Travelling to exotic places with cavity QED systems

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Abstract
Recent theoretical schemes for utilizing cavity QED models as quantum simulators are reviewed. By considering a quadrature representation for the fields, it is shown how Jahn–Teller models, effective Abelian or non-Abelian gauge potentials, transverse Hall currents and relativistic effects naturally arise in these systems. Some of the analytical predictions are verified numerically using realistic experimental parameters taking into account system losses thereby demonstrating their feasibility with current experimental setups.

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1. Introduction
Back in 2006, I was a co-author of a paper called ‘Travelling to exotic places with ultracold atoms’ where we discussed some new (at the time) ideas on how to mimic unusual phenomena occurring in condensed matter systems by using ultracold atoms [1]. Due to the controllability of system parameters and long decoherence times, cold atoms in optical lattices had shown to be suitable candidates for the simulation of different many-body theories developed in the community of condensed matter physics [2]. Around the same time, many theoretical predictions were presented in which high-energy and cosmological phenomena could be simulated by means of Bose–Einstein condensates [3]. In addition, ion-trap systems have recently drawn much attention in terms of simulating various spin chains as well as relativistic effects [4].

In this paper, I summarize some new directions employing cavity quantum electrodynamics (QED) systems as quantum simulators. On the experimental side, cavity QED systems have seen great progress during the last years. Pure quantum effects are nowadays not restricted to single few-level atom-cavity setups [5], but include both quantum-dots (qdots) or superconducting quantum interface devices (SQUIDs) [6] as well as Bose–Einstein condensates [7] coupled to single-cavity modes. These new experiments pave the way for reaching the superstrong coupling regime where the strength of the effective matter–field coupling exceeds the system losses by several orders of magnitude. One crucial outcome of reaching this regime is that the application of the rotating wave approximation (RWA) becomes questionable [8]. In most cases, going beyond the RWA implies that the corresponding system Hamiltonian is not integrable and other approximations or numerical methods are required. As outlined in [8], by expressing the cavity field(s) in its quadrature operators, it directly follows that beyond the RWA, typical cavity QED models describe a set of coupled harmonic potentials. In this representation, it is thereby possible to develop new physical intuition not easily extracted when the field is expressed in boson creation and annihilation operators. Moreover, in this picture the link between cavity QED models and systems from other fields of physics readily follows.

In the dipole approximation we assume the electric field $E(x)$ to be constant over the extent of the atom, hence letting $x = 0$, to
obtain
\[ \hat{E} = \sum_k \tilde{\varepsilon}_k \hat{E}_k, \]
where \( \tilde{\varepsilon}_k \) and \( \hat{E}_k \) are the polarization vector and the field amplitude of the \( k \)th mode, respectively. Denoting the dipole moment between the atomic states \( |i\rangle \) and \( |j\rangle \) by \( \hat{d}_{ij} \), the atom–field interaction takes the form
\[ H_I = \sum_{i,j} \hat{d}_{ij} \cdot \hat{E}. \]
Due to selection rules or large atom–field detunings, most terms in the above sums vanish either exactly or approximately. Explicitly, the dipole moment
\[ \hat{d}_{ij} = (d_{ij}^x, d_{ij}^y, d_{ij}^z), \]
with \( d_{ij}^\alpha = -e|\langle i| \alpha |j\rangle| |j\rangle + \text{h.c.} \) and \( e \) the electron charge. The full Hamiltonian, containing the free field and internal atomic energies, becomes
\[ H = H_f + H_a + H_I, \]
where
\[ H_f = \hbar \sum_k \omega_k \left( \frac{\hat{P}_k^2}{2} + \frac{\hat{X}_k^2}{2} \right), \quad H_a = \sum_{j=1}^N E_j |j\rangle \langle j|, \]
\( \omega_k \) the frequency of mode \( k \) and \( E_j \) being the energy of the \( j \)th initial atomic state.

