All-orders asymptotics of tensor model observables from symmetries of restricted partitions

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Abstract

The counting of the dimension of the space of $U(N) \times U(N) \times U(N)$ polynomial invariants of a complex 3-index tensor as a function of degree $n$ is known in terms of a sum of squares of Kronecker coefficients. For $n \leq N$, the formula can be expressed in terms of a sum of symmetry factors of partitions of $n$ denoted $Z_3(n)$. We derive the large $n$ all-orders asymptotic formula for $Z_3(n)$ making contact with high order results previously obtained numerically. The derivation relies on the dominance in the sum, of partitions with many parts of length 1. The dominance of other small parts in restricted partition sums leads to related asymptotic results. The result for the 3-index tensor observables gives the large $n$ asymptotic expansion for the counting of bipartite ribbon graphs with $n$ edges, and for the dimension of the associated Kronecker permutation centralizer algebra. We explain how the different terms in the asymptotics are associated with probability distributions over ribbon graphs. The large $n$ dominance of small parts also leads to conjectured formulae for the asymptotics of invariants for general $d$-index tensors. The coefficients of $1/n$ in these expansions involve Stirling numbers of the second kind along with restricted partition sums.

Key words: Tensor models, invariant theory, asymptotic combinatorics, Kronecker permutation centralizer algebras
1 Introduction

Tensor models are generalizations of random matrix theories where the random variables are multi-index tensors. New results on the large $N$ expansion of these models have attracted continuing active interest in theoretical physics [1, 2], in particular in connection with random discrete geometries [3, 4], quantum gravity [5], condensed matter physics, 2D topological field theories [6] and models of black hole physics [7, 8, 9].

We will consider complex tensor variables $\Phi_{i_1,\ldots,i_d}$ transforming as $V_N^{\otimes d}$ under a product of unitary groups $U(N)^{\times d}$, where $V_N$ is the fundamental representation of $U(N)$. A basis of invariants of $U(N)^{\times d}$ is built using index contractions between $n$ copies of $\Phi$ and $n$ copies of its complex conjugate, $\bar{\Phi}_{i_1,\ldots,i_d}$ which transforms $\bar{V}_N^{\otimes n}$, where $\bar{V}_N$ is the complex conjugate representation of $U(N)$. The invariant observables are useful as interaction terms in tensor model actions. Their enumeration is also of interest in the thermodynamics of quantum mechanical tensor theories [10] and the investigation of their holographic duals [11, 12, 13]. The leading order large $n$ asymptotics has been discussed in the physics literature in [10, 13, 14].
In [6] the counting and correlators of the rank-\(d\) complex tensor invariants have been given using permutation equivalences. A bijection of the permutation basis of invariants with branched covers of the 2-sphere was given and formulations of the counting and correlators in terms of 2D topological field theory were described. In [15, 16] it was shown that the invariants at degree \(n\) form a basis for an associative algebra \(\mathcal{K}(n)\), denoted the Kronecker permutation centralizer algebra, which has a decomposition into blocks of size equal to the Kronecker coefficient for triples of Young diagrams with \(n\) boxes. The algebra has implications for the structure of tensor model correlators [6, 16]. The algebraic perspectives on tensor model correlators have been developed in [17, 18, 19]. Similar techniques have been applied to orthogonal invariants [20]. Moreover, in computational complexity theory [21, 22, 23, 24, 25], Kronecker coefficients form a subject of active interest. Very recently, building on the results of [16], a question of Murnaghan [26, 27] (discussed among a class of positivity problems in representation theory in [28]) about the existence of a combinatorial interpretation of the Kronecker coefficient has motivated a construction based on bipartite ribbon graphs [29]: for every triple of Young diagrams with \(n\) boxes, the Kronecker coefficient counts vectors spanning a specified sub-lattice of the lattice of bipartite ribbon graphs of \(n\) edges. The vectors are constructed as null vectors of an integer matrix. The importance of Kronecker coefficients in mathematics gives additional motivations for detailed studies of the properties of \(\mathcal{K}(n)\). The asymptotics of the counting of tensor model observables gives the asymptotics of the dimension of \(\mathcal{K}(n)\).

The counting formulae in [6, 16] for 3-index tensor observables, in the case \(N \geq n\), is recalled as

\[
Z_3(n) = \sum_{p \vdash n} \text{Sym}(p) = \sum_{R_1,R_2,R_3 \vdash n} C(R_1, R_2, R_3)^2 \tag{1.1}
\]

with \(p = (p_1, \ldots, p_n)\) a partition of \(n\) with symmetry factor \(\text{Sym}(p)\), and \(C(R_1, R_2, R_3)\) the so-called Kronecker coefficient associated with three Young diagrams \(R_1, R_2, R_3\) with \(n\) boxes. For \(n > N\), the Young diagrams \(R_1, R_2, R_3\) are restricted to have no more than \(N\) rows. This counting has also been obtained with motivations from quantum entanglement in [30, 31]. The finite \(N\) cutoff is a feature related to Schur-Weyl duality which plays an important role in connection with the stringy exclusion principle [32] and giant gravitons [33] in the AdS/CFT correspondence [34, 35, 36] (for a review of the applications of Schur-Weyl duality in this context see [37]). The asymptotics of \(1.1\) at large \(n\) has been the subject of a very interesting study by Kotesovec up to high order using direct numerical fitting techniques and the results are available on the OEIS [38].

In this work, we investigate the asymptotic expansion of the counting of rank-\(d\) tensor invariants. We prove the asymptotic expansion series at all orders for \(d = 3\). Theorem [1] and Theorem [3] are our main results. We are able to match the series [38] and extend it to all orders. We find that the sum over partitions \(p\) in \(1.1\) is dominated by partitions in which most of the parts have length 1: these are partitions of the form \([1^{n-k}, q]\) where \(q\) is a partition of \(k\), with \(k\) fixed as \(n\) tends to infinity. Using this dominance of parts of
length 1, we also conjecture the form of the coefficients of the asymptotic series for rank $d$ invariants.

This plan of the paper is as follows. The next section introduces our notation, discusses main features of the asymptotic series and delivers our main result, namely the asymptotic expansion of the counting of rank $d = 3$ of tensor invariants. In the course of this proof, an important role is played by a partition of the set $\mathcal{P}(n)$ of all partitions of $n$ into subsets $\mathcal{P}_m(n)$, which are partitions of $n$ where the minimal part length is $m$. Corresponding to these subsets we define $Z_{3,m}(n)$. We show that large $n$ asymptotic series for $Z_3(n)$ is the same as that for $Z_{3,1}(n)$. At the conclusion of this section we explain how the different terms in the asymptotic expansion can be associated with different probability distributions over tensor invariants, or equivalently over bi-partite ribbon graphs. In section 3 we present formulae for the asymptotic series of $Z_{3,m}(n)$, with any finite $m$ which is kept fixed as $n$ tends to infinity. We then show that the coefficients of the $1/n$ expansion can be expressed in terms of the Stirling numbers of the second kind along with sums over symmetry factors of restricted partitions. Section 4 elaborates some conjectures about the expansion for arbitrary rank $d$ invariants, based again on the dominance of small parts. A conclusion follows in section 5 where we discuss future research directions motivated by this work. The paper closes with two appendices: Appendix A collects the proof of the main lemma which establishes the dominance of small cycles, while Appendix B provides the codes that yield the coefficients of the asymptotic series expansion of tensor invariants at $d = 3$ at arbitrary order $n$.

2 Asymptotic counting of 3-index tensor invariants

In tensor models we encounter a counting problem involving complex tensor variables $\Phi_{ijk}$ and $\bar{\Phi}_{ijk}$ which transform in $V_{N}^{\otimes 3}$ and $\bar{V}_{N}^{\otimes 3}$ of $U(N)$, where $V_{N}$ is the fundamental of $U(N)$ and $\bar{V}_{N}$ is the anti-fundamental of $U(N)$. The counting of degree $n$ invariant polynomials when $n \leq N$ is given by

$$Z_3(n) = \sum_{p \vdash n} \text{Sym}(p) = \sum_{R_1, R_2, R_3 \vdash n} C(R_1, R_2, R_3)^2$$  \hspace{1cm} (2.1)

$p$ is a partition of $n$. It is specified by non-negative integers $(p_1, p_2, \ldots, p_n)$ which are the numbers of parts of length 1, 2, $\ldots$, $n$. The symmetry factor of the partition is

$$\text{Sym}(p) = \prod_{i}^{n} i^{p_i} p_i!$$  \hspace{1cm} (2.2)

$p$ specifies the cycle structure of a permutation in $S_n$: $p_1$ is the number of cycles of length 1, $p_2$ is the number of cycles of length 2, etc. For a permutation $\sigma$ with cycle structure $p$, $\text{Sym}(p)$ is the number of permutations $\gamma \in S_n$ satisfying $\gamma \sigma \gamma^{-1} = \sigma$. Thus this is the
order of the centralizer of any $\sigma$ with a given cycle structure $p$. In the second equality of (2.1), $R_1, R_2, R_3$ are Young diagrams with $n$ boxes. $C(R_1, R_2, R_3)$ is the Kronecker coefficient for the triple of Young diagrams. The counting has been detailed in [6, 16], and has been generalized to arbitrary rank $d$ tensor $\Phi_{i_1i_2...i_d}$.

The asymptotics of $Z_3(n)$ at large $n$ has been calculated by evaluating the sum for $n$ up to 20000 and fitting to the form $n!P(1/n)$ where $P$ is a power series in $1/n$ [8]:

$$Z_3(n) \sim n!(1 + 2/n^2 + 5/n^3 + 23/n^4 + 106/n^5 + 537/n^6 + \cdots) \quad (2.3)$$

Remarkably, the coefficients in $P$ are all non-negative integers, this has been verified for the first 132 terms.

This remarkable integrality is suggestive of some underlying simplicity in the asymptotics. Below we propose a way to understand this simplicity. The idea is to identify a subset of the partitions in (2.1) which dominate in the large $n$ limit.

2.1 A partition of the set of partitions of integer $n$

$Z_3(n)$ is a sum over partitions of $n$, denoted $p \vdash n$. Let us call this set $\mathcal{P}(n)$. For example $\mathcal{P}(3)$ is the set

$$\mathcal{P}(3) = \{[1, 1, 1], [1, 2], [3]\} \quad (2.4)$$

Each square bracket contains positive integers adding to 3. The different entries within a bracket are the parts of the partition. We also use the exponent notation

$$\{[1^3], [1, 2], [3]\} \quad (2.5)$$

Each partition of $n$ has integers $i$ with multiplicities $p_i$.

In order to understand the asymptotics, it will be useful to describe $\mathcal{P}(n)$ as a disjoint union of subsets. We define $\mathcal{P}_m(n)$ to be the subset of $\mathcal{P}(n)$ consisting of partitions which have the smallest part equal to $m$. In the above case

$$\mathcal{P}_1(3) = \{[1^3], [1, 2]\}$$

$$\mathcal{P}_2(3) = \emptyset$$

$$\mathcal{P}_3(3) = \{[3]\} \quad (2.6)$$

These subsets are evidently disjoint: for any $p$ there is a unique integer $m$ which is the minimum part appearing in $p$. Hence

$$\mathcal{P}(3) = \mathcal{P}_1(3) \sqcup \mathcal{P}_2(3) \sqcup \mathcal{P}_3(3) = \mathcal{P}_1(3) \sqcup \mathcal{P}_3(3) \quad (2.7)$$

For $n = 4$ we have

$$\mathcal{P}(4) = \{[1^4], [1^2, 2], [1, 3], [2^2], [4]\}$$
\[ P_1(4) = \{[1^4], [1^2, 2], [1, 3]\} \]
\[ P_2(4) = \{[2, 2]\} \]
\[ P_3(4) = \emptyset \]
\[ P_4(4) = \{[4]\} \]

and
\[ P(4) = P_1(4) \uplus P_2(4) \uplus P_3(4) \uplus P_4(4) = P_1(4) \uplus P_2(4) \uplus P_4(4) \] (2.8)

Note that above \( m > \lfloor n/2 \rfloor \), \( P_m(n) = \emptyset \) unless \( m = n \), and then \( P_n(n) = \{[n]\} \). In general we have
\[ P(n) = P_1(n) \uplus P_2(n) \uplus \cdots \uplus P_{\lfloor n/2 \rfloor}(n) \uplus P_n(n) \] (2.9)

For each of the subsets in the decomposition (2.10), we can define the corresponding sum over \( \text{Sym}(p) \). First observe that we can write
\[ Z_3(n) = \sum_{p \vdash n} \text{Sym}(p) = \sum_{p \in P(n)} \text{Sym}(p) \] (2.11)

For the sums of symmetry factors of partitions restricted to the subsets \( P_m(n) \) of \( P(n) \), we define
\[ Z_{3,m}(n) = \sum_{p \in P_m(n)} \text{Sym}(p) = \sum_{p \vdash n, p_1 = \cdots = p_{m-1} = 0, p_m > 0} \text{Sym}(p) \] (2.12)

The conditions \( p \vdash n \), \( p_1 = \cdots = p_{m-1} = 0 \) and \( p_m > 0 \) give an equivalent way to express the restriction to \( P_m(n) \). In terms of these restricted partition sums, we have
\[ Z_3(n) = \sum_{m=1}^{n} Z_{3,m}(n). \] (2.13)

It is understood that, for \( \lfloor n/2 \rfloor < m \leq (n - 1) \), \( Z_{3,m}(n) = 0 \) since \( P_m(n) \) is empty in this range, as explained above.

