1. Introduction

The Schramm–Loewner evolution is so far the only model where a multi-fractal spectrum is not trivial and known explicitly (see e.g. [2, 3, 6, 7, 11, 17, 18]). Explicit computation of a harmonic spectrum is an almost impossible task for deterministic fractals and it is a difficult task even for random fractals. Apart from the case of the Schramm–Loewner evolution, two exact results for the second moment of harmonic measure were obtained in the study of Levy–Loewner evolution (LLE) [7, 19]. Extending this study, we will present an infinite set of non-trivial examples.

The bounded whole-plane LLE (also called exterior whole-plane LLE) is a stochastic process of the growth of the curve out of the origin \( z = 0 \) in the complex plane \( z \) (see figure 1).

The curve is described by time-dependent conformal mapping \( z = F(w, t) \) from the exterior of the unit circle \( |w| > 1 \) in the \( w \)-plane to the complement of the curve in the \( z \) plane. This mapping obeys the Levy–Loewner equation

\[
\frac{\partial F(w, t)}{\partial t} = w \frac{\partial F(w, t)}{\partial w} w + e^{iL(t)} w - e^{iL(t)} w - e^{iL(t)}, \quad \lim_{w \to \infty} F'(w, t) = e^t, \quad -\infty < t < \infty,
\]

(1)
where $L(t)$ is a Levy process and prime denotes the $w$-derivative. The limit $t \to -\infty$ corresponds to the origin of the curve.

Here we consider the Levy processes without a drift. Without loss of generality we set
\[
\langle L(t) \rangle = 0, \tag{2}
\]
where $\langle \rangle$ denote expectation (ensemble average).

The LLE is a conformally invariant stochastic process in the sense that the time evolution is consistent with the composition of conformal maps. When $L(t)$ is a continuous function of time, the conformal mapping $z = F(w, t)$ describes growth of a random continuous curve. The only continuous (modulo uniform drift) process of Levy type is the Brownian motion
\[
L(t) = \sqrt{\kappa} B(t), \quad \langle (B(t + \tau) - B(t))^2 \rangle = |\tau|, \tag{3}
\]
where the positive parameter $\kappa$ is the ‘temperature’ of the Brownian motion. The stochastic Loewner evolution driven by Brownian motion is called Schramm–Loewner Evolution or SLE$_\kappa$. Since it describes non-branching planar stochastic curves with a conformally-invariant probability distribution, SLE is a useful tool for the description of boundaries of critical clusters in two-dimensional equilibrium statistical mechanics. In this picture, different $\kappa$ correspond to different classes of models of statistical mechanics (a good introduction to SLE for physicists can be found e.g. in [4, 8] as well as mathematical reviews given e.g. in [15, 16]).

On the other hand, in the general case, when $L(t)$ is a discontinuous function of time the conformal mapping describes a stochastic (infinitely) branching curve.

The unbounded (or interior) version of the whole-plane LLE is an inversion $F(w, t) \to 1/F(1/w, t)$ of the above bounded version. Obviously, this mapping from an interior of the unit disc in the $w$-plane to the complement of the stochastic curve which grows from infinity towards the origin in the $z$-plane also satisfies the Loewner equation (1) with different asymptotic conditions given now at $w = 0$
\[
F'(w = 0, t) = e^{-\tau}.
\]

This version of LLE has been studied mainly due to its relationship with the problem of Bieberbach coefficients of conformal mappings [7, 17, 19].
To get the multi-fractal spectrum of LLE one has to find first the so-called ‘β-spectrum’\(^3\):
the integral means \(\beta(q)\)-spectrum of the domain is defined through the \(q\)th moment of a derivative of conformal mapping at the unit circle (i.e. at \(|w| \rightarrow 1\) as follows

\[
\beta(q) = \lim_{\epsilon \rightarrow 0, \pm} \frac{\log \int_0^{2\pi} |F'(e^{\epsilon+i\varphi})|^q \, d\varphi}{-\log \epsilon},
\]

(4)

where \(\pm\) signs correspond to bounded and unbounded whole-plane LLE (in what follows the upper/lower signs will correspond to the bounded/unbounded version respectively).

To find \(\beta(q)\)-spectrum (4) one needs to estimate moments of derivative: one can show that the value

\[
\rho = e^{\mp \epsilon q} \langle |F'(e^{i\varphi})|^{q} \rangle,
\]

is time independent and is a function of \(w, \bar{w} \) and \(q\) only [2, 3, 7, 17–19]. Moreover it satisfies the linear integro-differential equation

\[
\mathcal{L}^{\pm} \rho = 0,
\]

(5)

where the linear operator \(\mathcal{L}^{\pm}\) equals

\[
\mathcal{L}^{\pm} = -\dot{\eta} + w \frac{w+1}{w-1} \partial_w + \bar{w} \frac{\bar{w}+1}{\bar{w}-1} \partial_{\bar{w}} - \frac{q}{(w-1)^2} - \frac{q}{(\bar{w}-1)^2} + q \mp q.
\]

(6)

