Partial differential equations/Functional analysis

Brézis–Gallouet–Wainger-type inequality with critical fractional Sobolev space and BMO

**Inégalité de type Brézis–Gallouet–Wainger pour un espace de Sobolev fractionnaire critique et BMO**

Nguyen-Anh Dao, Quoc-Hung Nguyen

**A R T I C L E   I N F O**

**A B S T R A C T**

In this paper, we prove the Brézis–Gallouet–Wainger-type inequality involving the BMO norm, the fractional Sobolev norm, and the logarithmic norm of $\mathcal{C}^\eta$, for $\eta \in (0, 1)$.

**R É S U M É**

Dans cette Note, nous montrons l’inégalité de type Brézis–Gallouet–Wainger faisant intervenir la norme BMO, la norme fractionnaire de Sobolev et la norme logarithmique de $\mathcal{C}^\eta$, pour $\eta \in (0, 1)$.

1. Introduction and main results

The main purpose of this paper is to establish the $L^\infty$-bound by means of the BMO norm, or the critical fractional Sobolev norm with the logarithm of $\mathcal{C}^\eta$ norm. Such a $L^\infty$-estimate of this type is known as the Brézis–Gallouet–Wainger (BGW)-type inequality. Let us remind that Brézis–Gallouet [2], and Brézis–Wainger [3] considered the relation between $L^\infty$, $W^{k,r}$, and $W^{s,p}$, and proved that there holds

$$\|f\|_{L^\infty} \leq C \left(1 + \log \frac{\|f\|_{W^{s,p}}}{\|f\|_{W^{k,r}}} \right), \quad sp > n$$

(1.1)

provided that $\|f\|_{W^{k,r}} \leq 1$, for $kr = n$. Its application is to prove the existence of solutions to the nonlinear Schrödinger equations, see details in [2]. We also note that an alternative proof of (1.1) was given by H. Engler [4] for any bounded set in $\mathbb{R}^n$ with the cone condition. Similar embedding for vector functions $u$ with $\text{div} u = 0$ was investigated by Beale–Kato–Majda:

**E-mail addresses:** daonguyenanh@tdt.edu.vn (N.-A. Dao), quoc-hung.nguyen@sns.it (Q.-H. Nguyen).
\[ \| \nabla u \|_{L^n} \leq C \left( 1 + \| \text{rot} u \|_{L^n} \left( 1 + \log (1 + \| u \|_{W^{s, p}}) \right) + \| \text{rot} u \|_{L^2} \right), \]

for \( sp > n \), see \cite{1} (see also \cite{9} for an improvement of \eqref{2} in a bounded domain). An application of \eqref{2} is to prove the breakdown of smooth solutions to the 3-D Euler equations. After that, estimate \eqref{2} was enhanced by Kozono and Taniuchi \cite{5} in that \( \| \nabla u \|_{L^n} \) can be relaxed to \( \| \text{rot} u \|_{\text{BMO}} \):

\[ \| \nabla u \|_{L^n} \leq C \left( 1 + \| \text{rot} u \|_{\text{BMO}} \left( 1 + \log (1 + \| u \|_{W^{s+1, p}}) \right) \right). \]

To obtain \eqref{3}, Kozono–Taniuchi \cite{5} proved a logarithmic Sobolev inequality in terms of BMO norm and Sobolev norm, in which, for any \( 1 < p < \infty \), and for \( s > n/p \), there is a constant \( C = C(n, p, s) \) such that the estimate

\[ \| f \|_{L^n} \leq C \left( 1 + \| f \|_{\text{BMO}} \left( 1 + \log^+ (\| f \|_{W^{s, p}}) \right) \right) \]

holds for all \( f \in W^{s, p} \). Obviously, \eqref{4} is a generalization of \eqref{1}.

Besides, it is interesting to note that a Gagliardo–Nirenberg-type inequality with critical Sobolev space directly yields a BGW-type inequality. For example, H. Kozono and H. Wadade \cite{6} proved the Gagliardo–Nirenberg-type inequalities for the critical case and the limiting case of a Sobolev space as follows:

\[ \| u \|_{L^p} \leq C_{n} q^\frac{1}{p} \| u \|_{L^p} \| (\nabla)^s \tilde{u} \|_{L^r}^\frac{1}{r} \]

holds for all \( u \in L^p \cap H^\frac{s}{r} \) with \( 1 \leq p < \infty, 1 < r < \infty \), and for all \( q \) with \( p \leq q < \infty \) (see also Ozawa \cite{10}).

