Power corrections to $\alpha_s(M_\tau)$, $|V_{us}|$ and $\bar{m}_s$

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We re-examine recent determinations of power corrections from $\tau$-decay and confront the results with the existing ones from QCD spectral sum rules (QSSR). We conclude that contrary to the QSSR analysis, which lead to $\langle\alpha_s G^2\rangle = (6.8 \pm 1.3)10^{-2}$ GeV$^4$, $\tau$-decay is not a good place for extracting the gluon condensate due to its extra $\alpha_s^2$ coefficient which suppresses its contribution in this process. Results from $e^+e^-$ sum rules and $\tau$-decay: $\rho \alpha_s(\bar{u}u)^2 = (4.5 \pm 0.3)10^{-4}$ GeV$^6$, where $\rho = 3.0 \pm 0.2$ confirm the deviation from the vacuum saturation estimate of the four-quark condensate. "Non-standard" power corrections (direct instantons, duality violation and tachyonic gluon mass) beyond the SVZ-expansion, partially cancel out in the V-A hadronic $\tau$-decay channel, which gives at order $\alpha_s^2$: $\alpha_s(M_\tau) = 0.3249 (29)_{\text{ex}}(75)_{\text{th}}$ leading to $\alpha_s(M_Z) = 0.1192 (4)_{\text{ex}}(9)_{\text{th}}$, in remarkable agreement with (but more accurate than) $\alpha_s(M_Z) = 0.1191 (27)$ obtained at the same $\alpha_s^2$ order from the $Z$-width and the global fit of electroweak data. Finally, the rôle of the tachyonic gluon mass in the determinations of $|V_{us}|$ from $\tau$-decay and of $\bar{m}_s$ from $\tau$-decay, $e^+e^-$ and (pseudo)scalar channels is emphasized.

1. Introduction

Hadronic $V + A$ $\tau$-decay is expected to provide the most accurate determination of $\alpha_s$ once its value obtained at the $\tau$-mass is runned until the $Z$-mass \cite{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23}.

The extraction of $\alpha_s$, at this relatively low scale $M_\tau$, can become feasible, as theoretically, the $V + A$ channel is less sensitive to the QCD non-perturbative effects than the individual $V$ and $A$ channel (if one uses the OPE à la SVZ \cite{6}) and to the values of the quark and gluon condensates obtained from an overall fit of the hadronic channels \cite{7,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23}, heavy quarkonia sum rules \cite{20,21,22,23} and heavy-quarkonia mass-splittings \cite{24}. In addition, the dominant theoretical uncertainties on the $\alpha_s$ determination, though relatively modest at the $\tau$-mass (10%), becomes tiny at the $Z$-pole as it decreases as $1/\log^2$. More quantitatively, the non-strange $\Delta S = 0$ component of the $\tau$-hadronic width can be expressed as:

$$ R_\tau = \frac{\Gamma(\tau \to \nu_\tau + \text{hadrons}|\Delta S=0)}{\Gamma(\tau \to l + \nu_l + \nu_\tau)} = 3|V_{ud}|^2 S_{EW} \times (1 + \delta^{(0)} + \delta_{EW}^{(2)} + \delta_{svx}^{(2)} + \delta_{\text{int}}), $$

where $\delta^{(2)}_{svx}$ is the light quark mass corrections,

$$ \delta_{svx} \equiv \sum_{D=4}^{8} \delta^{(D)}, $$

is the sum of the non-perturbative (NP) contributions of dimension $D$ within the SVZ expansion \cite{6}, while $\delta_{\text{int}}$ are some eventual NP effects not included into $\delta_{svx}$. We shall use: $|V_{ud}| = 0.97418 \pm 0.00027$ \cite{25}. The electroweak corrections are:

$$ S_{EW} = 1.0198 \pm 0.0006 \text{ } \cite{20} \text{ and } \delta_{EW}^{(2)} = 0.001 \text{ } \cite{27}. $$

We shall use the complete QCD expression in BNP \cite{2} for the numerics.

$\delta^{(0)}$ is the perturbative correction, while $\delta_m$ is the light quark masses corrections. They will be discussed in details in the next paragraph.

2. SVZ power corrections in hadronic $\tau$-decay

Using the QCD expressions compiled in BNP \cite{2}, the light quark masses corrections read:

$$ M_{\tau}^2 \delta_{m,V/A}^{(2)} = -8 \left(1 + \frac{16}{3} a_s\right) \left(\bar{m}_u + \bar{m}_d\right) $$

$$ + 4 \left(1 + \frac{25}{3} a_s\right) \bar{m}_u \bar{m}_d, \quad (4) $$

where $\bar{m}_q$ and $a_s \equiv (\alpha_s/\pi)$ are respectively the light quark running masses and QCD running coupling evaluated at $M_\tau$. Using the recent determinations of the running mass in units of MeV and evaluated at 2 GeV \cite{2,29,30,31}:

$$ \bar{m}_s = 96.1 \pm 4.8 \text{ , } \bar{m}_d = 5.1 \pm 4 \text{ , } \bar{m}_u = 2.8 \pm 0.2 \text{ , } (5) $$

one can deduce the small and negligible corrections in $10^{-4}$:

$$ \delta_{m,V}^{(2)} = -(3.15 \pm 0.51) \text{ , } \delta_{m,V+A}^{(2)} = -(4.22 \pm 0.68). \quad (6) $$

