Morgan-Morgan-NUT disk space via the Ehlers transformation

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Abstract

Using the Ehlers transformation along with the gravitoelectromagnetic approach to stationary spacetimes we start from the Morgan-Morgan disk spacetime (without radial pressure) as the seed metric and find its corresponding stationary spacetime. As expected from the Ehlers transformation the stationary spacetime obtained suffers from a NUT-type singularity and the new parameter introduced in the stationary case could be interpreted as the gravitomagnetic monopole charge (or the NUT factor). As a consequence of this singularity there are closed timelike curves (CTCs) in the singular region of the spacetime. Some of the properties of this spacetime including its particle velocity distribution, gravitational redshift, stability and energy conditions are discussed.

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I. INTRODUCTION

Finding a new solution of Einstein field equations from a given solution of the same field
equations has a long history and mathematically is related to the problem of finding a solu-
tion to a set of nonlinear partial differential equations from another known solution. Here we
make use of a very special transformation introduced by Ehlers [1] through which one could
start from a static spacetime and end up with a stationary solution. We apply this trans-
formation to two static spacetimes, namely Schwarzschild metric and Morgan-Morgan disk
space [2] and in the meantime employ the concepts introduced in the gravitoelectromagnetic
approach (1+3 formalism) to spacetime decomposition to yield a physical interpretation of
the spacetime found through the transformation. The use of the gravitoelectromagnetic con-
cepts are justified by the fact that the main equation of the Ehlers transformation includes
the gravitomagnetic potential of the stationary spacetime. The outline of the paper is as
follows. In section II we introduce briefly gravitoelectromagnetism and the Ehlers transfor-
mation and use the latter to derive, as an example, the NUT solution [3] from Schwarzschild
spacetime and also Chazy-Curzon-NUT (CC-NUT) solution from Chazy-Curzom static met-
ric [4,5]. In section III we introduce briefly the static Morgan-Morgan disk space (without
radial pressure) and some of its properties. In section IV using Ehlers transformation we
find the corresponding stationary spacetime representing the spacetime of a finite disk of
mass $M$ and gravitomagnetic charge $l$. In section V we discuss some of the properties of this
stationary disk including its particle velocity distribution, gravitational redshift, stability
and energy conditions. In the last section we summarize the results obtained.

II. GRAVITOELECTROMAGNETISM AND EHLERS TRANSFORMATION

The $(1 + 3)$-decomposition (threading) of a spacetime by a congruence of timelike curves
(observer worldlines) leads to the following splitting of the spacetime interval element [6];

$$ds^2 = dT^2 - dL^2,$$

where $dL$ and $dT$ are defined to be the invariant spatial and temporal length elements of
two nearby events respectively. They are constructed from the normalized tangent vector
$u^a = \xi^a / |\xi|$ to the timelike curves in the following way [25];

$$dL^2 = h_{ab}dx^a dx^b,$$

(2)
\[ dT = u_a dx^a, \]  
\[ \text{(3)} \]

where
\[ h_{ab} = -g_{ab} + u_a u_b, \]
\[ \text{(4)} \]
is called the projection tensor. Taking \( h \equiv |\xi|^2 \) and \( A_a = -\frac{\xi_a}{|\xi|^2} \), equations (1) and (4) can be written in the following alternative forms:
\[ ds^2 = h(A_a dx^a)^2 - h_{ab} dx^a dx^b ; \quad h_{ab} = -g_{ab} + hA_a A_b. \]
\[ \text{(5)} \]

Using the preferred coordinate system in which the timelike curves are parameterized by the coordinate time \( x^0 \) of the comoving observers;
\[ \xi^a = (1, 0, 0, 0) \quad ; \quad A_a = (-1, -\frac{g_{0\alpha}}{g_{00}}), \]
\[ \text{(6)} \]
and the above spatial and spacetime distance elements will take the following forms [6,7];
\[ dL^2 \equiv dl^2 = \gamma_{\alpha\beta} dx^\alpha dx^\beta, \]
\[ \text{(7)} \]
\[ ds^2 = e^{2U} (dx^0 - A_\alpha dx^\alpha)^2 - dl^2, \]
\[ \text{(8)} \]
where
\[ e^{2U} \equiv g_{00} \quad ; \quad A_\alpha = -g_{0\alpha}/g_{00}, \]
\[ \text{(9)} \]
and
\[ \gamma_{\alpha\beta} = (-g_{\alpha\beta} + \frac{g_{0\alpha} g_{0\beta}}{g_{00}}). \]
\[ \text{(10)} \]