Also,

\[ \| u \|_{L^p} \leq C_{n} q^\frac{1}{p} \| u \|_{L^p} \| u \|_{\text{BMO}} \]

holds for all \( u \in L^p \cap \text{BMO} \) with \( 1 \leq p < \infty \), and for all \( q \) with \( p \leq q < \infty \).

As a result, \eqref{5} implies

\[ \| u \|_{L^n} \leq C \left( 1 + \| u \|_{L^p} + \| (\nabla)^s \tilde{u} \|_{L^r} \right) \left( \log (1 + \| (\nabla)^s \tilde{u} \|_{L^r}) \right)^\frac{1}{2} \]

for every \( 1 \leq p < \infty, 1 < r < \infty, 1 < q < \infty \) and \( n/q < s < \infty \).

Furthermore, \eqref{6} yields

\[ \| u \|_{L^n} \leq C \left( 1 + \| u \|_{L^p} + \| u \|_{\text{BMO}} \log (1 + \| (\nabla)^s \tilde{u} \|_{L^r}) \right) \]

for every \( 1 \leq p < \infty, 1 < q < \infty \), and \( n/q < s < \infty \).

Thus, \eqref{7} and \eqref{8} may be regarded as generalizations of the BGW inequality. Note that, in \eqref{7} and \eqref{8}, the logarithm term only contains the semi-norm \( \| u \|_{W^{s, p}} \).

Furthermore, Kozono, Ogawa, Taniuchi \cite{7} proved the logarithmic Sobolev inequalities in Besov space, generalizing the BGW inequality and the Beale–Kato–Majda inequality.

Motivated by the above results, we study in this paper the BGW-type inequality by means of the BMO norm, the fractional Sobolev norm, and the \( \hat{C}^\eta \) norm, for \( \eta \in (0, 1) \). Then, our first result is as follows.

**Theorem 1.1.** Let \( \eta \in (0, 1) \) and \( \alpha \in (0, n) \). Then, there exists a constant \( C = C(n, \eta) > 0 \) such that the estimate

\[ \| f \|_{L^n} \leq C + C \| f \|_{\text{BMO}} \left( 1 + \log^+ \left[ \sup_{z \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(y)|}{(|z - y| + 1)^\alpha} dy + \| f \|_{\hat{C}^\eta} \right] \right) \]

holds for all \( f \in \hat{C}^\eta \cap \text{BMO} \). We accept the notation \( \log^+ s = \log s \) if \( s \geq 1 \), and \( \log^+ s = 0 \) if \( s \in (0, 1) \).

**Remark 1.2.** It is clear that \( \left( \sup_{z \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(y)|}{(|z - y| + 1)^\alpha} dy \right) \) is finite if \( f \in L^1 \). On the other hand, if \( f \in L^r, r > 1 \), then for any \( \alpha \in (\frac{1}{r}, n) \), we have

\[ \sup_{z \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(y)|}{(|z - y| + 1)^\alpha} dy \leq C \| f \|_{L^r}, \]

where the constant \( C \) is independent of \( f \).
Remark 1.3. If $\text{supp } f \subset B_r$, then (1.9) implies
\[
\|f\|_{L^\infty} \leq C + C \|f\|_{BMO} \left(1 + \log^+ \left[ R^{n-\alpha+\eta} + \|f\|_{C^\eta} \right] \right).
\] (110)

**Remark 1.4.** Note that if $f \in W^{s,p}$ with $sp > n$, then (1.9) is stronger than (1.4) since $W^{s,p} \subset C^{0,\eta} \subset C^\eta$, with $\eta = \frac{sp-n}{p}$.

Concerning the BGW-type inequality involving the fractional Sobolev space, we have the following result.

**Theorem 1.5.** Let $s > 0$, $p \geq 1$ be such that $sp = n$. Let $\alpha > 0$, $\eta \in (0, 1)$. Then, there exists a constant $C = C(n, s, p, \eta, \alpha) > 0$ such that the estimate
\[
\|f\|_{L^\infty} \leq C + C \|f\|_{\dot{W}^{s,p}} \left(1 + \left( \log^+ \left[ R^{n-\alpha+\eta} + \|f\|_{C^\eta} \right] \right)^{\frac{p-1}{p}} \right)
\] (111)
holds for all $f \in C^\eta \cap \dot{W}^{s,p}$, where $\dot{W}^{s,p}$ is the homogeneous fractional Sobolev space, see its definition below.