A theoretical estimate of the quark masses contributions has been often used as input in existing analysis of $\tau$-decay data \cite{14,15,16,17,18,19,20,21,22,23}. The non-perturbative corrections à la SVZ \cite{6} read to leading order in $m_q$ and $\alpha_s$ \cite{2}:

$$ M_{\tau}^2 \delta_{V/A}^{(4)} = \frac{11}{4} \pi a_s^2 (\alpha_s G^2) $$

$$ + 16 \pi^2 \left(m_u + m_d\right) \langle \bar{\psi}_u \psi_u + \bar{\psi}_d \psi_d \rangle $$

$$ M_{\tau}^2 \delta_{V/A}^{(6)} = \left(7 - 11\right) \frac{256 \pi^3}{27} \rho \alpha_s \langle \bar{\psi} \psi \rangle^2 $$

$$ M_{\tau}^2 \delta_{V/A}^{(8)} \approx M_{\tau}^2 \delta_{V/A}^{(6)} \approx - \frac{26}{162} \pi^5 (\alpha_s G^2)^2. \quad (7) $$

$\rho$ is the strong coupling constant.
Table 1
SVZ power corrections from \( \tau \)-decay compared with the ones from \( e^+e^- \) and QSSR analysis of the other hadronic channels. FO and CI correspond to \( a_s(M_{\tau}) = 0.331 \pm 0.013 \) and \( 0.350 \pm 0.010 \). ALEPH fits come from [1210], while OPAL fits are from [15]. We take the average of different results and take the quadratic mean of the error (bold face) when the different fits are in good agreement, while in the case where some of the results are not significant, we only consider the most accurate fit (boldface). The final sum \( \delta_{\text{vs}} \) comes from the average or/and from the most accurate determinations.

| NP \( \times 10^3 \) | V -decay | \( e^+e^- \oplus \text{QSSR} \) | V +A -decay | \( e^+e^- \oplus \text{QSSR} \) |
|-----------------|----------|--------------------------|-------------|--------------------------|
| \( \delta^{(4)} \) | 0.87 ± 0.09 | -2.01 ± 0.11 |
| ALEPH | | |
| FO05 | 0.68 ± 0.10 | -2.4 ± 0.1 |
| CI05 | 0.41 ± 0.12 | -2.7 ± 0.1 |
| CI08 | 0.01 ± 0.15 | -3.0 ± 0.1 |
| Average | 41.3 ± 8.4 | -6.1 ± 0.9 |
| \( \delta^{(8)} \) | -15 ± 6 | |
| ALEPH | | |
| FO05 | -8.6 ± 0.6 | 0.1 ± 0.5 |
| CI05 | -9.0 ± 0.5 | -0.03 ± 0.05 |
| CI08 | -8.0 ± 0.5 | 0.81 ± 0.36 |
| OPAL | | |
| FO99 | -8.5 ± 1.8 | -1.5 ± 3.7 |
| CI99 | -8.0 ± 1.3 | -1.0 ± 3.3 |
| Average | -9.5 ± 1.1 | |
| \( \delta_{\text{vs}} \) | | |
| Average | -21.1 ± 1.9 | -7.8 ± 1.0 |

where \( \langle \bar{\psi}\psi \rangle \equiv \langle \bar{u}u \rangle \equiv \langle \bar{d}d \rangle \) and \( \rho \simeq 2 - 3 \) [7891011121321222324]. This can be due to the fact that its contribution in these \( \tau \)-decay channels is difficult to extract from the data as it acquires an extra \( \alpha_s^2 \) correction and then becomes relatively tiny compared to the other condensate effects. Indeed, from Eq. (7), one can notice that it is one order of magnitude smaller than the \( m(\bar{\psi}\psi) \) quark and of the four-quark condensate effects. Another problem may arise that in ALEPH and OPAL analysis, one has to do simultaneously a fit of many parameters in the \( \tau \)-decay analysis \( (\alpha_s, D = 4, 6 \) and 8 condensates). This is not the case of the QSSR analysis of \( e^+e^- \) within LSR, which has a stronger sensitivity to \( \langle \alpha_sG^2 \rangle \) and then permits its robust estimate. In these QSSR analysis, it has been found from different analysis a strong correlation between the \( D = 4 \) and \( D = 6 \) condensate contributions. Using the average of the ratio of the gluon and four-quark condensates \[8\] determined in [910111213],

\[
\frac{r_{46}}{\rho(\bar{\psi}\psi)^2} = (106 \pm 12) \text{ GeV}^{-2},
\]

which reduces the analysis to one-parameter fit [4] the LSR analysis of \( e^+e^- \) data gives \[8\]:

\[
\langle \alpha_sG^2 \rangle_{\text{heavy}} = (6.1 \pm 0.7) \times 10^{-2} \text{ GeV}^4,
\]

while the heavy quarkonia mass-splittings [24] lead to:

\[
\langle \alpha_sG^2 \rangle_{\text{heavy}} = (7.5 \pm 2.5) \times 10^{-2} \text{ GeV}^4,
\]

where the errors have been averaged quadratically. Though higher by about a factor 1.8 than the original SVZ value:

\[
\langle \alpha_sG^2 \rangle_{\text{svz}} \simeq 3.8 \times 10^{-2} \text{ GeV}^4,
\]

the previous estimate is at the border of the lower bound allowed by Bell & Bertlmann [21] in the analysis of the

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3Using arguments based on magnetic confinement, one can argue that \( \langle \alpha_sG^2 \rangle \) is positive [32]. For a review, see e.g. [33].

