Volterra mortality model: Actuarial valuation and risk management with long-range dependence

Ling Wang∗ Mei Choi Chiu† Hoi Ying Wong‡

September 4, 2020

Abstract

While abundant empirical studies support the long-range dependence (LRD) of mortality rates, the corresponding impact on mortality securities are largely unknown due to the lack of appropriate tractable models for valuation and risk management purposes. We propose a novel class of Volterra mortality models that incorporate LRD into the actuarial valuation, retain tractability and are consistent with the existing continuous-time affine mortality models. We derive the survival probability in closed-form solution by taking into account of the historical health records. The flexibility and tractability of the models make them useful in valuing mortality-related products such as death benefit, annuity, longevity bond and many others as well as offering optimal mean-variance mortality hedging rules. Numerical studies are conducted to examine the impact of LRD within mortality rates on various insurance products and the hedging efficiency.

Keywords: Stochastic mortality; Long-range dependence; Affine Volterra processes; Valuation; Mean-variance hedging.

∗Department of Statistics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong. (lingwang@link.cuhk.edu.hk)
†Department of Mathematics & Information Technology, The Education University of Hong Kong, Tai Po, N.T., Hong Kong. (mcchiu@eduhk.hk)
‡Corresponding author. Department of Statistics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong. (hywong@cuhk.edu.hk)
1 Introduction

Actuaries heavily rely on mortality modeling for mortality prediction, actuarial valuation and risk management. An accurate estimation and prediction of human mortality are essential building block for both insurance contract pricing and pension policy. The first study of this can be dated back to Gompertz (1825).

The arguably most well-received modern mortality model is the Lee and Carter (1992) model and its extensions using time series analysis. For instance, it has been generalized to multivariate populations with a common trend (Li and Lee, 2005), mortality forecasts using single value decomposition (Renshaw and Haberman, 2003), joint modeling of different national populations (Antonio et al., 2015) and sub-populations (Villegas and Haberman, 2014), multi-population stochastic mortality model (Danesi et al., 2015), Poisson regression model (Brouhns et al., 2002), and stochastic period and cohort effect (Toczydlowska et al., 2017) among others. A key advantage of the Lee-Carter model and its invariant is that statistical inference from time series analysis can be applied or generalized to estimate and test with a real mortality data set.

By incorporating fractionally integrated time series analysis into the Lee-Carter model, Yan et al. (2018) empirically show the existence of long-range dependence (LRD) (also known as long-memory pattern or fractional persistence) across age groups, gender and countries by using the dataset of 16 countries. When they apply their long-memory mortality model to forecast life expectancies, the mortality model ignoring LRD tends to underestimate life expectancy, leading to important implication to pension scheme and the funding issue. Yan et al. (2020) further extend the model to incorporating multivariate cohorts and document the existence of LRD. Yaya et al. (2019) show long-memory pattern in infant mortality rates of G7 countries. Delgado-Vences and Ornelas (2019) alternatively offer strong empirical evidence that mortality rates exhibit LRD using a fractional Ornstein-Uhlenbeck (fOU) process with Italian population data between the years 1950 to 2004.

Most stochastic mortality models focus on the mortality rate, or equivalently the Poisson intensity rate. We refer to the pioneering work of Milevsky and Promislow (2001) of introducing the Cox model to insurance applications. Biffis (2005) and Biffis and Millossovich (2006) further develop this idea of doubly stochastic mortality models with affine feature for exploiting the analytical tractability in actuarial valuation with both financial and mortality risks. Jevtić et al. (2013) extend it into cohort models while Wong et al. (2017) introduce continuous-time cointegration into the multivariate mortality rates.

Blackburn and Sherris (2013) advocate the use of continuous-time affine mortality models for longevity pricing and hedging because of its tractability and consistency with the market data. Jevtić and Regis (2019) propose a calibration to the multiple populations affine mortality models and demonstrate its empirical use with product price data. However, none of the aforementioned works provide an analytically tractable dynamic mortality model with the LRD feature.
The primary contribution of this paper is the proposal of a novel class of dynamic stochastic mortality models rendering actuarial valuation tractability and LRD property, simultaneously. As the proposed model is based on Volterra processes, we call them Volterra mortality models. Inspired by the affine Volterra process (Abi Jaber et al., 2019), our model preserves the affine structure for general actuarial valuation but still captures LRD. In terms of practical contribution, we derive closed-form solutions for the survival probability, death and survival benefits of insurance contracts, and longevity bonds with the model and then address the impact of LRD for these insurance products. To the best of our knowledge, the derived formulas constitute the first set of formulas for insurance products subject to LRD feature of mortality rates.

This study also contributes to risk management with LRD mortality rates. We rigorously develop the mean-variance (MV) strategy for hedging longevity risk with a longevity security subject to LRD. This later hedging strategy is highly non-trivial because the Volterra mortality rate is a non-Markovian and non-semimartingale process. Inspired by Han and Wong (2020), we derive the MV optimal hedging with the Volterra mortality models by means of linear-quadratic control with the backward stochastic differential equation (BSDE) framework similar to Wong et al. (2017) whereas Han and Wong (2020) solve the MV portfolio problem with rough volatility by constructing an auxiliary process. Our optimal hedging rule shows how to adjust the hedge for LRD of mortality rates.

The rest of paper is organized as follows. Section 2 introduces Volterra mortality model based on the doubly stochastic mortality models and explains how the model captures LRD. Section 3 offers some formulas for actuarial valuation. In section 4, we formulate an optimal hedging problem under Volterra mortality model and give explicit solution. To compare Volterra mortality model with LRD and Markovian mortality model, numerical study is conducted for both actuarial valuation and hedging problem in section 5. Section 6 gives concluding remarks. Some details and additional proof are given in appendix.

2 The model

Consider a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) where the filtration \(\mathbb{F} = \{\mathcal{F}_t : 0 \leq t \leq T\}\) satisfies the usual properties. We write \(\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t\) where \(\mathcal{H}_t\) represents the flow of information available as time goes by including the historical processes and the current states and \(\mathcal{G}_t\) contains the information whether an individual has died or not. We interpret \(\mathbb{P}\) as the physical probability measure. Alternatively, our model can be developed under a pricing measure so that model parameters are calibrated to insurance product prices available in the market. This enables actuarial valuation consistent with market prices. However, risk management strategies should be conducted under the physical probability measure. To avoid confusion, we denote the pricing measure by \(\mathbb{Q}\) and discuss relationship between \(\mathbb{P}\) and \(\mathbb{Q}\) in the next section. For the time being, we focus on the model development under \(\mathbb{P}\).

We begin with the classic doubly stochastic mortality models. To simplify matter, we
consider a group of people with homogeneous feature while individual differences certainly exist in this group at the same time. A counting process $N$ is a doubly stochastic process driven by subfiltration $\mathcal{G} = \{\mathcal{G}_t\}_{t \geq 0}$ of $\mathcal{F}$ and with $\mathcal{G}$-intensity $\mu_t$. Let $\tau$ be the first jump-time of the process $N$ with intensity $\mu_t$. In actuarial application, the process $\{N_t\}_{t \geq 0}$ records the number of death at each time $t \geq 0$. For any time $t \geq 0$ and state $\omega \in \Omega$ such that $\tau(\omega) > t$, we have:

$$P(\tau \leq t + \Delta|\mathcal{F}_t) \equiv \mu_t(\omega)\Delta,$$  

(1)

for a trajectory of $\mu_t(\omega)$ and a fixed $\omega \in \Omega$. Thus, the counting process $N$ associated with $\tau$ becomes an inhomogeneous Poisson with parameter $\int_0^\cdot \mu_s(\omega)ds$. In other words, for all $T \geq t \geq 0$ and integer $k$ ($k \geq 0$), we have:

$$P(N_T - N_t = k|\mathcal{F}_t \vee \mathcal{G}_T) = \frac{(\int_t^T \mu_s(\omega)ds)^k}{k!}e^{-\int_t^T \mu_s(\omega)ds}.$$ 

By the law of iterated expectations, the time-$t$ survival probabilities over the time interval $(t,T]$ (for fixed $T \geq t \geq 0$) can be expressed as follows:

$$P(\tau > T|\mathcal{F}_t) = \mathbb{E}\left[e^{-\int_t^T \mu_s(\omega)ds} \mid \mathcal{F}_t\right].$$  

(2)

If the intensity $\mu_t$ is a constant, then the doubly stochastic process reduces to the homogeneous Poisson process. However, the literature of mortality modeling is in favour of a stochastic intensity. Typically, the intensity is modeled through a stochastic differential equations (SDE). For instance, Biffis (2005) and Biffis and Millossovich (2006) postulate a Markovian process that $\mu_t = f(X_t)$, where $f$ is a continuous function on $\mathbb{R}$,

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,$$  

(3)

and $\{W_t\}_{t \geq 0}$ is the standard Brownian motion.

To incorporate LRD into the mortality rate, one simply replaces the Brownian motion in (3) with the fractional Brownian motion. In other words,

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t^H,$$  

(4)

where $W_t^H$ is a fractional Brownian motion (fBM) with the Hurst parameter $H \in [0.5, 1)$. For instance, the empirical study of Delgado-Vences and Ornelas (2019) uses $f(X_t) = h_0\exp(h_1t + h_2X_t)$, for constants $h_0, h_1, h_2 > 0$, and a fOU process in the form of (4) that the drift term $b(X_t)$ is a linear function of $X_t$ and the $\sigma(X_t) \equiv \sigma$ is a constant. However, the fractional Brownian motion is analytically intractable for actuarial valuation.

### 2.1 Volterra mortality

We propose a stochastic mortality model incorporating LRD such that key advantages of the works by Biffis (2005), Delgado-Vences and Ornelas (2019) and Leonenko et al. (2019) retain. More specifically, we want to maintain the affine nature of Biffis (2005), reflect

Electronic copy available at: https://ssrn.com/abstract=3666984
LRD with fBM as in Delgado-Vences and Ornelas (2019), and offers explicit expressions for some important Fourier-Laplace functional generalizing Leonenko et al. (2019) for actuarial valuation. Our model is highly inspired by the affine Volterra processes (Abi Jaber et al., 2019) and hence called Volterra mortality model.

