Entanglement Entropy in Random Quantum Spin-S Chains

A. Saguia,1‡ M. S. Sarandy,2 B. Boechat,3 and M. A. Continentino1

1Instituto de Física - Universidade Federal Fluminense,
Av. Gal. Milton Tavares de Souza s/n, Gragoatá, Niterói, 24210-346, RJ, Brazil
2Departamento de Ciências Exatas, Pólo Universitário de Volta Redonda,
Universidade Federal Fluminense, Av. dos Trabalhadores 420, Volta Redonda, 27255-125, RJ, Brazil

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We discuss the scaling of entanglement entropy in the random singlet phase (RSP) of disordered quantum magnetic chains of general spin-S. Through an analysis of the general structure of the RSP, we show that the entanglement entropy scales logarithmically with the size of a block and we provide a closed expression for this scaling. This result is applicable for arbitrary quantum spin chains in the RSP, being dependent only on the magnitude of the central charge of the associated conformal field theory. In experimental research (e.g., see [16, 17]). Moreover, disorder usually introduces a further effect, namely, the breaking of conformal symmetry in a critical model. Remarkably, Refael and Moore [11] have shown that, even in the absence of conformal invariance, due to disorder, the logarithmic scaling given by Eq. (2) holds for the spin-1/2 random exchange Heisenberg antiferromagnetic chain (REHAC) and for the Griffiths phase of the random transverse field Ising chain. In this case, however, it is governed by an effective central charge $\tilde{c} = c \ln 2$. For the pure, i.e., with no disorder, antiferromagnetic spin-1/2 Heisenberg chain, which presents conformal invariance with a central charge $c = 1$, any amount of disorder drives the ground state of this system to the so-called Random Singlet Phase (RSP). This is a gapless phase with broken conformal symmetry characterized by a collection of singlet pairs of spins randomly distributed throughout the chain. A schematic view of the RSP is displayed in Fig. 1. The RSP appears not only for spin-1/2 chains, but also for higher spin disordered chains. In the case of integer spin chains, the RSP will usually arise whenever disorder is strong enough to close the Haldane gap. From the point of view of quantum information applications, the RSP is potentially useful for implementing quantum channels with nearly perfect communication fidelity, since it exhibits pairs of distant spins in highly entangled states [18]. One important open question for understanding the behavior of entanglement in random quantum chains concerns how general the logarithmic scaling of the von Neumann entropy is in these systems.

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I - INTRODUCTION

Quantum information science provides fruitful connections between different branches of physics. In this context, the relationship between entanglement, which is a fundamental resource for quantum information applications [1], and the theory of quantum critical phenomena [2] has been a focus of intensive research. Specifically, entanglement measures have been proposed as a tool to characterize quantum phase transitions (See, e.g., Refs. [3, 4, 5, 6, 7, 8]). In this direction, a successful approach has been the analysis of bipartite entanglement in quantum systems as measured by the von Neumann entropy. Given a quantum system in a pure state $|\psi\rangle$ and a bipartition of the system into two blocks $A$ and $B$, entanglement between $A$ and $B$ can be measured by the von Neumann entropy $S$ of the reduced density matrix of either block, i.e.,

$$S = -\text{Tr} \left( \rho_A \log \rho_A \right) = -\text{Tr} \left( \rho_B \log \rho_B \right),$$

(1)

where $\rho_A = \text{Tr}_{B} \rho$ and $\rho_B = \text{Tr}_{A} \rho$ denote the reduced density matrices of blocks $A$ and $B$, respectively, with $\rho = |\psi\rangle \langle \psi|$. By evaluating $S$ for quantum spin systems, Ref. [6] numerically found that entanglement displays a logarithmic scaling with the size of the block in critical (gapless) chains and saturates to a constant value in chains with a gap for excitations. The logarithmic scaling was proven in general for one-dimensional quantum models exhibiting conformal invariance [9, 10], with the scaling governed by a universal factor, given by the central charge of the associated conformal field theory. Indeed, for a block of spins of length $L$ in a quantum chain, von Neumann entropy $S(L)$ scales as

$$S(L) = \frac{c}{3} \log_2 L + k,$$

(2)

