RIGIDITY FOR MEASURABLE SETS

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Abstract. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a set with finite Lebesgue measure such that, for a fixed radius $r > 0$, the Lebesgue measure of $\Omega \cap B_r(x)$ is equal to a positive constant when $x$ varies in the essential boundary of $\Omega$. We prove that $\Omega$ is a ball (or a finite union of equal balls) provided it satisfies a nondegeneracy condition, which holds in particular for any set of diameter larger than $r$ which is either open and connected, or of finite perimeter and indecomposable. The proof requires reinventing each step of the moving planes method by Alexandrov in the framework of measurable sets.

1. Introduction

In this paper we study the following rigidity problem for measurable sets of the Euclidean space $\mathbb{R}^d$: for a given radius $r > 0$, identify measurable sets $\Omega$ of finite Lebesgue measure such that, for a positive constant $c$,

$$|\Omega \cap B_r(x)| = c \quad \forall x \in \partial^* \Omega,$$

where $B_r(x)$ is the ball of radius $r$ centred at $x$, $|\cdot|$ denotes the Lebesgue measure, and $\partial^* \Omega$ indicates the essential boundary of $\Omega$ (i.e. the set of points $x \in \mathbb{R}^d$ at which both $\Omega$ and its complement $\Omega^c$ have a strictly positive $d$-dimensional upper density).

Such an easy-to-state geometric question conceals several relations with classical questions in Differential Geometry, as well as with recent advances in Convex Geometry and Geometric Measure Theory. We outline them below, before stating the results.

A quick historical overview. The question about rigidity criteria obtainable by measuring intersections of a domain $\Omega$ in $\mathbb{R}^d$ with balls rolling along its boundary dates back to almost one century ago. The idea is to look at the behaviour, for $x \in \partial \Omega$, of surface integrals $\mathcal{H}^{d-1}(\Omega \cap \partial B_r(x))$, or of volume integrals $|\Omega \cap B_r(x)|$. In its first grain, this idea can be found in a paper dating back to 1932 by Cimmino (see [14]), where he raised the following question: is it possible to characterize surfaces $\Gamma$ which bisect the $\mathcal{H}^2$-measure of the boundary of any ball which is centred on $\Gamma$ and has a sufficiently small radius? Cimmino’s problem was solved more than sixty years later by Nitsche, who proved that the only (smooth) surfaces with this property are the plane and the helicoid [31].

The problem reemerged in the first 2000s under different garments: motivated by the study of isothermic surfaces in heat conduction, also in relation with the so-called Matzoh ball soup problem [26]. Magnanini, Prajapat and Sakaguchi were led to consider $B$-dense
domains, namely subsets $\Omega$ in $\mathbb{R}^d$ such that, for any $r > 0$, there exists a positive constant $c(r)$ such that $|\Omega \cap B_r(x)| = c(r)$ for every $x \in \partial \Omega$. In particular they proved that, if the boundary of a $B$-dense domain $\Omega$ is a complete embedded minimal surface of finite total curvature in $\mathbb{R}^3$, $\Omega$ must be a plane [25]. Later in 2016, Magnanini and Marini proved that, in any space dimension and for any given convex body $K$, if a set $\Omega$ of positive and finite Lebesgue measure is $K$-dense (meaning that $|\Omega \cap (x + rK)| = c(r)$ for every $x \in \partial \Omega$), then $\Omega$ and $K$ are homothetic ellipsoids [24] (see also [23, 2]). Note that, as long as it is assumed to hold for any sufficiently small $r > 0$, the constancy of the volume measure $|\Omega \cap B_r(x)|$ is actually equivalent to that of the surface measure $\mathcal{H}^{d-1}(\Omega \cap \partial B_r(x))$: so the difference between $B$-dense domains and those considered by Cimmino is just that the volume fraction is no longer fixed to $\frac{1}{2}$ (in other words, the constant $c(r)$ may differ from $\frac{1}{2} |B_r(x)|$).

All the rigidity results mentioned so far are naturally related to a central question in Differential Geometry, namely the classification of hypersurfaces with constant mean curvature in $\mathbb{R}^d$. Indeed, in view of the asymptotic expansion

$$\tag{2} |\Omega \cap B_r(x)| = \frac{1}{2} \omega_d r^d - \frac{d-1}{2(d+1)} \omega_{d-1} H_\Omega(x) r^{d+1} + O(r^{d+2}),$$

where $\omega_d$ is the volume of the unit ball in $\mathbb{R}^d$ and $H_\Omega$ is the mean curvature of $\partial \Omega$ [21]. The Lebesgue measure of $\Omega \cap B_r(x)$ can be interpreted as an integral approximation of $H_\Omega(x)$; differentiating with respect to $r$, the same assertion is valid for the $(d-1)$-dimensional measure of $\Omega \cap \partial B_r(x)$.

In this perspective, the rigidity criteria quoted above can be read as counterparts of some cornerstone results for hypersurfaces with constant mean curvature. Thus, Magnanini-Marini criterion reminds the celebrated theorem proved in 1958 by Alexandrov [1]: if $\Omega$ is a bounded connected domain of class $C^2$ such that $\partial \Omega$ has constant mean curvature, then $\Omega$ is a ball (a generalization has been proved very recently by Delgadino-Maggi [17], showing that any set with finite Lebesgue measure and finite perimeter with constant distributional mean curvature is a finite union of equal balls). Likewise, Nitsche criterion, though not involving any topological constraint, reminds the much harder problem, settled only in 2005 by Meeks-Rosenberg [29], of classifying the plane and the helicoid as the unique simply connected minimal surfaces embedded in $\mathbb{R}^3$.

Comparing the constancy of the mean curvature with the constancy of one of the measures $\mathcal{H}^{d-1}(\Omega \cap \partial B_r(x))$ and $|\Omega \cap B_r(x)|$ for any sufficiently small $r > 0$, it is clear that the former is in principle weaker, as it concerns only one among the coefficients of the expansion [2] in powers of $r$; on the other hand, defining a notion of mean curvature requires some boundary regularity, even if done in distributional sense, while measuring intersections with balls requires no smoothness at all.

All in all, at present no rigidity result seems to be available for arbitrary measurable sets under a fairly weak condition such as the constancy of a single and well-defined quantity. Aim of this paper is to provide a first contribution in this direction, by considering sets which satisfy condition [1]. The fact that we work with one fixed radius makes the approach completely new. We call sets satisfying [1] r-critical. Note that, since the essential boundary $\partial^* \Omega$ is included in the topological boundary $\partial \Omega$, condition [1] is weaker than the constancy of $|\Omega \cap B_r(x)|$ along $\partial \Omega$. Incidentally let us also mention
that, at least for convex domains, \( r \)-criticality can be rephrased by saying that \( \partial \Omega \) is a level surface for the **cross-covariogram function** of \( \Omega \) and \( B_r(0) \). (The cross-covariogram function of two convex bodies \( K_1 \) and \( K_2 \) is defined as \( g(K_1, K_2)(x) := |K_1 \cap (x + K_2)| \), and the investigation of its level lines has attracted some attention in the literature on Convex Geometry, also in connection with the floating body problem, see for instance \([6, 30]\)).

The reason for the terminology “\( r \)-critical” is that, notably, this notion has still a variational interpretation. Actually, sets of constant mean curvature may be viewed as stationary sets for the perimeter functional under a volume preserving perturbation. From this point of view, Alexandrov result, along with its extension in \([17]\), allows to identify critical sets for the isoperimetric inequality proved in 1958 by De Giorgi \([16, 18]\).

An interpretation in the same vein can be given to \( r \)-critical sets, as soon as the isoperimetric inequality is replaced by another classical one, which is even more ancient, namely the **rearrangement inequality** proved in 1932 by Riesz \([33]\). In a simplified version it states that, for any radially symmetric, decreasing, non-negative function \( h \), balls maximize, under a constraint of prescribed Lebesgue measure, the integral functional

\[
J_h(\Omega) := \int_{\Omega} \int_{\Omega} h(x - y) \, dx \, dy.
\]

Given an integrand \( h \) as above, it is not difficult to check that balls maximize \( J_h \) provided they maximize \( J_{\chi_{B_r(0)}} \) for all \( r > 0 \). Hence the choice \( h = \chi_{B_r(0)} \) is of special relevance, and for such kernel stationary domains are precisely sets satisfying condition \([11]\).

Equivalently, in view of the equality \(|\Omega \cap B_r(x)| - |\Omega^c \cap B_r(x)| = 2|\Omega \cap B_r(x)| - \omega_d r^d|\), \( r \)-critical sets may be viewed as stationary domains, under volume preserving perturbations, for the **nonlocal perimeter**

\[
r-\text{Per}(\Omega) := \int_{\Omega} \int_{\Omega^c} \chi_{\{|x-y|<r\}} \, dx \, dy;
\]

in fact, the quantity \(|\Omega^c \cap B_r(x)| - |\Omega \cap B_r(x)| \) fits the definition of **generalized nonlocal mean curvature** proposed by Chambolle, Morini and Ponsiglione in \([13]\) (see also \([28]\)).

This nonlocal interpretation brings immediately to mind the fractional perimeter introduced in the seminal papers \([9, 11]\).

\[
P_s(\Omega) = \int_{\Omega} \int_{\Omega^c} \frac{1}{|x-y|^{d+2s}} \, dx \, dy, \quad s \in \left(0, \frac{1}{2}\right),
\]

and particularly two independent results appeared in 2018 by Ciraolo-Figalli-Maggi-Novaga \([15]\) and by Cabré-Fall-Morales-Weth \([8]\), which have identified balls as the unique stationary domains of class \( C^{1,\alpha} \) \((\alpha > 2s)\) for the perimeter \( P_s \).

The qualitative properties of the kernel \( \chi_{B_r(0)} \) make our problem dramatically different: it is bounded (allowing to deal with arbitrary measurable sets), compactly supported (producing short-range nonlocal effects), and discontinuous with level sets of positive measure (enhancing the need for some “transmission” issue, companion to \( r \)-criticality, in order to get rigidity). Fatally, notwithstanding the differences between the two questions, the investigation of \( r \)-critical measurable sets evokes Pompeiu problem.
Stationary domains for more general kernels $h$ could be considered, see Remark 19, but for the sake of clearness we prefer to focus on the choice $h = \chi_{B_r(0)}$, since it allows to capture all new relevant ideas.

To conclude, our way to rigidity appears to be very suitable to deal with from an applied point of view. As a matter of fact, in recent times the Lebesgue measure $|\Omega \cap B_r(x)|$ has been successfully exploited in Geometry Processing, under the name of \textit{volumetric integral invariant}, for an efficient computation avoiding noise of shape characteristics (including the mean curvature), with applications to feature extraction at multiple scales and automatic rasselling of fragments of broken objects (see \cite{27, 32}).

