On the uniqueness of a solution to a stationary convection-diffusion equation with a generalized divergence-free drift

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Abstract

Let $A$ be a skew-symmetric matrix in $L^2(\Omega)$, $\Omega$ — a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. The Dirichlet problem $-\text{div} (\nabla u + A \nabla u) = f$, $u \in H^1_0(\Omega)$, $f \in W^{-1,2}(\Omega)$ has at least one solution obtained by approximating $A$ and passing to the limit. In 2004 V.V. Zhikov constructed an example of nonuniqueness. In the same paper he proved the uniqueness of solutions if the $L^p(\Omega)$ norms of $A$ are $o(p)$ as $p$ goes to infinity. We prove the uniqueness of solutions if $\exp(\gamma |A|) \in L^1(\Omega)$ for some $\gamma > 0$, which generalizes Zhikov’s theorem.

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Dedicated to the memory of Academician V.I. Smirnov, One of the Founding Fathers of MathPhys in Russia

1 Introduction

Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, $f$ an element of $W^{-1,2}(\Omega)$ and $A$ a skew-symmetric matrix from $L^2(\Omega)$. In this paper we are concerned with the question of uniqueness of solutions to the Dirichlet problem

$$Lu = -\text{div} (\nabla u + A \nabla u) = f, \quad u \in W^{1,2}_0(\Omega).$$

By a solution we mean a function $u \in W^{1,2}_0(\Omega)$ such that the integral identity

$$\int_{\Omega} (\nabla u + A \nabla u) \nabla \varphi \, dx = (f, \varphi)$$

holds for any $\varphi \in C^\infty_0(\Omega)$.

Let us elucidate the term “generalized drift” in the paper title. Formally,

$$(A_{ij}u_{x_i})_{x_j} = A_{ij}u_{x_i,x_j} + u_{x_j}A_{ij,x_i} = (uA_{ij,x_i})_{x_j} - uA_{ij,x_i,x_j} = (uA_{ij,x_i})_{x_j}.$$

Here and below we use the Einstein convention of summation over repeated indices. For scalar and vector functional spaces we use the same notation, i.e. for $A : \Omega \to \mathbb{R}^{n \times n}$

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we write $A \in L^2(\Omega)$ instead of $A \in L^2(\Omega)^{n \times n}$, for a vector field $a : \Omega \to \mathbb{R}^n$ we write $a \in L^2(\Omega)$ instead of $a \in L^2(\Omega)^n$ etc.

More rigorously, if $A \in W^{1,2}(\Omega)$ and $\varphi \in C^\infty(\Omega)$,

$$
\langle -\text{div} (A \nabla u), \varphi \rangle = \int_{\Omega} A_{ij} u_{x_j} \varphi_{x_i} \, dx = -\int_{\Omega} A_{ij,x_j} u \varphi_{x_i} \, dx - \int_{\Omega} u A_{ij} \varphi_{x_i,x_j} \, dx
$$

$$
= \int_{\Omega} A_{ji,x_j} u \varphi_{x_i} \, dx = \int_{\Omega} (u \text{div } A) \nabla \varphi \, dx = \langle -\text{div} (u \text{ div } A), \varphi \rangle,
$$

where $(\text{div } A)_i = A_{ji,x_j}$. Since $A$ is skew-symmetric, $\text{div } A = A_{ji,x_j} = 0$. Thus, for a skew-symmetric $A \in W^{1,2}_0(\Omega)$ the Dirichlet problem (1) can be written in the form

$$
-\text{div} (\nabla u + a \nabla u) = f, \quad u \in W^{1,2}_0(\Omega), \quad f \in W^{-1,2}(\Omega),
$$

(3)

with the solenoidal vector field $a = \text{div } A (a_i = A_{ji,x_j})$. For nonsmooth $A$ one can say [1], [2] that (1) describes “diffusion in a turbulent flow” (in our case, stationary) since the flow velocity $a = \text{div } A$ exists only in the sense of distributions. A similar class of equations in “generalized divergence form” was studied in [3].

On the other hand, given a (smooth) solenoidal vector field $a$ we can construct (at least, locally) a skew-symmetric matrix $A$ such that $a = \text{div } A$. Indeed, solenoidal $a$ corresponds to the closed $(n-1)$ form $\omega = * (a_i dx^i)$ ($*$ — the Hodge star operator).

By the Poincaré lemma (for instance, [2]) it is also exact, $\omega = d\alpha$ for $(n-2)$ form $\alpha$, provided that $\Omega$ is star-shaped (or contractible to a point, or diffeomorphic to a ball). The coefficients of $\alpha$ give the coefficients of $A$. In the language of differential forms, the passage between (1) and (3) is equivalent to the relation $\int_{\partial D} u d\alpha = (-1)^{n-1} \int_{\partial D} \alpha \wedge du$, $D$ a subdomain of $\Omega$, which follows from $d(ud\alpha + (-1)^n \alpha \wedge du) = 0$.

The form $\alpha$ can be additionally normed by $d\alpha = 0$ ($\delta$ — codifferential), and sought in the form $\alpha = \delta \beta$, which eventually leads to the problem $(d\delta + \delta d) \beta = \omega$ with suitable boundary conditions. For the problem $a = \text{div } A$ the condition $d\alpha = 0$ is equivalent to $A_{ij,x_k} + A_{jk,x_i} + A_{ki,x_j} = 0$, which is necessary for the representation of $A$ in the form of the rotor of a vector field $V$, i.e. $A_{ij} = \text{curl}_{ij} V = V_{i,x_j} - V_{j,x_i}$. More on the Hodge decomposition for differential forms can be found in the famous Morrey’s monography [3, Chapter 7]. A rather complete theory of differential forms on Lipschitz domain was constructed in [6] in the framework of Besov spaces.

In dimension 2 this reduces to

$$
A = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}, \quad \text{div } A = (-\alpha_y, \alpha_x) = (a_1, a_2).
$$

Since $a$ is solenoidal the vector field $V = (a_2, -a_1)$ is potential. So one needs to find a function $\alpha$ with the given gradient $V$. In other words $a = \nabla^\perp \alpha$.

In dimension 3, any skew-symmetric matrix can be represented as $A x = a \times x$, and the problem of finding $A$ such that $a = \text{div } A$ reads as $\text{rot } w = a$, which is also easy to see from

$$
\text{div } (w \times \nabla u) = \nabla u \cdot \text{rot } w - w \cdot \text{rot } \nabla u = \text{div } (u \text{ rot } w).
$$

(4)

The problem of finding a vector field with prescribed rotor (and divergence) is a classical problem of vector calculus. For $\Omega = \{ |x| < R \}$ one of solutions obtained by the Poincaré lemma is $w(x) = \int_0^1 a(tx) \times tx \, dt$. 

