EFFICIENT LEGENDRE DUAL-PETROV-GALERKIN METHODS FOR ODD-ORDER DIFFERENTIAL EQUATIONS

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Abstract. Efficient Legendre dual-Petrov-Galerkin methods for solving odd-order differential equations are proposed. Some Sobolev bi-orthogonal basis functions are constructed which lead to the diagonalization of discrete systems. Accordingly, both the exact solutions and the approximate solutions can be represented as infinite and truncated Fourier-like series. Numerical results indicate that the suggested methods are extremely accurate and efficient, and suitable for the odd-order equations.

1. Introduction. Spectral methods have been popular for solving partial differential equations in many fields of science and engineering over the last few decades, see, e.g., [2, 3, 4, 5, 6, 7, 18]. Standard spectral Galerkin methods have been extensively investigated for solving the even-order differential equations, cf. [15, 16]. However, there are very few works concerning of spectral methods for odd-order equations. This is partly due to the fact that direct spectral methods for higher odd-order problems lead to much higher condition numbers (more precisely, of order \(N^{2k}\), where \(N\) is the number of modes and \(k\) is the order of the equation), and often exhibit unstable modes if the Gauss-Lobatto collocation points are chosen (see [8, 14, 17]). To this end, Huang and Sloan [8] presented a stable pseudospectral method for solving the third-order equations, which is based on the use of the zeros of Jacobi polynomial \(J^{(2,1)}_{N-2}(x)\) as the collocation points.

Since the main differential operators in odd-order differential equations are not symmetric, it is more reasonable to use the Petrov-Galerkin spectral method. Recently, Ma and Sun [12, 13] developed an efficient Legendre-Petrov-Galerkin and Chebyshev collocation method for the third-order differential equations and derived optimal rates of convergence. By choosing appropriate basis functions, the resulting linear system in the Legendre-Petrov-Galerkin method is sparse. Shen [17] presented a dual-Petrov-Galerkin spectral method for the third and higher odd-order equations with new choices of the test and trial function spaces and their basis functions, and obtained linear systems which are compactly sparse for problems with constant coefficients (a seven-diagonal matrix for the third-order equation and an eleven-diagonal matrix for the fifth-order equation) and well conditioned for problems with variable coefficients.

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For dual-Petrov-Galerkin method, the trial functions are chosen to satisfy the boundary conditions of the differential equation, and the test functions are chosen to satisfy the dual boundary conditions [17, 20]. Dual-Petrov-Galerkin method allows us to integrate by parts freely without introducing any additional boundary terms. Shen [17] refers to three important advantages of the dual-Petrov-Galerkin method: (i). a strongly coercive bilinear form, (ii). a well-conditioned linear system, (iii). the optimal error estimates.

It is generally known that the Fourier functions are the eigenfunctions of the Laplace operator, whose $k$th derivatives are orthogonal with respect to each other. This property results in the diagonalization of the resulting algebraic systems of the corresponding Fourier spectral methods for periodic problems, cf. [3, 4, 18]. For non-periodic problems, the usual spectral methods only lead to a highly sparse or full algebraic system. It is really meaningful and worth looking forward to find out a class of basis functions such that the resulting algebraic systems of the spectral methods for non-periodic problems are still diagonal, see [19]. Recently, Liu et al. [11, 10] proposed the diagonalized Laguerre spectral methods for second order problems on the half line. Ai et al. [1, 9] also presented the diagonalized Legendre spectral methods for even-order problems on bounded domains.

The purpose of this paper is to construct efficient Legendre dual-Petrov-Galerkin spectral methods for the third and fifth order problems on bounded domains. The main idea is to choose suitable test and trial function spaces and their basis functions, such that the trial functions satisfy the boundary conditions of the differential equations, and the test functions satisfy the dual boundary conditions. Particularly, the trial functions and the test functions are bi-orthogonal with respect to certain Sobolev inner product, which leads to the diagonalization of discrete systems.

This paper is organized as follows. In Section 2, we introduce the Legendre polynomials and their basic properties. In Section 3, we construct some Sobolev bi-orthogonal Legendre polynomials corresponding to the odd-order differential equations, and propose the diagonalized Legendre dual-Petrov-Galerkin spectral methods. Some numerical results are presented in Section 4 to demonstrate the effectiveness and accuracy.

2. Notations and preliminaries. $I = (-1, 1), \partial_{x}^{k}v = \frac{d^{k}v}{dx^{k}}, v'' = \frac{d^{2}v}{dx^{2}}$ and $v' = \frac{dv}{dx}$.

We first recall the Legendre polynomials. Denote by $L_{k}(x)$ the Legendre polynomial of degree $k$, which is the eigenfunction of the singular Strum-Liouville problem:

$$(1 - x^2)L''_{n}(x) - 2xL'_{n}(x) + n(n+1)L_{n}(x) = 0, \quad n \geq 0. \quad (1)$$

The Legendre polynomials satisfy the following recurrence relations (cf. [18]),

$$(n + 1)L_{n+1}(x) = (2n + 1)xL_{n}(x) - nL_{n-1}(x), \quad n \geq 1. \quad (2)$$

$$(2n + 1)L_{n}(x) = L'_{n+1}(x) - L'_{n-1}(x), \quad n \geq 1. \quad (3)$$

$$(1 - x^2)L'_{n}(x) = \frac{n(n+1)}{2n+1}(L_{n-1}(x) - L_{n+1}(x)), \quad n \geq 1. \quad (4)$$

$$L'_{n}(x) = \sum_{k=0}^{n-1} (2k + 1)L_{k}(x), \quad n \geq 1. \quad (5)$$
In particular, \( L_0(x) = 1 \) and \( L_1(x) = x \). The Legendre polynomials possess the following orthogonality,

\[
\int_{-1}^{1} L_n(x)L_m(x)dx = \frac{2}{2n+1} \delta_{mn}, \tag{6}
\]

\[
\int_{-1}^{1} L'_n(x)L'_m(x)(1-x^2)dx = \frac{2n(n+1)}{2n+1} \delta_{mn}, \tag{7}
\]

where \( \delta_{m,n} \) is the Kronecker symbol. Moreover,

\[
L_n(-x) = (-1)^n L_n(x), \quad L_n(\pm 1) = (\pm 1)^n. \tag{8}
\]

Next, let \((a)k = a(a+1) \cdots (a+k-1)\) be the Pochhammer symbol. In order to construct efficient Legendre spectral methods for the third-order differential equations defined on \( I \in [-1,1] \), we need to consider the following two kinds of polynomials:

\[
p_n(x) = (1-x)^2(1+x)L_n(x), \quad n \geq 0, \tag{9}
\]

\[
q_n(x) = (1-x)(1+x)^2L_n(x), \quad n \geq 0. \tag{10}
\]

It is clear that \( p_0(x) = (1-x)^2(1+x), q_0(x) = (1-x)(1+x)^2, p_n(\pm 1) = \partial_x p_n(1) = 0 \) and \( q_n(\pm 1) = \partial_x q_n(-1) = 0 \). Next, let \( L_n(x) \equiv 0 \) for any \( k < 0 \).

**Lemma 2.1.** For any integer \( n \geq 0 \), the following equalities hold:

\[
p_n(x) = \frac{(n-2)(n-1)n}{(2n-3)(2n-1)(2n+1)} L_{n-3}(x) - \frac{(n-1)n}{(2n-1)(2n+1)} L_{n-2}(x) - \frac{n(n^2 - 3)}{(2n-3)(2n+1)(2n+3)} L_{n-1}(x) + \frac{2n^2 + 2n - 2}{(2n-1)(2n+3)} L_n(x) - \frac{(n+1)(n^2 + 2n - 2)}{(2n-1)(2n+1)(2n+5)} L_{n+1}(x) + \frac{(n+1)(n+2)(n+3)}{(2n+1)(2n+3)(2n+5)} L_{n+3}(x), \tag{11}
\]

\[
q_n(x) = -\frac{(n-2)(n-1)n}{(2n-3)(2n-1)(2n+1)} L_{n-3}(x) - \frac{(n-1)n}{(2n-1)(2n+1)} L_{n-2}(x) + \frac{n(n^2 - 3)}{(2n-3)(2n+1)(2n+3)} L_{n-1}(x) + \frac{2n^2 + 2n - 2}{(2n-1)(2n+3)} L_n(x) + \frac{(n+1)(n^2 + 2n - 2)}{(2n-1)(2n+1)(2n+5)} L_{n+1}(x) - \frac{(n+1)(n+2)}{(2n+1)(2n+3)} L_{n+2}(x) - \frac{(n+1)(n+2)(n+3)}{(2n+1)(2n+3)(2n+5)} L_{n+3}(x). \tag{12}
\]

**Proof.** Clearly, by (3), (4) and (9), we get that for \( n \geq 0 \),

\[
p_n(x) = \frac{(1-x)^2(1+x)}{2n+1} (L'_{n+1}(x) - L'_{n-1}(x)) = \frac{(1-x)}{(2n+1)} \left[ (2n+1)(n^2 + n - 1) L_n(x) - \frac{(n+1)(n+2)}{2n+3} L_{n+2}(x) - \frac{(n-1)n}{2n-1} L_{n-2}(x) \right] = \frac{2n^2 + 2n - 2}{(2n-1)(2n+3)} L_n(x) - \frac{(n+1)(n+2)}{(2n+1)(2n+3)} L_{n+2}(x) - \frac{(n-1)n}{(2n-1)(2n+1)} L_{n-2}(x)
\]
This leads to the result (11). In the same manner, we obtain the result (12).