4The fit has been performed using a standard Mathematica least-square fit program.
heavy quark sum rules and is in line with results from finite energy sum rules (FESR) analysis \[10\] or other methods \[11,12\] in \(e^+e^-\) data and with the available lattice calculations \[13-17\].

\[ \langle \alpha_s G^2 \rangle_{\text{batt}} \approx (12.6 \pm 3.1) \times 10^{-2} \text{ GeV}^4. \] (14)

Using the value in Eq. (12) and the one of light quark masses and condensates into the QCD expressions of \(\delta^{(4)}_{V,V+A}\), one can deduce the QSSR predictions for \(\delta^{(4)}_V\) and \(\delta^{(4)}_{V+A}\) given in Table 1 which agree fairly with the range of values from \(\tau\)-decay \[14,15,16\].

**Four-quark condensate**

Using the ratio in Eq. (9) and the gluon condensate \(\rho \alpha_s \langle \bar{u}u \rangle\) in Eq. (12), one can deduce:

\[ \rho \alpha_s \langle \bar{u}u \rangle^2 = (6.4 \pm 1.3) \times 10^{-4} \text{ GeV}^6. \] (15)

Using this value into the theoretical expression of \(\delta^{(6)}_V\), one obtains the QSSR predictions in Table 1 \((e^+e^-\) column), where one can notice a nice agreement between the contributions of the \(D = 6\) condensates from the LSR in \(e^+e^-\) and from \(\tau\)-decay. Therefore, we can consider as a final result, their average:

\[ \delta^{(6)}_V = (29.4 \pm 2.0) \times 10^{-3} \implies \rho \alpha_s \langle \bar{u}u \rangle^2 = (4.5 \pm 0.3) \times 10^{-4} \text{ GeV}^6. \] (16)

Using the value of \(\alpha_s\) and \(\langle \bar{u}u \rangle\) at \(M_\tau\), one obtains:

\[ \rho = 3.0 \pm 0.2, \] (17)

confirming the deviations of a factor 2 \(\sim 3\) obtained in \[7,8,9,10,11,12,13,14,15,16\] from the vacuum saturation \[6\] of the value of the four-quark condensate. In the \(V+A\) channel, the existing results quoted in Table 1 are inaccurate except the most recent one from \[16\]. Considering as the final value, its average with the one obtained from Eq. (16) and the expression in Eq. (9), we obtain the value in Table 1:

\[ \delta^{(6)}_{V+A} = -(6.1 \pm 0.9) \times 10^{-3}. \] (18)

**Dimension \(D = 8\) condensates**

There is also a good agreement in the extraction of the \(D = 8\) condensates from the vector component of \(\tau\)-decay and QSSR in \(e^+e^-\), which allows us to take the average quoted in Table 1 \((e^+e^-\) column). In the \(V+A\) channel, most of the \(\tau\)-decay results are inaccurate but suggest a value much smaller than the one from the vector channel. As the theoretical relation in Eq. (4) is expected to be (a posteriori) very accurate, we shall consider as a value of \(\delta^{(8)}_{V+A}\), the most accurate estimate from \[16\]:

\[ \delta^{(8)}_{V+A} = -(0.81 \pm 0.36) \times 10^{-3}. \] (19)

Both results from the \(V\) and \(V+A\) channels indicate large deviations from the vacuum saturation suggested by Eq. (7). Analogous results have been obtained from the LSR and FESR analysis of the \(V-A\) channel \[17,18,19\].

**Sum of the SVZ power corrections**

Determinations of the NP corrections from the \(e^+e^-\) sum rules and \(\tau\)-decay data agree, in most case, indicating the consistency of the whole picture. However, the failure of \(\tau\)-decay for extracting \(\langle \alpha_s G^2 \rangle\), can be mainly due to the \(\alpha_s^2\) suppression of its effect making its extraction more delicate. From Table 1 we deduce:

\[ \delta_{svz} = \sum_{D=4}^{8} \delta(D) = -(7.8 \pm 1.0) \times 10^{-3}. \] (20)

This result is comparable with the direct fit from ALEPH and OPAL \[14,15\], where the new result is \[16\]:

\[ \delta_{svz} = \sum_{D=4}^{8} \delta(D) = -(5.9 \pm 1.4) \times 10^{-3}. \] (21)

