A pairing on the cuspidal eigenvariety for $GSp_{2g}$ and the ramification locus

Ju-Feng Wu

Abstract. In the present paper, we first construct a pairing on the space of analytic distributions associated with $GSp_{2g}$. By considering the overconvergent parabolic cohomology groups and following the work of Johansson–Newton, we construct the cuspidal eigenvariety for $GSp_{2g}$. The pairing on the analytic distributions then induces a pairing on some coherent sheaves of the cuspidal eigenvariety. As an application, we follow the strategy of Bellaïche to study the ramification locus of the cuspidal eigenvariety over the corresponding weight space.

Contents

1 Introduction 1

2 Analytic distributions 5

2.1 Algebraic and $p$-adic groups 6

2.2 Analytic distributions 7

2.3 A pairing on the analytic distributions 10

3 The overconvergent cohomology and the eigenvariety 14

3.1 A pairing on cohomology groups 14

3.2 Hecke operators 16

3.3 The cuspidal eigenvariety 18

4 The ramification locus of the cuspidal eigenvariety 22

4.1 Some commutative algebra 22

4.2 The ramification locus of the cuspidal eigenvariety 24

4.3 Non-degeneracy of the pairing 26

References 30

1 Introduction

1.1 Overview

After the introduction of the eigencurve by R. Coleman and B. Mazur in [CM98], there were many other mathematicians who contributed to its study. The eigencurve is a rigid analytic curve which parameterises overconvergent Hecke eigenforms of finite slope and its geometry is very interesting and quite mysterious. For example we don’t know even in an example if the eigencurve has finitely or infinitely many irreducible components.

It is a natural question to ask whether one can generalise the notion of the eigencurve to $p$-adic automorphic forms on other Shimura variety. In [AIP15], F. Andreatta, A. Iovita and V. Pilloni construct sheaves of (families of) overconvergent Siegel modular forms. As an application, they construct an eigenvariety $E^{AIP}$, parametrising overconvergent cuspidal Siegel eigenforms of finite slope and raise the following question:
Question 1.1.1 ([AIP13, Open Problem 1]). Let \( \mathcal{W} \) be the weight space. Is the weight map \( \mathcal{E}^{AIP} \rightarrow \mathcal{W} \) unramified at classical points?

On the other hand, G. Stevens introduced in [Ste94] the overconvergent modular symbols as a new tool to study the eigencurve, method of study which was taken over by other authors, for instance, [Bel12, Bel, Chapter VIII]. Following the philosophy presented in [Bel, Chapter VIII], we attempt to use Theorem 1.1.2. Fix a answer to Question 1.1.1.

We have the following (AIP15, Open Problem 1). Question 1.1.1 with level structure given by op. cit. VIII and JN19. More precisely, we construct a pairing on the overconvergent cohomology groups for \( p \) read the information over the further to give such a formalism in the language of adic spaces in JN19, which consequently allows one to read the information over the \( p = 0 \) locus of the weight space.

The work presented in this paper is motivated by Question 1.1.1 and is highly inspired by Bel, Chapter VIII and JN19. More precisely, we construct a pairing on the overconvergent cohomology groups for \( G_{Sp}^2 \) and adapt the formalism in op. cit. to construct the corresponding cuspidal eigenvariety \( \mathcal{E}_0 \) by working with the parabolic cohomology. Following the philosophy presented in Bel, Chapter VIII, we attempt to use such a pairing to detect the ramification locus of \( \mathcal{E}_0 \) over the weight space \( \mathcal{W} \), aiming to provide a (partial) answer to Question 1.1.1.

The results of the presenting paper are summarised in the following theorem.

Theorem 1.1.2. Fix a \( g \in \mathbb{Z}_{>0} \), an odd prime number \( p \) and an integer \( N > 3 \) such that \( p \nmid N \). Let \( X_{Iw}^+ (C) \) be the \( C \)-points of the Siegel modular variety parametrising principally polarised abelian varieties of genus \( g \) with level structure given by

\[
\Gamma(N) = \{ \gamma \in GSp_{2g}(\mathbb{Z}) : \gamma \equiv \mathbf{1}_g \mod N \} \quad \text{and} \quad Iw_{GSp_{2g}}^+ = \left\{ \gamma \in GSp_{2g}(\mathbb{Z}_p) : \gamma \equiv \begin{pmatrix} \ast & \ast & \cdots & \ast \\ \vdots & \vdots & \ddots & \vdots \\ \ast & \ast & \cdots & \ast \end{pmatrix} \mod p \right\}.
\]

We have the following

1. There exists a cuspidal eigenvariety \( \mathcal{E}_0 \rightrightarrows \mathcal{W} \), parametrising the eigenvectors of finite slope parabolic cohomology groups \( H^*_{par}(X_{Iw}^+(C), D_\lambda^\vee) \). Denote by \( \mathcal{E} \) the eigenvariety constructed in JN19 by using the algebraic group \( G_{Sp}^2 \), then there is a closed immersion \( \mathcal{E}_0 \rightarrow \mathcal{E} \) of adic spaces over \( \mathcal{W} \) and \( \mathcal{E}_0 \) is the cuspidal part of \( \mathcal{E} \). (see JN19)

2. Let \( Z \) be the Fredholm hypersurface in \( \mathbb{R}^N \) and let \( \pi_{\mathcal{E}_0} : \mathcal{E}_0 \rightarrow Z \) be the structure morphism. Let \( \mathcal{H}^{\text{tot}}_{par} \) be the coherent sheaf associated to (finite-slope) total parabolic cohomology groups \( H^{\text{tot}}_{par}(X_{Iw}^+(C), D_\lambda^\vee) \) on \( Z \). Then there is a pairing

\[
(\pi_{\mathcal{E}_0}^*)^* \mathcal{H}^{\text{tot}}_{par} \times (\pi_{\mathcal{E}_0}^*)^* \mathcal{H}^{\text{tot}}_{par} \rightarrow \mathcal{O}_{\mathcal{E}_0} \quad \text{(resp.} \mathcal{H}^{\text{tot}}_{par} \times \mathcal{H}^{\text{tot}}_{par} \rightarrow \mathcal{O}_Z\text{)}
\]
of coherent sheaves on $\mathcal{E}_0$ (resp., $\mathcal{Z}$). (see Corollary 3.3.9)

3. Suppose $x \in \mathcal{E}_0^\text{fl}$ is a good classical point (see Corollary 4.3.5). Then there exists a function $L^{\text{adj}}$ on a small enough clean neighbourhood $\mathcal{V}$ of $x$, determined uniquely by the above pairing up to a unit in the eigenalgebra, such that

$$ L^{\text{adj}}_\mathcal{V}(x) = 0 \text{ if and only if } w\text{t is ramified at } x. $$

(see Theorem 4.2.8 and Corollary 4.3.5)

4. Retain the situation as above and assume further that $x$ is a smooth point in $\mathcal{E}_0^\text{fl}$. Let $e(x)$ be the quantity depending on $x$ and $w\text{t}$ defined in Theorem 4.2.9, then

$$ \text{ord}_x L^{\text{adj}} = e(x). $$

(see Theorem 4.2.9 and Corollary 4.3.5)

1.2 Some remarks

The works presented in this paper have their connections to some known results. We summarise them in the following remarks:

1. We should remark first that the $p$-adic subgroup considered in this paper is slightly different from the other authors. This is due to an issue when constructing the pairing. However, the underlying distribution spaces $D^\infty_r(\mathcal{T}_0, R)$ appearing in the present paper are isomorphic to the ones considered in [JN19] (in the case of $GSp_{2g}$). Thus, the strategies of the known literature can go through after changing the underlying locally symmetric space. In particular, one can expect the comparisons of eigenvarieties in the next remark.

2. There are comparisons of the cuspidal eigenvariety $\mathcal{E}_0$ considered in this paper with the other eigenvarieties constructed by others. As mentioned above, $\mathcal{E}_0$ is the cuspidal part of the eigenvariety $\mathcal{E}$ constructed in [JN19] when considering $GSp_{2g}$. By [op. cit., Remark 4.1.9], the eigenvariety $\mathcal{E}^{\text{Han}}$ constructed by D. Hansen in [Han17] is the open locus of $\mathcal{E}$ on which $p \neq 0$. On the other hand, there is a closed immersion mapping from the eigenvariety $\mathcal{E}^{\text{Urb}}$ constructed by E. Urban in [Urb11] to $\mathcal{E}^{\text{Han}}$ by the introduction of [Han17]. Denote by $\mathcal{E}_0^{\text{Urb}}$ the cuspidal part of $\mathcal{E}^{\text{Urb}}$ and let $\mathcal{E}_0^{\text{Urb}, \text{red}}$ be the reduced cuspidal eigenvariety of $\mathcal{E}^{\text{Urb}}$, then it coincides with the eigenvariety $\mathcal{E}^{\text{AIP}}$ constructed in [AIP15] (see [AIP15] pp. 627). In conclusion, the comparisons among these eigenvarieties can be summarised in the following diagram

$$
\begin{array}{cccccc}
\mathcal{E}^{\text{AIP}} & \longrightarrow & \mathcal{E}^{\text{Urb}}_{0, \text{red}} & \longrightarrow & \mathcal{E}^{\text{Han}}_0 & \longrightarrow & \mathcal{E}_0 \\
\downarrow \text{closed immersion} & & \downarrow \text{open immersion} & & \downarrow \text{open immersion} & & \downarrow \text{closed immersion} \\
\mathcal{E}^{\text{Urb}} & \longrightarrow & \mathcal{E}^{\text{Han}} & \longrightarrow & \mathcal{E}_0 & \longrightarrow & \mathcal{E} \\
\end{array}
$$

During the study of the present work, we also encounter the following (natural) questions that are worth for further studies:

3. The function $L^{\text{adj}}$ in the $GL_2$ case was justified to $p$-adically interpolate the adjoint $L$-values associated to a family of eigen-newforms in [Kim06]. We analogously call $L^{\text{adj}}$ an “adjoint $p$-adic $L$-function” in our case, hence the justification is required. In [GT05 §12], A. Genestier and J. Tilouine established a pairing of the parabolic cohomology groups for $GSp_4$ by using the symplectic pairing on the algebraic
representations of $GSp_4$ and the cup product. Such a pairing is related to the cardinality of the Selmer group for the adjoint Galois representation attached to $GSp_4$ by \cite[, Théorème 12.0.1]{GT05}. Moreover, the cardinality of the Selmer group is suggested to be related to the adjoint $L$-value in the discussion after \emph{loc. cit.}. This suggested that once one can relate our pairing with the pairing introduced in \cite[, §12]{op.cit.}, then the justification of the name can be done. However, such a relation is unknown to us due to the fact that the authors of \cite, work with the Siegel modular variety without level structure at $p$ (in order to apply the Taylor–Wiles method) while the Siegel modular variety with $Iw_{GSp_{2g}}$-level structure at $p$ is considered in our case. The comparison between the Petersson norms of a Siegel modular form of prime-to-$p$ level and of a $p$-stabilised Siegel modular form is conjectured to be involved in solving this problem.

4. A key ingredient to obtain the results in Corollary \ref{cor:main} is the non-degeneracy of the pairing. We remark that the author of \cite can prove such a non-degeneracy for more general weights in the $GL_2$ case while we can only show this for classical weights. It is difficult to adapt the proof in \emph{op. cit.} since it relies on a straightforward computation and such a computation becomes messier and messier as $g$ grows.

5. Following the strategy in \cite, we defined the notion of “good points”. The author of \emph{op. cit.} could show that the set of good points is not empty in the case of $GL_2$ by using the Eichler–Shimura isomorphism. However, we do not know if the set of good points in our situation is nonempty.

**Outline of the paper.** The presenting work is organised as follows. In Section 2, we introduce the coefficients of the cohomology groups in our concern, the analytic distributions. Our modules of analytic distributions are defined by combining the formalisms in both \cite and \cite. Then, a pairing on the analytic distributions is constructed. Such a pairing is essential for the pairing on the overconvergent cohomology groups in Section 3. Additionally, we also construct the cuspidal eigenvariety in Section 3 after recalling a sufficient amount of terminologies from \cite. As mentioned before, our cuspidal eigenvariety sits inside the whole eigenvariety constructed in \emph{op. cit.} by considering the algebraic group $GSp_{2g}$. In the final section, we apply the pairing we constructed to detect the ramification locus of the cuspidal eigenvariety over the weight space by following the strategy of \cite, Chapter VIII closely.

**Acknowledgement.** The present work is part of the author’s Ph.D. project. He would like to thank his advisors Adrian Iovita and Giovanni Rosso for their enormously useful discussions regarding the materials in this paper. The author is grateful to Francisc Gispert for his careful reading and suggestions on the early drafts of this paper. The author would also like to thank Lennart Gehrmann for interesting conversations during the work of this article.

**Notations and conventions** Throughout this paper, we fix the following notations and conventions:

- $g \in \mathbb{Z}_{>0}$ (we are in particular interested in the case when $g > 1$)
- $p \in \mathbb{Z}_{>0}$ an odd prime number
- we fix once and forever an algebraic isomorphism $\mathbb{C}_p \cong \mathbb{C}$
- for any matrix $\alpha$, we write $^t\alpha$ for its transpose
• for any \( n \in \mathbb{Z}_{>0} \), we denote by \( \mathbb{I}_n \) the \( n \times n \) identity matrix and by \( \tilde{\mathbb{I}}_n \) the \( n \times n \)-matrix whose entries are 1 on the anti-diagonal positions and 0 elsewhere, i.e.,

\[
\tilde{\mathbb{I}}_n = \begin{pmatrix} \ddots & \ddots & \ddots \\ \ddots & 1 \\ \ddots & \ddots \\ 1 & \ddots & \ddots \\ \end{pmatrix}
\]

• in principle, symbols in Gothic font (e.g., \( \mathbf{X}, \mathbf{Y}, \mathbf{Z} \)) stand for formal schemes; symbols in calligraphic font (e.g., \( \mathcal{X}, \mathcal{Y}, \mathcal{Z} \)) stand for adic spaces; and symbols in script font (e.g., \( \mathcal{O}, \mathcal{F}, \mathcal{E} \)) stand for sheaves (over certain geometric object).

2 Analytic distributions

2.1 Algebraic and \( p \)-adic groups

Let \( V_\mathbb{Z} \) be the finite free \( \mathbb{Z} \)-module \( \mathbb{Z}^{2g} \). By viewing elements in \( V_\mathbb{Z} \) as column vectors, we equip \( V_\mathbb{Z} \) with the symplectic pairing

\[
\langle \cdot, \cdot \rangle : V_\mathbb{Z} \times V_\mathbb{Z} \to \mathbb{Z}, \quad (\vec{v}, \vec{v}') \mapsto \vec{v}^t \tilde{\mathbb{I}}_g \vec{v}'
\]

Then, the algebraic group \( \text{GSp}_{2g} \) is defined to be the subgroup of the automorphisms on \( V_\mathbb{Z} \) that preserves this pairing up to a unit. In particular, for any ring \( R \),

\[
\text{GSp}_{2g}(R) := \left\{ \gamma \in \text{GL}_{2g}(R) : \gamma \left( \begin{pmatrix} 1 \\ \tilde{\mathbb{I}}_g \end{pmatrix} \right) = \zeta(\gamma) \left( \begin{pmatrix} 1 \\ \tilde{\mathbb{I}}_g \end{pmatrix} \right) \text{ for some } \zeta(\gamma) \in R^* \right\}.
\]

Equivalently, for any \( \gamma = \begin{pmatrix} \gamma_a & \gamma_c \\ \gamma_c & \gamma_d \end{pmatrix} \in \text{GL}_{2g}, \ \gamma \in \text{GSp}_{2g} \) if and only if

\[
\begin{align*}
\gamma_a \tilde{\mathbb{I}}_g \gamma_c &= \gamma_c \tilde{\mathbb{I}}_g \gamma_a, \\
\gamma_b \tilde{\mathbb{I}}_g \gamma_d &= \gamma_d \tilde{\mathbb{I}}_g \gamma_b, \ \text{and} \\
\gamma_a \tilde{\mathbb{I}}_g \gamma_d - \gamma_c \tilde{\mathbb{I}}_g \gamma_b &= \zeta(\gamma) \tilde{\mathbb{I}}_g \text{ for some } \zeta(\gamma) \in \mathbb{G}_m.
\end{align*}
\]

In the present paper, we shall be considering the following algebraic and \( p \)-adic subgroups of \( \text{GL}_g \) and \( \text{GSp}_{2g} \):

• We consider the Borel subgroups \( B_{\text{GL}_g} \) and \( B_{\text{GSp}_{2g}} \) for \( \text{GL}_g \) and \( \text{GSp}_{2g} \) respectively, defined by

\[
B_{\text{GL}_g} := \text{the Borel subgroup of upper triangular matrices in } \text{GL}_g \\
B_{\text{GSp}_{2g}} := \text{the Borel subgroup of upper triangular matrices in } \text{GSp}_{2g}.
\]

Remark that one can take the Borel subgroup of upper triangular matrices in \( \text{GSp}_{2g} \) because of the choice of the symplectic pairing on \( V_\mathbb{Z} \).
The corresponding unipotent radicals are of the form
\[
U_{\text{GL}_g} := \text{the upper triangular } g \times g \text{ matrices whose diagonal entries are all } 1 \\
U_{\text{GSp}_{2g}} := \text{the upper triangular } 2g \times 2g \text{ matrices in GSp}_{2g} \text{ whose diagonal entries are all } 1.
\]

The maximal tori for both algebraic groups are considered to be the maximal algebraic tori of diagonal matrices. Then the Levi decomposition yields
\[
B_{\text{GL}_g} = U_{\text{GL}_g} T_{\text{GL}_g} \text{ and } B_{\text{GSp}_{2g}} = U_{\text{GSp}_{2g}} T_{\text{GSp}_{2g}}.
\]

Denote by $U^\text{opp}_{\text{GL}_g}$ and $U^\text{opp}_{\text{GSp}_{2g}}$ the opposite unipotent radical of $U_{\text{GL}_g}$ and $U_{\text{GSp}_{2g}}$ respectively.