**Lemma 2.2.** For any integer \( n \geq 0 \), the following equalities hold:

\[
\partial_x p_n(x) = -\frac{(n-2)(n-1)}{(2n-1)(2n+1)} L_{n-2}(x) + \frac{(n-1)n}{2n+1} L_{n-1}(x) + \frac{n(n+1)}{2n-1} L_n(x)
\]

(13)

\[
\partial_x q_n(x) = \frac{(n-2)(n-1)}{(2n-1)(2n+1)} L_{n-2}(x) - \frac{(n-1)n}{2n+1} L_{n-1}(x) - \frac{n(n+1)}{2n-1} L_n(x) + \frac{(n+1)(n+3)}{(2n+1)(2n+3)} L_{n+2}(x)
\]

(14)

\[
\partial_x^2 p_n(x) = \frac{(n+1)(n^2 + 4n - 2)}{2n+1} L_{n-1}(x) - (n+1)(n+2) L_n(x)
\]

(15)

\[
\partial_x^2 q_n(x) = -\frac{(n+1)(n^2 + 4n - 2)}{2n+1} L_{n-1}(x) - (n+1)(n+2) L_n(x)
\]

(16)

**Proof.** By (2) we have

\[
\partial_x p_n(x) = 3x^2 L_n(x) - 2x L_n(x) - L_n(x) + (1-x) \frac{n(n+1)}{2n+1} (L_n(x) - L_{n+1}(x))
\]

\[
= 3n + 3 \frac{n+2}{2n+1} \frac{L_{n+2}(x)}{2n+3} + \frac{n+1}{2n+1} L_n(x) \right] + \frac{3n}{2n+1} \left[ \frac{n}{2n-1} L_n(x) - \frac{n-1}{2n-3} L_{n-2}(x) \right] - 2 \left[ \frac{n+1}{2n+1} L_{n+1}(x) + \frac{n}{2n+1} L_{n-1}(x) \right] - L_n(x) + \frac{n(n+1)}{2n+1} \left[ L_{n-1}(x) - L_{n+1}(x) \right]
\]

\[
- \frac{n(n+1)}{2n+1} \left[ \frac{n}{2n-1} L_n(x) + \frac{n-1}{2n+1} L_{n-2}(x) - \frac{n+2}{2n+3} L_{n+2}(x) - \frac{n-1}{2n+3} L_n(x) \right].
\]

This yields (13). Next by (3) and (13),

\[
\partial_x^2 p_n(x) = -\frac{(n-2)(n-1)}{(n-1)n} L_{n-2}(x) + \frac{(n-1)n}{n} L_n(x) + \frac{n(n+1)}{(n-1)(n+1)} L_n'(x)
\]

\[
- (n+1)(n+2) \frac{L_{n+1}(x)}{n(n+1)} + \frac{(n+1)(n+2)(n+3)}{n(n+1)} L_{n+2}(x)
\]
\[ \begin{align*}
- \frac{(n-2)(n-1)n}{(2n-1)(2n+1)} + \frac{n(n+1)}{(2n-1)(2n+3)} + \frac{(n+1)(n+2)(n+3)}{(2n+1)(2n+3)} \right] L_n'(x) \\
+ \left[ \frac{(n-1)n}{2n+1} - \frac{(n+1)(n+2)}{2n+1} \right] L_{n-1}'(x) + \frac{(n-2)(n-1)n}{2n+1} L_{n-1}(x) \\
- (n+1)(n+2) L_n(x) + \frac{(n+1)(n+2)(n+3)}{2n+1} L_{n+1}(x) \\
= 2(L_n'(x) - L_{n-1}'(x)) + \frac{(n+1)(n+2)}{2n+1} L_{n-1}(x) \\
- (n+1)(n+2) L_n(x) + \frac{(n+1)(n+2)(n+3)}{2n+1} L_{n+1}(x).
\end{align*} \]

The above along with (15) leads to (15). Similarly, we can get the results (14) and (16).

In order to construct efficient Legendre spectral methods for the fifth-order differential equations defined on \( I \in [-1, 1] \), we need to consider the following two kinds of polynomials:

\[
\begin{align*}
  r_n(x) &= (1-x)^3(1+x)^2 L_n(x), \quad n \geq 0, \\
  s_n(x) &= (1-x)^2(1+x)^3 L_n(x), \quad n \geq 0.
\end{align*}
\]

It is clear that \( r_n(x) = (1-x^2)p_n(x), s_n(x) = (1-x^2)q_n(x), r_n(\pm 1) = \partial_x r_n(\pm 1) = \partial_x^2 r_n(1) = 0 \) and \( s_n(\pm 1) = \partial_x s_n(\pm 1) = \partial_x^2 s_n(-1) = 0. \)

**Lemma 2.3.** For any \( n \geq 0 \), the following equalities hold:

\[
\begin{align*}
r_n(x) &= \\
- \frac{(n-4)(n-3)(n-2)(n-1)n}{(2n-7)(2n-5)(2n-3)(2n-1)(2n+1)} L_{n-5}(x) \\
+ \frac{(n-3)(n-2)(n-1)n}{(2n-5)(2n-3)(2n-1)(2n+1)} L_{n-4}(x) \\
+ \frac{(n-2)(n-1)n(3n^2 - 6n - 17)}{4(n+1)n^2 - 2n - 4} L_{n-3}(x) \\
- \frac{2n(n^4 - 9n^2 + 20)}{(2n-5)(2n-3)(2n+1)(2n+3)} L_{n-2}(x) \\
+ \frac{2(3n^4 + 6n^3 - 11n^2 - 14n + 12)}{(2n-3)(2n-1)(2n+3)(2n+5)} L_{n-1}(x) \\
- \frac{2(n+1)(n^4 + 4n^3 - 3n^2 - 14n + 12)}{(2n-3)(2n-1)(2n+1)(2n+5)(2n+7)} L_{n+1}(x) \\
- \frac{4(n+1)(n+2)(n^2 + 3n - 2)}{(2n-1)(2n+1)(2n+3)(2n+7)} L_{n+2}(x) \\
+ \frac{(n+1)(n+2)(n+3)(3n^2 + 12n - 8)}{(2n-3)(2n-1)(2n+1)(2n+3)(2n+5)} L_{n+3}(x) \\
+ \frac{(n+1)(n+2)(n+3)(n+4)}{(2n+1)(2n+3)(2n+5)(2n+7)} L_{n+4}(x) \\
- \frac{(n+1)(n+2)(n+3)(n+4)(n+5)}{(2n+1)(2n+3)(2n+5)(2n+7)(2n+9)} L_{n+5}(x),
\end{align*}
\]
\(s_n(x) =\)
\[
\frac{(n - 4)(n - 3)(n - 2)(n - 1)n}{(2n - 7)(2n - 5)(2n - 3)(2n - 1)(2n + 1)} L_{n-5}(x)
\]
\[
+ \frac{(n - 3)(n - 2)(n - 1)n}{(2n - 5)(2n - 3)(2n - 1)(2n + 1)} L_{n-4}(x)
\]
\[
- \frac{(n - 2)(n - 1)n(3n^2 - 6n - 17)}{(2n - 7)(2n - 3)(2n - 1)(2n + 1)(2n + 3)} L_{n-3}(x)
\]
\[
- \frac{4(n - 1)n(n^2 - n - 4)}{(2n - 5)(2n - 1)(2n + 1)(2n + 3)} L_{n-2}(x)
\]
\[
+ \frac{2n(n^4 - 9n^2 + 20)}{(2n - 5)(2n - 3)(2n + 1)(2n + 3)(2n + 5)} L_{n-1}(x)
\]
\[
+ \frac{2(3n^4 + 6n^3 - 11n^2 - 14n + 12)}{(2n - 3)(2n - 1)(2n + 3)(2n + 5)} L_n(x)
\]
\[
+ \frac{2(n + 1)(n^4 + 4n^3 - 3n^2 - 14n + 12)}{(2n - 3)(2n - 1)(2n + 1)(2n + 5)(2n + 7)} L_{n+1}(x)
\]
\[
- \frac{4(n + 1)(n + 2)(n^2 + 3n - 2)}{(2n - 1)(2n + 1)(2n + 3)(2n + 7)} L_{n+2}(x)
\]
\[
- \frac{(n + 1)(n + 2)(n + 3)(3n^2 + 12n - 8)}{(2n - 3)(2n - 1)(2n + 1)(2n + 3)(2n + 5)} L_{n+3}(x)
\]
\[
+ \frac{(n + 1)(n + 2)(n + 3)(n + 4)}{(2n + 1)(2n + 3)(2n + 5)(2n + 7)} L_{n+4}(x)
\]
\[
+ \frac{(n + 1)(n + 2)(n + 3)(n + 4)(n + 5)}{(2n + 1)(2n + 3)(2n + 5)(2n + 7)(2n + 9)} L_{n+5}(x)
\]