For the subsequent discussion, it is also useful to define
\[ P_{m^+}(n) = P_m(n) \uplus P_{m+1}(n) \uplus \cdots \uplus P_n(n) \] (2.14)

Thus
\[ P_{1^+}(n) = P_1(n) \uplus P_2(n) \uplus P_3(n) \cdots \uplus P(n) = P(n) \]
\[ P_{2^+}(n) = P_2(n) \uplus P_3(n) \cdots \uplus P(n) = P(n) \setminus P_1(n) \]
\( \mathcal{P}_{3+}(n) = \mathcal{P}_3(n) \sqcup \mathcal{P}_4(n) \sqcup \cdots \sqcup \mathcal{P}(n) \)
\( = \mathcal{P}(n) \setminus \{\mathcal{P}_1(n) \sqcup \mathcal{P}_2(n)\} \)
\( \vdots \)
\( \mathcal{P}_{m+}(n) = \mathcal{P}(n) \setminus \{\mathcal{P}_1(n) \sqcup \mathcal{P}_2(n) \sqcup \cdots \sqcup \mathcal{P}_{m-1}(n)\} \)  \( (2.15) \)

\[ Z_{3;m+}(n) = \sum_{p \in \mathcal{P}_{m+}(n)} \text{Sym}(p) = \sum_{p \sqcup \mathcal{P}_{m-1}=0} \text{Sym}(p). \]  \( (2.16) \)

### 2.2 The leading asymptotics

Consider some of the terms in the sum \( (2.11) \). For \( p = [1^n] \), \( p_1 = n \) and all other \( p_i = 0 \), we have \( \text{Sym}(p) = n! \). This very simple fact has been used to very good effect, to bound Kronecker coefficients in \([25]\). Consider another \( p = [n] \), i.e. \( p_n = 1 \) and all other \( p_i = 0 \), then observe that \( \text{Sym}(p) = n \). The function \( \text{Sym}(p) \) is such that for a multiplicity \( p_i \) of cycles, we get factorials of the multiplicity, while the length \( i \) of the cycle we get a factor \( i \). So large multiplicities of cycles give dominant contributions at large \( n \). This will be a driving principle in our reasoning.

In the following, we will prove that the set of all \( p \) for fixed \( n \) can be organised in such a way as to identify the dominant subsets at large \( n \). The main idea is based on the fact that \( Z_3(n) \) is dominated by partitions of the form \( p = [1^{n-k}, q] \) of \( n \), where \( q \) is a partition of \( k \) with no parts of length 1. To get a finite order in the asymptotic expansion we need to keep \( k \) finite as \( n \) is taken to infinity. Noting that \( \text{Sym}([1^{n-k}, q]) = (n-k)!\text{Sym}(q) \), consider the finite sum

\[ S_{3;1;K}(n) = \sum_{k=0}^{K} (n-k)! \sum_{q \in \mathcal{P}_{2+}(k)} \text{Sym}(q) = \sum_{k=0}^{K} (n-k)! \sum_{q-k|q_1=0} \text{Sym}(q) \]  \( (2.17) \)

The subscript 1 in \( S_{3;1;K} \) indicates that we are isolating the parts of length 1 and treating the remaining parts as a restricted partition \( q \) with no parts of length 1, that is, \( q_1 = 0 \). Let us illustrate in greater detail that sum. Taking \( K = 0 \), we have

\[ S_{3;1;0}(n) = \text{Sym}([1^n]) = n! \]

Note that \( K = 1 \) does not allow any partition \( q \) with \( k = 1 \), since \( k = 1 \) means there are \( n-1 \) parts of length 1, which forces the remaining part to also be of length 1. Let us expand \( (2.17) \) taking \( K = 4 \), we obtain

\[ S_{3;1:4}(n) = \text{Sym}([1^n]) + \text{Sym}([1^{n-2}, 2]) + \text{Sym}([1^{n-3}, 3]) + \text{Sym}([1^{n-4}, 4]) + \text{Sym}([1^{n-4}, 2^2]) \]
\[ = n! + 2(n-2)! + 3(n-3)! + (n-4)!(4 + 8) \]
\[ \begin{align*}
= n! \left( 1 + \frac{2}{n(n-1)} + \frac{3}{n(n-1)(n-2)} + \frac{12}{(n)(n-1)(n-2)(n-3)} \right) \\
= n! \left( 1 + \frac{2}{n^2} + \frac{5}{n^3} + \frac{23}{n^4} + \cdots \right)
\end{align*} \] (2.19)

The \( k = 2 \) term gives corrections at order \( 1/n^2 \) and higher, the \( k = 3 \) gives a correction at order \( 1/n^3 \) and higher. The sum \( S_{3;1;4} \) agrees with the numerical asymptotics of (2.3) (from 38) up to \( 1/n^4 \) corrections.

We are in a position to formulate our first result. Consider the sum (2.17) re-expressed in the following form and the series: for \( K \leq n \),

\[
S_{3;1;K}(n) = n! \left( 1 + \sum_{k=2}^{K} \frac{1}{n(n-1)\cdots(n-k+1)} \sum_{q\vdash k:q_1=0} \text{Sym}(q) \right)
\]

\[
S'_{3;1;K}(n) = \frac{S_{3;1;K}(n)}{n!}
\]

(2.20)

Note that \( S'_{3;1;K}(n) \) has an expansion in \( 1/n \) at large \( n \). We regard this finite sum as a tool to construct a polynomial in \( 1/n \) of order \( K \), and by taking \( K \) arbitrarily large and finite (while \( n \) is taken to infinity) we obtain an infinite series in \( 1/n \). Consider \( S_{3;1}(n) \) as the sequence of functions \( S_{3;1;K}(n) \), for all \( K \), and introduce \( S'_{3;1}(n) \) as the sequence \( \frac{S_{3;1;K}(n)}{n!} \), for all \( K \).

The following statement holds:

**Theorem 1.** \( Z'_{3;1}(n) = Z_{3;1}(n)/n! \) is asymptotic to \( S'_{3;1}(n) \) in the large \( n \) limit.

This means we need to show that (see chap. 12.6 39), for fixed and arbitrary \( K \geq 0 \)

\[
\lim_{n \to \infty} n^K (Z'_{3;1}(n) - S'_{3;1;K}(n)) = 0
\]

(2.21)

where \( S_{3;1;K}(n) = S_{3;1;K}(n)/n! \).

We write

\[
(n^K)(Z'_{3;1}(n) - S'_{3;1;K}(n)) = \frac{n^K}{n!} \sum_{k=K+1}^{n} \sum_{q\vdash k:q_1=0} \text{Sym}([1^{n-k}, q])
\]

\[
= n^K \sum_{k=K+1}^{n} \frac{1}{n(n-1)\cdots(n-k+1)} \sum_{q\vdash k,q_1=0} \text{Sym}(q) \equiv R_{n,K}
\]

(2.22)

It is useful to express the theorem informally as

\[
\frac{Z_{3;1}(n)}{n!} \sim \left( 1 + \sum_{k=2}^{\infty} \frac{1}{n(n-1)\cdots(n-k+1)} \sum_{q\vdash k,q_1=0} \text{Sym}(q) \right)
\]

(2.23)

where it is understood that \( n \) is being taken to infinity and the precise meaning is the statement following the theorem.

To prove Theorem 1 we need to show that the remainder \( R_{n,K} \) goes to zero as \( n \) goes to infinity. This is the purpose of the next section.
2.3 Proof of the asymptotics

We first prove an important fact that partitions with small parts have a dominant $\text{Sym}(p)$.

**Lemma 1.** $\forall k \geq 0$ an integer

\[
\begin{align*}
\text{k even}, & \quad \text{Sym}[2^{k/2}] \geq \text{Sym}[q], \quad q \vdash k, \quad q_1 = 0 \quad (2.24) \\
\text{k odd and } k \geq 11, & \quad \text{Sym}[3, 2^{(k-3)/2}] \geq \text{Sym}[q], \quad q \vdash k, \quad q_1 = 0 \quad (2.25)
\end{align*}
\]

**Proof.** See Appendix A. \qed

**Lemma 2.** Let $K$ and $n$ be two positive integers, such that $K \ll n$, and $k \in \{K, \ldots, n\}$. Let $P_1(k)$ be the number of partitions of $k$ with no parts of length 1. The function $f(n, k)$ defined by

\[
\begin{align*}
 f(n, k) &= \frac{(n-k)!}{n!} P_1(k) 2^{k/2} (k/2)! , \quad \text{for } k \text{ even,} \\
 f(n, k) &= \frac{(n-k)!}{n!} P_1(k) 2^{(k-3)/2} ((k-3)/2)! , \quad \text{for } k \text{ odd,} \quad (2.26)
\end{align*}
\]

is maximised by $f(n, K)$.

**Proof.** Let us start with a few comments to explain our proof strategy. Numerical investigation shows that the function decreases as $k$ increases and reaches a minimum near $k = n$. In the following, we will estimate the position of the minimal as $n$ becomes large. For $k$ above that minimum, the function increases again but remains at $k = n$ well below the value at $k = K$.

We will prove that the slope of $f(n, k)$ as a function of $k$ is negative at $k = K$ and that there is just one minimum in the range $k = K$ to $k = n$. Further, we prove (easily) that the value at $k = K$ exceeds the value at $k = n$.

When $k$ is close to $K$, which is order 1 as $n \to \infty$, then we know that $f$ vanishes, simply because $f$ behaves like $\frac{(n-K)!}{n!} = O(1/n^K)$. When $k$ is close to $n$, i.e. $(n-k)$ is order 1, then we also know that $f(n, k)$ is vanishing at large $n$. To see this consider $f(n, n)$ in the even case of (2.26)

\[
f(n, n) = \frac{1}{n!} P_1(n) 2^{n/2} (n/2)! \sim \frac{e^{B \sqrt{n} 2^{n/2} (n/2)!}}{n!} \quad n \to \infty \to 0 \quad (2.27)
\]

where we used a standard result for the asymptotics of $P_1(n)$ where $B$ is a constant, which we give shortly. The odd case is similar.

We will consider $(n-k) \sim n^\alpha$ for $0 < \alpha < 1$, and thus interpolating between these two limits. For any $1 > \alpha > 0$, $k$ and $(n-k)$ both go to infinity as $n$ goes to infinity. So we can use the asymptotic form of $P_1(k)$. The asymptotic behaviour of $P_1(k)$ is easily
derived from the asymptotics of partition numbers because $P_1(k) = P(k) - P(k - 1)$ (see for example [43]):

$$P_1(k) = \frac{A}{k^{3/2}} e^{B \sqrt{k}} (1 + O(\frac{1}{k^a}))$$

(2.28)

where $A = \pi/(12\sqrt{2})$, $B = \pi \sqrt{2/3}$ and $a = 1/2$.

**k even** - We start by $n$ even and, given the above, approximate $f(n, k)$ by

$$f(n, k) = \frac{1}{n!} A k^{-3/2} e^{B \sqrt{k}} (1 + O(\frac{1}{k^a})) \Gamma(n - k + 1) 2^{k/2} \Gamma(\frac{k}{2} + 1)$$

(2.29)

As $k$ approaches $n$ for large $n$, the factor $\Gamma(n - k + 1)$ decreases whereas $\Gamma(\frac{k}{2} + 1)$ increases. As we will see, this results in a minimum of $f(n, k)$.

Taking the derivative of $f(n, k)$ with respect to $k$

$$\frac{\partial f}{\partial k} =$$

$$\left(\frac{-3}{2k} + \frac{B}{2 \sqrt{k}} - \left(\frac{1}{\Gamma(z)} \frac{d\Gamma(z)}{dz}\right)\right)_{z=n-k+1} + \frac{1}{2} \log(2) + \frac{1}{2} \left(\frac{1}{\Gamma(z)} \frac{d\Gamma(z)}{dz}\right)_{z=k/2+1} f$$

$$+ O\left(\frac{1}{(k+1)^{a+1}}\right) \frac{1}{n!} A k^{-3/2} e^{B \sqrt{k}} \Gamma(n - k + 1) 2^{k/2} \Gamma(\frac{k}{2} + 1)$$

(2.30)

The logarithmic derivative of $\Gamma$ is the di-gamma function $\Psi(z) = (1/\Gamma(z))(d\Gamma(z)/dz)$ (also denoted $\psi^{(0)}$). We can therefore write

$$\frac{\partial f}{\partial k} =$$

$$\left[\left(\frac{-3}{2k} + \frac{B}{2k^{1/2}} - \Psi(n - k + 1) + \frac{1}{2} \log(2) + \frac{1}{2} \Psi(\frac{k}{2} + 1)\right) \left(1 + O\left(\frac{1}{k^a}\right)\right)\right]$$

$$+ O\left(\frac{1}{(k+1)^{a+1}}\right) \frac{1}{n!} A k^{-3/2} e^{B \sqrt{k}} \Gamma(n - k + 1) 2^{k/2} \Gamma(\frac{k}{2} + 1)$$

(2.31)

As noted earlier with $(n - k) = n^\alpha$ for $0 < \alpha < 1$, $n - k$ and $k$ both tend to infinity as $n$ goes to infinity. This is what we are interested in, since the special cases of $\alpha = 0, 1$ are understood by direct calculation. We can use the asymptotic formula for $\Psi(z)$, for any large enough $z$:

$$\Psi(z) = \log(z) - \frac{1}{2z} + O\left(\frac{1}{z^2}\right)$$

(2.32)

Let us introduce $F(n, k) = \frac{1}{n!} A k^{-3/2} e^{B \sqrt{k}} \Gamma(n - k + 1) 2^{k/2} \Gamma(\frac{k}{2} + 1)$. Therefore, we approximate

$$\frac{\partial f}{\partial k} =$$

10
\[
\left( -\frac{3}{2k} + \frac{B}{2k^{1/2}} + \frac{1}{2} \log(2) - \left( \log(n-k) + \frac{1}{2(n-k)} + O\left( \frac{1}{(n-k+1)^2} \right) \right) \right).
\]

\[
\begin{align*}
&+ \frac{1}{2} \left( \frac{5}{2k} + \frac{B}{2k^{1/2}} + \frac{1}{2} \log(2) - \left( \log(n-k) + \frac{1}{2(n-k)} + O\left( \frac{1}{(n-k+1)^2} \right) \right) \right) F(n,k) \\
&= \left[ \left( -\frac{3}{2k} + \frac{B}{2k^{1/2}} + \frac{1}{2} \log(2) - \left( \log(n-k) + \frac{1}{2(n-k)} + O\left( \frac{1}{(n-k)^2} \right) \right) \right) \right. \\
&\left. + \frac{1}{2} \left( -\log 2 + \log k + \frac{1}{k} + O\left( \frac{1}{k^2} \right) \right) \right) \left( 1 + O\left( \frac{1}{k^a} \right) \right) + O\left( \frac{1}{k^{a+1}} \right) F(n,k) \\
&= \left[ \left( \frac{1}{2} \log k - \log(n-k) - \frac{1}{k} + \frac{B}{2k^{1/2}} - \frac{1}{2(n-k)} \right) + O\left( \frac{1}{(n-k)^2} \right) \right) \left( 1 + O\left( \frac{1}{k^a} \right) \right) + O\left( \frac{1}{k^{a+1}} \right) \right] F(n,k). \\
&= \right. \left( 1 + O\left( \frac{1}{k^{a+1}} \right) \right) F(n,k). \\
&\right. \\
&\end{align*}
\]

Given that \( k, (n-k) \to \infty \), the dominant terms above in the large \( n \) limit are \( \frac{1}{2} \log k - \log(n-k) \). Therefore the condition of vanishing derivative gives

\[
\log(n-k) = \frac{1}{2} \log(k) \iff \log(n-k) = \log(\sqrt{k}) \]

\[
(n-k) = \sqrt{k} \iff k + \sqrt{k} - n = 0 \\
\] (2.34)

Solving this quadratic equation for \( \sqrt{k} \), gives \( \sqrt{k} = \frac{-1 + \sqrt{1 + 2\sqrt{n}}}{2} \). The positive solution is \( \sqrt{k} = -1/2 + \sqrt{n} \) and yields

\[
k = n - \sqrt{n} + 1/4 \\
\] (2.35)

which confirms \( (n-k) \sim \sqrt{n} \) and also \( k^{1/2} < n-k \).