2. Jahn–Teller systems

Jahn–Teller systems frequently occur in both molecular/chemical physics and condensed matter physics [9]. The name dates back to works by Hermann Jahn and Edward Teller studying symmetry breaking for polyatomic molecules [10]. Common for the Jahn–Teller models is that two potential surfaces become degenerate in a single point forming a conical intersection. About two decades after the pioneering work by Jahn and Teller, Longuet-Higgins showed that by encircling the degeneracy, the electronic wave function changes sign [11]. This phase factor was later realized to be the so-called Berry phase [12].

2.1. The \( \beta \times E \) model

The simplest nontrivial atom–cavity system considers a two-level atom interacting with a single-cavity mode. Introducing the Pauli matrices \( \hat{\sigma}_z = |1\rangle \langle 1| + |2\rangle \langle 2| \), \( \hat{\sigma}_y = i(|1\rangle \langle 2| - |2\rangle \langle 1|) \), and \( \hat{\sigma}_x = |1\rangle \langle 1| - |2\rangle \langle 2| \), the Hamiltonian reads
\[ H_{\beta E} = \hbar \omega \left( \frac{\hat{P}^2}{2} + \frac{\hat{X}^2}{2} \right) + \hbar \Omega \hat{\sigma}_z + \hbar g \hat{\sigma}_x \hat{P}. \]
Here, \( \Omega \) is the atomic transition frequency and \( g \) the effective atom–field coupling. Application of the RWA with respect to the first two terms renders the well-known Jaynes–Cummings model [13]. For true atomic systems, it accurately explains several experimental observations [5]. In the terminology of molecular physics, \( H_{\beta E} \) is equivalent to the \( \beta \times E \) Jahn–Teller Hamiltonian by simply interchanging \( \hat{X} \) and \( \hat{P} \). For a deeper understanding of the dynamics, it is more convenient to think of \( \hat{X}^2/2 \) as the kinetic energy term and \( \hat{P} \) as the coordinate. In this picture, the adiabatic potentials are defined as
\[ V_{ad}^\pm (\hat{P}) = \frac{\hbar \omega \hat{P}^2}{2} \pm \hbar \sqrt{\frac{\Omega^2}{4} + g^2 \hat{P}^2}. \]
The diabatic potentials are
\[ V_{ad}^\pm (\hat{P}) = \frac{\hbar \omega \hat{P}^2}{2} \pm \hbar g \hat{P} \].

Figure 1 depicts examples of the different potentials. Whenever \(|g| > \sqrt{\omega \Omega^2/2}\), the lower adiabatic potential possesses two minima, while for \(|g| < \sqrt{\omega \Omega^2/2}\) a single global minimum occurs at \( P = 0 \). The Jahn–Teller effect [9, 10] is easily understood from the adiabatic potentials; the amplitude of the ground state wave function does not attain its maximum at \( P = 0 \), but around the two shifted minima. For polyatomic molecules it implies that the conical intersection causes the molecule to vibrate around a less symmetric atomic configuration, or in other words, the ground state energy is decreased by lowering the symmetry. In terms of cavity QED, the Jahn–Teller effect states that for sufficiently strong atom–field coupling the ground state is not the one with the atom in its ground state and the field in vacuum. This effect has long been known in cavity QED, and in the case of infinitely many atoms (the thermodynamical limit) it describes a second-order quantum phase transition [14]. However, only recently has it been shown that this phase transition is identified as a Jahn–Teller symmetry breaking [15].

Before proceeding, some words of caution. The presence of a Jahn–Teller effect in the cavity QED setting is not possible by considering a direct coupling between the true atomic ground state and an excited state. A full microscopic derivation of the Hamiltonian contains a ‘self-energy’ term that has been neglected. This is justified in most experimental situations, but in the \(|g| > \sqrt{\omega \Omega^2/2}\) regime it can no longer be left out. Once this term is considered, sum-rules or gauge arguments can be employed to show that the lower adiabatic potential indeed never attains two minima [16]. In typical experiments, however, the atomic state \(|g\rangle\) is not the true atomic ground state. Moreover, by utilizing Raman transitions the coupling is not direct and the above arguments
do not apply. Longer discussions on this topic can be found in [15, 17].

