Ding-Iohara-Miki symmetry of network matrix models

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Abstract

Ward identities in the most general “network matrix model” from [1] can be described in terms of the Ding-Iohara-Miki algebras (DIM). This confirms an expectation that such algebras and their various limits/reductions are the relevant substitutes/deformations of the Virasoro/W-algebra for $(q, t)$ and $(q_1, q_2, q_3)$ deformed network matrix models. Exhaustive for these purposes should be the Pagoda triple-affine elliptic DIM, which corresponds to networks associated with 6d gauge theories with adjoint matter (double elliptic systems). We provide some details on elliptic $qq$-characters.

1 Introduction

Recently, basing on the previous studies in [2]-[11], we introduced [1] a generic Dotsenko-Fateev (DF) [4] network conformal matrix model, associated with the most general brane web/network (the low-energy limit of toric Calabi-Yau compactifications). The first question to ask about this theory is what is the set of the relevant “Virasoro/W- constraints”: the Ward identities, which are satisfied by its partition function. In this paper, we argue that the substitute/deformation of the CFT stress tensor, which generates these identities, is now provided by the analogues of the $q$-characters [12] in the elliptic Ding-Iohara-Miki algebra (DIM) [13, 14], as anticipated at different deformation levels in [15]-[19].

We remind [2] that there are three equivalent ways to derive Ward identities in matrix models (and other quantum field and string theory models):

(i) by making a change of integration variable [20],
(ii) by considering an average of a total derivative [21] and
(iii) by building a matrix model from a free field correlator with a given symmetry [3, 22].

The first two methods can seem identical, but in fact this is not quite true: (ii) is technically simpler (more straightforward) than (i), but instead the emerging algebraic structure is more difficult to reveal. Ideal for this task is the method (iii), which we now briefly remind.

Given a symmetry generating operator (or a set of operators) $\hat{T}$ (say, the stress tensor and higher $W$-algebra generators), one gets a set of identities

$$\left\langle \Psi \right| \hat{G}(p) \hat{T} \hat{Q} \left| \text{vac} \right\rangle = 0 \iff L(\partial_p) \left\langle \Psi \right| \hat{G}(p) \hat{T} \left| \text{vac} \right\rangle = 0$$

(1)

where

- $\left| \text{vac} \right\rangle$ is a “vacuum” state annihilated by $\hat{T}$, $\hat{T} \left| \text{vac} \right\rangle = 0$,
- $\hat{Q}$ is any “screening” operator which commutes with $\hat{T}$, $[\hat{T}, \hat{Q}] = 0$.

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- $|Ψ⟩$ is an arbitrary state usually made out of vertex operators, and
- $\hat{G}(p)$ is an intertwiner with the property $L(∂p)\hat{G}(p) = \hat{G}(p)\hat{T}$, which can be used to convert operator(s) $\hat{T}$ into differential/difference operators $L$ acting on “the time variables” $p$.

This is a very general group theoretical construction describing a partition function

$$Z(p) = \langle Ψ | \hat{G}(p) \hat{Q} | \text{vac} \rangle$$

with a given ($T$-induced) symmetry as a matrix element. Conformal matrix models [3] arise in this way when matrix elements are correlators of 2d free fields and integrals or sums over partitions (interpreted as matrix model eigenvalue integrals/sums) emerge from an explicit description of screening charges $\hat{Q}$ (which are the centralizer of $\hat{T}$) in the free field Fock space.

Figure 1: a) Type IIA brane diagram consisting of two horizontal and three vertical intersecting lines representing NS5 and D4 branes. The low energy theory in this background is 4d $\mathcal{N} = 2$ gauge theory with $SU(2)^2$ gauge group. $Λ_i$ are exponentiated complexified gauge couplings, $a^{(a)}$ are Coulomb moduli and $m_a$ are the hypermultiplet masses. b) The toric diagram of the Calabi-Yau threefold, corresponding to the 5d gauge theory with the same matter content. Edges represent two-cycles with complexified Kähler parameters $Q_i$, which play the same role as the distances between the branes in a). c) The quiver encoding the matter content of the gauge theory. $SU(2)$ gauge groups live on each node and bifundamental matter on each edge. The squares represent pairs of (anti)fundamental matter hypermultiplets.

Reversing the logic, one can start from generic network matrix model [1], associated with the toric diagram in Fig.1b),

$$Z_Γ(q,t|p) = \sum_{\{R_E\}} \prod_{V} C_{R(E'_V),R(E''_V),R(E'''_V)}(q,t|p_V)$$

associated with a planar 3-valent graph $Γ$ with triples of edges $E'_V, E''_V, E'''_V$ merging at vertices $V$. The sum goes over Young diagrams $R_E$ on the edges and the topological vertices $C$ are provided by weighted sums over 3d partitions with given boundary conditions $R(E'_V), R(E''_V), R(E'''_V)$. The weights depend on compactification parameters $q_{1,2,3}$ and on auxiliary time variables $p_V$ (their background values can be used to develop the check-operator formalism a la [26]). Usually, time variables are ascribed to edges, not vertices (and we do so...
briefly encountered above:

the model belongs to the universality class \([27, 28]\) of the double elliptic integrable systems \([29]\) and is invariant

is the elliptic case, where infinite set of Kerov parameters form a geometric progression. For generic deformation

algebraical level, more Kerov parameters \([25]\) of the same nature can also be included \([24]\). Especially important

q

simplify the formula. Two of the deformation parameters are also parameterized as

in (4) below, but the right procedure remains disputable – and in \([3]\) we absorb propagators into vertices to

satisfy the matrix integral can be described in two ways. In the free field terms they

Using the screening charges given by single-variable integrals (perhaps, Jackson

one can realize the matrix model integral as an average

\[ Z_{\Gamma_\nu}(q|p) = \prod_{\alpha} dx^{(a)}_\alpha \prod_{\alpha,\beta} \Delta(x^{(a)}_\alpha, x^{(b)}_\beta) C_{\alpha\beta} \exp \left( \sum_{\alpha,\beta,k} \frac{q}{x^{(a)}_\alpha} u_k \left( x^{(b)}_\alpha \right) \right) \]  

Here (Jackson like) integrals over \(x^{(a)}_\alpha\) substitute the sums over Young diagrams for “vertical” edges \(E_{\text{vert}}^{(a)}\) of the web-diagram \(\Gamma\) (the choice is actually dictated by the 5d/4d limit and \(N^{(a)}\) may be interpreted as the number of lines in the Young diagram, which can be arbitrary). In the example from Fig. \([1]\) we have \(n = 2, m_1 = m_2 = 2\). We put an additional index on \(\Gamma\), to remind about additional vertical/horizontal (querier) structure on the graph \(\Gamma\), implicit in the formula. The set of “Casimir” functions \(u_k(x)\) is usually adjusted to simplify the differential/difference equations \([7]\) below (we will actually use the Miwa transform, converting Casimirs into vertex operator insertions). The sums over diagrams for “horizontal” edges \(E_{\text{hor}}^{(ab)}\) are substituted by the \(q_{1,2,3}\)-dependent Vandermonde-like quantities \(\Delta(x^{(a)}_\alpha, x^{(b)}_\beta)\) which can be realized as a free field pairwise correlator of screening currents \(S^{(a)}(x)\), and the product arises as a consequence of the Wick theorem.

Screening operators. Using the screening charges given by single-variable integrals (perhaps, Jackson

sums) of the screening currents (which are exponentials of free fields)

\[ S^{(a)}(\vec{x}) = \int dx : \exp \left( \sum_{k \in \mathbb{Z}} x^k \hat{a}_{k,a} \right) : \right] \]

one can realize the matrix model integral as an average

\[ Z_{\Gamma_\nu}(p) = \left\langle 0 \left| \hat{G}(p) \prod_{\alpha,a} \left( \hat{S}^{(a)}_\alpha \right)^{N^{(a)}_\alpha} \right| 0 \right\rangle_N \]

where the vacuum state is \(N = \sum_\alpha N^{(a)}_\alpha\)-charged vacuum with respect to the Heisenberg operators \(\hat{a}_{n,a}\) and \(\hat{G}(t)\). The number of free fields \(\{a_{n,a}\}\) actually depends on the number of horizontal edges (“D-branes”) in the original graph \(\Gamma\), i.e. on the ranks of gauge groups in 4d version of the model \([1]\). We implicitly include \(N\) into the set of time-variables.

