Theory of \textit{which path} dephasing in single electron interference due to trace in conductive environment

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A single-electron two-path interference (Young) experiment is considered theoretically. The decoherence of an electron wave packet due to the \textit{which path} trace left in the conducting (metallic) plate placed under the electron trajectories is calculated using the many-body quantum description of the electron gas reservoir.

\section{Introduction}

Understanding the interaction between a microscopic quantum system and its macroscopic environment is one of the most exciting challenges of the modern quantum mechanics. Such an interaction is known to perturb the phase relations between the components of quantum superpositions which damages the system ability to show quantum interference. This effect, known as decoherence or dephasing, is essential for our understanding of the origins of classical behavior. Moreover, it limits the feasibility of exploiting the quantum properties of microscopic and mesoscopic systems for performing useful tasks, out of which quantum computation is perhaps the widest known.

Dephasing phenomena manifest themselves in any quantum system. It is, however, natural to seek for their realization in the most clear form. Such an ultimate demonstration seems to be a realization of a “thought” experiment opening many quantum mechanics textbooks: vanishing of the single-particle interference fringes in a two-slit (Young) experiment due either to some perturbation of the environment state or to a controlled measurement. Such a trace (often referred to as \textit{which path} information) left in the measurement device or in the environment allows one (at least in principle) to identify the path chosen by the particle and destroys the interference to the extent depending on the distinguishability of the paths.

Experiments of this kind were realized with electrons in a semiconductor system as well as in optical and atomic interference setups. It was also suggested that an analogous demonstration in the time domain could be based on interference of which path information left in the measurement device or in the environment allows one (at least in principle) to identify the path chosen by the particle and destroys the interference to the extent depending on the distinguishability of the paths.

Another realization has been proposed involving free electron interference with a plate of finite conductivity placed below the electron paths and parallel to them. A model for the decoherence effect in such a setup has been formulated on phenomenological basis using the picture of image charges screening the external flying electron. The image charge would move parallel to the real electron inducing energy dissipation (Joule heat generation) due to finite conductivity of the material. The dissipation effect was quantitatively estimated using an earlier calculation of velocity fields penetrating a metal. The essential qualitative features of the resulting dephasing effect is the dependence on the distance to the plate (which governs the overall intensity of the \textit{which path} trace) and on the distance between the paths (on which the degree of distinguishability depends).

With the state-of-the-art experimental techniques, such a free-electron interference and controlled dephasing experiment is feasible. It is possible to observe interference fringes in experiments with single electrons. It has been proposed to use a single-electron interferometer to show the controlled dephasing effect due to the conducting plate.

The present paper aims at the development of a fully quantum description of the electron dephasing under the specific experimental conditions as described above. On the microscopic level, the \textit{which path} information is transferred from the electron (the system) to the conducting plate (the environment, or reservoir) by exciting the electrons in the plate. This effect may be described by analyzing the electron coupling to the electromagnetic modes affected by the presence of a conducting surface, as previously done for a superconducting and conducting plate. In order to capture the essential (dissipative) effect of the excitations in the reservoir it is, however, profitable to formulate the description in terms of the direct coupling between these two systems which is achieved by using the Coulomb gauge.

A description developed in a different context shows that such a \textit{which path} decoherence effect can be quantitatively represented as the overlap between the spectral density of the reservoir fluctuations and the appropriate spectral function related to the unperturbed evolution of the system. The present calculation follows the same path, with the reservoir fluctuations expressed in terms of the standard longitudinal dielectric function of the conductor. It turns out that the interaction between the flying electron and the charges in the plate involves low frequencies but high momentum transfer, which suggests that description going beyond the Debye model is necessary. Therefore, the present calculation includes a complete model of the quantum properties of the reservoir.

The paper is organized as follows. In the following Section the system under discussion and its model are presented. Next, in Sec. the general framework of the theory is described. In Sec. the problem is reduced to finding the low frequency longitudinal dielectric function. This is done for a metallic plate in the following Sec.
where the quantitative results are found. The final Sec. IV concludes the paper.

II. THE MODEL

In the proposed experimental setup, non-relativistic electrons with energies in the range of 150 eV to 3 keV are emitted by an electron gun. This corresponds to velocities $v \sim 7 \cdot 10^6$ to $3 \cdot 10^7$ m/s (i.e., roughly one order of magnitude higher than the Fermi velocities for metals). The electron beam is split into two paths separated by $D \sim 10 - 300 \mu m$. Both paths pass over a conducting plate of length $L \sim 1$ cm at a height $z_0 \sim 0.1$ mm and then interfere on a screen. The part of the experimental system relevant for the dephasing effect is shown in Fig. 1. It will be assumed that over the plate the electron paths are parallel. The plate is taken to have a length $L$ and a very large height $H$ and width $L_y$. The coordinate system is oriented in such a way that the $z = 0$ plane coincides with the plate surface and the electron paths are parallel to the $x$ axis. In order to refer the present calculations directly to the experiment, a single electron wave function will be chosen in the form of a Gaussian quantum wave packet travelling over the plate at a speed $v$ in a superposition of the two paths. We will assume that the electron wave packets corresponding to the two alternative paths do not overlap and that their extension is much smaller than any distance in the experiment geometry. For the electron velocities as given above, the dispersion of a minimum-uncertainty electron wave packet with initial width of a few micrometers leads to an additional spread of $\sim 0.01 \mu m$ per 1 cm of flight path, so the effects of dispersion may be neglected. It will also be assumed that the time interval between the consecutive electrons is much larger than the relaxation times relevant to the conducting plate so that each flight event is independent of the previous ones. The experiment is done at room temperature.

As mentioned above, contrary to the previous work, the present analysis uses the Coulomb gauge. In this way, the longitudinal electromagnetic field is eliminated in favor of direct and instantaneous coupling to the electronic degrees of freedom in the metal. Thus, the problem is of longitudinal screening type and may be treated using the well-established knowledge on the density-fluctuation spectra of the electron gas and its relation to the longitudinal dielectric function. This choice relates also the present calculation more closely to the original phenomenological concept (although the decoherence mechanism is described in a different way).

