OSCILLATORY SOLITONS OF U(1)-INVARIANT MKDV EQUATIONS I: 
ENVELOPE SPEED AND TEMPORAL FREQUENCY

STEPHEN C. ANCO¹, ABDUS SATTAR MIA¹,², MARK R. WILLOUGHBY¹,³

¹ DEPARTMENT OF MATHEMATICS
BROCK UNIVERSITY
ST. CATHARINES, ON CANADA

² DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF SASKATCHEWAN
SASKATOON, SK CANADA

³ INSTITUTE FOR APPLIED MATHEMATICS
UNIVERSITY OF BRITISH COLUMBIA
VANCOUVER, BC CANADA

Abstract. Harmonically modulated complex solitary waves which are a generalized type of 
envelope soliton (herein coined oscillatory solitons) are studied for the two U(1)-invariant 
integrable generalizations of the modified Korteweg-de Vries equation, given by the Hirota 
equation and the Sasa-Satsuma equation. A bilinear formulation of these two equations is 
used to derive the oscillatory 1-soliton and 2-soliton solutions, which are then written out 
in a physical form parameterized in terms of their speed, modulation frequency, and phase. 
Depending on the modulation frequency, the speeds of oscillatory waves (1-solitons) can be 
positive, negative, or zero, in contrast to the strictly positive speed of ordinary solitons. 
When the speed is zero, an oscillatory wave is a time-periodic standing wave. Properties 
of the amplitude and phase of oscillatory 1-solitons are derived. Oscillatory 2-solitons are 
graphically illustrated to describe collisions between two oscillatory 1-solitons in the case 
when the speeds are distinct. In the special case of equal speeds, oscillatory 2-solitons are 
shown to reduce to harmonically modulated breather waves.

1. Introduction

The modified Korteweg-de Vries (mKdV) equation

\[ u_t + au^2u_x + bu_{xxx} = 0 \]  

(1.1)

(where \( a \) and \( b \) are arbitrary positive constants) is an integrable evolution equation which 
arises in many physical applications, such as acoustic waves in anharmonic lattices [1] and 
Alfven waves in collision-free plasmas [2]. Its well-known integrability properties consist 
of multi-soliton solutions, a Lax pair, a bi-Hamiltonian structure, an infinite hierarchy of 
symmetries and conservation laws, and a bilinear formulation. Soliton solutions of the mKdV 
equation are solitary waves

\[ u(t, x) = \epsilon \sqrt{\frac{6c}{a}} \sech\left( \frac{c}{b} (x - ct) \right) \]  

(1.2)

Key words and phrases. mKdV equation, Hirota equation, Sasa-Satsuma equation, solitary wave, envelope 
soliton, oscillatory soliton, breather, overtake collision, head-on collision.
whose shape, wave speed $c > 0$, and up/down orientation $\epsilon = \pm 1$ are preserved after undergoing collisions. These waves have a mathematical characterization as stable travelling wave solutions that are single-peaked, unidirectional, and decaying for large $|x|$. Collisions of two or more mKdV solitary waves are described by multi-soliton solutions that reduce to a linear superposition of distinct solitary waves in the asymptotic past and future. All mKdV soliton solutions carry mass, momentum, energy, as well as Galilean energy associated with the motion of center of momentum, which are constants of motion arising from conservation laws for the mKdV equation (1.1). Remarkably, the only net effect of a collision is to shift the asymptotic positions of the solitary waves such that the center of momentum moves at a constant speed throughout the collision.

There are exactly two integrable complex versions of the mKdV equation [3, 4], given by the Hirota equation [5]

$$u_t + a|u|^2u_x + bu_{xxx} = 0$$  \hspace{1cm} (1.3)

and the Sasa-Satsuma equation [6]

$$u_t + \frac{1}{4}a(u\bar{u}_x + 3u_x\bar{u})u + bu_{xxx} = 0$$  \hspace{1cm} (1.4)

sharing the same scaling symmetry

$$t \rightarrow \lambda^3 t, \quad x \rightarrow \lambda x, \quad u \rightarrow \lambda^{-1} u$$  \hspace{1cm} (1.5)

admitted by the mKdV equation (1.1), and possessing an additional $U(1)$ phase symmetry

$$u \rightarrow \exp(i\phi)u.$$  \hspace{1cm} (1.6)

These two equations are interesting both physically and mathematically. In particular, under a Galilean transformation $t \rightarrow t$, $x \rightarrow x - vt$ combined with a phase-modulation transformation $u \rightarrow \exp(i(kx + \omega t))u$, the Hirota equation (1.3) and the Sasa-Satsuma equation (1.4) are each related to 3rd order generalizations of the nonlinear Schrodinger equation describing short wave pulses in optical fibers [7, 8] and deep water waves [9, 10]. Both equations have integrability properties similar to those of the mKdV equation, and their solitary wave solutions have the form of mKdV solitons up to a phase factor

$$u(t, x) = \exp(i\phi)f_{mKdV}(x - ct)$$  \hspace{1cm} (1.7)

where $c > 0$ is the speed and $-\pi \leq \phi \leq \pi$ is the phase angle, and where $f_{mKdV}$ is the envelope function

$$f_{mKdV}(x - ct) = \sqrt{\frac{6c}{a}} \sech\left(\sqrt{\frac{c}{b}}(x - ct)\right).$$  \hspace{1cm} (1.8)

For each equation (1.3) and (1.4), collisions of two or more solitary waves are described by multi-soliton solutions with the main feature that the net effect on the solitary waves is a shift in their asymptotic positions, while their asymptotic phases stay unchanged in the case of the Hirota equation (1.3) but undergo a shift in the case of the Sasa-Satsuma equation (1.4). The actual nonlinear interaction of these solitary waves during a collision exhibits interesting features which depend on the speed ratios and relative phase angles of the waves, as studied in recent work [11]. (See the animations at http://lie.math.brocku.ca/~sanco/solitons/mkdv_solitons.php)

Most interestingly, both the Hirota equation (1.3) and the Sasa-Satsuma equation (1.4) possess a more general type of soliton solution [12]

$$u(t, x) = \exp(i\phi)\exp(i(kx + \omega t))f(kx + wt)$$  \hspace{1cm} (1.9)
which has the form of a solitary wave \( \exp(i\phi)f(kx + wt) \), with speed \( c = -w/k \) and phase angle \( \phi \), modulated by a harmonic wave \( \exp(i(kx + \omega t)) \), with frequency \( \omega/(2\pi) \) and wave length \( 2\pi/\kappa \), satisfying the algebraic relations

\[
w = -bk(k^2 - 3\kappa^2), \quad \omega = -b\kappa(3k^2 - \kappa^2), \quad \kappa \neq 0.
\]

(1.10)

The envelope function \( f \) in this solution differs from \( f_{mKdV} \), specifically

\[
f_{\text{H}}(kx + wt) = \sqrt{\frac{6b}{a}}|k| \text{sech}(kx + wt)
\]

(1.11)

in the case of the Hirota equation (1.3), and

\[
f_{\text{SS}}(kx + wt) = \sqrt{\frac{6b}{a}}|k|(k^2 + \kappa^2 + (\kappa + ik)(\kappa/2) \exp(2(kx + wt)))[k^2 + \kappa^2 + (\kappa/8) \exp(3(kx + wt))]
\]

(1.12)

in the case of the Sasa-Satsuma equation (1.4). If \( \kappa = 0 \) (and hence \( \omega = 0 \)) then these envelope functions (1.11) and (1.12) reduce to \( f_{mKdV} \), whereby the soliton solution (1.9) reduces to the solitary wave (1.7). In contrast to an ordinary soliton (1.7), the envelope speed

\[
c = -w/k = b(k^2 - 3\kappa^2)
\]

(1.13)

can be positive, negative, or zero, depending on whether \( |\kappa| \) is less than, greater than, or equal to \( |k|/\sqrt{3} \), respectively. Consequently, harmonically modulated solitons can have three different types of collisions: (1) right-overtake — where a faster right-moving soliton overtakes a slower right-moving soliton or a stationary soliton; (2) left-overtake — where a faster left-moving soliton overtakes a slower left-moving soliton or a stationary soliton; (3) head-on — where a right-moving soliton collides with a left-moving soliton. All of these collisions can be expected to exhibit highly interesting new features compared to collisions of ordinary solitons. However, very little seems to be known about the explicit behaviour of colliding harmonically modulated soliton solutions in the literature on the Hirota equation (1.3) and the Sasa-Satsuma equation (1.4), although general formulas yielding the harmonically modulated multi-soliton solutions for both equations have been known for some time [5, 6, 13].

The present paper and a sequel paper will be devoted to studying the basic properties of harmonically modulated complex solitons and their nonlinear interactions.

In section 2 we use a bilinear formulation of general \( U(1) \)-invariant complex mKdV equations to derive explicit expressions for the harmonically modulated 2-soliton solutions

\[
u(t, x) = \exp(i\phi_1) \exp(i(k_1x + \omega_1t))f_1(k_1x + w_1t, k_2x + w_2t) + \exp(i\phi_2) \exp(i(k_2x + \omega_2t))f_2(k_1x + w_1t, k_2x + w_2t)
\]

(1.14)

of the Hirota equation (1.3) and the Sasa-Satsuma equation (1.4).

For the purpose of analytically and graphically understanding these solutions, in section 3 we rewrite the harmonically modulated 1-soliton solutions in a simpler physical form

\[
u(t, x) = \exp(i\phi) \exp(i\nu t)\tilde{f}(x - ct)
\]

(1.15)

involving only the envelope speed \( c = -w/k \), a temporal modulation frequency \( \nu = \omega + ck \), and the phase angle \( \phi \). We show that \( c \) and \( \nu \) obey a simple kinematic relation which gives a direct way to classify the cases for which \( c \) is positive, negative, or zero, depending only
on \( \nu \). In particular, when the envelope speed is \( c = 0 \), these solutions describe time-periodic standing waves

\[
u(t, x) = \exp(i\phi) \exp(i\nu t) \tilde{f}(x).
\]

(1.16)

We also rewrite the harmonically modulated 2-soliton solutions for the Hirota equation (1.3) and the Sasa-Satsuma equation (1.4) in an analogous physical form, given by

\[
u(t, x) = \exp(i\phi_1) \exp(i\nu_1 t) \tilde{f}_1(x - c_1 t, x - c_2 t) + \exp(i\phi_2) \exp(i\nu_2 t) \tilde{f}_2(x - c_1 t, x - c_2 t)
\]

(1.17)

if the envelope speeds \( c_1 = -w_1/k_1 \) and \( c_2 = -w_2/k_2 \) are distinct, or

\[
u(t, x) = \exp(i\phi_1) \exp(i\nu_1 t) \tilde{f}_1(x - ct, (\nu_1 - \nu_2) t) + \exp(i\phi_2) \exp(i\nu_2 t) \tilde{f}_2(x - ct, (\nu_1 - \nu_2) t)
\]

(1.18)

with \( c = -w_1/k_1 = -w_2/k_2 \) if the envelope speeds are equal.

We will call solitary wave solutions of the form (1.15) an oscillatory soliton, and solutions of the form (1.17) an oscillatory 2-soliton. The special case (1.18) describes a solitary wave solution that has speed \( c \) and involves two temporal frequencies \( \nu_1 \) and \( \nu_2 \). When these frequencies are related by \( \nu_1 + \nu_2 = 0 \), the resulting wave is a breather, which has the equivalent general form

\[
u(t, x) = \exp(i\phi_0) \tilde{f}(x - ct, \nu t + \phi)
\]

(1.19)

involving the frequency \( \nu = \nu_1 = -\nu_2 \) and the phase angles \( \phi = (\phi_1 - \phi_2)/2 \) and \( \phi_0 = (\phi_1 + \phi_2)/2 \). When the frequencies are independent, \( \nu_1 + \nu_2 \neq 0 \), we will write the solitary wave solution (1.18) in a harmonically modulated similar form

\[
u(t, x) = \exp(i(\nu_0 t + \phi_0)) \tilde{f}(x - ct, \nu t + \phi)
\]

(1.20)

which we call an oscillatory breather, with an envelope frequency \( \nu = (\nu_1 - \nu_2)/2 \) and phase angle \( \phi = (\phi_1 - \phi_2)/2 \), and a modulation frequency \( \nu_0 = (\nu_1 + \nu_2)/2 \neq 0 \) and phase angle \( \phi_0 = (\phi_1 + \phi_2)/2 \).

Our main new results are obtained in section 4. We first derive the basic properties of the amplitude \( |u| \) and phase \( \arg(u) \) for the oscillatory 1-solitons (1.15) of the Hirota equation (1.3) and the Sasa-Satsuma equation (1.4). Next we graphically illustrate that the oscillatory 2-solitons (1.17) of these two equations describe collisions of oscillatory 1-solitons with distinct speeds \( c_1 \neq c_2 \), and that the oscillatory breathers (1.20) of the equations describe solitary waves whose amplitude displays time-periodic oscillations with frequency \( \nu \) and speed \( c \).

Finally, in section 5 we make some concluding remarks.

All computations in the paper have been carried out by use of Maple. Hereafter, by scaling variables \( t, x, u \), we will put

\[
 a = 24, \quad b = 1
\]

(1.21)

for convenience.

2. Derivation of harmonically modulated soliton solutions

Consider a general \( U(1) \)-invariant complex mKdV equation

\[
u_t + (\alpha u\bar{u}_x + \beta u_x \bar{u})u + \gamma u_{xxx} = 0
\]

(2.1)

where \( \alpha, \beta, \gamma \) are constants. A bilinear formulation of this equation can be obtained by the following steps [14, 12]. First, we convert equation (2.1) into a rational form through the standard transformation

\[
u = G/F, \quad \bar{u} = \bar{G}/F
\]

(2.2)
where \( F(t, x) \) is a real function and \( G(t, x) \) is a complex function. Second, we express all derivatives of \( F \), \( G \) and \( \bar{G} \) in terms of Hirota’s bilinear operator defined by

\[
\begin{align*}
D(f, g) &= gDf - fDg \\
D^2(f, g) &= gD^2f + fD^2g - 2(Df)Dg \\
D^3(f, g) &= gD^3f - 3(Dg)D^2f + 3(Df)D^2g - fD^3g
\end{align*}
\]

where \( D \) denotes a total derivative. Last, we split the resulting rational equation

\[
0 = F^2(\gamma D^3_x(G, F) + D_t(G, F)) - \alpha GHD_x(G, \bar{G}) - (3\gamma D^2_x(F, F) - (\alpha + \beta)G\bar{G})D_x(G, F) \quad (2.6)
\]

into a system of bilinear equations

\[
\begin{align*}
D_t(G, F) + \gamma D^3_x(G, F) &= GH \\
3\gamma D^2_x(F, F) - (\alpha + \beta)G\bar{G} &= \lambda GH \\
\alpha D_x(G, \bar{G}) + \lambda D_x(G, F) &= FH
\end{align*}
\]

where \( H(t, x) \) is an auxiliary function and \( \lambda \) is a complex constant.

When the original equation (2.1) is integrable, \( N \)-soliton solutions can be derived via the Hirota ansatz

\[
\begin{align*}
\lambda &= 0 \\
G &= \text{complex polynomial of odd degree in } e^{\Theta_i}, e^{\bar{\Theta}_i} \\
F &= \text{real polynomial of even degree in } e^{\Theta_i}, e^{\bar{\Theta}_i} \\
H &= \text{complex polynomial of even degree in } e^{\Theta_i}, e^{\bar{\Theta}_i}
\end{align*}
\]

\( i = 1, \ldots, N \) (2.11)

with

\[
\Theta_i = k_i x + w_i t, \quad \bar{\Theta}_i = \bar{k}_i x + \bar{w}_i t, \quad i = 1, \ldots, N
\]

where \( k_i \) and \( w_i \) are complex constants when harmonically modulated solitons are sought or real constants if ordinary solitons are sought instead. Substitution of this ansatz (2.8)–(2.11) into the bilinear system (2.7) yields a system of algebraic equations given by polynomials in \( e^{\Theta_i}, e^{\bar{\Theta}_i} \), whose monomial coefficients must separately vanish. This system can be solved degree by degree, starting at the respective degrees 1, 2, 2 in equations (2.7a), (2.7b), (2.7c), and stopping at some degree such that the corresponding highest-degree coefficients in the polynomials \( G, F, H \) are found to vanish. (This termination generally will not occur unless the original equation is integrable.)

### 2.1. Harmonically modulated Hirota solitons. The bilinear system (2.7) combined with the ansatz equation (2.8) applied to the Hirota equation

\[
u_t + 24|u|^2u_x + u_{xxx} = 0 \quad (2.13)
\]

gives

\[
\begin{align*}
H &= 0 \\
D_t(G, F) + D^3_x(G, F) &= 0 \\
D^2_x(F, F) - 8G\bar{G} &= 0
\end{align*}
\]

To set up the soliton ansatz (2.9)–(2.10), let all monomial terms of fixed degree \( n \geq 0 \) in \( G, F \) be denoted by \( G^{(n)}, F^{(n)} \), so thus the ansatz is written as

\[
G = G^{(1)} + G^{(3)} + \cdots, \quad F = 1 + F^{(2)} + F^{(4)} + \cdots \tag{2.15}
\]

(\( F^{(0)} \) has been normalized to 1 without loss of generality). Then it is straightforward to split the bilinear system (2.14) into a hierarchy of equations indexed by degree:

\[
\begin{align*}
D_t(G^{(1)}, 1) + D_x^3(G^{(1)}, 1) &= 0 \tag{2.16a} \\
D^2_x(F^{(2)}, 1) - 4G^{(1)}G^{(1)} &= 0 \tag{2.16b} \\
D_t(G^{(3)}, 1) + D_x^3(G^{(3)}, 1) &= -D_t(G^{(1)}, F^{(2)}) - D_x^2(G^{(1)}, F^{(2)}) \tag{2.16c} \\
D^2_x(F^{(4)}, 1) - 4(G^{(3)}G^{(1)} + G^{(1)}G^{(3)}) &= -\frac{1}{2}D^2_x(F^{(2)}, F^{(2)}) \tag{2.16d}
\end{align*}
\]

etc.