The above treatment shows that for large \( n \) and \( (n-k) \sim n^\alpha, \alpha \in]0,1[, \) there is a single extremum of \( f(n,k) \), when \( k \leq n \).

**k odd -** This case can be handled in the similar way as above since

\[
\frac{\partial f}{\partial k} = \left[ \left( \frac{1}{2} \log k - \log(n-k) - \frac{5}{2k} + \frac{B}{2k^{1/2}} - \frac{1}{2(n-k)} \right) \right. \\
\left. + O\left( \frac{1}{(n-k)^2} \right) + O\left( \frac{1}{k^a} \right) \right) \left( 1 + O\left( \frac{1}{k^a} \right) \right) + O\left( \frac{1}{k^{a+1}} \right) \right] \tilde{F}(n,k) \\
\] (2.36)

with \( \tilde{F}(n,k) = \frac{1}{n!} Ak^{-3/2}e^{B\sqrt{\kappa}}\Gamma(n - k + 1)2^{(k-3)/2}\Gamma(\frac{k-3}{2} + 1) \), which departs \( (2.33) \) by an irrelevant term. Indeed, \(-5/2k\) does not contribute in the remaining analysis and we
arrive at the exact same result in the same approximation. Thus, (2.35) holds in both regimes \( k^{1/2} \leq n - k \) and \( k^{1/2} > n - k \).

Appendix B gathers numerical evaluations of the minimum value of \( f(n, k) \). It shows that the approximation \( \sqrt{k_{\min}} \sim n - k_{\min} \) holds for a range of values of \( n \). We have set \( n \in [20, 80] \) for \( n \) even, and \( n \in [21, 81] \), for \( n \) odd.

Now we address the slope of the function at \( k = K \) and at \( k = n \).

**Slope at \( K \).** We compare \( f(n, K) \) and \( f(n, K + 2) \), (notice that \( K \) and \( K + 2 \) share the same parity) at large \( n \gg K \),

\[
\frac{f(n, K)}{f(n, K + 2)} = (n - K)G(K) \geq 1 \tag{2.37}
\]

with a finite function \( G(K) \). Thus \( f \) is decreasing at \( K \).

**Slope at \( n \).** Consider \( n \) even, we compare \( f(n, n) \) and \( f(n, n - 2) \), at large \( n \gg K \). We get

\[
\frac{f(n, n)}{f(n, n - 2)} = \frac{1}{2} \frac{P_1(n)}{P_1(n-2)} \frac{2^{n/2} (n/2)!}{2^{(n-2)/2} ((n-2)/2)!} = \frac{P_1(n)}{P_1(n-2)} (n/2) \geq 1, \tag{2.38}
\]

with \( P_1(n) \geq P_1(n-1) \). Thus \( f \) is increasing between \( n - 2 \) and \( n \).

On the other hand, if \( n \) odd and large, we have

\[
\frac{f(n, n)}{f(n, n - 2)} = \frac{1}{2} \frac{P_1(n)}{P_1(n-2)} \frac{2^{(n-3)/2} ((n-3)/2)!}{2^{(n-5)/2} ((n-5)/2)!} = \frac{P_1(n)}{P_1(n-2)} (n/2) \geq 1, \tag{2.39}
\]

Again, we conclude that \( f \) increases between \( n - 2 \) and \( n \).

**\( f(n, K) \) is the max.** The last piece of information we need is the comparison between \( f(n, K) \) and \( f(n, n) \). As \( K \) is a finite small integer, we want to show that \( f(n, K) \geq f(n, n) \) at large \( n \). We note that \( k = K, \ldots, n \) could be even or odd, and therefore we could only compare \( K \) and \( n \) having the same parity.

Assuming that \( K \) and \( n \) are even, for \( n \) large and finite \( K \), using Stirling approximation and (2.28), we have

\[
\frac{f(n, K)}{f(n, n)} = (n - K)! \frac{P_1(K)}{P_1(n)} \frac{2^{K/2} (K/2)!}{2^{n/2} (n/2)!} \sqrt{2\pi(n - K)} \left( \frac{n-K}{e} \right)^{n-K} \left( 1 + O\left( \frac{1}{n} \right) \right)
\]

\[
= G(K) \frac{\Lambda n^{3/2} e^{-\sqrt{n}} (1 + O(1/n)) 2^{n/2} \sqrt{\pi n} (\frac{n}{2e})^{n/2} \left( 1 + O\left( \frac{1}{n} \right) \right)}{
\Lambda n^{3/2} e^{-\sqrt{n}} (1 + O(1/n)) 2^{n/2} \sqrt{\pi n} (\frac{n}{2e})^{n/2} \left( 1 + O\left( \frac{1}{n} \right) \right)}
\]

12
\[ G_1(K) \sqrt{2(n-K)n^{n/2-K}/(n-K-1)} \geq 1 \] (2.40)

because, at large \( n \) and finite \( K \), \( n^{n/2} \) dominates \( e^{B\sqrt{n}+n/2} \). Above \( G(K) \) and \( G_1(K) \) are finite functions of \( K \). In the same vein, considering that \( K \) and \( n \) are odd, we have

\[ \frac{f(n,K)}{f(n,n)} = (n-K)!P_1(K)2^{(K-3)/2}((K-3)/2)! \geq 1 \] (2.41)

and the inequality can be justified once by exactly the same argument at large \( n \) and finite \( K \). This ends the proof of the lemma. \( \square \)

**Theorem 1** is a straightforward corollary of on the following statement.

**Theorem 2.** The remainder \( (2.22) \) obeys the limit: \( \lim_{n \to \infty} R_{n,K} = 0 \).

**Proof.** Let us separate the remainder into two parts:

\[ R_{n,K} = R_{n,K}^+ + R_{n,K}^- \] (2.42)

where \( R_{n,K}^+ \) is a sum over even \( k \) and \( R_{n,K}^- \) is a sum over odd \( k \). To write explicit formulae for these, we will need to treat \( K \) even and odd separately.

**K is odd.** We seek an upper bound for

\[ R_{n,K}^+ = n^K \sum_{k=K+1; \; k \text{ even}}^{n} \frac{(n-k)!}{n!} \sum_{q=k \atop q_1=0} \Sym(q) \] (2.43)

Lemma 1 shows that \( \Sym(q) \) is maximised by \( q = [2^{k/2}] \) for \( k \) even. The number of terms in the sum over \( q \) is \( P_1(k) \). Hence, we claim

\[ R_{n,K}^+ < n^K \sum_{k=K+1; \; k \text{ even}}^{n} \frac{(n-k)!}{n!} P_1(k)2^{k/2}(k/2)! \equiv \tilde{R}_{n,K}^+ \] (2.44)

It is convenient to separate \( \tilde{R}_{n,K}^+ \) into two parts: the first term in the sum and the rest.

\[ \tilde{R}_{n,K}^+ = n^K \frac{(n-K-1)!}{n!} P_1(K+1)2^{(K+1)/2}((K+1)/2)! \]

\[ +n^K \sum_{k=K+3; \; k \text{ even}}^{n} \frac{(n-k)!}{n!} P_1(k)2^{k/2}(k/2)! \] (2.45)

For large \( n \) and \( K \) finite, the first term can be expressed as

\[ n^K \frac{(n-K-1)!}{n!} P_1(K+1)2^{(K+1)/2}((K+1)/2)! = C_Kn^K \frac{(n-K-1)!}{n!} \]

13
\[ C_K n^K \frac{1}{n(n-1)\ldots(n-K)} = C_K n^K \frac{1}{n^{K+1}} \left(1 + O\left(\frac{1}{n}\right)\right) = O\left(\frac{1}{n}\right) \]  

(2.46)

for \( C_K \) a constant. The first term scales as \( 1/n \), so vanishes at large \( n \).

Lemma 2 implies that the summand in the second term of (2.45) is maximised by the first term of that sum provided that \( n \gg K + 3 \). Therefore, the sum in (2.45) is bounded from above by

\[ n^K \frac{(n-K-3+2)(n-K-3)!}{2} P_1(K+3)2^{(K+3)/2}((K+3)/2)! \]

(2.47)

The factor \((n-K-3+2)/2\) bounds from above the number of terms in the sum over \( k \) (for \( n \) even it is exact, for \( n \) odd it exceeds the number of terms by \( 1/2 \)). At large \( n \) and fixed \( K + 3 \ll n \), we expand the previous expression as

\[ C_K n^K (n-K)(n-K-3)! n! P_1(k)2^{k/2}(k/2)! = C_K n^{K+1} \frac{1}{n^{K+3}} \left(1 + O\left(\frac{1}{n}\right)\right) = O\left(\frac{1}{n^2}\right) \]

(2.48)

for some finite positive constants \( C_K, C'_K > 0 \). This goes to zero at large \( n \).

**\( K \) is even.** Although in the following we keep the same notation, the reader should be aware that the expressions may designate different quantities. We adopt the same strategy as above and find upper bound for the remainder and show that it goes to 0 as \( n \) tends to infinity. We have

\[ \mathcal{R}_{n,K}^+ = n^K \sum_{k=K+2; \text{ even}}^{n} \frac{(n-k)!}{n!} \sum_{q+k=q_1=0}^{\text{Sym}(q)} \]

(2.49)

Once again, \( k \) is even, so Lemma 1 provides a bound on \( \mathcal{R}_{n,K}^+ \) as follows

\[ \mathcal{R}_{n,K}^+ < n^K \sum_{k=K+2; \text{ even}}^{n} \frac{(n-k)!}{n!} P_1(k)2^{k/2}(k/2)! = \tilde{\mathcal{R}}_{n,K}^+ \]

(2.50)

The rest of the proof is similar: we separate the first term \( n^K \frac{(n-K-2)!}{n!} P_1(K+2)2^{(K+2)/2}((K+2)/2)! = O(n^{-2}) \) and the remaining partial sum assumes the bound, by Lemma 2

\[ C_K n^K \frac{(n-(K+4)+2)(n-K-4)!}{2} n! = O\left(\frac{1}{n^3}\right) \]

(2.51)

for \( C_K > 0 \) a constant.

We now concentrate on the sum over \( k \) odd in the remainder. The above routine allows us to prove the statement.
\(K\) is odd. Consider the remainder

\[
\mathcal{R}_{n,K}^{-} = n^K \sum_{k=K+2;\ k \text{ odd}}^{n} \frac{(n-k)!}{n!} \sum_{q \vdash k, q_1=0} \text{Sym}(q)
\]

(2.52)

For this case, Lemma 1 leads us to the bound

\[
\mathcal{R}_{n,K}^{-} < n^K \sum_{k=K+2;\ k \text{ odd}}^{n} \frac{(n-k)!}{n!} P_1(k) 3 \cdot 2^{(k-3)/2}((k-3)/2)! \equiv \tilde{\mathcal{R}}_{n,K}^{-}
\]

(2.53)

The first term \(n^K \frac{(n-K-2)!}{n!} P_1(K+2) 3 \cdot 2^{(K-1)/2}((K-1)/2)!\) behaves like \(O(1/n^2)\), and, still by Lemma 2, we treat the remaining partial sum by the bound

\[
n^K \frac{(n-K-2)(n-K-4)!}{2n!} P_1(K+4) 3 \cdot 2^{(K+1)/2}((K+1)/2)! = O\left(\frac{1}{n^3}\right)
\]

(2.54)

\(K\) is even. This is the last case to deal with. We express the remainder as

\[
\mathcal{R}_{n,K}^{-} = n^K \sum_{k=K+1;\ k \text{ odd}}^{n} \frac{(n-k)!}{n!} \sum_{q \vdash k, q_1=0} \text{Sym}(q)
\]

(2.55)

Lemma 1 gives us

\[
\mathcal{R}_{n,K}^{-} < n^K \sum_{k=K+1;\ k \text{ odd}}^{n} \frac{(n-k)!}{n!} P_1(k) 3 \cdot 2^{(k-3)/2}((k-3)/2)! \equiv \tilde{\mathcal{R}}_{n,K}^{-}
\]

(2.56)

The first term \(n^K \frac{(n-K-1)!}{n!} P_1(K+1) 3 \cdot 2^{(K-2)/2}((K-2)/2)!\) behaves like \(O(1/n)\), and, Lemma 2 bounds the remaining partial sum with

\[
n^K \frac{(n-K-1)(n-K-3)!}{2n!} P_1(K+3) 3 \cdot 2^{K/2}(K/2)! = O\left(\frac{1}{n^2}\right)
\]

(2.57)

This ends the proof of the theorem.

