Causal continuity in degenerate spacetimes

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Abstract

A change of spatial topology in a causal, compact spacetime cannot occur when the metric is globally Lorentzian. One can however construct a causal metric from a Riemannian metric and a Morse function on the background cobordism manifold, which is Lorentzian almost everywhere except that it is degenerate at each critical point of the function. We investigate causal structure in the neighbourhood of such a degeneracy, when the auxiliary Riemannian metric is taken to be Cartesian flat in appropriate coordinates. For these geometries, we verify Borde and Sorkin’s conjecture that causal discontinuity occurs if and only if the Morse index is 1 or $n-1$.

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1 Introduction

Spatial topology change is incompatible with both a non-degenerate Lorentzian metric and a causal partial order on all spacetime points \[\text{[1]}\]. One or other of these conditions must be given up if topology change is to occur. However, even if the causal order is abandoned and closed timelike curves (CTCs) allowed, there are certain topology changes, including physically interesting ones such as the pair production of Kaluza-Klein monopoles, that still cannot occur via a globally Lorentzian spacetime \[\text{[2, 3]}\]. On the other hand if CTCs are excluded, all possible topology changes are permitted at a kinematical level as long as the metric is allowed to degenerate to zero at finitely many isolated “Morse” points \[\text{[4, 5]}\].

In the Sum-Over-Histories (SOH) framework for quantum gravity, the transition amplitude between two non-diffeomorphic spacelike hypersurfaces \(V_0\) and \(V_1\) is given as a sum over all interpolating geometries (see for example \[\text{[6, 7, 2, 8, 9, 10, 11]}\) ). These geometries are usually taken to be globally defined. As discussed in \[\text{[12, 13]}\), this prescription can be generalised to include the so-called “Morse geometries” on the interpolating cobordism \(M\), where \(\partial M = V_0 \amalg V_1\) (\(\amalg\) denoting disjoint union).

A Morse metric, \(g\), is defined to be
\[
g_{\mu\nu} \equiv h_{\mu\nu}(h^{\lambda\sigma}\partial_\lambda f \partial_\sigma f) - \zeta \partial_\mu f \partial_\nu f \tag{1}\]

where \(\zeta\) is a real number greater than one, \(h\) is a Riemannian metric on \(M\) and \(f\) is a Morse function \(f : M \rightarrow [0, 1]\) such that \(f^{-1}(0) = V_0\) and \(f^{-1}(1) = V_1\). A Morse function, \(f \in C^\infty(M)\) has critical points, at which \(\partial_\mu f = 0\), which are isolated and nondegenerate, i.e. the Hessian of \(f\) is non-singular. This metric is Lorentzian throughout the regular set of the Morse function and vanishes at its critical points \(\{p_k\}\). The index, \(\lambda_k\), of \(p_k\) is the number of negative eigenvalues of the Hessian of \(f\) at \(p_k\). We define a Morse geometry as the pair \((M, g)\), where \(M\) is a compact cobordism and \(g\) a Morse metric on it. We suggest that these Morse geometries should in fact be included in the SOH for quantum gravity.

Motivated by surgery theory, Borde and Sorkin conjectured that only those spacetimes which contain critical points of index 1 or \(n - 1\) have causal discontinuities \[\text{[14, 12]}\]. This conjecture, while on one hand of mathematical interest, in fact has potential importance in quantum gravity. It was shown by the authors of \[\text{[15, 16]}\) that scalar quantum field propagation on the 1 + 1 trousers is singular and hence it was suggested that such a topology will be suppressed in the SOH. Subsequently, the authors of \[\text{[17]}\) found that causally discontinuous topology changing processes in 1 + 1 dimensions are indeed suppressed, while causally continuous ones
are enhanced. This led Sorkin to further conjecture that singular propagation of quantum fields on such backgrounds is related to the causal discontinuity of the spacetime.

It is the purpose of this paper to investigate the first of these, the Borde-Sorkin conjecture. The standard analyses of causal structure however, posit the existence of a globally Lorentzian metric \[18, 19, 20\], and thus exclude Morse geometries with their isolated degeneracies. However, we may investigate the geometries induced in the regular set of the Morse function. Thus we define a Morse spacetime as the globally Lorentzian spacetime that results from excising the critical points from a Morse geometry. We note that a Morse spacetime is partially ordered by the causality relation, since the Morse function is a global time function and precludes CTCs. In the discussion section we return to this construction and suggest a way of extending the causal structure to the Morse geometry with the degenerate points left in.

A general theorem would run along the following lines: a Morse spacetime \((M, g)\) is causally continuous \(\text{iff} \) none of the excised Morse points \(\{p_k\}\) have index 1 or \(n - 1\). We do not prove the full theorem in this paper, but examine a specific class of spacetimes defined on a neighbourhood of a single critical point and are able to verify the conjecture in these cases.

In section 2 of this paper, we remind the reader of some standard definitions and properties of the causal structure of Lorentzian spacetimes, in particular the definition of causal continuity.

Section 3 contains preliminary general results on the causal continuity of two important classes of spacetimes. First we show that a Morse geometry with no degeneracies, which necessarily has topology \(\Sigma \times [0,1]\) (where \(\Sigma\) is a closed \(n - 1\) manifold), is causally continuous. This follows as a corollary to a general result on the equivalence of various causality conditions in the case of a compact cobordism. Another consequence of this general result is that imposing strong causality on the histories of the SOH in the case of a compact product cobordism is equivalent to restricting the sum to non-degenerate Morse metrics. This gives us additional confidence that the proposal to sum over Morse metrics is a good one. Second, the general sphere creation/destruction elementary cobordism, which we refer to as the “yarmulke” spacetime, is shown to be causally continuous.

In section 4 we briefly describe the Morse spacetimes that we study in the rest of the paper. These are simple ball-neighbourhoods of the critical points of a Morse
function $f$, with the critical point excised, on which the Morse metric is constructed from $f$ and a Riemannian metric which is flat and Cartesian in the coordinates in which $f$ takes its canonical form.

Section 5 contains a detailed analysis, in two spacetime dimensions, of the causal structure of these neighbourhood spacetimes for index $\lambda = 0$ (the yarmulke) and $\lambda = 1$ (the trousers). These are, up to time reversal, the two basic types of topology change in two dimensions. The general proof in Section 3 shows that the two dimensional yarmulke is causally continuous. We verify that the trousers neighbourhood geometry is causally discontinuous.

Section 6 contains our main results on the neighbourhood geometries. We show that a neighbourhood geometry is causally continuous if and only if its Morse point does not have index 1 or $n - 1$. The proof makes extensive use of the causal structure of the two dimensional yarmulke and trousers from the previous section.

We summarise our results in the Section 7 and comment on further aspects of this work that are currently under investigation.

## 2 Causal continuity

*Causal continuity* of a spacetime means, roughly, that the volume of the causal past and future of any point in the spacetime increases or decreases continuously as the point moves continuously around the spacetime. Hawking and Sachs [24] give six concrete characterisations of causal continuity, three of which are equivalent in any globally Lorentzian, time-orientable spacetime, while the equivalence to the remaining three further requires that the spacetime be distinguishing. A time orientation is defined in a spacetime $(M, g)$ by the choice, if possible, of a nowhere vanishing timelike vector field $u$. A spacetime is called distinguishing if any two distinct points have different chronological pasts and different chronological futures.

We take a timelike or null vector $v$ to be future pointing if $g(v, u) < 0$ and past pointing if $g(v, u) > 0$. We define a *future directed timelike curve* in $M$ to be a $C^1$ function $\gamma : [0, 1] \rightarrow M$ whose tangent vector is future pointing timelike at $\gamma(t)$ for each $t \in [0, 1]$. We also use the phrase future-directed timelike curve and the symbol $\gamma$ to denote the image, $\{x \in M : x = \gamma(t), t \in [0, 1]\}$, of such a function. Strictly we should call the function a “path”, say, and reserve “curve” for the image
set, but we will ignore this distinction for ease of notation, since no ambiguity arises in what follows. Future directed causal curves are defined similarly, but the future directed tangent vector can be null as well as timelike and the curves are allowed to degenerate to a single point. Past directed curves are similarly defined using past pointing tangent vectors.

We write $x << y$ whenever there is a future directed timelike curve $\gamma$ with $\gamma(0) = x$ and $\gamma(1) = y$ and $x < y$ whenever there is a future directed causal curve $\gamma$ with $\gamma(0) = x$ and $\gamma(1) = y$. The chronos relation $I \subset M \times M$ is defined by $I \equiv \{(x, y) : x << y\}$ while the causal relation $J \subset M \times M$ is defined by $J \equiv \{(x, y) : x < y\}$. The chronological future and past of a particular point $x \in M$ are, $I^+(x) \equiv \{y : (x, y) \in I\}$ and $I^-(x) \equiv \{y : (y, x) \in I\}$, respectively. The causal future $J^+(x)$ and the causal past $J^-(x)$ of a point are similarly defined.

It can be shown, using local properties of the lightcone, that the chronological relation is transitive (i.e., $x << y, y << z \Rightarrow x << z$) and that it is open as a subset of $M \times M$. The relation $J$ is transitive and reflexive (i.e., $x < x$) [21]. In simple spacetimes such as Minkowski, $J^+(x)$ is also closed as a set in $M$, but in general it is not so. Given a subset $U$ of the spacetime, $I^+(x, U)$ denotes the set of points in $U$ that can be reached from $x$ along future directed timelike curves totally contained in $U$. Note that $I^+(x, U) \subset I^+(x) \cap U$, but the converse is not true in general. $I^-(x, U)$ is similarly defined. Henceforth the dual or time-reversed definitions and statements are understood unless stated otherwise.