We will first review the derivation of the harmonically modulated 1-soliton solution from the split bilinear system (2.16). For \( N=1 \), the lowest degree terms in the ansatz (2.15) are given by

\[
G^{(1)} = Ae^{\Theta}, \quad F^{(2)} = Be^{\Theta+\bar{\Theta}} \tag{2.17}
\]

with

\[
\begin{align*}
\Theta &= kx + wt, \\
\bar{\Theta} &= \bar{k}x + \bar{w}t \tag{2.18}
\end{align*}
\]

where \( k, w, A \) are complex constants and \( B \) is a real constant. The two lowest degree equations (2.16a) and (2.16b) yield

\[
\begin{align*}
w &= -k^3, \\
B &= \frac{4AA}{(k+k)^2}. \tag{2.20}
\end{align*}
\]

The inhomogeneous terms in the next equation (2.16c) turn out to vanish, which determines \( G^{(3)} = 0 \). Likewise, the next equation (2.16d) determines \( F^{(4)} = 0 \). Hence the ansatz (2.15) terminates at degrees 1 and 2, respectively. This yields the 1-soliton solution

\[
u = \frac{Ae^{\Theta}}{1 + (|A|/\text{Re}k)^2e^{\Theta+\bar{\Theta}}}, \quad \Theta = k(x - k^2t). \tag{2.21}
\]

It can be written in the form of an harmonically modulated soliton (1.9), (1.10), (1.11) in the following way. By putting

\[
\text{Re}k = k, \quad \text{Im}k = \kappa, \quad \text{Re}w = w, \quad \text{Im}w = \omega, \tag{2.22}
\]

we see that the algebraic relations (1.10) and (2.19) match. Next expressing

\[
\exp(\Theta) = \exp(i\text{Im}\Theta)\exp(\text{Re}\Theta) = \exp(i(kx + \omega t))\exp(kx + wt), \quad A = |A|\exp(i\phi), \quad \phi = \arg A, \tag{2.23}
\]

we see that the solution (2.21) has the general harmonically modulated form (1.9) with

\[
f = |A|\exp(kx + wt)/(1 + (|A|/k)^2\exp(2kx + 2wt)). \tag{2.24}
\]

Then writing

\[
|A/k| = e^{-ak} \tag{2.25}
\]
and using the identity \( \text{sech}\theta = 2 \exp\theta/(1 + \exp 2\theta) \) where

\[
\theta = \text{Re} \Theta - ak = k(x - a) + wt, \tag{2.26}
\]

we find \( f \) matches the Hirota envelope function (1.11) up to a space translation \( x \to x - a \). Hence we have

\[
u(t, x) = \exp(i(\phi + \kappa x + \omega t))(|k|/2) \text{sech}(k(x - a) + wt)
\]

\[
= \exp(i(\varphi + \kappa(x - a) + \omega t))(|k|/2) \text{sech}(k(x - a) + wt)
\tag{2.27}
\]

after putting \( \varphi = \phi + ak \).

**Proposition 1.** The harmonically modulated 1-soliton (1.9) for the Hirota equation (2.13) has an explicit rational \( \cosh \) form given by the envelope function (up to space translations on \( x \) and phase shifts on \( \phi \))

\[
f_H(\theta) = \frac{|k|}{2 \cosh(\theta)}, \quad \theta = k(x - (k^2 - 3\kappa^2)t) \tag{2.28}
\]

which is invariant under reflection \( k \to -k \).

We will next derive the explicit form of the harmonically modulated 2-soliton solution. For \( N = 2 \), the lowest degree terms in the ansatz (2.15) are given by

\[
G^{(1)} = A_1e^{\Theta_1} + A_2e^{\Theta_2}, \quad F^{(2)} = B_1e^{\Theta_1+\Theta_1} + B_2e^{\Theta_2+\Theta_2} + Ce^{\Theta_1+\Theta_2} + \bar{C}e^{\Theta_2+\Theta_1} \tag{2.29}
\]

with

\[
\Theta_1 = k_1x + w_1t, \quad \Theta_2 = k_2x + w_2t, \quad \bar{\Theta}_1 = \bar{k}_1x + \bar{w}_1t, \quad \bar{\Theta}_2 = \bar{k}_2x + \bar{w}_2t, \tag{2.30}
\]

where \( k_1, k_2, w_1, w_2, A_1, A_2, C \) are complex constants and \( B_1, B_2 \) are real constants. Similarly to the \( N = 1 \) case, the two lowest degree equations (2.16a) and (2.16b) in the split bilinear system yield

\[
w_1 = -k_1^2, \quad w_2 = -k_2^2, \tag{2.31}
\]

\[
B_1 = \frac{4A_1\bar{A}_1}{(k_1 + k_2)^2}, \quad B_2 = \frac{4A_2\bar{A}_2}{(k_2 + k_2)^2}, \quad C = \frac{4A_1\bar{A}_2}{(k_1 + k_2)^2}. \tag{2.32}
\]

The inhomogeneous terms in the next two equations (2.16c) and (2.16d) no longer vanish. Instead, equation (2.16c) now contains monomial terms \( e^{\Theta_1+\Theta_2+\Theta_2} \) and \( e^{\Theta_2+\Theta_1+\Theta_1} \), while equation (2.16d) contains a single monomial term \( e^{\Theta_1+\Theta_2+\Theta_1+\Theta_2} \). To balance these degree 3 and 4 terms, the ansatz (2.15) needs to contain the corresponding monomial terms

\[
G^{(3)} = D_1e^{\Theta_2+\Theta_1+\Theta_1} + D_2e^{\Theta_1+\Theta_2+\Theta_2}, \quad F^{(4)} = Ee^{\Theta_1+\Theta_2+\Theta_1+\Theta_2} \tag{2.33}
\]

where \( D_1, D_2 \) are complex constants and \( E \) is a real constant. Equations (2.16c) and (2.16d) then yield

\[
D_1 = \frac{4A_1A_2\bar{A}_1(k_1 - k_2)^2}{(k_1 + k_2)^2(k_1 + k_1)^2}, \quad D_2 = \frac{4A_1A_2\bar{A}_2(k_1 - k_2)^2}{(k_1 + k_2)^2(k_2 + k_2)^2} \tag{2.34}
\]

and

\[
E = \frac{16A_1A_2\bar{A}_1\bar{A}_2(k_1 - k_2)^2(k_1 - k_2)^2}{(k_1 + k_1)^2(k_1 + k_2)^2(k_1 + k_2)^2(k_2 + k_2)^2}. \tag{2.35}
\]
The inhomogeneous terms in these equations are found to vanish, which determines $G^5 = 0$ and $F^6 = 0$. Hence the ansatz (2.15) terminates at degrees 3 and 4, respectively. This yields the 2-soliton solution

$$u = \frac{A_1 e^{\Theta_1}(1 + V_1) + A_2 e^{\Theta_2}(1 + V_2)}{1 + W}$$

(2.37)

with

$$V_1 = |A_2|^2 \Gamma_2 e^{2 \Re \Theta_2}, \quad V_2 = |A_1|^2 \Gamma_1 e^{2 \Re \Theta_1}$$

(2.38)

$$W = |A_1|^2 \Omega_1 e^{2 \Re \Theta_1} + |A_2|^2 \Omega_2 e^{2 \Re \Theta_2} + |A_1|^2 |A_2|^2 \Omega_1 \Omega_2 \Gamma^2 e^{2 \Re (\Theta_1 + \Theta_2)}$$

+ $8 \Re (A_1 \bar{A}_2 \Phi e^{i \Phi (\Theta_1 - \Theta_2)}) e^{\Re (\Theta_1 + \Theta_2)}$

(2.39)

where

$$\Theta_1 = k_1(x - k_1^2 t), \quad \Theta_2 = k_2(x - k_2^2 t),$$

(2.40)

$$\Gamma_1 = \frac{D_1}{A_2|A_1|^2} = \frac{(k_1 - k_2)^2}{(\Re k_1)^2(k_1 + k_2)^2}, \quad \Gamma_2 = \frac{D_2}{A_1|A_2|^2} = \frac{(k_1 - k_2)^2}{(\Re k_2)^2(k_1 + k_2)^2},$$

(2.41)

$$\Omega_1 = \frac{B_1}{|A_1|^2} = \frac{1}{(\Re k_1)^2}, \quad \Omega_2 = \frac{B_2}{|A_2|^2} = \frac{1}{(\Re k_2)^2},$$

(2.42)

$$\Phi = \frac{C}{4 A_1 \bar{A}_2} = \frac{1}{(k_1 + k_2)^2},$$

(2.43)

$$\Gamma = \frac{\sqrt{E}}{\sqrt{B_1 B_2}} = \frac{|k_1 - k_2|^2}{|k_1 + k_2|^2}.$$  

(2.44)

To write this solution (2.37) in the form of an harmonically modulated soliton, we proceed similarly to the 1-soliton case. First we put

$$\Re k_1 = k_1, \quad \Im k_1 = \kappa_1, \quad \Re w_1 = w_1, \quad \Im w_1 = \omega_1,$$

$$\Re k_2 = k_2, \quad \Im k_2 = \kappa_2, \quad \Re w_2 = w_2, \quad \Im w_2 = \omega_2,$$

(2.45)

so thus the algebraic relations (1.10) become

$$w_1 = -k_1(k_1^2 - 3 \kappa_1^2), \quad \omega_1 = -\kappa_1(3k_1^2 - \kappa_1^2),$$

$$w_2 = -k_2(k_2^2 - 3 \kappa_2^2), \quad \omega_2 = -\kappa_2(3k_2^2 - \kappa_2^2).$$

(2.46)

Next we write

$$A_1 = |A_1| \exp(i \phi_1), \quad \phi_1 = \arg A_1, \quad A_2 = |A_2| \exp(i \phi_2), \quad \phi_2 = \arg A_2,$$

(2.47)

$$\Gamma_1 = \Gamma \Omega_1 \exp(i \gamma_1), \quad \gamma_1 = \frac{\arg(\Gamma_1)}{2} = \arg \left(\frac{k_1 - k_2}{k_1 + k_2}\right),$$

$$\Gamma_2 = \Gamma \Omega_2 \exp(i \gamma_2), \quad \gamma_2 = \frac{\arg(\Gamma_2)}{2} = \arg \left(\frac{k_2 - k_1}{k_1 + k_2}\right).$$

(2.48)
We now express
\[ |A_1|\sqrt{\Gamma_1 \Omega_1} = e^{-a_1 k_1}, \quad |A_2|\sqrt{\Gamma_2 \Omega_2} = e^{-a_2 k_2} \] (2.49)
and
\[ \text{Re} \Theta_1 - a_1 k_1 = k_1(x - a_1) + w_1 t = \theta_1, \quad \text{Im} \Theta_1 = \kappa_1 x + \omega_1 t = \vartheta_1, \] \[ \text{Im} \Theta_2 = \kappa_2 x + \omega_2 t = \vartheta_2. \] (2.50)

The expressions in the numerator and denominator of the 2-soliton solution (2.37) are then given by
\[ V_1 = e^{i2\gamma_2} e^{2\theta_2}, \quad V_2 = e^{i2\gamma_1} e^{2\theta_1} \] (2.52)
\[ W = e^{2(\theta_1 + \theta_2)} + \frac{1}{\Gamma}(e^{2\theta_1} + e^{2\theta_2}) - 2 \frac{|\Gamma - 1|}{\Gamma} \text{Re} (e^{i(\vartheta_1 - \vartheta_2 - \gamma_1 + \gamma_2)} e^{i(\vartheta_1 - \vartheta_2)}) e^{\theta_1 + \theta_2} \] (2.53)
where we have used the identities
\[ \Phi/\bar{\Phi} = (\Gamma_2 \Omega_2)/(\Gamma_1 \Omega_1) \] (2.54)
\[ \Phi \bar{\Phi} = (\Gamma - 1)^2 \Omega_1 \Omega_2/16 \] (2.55)
\[ \arg(-\Phi) = \gamma_2 - \gamma_1, \quad |\Phi| = |\Gamma - 1|\sqrt{\Omega_1 \Omega_2}/4. \] (2.56)

Hence we have
\[ u = e^{i\phi_1} e^{i\theta_1} f_1 + e^{i\phi_2} e^{i\theta_2} f_2 \] (2.57)
with
\[ f_1 = X_1/Y, \quad f_2 = X_2/Y \] (2.58)
given by
\[ X_1 = (1/\sqrt{\Omega_1 \Gamma})(1 + V_1) e^{-\theta_2} = (2/\sqrt{\Omega_1 \Gamma}) \exp(i\gamma_2) \cosh(\theta_2 + i\gamma_2) \] (2.59)
\[ X_2 = (1/\sqrt{\Omega_2 \Gamma})(1 + V_2) e^{-\theta_1} = (2/\sqrt{\Omega_2 \Gamma}) \exp(i\gamma_1) \cosh(\theta_1 + i\gamma_1) \] (2.60)
\[ Y = (1 + W) e^{-\theta_1 - \theta_2} \]
\[ = (2/\Gamma)(\cosh(\theta_1 - \theta_2) + \Gamma \cosh(\theta_1 + \theta_2) - |\Gamma - 1| \cos(\vartheta_1 - \vartheta_2 + \phi_1 - \phi_2 + \gamma_2 - \gamma_1)). \] (2.61)

As expressed in this form, the 2-soliton solution (2.57)–(2.61) closely resembles a harmonically modulated 2-soliton (1.14) except for the presence of the shifts \(a_1, a_2\) on \(x\) in \(\theta_1, \theta_2\) and the appearance of \(\vartheta_1, \vartheta_2\) in the functions \(f_1, f_2\). However, under the assumption \(w_1/k_1 \neq w_2/k_2\), this solution can be converted exactly into the form (1.14). First we apply a combined space-time translation
\[ x \to x - x_0, \quad t \to t - t_0 \] (2.62)
such that
\[ 0 = k_1(x_0 + a_1) + w_1 t_0 = k_2(x_0 + a_2) + w_2 t_0 \] (2.63)
whereby
\[ \theta_1 \to k_1 x + w_1 t, \quad \theta_2 \to k_2 x + w_2 t \] (2.64)
absorbs the shifts \(a_1, a_2\) on \(x\). This transformation (2.62)–(2.63) exists provided \(k_1 w_2 \neq k_2 w_1\). It induces a corresponding transformation
\[ \theta_1 \to k_1 x + \omega_1 t - (\kappa_1 x_0 + \omega_1 t_0), \quad \theta_2 \to k_2 x + \omega_2 t - (\kappa_2 x_0 + \omega_2 t_0) \] (2.65)
producing additional phase angles which can be absorbed by shifts

$$\phi_1 \to \phi_1 - \varphi_1, \quad \phi_2 \to \phi_2 - \varphi_2$$

so that

$$\vartheta_1 + \phi_1 + \gamma_2 \to \kappa_1 x + \omega_1 t + \phi_1, \quad \vartheta_2 + \phi_2 + \gamma_1 \to \kappa_2 x + \omega_2 t + \phi_2$$

via \( \varphi_1 = \gamma_2 - (\kappa_1 x_0 + \omega_1 t_0) \), \( \varphi_2 = \gamma_1 - (\kappa_2 x_0 + \omega_2 t_0) \).

Next we use the identity

$$(\kappa_1 - \kappa_2)x + (\omega_1 - \omega_2)t = \mu_1(k_1x + w_1t) - \mu_2(k_2x + w_2t)$$

with

$$\mu_1 = \frac{w_2(k_1 - \kappa_2) - k_2(\omega_1 - \omega_2)}{k_1w_2 - k_2w_1}, \quad \mu_2 = \frac{w_1(k_1 - \kappa_2) - k_1(\omega_1 - \omega_2)}{k_1w_2 - k_2w_1}$$

which holds provided \( k_1 w_2 \neq k_2 w_1 \). This leads to the following result.

**Proposition 2.** When

$$w_1/k_1 \neq w_2/k_2,$$

the harmonically modulated 2-soliton (1.14) for the Hirota equation (2.13) has an explicit rational \( \cosh \) form given by the envelope functions (up to space-time translations on \( t, x \))

$$f_{1H}(\theta_1, \theta_2) = X_{1H}(\theta_1, \theta_2)/Y_{SS}(\theta_1, \theta_2), \quad f_{2H}(\theta_1, \theta_2) = X_{2H}(\theta_1, \theta_2)/Y_{SS}(\theta_1, \theta_2)$$

with

$$X_{1H}(\theta_1, \theta_2) = |k_1| \sqrt{\Gamma} \cosh(\theta_1 + i \gamma_1), \quad X_{2H}(\theta_1, \theta_2) = |k_2| \sqrt{\Gamma} \cosh(\theta_1 + i \gamma_1)$$

$$Y_{1H}(\theta_1, \theta_2) = \cosh(\theta_1 - \theta_2) + \Gamma \cosh(\theta_1 + \theta_2) - |\Gamma - 1| \cos(\mu_1 \theta_1 - \mu_2 \theta_2 + \phi_1 - \phi_2)$$

where \( w_1, w_2, \omega_1, \omega_2 \) are given by equation (2.46) and \( \mu_1, \mu_2 \) are given by equation (2.69), and where

$$\Gamma = \frac{(k_1 - k_2)^2 + (\kappa_1 - \kappa_2)^2}{(k_1 + k_2)^2 + (\kappa_1 - \kappa_2)^2},$$

$$\gamma_1 = \arg\left((k_1 + k_2)(k_1 - k_2) - (\kappa_1 - \kappa_2)^2 + i2k_1(\kappa_1 - \kappa_2)\right),$$

$$\gamma_2 = \arg\left((k_1 + k_2)(k_2 - k_1) - (\kappa_1 - \kappa_2)^2 - i2k_2(\kappa_1 - \kappa_2)\right),$$

$$\theta_1 = k_1 x + w_1 t = k_1 x - (k_1^2 - 3\kappa_1^2)t,$$

$$\theta_2 = k_2 x + w_2 t = k_2 x - (k_2^2 - 3\kappa_2^2)t.$$

The kinematic condition (2.70) will be seen later to have the interpretation that the harmonically modulated 2-soliton describes a collision of harmonically modulated 1-solitons with distinct speeds \( c_1 = -w_1/k_1 \neq c_2 = -w_2/k_2 \). These explicit expressions (2.71)–(2.76) for the harmonically modulated Hirota 2-soliton (1.14) have not previously appeared in the literature. They reduce to the ordinary 2-soliton solution [15, 11] for the Hirota equation in the case \( \kappa_1 = \kappa_2 = 0 \).

We remark that in the case of equal speeds \( c_1 = c_2 \), Proposition 2 remains valid if \( \mu_1 \theta_1 - \mu_2 \theta_2 \) is replaced by \( \vartheta_1 - \vartheta_2 \) through equation (2.68), and if \( k_1 x \) and \( k_2 x \) are respectively replaced by \( k_1 x + \chi \) and \( k_2 x - \chi \) through the equation \( k_1(x + a_1) + w_1 t_0 = -k_2(x + a_2) - w_2 t_0 = -\chi \), where \( \chi = (a_2 - a_1)k_1 k_2/(k_1 + k_2) \) is a shift which cannot be absorbed by a space-time translation (2.62).
Finally, we examine the properties of the Hirota envelope functions (2.71)–(2.73) under reflections

\[ k_1 \rightarrow \mp k_1, \quad k_2 \rightarrow \pm k_2 \quad (2.77) \]

which will be important when expressing the Hirota 2-soliton solution in oscillatory form. We first note that \( \theta_1 \rightarrow \mp \theta_1, \ \theta_2 \rightarrow \pm \theta_2 \) and that \( w_1 \rightarrow \mp w_1, \ w_2 \rightarrow \pm w_2 \) while \( \omega_1 \rightarrow \omega_1, \ \omega_2 \rightarrow \omega_2 \) from equation (2.46). We then have

\[ \mu_1 \rightarrow -\mu_1, \quad \mu_2 \rightarrow -\mu_2 \quad (2.78) \]

\[ \Gamma \rightarrow 1/\Gamma, \quad \gamma_1 \rightarrow -\gamma_1, \quad \gamma_2 \rightarrow -\gamma_2 \quad (2.79) \]

from equations (2.69), (2.74), (2.75), and thus

\[ X_{1H} \rightarrow (1/\Gamma)X_{1H}, \quad X_{2H} \rightarrow (1/\Gamma)X_{2H}, \quad Y_{1H} \rightarrow (1/\Gamma)Y_{1H}. \quad (2.80) \]

These transformations establish the following reflection property.