**Remark 1.6.** As Remark 1.4, we can see that (1.11) is stronger than (1.1). Furthermore, if $f \subset B_r$, then (1.9) implies
\[
\|f\|_{L^\infty} \leq C + C \|f\|_{\dot{W}^{s,p}} \left(1 + \left( \log^+ \left[ R^{n-\alpha+\eta} + \|f\|_{C^\eta} \right] \right)^{\frac{p-1}{p}} \right).
\] (112)

**Remark 1.7.** We consider $f_\delta(x) = -\log(|x| + \delta)\psi(|x|)$, where $\psi \in C^1_c([0, \infty))$, $0 \leq \psi \leq 1$, $\psi(|x|) = 1$ if $|x| \leq \frac{1}{4}$, and $\delta > 0$ is small enough. It is not hard to see that, for any $\delta > 0$ small enough,
\[
\|f_\delta\|_{L^\infty} \sim |\log(\delta)|, \quad \|f_\delta\|_{BMO} \sim 1, \quad \|f_\delta\|_{\dot{W}^{1,p}} \sim |\log(\delta)|^{\frac{1}{p}},
\]
and
\[
\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f_\delta(y)|}{(|y| + 1)^\alpha} \, dy \sim 1, \quad \|f_\delta\|_{C^\alpha} \lesssim \delta^{-1}.
\]
Therefore, the powers $1$ and $\frac{p-1}{p}$ of the term $\log^+ \left[ R^{n-\alpha+\eta} + \|f\|_{C^\eta} \right]$ in (1.9) and (1.11), respectively, are sharp, so there are no such estimates of the form:
\[
\|f_1\|_{\infty} \leq C + C \|f_1\|_{BMO} \left(1 + \left( \log^+ \left[ \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f_1(y)|}{(|y| + 1)^\alpha} \, dy + \|f_1\|_{C^\eta} \right] \right)^{\gamma} \right),
\]
and
\[
\|f_2\|_{L^\infty} \leq C + C \|f_2\|_{\dot{W}^{1,p}} \left(1 + \left( \log^+ \left[ \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f_2(y)|}{(|y| + 1)^\alpha} \, dy + \|f_2\|_{C^\eta} \right] \right)^{\gamma^{\frac{p-1}{p}}} \right),
\]
hold for all $f_1 \in BMO \cap C^\eta$, $f_2 \in C^\eta \cap \dot{W}^{1,p}$, for some $\gamma \in (0, 1)$.

Before closing this section, let us introduce some functional spaces that we use throughout this paper. First of all, we recall $C^\eta$, $\eta \in (0, 1)$, as the homogeneous Hölder continuous of order $\eta$, endowed with the semi-norm:
\[
\|f\|_{C^\eta} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\eta}}.
\]
Next, if $s \in (0, 1)$, then we recall $\dot{W}^{s,p}$ the homogeneous fractional Sobolev space, endowed with the semi-norm:
\[
\|f\|_{\dot{W}^{s,p}} = \left( \int \int_{\mathbb{R}^{2n}} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \right)^{\frac{1}{p}}.
\]
When $s > 1$, and $s$ is not an integer, we denote $\dot{W}^{s,p}$ as the homogeneous fractional Sobolev space endowed with the semi-norm:

$$\| f \|_{\dot{W}^{s,p}} = \sum_{|\alpha| = |s|} \| D^\alpha f \|_{L^p}.$$  

If $s$ is an integer, then

$$\| f \|_{\dot{W}^{s,p}} = \sum_{|\alpha| = |s|} \| D^\alpha f \|_{L^2}.$$  

We refer to [8] for details on the fractional Sobolev space.

After that, we accept the notation $(f)_{\Omega} := \int f = \frac{1}{|\Omega|} \int f(x)dx$ for any Borel set $\Omega$. Finally, $C$ is always denoted as a constant that can change from line to line. And $C(k, n, l)$ means that this constant merely depends on $k, n, l$.

2. Proof of the theorems

We first prove Theorem 1.1.

**Proof of Theorem 1.1.** It is enough to prove that

$$|f(0)| \leq C + C \| f \|_{BMO} \left( 1 + \log_2^+ \left( \int_{\mathbb{R}^n} \frac{|f(y)|}{(|y| + 1)^\alpha} dy + \| f \|_{C^\alpha} \right) \right). \quad (2.1)$$