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If $\Omega = \mathbb{R}^n$ and the solenoidal vector field $a$ is vanishing at infinity, a solution to $\text{div} A = a$ can be obtained as the curl of the newtonian potential of $a$:

$$A_{ij} = V_{i,x_j} - V_{j,x_i}, \quad V_i(x) = (n(n-2)\omega_n)^{-1} \int_{\mathbb{R}^n} a_i(y) |x-y|^{2-n} \, dy, \quad n \geq 3, \quad (5)$$

where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$, with the obvious modification for $n = 2$. If $\Omega$ is a bounded domain and the normal component of $a$ on the boundary of $\Omega$ is equal to zero, then a solution to $\text{div} A = a$ is given by the same formula (5), where $a$ is extended by zero outside $\Omega$ (this extension is also solenoidal).

In dimension 3 formula (5) represents the standard vector calculus solution to $\text{rot} w = a$ defined as $w = \text{rot} (4\pi)^{-1} \int_{\mathbb{R}^n} a(y)|x-y|^{-1} \, dy$, which follows from representing $w = \text{rot} v$ and using the vector calculus identity $\text{rot} \text{rot} v = \nabla \text{div} v - \Delta v$. Such representation is of course only possible under the condition $\text{div} w = 0$, which is equivalent to requiring $\delta \alpha = 0$ above.

If the normal component of $a$ on the boundary is not equal to zero, one can continue $a$ to a sufficiently large ball $B$ which contains $\Omega$ by solving the auxiliary Neumann problem $-\Delta u = 0$ in $B \setminus \Omega$, $\partial u/\partial n = a \cdot n$ on $\partial \Omega$, $\partial u/\partial n = 0$ on $\partial B$ ($n$ — the exterior unit normal to $B \setminus \Omega$). Then one sets $a = \nabla u$ in $B \setminus \Omega$, $a = 0$ in $\mathbb{R}^n \setminus B$, and a solution to $\text{div} A = a$ is given by (3). This construction assumes that either $\Omega$ does not have holes, or the flow of $a$ across the boundary of each hole is zero.

If $\Omega$ has holes, the representation $a = \text{div} A$ is obviously not always possible, but by the Hodge (Weyl in 3D) theorem there exists a harmonic (irrotational solenoidal) vector field $b$ such that $a = b + \text{div} A$. For instance, one can take $b = \sum_j c_j \nabla \Gamma(x - x_j)$ where $x_j$ is a point inside the $j$-th hole, $\Gamma$ is the fundamental solution of the Laplace, and the constants $c_j$ are chosen to balance the flux of $a$ across the boundary of the corresponding hole. In detail this construction is discussed in [15].

Another way is to directly solve the problem $-\Delta V = a$ in $\Omega$, $V \times n = 0$ and $\text{div} V = 0$ on $\partial \Omega$, and find $A = \text{curl} V$. Regarding the equation $\text{rot} u = f$ and corresponding boundary value problems see [7] (classical potential theory), [8] (modern potential theory) and recent papers [9, 10] (Galerkin’s method). For the closely related problem of finding a solenoidal vector field with prescribed boundary value (or a vector field with given divergence) we refer the reader to [11, 12, 13, 14].

2 Approximation solutions

It is easy to prove that (1) has at least one solution. Indeed, take a sequence $A_N$ of bounded skew-symmetric matrices converging to $A$ in $L^2(\Omega)$. Let $u_N$ be solutions to the corresponding problems

$$-\text{div} (\nabla u_N + A_N \nabla u) = f, \quad u_N \in W^{1,2}_0(\Omega),$$

i.e.

$$\int_{\Omega} (\nabla u_N + A_N \nabla u_N) \nabla \varphi \, dx = (f, \varphi) \quad \text{for all} \quad \varphi \in C_0^\infty(\Omega).$$
By the Lax-Milgram lemma such solutions exist and are uniquely defined. Using the test-function $u_N$ in the corresponding integral identity, we have
\[ \int_{\Omega} |\nabla u_N|^2 \, dx = (f, u_N), \tag{6} \]
wherefrom
\[ \int_{\Omega} |\nabla u_N|^2 \, dx \leq \int_{\Omega} f^2 \, dx. \]
Extracting from $u_N$ a weakly convergent in $W^{1,2}_0(\Omega)$ subsequence and passing to the limit in the integral identity
\[ \int_{\Omega} (\nabla u_N + A_N \nabla u_N) \nabla \varphi \, dx = (f, u_N), \]
we obtain a solution $u$ to (1). Passing to the limit in (6) we see that this solution satisfies the energy inequality
\[ \int_{\Omega} |\nabla u|^2 \, dx \leq (f, u). \tag{7} \]
Following Zhikov \[16\] we call a solution constructed by this procedure an approximation solution. In the same paper V.V. Zhikov constructed an example of nonapproximation solutions, which satisfy the “unnatural” energy inequality
\[ \int_{\Omega} |\nabla u|^2 \, dx > (f, u). \tag{8} \]
Denote
\[ [u, \varphi] = \int_{\Omega} A \nabla u \nabla \varphi, \]
so that (2) can be rewritten as
\[ \int_{\Omega} \nabla u \nabla \varphi \, dx + [u, \varphi] = (f, \varphi). \tag{9} \]
It is clear that
\[ \|[u, \varphi]\| \leq C \|\nabla \varphi\|_{L^2(\Omega)} \quad \text{for all} \quad \varphi \in C_0^\infty(\Omega), \tag{10} \]
and $[u, \varphi]$, initially defined for $\varphi \in C_0^\infty(\Omega)$, can be extended to a linear bounded functional on $W^{1,2}_0(\Omega)$. Accordingly, in (9) the set of admissible test functions can be extended to $W^{1,2}_0(\Omega)$. Substituting $u$ as a test function in (9) we obtain
\[ \int_{\Omega} |\nabla u|^2 \, dx + [u, u] = (f, u). \tag{11} \]
On the other hand, any $u \in W^{1,2}_0(\Omega)$ satisfying (10), is a solution to (1) with the right-hand side $f$ defined by (9). So, the set of functions $u \in W^{1,2}_0(\Omega)$ satisfying (10) is the set of all solutions to (1) when $f$ ranges over $W^{-1,2}(\Omega)$. For a given skew-symmetric matrix $A$ we denote this set by $D(A)$. When necessary to distinguish between different matrices, we add a subscript to $[\cdot, \cdot]$; for instance, $[u, v]_A$. 
The rest of this section is devoted to certain elementary observations. Inequality \((\ref{11})\) translates into \([u, u] \geq 0\) for approximation solutions. The idea of Zhikov was to find an example of \([u, u] < 0\). Since an approximation solution always exists this immediately implies nonuniqueness. On the other hand, if \([u, u] \geq 0\) for all \(u \in W_{0}^{1,2}(\Omega)\) then for any right-hand side \(f\) a solution is unique, and \((\ref{14})\) holds. Another easy observation is that \([u, u] = 0\) for all \(u \in W_{0}^{1,2}(\Omega)\) is equivalent to the uniqueness of solutions together with the energy identity

\[
\int_{\Omega} |\nabla u|^{2} \, dx = (f, u)
\]

for all \(f\).