3. **Sum of standard power corrections**

We add, to these SVZ power corrections, the ones from the pion and \(a_0(980)\) poles into the longitudinal part of the spectral function. It can be written as \[36\] in units of \(10^{-3}\):

\[ \delta_\tau = -16\pi^2 f_\pi^2 m_\pi^2 \left(1 - \frac{m_\pi^2}{M^2}\right)^2 = -(2.65 \pm 0.05) \] \[\delta_{a_0} = -(0.02 \pm 0.01), \] (22)

where we have used \(f_\pi = 93.28\) MeV and \(f_{a_0} = 1.6 \pm 0.5\) MeV \[7\]. Therefore, the total sum of the standard power corrections reads:

\[ \delta_{st} = \delta_{svz} + \delta_{a_0}, \] (23)

4. **\(\alpha_s(M_\tau)\) from \(\tau\)-decay using the SVZ expansion**

Here, we shall be concerned with the perturbative correction \(\delta^{(0)}\) which can be expressed in terms of \(R_{e^+e^-}\) as:

\[ 1 + \delta^{(0)} = \frac{2}{M^2} \int_0^{M^2} dt \left(1 - \frac{t}{M^2}\right)^2 \left(1 + \frac{2t}{M^2}\right) R_{ee}. \] (24)

The ratio \(R_{ee} = \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)\) is expressed through the absorptive part of the correlator of the electromagnetic current \(J_\mu\):

\[ R_{ee} = 12\pi \Im \Pi(-t - i\epsilon), \] (25)

\[ 3Q^2 \Pi(Q^2) = i \int \frac{d^4x}{e^{axz}} \langle 0|TJ_\mu(x)J_\mu(0)|0 \rangle, \] (26)

with \(Q^2 = -q^2\). At \(O(\alpha_s^4)\), its numerical form for \(n_f\) flavours is:

\[ R_{ee} = 1 + a_s + (1.9857 - 0.1152 n_f) a_s^2 \]
\[ + (-6.63694 - 1.20013 n_f - 0.00518 n_f^2) a_s^3 \]
\[ + (-156.61 + 18.77 n_f - 0.7974 n_f^2) a_s^4 \]
\[ + 0.0215 n_f^4, \] (27)
where the $\alpha_s^2$, $\alpha_s^4$ and $\alpha_s^6$ contributions have been computed respectively in [37,38,39]. The perturbative quantity $\delta^{(0)}$ can be evaluated using Fixed Order perturbation theory (FO) or using the so-called “Contour Improvement” (CI). To order $\alpha_s^4$, it reads [11,23,43]:

\[
\delta^{(0)}_\text{FO} = a_s + 5.202 a_s^2 + 26.366 a_s^3 + 127.079 a_s^4, \\
\delta^{(0)}_\text{CI} = 1.364 a_s + 2.54 a_s^2 + 9.71 a_s^3 + 64.29 a_s^4. 
\]

(28)

which, for a reference value $\alpha_s(M_\tau) = 0.34$, gives:

\[
\delta^{(0)}_\text{FO} = 0.2204 \quad \text{and} \quad \delta^{(0)}_\text{CI} = 0.1985, 
\]

(29)

where one can notice a slight difference in the evaluation of $\delta^{(0)}_\text{CI}$ from different authors [16,39,40,41]. This difference will only affect the last digit in the estimate of $\alpha_s$. We use the recent experimental value: $R_{\tau,v+A} = 3.479 \pm 0.011$ [16], and the value of the sum of standard power corrections $\delta_{\text{st}}$ in Eq. (23), from which we deduce:

\[
\delta^{(0)}_{\text{st}} = 0.2081 \times (38) \times \exp(11)_{\text{st}}. 
\]

(30)

Equating Eqs. (28) and (29) with Eq. (30), we obtain to order $\alpha_s^4$ within the SVZ-expansion:

\[
\alpha_s(M_\tau)_{\text{st}} = 0.3294 (33)_{\text{st}} (\text{FO}) = 0.3516 (44)_{\text{st}} (\text{CI}), 
\]

(31)

which, at the $M_Z$ scale, becomes:

\[
\alpha_s(M_Z)_{\text{st}} = 0.1197 (5)_{\text{st}} (\text{FO}) = 0.1223 (4)_{\text{st}} (\text{CI}). 
\]

(32)

At this stage, the theoretical errors do not take into account the estimate of the higher order terms which we shall discuss in the next section. One can compare these results with some other determinations from $\tau$-decay within the SVZ expansion given in Table 2 and the runned value at $M_Z$ with the one from the Z-width [39] and from a global fit of electroweak data at $O(\alpha_s^4)$ [16]:

\[
\alpha_s(M_Z)_{N^3LO} = 0.1191 (27)_{\text{exp}}. 
\]

(33)

and with the most recent world average [42]:

\[
\alpha_s(M_Z)_{\text{world}} = 0.1189 (10). 
\]

(34)

One can notice a quite good agreement within the errors but, like the ones from [16] and [39], these values are on the high side of the world average. Then, one may wonders on the possible effects on these results from some other (often unwanted) effects beyond the SVZ expansion. This will be the subject of the next section.

5. Power corrections beyond the SVZ expansion

To the previous standard contributions of the OPE, there are also other NP contributions which have been overlooked or/and not considered carefully in the existing literature. These contributions are expected to be present as we work with truncated (non) perturbative QCD series which will never describe exactly inclusive processes like e.g. $\tau$-decay, $e^+e^-$, but instead can only give a smearing of it [13,4]. Moreover, even if we are able to add the number of terms we want into the (non) perturbative QCD series, we will not be able to get exactly the QCD two-point correlator involved in an inclusive process such as $\tau$-decay and $e^+e^-$ because the QCD series are factorially divergent.

