Masslesslike minimal subtraction for massive scalar field theory

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Abstract – We introduce the simplest minimal subtraction method for massive \(\lambda \phi^4\) field theory with \(O(N)\) internal symmetry, which resembles the same method applied to massless fields by using two steps. First, the utilization of the partial-\(p\) operation in every diagram of the two-point vertex part in order to separate it into a sum of squared mass and external momentum, respectively, with different coefficients. Then, the loop integral which is the coefficient of the quadratic mass can be solved entirely in terms of the mass, no longer depending upon the external momentum, using the parametric dissociation transform. It consists in the choice of a certain set of fixed values of Feynman parameters replaced inside the remaining loop integral after solving the internal subdiagrams. We check the results in the diagrammatic computation of critical exponents at least up to two-loop order using a flat metric with Euclidean signature.

Introduction. – With the introduction of the renormalization group [1,2], scalar field theories have been explored in a wide range of physical situations. Some instances include elementary particle physics, unveiling important properties of the standard model [3] as well as semianalytical multi-loop calculations with massive particles in the context of deep inelastic scattering [4]. There are also examples in gravitation and cosmology. (It was realized long ago that classical gravitation can be formulated in terms of scalar fields [5]). Quantum gravity effects can be obtained at least at one-loop order through the linearization of the metric tensor in the Einstein-Hilbert action interacting with a free massive scalar field, resulting in a vertex with two scalars and the graviton. Resummation techniques in the infrared region can be utilized in order to improve the ultraviolet regime of the quantum theory [6]. Furthermore, scalar, spinors and vector massless fields described in a cosmological background with a dilaton scalar field playing the role of the cosmological constant for especial values of its vacuum expectation value and coupling with the quantum fields are examples of fine tuning in cosmology which generates the masses of those fields [7].

One of the most important applications of renormalization group ideas is perhaps the perturbative computations of critical properties of many-body critical systems undergoing phase transitions [8]. Minimal subtraction schemes [9,10] are particularly simple in dealing with massless fields, but become somewhat involved in the treatment of massive fields. Is it possible to enunciate a minimal subtraction technique for massive fields which captures the same essential pattern of the simplest version of its massless counterpart?

In this letter we commence to shed light on this issue by devising such a method for massive scalar fields in a \(\lambda \phi^4\) theory. It is appropriate to call it “masslesslike” massive minimal subtraction since it resembles that for massless fields. It requires a minimal number of diagrams, precluding diagrams which include tadpole insertions (unlike in the BPHZ method [11]). With this shortcut, we show that all primitively divergent vertex parts (see refs. [8,12]) that can be renormalized multiplicatively are rendered finite utilizing our method. Since the perturbative formulation of spinor and vector fields ultimately reduce to the computation of Feynman integrals of scalar fields [13], this perfected scheme of minimal subtraction has the potential of application in several instances of renormalized perturbative computations of quantum massive fields in the high-energy (ultraviolet or simply UV) regime.

We begin with the bare Lagrangian density in \(d\)-dimensional flat space (Euclidean or Minkowski space-time with index \(\nu\)) for scalar fields written as

\[
\mathcal{L} = \frac{1}{2} \partial_{\nu} \phi \partial^{\nu} \phi + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} (\phi^2)^2, \tag{1}
\]

