VELOCITY FIELD OF COMPRESSIBLE MAGNETOHYDRODYNAMIC TURBULENCE:
WAVELET DECOMPOSITION AND MODE SCALINGS

Grzegorz Kowal1,2,3 and A. Lazarian1
1 Department of Astronomy, University of Wisconsin, 475 North Charter Street, Madison, WI 53706, USA; kowal@astro.wisc.edu, lazarian@astro.wisc.edu
2 Astronomical Observatory, Jagiellonian University, Orla 171, 30-244 Kraków, Poland
3 Instituto de Astronomia, Geofísica e Ciências Atmosféricas, Universidade de São Paulo, Rua do Matão 1226, CEP 05508-900, São Paulo, Brazil

ABSTRACT

We study compressible magnetohydrodynamic turbulence, which holds the key to many astrophysical processes, including star formation and cosmic-ray propagation. To account for the variations of the magnetic field in the strongly turbulent fluid, we use wavelet decomposition of the turbulent velocity field into Alfvén, slow, and fast modes, which presents an extension of the Cho & Lazarian decomposition approach based on Fourier transforms. The wavelets allow us to follow the variations of the local direction of the magnetic field and therefore improve the quality of the decomposition compared to the Fourier transforms, which are done in the mean field reference frame. For each resulting component, we calculate the spectra and two-point statistics such as longitudinal and transverse structure functions as well as higher order intermittency statistics. In addition, we perform a Helmholtz–Hodge decomposition of the velocity field into incompressible and compressible parts and analyze these components. We find that the turbulence intermittency is different for different components, and we show that the intermittency statistics depend on whether the phenomenon was studied in the global reference frame related to the mean magnetic field or in the frame defined by the local magnetic field. The dependencies of the measures we obtained are different for different components of the velocity; for instance, we show that while the Alfvén mode intermittency changes marginally with the Mach number, the intermittency of the fast mode is substantially affected by the change.

Key words: ISM: structure – magnetohydrodynamics (MHD) – turbulence

1. INTRODUCTION

Astrophysical fluids are magnetized, and therefore, the astrophysical turbulence is magnetohydrodynamic (MHD) in nature. Compressible MHD turbulence is a key element for understanding star formation (see MacLow 2004; Elmegreen & Scalo 2004; McKee & Ostriker 2007; and references therein), and velocity fluctuations determine many of its properties, for instance, in the modern paradigm of star formation in which turbulent velocity sweeps up the matter from a large expanse of the interstellar space to create molecular clouds. Thus, it is important to know the statistical properties of the velocity field, e.g., its spectrum, which reflects how much energy is associated with the motions at a particular scale (see below).

A further insight into the properties of turbulence, including its generation, consequences, and dissipation, calls for the use of more sophisticated measures. For example, the processes of magnetic field generation depend on the velocity field vorticity associated with the solenoidal motions, while the processes of compressing gas are determined by the compressible motions. The problem of decomposing the velocity field into solenoidal and dilatational parts following Helmholtz–Hodge decomposition is frequently attempted (see Federrath et al. 2008, 2009). Another approach is the decomposition of the turbulent field in the MHD case into basic modes, i.e., Alfvén, slow, and fast waves. While this approach is trivial for the case of a strong magnetic field with infinitesimal fluctuations (see Dobrowolny et al. 1980), Cho & Lazarian (2002, 2003) proposed a statistical decomposition of modes in the Fourier space. The statistical nature of the procedure is clear when one considers its application to strongly perturbed magnetic fields. As the Fourier transform is defined in the reference frame related to the mean magnetic field, while the MHD motions happen with respect to the local magnetic field, there is an inevitable contribution of all types of motions to the decomposed modes. However, studying the cases when the real space decomposition was possible in real space, Cho & Lazarian (2003, henceforth CL03) showed that the cross-talk between the modes is small for sub-Alfvénic turbulence.

Testing of the results in CL03 and increasing the accuracy of the MHD mode decomposition of turbulence are the key goals of the present study. In doing so, in this paper we make use of wavelet transformations in addition to Fourier transformations. Wavelets (see Meneveau 1991a, 1991b) present a natural way of describing MHD turbulence. Indeed, while in the representation of the Goldreich & Sidhar (1995, henceforth GS95) model of turbulence the anisotropy is frequently described in terms of eddies with parallel \( k_{\parallel} \) and perpendicular \( k_{\perp} \) wave vectors, the actual description calls for choosing \( \parallel \) and \( \perp \) with respect to the local magnetic field.4 The latter is really easy to understand, as it is only the local magnetic field that influences fluid motions at a given point. Wavelets allow for a local description of magnetized turbulent eddies.

In this paper, we decompose the turbulent velocity fields using both wavelets and a more traditional Helmholtz–Hodge decomposition into solenoidal and compressible parts. We feel that the latter decomposition is more justified for the hydrodynamic turbulence than for the MHD turbulence that we study here. However, we feel that the use of a Helmholtz–Hodge decomposition provides an additional, although limited, insight into the properties of compressible motions.

4 This is due to the fact that closure relations used for the model justification in GS95 are doubtful. The importance of the local system of reference was clearly stressed in the works that followed the original GS95 study (Lazarian & Vishniac 1999; Cho & Vishniac 2000; Maron & Goldreich 2001; Cho et al. 2002, 2003; Lithwick & Goldreich 2001; Cho & Lazarian 2002).
The three major properties of the velocity field on which we focus our attention in this paper are turbulence spectra, anisotropies, and intermittency. These three measures require further description, which we provide below.

While turbulence is an extremely complex chaotic nonlinear phenomenon, it allows for a remarkably simple statistical description (see Biskamp 2003). If the injections and sinks of the phenomenon, it allows for a remarkably simple statistical further description, which we provide below.

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No. 1, 2010 VELOCITY FIELD OF COMPRESSIBLE MHD TURBULENCE 743

weak (see Galtier et al. 2000), while at sufficiently small scales it gets strong (see the discussion in Lazarian & Vishniac 1999).

spatial variations of any physical variable, X(r), is related to the amount of change of X between points separated by a chosen displacement I, averaged over the entire volume of interest. The result is usually given in terms of the Fourier transform of this average, with the displacement I being replaced by the wave number k parallel to I and |k| = 1/|I|. For example, for isotropic turbulence, the kinetic energy spectrum, E(k)dk, characterizes how much energy resides at the interval k, k + dk. At some large scale L (i.e., small k), one expects to observe features reflecting energy injection. At small scales, energy dissipation should be seen. Between these two scales, we expect to see a self-similar power-law scaling reflecting the process of nonlinear energy transfer. We shall attempt to get the power-law scalings for the components of the velocity field.

