the corresponding null isolines, and horizontal dashed lines in (a)-(b) are plotted at \( a = 0.26, 0.31 \) and \( a = 0.38 \). White domains correspond to the ejection of the trajectory to infinity. Other parameter values are set as in Fig. 1.

![FIG. 1. Dynamical characterization of a single Rössler system upon variation of the \( a \) parameter and fixing \( b = 0.2 \) and \( c = 5.7 \). Bifurcation diagram using the Poincaré section \( \mathcal{P}_x \) (a) and dynamical complexity \( C_D \) (b). The dashed vertical lines in (a) and (b) are located at \( a_1 = 0.26 \), \( a_2 = 0.31 \), and \( a_3 = 0.38 \). The corresponding first-return maps from \( y_n \) are shown in (c)-(e) and their complexity values at the bottom of each panel.](image)

![FIG. 2. Colormaps of the complexity values for a star of \( N = 16 \) coupled Rössler oscillators in the \( d-a \) parameter plane. Actual complexity change of the hub \( C_{\text{hub}} - C_0 \) (a)-(b) and the difference \( C_{\text{hub}} - (C_{\text{leaves}}) \) (c)-(d) when oscillators are coupled through the \( y \) (a,c), and \( x \) (b,d) variables, respectively. Bottom panels (e)-(f) show the synchronization error \( E \). The color scales are provided on the right sides. In (a)-(f) panels, white curves are the corresponding null isolines, and horizontal dashed lines in (a)-(b) are plotted at \( a = 0.26, 0.31 \) and \( a = 0.38 \). White domains correspond to the ejection of the trajectory to infinity. Other parameter values are set as in Fig. 1.](image)
in Fig. 2 show the time-averaged synchronization error computed as

$$E = \frac{1}{N(N-1)} \sum_{i \neq j} \| x_i - x_j \| .$$

which is in agreement with the prediction given by the master stability function (MSF) for Rössler systems coupled through the $y$ variable, type I synchronizability class -the MSF becomes negative above a critical value, and $x$ variable, type II class -the MSF is negative in a bounded interval. For the particular network structure of a star of size $N = 16$, complete synchronization is indeed reached for a larger number of pairs $(a,d)$ when Rössler systems are $y$-coupled than when $x$-coupled. In the latter case, full synchronization cannot be reached for $a \gtrsim 0.25$ before a boundary crisis ejects the trajectory to infinity (white domains in Fig. 2(f)). For this configuration, the location of $\lambda_i$, being $\lambda_i$ the $i$-th eigenvalue of the Laplacian matrix, is outside the range where the MSF is negative for any value of $d$, and the synchronous solution is always unstable.

Roughly, in the regions where the synchronization error is not null, for both $x$ and $y$ coupling, there is a large region of the parameter space with yellowish areas delimited by the white null isolines in panels in Figs. 2(a) and 2(b) where the effect of coupling is to render the dynamics of the hub more complex with respect to its uncoupled regime ($C_{\text{hub}} > C_0$). Outside this region, there are smaller islands (in blue) where the dynamics turns out to be less complex. However, when comparing this departure from the uncoupled dynamics between the hub and the leaves, Figs. 2(c) and 2(d), there are regions where the hub dynamics appears to be more complex than that of the leaves — for intermediate values of the coupling and of the parameter $a$ — and regions where the opposite is realized. For example, for the $y$ coupling, $C_{\text{hub}} < \langle C_{\text{leaves}} \rangle$ at the left and right side of the yellow region.