Introducing \( U \) and \( A \) as the gravitoelectric and gravitomagnetic potentials respectively, their corresponding fields are \([26, 7]\);
\[ E_g = -\nabla U, \]
\[ \text{(11)} \]
\[ B_g = \text{curl} A, \]
\[ \text{(12)} \]
where the differential operators are defined in the \( \gamma \) space. On the other hand Ehlers transformation, also called gravitational duality rotation, in its original form states that if
\[ g_{mn} dx^m dx^n = e^{2U} (dx^0)^2 - e^{-2U} dl^2, \]
\[ \text{(13)} \]
with \( dl^2 = e^{2U} dl^2 \) representing the conformal spatial distance, is the metric of a static exterior spacetime, then,
\[ \bar{g}_{mn} dx^m dx^n = (\alpha \cosh(2U))^{-1} (dx^0 - A_\beta dx^\beta)^2 - \alpha \cosh(2U) dl^2, \]
\[ \text{(14)} \]
(with $\alpha = constant > 0$, $U = U(x^\alpha)$ and $A_\beta = A_\beta(x^\alpha)$) would be the metric of a stationary exterior spacetime provided that $A_\alpha$ satisfies the following equation;

$$\alpha \sqrt{\tilde{\gamma}_{\alpha\bar{\beta}} U^{\bar{\gamma}}} = A_{[\alpha,\bar{\beta}]}$$

(15)

where $\tilde{\gamma} = \det \tilde{\gamma}_{\mu\nu}$, $\tilde{\gamma}_{\mu\nu}$ being the conformal spatial metric. Hereafter we call the above equation the Ehlers equation.

Considering the Ehlers transformation and in particular Ehlers equation (15) in the context of gravitomagnetism, then finding a stationary solution corresponding to a given static solution, amounts to finding the gravitomagnetic potential $A_\alpha$ related to a given static (potential) function $U$. In what follows, as an example of the procedure, we derive the NUT space [8] as the stationary spacetime with a radial gravitomagnetic field which is the gravitational dual of Schwarzschild space.

### A. NUT space from Schwarzschild through Ehlers transformation

Starting from the schwarzschild metric in the form (13) in which :

$$U_c = \frac{1}{2} \ln(1 - \frac{2m}{r}) + C$$

(16)

and

$$d\tilde{l}^2 = dr^2 + r^2 \exp(2U)(d\theta^2 + \sin^2 \theta d\phi^2),$$

(17)

we find that in this case the Ehlers equation (15) reduces to,

$$\alpha r^2 e^{2U} \sin \theta \tilde{\gamma}^{rr} U_{,r} = -\frac{1}{2} A_{\phi,\theta},$$

(18)

in which we set $A_{\theta,\phi} = 0$ to keep the resulted spacetime single valued and axially symmetric.

Using (16) the gravitomagnetic field is found to be,

$$A_\phi(\theta) = 2\alpha m \cos \theta.$$  

(19)

So the resulted stationary metric is given by (14) as;

$$ds^2 = \frac{r(r - 2m)}{\alpha f(r)} (dt - 2\alpha m \cos \theta d\phi)^2 - \frac{\alpha f(r)}{r(r - 2m)} dr^2 - \alpha f(r) d\Omega^2,$$

(20)

where

$$f(r) = r^2 \left(1 + \frac{c_1^2}{2c_1}\right) + 2m^2 c_1 - 2mrc_1, \quad c_1 = e^{2C}.$$  

