DYNAMICAL FRICTION IN CUSPY GALAXIES

M. ARCA-SEDDA AND R. CAPUZZO-DOLCETTA
Department of Physics, Sapienza, Università di Roma, P.le A. Moro 5, I-00185 Roma, Italy

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ABSTRACT
In this paper, we treat the problem of the dynamical friction decay of a massive object moving in an elliptical galaxy with a cuspidal inner distribution of the mass density. We present results obtained by both self-consistent, direct summation, N-body simulations, as well as by a new semi-analytical treatment of dynamical friction valid in such cuspy central regions of galaxies. A comparison of these results indicates that the proposed semi-analytical approximation is the only reliable one in cuspy galactic central regions, where the standard Chandrasekhar’s local approximation fails and also gives estimates of decay times that are correct at 1% with respect to those given by N-body simulations. The efficiency of dynamical friction in cuspy galaxies is found definitively higher than in core galaxies, especially on more radially elongated satellite orbits. As another relevant result, we find a proportionality of the dynamical friction decay time to the −0.67 power of the satellite mass, M, shallower than the standardly adopted M−1 dependence.

Key words: galaxies: elliptical and lenticular, cD – galaxies: star clusters: general – Galaxy: kinematics and dynamics – globular clusters: general – methods: numerical – stars: kinematics and dynamics

1. INTRODUCTION
Gravitational encounters between a massive body and a sea of light particles, such as a globular cluster moving in a galaxy, leads to a braking of the motion of the satellite, widely known as dynamical friction.

Dynamical friction plays a crucial role in several astronomical contexts: from large scales, since it drives the motion of galaxies in galaxy clusters, and to smaller scales, due to the consequences of this mechanism on the motion of black holes (BHs) and star clusters in galaxies.

The satellite mass and its orbit, together with the geometry of the system in which it moves, are relevant in the determination of the braking effect. As an example, the geometry of the galaxy plays a crucial role leading to significant different efficiency of this mechanism in spherical, axisymmetric, and triaxial galaxies (Chandrasekhar 1943a; Binney 1977; Pesce et al. 1992). Moreover, the presence of a cusp in the background matter distribution could affect it (Merritt 2006; Vicari et al. 2007).

Actually, the existence of “cuspy” density profiles of matter in galaxies has been argued in the last 20 yr as a result of high-resolution observations by the Hubble Space Telescope.

Many galaxies exhibit, indeed, in the inner region, a luminosity profile steeply increasing toward their geometrical center, at least within the telescope resolution.

In general, these luminosity profiles are well described by the so-called Sérsic profiles (Sérsic 1963)

\[ \ln I(R) = \ln I(0) - k R^{1/n}, \]

where \( R \) is the projected radial coordinate and \( n > 0 \), called the “Sérsic index,” controls the steepness of the profile. Brighter galaxies have larger best-fit values of \( n \) (\( n = 4 \) corresponds to the de Vaucouleurs (1948) fit to giant ellipticals profiles); dwarf galaxies are characterized by smaller values of \( n \).

Defining \( \Gamma \) as the logarithmic derivative of the luminosity profile:

\[ \Gamma(R) = \frac{d \ln I}{d \ln R}, \]

the brightness profile slope of Sérsic’s model is

\[ \frac{d I}{d R} = \frac{I(R)}{R} \Gamma(R) = -\frac{I(R)}{n} R^{-\frac{1}{n}}, \]

so that \( n > 1 \) correspond to a “true” cuspidal central brightness profile \( (d I/d R \rightarrow -\infty \text{ for } R \rightarrow 0) \).

In the innermost (3–10′′) regions of early type galaxies, it has been shown recently that the luminosity profile is well approximated by the core-Sérsic profile (Graham 2004; Dullo & Graham 2012):

\[ I(R) = I' \left[ 1 + \left( \frac{R_b}{R} \right)^{a - \gamma/\alpha} \right]^{-\frac{\gamma}{\alpha}} \exp\left[ -b \left( \frac{R_a^{\alpha} + R_b^{\alpha}}{R_a^{\alpha}} \right)^{1/(\alpha m)} \right], \]

where \( I' \) is given by:

\[ I' = I_b 2^{-\gamma/\alpha} \exp[b(2^{1/\alpha} R_b/R_a)^{1/\alpha}]. \]

\( I_b \) is the luminosity evaluated at the break radius \( R_b \), \( \gamma \) is the slope of the inner power-law region, and \( \alpha \) regulates the transition between the power-law and the Sérsic profile. Moreover, \( R_a \) is the half-light radius and \( b \) is a function of the shape parameter \( n \) and is defined as such to ensure that \( R_b \) actually encloses half of the total luminosity. Of course, the real existence of cuspy (infinite density) innermost profiles for galaxies is just an extrapolation below the resolution limit of the behavior of the observed distribution. On a theoretical point of view, numerical simulations of standard cold dark matter (CDM) halo dynamics predict density profiles with \( \rho \propto r^{-1} \) at small radii (Navarro et al. 1996); this prediction does not depend on particular cosmogonies or choice of initial conditions (Huss et al. 1999a, 1999b) or on the specific form of the dark matter power spectrum (Eke et al. 2001). Adding a dissipative baryon component makes mass distributions even more concentrated (Blumenthal et al. 1986; Dubinski 1994). Anyway, there is not a general consensus about the real existence of a cusp in the dark matter distribution because it could be an artifact of the finite resolution of the N-body simulations. While the CDM scenario is surely working on large scales, on smaller scales...
it encounters problems because observations seem to indicate that faint galaxies have cored profiles instead of real cuspy innermost densities (also the observed underabundance of dwarf satellites of large galaxies is a problem in the CDM scenario). For this reason and others, we notice that some authors proposed, relatively recently, finite density profiles as the Einasto models (Einasto 1965) as better suited to describe dark matter haloes (for a deep discussion about this matter see, for example, Merritt et al. 2006).

An important additional point is that many (if not all) galaxies host at their center a compact massive object, identified with a supermassive black hole (SMBH) in massive galaxies (well above $10^{11} M_\odot$), or a nuclear star cluster (NSC) in lower mass galaxies (around or below $10^9 M_\odot$). There are quite a few cases of galaxies where an SMBH coexists with an NSC.

Such objects shape the density profile of the host in the innermost region, down to and below $\sim 10$ pc. At present, it is still unclear whether NSCs have cuspy density profiles. Actually, while it is ascertained that some NSCs have a cored profile, as in the case of the Milky Way NSC (Do et al. 2013), it is not yet clear what is the innermost region of the majority of galactic nuclei because of resolution limits.

In this paper, we study a possible solution of the problem of giving a reliable, quantitative estimate of the dynamical friction effect on massive objects moving in a background of matter whose density profile is described by a cuspy central distribution.

This paper is organized as follows. In Section 2, we present the problems arising when applying the classic Chandrasekhar formula for dynamical friction to the case of centrally diverging galactic density distributions and give a possible solution. Section 3 contains the results of the calibration of the semi-analytical approach of Section 2 by means of N-body simulations. Sections 4 and 5 present the results obtained by our previously described approach regarding the actual modes of decay of massive objects in spherical, cuspy galaxies, also in the presence of a massive central galactic BH. Conclusions are drawn in Section 6.

2. DYNAMICAL FRICITION

Dynamical friction (df) indicates the collective deceleration exerted on a massive body by the fluctuating force field where it moves. The existence of such an effect was demonstrated by Chandrasekhar & von Neumann (1942, 1943) in their pioneering studies on this subject. Further on, Chandrasekhar (1943a, 1943b) developed a theory of dynamical friction which leads to a quantitative estimate of the braking in the simplified scheme of an infinite and homogeneous distribution of field stars.

In the astronomical context, the fluctuating force is given by the gravitational encounters between the test mass and the field objects, assumed as stochastic events. These encounters become significant over the mean field whenever they are close enough (strong encounters) or when the cumulative effect of weak encounters has grown sufficiently. The interest in the study of dynamical friction in astronomy is on various sides. For instance, dynamical friction can be a way to accumulate matter in the inner regions of galaxies, so to explain, for instance, the formation of central compact massive objects (Capuzzo-Dolcetta 1993; Capuzzo-Dolcetta & Miocchi 2008a, 2008b). An observable consequence of this braking process is the evolution of the Globular Cluster System (GCS) radial distribution in their hosting galaxies: the dynamical erosion would cause a flattening of the GCS radial distribution around the center of the galaxy, as actually seen in many galaxies (see discussions in Capuzzo-Dolcetta & Vignola 1997, Capuzzo-Dolcetta & Donnarumma 2001, and Capuzzo-Dolcetta & Mastrobuono-Battisti 2009).