**Proof.** Clearly, by (3), (4) and (9), we obtain
\[
r_n(x) = \frac{(1 - x)^3(1 + x)^2}{2n + 1}(L'_{n+1}(x) - L'_{n-1}(x))
\]
\[
= \frac{(n + 1)(n + 2)}{(2n + 1)(2n + 3)}(1 - x)^2(1 + x)(L_n(x) - L_{n+2}(x))
\]
\[
- \frac{(n - 1)n}{(2n - 1)(2n + 1)}(1 - x)^2(1 + x)(L_{n-2}(x) - L_n(x))
\]
\[
= \frac{(n - 1)n}{(2n - 1)(2n + 1)} p_{n-2}(x) + \frac{2(n^2 + n - 1)}{(2n - 1)(2n + 3)} p_n(x) - \frac{(n + 1)(n + 2)}{(2n + 1)(2n + 3)} p_{n+2}(x).
\]

This along with (11) leads to the first result. In the same manner, we can get the second one.

Using the same argument, we can derive the following three lemmas.

\[\square\]
Lemma 2.4. For any \( n \geq 0 \),
\[
\partial_x r_n(x) = \frac{(n-4)(n-3)(n-2)(n-1)n}{(2n-5)(2n-3)(2n-1)(2n+1)} L_{n-4}(x) - \frac{(n-3)(n-2)(n-1)n}{(2n-3)(2n-1)(2n+1)} L_{n-3}(x)
\]
\[
- \frac{(n-1)n(2n^3 - 5n^2 - 5n + 14)}{(2n-5)(2n-1)(2n+1)(2n+3)} L_{n-2}(x) + \frac{n(3n^3 - 2n^2 - 7n + 6)}{(2n-3)(2n+1)(2n+3)} L_{n-1}(x)
\]
\[
+ \frac{n(n^3 + 2n^2 - 5n - 6)}{(2n-3)(2n-1)(2n+3)(2n+5)} L_n(x) - \frac{(n+1)(3n^3 + 11n^2 + 6n - 8)}{(2n-1)(2n+1)(2n+5)} L_{n+1}(x)
\]
\[
+ \frac{(n+1)(n+2)(2n^3 + 11n^2 + 11n - 12)}{(2n-1)(2n+1)(2n+3)(2n+7)} L_{n+2}(x) + \frac{(n+1)(n+2)(n+3)(n+4)}{(2n+1)(2n+3)(2n+5)} L_{n+3}(x)
\]
\[
- \frac{(n+1)(n+2)(n+3)(n+4)(n+5)}{(2n+1)(2n+3)(2n+5)(2n+7)} L_{n+4}(x).
\]

Lemma 2.5. For any \( n \geq 0 \),
\[
\partial_x^2 r_n(x) = \frac{(n-4)(n-3)(n-2)(n-1)n}{(2n-3)(2n-1)(2n+1)} L_{n-4}(x) + \frac{(n-3)(n-2)(n-1)n}{(2n-3)(2n-1)(2n+1)} L_{n-3}(x)
\]
\[
- \frac{(n-3)(n-2)(n-1)n}{(2n-3)(2n+1)(2n+3)} L_{n-2}(x) - \frac{2(n-1)n(n+1)(n+2)}{(2n-1)(2n+3)} L_n(x)
\]
\[
+ \frac{(n-2)n(n+1)(n+2)(n+3)}{(2n-1)(2n+1)(2n+5)} L_{n+1}(x) + \frac{(n+1)(n+2)(n+3)(n+4)}{(2n+1)(2n+3)} L_{n+2}(x)
\]
\[
- \frac{(n+1)(n+2)(n+3)(n+4)(n+5)}{(2n+1)(2n+3)(2n+5)} L_{n+3}(x),
\]
\[
\partial_x^2 s_n(x) = \frac{(n-4)(n-3)(n-2)(n-1)n}{(2n-3)(2n-1)(2n+1)} L_{n-4}(x) + \frac{(n-3)(n-2)(n-1)n}{(2n-3)(2n-1)(2n+1)} L_{n-3}(x)
\]
\[
- \frac{(n-3)(n-2)(n-1)n}{(2n-3)(2n+1)(2n+3)} L_{n-2}(x) + \frac{2(n-1)n(n+1)(n+2)}{(2n-1)(2n+3)} L_n(x)
\]
\[
- \frac{(n-2)n(n+1)(n+2)(n+3)}{(2n-1)(2n+1)(2n+5)} L_{n+1}(x) + \frac{(n+1)(n+2)(n+3)(n+4)}{(2n+1)(2n+3)} L_{n+2}(x)
\]
\[
+ \frac{(n+1)(n+2)(n+3)(n+4)(n+5)}{(2n+1)(2n+3)(2n+5)} L_{n+3}(x).
\]
Lemma 2.6. For any \( n \geq 0 \),
\[
\frac{\partial^3 u_n(x)}{n + 1} = \frac{(n + 1)(n + 3)(n^3 - 14n^2 + 24n - 8)}{(2n - 1)(2n + 1)} L_{n-2}(x) - \frac{(n + 1)(n + 2)(n^2 - 9n + 4)}{2n + 1} L_{n-1}(x)
- \frac{(n + 1)(n + 2)(n + 3)(9n - 4)}{2n + 1} L_n(x) + \frac{(n + 1)(n + 2)(n + 3)(n + 4)}{2n + 1} L_{n+1}(x)
- \frac{(n + 1)(n + 2)(n + 3)(n^3 + 4)}{2n + 1} L_{n+2}(x) + \sum_{k=0}^{n-3} 8(2k + 1) L_k(x),
\]
\[
\frac{\partial^3 s_n(x)}{n + 1} = \frac{(n + 1)(n + 3)(n^3 - 14n^2 + 24n - 8)}{(2n - 1)(2n + 1)} L_{n-2}(x) - \frac{(n + 1)(n + 2)(n^2 - 9n + 4)}{2n + 1} L_{n-1}(x)
+ \frac{(n + 1)(n + 2)(n + 3)(9n - 4)}{2n + 1} L_n(x) + \frac{(n + 1)(n + 2)(n + 3)(n + 4)}{2n + 1} L_{n+1}(x)
+ \frac{(n + 1)(n + 2)(n + 3)(n^3 + 4)}{2n + 1} L_{n+2}(x) + \sum_{k=0}^{n-3} 8(2k + 1) L_k(x).
\]

3. Efficient Legendre dual-Petrov-Galerkin methods. In this section, we propose efficient Legendre dual-Petrov-Galerkin spectral methods for solving odd-order differential equations. The main idea is to find Sobolev bi-orthogonal basis functions with respect to the coercive bilinear form, such that both the exact solution and the approximate solution can be expressed explicitly.