The results. Before stating our main result, in order to introduce the key condition for rigidity, companion to $r$-criticality, we set the following definition: we say that a measurable set $\Omega$ in $\mathbb{R}^d$ is $r$-degenerate if

$$\inf_{x_1, x_2 \in \partial^* \Omega} \frac{|\Omega \cap (B_r(x_1) \Delta B_r(x_2))|}{\|x_1 - x_2\|} = 0.$$ 

A discussion about this notion is postponed to Section 2.1. Therein we shall provide, in particular, a measure theoretic condition sufficient for non degeneracy, which permits to show that bounded open connected sets, as well as bounded indecomposable sets with finite perimeter, are not degenerate for any $r$ smaller than their diameter.

Theorem 1. Let $\Omega$ be a measurable set with finite Lebesgue measure in $\mathbb{R}^d$, and let $r > 0$. Assume that $\Omega$ is $r$-critical and not $r$-degenerate. Then $\Omega$ is equivalent to a finite union of balls of the same radius $R > \frac{r}{2}$, at mutual distance larger than or equal to $r$.

Remark 2. Rigidity may fail if the finite measure assumption is dropped: any halfspace or any strip \{ $x \in \mathbb{R}^d : a < x_1 < b$ \} is critical and not degenerate for any $r > 0$. As well, rigidity may fail for $r$-critical sets of finite measure which are $r$-degenerate; for some examples in this respect, see Section 2.1.

Remark 3. A result analogue to Theorem 1 can be immediately deduced, by using the area formula, if balls are replaced by ellipsoids: if $E$ is a given ellipsoid, any set with finite measure which satisfies the criticality and degeneracy conditions with $x + E$ in place of $B_r(x)$, is a finite union of ellipsoids homotetic to $E$.

Remark 4. We point out that the initial choice of the radius $r$ produces a sort of \textit{tuned bubbling phenomenon}, which may occur only with a precise lower threshold both on the size of the balls and on their mutual distance. In accordance with the short-range nonlocal nature of our kernel, this behaviour should be compared with the local result in \cite{17}, where bubbling can occur at any scale, and the fractional results in \cite{15, 8}, where bubbling cannot occur at all. Let us also mention that a further motivation for characterising finite unions of equal balls is their appearance as optimal domains in spectral shape optimization problems (see \cite{20, 7}).

We now present some consequences of Theorem 1 for sets enjoying some kind of regularity. We begin by the case of open sets:
Corollary 5. Let $\Omega$ be an open set with finite Lebesgue measure in $\mathbb{R}^d$, and let $r > 0$. Assume that there exists a positive constant $c$ such that

$$|\Omega \cap B_r(x)| = c \quad \forall x \in \partial \Omega,$$

where $\partial \Omega$ denotes the topological boundary.

If $r < \inf_i \{\text{diam}(\Omega_i)\}$, where $\Omega_i$ are the open connected components of $\Omega$, then $\Omega$ is a finite union of balls of the same radius $R > \frac{r}{2}$, at mutual distance larger than or equal to $r$. In particular, if $\Omega$ is connected and $r < \text{diam}(\Omega)$, then $\Omega$ is a ball.

Next we turn to the case of sets with finite perimeter. Recall that, following [3], any set $\Omega$ with finite perimeter can be written as finite or countable family of indecomposable components $\Omega_i$. This means that each $\Omega_i$ is indecomposable in the sense that it does not admit a partition $(\Omega_i^+, \Omega_i^-)$, with $|\Omega_i^+| > 0$ and $\text{Per}(\Omega_i) = \text{Per}(\Omega_i^+) + \text{Per}(\Omega_i^-)$, and that the $\Omega_i$’s are maximal indecomposable sets. Recall also that the reduced boundary $\mathcal{F}\Omega$ is the collection of points $x \in \text{supp}(D\chi_{\Omega})$ such that the generalized normal $\nu_{\Omega}(x) := \lim_{\rho \to 0} D\chi_{\Omega}(B_\rho(x))/|D\chi_{\Omega}(B_\rho(x))|$ exists in $\mathbb{R}^d$ and satisfies $\nu_{\Omega}(x) = 1$. For sets of finite perimeter, a sufficient condition for $r$-criticality is the validity of condition (4) below, because the closure of $\mathcal{F}\Omega \setminus N$, for any $\mathcal{H}^{d-1}$-negligible set $N$, turns out to contain $\partial^* \Omega$.

Corollary 6. Let $\Omega$ be a set of finite perimeter and finite Lebesgue measure in $\mathbb{R}^d$, and let $r > 0$. Assume there exists a positive constant $c$ such that

$$|\Omega \cap B_r(x)| = c \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } x \in \mathcal{F}\Omega.$$

If $r < \inf_i \{\text{diam}(\Omega_i)\}$, where $\Omega_i$ are the indecomposable components of $\Omega$, then $\Omega$ is equivalent to a finite union of balls of the same radius $R > \frac{r}{2}$, at mutual distance larger than or equal to $r$. In particular, if $\Omega$ is indecomposable and $r < \text{diam}(\Omega)$, $\Omega$ is a ball.

About the proof of Theorem [1]. Alexandrov rigidity theorem was obtained by a very elegant proof, based on what he called reflection principle, nowadays commonly known as the moving planes method. His brilliant idea was destined to have a tremendous impact also in the field of Mathematical Analysis: its implications in PDEs were firstly enhanced in the seventies by Serrin [34] to get rigidity results for overdetermined boundary value problems, and afterwards enlivened to get symmetry and monotonicity properties of solutions to nonlinear elliptic equations, in particular by Gidas-Ni-Nirenberg [19], Berestycki-Nirenberg [4, 5], Caffarelli-Gidas-Spruck [10].

The proof of Theorem [1] is based on a reinvention of the moving planes method in the context of measurable sets. Alexandrov idea is that, if $\Omega$ has constant mean curvature, it must have a hyperplane of symmetry in every direction; this can be obtained starting from an arbitrary hyperplane, moving it in a parallel way until an appropriate stopping time, and reflecting $\Omega$ about such hyperplane. The conclusion is then reached by the qualitative behaviour of the constant mean curvature equation, specifically using the strong maximum principle and Hopf boundary point lemma. Our situation is completely different, for many reasons. First, no connectedness assumption is made on $\Omega$, so that the proof cannot be obtained just by observing that the choice of the initial hyperplane is arbitrary, but requires a new argument allowing to single out each ball and “extract” it from $\Omega$ once enough symmetries are detected. Second, no smoothness information is available: since the essential boundary does not admit a normal vector and is not locally
a graph, all the the steps of the method lose their meaning. Third, even in cases when the boundary is locally a graph and the contact with the reflected cap holds in classical sense, no PDE holds around the contact point, but merely the r-criticality condition on the essential boundary. Thus we need to conceive new arguments, in particular to decipher why the movement can start and especially when it has to stop. In the smooth setting, this occurs either when the boundary and the reflected cap become tangent, or when the boundary and the moving plane meet orthogonally; in the measurable setting, these two situations must be abandoned in favour of suitable notions of away or close contact. They are defined and handled relying on the concept of Steiner-symmetric sets, which plays a crucial role similarly as in De Giorgi’s proof of the isoperimetric theorem. A more detailed outline of the proof is given at the beginning of Section 3.

2. Preliminaries

In this section we discuss the main issues about degeneracy, and we prepare the proof of Theorem 1 by analyzing the structure of certain Steiner symmetric sets obtained by reflection.

2.1. About r-degeneracy. We start by showing some counterexamples of r-critical sets which escape from rigidity since they are degenerate.

Examples 7. Different kinds of bounded sets which, for some $r > 0$, are critical but degenerate:

(i) Small sets: any measurable set $\Omega$ with $\text{diam}(\Omega) \leq r$.

(ii) Unions of small sets at large mutual distance: any measurable set obtained as the union of a finite number of measurable sets $\Omega_j$, having the same measure, $\text{diam}(\Omega_j) \leq r$ $\forall j$, and $\text{dist}(\Omega_j, \Omega_l) \geq r$ $\forall j \neq l$.

(iii) Unions of spaced small sets at small mutual distance: for $d = 2$ and any fixed $n \in \mathbb{N}$, given $r > 0$ such that $|1 - e^{\frac{2\pi}{n}}| - r| \geq \varepsilon > 0 \forall j = 1, \ldots, n$, any set obtained as the union of $n$ measurable sets $\Omega_j$, having the same measure, such that $\Omega_j \subseteq B_{\frac{r}{2}}(e^{\frac{j2\pi}{n}})$ $\forall j = 1, \ldots, n$. This last example shows in particular that the connectedness of a $\frac{r}{2}$-neighbourhood of $\Omega$ is not sufficient to avoid r-degeneracy.

Next we establish a measure-theoretic sufficient condition for nondegeneracy (Proposition 8) which is useful, in particular, to deal with open sets and sets with finite perimeter (Proposition 10), and hence to deduce Corollaries 5 and 9 from Theorem 1. Such condition is expressed in terms of the total variation measure $|D\chi_{B_r(x)}|$, which is given by (see for instance [22], page 117)

\[ |D\chi_{B_r(x)}|(E) = \mathcal{H}^{d-1}(\partial B_r(x) \cap E) \quad \text{for any measurable set } E; \]

we are thus led back to handle the measure of spherical hypersurfaces considered by Cimmino.

Hereafter and in the sequel, we denote by $\Omega(t)$ the set of points $x \in \mathbb{R}^d$ at which $\Omega$ has $d$-dimensional density equal $t$. 

Proposition 8. Let $\Omega$ be a bounded measurable set, and let $r > 0$. Assume there exists $\varepsilon > 0$ such that

$$\inf_{x \in \mathcal{U}_\varepsilon(\partial^* \Omega)} |D\chi_{B_r(x)}|(\Omega^{(1)}) > 0$$

where $\mathcal{U}_\varepsilon(\partial^* \Omega)$ is the set of points at distance smaller than $\varepsilon$ from $\partial^* \Omega$. Then $\Omega$ is not $r$-degenerate.

Remark 9. (i) Taking points of density 1 in (6) is relevant in order to make the condition satisfied, for instance, by open sets deprived of a spherical hypersurface centred at a boundary point.

(ii) Condition (6) is not necessary for nondegeneracy. For instance, consider in the complex plane the union of the sets $\{z = \rho e^{i\theta} : 0 < \rho < \frac{1}{4}, \theta \in (-\frac{\pi}{4}, \frac{\pi}{4})\}$ and $\{z = \rho e^{i\theta} : 1 < \rho < 3, \theta \in (-\frac{\pi}{4}, \frac{\pi}{4})\}$. For $r = 1$, the set is not degenerate. However, the infimum (6) vanishes, by taking points $x$ arbitrarily close to 0 on the negative real axis.