where $c$ is the central charge and $k$ is a non-universal constant. Recently, the behavior of entanglement entropy in critical spin chains has also been discussed in presence of disorder [11, 12, 13, 14, 15]. Disorder appears as an essential feature in a number of condensed matter systems, motivating a great deal of theoretical and experimental research (e.g., see [16, 17]). Moreover, disorder usually introduces a further effect, namely, the breaking of conformal symmetry in a critical model. Remarkably, Refael and Moore [11] have shown that, even in the absence of conformal invariance, due to disorder, the logarithmic scaling given by Eq. (2) holds for the spin-1/2 random exchange Heisenberg antiferromagnetic chain (REHAC) and for the Griffiths phase of the random transverse field Ising chain. In this case, however, it is governed by an effective central charge $\tilde{c} = c \ln 2$. For the pure, i.e., with no disorder, antiferromagnetic spin-1/2 Heisenberg chain, which presents conformal invariance with a central charge $c = 1$, any amount of disorder drives the ground state of this system to the so-called Random Singlet Phase (RSP). This is a gapless phase with broken conformal symmetry characterized by a collection of singlet pairs of spins randomly distributed throughout the chain. A schematic view of the RSP is displayed in Fig. 1. The RSP appears not only for spin-1/2 chains, but also for higher spin disordered chains. In the case of integer spin chains, the RSP will usually arise whenever disorder is strong enough to close the Haldane gap. From the point of view of quantum information applications, the RSP is potentially useful for implementing quantum channels with nearly perfect communication fidelity, since it exhibits pairs of distant spins in highly entangled states [18]. One important open question for understanding the behavior of entanglement in random quantum chains concerns how general the logarithmic scaling of the von Neumann entropy is in these systems.
FIG. 1: A schematic picture of the RSP. Spin singlets are composed at arbitrary distances.

In this context, the RSP constitutes a rather interesting and universal disordered ground state, appearing in a number of half-integer and integer spin chains. In this letter, we investigate the scaling of entanglement in the RSP of arbitrary spin-$S$ chains. We find the logarithmic scaling holds in general and furthermore it is associated with an effective central charge \( c_R = \ln(2S+1) \) which depends only on the value of the spins in the chain. These conclusions apply even if the pure chain, i.e., with no disorder is described by a non-conformal theory. Renormalization group calculations presented here fully support our results.

II - RENORMALIZATION GROUP APPROACH FOR RANDOM QUANTUM SPIN CHAINS

Renormalization group techniques provide a convenient framework for the conceptualization of the quantum phases of random spin chains. As concerns the RSP, it was first obtained using a perturbative real-space renormalization group method introduced by Ma, Dasgupta and Hu (MDH) \[19, 20\] to treat the spin-1/2 REHAC. This approach was proven to be asymptotically exact, which allowed for a fully characterization of the properties of the RSP \[21\]. Consider a chain of spins described by the Heisenberg Hamiltonian

\[
H = \sum_i J_i \vec{S}_i \cdot \vec{S}_{i+1},
\]

where \( \{ \vec{S}_i \} \) is a set of spin-1/2 operators and \( \{ J_i \} \) is a positive random variable obeying some probability distribution \( P(J) \). The original MDH method consists in finding the strongest interaction \( \Omega \) between a pair of spins \( S_2 \) and \( S_3 \) in Fig. 2a) and treating the couplings of this pair with its neighbors \( J_1 \) and \( J_2 \) in Fig. 2a) as a perturbation. The singlet formed by the spins coupled by the strongest bond \( \Omega \) is decimated away and an effective interaction \( J' \) is perturbatively evaluated. By iteratively applying this procedure, the low-energy behavior of the ground state is obtained as a collection of singlet pairs formed over arbitrary distances. This RSP is pictorially displayed in Fig. 1.

