Theory of Orientational Pinning in Quantum Hall Nematics

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(Dated: May 11, 2014)

The orientation of the electron-nematic states discovered in the quantum Hall regime of GaAs [001] growth-direction quantum wells is pinned by a weak native source of anisotropy. In this Letter we explain that this property, which has remained mysterious over more than a decade of research, follows from the presence of both Rashba and Dresselhaus spin-orbit interactions. The hard transport direction of the nematic state is determined by the relative sign of the Rashba and Dresselhaus coefficients, and coincides with either the [110] or the [1¯10] crystallographic direction.

Our theoretical estimate of the pinning energy is in agreement with experimental studies of the competition between native pinning and intentional pinning by an in-plane magnetic field.

PACS numbers: 73.43.-f, 71.70.Ej, 71.27.+a, 71.10.-w, 71.20.+a.

Introduction—In the fractional quantum Hall regime two dimensional electron gases (2DEGs) host an unprecedented variety of strongly-correlated-electron states. It is now established that, starting with the \( n \) = 2 Landau level, Hall resistivities no longer exhibit the sequence of plateaus that are present in the vicinity of half-integer filling for \( n = 0 \), and that longitudinal resistivities become strongly anisotropic \([1, 2]\) at low temperature instead of vanishing. A considerable body of experimental \([1-3]\) and theoretical \([4, 5]\) evidence supports the view that these transport anisotropies signal the formation of electron nematics. Locally electron nematics are unidirectional charge-density-wave (stripe) states, but because of disorder and thermal fluctuations they may lack long-range positional order. Transport in nematics is easy along the stripe edges, which host chiral one-dimensional electron gases, but hard in the perpendicular direction. Experiment has shown that the easy transport direction is normally along the [110] axis of GaAs quantum wells grown in the [001] direction, although a reorientation to the [1¯10] axis has been observed when the density is increased above a critical value \([3]\). In wide quantum wells the stripe orientation has been observed to be different for majority and minority spin Landau levels \([6]\). The mechanism which selects these pinning directions is still undetermined after more than a decade of research.

In this Letter we propose a theory in which orientational pinning follows from the combined presence of Rashba and Dresselhaus spin-orbit interactions \([7]\). Our theory possesses the following three desirable features: (i) the natural pinning axes are predicted to be either [110] or [1¯10] without parameter fine tuning; (ii) the pinning energy scale estimated from typical values of the Rashba and Dresselhaus interactions in GaAs quantum wells is on the order of \( \sim 10^{-7} e V \), in agreement with experiment \([2, 3]\); and (iii) it is falsifiable with a relatively simple experimental test. Previous proposed explanations for orientational pinning, including ones based on band mass anisotropy \([8]\), piezoelectricity \([9]\), strain \([10]\), and external potential modulations on lengths scales longer than the stripe period \([11]\), do not share these features. We predict that the relative sign of the Rashba and Dresselhaus constants determines which of the two natural axes, [110] or [1¯10], is chosen by the stripes. Stripes should therefore flip between [110] and [1¯10] orientations when the sign of the perpendicular electric field which produces the Rashba effect is reversed.

Landau-Levels and Spin-Orbit Interactions— The starting point for our theory is an analysis of how spin-orbit interactions influence Landau level wavefunctions. We specialize to the narrow quantum well limit in which the magnetic length \( l = (\hbar e/B)^{1/2} \), greatly exceeds the well width \( w \), which affords several simplifications. In the narrow well limit we can restrict electronic states to the lowest subband, and neglect both finite-well-width corrections to the Coulomb interaction and the cubic Dresselhaus term. The single-particle Hamiltonian is then \( H = H_0 + H_{SO} \) where \( H_0 \) contains the cyclotron and Zeeman energies,

\[
H_0 = \frac{\pi^2}{2m^*} - \mu \cdot B = \hbar \omega_c (\hat{n} + 1/2) + \frac{1}{2} g^* \mu_B B \sigma_z, \tag{1}
\]

FIG. 1: Iso-probability contours of spin-orbit modified cyclotron wave-functions in the \( n = 2 \) Landau level: (a) \( \epsilon_R = \epsilon_D = 0.3 \); (b) \( \epsilon_R = -\epsilon_D = 0.3 \). The shaded area is the highest probability density region. c) Stripe state schematic showing areas occupied by orbit centers.
and $H_{SO}$ is the sum of the Rashba and Dresselhaus spin-orbit interaction terms \[^{[12]}\].

