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RAMIFICATION ESTIMATE FOR
FONTAINE-LAFFAILLE GALOIS MODULES

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Abstract. Suppose $K$ is unramified over $\mathbb{Q}_p$ and $\Gamma_K = \text{Gal}(\bar{K}/K)$. Let $H$ be a torsion $\Gamma_K$-equivariant subquotient of crystalline $\mathbb{Q}_p[\Gamma_K]$-module with HT weights from $[0, p-2]$. We give a new proof of Fontaine’s conjecture about the triviality of action of some ramification subgroups $\Gamma^{(v)}_K$ on $H$. The earlier author’s proof from [1] contains a gap and proves this conjecture only for some subgroup $s$ of index $p$ in $\Gamma^{(v)}_K$.

Introduction

Let $W(k)$ be the ring of Witt vectors with coefficients in a perfect field $k$ of characteristic $p$. Consider the field $K = W(k)[1/p]$, choose its algebraic closure $\bar{K}$ and set $\Gamma_K = \text{Gal}(\bar{K}/K)$. Denote by $\mathbb{C}_p$ the completion of $\bar{K}$ and use the notation $\mathbb{O}_{\mathbb{C}_p}$ for its valuation ring.

For $a \in \mathbb{Z}_{\geq 0}$, let $\mathcal{M}^{\text{cr}}_{\mathbb{Q}_p}(a)$ be the category of crystalline $\mathbb{Q}_p[\Gamma_K]$-modules with Hodge-Tate weights from $[0, a]$. Define the full subcategory $\mathcal{M}^{\text{cr}}_{N}(a)$ of the category of $\Gamma_K$-modules consisting of $H = H_1/H_2$, where $H_1, H_2$ are $\Gamma_K$-invariant lattices in $V \in \mathcal{M}^{\text{cr}}_{\mathbb{Q}_p}(a)$ and $p^N H_1 \subset H_2 \subset H_1$. J.-M. Fontaine conjectured in [5] that the ramification subgroups $\Gamma^{(v)}_K$ act on $H \in \mathcal{M}^{\text{cr}}_{N}(a)$ trivially if $v > N - 1 + a/(p-1)$. The author suggested in [1] a proof of this conjecture under the assumption $0 \leq a \leq p-2$.

It was pointed recently by Sh. Hattori to the author that the proof in [1] has a gap. More precisely, consider Fontaine’s ring $R = \lim_{\leftarrow} (\mathbb{O}_{\mathbb{C}_p}/p)_n$ where the projective limit is taken with respect to the maps induced by the $p$-power map in $\mathbb{C}_p$. Let $W_N$ be the functor of Witt vectors of length $N$. For $r = (o_n \mod p)_{n \geq 0} \in R$ and $m \in \mathbb{Z}$, set $r^{(m)} = \lim_{n \to \infty} o_n^{m+n} \in \mathbb{O}_{\mathbb{C}_p}$ and consider Fontaine’s map $\gamma : W_N(R) \to \mathbb{O}_{\mathbb{C}_p}/p^N$,

where $(r_0, \ldots, r_{N-1}) \mapsto \sum_{0 \leq i < N} p^i r_i^{(i)} \mod p^N$. Consider the projection $(\bar{o}_0, \ldots, \bar{o}_N, \ldots) \mapsto \bar{o}_N$ from $R$ to $\mathbb{O}_{\mathbb{C}_p}/p$ and denote the image of $\text{Ker} \gamma$ in $W_N(\mathbb{O}_{\mathbb{C}_p}/p)$ by $W_N^1(\mathbb{O}_{\mathbb{C}_p}/p)$. This is principal ideal and in order to apply Fontaine’s criterion about the triviality of the action of ramification subgroups from [5], we needed an element of $W_N(L)$, where $L$

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is a finite extension of $K$ with “small” ramification, which generates $W^1_N(O_{C_p}, p)$. Our “truncation” argument in [1] does not actually work: the resulting element does not belong to $W^1_N(O_{C_p}, p)$. In the moment the author is inclined to believe that such an element does not exist if $N > 1$. Nevertheless, our proof in [1] gives the Fontaine conjecture up to index $p$: the groups $\Gamma^{(e)}_K$ just should be replaced by the groups $\Gamma^{(e)}_K \cap \Gamma_K(\zeta_{N+1})$, where $\zeta_{N+1}$ is a primitive $p^{N+1}$-th root of unity.

The above difficulty appears in many other situations when we try to escape from “$R$-constructions” (e.g. $W(R), A_{cr}$, etc) to $p$-adic constructions inside $C_p$. In this paper we prove Fontaine’s conjecture by applying methods from [2]. These methods were used earlier by the author to study ramification properties in the characteristic $p$ case only. As a matter of fact, this is the first time when we applied them in the mixed characteristic situation.