To simplify the notation, we write
\[
T_{\text{GL}_g,0} = T_{\text{GL}_g}(\mathbb{Z}_p), \quad U_{\text{GL}_g,0} = U_{\text{GL}_g}(\mathbb{Z}_p), \\
T_{\text{GSp}_{2g},0} = T_{\text{GSp}_{2g}}(\mathbb{Z}_p) \quad U_{\text{GSp}_{2g},0} = U_{\text{GSp}_{2g}}(\mathbb{Z}_p).
\]
For any $s \in \mathbb{Z}_{>0}$, we define
\[
T_{\text{GL}_g,s} := \ker(T_{\text{GL}_g}(\mathbb{Z}_p) \to T_{\text{GL}_g}(\mathbb{Z}/p^s \mathbb{Z})), \\
U_{\text{GL}_g,s} := \ker(U_{\text{GL}_g}(\mathbb{Z}_p) \to U_{\text{GL}_g}(\mathbb{Z}/p^s \mathbb{Z})), \\
T_{\text{GSp}_{2g},s} := \ker(T_{\text{GSp}_{2g}}(\mathbb{Z}_p) \to T_{\text{GSp}_{2g}}(\mathbb{Z}/p^s \mathbb{Z})), \\
U_{\text{GSp}_{2g},s} := \ker(U_{\text{GSp}_{2g}}(\mathbb{Z}_p) \to U_{\text{GSp}_{2g}}(\mathbb{Z}/p^s \mathbb{Z})),
\]
where the all maps above are reduction maps.

The Iwahori subgroups of $\text{GL}_g(\mathbb{Z}_p)$ and $\text{GSp}_{2g}(\mathbb{Z}_p)$ are
\[
I_{\text{GL}_g} := \text{the preimage of } B_{\text{GL}_g}(\mathbb{F}_p) \text{ under the reduction map } \\
\text{GL}_g(\mathbb{Z}_p) \to \text{GL}_g(\mathbb{F}_p), \\
I_{\text{GSp}_{2g}} := \text{the preimage of } B_{\text{GSp}_{2g}}(\mathbb{F}_p) \text{ under the reduction map } \\
\text{GSp}_{2g}(\mathbb{Z}_p) \to \text{GSp}_{2g}(\mathbb{F}_p).
\]

Then the Iwahori decomposition gives
\[
I_{\text{GL}_g} = U_{\text{GL}_g,0} T_{\text{GL}_g,0} U_{\text{GL}_g,0} \text{ and } I_{\text{GSp}_{2g}} = U_{\text{GSp}_{2g},0} T_{\text{GSp}_{2g},0} U_{\text{GSp}_{2g},0},
\]
where $U_{\text{GL}_g,s}$ and $U_{\text{GSp}_{2g},s}$ are defined in the same way as above for any $s \in \mathbb{Z}_{>0}$.

We shall consider the “strict Iwahori subgroups” of $\text{GL}_g(\mathbb{Z}_p)$ and $\text{GSp}_{2g}(\mathbb{Z}_p)$, defined as
\[
I^*_{\text{GL}_g} := \text{the preimage of } T_{\text{GL}_g}(\mathbb{F}_p) \text{ under the reduction map } \\
\text{GL}_g(\mathbb{Z}_p) \to \text{GL}_g(\mathbb{F}_p), \\
I^*_{\text{GSp}_{2g}} := \{ \gamma \in \text{GSp}_{2g}(\mathbb{Z}_p) : \gamma \equiv \begin{pmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{pmatrix} \mod p \}
\]
We caution the readers that the strict Iwahori subgroups $\Iw_{\GL_n}$ and $\Iw_{\GSp_{2g}}^+$ are not defined analogously. We abuse the similar symbol to simplify the notations.

Observe that for any $\left( \gamma_a \quad \gamma_b \atop \gamma_c \quad \gamma_d \right) \in \Iw_{\GSp_{2g}}^+$, we have $\gamma_a \in \Iw_{\GL_n}$. Moreover, $\Iw_{\GL_n}$ is stable under transpose.

Obviously, we have $\Iw_{\GL_n}^+ \subseteq \Iw_{\GL_n}$ and $\Iw_{\GSp_{2g}}^+ \subseteq \Iw_{\GSp_{2g}}$. Thus, the Iwahori decompositions for $\Iw_{\GL_n}$ and $\Iw_{\GSp_{2g}}$ induce the Iwahori decompositions for $\Iw_{\GL_n}$ and $\Iw_{\GSp_{2g}}^+$:

\[
\begin{align*}
\Iw_{\GL_n}^+ &= U_{\GL_n}^{\text{opp}} \cdot T_{\GL_n, 0} U_{\GL_n, 1}^0 \\
\Iw_{\GSp_{2g}}^+ &= U_{\GSp_{2g}}^{\text{opp}} \cdot T_{\GSp_{2g}, 0} U_{\GSp_{2g}, 0}^0,
\end{align*}
\]

where $U_{\GSp_{2g}, 0}^+ = \Iw_{\GSp_{2g}}^+ \cap U_{\GSp_{2g}, 0}$.

### 2.2 Analytic distributions

The analytic functions and distributions that will be considered here are heavily inspired by the notions in [AIS15] and [JN19]; we indeed combine their ideas to define the objects that we are interested in. Before everything, we recall the terminology of Banach–Tate $\Z_p$-algebra defined in [JN19]:

**Definition 2.2.1.** A $\Z_p$-algebra $R$ (with the structure morphism $\Z_p \to R$) is a **Banach–Tate $\Z_p$-algebra** if and only if it satisfies the following properties

1. $R$ is a completed normed ring with norm $| \cdot |_R$;
2. there exists a multiplicative pseudouniformiser $\varpi \in R$, i.e., $\varpi \in R^\times$ such that $| \varpi |_R < 1$ and $| \varpi a | = | \varpi |_R | a |_R$ for any $a \in R$; and
3. the structure morphism $\Z_p \to R$ is norm-decreasing where we equip $\Z_p$ with the usual norm $| x | = p^{-\nu_p(x)}$.

Let $R$ be a complete Tate $\Z_p$-algebra, i.e., $R$ is a complete topological ring admitting a multiplicative pseudouniformiser $\varpi$. We assume that $R$ admits a noetherian ring of definition. Notice that $U_{\GSp_{2g}, 1}^{\text{opp}} \cong \Z_p^{d_0}$ as a $p$-adic manifold for some $d_0 \in \Z_{>0}$, then we consider the continuous functions and distributions

\[
\begin{align*}
\text{Cont}(U_{\GSp_{2g}, 1}^{\text{opp}}, R) := \{ f : U_{\GSp_{2g}, 1}^{\text{opp}} &\to R : f \text{ is continuous} \} \\
\text{Dist}(U_{\GSp_{2g}, 1}^{\text{opp}}, R) := \text{Hom}_{\text{cts}}^R (\text{Cont}(U_{\GSp_{2g}, 1}^{\text{opp}}, R), R).
\end{align*}
\]

On the other hand, define

\[ \mathbf{T}_0 := \{ (\gamma, \upsilon) \in \Iw_{\GL_n}^+ \times M_\gamma(p, \Z_p) : \gamma^t \upsilon \gamma = \gamma^t \upsilon = \upsilon \} \).
\]

The defining condition of $\mathbf{T}_0$ means that there exists $\alpha_0, \alpha_d \in M_\gamma(\Z_p)$ such that

\[
\begin{pmatrix}
\gamma & \alpha_0 \\
\upsilon & \alpha_d
\end{pmatrix} \in \GSp_{2g}(\Q_p) \cap M_{2g}(\Z_p).
\]
In fact, \((\gamma, \nu) \in T_0\) can be seen as an element in \(\text{Iw}^+_{\text{GSp}_{2g}}\) via
\[
T_0 \ni (\gamma, \nu) \mapsto \begin{pmatrix} \gamma \\ \nu \end{pmatrix} \tilde{I}_g \gamma^{-1} \tilde{I}_g \in \text{Iw}^+_{\text{GSp}_{2g}}.
\]
Moreover, one can view \(T_0\) as a \(p\)-adic closed submanifold of \(\text{Iw}^+_{\text{GL}_g} \times M_p(T_\mathbb{Z}_p)\).

There are two actions on \(T_0\):
1. The right action of \(B^+_{\text{GL}_g,0} := T_{\text{GL}_g,0} U_{\text{GL}_g,1}\) on \(T_0\) is defined by
\[
T_0 \times B^+_{\text{GL}_g,0} \to T_0, \quad ((\gamma, \nu), (\beta, \nu)) \mapsto (\gamma \beta, \nu \beta).
\]
   Indeed, by embedding \(B^+_{\text{GL}_g,0}\) into \(\text{Iw}^+_{\text{GSp}_{2g}}\) via \(\beta \mapsto \begin{pmatrix} \beta \\ \tilde{I}_g \beta^{-1} \tilde{I}_g \end{pmatrix}\), this action is given by
\[
\begin{pmatrix} \gamma \\ \nu \end{pmatrix} \begin{pmatrix} \beta \\ \tilde{I}_g \beta^{-1} \tilde{I}_g \end{pmatrix} = \begin{pmatrix} \gamma \beta \\ \nu \beta \end{pmatrix}.
\]
2. The left action of \(\Xi := \left( \begin{array}{c} \text{Iw}^+_{\text{GL}_g} \\ M_g(T_\mathbb{Z}_p) \\ M_g(T_\mathbb{Z}_p) \end{array} \right) \cap \text{GSp}_{2g}(\mathbb{Q}_p)\) on \(T_0\) is defined by
\[
\Xi \times T_0 \to T_0, \quad \left( \begin{pmatrix} \alpha_a \\ \alpha_b \\ \alpha_c \\ \alpha_d \end{pmatrix}, (\gamma, \nu) \right) \mapsto (\alpha_a \gamma + \alpha_b \nu, \alpha_c \gamma + \alpha_d \nu).
\]
   To verify this is indeed a left action, one considers
\[
\begin{pmatrix} \alpha_a & \alpha_b \\ \alpha_c & \alpha_d \end{pmatrix} \begin{pmatrix} \gamma \\ \nu \end{pmatrix} = \begin{pmatrix} \alpha_a \gamma + \alpha_b \nu \\ \alpha_c \gamma + \alpha_d \nu \end{pmatrix}.
\]
In particular, \(T_0\) admits a left action of \(\text{Iw}^+_{\text{GSp}_{2g}}\) as \(\text{Iw}^+_{\text{GSp}_{2g}} \subset \Xi\).

Inside \(T_0\), there is a special subset
\[
T_{00} := \{ (\gamma, \nu) \in T_0 : \gamma \in U_{\text{GL}_g,1}^{\text{opp}} \}.
\]
One can identify \(T_{00}\) with \(U_{\text{GSp}_{2g},1}^{\text{opp}}\) via
\[
T_{00} \sim U_{\text{GSp}_{2g},1}^{\text{opp}}, \quad (\gamma, \nu) \mapsto \begin{pmatrix} \gamma \\ \nu \end{pmatrix} \tilde{I}_g \gamma^{-1} \tilde{I}_g.
\]

Let \(\kappa : T_{\text{GL}_g,0} \to R^*\) be a \(p\)-adic weight and we assume that one can choose a norm \(|\cdot|_R\) on \(R\) making \(R\) a Banach–Tate \(\mathbb{Z}_p\)-algebra and that \(|\cdot|_R\) is adapted to \(\kappa\), i.e., the norm \(|\cdot|_R\) satisfies
- \(\kappa(T_{\text{GL}_g,0}) \subset R_0 := \text{the unit ball of } R\) with respect to \(|\cdot|_R\)
- \(|\kappa(\tau) - 1|_R < 1\) for all \(\tau \in T_{\text{GL}_g,1}\).
We write \( r_\kappa := \min \{ r \in [1/p, 1) : |\kappa(\tau)| - 1 |_R \leq r \text{ for all } \tau \in T_{\text{GL}_2,1} \} \). Finally, define
\[
A_\kappa(T_0, R) := \left\{ f : T_0 \to R : f \text{ is continuous } \right\}.
\]

One sees immediately that there is an isomorphism
\[
A_\kappa(T_0, R) \xrightarrow{\sim} \text{Cont}(U^{\text{opp}}_{\text{GSp}_{2g},1}, R), \quad f \mapsto f|_{T_0}.
\]

**Remark 2.2.2.** Our continuous functions \( A_\kappa(T_0, R) \) is the same as the continuous functions “\( A_\kappa \)” defined in [JN19] in the case of \( \text{GSp} \). We chose to work in this way due to some technicality when defining the pairing in §2.3. We remark that the continuous functions \( A_\kappa \) considered in op. cit. has its advantage for considering analytic functions and distributions for general reductive groups. We should also point out that we are not considering general weights associated to \( T_{\text{GS}_{2g},0} \) but weights associated to \( T_{\text{GL}_2,0} \) via the embedding
\[
T_{\text{GL}_2,0} \hookrightarrow \left( T_{\text{GS}_{2g},0}, \frac{1}{g} T_{\text{GL}_2,0} \frac{1}{g} \right) \in T_{\text{GS}_{2g},0}.
\]

This explains why we can use \( T_0 \) to rewrite the continuous functions \( A_\kappa \) considered in op. cit..

Evidently, we define
\[
D_\kappa(T_0, R) := \text{Hom}^{\text{cts}}(A_\kappa(T_0, R), R).
\]

Then we have a sequence of isomorphisms
\[
R[[U^{\text{opp}}_{\text{GSp}_{2g},1}]] \xrightarrow{\sim} \text{Dist}(U^{\text{opp}}_{\text{GSp}_{2g},1}, R) \xrightarrow{\sim} D_\kappa(T_0, R),
\]

where the last isomorphism obviously follows from the isomorphism \( A_\kappa(T_0, R) \approx \text{Cont}(U^{\text{opp}}_{\text{GSp}_{2g},1}, R) \). The first isomorphism follows from [JN19, Proposition 3.1.4] for which one sends each \( \gamma \in U^{\text{opp}}_{\text{GSp}_{2g},1} \) to
\[
\delta_\gamma := \text{the Dirac distribution at } \gamma,
\]
i.e., the evaluation at \( \gamma \). The ring structure on \( \text{Dist}(U^{\text{opp}}_{\text{GSp}_{2g},1}, R) \) is given by the usual convolution product; that is,
\[
\mu_1 * \mu_2 : f \mapsto \int_{\gamma_1 \in U^{\text{opp}}_{\text{GSp}_{2g},1}} \int_{\gamma_2 \in U^{\text{opp}}_{\text{GSp}_{2g},1}} f(\gamma_1 \gamma_2) \mu_2(\gamma_2) \mu_1(\gamma_1),
\]

which yields \( \delta_{\gamma_1} * \delta_{\gamma_2} = \delta_{\gamma_1 \gamma_2} \).