(21)
This is the well known NUT space \([3,11]\). To recover the Schwarzschild spacetime metric as \(\alpha \to 0\) we need
\[
\alpha = \frac{2c_1}{1 + c_1^2}, \quad |c_1| \leq 1. \tag{22}
\]
Now applying the following changes of variables and redefinitions of constants:
\[
M = m\left(\frac{1 - c_1^2}{1 + c_1^2}\right), \quad \alpha m = l, \quad (r - lc_1) = R, \tag{23}
\]
to the line element (20) we have,
\[
ds^2 = \frac{R^2 - 2MR - l^2}{R^2 + l^2}(dt - 2l \cos \theta d\phi)^2 - \frac{R^2 + l^2}{R^2 - 2MR - l^2}dR^2 - (R^2 + l^2)d\Omega^2, \tag{24}
\]
which is the NUT space in its more common form \([8,11,12]\). Note that having started from a potential with \(C = 0\) (or \(c_1 = 1\)) we would have ended up with the pure \((M = 0)\) NUT space. In other words although adding the constant \(C\) in the static case does not correspond to a different spacetime, in the Ehlers transformation it does. This actually answers another question of concern, namely, what is the seed metric (in Ehlers transformation) of pure NUT space?. By the above argument for both NUT and pure NUT spaces the seed metic is the Schwarzschild metric.

B. stationary spacetimes with \(A_\alpha = A_\alpha(r, \theta)\)

As another example of the application of the Ehlers transformation in this section we consider a family of solutions for a gravitomagnetic potential of the form
\[
A = A_\phi(r, \theta) \hat{\phi}, \tag{25}
\]
(In the spherical or the schwarzschild-type coordinates \((t, r, \theta, \phi)\) which according to the 1+3 formalism \([7-9]\) could, in principle, produce stationary spacetimes with gravitomagnetic field components \(B_g(r, \theta)\hat{r}\) and \(B_g(r, \theta)\hat{\theta}\)). Using Ehlers transformation one could easily show that for an \(A\) of the type (25) one ends up with the equation \(\nabla^2 U = 0\) for \(U\) with the general solution \([11]\)
\[
U = \sum_{n=0} a_n r^{-(n+1)} P_n(\cos \theta). \tag{26}
\]
For the simplest case, i.e. \(n = 0\) we have \(U \propto \frac{1}{r}\) which, in the static case corresponds to the Chazy-Curzon metric \([4,5]\) and in the stationary case (through Ehlers transformation)
to a spacetime dual to Chazy-Curzon with a gravitomagnetic potential of the form $A_\phi(\theta) = acos\theta + b$ [13,14]. It is notable that this has exactly the same form as the gravitomagnetic potential of NUT space and consequently the parameter $\alpha$ could be interpreted as the NUT factor (or magnetic mass). Another example in which Ehlers transformation is examined on a static spacetime to yield an stationary space is the case of cylindrical NUT space [15] where it is shown that it could be obtained from Levi-Civita’s static cylindrically symmetric solution.

III. STATIC SPACETIME OF A PRESSURELESS (MORGAN-MORGAN) DISK

Static axially symmetric spacetimes in the Weyl canonical coordinates are given by the following general form [11];

$$ds^2 = e^{-2U} \left[ e^{2k}(d\rho^2 + dz^2) + \rho^2 d\phi^2 \right] - e^{2U} dt^2,$$

in which $U$ and $k$ are functions of $\rho$ and $z$. Morgan and Morgan [2,11] found the following solution for a finite disk (of mass $M$ and radius $a$) without radial pressure in terms of the oblate ellipsoidal coordinates $(\xi, \eta)$;

$$U = -\frac{M}{a} \left( \arccot\xi + \frac{1}{4}(3\xi^2 + 1)\arccot\xi - 3\xi(3\eta^2 - 1) \right),$$

and

$$k = \frac{9}{4} M^2 \rho^2 a^{-4} \left[ (\frac{\rho}{a})^2 B^2(\xi) - (1 + \eta^2)A^2(\xi) - 2\xi(1 - \eta^2)A(\xi)B(\xi) \right],$$

where

$$A(\xi) = \xi \arccot\xi - 1 \hspace{1cm} B(\xi) = \frac{1}{2} \left[ \frac{\xi}{1 + \xi^2} - \arccot\xi \right],$$

in which the connection between the ellipsoidal $(\xi, \eta)$ and Weyl $(\rho, z)$ coordinates;

$$\rho^2 = a^2(1 + \xi^2)(1 - \eta^2), \hspace{1cm} z = a\xi\eta, \hspace{1cm} |\eta| \leq 1 \hspace{1cm} 0 \leq \xi \leq \infty,$$

infer that the disk is located at $\xi = 0, |\eta| \leq 1$. In [16] the authors have found the form of the same metric in Weyl canonical coordinates as the imaginary part of a complexified bar with the functions $U$ and $k$ as follows;

$$U = -\frac{M}{2ia} ln|Re[R] - ia|,$$
\[ k = -\frac{1}{2} \left( \frac{M}{2} \right)^2 |n| \frac{\text{Re}[R]^2 + a^2}{R^2}. \]  