Note also that if $X$ is a smooth proper scheme over $W(k)$ then our result gives the ramification estimates for the Galois equivariant subquotients of the etale cohomology groups $H^n(X_{\bar{k}}, Q_p)$. Since the appearance of paper [1] there was a considerable progress in the study of the appropriate ramification estimates in the case of schemes $X$ with semi-stable reduction modulo $p$ [7, 3] but in our case (i.e. the case of schemes with good reduction modulo $p$) the situation remained unchanged.

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1. Construction of torsion crystalline representations

The ring $R$ is perfect of characteristic $p$, it is provided with the valuation $v_R$ such that $v_R(r) := \lim_{n \to \infty} p^n v_p(o_n)$, where $r = (o_n \mod p)_{n \geq 0}$. With respect to $v_R$, $R$ is complete and the field $R_0 := \text{Frac}R$ is algebraically closed. Note that $R$ and $R_0$ are provided with natural $\Gamma_K$-action. Denote by $\sigma$ the Frobenius endomorphism of $R$ and $W(R)$ and by $m_R$ the maximal ideal of $R$.

1.1. Let $\mathcal{G} = \text{Spf} W(k)[[X]]$ be the Lubin-Tate 1-dimensional formal group over $W(k)$ such that $\text{pid}_G(X) = pX + X^p$. Then $\text{End}_{W(k)} \mathcal{G} = \mathbb{Z}_p$ and for any $l \in \mathbb{Z}_p$, $(l\text{id}_G)(X) \equiv lX \mod X^p$.

Fix $N \in \mathbb{N}$.

For $i \geq 0$, choose $o_i \in O_{C_p}$ such that $o_0 = 0$, $o_1 \neq 0$ and $\text{pid}_G(o_{i+1}) = o_i$. Set $\bar{u} = (o_{N+i} \mod p)_{i \geq 0} \in R$. Then $\mathcal{K} := k(\bar{u})$ is a complete discrete valuation closed subfield in $R_0$. If $\mathcal{K}_{sep}$ is the separable closure of $\mathcal{K}$ in $R_0$ then $\mathcal{K}_{sep}$ is separably closed and its completion coincides with $R_0$. The theory of the field-of-norms functor [9] identifies $\Gamma_K$ with a closed subgroup in $\Gamma_K$. The quotient $\Gamma_K/\Gamma_K$ acts strictly on $\mathcal{K}$. More precisely, there is a group epimorphism $\kappa : \Gamma_K \longrightarrow \text{Aut}_{W(k)} \mathcal{G} \simeq \mathbb{Z}_p$ such
that if \( g \in \Gamma_{K} \) then \( \kappa(g) \in \mathbb{Z}_p[[X]] \) and \( \kappa(g)(X) \equiv \chi(g)X \mod X^p \) with \( \chi(g) \in \mathbb{Z}_p^* \). (Actually, \( g \to \chi(g) \) is the cyclotomic character.) With this notation we have \( g(\bar{u}) = \kappa(g)(\bar{u}) \).

Use the \( p \)-basis \( \{ \bar{u} \} \) for separable extensions \( E \) of \( K \) in \( K_{sep} \) to construct the system of lifts \( O_N(E) \) of \( E \) modulo \( p^N \). Recall that \( O_N(E) = W_N(\sigma^{N-1}E)[u_N] \subset W_N(E) \) and \( O_N(K) = W_N(k)((u_N)) \), where \( u_N \) is the Teichmuller representative of \( \bar{u} \) in \( W_N(K) \). This construction essentially depends on a choice of \( p \)-basis in \( K \). If, say, \( \{ u' \} \) is another \( p \)-basis for \( K \) and \( O'_N(E) \) are the appropriate lifts then \( O_N(E) \) and \( O'_N(E) \) are not very much different one from another: they can be related by the natural embeddings \( \sigma^{N-1}O_N(E) \subset W(\sigma^{N-1}E) \subset O'_N(E) \). The lifts \( O_N(E) \) are provided with the endomorphism \( \sigma \) such that \( \sigma u_N = u_N^p \), and \( O_N(K_{sep}) \) is provided with continuous \( \Gamma_{K} \)-action.

