Effective Potential for Emergent Majorana Fermions in Superconductor Systems

A. W. Teixeira†, V. L. Carvalho-Santos‡, and J. M. Fonseca†
Departamento de Física, Universidade Federal de Viçosa, 36570-900, Viçosa, Brazil
(Dated: May 22, 2019)

Majorana fermions are particles that are its own antiparticles but they are not found in nature as a free fundamental particle, however, in condensed matter systems they emerge as a collective excitation. In this work, using functional integration techniques in complex time representation, we calculated the effective potential for emergent Majorana fermions in the Kitaev chain and showed how the superconductor parameter behave as function of temperature. We also found the particle number and showed the existence of both electrons and holes in the topological phase of the system. Using a surface induced superconductivity Hamiltonian in a Topological Insulator, we have calculated the effective potential for emergent Majorana fermions and showed the equivalence of the gap equation with the one in a quasi-two-dimensional Dirac electronic system which is a candidate to explain high-T_c superconductivity. Finally for the p-wave superconductor we have found a critical value of the electron-electron interaction determining the existence or not of induced superconductivity in the surface of the Topological Insulator, a remarkable result to guide experiments.

I. INTRODUCTION

Majorana Fermions [1, 2] are exotic particles studied in high energy physics for decades but have not been observed yet [3]. Proposed in 1937 by Ettore Majorana, they are associated with real solutions of the Dirac equation [1], and as a consequence, they are their antiparticles, without electric charge. Particles like electrons are described by their energy, momentum, and spin. In a solid an electron can occupy a energy level, and an unoccupied level is called a hole. Majorana fermions can emerge as a quantum superposition of an electron and a hole that move freely, with each one having the same direction, or spin. This Majorana fermion spin can interact with the spin of atomic nuclei in the material, so it ought to be seen using nuclear magnetic resonance techniques, they predict [3].

From a high energy physics perspective, Majorana fermions are essentially a half of an ordinary Dirac fermion. Due to the particle-hole redundancy, a single fermionic state is associated with each pair of ±E energy levels, being the presence or absence of a fermion in this state, defines a two-level system with energy splitting E. The existence of the Majorana fermions can be helpful to explain why the universe has a final asymmetry between matter and antimatter, once Majorana neutrinos obey all of the Sakharov requirements [4]. However, in a solid state perspective, Majorana fermions could be used to encode quantum information [5, 6] and also to tune the heat and charge transport [7], if they indeed emerge spatially separated the system must present topological properties, e.g., the Kitaev chain [10], topological insulator with induced superconductivity on the surface [11], topological superconductors [12], and others. In this way, it is an important issue to obtain and understand the effective potential of emergent Majorana fermions in different contexts, which could be used, for example, to obtain thermodynamic properties, study the dynamic of these quasiparticles in the presence of interaction as well as another physical properties of condensed matter systems in which such particle-like structures appear.

Many-body electronic states can be described by the formalism of second quantization. In this formalism, electrons are represented by creation and annihilation operators. Each electron of the system can be seen as a superposition of two Majorana fermions as

\[ \gamma_{f1} = c_j^\dagger + c_j, \quad \gamma_{f2} = i(c_j^\dagger - c_j), \]

where \( c_j^\dagger \) and \( c_j \) are respectively the creation and annihilation operators for electrons, with quantum numbers denoted by index \( j \). Additionally, \( \gamma \) annihilates or creates a Majorana fermion. Because \( \gamma^\dagger = \gamma \), to create or to destroy such particle has the same effect on the system. Furthermore, from inverting the transformation, we observe that a Majorana fermion consists of a superposition of electron and hole degrees of freedom. Systems with superconductor order can exhibit such kind of collective behavior, where their quasi-particles are indeed a product of such superposition [9]. Therefore, Majorana fermions are expected to emerge in superconductor materials. However, in most physical systems, the two Majorana fermions comprising the electron are interlaced in the space making no sense to describe them as isolated particles. To find it spatially separated the system must present topological properties, e.g., the Kitaev chain [10], topological insulator with induced superconductivity on the surface [11], topological superconductors [12], and others. In this way, it is an important issue to obtain and understand the effective potential of emergent Majorana fermions in different contexts, which could be used, for example, to obtain thermodynamic properties, study the dynamic of these quasiparticles in the presence of interaction as well as another physical properties of condensed matter systems in which such particle-like structures appear.