The following statement holds

**Theorem 3.** \(Z_3(n)\) is asymptotic to \(S_{3;1}(n)\) in the large \(n\) limit.

As in (2.23) it is useful to express this result informally as

\[
Z_3(n) \sim n! \left(1 + \sum_{k=2}^{\infty} \frac{1}{n(n-1)\cdots(n-k+1)} \sum_{q \vdash k: q_1=0} \text{Sym}(q)\right)
\]

(2.58)
Proof of Theorem 3. We show that, \( Z'_3(n) = Z_3(n)/n! \), for all \( K \)

\[
\lim_{n \to \infty} n^K (Z'_3(n) - S'_{3:1;K}(n)) = 0 \tag{2.59}
\]

Note first that \( Z_3(n) \) expands as

\[
Z_3(n) = Z_{3:1}(n) + Z_{3:2+}(n) \tag{2.60}
\]

\( Z_{3:2+}(n) \) sums over partitions with no parts of size 1. Lemma \[ \text{II} \] teaches that, for all \( n \geq 0 \), and even or odd, the maximal \( \text{Sym}[q] \), among \( q \vdash n \), with \( q_1 = 0 \), is known.

- Let us focus on the case \( n \) even:

\[
Z_{3:2+}(n) \leq \text{Sym}[2^{n/2}] P_1(n) \tag{2.61}
\]

where \( P_1(n) \) keeps its previous meaning as the number of the partitions of \( n \) without parts of size 1. It becomes obvious that

\[
Z_3(n) \leq Z_{3:1}(n) + \text{Sym}[2^{n/2}] P_1(n) \tag{2.62}
\]

We rewrite this under the light of Theorem \[ \text{I} \]

\[
n^K \left( Z'_3(n) - S'_{3:1;K}(n) \right) \leq n^K \left( Z'_{3:1}(n) - S'_{3:1;K}(n) + \frac{1}{n!} \text{Sym}[2^{n/2}] P_1(n) \right) \leq R_{n,K} + \frac{n^K}{n!} \text{Sym}[2^{n/2}] P_1(n) \tag{2.63}
\]

Taking the limit when \( n \to \infty \), using Theorem \[ \text{II} \] showing \( R_{n,K} \to 0 \) and, the fact that \( n^K \text{Sym}[2^{n/2}] P_1(n) \sim n^{n/2+K-3/2} e^{B \sqrt{n} - n/2} \) is suppressed by the denominator \( n! \sim n^n e^{-n} \), for any finite \( K \), we obtain the result.

- In the same vein, when \( n \) is odd, we have for \( n > 11 \tag{2.25} \),

\[
Z_{3:2+}(n) \leq \text{Sym}[3, 2^{(n-3)/2}] P_1(n) \tag{2.64}
\]

which leads, using the same above argument, to a vanishing remainder.

\[ \square \]

Remark: Note that \( Z_{3:1}(n) \) and \( Z_3(n) \) have the same large \( n \) asymptotic expansion. We will discuss a large \( n \) characterization of \( Z_3(n) \) which conjecturally fixes it uniquely in section \[ \text{III} \].

2.4 Interpretation in terms of probability distributions over bipartite ribbon graphs

It is natural to ask how we should interpret the asymptotic results we have found here. Which tensor invariants dominate in the large \( n \) limit? As explained in \[ 6, 16, 29 \] the
tensor invariants of degree $n$ correspond to bi-partite ribbon graphs with $n$ edges. These are in 1-1 correspondence with orbits of an action by $S_n$ on pairs $(\sigma_1, \sigma_2) \in S_n \times S_n$. The action of $\gamma \in S_n$ is given by

$$(\sigma_1, \sigma_2) \sim (\gamma \sigma_1 \gamma^{-1}, \gamma \sigma_2 \gamma^{-1}) \quad (2.65)$$

For each orbit there is a tensor invariant or bi-partite ribbon graph. Letting $r$ be an index running over the set of bipartite ribbon graphs, we can pick pairs $(\sigma_1^{(r)}, \sigma_2^{(r)}) \in S_n \times S_n$ in the orbit. One way to understand asymptotic results is to find configurations that dominate. For example the Plancherel distribution for Young diagrams is dominated by typical Young diagrams with shape close to a limit curve [10]. So is there a class of bi-partite graphs which dominate in the large $n$ limit? The short answer is that rather than dominant ribbon graphs, the explanation that follows from the derivation is that the leading asymptotics is determined by a probability distribution over ribbon graphs.

To understand this, recall the derivation using Burnside Lemma

$$Z_3(n) = \frac{1}{n!} \sum_{\gamma \in S_n} \sum_{\sigma_1, \sigma_2 \in S_n} \delta(\gamma \sigma_1 \gamma^{-1} \sigma_1^{-1}) \delta(\gamma \sigma_2 \gamma^{-1} \sigma_2^{-1}) = \frac{1}{n!} \sum_{p \vdash n} \frac{n!}{\text{Sym}(p)} (\text{Sym}(p))^2 = \sum_p \text{Sym}(p) \quad (2.66)$$

which is given and explained in more detail in [6]. The factor $\frac{n!}{\text{Sym}(p)}$ is the number of permutations $\gamma$ in the conjugacy class $p$. The two sums over $\sigma_1, \sigma_2$ give the factor $(\text{Sym}(p))^2$. Alternatively we can take the sum over $(\sigma_1, \sigma_2)$ outside and write it as a sum over orbits. We use $|\text{Orb}(r)|$ to denote the number of permutation pairs in the orbit of $(\sigma_1^{(r)}, \sigma_2^{(r)})$ and $|\text{Aut}(r)|$ is the number of permutations $\gamma$ leaving fixed the pair $(\sigma_1^{(r)}, \sigma_2^{(r)})$. By the orbit stabilizer theorem we have $n!/|\text{Orb}(r)| = |\text{Aut}(r)|$. We will use $\text{Aut}(r) \cap [p]$ to denote the subset of $\gamma \in \text{Aut}(r)$ which belong to the conjugacy class $[p]$ where the cycles of $\gamma$ define the partition $p$ of $n$ (we denote this as $[\gamma] = [p]$ below):

$$Z_3(n) = \sum_{\sigma_1, \sigma_2 \in S_n} \frac{1}{n!} \sum_{\gamma \in S_n} \delta(\gamma \sigma_1 \gamma^{-1} \sigma_1^{-1}) \delta(\gamma \sigma_2 \gamma^{-1} \sigma_2^{-1})$$

$$= \sum_r \frac{1}{n!} |\text{Orb}(r)| \sum_{\gamma} \delta(\gamma \sigma_1^{(r)} \gamma^{-1} (\sigma_1^{(r)})^{-1}) \delta(\gamma \sigma_2^{(r)} \gamma^{-1} (\sigma_2^{(r)})^{-1})$$

$$= \sum_r \frac{1}{n!} |\text{Orb}(r)| \sum_{p \vdash n} \sum_{\gamma : [\gamma] = [p]} \delta(\gamma \sigma_1^{(r)} \gamma^{-1} (\sigma_1^{(r)})^{-1}) \delta(\gamma \sigma_2^{(r)} \gamma^{-1} (\sigma_2^{(r)})^{-1})$$

$$= \sum_r \frac{1}{|\text{Aut}(r)|} \sum_{p \vdash n} |\text{Aut}(r) \cap [p]|$$

$$= \sum_{p \vdash n} \sum_r \frac{1}{|\text{Aut}(r)|} |\text{Aut}(r) \cap [p]| \quad (2.67)$$
For each fixed $p$, the sum over $r$ gives $\text{Sym}(p)$

$$\text{Sym}(p) = \sum_r \frac{|\text{Aut}(r) \cap [p]|}{|\text{Aut}(r)|}$$

(2.68)

from which the formula (2.66) for $Z_3(n)$ as a sum of the symmetry factors. This is a very interesting equation. The LHS is an integer defined entirely in terms of $S_n$. For any $\sigma$ in the conjugacy class $[p]$, it is the number of permutations $\gamma \in S_n$ such that $\gamma \sigma \gamma^{-1} = \sigma$. On the RHS we have a sum over bipartite ribbon graphs with $n$ edges (equivalently over tensor invariants). Each term is a positive rational number smaller or equal to 1. It is useful to spell out the derivation of (2.68):

$$\sum_r \frac{|\text{Aut}(r) \cap [p]|}{|\text{Aut}(r)|} = \sum_r \frac{|\text{Orb}(r)|}{n!} |\text{Aut}(r) \cap [p]|$$

$$= \sum_r \frac{|\text{Orb}(r)|}{n!} \sum_{\gamma;[\gamma]=[p]} \delta(\gamma \sigma_1^{(r)} \gamma^{-1})(\sigma_1^{(r)})^{-1}) \delta(\gamma \sigma_2^{(r)} \gamma^{-1}(\sigma_2^{(r)})^{-1})$$

$$= \sum_{\sigma_1,\sigma_2 \in S_n} \sum_{\gamma;[\gamma]=[p]} \frac{1}{n!} \delta(\gamma \sigma_1 \gamma^{-1} \sigma_1^{-1}) \delta(\gamma \sigma_2 \gamma^{-1} \sigma_2^{-1})$$

$$= \sum_{\gamma;[\gamma]=[p]} \frac{1}{n!} (\text{Sym}(p))^2 = \frac{n!}{\text{Sym}(p)} \frac{1}{n!} (\text{Sym}(p))^2$$

$$= \text{Sym}(p)$$

(2.69)

We used the fact the number of permutations in the class $[p]$ is $\frac{n!}{\text{Sym}(p)}$ and the structure of the proof is essentially reversing, at fixed $p$, the steps of (2.66).

The equation (2.68) means that, for each $p$, we can define a probability distribution $W(p, r)$ over ribbon graphs

$$W(p, r) = \frac{1}{\text{Sym}(p)} \frac{|\text{Aut}(r) \cap [p]|}{|\text{Aut}(r)|}$$

(2.70)

Since our asymptotic results have been derived by organising the set of $p$ in the sum for $Z_3(n)$ according to powers of $n$, each term can be interpreted using $W(p, r)$. Taking $p = [1^n]$ which contributes the leading term in the asymptotics of $Z_3(n)$, the equation (2.68) becomes

$$n! = \sum_r \frac{1}{|\text{Aut}(r)|}$$

(2.71)

The probability distribution over ribbon graphs is given by

$$W([1^n], r) = \frac{1}{n!|\text{Aut}(r)|}$$

(2.72)
Thus, the leading asymptotics comes from a probability distribution over all bi-partite ribbon graphs, where each contributes an inverse of the order of its automorphism group. The contribution at order $1/n^2$ in $Z_3(n)$ comes from the $[\gamma] = [1^{n-2}, 2]$. This contribution is associated with the probability distribution

$$W([1^{n-2}, 2], r) = \frac{1}{(n - 2)!2} \frac{|\text{Aut}(r) \cap [1^{n-2}, 2]|}{|\text{Aut}(r)|}$$  \hspace{1cm} (2.73)$$

As a generalization of this link to probability distributions, if we consider the coefficient of $1/(n)(n-1)\cdots(n-k+1)$ in (2.58), we have a sum of symmetry factors over a finite set of partitions of the form $p = [1^{n-k}, q]$ with $q \vdash k; q_1 = 0$. The contribution of a given ribbon graph equivalence class (labelled by $r$) to this sum is proportional to a probability distribution over ribbon graphs. Let $S_k$ be the set of partitions of this form specified $p = [1^{n-k}, q]$. For the subset $S_k$ there is a probability distribution

$$W(S_k, r) = \frac{1}{\sum_{p \in S_k} \text{Sym}(p)} \sum_{p \in S_k} \frac{|\text{Aut}(r) \cap [p]|}{|\text{Aut}(r)|}$$  \hspace{1cm} (2.74)$$

It is interesting to describe the geometrical characteristics of the ribbon graphs which lead to the largest contributions for each $p$. Since ribbon graphs also correspond to Belyi maps (see for example [41]), we may phrase this question in terms of characteristics such as Galois invariants of Belyi maps. We leave these as interesting questions for the future.

3 Asymptotics in terms of Stirling numbers and generalization to $Z_{3;m}$

In this section we show that the asymptotic expansion of $Z_3(n)$, which is the same as that of $Z_{3;1}(n)$, involves the well-known Stirling numbers of the second kind. This shows that integers obtained by [38] are expressible in terms of symmetry factors of restricted partitions multiplied by these Stirling numbers. The same structure holds true for asymptotic expansions of $Z_{3;m}$ for higher $m$.

3.1 $Z_3(n) \sim Z_{3;1}(n)$ in terms of Stirling numbers

We use the defining property of the Stirling numbers of the second kind $S(k + r, k)$ [42]

$$\frac{1}{(1-x)(1-2x)\cdots(1-kx)} = \sum_{r=0}^{\infty} S(k+r,k)x^r$$  \hspace{1cm} (3.1)$$

with the substitutions $x \to n^{-1}, k \to (k-1)$ to obtain the large $n$ expansion

$$\frac{1}{n(n-1)\cdots(n-k+1)} = \frac{1}{n^k(1-n^{-1})(1-2n^{-1})\cdots(1-(k-1)n^{-1})}$$

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\[ = \sum_{r=0}^{\infty} S(k - 1 + r, k - 1)(n^{-1})^{r+k} \]  
\[ (3.2) \]

We rewrite using \( K \leq n \)

\[ S_{3;1}(n, K) = n! \sum_{k=0}^{K} \frac{(n-k)!}{n!} \sum_{q \vdash k: q_1 = 0} \text{Sym}(q) \]
\[ = n! \left( 1 + \sum_{k=2}^{\infty} \sum_{r=0}^{\infty} S(k - 1 + r, k - 1)(n^{-1})^{k+r} \sum_{q \vdash k: q_1 = 0} \text{Sym}(q) \right) \]
\[ = n! \left( 1 + \sum_{r=0}^{\infty} \sum_{k=2}^{K} (n^{-1})^{k+r} S(k - 1 + r, k - 1) \sum_{q \vdash k: q_1 = 0} \text{Sym}(q) \right) \]
\[ = n! \left( 1 + \sum_{l=2}^{\infty} (n^{-1})^{l} \sum_{k=2}^{\min(K,l)} S(l - 1, k - 1) \sum_{q \vdash k: q_1 = 0} \text{Sym}(q) \right) \]  
\[ (3.3) \]

To understand the second line of the above, note that there no partitions \( q \) of \( k = 1 \) with \( q_1 = 0 \).