If \( \tau \) is a continuous automorphism of \( E \) then generally \( \tau \) can’t be lifted to an automorphism of \( O_N(E) \) (but it can always be lifted to \( W_N(E) \)). In many cases it is sufficient to use “the lift” \( \bar{\tau} : \sigma^{N-1}O_N(E) \longrightarrow O_N(E) \) induced by \( W_N(\tau) : W_N(E) \longrightarrow W_N(E) \). In other words, \( \bar{\tau} \) is defined only on a part of \( O_N(E) \), but \( \bar{\tau} \mod p = \sigma^{N-1} \circ \tau : \sigma^{N-1}E \longrightarrow \sigma^{N-1}E \) and, therefore, \( \tau \) can be uniquely recovered from the “lift” \( \bar{\tau} \).

On the other hand, any continuous automorphism \( \tau \) of \( K = k((\bar{u})) \) can be lifted to an automorphism \( \tau^{(N)} \) of \( O_N(K) = W_N(k)((u_N)) \) (use that \( u_N \mod p = \bar{u} \)). Taking into account the existence of a lift \( \tau_{sep} \) of \( \tau \) to \( K_{sep} \) we obtain a lift \( \tau_{sep}^{(N)} \) of \( \tau \) to \( O_N(K_{sep}) \).

Set \( O_N^0 := O_N(K_{sep}) \cap W_N(O_{sep}) \) and \( O_N^+ := O_N(K_{sep}) \cap W_N(m_{sep}) \), where \( m_{sep} \) is the maximal ideal of the valuation ring \( O_{sep} \) of \( K_{sep} \). Then \( \sigma(O_N^0) \subset O_N^0 \), \( \sigma(O_N^+) \subset O_N^+ \), and \( \bigcap_{n \geq 0} \sigma^n(O_N^+) = 0 \). Note that \( O_N^0(K) := O_N^0 \cap O_N(K) = W_N(k)((u_N)) \), \( O_N^+(K) := O_N^+ \cap O_N(K) = u_N W_N(k)((u_N)) \) and \( O_N^*(K) = O_N^0(K)[u_N^{-1}] = W_N(k)((u_N)) \).

For \( 0 \leq m \leq N \), introduce

\[
u_m = (p^{N-m}id_G)(u_N) \in O_N^0(K)
\]

Then \( u_0 = \sigma u_1 = pu_1 + u_1 \), \( t = u_0/u_1 = p + u_1^{p-1} \in O_N^0(K) \) and \( u_0^{p-1} = t^p - pt^{p-1} \). As a matter of fact, \( u_0, u_1, t \) depend only on \( \bar{u} \). Indeed, if \( u' \in W_N(R) \) and \( u' \mod p W_N(R) = \bar{u} \) then in \( O_N(K) \) we have \( u_1 = (p^{N-1}id_G)(u') \).

**Lemma 1.1.** Suppose \( g \in \Gamma_K \). Then

a) \( g(u_0) \equiv \chi(g)u_0 \mod u_0^p O_N^0(K); \)

b) \( \sigma(g(t)/t) \equiv 1 \mod u_0^{p-1} O_N^0(K). \)

**Proof.** \( g(u_1) = (p^{N-1}id_G)(g(u_N)) = (\kappa(g)(u_1) \equiv \chi(g)u_1 \mod u_0^p O_N^0(K) \)
implies a) because \( \sigma(u_1) = u_0 \). Then \( g(t)/t \equiv 1 \mod u_1^{p-1} O_N^0(K) \) and applying \( \sigma \) we obtain b). \( \square \)
1.2. Let $\mathcal{MF}$ be the category of $W(k)$-modules $M$ provided with decreasing filtration by $W(k)$-submodules $M = M^0 \supset \cdots \supset M^{p-1} \supset M^p = 0$ and $\sigma$-linear morphisms $\varphi_i : M^i \to M$ such that for all $i$, $\varphi_i|_{M^{i+1}} = p\varphi_{i+1}$.

For $0 \leq a \leq p - 2$, introduce the filtered module $S_a$ such that

- $S_a = O_N^0 / u_0^a O_N$;
- for $0 \leq i \leq a$, $\text{Fil}^i S_a = t^i S_a$;
- $\varphi_i : \text{Fil}^i S_a \to S_a$ is $\sigma$-linear morphism such that $\varphi_i(t^i) = 1$.

Clearly, $S_a \in \mathcal{MF}$ (use that $\sigma t \equiv p \mod u_0^{p-1}$). In addition, Lemma 1.1 implies also that the action of $\Gamma_K$ preserves the structure of an object of the category $\mathcal{MF}$ on $S_a$.

For $0 \leq a < p$, define the category of filtered Fontaine-Laffaille modules $MF^r_N(a)$ as the full subcategory in $\mathcal{MF}$ consisting of modules $M$ of finite length over $W_N(k)$ such that $M^{a+1} = 0$ and $\sum \text{Im} \varphi_i = M$.