The Hamiltonian for the Coulomb interaction of the electron in the beam with the electrons and ions in the solid is

$$V = \frac{e^2}{4\pi \epsilon_0} \int d^3 r \int d^3 r' \frac{\rho_B(r) \rho(r')}{|r - r'|},$$

where $\rho_B(r)$ is the density operator for the electron beam, $\rho(r)$ is the total charge density at a point $r$ in the conducting plate (including both electrons and ions), $e$ is the elementary charge, $\epsilon_0$ is the vacuum dielectric constant, and $r'$ is the coordinate of the electron above the plate.

The interaction between the electron and the charges in the plate (classically described as the image potential) will result from the microscopic theory, while the mutual screening of electrons in the plate must be included in the correct description of the longitudinal response of the interacting electron system.

Once we consider a flight of exactly one electron over the plate, the corresponding density operator may be written in the form $\rho_B(r) = |r|/|r|$, where $|r|$ is the position eigenstate of the electron. In the interaction picture, the Coulomb part of the interaction Hamiltonian reads

$$V_C(t) = \int d^3 r U_0^\dagger(t) |r\rangle \langle r| U_0(t) \frac{e^2}{4\pi \epsilon_0} \int d^3 r' \frac{\rho(r', t)}{|r - r'|},$$

where $\rho(r, t) = U_0^\dagger(t) \rho(r) U_0(t)$ and $U_0(t)$ is the evolution operator for noninteracting subsystems.

The electron may travel along one of two paths, with the corresponding quantum states

$$U_0(t)|0(1)\rangle = \int d^3 r \psi_0(1)(r, t)|r\rangle,$$

where $\psi_{0,1}(r, t)$ are the electron wave functions in the position representation. Since the essential decoherence effect is related to path distinguishability which affects only the relative phase between these two quantum states,
processes leading outside the two-dimensional subspace spanned by these states may be neglected. Projecting the electron states onto this subspace one gets

\[
\langle 0(1)|V_C(t)\rangle 0(1) = \epsilon^2 4\pi \varepsilon_0 \int d^3r |\psi_0(1)(r,t)|^2 \int d^3r' \frac{\rho(p',t)}{|r-r'|^3},
\]

\[
\langle 1(0)|V_C(t)\rangle 0(1) = \epsilon^2 4\pi \varepsilon_0 \int d^3r' \psi_1^{*}(0)(r,t)\psi_0(1)(r,t) \int d^3r' \frac{\rho(p',t)}{|r-r'|^3} = 0,
\]

under the assumption that the wave functions corresponding to the two basis states do not overlap. Hence, we have

\[
V_C(t) = |0\rangle\langle 0| \frac{\epsilon^2}{4\pi \varepsilon_0} \int d^3r \int d^3r' |\psi_0(r,t)|^2 \frac{\rho(p',t)}{|r-r'|^3} \quad (4)
\]

\[
+ |1\rangle\langle 1| \frac{\epsilon^2}{4\pi \varepsilon_0} \int d^3r \int d^3r' |\psi_1(r,t)|^2 \frac{\rho(p',t)}{|r-r'|^3}.
\]

III. THE WHICH PATH DEPHASING OF THE ELECTRON BEAM

In this Section the general framework for the description of the perturbation to the electron state (in the two-dimensional subspace defined above) is described. The approach is based on the second order expansion of the evolution equation for the density matrix in the presence of the instantaneous Coulomb interaction with the charges in the conducting plate. As a result one gets an expression for the visibility of interference fringes in a single-electron interference experiment in terms of a set of correlation constants depending on the electron evolution and on the reservoir properties. Qualitative discussion based on the general form of these correlation constants is also presented here, while the quantitative analysis is performed in the following Sections.

According to the standard theory the final reduced density matrix of the electron subsystem may be written as

\[
\rho(t) = U_0(t) \hat{\rho}(t) U_0^\dagger(t),
\]

where \(\hat{\rho}(t)\) is the reduced density matrix in the interaction picture. In the second order approximation (valid as long as the overall perturbation effect is weak) the latter reads

\[
\hat{\rho}(t) = \rho_0 + i \frac{\epsilon^2}{\hbar^2} \int_{t_0}^t d\tau Tr_R[V_C(\tau),\rho_0 \otimes \rho_R]
\]

\[
- i \frac{\epsilon^2}{\hbar^2} \int_{t_0}^t d\tau' Tr_R[V_C(\tau'),\rho_0 \otimes \rho_R].
\]

Here \(Tr_R\) denotes the trace over the reservoir degrees of freedom (i.e., electrons in the plate) and it has been assumed that at the initial time the system state is separable into the product \(\rho_0 \otimes \rho_R\), where \(\rho_0\) is the initial state of the electron and \(\rho_R\) is the thermal equilibrium state of the electron gas in the conducting plate (this results from the long time interval assumption).

Since the average fluctuation of the electron gas density in the plate vanishes in equilibrium, the leading order contribution to the dephasing is the second term. It may be written in the form

\[
\Delta \tilde{\rho} = -i[H_t, \rho_0] - \frac{1}{2} \{A, \rho_0\} + \tilde{\Phi}[\rho_0],
\]

where the first contribution is a unitary correction and the two other ones describe dephasing (\(\{.,.\}\) denotes an anti-commutator). In the present case it may be shown that \(H_t \sim I\), so that this term does not contribute. Physically, it contains the correction to the electron motion due to the interaction with the image charge induced in the metal. Due to the symmetry between the paths, these corrections are identical for both states and do not induce any nontrivial evolution within the restricted subspace. Their overall effect may be estimated by considering the attraction force \(\epsilon^2/(16\pi \varepsilon_0 \varepsilon_0^2)\) acting during the time of \(L/v\), which results in a negligible vertical shift of at most 100 nanometers.

The operator \(A\) and the superoperator \(\tilde{\Phi}\) are

\[
A = \sum_{i=0,1} R_{ij} |i\rangle\langle i|, \quad \tilde{\Phi}[\rho] = \sum_{ij=0,1} R_{ji} |i\rangle\langle i|\rho|j\rangle\langle j|,
\]

where the correlation constants are

\[
R_{ij} = \frac{\epsilon^4}{(4\pi \varepsilon_0 \hbar)^2} \times \int_{t_0}^t d\tau \int_{t_0}^t d\tau' \int d^3r_1 \int d^3r_2 \int d^3r_1' \int d^3r_2' \langle \rho(r_1', \tau - \tau') \rho(r_2') \rangle.
\]

Note that the integration in Eq. 11 is symmetric in the time variables, while the antisymmetric part has been separated into \(h_t\) in Eq. 6 (see Ref. 12 for technical details). Obviously, \(R_{ij} = R_{ji}^\dagger\). In fact, both these constant are real and equal to each other due to the symmetry between the beams which excludes any relative phase shifts.

