Diophantine properties of Brownian motion: recursive aspects

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We use recent results on the Fourier analysis of the zero set of Brownian motion to explore the diophantine properties of an algorithmically random Brownian motion (also known as a complex oscillation). We discuss the construction and definability of perfect sets which are linearly independent over the rationals directly from Martin-Löf random reals. Finally we explore the recent work of Tsirelson on countable dense sets to study the diophantine properties of local minimisers of Brownian motion.

1 Introduction

A Brownian motion on the unit interval is algorithmically random if it meets all effective (Martin-Löf) statistical tests, now expressed in terms of the statistical events associated with Brownian motion on the unit interval. The class of functions corresponds exactly, in the language of Weihrauch [25, 26], Gács [11] and specialised by Hoyrup and Rojas [13], in the context of algorithmic randomness, to the Martin-Löf random elements of the computable measure space 

$$\mathcal{R} = (C_0[0,1], d, B, W),$$

where $C_0[0,1]$ is the set of the continuous functions on the unit interval that vanish at the origin, $d$ is the metric induced by the uniform norm, $B$ is the countable set of piecewise linear functions vanishing at the origin with slopes and points of non-differentiability all rational numbers and where $W$ is the Wiener measure. We shall also refer to such a Brownian motion as a complex oscillation. This terminology was suggested to the author by the following Kolmogorov theoretic interpretation of this notion [2]: One can characterise a Brownian motion which is generic (in the sense just stated) as an effective and uniform limit of a sequence $(x_n)$ of “finite random walks”, where, moreover, each $x_n$ can be encoded by a finite binary string $s_n$ of length $n$, such that the (prefix-free) Kolmogorov complexity, $K(s_n)$, of $s_n$ satisfies, for some constant $d > 0$, the inequality $K(s_n) > n - d$ for all values of $n$. (See Definition [1] introduced by Asarin and Prokofskii [2, in Section 3 below.) We shall study the images of certain ultra-thin sets (perfect sets of Hausdorff dimension zero) under a complex oscillation. We have shown in [7] that these images are perfect sets whose elements are linearly independent over the field of rational numbers. In this paper we discuss the definability of these sets, within the recursion-theoretic hierarchy, by exploiting the recursive isomorphism constructed in [5] between the Kolmogorov-Chaitin random reals and the class of suitably encoded versions of complex oscillations. We shall also utilise Tsirelson’s theory of countable dense random sets [24] to study the diophantine properties of the local minimisers of Brownian motion. The local minimizers of a complex oscillation is studied in [8].

We shall utilise recent results by Mukeru and the author [9] on the rate of decay of the Fourier transform of the delta function of a continuous version of Brownian motion to identify some diophantine properties of the zero set of a complex oscillation. For more on the Fourier and consequent Diophantine properties of the sample paths of Brownian motion the reader is referred to the paper by Łaba and Pramanik [17]. We shall also show that some of these phenomena can be expressed within the hyperaritmetical hierarchy and pose the problem as to whether this is essentially so.

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2 Preliminaries from Brownian motion and geometric measure theory

A random variable $X$ with mean $\mu$ and variance $\sigma^2$ is normal if it has a density function of the form

$$\frac{1}{\sqrt{2\pi\sigma}} e^{-(t-\mu)^2/2\sigma^2}.$$ 

If $(\Omega, \mathcal{F}, P)$ is a probability space and $X$ is a real-valued random variable on $\Omega$, the measure $\mu$ on $\mathcal{F}$ given by $F \mapsto P(X^{-1}(F))$, $F \in \mathcal{F}$, is called the distribution of $X$.

A Brownian motion on the unit interval is a real-valued function $(\omega, t) \mapsto X_\omega(t)$ on $\Omega \times [0, 1]$, where $\Omega$ is the underlying space of some probability space, such that $X_\omega(0) = 0$ a.s. and for $t_1 < \ldots < t_n$ in the unit interval, the random variables $X_\omega(t_1), X_\omega(t_2) - X_\omega(t_1), \ldots, X_\omega(t_n) - X_\omega(t_{n-1})$ are statistically independent and normally distributed with means all 0 and variances $t_1, t_2 - t_1, \ldots, t_n - t_{n-1}$, respectively.

It is a fundamental fact that any Brownian motion has a “continuous version” (see, for example [10]). This means the following: Write $\Sigma$ for the $\sigma$-algebra of Borel sets of $C[0, 1]$ where the latter is topologised by the uniform norm topology. There is a unique probability measure $W$ on $\Sigma$ such that for $0 \leq t_1 < \ldots < t_n \leq 1$ and for a Borel subset $B$ of $\mathbb{R}^n$, we have

$$P(\{\omega \in \Omega : (X_\omega(t_1), \ldots, X_\omega(t_n)) \in B\}) = W(A),$$

where

$$A = \{x \in C[0, 1] : (x(t_1), \ldots, x(t_n)) \in B\}.$$ 

The measure $W$ is known as the Wiener measure. We shall usually write $X(t)$ instead of $X_\omega(t)$.