3.1. Third-order equations. Consider the third-order elliptic equation (cf. [17]):
\[
\begin{align*}
\alpha u - \beta \partial_x u - \gamma \partial^2_x u + \partial^3_x u &= f, \quad x \in I = (-1, 1), \\
u(\pm 1) &= \partial_x u(1) = 0,
\end{align*}
\]
where \( \alpha, \beta, \gamma \) are given constants.

Denote by \( P_N \) the space of polynomials of degree \( \leq N \), and set
\[
W_N = \{ u \in P_N : u(\pm 1) = \partial_x u(1) = 0 \}, \quad W_N^* = \{ u \in P_N : u(\pm 1) = \partial_x u(-1) = 0 \}.
\]

The Legendre dual-Petrov-Galerkin spectral approximation for (19) is to find \( u_N \in W_N \) such that
\[
\mathcal{A}(u_N, v_N) := (\alpha u_N - \beta \partial_x u_N - \gamma \partial^2_x u_N + \partial^3_x u_N, v_N) = (f, v_N), \quad \forall v_N \in W_N^*,
\]
where \( (u, v) = \int_I u v dx \).

To propose an efficient approximation scheme for (20), we need to construct two kinds of basis functions \( \{ \varphi_k \}_{0 \leq k \leq N-3} \in W_N \) and \( \{ \psi_k \}_{0 \leq k \leq N-3} \in W_N^* \), which are bi-orthogonal with respect to the bilinear operator \( \mathcal{A}(\cdot, \cdot) \).

Lemma 3.1. Let \( \varphi_k(x) = \psi_k(x) \equiv 0 \) for any \( k < 0 \), and
\[
\varphi_0(x) = p_0(x), \quad \psi_0(x) = q_0(x).
\]
Assume that \( \varphi_k(x) \in W_{k+3} \) and \( \psi_k(x) \in W^*_{k+3} \), whose leading coefficients are respectively the same as the polynomials \( p_k(x) \) and \( q_k(x) \), satisfying the biorthogonality with respect to \( \mathcal{A}(\cdot, \cdot) \),
\[
\mathcal{A}(\varphi_k, \psi_l) = \rho_{k,l}, \quad \forall k, l \geq 0.
\]
Then for $k \geq 1$, the following recurrence relations hold:

$$
\begin{align*}
\varphi_k(x) &= p_k(x) + a_k \varphi_{k-1}(x) + b_k \varphi_{k-2}(x) + c_k \varphi_{k-3}(x) \\
&\quad + d_k \varphi_{k-4}(x) + e_k \varphi_{k-5}(x) + f_k \varphi_{k-6}(x), \\
\psi_k(x) &= q_k(x) - a_k \psi_{k-1}(x) + b_k \psi_{k-2}(x) - c_k \psi_{k-3}(x) \\
&\quad + d_k \psi_{k-4}(x) - e_k \psi_{k-5}(x) + f_k \psi_{k-6}(x),
\end{align*}
$$

(23)

where the coefficients can be found in Appendix A.

Proof. Let

$$
\begin{align*}
\varphi_k(x) &= p_k(x) + \sum_{l=0}^{k-1} a_{k,l} \varphi_l(x),
\end{align*}
$$

(24)

\[ \psi_k(x) = q_k(x) + \sum_{l=0}^{k-1} b_{k,l} \psi_l(x). \]  

(25)

Then by (6), (25) and Lemmas 2.1 and 2.2, we deduce that for any $0 \leq l \leq k - 7$,

$$
\mathcal{A}(p_k, \psi_l) = (\alpha p_k - \beta \partial_x p_k - \gamma \partial_x^2 p_k + \partial_x^2 \psi_l, \psi_l)
\begin{align*}
&= \alpha(p_k, \psi_l) - \beta(\partial_x p_k, \psi_l) + \gamma(\partial_x p_k, \partial_x \psi_l) + (\partial_x p_k, \partial_x^2 \psi_l)
\end{align*}
$$

(26)

\[ = 0. \]

On the other hand, by (22) and (24), we get that for any $0 \leq l \leq k - 7$,

$$
\mathcal{A}(p_k, \psi_l) = \mathcal{A}(\varphi_k - \sum_{j=0}^{k-1} a_{k,j} \varphi_j, \psi_l) = \mathcal{A}(\varphi_k, \psi_l) - a_{k,l} \rho_l = -a_{k,l} \rho_l.
$$

(27)

Hence, we have $a_{k,l} = 0$ for $0 \leq l \leq k - 7$. Similarly, we get $b_{k,l} = 0$ for $0 \leq l \leq k - 7$.

For simplicity, we denote

$$
\begin{align*}
\varphi_l(x) &= p_l(x) + a_l \varphi_{l-1}(x) + b_l \varphi_{l-2}(x) + c_l \varphi_{l-3}(x) \\
&\quad + d_l \varphi_{l-4}(x) + e_l \varphi_{l-5}(x) + f_l \varphi_{l-6}(x),
\end{align*}
$$

(28)

and

$$
\begin{align*}
\psi_l(x) &= q_l(x) + A_l \psi_{l-1}(x) + B_l \psi_{l-2}(x) + C_l \psi_{l-3}(x) \\
&\quad + D_l \psi_{l-4}(x) + E_l \psi_{l-5}(x) + F_l \psi_{l-6}(x).
\end{align*}
$$

(29)

We next confirm the coefficients. Clearly, by (20) and (21) we know that

$$
\rho_0 = \mathcal{A}(\varphi_0, \psi_0) = (\alpha \varphi_0 - \beta \partial_x \varphi_0 - \gamma \partial_x^2 \varphi_0 + \partial_x^3 \varphi_0, \psi_0)
\begin{align*}
&= \alpha(p_0, q_0) - \beta(\partial_x p_0, q_0) + \gamma(\partial_x p_0, \partial_x q_0) + (\partial_x p_0, \partial_x^2 q_0)
\end{align*}
$$

(30)

\[ = \frac{96\alpha + 112\beta + 112\gamma}{105} + 8. \]

Moreover, by (22),

$$
\mathcal{A}(\varphi_1, \psi_0) = \mathcal{A}(p_1 + a_1 \varphi_0, q_0) = \mathcal{A}(p_1, \psi_0) + a_1 \rho_0 = 0.
$$

(31)

This along with (6) and Lemma 2.1 gives that

$$
\begin{align*}
a_1 &= -\frac{1}{\rho_0} \mathcal{A}(p_1, \psi_0) = -\frac{1}{\rho_0} (\alpha p_1 - \beta \partial_x p_1 - \gamma \partial_x^2 p_1 + \partial_x^3 p_1, q_0)
\end{align*}
$$

\[ = -\frac{1}{\rho_0} \left[ \alpha(p_1, q_0) - \beta(\partial_x p_1, q_0) + \gamma(\partial_x p_1, \partial_x q_0) + (\partial_x p_1, \partial_x^2 q_0) \right]
\]

\[ = -\frac{1}{\rho_0} (48\beta - 112\gamma + 168). \]
Similarly, we have
\[ A(\varphi_0, \psi_1) = A(p_0, q_1 + A_1 \psi_0) = A(p_0, q_1) + A_1 \rho_0 = 0, \]
and hence
\[ A_1 = -\frac{1}{\rho_0} A(p_0, q_1) = -\frac{1}{\rho_0} (-48 \beta + 112 \gamma - 168) = -a_1. \]
In the same manner, we can derive all the coefficients mentioned above for \( k \leq 6 \).
Next, we confirm the coefficients for \( k \geq 7 \). In fact, by (6) and Lemmas 2.1 and 2.2,
\[ A(p_k, q_{k-6}) = \alpha(p_k, q_{k-6}) - \beta(\partial_x p_k, q_{k-6}) + \gamma(\partial_x p_k, \partial_x q_{k-6}) + (\partial_x p_k, \partial_x^2 q_{k-6}) \]
\[ = \alpha(p_k, q_{k-6}) = -\frac{2 \cdot 2^7 \alpha(k-5)_6(k-5)_7}{(2k-11)_{14}}. \] (32)