Moreover, dynamical friction determines the decay of SMBHs in remnants of merged galaxies (Milosavljević & Merritt 2001) that leads to the formation of SMBH binaries. As a consequence of gravitational wave emission, the binary shrinks until the merging of the two components (Schutz 1999). In a non-spherical merging, the final SMBH gains a kick that pulls the object out of its original position (Bekenstein 1973), and the kicked BH could escape from the galaxy (Campanelli et al. 2007). However, for small kick velocity, the recoiled object tends to decay again into the galactic center because of dynamical friction (Gualandris & Merritt 2008; Villari et al. 2007). Also, the commonly observed presence of a giant elliptical galaxy at the center of galaxy clusters is attributed to this dynamical deceleration whose action is stronger on more massive galaxies.

The effect of dynamical friction in a galaxy depends on both the orbit of the massive test object and on the local phase space density along this orbital path. Regarding the overall matter distribution of the host galaxy, the lack of symmetry in the potential favors dynamical braking because of the loss of angular momentum conservation and the consequent closer approach of the massive objects to the central denser region of the galaxy (Pesce et al. 1992). Moreover, it is known that the central galactic regions are those of highest phase space density (as measured by the proxy $\rho/\sigma^2$) so to make low eccentricity orbits as the ones suffering most of the deceleration. Consequently, central regions of cuspy galaxies with a triaxial shape over the large spatial scale are candidates to be sites of strongly enhanced dynamical decay, so as to convince us of the importance of its correct evaluation.

2.1. Is Dynamical Friction Deceleration Diverging in the Central Density Cusp?

Letting $M$ and $m$ denote the mass of the test particle and of the generic field star, respectively, and identifying $v_M$ and $v_m$ as their velocities, given also the impact parameter vector $b$ (see Figure 1), the two-body hyperbolic interaction between the test mass and the field star induces the velocity variation for the test mass:

$$\Delta v_M = -\left(\frac{m}{m+M}\right) 2V \left[1 + \frac{b^2 V^4}{G^2(m+M)^2}\right]^{-1} \frac{v}{V},$$

Figure 1. Symbol $r_m$ indicates the position vector of the test particle of mass $M$, while $r_n$ is the position of the field particle of mass $m$; $b$ indicates the impact vector pointing to the field particle.
where $G$ is the Newton’s gravitational constant and $V = v_M - v_m$ is the two-body relative velocity.

The effective time duration of such a two-body interaction is the fly-by time, assumed to be $Δt \sim 2b/V$, so that the mean deceleration due to the single encounter in the direction of the initial motion is well approximated by $Δv_M / Δt$. Consequently, the global deceleration effect is simply given by an integral over the whole distribution of scatterers:

$$\left( \frac{dv_M}{dt} \right)_{df} = \int \frac{dv_M}{\Delta t} dN,$$

where $dN$ is the (infinitesimal) number of field stars in the elementary space volume centered in $r_m = r_m + b$ having velocities in the (infinitesimal) velocity volume centered in $v_m$.

The field stars’ steady state distribution function is known as $f(r_m, v_m)$, $dN$ is written as:

$$dN = f(r_m, v_m) d^3v_m d^3r_m.$$  

As a consequence, we can express the mean cumulative deceleration in Equation (7) as:

$$\left( \frac{dv_M}{dt} \right)_{df} = -\frac{m}{m + M} \int \int f(r_m + b, v_m) \frac{V}{b} \times \frac{V}{1 + b^2V^4G^2(M + m)^2}d^3v_m d^3b,$$

where $r_m = r_m + b$ and the integral is over the whole range of values of $b$ and $v_m$ allowed by self-consistency.

The integral in Equation (9) is, in general, quite complicated.

It can be performed in the hypothesis of a distribution function separated in the space and velocity coordinates, $f(r_m, v_m) = g(r_m)G(v_m)$, and under the additional “local” approximation, which means that in the evaluation of the integral in Equation (9), the substitution of $g(r_m)$ with $g(r_M)$ is done, that corresponds to weighting encounters at any generic distance $r_m$ from the test object with the local density (i.e., where the satellite is, $r = r_M$). The local approximation allows an integration over the impact parameters which leads to the function

$$\frac{1}{Q^2} \ln(1 + Q^2b_{max}),$$

where $Q^2 = Q^2(V) = V^4/[G(m + M)^2]$ and the cut at $b = b_{max}$ is needed to avoid the logarithmic divergence. Letting $\Lambda \equiv 1 + Q^2b_{max}$, the further assumption of isotropy in the velocity dependent function leads to the simple expression:

$$\left( \frac{dv_M}{dt} \right)_{loc} = -4\pi^2G^2(m + M)\ln\Lambda \rho(r_M, v_m \leq v_M) \frac{v_M}{v_M},$$

where $\rho(r_M, v_m \leq v_M)$ is the local, mass density of field stars slower than the test particle. At this last regard, it may be worth noting that often the logarithmic function of field star velocities is taken out of the integral over velocities also out of the above given, under which it can be correctly taken out. This happens, for instance, when, in spherical symmetry, a distribution function in the form $f(r_m, v_m) = f(r_m, v_m) = f(E)$, where $E = v_m^2/2 + \Phi(r_m)$ is the field star’s mechanical energy per unit mass, is assumed. Taking the logarithmic out of the velocity integral results in a dynamical friction deceleration in the form of Equation (11) where:

$$\rho(r_M, v_m \leq v_M) = 4\pi \int_0^{v_M} v_m^2 f(v_m^2/2 + \Phi(r_M)) dv_m.$$  

Several authors (e.g., Tremaine 1976) suggested that allowing a variation of the Coulomb logarithm, $ln \Lambda$, may be important for a good determination of its orbital evolution. Just & Pfennurrubia (2005) derived an expression for $ln \Lambda$ allowing the variation of the maximum impact parameter, $b_{max}$, and the $\phi(b)$ parameter (the typical impact parameter for a 90° deflection in a two-body encounter). The effect of this variation on the orbits of massive body traveling in cuspy galaxies are deeply discussed in Just et al. (2011).

Now, whenever the test particle is significantly off center with respect to the stellar system (star cluster, galaxy, etc.) where it moves, the local expression (Equation (11)) gives an acceptable approximation; on the contrary, it loses its validity in the neighborhood of the host system center. In this case, the local approximation is clearly an overestimate of the actual dynamical friction, because it corresponds to weighing the contribution of the gravitational encounters at any distance from the test particle, not with the correct density of target stars at that distance but rather with the density of targets evaluated at the location of the test particle itself, that is maximum at the origin of any self-gravitating system. This overestimate is a particularly serious problem when dealing with cuspy galaxies, where the spatial density of stars $\rho(r_m, v_m \leq v_M)$ at the galactic center. This divergence may be partially cured by introducing an artificial spatial cut-off in the density distribution but this, of course, implies a relevant dependence of dynamical friction on the choice of this radial cut-off.

We can better illustrate all this with the example of a distribution function as obtained using a $γ$ model (Dehnen 1993) around the central spatial cusp of a spherical galaxy, where the stellar density may indeed be represented as $ρ(r) \propto r^{-γ}$. As it is easily seen (see Appendix A), when $γ = 1$, the following expression for $ρ(r, v_m \leq v_M)$ in the high binding energy regime, i.e., around the galactic center, is obtained:

$$\rho(r, v_m \leq v_M) = \frac{4\pi}{3} A \frac{v_M^3}{(r/a)[(v_M^2/2)/(GM/a) + r/a]^{3/2}},$$

where $A$ is the multiplicative constant in the expression of the distribution function (see Appendix A). The resulting local approximation (Equation (11)) for the dynamical friction deceleration yields

$$\left( \frac{dv_M}{dt} \right)_{loc} = -\frac{16\pi^3}{3} AG^2(m + M)\ln\Lambda \frac{v_M}{(r/a)[(v_M^2/2)/(GM/a) + r/a]^{3/2}}.$$  

If, in the denominator of Equation (14), $r/a$ and $(v_M^2/2)/(GM/a)$ are (temporarily) set to zero with the same order of infinitesimal, the local dynamical friction deceleration diverges as $(r/a)^{-3}$ (or, equivalently, $[v_M^2/(GM/a)]^{-3}$). This divergence is due to the local approximation, while the correct (Equation (9)) expression for the deceleration does not diverge; on the contrary, deceleration goes to zero for particles of very high binding energy (see Appendix A). Therefore, the local approximation formula cannot be used to get astrophysically significant results when treating the motion of massive objects passing through (or close to) the center of a cuspy galaxy.

2.2. A Possible Solution of the Divergence Problem

In Appendix A, we show that the fully isotropic distribution functions of the Dehnen’s gamma model lead to a deceleration
which is finite around the galactic central density cusps, while its local approximation is not. This convinces us of the need to use, instead of the wrong local approximation, the complete (Equation (9)) expression for the dynamical friction process.

