Exact and quasi-exact solvability of two-dimensional superintegrable quantum systems.

I. Euclidean space

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Abstract

In this article we show that separation of variables for second-order superintegrable systems in two-dimensional Euclidean space generates both exactly solvable (ES) and quasi-exactly solvable (QES) problems in quantum mechanics. In this article we propose the another definition of ES and QES. The quantum mechanical problem is called ES if the solution of Schroedinger equation, can be expressed in terms of hypergeometrical functions $mF_n$ and is QES if the Schroedinger equation admit polynomial solutions with the coefficients satisfying the three-term or more higher order of recurrence relations.
1 Introduction

It is well known that $N$-dimensional nonrelativistic quantum systems described by the Hamiltonian

$$\mathcal{H} = -\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x^2_i} + V(x_1, x_2, ..., x_N)$$  \hspace{1cm} (1)$$

are integrable if there exist $N$ linearly independent and global integrals of motion $\mathcal{I}_\ell$, $\ell = 0, 1, .., N - 1$ and $\mathcal{I}_0 = \mathcal{H}$, commuting with the Hamiltonian (1) and with each other

$$[\mathcal{I}_\ell, \mathcal{H}] = 0, \quad [\mathcal{I}_\ell, \mathcal{I}_j] = 0, \quad \ell, j = 1, 2, ... N - 1. \hspace{1cm} (2)$$

This particular class of integrable systems is called superintegrable (this term was introduced first time by S.Rauch-Wojciechowski in [1]) if it is integrable and, in addition to this, possesses more integrals of motion than degrees of freedom. The additional integrals $\mathcal{L}_k$, commute with Hamiltonian

$$[\mathcal{L}_k, \mathcal{H}] = 0, \quad k = 1, 2, ... N, \hspace{1cm} (3)$$

but not necessarily with each other. If the number $D$ of linearly independent integrals takes the value $D = 2N - 1$ (N the number of degrees of freedom) then the system is called maximally superintegrable [2] and it is called minimally superintegrable if it has $D = 2N - 2$ integrals of motions. Three examples of this kind have been well-known a long time, namely the Kepler-Coulomb problem, the isotropic harmonic oscillator, and the nonisotropic oscillator with commensurable frequencies.

The existence of additional integrals of motion for these systems leads to many interesting properties unlike standard integrable systems. In particular, in quantum mechanics, there is phenomenon of accidental degeneracy when all energy eigenvalues are multiply degenerate. This property is intimately related to the existence of a dynamical symmetry group, a so-called hidden symmetry group, which contains properly the geometrical symmetry group describing this system. For instance, the dynamical group for the hydrogen atom is O(4) for the discrete spectrum [3], and the Lorentz group O(3,1) for the continuous spectrum [4]. For the isotropic harmonic oscillator it is SU(3) [5].

In classical mechanics the additional integrals of motion have the consequence that in the case of superintegrable systems in two dimensions and maximally superintegrable systems in three dimensions all finite trajectories are found to be periodic; in the case of minimally superintegrable systems in three dimensions all finite trajectories are found to be quasi-periodic [6].

One of the most important properties for superintegrable systems is multiseparability, the separation of variables for the Hamilton-Jacobi and Schrödinger equations in more than one orthogonal coordinate system [7, 8, 9]. For instance, the isotropic harmonic oscillator in three dimensions is separable in eight coordinate systems, namely in Cartesian, spherical, circular polar, circular elliptic, conical, oblate spheroidal, prolate spheroidal, and ellipsoidal coordinates. The Kepler-Coulomb potential is separable in four coordinate systems, namely in conical, spherical parabolic, and prolate spheroidal coordinates.
A systematic search for such systems in two- and three-dimensional Euclidean space was started in the pioneering work of Smorodinsky and Winternitz with collaborators in [10, 11, 12] and was continued in [2]. Particularly, in [11] it was shown that in two-dimensional Euclidean space there exist four superintegrable potentials (see Table 1.), three of them could be considered as the singular generalization of Kepler-Coulomb, circular oscillator and anisotropic oscillator systems. These results were extended for two- and thee-dimensional spaces with constant curvature (both positive and negative) [13], and on the complex two-dimensional sphere and Euclidean space [14, 15, 16, 17, 18]. The same program is continuing nowadays both in spaces with constant and nonconstant curvature [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30].

The wide applicability of superintegrable systems, both in physics and mathematical physics has stimulated further investigations. In the last fifteen years superintegrable systems (in spaces of constant curvature including flat Euclidean space) became a subject of investigation from many point of view: in [13, 18, 31, 32] via the path integral approach, in [19, 21, 22] by solving the Schrödinger equation with the help of the Niven ansatz [33], in [34, 35, 36, 37, 38, 39] from the purely algebraic approach. As has been shown by a number of authors, almost all superintegrable systems (excluding the pure N-dimensional Kepler-Coulomb and harmonic oscillator potentials) generate an algebraic structure which may be considered as a nonlinear extension of the Lie algebra (in classical mechanics Poisson algebras), namely a quadratic algebra. The general form of quadratic algebras, which are encountered in the case of two-dimensional quantum superintegrable systems has been investigated by Daskaloyannis [39].

In spite of all the above listed characteristics, superintegrable systems would not be so useful, except for the property of exact solvability of the all known superintegrable systems (at least in case of the second kind of superintegrability). More precisely this means that after any separation of variables each of the separated ordinary differential equations admits an exact solution. This question, again quite recently has been discussed in the literature [40, 41]. As mentioned in these papers the term exact solvability is defined quite differently by different authors. Indeed, in spite of an “intuitive” understanding of the term exactly solvable, no universal definition exists up to now.

On the other hand there are limiting cases of well-known one-dimensional exactly solvable systems, namely the harmonic oscillator and Coulomb problems with $\gamma/x^2$ ($\gamma > -1/4$) interaction, Morse potential, trigonometric and modified Pöschl-Teller potentials, trigonometric and hyperbolic Manning-Rosen potentials [43, 44], and the Natanson potential [45]. All these potentials have the general property that the Schrödinger spectral problem for bound state (or continuous state) has an explicit formula for the whole energy spectrum, and the eigenfunctions (up to the asymptotic ansatz or gauge transformations [40, 41]) are of hypergeometric type $1F_1, 2F_1$. For the bound states we have solutions in term of classical polynomials [46] whereas for continuous states just infinite series. Moreover, hypergeometric functions describe both the continuous quantum systems as well as the finite systems and appear also as solutions of related difference equations, for instance, the finite one- and two-dimensional oscillator expressed in terms of discrete variables polynomials: Krawchuk, Meixner and Hahn [47].

Thus, we propose another definition of exact-solvability: a quantum mechanical system

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1In [40, 41] (see also the recent paper [42]) we read that “an exactly solvable quantum mechanical system can be characterized by the fact that in its solution space one can indicate explicitly an infinite flag of functional linear spaces, which is preserved by the Hamiltonian” or the “Hamiltonian is exactly solvable if its spectrum can be calculated algebraically”.

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is called exactly solvable if the solutions of Schrödinger equation, can be expressed in terms of hypergeometric functions $\, mF_n$. (Basically, we are requiring that the coefficients in power series expansions of the solutions satisfy two-term recurrence relations, rather than recurrence relations of higher order.) It is obvious, that an $N$-dimensional Schrödinger equation is exactly-solvable if it is separable in some coordinate system and each of the separated equations is exactly-solvable. Further, We say that a superintegrable system is exactly solvable if is exactly solvable in at least one system of coordinates.

At first sight, such a definition of exactly-solvable problems may seem too narrow, but it leads us to distinguish two kind of models: 1) those which is possible to study analytically and 2) those which admit just numerical solutions (even if they admit polynomial solutions and their energy spectra can be calculated by solving an algebraic equation or system of equations).

The process of separation of variables in the $N$-dimensional Schrödinger equation leads to ordinary differential equations having as solutions many special functions of mathematical physics. A further complication of the separated equations involves the $N$ separation constants. In general we have a multiparameter eigenvalue problem [48]. It is possible to distinguish three different cases, namely when there is complete, partial or non-separability of the separation constants. It is obviously that in the case of complete separability (of separation constants) the initial $N$-dimensional Schrödinger equation splits into $N$ independent second order differential equations, each involving a single separation parameter. This situation occurs, for instance, in the case of separation of variables in the Helmholtz (free Schrödinger equation, which is also superintegrable) or the Schrödinger equation for the harmonic oscillator in Cartesian coordinates. The second "extremal" case, when complete non-separability exits, is realized, in separation of variables for the same problems but in ellipsoidal coordinates. In the last case the each separated second-order differential equation contains at once all separation constants (usually depending from dimensional or non-dimensional parameters) [8, 9], for which the simultaneous quantization becomes nontrivial.

The standard method of solution of a second order ordinary differential equation, obtained after separation of variables in $N$-dimensional Schrödinger equations, involves (after taking into account the asymptotic ansatz) expansions around one of the singular points of the differential equation (standard power-series method [49], or the so-called Hill-determinant method [50]). After that the problem reduces to solution of the recurrence relations for the expansion coefficients. If the coefficients obey a two-term recurrence relation, then the corresponding solution will be written in closed or analytic form or in terms of hypergeometric functions and we have an exactly-solvable problem. Such situations occur when separation of variables for superintegrable systems is possible in sub-group type coordinate (spherical, cylindrical and Cartesian) [51] and often in parabolic type coordinates. This method also powerfull when separation of variables possible in non-subgroup systems of coordinates as spheroidal or elliptic. In this case we arrive to high-order recurrence relation, the subsequent analysis of which, allow to investigate the behavior of the solution and to answer on the question whether exist the finite solution.

Actually there is another general approach for solving the Schrödinger equation by exploring the Niven-type ansatz [33] and essentially is based on existing of finite solutions. According to this methods the complete solution can be constricted without direct separation of variables and computed in terms of zeros of corresponding polynomial. This method have been using in papers [19, 21, 22] at the investigation of two- and three-dimensional superintegrable systems.
in Euclidean and curve spaces.

Consider now the problem of motion in the plane for a charged particle with two fixed Coulomb centers with coordinates \((\pm D/2, 0)\) (so-called plane two center problem)

\[
V(x, y) = -\frac{\alpha_1}{\sqrt{y^2 + (x + D/2)^2}} - \frac{\alpha_2}{\sqrt{y^2 + (x - D/2)^2}}
\]

This system is not superintegrable and separation of variables is possible only in two-dimensional elliptic coordinates (see eq. (92)). Upon the substitution \(\psi(\nu, \mu; D^2) = X(\nu; D^2)Y(\mu; D^2)\) and the separation constant \(A(D)\), the Schrödinger equation splits into a system of two ordinary differential equations

\[
\frac{d^2 X}{d\nu^2} + \left[ \frac{D^2 E}{2} \cosh^2 \nu + D(\alpha_1 + \alpha_2) \cosh \nu + A(D) \right] X = 0, \tag{5}
\]

\[
\frac{d^2 Y}{d\mu^2} - \left[ \frac{D^2 E}{2} \cos^2 \mu + D(\alpha_1 - \alpha_2) \cos \mu + A(D) \right] Y = 0. \tag{6}
\]

Both equations (5)-(6) belong to the class of non-exactly solvable problem. In generally the polynomial solutions do not exist even for the case of discrete spectrum \(E < 0^2\), and each of the wave functions \(X(\nu; D^2)\) and \(Y(\mu; D^2)\) is expressed as an infinite series with a three-term recurrence relation.

Let us now put \(\alpha_2 = 0\). Then the potential (4) transforms to the ordinary two-dimensional (2d) hydrogen atom problem, which is well-known as a superintegrable system [52, 53, 54] with dynamical symmetry group \(SO(3)\), and admits separation of variables in three systems of coordinates: polar, parabolic and elliptic. In this case we can see that the separation equations (5) and (6), namely

\[
\frac{d^2 X}{d\nu^2} + \left[ \frac{D^2 E}{2} \cosh^2 \nu + D\alpha_1 \cosh \nu + A(D) \right] X = 0, \tag{7}
\]

\[
\frac{d^2 Y}{d\mu^2} - \left[ \frac{D^2 E}{2} \cos^2 \mu + D\alpha_1 \cos \mu + A(D) \right] Y = 0. \tag{8}
\]

transform into each other by the change \(\mu \leftrightarrow i\nu\). Thus separation of variables in elliptic coordinates for the 2d hydrogen atom gives two functionally identical one-dimensional Schrödinger type equations with two parameters: coupling constant \(E\) and energy \(A(D)\) (correspondingly energy and separation constant for 2d), but one defined on the real and the other on the imaginary axis. In other word, instead the systems of differential equations (7)-(8), the task reduces to solving only to the one of equations (7) or (8) for what the "domain of definition" is complex plane. The requirement of finiteness for the wave functions in the complex plane permits only polynomial solutions (see for details [55]). As result we obtain simultaneously quantization of the energy spectrum

\[
E_n = -\frac{\alpha_1^2}{2(n + 1/2)^2}, \quad n = 0, 1, 2, \ldots \tag{9}
\]

\(^2\)To be completely correct let us note that the polynomial solution exist only for special values of parameters \(\alpha_1, \alpha_2\) and \(R\).
and the elliptic separation constant $A_s(D)$ where $s = 0, 1, 2, ... n$ (as a solution of an $n$th-degree algebraic equation). The polynomial solution defined by the help of finite series with the third-term recurrent relations for coefficients. They cannot be considered as exactly-solvable and maybe investigated only numerically. A similar situation occurs, for instance, in the case of the two-center problem in three-dimensional Euclidean space (the so-called prolate spheroidal radial and angular Coulomb wave functions) [56] and three-dimensional sphere (Heun wave functions) [57], where after eliminating one of the Coulomb centers the problems have reduced to superintegrable systems admitting only polynomial solutions. The presented above (and many others) examples let us to claim about deep connection of the notion of superintegrability and existing of polynomial solutions of the corresponding Schrödinger equation.