Proof of Proposition 8. Set $\alpha := \inf_{x \in \mathcal{U}_\varepsilon(\partial^* \Omega)} |D\chi_{B_r(x)}|(\Omega^{(1)})$. Assume by contradiction that $\alpha > 0$ but $\Omega$ is $r$-degenerate. Since $\Omega$ is bounded, we can find two sequences $\{x_n\}, \{y_n\} \subset \partial^* \Omega$, with $\lim_n \|x_n - y_n\| = 0$, such that

$$\lim_{n \to +\infty} \frac{\|\Omega \cap (B_r(x_n) \Delta B_r(y_n))\|}{\|x_n - y_n\|} = 0.$$  

Without loss of generality, we can assume that for every $n$ it holds

$$x_n, y_n \in \mathcal{U}_{\frac{\varepsilon}{2}}(\partial^* \Omega) \quad \text{and} \quad \|x_n - y_n\| \leq \frac{\varepsilon}{2}.$$  

This implies that, if $[x_n, y_n]$ is the closed segment with endpoints $x_n$ and $y_n$, we have

$$|D\chi_{B_r(x)}|(\Omega^{(1)}) \geq \alpha \quad \forall x \in [x_n, y_n].$$

For every $x \in \mathbb{R}^d$, we denote $x' = (x^1, \ldots, x^{d-1}) \in \mathbb{R}^{d-1}$, and we write $x = (x', x^d)$. We fix some $\delta_0 \in (0, \frac{\varepsilon}{4} \wedge r)$ such that

$$\mathcal{H}^{d-1}(\{x \in \partial B_r(0) : |x^d| \leq \delta_0\}) \leq \frac{\alpha}{4}.$$
For convenience, we position our system of coordinates so that \( x_n = (0, \delta_n) \) and \( y_n = (0, -\delta_n) \). Then, for \( n \) large enough so that \( \delta_n \leq \delta_0 \), we get the following estimate:

\[
\Omega \cap \left( B_r(x_n) \Delta B_r(y_n) \right) = \frac{1}{2\delta_n} \int_{B_r(x_n) \Delta B_r(y_n)} \chi_\Omega(x) \, dx
\]

\[
\geq \frac{1}{2\delta_n} \int_{-\delta_n}^{\delta_n} \left[ \int_{\|x'\| \leq \sqrt{r^2 - \delta_0^2}} \chi_{\Omega(1)}(x', \sqrt{r^2 - \|x'\|^2} + s) \, dx' \right] ds
\]

\[
+ \frac{1}{2\delta_n} \int_{-\delta_n}^{\delta_n} \left[ \int_{\|x'\| \leq \sqrt{r^2 - \delta_0^2}} \chi_{\Omega(1)}(x', -\sqrt{r^2 - \|x'\|^2} + s) \, dx' \right] ds
\]

\[
\geq \frac{1}{2\delta_n} \frac{\sqrt{r^2 - \delta_0^2}}{r} \int_{-\delta_n}^{\delta_n} \left[ \int_{\|x'\| \leq \sqrt{r^2 - \delta_0^2}} \chi_{\Omega(1)}(x', \sqrt{r^2 - \|x'\|^2} + s) \, dx' \right] ds
\]

\[
+ \frac{1}{2\delta_n} \frac{\sqrt{r^2 - \delta_0^2}}{r} \int_{-\delta_n}^{\delta_n} \left[ \int_{\|x'\| \leq \sqrt{r^2 - \delta_0^2}} \chi_{\Omega(1)}(x', -\sqrt{r^2 - \|x'\|^2} + s) \, dx' \right] ds
\]

\[
\geq \frac{\sqrt{r^2 - \delta_0^2} \alpha}{r} \frac{\delta_0}{2} ,
\]

where the last inequality follows from (8) and (9). This contradicts (7) and achieves the proof. \( \square \)

**Proposition 10.** Let \( \Omega \) be either a bounded open set or a bounded set of finite perimeter, and let \( \{\Omega_i\}_i \) denote the family respectively of its connected or indecomposable components. Then \( \Omega \) is not \( r \)-degenerate for any \( r < \inf_i \{\text{diam}(\Omega_i)\} \).

**Remark 11.** While sets \( \Omega \) with a single component are not \( r \)-degenerate if and only if \( r < \text{diam}(\Omega) \), for multiply connected domains the condition \( r < \inf_i \{\text{diam}(\Omega_i)\} \) is sufficient but clearly not necessary to avoid \( r \)-degeneracy: for instance, the disjoint union of two balls of radius \( R \) can be not \( r \)-degenerate also for radii \( r \geq 2R \) provided the balls are close enough.

**Remark 12.** In the light of Proposition 10 it is natural to ask if there exists some notion of “connectedness” avoiding \( r \)-degeneracy also for arbitrary measurable sets. To the best of our knowledge, the unique kind of such a general notion for Borel sets \( \Omega \) has been proposed in [12] under the name of essential connectedness, and amounts to ask that \( \mathcal{H}^{d-1}(\Omega^{(1)} \cap \partial^* \Omega_+ \cap \partial^* \Omega_-) > 0 \) for any nontrivial Borel partition \( (\Omega_+, \Omega_-) \) of \( \Omega \). However, if \( \Omega \) has not finite perimeter, relying on the possible lack of semicontinuity of the map \( t \mapsto \mathcal{H}^{d-1}(\Omega \cap B_t(x_i)) \) as \( x_i \to x \in \partial \Omega \), it is possible to construct examples of Borel sets \( \Omega \) which are essentially connected but degenerate for some \( r < \text{diam}(\Omega) \).

**Proof of Proposition 10.** We focus our attention on the case of finite perimeter sets, since for open sets the proof can be obtained in a similar way. Working component by component, we are reduced to prove that, if \( \Omega \) is indecomposable, then it is not \( r \)-degenerate for any \( r < \text{diam}(\Omega) \). To that aim, it is enough to prove that it satisfies,
for some $\varepsilon > 0$, condition $\Box$ in Proposition $\blacksquare$. Assume by contradiction this is not the case. Then, in view of $[\blacksquare]$, it would be possible to find a sequence $\{x_n\}$ converging to a point $x \in \partial^* \Omega$ such that

$$\lim_n \mathcal{H}^{d-1}(\partial B_r(x_n) \cap \Omega^{(1)}) = 0.$$  \hspace{1cm} (10)

Up to subsequences, we denote by $A$ and $C$ the limit in $L^1$ respectively of the characteristic functions of the sets $A_n := \Omega \cap B_r(x_n)$ and $C_n := \Omega \setminus B_r(x_n)$. We are going to show that they provide a nontrivial partition of $\Omega$ such that $\text{Per}(\Omega) = \text{Per}(A) + \text{Per}(C)$. We have

$$\begin{cases} 
\text{Per}(A_n) = \text{Per}(\Omega, B_r(x_n)) + \mathcal{H}^{d-1}(\partial B_r(x_n) \cap \partial^* A_n \cap \Omega^{(1)}) + \mathcal{H}^{d-1}(\partial B_r(x_n) \cap \Omega^{(1)}) \\
\text{Per}(C_n) = \text{Per}(\Omega, \mathbb{R}^d \setminus B_r(x_n)) + \mathcal{H}^{d-1}(\partial B_r(x_n) \cap \partial^* C_n \cap \Omega^{(1)}) + \mathcal{H}^{d-1}(\partial B_r(x_n) \cap \Omega^{(1)})
\end{cases}$$

Since perimeter is lower semicontinuous with respect to $L^1$-convergence, and we have $\mathcal{H}^{d-1}(\partial^* A_n \cap \partial^* C_n \cap \Omega^{(1)}) = 0$, by passing to the limit in the two relations above and summing, we get

$$\text{Per}(A) + \text{Per}(C) \leq \text{Per}(\Omega) + 2 \lim_n \mathcal{H}^{d-1}(\partial B_r(x_n) \cap \Omega^{(1)}).$$

The conclusion follows by condition $\Box$.

\hfill $\Box$

2.2. About some Steiner symmetric sets obtained by reflection. A measurable set $\omega$ is Steiner symmetric about a hyperplane $H$ with unit normal $\nu$ if the following equality holds as an equivalence between Lebesgue measurable sets:

$$\omega = \left\{ x \in \mathbb{R}^d : x = z + t\nu, \ z \in H, \ |t| < \frac{1}{2} \mathcal{H}^1(\omega \cap \{ z + t\nu : t \in \mathbb{R} \}) \right\}.$$

In Proposition $\blacksquare$ below, we focus our attention on a special kind of Steiner symmetric sets obtained by reflection, that we shall need to handle in the proof of Theorem $\blacksquare$.

To that aim and in the sequel, we shall make repeatedly use of the following elementary observation: given a measurable subset $\omega$ of $\mathbb{R}^d$, it holds

$$\omega^{(1)} \setminus \partial^* \omega = \text{int}(\omega^{(1)}), \quad \omega^{(0)} \setminus \partial^* \omega = \text{int}(\omega^{(0)}).$$  \hspace{1cm} (11)

In particular, $\mathbb{R}^d$ can be decomposed as a disjoint union,

$$\mathbb{R}^d = \text{int}(\omega^{(1)}) \sqcup \text{int}(\omega^{(0)}) \sqcup \partial^* \omega.$$  \hspace{1cm} (12)

Let us prove the first equality in $\Box$, the second one being analogous. The inclusion $\supseteq$ is immediate. Vice versa, let $x \in \omega^{(1)} \setminus \partial^* \omega$, and let $U$ be an open neighbourhood of $x$ which does not meet $\partial^* \omega$. Let us prove that $U \subset \omega^{(1)}$. By Federer’s Theorem, $\omega$ is of finite perimeter in $U$. Then, by the relative isoperimetric inequality, $\min\{|\omega^c \cap U|, |\omega \cap U|\} = 0$. But it cannot be $|\omega \cap U| = 0$, because $x \in \omega^{(1)}$. Hence $|\omega^c \cap U| = 0$. Then, $U$ cannot contain any point of density 0 for $\omega$, since such point would be of density 1 for $\omega^c$, against $|U \cap \omega^c| = 0$. Recalling that $U$ does not meet $\partial^* \omega$, we conclude that $U \subset \omega^{(1)}$.  \hfill $\Box$
Proposition 13. Let $H$ be a hyperplane with unit normal $\nu$, and let $\omega$ be a bounded measurable set contained into $H_\neq = \{H + tv : t \leq 0\}$ such that

$$\forall z \in H, \quad \partial \omega \cap \{z + tv : t < 0\}$$

so that $\omega$ can be viewed as the subgraph of the function $g : H \to \mathbb{R}_\neq$ defined by $g(z) = 0$ if the intersection in (13) is empty and $g(z) = t$ if such intersection is $z + tv$. The following properties hold:

1. $\partial^* \omega = 0$;
2. $\omega$ is essentially open;
3. the union of $\omega$ and its reflection about $H$ is Steiner-symmetric about $H$;
4. the function $g$ is continuous.