$$H_{SO} = \alpha (\sigma_x p_y - \sigma_y p_x) + \beta (\sigma_y p_y - \sigma_x p_x).$$ \[^{(2)}\]

In Eq. \[^{(1)}\] $\pi = p + eA/c$ is the mechanical momentum, $m^*$ and $g^*$ are the effective mass and $g$-factor of conduction electrons in GaAs, and $\omega_c = eB/m^* c$, where $B = \nabla \times A$ is the magnetic field. In Eq. \[^{(2)}\] $\alpha$ and $\beta$ are constants with units of velocity which specify the strengths of the Rashba and Dresselhaus interactions. It is convenient to define the dimensionless constants $\epsilon_R \equiv \sqrt{2} \alpha / \omega_c$ and $\epsilon_D \equiv \sqrt{2} \beta / \omega_c$, and to reexpress Eq. \[^{(2)}\] in terms of Landau level and spin raising and lowering operators, $a = l (\pi_x + i \pi_y) / \sqrt{2} h$, $s_+ = (\pi_x + i \sigma_y) / 2$ to obtain $H_{SO} = \hbar \omega_c (i \epsilon_R a^\dagger s_+ - \epsilon_D a s_- + h.c.)$.

Only the combined presence of Rashba and Dresselhaus spin-orbit interactions leads to anisotropy. This property follows from the observation that the Rashba Hamiltonian commutes with sum of the orbital and spin angular momenta along the z-axis $L_z + \sigma_z/2$, whereas the Dresselhaus Hamiltonian commutes with their difference $L_z - \sigma_z/2$. (Note that $L_z = b^\dagger b - a^\dagger a$, where $b$ is the guiding center lowering operator $b = (\pi_x + i \sigma_y) / \sqrt{2} i$, and $c = r - l^2 \hat{z} \times \pi / \hbar$ is the classical cyclotron-orbit guiding center coordinate \[^{[13]}\] \[^{[14]}\].) When only one of the spin-orbit coupling terms is present, a spatial rotation compensated by the appropriate spin-rotation leaves the Hamiltonian invariant. This conclusion survives interactions because Coulomb coupling is spin-independent and isotropic.

For quantitative calculations we must evaluate the Landau level wave functions explicitly. Provided that the dimensionless spin-orbit coupling constants $\epsilon_{R,D}$ are small we can safely use standard Raleigh-Schrödinger perturbation theory. When the spin-orbit Hamiltonian acts on an unperturbed wavefunction, it always reverses spin and changes the Landau level index by $\pm 1$. Second order processes therefore always preserve spin. The coherent addition of wave functions that describe cyclotron orbits with different radii yields anisotropic wavefunctions, as illustrated in Figs. \[^{[1]}(a)-(b)\]. Writing the eigenstates in the absence of spin-orbit interactions as $|n, \uparrow\rangle_0, |n, \downarrow\rangle_0$, the first and second order corrections to the wavefunctions (in the limit of zero Zeeman energy) are,

$$|n, \uparrow\rangle_2 = \frac{\epsilon_D}{\epsilon_R} \sqrt{n} |n, \downarrow\rangle_1 + \epsilon_D \sqrt{n + 1} |n + 1, \downarrow\rangle_0,$$

$$|n, \downarrow\rangle_2 = \frac{\epsilon_D}{\epsilon_R} \sqrt{n} |n, \uparrow\rangle_1 - \epsilon_D \sqrt{n + 1} |n + 1, \uparrow\rangle_0,$$

and

$$|n, \uparrow\rangle_2 = \frac{\epsilon_R \epsilon_D \sqrt{n(n - 1)}}{2} |n - 2, \uparrow\rangle_0 - \frac{\epsilon_R \epsilon_D \sqrt{(n + 2)(n + 1)}}{2} |n + 2, \uparrow\rangle_0 - \frac{\epsilon_D^2 n + \epsilon_D^2 (n + 1)}{2} |n, \uparrow\rangle_0.$$ \[^{(5)}\]

The $|n, \downarrow\rangle_2$ is obtained from the expression above by replacing $\uparrow$ by $\downarrow$ and interchanging $\epsilon_R \leftrightarrow \epsilon_D$. The finite Zeeman energy produces only weak corrections to the coefficients of Eqs. \[^{(3)}\], \[^{(4)}\], and \[^{(5)}\], proportional to its ratio to the cyclotron energy, which is small in GaAs except at extreme field tilt angles.