Although due to the finite size of the conducting plate \(\rho(r_1', \rho(r_2'))\) does not have the full translational symmetry, in the transverse (y) direction it may depend only on \(y_2 - y_1\). Hence, if the wave functions \(\psi_{0,1}(r)\) differ only by a shift along \(y\) then \(\rho_0 = R_{11}\). The evolution conserves the diagonal elements of the density matrix, while the off-diagonal ones change according to

\[
\langle 0|\Delta \tilde{\rho}|1\rangle = \Delta R\langle 0|\rho_0|1\rangle, \quad \Delta R = R_{01} - R_{11}.
\]

Neglecting the plate edge effects (homogeneous approximation: the decoherence effects simply accumulate while the electron is flying over the plate) and in the absence of reservoir memory (Markovian limit) the system state is separable at any time, so that the above
formula holds for any time step. Then one expects that \( \Delta R \) should be proportional to the path segment \( \Delta l \) traveled by the electron and the quantity \( \lambda^{-1} = -\Delta R/\Delta l \) becomes the dephasing rate per unit path length. In this case, starting from the equal superposition state \( \phi_0 = |\psi_0\rangle\langle\psi_0|, |\psi_0\rangle = (|0\rangle + |1\rangle)/\sqrt{2} \), one gets after the flight over the plate of length \( L \)

\[
\hat{\phi} = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) + \frac{1}{2} e^{-L/\lambda}(|0\rangle\langle 1| + |1\rangle\langle 0|).
\]

The detection probability for an electron at point \( r \) is proportional to

\[
I(r, t) = \langle r | \hat{\phi}(t) | r \rangle = \langle r | U_0(t) \hat{\phi}(t) U_0^\dagger(t) | r \rangle
\]

\[
= \frac{1}{2} \left\{ |\psi_0(r, t)|^2 + |\psi_1(r, t)|^2 + e^{-L/\lambda} |\psi_0'(r, t)\psi_1(r, t) + H.c.| \right\},
\]

where we used the position representation defined in Eq. (3). At a point where the intensity \( j(r) \) of the two beams is equal, i.e., \( \psi_{1,2}(r, t) = \sqrt{2} j(r) e^{i\phi_{1,2}(r, t)} \), one has

\[
I(r, t) = 2 j(r) \left[ 1 + e^{-L/\lambda} \cos \Delta \phi(r) \right],
\]

where \( \Delta \phi(r) = \phi_1(r, t) - \phi_2(r, t) \). In the two-slit experiment this phase difference does not depend on time but varies from point to point as a result of the difference of the corresponding paths lengths, which leads to the interference picture (it is assumed that \( j(r) \) varies slowly in space). The visibility of the interference fringes, defined in the standard way using the maximum and minimum values of \( I(r) \), is then

\[
\alpha = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}} = e^{-L/\lambda}.
\]

Before presenting the detailed calculations it may be worthwhile to discuss the effect qualitatively. First, if the correlations are strongly local,

\[
R(r_1, r_2, t) = \frac{e^t}{(4\pi\epsilon_0\hbar)^2} \int d^3r_1' \int d^3r_2' \frac{\langle \rho(r_1', t)\rho(r_2') \rangle}{|r_1 - r_1'| |r_2 - r_2'|} \delta(r_1 - r_2) \hat{R}(r_1, t),
\]

then

\[
R_{ij} = \int_{t_0}^{t} d\tau \int_{t_0}^{t} d\tau' \int d^3r_1 \times |\psi_i(r_1, \tau)|^2 |\psi_j(r_1, \tau)|^2 \hat{R}(r_1, \tau - \tau') \sim \delta_{ij},
\]

since different wave functions do not overlap. In this case, \( \Delta R \) does not vanish and a dephasing effect appears.

On the other hand, for infinitely long-range correlations, one can write

\[
R(r_1, r_2, t) = f(r_1)f(r_2)\hat{R}(t),
\]

where \( f(r) = f(x, z) \) describes the dependence of the interaction on the electron position relative to the plate. Then

\[
R_{ij} = \int_{t_0}^{t} d\tau \int_{t_0}^{t} d\tau' \left[ \int dx dz \Phi(x, z, \tau)f(x, z) \right] \times \hat{R}(\tau - \tau') \left[ \int dx dz \Phi(x, z, \tau')f(x, z) \right],
\]

where

\[
\Phi(x, z, \tau) = \int dx |\psi_0(r, \tau)|^2 = \int dy |\psi_1(r, \tau)|^2,
\]

since the eye functions differ only by a shift along \( y \). Therefore \( R_{01} = R_{11} \) and the dephasing effect vanishes.

In order to find quantitatively the decoherence path \( \lambda \) the correlation constants \( R_{ij} \) must be evaluated using the correct description of the properties of the electron gas in the conducting plate. This is the subject of the following Sections.

**IV. THE CORRELATION CONSTANTS FOR THE ELECTRON GAS SCREENING RESPONSE**

The goal of the present Section is to express the correlation constants defined in Eq. (4) in terms of a standard material response function (dielectric function) and a spectral function pertaining to the unperturbed evolution (free flight) of a single electron over the conducting plate. Since the dielectric function is expressed in the momentum space and frequency domain, Eq. (4) must be first Fourier-transformed.

As discussed in Sec. II, the electron states are described by Gaussian wave packets moving with the velocity \( v \), localized along two parallel paths, separated by a distance \( D \), at a fixed distance \( z_0 \) from the plate,

\[
\psi_i(r, t) = \frac{1}{\pi^{3/2} l_x l_y l_z} e^{-\frac{1}{\pi} \left( \frac{(x-x_0)^2}{l_x^2} + \frac{(y-y_0)^2}{l_y^2} + \frac{(z-z_0)^2}{l_z^2} \right)},
\]

for \( i = 0, 1 \), where \( y_{0,1} = \pm D/2 \) and \( l_{x,y,z} \) are the wave function widths in the three directions. For such a Gaussian state one has

