ON THE DISTRIBUTION OF GRAVITATIONAL ENERGY
IN THE DE SITTER SPACE

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Abstract

We calculate the total gravitational energy and the gravitational energy density of the de Sitter space using the definition of localized gravitational energy that naturally arises in the framework of the teleparallel equivalent of general relativity. We find that the gravitational energy can only be defined within the cosmological horizon and is largely concentrated in regions far from the center of spherical symmetry, i.e., in the vicinity of the maximal spacelike radial coordinate $R = \sqrt{\frac{3}{\Lambda}}$. The smaller the cosmological constant, the farther the concentration of energy. This result complies with the phenomenological features of the de Sitter solution, namely, the existence of a radial acceleration directed away from the center of symmetry experienced by a test particle in the de Sitter space. Einstein already contemplated the de Sitter solution as a world with a surface distribution of matter, a picture which is in agreement with the present analysis.

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I. Introduction

The difficulty in defining gravitational energy density within the framework of the Hilbert-Einstein Lagrangian formulation has led to the belief that the gravitational energy cannot be localized. It is widely assumed that an expression for the localized energy density of the gravitational field does not exist. However it is well known that the total energy of a given asymptotically flat spacetime can be calculated by means of pseudotensor methods, which make use of coordinate dependent expressions. A different approach to the construction of an energy expression for the gravitational field is based on the idea of quasilocal energy. The quasilocal definition of energy, momentum and angular momentum associates these quantities to an arbitrary spacelike two surface $S$ in an arbitrary spacetime manifold. The inexistence of an unequivocal definition of gravitational energy still remains an actual problem, important in its own right. Furthermore such definition may play a major role in the thermodynamics of self-gravitating systems. This problem has been recently addressed in ref. [1], where a comprehensive bibliography on quasi-local energy is presented. Although all attempts so far have led to interesting mathematical developments, they did not allow the achievement of a definite solution, either because of conceptual or mathematical difficulties.

Recently the problem of localization of energy in general relativity has been reconsidered in the framework of the teleparallel equivalent of general relativity (TEGR) [2]. The Lagrangian formulation of the TEGR is established by means of the tetrad field $e^a_\mu$ and the spin affine connection $\omega_{\mu ab}$, which are taken to be completely independent field variables, even at the level of field equations. This formulation has been investigated in the past in the context of Poincaré gauge theories [3, 4]. However, as we will explain ahead, this is not an alternative theory of gravity. This is just an alternative formulation
of general relativity, in which the curvature tensor constructed out of $\omega_{\mu ab}$ vanishes, but the torsion tensor is non-vanishing. The physical content of the theory is dictated by Einstein’s equations. As we will show, in this alternative geometrical formulation the gravitational energy density can be naturally defined.

The expression for the localized energy density of the gravitational field has arisen in the context of the Hamiltonian formulation of the TEGR\cite{5}. It has been demonstrated that under a suitable gauge fixing of $\omega_{\mu ab}$, already at the Lagrangian level, the Hamiltonian formulation of the TEGR is well defined\cite{5}. The resulting constraints are first class constraints. In fact the Hamiltonian formulation looks very much similar to the the usual ADM formulation\cite{3}. However there are crucial differences. The integral form of the Hamiltonian constraint equation $C = 0$ in the TEGR can be written in the form $C = H - E_{ADM} = 0$, when we restrict considerations to asymptotically flat spacetimes\cite{2}. The quantity $\varepsilon(x)$ which appears in the expression of $C$ and which under integration yields $E_{ADM}$ is recognized as the gravitational energy density. We have applied the expression of $\varepsilon(x)$ to the calculation of the energy inside a surface of constant radius, both for the Schwarzschild\cite{2} and for the Kerr metric\cite{7}, and the results are remarkably the same as those obtained by means of the quasilocal energy definition proposed by Brown and York\cite{1}. Moreover, the calculational scheme is rather simple, as we will see shortly, and is exempt of some complications inherent to the latter. Therefore for asymptotically flat spacetimes the gravitational energy density has a definite and unambiguous expression within the framework of the TEGR.