Let $m_0 \in \mathbb{N}$, set $B_\rho := B_\rho(0)$, we have

$$|f(0)| = \int_{B_{2^{-m_0}}} f - \int_{B_{2^{-m_0}}} f + \sum_{j = -m_0}^{m_0 - 1} \int_{B_{2^{j+1}}} f - \int_{B_{2^j}} f \right) + \int_{B_{2^{m_0}}} f \right)$$

$$\leq \int_{B_{2^{-m_0}}} |f - f(0)| + \sum_{j = -m_0}^{m_0 - 1} \int_{B_{2^{j+1}}} |f - f|_{B_{2^j}} + C2^{-m_0(n-\alpha)} \int_{B_{2^{m_0}}} \frac{|f(y)|}{(|y| + 1)^\alpha} dy$$

$$\leq \int_{B_{2^{-m_0}}} |y|^\alpha \| f \|_{C^\alpha} dy + 2m_0 \| f \|_{BMO} + C2^{-m_0(n-\alpha)} \int_{B_{2^{m_0}}} \frac{|f(y)|}{(|y| + 1)^\alpha} dy$$

$$\leq C2^{-m_0 \min(n-\alpha, l)} \left( \int_{\mathbb{R}^n} \frac{|f(y)|}{(|y| + 1)^\alpha} dy + \| f \|_{C^\alpha} \right) + Cm_0 \| f \|_{BMO}.$$  

Choosing

$$m_0 = \left[ \frac{\log_2^+ \left( \int_{\mathbb{R}^n} \frac{|f(y)|}{(|y| + 1)^\alpha} dy + \| f \|_{C^\alpha} \right) \right] + 1,$$

we get (2.1). The proof is complete. □

Next, we prove Theorem 1.5.

**Proof of Theorem 1.5.** To prove it, we need the following lemmas.

**Lemma 2.1.** Let $a_0 = 1$, and let $(a_0, a_1, ..., a_{k+1}) \in \mathbb{R}^{k+2}$. For any $k \geq 1$, be a unique solution to the following system:

$$\sum_{j=0}^{k+1} a_j 2^{jl} = 0, \quad \forall l = 0, ..., k.$$  

(2.2)
Then we have:
\[ a := \sum_{j=0}^{k} (k - j + 1)a_j \neq 0. \]  
(2.3)

Moreover, for any \( m \geq 1 \), and for \( b, b_i \in \mathbb{R}, l = -m, \ldots, m \), we have
\[ \sum_{l=-m}^{m-1} \left[ \sum_{j=0}^{k-1} a_j b_{j+l} \right] = \sum_{l=-m}^{m} \left[ \sum_{j=0}^{k-1} a_j \right] b_l + \sum_{l=-m}^{m} \left[ \sum_{j=0}^{m} a_j \right] (b_l - b) + ab. \]  
(2.4)

As a result, we obtain
\[ |b| \leq \frac{1}{|a|} \left| \sum_{j=0}^{k+1} |a_j| \right| \left| \sum_{l=-m}^{m-1} b_l \right| + \frac{1}{|a|} \left| \sum_{l=-m}^{m} \left[ \sum_{j=0}^{k+1} a_j b_{j+l} \right] \right| + \frac{1}{|a|} \left| \sum_{l=-m}^{m} \left[ \sum_{j=0}^{k+1} a_j \right] b_l \right|. \]  
(2.5)

**Proof.** First of all, we note that \( a_j \neq 0 \), for \( j = 0, \ldots, k + 1 \). Set
\[ Q(x) = \sum_{j=0}^{k+1} a_j x^j. \]

Then,
\[ Q'(1) = \sum_{j=1}^{k+1} ja_j. \]

On the other hand, by (2.2), we have \( Q(2^l) = 0 \), for \( l = 0, \ldots, k \). Thus,
\[ Q(x) = a_{k+1} \prod_{l=0}^{k} (x - 2^l), \quad \text{and} \quad Q'(1) = \prod_{l=1}^{k} (1 - 2^l). \]

This implies
\[ \sum_{j=1}^{k+1} ja_j = \prod_{j=1}^{k} (1 - 2^l) \neq 0. \]  
(2.6)

Next, we observe that
\[ 0 = (k + 1) \sum_{j=0}^{k+1} a_j = a + \sum_{j=1}^{k+1} ja_j = 0. \]

The last equation and (2.6) yield \( a = -\prod_{j=1}^{k} (1 - 2^j) \neq 0. \)

Now, we prove (2.4). We denote by LHS (resp. RHS) the left-hand side (resp. the right-hand side) of (2.4). It is not difficult to verify that
\[ \sum_{l=-m}^{m-1} \left[ \sum_{j=0}^{m} a_j \right] b_l = ab. \]