Also note that if there exists \(u\) with \([u, u] > 0\), then for problem \((\ref{11})\) with \(A\) replaced by \(-A\) there exists a nonapproximation solution. Analogously, if for a given matrix \(A\) there exists a solution with \([u, u] < 0\) then for problem \((\ref{11})\) with \(A\) replaced by \(-A\) there exists a solution which satisfies the strict energy inequality

\[
\int_{\Omega} |\nabla u|^{2} \, dx < (f, u).
\]

If for some right-hand side \(f\) there exist multiple solutions, then there exists a nontrivial solution \(u_{0}\) corresponding to \(f = 0\). For \(u_{0}\) identity \((\ref{14})\) gives

\[
[u_{0}, u_{0}] = -\int_{\Omega} |\nabla u_{0}|^{2} \, dx < 0.
\]

Since for any solution \(u\) there holds

\[
[u + tu_{0}, u + tu_{0}] = [u, u] + t[u, u_{0}] + t[u, u_{0}] + t^{2}[u_{0}, u_{0}] < 0 \quad \text{for large} \quad t,
\]

then nonuniqueness for some \(f\) implies nonuniqueness for all right-hand sides \(f\).

The same observation also allows us to single out an “extremal” solution from \(L^{-1}f\). Indeed, consider \(I[f] = \sup\{[u, u], \, u \in L^{-1}f\}\). Since an approximation solution always exists, \(0 \leq I[f]\). For solutions, satisfying \([u, u] \geq 0, \|u\| \leq \|f\|\) and \([u, u] \leq \langle f, u \rangle \leq \|f\|^{2}\). It follows that \(I[f] \leq \|f\|^{2}\). Take a sequence \(u_{k} \in L^{-1}f\) such that \([u_{k}, u_{k}]\) monotonically increases and converges to \(I[f]\). Then one can easily verify that

\[
\frac{1}{2} \int_{\Omega} |\nabla (u_{k} - u_{m})|^{2} \, dx = 2 \left[ \frac{u_{k} + u_{m}}{2}, \frac{u_{k} + u_{m}}{2} \right] - [u_{k}, u_{k}] - [u_{m}, u_{m}] \rightarrow 0
\]

as \(k, m \rightarrow \infty\). Thus, \(u_{k} \rightarrow u\) strongly in \(W_{0}^{1,2}(\Omega), Lu = f\) and

\[
[u, u] = (f, u) - \int_{\Omega} |\nabla u|^{2} \, dx = \lim_{k \rightarrow \infty} (f, u_{k}) - \int_{\Omega} |\nabla u_{k}|^{2} \, dx = \lim_{k \rightarrow \infty} [u_{k}, u_{k}] = I[f].
\]

For any \(z \in L^{-1}0, t \in \mathbb{R}\) we have \(u + tz \in L^{-1}f\), so \([u + tz, u + tz] \leq [u, u]\). Hence \(u\) satisfies

\[
[u, z] + [z, u] = 0 \quad \text{for any} \quad z \in L^{-1}0.
\]

From \((\ref{13})\), any function from \(L^{-1}f\) satisfying the latter property maximizes \(I[f]\) and is uniquely defined. Denote the special solution of \(Lu = f\) which maximizes \(I[f]\) by \(\tilde{L}^{-1}f\).
It is obvious that $\tilde{L}^{-1}f + \tilde{L}^{-1}g \in L^{-1}(f + g)$ and satisfies (14). Therefore $\tilde{L}^{-1}f + \tilde{L}^{-1}g = \tilde{L}^{-1}(f + g)$. So, $\tilde{L}^{-1}f : W^{-1,2}(\Omega) \to W^{1,2}_0(\Omega)$ is a linear bounded operator, which is the right inverse for $L$.

For any skew-symmetric matrix $B \in L^\infty(\Omega)$ there holds

$$|[u, \varphi]| \leq C\|B\|_\infty \|\nabla u\|_{L^2(\Omega)} \|\nabla \varphi\|_{L^2(\Omega)},$$

which implies

$$D(B) = W^{1,2}_0(\Omega), \quad [u, u]_B = \int_\Omega B \nabla u \nabla u \, dx = 0.$$

Thus, addition of any skew-symmetric matrix $B \in L^\infty(\Omega)$ to $A$ does not change $D(A)$ and $[u, u]$: $D(A + B) = D(A)$, $[u, u]_{A + B} = [u, u]_A + [u, u]_B = [u, u]_A$.

In certain sense, the information on uniqueness/nonuniqueness is contained in the set of large values of $A$. In [10] Zhikov proved the following

Theorem (Zhikov). Let

$$\lim_{p \to \infty} p^{-1} \|A\|_{L^p(\Omega)} = 0.$$  \hspace{1cm} (15)

Then (1) has a unique solution.

The aim of this paper is to clarify and refine this result.

3 Around BMO and $H^1$

Recall that $BMO$ is the set of locally integrable on $\mathbb{R}^n$ functions such that

$$\|f\|_{BMO} = \sup_{Q} \frac{1}{|Q|} \int_Q |f - f_Q| \, dx < \infty, \quad f_Q = \frac{1}{|Q|} \int_Q f \, dx,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with faces parallel to coordinate hyperplanes (or, alternatively, over all balls).

It is well known that $A \in BMO$ guarantees $D(A) = W^{1,2}_0(\Omega)$ and $[u, u]_A = 0$. Indeed, for $u, v \in C_0^\infty(\Omega)$ write

$$\int_\Omega A_{ij} u_{x_j} v_{x_i} \, dx = \frac{1}{2} \int_\Omega A_{ij} (u_{x_j} v_{x_i} - u_{x_i} v_{x_j}) \, dx.$$  

The crucial fact is that $u_{x_j} v_{x_i} - u_{x_i} v_{x_j}$ belongs to the Hardy space $H^1(\mathbb{R}^n)$, and

$$\|u_{x_j} v_{x_i} - u_{x_i} v_{x_j}\|_{H^1(\mathbb{R}^n)} \leq C \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}.$$

This fact can be proved using the commutator theorem from [17]. Much easier proof was given later in [18]. There is a number of different equivalent definitions of $H^1(\mathbb{R}^n)$, the proof of [18] used the following one. Let $\Phi$ be a smooth compactly supported function with $\int \Phi \, dx = 1$. Denote

$$M_\Phi f(x) = \sup_{t > 0} |\Phi_t * f|(x), \quad \Phi_t(x) = t^{-n} \Phi \left( \frac{x}{t} \right).$$