In the one dimensional case, Baudoin and Nualart (2003) show the equivalence between fBM and the Volterra process in the way that

$$W^H_t = c_H \int_0^t (t-s)^{H-\frac{1}{2}} dW_1(s),$$

where $c_H$ is a constant related to the Hurst parameter $H$, $W_1$ is the Wiener process and the integral process on the right-hand side is a standard Volterra process. To simplify matter and be consistent with the literature, we postulate the mortality rate $\mu$ of a group:

$$\mu_t = m(t) + \eta X_t, \quad (5)$$

where $m(t)$ is a bounded continuous deterministic function and $\eta$ is a constant. In other words, we require that $f(X_t)$ is a linear function of $X_t$. In addition, $X_t$ follows a stochastic Volterra integral equation (SVIE):

$$X_t = X_0 + \int_0^t K(t-s)b(X_s)ds + \int_0^t K(t-s)\sigma(X_s)dW_s \quad (6)$$

where $W = [W_1, \ldots, W_d]^\top$ is the standard $d$-dimensional Brownian motion under $\mathbb{P}$, and the coefficients $b$ and $\sigma$ are assumed to be continuous. The convolution kernel $K$ satisfies the following condition:

$$K \in L^{2}_{loc}(\mathbb{R}_+, \mathbb{R}), \quad \int_0^h K(t)^2 dt = O(h^\gamma) \text{ and } \int_0^T (K(t+h) - K(t))^2 dt = O(h^\gamma) \text{ for some } \gamma \in (0, 2] \text{ and every } T < \infty.$$

Although the process $X_t$ in (6) is generally of high-dimensional, we would like to illustrate it in a one-dimensional case. Table 1 exhibits some useful kernels $K$ in the one-dimensional case. We obtain the fBM by choosing $K$ as the fractional kernel in the table with a constant $\sigma(X_s)$ and $b = 0$ in (6). Therefore, Volterra processes can be applied to a wider class of LRD noise term. Note that the resolvent or resolvent of the second kind corresponding to $K$ shown in the table is defined as the kernel $R$ such that $K*R = R*K = K - R$. The convolutions $K*R$ and $R*K$ with $K$ a measurable function on $\mathbb{R}_+$ and $R$ a measure on $\mathbb{R}_+$ of locally bounded variation are defined by

$$(K*R)(t) = \int_{[0,t]} K(t-s)R(ds), \quad (R*K)(t) = \int_{[0,t]} R(ds)K(t-s)$$

for $t > 0$.

**Remark 1.** According to Biffis (2005), the deterministic function $m(t)$ in (5) may represent (i) a best-estimated assumption on $\mu$ enforcing unbiased expectations about the future based on the available information, (ii) pricing demographics basis, or (iii) an available mortality table for a population of insureds. In Section 5, we calibrate $m(t)$ to the table SIM92, a period table usually employed to price assurances.
Table 1: Examples of kernel function $K$ and the corresponding resolvent $R$. Here $E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha \cdot n + \beta)}$ denotes the Mittag-Leffler function.

In addition, when the convolution kernel $K$ is set to a constant $c$ in (6), the $X_t$ reduces to the solution of a SDE. Furthermore, once $b(X_t)$ is linear in $X_t$ and $\sigma(X_t)$ satisfies certain affine property, then our model in (6) becomes the affine stochastic mortality model of Biffis (2005). The possibly high-dimensional $X_t$ enables us to incorporate multi-factor mortality modelling too. However, we would like to highlight that the Volterra process in (6) is generally a non-Markovian and non-semimartingale process. The non-Markovian nature is obvious because the integrals in the SIVE take the whole realized sample path into account. The non-semimartingale feature is reflected by the fact that the time variable $t$ appears in both the integral limit and the kernel function, making it fail to define the Ito integral.

Fortunately, Abi Jaber et al. (2019) show that it is still possible to maintain the affine nature within the model (6). Let $a(x) = \sigma(x)\sigma(x)^\top$ be the covariance matrix.

**Definition 1.** The SVIE (6) is called an affine process (Abi Jaber et al., 2019) if

$$a(x) = A^0 + x_1A^1 + \cdots + x_dA^d,$$
$$b(x) = b^0 + x_1b^1 + \cdots + x_db^d,$$

for some $d$-dimensional symmetric matrices $A^i$ and vectors $b^i$. For simplicity, we set $B = (b^1, \ldots, b^d)$ and $A(u) = (uA1u^\top, \ldots, uAdu^\top)$ for any row vector $u \in \mathbb{C}^d$.

To draw insights from Definition 1, consider the one dimensional case. When $b(x) = b^0 - b^1x$, a linear function of $x$, and $a(x)$ is a constant, (6) is known as the Volterra type of Vasicek (VV) model which reduces to the classic Vasicek model by taking a constant kernel or, equivalently, $H = 1/2$ in the fractional kernel. When $b(x)$ is a linear in $x$ and $a(x)$ is directly proportional to $x$, our model in (6) reduces to the Volterra version of CIR (VCIR) model.

**2.2 Interest rate model**

Although we focus on the mortality modelling, actuarial valuation needs to specify the dynamic of the risk-free interest rate. We simply adopt a Markov affine model for the interest rate. Specifically, the short rate process $r$ satisfying $\int_0^t |r_s| ds < \infty$ for $t \geq 0$ and
we define the return of a risk-less asset as $\exp(\int_0^t r_s ds)$ for a unit dollar investment at time 0. In addition, the interest rate process is driven a Markov affine process $Z$ in $\mathbb{R}^k$:

$$dZ_t = \tilde{b}(Z_t)dt + \tilde{\sigma}(Z_t)dW'_t,$$

(7)

where $W'$ is a $k$-dimensional standard Brownian motion. The coefficients $\tilde{b}(Z_t)$ and $\tilde{\sigma}(Z_t)$ have affine dependence on $Z_t$ once they satisfy Definition 1 with the dimension $d$ replaced by $k$. Hence, it coincides with the definition of Markov affine process in Duffie et al. (2003). Furthermore, the short rate $r_t = r(t, Z_t) = \lambda_0(t) + \lambda_1(t) \cdot Z_t$ which is an affine function on $Z_t$ with coefficients $\lambda_0(t)$ and $\lambda_1(t)$ being bounded continuous functions on $[0, \infty)$. By the affine processes in Duffie et al. (2003) and Filipović (2005), at time $t$, we have

$$B(t, T) = \mathbb{E} \left[ e^{-\int_t^T r(s, Z_s)ds} \mid \mathcal{F}_t \right] = e^{\tilde{\alpha}(t, T) + \tilde{\beta}(t, T) Z_t},$$

(8)

where the functions $\tilde{\alpha}(\cdot, T)$ and $\tilde{\beta}(\cdot, T)$ are uniquely solved from the ordinary differential equations (ODEs) in Appendix A with boundary conditions $\tilde{\alpha}(T, T) = 0$ and $\tilde{\beta}(T, T) = 0$. If the interest rate model in (7) is defined under the pricing measure, i.e., $\mathbb{P} = \mathbb{Q}$, then the quantity $B(t, T)$ represents the price of a unit zero coupon bond.

3 Actuarial Valuation

We demonstrate the tractability of the proposed Volterra mortality model in actuarial valuation. Specifically, we derive closed-form solutions to the survival probability and prices of some standard life insurance products. The following theorem is the building block of the actuarial valuation.

**Theorem 1.** If the mortality rate $\mu_t$ follows (5) and (6) and has the affine structure specified in Definition 1, then, for any constant $c_0$ and $c_1$ and $T > t$, we have,

$$\mathbb{E} \left[ e^{-\int_t^T \mu_s ds} (c_0 + c_1 \mu_T) \mid \mathcal{F}_t \right] = c_0 g(t, T) - c_1 \frac{\partial g(t, T)}{\partial T},$$

(9)

where

$$g(t, T) = e^{-\int_t^T \eta(s)ds} e^{\int_t^T \mu_s ds} \exp(Y_t(T)),$$

$$Y_t(T) = Y_0 + \int_0^t \psi(T-s)\sigma(X_s)dW_s - \frac{1}{2} \int_0^t \psi(T-s)a(X_s)\psi(T-s)^T ds,$$

(10)

$$Y_0(T) = \int_0^T (\eta X_0 + \psi(s)b(X_0) + \frac{1}{2} \psi(s)a(X_0)\psi(s)^T) ds,$$

and $\psi \in \mathcal{L}^2([0, T], \mathbb{C}^d)$ solves the Riccati-Volterra equation:

$$\psi = (\eta + \psi B + \frac{1}{2} A(\psi)) * K,$$

(11)

with $A(\cdot)$ appearing in Definition 1. In addition, the $Y$ has an alternative expression:

$$Y_t(T) = -\eta \int_0^T \mathbb{E}[X_s | \mathcal{F}_t] ds + \frac{1}{2} \int_t^T \psi(T-s)a(\mathbb{E}[X_s | \mathcal{F}_t])\psi(T-s)^T ds,$$

(12)
where

\[
E[X_T | \mathcal{F}_t] = \left( \text{id} - \int_0^T R_B(s)ds \right) X_0 + \int_0^T E_B(T-s)b^0(s)ds + \int_0^T E_B(T-s)\sigma(X_s)dW_s
\]  

(13)

with \text{id} being the identity matrix, \(R_B\) the resolvent of \(-KB\), and \(E_B = K - R_B^* K\).

Proof. See Appendix A. \(\square\)

Remark 2. The partial derivative \(\frac{\partial \varphi(t,T)}{\partial T}\) does not admit a closed-form solution in general because the function \(\varphi(t,T)\) depends on \(Y_0(T)\) which depends on \(T\) through the \(\psi\) solved from the Riccati-Volterra equation (11). Fortunately, the partial derivative appears in insurance products related to the death benefit through an integration. We then get rid of computing it by means of integration by parts.

We highlight that the expression in (10) implies that \(Y_0(T)\) is a semimartingale because all of the integrants in (10) are independent of \(t\). It is important and interesting because this implies that insurance product prices can be expressed into SDE even though the mortality rate with LRD is not. This enables us to construct hedging strategy for longevity risk using longevity securities in a LRD mortality environment, implying the importance of the longevity securatization. For the time being, we apply Theorem 1 to obtain the survival probability of the Volterra mortality model in closed-form solution.

