Glassy properties of the Kawasaki dynamics of two-dimensional ferromagnets

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We study numerically the Kawasaki dynamics of the 2d Ising model. At large time we recover the coarsening growth as $L_c(t) \propto t^{1/3}$. At shorter time however, the system enters a metastable glassy regime that displays an extremely slow growth and non-trivial violations of the fluctuation dissipation theorem similar to those observed in spin glasses: this is one of the simplest system where such violations occur. We also consider Potts models, where a similar behavior is observed, and the model of Shore and Sethna where the domain growth is also slow, but where violations of the fluctuation dissipation theorem are trivial. We finally comment on these violations in the context of activated coarsening, and on similarities and differences with the glass transition phenomenology.

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Last decades have witnessed many progresses in the study of out-of-equilibrium dynamics and coarsening, or domain growth, processes [1]. In many simple models, the dynamics after a quench to the ordered phase is understood in terms of domains growing with time as $t^{1/2}$ with a Glauber (or non conserved magnetization) dynamics, and as $t^{1/3}$ with Kawasaki [2] (or conserved magnetization) dynamics [3]. The 2d Ising model plays an important role in the field as one of the simplest models for coarsening and phase separation. Recently, motivated by spin glass theory, many studies tried to go beyond these considerations to investigate two-time quantities such as correlation and response functions, the relation between them, and the way the dynamic Fluctuation-Dissipation Theorem (FDT) is violated in the out-of-equilibrium regime [2, 3, 4]. A generalization of the FDT can be formulated and amounts, within the mean field theory of aging, to the introduction of an effective temperature $T_{eff}$; it is widely accepted that $T_{eff} = \infty$ for coarsening process [2]. as confirmed by simulations in the Ising model with Glauber dynamics [2]. In glasses and spin glasses however, violations of the FDT are highly non trivial [2, 4, 5]; finding lattice models with such properties has recently triggered a lot of attention [3, 4, 8].

In this letter, we study the 2d Kawasaki Ising model [2]. Even if the domains grow as $t^{1/3}$ at large time [11, 12], we feel interesting to study the short time behavior as pre-asymptotic regimes are already non trivial in the 1d chain [13, 14]. We will demonstrate the existence of a glassy early regime, where domain growth is extremely slow, that displays a violation of the FDT similar to those of more complex frustrated models such as spin glasses or glass models. At large time we recover the Glauber-like behavior. The same phenomena arises with Potts variables but not in the model of Shore and Sethna [15], although it displays a logarithmic growth. We finally discuss our results in the context of glassy dynamics.

Model and methods — We consider the Ising model on a 2d $L^2$ lattice with Hamiltonian $H = -\frac{J}{2}\sum_{\langle i,j \rangle} S_i S_j$, whose critical temperature is $T_c \approx 2.27$. To increase the efficiency of the Monte-Carlo (MC) simulations, we use a few tricks: first, we use the method described in [14], a generalization to Kawasaki dynamics of an algorithm [8, 16] allowing a computation of the linear response to a field without physically adding a magnetic field in the simulation. Many two-time quantities for different waiting times were thus computed in the same run. Secondly, we speed up simulations using the continuous time method of [17]: instead of choosing at random a pair of neighboring spins and to try to exchange them at each MC step, we keep track of all the broken links in memory, choose one at random, increase the clock by the time needed for the system to find it (a Poissonian variable with mean $2L^2/N_b$; time units being in MC steps per site) and finally update the value of the corresponding pair of spins using the Heat Bath method. Since both tricks can be used at the same time, our approach allows a very good numerical determination of correlation and response functions at long time, even for large ($L \approx 10^3$) systems. Finally, we determined the coarsening length $L_c(t)$ by computing the excess energy [12, 14, 18] with respect to equilibrium $L_c(t) = -E_{eq}/(E(t) - E_{eq})$ (where $E_{eq}$ is given by the exact solution [21]; with this definition $L_c(t)$ is $O(1)$ in the high temperature phase) and checked that the length defined as the first zero of the spatial correlation function [11] gives similar results.

Growing length beyond the $t^{1/3}$ law — Let us first consider in details the domain growth process. It is now clear that it behaves as $t^{1/3}$ at late time but first numerical studies failed to find this exponent and, after initial claims in favor of a logarithmic coarsening [10], Huse stressed the importance of activated phenomena and energy barriers [11]: due to the spin exchange dynamics, some moves need activation before happening, which creates different time scales. For instance, zero temperature simulations converge toward highly non trivial configurations far from equilibrium [19], where they remain blocked forever and therefore, since first excitations cost...
an activation energy \( \delta E = 4 \), any finite temperature simulations for a time shorter than \( \tau_1 = e^{4.5} \) will behave as those at \( T = 0 \). Going at time larger than \( \tau_1 \) is not even enough to access the late time regime and in fact it is only when \( t \gg \tau_2 = e^{8.8} \) that the system reaches the proper power-law behavior.