Recall that \( U^{\text{opp}}_{\text{GSp}_{2g},1} \approx \mathbb{Z}_p^{d_0} \) as \( p \)-adic manifolds, thus we can fix topological generators \( \psi_1, \ldots, \psi_{d_0} \) for \( U^{\text{opp}}_{\text{GSp}_{2g},1} \). For \( i = (i_1, \ldots, i_{d_0}) \in \mathbb{Z}_{\geq 0}^{d_0} \), we write \( \psi_i := (\psi_1 - 1)^{i_1} \cdots (\psi_{d_0} - 1)^{i_{d_0}} \). Let \( r \in [r_\kappa, 1) \), we define the \( r \)-norm on \( R[[U^{\text{opp}}_{\text{GSp}_{2g},1}]] \) by
\[
\left\| \sum_{i \in \mathbb{Z}_{\geq 0}^{d_0}} a_i \psi_i \right\|_r := \sup \left\{ |a_i|_R \cdot r^{i_1} : i = (i_1, \ldots, i_{d_0}) \in \mathbb{Z}_{\geq 0}^{d_0} \right\}.
\]
Via the above isomorphisms, we defined an $r$-norm on $D_\kappa(T_0, R)$. Following [op. cit., §3], we define

\[ D^r_\kappa(T_0, R) := \text{the completion of } D_\kappa(T_0, R) \text{ with respect to the } r\text{-norm} \]

\[ D^s_\kappa(T_0, R) := \lim_{\tau \to r} D^r_\kappa(T_0, R). \]

Remark 2.2.3. By [JN19, Lemma 3.2.3], for any $r < s$, one has a compact inclusion

\[ D^s_\kappa(T_0, R) \to D^r_\kappa(T_0, R). \]

Hence one thinks of $D^\kappa_\kappa(T_0, R) = \cap_r D^r_\kappa(T_0, R)$.

Remark 2.2.4. Following [JN19], we denote by $D^{\kappa, \sigma}_\kappa(T_0, R)$ the unit ball of $D^r_\kappa(T_0, R)$. Moreover, we also consider their dual spaces

\[ A^{\kappa, \sigma}_\kappa(T_0, R) \quad \text{and} \quad A^{\kappa, \sigma}_\kappa(T_0, R), \]

which can be viewed as subspaces in $A_\kappa(T_0, R)$. We refer the readers to the end of [op. cit., §3] for more detail discussions.

2.3 A pairing on the analytic distributions

In this subsection, we establish a pairing on the analytic distributions $D^r_\kappa(T_0, R)$. Our strategy is the same as the strategy as in [Bel, Chapter VIII]. That is, we first build a map from $D^r_\kappa(T_0, R)$ to $A^{\kappa, \sigma}_\kappa(T_0, R)$ and then use the natural pairing between $D^r_\kappa(T_0, R)$ and $A^{\kappa, \sigma}_\kappa(T_0, R)$ to obtain the desired one.

An algebraic model. Our pairing is modelled on an algebraic version inspired by the one in [Han17, pp. 18], which we now explain.

Let $k = (k_1, \ldots, k_g) \in \mathbb{Z}_{\geq 0}^g$ with $k_1 \geq \cdots \geq k_g$. One can view $k$ as a character on $T_{GL_g}$ via

\[ k : T_{GSp_{2g}} \to \mathbb{G}_m, \quad \text{diag}(\tau_1, \ldots, \tau_g, \tau_0 \tau_g^{-1}, \ldots, \tau_0 \tau_1^{-1}) \mapsto \prod_{i=1}^g \tau_i^{k_i}. \]

One extends $k$ to $B_{GSp_{2g}}$ by setting $k(U_{GSp_{2g}}) = \{1\}$. Consider the irreducible representation for $GSp_{2g}$

\[ \mathbf{V}^{\text{alg}}_{GSp_{2g}, k} := \left\{ \phi : GSp_{2g} \to \mathbb{A}^1 : \phi(\gamma \beta) = k(\beta)\phi(\gamma) \text{ for any } (\gamma, \beta) \in GSp_{2g} \times B_{GSp_{2g}} \right\}. \]

One can consider the following actions of $GSp_{2g}$ on $\mathbf{V}^{\text{alg}}_{GSp_{2g}, k}$:

(i) The right action given by

\[ (\phi \cdot \gamma)(\gamma') = \phi(\gamma \gamma'). \]

(ii) The left action given by

\[ (\gamma \phi)(\gamma') = \phi(\gamma^{-1} \gamma). \]

(iii) The left action given by

\[ (\gamma \cdot \phi)(\gamma') = \phi(\gamma^{-1} \gamma'). \]
Notice that the second action is valid since $GSp_{2g}$ is stable under transpose. In fact, one deduces easily from the definition that
\[
{\gamma} = \zeta(\gamma) \left( \begin{array}{ccc}
\frac{1}{2} & 0 & 0 \\
0 & -\frac{1}{2} & 0 \\
0 & 0 & \frac{1}{2}
\end{array} \right) \gamma^{-1} \left( \begin{array}{ccc}
-\frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0 \\
0 & 0 & -\frac{1}{2}
\end{array} \right)
\]
for any $\gamma \in GSp_{2g}$. Therefore, the second action is nothing but a twisted action of the third one. In what follows, we equip $V_{\text{alg},k}^{\text{alg},v}$ with the left $GSp_{2g}$-action given by (ii).

Let $V_{GSp_{2g},k}^{\text{alg},v}$ be its linear dual. We equip $V_{GSp_{2g},k}^{\text{alg},v}$ with a left action induced by (i). Then, we have a morphism
\[
\Phi_{k} : V_{GSp_{2g},k}^{\text{alg},v} \rightarrow V_{GSp_{2g},k}^{\text{alg},v}, \quad \mu \mapsto \int_{\gamma \in GSp_{2g}} e_{k}^{\text{hst}}(\gamma^{\dagger} \gamma) \ d\mu,
\]
where $e_{k}^{\text{hst}} \in V_{GSp_{2g},k}^{\text{alg},v}$ is defined by
\[
e_{k}^{\text{hst}} : (X_{ij})_{1 \leq i,j \leq 2g} \mapsto X_{11}^{k_{1}} \cdots X_{1g}^{k_{g}} \times \cdots \times \det \left( \begin{array}{ccc}
X_{11} & \cdots & X_{1g} \\
\vdots & \ddots & \vdots \\
X_{g1} & \cdots & X_{gg}
\end{array} \right).
\]
One sees that $\Phi_{k}$ is $GSp_{2g}$-equivariant with respect to the left $GSp_{2g}$-actions on both spaces. Indeed, for any $\alpha, \gamma' \in GSp_{2g}$ and $\mu \in V_{GSp_{2g},k}^{\text{alg},v}$, we have
\[
\Phi_{k}^{\text{alg}}(\alpha \cdot \mu)(\gamma') = \int_{\gamma \in GSp_{2g}} e_{k}^{\text{hst}}(\gamma^{\dagger} \alpha \gamma) \ d\mu
\]
\[
= \int_{\gamma \in GSp_{2g}} e_{k}^{\text{hst}}(\gamma^{\dagger} \gamma') \ d\mu
\]
\[
= (\alpha \cdot \Phi_{k}^{\text{alg}}(\mu))(\gamma').
\]
Consequently, $\Phi_{k}$ defines a pairing on $V_{GSp_{2g},k}^{\text{alg},v}$ by
\[
(\mu_{1}, \mu_{2}) \mapsto \int_{\gamma_{1}, \gamma_{2} \in GSp_{2g}} e_{k}^{\text{hst}}(\gamma_{2} \gamma_{1}) \ d\mu_{1}(\gamma_{1})d\mu_{2}(\gamma_{2}).
\]

**Remark 2.3.1.** Notice that $V_{GSp_{2g},k}^{\text{alg},v}$ is an irreducible representation of $GSp_{2g}$, thus it admits a pairing induced by the symplectic pairing $\langle \cdot, \cdot \rangle$ on $V_{\mathbb{Z}}$. This pairing can be viewed by the following formula
\[
\langle \cdot, \cdot \rangle : (\mu_{1}, \mu_{2}) \mapsto \int_{\gamma_{1}, \gamma_{2} \in GSp_{2g}} e_{k}^{\text{hst}}(\gamma_{2} \gamma_{1}) \ d\mu_{1}(\gamma_{1})d\mu_{2}(\gamma_{2}).
\]
Indeed, for any $\alpha \in GSp_{2g}$, we have
\[
\langle \alpha \cdot \mu_{1}, \mu_{2} \rangle_{k} = \int_{\gamma_{1}, \gamma_{2} \in GSp_{2g}} e_{k}^{\text{hst}}(\gamma_{2} \gamma_{1}) \ d\mu_{1}(\gamma_{1})d\mu_{2}(\gamma_{2})
\]
\[
= \int_{\gamma_{1}, \gamma_{2} \in GSp_{2g}} e_{k}^{\text{hst}}(\gamma_{2} \zeta(\alpha) \gamma_{1}) \ d\mu_{1}(\gamma_{1})d\mu_{2}(\gamma_{2})
\]
\[
= \zeta(\alpha) \sum_{k_{1}} \int_{\gamma_{1}, \gamma_{2}} e_{k_{1}}^{\text{hst}}(\gamma_{2} \gamma_{1}) \ d\mu_{1}(\gamma_{1})d\mu_{2}(\gamma_{2})
\]
\[
= \zeta(\alpha) \sum_{k_{1}} \langle \mu_{1} \cdot \alpha^{-1} \mu_{2} \rangle_{k_{1}},
\]
where the second equality follows from the definition of $GSp_{2g}$. 
The pairing on the analytic distributions. Let \( \kappa : T_{GL_0}^* \to R^\times \) be a \( p \)-adic weight and we keep the assumption on the fixed norm on the Banach–Tate \( \mathbb{Z}_p \)-algebra \( R \) as before. Notice that we can write

\[
\kappa : T_{GL_0}^* \to R^\times, \quad \text{diag}(\tau_1, \ldots, \tau_g) \mapsto \kappa_1(\tau_1) \times \cdots \times \kappa_g(\tau_g)
\]

for some \( p \)-adic weight \( \kappa_i : \mathbb{Z}_p^\times \to R^\times \).

Define the function \( e^{\text{hst}} \) on \( Iw_{GL_0}^+ \) by

\[
e_{\kappa}^{\text{hst}}((X_{ij})_{1 \leq i, j \leq g}) = \frac{\kappa_1(X_{11})}{\kappa_2(X_{11})} \times \frac{\kappa_2(\det(X_{ij})_{1 \leq i, j \leq 2})}{\kappa_3(\det(X_{ij})_{1 \leq i, j \leq 2})} \times \cdots \times \frac{\kappa_g(\det(X_{ij})_{1 \leq i, j \leq g})}{\det(X_{ij})_{1 \leq i, j \leq g}}.
\]

One sees that \( e_{\kappa}^{\text{hst}} \in A_\kappa(T_0, R) \) via

\[
e_{\kappa}^{\text{hst}}(\gamma, \upsilon) = e_{\kappa}^{\text{hst}}(\gamma).
\]

Lemma 2.3.2. For \( r \in [r_\kappa, 1) \), \( e_{\kappa}^{\text{hst}} \in A_{\kappa}^{sr}(T_0, R) \)

Proof. Note that \( e_{\kappa}^{\text{hst}}(\gamma, \upsilon) = 1 \) for any \( (\gamma, \upsilon) \in T_{00} \). The assertion then follows from the explicit description of \( A_{\kappa}^{sr}(T_0, R) \) in [JN19, §3.2 & 3.3].

Therefore, we can define

\[
\Phi_{\kappa} : D_{\kappa}^r(T_0, R) \to A_{\kappa}^{sr}(T_0, R),
\]

\[
\mu \mapsto (\gamma', \upsilon') \mapsto \int_{(\gamma, \upsilon) \in T_{00}} e_{\kappa}^{\text{hst}}(\gamma', \upsilon') \left( \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}^{-1} \right)(\gamma, \upsilon) \ d\mu.
\]

Notice that

\[
e_{\kappa}^{\text{hst}}(\gamma', \upsilon') \left( \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}^{-1} \right)(\gamma, \upsilon) = e_{\kappa}^{\text{hst}}(\gamma' + \upsilon'/p)
\]

is valid since \( (\gamma', \gamma + \upsilon'/p) \in Iw_{GL_0}^+ \) (this is because \( Iw_{GL_0}^+ \) is stable under transpose and both \( \upsilon \) and \( \upsilon' \) are divisible by \( p \)).

Consequently, we have the pairing

\[
[\cdot, \cdot]^\circ : D_{\kappa}^r(T_0, R) \times D_{\kappa}^r(T_0, R) \to R
\]

given by the formula

\[
[\mu_1, \mu_2]^\circ_{\kappa} = \int_{T_{00}} e_{\kappa}^{\text{hst}}(\gamma_2, \upsilon_2) \left[ \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}^{-1} \right](\gamma_1, \upsilon_1) \ d\mu_1(\gamma_1, \upsilon_1)d\mu_2(\gamma_2, \upsilon_2)
\]

Proposition 2.3.3. For any \( \alpha = \begin{pmatrix} \alpha_a & \alpha_b \\ \alpha_c & \alpha_d \end{pmatrix} \in \Xi \), write

\[
\alpha^\perp = \begin{pmatrix} \alpha_a & \alpha_c/p \\ p \alpha_b & \alpha_d \end{pmatrix} \in \Xi.
\]

Then, for any \( \mu_1, \mu_2 \in D_{\kappa}^r(T_0, R) \), we have

\[
[\alpha \mu_1, \mu_2]^\circ_{\kappa} = [\mu_1, \alpha^\perp \mu_2]^\circ_{\kappa}.
\]
Proof. The assertion follows from the computation

\[
\begin{align*}
(\tau \gamma_2 & \cdot \tau v_2) \left( \mathbb{1}_g \cdot p^{-1} \mathbb{1}_g \right) \left( \begin{array}{c}
\alpha_a \\
\alpha_c \\
\alpha_d \\
\end{array} \right) \left( \begin{array}{c}
\gamma_1 \\
\gamma_1 \\
v_1 \\
\end{array} \right) \\
= (\tau \gamma_2 & \cdot \tau v_2) \left( \begin{array}{c}
\alpha_a \\
\alpha_c / p \\
\alpha_d \\
\end{array} \right) \left( \begin{array}{c}
\mathbb{1}_g \\
p \mathbb{1}_g \\
p^{-1} \mathbb{1}_g \\
\end{array} \right) \left( \begin{array}{c}
\gamma_1 \\
v_1 \\
\end{array} \right) \\
= (\tau & (\tau \alpha_a \\
\tau p \alpha_b \\
\tau p \alpha_c / p \\
\tau \alpha_d) \left( \begin{array}{c}
\gamma_2 \\
v_2 \\
\end{array} \right) \left( \begin{array}{c}
\mathbb{1}_g \\
p^{-1} \mathbb{1}_g \\
\end{array} \right) \left( \begin{array}{c}
\gamma_1 \\
v_1 \\
\end{array} \right)
\end{align*}
\]

\[
\square
\]

Remark 2.3.4. When comparing with our algebraic model, one notices that the definition of the pairing \([\cdot, \cdot]_k^\circ\) involves a “normalisation” by \(p^{-1}\). Such a normalisation is due to our model for \(g = 1\). More precisely, when \(g = 1\), elements in \(T_0\) can be written as \((1, pc)a\) for some \(a \in Z_p^*\) and \(c \in Z_p\). Then, for any \(\mu_1, \mu_2 \in D_\kappa(T_0, R)\), we have

\[
[\mu_1, \mu_2]_k^\circ = \int_{T_0} \kappa(1 + pc_1 c_2) \, dm_1(1, c_1) dm_2(1, c_2),
\]

which then coincides with the interpretation in Hansen’s unpublished notes [Han12]. In particular, by applying [Bel, Definition VIII.2.4], we have the formula

\[
[\mu_1, \mu_2]_k^\circ = \sum_{i=0}^\infty p^i (\gamma_i) \mu_1(c_1^i) \mu_2(c_2^i),
\]

which is (almost) the same formula given by [op. cit., (VIII.2.4)]. Here, for \(j = 1, 2\), we view \(c_j^i\) as a function on \(T_0\) via

\[
c_j^i : T_0 \ni (a, pc) \mapsto \kappa(a)(c/a)^i.
\]