**Small size instantons**

Direct instantons are expected to be present in QCD for explaining the $\eta' - \pi$ mass shift (the so-called $U(1)_A$ axial problem [15]). At large $Q^2$, it will be highly suppressed as it can be parametrized by an operator of high-dimension $D = 9$. Its quantitative effect has been discussed in previous QSSR literature and has lead to some controversy [7]. A phenomenological fit from $e^+e^-$ data leads to [24]:

\[
\delta_{\text{V,inst}} \simeq 20\delta_{\text{V+A,inst}} \approx -(0.7 \pm 2.7) \times 10^{-3}. 
\]

(35)

Taking for a reference CI perturbative series and the corresponding value $\alpha_s(M_\tau)$ in Eq. (31), it would induce a shift on the value of $\alpha_s$:

\[
\delta \alpha_s(M_\tau)_{\text{inst}} = (0.4 \pm 1.6) \times 10^{-4}, 
\]

(36)

which is invisible at $M_Z$ within the present accuracy.

**Quark-hadron duality violation**

A description of the measured spectral function which is not possible using PT QCD alone has been initially discussed in [44]. This so-called “duality violation” (DV)

\footnote{As argued in [44], lattice calculations may cannot help for solving this issue.}

Table 2

| PT | $\alpha_s(M_\tau)$ | $\alpha_s(M_Z)$ | Ref. |
|----|------------------|-----------------|-----|
| CI | 0.3440 (50)_{th} | 0.1212 (5)_{th} | [16] |
| FO | 0.3156 (30)_{th} | 0.1180 (4)_{th} | [19] |
| CI | 0.3209 (46)_{th} | 0.1187 (6)_{th} | [11] |

This work

| PT | $\alpha_s(M_\tau)$ | $\alpha_s(M_Z)$ | Ref. |
|----|------------------|-----------------|-----|
| FO | 0.3294 (39)_{th} | 0.1197 (5)_{th} | Eq. [31] |
| CI | 0.3516 (44)_{th} | 0.1223 (4)_{th} | Eq. [31] |
| FO | 0.3274 (36)_{th} | 0.1195 (4)_{th} | Eq. [31] |
| CI | 0.3221 (48)_{th} | 0.1188 (6)_{th} | Eq. [31] |
| FO | 0.3249 (29)_{th} | 0.1192 (4)_{th} | Eq. [31] |

| PT | $\alpha_s(M_Z)$ | Ref. |
|----|-----------------|-----|
| Z  | 0.1191 (27)_{th} | [39,10] |
| CI | 0.1189 (10)     | [12,29] |
ImΠ(t)|_{DV} = κ e^{−\gamma t} \sin(\alpha + \beta t)(t − t_{\min})
\tag{37}
\end{equation}

where \(κ, γ, α\) and \(β\) come from the fit of the \(τ\)-data for \(t_{\min} \simeq 1.1 \text{GeV}^2 \leq t \leq M^2_{\tau}\), and depend on the channels studied. DV induces an effect \([40]\):
\[\delta_{DV,V} = -(15 \pm 9) 10^{-3}, \ \delta_{DV,A} \simeq (2 \pm 2) 10^{-4}, \ \tag{38}\]

respectively in the \(V\) and \(A\) channels, where \(δ_{DV,V}\) (which is model dependent) can be relatively large in agreement with the rough estimate \(|δ_{DV,V}| \approx 3\%\) obtained in \([44]\), while \(δ_{DV,A}\) is almost negligible and remains to be understood. DV would induce an effect in units of \(10^{-4}\):
\[\delta α_s(M_\phi)|_{DV} = 175 \pm 101, \ \delta α_s(M_Z)|_{DV} = 18 \pm 17. \ \tag{39}\]

**• Tachyonic gluon from \(e^+e^-\) and \(π\) sum rules**

This mass can induce a new \(1/Q^2\)-term not present in the original SVZ expansion \([8]\). Some eventual origins of this term \([33,47,49,60]\) and its phenomenological applications \([52,53,54,17,28,29,30,34]\) have been discussed in the literature, as well as its relation with the short distance linear part of the QCD heavy quark potential. Here, we shall consider that the \(1/Q^2\)-term is purely of short distance nature and can mainly emerge from the resummation of the infinite terms of the pQCD series (UV renormalon). It is expected to provide a phenomenological parametrization of the unknown higher order terms of the PT series as an alternative to the estimates in the existing literature \([23,31,45,51,55]\) and to the large \(β_0\)-approximation of the UV renormalon contribution \([33,47,10]\). However, its size can depend on the order at which the PT series is truncated and may (in principle) disappear if infinite terms of the series are known \([33,47,10]\). Its contribution to \(R_τ\) is \([53]\):
\[M^2_{\tau}\delta_{V,tach}^{(2)} = M^2_{\tau}\delta_{λ,tach}^{(2)} = -2 \times 1.05α_sλ^2, \ \tag{40}\]

where \(λ^2\) is the tachyonic gluon mass estimated from an overall fit of the pion sum rule \([53,54]\) and of the \(I=1\) part of the \(e^+e^- → \text{hadrons}\) data \([52,53]\). The average of the two determinations gives \([53]\):
\[α_sλ^2 = -(0.07 \pm 0.03) \text{GeV}^2. \ \tag{41}\]