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Then
\[ \mathcal{H}^1(\mathbb{R}^n) = \{ f \in L^1(\mathbb{R}^n) : M_{\Phi} f \in L^1(\mathbb{R}^n) \} \]

Since BMO is dual to \( \mathcal{H}^1(\mathbb{R}^n) \) \cite{19}, we arrive at
\[ \int_{\Omega} A_{ij} u_{x_j} v_{x_i} \, dx \leq C \| A \|_{BMO} \| \nabla u \|_{L^2(\Omega)} \| \nabla v \|_{L^2(\Omega)}, \quad C = C(n). \]

Thus, the skew-symmetric bilinear form \([ \cdot, \cdot ]_A\) defined on \( W^{1,2}_0(\Omega) \times W^{1,2}_0(\Omega) \) is continuous with respect to both arguments in the norm of \( W^{1,2}_0(\Omega) \) and can be extended to the form on \( W^{1,2}_0(\Omega) \times W^{1,2}_0(\Omega) \) satisfying
\[ |[u, v]_A| \leq C \| \nabla u \|_{L^2(\Omega)} \| \nabla v \|_{L^2(\Omega)}, \quad [u, v]_A = -[v, u]_A \]

for all \( u, v \in W^{1,2}_0(\Omega) \). Then the existence and uniqueness of a solution to (1) follows from the Lax-Milgram lemma.

For other useful properties of BMO and Hardy spaces we refer the reader to \cite{20} (see also the excellent expository article \cite{21}).

A decade ago Maz’ya and Verbitsy \cite{22} proved a reverse result. This result is formulated for a wide class of equations with lower-order terms. We cite here only the basic part which relates to (1). Let \( L^{1,2}(\mathbb{R}^n) \) be the closure of smooth finite functions with respect to the norm \( \| \nabla u \|_{L^2(\mathbb{R}^n)} \), and \( L^{-1,2}(\mathbb{R}^n) \) be its dual. The operator
\[-\text{div} (A \nabla u) : L^{1,2}(\mathbb{R}^n) \rightarrow L^{-1,2}(\mathbb{R}^n)\]
is bounded if and only if
\[ A^s = \frac{A + A^T}{2} \in L^\infty(\mathbb{R}^n), \quad \text{and} \]
\[ \text{div} A^c \in BMO^{-1}(\mathbb{R}^n)^n, \quad A^c = \frac{A - A^T}{2}. \]

Here \( BMO^{-1}(\mathbb{R}^n) \) denotes the set of distributions which can be represented as the divergence of a BMO vector field. So, there exists a matrix \( \Phi \) with BMO entries such that \( \text{div} A^c = \text{div} \Phi \). In the sense of generalized functions, for \( u, v \in C^\infty_0(\mathbb{R}^n) \) we have
\[ \langle -\text{div} (A \nabla u), v \rangle = \langle A^s \nabla u, \nabla v \rangle - \langle \text{div} A^c \nabla u, v \rangle \]
\[ = \langle A^s \nabla u, \nabla v \rangle - \langle \Phi \nabla u, v \rangle = -\langle \text{div} ((A^s + \Phi) \nabla u), v \rangle. \]

This means that on smooth finite functions the operator is identical to an analogous operator with symmetric part of the matrix bounded and skew-symmetric part from BMO. The skew-symmetric part \( \Phi \) can be found from \( \Phi = -\Delta^{-1} \text{curl} \text{div} A^c \). Here the divergence operator acts on \( a = a_{ij} \) as \( \text{div}_j a = \partial_{x_j} a_{ij} \), and the curl of \( f = \{ f_i \} \) is \( \text{curl}_i f = \partial_{x_j} f_i - \partial_{x_i} f_j \). In dimension 2 the matrix \( A^c \) itself belongs to BMO.

The functions from BMO are exponentially summable (the John-Nirenberg lemma \cite{23}), and satisfy
\[ \frac{1}{|Q|} \int_Q |f - f_Q|^p \, dx \leq (C p \| f \|_{BMO})^p, \quad C = C(n) \] (16)
for any \( f \in BMO \) and cube \( Q \subset \mathbb{R}^n \). Thus, for \( A \in BMO \) the limit in (15) is always finite, but need not be zero, as can be demonstrated by the example of \( \log |x| \).

For \( A \in BMO \) Zhikov proved the uniqueness of approximation solutions without using the BMO–\( H^1 \) duality. In this case, it is sufficient to prove uniqueness for solutions corresponding to the set of bounded right-hand sides, which is dense in \( W^{-1,2}(\Omega) \) (see [16] for details). If \( A \in BMO \cap L^\infty(\Omega) \), one can obtain the Meyers type estimate

\[
\| \nabla u \|_{L^q(\Omega)} \leq C \| f \|_{L^\infty(\Omega)}
\]

for some \( q > 2 \) and \( C \) which depend only on \( \| A \|_{BMO} \) and \( \Omega \). Since BMO functions are summable to any power, \( A \nabla u \nabla u \in L^1(\Omega) \). By Hölder’s inequality

\[
|[u, \varphi]_A| \leq C \| \nabla u \|_{L^q(\Omega)} \| A \|_{L^r(\Omega)} \| \nabla \varphi \|_{L^2(\Omega)}, \quad r^{-1} = 2^{-1} - q^{-1},
\]

for \( \varphi \in C^\infty_0(\Omega) \). Approximating \( u \) by such \( \varphi \) we arrive at

\[
[u, u]_A = \int_\Omega A \nabla u \nabla u = 0,
\]

which implies uniqueness for approximation solutions corresponding to bounded right-hand sides.

There is a variety results on equations of type (1) with \( A \in BMO \) (or equations of type (3) with divergence-free \( a \in BMO^{-1} \)). See, for instance, the survey article [24] on the magnetogeostrophic equation and [25, 26] for results on regularity and qualitative theory of solutions.

### 4 Main result

Now we are ready to state the main result of this paper.

**Theorem 4.1.** Let the matrix \( A \) satisfy the condition

\[
\lim_{p \to \infty} p^{-1} \| A \|_{L^p(\Omega)} < \infty.
\]

Then (1) has a unique solution.

By the John-Nirenberg estimate (16), matrices with BMO elements satisfy (17). It is easy to see that (17) is equivalent to the exponential summability of \( A \):

\[
\int_\Omega \exp(\gamma |A|) \, dx = \sum_{p=0}^\infty \frac{\gamma^p}{p!} \int_\Omega |A|^p \, dx,
\]

and by the Stirling formula

\[
\frac{\gamma^p}{p!} \int_\Omega |A|^p \, dx \sim \frac{1}{\sqrt{2\pi p}} (Le\gamma)^p, \quad L = \lim_{p \to \infty} p^{-1} \| A \|_{L^p(\Omega)}.
\]

The series on the right-hand side of (18) converge if \( \gamma < (Le)^{-1} \).
Let \( M \exp(\gamma|A|) \) be the Hardy-Littlewood maximal function of \( \exp(\gamma|A|) \in L^1(\Omega) \), which is continued by zero outside \( \Omega \). Clearly,
\[
|A| \leq \frac{1}{\gamma} \log M \exp(\gamma|A|).
\]

By the result of Coifman and Rochberg [27], the right-hand side of the last expression is in \( BMO \) with the \( BMO \) “norm” bounded by \( \gamma^{-1}C(n) \). So, (17) is equivalent to \( |A| \) having a \( BMO \) majorant.