\[
\int d^3r |\psi_i(r, t)|^2 = \frac{1}{|r-r'|} = \frac{1}{2\pi^2} \int \frac{d^3p}{p^2} e^{-ip(r'-r)} e^{i(p_x v + p_y y + p_z z_0)} e^{-\frac{1}{4}[l_x(p_x)^2 + (l_y p_y)^2 + (l_z p_z)^2]}.
\]
The conducting plate occupies the volume defined by \(-L/2 < x < L/2, -L_y/2 < y < L_y/2, -H < z < 0\), where \(L\) is the finite plate length while \(L_y\) and \(H\) are assumed to be very large. Using the fluctuation-dissipation theorem\(^{36,37}\), the density-density correlation function may be expressed by the imaginary part of the full dielectric function of the conducting plate

\[
\langle \rho(r_1', t) \rho(r_2') \rangle = \frac{e^2}{16\pi^2 \varepsilon_0 V} \sum_q q^2 \int_0^\infty d\omega \coth \frac{h\beta\omega}{2} \Im \left[ \frac{1}{\varepsilon(q, \omega)} \right] \int d^3r_1' \int d^3r_2' \int \frac{d^3p}{p^2} \int \frac{d^3p'}{p'^2} \delta(p_x - \omega)e^{i[-(p+q) \cdot r_1' + p_y y + p_z z]} e^{-\frac{1}{4}[l_x p_x^2 + (l_y p_y + l_z p_z)^2]} \times \delta(p_y' - \omega)e^{i[-(p'-q) \cdot r_2' + p_y' y + p_z' z]} e^{-\frac{1}{4}[l_x p_x'^2 + (l_y p_y' + l_z p_z')^2]}.
\]

where the expansion in space is made in terms of the discrete set of plane waves in the finite volume of the plate (with periodic boundary conditions).

Let us assume that the total dephasing accumulated during the whole flight over the plate is weak. In this case, the perturbative Eq. (5) yields a good approximate description of the effect. Since the interaction takes place only when the electron is over the plate, one may shift the initial and final times of the evolution \(-\infty\) to \(-\infty\) and \(+\infty\), respectively. Then, substituting Eqs. (8) and (9) into Eq. (7) and performing the integral over times we get

\[
R_{ij} = \frac{e^2}{4\pi^3 \varepsilon_0 \hbar^2 V} \sum_q q^2 \int_0^\infty d\omega \coth \frac{h\beta\omega}{2} \Im \left[ \frac{1}{\varepsilon(q, \omega)} \right] \int d^3r_1' \int d^3r_2' \int \frac{d^3p}{p^2} \int \frac{d^3p'}{p'^2} \times \delta(p_x - \omega)e^{i[-(p+q) \cdot r_1' + p_y y + p_z z]} e^{-\frac{1}{4}[l_x p_x^2 + (l_y p_y + l_z p_z)^2]} \times \delta(p_y' - \omega)e^{i[-(p'-q) \cdot r_2' + p_y' y + p_z' z]} e^{-\frac{1}{4}[l_x p_x'^2 + (l_y p_y' + l_z p_z')^2]} \times \delta(p_z' - \omega)e^{i[-(p_z - q) \cdot z + p_z z]} e^{-\frac{1}{4}[l_x p_x'^2 + (l_y p_y' + l_z p_z')^2]}.
\]

Since in the \(y\) direction the plate is very long, the integrals over \(y_1, y_2\) yield a Dirac \(\delta\). On the other hand, in the \(x\) direction, the plate length is limited. Performing the four integrations over \(x_1, x_2, y_1, y_2\), followed by those over \(p_x, p_y, p_z\), one arrives at

\[
R_{ij} = \frac{e^2}{4\pi^3 \varepsilon_0 \hbar^2 V} \sum_q q^2 \int_0^\infty d\omega \coth \frac{h\beta\omega}{2} \Im \left[ \frac{1}{\varepsilon(q, \omega)} \right] \times e^{i\varphi_y(y_1-y_2)} e^{-\frac{1}{4}[(\frac{1}{4\pi^2})^2 + (l_y p_y)^2]} 2\pi L \delta_{L_y}(q_x + \frac{\omega}{v}) |I_z|^2.
\]

Here \(\delta_{L_y}(q) = \frac{4\sin^2 \frac{\pi q}{2L_y}}{2L_y q^2}\) and

\[
I_z = \left[ \int dp_z \frac{1}{(\omega/v)^2 + q_y^2 + p_z^2} e^{-\frac{1}{4}(l_x p_x^2 + (l_y p_y)^2)^2} \int_0^\infty d\omega \coth \frac{h\beta\omega}{2} \Im \left[ \frac{1}{\varepsilon(q, \omega)} \right] \times e^{i\varphi_y(q_y)^2 + \frac{q_y}{2\hat{q}}} \times e^{i\psi_z(z_0 - \frac{z_0}{l_z})} \left[ 1 - \text{Erf} \left( \frac{\hat{q}l_z}{2} - \frac{z_0}{l_z} \right) \right] \right]_{H<\infty}.
\]

where \(\hat{q} = (q_y^2 + (\omega/v)^2)^{1/2}\). In this result, terms proportional to \((l_z/z_0)\) have been neglected, since \(l_z \ll z_0\).

Since the characteristic momentum scales of the system are at least of the order of \(1/z_0\), the broadening of the function \(\delta_L(q)\), which is of the order of \(1/L_z\), may be neglected and one can write \(\delta_L(q) \approx \delta(q)\). In this way one neglects the corrections related to the approach to the edge of the finite plate and to the fly-away phase after crossing it, compared to the dephasing accumulated during the flight directly over the plate. It should be noted that Eq. (10) is limited to weak dephasing but it does not involve any Markovian approximations. On the other hand, the memory time of the screening response of the electrons in metals is of order of inverse Fermi energy, i.e., femtoseconds, which is many orders of magnitude shorter than any time scale of the problem. Vanishing memory together with the proportionality of dephasing to the length travelled over the plate allow us to interpret the perturbative result as a dephasing rate (per unit path length) and to obtain an exponential decay as a solution of the corresponding rate equation. As discussed in the Appendix A the same result is obtained by coarse-graining the flight over the plate and explicitly using the short memory assumption, which directly leads to correlation constants \(R_{ij}\) proportional to time over each small time step.