Unfortunately, the integral in Equation (9) is of overwhelming complexity, unless some simplifications are adopted. An intuitive, immediate approximation comes from letting both \( r_M = 0 \) and spatial isotropy (i.e., spherical symmetry) for the distribution function \( (f(r_M, v_m) = f(r_M, 0)) \) to get the simpler expression for the deceleration:

\[
\left( \frac{d\mathbf{v}_M}{dt} \right)_{cen} = - \frac{4\pi m}{m + M} \int_{b_{\text{max}}}^{b_{\text{min}}} \frac{f(b, v_m)}{V} \int_{r_M}^{\infty} \frac{v^3 b db}{1 + b^2V^2G^2(m + M)^2},
\]

where \( b_{\text{min}} \) and \( b_{\text{max}} \) are, respectively, the minimum and maximum impact parameters allowed. The lower minimum cannot be zero because this would correspond to a front collision, i.e., to a radial relative motion which does not fulfill the basic condition of positive mechanical energy for the idealized two-body encounter. On the other hand, the upper limit, \( b_{\text{max}} \), is usually chosen to be large enough to guarantee that the stellar density at distance \( b_{\text{max}} \) from the center is much smaller than in the neighborhood of the test object.

For a large set of distribution functions, the vector integral in Equation (15) is both convergent (see Appendix A) and suited to a proper numerical integration. The integration over field stars’ velocities in Equation (15) has been done over the entire interval allowed, i.e., limited to the central escape velocity.

Of course, the dynamical friction evaluated this way gives a good result along the motion of the test mass in the neighborhood of the galactic center but cannot be used on a larger spatial scale. Consequently, our choice was that of an interpolation between the “central” dynamical friction evaluation and the “local” approximation, by means of a proper interpolation formula of the type:

\[
\left( \frac{d\mathbf{v}_M}{dt} \right)_{df} = p(r) \left( \frac{d\mathbf{v}_M}{dt} \right)_{cen} + [1 - p(r)] \left( \frac{d\mathbf{v}_M}{dt} \right)_{loc},
\]

where the interpolation function, \( 0 \leq p(r) \leq 1 \), is assumed to be monotonically decreasing from \( p(0) = 1 \) outward. Within these constraints, the interpolation function is a priori arbitrary; the only way to tune it is through a careful comparison with \( N \)-body simulations of the decay of massive objects under different initial conditions. With this comparison, we found that a good interpolation expression is \( p(r) = e^{-r/r_{\text{rc}}} \), where \( r_{\text{rc}} \) is the size of the region of dominance, in the contribution to the dynamical friction of the central cusp. The actual \( r_{\text{rc}} \) values are determined in Section 3. It is relevant to note that although the exponential choice is not unique, the simpler, linear interpolation can be excluded because our results show that a linear function weighs the central contribution too heavily, giving an unrealistically high deceleration.

The computation of the scattering integral in Equation (15) presents numerical difficulties due to the singularity in the integrand. These difficulties can be overcome by using a proper integration algorithm; in particular, we used DECUHR, an algorithm which combines an adaptive subdivision strategy with extrapolation (Espelid & Genz 1994).

In Figure 2, the departure of the local friction evaluated via Equation (11) is evident with respect to the central estimate given by Equation (15).

![Figure 2](image)

**Figure 2.** Ratio, \( f \), of the dynamical friction acceleration evaluated with the local approximation formula in Equation (11) to the central given by Equation (15), in the cases \( \gamma = 0 \) (dashed line) and \( \gamma = 1 \) (solid line).

We can note that, in the regime of very low or very high speed for the test particle and/or when its mass, \( M \), is small, the above integration algorithm requires an exceedingly large number of iterations to reach convergence. In such cases, to speed up computations, we looked for an appropriate approximation formula.

We actually found that the linear dependence of dynamical friction on \( v_M \) is recovered, while at high velocities, the dependence is a power law with a spectral index, \( \alpha \), that depends both on \( \gamma \) and on \( b_{\text{min}} \):

\[
\alpha = \begin{cases} 
2(\gamma - 1) & \text{if } b_{\text{min}} = 0, \\
-2 & \text{if } b_{\text{min}} > 0,
\end{cases}
\]

in the range \( 0 \leq \gamma \leq 2 \).

### 3. CALIBRATION BY MEANS OF \( N \)-BODY SIMULATIONS

A fully self-consistent study of the dynamical friction caused by environment stars on the motion of a (massive) object of mass \( M \) requires the numerical integration of an \( N \)-body problem where \( N_f \) particles sample the galactic field, and \( N_M \) particles represent the massive star system \((N_f + N_M = N)\). In principle, to have results of high reliability in the astrophysical context, high-resolution simulations are needed, which require both a large value of \( N_f \) and \( N_M \).

This may be unfeasible when aiming of a statistically complete set of simulations over a huge set of initial conditions. On the other hand, an analytical or semi-analytical approach, although much more suited to an extensive analysis, suffers of its intrinsic, more or less severe approximations. The natural way to treat the topic of dynamical friction of massive objects in a stellar background in a simplified scheme is that of the integration of the equations of motion of the single, massive object in a given external potential \( \Phi(r) \) with the inclusion of the drag term given by Equation (16) (dragged one-body problem).

Taking all this into account, a good choice can be that of a proper calibration of the free parameters in the dragged one-body problem by means of a set of reliable, high-precision \( N \)-body simulations.
3.1. The Dragged One-body Problem

In the dragged one-body problem, the equations of motion to solve are written as:

$$\ddot{r}_M = \nabla \Phi(r_M) + \left( \frac{dV_M}{dt} \right)_{df},$$

(17)

with the proper initial conditions. To solve this set of differential equations, we use a high-precision sixth (seventh) order Runge–Kutta–Nyström method with variable time step (Fehlberg 1972). The time step size, $\Delta t$, was varied according to

$$\Delta t = \eta \min \left( \frac{|r_M^1|}{|r_M^0|}, \frac{|r_M^2|}{|r_M^0|} \right),$$

which, with the choice of $\eta = 0.01$, allows both a fast integration and an energy and angular momentum conservation at a fractionary $10^{-11}$ level (per time step).

In this paper, we choose as units of mass and length the galactic mass and scale length of its density distribution, denoted by $M_G$ and $a$. The further choice of setting the gravitational constant $G = 1$ leads to

$$T = \frac{a^{3/2}}{\sqrt{GM_G}}$$

(18)
as unit of time.

Once the expression for the interpolation function, $p(r) = e^{-r/r_c}$, cited in Section 2.2, is given, the free parameters in the semi-analytical evaluation of dynamical friction are the scale length $r_c$ in $p(r)$, and the values for $b_{\text{min}}$ and $b_{\text{max}}$ are as in the local (Equation (11)) and central (Equation (15)) expressions of the deceleration.

We made several simulations using both constant and variable $b_{\text{max}}$ to conclude that the advantages in accuracy given by a somewhat arbitrary variation in $b_{\text{max}}$ are not such to overcome the simplicity of the choice of $b_{\text{max}}$ set at the constant value $R$, the assumed radius of the spherical galaxy. On the other hand, due to its undoubted relevance in a cuspy galaxy, we let $b_{\text{min}}$ vary. Also, the length scale $r_c$, which determines the size of the region of dominance of the central to the local friction term, is allowed to vary.

An unambiguous way to select their optimal values is through a comparison of results given via the integration of Equation (17) by varying the pair $(r_c, b_{\text{min}})$ and the, supposedly “exact,” results coming from the integration of motion of a single, point-like massive object of mass, $M$, interacting with $N$ bodies of mass, $m$, representing the galactic field. At this scope, we used our direct summation, high-precision sixth order Hermite’s integrator with individual block time steps called HiGPUs (Capuzzo-Dolcetta & Spera 2013). HiGPUs runs on composite platforms where the host governs the activity of Graphic Processing Units (GPUs) as computing accelerators. The code exploits all the potential of such architectures, since it uses Message Passing (MPI), Open Multiprocessing (Open MP) on the host CPUs, and Compute Unified Device Architecture (CUDA) or Open Computing Language (OpenCL) on the GPUs (Capuzzo-Dolcetta & Spera 2013) at the same time. HiGPUs has been extensively checked in its accuracy; for the purposes of this paper (once the optimal number of particles has been set), we performed several simulations to check its accuracy. In particular, we verified that over the time lengths of relevance for our purposes, the code conserves the total energy, linear, and angular momentum of the system with a relative error down to $10^{-8}$ (for energy) and down to $10^{-10}$ for momentum. Moreover, we checked that the simulated systems do not expand or contract significantly during their evolution, as a guaranty of both correct choice of initial conditions and quality of time integration. The system stability has been verified by also looking at the Lagrangian radii and density profiles, which remain substantially constant during the whole orbital evolution of the satellite, apart from local wake effects induced by the satellite motion.