Unfortunately, when generalized to higher spins, this method, at least in its simplest version, revealed to be ineffective. The reason is that, after the elimination procedure of the strongest bond \( \Omega \), the effective interaction \( J' \) may be greater than \( \Omega \). Then, the problem becomes essentially non-perturbative for arbitrary distributions of exchange interactions. For instance, considering an arbitrary spin-$S$ REHAC, the renormalized coupling is given by the recursive relation \[22\]

\[
J' = \frac{2}{3} S(S+1) \frac{J_1 J_2}{\Omega}.
\]

Notice that, for \( S \geq 1 \), the renormalization factor is \((2/3)S(S+1) > 1\), resulting in a possible breakdown of perturbation theory depending on the values of \( J_1 \) and \( J_2 \). In order to solve this problem, a generalization of the MDH method was proposed in Refs. \[23, 24\]. This modified MDH method consists in either of the following procedures shown in Fig. 2. Taking the case of the Heisenberg chain as an example, if the largest neighboring interaction to \( \Omega \), say \( J_1 \), is \( J_1 < 3/(2S(S+1))\Omega \), then we eliminate the strongest coupled pair obtaining an effective interaction between the neighbors to this pair which is given by Eq. 3 (see Fig. 2a). This new effective interaction is always smaller than those eliminated. Now suppose \( J_1 > J_2 \) and \( J_1 > 3/(2S(S+1))\Omega \). In this case, using Eq. 4 would give rise to an interaction larger than those eliminated. In order to avoid that, we consider the trio of spins-$S$ coupled by the two strongest interactions of the trio, \( J_1 \) and \( \Omega \) and solve it exactly (see Fig. 2b). The trio of spins is then substituted by one effective spin-$S$ interacting with its neighbors through new renormalized interactions obtained by degenerate perturbation theory for the ground state of the trio. This method has been successfully applied to investigate the quantum phase diagram of the spin-1 \[23\] and spin-3/2 \[24\] REHACs. As we show below, it turns out to be also essential for the computation of block entanglement in high spin chains.

III - ENTANGLEMENT ENTROPY IN THE RSP

Let us consider the general structure of the RSP shown schematically in Fig. 1. The ground state of the system consists of pairs of spins coupled into singlets over arbitrary distances. These can be represented at zeroth order by the tensor product \( |\psi^{(0)}\rangle = |\psi_{i_1 i_2}^{(0)}\rangle \otimes |\psi_{i_3 i_4}^{(0)}\rangle \otimes \cdots \otimes |\psi_{i_k i_{k+1}}^{(0)}\rangle \), where \( |\psi_{i_k i_{k+1}}^{(0)}\rangle \) denotes the singlet state \((1/\sqrt{2})(|\uparrow \!\!\!\!\downarrow\rangle - |\downarrow \!\!\!\!\uparrow\rangle)\) for spins at arbitrary distant sites labelled by integer numbers \( i_k \) and \( i_{k+1} \). In order to evaluate entanglement, we then observe that von
Neumann entropy of a tensor product $\rho \otimes \sigma$ for density matrices $\rho$ and $\sigma$ is simply $\mathcal{S}(\rho \otimes \sigma) = \mathcal{S}(\rho) + \mathcal{S}(\sigma)$. Therefore, the entanglement entropy of a block of length $L$ can be obtained by counting singlet pairs which cross the boundary of the block. This yields

$$\mathcal{S}(L) = S_p \langle N_S(L) \rangle,$$  \hspace{1cm} (5)

where $S_p$ is the entanglement entropy of a pair of spins and $\langle N_S(L) \rangle$ is the configurational average of the number of singlets connecting the two blocks. For a pair of spin-$S$ particles in a singlet state, von Neumann entropy of the pair is maximal and given by $S_p = \log_2 (2S + 1)$. Concerning $\langle N_S(L) \rangle$, it has been evaluated analytically for the spin-1/2 random Heisenberg antiferromagnetic chain in Ref. [11] via counting of singlets from the explicit solution of the renormalization group flow equation for the distribution of couplings. The value obtained for $\langle N_S(L) \rangle$ reads

$$\langle N_S(L) \rangle = \frac{\ln \frac{2}{3}}{3} \log_2 L + k,$$  \hspace{1cm} (6)

with $k$ a non-universal constant. Although Eq. (6) was derived for the specific case of spin-1/2, it should actually hold for any spin-$S$ chain in the RSP. The reason is that the average number of singlet pairs should be the same for any ground state which is a completely random collection of singlets along the chain. Indeed, $\langle N_S(L) \rangle$ is a number which should be a general property of the structure of the RSP, being independent of the magnitude $S$ of the spins composing the chain. Therefore, by inserting Eq. (6) into Eq. (5), we obtain for any random quantum spin-$S$ chain in the RSP the following relationship:

$$\mathcal{S}(L) = \frac{c_R}{3} \log_2 L + k,$$  \hspace{1cm} (7)

where

$$c_R = \ln \frac{2}{3} \log_2 (2S + 1) = \ln (2S + 1).$$  \hspace{1cm} (8)