FIG. 1. (Color online) 3D Topological Insulator covered with a thin layer of a p-wave superconductor film presenting a Majorana zero mode bound to the vortex. The Majorana zero mode encircles a vortex in the surface.

In this work, we obtain the effective potential for emergent Majorana fermions on both, the one-dimensional Kitaev Model (KM) and a TI surface with induced superconductivity. The analytical calculations have been performed by using the
functional integral techniques in the complex time representation, extracting the natural logarithm of the partition function \([13]\). In the case of KM, the calculation of the effective potential allows us to determine the dependence on the temperature of the superconductor parameter (SP). Additionally, we show the existence of a phase transition from the trivial to the topological phase by analyzing the mixture of electrons and holes into the system. These obtained results motivate us to use similar techniques to find the effective potential for a TI surface with induced superconductivity (See Fig. 1) \([11, 12]\). We have then shown that this effective potential depends on the type of superconductor gap. For \(s\)-wave, the obtained result is in agreement with high-\(T_c\) superconductivity theory for cuprates, which assumes that superconductivity appears in the \(CuO\) planes. This result is very interesting because it gives some insights on the physical mechanism behind the superconductivity in high-\(T_c\). For \(p\)-wave, the obtained effective potential at zero temperature shows the existence of a continuous quantum phase transition separating the trivial and the superconductor states as a function of the electron-electron interaction. For non-zero temperature, the gap equation also produces the solutions show separated Majorana zero modes. For \(s\)-wave, the obtained results of the effective potential of a KM. Section III brings the discussions on the effective potential of KM can be written as

\[
\mathcal{H} = \frac{i}{2} \sum_j \left[ -\mu c_j^\dagger c_{j+1} + c_j^\dagger c_j - \frac{1}{2} \right] + \Delta \left( c_j^\dagger c_{j+1}^\dagger + c_{j+1} c_j \right) - \frac{\Delta^2}{g}, \tag{2}
\]

where \(t\) and \(\Delta\) are respectively the hopping and SP, \(\mu\) is the electronic chemical potential, and \(g\) is the electron-electron interaction. Let us assume that \(\Delta\) is real and consider a system with open boundary conditions. Using the inverse representation of Eq. (1), given by

\[
c_j = \frac{1}{2} (\gamma_{j,1} + \phi_{j,2}), \quad c_j^\dagger = \frac{1}{2} (\gamma_{j,1} - \phi_{j,2}), \tag{3}
\]

the Hamiltonian of KM can be rewritten in the Majorana basis as

\[
\mathcal{H} = \frac{i}{2} \sum_j \left[ -\mu c_j^\dagger c_{j+1} + (t + \Delta) c_j^\dagger c_{j+1} + (t - \Delta) c_{j+1}^\dagger c_j - \frac{\Delta^2}{g} \right]. \tag{4}
\]

The partition function for Majorana fermions can be then obtained by performing a functional integral in time complex representation \([13]\), that is,

\[
Z = \int \left[ id \gamma \right] \left[ d \phi \right] \exp \left[ -\int_0^\beta d\tau \sum_j \left( \gamma_j \partial_\tau \gamma_j + \mathcal{H} \right) \right],
\]

\[
\mathcal{H} = \begin{pmatrix}
\gamma_1 \\
\gamma_2
\end{pmatrix}
\]

where \(\gamma_1 = (\gamma_{1,1} \gamma_{2,2})\) and \(\gamma = (\gamma_1 \gamma_2)^T\) in the real space and, \(\beta = 1/T\). It is worth notice that those fields are independent and they can be integrated separately. To calculate this integral, we can proceed with a Fourier transformation of the Majorana operator in both, space and time, in order to obtain a partition function in the momentum space. That is,