Its presence will only affect slightly the previous determinations of the condensates having higher dimensions, where its correlation to the estimate of \(⟨α_sG^2⟩\) from \(e^+e^-\) has been studied in \([8]\). As originally introduced,

\(^7\)An extension of the model to fit \(e^+e^-\) data above the \(τ\) mass would lead to a slightly lower value of about \(−(2/3)(6.5 \pm 4.2) 10^{-3}\) though consistent with the former within the errors.

\(^8\)An earlier phenomenology of the \(1/Q^2\)-term in \(τ\)-decay and \(e^+e^-\) has been discussed in \([51,52]\).

\(^9\)More discussions on its motivation and rôle in some other hadronic channels and QCD phenomena will be reported in a forthcoming work.

it will be, instead, relevant to the estimate of unknown higher order PT series, which will be discussed in the next section. It would induce, in the \(V+A\) channel, an effect:
\[δ_{tach,\text{pheno}} = (46 \pm 20) \times 10^{-3}. \ \tag{42}\]

**• Tachyonic gluon from large \(β_0\)-approximation**

It is instructive to compare the previous value of the tachyonic gluon (TACH) contribution from the one which would be obtained from the large \(β_0\)-approximation. In so doing, we take advantage of the analysis in \([40]\), where, for a reference value \(α_s(M_\phi) = 0.34\), one can predict for the Borel summed PT series:
\[δ_β^{(2)} = 0.2371. \ \tag{43}\]

Truncating the PT series known until \(n = 4\), one can deduce from Eqs. \([28]\) and \([29]\) in units of \(10^{-3}\):
\[δ_{tach}^{(2)}|_β = δ_β^{(0)} - δ_α^{(0)} ≃ 17 \pm 0.5 \ (\text{FO}) \ \tag{44}\]
\[≃ 39 \pm 5 \ (\text{CI}), \]

which agrees nicely with the phenomenological fit in Eq. \([42]\). We have estimated the error from the deviation of the large \(β_0\)-approximation from the sum of the calculated terms of the series truncated at \(n = 4\). However, as can be observed from \([10,40]\), though the CI converges faster than FO, it will never reach the sum of the large \(β_0\)-approximate result, because the CI series stabilizes for \(n \geq 5 - 6\) well below the exact value, while FO slowly reaches the exact result for \(n \geq 7\). Therefore, the inclusion of TACH with the value in Eq. \([14]\) is necessary for the PT series to reach the “exact result”. In the case of FO, the TACH contribution will decrease for increasing numbers of term in the PT series if one follows the analysis in \([40]\), while for CI, it will remain almost constant. For definiteness, we shall use the large \(β_0\)-approximate result in Eq. \([14]\), which can be more appropriate and accurate for this specific channel than the phenomenological fit. For CI, it would induce an effect in units of \(10^{-4}\):
\[δ α_s(M_\phi)|_{tach} ≃ -491 \pm 63, \ \delta α_s(M_Z)|_{tach} ≃ -61 \pm 8. \ \tag{45}\]

**• Total contributions**

Adding the previous contributions beyond the SVZ expansion, we obtain an estimate of the total “new” contributions in the \(V+A\) channel (in units of \(10^{-3}\)):
\[δ_{\text{inst}} = δ_{\text{inst}} + δ_{DV} + δ_{tach}^{(2)} = (2.0 \pm 9.4) \ (\text{FO}) \ \tag{46}\]
\[= (24.0 \pm 10.6) \ (\text{CI}), \]

where there is a partial cancellation between the DV and TACH contributions. These effects can be about a factor 3 larger than the standard SVZ non-perturbative contributions given in Table \([1]\) but remain relatively small compared to the PT contributions of about 20% which we shall discuss in the next section.
6. \( \delta_{\text{st+nst}} \) corrections to \( \alpha_s(M_t) \)

Adding the sum of \( \delta_{\text{nst}} \) corrections in Eq. (16) to the value of \( \delta_{\text{st}} \) in Eq. (28), the estimates in Eqs. (30) to (32) become to \( O(\alpha_s^4) \):

\[
\delta_{\text{st+nst}}^{(0)}_{\exp} = 0.2061 \ (38)_{\text{exp}}(11)_{\text{st}}(94)_{\text{nst}} \ (\text{FO}) = 0.1841 \ (38)_{\text{exp}}(11)_{\text{st}}(94)_{\text{nst}} \ (\text{CI}) . \tag{47}
\]

implying:

\[
\alpha_s(M_t) = 0.3276 \ (34)_{\text{exp}}(10)_{\text{st}}(85)_{\text{nst}} \ (\text{FO}) = 0.3221 \ (48)_{\text{exp}}(14)_{\text{st}}(121)_{\text{nst}} \ (\text{CI}) . \tag{48}
\]

However, as explained in the previous section, the result from CI is preferred than from FO due to the faster convergence of the PT series, where the difference between the Borel summed series and the known PT series is expected to be only reproduced by \( \delta_{\text{tach}}^{(2)} \) given in Eq. (42) which remains (almost) constant. When higher order terms of the series will be in the available future.