At the other hand side the each of equations (7) or (8) have the form of one dimensional Schrödinger equation with the parameter $E$ and for eigenvalue $A(D)$, and could be separately considered in the regions $\mu \in [0, 2\pi]$ or $\nu \in [0, \infty)$ correspondingly. Then for the arbitrary values of constant $E$ the solutions of eqs. (7) or (8) expressed via infinite series and only on the “surface energy” corresponding of 2d hydrogen atom (9) the solutions split onto polynomial and nonpolynomial sectors (each of these sectors are non complete) and for fixed number $n$, only part of eigenvalue $A_s(D)$, $(s = 0, 1, 2...n)$ are possible to calculate as $n$th-degree algebraic equation. We can say that the eqs. (7) and (8) ”remember” for its polynomial solutions. It is obviously that the spectrum of $A_s(D)$, $(s = 0, 1, 2...n)$ and solutions in polynomial form of the each of equations (7)-(8) coincide with the eigenvalue of separation constant and the wave function after the reduction to one of the regions $\mu \in [0, 2\pi]$ or $\nu \in [0, \infty]$) for 2d hydrogen atom.

This phenomena have been intensively discussed in literature in the late 1980’s and was determined as quasi-exactly solvability (this term first time introduced by Turbiner and Ushveridze in [58]) and such models called as quasi-exactly solvable systems [59, 60, 61] (see also [62] and references therein). The crucial example stimulated the investigation of quasi-exactly solvable systems is the hamiltonian (1) with the anharmonic potential

$$V(x) = \frac{1}{2} \omega^2 x^6 + 2\beta \omega^2 x^4 + (2\beta^2 \omega^2 - 2\delta \omega - \lambda)x^2 + 2(\frac{\delta - \frac{1}{4}}{\omega^2} - \frac{\omega}{8} - \frac{\lambda}{16})x^2,$$  \hspace{1cm} (10)

where $\omega, \beta, \delta > 1/2$ and $\lambda$ are the constants. As it was notice of many authors [63, 64, 65], this systems only for special values of constant $\lambda = \omega(2n + 1)$, $(n = 0, 1, 2...)$ admit partial polynomial solutions

$$\Psi(x) \approx x^{2\delta - \frac{1}{4}} e^{-x^2 - \beta \omega x^2} P_n(x^2).$$  \hspace{1cm} (11)

There are two different approaches in investigation of quasi-exactly solvable systems. In the base of algebraic approach formulated by Turbiner in [59] the fact of quasi-exactly solvability was explain in term of “hidden symmetry algebra” $sl(2, R)$. More precisely it means following: the one-dimensional Hamiltonian (1) after suitable changes of variable $z = \xi(x)$ and ”gauge transformation" $H = e^{-\alpha(z)} H e^{\alpha(z)}$ can be written in form

$$H = \sum_{a,b=0,\pm} C_{ab} J_a J_b + \sum_{a=0,\pm} C_a J_a$$  \hspace{1cm} (12)

\text{3For example when $E_n = 0$ ($n \to \infty$) the equations (7) and (8) transforms to periodic and modifying Matie equations, which are non-exactly-solvable.}

\text{4Really this is not hidden dynamical symmetry in usual sence because the hamiltonian (12) belong to the enveloping algebra but not a Casimir.}
where the first-order differential operators \( \{ J_\pm, J_0 \} \) span to the commutation relations for algebra \( sl(2, R) \) [59].

The mentioned above analysis for the 2d hydrogen atom have shown, that in spite of the elegance of algebraic approach, the phenomena of quasi-exactly solvability have really more deeper roots that it can be explain via "one-dimensional" model (12). The other examples are hydrogen atom and oscillator problems on two and three-dimensional spheres [19, 66] and two-dimensional hyperboloids [22], which generate not only hyperbolic and trigometric but elliptic type of quasi-exactly solvable systems (see also oldest articles [67, 55, 68, 69]). We can also mention Lamè polynomials. They comes from separation of variables for the Helmholtz (also superintegrable!) or Schrödinger equation in elliptic coordinates on the two-dimensional sphere. As it have been also determined in [37] (without showing of the mechanism of this phenomena) some of the quasi-exactly solvable systems can be obtained through dimensional reduction from the two and three-dimensional superintegrable models with quadratic invariants (second-order superintegrability).

The second approach, known as analitic, was formulated by Uschveridze (see for example [60, 61, 62]) and really represent a one-dimensional reduction of the Niven-Stilites method for solving of many-parametric differential equation as generalized Lame equation (or ellipsoidal equation) [33]. The solution in this method are presented in the term of zeros of polynomials \( P_n(x^2) \). Then the wave function (11) can be rewritten in the form

\[
\Psi(x) \approx x^{2\delta - \frac{1}{2}} e^{-\frac{1}{2}x^4 - \beta \omega x^2} \Pi_{i=0}^{n_2}(x^2 - \xi_i),
\]

where the numbers \( (\xi_1, \xi_2, ... \xi_n) \) satisfying the systems of \( n \) algebraic equations (see section 2.3). Accordingly to the oscillation theorem the number of zeros in physical interval \( \xi_i \in [0, \infty) \) enumerate the ground state and first \( n \) - excitations, describing in terms of all zeros (complete solutions of the systems of algebraic equations and including non physical section \( \xi_i \in (-\infty, 0] \)) as

\[
E = 4\delta \left[ \beta \omega + \sum_{i=1}^{n} \frac{1}{\xi_i} \right].
\]

The two natural questions appering in such approach: what is the physical meaning of the negative zeros \( \xi_i \), and why in the correct formula of energy spectrum (14) are participating all \( n \) zeros of the polynomial \( P_n(x^2) \)?

With this article we begin the new investigation of all known two- and three-dimensional second order superintegrable systems on curved spaces (Euclidean, sphere, hyperboloid and pseudo-euclidean) base on the separation of variables and direct solutions of the Schrödinger equation. We pay special attention to non-subgroup type coordinates and prove the existence the polynomial solution for this systems, which we found surprisingly that never have been done before.

We also demonstrate that the fact of quasi-exactly solvability are directly related with the multiseparability of the second-order superintegrable systems, from one hand side, and with presence of polynomial solution in this systems on the other hand side.

The present work devoted only two (singular anisotropic and singular circular oscillators) from four famous superintegrable systems in two-dimensional Euclidean space (see first two
potentials in Table 1.). The next two systems maybe transformed (only for the discrete spectrum!) to singular circular oscillator (for $V_3$) or ordinary shifted oscillator (for $V_4$) systems by the help of Levi-Civita mapping [70].

Table 1: Superintegra potentials in two-dimensional Euclidean space.

| Potential $V(x, y)$ | Coordinate System |
|---------------------|-------------------|
| $V_1 = \frac{1}{2}\omega^2(4x^2 + y^2) + k_1x + \frac{1}{2}k_2^2 - \frac{1}{4}$ | Cartesian |
| $V_2 = \frac{1}{2}\omega^2(x^2 + y^2) + \frac{1}{2}\left(\frac{k_1^2 - \frac{1}{4}}{x^2} + \frac{k_2^2 - \frac{1}{4}}{y^2}\right)$ | Parabolic |
| $V_3 = -\frac{\alpha}{\sqrt{x^2 + y^2}} + \frac{1}{4}\frac{1}{\sqrt{x^2 + y^2}}\left(\frac{k_1^2 - \frac{1}{4}}{\sqrt{x^2 + y^2} + x} + \frac{k_2^2 - \frac{1}{4}}{\sqrt{x^2 + y^2} - x}\right)$ | Polar |
| $V_4 = -\frac{\alpha}{\sqrt{x^2 + y^2}} + \frac{1}{4}\frac{1}{\sqrt{x^2 + y^2}}\left(\beta_1\sqrt{\sqrt{x^2 + y^2} + x} + \beta_2\sqrt{\sqrt{x^2 + y^2} - x}\right)$ | Mutually Parabolic |
2  Singular anisotropic oscillator

Let us first consider the potential \((k_2 > 0)\)

\[
V_1(x, y) = \frac{1}{2} \omega^2 (4x^2 + y^2) + k_1 x + \frac{k_2^2 - \frac{1}{4}}{2y^2}
\]

(second potential on the table 1.), which we will call the singular anisotropic oscillator. The Schrödinger equation has the form

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \Psi + \left[ 2E - \omega^2 (4x^2 + y^2) - 2k_1 x - \frac{k_2^2 - \frac{1}{4}}{y^2} \right] \Psi = 0.
\]

(16)

For \(k_2 > 1/2\) the singular term at \(y = 0\) is repulsive and the motion takes place only on one of the half planes \((-\infty < x < \infty, y > 0)\) or \((-\infty < x < \infty, y < 0)\), whereas for \(0 < k_2 < 1/2\) in whole plane \((x, y)\). There are two coordinate systems of relevance here: Cartesian and parabolic coordinates.

2.1  Cartesian bases

Separation of variables for the eq. (16) in Cartesian coordinates leads to the two independent one-dimensional Schrödinger equations

\[
\frac{d^2 \psi_1}{dx^2} + \left( 2\lambda_1 - 4\omega^2 x^2 - 2k_1 x \right) \psi_1 = 0.
\]

(17)

\[
\frac{d^2 \psi_2}{dy^2} + \left( 2\lambda_2 - \omega^2 y^2 - \frac{k_2^2 - \frac{1}{4}}{y^2} \right) \psi_2 = 0.
\]

(18)

where

\[
\Psi(x, y; k_1, \pm k_2) = \psi_1(x; k_1) \psi_2(y; \pm k_2)
\]

(19)

and \(\lambda_1, \lambda_2\) are two Cartesian separation constant with \(\lambda_1 + \lambda_2 = E\).

The equation (18) represents the well-known linear singular oscillator system studied in detail (see for instance the books [71, 72] and articles [11, 20, 73]). It is an exactly-solvable problem and has been used in many applications, for example as a model in \(N\) - body problems [74], or fractional statistics and anyons [75, 76]. The complete set of orthonormalized eigenfunctions, (on 1/2) in the interval \(0 < y < \infty\) of eq. (18), can be express in terms of finite confluent hypergeometric series or Laguerre polynomials

\[
\psi_{n_2}(y; \pm k_2) = \frac{2\omega(1\pm k_2)n_2!}{\Gamma(n_2 \pm k_2 + 1)} y^{\frac{1}{2} \pm k_2} e^{-\frac{1}{2} \omega y^2} L_{n_2}^{\pm k_2}(\omega y^2)
\]

(20)

and \(\lambda_2 = \omega(2n_2 + 1 \pm k_2)\). We assume that the positive sign at the \(k_2\) has to be taken if \(k_2 > \frac{1}{2}\) and both the positive and the negative sign must be taken if \(0 < k_2 < \frac{1}{2}\), so that the polynomials have finite norm. Let us also note that unlike the potential (15) the wave function
is not invariant under the replacement $k_2 \rightarrow -k_2$ and splits into two families of solutions that transform to one another under this change.

The second equation (17) easily transforms to the ordinary one-dimensional oscillator problem. In terms of Hermite polynomials the orthonormal solution (in region $-\infty < x < \infty$) is

$$\psi_{n_1}(x; k_1) = \left(\frac{2\omega}{\pi}\right)^{1/4} \frac{e^{-\omega z^2}}{\sqrt{2^{n_1} n_1!}} H_{n_1}(\sqrt{2\omega} z), \quad z = x + \frac{k_1}{4\omega^2}, \quad (21)$$

and where $\lambda_1 = \omega(2n_1 + 1) - \frac{k_1^2}{8\omega^2}$. Thus the complete energy spectrum is

$$E = \lambda_1 + \lambda_2 = \omega[2n + 2 \pm k_2] - \frac{k_1^2}{8\omega^2}, \quad n = n_1 + n_2 = 0, 1, 2, ... \quad (22)$$

and the degree of degeneracy for fixed principal quantum number $n$ is $(n + 1)$. Finally note that the separation of variables in Cartesian coordinates leads to two exactly solvable one-dimensional Schrödinger equations and the complete wave function may be constructed with the help of formulas (20), (21) and (19).

