BISPECTRALITY OF KP SOLITONS

ALEX KASMAN

Abstract. It is by now well known that the wave functions of rational solutions to the KP hierarchy which can be achieved as limits of the pure \( n \)-soliton solutions satisfy an eigenvalue equation for ordinary differential operators in the spectral parameter. This property is known as “bispectrality” and has proved to be both interesting and useful. In this note, it is shown that certain (non-rational) soliton solutions of the KP hierarchy satisfy an eigenvalue equation for a non-local operator constructed by composing ordinary differential operators in the spectral parameter with translation operators in the spectral parameter, and therefore have a form of bispectrality as well. Considering the results relating ordinary bispectrality to the self-duality of the rational Calogero-Moser particle system, it seems likely that this new form of bispectrality should be related to the duality of the Ruijsenaars system.

1. Introduction

1.1. The KP Hierarchy and Bispectrality. Let \( \mathbb{D} \) be the vector space spanned over \( \mathbb{C} \) by the set

\[ \{ \Delta(\lambda, n) \mid \lambda \in \mathbb{C}, n \in \mathbb{N} \} \]

whose elements differentiate and evaluate functions of the variable \( z \):

\[ \Delta(\lambda, n)[f(z)] := f^{(n)}(\lambda). \]

The elements of \( \mathbb{D} \) are thus finitely supported distributions on appropriate spaces of functions in \( z \). For lack of a better term, we will continue to call them distributions even though their main use in this paper will be their application to functions of two variables. (Such distributions were called “conditions” in [17] since a KP wave function was specified by requiring that it be in their kernel.) Note that if \( c \in \mathbb{D} \) and \( f(x, z) \) is sufficiently differentiable in \( z \) on the support of \( c \), then \( f(x) = c[f(x, z)] \) is a function of \( x \) alone. Furthermore, note that one may “compose” a distribution with a function of \( z \), i.e. given \( c \in \mathbb{D} \) and \( f(z) \) (sufficiently differentiable on the support of \( c \) then there exists a \( c' := c \circ f \in \mathbb{D} \) such that

\[ c'(g(z)) = c(f(z)g(z)) \quad \forall g. \]

The subspaces of \( \mathbb{D} \) can be used to generate solutions to the KP hierarchy [14] in the following way. Let \( C \subset \mathbb{D} \) be an \( n \) dimensional subspace with basis \( \{ c_1, \ldots, c_n \} \). Then, if \( K = K_C \) is the unique, monic ordinary differential operator in \( x \) of order \( n \) having the functions \( c_i(e^{xz}) \) in its kernel (see [15]) we define \( L_C = K \frac{1}{n!} K^{-1} \) and \( \psi_C = \frac{1}{\Delta^2} Ke^{xz} \). The connection to integrable systems comes from the fact that adding dependence to \( C \) on a sequence of variables \( t_j \ (j = 1, 2, \ldots) \) by letting \( C(t_j) \) be the space with basis

\[ \{ c_1 \circ e^{\sum -t_j z^j}, c_2 \circ e^{\sum -t_j z^j}, \ldots, c_n \circ e^{\sum -t_j z^j} \} \]
it follows that the “time dependent” pseudo-differential operator $L = L(t)$ satisfies the equations of the KP hierarchy \cite{2, 9, 10, 13, 17}.

$$\frac{\partial}{\partial t_j} L = ([L^j]_+, L).$$

The wave function $\psi_C(x, z)$ generates the corresponding subspace of the infinite dimensional grassmannian $Gr$ \cite{15} which parametrizes KP solutions and thus it is not difficult to see that this construction produces precisely those solutions associated to the subgrassmannian $Gr_1 \subset Gr$ \cite{15, 17}.