\[
Z = \int \left[ id \gamma \right] \exp \left[ -\sum_{n,p} \beta \left( \gamma_{n,p}^\dagger A_{n,p} \gamma_{n,p} + \frac{\Delta^2}{g} \right) \right], \tag{5}
\]

where,

\[
\gamma_{n,p} = \left( \gamma_{1,n,p} \gamma_{2,n,p} \right); \quad A = \begin{pmatrix}
\omega_n & D^* \\
-D & \omega_n
\end{pmatrix}; \quad D = [\mu + 2t \cos(p) + 2\Delta \sin(p)]/4, \tag{6}
\]

\(\omega_n\) are the Matsubara frequencies for fermions \([13]\). Eq. (5) is not merely the electronic partition function of the Hamiltonian \([2]\) with a transformation \([3]\) implemented. Indeed, it is the Majorana partition function obtained from Hamiltonian \([4]\). A simple change of basis in the electronic partition function does not provide the correct answer to the Majorana one. This model presents a topological phase when \(|2t| > |\mu|\) and otherwise, it is trivial. In the topological phase, the solutions show separated Majorana zero modes.

Since in the momenta space, the Majorana operator works as a Grassmann variable, the integral given in Eq. (5) is reduced to a Gaussian integral, whose effective potential can be promptly calculated. Therefore, after performing the sum on the Matsubara frequencies, we have that the effective potential is given by

\[
V_{eff} = -\frac{1}{\beta} \ln Z = \frac{\Delta^2}{g} - \frac{1}{\beta} \int dp \left[ \beta |D| + 2 \ln \left( 1 + e^{-\beta |D|} \right) \right]. \tag{7}
\]

It must be highlighted that the sign of the chemical potential distinguishes electrons (negative chemical potential) and holes (positive chemical potential). Nevertheless, unlike calculations performed for free fermions \([13]\), the contributions of electrons and holes to the effective potential must not be

\[
\text{II. THE KITAEV MODEL}
\]

The Kitaev chain is the simplest model exhibiting Majorana fermions. It consists of a 1D system of spinless fermions that interacts with the nearest neighbor. It is an exactly soluble model and provides a useful place to study Majorana fermions in 1D space \([3]\). This model allows considering a 1D superconducting order with spinless fermions. The Hamiltonian of KM can be written as

\[
\mathcal{H} = \sum_j \left[ -t \left( c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j \right) - \mu \left( c_j^\dagger c_j - \frac{1}{2} \right) + \Delta \left( c_j^\dagger c_{j+1}^\dagger + c_{j+1} c_j \right) \right] - \frac{\Delta^2}{g}, \tag{2}
\]

where \(t\) and \(\Delta\) are respectively the hopping and SP, \(\mu\) is the electronic chemical potential, and \(g\) is the electron-electron interaction. Let us assume that \(\Delta\) is real and consider a system with open boundary conditions. Using the inverse representation of Eq. (1), given by

\[
c_j = \frac{1}{2} (\gamma_{j,1} + \phi_{j,2}), \quad c_j^\dagger = \frac{1}{2} (\gamma_{j,1} - \phi_{j,2}), \tag{3}
\]

the Hamiltonian of KM can be rewritten in the Majorana basis as

\[
\mathcal{H} = \frac{i}{2} \sum_j \left[ -\mu c_j^\dagger c_{j+1} + (t + \Delta) c_j^\dagger c_{j+1} + (t - \Delta) c_{j+1}^\dagger c_j - \frac{\Delta^2}{g} \right]. \tag{4}
\]