We note that the Morse spacetimes (1) are time-oriented and distinguishing. The time orientation is given by the timelike vector field normal to the level surfaces of the Morse function $f$. Distinguishability is guaranteed by the fact that $f$ is a global time function: if $y$ had the same future set as $x$, then $y$ would be in $I^+(x)$ and it would have to lie in the same level surface $f^{-1}(a)$; but looking at a convex normal neighbourhood $[18]$ of $x$, we see that $x$ is the only point in $I^+(x) \cap f^{-1}(a)$.

Thus, all six characterisations of causal continuity given in [20] are equivalent for Morse geometries. Before we list four of these we first give a few more definitions.

The common past, $\downarrow U$ (common future, $\uparrow U$) of an open set $U$ is the interior of the set of all points connected to each point in $U$ along a past (future) directed timelike curve, i.e.,

$$\downarrow U \equiv \text{Int} \left( \{x : x << u, \forall u \in U\} \right)$$

$$\uparrow U \equiv \text{Int} \left( \{x : x >> u, \forall u \in U\} \right).$$

(2)
We state two properties of the past and future sets are used later. If \( x \) is a point in the time-oriented spacetime \( M \), then:

(i) \( J^\pm(x) \subset \overline{I^\pm(x)} \) (Proposition 3.9[18])

(ii) \( I^+(x) \subset \uparrow I^-(x) \) and \( I^-(x) \subset \downarrow I^+(x) \) (Proposition 1.1[20]).

Let \( F \) be a function which assigns to each event \( x \) in \( M \) an open set \( F(x) \subset M \).

Then \( F \) is said to be outer continuous if for any \( x \) and any compact set \( K \subset M - F(x) \), there exists a neighbourhood \( U \) of \( x \) with \( K \subset M - \overline{F(y)} \forall y \in U \).

A time-orientable distinguishing spacetime, \((M,g)\) is said to be causally continuous if it satisfies any of the equivalent properties:

1. for all events \( x \) in the interior of \( M \) we have \( I^+(x) = \uparrow I^-(x) \) and \( I^-(x) = \downarrow I^+(x) \).

2. \((M,g)\) is reflecting, i.e., for all events \( x \) and \( y \) in \( M \), \( I^-(x) \subset I^-(y) \iff I^+(y) \subset I^+(x) \);

3. for all events \( x \) and \( y \), \( x \in \overline{J^+(y)} \iff y \in \overline{J^-(x)} \);

4. for all events \( x \in M \), \( I^+(x) \) and \( I^-(x) \) are outer continuous.

Although the last characterisation might seem to capture better our intuitive understanding of causal continuity, it is the first one that we use widely in this paper\(^1\). For this reason, we introduce the following point-by-point criterion. A spacetime \((M,g)\) is causally continuous at point \( x \) if \( I^+(x) = \uparrow I^-(x) \) and \( I^-(x) = \downarrow I^+(x) \). Thus \((M,g)\) is causally continuous iff it is causally continuous at every point of \( M \).

We also require the definitions of causality, strong causality, stable causality, causal simplicity and globally hyperbolicity. Causality holds in a subset \( S \) of \( M \) if there are no causal loops based at points in \( S \). Strong causality holds in a subset \( S \) of \( M \) if for every point \( s \in S \) any neighbourhood \( U \) of \( s \) contains another neighbourhood \( V \) of \( s \) that no causal curve intersects more than once. A spacetime \((M,g)\) is said to be stably causal if there is a metric \( g' \), whose lightcones are strictly broader than

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\(^1\)Note that the set identities in condition 1 would trivially fail at the initial and final boundaries of a Morse spacetime, while the other conditions hold everywhere. In [20] condition 1 appears without the specification “the interior of”. There a spacetime is implicitly assumed to be a genuine manifold without boundary. With our specification we are ensuring that the conditions above are still equivalent in the case of a Morse spacetime and preventing the trivial causal discontinuities at the boundaries.
those of $g$ and for which the spacetime $(M, g')$ is causal. A spacetime is stably causal if and only if it admits a global time function, i.e., a function whose gradient is everywhere timelike. A spacetime is said to be \textit{causally simple} if $J^+(q)$ and $J^-(q)$ are closed for every point $q$. This is equivalent to the conditions $J^+(q) = I^+(q)$ and $J^-(q) = I^-(q)$. A spacetime is said to be \textit{globally hyperbolic} if it contains a spacelike hypersurface which every inextendible causal curve in the spacetime intersects exactly once. There is a standard sequence of implications amongst these causality conditions \cite{20}: global hyperbolicity of a spacetime $\Rightarrow$ causal simplicity $\Rightarrow$ causal continuity $\Rightarrow$ stable causality $\Rightarrow$ strong causality $\Rightarrow$ causality.

Finally, we recall a result that we use in our proofs: the Causal Curve Limit Theorem (CCLT) \cite{18}, which states that if $K$ is the set of all points in $M$ where strong causality holds and $C \subset K$ is compact, then for any closed subsets $A, B$ of $C$ the space $\mathcal{C}(A, B)$ of causal curves in $C$ from $A$ to $B$ is compact. Strictly speaking, this holds only if one uses a weaker definition of causal curve than the one we are using since the limit curve $\gamma$ to which a sequence of $C^1$ causal curves converges is not in general $C^1$. The existence of the limit curve nevertheless ensures the existence of some $C^1$ causal curve between the endpoints of the limit curve, and this subtlety is ignorable for our purposes.

\section{Causal continuity of non-degenerate and yarmulke spacetimes}

By “non-degenerate spacetime” we mean a compact Morse spacetime $(M, g)$, i.e. a Morse geometry with no degeneracies. A yarmulke spacetime is one arising from a Morse geometry in an $n$-dimensional elementary cobordism of index $\lambda = 0$ ($n$), whose initial (final) boundary is empty and final (initial) boundary is $S^{n-1}$.

In what follows all spacetimes are assumed to be time-orientable and distinguishing. We start by establishing a general result on the behavior of compact, globally Lorentzian spacetimes:

\textbf{Proposition 1} For a compact spacetime $(M, g)$, the following properties are equivalent:

1. It is causally simple.
2. It is causally continuous.

3. It is stably causal.

4. It is strongly causal.

Proof: Since each item implies the following we only need to prove that the last item implies the first. (Indeed, one can readily check that (1) ⇒ (2). That (2) ⇒ (3) is the content of proposition 2.3 \[20\] and that (3) ⇒ (4) is shown in for example \[19\].)

So suppose that strong causality holds on \((M, g)\). For any spacetime, \(J^+(q) \subset I^+(q)\). Let \(p \in I^+(q)\), and consider a sequence of points \(p_k \in I^+(q)\), which converges to \(p\). There must be a sequence of future-directed timelike curves, \(\gamma_k\) from \(q\) to \(p_k\). Since \((M, g)\) is strongly causal, we can use the CCLT by taking \(C = M\), \(A = \{q\}\), \(B = \{p_k : k = 1, 2, \ldots\} \cup \{p\}\), so that there is a causal limit curve \(\gamma\) from \(q\) to \(p\). Thus, \(p \in J^+(q)\), or \(I^+(q) \subset J^+(q)\) which implies that \(J^+(q) = I^+(q)\). \(J^-(q) = I^-(q)\) is proved similarly. So \((M, g)\) is causally simple \(\Box\).

This proposition provides us with a proof of the causal continuity of non-degenerate Morse spacetimes \((M, g)\), since they are compact and possess a global time function:

**Corollary 1** A non-degenerate Morse spacetime \((M, g)\) is causally continuous.

**Proposition 2** Let \((M, g)\) be a compact spacetime with boundary \(\partial M = V_0 \sqcup V_1\) such that \(V_0\) and \(V_1\) are closed \(n-1\) manifolds which are the initial and final spacelike boundaries of \(M\) respectively. Then the following properties of \((M, g)\) are equivalent:

1. It is a non-degenerate Morse spacetime.

2. It is stably causal.

3. It is globally hyperbolic.

This result, as stated, does not ask that \(V_0\) and \(V_1\) be non-empty or have the same topology but the known causality violations in those cases makes the result
relevant only to spacetimes with these properties. Its significance is that for this class of spacetimes global hyperbolicity can be added to the equivalent conditions in proposition 4 and that any Lorentz metric in a strongly causal product cobordism can be written as a Morse metric.

**Proof:**

a) $1 \Rightarrow 2$. For any Morse spacetime $(M, g)$ with its critical points excised the Morse function is a global time function and hence $[19]$ the spacetime is stably causal.

b) $2 \Rightarrow 1$. Suppose that $(M, g)$ is stably causal with time function $f$. Let $b^2 = -g^{\mu\nu}\partial_\mu f \partial_\nu f$. Define the positive definite metric $h$ by

$$h_{\mu\nu} \equiv g_{\mu\nu} + (1 + 1/b^2)\partial_\mu f \partial_\nu f. \quad (3)$$

(That $h$ is positive definite may be checked by using the basis $\{T^\mu, S^\mu_i\}$, where $T^\mu = g^{\mu\nu}\partial_\nu f$ is timelike with respect to $g$ and the $S^\mu_i$ are chosen to be spacelike and orthogonal to each other and to $T^\mu$ (with respect to $g$). Then $\{T^\mu, S^\mu_i\}$ forms an orthogonal basis with respect to $h$, and all the vectors have positive norm.) Since $h^{\mu\nu}\partial_\mu f \partial_\nu f = 1$, we see that we may invert the above expression and express $g$ as a Morse metric of the form (1).

