On the structure of $S_2$-ifications of complete local rings

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Assumption

\((R, m, k)\) is a commutative noetherian local ring that is complete, equidimensional, and unmixed with canonical module \(\omega\) and total ring of fractions \(Q(R)\).

Definition (Hochster and Huneke, ’94)

An \(R\)-subalgebra \(T \subseteq Q(R)\) is an \(S_2\)-ification of \(R\) if:

1. \(T\) is module finite and \((S_2)\) over \(R\); and
2. the inclusion \(R \rightarrow T\) is an isomorphism in codimension 2.

Fact (HH)

(a) \(R\) has a unique \(S_2\)-ification \(T\).
(b) If \(R\) is \((R_1)\), then \(T\) is the integral closure of \(R\) in \(Q(R)\).
(c) In general, one has \(T \cong \text{Hom}_R(\omega, \omega)\).
Local $S_2$-ifications

**Definition (HH)**

$Γ_R$ is the graph with vertex set $\text{Min}(R)$ such that distinct vertices $p$ and $q$ are adjacent in $Γ_R$ if and only if $\text{ht}_R(p + q) = 1$.

**Fact (HH)**

The following conditions are equivalent:

(i) $T$ is local;
(ii) $ω$ is indecomposable;
(iii) $H^{\dim(R)}_m(R)$ is indecomposable;
(iv) For every ideal $J$ of height at least two, $\text{Spec}(R) - V(J)$ is connected; and
(v) $Γ_R$ is connected.

**Question**

Can one similarly obtain more information about $m$-$\text{Spec}(T)$?
Maximal Ideals of $S_2$-ifications

**Theorem (SW-Spiroff)**

The following quantities are equal:

(i) $|m\text{-}\text{Spec}(T)|$;

(ii) the number of summands in an indecomposable decomposition of $\omega$;

(iii) the number of summands in an indecomposable decomposition of $H_{m}^{\dim(R)}(R)$;

(iv) the maximum number of components of $\text{Spec}(R) - V(J)$ where $J$ ranges through the ideals of height at least 2; and

(v) the number of connected components of $\Gamma_R$.

**Remark (Lyubeznik ‘06, Zhang ‘07)**

If $R$ is the completion of the strict henselization of an equicharacteristic local ring $A$, then the above quantity is also the top “Lyubeznik number” $\lambda_{d,d}(A)$. 

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### Examples of $\Gamma_R$

#### Definition

$\Gamma_R$ is the graph with vertex set $\text{Min}(R)$ such that distinct vertices $p$ and $q$ are adjacent in $\Gamma_R$ if and only if $\text{ht}_R(p + q) = 1$.

#### Example (complete graphs)

If $R$ is a hypersurface or $\text{dim}(R) \leq 1$, then $\Gamma_R = K_{|\text{Min}(R)|}$.

#### Example (2-vertex graphs)

\[
k[[X_1, X_2]]/(X_1 X_2)
\]

\[
(X_1) \quad \quad \quad \quad \quad \quad \quad \quad \quad (X_2)
\]

\[
k[[X_1, X_2, X_3, X_4]]/(X_1 X_2, X_2 X_3, X_3 X_4, X_1 X_4)
\]

\[
(X_1, X_3) \quad \quad (X_2, X_4)
\]
More Examples of $\Gamma_R$

**Example (paths)**

Let $n \geq 1$ and $R = k[[X_1, \ldots, X_n]]/J$ where

$$J = (X_1, X_2) \cap (X_2, X_3) \cap \cdots \cap (X_{n-1}, X_n).$$

Then $\Gamma_R \cong P_n$.

$$(X_1, X_2) \quad (X_2, X_3) \quad \cdots \quad (X_{n-1}, X_n)$$

**Example (cycles)**

Let $n \geq 3$ and $R = k[[X_1, \ldots, X_n]]/J$ where

$$J = (X_1, X_2) \cap (X_2, X_3) \cap \cdots \cap (X_{n-1}, X_n) \cap (X_n, X_1).$$

Then $\Gamma_R \cong C_n$. 
**Question**

How to decide whether a graph $G$ is of the form $\Gamma_R$?

**Definition (address labeling of $G$, intuitive version)**

Each vertex of $G$ is assigned a distinct “address” of $s$ distinct numbers from $[n] = \{1, \ldots, n\}$, so that two vertices are adjacent if and only if their addresses differ by exactly one number.

**Example (paths)**

$$\{1, 2\} \rightarrow \{2, 3\} \rightarrow \cdots \rightarrow \{n - 1, n\}$$
Theorem (SW-Spiroff)

If \( G \) admits an address labeling, then there is a complete local equidimensional unmixed ring \( R \) such that \( \Gamma_R \cong G \). Moreover, the ring \( R \) is of the form \( k[[X_1, \ldots, X_n]]/I \) where \( I \) is a square-free monomial ideal.

Example (paths)

\[
\begin{align*}
\{1, 2\} & \quad \{2, 3\} \quad \cdots \quad \{n-1, n\} \\
(X_1, X_2) & \quad (X_2, X_3) \quad \cdots \quad (X_{n-1}, X_n)
\end{align*}
\]

\( R = k[[X_1, \ldots, X_n]]/J \) where

\[
J = (X_1, X_2) \cap (X_2, X_3) \cap \cdots \cap (X_{n-1}, X_n).
\]
Graph Labelings, cont.

Example

The following graphs do not have address labelings.

Thus, one cannot realize these graphs as $\Gamma_R$ for any unmixed equidimensional monomial ideal. Note that the first graph is chordal and the second one is complete bipartite.

Question

Can these graphs be realized as $\Gamma_R$?
Notation

Let $G$ be a graph with vertex set $V$. Fix positive integers $n$ and $s$, and let $\binom{[n]}{s}$ denote the set of subsets of $[n]$ with cardinality $s$.

Definition

An address labeling of $G$ is an injective function $\phi : V \rightarrow \binom{[n]}{s}$, for some choice of $n$ and $s$ such that for all $v, w \in V$, we have $v$ adjacent to $w$ in $G$ if and only if $|\phi(v) \cap \phi(w)| = s - 1$, that is, if and only if $|\phi(v) \cup \phi(w)| = s + 1$. 