SINGULARITY VERSUS EXACT OVERLAPS FOR SELF-SIMILAR MEASURES

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Abstract. In this note we present some one-parameter families of homogeneous self-similar measures on the line such that

- the similarity dimension is greater than 1 for all parameters and
- the singularity of some of the self-similar measures from this family is not caused by exact overlaps between the cylinders.

We can obtain such a family as the angle-α projections of the natural measure of the Sierpiński carpet. We present more general one-parameter families of self-similar measures $\nu_\alpha$, such that the set of parameters $\alpha$ for which $\nu_\alpha$ is singular is a dense $G_\delta$ set but this "exceptional" set of parameters of singularity has zero Hausdorff dimension.

1. Introduction

1.1. Description of the problem investigated. In recent years there has been a rapid development in the field of self-similar Iterated Function Systems (IFS) with overlapping construction. Most importantly, Hochman [3] proved for any self-similar measure $\nu$ that we can have dimension drop (that is $\dim_H \nu < \min \{1, \dim_S \nu\}$) only if there is a super-exponential concentration of cylinders (see Section 1.4.1 for the definitions of the various dimensions used in the paper). Consequently, for a one-parameter family of self-similar measures $\{\nu_\alpha\}$ on $\mathbb{R}$, satisfying a certain non-degeneracy condition (Definition 5) the Hausdorff dimension of the measure $\nu_\alpha$ is equal to the minimum of its similarity dimension and 1 for all parameters $\alpha$ except for a small exceptional set of parameters $E$. This exceptional set $E$ is so small that its packing dimension (and consequently its Hausdorff dimension) is zero. The corresponding problem for the singularity versus absolute continuity of self-similar measures was treated by Shmerkin and Solomyak [13]. They considered one-parameter families of self-similar measures constructed by one-parameter families of homogeneous self-similar IFS, also satisfying the non-degeneracy condition of Hochman Theorem. It was proved in [13, Theorem A] that for such families $\{\nu_\alpha\}$ of self-similar measures if the similarity dimension of the measures in the family is greater than 1 then for all but a set of Hausdorff dimension zero of parameters $\alpha$, the measure $\nu_\alpha$ is absolute continuous with respect to the Lebesgue measure. The results presented in this note imply that in this case it can happen that the set of exceptional parameters have packing dimension 1 as opposed to Hochman’s Theorem where we remind that the packing dimension of the set of exceptional parameters is equal to 0.

Still, we do not know what causes the drop of dimension or the singularity of a self-similar measure on the line of similarity dimension greater than 1. In particular it is a natural question whether the only reason for the drop of the dimension or singularity of self-similar measures having similarity dimension larger than 1 is the "exact overlap". More precisely, let $\{\varphi_i\}_{i=1}^m$ be a self-similar IFS and $\nu$ be a corresponding self-similar

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measure. We say that there is an exact overlap if we can find two distinct \( \mathbf{i} = (i_1, \ldots, i_k) \) and \( \mathbf{j} = (j_1, \ldots, j_\ell) \) finite words such that

\[
\varphi_{i_1} \circ \cdots \circ \varphi_{i_k} = \varphi_{j_1} \circ \cdots \circ \varphi_{j_\ell}.
\]

The following two questions have naturally arisen for a long time (e.g. Question 1 below appeared as [8, Question 2.6]):

**Question 1:** Is it true that a self-similar measure has Hausdorff dimension strictly smaller than the minimum of 1 and its similarity dimension only if we have exact overlap?

**Question 2:** Is it true for a self-similar measure \( \nu \) having similarity dimension greater than one, that \( \nu \) is singular only if there is exact overlap?

Most of the experts believe that the answer to Question 1 is positive and it has been confirmed in some special cases [3]. On the other hand, a result of Nazarov, Peres and Shmerkin indicated that the answer to Question 2 should be negative. Namely, they constructed in [6] a planar self-affine set having dimension greater than one, such that the angle-\( \alpha \) projection of its natural measure was singular for a dense \( G_\delta \) set of parameters \( \alpha \).

However, this was not a family of self-similar measures. Up to our best knowledge before this note, Question 2 has not yet been answered.

### 1.2. New results.

We consider one-parameter families of homogeneous self-similar measures on the line, having similarity dimension greater than 1. We call the set of those parameters for which the measure is singular, set of parameters of singularity.

(a): We point out that the answer to Question 2 above is negative. (Theorem 14).

(b): We consider one-parameter families of self-similar measures for which the set of parameters of singularity is big in the sense that it is a dense \( G_\delta \) set but in the same time the parameter set of singularity is small in the sense that it is a set of Hausdorff dimension zero. We call such families antagonistic. We point out that there are many antagonistic families. Actually, we show that such antagonistic families are dense in a natural collection of one parameter families. (Proposition 17.)

(c): As a corollary, we obtain that it happens quite frequently that in Shmerkin-Solomyak Theorem (Theorem 7) the exceptional set has packing dimension 1. (Corollary 18.)

(d): We extend the scope of [7, Proposition 8.1] from infinite Bernoulli convolution measures to very general one-parameter families of (not necessarily self-similar, or self-affine) IFS, and state that the parameter set of singularity is a \( G_\delta \) set (Theorems 9, 10).

### 1.3. Comments.

The main goal of this note is to make the observation that the combination of an already existing method of Peres, Schlag and Solomyak [7] and a result due to Manning and the first author of this note [4] yields that the answer to Question 2 is negative.

There are two ingredients of our argument:

(i): The fact that the set of parameters of singularity is a \( G_\delta \) set in any reasonable one-parameter family of self-similar measures on the line.

(ii): The existence of a one-parameter family of self-similar measures having similarity dimension greater than one (for all parameters) with a dense set of parameters of singularity.
It turned out that both of these ingredients have been available for a while in the literature. Although in an earlier version of this note the authors had their longer proof for (i), we learned from B. Solomyak that (i) has already been proved in [7, Proposition 8.1] in the special case of infinite Bernoulli convolutions. Actually, the authors of [7] acknowledged that the short and elegant proof of [7, Proposition 8.1] is due to Elon Lindenstrauss. We extend the scope of [7, Proposition 8.1] to a more general case. Then following the supposition of the anonymous referee we finally got a very general case. So, to prove (i), we will present here a more detailed and very general extension of the proof of [7, Proposition 8.1].