Corollary 1. (Survival Probability) Under the Volterra mortality model of (5), (6) and Definition 1, for any \(t < T\), the survival probability reads,

\[
P(\tau > T | \mathcal{F}_t) = E \left[ e^{-\int_t^T \mu_s ds} | \mathcal{F}_t \right] = g(t, T) = e^{-\int_0^T m(s)ds + \int_0^T \mu_s ds} \exp(Y_t(T)),
\]  

(14)

where \(Y_t(T)\) is defined in (10) or, equivalently, (12).

Proof. The result follows by taking \(c_0 = 1\) and \(c_1 = 0\) in Theorem 1. \(\square\)

The survival probability in Corollary 1 captures LRD because it depends on the whole historical path of the mortality rate. This is reflected by the terms \(e^{-\int_0^T m(s)ds + \int_0^T \mu_s ds}\) and \(Y_0(T)\). However, when compared our survival probability with LRD with that of the corresponding Markovian mortality model, we find consistency between the two. Consider the case of fractional kernel \(K(t) = \frac{\alpha^{-1}}{\Gamma(\alpha)} \text{id}\), where \(\alpha = H + 1/2\) and \(H\) is the Hurst parameter \(H\). The process \(X_t\) becomes,

\[
X_t = X_0 + \lambda \int_0^t (t-s)^{\alpha-1} (\theta - X_s)ds + \int_0^t (t-s)^{\alpha-1} \sigma(X_s)dW_s. 
\]  

(15)

When \(\alpha = 1\), the \(K(t) \equiv \text{id}\) and

\[
dX_t = \lambda(\theta - X_t)dt + \sigma(X_t)dW_t,
\]

which is the Vasicek mortality rate model for a constant \(\sigma(X_t)\) and CIR model for \(\sigma(X_t) = \sigma \sqrt{X_t}\). Both are investigated by Biffis (2005). In such a situation, a part of the \(Y_0(T)\) in
which implies: $1 = \frac{\alpha}{\alpha}$. The Volterra-Riccatti equation (11) reduces to the ordinary Riccati equation. This makes our solution the same as the results in Biffis (2005) for $\alpha = 1$ or $H = 1/2$. However, once $\alpha > 1$, the process $X_t$ has LRD feature. The empirical study in Yan et al. (2018) shows that the survival probability is underestimated without taking LRD into account.

3.1 Standard Insurance contracts

To streamline the presentation, we assume that mortality rates are independent of the interest rate. Although this assumption could be considered as restrictively mathematically, it is a usual assumption in the actuarial and insurance literature. Two basic payoffs in insurance contracts are the survival benefit and the death benefit from an insurance contract.

Let $C_T$ be a bounded random payoff for a survivor at time $T$ independent of the mortality. The time-$t$ fair value of the survival benefit $SB_t(C_T; T)$ of the terminal amount $C_T$, with $0 \leq t \leq T$ under the pricing measure $Q$ is given by:

$$SB_t(C_T; T) = 1_{\{T>t\}} E^Q \left[ e^{-\int_t^T \beta_u ds} C_T G^T_t \right] E^Q \left[ e^{-\int_t^T \mu_u ds} G^X_t \right].$$

To draw some insights from (16), let us consider the situation that the mortality model of (5) and (6) and interest rate process of (7) are constructed under the pricing measure $Q$ or, equivalently, that $P = Q$ in Section 2. We refer the results under such an assumption as the baseline case in this paper and the corresponding valuation becomes simple.

**Proposition 1.** (Survival Benefit: The Baseline Valuation.) If $P = Q$ and the mortality and interest rate are independent, then the Volterra mortality model of (5), (6) and Definition 1 and the affine interest rate model imply that

$$SB_t(C_T; T) = 1_{\{T>t\}} B(t, T) E^{Q_T} \left[ C_T | G^T_t \right] g(t, T),$$

where $g(t, T)$ is presented in Theorem 1, $B(t, T)$ is the zero coupon bond price in (8) and $Q_T$ is the forward pricing measure:

$$\frac{dQ_T}{dQ} \bigg|_{F_t} = \exp \left( -\frac{1}{2} \int_0^T \tilde{\beta}^2(u, T) \tilde{\sigma}^2(Z_u) du - \int_0^T \tilde{\beta}(u, T) \tilde{\sigma}(Z_u) dW^u_t \right).$$

**Proof.** By Corollary 1,

$$E^Q \left[ e^{-\int_t^T \mu_u ds} G^X_t \right] = g(t, T).$$

By the affine short-rate model (7) and equation (8), we have

$$dB(t, T) = B(t, T) \beta(T, T) dW^u_t - B(t, T) \tilde{\beta}(T, T) \tilde{\sigma}(Z_t) dW^u_t,$$

which implies: $1 = B(T, T) = B(t, T) e^{\int_t^T \beta_u du - \frac{1}{2} \tilde{\beta}^2(u, T) \tilde{\sigma}^2(Z_u) du - \int_t^T \tilde{\beta}(u, T) \tilde{\sigma}(Z_u) dW^u_u}$. Hence,

$$e^{-\int_t^T \mu_u ds} = B(t, T) \exp \left( -\frac{1}{2} \int_t^T \tilde{\beta}^2(u, T) \tilde{\sigma}^2(Z_u) du - \int_t^T \tilde{\beta}(u, T) \tilde{\sigma}(Z_u) dW^u_u \right).$$
An application of the Girsanov theorem shows that
\[
E^Q \left[ e^{-\int_T^t r_s ds} C_T \big| G^Z_t \right] = B(t,T) E^{Q^T} \left[ C_T \big| G^Z_t \right],
\]
where the forward measure \( Q^T \) is presented in the Proposition.

Another important basic payoff is the death benefit. Let \( C_t \) be a bounded \( G^Z \)-predictable process, representing a cash flow stream independent of the mortality rate. Then, the time-\( t \) fair value of the death benefit with a cash flow stream \( C_t \), payable in case the insured dies before time \( T \) and \( 0 \leq t \leq T \), is given by:
\[
DB_t(C_T; T) = 1_{\{\tau > t\}} \int_t^T E^Q \left[ e^{-\int_T^u r_s ds} C_u \big| G^Z_t \right] E^Q \left[ e^{-\int_u^T \mu_s ds} \big| G^X_t \right] du.
\]
Then, we also have an explicit baseline valuation formula for the death benefit.

**Proposition 2.** (Death Benefit: The Baseline Valuation.) If \( \mathbb{P} = Q \) and the mortality and interest rate are independent, then the Volterra mortality model of (5), (6) and Definition 1 and the affine interest rate model imply that
\[
DB_t(C_T; T) = -1_{\{\tau > t\}} \int_t^T B(t,u) E^{Q_u} \left[ C_u \big| G^Z_t \right] \frac{\partial g(t,u)}{\partial u} du
\]
where \( Y_t(u) \) is defined in (10), \( B(t,T) \) in (8), \( g(t,T) \) in Theorem 1, and the forward pricing measure \( Q^u \) in Proposition 1.

**Proof.** The proof is similar to that of Proposition 1 except for the second expectation appearing in the representation of \( DB_t(C_T; T) \). By Theorem 1, it is clear that
\[
E^Q \left[ e^{-\int_T^u \mu_s ds} \big| G^X_t \right] = - \frac{\partial g(t,u)}{\partial u}.
\]

Applying integration by parts to \( DB \) in Proposition 2 yields an alternative expression:
\[
DB_t(C_T; T) = -1_{\{\tau > t\}} \left\{ B(t,u) E^{Q^T} \left[ C_T \big| G^Z_t \right] g(t,T) - E^{Q^T} \left[ C_T \big| G^Z_t \right] \right\} - \int_t^T \frac{\partial (B(t,u) E^{Q_u} \left[ C_u \big| G^Z_t \right])}{\partial u} g(t,u) du.
\]
In this way, as the interest rate model follows the Markovian affine model, the partial derivative term in (17) admits a closed-form solution in many cases and we get rid of computing a \( T \)-partial derivative of \( g(t,T) \), which is rather more complicated.

### 3.1.1 Examples of concrete insurance contracts

If the formulas for survival and death benefits are still considered abstract, we apply them to some concrete insurance or pension products.

Electronic copy available at: https://ssrn.com/abstract=3666984
**Longevity Bond:** Consider a unit zero-coupon longevity bond which pays $1 times $e^{-\int_{t}^{T} \mu_s ds}$, the percentage of survivors in a population during $t$ to $T$. Blake et al. (2006) show that the longevity bond takes the form:

$$B_L(t, T) = E_Q \left[ e^{-\int_{T}^{t} r_s + \mu_s ds} \bigg| \mathcal{F}_t \right].$$

Under the Volterra mortality model with LRD, Proposition 1 immediately implies that

$$B_L(t, T) = B(t, T) g(t, T),$$

by setting $C_T \equiv 1$ once the financial market is independent of the human mortality.

**Annuity:** Consider a $t'$-years deferred annuity involving a continuous payment of an indexed benefit from time $t$ onwards, conditional on survival of the policyholder at that time. Suppose that the payoff is made of a unit amount each year. Denote $x^{*}$ as the maximum age humans can live. The fair value of such an annuity is given by:

$$A_{N_t}(t') = \sum_{h=t'}^{x^{*}-t-1} SB_t(1; t+h) = \sum_{T=t+t'}^{x^{*}-1} B(t, T) g(t, T)$$

$$= \sum_{T=t+t'}^{x^{*}-1} e^{\tilde{\alpha}(t, T)+\tilde{\beta}(t, T)Z_t} e^{-\int_{0}^{T} m(s) ds + \int_{t}^{T} \mu_s ds} \exp(Y_t(T))$$

where $Y_t(T)$ is defined in (10) and $\tilde{\alpha}(t, T)$ and $\tilde{\beta}(t, T)$ are in (8).

**Assurances:** Consider an assurance guaranteeing a unit amount benefit in case of death in the period $(t, T]$. By setting $C \equiv 1$ in (17), the fair value of such an assurance is given by:

$$A_{S_t}(T) = 1 - B(t, T) g(t, T) + \int_{t}^{T} \frac{\partial B(t, u)}{\partial u} g(t, u) du$$

where $B(t, T)$ is defined in (8) and $g(t, T)$ in Theorem 1.