(39)

where \( R = \sqrt{\rho^2 + (z - ia)^2} \). Exact thin disk solutions of Einstein field equations are discussed extensively in different contexts in the literature. The above static solution, first derived by Morgan and Morgan [2], is considered to represent the spacetime of a finite disk of counterrotating particles. Properties of static counterrotating disks have been studied in [17,18]. Superposition of a central black hole with a surrounding thin disk of counterrotating particles is discussed in [19] and infinite self-similar counterrotating disks were studied in [20,21]. In the following section using the Ehlers transformation we obtain the stationary spacetime corresponding to the Morgan-Morgan static disk space. To our knowledge the exact NUT extension of the above Morgan-Morgan disk space has not appeared in the literature.

IV. STATIONARY SPACETIME OF A FINITE NUTTY DISK

Starting with the Morgan-Morgan static disk space given by equations (34-36) and using the fact that \( 3\eta^2 - 1 = 2P_2(\eta) \) we write (34) as follows

\[ U_c = -\frac{M}{a} \left( \text{arccot} \xi + \frac{1}{2} \left[ (3\xi^2 + 1)\text{arccot} \xi - 3\xi \right] P_2(\eta) \right) + C. \]  

(40)

Now choosing an axially symmetric gravitomagnetic potential \( A = A_\phi(\xi, \eta) \hat{\phi} \), by Ehlers transformation we have;

\[ A_{\phi,\eta} = 2\alpha a(1 + \xi^2)U_\xi, \]  

(41)

\[ A_{\phi,\xi} = -2\alpha a(1 - \eta^2)U_\eta, \]  

(42)

where \( \alpha \) is the Ehlers transformation parameter (duality rotation parameter) and \( a \) is the radius of the disk with mass \( M \). Since the right hand sides of the above two equations include Legendre polynomials \( P_0(\eta) \) and \( P_2(\eta) \) we can expand the gravitomagnetic potential as follows;

\[ A_\phi(r, \theta) = f_1(\eta)P_1(\xi) + f_3(\eta)P_3(\xi). \]  

(43)

By substituting the above potential into the equations (41) and (42) we end up with the following solution for the gravitomagnetic potential;

\[ A_\phi(\xi, \eta) = 3\xi \eta(1 - \eta^2)(1 + \xi^2) \text{arctan} \xi + \frac{3}{2} \eta \left[ (\pi \xi^3 - 2\xi^2 - 4/3 + \pi \xi)\eta^2 - (\pi \xi^2 - 2\xi + \pi)\xi \right] + C_1, \]  

(44)
where as before $l = M \alpha$ is the NUT factor. To regain the Morgan-Morgan space as $l \to 0$, we choose $\alpha = 2c_1$ where $c_1 = e^{2C}$, so that the line element of the stationary spacetime of a finite disk with mass $M$ and magnetic mass (NUT factor) $l$ is given by:

$$
\begin{align*}
\quad & ds^2 = \left( \frac{l^2}{4m^2} e^{2U} + e^{-2U} \right)^{-1} \left[ dt - A_\phi(\xi, \eta) d\phi \right]^2 \\
\quad & - a^2 \left( \frac{l^2}{4m^2} e^{2U} + e^{-2U} \right) \left( e^{2k}(\xi^2 + \eta^2)(\frac{d\xi^2}{1+\xi^2} + \frac{d\eta^2}{1-\eta^2}) + (1 + \xi^2)(1 - \eta^2) d\phi^2 \right).
\end{align*}
$$

(45)