For a compact subset $A$ of Euclidean space $\mathbb{R}^d$ and real numbers $\alpha, \epsilon$ with $0 \leq \alpha < d$ and $\epsilon > 0$, consider all coverings of $A$ by balls $B_n$ of diameter $\leq \epsilon$ and the corresponding sums

$$\sum_n |B_n|^{\alpha},$$

where $|B|$ denotes the diameter of $B$. All the metric notions here are to be understood in terms of the standard $\ell^2$ norms on Euclidean space. The infimum of the sums over all coverings of $A$ by balls of diameter $\leq \epsilon$ is denoted by $H^\epsilon_{\alpha}(A)$. When $\epsilon$ decreases to 0, the corresponding $H^\epsilon_{\alpha}(A)$ increases to a limit (which may be infinite). The limit is denoted by $H_{\alpha}(A)$ and is called the Hausdorff measure of $A$ in dimension $\alpha$.

If $0 < \alpha < \beta \leq d$, then, for any covering $(B_n)$ of $A$,

$$\sum_n |B_n|^{\beta} \leq \sup_n |B_n|^{\beta - \alpha} \sum_n |B_n|^{\alpha},$$

from which it follows that

$$H^\epsilon_{\beta}(A) \leq \epsilon^{\beta - \alpha} H^\epsilon_{\alpha}(A).$$

Hence if $H_{\alpha}(A) < \infty$, then $H_{\beta}(A) = 0$. Equivalently,

$$H_{\beta}(A) > 0 \implies H_{\alpha}(A) = \infty.$$ 

Therefore,

$$\sup\{\alpha : H_{\alpha}(A) = \infty\} = \inf\{\beta : H_{\beta}(A) = 0\}.$$ 

This common value is called the Hausdorff dimension of $A$ and denoted by $\text{dim}_h A$.

If $\alpha$ is such that $0 < H_{\alpha}(A) < \infty$, then $\alpha = \text{dim}_h A$. However, if $\alpha = \text{dim}_h A$, we cannot say anything about the value of $H_{\alpha}(A)$.

It is easy to check that $A \mapsto H_{\alpha}(A)$ defines an outer measure which is invariant under translations and rotations, and homogeneous of degree $\alpha$ with respect to dilations.

If $A$ is a Borel subset of Euclidean space, the set of non-zero Radon measures with support contained in $A$ is denoted by $M^+(A)$. For a given $\mu \in M^+(A)$, the energy integral of $\mu$ with respect to the kernel $|x|^{-\alpha}$ is given by

$$I_{\alpha}(\mu) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{d\mu(x)d\mu(y)}{|x-y|^{\alpha}}.$$
We say that \( \mu \) has finite energy with respect to \(|x|^{-\alpha}\) when \( I_\alpha(\mu) < \infty \). If \( A \) carries positive measures of finite energy with respect to \(|x|^{-\alpha}\) we say that \( A \) has positive capacity with respect to \(|x|^{-\alpha}\) and we write
\[
\text{Cap}_\alpha(A) > 0.
\]
If \( A \) carries no positive measure of finite energy with respect to \(|x|^{-\alpha}\), we say that \( A \) has capacity zero with respect to this kernel and we write \( \text{Cap}_\alpha(A) = 0 \).

It follows from the Fourier analysis of temperate distributions that
\[
I_\alpha(\mu) = C(\alpha, d) \int_{\mathbb{R}^d} |\hat{\mu}(\xi)|^2 |\xi|^{\alpha} \frac{d\xi}{|\xi|^d},
\]
when \( 0 < \alpha < d \), where \( C(\alpha, d) \) is a positive constant and where, moreover,
\[
\hat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{i\xi s} d\mu(s),
\]
is the Fourier transform of the measure \( \mu \). (For a proof see [19].)

We shall make frequent use of the following very fundamental fact.

**Proposition 1.** For a compact subset \( A \) of \( \mathbb{R}^d \) and \( 0 < \alpha < \beta < d \),
\[
H_\beta(A) > 0 \Rightarrow \text{Cap}_\alpha(A) > 0 \Rightarrow H_\alpha(A) > 0.
\]

Hence
\[
\sup\{ \alpha : \text{Cap}_\alpha(A) > 0 \} = \dim_h A,
\]
or, equivalently,
\[
\sup\{ \alpha : I_\alpha(\mu) < \infty \} = \dim_h A.
\]

The Fourier dimension of a compact set is the supremum of positive real numbers \( \alpha < 1 \) such that for some non-zero Radon measure \( \mu \) supported by \( E \), it is the case that
\[
|\hat{\mu}(\xi)|^2 \leq \frac{1}{|\xi|^{\alpha}},
\]
for \( |\xi| \) sufficiently large. The Fourier dimension of \( E \) is denoted by \( \dim_f(E) \). Clearly, by [1],
\[
\dim_f(E) \leq \dim_h(E),
\]
for all compact sets \( E \). The set is called a Salem set if \( \dim_f(E) = \dim_h(E) \).

The following question posed by Beurling was addressed and solved in the positive by Salem in 1950. (On singular monotonic functions whose spectrum has a given Hausdorff dimension By R. Salem (1950), Ark Mat 1,353-365.)

Given a number \( \alpha \in (0, 1) \), does there exist a closed set on the line whose Hausdorff dimension is \( \alpha \) that carries a Borel measure \( \mu \) whose Fourier transform
\[
\hat{\mu}(u) = \int_{\mathbb{R}} e^{iu x} d\mu(x)
\]
is dominated by \(|u|^{-\alpha/2}\) as \(|u| \to \infty|\)?