We can assume that $M$ is given together with a functorial splitting of its filtration, i.e. there are submodules $N_i$ in $M$ such that for all $i$, $M^i = N_i \oplus M^{i+1}$.

Let $M \in MF^r_N(a)$ and $\tilde{U}_a(M) = \text{Hom}_{\mathcal{MF}}(M, S_a)$. Then the correspondence $M \mapsto \tilde{U}_a(M)$ determines the functor $\tilde{U}_a$ from $MF^r_N(a)$ to the category of $\Gamma_K$-modules.

**Proposition 1.2.** If $0 \leq a \leq p - 2$ and $H \in M\Gamma^r_N(a)$ then there is $M \in MF^r_N(a)$ such that $\tilde{U}_a(M) = H$.

**Proof.** Recall briefly the main ingredients of the Fontaine-Laffaille theory [6]. The $p^N$-torsion crystalline ring $A_{cr,N} := A_{cr}/p^N$ appears as the divided power envelope of $W_N(R)$ with respect to $\text{Ker} \gamma$. We need the following construction of a generator of $\text{Ker} \gamma$. (Note that we have a natural inclusion of $W(k)$-modules $O_N^0 \subset W_N(R)$.)

**Lemma 1.3.** $\text{Ker} \gamma = t W_N(R)$.

**Proof.** We have $\gamma(u_N) \equiv o_N \mod p O_{\mathbb{C}_p}$, therefore, $\gamma(u_0) \equiv 0 \mod p N O_{\mathbb{C}_p}$ and $t \in \text{Ker} \gamma$. On the other hand, $t \equiv u_1^{p-1} \equiv [r] \mod p W(R)$, where $r \in R$ is such that $r^{(0)} \equiv o_1^{p-1} \equiv -p \mod p^{p/(p-1)} O_{\mathbb{C}_p}$. Therefore, $v_p(r^{(0)}) = 1$ and $t$ generates $\text{Ker} \gamma$, cf. [6].

By above Lemma, $A_{cr,N} = W_N(R)[\{ \gamma_i(t) | i \geq 1 \}]$, where $\gamma_i(t)$ are the $i$-th divided powers of $t$. Then the identity $\gamma_p(t) = t^{p-1} + u_0^{p-1} / p$ implies that $A_{cr,N} = W_N(R)[\{ \gamma_i(u_0^{p-1} / p) | i \geq 1 \}]$.

Recall that $A_{cr,N} \in \mathcal{MF}$ with:

- the filtration $\text{Fil}^i A_{cr,N}$, $0 \leq i < p$, generated as ideal by $t^i$ and all $\gamma_j(u_0^{p-1} / p)$, $j \geq 1$;
— the \( \sigma \)-linear morphisms \( \varphi_i : \text{Fil}^iA_{cr,N} \to A_{cr,N} \) (which come from \( \sigma/p^i \) on \( A_{cr} \)) such that \( \varphi_i(t^i) = (1 + u_0^{p-1}/p)^i \) and \( \varphi_i(u_0^{p-1}/p) = p^{p-1-i}(u_0^{p-1}/p)(1 + u_0^{p-1}/p)^{p-1} \).

Then the Fontaine-Laffaille functor \( U_a \) attaches to \( M \in \text{MF}_N(a) \) the \( \Gamma_K \)-module \( \text{Hom}_{\mathcal{M} \mathcal{F}}(M, A_{cr,N}) \). This functor is fully-faithful (we assume that \( a < p - 2 \)) and, therefore, there is \( M \in \text{MF}_N(a) \) such that \( U_a(M) = H \).

Consider the \( W(k) \)-module \( \mathcal{W}_N^a = W_N(R)/u_0^a W_N(m_R) \) with the filtration induced by the filtration \( W_N^i(R) = t^i W_N(R) \) and \( \sigma \)-linear morphisms \( \varphi_i \) such that \( \varphi_i(t^i) = 1 \). Prove that we have an identification of \( \Gamma_K \)-modules \( H = \text{Hom}_{\mathcal{M} \mathcal{F}}(M, \mathcal{W}_N^a) \).

Indeed, let \( T_a \) be the maximal element in the family of all ideals \( I \) of \( A_{cr,N} \) such that \( \varphi_a \) induces a nilpotent endomorphism of \( I \). Then for any \( M \in \text{MF}_N(a) \), \( U_a(M) = \text{Hom}_{\mathcal{M} \mathcal{F}}(M, A_{cr,N}/T_a) \). By straightforward calculations we can see that \( T_a \) is generated by the elements of \( u_0^a W_N(m_R) \) and all \( \gamma_j(u_0^{p-1}/p), j \geq 1 \). It remains to note that we have a natural identification \( A_{cr,N}/T_a = \mathcal{W}_N^a \) in the category \( \mathcal{M} \mathcal{F} \).