The presence of a magnetic field makes MHD turbulence anisotropic (Montgomery & Turner 1981; Matthaeus et al. 1983; Shebalin et al. 1983; Higdon 1984; Goldreich & Sridar 1995; see Oughton et al. 2003 for review). The relative importance of hydrodynamic and magnetic forces changes with scale, and so the anisotropy of MHD turbulence does too. Many astrophysical results, e.g., the dynamics of dust, scattering and acceleration of energetic particles, and thermal conduction, can be obtained if the turbulence spectrum and its anisotropy are known (see Lazarian et al. 2009 for review). The knowledge of the anisotropy of the Alfvénic mode of MHD turbulence determines the extent of magnetic field wandering that influences heat transfer (Narayan & Medvedev 2001; Lazarian 2006) and magnetic reconnection (Lazarian & Vishniac 1999; Kowal et al. 2009).

In the following, we would like to stress that we discuss the properties of strong MHD turbulence. This type of turbulence is not directly related to the amplitude of the magnetic perturbations, however. The low-amplitude turbulence can be strong, and isotropically driven turbulence with δB ≪ B at the injection scale exhibits only a limited range of scales in which it is weak (see Galtier et al. 2000), while at sufficiently small scales it gets strong (see the discussion in Lazarian & Vishniac 1999).

An anisotropic spectrum alone, say, E(k)dk, cannot characterize MHD turbulence in all its complexity because it involves only the averaged energy in motions along a particular direction. To have a full statistical description, one needs to know not only the averaged spectrum of a physical variable but also higher orders. The tendency of fluctuations to become relatively more violent but increasingly sparse in time and space as the scales decrease, so that their influence remains appreciable, is called intermittency. Intermittency increases with the ratio of the size scales of injection and dissipation of energy, so the very limited range of scales within numerical simulations may fail to reflect the actual small-scale processes. The turbulence intermittency can result in an important intermittent heating of the interstellar medium (ISM; Falgarone et al. 2005, 2006, 2007).

In this paper, we investigate the scaling properties of the structure functions of the velocity and its components for compressible MHD turbulence with different sonic and Alfvénic Mach numbers. In Section 2, we describe the numerical models of compressible MHD turbulence. We decompose the velocity into a set of components including the incompressible and compressible parts, and MHD waves: Alfvén, slow, and fast using the methods described in Section 3. In Section 4 we study the spectra of the velocity and its components. In Section 5 we study the anisotropy of dissipative structures. We show differences in the structures for different components. In Section 6, we study the scaling exponents and the intermittency of the velocity structures. We show their dependence on the sonic and Alfvén regime of turbulence. In Section 7, we discuss our results and their relation to the previous studies. In Section 8 we draw our conclusions.

2. NUMERICAL SIMULATIONS

We used a second-order-accurate essentially non-oscillatory (ENO) scheme (see Cho & Lazarian 2002) to solve the ideal isothermal MHD equations in a periodic box,

where ρ is the density, v is the velocity, B is the magnetic field, and a is the isothermal speed of sound. We incorporated the field interpolated constrained transport (CT) scheme (see, e.g., Tóth 2000) into the integration of Equation (3) to maintain the v · B = 0 constraint numerically. On the right-hand side, the source term f represents a random solenoidal large-scale driving force. The rms velocity δv is maintained to be approximately unity, so that v can be viewed as the velocity measured in units of the rms velocity of the system and B/(4πρ)1/2 as the Alfvén velocity in the same units. The time t is in units of the large eddy turnover time (∼L/δv) and the length in units of L, the scale of the energy injection. The magnetic field consists of the uniform background field and a fluctuating field: B = Bext + b. Initially, b = 0. We use units in which the Alfvén speed vA = Bext/(4πρ)1/2 = 1 and ρ = 1 initially. The values of Bext have been chosen to be similar to those observed in ISM turbulence.

For our calculations, similar to our earlier studies (Kowal et al. 2007), the sound speed and the strength of the external field Bext are the controlling parameters defining the sonic Mach number Ms = (δv/a) and the Alfvénic Mach number MA = (δv/vA), respectively. The angle brackets ⟨⟩ signify the averaging over the volume. Ms < 1 and Ms > 1 define subsonic and supersonic regimes, respectively, and MA < 1 and MA > 1 define another two regimes, sub-Alfvénic and super-Alfvénic, respectively. Since these two parameters are independent we can analyze, e.g., supersonic sub-Alfvénic turbulence, which signifies that Ms > 1 and MA < 1. Note that within our model the β-plasma is controlled by the ratio of Mach number, since $\beta \equiv p/p_B \sim (M_A/M_s)^2$.

We present three-dimensional numerical experiments of compressible MHD turbulence for a broad range of Mach numbers (0.7 ≤ Ms ≤ 7.5 and 0.5 ≤ MA ≤ 2.1; see Table 1). The model name contains two letters “P” and “B” followed by a number. The letters “B” and “P” mean the external magnetic
field and the initial gas pressure, respectively, and the numbers designate the value of the corresponding quantity. For example, a name “B.1P.01” points to an experiment with $B_{\text{ext}} = 0.1$ and $P = 0.01$.

We drove the turbulence at wave scale $k$ equal to about 2.5 (2.5 times smaller than the size of the box). This scale defines the injection scale in our models. We did not set the viscosity (2.5 times smaller than the size of the box). This scale defines the injection scale in our models. We did not set the viscosity explicitly in our models. The scale at which the dissipation starts to act is defined by the numerical diffusivity $(2.5 \text{ times smaller than the size of the box})$. This scale defines the dissipation scale from the velocity spectra. In the case of our models, we estimated the dissipation scale approximately equal to about 2.5.

Table 1

| Model   | $B_{\text{ext}}$ | $a$ | $M_f$ | $M_A$ | Resolution | Maximum Time | $\delta p$ | $\delta |\rho|$ |
|---------|------------------|-----|-------|-------|------------|--------------|------------|----------|
| B1P1    | 1.0              | 1.0 | 0.75  | 0.04  | 512³       | 7.0          | ~0.37     | ~0.79    |
| B1P1    | 1.0              | 0.3 | 2.52  | 0.05  | 512³       | 7.0          | ~0.14     | ~0.76    |
| B1P01   | 1.0              | 0.1 | 1.95  | 0.3   | 512³       | 7.0          | ~2.22     | ~0.75    |
| B1P01   | 0.1              | 1.0 | 0.74  | 0.02  | 512³       | 7.0          | ~0.26     | ~0.74    |
| B1P1    | 0.1              | 0.3 | 2.52  | 0.05  | 512³       | 7.0          | ~0.92     | ~0.76    |
| B1P01   | 0.1              | 1.0 | 7.52  | 0.09  | 512³       | 7.0          | ~1.85     | ~0.75    |

Figure 1. Graphical representation of the mode separation method. We separate the Alfvén, slow, and fast modes by the projection of the velocity Fourier component $v_k$ on the bases $\hat{\xi}_A$, $\hat{\xi}_s$, and $\hat{\xi}_f$, respectively. Figure taken from CL03.