On the other hand (ii) was proved in [4].

1.4. Notation. First we introduce the Hausdorff and similarity dimensions of a measure and then we present some definitions related to the singularity and absolute continuity of the family of measures considered in the paper.

1.4.1. The different notions of dimensions used in the paper.
- The notion of the Hausdorff and box dimension of a set is well known (see e.g. [2]).
- Hausdorff dimension of a measure: Let $m$ be a measure on $\mathbb{R}^d$. The Hausdorff dimension of $m$ is defined by
  \[ \dim_H m := \inf \{ \dim_H A : m(A) > 0, \text{ and } A \text{ is a Borel set} \}, \]
  see [2, p. 170] for an equivalent definition.
- We will use the following definition of the Packing dimension of a set $H \subset \mathbb{R}^d$ [2, p. 23.]:
  \[ \dim_P H = \inf \{ \sup \overline{\dim_B} E_i : H \subset \bigcup_{i=1}^{\infty} E_i \}, \]
  where $\overline{\dim_B}$ stands for the upper box dimension. The most important properties of the packing dimension can be found in [2].
- Similarity dimension of a self-similar measure: Consider the self-similar IFS on the line: $\mathcal{F} := \{ \varphi(x) := r_i \cdot x + t_i \}_{i=1}^{m}$, where $r_i \in (-1, 1) \setminus \{0\}$. Further we are given the probability vector $w := (w_1, \ldots, w_m)$. Then there exists a unique measure $\nu$ satisfying $\nu(H) = \sum_{i=1}^{m} w_i \cdot \nu (\varphi_i^{-1}(H))$. (See [2].) We call $\nu = \nu_{\mathcal{F}, w}$ the self-similar measure corresponding to $\mathcal{F}$ and $w$. The similarity dimension of $\nu$ is defined by
  \[ \dim_S (\nu_{\mathcal{F}, w}) := \frac{\sum_{i=1}^{m} w_i \log w_i}{\sum_{i=1}^{m} w_i \log r_i}. \]

1.4.2. The projected families of a self-similar measure. Let
  \[ \mathcal{F}_\alpha := \left\{ \varphi_\alpha^\alpha(x) := r_{\alpha,i} \cdot x + t_i^{\alpha} \right\}_{i=1}^{m}, \quad \alpha \in A, \]
be a one-parameter family of self-similar IFS on $\mathbb{R}$ and let $\mu$ be a measure on the symbolic space $\Sigma := \{1, \ldots, m\}^\mathbb{N}$. We write
  $\varphi_{i_1 \ldots i_n} := \varphi_{i_1} \circ \cdots \circ \varphi_{i_n}$ and $r_{\alpha,i_1 \ldots i_n} := r_{\alpha,i_1} \cdots r_{\alpha,i_n}$. 
The natural projection $\Pi_\alpha : \Sigma \to \mathbb{R}$ is defined by

\[
\Pi_\alpha(i) := \lim_{n \to \infty} \varphi_{i_1, \ldots, i_n}^\alpha(0) = \sum_{k=1}^{\infty} t_{i_k}^{(\alpha)} r_{i_1, \ldots, i_{k-1}},
\]

where $r_{i_1, \ldots, i_{k-1}} := 1$ when $k = 1$. Let $\mu$ be a probability measure on $\Sigma$. We study the family of its push forward measures $\{\nu_\alpha\}_{\alpha \in A}$:

\[
\nu_\alpha(H) := (\Pi_\alpha)_* \mu(H) := \mu(\Pi_\alpha^{-1}(H)),
\]

where $H$ is a Borel subset of $\Sigma$.

The elements of the symbolic space $\Sigma := \{1, \ldots, m\}^\mathbb{N}$ are denoted by $i = (i_1, i_2, \ldots)$. If $w := (w_1, \ldots, w_m)$ is a probability vector and $\mu$ is the infinite product of $w$, that is $\mu = \{w_1, \ldots, w_m\}^{\mathbb{N}}$ then the corresponding one-parameter family of self-similar measures defined in (7) is denoted by $\{\nu_\alpha, w\}_{\alpha \in A}$.

The set of parameters of singularity and the set of parameters of absolute continuity with $L^q$-density are denoted by

\[
\text{Sing}(\mathcal{F}_\alpha, \mu) := \{\alpha \in A : \nu_\alpha \perp \text{Leb}\}.
\]

and

\[
\text{Cont}_Q(\mathcal{F}_\alpha, \mu) := \{\alpha : \nu_\alpha \ll \text{Leb} \text{ with } L^q \text{ density for a } q > 1\}.
\]

**Definition 1.** Using the notation introduced in (5)-(9) we say that the family $\{\nu_\alpha\}_{\alpha \in A}$ is antagonistic if both of the two conditions below hold:

\[
\dim_H \text{Sing}(\mathcal{F}_\alpha, \mu) = \dim_H (\text{Cont}_Q(\mathcal{F}_\alpha, \mu))^c = 0
\]

and

\[
\text{Sing}(\mathcal{F}_\alpha, \mu) \text{ is a dense } G_\delta \text{ subset of } A.
\]

Clearly, $\text{Sing} \subset (\text{Cont}_Q)^c$. Our aim is to prove that the angle-$\alpha$ projections of the natural measure of the Sierpiński-carpet is an antagonistic family. This implies that in Shmerkin-Solomyak’s Theorem, [13], Theorem A] (this is Theorem 7 below) the exceptional set has packing dimension 1.

1.5. Regularity properties of $\mathcal{F}_\alpha$. Whenever we say that $\{\nu_\alpha\}_{\alpha \in A}$ is a one-parameter family of self-similar IFS we always mean that $\{\nu_\alpha\}_{\alpha \in A}$ is constructed from a pair $(\mathcal{F}_\alpha, \mu)$ as in [7] for a $\mu = w^\mathbb{N}$, where $w = (w_1, \ldots, w_m)$ is a probability vector.