The partition function for Majorana fermions can be then obtained by performing a functional integral in time complex representation \([13]\), that is,

\[
Z = \int \left[ id \gamma \right] \left[ d \phi \right] \exp \left[ -\int_0^\beta d\tau \sum_j \left( \gamma_j \partial_\tau \gamma_j + \mathcal{H} \right) \right],
\]

\[
\gamma_j = \begin{pmatrix}
\gamma_{1,j} \\
\gamma_{2,j}
\end{pmatrix}
\]

where \(\gamma_1 = (\gamma_{1,1} \gamma_{2,2})\) and \(\gamma = (\gamma_1 \gamma_2)^T\) in the real space and, \(\beta = 1/T\). It is worth notice that those fields are independent and they can be integrated separately. To calculate this integral, we can proceed with a Fourier transformation of the Majorana operator in both, space and time, in order to obtain a partition function in the momentum space. That is,

\[
Z = \int \left[ id \gamma \right] \exp \left[ -\sum_{n,p} \beta \left( \gamma_{n,p}^\dagger A_{n,p} \gamma_{n,p} + \frac{\Delta^2}{g} \right) \right], \tag{5}
\]

where,

\[
\gamma_{n,p} = \left( \gamma_{1,n,p} \gamma_{2,n,p} \right); \quad A = \begin{pmatrix}
\omega_n & D^* \\
-D & \omega_n
\end{pmatrix}; \quad D = [\mu + 2t \cos(p) + 2\Delta \sin(p)]/4, \tag{6}
\]

\(\omega_n\) are the Matsubara frequencies for fermions \([13]\). Eq. (5) is not merely the electronic partition function of the Hamiltonian \([2]\) with a transformation \([3]\) implemented. Indeed, it is the Majorana partition function obtained from Hamiltonian \([4]\). A simple change of basis in the electronic partition function does not provide the correct answer to the Majorana one. This model presents a topological phase when \(|2t| > |\mu|\) and otherwise, it is trivial. In the topological phase, the solutions show separated Majorana zero modes.

Since in the momenta space, the Majorana operator works as a Grassmann variable, the integral given in Eq. (5) is reduced to a Gaussian integral, whose effective potential can be promptly calculated. Therefore, after performing the sum on the Matsubara frequencies, we have that the effective potential is given by

\[
V_{eff} = -\frac{1}{\beta} \ln Z = \frac{\Delta^2}{g} - \frac{1}{\beta} \int dp \left[ \beta |D| + 2 \ln \left( 1 + e^{-\beta |D|} \right) \right]. \tag{7}
\]

It must be highlighted that the sign of the chemical potential distinguishes electrons (negative chemical potential) and holes (positive chemical potential). Nevertheless, unlike calculations performed for free fermions \([13]\), the contributions of electrons and holes to the effective potential must not be
separated, as a consequence of the system’s topology. This fact will be evidenced when we are performing the calculation of the number of particles. The characteristic behavior of the SP as a function of temperature can be obtained using the minimum of the $V_{\text{eff}}$. Additionally, $\Delta$ must not change if we are considering electronic or Majorana effective potential. In this way, by assuming that $t = \Delta$ and $\mu = 0$, we obtain

$$T' = \frac{\Delta'}{\arctanh (\Delta')},$$

where we have defined $T \equiv T'\Delta_0/4$ and $\Delta \equiv \Delta'\Delta_0$, with $\Delta_0 = \frac{1}{4} \int dp \sin^2 p$. $\Delta_0$ can be interpreted as a finite value that does not contribute qualitatively to the obtained results and it can be determined by assuming the existence of an energy cutoff, $\Lambda$, considered as the limit of the first Brillouin zone, $\Lambda = \pi/a$, where $a$ is the lattice spacing. In this context, we can notice that Eq. (5) agrees qualitatively with experimental data for the behavior of the SP in such a way that $\Delta$ has a finite and positive value when $T = 0$, decreasing to zero as $T$ increases, as shown in Fig. 2.