**Form Factors and Anisotropy**—We will assume that stripe states are always maximally spin-polarized. With this simplification the Hamiltonian is given up to a constant by the Coulomb interaction projected onto a single spin sublevel $|n, \sigma\rangle$ that has been perturbed by spin-orbit interactions:

$$\tilde{V} = \frac{1}{A} \sum_{i < j} \sum_{q \neq 0} v_q |F_{n\sigma}(q)|^2 e^{iq(c_i - c_j)},$$ \[^{(6)}\]

where $v_q = 2 \pi e^2 / eq$ is the 2D Coulomb interaction and

$$F_n(q) = F_n^0(q) + n^2 \epsilon_D^2 \left[ F_{n-1}^0(q) - F_n^0(q) \right] + (n + 1) \epsilon_D^2 \left[ F_n^0(q) - F_{n+1}^0(q) \right]$$

$$+ \frac{1}{2} \epsilon_R \epsilon_D q^2 l^2 \sin(2\theta_q) e^{-|q|^2 l^2 / 4} \left[ 2 L_{n-1}^2 \left( |q|^2 l^2 / 2 \right) - L_n^2 \left( |q|^2 l^2 / 2 \right) - L_{n-2}^2 \left( |q|^2 l^2 / 2 \right) \right].$$ \[^{(8)}\]
where $\theta_q$ is a momentum-space orientation angle. The final term in Eq. (8) is responsible for anisotropy. Form factor contributions that are dependent on $\theta_q$ appear at any order in perturbation theory only if both Rashba and Dresselhaus spin-orbit interactions are present. The form factor for $\downarrow$ states can be obtained from Eq. (8), by interchanging the Rashba and Dresselhaus coefficients $\epsilon_R \leftrightarrow \epsilon_D$.

**Orientalional Pinning**— We are now in a position to estimate the pinning energy of the stripes. To construct the stripe state, we consider the rotated guiding center operator,

$$c^a_\theta = -\sin \theta \, c_x + \cos \theta \, c_y,$$

(9)

The eigenstates of $c^a_\theta$ are localized as a function of $y^\theta = -\sin \theta x + \cos \theta y$, but extended along the line $x^\theta = x \cos \theta + y \sin \theta$. We construct a single-Slater-determinant electron–magnon trial wave function by occupying eigenstates of $c^a_\theta$ with eigenvalue $k$ inside the region $K$ defined by the periodically repeated strips depicted in Fig. 1(c). The many-body wavefunction of a $(n, \sigma)$-Landau-level stripe state that has its hard transport direction along $\hat{y}^\theta$, can therefore be written as,

$$|\Psi_{n,\sigma}(\theta, a)\rangle = \prod_{k \in K} C_{n,\sigma,k} |\Psi^0_{n,\sigma}\rangle,$$

(10)

where $|\Psi^0_{n,\sigma}\rangle$ is a vacuum in which the lower Landau levels are completely filled and $C_{n,\sigma,k}$ creates electrons $(n, \sigma)$ with $c^a_\theta$ eigenvalue $k$. For a given valence Landau level filling factor $\nu$, the stripe state, $|\Psi_{n,\sigma}(\theta, a)\rangle$ has two variational parameters: $\theta$ which characterizes the direction measured from the [100] axis along which the stripes run, and the stripe period $a$. These two free parameters must be optimized to minimize the stripe energy.