Notice that $c_R$ depends only on the value of $S$, which is therefore the key object to set the effective central charge of random spin chains in the RSP. Notice that Eq. (8) does not involve any central charge associated with integer spin chains exhibiting a Haldane gap. Let us turn now to the explicit evaluation of entanglement entropy for the spin-1 and spin-3/2 REHACs via numerical implementation of the renormalization group equations. As it will be seen in the next sections, this will provide a full support to our scaling law for $\mathcal{S}(L)$.

III.1 - Spin-1 and spin-3/2 REHACs

We apply now the generalized MDH method to discuss entanglement in the spin-1 and spin-3/2 Heisenberg chains. The Hamiltonian for these chains will be given by Eq. (3), but with the set $\{\vec{S}_i\}$ denoting now spin-1 and spin-3/2 operators, respectively. Decimation of the chain follows the procedure previously described, with the renormalized couplings given by Eq. (4). Then, block entanglement can be numerically computed in the following way. If a singlet is decimated and the spins composing the singlet are in different blocks, this singlet adds $\log_2 (2S + 1)$ to the von Neumann entropy. On the other hand, in the case of a trio elimination, we have seen that one effective spin is returned to the chain. Then, if the spins composing the trio are in different blocks, the effective spin is introduced, by a majority vote strategy, in the block contributing with more spins to the trio, such that, each block contributes with one spin for the elimination process. At the time the effective spin forms a singlet with another spin (either effective or not), entanglement will then be counted as in the singlet case previously described. Hence, notice that von Neumann entropy is counted only when a singlet decimation occurs. This is somewhat similar to the procedure adopted in Ref. [11] for the transverse field Ising model, where entanglement was counted only for effective spins decimated by the transverse field. In our numerical procedure, we averaged entanglement over 1000 random coupling configurations following a power-law probability distribution of couplings $J_i$ given by $P(J) \sim J^{-0.8}$, for which, trio renormalizations are in a negligible number in the RSP [25]. Numerical analysis of entanglement as a function of the length of the block is plotted in Fig. 3 where we considered blocks of spins up to 10000 sites in a chain initially with 200000 sites. Notice that: (i) block entanglement scales logarithmically with the length $L$ of the block; (ii) the logarithmic scaling follows our Eq. (7), with $c_R = \ln 3$ for spin-1 and $c_R = \ln 4$ for spin-3/2. The case of spin-1 is particularly remarkable, since the scaling is governed by an effective central charge although the model does not exhibit conformal invariance even in the limit of vanishing disorder.

III.2 - Random biquadratic spin-1 chain

As shown by Eq. (3), scaling in the RSP depends only on the magnitude of the spins composing the chain, being independent of the particular properties of the model. Indeed, this can be illustrated by considering the spin-1 biquadratic chain, whose Hamiltonian is given by

$$H = \sum_i \Delta_i \left( \vec{S}_i \cdot \vec{S}_{i+1} \right)^2,$$  \hspace{1cm} (9)

where $\{\vec{S}_i\}$ is a set of spin-1 operators and $\{\Delta_i\}$ is a positive random variable obeying some probability distribution $P(J)$. Application of the generalized MDH procedure here results only in the formation of singlets, with
the renormalized exchange coupling reading [22]

\[ \Delta' = \frac{2}{9} \frac{\Delta_1 \Delta_2}{\Omega}, \]