Remark 2.3.5. Following Remark 2.3.1, for any dominant \(k \in Z_{\geq 0}^g\), we may consider the pairing \([\cdot, \cdot]_k^\circ\) to be the twist of \([\cdot, \cdot]_k\) by an Atkin–Lehner operator. More precisely, let

\[
\mathbf{w}_p := \left( \begin{array}{c}
\mathbb{1}_g \\
-p^{-1} \mathbb{1}_g \\
\end{array} \right),
\]

then

\[
[\mu_1, \mu_2]_k^\circ = \int_{T_{0k}} ^{\mathbf{w}_p} c_k \left( (\tau \gamma_2 \cdot \tau v_2) \left( \mathbb{1}_g \cdot p^{-1} \mathbb{1}_g \right) \left( \begin{array}{c}
\gamma_1 \\
\gamma_1 \\
v_1 \\
\end{array} \right) \right) \, dm_1(\gamma_1, v_1) dm_2(\gamma_2, v_2)
\]

\[
= \int_{T_{0k}} ^{\mathbf{w}_p} c_k \left( (\tau \gamma_2 \cdot \tau v_2) \left( -p^{-1} \mathbb{1}_g \mathbb{1}_g \right) \left( \begin{array}{c}
\gamma_1 \\
\gamma_1 \\
v_1 \\
\end{array} \right) \right) \, dm_1(\gamma_1, v_1) dm_2(\gamma_2, v_2)
\]

\[
= \int_{T_{0k}} ^{\mathbf{w}_p} c_k \left( \mathbf{w}_p (\gamma_2 \cdot \tau v_2) \left( -p^{-1} \mathbb{1}_g \mathbb{1}_g \right) \left( \begin{array}{c}
\gamma_1 \\
\gamma_1 \\
v_1 \\
\end{array} \right) \right) \, dm_1(\gamma_1, v_1) dm_2(\gamma_2, v_2).
\]

In particular, this viewpoint coincides with the perspectives in [Kim06; Bel; Han12] when \(g = 1\).
3 The overconvergent cohomology and the eigenvariety

3.1 A pairing on cohomology groups

Let \( N \in \mathbb{Z}_{\geq 3} \) such that \( p \nmid N \) and we fix an \( N \)-th primitive root of unity \( \zeta_N \) (and so we fixed an isomorphism \( \mu_N \cong \mathbb{Z}/N\mathbb{Z} \)). Equip on \((\mathbb{Z}/N\mathbb{Z})^{2g}\) a symplectic pairing induced by the pairing on \( \mathbf{V}_\mathbb{Z} \) in the previous section. Let \( \text{Sch}_{\mathbb{Z}_p[\zeta_N]} \) be the category of locally noetherian schemes over \( \mathbb{Z}_p[\zeta_N] \), then the moduli problem

\[
\text{Sch}_{\mathbb{Z}_p[\zeta_N]} \to \text{Sets},
\]

\[
S \mapsto \left\{ \left( A_{[S]} : (A_{[S]}, \lambda, \alpha_N) : \begin{array}{ll}
A_{[S]} & \text{is a principally polarised abelian scheme over } S \\
\lambda : A & \to A^\vee \text{ is the principal polarisation} \\
\alpha_N : A[N] & \to (\mathbb{Z}/N\mathbb{Z})^{2g} \text{ is a symplectic isomorphism} 
\end{array} \right) / \sim 
\right\}
\]

is representable by a scheme \( X_{\mathbb{Z}_p[\zeta_N]} \), where the symplectic isomorphism \( \alpha_N \) is taken with respect to the Weil pairing on \( A[N] \) and the pairing induced by \( \langle \cdot, \cdot \rangle \) on \((\mathbb{Z}/N\mathbb{Z})^{2g}\). We let

\[
X = X_{C_p} := X_{\mathbb{Z}_p[\zeta_N]} \times_{\mathbb{Z}_p[\zeta_N]} \text{Spec } C_p.
\]

Let \( \zeta_p \) be a primitive \( p \)-th root of unity. We also consider the scheme \( X_{2g}, Q_p[\zeta_p, \zeta_N] \), paramatrising the isomorphism classes of tuples \( (A, \lambda, \alpha_N, \text{Fil}^i A[p], \{ C_i : i = 1, ..., g \}) \), where

- \( (A, \lambda, \alpha_N) \in X_{Q_p[\zeta_p, \zeta_N]} = X_{\mathbb{Z}_p[\zeta_N]} \times_{\mathbb{Z}_p[\zeta_N]} \text{Spec } Q_p[\zeta_p, \zeta_N] \),
- \( \text{Fil}^i A[p] \) is a full flag of \( A[p] \) such that
  \[
  (\text{Fil}^i A[p])^i \cong \text{Fil}_{2g-i} A[p]
  \]
with respect to the Weil pairing, and
- \( \{ C_i : i = 1, ..., g \} \) is a collection of subgroups of \( A[p] \) of order \( p \) such that \( \text{Fil}_i A[p] = \langle C_1, ..., C_i \rangle \) for \( i = 1, ..., g \).

Again, we write \( X_{2g} := X_{2g, Q_p[\zeta_p, \zeta_N]} \times_{Q_p[\zeta_p, \zeta_N]} \text{Spec } C_p \). Obviously, we have a natural forgetful map

\[
\pi_{2g} : X_{2g} \to X, \quad (A, \lambda, \alpha_N, \text{Fil}^i A[p], \{ C_i : i = 1, ..., g \}) \mapsto (A, \lambda, \alpha_N).
\]

Via the fixed (algebraic) isomorphism \( C_p \cong C \), we consider the locally symmetric space \( X_{2g}(C) \) in the rest of this article, which admits an alternative description

\[
X_{2g}(C) = GSp_{2g}(Q) \backslash GSp_{2g}^+(A_f) \times \mathbb{H}_g/\Gamma(N),
\]

where

- \( A_f \) is the ring of finite adeles of \( Q \),
- \( \mathbb{H}_g \) is the disjoint union of the Siegel upper- and lower-half spaces of genus \( g \),
- \( \Gamma(N) := \{ \gamma \in GSp_{2g}^+(\mathbb{Z}) : \gamma \equiv \mathbb{I}_{2g} \mod N \} \).
Fix a $p$-adic weight $\kappa : T_{\GL_n,0} \to R^\times$ satisfying the assumptions on the Banach–Tate $\Z_p$-algebra norm $| \cdot |_p$ on $R$ together with a fixed multiplicative pseudouniformiser $\varpi \in R$. Recall the analytic distributions $D^\kappa(T_0, R)$ that we introduced in the previous section. From now on, we simplify the notation by writing

$$D^\kappa = D^\kappa(T_0, R).$$

Since $D^\kappa$ admits a left $Iw^+_\GSp_{2g}$-action, we can follow the strategy in $\cite{Han17}$ to compute the Betti cohomology groups $H^i(X_{1w^+}(C), D^\kappa)$.

That is, we consider the so-called Borel–Serre cochain complex $C^\bullet(Iw^+_\GSp_{2g}, D^\kappa)$, constructed by fixing a triangulation on the Borel–Serre compactification $\overline{X}_{1w^+}(C)$ of the locally symmetric space $X_{1w^+}(C)$ (see $\cite{BS73}$). The Borel–Serre cochain complex admits the following nice properties (see also $\cite{Han17}$ §2.1):

1. There is a homotopy between the singular cochain complex and the Borel–Serre cochain complex and hence the reason why one can compute the cohomology groups by considering the Borel–Serre cochain complex.

2. The total space $C^\text{tot}_{\kappa, r} := \oplus t C^t(Iw^+_\GSp_{2g}, D^\kappa)$ is a potentially ON-able Banach module over $R$ since $C^\bullet(Iw^+_\GSp_{2g}, D^\kappa)$ is a finite cochain complex and $D^\kappa(T_0, R)$ is potentially ON-able with an explicit potential ON-basis described in $\cite{JN19}$ §3.2.

The fixed triangulation on $\overline{X}_{1w^+}(C)$ provides also a triangulation on the boundary $\partial \overline{X}_{1w^+}(C) := \overline{X}_{1w^+}(C) \setminus X_{1w^+}(C)$ and hence defines a cochain complex $C^\bullet_0(Iw^+_\GSp_{2g}, D^\kappa_0)$ that computes the cohomology groups at the boundary. The natural closed embedding $\partial \overline{X}_{1w^+}(C) \to \overline{X}_{1w^+}(C)$ then induces a morphism of cochain complexes

$$\pi : C^\bullet(Iw^+_\GSp_{2g}, D^\kappa) \to C^\bullet_0(Iw^+_\GSp_{2g}, D^\kappa_0).$$

Following $\cite{Bar18}$ §3.1.3, we define $C^\bullet_c(Iw^+_\GSp_{2g}, D^\kappa) := \text{Cone}(\pi)$ the mapping cone of $\pi$, i.e.,

$$\text{Cone}(\pi)^t = C^t(Iw^+_\GSp_{2g}, D^\kappa) \oplus C^{t-1}_0(Iw^+_\GSp_{2g}, D^\kappa_0)$$

$$d^t : \text{Cone}(\pi)^t \to \text{Cone}(\pi)^{t+1}, \quad (\sigma, \sigma_0) \mapsto (-d^t \sigma, -\varpi^t \sigma + d^{t-1}_0 \sigma_0),$$

where $d$ and $d_0$ are differentials on $C^\bullet(Iw^+_\GSp_{2g}, D^\kappa)$ and $C^\bullet(Iw^+_\GSp_{2g}, D^\kappa_0)$ respectively. The strategy of the proof of $\cite{Bar18}$ Proposition 3.5 applies here and one sees that $C^\bullet_c(Iw^+_\GSp_{2g}, D^\kappa)$ computes the compactly supported cohomology groups $H^c(X_{1w^+}(C), D^\kappa)$. Moreover, the natural morphism

$$C^\bullet_c(Iw^+_\GSp_{2g}, D^\kappa) \to C^\bullet(Iw^+_\GSp_{2g}, D^\kappa)$$

induces a morphism on the cohomology groups

$$H^c(X_{1w^+}(C), D^\kappa) \to H^i(X_{1w^+}(C), D^\kappa).$$

For each $t$, we let

$$H^t_{\text{par}}(X_{1w^+}(C), D^\kappa) := \text{image} \left( H^t_c(X_{1w^+}(C), D^\kappa) \to H^i(X_{1w^+}(C), D^\kappa) \right),$$

and call them the parabolic cohomology groups.
Proposition 3.1.1. Let \( n_0 = g(g + 1)/2 \) be the \( C \)-dimension of \( X_{1w^*}(C) \). Then, we have a well-defined pairing

\[
\cdot \cdot \cdot : H^t_{\text{par}}(X_{1w^*}(C), D^\kappa_r) \times H^{2n_0-t}_{\text{par}}(X_{1w^*}(C), D^\kappa_r) \to R
\]

for any \( 0 \leq t \leq 2n_0 \).

Proof. Recall the pairing \([\cdot \cdot \cdot]_c\) defined in §2.3. Together with the cup product on cohomology groups, one obtains a pairing \([\cdot \cdot \cdot]_c\) defined as the composition

\[
\begin{array}{ccc}
H^t_c(X_{1w^*}(C), D^\kappa_r) \times H^{2n_0-t}_c(X_{1w^*}(C), D^\kappa_r) & \longrightarrow & H^{2n_0}_c(X_{1w^*}(C), D^\kappa_R \otimes D^\kappa_r) \\
\downarrow \cdot \cdot \cdot & & \downarrow \cdot \cdot \cdot \\
H^t_c(X_{1w^*}(C), D^\kappa_r) \otimes H^{2n_0-t}_c(X_{1w^*}(C), D^\kappa_r) & = & H^{2n_0}_c(X_{1w^*}(C), D^\kappa_R \otimes D^\kappa_r).
\end{array}
\]

where \( \cdot \cdot \cdot \) denotes the cup product.

The compatibility of cup products (see, for example, [Mum84, Chapter 5, §48, Exercise 2]) yields the commutative diagram

\[
\begin{array}{ccc}
H^t_c(X_{1w^*}(C), D^\kappa_r) \times H^{2n_0-t}_c(X_{1w^*}(C), D^\kappa_r) & \longrightarrow & H^{2n_0}_c(X_{1w^*}(C), D^\kappa_R \otimes D^\kappa_r) \\
\downarrow & & \downarrow \\
H^t_c(X_{1w^*}(C), D^\kappa_r) \otimes H^{2n_0-t}_c(X_{1w^*}(C), D^\kappa_r) & = & H^{2n_0}_c(X_{1w^*}(C), D^\kappa_R \otimes D^\kappa_r).
\end{array}
\]

In particular, if \([\mu_1] \in H^t_c(X_{1w^*}(C), D^\kappa_r)\) and \([\mu_2] \in H^{2n_0-t}_c(X_{1w^*}(C), D^\kappa_r)\) with \([\mu'_1] \in H^t_c(X_{1w^*}(C), D^\kappa_r)\) and \([\mu'_2] \in H^{2n_0-t}_c(X_{1w^*}(C), D^\kappa_r)\) such that \([\mu'_1] \mapsto [\mu_1]\) for \( i = 1, 2 \), then

\[
[\mu_1] \cdot \cdot [\mu_2] = [\mu'_1] \cdot \cdot [\mu'_2] = [\mu'_1] \cdot [\mu_2].
\]

Hence we define

\[
[[\mu_1], [\mu_2]]_c = [[\mu'_1], [\mu'_2]]_c = [[\mu_1], [\mu_2]]_c^*.
\]

We see that \([\cdot \cdot \cdot]_c\) is well-defined, i.e., independent of the choice of the lifting, due to the commutativity of the above diagram.

\[\square\]

### 3.2 Hecke operators

**Hecke operators outside \( pN \).** Let \( q \) be a prime number not dividing \( pN \). We consider the set of double cosets

\[
\mathcal{T}_q := \{ [\text{GSp}_{2g}(\mathbb{Z}_q) \delta \text{GSp}_{2g}(\mathbb{Z}_q)] : \delta \in \text{GSp}_{2g}(\mathbb{Q}_q) \cap M_{2g}(\mathbb{Z}_q) \}.
\]

For any fixed \( \delta \), we have the coset decomposition

\[
\text{GSp}_{2g}(\mathbb{Z}_q) \delta \text{GSp}_{2g}(\mathbb{Z}_q) = \sqcup_j \delta_j \text{GSp}_{2g}(\mathbb{Z}_p)
\]

16
for finitely many \( \delta_j \in \text{GSp}_{2g}(Q_p) \cap M_{2g}(Z_q) \). By letting \( \delta_j \)'s act trivially on \( D_\kappa^r \), we have a left action of the double coset \([\text{GSp}_{2g}(Z_q) \delta \text{GSp}_{2g}(Z_q)]\) on \( C^*(\text{Iw}^+_{\text{GSp}_{2g}}, D_\kappa^r) \) by
\[
[\text{GSp}_{2g}(Z_q) \delta \text{GSp}_{2g}(Z_q)] \cdot \sigma = \sum_j \delta_j \cdot \sigma
\]
for any \( \sigma \in C^*(\text{Iw}^+_{\text{GSp}_{2g}}, D_\kappa^r) \). Then the Hecke algebra at \( q \) (over \( Z_p \)) is defined to be \( \mathbb{T}_q = \mathbb{T}_q Z_p = Z_p[\Upsilon_q] \).

**Hecke operators at \( N \).** We ignore the Hecke actions at \( N \), i.e., for \( \ell | N \), we only consider the trivial action and hence the Hecke algebra at \( \ell \) is \( \mathbb{T}_\ell = \mathbb{T}_\ell Z_p = Z_p \).