Proof. Statement (i) is an immediate consequence of the assumption (13) and Fubini Theorem. To obtain statement (ii), it is enough to show that $\omega^{(1)}$ is essentially open. Such property follows from statement (i), after applying (11). To prove statement (iii), it is enough to show that the union of $\omega^{(1)}$ and its reflection about $H$ is Steiner-symmetric about $H$. To that aim, let us fix $z \in H$ such that $\partial^* \omega \cap \{z + tv : t < 0\} = \{p\}$, and let us show that the open segment $(p, z)$ is contained into $\omega^{(1)}$. Recalling the decomposition (12) and assumption (13), we infer that the open segment $(p, z)$ is either in $\text{int}(\omega^{(1)})$ or in $\text{int}(\omega^{(0)})$. In the first case we are done. It remains to exclude that it is entirely contained in $\text{int}(\omega^{(0)})$. Assume by contradiction this is the case. We observe that, by the first equality in (11), since $p \in \partial^* \omega$, $p$ is the limit of a sequence of points $\{p_n\} \subset \omega^{(1)}$. Since we are assuming that the open segment $(p, z)$ is entirely contained in $\text{int}(\omega^{(0)})$, the points $p_n$ do not belong to such segment, so that they belong to straight lines of the form $\{z_n + t\nu : t \in \mathbb{R}\}$, for a sequence of points $\{z_n\} \subset H$ converging to $z$. But then some of these straight lines would necessarily contain at least two points of $\partial^* \omega$ (otherwise the segments $(p_n, z_n)$ would be entirely contained into $\text{int}(\omega^{(1)})$ and could not converge to $(p, z)$ which is entirely contained into $\text{int}(\omega^{(0)})$). Let $g$ be the function defined via (13) as in the statement, so that $\omega$ can be viewed as the subgraph of $g$. To show the continuity of $g$ at a fixed point $z_0 \in H$, we consider separately the cases $g(z_0) = 0$ and $g(z_0) < 0$. If $g(z_0) = 0$, we have to prove that, for any sequence $\{z_n\} \subset H$ converging to $z_0$, the sequence $\{g(z_n)\}$ converges to 0. Up to a subsequence, we may assume $g(z_n) \to \lambda$, with $\lambda \leq 0$. If $\lambda < 0$, for $n$ large enough we have $g(z_n) < 0$, which by definition of $g$ means that $z_n + g(z_n)\nu \in \partial^* \omega$; passing to the limit in the last relation, we get $z_0 + \lambda \nu \in \partial^* \omega$, against $g(z_0) = 0$.

Assume now $g(z_0) < 0$, and let $\{z_n\} \subset H$ be any sequence converging to $z_0$. We may assume that $g(z_n) \to \lambda$, with $\lambda \leq 0$. If $\lambda < 0$, we get as above $z_0 + \lambda \nu \in \partial^* \omega$; by (14), we conclude that $g(z_0) = \lambda$. It remains to show that the case $\lambda = 0$ cannot occur. Assume $\lambda = 0$. We consider the open segment $S := (z_0 + g(z_0)\nu, z_0)$. By (13), $S \cap \partial^* \omega = \emptyset$, and hence $S$ is entirely contained either into $\text{int}(\omega^{(1)})$ or into $\text{int}(\omega^{(0)})$. If $S \subset \text{int}(\omega^{(1)})$, we pick a point $x_0 \in S$ and a small ball $B_\varepsilon(x_0) \subset \omega^{(1)}$. For every $x \in B_\varepsilon(x_0)$, denoting by $z_x$ its projection onto $H$ (in particular, $z_{x_0} = z_0$), by statement (iii) we have that the segment $(x, z_x)$ lies into $\omega^{(1)}$. This property leads to a contradiction, as it implies on one hand that $z_0 \in \text{int}(\omega^{(1)})$ and on the other hand that $g(z_n) < 0$ for $n$ large enough,
which in turn gives \( z_0 \in \overline{\partial^* \omega} \) (passing to the limit in the relation \( z_n + g(z_n)\nu \in \overline{\partial^* \omega} \)). If \( S \subseteq \text{int}(\omega(0)) \), we can pick a point \( x_0 \in S \) and a small ball \( B_\varepsilon(x_0) \subseteq \omega(0) \). This contradicts statement (iii) and the fact that, since the point \( z_0 + g(z_0)\nu \) belongs to \( \overline{\partial^* \omega} \), it is the limit of a sequence of points of density 1 for \( \omega \). \( \square \)

3. Proof of Theorem 1

Outline of the proof. We observe first of all that the equality (1) continues to hold at every point \( x \in \overline{\partial^* \Omega} \). Then we fix a direction \( \nu \in S^{d-1} \), and we consider an initial hyperplane \( H_0 \) with unit normal \( \nu \), not intersecting \( \overline{\partial^* \Omega} \). Such an initial hyperplane exists because, since \( \Omega \) has finite measure and is \( r \)-critical, it is necessarily bounded. We start moving \( H_0 \) in the direction of its normal \( \nu \) to new positions, so that at a certain moment of the process it starts intersecting \( \overline{\partial^* \Omega} \). We continue the movement in direction \( \nu \), and we denote by \( H_t \) the hyperplanes thus obtained. We set:

\[
\begin{align*}
H_t^- & := \text{the closed halfspace determined by} \ H_t \text{ containing} \ H_0 \\
H_t^+ & := \text{the closed halfspace determined by} \ H_t \text{ not containing} \ H_0 \\
\Omega_t & := \Omega \cap H_t^- \\
R_t & := \text{the reflection of} \ \Omega_t \text{ about} \ H_t.
\end{align*}
\]

- We say that symmetric inclusion holds at \( t \) if

\( R_t \subset \Omega \) and \( \Omega_t \cup R_t \) is Steiner symmetric about \( H_t \).

- We say that symmetric inclusion occurs at \( t \) if with away contact if (14) holds and there exists an “away contact point”, namely a point

\( p' \in [\overline{\partial^* R_t} \cap \overline{\partial^* \Omega}] \setminus H_t. \)

when (14) holds but (15) is false, we say that symmetric inclusion at \( t \) holds without away contact.

- We say that symmetric inclusion occurs at \( t \) with close contact if (14) holds and there exists a “close contact point”, namely a point

\( H_t \ni q = \lim_{n} q_{1,n} = \lim_{n} q_{2,n}, \quad q_{i,n} \in \overline{\partial^* \Omega} \cap \{ q + t\nu : t \in \mathbb{R} \}, \quad q_{1,n} \neq q_{2,n}, \)

(whence one among \( q_{1,n} \) and \( q_{2,n} \) will always happen to belong to \( H_t^+ \), while the other one may fall in \( H_t^+ \) as well as in \( H_t^- \)). Notice that symmetric inclusion can occur at the same \( t \) with both away contact and close contact.

The statement will be obtained in the following steps, which are carried over separately in the next subsections.

Step 1 (start)
There exists \( \varepsilon > 0 \) such that, for every \( t \in [0, \varepsilon] \), symmetric inclusion holds.

Step 2 (the stopping time: no close contact without away contact)
Setting

\( T := \sup \left\{ t > 0 : \text{for all } s \in [0, t), \text{ symmetric inclusion occurs without away contact} \right\}, \)
we have $T < +\infty$, and symmetric inclusion occurs at $T$ with away or with close contact. But we are able to rule out the case of close contact without away contact, so necessarily at $t = T$ we are in the situation of away contact.

**Step 3 (decomposition of $\Omega$ into symmetric and non-symmetric part)**

We show that $\Omega$ can be decomposed as

$$\Omega = \Omega^s \sqcup \Omega^{ns},$$

where $\Omega^s$ is an open set representing the Steiner symmetric part of $\Omega$, given by

$$\Omega^s := \bigcup \left\{ \right(p, p'\right) : p' \text{ is an away contact point, } p \text{ is its symmetric about } H_T \right\},$$

$(p, p')$ being the open segment with endpoints $p$ and $p'$, and $\Omega^{ns} := \Omega \setminus \Omega^s$ represents the non-symmetric part. Moreover, denoting by $\Omega^s_i$ the open connected components of $\Omega^s$, we prove that:

$$\partial^\ast \Omega^s_i \cap (H_T^\pm \setminus H_T) \text{ are connected sets;}$$

(17)

$$\partial^\ast \Omega^s \cap \partial^\ast \Omega^{ns} \subset H_T.$$  

**Step 4 (conclusion)** We show that the open connected components of $\Omega^s$ are balls of the same radius $R > r/2$, lying at distance larger than or equal to $r$, while the set $\Omega^{ns}$ is Lebesgue negligible.

### 3.1. Proof of Step 1.

The proof is based on the following lemma.

**Lemma 14** (no converging pairs). Let $\Omega \subset \mathbb{R}^d$ be a measurable set which is $r$-critical and not $r$-degenerate. Assume that $\Omega$ is contained into $H_0^+ := \{z + t\nu : z \in H_0, \ t \geq 0\}$, $H_0$ being a hyperplane with unit normal $\nu$. Then there cannot exist two sequences of points $\{p_{1,n}\}, \{p_{2,n}\}$ in $\partial^\ast \Omega \cap H_0^+$ which for every fixed $n$ are distinct, with the same projection onto $H_0$, and at infinitesimal distance from $H_0$ as $n \to +\infty$.

**Proof.** We argue by contradiction. Setting $t_{i,n} := \text{dist}(p_{i,n}, H_0)$, we can assume up to a subsequence that $t_{1,n} > t_{2,n}$ for every $n$. We are going to show that

$$\liminf_{n \to +\infty} \frac{|\Omega \cap B_r(p_{1,n})| - |\Omega \cap B_r(p_{2,n})|}{t_{1,n} - t_{2,n}} > 0,$$

against the fact that $\Omega$ is $r$-critical. We have

(20) $|\Omega \cap B_r(p_{1,n})| - |\Omega \cap B_r(p_{2,n})| = |\Omega \cap (B_r(p_{1,n}) \setminus B_r(p_{2,n})))| - |\Omega \cap (B_r(p_{2,n}) \setminus B_r(p_{1,n})))|.$

Since $\Omega$ is not $r$-degenerate, there exists a positive constant $C$ such that

$$\frac{|\Omega \cap (B_r(p_{1,n}) \setminus B_r(p_{2,n})))| + |\Omega \cap (B_r(p_{2,n}) \setminus B_r(p_{1,n})))|}{t_{1,n} - t_{2,n}} \geq C.$$  

(21)

In view of (20) and (21), the inequality (19) holds true provided

$$\frac{1}{4}.$$  

(22)
In turn, by the inclusion $\Omega \subset H_t^+$, the inequality \((23)\) is satisfied as soon as
\[
\frac{|H_t^+ \cap (B_r(p_2, n) \setminus B_r(p_1, n))|}{t_{1,n} - t_{2,n}} \leq \frac{C}{4}.
\]
Such inequality follows from elementary geometric arguments. Indeed, for every fixed $n$, the set $H_t^+ \cap (B_r(p_2, n) \setminus B_r(p_1, n))$ has volume not larger than the region $D_n$ obtained as the difference between two right cylinders having the same axis, given by the straight line orthogonal to $H_0$ through the common projection $z_n$ of $p_{1,n}$ and $p_{2,n}$ onto $H_0$, the same height equal to $t_{2,n} + (1/2)(t_{1,n} - t_{2,n})$, and as bases the $(d - 1)$-dimensional balls contained into $H_0$, with center at $z_n$ and radii respectively equal to $(r^2 - t_{2,n}^2)^{1/2}$ and $(r^2 - t_{1,n}^2)^{1/2}$. Hence, to get \((23)\) it is enough to show that $|D_n| = o(t_{1,n} - t_{2,n})$. This is readily checked since, setting $\gamma_n := (t_{1,n} - t_{2,n})/2$, we have
\[
|D_n| = \omega_{d-1}((r^2 - t_{2,n}^2)^{d/(d+1)} - (r^2 - t_{1,n}^2)^{d/(d+1)}) (t_{2,n} + \frac{1}{2}(t_{1,n} - t_{2,n}))
\]
\[
\sim 2(d-1)\omega_{d-1}r^{d-3}(t_{2,n}\gamma_n + \gamma_n^2)(t_{2,n} + \gamma_n).
\]

Assume the claim in Step 1 false. Then, at least one of the following assertions holds:

(i) $\exists \{t_n\} \to 0$ such that $\forall n$ $\Omega_{t_n} \cup \mathcal{R}_{t_n}$ is not Steiner symmetric about $H_{t_n}$;

(ii) $\exists \{t_n\} \to 0$ such that $\forall n$ $|\mathcal{R}_{t_n} \setminus \Omega| > 0$.