To analyse with further detail the dynamical differentiation of hub and leaves along the route to synchronization, we show in Fig. 3 cuts of the complexity difference $C_{j0} = C_j - C_0$, with $j = \{\text{hub, leaf}\}$ (black for the hub and red for one leaf) along the three horizontal dashed lines shown in Fig. 2(a). These curves clearly show that hub and leaf experience unlike differentiation routes before either the star synchronizes at $d_c$ to the same dynamical state as in $d = 0$ for $y$-coupling [Fig. 3(a)-(d)] or it reaches a boundary crisis at $d_b$ for the $x$ coupling [Fig. 3(e)-(h)]. In all cases, the maximal differentiation occurs always before for the hub than for the leaf, that is, the maximum of the $C_{j0}$ curves is located at a coupling that is lower for the hub than for the leaf. As it is shown at the inset of each panel, this shift coincides with the earlier synchronization of the hub to the mean field, here measured as

$$S_j = \langle \text{Re}(e^{i(\theta_j - \Phi)}) \rangle_t,$$

FIG. 3. Evolution of the relative complexity $C_{j0}$ as a function of the coupling strength $d$ for the hub (in black) and a leaf (in red) of a star network of $y$ coupled (a)-(d) and $x$ coupled (e)-(h) Rössler oscillators. (a,e) $a = 0.26$, (b,f) $a = 0.31$, and (c,g) $a = 0.38$ and $N = 16$. (d,h) $a = 0.31$ and $N = 32$. Insets in all panels show the phase synchronization $S_j$ of the hub (black) and the leaf (red) with the mean field. Other parameter values: $b = 0.2$, and $c = 5.7$. One run per $d$-value.
being \((\ldots)\) the time average, \(\Re\) stands for the real part, \(\theta_j = \arctan(y_j/x_j)\) the phase of the \(j = \text{hub/leaf}\), and \(\Phi\) the mean-field global phase. By definition, \(S_j \in [0, 1]\), and approaches 1 as the phase of the \(j\)-th oscillator is locked to the phase of the mean field.

Another observed systematic behavior is that, for increasing values of \(a\), as the uncoupled dynamics is more developed (see Fig. 1), the relative complexity of the hub diminishes while increases for the leaf. Therefore, the relative position between the hub and leaf curves changes such that for \(a = 0.26\) the hub’s curve is almost always above the leaf’s one, \(C_{\text{hub},0} > C_{\text{leaf},0}\), while for \(a = 0.38\), when the isolated dynamics is the most developed, the hub’s curve is always below, \(C_{\text{hub},0} < C_{\text{leaf},0}\). Another salient feature is that in the very weak coupling regime \((d < 0.05)\), just after oscillators start interacting, there is a marked fall of the complexity with respect to \(C_0\) for both hub and leaf. Finally, the impact of the size of the star is analysed in Fig. 3(d,h) for \(N = 32\) and \(a = 0.31\) which has to be compared with the panels (b) and (f) of the same figure for \(N = 16\). Precisely, the hub complexity is slightly affected and only in the region of low coupling. As predicted by the MSF, for the \(y\) coupling, the threshold for synchronization \(d_c\) does only depend on the smallest non-zero eigenvalue \(\lambda_2\), which equals \(\lambda_2 = 1\) in both configurations. On the other hand, for the \(x\) coupling, the boundary crisis is anticipated to smaller coupling values as the largest eigenvalue \(\lambda_N = N\) increases with the size.

To picture the different dynamical states the hub and the leaf are visiting, we monitored their first-return maps at a specific points along the route to synchronization as the leaves are visiting, we monitored their first-return maps with the size. \(\lambda\) is the largest eigenvalue \(\lambda_1\) and \(\lambda_2\) the smallest non-zero eigenvalue \(\lambda_0\). The two increasing branches correspond to the reinjection of the trajectory into the wing from which it is issued, while the two decreasing branches are associated with the transition from one wing to the other. The two left (right) branches correspond to the \(n\)th intersection in the left (right) wing and to the \((n + 1)\)th intersection in the left (right) or right (left) wing depending on the sign of the slope.