Ward identities satisfied by the matrix integral can be described in two ways. In the free field terms they are provided by the free field operators \(W_a\) that are defined to commute with the screening charges, while in terms of time variables (i.e. literally as a set of constraints imposed on the time-dependent integral) they are expressed with the help of the intertwiner \(\hat{G}(p)\):

\[ W_a(p, \partial_p) Z_{\Gamma_\nu}(p) = \left\langle 0 \left| \hat{G}(p) \hat{W}_a \hat{Q} \right| 0 \right\rangle_N = 0 \]

The question is what is the algebra formed by this set of constraints on a matrix integral. In simplest examples this is just a Borel subalgebra of Virasoro or various \(W_{m_n}\) algebras, where \(m_n\) are related by the number of horizontal edges in \(\Gamma\) (e.g. \(m_1 = m_2 = 2\) in Fig. \([1]\).
• **Toroidal algebra.** One can embed all the $W_m$-algebras associated with the set of matrix integrals of a given type into a larger algebra. For instance, in the case of Dotsenko-Fateev integrals associated with Nekrasov functions and topological vertices corresponding to all graphs, this gives rise to toroidal algebras: affine Yangians in the case of 4d Nekrasov functions \[30, 19\], Ding-Iohara-Miki (DIM) (quantum toroidal) algebra in the 5d case \[13, 14, 31\] and elliptic DIM algebra in the 6d case \[32, 9\]. Concrete quiver corresponds to a set of representations of the toroidal algebra given by a fixed number of Young diagrams. Moreover, the (refined) topological vertex can be obtained as a matrix element of the intertwining operators of the DIM algebra \[33\]. We concentrate below on the level 1 representation of DIM algebra so that there always exists a simple bosonization \[32\]. For generic levels an analogue of the free-field representation of Kac-Moody algebras \[34\] will be needed.

• **$qq$-characters.** Generalized stress tensor operators $\hat{W}_n$ can be actually understood in terms of the DIM $R$-matrices, and from this perspective they give abstract algebraic description of the $qq$-characters \[16, 17, 18\]. This construction generalizes ordinary $q$-characters for quantum groups introduced in \[12\].

• **Systems of symmetric functions.** One can associate with the set of matrix integrals/algebra a set of symmetric functions in two different ways. One option is to construct them directly from the integral, omitting one set of integrations \[35, 36\]. In the simplest example of the matrix integral with $n = 1$ we fix the Young diagram $\lambda$ with lengths of lines $\lambda_\alpha$, $\alpha = 1 \ldots (m - 1)$ and the corresponding symmetric function of variables $x_i \equiv x_i^{(N)}$ corresponding to $\lambda$ is given by the matrix integral

$$P_{\lambda, T}(x_i) \sim \int \prod_{\alpha=1}^{m-1} \lambda_\alpha \prod_{i} d_\alpha^{(n)} \prod_{\alpha, \beta=1}^{m} \Delta(\bar{x}_\alpha, \bar{x}_\beta)^C_{\alpha \beta} \quad (8)$$

where $\lambda_m$ is put equal to $m$. This is a generalization of old formulas from \[37\] and and it can be considered as an extension of the degenerate field insertion into the conformal block \[38\] in the DF approach.

Another way to construct symmetric functions \[36, 10\] is to consider a level 1 representation of the algebra so that it is realized by the Heisenberg algebra. Choose a Hamiltonian as an element of the algebra, it is a function of generators $a_n$. Realizing them in terms of time variables, $a_{n<0} \sim p_n$, $a_{n>0} \sim \partial_{p_n}$, one obtains a set of eigenfunctions of the Hamiltonian as functions of $p_n$. After the Miwa transformation, $p_n = \sum x_i^n$ they give rise to symmetric functions of $x_i$. For instance, the level one representation of the DIM $gl_1$ algebra leads to the set of $gl_1$ Macdonald polynomials. As a next step, one can consider the sets of eigenfunctions which diagonalize co-products (of degree $N - 1$) of the Hamiltonian (i.e. representations of higher levels), which, in this concrete example, leads to the generalized $gl_N$ Macdonald polynomials.

• **Lift to the graph level.** The next step is restoration of the vertical/horizontal symmetry and lifting the symmetry (Ward identities) to the original network matrix model \[4\]. Important at this step is that topological vertices are associated with matrix elements of the intertwining operators of the DIM algebra \[33\].

The crucial ingredient of this construction is the centralizer of the algebra of constraints, defining the screening operators and the matrix model. The centralizer depends on the representation, and it is this dependence that leads to a variety of different matrix models, encoded by the graph $\Gamma$. Once the graph (with some additional decorations: preferred direction, etc.) is chosen, the particular representation of DIM is fixed and so is the particular matrix model. However, traces of the larger DIM symmetry remain in various forms, the most notable example being the spectral duality \[39, 11, 40\] connecting multi-matrix models with different numbers of matrices and vertex operator insertions.

In the simplest case, associated with 4d Seiberg-Witten theory, the role of $\hat{T}$ is played by the stress tensor $T(z) = \frac{1}{2} \partial \phi(z)^2 + Q \partial^2 \phi(z)$, which generates the ordinary Virasoro algebra (and its $W_N$-algebra generalizations), and $Z(t)$ is just the ordinary Dotsenko-Fateev (DF) matrix model of \[4\]. Various types of $q/t/q_{123}$-deformations, associated with reviving of the hidden compactification moduli, i.e. revealing the hidden 6d and M-theory nature of the theory, require a lifting/resolution of $\frac{1}{2} \partial \phi(z)^2 + Q \partial^2 \phi(z)$ of a peculiar Toda-like combination of vertex operators:

$$\mathcal{T}(z) = e^{\Phi(z)} e^{-\Phi(t^{-1} z)} + t e^{-\Phi(tz/q)} e^{\Phi(z/q)}$$

where

$$\Phi(z) = \sum_{n \geq 1} \frac{z^n}{n} \alpha_n + \Phi_0 - \sum_{n \geq 1} \frac{z^n}{n} \alpha_n, \quad (10)$$
and the modes of $\Phi(z)$ satisfy the $q$-deformed commutation relations:
\[
[\alpha_n, \alpha_m] = \frac{n}{1 + (\frac{q}{t})^{|n|-|m|}} \frac{1 - q^{|n|}}{1 - t^{|n|}} \delta_{n+m,0}.
\] (11)

Given these $q$-boson relations, $\mathcal{T}(z)$ generates the $q$-deformed Virasoro algebra $\text{Vir}_{q,t}$. Deformed stress-energy tensor $\mathcal{T}$ can be guessed from the requirement that it commutes with the screening current. The expression for the screening current essentially determines the matrix model and its symmetry. Concretely, the $q$-deformed screening current is given by
\[
S(x) = \prod_{k \geq 0} \exp \left( -\Phi(q^k x) + \Phi(q^k tx) + \Phi(q^{k+1} x) - \Phi \left( \frac{q^{k+1}}{t} x \right) \right):
\] (12)

We will derive the formulas for $\mathcal{T}$ and $S$ in detail in sec. 2.2.5 in particular we obtain the form (9) of stress-energy tensor in Eq. (11).

One can see that the expressions for $\mathcal{T}$ and $S$ are not symmetric under the exchange of $t$ and $q^{-1}$, which, as we will see, is natural symmetry of the DIM algebra. This is another artifact of the choice of a concrete representation/matrix model description of the object with larger symmetry. All the essential quantities of each particular model should be symmetric w.r.t. $q \leftrightarrow t^{-1}$, though the intermediate results do not respect this symmetry.

In the double-scaling limit $q = e^h \to 1$, $t = q^\beta$ the ordinary Virasoro stress-energy tensor $T(z) = \frac{1}{2} \partial \phi(z)^2 + Q \partial^2 \phi(z)$ (with $Q = \sqrt{\beta} - \sqrt{\beta}^{-1}$) is recovered from (9):
\[
T(z) \to 2 + \hbar(1 - \beta) + \frac{\hbar^2}{2} \left[ (\beta - 1)^2 + 2T(z) \right] + \mathcal{O}(\hbar^3),
\] (13)

where
\[
\phi(z) = 2\sqrt{\beta} \lim_{q,t \to 1} \Phi(z) = \sum_{n \geq 1} \frac{z^n}{n} \hat{\alpha}_{-n} + \Phi_0 - \sum_{n \geq 1} \frac{z^{-n}}{n} \hat{\alpha}_n,
\] (14)

and $\hat{\alpha}_n$ are ordinary boson generators, satisfying $[\hat{\alpha}_n, \hat{\alpha}_m] = 2n \delta_{n+m,0}$.

As already mentioned, a nice bonus is that multi-field generalization of (9), which in 4d leads to substitution of Virasoro by $W$-algebras, is now just another representation of the same symmetry algebra. In other words, after the deformation the Sugawara-like bi- and multi-linear combinations of currents can be obtained from comultiplication of the deformed current algebra, without a need to consider its universal enveloping.

The purpose of this paper is a sketchy survey of this remarkable DIM symmetry of (4). Various details will be presented in separate publications. We will discuss here the 5d and 6d DIM $\mathfrak{gl}_1$ algebras which correspond to the quiver gauge theories with fundamental matter. The most interesting case of the DIM affine algebras which describe, in particular, the 6d gauge theory with adjoint matter and correspond to the double elliptic systems will be touched only briefly. This issue, and also various details of other cases will be presented in separate publications.