In case (i), for every $n$ we can apply Proposition \([13]\) with $H = H_{t_n}$ and $\omega = \Omega_{t_n}$ to infer that, for some $z_n \in H_{t_n}$, the set $\overline{\partial^\Omega H_{t_n}} \cap \{z_n + t\nu : t < 0\}$ contains at least two distinct points. Then $\overline{\partial^\Omega H_{t_n}} \cap H_0^+$ contains two sequences of points $\{p_{1,n}\}$, $\{p_{2,n}\}$ which for every $n$ are distinct, with the same projection onto $H_0$, and at infinitesimal distance from $H_0$ as $n \to +\infty$, against Lemma \([14]\).

In case (ii), we may assume that $\Omega_{t_n} \cup \mathcal{R}_{t_n}$ is Steiner symmetric about $H_{t_n}$. For every $n$ let $y_n' \in \Omega(0) \cap \mathcal{R}_{t_n}^{(1)}$, and let $z_n$ be the orthogonal projection of $y_n'$ on $H_{t_n}$. If on the segment $(z_n, y_n')$ there is some point in $\overline{\partial^\Omega H_{t_n}}$, since $\Omega_{t_n} \cup \mathcal{R}_{t_n}$ is Steiner symmetric about $H_{t_n}$, we would have a pair of distinct points belonging to $\overline{\partial^\Omega H_{t_n}} \cap H_0^+$, with the same projection on $H_0$, and infinitesimal distance from $H_0$, against Lemma \([14]\).

If on the segment $(z_n, y_n')$ there is no point in $\overline{\partial^\Omega H_{t_n}}$, invoking \([12]\) and recalling that $y_n' \in \Omega(0)$, we infer that the whole segment $(z_n, y_n')$ is contained into $\text{int}(\Omega(0))$. On the other hand, since $y_n' \in \mathcal{R}_{t_n}^{(1)}$, denoting by $y_n$ the reflection of $y_n'$ about $H_{t_n}$, we have that the whole segment $[y_n, z_n]$ is contained into $\text{int}(\Omega(1))$. We conclude that the point $z_n$ belongs to $\overline{\partial^\Omega H_{t_n}}$. Then, by arguing in the same way as in the last part of the proof of Proposition \([13]\) it would be possible to find some straight line of the form $\{\tilde{z}_n + t\nu : t \in \mathbb{R}\}$, with $\tilde{z}_n \in H_{t_n}$ arbitrarily close to $z_n$, containing at least two points of $\overline{\partial^\Omega H_{t_n}}$. Again, this would contradict Lemma \([14]\).
3.2. Proof of Step 2. Since $\Omega$ is bounded, we have $T < +\infty$. Then the proof of Step 2 is obtained by showing the following claims:

- Claim 2a. Symmetric inclusion holds at $T$ with away contact or with close contact.
- Claim 2b. Symmetric inclusion cannot hold with close contact and no away contact.

Proof of Claim 2a. Symmetric inclusion clearly continues to hold at $T$. Moreover, by definition of $T$, at least one of the following assertions is true:

(i) $\exists \{ t_n \} \to T^+$ such that $\forall n$ symmetric inclusion with away contact holds at $t_n$;

(ii) $\exists \{ t_n \} \to T^+$ such that $\forall n$ symmetric inclusion does not hold at $t_n$.

In case (i), for every $n$ there exists an away contact point at $t_n$, namely a point $p_n' \in \overline{\partial^* R_{t_n} \cap \partial^* \Omega} \setminus H_{t_n}$. Up to a subsequence, denote by $p_n'$ the limit of $p_n'$. Two cases may occur. If $p_n' \notin H_T$, then $p_n'$ is an away contact point at $T$. If $p_n' \in H_T$, denoting by $p_n$ the symmetric of $p_n'$ about $H_{t_n}$, taking $q_{1,n} = p_n$ and $q_{2,n} = p_n'$ in (16), we see that $p_n'$ is a close contact point.

To deal with case (ii), we point out the validity of the following

Away inclusion property: If symmetric inclusion occurs without away contact at $T$, for every $\delta > 0$, there exists $s_\delta > 0$ such that, for every $s \in [0, s_\delta]$ the set

$$U_{T-\delta}^s := \{ x + (2\delta + 2s)\nu : x \in \partial^* R_{T-\delta} \}$$

is contained into $\Omega$.

The away inclusion property can be easily proved by contradiction. If it was false, we could find an infinitesimal sequence $\{ s_n \}$ of positive numbers, and a sequence of points $\{ x_n' \}$ of density 1 for $U_{T-\delta}^s$ but of density 0 for $\Omega$. Up to a subsequence, there exists $x' := \lim_n x_n'$. By construction, we have $x' \in \overline{\partial^* R_{T-\delta}} \subset \partial^* R_T$. But, since we are assuming that symmetric inclusion occurs without away contact at $T$, it is readily checked that $\overline{\partial^* R_T} \subseteq \text{int}(\Omega)$. Then $x' \in \text{int}(\Omega)$, against the fact that $x_n'$ are points of density 0 for $\Omega$.

Now, going back to case (ii), we can assume that symmetric inclusion occurs at $T$ without away contact (otherwise Claim 2a holds for free). Then, in view of the away inclusion property, the failure of symmetric inclusion at $t_n$ implies that, for every $n$ and every $\delta > 0$, there exist at least two distinct points in $\overline{\partial^* \Omega}$, say $q_{1,\delta,n}$ and $q_{2,\delta,n}$, which have the same orthogonal projection onto $H_T$ and have distance less than $\delta$ from $H_T$. By the arbitrariness of $\delta > 0$, we can choose a diagonal sequence, and passing to the limit we get a close contact point at $T$ according to definition (16).

Proof of Claim 2b. Assume by contradiction that symmetric inclusion holds at $T$ with close contact. We are going to contradict (11) by showing that, if $\{ q_{1,n} \}$ and $\{ q_{2,n} \}$ are sequences converging to a point $q \in H_T$ as in (16), it holds

$$\liminf_{n \to +\infty} \frac{|\Omega \cap B_r(q_{1,n})| - |\Omega \cap B_r(q_{2,n})|}{\|q_{1,n} - q_{2,n}\|} > 0.$$
We have
\[
|\Omega \cap B_r(q_{1,n})| - |\Omega \cap B_r(q_{2,n})| = \\
(25)
|\Omega \cap (B_r(q_{1,n}) \setminus B_r(q_{2,n}))| - |\Omega \cap (B_r(q_{2,n}) \setminus B_r(q_{1,n}))|.
\]
In order to estimate the two terms at the r.h.s. of (25), we fix \(\delta > 0\) (to be chosen later),
and we let \(s_n > 0\) be such that \(H_{T+s_n}\) contains the midpoint of the segment \((q_{1,n}, q_{2,n})\).
Up to working with \(n\) large enough, since \(q_{1,n}\) and \(q_{2,n}\) converge to a point of \(H_T\), thanks
to the away inclusion property we can assume that
\[
U_T^{s_n} \subset \Omega.
\]
Hence,
\[
|\Omega \cap (B_r(q_{1,n}) \setminus B_r(q_{2,n}))| = |U_T^{s_n} \cap (B_r(q_{1,n}) \setminus B_r(q_{2,n}))| \\
(27)
+ |(\Omega \setminus U_T^{s_n}) \cap (B_r(q_{1,n}) \setminus B_r(q_{2,n}))|.
\]
On the other hand,
\[
|\Omega \cap (B_r(q_{2,n}) \setminus B_r(q_{1,n}))| = |\Omega_{T-\delta} \cap (B_r(q_{2,n}) \setminus B_r(q_{1,n}))| \\
(28)
+ |\Omega \cap (H_{T+s_n} \oplus B_\delta_{s_n}) \cap (B_r(q_{2,n}) \setminus B_r(q_{1,n}))|
\]
Here and below, we denote by \(H \oplus B_R(0)\) the strip given by points of \(\mathbb{R}^d\) with distance
less than \(R\) from a hyperplane \(H\).
The two sets \(\Omega_{T-\delta} \cap (B_r(q_{2,n}) \setminus B_r(q_{1,n}))\) and \(U_T^{s_n} \cap (B_r(q_{1,n}) \setminus B_r(q_{2,n}))\) have the same
measure as they are symmetric about the hyperplane \(H_{T+s_n}\). Therefore, by subtracting (28) from (27), and recalling (25), we obtain
\[
|\Omega \cap B_r(q_{1,n})| - |\Omega \cap B_r(q_{2,n})| = I_n - J_n,
\]
with
\[
\begin{cases}
I_n := |(\Omega \setminus U_T^{s_n}) \cap (B_r(q_{1,n}) \setminus B_r(q_{2,n}))| \\
J_n := |\Omega \cap (H_{T+s_n} \oplus B_\delta_{s_n}) \cap (B_r(q_{2,n}) \setminus B_r(q_{1,n}))|
\end{cases}
\]
We are now going to estimate \(J_n\) from above and \(I_n\) from below. We have
\[
J_n \leq |(H_{T+s_n} \oplus B_\delta_{s_n}) \cap (B_r(q_{2,n}) \setminus B_r(q_{1,n}))|.
\]
In turn, the right hand side of the above inequality does not exceed the measure of the
region \(D_n\) obtained as the difference between two right cylinders having both as axis
the straight line containing \(q_{1,n}\) and \(q_{2,n}\), as height \(\delta + s_n\), and as bases the \((d-1)\)-
dimensional balls obtained as intersecting \(H_{T-\delta}\) respectively with \(B_r(q_{2,n})\) and \(B_r(q_{1,n})\).
The measure of such region \(D_n\) satisfies (cf. the proof of Lemma 14)
\[
|D_n| \sim 2(d-1)\omega_{d-1} r^{d-3}(t_{2,n} \gamma_n + \gamma_n^2) (t_{2,n} + \gamma_n),
\]
where \(\gamma_n\) is the distance of \(q_{1,n}\) and \(q_{2,n}\) from \(H_{T+s_n}\) (or equivalently, \(2\gamma_n\) is the distance
between \(q_{1,n}\) and \(q_{2,n}\)), and \(t_{2,n} = \delta + s_n - \gamma_n\) is the distance of \(q_{2,n}\) from \(H_{T-\delta}\). We infer that
\[
J_n \leq 8(d-1)\omega_{d-1} r^{d-3} \delta^2 \gamma_n,
\]
where the last inequality holds because \(s_n \leq \delta\) for \(n\) large enough.
We now turn to estimate \( I_n \).
Let us begin by proving that:
\[
\exists \delta_0 > 0 : \inf_n |\Omega \cap H^+_{T+\delta_0+2s_n} \cap (B_r(q_{1,n}) \setminus B_r(q_{2,n}))| > 0, \tag{31}
\]
\[
\exists \eta > 0 : \inf_n \text{dist}(U^s_{T-\delta_0}, \partial^\prime \Omega \cap (B_r(q_{1,n}) \setminus B_r(q_{2,n}))) \geq \eta, \tag{32}
\]
where \( \text{dist} \) stands for the distance in the halfspace \( H^+_{T+\delta_0+2s_n} \).
If the infimum in (31) is zero, by virtue of (26) we obtain
\[
\delta \text{ equality is infinitesimal as } \delta \to 0^+, \text{ while the right hand side is controlled from below by a positive constant thanks to the nondegeneracy assumption.}
\]
If (32) was false we could find a sequence \( \{x_n\} \subset \partial^\prime \Omega \cap (B_r(q_{1,n}) \setminus B_r(q_{2,n})) \) such that \( \lim_n \text{dist}(U^s_{T-\delta_0}, x_n) = 0 \). This is not possible because, up to a subsequence, the limit of \( \{x_n\} \) would provide an away contact point at \( T \), against our assumption.