3.2. Sampling Effects

In order to make an optimal selection of the two free parameters needed to set the drag term in the one-body scheme, we perform an adequate set of direct $N$-body integrations, as explained above. To calibrate these parameters, it is of course important to be sure of the reliability of such $N$-body simulations. The main problem is the sampling. Actually, the $N$-body sampling acts on both small (“granularity”) and large (deviation from spherical symmetry) scales. This makes initial circular orbits evolve into precessing ellipses of moderate eccentricity (see Figure 3). This is one of the unavoidable causes of departure in the decay times for the $N$ case and semi-analytical case. To reduce spurious sampling effects, we tried to determine an acceptable threshold value of $N$ above which fluctuations are kept relatively small. To do this, we followed the orbital evolution of a particle of the same mass of the generic particle of the $N$-body representation of the galaxy, starting from initial conditions corresponding to the extreme (in eccentricity) cases of circular and radial orbits.

As it can be seen from Figures 4 and 5, in both of these extreme cases, the quadratic deviation of the actual trajectory computed in a finite $N$-body representation of a Dehnen’s $\gamma = 1$ galactic density law with respect to the ideal (infinite $N$) circular and radial trajectories decreases significantly when $N$ is in the range $10^5 < N < 10^6$. Actually, the reduction of fluctuations passing from $N = 131,072$ to $N = 524,288$ suggested that we choose this latter value as a good compromise to achieve an acceptable smoothness at a reasonable computational cost.

3.3. Determination of the Free Parameters in the One-body Scheme

After determining the threshold in $N$ over which an acceptable fit between the $N$-body test object integration and that obtained...
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Figure 4. Squared fractional departure of the distance to the center of a test particle of same mass of the field particles along its motion as integrated in an N-body sampled $\gamma = 1$ model respect to the ideal radial orbit.

Figure 5. Squared fractional departure of the distance to the center of a test particle of same mass of the field particles along its motion as integrated in an N-body sampled $\gamma = 1$ model respect to the ideal circular orbit.

by the solution of the single body motion in the external smooth galactic field, the following step was obtaining the best pairs of values ($r_{cr}$, $b_{min}$) in each $\gamma$-model we investigated. We proceeded in the following manner. (1) Perform N-body integrations of the motion of a massive point mass, starting from an initial distance $r_{cr}$ from the galactic center with the local circular velocity. (2) Perform a similar time evolution in the simplified one-body scheme of Equation (17), where the dynamical friction dissipation term is given in the standard local approximation form. (3) Reduce $r_{cr}$ until the difference between the orbit self-consistently evaluated in (1) and that obtained as explained in (2) changes significantly. (4) Take this latter value of $r_{cr}$ as the optimal value for the $p(r)$ function in the interpolation formula (Equation (16)). To do this, we set $M = 10^{-3}$ as the value for the test particle mass.

An idea of the quality of this fitting procedure to determine the pair ($r_{cr}$, $b_{min}$) that would achieve the results of interest here is given by Figure 6. It reports the ratio between the test particle orbital energy evaluated in the one-body, semi-analytical case and that computed in the N-body sampled galaxy for a radial (solid line) and a circular orbit (dotted line).

Figure 6. Time evolution of the ratio between the test particle energy evaluated in the one-body, semi-analytical case and that computed in the N-body sampled galaxy for a radial (solid line) and a circular orbit (dotted line).

Table 1

| $\gamma$ | $r_{cr}$ | $r(0.1)$ | $\Delta r/r\%$ |
|----------|----------|----------|----------------|
| 0.5      | 0.7      | 0.661    | 0.059          |
| 1.0      | 0.5      | 0.463    | 0.080          |
| 1.5      | 0.3      | 0.275    | 0.091          |

Note. The last column reports the relative variation between the two.

units in the radial case and within 2% in the circular case over the same time interval. In the circular case, we extended the comparison up to 80 time units, finding a relative maximum of the fractional difference of about 12%.

We found that the greater the $\gamma$ the smaller the $r_{cr}$, as expected. Actually, higher values of $\gamma$ represent steeper profiles toward the center, with a large part of the total mass enclosed within a relatively small radius. On the other hand, the result of $r_{cr}$ is less intuitive as it is very similar to the radius enclosing 10% of the mass of the system. A simple inversion of the mass-radius profile for Dehnen’s models gives:

$$r(x_M) = \frac{x_M}{1 - x_M^{1/(3-\gamma)}}$$

with $x_M = M(r)/M_G$. The value of $r(0.1)$ is found (see Table 1) to be in good agreement with those of $r_{cr}$ obtained in the way indicated above.

Once the $r_{cr}$ values are obtained for different $\gamma$, to get the best minimum impact parameter $b_{min}$, we vary it in a set of one-body integrations covering circular and radial cases to find those best fitting results of direct N-body computations. In Figure 7, we show the $b_{min}$ selected this way, as a function of $\gamma$.

4. RESULTS

The main scope of this paper was to obtain reliable estimates of the role of dynamical friction in cuspy galaxies, as previously explained.

This aim has been reached by means of both direct numerical integrations of the motion of a massive test particle in an N-body representation of the host cuspy galaxies and of the simpler, and much faster, one-body representation given by Equation (17) together with Equations (11), (15), and (16).
Figure 7. Minimum impact parameter as a function of the initial galactocentric distance, for initial radial (r) and circular (c) orbits at various values of $\gamma$.

Table 2

| $r_0$ | $e = 0$ | $e = 0.50$ | $e = 1$ |
|-------|---------|------------|---------|
| $M = 10^{-3}$ | $M = 10^{-3}$ | $M = 5 \times 10^{-4}$ | $M = 10^{-3}$ | $M = 5 \times 10^{-3}$ |
| 0.2 | ✓ | ... | ... | ✓ | ...
| 0.3 | ✓ ... | ✓ ... | ...
| 0.5 | ... ... | ✓ ... | ...
| 0.7 | ... ... | ✓ ... | ...
| 0.8 | ... ... | ... ... | ...
| 1.0 | ... ... | ✓ ... | ...
| 1.44 | ... ... | ... ... | ...
| 1.5 | ... ... | ✓ ... | ...
| 1.67 | ... ... | ... ... | ...
| 2.0 | ... ... | ✓ ... | ...

Notes. In this table, the ✓ symbol indicates the exploited values for the initial galactocentric distance ($r_0$), eccentricity ($e$), and satellite mass ($M$) in the N-body simulations performed.

Table 3

| $r_0$ | $e = 0$ | $e = 0.50$ | $e = 1$ |
|-------|---------|------------|---------|
| $M = 10^{-3}$ | $M = 10^{-3}$ | $M = 5 \times 10^{-4}$ | $M = 10^{-3}$ | $M = 5 \times 10^{-3}$ |
| 0.2 | ✓ ✓ ✓ ✓ ✓ ✓ |
| 0.3 | ✓ ✓ ... ✓ ✓ ✓ |
| 0.31 | ... ... ✓ ✓ ✓ |
| 0.5 | ✓ ✓ ✓ ✓ ✓ ✓ |
| 0.7 | ✓ ✓ ✓ ✓ ✓ ✓ |
| 0.8 | ✓ ✓ ✓ ✓ ✓ ✓ |
| 1.0 | ✓ ✓ ✓ ✓ ✓ ✓ |
| 1.44 | ✓ ✓ ✓ ✓ ✓ ✓ |
| 1.5 | ✓ ✓ ✓ ✓ ✓ ✓ |
| 1.67 | ✓ ✓ ✓ ✓ ✓ ✓ |
| 2.0 | ✓ ✓ ✓ ✓ ✓ ✓ |

Note. All symbols as in Table 2.

Tables 2–4 give the fundamental data of the whole set of N-body simulations performed.

Using these N-body simulations as reference, the quality of the one-body treatment is given in Figures 8–11, where the time evolution of the test mass galactocentric distance is reported.

Figure 8. Damped oscillations along the x axis for the test object with mass $M = 10^{-3}$ in the $\gamma = 1$ model. The darker line refers to the N-body simulation, while the gray line refers to the semi-analytical.