It follows from [1] that given a compact subset \( E \) of \([0, 1]\) with Hausdorff dimension \( \alpha \in (0, 1) \), the number \( \alpha/2 \) is critical for this question to have an affirmative answer.

Salem proved this result by constructing for every \( \alpha \) in the unit interval, a random measure \( \mu \) (over a convenient probability space) whose support has Hausdorff dimension \( \alpha \) and which satisfies the Beurling-requirement with probability one.

It was recently shown by the author in collaboration with George Davie and Safari Mukeru that such sets can also be constructed by looking at Cantor type sets \( E \) with computable ratios \( \xi \) and then to consider the image of \( E \) under a complex oscillation.

The following theorem illustrates the rich diophantine structure of sets \( E \) of non-zero Fourier dimension. Even though the proof method is well-known in geometric measure theory, we give a full proof, for we need sharper estimates than waht the author could have found in the literature.
**Theorem 2.** (folklore) Suppose \( E \) is a compact subset of reals such that, for every \( \epsilon > 0 \), there is some \( \mu \in M_+(E) \) and \( 0 < \alpha < 1 \), such that, for some constant \( C = C(\epsilon) \), it is the case that

\[
|\hat{\mu}(\xi)|^2 \leq C|\xi|^{-\alpha+\epsilon},
\]

as \( |\xi| \to \infty \). Then, if \( k \) is a natural number such that \( k\alpha > 1 \), it will follow, upon writing

\[
E_k = E + \cdots + E \ (k \text{ times}),
\]

that

\[
\mathbb{R} = \bigcup_{n<\omega} n(E_k - E_k).
\]

Moreover, if \( A \) is any finite set of real numbers, then \( E_k \) will contain an affine (a translated and rescaled) copy of \( A \).

**Proof.** Set

\[
\nu = \mu \ast \cdots \ast \mu \ (k \text{ times}).
\]

(Here * denotes the convolution product.) Clearly, by choosing \( \epsilon > 0 \) such that \( k(\alpha - \epsilon) > 1 + \epsilon \), we have, for \( |\xi| \) large,

\[
|\hat{\nu}(\xi)|^2 = |\hat{\mu}(\xi)|^k \leq C^k|\xi|^{-k\alpha+k\epsilon} \leq C^k|\xi|^{-1-\epsilon}.
\]

It follows that the function \( \hat{\nu} \) is in \( L^2(\mathbb{R}) \). Since \( \nu \) is a non-zero measure, it follows from Parseval’s theorem that \( \nu \) is absolutely continuous with respect to Lebesgue measure. In particular, \( \text{supp } \nu \) has non-zero Lebesgue measure. Since

\[
\text{supp } \nu \subseteq E_k,
\]

we conclude that \( E_k \) has non-zero Lebesgue measure. It follows from Steinhaus’s theorem \([23]\) that \( E_k - E_k \) has zero as an interior point. This concludes the first part of the theorem.

The second part follows from the following beautiful remark \([17]\): If \( F \) is any set of positive Lebesgue measure, then \( F \) will contain an affine copy of any finite set \( A \) of real numbers. This is, as noted by Łaba and Pramanik \([17]\), a consequence of Lebesgue’s density theorem. \( \square \)

### 3 Complex oscillations

The set of non-negative integers is denoted by \( \omega \) and we write \( \mathcal{B} \) for the Cantor space \( \{0, 1\}^\omega \). The set of words over the alphabet \( \{0, 1\} \) is denoted by \( \{0, 1\}^* \). If \( a \in \{0, 1\}^* \), we write \( |a| \) for the length of \( a \). If \( \alpha = a_0a_1\ldots \) is in \( \mathcal{B} \), we write \( \overline{\alpha}(n) \) for the word \( \prod_{j<n} a_j \). We use the usual recursion-theoretic terminology \( \Sigma^0_k \) and \( \Pi^0_k \) for the arithmetical subsets of \( \omega^k \times \mathcal{B}^l \), \( k, l \geq 0 \). (See, for example, \([12]\)). We write \( \lambda \) for the Lebesgue probability measure on \( \mathcal{B} \). For a binary word \( s \) of length \( n \), say, we write \( [s] \) for the “interval” \( \{ \alpha \in \mathcal{B} : \overline{\alpha}(n) = s \} \). A sequence \( (a_n) \) of real numbers converges effectively to 0 as \( n \to \infty \) if for some total recursive \( f : \omega \to \omega \), it is the case that \( |a_n| \leq (m+1)^{-1} \) whenever \( n \geq f(m) \).

For any finite binary word \( a \) we denote its (prefix-free) Kolmogorov complexity by \( K(a) \). Recall that an infinite binary string \( \alpha \) is Kolmogorov-Chaitin complex if

\[
\exists \forall n \ K(\overline{\alpha}(n)) \geq n - d.
\]

In the sequel, we shall denote this set by \( KC \) and refer to its elements as \( KC \)-strings. (See, e.g., \([3, 18]\) or \([21]\) for more background.)