**Principal Assumption 1.** Throughout this note, we always assume that the one-parameter family of self-similar IFS $\{\mathcal{F}_\alpha\}_{\alpha \in A}$ satisfies properties P1-P4 below:

**P1:** The parameter domain is a non-empty, proper open interval $A$.

**P2:** $0 < r_{\min} := \inf_{\alpha \in A, i \leq m} |r_{\alpha,i}| \leq \sup_{\alpha \in A, i \leq m} |r_{\alpha,i}| =: r_{\max} < 1$.

**P3:** $t_{\max} := \sup_{\alpha \in A, i \leq m} |t_{i}^{(\alpha)}| < \infty$.

**P4:** Both of the functions $\alpha \mapsto t_{i}^{(\alpha)}$ and $\alpha \mapsto r_{\alpha}$, $\alpha \in A$, can be extended to $\overline{A}$ (the closure of $A$) such that these extensions are both continuous.

Note that P4 implies P3. It follows from properties P2 and P3 that there exists a big $\xi \in \mathbb{R}^+$ such that

\[
\text{spt}(\nu_\alpha) \subset (-\xi, \xi), \quad \forall \alpha \in A.
\]

We always confine ourselves to this interval $(-\xi, \xi)$. In particular, whenever we write $H^c$ for a set $H \subset \mathbb{R}$ we mean $(-\xi, \xi) \setminus H$. It will be our goal to prove that additionally the following properties also hold for some of the families under consideration:
P5A: Sing($\mathcal{F}_\alpha, \mu$) is dense in $A$.
P5B: Sing($\mathcal{F}_\alpha, \mu$) is a $G_\delta$ dense subset of $A$.

We will prove below that Properties P5A and P5B are equivalent. Our motivating example, where all of these properties hold is as follows.

1.6. Motivating example. Our most important example is the family of angle-$\alpha$ projection of the natural measure of the usual Sierpiński carpet. We will see that the set of angles of singularity is a dense $G_\delta$ set which has Hausdorff dimension zero and packing dimension 1. First we define the Sierpiński carpet.

Definition 2. Let $t_1, \ldots, t_8 \in \mathbb{R}^2$ be the 8 elements of the set
\[ \{(0,1,2) \times \{0,1,2\} \setminus \{(1,1)\}\} \] in any particular order. The Sierpiński carpet is the attractor of the IFS
\[
S := \left\{ \varphi_i(x,y) := \frac{1}{3}(x,y) + \frac{1}{3} t_i \right\}_{i=1}^8.
\]

Example 1 (Motivating example). Let $S$ be the IFS given in (13). Let $\mu := \left(\frac{1}{8}, \ldots, \frac{1}{8}\right)^N$ be the uniform distribution measure on the symbolic space $\Sigma := \{1, \ldots, 8\}^N$. Further we write $\Pi$ for the natural projection from $\Sigma$ to the attractor $\Lambda$. Let $\nu := \Pi_* \mu$. Let $\ell_\alpha \subset \mathbb{R}^2$ be the line having angle $\alpha$ with the positive half of the $x$-axis (see Figure 2). Let $\text{proj}_\alpha$ be the angle-$\alpha$ projection from $\mathbb{R}^2$ to the line $\ell_\alpha$. For each $\alpha$, identifying $\ell_\alpha$ with the $x$-axis, $\text{proj}_\alpha$ defines a one parameter family of self-similar IFS on the $x$-axis:
\[
S_\alpha := \left\{ \varphi_i^{(\alpha)} \right\}_{i=1}^8,
\]
where $\alpha \in A := (0, \pi)$ and $\varphi_i^{(\alpha)}(x) = r_{\alpha,i} x + t_i^{(\alpha)}$ with $r_{\alpha,i} \equiv 1/3$ and $t_i^{(\alpha)} = t_i \cdot (\cos(\alpha), \sin(\alpha))$. For an $i \in \Sigma$ we define the natural projection $\Pi_{\alpha}(i)$ as in (6). Clearly, $\Pi_{\alpha} := \text{proj}_\alpha \circ \Pi$. The natural invariant measure for $S_\alpha$ is $\nu_\alpha := (\Pi_{\alpha})_* \mu$. Obviously, $\nu_\alpha = (\text{proj}_\alpha)_* \nu$. 

Figure 1. The first three approximations of the Sierpiński carpet.
The fact that Property P5A holds for the special case in the example was proved in [4, p.216]. It follows from the proof of Bárány and Rams [1, Theorem 1.2] that property P5A holds also for the projected family of the natural measure for most of those self-similar carpets, which have dimension greater than one.

**Remark 1** (The cardinality of parameters of exact overlaps). *It is obvious that in the case of the angle-\(\alpha\) projections of a general self-similar carpet, exact overlap can happen only for countably many parameters. However, this is not true in general. To see this, we follow the ideas in the paper of Cs. Sándor [10] and construct the one parameter family of self-similar IFS \(\{S_i^{(u)}\}_{i=1}^3, u \in U\), where \(S_i^{(u)} := \lambda_i^{(u)}(x + 1)\) and \((\lambda_1^{(u)}, \lambda_2^{(u)}, \lambda_3^{(u)}) = (\frac{u^3 + \varepsilon}{1 + \varepsilon}, u, u + \varepsilon)\), further \(U := [\frac{1}{3} + \varepsilon, \frac{1}{3} + \eta - \varepsilon]\) for sufficiently small \(\eta > 0\) and \(0 < \varepsilon < \frac{3}{4}\). Then for all \(u \in U\) we have:

(a): there is an exact overlap, namely: \(S_{132}^{(u)} \equiv S_{213}^{(u)}\),
(b): the similarity dimension of the attractor is greater than 1,
(c): the Hausdorff dimension of the attractor is smaller than 1.

2. **Theorems we use from the literature**

For the ease of the reader here we collect those theorems we refer to in this note. We always use the notation of Section 1. The theorems below are more general as stated here. We confine ourselves to the generality that matters for us.