Now, thanks to (32), we can consider a geodesic curve minimizing the distance between \( U^s_{T-\delta_0} \) and \( \partial^\prime \Omega \cap (B_r(q_{1,n}) \setminus B_r(q_{2,n})) \) inside \( (B_r(q_{1,n}) \setminus B_r(q_{2,n})) \cap H^+_{T+\delta_0+2s_n} \). We take the mid-point, say \( y_n \), and we consider the ball \( B_{2}\gamma(y_n) \). The set \( B_{2}\gamma(y_n) \cap (B_r(q_{1,n}) \setminus B_r(q_{2,n})) \) does not intersect \( U^s_{T-\delta_0} \) and, by (26) and (31), it is contained into \( \Omega \cap (B_r(q_{1,n}) \setminus B_r(q_{2,n})) \). After noticing that this construction continues to work for all \( \delta < \delta_0 \), we infer that there exists a positive constant \( K \) such that
\[
(33) \quad I_n \geq K \gamma_n.
\]
By (29), (30), and (33) we conclude that, up to taking \( \delta_0 \) smaller, it holds
\[
\lim_{n \to +\infty} \inf \frac{|\Omega \cap B_r(q_{1,n})| - |\Omega \cap B_r(q_{2,n})|}{\|q_{1,n} - q_{2,n}\|} > \frac{3K}{4}.
\]
Thus (24) holds true and the proof of Step 2 is achieved.

### 3.3. Proof of Step 3

By Step 2, we know that at \( t = T \) symmetric inclusion occurs with away contact. The proof of Step 3 is obtained by showing the following claims:

- **Claim 3a.** \( p' \) is an away contact point and \( p \) is its symmetric about about \( H_T \).

  \[
  (34) \quad |[B_r(p') \setminus B_r(p)] \cap [\Omega \setminus R_T]| = 0, \text{ and hence } |(B_r(p') \setminus B_r(p)) \cap \Omega^s| = 0;
  \]

- **Claim 3b.** Properties (17) and (18) hold.

**Proof of Claim 3a.** We have:
\[
[B_r(p') \setminus B_r(p)] \cap (\Omega \setminus R_T)
= [B_r(p') \cap (\Omega \setminus (\Omega_T \cup R_T))] - [B_r(p) \cap (\Omega \setminus (\Omega_T \cup R_T))]
= [B_r(p') \cap \Omega] - [B_r(p') \cap (\Omega_T \cup R_T)] - [B_r(p) \cap \Omega] + [B_r(p) \cap (\Omega_T \cup R_T)] = 0,
\]
where the first equality holds since \(B_r(p') \setminus B_r(p)\) does not intersect \(\Omega_T\), while in the last one we have used \(r\)-criticality and the fact that the two sets \(B_r(p') \cap (\Omega_T \cup R_T)\) and \(B_r(p) \cap (\Omega_T \cup R_T)\) are reflected of each other about \(H_T\). We have thus proved (34).

In view of (31), the equality (35) is immediate in case \(p' \notin \overline{B}_r(p)\). Therefore, we may prove it having in mind that \(p' \in \overline{B}_r(p)\). We claim that

\[
0 < |[B_r(p') \setminus B_r(p)] \cap \Omega| < |B_r(p') \setminus B_r(p)|.
\]

Indeed, let us exclude both the equalities

\[
|[B_r(p') \setminus B_r(p)] \cap \Omega| = 0 \quad \text{and} \quad |[B_r(p') \setminus B_r(p)] \cap \Omega| = |B_r(p') \setminus B_r(p)|.
\]

The former cannot hold since \(\Omega\) is not \(r\)-degenerate. The latter, in view of (31), would imply that \(B_r(p') \setminus B_r(p)\) is contained into \(R_T\), and hence \(B_r(p) \setminus B_r(p')\) is contained into \(\Omega_T\). Since \(\Omega_T \cup R_T\) is Steiner-symmetric about \(H_T\), this would give (via Fubini Theorem) that \(p\) and \(p'\) belong to \(\text{int}(\Omega(1))\), contradicting the fact that they belong to \(\partial^* \Omega\).

As a consequence of (36), we observe that

\[
\exists y' \in [B_r(p') \setminus \overline{B}_r(p)] \cap \overline{\partial^* \Omega}.
\]

Indeed, if (37) was false, \(B_r(p') \setminus \overline{B}_r(p)\) would be contained either into \(\text{int}(\Omega(1))\) or into \(\text{int}(\Omega(0))\), against (36). Next we observe that, in view of (31), the two sets \(\Omega\) and \(R_T\) have the same density at every point of \(B_r(p') \setminus \overline{B}_r(p)\), and hence

\[
[B_r(p') \setminus \overline{B}_r(p)] \cap \partial^* \Omega = [B_r(p') \setminus \overline{B}_r(p)] \cap \partial^* R_T;
\]

consequently, since the set \(B_r(p') \setminus \overline{B}_r(p)\) is open, we have

\[
[B_r(p') \setminus \overline{B}_r(p)] \cap \partial^* \Omega = [B_r(p') \setminus \overline{B}_r(p)] \cap \partial^* R_T.
\]

By (37) and (38), it turns out that \(y'\) is itself an away contact point. Therefore, denoting by \(y\) its symmetric about \(H_T\), in the same way as we obtained (31), replacing the pair \(p,p'\) by the pair \(y,y'\), we get

\[
|[B_r(y') \setminus B_r(y)] \cap (\Omega \cap R_T)| = 0.
\]

Moreover, since the set \(B_r(p') \setminus \overline{B}_r(p)\) is open, for \(\varepsilon > 0\) sufficiently small the ball \(B_\varepsilon(y')\) is contained into \(B_r(p') \setminus \overline{B}_r(p)\), and hence

\[
\exists \varepsilon > 0 : B_\varepsilon(p') \subset [B_r(y') \setminus \overline{B}_r(y)].
\]

By (39) and (40), (35) is proved.

**Proof of Claim 3b.** In order to prove (17) - (18), we consider the subsets of \(H_T\) defined by

\[
C_T := \left\{ m_{(p,p')} : p' \text{ is an away contact point, } p \text{ is its symmetric about } H_T \right\},
\]

\[
A_T := \left\{ z \in H_T : g(z) < 0 \right\},
\]

where \(m_{(p,p')} \in H_T\) denotes the mid-point of the segment \((p,p')\), and the function \(g\) is defined as in Proposition (13) (iv) (applied with \(\omega := \Omega_T\) and \(H := H_T\)). By Proposition (13) (iv), we know that \(g\) is continuous and hence the set \(A_T\) turns out to be open. Since Claim 3a. implies that \(C_T\) is open in \(H_T\), we infer that \(C_T\) is relatively
open in \(A_T\). On the other hand, since \(\partial^\nu \Omega\) is a closed set, it is easy to check that \(C_T\) is also relatively closed in \(A_T\). Hence, \(C_T\) consists in a non-empty union of connected components of \(A_T\). Accordingly, \(\partial^\nu \Omega \cap (H^-_T \setminus H_T)\) (resp., \(\partial^\nu \Omega \cap (H^+_T \setminus H_T)\)) is a union of connected sets, which are the images of the open connected components of \(C_T\) through the continuous function \(g\) (resp., the reflections of such images about \(H_T\)). Each of these connected sets corresponds to \(\partial^\nu \Omega_r^{\nu, s}_i \cap (H^-_T \setminus H_T)\) (resp., \(\partial^\nu \Omega_r^{\nu, s}_i \cap (H^+_T \setminus H_T)\)) for some open connected component \(\Omega^{\nu, s}_i\) of \(\Omega^s\). This proves (17). Since by (55) none of the sets \(\partial^\nu \Omega^{\nu, s}_i \cap (H^\pm_T \setminus H_T)\) can intersect \(\partial^\nu \Omega^{\nu, ns}_i\), (18) follows.

### 3.4. Proof of Step 4

Relying on decomposition \(\Omega = \Omega^s \sqcup \Omega^{\nu, ns}\) made in Step 3, we are going to analyze in detail the behaviour of the open connected components \(\Omega^s_i\) of \(\Omega^s\). To that aim, we need to set up some additional definitions and notation.

Given two two different open connected components \(\Omega_i^s, \Omega_j^s\) of \(\Omega^s\), we say that \(\Omega_i^s\) is in \(r\)-contact with \(\Omega_j^s\) if there exists an away contact point \(p' \in \partial^\nu \Omega_i^s \setminus H_T\) such that, denoting by \(p\) its symmetric about \(H_T\), it holds

\[
| \left( (B_r(p) \Delta B_r(p')) \cap \Omega^s_j \right) | > 0.
\]

It is not difficult to check that, if \(\Omega_i^s\) is in \(r\)-contact with \(\Omega_j^s\), \(\Omega_j^s\) is in \(r\)-contact with \(\Omega_i^s\).

If \(\Omega_i^s\) is not in contact with any other component of \(\Omega^s\), we say that \(\Omega_i^s\) is \(r\)-isolated. Since our strategy will require to let the initial hyperplane vary, we will write

\[
\Omega = \Omega^{\nu, s} \sqcup \Omega^{\nu, ns},
\]

where the additional superscript \(\nu\) indicates the direction of the parallel movement, namely the normal to the initial hyperplane \(H_0\) (and the decomposition is always meant with respect to the parallel hyperplane \(H_T\) at the stopping time \(T\) defined in Step 2).