Moreover, the ring $AC = \{ p \in \mathbb{C}[z] | c_i \circ p \in C \ 1 \leq i \leq n \}$ is necessarily non-trivial (i.e. contains non-constant polynomials) and the operator $L_p = p(L)$ is an ordinary differential operator for every $p \in AC$ and satisfies

$$L_p \psi_C(x, z) = p(z) \psi_C(x, z). \quad (1)$$

The subject of this paper is the existence of additional eigenvalue equations satisfied by $\psi_C(x, z)$. In particular, we wish to consider the question of whether there exists an operator $\Lambda$ acting on functions of the variable $z$ such that

$$\Lambda \psi_C(x, z) = \pi(x) \psi_C(x, z) \quad (2)$$

where $\pi(x)$ is a non-constant function of $x$. For example, the following theorem is due to G. Wilson in \cite{17}:

**Theorem 1.1.** In addition to (1) the wave function $\psi_C(x, z)$ is also an eigenfunction for a ring of ordinary differential operators in $z$ with eigenvalues depending polynomially on $x$ if and only if $C$ has a basis of distributions each of which is supported only at one point.

In other words, for this special class of KP solutions for which the coefficients of $L$ are rational functions of $x$, the wave function $\psi_C$ satisfies an additional eigenvalue equation of the form (2) where $\Lambda$ is an ordinary differential operator in $z$ and $\pi(x)$ a non-constant polynomial in $x$. Together (1) and (2) are an example of bispectrality \cite{5, 7}. The bispectral property is already known to be connected to other questions of physical significance such as the time-band limiting problem in tomography \cite{3}. Huygens’ principle of wave propagation \cite{4}, quantum integrability \cite{8, 16} and, especially in the case described above, the self duality of the Calogero-Moser particle system \cite{10, 17, 18}.

It is known that the only subspaces $C$ for which the corresponding wave function satisfies (1) and (2) with $L_p$ and $\Lambda$ ordinary differential operators in $x$ and $z$ respectively are those described in Theorem 1.1. However, suppose we allow $\Lambda$ to involve not only differentiation and multiplication in $z$ but also translation in $z$ and call this more general situation t-bispectrality\cite{2}. It will be shown below that there are more KP solutions which are bispectral in this sense. In particular, a class of (non-rational) $n$-soliton solutions of the KP hierarchy will be shown to be t-bispectral.

\footnote{Moreover, he demonstrated that up to conjugation or change of variables, the operators $L_p$ found in this way are the only bispectral operators which commute with differential operators of relatively prime order, but this fact will not play an important role in the present note.}

\footnote{It should be noted that the term “bispectrality” already applies to more general situations than simply differential operators \cite{4}, but in the case of the KP hierarchy I believe only differential bispectrality has thus far been considered.}
1.2. Notation. Using the shorthand notation $\partial = \frac{\partial}{\partial x}$ any ordinary differential operator in $x$ can be written as

$$L = \sum_{i=0}^{N} f_i(x) \partial^i \quad (N \in \mathbb{N}).$$

All ordinary differential operators considered in this note will have only coefficients that are rational functions of $x$ and of functions of the form $e^{\lambda x}$. Similarly, we will write $\partial_z = \frac{\partial}{\partial z}$ but will need to consider only differential operators in $z$ with rational coefficients.

For any $\lambda \in \mathbb{C}$ let $S_\lambda = e^{\lambda \partial_z}$ be the translational operator acting on functions of $z$ as

$$S_\lambda(f(z)) = f(z + \lambda).$$

Then consider the ring of translational-differential operators $T$ generated by these translational operators and ordinary differential operators in $z$. Any translational-differential operator $\hat{T} \in T$ can be written as

$$\hat{T} = \sum_{i=1}^{N} p_i(z, \partial_z)S_{\lambda_i}$$

where $p_i$ are ordinary differential operators in $z$ with rational coefficients and $N \in \mathbb{N}$. Note that the ring of ordinary differential operators in $z$ with rational coefficients is simply the subring of $T$ of all elements which can be written as $pS_0$ for a differential operator $p$.