For \( n \geq 1 \), we write \( C_n \) for the class of continuous functions on the unit interval that vanish at 0 and are linear with slopes \( \pm \sqrt{n} \) on the intervals \( [(i-1)/n, i/n] \), \( i = 1, \ldots, n \). With every \( x \in C_n \), one can associate a binary string \( a = a_1 \cdots a_n \) by setting \( a_i = 1 \) or \( a_i = 0 \) according to whether \( x \) increases or decreases on the interval \( [(i-1)/n, i/n] \). We call the sequence \( a \) the code of \( x \) and denote it by \( c(x) \). The following notion was introduced by Asarin and Prokopenko in \([2]\).

**Definition 1.** A sequence \( (x_n) \) in \( C[0, 1] \) is complex if \( x_n \in C_n \) for each \( n \) and there is a constant \( d > 0 \) such that \( K(c(x_n)) \geq n - d \) for all \( n \). A function \( x \in C[0, 1] \) is a complex oscillation if there is a complex sequence \( (x_n) \) such that \( ||x - x_n|| \) converges effectively to 0 as \( n \to \infty \).

The class of complex oscillations is denoted by \( \mathcal{C} \).

In \([5]\) the author constructed a bijection \( \Phi : KC \to \mathcal{C} \) which is effective in the following sense: If \( \alpha \in KC \) and \( m < \omega \), one can effectively construct from the first \( m \) bits of \( \alpha \), a function \( p_m \), where \( p_m \) is a finite linear combination of piecewise linear functions
with rational coefficients, such that, for some absolute positive constant $C$, the complex oscillation $\Phi(\alpha)$ is approximated by the sequence $(p_m(\alpha))$ as follows:

$$\sup_{t \in [0,1]} |\Phi(\alpha)(t) - p_m(t)| \leq C \log m / \sqrt{m}$$  \hspace{1cm} (3)

for all $m > M_\alpha$, where $M_\alpha$ is a constant that depends on $\alpha$ only. Conversely, if $x \in C$, then one can compute, relative to an infinite binary string which encodes the values of a complex oscillation $x$ at the rational numbers in the unit interval, the KC-string $\alpha$ such that $\Phi(\alpha) = x$.

In [3] the author proved:

**Theorem 3.** There is a uniform algorithm that, relative to any KC-string $\alpha$, with input a rational number $t$ in the unit interval and a natural number $n$, will output the first $n$ bits of the value of the complex oscillation $\Phi(\alpha)$ at the value $t$.

This result plays a crucial rôle in this paper, for it will enable us to show how the sample path properties of a complex oscillation $\Phi(\alpha)$ (and hence of a typical Brownian motion) can be described within the arithmetical hierarchy relative to the associated KC-string $\alpha$. In this way, as was stated in the introduction of this paper, one finds an explicit unfolding of the incredibly rich geometry that is enfolded in every KC-string $\alpha$ by merely regarding such an $\alpha$ as an encoding of a complex oscillation or, equivalently, of an (effectively) generic Brownian motion.

The mapping $\Phi$ is also a measure-theoretic isomorphism in the following (standard) sense: Write $\lambda$ for the Lebesgue measure on the space $\{0, 1\}^\omega$ and write $W$ for the Wiener measure on $C[0,1]$. Then, for any Borel subset $A$ of $C[0,1]$ with the uniform norm topology, we have

$$\lambda(\Phi^{-1}(A)) = W(A).$$

In other words, $W$ is the pushout of $\lambda$ under $\Phi$. We shall frequently denote $\Phi(\alpha)$ by $x_\alpha$.

We follow [4] to define an analogue of a $\Pi^0_1$ subset of $C[0,1]$ which is of constructive measure 0. If $F$ is a subset of $C[0,1]$, we denote by $\overline{F}$ its topological closure in $C[0,1]$ with the uniform norm topology. For $\epsilon > 0$, we let $O_\epsilon(F)$ be the $\epsilon$-ball $\{f \in C[0,1] : \exists x \in F \parallel f - g \parallel < \epsilon\}$ of $f$. (Here $\parallel \cdot \parallel$ denotes the supremum norm.) We write $F^0$ for the complement of $F$ and $F^1$ for $F$.

**Definition 2.** A sequence $F_0 = (F_i : i < \omega)$ in $\Sigma$ is an effective generating sequence if

1. for $F \in F_0$, for $\epsilon > 0$ and $\delta \in [0,1]$, we have, for $G = O_\epsilon(F^\delta)$ or for $G = F^\delta$, that $W(\overline{G}) = W(G)$.
2. there is an effective procedure that yields, for each sequence $0 \leq i_1 < \ldots < i_n < \omega$ and $k < \omega$ a binary rational number $\beta_k$ such that

$$|W(F_{i_1} \cap \ldots \cap F_{i_n}) - \beta_k| < 2^{-k},$$

3. for $n, i < \omega$, a strictly positive rational number $\epsilon$ and for $x \in C_n$, both the relations $x \in O_\epsilon(F_i)$ and $x \in O_\epsilon(F^n_i)$ are recursive in $x, \epsilon, i$ and $n$, relative to an effective representation of the rationals.