On the other hand, by (22), (28) and (29), we get
\[ A(p_k, q_{k-6}) = A(\varphi_k - a_k \varphi_{k-1} - b_k \varphi_{k-2} - c_k \varphi_{k-3} - d_k \varphi_{k-4} - e_k \varphi_{k-5} - f_k \varphi_{k-6}, \psi_{k-6} - A_k \psi_{k-7} - B_k \psi_{k-8} - C_k \psi_{k-9} - D_k \psi_{k-10} - E_k \psi_{k-11} - F_k \psi_{k-12}) = -f_k \rho_{k-6}. \]
Therefore
\[ f_k = -\frac{A(p_k, q_{k-6})}{\rho_{k-6}} = \frac{1}{\rho_{k-6}} \frac{2 \cdot 2^7 \alpha(k-5)_6(k-5)_7}{(2k-11)_{14}}. \] (34)

Similarly, we have
\[ A(p_k, q_{k-5}) = A(\varphi_k - a_k \varphi_{k-1} - b_k \varphi_{k-2} - c_k \varphi_{k-3} - d_k \varphi_{k-4} - e_k \varphi_{k-5} - f_k \varphi_{k-6}, \psi_{k-5} - A_k \psi_{k-6} - B_k \psi_{k-7} - C_k \psi_{k-8} - D_k \psi_{k-9} - E_k \psi_{k-10} - F_k \psi_{k-11}) = -e_k \rho_{k-5} + A_k f_k \rho_{k-6}, \] (35)
and
\[ A(p_k, q_{k-5}) = \alpha(p_k, q_{k-5}) - \beta(\partial_x p_k, q_{k-5}) + \gamma(\partial_x p_k, \partial_x q_{k-5}) + (\partial_x p_k, \partial_x^2 q_{k-5}) \]
\[ = -\beta(\partial_x p_k, q_{k-5}) = -\frac{2 \cdot 2^6 \beta(k-4)_5(k-4)_6(k-2)}{(2k-9)_{12}}. \]
A combination of the previous two formulae leads to
\[ e_k = \frac{1}{\rho_{k-5}} \frac{2 \cdot 2^6 \beta(k-4)_5(k-4)_6(k-2)}{(2k-9)_{12}} + \frac{1}{\rho_{k-5}} A_k f_k \rho_{k-6}. \]
In the same manner, we derive the expressions of \( d_k, c_k, b_k \) and \( a_k \). In addition, we can also deduce that \( A_k = -a_k, B_k = b_k, C_k = -c_k, D_k = d_k, E_k = -e_k \) and \( F_k = f_k \).
Next, by (22), (28) and (29), we know
\[ A(p_k, q_k) = \rho_k + a_k A_k \rho_{k-1} + b_k B_k \rho_{k-2} + c_k C_k \rho_{k-3} + d_k D_k \rho_{k-4} + e_k E_k \rho_{k-5} + f_k F_k \rho_{k-6}. \] (36)
Moreover, by (6) and Lemmas 2.1 and 2.2 we have
\[
\mathcal{A}(p_k, q_k) = \alpha(p_k, q_k) - \beta(\partial_x p_k, q_k) + \gamma(\partial_x p_k, \partial_x q_k) + (\partial_x p_k, \partial_x^2 q_k)
\]
\[
= 8 \cdot 2^7 (k-2) \gamma (5k^6 + 15k^5 - 52k^4 - 129k^3 + 155k^2 + 22k - 180)
\]
\[
+ 4 \cdot 2^5 \beta(k-1)^5(3k^4 + 6k^3 - 11k^2 - 14k + 12)
\]
\[
+ 4 \cdot 2^5 \gamma(k-1)^5(3k^6 + 9k^5 - k^4 - 17k^3 - 18k^2 - 8k + 12)
\]
\[
+ 12\beta \cdot 2^3 (k^4 + 2k^3 + 3k^2 + 2k - 2).
\]

The combination leads to the desired result of \(\rho_k\). \(\square\)

By (19), (20) and the biorthogonality of \(\varphi_k\) and \(\psi_k\), we further derive the following main lemma in this subsection.

**Theorem 3.2.** Let \(u(x)\) and \(u_N(x)\) be the solutions of (19) and (20), respectively. Then both \(u(x)\) and \(u_N(x)\) have the explicit representations in \(\{\varphi_k(x)\}\),

\[
u(x) = \sum_{k=0}^{\infty} \hat{\varphi}_k \varphi_k(x), \quad u_N(x) = \sum_{k=0}^{N-3} \hat{\varphi}_k \varphi_k(x),
\]

\[
\hat{\varphi}_k = \frac{1}{\rho_k} \mathcal{A}(u, \varphi_k) = \frac{1}{\rho_k} (f, \varphi_k), \quad k \geq 0.
\]

**Remark 1.** Shen [17] studied the convergence of scheme (20) and obtained that for \(\alpha, \beta \geq 0\) and \(-\frac{1}{3} < \gamma < \frac{1}{6}\),

\[
\alpha\|u - u_N\|_{L^1} + N^{-1}\|\varphi_k(x)\|_{L^{\infty}} \leq c(1 + |\gamma|N)N^{-m}\|\partial_x^m u\|_{L^{m-2, m-1}}, \quad m \geq 1,
\]

where \(\omega^a(x) = (1 - x)^a (1 + x)^b\).

3.2. Fifth-order equations. Consider the fifth-order equation (cf. [17]):

\[
\begin{cases}
\alpha u + \beta \partial_x^5 u - \partial_x^2 u = f, & x \in I = (-1, 1), \\
u(\pm1) = \partial_x u(\pm1) = \partial_x^2 u(1) = 0,
\end{cases}
\]

where \(\alpha\) and \(\beta\) are given constants. Define

\[
V_N = \{u \in P_N : u(\pm1) = \partial_x u(\pm1) = \partial_x^2 u(1) = 0\},
\]

\[
V_N^* = \{u \in P_N : u(\pm1) = \partial_x u(\pm1) = \partial_x^2 u(-1) = 0\}.
\]

We consider the following Legendre dual-Petrov-Galerkin approximation for (39):

Find \(u_N \in V_N\) such that

\[
\mathcal{B}(u_N, v_N) := (\alpha u_N + \beta \partial_x^5 u_N - \partial_x^2 u, v) = (f, v_N), \quad \forall v_N \in V_N^*.
\]

To propose an efficient approximation scheme for (41), we also need to construct two kinds of basis functions \(\{\Phi_k\}_{0 \leq k < N-5} \in V_N\) and \(\{\Psi_k\}_{0 \leq k < N-5} \in V_N^*\), which are bi-orthogonal with respect to the bilinear operator \(\mathcal{B}(\cdot, \cdot)\).

**Lemma 3.3.** Let \(\Phi_k(x) = \Psi_k(x) = 0\) for any \(k < 0\), and

\[
\Phi_0(x) = r_0(x), \quad \Psi_0(x) = s_0(x).
\]
Assume that $\Phi_k(x) \in V_{k+5}$ and $\Psi_k(x) \in V_{k+5}^*$, whose leading coefficients are respectively the same as the polynomials $r_k(x)$ and $s_k(x)$, satisfying the biorthogonality with respect to $\mathcal{B}(\cdot, \cdot)$,

$$\mathcal{B}(\Phi_k, \Psi_l) = \eta_k \delta_{k,l}, \quad \forall k, l \geq 0. \quad (43)$$

Then for $k \geq 1$, we have

$$\Phi_k(x) = r_k(x) + a_k \Phi_{k-1}(x) + b_k \Phi_{k-2}(x) + c_k \Phi_{k-3}(x) + d_k \Phi_{k-4}(x) + e_k \Phi_{k-5}(x) + f_k \Phi_{k-6}(x) + g_k \Phi_{k-7}(x) + h_k \Phi_{k-8}(x) + i_k \Phi_{k-9}(x) + j_k \Phi_{k-10}(x),$$

$$\Psi_k(x) = s_k(x) - a_k \Psi_{k-1}(x) + b_k \Psi_{k-2}(x) - c_k \Psi_{k-3}(x) + d_k \Psi_{k-4}(x) - e_k \Psi_{k-5}(x) + f_k \Psi_{k-6}(x) - g_k \Psi_{k-7}(x) + h_k \Psi_{k-8}(x) - i_k \Psi_{k-9}(x) + j_k \Psi_{k-10}(x),$$

where the coefficients can be found in Appendix B.