Thus, we have the expansion

\[ S_{3;1;K}(n) = n! \sum_{l=0}^{\infty} a_l(K) (n^{-1})^{l} \]  
\[ (3.4) \]

where the coefficients \( a_l(K) \) are given by

\[ a_0(K) = 1, \quad a_1(K) = 0 \]
\[ a_l(K) = \sum_{k=1}^{\min(K,l)} S(l - 1, k - 1) \sum_{q \vdash k: q_1 = 0} \text{Sym}(q) \quad l \geq 2 \]  
\[ (3.5) \]

It is useful, as in \( (2.23) \) to express the result \( (3.3) \) for the asymptotic \( 1/n \) expansion informally as

\[ \frac{Z_{3;1}(n)}{n!} \sim \left( 1 + \sum_{r=0}^{\infty} \sum_{k=2}^{\infty} (n^{-1})^{k+r} S(k - 1 + r, k - 1) \sum_{q \vdash k: q_1 = 0} \text{Sym}(q) \right) \]
\[ \sim \left( 1 + \sum_{l=2}^{\infty} (n^{-1})^{l} \sum_{k=2}^{l} S(l - 1, k - 1) \sum_{q \vdash k: q_1 = 0} \text{Sym}(q) \right) \]  
\[ (3.6) \]

It also follows from Theorem 1 and Theorem 3 that \( \frac{Z_3(n)}{n!} \) has the same asymptotics as \( \frac{Z_{3;1}(n)}{n!} \) so that we have

\[ \frac{Z_3(n)}{n!} \sim \left( 1 + \sum_{l=2}^{\infty} (n^{-1})^{l} \sum_{k=2}^{l} S(l - 1, k - 1) \sum_{q \vdash k: q_1 = 0} \text{Sym}(q) \right) \]  
\[ (3.7) \]
Examples. We illustrate the above formulae and check if the coefficient appears in the series (2.19).

- For \( l = 2, K = n > 2 \)

\[
\begin{align*}
a_2 &= \sum_{k=1}^2 S(1, k-1) \sum_{q^i \cdot q_1 = 0} \text{Sym}(q) \\
&= S(1, 0) \sum_{q^i \cdot q_1 = 0} \text{Sym}(q) + S(1, 1) \sum_{q^2 \cdot q_1 = 0} \text{Sym}(q) = S(1, 1) \text{Sym}([2]) = 2
\end{align*}
\]

That agrees with (2.19).

- For \( l = 3, K = n > 3 \)

\[
\begin{align*}
a_3 &= \sum_{k=1}^3 S(2, k-1) \sum_{q^i \cdot q_1 = 0} \text{Sym}(q) \\
&= S(2, 0) \sum_{q^i \cdot q_1 = 0} \text{Sym}(q) + S(2, 1) \sum_{q^2 \cdot q_1 = 0} \text{Sym}(q) + S(2, 2) \sum_{q^3 \cdot q_1 = 0} \text{Sym}(q) \\
&= S(2, 1) \text{Sym}([2]) + S(2, 2) \text{Sym}([3]) = 2 + 3 = 5
\end{align*}
\]

that once again agrees with (2.19).

More generally, the formula matches with asymptotic expansion as given in OEIS A279819.

3.2 \( Z_{3;m}(n) \) in terms of Stirling numbers

We conjecture here the asymptotic series expansion for \( Z_{3;m}(n) \), for general finite \( m \), according to similar arguments given above. We recall that

\[
Z_{3;m}(n) = \sum_{p \in P_m(n)} \text{Sym}(p) = \sum_{p_1=\ldots=p_{m-1}=0; \ p_m>0} \text{Sym}(p)
\]

with \( m \leq n \).

For \( m > 1 \), there are in fact more constraints on the partition than \( p \vdash n \), \( p_1 = \cdots = p_{m-1} = 0 \), \( p_m > 0 \), in the above sum (3.10). Indeed, consider the Euclidean division

\[
n = ml_1 + l_2, \quad 0 \leq l_2 < m, \quad l_1 \geq 1
\]

Two cases should be discussed pertaining to the value of the remainder: either \( l_2 \) equals 0 or does not.
If $l_2 = 0$, then $n = ml_1$ and we claim the dominant term in (3.10) is given by

$$p = [ml_1]$$

(3.12)

If $l_2 > 0$, then $n - l_2 = ml_1$, then the following term should be the dominant one:

$$p = [ml_1^{-1}, q], \quad q \vdash m + l_2$$

(3.13)

where $q$ should also obey $q_1 = q_2 = \cdots = q_m = 0$. A quick inspection shows the unique possibility $q = [m + l_2]$, hence $p = [ml_1^{-1}, m + l_2]$.

Depending on $l_2$, we use the notation $\delta_{l_2=0} = 1$, if $l_2 = 0$ and $\delta_{l_2=0} = 0$, otherwise, and $\delta_{l_2>0} = 1$, if $l_2 > 0$ and $\delta_{l_2>0} = 0$, otherwise. We expand the partial sum

$$S_{3;m;K}(n) = \delta_{l_2=0} l_1!m^{l_1} + \delta_{l_2>0} (l_1 - 1)!m^{l_1^{-1}}(m + l_2)$$

$$+ \sum_{k=2}^{K} (l_1 - k)!m^{l_1-k} \sum_{q^\vdash mk + l_2, q_1 = q_2 = \cdots = q_m = 0} \text{Sym}(q)$$

(3.14)

where it is understood that $K \leq l_1$, as $n$ and, therefore, $l_1 = (n - l_2)/m$ go to infinity.

For $l_2 = 0$, we can further expand $S_{3;m;K}(n)$ and obtain the coefficients of the conjectured asymptotic expansion of $Z_{3,m}(n)$. In an analogous way to the steps leading to (3.3), we introduce

$$F_{m,k}(l_2) := \sum_{q^\vdash mk + l_2, q_1 = q_2 = \cdots = q_m = 0} \text{Sym}(q)$$

(3.15)

and write:

$$S_{3;m;K}(n) = l_1!m^{l_1} \left[1 + \sum_{k=2}^{K} m^{-k} \frac{(l_1 - k)!}{l_1!} F_{m,k}(0)\right]$$

(3.16)

$$= l_1!m^{l_1} \left[1 + \sum_{k=2}^{K} m^{-k} \sum_{r=0}^{\infty} S(k - 1 + r, k - 1) (l_1^{-1})^{k+r} F_{m,k}(0)\right]$$

$$= l_1!m^{l_1} \left[1 + \sum_{r=2}^{\infty} (l_1^{-1})^r \sum_{k=2}^{\min(K,r)} m^{-k} S(r - 1, k - 1) F_{m,k}(0)\right]$$

Thus, the coefficients of the expansion of $S_{3;m;K}(n)/(l_1!m^{l_1})$ read off

$$a_0(m, K) = 1$$
$$a_1(m, K) = 0$$
$$a_r(m, K) = \sum_{k=2}^{\min(K,r)} m^{r-k} S(r - 1, k - 1) \text{Sym}_{m,k}(0)$$

(3.17)
where in the last line \( r \geq 2 \). This covers the case \( m = 1 \) in (3.5). Our conjecture is expressed as

\[
Z_{3;m}(n) \sim \frac{1}{(2m)!m^{3/2}} \sum_{r=2}^{\infty} (n^{-1})^r \sum_{k=2}^{\min(K,r)} m^{r-k} S(r-1, k-1) \text{Sym}_{m,k}(0)
\]

(3.18)

Applying (3.17) to \( m = 2 \), the procedure computing the coefficients of the expansion of \( S_{3,2}(n)/(2^{n/2}(n/2)!) \) yield at \( n = 50 \)

\[
\begin{align*}
r &= 2, & 4 \\
r &= 3, & 32 \\
r &= 4, & 215 \\
r &= 5, & 1541 \\
r &= 6, & 14658 \\
r &= 7, & 180246 \\
r &= 8, & 2425061 \\
r &= 9, & 33315155 \\
r &= 10, & 478703544
\end{align*}
\]

(3.19)

a sequence unlisted in OEIS.

We now address the case \( l_2 > 0 \) and express \( S_{3;m,K}(n) \) as

\[
S_{3;m,K}(n) = (l_1 - 1)!m^{l_1-1} \left[ (m + l_2) + \sum_{k=2}^{K} m^{-(k-1)} \frac{(l_1 - 1 - (k-1))!}{(l_1 - 1)!} F_{m,k}(l_2) \right]
\]

\[
= (l_1 - 1)!m^{l_1-1} \left[ (m + l_2) + \sum_{k=2}^{K} m^{-(k-1)} \sum_{r=0}^{\infty} S(k - 2 + r, k - 2)(l_1^{-1})^{k-1+r} F_{m,k}(l_2) \right]
\]

\[
= (l_1 - 1)!m^{l_1-1} \left[ (m + l_2) + \sum_{r=2}^{\infty} (l_1^{-1})^{r-1} \sum_{k=2}^{\min(K,r)} m^{-(k-1)} S(r-2, k-2) F_{m,k}(l_2) \right]
\]

\[
= (l_1 - 1)!m^{l_1-1} \times \left[ (m + l_2) + \sum_{r=1}^{\infty} (n - l_2)^{-r} \sum_{k=2}^{\min(K,r+1)} m^{r-(k-1)} S(r-1, k-2) F_{m,k}(l_2) \right]
\]

\[
= (l_1 - 1)!m^{l_1-1} \times \left[ (m + l_2) + \sum_{r=1}^{\infty} n^{-r} \left( \sum_{i=0}^{\infty} C_{r,i} \left( \frac{l_2}{n} \right)^i \right)^{\min(K,r+1)} \sum_{k=2}^{\min(K,r+1)} m^{r-(k-1)} S(r-1, k-2) F_{m,k}(l_2) \right]
\]

where we used the notation \( C_{r,i} = \frac{r(r-1)...(r-i+1)}{i!} \) for the generalized binomial coefficient. We obtain

\[
S_{3;m,K}(n) = (l_1 - 1)!m^{l_1-1}
\]
\[
\times \left[ (m + l_2) + \sum_{r=1}^{\infty} \sum_{i=0}^{\infty} C_{r,i} \frac{l_2^r}{n^{r+i}} \sum_{k=2}^{\text{min}(K,r+1)} m^{r-(k-1)} S(r-1, k-2) F_{m,k}(l_2) \right]
\]
\[
= (l_1 - 1)! m^{l_1-1} \times \left[ (m + l_2) + \sum_{s=1}^{\infty} n^{-s} \sum_{r=0}^{s} l_2^{s-r} C_{r,s-r} \sum_{k=2}^{\text{min}(K,r+1)} m^{r-(k-1)} S(r-1, k-2) F_{m,k}(l_2) \right]
\]

The coefficients of the expansion of \( S_{3;m;K}(n)/((l_1 - 1)! m^{l_1-1}(m + l_2)) \) are given by,
\[
a_0(m, K) = 1 \tag{3.20}
\]
\[
a_s(m, K) = \frac{1}{(m + l_2)} \sum_{r=0}^{s} l_2^{s-r} C_{r,s-r} \sum_{k=2}^{\text{min}(K,r+1)} m^{r+1-k} S(r-1, k-2) F_{m,k}(l_2)
\]

where the last line holds for \( s \geq 1 \).

In this case, \( l_2 > 0 \), we conjecture the asymptotics,
\[
\frac{Z_{3;m}(n)}{(\frac{n}{m} - 1)! m^{\frac{n}{m}-1}(m + l_2)} \sim 1 + \sum_{s=1}^{\infty} n^{-s} \sum_{r=0}^{s} l_2^{s-r} C_{r,s-r} \sum_{k=2}^{\text{min}(K,r+1)} m^{r-(k-1)} S(r-1, k-2) F_{m,k}(l_2) \tag{3.21}
\]

### 3.3 Discussion: Non-perturbative asymptotics of \( Z_3(n) \)

We have established the large \( n \) asymptotic series for \( Z_3, Z_{3;1} \), both of which have the same large \( n \) series. We have argued for and conjectured the large \( n \) series for \( Z_{3;m} \), for any \( m \sim \mathcal{O}(1) \) as \( n \to \infty \). Drawing on analogies with non-perturbative expansions in QFT and quantum mechanics, it is natural to ask whether our knowledge of the asymptotic expansions of \( Z_{3;m} \) can be collected into a non-perturbative expansion for \( Z_3 \). Although \( Z_3 = \sum_m Z_{3;m} \) this is not straightforward since the number of terms in the sum over \( m \) goes to infinity as \( n \to \infty \).

Let us first explain the analogy in more detail. Instanton expansions in QFT where QFT observables are expressed as an approximation of the form
\[
F(g) \sim S_0(g) + e^{\frac{1}{\sigma}} S_1(g) + e^{\frac{2}{\sigma}} S_2(g) + \cdots \tag{3.22}
\]
Here \( S_0(g), S_1(g), \cdots \) are power series in powers of \( g \), the coupling constant, which are asymptotic expansions in the limit \( g \to 0 \). The successively higher instanton numbers are exponentially suppressed in the limit. See [44] for a review of this subject.

Based on the analogy, we can ask if it is possible to make sense of an expansion of the form
\[
\frac{Z_3}{n!} \sim S_{3;1} + f_2(n) S_{3;2} + f_3(n) S_{3;3} + \cdots \tag{3.23}
\]
$S_{3;1}$ is a power series analogous to $S_0(g)$, the perturbative term in QFT. $S_{3;2}, S_{3;3} \cdots$ are likewise power series in $\frac{1}{n}$ analogous to $S_1(g), S_2(g) \cdots$ in QFT. $f_2(n)$ is super-exponentially suppressed compared to 1, $f_3(n)$ is super-exponentially suppressed compared to $f_2(n)$ etc. As we saw, the series $S_{3;1}$ is obtained from $Z_{3;1}$ where $Z_{3;1}$ is the sum of $\text{Sym}(p)$ for partitions where the minimum part has length 1. Finite truncations $S_{3;1;K}$ are obtained by taking $n \gg K$ and summing terms in $\frac{Z_{3;1}}{n!}$ with $n-K$ parts of length 1.