**Endowment:** Consider an endowment given the survival on time $t$ with maturity time $T$, which includes a survival benefit $C_1$ given the survival on time $T$ and a death benefit $C_2$ in case of the death in the period $(t, T]$. $C_1$ and $C_2$ are constants. By Propositions 1 and 2 and (17), the fair value of such an endowment is given by:

$$E_{N_t}^{T}(C_1, C_2) = SB_t(C_1; T) + DB_t(C_2; T)$$

$$= (C_1 - C_2) B(t, T) g(t, T) + C_2 \left( 1 + \int_{t}^{T} \frac{\partial B(t, u)}{\partial u} g(t, u) du \right),$$

where $B(t, T)$ is defined in (8) and $g(t, T)$ in Theorem 1.

### 3.2 Esscher transform

Although Propositions 1 and 2 facilitate the model development under the pricing measure and calibration to market prices of insurance products, the insurance practice may not have sufficient market prices for calibration. In addition, risk management strategy requires the connection between the physical and pricing measures as demonstrated in the
next section. Therefore, we present two possible ways to link the measures of $\mathbb{P}$ and $\mathbb{Q}$ with limited observed prices. For the time being, we focus on the situation that the Volterra mortality model is estimated with historical mortality table and hence built under the physical measure $\mathbb{P} \neq \mathbb{Q}$.

The first approach commonly used to identify a pricing measure in the actuarial literature is the Esscher transform. Chuang and Brockett (2014) apply Esscher transform on the mortality rate to find a related martingale measure for pricing longevity derivatives. Wang et al. (2019) also use Esscher transformation for pricing longevity derivatives based on an improved Lee–Carter model. Although the mortality rate $\mu_t$ is non-Markovian and non-semimartingale under our framework, the advantage is that we have an explicit Laplace-Fourier functional representation in Theorem 1. For a random variable $\gamma$ with a well-defined moment-generating function (MGF) under $\mathbb{P}$, an equivalent probability measure $\mathbb{Q}(\theta)$ derived from the Esscher transform with parameter $\theta$ is defined as

$$\frac{d\mathbb{Q}(\theta)}{d\mathbb{P}} = \frac{e^{\theta \gamma}}{\mathbb{E}[e^{\theta \gamma}]}.$$  \hspace{1cm} (19)

By setting $c_0 = 1$ and $c_1 = 0$ in Theorem 1, the MGF for the random variable $-\int_t^T \mu_s \, ds$ is well-defined and can be obtained in an explicit form. Specifically, as we assume $\mu_t = m(t) + \eta X_t$, the MGF defined as

$$M(\theta T) = \mathbb{E}[e^{-\theta T \int_t^T \mu_s \, ds}],$$

which corresponds to the $g(t, T)$ in Theorem 1 with parameters $m(t)$ and $\eta$ replaced with $\theta T m(t)$ and $\theta T \eta$ for the constant $\theta T$ and a fixed $T$. For instance, we observe a risk-free zero coupon bond and a zero coupon longevity bond with the same maturity. Then, we can deduce the synthetic value of

$$\mathbb{E}_t^{\mathbb{Q}(\theta T)}[e^{-\theta T \int_t^T \mu_s \, ds}] = \frac{\mathbb{E}_t[e^{-(\theta T + 1) \int_t^T \mu_s \, ds}]}{\mathbb{E}_t[e^{-\theta T \int_t^T \mu_s \, ds}]} = \frac{M(\theta T + 1)}{M(\theta T)}.$$  \hspace{1cm} (20)

While the left-hand quantity is deduced from market prices, the $M(\theta T)$ achieves a closed-form solution from our model through Theorem 1. Specifically, $M(\theta)$ is the $g(t, T)$ in Theorem 1 with $m(t)$ and $\eta$ replaced with $\theta m(t)$ and $\theta \eta$, respectively. One can then calibrate $\theta T$ to the term structure of longevity bonds, or longevity bond prices for different maturity $T$, after estimating physical model parameters, including the LRD feature, with historical data.

From (20), when $\theta T = 0$, the longevity bond is priced under $\mathbb{P}$ and our previous valuation formulas hold. For a nonzero $\theta T$, a slight adjustment can be made through (20) as the MGF is explicitly known.

### 3.3 Affine retaining transform

Although the Esscher transform provides us with a powerful and convenient framework to identify a pricing measure, it does not offer us an explicit stochastic process under
the pricing measure. When we perform risk management strategy, we need the stochastic process of the mortality rate under both $P$ and $Q$. It is desirable that the Volterra mortality model retains the affine nature in Definition 1. Therefore, we propose the following affine retaining transform based on the Girsanov theorem.

**Definition 2.** Given an affine SIVE of (6) satisfying Definition 1, an affine retaining transform for measure change is based on shifting the Wiener process as follows.

\[ dW^Q_t = dW_t - \sigma(X_t)^T \varphi(t) \, dt, \]

for a deterministic function $\varphi(t) \in \mathbb{R}^d$ satisfying

\[ \mathbb{E}_t \left[ e^{\frac{1}{2} \int_0^T |\sigma(X_s)^T \varphi(s)|^2 \, ds} \right] < \infty. \] (21)

Under Definition 2, we identify a pricing measure $Q$ equivalent to $P$:

\[ \frac{dQ}{dP} = \exp \left( -\frac{1}{2} \int_0^T |\sigma(X_s)^T \varphi(s)|^2 \, ds + \int_0^T \varphi(s)^T \sigma(X_s) \, dW^Q_s \right). \]

where $\varphi(t)$ is calibrated to observed prices. In addition, the mortality process $\mu_t = m(t) + \eta X_t$ in (6) under $Q$ has the $X_t$ changed to,

\[ X_t = X_0 + \int_0^t K(t-s)(b(X_s) + a(X_s)\varphi(s)) \, ds + \int_0^t K(t-s)\sigma(X_s) \, dW^Q_s, \] (22)

where $b(X_s) + a(X_s)\varphi(s)$ and $a(X_s)$ still satisfy the affine nature in Definition 1. Hence, the pricing formulas of Propositions 1 and 2 remain the same except for that the $b(X_s)$ there is replaced with $b(X_s) + a(X_s)\varphi(s)$ once the affine retaining transform in Definition 2 is adopted.

**Remark 3.** Although the Esscher and affine retaining transforms presented in Sections 3.2 and 3.3 are applied to the Volterra mortality model, these techniques have been widely used in the actuarial science literature, including the measure change with the affine interest rate models. Therefore, we do not repeat the detail for the interest rate. We mention them for highlighting the advantage of the proposed LRD mortality model in sense of calibrating to the pricing measure.

# 4 Optimal hedging of longevity risk

We further investigate optimal hedging with the proposed LRD mortality model as hedging is a typical risk management. The intent is to demonstrate the tractability of the LRD mortality model in hedging problems. As hedging should be performed under the physical probability measure $P$ whereas longevity securities such as the longevity bonds and swaps are valued in the market-implied pricing measure $Q$, we adopt the affine retaining transform detailed in Section 3.3 to bridge to the two probability measures in this section.
Let us sketch the conceptual framework prior to detailing the mathematics. As insurance product prices under the Volterra mortality model are semimartingales and hence can be expressed in SDE, the insurer’s wealth also satisfies a SDE with stochastic coefficients, which are possibly non-Markovian. In the notion of stochastic control theory, the insurer’s wealth takes the role as the state process. Therefore, theory of backward SDE (BSDE) is useful to solve the stochastic optimal control problem for state process with stochastic coefficients. Typically, the mean-variance (MV) hedging problem is closely related to the linear-quadratic (LQ) control problem that the formulation of BSDE approach is classic. In the following, we leverage on this well-received theoretical result to show the application of the LRD mortality model although the optimal hedging derived is novel and has remarkable performance in reducing risk with the LRD mortality. The performance is however shown in the next section numerically.

4.1 Problem formulation

Consider an insurer offering pension scheme wants to hedge the longevity risk by a longevity security. Specifically, the insurer allocates her capital among a bank account, risk-free zero-coupon bond and zero-coupon longevity bond. Let us concentrate on the one-dimensional case so that \( d = k = 1 \) from now on.

To simplify discussion, we adopt the VV mortality rate and assume \( m(t) = 0 \) and \( \eta = 1 \) in (5). In other words, \( \mu(t) = X(t) \) and

\[
\mu_t = X_t = X_0 + \int_0^t K(t-s)(b^0 - b^1 X_s)ds + \int_0^t K(t-s)\sigma dW_s, \tag{23}
\]

where \( b^0, b^1 \) and \( \sigma \) are constants and \( K \) is the Volterra kernel. In addition, the interest rate \( r_t = Z_t \) follows the Vasicek model:

\[
dr(t) = (\tilde{b}^0 - \tilde{b}^1 r_t)dt + \sigma_r dW'_t, \tag{24}
\]

where \( \tilde{b}^0, \tilde{b}^1 \) and \( \sigma_r \) are constant parameters. \( W_t \) and \( W'_t \) are independent Wiener processes under \( P \). Let \( W(t) = (W_t, W'_t)^\top \). Using the affine retaining transform in Definition 2, the Weiner process under the pricing measure is given by,

\[
dW_t^Q = dW_t - \sigma \frac{\varphi(t)}{\sigma} dt, \quad dW'_t^Q = dW'_t - \sigma_r \frac{\vartheta(t)}{\sigma_r} dt,
\]

where \( \vartheta \) and \( \varphi \) are deterministic functions satisfying the condition (21). Under the pricing measure, the mortality and interest rates are, respectively,

\[
X_t = X_0 + \int_0^t K(t-s)(b^0 + \varphi(s)\sigma - b^1 X_s)ds + \int_0^t K(t-s)\sigma dW_s^Q; \quad dr(t) = (\tilde{b}^0 + \vartheta(t)\sigma - \tilde{b}^1 r_t)dt + \sigma_r dW'_t^Q.
\]