### 2.2 Parabolic bases

#### 1.2.1. Separation of variables.

Parabolic coordinates $\xi$ and $\eta$ are connected with the Cartesian $x$ and $y$ by

$$x = \frac{1}{2}(\xi^2 - \eta^2), \quad y = \xi \eta, \quad \xi \in \mathbb{R}, \eta > 0. \quad (23)$$

The Laplacian and two-dimensional volume element are given by

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{\xi^2 + \eta^2} \left( \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right), \quad dv = dxdy = (\xi^2 + \eta^2)d\xi d\eta. \quad (24)$$

The Schrödinger equation in parabolic coordinates (23) is

$$\frac{1}{\xi^2 + \eta^2} \left( \frac{\partial^2 \Psi}{\partial \xi^2} + \frac{\partial^2 \Psi}{\partial \eta^2} \right) + \left[ 2E - \omega^2(\xi^4 - \xi^2\eta^2 + \eta^4) - k_1(\xi^2 - \eta^2) - \frac{k_2^2 - \frac{1}{4}}{\xi^2\eta^2} \right] \Psi = 0. \quad (25)$$

Upon substituting

$$\Psi(\xi, \eta) = X(\xi)Y(\eta)$$

and introducing the parabolic separation constant $\lambda$, the equation (25) split into two ordinary differential equations:

$$\frac{d^2 X}{d\xi^2} + \left( 2E\xi^2 - \omega^2\xi^6 - k_1\xi^4 - \frac{k_2^2 - \frac{1}{4}}{\xi^2} \right) X = -\lambda X, \quad (26)$$

$$\frac{d^2 Y}{d\eta^2} + \left( 2E\eta^2 - \omega^2\eta^6 + k_1\eta^4 - \frac{k_2^2 - \frac{1}{4}}{\eta^2} \right) Y = +\lambda Y. \quad (27)$$
Equations (26) and (27) are transformed into one another by change $\xi \leftrightarrow i\eta$. We have

$$\Psi(\xi, \eta; E, \lambda) = C(E, \lambda)Z(\xi; E, \lambda)Z(i\eta; E, \lambda) \quad (28)$$

where $C(E, \lambda)$ is the normalization constant determined by the condition

$$\int_0^\infty d\eta \int_{-\infty}^\infty d\xi (\xi^2 + \eta^2)|\Psi(\xi, \eta; E, \lambda)|^2 = 1 \quad (29)$$

and the function $Z(\mu; E, \lambda)$ is a solution of the equation

$$\left[-\frac{d^2}{d\mu^2} + \left(\omega^2 \mu^6 + k_1 \mu^4 - 2E \mu^2 + \frac{k_2}{\mu^2} - \frac{1}{4}\right)\right]Z(\mu; E, \lambda) = \lambda Z(\mu; E, \lambda). \quad (30)$$

Thus, at $\mu \in (-\infty, \infty)$ we have eq. (26) and at $\mu \in [0, i\infty)$ - the eq. (27). Note that in the complex $\mu$ domain the “physical” region is just the two lines $\text{Im} \mu = 0$ and $\text{Re} \mu = 0, \text{Im} \mu > 0$.

1.2.2. Recurrence relations.

Consider now the equation (30). To solve it we make the substitution

$$Z(\mu; E, \lambda) = \exp \left(-\frac{\omega}{4} \mu^4 - \frac{k_1}{4\omega} \mu^2\right) \mu^{\frac{1}{2}k_2} \psi(\mu; E, \lambda), \quad (31)$$

and obtain the differential equation

$$z \frac{d^2 \psi}{dz^2} + \left[\frac{2(1 \pm k_2)}{\mu} - 2\omega \left(\mu^2 + \frac{k_1}{2\omega^2}\right)\right] \frac{d\psi}{dz} + \left[2\tilde{E} \mu^2 + \tilde{\lambda}\right] \psi = 0 \quad (32)$$

where

$$\tilde{E} = E + \frac{k_1^2}{8\omega^2} - \omega(2 \pm k_2), \quad \tilde{\lambda} = \lambda - \frac{k_1}{\omega}(1 \pm k_2). \quad (33)$$

Passing to a new variable $z = \mu^2$ in eq. (32), we have

$$z \frac{d^2 \psi}{dz^2} + \left[(1 \pm k_2) - \omega z \left(z + \frac{k_1}{2\omega^2}\right)\right] \frac{d\psi}{dz} + \left[\frac{1}{2} \tilde{E} z + \frac{1}{4} \tilde{\lambda}\right] \psi = 0 \quad (34)$$

We express the wave function $\psi(z)$ in the form

$$\psi(z; E, \lambda) = \sum_{s=0}^{\infty} A_s(E, \lambda) z^s \quad (35)$$

The substitution (35) in eq. (34) leads to the following three-term recurrence relation for the expansion coefficients $A_s \equiv A_s(E, \lambda),$

$$(s + 1)(s + 1 \pm k_2) A_{s+1} + \frac{1}{4} \left[\lambda - \frac{k_1}{\omega}(2s + 1 \pm k_2)\right] A_s$$

$$+ \frac{1}{2} \left[E + \frac{k_1}{8\omega^2} - \omega(2s \pm k_2)\right] A_{s-1} = 0, \quad (36)$$

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with the initial conditions $A_{-1} = 0$ and $A_0 = 1$.

1.2.3. Asymptotic behavior.

To understand the behavior of the coefficients for large $s$ we use continued fractions theory [77]. Setting

$$\frac{A_s}{A_{s-1}} = \xi_s f(s)$$

$$f(s) = \sqrt{\frac{\Gamma(\frac{s}{2} + \frac{1}{2}) \Gamma(\frac{s}{2} + \frac{1}{2} + \frac{k_2}{2} + \frac{k_1}{2} + \frac{1}{2}) \Gamma(\frac{\alpha}{2\omega} \pm \frac{k_2}{2} + \frac{s}{2} + \frac{1}{2})}{\Gamma(s + 1) \Gamma(\frac{s}{2} + \frac{1}{2} + \frac{k_1}{2} + \frac{1}{2}) \Gamma(\frac{\alpha}{2\omega} \pm \frac{k_2}{2} + \frac{s}{2} + \frac{1}{2})}},$$

where $\Gamma(z)$ is the Gamma function, we can write the recurrence relation (36) in the standard form

$$\xi_s = 1 + \frac{1}{b_s + \xi_{s+1}},$$

$$b_s = \sqrt{\frac{2 \Gamma(\frac{s}{2} + \frac{1}{2}) \Gamma(\frac{s}{2} + \frac{1}{2} + \frac{k_1}{2} + \frac{k_2}{2} + \frac{1}{2}) \Gamma(\frac{\alpha}{2\omega} \pm \frac{k_2}{2} + \frac{s}{2} + \frac{1}{2})}{\Gamma(s + 1) \Gamma(\frac{s}{2} + \frac{1}{2} + \frac{k_1}{2} + \frac{1}{2}) \Gamma(\frac{\alpha}{2\omega} \pm \frac{k_2}{2} + \frac{s}{2} + \frac{1}{2})^{2^3}},$$

where $\alpha = -(E + k_1/(8\omega^2))/2$. Note that

$$f(s + 1) f(s) = \omega \left( \frac{\alpha}{\omega} \pm k_2 + s + 1 \right) \left( \frac{\alpha}{\omega} \pm k_2 + s + 1 \right).$$

Stirling’s formula for the Gamma function $\Gamma(z) = z^{-1/2} e^{-z} \sqrt{2\pi} (1 + O(1/z))$ as $|z| \to \infty$ with $|\arg z| < \pi$, gives $f(s) = -\sqrt{\frac{\pi}{s}} (1 + O(1/s))$ and $b_s = \pm \frac{k_2}{2\sqrt{s\omega}} (1 + O(1/s))$. In the following we take $k_1 \leq 0$, $k_2$, $\lambda$, $E$ real and $\omega > 0$. Without loss of generality we can assume $b_s$ is positive for sufficiently large $s$ since, otherwise, we could make the replacements $b_s \to -b_s$, $\xi_s \to -\xi_s$.

Since $\sum b_s = \infty$, it is a consequence of the Seidel-Stern theorem that the formal continued fraction expressions for the $\xi_s$ converge:

$$\xi_s = \lim_{n \to \infty} \frac{A_n(s)}{B_n(s)},$$

where

$$\begin{pmatrix} A_{-1}(s) \\ B_{-1}(s) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} A_0(s) \\ B_0(s) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and

$$\begin{pmatrix} A_n(s) \\ B_n(s) \end{pmatrix} = b_{n+s} \begin{pmatrix} A_{n-1}(s) \\ B_{n-1}(s) \end{pmatrix} + \begin{pmatrix} A_{n-2}(s) \\ B_{n-2}(s) \end{pmatrix}, \quad n \geq 1.$$
Furthermore the relation \( A_n B_{n-1}^{(s)} - A_{n-1}^{(s)} B_n = (-1)^{n-1} \) holds for all \( n \geq 0 \), which implies
\[
\frac{A_n^{(s)}}{B_n^{(s)}} = \frac{A_{n-1}^{(s)}}{B_{n-1}^{(s)}} = (-1)^{n-1} \frac{B_{n-1}^{(s)}}{B_n^{(s)}}.
\]
This result in turn implies that the sequence \( A_{2n}^{(s)} / B_{2n}^{(s)} \) is, for large \( s \) and \( n \) monotone increasing in \( n \) and goes to \( \xi_s \) in the limit, whereas \( A_{2n+1}^{(s)} / B_{2n+1}^{(s)} \) is monotone decreasing in \( n \) and goes to \( \xi_s \) in the limit. For example,
\[
\frac{A_{2n+2}^{(s)}}{B_{2n+2}^{(s)}} - \frac{A_{2n}^{(s)}}{B_{2n}^{(s)}} = \frac{b_{2n+2+s}}{B_{2n}^{(s)} B_{2n+2}^{(s)}}. \tag{41}
\]
It follows from (39), (41) that
\[
\xi_s = \frac{A_0^{(s)}}{B_0^{(s)}} + \sum_{n=1}^{\infty} \frac{b_{2n+2+s}}{B_{2n}^{(s)} B_{2n+2}^{(s)}}. \tag{42}
\]
Simple estimates using the recurrence relations (40) give
\[
B_{2n}^{(s)} > 1 + b_{s+1} \sum_{m=1}^{n} b_{2m+s}, \quad B_{2n+1}^{(s)} > \sum_{m=0}^{n} b_{2m+1+s},
\]
Substituting these results into the identities
\[
B_{2n}^{(s)} = \sum_{m=1}^{n} b_{2m+s} B_{2m-1}, \quad B_{2n+1}^{(s)} = \sum_{m=0}^{n} b_{2m+s+1} B_{2m}
\]
we get refined upper bounds for \( B_{2n}^{(s)}, B_{2n+2}^{(s)} \). We can approximate the sum \( \sum_{m=s}^{n} 1/\sqrt{m} \) by the integral \( \int_{s}^{n} \frac{1}{\sqrt{x}} \, dx \) and use similar approximations to get an upper bound for the series (42):
\[
|\xi_s| < \kappa_1 \int_{0}^{\infty} \frac{dy}{\sqrt{y + s(y^2 + 1)}} + \kappa_2
\]
for positive constants \( \kappa_j \) independent of \( s \). This shows that \( |\xi_s| \) is uniformly bounded in \( s \). Since \( \xi_{s+1} = -b_s + 1/\xi_s \) and \( b_s \rightarrow 0 \) as \( s \rightarrow \infty \) it is also true that \( |1/\xi_s| \) is uniformly bounded in \( s \).

It follows from (38) that
\[
\xi_{s+1} - \xi_{s-1} = \frac{\xi_{s-1} - (b_s + \xi_{s-1})(1 - b_{s-1}\xi_{s-1})}{1 - b_{s-1}\xi_{s-1}}.
\]
Now choose \( s_0 \) so large that \( b_{s+1} < b_s \) and \( b_s \xi_s < 1 \) for all \( s \geq s_0 \). Note from this identity that if \( \xi_{s_1-1} > 1 \) for some \( s_1 > s_0 \) then \( \xi_{s+1} > \xi_{s_1-1} > 1 \). Thus the sequence \( \xi_{s_1+2k-1} \) is monotonically increasing for all \( k \geq 0 \). Since \( |\xi_s| \) is bounded, it follows that in this case \( \lim_{k \rightarrow \infty} \xi_{s_1+2k-1} = \xi_+ \) exists, and \( \xi_+ > 1 \). Since \( \xi_{s+1} = -b_s + 1/\xi_s \), \( b_s \rightarrow 0 \) as \( s \rightarrow \infty \) and \( |1/\xi_s| \) is uniformly bounded in \( s \), then the sequence \( \xi_{s_1+2k} \) is also convergent, \( \lim_{k \rightarrow \infty} \xi_{s_1+2k} = \xi_- \) where \( 0 < \xi_- < 1 \).