Table 4

| $r_0$ | $e = 0$ | $e = 0.50$ | $e = 1$ |
|-------|---------|------------|---------|
| $M = 10^{-3}$ | $M = 10^{-3}$ | $M = 5 \times 10^{-4}$ | $M = 10^{-3}$ | $M = 5 \times 10^{-3}$ |
| 0.2 | ✓ ✓ ✓ ✓ ✓ ✓ |
| 0.3 | ✓ ✓ ... ✓ ✓ ✓ |
| 0.31 | ... ... ✓ ✓ ✓ |
| 0.5 | ✓ ✓ ✓ ✓ ✓ ✓ |
| 0.7 | ✓ ✓ ✓ ✓ ✓ ✓ |
| 0.8 | ✓ ✓ ✓ ✓ ✓ ✓ |
| 1.0 | ✓ ✓ ✓ ✓ ✓ ✓ |
| 1.44 | ✓ ✓ ✓ ✓ ✓ ✓ |
| 1.5 | ✓ ✓ ✓ ✓ ✓ ✓ |
| 1.67 | ✓ ✓ ✓ ✓ ✓ ✓ |
| 2.0 | ✓ ✓ ✓ ✓ ✓ ✓ |

Note. All symbols as in Table 2.

The role of the geometrical shape of the orbit is evident in Figure 12, which shows the energy decay of the test mass for different initial eccentricities at fixed initial orbital energy. As expected, circular ($e = 0$) orbits decay slower than radial ($e = 1$), while orbits with $e = 0.5$ have a decay time between these two extreme cases. We also note that, for higher initial orbital energies, the decay time of the $e = 0.5$ orbit approaches that of the circular orbits, indicating a clear nonlinearity of the decay time with $e$ in this high-energy regime.

Actually, the most important astrophysical parameter that can be inferred in this framework is the dynamical friction decay time, $\tau_{df}$, which we define as the time needed to reduce the test particle orbital energy to $E(\tau_{df}) = \Phi(5 \times 10^{-5}a)$. A correct evaluation of this time, which depends on both small and large scale characteristics of the galaxy where the test mass moves, as well on the test particle mass, is crucial in determining the actual role of dynamical friction in carrying matter toward the center of galaxies with the consequent, relevant astrophysical implications.

Figure 13 shows the $\tau_{df}$ dependence on the initial radial distance of circular and radial orbits in the $\gamma = 1$ model. The relations are two power laws with a slightly different slope.
Figure 9. As in Figure 8, but for the model with $\gamma = 1/2$.

Figure 10. Time evolution of the galactocentric distance of an $M = 10^{-3}$ test mass on initial circular orbit in the $\gamma = 1/2$ model.

is evident again in Figure 14, where we compare $\tau_{df}$ for circular and radial orbits with same the initial energy.

Consider that both circular and radial trajectories of the same apocenter allow us to obtain an upper and lower limit, respectively, for decay time of any orbit at fixed position but with different velocity. Simulating orbits with the same initial energy, instead, we can study the efficiency of dynamical friction with respect to the shape of the orbit.

The dynamical friction time depends, obviously, on the model considered: the steeper the density profile (large $\gamma$) the shorter the decay time. This is clear in Figure 15, which shows that when increasing $\gamma$ by a factor of three (from $\gamma = 1/2$ to $\gamma = 3/2$), the decay time decreases by almost the same factor.

4.1. Dynamical Friction Dependence on the Test Mass

Besides the dependence from initial position, eccentricity, and model, another important parameter that affects the dynamical friction effect is the mass of the satellite. This dependence deserves some considerations. Actually, a direct, linear proportionality of the dynamical friction braking deceleration to the test mass, $M$, is generally assumed. This comes, in Equation (9), by the contemporary assumption $m \ll M$ and $(b^2 V^4)/(G^2 (m + M)^2) \gg 1$. The opposite limit $(b^2 V^4)/(G^2 (m + M)^2) \ll 1$ would lead to an inverse linear proportionality. So it is logically inferred that performing the integrals in Equation (9) over the whole integration ranges lead to a dependence on $M^\alpha$ with $-1 < \alpha < 1$, even taking into account a possible dependence of the integration limits on $m$ and $M$.

We refer to Appendix B for details.

While it is confirmed that a higher mass of the test object leads to a shorter decay time (see Figure 16), we see that by varying the satellite mass in the range $[5 \times 10^{-5}, 5 \times 10^{-3}]$, the relation between $\tau_{df}$ and $M$ is shallower:

$$\tau_{df} \propto M^{-0.67 \pm 0.1},$$

as obtained by a least square fit to the data of Figure 17, coming from direct N-body integrations and confirmed by the simplified one-body scheme.

Assuming the minimum impact parameter independent of the test mass, we performed semi-analytical simulations in a wide range of masses $[10^{-5}, 5 \times 10^{-3}]$, setting the initial position $r_0$ to the values $0$, $1$, and $2$ for initially circular and radial orbits, and find that the decay time-mass relation depends strongly on the starting position of the satellite (see Figure 18), as expected.

4.2. A Fitting Formula for Dynamical Friction Decay Time

A deep analysis of all the simulations performed allowed us to obtain a useful analytical approximation to $\tau_{df}$ as a function of $M$, $r_0$, $e$, and $\gamma$, as

$$\tau_{df} = \tau_0 (1 + g(e))(2 - \gamma)M^{-0.67}r_0^{1.76},$$

where $\tau_0 = 0.2$ is an adimensional time constant and $g(e)$ is an adimensional function of the eccentricity:

$$g(e) = 3.93(1 - e).$$
Figure 12. Time evolution of the fractional variation of the test particle energy in the circular \(e = 0\), solid black line), radial \(e = 1\), dashed line), and eccentric \(e = 0.5\), gray line) cases of same initial energy, \(E(0)\), in the \(N\)-body sampled, \(\gamma = 1\), galaxy. The \(r_0\) values refer to the initial distances of the test particle from the galactic center.

Figure 13. Dynamical friction decay time vs. initial galactocentric distance for circular (filled circles) and radial (triangles) orbits in the \(\gamma = 1\) model.

Figure 14. As in Figure 13, limiting the comparison to pair of orbits of same initial energy, \(E(0)\).

4.3. A Straightforward Application to a Galactic Satellite Population Sinking

By means of this formula and assuming a population of galaxy satellites (that may represent globular clusters in a galaxy) initially distributed following either the same \(\gamma\) density law of the background stars or accordingly to a Plummer profile, we

\[
 r_{\text{max}} = 2.5 \left[ \frac{t}{(1 + g(\bar{e}))(2 - \bar{\gamma})} \right]^{0.57} \bar{M}^{0.375}. \tag{23}
\]
of the galaxy at different physical times (500 Myr, 1 Gyr, and 13.7 Gyr), estimated the fraction to the total of satellites sunk to the center of the galaxy at different physical times (500 Myr, 1 Gyr, and 13.7 Gyr), and synthesized some results in Tables 5 and 6. The fundamental role of the steepness of the galaxy density profile on the depletion of the satellite population is clearly seen, as well as that of the satellite mass. The cuspy, \( \gamma = 3/2 \), galactic profile is able to erode around 40\% of the initial satellite population of masses larger than \( M = 10^6 M_\odot \) within 1 Gyr, assuming satellite moving on circular orbits, and up to 63\%–83\% of the initial population (the larger erosion for an initial satellite profile following the Plummer’s law) in the case of radial (\( \epsilon = 1 \)) orbits. This erosion reduces to 4\%–9\% of the initial satellite circular orbits and to 18\%–49\% of the initial satellite radial orbits, when the galaxy profile follows the innermost flat \( \gamma = 0 \) profile (also here the percentage intervals refer to the satellites distributed as a \( \gamma = 0 \) profile or as a Plummer’s model). As a general conclusion, the dynamical friction effect is maximized for massive satellites (\( M/M_\odot \geq 10^{-6} \)) of cuspy, massive and compact galaxies (\( M_\odot \geq 10^9 M_\odot , a \lesssim 500\) pc) whose satellites systems evolve faster in a given physical time due to the \( \propto a^{-3/2} M_\odot^{-1/2} \) scaling of the time unit (Figure 19). In few Gyr, such galaxies remain with a low abundant satellite population, having packed most of their mass (up to 90\%, or more) into the galactic nuclear region.

### 4.4. Massive Object Stalling in Core Galaxies

The approximation formula given by Equation (21) was obtained by fitting results of \( N \)-body integrations in cuspy density profiles. To check its application to cored models, we performed two \( N \)-body simulations of the evolution of a radial and a circular orbit in a Dehnen model with \( \gamma = 0 \). The orbits have the same initial energy with the circular orbit starting at \( r_0 = 0.5 \). Figure 20 reports the evolution of the test mass orbital energy in the two cases studied.

We see that the extrapolation of Equation (21) to the \( \gamma = 0 \) case gives a decay time correct within 10\% for the radial orbit. On the other hand, the \( N \)-body evolution of the circular orbit shows that the decay stops when the test particle galactocentric distance reduces to \( r \lesssim 0.1 \); then, the orbit “stalls,” in the sense that the test particle oscillates without appreciable further decay, as indicated by Figure 21. This orbit stalling in cored profiles was already found by previous authors (Kalnajs 1972; Read et al. 2006); in particular, Antonini & Merritt (2012) found evidence that the stall is due to a lack of slow stars within the orbit size. Although it is not exactly true that dynamical...
friction is contributed by field stars slower than the decaying object, this interpretation is substantially correct as shown by Figure 22, where the fraction (to the total) of stars slower than the decaying object and enclosed within its actual position is reported as a function of time.

