Absolute Bounds for Ergodic Deviations of Linear Form Sequences Relative to Intervals in $\mathbb{T}^1$

Hao Wu

Abstract. Given a positive increasing function $\varphi$, we show that for a full measure set of vectors $\alpha \in \mathbb{R}^d$, the maximal ergodic discrepancy of the $d$-linear form sequence $\left\{\sum_{1 \leq i \leq d} k_i \alpha_i \mod 1\right\}_{1 \leq k_i \leq N, 1 \leq i \leq d}$ relative to intervals in $[0, 1)$ has an absolute upper bound of $C(\alpha, \varphi) \cdot (\log N)^d \varphi^{\max\{d, 3\}} (\log \log N)$ if $\sum_{n=1}^{\infty} \frac{1}{\varphi(n)}$ converges.
1 Introduction

Given an irrational number \( \alpha \in \mathbb{R} \), the irrational rotation \( T_\alpha \) over \( \mathbb{T} = (\mathbb{R}/\mathbb{Z}) \cong [0,1) \) is defined by \( x \mapsto x + \alpha \mod 1 \) for \( x \in \mathbb{T} \). By Weyl’s criterion (see \[2\], section 1.2.1), the sequence \( \{n\alpha \mod 1\}_{1 \leq n \leq N} \) becomes equidistributed over \( \mathbb{T} \) as \( N \to \infty \), i.e. for any measurable set \( B \subset \mathbb{T} \),

\[
\sum_{n=1}^{N} \frac{\chi_B(n\alpha)}{N} \to \text{Vol}(B), \quad N \to \infty.
\]

To measure the rate of convergence, we introduce the discrepancy function defined as the difference between the actual number of hits in \( B \) before time \( N \) and the expected number of hits \( N \cdot \text{Vol}(B) \):

\[
D_B(\alpha; N) = \sum_{n=1}^{N} \chi_B(n\alpha \mod 1) - N\text{Vol}(B).
\]

In 1920s, Khintchine \[3\] proved that the maximal discrepancy for an irrational rotation \( T_\alpha \) relative all possible intervals in \([0,1)\):

\[
\Delta(\alpha; N) = \max_{x \in [0,1]} |D_{[0,x]}(\alpha; N)|
\]

is exactly between \( C(\alpha, \epsilon)(\log N)(\log \log N) \) and \( C(\alpha, \epsilon)(\log N)(\log \log N)^{1+\epsilon} \) for any \( \epsilon > 0 \). His proof used the continued fraction algorithm for irrationals. Due to the absence of the continued fraction algorithm in higher dimensions, the research about the higher-dimensional counterpart of Khintchine’s theorem proved to be difficult. Schmidt \[4\] proved that the maximal discrepancy \( \Delta(\alpha; N) \) (see the definition below) in dimension \( k \) has an upper bound of \( C(\alpha, \epsilon)(\log N)^{k+1+\epsilon} \) for almost every \( \alpha \in \mathbb{R}^d \), by using the Erdős–Turán inequality. In 1994, by a surprising method which consists of a combination of Fourier analysis, the “second-moment method” and combinatorics, J. Beck \[1\] successfully got rid of the extra log \( N \) factor and proved the following multidimensional analogue of Khintchine’s theorem:

**Theorem 1.1.** Let \( k \geq 2 \), \( \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k \) be the translation vector, and \( B(x) = [0, x_1) \times \cdots \times [0, x_k) \subset [0,1)^k \), define the ergodic discrepancy:

\[
D(\alpha, x; m) = \sum_{1 \leq n \leq m} \chi_{B(x)}(n\alpha) - m\text{Vol}(B(x))
\]

and the maximal discrepancy:

\[
\Delta(\alpha; N) = \max_{1 \leq m \leq N, x \in [0,1]^d} |D(\alpha, x; m)|.
\]

Then for arbitrary positive increasing function \( \varphi(n) \) of \( n \),

\[
\Delta(\alpha; N) \ll (\log N)^k \cdot \varphi(\log \log N) \iff \sum_{n=1}^{\infty} \frac{1}{\varphi(n)} < \infty
\]

[1] Beck, J. (1994). On the discrepancy of o(n) points. Acta Arithmetica, 64(2), 179-183.