The other possibility is that \( \xi \leq 1 \) for all \( s \geq s_0 \). Since \( 1/\xi_s - \xi_{s+1} = b_s \rightarrow 0 \) as \( s \rightarrow \infty \), and \( 1/\xi_s \geq 1 \), \( \xi_{s+1} \leq 1 \) for all \( s \geq s_0 \) it follows that \( \lim_{k \rightarrow \infty} \xi_k = \xi_+ = \xi_- = 1 \). Thus in all cases the sequences \( \xi_{2k} \) and \( \xi_{2k+1} \) converge.
We conclude that
\[ \frac{A_{s+1}}{A_s} = f(s + 1)\xi_s = \sqrt{\frac{\omega\xi_+}{s}}(1 + O(1/s)), \quad \xi_+\xi_- = 1, \]
depending on whether \( s \) is even or odd. Thus asymptotically \( A_s \sim \sqrt{\xi_+\sqrt{\frac{\omega}{s^2}}} \), depending on whether \( s \) is even or odd, and
\[ \psi(z) \sim \sum \frac{\sqrt{\xi_+(\sqrt{\omega z})^s}}{s!}. \] (43)

Then we have for \( z > 0 \) [the case of eq. (26)]
\[ \sum \frac{\sqrt{\xi_+(\sqrt{\omega z})^s}}{s!} > \sqrt{\sum \frac{\xi_+(\omega z^2)^s}{s!}} = \xi_+ \cosh \left( \frac{\omega}{2} z^2 \right) + \xi_- \sinh \left( \frac{\omega}{2} z^2 \right). \] (44)

This function does not belong to the Hilbert space. If \( k_1 > 0 \) then we must make the replacements \( b_s \to -b_s \) and \( \xi_s \to -\xi_s \). This has the effect of replacing \( z \) by \( -z \) in (43). Now the asymptotic solution is oscillatory. However, then for \( z < 0 \) [the case of eq. (27)] the solution doesn’t belong to the Hilbert space. The solution we have found is the minimal solution of the three-term recurrence relations. There is a linearly independent solution, but the coefficients grow more rapidly than the minimal solution coefficients.

1.2.4. Energy spectrum and separation constant.

Thus the function \( Z(\mu) \) cannot converge simultaneously at large \( \mu \) for real and imaginary \( \mu \) and therefore the series (35) should be truncated. The condition for series (35) to be truncated results in the energy spectrum giving the same formula (22) and now the coefficients \( A_s \equiv A_{nq}(k_1, \pm k_2) \) satisfy the following relation
\[ (s + 1)(s + 1 \pm k_2)A_{s+1} + \beta_s A_s + \omega(n + 1 - s)A_{s-1} = 0, \] (45)
\[ \beta_s = \frac{\lambda}{4} - \frac{k_1}{4\omega}(2s + 1 \pm k_2). \]

The three-term recurrence relation (45) represents a homogeneous system of \( n + 1 \) - algebraic equations for \( n + 1 \) - coefficients \( \{A_0, A_1, A_2, ... A_n\} \). The requirement for the existence of a non-trivial solution leads to a vanishing of the determinant
\[ D_n(\lambda) = \begin{vmatrix} \beta_0 & 1 \pm k_2 \\ \omega n & \beta_1 & 2(2 \pm k_2) \\ . & . & . & . \\ . & . & . & . \\ 2\omega & \beta_{n-1} & n(n \pm k_2) \\ . & \omega & \beta_n \end{vmatrix} = 0 \] (46)

The roots of the corresponding algebraic equation give us the \( (n+1) \) eigenvalues of the parabolic separation constant \( \lambda_n(k_1, \pm k_2) \). It is known that all roots for a such determinants are real and distinct [78]. Thus all values of the separation constant are real and can be enumerated with the help of the integer \( q \), namely the values are \( \lambda_n(k_1, \pm k_2) \to \lambda_{nq}(k_1, \pm k_2) \), where \( 0 \leq q \leq n \). The degeneracy for the \( n \) - energy state, as in the Cartesian case, equals \( n + 1 \).
Note that eq. (46) is invariant under the simultaneous transformation $k_1 \to -k_1$ and $\lambda \to -\lambda$. Thus if one of the $\lambda = \lambda_n(k_1, \pm k_2)$ is root of eq. (46), then $\lambda = -\lambda_n(-k_1, \pm k_2)$ is also a root of the same equation. We see that for the odd energy state ($n$-odd) the range of $\lambda_{nq}(k_1, \pm k_2)$ splits into two subset $\lambda_{nq}^{(1)}$ and $\lambda_{nq}^{(2)}$ connected with each to other by the relation $\lambda_{nq}^{(1)}(k_1, \pm k_2) \leftrightarrow -\lambda_{nq}^{(2)}(-k_1, \pm k_2)$. For $n$-even, always there exists the additional root $\lambda_{nq}(k_1, \pm k_2) = -\lambda_{nq}(-k_1, \pm k_2)$, which equals zero when $k_1 = 0$.

1.2.5. Wave functions.
We will term the polynomial solutions of eq.(34), or eq.(32), as $M_{nq}(z; k_1, \pm k_2)$, and the function (31) as $T_{nq}(z; k_1, \pm k_2)$ 5. Then the physical admissible solutions of eq. (34) have the form

$$M_{nq}(z; k_1, \pm k_2) \equiv \psi(z; E, \lambda) = \sum_{s=0}^{n} A_{s}^{nq}(k_1, \pm k_2) z^s,$$

(47)

and the corresponding solution of eq. (31) is

$$T_{nq}(\mu; k_1, \pm k_2) = exp \left( -\frac{\omega}{4} \mu^4 - \frac{k_1}{4\omega} \mu^2 \right) \mu^\frac{1}{2} \pm k_2 M_{nq}(\mu^2; k_1, \pm k_2)$$

(48)

Observe that parabolic wave functions (as also Cartesian wave functions) split into two classes and transform to each other via $k_2 \to -k_2$. In the case $k_2 = 0$ (when the centrifugal term disappears), the solution (48) becomes an even and odd parity wave function under exchange $\mu \to -\mu$.

It is known that there exists a direct connection between the quantum numbers $q$ and numbers of zeros of the polynomial (47) and, therefore, the eigenvalues of the separation constant $\lambda_{nq}(k_1, \pm k_2)$ may be ordered by the numbers of nodes of the wave function $T_{nq}(\mu; k_1, \pm k_2)$. Indeed by we will see that these are orthogonal polynomials, hence [49], all the $n$ - zeros of the $M_{nq}(z; k_1, \pm k_2)$ are situated on the real axis $-\infty < z < \infty$, and all zeros have multiplicity one. Assume that the separation constants $\lambda_{nq}(k_1, \pm k_2)$ are numerated in ascending order, i.e

$$\lambda_{n0}(k_1, \pm k_2) < \lambda_{n1}(k_1, \pm k_2) < \ldots < \lambda_{n,n-1}(k_1, \pm k_2) < \lambda_{n,n}(k_1, \pm k_2)$$

(49)

then according to the oscillation theorem [79], the quantum number $q$ also enumerates the zeros of polynomials $M_{nq}(z; k_1, \pm k_2)$ in the region $z > 0$, or the real axis of $\mu$, (see eqs. (52) - (60)). Let us now introduce two quantum number $q_1$ and $q_2$, which determine the zeros of polynomials $M_{nq}(z; k_1, \pm k_2)$ for $z > 0$ and $z < 0$, correspondingly. Then $q_1 + q_2 = n$, and

$$\lambda_{nq_1}(k_1, \pm k_2) = -\lambda_{nq_2}(-k_1, \pm k_2)$$

(50)

For $\mu = \xi$ the function (48) gives the solution of equation (26), and for $\mu = i\eta$ the solution for equation (27). Thus the parabolic wave function (28) can be written in following way

$$\Psi_{nq_1q_2}(\xi, \eta; k_1, \pm k_2) = C_{nq_1q_2}(k_1, \pm k_2) T_{nq_1}(\xi; k_1, \pm k_2) T_{nq_2}(i\eta; k_1, \pm k_2).$$

(51)

1.2.7. The particular cases.

5The notation $Ta$ - devoted to the memory of Professor V.Ter-Antonyan (1942-2003)
Let us consider some low energy state \( n = 0, 1, 2 \). In case of \( n = 0 \) we have
\[
Mk_{00}(z) = 1, \quad \lambda_{00} = \frac{k_1}{\omega}(1 \pm k_2)
\] (52)

For \( n = 1 \) we have that \( q = 0, 1 \) and equation (46) admits two solutions
\[
\lambda_{10} = \frac{k_1}{\omega}(2 \pm k_2) - \sqrt{\frac{k_1^2}{\omega^2} + 16\omega(1 \pm k_2)}, \quad \lambda_{11} = \frac{k_1}{\omega}(2 \pm k_2) + \sqrt{\frac{k_1^2}{\omega^2} + 16\omega(1 \pm k_2)}.
\] (53)

Therefore there are two polynomials
\[
Mk_{10}(z; k_1, \pm k_2) = 1 - \frac{k_1 - \sqrt{k_1^2 + 16\omega^3(1 \pm k_2)}}{4\omega(1 \pm k_2)}z,
\] (54)
\[
Mk_{11}(z; k_1, \pm k_2) = 1 - \frac{k_1 + \sqrt{k_1^2 + 16\omega^3(1 \pm k_2)}}{4\omega(1 \pm k_2)}z.
\] (55)

For \( n = 2 \) the equation (46) is equivalent to a cubic algebraic equation
\[
\lambda^3 - \frac{3k_1^2}{2\omega}\lambda^2 + \left[\frac{k_1^2}{2\omega^2} - 2\omega(3 \pm k_2)\right]\lambda + 2k_1(1 \pm k_2) = 0.
\] (56)

At \( k_1 = 0 \) this equation can be simplified, and we have following three solutions
\[
\lambda_{20} = -\sqrt{32\omega(3 \pm k_2)}, \quad \lambda_{21} = 0, \quad \lambda_{22} = \sqrt{32\omega(3 \pm k_2)}
\] (57)

with
\[
Mk_{20}(z) = 1 + \frac{\sqrt{2\omega(3 \pm k_2)}}{(1 \pm k_2)}z + \frac{\omega}{(1 \pm k_2)}z^2
\] (58)
\[
Mk_{21}(z) = 1 - \frac{\omega}{(2 \pm k_2)}z^2
\] (59)
\[
Mk_{22}(z) = 1 - \frac{\sqrt{2\omega(3 \pm k_2)}}{(1 \pm k_2)}z + \frac{\omega}{(1 \pm k_2)}z^2.
\] (60)

1.2.6. Orthogonality relations and normalization constant.

The wave functions (51) as eigenfunctions of Hamiltonians are orthogonal for quantum number \( n \), or for \( n \neq n' \)
\[
\int_0^\infty d\eta \int_{-\infty}^\infty d\xi (\xi^2 + \eta^2)\Psi^*_nq_1q_2(\xi, \eta; k_1, \pm k_2)\Psi^*_nq_1q_2(\xi, \eta; k_1, \pm k_2) = 0.
\] (61)

Because the energy spectrum is degenerate there exist additional orthogonality relations for quantum number \( q \). Using the equations (26) and (27) it is easy to prove that for \( q_1 \neq q'_1 \) and \( q_2 \neq q'_2 \)
\[
\int_{-\infty}^\infty d\xi Ta^*_nq_1(\xi; k_1, \pm k_2) T a_{nq_1}(\xi; k_1, \pm k_2) = 0
\] (62)
\[
\int_0^\infty d\eta Ta^*_nq_2(i\eta; k_1, \pm k_2) T a_{nq_2}(i\eta; k_1, \pm k_2) = 0.
\] (63)
Thus we have for \( q \neq q' \)
\[
\int_0^\infty d\eta \int_0^\infty d\xi (\xi^2 + \eta^2) \Psi_{nq'q_2}(\xi, \eta; k_1, \pm k_2) \Psi_{nq'q_2}(\xi, \eta; k_1, \pm k_2) = 0. \tag{64}
\]

Let us now calculate the normalization constant \( C_{nq_1q_2}(k_1, \pm k_2) \). From the explicit form of the wave function \( \Psi_{nq_1q_2}(\xi, \eta; k_1, \pm k_2) \) and the normalization condition (29), it follows that
\[
\frac{1}{8} |C_{nq_1q_2}(k_1, \pm k_2)|^2 \sum_{s,s',t,t'=0} (-1)^{t+t'} A^2_s(k_1, \pm k_2) A^2_{s'}(k_1, \pm k_2)
\times A^2_t(k_1, \pm k_2) A^2_{t'}(k_1, \pm k_2) \left\{ F^{-1/4}_{t,t'} F^{+1/4}_{s,s'} + F^{+1/4}_{t,t'} F^{-1/4}_{s,s'} \right\} = 1 \tag{65}
\]

where
\[
F_{t,t'}^{\pm1/4} = \sum_{m=0}^{\infty} \frac{\Gamma \left( \frac{m+t+t'+k_2+1}{2} + \frac{1}{4} \pm \frac{1}{4} \right) \left( \frac{k_1}{2\omega} \right)^m}{m!}. \tag{66}
\]

2.3 Niven approach

Let us express solution of the Schrödinger equation (16) in the following form [19]
\[
\Psi(x, y) = e^{-\frac{\omega(x+k_1)}{2} - \frac{1}{2} \omega y^2 + \frac{1}{2} (x^2 - k_2 y^2 + k_2)} \Phi(x, y). \tag{67}
\]