## 2 $A_1 (q, t)$-matrix model

Let us start with the prototypical example of the $A_1 (q, t)$-deformed conformal matrix model. This is the simplest model where $q$-Virasoro symmetry arises and, therefore, serves as an accessible port of entry to the land of DIM algebras.

In this section we describe the general scheme for investigating a network-type matrix model. We start by writing down the conventional definition of the model in terms of matrix integral. However, one should remember, that this is just a particular representation of the network of topological vertices, as in Eq. (3). We next describe the algebraic face of the matrix model more concretely by specifying the screening operators, which OPE gives the actual matrix model integrals. The $\text{centralizer}$ of the screenings inside the representation of DIM gives the $W$-algebra corresponding to the matrix model, which also generates the $qq$-characters in the gauge theory. This description was used in [18] to introduce the $W$-algebras corresponding to an arbitrary (affine) ADE-type quiver. Our aim in this paper is more general (though in this section we study it on a very humble example). We would like to elucidate the hidden symmetries, which are only visible in the network-type formalism (3) (see [11] for an example of such an approach). The symmetries of the network/topological string/toric diagram are described
by DIM algebra, of which different $W$-algebras are only particular representations/subalgebras. In this part of the paper we will demonstrate explicitly how various concepts in matrix models and gauge theories, such as $qq$-characters and generalized Macdonald polynomials, are tied together with the help of the DIM algebra.

We introduce the DIM algebra generators and relations in sec. 2.2.1. We describe the simplest representations of DIM algebra in sec. 2.2.3 and how they give rise to generalized Macdonald polynomials. In sec. 2.2.4 with the help of dressing operators, we build the deformed Virasoro subalgebra of the DIM algebra and show its connection to $qq$-characters in the gauge theory. In sec. 2.2.6 we focus on the details of the dressing procedure and identify it with the reduction of the “$U(1)$ part” in the Nekrasov function/conformal block. We also describe the relation with Benjamin-Ono integrable system.

### 2.1 Free-field description

The matrix model can be described in two different ways: as a Jackson or contour integrals respectively. Here we adopt the latter form:

$$Z_{A_1} = \int d^N x \Delta^{(q,t)}(x) V_1(z_1, x) \cdots V_M(z_M, x),$$

where

$$\Delta^{(q,t)}(x) = \prod_{i \neq j} \left( \frac{x_i}{x_j}; q \right)_\infty,$$

$$V_a(z_a, x) = \prod_{i=1}^N \left( \frac{z_{a_i} x_{a_i}}{x_i} q \right)_\infty,$$

and the Pochhammer symbol ($q$-exponential) is defined as $(x; q)_\infty = \prod_{k \geq 0} (1 - q^k x)$. Time variables are traded for a product of vertex operators $V(z)$: this can be understood/interpreted as a Miwa transform.

Following the general recipe given in the introduction, we would like to interpret the matrix model as an average of screening currents $S(x)$. One can see explicitly that the necessary choice is

$$S(x) = \exp \left[ - \sum_{n \geq 1} x^n \frac{1 - t^n}{1 - q^n} \left( 1 + \left( \frac{q}{t} \right)^n \right) \alpha_n + \sum_{n \geq 1} x^{-n} \frac{1 - t^{-n}}{1 - q^{-n}} \left( 1 + \left( \frac{q}{t} \right)^{-n} \right) \alpha_n \right]:$$

where $\Phi(x)$ is defined in Eq. (10). From the $q$-boson commutation relations (11) one get the following OPE for the screenings currents

$$S(x_1)S(x_2) = \frac{\left( \frac{x_1}{x_2}; q \right)_\infty}{\left( \frac{t x_1 / x_2}{x_1}; q \right)_\infty} \left( \frac{q x_2 / x_1}{x_1}; q \right)_\infty \cdot S(x_1)S(x_2) = \prod_{k=0}^{\beta-1} \left( 1 - q^k \frac{x_1}{x_2} \right) \left( 1 - q^k \frac{x_2}{x_1} \right) : S(x_1)S(x_2) :$$

From the OPE (18) we can immediately see that the matrix model (19) is indeed the correlator of screenings with vertex operators:

$$Z_{A_1} = \langle 0 | \int d^N x \prod_{i=1}^N S(x_i) V_1(z_1) \cdots V_M(z_M) | 0 \rangle$$

What are the Ward identities for the $(q,t)$-matrix model? To obtain them let us perform the steps we discussed in the Introduction: first, we introduce time variables $p_k$ into the matrix integral inserting into the average (19) the operator

$$G(p) = \exp \left( \sum_{k > 0} p_k \alpha_{-k} \right)$$

and, second, we verify that the deformed stress-energy tensor $T(z)$ commutes with the integral of the screening current (17). $\int S(x) dx/x$. This means that $T(z)$ commutes with $S(x)$ up to total derivative (or total
While obviously following from commutativity of $T$, the deformed stress-energy tensor, written in the bosonized form, as in Eq. (9), or in the form of matrix model

\[ e^{\Phi(z)-\Phi(z/t)} S(x) : + t \quad e^{-\Phi(tz/q)+\Phi(z/q)} S(x) := \]

\[ = \left\{ \left( 1 - t^2 \frac{x}{z} \right) e^{\Phi(z)-\Phi(x)} + e^{\Phi(tz/q)-\Phi(tz/q)+\Phi(z/q)-\Phi(z/q)} \right\} e^{\Phi(x)-\Phi(z/t)} + \left( q^z \frac{\partial}{\partial z} - 1 \right) \left( 1 - t^2 \frac{x}{z} \right) e^{\Phi(tz/q)-\Phi(tz/q)+\Phi(z/q)-\Phi(z/q)} \right\} S(x) : \]  

(21)

Remarkably, the pole at $z = x$ is exactly canceled in both terms in the first line: the shift in the infinite sum of operators inside $S(x)$ plays a crucial role in this cancellation, and only the total difference remains singular. Since the poles are canceled up to total $q$-difference, the commutator with screening charge, i.e. with the integral of $S(x)$ vanishes. This fact was used in [18] to derive the regularity of the $qq$-characters.

This implies that $T(z)$ is a symmetry of the model: negative modes of its Laurent expansion in $z$ annihilate the vacuum and thus annihilate the entire matrix integral. Inserting the $T$-$S$ OPE into the matrix integral, we get:

\[ \mathcal{T}_p(z) Z_A(p) = \langle T(z) \rangle = \frac{\langle [G(p)V_1(z_1) \cdots V_M(z_M)] T(z) \rangle d^N x \prod_{i=1}^N S(x_i)|0\rangle}{\langle 0| V_1(z_1) \cdots V_M(z_M) |0\rangle d^N x \prod_{i=1}^N S(x_i)|0\rangle} = \]

\[ \int d^N x z^{M(t)}(x) \left( \prod_i V_1(z_1, x_i) \cdots V_M(z_M, x_i) U(x, p) \right) \prod_{i=1}^N 1 - t \frac{x_i}{z_i} + P(z|\{z_a\}, \{v_a\}) \prod_{i=1}^N 1 - \frac{q z_i}{t z_i} = \text{Pol}(z) \]  

(22)

where $P(z|\{z_a\}, \{v_a\})$ is the contribution of vertex operators, a $z$-polynomial factor and the time dependence of the partition function is encoded in the potential

\[ U(x, p) = \exp \left( - \sum_{n \geq 1} \frac{x^n}{n} \frac{1 - t^n}{1 - q^n} \left( 1 + \frac{q}{t} \right)^n p_n \right) \]  

(23)

The deformed stress-energy tensor, written in the bosonized form, as in Eq. (9), or in the form of matrix model average, can also be realized as a difference operator upon identification

\[ \hat{\alpha}_n = p_n, \quad \hat{\alpha}_n = \frac{n}{1 + \left( \frac{q}{t} \right)^n} \frac{1 - q^n}{1 - t^n} \frac{\partial}{\partial p_n} \]  

(24)

leading to a difference equation on the partition function, a counterpart of the Baxter equation. It is sometime called a $qq$-character [16, 18, 17, 19], since it can be considered as a deformation of the Frenkel-Reshetikhin $q$-character [12] (trace over Cartan part of the quantum $R$-matrix). Virasoro symmetry of the matrix model implies that this average has no negative modes in its $z$-expansion, i.e. is regular (and therefore polynomial) in $z$:

regularity of the $qq$-character = polynomiality of the average $\langle T(z) \rangle =$ Ward identity (DIM/Virasoro constraint)  

(25)

While obviously following from commutativity of $T(z)$ with $S$, this looks like a non-trivial property of the r.h.s. in (22).