Applying the divergence operation on both sides and using the divergence-free property of the solenoidal and Laplace components we obtain

$$\nabla \cdot \mathbf{u} = \nabla^2 \phi = \Delta \phi. \tag{8}$$

To find the scalar potential $\phi$ and the potential field $\mathbf{u}_p$ we have to solve the Poisson equation with a source term $\nabla \cdot \mathbf{u}$.

To calculate the vector potential $\mathbf{A}$ we apply the curl operation on both sides of Equation (7). Similarly, using the divergence-free property of the potential and Laplace fields results in the equation

$$\nabla \times \mathbf{u} = \nabla \times \nabla \times \Phi = \Delta \Phi. \tag{9}$$

Here, the calculation of the vector potential $\mathbf{A}$ requires solving a triple set of Poisson equations—one equation for each component of $\Phi$.

The simulations with periodic boundary conditions have the advantage of allowing us to solve the Poisson equation using Fourier methods. The Fourier components of the velocity field are then transformed back into the real space, and are further analyzed.

3.2. Separation into the Alfvén, Slow, and Fast Modes

Another very important type of decomposition is the separation of velocity into the MHD waves: Alfvén, slow, and fast. In this paper, we use an extended mode based on a technique described in CL03. The procedure of decomposition is performed in Fourier space by a simple projection of the velocity’s Fourier components $\mathbf{u}$ on the direction of the displacement vector for each mode (see Figure 1). The directions of the displacement vectors $\hat{\xi}_s$, $\hat{\xi}_f$, and $\hat{\xi}_A$ corresponding to the slow, fast, and Alfvén modes, respectively, are defined by their unit vectors

$$\hat{\xi}_s \propto (1 + \alpha - \sqrt{D})\hat{k}_\parallel \hat{k}_\perp + (1 + \alpha - \sqrt{D})\hat{k}_\parallel \hat{k}_\perp, \tag{10}$$

$$\hat{\xi}_f \propto (1 + \alpha + \sqrt{D})\hat{k}_\parallel \hat{k}_\perp + (1 + \alpha + \sqrt{D})\hat{k}_\parallel \hat{k}_\perp, \tag{11}$$

and

$$\hat{\xi}_A = -\hat{\phi} = \hat{k}_\perp \times \hat{k}_\parallel, \tag{12}$$
where $k_\parallel$ and $k_\perp$ are the parallel and perpendicular to $B_{\text{ext}}$ components of the wave vector, respectively; $D = (1 + \alpha)^2 - 4\alpha \cos^2 \theta$; $\alpha = a^2 / V_0^2$; $\theta$ is the angle between $k$ and $B_{\text{ext}}$; and $\varphi$ is the azimuthal basis in the spherical polar coordinate system. The Fourier components of each mode can be directly used to calculate spectra. For other measures, such as structure functions, we transform them back into real space.

We extend the CL03 technique by introducing an additional step before the Fourier separation, in which we decompose each component of the velocity field into orthogonal wavelets using the discrete wavelet transform (see, e.g., Antoine 1999),

$$U(a, \mathbf{w}_{\text{lmm}}) = a^{-N/2} \sum_{\mathbf{x}_{i,j,k}} \psi \left( \frac{x_{ijk} - \mathbf{w}_{\text{lmm}}}{a} \right) u(x_{ijk}) \Delta^N \mathbf{x},$$

where $x_{ijk}$ and $\mathbf{w}_{\text{lmm}}$ are the $N$-dimensional position and translation vectors, respectively; $a$ is the scaling parameter; $u(x_{ijk})$ is the velocity vector field in the real space; $U(x_{ijk})$ is the velocity vector field in the wavelet space; and $\psi$ is the orthogonal analyzing function called the wavelet. The summation in the equation is taken over all position indices. We use the 12-tap Daubechies wavelet as an analyzing function and the fast discrete version of the wavelet transform (Antoine 1999); thus, as a result, we obtain a finite number of wavelet coefficients.

After the wavelet transform of the velocity, we calculate the Fourier representation of each wavelet coefficient and perform its individual separation into the MHD waves in the Fourier space using the CL03 method and then update the Fourier coefficients of all MHD waves, iterating over all wavelets. In this way, we obtain a Fourier representation of the Alfvén, slow, and fast waves. The final step is the inverse Fourier transform of all wave components.

This additional step allows for an important extension of the CL03 method, namely, it allows for the local definition of the mean magnetic field and density used to calculate the $\alpha$ and $D$ coefficients. Since the individual wavelets are defined locally both in the real and Fourier spaces, the averaging of the mean field and density is done only within the space of each wavelet.

We can summarize the extended version of the MHD wave decomposition through the following steps.

1. Perform a wavelet transform of all velocity components.
   2. Iterate over all wavelet coefficients.

   (a) Calculate the Fourier representation of the current wavelet.

   (b) Calculate the mean density and magnetic field within the space occupied by the wavelet.

   (c) Perform the CL03 separation in the Fourier space.

   (d) Add the contribution from the separated wavelet to the Fourier representation of the Alfvén, slow, and fast modes.

3. Perform the inverse Fourier transform of the Alfvén, slow, and fast wave components.

4. SPECTRA OF THE VELOCITY COMPONENTS

In order to obtain one-dimensional (1D) spectra, we first calculate the Fourier transform of a quantity and then multiply it by its conjugate. We can use this procedure because we used a fully periodic domain. The three-dimensional spectra must be averaged or integrated over shells $k_n \leq k < k_{n+1}$. We used a simple integration by summing all squared amplitudes at the given shell. For each model, we have collected data at several time steps. We used them to increase the size of the sample and to measure the time departures of the spectra from their mean profiles. We should note here, however, that the time averaging and standard deviations were calculated in log–log space. Otherwise, it could result in taking the logarithm of a negative number, e.g., for ranges of $k$ where the power spectrum has very small values (of order $10^{-6}$ or less).