From the eqs. (20), (21) and (31) follows that the function \( \Phi(x, y) \) is polynomial (product of two polynomials) from variables \((x, y^2)\) in Cartesian coordinatees and \((\xi^2, \eta^2)\) for parabolic ones. It satisfy the equation
\[
\mathcal{R} \Phi(x, y) = -2E \Phi(x, y), \tag{68}
\]

where the operator \( \mathcal{R} \) is
\[
\mathcal{R} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left[ (1 + 2k_2) \frac{1}{y} - 2\omega y \right] \frac{\partial}{\partial y} - 4\omega \left[ x + \frac{k_1}{4\omega} \right] \frac{\partial}{\partial x} - \omega(2 \pm k_2) + \frac{k_2^2}{8\omega^2}. \tag{69}
\]

Taking into account that
\[
Mk_{nq}(z; k_1, \pm k_2) = \sum_{s=0}^{n} A^2_s(k_1, \pm k_2) z^s = \Pi_{\ell=1}^{n} \left( z - \alpha_\ell \right), \tag{70}
\]

where \( \alpha_\ell, \ell = 1, 2, \ldots n \) are zeros of polynomials \( Mk_{nq}(z) \) on the real axis \(-\infty < z < \infty\), and that in parabolic coordinates
\[
\frac{y^2}{\alpha} + 2x - \alpha = \frac{(\xi^2 - \alpha)(\eta^2 + \alpha)}{\alpha}, \tag{71}
\]

we can choose a solution of eq. (68) in the form
\[
\Phi(x, y) = Mk_{nq_1}(\xi^2; k_1, \pm k_2) Mk_{nq_2}(\eta^2; k_1, \pm k_2) \equiv \Pi_{\ell=1}^{n} \left( \frac{y^2}{\alpha_\ell} + 2x - \alpha_\ell \right). \tag{72}
\]
Then from (68) follows that zeros $\alpha_\ell$ must satisfy the systems of $n$ - algebraic equations
\[
\sum_{m \neq \ell}^{n} \frac{2}{\alpha_\ell - \alpha_m} + \frac{(1 \pm k_2)}{\alpha_\ell} - \omega \alpha_\ell = \frac{k_1}{2\omega}, \quad \ell = 1, 2, \ldots, n,
\] (73)
and for energy spectrum we again have a formula (22). The system algebraic equation (73) contains $n$ - set of solutions (zeros) $(\alpha_1^{(q)}, \alpha_2^{(q)}, \ldots, \alpha_n^{(q)}), \ q = 1, 2, \ldots n$ and all zeros are real. The positive zeros $\alpha_\ell > 0$ define the nodes of wave functions for equation (26), whereas negative zeros $\alpha_\ell < 0$ define the nodes of wave functions for equation (27).

The eigenvalues of parabolic separation constant can be calculated the same way via the operator equation $\Lambda \Phi(x, y) = \lambda \Phi(x, y)$ (see for details [19]). More elegant way is direct to use the differential equation (34) [62]. Rewriting first the eq. (34) in following form
\[
\left\{ 4z \frac{d^2}{dz^2} + 4 \left[ (1 \pm k_2) - \omega z \left( z + \frac{k_1}{2\omega^2} \right) \right] \frac{d}{dz} + \left[ 4n\omega z - \frac{k_1}{\omega} (1 \pm k_2) \right] \right\} M_{nq}(z; k_1, \pm k_2) = \lambda M_{nq}(z; k_1, \pm k_2)
\] (74)
Putting now the wave function $M_{nq}(z; k_1, \pm k_2)$ in form of (70), we arrive to the following result
\[
\lambda_{nq}(k_1, \pm k_2) = 4(1 \pm k_2) \left[ \frac{k_1}{4\omega} + \sum_{\ell=1}^{n} \frac{1}{\alpha_\ell^{(q)}} \right],
\] (75)
(in case of $n = 0$ the sum must be eliminated) where the quantum number $q = 1, 2, \ldots n$ labeled the eigenvalue of parabolic separation constant.

### 2.4 Interbasis expansions between Cartesian and parabolic bases

We determine the interbasis expansion relating Cartesian and parabolic wave functions at the fixed energy $E_n$
\[
\Psi_{n_{1}q_{1},q_{2}}(\xi, \eta; \pm k_2) = \sum_{n_{1}=0}^{n_{1}+n_{2}} W_{n_{1}q_{1},q_{2}}^{n_{1}n_{2}}(\pm k_2) \Psi_{n_{1}n_{2}}(x, y; \pm k_2).
\] (76)
For simplicity we consider only the case when $k_1 = 0$. To calculate the interbasis coefficients $W_{nq}^{n_{1}n_{2}}$ we use asymptotic methods [80]. First, we change in $\Psi_{n_{1}q_{1},q_{2}}(\xi, \eta; \pm k_2)$ from parabolic coordinates to Cartesian ones
\[
\xi^2 = \sqrt{x^2 + y^2 + x}, \quad \eta^2 = \sqrt{x^2 + y^2 - x},
\] (77)
and then let $y$ tend to zero on the both sides of (76). As a result
\[
\xi^2 \to |x| + x, \quad \eta^2 \to |x| - x,
\] (78)
and the dependence on variable $y$ in (76) is removed. Using the orthogonality conditions for Hermite polynomials we get the equality
\[
W_{n_{1}q_{1},q_{2}}^{n_{1}n_{2}}(\pm k_2) = C_{n_{1}q_{1},q_{2}}(\pm k_2) \left( \frac{2\omega}{\pi} \right)^{\frac{1}{4}} \sqrt{\frac{1}{2^{n_{1}+1}(n_{1})!(n_{2})!(1 \pm k_2)}} I_{n_{1}q_{1},q_{2}}^{n_{1}n_{2}}(\pm k_2)
\] (79)
where
\[ L_{nq_1q_2}^n(\pm k_2) = \int_{-\infty}^{\infty} M_{nq_1}(|x|+x; \pm k_2)M_{nq_1}(ix-i; \pm k_2) e^{-2\omega x^2} H_{n_1}(\sqrt{2} \omega x) \, dx \]
\[ = \int_0^{\infty} [M_{nq_1}(2x; \pm k_2) + (-1)^{n_1} M_{nq_2}(2ix; \pm k_2)] e^{-2\omega x^2} H_{n_1}(\sqrt{2} \omega x) \, dx. \]

With the formula (47) the latter integral can be expressed via the coefficients \( A_s^n(\pm k_2) \):
\[ L_{nq_1q_2}^n(\pm k_2) = \frac{1}{\sqrt{2\omega}} \sum_{s=0}^{n} \left[ 1 + (-1)^{s+n_1} \right] A_s^n(\pm k_2) \left( \frac{2}{\omega} \right)^s \int_0^{\infty} z^{2s} e^{-z^2} H_{n_1}(z) \, dz. \] (80)

It is more convenient to calculate \( L_{nq_1q_2}^n(\pm k_2) \) independently for even and odd \( n_1 \). By integration by parts we find that integral in the sum is nonzero only at \( n/2 \leq s \leq n/2 + 1 \), for \( n_1 \) - even and \((n_1+1)/2 \leq s \leq n_1 + n_2 \), for \( n_1 \) - odd.

\[ W_{nq_1q_2}^n(\pm k_2) = C_{nq_1q_2}(\pm k_2) \left( \frac{\pi}{2\omega} \right)^{1/4} \sqrt{\frac{\Gamma(n_2+1 \pm k_2)}{2\omega(1+n_1 \pm k_2)!(n_2)!}} \]
\[ \times \sum_{s=0}^{n-n_1/2} \left[ 1 + (-1)^{s+n_1} \right] A_s^n(\pm k_2) \left( \frac{\Gamma(2s+n_1+1)}{(2\omega)^{s+1}} \right), \]
\[ \sum_{s=0}^{n-n_1+1/2} \left[ 1 + (-1)^{s+n_1+1} \right] A_s^n(\pm k_2) \left( \frac{\Gamma(2s+n_1+2)}{(2\omega)^{s+3/2}} \right) \] (81)

for \( n \) even and odd respectively.

3 Singular circular oscillator

The second potential of the singular oscillator is \((k_1, k_2 > 0)\)
\[ V_2(x, y) = \frac{1}{2} \omega^2 (x^2 + y^2) + \frac{1}{2} \left( \frac{k_1^2 - \frac{1}{4}}{x^2} + \frac{k_2^2 - \frac{1}{4}}{y^2} \right). \] (82)

The corresponding Schrödinger equation separates in three different orthogonal coordinate systems: Cartesian, polar and elliptical coordinates.

3.1 Cartesian bases

Let us first consider the separation of variables in Cartesian coordinates. From the asymptotic ansatz
\[ \Psi(x, y) = x^{1/2 \pm k_1} y^{1/2 \pm k_2} \exp[-\omega(x^2 + y^2)] X(x)X(y) \] (83)
we obtain two independent and identical separation equations
\[ \left[ \frac{\partial^2}{\partial z_i^2} + \left( -2\omega + \frac{1 \pm 2k_i}{x_i} \right) \frac{\partial}{\partial x_i} - (1 \pm 2k_i)\omega \right] X(x_i) = 2\lambda_i X(x_i), \quad i = 1, 2 \] (84)
where \( x_1 = x, x_2 = y \) and \( \lambda_1 + \lambda_2 = -E \). As in the case of singular anisotropic oscillator we assume that the positive sign at the \( k_i \) has to be taken if \( k_i > \frac{1}{2} \) and both the positive and the negative sign must be taken if \( 0 < k_i < \frac{1}{2} \).

The last equation is exactly the equation for confluent hypergeometric functions. The quantization rule gives

\[
\lambda_i = -\omega(2n_i \pm k_i + 1), \quad n_i = 0, 1, 2, \ldots
\]

and the solution of eq. (84) in terms of Laguerre polynomials \( X(x_i) = L_{n_i}^{\pm k_i}(\omega x_i^2) \). Thus the corresponding set of orthonormal eigenfunctions which are normalized in quadrant \( x > 0, y > 0 \) (on 1/4) is

\[
\Psi_{n_1, n_2}^{(\pm k_1, \pm k_2)}(x, y) = C_{n_1, n_2}^{(\pm k_1, \pm k_2)}(x)\frac{\omega^{\pm k_1 \pm k_2} e^{-\frac{\omega}{2}(x^2+y^2)} L_{n_1}^{\pm k_1}(\omega x^2) L_{n_2}^{\pm k_2}(\omega y^2)}{\sqrt{\Gamma(n_1 \pm k_1 + 1) \Gamma(n_2 \pm k_2 + 1)}}
\]

From (85) we have

\[
E_n = \omega(2n + 2 \pm k_1 \pm k_2),
\]

where \( n = n_1 + n_2 = 0, 1, 2, \ldots \) is principal quantum number and the degree of degeneracy is \( n + 1 \).

### 3.2 Polar bases

The separation of variables in the Schrödinger equation for potential (83) in polar coordinates

\[
x = r \cos \phi, \quad y = r \sin \phi, \quad 0 \leq r < \infty, \quad 0 \leq \phi < 2\pi
\]

gives us the orthonormal solution in polynomial form

\[
\Psi_{n, m}^{(\pm k_1, \pm k_2)}(r, \phi) = \left( \frac{2n_r!}{\Gamma(n_r + 2m \pm k_1 \pm k_2 + 2)} \right)^{1/2} \sqrt{\omega r} (2m \pm k_1 \pm k_2 + 1) L_{n_r}^{2q \pm k_1 \pm k_2 + 1}(\omega r) P_m^{(\pm k_2, \pm k_1)}(\cos 2\phi)
\]

where \( P_m^{(\alpha, \beta)}(x) \) is a Jacobi polynomials and \( E = \omega(2n \pm k_1 \pm k_2 + 2) \), with \( n = n_r + m \) and with the same degree of degeneracy \( n + 1 \).