Figure 5 shows the relative complexity \(C_{j,0}\) of the hub and a leaf of a \(N = 32\) star network \(y\)-coupled Lorenz systems for two values of the parameter \(R\). For \(R = 28\), full synchronization is reached for \(d_c \approx 2.22\). As with the Rössler systems, the complexity of the hub and leaves drop below \(C_0\) in the weak coupling regime \((d \lesssim 1)\) and beyond that point the hub becomes more complex than the nominal dynamics before full synchronization is eventually reached. For this parameter setting, the Lorentz hub experiences a sudden drop in complexity as illustrated in the first-return map [Fig. 5(A)]: while the hub dynamics is characterized by small fluctuations around the singular point in the centre of one of the wings, the leaves are nearly unperturbed with their four branches.
The nominal complexity is $C = 0.82$.

When $d$ is further increased [Fig. 5(B)], the leaf dynamics also collapses, with all nodes surprisingly exhibiting a quasi-periodic dynamics, a regime not observed in an isolated Lorenz system. Notice that the size of the two tori are different. When the nodes’ dynamics have topologically equivalent attractors (here, tori) but with a scaling factor, the phenomenon is known as *amplitude envelope synchronization*.[16] Finally, beyond this point of reduced dynamics and just before full synchronization [Fig. 5(C)], the interaction between hub and leaves moves the dynamics of the leaves again to the four paired branches although slightly more noisy than the uncoupled one, while the hub is strongly perturbed with a very disorganized dynamics. This scenario resembles the one observed for the Rössler system depicted in panel D of Fig. 4. Finally, we explored a second regime with an even more developed uncoupled dynamics for $R = 175$ whose first-return map has six branches (not shown). As in Fig. 3(c) for the Rössler system, the switch between the relative complexities of hub and leaf is no longer observed in Fig. 5 and the leaf curve is always above $C_{leaf,0} > C_{hub,0}$.

### B. Mackey-Glass delay differential equation

It is possible to produce a chaotic attractor characterized by a smooth unimodal map via one-dimensional delay differential equations, for instance, the Mackey-Glass (MG) equation\[10,11\]

\[
\dot{x} = \mu \frac{x_{\tau}}{1 + x_{\tau}^p} - x_t
\]  

(11)

where $x$ is the population of blood cells, $x_t = x(t)$ and $x_{\tau} = x(t - \tau)$. This equation was initially proposed for the control of hematopoiesis (the production of blood cells). Typically, the delay $\tau$ is the time-scale for proliferation and maturation of these blood cells. The di-

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**FIG. 4.** First-return maps to the Poincaré section $\mathcal{P}$ for the hub (top row) and one of the leaves (bottom row) in a star network of $N = 16 \gamma$-coupled Rössler oscillators for $a = 0.31$ and for different $d$ values. The corresponding complexities $C_D$ are reported. The nominal complexity is $C_0 = 0.82$.

**FIG. 5.** (a)-(b) Relative complexity for the hub (black) and one of the leaves (red) as a function of the coupling strength $d$ for a star network with $N = 32 \gamma$-coupled Lorenz systems for two $R$-values. First-return maps of the hub (top) and of one leaf (bottom) for the cases A, B and C indicated by the blue dashed lines in the top panel. Other parameter values: $\sigma = 10$, $b = \frac{8}{3}$. A magnified view of the maps is plotted for the case B: the range used is not the same between the hub and leaf.
mension of the effective state space associated with a delay differential equation is dependent on the delay. Parameter values for parameters $\mu$ and $p$ are such that a period-doubling cascade as a route to chaos is observed [Fig. 6(a)]. We chose a delay $\tau = 4.8$ (red dashed line in Fig. 6(a)) whose attractor is characterized by a smooth unimodal map with equivalent topological properties to the Rössler system. Increasing $\tau$ further leads to a much more complex behavior and computing a reliable Poincaré section is rather tricky. Our goal here is to investigate the route to synchronization for unimodal dynamics produced by a potentially high-dimensional system. A network of MG systems can be fully synchronized using bidirectional coupling.