Proof. Let

$$\Phi_k(x) = r_k(x) + \sum_{l=0}^{k-1} a_{k,l} \Phi_l(x), \quad (44)$$

$$\Psi_k(x) = s_k(x) + \sum_{l=0}^{k-1} b_{k,l} \Psi_l(x). \quad (45)$$

Then by (6), (45) and Lemmas 2.3-2.6, we get that for $1 \leq l \leq k - 1$,

$$\mathcal{B}(r_k, \Psi_l) = (\alpha r_k + \beta \partial_x^3 r_k - \partial_x^2 r_k, \Psi_l)$$

$$= \alpha(r_k, \Psi_l) + \beta(\partial_x r_k, \partial_x^2 \Psi_l) + (\partial_x^2 r_k, \partial_x^3 \Psi_l)$$

$$= 0, \quad (46)$$

On the other hand, by (44) and (43), we know that for $1 \leq l \leq k - 1$,

$$\mathcal{B}(r_k, \Psi_l) = \mathcal{B}(\Phi_k - \sum_{j=0}^{k-1} a_{k,j} \Phi_j, \Psi_l) = \mathcal{B}(\Phi_k, \Psi_l) - a_{k,l} \eta_l = -a_{k,l} \eta_l. \quad (47)$$

Thus, $a_{k,l} = 0$ for $0 \leq l \leq k - 11$. Similarly, we have $b_{k,l} = 0$ for $0 \leq l \leq k - 11$. Therefore, we denote

$$\Phi_k(x) = r_k(x) + A_k \Phi_{k-1}(x) + B_k \Phi_{k-2}(x) + C_k \Phi_{k-3}(x) + D_k \Phi_{k-4}(x) + E_k \Phi_{k-5}(x) + F_k \Phi_{k-6}(x) + G_k \Phi_{k-7}(x) + H_k \Phi_{k-8}(x) + I_k \Phi_{k-9}(x) + J_k \Phi_{k-10}(x),$$

and

$$\Psi_k(x) = s_k(x) + A_k \Psi_{k-1}(x) + B_k \Psi_{k-2}(x) + C_k \Psi_{k-3}(x) + D_k \Psi_{k-4}(x) + E_k \Psi_{k-5}(x) + F_k \Psi_{k-6}(x) + G_k \Psi_{k-7}(x) + H_k \Psi_{k-8}(x) + I_k \Psi_{k-9}(x) + J_k \Psi_{k-10}(x).$$

We next confirm the coefficients. As in the proof of Lemma 3.1, we easily deduce the coefficients for $k \leq 10$. We next verify the result for $k \geq 11$. In fact, by (6) and Lemmas 2.3-2.6,

$$\mathcal{B}(r_k, s_{k-10}) = \alpha(r_k, s_{k-10}) + \beta(\partial_x r_k, \partial_x^2 s_{k-10}) + (\partial_x^2 r_k, \partial_x^3 s_{k-10})$$

$$= \alpha(r_k, s_{k-10}) = -\frac{2 \cdot 2^{11} \alpha(k - 9)_{10}(k - 9)_{11}}{(2k - 19)_{22}}. \quad (50)$$
On the other hand, by \((43), (48)\) and \((49)\) we have
\[
B(r_k, s_{k-10}) = B(\Phi_k - a_k \Phi_{k-1} - b_k \Phi_{k-2} - c_k \Phi_{k-3} - d_k \Phi_{k-4} - e_k \Phi_{k-5} - f_k \Phi_{k-6} - g_k \Phi_{k-7} - h_k \Phi_{k-8} - i_k \Phi_{k-9} - j_k \Phi_{k-10} - A_{k-10} \Psi_{k-11} - B_{k-10} \Psi_{k-12} - C_{k-10} \Psi_{k-13} - D_{k-10} \Psi_{k-14} - E_{k-10} \Psi_{k-15} - F_{k-10} \Psi_{k-16} - G_{k-10} \Psi_{k-17} - H_{k-10} \Psi_{k-18} - I_{k-10} \Psi_{k-19} - J_{k-10} \Psi_{k-20})
\]
\[
= j_k \eta_{k-10}.
\]
Therefore
\[
j_k = \frac{1}{\eta_{k-10}} \frac{2 \cdot 2^{11} (k - 9)_{10} (k - 9)_{11}}{(2k - 19)_{22}}.
\]
Similarly, we get
\[
B(r_k, s_{k-9}) = 0 \quad \text{and} \quad B(r_k, s_{k-9}) = -i_k \eta_{k-9} + A_{k-9} j_k \eta_{k-10}.
\]
Hence
\[
i_k = \frac{1}{\eta_{k-9}} A_{k-9} j_k \eta_{k-10}.
\]
In the same manner, we can obtain the other coefficients. In addition, we can also deduce that \(A_k = -a_k, B_k = b_k, C_k = -c_k, D_k = d_k, E_k = -e_k, F_k = f_k, G_k = -g_k, H_k = h_k, I_k = -i_k \) and \(J_k = j_k\).

Next, by \((43), (48)\) and \((49)\), we get
\[
B(r_k, s_k) = \eta_k + a_k A_k \eta_k - 1 + b_k B_k \eta_k - 2 + c_k C_k \eta_k - 3 + d_k D_k \eta_k - 4 + e_k E_k \eta_k - 5 + f_k F_k \eta_k - 6 + g_k G_k \eta_k - 7 + h_k H_k \eta_k - 8 + i_k I_k \eta_k - 9 + j_k J_k \eta_k - 10.
\]
Moreover, by \((6)\) and Lemmas 2.3-2.6,
\[
B(r_k, s_k) = \alpha(r_k, s_k) + \beta(\partial_x r_k, \partial_x^2 s_k) + (\partial_x^2 r_k, \partial_x^3 s_k)
\]
\[
= \frac{8 \cdot 2^{11} \eta_k (k - 4)_{11}}{(2k - 9)_{22}} + \frac{24 \cdot 2 \cdot \eta_k (k - 2)_{11}}{(2k - 5)_{14}} + \frac{60 \cdot \eta_k (k - 1)_{15}}{(2k - 3)_{10}}.
\]
The combination leads to the desired result of \(\eta_k\).

Thus, by \((39), (41)\) and the biorthogonality of \(\Phi_k\) and \(\Psi_k\), we obtain the following main lemma in this subsection.

**Theorem 3.4.** Let \(u(x)\) and \(u_N(x)\) be the solutions of \((39)\) and \((41)\), respectively. Then both \(u(x)\) and \(u_N(x)\) have the explicit representations in \(\{\Phi_k(x)\}\),
\[
u(x) = \sum_{k=0}^{\infty} \hat{u}_k \Phi_k(x), \quad u_N(x) = \sum_{k=0}^{N-5} \hat{u}_k \Phi_k(x),
\]
\[
\hat{u}_k = \frac{1}{\eta_k} B(u, \Phi_k) = \frac{1}{\eta_k} (f, \Psi_k), \quad k \geq 0.
\]

**Remark 2.** Shen \([17]\) also studied the convergence of scheme \((41)\) and obtained that for \(\alpha, \beta \geq 0\) and \(m \geq 2,\)

\[
\alpha \|u - u_N\|_{\omega^{m+1}} + \beta \|\partial_x (u - u_N)\|_{\omega^{m+2}} + N^{-2} \|\partial_x^2 (u - u_N)\|_{\omega^{m+1}}
\]
\[
\leq c(1 + \beta N^{-m}) \|\partial_x^m u\|_{\omega^{m-3}},
\]
\[
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\]
3.3. Application to the KdV equation. Consider the third-order KdV equation (cf. [17]):

\[
\begin{align*}
\epsilon \partial_t u + \lambda \partial_x u + \mu u \partial_x u + \partial_x^3 u &= f, & \quad x \in I = (-1, 1), & \quad t \in (0, T], \\
u(\pm 1, t) &= \partial_x u(\pm 1, t) = 0, & \quad t \in (0, T], \\
u(x, 0) &= \nu_0(x), & \quad x \in I,
\end{align*}
\]

where \(\epsilon, \lambda, \mu\) are positive constants.