Thus we have seen that quantum system (82) is **exactly-solvable** in the Cartesian and polar systems of coordinates.
3.3 Elliptic bases

3.3.1. Separation of variables.
The elliptic coordinate \((\nu, \mu)\) connected with Cartesian one by \(0 \leq \nu < \infty, 0 \leq \mu < 2\pi\)

\[
x = \frac{D}{2} \cosh \nu \cos \mu, \quad y = \frac{D}{2} \sinh \nu \sin \mu,
\]

where \(D\) is the interfocal distance. As \(D \to 0\) and \(D \to \infty\), the elliptic coordinate degenerate into the polar and Cartesian coordinates

\[
\cosh \nu \to \frac{2r}{D}, \quad \cos \mu \to \cos \phi, \quad (D \to 0),
\]
\[
\sinh \nu \to \frac{2y}{D}, \quad \cos \mu \to \frac{2x}{D}, \quad (D \to \infty).
\]

The Laplacian and volume element are

\[
\Delta = \frac{8}{D^2(\cosh 2\nu - \cos 2\mu)} \left( \frac{\partial^2}{\partial \nu^2} + \frac{\partial^2}{\partial \mu^2} \right), \quad dV = \frac{D^2}{8} (\cosh 2\nu - \cos 2\mu) d\nu d\mu.
\]

The Schrödinger equation with (82) can be rewritten as

\[
\frac{\partial^2 \psi}{\partial \nu^2} + \frac{\partial^2 \psi}{\partial \mu^2} + \left\{ \frac{D^2 E}{4} (\cosh 2\nu - \cos 2\mu) - \frac{D^4 \omega^2}{64} (\cosh^2 2\nu - \cos^2 2\mu) 
\right.
\]
\[
- \left[ \frac{(k_1^2 - \frac{1}{4})}{\cos^2 \mu} + \frac{(k_2^2 - \frac{1}{4})}{\sin^2 \mu} \right] - \left[ \frac{(k_1^2 - \frac{1}{4})}{\sinh^2 \nu} - \frac{(k_2^2 - \frac{1}{4})}{\cosh^2 \nu} \right] \psi = 0.
\]

and after the separation ansatz

\[
\psi(\nu, \mu; D^2) = X(\nu; D^2)Y(\mu; D^2)
\]
transforms to two ordinary differential equations

\[
\frac{d^2 X}{d\nu^2} + \left[ \frac{D^2 E}{4} \cosh 2\nu - \frac{D^4 \omega^2}{64} \cosh^2 2\nu - \frac{k_2^2 - \frac{1}{4}}{\sinh^2 \nu} + \frac{k_1^2 - \frac{1}{4}}{\cosh^2 \nu} \right] X = -\lambda(D^2)X, \tag{98}
\]

\[
\frac{d^2 Y}{d\mu^2} - \left[ \frac{D^2 E}{4} \cos 2\mu - \frac{D^4 \omega^2}{64} \cos^2 2\mu + \frac{k_2^2 - \frac{1}{4}}{\cos^2 \mu} + \frac{k_1^2 - \frac{1}{4}}{\sin^2 \mu} \right] Y = +\lambda(D^2)Y, \tag{99}
\]

where \(\lambda\) is the elliptic separation constant. These equations can be written in the unit form

\[
\frac{d^2 Z(\zeta)}{d\zeta^2} + \left[ \frac{D^4 \omega^2}{64} \cos^2 2\zeta - \frac{D^2 E}{4} \cos 2\zeta - \frac{k_1^2 - \frac{1}{4}}{\cos^2 \zeta} - \frac{k_2^2 - \frac{1}{4}}{\sin^2 \zeta} \right] Z(\zeta) = \lambda(D^2)Z(\zeta) \tag{100}
\]

where at \(\zeta \in [0, 2\pi]\) we have the equation (99) but at \(\zeta \in [0, i\infty)\) - equation (98). In other words, as we see from Fig. 1, in the complex plane \(\zeta\) the "physical" are only the shaded domain on the two lines \(\text{Im } \zeta = 0\) and \(\text{Re } \zeta = 0\).

For \(k_{1,2} > \frac{1}{2}\) the centrifugal barrier is repulsive and motion takes place in only one of the quadrants, as \(\zeta \in [0, \pi/2]\), whereas for \(0 < k_{1,2} < \frac{1}{2}\) it takes place in the whole region \(\zeta \in [0, 2\pi]\). For the particular case \(k_1 = k_2 = \frac{1}{2}\) the equation (100) transforms to the problem of the ordinary two dimensional oscillator and has been investigated in detail in the paper [68]. In this article have shown that the solution of eq. (100) (for \(k_1 = k_2 = 1/2\)) is described by the Ince polynomials [81].

In the case where \(k_1\) and \(k_2\) are integers, eqs. (98) and (99) coincide with those that have been found via separation of variables in the Schrödinger equation for the four dimensional isotropic oscillator in spheroidal coordinates [69].

### 3.3.2. Recurrence relations.

Let us now consider the equation (100). First, introducing the function \(W(\zeta; D^2)\) according to

\[
Z(\zeta; D^2) = \exp \left[ -\frac{D^2 \omega}{16} \cos 2\zeta \right] W(\zeta; D^2), \tag{101}
\]

we have the equation

\[
\frac{d^2 W}{d\zeta^2} + \frac{D^2 \omega}{4} \sin 2\zeta \frac{dW}{d\zeta} + \left[ \frac{D^2 \omega}{4} \cos 2\zeta - \frac{D^2 E}{2} \cos^2 \zeta - \frac{k_1^2 - \frac{1}{4}}{\cos^2 \zeta} - \frac{k_2^2 - \frac{1}{4}}{\sin^2 \zeta} - \lambda \right] W = 0. \tag{102}
\]

For \(k_1 = k_2 = 1/2\) this is the Ince equation, [49].

Next the substitution

\[
W(\zeta; D^2) = (\sin \zeta)^{\frac{1}{2} k_2} (\cos \zeta)^{\frac{1}{2} k_1} U(\zeta; D^2) \tag{103}
\]

yields the equation

\[
\frac{d^2 U}{d\zeta^2} + \left[ (1 \pm 2k_2) \cot \zeta - (1 \pm 2k_1) \tan \zeta + \frac{D^2 \omega}{4} \sin 2\zeta \right] \frac{dU}{d\zeta} + \left[ p \cos^2 \zeta - \tilde{\lambda} \right] U = 0, \tag{104}
\]

where

\[
p = \frac{D^2}{2} [\omega(2 \pm k_1 \pm k_2) - E], \quad \tilde{\lambda} = \lambda + \frac{D^2 \omega}{2} (1 \pm k_1) + (1 \pm k_1 \pm k_2)^2 - \frac{D^2 E}{4} - \frac{D^4 \omega^2}{64}. \tag{105}
\]
Passing to a new variable $t = \cos^2 \zeta$ we find
\[
t(1-t)\frac{d^2U}{dt^2} + \left\{ (1 \pm k_1)(1-t) - (1 \pm k_2)t + \frac{D^2\omega}{4}t(t-1) \right\} \frac{dU}{dt} + \frac{1}{4}[pt - \tilde{\lambda}]U = 0, \quad (106)
\]
Finally, looking for the solution of the last equation in the form
\[
U(t; D^2) = \sum_{s=0}^{\infty} A_s(D^2)t^s, \quad (107)
\]
for coefficients $A_s(D^2)$ we have the three-term recurrence relation
\[
(s + 1)(s + 1 \pm k_1)A_{s+1} = \left[ s(s + 1 \pm k_1 \pm k_2) + \frac{D^2\omega}{4}s + \frac{\tilde{\lambda}}{4} \right] A_s + \frac{1}{4}[p + D^2\omega(s - 1)]A_{s-1} = 0, \quad (108)
\]
with $A_1 = 0$ and initial condition $A_0 = 1$.

### 3.3.3. Energy spectrum and separation constant.
In analogy with our asymptotic solution of the recurrence relation for the singular anisotropic operator in the parabolic basis we use continued fractions. For the minimal solution of the recurrence relations we find for $s^{-1} \ll 1$
\[
\frac{A_{s+1}}{A_s} \sim \frac{D^2\omega}{4s} \left( 1 + O\left( \frac{1}{\sqrt{s}} \right) \right). \quad (109)
\]
Thus we have
\[
A_s \sim \left( \frac{D^2\omega}{4} \right)^s s!,
\]
and
\[
U(\cos \zeta) \sim \sum \left( \frac{D^2\omega}{4} \right)^k \frac{k!}{k!} \cos^{2k} \zeta \sim \exp\left( \frac{D^2\omega}{8} \cos 2\zeta \right). \quad (110)
\]
Therefore we see that for this case the function $Z(\cos \zeta; D^2)$ as $\zeta \to i\infty$ is not normalizable. There is a linearly independent solution of the recurrence relations, but the coefficients grow even faster. Hence it follows that the series (107) should be truncated. The condition for series (107) be truncated give us already known formula for energy spectrum (88) and reduce to the polynomials:
\[
U_n(\pm k_1, \pm k_2) (t; D^2) = \sum_{s=0}^{n} A_s^{(\pm k_1, \pm k_2)}(D^2)t^s, \quad (111)
\]
where now the coefficients $A_s \equiv A_s^{(\pm k_1, \pm k_2)}(D^2)$ satisfy the following three-term recurrent relations
\[
(s + 1)(s + 1 \pm k_1)A_{s+1} + \beta_s A_s - \frac{D^2\omega}{4}(n - s + 1)A_{s-1} = 0, \quad s = 0, 1, \ldots n \quad (112)
\]
with

\[
\beta_s = -\frac{1}{4}\left[ (2s + 1 \pm k_1 \pm k_2)^2 + \frac{D^2\omega}{2}(2s - n + 4 \pm 4k_1) \right.
- \left. \frac{D^2\omega}{4}(2 \pm k_1 \pm k_2) - \frac{D^4\omega^2}{64} + \lambda(D^2) \right] \tag{113}
\]

and \(A_{-1} = A_{n+1} = 0\).

The recurrent relations (112) becomes a system of \((n+1)\) linear homogeneous equations for the coefficients \(A_s\). Equating the corresponding determinat to zero

\[
D_n(\lambda) = \begin{vmatrix}
\beta_0 & (1 \pm k_1) & \cdot & \cdot & \cdot \\
-\frac{D^2\omega}{3}n & \beta_1 & 2(2 \pm k_1) & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -\frac{D^2\omega}{2} & \beta_{n-1} \\
-\frac{D^2\omega}{4} & \cdot & \cdot & \cdot & \beta_n
\end{vmatrix} = 0 \tag{114}
\]

leads to the algebraic equation of a \((n+1)\) degree which determine the eigenvalues of elliptic separation constant \(\lambda_{nq}^{(\pm k_1, \pm k_2)}(D^2)\). The quantum number \(q = 0, 1, 2, \ldots, n\) enumerate \((n+1)\) roots of eq. (114) and therefore the degree of degeneracy as in the polar and Cartesian cases for \(n\)-th energy state equal \(n + 1\). It is also known at the corresponding numeration the quantum number \(q\) defines the numbers of zeros of polynomial (111), which has exactly \(n\) - zeros situated in the open interval \(0 < t < \infty\), and therefore, the elliptic separation constant \(\lambda_{nq}^{(\pm k_1, \pm k_2)}(D^2)\) may be ordered also by the numbers of the nodes of the eigenfunction of equation (100).

### 3.3.5. Wave functions.

Thus the the condition of finitness of the solution of eq. (102) allows the following polynomials:

\[
\mathcal{T}_{nq}^{(\pm k_1, \pm k_2)}(\zeta; D^2) = (\sin \zeta)^{\frac{1}{2} \pm k_2} (\cos \zeta)^{\frac{1}{2} \pm k_1} \sum_{s=0}^{n} A_{s}^{(\pm k_1, \pm k_2)}(D^2)(\cos \zeta)^{2s}, \tag{115}
\]

while the corresponding solution of eq. (100) is

\[
\mathcal{Z}_{nq}^{(\pm k_1, \pm k_2)}(\zeta; D^2) = e^{-\frac{D^2\omega}{48}\cos 2\zeta} \mathcal{T}_{nq}^{(\pm k_1, \pm k_2)}(\zeta; D^2) \tag{116}
\]

We will term the polynomials \(\mathcal{T}_{nq}^{(\pm k_1, \pm k_2)}(\zeta; D^2)\) as the **associated Ince polynomials**. In the case of \(k_1 = k_2 = 1/2\) these polynomials transforms to the four type of the ordinary **Ince polynomials**, which are even or odd with respect to the changes \(\zeta \rightarrow -\zeta\) and \(\zeta \rightarrow \zeta + \pi\) [81, 68].