where \( \Delta_1 \) and \( \Delta_2 \) are the neighbors to the strongest bond. Notice that trio renormalizations are completely absent here, since the renormalized couplings generated are always smaller than the energy scale \( \Omega \). Similar behavior occurs for the spin-1/2 Heisenberg chain, with \( J' = (1/2)J_1J_2/\Omega \) and, therefore, the original MDH method can be straightforwardly applied. In this simpler situation, entanglement between a block of length \( L \) and the rest of the chain can be computed by exclusively singlet renormalizations (there is no possibility of trio renormalizations). Block entanglements for both the biquadratic spin-1 chain and the spin-1/2 Heisenberg chain are shown in Fig. 4. As before, we considered blocks of spins up to 10000 sites in a chain initially with 200000 sites, with entanglement averaged over 1000 random coupling configurations following a power-law probability distribution of couplings \( J_i \) given by \( P(J) \sim J^{-0.8} \). For the spin-1/2 Heisenberg chain, the result for the scaling is in agreement with Ref. [11] and reproduces Eq. [5]. Moreover, observe that both the spin-1 REHAC in Fig. 3 and the random biquadratic chain exhibit the same effective central charge \( c_R \), in complete agreement with Eq. [8].

IV - CONCLUSION

In summary, we have characterized the logarithmic scaling of entanglement entropy in the RSP of random quantum chains of general spin-\( S \). In particular, we have shown that an effective central charge governing the scaling can be defined, which depends exclusively on the magnitude \( S \) of the spin, as given by Eq. (8). In order to give support to these conclusions, we evaluated entanglement for different spin-1 and spin-3/2 chains using a real space renormalization group approach. These results are encouraging for the pursuit of a complete characterization of entanglement entropy in general random critical spin-\( S \) chains. In particular, characterization of the complete phase diagram, i.e., for different degrees of disorder of spin-1 and spin-3/2 REHACS, is challenging. This topic is left for a future investigation.

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* Electronic address: amen@if.uff.br

[1] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, UK, 2000).
[2] S. Sachdev, Quantum Phase Transitions (Cambridge University Press, Cambridge, UK, 2001).
[3] M. A. Continentino, Quantum Scaling in Many-Body Systems (World Scientific, Singapore, 2001).
[4] A. Osterloh, L. Amico, G. Falci, and R. Fazio, Nature 416, 608 (2002).
[5] T. J. Osborne and M. A. Nielsen, Phys. Rev. A 66, 032110 (2002).
[6] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Phys. Rev. Lett. 90, 227902 (2003).
[7] L.-A. Wu, M. S. Sarandy, and D. A. Lidar, Phys. Rev. Lett. 93, 250404 (2004).
[8] T. R. de Oliveira, G. Rigolin, M. C. de Oliveira, and E. Miranda, Phys. Rev. Lett. 97, 170401 (2006).
[9] V. E. Korepin, Phys. Rev. Lett. 92, 096402 (2004).
[10] P. Calabrese and J. Cardy, J. Stat. Mech. P06002 (2004).
[11] G. Refael and J. E. Moore, Phys. Rev. Lett. 93, 260602 (2004).
[12] N. Laflorencie, Phys. Rev. B 72, 140408 (2005).
[13] G. De Chiara, S. Montangero, P. Calabrese, et al. J. Stat. Mech. P03001 (2006).
[14] R. Santachiara, J. Stat. Mech. L06002 (2006).
[15] F. Iglói, R. Juhasz, Z. Zimboras, e-print cond-mat/0701527 (2007).
[16] A. P. Young (Ed.), Spin Glasses and Random fields, World Scientific, Singapore, 1998.
[17] F. Iglói and C. Monthus, Phys. Rep. 412, 277 (2005).

[18] J. A. Hoyos and G. Rigolin, Phys. Rev. A 74, 062324 (2006).
[19] S. K. Ma, C. Dasgupta, and C. K. Hu, Phys. Rev. Lett. 43, 1434 (1979).
[20] S. K. Ma and C. Dasgupta, Phys. Rev. B 22, 1305 (1980).
[21] D. S. Fisher, Phys. Rev. B 50, 3799 (1994).
[22] B. Boechat, A. Suguia e M. A. Continentino, Sol. State Comm. 98, 411 (1996).
[23] A. Suguia, B. Boechat, and M. A. Continentino, Phys. Rev. Lett. 89, 117202 (2002).
[24] A. Suguia, B. Boechat, and M. A. Continentino, Phys. Rev. B 68, 020403(R) (2003).
[25] Power-law probability distributions are numerically convenient (but not fundamental) to make evident the scaling of entropy in the RSP, since trio renormalizations get rather negligible in this scenario. Indeed, our results were checked for a number of negative exponents, even for very small ones and also zero (square distribution case).