We recall that the gravitational energy can also be calculated by means of the surface term that appears in the expression of the gravitational Hamiltonian\cite{8, 4}. However, such surface term yields only the total gravitational energy, as the integration has to be necessarily carried out over the whole three dimensional spacelike hypersurface, in which
case the lapse function $N(x)$ goes over into its asymptotic value $N \to 1$ at spatial infinity.

The action integrals for spacetimes with different topologies require surface terms that are specific to each topology. Thus the corresponding Hamiltonian also acquires a surface term that is determined by the topological boundary conditions\[10\]. However the Hamiltonian constraint for a spacetime foliated by spacelike hypersurfaces always has the same basic structure, irrespective of boundary conditions (additional terms such as the cosmological constant may appear in the Hamiltonian constraint, as we will see ahead in eq.(10)).

Therefore the question immediately arises as to whether the Hamiltonian constraint equation in the TEGR can always be written as $C = H - E = 0$, in which case $\varepsilon(x)$ would be the gravitational energy density for any curved spacetime.

One of the simplest deviations from asymptotically flat geometries are spacetimes with conical defects. We have applied our expression of gravitational energy density to the calculation of the energy per unit length of defects of topological nature, which include disclinations, i.e., cosmic strings, and dislocations\[11\]. The result is quite encouraging. We arrive at precisely the same well known expression for the energy per unit length of a cosmic string (not even multiplicative factors have to be adjusted). Moreover the total energy of a dislocation is zero, a result which is in close analogy with the statements of the theory of elasticity, which asserts that disclinations and dislocations are defects which require high energy and low energy, respectively.

In this paper we consider the de Sitter space, which is topologically of the type $S^3 \times R$. We restrict the considerations to the static region within the cosmological horizon (i.e., the region for which $-g_{00} > 0$) and calculate both the total energy and the distribution of energy along the radial direction. Again the result is rather remarkable. We will show that the cosmological constant induces a distribution of gravitational energy in such a
way that the energy is largely concentrated in the peripheral region, i.e., in the vicinity of the maximal spacelike radial coordinate \( R = \sqrt{\frac{3}{\Lambda}} \). As we will show in section III, this picture is in total agreement with the phenomenological features of the de Sitter solution, and is as well in agreement with Einstein’s belief, according to which the de Sitter’s solution represents a spacetime with a surface distribution of matter. This fact strongly supports the validity of our expression for the gravitational energy density and also represents a clear indication that the Hamiltonian constraint equation in the TEGR can be unambiguously interpreted as an energy equation of the type \( H - E = 0 \).

Notation: spacetime indices \( \mu, \nu, \ldots \) and local Lorentz indices \( a, b, \ldots \) run from 0 to 3. In the 3+1 decomposition Latin indices from the middle of the alphabet indicate space indices according to \( \mu = 0, i, \ a = (0), (i) \). The tetrad field \( e^a{}_{\mu} \) and the spin connection \( \omega_{\mu a b} \) yield the usual definitions of the torsion and curvature tensors: \( R^a{}_{b\mu\nu} = \partial_{\mu} \omega^a{}_{\nu b} + \omega^a{}_{\mu c} \omega^c{}_{\nu b} - \ldots \), \( T^a{}_{\mu\nu} = \partial_{\mu} e^a{}_{\nu} + \omega^a{}_{\mu b} e^b{}_{\nu} - \ldots \). The flat spacetime metric is fixed by \( \eta_{(0)(0)} = -1 \).

II. The Lagrangian and Hamiltonian formulations of the TEGR

In the TEGR the tetrad field \( e^a{}_{\mu} \) and the spin connection \( \omega_{\mu a b} \) are independent field variables, not related by any of the field equations. The spin connection is enforced to satisfy the condition of zero curvature. The Lagrangian density in empty spacetime is given by

\[
L(e, \omega, \lambda) = -ke\left(\frac{1}{4} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{bac} - T^{aT_a}\right) + e\lambda^{ab\mu} R_{ab\mu
u}(\omega).
\]
where \( k = \frac{1}{16\pi G} \), \( G \) is the gravitational constant; \( e = \det(e^a{}^\mu) \), \( \lambda^{ab\mu\nu} \) are Lagrange multipliers and \( T_a \) is the trace of the torsion tensor defined by \( T_a = T^b{}_{ba} \).