Using Eq. (10) in the above approximation and replac-
ing the summation over \( q \) by integration one may write

\[
\Delta R = -\frac{e^2 L}{\epsilon_0 \hbar v (2\pi)^3} \int \frac{d\omega}{\omega} \coth \frac{\hbar \beta \omega}{2} \int d^3q \left[ -\frac{1}{\epsilon(q, \omega)} \right] S(q, \omega),
\]

where

\[
S(q, \omega) = \frac{\omega}{8vq} \delta \left( \sin \theta + \frac{\omega}{qv} \right) \left[ 1 - e^{iDq \cos \varphi \cos \theta_0} \right]
\]

\[
\times e^{-\frac{1}{2} \left( \frac{\omega}{2v} \right)^2 - \frac{1}{4} \left( \frac{\omega}{2v} \right)^2 \cos^2 \varphi \cos^2 \theta_0 - \frac{1}{4} \left( \frac{\omega}{2v} \right)^2 \sqrt{\cos^2 \varphi \cos^2 \theta_0 + \left( \frac{\omega}{2v} \right)^2} \}
\]

\[
\times \frac{1}{\cos^2 \varphi \cos^2 \theta_0 + \left( \frac{\omega}{2v} \right)^2} \left[ 1 - \text{Erf} \left( \frac{q l_z}{2} \sqrt{\cos^2 \varphi \cos^2 \theta_0 + \frac{\omega^2}{v^2 q^2} - \frac{z_0}{l_z}} \right) \right]^2,
\]

where we write \( q = q(\sin \theta, \cos \theta \cos \varphi, \cos \theta \sin \varphi) \) and denote \( \theta_0 = \arcsin[\omega/(vq)]. \) Since \( z_0 \) is a large (macroscopic) distance, for \( q \gg \omega/v \sim 1/z_0 \) the above function is strongly peaked around \( \varphi = \pm \pi/2. \) On the other hand, the dielectric function of a conductor extends over momenta of the order of the Fermi momentum, \( q \sim k_F \gg 1/z_0 \) and its imaginary part vanishes for low momenta, so that for the relevant momentum values one can write

\[
S(q, \omega) = \frac{1}{2} S(q, \omega) \delta \left( \sin \theta + \frac{\omega}{qv} \right) \left[ \delta \left( \varphi - \frac{\pi}{2} \right) + \delta \left( \varphi + \frac{\pi}{2} \right) \right],
\]

where

\[
S(q, \omega) = \int_0^{2\pi} d\varphi \int_{-\pi/2}^{\pi/2} \cos \theta d\theta S(q, \omega).
\]

Physically, the above formulas mean that the momentum transfer from the flying electron to the electron gas excitations in the \((x, y)\) plane may be at most of order of \(1/z_0\) and the energy transfer of order of \(\hbar \omega/z_0.\)

The integral over \( \theta \) is trivial while that over \( \varphi \) may be performed by a saddle point approximation around \( \varphi = \pm \pi/2, \)

writing \( \cos(\varphi \pm \pi/2) \approx \pm \varphi \) which, upon further substitution \( \varphi = [(qv/\omega)^2 - 1]^{-1/2}u \) and extending the integration limits, leads to

\[
S(q, \omega) = \left[ 1 - \left( \frac{\omega}{qv} \right)^2 \right]^{-1/2} \int_{-\infty}^{\infty} \left[ 1 - e^{-i \frac{D}{2} u} \right] \frac{1}{1 + u^2 e^{-2 \frac{z_0}{u^2} \sqrt{1 + u^2}}} \]

\[
\times e^{-\frac{1}{4} \left( \frac{\omega}{2v} \right)^2 - \frac{1}{4} \left( \frac{\omega}{2v} \right)^2 u^2 - \frac{1}{4} \left( \frac{\omega}{2v} \right)^2 \sqrt{1 + u^2} \left( \frac{1}{4} \left[ 1 - \text{Erf} \left( \frac{l_z \omega}{2v} \sqrt{1 + u^2 - \frac{z_0}{l_z}} \right) \right] \right)^2. \]

The analysis of the spectral function \( S(q, \omega) \) (see Fig. 2) shows that it decays exponentially for \( \omega \gtrsim v/z_0 \) while for a fixed \( \omega \) it varies with \( q \) on an interval \( \sim 1/z_0 \) to reach a plateau for \( q \gg \omega/v, \) where it attains a constant value, dependent only on \( \omega, \)

\[
S(q, \omega) \xrightarrow{q \gg \omega/v} S(\omega).
\]

The description simplifies considerably if one assumes that the size of the electron wave packet is much smaller than the distance to the conducting plate, i.e., \( l_{x,y,z} \ll z_0. \) Note that keeping the finite wave packet size leads only to negligible quantitative corrections. In particular, it does not introduce any momentum cut-off. In the approximation \( l_{x,y,z} \to 0, \) the asymptotic value may be written in the form

\[
S(\omega) = \int_{-\infty}^{\infty} du \left[ 1 - e^{-i \frac{D}{2} u} \right] \frac{1}{1 + u^2 e^{-2 \frac{z_0}{u^2} \sqrt{1 + u^2}}} \]

\[
\times \left[ 1 - \text{Erf} \left( \frac{l_z \omega}{2v} \sqrt{1 + u^2 - \frac{z_0}{l_z}} \right) \right]^2. \]

In the low frequency range selected by the function \( S(\omega), \) one has always \( \text{Im} e^{-i(q, \omega)} \sim \omega \) (this is a general property see also below). Moreover, \( \hbar \omega/(k_B z_0) \sim 1 \) K, so that at room temperature one may write \( \coth(h\beta \omega/2) \approx 2k_B T/(\hbar \omega) \) for all relevant frequencies \( \omega \lesssim v/z_0. \) Therefore, the frequency integral in Eq. (11)
partly performed using Eq. (12), structure. In any case, however, this integral may be very different depending, for instance, on the relation between the thermal energy $k_B T$ and the Fermi energy, as well as on the conduction band structure. In any case, however, this integral may be partly performed using Eq. (12),

$$\int_0^\infty d\omega \frac{S(\omega)}{\omega} = \frac{D}{z_0},$$

where

$$\gamma(x) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{du}{1 + u^2} \ln \left[ 1 + \frac{x^2}{4 \left( 1 + u^2 \right)} \right],$$

$$= \frac{\pi}{16} x^2 + O(x^4),$$

[using the asymptotic form given in Eq. (14)]. This function depends only on the geometry of the system. More specifically, the only relevant parameter is the ratio $D/z_0$. The function $\gamma(x)$ is plotted in Fig. 3.