7. Comparison of \( \alpha_s \) from \( \tau \)-decay and \( Z \)-data.

We run our previous result in Eq. (48) at \( M_Z \) and obtains:

\[
\alpha_s(M_Z) = 0.1195 \ (4)_{\text{exp}}(1)_{\text{st}}(10)_{\text{nst}}(2)_{\text{ev}} \ (\text{FO}) = 0.1188 \ (6)_{\text{exp}}(2)_{\text{st}}(15)_{\text{nst}}(2)_{\text{ev}} \ (\text{CI}) . \tag{49}
\]

Both results agree quite well each other and with the one from \( Z \)-data in Eq. (33) and from the world average in Eq. (41). They indicate that the presence of the TACH reduces the difference between the FO and CI results by a factor 4 [comparison of Eqs. (31), (32) and (49)], which is not the case of the ones in the existing literature. We consider, as a final result, the average of the two determinations FO and CI to \( O(\alpha_s^4) \) and in the \( \overline{\text{MS}} \)-scheme:

\[
\langle \alpha_s(M_t) \rangle = 0.3249 \ (29)_{\text{exp}}(9)_{\text{st}}(74)_{\text{nst}} \quad \text{and} \quad \langle \alpha_s(M_Z) \rangle = 0.1192 \ (4)_{\text{exp}}(1)_{\text{st}}(9)_{\text{nst}}(2)_{\text{ev}} \tag{50}
\]

to which correspond:

\[
\begin{align*}
\Lambda_3 &= 353 \ (6)_{\text{exp}}(2)_{\text{st}}(14)_{\text{nst}} \text{ MeV} \quad \text{and} \quad \\
\Lambda_4 &= 307 \ (6)_{\text{exp}}(2)_{\text{st}}(14)_{\text{nst}}(2)_{\text{ev}} \text{ MeV} , \\
\Lambda_5 &= 223 \ (5)_{\text{exp}}(1)_{\text{st}}(11)_{\text{nst}}(1)_{\text{ev}} \text{ MeV} .
\end{align*} \tag{51}
\]

In Table 2 we compare our results with the ones obtained using models which differ in the estimate of the strength of the unknown higher order PT terms:

- The most popular [31, 41, 15, 16] is the estimate of the coefficient of the next order term either as equal to the last term of the PT series or a prediction based on a geometric growth of the PT coefficient [2] or using [55] the principles of “Fastest Apparent Convergence” (FAC) [60] of “Minimal Sensitivity” (PMS) [57, 10]. In [10] CI is preferred over FO as the series converge faster.
- Using an analogous estimate of the higher order terms, Ref. [30] takes the average of the FO and CI results, where 1/3 of the theoretical error (-0.005) comes from the estimate of the unknown \( \alpha_s^2 \) term.
- An alternative approach is the use of a toy model for the Borel transform of the Adler function beyond the \( n = 4 \) order. This model favours FO over the CI [40], where the main theoretical error comes from the renormalisation scale dependence of the FO.
- In [11], CI has been used in connection with Finite Energy Sum Rules having some specific weights.
- In the present work, all possible (standard \( \delta_{\text{st+nst}} \) and non-standard \( \delta_{\text{nst}} \)) QCD power corrections are considered in the extraction of \( \alpha_s(M_t) \). One can notice, from Table 2, a good agreement between different determinations, despite various appreciations on the estimate of higher order terms. However, according to the analysis in Ref. [40], none of the previous estimates have carefully considered the quark-hadron duality violation (DV) effects which cannot be neglected. The DV effects shift the results of [10] and [30] to the high-side of the world average. In this paper, we point out, that the tachyonic gluon mass partially cancels the DV effects and reduces the usual systematic difference between the FO and CI results by a factor 4 [comparison of Eqs. (41), (42) and (49)].

- There is a remarkable agreement between the two determinations from \( \tau \) [Eqs. (31) and (32)] and \( Z \) [Eq. (33)] decays obtained at the same \( N^3\text{LO} \) \( O(\alpha_s^3) \) level of accuracy and at different regions of energy which demonstrates the running strong coupling \( \alpha_s \) level of QCD predicted by asymptotic freedom. At present, the two values of \( \alpha_s(M_Z) \) from \( \tau \) and \( Z \)-decays are the most accurate determinations available today compared to the other determinations compiled in [25, 42]. Though the non-standard power corrections give the largest errors of \( \alpha_s(M_t) \), its accuracy is not largely affected by these new contributions.