In Figure 2, we present the spectra for the velocity field (top row) and its incompressible and compressible parts (middle and bottom rows, respectively) for models with different sonic Mach numbers. The left column in Figure 2 shows models with a strong magnetic field ($M_A \sim 0.5–0.7$), while the right column shows the corresponding models but with a weak magnetic field ($M_A \sim 1.9–2.1$). The spectra are averaged over several time snapshots taken for a fully developed turbulence. This allowed us to estimate the spectra time variance which is shown as the gray area plotted around the mean profiles. We see that in all cases those variances are small.

Within the inertial range, which we estimated to be $k \in [4, 20]$, the spectra of the velocity field slightly change their spectral indices with the value of $M_s$. In the case of sub-Alfvénic models, the spectral index is close to $-5/3$. In the case of super-Alfvénic turbulence, the indices do not change with $M_s$ as well, and all models show the same, close to $-2$, value of the spectral index at low wave numbers, which is a reflection of the presence of shocks in the system. Proceeding toward smaller scales we observe a bump with the spectral index approaching $-5/3$, which could be explained by the growing importance of the mean field at these scales and the dominance of Alfvén waves. The velocity fluctuations have comparable amplitudes for corresponding scales for all models. It means that the strength of fluctuations at a particular scale $k$ within the inertial range does not depend on $M_s$. The total velocity field contains two components: solenoidal, which is equivalent to the incompressible part, and potential, which contains the compressible part of the field and the remaining part which is curl and divergence free. In Table 2, we show the percentage contribution of each component to the total velocity field. We see that the compressible part constitutes only a fraction of the total field. However, the magnitude of this fraction is different for sub and supersonic models. In the case of sub-Alfvénic turbulence, it is about 3% in subsonic models and about 7% in supersonic models, which confirms a higher efficiency of the compression in the presence of supersonic flows. Furthermore, the fraction also changes when we compare models with strong and weak magnetic fields. In the presence of a weak magnetic field, the velocity field contains about 5% of the compressible part in the model with $M_s \sim 0.7$ and even up to 16% in models with $M_s > 1$. The consequence of the presence of a strong magnetic field results in a reduction of the compressible part of the velocity field by a factor of two. This indicates a substantial role of the magnetic field in the damping of the generation of the compressible flows.

Due to a substantial dominance of the incompressible part of the velocity field, we expect that its spectra should follow the spectra for the total velocity field. In the middle row of Figure 2, we present the spectra for the incompressible part of the velocity field. We see that the spectra, at least within the inertial range, are very similar to those observed for the velocity field. The spectral indices also have very close values for models with strong and weak magnetic fields.

In the case of compressible turbulence, the spectra of the potential component (the bottom row in Figure 2) are more
interesting. We see that these spectra are different from those for the incompressible part. The strength of the compressible part confirms the change of the contribution of the compressible part. It is smaller for subsonic models than for supersonic models, which can be seen in the bottom row of Figure 2. However, both supersonic models have almost the same spectra with the spectral indices about $-2$ for sub-Alfvénic turbulence and for super-Alfvénic turbulence. Both supersonic spectra show almost exactly the same profiles and amplitudes at all scales, even within the dissipation range. This means that the amount of the compressible part of the velocity field does not change with $M_s$ when its value is larger than 1.

The second decomposition, which is more important for MHD turbulence, separates the velocity field into three different MHD waves: an incompressible Alfvén wave and slow and fast magneto acoustic waves, both of which are compressible. In
Table 2

| $M_e$ | $M_A$ | $V_{\text{incomp}}$ | $V_{\text{comp}}$ | $V_A$ | $V_{\text{f}}$ |
|-------|-------|---------------------|-------------------|-------|--------------|
| $-0.8$ | $-0.7$ | $96.5^{+0.8}_{-0.7}$ | $3.2^{+0.8}_{-0.7}$ | $58^{+4}_{-3.7}$ | $3^{+3}_{-2.9}$ | $4.8^{+0.7}_{-0.9}$ |
| $-2.5$ | $-2.6$ | $93^{+3}_{-2.7}$ | $7^{+2}_{-2.7}$ | $58^{4}_{-3.4}$ | $3^{+3}_{-2.9}$ | $9^{+2}_{-2.4}$ |
| $-7.5$ | $-7.5$ | $92^{+2}_{-2.5}$ | $7^{+2}_{-2.5}$ | $56^{4}_{-3.4}$ | $3^{+3}_{-2.9}$ | $8.0^{+0.7}_{-0.9}$ |
| $-0.7$ | $-2.1$ | $95^{+2}_{-2.5}$ | $5^{+2}_{-2.5}$ | $52^{+4}_{-3.4}$ | $4^{+3}_{-2.9}$ | $6.2^{+0.8}_{-0.9}$ |
| $-2.5$ | $-2.1$ | $86^{+1}_{-2.5}$ | $14^{+2}_{-2.5}$ | $47^{+3}_{-3.4}$ | $3^{+3}_{-2.9}$ | $16^{+2}_{-2.4}$ |
| $-7.5$ | $-1.9$ | $84^{+2}_{-2.5}$ | $16^{+2}_{-2.5}$ | $47^{+4}_{-3.4}$ | $3^{+3}_{-2.9}$ | $20^{+2}_{-2.4}$ |

Note. Errors correspond to a measure of the time variation.

Table 2, we included the percentage amount of these components in the total velocity field. As we see, most of the energy is contained in the Alfvén wave. It is almost 60% in the case of sub-Alfvénic turbulence, and about 50% for super-Alfvénic turbulence. The slow wave contains approximately 1/3 of the total energy. However, for the super-Alfvénic case, this amount is slightly higher. Table 2 suggests that the slow wave is weaker when the turbulence becomes supersonic. We do not see a similar behavior for the Alfvén wave in the case of models with a strong magnetic field. This effect could also take place in the super-Alfvénic models, but it is weakened by relatively large errors. An interesting dependence is observed in the case of the fast wave. Although the fast wave is the weakest among all MHD waves, it strongly depends on the regime of turbulence. Similarly to the compressible part of the velocity field, it is stronger for models with a weak magnetic field. In addition, it is much stronger when turbulence is supersonic, but this strength seems to be weakly dependent on the sonic Mach number.

In Figure 3, we present spectra for the MHD waves for all models from Table 1. In the left column we show spectra for models with a strong magnetic field, while in the right column spectra, we show for models with a weak magnetic field. The top, middle, and bottom rows show the Alfvén, slow, and fast mode spectra, respectively. In the case of the Alfvén wave, we see that the spectral indices weakly depend on the sonic Mach number. All indices lie between $-2$ and $-5/3$ with a slight dependence on $M_A$. This situation is similar to that of the spectra of the velocity field. For super-Alfvénic models we observe a similar situation; however, since the mean field is weaker, the power spectra are shifted down to smaller amplitudes. The similarities in spectra between the Alfvén mode and the velocity field are due to the fact that the Alfvén mode constitutes a major part of the velocity field.