2. Translational Bispectrality of KP Solutions

Let us say that a finite dimensional subspace $C \subset \mathbb{D}$ is $t$-bispectral if there exists a translational-differential operator $\hat{\Lambda} \in T$ satisfying equation (2) for the corresponding KP wave function $\psi_C(x, z)$. By Theorem 1.1 and the fact that the ring of rational coefficient ordinary differential operators in $z$ is contained in $T$, we know that $C$ is $t$-bispectral if it has a basis of point supported distributions. In order to prove that there are other $t$-bispectral subspaces $C$, this section will determine a necessary and sufficient condition for $t$-bispectrality.

It will be convenient for us to refer to the following easily verified facts:

**Lemma 2.1.** A translational-differential operator $\hat{T} \in T$ satisfies $\hat{T}(e^{xz}) \equiv 0$ if and only if $T \equiv 0$ is the zero operator.

**Lemma 2.2.** There exists a translational-differential operator $\hat{T} \in T$ satisfying $\hat{T}(e^{xz}) = \pi(x)e^{xz}$ for some $z$-independent function $\pi(x)$ if and only if $\hat{T}$ is a constant coefficient operator and

$$\pi(x) = \sum_{i=1}^{N} p_i(x)e^{\lambda_i x} \quad p_i \in \mathbb{C}[x], \ \lambda_i \in \mathbb{C}. \quad (3)$$

**Lemma 2.3.** Two translational-differential operators $\hat{\Lambda}_1, \hat{\Lambda}_2 \in T$ satisfy

$$\hat{\Lambda}_1(e^{xz}) = \pi(x)\hat{\Lambda}_2(e^{xz}) \quad (4)$$

...and that the ring $A_C$ is bispectral in the sense of [15].
if and only if there exist constant coefficient operators $\hat{\Gamma}_1, \hat{\Gamma}_2 \in T$ such that
\begin{equation}
\hat{\Lambda}_1 \circ \hat{\Gamma}_1 = \hat{\Lambda}_2 \circ \hat{\Gamma}_2.
\end{equation}

**Proof.** First suppose that (4) is satisfied. Then, since
\[ \pi(x) = \frac{\hat{\Gamma}_2(e^{xz})}{\hat{\Gamma}_1(e^{xz})} \]
it is necessarily of the form $\pi(x) = \hat{\Pi}_i(x)$ where $\pi_i(x)$ are both functions of the form (3). Then let $\hat{\Gamma}_i$ be the constant coefficient translational-differential operator such that $\hat{\Pi}_i(x) = e^{-xz} \hat{\Gamma}_i(e^{xz})$. Then, the operator $\hat{T}_0 = \hat{\Lambda}_1 \circ \hat{\Gamma}_1 - \hat{\Lambda}_2 \circ \hat{\Gamma}_2$ satisfies
\[ \hat{T}_0(e^{xz}) = \pi_1(x) \frac{\pi_2(x)}{\pi_1(x)} \hat{\Lambda}_2(e^{xz}) - \pi_2(x) \hat{\Lambda}_2(e^{xz}) = 0 \]
and so (5) follow from Lemma 2.1.

Conversely, it is clear that if (5) is satisfied then $\hat{\Lambda}_1(e^{xz}) = 1$, $\hat{\Lambda}_2(e^{xz}) = 1$ and
\[ \pi_1(x) \frac{\pi_2(x)}{\pi_1(x)} \hat{\Lambda}_2(e^{xz}) - \pi_2(x) \hat{\Lambda}_2(e^{xz}) = 0 \]
follow from Lemma 2.1.

Using these lemmas, it is possible to prove the following statement which characterizes those subspaces $C$ which can be t-bispectral.

**Theorem 2.1.** The operator $\hat{\Lambda}$ satisfies (4) if and only if there exist constant coefficient translational-differential operators $\hat{\Gamma}_1, \hat{\Gamma}_2 \in T$ such that
\begin{equation}
\hat{\Lambda} \circ \hat{K}_C \circ \Gamma_1 = \hat{K}_C \circ \Gamma_2.
\end{equation}
Proof. Supposing that (8) holds, it follows from Lemmas 2.3 and 2.4 that
\[ \hat{\Lambda} \psi_C(x, z) = \frac{1}{\tau_C(x)} \hat{\Lambda} \circ \hat{K}_C(e^{xz}) = \frac{\pi(x)}{\tau_C(x)} \hat{K}_C(e^{xz}) = \pi(x) \psi_C(x, z). \]
Conversely, if (2) is satisfied then
\[ \hat{\Lambda} \circ \hat{K}_C(e^{xz}) = \tau_C(x) \hat{\Lambda} \psi_C(x, z) = \tau_C(x) \pi(x) \psi_C(x, z) = \pi(x) \hat{K}_C(e^{xz}) \]
and then (8) follows from Lemma 2.3.