If $F_0 = (F_i : i < \omega)$ is an effective generating sequence and $F$ is the Boolean algebra generated by $F_0$, then there is an enumeration $(T_i : i < \omega)$ of the elements of $F$ (with possible repetition) in such a way, for a given $i$, one can effectively describe $T_i$ as a finite union of sets of the form

$$F = F^{\delta_{i_1}}_{i_1} \cap \ldots \cap F^{\delta_{i_n}}_{i_n},$$

where $0 \leq i_1 < \ldots < i_n$ and $\delta_i \in [0,1]$ for each $i \leq n$. We call any such sequence $(T_i : i < \omega)$ a recursive enumeration of $F$. We say in this case that $F$ is effectively generated by $F_0$ and refer to $F$ as an effectively generated algebra of sets.

Let $(T_i : i < \omega)$ be a recursive enumeration of the algebra $F$ which is effectively generated by the sequence $F_0 = (F_i : i < \omega)$ in $\Sigma$. It is shown in [4] that there is an effective procedure that yields, for $i, k < \omega$, a binary rational $\beta_k$ such that

$$|W(T_i) - \beta_k| < 2^{-k},$$

in other words, the function $i \mapsto W(T_i)$ is computable.

A sequence $(A_n)$ of sets in $F$ is said to be $F$-semirecursive if it is of the form $(T_{\phi(n)})$ for some total recursive function $\phi : \omega \rightarrow \omega$ and some effective enumeration $(T_i)$ of $F$. (Note that the sequence $(A^\epsilon_n)$, where $A^\epsilon_n$ is the complement of $A_n$, is also an $F$-semirecursive sequence.) In this case, we call the union $\cup_n A_n$, a $\Sigma^0_1(F)$ set. A set is a $\Pi^0_1(F)$-set if it is the complement of a $\Sigma^0_1(F)$-set. It is of the form $\cap_n A_n$ for some $F$-semirecursive sequence $(A_n)$. A sequence $(B_n)$ in $F$ is a uniform sequence of $\Sigma^0_1(F)$-sets if, for some total recursive function $\varphi : \omega^2 \rightarrow \omega$ and some effective enumeration $(T_i)$ of $F$, each $B_n$ is of the form

$$B_n = \bigcup_m T_{\varphi(n,m)}.$$

In this case, we call the intersection $\cap_n B_n$ a $\Pi^0_1(F)$-set. If, moreover, the Wiener-measure of $B_n$ converges effectively to 0 as $n \rightarrow \infty$, we say that the set given by $\cap_n B_n$ is a $\Pi^0_2(F)$-set of constructive measure 0.

The proof of the following theorem appears in [4].
Theorem 4. Let $\mathcal{F}$ be an effectively generated algebra of sets. If $x$ is a complex oscillation, then $x$ is in the complement of every $\Pi^0_2(\mathcal{F})$-set of constructive measure 0.

This means, that every complex oscillation is, in an obvious sense, $\mathcal{F}$-Martin-Löf random.

Definition 3. An effectively generated algebra of sets $\mathcal{F}$ is universal if the class $\mathcal{C}$ of complex oscillations is definable by a single $\Sigma^0_2(\mathcal{F})$-set, the complement of which is a set of constructive measure 0. In other words, $\mathcal{F}$ is universal iff a continuous function $x$ on the unit interval is a complex oscillation iff $x$ is $\mathcal{F}$-Martin-Löf random.

We introduce two classes of effectively generated algebras $\mathcal{G}$ and $\mathcal{M}$ which are very useful for reflecting properties of one-dimensional Brownian motion into complex oscillations.

Let $\mathcal{G}_0$ be a family of sets in $\Sigma$ each having a description of the form:

$$a_1X(t_1) + \cdots + a_nX(t_n) \leq L \tag{4}$$

or of the form $\{\}$ with $\leq$ replaced by $<$, where all the $a_j, t_j \ (0 \leq t_j \leq 1)$ are non-zero rational numbers, $L$ is a recursive real number and $X$ is one-dimensional Brownian motion.

We require that it be possible to find an enumeration $(G_i : i < \omega)$ of $\mathcal{G}_0$ such that, for given $i$, if $G_i$ is given by (4), we can effectively compute the sign, the denominators and numerators of the rational numbers $a_j, t_j$ and, moreover, that the recursive real $L$ can be computed up to arbitrary accuracy.

It is shown in [5] that $\mathcal{G}_0 = (G_i : i < \omega)$ is an effective generating sequence in the sense of Definition 2. The associated effectively generated algebra of sets $\mathcal{G}$ will be referred to as a gaussian algebra.

It is shown in [4] that if $\mathcal{G}_0$ is defined by events of the form (4) with $n = 1$ and $a_1 = 1$, then the associated $\mathcal{G}$ is in fact universal in the sense of Definition 3.

We shall also make frequent use of the following result from [4] which is an easy consequence of Theorem 4. It is the analogue, for continuous functions, of the well-known fact that Kurtz-random reals are in fact Martin-Löf random.

Theorem 5. If $B$ is a $\Sigma^0_2(\mathcal{F})$ set and $W(B) = 1$, then $C$, the set of complex oscillations, is contained in $B$.

4 Diophantine properties of zero sets of Brownian motion and complex oscillations

The following result is proven in [9].