The equivalence of the TEGR with Einstein’s general relativity is guaranteed by the identity

\[
eR(e, \omega) = eR(e) + e\left(\frac{1}{4} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{acb} - T^a T_a \right) - 2\partial_\mu (e T^\mu) ,
\]

which is obtained by just substituting the arbitrary spin connection \( \omega_{\mu ab} = \omega_{\mu ab}(e) + K_{\mu ab} \) in the scalar curvature tensor \( R(e, \omega) \) in the left hand side of (2); \( \omega_{\mu ab}(e) \) is the Levi-Civita connection and \( K_{\mu ab} = \frac{1}{2} e_a \lambda e_b \nu (T_{\lambda \mu \nu} + T_{\nu \lambda \mu} - T_{\mu \nu \lambda}) \) is the contorsion tensor.

The vanishing of \( R^a{}_{b\mu\nu}(\omega) \), which is one of the field equations derived from (1), implies the equivalence of the scalar curvature \( R(e) \), constructed out of \( e^a{}^\mu \) only, and the quadratic combination of the torsion tensor. It also ensures that the field equation arising from the variation of \( L \) with respect to \( e^a{}^\mu \) is strictly equivalent to Einstein’s equations in tetrad form. Let \( \frac{\delta L}{\delta e_{\mu a}} = 0 \) denote the field equation satisfied by \( e_{\mu a} \). It can be shown by explicit calculations that

\[
\frac{\delta L}{\delta e_{\mu a}} = \frac{1}{2} \{ R_{\mu a} - \frac{1}{2} e_{\mu a} R(e) \} .
\]

(we refer the reader to ref.\[5\] for additional details).

For asymptotically flat spacetimes the total divergence in (2) does not contribute to the action integral. Therefore the latter does not require additional surface terms, as it is already invariant under coordinate transformations that preserve the asymptotic structure of the field quantities\[9\]. It is well known that for compact geometries a surface term has to be included in the action, in order to make the variations of the field variables well defined. This surface term is constructed out of the trace of the extrinsic curvature on
the boundary. However we will no longer worry about surface terms in the Lagrangian or in the Hamiltonian, as we will be interested only in the constraint structure of the theory.

The Hamiltonian formulation of the TEGR can be successfully implemented if we fix the gauge $\omega_{0ab} = 0$ from the outset, since in this case the constraints constitute a first class set. The condition $\omega_{0ab} = 0$ is achieved by breaking the local Lorentz symmetry of (1). We still make use of the residual time independent gauge symmetry to fix the usual time gauge condition $e_{(k)0} = e_{(0)i} = 0$. Because of $\omega_{0ab} = 0$, $H$ does not depend on $P^{kab}$, the momentum canonically conjugated to $\omega_{kab}$. Therefore arbitrary variations of $L = p\dot{q} - H$ with respect to $P^{kab}$ yields $\dot{\omega}_{kab} = 0$. Thus in view of $\omega_{0ab} = 0$, $\omega_{kab}$ drops out from our considerations. The above gauge fixing can be understood as the fixation of a global reference frame.

Under the above gauge fixing the canonical action integral obtained from (1) becomes

$$A_{TL} = \int d^4x \{ \Pi_{(j)k} \dot{e}_{(j)k} - H \},$$

(3)

$$H = NC + N^i C_i + \Sigma_{mn} \Pi^{mn}. \quad (4)$$

In expression (4) above we are omitting surface terms. $N$ and $N^i$ are the lapse and shift functions, $\Pi^{mn} = e_{(j)m} \Pi^{(j)n}$ and $\Sigma_{mn} = -\Sigma_{nm}$ are Lagrange multipliers. The constraints are defined by

$$C = \partial_j (2keT^j) - ke\Sigma^{kij} T_{kij} - \frac{1}{4ke}(\Pi^{ij}\Pi_{ji} - \frac{1}{2}\Pi^2),$$