2.1. **Hochman Theorems**.
Theorem 3. [3, Theorems 1.7, Theorems 1.8] Given the one-parameter family \( \{ F_\alpha \}_{\alpha \in A} \) in the form as in [5]. For \( i, j \in \Sigma^n := \{1, \ldots, m\}^n \) we define

\[
\Delta_{ij}(\alpha) := \varphi_i^\alpha(0) - \varphi_j^\alpha(0) \quad \text{and} \quad \Delta_n(\alpha) := \min_{i,j \in \Sigma^n} \{ \Delta_{ij}(\alpha) \}.
\]

Moreover, we define the exceptional set of parameters \( E \subset A \)

\[
E := \bigcap_{\varepsilon > 0} \bigcup_{N=1}^{\infty} \bigcap_{n>N} \Delta^{-1}_n(-\varepsilon^n, \varepsilon^n).
\]

Then for an \( \alpha \in E^c \) and for every probability vector \( w \) the Hausdorff dimension of the corresponding self-similar measure \( \nu_{\alpha, w} \) is

\[
\dim_H(\nu_{\alpha, w}) = \min \{1, \dim_S(\nu_{\alpha, w})\}
\]

The following Condition will also be important:

Definition 4. We say that for an \( \alpha \in A \), \( F_\alpha \) satisfies Condition H if

\[
\exists \rho = \rho(\alpha) > 0, \exists n_k = n_k(\alpha) \uparrow \infty, \quad \Delta_{n_k}(\alpha) > \rho^{n_k}.
\]

Observe that \( \alpha \in E^c \) if and only if \( F_\alpha \) satisfies Condition H.

Definition 5. We say that the Non-Degeneracy Condition holds if

\[
\forall i, j \in \Sigma, \ i \neq j, \ \exists \alpha \in A \ s.t. \ \Pi_\alpha(i) \neq \Pi_\alpha(j).
\]

Theorem 6. [3, Theorems 1.7, Theorems 1.8] Assume that the Non-Degeneracy Condition holds and the following functions are real analytic:

\[
\alpha \mapsto r_\alpha, i = 1, \ldots, m \quad \text{and} \quad \alpha \mapsto t_i(\alpha).
\]

Then

\[
\dim_H E = \dim_P E = 0.
\]

2.2. Shmerkin-Solomyak Theorem.

Theorem 7. [13, Theorem A] We assume that the conditions of Theorem 6 hold. Here we confine ourselves to homogeneous self-similar IFS on the line of the form

\[
F_\alpha := \left\{ \varphi_i^\alpha(x) := r_\alpha \cdot x + t_i(\alpha) \right\}_{i=1}^m, \quad \alpha \in A.
\]

Then there exists an exceptional set \( E \subset A \) with \( \dim_H E = 0 \) such that for any \( \alpha \in E^c \) and for any probability vector \( w = (w_1, \ldots, w_m) \) with \( \dim_S(\nu_{\alpha, w}) > 1 \) we have

\[
\nu_{\alpha, w} \ll \text{Leb with } L^q \text{ density, for some } q > 1.
\]

2.3. An extension of Bárány-Rams Theorem. Lídia Torma realized in her Master’s Thesis [14] that the proof of Bárány and Rams [1, Theorem 1.2], related to the projections of general self-similar carpets, works in a much more general setup, without any essential change.

Theorem 8. (Extended version of Bárány-Rams Theorem). Given an \( a \in \mathbb{R} \setminus \{0\} \). Let \( T = \{ n \cdot a \}_{n \in \mathbb{Z}} \) be the corresponding lattice on \( \mathbb{R} \). Moreover, given the self-similar IFS on the line of the form:

\[
S := \left\{ S_i(x) := \frac{1}{L} \cdot x + t_i \right\}_{i=1}^m,
\]
where \( L \in \mathbb{N}, L \geq 2 \) and \( t_i \in T \) for all \( i \in \{1, \ldots, m\} \). We are also given a probability vector \( \mathbf{w} = (w_1, \ldots, w_m) \) with rational weights \( w_i = p_i/q_i, p_i, q_i \in \mathbb{N} \setminus \{0\} \) satisfying

\[
L \mid Q := \text{lcm}\{q_1, \ldots, q_m\}, \quad s := \dim_S \nu = \frac{-\sum_{i=1}^{m} w_i \log w_i}{\log L} > 1,
\]

where \( \nu \) is the self-similar measure corresponding to the weights \( \mathbf{w} \). That is \( \nu = \sum_{i=1}^{m} w_i \cdot \nu \circ S_i^{-1} \). Then we have

\[
\dim_H \nu < 1.
\]

3. \( \mathcal{S}(F_\alpha, \mu) \) is a \( G_\delta \) set

As we have already mentioned the following result appeared as \cite{7, Proposition 8.1} in the special case when the family of self-similar measures is the Bernoulli convolution measures. We extend the original proof of \cite{7, Proposition 8.1} to the following much more general situation.

**Theorem 9.** Let \( R \subset \mathbb{R}^{d} \) be a non-empty bounded open set. Let \( U \) be a metric space (the parameter domain). Let \( \lambda \) be a finite Radon measure with \( \text{spt}(\lambda) \subset R \) (the reference measure). For every \( \alpha \) we are given a probability Radon measure \( \nu_\alpha \) such that \( \text{spt}(\nu_\alpha) \subset R \).

For every \( F \in \mathcal{C}_R \) we define \( \Phi_f : U \to \mathbb{R} \)

\[
\Phi_f(\alpha) := \int_R f(x)d\nu_\alpha(x).
\]

Finally, we define

\[
\mathcal{S}(\lambda)(\{\nu_\alpha\}_{\alpha \in U}) := \{\alpha \in U : \nu_\alpha \perp \lambda\}.
\]

If \( \alpha \mapsto \Phi_f(\alpha) \) is lower semi-continuous then \( \mathcal{S}(\lambda)(\{\nu_\alpha\}_{\alpha \in U}) \) is a \( G_\delta \) set.