The term $f_2(n)S_{3;2}$ is obtained from $Z_{3;2}/n!$ which is the sum of $\text{Sym}(p)$ for partitions that have no cycles of length 1 and a non-zero number of parts of length 2. Finite truncations $f_2(n)S_{3;2;K}$ are obtained by taking $n \gg K$ and considering terms in $Z_{3;2}$ with at least $(n-K)$ parts of length 2:

$$f_2(n) = \frac{([n/2])!2^{[n/2]}}{n!}$$  \hspace{1cm} (3.24)

We see that $f_2(n)$ is super-exponentially suppressed compared to $f_1(n) = 1$. Similarly $f_3(n)S_{3;3}$ is obtained from $Z_{3;3}/n!$ which is the sum of $\text{Sym}(p)$ for partitions having no cycles of length 1, 2 and a minimum part of length 3. $f_3(n)$ is the symmetry factor for a partition with the largest number of 3, divided by $n!$.

$$f_3(n) = \frac{([n/3])!3^{[n/3]}}{n!}$$  \hspace{1cm} (3.25)

The series $S_{3;3}$ starts with 1 and any finite order truncation $S_{3;3;K}$ is obtained by taking $n \gg K$ and summing $\text{Sym}(p)$ over partitions having $[n/3] - K$ parts of length 3. It is easy to see that $f_3(n)$ is super-exponentially suppressed compared to $f_1(n)$ and $f_2(n)$.

One approach to making sense of (3.23) is to interpret it as a sequence of asymptotic expansions related to $Z_3(n)$

$$Z_3(n) \sim S_{3;1}(n)$$  \hspace{1cm} (3.26)

After subtracting $Z_{3;1}(n)$ we have

$$\frac{Z_3(n) - Z_{3;1}(n)}{n!f_2(n)} \sim S_{3;2}(n)$$  \hspace{1cm} (3.27)

After further subtracting $Z_{3;2}(n)$ we have

$$\frac{Z_3(n) - Z_{3;1}(n) - Z_{3;2}(n)}{n!f_3(n)} \sim S_{3;3}(n)$$  \hspace{1cm} (3.28)

Indeed for any finite $m$,

$$\frac{Z_3(n) - Z_{3;1}(n) - \cdots - Z_{3;m}(n)}{n!f_{m+1}(n)} \sim S_{3;m+1}(n)$$  \hspace{1cm} (3.29)
$Z_3(n)/n!$ is not unique in having the asymptotic expansion $S_{3,1}(n) : (Z_3(n) - Z_{3;2})/n!$ has the same expansion, it is tempting to conjecture that the above equations uniquely determine $Z_3(n)$. Thus we present a conjecture.

**Conjecture.** If a function $F(n)$ obeys the properties

$$\frac{F(n)}{n!} \sim S_{3,1}(n)$$

$$\frac{F(n) - Z_{3,1}(n) - Z_{3,2}(n)}{n!} \sim S_{3,2}(n)$$

$$\frac{F(n) - Z_{3,1}(n) - Z_{3,2}(n) - Z_{3,3}(n)}{n!f_3(n)} \sim S_{3,3}(n)$$

$$\vdots$$

$$\frac{F(n) - Z_{3,1}(n) - \cdots - Z_{3,m}(n)}{n!f_{m+1}(n)} \sim S_{3,m+1}(n)$$

for all finite $m$, then $F(n) = Z_3(n)$.

### 4 Higher rank tensors

There exists an enumeration formula for higher rank $d$ tensor model observables in terms of sums of powers of symmetry factors [6]. It is then natural to ask, in full generality, the question of the asymptotic expansion of that counting. At this point, from the asymptotic dominance of small parts, and since one easily realizes that this should hold independently of the rank of the tensor invariant, we conjecture below the series expansion of that counting of rank $d$ tensor.

For rank $d$ tensors, $Z_d(n)$ counts the number of rank $d$ tensor invariants. We have in a similar way as above

$$Z_d(n) = \sum_{p\triangleright n} (\text{Sym } p)^{d-2} = \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} Z_{d;m}(n) + Z_{d,n}(n)$$

with, for $1 \leq m \leq \lfloor \frac{n}{2} \rfloor$.

For the next developments, we assume for simplicity that $n = ml$, i.e. that $n$ is a multiple of $m$. The generic case should require a bit more work.

We introduce the partial sum, assuming that $n$ and $l$ are large enough,

$$S_{d,m;K}(n) = \sum_{k=0}^{K} \sum_{q^i = m; q_1 = q_2 = \cdots = q_m = 0} (\text{Sym}([m^{f-k},q]))^{d-2}, \quad 1 \leq m \leq \lfloor \frac{n}{2} \rfloor$$

$$S_{d;n;K}(n) = (\text{Sym}([n]))^{d-2} = n^{d-2} = Z_{d,n}(n)$$

(4.2)
Thus we conjecture that

\[ Z \]

\[ \text{Using a change of variables } \tilde{p} \]

By similar techniques previously introduced, we can work out the following expansions:

When \( n \to \infty \), the coefficient \( (m^{l-k}(l-k)!)^{d-2} \) becomes less and less dominant as soon as \( m > 1 \).

Let us restrict to \( m = 1 \), and conjecture an asymptotic expansion of that sector (we conjecture to be the dominant order is given by the fixed order \( m = 1 \)):

\[
S_{d;1;K}(n) = \sum_{q^r m k; q_1 = q_2 = \cdots = q_m = 0}^{K} \left( m^{l-k}(l-k)! \operatorname{Sym}(q) \right)^{d-2} = \sum_{q^r m k; q_1 = q_2 = \cdots = q_m = 0}^{K} \left( (n-k)! \operatorname{Sym}(q) \right)^{d-2}
\]

\[
= (n!)^{d-2} \left[ 1 + \sum_{k=2}^{K} \left( \frac{(n-k)!}{n!} \right)^{d-2} \sum_{q^r m k; q_1 = q_2 = \cdots = q_m = 0}^{K} \left( \operatorname{Sym}(q) \right)^{d-2} \right]
\]

\[
= (n!)^{d-2} \left[ 1 + \sum_{k=2}^{K} \left( \sum_{r=0}^{\infty} S(k - 1 + r, k - 1)(n^{-1})^{k+r} \right) \sum_{q^r m k; q_1 = q_2 = \cdots = q_m = 0}^{K} \left( \operatorname{Sym}(q) \right)^{d-2} \right]
\]

\[
= (n!)^{d-2} \left[ 1 + \sum_{k=2}^{K} \sum_{p=0}^{\infty} n^{-p-(d-2)k} \frac{\left( \prod_{i=1}^{d-2} S(k - 1 + r_i, k - 1) \right) \sum_{q^r m k; q_1 = q_2 = \cdots = q_m = 0}^{K} \left( \operatorname{Sym}(q) \right)^{d-2}}{\sum_{i=1}^{d-2} r_i = p} \right]
\]

Using a change of variables \( \tilde{p} = p + (d-2)k \) (and rename \( \tilde{p} \to p \)), and \( \tilde{r}_i = r_i + k \) (and rename \( \tilde{r}_i \to r_i \)), then swapping the two sums over \( k \) and \( p \), we obtain an expression generalizing \( (3.3) \),

\[
S_{d;1;K}(n) = (n!)^{d-2} \left[ 1 + \sum_{p=2(d-2)}^{\min(K, \frac{p}{d-2})} n^{-p} \right]
\]

\[
\times \sum_{k=2}^{\min(K, \frac{p}{d-2})} \sum_{\sum_{i=1}^{d-2} r_i = p; \text{ and } r_i \geq k}^{\sum_{i=1}^{d-2} r_i = p} \left( \prod_{i=1}^{d-2} S(r_i - 1, k - 1) \right) \sum_{q^r m k; q_1 = q_2 = \cdots = q_m = 0}^{K} \left( \operatorname{Sym}(q) \right)^{d-2}
\]

Thus we conjecture that \( \frac{Z_{d;1}}{(n!)^{d-2}} \sim \sum_{p=0}^{\infty} A_{d,p}/n^p \) has the coefficients

\[ A_{d,0} = (n!)^{d-2} \]
\[ A_{d,p} = \min(K, \frac{p}{d-2}) \sum_{k=2}^{d-1} \sum_{r_i=p; \text{ and } r_i \geq k}^{\sum_{i=1}^{d-2} r_i = p} \left( \prod_{i=1}^{d-2} S(r_i - 1, k - 1) \right) \sum_{q^k q_1 = 0} \left( \text{Sym}(q) \right)^{d-2} \] 

(4.6)

for \( p \geq 2(d - 2) \). The same asymptotic series should hold for \( \frac{Z_d}{(n!)^{d-2}} \). Restricted to \( d = 3 \), we recover (3.3) as expected.

5 Conclusion

We have determined the asymptotic expansion of the counting of rank 3 (unitary) tensor invariants. We have exploited the counting formula in terms of a sum of symmetry factors of partitions. A general principle we have found useful is that these sums of symmetry factors are dominated by partitions with a large multiplicity of a small part. The asymptotic series has been provided and its coefficients determined at all orders: the key results are (2.23)(3.7). As an interesting feature, we express these coefficients as a sum of Stirling numbers of the second kind. We also conjecture similar formulae for the enumeration of any rank \( d \) tensor invariants, and expect similar proofs will work. The same general principle allows formulae for sums of symmetry factors of restricted partitions, which we have denoted \( Z_{3,m} \).

It would be interesting to investigate the asymptotics of connected tensor invariants. The connected invariants are obtained from the disconnected ones, \( Z_3(n) \) (for which we have derived the asymptotics), by taking a plethystic logarithm [6]. The sequence of connected invariants is known to high orders [45]. By inserting the asymptotic expansion of \( Z_3(n) \) into the plethystic logarithm (PLOG) function, it should be possible to obtain the asymptotic expansion of [45]. Finally, we may ask if the same ideas developed in this work could be applied to the analysis of orthogonal tensor invariants [20]. That series would be slightly different but we expect that the main principle discovered in this work, i.e. partitions with a large multiplicity of a small part will dominate sums over symmetry factors, would apply again in that situation. This deserves to be addressed thoroughly.

It is instructive, in the context of holography and brane physics, to compare the asymptotics of tensor model counting with that of multi-matrix models, and to develop interpretations of the asymptotic results (2.23)(3.7) in these contexts. These asymptotic results have implications for the thermodynamics of quantum mechanical models based on tensor or multi-matrix models respectively. The multi-matrix case has been discussed in the context of AdS5/CFT4 and related gauge theories in [46] and more recently in [47, 48, 49, 50]. The asymptotic counting we have done in this paper holds at large \( N \). We are considering large \( n \) invariants when \( N \gg n \). The super-exponential growth of \( Z_3(n) \) has the consequence of a vanishing Hagedorn temperature in this large \( n \) limit [10]. It will be very interesting to investigate the fate of this Hagedorn behaviour in the
finite $N$ tensor systems. The analogous investigation has been investigated for multi-matrix models in [47, 50]. The role of tensor models in connection with M5-branes has been discussed in [10]. Beyond the counting of observables it is also interesting to look at the asymptotic behaviour of correlators, for example with motivations from quantum information theoretic aspects of holography [12].

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Appendix

A Proof: of Lemma 1

In this appendix, the proof of Lemma 1 is given. It divides into several cases that must be carefully checked.

A.1 Case $k$ even: Proof of $\text{Sym}[2^{k/2}] \geq \text{Sym}(q)$, $q \vdash k$, $q_1 = 0$

We want to prove that $\text{Sym}[2^{k/2}] > \text{Sym}(q)$, for $q \vdash k$, with $q_1 = 0$, for $k$ sufficiently large. We proceed by induction on $k$.

Assume $k = 2$. It is easy to see that $\text{Sym}[2^{k/2}] \geq \text{Sym}(q)$ for $q \vdash 2$, $q_1 = 0$. The condition $q_1 = 0$ means that $q = [2]$, so that $\text{Sym}(q) = 2 = \text{Sym}[2^{2/2}]$. Let us assume this to be true for all even $k'$ up to $k$, that is

$$\text{Sym}[2^{k'/2}] \geq \text{Sym}(q), \quad q \vdash k', \quad q_1 = 0, \text{ for } k' \text{ even and } k' \leq k \quad (A.1)$$

We now prove for $k + 2$ that $\text{Sym}[2^{(k+2)/2}] \geq \text{Sym}(q)$, $q \vdash k + 2$, $q_1 = 0$.

Consider $q \vdash k + 2$, we decompose $q = [2^m, q']$, with $q' \vdash k + 2 - 2m$, $q'_2 = 0$ (no part of size 2 in $q'$).