(5)

$$C_k = -e_{(j)k} \partial_1 \Pi_{(j)i} - \Pi^{(j)i} T_{(j)ik},$$

(6)

with $e = \text{det}(e_{(j)k})$ and $T^i = g^{ik} e^{(j)l} T_{(j)lk}, \ T_{(j)lk} = \partial_i e_{(j)k} - \partial_k e_{(j)i}$. We remark that (3)
and (4) are invariant under global SO(3) and general coordinate transformations (in eqs. (1) and (2) $e$ is the determinant of the spacetime tetrad field; from eq.(3) on $e$ stands for the determinant of the triads restricted to the three dimensional spacelike hypersurface).

If we assume the asymptotic behaviour $e_{i j} \approx \eta_{i j} + \frac{1}{2} h_{i j} \left( \frac{1}{r} \right)$ for $r \to \infty$, which is appropriate for an asymptotically flat spacetime, then in view of the relation

$$\frac{1}{8\pi G} \int d^3 x \partial_j (e T^j) = \frac{1}{16\pi G} \int_S dS_k (\partial_r h_{ik} - \partial_k h_{ri}) \equiv E_{ADM}$$

(7)

where the surface integral is evaluated for $r \to \infty$, we note that the integral form of the Hamiltonian constraint $C = 0$ may be rewritten as

$$\int d^3 x \left\{ k e \Sigma^{kij} T_{kij} + \frac{1}{4 ke} (\Pi^{ij} \Pi_{ji} - \frac{1}{2} \Pi^2) \right\} = E_{ADM}.$$ 

(8)

The integration is over the whole three dimensional space. Given that $\partial_j (e T^j)$ is a scalar density, from (7) and (8) we define the gravitational energy density enclosed by a volume $V$ of the space as

$$E_g = \frac{1}{8\pi G} \int_V d^3 x \partial_j (e T^j).$$

(9)

It must be noted that this expression is also invariant under global SO(3) transformations.

One is immediately led to ask whether the Hamiltonian constraint for topologically different spacetimes can also be written as eq.(8). In the next section we will consider the de Sitter space. Before addressing the latter, let us recall here some applications of $E_g$. We have calculated the gravitational energy inside a surface of constant radius $r_o$ both for the Schwarzschild[2] and for the Kerr solution[7], using Boyer and Lindquist coordinates[13, 14]. These quantities have also been calculated by means of Brown and York’s procedure, in refs.[1] and [15], respectively. The expressions found by using (9)
are in total agreement with those obtained via the method of ref.\[1\]. Moreover $E_g$ can be calculated for any volume in the three-dimensional spacelike hypersurface, as least through numerical integration, whereas the evaluation of the energy in ref.\[15\] can only be carried out in the limit of slow rotation of the black hole (the application of Brown and York’s procedure to the Kerr solution with arbitrary parameters meets some technical difficulties, as discussed in ref.\[15\]).

Definition (9) has also been applied to a class of conical spacetime defects, in which disclinations (cosmic strings) and dislocations are considered altogether. For the spacetime of a single cosmic string, i.e., for a pure disclination, we obtain precisely the well known value of energy per unit length of the string\[11\]. Furthermore the total gravitational energy for a pure dislocation vanishes. This is a very interesting result, because we know from the theory of elasticity that disclinations are defects that require a large amount of energy to be formed, whereas dislocations require low energy (see sections 6.3.2 and 6.5 of ref.\[16\] for a discussion as to why the energy demanded for the formation of a disclination in a crystal is very high). Therefore the above calculations of energy are in close agreement with the statements of the theory of elasticity (in this respect we recall that attempts were made long time ago which envisaged the spacetime as a continuum with microstructure (see\[4\], section 1.2)).