8. Tachyonic gluon mass to \( |V_{us}| \) and \( m_s \)

- The CKM angle \( |V_{us}| \) can be determined accurately [5]:

\[
|V_{us}| = 0.2212 \ (31)_{\text{exp}}(5)_{\text{th}} , \tag{52}
\]

from the observable:

\[
|V_{us}|^2 = \frac{R_{\tau,S}}{(R_{\tau}/|V_{ud}|^2) - \delta R_{\tau}} , \tag{53}
\]

if one uses the experimental values:

\[
\frac{R_{\tau}}{|V_{ud}|^2} = 3.6661 \ (12) , \quad \frac{R_{\tau,S}}{|V_{us}|^2} = 0.1686 \ (47) , \tag{54}
\]

while the theoretical input only enters into the SU(3) breaking quantity:

\[
\delta R_{\tau} = 0.227(54) , \tag{55}
\]

which is one order of magnitude smaller than \( R_{\tau} \). About 87% of the error in \( \delta R_{\tau} \) is mainly due to the value of
m_s, which has been taken in [59] to be: m_s(2 GeV) = (96±10) MeV. This error reduces further by a factor 2 if one uses the average quoted in Eq. (5). One may wonder on the TACH contribution to δR_τ. TACH effects to the SU(3) breaking part of the hadronic correlator are [53]:

$$\delta\Pi(Q^2)_{V+A}^{(1)} = \frac{1}{16\pi^2} \frac{a_s\lambda^2\bar{m}^2_s}{Q^4} \left( -\frac{100}{3} + 16\zeta(3) \right)$$

$$\delta\Pi(Q^2)_{V+A}^{(0)} = \frac{3}{8\pi^2} \frac{a_s\lambda^2\bar{m}^2_s}{Q^4},$$

where ζ(3) = 1.2020569... is the Riemann zeta function. Its contribution to δR_τ vanishes to leading order in α_s like other dimension D = 4 operators can and be neglected in the determination of |V_{us}| within the present accuracy of δR_τ.

- **m_s from τ-decay and e^+e^-**

It has been determined from the SU(3)-breaking τ-decay widths [56,99]:

$$\delta R_\tau \equiv \frac{R_\tau}{|V_{ud}|^2} - \frac{R_{\tau,s}}{|V_{us}|^2}$$

or from τ-like decays in e^+e^- using the difference between the isovector and φ-meson channels [29,28]:

$$\Delta_{1\phi} \equiv R_{\tau,1} - R_{\tau,\phi},$$

or with the difference of the usual exponential inverse Laplace α/and FESR sum rules [30,28]. However, TACH contribution is larger in the case of LSR and FESR used in [30,28] for determining m_s, because in these observables, there is no more a cancellation of the D = 4 contribution occured in R_τ. A correlation between the values of m_s and α_sλ^2 has been studied. FESR analysis with the value of α_sλ^2 in Eq. (11) gives to order α_s^2:

$$\bar{m}_s(2 \text{ GeV}) = 100 \ (34) \ \text{ MeV : e^+e^-}$$

$$= 93 \ (30) \ \text{ MeV : τ-decay},$$

in good agreement with the weighted average in Eq. (5), but disfavours the value λ = 0 to which would correspond a too small m_s mass but having larger errors.

- **m_s from the (pseudo)scalar channels**

One should also note that TACH contribution in the pseudoscalar pion and kaon sum rule channels reduces the estimate obtained for λ = 0 to about 5% [53]:

$$(\bar{m}_u + \bar{m}_d)(2 \text{ GeV}) = 8.6 \ (2.1) \ \text{ MeV} \ , \ \Rightarrow$$

$$\bar{m}_s(2 \text{ GeV}) = 105 \ (26) \ \text{ MeV} ,$$

which is, in better agreement with the average in Eq. (5). In the same time, it improves the duality between the experimental and theoretical side of the sum rules.

9. Summary and conclusions

We have reexamined the determinations of the SVZ power corrections and their effects on the extraction of α_s from τ-decay data:

- (α_sG^2) is better determined from QSSR [78,24] than from τ-decay with the robust estimate in Eq. (12).
- In the V channel, τ-decay and e^+e^- sum rules lead to a value of the four–quark condensate ρα_s(⟨uu⟩^2 in Eq. (10) which violates the vacuum saturation estimate by a factor 3 [Eq. (17)] and confirms previous QSSR results from different hadronic channels [78,10,17,18,19,20].
- Using the V+A τ-decay channel, which is the most sensible one for extracting α_s, we show in Eqs. (19) and (50), our estimate to O(α_s^3) which takes into account the effects of power corrections beyond the standard SVZ expansion. Instead of ruining the existing estimates without these non standard effects, these new contributions improve the agreement of the α_s value in Eqs. (19) and (50) with the one in Eq. (33) from the direct determinations at the Z-mass from the Z-width and global fit of electroweak data as well as from the recent world average in Eq. (34). The determinations of α_s(M_Z) from τ-decay lead to the most accurate value of α_s(M_Z) available today and demonstrates with high accuracy the running of the QCD coupling as expected from asymptotic freedom!
- Finally, we have discussed the rôle of the tachyonic gluon mass by taking the example of the determinations of |V_{us}| from τ-decay and of m_s from τ-decay, e^+e^- and pseudoscalar channels. Though its effect is negligible in the τ-like decay sum rules due to the vanishing of the D = 4 contribution, to leading order, in this particular observable, it plays an important rôle in the LSR and FESR approaches.

Acknowledgement

It is a pleasure to thank Santi Peris and Valya Zakharov for communications and for reading the manuscript.

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