Doing long quenches at many low temperatures, we investigate precisely the behavior of \( l_c(t) \) and summarized our results in FIG.1, where three different regimes for the dynamics after a quench to \( T \ll T_c \) can be observed: (A) the Zero Temperature regime: after a finite time, one reaches the plateau characteristic of the \( T = 0 \) dynamics, (B) the Glassy regime: after an activation time \( \tau_1 \propto e^{4.5} \), the dynamics leaves the plateau and enters a regime with a very slow domain growth. Actually, \( l_c(t) \) seems even sub-logarithmic at large time, suggesting, as first intuited by \([11]\), that \( l_c(t) \) saturates and reaches a second plateau at very large \( t \) (this is however quite invisible at the time scales considered here), and (C) the Asymptotic regime: after a time \( \tau_2 \propto e^{8.8} \), the dynamics enters the canonical regime where \( l_c(t) \approx A + Bt^{1/3} \), as can be seen from the behavior of its effective growth exponent (inset in FIG.1).

This glassy regime, whose name will be justified later, thus corresponds to a metastable state with lifetime \( O(\tau_2) \) that lasts extremely long at low temperature. That \( l_c(t) \) seems to saturate indicates that the system is growing small organized domains that then remain blocked because further growth would need a move costing \( \delta E = 8 \) (i.e. flipping a spin on domain edges). This looks very similar to the polycrystalline picture (a collection of small crystalline zone) observed in the early stages of the dynamics of some model glass \([12]\). We also noticed that the way the system escapes from these metastable states may involve cooperative phenomena and non just a single spin flip of energy 8: modifying the dynamics in such a way that only the moves that cost \( \delta E \leq 4 \) are allowed, we found that the dynamics still converge to the \( t^{1/3} \) regime after a time, although a bit larger, still scaling as \( e^{8.8} \). To emphasize the differences between the dynamics in this glassy regime and the late time coarsening, we turn now to FDT violations.

**Violation of the FDT** — Studies of the Fluctuation-Dissipation Ratio (FDR) \([3, 4]\) are based on the comparison of how spontaneous and induced fluctuations do relax: one measures the two-time correlation function \( C(t, s) = \langle S(t)S(s) \rangle \) and the associated response to a field function \( R(t, s) = \partial \langle S(t) \rangle / \partial h(s) \), and defines the FDR \( X(t, s) \) through the formula \( T R(t, s) = X(t, s) \partial_c C(t, s) \). At equilibrium the FDT holds, thus \( X = 1 \). In the off-equilibrium regime however, the FDT is violated. In the large time limit \( (s, t \to \infty \) with \( C(t, s) \to q \) the FDR \( X(t, s) \) converges to a limiting function \( X(q) \) whose physical meaning has been derived within the mean field theory of aging, where it has been shown \([5]\) that it can be computed using the overlap pdf \( P(q) \) in the thresh-
old states (the states reached by the dynamics on very large time \( t_w \)) using \( X(q) = x(q) \equiv \int_0^q P(q')dq' \). The interpretation of \( X(q) \) in finite \( d \) models, the meaning or even the existence of a unique effective temperature \( T_{eff} = T/X \) play a central role in present research in off-equilibrium physics \[2\]. Experiments and simulations consider integrated quantities, in which case one gets

\[
\chi(t, t_w) = \frac{1 - C(t, t_w)}{T_{eff}},
\]

where \( \chi(t, t_w) \) is the linear susceptibility at time \( t \) that results from a magnetic field switched on from times \( t_w \) to \( t \). A popular presentation of such data \[2\] is a plot of \( T \chi(t, t_w) \) versus the \( C(t, t_w) \): one finds that FDT holds at short time, when \( C(t, t_w) \) is large (in the so-called quasi-equilibrium regime) so that \( T_{eff} = T \) until \( C(t, t_w) \) drops and reaches a given value \( q_{EA} \) \[4\] (equal to the exactly known equilibrium magnetization in the Ising model with Glauber dynamics) and FDT is violated for \( C(t, t_w) < q_{EA} \) where the introduction of \( T_{eff} > T \) is thus needed.

It is widely believed that any coarsening processes is characterized by \( T_{eff} = \infty \). This is quite expected since the \( P(q) \) corresponding to ferromagnetic equilibrium is trivial. Here, however, there is the early regime where the dynamics is reaching metastable states far from the equilibrium ones, so, inspired by the mean field interpretation of FDT violations, we expect that in the glassy regime they are not ruled by the equilibrium ferromagnetic \( P(q) \) but by the glassy metastable states so that the FDR is non trivial. We present our data for \( T = 0.7T_c \) in FIG. 3a: at large time, the curves converge very slowly toward the expected limit, approaching it downward as in the Glauber case (a direct comparison with \[6\] is instructive). This excess response signal can be understood by noticing that spins on domain walls have low local fields and thus participate a lot to the global magnetic response. At short time \( l_c(t) \) is quite low so there is a large excess density of domains that eventually decreases at larger time as domains annihilate: this gives rise to the bump in the FDT plot and explain the slow downward convergence of the susceptibility as \( t_w \) increases. All this is similar with what happens with Glauber dynamics \[7\].