Also $qq$-characters can be thought of as the recurrence relation on the matrix model correlators, obtained by expanding the average of $T(z)$ in powers of $z$. The recurrence relations can also be derived by considering the vanishing total difference under the matrix model integral [11]. Of course, this only means that the commutator of $T(z)$ and $S(x)$ is given by the corresponding total difference. In the case at hand the relevant total difference
is given by

\[
0 = \oint d^N x \sum_{i=1}^{N} \frac{1}{x_i} (1 - q^{x_i}) \left[ \frac{x_i}{z - x_i} \prod_{j \neq i} \frac{x_i - tx_j}{x_i - x_j} \Delta(q,t)(x) \right] = \int d^N x \sum_{i=1}^{N} \left[ 1 - \frac{x_i}{z - x_i} \prod_{j \neq i} \frac{x_i - tx_j}{x_i - x_j} \right] \Delta(q,t)(x) = \int d^N x \left[ \prod_{j=1}^{N} \frac{1 - \frac{tx_j}{z - x_j}}{1 - \frac{x_j}{z}} + t^{2N-1} q \prod_{j=1}^{N} \frac{1 - \frac{qx_j}{z}}{1 - \frac{tq}{z}} - Q_N(z) \right]
\]  

(26)

where \(Q_N(z)\) is degree \(N\) polynomial in \(z\) and \(x_i\), and in the last line we have summed over poles in \(z\) to obtain the products. The identity (26) is precisely the regularity constraint on the \(qq\)-character telling that \(\langle T(z) \rangle = \langle P(z) \rangle\) is regular in \(z\). For details of derivation along this route see [11]. We will employ similar technique to get the symmetry constraints for the elliptic matrix model in sec. 3.

In the next section we show how to obtain the deformed energy-momentum tensor from the representation of the abstract DIM algebra.

2.2 Abstract algebraic description

We now describe the algebraic structures of DIM algebra governing the network-type matrix model. Let us first recall the definition of the DIM algebra \(U_q(\hat{\mathfrak{gl}_2})\) and its simplest representations and then demonstrate the connections of this algebra with deformed Virasoro algebra, \(qq\)-characters, generalized Macdonald polynomials and integrable systems.

2.2.1 DIM algebra

This looks like a deformation of the affine quantum algebra \(U_q(\hat{\mathfrak{gl}_2})\) with the positive/negative root generators \(x^\pm(z)\), two exponentiated Cartan generators \(\psi^\pm(z)\) and the central element \(\gamma\).

Commutation relations are

\[
G^\mp(z/w)x^\pm(z) x^\pm(w) = G^\pm(z/w)x^\pm(w) x^\pm(z)
\]

\[
[x^+(z), x^-(w)] = \frac{(1-q)(1-t^{-1})}{1-qt} \left( \delta(\gamma^{-1/2}z/w) \psi^+(\gamma^{1/2}w) - \delta(\gamma z/w) \psi^-(\gamma^{-1/2}w) \right)
\]

(27)

\[
\psi^+(z) \psi^-(w) = \psi^-(w) \psi^+(z)
\]

\[
\psi^+(z) \psi^-(w) = g(\gamma w/z) g(\gamma^{-1} w/z) \psi^-(w) \psi^+(z)
\]

\[
\psi^+(z) x^\mp(w) = g(\gamma^{1/2} w/z) \psi^+(w) x^\pm(w)
\]

\[
\psi^-(z) x^\mp(w) = g(\gamma^{1/2} z/w) \psi^-(w) x^\pm(w)
\]

\[
\text{Sym}_{z_1, z_2, z_3} [x^\pm(z_1), [x^\pm(z_2), x^\pm(z_3)]] = 0
\]

DIM algebra is a Hopf algebra with comultiplication

\[
\Delta(\psi^\pm(z)) = \psi^\pm(\gamma_2^{1/2} z) \otimes \psi^\pm(\gamma_1^{1/2} z)
\]

\[
\Delta(x^\pm(z)) = \psi^-(\gamma_1^{1/2} z) \otimes x^+(\gamma_1 z) + x^+(z) \otimes 1
\]

\[
\Delta(x^-(z)) = 1 \otimes x^-(z) + x^-(\gamma_2 z) \otimes \psi^+(\gamma_2^{1/2} z)
\]

(28)

where \(\gamma_1^{1/2} = \gamma^{1/2} \otimes 1\), \(\gamma_2^{1/2} = 1 \otimes \gamma^{1/2}\). The function \(g(z) = \frac{G^+(z)}{G^-(z)}\) is restricted by the requirement \(g(z) = g(z^{-1})^{-1}\) and \(\delta(z) = \sum_{n \in \mathbb{Z}} z^n\). We omit expression for the counit and antipode, since we will not need them.
2.2.2 Specification of the structure function

The structure of the algebra is encoded in the function $G(z)$ which is often chosen to be cubic in $z$ with additional restriction $q_1 q_2 q_3 = 1$:

$$G^\pm (z) = (1 - q_1 z)(1 - q_2 z)(1 - q_3 z) = (1 - q^\pm 1 z) (1 - t^{\mp 1} z) (1 - (t/q)^{\pm 1} z)$$  \(29\)

Without any harm to commutation relations and comultiplication it can be further promoted to unrestricted $q$ and more general Kerov deformations, and even to elliptic function, though details of bosonization procedure below should still be worked out in these cases. We describe the elliptic version in sec. 3.

2.2.3 Level one Fock representation

The simplest representation of DIM algebra is the level one representation $\rho_n$ acting on the Fock module $F_u$, generated by the $q$-deformed Heisenberg creation operators $a_{-n}$ from the vacuum $|u\rangle$ annihilated by the annihilation operators $a_n$. The Heisenberg generators satisfy

$$[a_n, a_m] = n \frac{1 - q^{|n|}}{1 - t^n} \delta_{n+m,0}$$  \(30\)

Note that $a_n$ are normalized differently from $\alpha_n$ in eqs. (10), (11) (that normalization was chosen to maximally simplify the final expressions). Of course the $a_n$ generators are related to $\alpha_n$ generators in a simple way:

$$\alpha_n = \frac{1}{1 + (q/t)^n} a_n \quad n \geq 1$$  \(31\)

$$\alpha_{-n} = a_{-n} \quad n \geq 1$$  \(32\)

The generators of the DIM algebra are expressed in terms of the Heisenberg generators:

$$\rho_u (x^+(z)) = u \eta(z) = u : \exp \left( \sum_{n \geq 1} \frac{1 - t^{-n}}{n} a_{-n} z^n - \sum_{n \geq 1} \frac{1 - t^n}{n} a_n z^{-n} \right) :$$

$$\rho_u (x^-(z)) = u^{-1} \xi(z) = u^{-1} : \exp \left( \sum_{n \geq 1} \frac{1 - t^{-n}}{n} \left( \frac{t}{q} \right)^{n/2} a_{-n} z^n - \sum_{n \geq 1} \frac{1 - t^n}{n} \left( \frac{t}{q} \right)^{n/2} a_n z^{-n} \right) :$$

$$\begin{align*}
\rho_u (\psi^\pm(z)) &= \varphi^\pm(z) = \exp \left( \pm \sum_{n \geq 1} \frac{1 - t^{\pm n}}{n} \left( 1 - \left( \frac{t}{q} \right)^{\pm n} \right) a_{\pm n} z^{\mp n} \right) \\
\rho_u (\gamma) &= \left( \frac{t}{q} \right)^{1/2}
\end{align*}$$  \(33\)

Let us see an example how OPE of these operators reproduces the DIM commutation relation:

$$\eta(z)\eta(y) = \frac{1 - \frac{y}{t}}{1 - \frac{t}{q}} \left( \frac{1 - \frac{y}{t}}{1 - \frac{t}{q}} \right) : \eta(z)\eta(y) := \left( \frac{1 - \frac{y}{t}}{1 - \frac{t}{q}} \right) \left( \frac{1 - \frac{y}{t}}{1 - \frac{t}{q}} \right) \eta(z)\eta(y) = \frac{G^-(\frac{z}{y})}{G^+(-\frac{z}{y})} \eta(z)\eta(y)$$  \(34\)

2.2.4 Level two Fock representation and generalized Macdonald polynomials

Tensor product of $m$ Fock representations $F_{u_1} \otimes \cdots \otimes F_{u_m}$ can be easily obtained from the comultiplication and will be called the level $m$ Fock representation. In this tensor product the generators of DIM algebra are expressed in terms of $m$ $q$-Heisenberg generators $a_{n}^{(a)}$, $a = 1, \ldots, m$. In particular, we will need the expression for $x^+(z)$ in this representation:

$$\rho_{u_1, u_2}^{(2)} (x^+(z)) = u_1 \tilde{\Lambda}_1(z) + u_2 \tilde{\Lambda}_2(z) = u_1 \eta_1(z) + u_2 \varphi^{-1}_{1} \left( (t/q)^{1/4} z \right) \eta_{2} \left( \left( t/q \right)^{1/2} z \right)$$  \(35\)

where we use the shorthand notation $\tilde{\Lambda}_{1,2}$ for the components of the level two representation $\rho_{u_1, u_2}^{(2)} = (\rho_{u_1} \otimes \rho_{u_2}) \Delta$, and the subscript denotes the number of term in the tensor product, e.g. $\eta_1(z) = \eta(z) \otimes 1$. 