As for the momentum integral in Eq. (11), it obviously depends on the specific properties of excitations in the electron gas and may be very different depending, for instance, on the relation between the thermal energy $k_B T$ and the Fermi energy, as well as on the conduction band structure. In any case, however, this integral may be partly performed using Eq. (12),

$$\int \frac{d^3q}{q^2} \text{Im} \left[ -\frac{1}{\varepsilon(q, \omega)} \right]$$

$$\times \frac{1}{2} \delta \left( \sin \theta + \frac{\omega}{q v} \right) \left[ \delta \left( \varphi - \frac{\pi}{2} \right) + \delta \left( \varphi + \frac{\pi}{2} \right) \right]$$

$$= \int dq_z \text{Im} \left[ -\frac{1}{\varepsilon(q, \omega)} \right]_{q_x = q_y = 0},$$

where it is assumed again that $qv \gg \omega$. Hence, from Eq. (11),

$$\Delta R = -\frac{k_B T \epsilon^2 L}{4\pi^3 \epsilon_0 \hbar^2 v} \frac{D}{z_0} \frac{1}{\omega} \int dq_z \text{Im} \left[ -\frac{1}{\varepsilon(q, \omega)} \right]_{q_x = q_y = 0}.$$ 

Since, at low frequencies, $\epsilon_2(q, \omega) \ll \epsilon_1(q, \omega)$, one may write

$$\text{Im} \left[ -\frac{1}{\varepsilon(q, \omega)} \right] \approx \frac{\epsilon_2(q, \omega)}{\epsilon_1^2(q, \omega)}.$$ 

Below, the results for metallic plates are presented.

V. DECOHERENCE OVER A METALLIC PLATE

In this Section, the decoherence effect for a metallic plate is considered. For the description of the reservoir response properties, we will use the quasi-free electron (isotropic) longitudinal dielectric function in the random phase approximation (the Lindhard theory). By assuming periodic boundary conditions this dielectric function may be applied to a finite system with negligible corrections as long as the typical momentum transfers are much larger than the inverse system size, which is the case in our problem. Obviously, the screening response originates only from the finite volume of the plate, which has already been included in the theory developed above.

On the other hand, the Lindhard theory assumes that single-particle states are just plain waves confined in a box with infinite potential walls, which is not quite the case for a real piece of metal. In particular, the dielectric discontinuity at the surface acts as a trapping potential for carriers. Since the surface states are confined in the vertical direction, low energy excitations must involve in-plane momentum transfer, which is not favorable in our problem. Thus, one should not expect large dissipation (dephasing) from these modes an their contribution will be dominated by virtual high-frequency excitations (surface plasmons) which will tend to screen the potential of the external charge, thus decreasing the dephasing. This effect will be neglected in the present analysis.

Even in this quasi-homogeneous approximation, the strictly correct description of the screening response requires solving the complicated many-body problem, involving the electrons in the metal, the positive ions, and the mutual screening interaction between these subsystems. Under the conditions of the present study the treatment may be, however, greatly simplified with some loss of precision but retaining the essential features and yielding reasonable quantitative estimations.

Thus, the calculation of the total dielectric function of a metal will be done within the approximation consisting in simply adding the susceptibilities of the electron
and ion subsystems,
\[
\epsilon(q, \omega) = 1 + \chi_{el}(q, \omega) + \chi_{ph}(q, \omega) = [1 + \chi_{ph}(q, \omega)] \left[ 1 + \frac{1}{1 + \chi_{ph}(q, \omega)} \chi_{el}(q, \omega) \right],
\]
where \(\chi_{el}(q, \omega), \chi_{ph}(q, \omega)\) are the electron and phonon electric susceptibilities, respectively.

The phonon susceptibility, including the screening of the ion interaction by electrons, is given by
\[
\chi_{ph}(q, \omega) = \frac{\omega^2_{pi}}{\omega^2(q) - \omega^2},
\]
where \(\omega_{pi} = Ze^2 n/(\epsilon_0 M)^{1/2}\) is the ion plasma frequency (here \(Z\) is the ion charge, \(n\) is the ion concentration, and \(M\) is the ion mass). Since there is no momentum cut-off in the problem under discussion, the system response is dominated by excitations with \(q\) of order of the Debye wave vector \(k_D\), so that we may replace \(\omega(q)\) by its short-wavelength value for which the standard simple theory yields \(\omega_{pi}\). Also, since \(\omega \sim v/\omega_{pi} \ll \omega_{pi}\), we may put \(\omega \to 0\) in the denominator. In this approximation, the effect of screening by the lattice excitations is fully contained in the ion dielectric constant
\[
\epsilon_i = 1 + \chi_{ph}(k_D, 0) \approx 2.
\]

In this discussion the imaginary part of the lattice response describing the dissipative effects has been neglected. Such effects must always be resonant, i.e., they involve phonons with low frequencies \(\omega \sim v/\omega_{pi}\). The density of states for such long-wavelength phonons is very low. Moreover, at the corresponding low momenta \(v/(\epsilon_0\omega_{pi}) \ll k_F\), where \(\epsilon_0\) is the sound speed, the real part of the total dielectric function is very large due to electron contribution (see below), so that such low-frequency lattice excitations are additionally very strongly screened. Therefore, the lattice contribution to the dissipative processes [i.e., to the \(\varepsilon_2(q, \omega)\) function] is negligible. In terms of the general picture of the dephasing due to the trace in the environment this means that most of the phonons have frequencies much higher than that characteristic of the external field so that they follow the perturbation adiabatically, returning to the original state after the flying electron is away and thus registering no trace of its passage.