While in the radial case, the fraction of “slow” increases when the test mass crosses the center of the system, resulting in an enhancement of the dynamical friction effect that induces a progressive decay until the particle reaches the center of the system, in the circular case, the fraction decreases continuously until \( t \sim 30 \), which is roughly the time at which the decay ends and the test mass reaches an almost steady eccentric orbit.

Since the spatial distribution of background stars is not significantly altered on all scales by the satellite motion, as it is shown in Figure 23 where we compare the background density at the beginning and at the end of the simulation, it is argued that the key parameter in the modes of braking is actually the variation in the fraction of slow stars.
Figure 22. Local fraction of field stars slower than the test mass as a function of time for a circular (straight line) and a radial (dotted line) orbit.

Figure 23. Density profile of the background distribution of particles at the beginning (solid line) and at the end of the simulation (dashed).

Looking at the position at which stall begins, we found that the radius at which the dynamical friction action becomes negligible encloses a mass roughly equal to the test mass \( M \), in agreement with the conclusion in Gualandris & Merritt (2008). Obviously, in flattened density cores, this “critical mass” is reached at a greater radius with respect to cuspy profiles, enlarging the region of motion stalling. Of course, the stalling radius is smaller for centrally peaked profiles; for example, if \( \gamma = 1 \), it shrinks to \( r \approx 0.035 \), as seen in Figure 3.

5. INDIRECT EFFECTS OF A CENTRAL BLACK HOLE ON THE SATELLITE DECAY

It is well known that galaxies in a wide range of luminosities and Hubble types host massive or even SMBHs at their center, whose masses range in the \( 10^6–10^{10} M_\odot \) interval (Antonucci 1993; Urry & Padovani 1995), strongly influencing the environment.

As an example of such an influence, Antonini & Merritt (2012) noted that a hypothetical stellar-mass BH population would significantly enlarge the time to reach the center of the Milky Way by the presence of the central SMBH.

Figure 24. Test mass, \( M \), orbital decay in the presence of a central black hole, whose mass, \( M_{\text{BH}} \), is labeled. The galaxy is modeled as a Hernquist sphere and the test mass motion computed in the complete \( N \)-body framework.

Figure 25. Apocenter distance after the first oscillation through the galactic center of the test mass in the presence of a central massive black hole of mass \( M_{\text{BH}} \).

Actually, the presence of an SMBH also affects larger space and time scales through, for instance, its indirect role on the dynamical friction efficiency.

To check this role, we performed some specific \( N \)-body simulations of the motion of a point-like object which starts on an initial radial orbit in a \( \gamma = 1 \) sphere sampled with \( N = 524,288 \) particles and in the presence of a central SMBH with mass \( M_{\text{BH}} \).

In this framework, each background star has a mass, \( m_\ast \approx 2 \times 10^{-6} \).

We chose three different values for \( M_{\text{BH}} \), namely, \( M_{\text{BH}} = M, 4M, \) and \( 10M \), where the mass of the test object is set to \( M = 10^{-3} \gg m_\ast \).

Initial conditions for the test object are those of null initial velocity and of an initial position \( \mathbf{r}_0 = (x_0 > 0, 0, 0) \), such that the initial orbital energy of the test object is the same in the three cases, \( E_0 = -5 \times 10^{-4} \). This choice leads to about the
same speed at the closest approach of the test particle to the center, a condition needed to appreciate differences in the decay as mainly due to the presence of the BH.

The time evolution of the test object distance to the galactic center, shown in Figure 24, indicates that the presence of a SMBH does affect the dynamical friction decay time. More massive BH determines a longer decay time of the infalling object. It should not be a surprise that the behavior of $r(t)$ in the case of absence of a SMBH is more similar to the behavior in the case of the most massive SMBH considered. This is due to the fact that the apocentric distance reached after the first crossing through the center is much more similar in these two extreme cases than in the others because the very massive BH gains just a small velocity after the close encounter with the test particle. Less massive SMBHs, on the other hand, move more and the test mass apocenter consequently reduces, making it move in an innermost region where the galactic dynamical friction effect is larger. This is made clear by Figures 25–27.

This effect dominates on the other opposite effect of deviation from the unperturbed radial trajectory as quantified in Figure 28. This figure shows a very similar time for the closest approach to the galactic center (and so to the SMBH therein) in all the cases studied ($t \simeq 2.5$), as a consequence of the same value of initial orbital energy. After this closest approach, the time evolution of the distance to the center is quite different, and differences cumulate over the following closest approaches.

The effect of the interaction BH-test mass is clearly shown in Figure 26, which draws the trajectories (labeled with times) of the test mass and of the SMBHs in the case $M_{\text{BH}}/M = 1$ with the clear departure of the central BH from its initial central position.

The effects induced by the presence of a central BH on the motion of the satellite can significantly change the time needed

Figure 26. Trajectories of the radially falling test mass (empty squares) and of the perturbed central BH (filled squares), in the case of equal mass. Some of the apocenter positions are labeled with their times.

Figure 27. Test mass (dotted line) and BH (straight line) galactocentric distances vs. time in the case $M_{\text{BH}} = M$ (upper panel) and $M_{\text{BH}} = 10M$ (lower panel).

Figure 28. Cumulative standard deviation from the unperturbed radial motion of the test mass $M$ as a function of the BH mass, which shows the standard deviation of the distance of the test mass from the direction of unperturbed radial motion evaluated over the whole orbital evolution of the test mass until its total decay, as function of $M_{\text{BH}}$. 

to carry the satellite toward the center of the system. Here, we have shown that the decay time is at a minimum when the sinking satellite and the central BH have about the same mass, while when the BH mass exceeds several times the mass of the infalling object, the decay time tends to be the same value estimated in the absence of a central BH. This implies that results presented in Section 4.2 are still valid whenever the central body is significantly more massive than the incoming satellite. This is often the case of real galaxies, at least for galaxies more massive than \( \sim 10^{10} M_\odot \) (Scott & Graham 2013). In these massive hosts, the fitting formula given in Equation (21) represents a valid way to measure the amount of mass deposited in time within the central region of a galaxy, if it hosts a central, massive BH.

6. CONCLUSIONS

In this paper, we studied dynamical friction in cuspy density profiles of spherical (E0) galaxies, both on a theoretical and a numerical point of view.

The main results are summarized as follows.

1. The classic Chandrasekhar (1943a) formula in its local approximation does not work in the central region of a cuspy distribution because it diverges at the center and overestimates the actual dynamical friction in the vicinity of the density singularity.

2. An alternative, semi-analytic expression for the dynamical friction formula (Equation (16)) which is finite at the center of density for diverging galaxies (as mathematically shown in this paper Appendix A in the case of the family of Dehnen 1993 \( \gamma \) models) and smoothly connected to the usual local approximation is given and discussed.

3. The free parameters in the semi-analytic formula are tuned via comparison with high precision N-body simulations of massive object decay in a self consistent particle representation of the cuspy host galaxy (Section 3); the best values of the minimum impact parameter is systematically larger for circular \((e = 0)\) orbits than for radial \((e = 1)\).

4. An extensive set of orbits of different initial eccentricities for a massive test object in the N-body representation of the parent galaxy has been computed, showing both a good agreement with the semi-analytic formula as shown by Figures 6–11.

5. For any given initial orbital energy, the decay times of orbits of different eccentricities range within the interval defined by the radial (shortest) and circular (longest) case.

6. The ratio of the radial to circular decay times in the case of the \( \gamma = 1 \) density slope is about 1/2.

7. Global approximation formulas for the dynamical friction decay time in function of the relevant structural parameters are obtained, which show clearly how dynamical friction is maximized in massive host galaxies with a steeper central density profile, for higher eccentricity orbits of massive satellites.

8. As an example, our Milky Way, if represented in its central region as a moderate cuspy density \((\gamma = 1/2)\), should have lost, in a Hubble time, about 75% of the initial population of massive \((\geq 10^5 M_\odot)\) globular clusters, decayed into the innermost region.

9. The dynamical friction decay of test objects is altered significantly by the presence of a central massive BH if it has a mass comparable to the satellite BH; the decay time of the initial radial orbits is an increasing function of \( M_{\text{BH}} \).

10. On the other hand, when the central BH has a mass significantly greater than the satellite mass, the decay time is well estimated by our general formulas.