Theorem 6. (Fouché and Mukeru) (2013). Let $X$ be a continuous version of one-dimensional Brownian motion on the unit interval. Then, almost surely, there exists a nonzero Radon measure $\mu$ with support on $Z_X$, the zero set of $X$, such that its Fourier transform $\hat{\mu}$ satisfies the inequality

$$|\hat{\mu}(\xi)|^2 \ll |\xi|^{-\frac{1}{2} + \varepsilon}, \tag{5}$$

as $|\xi| \to \infty$. In particular, the zero-set of Brownian motion is a Salem set.

It would be interesting to study the existence of arithmetic progressions in the zero sets of $X$. This question is related to the results obtained in Section 8 of [17] by Laba en Pramanik.

By Theorem 2 the preceding theorem has the following consequence:

Theorem 7. For a continuous version $X$ of Brownian motion over the unit interval, we have, almost surely,

$$\mathbb{R} = \bigcup_{n=1}^{\infty} n(Y_X - Y_X),$$

where

$$Y_X = Z_X + Z_X + Z_X,$$

and $Z_X$ is the zero set of $X$. Moreover, almost surely, for any finite set $A$ of real numbers, the set $Y_X$ will contain an affine (rescaled and translated) copy of $A$.

We now investigate the extent to which this result can be reflected in every complex oscillation. For a fixed $r \in \mathbb{R}$ define the subset $\Omega_r$ of $C[0, 1]$ by:

$$X \in \Omega_r \iff \exists n \exists z_1, \ldots, z_8 \in Z_X \ [r = n(\ell(1 + z_2 + z_3) - (z_4 + z_5 + z_6))].$$

It follows from the preceding that each $\Omega_r$ has Wiener measure one. For a real $r$ and a natural number $\ell$, let $I_{r, \ell}$ be any interval of length $\leq \frac{1}{\ell}$ with rational endpoints which contains $r$. 
For a real \( r \), a continuous function \( X \) on the unit interval and an natural number \( \ell \) define the predicate \( P(r, \ell, X) \) by:

\[
P(r, \ell, X) \leftrightarrow \exists n \exists t_1, \ldots, t_6 \in [0,1] \cap \mathbb{Q} \left[ n((t_1 + t_2 + t_3) - (t_4 + t_5 + t_6)) \in I_r \right] \land \forall 1 \leq i \leq 6 |X(t_i)| < \frac{1}{\ell}.
\]

Note that for fixed \( r \) and \( \ell \) the predicate \( P(r, \ell, X) \) is \( \Sigma_1^0(\mathcal{G}) \) for some (fixed) gaussian algebra \( \mathcal{G} \).

Our next aim is to show, for nonzero \( r \):

\[
X \in \Omega_r \rightarrow \forall \ell P(r, \ell, X).
\]

This will have the implication that for fixed \( r, \ell \), the predicate \( P(r, \ell, X) \) defines a \( \Sigma_1^0(\mathcal{G}) \)-set of Wiener measure one so that in particular \( P(r, \ell, x) \) will also hold for each complex oscillation \( x \).

For \( X \in \Omega_r \) and \( \ell \geq 1 \) let \( n \) be a natural number and \( z_1, \ldots, z_6 \) be zeroes of \( X \) such that \( r = n((z_1 + z_2 + z_3) - (z_4 + z_5 + z_6)) \).

Next choose \( t_1, \ldots, t_6 \in [0,1] \cap \mathbb{Q} \) sufficiently close to \( z_1, \ldots, z_6 \) to ensure that both \( |n((t_1 + t_2 + t_3) - (t_4 + t_5 + t_6)) - r| < \frac{1}{\ell} \) and \( |X(t_i)| < \frac{1}{\ell} \) for \( i = 1, \ldots, 6 \) holds. Consequently, we can deduce \( P(r, \ell, X) \) for all \( \ell \).

We have proven

**Theorem 8.** If \( x \) is a complex oscillation and \( r \) is a real number then

\[
\forall \ell \exists n \exists t_1, \ldots, t_6 \in [0,1] \cap \mathbb{Q} \left[ |n((t_1 + t_2 + t_3) - (t_4 + t_5 + t_6)) - r| < \frac{1}{\ell} \right] \land \forall 1 \leq i \leq 6 |x(t_i)| < \frac{1}{\ell}.
\]

Denote the predicate in Theorem 8 by \( P(x, r) \). It follows that the set defined \( B \) by

\[
x \in B \leftrightarrow \forall r P(x, r)
\]

contains all the complex oscillations. Define \( Q(x, r) \) as \( P(x, r) \) but with the first two quantifiers interchanged. Then

\[
\mathbb{R} = \bigcup_{n<\omega} n(Y_x - Y_x) \leftrightarrow \forall r Q(x, r).
\]

It is an open problem whether the predicate \( \forall r Q(x, r) \) defines a set that will contain all complex oscillations.

## 5 Hamel sets generated by complex oscillations

For the historical background to and a Fourier-analytical perspective on the results of this section, the reader is referred to Chapter 5 of the book by Rudin [22].

A perfect subset of the unit interval is called a *Hamel set*, if its elements are linearly independent over the field of rational numbers, or, equivalently, if it is a perfect subset of some Hamel basis of the reals over the rationals. Our aim is to show how Hamel sets can be generated by complex oscillations. Our results are inspired by the arguments on pp 255-257 of Kahane [14].