The attractor is bounded by a genus-1 torus and the single-component Poincaré section can be defined as

$$\mathcal{P}_{MG} \equiv \{x_n, \in \mathbb{R} \mid \dot{x}_n = 0.025, \bar{x}_n > 0, x_n < 0.9\}. \quad (12)$$

The first-return map is one-dimensional and slightly foliated and quite similar to the one shown for the leaf in Fig. 7(A) for $N = 16$ coupled MG systems. Figure 7(a) shows the relative complexity of the hub and leaf as a function of the coupling strength. As observed in the Rössler and Lorenz systems, for low $d$-values, the hub experiences a sudden complexity drop while the leaves keep almost their nominal dynamics. As the coupling is increased, the hub starts synchronizing with the mean field and its relative complexity rises above 0. Beyond point B, all nodes reduce their complexity down to a minimal value (point C) before finally increasing up to full synchronization is reached. The noisy curves in that region is due to the extreme sensitivity to initial conditions found in this system. The maps for the hub and leaf [Figs. 7(A)-(C)] corresponding to the setting points A,B, and C marked in Fig. 7(a) sketch the dynamical differentiation route.

The bifurcation diagrams for the hub and leaf respectively are computed as a function of the coupling strength $d$ [Figs. 6(b)-(c)]. In both cases, an inverse period-doubling cascade is observed up to a period-2 limit cycle is settled at $d = 0.18$. Curiously, the size of both limit cycles is different which, again, is an example of an amplitude envelope synchronization. The crisis leading to a larger chaotic attractor is strongly dependent on the initial conditions. In summary, the prevalent lines of the route to synchronization previously sketched are visible observed, albeit with some differences. Further investigations are necessary to fully understand their origin.

### C. Four-dimensional Saito model

In this section, to confirm the generality of the nodal dynamical differentiation route, we will investigate a completely different model of nonlinear dynamics, the Saito model which is able to generate a very rich dynamical behavior including toroidal chaos. The vector...
flow in Eq. (13) reads

$$\mathbf{f}(\mathbf{x}) = \left[ \rho(\mathbf{y} + \mathbf{z}), x + 2\mathbf{\delta}y, -x - \mathbf{w}, (z - \Phi(\mathbf{w}))/\epsilon \right]$$

where $\mathbf{x} = (x, y, z, w)$ and

$$\Phi(\mathbf{w}) = \begin{cases} w - (1 + \eta) & w \geq \eta \\ \frac{-w}{\eta} & \text{if } |w| < \eta \\ w + (1 + \eta) & w \leq -\eta \end{cases}$$

This is a four-dimensional system involving a linear piecewise function as a switch mechanism. The dynamics produced by this model can be chaotic, quasi-periodic, toroidal chaotic, or even hyperchaotic, also structured around a torus. Parameters $\rho$, $\eta$, and $\epsilon$ are fixed and $\delta$ is used as a bifurcation parameter. Here we chose $\delta = 0.58$ for toroidal chaos, and $\delta = 82$ to produce hyperchaotic toroidal chaos. Close examples of the Poincaré sections of these attractors can be grasped, respectively, in Fig. 8(A) and 8(B) for a leaf of a star of $N = 16$ $y$-coupled Saito models whose dynamics are almost identical to the uncoupled scenarios. The main difference between these two dynamics is the “thickness” of the Poincaré section being much thinner for $\delta = 0.58$ than for $\delta = 0.82$. The hyperchaotic nature is revealed by the overlapping structures of the Poincaré section as observed in the folded-towel map introduced by Rössler. This is partly confirmed with the Lyapunov exponents which for $\delta = 0.58$ are

$$\lambda_1 = 0.047 > \lambda_2 = 0.012 \approx \lambda_3 = -0.022 > \lambda_4 = -94.79,$$

with two null exponents as expected for toroidal chaos structured around a torus $T^2$ and for $\delta = 0.82$ are

$$\lambda_1 = 0.164 > \lambda_2 = 0.069 \approx \lambda_3 = -0.033 > \lambda_4 = -94.70,$$

with two positive and one null exponents as needed for toroidal hyperchaos. Note that in the latter case it is still unclear whether the second positive exponent is merged or not with the third null exponent and this is an issue currently under study. Investigations, which are out of the scope of the present work, would allow determining whether the fourth dimension is required for embedding the toroidal chaos ($\delta = 0.58$). Being hyperchaotic for $\delta = 0.82$, the dynamics are necessarily four-dimensional.