This is verified to be an exact solution to the vacuum Einstein field equations using the Maple tensor package [28]. As the original Morgan-Morgan disk space is asymptotically flat, it is not difficult to see that the above stationary spacetime shares the same property. Hereafter we call the above solution the MM-NUT disk space. The presence of the Dirac-type (NUT-type) singularity could be inferred from the form of the gravitomagnetic potential (44), as one could not make it vanish simultaneously for both the positive ($z > 0 \equiv \eta = 1$) and negative ($z > 0 \equiv \eta = -1$) half-axes ($\rho = 0$) by any choice of the constant $C_1$. In other words only half of the axis could be made regular, either the positive half-axis (for $C_1 = 2l$) or the negative half-axis (for $C_1 = -2l$). Therefore a shrinking loop around the singular half-axis will have a non-zero circumference even when $\rho \to 0$. Now consider a $t = \text{Constant}$ hypersurface and take the lower half-axis ($\eta = -1$) to be singular (i.e. $C_1 = 2l$). In this case the coefficient of $d\phi^2$ in (45) becomes positive at sufficiently small $\rho (\eta \approx -1)$; hence $\phi$ becomes a timelike coordinate and being a cyclic variable with a period of $2\pi$, this means that the spacetime contains closed timelike curves in the singular section; another characteristic of NUT-type singularity.

V. PARTICLE VELOCITY, STABILITY, ENERGY CONDITIONS AND GRAVITATIONAL REDSHIFT

In this section we calculate particle velocity distribution of the MM-NUT (stationary) disk found in the previous section. We also discuss its stability and energy conditions and obtain the gravitational redshift suffered by the photons emitted from different radii on the disk. Hereafter we take a disk with unit radius i.e. we set $a = 1$ for convenience.
A. Particle velocity

The general form of the metric for stationary axially symmetric spacetimes (in cylindrical coordinates) is given by;

\[ ds^2 = g_{tt}(\rho, z)dt^2 + 2g_{t\phi}(\rho, z)dtd\phi + g_{\alpha\alpha}(\rho, z)(dx^\alpha)^2 \quad \alpha = \rho, z, \phi. \tag{46} \]

For a zero angular momentum observer (ZAMO) in these spacetimes, the velocity of a particle in a circular motion (in the \( \rho = \text{Constant}, z = 0 \) plane), is given by [22];

\[ v^2_\phi = \frac{g_{\phi\phi}^2}{g_{\phi\phi}^2 g_{tt}} \left( \frac{d\phi}{dt} + \frac{g_{tt}}{g_{\phi\phi}} \right)^2. \tag{47} \]

Therefore we need to find \( \frac{d\phi}{dt} \), for which, we use the energy-momentum tensor of the particle distribution which in turn could be calculated by the formalism of distributions [19,23]. Using the general form of the energy-momentum tensor of a system of particles [7],

\[ T_{\mu\nu} = \sum m_i (-g)^{-\frac{1}{2}} \frac{dx^\mu}{dt} \frac{dx^\nu}{ds} \delta(x^i - x), \tag{48} \]

and the fact that \( \frac{d\phi}{dt} = \left( \frac{T^{\phi\phi}}{T^{tt}} \right)^{1/2} \) we end up with the following final result;

\[ \left( \frac{d\phi}{dt} \right)^2 = \frac{T^{\phi\phi}}{T^{tt}} \frac{b_{\rho\rho}(g^{\rho\rho}g^{\phi\phi}) + b_{tt}(g^{tt}g^{\rho\rho} - g^{\phi\phi}g^{tt})}{b_{\rho\rho}(g^{tt}g^{\rho\rho} - g^{\phi\phi}g^{tt})}, \tag{49} \]

in which, through the Einstein field equations, the components of the energy-momentum tensor are given by [29],

\[ T^a_b = \frac{1}{2} \left[ b_{az} ^b \delta^z_b - b_{zz} ^b \delta^z_b + g^{az} ^b \delta^z_b - g^{zz} ^b \delta^z_b + b_c ^b (g^{zz} ^a \delta^a_b - g^{az} ^a \delta^z_b) \right] \delta(z), \tag{50} \]

where \( b_{ab} \) are the components of the discontinuity in the first derivative of the metric tensor (at \( z = 0 \)),

\[ b_{ab} = g_{ab,z}|_{z=0^+} - g_{ab,z}|_{z=0^-} = 2g_{ab,z}|_{z=0^+}. \tag{51} \]