Set

\[
E = \left\{ \frac{1}{2} + \sum_{k=2}^{\infty} \epsilon_k \frac{1}{2^k} : \epsilon_k \in \{-1,1\} \text{ for all } k \right\}.
\]  

(6)

In [7], the author proved:

**Theorem 9.** If \( x \) is a complex oscillation then the elements of the image \( x(E) \) of the set \( E \) under \( x \) will be linearly independent over the field of rational numbers.

Our next aim is to show how one can use this theorem together with Theorem 3 to find definitions of Hamel sets within the arithmetical hierarchy. For \( \ell \geq 2 \), set

\[
D_\ell = \left\{ \frac{1}{2} + \sum_{k=2}^{\ell} \epsilon_k \frac{1}{2^k} : \epsilon_k \in \{-1,1\} \text{ for all } k = 2, \ldots, \ell \right\}.
\]

Write \( D = \cup_{\ell \geq 2} D_\ell \). Note that the topological closure of \( D \) is \( D \cup E \) We begin by proving

**Proposition 10.** If \( \alpha \in KC \), then

\[
z \in x_\alpha(E) \leftrightarrow \forall n \exists m \exists t_{1 \leq i \leq n} |x_\alpha(t) - z| < \frac{1}{2^n}.
\]  

(7)
Proof: Suppose \( z = x_\alpha(t) \) where \( \alpha \in KC \) and \( t \in E \) is given by
\[
t = \frac{1}{2} + \sum_{k=2}^{\infty} \epsilon_k \frac{1}{2k^2}.
\]
For \( n \geq 1 \) set
\[
t_n = \frac{1}{2} + \sum_{k=2}^{n} \epsilon_k \frac{1}{2k^2}.
\]
It follows from Proposition 1 in [5] that for some constant \( C > 1 \) and \( n \) sufficiently large it is the case
\[
|x_\alpha(t_n) - z| \leq C|t_n - t|^{\frac{2}{3}} \log \frac{1}{|t_n - t|}.
\]
Since \( |t_n - t| \leq \frac{1}{2^n} \) we conclude that for all \( n \) sufficiently large
\[
|x_\alpha(t_n) - z| < \frac{1}{2^n}.
\]
Conversely, suppose that \( z, \alpha \) satisfy the predicate on the right-hand side of (7). With each \( m \), we associate an \( n = n(m) > m \) such that \( |x_\alpha(t_n) - z| < \frac{2^m}{m} \) for some \( t_n \in D_n \). The sequence \( (t(m)) \) has some convergent sequence with a limit \( \tau \) say. Clearly \( \tau \in E \), and by the continuity of \( x_\alpha \), we can conclude that \( x_\alpha(\tau) = z \). This concludes the proof of the Proposition.

Note that
\[
|\sigma(x_\alpha(t) - z| < \frac{1}{2^n} \iff \exists k |\sigma(x_\alpha(t)(k) - z(k))| < \frac{1}{2^n} - \frac{2}{2^n},
\]
the right-hand side being \( \Sigma_1^0 \) in \( \alpha, z, t \) and \( n \). Consequently

Theorem 11. There is a \( \Pi^0_3 \)-formula \( Q(\alpha, z) \) defined over \( KC \times \{0, 1\}^\omega \) such that
\[
z \in x_\alpha(E) \iff Q(\alpha, z).
\]

Let \( \Omega \) be any \( \Delta^0_2 \)-element of \( KC \) (a Chaitin real). Then \( Q(\lambda i \Omega(i), z) \) is a \( \Pi^0_3 \)-predicate in \( z \) that defines a Hamel set. We have proven

Theorem 12. There is a \( \Pi^0_3 \)-predicate \( R(z) \) over \( \{0, 1\}^\omega \) and a Hamel set \( K \) such that
\[
R(z) \iff z \in K.
\]

6 Further developments and an open problem

We write \( S_\infty \) for the symmetric group of a countable set. We place on \( S_\infty \) the pointwise convergence topology thus giving \( S_\infty \) the subspace topology under its embedding into the Baire space \( \mathbb{N}^\mathbb{N} \). The group \( S_\infty \) acts naturally (and continuously) on \( (0, 1)^\infty_\mathbb{R} \):
\[
\sigma.(u_j : j \geq 1) := (u_{\sigma^{-1}(j)} : j \geq 1),
\]
for all \( (u_j) \in (0, 1)^\infty_\mathbb{R} \) and \( \sigma \in S_\infty \). The orbit space under this action is denoted by \( (0, 1)^\infty_\mathbb{R} / S_\infty \). The Borel structure on this space is given by the topology induced by the canonical mapping
\[
\pi : (0, 1)^\infty_\mathbb{R} \longrightarrow (0, 1)^\infty_\mathbb{R} / S_\infty.
\]
If \( X \) is a continuous function on the unit interval, then a local minimizer of \( X \) is a point \( t \) such that there is some closed interval \( I \subset [0, 1] \) containing \( t \) such that the function \( X \) assumes a minimum value on \( I \) at the point \( t \). We denote by \( MIN(X) \) the set of local minimizers of \( X \).