**Hecke operator at \( p \).** Let
\[
\begin{align*}
\mathbf{u}_{p,0} & := \begin{pmatrix} 1_g & p \cdot 1_g \\ p & 1_g \end{pmatrix} \\
\mathbf{u}_{p,i} & := \begin{pmatrix} 1_g & p \cdot 1_i \\ p & 1_i \\ p^2 & 1_{g-i} \end{pmatrix} \in T_{\text{GSp}_{2g}}(Q_p) \cap M_{2g}(Z_p) \text{ for } 1 \leq i \leq g - 1
\end{align*}
\]
and consider the set of double cosets
\[
\Upsilon_p := \{ [\text{Iw}^+_{\text{GSp}_{2g}} \mathbf{u}_{p,i} \text{Iw}^+_{\text{GSp}_{2g}}] : i = 0, \ldots, g - 1 \}.
\]
We immediately see that \([\text{Iw}^+_{\text{GSp}_{2g}} \mathbf{u}_p \text{Iw}^+_{\text{GSp}_{2g}}] = \prod_{i=0}^{g-1} [\text{Iw}^+_{\text{GSp}_{2g}} \mathbf{u}_{p,i} \text{Iw}^+_{\text{GSp}_{2g}}] \). A direct computation shows that the coset decomposition of \([\text{Iw}^+_{\text{GSp}_{2g}} \mathbf{u}_{p,i} \text{Iw}^+_{\text{GSp}_{2g}}] \) can be given by
\[
[\text{Iw}^+_{\text{GSp}_{2g}} \mathbf{u}_{p,i} \text{Iw}^+_{\text{GSp}_{2g}}] = \bigcup_j \delta_{i,j} \text{Iw}^+_{\text{GSp}_{2g}}
\]
for some \( \delta_{i,j} \in \text{GSp}_{2g}(Q_p) \cap M_{2g}(Z_p) \); in particular, \( \delta_{i,j} = \lambda_{i,j} \mathbf{u}_{p,i} \) for some \( \lambda_{i,j} \in \text{Iw}^+_{\text{GSp}_{2g}} \).

For any \( (\gamma, \nu) \in T_0 \), write \( (\gamma, \nu) = (\gamma_0, \nu_0) \beta \) for some \( \beta \in R_{\text{GL}_{2g},0}^+ \) such that \( \gamma_0 \in U_{\text{GL}_{2g},0}^{\text{opp}} \). Then, the left action of \( \mathbf{u}_{p,i} \) on \( T_0 \) is defined by the formula
\[
\mathbf{u}_{p,i} \cdot (\gamma, \nu) = (\gamma_0, \nu_0, \nu_{p,i}^\gamma, \mathbf{u}_{p,i} \nu_{p,i}^\nu, 0, \nu_{p,i}^{(i-1)}) \beta,
\]
where we write
\[
\mathbf{u}_{p,i} = \begin{pmatrix} \gamma \nu_{p,i}^\gamma \nu_{p,i}^{(i-1)} \mathbf{u}_{p,i} \nu_{p,i}^{(i-1)} \end{pmatrix}.
\]
Consequently, this defines a left action of \( \mathbf{u}_{p,i} \) on \( D_\kappa^r \). On the other hand, the right multiplication of \( \delta_{i,j} \) on \( \text{GSp}_{2g}(A_f) \) gives a left action on the homomorphisms between the free abelian group of simplical complexes on \( \tilde{X}_{\text{Iw}}(C) \). The two actions then combine to an action on \( C^*(\text{Iw}^+_{\text{GSp}_{2g}}, D_\kappa^r) \) and hence defines the action of \([\text{Iw}^+_{\text{GSp}_{2g}} \mathbf{u}_{p,i} \text{Iw}^+_{\text{GSp}_{2g}}] \) on \( C^*(\text{Iw}^+_{\text{GSp}_{2g}}, D_\kappa^r) \) by
\[
[\text{Iw}^+_{\text{GSp}_{2g}} \mathbf{u}_{p,i} \text{Iw}^+_{\text{GSp}_{2g}}] \cdot \sigma := \sum_j \delta_{i,j} \sigma = \sum_j \lambda_{i,j} \cdot (\mathbf{u}_{p,i} \sigma)
\]
17
for any $\sigma \in C^\bullet(Iw^+_{GSp_2}, D^\nu_\kappa)$. We shall denote by $U_{p,i}$ and $U_p$ the operators on $C^\bullet(Iw^+_{GSp_2}, D^\nu_\kappa)$ corresponding to the classes of double cosets $[Iw^+_{GSp_2} u_{p,i} Iw^+_{GSp_2}]$ and $[Iw^+_{GSp_2} u_p Iw^+_{GSp_2}]$. The Hecke algebra at $p$ is then defined to be $T_p = T_p Z_p = Z_p[T_p]$.

**Definition 3.2.1.** We define

$$T^p := \mathcal{O}_{q=p} T_q = \text{the Hecke algebra outside } p$$

$$T := T^p \otimes_{Z_p} T_p = \text{the total Hecke algebra}.$$

**Lemma 3.2.2.** The parabolic cohomology groups $H^i_{par}(X_{Iw^+}(C), D^\nu_\kappa)$ are $T$-stable.

**Proof.** Due to the nature of the Borel–Serre compactification, $C^\bullet(Iw^+_{GSp_2}, D^\nu_\kappa)$ admits Hecke actions as the ones defined above. Hence

$$\pi : C^\bullet(Iw^+_{GSp_2}, D^\nu_\kappa) \to C^\bullet_0(Iw^+_{GSp_2}, D^\nu_\kappa)$$

is a Hecke equivariant morphism of cochain complexes and hence

$$C^\bullet_c(Iw^+_{GSp_2}, D^\nu_\kappa) \to C^\bullet(Iw^+_{GSp_2}, D^\nu_\kappa)$$

is also Hecke equivariant and induces a Hecke equivariant map on cohomology groups

$$H^i_c(X_{Iw^+}(C), D^\nu_\kappa) \to H^i(X_{Iw^+}(C), D^\nu_\kappa).$$

This then shows the desired result. \hfill \square

### 3.3 The cuspidal eigenvariety

In this subsection, we extract out the cuspidal part of the eigenvarieties constructed in \[JN19\]. Although this is an easy consequence of op. cit. for experts, we write down the construction after recalling sufficiently amount of materials.

**Lemma 3.3.1.** The functor assigning each sheafy complete affinoid $(Z_p, Z_p)$-algebra $(R, R^\ast)$ to the set $\text{Hom}_{cts}(T_{GL_2,0}, R^\ast)$ is represented by the affinoid algebra $(Z_p[[T_{GL_2,0}]], Z_p[[T_{GL_2,0}]])$.

**Proof.** For any sheafy complete affinoid $(Z_p, Z_p)$-algebra $(R, R^\ast)$, we have a bijection

$$\text{Hom}_{cts}(T_{GL_2,0}, R^\ast) \cong \text{Hom}_{Z_p}^{cts}(Z_p[[T_{GL_2,0}]], R).$$

The bijection is obtained by extending the characters on the left-hand-side $Z_p$-linearly. Note that any continuous character $T_{GL_2,0} \to R$ automatically lands in $(A^\ast)^\ast$ as discussed in \[JN19\] Definition 4.1. \hfill \square

**Definition 3.3.2.** The weight space in our concern is then defined to be

$$\mathcal{W} := \text{Spa}(Z_p[[T_{GL_2,0}]], Z_p[[T_{GL_2,0}]]^{an}),$$

where the superscript $^{an}$ means that we are taking the analytic locus of the corresponding adic space.

For any open affinoid $U \subset \mathcal{W}$, we will always write the corresponding affinoid algebra to be $(R_U, R^\ast_U)$ and the universal weight on $U$ to be $\kappa_U : T_{GL_2,0} \to R^\ast_U$. In the following, we will always assume that the universal weight $\kappa_U$ and $R_U$ admit the previous assumptions that we made for $p$-adic weights. That is, we assume that one can choose a norm $|\cdot|_{R_U}$ on $R_U$ so that $R_U$ is a Banach–Tate $Z_p$-algebra and that $|\cdot|_{R_U}$ is adapted
to $\kappa_U$.  

Let $A^1_{Z_p} := \operatorname{Spa}(Z_p[T], Z_p)$, we write $A^1_U := U \times_{\operatorname{Spa}(Z_p, Z_p)} A^1_{Z_p}$ for any open affinoid $U \subset \mathcal{W}$. We have the following explicit description

$$A^1_U = \cup_m \operatorname{Spa}(R_U(\varpi^m T), R_U^+(\varpi^m T)),$$

where $\varpi$ is a fixed pseudouniformiser of $R_U$. Moreover, the global functions on $A^1_U$ is the ring

$$R_U \{\{T\}\} := \left\{ \sum_{n \geq 0} a_n T^n \in R_U[[T]] : |a_n|_R M^n \to 0 \text{ for all } M \in \mathbb{R}_{\geq 0} \right\}.$$

Fix an open affinoid $U \subset \mathcal{W}$, recall that the total Borel–Serre cochain complex $C^\text{tot}_{\kappa_U}$ is a potentially ON-able Banach $R_U$-module. Moreover, $U_p$ acts on $C^\text{tot}_{\kappa_U}$ compactly by [IN19, Corollary 3.3.10], hence we can consider the Fredholm determinant

$$F^r_{\kappa_U}(T) := \det \left( 1 - T U_p^{op} \right) \in R_U \{\{T\}\}.$$  

According to [op. cit., Proposition 4.1.2 & Proposition 4.1.4], the Fredholm determinant doesn’t depend on $r \in [r_\kappa, 1)$ and the chosen norm on $R_U$, thus we write $F_{\kappa_U}$. By [op. cit., Corollary 4.1.5], the Fredholm determinants $(F_{\kappa_U})_U$, where $U$ ranges over all open affinoid $U \subset \mathcal{W}$, glue together to $F_{\mathcal{W}} \in \mathcal{O}_{\mathcal{W}}(\mathcal{W})\{\{T\}\}$ and hence we define the **Fredholm hypersurface** (or the **spectral variety**)

$$\mathcal{Z} := \text{the zero locus of } F_{\mathcal{W}} \text{ in } A^1_{\mathcal{W}}$$

and denote by $\operatorname{wt}_\mathcal{Z} : \mathcal{Z} \to \mathcal{W}$ the structure morphism.

**Definition 3.3.3.** Let $h = \frac{m}{n} \in \mathbb{Q}_{\geq 0}$ and define

$$B_{U, h} := \{ x \in A^1_U : |T^n|_x \leq |\varpi^{-m}|_x \}.$$  

We also define $\mathcal{Z}_U$ to be the zero locus of $F_{\kappa_U}$ in $A^1_U$. Then, we say the pair $(U, h)$ is a **slope datum** if and only if

$$\mathcal{Z}_{U, h} := \mathcal{Z}_U \cap B_{U, h} \to U$$

is finite of constant degree.

**Proposition 3.3.4 ([IN19 Theorem 2.3.2]).** Keep the above notations. We have the following

1. The pair $(U, h)$ is a slope data if and only if $F_{\kappa_U}$ admits a factorisation $F_{\kappa_U} = QS$, where

   - $Q$ is a polynomial whose leading coefficient is a unit in $R_{\kappa_U}$ and its corresponding Newton polygon has slope $\leq h$ (see [IN19, Definition 2.2.4]),
   - $S = 1 + \sum_{n>0} a_n T^n \in R_{\kappa_U}\{\{T\}\}$ and
   - the ideal generated by $Q$ and $S$ in $R_{\kappa_U}\{\{T\}\}$ it the unit ideal.

2. The collection $\operatorname{Cov}_{\text{ad}}(\mathcal{Z}) := \{ \mathcal{Z}_{U, h} : (U, h) \text{ is a slope datum} \}$ is an open cover for $\mathcal{Z}$.

**Remark 3.3.5.** Recall that the cochain complex $C^\text{C}_c(\Gamma^+_\text{Sp}_{2g}, D^+_\delta)$ computes the compactly supported cohomology groups. By definition, the total complex $C^\text{tot}_{c, r} := \oplus C^c(\Gamma^+_\text{Sp}_{2g}, D^+_\delta)$ is a finite number of copies of $D^+_\delta(\mathcal{T}_0, R)$ as an $R$-module. Therefore, it is a potentially ON-able module and the above also applies to $C^\text{tot}_{c, r}$. In particular, we have a Fredholm hypersurface $\mathcal{Z}^c$ when considering the compactly supported cohomology groups and for any slope datum $(U, h)$ for $\mathcal{Z}^c$, we have a slope decomposition $C^\text{tot}_{c, r} = C^\text{C}_c, h \oplus C^\text{C}_c, > h$.  

19
Let \((U, h)\) be a slope datum for \(Z\) and let \(C_{\kappa_U}^{\text{tot}} = \oplus_i C_i^{\text{t}}(I_{\text{GSp}}^*, D_{\kappa_U}^i)\). As the above discussions hold for all \(r \in [r_\kappa, 1)\), the results also apply to \(C_{\kappa_U}^{\text{tot}}\). In particular, when considering the \(U_{\text{p}}\)-operator acting on \(C_{\kappa_U}^{\text{tot}}\), we have the factorisation of the corresponding Fredholm determinant \(F_{\kappa_U}\) and the decomposition \(C_{\kappa_U}^{\text{tot}} = C_{\kappa_U}^{\text{tot}, \leq h} \oplus C_{\kappa_U}^{\text{tot}, > h}\). Define \(H^i(X_{\text{Iw}}^\ast(C), D_{\kappa_U}^i)^{\leq h}\) to be the \(t\)-th cohomology group of the cochain complex \(C_{\kappa_U}^{\text{tot}, \leq h}\) and let \(H_{\kappa_U}^{\text{tot}, \leq h} = \oplus_i H^i(X_{\text{Iw}}^\ast(C), D_{\kappa_U}^i)^{\leq h}\). From the construction of the eigenvarieties in [IN19], we know that the assignment
\[
\text{Cov}_{\text{sd}}(Z) \ni Z_{U, h} \mapsto H_{\kappa_U}^{\text{tot}, \leq h}
\]
is a coherent sheaf on \(Z\).

We use the similar notation when considering the cohomology groups with compact supports. If \((U, h)\) is a slope datum for both \(Z\) and \(Z^c\), we have
\[
C_{c, \kappa_U}^{\text{tot}, \leq h} \to C_{\kappa_U}^{\text{tot}, \leq h}
\]
due to the definition of the slope \(\leq h\)-decomposition and the Hecke-equivariance of the map \(C_{c, \kappa_U}^{\text{tot}} \to C_{\kappa_U}^{\text{tot}}\). Hence we have a Hecke-equivariant map
\[
H_{c, \kappa_U}^{\text{tot}, \leq h} \to H_{\kappa_U}^{\text{tot}, \leq h}
\]
which preserves the degrees. We then define
\[
H_{\text{par}, \kappa_U}^{\text{tot}, \leq h} = \text{image}(H_{c, \kappa_U}^{\text{tot}, \leq h} \to H_{\kappa_U}^{\text{tot}, \leq h})
\]
as well as the analogous notation for each degree.

**Proposition 3.3.6.** The assignment
\[
\text{Cov}_{\text{sd}}(Z) \ni Z_{U, h} \mapsto H_{\text{par}, \kappa_U}^{\text{tot}, \leq h}
\]
defines a coherent sheaf on \(Z\), denoted by \(\mathcal{X}_{\kappa_U}^{\text{tot}, \leq h}\).

**Proof.** We need to show that for any slope data \((U, h)\) and \((V, h)\) for \(Z\) with \(V \subset U\) being a rational open subset, we have \(H_{\text{par}, \kappa_U}^{\text{tot}, \leq h} \otimes_{R_{\kappa_U}} R_V = H_{\text{par}, \kappa_V}^{\text{tot}, \leq h}\). This is the same to show the existence of the commutative diagram
\[
\begin{array}{ccc}
H_{c, \kappa_U}^{\text{tot}, \leq h} \otimes_{R_{\kappa_U}} R_V & \longrightarrow & H_{\kappa_U}^{\text{tot}, \leq h} \otimes_{R_{\kappa_U}} R_V \\
\downarrow & & \downarrow \\
H_{c, \kappa_V}^{\text{tot}, \leq h} & \longrightarrow & H_{\kappa_V}^{\text{tot}, \leq h}
\end{array}
\]
whose vertical arrows are isomorphisms.