Computing these elements for the metric (45) we have;

\[ b_{tt} = \frac{30M \sqrt{1 - \rho^2 F_-(\rho)}}{F_+(\rho)^2}, \tag{52} \]

\[ b_{\rho\rho} = 2e^{2M^2\rho^2} \rho^2 \left( \frac{1}{\pi^2 \rho^2 - 2 + \rho^2} \right)^2 \left( 15M \sqrt{1 - \rho^2 F_-(\rho)} + \frac{45}{2} \pi M^2 \rho^2 \sqrt{1 - \rho^2 F_+(\rho)} \right), \tag{53} \]

\[ b_{\phi\phi} = 30M \sqrt{1 - \rho^2 F_-(\rho)} \left( \rho^2 + \frac{G(\rho)}{F_+^2(\rho)} \right), \tag{54} \]
where
\[ F_\pm(\rho) = \frac{l^2}{4M^2} e^{\frac{3}{4}M\pi(\rho^2-2)} \pm e^{-\frac{3}{4}M\pi(\rho^2-2)}, \] (55)

and
\[ G(\rho) \equiv A_\phi^2(z=0) = (2l - 2l[1 - \rho^2]^\frac{3}{2}). \] (56)

Now substituting (52-56) into (50) and use the outcome in (49) and (47) we obtain the following expression for the particle velocity,
\[ v_\phi = \frac{1}{\rho F_+(\rho)} \left( F_+^2(\rho)\rho^2 - G(\rho) \right) \left( \Omega(\rho) + \frac{G^{1/2}(\rho)}{F_+^2(\rho)\rho^2 - G(\rho)} \right), \] (57)

where
\[ \Omega(\rho) = 4\sqrt{3\pi} \left( 3\pi \frac{l^4}{M^4}\rho^2 \exp\left(\frac{3M\pi(\rho^2-2)}{2}\right) + 4 \frac{l^4}{M^5} \exp\left(\frac{3M\pi(\rho^2-2)}{2}\right) \right. \]
\[ + \ 24\pi \frac{l^2}{M^2}\rho^2 - 192\pi G^\frac{1}{2}(\rho) + 576\pi l^2 \rho^2 (1 - \rho^2) + 192\pi l^2 \rho^6 \]
\[ + \ 48\pi \rho^2 \exp\left(-\frac{3M\pi(\rho^2-2)}{2}\right) - \frac{64}{M} \exp\left(-\frac{3M\pi(\rho^2-2)}{2}\right) \right) \]^{-\frac{1}{2}}. \] (58)

For the disk to be physical we impose the condition that the velocity of the particles at the rim of the disk do not exceed that of the light i.e.,
\[ v_\phi(\rho = 1) \leq 1. \] (59)

Imposing the above constraint in its extreme form (i.e. with the equal sign) on (57), we find the following relation between \( l \) and \( M \)
\[ l_\pm = \frac{M}{3\pi M + 2} \exp\left(\frac{3\pi M}{4}\right) \left( -12\pi M^2 \pm 2\sqrt{36\pi^2 M^4 - 9\pi^2 M^2 + 4} \right), \] (60)

and consequently the following constraint on the maximum value of the disk mass,
\[ M_{\text{max}}^{MM-NUT} = \frac{\sqrt{6}}{12\pi} \left( 3\pi^2 - \sqrt{9\pi^4 - 64\pi^2} \right) \frac{1}{2} \approx 0.2427, \] (61)

for \( l = -0.2204 \). This is a higher maximum value for the disk mass compared to that obtained by Morgan and Morgan [2],
\[ M_{\text{max}}^{MM} = \frac{2}{3\pi} \approx 0.2122. \] (62)

In other words, addition of the NUT factor as an extra parameter, not only changes the static character of the spacetime (to stationary) but also allows for a higher mass of the
FIG. 1: $l$ as a function of $M$ by velocity constraint (60). The allowed values of $l$ and $M$ lie inside the loop formed by the two curves.

Having discussed the velocity distribution of the disk particles one could also discuss stability of particle orbits under radial perturbations. To do so we use the extended Rayleigh
FIG. 2: Velocity of disk particles vs radius for a fixed mass $M = 0.15$ and different $l$ values including $l = 0$ i.e. Morgan-Morgan disk. For two forbidden values of $l$ the velocity diverges and exceeds that of light.

criterion of stability under radial perturbations given by [23,24],

$$\frac{dL^2}{d\rho} > 0.$$  \hspace{1cm} (63)

where $L$ is the angular momentum of disk particles. For circular geodesics in $z = 0$ plane the specific angular momentum is given by;

$$L = g_{\phi\phi}\dot{\phi} + g_{t\phi}\dot{t} = \frac{1}{\sqrt{g_{tt}}}(g_{t\phi} + g_{\phi\phi}\frac{d\phi}{dt}).$$ \hspace{1cm} (64)