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where $\mu_0$ and $\lambda$ are the bare mass and coupling constant, respectively. We omitted the index corresponding to the $O(N)$ internal symmetry since they appear only as $N$-dependent coefficients in the diagrammatic expansion. The one-particle irreducible (1PI) primitive divergent vertex parts are $\Gamma^{(2)}(p, \mu_0, \lambda, \Lambda)$, $\Gamma^{(4)}(p, \mu_0, \lambda, \Lambda)$ and the composite one $\Gamma^{(2,1)}(p_1, p_2; Q_1, \mu_0, \lambda, \Lambda)$. The parameter $\Lambda$ is the cutoff which characterizes the bare theory under consideration [14]. The $p_i$ in the argument of the several vertex parts stand for external momenta, whereas $Q$ is the momentum of the inserted composite operator. Note that a vertex part with an arbitrary number $N$ of external legs and of composite operators $L$, is represented by $\Gamma^{(N,L)}(p_1, \ldots, p_N; Q_1, \ldots, Q_L, \mu_0, \lambda, \Lambda)$. Let us summarize some basic facts. The starting point is to define a three-loop bare mass $\mu^2 = \Gamma^{(2)}(p = 0, \mu_0, \lambda, \Lambda) = \mu_0^2 + O(\lambda)$. Next we write $\mu_0^2 = \mu^2 - O(\lambda)$ and replace it in all the vertex parts. Consequently, the vertex parts now depend only upon $\mu$ in their arguments. This procedure has the virtue of eliminating all diagrams including tadpole insertions inside all vertex parts at arbitrary loop order. For the aim we have in mind, we expand $\Gamma^{(2)}$ up to three-loop order, whereas $\Gamma^{(4)}$ and $\Gamma^{(2,1)}$ are expanded up to the two-loop level. The diagrams that are left in $\Gamma^{(2)}$ have the peculiarity that they must be subtracted from their value at $p = 0$ order by order at the loop expansion (see the conventions in [15]). Without loss of generality, we shall consider a Euclidean metric that will be useful for our purposes in what follows. The results, however, are valid (with minor modifications) to include Minkowski spacetime. Henceforth we shall drop the cutoff from the arguments of all vertex parts: dimensional regularization of the divergent integrals will be expressed as poles in $\epsilon = 4 - d$ throughout. The aforementioned bare primitive divergent vertex parts have the following perturbative expansions:

\[
\Gamma^{(2)}(p, \mu, \lambda) = p^2 + \mu^2 - \frac{\lambda^2(N + 2)}{6}[I_3(p, \mu) - I_3(0, \mu)],
\]
\[
\frac{\lambda^2(N + 2)(N + 8)}{27}[I_5(p, \mu) - I_5(0, \mu)], \quad (2a)
\]

\[
\Gamma^{(4)}(p_1, \mu, \lambda) = \lambda - \frac{\lambda^2(N + 8)}{2}[I_2(p_1 + p_2, \mu) + 2\text{perms.}],
\]
\[
\frac{\lambda^3(N + 6N + 20)}{27}[I_2(p_1 + p_2, \mu) + 2\text{perms.}], \quad (2c)
\]

\[
\Gamma^{(2,1)}(p_1, p_2; Q, \mu, \lambda) = 1 - \frac{\lambda(N + 2)}{18}[I_2(p_1 + p_2, \mu) + 2\text{perms.}],
\]
\[
\frac{\lambda^2(N + 2)^2}{108}[I_2(p_1 + p_2, \mu) + 2\text{perms.}], \quad (2c)
\]

The integrals $I_2$ and $I_4$ are not particularly important to the manipulations we are going to make. They are given by

\[
I_2(p, \mu) = \int \frac{d^dq}{(q^2 + \mu^2)[(q + P)^2 + \mu^2]},
\]
\[
I_4(p_1, \mu) = \int \frac{d^dq d^dq_2 d^dq_3}{(q_1^2 + \mu^2)(q_2^2 + \mu^2)} \times \frac{1}{[(P - q_1)^2 + \mu^2][(q_1 - q_2 + q_3)^2 + \mu^2]}.
\]

We simply give their expressions in terms of $\epsilon$ as

\[
I_2(p, \mu) = \frac{d^d[q^2 + q \cdot (1 + L(P, \mu))]}{[1 - \epsilon(L(P, \mu) + \frac{1}{2})]},
\]
\[
I_4(p_1, \mu) = \frac{d^d[q_1^2 + q_2^2 + q_3^2]}{[1 - \epsilon(L(P, \mu) + \frac{1}{2})]},
\]
\[
\frac{1}{[(q_1 + q_3 + q_4)^2 + \mu^2]}.
\]

We now apply the “partial-$p$” operation in the two-point vertex part $\Gamma^{(2)}$ in the two- and three-loop diagrams. Our paradigmatic example to be discussed here is the two-loop diagrams which we are left with. We apply it in the form

\[
\frac{\partial }{\partial q^2} + \frac{\partial }{\partial q^2_q_2},
\]

where $q_i$ are the loop momentum. For the three-loop graphs we have to use the partial-$p$ inside the integrand in a different form, involving all loop momenta (with $3d$ in the denominator of the operation).