**Lemma 1.** The Hirota envelope functions (2.71) are invariant under reflections (2.77).

### 2.2. Harmonically modulated Sasa-Satsuma solitons.

For the Sasa-Satsuma equation

\[ u_t + 6(uu_x + 3uu)u + u_{xxx} = 0 \quad (2.81) \]

the bilinear system (2.7) combined with the ansatz equation (2.8) gives

\[ D_t(G, F) + D_x^2(G, F) = GH \quad (2.82a) \]

\[ D_x^2(F, F) - 8GG = 0 \quad (2.82b) \]

\[ 6D_x(G, \bar{G}) = FH \quad (2.82c) \]

which is more complicated than in the case of the Hirota equation. To set up the soliton ansatz (2.9)–(2.10), let all monomial terms of fixed degree \( n \geq 0 \) in \( G, F, H \) be denoted by \( G^{(n)}, F^{(n)}, H^{(n)} \). The ansatz is thus written as

\[ G = G^{(1)} + G^{(3)} + \cdots, \quad F = 1 + F^{(2)} + F^{(4)} + \cdots, \quad H = H^{(2)} + H^{(4)} + \cdots \quad (2.83) \]

(where \( F^{(0)} \) has been normalized to 1 without loss of generality). Then the bilinear system (2.82) splits into a hierarchy of equations indexed by degree:

\[ D_t(G^{(1)}, 1) + D_x^3(G^{(1)}, 1) = 0 \quad (2.84a) \]

\[ D_x^2(F^{(2)}, 1) - 4G^{(1)}G^{(1)} = 0 \quad (2.84b) \]

\[ H^{(2)} = 6D_x(G^{(1)}, \bar{G}^{(1)}) \quad (2.84c) \]

\[ D_t(G^{(3)}, 1) + D_x^2(G^{(3)}, 1) = -D_t(G^{(1)}, F^{(2)}) - D_x^2(G^{(1)}, F^{(2)}) + G^{(1)}H^{(2)} \quad (2.84d) \]

\[ D_x^2(F^{(4)}, 1) - 4(G^{(3)}\bar{G}^{(1)} + G^{(1)}\bar{G}^{(3)}) = -\frac{1}{2}D_x^2(F^{(2)}, F^{(2)}) \quad (2.84e) \]

\[ H^{(4)} = 6D_x(G^{(1)}, G^{(3)}) + 6D_x(G^{(3)}, G^{(1)}) - F^{(2)}H^{(2)} \quad (2.84f) \]

\[ D_t(G^{(5)}, 1) + D_x^3(G^{(5)}, 1) = -D_t(G^{(3)}, F^{(2)}) - D_x^3(G^{(3)}, F^{(2)}) + G^{(3)}H^{(2)} \]

\[ -D_t(G^{(1)}, F^{(4)}) - D_x^3(G^{(1)}, F^{(4)}) + G^{(1)}H^{(4)} \quad (2.84g) \]

\[ D_x^2(F^{(6)}, 1) - 4(G^{(5)}\bar{G}^{(1)} + G^{(1)}\bar{G}^{(5)}) = 4G^{(3)}\bar{G}^{(3)} - D_x^2(F^{(4)}, F^{(2)}) \quad (2.84h) \]

e etc.
Note that $H^{(2)}$, $H^{(4)}$, etc. can be successively eliminated through equations (2.84c), (2.84f), and so on.

We will first summarize the derivation of the harmonically modulated 1-soliton solution from the split bilinear system (2.84). For $N=1$, the lowest degree terms in the ansatz (2.83) are given by

$$G^{(1)} = Ae^\Theta, \quad F^{(2)} = Be^{\Theta+\bar{\Theta}}, \quad H^{(2)} = Ce^{\Theta+\bar{\Theta}},$$

with

$$\Theta = kx + wt, \quad \bar{\Theta} = \bar{k}x + \bar{w}t$$

where $k, w, A, C$ are complex constants and $B$ is a real constant. The two lowest degree equations (2.84a) and (2.84b) yield exactly the results (2.19) and (2.20) obtained for the Hirota solution, while the next equation (2.84c) determines

$$C = 6A\bar{A}(k - \bar{k}).$$

In the next two lowest degree equations (2.84d) and (2.84e), the inhomogeneous terms are found to consist of the respective monomials $e^{2\Theta+\bar{\Theta}}$ and $e^{2\Theta+2\bar{\Theta}}$. These degree 3 and 4 terms need to be balanced by having the ansatz (2.83) for $G$ and $F$ contain the corresponding monomial terms

$$G^{(3)} = De^{2\Theta+\bar{\Theta}}, \quad F^{(4)} = Ee^{2\Theta+2\bar{\Theta}},$$

where $D$ is a complex constant and $E$ is a real constant. Equations (2.84d) and (2.84e) then yield

$$D = \frac{A^2\bar{A}(k - \bar{k})}{k(k + k)^2}, \quad E = \frac{-A^2\bar{A}^2(k - \bar{k})^4}{kk(k + k)^4}.$$  

(2.89)

Next, the inhomogeneous terms in equation (2.84f) turn out to vanish, which determines $H^{(4)} = 0$. Likewise, the next two higher degree equations (2.84e) and (2.84g) determine $G^{(5)} = 0$ and $F^{(6)} = 0$. Hence the ansatz (2.83) terminates at degrees 3, 4, and 2, respectively. This yields the 1-soliton solution

$$u = \frac{Ae^\Theta \left(1 + \frac{|A|^2}{(\text{Re} k)^2} \frac{i\text{Im} k}{2k} e^{\Theta+\bar{\Theta}}\right)}{1 + \frac{|A|^2}{(\text{Re} k)^2} e^{\Theta+\bar{\Theta}} + \frac{|A|^4}{(\text{Re} k)^4} \frac{(\text{Im} k)^2}{4|k|^2} e^{2\Theta+2\bar{\Theta}}}, \quad \Theta = k(x - k^2 t)$$

(2.90)

which can be written in the form of a harmonically modulated soliton (1.9), (1.10), (1.12) similarly to the Hirota case. In particular, using expressions (2.22) and (2.23), we see that the solution (2.90) has the general harmonically modulated form (1.9) with

$$f = \frac{|A| \exp(kx + wt)(1 + i(|A|/k)^2\Lambda \exp(2kx + 2wt))}{1 + \exp(2kx + 2wt) + (|A|/k)^4 |A|^2 \exp(4kx + 4wt)}$$

(2.91)

where

$$\Lambda = \frac{i\kappa}{2(k + i\kappa)} = \frac{\kappa(\kappa + ik)}{2(k^2 + \kappa^2)}.$$  

(2.92)

After we simplify this function $f$ in terms of $|A/k| = e^{-ak}$ and $\Theta = \text{Re} \Theta - ak = k(x - a) + wt$, it matches the Sasa-Satsuma envelope function (1.12) up to a space translation $x \rightarrow x - a$. 


The harmonically modulated Proposition 3. Thus we have the following result.

\begin{align}
\kappa \text{ has an explicit rational (2.81)} \\
\text{becomes} \phi \text{ is absorbed into a space translation } x \to x - \tilde{a}. \text{ The envelope function (2.91) then becomes} \\
f = (|k|/\sqrt{|\Lambda|})e^{\tilde{\phi}}(1 + e^{i\lambda e^{2\tilde{\phi}}}/(1 + e^{2\tilde{\phi}} + (1/|\Lambda|)e^{2\tilde{\phi}}) \\
\text{where} \\
|\Lambda| = \frac{|k|}{2\sqrt{k^2 + k'^2}}, \quad \lambda = \arg(\kappa + ik)). \\
\end{align}

We next use the identity \(1 + \exp(4\tilde{\phi}) = 2\exp(2\tilde{\phi})\cosh(2\tilde{\phi}),\) which yields

\begin{align}
f = (|k|/\sqrt{|\Lambda|})e^{i\lambda/2} \cosh(\tilde{\phi} + i\lambda/2)/((\cosh(2\tilde{\phi}) + 1/(2|\Lambda|)) \\
\text{By now absorbing } \lambda/2 \text{ into the phase angle } \phi \to \varphi = \phi + \tilde{a}\kappa + \lambda/2, \text{ we obtain} \\
u(t, x) = \frac{\exp(i\varphi)\exp(i(\kappa(x - \tilde{a}) + \omega t))(|k|/\sqrt{|\Lambda|})\cosh(k(x - \tilde{a}) + \omega t)}{\cosh(2(k(x - \tilde{a}) + \omega t)) + 1/(2|\Lambda|)} \\
\text{Thus we have the following result.}
\end{align}

**Proposition 3.** The harmonically modulated 1-soliton (1.9) for the Sasa-Satsuma equation (2.81) has an explicit rational \(\cosh\) form when \(\kappa \neq 0\) given by the envelope function (up to space translations on \(x\) and phase shifts on \(\phi\))

\begin{align}
f_{SS}(\theta) = \frac{|k|/2|k|/2}((k^2 + k'^2)^{1/4}\cosh(\theta + i\lambda/2)}{\sqrt{|k|} \cosh(2\theta) + (k^2 + k'^2)^{1/2}}, \quad \theta = k(x - (k^2 - 3k'^2)t)} \\
\text{where } \lambda \text{ is given by equation (2.98). This function is invariant under reflections } k \to -k.
\end{align}

We will next derive the explicit form of the envelope 2-soliton solution. For \(N=2,\) the lowest degree terms in the ansatz (2.83) are given by

\begin{align}
G^{(1)} = A_1e^{\Theta_1} + A_2e^{\Theta_2}, \quad F^{(2)} = B_1e^{\Theta_1} + B_2e^{\Theta_2} + C e^{\Theta_1} + D e^{\Theta_2} + E e^{\Theta_2}, \\
H^{(2)} = D_1e^{\Theta_1} + D_2e^{\Theta_2} + E_1e^{\Theta_1} + E_2 e^{\Theta_2} \\
\text{with} \\
\Theta_1 = k_1x + w_1t, \quad \Theta_2 = k_2x + w_2t, \quad \Theta_1 = \tilde{k}_1x + \tilde{w}_1t, \quad \Theta_2 = \tilde{k}_2x + \tilde{w}_2t, \\
\end{align}
where $k_1, k_2, w_1, w_2, A_1, A_2, C, D_1, D_2, E_1, E_2$ are complex constants and $B_1, B_2$ are real constants. Similarly to the $N = 1$ case, the two lowest degree equations (2.16a) and (2.16b) in the split bilinear system yield exactly the results (2.31) and (2.32) obtained for the Hirota solution, while the next equation (2.84c) determines

$$D_1 = 6A_1\bar{A}_1(k_1 - \bar{k}_1), \quad D_2 = 6A_2\bar{A}_2(k_2 - \bar{k}_2), \quad E_1 = 6A_1\bar{A}_2(k_1 - \bar{k}_2) = -\bar{E}_2. \quad (2.105)$$

In contrast the next two lowest degree equations (2.84d) and (2.84e) now contain many more inhomogeneous monomial terms than in the Hirota case. As a consequence, the ansatz (2.83) continues past degrees 3 and 4 for $G$ and $F$ and finally turns out to terminate at degree 7 for $G$, degree 8 for $F$, and degree 6 for $H$. The higher degree equations that determine these terms are given by

$$H_6 = 6D_x(G^{(1)}, \bar{G}^{(5)}) + 6D_x(G^{(5)}, \bar{G}^{(1)}) + 6D_x(G^{(3)}, \bar{G}^{(3)}) - F^{(4)}H^{(2)} - F^{(2)}H^{(4)} \quad (2.106a)$$

$$D_t(G^{(7)}, 1) + D^3_x(G^{(7)}, 1) = -D_t(G^{(5)}, F^{(2)}) - D^3_x(G^{(5)}, F^{(2)}) + G^{(5)}H^{(2)}$$

$$- D_t(G^{(3)}, F^{(4)}) - D^3_x(G^{(3)}, F^{(4)}) + G^{(3)}H^{(4)} \quad (2.106b)$$

$$D^2_x(F^{(8)}, 1) - 4(G^{(7)}\bar{G}^{(1)} + G^{(1)}\bar{G}^{(7)}) = 4(G^{(5)}\bar{G}^{(3)} + G^{(3)}\bar{G}^{(5)}) - \frac{1}{2}D^2_x(F^{(4)}, F^{(4)}) \quad (2.106c)$$

Omitting all details, we find that equations (2.84f)–(2.84h) and (2.106a)–(2.106c) in the split bilinear system lead to the following results. The higher degree terms in $G$ and $F$ consist of

$$G^{(3)} = F_1e^{2\theta_1+\theta_1} + F_2e^{2\theta_2+\theta_2} + F_3e^{2\theta_1+\theta_2} + F_4e^{2\theta_2+\theta_1} + G_2e^{\theta_2+\theta_1+\theta_1} \quad (2.107)$$

$$G^{(5)} = H_1e^{\theta_1+2\theta_2+2\bar{\theta}_1} + H_2e^{2\theta_1+\theta_2+2\bar{\theta}_2} + H_3e^{\theta_1+2\theta_2+2\bar{\theta}_2} + H_4e^{2\theta_1+\theta_2+2\bar{\theta}_1}$$

$$+ I_1e^{2\theta_1+\theta_2+\bar{\theta}_1+\bar{\theta}_2} + I_2e^{\theta_1+2\theta_2+\bar{\theta}_1+\bar{\theta}_2} \quad (2.108)$$

$$G^{(7)} = J_1e^{2\theta_1+2\theta_2+\theta_1+2\theta_2} + J_2e^{2\theta_1+2\theta_2+\theta_2+2\theta_1} \quad (2.109)$$

and

$$F^{(4)} = K_1e^{2\theta_1+2\bar{\theta}_1} + K_2e^{2\theta_2+2\bar{\theta}_2} + K_3e^{2\theta_1+2\theta_2} + K_3e^{2\theta_2+2\theta_1} + L_1e^{\theta_1+\theta_2+2\bar{\theta}_1} + L_1e^{\theta_1+2\theta_2+2\bar{\theta}_1}$$

$$+ L_2e^{\theta_1+\theta_2+2\theta_2} + L_2e^{\theta_2+\theta_1+\bar{\theta}_1+\bar{\theta}_2} + M_1e^{\theta_1+\theta_2+\bar{\theta}_1+\bar{\theta}_2} \quad (2.110)$$

$$F^{(6)} = N_1e^{2\theta_1+\theta_2+2\bar{\theta}_1+\bar{\theta}_2} + N_2e^{\theta_1+2\theta_2+\bar{\theta}_1+2\bar{\theta}_2} + Oe^{2\theta_1+\theta_2+2\bar{\theta}_1+\bar{\theta}_2} + Oe^{\theta_1+2\theta_2+2\bar{\theta}_1+\bar{\theta}_2} \quad (2.111)$$

$$F^{(8)} = P_1e^{2\theta_1+2\theta_2+2\bar{\theta}_1+2\bar{\theta}_2} \quad (2.112)$$
where \( F_1, F_2, F_3, F_4, G_1, G_2, H_1, H_2, H_3, H_4, I_1, I_2, J_1, J_2 \) are complex constants given by

\[
F_1 = \frac{A_1^2 \bar{A}_1(k_1 - \bar{k}_1)}{k_1(k_1 + k_1)^2}, \quad F_2 = \frac{A_2^2 \bar{A}_2(k_2 - \bar{k}_2)}{k_2(k_2 + k_2)^2}, \tag{2.113}
\]

\[
F_3 = \frac{A_1^2 \bar{A}_2(k_1 - k_2)}{k_1(k_1 + k_2)^2}, \quad F_4 = \frac{A_2^2 \bar{A}_1(k_2 - k_1)}{k_2(k_2 + k_1)^2}, \tag{2.114}
\]

\[
G_1 = \frac{2A_1A_2 \bar{A}_2((k_1 + k_2)(k_1 - k_2)^2 + (k_1 - \bar{k}_2)(k_1 + \bar{k}_2)^2 + (k_2 - \bar{k}_1)(k_2 + \bar{k}_1)^2)}{(k_1 + k_2)(k_1 + k_2)^2(k_2 + k_2)^2}, \tag{2.115}
\]

\[
G_2 = \frac{2A_1A_2 \bar{A}_1((k_1 + k_2)(k_1 - k_2)^2 + (k_1 - \bar{k}_1)(k_1 + \bar{k}_1)^2 + (k_2 - \bar{k}_1)(k_2 + \bar{k}_1)^2)}{(k_1 + k_2)(k_1 + k_1)^2(k_2 + k_1)^2}, \tag{2.116}
\]

\[
H_1 = \frac{-A_1A_2^2 \bar{A}_1^2(k_1 - \bar{k}_1)(k_1 - k_2)^2(k_2 - \bar{k}_1)^2}{k_2(k_1 + k_1)^2(k_1 + k_1)^2(k_2 + k_1)^4}, \tag{2.117}
\]

\[
H_2 = \frac{-A_2A_1^2 \bar{A}_2^2(k_2 - \bar{k}_2)(k_1 - k_2)^2(k_1 - k_2)^2}{k_1(k_1 + k_2)^2(k_2 + k_2)^2(k_1 + k_2)^4}, \tag{2.118}
\]

\[
H_3 = \frac{-A_1A_2^2 \bar{A}_2^2(k_1 - \bar{k}_2)(k_1 - k_2)^2(k_2 - \bar{k}_2)^2}{k_2(k_1 + k_2)^2(k_1 + k_1)^2(k_2 + k_2)^4}, \tag{2.119}
\]

\[
H_4 = \frac{-A_2A_1^2 \bar{A}_1^2(k_2 - \bar{k}_1)(k_1 - k_2)^2(k_1 - k_2)^2}{k_1(k_1 + k_2)^2(k_2 + k_2)^2(k_1 + k_1)^4}, \tag{2.120}
\]

\[
I_1 = -2A_1^2A_2 \bar{A}_1 \bar{A}_2(k_1 - \bar{k}_1)(k_1 - k_2)^2 \times \frac{(k_2 - \bar{k}_1)(k_2 + \bar{k}_1)^2 + (k_2 - \bar{k}_2)(k_2 + \bar{k}_2)^2 - (k_1 + k_2)(k_1 - k_2)^2}{k_1(k_1 + k_2)(k_1 + k_2)(k_1 + k_1)^2(k_2 + k_2)^2(k_1 + k_1)^2(k_2 + k_2)^2}, \tag{2.121}
\]

\[
I_2 = -2A_1^2A_2 \bar{A}_1 \bar{A}_2(k_2 - \bar{k}_1)(k_2 - k_2)^2 \times \frac{(k_1 - \bar{k}_1)(k_1 + k_1)^2 + (k_1 - \bar{k}_2)(k_1 + k_2)^2 - (k_1 + k_2)(k_1 - k_2)^2}{k_2(k_1 + k_2)(k_1 + k_2)(k_1 + k_1)^2(k_2 + k_2)^2(k_2 + k_1)^2(k_2 + k_2)^2}, \tag{2.122}
\]

\[
J_1 = \frac{A_1^2A_2^2 \bar{A}_1 \bar{A}_2^2(k_2 - \bar{k}_1)(k_2 - \bar{k}_2)(k_2 - k_2)^2(k_2 - \bar{k}_2)^2(k_1 - k_2)^2(k_1 - k_2)^4}{k_1k_2k_2(k_1 + k_2)(k_1 + k_2)^2(k_1 + k_1)^2(k_1 + k_2)^4(k_1 + k_2)^4(k_1 + k_1)^4}, \tag{2.123}
\]

\[
J_2 = \frac{A_1^2A_2^2 \bar{A}_1 \bar{A}_2^2(k_1 - \bar{k}_2)(k_2 - \bar{k}_2)(k_1 - \bar{k}_2)^2(k_2 - \bar{k}_2)^2(k_2 - k_2)^2(k_2 - k_2)^4}{k_1k_2k_1(k_1 + k_2)(k_1 + k_2)^2(k_1 + k_2)^2(k_2 + k_2)^2(k_2 + k_2)^4(k_1 + k_1)^4}, \tag{2.124}
\]

and where \( K_1, K_2, M, N_1, N_2, P \) are real constants and \( K_3, L_1, L_2, O \) are complex constants given by

\[
K_1 = \frac{-A_1^2 \bar{A}_1^2(k_1 - \bar{k}_1)^2}{k_1(k_1 + k_1)^4}, \quad K_2 = \frac{-A_2^2 \bar{A}_2^2(k_2 - \bar{k}_2)^2}{k_2k_2(k_2 + k_2)^4}, \tag{2.125}
\]

\[
K_3 = \frac{-A_2^2 \bar{A}_2^2(k_1 - \bar{k}_2)^2}{k_1k_2(k_1 + k_2)^4}, \tag{2.126}
\]

\[
L_1 = \frac{-4A_1A_2 \bar{A}_1^2(k_1 - \bar{k}_1)(k_2 - \bar{k}_1)}{k_1(k_1 + k_2)(k_1 + k_1)^2(k_2 + k_1)^2}, \tag{2.127}
\]
For completeness, we also list the higher degree terms in $H$:

$H^{(4)} = Q_1 e^{\Theta_1 + \Theta_2 + 2\Theta_1} + Q_2 e^{\Theta_1 + \Theta_2 + 2\Theta_2} + Q_3 e^{2\Theta_1 + \Theta_1 + \Theta_2} + Q_4 e^{2\Theta_2 + \Theta_1 + \Theta_2} + Re^{\Theta_1 + 2\Theta_2 + \Theta_1 + \Theta_2}$

$H^{(6)} = S_1 e^{2\Theta_1 + \Theta_2 + 2\Theta_1 + \Theta_2} + S_2 e^{\Theta_1 + 2\Theta_2 + \Theta_1 + \Theta_2} + S_3 e^{\Theta_1 + 2\Theta_2 + \Theta_1 + \Theta_2} + S_4 e^{2\Theta_1 + \Theta_2 + \Theta_1 + \Theta_2}$

where $Q_1, Q_2, Q_3, Q_4, R, S_1, S_2, S_3, S_4$ are complex constants given by

$Q_1 = -\frac{6A_1 A_2 A_2^2 (k_1 - \tilde{k}_1)(k_2 - \tilde{k}_2)(k_1 - k_2)^2}{k_1(k_1 + k_2)(k_1 + k_2)^2(k_2 + k_1)^2(k_2 + k_2)^2} = -\tilde{Q}_3$, \hspace{1cm} (2.139)

$Q_2 = -\frac{6A_1 A_2 A_2^2 (k_1 - \tilde{k}_1)(k_2 - \tilde{k}_2)(k_1 - k_2)^2}{k_2(k_1 + k_2)(k_1 + k_2)^2(k_2 + k_1)^2(k_2 + k_2)^2} = -\tilde{Q}_4$, \hspace{1cm} (2.140)

$R = -\frac{12A_1 A_2 A_1 A_2 (R_1 + R_2 + R_3 + R_4 + R_4)}{(k_1 + k_2)(k_1 + k_2)^2(k_1 + k_1)^2(k_2 + k_1)^2(k_2 + k_2)^2}$, \hspace{1cm} (2.141)

$R_1 = -k_1 \tilde{k}_1 (k_1 - k_1)(k_1 + k_1)^2((k_2 - \tilde{k}_2)^2(k_2 - \tilde{k}_2)^2 + (k_2 - k_2)^2(k_2 - k_2)^2 + (k_2 - \tilde{k}_2)^2(k_2 - \tilde{k}_2)^2)$, \hspace{1cm} (2.142)

$R_2 = -k_2 \tilde{k}_2 (k_2 - \tilde{k}_2)(k_2 + k_2)^2((k_1 - \tilde{k}_1)^2(k_1 - \tilde{k}_1)^2 + (k_1 - k_1)^2(k_1 - k_1)^2 + (k_1 - \tilde{k}_1)^2(k_1 - \tilde{k}_1)^2)$, \hspace{1cm} (2.143)

$R_3 = k_1 k_2 (k_1 + k_2)(k_1 - k_2)^2((k_2 - \tilde{k}_2)^2(k_2 - \tilde{k}_2)^2 + (k_2 - k_2)^2(k_2 - k_2)^2 + (k_2 - \tilde{k}_2)^2(k_2 - \tilde{k}_2)^2)$, \hspace{1cm} (2.144)

$R_4 = -k_2 \tilde{k}_2 (k_1 - \tilde{k}_1)(k_1 + k_2)^2((k_2 - \tilde{k}_2)^2(k_2 - \tilde{k}_2)^2 - (k_2 - k_2)^2(k_2 - k_2)^2 - (k_2 - \tilde{k}_2)^2(k_2 - \tilde{k}_2)^2)$, \hspace{1cm} (2.145)

$S_1 = \frac{6A_1^2 A_2 A_2^2 (k_1 - \tilde{k}_1)(k_2 - \tilde{k}_2)(k_1 - k_2)^2(k_1 - \tilde{k}_2)^2}{k_1 k_2 (k_1 + k_2)(k_1 + k_2)^2(k_1 + k_1)^2(k_2 + k_2)^2}$, \hspace{1cm} (2.146)

$S_3 = \frac{6A_1 A_2^2 A_1 A_2^2 (k_1 - \tilde{k}_1)(k_2 - \tilde{k}_2)(k_1 - k_2)^2(k_1 - \tilde{k}_2)^2}{k_2 k_2 (k_1 + k_2)(k_1 + k_2)^2(k_1 + k_1)^2(k_2 + k_2)^2}$, \hspace{1cm} (2.147)
\[ S_4 = \frac{6A_1^2A_2\hat{A}_1\hat{A}_2(k_1 - \bar{k}_2)(k_2 - \bar{k}_1)(k_2 - \bar{k}_2)(k_1 - \bar{k}_1)^2(k_1 - \bar{k}_2)^2}{k_1k_1(k_1 + k_2)(k_1 + k_2)(k_1 + k_2)^2(k_2 + k_1)^2(k_1 + k_1)^4}. \] (2.148)

Expressions (2.113)–(2.136) yield the 2-soliton solution
\[ u = \frac{A_1e^{\Theta_1}(1 + V_1) + A_2e^{\Theta_2}(1 + V_2)}{1 + W}. \] (2.149)

with
\begin{align*}
V_1 &= |A_1|^2\Lambda_1e^{2\text{Re} \Theta_1} + |A_2|^2(\Pi_1 + |A_2|^2|\Lambda_2|^2\psi_2e^{2\text{Re} \Theta_2})e^{2\text{Re} \Theta_2} \\
&+ |A_1|^2|A_2|^2\Lambda_1\psi_2(\Pi_1 + |A_2|^2|\Lambda_2|^2\psi_2e^{2\text{Re} \Theta_2})e^{2\text{Re} (\Theta_1 + \Theta_2)} \\
&+ A_1\bar{A}_2\Sigma_1(1 + |A_2|^2\Lambda_2\psi_2e^{2\text{Re} \Theta_2})e^{i\text{Im} (\Theta_1 - \Theta_2)}e^{\text{Re} (\Theta_1 + \Theta_2)},
\end{align*}
(2.150)
\begin{align*}
V_2 &= |A_2|^2\Lambda_2e^{2\text{Re} \Theta_2} + |A_1|^2(\Pi_2 + |A_1|^2|\Lambda_1|^2\psi_1e^{2\text{Re} \Theta_1})e^{2\text{Re} \Theta_1} \\
&+ |A_1|^2|A_2|^2\Lambda_2\psi_1(\Pi_2 + |A_1|^2|\Lambda_1|^2\psi_1e^{2\text{Re} \Theta_1})e^{2\text{Re} (\Theta_1 + \Theta_2)} \\
&+ A_2\Lambda_1\Sigma_2(1 + |A_1|^2\Lambda_1\psi_1e^{2\text{Re} \Theta_1})e^{i\text{Im} (\Theta_2 - \Theta_1)}e^{\text{Re} (\Theta_1 + \Theta_2)},
\end{align*}
(2.151)
\begin{align*}
W &= |A_1|^2(\Omega_1 + |A_1|^2|A_1|^2e^{2\text{Re} \Theta_1})e^{2\text{Re} \Theta_1} + |A_2|^2(\Omega_2 + |A_2|^2|\Lambda_2|^2e^{2\text{Re} \Theta_2})e^{2\text{Re} \Theta_2} \\
&+ |A_1|^2|A_2|^2(\Pi + |A_1|^2\Omega_2|A_2|^2\psi_2e^{2\text{Re} \Theta_1} + |A_2|^2\Omega_1|A_2|^2\psi_2e^{2\text{Re} \Theta_2})e^{2\text{Re} (\Theta_1 + \Theta_2)} \\
&+ |A_1|^2|A_2|^2\Lambda_1^2\psi_2e^{4\text{Re} (\Theta_1 + \Theta_2)} + 2\text{Re} \left( (A_1\bar{A}_2)^2\Sigma_1\Sigma_2e^{2\text{Im} (\Theta_1 - \Theta_2)} \right) e^{2\text{Re} (\Theta_1 + \Theta_2)} \\
&+ 8\text{Re} \left( A_1\bar{A}_2\Phi (1 + |A_1|^2\Lambda_1\Delta_1e^{2\text{Re} \Theta_1} + |A_2|^2\Lambda_2\Delta_2e^{2\text{Re} \Theta_2})e^{i\text{Im} (\Theta_1 - \Theta_2)} \right) e^{\text{Re} (\Theta_1 + \Theta_2)} \\
&+ 8|A_1|^2|A_2|^2\text{Re} \left( A_1\bar{A}_2\Phi A_1\bar{A}_2\psi_2e^{i\text{Im} (\Theta_1 - \Theta_2)} \right) e^{3\text{Re} (\Theta_1 + \Theta_2)}
\end{align*}
(2.152)

where
\begin{align*}
\Theta_1 &= k_1(x - k_1^2t), & \Theta_2 &= k_2(x - k_2^2t), \\
\Omega_1 &= \frac{B_1}{|A_1|^2} = \frac{1}{(\text{Re} k_1)^2}, & \Omega_2 &= \frac{B_2}{|A_2|^2} = \frac{1}{(\text{Re} k_2)^2}, \\
\Lambda_1 &= \frac{F_1}{A_1|A_1|^2} = \frac{i\text{Im} k_1}{2k_1(\text{Re} k_1)^2}, & \Lambda_2 &= \frac{F_2}{A_2|A_2|^2} = \frac{i\text{Im} k_2}{2k_2(\text{Re} k_2)^2}, \\
\Sigma_1 &= \frac{F_3}{A_1^2A_2} = \frac{k_1 - k_2}{k_1 + k_2}, & \Sigma_2 &= \frac{F_4}{A_2^2A_1} = \frac{k_2 - k_1}{k_2 + k_1}, \\
\Delta_1 &= \frac{\bar{A}_1L_1}{F_1C} = \frac{k_2 - k_1}{k_1 + k_2}, & \Delta_2 &= \frac{\bar{A}_2L_2}{F_2C} = \frac{k_1 - k_2}{k_1 + k_2}, \\
\Psi_1 &= \frac{\bar{A}_1H_1}{F_1F_4} = \frac{|A_1|^2H_4}{A_2|A_1|^2} = \frac{(k_2 - k_1)(k_1 - k_2)^2}{(k_1 + k_2)(k_1 + k_2)^2}, \\
\Psi_2 &= \frac{\bar{A}_2H_2}{F_2F_3} = \frac{|A_2|^2H_3}{A_1|F_2|^2} = \frac{(k_1 - k_2)(k_1 - k_2)^2}{(k_1 + k_2)(k_1 + k_2)^2}, \\
\Xi_1 &= \frac{2\bar{A}_2F_1L_2}{F_2F_3B_1} = \frac{4i\text{Im} k_1}{k_1 + k_2}, & \Xi_2 &= \frac{2\bar{A}_1F_2L_1}{F_1F_4B_2} = \frac{4i\text{Im} k_2}{k_1 + k_2}, \\
\Pi_1 &= \frac{G_1}{A_1|A_2|^2} = \frac{A_2F_1\bar{I}_1}{A_2F_1\bar{H}_2} \\
&= \frac{(k_1 - k_2)^2}{2(\text{Re} k_2)^2(k_1 + k_2)^2} + \frac{k_1 - k_2}{2(\text{Re} k_2)^2(k_1 + k_2)} + \frac{4i\text{Im} k_2}{(k_1 + k_2)(k_1 + k_2)^2}. \tag{2.161}
\end{align*}
\[ \Pi_2 = \frac{G_2}{A_2|A_1|^2} = \frac{F_1 F_4 I_2}{A_1 F_2 H_1} \]
\[ = \frac{(k_1 - k_2)^2}{2(\text{Re} k_1)^2(k_2+k_1)^2} + \frac{k_2 - \bar{k}_1}{2(\text{Re} k_1)^2(k_1+k_2)^2} + \frac{4i\text{Im} k_1}{(k_1+k_2)(k_2+k_1)^2}, \tag{2.162} \]
\[ \Phi = \frac{C}{4A_1 A_2} = \frac{1}{(k_1+k_2)^2}, \tag{2.163} \]
\[ \Psi^2 = |\Psi_1|^2 = |\Psi_2|^2 = \frac{|A_1||A_2|\sqrt{P}}{|F_1||F_2|} = \frac{|A_1|^2 N_1}{|F_1|^2 B_2} = \frac{|A_2|^2 N_2}{|F_2|^2 B_1} = \frac{A_1 J_2}{F_1 H_3} = \frac{A_2 J_2}{F_2 H_4} \]
\[ = \frac{|k_1 - k_2|^4}{|k_1 + k_2|^4}; \tag{2.164} \]
\[ \Delta = |\Delta_1| = |\Delta_2| = \frac{|A_1||L_1|}{|F_1||C|} = \frac{|A_2||L_2|}{|F_2||C|} = \frac{|k_1 - \bar{k}_2|}{|k_1 + k_2|^4}; \tag{2.165} \]
\[ \Xi = 2\text{Re} (\Xi_1 \bar{\Xi}_2) = \text{Re} \left( \frac{A_1 \bar{A}_2 L_2 \bar{L}_1}{F_3 F_4} \right) \frac{8}{B_1 B_2} = \frac{32\text{Im} k_1 \text{Im} k_2}{|k_1 + k_2|^2}, \tag{2.166} \]
\[ \Pi = \frac{M}{|A_1|^2|A_2|^2} \]
\[ = \frac{|k_1 - k_2|^4}{2(\text{Re} k_1 \text{Re} k_2)^2|k_1 + k_2|^4} + \frac{|k_1 - \bar{k}_2|^2}{2(\text{Re} k_1 \text{Re} k_2)^2|k_1 + k_2|^2} + \frac{32\text{Im} k_1 \text{Im} k_2}{|k_1 + k_2|^4}. \tag{2.167} \]

This solution (2.149) can be shown to reduce to the ordinary 2-soliton solution [12, 11] for the Sasa-Satsuma equation when \( \kappa_1 = \kappa_2 = 0 \). For \( \kappa_1 \neq 0 \) and \( \kappa_2 \neq 0 \), it can be written in the form of a harmonically modulated soliton similarly to the Hirota case by use of the notation (2.45), (2.46), (2.47). To proceed, we first we observe that the highest-degree monomial terms in the numerator and denominator of expression (2.149) consist of |\( \Lambda_1|^2 \Psi_2 \Lambda_2 \Psi_1 e^{4\text{Re} \Theta_1} e^{2\text{Re} \Theta_2}, |\Lambda_2|^2 \Psi_2 \Lambda_1 \Psi_2 e^{4\text{Re} \Theta_2} e^{2\text{Re} \Theta_1} \). Based on the form of their coefficients, we write
\[ |A_1|\sqrt{|\Lambda_1|} \Psi = e^{-a_1 k_1}, \quad |A_2|\sqrt{|\Lambda_2|} \Psi = e^{-a_1 k_2}, \tag{2.168} \]
and
\[ \text{Re} \Theta_1 - a_1 k_1 = k_1(x - a_1) + w_1 t = \theta_1, \quad \text{Re} \Theta_2 - a_2 k_2 = k_2(x - a_2) + w_2 t = \theta_2, \tag{2.169} \]
\[ \text{Im} \Theta_1 = \kappa_1 x + \omega_1 t = \vartheta_1, \quad \text{Im} \Theta_2 = \kappa_2 x + \omega_2 t = \vartheta_2. \tag{2.170} \]
Then, in the denominator of the 2-soliton solution (2.149), we have
\[ W = e^{4\theta_1 + 4\theta_2} + \frac{1}{\Psi^2} \left( e^{4\theta_1} + e^{4\theta_2} \right) + \frac{\Omega_1}{\Psi|\Lambda_1|} \left( 1 + e^{4\theta_2} \right) e^{2\theta_1} + \frac{\Omega_2}{\Psi|\Lambda_2|} \left( 1 + e^{4\theta_1} \right) e^{2\theta_2} \]
\[ + \frac{1}{\Psi^2 |\Lambda_1||\Lambda_2|} \left( \Pi + 2\text{Re} \left( \Sigma_1 \bar{\Sigma}_2 e^{i(\phi_1 - \phi_2)} e^{i(\theta_1 - \theta_2)} \right) \right) e^{2\theta_1 + 2\theta_2} \]
\[ + \frac{8\Delta}{\Psi^2 \sqrt{|\Lambda_1||\Lambda_2|}} \text{Re} \left( \Phi e^{i(\phi_1 - \phi_2)} e^{i(\theta_1 - \theta_2)} \left( \frac{\bar{\Lambda}_2 \Delta_2}{|\Lambda_2|^2 \Delta} e^{2\theta_2} + \frac{\Lambda_1 \bar{\Delta}_1}{|\Lambda_1|^2 \Delta} e^{2\theta_1} \right) \right) e^{\theta_1 + \theta_2} \]
\[ + \frac{8}{\Psi \sqrt{|\Lambda_1||\Lambda_2|}} \text{Re} \left( e^{i(\phi_1 - \phi_2)} e^{i(\theta_1 - \theta_2)} \left( \Phi + \Phi \frac{\bar{\Lambda}_2 \Phi}{|\Lambda_2|\Psi} \frac{\Lambda_1 \bar{\Psi}}{|\Lambda_1|\Psi} e^{2\theta_1 + 2\theta_2} \right) \right) e^{\theta_1 + \theta_2}. \tag{2.171} \]
In the numerator of the 2-soliton solution (2.149), we have