Of course, since it is already known that any subspace \( C \) with a basis of point supported distributions is bispectral, Theorem 2.1 holds in those cases. Moreover, in all of those cases one is able to take \( \hat{\Gamma}_1 \) in (8) to be the identity operator. So, this special case is clearly useful:

**Corollary 2.1.** The subspace \( C \subset D \) is t-bispectral if there exist operators \( \hat{\Lambda}, \hat{\Gamma} \in T \) such that
\[ \hat{\Lambda} \circ \hat{K}_C = \hat{K}_C \circ \hat{\Gamma} \] and such that \( \hat{\Gamma} \) is a constant coefficient operator.

In the context of bispectral ordinary differential operators, equation (8) describes a bispectral Darboux transformation \[1, 3, 11, 12\] of the operator \( \hat{\Gamma} \) into the operator \( \hat{\Lambda} \). The next section will demonstrate that the same procedure may be used to construct examples of t-bispectral subspaces corresponding to non-rational KP solutions.

### 3. Non-rational t-Bispectral KP Solutions

Let
\[ c = \sum_{k=1}^{N} \alpha_k \Delta(\mu_k, n_k) \in D \quad \alpha_k \in \mathbb{C} \]
be an arbitrary finitely supported distribution and pick any polynomial \( q(z) \in \mathbb{C}[z] \). We write \( n = \deg q \) and label its roots and multiplicities by
\[ q(z) = \prod_{i=1}^{m} (z - \lambda_i)^{m_i}, \quad \sum m_i = n. \]
Then consider the \( n \) dimensional subspace \( C \) spanned by the distributions
\[ c_{ij} = \sum_{k=1}^{N} \alpha_k \Delta(\mu_k + \lambda_i, n_k + j - 1) \quad 1 \leq i \leq m, \ 1 \leq j \leq m_i. \]
The main result of this note is the following:

**Theorem 3.1.** Every \( C \subset D \) determined from \( c \in D \) and \( q \in \mathbb{C}[z] \) as described above is t-bispectral. That is, there exists a translational-differential operator \( \hat{\Lambda} \in T \) in the variable \( z \) such that the wave function \( \psi_C \) satisfies (8).

In the case that \( c \) has support at a single point, this merely reproduces the known result that there exists an ordinary differential operator in \( z \) having \( \psi_C \) as an eigenfunction. However, for general \( c \) the other operator cannot be simply a differential operator. In particular, in the case that \( c \) is chosen to be of the form \( c = \Delta(\lambda_1, 0) + \hat{\Gamma} \Delta(\lambda_2, 0) \) (\( \lambda_1 \neq \lambda_2 \)) and \( q \) is a polynomial with distinct roots, then the corresponding solution is a pure \( n \)-soliton solution of the KP hierarchy.
3.1. **Proof of Theorem** 3.1. The following lemma demonstrates that \( K_C(x, \partial) \) can be written in a very simple form for this particular choice of \( C \) in terms of the polynomial \( q \) and the function \( \phi(x) := c(e^{xz}) \):

**Lemma 3.1.** For \( C, q \) and \( \phi \) as above, the dressing operator \( K_C \) can be written as

\[
K_C = \frac{\phi(x)q(\partial)}{\phi(x)} \frac{1}{x^j} e^{x \lambda_i}.
\]