Case $0 < m \leq (k + 2)/2$: Then $q' \vdash k + 2 - 2m = k - 2(m - 1) \leq k$, so as $k - 2(m - 1)$ is even then we write using our induction hypothesis

$$\text{Sym}(q) = \text{Sym}[2^m, q'] = 2^m m! \text{Sym}(q') \leq 2^m m! \text{Sym}(2^{(k-2(m-1))/2})$$
\[ = 2^m m! 2^{(k-2(m-1))/2} \left( \frac{k - 2(m - 1)}{2} \right)! \]
\[ = 2^{(k+2)/2} m! \left( \frac{k - 2(m - 1)}{2} \right)! \leq 2^{(k+2)/2} \left( \frac{k + 2}{2} \right)! = \text{Sym}[2^{(k+2)/2}] \]  
(A.2)

**Case** \( m = 0 \): Let \( l_0 > 2 \) be the minimum part such that \( q_{l_0} > 0 \), thus \( q = [l_0^{q_{l_0}}, q'] \) with \( q' \) only containing parts of size strictly larger than \( l_0 \); \( k + 2 = \sum_{l \geq l_0} l q_l \). Then

\[ \sum_{l > l_0} l q_l = k + 2 - l_0 q_{l_0} < k + 2 - 2 q_{l_0} = k - 2 (q_{l_0} - 1) \leq k \]  
(A.3)

Then we write using the induction hypothesis on \( q' \)

\[ \text{Sym}[l_0^{q_{l_0}}, q'] = l_0^{q_{l_0}} q_{l_0}! \text{Sym}[q'] \leq l_0^{q_{l_0}} q_{l_0}! \text{Sym}[2^{(k-2(q_{l_0} - 1))/2}] \]  
(A.4)

We want to show that

\[ l_0^{q_{l_0}} q_{l_0}! \text{Sym}[2^{(k-2(q_{l_0} - 1))/2}] \leq \text{Sym}[2^{(k+2)/2}] . \]  
(A.5)

We evaluate the ratio

\[ \frac{2^{(k+2)/2} ((k + 2)/2)!}{l_0^{q_{l_0}} q_{l_0}! 2^{(k-2(q_{l_0} - 1))/2} ((k - 2(q_{l_0} - 1))/2)!} = \frac{2^{(k+2)/2 - (k-2(q_{l_0} - 1))/2} ((k + 2)/2)!}{l_0^{q_{l_0}} q_{l_0}! ((k - 2(q_{l_0} - 1))/2)!} \]

\[ = \frac{\prod_{i=0}^{q_{l_0} - 1} (k + 2 - 2i)}{l_0^{q_{l_0}} q_{l_0}!} = \frac{\prod_{i=0}^{q_{l_0} - 1} (k + 2 - 2i)}{\prod_{i=0}^{q_{l_0} - 1} (l_0 q_{l_0} - l_0 i)} \geq \frac{\prod_{i=0}^{q_{l_0} - 1} (k + 2 - 2i)}{\prod_{i=0}^{q_{l_0} - 1} (l_0 q_{l_0} - 2i)} \]  
(A.6)

Using \( k + 2 \geq l_0 q_{l_0} \) and \( (k + 2 - 2i) \geq (l_0 q_{l_0} - 2i) \), for all \( i = 0, ..., q_{l_0} - 1 \), the proof is completed.

**A.2 Case** \( k \geq 11 \) odd: Proof of \( \text{Sym}[3, 2^{(k-3)/2}] \geq \text{Sym}(q) \), \( q \vdash k \), \( q_1 = 0 \)

We proceed again by induction on \( k \).

Let \( k = 11 \), the list of partitions of \( q \vdash k \) with parts \( \geq 2 \) and their corresponding \( \text{Sym}(q) \) are given by

\[
\begin{align*}
\text{Sym}[11] &= 11, & \text{Sym}[9, 2] &= 18, & \text{Sym}[8, 3] &= 24, & \text{Sym}[7, 4] &= 28, \\
\text{Sym}[7, 2^3] &= 56, & \text{Sym}[6, 5] &= 30, & \text{Sym}[6, 3, 2] &= 36, & \text{Sym}[5, 4, 2] &= 40, \\
\text{Sym}[5, 3^2] &= 90, & \text{Sym}[5, 2^3] &= 240, & \text{Sym}[4^2, 3] &= 96, & \text{Sym}[4, 3, 2^2] &= 96, \\
\text{Sym}[3^3, 2] &= 324, & \text{Sym}[3, 2^4] &= 1152.
\end{align*}
\]  
(A.7)
Hence \( \text{Sym}[3, 2^{(11-3)/2}] \geq \text{Sym}(q) \), for any other \( q \vdash k \) in the above list. Let us assume that the statement is true at order \( k \geq k' \geq 11 \)

\[
\text{Sym}[3, 2^{(k'-3)/2}] \geq \text{Sym}(q), \quad q \vdash k', \quad q_1 = 0.
\]

(A.8)

Let us prove it at order \( k + 2 \).

Consider the partition \( q = [2^{q_2}, 3^{q_3}, q'] \vdash k + 2 \), where \( q' \vdash k + 2 - 2q_2 - 3q_3 \). Either \( q' \) is empty in which case \( k + 2 - 2q_2 - 3q_3 = 0 \), or \( q' \) is a non-empty partition with parts of size 4 or greater, in which case \( k + 2 - 2q_2 - 3q_3 \geq 4 \).

**Case** \( k + 2 - 2q_2 - 3q_3 = 0 \). Then \( q' \) is an empty partition. Then

\[
k + 2 = 2q_2 + 3q_3,
\]

and we should compare \( \text{Sym}(q) = 2^{q_2} q_2! 3^{q_3} q_3! \) (for \( q_3 > 1 \)) and \( \text{Sym}[3, 2^{(k+2-3)/2}] \). If \( k + 2 \geq 13 \), and is odd, then \( q_3 > 1 \) and should be an odd number. From now, \( q_3 \geq 3 \).

We write the ratio (with \( k + 2 - 3 = k - 1 \geq 10 \))

\[
\frac{\text{Sym}[3, 2^{(k+2-3)/2}]}{\text{Sym}(q)} = \frac{3 \cdot 2^{q_2 + 3(q_3 - 1)/2} (2q_2 + 3q_3 - 3)/2)!}{2^{q_2} q_2! 3^{q_3} q_3!} = \frac{3 \cdot 2^{q_2 + (q_3 - 3)/2} (q_2 + q_3 + (q_3 - 3)/2)!}{3^{q_3} q_3!} \quad \text{(A.10)}
\]

Assuming \( q_3 = 3 \), then \( k + 2 = 2q_2 + 9 \), so \( k - 7 = 2q_2 \geq 4 \), \( q_2 \geq 2 \), such that

\[
\frac{\text{Sym}[3, 2^{(k+2-3)/2}]}{\text{Sym}[q]} = \frac{3 \cdot 2^3 (q_2 + 3)!}{q_2! 3^3 3!} = \frac{8 \cdot (q_2 + 3)(q_2 + 2)(q_2 + 1)}{9 \cdot 6} \quad \text{(A.11)}
\]

Note that \( q_2 = 0 \) would compromise this result. Indeed, this is what is happening for \( k = 9 \) such that \( \text{Sym}[3^3] \geq \text{Sym}[3, 2^3] \). Our condition \( k \geq 11 \) ensures that this does not happen.

Now, assume \( q_3 > 3 \). As \( q_3 \) is odd, we must have \( q_3 \geq 5 \), so that \((q_3 - 3)/2 - 1 \geq 0\). Then

\[
\frac{\text{Sym}[3, 2^{(k+2-3)/2}]}{\text{Sym}[q]} \geq \frac{2^{(3q_3-1)/2} \prod_{i=0}^{(q_3-3)/2-1} (q_2 + q_3 + (q_3 - 3)/2 - i)}{3^{q_3-1}} \cdot \frac{(q_2 + q_3)!}{q_2! q_3!}
\]

\[
\geq \frac{2^{(3q_3-1)/2}}{3^{q_3-1} q_3!} \prod_{i=0}^{(q_3-3)/2-1} (q_2 + q_3 + (q_3 - 3)/2 - i)
\]

\[
\geq \frac{2^{(3q_3-1)/2}}{3^{q_3-1} q_3!} \prod_{i=0}^{(q_3-3)/2-1} (q_2 + q_3 + 1) \geq \frac{2^{(3q_3-1)/2} (2q_2 + q_3 + 3 - 2)}{3^{q_3-1} q_3!} = \frac{2^{(3q_3-1)/2} (3q_3-3-2)}{3^{q_3-1}} \quad \text{(A.12)}
\]
Checking the exponent, we get for any \( q \geq 5 \),

\[
(5q - 11) \ln 2 - 2(q - 1) \ln 3 = -11 \ln 2 + 2 \ln 3 + q(5 \ln 2 - 2 \ln 3)
\]

\[
= (q - 5)(\ln 2^5 - \ln 3^2) + \ln 2^{14} - \ln 3^8 \geq 0
\]

and this ends the proof of the current case.

**Case** \( k + 2 - 2q_2 - 3q_3 \geq 4 \). In this case, \( q' \vdash k + 2 - 2q_2 - 3q_3 \) is non-empty.

- **Subcase 1:** \( q_2 = q_3 = 0 \): Consider the smallest part \( l_0 \geq 4 \), such that \( q_{l_0} > 0 \), and write \( k + 2 = l_0 q_{l_0} + \sum_{l > l_0} l q_l \). Then, \( k + 2 - l_0 q_{l_0} \leq k + 2 - 4q_{l_0} \leq k - 2 \leq k \). Define \( q'' \vdash k + 2 - l_0 q_{l_0} \), such that \( q' = [l_0^{q_{l_0}}, q''] \) where \( q'' \vdash k + 2 - l_0 q_{l_0} \leq k \) and the smallest part of \( q'' \) is of size \( \geq l_0 + 1 \).

  a) If \( l_0 \) is even, then \( k + 2 - l_0 q_{l_0} \) is odd and \( \leq k - 2 \), and so the induction hypothesis applies to it.

  b) If \( l_0 \) is odd, \( l_0 \geq 5 \), and \( q_{l_0} \) is even, then \( k + 2 - l_0 q_{l_0} \) is odd and \( \leq k - 2 \), we can still apply the induction hypothesis to it.

  c) If \( l_0 \) is odd, \( l_0 \geq 5 \), and \( q_{l_0} \geq 1 \) is odd, then \( k + 2 - l_0 q_{l_0} \) is even and \( \leq k - 2 \). We infer that \( k + 2 - l_0 q_{l_0} \leq k + 2 - 5q_{l_0} \leq k - 3 \). We rather use in this situation \( (2.23) \), for \( k = k + 2 - l_0 q_{l_0} \) is even.

  Let us focus on a) and b) and we write using our induction hypothesis on \( q'' \),

\[
\text{Sym}[q'] = l_0^{q_{l_0}} q_{l_0}! \text{Sym}[q''] \leq l_0^{q_{l_0}} q_{l_0}! \text{Sym}[3, 2^{(k+2-l_0 q_{l_0} - 3)/2}]
\]

\[
\leq l_0^{q_{l_0}} q_{l_0}! 3 \cdot 2^{(k+2-l_0 q_{l_0} - 3)/2}((k + 2 - l_0 q_{l_0} - 3)/2)!
\]

\[
\leq 3 \cdot 2^{(k+2-3)/2}((k + 2 - 3)/2)! = \text{Sym}[3, 2^{(k+2-3)/2}]
\]

where at an intermediate step we use \( \frac{2}{3} \geq 2 \), and \( \frac{\ln 2}{2^{l_0/2}} \leq 1 \), \( l_0 \geq 4 \), the case where \( \frac{l_0}{2^{l_0/2}} = 1 \), is precisely when \( l_0 = 4 \).

We deal with the case c). Note \( l_0 \geq 5 \), and \( (l_0 - 5)q_{l_0} \geq 0 \), then using \( (2.24) \), we write

\[
\text{Sym}[q'] = l_0^{q_{l_0}} q_{l_0}! \text{Sym}[q''] \leq l_0^{q_{l_0}} q_{l_0}! \text{Sym}[2^{(k+2-l_0 q_{l_0})/2}]
\]

\[
\leq l_0^{q_{l_0}} q_{l_0}! 2^{(k+2-l_0 q_{l_0})/2}((k + 2 - (l_0 - 2)q_{l_0})/2 - q_{l_0})!
\]

\[
\leq 2^{(k+2-3)/2}((k + 2 - (l_0 - 2)q_{l_0})/2 - q_{l_0})!
\]

\[
\leq 3 \cdot 2^{(k+2-3)/2}3^{3/2} \frac{l_0}{2^{l_0/2}} q_{l_0}! ((k + 2 - (l_0 - 2)q_{l_0})/2 - q_{l_0})!
\]
that completes the proof of case c).

- Subcase 2: $q_2 > 0$ or $q_3 > 0$.
  a) Let us assume that $q_2 > 0$, then we write $q = [2^{q_2}, q'']$, where $q'' \vdash k + 2 - 2q_2 = k + 2 - 2 \leq k$. We write $q'' \vdash k - 2(q_2 - 1) \leq k$ and since $k - 2(q_2 - 1)$ is odd, the induction hypothesis applies to $q''$.

  Then, we obtain

  \[
  \text{Sym}(q) = 2^{q_2}q_2! \text{Sym}[q''] \leq 2^{q_2}q_2! \text{Sym}[3, 2^{(k-2(q_2-1)-3)/2}]
  \]

  \[
  \leq 2^{q_2}q_2! 3 \cdot 2^{(k-2(q_2-1)-3)/2}((k - 2(q_2 - 1) - 3)/2)! \\
  \leq 3 \cdot 2^{q_2}2^{(k+2-3)/2-2q_2}q_2!((k + 2 - 3)/2 - q_2)! \\
  \leq \text{Sym}[3, 2^{(k+2-3)/2}]
  \]

  which is the expression we sought.

  b) We now consider the case $q_3 > 0$. If $q_2 > 0$ then we can conclude by the just above argument. Hence, only the situation $(q_2 = 0, q_3 > 0)$ remains to be dealt with. Then $q = [3^{q_3}, q'']$ with $q'' \vdash k + 2 - 3q_3 \leq k + 2 - 3 \leq k - 1$ and the smallest part in $q''$ is of minimal size 4. At this moment, we must study some cases.

  b1) If $q_3 > 0$ is even, then $q_3 \geq 2$ and $q'' \vdash k + 2 - 3(q_3 - 2) - 6 = k - 4 - 3(q_3 - 2) \leq k - 4$. Since $k - 4 - 3(q_3 - 2)$ is odd, the induction hypothesis applies to $q''$. We get

  \[
  \text{Sym}(q) = 3^{q_3}q_3! \text{Sym}[q''] \leq 3^{q_3}q_3! \text{Sym}[3, 2^{(k-4-3(q_3-2)-3)/2}]
  \]

  \[
  \leq 3^{3q_3}2^{(k-1)/2-3q_3/2}q_3!((k - 1)/2 - q_3 - q_3/2)! \\
  \leq 3^{3q_3}2^{(k+2-3)/2-3q_3/2}q_3!((k + 2 - 3)/2 - q_3)! \\
  \leq (3.2^{-3/2})q_3 \prod_{i=0}^{q_3-2}((k - 1)/2 - q_3 - i) \\
  \leq (3.2^{-3/2})q_3 \prod_{i=0}^{q_3-2}((k + 1 - 3q_3)/2q_i/2) \text{Sym}[3, 2^{(k+2-3)/2}] \\
  \leq (3.2^{-3/2})q_3 2^{-q_i/2} \text{Sym}[3, 2^{(k+2-3)/2}] = (3.2^{-2})q_3 \text{Sym}[3, 2^{(k+2-3)/2}] \\
  \leq \text{Sym}[3, 2^{(k+2-3)/2}]
  \]

  where at an intermediate step, we use $k + 2 \geq 13$, $k + 1 - 3q_3 > 1$. Indeed, since $k + 2 = 3q_3 + |q''|$, where $|q''| \geq 4$ is the sum of parts of $q''$ that is non empty with smallest
part at least 4. Then

\[ 0 = k + 2 - 3q_3 - (|q'| - 4) - 4 = k - 2 - 3q_3 - (|q'| - 4) \]
\[ 3 = k + 1 - 3q_3 - (|q'| - 4) \leq k + 1 - 3q_3, \]  
(A.18)

and since \( k + 1 - 3q_3 \) is even we have \( k + 1 - 3q_3 \geq 4 \).

b2) If \( q_3 > 0 \) is odd, \( q_3 \geq 1, q'' \vdash k + 2 - 3(q_3 - 1) - 3 = k - 1 - 3(q_3 - 1) \leq k - 1 \). There are three subcases to be treated.