As the unit zero coupon bond price takes the form:

\[
B(t,T) = \mathbb{E}^Q \left[ e^{-\int_T^T r(s)ds} \big| \mathcal{F}_t \right] = e^{\tilde{\alpha}(t,T) + \tilde{\beta}(t,T)r_t},
\]
with \( \tilde{\alpha}(t, T) \) and \( \tilde{\beta}(t, T) \) defined in Appendix A, the P-dynamics of the bond reads

\[
dB(t, T) = B(t, T)(r(t) + \nu_B(t))dt + B(t, T)\sigma_B(t)dW_t^I,
\]

where \( \nu_B = \vartheta(t)\sigma_B(t) \) and \( \sigma_B(t) = -\tilde{\beta}(t, T)\sigma_r \). Similarly, using the expression for a zero coupon longevity bond:

\[
B_L(t, T) = \mathbb{E}^Q \left[ e^{-\int_t^T r(s) + \mu(s)ds} \bigg| \mathcal{F}_t \right] = B(t, T)e^{\int_t^T \mu(s)ds}\exp(Y_t^1(T)),
\]

where \( Y_t^1(T) \) is equivalent to the \( Y_t(T) \) in (10) with \( b(x) = b^0 + \varphi(s)\sigma_r - b^1 x \), \( \sigma(x) = \sigma_r \) and \( W \) replaced by \( W^Q \), we obtain the P-dynamics of the longevity bond prices:

\[
dB_L(t, T) = B_L(t, T)(r(t) + \mu(t) + \nu_L(t))dt + B_L(t, T)\sigma_L(t)dW_t + B_L(t, T)\sigma_BdW_t^I,
\]

where \( \nu_L = \nu_B + \varphi(t)\sigma_t \), \( \sigma_t = -\psi_1(T - t)\sigma_r \) and \( \psi_1 \in L^2([0, T], \mathbb{C}) \) is the solution of the Riccati equation \( \psi_1 = (-1 - b^1 \psi_1) * K \). As an investment amount of \( B_L(t, T) \) in the longevity bond at time \( t \) becomes \( e^{-\int_t^T \mu(s)ds}B_L(t, \tau) \) at \( \tau > t \), the value of holding one unit of zero coupon longevity bond \( B_L(t) \) satisfies:

\[
dB_L(t, T) = B_L(t, T)(r(t) + \nu_L(t))dt + B_L(t, T)\sigma_L(t)dW_t + B_L(t, T)\sigma_BdW_t^I.
\] (25)

The quantities \( \nu_L - \nu_B \) and \( \nu_B \) are often known as the market prices of mortality and interest rate risks, respectively. From (25), the zero coupon longevity bond price still satisfies a SDE due to the semimartingale nature of \( Y_t(T) \). This fact enables us to deal with the optimal hedging problem with a LRD mortality rate. Note that the LRD feature is reflected by the volatility term of \( B_L(t) \) through a Riccati-Volterra equation.

Let \( u_0(t), u_1(t), \) and \( u_2(t) \) denote the investment amounts in the bank account, zero-coupon longevity bond and zero-coupon bond respectively. Denote \( \tilde{N}(t) \) as a stochastic Poisson process with intensity \( k_1(t) \) and \( \{z_i\}_{i=1}^\infty \) as independent identically distributed (iid) insurance claims. Consider a hedging horizon of \( T_0 < T \). Then, the wealth process of the insurer reads

\[
M(t) = u_0(t) + u_1(t) + u_2(t) - \sum_{i=1}^{\tilde{N}(t)} z_i - \Pi(t), \quad t \in [0, T_0],
\] (26)

where \( \Pi = \int_0^t \pi(s)ds, \ t \in [0, T_0] \) and \( \pi(t) \) is a \( \mathcal{F}_t \)-adapted, square integrable process representing the pension annuity net cash outflow. We denote the filtration generated by \( \{M(s) : 0 \leq s \leq t\} \) by \( \mathcal{H}_t \supseteq \mathcal{F}_t \). The insurer’s wealth \( M(t) \) satisfies the following SDE:

\[
dM(t) = (M(t)r(t) + u(t)\top \nu(t) - \pi(t))dt + u(t)\top \sigma_S(t) \top dW(t) - zd\tilde{N}(t),
\] (27)

where \( z \) has the same distribution as \( z_1 \), \( u(t) = (u_1(t), u_2(t))\top \), \( \nu(t) = (\nu_L(t), \nu_B(t))\top \), and

\[
\sigma_S(t)\top = \begin{pmatrix} \sigma_t & \sigma_b \\ 0 & \sigma_b \end{pmatrix}.
\]

If a hedging strategy \( u(t) \) is a \( \mathcal{F}_t \)-adapted process and \( \mathbb{E}[\int_0^{T_0} |u(s)|^2ds] < \infty \), then it is said to be admissible. We denote the set of admissible controls as \( U \).
Definition 3. The classic mean-variance (MV) hedging problem is defined as

\[ V(\phi) = \min_{u(\cdot) \in \mathcal{U}} \text{Var}(M(T_0)) - \frac{\phi}{2} \mathbb{E}[M(T_0)], \tag{28} \]

where the parameter \( \phi \) measures the insurer’s risk averseness.

When \( \phi = 0 \), problem (28) refers to the minimum-variance hedging. For any given \( \bar{M} = \mathbb{E}[M(T_0)] \),

\[ \mathbb{E}[(M(T_0) - \bar{M})^2] - \frac{\phi}{2} \mathbb{E}[M(T_0)] = \mathbb{E}[(M(T_0) - (\bar{M} + \frac{\phi}{4}))^2] - \frac{\phi}{2} \bar{M} - \frac{\phi^2}{16}. \]

In addition, the MV hedging problem can be embedded into a target-based objective. Specifically, the problem (28) is equivalent to:

\[ \min_{\bar{M}} \min_{u(\cdot) \in \mathcal{U}} \mathbb{E}[(M(T_0) - c)^2] - \frac{\phi}{2} \bar{M} - \frac{\phi^2}{16}, \tag{29} \]

where \( c = \bar{M} + \frac{\phi}{4} \). The inner minimization problem there refers to a target-based objective that aims to make the wealth gets close to the target \( c \).

4.2 Hedging mortality with LRD

Let \( \pi(t) = k_2 e^{-\int_0^t \mu(s) ds} \) and \( \Sigma(t) = \sigma_S(t)^T \sigma_S(t) \). To solve the optimal hedging problem, we introduce two additional probability measures:

\[ \frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = e^{-\int_0^t \xi(s)^T dW(s) - \frac{1}{2} \xi(s)^T \xi(s) ds}, \quad \frac{d\check{\mathbb{P}}}{d\mathbb{P}} = e^{-\int_0^t \zeta(s)^T dW(s) - \frac{1}{4} \zeta(s)^T \zeta(s) ds} \]

with \( \xi(t) = (2\varphi(t), 2\vartheta(t))^T \) and \( \zeta(t) = (\varphi(t), \vartheta(t))^T \). By the Girsanov theorem, \( \check{W}_t \triangleq W_t + \int_0^t \xi(s) ds \) and \( \hat{W}_t \triangleq W_t + \int_0^t \zeta(s) ds \) are Wiener processes under \( \hat{\mathbb{P}} \) and \( \check{\mathbb{P}} \), respectively. Denote \( \hat{\mathbb{E}}[\cdot] \) and \( \check{\mathbb{E}}[\cdot] \) as expectations under \( \hat{\mathbb{P}} \) and \( \check{\mathbb{P}} \), respectively. By Theorem 1,

\[ \hat{\mathbb{E}}\left[ e^{-\int_0^t \mu_v dr} \bigg| \hat{\mathcal{H}}_t \right] = \exp(Y_T^2(T)), \]

where \( Y_T^2(T) \) is equivalent to the \( Y_T(T) \) in (10) with \( b(x) = b^0 - \varphi(x)\sigma_\mu - b^1 x \), \( \sigma(x) = \sigma_\mu \) and \( W \) replaced by \( \check{W} \); \( \hat{\mathbb{E}}[\mu_\nu][\hat{\mathcal{H}}_t] = \check{\mathbb{E}}[X_\nu][\check{\mathcal{H}}_t] \) is equivalent to \( \mathbb{E}[X_\nu][\check{\mathcal{H}}_t] \) defined in (13) with \( B = -b^1 \), \( b^0(\cdot) \) replaced by \( b^0 - \varphi(\cdot)\sigma_\mu \), and \( W \) replaced by \( \check{W} \). In addition, we have the following expressions.

\[ \hat{\mathbb{E}}\left[ e^{-2\int_0^\tau r(s) ds} \bigg| \hat{\mathcal{F}}_r \right] = \exp(\alpha_1(t, T_0) + \beta_1(t, T_0) r(t)), \tag{30} \]

\[ \check{\mathbb{E}}(t, s) = \check{\mathbb{E}}\left[ e^{-2\int_s^\tau r(s) ds} \bigg| \check{\mathcal{F}}_s \right] = \exp(\alpha_2(t, s) + \beta_2(t, s) r(t)), \tag{31} \]

where \( \alpha_1(t, T_0) \), \( \beta_2(t, T_0) \), \( \alpha_2(t, s) \) and \( \beta_2(t, s) \) solve the ODEs in Appendix A. The following theorem provides the optimal hedging strategy.

Theorem 2. Consider two stochastic processes

\[ P(t) = \frac{e^{-\int_0^t \varphi^2(s) + \vartheta^2(s) ds}}{\check{\mathbb{E}}\left[ e^{-2\int_0^\tau r(s) ds} \bigg| \check{\mathcal{F}}_r \right]}. \tag{32} \]
Proposition 4. \( Q(t) = -P(t)[Q_0(t) + c \mathcal{B}(t, T_0)] \), (33)

where

\[
Q_0(t) = \int_t^{T_0} \mathcal{B}(t, s) (k_1 E[z]E[t_0] \mu_r [\tilde{t}_r] + k_2 \mathcal{E} e^{-\int_0^r \mu_r dr} [\tilde{t}_r]) ds,
\]

\[
\mathcal{B}(t, s) = \mathcal{E} \left[ e^{-\int_0^r r(u) du} \right] \mathcal{F}_t \text{, } 0 \leq t \leq s.
\]

Once

\[
dP(t) = \mu_p(t) dt + \eta_1^T dW(t) \text{ and } dQ(t) = \mu_Q(t) dt + \eta_2^T dW(t)
\]

under \( \mathbb{P} \), the inner minimization problem in (29) has an optimal feedback control:

\[
u_* = -\Sigma(t)^{-1} \left( \mu + \frac{\sigma_S(t)^T \eta_1(t)}{P(t)} + \frac{Q(t) + \sigma_S(t)^T \eta_2(t)}{P(t)} \right).
\]

In addition, the optimal objective value is \( P(0)(M(0) + \frac{Q(0)}{P(0)})^2 + I(0) \), where

\[
I(t) = E \left[ \int_t^{T_0} \mu z^2 + \frac{(n_2 - Q \eta_1)^T}{P^2} \Sigma \left( \frac{n_2 - Q \eta_1}{P^2} \right) \right] (s) ds \bigg| \tilde{H}_t
\]

in which \( \Sigma = \text{id} - \sigma_S(t) \Sigma(t)^{-1} \sigma_S(t)^T \).