It is possible to choose $f$ such that $f^{-1}(0) = V_0$ and $f^{-1}(1) = V_1$. The function $\tilde{f}(p) \equiv Volume(I^-(p))$ is a time function on $(M, g)$. Let $T^\mu \equiv g^{\mu\nu}\partial_\nu \tilde{f}$. For any point $p \in M$, let $q_p \in V_1$ be the future endpoint of the integral curve of $T^\mu$ through $p$. Define $v(p) \equiv Volume(I^-(q_p))$. Then $f(p) \equiv (1/v(p))\tilde{f}(p)$ is a time function on $M$ with $f^{-1}(0) = V_0$ and $f^{-1}(1) = V_1$.

c) $2 \Rightarrow 3$. Suppose that $(M, g)$ is stably causal, with time function $f$. Let $S$ be any $f = \text{constant}$ surface. Let $p$ lie on some later time surface, and suppose that a past inextendible causal curve from $p$, say $\lambda$, does not intersect $S$. Then $\lambda$ is trapped in the region of $M$ between the two constant-time surfaces mentioned above. Since this region is compact $[22]$ $\lambda$ must accumulate at some point $q$ and therefore it intersects the constant-time surface through $q$ more than once. This is not possible. $\square$

Proposition 3 shows that the restriction to Morse metrics in a Lorentzian compact spacetime with initial and final spacelike boundaries is a reasonable one, since that restriction is equivalent to stable causality which is in turn equivalent to strong causality.
We now prove causal continuity for yarmulke spacetimes in \( n \) dimensions. The Morse function has a single critical point \( p \). Suppose that \( f : M \rightarrow [0,1] \) is the Morse function and that the boundary of \( M \) is a final boundary. Then we must have \( f(p) = 0 \). Excising \( p \) from our manifold, the associated Morse spacetime has topology \( S^{n-1} \times (0,1] \).

**Lemma 1** The yarmulke spacetime \((M,g)\), with topology \( S^{n-1} \times (0,1] \), is causally continuous.

**Proof:** Using arguments similar to those in used in the above proposition, we see that since the spacetime is stably causal, and therefore strongly causal, the surfaces of constant \( f \) are Cauchy surfaces. (The full spacetime is not compact here, but the region between any two level surfaces of \( f \) is.) Thus the spacetime is globally hyperbolic and hence causally continuous. \(\Box\)

### 4 The neighbourhood of a Morse point

In this section we define the neighbourhood Morse spacetimes for which we verify the Borde-Sorkin conjecture. Consider an open ball, \( D_\epsilon \), of radius \( \epsilon \) in \( \mathbb{R}^n \) centred on the point \( p \). Let \( \{x^i, y^j : i = 1, \ldots, \lambda, j = 1, \ldots, n - \lambda\} \), be local coordinates with \( x^i(p) = y^j(p) = 0, \sum_i (x^i)^2 + \sum_j (y^j)^2 < \epsilon^2 \) and in which the Morse function \( f \) takes the following canonical form (Morse lemma \[22\]):

\[
f = f(p) - \sum_i (x^i)^2 + \sum_j (y^j)^2.
\]

The spacetime manifold we are concerned with is then \( N_\epsilon \equiv D_\epsilon - \{p\} \). Henceforth we consider all set closures, etc., to be taken in the manifold \( N_\epsilon \) except when explicitly stated otherwise. We frequently refer to \( p \) as though it is in the spacetime: for example we refer to sets of curves that are “bounded away from \( p \)” the meaning of which should be clear.

We define the polar coordinates, \((\rho, \Theta, r, \Phi)\), where \( \rho^2 = \sum_1^\lambda x_i^2, \ r^2 = \sum_1^{n-\lambda} y_j^2 \), and the collective coordinates \( \Theta \) and \( \Phi \) stand for the angles \( \theta_i, \ i = 1 \cdots \lambda - 1 \) and \( \phi_j, \ j = 1 \cdots n - \lambda - 1 \) that coordinatise the \((\lambda - 1)\)-sphere and the \((n - \lambda - 1)\)-sphere, respectively. When \( \lambda = 1 \) the sphere \( S^0 \) is disconnected and our convention for such cases is to adopt a single discrete “coordinate” \( \theta_0 \) with two possible values: 0 and \( \pi \).
π. Then the transformation between $x^1$ and $(\rho, \theta_0)$ is $x^1 = \rho \cos \theta_0$. Similarly, when $\lambda = n-1$, the disconnected sphere would be parameterised by the single discrete angle $\phi_0$ and $y^1 = r \cos \phi_0$. We will often use the notation whereby the coordinates of a point $q$ are referred to as $(x_q^i, y_q^j)$ or $(\rho_q, \Theta_q, r_q, \Phi_q)$ except that the discrete angles are written $\theta_0(q)$ or $\phi_0(q)$ for ease.

The particular Morse metrics that we study in this paper are those for which the auxiliary Riemannian metric $h$ is the flat Cartesian metric in the coordinates $\{x^i, y^j\}$ introduced above. In that case we find, transforming to polar coordinates, the metric (1) becomes

$$ds^2 = 4\{ (\rho^2 + r^2)(\rho^2 d\Theta_{\lambda-1}^2 + r^2 d\Phi_{n-\lambda-1}^2) + (r^2 - (\zeta - 1) \rho^2) d\rho^2 + (\rho^2 - (\zeta - 1) r^2) dr^2 + 2\rho r d\rho dr \}. \quad (5)$$

In these coordinates $ds^2$ is seen to be a warped product metric since it decomposes into a radial and angular part, i.e., $ds^2 = ds_R^2 + ds_A^2$, with the radial coordinates $(r, \rho)$ warping the angular part. An important property of such a warped product form, is that the geodesics of the metric $ds_R^2$ are also geodesics of the full metric. We note also, that $ds_A^2$ is never negative, so that for any timelike (causal) curve $\gamma(t) = (\rho(t), \Theta(t), r(t), \Phi(t))$, the related curve $\gamma'(t) = (\rho(t), \Theta_a, r(t), \Phi_b)$ at any fixed angles $\Theta_a$ and $\Phi_b$, is also timelike (causal). The Morse function $f(r, \rho) = f(p) - \rho^2 + r^2$ must increase along future directed timelike curves so that the future time direction is, roughly speaking, decreasing $\rho$ and increasing $r$.

We call $(N, g)$, where $g$ is given by (5), the *neighbourhood geometry* of type $(\lambda, n-\lambda)$.

### 5 Causal structure for $n = 2$

We study in detail the two elementary neighbourhood geometries —up to time reversal— in $n = 2$, namely the trousers ($\lambda = 1$) and the yarmulke ($\lambda = 0$.) The causal structure of these two cases turns out to be crucial in studying the general $n$-dimensional neighbourhood geometries. (A study of the causal structure of the $1+1$ trousers for a particular choice of flat metric was first carried out by [23].)

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2A metric $g(x^a, y^A)$ is a warped product metric if its interval splits as $ds^2 = g_{ab}(\vec{x}, \vec{y})dx^a dx^b + g_{AB}(\vec{y})dy^A dy^B$. 
Figure 1: Global topology of the trousers and yarmulke cobordisms. In the trousers $N_\epsilon$ is to be regarded as a little region around the saddle point, while in the yarmulke $N_\epsilon$ has the same topology as the whole cobordism, which is just a disc.

5.1 Causal structure in the trousers

The neighbourhood Morse metric for the trousers is

$$ds^2 = 4\{(y^2 - (\zeta - 1)x^2)dx^2 + (x^2 - (\zeta - 1)y^2)dy^2 + 2\zeta xydxdy\}. \quad (6)$$

Any radial line with endpoint at the origin is a geodesic so the norm of its tangent vector has the same sign all along. Thus the disc can be partitioned into sectors that are loosely speaking, either future time-like, past time-like or space-like related to the origin (see figure 2). Indeed by substituting for $y = mx$ in the interval one obtains:

$$ds^2 = -4((\zeta - 1)m^4 - 2(\zeta + 1)m^2 + (\zeta - 1))x^2dx^2 \begin{cases} > 0 & \text{if } (m_1)^2 < m^2 < (m_2)^2 \\ \leq 0 & \text{otherwise} \end{cases} \quad (7)$$

where the gradients $m_1 = \frac{\sqrt{\zeta - 1}}{\sqrt{\zeta - 1}} < 1$ and $m_2 = \frac{\sqrt{\zeta + 1}}{\sqrt{\zeta - 1}} = m_1^{-1} > 1$ mark the transition between the spacelike and timelike character of the radii, with the lines $y = \pm m_1 x$ and $y = \pm m_2 x$ being null and separating the sectors.

We define sets $\mathcal{P}_1, \mathcal{P}_2, \mathcal{F}_1$ and $\mathcal{F}_2$, via

$$\mathcal{P}_1 \equiv \{(x, y) \in N_\epsilon : |y| < m_1 |x| \text{ and } x > 0\}$$
$$\mathcal{P}_2 \equiv \{(x, y) \in N_\epsilon : |y| < m_1 |x| \text{ and } x < 0\}$$
$$\mathcal{F}_1 \equiv \{(x, y) \in N_\epsilon : |y| > m_2 |x| \text{ and } y > 0\}$$
\[ \mathcal{F}_2 \equiv \{(x, y) \in N_e : |y| > m_2|x| \text{ and } y < 0\} \]  
\[ (8) \]

Also \( \mathcal{P} \equiv \mathcal{P}_1 \cup \mathcal{P}_2 \), and \( \mathcal{F} \equiv \mathcal{F}_1 \cup \mathcal{F}_2 \). We define \( \partial \mathcal{P} \) and \( \partial \mathcal{F} \):
\[
\partial \mathcal{P} \equiv \{(x, y) \in N_e : y = m_1x \text{ or } y = -m_1x\}
\]
\[
\partial \mathcal{F} \equiv \{(x, y) \in N_e : y = m_2x \text{ or } y = -m_2x\}
\]  
\[ (10) \]

and \( \mathcal{S} \equiv N_e - (\mathcal{P} \cup \mathcal{F} \cup \partial \mathcal{P} \cup \partial \mathcal{F}) \). We see that \( \mathcal{F} \) is what we’d expect for the chronological future of the Morse point, \( p \), \( \partial \mathcal{F} \) is what we might want to call the future lightcone of \( p \) and similarly for the past; \( \mathcal{S} \) is the “elsewhere” of \( p \). The status of these sets is discussed further in a later subsection.