Then, a direct computation shows
\[ \text{RHS} = a_0 b_{-m} + (a_0 + a_1) b_{-m+1} + \ldots + (a_0 + \ldots + a_k) b_{-m} \\
+ (a_1 + \ldots + a_{k+1}) b_m + (a_2 + \ldots + a_{k+1}) b_{m+1} + \ldots + a_{k+1} b_{k+m} = a_0 \sum_{l=-m}^{k-m} b_l \\
+ a_1 \left( \sum_{l=-m}^{m} b_l + \sum_{l=m}^{m} b_l \right) + \ldots + a_k \left( \sum_{l=-m}^{m-k} b_{k-m} + \sum_{l=m}^{m+k-1} b_l \right) + a_{k+1} \left( \sum_{l=-m}^{m+k} b_l \right). \]
Note that \( \left( \sum_{j=0}^{k+1} a_j \right) \sum_{l=k+1-m}^{m-1} b_l = 0 \). Thus,

\[
\begin{align*}
RHS &= RHS + \left( \sum_{j=0}^{k+1} a_j \right) \sum_{l=k+1-m}^{m-1} b_l \\
&= \sum_{j=0}^{k+1} a_j \left( \sum_{l=j-m}^{j+m-1} b_l \right) \\
&= \sum_{l=m}^{k+m} \left( \sum_{j=l-m+1}^{k+1} a_j \right) b_l + \sum_{l=m}^{k-m} \left( \sum_{j=0}^{k+1} a_j \right) b_l \\
&= \sum_{l=m}^{k+m} \left( \sum_{j=l-m+1}^{k+1} a_j \right) b_l + \sum_{l=-m}^{k-m} \left( \sum_{j=0}^{k+1} a_j \right) b_l \\
&= LHS.
\end{align*}
\]

We get (2.4).

Finally, (2.5) follows from (2.4) by using the triangle inequality. In other words, we get Lemma 2.1. \( \square \)

Next, we have the following lemma.

**Lemma 2.2.** Assume \( a_0, a_1, \ldots, a_{k+1} \) as in Lemma 2.1. Let \( \Omega_j = B_{2j+1} \setminus B_{2j} \), where \( B_\rho := B_\rho (0) \) for any \( \rho > 0 \). Then, there holds:

\[
\left| \sum_{j=0}^{k+1} a_j \right|_{\Omega_j} \leq C \int_{B_{2k+3} \setminus B_{2k-1}} \left( D^k f (y) - \left( D^k f \right)_{B_{2k+3} \setminus B_{2k-1}} \right) dy.
\]

(2.7)

For any \( l \in \mathbb{R} \), we set \( E_l = B_{2l+3} \setminus B_{2l-1} \). As a consequence of (2.7), we obtain:

\[
\left| \sum_{j=0}^{k+1} a_j \right|_{\Omega_{j+l}} \leq C 2^l \int_{E_l} \int_{E_l} \left| D^k f (y) - D^k f (y') \right| dy dy'.
\]

(2.8)

Moreover, by the triangle inequality, we get from (2.8):

\[
\left| \sum_{j=0}^{k+1} a_j \right|_{\Omega_{j+l}} \leq C 2^l \int_{E_l} \left| D^k f (y) \right| dy.
\]

(2.9)

**Proof.** Assume that (2.7) is not true, which is a contradiction. There exists then a sequence \( (f_m)_{m \geq 1} \subset W^{k,1}(B_{2k+3} \setminus B_{2k-1}) \) such that

\[
\int_{B_{2k+3} \setminus B_{2k-1}} \left| D^k f_m (y) - \left( D^k f_m \right)_{B_{2k+3} \setminus B_{2k-1}} \right| dy \leq \frac{1}{m},
\]

(2.10)

and

\[
\left| \sum_{j=0}^{k+1} a_j \right|_{\Omega_j} f_m | = 1, \ \forall m \geq 1.
\]

Let us put

\[
\tilde{f}_m (x) = f_m (x) - P_{k,m} (x), \ \text{with} \ P_{k,m} (x) = \sum_{l=0}^{k} \sum_{\alpha_1 + \ldots + \alpha_n = l} c_{l,k,m} (\alpha_1, \ldots, \alpha_n) x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}.
\]
and $c_{i,k,m}(\alpha_1, ..., \alpha_n)$ is a constant such that
\[
\left( D_l^k \tilde{f}_m \right)_{B_{2^{k+3}} \setminus B_2} = 0, \quad \forall \; l = 0, ..., k.
\] (2.11)

By a sake of brief, we denote $c_{i,m} = c_{i,k,m}(\alpha_1, ..., \alpha_n)$. Since $p_{k,m}$ is a polynomial of at most degree $k$, then $D^k p_{k,m} = \text{const.}$ This, (2.10), and (2.11) imply
\[
\int_{B_{2^{k+3}} \setminus B_2} |D^k \tilde{f}_m(y)| \, dy = \int_{B_{2^{k+3}} \setminus B_2} |D^k f_m(y) - \left( D^k f_m \right)_{B_{2^{k+3}} \setminus B_2} | \, dy \leq \frac{1}{m}.
\]