Consider the natural embedding \( \Omega_K^0 \to W_N(R) \) and the induced natural map \( \iota_a : S_a \to \mathcal{W}_N^a \) in \( \mathcal{M} \mathcal{F} \). Prove that \( \iota_a : \tilde{U}_a(M) \to H \) is isomorphism of \( \Gamma_K \)-modules.

Choose \( W(k) \)-submodules \( N_i \) in \( M \) such that \( M = N_i \oplus M^{i+1} \) and choose vectors \( \bar{n}_i \) whose coordinates give a minimal system of generators of \( N_i \). Then the structure of \( M \) can be given by the matrix relation \( (\varphi_a(n_a), \ldots, \varphi_0(n_0)) = (\bar{n}_a, \ldots, \bar{n}_0)C \), where \( C \) is an invertible matrix with coefficients in \( W(k) \). The elements of \( H \) are identified with the residues \( (\bar{u}_a, \ldots, \bar{u}_0) \mod u_0^a W_N(m_{\text{sep}}) \) where the vectors \( (\bar{u}_a, \ldots, \bar{u}_0) \) have coefficients in \( W_N(K_{\text{sep}}) \) and satisfy the following system of equations (use that \( \varphi_a \) is topologically nilpotent on \( u_0^a W_N(m_{\text{sep}}) \))

\[ \left( \frac{\sigma \bar{u}_a}{\sigma t^a}, \ldots, \frac{\sigma \bar{u}_i}{\sigma t^i}, \ldots, \sigma(\bar{u}_0) \right) = (\bar{u}_a, \ldots, \bar{u}_0)C \]

In particular, if \( \bar{u} = (\bar{u}_a, \ldots, \bar{u}_0) \) then there is an invertible matrix \( D \) with coefficients in \( O_N(K) \) such that

\[ \sigma(\bar{u})D = \bar{u} \]

We know that all coordinates of \( \sigma^{N-1} \bar{u} \) belong to \( \sigma^{N-1} W_N(K_{\text{sep}}) \subset O_N(K_{\text{sep}}) \). Then (1.1) implies step-by-step that the vectors \( \sigma^{N-2} \bar{u}, \ldots, \bar{u} \) have coordinates in \( O_N(K_{\text{sep}}) \). It remains to note that \( O_N^0 = O_N(K_{\text{sep}}) \cap W_N(O_{\text{sep}}) \) and \( O_N^+ = O_N(K_{\text{sep}}) \cap W_N(m_{\text{sep}}) \). The proposition is proved.

\[ \square \]

2. Reformulation of the Fontaine conjecture

2.1. Review of ramification theory. Let \( L_K \) be the group of all continuous automorphisms of \( K_{\text{sep}} \) which keep invariant the residue field of \( K_{\text{sep}} \) and preserve the extension of the normalised valuation \( v_K \)
of $\mathcal{K}$ to $\mathcal{K}_{\text{sep}}$. This group has a decreasing filtration by its ramification subgroups $I_{\mathcal{K}}^{(v)}$ in upper numbering $v \geq 0$. Recall basic ingredients of the definition of this filtration following the papers [4, 9, 10].

For any field extension $\mathcal{E}$ of $\mathcal{K}$ in $\mathcal{K}_{\text{sep}}$, set $\mathcal{E}_{\text{sep}} = \mathcal{K}_{\text{sep}}$, in particular, $I_{\mathcal{E}} = I_{\mathcal{K}}$. All elements of $I_{\mathcal{K}}$ preserve the extension $v_{\mathcal{E}}$ of the normalised valuation on $\mathcal{E}$ to $\mathcal{K}_{\text{sep}}$.

For $x \geq 0$, set $I_{\mathcal{E},x} = \{ \iota \in I_{\mathcal{E}} \mid v_{\mathcal{E}}(\iota(a) - a) \geq 1 + x \ \forall a \in m_{\mathcal{E}} \}$, where $m_{\mathcal{E}}$ is the maximal ideal in $O_{\mathcal{E}}$.

Denote by $I_{\mathcal{E}/\mathcal{K}}$ the set of all continuous embeddings of $\mathcal{E}$ into $\mathcal{K}_{\text{sep}}$ which induce the identity map on $\mathcal{K}$ and the residue field $k_{\mathcal{E}}$ of $\mathcal{E}$. For $x \geq 0$, set $I_{\mathcal{E}/\mathcal{K},x} = I_{\mathcal{E},x} \cap I_{\mathcal{E}/\mathcal{K}}$.