\( q_3 = 1: q'' \vdash k - 1, \) with \( k - 1 \) even so the induction hypothesis \( \text{(2.24)} \) applies to \( q'' \), and so we write:

\[
\text{Sym}(q) = 3 \text{Sym}(q'') \leq 3 \text{Sym}(2^{(k-1)/2}) = 3 \text{Sym}(2^{(k+2-3)/2}) = \text{Sym}(3, 2^{(k+2-3)/2}) \quad (A.19)
\]

\( q_3 = 3: q'' \vdash k + 2 - 3 \times 3 = k - 7, \) with \( k - 7 \) even. Thus, the induction \( \text{(2.24)} \) applies to \( q'' \):

\[
\text{Sym}(q) = 3^3 3! \text{Sym}(q'') \leq 3^3 3! \text{Sym}(2^{(k-7)/2}) = 3^3 3! 2^{(k-7)/2} ((k - 7)/2)!
\]

\[
\leq 3^3 3! 2^{(k-1)/2 - 3} ((k - 1)/2 - 3)! = 3^3 3! 2^{-3}
\]

\[
\times \frac{1}{((k - 1)/2 - 2)((k - 1)/2 - 1)((k - 1)/2) 2((k - 1)/2 - 1)(k - 1)/2)
\]

\[
\leq 3^3 3! 2^{-3}
\]

\[
\times \frac{1}{((k - 1)/2 - 2)((k - 1)/2 - 1)((k - 1)/2) 2((k - 1)/2 - 1)(k - 1)/2)} \text{Sym}(3, 2^{(k+2-3)/2}) \quad (A.20)
\]

Using \( k \geq 11 \), we have

\[
\text{Sym}(q) \leq 3^2 3! 2^{-3} \frac{1}{((11 - 1)/2 - 2)((11 - 1)/2 - 1)((11 - 1)/2) 2((11 - 1)/2 - 1)(11 - 1)/2)} \text{Sym}(3, 2^{(k+2-3)/2})
\]

\[
\leq 3^2 3! 2^{-3} \frac{1}{(3)(4)(5)} \text{Sym}(3, 2^{(k+2-3)/2}) \leq \text{Sym}[3, 2^{(k+2-3)/2}] \quad (A.21)
\]

\( q_3 \geq 5: \) Then \( k + 2 \geq 19 \) and \( q'' \vdash k + 2 - 3(q_3 - 5) - 15 = k - 13 - 3(q_3 - 5) \leq k - 13 \).

Considering that \( k - 13 - 3(q_3 - 5) \) is even, we use the bound \( \text{(2.24)} \) on \( q'' \) and write

\[
\text{Sym}(q) = 3^q_3 q_3! \text{Sym}(q'') \leq 3^q_3 q_3! \text{Sym}[2^{(k-13-3(q_3-5))/2}]
\]

\[
\leq 3 \cdot 2^{(k-13)/2} \frac{3^{q_3-1}}{2^{3(q_3-5)/2}} q_3! ((k - 13)/2 - (q_3 - 5) - (q_3 - 5)/2)!
\]

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\begin{align*}
\leq 3 \cdot 2^{(k-13)/2} (\frac{3q_3-1}{2(3q_3-6)/2}) q_3! \cdot ((k-1)/2 - 1 - q_3 - (q_3 - 5)/2)!
\leq 3 \cdot 2^{(k-1)/2-6} (\frac{3q_3-1}{2(3q_3-5)/2}) ((k-1)/2 - 1 - (q_3 - 5)/2)!
\leq 3 \cdot 2^{(k-1)/2} \frac{1}{2^6(\frac{3q_3-1}{2(3q_3-5)/2})} \frac{((k-1)/2)!}{\prod_{i=0}^{(q_3-5)/2} ((k-1)/2 - i)}
\leq \frac{1}{2^6(\frac{3q_3-1}{2(3q_3-5)/2})} \frac{1}{\prod_{i=0}^{(q_3-5)/2} ((k-1)/2 - (q_3 - 5)/2)} \text{Sym}[3, 2^{(k+2-3)/2}]
\leq \frac{1}{2^6(\frac{3q_3-1}{2(3q_3-5)/2})} \frac{1}{((k-1)/2 + 2q_3)/2 + 1} \text{Sym}[3, 2^{(k+2-3)/2}]
\end{align*}

where we used $(k-a)! \prod_{i=0}^{a-1} (k-i) = k!$, $\forall k \geq a \geq 0$. We have $k + 2 - 3q_3 \geq 4$, therefore $(k - q_3)/2 \geq 4/2 = 2$. The above expression finds the bound

\begin{align*}
\text{Sym}(q) & \leq \frac{1}{2^6(\frac{3q_3-1}{2(3q_3-5)/2})} \frac{1}{2(q_3-5)+2} \text{Sym}[3, 2^{(k+2-3)/2}]
\leq \frac{3q_3-1}{2(q_3-5)/2+2q_3/2+3} \text{Sym}[3, 2^{(k+2-3)/2}]
\leq \frac{3q_3-1}{2(q_3-9)/2} \text{Sym}[3, 2^{(k+2-3)/2}] < \text{Sym}[3, 2^{(k+2-3)/2}]
\end{align*}

that ends the proof of the case and of the lemma.

**B Sage codes for coefficients of the asymptotic series**

The coefficients $a_l$, $l \geq 1$ of the series expansion of $S_{3,m,K}(n)$ as $n \to \infty$ is given by the following program.

We use the built-in methods
- p.centralizer_size() to compute Sym($p$) for a given partition $p$,
- Partitions($k$, min_part = p).list() that produces the list of partitions of $k$, each with parts larger or equal $q$ involved in the constrained sum $\sum_{q(l, q_1=q_2=\ldots=q_l=0}$
- and stirling_number2(1,$k$) to evaluate the Stirling number of second kind with parameter $(l,k)$.

**Code for $S_{3,1,K}(n)$**

```python
def coeff ( n , 1):
    som2 = 0
```

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\[ m = \min(n, 1) \]

for k in range(1, m+1) :  # k = 1 ... m
    som1 = 0
    lsk = Partitions(k, min_part = 2).list()
    for i in range(len(lsk)) :  # i = 0 .. len(lsk) -1
        som1 = som1 + lsk[i].centralizer_size()
    som2 = som2 + som1*stirling_number2(l-1,k-1)
return som2

Code for \( S_{3,m,K}(n) \)

```python
def coeffm (m, n , l):
    som2 = 0
    if m> floor(n/2):
        print "m index out of range"
        return 0
    else:
        mo = min(floor(n/m),l)

        for k in range(1,mo+1) :  # k = 1 ... m
            som1 = 0
            lsk = Partitions(m*k, min_part = m+1).list()
            for i in range(len(lsk)) :  # i =0 .. len(lsk) -1
                som1 = som1 + lsk[i].centralizer_size()
            som2 = som2 + som1*m^(l-k)*stirling_number2(l-1,k-1)
        return som2
```

Code for tabulating the minimal value of \( f(n, k) \)

In the proof of lemma \(^2\) we use an approximation of \( k_{min} \) the minimum of the function \( f(n, k) \), for \( k \in [K, n] \). Here, we show by numerics that, at large and various values of \( n \), the following approximation of the minimal value of \( f(n, k) \) holds:

\[ k_{min} = n - \sqrt{n} + 1/4 \]  \hspace{1cm} (B.1)

so that \( (n - k_{min}) \sim \sqrt{n} \) and also \( k_{min}^{1/2} < n - k \) holds, for large \( n \).
The function TabMin(nmin, nmax) tabulates the minimal values of $f(n, k)$, for $n \in [nmin, nmax]$, and $k \in [K = n - 13, n]$. It outputs 4-tuples

$$[n, kmin, n - kmin, n - \sqrt{n} + 1/4 - kmin],$$

where $kmin$ is the index of the minimum of $f(n, k)$ for $k \in [K = n - 13, n]$. The claim is that $n - \sqrt{n} + 1/4 \sim kmin$, for large $n$, therefore the last entry of the 4-tuple should be small. The specific value 14 is chosen as we run a calculation for even $n \in [20, 80]$, so that $\sqrt{n} \in [4, 8]$, and therefore $kmin \sim n - \sqrt{n} \in [n - 14, n]$. Using this we are investigating the neighborhood of $n - \sqrt{n}$. For odd $n \in [21, 79]$, we perform the analogue calculations as well with the hinge value of 13.

# The function $f(n,k)$ for $n$ and $k$ even

```python
def feven(n,k):
    return 1.0*factorial (n-k)/factorial(n)*
        len(Partitions (k , min_part=2).list())*2^(k/2)*factorial (k/2)
```

# The function $f(n,k)$ for $n$ and $k$ odd

```python
def fodd(n,k):
    return 1.0*factorial (n-k)/factorial(n)*
        len(Partitions (k , min_part=2).list())*2^((k-3)/2)*factorial ((k-3)/2)
```

# Table of values of feven

```python
def TabEvenCoefksum ( n , K ):
    cnk = [ 0 for i in range( ( n - K )/2 +1 ) ]
    for i in range( ( n - K )/2 +1 ):
        ki = K + 2*i
        cnk[i] = N(feven( n , ki ))
    return cnk
```

# Table of values of fodd

```python
def TabOddCoefksum ( n , K ):
    cnk = [ 0 for i in range( ( n - K )/2 +1 ) ]
    for i in range( ( n - K )/2 +1 ):
        ki = K + 2*i
        cnk[i] = N(fodd( n , ki))
    return cnk
```

# Table of values $[ n, kmin, n-kmin, n- \sqrt{(n) + 1/4 - kmin } ]$

# the parity of nmin will determine which function one chooses
def TabMin ( nmin , nmax ) :
    cnk = [ 0 for i in range ( ( nmax - nmin +2)/2 ) ]
    for i in range ( ( nmax - nmin +2)/2 ) :
        n_min = nmin + 2*i
        if nmin%2 == 0 :
            tab = TabEvenCoefksum ( n_min , n_min - 14 )
            kmin = n_min - 14 + 2*tab.index( min (tab ) )
        if nmin%2 == 1 :
            tab = TabOddCoefksum ( n_min , n_min + 1 - 13 )
            kmin = ( n_min - 12 ) + 2*tab.index( min (tab ) )
        cnk[i] = [ n_min , kmin, n_min - kmin,
                   N (( n_min - kmin ) - sqrt (n_min) + 1/4 ) ]
    return cnk

- For the case \( k \) even we obtain \( n \in [20, 80] \)

[[20, 16, 4, -0.222135954999580],
 [22, 18, 4, -0.44041579823430],
 [24, 18, 6, 1.35102051443364],
 [26, 20, 6, 1.15098048640722],
 [28, 22, 6, 0.985497377870819],
 [30, 24, 6, 0.77277424948339],
 [32, 26, 6, 0.593145750507619],
 [34, 28, 6, 0.419048105154699],
 [36, 30, 6, 0.250000000000000],
 [38, 32, 6, 0.085585997031024],
 [40, 34, 6, -0.0745553203367590],
 [42, 36, 6, -0.230740698407860],
 [44, 38, 6, -0.383249580710800],
 [46, 38, 8, 1.46767001687473],
 [48, 40, 8, 1.32179676972449],
 [50, 42, 8, 1.17893218813452],
 [52, 44, 8, 1.03889744907202],
 [54, 46, 8, 0.901530771650466],
 [56, 48, 8, 0.766885226452117],
 [58, 50, 8, 0.634226894136091],
 [60, 52, 8, 0.504033307585166],

For the case $k$ odd we obtain $n \in [21, 81]$

\[[21, 17, 4, -0.332575694955840],
[23, 19, 4, -0.545831523312719],
[25, 21, 4, -0.750000000000000],
[27, 21, 6, 1.05384757729337],
[29, 23, 6, 0.86435192865496],
[31, 25, 6, 0.682235637169978],
[33, 27, 6, 0.50543753461971],
[35, 29, 6, 0.333920216900384],
[37, 31, 6, 0.167237469701781],
[39, 33, 6, 0.00500200160160169],
[41, 35, 6, -0.153124237432849],
[43, 37, 6, -0.307438524302000],
[45, 39, 6, -0.458203932499369],
[47, 41, 6, -0.605654600401044],
[49, 43, 6, -0.750000000000000],
[51, 43, 8, 1.10857157145715],
[53, 45, 8, 0.969890110719482],
[55, 47, 8, 0.833801512904337],
[57, 49, 8, 0.700165564729250],
[59, 51, 8, 0.568854252131392],
[61, 53, 8, 0.439750324093346],
[63, 55, 8, 0.312746066806228],
[65, 57, 8, 0.18774251701451],
[67, 59, 8, 0.0646472281275496],
[69, 61, 8, -0.0566238629180749],
[71, 63, 8, -0.176149773176359],
[73, 65, 8, -0.294003745317530],
[75, 67, 8, -0.410254037844386],
]
[77, 69, 8, -0.524964387392123],
[79, 71, 8, -0.638194417315589],
[81, 71, 10, 1.250000000000000]]
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