The slow wave, however, also constitutes a substantial part of the velocity field. Looking closer at the spectral indices for the slow mode plotted in Figure 3, we see that their values do not change significantly with $M_e$ when the mean magnetic field is strong, although it is difficult to determine one spectral index due to the changing profile of the spectrum with the wave number. There seems to be a stronger dependence of the spectral indices on the sonic Mach number when the mean field is weak, where the inertial region is easier to determine.

The fast wave, the weakest mode of the velocity field, shows two types of spectra depending on the sonic and Alfvénic regime of turbulence. In the case of subsonic models, we see that the fast mode spectrum has an index close to $-2$. When the field is weak, the value of the index is a bit flatter than $-2$. This indicates a clear dissimilarity of spectra for subsonic turbulence for different strengths of the magnetic field. In supersonic models, however, the spectral indices are closer to the value of $-2$, independent of the sonic Mach number. This signifies the growing role of shock compression in supersonic turbulence.

5. ANISOTROPY

If we want to study the anisotropy of turbulent structures, we need to introduce a reference frame. In the case of magnetized turbulence, the reference frame is defined in the natural way by the local mean magnetic field. The local magnetic field is computed using the procedure of smoothing by a three-dimensional Gaussian profile with the width equal to the separation length. Since the volume of smoothing grows with the separation length $l$, the direction of the local magnetic field might change with $l$ at an arbitrary point. This is an extension of procedures employed in Cho & Vishniac (2000) and Cho et al. (2002).

To analyze the anisotropy of the different components of the velocity field, we use the total second-order structure function

$$ S_2(l) = \langle (v(x + l) - v(x))^2 \rangle, $$

where $\langle \cdot \cdot \cdot \rangle$ denotes an ensemble averaging. For each component, we evaluate the parallel and perpendicular structure functions taking respectively the directions of the separation length $l$ parallel and perpendicular to the local magnetic field. To show the degree of anisotropy, we plot the perpendicular structure function as a function of the parallel structure function for corresponding separation lengths. In this way we also show how the anisotropy changes with the scale of structures.

In Figure 4, we present the degree of anisotropy for the velocity field (top row) and for the solenoidal (incompressible) and potential (compressible) parts of the velocity fields (middle and bottom rows, respectively). The left and right columns correspond to the sub- and super-Alfvénic turbulence, respectively. The gray areas behind the points show the degree of time variances of the structure functions used to plot anisotropies. In each panel of Figure 4, we also plot lines corresponding to the isotropic structure, $l_0 \sim l_1$, and theoretical, $l_0 \sim l_1^{2/3}$ (Goldreich & Sidhar 1995, GS95). We see that the structures of velocity and its incompressible part show the GS95-type anisotropy for both Alfvénic regimes, although the degree of anisotropy is reduced in the case of weakly magnetized turbulence (see the top right in Figure 4). Another conclusion coming from these plots is that the anisotropy is insensitive to the value of the sonic Mach number. All curves have the same shape in the presented plots. The one noticeable difference is that in the case of sub-Alfvénic turbulence the anisotropy slightly changes with the values of $S_2$, which are related to the scale of structures. Lower values of $S_2$ correspond to the small-scale structures; larger values correspond to the large-scale structures. In this description we can see that the small-scale structures are highly anisotropic, and the degree of anisotropy decreases with the scale. In the case of a weak external magnetic field, the degree of anisotropy scales according to the GS95 model, but isotropy is reached at scales smaller than in the case of strongly magnetized turbulence. The compressible component of the velocity field behaves differently. For sub-Alfvénic turbulence it is almost perfectly isotropic and independent of the sonic Mach number. In super-Alfvénic turbulence, however, we see that structures of the subsonic potential field are isotropic on average, although there are relatively large time departures from isotropy. On the contrary, in the case of supersonic turbulence, the potential field contains a large amount of the anisotropic structures, and the degree of anisotropy follows the GS95 law.
Figure 3. Spectra of the Alfvén, slow, and fast modes (top, middle, and bottom rows, respectively) of the velocity field for experiments with different sonic Mach numbers in two regimes: sub-Alfvénic (left column) and super-Alfvénic (right column). The gray area denotes the degree of departures at single-time snapshots from the mean profile.

In Figure 5, we show the anisotropy of structures for the Alfvén, slow, and fast modes of velocity. The separation of the modes is performed with respect to the local magnetic field. The first comparison of these plots shows that the degree of anisotropy does not depend on the sonic Mach number for the Alfvén and slow modes. They show very similar anisotropies to those observed in the velocity and its incompressible part. We see that basically the degree of anisotropy of structures in these two modes follows the GS95 scaling, which decreases with the scale. In addition, the presence of a strong magnetic field results in the increase in the degree of anisotropy and much stronger bending of the curves in the left panels of Figure 5, signifying changes of the anisotropy with the scale of the structure. The curves in the plots for the super-Alfvénic models are almost straight and independent of the values of $S_L$. The anisotropy of the fast mode, similarly to the case of the compressible part of
the velocity, depends on the regime of sonic motions. If the fluid motions are subsonic, the structures of the fast waves are more isotropic. When the fluid motions are supersonic, more structures become anisotropic. We should note here that individual structures in the turbulence evolve and yield different deformations, resulting in a constant change in the degree of the anisotropy; thus, in this situation, speaking about anisotropy, we are relating to the mean dominant anisotropy of all structures in the system.

6. SCALING EXPONENTS AND INTERMITTENCY

Intermittency is an essential property of astrophysical fluids. As intermittency violates the self-similarity of motions, it is impossible to naively extrapolate the properties of fluids obtained computationally with a relatively low resolution to the actual astrophysical situations. In astrophysics, intermittency affects turbulent heating; momentum transfer; and interaction with cosmic rays, radio waves, and many other essential
processes. Physical interpretation of intermittency started after the work by Kolmogorov, but the first successful model was presented by She & Léveque (1994). The scaling relations suggested by She & Léveque (1994) relate $\zeta(p)$ to the scaling of the velocity $v_1 \sim l^{1/g}$, the energy cascade rate $t_1^{-1} \sim l^{-x}$, and the co-dimension of the dissipative structures $C$:

$$
\zeta(p) = \frac{P}{g} (1 - x) + C \left( 1 - (1 - x/C)^{p/g} \right). 
$$

Parameter $C$ is related to the dimension of the dissipative structures $D$ through the relation $C = 3 - D$ (Müller & Biskamp 2000). In hydrodynamical turbulence, according to the Kolmogorov scaling, we have $g = 3$ and $x = 2/3$. Vortex filaments, which are 1D structures, correspond to $C = 2$ ($D = 1$). In the MHD turbulence we also observe current sheets, which are two-dimensional dissipative structures and correspond to $C = 1$ ($D = 2$). For these two types of dissipative structures we obtain two different scaling relations (substituting
Relation (16) is often called the She & Lévéque scaling (She & Lévéque 1994), while relation (17), the Müller–Biskamp scaling (Müller & Biskamp 2000). There are theoretical arguments against the She–Lévéque (see Novikov 1994), but so far the She–Lévéque scaling is the best for reproducing the intermittency of incompressible hydrodynamic turbulence.