In (6), the linear operator \(\dot{\eta}\) acts on functions of \(w, \bar{w}\) as follows (for details see e.g. [2, 7] or [19])

\[
\dot{\eta} \rho(w, \bar{w}) = \lim_{t \rightarrow 0} \frac{1}{t} \int_0^{2\pi} \left( \rho(w, \bar{w}) - \rho(e^{i\varphi} w, e^{-i\varphi} \bar{w}) \right) P(\varphi, t) \, d\varphi,
\]

where \(P(\varphi, t)\) is the probability density that \(L(t) = \varphi\) under condition \(L(0) = 0\). One can present operator \(\dot{\eta}\) in the more convenient (integro-differential) form as

\[
\dot{\eta} \rho(w, \bar{w}) = \frac{\kappa}{2} \left( w \partial_w + \bar{w} \partial_{\bar{w}} \right) \rho(w, \bar{w}) + \int_{-\pi}^{\pi} \left( \rho(w, \bar{w}) - \rho(e^{i\varphi} w, e^{-i\varphi} \bar{w}) \right) \eta(\varphi) \, d\eta(\varphi), \quad d\eta(\varphi) \geq 0
\]

(7)

where \(d\eta(\varphi)\) is a symmetric Levy measure. In polar coordinates \((r, \phi)\), such that \(w = re^{i\phi}, \bar{w} = re^{-i\phi}\), the operator \(\dot{\eta}\) acts only wrt angular variable \(\phi\), i.e. \(\dot{\eta}\) commutes with \(r\) and

\[
\dot{\eta} f(\phi) = -\frac{\kappa}{2} \partial_{\phi}^2 f(\phi) + \int_{-\pi}^{\pi} (f(\phi) - f(\phi + \varphi)) \, d\eta(\varphi)
\]

whenever is dependence of \(f\) on \(r\). In other words, it acts diagonally on the basis \(w^n \bar{w}^m\)

\[
\dot{\eta} w^n \bar{w}^m = \eta_{n-m} w^n \bar{w}^m, \quad n, m \in \mathbb{Z}.
\]

(8)

Since we consider only Levy processes without drift (2) (i.e. with Levy measure symmetric wrt reflection \(\phi \rightarrow -\phi\)), the characteristic coefficients \(\eta_n\) are real, non-negative and symmetric

\[
\eta_n = \bar{\eta}_n = \eta_{-n}, \quad \eta_0 = 0.
\]

\(^3\)For an introduction to multi-fractal analysis and the relationship between different kinds of multi-fractal spectra see e.g. [2, 9–11, 23].

\(^4\)\(\bar{w}\) denotes complex conjugate of \(w\).
They express through the Levy measure as
\[ \eta_n = \frac{\kappa n^2}{2} + \int_{-\pi}^{\pi} (1 - \cos n\phi) d\eta(\phi). \] (9)

The above equation is a particular case of Levy–Khintchine formula for a class of Levy processes we deal with.

The probability density \( P(\phi, t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{-\eta_n t + in\phi} \) is a fundamental solution of the integro-differential equation of the parabolic type on the unit circle
\[ \partial_t P(\phi, t) = -\hat{\eta} P(\phi, t), \quad P(\phi, 0) = \delta(\phi). \]

Note that the particular case \( d\eta(\phi) = 0 \) corresponds to the Schramm–Loewner evolution SLE\(_\kappa\). Here, equations (5) and (6) becomes the second-order differential equation with
\[ \hat{\eta} = -\frac{\kappa \eta^2}{2\phi}, \quad \eta_n = \frac{\kappa n^2}{2}, \]
which allows us to find exact form of the multi-fractal spectrum of the SLE\(_\kappa\) [2, 3, 11, 18].

In the case of the bounded whole-plane LLE one can use analyticity of \( \rho(w, \bar{w}) \) at infinity as boundary conditions for linear equations (5) and (6). Namely (see e.g. [19] for details), in this version of LLE \( \rho \) has the following asymptotic expansion at \( w \to \infty \)
\[ \rho = \sum_{i=0}^{\infty} \rho_{ij} w^i \bar{w}^j, \quad \rho_{0,0} = 1 \]
with expansion coefficients \( \rho_{ij} \) fixed uniquely by equations (5) and (6). In other words, once an analytic, non-vanishing at infinity solution of (5) and (6) is found, it gives (modulo constant factor) expectation of the moment of the derivative of conformal mappings of the bounded whole-plane LLE.

Similarly, for the unbounded version of LLE one has to look for a non-vanishing analytic solution of the corresponding equation, this time at the origin \( w = 0 \)
\[ \rho = \sum_{i=0}^{\infty} \rho_{ij} w^i \bar{w}^j, \quad \rho_{0,0} = 1. \]
Similarly to the bounded case, such a solution is unique.