The proof of Step 4 is achieved by showing the following claims:

- **Claim 4a.** Given \(\nu \in S^{d-1}\), let \(\Omega\) be a \(r\)-isolated open connected component of \(\Omega^{\nu, s}\), then \(\Omega\) is a ball of radius at least \(r/2\), and \(\Omega \setminus \Omega\) is \(r\)-critical and not \(r\)-degenerate, unless it has measure zero.

- **Claim 4b.** The following family is empty:

\[
\mathcal{F} := \bigcup_{\nu \in S^{d-1}} \left\{ \text{open connected components not } r\text{-isolated of } \Omega^{\nu, s} \right\}.
\]

- **Claim 4c (conclusion).** \(\Omega\) is equivalent to a finite union of balls of radius \(R > r/2\), at mutual distance larger than or equal to \(r\).

**Proof of claim 4a.** Given \(\nu \in S^{d-1}\), let \(\Omega\) be a \(r\)-isolated open connected component of \(\Omega^{\nu, s}\). Assume by a moment to know that

\[
\Omega \setminus \Omega\text{ is } r\text{-critical and not } r\text{-degenerate.}
\]

In this case, we can restart our proof, with \(\Omega\) in place of \(\Omega\). Given an arbitrary direction \(\tilde{\nu} \in S^{d-1}\), we make the decomposition

\[
\Omega = \Omega^{\tilde{\nu}, s} \sqcup \Omega^{\tilde{\nu}, ns}.
\]

We are going to show that, unless \(\Omega^{\tilde{\nu}, ns}\) is empty, this decomposition splits \(\Omega\) into two open sets, contradicting the connectedness of \(\Omega\). Hence \(\Omega\) is Steiner symmetric about
a hyperplane with unit normal \( \tilde{\nu} \). By the arbitrariness of \( \tilde{\nu} \), we deduce that \( \Omega_\nu \) is a ball. (Indeed, since \( \overline{\Omega}_\nu \) is a compact set, following [35], there exists a sequence of Steiner symmetrizations of it converging to a ball; but since \( \Omega_\nu \) is already Steiner symmetric in every direction, it must coincide with such ball). Since \( \Omega_\nu \) is not \( r \)-degenerate, the radius of the ball is strictly larger than \( r/2 \).

Assuming that \( \Omega_\nu^{\tilde{\nu},ns} \) is not empty, let us show that every point of \( \Omega_\nu \) is in the interior of one among the two sets \( \Omega_\nu^\tilde{\nu},s \) and \( \Omega_\nu^\tilde{\nu},ns \). Let us denote by \( \tilde{T} \) the stopping time defined as in Step 2 for the parallel movement with normal \( \tilde{\nu} \). Recall from [15] that

\[
\partial^*\Omega_\nu^{\tilde{\nu},s} \cap \partial^*\Omega_\nu^{\tilde{\nu},ns} \subseteq H_{\tilde{T}},
\]

Let us consider separately the cases when \( x \in \Omega_\nu \setminus H_{\tilde{T}} \) and when \( x \in \Omega_\nu \cap H_{\tilde{T}} \).

Let \( x \in \Omega_\nu \setminus H_{\tilde{T}} \). Since \( \Omega_\nu \) is open, there exists a ball \( B_\epsilon(x) \) contained into \( \Omega_\nu \setminus H_{\tilde{T}} \). It cannot be \( 0 < |\Omega_\nu^{\tilde{\nu},s} \cap B_\epsilon(x)| < |B_\epsilon(x)| \). Otherwise, by Federer’s Theorem, \( B_\epsilon(x) \) would contain points of \( \partial^*\Omega_\nu^{\tilde{\nu},s} \cap \partial^*\Omega_\nu^{\tilde{\nu},ns} \), against (42). We deduce that \( B_\epsilon(x) \) is contained either into \( \Omega_\nu^{\tilde{\nu},s} \) or into \( \Omega_\nu^{\tilde{\nu},ns} \), namely \( x \) is an interior point for one among \( \Omega_\nu^{\tilde{\nu},s} \) and \( \Omega_\nu^{\tilde{\nu},ns} \).

Let now \( x \in \Omega_\nu \cap H_{\tilde{T}} \), and let \( B_\epsilon(x) \) be a ball contained into \( \Omega_\nu \). By the same arguments as above, each of the two sets \( B_\epsilon(x) \cap (H_{\tilde{T}} \setminus H_{\tilde{T}}) \) and \( B_\epsilon(x) \cap (H_{\tilde{T}} \setminus H_{\tilde{T}}) \) must be entirely contained either into \( \Omega_\nu^{\tilde{\nu},s} \) or into \( \Omega_\nu^{\tilde{\nu},ns} \). Recalling that \( \Omega_\nu^{\tilde{\nu},s} \) is Steiner symmetric about \( H_{\tilde{T}} \), we infer that either both sets are contained into \( \Omega_\nu^{\tilde{\nu},s} \), or both sets are contained into \( \Omega_\nu^{\tilde{\nu},ns} \). Then, also in this case \( x \) is an interior point for one among \( \Omega_\nu^{\tilde{\nu},s} \) and \( \Omega_\nu^{\tilde{\nu},ns} \).

To conclude the proof of Claim 4a., it remains to show that (41) holds true and that the same property is valid for \( \Omega \setminus \Omega_\nu \), unless it has measure zero. For the sake of clearness, this will be obtained as the final product of three consecutive lemmas.

**Lemma 15.** Given \( \nu \in S^{d-1} \), let \( \Omega_\nu \) be a \( r \)-isolated open connected component of \( \Omega^{\nu,s} \). Then

\[
\inf_{x_1,x_2 \in \partial^*\Omega_\nu} \frac{|\Omega^{\nu,s} \cap (B_r(x_1) \Delta B_r(x_2))|}{\|x_1 - x_2\|} > 0.
\]

**Proof.** Assume by contradiction that

\[
\inf_{x_1,x_2 \in \partial^*\Omega_\nu} \frac{|\Omega^{\nu,s} \cap (B_r(x_1) \Delta B_r(x_2))|}{\|x_1 - x_2\|} = 0.
\]

Then there exist sequences of distinct points \( \{x_{1,n}\}, \{x_{2,n}\} \subset \partial^*\Omega_\nu \), with \( \|x_{1,n} - x_{2,n}\| \to 0 \), such that

\[
\frac{|\Omega^{\nu,s} \cap (B_r(x_{1,n}) \Delta B_r(x_{2,n}))|}{\|x_{1,n} - x_{2,n}\|} \to 0.
\]

Up to subsequences, we may assume that \( \|x_{1,n} - x_{2,n}\| \) converges to 0 decreasingly, and that \( \{x_{1,n}\} \) and \( \{x_{2,n}\} \) converge to some point \( \pi \in \partial^*\Omega_\nu \), which may belong or not to \( H_{\tilde{T}} \), being as usual \( T \) the stopping time defined as in Step 2 for the parallel movement with normal \( \nu \). Let us examine the two cases separately.
In case $\mathfrak{T} \notin H_T$, we may assume without loss of generality that $\{x_1,n\}, \{x_2,n\} \in H^+_T \setminus H_T$. Recall that, by (17), the set $\partial^+\Omega_2 \cap (H^+_T \setminus H_T)$ is connected. Hence for every $n \geq 1$ we can join $x_{1,n}$ to $x_{1,n+1}$ by a continuous arc $\gamma_{1,n}(s)$ contained into $\partial^+\Omega_1 \cap (H^+_T \setminus H_T)$. We can repeat the same procedure for the second sequence, constructing a family of continuous arcs $\gamma_{2,n}(s)$ joining $x_{2,n}$ to $x_{2,n+1}$ for every $n \geq 1$.

We look at the boundaries of the balls of radius $r$ whose centre moves along $\gamma_{1,n}(s)$ and $\gamma_{2,n}(s)$. Clearly these balls tends to superpose in the limit as $n \to +\infty$, since $\|x_{1,n} - x_{2,n}\|$ decreases to 0. Moreover, we know from (34) that, during the continuous movement of their centre along along $\gamma_{1,n}(s)$ and $\gamma_{2,n}(s)$, the boundary of these balls cannot cross points of density 1 for $\Omega^{\nu,ns}$. This property will give us the required contradiction. More precisely, we argue as follows. Since $\Omega$ is not $r$-degenerate, (13) implies

$$\inf_{x_1, x_2 \in \partial^+\Omega_r} \frac{|\Omega^{\nu,ns} \cap (B_r(x_1) \Delta B_r(x_2))|}{\|x_1 - x_2\|} > 0.$$  

In particular, for $n = 1$, we have $|\Omega^{\nu,ns} \cap (B_r(x_{1,1}) \Delta B_r(x_{2,1}))| > 0$. Hence we can pick a point $p \in \text{int}(B_r(x_{1,1}) \Delta B_r(x_{2,1}))$ of density 1 for $\Omega^{\nu,ns}$, and a radius $\varepsilon > 0$ sufficiently small so that

$$|B_{\varepsilon}(p) \cap \Omega^{\nu,ns}| \geq \frac{1}{2} |B_{\varepsilon}(p)|.$$  

Possibly reducing $\varepsilon$ we can also assume that $B_{\varepsilon}(p) \subseteq (B_r(x_{1,1}) \Delta B_r(x_{2,1}))$. Recalling that the boundaries of the balls of radius $r$ whose centre moves along the continuous arcs $\gamma_{1,n}(s)$ and $\gamma_{2,n}(s)$ cannot meet $\Omega^{\nu,ns}$, we infer that, for $n$ large,

$$B_{\varepsilon}(p) \cap \Omega^{\nu,ns} \subseteq B_r(x_{1,1}) \Delta B_r(x_{2,n});$$

hence, still for $n$ sufficiently large,

$$|B_{\varepsilon}(p) \cap \Omega^{\nu,ns}| \leq |B_r(x_{1,1}) \Delta B_r(x_{2,n})| < \frac{1}{4} |B_{\varepsilon}(p)|,$$

against (13).

In case $\mathfrak{T} \in H_T$, we proceed in the same way, except that we cannot ensure any more that both sequences $\{x_{1,n}\}$ and $\{x_{2,n}\}$ belong to the same halfspace $H^+_T$ or $H^-_T$. Thus, when we construct the continuous arcs $\gamma_{1,n}$ and $\gamma_{2,n}$, they may belong indistinctly to $\partial^+\Omega_2 \cap (H^+_T \setminus H_T)$ or to $\partial^+\Omega_1 \cap (H^+_T \setminus H_T)$, but this does not affect the validity of the proof since the contradiction follows as soon as $x_{1,n}$ and $x_{2,n}$ are close enough.

Lemma 16. Given $\nu \in \mathbb{S}^{d-1}$, let $\Omega_2$ be a $r$-isolated open connected component of $\Omega^{\nu,s}$. There exists a constant $c_0 > 0$ such that

$$|\Omega^{\nu,s} \cap B_r(x)| = c_0 \quad \forall x \in \partial^+\Omega_2.$$  

Moreover, the constant is the same for any other open connected component of $\Omega^{\nu,s}$ such that the closure of its essential boundary intersects $\partial^+\Omega_2$.