When these models are coupled through the variable $y$, they synchronize as shown in Figs. 8(a) and 8(b) with the convergence of their relative complexity to zero, being the critical coupling $d_c$ larger for the hyperchaotic dynamics. Nevertheless, in both cases, what is preserved along the route to synchronization with these toroidal chaotic and hyperchaotic dynamics is the node differentiation mainly of the hub whose dynamics turns less developed than the uncoupled one while the leaves sustain it over the whole range of coupling strengths [compare the first-return maps in Fig. 3(A), and in Fig. 3(B)]. The route to synchronization appears very simple, most likely due to the constrained toroidal structure of the nominal dynamics.

V. NETWORKS OF RÖSSLER OSCILLATORS

Having characterized the relationship between the degree centrality and dynamical complexity in star networks, we move forward to generalize our results to networks with a broader degree distribution. This issue was partially tackled in Refs. [19,20] where a strong correlation between degree $k$ and complexity $C_D$ allowed establishing a node hierarchy. However, given the present novel results revealing the important role played by the nominal nodal dynamics, we extend our present study to larger networks.

We maximize the degree heterogeneity by using Barabasi-Albert scale-free (SF) networks of $N$ identical
In the context of the synchronization of $y$-coupled Rössler oscillators retaining the parameter settings provided in Section III, we calculate the evolution of $C_k$ within a degree class by averaging over the $N_k$ nodes having degree $k$, that is,

$$C_k = \frac{1}{N_k} \sum_{[j|k_j=k]} C_j,$$

where $C_j$ is the dynamical complexity of the $j$th node. Here, we use the relative complexity $C_{k0} = C_k - C_0$ which helps to better assess the effects of both the coupling and the topology in the complexity of the $k$-class nodes. In addition, to evaluate the impact of the nodal environment, we perform our calculations for networks made of $N = 300$ nodes, wherein $\langle k \rangle = 4$. All the results are averaged over 10 different networks realizations.

First, in Fig. 9(a-b) we plot the time averaged phase synchronization $S_k$ of $k$-class nodes with respect to the phase of the mean field for $N = 300$ SF networks as a function of the coupling $d$. We define $S_k$ as

$$S_k = \frac{1}{N_k} \sum_{[j|k_j=k]} S_j,$$

where $S_j$ is the time averaged phase synchronization of node $j$ with the mean-field global phase, defined in Eq. (8). The result is then ensemble averaged over 10 network realizations. Along the route to synchronization, the $k$-classes synchronize hierarchically to the mean field before all of them lock at the critical coupling ($d_c = 0.37$ for $a = 0.26$ and $d_c = 0.52$ for $a = 0.31$). Therefore existing node differentiation also in heterogeneous networks. While the phase synchronization $S_k$ is monotonously increasing for most of the coupling range and classes, the weakly coupled regime presents anomalous synchronization over which most of the $S_k$ are below the basal, uncoupled level. This anomalous range, more prominent for $a = 0.31$, is associated with the maximal node differentiation [compare Fig. 9(b) and 9(d)].

As already observed in Section III the less developed the dynamics, the smaller the critical value $d_c$ [Fig. 9(c)-(d)]. In the weakly coupled regime, the relative complexity $C_{k0}$ shows a strong node differentiation: while the hubs present a markedly reduced complexity with respect to the nominal value (with a clear negative minimal), for the smaller degrees the complexity increases. This increment in the less connected nodes ($k = 2$ and $k = 5$ in the example) is non-monotonic with $d$ and $k$, as observed in the stars in Sections III and IV.