**Proof.** Recall that \( \nu_\alpha \) is a probability measure for all \( \alpha \). Note that without loss of generality we may assume that \( \lambda \) is also a probability measure on \( R \). For every \( \varepsilon > 0 \) we define

\[
\mathcal{A}_\varepsilon := \left\{ f \in \mathcal{C}_R : \int_R f(x)d\lambda(x) < \varepsilon \right\}.
\]

We follow the proof of \cite{7, Proposition 8.1} and a suggestion of an unknown referee. First we fix an arbitrary sequence \( \varepsilon_n \downarrow 0 \) and then define

\[
S_{\perp} := \bigcup_{n=1}^{\infty} \bigcup_{f \in \mathcal{A}_{\varepsilon_n}} \{\alpha \in U : \Phi_f(\alpha) > 1 - \varepsilon_n\}.
\]

Since we assumed that \( \alpha \mapsto \Phi_f(\alpha) \) is lower semi-continuous, the set \( \{\alpha \in U : \Phi_f(\alpha) > 1 - \varepsilon_n\} \) is open. That is \( S_{\perp} \) is a \( G_\delta \) set. Hence it is enough to prove that

\[
\mathcal{S}(\lambda)(\{\nu_\alpha\}_{\alpha \in U}) = S_{\perp}.
\]

First we prove that \( \mathcal{S}(\lambda)(\{\nu_\alpha\}_{\alpha \in U}) \subset S_{\perp} \). Let \( \beta \in \mathcal{S}(\lambda)(\{\nu_\alpha\}_{\alpha \in U}) \). Fix an arbitrary \( \varepsilon > 0 \). Then by definition we can find a \( T \subset R \) such that

\[
\nu_\beta(T) = 1, \quad \lambda(T) = 0.
\]
Recall that both $\lambda$ and $\nu_\beta$ are Radon probability measures. So we can choose a compact $C_\varepsilon \subset T$ such that
\[(30) \quad \nu_\beta(C_\varepsilon) > 1 - \varepsilon, \quad \lambda(C_\varepsilon) = 0.\]
Using that $\lambda$ is a Radon measure, we can choose an open set $V_\varepsilon \subset R$ such that $C_\varepsilon \subset V_\varepsilon$ and $\lambda(V_\varepsilon) < \varepsilon$. We can choose an $f_\varepsilon \in C_R$ such that $\text{spt}(f_\varepsilon) \subset V_\varepsilon$ and $f_\varepsilon|_{C_\varepsilon} \equiv 1$ (see [9] p. 39).

Then $\int f_\varepsilon d\lambda(x) \leq \lambda(V_\varepsilon) < \varepsilon$ (that is $f_\varepsilon \in A_\varepsilon$) and $\int f_\varepsilon(x) d\nu_\beta(x) \geq \nu_\beta(C_\varepsilon) > 1 - \varepsilon$. Since $\varepsilon > 0$ was arbitrary we obtain that $\beta \in S_\perp$.

Now we prove that $S_\perp \subseteq \text{Sing}_\lambda(\{\nu_\alpha\}_{\alpha \in U})$. Let $\beta \in S_\perp$. Then for every $n$ there exists an $f_n \in C_R$ such that
\[(31) \quad \int f_n(x) d\nu_\beta(x) > 1 - \varepsilon_n \quad \text{and} \quad \int f_n d\lambda(x) < \varepsilon_n.\]
Let $C_\beta := \text{spt}(\nu_\beta)$. Clearly, $C_\beta$ is compact and $C_\beta \subset R$. We define $g_n := f_n 1_{C_\beta}$, and $g := 1_{C_\beta}$.

Clearly, $0 \leq g_n(x) \leq g(x)$ for all $x \in C_\beta$ and \[
\int g(x) d\nu_\beta(x) = 1, \quad \int g_n(x) d\nu_\beta(x) > 1 - \varepsilon_n \quad \text{and} \quad \int g_n d\lambda(x) < \varepsilon_n.
\]
Hence,\[
g_n \xrightarrow{L_1(\nu_\beta)} g.
\]
Thus, we can select a subsequence $g_{n_k}$ such that $g_{n_k}(x) \to g(x)$ for $\nu_\beta$-almost all $x \in C_\beta$. Let $D_\beta := \{x \in C_\beta : g_{n_k}(x) \to g(x)\}$.

Then on the one hand we have \[(32) \quad \nu_\beta(D_\beta) = 1.\]
On the other hand using the Lebesgue Dominated Convergence Theorem:
\[(33) \quad \lambda(D_\beta) = \int_{D_\beta} g(x) d\lambda(x) = \int_{D_\beta} \lim_{k \to \infty} g_{n_k}(x) d\lambda(x) = \lim_{k \to \infty} \int_{D_\beta} g_{n_k}(x) d\lambda(x) \leq \lim_{k \to \infty} \varepsilon_{n_k} = 0.
\]
Putting together (32) and (33) we obtain that $\beta \in \text{Sing}_\lambda(\{\nu_\alpha\}_{\alpha \in U})$. \hfill \Box

**Theorem 10.** We consider one-parameter families of measures $\nu_\alpha$ on $\mathbb{R}^d$ for some $d \geq 1$, which are constructed as follows: The parameter space $U$ is a non-empty compact metric space. We are given a continuous mapping
\[(34) \quad \Pi : U \times \Omega \to R \subset \mathbb{R}^d,
\]
where $R$ is an open ball in $\mathbb{R}^d$ and $\Omega$ is a compact metric space (in our applications $U$ is a compact interval, $\Omega = \Sigma$ and $\Pi_\alpha$ is the natural projection corresponding to the parameter $\alpha$). Moreover let $\mu$ be a probability Radon measure on $\Omega$. (In our applications $\mu$ is Bernoulli measure on $\Sigma$.) For every $\alpha \in U$ we define
\[(35) \quad \nu_\alpha := (\Pi_\alpha)_* \mu.
\]
Clearly, $\nu_\alpha$ is a Radon measure whose support is contained in $R$. Finally let $\lambda$ be a Radon (reference) measure whose support is also contained in $R$. (In our applications $\lambda$ is the Lebesgue measure $\text{Leb}_d$ restricted to $R$.)
Then the set of parameters of singularity
\[ \text{Sing}_\lambda(\Pi, \mu) := \{ \alpha \in U : \nu_\alpha \perp \lambda \} \]
is a $G_\delta$ set.