Let us note that the condition of exponential summability naturally arises in the theory of quasiharmonic vector fields with unbounded distortion [28].

It is easy to give an example of function satisfying (17) but not in \( BMO \). It follows from the definition of \( BMO \) that for two touching cubes of the same size there holds
\[
|f_{Q_1} - f_{Q_2}| \leq 2^{n+1}||f||_{BMO}.
\]

Let \( n = 2 \), \( x = (x_1, x_2) \). Take \( f(x) = \log |x| \) if \( x_1x_2 > 0 \) and \( f = 0 \) otherwise. Clearly, for such function (19) is not satisfied.

The condition of theorem (4.1) is sufficient for the uniqueness but far from necessary. It is worth to note that the addition of a skew-symmetric matrix with zero divergence to matrix \( A \) does not change the equation. Let \( C \in L^2(\Omega) \) be a skew-symmetric matrix with \( \text{div} \, C = 0 \), \( u \in W^{1,2}_0(\Omega) \) and \( \varphi \in C^\infty_0(\Omega) \). We have
\[
\langle -\text{div} (C\nabla u), \varphi \rangle = \int_{\Omega} C_{ij}u_{x_j}\varphi_{x_i} \, dx = \int_{\Omega} C_{ij}(u\varphi)_{x_j} \, dx - \int_{\Omega} uC_{ij}\varphi_{x_i}x_j \, dx = 0.
\]

In dimension 2 this does not bring anything new since any skew-symmetric matrix \( 2 \times 2 \) with zero divergence is of the form
\[
\begin{pmatrix}
0 & c \\
-c & 0
\end{pmatrix}, \quad c = \text{const},
\]
and the addition of any bounded matrix to \( A \) does not affect (17). In dimension 3 the situation is more interesting. Write the skew-symmetric matrix \( A \) as
\[
A = \begin{pmatrix}
0 & -a_3 & a_2 \\
-a_3 & 0 & -a_1 \\
a_2 & a_1 & 0
\end{pmatrix}, \quad A\xi = a \times \xi, \quad a = (a_1, a_2, a_3).
\]

The condition of zero divergence leads to \( \nabla \times a = 0 \), which is satisfied by \( a = \nabla \varphi \). This can be also seen from (4). We can add any matrix of the form
\[
C(\varphi) = \begin{pmatrix}
0 & -\varphi_{x_3} & \varphi_{x_2} \\
\varphi_{x_3} & 0 & -\varphi_{x_1} \\
-\varphi_{x_2} & \varphi_{x_1} & 0
\end{pmatrix}, \quad \varphi \in W^{1,2}(\Omega),
\]
to \( A \) and the equation basically stays the same. This is the reason why in [22] the result is given in terms of equivalence classes for \( n \geq 3 \).
5 Lipschitz truncations. The proof of the main result

In this section we prove Theorem 4.1. The proof relies on the technique of Lipschitz truncations. For the reader’s convenience we briefly remind the details. Let \( u \in W^{1,1}_0(\Omega) \) and \( g = M|\nabla u| \), where \( M \) stands for the standard Hardy-Littlewood maximal function:

\[
Mf(x) = \sup_{|B|} \frac{1}{|B|} \int_B |f| \, dx,
\]

where the supremum is taken over all balls \( B \subset \mathbb{R}^n \) which contain \( x \) (uncentered maximal function) or are centered at \( x \) (centered maximal function). Then for almost all \( x, y \in \Omega \) there holds

\[
|u(x) - u(y)| \leq Cn|x - y|(g(x) + g(y)), \quad |u(x)| \leq C\text{dist}(x, \partial\Omega)g(x).
\]

From these estimates it follows that on the set \( F(\lambda) = \{ g \leq \lambda \} \cup (\mathbb{R}^n \setminus \Omega) \) the function \( u \) is Lipschitz with the Lipschitz constant \( C\lambda \). Using the McShane theorem [29], we can extend \( u|_{F(\lambda)} \) to the whole space \( \mathbb{R}^n \) with the same Lipschitz constant \( C\lambda \). The resulting extension \( u_\lambda \) is called the Lipschitz truncation of \( u \). For further details on Lipschitz truncations and their applications we recommend [30].

Let \( u \) be a solution to (1) with \( f = 0 \), i.e.

\[
\int_\Omega (\nabla u + A\nabla u)\nabla \varphi \, dx = 0 \quad \text{for all } \varphi \in C^\infty_0(\Omega).
\]  

(20)

By approximation, one can take here Lipschitz \( \varphi \) vanishing on \( \partial\Omega \).

Take the test function \( \varphi = u_\lambda \) in (2). Using the skew-symmetry of \( A \) we obtain

\[
\int_{\{g \leq \lambda\}} |\nabla u|^2 \, dx = \int_{\{g > \lambda\}} (A + I)\nabla u\nabla u_\lambda \, dx \leq C\lambda \int_{\{g > \lambda\}} (|A| + 1)|\nabla u| \, dx.
\]

Next, multiply this inequality by \( \varepsilon\lambda^{-1 - \varepsilon} \), \( \varepsilon > 0 \), and integrate with respect to \( \lambda \) from 1 to \( \infty \). Fubini’s theorem yields

\[
\int_\Omega |\nabla u|^2(\max(1, g))^{-\varepsilon} \, dx \leq \frac{C\varepsilon}{1 - \varepsilon} \int_\Omega (|A| + 1)|\nabla u|(g^{1-\varepsilon} - 1)_+ \, dx.
\]

Using Hölder’s inequality and the boundedness of the maximal function in \( L^2 \), for small \( \varepsilon \) we obtain

\[
\int_\Omega |\nabla u|^2(\max(1, g))^{-\varepsilon} \, dx \\
\leq C\varepsilon \left( \int_\Omega |\nabla u|^2 \, dx \right)^{1/2} \left( \int_\Omega g^2 \, dx \right)^{(1-\varepsilon)/2} \left( \int_\Omega (|A| + 1)^{2/\varepsilon} \right)^{\varepsilon/2} \\
\leq C\varepsilon \left( \int_\Omega (|A| + 1)^{2/\varepsilon} \, dx \right)^{\varepsilon/2} \left( \int_\Omega |\nabla u|^2 \, dx \right)^{1-\varepsilon/2}.
\]