Structure functions can be calculated with respect to global or local reference frames. With the scaling exponents calculated in the global reference frame, we understand the scaling exponents calculated from the structure functions averaged over all directions. In the local reference, we distinguish between directions parallel and perpendicular to the local mean magnetic field. In this way, we define the reference frame locally by the direction of the local magnetic field.

In Figures 6 and 7, we show scaling exponents for the velocity and all its parts and waves calculated in the global reference frame. In the top left panel of Figure 6, we see that for the sub-Alfvénic turbulence the scaling exponents of velocity follow the She–Lévéque (S–L) scaling with $D = 1$. Supported by theoretical considerations, we can say that most of the dissipative structures are 1D. Even though the scalings are not perfectly independent of the value of $M_s$, since we see somewhat lower values of $\zeta$ for higher $p$, the differences between these values for models with different sonic Mach numbers are within their error bars, and thus it is relatively difficult to state that the scalings are completely independent of or only weakly dependent on the values of $M_s$. Looking in the corresponding plot for models with a weak magnetic field we clearly see that the spread of curves for different sonic Mach numbers is much higher than in the previous case. For the subsonic model, the scaling exponents of the velocity follow the theoretical curve defined by the S–L scaling with parameter $D$ corresponding to 1D structures very well. The model with $M_s \sim 2.3$, however, follows perfectly the S–L scaling with $D = 2$ corresponding to the two-dimensional dissipative structures. Moreover, models with even higher values of the sonic Mach number have the scaling exponents for $p > 3$ somewhat below the S–L scaling with $D = 2$. These observations suggest that the scaling exponents of the velocity change with the sonic Mach number but only in the case of weak magnetic field turbulence. The presence of a strong magnetic field significantly reduces these changes and prevents the generation of dissipative structures higher than one dimensions.

After the decomposition of velocity into its incompressible and compressible parts, we also calculate their scaling exponents. In the middle and right columns of Figure 6, we show the incompressible and compressible parts of the velocity field, respectively. The incompressible part is strong. It constitutes most of the velocity field, and thus it is not surprising that its scaling exponents are very similar to those observed in the velocity. This is true in the case of sub-Alfvénic models because all curves in the middle left panel in Figure 6 tightly cover the S–L scaling with $D = 1$. The similarity between the velocity and its solenoidal part is also confirmed in the case of super-Alfvénic models but only for the subsonic case, when the role of shocks is strongly diminished. Two supersonic models show exponents following a scaling closer to the S–L one with $D = 1$, still with lower values for $p > 3$. The scalings of the structure of the compressible part, shown in the right column of Figure 6, cannot be compared to any of the theoretical models. Their scalings signify dissipative structures with dimensions higher than 2, but the theory may not be applicable here. In order to explain what these scaling exponents represent, we should describe what could be a physical picture of the compressible part of the velocity field. First, the compressible part is much weaker than the incompressible one, and thus structure functions of higher
orders amplify rare events in the structure, such as individual regions compressed by shocks. This is supported by the huge error bars that increase with the value of $p$ signifying that these rare events may have poor statistics or can change rapidly, contributing differently at different times and in different models. In such situations, the scaling exponents for higher values of $p$ are not reliable.

Another decomposition, which separates the velocity into MHD waves, gives an opportunity to calculate their scaling relations as well. In Figure 7, we present the scaling exponents for the Alfvén, slow, and fast waves. The Alfvén wave is shown in the plots of the left column. Comparing these plots with the corresponding plots for the solenoidal part of the velocity reveals the obvious conclusion that the Alfvén wave, due to its incompressibility, should follow exactly the same scaling as the incompressible part. This is clearly visible in these two plots. The scaling exponents of the solenoidal part and the Alfvén mode match nicely in plots for sub-Alfvénic models with a weaker dependency of the sonic Mach number, and for super-Alfvénic models where the dependency of $M_s$ is stronger. Two other modes, the slow and fast waves presented in the middle and right columns of Figure 7, respectively, show similar scaling relations. For example, the scaling exponents for slow and fast waves in the sub-Alfvénic turbulence follow the S–L scaling with $D = 2$ and depend on the sonic Mach number marginally. This signifies that the two-dimensional structures dominate in both components. For super-Alfvénic turbulence, the scalings of these two waves show a somewhat different situation. Both components, the slow and fast waves, have similar values of scaling exponents $\zeta$, but they change with the sonic Mach number. We see that for subsonic turbulence most of the dissipative structures of the slow and fast waves are 1D. Scaling exponents follow very precisely the S–L scaling with $D = 1$. In the case of supersonic turbulence, however, these scalings suggest the two-dimensional dissipative structures again, similarly to the sub-Alfvénic turbulence. This signifies an important role of the magnetic field in the generation of the structures of different dimensions in the subsonic turbulence. The above considerations were carried out for the scaling relations in the global reference frame, i.e., when we do not take the direction of the local magnetic field into account. The strong magnetic field is dynamically important in sub-Alfvénic turbulence and can greatly influence the generation of the structure. Thus, the statistical methods, which also include its local direction in the analysis, can give substantial insight into the physics of turbulence with the presence of the magnetic field. In Figure 8, we show the scaling exponents calculated in the local reference frame for velocity and its incompressible and compressible parts. Starting from the sub-Alfvénic turbulence, which is shown in the two upper rows (for the parallel and perpendicular directions, respectively), we see that the scaling exponents are different depending on the direction. For example, the structures of the velocity in the direction perpendicular to the local field are 1D, as suggested by the plot in the second row on the left in Figure 8. These scalings are very similar to those calculated in the global reference frame. However, the same velocity field in the direction along the local magnetic field has different structures with dimensions higher than 2 according to the corresponding plots. Both scalings, in the parallel and perpendicular directions, depend only marginally on the sonic Mach number. Next, the solenoidal component shown in the middle column of Figure 8 has a similar property. Its structures in the perpendicular direction are very similar to that observed in the global reference frame while in the parallel direction its structures are two dimensional. Again, the potential component, although its scaling relation cannot be explained by the current theoretical models, has scaling relations in the perpendicular direction very similar to that in the global reference frame. All these observations signify that the dominant structures are created in the directions perpendicular to the local field, while the parallel structures are less significant and usually have more dimensions. We see that in these models the role of the
magnetic field in the generation of the structure is very clear. What do we expect when the magnetic field is weaker in such turbulence?