Observe that
\[
D_{\kappa_U}^i(T_0, R_U) \otimes_{R_{\kappa_U}} R_V \simeq R_U[[t_{\text{GSp}_{2g}, 1}]]^{n \cdot 1} \otimes_{R_{\kappa_U}} R_V \simeq R_V[[t_{\text{GSp}_{2g}, 1}]]^{n \cdot 1} \simeq D_{\kappa_V}^i(T_0, R_V),
\]
where the superscript \(\otimes^{\cdot, \cdot}\) means taking the completion with respect to the family of norms \(\| \cdot \|_r\). Since both \(C_{\kappa_U}^{\text{tot}, \leq h}\) and \(C_{c, \kappa_U}^{\text{tot}, \leq h}\) are of finite presentation over \(R_{\kappa_U}\), taking cohomology commutes with flat base change, so
\[
\begin{align*}
H^i(X_{\text{Iw}}^\ast(C), D_{\kappa_U}^i)^{\leq h} \otimes_{R_{\kappa_U}} R_V & \simeq H^i(X_{\text{Iw}}^\ast(C), D_{\kappa_V}^i(T_0, R_U) \otimes_{R_{\kappa_U}} R_V)^{\leq h} \\
& \simeq H^i_c(X_{\text{Iw}}^\ast(C), D_{\kappa_V}^i)^{\leq h} \\
H^i_c(X_{\text{Iw}}^\ast(C), D_{\kappa_U}^i)^{\leq h} \otimes_{R_{\kappa_U}} R_V & \simeq H^i_c(X_{\text{Iw}}^\ast(C), D_{\kappa_V}^i(T_0, R_U) \otimes_{R_{\kappa_U}} R_V)^{\leq h} \\
& \simeq H^i_c(X_{\text{Iw}}^\ast(C), D_{\kappa_V}^i)^{\leq h},
\end{align*}
\]

20
where the first isomorphisms for both rows follow from that we are considering the finite slope parts.

For any slope datum \((U, h)\), the action of \(T\) on \(H^{t,\text{fin}}_{\text{par}}(U)\) yields a morphism of commutative algebras \(T \to \text{End}_{\mathcal{O}W}(U)\) whose image is denoted by \(T_{\text{fin}}\), which is a finite algebra over \(R_U\) since \(H^{t,\text{fin}}_{\text{par}}(U)\) is finitely generated. Since \(\mathcal{H}_{\text{fin}}\) is a coherent sheaf, the assignment

\[
\mathcal{F}_{\text{fin}} : \text{Cov}_{\text{fin}}(Z) \ni (Z, \varpi) \mapsto T_{\text{fin}}(Z, \varpi)
\]

is a coherent sheaf of \(\mathcal{O}_Z\)-algebras. Then the cuspidal eigenvariety \(E_0\) is defined to be

\[
E_0 := \text{Spa}_{\mathcal{O}_Z}(\mathcal{F}_{\text{fin}}, \mathcal{F}_{\text{fin}}^\circ),
\]

where the sheaf of integral elements \(\mathcal{F}_{\text{fin}}^\circ\) is determined by [JN19, Lemma A.3]. We name the structure morphisms

\[
E_0 \xrightarrow{\pi E_0} Z \xrightarrow{\text{wt}} W.
\]

We further let \(Z_{\text{par}} := \text{image } \pi E_0\) to be the Fredholm hypersurface corresponding to \(E_0\).

**Remark 3.3.7.** By applying the eigenvariety construction in [JN19] directly to GSp\(_{2g}\), one obtains the eigenvariety \(E\), parametrising the Hecke eigenvectors of finite slope cohomology groups \(H^t(X_{1w^*}(C), D_k^t)\). Recall that we have a Hecke-equivariant diagram

\[
\begin{align*}
H^1_t(X_{1w^*}(C), D_k^1) &\longrightarrow H^t(X_{1w^*}(C), D_k^t) \xrightarrow{\pi} H^1_0(X_{1w}(C), D_k^1) \\
H^t_{\text{par}}(X_{1w^*}(C), D_k^t) &\quad \uparrow \\
H^1_{\text{par}}(X_{1w^*}(C), D_k^1)
\end{align*}
\]

for each \(t\) such that

\[
H^1_{\text{par}}(X_{1w^*}(C), D_k^1) = \text{image}(H^1_t(X_{1w^*}(C), D_k^t) \to H^t(X_{1w^*}(C), D_k^t)) = \ker \pi,
\]

where the last equation is given by the exactness of the long exact sequence of cohomology groups. Let \(\mathbb{T}^{t, h}\) be the image of \(T\) in \(\text{End}_{\text{Cov}}(U)\), then there is a surjection \(\mathbb{T}^{t, h} \to \mathbb{T}^{t, h}_{\text{par}}\) given by the restriction. Hence, one sees that there is a closed immersion \(E_0 \to \mathcal{E}\) of adic spaces over \(W\) and one views \(E_0\) as the “cuspidal part” of \(\mathcal{E}\) since \(\pi|_{H^1_{\text{par}}(X_{1w^*}(C), D_k^1)}\) is the zero map.

**Remark 3.3.8.** As pointed out in [JN19, Remark 4.1.9], the eigenvariety constructed in [Han17] is the open locus of \(p \neq 0\) inside \(\mathcal{E}\). Consequently, the cuspidal part of the eigenvariety in op. cit. is the open locus of \(p \neq 0\) inside our cuspidal eigenvariety \(E_0\).

**Corollary 3.3.9.** The pairing in Proposition 3.1.1 induce pairings

\[
[\cdot, \cdot] : \mathcal{H}_{\text{par}}^{\text{fin}} \times \mathcal{H}_{\text{par}}^{\text{fin}} \to \mathcal{O}_Z \quad \text{and} \quad [\cdot, \cdot] : \pi_{\mathcal{O}_Z}^* \mathcal{H}_{\text{par}}^{\text{fin}} \times \pi_{\mathcal{O}_Z}^* \mathcal{H}_{\text{par}}^{\text{fin}} \to \mathcal{O}_{E_0}
\]

of coherent sheaves on \(Z\) and \(E_0\) respectively. Moreover, the first pairing is \(T\)-equivariant.
Proof. First of all, we claim that the pairing \( [\cdot, \cdot]_p^\kappa \) is \( u_{p,i} \)-equivariant for any \( i = 0, 1, ..., g - 1 \) and for any \( p \)-adic weight \( \kappa : T_{GL_n,0} \to R \). Take any \( \mu_1, \mu_2 \in D_k^\kappa \), we have

\[
[u_{p,i} \cdot \mu_1, \mu_2]_p^\kappa = \int_{T_{\kappa(in)}} e_{\kappa}^{\text{pst}} \left( (\tau \gamma_2 \tau v_2) \left( \frac{\mathbb{I}_g}{p^{-1}} \right) (\gamma_1) (v_1) \right) \ d u_{p,i} \cdot \mu_1(\gamma_1, v_1) d \mu_2(\gamma_2, v_2)
\]

and write

\[
\text{mult} \left( \gamma_1 \gamma_2 \gamma_3 \cdots \gamma_n \right) = \int_{T_{\kappa(in)}} e_{\kappa}^{\text{pst}} \left( (\tau \gamma_2 \gamma_3 \cdots \gamma_n \gamma_1) (v_1) \right) \ d \mu_1(\gamma_1, v_1) d \mu_2(\gamma_2, v_2)
\]

Thus, by gluing, one obtains the first desired pairing. It is furthermore \( T^p \)-equivariant since the Hecke operators outside \( p \) acts on the analytic distributions trivially. The second one follows immediately.

\[ \square \]

4 The ramification locus of the cuspidal eigenvariety

In this section, we apply our pairing to the study of the ramification locus of the cuspidal eigenvariety for \( \text{GSp}_{2g} \). We will first set up a formalism by following the strategy in [Bel]. Then, the main results of this paper are proven in Theorem 4.2.8, Theorem 4.2.9 and Corollary 4.3.5.

4.1 Some commutative algebra

In this subsection, we recollect some ingredients of commutative algebra from [Bel].

Let \( A \) be a noetherian domain and \( B \) be a finite flat \( A \)-algebra. Consider \( \text{mult} : B \otimes_A B \to B, \ b \otimes b' \mapsto bb' \) and write \( \text{mult} = \ker(\text{mult}) \). Let \( (B \otimes_A B)[\text{mult}] := \{ x \in B \otimes_A B : y \cdot x = 0 \ \forall y \in \text{mult} \} \), then the Noether’s different of \( B \) over \( A \) is defined to be the ideal

\[ \mathfrak{d}(B/A) := \text{image} \left( (B \otimes_A B)[\text{mult}] \xrightarrow{\text{mult}} B \right) \]

in \( B \).
We claim first that for any
...2
Proof. See [AB59, Theorem 2.7].

Suppose \( M, N \) are two \( B \)-modules which are finite flat over \( A \) and assume we are in the following situation:

- There exists an \( A \)-linear pairing
  \[ \beta : M \times N \to A \]
  such that \( \beta \) is \( B \)-equivariant.

- We have isomorphisms \( M \cong N \cong B^\vee := \text{Hom}_A(B, A) \) of \( B \)-modules.

**Lemma 4.1.2** ([Be], Proposition VIII.1.11]). Denote by \( \beta_B \) the base change of \( \beta \) to \( B \) on \( M \otimes_A B \times N \otimes_A B \).

Let

\[
(M \otimes_A B)[\text{mult}] = \{ x \in M \otimes_A B : y \cdot x = 0 \ \forall y \in \text{mult} \}
\]

\[
(N \otimes_A B)[\text{mult}] = \{ x \in N \otimes_A B : y \cdot x = 0 \ \forall y \in \text{mult} \}.
\]

Then the ideal

\[ \mathfrak{L}_\beta := \text{image} (\beta_B : (M \otimes_A B)[\text{mult}] \times (N \otimes_A B)[\text{mult}] \to B) \]

is a principal ideal in \( B \).

**Proof.** We claim first that for any \( B \)-module \( M \) which is finite flat over \( A \), we have an isomorphism \( M^\vee \otimes_A B[U] \cong \text{Hom}_B(M, B) \), where \( M^\vee = \text{Hom}_A(M, A) \). Notice that \( M^\vee \) also admits a \( B \)-module structure by \( b\psi : m \mapsto \psi(bm) \) for all \( b \in B, \psi \in M^\vee \) and \( m \in M \). We have a natural isomorphism

\[ M^\vee \otimes_A B = \text{Hom}_A(M, A) \otimes_A B \to \text{Hom}_A(M, B), \quad \psi \otimes b \mapsto (m \mapsto \psi(m)b). \]

Since \( \text{mult} = \sum_{b \in B}(b \otimes 1 - 1 \otimes b)B \otimes_A B \), thus

\[ \psi \otimes b \in M^\vee \otimes_A B[U] \iff (b' \otimes 1 - 1 \otimes b')\psi \otimes b = 0 \ \forall b' \in B \]

\[ \iff b'\psi \otimes b = \psi \otimes bb' \ \forall b' \in B \]

\[ \iff \psi(b'm)b = \psi(mb)bb' \ \forall b' \in B, m \in M \]

\[ \iff (m \mapsto \psi(m)b) \in \text{Hom}_B(M, B). \]

Apply the claim in our situation, we have isomorphisms of \( B \)-modules

\[ (M \otimes_A B)[\text{mult}] \cong (B^\vee \otimes_A B)[\text{mult}] \cong \text{Hom}_B(B, B) \cong B \]

and same for \( (N \otimes_A B)[\text{mult}] \). Hence, let \( \bar{m} \) and \( \bar{n} \) be generators of \( (M \otimes_A B)[\text{mult}] \) and \( (N \otimes_A B)[\text{mult}] \) respectively as \( B \)-modules. Then \( \mathfrak{L}_\beta = \beta_B(\bar{m}, \bar{n})B \).

**Proposition 4.1.3** ([Be], Corollary VIII.1.13]). Suppose \( B \) is Gorenstein over \( A \), i.e., \( B^\vee \) is flat of constant rank 1 over \( B \), and \( M, N \) are \( B \)-modules which are finite flat over \( A \) and flat of rank 1 over \( B \). Assume there is an \( A \)-linear pairing \( \beta : M \times N \to A \) which is \( B \)-equivariant. We retain the notation \( \beta_B \) and \( \mathfrak{L}_\beta \) as in Lemma 4.1.2. Then

1. Both ideals \( \mathfrak{d}(B/A) \) and \( \mathfrak{L}_\beta \) are locally principal. Moreover, there exists \( b_0 \in B \) such that \( \mathfrak{L}_\beta = b_0 \mathfrak{d}(B/A) \).
2. We have $\mathcal{L}_\beta = \mathfrak{d}(B/A)$ if and only if $\beta$ is non-degenerate.

**Proof.** We are in a special case of Lemma 4.1.2 that we can identify (locally) $M \simeq N \simeq B^\vee \simeq B$ and hence we know $\mathcal{L}_\beta$ is principal. Moreover, the identification $B \otimes_A B[\text{mult}] \simeq \text{Hom}_B(B, B) \simeq B$ implies that $\mathfrak{d}(B/A)$ is also principal.

Observe that we can identify $\beta : M \times N \to A$ as a linear morphism $B^\vee \otimes_A B^\vee \to A$. Hence by duality, we identify $\beta$ with an element $b \in B \otimes_A B$. We claim that $\mathcal{L}_\beta = \text{mult}(b)B$. As we are working locally, we assume $b_1, \ldots, b_n$ is a basis of $B$ over $A$, then $b_1^\vee, \ldots, b_n^\vee$ is a basis of $B^\vee$ over $A$. Observe that $b^\vee := \sum b_i^\vee \otimes b_i$ is a generator of $B^\vee \otimes_A B[\text{mult}] \simeq \text{Hom}_B(B, B)$ as it maps to the identity in $\text{Hom}_B(B, B)$. Hence by definition

$$\mathcal{L}_\beta = \beta_B(b^\vee, b^\vee)B = \left( \sum_{i,j} \beta(b_i^\vee, b_j^\vee)b_ib_j \right)B.$$ 

On the other hand, by the above construction, we see that $b = \sum_{i,j} \beta(b_i^\vee, b_j^\vee)b_ib_j$ with $\text{mult}(b) = \sum_{i,j} \beta(b_i^\vee, b_j^\vee)b_ib_j$.

Let $\widetilde{b^\vee} = \sum b_i \otimes b_i$, then it is a generator of $B \otimes_A B[\text{mult}] \simeq B$. Thus, there exists $b_0 \in B$ such that $b_0\widetilde{b^\vee} = b$. We conclude that

$$\mathcal{L}_\beta = \text{mult}(b)B = \text{mult}(b_0\widetilde{b^\vee})B = b_0 \text{mult}(\widetilde{b^\vee})B = b_0 \mathfrak{d}(B/A).$$

Finally, we have

$$\mathcal{L}_\beta = \mathfrak{d}(B/A) \iff b_0 \in B^\times$$

$$\iff \beta(b_i^\vee, b_j^\vee) = \begin{cases} b_0 \in B^\times & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\iff \beta \text{ is non-degenerate.}$$

3.2 The ramification locus of the cuspidal eigenvariety

Recall the weight map $\text{wt} : \mathcal{E}_0 \to \mathcal{W}$ and $\pi_0^{\text{par}} : \mathcal{E}_0 \to \mathcal{Z}$. For each slope datum $(\mathcal{U}, h)$, let $\mathcal{E}_{\mathcal{U}, h} := (\pi_0^{\text{par}})^{-1}(\mathcal{Z}_{\mathcal{U}, h})$. We adapt the definitions of “clean neighbourhoods” and “good points” in [Bel] in our situation:

**Definition 4.2.1.**

1. Let $x \in \mathcal{E}_0$ and $\mathcal{V} = \text{Spa}(R_{\mathcal{V}}, R_{\mathcal{V}}^\vee)$ be an open affinoid neighbourhood of $x$. We say $\mathcal{V}$ is a clean neighbourhood of $x$ if it satisfies the following properties:

   - $\text{wt}(\mathcal{V}) = \mathcal{V} \subseteq \text{Spa}(R_{\mathcal{V}}, R_{\mathcal{V}}^\vee) \subset \mathcal{W}$ is an open affinoid subset of $\mathcal{W}$ and there exists a slope datum $(\mathcal{U}, h)$ for $\mathcal{Z}$ such that $\mathcal{V}$ is the connected component of $x$ in $\mathcal{E}_{\mathcal{U}, h}$;
   
   - $x$ is the only point of $\mathcal{V}$ sitting above $\text{wt}(x)$;
   
   - the map $\text{wt} : \mathcal{V} \to \mathcal{W}$ is flat and is moreover étale except perhaps at $x$.