III. Gravitational Energy in the de Sitter Space

We will consider now the theory defined by the Lagrangian density (1) supplemented by a term containing the cosmological constant $\Lambda$. Thus we add to (1) the quantity $2^4e\Lambda$, where $^4e = Ne$ is the determinant of the spacetime tetrad field $e_{\mu\nu}$. This additional term will contribute to the action integral (3) only as an extra term of the Hamiltonian
constraint. The new Hamiltonian constraint reads

\[ C = \partial_j(2keT^j) - ke\Sigma^{kij}T_{kij} - \frac{1}{4ke}(\Pi^{ij}\Pi_{ji} - \frac{1}{2}\Pi^2) - 2e\Lambda, \quad (10) \]

The most general spherically symmetric solution of the field equations with a positive cosmological constant is the Schwarzschild-de Sitter solution (throughout this section we will make \( G = 1 \)):

\[ ds^2 = -(1 - \frac{2m}{r} - \frac{r^2}{R^2})dt^2 + \left(1 - \frac{2m}{r} - \frac{r^2}{R^2}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2. \quad (11) \]

This metric represents the gravitational field of a particle of mass \( m \) located at the origin of a globally hyperbolic spacetime. The vacuum solution, obtained by setting \( m = 0 \) in (11), is the de Sitter solution. \( R \) is the maximal spacelike radial coordinate for the (vacuum) de Sitter space and is given by \( R = \sqrt{\frac{2}{\Lambda}}. \)

Strictly speaking de Sitter spacetime corresponds to a four-dimensional surface in a flat five-dimensional space with metric \((-,-,+,+,-,+)\) described by

\[-z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_4^2 = \frac{3}{\Lambda}, \quad \Lambda > 0.\]

The coordinates \((t, r, \theta, \phi)\) in (11) cover only half of the space defined by the relation above. However we will be interested just in (11), as it suffices for our purposes. Moreover we will restrict the considerations to the physical region between the Schwarzschild (black hole) and the cosmological horizons.

Expression (9) allows us to calculate the gravitational energy for any volume in space. We wish to obtain the energy contained within a surface of constant radius \( r_o \). For this purpose we will calculate \( eT^1 = eT^r \) for a spacetime whose spacelike section is described by the line element
\[ dl^2 = \alpha^2 dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \]  

(12)

where \( \alpha \) is a function of the coordinate \( r \). The triads that correspond to (12) are given by

\[
e_{(k)ij} = \begin{pmatrix}
\alpha \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\
\alpha \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\
\alpha \cos \theta & -r \sin \theta & 0
\end{pmatrix}.
\]

(13)

\((k)\) is the line index, and \( i \) is the column index.

The determinant \( e \) of (13) reads \( e = \alpha r^2 \sin \theta \). After a lengthy but otherwise straightforward calculation of

\[
e T^1 = e \ g^{1j} g^{im} e_{(k)mT(k)ij}
\]

we arrive at

\[
e T^1 = 2 \ r \sin \theta \left( 1 - \frac{1}{\alpha} \right) .
\]

(14)

Therefore for the Schwarzschild-de Sitter solution we have

\[
e T^1 = 2 \ r \sin \theta \left( 1 - \sqrt{1 - \frac{2m}{r} - \frac{r^2}{R^2}} \right).
\]

(15)

The energy contained within a surface of constant radius \( r_o \) is thus given by

\[
E_g = \frac{1}{8\pi} \int_S d\theta d\phi e T^1 = r_o \left( 1 - \sqrt{1 - \frac{2m}{r_o} - \frac{r_o^2}{R^2}} \right),
\]

(16)

where \( S \) is a surface of constant radius \( r_o \).

Let us evaluate expression (16) for the range of values of \( r_o \) such that
in which case we assume the cosmological constant to be very small. Expanding (16) and
neglecting all powers of both \( \frac{2m}{r_o} \) and \( \frac{r_o^2}{R^2} \) we arrive at

\[
E_g = r_o + m = E_{dS} + m .
\]

(17)

\( E_{dS} \) is the value of energy we would obtain in the absence of the mass \( m \), and therefore
it represents the background (vacuum) energy. Upon subtraction of this term we obtain
the standard ADM value of energy for a particle of mass \( m \). Of course in (17) we expect
\( r_o \) to be much larger than \( m \).