Consider $I_3$. First apply the “partial-$p$” operation in the form given above. A preliminary result is

\[
I_3(p, \mu) = \frac{1}{(d - 3)} \left[ 3\mu^2 A(p, \mu) + B(p, \mu) \right],
\]
\[
A(p, \mu) = \int \frac{d^dq d^dq_2}{(q_1^2 + \mu^2)(q_2^2 + \mu^2)} \times [(q_1 + q_3 + q_4)^2 + \mu^2]},
\]
\[
B(p, \mu) = \int \frac{d^dq d^dq_2 d^dq_3}{(q_1^2 + \mu^2)(q_2^2 + \mu^2)} [(q_1 + q_2 + q_3)^2 + \mu^2].
\]

The issue is how to get rid of the $p$-dependence of the integral $A(p, \mu)$, since its dependence is there to stay. This can be done using the “parametric dissociation transform” (PDT) to be described now. Observe that $I_2(q_1 + p, \mu)$ appears as a subdiagram of $I_3(p, \mu)$. Solving for this internal
We then find
\[ C_1(p, \mu) = \int \frac{d^d q_1 d^d q_2 d^d q_3}{(q_1^2 + \mu^2)(q_2^2 + \mu^2)(q_3^2 + \mu^2)} \times \left[ \frac{1}{(q_1 + q_3 + p)^2 + \mu^2} \right]. \] (8b)

\[ C_2(p, \mu) = \int \frac{d^d q_1 d^d q_2 d^d q_3}{(q_1^2 + \mu^2)(q_2^2 + \mu^2)(q_3^2 + \mu^2)} \times \left[ \frac{1}{((q_1 + q_3 + p)^2 + \mu^2)} \right]. \] (8c)

\[ D(p, \mu) = \int \frac{d^d q_1 d^d q_2 d^d q_3 \cdot (q_1 + q_3 + p)}{(q_1^2 + \mu^2)(q_2^2 + \mu^2)(q_3^2 + \mu^2)} \times \left[ \frac{1}{((q_1 + q_3 + p)^2 + \mu^2)} \right]. \] (8d)

Let us analyze \( C_1(p, \mu) \) (\( C_2(p, \mu) \) can be studied analogously). Integrating independently the two internal bubbles (each one turns out to be \( I_2(q_1 + p, \mu) \)) and employing extra Feynman parameters, we are left with

\[ see \; eq. \; (9) \; above \]

The PDT implementation in the loop integral follows the same principle: replace into the momentum integral the values \( z = w = 1 \) (endpoint singularities of their parametric integrals). The principle is the same for arbitrary loops. We then find
\[ C_1(p, \mu) = \frac{\mu^2 - 2 \epsilon \mu^2}{2 \epsilon} \left[ 1 - \frac{\epsilon}{4} + \frac{\epsilon^2}{12} \right]. \] (6)

If we employ the same PDT in \( I_3(0, \mu) \), then we consistently obtain \( \langle A(p, \mu) - A(0, \mu) \rangle_{PDT} = 0 \). This asset is now available and makes the connection with the simplest version of the minimal subtraction scheme for massless fields [12]. Without loss of generality, we shall omit \( \mu \) from the arguments of the several diagrams, since it is obvious from the beginning that this parameter is left as a function of \( \mu \) only. Note that this value of the parameter corresponds to the endpoint singularity of the \( \gamma \) integration, which will maximize the loop momentum contribution as far as the \( y \)-dependence is concerned keeping, therefore, the correct pole structure of the integral \( A(p, \mu) \). We then find

\[ A(p, \mu) = \frac{\mu^2 - 2 \epsilon \mu^2}{2 \epsilon} \left[ 1 - \frac{\epsilon}{4} + \frac{\epsilon^2}{12} \right]. \] (5)