To summarise, through all the points in the shaded regions of figure (2) there passes a radial timelike geodesic; \( \mathcal{S} \) denotes the points outside the shaded regions through which the radial geodesics are spacelike, while the boundary radial geodesics, with gradient \( \pm m_1 \) and \( \pm m_2 \) are null.

\[ \sqrt{\zeta - 1(x^2 - y^2)} = 2xy + c_+ \]  
\[ (11) \]
\[ \text{or} \quad \sqrt{\zeta - 1(x^2 - y^2)} = -2xy + c_- \]  
\[ (12) \]

Figure 2: The trousers. Partition of the disc by radial geodesics. The timelike geodesics are in the shaded regions, future outward about the y-axis and future inward about the x-axis. The radial geodesics outside the shaded regions are spacelike.

We return to the general null geodesics which give us the lightcones for an arbitrary point. This is a generalisation to arbitrary \( \zeta \) of the analysis of [12]. From equation (3) one obtains an implicit expression for the null curves:

\[ \sqrt{\zeta - 1(x^2 - y^2)} = 2xy + c_+ \]  
\[ (11) \]
\[ \text{or} \quad \sqrt{\zeta - 1(x^2 - y^2)} = -2xy + c_- \]  
\[ (12) \]
where \( c_\pm \) are constants. To get a clearer idea of what these curves are consider a rotated coordinate system \((x', y')\) with the \( x' \) axis being the line \( y = m_1 x \) at an angle of \( \alpha = \tan^{-1} m_1 \) with the \( x \) axis:

\[
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]

Then (11) takes the form

\[
x' y' = \frac{-c_+}{2\sqrt{\zeta}}
\]

which shows that they are hyperbolae with \( y = m_1 x \) and \( y = -m_2 x \) as asymptotes. Rotating to coordinates \((x'', y'')\) by \(-\alpha\) instead, (12) becomes

\[
x'' y'' = \frac{c_-}{2\sqrt{\zeta}}
\]

so these are hyperbolae with \( y = -m_1 x \) and \( y = m_2 x \) as asymptotes.

Through every point \( q \) in \( N_\epsilon \) there passes a curve which satisfies eq. (14) with a particular \( c_+ \), we call it \( \nu^+_q \) and another curve which satisfies eq. (15), we call this \( \nu^-_q \). As we see shortly, these null geodesics through \( q \) suffice to bound its past and future provided \( q \) lies in \( S \), but not otherwise. To determine systematically the chronological pasts and futures of all points in the disc, we start by noting that in the rotated coordinate systems,

\[
\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \cos \psi & \sin \psi \\ -\sin \psi & \cos \psi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]

the hyperbolae given by \( \tilde{x} \tilde{y} = \text{constant} \) are timelike if and only if \(-\alpha < \psi < \alpha \) or \( \frac{\pi}{2} - \alpha < \psi < \frac{\pi}{2} + \alpha \), that is when the \( \tilde{x} \) and \( \tilde{y} \) axis fall in the interior of the shaded regions in figure 2. We will use segments of such timelike hyperbolae to determine the chronological relation. By symmetry and time-reversal invariance, we need consider only three representative points in the upper right quadrant: (i) \( q \in S \), (ii) \( q \in F_1 \) and (iii) \( q \in \partial F_1 \).

(i) For \( q \in S \), we claim that \( I^+(q) \) is the interior of the horizontally shaded region in figure 3, bounded by the two null hyperbolae through \( q \), \( \nu^+_q \) and \( \nu^-_q \) and that \( I^-(q) \) is the interior of the vertically shaded region between the same hyperbolae. Those regions are contained in \( I^+(q) \) and \( I^-(q) \), because they are swept out by the timelike hyperbolae through \( q \) between \( \nu^+_q \) and \( \nu^-_q \). To see that such regions exhaust all of \( I^+(q) \) and \( I^-(q) \) is also straightforward. Any future directed timelike curve from \( q \) must begin by heading into the horizontally shaded region. If there was one such
curve ending at a point \( s \) outside the region, it would have to intersect one of the bounding hyperbolae at some point \( q' \), but the tangent vector there could not point out of the region and be both timelike and future directed according to the local lightcone at \( q' \). Similarly for \( I^-(q) \).

![Figure 3: The trousers.](image)

Figure 3: The trousers. \( q \in S \) and \( I^+(q) \) \( (I^-(q)) \) is the horizontally (vertically) shaded region.

(ii) For \( q \in \mathcal{F}_1 \), \( I^+(q) \) is the interior of the horizontally shaded region shown in figure 4 bounded by the two null hyperbolae through \( q \), by the same arguments as in case (i), and we claim that \( I^-(q) \) is the interior of the vertically shaded region bounded by the hyperbolae and by lines \( y = m_1 x \) and \( y = -m_1 x \) with \( y < 0 \).

To show that this region is indeed contained in \( I^-(q) \) we find sequences of timelike hyperbola segments from all points in the region to \( q \). Let us first divide the region in two zones. If \( y_q = m_q x_q \), zone 1 consists of points with \( y > -\frac{1}{m_q} x \) and zone 2 of points with \( y \leq -\frac{1}{m_q} x \) (see figure 5).

A point \( s \) in zone 1 can be joined to \( q \) by a single arc of timelike hyperbola whose asymptotes are the \( \tilde{x} \) and \( \tilde{y} \) axes defined by (16) with \( -\frac{1}{m_q} < \tan \psi < m_1 \), except when \( y_s = m_q x_s \) in which case the timelike curve to \( q \) is \( y = m_q x \).

For a point \( t \) in zone 2 two hyperbolic arcs are needed: the first one takes it into a point \( s' \) in zone 1 via a hyperbola with asymptotes given by (17) with
Figure 4: The trousers. \( q \in \mathcal{F} \) and \( I^+(q) \) \((I^-(q))\) is the horizontally (vertically) shaded region.

\[ -m_1 < \tan \psi < m_t \] where \( m_t = y_t/x_t \). Then \( s' \) can be connected to \( q \) as before.

The argument that these regions comprise all of \( I^\pm(q) \) is as in (i).

(iii) For \( q \in \partial \mathcal{F}_1 \), similar arguments show that \( I^+(q) \) is the interior of the horizontally shaded region bounded by the null hyperbola \( x'y' = x'_q y'_q \) and the null line \( y = m_2x \). \( I^-(q) \) is the interior of the vertically shaded region bounded by the hyperbola, \( y = m_2x \) and \( y = -m_1x \) for \( x > 0 \) (see figure 4). Note that \( I^-(q) \) does not contain any point with \( x < 0 \).

Using \( J^\pm(q) \subset \overline{I^\pm(q)} \) the causal pasts and futures of these representative points are easy to find. Since all the points in a null geodesic through \( q \) are in its causal past or future, we just need to decide whether the additional straight lines bounding the \( I^\pm \) of points type (ii) and (iii) are in \( J^\pm \). For definiteness, consider \( q \in \mathcal{F}_1 \) and a point \( s \) on \( y = -m_1x, y < 0 \). Now \( q \) is not in \( \overline{I^+(s)} \), so it doesn’t belong to \( J^+(s) \) either: there is no causal curve from \( s \) to \( q \). It follows that the lines that bound \( I^-(q) \)
Figure 5: A point \( s \in I^- (q) \) for which \( y_s > -x_s/m_q \) can be joined to \( q \) with a single arc of timelike hyperbola. A point \( t \in I^- (q) \) with \( y_t < -x_t/m_q \) requires two such arcs to reach \( q \).

in the lower hemiplane are not in \( J^- (q) \). Neither is the line \( y = -m_1 x \) contained in \( J^- (q) \) when \( q \in \partial \mathcal{F}_1 \). Summarising, for points \( q \in S \) we have \( J^\pm (q) = \overline{I^\pm (q)} \), otherwise the causal sets \( J^\pm (q) \) are not closed.

This completes our analysis of the causal structure around the crotch singularity in the 1 + 1 trousers, which we use extensively later. For the moment it allows us to establish the following.

**Lemma 2** The neighbourhood geometry \((N_\epsilon, g)\) of type \((1, 1)\) is a causally discontinuous spacetime.

**Proof:** Let \( q \in \partial \mathcal{F}_1 \) with \( x_q > 0 \), then \( \downarrow I^+ (q) \neq I^- (q) \) since any \( s \) on the negative \( x \)-axis satisfies \( s \in \downarrow I^+ (q) \) but \( s \notin I^- (q) \).