Substituting for $\frac{d\phi}{dt}$ from (49) it could be seen that $L^2$ is an increasing function of $\rho$ and satisfies the above criterion leading to the stability of circular orbits (Fig. 3). Since we have already confined the domain of the allowed values of $l$ and $M$ to those respecting the velocity of light barrier, it is expected that the same domain of values respect the energy conditions as well. This is indeed the case for strong and weak energy conditions as we will see below but not for the dominant energy condition. To consider the energy conditions for MM-NUT disk space we diagonalize the stress-energy tensor to find the principal pressures
FIG. 3: Specific angular momentum as a function of $\rho$.

[23]. The Eigenvalue equation for the stress-energy tensor;

$$T^a_b \xi^b = \lambda \xi^a \quad a, b = t, \phi.$$  \hspace{1cm} (65)

constitutes the following eigenvalues

$$\lambda_{\pm} = \frac{1}{2} (T \pm \sqrt{D}),$$ \hspace{1cm} (66)

where

$$D = (T^t_t - T^\phi_\phi)^2 + 4T^t_\phi T^\phi_t \quad \text{and} \quad T = T^t_t + T^\phi_\phi.$$ \hspace{1cm} (67)

It could be easily checked that for the discriminant $D > 0$ the eigenvalues $\lambda_{\pm}$ are the azimuthal pressure $P$ and energy density $\epsilon$ respectively [23]. In our case, using the non-zero components of the energy-momentum tensor (50), the discriminant,

$$D = 14400(1 - \rho^2)M^6 e^{(-\frac{9M^2\rho^2(\pi^2\rho^2 - 32 + 16\rho^2)}{16})} \left[\frac{[p^2 e^{(\frac{3\pi M}{4}(2 - \rho^2))} - 4M^2 e^{(\frac{3\pi M}{4}(2 - \rho^2))}]^2}{[p^2 e^{(\frac{3\pi M}{4}(2 - \rho^2))} + 4M^2 e^{(\frac{3\pi M}{4}(2 - \rho^2))}]^4}\right],$$ \hspace{1cm} (68)

is positive definite and the strong energy condition (SEC),

$$\text{SEC} \equiv \epsilon + P = \lambda_+ - \lambda_- = |D|^\frac{1}{2} \geq 0.$$ \hspace{1cm} (69)
is always satisfied. On the other hand the weak energy condition (WEC) and the dominant energy condition (DEC) could be written equivalently as the following conditions on the magnitude of the gravitomagnetic charge $l$,

$$ \text{WEC} \equiv \epsilon = -\lambda_\pm \geq 0 \implies |l| < 2M \sqrt{\frac{4 - 3\pi M \rho^2}{4 + 3\pi M \rho^2} e^{\frac{3\pi M}{4}(2 - \rho^2)}}. \quad (70) $$

$$ \text{DEC} \equiv \epsilon \geq |P| \implies |l| < 2M \sqrt{\frac{2 - 3\pi M \rho^2}{2 + 3\pi M \rho^2} e^{\frac{3\pi M}{4}(2 - \rho^2)}}. \quad (71) $$

It could be seen that the most restricted situation happens at the rim of the disk, so, to ensure the above conditions all over the disk we apply them at $\rho = 1$ i.e.,

$$ \text{WEC} \equiv |l| \leq 2M \sqrt{\frac{4 - 3\pi M}{4 + 3\pi M} \exp\left(\frac{3\pi M}{4}\right)}, \quad (72) $$

$$ \text{DEC} \equiv |l| \leq 2M \sqrt{\frac{2 - 3\pi M}{2 + 3\pi M} \exp\left(\frac{3\pi M}{4}\right)}. \quad (73) $$

In Fig. 4 the DEC, WEC and velocity conditions are shown in a diagram for the same range of values of $l$ and $M$ as in Fig. 1. It is noted that the WEC is already satisfied by those values of $l$ and $M$ respecting the velocity condition, but DEC could be violated for some values of $l$ and $M$ already allowed by the velocity condition and vice versa. We also note from (73) that, for the static Morgan-Morgan disk ($l = 0$), DEC is equivalent to the velocity condition, in other words, the maximum allowed mass is given by $M_{\text{max}} = \frac{2}{3\pi}$ as in (62).