The chemical potential, can be then determined in the limit of infinite volume, with the number of electrons given by

$$N = \frac{1}{\beta} \frac{\partial}{\partial \mu} \mathcal{Z} = \frac{1}{16} \int dp \mu + \frac{2\mu \cos(p)}{D} \tanh \left( \frac{\beta |D|}{2} \right).$$

The integrand in Eq. (9) is the density of electrons in the momentum space, in such a way that a positive density corresponds to electrons while a negative one corresponds to holes. If $|2\mu| < |\mu|$ (trivial phase), the electronic density is always positive or negative, meaning the system has only electrons or holes depending on if the chemical potential is positive or negative. On the other hand, if $|2\mu| > |\mu|$ (topological phase), the electronic density can be positive and negative, showing a mixture of electrons and holes in the system. This fact evidences the existence of a topological phase and spatially separated Majorana zero modes because both electrons and holes are constituents of the system. In Fig. 3 we show the behavior of the number of electrons and holes for $\Delta = 1$ and different values of $\mu$ and $t$, for $T = 0$. In the topological phase, occurring when $\mu = 0$ and $t = 1$, the density of particles changes from positive to negative values. On the other hand, for the trivial phase, appearing when $\mu = 2$, $t = 1$, and when $\mu = -2$, $t = -1$, the density of particles is always positive or negative, respectively referring to electrons (blue-dashed line) and holes (red-dashed line). The topology of the considered system is independent on the SP, although we have assumed an exact value for $\Delta$, its magnitude is not relevant for the phenomenon, but only for the amplitude of the particle number. The same effect is observed for non-zero temperature.

### III. INDUCED SUPERCONDUCTIVITY IN TI SURFACE

In a previous work, Fu and Kane [11] have considered an s-wave superconductor covering a TI, showing that the superconductivity can be induced in the surface of a 3D TI by proximity effects. In this context, aiming to perform a more general analysis, we consider a superconductor with parameter $\Delta(r, r')$ inducing superconductivity in the surface states of 3D TI. The interface between the superconductor and the TI can be described by the Hamiltonian

$$H = \int dr dr' \left\{ \frac{c_{\sigma}^\dagger \Delta(r, r') c_{\sigma}^\dagger - c_{\sigma} \Delta^*(r, r') c_{\sigma}}{\Delta(r, r')} \right\} + \frac{\delta(r-r')}{2} \left[ c_{\sigma}^\dagger p_+ c_{\sigma}^\dagger + c_{\sigma} c_{\sigma}^\dagger p_+ + c_{\sigma} p_+ c_{\sigma}^\dagger - 2 \sum_{\alpha=\uparrow, \downarrow} \mu \left( c_{\sigma}^\dagger c_{\sigma} - \frac{1}{2} \right) \right],$$

where $p_\pm = p_x \pm ip_y$ depends on the momentum operator, $c_{\sigma}^\dagger$ and $c_{\sigma}$ are the creation and annihilation operators for electrons, and $g$ is the electron coupling constant. The presence of a vortex in the SP, which depends on the nature of the superconductor material, can yield zero mode states with the spinor structure having components that obey the Majorana fermions.
conditions. This fact implies the existence of Majorana bound states in the surface of the TI. Using Eq. (10) we can determine the effective potential for emergent Majorana fermions using the partition function. For such purpose, the Hamiltonian has to be rewritten as a function of Majorana operators in the momentum space, using the inverse of relation (1). The partition function, as a functional integral in time complex representation, has the form

\[
Z = \left[ \prod_n \prod_{p,p'} \alpha \right] \int \prod \alpha \eta_{n} \eta_{n} D \eta_{n} \exp \left\{ \sum_{p,p'} \left[ \sum_n \delta_{n}^{\alpha} \tilde{D} \eta_{n} \eta_{n} - |\Delta(p, -p')|^2 \right] \right\},
\]

where the Majorana Nambu field is defined as

\[
\gamma = (\gamma_{1}(p) \gamma_{2}(p)) \gamma_{1}(p') \gamma_{2}(p') \right)^T,
\]

and the matrix \( \tilde{D} \) is given by

\[
\tilde{D} = \frac{\delta_{p, p'}}{4} \left( \begin{array}{cc} \mathcal{W} + \tilde{\mu} & \mathcal{P}_+ - \tilde{\Delta} \\ \mathcal{P}_- + \tilde{\Delta} & \mathcal{W} + \tilde{\mu} \end{array} \right),
\]