In figure 7 we illustrate the failure of causal continuity in each of the remaining three equivalent definitions given in section 2, in the hope to give the reader an intuition for their meaning.
5.2 Causal structure in the yarmulke

We now undertake a similar analysis for the neighbourhood geometry $(0,2)$. The Morse metric on the punctured disc is

$$ds^2 = 4(-\zeta - 1)r^2 dr^2 + r^4 d\phi^2 .$$

(17)

Because of the $U(1)$ symmetry of this metric, there is only one class of points. Since $f(r) = r^2$ is the time function, timelike and null tangent vectors are past pointing if their radial component is inwards and future pointing if it is outwards. As before, in two dimensions the geodesic equation is not really needed to find the null geodesics through a point; solving for a tangent vector with vanishing norm suffices. The solutions are the null “spiraling” curves $\sigma_0^\pm$ given by,

$$r(\phi) = r_0 e^{\frac{\phi}{\sqrt{\zeta - 1}}}$$

(18)
Figure 7: On the left, the futures and pasts of a pair of points, \((s, q)\) are shown. This pair violates the reflecting property, since \(I^+(q) \subset I^+(s)\) but \(I^-(s) \not\subset I^-(q)\). Also \(q \in J^+(s)\) but \(s \not\in J^-(q)\). On the right the compact set \(C \subset M - I^-q\) intersects \(I^-q'\) for a point \(q' \in \mathcal{F}_1\) which can be taken arbitrarily close to \(q\).

Again these spirals can be shown to be consistent with the geodesic equations. For geometries with \(\zeta \to 1\), the lightcones are squeezed onto the radial direction, while for \(\zeta \to \infty\), the lightcones widen to become circles.

The future direction along the spirals \(\sigma^+_q\) and \(\sigma^-_q\) through a point \(q\) corresponds to an increase in the radial coordinate, which, along \(\sigma^+_q\) is achieved by increasing \(\phi\) and along \(\sigma^-_q\) by decreasing \(\phi\). The radial straight lines \(\phi = \text{constant}\) also satisfy the geodesic equations. These geodesics are timelike, since \(d\phi = 0\) along them.

Forgetting for the moment that we are confined to a disc of finite radius, the two spirals through a point \(q\) converge again at both an earlier \(q'\) and at a later \(q''\), beyond which these null geodesics no longer bound the past or future of \(q\) (see figure 8). Let \(L_q\) be the interior of the little heart-shaped region bounded by the two past directed null spirals from \(q\) to \(q'\) and \(B_q\) the big heart-shaped region bounded by the future directed null spirals from \(q\) to \(q''\). Then (i) \(I^-(q) = L_q\) and (ii) \(I^+(q) = N_e - B_q\).

By the symmetry of this metric, all points are equivalent so that our task reduces to considering the representative point \(q = (r_q, 0)\). We only give the argument for (i) since (ii) is similar. Let \(s = (r_s, \phi_s) \in L_q\). By the symmetry of \(L_q\), we can assume without loss of generality that \(0 \leq \phi_s \leq \pi\). Thus, \(r_s < r_q e^{\frac{2\phi_s}{\sqrt{\zeta - 1}}}\), where \(Z = \sqrt{\zeta - 1}\). In particular for points \(s\) such that \(\phi_s = 0\), the radius \(\phi = 0\) is a timelike curve from \(s\) to \(q\) and therefore they belong to \(I^-(q)\). If \(\phi_s \neq 0\), consider the curve \(\gamma\) from \(s\) to
Given by
\[ r(\phi) = u(\phi)e^{\frac{\phi}{Z}} \]  
(19)
where the function \( u(\phi) \),
\[ u(\phi) = r_q \frac{\phi_s - \phi}{\phi_s} + r_s e^{\frac{\phi}{Z}} \frac{\phi}{\phi_s} \]  
(20)
decreases smoothly from \( r_q \) to \( r_s e^{\frac{\phi}{Z}} \). Then
\[ \frac{dr}{d\phi} = -\left( r_q - \frac{\phi}{\phi_s} (r_q - r_s e^{\frac{\phi}{Z}}) \right) e^{\frac{\phi}{Z}} - \frac{1}{\phi_s} (r_q - r_s e^{\frac{\phi}{Z}}) e^{\frac{\phi}{Z}} \]
(21)
so that, along \( \gamma \),
\[ g(\dot{\gamma}, \dot{\gamma}) = 4r^2(-Z^2\dot{r}^2 + r^2\dot{\phi}^2) = 4r^2\dot{\phi}^2Z^2 \left( -\left( \frac{dr}{d\phi} \right)^2 + \frac{r^2}{Z^2} \right) \]  
(22)
Since \( r_q - r_s e^{\frac{\phi}{Z}} > 0 \) we have \( \left| \frac{dr}{d\phi} \right| > \frac{|r|}{Z} \), so that the tangent to \( \gamma \) is everywhere timelike. We can choose the direction of parameterisation so that \( \gamma \) is future directed. Thus all points in \( L_q \) belong to \( I^- (q) \). Moreover, no point outside \( L_q \) belongs to \( I^- (q) \), since the curve would have to cross the small heart boundary at some point \( s \); but according to the local lightcone at \( s \) any vector in its tangent space \( T_s \) pointing into \( L_q \) is either spacelike or past directed.

The causal past and future of \( q \) in this case are simply the closure of their chronological analogues since the bounding curves of the lightcones are always geodesics through \( q \).

That the above geometry is causally continuous follows from the general result lemma \( \Box \). But the reader can verify it graphically by finding the intersections of the past of all points to the future of \( q \) and vice-versa.

### 5.3 The “chronos” of the Morse point

Violation of causal continuity in the trousers occurs at points on the null geodesics that approach the origin. Looking back at figure (3) one is inclined to regard these
Figure 8: The yarmulke. Segments of the null geodesics through \( q = (a, 0) \) that bound \( q \)'s past and future. Their first intersection in the past occurs at the point \( q' = (ae^{-\pi}, \pi) \) and their first intersection in the future occurs at \( q'' = (ae^{+\pi}, \pi) \).

lines as the null cone of the critical point \( p \), with \( \mathcal{F} \) constituting its chronological future and \( \mathcal{P} \) its chronological past. Although this is indeed the case in some scheme where the critical point is retained, we have excised it for present purposes and therefore need a few more definitions to give \( \mathcal{F} \) and \( \mathcal{P} \) a more formal status.

An **indecomposable past set (IP)** in a geometry \((M, g)\) is a subset \( W \) of \( M \) possessing the following two properties: 1. \( I^{-}(W) = W \), a set with this property is deemed to be a *past set*. 2. \( W \) cannot be decomposed as the union of two proper past sets. An indecomposable future set (IF) is defined dually.

The chronological past and future of any point in a Lorentzian spacetime are the standard examples; that is for any \( q \), \( I^{-}(q) \) is an IP and \( I^{+}(q) \) is an IF. The IP’s and IF’s that are not of this form were introduced to probe the causal boundary of spacetime. They are called terminal indecomposable pasts (TIP) and futures (TIF)[19].

It can be shown that a set \( W \) is a TIP if and only if there is a future inextendible timelike curve \( \gamma \) such that \( W = I^{-}(\gamma) \). In general, a whole class of such curves

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\[3\text{A causal curve in some spacetime is said to be future (past) inextendible if it has no future} \]

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generates the same TIP \( \mathcal{P} \). These concepts have an immediate application to our elementary cobordisms, since the removed critical point is, in this sense, a boundary of the spacetime.

An illustration is furnished by the \((1+1)\) dimensional examples from the previous sections. In the trousers it is straightforward to show there are two distinct TIP’s, \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \), corresponding to each of the future inextendible timelike curves \( \omega_1(t) = (x(t), y(t)) = (\frac{\epsilon}{2}e^{-t}, 0) \) and \( \omega_2(t) = (-\frac{\epsilon}{2}e^{-t}, 0), \ t \in [0, \infty) \). Similarly, there are two distinct TIF’s, \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \), generated by the past inextendible timelike curves \( \gamma_1(t) = (0, \frac{\epsilon}{2}e^t) \) and \( \gamma_2(t) = (0, -\frac{\epsilon}{2}e^t), \ t \in (-\infty, 0]. \)

In the \(1+1\) yarmulke, on the other hand, every radial geodesic is a past inextendible timelike curve. In fact they are all equivalent as future set generators, since any such curve \( \gamma \) generates \( I^+(\gamma) = N_\epsilon \), the whole punctured disc. Thus we say that the TIF representing the singularity is the entire \( N_\epsilon \). But no timelike curve is future inextendible towards the origin: the TIP associated with the creation point is empty. So in this case \( \mathcal{F} = N_\epsilon \) and \( \mathcal{P} = \emptyset \), as expected.

To summarise, we see that in these 2-dimensional geometries we can use inextendible timelike curves whose endpoint would be the origin to view the singularity as an ideal point \( \mathbb{P} \) in the spacetime and thus talk about its past \( \mathcal{P} \) and its future \( \mathcal{F} \). The topology of these sets plays a role in determining the causal structure around the critical point.

In the causally discontinuous \((1 + 1)\) trousers both \( \mathcal{P} \) and \( \mathcal{F} \) consist of two disconnected components. Moreover \( \mathcal{F} \) “separates” \( \mathcal{P} \), in the sense that any curve joining the two disconnected components of \( \mathcal{P} \) necessarily traverses \( \mathcal{F} \). Similarly \( \mathcal{P} \) separates \( \mathcal{F} \). The discontinuity of \( I^- \) is manifest in that the past of any point on \( \partial \mathcal{F} \) intersects only one of the two components of \( \mathcal{P} \), while any point in \( \mathcal{F} \) contains the whole of \( \mathcal{P} \) in its past. Dual statements can be made regarding the discontinuity of \( I^+ \).

In the causally continuous \((1 + 1)\) yarmulke and its time reverse, both \( \mathcal{F} \) and \( \mathcal{P} \), (past) endpoint. Inextendibility of a geodesic shows up as it approaches the edge of spacetime or a singularity.