Except for the Landau levels being perturbed by spin orbit interactions, these variational wave functions are identical to those conventionally employed to perform Hartree-Fock studies of stripe states, in particular those which address the influence of an in-plane magnetic field [5]. For given values of $\nu$, $a$, and $\theta$ the variational energies can be expressed as:

$$E = \frac{N_\phi}{4\pi^2 L} \sum_{n=-\infty}^{\infty} v_{HF}(2n\pi/a) \left[ \frac{\sin(n\pi \nu)}{n\pi} \right]^2,$$

(11)

where $N_\phi = A/2\pi L^2$ is the orbital Landau level degeneracy and $v_{HF}(p)$ is the sum of a Hartree ($v_H(p)$) and a Fock ($v_F(p)$) contribution:

$$v_H(p) = v_q |F_{n,\sigma}(q)|^2 \delta_{q^\theta=0,q^a=p},$$

$$v_F(p) = -\frac{A}{N_\phi} \int \frac{d^2q}{(2\pi)^2} v_q |F_{n,\sigma}(q)|^2 e^{i p q^\theta}.$$

(12)

Here $(q^a, q^\theta)$ are measured along the stripe axes $(x^\theta, y^\theta)$ (see Fig. 1(c)), and the Hartree potential is understood to vanish for $p = 0$ (i.e. $v_H(p = 0) = 0$), to account for the neutralizing background charge.

The leading order anisotropic contribution to the energy in Eq. (11), which is entirely due to spin-orbit coupling and comes from the anisotropic/isotropic cross term in $|F_{n,\sigma}(q)|^2$, has the form

$$E^{ani}(\theta, a, \nu) = N_\phi \sin 2\theta \, \epsilon(\nu, \epsilon) \, \epsilon_{R,D} e^{2 \epsilon_i},$$

(13)

where $\epsilon(\nu, \epsilon)$ is a dimensionless number. Eq. (13) is the key result of this study and predicts that the minimum of energy is reached when the stripes are aligned along the [110] axis when $\epsilon(\nu, \epsilon) \epsilon_{R,D} < 0$ and along the [101] axis when $\epsilon(\nu, \epsilon) \epsilon_{R,D} > 0$. Numerical values for $\epsilon$ are listed in Table 1.

We can estimate spin-orbit parameters for the samples of Ref [2] from their carrier densities $n_0$ and well widths $w$ using a simple capacitor model: $\hbar a = r_{41}^6 \epsilon_{D} v^2_c \approx 4.4 \times 10^{-4}\epsilon_{D} v^2_c$, and $\hbar^2 = b_{41}^6 v^2_c \approx 3.6 \times 10^{-5}\epsilon_{D} v^2_c$. Here $r_{41}^6 \approx 5.2 e \AA^2$ and $b_{41}^6 \approx 27.6 e \AA^3$ are material parameters for GaAs [12], and $a^* \approx 103 \AA$ is the effective Bohr radius of GaAs. Using these estimates the stripe pinning energy scale is found to be $\epsilon_{R,D} e^2/\epsilon_i \sim 5 \times 10^{-7} eV \sim 6.1 mA$ at $B \sim 1 T$. This value agrees with that determined from experiments in which the preferred stripe orientation is changed by tilting the applied magnetic field away from the normal to the 2DEG plane [2] [5]. Agreement with experiment for both qualitative and quantitative characteristics of the native pinning effect strongly suggests that we have identified the mechanism that is responsible.

**Discussion**— Our theory does not account for Landau-level mixing even though stripe states are normally studied experimentally at relatively weak magnetic fields. We can partially assess its importance by comparing estimates made with bare Coulomb interactions with those in which inter-Landau level contributions to polarization functions are used to construct a statically screened RPA Coulomb interaction. In this approximation, we replace the Coulomb interaction $v_q$ by $v_q^{RPA} = v_q/(1 - v_q^2)$ where

$$\chi_q = \frac{1}{2\pi L^2} \sum_{\sigma,\sigma',n',\sigma \neq n} \frac{f_{\sigma \sigma'} - f_{n',\sigma'}}{E_{\sigma \sigma'} - E_{n',\sigma'}} |F_{n,\sigma,(q)}|^2,$$

(14)

Here $f_{\sigma \sigma'}$ is a Fermi occupation factor which equals $1$ for the fully occupied Landau sublevels and $0$ for the fully empty ones, and it equals the fractional part of the total filling factor for partially occupied Landau sublevel in which the stripe is constructed. The form factors $F_{n,\sigma,(q)}$ in Eq. (14) are the off-diagonal generalizations of the density form factors defined in Eq. (7). This susceptibility applies for a translationally invariant state and therefore neglects modifications to screening arising from the stripe state itself. In practice one does not need to include the spin-orbit modifications to the energies or to
TABLE I: Stripe period $a_0$ and anisotropy energy parameter $\varepsilon$ (Eq. [13]) for half-filled higher Landau levels. The calculations were carried out neglecting and including RPA screening evaluated in GaAs at $B \approx 1T$.