The total dielectric function including electron transitions within the conduction band and the lattice screening as described above is therefore described by the formula
\[
\varepsilon(q, \omega) = \epsilon_i \left[ 1 + \frac{1}{\epsilon_i} \chi_{el}(q, \omega) \right],
\]
where we take into account that in the isotropic model the dielectric function may depend only on the value but not on the direction of \(q\). The electron susceptibility in

![FIG. 4:](image)

**FIG. 4:** (a) Solid line: the “material” function \(\mu(x)\) for metals with the values corresponding to a few metals (assuming \(\epsilon_i = 2\)); dashed and dotted lines: first and second order approximation according to Eq. (14), respectively; points: the values for the five metals corrected for the Coulomb and exchange correlations within the Hubbard model. (b) The inverse decoherence length as a function of the \(D/\omega_0\) ratio for gold at \(T = 293\ K\), for three values of the electron energy

the quasi-free electron model valid for arbitrary \(q\) is given by the Lindhard formula
\[
\chi_{el}(q, \omega) = \frac{e^2}{4\pi^3 \epsilon_0 q^2} \int d^3p \frac{n_F(E_p) - n_F(E_{p+q})}{E_p - E_{p+q} + \omega + i0^+},
\]
where \(E_p = p^2/(2m)\), \(m\) is the electron mass and \(n_F(E)\) is the Fermi-Dirac distribution. Since the function \(S(q, \omega)\) has an exponential spectral cut-off at frequencies much lower than the Fermi energy, one needs only the low-frequency part of the dielectric function (as already remarked in Sec. [IV]). To be specific, \(h\varepsilon/\omega_{pi} \approx 10^{-5}\omega_F\), where \(E_F\) is the Fermi energy (the relevant frequency range corresponds to radio frequencies). Thus, for the imaginary part of the susceptibility [Eq. (15)] we write, using the standard results
\[
\text{Im} \chi_{el}(q, \omega) = \begin{cases} 
\frac{e^2 q^2}{4\pi^2 \epsilon_0} \frac{\omega}{\omega_F} & \text{for } \omega/\nu_F < q < 2k_F, \\
0 & \text{otherwise}.
\end{cases}
\]

The electron velocities used in the experiment satisfy \(v > \nu_F\) for practically all metals. Thus, the condition \(q > \omega/v\) is automatically satisfied and the asymptotic form of the spectral function given by Eq. (15) may always be used.

The leading low-frequency term of the real part of Eq. (15) is
\[
\text{Re} \chi_{el}(q, 0) = \frac{me^2 k_F}{2\pi^2 \epsilon_0^2 q^2} \left( 1 + \frac{4k_F^2 - q^2}{4qk_F} \ln \left( \frac{q + 2k_F}{q - 2k_F} \right) \right),
\]
where \(k_F\) is the Fermi wave vector.

The momentum integral in Eq. (15) may be written, using the explicit expression (20), in the form
\[
\int dq_2 \varepsilon_2(q_2, \omega) = \frac{2\pi\epsilon_0 \hbar}{\epsilon^2} \left( \frac{me^2}{2\pi\epsilon_0 \epsilon_1^2 \hbar^2 k_F} \right).
\]

The lower value of the integral over \(q\) may be extended from \(\omega/v\) to 0. This is a negligible correction since \(\omega/v \ll
k_F and the expression under the integral is regular at q → 0. The function
\[ \mu(x) = \frac{x^2}{4} \int_0^1 \frac{du}{u^3} \left[ 1 + \frac{x}{4\pi u^2} \left( 1 + \frac{1 - u^2}{2u} \ln \left( 1 + u \right) \right) \right]^{-2} \]
describes the properties of the metallic reservoir (the material properties of the system). It is depicted in Fig. 4a. Inserting the above result into Eq. (10) and using Eq. (11), one gets the characteristic dephasing path length for a metallic reservoir
\[ \lambda^{-1} = -\frac{\Delta R}{L} = \frac{k_B T}{2\pi^2 \hbar v} \left( \frac{m e^2}{2\pi \epsilon_0 \hbar^2 k_F} \right) \gamma \left( \frac{D}{z_0} \right). \] (22)
This result is shown for gold in Fig. 4b for a few electron energies. The dependence on material parameters is contained in the function \( \mu \) which, for many metals, differs only slightly from that corresponding to gold (see Fig. 4a). For electrons with energy 150 eV the decoherence effect should be noticeable with a 1 cm plate already for \( D/z_0 \sim 0.1 \). In Fig. 4 the visibility of interference fringes for a specific system setup is shown, along with a simulation of the fringes.

A standard way to quantitatively improve the description of the screening response of an electron gas is to include correlations between electrons leading to Coulomb and exchange hole around an electron. This can be done within the Hubbard model, where the electron susceptibility is replaced by
\[ \chi^{(H)}_{\text{el}}(q, \omega) = \frac{\chi_{\text{el}}(q, \omega)}{1 - G(q) \chi_{\text{el}}(q, \omega)}, \] (23)
with
\[ G(q) = \frac{1}{2} \frac{q^2}{q^2 + k_{\text{TF}}^2}, \]
where \( k_{\text{TF}} \) is the Thomas–Fermi wave vector for a given metal. It turns out that this correction affects the value of \( \mu \) for typical metals only to a very little degree, as shown in Fig. 4a (points).

It may be interesting to obtain a closed formula for the decoherence length \( \lambda \) or for the decoherence time \( \tau_d \) and compare it to the phenomenological prediction of Ref. 20. As can be seen in Fig. 4b, for most metals one has
\[ \frac{m e^2}{2\pi \epsilon_0 \hbar^2 k_F^3} \sim 1. \]
In this range of values the asymptotic expansion is valid
\[ \mu(x) \approx \frac{\pi}{4} x - \frac{1}{12} x^2 \] (24)
(see Fig. 4b). Retaining only the linear term one finds
\[ \int dq \frac{\varepsilon_2(q, \omega)}{\varepsilon_1(q, 0)} = \frac{\pi}{4} \frac{m}{\epsilon_0 \hbar v} \omega, \]
which leads to the approximate formula for the decoherence length \( \lambda \)
\[ \lambda^{-1} = \frac{k_B T}{8\pi^2 \hbar v \epsilon_0 \hbar v} \gamma \left( \frac{D}{z_0} \right). \] (25)
For \( D/z_0 \) we use the expansion from Eq. 15 and write for the decoherence time \( \tau_d = \lambda/v \)
\[ \tau_d^{-1} = \frac{\pi}{32} \tau^{-1} \left( \frac{D}{\lambda_{\text{dB}}} \right)^2, \] (26)
where \( \lambda_{\text{dB}} = 2\pi \hbar / \sqrt{2m \epsilon_0 \hbar v} \) is the thermal de Broglie wave length of an electron and
\[ \tau_r^{-1} = \frac{\epsilon^2}{2\pi \epsilon_0 \hbar v \lambda_{\text{dB}}^2}. \] (27)
The formula 20 is essentially of the same form as that given in Ref. 20. However, here \( \lambda_{\text{dB}} \) refers to the electrons in the metal and involves the mass of these electrons. In fact, decoherence results from the perturbation of the reservoir state and not that of the system. Since the reservoir responds to the electromagnetic field of the flying electron, its state may depend on its charge but not on the mass. Therefore, it should be expected that particles of different mass but equal charge will undergo the same decoherence.