We start with PDT by setting \( y = 0 \) inside the momentum loop integral, since this has the virtue of eliminating all dependence on the external momentum \( p \) and what is left is a function of \( \mu \) only. The principle is the same for arbitrary loops. We then find
\[ I_3(p, \mu) - I_3(0) = -\frac{\mu^2 \epsilon^2}{4 \epsilon} \left[ 1 - \frac{\epsilon}{4} + \frac{\epsilon^2}{12} \right]. \] (7a)

\[ L_3(p, \mu) = \int_0^1 dx \delta y (1 - y) \ln \left[ \frac{\mu^2}{\epsilon^2} y (1 - y) \right]. \] (7b)

The same procedure can be applied to \( I_5 \). Using the partial-\( p \) operation, the integral \( I_5(p) \) now reads

\[ I_5(p) = -\frac{2}{3d - 10} \left[ \mu^2 (C_1(p, \mu) + 4C_2(p, \mu)) + 2D(p, \mu) \right]. \] (8a)
the extra subtraction, it turns out that it no longer satis-
ifies the nonperturbative Callan-Symanzik (CS) equation.

The same vertex part produces composite operators which
are not identical to the standard ones because of the ex-
tra subtraction. Consequently, the scaling limit of the CS
equation in the ultraviolet regime for the massive theory
is never attained in the context of that work. It is oppor-
tune to point out that, as it is going to be shown below,
the standard argument of minimal subtraction applies in
a straightforward manner to the present method. This is
in stark contrast with the problems plaguing the method
previously mentioned.

Application. — Let us express the bare and renor-
malized coupling constants in terms of dimensionless ones
through $\lambda = u_0 m^2$ and $g = u \mu^2$, respectively. In
the four-point vertex part, whose first term is put in
evidence, the remaining loop terms only depend on $\mu$
through the logarithmic integrations. In the other
vertex parts, those definitions suppress the overall de-
pendence on $\mu$ in every diagram. This dependence
only occurs through the parametric logarithmic inte-
grals as well, resulting in a bare perturbative expansions
in terms of the bare dimensionless coupling constant $u_0$.

The renormalized vertex parts built out of the
primatively divergent can be written as $\Gamma_R^{(2)}(p, m, u) = Z_0 \Gamma^{(2)}(p, \mu, u)$, $\Gamma_R^{(4)}(p_1, m, u) = Z_0^2 \Gamma^{(4)}(p_1, \mu, u)$ and
$\Gamma_R^{(2)}(p_2; Q, m, u) = Z_0 \Gamma_R^{(2)}(p_2; Q, \mu, u)$, respectively ($Z_\phi \equiv Z_0 Z_{\phi^2}$). Therefore, any vertex part
which is multiplicatively renormalizable satisfies the equa-
tion

$$\Gamma_R^{(N,L)}(p_i; Q_j, m, u) = Z_{\phi^2}^N Z_{\phi^2}^L \Gamma^{(N,L)}(p_i; Q_j, \mu, u)$$

\((i = 1, \ldots, N; j = 1, \ldots, L)\), where the normalization
functions $Z_\phi$ and $Z_{\phi^2}$ are determined entirely from the
finiteness of the renormalized vertex parts obtained from
the primively divergent vertex parts. The expansions

$$u_0 = u(1 + a_1 u + a_2 u^2),$$

$$Z_\phi = 1 + b_1 u^2 + b_2 u^4,$$

$$\bar{Z}_{\phi^2} = 1 + c_1 u + c_2 u^3,$$

\(10a, 10b, 10c\)

in terms of the renormalized coupling constant will suffice
for our program.

Let us state the renormalization by starting with the
two-point function up to three loops, but only the com-
putation of $b_2$ will be made explicit. The bare vertex
function can be written as $\Gamma^{(2)} = \rho^2 + \mu^2 - B_2 u_0^2 + B_3 u_0^3$. Note
that $B_2 = \rho^2[\delta I_3(p) - I_3(0)]$ and $B_3 = \mu^2[\delta I_5(p) - I_5(0)]$.