For stronger coupling all the nodes increase their complexity well above the nominal value, ordered following the reverse degree ranking [Fig. 9(c)-(d)], recovering the scenario observed for the hub in Section III. All these features are qualitatively shared for the two different $a$-values, but the deviations from the uncoupled value are larger for the more developed chaos, $a = 0.31$.

Plotting the relative complexity $C_{k0}$ as a function of the phase synchronization $S_k$ reveals a dependency which is stronger for nodes with a larger degree [Fig. 9(c)-(f)]. Typically, low-degree nodes exhibit a dynamics which is nearly independent of the synchronization while it is the opposite for large-degree nodes. It also clearly shows that the relative complexity converges to zero for larger $d$-values when $k$ increases. This delineates another signature of node differentiation.

Therefore, we conclude that in most of the regimes it is possible to correlate the node degree with the relative complexity. Furthermore, the degree centrality is the single structural parameter that affects the node behaviour, while the rest of environmental topological features has no impact. This is shown in Fig. 10 where we plot the value of the relative complexity $C_{k0}$ as a function of $k$ for three representative values of $d$, both for SF networks and ER networks with $\langle k \rangle = 4$. The ER and SF curves overlap and, therefore, the dependence of $C_k$ on $a$ and $d$ are the same regardless of the topology. This is quite remarkable since the ER and SF networks have a different critical coupling $d_c$: for a given $d$-value, their global dynamics are different, but the nodes of degree $k$ have equivalent dynamics independently of the environment. The same result is obtained for different sizes of both ER and SF networks (not shown).

To better illustrate the node differentiation in larger networks, we plotted the first-return maps for three dif-
VI. CONCLUSION

Routes to synchronization need to be elucidated to attain a better understanding and knowledge of the possible scenarios that may be encountered depending on the coupling, nominal node dynamics, topology, and network size. Here, we confirmed and extended previous work depicting a non-trivial effect of connectedness on node dynamics, particularly the existence of a non-monotonic relationship between the complexity of node dynamics and coupling strength. There is, indeed, a node differentiation that evolves with the coupling strength. Typically, when the coupling function provides type-1 synchronizability, low coupling strengths induce a significant reduction in the complexity of the large-degree nodes, while those having a small degree are left nearly unaffected. Consequently, increasing the \( d \)-value, all the nodes in small networks or those with a large degree in large networks present a minimal dynamical complexity. In every network, all nodes have a complexity that increases, often reaching a greater level than the nominal one, before the onset of full synchronization. This sketch for the route to synchronization is clearly observed for the two

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig10.png}
\caption{Relative complexity \( C_{k0} \) as a function of the degree \( k \) for (a) \( a = 0.26 \), and (b) \( a = 0.31 \). In each panel, curves correspond to SF (void symbols) and ER (full symbols) with the coupling strengths marked in Fig. 10(c,d) with vertical lines: \( d = 0.04 \) (circles), \( d = 0.08 \) (squares), and \( d = 0.22 \) (triangles). Results are averaged over 10 different network realizations of \( N = 300 \). Other parameter values as in Fig. 1.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig11.png}
\caption{Different first-return maps to a Poincaré section for three \( k \)-classes of nodes along a route to synchronization in a \( N = 300 \left( \langle k \rangle = 4 \right) \) SF network of \( y \)-coupled Rössler nodes \( a = 0.31 \) and other parameters as in Fig. 1. The complexity \( C_D \) is reported in each case.}
\end{figure}
three-dimensional systems (Rössler and Lorenz). With more complex node dynamics (Mackey-Glass and Saito), the decrease towards a minimal complexity is only observed in the hub of star networks; further studies with other network types are still needed for attaining a more general view of the latter system. The node differentiation is not a monotonic function of the coupling nor synchronizability. The route to synchronization is an abridged version of the route observed with a type-I synchronizability, the route to synchronization is not a monotonic function of the coupling nor synchronizability, the route to synchronization is an abridged version of the route observed with a type-I synchronizability.

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