If $\iota_1, \iota_2 \in I_{\mathcal{E}/\mathcal{K}}$ and $x \geq 0$ then $\iota_1$ and $\iota_2$ are $x$-equivalent iff for any $a \in m_{\mathcal{E}}$, $v_{\mathcal{E}}(\iota_1(a) - \iota_2(a)) \geq 1 + x$. Denote by $(I_{\mathcal{E}/\mathcal{K}} : I_{\mathcal{E}/\mathcal{K},x})$ the number of $x$-equivalent classes in $I_{\mathcal{E}/\mathcal{K}}$. Then the Herbrand function $\varphi_{\mathcal{E}/\mathcal{K}}$ can be defined for all $x \geq 0$, as

$$\varphi_{\mathcal{E}/\mathcal{K}}(x) = \int_0^x (I_{\mathcal{E}/\mathcal{K}} : I_{\mathcal{E}/\mathcal{K},x})^{-1} dx.$$  

This function has the following properties:

- $\varphi_{\mathcal{E}/\mathcal{K}}$ is a piece-wise linear function with finitely many edges;
- if $\mathcal{K} \subset \mathcal{E} \subset \mathcal{H}$ is a tower of finite field extensions in $\mathcal{K}_{\text{sep}}$ then for any $x \geq 0$, $\varphi_{\mathcal{H}/\mathcal{K}}(x) = \varphi_{\mathcal{E}/\mathcal{K}}(\varphi_{\mathcal{H}/\mathcal{E}}(x))$.

The ramification filtration $\{I_{\mathcal{K}}^{(v)}\}_{v \geq 0}$ appears now as a decreasing sequence of the subgroups $I_{\mathcal{K}}^{(v)}$ of $I_{\mathcal{K}}$, where $I_{\mathcal{K}}^{(v)}$ consists of $\iota \in I_{\mathcal{K}}$ such that for any finite extension $\mathcal{E}$ of $\mathcal{K}$, $\iota \in I_{\mathcal{E},v_{\mathcal{E}}}$ with $\varphi_{\mathcal{E}/\mathcal{K}}(v_{\mathcal{E}}) = v$.

If we replace the lower indices $\mathcal{K}$ to $\mathcal{E}$, the ramification filtration $\{I_{\mathcal{K}}^{(v)}\}_{v \geq 0}$ is not changed as a whole, just only individual subgroups change their upper indices, that is $I_{\mathcal{K}}^{(v)} = I_{\mathcal{E}}^{(v)}$.

Note that the inertia subgroup $\Gamma_{\mathcal{E}}^{0}$ of $\Gamma_{\mathcal{E}} = \text{Gal}(\mathcal{K}_{\text{sep}}/\mathcal{E})$ is a subgroup in $I_{\mathcal{E}}$ and for any $v \geq 0$, the appropriate subgroup $\Gamma_{\mathcal{E}}^{(v)} = \Gamma_{\mathcal{E}} \cap I_{\mathcal{E}}^{(v)}$ is just the ramification subgroup of $\Gamma_{\mathcal{E}}$ with the upper number $v$ from $[8]$.

### 2.2. Statement of the main theorem. The main idea of our approach to the $\Gamma_{\mathcal{K}}$-modules $\tilde{U}_a(M)$ is related to the following fact. The filtered module $S_a$ depends only on the field $\mathcal{K}$ and its uniformizer $\tilde{u}$. Therefore, $S_a$ can be identified with its analogue $S'_a$ constructed for any ramified extension $\mathcal{K}'$ of $\mathcal{K}$ together with its uniformizer $\tilde{u}'$. The whole group $I_{\mathcal{K}}$ does not preserve the structure of $S_a$ but the ramification subgroups $I_{\mathcal{K}}^{(v)}$, where $a > a_N^* := (a + 1)p^{N-1} - 1$ do preserve this structure because of the following proposition.

**Proposition 2.1.** If $v > a_N^*$ and $M \in \text{MF}_N(a)$ then a natural action of $I_{\mathcal{K}}$ on $W_N(\mathcal{K}_{\text{sep}})$ induces the $I_{\mathcal{K}}^{(v)}$-module structure on $\tilde{U}_a(M)$.

**Proof.** All we need is just the following lemma.
Lemma 2.2. If $\tau \in \mathcal{I}^{(v)}_K$ with $v > a_N^*$ then

a) $\tau(u_0)/u_0 \in O_N^+(K_{sep})$;

b) for $0 \leq i \leq a$, $\varphi_i(\tau t^i) = 1$.