2.2. The ε × E model

By considering a degenerate bimodal cavity, the β × E model can be extended to the ε × E one provided the atom–field couplings obey $d_{i\epsilon} \cdot \varepsilon_i \hat{E}_\epsilon = h g \hat{\sigma}_\epsilon$ and $d_{i\sigma} \cdot \varepsilon_i \hat{E}_\sigma = h g \hat{\sigma}_\sigma$. We label the two modes by $x$ and $y$. The Hamiltonian then takes the form

$$H_{\epsilon E} = \hbar \omega \left( \frac{\hat{P}_x^2 + \hat{P}_y^2}{2} + \frac{\hat{X}^2 + \hat{Y}^2}{2} \right) + \sum_{j=1,2,3} E_j |j\rangle \langle j| + g \left( |3\rangle \langle 3| \hat{P}_x + |3\rangle \langle 2| \hat{P}_y + h.c. \right).$$

with the adiabatic potential surfaces

$$V^{\pm}_{ad}(\hat{P}_x, \hat{P}_y) = \hbar \omega \left( \frac{\hat{P}_x^2 + \hat{P}_y^2}{2} \right) \pm h \sqrt{\frac{\Omega^2}{4} + g^2 \left( \hat{P}_x^2 + \hat{P}_y^2 \right)},$$

(8)

Instead of having the double well structure as in figure 1, the lower adiabatic potential has a sombrero shape. This model was extensively studied in [15], focusing on the effects due to the non-zero Berry phase acquired when encircling the conical intersection. Initializing a coherent state in one of the cavity modes such that it predominantly populates the lower adiabatic potential, it was demonstrated that over longer time periods the population will periodically swap between the two modes and that this period greatly depends on the Berry phase.

2.3. Renner–Teller model

If the degeneracy is not conical it is glancing, meaning that the tangents of the two potential surfaces are identical. This defines the Renner–Teller models. The same type of Jahn–Teller effect is possible for Renner–Teller models, but the Berry phase vanishes whenever the intersection is in configuration space. From a physical point of view, the model we consider is thereby different since the intersection occurs in momentum space rather than in configuration space.

To achieve a glancing intersection we look at a three-level $\Lambda$-atom with the two lower atomic states $|1\rangle$ and $|2\rangle$ coupled via two-cavity modes to an excited state $|3\rangle$. The model Hamiltonian is taken as

$$H_{RT} = \hbar \omega \left( \frac{\hat{P}_x^2 + \hat{P}_y^2}{2} + \frac{\hat{X}^2 + \hat{Y}^2}{2} \right) + \sum_{j=1,2,3} E_j |j\rangle \langle j|$$

$$+ g \left( |3\rangle \langle 1| \hat{P}_x + |3\rangle \langle 2| \hat{P}_y + h.c. \right).$$

As a three-level system, there are three adiabatic and diabatic potential surfaces. Assuming degenerate modes and degenerate ground atomic states, $E_1 = E_2 = 0$, the adiabatic potentials become

$$V^{\pm}_{ad}(\hat{P}_x, \hat{P}_y) = \hbar \omega \left( \frac{\hat{P}_x^2 + \hat{P}_y^2}{2} \right) \pm \hbar g \sqrt{\frac{E_3^2}{2} + g^2 \left( \hat{P}_x^2 + \hat{P}_y^2 \right)},$$

(11)

from which it is clear that $V^{\pm}_{ad}(\hat{P}_x, \hat{P}_y)$ and $V^0_{ad}(\hat{P}_x, \hat{P}_y)$ possess a glancing intersection at the origin.

3. Effective gauge potentials

Gauge theories arise in a variety of fields in physics. For the most familiar case of a charged particle in an electromagnetic field the theory is Abelian. An example of a non-Abelian gauge theory is the one of Yang and Mills describing strong interaction. Getting an experimental handle of a system showing non-Abelian characteristics is therefore very attractive and has led to many suggestions. One possibility is adiabatically evolving systems either by means of external changes of the Hamiltonian or for ultracold atoms subjected to spatially varying light fields [18].