2. Beta spectrum of the unbounded whole-plane LLE and Fuchsian systems

Let us try to find the value of \( \beta \)-spectrum at \( q = 2 \): representing \( \rho(w, \bar{w}) \) in the form
\[ \rho(w, \bar{w}) = (1-w)(1-\bar{w})\Theta(w, \bar{w}), \] (10)
from (5) and (6) with \( q = 2 \) we get
\[ -\hat{\eta}[(1-w)(1-\bar{w})\Theta] + (w+1)(\bar{w}-1)w \frac{\partial \Theta}{\partial w} + (\bar{w}+1)(w-1)\bar{w} \frac{\partial \Theta}{\partial \bar{w}} + 3(2w\bar{w} - w - \bar{w})\Theta = 0, \] (11)
where \( \Theta \) is the series
\[ \Theta = \theta_0(\xi) + \sum_{i=1}^{\infty} (w^i + \xi w^{-i}) \theta_i(\xi), \quad \xi = r^2 = w\bar{w}. \] (12)
Substituting it into (11), and taking into account the facts that operator \( \hat{\eta} \) commutes with \( \xi = r^2 = \bar{w} w \) and that, according to (8), \( \hat{\eta}[w^i] = \eta_i w^i \), from (11) and (12) we get a three-term differential recurrence relation for \( \theta_i(\xi) \).

\[
2\xi(\xi - 1)\theta_i'(\xi) - (\eta_i + i + (\eta_i - i - 6)\xi)\theta_i(\xi) + \xi(\eta_i + i - 2)\theta_{i+1}(\xi) + (\eta_i - i - 2)\theta_{i-1}(\xi) = 0,
\]

\[\theta_{-1}(\xi) = \xi^2 \theta_1(\xi). \tag{13}\]

It is easy to see that this recurrence relation has a solution that truncates at \( i = N \), i.e. \( \theta_i = 0 \) for \( i \geq N \), when \( \eta_N = N + 2 \) for some \( N \).

The simplest truncation happens when \( N = 1 \), i.e. when \( \eta_1 = 3 \), which corresponds to the case of conjecture by authors of [7] proved by us in [19]. Before studying the general case of an arbitrary \( N \) we will consider separately the case of truncation at \( N = 2 \). Here not only \( \beta(2) \), but also \( \rho(w, \bar{w}) \) can be found explicitly for arbitrary \( \eta_i, i \neq 2 \). We have the following

**Theorem 1.** For the unbounded whole plane LLE with \( \eta_2 = 4 \) the integral means \( \beta \)-spectrum has the following value at \( q = 2 \)

\[
\beta(2) = \frac{6 - \eta_1 + \sqrt{\eta_1^2 - 4\eta_1 + 12}}{2}. \tag{14}\]

**Proof.** The proof is based on explicit computation of \( \rho(w, \bar{w}) \). It is convenient to re-parametrize \( \eta_1 \geq 0 \) through \( \beta \) given by l.h.s of equation (14), i.e.

\[
\eta_1 = \frac{\beta^2 - 6\beta + 6}{2 - \beta}, \quad 2 < \beta < 3 + \sqrt{3}, \tag{15}\]

and introduce new dependent variables \( f_0, f_1 \) such that \( \theta_i(\xi) = (1 - \xi)^{-\beta} f_i(\xi) \).

In the case of truncation at \( N = 2 \) (i.e. when \( \eta_2 = 4 \)), system (13) writes as

\[
\xi(\xi - 1)f''_0 + (3 - \beta)\xi f_0 - 2\xi f_1 = 0 \tag{16}\]

\[
2\xi(\xi - 1)f''_1 - (\eta_1 + 1 + (\eta_1 - 7 + 2\beta)\xi) f_1 + (\eta_1 - 3) f_0 = 0. \tag{17}\]

Then expressing \( f_1 \) through \( f_0, f'_0 \) into (16) and substituting the result in (17) we get the hypergeometric equation for \( f_0 \)

\[
\xi(\xi - 1)f''_0 + ((a + b + 1)\xi - c) f'_0 + ab f_0 = 0
\]

where

\[
a = \frac{\beta^2 - 5\beta + 8}{2(2 - \beta)}, \quad b = 3 - \beta, \quad c = \frac{\beta^2 - 7\beta + 8}{2(2 - \beta)}. \tag{18}\]

Since we are looking for \( \rho(w, \bar{w}) \) which is non-vanishing and analytic at \( w = 0 \), from all linearly independent solutions of the above hypergeometric equation we choose the one which is non-vanishing and analytic at \( \xi = 0 \), i.e.

\[
f_0 = \text{hyper}_1(a, b, c; \xi).
\]
Then from (16) it follows that

\[ f_1 = \frac{\xi - 1}{2} f_0' + \frac{3 - \beta}{2} f_0 = \frac{ab}{2c} (\xi - 1) \, _2F_1(a + 1, b + 1, c + 1; \xi) + \frac{3 - \beta}{2} \, _2F_1(a, b; c; \xi) \]

and according to (10) and (12)

\[ \rho(w, \bar{w}) = (1 - w)(1 - \bar{w})(1 - w\bar{w})^{-\beta} (f_0(w\bar{w}) + (w + \bar{w}) f_1(w\bar{w})). \]

According to (4), (10) and (12), the value of the \( \beta \)-spectrum equals the blowup rate (also called 'polynomial' growth rate) of \( \theta_0 - \theta_1 = (1 - \xi)^{-\beta} (f_0(\xi) - f_1(\xi)) \) at \( \xi \to 1 \). Using the Gauss identity

\[ _2F_1(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}, \quad \text{Re}(c - a - b) > 0, \]

and taking equations (15) and (18) into account, it is easy to see that \( f_0 - f_1 \) is finite and non-vanishing at \( \xi = 1 \). Therefore, the blowup rate of \( \theta_0 - \theta_1 \) equals \( \beta \) and is given by lhs of equation (14). This completes the proof. \( \square \)

To consider the case of truncation at \( i = N \) (i.e. when \( \eta_N = N + 2 \)), we note that in such a case the system of ODEs (13) is a Fuchsian system with three singular points \( \xi = 0, 1, \infty \).