\[ V_1 = \frac{\Lambda_1 \Psi_2}{|\Lambda_1| \Psi} e^{2\theta_1 + \delta_2} + \frac{1}{\Psi} \left( \frac{\Lambda_1}{|\Lambda_1|} e^{\theta_1} + \frac{\Psi_2}{\Psi} e^{\delta_2} \right) + \frac{1}{\Psi |\Delta_2|} \left( \Pi_1 + \frac{\Lambda_1 \Psi_2 \bar{\Pi}_1 e^{2\theta_1}}{|\Lambda_1| \Psi} \right) e^{2\theta_2} \]

\[ + \frac{\Sigma_1}{\Psi \sqrt{|\Lambda_1||\Delta_2|}} \left( 1 + \frac{\Lambda_2 \Psi_2}{|\Lambda_2| \Psi} e^{2\theta_2} \right) e^{i(\phi_1 - \phi_2)} e^{i(\phi_1 - \phi_2) e^{\theta_1 + \theta_2}} \]

and

\[ V_2 = \frac{\Lambda_2 \Psi_1}{|\Lambda_2| \Psi} e^{2\theta_2 + \delta_1} + \frac{1}{\Psi} \left( \frac{\Lambda_2}{|\Lambda_2|} e^{\theta_2} + \frac{\Psi_1}{\Psi} e^{\delta_1} \right) + \frac{1}{\Psi |\Lambda_1|} \left( \Pi_2 + \frac{\Lambda_2 \Psi_1 \bar{\Pi}_2 e^{2\theta_2}}{|\Lambda_2| \Psi} \right) e^{2\theta_1} \]

\[ + \frac{\Sigma_2}{\Psi \sqrt{|\Lambda_1||\Lambda_2|}} \left( 1 + \frac{\Lambda_1 \Psi_1}{|\Lambda_1| \Psi} e^{2\theta_1} \right) e^{i(\phi_2 - \phi_1)} e^{i(\phi_2 - \phi_1) e^{\theta_1 + \theta_2}} \]

where

\[ \Pi_1 = \frac{\Omega_2}{2} \left( \frac{\Psi_2}{\Delta_2} + \Delta_2 \right) + \Xi_2 \Phi, \quad \Pi_2 = \frac{\Omega_1}{2} \left( \frac{\Psi_1}{\Delta_1} + \Delta_1 \right) + \Xi_1 \bar{\Phi}, \]

\[ \Pi = \Xi |\Phi|^2 + \frac{\Omega_1 \Omega_2}{2} \left( \frac{\Psi^2}{\Delta_2} + \Delta_2^2 \right). \]

Next we write

\[ \lambda_1 = \arg(\Lambda_1) = \arg(i \bar{k}_1 \text{Im } k_1), \quad \lambda_2 = \arg(\Lambda_2) = \arg(i \bar{k}_2 \text{Im } k_2), \]

\[ \delta_1 = \arg(\Delta_1) = \arg((\bar{k}_1 + \bar{k}_2)(k_2 - \bar{k}_1)), \]

\[ \delta_2 = \arg(\Delta_2) = \arg((\bar{k}_1 + \bar{k}_2)(k_1 - \bar{k}_2)), \]

\[ \sigma_1 = \arg(\Sigma_1) = \arg((\bar{k}_1 + \bar{k}_2)(k_1 - \bar{k}_2)), \]

\[ \sigma_2 = \arg(\Sigma_2) = \arg((k_2(k_2 - \bar{k}_1)(k_1 + \bar{k}_2))^2), \]

\[ \psi_1 = \arg(\Psi_1) = \arg((\bar{k}_1 + \bar{k}_2 - k_1)(k_1 - \bar{k}_2)), \]

\[ \psi_2 = \arg(\Psi_2) = \arg((\bar{k}_1 + \bar{k}_2 - k_2)(k_2 + \bar{k}_2))^2, \]

\[ \zeta_1 = \arg(\Xi_1 \bar{\Phi}) = \arg(i(\bar{k}_1 + \bar{k}_2)(k_1 + \bar{k}_2))^2 \text{Im } k_1, \]

\[ \zeta_2 = \arg(\Xi_2 \Phi) = \arg(i(\bar{k}_1 + \bar{k}_2)(k_1 + \bar{k}_2))^2 \text{Im } k_2, \]

and use the identities

\[ \frac{\Phi}{\Phi} = \frac{\Delta^2 \bar{\Psi}_1 \Psi_2}{\Psi^2 \Delta_1 \Delta_2}, \quad |\Phi| = -\Phi \left( \frac{\Delta_2 \Psi_1}{\Delta_1 \Psi_2} \right)^{1/2} = -\bar{\Phi} \left( \frac{\Delta_1 \Psi_2}{\Delta_2 \Psi_1} \right)^{1/2}. \]

Hence the 2-soliton solution takes the form

\[ u = e^{i\phi_1} e^{i\theta_1} f_1 + e^{i\phi_2} e^{i\theta_2} f_2 \]

with

\[ f_1 = X_1/Y, \quad f_2 = X_2/Y \]
given by

\[ X_1 = \frac{2}{\sqrt{|\Lambda_1||\Psi|}} \left( \exp(i\frac{1}{2}(\lambda_1 + \psi_1)) \left( \Psi \cosh(\theta_1 + 2\theta_2 + i\frac{1}{2}(\psi_2 + \lambda_1)) + \cosh(\theta_1 - 2\theta_2 + i\frac{1}{2}(\lambda_1 - \psi_2)) + \frac{|\Xi_2|\Phi}{|\Lambda_2|} \cosh(\theta_1 + i(\frac{1}{2}(\lambda_1 + \psi_1) - \zeta_2)) + \frac{\Omega_2}{2|\Lambda_2|} \left( \frac{\Psi}{\Delta} \cosh(\theta_1 + i(\frac{1}{2}(\lambda_1 - \psi_2) + \delta_2)) + \Delta \cosh(\theta_1 + i(\frac{1}{2}(\lambda_1 + \psi_2) - \delta_2)) \right) \right) + \exp(i(\sigma_1 + \frac{1}{2}(\psi_2 - \lambda_2))) \left( \frac{|\Sigma_1|}{\sqrt{|\Lambda_1||\Lambda_2|}} \exp(i(\vartheta_1 - \vartheta_2 + \phi_1 - \phi_2)) \times \cosh(\theta_2 + i\frac{1}{2}(\psi_2 - \lambda_2)) \right) \right) \]  

(2.184)

\[ X_2 = \frac{2}{\sqrt{|\Lambda_2||\Psi|}} \left( \exp(i\frac{1}{2}(\lambda_2 + \psi_1)) \left( \Psi \cosh(\theta_2 + 2\theta_1 + i\frac{1}{2}(\psi_1 + \lambda_2)) + \cosh(\theta_2 - 2\theta_1 + i\frac{1}{2}(\lambda_2 - \psi_1)) + \frac{|\Xi_1|\Phi}{|\Lambda_1|} \cosh(\theta_2 + i(\frac{1}{2}(\lambda_2 + \psi_1) - \zeta_1)) + \frac{\Omega_1}{2|\Lambda_1|} \left( \frac{\Psi}{\Delta} \cosh(\theta_2 + i(\frac{1}{2}(\lambda_2 - \psi_1) + \delta_1)) + \Delta \cosh(\theta_2 + i(\frac{1}{2}(\lambda_2 + \psi_1) - \delta_1)) \right) \right) + \exp(i(\sigma_2 + \frac{1}{2}(\psi_1 - \lambda_1))) \left( \frac{|\Sigma_2|}{\sqrt{|\Lambda_1||\Lambda_2|}} \exp(i(\vartheta_2 - \vartheta_1 + \phi_2 - \phi_1)) \times \cosh(\theta_1 + i\frac{1}{2}(\psi_1 - \lambda_1)) \right) \right) \]  

(2.185)

\[ Y = (1 + W)e^{-\theta_1 - 2\theta_2} \]

\[ = \frac{2}{\Psi} \left( \frac{\Omega_1}{|\Lambda_1|} \cosh(2\theta_2) + \frac{\Omega_2}{|\Lambda_2|} \cosh(2\theta_1) + \Psi \cosh(2(\theta_1 + \theta_2)) + \frac{1}{\Psi} \cosh(2(\theta_1 - \theta_2)) \right) - 8 \frac{|\Phi|}{\sqrt{|\Lambda_1||\Lambda_2|}} \text{Re} \left( \exp(i(\vartheta_1 - \vartheta_2 + \phi_1 - \phi_2 + \frac{1}{2}(\lambda_1 - \lambda_2 + \psi_2 - \psi_1))) \times \left( \cosh(\theta_1 + \theta_2 + i\frac{1}{2}(\delta_2 - \delta_1 + \lambda_1 - \lambda_2)) + \frac{\Delta}{\Psi} \cosh(\theta_1 - \theta_2 + i\frac{1}{2}(\lambda_1 + \lambda_2 - \delta_2 - \delta_1)) \right) \right) + \frac{1}{\Psi|\Lambda_1||\Lambda_2|} \left( |\Sigma_1||\Sigma_2| \cos(2(\vartheta_1 - \vartheta_2) + 2(\phi_1 - \phi_2) + \sigma_1 - \sigma_2) + \frac{\Xi|\Phi|^2}{2} + \frac{\Omega_1\Omega_2}{4}(\Delta^2 + \Psi^2) \right). \]  

(2.186)
In this form, the 2-soliton solution (2.182)–(2.186) closely resembles a harmonically modulated 2-soliton (1.14), except that \(a_1, a_2\) occur in \(\theta_1, \theta_2\) while \(\vartheta_1, \vartheta_2\) appear in the functions \(f_1, f_2\). The same space-time translation (2.62)–(2.63) used in the Hirota case can be applied here to absorb \(a_1, a_2\) on \(x\), and then the same identity (2.68) expressing \(\vartheta_1 - \vartheta_2\) as a linear combination of \(\theta_1\) and \(\theta_2\) can be used to convert \(f_1, f_2\) into the proper harmonically modulated form. Finally, the phase angles in the resulting expressions can be substantially simplified through phase shifts (2.66) given by

\[
\varphi_1 = (\lambda_1 + \psi_2)/2, \quad \varphi_2 = (\lambda_2 + \psi_1)/2,
\]

and through the angle identities

\[
\begin{align*}
\zeta_1 - \delta_2 - \rho_1 &= \sigma_2 - \lambda_2 + \rho_2 = \arg(i(k_1 - k_2)(k_1 + k_2)^2), \\
\zeta_2 - \delta_1 - \rho_2 &= \sigma_1 - \lambda_1 + \rho_1 = \arg(i(k_2 - k_1)(k_1 + k_2)^2), \\
(\psi_1 - \delta_1)/2 &= \arg((k_1 - k_2)(k_1 + \bar{k}_2)), \\
(\psi_2 - \delta_2)/2 &= \arg((k_2 - k_1)(\bar{k}_1 + k_2)),
\end{align*}
\]

(2.192) where

\[
\rho = \begin{cases} 
0, & |\text{Re } k_1| \neq |\text{Re } k_2|, \\
\arg(\text{Im } k_1 + \text{Im } k_2), & |\text{Re } k_1| = |\text{Re } k_2|.
\end{cases}
\]

This leads to the following result.

**Proposition 4.** When

\[
w_1/k_1 \neq w_2/k_2, \quad \kappa_1 \neq 0, \quad \kappa_2 \neq 0
\]

the harmonically modulated 2-soliton (1.14) for the Sasa-Satsuma equation (2.81) has an explicit rational \(\cosh\) form given by the envelope functions (up to space-time translations on \(t, x\))

\[
f_{1SS}(\theta_1, \theta_2) = X_{1SS}(\theta_1, \theta_2)/Y_{SS}(\theta_1, \theta_2), \quad f_{2SS}(\theta_1, \theta_2) = X_{2SS}(\theta_1, \theta_2)/Y_{SS}(\theta_1, \theta_2)
\]

(2.196) with

\[
X_{1SS}(\theta_1, \theta_2) = |k_1|(k_1^2 + \kappa_1^2)^{1/4}|2\kappa_1|^{1/2} \times
\]

\[
\left( |\kappa_2| \left( \sqrt{\Delta \Gamma} \cosh(\theta_1 + 2\theta_2 + i(\alpha_2 + \gamma_2)) + \frac{1}{\sqrt{\Delta}} \cosh(\theta_1 - 2\theta_2 + i(\nu_2 - \gamma_2)) \right) \\
+ (k_2^2 + \kappa_2^2)^{1/2} \left( -8k_2^2|\kappa_2| \frac{1}{\sqrt{\Delta \Gamma}} \cosh(\theta_1 + i(\rho + \nu_2 + \omega_1 + \gamma_1)) \\
+ \frac{\sqrt{\Gamma}}{\sqrt{\Delta}} \cosh(\theta_1 + i(\alpha_2 - \gamma_2)) + \frac{\sqrt{\Delta}}{\sqrt{\Gamma}} \cosh(\theta_1 + i(\nu_2 + \gamma_2)) \right) \right) \\
- k_1^2|k_2|(k_2^2 + \kappa_2^2)^{1/4}|8\kappa_2|^{1/2} \times
\]

\[
\left( \frac{\sqrt{\Gamma}}{\sqrt{\Delta}} \exp(i(\mu_1\theta_1 - \mu_2\theta_2 + \phi_1 - \phi_2)) \cosh(\theta_2 - i(\rho + \nu_1 + \omega_2 - \gamma_2)) \right),
\]

(2.197)
\[ X_{SS}(\theta_1, \theta_2) = |k_2| (k_2^2 + \kappa_2^2)^{1/4}|2\kappa_2|^{1/2} \times \]
\[ \left( |\kappa_1| \left( \sqrt{\Delta} \cosh(\theta_2 + 2\theta_1 + i(\alpha_1 + \gamma_1)) + \frac{1}{\sqrt{\Delta}} \cosh(\theta_2 - 2\theta_1 + i(v_1 - \gamma_1)) \right) \right. \]
\[ + (k_2^2 + \kappa_2^2)^{1/2} \left( -8k_1^2|\kappa_1| \frac{1}{\sqrt{\Delta}} \cosh(\theta_2 + i(\rho + \rho_1 + \omega_2 + \gamma_2)) \right) \]
\[ + \frac{\sqrt{\Gamma}}{\sqrt{\Delta}} \cosh(\theta_2 + i(\alpha_1 - \gamma_1)) + \frac{\sqrt{\Delta}}{\sqrt{\Gamma}} \cosh(\theta_2 + i(v_1 + \gamma_1)) \right) \]
\[ - k_2^2|\kappa_1|(k_2^2 + \kappa_2^2)^{1/4}|8\kappa_1|^{1/2} \times \]
\[ \left( \frac{\sqrt{\Omega}}{\sqrt{Y}} \exp(i(\mu_2 \theta_2 - \mu_1 \theta_1 + \phi_2 - \phi_1)) \cosh(\theta_1 - i(\rho + \rho_2 + \omega_1 - \gamma_1)) \right), \]

\[ Y_{SS}(\theta_1, \theta_2) = (k_2^2 + \kappa_2^2)^{1/2}(k_2^2 + \kappa_2^2)^{1/2} \left( \frac{\Gamma}{\Delta} + \frac{\Delta}{\Gamma} + 64k_1^2k_2^2k_1\kappa_2 \frac{1}{\Omega Y} \right) \]
\[ + |\kappa_1\kappa_2| \left( \Delta \Gamma \cosh(2(\theta_1 + \theta_2)) + \frac{1}{\Delta \Gamma} \cosh(2(\theta_1 - \theta_2)) \right) \]
\[ + 2|\kappa_2|(k_2^2 + \kappa_2^2)^{1/2} \cosh(2\theta_2) + 2|\kappa_1|(k_2^2 + \kappa_2^2)^{1/2} \cosh(2\theta_1) \]
\[ + 4k_1^2k_2^2 \frac{\Omega}{Y} \cos(2(\mu_1 \theta_1 - \mu_2 \theta_2) + 2(\phi_1 - \phi_2) + \rho_2 - \rho_1) \]
\[ - 16|k_1k_2||\kappa_1\kappa_2|^{1/2}(k_1^2 + \kappa_1^2)^{1/4}(k_2^2 + \kappa_2^2)^{1/4} \Re \left( \exp(i(\mu_1 \theta_1 - \mu_2 \theta_2 + \phi_1 - \phi_2)) \times \right) \]
\[ \left( \frac{\sqrt{\Gamma}}{\sqrt{\Delta}} \cosh(\theta_1 + \theta_2 + i(\omega_1 - \omega_2)) + \frac{1}{\sqrt{\Delta \Gamma}} \cosh(\theta_1 - \theta_2 + i(\omega_1 + \omega_2)) \right), \]

where \( w_1, w_2, \omega_1, \omega_2 \) are given by equation (2.46) and \( \mu_1, \mu_2 \) are given by equation (2.69), and where

\[ \Omega = \sqrt{((k_1 + k_2)^2 + (\kappa_1 + \kappa_2)^2)((k_1 - k_2)^2 + (\kappa_1 + \kappa_2)^2)}, \]

\[ \gamma = ((k_1 - k_2)^2 + (\kappa_1 - \kappa_2)^2)((k_1 + k_2)^2 + (\kappa_1 - \kappa_2)^2), \]

\[ \Delta = \sqrt{(k_1 - k_2)^2 + (\kappa_1 + \kappa_2)^2}, \]

\[ \Gamma = \sqrt{(k_1 - k_2)^2 + (\kappa_1 - \kappa_2)^2}, \]

\[ \alpha_1 = (\lambda_2 + \delta_1)/2, \quad \alpha_2 = (\lambda_1 + \delta_2)/2, \]

\[ \nu_1 = (\lambda_2 - \delta_1)/2, \quad \nu_2 = (\lambda_1 - \delta_2)/2, \]

\[ \omega_1 = (\lambda_1 - \delta_1)/2, \quad \omega_2 = (\lambda_2 - \delta_2)/2, \]

\[ \gamma_1 = \arg(k_1^2 - k_2^2 - (\kappa_1 - \kappa_2)^2 + i2k_1(\kappa_1 - \kappa_2)), \]

\[ \gamma_2 = \arg(k_2^2 - k_1^2 - (\kappa_1 - \kappa_2)^2 - i2k_2(\kappa_1 - \kappa_2)), \]

\[ \delta_1 = \arg(k_2^2 - k_1^2 + (\kappa_1 + \kappa_2)^2 + i2k_1(\kappa_1 + \kappa_2)), \]

\[ \delta_2 = \arg(k_1^2 - k_2^2 + (\kappa_1 + \kappa_2)^2 + i2k_2(\kappa_1 + \kappa_2)), \]
\[ \lambda_1 = \arg(\kappa_1(\kappa_1 + ik_1)), \quad \lambda_2 = \arg(\kappa_2(\kappa_2 + ik_2)), \quad (2.209) \]
\[ \theta_1 = k_1x + w_1 t = k_1(x - (k_2^2 - 3\kappa_1^2)t), \quad (2.210) \]
\[ \theta_2 = k_2x + w_2 t = k_2(x - (k_2^2 - 3\kappa_2^2)t). \]