Forget for the time being the last term which will be
important in the computation of $b_3$. Multiplicative renor-
malizability implies that

$$\Gamma^{(2)}(p, m, u) = Z_{\phi^2} \Gamma^{(2)}(p, \mu, u) = \rho^2 + Z_{\phi^2} \rho^2 = (b_2 u_0)^2$$

is finite (at this order $u_0^2 = u^2$). Since we analyze the
two-point vertex part at three-loop order, we define the
renormalized mass at third order in perturbation theory as

$$m^2 = Z_{\phi^2} m^2.$$ 

Had we worked at the $l$-loop order, we would have defined the renormalized
mass at the $l$-loop order in the same manner, with $Z_\phi$ computed at the $l$-loop order. (Without loss of generality
we restrict ourselves only up to three-loop level.) This
is similar to what happens in the massless theory: there
are no tadpoles if we impose that the renormalized mass
is zero to all orders in perturbation theory, which follows
from the same equation at arbitrary loop order when set-
ing $\mu = 0$. By demanding that the renormalized two-
point vertex part to be finite at two-loop level, we find

$$b_2 = \frac{(N + 2)}{144} \mu.$$

Now, $\Gamma_R^{(2)}(p_1; \mu, u) = Z_0^2 \Gamma^{(2)}(p_1; \mu, u)$ and the logarith-
mic integrals whose coefficients contain poles in $\epsilon$ cancel
out in the perturbative expansion after we expand the
dimensionless bare coupling constant in terms of the di-
mensionless renormalized one (in the computation of $a_2$).

This yields $a_1 = \frac{(N + 4)}{6\epsilon^4}, a_2 = \frac{(N + 8)}{6\epsilon^6} - \frac{(N + 14)}{24\epsilon^6}$. Us-

ing a similar reasoning for the composite vertex, namely
that the explicit cancellation of the logarithmic integrals
takes place in a similar manner as occurred with the four-
point function just discussed, and including the computa-
tion of three-loop contribution belonging to the two-point
function, we can write the normalization functions in the form

$$Z_{\phi^2} = 1 + \frac{(N + 2)}{6\epsilon} u + \frac{(N + 2)(N + 5)}{36\epsilon^2} u^2,$$

$$Z_{\phi} = 1 + \frac{(N + 2)}{144\epsilon} u^2 - \frac{(N + 2)(N + 8)}{1296\epsilon^2} u^3,$$

\(11a, 11b\)

The Euclidean metric chosen can be utilized to check
universality in critical phenomena through the diagram-
ic calculation of critical exponents using the CS frame-
work [16–18], since in the Lagrangian (1), the mass
is proportional to $|T - T_C|$. Therefore, the tempera-
ture of the system is away from the critical tempe-
ratue $T_C$ characterizing the phase transition in the present
setting.

By considering the vertex parts in terms of the
dimensionless coupling constants and applying
the operator $m_0 \frac{\partial}{\partial m_0}$ on $\Gamma^{(N,L)}$, we find

$$[m_0 \frac{\partial}{\partial m_0} + \beta(u) \frac{\partial}{\partial u} - \frac{N}{2} \gamma_0(u) + L \gamma_0(u)]\Gamma^{(N,L)}(p_i; Q_j, m, u) = \frac{2 - \gamma_0(u)}{Z_{\phi}} \Gamma^{(N,L+1)}(p_i; Q_j, 0, m, u).$$

In the original argument, the right-hand side (r.h.s.)
of this equation was obtained using normalization conditions for
the renormalized mass by setting $N = 2, L = 0$ [12]. Since the
renormalized mass in our method is not obtained from a
fixed value of external momenta as in normalization condi-
tions, it results in the appearance of the term $\frac{1}{Z_{\phi}}$ in
in the CS equation. Then, by demanding independence of
the renormalization scheme, we simply have to set the
tree-level value $\bar{Z}_{\phi^2} = 1$ within the context of our method
and the CS equation turns out to be the same either
using normalization conditions or the present minimal subtraction method, namely
\[\left[ m \frac{\partial}{\partial m} + \beta(u) \frac{\partial}{\partial u} - N \frac{\gamma(u)}{2} + L \phi^2 \right] \times R_\phi^{(N,L)}(p_i ; Q_j , m , u) = (2 - \gamma(u))m^2 R_\phi^{(N,L+1)}(p_i ; Q_j , 0 , m , u). \tag{12}\]

After that, in the UV regime, the right-hand side (r.h.s.) appearance of $\gamma(u)$ but defined as discussed above, the normalization conditions of the squared bare mass in two-point vertex parts, but they do not contribute due to the special properties of the perturbative expansion chosen herein.