The general Hamiltonian (4) may be rewritten as

$$H = \hbar \sum_k \epsilon_k \left( \frac{\hat{P}_k^2}{2} + \frac{\hat{X}_k^2}{2} \right) + \sum_j E_j |j\rangle \langle j| + \hat{\Psi},$$

(12)

where

$$\hat{A}_k = -\epsilon_k \hat{E}_k / \omega_k, \quad \hat{\Phi} = -h \sum_k \epsilon_k \hat{A}_k^2.$$

(13)

The operators $\hat{A}_k$ and $\hat{\Phi}$ have the properties as vector and scalar potential, respectively. That is, they transform appropriately under unitary transformations [19]. For any atomic basis, these gauge potentials are matrices and they are said to be Abelian if $[\hat{A}_k, \hat{A}_l] = 0 \forall k$ and $l$ and non-Abelian for non-commuting operators. As we now demonstrate, all three examples of the previous section render different types of gauge potentials.

3.1. Abelian gauge potential

As the atom–field interaction only includes a single mode for the $\beta \times E$ model, the vector potential consists of a single component, which reads

$$A = -\frac{g}{\omega} \hat{\sigma}_y.$$

(14)

Consequently, the gauge potential is Abelian.

3.2. Non-Abelian SU(2) gauge potential

For the second model, the $\epsilon \times E$ one, we have

$$(\hat{A}_x, \hat{A}_y) = -\frac{g}{\omega} (\hat{\sigma}_x, \hat{\sigma}_y)$$

(15)

and hence $[\hat{A}_x, \hat{A}_y] = g^2 \hat{\sigma}_x / \omega^2$ showing that the gauge potential is non-Abelian.

Naturally, the dynamics is considerably richer for a system exhibiting non-Abelian structures. In general, time-ordering becomes important, e.g. enclosing a ‘loop’ clockwise or anti-clockwise will not result in the same final system state. In [19], the time evolution of an initial state consisting of one empty mode and the other with a coherent state was numerically simulated. By properly choosing the phase of the coherent state, it will either set off clockwise or anti-clockwise around the conical intersection. Figure 2 displays an example of the atomic inversion $W(t) = \langle \hat{\sigma}_z |$ during such time evolutions. The simulation utilizes realistic parameters of the qdot cavity QED experiment presented in [6], and furthermore takes into account both cavity losses.
and atomic spontaneous emission. Solid and dotted lines correspond to the different directions around the conical intersection of the coherent state. The difference between the two curves of the plot demonstrates the non-Abelian property.

3.3. Non-Abelian SU(3) gauge potential

The last of our models, the Renner–Teller one, turns out to be non-Abelian as well. As a three-level system we express the gauge potential in Gell–Mann matrices

\[
(\hat{A}_x, \hat{A}_y) = -\frac{g}{\alpha} (\hat{\lambda}_4, \hat{\lambda}_6),
\]

where \(\hat{\lambda}_4 = |3\rangle\langle 1| + |1\rangle\langle 3|\) and \(\hat{\lambda}_6 = |3\rangle\langle 2| + |2\rangle\langle 3|\). We note that \([\hat{A}_x, \hat{A}_y] = ig^2 \hat{\lambda}_2/\alpha^2\), with \(\hat{\lambda}_2 = -i|1\rangle\langle 2| + i|2\rangle\langle 1|\).

4. Intrinsic anomalous Hall effect

In the condensed matter community, the coupling of our \(\varepsilon \times E\) Hamiltonian is said to be on Rashba spin–orbit form [20]. It is known that such coupling gives rise to an effective Lorentz force inducing a transverse Hall current. The force is state dependent; opposite sign for the appropriate two internal states. When \(\Omega = 0\), the population ratio between the two internal states is balanced and therefore the Rashba coupling brings about a transverse spin current [21]. A non-zero \(\Omega\) breaks this symmetry and thereby it causes a net transverse particle current. Since the phenomenon derives from an intrinsic spin–orbit coupling, and not from an externally applied magnetic field as in the regular Hall effect, it has been termed the anomalous Hall effect [22].