### C. Gravitational redshift

Using the usual formula for the gravitational redshift in an asymptotically flat spacetime, for the MM-NUT disk space (which is asymptotically flat) we have,

$$ 1 + z = \sqrt{\frac{g_{tt}(\infty)}{g_{tt}(\rho)}} = \frac{1}{\sqrt{1 + \frac{\rho^2}{4M^2}}} \left(\exp\left(\frac{3\pi M}{4}(\rho^2 - 2)\right) + \frac{l^2}{4M^2} \exp\left(-\frac{3\pi M}{4}(\rho^2 - 2)\right)\right)^{\frac{1}{2}}. \quad (74) $$

It is seen that the largest redshift occurs for an emitter at the centre of the disk;

$$ 1 + z_{\text{max}} = \sqrt{\frac{g_{tt}(\infty)}{g_{tt}(0)}} = \frac{1}{\sqrt{1 + \frac{\rho^2}{4M^2}}} \left(\exp\left(-\frac{3\pi M}{2}\right) + \frac{l^2}{4M^2} \exp\left(-\frac{3\pi M}{2}\right)\right)^{\frac{1}{2}}, \quad (75) $$

which compared to the Morgan-Morgan disk has the extra $l$-dependent term in the right hand side. In Fig. 5 redshift of photons emitted from the center of the disk in terms of NUT
parameter for two different disk masses are given. We note from the above discussions that in general for each value of $M$ there is a range of allowed values of $l$ and indeed in Fig. 5 the masses and their corresponding NUT parameters are chosen such that all are physically acceptable.

VI. SUMMARY AND DISCUSSION

In this article using the Ehlers transformation as a tool for finding a stationary spacetime from a static spacetime and the $1 + 3$ approach to spacetime decomposition which yields a physical interpretation of the stationary spacetime in terms of the concepts introduced under the general name of gravitomagnetism, we found the stationary spacetime of a finite disk dual to the static spacetime of a non-rotating disk (without radial pressure) introduced by Morgan and Morgan [2]. The exact form of this metric, which we called MM-NUT disk space, is given in oblate ellipsoidal coordinates in (45). The axially symmetric nature of
the spacetime and the form of the gravitomagnetic potential (44), infers a gravitomagnetic field with components along the polar directions $\rho$ and $\theta$. The fact that we find the extra parameter of the spacetime in the combination $\alpha M$ suggests that the parameter $\alpha$ could be interpreted as the gravitomagnetic monopole charge per unit mass. It is shown that the MM-NUT stationary disk admits, through the introduction of the NUT factor, a higher maximum disk mass than the static Morgan-Morgan disk for a negative value of $l$. It is found that this spacetime is singular either in the positive or the negative half axis, leading to the existence of closed timelike curves (CTCs) in the singular sector. We also showed that MM-NUT disk has stable particle orbits and satisfies the strong energy condition. The weak energy condition on the other hand holds as long as the values of mass and NUT parameter are such that the particle velocities do not exceed that of light. Interesting enough, it is shown that the dominant energy condition could be violated for some values of $l$ and $M$ which respect the velocity barrier condition and vice versa. Also as a by-product and as an example of the application of the Ehlers transformation, we showed that the seed metric for both NUT and pure NUT ($M = 0$) spaces is the Schwarzschild metric.
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[25] Note that Latin indices run from 0 to 3 while the Greek ones from 1 to 3 and throughout we use gravitational units where c=G=1.

[26] For reviews on the subject of Gravitoelectromagnetism see references [8-10].

[27] Note that we have added a constant to the potential using the fact that solutions with potentials differing in additive constants are equivalent.

[28] We used Maple 8, Waterloo Maple Inc., 2002.

[29] For an axially symmetric metric the non-zero components of this tensor (at \( z = 0 \) plane) are given by

\[
T_\phi^\phi = \frac{g^{zz}}{2}(g^{\rho\rho} b_{\rho\rho} + g^{\phi\phi} b_{\phi\phi} + g^{t\phi} b_{t\phi}) , \quad T_t^t = \frac{g^{zz}}{2}(g^{\rho\rho} b_{\rho\rho} + g^{tt} b_{tt} + g^{t\phi} b_{t\phi}) , \quad T_t^\phi = -\frac{g^{zz}}{2}(g^{tt} b_{t\phi} + g^{t\phi} b_{t\phi})
\]