It will be useful to write out the half-angle expressions for \( \lambda_1/2 \), \( \lambda_2/2 \), \( \delta_1/2 \), \( \delta_2/2 \) appearing in the envelope functions (2.197)–(2.199). From the angle expressions (2.209) and (2.208), we obtain by a direct calculation
\[ \lambda_1/2 = \arg\left(\sqrt{1 + k_1^2/\kappa_1^2} + 1 + i\varepsilon_1\sqrt{1 + k_1^2/\kappa_1^2 - 1}\right), \quad (2.211) \]
\[ \lambda_2/2 = \arg\left(\sqrt{1 + k_2^2/\kappa_2^2} + 1 + i\varepsilon_2\sqrt{1 + k_2^2/\kappa_2^2 - 1}\right), \quad (2.212) \]
where
\[ \varepsilon_1 = \text{sgn}(\kappa_1), \quad \varepsilon_2 = \text{sgn}(\kappa_2), \quad (2.213) \]
and
\[ \delta_1/2 = \arg\left((\epsilon^2\varepsilon_- + (1 - \epsilon^2\varepsilon_-)\text{sgn}(\kappa_1 + \kappa_2))\sqrt{k_2^2 - k_1^2 + (\kappa_1 + \kappa_2)^2 + \Omega} \right. \]
\[ + i\varepsilon_1(1 + \epsilon^2\varepsilon_- (\text{sgn}(\kappa_1 + \kappa_2) - 1))\sqrt{k_2^2 - k_1^2 + (\kappa_1 + \kappa_2)^2 - \Omega}, \]
\[ \delta_2/2 = \arg\left((\epsilon^2\varepsilon_+ + (1 - \epsilon^2\varepsilon_+)\text{sgn}(\kappa_1 + \kappa_2))\sqrt{k_2^2 - k_1^2 + (\kappa_1 + \kappa_2)^2 + \Omega} \right. \]
\[ + i\varepsilon_2(1 + \epsilon^2\varepsilon_+ (\text{sgn}(\kappa_1 + \kappa_2) - 1))\sqrt{k_2^2 - k_1^2 + (\kappa_1 + \kappa_2)^2 - \Omega}, \quad (2.214) \]
where
\[ \epsilon = (1 \pm \epsilon)/2, \quad \epsilon = \text{sgn}(|k_1| - |k_2|) = \begin{cases} 1, & |k_1| > |k_2| \\ -1, & |k_1| < |k_2| \\ 0, & |k_1| = |k_2| \end{cases}. \quad (2.215) \]

Then, combining expressions (2.214) and (2.215), we have
\[ \left(\delta_1 \pm \delta_2\right)/2 = \arg \left(\varepsilon(\kappa_1 + \kappa_2 + i(k_1 \pm k_2))\right) \quad (2.216) \]
where
\[ \varepsilon = \epsilon^2 + (1 - \epsilon^2)\text{sgn}(\kappa_1 + \kappa_2) = \begin{cases} 1, & |k_1| \neq |k_2| \\ \text{sgn}(\kappa_1 + \kappa_2), & |k_1| = |k_2| \end{cases}. \quad (2.217) \]

Additionally, we note
\[ \exp(i\rho_1) = \varepsilon_1, \quad \exp(i\rho_2) = \varepsilon_2, \quad \exp(i\rho) = \varepsilon. \quad (2.218) \]

These explicit expressions (2.196)–(2.210) and (2.211)–(2.219) for the harmonically modulated Sasa-Satsuma 2-soliton (1.14) have not previously appeared in the literature. The kinematic condition (2.195) will be seen later to imply that this solution describes a collision of harmonically modulated 1-solitons with distinct speeds \( c_1 = -w_1/k_1 \neq c_2 = -w_2/k_2 \).

In the case of equal speeds \( c_1 = c_2 \), we remark that Proposition 4 remains valid if \( \mu_1\theta_1 - \mu_2\theta_2 \) is replaced by \( \theta_1 - \theta_2 \) through equation (2.68), and if \( k_1x \) and \( k_2x \) are respectively replaced by \( k_1x + \chi \) and \( k_2x - \chi \) through the equation \( k_1(x_0 + a_1) + w_1 t_0 = -k_2(x_0 + a_2) - w_2 t_0 = -\chi \), where \( \chi = (a_2 - a_1)k_1k_2/(k_1 + k_2) \) is a shift which cannot be absorbed by a space-time translation.
The solution in this particular case has previously appeared in an equivalent envelope form in Ref. [16, 17].

For expressing the Sasa-Satsuma 2-soliton in oscillatory form, the properties of the envelope functions (2.196)–(2.199) under reflections (2.77) will be important. As in the Hirota case, we have

\[ \theta_1 \rightarrow \mp \theta_1, \quad \theta_2 \rightarrow \pm \theta_2, \]
\[ \mu_1 \rightarrow -\mu_1, \quad \mu_2 \rightarrow -\mu_2. \]

Also, from equations (2.200)–(2.209), we have

\[ \Omega \rightarrow \Omega, \quad \Upsilon \rightarrow \Upsilon, \]
\[ \Gamma \rightarrow 1/\Gamma, \quad \Delta \rightarrow 1/\Delta, \]
\[ \lambda_1 \rightarrow \mp \lambda_1, \quad \lambda_2 \rightarrow \pm \lambda_2, \]
\[ \gamma_1 \rightarrow \mp \gamma_1, \quad \gamma_2 \rightarrow \pm \gamma_2, \]
\[ \delta_1 \rightarrow \mp \delta_1, \quad \delta_2 \rightarrow \pm \delta_2. \]

These transformations yield

\[ X_{1SS} \rightarrow X_{1SS}, \quad X_{2SS} \rightarrow X_{2SS}, \quad Y_{SS} \rightarrow Y_{SS}. \]

which establishes the following reflection property.

**Lemma 2.** The Sasa-Satsuma envelope functions (2.196) are invariant under reflections (2.77).

### 3. Oscillatory parameterization

The harmonically modulated 1-soliton solutions shown in Proposition 1 for the Hirota equation (2.13) and Proposition 3 for the Sasa-Satsuma equation (2.81) will now be expressed in oscillatory form (1.15).

We write

\[ kx + wt = k(x - ct), \quad \kappa x + \omega t = \kappa(x - ct) + \nu t, \]

where

\[ c = -w/k, \quad \nu = \omega - w\kappa/k. \]

From relations (1.10), (1.21) for \( w \) and \( \omega \), we get

\[ c = k^2 - 3\kappa^2, \]
\[ \nu = -2\kappa(k^2 + \kappa^2), \]

with

\[ \kappa \neq 0. \]

By combining these equations (3.3) and (3.4), we have a cubic equation that determines \( \kappa \),

\[ 8\kappa^3 + 2\kappa c + \nu = 0 \]

and an elementary quadratic equation that determines \( |k| \),

\[ k^2 = \frac{3\kappa^2 + c}{24}. \]
The discriminant of the cubic equation (3.6) is
\[ \Delta = -\frac{64}{81}(\bar{c}^3 + \bar{\nu}^2) \] (3.8)

where
\[ \bar{c} = c/3, \quad \bar{\nu} = \nu/2 \neq 0. \] (3.9)

There are three cases to consider.

First, if \( \Delta < 0 \), i.e. \( \bar{c}^3 > -\bar{\nu}^2 \), then equation (3.6) has only one real root,
\[ \kappa = \frac{\beta_- - \beta_+}{2} \] (3.10)

where
\[ \beta_\pm = \frac{3}{2} \sqrt{\bar{c}^3 + \bar{\nu}^2} \] (3.11)

which is defined as the real cube root. Equation (3.7) then becomes
\[ k^2 = 3(\kappa^2 + \bar{c}) = \frac{3(\beta_- + \beta_+)^2}{4} \] (3.12)

so thus
\[ |k| = \frac{\sqrt{3}(\beta_- + \beta_+)}{2} \] (3.13)

where \( \beta_+ > -\beta_- > 0 \) when \( \bar{\nu} > 0 \) and \( \beta_- > -\beta_+ > 0 \) when \( \bar{\nu} < 0 \).

Next, if \( \Delta = 0 \), i.e. \( \bar{c}^3 = -\bar{\nu}^2 \), then equation (3.6) has three real roots, two of which are repeated,
\[ \kappa = \beta_- , \quad \kappa = \beta_+ / 2 \quad \text{(repeated)} \] (3.14)

where \( \beta_\pm = \pm \sqrt{-\bar{\nu}} \neq 0 \). The single root coincides with the previous real root (3.10), while the repeated roots violate equation (3.7) because
\[ 0 \leq k^2 = 3(\kappa^2 + \bar{c}) = 3((\beta_+/2)^2 - \beta_+^2) = -(3\beta_+/2)^2 < 0. \] (3.15)

Last, if \( \Delta > 0 \), i.e. \( \bar{c}^3 < -\bar{\nu}^2 \), then equation (3.6) has three distinct real roots,
\[ \kappa = \beta_\pm \cos(\psi), \quad \psi = \arg \beta_+, \quad \psi = \arg \beta_+ + 2\pi/3, \quad \psi = \arg \beta_+ - 2\pi/3 \] (3.16)

where \( |\beta_\pm| = \sqrt{-\bar{c}} \neq 0 \) and \( \tan(\arg \beta_\pm) = \pm \sqrt{|\bar{c}^3 + \bar{\nu}^2|}/\bar{\nu} \neq 0 \). But all three roots violate equation (3.7),
\[ 0 \leq k^2 = 3(\kappa^2 + \bar{c}) = 3|\beta_\pm|^2(\cos^2(\psi) - 1) < 0 \] (3.17)

due to \( |\cos(\psi)| \neq 1 \) which is the condition for no roots to be repeated.

Hence we have established the following main identities.

**Lemma 3.** (i) Let \( w = -k(k^2 - 3\kappa^2), \ \omega = -\kappa(3k^2 - \kappa^2) \), and \( \kappa \neq 0 \). Then
\[ |w| = -c|k|, \quad \omega = c\kappa + \nu \] (3.18)

is an identity, where \( \kappa \) and \( |k| \) are given by equations (3.9), (3.10), (3.13). (ii) Suppose \( f \) is an even function. Then, \( \exp(i(\kappa x + \omega t))f(kx + wt) \) is invariant under reflection \( k \to -k \) and hence it can be expressed in the equivalent form
\[ \exp(i(\kappa x + \omega t))f(kx + wt) = \exp(ivt)\hat{f}(x - ct), \quad \hat{f}(\xi) = \exp(i\kappa \xi)f(k\xi) \] (3.19)
in terms of

\[
k = \frac{\sqrt{3}}{2} \left( \sqrt[3]{\frac{(c/3)^3 + (\nu/2)^2 - \nu/2}{{}} + \sqrt[3]{\frac{(c/3)^3 + (\nu/2)^2 + \nu/2}{{}}} \right),
\]

(3.20)

\[
\kappa = \frac{1}{2} \left( \sqrt[3]{\frac{(c/3)^3 + (\nu/2)^2 - \nu/2}{{}} - \sqrt[3]{\frac{(c/3)^3 + (\nu/2)^2 + \nu/2}{{}}} \right),
\]

(3.21)

where

\[(c/3)^3 + (\nu/2)^2 \geq 0.
\]

(3.22)

Applying Lemma 3 to the Hirota and Sasa-Satsuma harmonically modulated 1-solitons, which are given by the reflection-invariant envelope functions (2.28) and (2.101), we obtain the following result.

**Theorem 1.** The oscillatory form (1.15) of a harmonically modulated 1-soliton (1.9) is parameterized by a phase angle \( \phi \), a temporal frequency \( \nu \), and a speed \( c \), which satisfy condition (3.22), where \( k, \kappa \) are given in terms of \( c, \nu \) by equations (3.20), (3.21). For the Hirota equation (2.13) and the Sasa-Satsuma equation (2.81), the oscillatory form of the harmonically modulated 1-soliton solutions (2.28) and (2.101) expressed using a travelling wave coordinate \( \xi = x - ct \) is given by

\[u(t,x) = \exp(i\phi) \exp(i\nu t) \tilde{f}(x - ct)\]

in terms of the respective functions

\[\tilde{f}_H(\xi) = \frac{k \exp(i\kappa \xi)}{2 \cosh(k \xi)},\]

(3.24)

\[\tilde{f}_{SS}(\xi) = \frac{k(2|\kappa|)^{1/2}(k^4 + \kappa^4)^{1/4} \exp(i\kappa \xi) \cosh(k \xi + i\lambda/2)}{|\kappa| \cosh(2k \xi) + (k^2 + \kappa^2)^{1/2}}, \quad \kappa \neq 0,\]

(3.25)

where \( \lambda \) is given by equation (2.98). When \( c = 0 \) (and \( \kappa \neq 0 \)), these 1-soliton solutions are standing waves (i.e. harmonically modulated stationary solitons).

From equations (3.21) and (3.20), note that \( \kappa = 0 \) iff \( \nu = 0 \) and that \( k = 0 \) iff \( (c/3)^3 + (\nu/2)^2 = 0 \). Consequently, Theorem 1 implies the following result.

**Corollary 1.** For the Hirota and Sasa-Satsuma equations, a harmonically modulated 1-soliton solution

\[u(t,x) = \exp(i\phi) \exp(i(\kappa x + \omega t)) f(kx + \omega t) = \exp(i\phi) \exp(i\nu t) \tilde{f}(x - ct)\]

(3.26)

is distinguished from an ordinary 1-soliton solution

\[u(t,x) = \exp(i\phi) f(x - ct)\]

(3.27)

by the kinematic conditions

\[\nu \neq 0, \quad (c/3)^3 + (\nu/2)^2 > 0,\]

(3.28)

or equivalently

\[c > -(3/\sqrt{4})(\sqrt{\nu})^2 \neq 0.\]

(3.29)
Next, the harmonically modulated 2-soliton solutions shown in Proposition 2 for the Hirota equation (2.13) and Proposition 4 for the Sasa-Satsuma equation (2.81) will be expressed in the analogous oscillatory form (1.17).

From Lemmas 1 and 2, the envelope functions (2.71) and (2.196) in these 2-soliton solutions are invariant under reflections (2.77), allowing Lemma 3 to be applied as follows. We define the analogous oscillatory form (1.17).

\[ k_1 = \sqrt{3}(\beta_{1-} + \beta_{1+})/2, \quad \kappa_1 = (\beta_{1-} - \beta_{1+})/2, \quad (3.30) \]
\[ k_2 = \sqrt{3}(\beta_{2-} + \beta_{2+})/2, \quad \kappa_2 = (\beta_{2-} - \beta_{2+})/2, \quad (3.31) \]

where

\[ \beta_{1\pm} = 3/\sqrt{(c_1/3)^3 + (\nu_1/2)^2 \pm \nu_1/2} \quad (3.32) \]
\[ \beta_{2\pm} = 3/\sqrt{(c_2/3)^3 + (\nu_2/2)^2 \pm \nu_2/2} \quad (3.32) \]

with the conditions

\[ (c_1/3)^3 + (\nu_1/2)^2 \geq 0, \quad (3.33) \]
\[ (c_2/3)^3 + (\nu_2/2)^2 \geq 0, \quad (3.33) \]

which imply \( k_1 \geq 0 \) and \( k_2 \geq 0 \). We then have the identities

\[ c_1 = -w_1/k_1, \quad \nu_1 = \omega_1 - w_1\kappa_1/k_1, \]
\[ c_2 = -w_2/k_2, \quad \nu_2 = \omega_2 - w_2\kappa_2/k_2, \quad (3.34) \]

holding from the algebraic relations (2.46). Through these expressions, the identity (2.68)–(2.69) becomes

\[ (\kappa_1 - \kappa_2)x + (\omega_1 - \omega_2)t = (\kappa_1 - \mu)(x - c_1t) - (\kappa_2 - \mu)(x - c_2t), \quad c_1 \neq c_2 \quad (3.35) \]

where

\[ \mu = (\nu_1 - \nu_2)/(c_1 - c_2). \quad (3.36) \]

This leads to the following main result.