Proof of Lemma. For $\tau \in \mathcal{I}^{(v)}_K$, we have $\tau(u_N) = u_N + \eta_N + pw$, where $\eta_N \in u_1^{a+1}O_N^+$ and $w \in W_N(K_{sep})$. For $1 \leq i \leq N$, this implies

$$\tau(u_i) = u_i + \eta_i + p^{N-i+1}w_i,$$

where $\eta_i \in u_i^{a+1}O_N^+$ and $w_i \in W_N(K_{sep})$. Therefore,

$$\tau(u_1) \equiv u_1 \text{ mod } u_1^{a+1}O_N^+.$$ 

This implies part a) because $\tau(u_0) \equiv u_0 \text{ mod } u_0^{a+1}O_N^+$ and part b) because $\sigma(\tau t) / \sigma(t) \equiv 1 \text{ mod } u_0^aO_N^+$. \hfill \square

With the relation to the original problem of estimating the upper ramification numbers of the $\Gamma_K$-module $H$ notice now that $K = k((\tilde{u}))$ coincides with $\sigma^{-N}\mathcal{K}_0$, where $\mathcal{K}_0$ is the field-of-norms of the $p$-cyclotomic extension $\tilde{K}$ of $K$. Then for any $v \geq 0$, $\Gamma_K^{(v)} = \Gamma_K \cap \mathcal{I}^{(v^*)}_K$, where $\varphi_{\tilde{K}/K}(v^*) = v$. In particular, $v > N - 1 + a/(p - 1)$ if and only if $v^* > a_N^*$.

So, the proof of Fontaine’s conjecture is reduced to the proof of the following theorem stated exclusively in terms of the field $K$ of characteristic $p$.

Theorem 2.3. For any $v > a_N^*$, the group $\mathcal{I}^{(v)}_K$ acts trivially on $\tilde{U}_a(M)$.

3. Proof of Theorem 2.3

3.1. Auxiliary field $K'$. Let $N^* \in \mathbb{N}$ and $r^* \in \mathbb{Q}$ be such that for $q := p^{N^*}$, $r^*(q - 1) := b^* \in \mathbb{N}$ and $v_p(b^*) = 0$.

Consider the field $K' = K(N^*, r^*)$ from [2]. Remind that

— $[K' : K] = q$;

— $K' = k((\tilde{u}'))$, where $\tilde{u} = \tilde{u}^aE(\tilde{u}^{b^*})^{-1}$ (here $E$ is the Artin-Hasse exponential);

— the Herbrand function $\varphi_{K'/K}$ has only one edge point $(r^*, r^*)$. (In particular, $\varphi_{K'/K}(x) < x$ for all $x > r^*$.)

For $K'$ and its above uniformiser $\tilde{u}'$ proceed as earlier to construct the lifts $O_N'(K')$ and $O_N'(K_{sep})$ obtained with respect to the $p$-basis $\tilde{u}'$. Introduce similarly the modules $O_N'^0$, $O_N'^+$, the elements $u_0'$, $t' \in O_N(K')$ and the filtered module $\mathcal{S}'_0$. 


3.2. Compare the old and the new lifts using their canonical embeddings into \( W_N(K_{\text{sep}}) \). Note that \( u_N \) is not generally an element of \( O'_N(K') \) because the Teichmüller representative \( u_N = [\tilde{u}] \) can't be written as a power series in \( u \leq \sigma(3.1) \)

If \( \sigma(3.1) \)

Now suppose \( \nu^* \geq a_N^\ast \), \( \mathcal{I}_{\xi}^{(v)} \) acts trivially on \( \tilde{U}_a(M) \) for all \( v > v^* \) and \( v^* \) is the minimal with this property. The existence of \( v^* \) follows from the left-continuity of the ramification filtration with respect to the upper numbering.

If \( v^* = a_N^\ast \) then our theorem is proved.

Suppose that \( v^* > a_N^\ast \). Choose the parameters \( r^\ast \) and \( N^\ast \) from Subsection 3.1 such that \( a_N^\ast q/(q - 1) < r^\ast < v^* \).