This system rewrites in the standard form as

\[ \theta'(\xi) = \frac{A\theta(\xi)}{\xi} - \frac{B\theta(\xi)}{\xi - 1}, \quad (19) \]

where \( \theta \) is \( N \)-dimensional vector \( \theta(\xi) = (\theta_0(\xi), \theta_1(\xi), \ldots, \theta_{N-1}(\xi))^T \) and \( A \) and \( B \) are correspondingly two and three-diagonal constant \( N \times N \) matrices (matrix indexes are running from 0 to \( N - 1 \)):

\[
A = -\frac{1}{2} \left[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
3 - \eta_1 & \eta_1 + 1 & 0 & 0 \\
4 - \eta_2 & \eta_2 + 2 & 0 & 0 \\
& \ddots & \ddots & \ddots \\
& & N - \eta_{N-2} & \eta_{N-2} + N - 2 & 0 \\
& & & N + 1 - \eta_{N-1} & \eta_{N-1} + N - 1 \\
\end{array}
\right],
\]

\[
B = -\frac{1}{2} \left[
\begin{array}{cccc}
-6 & 4 & 1 - \eta_1 & \eta_1 - 1 \\
3 - \eta_1 & 2\eta_1 - 6 & 2\eta_2 - 6 & -\eta_2 \\
& \ddots & \ddots & \ddots \\
& & 2 + i - \eta_i & 2\eta_i - 6 & 2 - i - \eta_i \\
& & & \ddots & \ddots \\
& & & & N - \eta_{N-2} & 2\eta_{N-2} - 6 \\
& & & & & N + 1 - \eta_{N-1} & 2\eta_{N-1} - 6 \\
\end{array}
\right]. \quad (20)
\]
Note that the two-diagonal matrix $A$ has a zero eigenvalue (zero Fuchsian exponent at $\xi = 0$) which is a consequence of existence of an analytic at $w = 0$ solution of (19) (the one corresponding to solution of (5) and (6) we are looking for). This zero eigenvalue is non-degenerate, while other eigenvalues are negative, which confirms uniqueness of the above analytic solution (modulo constant factor). Indeed, this solution can be constructed starting from $\theta(0)$

$$\theta(\xi) = T(\xi)\theta(0), \quad T(\xi) = \sum_{n=0}^{\infty} T_n \xi^n, \quad A\theta(0) = 0,$$

where matrices $T_n$ are defined recurrently as

$$T_0 = 1, \quad T_{n+1} = (A - n - 1)^{-1}(A - B - n)T_n.$$  

Since all eigenvalues of $A - n - 1$ are negative, the Taylor series (21) exists and is unique (and converges absolutely for $\xi < 1$).

The eigenvalues of the matrix $B$ are minus Fuchsian characteristic exponents at $\xi = 1$. They are roots of the characteristic polynomial $P_n(\beta) = \det(\beta - B)$ which is determined by the three-term recurrence relation

$$\beta P_n = P_{n+1} + a_n P_{n-1} + b_n P_n, \quad P_0 = 1, \quad P_{-1} = 0,$$

where

$$a_1 = 3 - \eta_1, \quad a_n = (\eta_n - n - 2)(\eta_{n-1} + n - 3)/4, \quad n > 1, \quad b_n = 3 - \eta_n, \quad n \geq 0.$$  

When $a_n > 0$, $0 < n < N - 1$, all eigenvalues of $B$ are real and non-degenerate and polynomials $P_n(\beta)$, $0 \leq n \leq N$ are orthogonal wrt a non-signed measure (see e.g. [5]). Although one can find a wide range of the Levy processes for which a condition of positivity of $a_n$ holds, it is not always the case and $P_n$ are not necessarily orthogonal wrt a non-signed measure (i.e. $P_n(\beta)$ can be so-called ‘formal’ orthogonal polynomials).

**Lemma 1.** Matrix $B$, given by equation (20), has at least one real non-negative eigenvalue and $\beta(2)$ equals a non-negative eigenvalue of $B$.

**Remark 1.** A similar statement is valid for the bounded version of LLE for the matrix given by (29). We will refer to both statements as lemma 1.