\(^4\)Hawking and Ellis identify IP’s with IF’s so that the boundaries of space, both at infinity or singularities, are in some sense attached to the spacetime. They take the classes of such past and future sets to be the elements among which a new causal space is defined. It may be interesting to pursue this approach in our cases and confirm that it leads to the same conclusions about causal continuity.

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respectively, are trivially connected.

This seems to indicate that the source of causal discontinuity for the neighbourhood geometries resides on the boundaries of the TIFs and TIPs, and suggests a scheme to investigate the neighbourhood of a critical point of arbitrary index $\lambda$ in general dimension $n$.

6 Causal continuity in higher dimensional Morse geometries

We now examine the causal continuity of the neighbourhood geometries with arbitrary $\lambda$ for $n \geq 3$. For the yarmulke neighbourhoods lemma 4 shows that they are causally continuous without further work. Thus in the remainder of this section we impose the condition $\lambda \neq 0, n$. The causal structure in these cases can be analysed by identifying higher dimensional analogues of the TIP’s and TIF’s discussed in Section 5.3 and appropriately combining the causal structures of the 2-dimensional spacetimes studied in Section 4.

6.1 TIP’s and TIF’s in higher dimensions.

Guided by the form of $F$ and $P$ in the trousers, we start by guessing at their analogues in the higher dimensional neighbourhood geometries. For the moment, we mark these sets with a tilde, but then show that they are in fact TIP’s and TIF’s associated with the singularity. We define them in the spherical coordinates used in equation (5). The angular coordinates (including the discrete angle if $\lambda = 1$ or $n - 1$) are understood to vary over all their possible values.

$$\tilde{P} \equiv \{ q \in N_\epsilon : r_q < m_1 \rho_q \}$$

$$\tilde{F} \equiv \{ q \in N_\epsilon : r_q > m_2 \rho_q \}$$

(23)

where $\rho_q$ is the $\rho$ coordinate of $q$, etc., and $m_1$ and $m_2$ are as in (5).

Let us define the quadrant $\chi_q$ to be the set of points with the same angular coordinates as $q$. The warped product form of (5) with respect to the pair $(\rho, r)$, ensures that the geodesics in this quadrant are geodesics of the full metric.
We introduce a projection operator $P_q$ which acts on points and curves by projecting them into the quadrant $\chi_q$, so that for $x \in N_\varepsilon$ with $x = (\rho_x, \Theta_x, r_x, \Phi_x)$, $P_q x = (\rho_x, \Theta_q, r_x, \Phi_q)$. Similarly, if $\gamma(t)$ is a curve with coordinates $\gamma(t) = (\rho(t), \Theta(t), r(t), \Phi(t))$ then $P_q \gamma(t) = (\rho(t), \Theta_q, r(t), \Phi_q)$. A timelike curve $\gamma$ projects to a timelike curve $P_q \gamma$ in the quadrant and a causal curve projects to a causal curve. It follows that $I^\pm(q, \chi_q) = I^\pm(q) \cap \chi_q$ and moreover that for any point $y$ in $I^\pm(q)$ its projection $P_q y$ to $\chi_q$ must lie in $I^\pm - (q, \chi_q)$. Hence, the causal structure in $\chi_q$ is that of the upper right quadrant of the trousers and this is illustrated in figures 9–11 in which we give the chronological pasts and futures of three representative points.

We further define the set $\tilde{P}_q$ to be the intersection $\tilde{P} \cap \chi_q$ and $\tilde{F}_q = \tilde{F} \cap \chi_q$.

If $\lambda$ is neither 1 nor $n-1$, $\tilde{P}$ and $\tilde{F}$ are connected sets. However when $\lambda = 1$ $\tilde{P}$ comprises two disjoint components $\tilde{P}_1$ which has $\theta_0 = 0$ and $\tilde{P}_2$ with $\theta_0 = \pi$. Moreover $\tilde{F}$, which is connected ($n \geq 3$), then separates $\tilde{P}$ in the sense that any curve between $\tilde{P}_1$ and $\tilde{P}_2$ necessarily intersects the $r = 0$ hyperplane, which is contained in $\tilde{F}$. When $\lambda = n-1$, in the time-reversed geometry, $\tilde{P}$ separates $\tilde{F} = \tilde{F}_1 \cup \tilde{F}_2$ similarly.

Now we define a terminal past set and a terminal future set, generated by inextendible timelike curves that approach the Morse point. Let the curve $\omega_a$ in $N_\varepsilon$ be
Figure 10: \( \chi_q. \) \( q \in \partial \mathcal{F}. \) \( I^+(q, \chi_q) \) (\( I^-(q, \chi_q) \)) is the horizontally (vertically) shaded region.

given in polar coordinates by

\[
\omega_a(t) = \left( \frac{\epsilon}{2} e^{-t}, \Theta_a, 0, 0 \right), \quad t \in [0, \infty), \tag{24}
\]

where \( \Theta_a \) denotes some fixed point on the \( (\lambda - 1) \) sphere and \( \Phi \) is strictly undefined since \( r = 0 \) though we have written it as 0, and the curve \( \gamma_b \) be given by

\[
\gamma_b(t) = (0, 0, \frac{\epsilon}{2} e^t, \Phi_b), \quad t \in (-\infty, 0]. \tag{25}
\]

These are both timelike curves; \( \omega_a \) is future inextendible and \( \gamma_b \) is past inextendible.

For \( \lambda \neq 1 \), we define \( \mathcal{P} \equiv I^-(\omega_a). \) We show that \( \mathcal{P} \) does not depend on the choice of \( \Theta_a. \) Consider \( \omega_a'(t) = (\frac{\epsilon}{2} e^{-t}, \Theta_a', 0, 0) \) and let \( \mathcal{P}' \equiv I^-(\omega_a'). \) Consider a point \( z \) in \( \mathcal{P} \) so that there exists a future directed timelike curve from \( z = (\rho_z, \Theta_z, \tau_z, \Phi_z) \) to some point of \( \omega_a. \) The lines \( \omega_a \) and \( \omega_a' \) lie in a unique 2-plane on which the induced metric is that of the \((2, 0)\) yarmulke and in which the two curves are radial timelike geodesics. It follows from the analysis of Section 4.2 that within that 2-plane there are future directed timelike spiraling curves that start at any given point of \( \omega_a \) and end at some point of \( \omega_a'. \) Therefore there’s a future directed timelike curve from \( z \) to some point of \( \omega_a' \) and so \( z \in \mathcal{P}' \). Reversing the roles of \( a \) and \( a' \) we also have \( \mathcal{P}' \subset \mathcal{P} \) and so \( \mathcal{P}' = \mathcal{P}. \)
Figure 11: $\chi_q$. $q \in \mathcal{F}$ and $I^+(q, \chi_q)$ ($I^-(q, \chi_q)$) is the horizontally (vertically) shaded region.

Similarly for $\lambda \neq n - 1$ we define $\mathcal{F} \equiv I^+(\gamma_b)$. An identical argument shows that $\mathcal{F}$ does not depend on the choice of $\Phi_b$: any $\gamma_b$ generates it.

When $\lambda = 1$, instead of one generator for $\mathcal{P}$ we require two, one for each possible value of the $S^0$ coordinate and assign them fixed labels 1 and 2:

$$\omega_1(t) = \left(\frac{\epsilon}{2}e^{-t}, \theta_0 = 0, 0, 0\right), \quad t \in [0, \infty) \quad (26)$$

$$\omega_2(t) = \left(\frac{\epsilon}{2}e^{-t}, \theta_0 = \pi, 0, 0\right), \quad t \in [0, \infty). \quad (27)$$

Then let $\mathcal{P}_i \equiv I^-(\omega_i), i = 1, 2$ and $\mathcal{P} \equiv \mathcal{P}_1 \sqcup \mathcal{P}_2$.

Finally when $\lambda = n - 1$ there are two generators for $\mathcal{F}$:

$$\gamma_1(t) = (0, 0, \frac{\epsilon}{2}e^t, \phi_0 = 0) \quad t \in (-\infty, 0] \quad (28)$$

$$\gamma_2(t) = (0, 0, \frac{\epsilon}{2}e^t, \phi_0 = \pi) \quad t \in (-\infty, 0] \quad (29)$$

with $\mathcal{F}_i \equiv I^+(\gamma_i), i = 1, 2$ and $\mathcal{F} \equiv \mathcal{F}_1 \sqcup \mathcal{F}_2$.

We now show that $\mathcal{P} = \tilde{\mathcal{P}}$. The proof that $\mathcal{F} = \tilde{\mathcal{F}}$ is analogous. There are two cases.
a) $\lambda \neq 1$

Let $z \in \mathcal{P}$. Take the curve $\omega_z$ with $S^3$ coordinate $\Theta_z$ as the generator of $\mathcal{P}$. Consider the 2-d quadrant $\chi_z$. $\chi_z$ has the causal structure of the 1+1 trousers as represented by figures [●]. In $\chi_z$, $\omega_z$ is the $\rho$-axis (horizontal or past axis). $z \in I^-(\omega_z, \chi_z)$ and so it must satisfy $r_z < m_1 \rho_z$ and so $z \in \mathcal{P}$. Now suppose $u \in \mathcal{P}$. Let $\omega_u$ be a generator of $\mathcal{P}$ with $\Theta = \Theta_u$. Then $u \in I^+(\omega_u, \chi_u)$ by the causal structure of $\chi_u$ and so $u \in \mathcal{P}$.

b) $\lambda = 1$

Wlog let $z \in \mathcal{P}_1$. Then $\omega_1$, the generator of $\mathcal{P}_1$ is the $\rho$-axis in the quadrant $\chi_z$ and $r_z < m_1 \rho_z$ by the same argument as above and so $\mathcal{P}_1 \subset \mathcal{P}_1$. Finally let $u \in \mathcal{P}_1$. In $\chi_u$, $u \in I^-(\omega_1, \chi_u)$ and we are done.