At \(\zeta = \mu\) the wave functions (116) give us the solution of angular equation (99), and for \(\zeta = \imath \nu\) the solution of radial equation (98). For the each of wave functions, radial or angular, corresponds definite numbers of zeros which could be presented by two quantum numbers \(q_1\) and \(q_2\), obeying the condition \(q_1 + q_2 = n\). Then the complete elliptic wave function (97) may be written as

\[
\Psi_{nq_1q_2}(\nu, \mu; D^2) = C_{nq_1q_2}(\pm k_1, \pm k_2; D^2) \mathcal{Z}_{nq_1}^{(\pm k_1, \pm k_2)}(\mu; D^2) \mathcal{Z}_{nq_2}^{(\pm k_1, \pm k_2)}(\imath \nu; D^2), \tag{117}
\]
where \( C_{nq_{1}q_{2}}(\pm k_{1}, \pm k_{2}; D^{2}) \) is the normalization constant. It could be calculated from the condition

\[
\frac{D^{2}}{4} \int_{0}^{\infty} d\nu \int_{0}^{\frac{\pi}{2}} d\mu (\cosh^{2} \nu - \cos^{2} \mu) \Psi_{nq_{1}q_{2}}^{(\pm k_{1}, \pm k_{2})}(\nu, \mu; D^{2}) \Psi_{nq_{1}q_{2}}^{(\pm k_{1}, \pm k_{2})}(\nu, \mu; D^{2}) = \frac{1}{4}
\]

(118)

In some cases it is more convenient to have alternative form for the wave functions (117). Equation (100) represents the eigenvalue problem for the separation constant \( \lambda(D^{2}) \) and the corresponding eigenfunction. It easy to see that the operator in the left-hand side of eq. (100)

\[ \text{is invariant under the simultaneous transformations} \]

\[ \zeta \rightarrow \zeta + \frac{\pi}{2}, \quad D^{2} \rightarrow -D^{2}, \quad k_{1} \leftrightarrow k_{2} \]

and therefore the substitution \( D^{2} \rightarrow -D^{2} \) and \( k_{1} \leftrightarrow k_{2} \) does not change the set of eigenvalues of the separation constant \( \lambda(D^{2}) \), but change only their numeration:

\[ \lambda_{nq_{1}}^{(\pm k_{1}, \pm k_{2})}(D^{2}) = \lambda_{nq_{2}}^{(\pm k_{2}, \pm k_{1})}(-D^{2}). \]

(120)

As a result transformation (119) transform the solutions (116) into each other

\[ Z_{nq_{1}}^{(\pm k_{1}, \pm k_{2})}(\zeta; D^{2}) \rightarrow Z_{nq_{2}}^{(\pm k_{2}, \pm k_{1})}(\zeta + \frac{\pi}{2}; -D^{2}) \]

(121)

The coefficients \( A_{4}^{(\pm k_{2}, \pm k_{1})}(-D^{2}) \) are calculated from the three-term recurrent relations (112) after the changing \( D^{2} \rightarrow -D^{2} \) and \( k_{1} \leftrightarrow k_{2} \). From (121) we conclude that elliptic basis (117) can be represented in the form

\[ \Psi_{nq_{1}q_{2}}^{(\pm k_{1}, \pm k_{2})}(\nu, \mu; D^{2}) = C_{nq_{1}q_{2}}^{(\pm k_{1}, \pm k_{2})}(\nu_{1}, \pm k_{2}; D^{2}) Z_{nq_{1}}^{(\pm k_{1}, \pm k_{2})}(\mu; D^{2}) Z_{nq_{2}}^{(\pm k_{2}, \pm k_{1})}(i\nu + \frac{\pi}{2}; -D^{2}), \]

(122)

where the quantum number \( q_{1}, q_{2} \) \((q_{1} + q_{2} = n)\) again have the meaning of zeros of the angular and radial wave functions and the constant \( C_{nq_{1}q_{2}}^{(\pm k_{1}, \pm k_{2})}(\pm k_{1}, \pm k_{2}; D^{2}) \) satisfy the normalization condition (118).

### 3.3.6. Orthogonality relations.

The wave function (117) as eigenfunction of the Hamiltonians are orthogonal \( n \neq n' \)

\[ \int_{0}^{\infty} \int_{0}^{\frac{\pi}{2}} \Psi_{nq_{1}q_{2}}^{(\pm k_{1}, \pm k_{2})}(\nu, \mu; D^{2}) \Psi_{n'q_{1}q_{2}}^{(\pm k_{1}, \pm k_{2})}(\nu, \mu; D^{2}) dV = 0 \]

(123)

Eqs. (98) and (99) allow to prove the property of double orthogonality for wave functions \( Z_{nq_{1}q_{2}}^{(\pm k_{1}, \pm k_{2})}(\zeta; D^{2}) \), namely

\[ \int_{0}^{\infty} Z_{nq_{2}}^{(\pm k_{1}, \pm k_{2})}(i\nu; D^{2}) Z_{nq_{1}}^{(\pm k_{1}, \pm k_{2})}(i\nu; D^{2}) d\nu = 0 \]

(124)

\[ \int_{0}^{\frac{\pi}{2}} Z_{nq_{1}}^{(\pm k_{1}, \pm k_{2})}(\mu; D^{2}) Z_{nq_{2}}^{(\pm k_{1}, \pm k_{2})}(\mu; D^{2}) d\mu = 0 \]

(125)

for \( q_{1} \neq q'_{1} \) and \( q_{2} \neq q'_{2} \), and therefore when \( q \neq q' \)

\[ \int_{0}^{\infty} d\nu \int_{0}^{\frac{\pi}{2}} d\mu (\cosh^{2} \nu - \cos^{2} \mu) \Psi_{nq_{1}q_{2}}^{(\pm k_{1}, \pm k_{2})}(\nu, \mu; D^{2}) \Psi_{nq_{1}q_{2}}^{(\pm k_{1}, \pm k_{2})}(\nu, \mu; D^{2}) = 0 \]

(126)
3.3.7. Limit $D \to 0$.
Now we will show that in the limit $D \to 0$ the elliptic bases transforms to polar one. At the
limit $D \to 0$ in eq. (114) it is possible to neglect the terms depending of $D^2$, then

$$D_n(\lambda) = \beta_0 \beta_1 \ldots \beta_n = 0$$  \hspace{1cm} (127)

Let one of the $\beta_s$ at $s = m$ equal zero: $\beta_m = 0$. Then $\lambda(0) = -(2m + 1 \pm k_1 \pm k_2)^2$ and
therefore at the limit $D \to 0$ the quantum numbers $q_1$ and $q_2$ transforms to $m$ and $n - m = n_r$
correspondingly. For $s \neq m$, we have

$$\beta_s(D^2) \rightarrow \beta_s(0) = -(s - m)(s + m + 1 \pm k_1 \pm k_2)$$  \hspace{1cm} (128)

It follows from (128) that the three term recurrent relation (112) at $D \to 0$ split to two term
recurrent relations

$$(s + 1)(s + 1 \pm k_1 \pm k_2)A_{s+1} + \beta_s A_s = 0, \quad s = 0, 1, \ldots, m - 1$$  \hspace{1cm} (129)

and

$$\beta_s A_s - \frac{D^2 \omega}{4} (n - s + 1)A_{s-1} = 0, \quad s = m + 1, m + 2, \ldots, n$$  \hspace{1cm} (130)

From two-term recurrent relations (129) and (129), we get

$$A_s(D^2) \xrightarrow{D \to 0} \frac{(-m)_s (m + 1 \pm k_1 \pm k_2)_s}{s!(1 \pm k_1)_s},$$  \hspace{1cm} (131)

for $s = 0, 1, \ldots, m - 1$ and

$$A_{m+s}(D^2) \xrightarrow{D \to 0} (-1)^m \frac{(m + 1 \pm k_1 \pm k_2)_m (m - n)_s}{s!(2m + 2 \pm k_1 \pm k_2)_s (1 \pm k_1)_m} \left(\frac{D^2 \omega}{4}\right)^s,$$  \hspace{1cm} (132)

for $s = 1, 2, \ldots n - m$, and $(m)_s = \Gamma(m + s)/\Gamma(m)$. Then according to eq. (93)

$$Z^{(\pm k_1, \pm k_2)}_{q_1}(\mu; D^2) \xrightarrow{D \to 0} (\cos \phi)^{\frac{1}{2} \pm k_1} (\sin \phi)^{\frac{1}{2} \pm k_2} \sum_{s=0}^{n} A_s(D^2) (\cos \phi)^{2s},$$  \hspace{1cm} (133)

$$Z^{(\pm k_1, \pm k_2)}_{q_2}(i\nu; D^2) \xrightarrow{D \to 0} (i)^{\frac{1}{2} \pm k_2} e^{-\frac{\nu}{2}} \frac{(2r)^{1 \pm k_1 \pm k_2}}{2s} \sum_{s=0}^{n} \frac{A_s(D^2)}{D^{2s}} (2r)^{2s},$$  \hspace{1cm} (134)

Substituting here the eqs. (131) and (132), we get

$$Z^{(\pm k_1, \pm k_2)}_{q_1}(\mu; D^2) \xrightarrow{D \to 0} (\cos \phi)^{\frac{1}{2} \pm k_1} (\sin \phi)^{\frac{1}{2} \pm k_2} 2F_1(-m, m + 1 \pm k_1 \pm k_2; 1 \pm k_1; \cos^2 \phi)$$

$$= \frac{(-1)^m m!}{(1 \pm k_1)_m} (\cos \phi)^{\frac{1}{2} \pm k_1} (\sin \phi)^{\frac{1}{2} \pm k_2} P^{(\pm k_2, \pm k_1)}_m(\phi)$$  \hspace{1cm} (135)

$$Z^{(\pm k_1, \pm k_2)}_{q_2}(i\nu; D^2) \xrightarrow{D \to 0} (i)^{(2m + \frac{1}{2} \pm k_2)} \frac{(m + 1 \pm k_1 \pm k_2)_m}{(1 \pm k_1)_m}$$

25
\[
\times \quad e^{-\frac{\omega^2}{2} \left(\frac{2r}{D}\right)^{2m+1\pm k_1 \pm k_2}} 1_F\left(-n_r; 2m + 2 \pm k_1 \pm k_2; \omega r^2\right)
\]

\[
= (i)^{2m+\frac{1}{2} \pm k_2} \frac{n_r!(m + 1 \pm k_1 \pm k_2)_m}{(1 \pm k_1)_m(2m + 1 \pm k_1 \pm k_2)_m}
\times e^{-\frac{\omega^2}{2} \left(\frac{2r}{D}\right)^{2m+1\pm k_1 \pm k_2}} L_n^{2m+1\pm k_1 \pm k_2}(\omega r^2)
\]

(136)

From these formulas and normalization condition (118) we finally get that the elliptic basis (117) at the limit \(D \to 0\) becomes the polar basis

\[
\Psi_{nq_1q_2}(\nu, \mu; D^2) \xrightarrow{D \to 0} \Psi_{n\nu m}(r, \phi)
\]

(137)

3.3.8. Limit \(D \to \infty\).

Let us now to investigate the Cartesian limit. At \(D \to \infty\) we can neglect all final terms and again the determinat (114) transforms to product of diagonal elements. Let now one of the \(\beta_s\) for \(s = n_1\) equal zero: \(\beta_{n_1} = 0\), then

\[
\lambda_{nq_1}^{(\pm k_1, \pm k_2)}(D^2) \xrightarrow{D \to \infty} \lambda_{n_1n_1}^{(\pm k_1, \pm k_2)}(D^2) = \frac{D^4\omega^2}{64} - \frac{D^2\omega}{4}(4n_1 - 2n + 6 \pm 7k_1 \mp k_2)
\]

(138)

here \(n_1 = 0, 1, 2, \ldots\). Analogically it is easy obtain

\[
\lambda_{nq_2}^{(\pm k_2, \pm k_1)}(-D^2) \xrightarrow{D \to \infty} \lambda_{n_1n_1}^{(\pm k_2, \pm k_1)}(-D^2) = \frac{D^4\omega^2}{64} + \frac{D^2\omega}{4}(4n_2 - 2n + 6 \pm 7k_2 \mp k_1)
\]

(139)

with \(n_2 = 0, 1, 2, \ldots\), and from the condition (120) follows that \(n_1 + n_2 = n\) and \(n_1, n_2\) are the Cartesian quantum number. From the eqs. (138) and (138), and (113) it is easy to get \((s \neq n_1,\) and \(s \neq n_2)\)

\[
\beta_s(D^2) \xrightarrow{D \to \infty} D^2\omega(s - n_1), \quad \beta_s(-D^2) \xrightarrow{D \to \infty} -D^2\omega(s - n_2).
\]

(140)

These formulas and the condition \(A_{-1} = 0\) shows that in the limit \(D \to \infty\) the three-term recurrent relation (112) transforms to two-term recurrent relations

\[
(s + 1)(s + 1 \pm k_1)A_{s+1}(D^2) - \frac{D^2\omega}{4}(s-n_1)A_s(D^2) = 0, \quad 0 \leq s \leq n_1 - 1
\]

(141)

\[
(s + 1)(s + 1 \pm k_2)A_{s+1}(-D^2) + \frac{D^2\omega}{4}(s-n_2)A_s(-D^2) = 0, \quad 0 \leq s \leq n_2 - 1
\]

(142)

Analogically using the condition \(A_{n+1} = 0\), we obtain

\[
(s - n_1)A_s(D^2) + \frac{1}{4}(n - s + 1)A_{s-1}(D^2) = 0, \quad n_1 + 1 \leq s \leq n
\]

(143)

\[
(s - n_2)A_s(-D^2) + \frac{1}{4}(n - s + 1)A_{s-1}(-D^2) = 0, \quad n_2 + 1 \leq s \leq n
\]