**Proof.** It is known (see e.g. [13]) that an asymptotic of a general solution of the Fuchsian system (19) in a neighborhood of singular point $\xi = 1$ has the form

$$\theta \to \sum_{l=0}^{N-1} (1 - \xi)^{-\beta_l} f^{(l)}(C, \xi), \quad \xi \to 1,$$

where $\beta_l$ is an eigenvalue of $B$ and $C = \{C_0, \ldots, C_{N-1}\}$ are arbitrary constants. If matrix $B$ is diagonalizable and non-resonant, the vector functions $f^{(l)}$ are finite at $\xi = 1$ and $f^{(l)}(C, \xi) = C_l f^{(l)}$, with $f^{(l)}$ being an eigenvector of $B$ corresponding to an eigenvalue $\beta_l$. If $B$ is not diagonalizable or/and resonant, $f^{(l)}(C, \xi)$ are polynomials in $x = \log(1 - \xi)$ (i.e. they are of finite non-negative integer degrees in $x$).

From (24) and (12) it follows that the corresponding solution of (11) has the following asymptotics at $\xi \to 1$ (or $r \to 1$ in polar coordinates $(r, \phi), w = re^{i\phi}$)

5 Matrix is resonant if it has eigenvalues which differ by a non-zero integer.
\[ \Theta(r, \phi) \to \sum_{j=0}^{N-1} (1-r)^{-\Re\beta_j} \sum_{m=0}^{M_j} G_{lm}(\phi) x^m \cos \left( x \Im \beta_j - s_{lm}(\phi) \right), \quad x = \log(1-r^2), \quad r \to 1, \]

where all functions \( G_{lm}(\phi) \) are bounded. As \( r \to 1 \), the above finite sum will be dominated by term(s) with the maximal \( \Re \beta_j \) and, if there are several of them, one has to choose the one(s) with the maximal \( M_j \) among them. In other words

\[ \Theta \to (1-r)^{-\max \Re \beta_j} x^M \mathcal{G}(x, \phi), \quad r \to 1, \tag{25} \]

where \( M \) is some non-negative integer and \( \mathcal{G}(x, \phi) \) is a sum of a finite number of the cosine terms

\[ \mathcal{G}(x, \phi) = \sum_{\alpha_i \in \{ \Im \beta_1, \ldots, \Im \beta_{N-1} \}} G_i(\phi) \cos \left( \alpha_i x - \gamma_i(\phi) \right). \tag{26} \]

From (25) and (26) it follows that at least one eigenvalue with maximal real part has zero imaginary part. Indeed, let us suppose that the above is not true. Then all \( \alpha_i \) in (26) are non-zero. Since (26) is the sum of a finite number of the cosine terms with \( x \) running along the semi-infinite interval, \( \mathcal{G}(x, \phi) \) is not of a constant sign on this interval. Therefore, any real solution of (11) that truncates at \( N \) in (12), including an analytic at \( \xi = 0 \) solution we are looking for, oscillates, changing its sign as \( \xi \to 1 \). However, an analytic at \( \xi = 0 \) solution cannot change sign at \( \xi < 1 \), since both \( \rho(w, \bar{w}) \) and \( (1 - w)(1 - \bar{w}) \) are not negative at \( \xi < 1 \). Therefore, there exists a real eigenvalue which is not less than the real part of any non-real eigenvalue.

Next, if all real \( \beta_j \) were negative, then (according to (25)) any non-negative solution would have a negative blow-up rate and therefore \( \beta(2) \) given by (4) would be negative. This is also a contradiction, since the integral means \( \beta \)-spectrum must be non-negative (see e.g. [20]). Thus, matrix \( B \) must have at least one non-negative eigenvalue, and \( \beta(2) \) equals one of the non-negative eigenvalues of \( B \).

By consequence, in the \( N = 2 \) case all eigenvalues are real\(^6\) (as confirmed by theorem 1). Note that here both eigenvalues can be positive, but \( \beta(2) \) always equals the maximal one.

Another useful example where \( \beta(2) \) and the spectrum of \( B \) can be found explicitly is the \( N \)-truncated unbounded SLE, i.e. the unbounded whole-plane SLE\( \kappa \) with \( \kappa = 2(N + 2)/N^2 \) (for details see [18])

\[ \eta_n = \frac{N + 2}{N^2} n^2, \quad n = 0..N, \]

\[ \beta_l = \frac{N + 2 - (2N^2 - 3N - 6)l + 2(N + 2)l^2}{N^2}, \quad l = 0..N - 1. \tag{27} \]

Here all eigenvalues \( \beta_l \) are real, while not all \( a_n \) are positive. The value of the integral means \( \beta \) spectrum at \( q = 2 \) equals the maximal eigenvalue \( \beta_{N-1} \), i.e.

\[ \beta(2) = \max \{ \beta_l, l = 0 \ldots N - 1 \} = (5N - 2)/N \quad (\text{see} \ [7, 18]). \]

The above example shows that the spectrum of \( B \) can be degenerate and resonant. For instance, the spectrum is resonant for \( N = 2, 3, 10, 12, 14, 15, \ldots \).