Passing to the limit as \( \varepsilon \to 0 \) we arrive at

\[
\int_\Omega |\nabla u|^2 \, dx \leq C \lim_{p \to \infty} \frac{\|A\|_{L^p(\Omega)}}{p} \int_\Omega |\nabla u|^2 \, dx, \quad C = C(n).
\]

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Therefore, \( u = 0 \) provided that the limit in (17) is small enough. The theorem is thus proved for \( A \) such that
\[
\lim_{p \to \infty} p^{-1} \| A \|_{L^p(\Omega)} < C(n)
\] (21)
for some positive constant \( C(n) \). Let \( A \) be a skew-symmetric matrix satisfying (17). Consider (1) with \( A \) replaced by \( \pm tA \) with \( t > 0 \) such that \( tA \) satisfies (21). Clearly, \( D(\pm tA) = D(A) \) and \( [u, u]_{\pm tA} = \pm t[u, u]_A \). For \( tA \) we have uniqueness, so \( [u, u]_{tA} \geq 0 \) for all \( u \in D(A) \). Similarly, \( [u, u]_{-tA} \geq 0 \) for all \( u \in D(A) \). Thus, \( [u, u] = 0 \) for all \( u \in D(A) \).

This immediately implies the uniqueness of solutions and validity of (12). The proof of Theorem 4.1 is complete.

6 Corollaries.

In this section we focus on problem (3) with “standard” solenoidal drift. A vector field \( a \in L^1(\Omega) \) is called solenoidal (or divergence free) if \( \text{div} \, a = 0 \) in the sense of distributions, i.e.
\[
\int_{\Omega} a \nabla \varphi \, dx = 0 \quad \text{for all} \quad \varphi \in C_0^\infty(\Omega).
\]
A solution to (3) is a function \( u \in W^{1,2}_0(\Omega) \) which satisfies
\[
\int_{\Omega} (\nabla u + au) \nabla \varphi \, dx = (f, \varphi) \quad \text{for all} \quad \varphi \in C_0^\infty(\Omega).
\] (22)

Using the same reasoning as above, one can show the existence of approximation solutions if the solenoidal vector field
\[
a \in L^{2n/(n+2)}(\Omega) \quad \text{for} \quad n \geq 3, \quad \text{and} \quad a \in L^{\log 1/2} L(\Omega) \quad \text{for} \quad n = 2.
\] (23)

In view of the embedding theorem this condition guarantees \( au \in L^1(\Omega) \). Denote
\[
[u, \varphi]_a = \int_{\Omega} au \nabla \varphi \, dx, \quad u \in W^{1,2}_0(\Omega), \quad \varphi \in C_0^\infty(\Omega),
\]

\[
D(a) = \{ u \in W^{1,2}_0(\Omega) : ||u, \varphi||_a \leq C \| \nabla \varphi \|_{L^2(\Omega)} \quad \text{for all} \quad \varphi \in C_0^\infty(\Omega) \}.
\]

As above, the set \( D(a) \) coincides with the set of all solutions to (3), for a solution \( u \) the form \([u, \varphi]_a \) is extended to \( \varphi \in W^{1,2}_0(\Omega) \), (22) can be written in the form (11), substituting \( u \) as a test-function one obtains (11). Further on, if there is no ambiguity, we drop the subscript \( a \) in the form \([u, \varphi]\).

The simplest condition (apart from the trivial \( a \in L^\infty(\Omega) \)) which guarantees the existence and uniqueness of a solution is \( a \in L^n(\Omega) \). For \( n > 2 \), by the Sobolev embedding theorem,
\[
\left| \int_{\Omega} au \nabla \varphi \, dx \right| \leq C \| a \|_{L^n(\Omega)} \| u \|_{L^2(\Omega)} \| \nabla \varphi \|_{L^2(\Omega)},
\] (24)
so the form \([u, \varphi]\) is continuous with respect to both arguments in the norm of \( W^{1,2}_0(\Omega) \times W^{1,2}_0(\Omega) \), and the existence and uniqueness of a solution follows from the Lax-Milgram lemma.
For $n = 2$, let $\Omega$ be a simply-connected domain. Since $a = (a_1, a_2)$ is solenoidal, we can find $Q \in W^{1,2}(\Omega)$ such that $a_1 = -Q_y, a_2 = Q_x$. Rewrite (4) in the form (1):

$$
\int_{\Omega} u(a_1 \varphi_x + a_2 \varphi_y) \, dx \, dy = \int_{\Omega} Q(u_y \varphi_x - u_x \varphi_y) \, dx \, dy
$$

for all $u \in W^{1,2}_0(\Omega)$ and $\varphi \in C_0^{\infty}(\Omega)$. Extend the function $Q$ to the whole plane so that $\|Q\|_{W^{1,2}(\mathbb{R}^n)} \leq \|Q\|_{W^{1,2}(\Omega)}$. By the Poincaré inequality, for any ball $B \subset \mathbb{R}^2$ there holds

$$
|B|^{-1} \int_{B} |Q - \bar{Q}|^2 \, dx \leq \int_{B} |a|^2 \, dx.
$$

Hence $Q \in BMO$ and $\|Q\|_{BMO} \leq C\|a\|_{L^2(\Omega)}$. Using the duality of $BMO$ and $\mathcal{H}^1$, we obtain (24) for $n = 2$.

A thorough study of regularity properties (boundedness, strong maximum principle, continuity, Harnack’s inequality) of solutions of second-order linear elliptic and parabolic equations with “rough” divergence free drifts from $L^n$ and Morrey spaces generalizing $L^n$ was done by Nazarov and Ural’tseva in [31]. Interesting examples are due to Filonov [32].

It is not hard to prove [2] that $a \in L^2(\Omega)$ guarantees the uniqueness of solutions and validity of the energy identity (12). Indeed, approximating $u$ by smooth functions one can prove that for $a \in L^2(\Omega)$ there holds

$$
\int_{\Omega} a u \nabla \varphi \, dx = - \int_{\Omega} a \varphi \nabla u \, dx, \quad u \in W^{1,2}_0(\Omega), \; \varphi \in C_0^{\infty}(\Omega),
$$

so (22) acquires the form

$$
\int_{\Omega} \nabla u \nabla \varphi \, dx = \int_{\Omega} a \varphi \nabla u \, dx + (f, \varphi) \quad \text{for all} \quad \varphi \in C_0^{\infty}(\Omega).
$$

Approximating $T_k(u) = \max(\min(u, k), -k)$, $k > 0$, by bounded smooth functions, we can set $\varphi = T_k(u)$ in (25), which gives

$$
\int_{\Omega} |\nabla T_k(u)|^2 \, dx - (f, T_k(u)) = \int_{\Omega} a T_k(u) \nabla u \, dx = \int_{\Omega} a \nabla \left( \int_{0}^{u} T_k(s) \, ds \right) \, dx = 0.
$$

Sending $k$ to infinity, and using $\nabla T_k(u) = \chi_{\{|u| < k\}} \nabla u$ a.e. in $\Omega$, we finally obtain energy identity (12) which implies the uniqueness.