The net effective force acting on the state can be derived from

\[
\vec{F} = \frac{d\vec{\hat{r}}}{dt} = \frac{1}{\hbar^2} [\hat{H}, \hat{\vec{r}}] = g(\hat{\vec{P}} \times \hat{\vec{Z}}),
\]

where \(\hat{\vec{r}} = (\hat{X}, \hat{Y})\), \(\hat{\vec{P}} = (\hat{\vec{P}}_x, \hat{\vec{P}}_y)\) are the momentum operators in the Heisenberg representation, and dot indicates the time derivative. In quantum optics, the field is conveniently examined in phase space. The force (17) acts on the motion of the phase space distribution. Thereby, the Hall current occurs in phase space and hence does not involve a true particle current. In the quadrature representation it is clear that any field states evolving under Hamiltonian (8) are bounded by an harmonic potential. Without the Rashba coupling a Gaussian coherent state will bunch back and forth in the two-dimensional potential maintaining its shape. On the other hand, if the Rashba coupling is non-zero, the motion of the Gaussian wave packet will start to bend [23]. Hence, the rocking motion will be accompanied by a rotation around the Z-axis. This rotation is either clockwise or anti-clockwise depending on the internal atomic state. These assumptions are verified by propagating an initial state with mode \(x\) in a coherent state with amplitude \(\alpha = 10/\sqrt{2}\), mode \(y\) in vacuum, and the atom in its lower state. The results are presented in figure 3 showing the trajectories of the averages \(\langle \hat{X} \rangle\) and \(\langle \hat{Y} \rangle\).

It is seen that after hundreds of oscillations in the harmonic trap, all population has been transferred from the \(x\) to the \(y\) mode. The fact that the oscillating amplitude decreases during the evolution results from system losses.

5. Relativistic effects

When conical intersections as the ones of the \(\varepsilon \times E\) model appear in momentum space, they are frequently referred to as Dirac cones. Such Dirac cones have attracted a great amount of interest, especially in graphene research. In the vicinity of the cones, the dispersions are linear as for free relativistic particles. The electrons of graphene may therefore show relativistic effects like for example Zitterbewegung. For avoided intersections the relativistic electrons have a non-zero effective mass, while for unavoidable intersections the electrons are massless.

For small \(\hat{P}_x\) and \(\hat{P}_y\), we neglect the quadratic terms of the \(\varepsilon \times E\) Hamiltonian

\[
H_{rel} = \hbar g \sum_{k=x,y} \sigma_k \hat{P}_k + \hbar \beta \vec{\sigma} \cdot \vec{\Omega} + V(\hat{X}, \hat{Y}).
\]

This has the form of a Dirac equation for a spin-less particle moving within a two-dimensional harmonic potential \(V(\hat{X}, \hat{Y})\) [24]. The effective masses are equal but with opposite signs for the positive and negative energy solutions. In the present model, small values of \(\hat{P}_x\) and \(\hat{P}_y\) imply that the field amplitudes should be small, something easily achieved with ultracold high-Q cavities.

One interesting observation presented in [25] is that in the non-relativistic limit, the Dirac oscillator in 2+1 dimensions

\[
H_{\text{Dirac}} = c \alpha \cdot (\vec{p} - i m \beta \omega) + \beta mc^2,
\]

Figure 2. Time-evolved atomic inversion for an initial coherent state boosted clockwise (dotted) or anti-clockwise (solid) around the conical intersection of the \(\varepsilon \times E\) model. Parameters can be found in [19].