\(^6\) Also, one can check that in the \( N = 3 \) case all roots of cubic characteristic polynomial are always real for any \( \eta_1, \eta_2 \geq 0, \) whatever the signs of \( a_n \).
To present an example of $B$ with complex eigenvalues, first consider the case when $N = 6$ in (27), i.e. for $\eta_n = 4n^2/18$, $n \leq 6$. Here the spectrum is degenerate and $\beta_0 = \beta_1 = 2/9$, $\beta_1 = \beta_2 = -2/3$ (two other eigenvalues $\beta_4 = 2$ and $\beta_5 = 14/3$ are non-degenerate). Let us now introduce the following perturbation of the above LLE

$$\eta_n = \left(\frac{4}{9} - \delta \kappa\right) \frac{n^2}{2} + 18\delta \kappa, \quad \delta \kappa > 0 \quad 0 < n \leq 6,$$

which can be, for instance, the LLE driven by a combination of the Brownian motion with $\kappa = 4/9 - \delta \kappa$ and a compound Poisson process with jumps uniformly distributed over the circle. The Levy measure of the latter equals $d\eta(\phi) = (9\delta \kappa/\pi)d\phi$ (see equation (9)). When $\delta \kappa$ are small, the spectrum of $B$ has two positive and four complex eigenvalues. The latter four, up to $O(\delta \kappa)$, equal

$$\frac{2}{9} \pm \frac{21}{128} \sqrt{-30\delta \kappa}, \quad -\frac{2}{3} \pm \frac{5}{128} \sqrt{-70\delta \kappa}, \quad \delta \kappa > 0.$$

Returning to the generic LLE, we note that since the analytic at $\xi = 0$ solution of (19) is a linear combination of solutions corresponding to different Fuchsian exponents at $\xi = 1$, one may assume that the $\beta$-spectrum corresponds to the lowest real exponent at $\xi = 1$ (i.e. maximal real eigenvalue of $B$). Indeed, an absence of the corresponding component in the above linear combination could happen only by an ‘accident’. Although, as we have seen above, there is no such ‘accident’ in the $N = 2$ case as well in the case of the $N$-truncated SLE, its absence for the generic $N > 2$ case (or generic $N > 4$ case for the bounded LLE, see next section) requires a proof that does not seem to be elementary.

Next, we will consider a family of LLEs which are small deformations of the $N$-truncated SLE, its $N$-truncated SLEs. It is more convenient to consider deformation of the bounded version where, in contrast to (27), all eigenvalues but one are negative. The latter fact facilitates the study.

3. Bounded LLE

For $q = 2$ one looks for a solution of (5) and (6) that is analytic at $w \rightarrow \infty$, representing $\rho(w, \bar{w})$ in the following form

$$\rho(w, \bar{w}) = (1 - w^{-1})(1 - \bar{w}^{-1})\Theta(w, \bar{w}).$$

Then for $q = 2$ we get

$$-\bar{\eta} \left[ \left(1 - \frac{1}{w}\right) \left(1 - \frac{1}{\bar{w}}\right) \Theta\right] + \left(1 + \frac{1}{w}\right) \left(1 - \frac{1}{\bar{w}}\right) w \frac{\partial \Theta}{\partial w} + \left(1 + \frac{1}{\bar{w}}\right) \left(1 - \frac{1}{w}\right) \bar{w} \frac{\partial \Theta}{\partial \bar{w}} + \left(\frac{1}{w} + \frac{1}{\bar{w}} - \frac{2}{w\bar{w}}\right) \Theta = 0. \quad (28)$$

For $\Theta$ given by the Fourier series

$$\Theta = \theta_0(\xi) + \sum_{i=1}^{\infty} (w^{-i} + \xi^{-i} w^i) \theta_i(\xi), \quad \xi = r^2 = w\bar{w}$$

we get the following recurrence relation for $\theta_i(\xi)$

$$2\xi(\xi - 1)\theta_1^i(\xi) - (\eta + 2 - i + (\eta + i)\xi) \theta_i(\xi) + (\eta + i + 2)\theta_{i+1}(\xi) + \xi(\eta - i + 2)\theta_{i-1}(\xi) = 0,$$

$$\theta_{-i}(\xi) = \xi^{-i} \theta_i(\xi).$$
This relation truncates if \( \eta_N = N - 2 \). In the case of truncation it can be rewritten in the Fuchsian form (19) with

\[
A = -\frac{1}{2} \begin{bmatrix}
2 & -4 & 0 & \cdots & 0 \\
0 & \eta_1 + 1 & -3 - \eta_1 & \cdots & 0 \\
0 & \eta_2 & -4 - \eta_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \eta_i + 2 - i & -2 - i - \eta_i & \cdots & 0 \\
& & & & & \eta_N - 2 - N + 4 & -N - \eta_N - 2 & \eta_{N+1} + 3 - N
\end{bmatrix},
\]

\[B = -\frac{1}{2} \begin{bmatrix}
2 & -4 & 0 & \cdots & 0 \\
-1 - \eta_1 & 2\eta_1 + 2 & -3 - \eta_1 & \cdots & 0 \\
-\eta_2 & 2\eta_2 + 2 & -4 - \eta_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\eta_N + 2 - N - \eta_{N-2} & 2\eta_{N-2} + 2 & -N - \eta_{N-2} & \cdots & 0 \\
N - 3 - \eta_{N-1} & 2\eta_{N-1} + 2 & \eta_{N+1} + 3 - N
\end{bmatrix}.
\]

(29)

Now matrix \( A - B \) (i.e. residue matrix at \( \xi = \infty \)) has a single zero eigenvalue and \( N - 1 \) positive eigenvalues which confirms that an analytic at \( w = \infty \) solution is an \( N \)-truncated solution (demonstration of this fact repeats arguments used for unbounded LLE in the previous section).