[2] Weyl, H. (1916). Über die Gleichverteilung von Zahlen mod 1. Hannover: Zentralblatt für Mathematik und ihre Grenzgebiete.

[3] Khintchine, A. Y. (1924). Über dem Maximum der Diskrepanz einiger Zahlensysteme. Mathematische Annalen, 92(3), 239-252.

[4] Schmidt, W. M. (1972). On the discrepancy of infinite sequences. Acta Arithmetica, 21(1), 107-113.
for almost every $\alpha \in \mathbb{R}^k$, where $\ll$ denotes the Vinogradov symbol, e.g. $f(N) \ll g(N)$ means that $|f(N)| < c \cdot g(N)$ for all $N$ with a uniform constant $c$.

In this paper, we consider the sequence of linear forms \( \{\sum_{1 \leq i \leq d} k_i \alpha_i \mod 1\}_{1 \leq k_i \leq N, 1 \leq i \leq d} \) relative to intervals in $[0,1)$. Define the ergodic discrepancy:

\[
D(\alpha, x; N) = \sum_{1 \leq k_i \leq N, 1 \leq i \leq d} \chi_{[0,x)} \left( \sum_{1 \leq i \leq d} k_i \alpha_i \mod 1 \right) - N^d x,
\]

and the maximal discrepancy:

\[
\Delta(\alpha; N) = \max_{0 < x \leq 1} |D(\alpha, x; N)|.
\]

The term $N^d x$ is the expected value of the number of terms in the sequence \( \{\sum_{1 \leq i \leq d} k_i \alpha_i \}_{1 \leq k_i \leq N, 1 \leq i \leq d} \) whose fractional parts visit the interval $[0, x)$.

The main result of this paper is the following:

**Theorem 1.2.** Let $\varphi(n)$ be an arbitrary positive increasing function of $n$, then for almost every $\alpha \in \mathbb{R}^d$, we have:

\[
\sum_{n=1}^{\infty} \frac{1}{\varphi(n)} < \infty \implies \Delta(\alpha; N) \ll (\log N)^d \cdot \varphi^{\max\{3,d\}}(\log \log N), \tag{1.2}
\]

**Remark.** Our result successfully keep the main factor $(\log N)^d$, but due to an absence of a second moment estimation as in Beck[1], we used an $L_1$ estimation, and the additional factor $\varphi(\log \log N)^{\max\{2,d-1\}}$ is needed in our proof for controlling the small divisor $n \prod_{i=1}^{d} \|n\alpha_i\|$.

This paper is organized as the following: in Section 2, for the convenience of following estimations, we transform the ergodic discrepancy to the Fourier series using Poisson’s summation formula. In Section 3 we estimate the contribution of the ”tail” of the Fourier series, i.e. the high frequency nodes. Section 4 and 5 deal with the main part of the discrepancy, Section 4 is about the constant part and Section 5 deals with the exponential part, both of which will be properly defined later. Combining the section 3-5, we have an overall estimation of the discrepancy, which gives the desired result above.

## 2 Poisson’s summation formula

In this section, following [4], we use the Poisson formula to transform the ergodic discrepancy into a Fourier series. The main result of this section, Proposition 2.1, gives a better version of the Fourier series by taking a roof-like average, which has better convergent properties. First we introduces the Fourier series of the ergodic discrepancy by the following Lemma.
Lemma 2.1. By the Poisson formula, the ergodic sum $D(\alpha, x; N)$ becomes:

$$D(\alpha, x; N) = \sum_{n \in \mathbb{Z}^{d+1} \setminus \{0\}} \frac{1 - e^{2\pi i x}}{2\pi n_1} \prod_{i=1}^{d} \frac{1 - e^{-2\pi i N(n_1 \alpha_i - n_{i+1})}}{2\pi (n_1 \alpha_i - n_{i+1})}.$$ 