Proof. This theorem immediately follows from Theorem 9 if we prove that for every $f \in C_R$ the function $\Phi_f(\cdot)$ is continuous. To see this we set $\psi : U \times \Omega \to \mathbb{R}$,
\[
\psi(\alpha, \omega) = f(\Pi(\omega)), \quad \text{then } \Phi_f(\alpha) := \int f(x)d\nu_\alpha(x) = \int \psi(\alpha, \omega)d\mu(\omega),
\]
where the last equality follows from the change of variables formula. By compactness, $\psi$ is uniformly continuous. Hence for every $\varepsilon > 0$ we can choose $\delta > 0$ such that whenever $\text{dist}((\alpha_1, \omega), (\alpha_2, \omega)) < \delta$ then $|\psi(\alpha_1, \omega) - \psi(\alpha_2, \omega)| < \varepsilon$, where $\text{dist}((\alpha_1, \omega), (\alpha_2, \omega)) := \max\{\text{dist}_U(\alpha_1, \alpha_2), \text{dist}_\Omega(\omega_1, \omega_2)\}$. Using that $\mu$ is a probability measure, we obtain that $|\Phi_f(\alpha_1) - \Phi_f(\alpha_2)| < \varepsilon$ whenever $\text{dist}_U(\alpha_1, \alpha_2) < \delta$. \hfill $\square$

Corollary 11. Using the notation of Section 1.4 and assuming our Principal Assumption (defined on page 3) we obtain that the set of parameters of singularity $\text{Sing}(F_\alpha, \mu)$ is a $G_\delta$ set.

The proof is obvious since our Principal Assumptions imply that the conditions of Theorem 10 hold.

To derive another corollary we need the following fact. It is well known, but we could not look it up in the literature, therefore we include its proof here.

Fact 12. Let $H \subset \mathbb{R}^d$ be a $G_\delta$ set which is not a nowhere dense set. Then $\dim_P H = d$.

Proof. Since $H$ is not a nowhere dense set, there exist a ball $B$ such that $B \subset H$. That is $V := B \cap H$ is a dense $G_\delta$ set in $B$, that is by Banach's theorem $V$ is not a set of first category. So, if $V \subset \bigcup_{i=1}^\infty E_i$ then there exists an $i$ such that $E_i$ is not nowhere dense in $B$. That is there exists a ball $B' \subset B$ such that $B' \subset E_i$. Then $\dim_B E_i = d$. Hence by (3) we have $\dim_P H \geq \dim_P V = d$. On the other hand, $\dim_P H \leq d$ always holds. \hfill $\square$

Applying this for $\text{Sing}(F_\alpha, \mu)$ we obtain that

Corollary 13. Under the conditions of Theorem 10, for the set of parameters of singularity $\text{Sing}(F_\alpha, \mu)$ the following holds:

(i): Either $\text{Sing}(F_\alpha, \mu)$ is nowhere dense or

(ii): $\dim_P (\text{Sing}(F_\alpha, \mu)) = d$.

Henna Koivusalo called the attention of the authors for the following immediate corollary of Theorem 10.

Remark 2. Let $\mu$ be a compactly supported Borel measure on $\mathbb{R}^2$ with $\dim_H \mu > 1$. Let $\nu_\alpha := (\text{proj}_\alpha)_* \mu$. Then Theorem 10 immediately implies that either the singularity set
\[ \text{Sing}_{\text{Leb}} \left( \{ \nu_\alpha \}_{\alpha \in [0, \pi]} \right) = \{ \alpha \in [0, \pi) : \nu_\alpha \perp \text{Leb}_1 \} . \]
or its complement is big in topological sense. More precisely,

(a): Either $\text{Sing}_{\text{Leb}} \left( \{ \nu_\alpha \}_{\alpha \in [0, \pi]} \right)$ is a residual subset of $[0, \pi)$ or

(b): $\left( \text{Sing}_{\text{Leb}} \left( \{ \nu_\alpha \}_{\alpha \in [0, \pi]} \right) \right)^c$ contains an interval.

We remind the reader that a set is called residual if its complement is a set of first category and residual sets are considered as "big" in topological sense.

In contrast we recall that by Kaufman's Theorem (see e.g. [5, Theorem 9.7]) we have
\[ \nu_\alpha \ll \text{Leb}_1 \text{ for } \text{Leb}_1 \text{ almost all } \alpha \in [0, \pi) . \]
The following theorem shows that there are reasons other than exact overlaps for the singularity of self-similar measures having similarity dimension greater than one.

**Theorem 14.** Using the notation of our Example 1 (angle-\(\alpha\) projections of the Sierpiński carpet), we obtain that

\[(38) \quad \text{Sing}(S_\alpha, \mu) = \{\alpha \in A : \nu_\alpha \perp \text{Leb}\} \text{ is a dense } G_\delta \text{ set}\]

and

\[(39) \quad \dim_H(\text{Cont}_Q(S_\alpha, \mu)^c) = 0.\]

That is \((S_\alpha, \mu)\) is antagonistic in the sense of Definition 1.

**Proof.** The first part follows from Corollary 11 and from the fact that property P5A holds for the projections of the Sierpiński-carpet. This was proved in [4].

Now we turn to the proof of the second part of the Theorem. This assertion would immediately follow from Shmerkin and Solomyak [13, Theorem A] if we could guarantee that the Non-Degeneracy Condition holds. Unfortunately in this case it does not hold. Still it is possible to gain the same conclusion not from the assertion of [13, Theorem A] but from its proof, combined with [13, Lemma 5.4] as it was explained by P. Shmerkin to the authors [11]. For completeness we point out the only two steps of the original proof of [13, Theorem A] where we have to make slight modifications.