The total gravitational energy contained within the cosmological horizon can be easily
calculated, but for this purpose one has to find the roots of the equation
\( 1 - \frac{2m}{r} - \frac{r^2}{R^2} = 0 \).

The result is not illuminating. If \( R \gg m \) we find that \( E_g^{total} = r_{max} \), where \( r_{max} \) is
slightly smaller than \( R \) and is a solution of the equation above. In what follows we will
rather consider the vacuum de Sitter solution only, since in this case the analysis of the
energy density is most easily carried out, and the main features are not altered by the
introduction of a mass \( m \) at \( r = 0 \).

Before proceeding we mention that the present analysis is different from that carried
out by Abbott and Deser [17]. These authors provide an expression for the energy of the
gravitational field \( about \) the de Sitter background, i.e., they calculate the energy of a
field configuration that deviates from the de Sitter metric and which vanish at infinity. In
contrast, by means of expression (9) we can compute the energy of the whole gravitational
field configuration, including the background.

The total gravitational energy \( E_{dS} \) contained in the physical region of the vacuum de
Sitter space is obtained from (16) by making $m = 0$ and $r_o = R$:

$$E_{ds} = R = \sqrt{\frac{3}{\Lambda}}. \quad (18)$$

The total volume of the compact spacelike section equals $2\pi^2 R^3$. Therefore the average energy density is given by

$$\frac{E_{ds}}{2\pi^2 R^3} = \frac{\Lambda}{6\pi^2}. \quad (19)$$

Let us obtain now the distribution of gravitational energy in the de Sitter space. In view of the spherical symmetry we will be interested in calculating the density of energy per unit radial distance $\varepsilon(r)$, which is obtained by integrating $\frac{1}{8\pi} \partial_r (e T^1)$ in $\theta$ and $\phi$. Thus $\varepsilon(r)$ yields the gravitational energy contained between the spherical shells of radii $r$ and $r + dr$. Considering $m = 0$ in (15) we obtain upon integration in the angular variables and differentiation in $r$:

$$\varepsilon(r) = 1 + \frac{2\beta^2 - 1}{\sqrt{1 - \beta^2}}. \quad (20)$$

where we have set $\beta^2 = \frac{r^2}{R^2}$. The function $\varepsilon(r)$ has the following properties. In the range $0 \leq \beta \leq 1$ $\varepsilon(r)$ vanishes only for $r = 0$. Moreover for $\beta = 1$ it diverges: $\varepsilon(\beta = 1) \to \infty$.

It is straightforward to check that this is a monotonically increasing function, largely concentrated in the vicinity of $\beta = 1$: $\varepsilon(\beta = 0.1) = 0.015$; $\varepsilon(\beta = 0.5) = 0.423$; $\varepsilon(\beta = 0.9) = 2.422$. The total energy contained inside the surfaces of radii $0.1R$, $0.5R$, $0.9R$ are given by $E_g = 5.01 \times 10^{-4} R$, $E_g = 0.067 R$, $E_g = 0.51 R$, respectively.

Therefore almost half of the gravitational energy is located between $\beta = 0.9$ and $\beta = 1$. This result is in total agreement with the phenomenological features of the de Sitter solution, and can be verified in the following way. The $g_{00}$ component of (11) can
be written as

\[ g_{00} = 1 + 2\phi , \]

where \( \phi \) is given by

\[ \phi = -\frac{m}{r} - \frac{1}{6}\Lambda r^2 . \]

\( \phi \) is the potential in classical mechanics which would induce motion of a test particle approximately along the geodesics of (11). Therefore even in the absence of a mass \( m \) a test particle would be subject to a radial acceleration

\[ a = \frac{1}{3}\Lambda r , \]

directed away from \( r = 0 \).

The acceleration increases with the distance \( r \), indicating that the gravitational field is more intense at points far from the origin. Therefore when \( m = 0 \) the gravitational given by (11) acts on physical bodies as if there were a radially symmetric distribution of matter about the origin, beyond the cosmological horizon, just as \( m \) represents the mass of a black hole, concentrated inside the black hole horizon.