Proof. The ergodic sum could be calculated as follows, note that the condition: $0 \leq \{\sum_{1 \leq i \leq d} k_i \alpha_i\} < x$ is equivalent to $\exists m \in \mathbb{Z}$ such that:

$$0 \leq \sum_{1 \leq i \leq d} k_i \alpha_i - m < x;$$

$$1 \leq k \leq N;$$

$$1 \leq l \leq N.$$ 

Consider the lattice inside $\mathbb{R}^{d+1}$,

$$\left\{ \left( \sum_{1 \leq i \leq d} k_i \alpha_i - m, k_1, \ldots, k_d \right) \mid (k_1, \ldots, k_d, m) \in \mathbb{Z}^{d+1} \right\},$$

note that the fractional part of $\sum_{1 \leq i \leq d} k_i \alpha_i$ lying in $[0, x)$ is equivalent to the lattice point inside box

$$B = [0, x) \times \prod_{d \text{ copies}} (0, N].$$

So the sum becomes

$$\sum_{1 \leq k \leq N} \chi_{[0,x)} \left( \sum_{1 \leq i \leq d} k_i \alpha_i \mod 1 \right) = \sum_{(k_1, \ldots, k_d, m) \in \mathbb{Z}^{d+1}} \chi_B \left( \sum_{1 \leq i \leq d} k_i \alpha_i - m, k_1, \ldots, k_d \right)$$

On the other hand, we see that:

$$\left( \sum_{1 \leq i \leq d} k_i \alpha_i \mod 1 \right) = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_d & -1 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_d \\ m \end{pmatrix} = \mathbf{A} \cdot \mathbf{y}$$

where

$$\mathbf{A} = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_d & -1 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_d \\ m \end{pmatrix}$$
Apply the Poisson formula to the function $f(y) = \chi_B(A \cdot y)$, $y = (k_1, \ldots, k_d, m) \in \mathbb{Z}^{d+1}$, we have

$$
\sum_{y \in \mathbb{Z}^{d+1}} \chi_B(A \cdot y) = \sum_{y \in \mathbb{Z}^{d+1}} f(y) = \sum_{\nu \in \mathbb{Z}^{d+1}} \int_{\mathbb{R}^{d+1}} f(y) \cdot e^{-2\pi i \nu \cdot y} dy
$$

$$
= \sum_{\nu \in \mathbb{Z}^{d+1}} \int_{\mathbb{R}^{d+1}} \chi_B(A \cdot y) \cdot e^{-2\pi i \nu \cdot y} dy
$$

$$
= \frac{1}{\det A} \sum_{\nu \in \mathbb{Z}^{d+1}} \int_{\mathbb{R}^{d+1}} \chi_B(z) e^{-2\pi i \nu \cdot (A^{-1} z)} dz
$$

Integrate with respect to each coordinate of $z$ and we get

$$
D(\alpha, x; N) = \sum_{y \in \mathbb{Z}^{d+1}} f(y) - N^d x
$$

$$
= 1^{d+1} \sum_{n \in \mathbb{Z}^{d+1} \backslash \{0\}} \frac{1 - e^{2\pi i n_1 x}}{2\pi n_1} \prod_{i=1}^{d} \frac{1 - e^{-2\pi i N(n_1 \alpha_i - n_{i+1})}}{2\pi (n_1 \alpha_i - n_{i+1})}.
$$

In order to avoid technical problems with the convergence, we will not study $D(\alpha, x; N)$ directly, instead, we will follow Beck[1] and use a special weighted average of $D(\alpha, x; N)$ over a $\frac{1}{N^2}$ neighborhood. To this end, we oscillate the target interval $[0, x]$ with an amplitude of $\frac{1}{N^2}$, and also the range for summation $\{1, \ldots, N\}$ with amplitude of 2, specifically, let $u = (u_2, \ldots, u_{d+1})$, $u + N = (u_2 + N, \ldots, u_{d+1} + N)$, and define:

$$
D(\alpha; a, b; u, u + N) = \sum_{u_{i+1} < k_i < N + u_{i+1}} \chi_{[a, b)} \left( \sum_{1 \leq i \leq d} k_i \alpha_i \mod 1 \right) - N^d (b - a) \quad (2.1)
$$