Note that, since \( \eta_n > 0 \) when \( n > 0 \), while \( \eta_N = N - 2 \), the first non-trivial case of truncation takes place at \( N = 3 \) (i.e. for \( N > 2 \)). This differs from the unbounded case considered in the previous section, where non-trivial truncations take place for all \( N > 0 \). The characteristic polynomial \( P_N(\beta) \) can be found with the help of the three-term recurrence relation (22) with the following coefficients

\[
a_1 = \eta_1 + 1, \quad a_n = (\eta_n - n + 2)(\eta_{n+1} + n + 1)/4, \quad n > 1, \quad b_n = -\eta_n - 1, \quad n \geq 0.
\]

(30)

Similarly to the unbounded version an analog of lemma 1 for matrix \( B \), given by (29) holds. Let us now prove the following

**Theorem 2.** For the bounded whole-plane LLE with

\[\eta_N = N - 2\]

and

\[\eta_n = \frac{\kappa_n \eta_n^2}{2} + \delta \eta_n, \quad \kappa_N = \frac{2N - 2}{N^2}, \quad 0 < n < N,\]

where \( \delta \eta_n \neq 0 \) are sufficiently small\(^7\), the value of the integral means \( \beta \)-spectrum at \( q = 2 \) is given by the greatest eigenvalue of the matrix (29).

\(^7\)One can easily check with the help of the Levy-Khinchine formula (9) that such families of Levy processes exist, e.g. by setting \( \kappa = \kappa_N - \delta, \delta \kappa > 0 \) and \( d\eta(\phi) > 0 \) a such that \( \int_0^{2\pi} (1 - \cos N \phi) d\eta(\phi) = \delta \kappa N^2/2. \)
Proof. In the case of the bounded SLE\(_k\) at \(q = 2\) and \(\kappa = \kappa_N = 2(N - 2)/N^2\), the spectrum of the matrix \(B\) reads as [18]

\[
\beta_l = \frac{2\kappa_N + (3\kappa_N - 4)l + \kappa_N l^2}{4}, \quad l = 0..N - 1
\]  

(31)

with maximal eigenvalue \(\beta(2) = \beta_0\) being the only non-negative eigenvalue (for derivation of the spectra see [2, 18]).

Since the spectrum is real and non-degenerate, it remains such when we add small perturbations \(\delta\eta_n\) to \(\eta_n = \kappa_N n^2/2, n < N\). Indeed, let \(P_N(\beta)\) be a characteristic polynomial of matrix \(B\)

\[
P_N = \beta^N + \sum_{n=0}^{N-1} p_n\beta^n, \quad p_n = p_n(\eta_1, \ldots, \eta_{N-1}),
\]

then

\[
\frac{\partial \beta_l}{\partial \eta_j} = -\sum_{n=0}^{N-1} \frac{\partial p_n}{\partial \eta_j} \beta^n \prod_{k \neq l} (\beta_l - \beta_k).
\]

Since \(\beta_i \neq \beta_j\) when \(i \neq j\), all derivatives of the spectrum wrt \(\eta_n\) are finite and real and therefore the spectrum remains real and non-degenerate for small perturbations \(\eta_n \to \eta_n + \delta\eta_n, \delta\eta_n \neq 0, 0 < n < N\).

In equation (31) the maximal eigenvalue \(\beta_0\) is positive while the rest of the eigenvalues are negative. Since \(\partial \beta_i/\partial \eta_j\) are finite, the sign of the eigenvalues does not change for small deformations \(\eta_n \to \eta_n + \delta\eta_n\), so the maximal eigenvalue still remains the only non-negative eigenvalue. Thus, from lemma 1 it follows that \(\beta(2)\) equals the maximal eigenvalue, which completes the proof. □

The next proposition considers generic nontrivial \(N < 5\) cases

**Theorem 3.** For the bounded whole-plane LLE with \(\eta_N = N - 2\), where \(N < 5\), the value of the integral means \(\beta\)-spectrum at \(q = 2\) is the maximal real eigenvalue of the three-diagonal matrix (29).