being

\[
\mathcal{W} = 2 \begin{pmatrix} \omega_n & 0 \\ 0 & \omega_n \end{pmatrix}, \tilde{\mu} = \begin{pmatrix} 0 & -\mu \\ \mu & 0 \end{pmatrix},
\]

\[
\mathcal{P}_\pm = \frac{1}{4} \begin{pmatrix} -i(p_\pm + p'_\pm) & (p_\pm + p'_\pm) \\ -(p_\pm + p'_\pm) & i(p_\pm + p'_\pm) \end{pmatrix},
\]

\[
\tilde{\Delta} = \frac{1}{\delta_{p, p'}} \begin{pmatrix} i\Delta(p, -p') & \Delta(p, -p') \\ -\Delta(p, -p') & i\Delta(p, -p') \end{pmatrix}.
\]

\( \Delta(p, -p') \) is the Fourier transform of \( \Delta(r, r') \), and \( \mu \) is the chemical potential for Majorana fermions. The SP can assume different wave configurations, as s-wave and p-wave for example. In both cases, we can write this parameter as \( \Delta(p, -p') = i|\Delta(p)|d_{p, p'} = i|\Delta(p)|d_{p, p'} \), where \( f(p) \) characterizes the symmetry and \( |\Delta(p)| \) is a constant. Additionally, the integration in Eq. (11) can also be solved using a Gaussian integration.

After performing the summation over the Matsubara frequencies, the effective potential is (assuming \( \mu = 0 \) for simplicity)

\[
V_{\text{eff}} = \sum_{p} \left\{ \frac{|\Delta(p)|}{g} - \sqrt{|p|^2 + |\Delta(p)|^2} \right\} - \frac{2}{\beta} \ln \left[ 1 + e^{-\beta \sqrt{|p|^2 + |\Delta(p)|^2}} \right].
\]

(14)

At first place, we will consider the s-wave case. Therefore, by assuming that \( \Delta(r, r') = |\Delta| \delta(r - r') \), which results in \( f(p) = -i \). The effective potential in the continuum momentum space then becomes

\[
V_{\text{eff}} = -2\pi \int_{0}^{\Lambda} \frac{d p}{p} \int \left\{ \frac{2}{\beta} \ln \left[ 1 + e^{i\beta k} \right] - k \right\} + \frac{\pi \Lambda^2 |\Delta(p)|^2}{g},
\]

(15)

where \( k = \sqrt{p^2 + |\Delta(p)|^2} = \sqrt{p^2 + |\Delta|^2} \). To evaluate this integral, we use polar coordinates, whereby the energy cutoff \( \Lambda \) depends on the lattice parameter of the TI, we use a approximate circular first Brillouin zone (good approximation for several TIs) and we can perform the integration using the cut-off as the value of the momentum in the limit of the Brillouin zone [14][15].

Following the same procedure used in Section II, we can analyze the minimum of the potential in order to obtain the behavior of the SP as a function of the temperature, which leads to the following gap equation

\[
\frac{\partial}{\partial \Delta} V_{\text{eff}} = -\int_{0}^{\Lambda} dx \frac{2\pi|\Delta|}{\sqrt{|\Delta|^2 + x}} \frac{\partial}{\partial \Delta} \left( \frac{\beta}{2} \sqrt{|\Delta|^2 + x} \right) + \frac{2\pi \Lambda^2 |\Delta|}{g}.
\]

(16)

Apart from constants, the equation (16) agrees with previously obtained results describing the behavior of \( \Delta \) in quasi-two-dimensional Dirac electronic systems, as a function of the temperature [16]. Indeed, this result was expected once the present work is considering an electronic system, although described by the effective potential of Majorana fermions with \( \Delta \) being the same parameter in both effective potentials, for Majorana or electrons. Additionally, the obtained result corresponds to the same one obtained by the spin fermion model, candidate to explain how high-Tc superconductivity emerges in CuO planes of cuprates systems [17][19]. This fact may suggest the existence of a natural emergence of Majorana fermions in the CuO planes in cuprates, which is an opened issue under investigation.