9
There is an distinguished basis in $\mathcal{F}_{a_1} \otimes \cdots \otimes \mathcal{F}_{a_m}$, the basis of generalized Macdonald polynomials [10] obtained by diagonalizing the action of the zero mode of $x^+(z)$. Representation of this zero mode was called generalized Macdonald Hamiltonian in:

$$H_1^{\text{gen}} = \rho_{u_1, u_2}^{(2)}(x^+_0) = \oint_{\mathcal{C}_0} \frac{dz}{z} \rho_{u_1, u_2}^{(2)}(x^+(z)), \quad (36)$$

In those papers the following definition of the generalized Macdonald polynomials was given:

$$H_1^{\text{gen}} M_{AB}(a_{-n}^{(1)}, a_{-n}^{(2)}) |u_1 \otimes u_2\rangle = [u_1 \kappa_A(q, t) + u_2 \kappa_B(q, t)] M_{AB}(a_{-n}^{(1)}, a_{-n}^{(2)}) |u_1 \otimes u_2\rangle, \quad (37)$$

where

$$\kappa_A = (1 - t) \sum_{i \geq 1} q^{A_i} t^{-i}. \quad (38)$$

These polynomials were instrumental in demonstrating the 5d version of the AGT conjecture [11]. Matrix elements of Virasoro primary fields in this basis turned out to coincide with fixed point contributions in the Nekrasov partition function. Thus, after decomposition of conformal blocks in terms of generalized Macdonald polynomials, the AGT relation becomes explicit. In the 4d limit this special basis degenerates into the basis of generalized Jack polynomials [10], with similar properties.

### 2.2.5 W-algebra, Ward identities and $qg$-characters from DIM

As we have announced in the introduction, the great benefit of DIM approach is that it describes different matrix models from a unified viewpoint. In particular, $m$-multimatrix models have $W_m$-algebra symmetries, and these algebras are all particular representations of subalgebras of DIM algebra.

$q$-deformed $W_m$-algebra, which is also called $W_{q, t}(a_{1, m})$, is obtained from level $m$ Fock representation of the DIM algebra as follows. The stress-energy tensor of the $W_m$-algebra is obtained from the dressing of the $x^+$ generator of DIM. More concretely, we have:

$$t(z) = A(z)x^+(z)B(z), \quad (39)$$

where

$$A(z) = \exp \left( - \sum_{n \geq 1} \frac{1}{\gamma^n} \gamma^{-n} b_n z^n \right), \quad B(z) = \exp \left( \sum_{n \geq 1} \frac{1}{\gamma^n} \gamma^{-n} b_n z^{-n} \right) \quad (40)$$

and $b_n$ are the modes of the $\psi^\pm$ generators:

$$\psi^\pm(z) = \psi^\pm_0 \exp \left( \pm \sum_{n \geq 1} b_{\pm n} z^{\pm n} \right). \quad (41)$$

The stress-energy tensor $\mathcal{T}$ of the $W_M$-algebra is the representation of the dressed current $t(z)$ in the level $m$ Fock module. For the Virasoro case ($m = 2$), using Eq. (35), we get

$$\mathcal{T}(z) = \rho_{u_1, u_2}^{(2)}(t(z)) = u_1 \Lambda_1(z) + u_2 \Lambda_2(z) = \rho_{u_1, u_2}^{(2)}(A(z)) \left( u_1 \eta_1(z) + u_2 \varphi_1^- \left( (t/q)^{1/4} z \right) \eta_2 \left( (t/q)^{1/2} z \right) \right) \rho_{u_1, u_2}^{(2)}(B(z)). \quad (42)$$

where $\Lambda_i(z)$ are dressed versions of the components $\tilde{L}_i(z)$. From Eq. (42) we see that $\mathcal{T}(z)$ depends on two sets of Heisenberg generators (hidden inside $\eta_1$, $\eta_2$ and $\varphi^-$) acting on the tensor product of two Fock modules. However, as we will see explicitly in the next section, the expression for $\mathcal{T}$ actually depends only on one linear combination of $a_{n}^{(1)}$ and $a_{n}^{(2)}$. Related to this fact is that in the level two representation the product of $\Lambda_{1, 2}$ elements is equal to identity:

$$: \Lambda_1(z) \Lambda_2 \left( zq/t \right) : = 1. \quad (43)$$

To see this fact we should write explicit (though lengthy) expressions for $\Lambda_{1, 2}$ in the level two representation:

$$\Lambda_1(z) = : \exp \left( \sum_{n \geq 1} \frac{1}{n} \left( - t^n \right) \frac{1}{1 + (q/t)^n} z^n \left( \alpha_{-n}^{(1)} - (q/t)^{n/2} \alpha_{-n}^{(2)} \right) - \sum_{n \geq 1} \frac{1 - t^n}{n} z^{-n} \left( \alpha_{n}^{(1)} - (q/t)^{n/2} \alpha_{n}^{(2)} \right) \right) : \quad (44)$$

$$\Lambda_2(z) = : \exp \left( - \sum_{n \geq 1} \frac{1}{n} \left( - t^n \right) \frac{1}{1 + (q/t)^n} (zt/q)^n \left( \alpha_{-n}^{(1)} - (q/t)^{n/2} \alpha_{-n}^{(2)} \right) + \sum_{n \geq 1} \frac{1 - t^n}{n} (z^{-1} q/t)^n \left( \alpha_{n}^{(1)} - (q/t)^{n/2} \alpha_{n}^{(2)} \right) \right) : \quad (45)$$
From these expressions we see that indeed : $\Lambda_1(z)\Lambda_2(zq/t) = 1$. We also identify the combinations of creation and annihilation operators, on which $\mathcal{T}$ depends, and denote these combinations by $\tilde{\alpha}$. They are given by

$$\tilde{\alpha}_{-n} = \frac{1}{1 + (q/t)^n} \left( \alpha_{-n}^{(1)} - (q/t)^{n/2} \alpha_{-n}^{(2)} \right), \quad n \geq 1$$

$$\tilde{\alpha}_n = \left( \alpha_n^{(1)} - (q/t)^{n/2} \alpha_n^{(2)} \right), \quad n \geq 1$$

One can see that the commutation relations for $\tilde{\alpha}_n$ are the same as for $\alpha_n^{(1)}$. Now $\Lambda_{1,2}$ and the stress-energy tensor $\mathcal{T}$ are all nicely written in terms of these combinations:

$$\mathcal{T}(z) = u_1 : e^{\tilde{\Phi}(z)} e^{\tilde{\Phi}(z^{-1})} + u_2 : e^{-\tilde{\Phi}(iz/q)} e^{\tilde{\Phi}(z/q)} :$$

where the definition of $\tilde{\Phi}$ is similar to that of $\Phi$ from Eq. (10), only the role of bosons $\alpha_n$ is now played by $\tilde{\alpha}_n$. The constants $u_1$ and $u_2$ can be absorbed into the definition of zero modes, which brings Eq. (48) into the form of the deformed stress-energy tensor identity (9). The zero modes of the screening operators are omitted to simplify the formulas.

Finally, the operators $\tilde{\alpha}_n$ are in fact precisely those bosonic operators, in terms of which we have defined our matrix model [13]. We have, therefore, identified the Ward identities/Virasoro constraints of the matrix model with the particular combination of the DIM operators in the level two Fock representation. We can also make the identification with $qq$-character more explicit by introducing the usual notation:

$$\Lambda_1(z) = \mathcal{Y}(z), \quad \Lambda_2(z) = \mathcal{Y}^{-1}\left( \frac{q}{t}z \right) .$$

The last definition follows from the condition (43). Now we would like to understand where the other combination of the bosonic generators is hidden. To see this we have to revisit the dressing procedure, for the current $t(z)$.

2.2.6 Vir$_{q,t} \oplus$ Heis$_{q,t}$ reduction is equivalent to dressing

In this section we show how the dressing operators $\alpha(z)$ and $\beta(z)$ are in fact performing the reduction of the algebra Vir$_{q,t} \oplus$ Heis$_{q,t}$ acting in the level two Fock representation to its Vir$_{q,t}$ part. The condition (43) can be thought of as a gauge condition used to kill the Heis$_{q,t}$ degrees of freedom, which enter both $\Lambda_1$ and $\Lambda_2$ multiplicatively. This separation of variables is usual for description of Hamiltonian reductions in the free field formalism [42].