We again define the sets

\[
\partial \mathcal{P} \equiv \{ x \in N_\epsilon : r_x = m_1 \rho_x \} \\
\partial \mathcal{F} \equiv \{ x \in N_\epsilon : r_x = m_2 \rho_x \} \\
S \equiv N_\epsilon - (\mathcal{P} \cup \mathcal{F} \cup \partial \mathcal{P} \cup \partial \mathcal{F})
\]

and let $\mathcal{F}_q \equiv \mathcal{F} \cap \chi_q$, etc.

Claim 1 (i) If $q \in \mathcal{F}$, then $\mathcal{P} \subset I^-(q)$. (ii) If $q \in \partial \mathcal{F}$ then $I^+(q) \subset \mathcal{F}$. (iii) If $q \in \partial \mathcal{F}$, then (a) $\lambda \neq 1$ implies $\mathcal{P} \subset I^-(q)$, while (b) $\lambda = 1$ implies either $I^-(q) \cap \mathcal{P}_2 = \emptyset$ or $I^-(q) \cap \mathcal{P}_1 = \emptyset$.

Proof: (i) Let $q \in \mathcal{F}$ and $z \in \mathcal{P}$. $\mathcal{F}$ is generated by $\gamma_q$ with $\Phi = \Phi_q$ and $q \in I^+(\gamma_q)$. $\mathcal{P}$ is generated by $\omega_z$ with $\Theta = \Theta_z$ and $z \in I^-(\omega_z)$. The curves $\omega_z$ and $\gamma_q$ lie in the 2-d quadrant defined by $\Theta = \Theta_z$ and $\Phi = \Phi_q$ in which $\omega_z$ is the $\rho$-axis and $\gamma_q$ is the $r$-axis. By the causal structure of $\chi_q$, for every point $q' \in \gamma_q$ we have $\omega_z \in I^-(q')$. Hence result.

(ii) Let $q \in \partial \mathcal{F}$ and $z \in I^+(q)$. Consider the projection $P_q \omega_z$ into $\chi_q$ which must lie in $I^+(q, \chi_q)$. This gives us $r_z > m_2 \rho_z$ and so $z \in \mathcal{F}$. Note that we didn’t actually need to explicitly prove this point, since in general, for any future set $A = I^+(B)$ we have $I^+(\partial A) \subset A$ [18].

(iii) Let $q \in \partial \mathcal{F}$.
(a) $\lambda \neq 1$. Let $z \in \mathcal{P}$. $\mathcal{P}$ is generated by $\omega_q$ with $\Theta = \Theta_q$ and $z \in I^-(\omega_q)$. Now $\omega_q$ is the $\rho$ axis of the quadrant $\chi_q$ and again the causal structure in this quadrant implies that $\omega_q \subseteq I^-(q)$ and so $z \in I^-(q)$.

(b) $\lambda = 1$. Suppose wlog $\theta_0(q) = 0$ and let $z \in I^-(q)$. If $\theta_0(z) = \pi$ then there exists a future directed timelike curve from $z$ to $q$ which passes through $\rho = 0$. Considering the projection of this curve into $\chi_q$ shows this to be a contradiction. So $\theta_0(z) = 0$ and $I^-(q) \cap \mathcal{P}_2 = \emptyset$. □

6.2 Causal continuity in neighbourhood geometries of general index and dimension

We use the following notation to denote scale transformations by a real number $a$: given a point $q = (\rho_q, \Theta_q, r_q, \Phi_q)$ and a curve $\gamma(t) = (\rho(t), \Theta(t), r(t), \Phi(t))$, we write $aq$ for the point $aq = (a \rho_q, \Theta_q, ar_q, \Phi_q)$ and $a \gamma$ for the curve $a\gamma(t) = (a \rho(t), \Theta(t), ar(t), \Phi(t))$. Notice that the timelike character of a curve is preserved under this scaling. We refer to $R^2 = \rho^2 + r^2$ as the squared distance from the origin.

**Lemma 3** For any index $\lambda$, causal continuity holds at all points $q \in \mathcal{P} \cup \mathcal{F}$.

**Proof:** Let $q \in \mathcal{F}$. First, we prove that $\downarrow I^+(q) = I^-(q)$. Suppose not. Then $\downarrow I^+(q) \not\subseteq I^-(q)$ and by claim 2 (appendix) there is a point $y$ and a neighbourhood $U_y$ of $y$ such that $U_y \subseteq \downarrow I^+(q)$ and $U_y \cap I^-(q) = \emptyset$. Define the sequence of points $q_k = a_k q$ where $a_k = (1 + \frac{\delta}{k})$, $k = 1, 2, \ldots$ and $\delta > 0$ is small enough that $q_1 \in N_c$. The $q_k$ tend to $q$ and lie along the radial timelike line from the origin through $q$. So $q_k \in I^+(q)$, $\forall k$. Thus there exists a future directed timelike curve $\gamma_k$ from $y$ to $q_k$. Let $\gamma'_k = a_k^{-1} \gamma_k$, again a future directed timelike curve. The final point of each of these scaled curves is $q$ and the initial point of $\gamma'_k$ is $y_k = a_k^{-1} y$. Choose $k$ large enough so that $y_k \in U_y$. This is a contradiction.

To prove that $\uparrow I^-(q) = I^+(q)$ we again assume it does not hold and so there is a point $y$ and a neighbourhood $U_y$ of $y$ such that $U_y \subseteq \uparrow I^-(q)$ and $U_y \cap I^+(q) = \emptyset$. Let $q_k = b_k q$ where $b_k = 1 - \frac{\delta}{k}$, $k = 1, 2, \ldots$. Then each $q_k \in I^-(q)$ and so $\exists$ a future directed timelike curve $\gamma_k$ from $q_k$ to $y$. Let $\gamma'_k = b_k^{-1} \gamma_k$ which is again future directed and timelike. The initial point of each $\gamma'_k$ is $q$ and the final point of $\gamma'_k$ is $y_k = b_k^{-1} y$. 27
We must now check that for large enough $k$, the curve $\gamma_k'$ lies entirely within the disc $N_\varepsilon$. $\exists K > 0$ such that $k > K$ implies that $y_k \in N_\varepsilon$. Also we have $q_k \in F$, $\forall k$. By claim (3) (appendix) we therefore know that $R^2$ reaches its maximum along $\gamma_k'$ at its future endpoint $y_k$ and so $\gamma_k'$ remains in $N_\varepsilon$ for $k > K$. Now choose some $k > K$ large enough so that $y_k \in U_y$ which is a contradiction. Hence the result.

Causal continuity at points in $\mathcal{P}$ is proved similarly. $\square$

**Lemma 4** For any index $\lambda$, causal continuity holds at any point $q \in S$.

**Proof:** We show that $I^-(q) = \downarrow I^+(q)$. Suppose not. Then, as before, there exists a $y \in \downarrow I^+(q)$ with a neighbourhood $U_y$ contained in $\downarrow I^+(q)$ which doesn’t intersect $I^-(q)$. Choose some sequence of points $q_k \in I^+(q, \chi_q) \cap S$, $k = 1, 2, \ldots$, that converges to $q$ from the future so that $q_k \in I^+(q_{k+1})$. For example they could lie on the timelike hyperbola $r \rho = r_q \rho_q$ through $q$ in $\chi_q$. There exist future directed timelike curves, $\gamma_k$ from $y$ to $q_k \forall k$. We plan to use the CCLT to construct a causal curve from $y$ to $q$ and to do so we must identify a compact region in which all (or at least infinitely many of) the $\gamma_k$ are contained. The set of all the $\gamma_k$ is bounded away from the origin since each $\gamma_k$ must lie in $I^-(q_1)$ which is seen to be bounded away from the origin by considering $I^-(q_1) \cap \chi_{q_1}$.

However it might be that the set is not bounded away from the edge of the punctured disc, $N_\varepsilon$. So consider the geometry with metric (4) on the larger punctured disc $N_{2\varepsilon} \equiv D_{2\varepsilon} - \{p\}$. Our neighbourhood geometry $(N_\varepsilon, g)$ is embedded in it in the obvious way. In $N_{2\varepsilon}$ we can find a compact set in which all the $\gamma_k$ lie and which is $C = \overline{D_\varepsilon} - D_{\varepsilon'}$ where $\overline{D_\varepsilon}$ is the closed (in $\mathbb{R}^n$) ball of radius $\varepsilon$ and $\varepsilon' > 0$ is small enough.

The CCLT now implies that $\exists$ a causal curve in $C$ from $y$ to $q$.

This is true for all points $y' \in U_y$. So that $U_y \subset J^-(q, C)$. Since $\text{Int}(J^-(q, C)) \subset \text{Int}(I^-(q, C)) = I^-(q, C)$ this implies that there exists a timelike curve, $\gamma$, from $y$ to $q$ in $C$. If the radial distance $R$ is bounded away from $\varepsilon$ along $\gamma$ we are done, since then $\gamma$ lies in $N_\varepsilon$. So suppose it is not. We construct a new curve $\tilde{\gamma}$ by rescaling $\gamma$ down away from the boundary almost everywhere, except in a small neighbourhood of $q$ where we fix it so that it still ends at $q$.