This is precisely the picture we obtain from (20). By applying (9) to the de Sitter solution we find that the cosmological constant induces a spherically symmetric distribution of gravitational energy, concentrated in regions distant from the origin, due to the gravitational field that acts on a test particle placed in the vacuum de Sitter space. Such a field can be thought as due to some matter distribution.

One may think of (11) as representing the gravitational field of a spherical cavity inside some spherically symmetric distribution of matter (this idea is discussed, for instance, in
ref.\textsuperscript{18}). In this respect we recall that Einstein already conjectured that de Sitter solution would correspond to a world with a surface distribution of matter\textsuperscript{12}. Such conjecture has found a natural explanation within the present geometrical framework, and shows that (9) yields a consistent expression for the gravitational energy in the de Sitter space.

We will briefly discuss how our procedure applies to the anti-de Sitter solution. The latter is obtained by making the replacement \( \frac{r^2}{R^2} \rightarrow -\frac{r^2}{R^2} \) in (11). The anti-de Sitter space is a non-compact manifold with constant negative curvature. The energy contained within a surface of constant radius \( r_o \) can be easily calculated and reads

\[
E_g = r_o \left( 1 - \sqrt{1 + \frac{r_o^2}{R^2}} \right),
\]

where we have ignored the mass term \( m \). \( r_o \) ranges from 0 to \( \infty \). Therefore as \( r_o \rightarrow \infty \), we find that \( E_g \rightarrow -\infty \). This is an expected result, since the anti-de Sitter space is non-compact. The density of energy per unit radial distance \( \varepsilon(r) \) in this case is given by

\[
\varepsilon(r) = 1 - \frac{1 + \frac{2r^2}{R^2}}{\sqrt{1 + \frac{r^2}{R^2}}}. \tag{22}
\]

We find that \( \varepsilon(r) = 0 \) only for \( r = 0 \). This point is also the only global maximum for \( \varepsilon(r) \); for \( r \rightarrow \infty \) we clearly see that \( \varepsilon(r) \rightarrow -\infty \). Thus \( \varepsilon(r) \) is a non-positive monotonically decreasing function.

\textbf{IV. Discussion}

The definition of gravitational energy is a long-standing problem in the theory of general relativity. Numerous attempts have been made in the past for a solution. This problem still attracts considerable attention in the literature, and remains an important issue to be settled. Essentially all of these previous attempts are in one or another way unsatisfactory. In particular it is widely claimed that the gravitational energy cannot be localized. We
do not share this opinion. The mathematical structure of the TEGR shows that not only we do have a consistent and unambiguous definition of gravitational energy for asymptotically flat spacetimes, naturally built in the Hamiltonian formulation, but also that the gravitational energy is localized. The gravitational energy in the framework of the TEGR is given by expression (9). This expression has been successfully applied to a number of spacetimes, as we mentioned in section II, whose gravitational energy is already known. A justification for the extension of this definition to more general spacetimes is not straightforward. In the case of asymptotically flat spacetimes the Hamiltonian constraint equation can be written as $C = H - E_{ADM} = 0$. We assume that this form of the constraint is a general feature of the theory, namely, that we can write the Hamiltonian constraint as $C = H - E$ for an arbitrary spacetime, since the constraint structure in general relativity is fixed and does not depend on any particular topology.

In the above we considered the de Sitter solution and concluded that the cosmological constant induces a distribution of gravitational energy largely concentrated in the vicinity of the maximal spacelike radial distance $R$. This result is in total agreement with the fact that a test particle in the de Sitter space is subject to a radial acceleration directed away from the center of symmetry. Therefore the outcome of our analysis complies with the phenomenological behaviour of a test particle in the de Sitter space. To our knowledge this is the first time that such analysis has been provided.

By inspecting equation (18) we see that if we make $\Lambda \to 0$ the total energy $E_g$ diverges. The vanishing of $\Lambda$ in (11) amounts to a change from a compact to a non-compact topology. Therefore such a change is not smooth, as it requires an infinite amount of energy. This fact seems to indicate that, at the classical level, topology changing processes are forbidden.
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