11. The dynamical friction time, \( \tau_{\text{df}} \), depends on the test object mass in a non-trivial manner, which is different from the usually adopted inverse linearity, \( \tau_{\text{df}} \propto M^{-1} \), resulting in \( \tau_{\text{df}} \propto M^{-0.67} \), instead.

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APPENDIX A

In this paper, we use as self consistent models of spherical, cuspy galaxies the distribution functions that represent the so-called Dehnen’s (or gamma) models (Dehnen 1993) that are the three-parameter density distributions following the laws

\[
\rho(r) = \frac{(3 - \gamma) M}{4\pi a^3} \frac{1}{(r/a)^\gamma (1 + r/a)^{4-\gamma}}, \tag{A1}
\]

where \( 0 \leq \gamma \leq 3 \) gives the slope of the centrally diverging (if \( \gamma > 0 \)) profile, \( a \) is the length scale, and \( M \) is the total mass of the model. The case \( \gamma = 0 \) corresponds to a central core, where density flattens. The cases \( \gamma = 1 \) and \( \gamma = 2 \) correspond to the classic Hernquist (1990) and Jaffe (1983) models, respectively.

The density profile \( \rho(r) \) of Equation (A1) can be expressed as a function of the potential \( \Psi(r) \) so that it is possible to apply the Eddington (1915) inversion formula to obtain the unknown distribution function \( f(\xi) \) as:

\[
f(\xi) = \frac{(3 - \gamma)}{2(2\pi^2 GMa)^3/2} \left\{ \int_0^\xi (1-x)^2 [\gamma + 2x + (4-\gamma)x^2] \sqrt{x^\gamma - \Psi} \right\} dx, \tag{A2}
\]

where

\[
x \equiv x(\Psi) = \begin{cases} e^{-\Psi} & \gamma = 2 \\ (1 - (2 - \gamma)\Psi)^{1/(2-\gamma)} & \gamma \neq 2\end{cases} \tag{A3}
\]

If \((2 - \gamma)^{-1}\) is an integer or half-integer, the integral in Equation (A2) can be calculated in terms of linear combination of hypergeometric series, easily reduced to elementary functions (Gradshteyn & Ryzhik 2007).
where cuspy matter density distributions, such as the case of the family of the “gamma” laws given by Equation (A1). The dynamical friction expression for the isotropic distribution function:

\[ f(\mathcal{E}) = \frac{M}{(GMa)^{3/2}} \left( \Psi(0) - \mathcal{E} \right)^{(\delta - \gamma)/(2(\delta - \gamma))} \left[ g_{\gamma}(\mathcal{E}) + B_{\gamma} \sqrt{\mathcal{E}} \left( \sum_{i=0}^{(2\gamma)/(2-\gamma)} b_i \mathcal{E}^i \right) \right], \]

(A4)

where \( g_{\gamma}(\mathcal{E}) \) is

\[
\begin{align*}
&\left(3 - 4\gamma\right)\sqrt{2\mathcal{E}} \sqrt{\Psi(0) - \mathcal{E}} - 3\sqrt{\Psi(0) - \mathcal{E}}^3 \log \left( \frac{1 + \sqrt{2\mathcal{E}}}{1 - \sqrt{2\mathcal{E}}} \right), \quad \gamma = 0 \\
&3 \arcsin \sqrt{\mathcal{E}}, \quad \gamma = 1 \\
&\frac{54675}{\sqrt{2}} \arcsin \sqrt{\frac{\mathcal{E}}{2}} - 450\sqrt{6} (3 - 2\mathcal{E})^{3/2} \log \left( \frac{3 + \sqrt{2\mathcal{E}} + \sqrt{2\mathcal{E}}^3}{3 - 2\mathcal{E}} \right), \quad \gamma = 4/3 \\
&3(3 + 32\mathcal{E} - 8\mathcal{E}^2) \arcsin \sqrt{\frac{\mathcal{E}}{2}}, \quad \gamma = 3/2 \\
&-33633600 \left( 83 - 512\mathcal{E} + 192\mathcal{E}^2 - 32\mathcal{E}^3 + 2\mathcal{E}^4 \right) \arcsin \sqrt{\frac{\mathcal{E}}{2}}, \quad \gamma = 7/4
\end{align*}
\]

and the values of \( A_{\gamma} \) and \( b_i \) are reported in Table 7.

Finally, the \( \gamma = 2 \) case (Jaffe’s model) has a formal expression that is not easily reduced into the form of Equation (A4); as known (Jaffe 1983) it is given by

\[ f(\mathcal{E}) = \frac{M}{2\pi^3(GMa)^{3/2}} \left[ F_{\gamma}(\sqrt{2\mathcal{E}}) - \sqrt{2} F_{\gamma}(\sqrt{2\mathcal{E}}) - \sqrt{2} F_{\gamma}(\sqrt{2\mathcal{E}}) + F_{\gamma}(\sqrt{2\mathcal{E}}) \right] \quad \text{(A6)} \]

where

\[ F_{\pm}(\eta) = e^{\mp \eta^2} \int_{0}^{\eta} e^{\pm \eta^2} d\eta. \]

(A7)

### A.1. The Convergence of the Dynamical Friction Integral

We study here the convergence of the dynamical friction integral in Equation (9), which is an improper integral in the case of the cuspy matter density distributions, such as the case of the family of the “gamma” laws given by Equation (A1). The dynamical friction integral is not singular only when \( \gamma = 0 \) (which means a central core), while it is for any \( \gamma > 0 \). In these cases, the adoption of the distribution functions in the case of high-binding energies, as expressed by Equation (A5), leads to a dynamical friction integral, which in a neighborhood of the origin of the phase-space (that is for a slow motion around the galactic center), assumes the form:

\[ \frac{d\mathbf{v}_m}{dt} = -A \int_{b_{\min}}^{b_{\max}} \int \left[ \frac{v^2}{(GMa)/a + (r/(r + a))^2} \right] \frac{Vb}{1 + b^2V^2G^{-2}(m + M)^{-2}} dV dM db, \]

(A8)

for \( \gamma = 0 \), and

\[ \frac{d\mathbf{v}_M}{dt} = -A \int_{b_{\min}}^{b_{\max}} \int \left[ \frac{v^2}{2 (GMa)/a + 1 - \gamma (r/(r + a))^{2-\gamma}} \right]^{-1} \frac{Vb}{1 + b^2V^2G^{-2}(m + M)^{-2}} dV dM db, \]

(A9)
for \( 0 < \gamma < 2 \), and
\[
\frac{d\mathbf{v}_M}{dt} = -A \int_{b_{\text{min}}}^{b_{\text{max}}} e^{-v^2/(GM)/a} \left( \frac{r}{r+a} \right)^{-2} \frac{Vb}{1+b^2V^4G^{-2}(m+M)^{-2}} \mathbf{v}_m \, db,
\]
(A10)
for \( \gamma = 2 \), and
\[
\frac{d\mathbf{v}_M}{dt} = -A \int_{b_{\text{min}}}^{b_{\text{max}}} \left\{ \frac{1}{2-\gamma} \left[ 1 - \left( \frac{r}{r+a} \right)^{2-\gamma} \right] - \frac{v^2}{2b} \right\}^{(6-\gamma)/(2(2-\gamma))} \frac{Vb}{1+b^2V^4G^{-2}(m+M)^{-2}} \mathbf{v}_m \, db,
\]
(A11)
for \( 2 < \gamma < 3 \).

The convergence of the above improper integrals can be studied by analyzing the properties of the integrands (which we call \( I_1 \) and \( I_2 \), respectively) for \( r/a \) and \( (1/2)v^2/(GM)/a \) going temporarily to zero (i.e., with the same order), introducing the auxiliary infinitesimal variable \( x \equiv r/a = v^2/(2b) \). This way, it is easily seen that for \( x \ll 1 \), the four integrands behave as:
\[
I_1 \approx x \left( 1 + \frac{1}{2}x \right)^{-1}, \quad \gamma = 0,
\]
(A12)
\[
I_2 \approx x^2 [x(1+x^{1-\gamma})]^{(6-\gamma)/(2(2-\gamma))}, \quad 0 < \gamma < 2,
\]
(A13)
\[
I_3 \approx e^{-x}, \quad \gamma = 2,
\]
(A14)
\[
I_4 \approx x^{(10-\gamma)/2}, \quad 2 < \gamma < 3
\]
(A15)
In the case of Equation (A9), if \( 0 < \gamma \leq 1 \), the behavior is \( x^{(2-3\gamma)/(2(2-\gamma))} \) whose exponent is \( \geq -1/2 \), implying the integral convergence; if \( 1 < \gamma < 2 \), the behavior is \( x^{-(2-\gamma)/2} \) whose exponent is \( \geq -1 \), which again guarantees convergence. In the cases of Equation (A10) and Equation (A11), the limits are again finite, different from zero when \( \gamma = 2 \) and equal to zero for \( 2 < \gamma < 3 \). Note that this latter case has not always had an acceptable physical meaning because it may give negative values for the distribution function around the origin of the phase space.