To this end let us look at the bosonization of the dressing operators $A(z)$ and $B(z)$. From Eqs. (33), (40) we get

$$\rho^{(2)}_{u_1, u_2}(A(z)) = \exp \left( - \sum_{n \geq 1} \frac{1}{n} \frac{1 - t^{-n}}{1 + (q/t)^n} (q/t)^{n/2} z^n \left( (q/t)^{n/2} \alpha_{-n}^{(1)} + \alpha_{-n}^{(2)} \right) \right),$$

$$\rho^{(2)}_{u_1, u_2}(B(z)) = \exp \left( \sum_{n \geq 1} \frac{1 - t^n}{n} (q/t)^{n/2} z^n \left( (q/t)^{n/2} \alpha_{n}^{(1)} + \alpha_{n}^{(2)} \right) \right).$$

In the exponent, these two operators contain precisely the linear combination of $\alpha_{n}^{(1,2)}$ orthogonal to $\tilde{\alpha}_n$. We denote the new bosons by $\bar{\alpha}_n$:

$$\bar{\alpha}_{-n} = \frac{(q/t)^{n/2}}{1 + (q/t)^n} \left( (q/t)^{n/2} \alpha_{-n}^{(1)} + \alpha_{-n}^{(2)} \right),$$

$$\bar{\alpha}_n = \frac{(q/t)^{n/2}}{1 + (q/t)^n} \left( (q/t)^{n/2} \alpha_{n}^{(1)} + \alpha_{n}^{(2)} \right).$$

These bosons commute with $\tilde{\alpha}_n$ and satisfy slightly modified (compared to (11)) commutation relations among themselves:

$$[\bar{\alpha}_n, \bar{\alpha}_m] = \frac{n(1 - q^n)(q/t)^n}{1 + (q/t)^n(1 - t^n)} \delta_{n+m,0}.$$
Virasoro algebra times an additional “\(U(1)\) factor”, which corresponds to an extra boson, forming the \(\text{Heis}\) algebra \cite{47}. Here we get the extra boson for similar reasons: we are working in the tensor product of two Fock modules, and have to eliminate the “diagonal part” of the bosonized algebra. This elimination corresponds to the dressing transformation, which is nothing but the transformation to the “center of mass frame” for the two bosons \(\alpha_n^{(1,2)}\).

Finally, we can write down a compact expression for the \textit{undressed} current \(x^+(z)\) in the level two Fock representation:

\[
\rho^{(2)}_{u_1,u_2}(x^+(z)) = u_1 \tilde{\Phi}_1(z) + u_2 \tilde{\Phi}_2(z) = \mathcal{T}(z) \mathcal{Z}(z) = \mathcal{T}(z) : e^{\hat{\Phi}(z) - \hat{\Phi}(z/t)} : \]

where \(\hat{\Phi}(z)\) is again the bosonic field defined analogously to \cite{10} using \(\hat{a}_n\) generators. We introduced the \(\text{Heis}_{q,t}\) \(qq\)-character \(\mathcal{Z}(z)\), in terms of which the undressed current factorizes into the product of two terms corresponding to algebras in \(\mathcal{V}_{q,t} \oplus \mathcal{Heis}_{q,t}\).

The factorized form of the current \(x^+(z)\) is also reflected in structure of its zero mode: the \(H^\text{gen}_i\) operator. Written in this form it gives the trigonometric generalization of the Benjamin-Ono (BO) equation \cite{18}, the continuous integrable model also related to the AGT correspondence. It is easy to see the structure of the BO Hamiltonians in the double scaling limit \(q \to 1, t = q^\beta\):

\[
\oint_{C_0} \frac{dz}{z} \rho^{(2)}_{u_1,u_2}(x^+(z)) = 2 + \hbar(1 - \beta) + \frac{\hbar^2}{2}(I_1 + C_1) + \frac{\hbar^3 \beta}{2}(I_2 + C_2) + \mathcal{O}(\hbar^4),
\]

where \(C_{1,2}\) are constants,

\[
I_1 = L_0 + 2 \sum_{n \geq 1} \hat{a}_{-n} \hat{\alpha}_n - \frac{1 - 3Q^2}{6},
\]

\[
I_2 = \sum_{k \neq 0} \hat{a}_{-k} L_k + 2Q \sum_{n \geq 1} n \hat{a}_{-n} \hat{\alpha}_n + \frac{1}{3} \sum_{n+m+k=0} \hat{a}_n \hat{\alpha}_m \hat{\alpha}_k,
\]

and \(\hat{\alpha}_n\) are the ordinary Heisenberg generators, obtained from \(\tilde{a}_n\) in the double scaling limit. All higher BO Hamiltonians appear in the higher terms. What we have found is that generalized Macdonald polynomials are in fact joint polynomial eigenfunctions of the quantum \(\text{BO}\) system.

3 \textbf{Elliptic DIM algebra and elliptic matrix model}

In this section we describe the elliptic generalization of the matrix model and DIM algebra governing it. As we will see, most of the discussion is exactly parallel to the trigonometric case. This is another manifestation of the universality of network type matrix models and the DIM algebra. The fact that the description of the elliptic case is so similar to the trigonometric one gives one the hope that the corresponding structure in the double elliptic case might also be tractable.

3.1 \textbf{Elliptic matrix model}

This matrix model has been described in \cite{9,11}, and we follow the notations of this paper.

\[
Z_{A_1}^{\text{ell}} = \oint d^N x \Delta_\text{ell}^{(q,q',t)}(x) V_1(z_1, x) \cdots V_M(z_M, x),
\]

where

\[
\Delta_\text{ell}^{(q,q',t)}(x) = \prod_{i \neq j} \left( \frac{z_i}{x_j}; q, q' \right)_\infty \left( \frac{q q' x_j}{x_i}; q, q' \right)_\infty,
\]

\[
V_a(z_a, x) = \prod_{i=1}^N \left( \frac{1 - q^{1-x_i}}{x_i}; q, q' \right) \left( q q' x_i^{1-x_i}; q, q' \right),
\]

where the double \(q\)-Pochhammer symbol is \((z; q, q')_\infty = \prod_{k,l \geq 0} (1 - z q^k q^l)\).
This elliptic integral arises from the following screening currents:

\[ S(x) = \prod_{k \geq 0} \exp \left( -\Phi(q^k x) + \Phi(q^k t x) + \Phi(q^{k+1} x) - \Phi \left( \frac{q^{k+1}}{t} x \right) \right) = \exp \left[ -\sum_{n \neq 0} \frac{x^n}{n(1-q^n)(1-q'^n)} (1 + \left( \frac{q}{t} \right)^n) \hat{\alpha}_{-n} + \sum_{n \neq 0} \frac{x^{-n}}{n(1-q^n)(1-q'^n)} (1 + \left( \frac{q}{t} \right)^n) \hat{\beta}_{-n} \right] \] (60)

where the bosons \( \hat{\alpha}_n \) and \( \hat{\beta}_n \) obey the commutation relations:

\[
[\hat{\alpha}_n, \hat{\alpha}_m] = \frac{n(1-q'^{|n|})}{1 + \left( \frac{q}{t} \right)^{|n|}} \frac{1-q{|n|}}{1-t{|n|}} \delta_{n+m,0}, \\
[\hat{\beta}_n, \hat{\beta}_m] = -\frac{nq'^{|n|}}{1 + \left( \frac{q}{t} \right)^{|n|}} \frac{1-q{|n|}}{1-t{|n|}} \delta_{n+m,0},
\]

and \( \hat{\Phi}(z) \) is the field built out of \( \hat{\alpha}_n \) and \( \hat{\beta}_n \):

\[
\hat{\Phi}(z) = \sum_{n \neq 0} \frac{z^n}{n(1-q'^{|n|})} \hat{\alpha}_{-n} - \sum_{n \neq 0} \frac{z^{-n}}{n(1-q'^{|n|})} \hat{\beta}_{-n}
\] (63)

Notice the presence of two sets of boson generators \( \hat{\alpha}_n \) and \( \hat{\beta}_n \), which is related to the modular invariance of the elliptic model. More concretely two bosons produce two terms in the product representation the theta-function: \( \prod_{k \geq 0} (1-q^k z)(1-q^k q'/z) \), the elliptic version of the free field correlator \( 1-z \). This explains why the powers of \( z \) in front of \( \hat{\alpha}_n \) and \( \hat{\beta}_n \) are opposite, and also why their commutation relation differ by \( q'^{|n|} \).