| $\varepsilon_{\text{total}}$ | 9/2 | 11/2 | 13/2 | 15/2 | 17/2 | 19/2 |
|-----------------------------|-----|------|------|------|------|------|
| $a_0/l$                     | 6.19| 6.19 | 7.20 | 7.20 | 8.08 | 8.08 |
| $\varepsilon$               | $-0.093$ | $-0.093$ | $-0.102$ | $-0.102$ | $-0.113$ | $-0.113$ |
| $a_0^{\text{RPA}}/l$        | 6.51| 6.57 | 7.51 | 7.54 | 8.37 | 8.41 |
| $\varepsilon^{\text{RPA}}$ | $-0.072$ | $-0.075$ | $-0.089$ | $-0.091$ | $-0.102$ | $-0.105$ |

the density form factors appearing in the density-density response function of Eq. (14) because they make a small relative contribution. We see in Table I that at typical fields, screening reduces the estimated anisotropy energy by $\sim 10 - 20\%$. As long as other Landau level mixing effects, which cannot be accounted for simply by changing the effective interaction [15], have similar importance, our main conclusions should be reliable.

In Table I we compare stripe periods and anisotropy energies for several half-filled Landau levels. We have found that $\varepsilon(a_0, \nu)$ is dominated by its exchange energy contribution and that it is negative with a typical value $\varepsilon \sim -0.1$ near the optimal stripe period $a_0$ at half-filled Landau levels. We have estimated the optimal stripe period $a_0$ in the absence of of spin-orbit coupling. The small modifications to stripe period arising from the spin-orbit coupling terms can be safely neglected because of the small values of $\varepsilon_{R,D}$. The main role of spin-orbit interactions is simply to choose the preferred stripe orientation.

Since the pinning parameter, $\varepsilon(a_0, \nu)$, is negative, we predict that stripes pin along the [110] axis when $\varepsilon_{R,D} > 0$, and along the [110] axis when $\varepsilon_{R,D} < 0$. Because $\varepsilon_R$ is an odd function of the effective electric field associated with structural inversion asymmetry whereas $\varepsilon_D$ is even, we also predict that $\text{sign}(\varepsilon_{R,D}) = \text{sign}(\varepsilon_{\text{eff}}^z)$. Some caution must be exercised in applying this last conclusion since the effective electric fields are non-trivial [12]. Experiments have revealed that the Rashba constant can be finite even in a nominally symmetric quantum wells [16]. Therefore, a test of the conclusion that the stripes rotate upon a change of sign of this effective electric field should ideally be accompanied by an independent determination of the sign of the Rashba and Dresselhaus spin-orbit coupling constants.

Our theory has been specialized to the case in which the quantum well width is much smaller than the magnetic length $w \ll l \approx 26nm/\sqrt{B[T]}$. The widths of the wells employed in early observations of the stripes [12], can be estimated to be on the order of $10nm$ using the simple capacitor model. Thus it is reasonable to expect that our theory is not accurate at a quantitative level for these experiments. One of the terms that is neglected in the narrow well limit, and that could have a significant impact on the stripe pinning is the cubic Dresselhaus term, which alone breaks rotational symmetry. Additionally we have not explored the interplay of spin-orbit coupling and in-plane magnetic fields which may be important in wider quantum wells.

Our study underscores the importance of the rotational symmetry breaking induced by spin-orbit coupling. At moderate magnetic fields ($B \approx 1T$), the Zeeman energy scale is typically of the same order of magnitude of the spin-orbit coupling terms $\sim 10^{-5}eV$ in GaAs. The interplay of these terms remains a relatively unexplored subject, and should be of special importance for situations where broken rotational invariance plays a role.

We are grateful to A. Croxall, J. Eisenstein, M. Fogler, B. Frieß, Y. Liu and M. Shayegan for estimulating discussions. This work was supported by the DOE Division of Materials Sciences and Engineering under grant DE-FG03-02ER45958 and by the Welch foundation under grant TBF1473.

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