It is well-known that if \( X \) is a continuous version of Brownian motion on the unit interval, then \( MIN(X) \) is almost surely a dense and countable set and that all the local minimizers of \( X \) are strict. This means that, for each closed subinterval \( I \) of the closed unit interval, there is a unique \( \nu \in I \) where the minimum of \( X \) on \( I \) is assumed.

This has the implication that there is a subset \( \Omega_0 \) of \( C[0, 1] \) of full Wiener measure such that one can define a measurable mapping \( min : C[0, 1] \supset \Omega_0 \longrightarrow (0, 1)^\infty_\mathbb{R} \) in such a way that the composition of \( min \) with the projection \( \pi \) will define a mapping \( X \mapsto \).
MIN(X). In the sequel this strongly random set will be denoted by MIN. To summarise, we have the following commutative diagram:

\[
\begin{array}{ccc}
C[0,1] \supset \Omega_0 & \xrightarrow{\min} & (0,1)_\neq^\infty \\
\downarrow{\text{MIN}} & \searrow{\pi} & \\
(0,1)_\neq^\infty / S_\infty & & \\
\end{array}
\]

Let \((m_k)\) be any random enumeration of the local minimizers of a continuous version of Brownian motion in the unit interval. Let \(q > 2\) and for \(k \geq 1\) set \(s_k = (1 + m_k)\).

**Theorem 13.** The sequence \((s_k)\) is linearly independent over \(\mathbb{Q}\).

Proof: Let \(\Omega\) be a standard Borel space. A strongly countable set in the unit interval is a measurable mapping \(X : \Omega \to (0,1)_{\neq}^\infty / S_{\infty}\) that factors through some (traditional) random sequence \(Y\) as shown:

\[
\begin{array}{ccc}
\Omega & \xrightarrow{Y} & (0,1)_{\neq}^\infty \\
\downarrow{X} & \searrow{\pi} & \\
(0,1)_{\neq}^\infty / S_{\infty} & & \\
\end{array}
\]

One can think of \(X\) as a random countable set induced via \(S_{\infty}\)-equivalence, by a random sequence \(Y\), both in the unit interval. Denote the Borel space \((0,1)_{\neq}^\infty / S_{\infty}\) by \(CS(0,1)\).

For standard measure spaces \((\Omega_1, P_1)\) and \((\Omega_2, P_2)\), let there be some \(P_i\)-measurable strongly random variable \(X_i : \Omega_i \to CS(0,1)\) such that the induced probability distributions on \(CS(0,1)\) are the same. We say in this case that the strongly random sets \(X_1\) and \(X_2\) are statistically similar relative to the probabilities \(P_1, P_2\) and we write \(X_1 \sim X_2\). This means exactly that

\[
P_1(X_1^{-1}(\Sigma)) = P_2(X_2^{-1}(\Sigma)),
\]

For all Borel subsets \(\Sigma\) of \(CS(0,1)\).

Write \(\lambda^\infty\) for the product measure on \((0,1)_{\neq}^\infty\) which is the countable product of the Lebesgue measure \(\lambda\) on the unit interval and write \(\Lambda\) for the measure on \(CS(0,1)\) which is the pushout of \(\lambda^\infty\) under \(\pi\). In other words, for a Borel subset \(\Sigma\) of \(CS(0,1)\),

\[
\Lambda(\Sigma) = \lambda^\infty(\pi^{-1}\Sigma).
\]

Write \(U : (0,1)_{\neq}^\infty \to CS(0,1)\) for the strictly random set as defined by the following commutative diagram:

\[
\begin{array}{ccc}
(0,1)_{\neq}^\infty & \xrightarrow{Id} & (0,1)_{\neq}^\infty \\
\downarrow{U} & \searrow{\pi} & \\
CS(0,1) = (0,1)_{\neq}^\infty / S_{\infty} & & \\
\end{array}
\]

In statistics \(U\) is a model of an unordered uniform infinite sample. Moreover, it follows from the Hewitt-Savage theorem, that for every Borel subset \(\Sigma\) of \(CS(0,1)\), it is the case that

\[
\Lambda(\Sigma) \in \{0,1\}.
\]

(8)

Note that \(\Lambda\) is non-atomic.

In [24] Tsirelson proved the truly remarkable result that

\[
MIN \sim U.
\]

(9)

The theorem with the uniform sequence \((u_k)\) replacing the local minimizers \((m_k)\) is known to be true. (See pp 256-260 in Meyer [20].) The set \(E\) remains invariant under permutations of the indices \(k\). Hence the theorem follows from the statistical similarity of \(MIN\) and \(U\).
Open problem. In [8] the author showed how the local minimizers of a complex oscillation $\Phi(\alpha)$ can be computed from a KC-string $\alpha$. This opens the possibility of finding analogues of Theorem 13 for complex oscillations.

Let us call a continuous function $x$ on the unit interval strongly random if it belongs to every $\Sigma^0_2(G)$ set of Wiener measure one, for some gaussian algebra $G$. The set of strongly random functions is a subclass of the complex oscillations. By using the constructions in [8], it can be shown that the associated with a strongly random function will be linearly independent over the rationals. Whether this result can be extended to complex oscillations, is an open problem.

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