Now, we compare the scaling exponents of the MHD waves obtained in the local reference frame which are presented in Figure 9. The Alfvén wave, which is incompressible and directed perpendicular to the local magnetic field, is presented in the left column of Figure 9. In the sub-Alfvénic turbulence when the magnetic field is strong, we expect that most of the structure of the Alfvén mode should be generated in the direction perpendicular to the local magnetic field. This is confirmed by the corresponding plots in Figure 9. The scaling exponents in the perpendicular direction are consistent with those in the global reference frame (Figure 7). The scalings suggest that the 1D dissipative structures in all sub-Alfvénic models are independent of the sonic Mach number. In the parallel direction, the plot suggests dissipative structures with higher dimension, but at the same time we see very large error bars which signify very poor statistics. It means that the structures of the Alfvén wave created in the direction parallel to the local field are marginal. On the contrary, the slow wave has a direction parallel to the local mean magnetic field, which signifies that its dissipative structure
should be more dominant in the parallel direction. Indeed, the slow wave in the parallel direction has structures consistent with those observed in the global mean magnetic field while in the perpendicular direction it shows some random, rare events in the structure.

7. DISCUSSION

7.1. Major Accomplishments and Limitations of the Present Study

In this paper, we have introduced a new procedure which uses wavelets, for decomposing the MHD turbulence field into the Alfvénic, slow, and fast modes. Compared to the decomposition procedure based on Fourier transforms described in CL03, the wavelet decomposition is more local, and thus it better follows the local magnetic field direction with respect to which the decomposition into modes takes place. As a result, we expect the wavelet decomposition procedure to be more accurate for larger amplitudes of turbulence, i.e., larger perturbations of the magnetic field.

Our decomposition of the MHD turbulence confirmed the results in CL03 in terms of spectra, namely, that the Alfvénic and slow modes are anisotropic and consistent with the predictions of the GL95 model of incompressible turbulence, while the fast modes are mostly isotropic and form an acoustic turbulence cascade (Lithwick & Goldreich 2001; Cho & Lazarian 2002). As these results were used in studies of cosmic-ray scattering and acceleration (Yan & Lazarian 2002, 2004, 2008; Cho & Lazarian 2005; Brunetti & Lazarian 2007) as well as charged dust acceleration (Lazarian & Yan 2002; Yan & Lazarian 2003; Yan et al. 2004; Yan 2009), this is an encouraging development.

At the same time, the intermittency of the different MHD modes was shown to be very different. We clearly see the dependence of high-order statistics of compressible motions on the Mach number. We interpret this dependence as the result of shock formation, which eventually changes the nature of the compressible motion cascade compared to the CL03 assumptions.

The limitations of the present study arise from the yet unclear nature of the turbulent cascade. For instance, it was shown in Beresnyak & Lazarian (2009) that the degree of locality of the interactions in hydrodynamic and MHD cascade is different. Thus, even the largest available MHD simulations may not present the actual inertial range of the cascade, but the measured slope may be strongly affected by the extended bottle-neck effect of the simulations. In addition, the limited range over which Alfvénic turbulence is weak may exhibit a rather different scaling of fast modes as a result of the interactions of the Alfvénic and fast modes (Chandran 2005).

In addition, within the present study we intentionally do not consider the scaling of magnetic perturbations. The velocity and magnetic perturbations for sub-Alfvénic turbulence show some differences, which are rather difficult to study reliably with the available numerical simulations. These differences are not a part of the GS95 picture, but may reflect additional yet unclear properties of the MHD cascade (see Müller & Biskamp 2000).

In our study, we only used incompressible driving. In the presence of the compressible supersonic driving (Federrath et al. 2009) the scaling looks different, but the existence of the inertial range is then questionable due to the existence of the bottle-neck effect (see Beresnyak & Lazarian 2009). However, the later studies by Federrath et al. (2010) showed that there is no problem whatsoever with the inertial range in their simulations. Kritsuk et al. (2010) claimed that by combining compressible and incompressible driving in a Mach number-dependent fashion, one can obtain a better power-law inertial range. This issue requires further studies.

Turbulence driving in our study is balanced, in the sense that the energy flows in opposite directions are equal. In the presence of sources and sinks of turbulent energy, astrophysical turbulence is expected to be imbalanced. Our numerical studies of imbalanced turbulence in Beresnyak & Lazarian (2010) show that the properties of Alfvénic turbulence change substantially in the presence of imbalance. However, the degree of sustainable imbalance in compressible turbulence is still unclear. One expects the density fluctuation in turbulent fluid to reflect the incoming waves, altering the imbalance. We believe that in high Mach number fluids, the imbalance is low due to the existence of substantial density contrasts.

7.2. Astrophysical Implications of the Turbulence Anisotropy and Intermittency

Depending on driving, astrophysical turbulence may be sub-Alfvénic, if the injection velocity $V_L$ is less than Alfvén speed $V_A$; Alfvénic, if $V_L = V_A$; and super-Alfvénic, if $V_L > V_A$. This is also frequently described by the Alfvén Mach number $M_A = V_L / V_A$. Formally, the GS95 model applies only to incompressible motions with $V_L = V_A$, or equivalently, $M_A = 1$. Some of the astrophysical applications of the model, indeed, use the original form of the theory, which substantially limits the applications of the theory (see Narayan & Medvedev 2001). However, the model can easily be generalized to cover extensive ranges of super-Alfvénic and sub-Alfvénic turbulence (see Lazarian & Vishniac 1999; Lazarian 2006). For sub-Alfvénic turbulence with isotropic driving at the scale $L$, an initial weak cascade, in which the parallel scale of motions stays the same and the spectrum $E(k_\perp) \sim k_\perp^{-2}$ is applicable, transfers to the regime of strong turbulence at the scale of $L M_A^3$, for which the GS95 critical balance arguments are applicable. For super-Alfvénic turbulence, it approaches the GS95-type regime for smaller scales, while up to the scale $L M_A^3$ the turbulence is hydrodynamic. Therefore, the relations obtained for MHD turbulence that we have studied above can be generalized for cases of different driving intensities.