**Theorem 2.** The oscillatory form (1.17) of a harmonically modulated 2-soliton (1.14) has parameters \( \phi_1, \phi_2, \nu_1, \nu_2, c_1, c_2 \), satisfying condition (3.33), where \( k_1, \kappa_1 \) are given in terms of \( c_1, \nu_1 \) by equation (3.30), and \( k_2, \kappa_2 \) are given in terms of \( c_2, \nu_2 \) by equation (3.31). As expressed using travelling wave coordinates \( \xi_1 = x - c_1t \) and \( \xi_2 = x - c_2t \) when \( c_1 \neq c_2 \), the oscillatory form for the 2-soliton solutions (2.71)–(2.76) for the Hirota equation (2.13) and (2.196)–(2.210) for the Sasa-Satsuma equation (2.81) is given by

\[ u(t, x) = \exp(i\phi_1)\exp(i\nu_1 t)\tilde{f}_1(\xi_1, \xi_2) + \exp(i\phi_2)\exp(i\nu_2 t)\tilde{f}_2(\xi_1, \xi_2) \quad (3.37) \]

with

\[ \tilde{f}_1(\xi_1, \xi_2) = \tilde{X}_1(\xi_1, \xi_2)/\tilde{Y}(\xi_1, \xi_2), \quad \tilde{f}_2(\xi_1, \xi_2) = \tilde{X}_2(\xi_1, \xi_2)/\tilde{Y}(\xi_1, \xi_2) \quad (3.38) \]

in terms of the respective functions

\[ \tilde{X}_{1H}(\xi_1, \xi_2) = k_1 \exp(i\kappa_1\xi_1)\cosh(k_2\xi_2 + i\gamma_2), \quad (3.39) \]
\[ \tilde{X}_{2H}(\xi_1, \xi_2) = k_2 \exp(i\kappa_2\xi_2)\cosh(k_1\xi_1 + i\gamma_1), \quad (3.40) \]
\[ \tilde{Y}_H(\xi_1, \xi_2) = \sqrt{T}\cosh(k_1\xi_1 + k_2\xi_2) + \frac{1}{\sqrt{T}}\cosh(k_1\xi_1 - k_2\xi_2) \]
\[ - \frac{4k_1k_2}{\sqrt{T}}\cos(k_1\xi_1 - k_2\xi_2 + \mu(\xi_2 - \xi_1) + \phi_1 - \phi_2) \quad (3.41) \]
in the Hirota case, and in the Sasa-Satsuma case when $\kappa_1 \neq 0$ and $\kappa_2 \neq 0$,

\[ \tilde{X}_{1SS}(\xi_1, \xi_2) = \exp(i\kappa_1\xi_1) \left( k_1(k_1^2 + \kappa_1^2)^{1/4}|2\kappa_1|^{1/2} \left( |\kappa_2| \left( \sqrt{\Delta \Gamma} \cosh(k_1\xi_1 + 2k_2\xi_2 + i(\alpha_2 + \gamma_2)) + \frac{1}{\sqrt{\Delta \Gamma}} \cosh(k_1\xi_1 - 2k_2\xi_2 + i(v_2 - \gamma_2)) \right) + (k_1^2 + \kappa_1^2)^{1/2} \left( -8k_1\kappa_2 \frac{1}{\sqrt{\Omega \Gamma}} \cosh(k_1\xi_1 + i(\omega_1 + \gamma_1)) \right) + \sqrt{\frac{\Gamma}{\Delta}} \cosh(k_1\xi_1 + i(\alpha_2 - \gamma_2)) + \sqrt{\frac{\Delta}{\Gamma}} \cosh(k_1\xi_1 + i(\gamma_2 + \omega_2)) \right) \right) \]

\[ + k_1k_2\sqrt{\frac{\Omega}{\Gamma}} \left( k_2(k_2^2 + \kappa_2^2)^{1/4}|8\kappa_2|^{1/2} \cosh(k_1\xi_1 + i(\omega_1 - \gamma_1)) + k_1(k_2^2 + \kappa_2^2)^{1/4}|32\kappa_1|^{1/2} \cosh(k_2\xi_2 + i(\gamma_2 - \omega_2)) \times \exp(i(\kappa_1\xi_1 - \kappa_2\xi_2 + \mu(\xi_2 - \xi_1) + \phi_2 - \phi_1)) \right) \right) \right), \]

(3.42)

\[ \tilde{X}_{2SS}(\xi_1, \xi_2) = \exp(i\kappa_2\xi_2) \left( k_2(k_2^2 + \kappa_2^2)^{1/4}|2\kappa_2|^{1/2} \left( |\kappa_1| \left( \sqrt{\Delta \Gamma} \cosh(k_2\xi_2 + 2k_1\xi_1 + i(\alpha_1 + \gamma_1)) + \frac{1}{\sqrt{\Delta \Gamma}} \cosh(k_2\xi_2 - 2k_1\xi_1 + i(v_1 - \gamma_1)) \right) + (k_2^2 + \kappa_2^2)^{1/2} \left( -8k_2\kappa_1 \frac{1}{\sqrt{\Omega \Gamma}} \cosh(k_2\xi_2 + i(\omega_2 + \gamma_2)) \right) + \sqrt{\frac{\Gamma}{\Delta}} \cosh(k_2\xi_2 + i(\alpha_1 - \gamma_1)) + \sqrt{\frac{\Delta}{\Gamma}} \cosh(k_2\xi_2 + i(\gamma_2 - \omega_2)) \right) \right) \]

\[ + k_1k_2\sqrt{\frac{\Omega}{\Gamma}} \left( k_1(k_2^2 + \kappa_2^2)^{1/4}|8\kappa_1|^{1/2} \cosh(k_1\xi_1 + i(\gamma_1 - \omega_1)) \times \exp(i(\kappa_2\xi_2 - \kappa_1\xi_1 + \mu(\xi_1 - \xi_2) + \phi_1 - \phi_2)) \right) \right) \right), \]

(3.43)
\[ Y_{SS}(\xi_1, \xi_2) = |\kappa_1 \kappa_2| \left( \Delta \Gamma \cosh(2(k_1 \xi_1 + k_2 \xi_2)) + \frac{1}{\Delta \Gamma} \cosh(2(k_1 \xi_1 - k_2 \xi_2)) \right) \\
+ 2(k_1^2 + \kappa_1^2)^{1/2}|\kappa_2| \cosh(2k_2 \xi_2) + 2(k_2^2 + \kappa_2^2)^{1/2}|\kappa_1| \cosh(2k_1 \xi_1) \\
+ 4k_1^2 k_2^2 \varepsilon_1 \varepsilon_2 \frac{\Omega}{\Gamma} \cos(2(\kappa_1 \xi_1 - \kappa_2 \xi_2 + \mu(\xi_2 - \xi_1)) + 2(\phi_1 - \phi_2)) \\
+ (k_1^2 + \kappa_1^2)^{1/2}(k_2^2 + \kappa_2^2)^{1/2} \left( \frac{\Gamma}{\Delta} + \frac{\Delta}{\Gamma} + 64k_1^2 k_2^2 \kappa_1 \kappa_2 \frac{1}{\Omega \Gamma} \\
- 16k_1 k_2 |\kappa_1 \kappa_2|^{1/2} \text{Re} \left( \exp(i(\kappa_1 \xi_1 - \kappa_2 \xi_2 + \mu(\xi_2 - \xi_1) + \phi_1 - \phi_2)) \right) \times \\
\left( \sqrt{\frac{\Gamma}{\Omega}} \cosh(k_1 \xi_1 + k_2 \xi_2 + i(\varpi_1 - \varpi_2)) \\
+ \frac{1}{\sqrt{\Omega \Gamma}} \cosh(k_1 \xi_1 - k_2 \xi_2 + i(\varpi_1 + \varpi_2)) \right) \right), \]

where, in both cases, \( \mu \) is given by equation (3.36), \( \Omega, \Gamma, \Delta, \Gamma \) are given by equations (2.200)–(2.203), \( \alpha_1, \alpha_2, \nu_1, \nu_2, \varpi_1, \varpi_2, \gamma_1, \gamma_2 \) are given by equations (2.204)–(2.207), and \( \lambda_1, \lambda_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \varepsilon \) are given by equations (2.211)–(2.218).

Similarly to the 1-soliton case, Theorem 2 implies the following result.

**Corollary 2.** For the Hirota and Sasa-Satsuma equations, a harmonically modulated 2-soliton solution

\[
u(t, x) = \exp(i\varphi_1) \exp(i(\kappa_1 x + \omega_1 t)) f_1(k_1 x + w_1 t, k_2 x + w_2 t) \\
+ \exp(i\varphi_2) \exp(i(\kappa_2 x + \omega_2 t)) f_2(k_1 x + w_1 t, k_2 x + w_2 t) \\
= \exp(i\varphi_1) \exp(i\nu_1 t) f_1(x - c_1 t, x - c_2 t) \\
+ \exp(i\varphi_2) \exp(i\nu_2 t) f_2(x - c_1 t, x - c_2 t), \quad c_1 \neq c_2
\]

is distinguished from an ordinary 2-soliton solution

\[
u(t, x) = \exp(i\varphi_1) f(x - c_1 t, x - c_2 t) + \exp(i\varphi_2) f(x - c_1 t, x - c_2 t), \quad c_1 \neq c_2
\]

by the kinematic conditions

\[
u_1 \neq 0, \quad (c_1/3)^3 + (\nu_1/2)^2 > 0, \quad (3\sqrt{4}(\sqrt{\nu_1})^2 \neq 0 \\
\nu_2 \neq 0, \quad (c_2/3)^3 + (\nu_2/2)^2 > 0, \quad (3\sqrt{4}(\sqrt{\nu_2})^2 \neq 0.
\]

or equivalently

\[
(3\sqrt{4}(\sqrt{\nu_1})^2) \neq 0, \quad (3\sqrt{4}(\sqrt{\nu_2})^2) \neq 0.
\]

Note the kinematic conditions (3.49) imply the property

\[
k_1 > 0, \quad k_2 > 0
\]

which holds in the functions \( \tilde{X}_1(\xi_1, \xi_2), \tilde{X}_2(\xi_1, \xi_2), \tilde{Y}(\xi_1, \xi_2) \) appearing in the oscillatory 2-soliton solutions in Theorem 2.
3.1. Breathers. From the remarks made after Proposition 2 for the Hirota equation (2.13) and Proposition 4 for the Sasa-Satsuma equation (2.81), we will now state a counterpart of Theorem 2 for harmonically modulated breather solutions.

Let

\[ \beta_{1\pm} = \frac{3}{\sqrt[3]{2}} \left( \frac{(c/3)^3 + ((\nu_0 + \nu)/2)^2 \pm (\nu_0 + \nu)/2}{(\nu_0 - \nu)/2 \pm (\nu_0 - \nu)/2} \right) \]

(3.51)

with the condition

\[ (c/3)^3 + ((|\nu_0| + |\nu|)/2)^2 \geq 0. \]

(3.52)

**Theorem 3.** The oscillatory form (1.20) of a harmonically modulated breather (1.18) has parameters \( \chi, \phi_0, \nu_0, c, \phi, \nu \neq 0 \), satisfying condition (3.52). As expressed using a travelling wave coordinate \( \xi = x - ct \) and an oscillation coordinate \( \tau = \nu t + \phi \), the oscillatory form for the breather solutions for the Hirota equation (2.13) and the Sasa-Satsuma equation (2.81) is given by

\[
\begin{align*}
\hat{u}(t, x) &= \exp(i(\nu_0 t + \phi_0)) \tilde{f}(\xi, \tau) \\
\tilde{f}(\xi, \tau) &= \exp(i\tau) \tilde{X}_1(\xi, \tau) / \tilde{Y}(\xi, \tau) + \exp(-i\tau) \tilde{X}_2(\xi, \tau) / \tilde{Y}(\xi, \tau)
\end{align*}
\]

(3.53)

in terms of the respective functions

\[
\begin{align*}
\tilde{X}_{1H}(\xi, \tau) &= k_1 \exp(i k_1 \xi) \cosh(k_2 \xi - \chi + i \gamma_2), \\
\tilde{X}_{2H}(\xi, \tau) &= k_2 \exp(i k_2 \xi) \cosh(k_1 \xi + \chi + i \gamma_1), \\
\tilde{Y}_{1H}(\xi, \tau) &= \sqrt{\Gamma} \cosh((k_1 + k_2) \xi) + \frac{1}{\sqrt{\Gamma}} \cosh((k_1 - k_2) \xi + 2\chi) - \frac{4k_1 k_2}{\sqrt{\Gamma}} \cos((\kappa_1 - \kappa_2) \xi + 2\tau)
\end{align*}
\]

(3.54)

(3.55)

(3.56)

in the Hirota case, and in the Sasa-Satsuma case when \( \kappa_1 \neq 0 \) and \( \kappa_2 \neq 0 \),

\[ \begin{align*}
\tilde{X}_{1SS}(\xi, \tau) &= \exp(i k_1 \xi) \left( (k_1^2 + \kappa_1^2)^{1/4} |2k_1|^{1/2} \left( |\kappa_2| \left( \sqrt{\Delta \Gamma} \cosh((k_1 + 2k_2) \xi - \chi + i(\alpha_2 + \gamma_2)) 
+ \frac{1}{\sqrt{\Delta \Gamma}} \cosh((k_1 - 2k_2) \xi + 3\chi + i(\nu_2 - \gamma_2)) \right) 
+ (k_2^2 + \kappa_2^2)^{1/2} \left( -8k_2^2 \kappa_2 \epsilon \frac{1}{\sqrt{\Delta \Gamma}} \cosh(k_1 \xi + \chi + i(\omega_1 + \gamma_1)) 
+ \sqrt{\frac{\Delta}{\Gamma}} \cosh(k_1 \xi + \chi + i(\nu_2 + \gamma_2)) \right) \right) 
+ k_1 k_2 \epsilon \sqrt{\frac{\Omega}{\Gamma}} \left( (k_2^2 + \kappa_2^2)^{1/8} 32 \kappa_1^{1/2} \varepsilon_2 \cosh(k_1 \xi + \chi + i(\omega_1 - \gamma_1)) 
- k_1 (k_2^2 + \kappa_2^2)^{1/8} 32 \kappa_1^{1/2} \varepsilon_1 \Re \left( \cosh(k_2 \xi - \chi + i(\gamma_2 - \omega_2)) \times \exp(i((\kappa_1 - \kappa_2) \xi + 2\tau)) \right) \right) \right). \end{align*} \]

(3.57)
\[ X_{2SS}(\xi, \tau) = \exp(i\kappa_2 \xi) \left( k_2(k_2^2 + \kappa_2^2)^{1/4}|2\kappa_2|^{1/2} \left( |\kappa_1| \left( \sqrt{\Delta \Gamma} \cosh((k_2 + 2k_1)\xi + \chi + i(\alpha_1 + \gamma_1)) + \frac{1}{\sqrt{\Delta \Gamma}} \cosh((k_2 - 2k_1)\xi - 3\chi + i(v_1 - \gamma_1)) \right) + (k_1^2 + \kappa_1^2)^{1/2} \left( -8k_1^2\kappa_1\varepsilon \frac{1}{\sqrt{\Omega \Gamma}} \cosh(k_2\xi - \chi + i(\omega_2 + \gamma_2)) + \sqrt{\frac{\Gamma}{\Delta}} \cosh(k_2\xi - \chi + i(v_1 + \gamma_1)) \right) \right) + k_1k_2\varepsilon \sqrt{\frac{\Omega}{\Gamma}} \left( k_1(k_1^2 + \kappa_1^2)^{1/4}|8\kappa_1|^{1/2}\varepsilon_1 \cosh(k_2\xi - \chi + i(\omega_2 - \gamma_2)) \right) - k_2(k_1^2 + \kappa_1^2)^{1/4}|32\kappa_2|^{1/2}\varepsilon_2 \Re \left( \cosh(k_1\xi + \chi + i(\gamma_1 - \omega_2)) \times \exp(i((\kappa_2 - \kappa_1)\xi + 2\tau)) \right) \right) \right), \]

(3.58)

\[ Y_{SS}(\xi, \tau) = |\kappa_1\kappa_2| \left( \Delta \Gamma \cosh(2(k_1 + k_2)\xi) + \frac{1}{\Delta \Gamma} \cosh(2((k_1 - k_2)\xi + 2\chi)) \right) + 2(k_1^2 + \kappa_1^2)^{1/2}|\kappa_2| \cosh(2(k_2\xi - \chi)) + 2(k_2^2 + \kappa_2^2)^{1/2}|\kappa_1| \cosh(2(k_1\xi + \chi)) + 4k_1^2k_2^2\varepsilon_1\varepsilon_2 \Omega \Gamma \cos(2(\kappa_1 - \kappa_2)\xi + 4\tau) + (k_1^2 + \kappa_1^2)^{1/2}(k_2^2 + \kappa_2^2)^{1/2} \left( \frac{\Gamma}{\Delta} + \frac{\Delta}{\Gamma} + 64k_1^2k_2^2\kappa_1\kappa_2 \frac{1}{\Omega \Gamma} \right)

- 16k_1k_2|\kappa_1\kappa_2|^{1/2} \Re \left( \exp(i((\kappa_1 - \kappa_2)\xi + 2\tau)) \times \left( \sqrt{\frac{\Gamma}{\Delta}} \cosh((k_1 + k_2)\xi + i(\omega_1 - \omega_2)) + \frac{1}{\sqrt{\Delta \Gamma}} \cosh((k_1 - k_2)\xi + 2\chi + i(\omega_1 + \omega_2)) \right) \right) \right), \]

(3.59)

where, in both cases, \( k_1, k_2, \kappa_1, \kappa_2 \) are given in terms of \( c, \nu \neq 0, \nu_0 \) by equations (3.30)–(3.31), \( \Omega, \gamma, \Delta, \Gamma \) are given by equations (2.200)–(2.203), \( \alpha_1, \alpha_2, \nu_1, \nu_2, \omega_1, \omega_2, \gamma_1, \gamma_2 \) are given by equations (2.204)–(2.207), and \( \lambda_1, \lambda_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \varepsilon \) are given by equations (2.211)–(2.218).

Note the kinematic condition (3.52) implies that the property (3.50) holds in the functions \( \hat{X}_1(\xi, \tau), \hat{X}_2(\xi, \tau), \hat{Y}(\xi, \tau) \).

In the special case \( \nu_0 = 0 \), the oscillatory breathers (3.53) reduce to ordinary breathers (1.19), which are given by much simpler expressions.

**Proposition 5.** For the Hirota equation (2.13) and the Sasa-Satsuma equation (2.81), the ordinary breather solutions expressed using a travelling wave coordinate \( \xi = x - ct \) and an
oscillation coordinate \( \tau = \nu t + \phi \) are given by
\[
u(t, x) = \exp(i\phi_0)f(\xi, \tau)
\]
\[
f(\xi, \tau) = \exp(i\tau)X_1(\xi, \tau)/Y(\xi, \tau) + \exp(-i\tau)X_2(\xi, \tau)/Y(\xi, \tau)
\] (3.60)
in terms of the respective functions
\[
X_{1H}(\xi, \tau) = k|\kappa|\sqrt{k^2 + \kappa^2} \exp(i\kappa \xi) \sinh(k\xi - \chi - i\gamma),
\] (3.61)
\[
X_{2H}(\xi, \tau) = k|\kappa|\sqrt{k^2 + \kappa^2} \exp(-i\kappa \xi) \sinh(k\xi + \chi + i\gamma),
\] (3.62)
\[
Y_H(\xi, \tau) = k^2 \cosh(2k\xi) + (k^2 + \kappa^2) \cosh(2\chi) + k^2 \cos(2(k\xi + \tau))
\] (3.63)
in the Hirota case, and in the Sasa-Satsuma case
\[
X_{1SS}(\xi, \tau) = (k/\Sigma) \exp(i\kappa \xi) \left(k|\sqrt{k^2 + \kappa^2} \sinh(k\xi - \chi - i\gamma)
\right.
\]
\[
- \left.\frac{(\gamma/2) \Lambda}{|k|(1 + \Sigma)^{1/2}} \exp(\chi + i(\lambda + \gamma/2)) \cosh(k\xi + \varpi + i(\lambda + \gamma/2))\right),
\] (3.64)
\[
X_{2SS}(\xi, \tau) = (k/\Sigma) \exp(-i\kappa \xi) \left(k|\sqrt{k^2 + \kappa^2} \sinh(k\xi + \chi + i\gamma)
\right.
\]
\[
- \left.\frac{(\gamma/2) \Lambda}{|k|(1 + \Sigma)^{1/2}} \exp(-\chi + i(\lambda - \gamma/2)) \cosh(k\xi + \varpi + i(\lambda - \gamma/2))\right),
\] (3.65)
\[
Y_H(\xi, \tau) = k^2 \Sigma \cosh(2k\xi) + (k^2 + \kappa^2) \cosh(2\chi) + k^2 \cos(2(k\xi + \tau)),
\] (3.66)
where
\[
\Lambda = \sinh(2|\chi|), \quad \Sigma = \sqrt{1 + (1 + k^2/\kappa^2)\Lambda^2}, \quad \varpi = \frac{1}{4} \ln \left(\frac{\Sigma + 1}{\Sigma - 1}\right)
\] (3.67)
\[
\gamma = \arg(k + i\kappa), \quad \lambda = \arg(1 + i\text{sgn}(\kappa \chi))
\] (3.68)
and where \( k, \kappa \neq 0 \) are given in terms of \( c, \nu \neq 0 \) by equations (3.20) and (3.21), such that the kinematic condition \((c/3)^3 + (\nu/2)^2 > 0\) holds.