From considering a p-wave superconductor, we have that \( \Delta(p) = |\Delta|(p_\perp + ip_\parallel) \) in the effective potential [14]. After perform the Matsubara summation, we obtain the same result given in Eq. (15), with the new p-wave parameter, \( \Delta(p) \), instead the s-wave parameter.

In the equilibrium of the system, occurring when it is in the minimum of the equation (15) (derivative equal zero), one can observe the existence of a critical value of the electron coupling constant \( g_c \). Indeed, if \( g < g_c \), there is no induced superconductivity on the TI surface. Thus, it can be observed that there is a critical \( g \) value allowing an induced superconductivity in the surface states of the topological insulators by proximity effect. This condition leads to the SP at zero temperature, \( \Delta_0 \), as

\[
|\Delta_0| = \begin{cases} \frac{\Lambda}{2\pi} & \text{if } g \leq 4\Lambda/3 \\ \sqrt{\left(\frac{\Lambda}{2\pi}\right)^2 - 1} & \text{if } g > 4\Lambda/3. \end{cases}
\]

(17)

Then, a continuous quantum phase transition occurs in the critical value \( g_c = 4\Lambda/3 \), which separates the normal from the superconductor phase. Additionally, the second derivative shows that it is indeed a minimum of the system.

Aiming to obtain the magnitude of \( g_c \), we must multiply \( p_\perp \) in Eq. (10) by \( \hbar v_f \), where \( v_f \) corresponds to the Fermi velocity of the surface carriers of the topological insulator. In this way, we obtain that \( g_c = 4\hbar v_f \Lambda/3 \), evidencing that \( g_c \) depends only on the TI parameters, since \( \Lambda = 1/a \) is related...
been observed in Bi experimentual results in which induced superconductivity has conductivity in Bi Since

\[ \Theta_T \]

where \( \Delta \) is the critical temperature for superconductors transition, \( \Theta_D \) is the Debye temperature, and \( N(0) \) is the density of states of the superconductor. If we consider, for example, the Pb parameters, that is, \( T_c = 7.19 \text{ K}, \Theta_D = 105 \text{ K} \) and \( N(0) = 0.49 \text{ states/eV} \) [21][22], we obtain \( g_{sc} \approx 0.76 \text{ eV}. \) Since \( g_{sc} \) is the electron-electron interaction in the superconductor, it will be expected that Pb can induce superconductivity in Bi\(_2\)Se\(_3\). Indeed, this result is corroborated by experimental results in which induced superconductivity has been observed in Bi\(_2\)Se\(_3\) in the presence of Pb [23].

At finite temperature, the minimum of the effective potential for \( p \)-wave produces a gap equation as follow

\[
g_{sc} \approx -\frac{1}{N(0) \ln \left( \frac{T_c}{\Theta_D} \right)}, \tag{18}
\]

where \( T_c \) is the critical temperature for superconductors transitions, \( \Theta_D \) is the Debye temperature, and \( N(0) \) is the density of states of the superconductor. If we consider, for example, the Pb parameters, that is, \( T_c = 7.19 \text{ K}, \Theta_D = 105 \text{ K} \) and \( N(0) = 0.49 \text{ states/eV} \) [21][22], we obtain \( g_{sc} \approx 0.76 \text{ eV}. \) Since \( g_{sc} \) is the electron-electron interaction in the superconductor, it will be expected that Pb can induce superconductivity in Bi\(_2\)Se\(_3\). Indeed, this result is corroborated by experimental results in which induced superconductivity has been observed in Bi\(_2\)Se\(_3\) in the presence of Pb [23].

\[ \frac{3g}{2\Lambda^3 \sqrt{1 + \text{abs}\Delta^2}} \int_0^\Lambda dp \ p \ \tanh \left( \frac{\beta p}{2} \sqrt{1 + \text{abs}\Delta^2} \right) = 1. \tag{19} \]

The integration of this equation in momentum space results in an SP dependence on the temperature. By assuming \( \Lambda = 1, \) Fig. 4 presents the characteristic behavior of \( \Delta, \) which decreases when the temperature increases. Furthermore, it can be noticed that an increase of \( g \) yields a decrease in the critical temperature below which the system becomes a superconductor. Besides that, the value of the SP in zero temperature corresponds to \( |\Delta_0| \) calculated previously.