Proof. We argue in a similar way as in the proof of the previous lemma. Given $x_1, x_2 \in \partial^+\Omega_2 \cap (H^+_T \setminus H_T)$, by (17), they can be joined by a continuous arc $\gamma(s)$ contained into $\partial^+\Omega_2 \cap (H^+_T \setminus H_T)$. By (34), the boundary of the ball of radius $r$ centred at any point along $\gamma(s)$ cannot cross points of density 1 for $\Omega^{\nu,ns}$. We deduce that $B_r(x_1) \Delta B_r(x_2)$ cannot
contain points of density 1 for $\Omega^{\nu,s}$. Since $\Omega$ is $r$-critical, it follows that $|\Omega^{\nu,s} \cap B_r(x_1)| = |\Omega^{\nu,s} \cap B_r(x_2)|$. By the arbitrariness of $x_1, x_2$, we infer that there exists a constant $c_{s}^+ > 0$ such that $|\Omega^{\nu,s} \cap B_r(x)| = c_{s}^+$ for every $x \in \partial^s \Omega \cap (H^+_T \setminus H_T)$. In the same way, we obtain that there exists a constant $c_{s}^- > 0$ such that $|\Omega^{\nu,s} \cap B_r(x)| = c_{s}^-$ for every $x \in \partial^s \Omega \cap (H^-_T \setminus H_T)$. Since the two sets $\partial^s \Omega \cap H^\pm_T$ have common points on $H_T$, we conclude that $c_{s}^+ = c_{s}^-$. The same argument proves also the last assertion of the lemma.

\textbf{Lemma 17.} Given $\nu \in \mathbb{S}^{d-1}$, let $\Omega_\nu$ be a $r$-isolated open connected component of $\Omega^{\nu,s}$. Then $\Omega_\nu$ is and $r$-critical and not $r$-degenerate. The same assertions hold true for its complement $\Omega \setminus \Omega_\nu$, unless it is of measure zero.

\textbf{Proof.} The fact that $\Omega_\nu$ is not $r$-degenerate follows from Lemma 15 and the assumption that $\Omega_\nu$ is $r$-isolated. From equality (45) in Lemma 16, and the assumption that $\Omega$ itself is not, combined with the fact that points of $\Omega_\nu$ and of $\Omega \setminus \Omega_\nu$ cannot lie at distance smaller than $r$ (again by (45) and the assumption that $\Omega_\nu$ is $r$-isolated). Finally, it holds $|(\Omega \setminus \Omega_\nu) \cap B_r(x)| = c$ for every $x \in \partial^r(\Omega \setminus \Omega_\nu)$, namely $\Omega \setminus \Omega_\nu$ is $r$-critical. \qed

\textbf{Proof of Claim 4b.} As a preliminary remark, we observe that the family $\mathcal{F}$ is at most countable. This is an immediate consequence of the fact that any open set of $\mathbb{R}^d$ has at most countable connected components, and of the fact that, for two different directions $\nu_1$ and $\nu_2$, it is not possible that a connected component of $\Omega^{\nu_1,s}$ intersects a connected component of $\Omega^{\nu_2,s}$ without being equal.

We now prove Claim 4b. by contradiction.

First of all let us show that, if the family $\mathcal{F}$ is not empty, it contains an element $\Omega_\nu$ which is Steiner symmetric about $d$ hyperplanes with linearly independent normals $\nu_1, \ldots, \nu_d$. Indeed, let $S_k$ denote the family of linear subspaces of dimension $k$ in $\mathbb{R}^d$. For every $k = d-2, d-3, \ldots, 1$, we are going to associate with a given subspace $V \in S_k$ an element of $\mathcal{F}$, which will be denoted by $\Omega^V_k$. These mappings

\begin{equation}
S_k \ni V \rightarrow \Omega^V_k \in \mathcal{F}
\end{equation}

are constructed as follows.

For $k = d-2$, given $V \in S_{d-2}$, we consider all the subspaces $\widetilde{V} \in S_{d-1}$ containing $V$. For every such $\widetilde{V}$, denoting by $\tilde{\nu}$ the normal direction to $\widetilde{V}$, we perform the decomposition $\Omega^{\tilde{\nu},s} \cup \Omega^{\tilde{\nu},ns}$. Since $\mathcal{F}$ is at most countable, there exist two distinct subspaces $\tilde{V}_1$ and $\tilde{V}_2$ in $S_{d-1}$ such that the corresponding symmetric parts $\Omega^{\tilde{V}_1,s}$ and $\Omega^{\tilde{V}_2,s}$ share some open connected component. We pick one among such shared connected components and we associate it with $V$, denoting it by $\Omega^V_k$. Notice that neither the spaces $\tilde{V}_1$, $\tilde{V}_2$ nor the shared connected component are unique, so the definition is made by choice.
For $k = d - 3$, given $V \in \mathcal{S}_{d-3}$, we consider all subspaces $\tilde{V} \in \mathcal{S}_{d-2}$ containing $V$. Since the image of the mapping in (40) previously defined for $k = d - 2$ is at most countable, there exist two distinct subspaces $\tilde{V}_1$ and $\tilde{V}_2$ in $\mathcal{S}_{d-2}$ such that $\Omega^d_{\tilde{V}_1} = \Omega^d_{\tilde{V}_2}$. We set
\[ \Omega^d_{\tilde{V}} := \Omega^d_{\tilde{V}_1} = \Omega^d_{\tilde{V}_2}. \]
We continue the process until we define the map in (41) for $k = 1$. Arguing as above, we find two distinct $\tilde{V}_1$ and $\tilde{V}_2$ in $\mathcal{S}_1$ such that $\Omega^1_{\tilde{V}_1} = \Omega^1_{\tilde{V}_2}$. We set
\[ \Omega := \Omega^1_{\tilde{V}_1} = \Omega^1_{\tilde{V}_2}. \]
By construction $\Omega$ is Steiner symmetric with respect to $d$ hyperplanes with independent normals $\nu_1, \ldots, \nu_d$.

Next we consider any other element $\Omega_{d^*}$ of $\mathcal{F}$ which is in $r$-contact with $\Omega$ in the decomposition with respect to one among the directions $\nu_1, \ldots, \nu_d$, say $\nu_1$. If $T_1$ is the stopping time for the parallel movement with normal $\nu_1$, there exist $p, p' \in \partial^\ast\Omega_{d^*} \cap H_{T_1}$, symmetric about $H_{T_1}$, such that
\[ |(B_r(p)\Delta B_r(p')) \cap \Omega_{d^*}| > 0. \]
Since we are assuming that $\Omega_{d^*}$ is Steiner symmetric with respect to $H_{T_1}$, the above inequality implies that $\partial B_r(p)$ contains points of density 1 for $\Omega_{d^*}$. In particular, this implies that $\Omega_{d^*}$ is itself Steiner symmetric about the same hyperplanes as $\Omega$ is.

Then, Lemma 18 below implies that the set $\Omega \cup \Omega_{d^*}$ is connected, yielding a contradiction.

**Lemma 18.** Assume that $\omega \subseteq \mathbb{R}^d$ is a bounded open set, Steiner symmetric about $d$ hyperplanes whose normals are linearly independent. Then $\omega$ is a connected set containing its centre of mass.

**Proof.** Let us denote the hyperplanes by $H_1, \ldots, H_d$, and by $\Pi_{H_k}(x)$ the orthogonal projection from $\mathbb{R}^d$ onto $H_k$, for $k = 1, \ldots, d$. Starting from a fixed point $x_0 \in \omega$, let us consider the sequence of points defined by $x_n := \Pi_{H_k}(x_{n-1})$ if $n = k \mod d$. It is easy to check that $\{x_n\}$ converges to the centre of mass $G$ of $\omega$. In fact, let us assume without losing generality that $G$ is at the origin. If $\alpha_n \in (0, \pi/2]$ is the angle between the normals to the hyperplanes $H_n$ and $H_{n+1}$ (obtained by cyclically repeating $H_1, \ldots, H_d$), and $d_n$ is the distance of $x_n$ to $H_n \cap H_{n+1}$, we have that $\|x_{n+1}\|^2 = \|x_n\|^2 - d_n^2 \sin^2(\alpha_n)$. Then $\|x_n\|^2$ is decreasing and $\{d_n\}$ converges to 0, since $\{\alpha_n\}$ is a periodic sequence of strictly positive numbers. This readily implies that $\text{dist}(x_n, H_k) \to 0$, for every $k = 1, \ldots, d$, and hence $x_n \to 0$.

Next we observe that, since $\omega$ is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(x_0) \subseteq \omega$. From the assumption that $\omega$ is Steiner symmetric about $H_1, \ldots, H_d$, we get that $B_\varepsilon(x_n) \subseteq \omega$ for every $n$ and, more in general, that
\[ \bigcup_{n \geq 1} \left( [x_{n-1}, x_n] \oplus B_\varepsilon(0) \right) \subseteq \omega. \]
By (47), it turns out that $B_\varepsilon(G)$ is contained into $\omega$. Moreover, $\omega$ is connected because the initial point $x_0$ was arbitrarily chosen, and by (47) it can be joined to $G$ by a continuous path contained into $\omega$. \qed
Proof of Claim 4c. We start the procedure by choosing a direction $\nu \in S^{d-1}$. By Claim 4b., we can pick a $r$-isolated open connected component of $\Omega^\nu$, which by Claim 4a. turns out to be a ball of radius $R_1 > r/2$. We remove this ball from $\Omega$. By Claim 4a., we are left with a set $\Omega'$ which is still $r$-critical and not $r$-degenerate (unless it has measure zero). So we can restart the process with $\Omega'$ in place of $\Omega$. Again, by Claim 4b., we can pick a $r$-isolated open connected component of $(\Omega')^\nu$, which by Claim 4a. turns out to be a ball of radius $R_2 > r/2$. We remove this ball from $\Omega'$. We observe that, since the two balls of radii $R_1$ and $R_2$ that we have extracted from $\Omega$ are $r$-isolated and $r$-critical, necessarily $R_1 = R_2 =: R$, and the balls lie at distance larger than or equal to $r$ from each other. Since $\Omega$ has finite measure, we can repeat this process a finite number of times, until when we are left with a set of measure zero. $\square$

Remark 19. A technical extension of Theorem 1 is expected to hold when the kernel $\chi_{B_r(0)}$ is replaced by a radially symmetric, decreasing, non negative function $h$ satisfying suitable assumptions: any set $\Omega$ with finite measure satisfying the criticality and nondegeneracy conditions, meant as

$$\int_{\Omega} h(x-y) dy = c \quad \forall x \in \partial^* \Omega \quad \text{and} \quad \inf_{x_1, x_2 \in \partial^* \Omega} \frac{\int_{\Omega} |h(x_1-y) - h(x_2-y)| dy}{\|x_1 - x_2\|} > 0,$$

will be a finite union of balls or a single ball, depending on the structure of the level sets of $h$.

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