The time constant \( \tau_r \) was originally interpreted as the energy dissipation rate, i.e. the rate at which energy is transferred from the electron to the reservoir. A similar interpretation is possible here, within a semi-classical approach. The electrons in the metal involved in interactions with external fields are those near the Fermi surface. The relative rate at which their energy changes is therefore
\[ \tau_r^{-1} = \frac{1}{E_F} \frac{dE}{dt}. \]
The variation of the energy of electrons is due to their acceleration by the electric field \( \mathcal{E} \) of the flying electron the distance to which is roughly \( z_0 \), so that

\[
\frac{dE}{dt} = m\mathbf{v}\cdot \mathcal{E} = mv\cdot \frac{e\mathcal{E}}{m} \approx v_F \frac{e^2}{4\pi\varepsilon_0 c_0 z_0},
\]

where \( v_F \) is the electron speed at the Fermi surface and the screening by ions has been included. Using the equal-acceleration by the electric field \( \mathcal{E} \)

\[
\text{where } F = \frac{\hbar^2 k_F^2}{2m} \text{ and } v_F = \frac{\hbar k_F}{m} \text{ one arrives at the formula (24).}
\]

In spite of the formal similarity of Eq. (26) to the phenomenological decoherence rate obtained from the energy losses due to dc resistance and Joule heating, the present theory actually describes a different mechanism of decoherence. The dc resistance-based description assumes implicitly that formation of the screening image charge is dissipationless (adiabatic), involving only virtual transitions, and therefore reversible. The irreversible, dissipative processes take place only due to carrier scattering as the image charge moves beneath the metal surface. Since, at high temperature, such scattering is mostly due to phonons, the which path information is effectively stored in lattice excitations. In contrast, the quantum description presented here shows that already the process of formation of the image charge is to a large extent dissipative, even in the absence of carrier-phonon scattering. Since the reservoir response to the external charge involves only excitations of low frequency but of arbitrary momenta the appropriate dielectric function is different from the commonly used Drude limit. As a result, there is no direct correlation between the decoherence effect and the dc conductivity or resistivity. Instead, for metals the theory predicts roughly inverse proportionality to the Fermi momentum which is the only material parameter entering the result (apart from the electron mass). Quantitative comparison shows that for noble metals the rate given by Eq. (26) is many orders of magnitude higher than that resulting from resistive dissipation. Therefore, the Ohmic resistivity effect is of minor importance.

VI. CONCLUSIONS

The decoherence effect on a single electron travelling over a conducting plate was calculated in a model reflecting the conditions of a currently performed experiment. The dissipative response of the electron gas in the plate, tending to screen the external charge, generates a trace in the plate. The resulting distinguishability of the electron paths destroys the electron’s ability to interfere. In order to describe this effect quantitatively a fully quantum model of the mutual interaction between the electron and the conductive reservoir was formulated. The decoherence effect may be expressed by the spectral density of the reservoir fluctuations which, in turn, is related to its dissipative properties, i.e., to the imaginary part of the inverse longitudinal dielectric function.

The resulting decoherence was described for a metallic reservoir but the qualitative form of the result depends only on the universal linear frequency dependence of its imaginary part. Therefore, the decoherence effect will be qualitatively similar for any plate material (e.g., for semiconductors) as long as the electron gas response is dominated by the low-frequency sector. Whether this is the case for a specific system, will depend on the interplay of the experimental conditions and material parameters.

The results presented in the paper explain the mechanism of electron dephasing on a fully quantum level and give quantitative estimations of the effect, depending on the material parameters and the geometry of the experimental setup. Therefore, they should be helpful for experimental studies in this fundamental field.

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APPENDIX A: EXPLICIT MARKOVIAN APPROXIMATION

In this Appendix, Eq. (10) is re-derived in a way that clearly displays the Markovian approximation by making an explicit use of the short memory time \( \tau_{\text{mem}} \) of the screening response which leads to correlation constants which are directly proportional to time. Now we consider the evolution during the time segment \( t_0 < t < t_0 + \Delta t \) during which the electron is moving over the plate. We assume that \( \Delta t \) is much longer than \( \tau_{\text{mem}} \) but short enough for the accumulated perturbation to be small. Since \( \tau_{\text{mem}} \sim h/E_F \) is of order of femtoseconds and the time of flight over a 1 cm plate is in the nanosecond range such time-slicing is always possible in the experimentally interesting situation of non-complete dephasing.

Let us start again by substituting Eqs. 3 and 4 into Eq. 11. Now, however, we first integrate over \( y_{1,2} \), which yields \( 4\pi \delta(p_y + q_y)\delta(p_y' - q_y) \), and over \( x'_{1,2} \), which yields \( 4\pi^2 \delta_{L'}(p_x + q_x)\delta_{L'}(p_x' - q_x) \), where \( \delta_{L'}(q) = \sin(qL/2)/(\pi q) \). Based on the same argument as before, for \( L \gg 2z_0 \) the latter may me replaced with a Dirac delta. The trivial integration over \( p_x \) and \( p_{x,y} \) yields a formula containing the time integration in the
form
\[ \int_{t_0}^{t_1} dt \int_{t_0}^{t_1} dt' e^{-i q \cdot v (t - t')} M_q(t - t') \approx \int_{t_0}^{t_1} dt \int_{-\infty}^{\infty} du e^{-i q \cdot v u} M_q(u) = -2\pi \Delta t \coth \frac{\hbar \beta \omega}{2} \text{Im} \varepsilon^{-1}(q, -q, v), \]

where we defined the memory function
\[ M_q(t) = \int_{-\infty}^{\infty} d\omega e^{-i \omega t} \coth \frac{\hbar \beta \omega}{2} \text{Im} \varepsilon^{-1}(q, \omega) \]

and extended the limits of the second integration because \( M_q(t) \) decays on time scales much shorter than \( \Delta t \).

Using this result one obtains
\[ R_{ij} = \frac{\Delta t e^2}{2\pi^2 \epsilon_0 \hbar V} \sum_{q_x < 0, q_y, q_z} q^2 \coth \left( \frac{\hbar \beta |q_x| v}{2} \right) \text{Im} \left[ \frac{1}{\varepsilon(q_x, |q_x| v)} \right] \]

which is identical to the \( L \to \infty \) limit of Eq. (10) with \( L = v \Delta t \).