Figure 3. Time evolution of the expectations \(\langle \hat{X} \rangle\) and \(\langle \hat{Y} \rangle\). Due to the transverse Hall current, the population initially residing in mode \(x\) is swapped to the \(y\) mode at the final time of propagation, which is after approximately 80 ns. The parameters are the same as those used in figure 2, and can be found in [23].
with \( c \) the speed of light, \( \mathbf{p} \) and \( \mathbf{r} \) momentum and position, respectively, \( \alpha_k = \text{off} - \text{diag}(\hat{\sigma}_1, \hat{\sigma}_2) \) and \( \beta = \text{diag}(1, -1) \), becomes identical to an \( \varepsilon \times E \) Jahn–Teller Hamiltonian. Somewhat surprising is that the spin-less version of \( H_{\text{Dy}} \), in which the Dirac four-component matrices \( \alpha_k \rightarrow \hat{\sigma}_k \) and \( \beta \rightarrow \hat{\sigma}_z \), can be mapped onto the Jaynes–Cummings model [26]. Furthermore, even in the non-relativistic limit the trembling ‘motion’ characterizing Zitterbewegung survives and is interpreted as the Ramsey interferometric effect [27].

6. Conclusions

In this paper, we have given a short summary how different cavity QED settings may serve as quantum simulators in various fields of physics. By working in a quadrature representation for the fields, the dynamics of the combined atom–field system can be thought of as an artificial particle moving on a set of coupled potential surfaces. From thereon it is easy to identify different model Hamiltonians as Jahn–Teller ones. By rewriting these Hamiltonians, we defined effective gauge potentials, both Abelian and non-Abelian. Finally, we also demonstrated that relativistic effects should appear for weak field amplitudes.

From the list of references it is clear that it is only recently that these systems have been considered as quantum simulators. A natural conclusion therefore is that much more is to be discovered within this topic. To mention a few possibilities: spin Hall effects in, for example, the Renner–Teller model, Dicke models with multiple number of atoms, relativistic effects and spintronics.

This work did not cover current proposals for simulation of many-body physics by means of cavity QED with a single cavity or an array of coupled cavities [28].

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References

[1] Lewenstein M et al 2006 At. Phys. 20 869 201
[2] Lewenstein M et al 2007 Adv. Phys. 56 243
[3] Garay L P et al 2000 Phys. Rev. Lett. 85 4643
[4] Stoof H T C et al 2001 Phys. Rev. Lett. 87 120407
[5] Joffe M et al 2009 J. Phys. B: At. Mol. Opt. Phys. 42 154009
[6] Raimond J M and Haroche S 2006 Exploring the Quantum (Oxford: Oxford University Press)
[7] Schuster D I 1972 The Jahn–Teller Effect in Molecules and Crystals (New York: Wiley)
[8] Jahn H and Teller E 1937 Proc. R. Soc. A 161 220
[9] Longuet-Higgins H C et al 1958 Proc. R. Soc. A 244 1
[10] Berry M V 1984 Proc. Roy. Soc. A 392 45
[11] Berry M V 1984 Proc. Roy. Soc. A 392 45
[12] Jaynes E T and Cummings F W 1963 Proc. IEEE 51 19
[13] Shore B W and Knight P L 1993 J. Mod. Opt. 40 1195
[14] Wang Y K and Hioe F T 1973 Phys. Rev. A 7 831
[15] Colombe Y et al 2007 Nature 450 272
[16] Larsson J 2007 Phys. Scr. 76 146
[17] Larsson J and Sjöqvist E 2009 Phys. Rev. Lett. 103 013602
[18] Bychkov Y A and Rashba E I 1984 J. Phys. C: Solid State Phys. 17 6039
[19] Sinova J et al 2004 Phys. Rev. Lett. 92 126603
[20] Nagaosa et al 2004 Rev. Mod. Phys. 82 1539
[21] Larsson J and Levin S 2009 Phys. Rev. Lett. 103 013602
[22] Moshinski M and Szczepaniak A 1989 J. Phys. A: Math. Gen. 22 L817
[23] Bermudez A et al 2007 Phys. Rev. A 76 041801
[24] larson J and Stenholm S 2000 Adv. Phys. 22 1817
[25] Bermudez A et al 2008 Phys. Rev. A 77 033832
[26] Morrison S and Parksins A S 2008 Phys. Rev. Lett. 100 040403
[27] Morrison S and Parksins A S 2008 Laser Phot. Rev. 2 527