Proof. See Appendix B.

\( \Box \)

Proposition 3. Then diffusion coefficients in (34) are: \( \eta_1 = (0, \eta_{12})^T \) where \( \eta_{12} = -P(t) \beta_1(t, T_0) \sigma_r \) and \( \eta_2 = (\eta_{21}, \eta_{22})^T \) in which

\[
\eta_{21} = -P(t) \int_t^{T_0} \mathcal{B}(t, s) \left( k_1 E[z]E[B(s - t) \sigma_r + k_2 \mathcal{E} e^{-\int_0^r \mu_r dr} [\tilde{t}_r] \psi_2(s - t) \sigma_r \right. ds
\]

\[
\eta_{22} = -P(t) \left\{ \int_t^{T_0} \mathcal{B}(t, s) \left( k_1 E[z]E[t_0] \mu_r [\tilde{t}_r] + k_2 \mathcal{E} e^{-\int_0^r \mu_r dr} [\tilde{t}_r] \right) \beta_2(t, s) \sigma_r ds \right. + c \mathcal{B}(t, T_0) \beta_2(t, T_0) \sigma_r
\]

\[
+ P(t) Q_0(t) + c \mathcal{B}(t, T_0) \beta_1(t, T_0) \sigma_r \right\} + P(t) Q_0(t) + c \mathcal{B}(t, T_0) \beta_1(t, T_0) \sigma_r
\]

(37)

where \( \beta_1(t, T_0) \) is defined in (30), \( \beta_2(t, s) \) in (31), \( E_B \) in Theorem 1 with \( B = -b^1 \) and \( \psi_2 \in \mathcal{L}^2([0, s], \mathbb{C}) \) solves the Riccati equation \( \psi_2 = (-1 - \psi_2 b^1) \ast K \).

Proposition 4. The optimal hedging strategy \( u^*(t) = (u_1^*(t), u_2^*(t))^T \) to problem (28) is given by

\[
u_1^*(t) = -\frac{1}{\sigma(t)} \left\{ \left[ M(t) - Q_0(t) - \left( \tilde{M}^* + \frac{\phi}{4} \right) \mathcal{B}(t, T_0) \right] \varphi(t) + \frac{\eta_{21}(t)}{P(t)} \right\}
\]

\[
u_2^*(t) = -\frac{1}{\sigma(t)} \left\{ \left[ M(t) - Q_0(t) - \left( \tilde{M}^* + \frac{\phi}{4} \right) \mathcal{B}(t, T_0) \right] \varphi(t) + \frac{M(t) \eta_{12}(t) + \eta_{22}(t)}{P(t)} \right\}
\]

where

\[
\tilde{M}^* = \frac{\phi}{4} (1 - P(0) \mathcal{B}^2(0, T_0)) + P(0) \mathcal{B}(0, T_0) (M(0) - Q_0(0))
\]

\[
\left\{ \frac{P(0) \mathcal{B}^2(0, T_0)}{P(0) \mathcal{B}^2(0, T_0)} \right\}
\]

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The explicit optimal hedging strategy in Proposition 4 incorporates the LRD feature through $\eta$ which depends on the mortality rate path and the kernel $K$ as shown in Proposition 3. In addition, the Hurst parameter is contained in the kernel function $K$.

5 Impact of LRD: Numerical studies

In this section, we numerically examine the impact of long-range dependence on prices of insurance products and the hedging effectiveness. To do so, we contrast between the LRD mortality model against its Markovian counterpart. For the latter case, the Hurst parameter $H$ is set to $1/2$. As the LRD appears when $H > 1/2$, we examine the effect for $H$ falling into this range.

5.1 Survival probability

As the basic quantity, we begin with the survival probability. Under Volterra mortality model, we assume that process $X$ satisfy equation (15) which is a Volterra type of Vasicek model. The Vasicek model is a special case with $\alpha = 1$ or $H = 1/2$. We compare Vasicek and VV mortality model by using two different values of $H$ while other parameters are kept constant. It is empirically estimated by Yan et al. (2018) that the $H$ is around 0.83 using mortality data. Thus, we choose $\alpha$ as 1.33 for the VV mortality model. Table 2 summarizes remaining parameters used in this numerical study. The parameters chosen have similar magnitudes as those in Biffis (2005) for the case of Markovian model.

\[
\begin{array}{cccccccc}
\text{Projection} & \alpha & m(t) & \eta & \lambda & \theta & \sigma & t & X_0 \\
A & 1.33 & \text{SIM92} & 0.2 & 0.5 & 0.0009 & 0.01 & 40 & 0.001 \\
B & 1 & \text{SIM92} & 0.2 & 0.5 & 0.0009 & 0.01 & 40 & 0.001 \\
\end{array}
\]

Table 2: Parameter values for the mortality model

Remark 4. The SIM92 in Table 2 is a dataset coming from the Italian national statistical bureau (ISTAT) which reports Italian population life tables. SIM92 is usually employed to price assurance. Such a setting for $m(t)$ has been adopted in Biffis (2005). Specifically, the $m(t)$ is calibrated to fit the SIM92 table so that the functional form of $m(t)$ is not explicitly shown here.

Although parameter values are assigned in this numerical experiment, we stress that, in reality, the parameters can be calibrated to observed prices of actuarial products using the set of the closed-form pricing formulas derived in this paper. In addition, the parameter $\theta$ in (14) or $b^0$ in Definition 1 can be set as a bounded measurable function of time $t$ rather than a constant in our example.
In Table 2, the symbol $t$ stands for the age group. For example, when we set $t = 40$, it corresponds to a group of survival population at the age of 40. In Figures 1(a) and 2(a), we simulate two different sample paths of $X$ for the group of individuals over the time interval $[0, t]$. Under VV mortality model, the historical sample paths of $X$ affect the estimated survival probability whereas the Vasicek model is not due to the Markovian nature. Given the parameters in Table 2 and (14), we directly calculate survival probabilities from the two models. By (14) and Theorem 1,

\[
P(\tau > T | \mathcal{F}_t) = e^{- \int_t^T m(s) ds} \exp \left( -n \int_t^T \mathbb{E}[X_s | \mathcal{F}_t] ds + \frac{1}{2} \int_t^T \psi(T - s) \alpha(\mathbb{E}[X_s | \mathcal{F}_t]) \psi(T - s)^\top ds \right),
\]

for $T > t$. Under the Vasicek mortality model, the survival probability only depends on $X_t$ ($t = 40$) as $\mathbb{E}[\mu_s | \mathcal{F}_t] = \mu_t$. Whereas under the VV mortality model, the expression of $\mathbb{E}[\mu_s | \mathcal{F}_t]$ given in (13) depends on the whole historical path of $X$. Based on the simulated sample paths, we calculate the survival probabilities on the interval $T \in [t, x^*]$, where we set the maximum age at $x^* = 109$.

Figures 1(b) and 2(b) show the survival probabilities corresponding to the historical records in Figure 1(a) and 2(a), respectively. The solid line is the survival probability curve with LRD while the dash line is that of the Markovian model. Depending on the historical record, the LRD survival probability can be higher or lower than the Markovian survival probability. This indicates that the historical sample path has impact on the survival probability when LRD is present. The effect is more pronounced for the middle age group. It is reasonable because the young age group has a shorter historical record while old age group may have to obey the human age limit. This kind of middle-age effect may result in significant effect for insurance pricing. We further examine it with a concrete insurance product.
Figure 1: A sample historical path of $X$ making the survival probability with LRD higher than its Markovian counterpart.

Figure 2: A sample of historical path of $X$ making the survival probability with LRD lower than its Markovian counterpart.
5.2 Impact on annuity

To examine the effect of LRD on annuity prices, we compare the prices calculated from the two models. We are interested in annuity because it is a popular insurance and pension product around the globe.

The numerical experiment is constructed as follows. Consider a 20-year deferred annuity and its payoff is a unit amount each year. To simplify matter, we assume that $\mathbb{Q} = \mathbb{P}$ in this part so that no additional effort is required on identifying the pricing measure. Simulation and calculation are made with parameters in Table 2. In addition, we specify the short interest rate $r_t = Z_t$ as follows.

$$dZ_t = (\tilde{b}^0 - \tilde{b}^1 Z_t)dt + \sigma_r dW'$$

where $\tilde{b}^0 = 0.01$, $\tilde{b}^1 = 0.5$, $\sigma_r = 0.3$, and $Z(40) = 0.01$. Then we use equation (18) directly to calculate the price of the annuity and $t' = 20$.

![Figure 3: Examples of historical paths for X and histogram of percentage difference in annuity prices between the two models](image_url)

To demonstrate the LRD effect, we generate 15,000 sample paths of $X$ over the time interval $[0, t]$. In Figure 3(a), we illustrate that the last two sample paths meet at time $t$. The classic Markovian model ignores how they come to this point and assigns the same price to the two scenarios as explained in (40). However, our LRD mortality model takes the historical record into account and assigns two different prices as shown in (18) and Theorem 1. The problem is how much difference between these two models. Clearly, the difference is not a single number as there are uncountably many ways to reach the same
point. Therefore, we examine the distribution of the price difference for different historical paths.

To do so, Figure 3(b) plots a histogram of percentage difference of the annuity prices between the LRD and Markovian models. First of all, the mean of the distribution nears zero, implying that the Markovian mortality model offers an appropriate estimate to the averaged price even under the LRD feature. However, the dispersion of the histogram is still obvious. The price difference between the two models can reach 4% just for a linear annuity product while this 4% difference seems not negligible in practice. The discrepancy may be amplified for products with leveraging effects such as those with optionality. Even for this annuity product, we can see the volatility could be higher compared with the Markovian model due to the incorrect prediction of the mortality rate if the realized mortality has LRD feature.