APPENDIX B

While the hypothesis of dynamical friction as cumulative effect of multiple hyperbolic encounters implies a growth of its effect at increasing values of \( M \), the integral in Equation (9) is such that the final dependence on \( M \) may be different than a simple proportionality to \( M \), although in the limit \( m \ll M \).

Actually, the expression for dynamical friction given by Equation (16) contains two additive terms. The local term (Equation (11)) has an explicit, dominant linear dependence on \( M \) factor of the integral and depends on proportionality to \( M \) and \( I_2 \) for \( 0 < \gamma < 1 \), \( I_3 \) for \( 2 < \gamma < 3 \). While in the opposite (strong encounter) regime which tends to a mass independent value (i.e., linearity of dynamical friction deceleration in \( m+M \)) only in the weak encounter regime
\[
\frac{b^2V^4}{G^2(m+M)^2} \gg 1,
\]
(B3)
while in the opposite (strong encounter) regime
\[
\frac{b^2V^4}{G^2(m+M)^2} \ll 1,
\]
(B4)
it shows an inverse quadratic dependence on \( m+M \) (lighter test masses would be more strongly decelerated).
Regarding the other two terms in Equation (B1), the first is usually set to 0 by the assumption of \( b_{\text{max}} \) as the fixed, characteristic length size of the system, while the second depends on the choice for \( b_{\text{min}} \). For a generic dependence of \( b_{\text{min}} \) on \( M \), the dependence on \( M \) through the explicit derivative of \( b_{\text{min}} \) with respect to \( M \) is modulated by the dependence on \( b_{\text{min}} \) in the integrand. If we impose on \( b_{\text{min}} \) the logical constraint large enough to allow 2-body hyperbolic encounters, only something like \( b_{\text{min}} = G(m + M)/v_\infty^2 \) (where \( v_\infty \) is the speed of the free test mass) is obtained, whose derivative respect to \( M \) is \( G/v_\infty^2 \). Hence, the second term has the likely dominant dependence on \( M \) in its explicit part in the integrand and, thus neglecting the dependence on \( M \) through \( f(b_{\text{min}}, v_m) \), we have that the regime:

\[
\frac{b_{\text{min}}^2 v^4}{G^2(m + M)^2} \gg 1, \tag{B5}
\]

gives a direct linear dependence on \( m + M \) (i.e., quadratic in dynamical friction), while in the opposite regime

\[
\frac{b_{\text{min}}^2 v^4}{G^2(m + M)^2} \ll 1, \tag{B6}
\]

the dependence is inversely linear in \( m + M \) (i.e., a logarithmic dependence of dynamical friction on \( m + M \)).

From the above, also assuming that dynamical friction is mainly contributed by the cumulation of many weak encounters, its dependence on mass is not simply linear in \( m + M \) but is altered by an additive \( \ln(m + M) \) dependence whose amplitude is modulated by the degree of spatial divergence of the distribution function. In any case, the expected dynamical friction dependence on \( m + M \) is something like \( \propto (m + M)^{\alpha} \), with \( 0 < \alpha < 1 \).

REFERENCES

Antonini, F., & Merritt, D. 2012, ApJ, 745, 83
Antonucci, R. 1993, ARAA, 31, 473
Bekenstein, J. D. 1973, ApJ, 183, 657
Binney, J. 1977, MNRAS, 181, 735
Blumenthal, G. R., Faber, S. M., Flores, R., & Primack, J. R. 1986, ApJ, 301, 27
Campanelli, M., Lousto, C., Zlochower, Y., & Merritt, D. 2007, ApJL, 659, L5
Capuzzo-Dolcetta, R. 1993, ApJ, 415, 616
Capuzzo-Dolcetta, R., & Donnarumma, I. 2001, MNRAS, 328, 645
Capuzzo-Dolcetta, R., & Mastrobuno-Battisti, A. 2009, A&A, 507, 183
Capuzzo-Dolcetta, R., & Miocchi, P. 2008a, ApJ, 681, 1136
Capuzzo-Dolcetta, R., & Miocchi, P. 2008b, MNRAS, 388, L69
Capuzzo-Dolcetta, R., & Spera, M. 2013, CoPhC, 184, 2528
Capuzzo-Dolcetta, R., & Vignola, L. 1997, A&A, 327, 130
Chandrasekhar, S. 1943, ApJ, 97, 255
Chandrasekhar, S. 1943a, ApJ, 97, 263
Chandrasekhar, S., & von Neumann, J. 1942, ApJ, 95, 489
Chandrasekhar, S., & von Neumann, J. 1943, ApJ, 97, 1
Dahlen, W. 1993, MNRAS, 265, 250
de Vaucouleurs, G. 1948, AnAp, 11, 247
Do, T., Martinez, G. D., Yelda, S., et al. 2013, ApJL, 779, L6
Dubinski, J. 1994, ApJ, 431, 617
Dullo, B. T., & Graham, A. W. 2012, ApJ, 755, 163
Eddington, A. S. 1915, MNRAS, 75, 366
Einasto, J. 1965, TrAlm, 5, 87
Eke, V. R., Navarro, J. F., & Steinmetz, M. 2001, ApJ, 554, 114
Epinat, B., & Genz, A. 1994, NuAlg, 8, 201
Fehlberg, E. 1972, NASA Technical Report, 381, 1
Gradsteyn, I., & Ryzhik, I. 2007, in Table of Integrals, Series and Products, ed. A. Jeffrey & D. Zwillinger (7th ed.; Amsterdam: Elsevier)
Graham, A. W. 2004, ApJL, 613, L33
Gualandris, A., & Merritt, D. 2008, ApJ, 678, 780
Hernquist, L. 1990, ApJ, 356, 359
Huss, A., Jain, B., & Steinmetz, M. 1999a, ApJ, 517, 64
Huss, A., Jain, B., & Steinmetz, M. 1999b, MNRAS, 308, 1011
Jaffe, W. 1983, MNRAS, 202, 995
Just, A., Khan, F. M., Berczik, P., Ernst, A., & Spurzem, R. 2011, MNRAS, 411, 653
Just, A., & Peñarrubia, J. 2005, A&A, 431, 861
Kalnajs, A. J. 1972, in IAU Colloq. 10: Gravitational N-Body Problem, ed. A. Jeffrey & D. Zwillinger (7th ed.; Amsterdam: Elsevier)
Kelement, S. D. 1979, CQGra, 16, A131
Kimmel, S., & Merritt, D. 2001, ApJ, 554, 114
Kroupa, P., & Merritt, D. 2002, MNRAS, 336, 46
Kroupa, P., & Merritt, D. 2005, ApJ, 631, 725
Kroupa, P., & Merritt, D. 2006, MNRAS, 372, 1451
Kroupa, P., & Merritt, D. 2007, MNRAS, 377, 371
Kroupa, P., & Merritt, D. 2008, MNRAS, 385, 234
Kroupa, P., & Merritt, D. 2009, New Astron., 15, 521
Kroupa, P., & Merritt, D. 2010, MNRAS, 405, 338
Kroupa, P., & Merritt, D. 2011, MNRAS, 418, 1258
Kroupa, P., & Merritt, D. 2012, MNRAS, 424, 2858
Liebtag, T., & Merritt, D. 2007, ApJ, 674, 1062
Llinares, J., & Merritt, D. 2006, MNRAS, 371, 1267
Lucas, S. A., & Merritt, D. 2006, ApJ, 653, 280
Merritt, D. 1991, ApJ, 372, 537
Merritt, D. 2000, ApJ, 531, L99
Merritt, D., Graham, A. W., Moore, B., Diemand, J., & Terzić, B. 2006, AJ, 132, 2685
Miosavlević, M., & Merritt, D. 2001, ApJ, 563, 34
Navarro, J. F., Frenk, C. S., & White, S. D. M. 1996, ApJ, 462, 563
Pesce, E., Capuzzo-Dolcetta, R., & Vietri, M. 1992, MNRAS, 254, 466
Read, J. I., Goerdt, T., Moore, B., et al. 2006, MNRAS, 373, 1451
Schutz, B. F. 1999, CQGra, 16, A131
Scott, N., & Graham, A. W. 2013, ApJ, 763, 76
Sérsic, J. L. 1963, BAAA, 6, 41
Tremaine, S. D. 1976, ApJ, 203, 72
Urry, C. M., & Padovani, P. 1995, PASP, 107, 803
Vicari, A., Capuzzo-Dolcetta, R., & Merritt, D. 2007, ApJ, 662, 797