For any \( \alpha \in O'_N(K_{\text{sep}}) \), set \( \alpha^{(q)} = \sigma^{N^\ast} \alpha. \)

Lemma 3.2. \( u_1/u_1^{(q)} \equiv 1 \mod u_1^{(q)a}O_N^+(K'). \)

Proof. Consider \( b^\ast = r^\ast(q - 1) \in \mathbb{N} \) from Subsection 3.1. Then \( b^\ast + q > q(a_N^\ast + 1) = q(a + 1)p^{N^\ast - 1} \) and

\[
u_N \equiv u_1^{(q)} \mod \left( u_1^{(q)a}O_N^+(K') + pO_N(K') \right)
\]

This implies \( u_1 \equiv u_1^{(q)} \mod u_1^{(q)a}O_N^+(K') \) and the lemma is proved. \( \square \)

Corollary 3.3. a) \( u_0/u_0^{(q)} \) is invertible in \( O_N^{(q)}(K') \);

b) \( \sigma(t/t^{(q)}) \equiv 1 \mod u_0^{(q)a}O_N^+(K') \).

3.4. \( \mathcal{I}_{\xi}^{(v^*)} \)-action. Introduce the filtered module \( S_a^{(q)} \) as follows.

\[
- S_a^{(q)} = O_N^{(q)} / u_0^{(q)a}O_N^+;
- \text{Fil}^i S_a^{(q)} = t^{(q)i} S_a^{(q)};
- \varphi_i^{(q)} : \text{Fil}^i S_a^{(q)} \rightarrow S_a^{(q)} \text{ is } \sigma\text{-linear such that } \varphi_i^{(q)}(t^{(q)i}) = 1.
\]
Suppose $M' \in \text{MF}_N(a)$ is given similarly to $M$ by the relation
\[(\varphi_a(\tilde{n}_a), \ldots, \varphi_0(\tilde{n}_0)) = (n_a, \ldots, n_0)\sigma^{-N}C\]
Then we can use $\sigma^{N^*}$ to identify the modules $\tilde{U}'_a(M') := \text{Hom}_{\text{MF}}(M', S'_a)$ and $\tilde{U}'_a(q)(M) := \text{Hom}_{\text{MF}}(M, S'_a(q))$. This identification is compatible with the action of the subgroups $\mathcal{I}_{K'}^{(v)}$, where $v > a^*_N$.

Note that the fields $K$ and $K'$ are isomorphic (as any two fields of formal power series with the same residue field). Choose an isomorphism $\kappa : K \rightarrow K'$ such that $\kappa(\tilde{u}) = \tilde{u}'$ and $\kappa|_k = \sigma^{-N^*}$. We can extend $\kappa$ to an isomorphism of separable closures of $K$ and $K'$. This allows us to identify the groups $\mathcal{I}_{K}$ and $\mathcal{I}_{K'}$ and this identification is compatible with the appropriate ramification filtrations. Even more, we obtain an identification of $\tilde{U}_a(M)$ with $\tilde{U}'_a(M')$ and this identification respects the action of $\mathcal{I}_{K}^{(v)}$ on $\tilde{U}_a(M)$ and the action of $\mathcal{I}_{K'}^{(v)}$ on $\tilde{U}'_a(M')$ for any $v > a^*_N$. Therefore, $v^*$ is the maximal number such that $\mathcal{I}_{K'}^{(v^*)}$ acts non-trivially on $\tilde{U}'_a(M')$ and

- $v^*$ is the maximal such that $\mathcal{I}_{K'}^{(v^*)}$ acts non-trivially on $\tilde{U}'_a(q)(M)$.

3.5. $\mathcal{I}_{K}^{(v^*)}$-action. Introduce the filtered module $S_a^*$ as follows:
- $S_a^* = O_N^0 \cap O_N(K_{\text{sep}})/u_0^{a}O_N^0 \cap O_N'(K_{\text{sep}})$;
- $\text{Fil}^iS_a^* = t^iS_a^* \cap S_a^*$;
- $\varphi_i^* = \varphi_i|_{\text{Fil}^iS_a^*} : \text{Fil}^iS_a^* \rightarrow S_a^*$.

The results from Subsection 3.2 allow us to identify $\tilde{U}_a(M)$ with $U_a^*(M) = \text{Hom}_{\text{MF}}(M, S_a^*)$. By the results from Subsection 3.3, there is a natural embedding of filtered modules $S_a^* \rightarrow S'_a(q)$ and, therefore, we can identify $\tilde{U}_a(M)$ with $\tilde{U}'_a(q)(M')$. This identification is compatible with the action of ramification subgroups $\mathcal{I}_{K}^{(v)}$ for all $v > a^*_N$. So,

- $v^*$ is the maximal such that $\mathcal{I}_{K}^{(v^*)}$ acts non-trivially on $\tilde{U}'_a(q)(M)$.

3.6. The end of proof of Theorem. It remains to notice that $\mathcal{I}_{K'}^{(v^*)} = \mathcal{I}_{K}^{(v_0^*)}$, where $v_0^* = \varphi_{K'/K}(v^*) < v^*$.

The contradiction.

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