When \( \chi = \phi_0 = 0 \), we remark that both the Hirota and Sasa-Satsuma breather solutions in Proposition 5 reduce to the well-known mKdV breather solution [18]
\[
u(t, x) = k|\kappa|\sqrt{k^2 + \kappa^2} \sinh(k\xi) \cos(\gamma) \cos(k\xi + \tau) + \cosh(k\xi) \sin(\gamma) \sin(k\xi + \tau)
\]
\[
\kappa^2 \cosh(k\xi)^2 + k^2 \cos(\kappa \xi + \tau)^2.
\] (3.69)

4. Properties of oscillatory soliton solutions

We begin by discussing some basic properties of the oscillatory 1-soliton solutions from Theorem 1 for the Hirota equation (2.13) and the Sasa-Satsuma equation (2.81).

4.1. Oscillatory 1-solitons. An oscillatory wave (3.26) has amplitude \(|u| = |\tilde{f}(\xi)|\) where \( \xi = x - ct \) is a moving coordinate centered at \( x = ct \). Hence the spatial shape of \(|u|\) is determined by the properties of the function \(|\tilde{f}(\xi)|\). In both the Hirota and Sasa-Satsuma oscillatory 1-soliton solutions, these functions \(|\tilde{f}(\xi)|\) share two main properties, as seen from expressions (3.24) and (3.25). First, for large \(|\xi|\), both functions exhibit exponential decay
\[ |\tilde{f}(\xi)| \sim O(\exp(-k|\xi|)) \]. Second, both functions exhibit reflection-conjugation invariance \( \tilde{f}(-\xi) = \overline{f(\xi)} \) (where a bar denotes complex conjugation), implying that \( |\tilde{f}(\xi)| \) is an even function of \( \xi \) and thus \( \text{Re}(\tilde{f}'(0)) = 0 \).

In the case of the Hirota function \( |\tilde{f}_H(\xi)| \), from expression (3.24) we find that \( \text{Re}(\tilde{f}_H(\xi)) \neq 0 \) when \( \xi \neq 0 \). Hence the function \( |\tilde{f}_H(\xi)| \) has a peak at \( \xi = 0 \). In contrast, in the case of the Sasa-Satsuma function \( |\tilde{f}_{SS}(\xi)| \), from expression (3.25) we find that \( \text{Re}(\tilde{f}_{SS}(\xi)) = 0 \) has roots \( \xi \neq 0 \) when \( \kappa \neq 0 \) and (only when) \( \cosh(2k\xi) = (k^2 - \kappa^2)/(|\kappa|\sqrt{k^2 + \kappa^2}) \), which requires the condition \( (k^2 - \kappa^2)/(|\kappa|\sqrt{k^2 + \kappa^2}) \geq 1 \) on \( k, \kappa \). This condition is equivalent to \( k^2 \geq 3\kappa^2 \). Hence in this case the function \( |\tilde{f}_{SS}(\xi)| \) has a pair of peaks centered symmetrically around \( \xi = 0 \).

From these properties, we obtain the following two results about the spatial shape of \( |u| \), stated in terms of the notation \( \beta_{\pm} = \frac{3}{2}\sqrt{(c/3)^3 + (\nu/2)^2 \pm \nu/2} \). Cases \( c > 0, \ c < 0, \ c = 0 \) are illustrated in Fig. 1a, Fig. 2a, Fig. 5a for the Hirota oscillatory 1-soliton, and in Fig. 3a, Fig. 4a, Fig. 6a for the Sasa-Satsuma oscillatory 1-soliton.

**Proposition 6.** For both the Hirota oscillatory 1-soliton (3.23), (3.24), and the Sasa-Satsuma oscillatory 1-soliton (3.23), (3.25), the amplitude \( |u| \) is an even function of \( x - ct \) and decays exponentially for \( |x - ct| \gg 1/k = 2/(\sqrt{3}\beta_- + \beta_+) \). The Hirota oscillatory 1-soliton and the Sasa-Satsuma oscillatory 1-soliton for \( c \leq 0 \) each have a single peak centered at \( x = ct \), with the height of the respective peaks given by

\[
|u||_{x = ct} = \frac{k}{2} = \frac{\sqrt{3}}{4}(\beta_- + \beta_+) \tag{4.1}
\]

and

\[
|u||_{x = ct} = \frac{k|\kappa|^{1/2}}{(|\kappa| + (k^2 + \kappa^2)^{1/2})^{1/2}} = \frac{\sqrt{3}(\beta_- + \beta_+)|\nu + (\beta_- - \beta_+)c|^{1/4}}{2(2|\nu|^{1/2} + |\nu + (\beta_- - \beta_+)c|^{1/2})^{1/2}}. \tag{4.2}
\]

For \( c > 0 \), the Sasa-Satsuma oscillatory 1-soliton instead has a symmetrical pair of peaks at \( x = ct \pm x_0 \), where

\[
x_0 = \frac{k^2 - \kappa^2 + k(k^2 + \kappa^2)^{1/2}}{|\kappa|(k^2 + \kappa^2)^{1/2}} = \frac{\beta_+ - \beta_-c + |\beta_+^2 - \beta_-^2|^{1/2}}{|\nu|^{1/2}|\beta_+ - \alpha_1|^{3/2}} \geq 0, \tag{4.3}
\]

with the height of the peaks given by

\[
|u||_{x = ct \pm x_0} = \frac{1}{2}(k^2 + \kappa^2)^{1/2} = \frac{|\nu|^{1/2}}{2|\beta_+ - \beta_-|}. \tag{4.4}
\]

In general, an oscillatory wave (3.26) can be factorized into a harmonic wave part \( \exp(i(\kappa x + \omega t)) \) and a travelling wave part \( f(kx + \omega t) \) through equation (3.19). The envelope function \( f \) in this factorization will be real-valued when (and only when) it satisfies \( f = |f| \) or \( \arg(f) = 0 \), corresponding to \( \arg(u) = \phi + \kappa x + \omega t \) being linear in \( x, t \) (in which case \( u \) is a harmonically modulated travelling wave). This condition is equivalent to the relation \( \arg(\tilde{f}(\xi)) = \kappa \xi \) given in terms of the function \( \tilde{f}(\xi) \) in the oscillatory wave (3.26).

It is thereby convenient to decompose \( \arg(\tilde{f}(\xi)) \) into a homogeneous linear part and an inhomogeneous nonlinear part with respect to the moving coordinate \( \xi \) by writing

\[
\arg(\tilde{f}(\xi)) = \kappa \xi + \varphi(\tilde{f}(\xi)) \mod 2\pi. \tag{4.5}
\]
whereby $\varphi(\kappa \xi) = 0$. Then the condition $\arg(\tilde{f}(\xi)) = \kappa \xi$ can be formulated simply as

$\varphi(\tilde{f}(\xi)) = 0$. Since an oscillatory wave (3.26) has $\arg(u) = \phi + \nu t + \arg(\tilde{f}(\xi))$, we see that the wave will have a linear phase when (and only when)

$$\varphi(u) = \arg(u) - \kappa x - (\nu - \kappa c)t \mod 2\pi$$

is constant. The condition $\varphi(u) \neq \text{const.}$ therefore distinguishes a general oscillatory wave from a harmonically modulated travelling wave with a real envelope function. Correspondingly, it is natural to define

$$\ell = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(u)_x \, dx = \frac{\varphi(u)}{2\pi} \bigg|_{x=-\infty}^{x=\infty}$$

which can be regarded as measuring the net winding contributed by the envelope function of an oscillatory wave (3.26). (Note that the net winding will be $\ell = 0$ when the envelope function is real-valued.) We will thus refer to the expressions (4.6) and (4.7) as the *envelope phase* and *envelope winding number*, respectively.

It is straightforward to derive the following phase properties of the Hirota and Sasa-Satsuma oscillatory 1-soliton solutions.

**Proposition 7.** For both the Hirota oscillatory 1-soliton (3.23), (3.24), and the Sasa-Satsuma oscillatory 1-soliton (3.23), (3.25), the envelope phase $\varphi(u)$ is an even function of $x - ct$ and equals $\phi$ at $x = ct$. Away from $x = ct$, the envelope phase has the features

$$\varphi(u) = \phi, \quad \pm (x - ct) > 0$$

(4.8)

$$\ell = 0$$

(4.9)

for the Hirota 1-soliton, and

$$\varphi(u) \sim \phi \pm \lambda/2, \quad \pm (x - ct) \gg 1/k = 2/(\sqrt{3}(\beta_- + \beta_+))$$

(4.10)

$$\ell = \lambda/(2\pi)$$

(4.11)

for the Sasa-Satsuma 1-soliton, where $\lambda$ is given by equation (2.98).

Cases $c > 0$, $c < 0$, $c = 0$ are shown in Fig. 1b, Fig. 2b, Fig. 5b, for the Hirota oscillatory 1-soliton, and in Fig. 3b, Fig. 4b, Fig. 6b for the Sasa-Satsuma oscillatory 1-soliton.

![Figure 1](image-url)

**Figure 1.** Hirota oscillatory 1-solitons with $c = 4$, $|\nu|$ = 15, 100, 250 and $\nu = 0$ (dotted line), $\phi = \pi/2$
Figure 2. Hirota oscillatory 1-solitons with $c = -4$, $|\nu| = 4, 15, 100, 250$, $\phi = \pi/2$

Figure 3. Sasa-Satsuma oscillatory 1-solitons with $c = 4$, $|\nu| = 15, 100, 250$ and $\nu = 0$ (dotted line), $\phi = \pi/2$

Figure 4. Sasa-Satsuma oscillatory 1-solitons with $c = -4$, $|\nu| = 4, 15, 100, 250$, $\phi = \pi/2$
4.2. Oscillatory 2-solitons. We now illustrate some properties of the oscillatory 2-soliton solutions from Theorem 2, shown graphically by the amplitude $|u|$ and the phase gradient $(\arg(u))_x$.

An oscillatory 2-soliton (3.45) is parameterized by phases $\phi_1, \phi_2$, frequencies $\nu_1, \nu_2$, and speeds $c_1 \neq c_2$, satisfying the kinematic conditions (3.33). These solitons are symmetric under simultaneously interchanging $c_1 \leftrightarrow c_2$, $\nu_1 \leftrightarrow \nu_2$, $\phi_1 \leftrightarrow \phi_2$, and thus we can assume $c_1 > c_2$ without loss of generality.

As seen from Fig. 7–Fig. 9 for the Hirota equation (2.13) and Fig. 10–Fig. 12 for the Sasa-Satsuma equation (2.81), oscillatory 2-solitons describe collisions between two oscillatory waves. The collision is a right-overtake when $c_1 > c_2 \geq 0$, a left-overtake when $0 \geq c_1 > c_2$, and a head-on when $c_1 > 0 > c_2$. Figures show $t = t_0, 0, -t_0$. 

Figure 5. Hirota standing wave $(c = 0)$ with $|\nu| = 1, 15, 100, 250$

Figure 6. Sasa-Satsuma standing wave $(c = 0)$ with $|\nu| = 1, 15, 100, 250$
Figure 7. Hirota oscillatory 2-soliton right-overtake with $c_1 = 4$, $c_2 = 2$, $\nu_1 = 2$, $\nu_2 = 5$, $\phi_1 = 0$, $\phi_2 = \pi/2$ ($t_0 = -4$)

Figure 8. Hirota oscillatory 2-soliton left-overtake with $c_1 = -2$, $c_2 = -4$, $\nu_1 = 2$, $\nu_2 = 5$, $\phi_1 = 0$, $\phi_2 = \pi/2$ ($t_0 = -10$)
Figure 9. Hirota oscillatory 2-soliton head-on with $c_1 = 4$, $c_2 = -2$, $\nu_1 = 2$, $\nu_2 = 5$, $\phi_1 = 0$, $\phi_2 = \pi/2$ ($t_0 = -1.5$)

Figure 10. Sasa-Satsuma oscillatory 2-soliton right-overtake with $c_1 = 4$, $c_2 = 2$, $\nu_1 = 2$, $\nu_2 = 5$, $\phi_1 = 0$, $\phi_2 = \pi/2$ ($t_0 = -4$)
Figure 11. Sasa-Satsuma oscillatory 2-soliton left-overtake with $c_1 = -2$, $c_2 = -4$, $\nu_1 = 2$, $\nu_2 = 5$, $\phi_1 = 0$, $\phi_2 = \pi/2$ ($t_0 = -10$)

Figure 12. Sasa-Satsuma oscillatory 2-soliton head-on with $c_1 = 4$, $c_2 = -2$, $\nu_1 = 2$, $\nu_2 = 5$, $\phi_1 = 0$, $\phi_2 = \pi/2$ ($t_0 = -1.5$)
In all collisions, the speed and the shape of both oscillatory waves are preserved. We will show analytically in a sequel paper [19] that a collision produces a shift in the asymptotic phase and position of each oscillatory wave.

4.3. Oscillatory breathers. Last we discuss a few aspects of the oscillatory breather solutions from Theorem 3.

An oscillatory breather (3.53) is parameterized by a speed \( c \), an envelope frequency \( \nu \neq 0 \) and phase \( \phi \), and a modulation frequency \( \nu_0 \) and phase \( \phi_0 \), where the speed and frequencies satisfy the kinematic condition \((c/3)^2 + (\nu/2)^2 > 0\). Depending on the combined frequency \(|\nu| + |\nu_0|\), the speed of a breather can be positive, negative, or zero. The additional parameter \( \chi \) controls the size of the oscillations, such that the oscillations disappear in the limit \(|\chi| \gg 1\).

From the expressions (3.54)–(3.56) for the Hirota oscillatory breather and (3.57)–(3.59) for the Sasa-Satsuma oscillatory breather, we see that the amplitude \(|u| = |\tilde{f}(\xi, \tau)|\) has exponential decay in the moving coordinate \(|\xi| = |x - ct|\) and is periodic in the oscillation coordinate \(\tau = \nu t + \phi\). In particular, these breathers are distinguished from oscillatory waves by having an oscillating amplitude with a temporal period of \(T = \pi/\nu\). We also see that the phase \(\text{arg}(u)\) of these breathers is time-periodic if either the frequency \(\nu_0\) vanishes, or the two frequencies \(\nu_0 \neq 0\) and \(\nu \neq 0\) are commensurate, where the condition \(\nu_0 \neq 0\) distinguishes an oscillatory breather from an ordinary breather.

The amplitude and phase gradient of the breathers solutions (3.54)–(3.56) and (3.57)–(3.59) are illustrated in Fig. 13–Fig. 18 for the Hirota equation (2.13) and Fig. 19–Fig. 24 for the Sasa-Satsuma equation (2.81). Figures show \(t = t_0(= t_0 + T), t_0 + T/3, t_0 + 2T/3\).

![Graphs showing amplitude and phase gradient of oscillatory breathers](image_url)

**Figure 13.** Hirota stationary breather with \(\chi = 0.5, c = 0, \nu = 2, \phi = 0, \phi_0 = 0 (t_0 = -2)\)
Figure 14. Hirota moving breather with $\chi = 0.5$, $c = 3$, $\nu = 2$, $\phi = 0$, $\phi_0 = 0$ ($t_0 = -2$)

Figure 15. Hirota moving breather with $\chi = 0.5$, $c = -2$, $\nu = 3$, $\phi = 0$, $\phi_0 = 0$ ($t_0 = -2$)
Figure 16. Hirota stationary oscillatory breather with $\chi = 0$, $c = 0$, $\nu = 2$, $\phi = 0$, $\nu_0 = 3$, $\phi_0 = 0$ ($t_0 = -2$)

Figure 17. Hirota moving oscillatory breather with $\chi = 0$, $c = 3$, $\nu = 2$, $\phi = 0$, $\nu_0 = 3$, $\phi_0 = 0$ ($t_0 = -2$)
Figure 18. Hirota moving oscillatory breather with $\chi = 0$, $c = -2$, $\nu = 1$, $\phi = 0$, $\nu_0 = 9$, $\phi_0 = 0$ ($t_0 = -2$)

Figure 19. Sasa-Satsuma stationary breather with $\chi = 0.5$, $c = 0$, $\nu = 2$, $\phi = 0$, $\nu_0 = 0$ ($t_0 = -2$)
Figure 20. Sasa-Satsuma moving breather with $\chi = 0.5$, $c = 3$, $\nu = 2$, $\phi = 0$, $\phi_0 = 0$ ($t_0 = -2$)

Figure 21. Sasa-Satsuma moving breather with $\chi = 0.5$, $c = -2$, $\nu = 3$, $\phi = 0$, $\phi_0 = 0$ ($t_0 = -2$)
Figure 22. Sasa-Satsuma stationary oscillatory breather with $\chi = 0$, $c = 0$, $\nu = 2$, $\phi = 0$, $\nu_0 = 3$, $\phi_0 = 0$ ($t_0 = -2$)

Figure 23. Sasa-Satsuma moving oscillatory breather with $\chi = 0$, $c = 3$, $\nu = 2$, $\phi = 0$, $\nu_0 = 3$, $\phi_0 = 0$ ($t_0 = -2$)
Figure 24. Sasa-Satsuma moving oscillatory breather with $\chi = 0$, $c = -2$, $\nu = 1$, $\phi = 0$, $\nu_0 = 9$, $\phi_0 = 0$ ($t_0 = -2$)

5. CONCLUDING REMARKS

In the literature, harmonically modulated solitons (1.9) and (1.14) are commonly called an envelope soliton. Strictly speaking, however, the factorization of such solitons (1.9) into a solitary wave part $f(kx + wt)$ and a harmonic wave part $\exp(i(\kappa x + \omega t))$ is well-defined only if the function $f$ is real-valued, so that the modulation of the solitary wave envelope is fully contained in the phase $\arg(u(t, x)) = \kappa x + \omega t$, as seen in the Hirota envelope soliton (1.11). Otherwise, when the function $f$ is complex-valued, as happens in the Sasa-Satsuma harmonically modulated soliton (1.12) as well as in the harmonically modulated 2-solitons for both the Hirota and Sasa-Satsuma equations, the solitary wave envelope is given by the modulus $|f(kx + wt)|$ while its modulation comes from the combined phase $\kappa x + \omega t + \arg(f(kx + wt))$. In general the only mathematically and physically meaningful way to decompose this phase is to write it as a travelling wave part $\arg(f(kx + wt))$ plus a temporal part $\nu t$, corresponding to the oscillatory forms (1.15) and (1.17). Thus the oscillatory parameterization that we have introduced in this paper for harmonically modulated solitons is mathematically clearer and physically simpler than the usual envelope parameterization (1.9) and (1.14).

In a sequel paper [19], we will study the main features of the amplitude and phase of colliding oscillatory waves as described by the oscillatory 2-soliton solutions (3.37)–(3.44) presented in Theorem 2 for the Hirota equation (2.13) and the Sasa-Satsuma equation (2.81).

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Email: sanco@brocku.ca, sattar_ju@yahoo.com, markw@math.ubc.ca

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