It follows from the compact embeddings that there exists a subsequence of $(\tilde{f}_m)_{m \geq 1}$ (still denoted as $(\tilde{f}_m)_{m \geq 1}$) such that $\tilde{f}_m \to \tilde{f}$ strongly in $L^1(B_{2^{k+3}} \setminus B_2)$, and
\[
D^k \tilde{f} = 0, \quad \text{in} \; B_{2^{k+3}} \setminus B_2.
\]

This implies that $\tilde{f}$ is a polynomial of at most degree $(k - 1)$, i.e.: \[
\tilde{f}(x) = \sum_{l=0}^{k-1} \sum_{\alpha_1 + \ldots + \alpha_n = l} c_{l,k}(\alpha_1, ..., \alpha_n) x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}, \quad \forall \; x \in B_{2^{k+3}} \setminus B_2.
\]

On the other hand, we observe that, for any $l = 0, ..., k$,
\[
\sum_{j=0}^{k+1} a_j \int_{\Omega_j} \sum_{\alpha_1 + \ldots + \alpha_n = l} c(\alpha_1, ..., \alpha_n) x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \, dx_1 \, dx_2 \ldots \, dx_n
\]
\[
= \sum_{j=0}^{k+1} a_j \int_{\Omega_j} \sum_{\alpha_1 + \ldots + \alpha_n = l} c(\alpha_1, ..., \alpha_n)(2^j x_1)^{\alpha_1} (2^j x_2)^{\alpha_2} \ldots (2^j x_n)^{\alpha_n} \, dx_1 \, dx_2 \ldots \, dx_n
\]
\[
= \sum_{j=0}^{k+1} a_j \left( \sum_{j=0}^{k+1} a_j 2^{jl} \right) x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \, dx_1 \, dx_2 \ldots \, dx_n = 0,
\]
by (2.2). This implies
\[
\sum_{j=0}^{k+1} a_j \int_{\Omega_j} \tilde{f} = 0, \quad \text{(2.12)}
\]
and
\[
\sum_{j=0}^{k+1} a_j \int_{\Omega_j} \tilde{f}_m = \sum_{j=0}^{k+1} a_j \int_{\Omega_j} f_m = 1.
\]

Remind that $\tilde{f}_m \to \tilde{f}$ strongly in $L^1(B_{2^{k+3}} \setminus B_2)$; then we have
\[
\sum_{j=0}^{k+1} a_j \int_{\Omega_j} \tilde{f} = 1.
\]

Now we complete the proof of (2.7).

The proof of (2.8) (resp. (2.9)) is trivial; we leave it to the reader. This puts an end to the proof of Lemma 2.2. \hfill \Box

Now, we are ready to prove Theorem 1.5.

It is enough to show that
\[
|f(0)| \leq C + C \|f\|_{W^{s,p}} \left( 1 + \log^2 \left( \int_{\mathbb{R}^n} \frac{|f(y)|}{|y| + 1} \, dy + \|f\|_{C^2} \right) \right)^{\frac{p-1}{p}}.
\] (2.13)
Set $s_1 = s - k$, $s_1 \in [0, 1)$. Then, we divide our study into the two cases.

**i) Case 1**: $s_1 \in (0, 1)$. We apply Lemma 2.1 with $b = f(0)$, $b_j = \int f$. Then, for any $m_0 \geq 1$, there is a constant $C = C(k) > 0$ such that

$$|f(0)| \leq C \left( \sum_{l=-m_0}^{k-m_0} \left| \sum_{l=-m_0}^{k-1} a_j f \right| + \sum_{l=-m_0}^{k+1} \left| \sum_{l=-m_0}^{k+m_0} \int f \right| \right). \quad (2.14)$$

Concerning the first term on the right-hand side of (2.14), we have

$$\sum_{l=-m_0}^{k-m_0} \left| \sum_{l=-m_0}^{k-1} a_j f \right| \leq \sum_{l=-m_0}^{k-1} \int \left| f - f(0) \right| \leq \sum_{l=-m_0}^{k-1} \int_{\Omega_l} |x|^\eta \| f \|_{\mathcal{C}_\eta} \, dx.$$ 

Thus,

$$\sum_{l=-m}^{k-m} \left| \sum_{l=-m_0}^{k-1} a_j f \right| \leq \sum_{l=-m}^{k-1} 2^{(l+1)\eta} \| f \|_{\mathcal{C}_\eta} \leq C(\eta, k) 2^{-m\eta} \| f \|_{\mathcal{C}_\eta}, \quad (2.15)$$