More precisely let $U_q \subset N_\varepsilon$ be a neighbourhood of $q$ and let $s \subset U_q$ be a point on $\gamma$. In particular $s \in I^-(q)$. Choose $\delta > 0$ small enough so that $s' = (1 - \delta)s$ is
also in $I^-(q)$ and $y' = (1 - \delta)y$ is in $U_y$. Then the initial segment of $\tilde{\gamma}$ is $(1 - \delta)\gamma$ from $y'$ to $s'$ and then we smooth it into a timelike curve from $s'$ to $q$. $\tilde{\gamma}$ then lies entirely within $N_\epsilon$ and so $y' \in I^-(q)$ which is a contradiction.

$I^+(q) = \uparrow I^-(q)$ is proved similarly. □

The only potential obstructions to global causal continuity therefore lie in the remaining region $\partial F \cup \partial P$. Combining the method of the previous lemma and the results gathered in the last section we can prove.

**Lemma 5** (a) When $\lambda \neq 1$ causal continuity holds at all points $q \in \partial F$. (b) When $\lambda = 1$, if $q \in \partial F$ then $I^-(q) \neq \downarrow I^+(q)$. (c) When $\lambda \neq n - 1$ causal continuity holds at all points $q \in \partial P$. (d) When $\lambda = n - 1$, if $q \in \partial P$ then $I^+(q) \neq \uparrow I^-(q)$.

**Proof:** Consider $q \in \partial F$.

(a) $\lambda \neq 1$. We first show that $\downarrow I^+(q) = I^-(q)$. Suppose not. Then as usual, there is a point $y$ with a neighbourhood, $U_y$, such that $U_y \subset \downarrow I^+(q)$ and $U_y \cap I^-(q) = \emptyset$. First $y \notin \mathcal{P}$ by claim [1](iiia); also $y \notin \partial \mathcal{P}$ since otherwise some other point in $U_y$ would be in $\mathcal{P}$. Secondly $y \notin \mathcal{F}$, since otherwise considering a sequence of points $q_k \to q$, with $q_k \in I^+(q, \chi_q)$ and the fact that $P_qy \in I^-(q_k, \chi_q) \forall k$ would lead to a contradiction with the known causal structure of $\chi_q$; again $y \notin \partial \mathcal{F}$ since otherwise $U_y \cap \mathcal{F} \neq \emptyset$ and the same contradiction would arise. Finally if $y$ lies in $S$ then using arguments similar to those of the previous lemma we would obtain a contradiction also.

(b) $\lambda = 1$. Wlog let $\theta_0(q) = 0$. Let $y \in \mathcal{P}_2$. Then $x \in I^+(q) \Rightarrow x \in \mathcal{F}$ by claim [1](ii) $\Rightarrow y \in I^-(x)$ by claim [1](i) and similarly for any point $y'$ in a neighbourhood, $U_y$, of $y$ such that $U_y \subset \mathcal{P}_2$. Thus $y \in \downarrow I^+(q)$. But $y \notin I^-(q)$ by the proof of claim [1](iia).

Parts (c) and (d) are proved similarly. □

Putting together the partial results in lemmas [1], [2], [3], [4] and [5], we have a proof of the following proposition.

**Proposition 3** Let $(N_\epsilon, g)$ be the neighbourhood geometry type $(\lambda, n - \lambda)$. Then

(i) If $\lambda \neq 1, n - 1$, $(N_\epsilon, g)$ is causally continuous.
(ii) If $\lambda = 1$ or $\lambda = n - 1$, $(N, g)$ is not causally continuous.

7 Discussion

We have made progress on the way to proving the Borde-Sorkin conjecture that causal continuity in Morse geometries for topology change is associated only with indices $\lambda \neq 1, n - 1$. In particular we have proved that the conjecture holds for certain geometries that are neighbourhoods of single Morse points. We have also proved some more general results. The causal continuity of the yarmulke spacetimes has been demonstrated for any Morse metric on these cobordisms. We have also shown that any strongly causal compact product spacetime must be a Morse metric. This shows that the class of Morse spacetimes is in fact quite general and hence supports the proposal to sum over Morse metrics in the SOH.

In order to obtain a full proof, first our main results in the neighbourhood need to be extended to more general metrics than those built from the Cartesian flat Riemannian auxiliary metric. The next step would be to understand how the causal properties of the individual neighbourhoods affect the causal properties of the entire spacetime. We address these issues in a forthcoming paper [25].

We mentioned in the introduction our intention, eventually, to consider the Morse points as part of the spacetime and not to excise them. A causal order that includes the critical points is desirable, especially since it does not seem plausible that isolated points should be relevant in the quantum context. Most simply we can add in the degenerate point, $p$, and extend the causal relation by hand as follows: the causal past and future of $p$ are taken to be the closure of the union of respectively the TIPs and TIFs associated with $p$. For our neighbourhood spacetimes, then $J^-(p)$ would be $\overline{P}$, and $J^+(p) = \overline{F}$, where the closure is taken in the unpunctured disc. The full causal relation on the spacetime is then completed by transitivity.

Though simple, this prescription may appear a little ad hoc. We believe, however, that the causal order obtained in this way coincides with a robust generalisation of the usual causal relation proposed in [24]. This new relation, called $K$, is defined in terms of the chronological relation $I$, which can be extended to the Morse geometries by using the strict definition for chronology, i.e., by putting $I^\pm(p) = \phi$. One can immediately verify that the causal relation proposed above is the same as $K$ for the neighbourhood geometries. Our trivial extension of the chronology to the critical
points seems justified by the robustness of $K$, which is indifferent to the presence of isolated points in a spacetime. A remarkable feature of this new setting is that the property of causal continuity appears as the condition that the pair $(I, K)$ constitute a genuine causal structure, in the axiomatic sense of [21]. Details of this analysis will appear elsewhere [27].

Even if further work is required before we can understand the physical relevance of our results, in particular the effect of causal (dis)continuity on the propagation of quantum fields, we can already make an interesting observation: if the universe did begin in a big bang that could be described in its earliest moments by a Morse metric with an index 0 point, then the causal structure of the yarmulke is such that there are no particle horizons: all points on a given level surface of the Morse function have past points in common. That would mean that there would no longer be a cosmological horizon problem.

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9 Appendix

We prove two results that we use in the text.

Claim 2 If $\exists y \in \downarrow I^+(x)$ such that $y \notin I^-(x)$ then there is also exists a $z \in \downarrow I^+(x)$ such that $z \notin I^-(x)$. Dually, if $\uparrow I^-(x) - I^+(x)$ is not empty, then there is a point $z$ in $\uparrow I^-(x)$ with $z \notin I^+(x)$. 
Proof: Suppose not, namely suppose \( \downarrow I^+(x) - I^-(x) \) is not empty and all of its points lie in \( \partial I^-(x) \). Pick one such \( y \). By the obvious generalisation of Proposition 6.3.1[19] to a Lorentzian metric in general dimension, the boundary \( \partial I^-(x) \) is an \( n-1 \) dimensional submanifold, so that any ball centered on the boundary intersects both \( I^-(x) \) and the complement of \( \overline{I^-(x)} \). Thus every neighbourhood of \( y \) intersects the complement of \( \overline{I^-(x)} \); hence no neighbourhood of \( y \) is contained in \( \downarrow I^+(x) \), a contradiction. \( \square \)

Claim 3 Any future directed timelike curve, \( \gamma \) in \( N_\varepsilon \) that begins in \( \mathcal{F} \) remains in \( \mathcal{F} \) and moreover the function \( R^2 = \rho^2 + r^2 \) increases monotonically along it.

Proof: Let \( \gamma \) have initial point \( q \in \mathcal{F} \). Consider the projected curve \( P_q \gamma \). It has the same \( r \) and \( \rho \) behaviour as \( \gamma \) and so if we prove the result for \( P_q \gamma \) then it follows for \( \gamma \) itself.

\[ P_q \gamma \text{ lies in } \chi_q \text{ and starts at } q \in \mathcal{F}_q. \text{ The 1+1 trousers causal structure of } \chi_q \text{ shows that } P_q \gamma \text{ remains in } \mathcal{F}_q. \text{ So we have} \]

1. \( r\dot{r} - \rho\dot{\rho} > 0 \)
2. \( (r\dot{r} - \rho\dot{\rho})^2 > (r\dot{\rho} + \rho\dot{r})^2 \).
3. \( r > m_2 \rho \)

Their combination forces \( r\dot{r} + \rho\dot{\rho} \) to be positive along \( P_q \gamma \). We check this for the two possible signs of \( r\dot{\rho} + \rho\dot{r} \) in condition 2.

a) If \( r\dot{\rho} + \rho\dot{r} \geq 0 \), using \( r > 0 \) we can reduce this inequality and 1 to the conditions \( \dot{r} > 0 \) and \( \dot{\rho} > -\frac{\rho}{r} \). Then

\[ r\dot{r} + \rho\dot{\rho} > \frac{1}{r}(r^2 - \rho^2)\dot{r} > 0 \]

where we have used 3 in the last inequality.

b) If \( r\dot{\rho} + \rho\dot{r} < 0 \), conditions 1 and 2 reduce to \( \dot{\rho} < 0 \) and \( \dot{r} > \frac{\rho - r}{r + \rho} \). Then

\[ r\dot{r} + \rho\dot{\rho} > \frac{1}{r + \rho}(-r^2 + 2r\rho + \rho^2)\dot{\rho} \]

Now the bracket is negative precisely when \( r > m_2 \rho \). This together with \( \dot{\rho} < 0 \) gives the result. \( \square \)

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