Denote by \(\tau\) the time step size, \(M = \lfloor \frac{T}{\tau} \rfloor\) and \(u^k(x) = u(x, k\tau), 0 \leq k \leq M\). Then a standard centered difference scheme in time is given by

\[
\begin{align*}
\epsilon \frac{u^{k+1} - u^k}{\tau} + \lambda \partial_x u^{k+1} + \mu \partial_x^2 u^k + \partial_x^3 u^{k+1} + \mu \partial_x u^{k+1} + \partial_x^2 u^k &= \frac{f^{k+1} - f^k}{2}, & \quad x \in I, & \quad 0 \leq k \leq M - 1, \\
u^{(k)}(\pm 1) &= \partial_x u^{(k)}(1) = 0, & \quad 0 \leq k \leq M, \\
u^{(0)}(x) &= \nu_0(x), & \quad x \in I.
\end{align*}
\]

The Legendre spectral scheme for (58) is to find \(u_N^{(k+1)} \in W_N\) such that

\[
\mathcal{D}(u_N^{(k+1)}, \phi) = (g_N^{(k)}, \phi), \quad \phi \in W_N^*,
\]

where \(\mathcal{D}(u, v) := 2\epsilon(u, v) + \lambda \tau (\partial_x u, v) + \tau (\partial_x^2 u, v)\), and

\[
g_N^{(k)} = \tau f^{(k+1)} + \tau f^{(k)} + 2\epsilon u_N^{(k)} - \lambda \tau \partial_x^2 u_N^{(k)} - \mu \tau (\partial_x u_N^{(k+1)} + \partial_x u_N^{(k+1)} + \partial_x^2 u_N^{(k)} - \tau \partial_x^3 u_N^{(k)}).
\]

Next, take \(\alpha = \frac{2\epsilon}{\tau}, \beta = -\lambda\) and \(\gamma = 0\) in (20), and denote by \(\varphi_k\) and \(\psi_k\) the corresponding bi-orthogonal basis functions defined in Lemma 3.1. It is clear that

\[
\mathcal{D}(\varphi_k, \psi_l) = \tau A(\varphi_k, \psi_l) = \tau \rho_{k,l}, \quad \forall k, l \geq 0.
\]

**Theorem 3.5.** Let \(u_N^{(k+1)}(x)\) be the solution of (59). Then we have

\[
u_N^{(k+1)}(x) = \sum_{l=0}^{N-3} \tilde{u}_l^{(k+1)} \varphi_l(x), \quad \tilde{u}_l^{(k+1)} = \frac{1}{\tau \rho_l}(g_N^{(k)}, \psi_l), \quad l \geq 0.
\]

**Remark 3.** This is an implicit scheme. In actual computation, an iterative process should be employed to evaluate the expansion coefficients.

4. Numerical experiments. In this section, we examine the effectiveness and accuracy of the suggested Legendre dual-Petrov-Galerkin spectral method for solving odd-order differential equations.

4.1. Experimental results. We first use (38) to solve the third-order equation (19). We take \(\alpha = \beta = \gamma = 1\) in (19) and consider the following two cases of the smooth solutions:

- \(u(x) = (1 + x)(1 - x)^2 e^{\tau}\). In Figure 1, we plot the \(\log_{10}\) of the discrete \(L^2\)- and \(H^1\)- errors versus \(N\). Clearly, a geometric convergence rate is observed in this case.
- \(u(x) = (1 + x)(1 - x)^2 e^{\tau} \sin(x)\). In Figure 2, we plot the \(\log_{10}\) of the discrete \(L^2\)- and \(H^1\)- errors versus \(N\). The two near straight lines show a geometric convergence rate.
We next use (56) to solve the fifth-order equation (39). We take $\alpha = \beta = 1$ in (39) and consider the following two cases of the smooth solutions:

- $u(x) = (1 + x)^2(1 - x)^3e^x$. In Figure 3, we plot the $\log_{10}$ of the discrete $L^2$- and $H^1$- errors versus $N$. Clearly, a geometric convergence rate is observed in this case.

- $u(x) = (1 + x)^2(1 - x)^3e^x \sin(x)$. In Figure 4, we plot the $\log_{10}$ of the discrete $L^2$- and $H^1$- errors versus $N$. The two near straight lines show a geometric convergence rate.

We now use (61) to solve the KdV equation (57). We take $\epsilon = \lambda = \mu = 1$ in (57) and consider the smooth solution:

$$u(x) = (1 - x)^2(1 + x) \sin(x + t).$$

In Figures 5 and 6, we plot the $\log_{10}$ of the discrete $L^2$- and $H^1$- errors versus $N$ at $T = 1$ with $\tau = 0.1, 0.01, 0.001, 0.0001$. Clearly, a geometric convergence rate is
observed in this case. They also indicate that the smaller the time step size $\tau$, the smaller the numerical errors would be.

4.2. **Comparisons of condition numbers.** To demonstrate the essential superiority of our new Legendre dual-Petrov-Galerkin method to the usual Legendre dual-Petrov-Galerkin method, we examine the issues on the 2-norm condition number for the resulting algebraic systems.

We first consider the third-order problem (19).

For the usual Legendre dual-Petrov-Galerkin method, the basis functions are chosen as $\{p_k(x)\}_{k=0}^{N-3}$ and $\{q_k(x)\}_{k=0}^{N-3}$. The corresponding matrices have off-diagonal entries. In Tables 1 and 2, we list the condition numbers of two kinds of numerical methods for problem (19) with $\alpha = \beta = \gamma = 1$. Notice that the condition numbers of the mass and stiff matrices are different. Particularly, the condition numbers of the total matrices $\{A(p_k, q_l)\}$ for the usual Legendre dual-Petrov-Galerkin method increase asymptotically as $O(N^3)$, while the condition numbers of the total matrices $\{A(p_k, q_l)\}$ for our new Legendre dual-Petrov-Galerkin method increase asymptotically as $O(N^2)$. This demonstrates the essential superiority of our new method.
\{A(\varphi_k, \psi_l)\} for our new Legendre dual-Petrov-Galerkin method with respect to the basis functions \(\{\frac{1}{\sqrt{\rho^k}} \varphi_k(x)\}_{k=0}^{N-3}\) and \(\{\frac{1}{\sqrt{\rho^k}} \psi_k(x)\}_{k=0}^{N-3}\) are equal to 1.

| Matrices | \(N = 20\) | \(N = 40\) | \(N = 60\) | \(N = 80\) | \(N = 100\) |
|----------|-------------|-------------|-------------|-------------|-------------|
| \alpha(p_k, q_l) | 3.1644e+05 | 2.3889e+07 | 3.3847e+08 | 2.3036e+09 | 1.0362e+10 |
| \beta(p_k, \partial_x q_l) | 1.5106e+04 | 3.6487e+05 | 2.5120e+06 | 1.0070e+07 | 2.9818e+07 |
| \gamma(\partial_x p_k, \partial_x q_l) | 2.4015e+03 | 3.9365e+04 | 2.0296e+05 | 6.4966e+05 | 1.6010e+06 |
| \partial_{x^2} p_k, \partial_{x^2} q_l | 2.5926e+02 | 2.3471e+03 | 8.4079e+03 | 2.0652e+04 | 4.1318e+04 |
| \partial_{x^2} \beta(p_k, q_l) | 2.5961e+02 | 2.3422e+03 | 8.3910e+03 | 2.0616e+04 | 4.1257e+04 |

Table 1. Condition numbers of the usual Legendre dual-Petrov-Galerkin method for problem (19).