Next, we use (2.8) in Lemma 2.2 to obtain

$$\sum_{l=-m}^{m-1} \sum_{j=0}^{k+1} a_j f \int f \left| D^k f(y) - D^k f(z) \right| dy \, dz \leq C \sum_{l=-m}^{m-1} 2^{k^2} \left| D^k f(y) - D^k f(z) \right| dy \, dz, \quad (2.16)$$

where $E_l = B_{2^k+1} \setminus B_{2^k-1}$. It follows from Hölder’s inequality:

$$\sum_{l=-m}^{m-1} 2^{k^2} \int \left| D^k f(y) - D^k f(z) \right| dy \, dz \leq C(n, p, k) 2^{k+1} \frac{m+1}{p}.$$ 

Since $y, z \in E_l$, we have $|y - z| \leq |y| + |z| \leq 2^{k+4}$. Thus, the right-hand side of the indicated inequality is less than

$$C(n, p, k) 2^{k+1} \frac{m+1}{p} \frac{m_0}{|E_l|^p} \sum_{l=-m_0}^{m_0-1} \left( \int \int \left| D^k f(y) - D^k f(z) \right|^p |y - z|^{|n+1|p} dy \, dz \right)^{1/p}.$$ 

Note that $n = sp = (k + s_1)p$, and $|E_l|^p \leq C(n, p, k) 2^{-k\frac{n}{p}}.$

Then, there is a constant $C = C(k, s, n) > 0$ such that

$$\sum_{l=-m_0}^{m_0-1} \left( \int \int \left| D^k f(y) - D^k f(z) \right|^p |y - z|^{|n+1|p} dy \, dz \right)^{1/p} \leq C \sum_{l=-m_0}^{m_0-1} \left( \int \int \left| D^k f(y) - D^k f(z) \right|^p |y - z|^{|n+1|p} dy \, dz \right)^{1/p}. \quad (2.17)$$

Thanks to the inequality

$$\sum_{j=-m_0}^{m_0-1} c_j \geq (2m_0)^{p-1} \left( \sum_{j=-m_0}^{m_0-1} c_j \right)^{1/p}, \quad (2.18)$$

we have
\[
\sum_{l=-m_0}^{m_0-1} \left( \int_{E_l} \int_{E_l} \frac{|D^k f(y) - D^k f(z)|^p}{|y-z|^{n+p} \eta} \, dy \, dz \right)^{\frac{1}{p}} \leq (2m_0)^{\frac{p-1}{p}} \left( \sum_{l=-m_0}^{m_0-1} \int_{E_l} \int_{E_l} \frac{|D^k f(y) - D^k f(z)|^p}{|y-z|^{n+p} \eta} \, dy \, dz \right)^{\frac{1}{p}}.
\]

(2.19)

Moreover, we observe that \( \sum_{l=-\infty}^{\infty} \chi_{E_l \times E_l}(y_1, y_2) \leq k + 4 \), for all \((y_1, y_2) \in \mathbb{R}^n \times \mathbb{R}^n \). Thus,

\[
\sum_{l=-m_0}^{m_0-1} \int_{E_l} \int_{E_l} \frac{|D^k f(y) - D^k f(z)|^p}{|y-z|^{n+p} \eta} \, dy \, dz \leq (k + 4) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|D^k f(y) - D^k f(z)|^p}{|y-z|^{n+p} \eta} \, dy \, dz.
\]

(2.20)

Combining (2.17), (2.19) and (2.20) yields

\[
\sum_{l=-m_0}^{m_0-1} 2^k \int_{E_l} \int_{E_l} |D^k f(y) - D^k f(z)| \, dy \, dz \leq C(k, s, n) m_0^{\frac{p-1}{p}} \| f \|_{W^{s,p} \eta}.
\]

(2.21)

It remains to treat the last term. Then, it is not difficult to see that, for any \( \alpha > 0 \),

\[
\sum_{l=m_0}^{k+m_0} \int_{\Omega_l} |f| \leq C(k, n) 2^{-m_0} \int_{B_{2^{k+m_0}}} |f| \leq C(k, n, \alpha) 2^{-m_0(n-\alpha)} \int_{B_{2^{k+m_0}}} \frac{|f(x)| \, dx}{(|x| + 1)^{\alpha}}.
\]

(2.22)

Inserting (2.15), (2.21), and (2.22) into (2.14) yields

\[
|f(0)| \leq C 2^{-m_0 \min(n-\alpha, \eta)} \left( \int_{\mathbb{R}^n} \frac{|f(y)|}{(|y| + 1)^{\alpha}} \, dy + \| f \|_{L^p} \right) + C m_0 \| f \|_{W^{s,p} \eta}.
\]

(2.23)

By choosing

\[
m_0 = \left[ \log_2 \left( \int_{\mathbb{R}^n} \frac{|f(y)|}{(|y| + 1)^{\alpha}} \, dy + \| f \|_{L^p} \right) \right] \min(n-\alpha, \eta) + 1,
\]

we obtain (2.13).