However, a new phenomenon arises at shorter time: in FIG. 3a, curves are first drifting up as the waiting time \( t_w \) increases. This is in sharp contrast with what is observed with Glauber dynamics and quite similar to spin glass behavior \[8\]: in this regime the system is somehow creating more response as \( t_w \) increases. These phenomena arise for time scale \( O(t_2) \approx 10^2 \); this suggests that, staying long enough in the glassy regime, by working at a lower temperature, we should observe clearer FDT violations. This is indeed the case (see FIG. 2b) at \( T = 0.25T_c \) (where \( t_2 \approx 10^6 \)) where we observe a superposition of FDR curves on more than two decades and obtain a plot very similar to those of complex systems as glasses and spin glasses. This shows that, between times \( \tau_1 \) and \( \tau_2 \), the dynamics of the Kawasaki model in the glassy regime is very sensitive to the metastable states and, as expected, exhibits a non trivial FDR. It is interesting that ideas arising in mean field disordered complex systems are realized in a certain time range in the 2d Ising model, which as opposed to other simple models where FDT violations where observed \[3,10,20\], is not critical or close to a critical point, not disordered and not even statically frustrated. Despite that, it has the striking properties of behaving like a ferromagnet at long time but as a spin glass at short time, suggesting that any coarsening model with similar activated regimes could behave like a glass at short times.

**Logarithmic coarsening** — To investigate the generality of our results, we followed Shore and Sethna (SS) \[15\] who introduced a 3d Ising model with ferromagnetic first neighbors but weak anti-ferromagnetic second neighbors interactions, and argued that it has barriers growing with the size \( l_c(t) \) of the ferromagnetic domains, thus leading to an activated logarithmic coarsening \[13\]. We repeated our FDT study in the range of temperature and parameters where SS claimed to observe a logarithmic growth and computed \( l_c(t) \) using the same definition as \[15\]. We present our results in FIG. 3 although we find data compatible with a logarithmic \( l_c(t) \) at large time, \( \chi \) is very small, the FDR seems trivial and all our FDT plots look like what is obtained for usual coarsening process \[2\]; in particular all curves are shifting downward as \( t_w \) increases, as opposed to FIG. 2a. This demonstrates that it is not just the slowly growing length that creates the non trivial FDR in FIG. 2. There, the dynamics clearly differs from usual coarsening, even logarithmic, and reaches some non trivial quasi-equilibrium states that seem stable until times of \( O(t_2) \); this is in contrast with the logarithmic coarsening of the SS model which is just going slowly toward the ferromagnetic state, and where no such equilibration in a metastable state could be defined.
Potts variables and glasses — The present study is easily extended to the q-states Potts model \[21\] (where \(T_c = q/\log (1 + \sqrt{q})\)) and we checked that a similar phenomenology for the FDR and the slow dynamics arises: the same activated processes give rise to glassy behavior (see FIG 4). This is interesting as Potts models have a same activated processes give rise to glassy behavior for \(q > 1\), hence we obtain a model with many features of the glass phenomenology such as FDT violations, melting transition and metastable states. Moreover, like other models with much more complicated Hamiltonians recently developed to study analytically the glass transition \[22\], our Ising or Potts magnets with Kawasaki dynamics have a non trivial mean-field limit on Bethe lattices, where they display a spin glass behavior (within the cavity method \[23\], the spin glass solution applies to ferromagnetic couplings when the magnetization is fixed to zero) so we may ask if this model is pertinent as a model glass and how it compares with \[22\] ? As in any realistic models, the low \(T\) phase eventually crystalizes at large time once the free energy barriers to crystal nucleation are crossed (if the nucleation time is shorter than the equilibration time of the metastable glass state, the later cannot be thermodynamically defined: this is the solution Kautzmann originally proposed to its own paradox \[24\]). A lifetime of \(\mathcal{O}(e^{8.3})\) being not very large close to \(T_c\), it is quite easy in this model to overcome these barriers in a cooling experiment so that no supercooled state and no glass transition can be observed (this is also what happens in the model of \[4\], obtained with a very frustrated plaquette Hamiltonian): here, it is only by quenching the system fast enough to low \(T\) that it shows its glassy nature. Precautions are thus needed when using mean field methods; it suggests however that a similar model with barriers large enough not be overcome in the time scale of the simulation would be a good model; this is probably what happens in models \[22\] which are believed to be good glass formers.

Discussion — We studied the off-equilibrium dynamics of 2d Kawasaki magnets and demonstrated the existence of a early extremely slow glassy regime with a non trivial violation of the FDT similar to those of spin glasses. We observed the same phenomenology with Potts variables but not in the SS model, pointing out the difference between this dynamical behavior and the usual coarsening, however slow it may be. We believe that this behavior is quite generic and should be observed (as it has already been in some specific cases \[4, 10\]) in the early stage of other coarsening model (provided that they share similar activation properties) or in some lattice model for glasses. Since non trivial FDT violations have been observed in many systems \[4\], it is interesting that a simple model such as the 2d Ising ferromagnet, whose statics solution is analytically known for decades, has a dynamics that already shares many properties with those of much more complex materials: very few ingredients are needed for that. Finally our results show, as was pointed out in \[20\], that great cares should be taken when using dynamical simulations to probe the statics.

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