The details will be reported elsewhere.

The Wilson function $\beta(u) = m \left( \frac{\partial m}{\partial m} \right)_\lambda = -m \left( \frac{\partial m}{\partial m} \right)_\mu$, that can be rewritten as
\[\beta(u) = - \epsilon \left( \frac{\partial \ln u_0}{\partial u} \right)^{-1} u \left[ -\epsilon + \frac{(N + 8)}{6} u - \frac{(3N + 14)}{12} u^2 \right], \tag{13}\]
has a nontrivial (repulsive) UV fixed point $\left( \beta(u_\infty) = 0 \right)$, namely $u_\infty = \frac{6}{1 + (N + 14) \epsilon}$. The function $\gamma(u) = \beta(u) \frac{\partial \ln Z_{m,u}}{\partial u} = u \left[ \frac{N + 2}{u} - \frac{(N + 2)(N + 8)}{128} \epsilon^2 \right]$ when computed at the fixed point, yields the (anomalous dimension of the field) exponent $\eta(\epsilon)$ up to three-loop order, namely
\[\eta = \frac{(N + 12)}{2(N + 8)^2} \left[ 1 + \frac{6(N + 14)}{(N + 8)^2} - \frac{1}{4} \epsilon \right]. \tag{14}\]

Moreover, the function $\gamma(\phi^2)(u) = - \beta(u) \frac{\partial \ln Z_{\phi^2}}{\partial u} = \frac{(N + 2)}{u} \left[ 1 + \frac{6}{4(N + 8)} \epsilon + \frac{(N + 2)(N^2 + 23N + 60)}{8(N + 8)^2} \epsilon^2 \right]. \tag{15}\]
The details will be reported elsewhere.

**Discussion and conclusions.** It turns out that the present method has several advantages over all previously massive renormalization schemes. First, in comparison with normalization conditions [20] it is much simpler. Second, when compared with the BPHZ minimal subtraction method [21–23], a minimal number of diagrams is required. Third, in the case studied here, the renormalized mass receives no “radiative corrections” but is defined by the product of the three-loop bare mass, which is an arbitrary parameter, multiplied by the normalization function $Z_\phi$.

Since the renormalized mass is not obtained from normalization conditions but defined as discussed above, the appearance of $Z_\phi$ in the CS equation is a residual effect of this definition and can be neutralized by setting its tree-level value $Z_\phi = 1$. This results in the “covariance” of the CS equation by using either normalization conditions or the present minimal subtraction scheme. Since the CS equation is a nonperturbative tool, valid order by order in perturbation theory, this general feature will be maintained in higher-loop orders.

The nontrivial determination of the renormalization functions is almost the same as in massless theories. Moreover, the method keeps the pole structure of the coefficients of the squared bare mass in two-point vertex parts, but they do not contribute due to the special properties of the perturbative expansion chosen herein.

The present framework might be able to address the renormalization of perturbative expansions of quantum massive scalar fields in particle physics. For instance, in the scalar sector of the Higgs doublet model [24]. A recent study of minimal subtraction renormalization [25] beyond one-loop level would be feasible within the context of our method. Moreover, it might offer a simple alternative to tackling massive scalar field renormalization in an external potential [26].

In critical phenomena, systems confined in a parallel plate geometry represented by massive fields can now be treated within this minimal subtraction generalizing the treatment in the massless scheme for periodic and antiperiodic boundary conditions [27] for the field. They remain to be investigated in the massive theory and with more general boundary conditions [28].

Curiously, the cancellation of tadpoles in the massive formulation of ref. [28] with a more complicated internal tensor structure is analogous to the perturbative expansion of the vertex parts discussed in the present proposal. Indeed, the finite-size (FS) effect is implemented as an internal symmetry as $\left( O(N) \times (FS) \right)$ and works in the same manner as presented here, but using normalization conditions. It will be interesting to apply the present formalism in the same problem and see whether it needs any adaptation. Its application in the renormalization of generic Lifshitz competing systems [29] is left for future research.

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