Of course, the stress-energy tensor, which generates the centralizer of the screening charge \( \hat{Q} = \oint S(x) dx/x \) also depends on two sets of bosonic variables. It is very analogous to the trigonometric case:

\[
\mathcal{T}(z) = : e^{\hat{\Phi}(t^{-1} z)} : + t : e^{-\hat{\Phi}(t z/q)} e^{\hat{\Phi}(z/q)} :
\] (64)

This elliptic stress-energy tensor generates the elliptic deformation of the Virasoro algebra, which has been considered in many works [9]. We proceed along the lines of the previous section and move to the corresponding DIM algebra, which gives tensor \( \mathcal{T} \) in the level two representation.

### 3.2 Elliptic DIM algebra, elliptic Virasoro and ILW equation

Elliptic version of DIM algebra is generated by the same set of operators as the ordinary DIM: \( x^\pm(z), \psi^\pm(z) \) and the central element \( \gamma \). The relations are a copy of Eq. (27), except for the \( [x^+, x^-] \) relation, which changes to

\[
[x^+(z), x^-(w)] = \frac{\Theta_q(q; q') \Theta_{q'}(t^{-1}; q')}{(q'; q')_z \Theta_q(q/t; q')} \left( \delta(\gamma^{-1} z/w) \psi^+(\gamma^{1/2} w) - \delta(\gamma z/w) \psi^-(\gamma^{-1/2} w) \right)
\] (65)

where \( \Theta_p(z) = (p; p)_\infty (z; p)_\infty (p/z; p)_\infty \) is the theta-function. Also, most importantly, the structure function \( G^\pm(z) \) is now not trigonometric, but elliptic:

\[
G^\pm(z) = \Theta_p(q^{\pm 1} z) \Theta_p(t^\pm 1 z) \Theta_p(t^{\mp 1} q^{\pm 1} z),
\] (66)

The comultiplication \( \Delta \) is exactly the same as in the trigonometric case, given by Eqs. (28). As with the matrix model in the previous section, the essential difference with the trigonometric case appears when one tries to build Fock representation of elliptic DIM: one set of bosons turns out not to be enough. We need at least two sets of Heisenberg generators \( \hat{a}_n \) and \( \hat{b}_n \) to reproduce the commutation relations of the elliptic algebra. Concretely, we
have for the level one representation:

\[
\rho_u(x^+(z)) = u\eta(z) = u : \exp \left( - \sum_{n \neq 0} \frac{(1-t^n)z^{-n}}{n(1-q^{1/2})} \hat{a}_n \right) \exp \left( - \sum_{n \neq 0} \frac{(1-t^{-n})q^{1/2}z^{-n}}{n(1-q^{1/2})} \hat{b}_n \right) : \]

\[
\rho_u(x^-(z)) = u^{-1}\xi(z) = u^{-1} : \exp \left( \sum_{n \neq 0} \frac{(1-t^n)p^{-1/2}z^{-n}}{n(1-q^{1/2})} \hat{a}_n \right) \exp \left( \sum_{n \neq 0} \frac{(1-t^{-n})p^{1/2}q^{1/2}z^{-n}}{n(1-q^{1/2})} \hat{b}_n \right) : \]

\[
\rho_u(\psi^+(z)) = \psi^+(z) = \exp \left( \sum_{n>0} \frac{(1-t^n)(p^{-1/2}-p^{1/2})p^{-n/4}}{n(1-q^{1/2})} \right) \left( z^{-n} \partial_n - p^2 q^n z^n \hat{b}_n \right) \]

\[
\rho_u(\psi^-(z)) = \psi^-(z) = \exp \left( - \sum_{n>0} \frac{(1-t^{-n})(p^{-1/2}-p^{1/2})p^{-n/4}}{n(1-q^{1/2})} \right) \left( z^n \partial_n - p^2 q^n z^n \hat{b}_n \right) \]

\[
\rho_u(\gamma) = (t/q)^{1/2},
\]

where \( p = \frac{q}{2} \) and the bosons \( \hat{a}_n \) and \( \hat{b}_n \) satisfy the following commutation relations:

\[
[\hat{a}_m, \hat{a}_n] = m \frac{(1-q^{1/2})(1-q^{1/2})}{1-t^{1/2}} \delta_{m+n,0},
\]

\[
[\hat{b}_m, \hat{b}_n] = m \frac{(1-q^{1/2})(1-q^{1/2})}{(pq)^{1/2}(1-t^{1/2})} \delta_{m+n,0},
\]

\[
[\hat{a}_m, \hat{b}_n] = 0. \tag{68}
\]

Again, the fields \( \hat{a}_n, \hat{b}_n \) are related to \( \hat{a}_n, \hat{b}_n \) by a simple redefinition.

The dressed current \( t(z) = A(z)x^+(z)B(z) \), corresponding to the stress energy tensor is given by exactly the same expression \([69]\), as in the ordinary DIM case. Moreover, the dressing operators \( A(z) \) and \( B(z) \) are constructed from the \( \psi^k \) generators of the elliptic DIM algebra using the same formulas \([40]\) as given above. In the level two representation \( \rho_u^{(2)} \), the element \( t(z) \) produces the elliptic Virasoro stress-energy tensor \([64]\).

Let us also mention that the undressed elliptic DIM charge \( \oint x^+(z)dz/z \) also leads to several very interesting objects. In the level one representation it gives elliptic Ruijsenaars Hamiltonian, while in the second level representation it is the difference version of the intermediate long-wave (ILW) Hamiltonian \([49]\), which itself is a generalization of the Benjamin-Ono system.

### 3.3 Ward identities and \( qq \)-characters

One can derive Ward identities in the same algebraic fashion as for the trigonometric case. The OPE of the stress-energy tensor \([64]\) with the screening current \([60]\) is given by:

\[
\mathcal{T}(z)S(x) = \frac{\Theta_{q'}(\frac{tx}{z})}{\Theta_q(\frac{x}{z})} : e^{\bar{\phi}(z)}e^{-\bar{\phi}(z/t)}S(x) : + \frac{\Theta_{q'}(\frac{tx}{2})}{\Theta_q(\frac{q}{2})} : e^{-\bar{\phi}(tq/z)}e^{\bar{\phi}(z/t)}S(x) :, \tag{69}
\]

which is non-singular up to total \( q \)-difference due to the same cancellation, as in Eq. (21), and we use the same time insertion operator \([20]\), but this time depending on two sets of times, \( p_k \) and \( \bar{p}_k \) related to sets of Heisenberg operators \( a_- \) and \( \hat{b}_n \):

\[
\mathcal{G}(p) = \exp \left( \sum_{k>0} p_k a_- \left( \sum_{k>0} \bar{p}_k \hat{b}_n \right) \right) \tag{70}
\]

Thus the insertion of \( \mathcal{T}(x) \) into the correlator corresponds to the insertion of the following expression under the matrix model integral:

\[
\mathcal{T}_{\rho_u}(z)Z_{M,N}^{ll}(p) = \langle \mathcal{T}(z) \rangle = \oint d^N x \Delta_{ell}(q,q',t)(x) \left( \prod_{a=1}^{M} \prod_{i=1}^{N} V_a(z_a, x_i) U(x_i, p, \bar{p}) \right) \times \left[ \prod_{j=1}^{N} \Theta_{q'}(\frac{tx_j}{z}) \right] \left[ \hat{P}(z, \{ v_a \}) \prod_{j=1}^{N} \Theta_{q'}(\frac{q x_j}{2}) \right] = \hat{Q}_N(z), \tag{71}
\]
where $\hat{P}(z)\{z_a\}, \{u_a\}$ and $\hat{Q}_N(z)$ are products of theta functions of the form $\prod_a \Theta_{q'}(z/\lambda_a)$, and the potential now has the form

$$U(x,p,\bar{p}) = \exp \left[ -\sum_{n>0} \frac{x^n}{n} \left( 1 - t^n \right) (1 + \left( \frac{q}{q'} \right)^n) p_n - \sum_{n>0} \frac{x^n}{n} \left( 1 - q^{-n} \right) (1 - q^n) \left( 1 + \left( \frac{t}{q'} \right)^n \right) \bar{p}_n \right]$$  \hspace{1cm} (72)

while the difference realization of the operator $T_{p_n}(z)$ is given by the substitution

$$\hat{\alpha}_{-n} = p_n, \quad \hat{\alpha}_n = \frac{n(1-q^n)}{1 + \left( \frac{q}{q'} \right)^n} \frac{1 - q^n}{1 - t^n} \frac{\partial}{\partial p_n}, \quad \hat{\beta}_n = \bar{p}_n, \quad \hat{\beta}_{-n} = -nq^n(1-q^n) \frac{1 - q^n}{1 - t^n} \frac{\partial}{\partial \bar{p}_n}$$ \hspace{1cm} (73)

This gives the elliptic $qq$-character corresponding to the $6d$ gauge theory corresponding to the $A_1$ quiver, i.e. the gauge group should consist of single $SU(n)$ factor possibly with some fundamental matter hypermultiplets.