It was in fact the convection-diffusion equation in form (3) for which Zhikov’s example in [10] was constructed. The example had the following form: $\Omega = B_1 \subset \mathbb{R}^3$, $a = a_0(x|x|^{-1})x|x|^{-3}$, $u = (1 - |x|^4)u_0(x|x|^{-4})$, where $\int_S a_0 \, d\sigma = \int_S u_0 a_0 \, d\sigma = 0$ and $\int_S a_0 u_0^2 \, d\sigma = -2$, $S = \{|x| = 1\}$. Using $u$, $a \nabla u \in L^\infty(\Omega)$ one can verify that $[u, u] = -\int_{\Omega} a u \nabla u \, dx = -1$. A similar example for the problem $-\Delta u + b \nabla u + \text{div}(bu) = f \in W^{-1,2}(\Omega)$, $u \in W_0^{1,2}(\Omega)$ was constructed in [33], where the question of existence and uniqueness was studied in the framework of renormalized solutions.

In [16] Zhikov proved the following result which improves the $L^2(\Omega)$ condition.

**Theorem (Zhikov).** If the solenoidal vector field $a$ satisfies $\lim_{\varepsilon \to 0} \varepsilon \|a\|_{L^{2-\varepsilon}(\Omega)} = 0$, then the approximation solution of (3) is unique for each $f \in W^{-1,2}(\Omega)$. 

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In dimension 2 this result can be strengthened.

**Theorem 6.1.** Let \( n = 2 \) and the solenoidal vector field \( a = (a_1, a_2) \) satisfy
\[
\lim_{\varepsilon \to 0} \sqrt{\varepsilon} \|a\|_{L^{2-\varepsilon}(\Omega)} = 0.
\] (26)

Then for any \( f \in W^{-1,2}(\Omega) \) equation (3) has a unique solution.

We shall obtain this theorem as a partial case of a more general statement.

Recall that the Morrey space \( M^p(\Omega), 1 \leq p \leq \infty \), is the set of all integrable functions such that
\[
\|f\|_{M^p(\Omega)} := \sup_{R>0} R^{-n(1-1/p)} \int_{\Omega \cap B_R} |f| \, dx < \infty,
\]
where the supremum is taken over all balls \( B_R \) of radius \( R \). It is well known that for \( f \in M^p(\Omega) \) the Riesz potential
\[
I_{\Omega}f(x) = \int_{\Omega} |x-y|^{-n} |f(y)| \, dy
\]
is exponentially summable and satisfies \[\text{[34, proof of Lemma 7.20]}\]
\[
\int_{\Omega} |I_{\Omega}f|^q \, dx \leq n(n-1)^{q-1} \omega_n q^q (\text{diam } \Omega)^n \|f\|_{M^q(\Omega)},
\] (27)
where positive constants \( c_1 \) and \( c_2 \) depend only on \( n, p \). We shall also use the following simple potential estimate \[\text{[34, Lemma 7.12]}\]. Let \( 1 \leq q \leq \infty \), \( 0 \leq \delta = p^{-1} - q^{-1} < \mu \), \( \mu = n^{-1} \). Then
\[
\|I_{\Omega}f\|_{L^q(\Omega)} \leq \left( \frac{1-\delta}{\mu-\delta} \right)^{1-\delta} \omega_n^{1-\mu} |\Omega|^\mu |f|_{L^p(\Omega)}.
\] (28)

**Theorem 6.2.** Let \( a \in M^n(\Omega) \) and satisfy (23). Then (3) has a unique solution. The same conclusion also holds if
\[
\lim_{\varepsilon \to 0} \varepsilon^{1/n} \|a\|_{L^{n-\varepsilon}(\Omega)} < \infty.
\] (29)

**Proof.** Here we use the same notation as in the proof of Theorem 4.1. Let \( u \) be a solution to (3) with \( f = 0 \) and \( u_\lambda \) be the Lipschitz truncation of \( u \). Using \( u_\lambda \) as a test function in (22), we obtain
\[
\int_{\{g \leq \lambda\}} \|
abla u\|^2 \, dx = - \int_{\Omega} au \nabla u_\lambda \, dx - \int_{\{g > \lambda\}} \nabla u \nabla u_\lambda \, dx \\
= \int_{\{g > \lambda\}} (a(u_\lambda - u) - \nabla u) \nabla u_\lambda \, dx \leq C\lambda \int_{\{g > \lambda\}} (|a| \cdot |u - u_\lambda| + |
abla u|) \, dx.
\] (30)

For almost all \( x \in \Omega \) by the Poincaré inequality \[\text{[34, Lemma 7.16]}\] there holds
\[
|u - u_\lambda|(x) \leq \frac{(\text{diam } \Omega)^n}{|\{g \leq \lambda\} \cap \Omega|} \int_{\{g > \lambda\}} \frac{\|
abla (u - u_\lambda)|}{|x-y|^{n-1}} \, dy.
\] (31)
Let $\lambda_0$ be such that $\{g \leq \lambda_0\} \cap \Omega > \delta (\text{diam } \Omega)^n$, $\delta$ a small positive number. Let $V(x) = \int_{\Omega} |x-y|^{1-n}|a(x)| \, dx$. From (30), (31), for $\lambda \geq \lambda_0$ we have

$$
\int_{\{g \leq \lambda\}} |\nabla u|^2 \, dx \leq C \lambda \int_{\{g > \lambda\}} |\nabla u| \, dx + C \lambda \int_{\{x \in \Omega : g(x) > \lambda\}} \int_{\{g(y) > \lambda\}} |a(x)| \cdot |x-y|^{1-n} \cdot (|\nabla u(y)| + \lambda) \, dy \, dx \\
+ \int_{\{g > \lambda\}} (|\nabla u| + \lambda) V \, dx.
$$

Multiply this relation by $\varepsilon^\lambda \cdot |\cdot|$, 0 < $\varepsilon$ < 1/2, and integrate with respect to $\lambda$ from $\lambda_0$ to $\infty$. Fubini’s theorem yields

$$
\int_{\Omega} |\nabla u|^2 (\max(\lambda_0, g))^{-\varepsilon} \, dx \leq C \varepsilon \int_{\Omega} (|\nabla u| g^{1-\varepsilon} + g^{2-\varepsilon}) V \, dx + C \varepsilon \int_{\Omega} g^{1-\varepsilon} |\nabla u| \, dx.
$$

By Hölder’s inequality,

$$
\int_{\Omega} |\nabla u|^2 (\lambda_0 + g)^{-\varepsilon} \, dx \leq C \varepsilon \left[ \left( \int_{\Omega} g^2 \, dx \right)^{\frac{2-\varepsilon}{2}} \left( \int_{\Omega} V^{2/\varepsilon} \, dx \right)^{\varepsilon/2} + \int_{\Omega} g^{1-\varepsilon} |\nabla u| \, dx \right]. \tag{32}
$$