Proof. By direct computation, from (22) and (30) we get

\[
P_3 = \beta^3 + (\eta_1 + \eta_2 + 3) \beta^2 + \left(2 + \eta_1 + \frac{5}{4} \eta_2 + \frac{3}{4} \eta_2 \eta_1\right) \beta - \frac{1}{4} \eta_2 \eta_1 - \frac{3}{4} \eta_2,
\]

\[
P_4 = \beta^4 + (\eta_2 + \eta_1 + \eta_3 + 4) \beta^3 + \left(\frac{3}{4} \eta_2 \eta_1 + \frac{5}{2} \eta_2 + \eta_3 \eta_1 + \frac{3}{4} \eta_2 \eta_2 + 2 \eta_3 + 6 + 2 \eta_1\right) \beta^2
\]

\[+
\left(\frac{1}{2} \eta_3 \eta_2 \eta_1 + \frac{3}{4} \eta_2 \eta_2 + \eta_2 + 4 + \frac{3}{4} \eta_2 \eta_1 + 2 \eta_1\right) \beta - \frac{3}{4} \eta_2 \eta_2 - \frac{1}{4} \eta_3 \eta_2 \eta_1 - \frac{3}{4} \eta_2 - \frac{1}{4} \eta_2 \eta_1.
\]
Since \( \eta_n > 0, n > 0 \), the polynomial \( P_4 \) has a single negative coefficient. By Descartes’ rule of sign it has only one positive root and the rest of the roots are negative. The same applies to \( P_3 \). By lemma 1, \( \beta(2) \) equals the maximal root of characteristic polynomial.

Concluding this section we would like to mention another set of examples which are similar to the Hastings–Levitov (HL) models [12], namely, LLE driven by a compound Poisson process with jumps distributed uniformly over the circle (we call them PLE\( _\lambda \), where \( \lambda \) is the arrival rate of the Poisson process). Here, \( \eta_n = \lambda = N - 2, n > 0 \) and the recurrence coefficients \( a_{n_0}, n = 1 \ldots N - 1 \) are positive. Therefore all roots of \( P_N(\beta) \) are real and simple. We have verified for \( N \) up to several hundred that a polynomial \( P_N(\beta) \) has only one change of sign in the sequence of its coefficients. By consequence, in these (verified) models, \( \beta(2) \) equals maximal eigenvalue which is the only non-negative eigenvalue (see figure 2).

4. Conclusions

In the present paper we considered a wide class of the whole-plane LLEs driven by a generic Levy process on a circle, without the drift, restricted by a condition that there exist \( N \) for which \( \eta_N = N + 2 \) in the unbounded version of LLE and \( \eta_N = N - 2 \) in the bounded version. We showed that the second moment of derivative of the stochastic conformal mapping for the whole-plane LLE is expressed in terms of solution of \( N \)-dimensional Fuchsian system with three singular points. The value of the integral means \( \beta \) spectrum at \( q = 2 \) (i.e. \( \beta(2) \)) is a non-negative eigenvalue of a three-diagonal matrix. We also were able to show that \( \beta(2) \) is actually the maximal eigenvalue for several wide classes of processes.

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\( ^8 \) Actually, there is no need to consider \( P_3 \) separately since the existence of only one non-negative eigenvalue for \( N \) automatically implies the same for \( N' < N \).

\( ^9 \) Similarly to the HL model, the PLE\( _\lambda \) is a composition of random elementary ‘spike’ mappings uniformly distributed over the circle. Note, however, that the HL models are not LLEs. LLEs driven by compound Poisson processes have been considered in [14].
Therefore, one might try to generalize the theorems 1–3 by dropping the condition of smallness of $\delta \eta_n$ in the theorem 2. We recall that in the case of theorems 1 and 3 the condition of smallness is absent, so we may put forward the following

**Conjecture 1.** Let the whole-plane LLE be driven by a Levy process without drift for which there exists an integer $N$, such that

- $\eta_N = N + 2$ in the unbounded version of LLE
- $\eta_N = N - 2$ in the bounded version of LLE.

Then, the value of the integral means $\beta$-spectrum at $q = 2$ equals the maximal real eigenvalue of the three-diagonal matrix $B$ given by (20) or (29) correspondingly.

Moreover, one can generalize the above conjecture for the case when the truncation conditions do not necessarily hold. Indeed, in this case matrix $B$ is infinite and we can look for a maximal eigenvalue of this infinite matrix. In more detail, one has to take an infinite set of ‘formally orthogonal’ polynomials and look at the sequence of maximal roots (or equally at the sequence of maximal eigenvalues $\beta_{\text{max}}(N)$ of the sub-matrices of dimension $N$ of an infinite matrix $B$). One expects this sequence to converge to $\beta(2)$ as $N \to \infty$.

We have checked this hypothesis numerically for SLE$_\kappa$ (where $\beta(q)$-spectrum is known for any $\kappa$ and $q$) for different $\kappa$. Also, for PLE$_\lambda$ numerical computations suggest that the maximal eigenvalue is a smooth function of $\lambda$ (see figure 2). When the $N$-truncation takes place, the maximal eigenvalue of the infinite matrix equals the one of its sub-matrix of size $N$, i.e. the sequence $\beta_{\text{max}}(M)$ reaches its limit at $M = N$.

Therefore, one may state the following

**Conjecture 2.** Let the whole-plane LLE be driven by a Levy process without drift. Then $\beta(2)$ is a limit of the sequence of maximal real eigenvalues of matrices $B_N$, where $B_N$ is given by (20) for the unbounded version of LLE or (29) for the bounded version (or, equivalently, the sequence of maximal roots of polynomials set by the recurrence relation (22) and (23) or (22) and (30) correspondingly).

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