**Proof.** This follows from the fact that \( K \) is the unique monic operator of order \( n \) which annihilates all of the functions \( c_{ij}(e^{xz}) \). Using the easily derived formula

\[
\frac{c_{ij}(e^{xz})}{\phi(x)} = x^{j-1} e^{x \lambda_i}
\]

it follows that

\[
\left( \frac{\phi(x)q(\partial)}{\phi(x)} \right) c_{ij}(e^{xz}) = \phi(x)q(\partial) x^{j-1} e^{x \lambda_i} = \phi(x)q(\partial)(\partial - \lambda_i)^{m_1} x^{j-1} e^{x \lambda_i} = 0
\]

since \( j - 1 < m_i \). Clearly the operator applied above has the correct order and leading coefficient and so it must be \( K_C \). \( \square \)

Note that the translational-differential operator

\[
\hat{Q}_C = \frac{1}{q(z)} \sum_{j=1}^N \alpha_j \partial_z^{n_j} S_{\mu_j}
\]

satisfies

\[
\hat{Q}_C(e^{xz}) = \frac{\phi(x)}{q(z)} e^{xz}.
\]

Finally, using \( \hat{Q}_C \) along with \( \hat{K}_C \) from Lemma 2.4 we are able to construct operators \( \hat{\Gamma}, \Lambda \in \mathbb{T} \) satisfying (1).

**Lemma 3.2.** \( \hat{\Gamma} = \hat{Q}_C \circ \hat{K}_C \) is a constant coefficient translational-differential operator.

**Proof.**

\[
\hat{\Gamma}(e^{xz}) = \hat{Q}_C \circ \hat{K}_C e^{xz} = \hat{Q}_C(\tau_C(x) \psi_C) = \tau_C(x) \hat{Q}_C K_C e^{xz} = \tau_C(x) \phi(x) \circ q(\partial) \circ \frac{1}{\phi(x)} \hat{Q}_C(e^{xz}) = \tau_C(x) \phi(x) \circ q(\partial) \circ \frac{1}{\phi(x)} q(z)(e^{xz}) = \tau_C(x) \phi(x) e^{xz}.
\]

Then, by Lemma 2.2 we see that \( \hat{\Gamma} \) must be a constant coefficient operator. \( \square \)
Letting \( \hat{\Lambda} = \hat{K}_C \circ \hat{Q}_C \in \mathbb{T} \) it is clear that
\[
\hat{K}_C \circ \hat{\Gamma} = \hat{K}_C \circ \hat{Q}_C \circ \hat{K}_C = \hat{\Lambda} \circ \hat{K}_C.
\]
Thus, Theorem 3.1 follows from Corollary 2.1 and we see that
\[
\hat{\Lambda} \psi_C(x, z) = \tau_C(x) \phi(x) \psi_C(x, z).
\]

4. Examples

If we choose \( c = \Delta(1, 0) \) (corresponding to the stationary rational KdV solution \( u = -2/x^2 \)) and \( q = z(z - 1) \) then \( C \) is spanned by \( c_1 = c \) and \( c_2 = \Delta(1, 1) \) (a “two-particle” Calogero-Moser type solution). Now \( \phi(x) = x \) and then
\[
\psi_C(x, z) = (1 + 2 + \frac{x^2}{x^2 z^2}) e^{xz}.
\]
Obviously if \( \hat{Q}_C = \frac{1}{q(z)} \partial_z S_0 \) then \( \hat{Q}_C e^{xz} = \frac{e^{xz}}{q(z)} e^{xz} \) and if \( \hat{K}_C = \frac{1}{z^2}((z^2 - z) \partial_z^2 + (1 - 2z) \partial_z + 2) S_0 \) then \( \hat{K}_C(e^{xz}) = x^2 \psi_C(x, z) \). Then it turns out that the operator \( \hat{\Lambda} \) given by
\[
\hat{\Lambda} = \partial_z^3 + \frac{3}{z - z^2} \partial_z^2 - \frac{6z^2 - 12z + 3}{z^3(z - 1)^2} \partial_z + \frac{12z - 6}{z^2(z - 1)^2}
\]
which satisfies \( \hat{\Lambda} \psi_C(x, z) = x^3 \psi_C(x, z) \) (as we would expect from earlier results on bispectrality.)