To illustrate the influence of LRD on products with optionality, consider a European call option on a zero-coupon longevity bond $B_L(t,T)$ with strike $D$ and expiration time $T_1$, where $T$ is the fixed maturity of the bond whereas $T_1$ is the expiration date of the option so that $0 \leq t \leq T_1 < T$. Specifically, the call option payoff reads $V_0(B_L(t_1, T)) = \max(B_L(t_1, T) - D, 0)$. We want to focus on the effect of LRD mortality rate and, therefore, assume a constant interest rate $r$ and $m(\cdot) = 0$. By (15) and (25), we have

\[ dB_L(t,T) = B_L(t,T) \left[ r dt + \psi(T-t) \sigma dW^Q_t \right], \tag{41} \]

under the pricing measure, where $\psi$ solves $\psi = (-\eta - \lambda \psi) \ast K$. As (25) is the dynamic of $B_L(t,T)$ under $P$, the corresponding $Q$ dynamics in (41) is one which the term $\nu_L$ in (25) is absorbed into the $P$-Brownian motion to form a $Q$-Brownian motion. Hence, the call value function $V_0(B_L, t)$ resembles the Black-Scholes formula. Specifically,

\[
\begin{align*}
V_0(B_L, t) &= \Phi(d_1)B_L(t,T) - \Phi(d_2)De^{-r(T_1-t)} \\
\Phi(d_1) &= \frac{1}{\psi(T-t)\sigma\sqrt{T_1-t}} \left[ \ln \left( \frac{B_L(t,T)}{D} \right) + \left( r + \frac{1}{2} \psi^2(T-t)\sigma^2 \right)(T_1-t) \right], \\
d_1 &= \frac{1}{\psi(T-t)\sigma\sqrt{T_1-t}} \left[ \ln \left( \frac{B_L(t,T)}{D} \right) + \left( r + \frac{1}{2} \psi^2(T-t)\sigma^2 \right)(T_1-t) \right], \\
d_2 &= d_1 - \psi(T-t)\sigma\sqrt{T_1-t},
\end{align*}
\]

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution.

Let us make numerical comparison in terms of percentage difference in option price between the VV and Markovian models. Let $r = 0.01$, $T = 5$, $T_1 = 2$ and other parameters are set as those in Table 2. Assume $B_L(t,T) = 0.8 \triangleq D_0$, the benchmarking at-the-money (ATM) strike, at the option issuance time. Note that the historical path of the mortality rate is subsumed into the longevity bond price $B_L(t,T)$. By varying the strike $D$ from 0.8 (ATM) to 0.832 (4% in-the-money), option prices under the two models are shown in Figure 4(a) while the percentage difference in price in Figure 4(b). When the strike increases by 4%, the percentage difference in option price could reach 20% which is quite significant. We mention the 4% increase in strike because the price of annuity can reach 4% difference in price from the former analysis. When the strike is set to make the option ATM, the difference in the longevity bond price results in a 4% difference in setting.

Electronic copy available at: https://ssrn.com/abstract=3666984
the ATM strike. This example shows that optionality may further amplify the pricing difference.

![Diagram of mortality and interest rate paths](image)

Figure 4: Option prices and difference of the prices under two models

5.3 **LRD effect on longevity hedging**

We further examine the hedging with LRD. In this part, we still consider the fractional kernel in (23) so that $K(t) = t^{\alpha - 1}/\Gamma(\alpha)$. Again, we first simulate the a pair of sample paths of mortality and interest rates as shown in Figure 5. The model parameters used are $\mu(0) = 0.15, b^1 = 0.5, b^0 = 0.1, \sigma_{\mu} = 0.05, r(0) = 0.04, \tilde{b}^1 = 0.6, \tilde{b}^0 = 0.02, \sigma_r = 0.01, T_0 = 5, \alpha = 1.33, k_1 = 1, k_2 = 10, \mathbb{E}[z] = 2$.

![Diagram of mortality and interest rate paths](image)

Figure 5: A pair of sample paths of mortality rate and interest rate

We hedge with the following two models.
- Model 1: Above assumption with $K(t) = t^{\alpha - 1}/\Gamma(\alpha)$ (Volterra mortality model);
- Model 2: Above assumption with $K(t) = 1$ (Markovian mortality model).

Our objective is to hedge with $\phi = 3000$ over a horizon of 5 years using a zero-coupon longevity bond and a zero-coupon bond with a maturity time $T = 15$, and the initial value of wealth process is set to be 2000. The optimal hedging strategies are calculated according to (38) and (39). The longevity bond price and bond price are calculated by assuming constant market price of risks $\varphi = 0.1$ and $\vartheta = 0.1$.

The optimal hedging strategies and corresponding wealth processes under two models are plotted in Figures 6 and 7, respectively. Once the mortality rate has the LRD feature, our hedging strategy significantly outperforms its Markovian counterpart and the unhedged position. Numerically, the objective function value for Model 1 is -3622443 which is less than -3620889, the value for Model 2. As our goal is to minimize the MV objective, the smaller the number the better performance in terms of the objective function. If one concerns about the risk level or the variance here, we report that the variance of terminal wealth is 66120 under Model 1 and 66317 under Model 2. The LRD hedging strategy prevails, too. We stress that it does not mean that the LRD hedging must be better in reality. Instead, we want to demonstrate the potential loss in hedging effectiveness with the Markovian model once the mortality rate has LRD feature.

Although we set $\alpha = 1.33$ (or $H = 0.83$) in this numerical experiment, the value of $\alpha$ can be calibrated or estimated in practice by using the pricing formulas we provide. Therefore, this study offers an option to choose between Volterra and Markovian mortality models when dealing with longevity hedging in reality. Our proposed model renders a practically flexible approach for the choice of $\alpha$.

![Figure 6: Optimal hedging strategy $u_1(t)$ and $u_2(t)$](https://ssrn.com/abstract=3666984)
6 Conclusion

In this paper, we propose a tractable continuous-time mortality rate model that incorporates the LRD feature. Using our model, we derive novel closed-form solutions to the survival probability and prices of several basic insurance products. In addition, our model also enables us to investigate optimal longevity hedging strategy via the BSDE framework. Therefore, the key advantage of our model is its tractability for pricing and risk management as well as capturing the LRD feature. Our numerical experiments show that LRD has significant effects for insurance pricing and hedging. The new longevity hedging strategy improves the hedging effectiveness when mortality rate observes LRD feature.

A Transformation of Markov affine processes

We now give the ODEs which the coefficients \( \tilde{\alpha} \) and \( \tilde{\beta} \) solve appearing in Section 2 and Section 4. A \( \mathbb{R}^k \)-valued affine diffusion \( Z \) is a \( \mathbb{F} \)-Markovian process specified as the strong solution to the following SDE:

\[
dZ_t = \tilde{b}(Z_t)dt + \tilde{\sigma}(Z_t)dW'_t,
\]

where \( W'_t \) is a \( \mathbb{F} \)-standard \( k \)-dimensional Brownian motion. We require covariance matrix \( \tilde{\alpha}(Z) = \tilde{\sigma}(Z)\tilde{\sigma}(Z)^\top \) and the drift \( \tilde{b}(Z) \) to have affine dependence on \( Z \) as in Definition 1. That is:

\[
\begin{align*}
\tilde{\alpha}(Z) &= \tilde{A}_0 + Z_1\tilde{A}_1 + \cdots + Z_k\tilde{A}_k, \\
\tilde{b}(Z) &= \tilde{b}_0 + Z_1\tilde{b}_1 + \cdots + Z_k\tilde{b}_k.
\end{align*}
\]
for some $k$-dimensional symmetric matrices $\tilde{A}^i$ and vectors $\tilde{b}^i$. For convenience, we set $\tilde{A}_1 = (\tilde{A}^1, \ldots, \tilde{A}^k)$ and $\tilde{b}_1 = (\tilde{b}^1, \ldots, \tilde{b}^k)$. As shown in Duffie et al. (2000), for any $c_1, c_2 \in \mathbb{C}^k$ and $c_3 \in \mathbb{C}$, given $T > t$ and affine function $\Lambda(t, x) = \lambda_0(t) + \lambda_1(t) \cdot Z$ ($\lambda_0$ and $\lambda_1$ are bounded continuous functions), under technical conditions we have:

$$
E[e^{-\int_t^T \Lambda(s, Z_s)ds}e^{c_1 \cdot Z_T + c_3}] = e^{\hat{\alpha}(t) + \hat{\beta}(t) \cdot Z_t}E[\alpha(t) + \beta(t) \cdot Z_t],
$$

where the functions $\hat{\alpha}() \equiv \hat{\alpha}()$, $T$ and $\hat{\beta}() \equiv \hat{\beta}()$, $T$ solve the following ODEs:

$$
\hat{\beta}(t) = \lambda_1(t) - \tilde{b}_1(t)^\top \hat{\beta}(t) - \frac{1}{2} \hat{\beta}(t)^\top \tilde{A}_1(t) \hat{\beta}(t),
$$

$$
\hat{\alpha}(t) = \lambda_0(t) - \tilde{b}^0(t) \cdot \hat{\beta}(t) - \frac{1}{2} \hat{\beta}(t)^\top \tilde{A}^0(t) \hat{\beta}(t),
$$

with boundary conditions $\hat{\alpha}(T) = 0$ and $\hat{\beta}(T) = c_1$; the functions $\hat{\alpha}() \equiv \hat{\alpha}()$, $T$ and $\hat{\beta}() \equiv \hat{\beta}()$, $c_1, c_2, c_3, T$ are the solutions of following ODEs:

$$
\hat{\beta}(t) = -\tilde{b}_1(t)^\top \hat{\beta}(t) - \hat{\beta}(t)^\top \tilde{A}_1(t) \hat{\beta}(t),
$$

$$
\hat{\alpha}(t) = -\tilde{b}^0(t) \cdot \hat{\beta}(t) - \hat{\beta}(t)^\top \tilde{A}^0(t) \hat{\beta}(t),
$$

with boundary conditions $\hat{\alpha}(T) = c_3$ and $\hat{\beta}(T) = c_2$.