Let \(\mathcal{P}\) be the set of probability Borel measures on the line. We write

\[(40) \quad D := \{\mu \in \mathcal{P} : |\hat{\mu}(\xi)| = O(\xi^{-\sigma}) \text{ for some } \sigma > 0\}.\]

The elements of \(D\) are the probability measures on the line with power Fourier-decay. Let \(\{\varphi_i(\alpha)\}^8_{i=1}\) be the IFS defined in Example 1. Now we write the projected self-similar natural measure \(\nu_\alpha\) of the Sierpiński carpet in the infinite convolution form. That is we consider \(\nu_\alpha\) as the distribution of the following infinite random sum:

\[\nu_\alpha \sim \sum_{n=1}^{\infty} (1/3)^{n-1} A_n,\]

where \(A_n\) are independent Bernoulli random variables with \(\mathbb{P}(A_n = \varphi_i(\alpha)(0)) = 1/8\). For \(k \geq 2\) integers we decompose the random sum on the right hand side as

\[\nu_\alpha \sim \sum_{n=1}^{\infty} (1/3)^{n-1} A_n + \sum_{k \mid n} (1/3)^{n-1} A_n.\]

Writing \(\eta_{\alpha, k}^{'}\) and \(\eta_{\alpha, k}^{''}\) for the distribution of the first and the second random sum, respectively, we get \(\nu_\alpha = \eta_{\alpha, k}^{'} \ast \eta_{\alpha, k}^{''}\). Our goal is to show that with appropriately chosen \(k\) we can apply [13, Corollary 5.5] to \(\eta_{\alpha, k}^{'}\) and \(\eta_{\alpha, k}^{''}\) which would conclude the proof. To this end it is enough to show that on the one hand

\[(41) \quad \dim_H \eta_{\alpha, k}^{'} = 1 \quad \text{for every } k \text{ large enough}\]

and on the other hand we have

\[(42) \quad \eta_{\alpha, k}^{''} \in D, \quad \forall k \geq 2.\]

This is the first place where we depart from the proof of [13, Theorem A]. According to [12, Theorem 5.3] if \(\dim_S \eta_{\alpha, k}^{'} > 1\) (which holds if \(k\) is big enough), then there exists a countable set \(E'_k\) such that \(\dim_H \eta_{\alpha, k}^{'} = 1\) for all \(\alpha \notin E'_k\). Note that the original proof at this point relies on the non-degeneracy condition, what we do not use here.
To get the Fourier decay of \( \eta''_{\alpha,k} \) we follow the proof of \([13, \text{Theorem A}]\). In our special case, we may choose the function \( f \) in the middle of page 5147 in \([13]\) as
\[
 f(\alpha) = \frac{\text{proj}_\alpha \left( \frac{2}{3}, 0 \right) - \text{proj}_\alpha \left( \frac{1}{3}, \frac{2}{3} \right)}{\text{proj}_\alpha \left( 0, \frac{2}{3} \right) - \text{proj}_\alpha \left( \frac{1}{3}, \frac{2}{3} \right)} = 2 \tan(\alpha) - 1.
\]
Clearly \( f \) is non-constant and \( f^{-1} \) preserves the Hausdorff dimension. Hence by \([13, \text{Lemma 6.2 and Proposition 3.1}]\) there is a set \( E''_k \) of Hausdorff dimension 0 such that \( \eta''_{\alpha,k} \) has power Fourier-decay for all \( \alpha \notin E''_k \). Altogether, setting the 0-dimensional exceptional set of parameters \( E = \bigcup_{k=2}^{\infty} E'_k \cup E''_k \), by \([13, \text{Corollary 5.5}]\) we have that \( \nu_\alpha \) is absolutely continuous with an \( L^q \) density for some \( q > 1 \) for all \( \alpha \notin E \) exactly as in the proof of \([13, \text{Theorem A}]\) with no further modifications.

In Theorem 14 we have proved that the family of the angle-\( \alpha \) projection of the Sierpiński-carpet is antagonistic in the sense of Definition 1. In the rest of this note we prove that there are many antagonistic families.

4. AN EQUI-HOMOGENEOUS FAMILY FOR WHICH THE NON-DEGENERACY CONDITION HOLDS

First of all we remark that the Non-Degeneracy Condition does not hold for all families. For example let
\[
 F_\alpha := \left\{ \frac{1}{2} \cdot x + t^{(\alpha)}_i \right\}_{i=1}^{m}, \quad m \geq 2.
\]
Then for every \( \alpha \), \( \Pi_\alpha(i) = \Pi_\alpha(j) \) for \( i = (1, 2, 2, 2, \ldots) \) and \( j = (2, 1, 1, 1, \ldots) \). So, the non-degeneracy condition does not hold.

However, if the contraction ratio is the same \( \lambda \in (0, \frac{1}{2}) \) for all maps of all IFS in the family (the family is equi-homogeneous) and the translations are independent real-analytic functions then the Non-Degeneracy Condition holds:

**Proposition 15.** Given
\[
 F_\alpha := \left\{ \lambda \cdot x + t^{(\alpha)}_i \right\}_{i=1}^{m}, \quad m \geq 2, \quad \alpha \in A,
\]
where
(a): \( \lambda \in (0, \frac{1}{2}) \) and  
(b): For \( \ell = 1, \ldots, m \), the functions \( \alpha \mapsto t^{(\alpha)}_\ell = \sum_{k=0}^{\infty} a_{\ell,k} \cdot \alpha^k \), are independent real-analytic functions:
\[
 \forall \alpha \in A, \sum_{i=1}^{m} \gamma_i \cdot t^{(\alpha)}_i \equiv 0 \text{ iff } \gamma_1 = \cdots = \gamma_m = 0.
\]

Then \( \{F_\alpha\}_{\alpha \in A} \) satisfies the Non-Degeneracy Condition.