   In this case, there exists an idempotent $\eta_{\mathcal{V}} \in \pi_{\text{par}}^{\mathcal{U}, h}$ such that $\mathcal{V}$ is defined by the equation $\eta = 1$ and the module $\eta H_{\text{par,}\mathcal{U}, h}^{\text{top,sh}}$ is a direct summand of $H_{\text{par,}\mathcal{U}, h}^{\text{top,sh}}$.

2. A point $x \in \mathcal{E}_0$ is said to be a good point if it admits a sufficiently small clean neighbourhood $\mathcal{V}$ with $\text{wt}(\mathcal{V}) = \mathcal{V}$ such that the modules $\eta_{\mathcal{V}} H_{\text{par,}\mathcal{U}, h}^{\text{top,sh}}$ and $(\eta_{\mathcal{V}} H_{\text{par,}\mathcal{U}, h}^{\text{top,sh}})^\vee$ are free of rank one over $R_{\mathcal{V}}$, where the dual is taken to be an $R_{\mathcal{V}}$-dual.

**Remark 4.2.2.** We remark the following:
1. In the GL_2 case, the eigencurve is finite flat over the weight space ([Bel. §VI.1.4]) and so the author of op. cit. can consequently deduce that the collection of clean neighbourhoods of points on the eigencurve gives a open cover of the eigencurve. In our case, the Fredholm hypersurface \( Z \) is finite flat over \( \mathcal{W} \) by [AIP18, Theorem B.1]. However, we don’t know if \( \mathcal{E}_0 \) is flat over \( Z \). Therefore, instead of considering \( \mathcal{E}_0 \), we consider \( \mathcal{E}_0^\mathfrak{fl} \oplus \mathcal{E}_0^\mathfrak{gl} \) the flat locus over \( \mathcal{W} \), which is open over \( \mathcal{W} \), and let \( \text{Cov}_{\mathfrak{fl}}(\mathcal{E}_0^\mathfrak{fl}) \) be the open cover of clean neighbourhoods.

2. In the definition of good points, we see immediately that \( R_{\mathcal{W}} \) is Gorenstein over \( R_{\mathcal{Y}} \).

Following [Bel. §VIII. 4], we study the adjoint \( L \)-ideal and define the \( p \)-adic adjoint \( L \)-function here. Let \( x \in \mathcal{E}_0^\mathfrak{fl} \) and \( \mathcal{V} \) be a clean neighbourhood of \( x \) with weight \( \text{wt}(\mathcal{V}) = \mathcal{Y} \). There is a natural multiplication map

\[
\text{mult}: R_{\mathcal{W}} \widehat{\otimes} R_{\mathcal{Y}} \to R_{\mathcal{Y}}, \quad b \otimes b' \mapsto bb'.
\]

Let \( \text{mult} := \ker \text{mult} \) and define

\[
M \widehat{\otimes} R_{\mathcal{Y}}[\text{mult}] := \{ m \in M \widehat{\otimes} R_{\mathcal{Y}} : \text{mult} \cdot m = 0 \}
\]

for any Banach \( R_{\mathcal{Y}} \)-module \( M \).

**Definition 4.2.3.** Keep the above notation. The adjoint \( L \)-ideal of \( \mathcal{V} \) is defined to be

\[
\mathcal{L}^{\text{adj}}(\mathcal{V}) := \text{image}(\cdot \kappa, \eta_\mathcal{Y} : H^{\text{par}, \mathfrak{sch}}_{\mathfrak{fl}} \widehat{\otimes} R_{\mathcal{Y}}[\text{mult}] \times \eta_\mathcal{Y} H^{\text{par}, \mathfrak{sch}}_{\mathfrak{fl}} \widehat{\otimes} R_{\mathcal{Y}}[\text{mult}] \to R_{\mathcal{Y}}).
\]

**Remark 4.2.4.** Since the clean neighbourhoods cover \( \mathcal{E}_0^\mathfrak{fl} \), the collection \( \{ \mathcal{L}^{\text{adj}}(\mathcal{V}) : \mathcal{V} \in \text{Cov}_{\mathfrak{fl}}(\mathcal{E}_0^\mathfrak{fl}) \} \) glues to a coherent sheaf \( \mathcal{L}^{\text{adj}} \) on \( \mathcal{E}_0^\mathfrak{fl} \).

**Proposition 4.2.5.** Let \( x \in \mathcal{E}_0^\mathfrak{fl} \) be a good point. Then there exists a sufficiently small clean neighbourhood \( \mathcal{V} \) of \( x \) with \( \text{wt}(\mathcal{V}) = \mathcal{Y} \) such that \( \mathcal{L}^{\text{adj}}(\mathcal{V}) \) is a principal ideal in \( R_{\mathcal{Y}} \).

**Proof.** The assertion follows from Lemma 4.1.2.

**Definition 4.2.6.** Let \( x \in \mathcal{E}_0^\mathfrak{fl} \) be a good point and \( \mathcal{V} \) be a sufficiently small clean neighbourhood such that \( \mathcal{L}^{\text{adj}}(\mathcal{V}) \) is principal. We define the adjoint \( p \)-adic \( L \)-function on \( \mathcal{V} \) to be \( L^{\text{adj}}(\mathcal{V}) \in R_{\mathcal{Y}} \) such that \( L^{\text{adj}} \) generates \( \mathcal{L}^{\text{adj}}(\mathcal{V}) \). The value of \( L^{\text{adj}}(x) \) at \( x \) is denoted by \( L^{\text{adj}}(x) \) as it doesn’t depend on the clean neighbourhood.

**Remark 4.2.7.** We point out that the adjoint \( p \)-adic \( L \)-function \( L^{\text{adj}} \) is defined up to a unit in \( R_{\mathcal{Y}} \). In the case of \( \text{GL}_2 \), the name “adjoint \( p \)-adic \( L \)-function” is justified in [Kim06, Proposition 3.9.2] and [Bel. §VIII.5]. However, the justification of the name is unknown to us in our situation as discussed in the introduction.

Let \( x \in \mathcal{E}_0^\mathfrak{fl} \) be a good point and let \( \mathcal{V} \) be a sufficiently small clean neighbourhood of \( x \) such that \( L^{\text{adj}} \) is defined. Let \( (\mathcal{U}, h) \) be the slope datum that defines \( \mathcal{V} \) and let \( \text{wt}(\mathcal{V}) = \mathcal{Y} \). Corollary 3.3.9 yields an \( R_{\mathcal{Y}} \)-equivariant pairing

\[
[\cdot, \cdot]_{\eta_\mathcal{Y} : H^{\text{par}, \mathfrak{sch}}_{\mathfrak{fl}} \times H^{\text{par}, \mathfrak{sch}}_{\mathfrak{fl}} \to R_{\mathcal{Y}}},
\]

Together with the definition of good points, we are in the situation of Proposition 4.1.3.

**Theorem 4.2.8.** Let \( x \in \mathcal{E}_0^\mathfrak{fl} \) be a good point and let \( \kappa = \text{wt}(x) \). Suppose the pairing

\[
[\cdot, \cdot]_{\eta_\mathcal{Y} : H^{\text{par}, \mathfrak{sch}}_{\mathfrak{fl}} \times H^{\text{par}, \mathfrak{sch}}_{\mathfrak{fl}} \to R_{\mathcal{Y}}}
\]

is non-degenerate at \( \text{wt}(x) \), then

\[
L^{\text{adj}}(x) = 0 \text{ if and only if } \text{wt } \text{ is ramified at } x.
\]

25
Proof. Let $\mathcal{V}$ be a sufficiently small clean neighbourhood of $x$ which is defined by the slope datum $(\mathcal{U}, h)$ and $\text{wt}(\mathcal{V}) = \mathcal{V}$. Since the pairing
\[ [\cdot, \cdot]_{\kappa_{\mathcal{U}}} : \eta_{\mathcal{V}} H_{\text{par}, \kappa_{\mathcal{U}}}^{\text{hol}, \leq h} \times \eta_{\mathcal{V}} H_{\text{par}, \kappa_{\mathcal{U}}}^{\text{hol}, \leq h} \rightarrow \mathcal{V} \]
is assumed to be non-degenerate, then by Proposition 4.1.3, $\mathcal{L}^{\text{adj}}(\mathcal{V}) = \partial(R_{\mathcal{V}}/R_{\mathcal{Y}})$. Thus,
\[ L^{\text{adj}}(x) = 0 \iff L^{\text{adj}} \in \supp x \iff \partial(R_{\mathcal{V}}/R_{\mathcal{Y}}) \subset \supp x \iff \text{wt is ramified at } x, \]
where the last equivalence is due to Auslander–Buchsbaum’s theorem.

**Theorem 4.2.9.** Let $x \in \mathcal{E}_0$ be a good and smooth point and let $\kappa = \text{wt}(x)$. We further assume $x$ lives in the open locus of $\mathcal{E}_0$ where $p \neq 0$. Assume again that the pairing
\[ [\cdot, \cdot]_{\kappa_{\mathcal{U}}} : \eta_{\mathcal{V}} H_{\text{par}, \kappa_{\mathcal{U}}}^{\text{hol}, \leq h} \times \eta_{\mathcal{V}} H_{\text{par}, \kappa_{\mathcal{U}}}^{\text{hol}, \leq h} \rightarrow \mathcal{V} \]
is non-degenerate at $\text{wt}(x)$. Let $R_{\text{wt}(x)}$ and $R_x$ be the local rings at $\text{wt}(x)$ and $x$ respectively and denote by $m_{\text{wt}(x)}$, $m_x$ their maximal ideals respectively. Let $\text{Fitt}_x$ be the 0-th Fitting ideal of $\Omega_{R_x/R_{\text{wt}(x)}}^{1}$ and define
\[ e(x) := \max\{ e \in \mathbb{Z}_{\geq 0} : \text{Fitt}_x \subset m_x^e \}. \]
Then, we have
\[ \text{ord}_x L^{\text{adj}} = e(x). \]

Proof. By [Bel, Theorem VIII. 1.4], we have $\partial(R_x/R_\kappa) = \text{Fitt}_x$ (since $x$ is a smooth point) and so
\[ e(x) = \max\{ e \in \mathbb{Z}_{\geq 0} : \partial(R_x/R_\kappa) \subset m_x^e \}. \]
On the other hand,
\[ \text{ord}_x L^{\text{adj}} := \max\{ e \in \mathbb{Z}_{\geq 0} : L^{\text{adj}}(x) \in m_x^e \}. \]
In our situation, we see that
\[ m_x^{e(x)} \supset \partial(R_x/R_\kappa) = L^{\text{adj}}(x) R_x \subset m_x^{\text{ord}_x L^{\text{adj}}}. \]
As the inclusions on both sides satisfy the same condition, the exponents coincide.

**Remark 4.2.10.** We remark that the above two theorems have their roots in the $GL_2$ case. Theorem 4.2.8 is an analogue of [Bel, Theorem VIII.4.7] while Theorem 4.2.9 is inspired by [op. cit., Theorem VIII.4.8(i)].

### 4.3 Non-degeneracy of the pairing

In the statements of Theorem 4.2.8 and Theorem 4.2.9 we assumed that the pairing $[\cdot, \cdot]_{\text{wt}(x)}$ is non-degenerate at $\text{wt}(x)$. In this subsection, we justify that such an assumption is not vacuous.

Let $\kappa : T_{GL_2, 0} \rightarrow R$ be any $p$-adic weight. Recall the pairing $[\cdot, \cdot]_\kappa$ on the parabolic cohomology groups is defined by the pairing
\[ [\cdot, \cdot]_\kappa : D^l_\kappa(T_0, R) \times D^l_\kappa(T_0, R) \rightarrow R, \]
\[ (\mu_1, \mu_2) \mapsto \int_{T_{\text{int}}}^\text{hast} \left( \tau \gamma_2 \overset{\tau}{\nu} \left( \begin{array}{c} \mathbb{1}_g \\ p^{-1} \mathbb{1}_g \end{array} \right) \right) \cdot d\mu_1 d\mu_2. \]
When $\kappa = k \in \mathbb{Z}_{\geq 0}$ is a dominant algebraic weight, recall that we also have algebraic representations $V_{\text{GSp}_{2g}, k}^{\text{alg}, \vee}$ and $V_{\text{GSp}_{2g}, k}^{\text{alg}, \circ}$ defined in §2.3. From now on, we abuse the notation, writing $V_{\text{GSp}_{2g}, k}^{\text{alg}, \vee}$ and $V_{\text{GSp}_{2g}, k}^{\text{alg}, \circ}$ for their $Q_p$-realisation. That is,

$$V_{\text{GSp}_{2g}, k}^{\text{alg}, \vee} = \left\{ \phi : \text{GSp}_{2g}(Q_p) \to Q_p : \begin{array}{l}
\bullet \phi \text{ is a polynomial function} \\
\bullet (\gamma, \beta) = k(\beta) \phi(\gamma) \\
\forall (\gamma, \beta) \in \text{GSp}_{2g}(Q_p) \times B_{\text{GSp}_{2g}}(Q_p)
\end{array} \right\}$$

$$V_{\text{GSp}_{2g}, k}^{\text{alg}, \circ} = \text{Hom}_{Q_p}(V_{\text{GSp}_{2g}, k}^{\text{alg}}, Q_p).$$

There is an obvious injective morphism

$$V_{\text{GSp}_{2g}, k}^{\text{alg}, \circ} \to A_{k}(T_0, Q_p), \quad \phi \mapsto \left( (\gamma, \nu) \mapsto k(\beta) \phi \left( \begin{pmatrix} \gamma_0 \\ \nu_0 \\ \tilde{I}_g \end{pmatrix} \right) \right)$$

for any $r$, where $(\gamma, \nu) = (\gamma_0, \nu_0) \beta$ with $\gamma_0 \in U_{\text{GSp}_{2g}, 1}^{\text{opp}}$ and $\beta \in B_{\text{GSp}_{2g}}^{\text{opp}}$. Therefore, there is a natural surjection $D_k(T_0, Q_p) \to V_{\text{GSp}_{2g}, k}^{\text{alg}, \circ}$, which is $Iw_{\text{GSp}_{2g}}^{\text{opp}}$-equivariant. We then descend the pairing $[\cdot, \cdot]_k$ to $V_{\text{GSp}_{2g}, k}^{\text{alg}, \circ}$ by the same formula

$$[\cdot, \cdot]_k^{\circ} : V_{\text{GSp}_{2g}, k}^{\text{alg}, \circ} \times V_{\text{GSp}_{2g}, k}^{\text{alg}, \circ} \to Q_p^*,$$

$$(\mu_1, \mu_2) \mapsto \int_{T_1, T_2 \in U_{\text{GSp}_{2g}}^{\text{opp}}} \phi_{h k} \left( \gamma_2 \left( \begin{pmatrix} I_g & -\tilde{I}_g \\ \tilde{I}_g & I_g \end{pmatrix} \right) \gamma_1 \right) d\mu_1(\gamma_1) d\mu_2(\gamma_2).$$

**Proposition 4.3.1.** Let $k \in \mathbb{Z}_{\geq 0}$ be a dominant weight. Then the pairing $[\cdot, \cdot]_k^{\circ}$ on $V_{\text{GSp}_{2g}, k}^{\text{alg}, \circ}$ is non-degenerate.

**Proof.** Recall the symplectic pairing $\langle \cdot, \cdot \rangle_k$ on $V_{\text{GSp}_{2g}, k}^{\text{alg}, \circ}$ from Remark 2.3.1

$$\langle \mu_1, \mu_2 \rangle_k = \int_{T_1, T_2 \in \text{GSp}_{2g}(Q_p)} \phi_{h k} \left( \gamma_2 \left( \begin{pmatrix} I_g & -\tilde{I}_g \\ \tilde{I}_g & I_g \end{pmatrix} \right) \gamma_1 \right) \, d\mu_1(\gamma_1) d\mu_2(\gamma_2).$$

Since the symplectic pairing $\langle \cdot, \cdot \rangle$ on $V_{\text{Z}}$ is non-degenerate, we know that $\langle \cdot, \cdot \rangle_k$ is non-degenerate.