However, if we had chosen instead \( c = \Delta(0, 1) + \Delta(0, -1) \), then the KP solution corresponding to the space spanned by \( c \) alone is a standard 1-soliton solution of the KdV hierarchy. Letting \( q = z(z - 1) \) again we consider the space \( C \) spanned by \( c_1 = c \) and \( c_2 = \Delta(0, 2) + \Delta(0, 0) \) which corresponds to a special case of the KP 2-soliton solution.

In this case we find that \( \phi(x) = e^x + e^{-x} \) and so
\[
\psi_C(x, z) = (1 - \frac{6}{x^2} + \frac{3z - 2}{x^2} e^{2x} + 2z - ze^{-2x}) e^{xz}.
\]

One may easily check that \( \hat{Q}_C = \frac{1}{q(z)} S_1 + \frac{1}{q(z)} S_{-1} \) satisfies (10). Moreover,
\[
\hat{K}_C = (1 - \frac{3}{z} + \frac{2}{z^2}) S_2 - (1 + \frac{1}{z}) S_{-2} + (2 - \frac{2}{z} - \frac{6}{z^2}) S_0
\]
satisfies (9).

Finally, one finds that
\[
\hat{\Lambda} = (1 + \frac{1}{z}) S_{-2} + (3 - \frac{6}{z^2} - \frac{1}{z}) S_{-1}
\]
\[
+ (3 - \frac{4}{z^2} - \frac{5}{z}) S_0 + (1 + \frac{2}{z^2} - \frac{3}{z}) S_3
\]
satisfies \( \hat{\Lambda} \psi_C(x, z) = \phi^3(x) \psi_C(x, z) \).
5. Conclusions

In addition to being a generalization of the results of [5, 17] on bispectral ordinary differential operators, the present note may be seen as a generalization of [14] in which wave functions of $n$-soliton solutions of the KdV equation are shown to satisfy difference equations in the spectral parameter. (In fact, the methods of that paper are quite similar in many ways to the methods used here.)

As in [5, 17], the equations (1) and (2) lead to the well known “ad” relations associated to bispectral pairs. That is, defining the ordinary differential operator $A_m$ in $x$ and the translational-differential operator $\hat{A}_m$ in $z$ by

$$A_m = \text{ad}^m_{\pi(x)}$$
$$\hat{A}_m = (-1)^m \text{ad}^m_{\rho(z)}$$

one finds that $A_m \psi_C(x,z) = \hat{A}_m \psi_C(x,z)$. Similarly, if

$$B_m = \text{ad}^m_{\pi(x)}(L_p)$$
$$\hat{B}_m = (-1)^m \text{ad}^m_{\rho(z)}$$

then $B_m \psi_C(x,z) = \hat{B}_m \psi_C(x,z)$. Since the order of the operator $B_m$ is at least one less than the order of the operator $B_{m-1}$, the familiar result that $B_m \equiv 0$ and $\hat{B}_m \equiv 0$ for $m > \text{ord } L_p$ holds, which is clearly a strong restriction on the operator $\hat{A}$. However, unlike the case of bispectral ordinary differential operators, one cannot conclude that $A_m \equiv 0$ for sufficiently large $m$ since the order of $\hat{A}_m$ may not be reduced by increasing $m$.

Finally, although the results of the preceding section demonstrate that the generalization from ordinary bispectrality to $t$-bispectrality is not a trivial one, they have not completely addressed the problem posed above. Namely, it remains to be determined whether there are other $t$-bispectral subspaces $C$ and more significantly whether this non-local form of bispectrality for KP solutions has any dynamical significance. In particular, just as Wilson’s bispectral involution is a manifestation of the self-duality of the Calogero-Moser system [10, 17, 18], it seems reasonable to conjecture that the form of bispectrality presented above should also be related to the dualities of classical integrable particle systems.

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Department of Mathematics and Statistics, Concordia University, and, Centre de recherches mathématiques, Université de Montréal

Current address: Mathematical Sciences Research Institute, Berkeley, CA