**Proof.** Fix two distinct \( i, j \in \Sigma \). For every \( \ell = 1, \ldots, m \), define \( q_\ell := q_\ell(i,j) \) by
\[
 q_\ell := \sum_{\{k; i_k = \ell\}} \lambda^{(k-1)} - \sum_{\{k; j_k = \ell\}} \lambda^{(k-1)}.
\]
Then
\[
 \Pi_\alpha(i) - \Pi_\alpha(j) = \sum_{k=0}^{\infty} \alpha^k \cdot b_k,
\]
where

\[ b_k := \sum_{\ell=1}^{m} a_{\ell,k} \cdot q_{\ell} \]

for all \( k \in \mathbb{N}^+ \), where \( \mathbb{N}^+ := \mathbb{N} \setminus \{0\} \). Observe that for \( b := (b_0, b_1, \ldots) \) and \( \forall \ell = 1, \ldots, m \) for \( a_{\ell} := (a_{\ell,0}, a_{\ell,1}, a_{\ell,2}, \ldots a_{\ell,k}, \ldots) \) we have that (48) can be written as

\[ \sum_{\ell=1}^{m} q_{\ell} \cdot a_{\ell} = b. \]

Assume that

\[ (50) \quad \forall \alpha \in A, \quad \Pi_{\alpha}(i) - \Pi_{\alpha}(j) \equiv 0. \]

To complete the proof it is enough to verify that \( i = j \). Using (47), we obtain from (50) that \( b_{k} = 0 \) for all \( k \in \mathbb{N}^+ \). Note that (45) states that the vectors \( \{a_{\ell}\}_{\ell=1}^{m} \) are independent. So, from \( b = 0 \) and from (49) we get that \( q_1 = \cdots = q_m = 0 \). This and \( \lambda \in (0, \frac{1}{2}) \) implies that \( i = j \).

\[ \square \]

5. ANTAGONISTIC FAMILIES OF SELF-SIMILAR IFS

Here we prove the following assertion: The collection of one-parameter families of IFS and self-similar measures are dense in the collection of equi-homogeneous IFS having contraction ratio \( 1/L \) \((L \in \mathbb{N}^+)\) equipped with invariant measures with similarity dimension greater than one. To state this precisely, we need some definitions:

**Definition 16.** First we consider collections of equi-homogeneous self-similar IFS having at least 4 functions.

(i): Let \( \mathcal{F}_L \) be the collection of all pairs \((\mathcal{F}_\alpha, \mu)\) satisfying the conditions below:

- \( \{\mathcal{F}_\alpha\}_{\alpha \in A} \) is of the form:
  \[ \mathcal{F}_\alpha := \left\{ \varphi_{i}^{(\alpha)}(x) := \frac{1}{L} : x + t_{i}^{(\alpha)} \right\}_{i=1}^{m}, \quad \alpha \in A, \]

  where \( m \geq 4 \), \( A \subset \mathbb{R} \) is a proper interval (\( A \) is compact) and \( L \in \mathbb{N}, \quad 3 \leq L \leq m - 1 \).

  Moreover, the functions \( \alpha \mapsto t_{i}^{(\alpha)} \) are continuous on \( \overline{A} \) for all \( \ell = 1, \ldots, m \).

- Let \( \mu \) be an infinite product measure \( \mu := (w_1, \ldots, w_m)^{\mathbb{N}} \) on \( \Sigma := \{1, \ldots, m\}^{\mathbb{N}} \) satisfying:
  \[ s := \frac{- \sum_{i=1}^{m} w_i \log w_i}{\log L} > 1, \]

(ii): Now we define a rational coefficient sub-collection \( \mathcal{F}_{L,\text{rac}} \subset \mathcal{F}_L \) satisfying a non-resonance like condition (54) below:

- \( \alpha \mapsto t_{i}^{(\alpha)} \) are polynomials of rational coefficients. We assume that \( \left\{t_{i}^{(\alpha)}\right\}_{i=1}^{m} \) are independent, that is (45) holds. Moreover,

- The weights \( w_i \) are rational: \( \{w_i\}_{i=1}^{m} \), \( w_i = r_i/q_i \), with \( r_i, q_i \in \mathbb{N} \setminus \{0\} \) satisfying:
  \[ L \nmid \text{lcm} \{q_1, \ldots, q_m\}, \]

  where lcm is the least common multiple. Let \( \nu_{\alpha} := \left(\Pi_{\alpha}\right)_{\ast} \mu. \)
Proposition 17.

(a): All elements \{\nu_\alpha\} of \mathcal{F}_{L,\text{rac}} are antagonistic.

(b): \mathcal{F}_{L,\text{rac}} is dense in \mathcal{F}_L in the sup norm.

Proof. (a) It follows from Proposition \[15\] that we can apply Shmerkin-Solomyak Theorem (Theorem \[7\]). This yield that \text{Cont}_Q (defined in \[9\]) satisfies \(\dim_H(\text{Cont}_Q(F_\alpha, \mu)) < 0\). On the other hand, for every rational parameter \(\alpha\), \(\langle F_\alpha, \mu \rangle\) satisfies the conditions of Theorem \[9\]. So, for every \(\alpha \in Q\) we have \(\dim_H \nu_\alpha < 1\). Using this and Corollary \[11\] we get that \(\text{Sing}(F_\alpha, \mu)\) is a dense \(G_\delta\) set. So, \(\{\nu_\alpha\}_{\alpha \in A}\) is antagonistic.

(b) Let \((\widetilde{F}_\alpha, \widetilde{\mu}) \in \mathcal{F}_L\), with \(\widetilde{F}_\alpha := \left\{ x : \frac{1}{L} \cdot x + i^{(1)} \right\}_{i=1}^m\) and \(\widetilde{\mu} = (\widetilde{w}_1, \ldots, \widetilde{w}_m)^N\). Fix an \(\varepsilon > 0\). We can find independent polynomials \(\alpha \mapsto t^{(1)}_i\) such that \(\|t^{(1)}_i - t^{(1)}_i\| < \varepsilon\) for all \(\alpha \in A\) and \(i = 1, \ldots, m\). Moreover, we can find a product measure \(\mu = (w_1, \ldots, w_m)^N\) such that for \(w = (w_1, \ldots, w_m)\) we have \(\|w - \tilde{w}\| < \varepsilon\) and \(w\) has rational coefficients \(w_i = p_i/q_i\) satisfying \(54\).

Corollary 18. Let \(\langle F_\alpha, \mu \rangle \in \mathcal{F}_{L,\text{rac}}\) Then

\[
\dim_H(\text{Sing}(F_\alpha, \mu)) = 1.
\]

Proof. From Solomyak-Shmerkin Theorem, we obtain that \(\text{Sing}(F_\alpha, \mu)\) is dense. Then the assertion follows from Corollary \[13\].

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