Define

$$\langle \cdot, \cdot \rangle_k' : V_{\text{GSp}_{2g}, k}^{\text{alg}, \circ} \times V_{\text{GSp}_{2g}, k}^{\text{alg}, \circ} \to Q_p^*,$$

$$(\mu_1, \mu_2) \mapsto \int_{T_1, T_2 \in U_{\text{GSp}_{2g}}^{\text{opp}}} \phi_{h k} \left( \gamma_2 \left( \begin{pmatrix} I_g & -\tilde{I}_g \\ \tilde{I}_g & I_g \end{pmatrix} \right) \gamma_1 \right) \, d\mu_1(\gamma_1) d\mu_2(\gamma_2).$$

Then $\langle \cdot, \cdot \rangle_k'$ is a non-degenerate pairing. Indeed, we have

$$\langle \mu_1, \mu_2 \rangle_k' = \int_{T_1, T_2 \in \text{GSp}_{2g}(Q_p)} \phi_{h k} \left( \gamma_2 \left( \begin{pmatrix} I_g & -\tilde{I}_g \\ \tilde{I}_g & I_g \end{pmatrix} \right) \gamma_1 \right) \, d\mu_1(\gamma_1) d\mu_2(\gamma_2)$$

$$= \int_{T_1, T_2 \in \text{GSp}_{2g}(Q_p)} k(\beta_1) k(\beta_2) \phi_{h k} \left( \gamma_2 \left( \begin{pmatrix} I_g & -\tilde{I}_g \\ \tilde{I}_g & I_g \end{pmatrix} \right) \gamma_1 \right) \, d\mu_1(\gamma_1) d\mu_2(\gamma_2),$$

27
where $\gamma_i = \gamma'_i \beta$ with $\gamma'_i \in U^{opp}_\text{GSp}_{2g}(Q_p)$ and $\beta_i \in B_{\text{GSp}_{2g}}(Q_p)$ for $i = 1, 2$. As $k$ is non-zero on $B_{\text{GSp}_{2g}}(Q_p)$, we see that $\{\mu_1, \mu_2\}_k = 0$ if and only if $\{\mu_1, \mu_2\}_k' = 0$.

Now, let $[\cdot, \cdot]_k'$ be the pairing on $V^{\text{alg},v}_{\text{GSp}_{2g},k}$ defined by
\[
[\mu_1, \mu_2]' := \left[\mu_1, w_p \cdot \mu_2\right]'_k = \int_{\gamma_1, \gamma_2 \in U^{opp}_\text{GSp}_{2g}(Q_p)} e^{ib_0} \left(t \gamma_2 \left(\frac{1}{g} p^{-1} \mathbb{I}_g\right) \gamma_1\right) d\mu_1(\gamma_1) d\mu_2(\gamma_2).
\]

Then, $[\cdot, \cdot]_k'$ is again a non-degenerate pairing since $w_p \in \text{GSp}_{2g}(Q_p)$. Recall that $U^{opp}_{\text{GSp}_{2g},1} \cong \mathbb{Z}_p^{d_0}$, for some $d_0 \in \mathbb{Z}_{>0}$, as $p$-adic manifolds, thus $U^{opp}_{\text{GSp}_{2g},1}(Q_p) \cong Q_p^{d_0}$. However, $V^{\text{alg},v}_{\text{GSp}_{2g},k}$ is defined algebraically and $\mathbb{Z}_p^{d_0} \subset Q_p^{d_0}$ is Zariski dense, thus the non-degeneracy of $[\cdot, \cdot]'_k$ implies the non-degeneracy of $[\cdot, \cdot]_k'$. □

**Theorem 4.3.2** (Control theorem). For $g \in \mathbb{Z}_{>0}$, let $k = (k_1, \ldots, k_g) \in \mathbb{Z}_p^{d_0}$ be a dominant algebraic weight. Then, there exists $h_k \in Z_{\geq 0}$ (depending on $k$) such that for any $Q_{>0} \ni h < h_k$, we have a canonical isomorphism
\[
H^t_{\text{par}}(X_{1w^*}(C), D_k^+) \cong H^t_{\text{par}}(X_{1w^*}(C), V^{\text{alg},v}_{\text{GSp}_{2g},k}) \leq h.
\]

*(see also [AS08, Theorem 6.4.1]*)

**Proof.** Let $K := \ker(D^1_k(T_0, Q_p) \to V^{\text{alg},v}_{\text{GSp}_{2g},k})$ and so we have an exact sequence
\[
0 \to C^*(Iw^+_{\text{GSp}_{2g},k}) \to C^*(Iw^+_k, D^1_k) \to C^*(Iw^+_{\text{GSp}_{2g},k}) \to 0.
\]

Define the $[Iw^+_{\text{GSp}_{2g},k} u_{p,i} Iw^+_{\text{GSp}_{2g}}]$-action on $V^{\text{alg},v}_{\text{GSp}_{2g},k}$ as the same formula on $D^1_k(T_0, Q_p)$ but with $u_{p,i}$ acting on $V^{\text{alg},v}_{\text{GSp}_{2g},k}$ via the conjugation
\[
u_{p,i} \cdot \gamma = u_{p,i} \gamma u_{p,i}^{-1}
\]
on $\text{GSp}_{2g}(Q_p)$. We then see that the map
\[
C^*(Iw^+_{\text{GSp}_{2g},k}) \to C^*(Iw^+_k, V^{\text{alg},v}_{\text{GSp}_{2g},k})
\]
is Hecke equivariant and so $C^*(Iw^+_{\text{GSp}_{2g},k})$ is Hecke stable. Denote by $C_{K}^{\text{tot}}$ and $C_{k,\text{alg}}^{\text{tot}}$ the total cochain complexes of $C^*(Iw^+_{\text{GSp}_{2g},k})$ and $C^*(Iw^+_k, V^{\text{alg},v}_{\text{GSp}_{2g},k})$ respectively. Additionally, let $F_k$ and $F_{k,\text{alg}}$ be the Fredholm determinant of $U_p$ acting on $C_k^{\text{tot}}$ and $C_{k,\text{alg}}^{\text{tot}}$ respectively. We define
\[
h_K := \sup \left\{ h \in Q_{>0} : \|U_p\|_K := \sup \left\{ \left\| U_p \cdot \sigma \right\|_i : \|\sigma\|_i \in C_k^{\text{tot}} \right\} \leq p^{-h} \right\},
\]
where $\|\cdot\|_i$ is the norm on $C_k^{\text{tot}}$.
\[
h_1 := \sup \left\{ h \in Q_{>0} : F_k = Q^1 \mathbb{S}_1 \right\} \text{ satisfying conditions in Proposition 3.3.3 w.r.t. } h
\]
\[
h_{\text{alg}} := \sup \left\{ h \in Q_{>0} : F_{k,\text{alg}} = Q^{\text{alg}} \mathbb{S}^{\text{alg}} \right\} \text{ satisfying conditions in Proposition 3.3.3 w.r.t. } h
\]
\[
h_k := \min(h_K, h_1, h_{\text{alg}}).
\]

Now, we claim the following: Fix $Q_{>0} \ni h < h_K$, if $Q \in Q_p[X]$ with $Q^*(0) \in Q_p^*$ and the slope of $Q$ is $\leq h$, then $Q^*(U_p)$ acts on $C^*_K^{\text{tot}}$ invertibly. Write $Q = a_0 + a_1 X + \cdots + a_n X^n$. The two conditions on $Q$ means
• $a_n \in \mathbb{Q}_p$

• $v_p(a_n) - v_p(a_i) \leq (n - i)h$ for all $i = 0, ..., n - 1$.

Therefore, we have

$$|a_i/a_n| < p^{(n-i)h} \quad \text{and} \quad \frac{a_i}{a_n} U_p^{n-1} < 1.$$ 

Let $P(X) = -\frac{a_0}{a_n} X^n - \frac{a_1}{a_n} X^{n-1} - \cdots - \frac{a_{n-1}}{a_n} X$, then $\frac{1}{a_n} Q^*(X) = 1 - P(X)$. We can deduce that $\|P(U_p)\|_K < 1$ and so $Q^*(U_p)$ acts on $C_{K}^{\text{alg}}$ invertibly with inverse given explicitly by

$$Q^*(U_p)^{-1} = \frac{1}{a_n} \sum_{j \geq 0} P(U_p)^j.$$ 

Now fix $h < h_k$, then we have the corresponding decomposition $F_k = Q_h^1 \sigma^1_h$ and $F_{k}^{\text{alg}} = Q_h^1 C_{k}^{\text{alg}}$ and

$$C_{k}^{\text{tot} \cdot h \ast} \rightarrow C_{k, \text{alg} \ast}$$

with $C_{k}^{\text{tot} \cdot h \ast} = \ker Q_h^1 (U_p |C_{k}^{\text{alg}})$ and $C_{k, \text{alg} \ast} = \ker Q_h^1 (U_p |C_{k, \text{alg}})$. Let $C_{k}^{\text{tot} \cdot h}$ be the kernel of the surjection, then, by taking cohomology, we have the corresponding long exact sequence

$$\cdots \rightarrow H^t(X_{Iw^*}(C), K)^{\geq h} \rightarrow H^t(X_{Iw^*}(C), D_k^1)^{\geq h} \rightarrow \cdots$$

The above claim shows that both $Q_h^1 (U_p)$ and $Q_h^{\text{alg} \ast} (U_p)$ act on $H^t(X_{Iw^*}(C), K)^{\geq h}$ invertibly. Take any $\sigma \in H^t(X_{Iw^*}(C), K)^{\geq h}$, the image of $Q_h^1 (U_p) \sigma$ in $H^t(X_{Iw^*}(C), D_k^1)^{\geq h}$ is zero, thus there exists $\sigma' \in H^{t-1}(X_{Iw^*}(C), \mathcal{V}_{\text{alg}, k}^{\text{alg}, \ast})^{\geq h}$ whose image in $H^t(X_{Iw^*}(C), K)^{\geq h}$ is $Q_h^{1 \ast} (U_p) \sigma$. Since $Q_h^1 (U_p) \sigma' = 0$, thus $Q_h^{\text{alg} \ast} (U_p) Q_h^1 (U_p) \sigma = 0$. This implies $\sigma = 0$ so the desired isomorphism follows.

**Remark 4.3.3.** The above control theorem is basically [AS08, Theorem 6.4.1] with only a slight modification. There is another version of the control theorem by [Urb11, Proposition 4.3.10] (see also [Han17, Theorem 3.2.5]), which gives a more explicit description of the bound $h_k$. However, the control theorem in [Urb11] requires a modification on the Shimura varieties while this is not the case in our version.

**Corollary 4.3.4.** Let $\kappa = k \in \mathbb{Z}_{>0}$ be a dominant algebraic weight. Then the pairing

$$[\cdot, \cdot]_k : H_{\text{par}, k}^{\text{tot} \cdot h} \times H_{\text{par}, k}^{\text{tot} \cdot h} \rightarrow \mathbb{Q}_p$$

is non-degenerate when $h < h_k$.

**Proof.** This is an easy consequence of Proposition 4.3.1 and Theorem 4.3.2.

We conclude the paper by the following immediate consequence of Theorem 4.2.8, Theorem 4.2.9 and Corollary 4.3.4.

**Corollary 4.3.5.** Suppose $x \in \mathcal{C}_0$ is a good classical point, i.e., $x$ satisfies the following conditions

• $x$ is a good point;
• $\text{wt}(x) = k \in \mathbb{Z}_{>0}$ is a dominant algebraic weight; and
• there is a slope datum $(U, h)$ such that $x \in U$ and $h < h_k$.

Then

1. The adjoint $p$-adic $L$-function $L^{\text{adj}}$ vanishes at $x$ if and only if the weight map $\text{wt} : \mathcal{E}_0 \to W$ is ramified at $x$.

2. If $x$ is furthermore a smooth point of $\mathcal{E}_0$, let $e(x)$ be as defined in Theorem 4.2.9, then we have $\text{ord}_x L^{\text{adj}} = e(x)$.

References

[AB59] Maurice Auslander and David Buchsbaum. “On Ramification Theory in Noetherian Rings”. In: American Journal of Mathematics 81.3 (1959).

[AIP15] Fabrizio Andreatta, Adrian Iovita, and Vincent Pilloni. “$p$-adic families of Siegel modular cusp-forms”. In: Annals of Mathematics 181.2 (2015), pp. 623–697.

[AIP18] Fabrizio Andreatta, Adrian Iovita, and Vincent Pilloni. “Le Halo Spectral”. In: Ann. Sci. ENS 51.3 (2018).

[AIS15] Fabrizio Andreatta, Adrian Iovita, and Glenn Stevens. “Overconvergent Eichler–Shimura isomorphisms”. In: Journal of the Institute of Mathematics of Jussieu 14.2 (2015), pp. 221–274. doi: 10.1017/S1474748013000364

[AS08] Avner Ash and Glenn Stevens. $p$-adic deformations of arithmetic cohomology. Preprint. Available at http://math.bu.edu/people/ghs/preprints/Ash-Stevens-02-08.pdf 2008.

[Bar18] Daniel Barrera Salazar. “Overconvergent cohomology of Hilbert modular varieties and $p$-adic $L$-functions”. In: Annales de l’Institut Fourier 68.5 (2018), pp. 2177–2213. doi: 10.5802/aif.3206

[Bel] Joël Bellaïche. The Eigenbook: Eigenvarieties, families of Galois representations, $p$-adic $L$-functions. http://people.brandeis.edu/~jbellaic/preprint/Eigenbook.pdf

[Bel12] Joël Bellaïche. “Critical $p$-adic $L$-functions”. In: Inventiones mathematicae 189 (2012), pp. 1–60. doi: https://doi.org/10.1007/s00222-011-0358-z

[BS73] Armand Borel and Jean-Pierre Serre. “Corners and Arithmetic Groups”. In: Commentarii mathematici Helvetici 48 (1973), pp. 436–483.

[CM98] R. Coleman and B. Mazur. “The Eigencurve”. In: Galois Representations in Arithmetic Algebraic Geometry. Ed. by A. J. Scholl and R. L. Editors Taylor. London Mathematical Society Lecture Note Series. Cambridge University Press, 1998, pp. 1–114. doi: 10.1017/CBO9780511662010.003

[GT05] Alain Genestier and Jacques Tilouine. “Systèmes de Taylor-Wiles pour $\text{GSp}_4$”. In: Formes automorphes (II) - Le cas du groupe $\text{GSp}(4)$. Ed. by Tilouine Jacques et al. Astérisque 302. Société mathématique de France, 2005, pp. 177–290.

[Han12] David Hansen. Pairings on modules of analytic distributions. Unpublished notes. http://www.davidrenshavhansen.com 2012.

[Han17] David Hansen. “Universal eigenvarieties, trianguline Galois representations, and $p$-adic Langlands functoriality”. In: Journal für die reine und angewandte Mathematik (730 2017), pp. 1–64. doi: https://doi.org/10.1515/crelle-2014-0130
[JN19] Christian Johansson and James Newton. “Extended eigenvarieties for overconvergent cohomology”. In: *Algebra and Number Theory* 13.1 (Feb. 2019), pp. 93–158. DOI: [10.2140/ant.2019.13.93](https://doi.org/10.2140/ant.2019.13.93).

[Kim06] Walter Kim. “Ramification Points on the Eigencurve and the Two Variable Symmetric Square $p$-adic $L$-Function”. PhD thesis. 2006.

[Mun84] James R. Munkres. *Elements of Algebraic Topology*. Addison Wesley Publishing Company, 1984.

[Par10] Jeehoon Park. “$p$-Adic family of half-integral weight modular forms via overconvergent Shintani lifting”. In: *manuscripta mathematica* 131 (2010), pp. 355–384. DOI: [https://doi.org/10.1007/s00229-009-0323-y](https://doi.org/10.1007/s00229-009-0323-y).

[Ste94] Glenn Stevens. *Rigid Analytic Modular Symbols*. Preprint. Available at [http://math.bu.edu/people/ghs/research.html](http://math.bu.edu/people/ghs/research.html). 1994.

[Urb11] Eric Urban. “Eigenvarieties for reductive groups”. In: *Annals of Mathematics* 174.3 (2011), pp. 1685–1784.

Concordia University
Department of Mathematics and Statistics
Montréal, Québec, Canada

*E-mail address:* ju-feng.wu@mail.concordia.ca