**ii) Case 2:** \( s_1 = 0 (s = k) \). The proof is similar to the one of the case \( s_1 \in (0, 1) \). There is just a difference of estimating the second term on the right-hand side of (2.14) as follows.

Using (2.9), we get:

\[
\sum_{l=-m_0}^{m_0-1} \sum_{j=0}^{k+1} \alpha_j \int_{\Omega_{l+j}} f \leq C \sum_{l=-m_0}^{m_0-1} 2^j \int_{E_l} |D^k f|.
\]

(2.24)

Applying Hölder’s inequality, we have
\[
\sum_{l=-m_0}^{m_0-1} 2^{kl} \int_{E_l} |D^k f| \leq \sum_{l=-m_0}^{m_0-1} 2^{kl} |E_l|^{1/p} \left( \int_{E_l} |D^k f|^p \right)^{1/p} \\
\leq C(n, k) \sum_{l=-m_0}^{m_0-1} \left( \int_{E_l} |D^k f|^p \right)^{1/p} \\
\leq C m_0^{\frac{p-1}{p}} \left( \sum_{l=-m_0}^{m_0-1} \int_{E_l} |D^k f|^p \right)^{1/p} .
\]

(2.25)

We utilize the fact \( \sum_{l=-\infty}^{\infty} \chi_{E_l}(y) \leq k + 4, \forall y \in \mathbb{R}^n \) again in order to get
\[
\left( \sum_{l=-m_0}^{m_0-1} \int_{E_l} |D^k f|^p \right)^{1/p} \leq (k + 4) \left( \int_{\mathbb{R}^n} |D^k f|^p \right)^{1/p} .
\]

(2.26)

From (2.26), (2.25), and (2.24), we get
\[
\sum_{l=-m_0}^{m_0-1} \sum_{j=0}^{k+1} a_j \int_{\Omega_{j+1}} f \leq C(k, n) \| f \|_{W^{s,p}} .
\]

(2.27)

Thus, we obtain another version of (2.23) as follows:
\[
|f(0)| \leq C 2^{-m_0 \min(n-\alpha, n)} \left( \int_{\mathbb{R}^n} \frac{|f(y)|}{(|y| + 1)^{\alpha}} dy + \| f \|_{C_0} \right) + C m_0^{\frac{p-1}{p}} \| f \|_{W^{s,p}} .
\]

(2.28)

By the same argument as above (after (2.23)), we get the proof of the case \( s_1 = 0 \). This completes the proof of Theorem 1.5. \( \square \)

References

[1] J.T. Beale, T. Kato, A. Majda, Remarks on the breakdown of smooth solutions for the 3-D Euler equations, Commun. Math. Phys. 94 (1984) 61–66.
[2] H. Brézis, T. Gallouet, Nonlinear Schrödinger evolution equations, Nonlinear Anal. 4 (1980) 677–681.
[3] H. Brézis, S. Wainger, A note on limiting cases of Sobolev embeddings and convolution inequalities, Commun. Partial Differ. Equ. 5 (1980) 773–789.
[4] H. Engler, An alternative proof of the Brézis-Wainger inequality, Commun. Partial Differ. Equ. 14 (1989) 541–544.
[5] H. Kozono, Y. Taniuchi, Limiting case of the Sobolev inequality in BMO with application to the Euler equations, Commun. Math. Phys. 214 (2000) 191–200.
[6] H. Kozono, H. Wadade, Remarks on Gagliardo–Nirenberg type inequality with critical Sobolev space and BMO, Math. Z. 295 (2008) 935–950.
[7] H. Kozono, T. Ogawa, Y. Taniuchi, The critical Sobolev inequalities in Besov spaces and regularity criterion to some semi-linear evolution equations, Math. Z. 242 (2002) 251–278.
[8] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, Bull. Soc. Math. Fr. 136 (2012) 521–573.
[9] T. Ogawa, Y. Taniuchi, On blow-up criteria of smooth solutions to the 3-D Euler equations in a bounded domain, J. Differ. Equ. 190 (2003) 39–63.
[10] T. Ozawa, On critical cases of Sobolev’s inequalities, J. Funct. Anal. 127 (1995) 259–269.