IV. CONCLUSIONS

We used the functional integral techniques in order to obtain the effective potential of Majorana fermions for two different systems which may present Majorana fermions modes spatially separated. Those effective potentials describe, in both cases, the expected behavior of the SP as a function of the temperature.

In the case of the Kitaev chain, we showed that the SP has a positive and finite value when \( T = 0, \) decreasing when \( T \) increases. We have also obtained the density of particle in momentum space as a function of \( p, \) showing the differences between topological and trivial phases concerning the presence of holes and electrons in the system.

From considering a topological insulator surface with induced \( s \)-wave superconductivity, we have obtained a gap equation similar to that obtained for superconductivity in quasi-two-dimensional Dirac electronic systems. Our results suggest the natural emergence of Majorana fermions in CuO planes. For \( p \)-wave and zero temperature, it was showed the existence of a continuous quantum phase transition separating the normal and the superconductor states. The obtained critical electron-electron interaction depends on the parameters of the TI. It is shown that superconductivity will be induced on the surface of TI only if their electron-electron interaction is higher than a critical value of \( g_c. \) For non zero temperature, we have obtained that the gap equation exhibits the expected behavior of the \( \Delta \) as a function of the temperature. As prospects for future investigations, we are considering the possibility of the emergence of Majorana fermions in the CuO planes in cuprates, an important issue to be responded if we think in applications such as quantum computation based on these elusive particles.

ACKNOWLEDGMENTS

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001. The authors also thank CNPq (Grant Nos. 401132/2016-1 and 309484/2018-9) and FAPEMIG for financial support.
D. Litinski and F. von Oppen, Phys. Rev. B 97 (2018) 205404;
R.L.S. et al, Scientific Reports 8 (2018) 2790;
V. Mourik, et al, Science 336 (2012) 1003.
J. Bardeen, L. N. Cooper and J. R. Schrieffer, Phys. Rev. 108 (1957) 1175;
A. Kitaev, Phys. Usp. 44 (2001) 131;
Fu L., and C. L. Kane, Phys. Rev. Lett. 100 (2008) 096407;
B. A. Bernevig, T. L. Hughes; Topological Insulators and Topological Superconductors - Princeton University Press 2013;
J. I. Kapusta, C. Gale; Finite-temperature Field Theory, Principles and Applications - Cambridge University Press, 1989;
M. Z. Hasan and C. L. Kane, Rev. Mod. Phy. 82 (2010) 3045;
Xiao-Liang Qi and Shou-Cheng Zhang, Rev. Mod. Phys 83 (2011) 1057;
E. C. Marino and L. H. C. M. Nunes, Nuc. Phy. B, 741 (2006) 404;
E. C. Marino and L. H. C. M. Nunes, arXiv:1109.2151 [cond-mat.str-el];
L. H. C. M. Nunes, A. W. Teixeira and E. C. Marino, Europhys. Lett., 110 (2015) 27008;
L. H. C. M. Nunes, A. W. Teixeira and E. C. Marino, Sol. State Comm. 251 (2017) 5;
R. W. G. Wyckoff, Crystal Structures, Vol. 2. John Wiley and sons, New York (1964);
T. Koretsune and R. Arita, arXiv:1610.09441 [cond-mat.supr-con].
P. B. Allen, Handbook of Superconductivity - Academic Press, New York, 1999;
P. Arevalo-López, R. E. López-Romero and R. Escudero, arXiv:1512.09098 [cond-mat.supr-con]