### B Some Proofs

#### Proof of Theorem 1

Under our model, from (5),

$$
E[e^{-\int_t^T \mu_s ds} | \mathcal{F}_t] = E[e^{-\int_t^T m(s) + \eta X_s ds} | \mathcal{F}_t] = e^{-\int_t^T m(s) ds}E[e^{-\int_t^T \eta X_s ds} | \mathcal{F}_t].
$$

Since $X_t$ has the affine structure specified in Definition 1, by application of Lemma 4.2 and Theorem 4.3 provided in (Abi Jaber et al., 2019), we have:

$$
E[e^{-\int_t^T \eta X_s ds} | \mathcal{F}_t] = e^{\int_t^T \eta X_s ds}E[e^{-\int_t^T \eta X_s ds} | \mathcal{F}_t] = e^{\int_t^T \eta X_s ds \exp(Y_t(T))},
$$

where $Y_t(T)$ is the Markovian process defined in (10) or equivalently (12) in Theorem 1. Then, for $T > t \geq 0$, we have

$$
E[e^{-\int_t^T \mu_s ds} | \mathcal{F}_t] = e^{-\int_t^T m(s) ds}e^{\int_t^T \eta X_s ds \exp(Y_t(T))}.
$$

Notice that $-\int_t^T m(s) ds + \int_t^T \eta X_s ds = -\int_t^T m(s) ds + \int_t^0 \mu_s ds$. Hence,

$$
E[e^{-\int_t^T \mu_s ds} | \mathcal{F}_t] = e^{-\int_t^T m(s) ds}e^{\int_t^T \mu_s ds \exp(Y_t(T))} = g(t, T).
$$

By taking derivative to $g(t, T)$ with respect to $T$, we get:

$$
\frac{\partial g(t, T)}{\partial T} = E[e^{-\int_t^T \mu_s ds} \mu_T | \mathcal{F}_t], \ T > t.
$$

Then, by combining the equations (42) and (43), the result in (9) follows.
Proof of Theorem 2 and Proposition 3

For $P(t)$, it’s obvious that $P(t) > 0$, $P(T_0) = 1$ and
\[
P^{-1}(t) = e^{\int_{T_0}^t \varphi^2(s)ds} \mathbb{E}[e^{-\int_{T_0}^t \varphi(s)sds} | F_t] = e^{\int_{T_0}^t \varphi^2(s)ds} \exp(\alpha_1(t, T_0) + \beta_1(t, T_0)r(t)).
\]
Under our setting, $\nu(t)\Sigma(t)^{-1}\nu(t) = \varphi^2(t) + \varphi^2(t)$. Then, by applying Ito’s formula,
\[
dP^{-1}(t) = P^{-1}(t)(2r(t) - \varphi^2(t) - \varphi^2(t))dt - P^{-1}(t)\tilde{\eta}_1(t)^\top d\tilde{W}(t)
\]
where $\tilde{\eta}_1 = -\beta_1(t, T_0)\sigma_r = \eta_1(P(t)$ and $\eta_1(t)$ is defined in Proposition 3. Notice that $\xi(t) = 2\sigma_S\Sigma(t)^{-1}\nu(t)$ and $\sigma^+\tilde{\eta}_1 = 0$. Then by Ito’s lemma again, $P(t)$ satisfies:
\[
dP(t) = \left\{ -2r(t) + \nu(t)^\top \Sigma(t)^{-1}\nu(t) \right\} P(t) + 2\nu(t)^\top \Sigma(t)^{-1}\sigma_S(t)^\top \eta_1(t) + \eta_1(t)^\top \sigma_S(t)\Sigma(t)^{-1}\sigma_S(t)^\top \eta_1(t) - \frac{1}{P(t)} \right\} dt + \eta_1(t)^\top d\tilde{W}(t).
\]
For $Q(t)$, it’s obvious that $Q(T_0) = -c$ and $\frac{Q(t)}{P(t)} = -[Q_0(t) + c\tilde{B}(t, T_0)]$. By applying Ito’s lemma to $\tilde{E}[\mathbb{E}^{\frac{Q(t)}{P(t)}}\tilde{\mathcal{H}}_t]$ on time $t$, we have:
\[
d\left( \tilde{E}[\mathbb{E}^{\frac{Q(t)}{P(t)}}\tilde{\mathcal{H}}_t] \right) = E_B(s - t)\sigma_{B}d\tilde{W}_t,
\]
where $E_B$ is defined in Theorem 1 with $B = -b^1$. By applying Ito’s lemma to $\tilde{E}[e^{-\int_{s}^{t} \mu_r d\tau} | \tilde{\mathcal{H}}_t] = \exp(Y_2(T))$ on time $t$, we get
\[
d\left( \tilde{E}[e^{-\int_{s}^{t} \mu_r d\tau} | \tilde{\mathcal{H}}_t] \right) = \tilde{E}[e^{-\int_{s}^{t} \mu_r d\tau} | \tilde{\mathcal{H}}_t] \psi_2(s - t)\sigma_r d\tilde{W}_t
\]
with $\psi_2 \in \mathcal{C}^2([0, s], \mathbb{C})$ solving the Riccati equation $\psi_2 = (1 - \psi_2 b^1) * K$. From (31), $d\tilde{B}(t, s) = \tilde{B}(t, s) r(t) dt - \tilde{B}(t, s) \beta_2(t, s) \sigma_r d\tilde{W}_t$. Then, by applying Ito’s lemma to $\frac{Q(t)}{P(t)}$, we have:
\[
d\left[ \frac{Q(t)}{P(t)} \right] = \left[ \frac{Q(t)}{P(t)} r(t) + k_1 \mu(t) z + \pi(t) \right] dt + \left[ \tilde{\eta}_2(t)^\top - \frac{Q(t)}{P(t)} \tilde{\eta}_1(t)^\top \right] d\tilde{W}(t)
\]
where $\tilde{\eta}_2 = \eta_2/\sigma_r$ and $\eta_2$ is shown in Proposition 3. Notice that $\zeta(t) = \sigma_S\Sigma(t)^{-1}\nu(t)$ and $\sigma^+\tilde{\eta}_1 = 0$. Then, by Ito’s lemma again, $Q(t)$ satisfies:
\[
dQ(t) = \left\{ -r(t) + \nu(t)^\top \Sigma(t)^{-1} \left( \Sigma(t)^{-1} \nu(t) + \frac{\sigma_S(t)^\top \eta_1(t)}{P(t)} \right) \right\} Q(t) + P(t)(k_1 \mu(t) z + \pi(t)) + \eta_2(t)^\top \sigma_S(t)\Sigma(t)^{-1} \left( \Sigma(t)^{-1} \nu(t) + \frac{\sigma_S(t)^\top \eta_1(t)}{P(t)} \right) dt + \eta_2(t)^\top d\tilde{W}(t).
\]
Finally, we consider the process $P(t) \left( M(t) + \frac{Q(t)}{P(t)} \right)^2 + I(t)$. By Ito’s formula, we have:

$$
\begin{align*}
    d \left[ P(t) \left( M(t) + \frac{Q(t)}{P(t)} \right)^2 + I(t) \right] &= d[P(t)M^2(t) + 2d[Q(t)M(t)] + d[Q^2(t)P^{-1}(t)] + dI(t) \\
    &= P(t)(u(t) - u^*_c(t))^\top \sigma_S(t)^\top \sigma_S(t)(u(t) - u^*_c(t))dt + \{\cdots\}dW(t) + \{\cdots\}d\mathcal{K}(t) \\
    &= P(t)|\sigma_S(t)(u(t) - u^*_c(t))|^2dt + \{\cdots\}dW(t) + \{\cdots\}d\mathcal{K}(t),
\end{align*}
$$

where $u^*_c(t)$ is defined in (35) and $\mathcal{K}(t) = \bar{N}(t) - k_1 \int_0^t \mu(s)ds$ is a martingale with respect to the filtration $\mathcal{F}_t$. Then, there exists an increasing sequence of stopping times $\{\tau_i\}$ such that $\tau_i \uparrow T_0$ as $i \to \infty$ and

$$
\begin{align*}
    \mathbb{E} \left[ P(T_0 \wedge \tau_i) \left( M(T_0 \wedge \tau_i) + \frac{Q(T_0 \wedge \tau_i)}{P(T_0 \wedge \tau_i)} \right)^2 + I(T_0 \wedge \tau_i) \right] \\
    &= P(0)(Y(0) + \frac{Q(0)}{P(0)})^2 + I(0) + \mathbb{E} \left[ \int_0^{T_0 \wedge \tau_i} P(t)|\sigma_S(t)(u(t) - u^*_c(t))|^2dt \right].
\end{align*}
$$

From (32) and (33), we can see $P(t)$ and $Q(t)$ are bounded. From (36), $I(t)$ is also bounded. Since $\mathbb{E}[\sup_{t \leq T_0}|Y(t)|^2] < \infty$, according to the Dominance Covergence Theorem and Monotone Convergence Theorem as $i \to \infty$, we have:

$$
\begin{align*}
    \mathbb{E} \left[ P(T_0) \left( M(T_0) + \frac{Q(T_0)}{P(T_0)} \right)^2 + I(T_0) \right] \\
    &= P(0)(Y(0) + \frac{Q(0)}{P(0)})^2 + I(0) + \mathbb{E} \left[ \int_0^{T_0} P(t)|\sigma_S(t)(u(t) - u^*_c(t))|^2dt \right].
\end{align*}
$$

Thus, the objective function $\mathbb{E}[(M(T_0) - c)^2] = \mathbb{E} \left[ P(T_0) \left( M(T_0) + \frac{Q(T_0)}{P(T_0)} \right)^2 + I(T_0) \right]$ is minimized when $u(t) = u^*_c$. $P(0)(Y(0) + \frac{Q(0)}{P(0)})^2 + I(0)$ is the optimal objective value.

**Proof of Proposition 4**

By Theorem 2, the optimal objective value is given by $P(0)(M(0) + \frac{Q(0)}{P(0)})^2 + I(0)$ for any given $c$. Take $c = \bar{M} + \frac{\phi}{4}$ and substitute $Q(0) = -P(0)[Q_0(0) + c\mathcal{B}(0,T_0)]$, then the external minimization problem in (29) becomes

$$
\begin{align*}
    \min_{\bar{M} \in \mathbb{R}} P(0)(M(0) - (\bar{M} + \frac{\phi}{4})\mathcal{B}(0,T_0) - Q_0(0))^2 + I(0) - \frac{\phi}{2} \bar{M} - \frac{\phi^2}{16},
\end{align*}
$$

which is a quadratic function attaining its minimum at

$$
\bar{M}^* = \frac{\phi}{4}(1 - P(0)\mathcal{B}^2(0,T_0)) + P(0)\mathcal{B}(0,T_0)(M(0) - Q_0(0)).
$$

By substituting $c = \bar{M}^* + \frac{\phi}{4}$, the result follows.
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