(144)
From the two-term recurrent relations (141) - (142) we have
\[ A_s(D^2) \xrightarrow{D \to \infty} \frac{(D^2 \omega)^s}{4^s} \frac{(-n_1)_s}{s!(1 \pm k_1)_s}, \quad 0 \leq s \leq n_1 \]
\[ A_{n_1+s}(D^2) \xrightarrow{D \to \infty} \frac{A_{n_1}(D^2)}{4^s} \frac{(n_1 - n)_s}{s!}, \quad 1 \leq s \leq n - n_1 \]
and the same for \( A_s(-D^2) \)
\[ A_s(-D^2) \xrightarrow{D \to \infty} \frac{(-D^2 \omega)^s}{4^s} \frac{(-n_2)_s}{s!(1 \pm k_2)_s}, \quad 0 \leq s \leq n_2 \]
\[ A_{n_2+s}(-D^2) \xrightarrow{D \to \infty} \frac{A_{n_2}(-D^2)}{4^s} \frac{(n_2 - n)_s}{s!}, \quad 1 \leq s \leq n - n_2 \]
We can investigate now the limit \( D \to \infty \) in the wave functions. Taking into account the conditions (94) and formulas (145)-(148), we obtain the following results
\[ T_{nq_1}^{(\pm k_1, \pm k_2)}(\mu; D^2) \xrightarrow{D \to \infty} \left(\frac{2\pi}{D}\right)^{\frac{1}{2} \pm k_1} \sum_{s=0}^{n} A_s^{(\pm k_1, \pm k_2)}(D^2) (4x^2)^s \]
\[ = \left(\frac{2\pi}{D}\right)^{\frac{1}{2} \pm k_1} {}_1F_1(-n_1; 1 \pm k_1; \omega x^2), \quad (149) \]
\[ T_{nq_2}^{(\pm k_2, \pm k_1)}(i\nu \mp \frac{\pi}{2}; -D^2) \xrightarrow{D \to \infty} \left(\frac{2iy}{D}\right)^{\frac{1}{2} \pm k_2} \sum_{s=0}^{n} A_s^{(\pm k_2, \pm k_1)}(-D^2) (4y^2)^s \]
\[ = \left(\frac{2iy}{D}\right)^{\frac{1}{2} \pm k_2} {}_1F_1(-n_2; 1 \pm k_2; \omega y^2). \quad (150) \]
Using the connection between hypergeometrical functions and Laguerre polynomials [46]
\[ L_n^\alpha(x) = \frac{(\alpha + 1)_n}{n!} {}_1F_1(-n; 1 + \alpha; x). \quad (151) \]
finally we get
\[ \Psi_{nq_1q_2}^{(\pm k_1, \pm k_2)}(\nu, \mu; D^2) \xrightarrow{D \to \infty} \frac{\tilde{C}_{nq_1q_2}(\pm k_1, \pm k_2; D^2)}{D^{1 \pm k_1 \pm k_2}} e^{-\frac{\pi}{4}(x^2+y^2)} \]
\[ \times (2x)^{\frac{1}{2} \pm k_1} (2y)^{\frac{1}{2} \pm k_2} L_n^{\pm k_1}(\omega x^2) L_n^{\pm k_2}(\omega y^2) \quad (152) \]
which after taking the limit in the normalization constant coincide with the Cartesian basis of singular oscillator.

### 4 Conclusions and summary

Let us here to summarize the presented investigation of superintegrable potentials \( V_1 \) and \( V_2 \).
We have determined that solution of Schrödinger equation for the potentials $V_1$ maybe constructed via separation of variables with two different ways. One of them is expolting the separation of variables in Cartesian coordinates, which lead two independent exactly-solvable equations (17) and (18), the each of them represent the one-dimensional non-parametric spectral problem where the cartesian separation constants $\lambda_i$ play the role of energy. It admit to get the solution in form of Lagerre and Hermite polynomials, quantize both separation constants and as result to obtain energy spectrum for two-dimensional Schrödinger equation. The second way is the separation of variables in parabolic coordinate. We have shown that the separation procedure reduce to one-parametric (parabolic separation constant) Schrödinger type differential equation (30), living in the complex plane. It have been proven that the requirement of convergence for solutions of eq. (30) at the singular points $\mu = \pm \infty$ and $\mu = i \infty$ lead only to polynomial solutions (48) with restriction for energy spectrum $E$ in form (22) and at the fixed energy (or quantum number $n$) give the spectrum of separation constant as root of nth-degree algebraic equation. In difference of the previous case, the power expansions of polynomial solutions satisfy three-term recurrence relations and cannot be rewriting in explicit form and admit only numerical description. For this reason no sence to term the equation (30) as exactly-solvable.

At the other hand side, the substitution of the formula of energy spectrum in the eq. (30) bring us to equation

$$
\left[ -\frac{d^2}{d\mu^2} + \left( \omega^2 \mu^6 + k_1 \mu^4 + \left[ \frac{k_1^2}{4\omega^2} - \omega (4n + 4 \pm 2k_2) \right] \mu^2 + \frac{k_1^2 - \frac{1}{4}}{\mu^2} \right) \right] Z_n(\mu) = \lambda Z_n(\mu),
$$

(153)

which on the real axis completely coincides for $k_1 = 4\beta \omega^2$ and $1 \pm k_2 = 2\delta$, with the one-dimensional spectral problem (10), and term as quasi-exactly solvable problem. Now it is easy to understand the birth mechanism of quasi-exactly solvable systems. The requirement of convergence just in real space (which is possible to determin following the [37] as the dimensional reduction) in singular points $\mu = \pm \infty$ perfotne together with the polynomial solutions in form (47) (with restriction on real or imaginery axis), which is already non complete, also non polynomial part. We also can shed a light on the mistery of zeros of polynomial $P_n(x^2)$. Indeed, the substitution of the wave function (11) in Schrödinger equation with potential (10) lead to the differential equation for polynomial $P_n(x^2)$ in the same form as equation (74) (in variable $x^2 = z$), but with difference that the physical region of eq. (74) is whole real axis $z \in (-\infty, \infty)$, and therefore all zeros (for positive and negative $x^2$) of $P_n(x^2)$ corresponds to the zeros of two-dimensional eigenfunction of singular anisotropic oscillator in parabolic coordinates.

We also shown the situation have repeated in the case of second potential (82). We have determined that the separation of variables in two-dimensional elliptic coordinates lead to Schrödinger type equation (100) in complex plane and requirement of convergence at the point $\zeta = 0, 2\pi$ and $\zeta = i \infty$ deduce to polynomial solutions and difine the energy spectrum (88). So, it generate trigonometric and hyperbolic quasi-exactly solvable systems (see potentials 5 and 8 in [60]) in form of

$$
\frac{d^2 X}{d\nu^2} + \left[ \left( \frac{\alpha^2}{4} + \alpha(2n + 2 \pm k_1 \pm k_2) \right) \cosh^2 \nu - \frac{\alpha^2}{4} \cosh^4 \nu - \frac{k_1^2 - \frac{1}{4}}{\sinh^2 \nu} + \frac{k_2^2 - \frac{1}{4}}{\cosh^2 \nu} + \lambda \right] X = 0,
$$

$$
\frac{d^2 Y}{d\mu^2} - \left[ \left( \frac{\alpha^2}{4} + \alpha(2n + 2 \pm k_1 \pm k_2) \right) \cos^2 \mu - \frac{\alpha^2}{4} \cos^4 \mu + \frac{k_1^2 - \frac{1}{4}}{\cos^2 \mu} + \frac{k_2^2 - \frac{1}{4}}{\sin^2 \mu} + \lambda \right] Y = 0,
$$

28
where $\alpha = D^2 \omega / 2$. Thus we have proven that in the base of quasi-exactly solvability phenomena stay the dimensional reduction of superintegrable systems on the one-dimension.

This analogy prompt us to use the term of quasi-exactly solvability for the equations of type (30) or (100), define in the complex plane and which are not exactly-solvable but also admit complete set of polynomial solution. Thus we suggest to call the quantum mechanical systems **first-order quasi-exactly solvable** if the polynomial solution one-parametric differential equation of the kind of Schrödinger equation or $N$-dimensional equation after separation of variables is defined through three-term recurrence relations and the discrete eigenvalues could be calculated only numerically as the solutions of algebraic equations. Accordingly this definition systems (30) and (100) are first order quasi-exactly solvable. What about quasi-exactly solvable systems in his old sense, it more convenient to call as **partially quasi-exactly solvable**, because only part of solution is polynomial.

It is easy to generelize our definition for $N$-dimensional quasi-exactly solvable systems. Let us consider the Helmholtz equation on $N$-dimensional sphere $S_N$: $u_1^2 + u_2^2 + ... u_{N+1}^2 = R^2$

$$\frac{1}{2} \Delta_{LB} \Psi + E \Psi = 0, \quad (154)$$

where the Laplace-Beltrami operator is

$$\Delta_{LB} = -\frac{1}{2R^2} \sum_{i,k=1}^{N+1} L_{ik}^2, \quad L_{ik} = -i \left( u_i \frac{\partial}{\partial u_k} - u_k \frac{\partial}{\partial u_i} \right), \quad (155)$$

The separation of variables in $N$-dimensional ellipsoidal coordinates [82, 83, 84]

$$u_i^2 = \frac{\Pi_{j=1}^{N} (\rho_j - a_i)}{\Pi_{j \neq i}^{N+1} (a_j - a_i)} \quad i = 1, 2, ..., N + 1 \quad (156)$$

with

$$a_1 < \rho_1 < a_2 < \rho_2 < a_3 < ...... < \rho_N < a_{N+1}, \quad (157)$$

lead to system of $N$ - equivalent differential equations each in the region $a_i < \rho_i < a_{i+1}$. They have the following form

$$4 \sqrt{\Pi_{i=1}^{N+1} (\rho - a_i)} \frac{\partial}{\partial \rho} \sqrt{\Pi_{i=1}^{N+1} (\rho - a_i)} \frac{\partial \psi}{\partial \rho} + \left( E \rho^{N-1} + \sum_{j=2}^{N} \lambda_j \rho^{N-j} \right) \psi = 0 \quad (158)$$

and together with the energy $E$ contain the $(N - 1)$ separation constants $\lambda_i$ $(i = 1, 2, ..., N - 1)$ as parameter. This is the case of complete nonseparation of separation constants. Equation (158) is the generalized Lame’ equation and falls into a class of equations of the Fuchsian type with $N + 2$ singularities in the points: $\{a_1, a_2, ....a_N, a_{N+1}, \infty \}$.\(^7\)

\(^6\) Really we can express our observation in the form of following hypothesis: **all quantum-mechanical problems which known for today as one-dimensional quasi-exactly solvable systems could be determine via separation of variables in $N$-dimensional Schrödinger equation for superintegrable systems.**

\(^7\) $(a_1, a_2,....a_N, a_{N+1})$ are elementary singularities with indices $(0,1/2 )$ and a point at infinity is regular.
Consider $N = 2$ when eq. (158) reduce to the famous Lam’e equation. It is possible to prove that condition of finiteness for eigenfunctions $\psi$ in the whole region for variable $\rho$ including the singular point $(a_1, a_2, a_3)$ perform only the polynomial solutions (Lam’e polynomials) which provide the quantization of energy in form $E = \ell(\ell+1)/R^2$, where $\ell = 0, 1, 2...$ and simultaneously determine the spectrum of separation constant $\lambda_i(R)$ as a solution of algebraic equation. It is well known that Lam’e polynomials can be represented by the series expansion around one of the singularities with the three-term recurrence relation for corresponding coefficients. Accordingly of our definition the equation (158) at $N = 2$ in interval $(a_1, a_3)$ give as the example of first order quasi-exactly solvable system. It is also obviously that reduction into intervals $(a_1, a_2)$ or $(a_2, a_3)$ for the fixed values of $E = \ell(\ell + 1)/R^2$ admit both polynomial and non-polynomial solutions. The system become partially quasi-exactly solvable.

Let us now consider the next dimension $N = 3$ which is more complicated. For $N = 3$ equation (158) already contain together energy parameter $E$ also two separation constants $\lambda_1$ and $\lambda_2$. As in previous case it is also possible to prove that the requirement of finiteness for wave function simultaneously in three intervals $(a_1, a_2)$, $(a_2, a_3)$ and $(a_3, a_4)$ lead to polynomial solutions and provide quantization of energy according the formula $E = J(J+2)/R^2$. Accordingly to [67] the generalized Lam’e polynomial is derived as expansion around one of the singularities $a_i$ ($i = 1, 2, 3, 4$) of the equation (158) where the coefficients of this expansion obey the four-term recurrence relations. The obtained from this recurrence relations system of homogeneous algebraic equations is overcomplete since the number of equations is larger than the number of unknowns (expansions coefficients), and the corresponding matrix is rectangular. Concerning a homogeneous systems of this type, it is known that a necessary and sufficient condition for the existence of a nontrivial solution is equality to zero of all determinants, however, as it is proved in [67], for our system it is sufficient that only two determinants, resulting from this system by eliminating the last and the next to last rows, be equal to zero. Therefore we are coming to the system of two algebraic equations which give a quantization of two separation constant $\lambda_1(R)$ and $\lambda_2(R)$. So, we term the equation (158) for $N = 3$ in the whole interval of variable $\rho \in (a_1, a_4)$ as second order quasi-exactly solvable.

Now we can determine the $N$th order quasi-exactly solvable systems following way: the quantum mechanics system is $N$th - order quasi-exactly solvable if the polynomial solution of the $N$ parametric differential equation of the Schrödinger equation type (or $N+1$ - dimensional Schrödinger equation after separation of variables) is defined through $N+1$-term recurrent relations and the spectrum of eigenvalues of separation constant could be calculated only numerically as the solutions of algebraic (or systems of algebraic) equations.

Thus in this work we classify four kind of solvability: exactly-solvable, quasi-exactly solvable, partially quasi-exactly solvable, and non-exactly solvable systems.

The interesting question which we not consider here is following: is it possible to classify all polynomials which coefficients in power series expansions can be determine through three-term or more higher order recurrence relations as it known for classical polynomials as Jacoby, Gegenbauer, Legandre, Lagerre and Hermite polynomials?
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