We next consider the fifth-order problem (39).
For the usual Legendre dual-Petrov-Galerkin method, the basis functions are chosen as \( \{ r_k(x) \}_{k=0}^{N-5} \) and \( \{ s_k(x) \}_{k=0}^{N-5} \). The corresponding matrices have off-diagonal entries. In Tables 3 and 4, we list the condition numbers of two kinds of numerical methods for problem (39) with \( \alpha = \beta = 1 \). Notice that the condition numbers of the mass and stiff matrices are different. Particularly, the condition numbers of the total matrices \( \{ \mathcal{B}(r_k, s_l) \} \) for the usual Legendre dual-Petrov-Galerkin method increase asymptotically as \( O(N^5) \), while the condition numbers of the total matrices \( \{ \mathcal{B}(\Phi_k, \Psi_l) \} \) for our new Legendre dual-Petrov-Galerkin method with respect to the basis functions \( \{ \frac{1}{\sqrt{k}} \Phi_k(x) \}_{k=0}^{N-5} \) and \( \{ \frac{1}{\sqrt{k}} \Psi_k(x) \}_{k=0}^{N-5} \) are equal to 1.

| Matrices | \( N = 20 \) | \( N = 40 \) | \( N = 60 \) | \( N = 80 \) | \( N = 100 \) |
|----------|--------------|--------------|--------------|--------------|--------------|
| \( \alpha(r_k, s_l) \) | 2.2919e+07 | 1.4122e+10 | 7.8699e+11 | 1.4829e+13 | 1.5013e+14 |
| \( \beta(\partial_r r_k, \partial_s s_l) \) | 1.0527e+05 | 1.1264e+07 | 1.8445e+08 | 1.3612e+09 | 6.4521e+09 |
| \( (\partial_r^2 r_k, \partial_s^2 s_l) \) | 3.7507e+03 | 1.3987e+05 | 1.6553e+06 | 5.2074e+06 | 1.6555e+07 |
| \( B(r_k, s_l) \) | 3.7593e+03 | 1.3996e+05 | 1.6556e+06 | 5.2083e+06 | 1.6555e+07 |

Table 3. Condition numbers of the usual Legendre dual-Petrov-Galerkin method for problem (39).

| Matrices | \( N = 20 \) | \( N = 40 \) | \( N = 60 \) | \( N = 80 \) | \( N = 100 \) |
|----------|--------------|--------------|--------------|--------------|--------------|
| \( \alpha(\Phi_k, \Psi_l) \) | 6.1180e+06 | 2.2260e+09 | 8.8437e+10 | 1.2957e+12 | 1.0730e+13 |
| \( \beta(\partial_r \Phi_k, \partial_s \Psi_l) \) | 5.1022e+02 | 5.8507e+03 | 2.0067e+04 | 7.7483e+04 | 1.8226e+05 |
| \( (\partial_r^2 \Phi_k, \partial_s^2 \Psi_l) \) | 1.0683e+00 | 1.0684e+00 | 1.0684e+00 | 1.0684e+00 | 1.0684e+00 |
| \( B(\Phi_k, \Psi_l) \) | 1.0000e+00 | 1.0000e+00 | 1.0000e+00 | 1.0000e+00 | 1.0000e+00 |

Table 4. Condition numbers of our new Legendre dual-Petrov-Galerkin method for problem (39).

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Appendix A. The coefficients in (23). Let \( \rho_k = 0 \) for \( k < 0 \), \( a_k = 0 \) for \( k < 1 \), \( b_k = 0 \) for \( k < 2 \), \( c_k = 0 \) for \( k < 3 \), \( d_k = 0 \) for \( k < 4 \), \( e_k = 0 \) for \( k < 5 \), \( f_k = 0 \) for \( k < 6 \), then the coefficients in (23) can be expressed as

\[
\rho_k = \frac{8 \cdot 2^7 (k - 2) \tau (5k^6 + 15k^5 - 52k^4 - 129k^3 + 155k^2 + 22k - 180)}{(2k - 5)_{14}} + \frac{4 \cdot 2^5 \beta(k - 1)_{13} (3k^4 + 6k^3 - 11k^2 - 14k + 12)}{(2k - 3)_{10}}
\]
The coefficients in (3.3) can be expressed as

\[ a_k = \frac{1}{\rho_{k-1}} \left( \frac{4 \cdot 2^5 \gamma (k-1) \beta (k-2) \eta (5k^4 + 43k^3 - 17k^2 - 8k + 12)}{(2k - 3)_{10}} + \frac{4 \cdot 2^4 \gamma (k-1) \beta (k-2) \eta (5k^2 - 7)}{(2k - 3)_{8}} \right) \]

\[ b_k = \frac{1}{\rho_{k-2}} \left( \frac{8 \cdot 2^5 \beta (k-2) \eta (k-1) \eta (5k^4 - 10k^3 - 50k^2 + 64k + 180)}{(2k - 7)_{14}} + \frac{8 \cdot 2^5 \beta (k-2) \eta (k-1) \eta (k^4 - 2k^3 - 2k^2 + 3k - 3)}{(2k - 5)_{10}} \right) \]

\[ c_k = \frac{1}{\rho_{k-3}} \left( \frac{2 \cdot 2^6 \beta (k-2) \eta (k-1) \eta (5k^3 - 15k^2 - 17k + 27)}{(2k - 7)_{12}} + \frac{4 \cdot 2^4 \gamma (k-2) \beta (k-1) \eta (k^4 - 2k^3 - 2k^2 + 3k - 3)}{(2k - 5)_{8}} \right) \]

\[ d_k = \frac{1}{\rho_{k-4}} \left( \frac{12 \cdot 2^7 \alpha (k-4) \eta (k-3) \eta (k^2 - 3k + 1)}{(2k - 9)_{14}} + \frac{2 \cdot 2^5 \gamma (k-3) \beta (k-3) \eta (k^2 - 3k + 7)}{(2k - 7)_{10}} \right) \]

\[ e_k = \frac{1}{\rho_{k-5}} \left( \frac{2 \cdot 2^6 \beta (k-4) \eta (k-4) \eta (k^2 - 3k + 7)}{(2k - 9)_{12}} + \frac{1}{\rho_{k-5}} \eta (k^2 - 3k + 7) \right) \]

\[ f_k = \frac{1}{\rho_{k-6}} \left( \frac{2 \cdot 2^7 \alpha (k-5) \eta (k-5) \eta (k^2 - 3k + 7)}{(2k - 11)_{14}} \right), \quad k \geq 6. \]

Appendix B. The coefficients in (3.3). Let \( \eta_k = 0 \) for \( k < 0 \), \( a_k = 0 \) for \( k < 1 \), \( b_k = 0 \) for \( k < 2 \), \( c_k = 0 \) for \( k < 3 \), \( d_k = 0 \) for \( k < 4 \), \( e_k = 0 \) for \( k < 5 \), \( f_k = 0 \) for \( k < 6 \), \( g_k = 0 \) for \( k < 7 \), \( h_k = 0 \) for \( k < 8 \), \( i_k = 0 \) for \( k < 9 \), \( j_k = 0 \) for \( k < 10 \), then the coefficients in (3.3) can be expressed as

\[ \eta_k = \frac{8 \cdot 2^3 \alpha \beta \eta (k-4) \eta (k-3) \eta (k^2 - 3k + 7)}{(2k - 9)_{22}} + \frac{24 \cdot 2^4 \beta \eta (k-2) \eta (k^2 - 3k + 7)}{(2k - 5)_{14}} + \frac{60 \cdot 2^4 \eta (k^2 - 3k + 7)}{(2k - 3)_{10}} \]

\[ \left[ \frac{a_k}{\eta_k} \right] = \frac{1}{(2k - 5)_{14}} \left( \frac{2 \cdot 2^7 \beta \eta (k-2) \eta (k^2 - 3k + 7)}{(2k - 9)_{12}} + \frac{20 \cdot 2^6 \eta (k-2) \eta (k^2 - 3k + 7)}{(2k - 7)_{14}} + \frac{2 \cdot 2^6 \eta (k-2) \eta (k^2 - 3k + 7)}{(2k - 5)_{12}} \right) - \frac{a_{k-1} b_k \eta (k-2) \eta (k-1) \eta (k^2 - 3k + 7)}{(2k - 9)_{12}} \]

\[ \left[ \frac{b_k}{\eta_k} \right] = \frac{1}{(2k - 11)_{22}} \left( \frac{2 \cdot 2^7 \beta \eta (k-2) \eta (k-1) \eta (k^2 - 3k + 7)}{(2k - 11)_{22}} + \frac{6 \cdot 2^7 \beta \eta (k-1) \eta (k^2 - 3k + 7)}{(2k - 7)_{14}} \right) + \frac{4 \cdot 2^7 \beta \eta (k-1) \eta (k^2 - 3k + 7)}{(2k - 5)_{10}} \]
\[\begin{align*}
& d_k = \frac{1}{\eta_k - 3} \left( - \frac{2 \cdot 2^8 \beta_k^3 (k - 9)(k - 11)}{(2k - 9)_{16}} - \frac{10 \cdot 2^6 \beta_k^2 (k - 3)(k - 5)}{(2k - 7)_{12}} \\
& - a_k - 3d_k \eta_k - 4b_k - 3c_k \eta_k - 5c_k - 3f_k \eta_k - 6d_k - 3g_k \eta_k - 7e_k - 3h_k \eta_k - 8f_k - 3i_k \eta_k - 9 - g_k - 3j_k \eta_k - 10 \right), \\
& k \geq 3,
\end{align*}\]
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