While the GS95 model is a model of incompressible turbulence, our simulations confirm the numerical findings in Cho & Lazarian (2002) and CL03 that the scaling of the Alfvénic mode in the compressible turbulence is very similar to its scaling in the incompressible case. In particular, the GS95 anisotropy of MHD turbulence determines the rate of magnetic field wandering, which is important for many astrophysical processes, including the ubiquitous process of magnetic reconnection (Lazarian & Vishniac 1999). Additional implications of magnetic field wandering include the diffusion of heat and cosmic rays, MHD acceleration of dust, etc. (see Lazarian et al. 2009 for a review). The wavelet approach has the potential of increasing accuracy while studying small-scale anisotropy in simulations with strongly perturbed magnetic fields.

Falgarone et al. (2005, 2006, 2007) and collaborators (Hily-Blant & Falgarone 2007; Hily-Blant et al. 2007, and references therein) attracted the attention of the interstellar research community to the potentially important implications of intermittency. A small and transient volume with high temperatures or violent turbulence can have significant effects on the net rates of processes within the ISM. For instance, many
interstellar chemical reactions (e.g., the strongly endothermic formation of CH⁺) might take place within very intensive intermittent vortices. The aforementioned authors claimed the existence of the observational evidence for such reactions and heating, but a more quantitative approach to the problem is possible. Beresnyak & Lazarian (2007; see also Lazarian et al. 2009) used the intermittency scaling and calculated the distribution of the dissipation rate in the turbulent volumes. In doing so, they used the fact that the She & Lévêque (1994) model of intermittency corresponds to the generalized log–Poisson distribution of the local dissipation rates (Dubrulle 1994; She & Waymire 1995). The obtained rates of enhancement were not sufficient to explain the heating required for inducing interstellar chemistry (Beresnyak & Lazarian 2007). The same approach was used by Pan et al. (2009); however, they obtained a different result. We believe that one should distinguish shocks from vortical motions while calculating the heating induced by intermittency. Our present study shows very different scalings relevant to these types of motions.

7.3. Studies of Compressible MHD Turbulence in the Astrophysical Context

Numerous studies of compressible MHD turbulence are done in the context of star formation (see reviews by Mac Low & Klessen 2004; McKee & Ostriker 2007; and references therein). Most of these simulations are focused on the large-scale appearances of turbulence, which are determined by turbulent driving, and do not exhibit any extended inertial range of turbulence.

Search for the universal relations for compressible turbulence resulted in the rise of interest in the Kritsuk et al. (2007) idea of searching the universality not for the velocity, but for the combination of the velocity and density in the form \( \rho^{1/3} v \). The density-weighted velocity \( \rho^{1/3} v \) was studied in the hierarchical models of compressible turbulence (see Fleck 1983 and references therein). The numerical study of hydrodynamic compressible turbulence revealed that, indeed, the density-modified velocity shows the same Kolmogorov scaling both for low and high Mach number turbulence (Kritsuk et al. 2007). A similar effect was confirmed in our MHD simulations (Kowal & Lazarian 2007). However, the physical justification of this universality is unclear and it may result just from the coincidental compensation of the change of velocity and density indexes as shocks develop at high Mach number turbulence. The testing of \( \rho^{1/3} v \) showed that it also provides universality for different driving as well, but only for the extended self-similarity structure functions (see Schmidt et al. 2008).

We report on the steepening of the spectra of compressible motions at high Mach numbers. High-resolution hydro simulations (Kritsuk et al. 2007; Schmidt et al. 2009; Federrath et al. 2010) show that the velocity spectrum becomes steeper for high Mach number simulations. This corresponds to the observational studies of the supersonic velocity turbulence in Padoan et al. (2006, 2009) and Chepurnov et al. (2006). These studies are done with the velocity channel analysis and velocity coordinate spectrum techniques, which are theory-motivated and tested techniques (Lazarian & Pogosyan 2000, 2004, 2006, 2008; Chepurnov et al. 2008). The application of these techniques should enhance the range of astrophysical turbulent velocity fields that can be studied observationally.5 It compares the numerics, observations, and theory, which the progress of understanding turbulence requires.

8. SUMMARY

In this paper, we presented a new technique of decomposing turbulent MHD motions into Alfvén, slow, and fast modes. The technique is based on the use of wavelets, which allow a more local decomposition compared to the Fourier approach in CL02 and CL03. This enables one to have better accuracy of the decomposition of MHD turbulence into the fundamental modes for higher amplitudes of magnetic perturbations.

By applying the wavelet decomposition to the results of our simulations of compressible MHD turbulence, we investigated the scaling properties of velocity in compressible MHD turbulence for different sonic \( M_s \) and Alfvénic \( M_A \) Mach numbers. We analyzed the spectra, anisotropy, scaling exponents and intermittency of the total velocity and its components corresponding to the Alfvén, slow, and fast modes. We found the following.

1. The amplitude of velocity fluctuations depends only marginally on \( M_s \). The lack of significant dependence of the velocity fluctuations is also observed for its incompressible part, as well as for the Alfvén and slow waves. The compressible part of the velocity and the fast wave show a dependence on \( M_s \), but only for subsonic turbulence. In the case of supersonic modes, the fluctuations of the compressible part and the fast mode of the velocity have comparable amplitudes.

2. The spectral indices depend on \( M_A \) in turbulence with a strong magnetic field. In the case of turbulence with a weak magnetic field, only the indices of the spectra of the fast wave change between sub- and supersonic models. For the other components, the spectral indices do not change appreciably with the sonic Mach number. While our conclusions about the spectra of fast modes for subsonic turbulence agree with the CL03 conclusion about the acoustic cascade of these modes, we feel that for a high Mach number, we get a spectrum of shocks. The anisotropy of Alfvénic turbulence and slow modes is in agreement with the GS95 theory for both the cases of high and low \( \beta \)-plasmas. The velocity fluctuations of the fast modes demonstrate isotropy.

3. In the global reference frame, we observe stronger changes of the scaling exponents and intermittency for the velocity and all its components with \( M_s \) in the case of turbulence with a weak magnetic field. The intermittency of structures grows with the value of \( M_s \). However, when the external magnetic field is strong, the intermittency for all components only marginally depends on the sonic Mach number. In the local reference frame, the scaling exponents in turbulence depend on direction with respect to the direction of the local mean magnetic field. The dependence is stronger for sub-Alfvénic turbulence.

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5 Recent examples of techniques of observational studies of the turbulent density field can be found in Kowal et al. (2007) and Burkhart et al. (2009, 2010). The velocity is a more covered statistic, but it is more difficult to study (see Lazarian 2009 for a review).