If $a \in M^n(\Omega)$, from (27) we obtain

$$
\left( \int_{\Omega} V(x)^{2/\varepsilon} \, dx \right)^{\varepsilon/2} \leq C(\text{diam } \Omega)^{n \varepsilon/2} \|a\|_{M^n(\Omega)} / \varepsilon^{\varepsilon/2}.
$$

If $a$ satisfies (28), then using (28) with $p = 2n/(2 + \varepsilon)$ and $q = 2/\varepsilon$ we have

$$
\left( \int_{\Omega} V(x)^{2/\varepsilon} \, dx \right)^{\varepsilon/2} \leq C \varepsilon^{(1-n)/n} |\Omega|^{\varepsilon(n-1)/2n} \|a\|_{L^{1-n}(\Omega)}.
$$

Substituting this estimate into (32), using the boundedness of the Hardy-Littlewood maximal function in $L^2$ and sending $\varepsilon \to 0$, we arrive at

$$
\int_{\Omega} |\nabla u|^2 \, dx \leq CJ \int_{\Omega} g^2 \, dx \leq CJ \int_{\Omega} |\nabla u|^2 \, dx
$$

where $J$ is either $\|a\|_{M^n(\Omega)}$ or $\lim_{\varepsilon \to 0} \varepsilon^{1/2} \|a\|_{L^{1-n}(\Omega)}$. Hence, $\int_{\Omega} |\nabla u|^2 \, dx = 0$ provided that $J$ is sufficiently small. This assumption can be removed by the same argument as in the proof of Theorem 4.1.1.

Theorem 4.2 can be obtained as a corollary of Theorem 4.1 if we find a suitable representation of a solution to $\text{div } A = a$ in terms of integral potentials with kernels $K(x, y) = O(|x-y|^{-n})$. In this case applying (27) (or (28)) we would obtain (17) for $A$. In dimension 2 this is easy. Also this is simple provided that the normal component of $a$ on $\partial \Omega$ is zero, in which case a solution is given by (3). In the general case, this is also possible, but requires certain analytical work. Let $n = 3$ and $a$ be a smooth solenoidal vector field. Let $\Omega$ be star-shaped with respect to a ball $B \subset \Omega$. For $y \in B$ and $x \in \Omega$ the
function \( w(x, y) = \int_0^1 \nabla a(y + t(x - y)) \times t(x - y) \, dt \) satisfies \( \text{rot}_x w = a \). Let \( \psi \in C_c^\infty(B) \), \( \int_B \psi \, dx = 1 \). The function \( W(x) = \int_B \psi(y) w(x, y) \, dy \) solves \( \text{rot} W = a \). Interchanging the order of integration, we arrive at

\[
W(x) = \int_\Omega a(y) \times \frac{x - y}{|x - y|^n} \int_{|x-y|}^{+\infty} \left( 1 - \frac{|x - y|}{r} \right) r^2 \psi \left( x + r \frac{y - x}{|y - x|} \right) \, dr \, dy.
\]

There remains the task of checking the validity of this formula (say, in the spirit of [13]), and for domains of more complex geometry this is not directly applicable. In the proof of Theorem 6.2 we circumvent these problems.

Condition (29) means that \( a \) is from the grand Lebesgue space \( L^n(\Omega) \) introduced by Iwaniec and Sbordone [35], which is the set of functions \( f \) integrable to any power less than \( n \) with the finite norm \( \| f \|_{L^n(\Omega)} = \sup_{1 \leq s < n} \left( (n - s)|\Omega|^{-1} \int_\Omega |f|^s \, dx \right)^{1/s} \). Clearly, this condition is satisfied for \( a \) from the Marcinkiewicz weak-\( L^n(\Omega) \) space, i.e. \( |\{ |a| > t \}| \leq C t^{-n} \). The Orlicz space \( L^n \log^{-1} L(\Omega) \) is also contained in \( L^n(\Omega) \) and \( L^n(\Omega) \subset L^n \log^a L(\Omega) \) for all \( \alpha < -1 \). Further account of properties of grand Lebesgue spaces and their investigation by methods of interpolation theory can be found in [36]. The closure of \( L^n(\Omega) \) in \( L^n(\Omega) \) is strictly less than the latter space and is characterized by \( \lim \sup_{\varepsilon \to 0} \varepsilon^{1/n} \| f \|_{L^{n-\varepsilon}(\Omega)} = 0 \).

For a solenoidal vector field \( a \in L^1(\Omega) \) one can easily construct an approximation solution of (3) for bounded right-hand sides, \( f \in L^\infty(\Omega) \) (or, say, \( f = f_{i,x} + g, g \in L^{n/2}(\Omega), f_i \in L^q(\Omega), q > n \)). This fact follows from the supremum estimate \( \| u \|_{L^\infty(\Omega)} \leq C \| f \|_{L^\infty(\Omega)} \). The Orlicz space \( L^n \log^{-1} L(\Omega) \) which is valid for bounded solenoidal \( a \) with the constant \( C \) independent of \( a \). Applying the same reasoning as in Theorem 6.2 one can prove the uniqueness of approximation solution of (3) with \( a \) from the Morrey space \( M^n(\Omega) \) and the right-hand side \( f \) from \( L^\infty(\Omega) \) without requiring (23). Now, let \( f \) be an arbitrary element of \( W^{-1,2}(\Omega) \). It can be approximated by bounded \( f_j \). Let \( u_j \) be approximation solutions of (3) corresponding to \( f_j \). Since in this case approximation solutions are uniquely defined, the difference of any two approximation solutions is also an approximation solution, satisfying \( \| u_j - u_k \|_{W^{1,2}(\Omega)} \leq C \| f_j - f_k \|_{W^{-1,2}(\Omega)} \). Therefore, the sequence \( u_j \) has a strong limit \( u \), which does not depend on the choice of approximation of \( f \). It would be natural to call this limit a solution to (3) corresponding to the right-hand side \( f \). The limit function can be unbounded, so the term \( au \) need not be integrable here. The question is how to understand the equation. For instance, using the Sobolev representation and Fubini’s theorem, for bounded \( u \) we can transform the drift term in the integral identity as follows:

\[
\int_\Omega au \nabla \varphi \, dx = (n\omega_n)^{-1} \int_\Omega \nabla u(y) \, dy \int_\Omega \frac{(x - y)a \nabla \varphi(x)}{|x - y|^n} \, dx,
\]

which is well defined (\( a \nabla \varphi \in M^n(\Omega) \) if \( \nabla \varphi \) is bounded) and allows the passage to the limit with respect to the convergence of \( \nabla u \) in \( L^2(\Omega) \).

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