As we have seen in the trigonometric case, there is another very explicit way to derive the Ward identities: consider the vanishing integral of a cleverly chosen total difference. In the elliptic case this method work as well, provided the total difference is

$$0 = \oint d^N x \sum_{i=1}^N \frac{1}{x_i} (1 - q^{x_i}) \left[ \sum_{k \in \mathbb{Z}} x_i t^{kN} z^{-q^k x_i} \prod_{j \neq i} \Theta'_{q'} \left( \frac{t x_j}{z} \right) \Delta_{\text{ell}}^{(q,q',t)}(x) \right] \sim$$

$$\sim \oint d^N x \Delta_{\text{ell}}^{(q,q',t)}(x) \left[ \prod_{j=1}^N \frac{\Theta'_{q'} \left( \frac{x_j}{z} \right)}{\Theta'_{q'} \left( \frac{t x_j}{z} \right)} + t^{2N-1} q \prod_{j=1}^N \frac{\Theta'_{q'} \left( \frac{x_j}{z} \right)}{\Theta'_{q'} \left( \frac{t x_j}{z} \right)} - \hat{Q}_N(z) \right].$$ \hspace{1cm} (74)

The resulting equation is, of course the same as Eq. (71). The meaning of the identity (74) in the elliptic matrix model is the same as in the $(q,t)$-matrix model: it provides the recurrence relations for the correlators of arbitrary symmetric functions of $x_i$. It would be interesting to obtain the factorization formulas for the averages in this model similar to those for the averages of (generalized) Macdonald polynomials in the $(q,t)$-model. Let us also mention that in the Nekrasov-Shatashvili limit Eq. (74) reduces to the quantum spectral curve of the XYZ spin chain, to the Seiberg-Witten integrable system corresponding to the $6d$ gauge theory.

This concludes our brief tour into the realm of elliptic matrix models and elliptic DIM algebras. The most important lesson to learn here is that the DIM description indeed seems to be universal: the elliptic case is almost literally the same as the trigonometric one.

4 Conclusions and further directions

We have worked out the connection between a large class of network matrix models associated with toric diagrams and the DIM algebra. The algebra provides a unified description of the symmetry behind all such matrix models giving rise to $qq$-characters, generalized polynomials and Ward identities.

Application of the algebraic description to a matrix model such as (4) requires:

(i) identification of a particular free field representation of the appropriate DIM associated with the given model,

(ii) building explicit expressions for the screening operators expressed as integrals of screening currents $S(x)$,

(iii) constructing the symmetry generators (generalized stress-tensors) $T(z)$ for which the screening operators are the centralizers,

(iv) representing the correlators of screening currents as Vandermonde measures and stress-tensor insertions as $qq$-characters which can be converted into the action of differential/difference operators. This step relies on the $T$-$S$ OPE, which should be nonsingular up to a total difference, and the $S$-$S$ OPE, which should give the desired version of the Vandermonde determinant.

Schematically, one should have

$$T(z)S(x) = \text{Regular}(z,x) + (1 - q^{x \partial_x}) \text{Singular}(z/x)$$

$$S(x_1)S(x_2) = f(x_1/x_2) : S(x_1)S(x_2) :$$ \hspace{1cm} (75)
and the function \( f(x) \) defines the Vandermonde factor through \( \Delta(x) = \prod_{i \neq j} f(x_i/x_j) \). For the concrete examples of OPEs like \(^{(3)}\rho \) see Eqs. \(^{[18, 21]}\).

It is still unclear how to separate the contributions of screening currents and vertex operators in the network matrix model formalism since both objects are packed into a single intertwiner/topological vertex. Probably, the technical answer to this question should depend on the “star-chain” duality for conformal blocks.

This procedure is supposed to associate a D-module structure with each particular network matrix model or, what is the same, with representation of DIM. A non-trivial feature of actual construction, already seen in \(^{[9]}\) and \(^{[17]}\) is that the stress tensors are actually build from roots of algebra, while the screening operators from Cartan generators of DIM, which is somewhat against a naive intuition coming from their realization as powers \( \partial \phi \) and \( \int e^{\pm \phi} \) in the simplest free field conformal theories. General understanding of this phenomena includes relation between the Sugawara construction and the DIM comultiplication and between the screening charges and the action of the Weyl group. Remarkably, the Weyl group of elliptic DIM should be the elliptic DAHA, of which the elliptic Macdonald functions explicitly provided by formulas like \(^{[8]}\) in elliptic matrix model \(^{[57]}\), are eigenfunctions.

An interesting question here is interpretation of the BPZ equations \(^{[40]}\) for such insertions as the Baxter equations for symmetric functions of Macdonald family, especially in elliptic case, where there exist alternative approaches \(^{[49]}\).

\[
\rho_u(x+(z)) = \frac{\rho}{-uv \otimes \rho} \Delta(x+(z))
\]

\[
\rho(2)_{u1,u2}(x+(z)) = (\rho/(-u_1v \otimes \rho) \otimes \rho_{u2}) \Delta_2(x+(z)) = (\rho/(-u_1v \otimes \rho) \otimes \rho_{u2}) \Delta(x+(z))
\]

Figure 2: Topological vertex as the intertwiner of DIM representations. a) The action of the generator \( x^+(z) \) on the level one Fock representation \( \rho_u \) sitting on the horizontal leg of the topological vertex (denoted by the dashed line) is the same as its action on the product of two representations — the "vertical" \( \rho_v \) and "diagonal" \( \rho^l(-uv) \). b) Appropriate contraction of two intertwiners is also an intertwiner. This gives the vertex operator of the corresponding conformal field theory with deformed Virasoro symmetry, corresponding to a single vertical brane in Fig. 1.

At the level of network matrix model \(^{[3]}\) DIM symmetry generators act on any section which cuts \( M \) edges to separate the diagram into disconnected parts (see Fig. 2). They act as \((M - 1)\)-th coproduct of the original DIM generators. As was shown in \(^{[33]}\), topological vertices are intertwiners of DIM representations, i.e. the action on one of the legs is equal to the action on two others — this allows to pull the generator through the vertex (Fig. 2 a)). Moreover, contraction of the legs is consistent with this procedure (Fig. 2 b)). In the result DIM generators can be pulled from the original section to the right of the diagram, where negative modes annihilate the Fock vacuum, or to the left, where the positive modes act trivially. This provides the constraints on the matrix model averages (or, equivalently, gives the \( qq \)-characters), which are the 5d generalization of the constraints obtained in \(^{[19]}\). Fig. 2 actually gives the uplift of the setup considered in \(^{[19]}\) to the level of the topological string (or network matrix model). Further developments of this approach and its applications to compactified toric diagrams will be reported elsewhere.

Network matrix model is naturally built from the Seiberg-Witten integrable system — which is a spin chain in the simplest cases \(^{[50, 40]}\). The network is the tropicalization of its spectral curve, and the vertical and horizontal branes encode the rank and number of chain sites respectively. The structure of intertwiners/R-matrices forming a network can be understood as a lift of ordinary trigonometric R-matrices similar to the tetrahedron equation \(^{[51]}\). This will give the connection between the algebraic and integrable parts of the story \(^{[31]}\).

After the basic structure of network matrix model constraints is understood, we face a multitude of different paths, each one worth following. First of all, since DIM algebras involves double affinization of any Lie algebra (we have only considered \( \mathfrak{sl}_1 \) case) it can be applied to the affine algebra \( \widehat{\mathfrak{sl}_1} \). This should provide a triply affine algebra \( U_{q,t,t}(\widehat{\mathfrak{sl}_1}) \) with three parameters. In this notation it seems appropriate to name this crucially important
structure the Pagoda Algebra. This algebra should have remarkable properties, one of which is the presence of an \( SL(3, \mathbb{Z}) \) automorphism group \([52]\), corresponding to the automorphisms of the compactification torus \( T^3 \).

In the second part of this paper we have considered elliptic DIM algebra, corresponding to 6d gauge theory with matter content given by a linear quiver. The elliptization of the triply affine Pagoda algebra should, therefore, describe the 6d gauge theory with adjoint matter, the most mysterious of all Seiberg-Witten systems, corresponding to double-elliptic integrable systems and affine elliptic Selberg integrals. However, even without extra deformations, already the case of elliptic DIM poses interesting questions.

To summarize, the main idea of this paper is that DIM provides a functor, which lifts the picture — a network — to formulas made out of Nekrasov functions, 3d partitions or topological vertices. In other words, the input is a tropical spectral curve (associated with the underlying Seiberg-Witten integrable system) and the output is the partition function of the associated topological string theory, which is provided by one and the same universal procedure. At the algebraic level the input should be the algebra \( \hat{\mathfrak{gl}}_1 \) which, treated as \( \mathfrak{gl}_\infty \) or \( W_{1+\infty} \), incorporates various \( \mathfrak{gl}_n \)’s, and the output is described by the Pagoda algebra, which still needs to be fully investigated.

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