and define the $\frac{1}{N^2}$ average:

$$
\bar{D}(\alpha, x; N) = \frac{N^2}{2} \left( \frac{1}{2} \right)^d \int_{-\frac{N^2}{2}}^{\frac{N^2}{2}} \int_{-\frac{N^2}{2}}^{\frac{N^2}{2}} \cdots \int_{-\frac{N^2}{2}}^{\frac{N^2}{2}} \left( 1 - \frac{N^2}{2} |u_1| \right) \prod_{i=1}^{d} \left( 1 - \frac{|u_{i+1}|}{2} \right)
$$

$$
\times D(\alpha; u_1, x + u_1; u, u + N) du_1 du_2 \ldots du_{d+1}.
$$

Using the Féjer kernel identity

$$
\frac{N^2}{2} \int_{-\frac{N^2}{2}}^{\frac{N^2}{2}} \left( 1 - \frac{N^2 |y|}{2} \right) e^{\pi i k y} dy = \left( \frac{\sin(2\pi k \frac{y}{N^2})}{2\pi k \frac{y}{N^2}} \right)^2,
$$
\[ \int_{-2}^{2} \left(1 - \frac{|y|}{2}\right) e^{2\pi i ky} dy = \left(\frac{\sin 2\pi k}{2\pi k}\right)^2. \]

We arrive at:

\[ \tilde{D}(\alpha, x; N) = i^{d+1} \sum_{n \in \mathbb{Z}^{d+1}} \frac{1 - e^{2\pi i n_1 x}}{2\pi n_1} \left(\frac{\sin 2\pi (\frac{n_1}{N^2})}{2\pi (\frac{n_1}{N^2})}\right)^2 \prod_{i=1}^{d} \left(1 - e^{-2\pi i (n_1 \alpha_i - n_{i+1})}\right) \left(\frac{\sin 2\pi (n_1 \alpha_i - n_{i+1})}{2\pi (n_1 \alpha_i - n_{i+1})}\right)^2. \]

We claim that if \( \tilde{D}(\alpha, x; N) \) is bounded from above by \( (\log N)^d \varphi^{d+1} (\log \log N) \), then so is \( D(\alpha, x; N) \).

**Proposition 2.1.** For almost every \( \alpha \in [0, 1)^d \), we have

\[ |\tilde{D}(\alpha, x; N) - D(\alpha, x; N)| \ll (\log N)^{1+\epsilon}, \]

**Proof.** By a standard application of Borel-Cantelli Lemma, there exists a constant \( C(\alpha) \) for almost every \( \alpha \in [0, 1)^d \), and every \( k = (k_1, \ldots, k_d) \in \mathbb{Z}^d \) such that \( |k| > 1 \),

\[ \| \sum_{1 \leq i \leq d} k_i \alpha_i \| > \frac{C(\alpha)}{\left(\sum_{1 \leq i \leq d} k_i^2\right)^{1+\epsilon}}, \]

therefore \( \| \sum_{1 \leq i \leq d} k_i \alpha_i \| \gg \frac{1}{N^{(\log N)^{1+\epsilon}}} \) for \( k_1, \ldots, k_d \leq N \). Since the numbers of elements of the sequence \( \{ \sum_{1 \leq i \leq d} k_i \alpha_i \} \) which lie inside \([0, x]\) and \([-2/N^2, x + 2/N^2]\) differ only by the number of elements which lie inside the intervals \([-2/N^2, 0]\) and \([x, x + 2/N^2]\), by using Dirichlet principle, we have

\[ D(\alpha, u_1, x + u_1, u_2, N + u_2; u_3; N + u_3) = D(\alpha, x; N) + O \left((\log N)^{1+\epsilon}\right), \]

for almost every \( \alpha \in [0, 1)^d \), and every \( u_1 \in [-\frac{2}{N^2}, \frac{2}{N^2}], u_2, u_3 \in [-2, 2] \). The constant above only depend on \( \alpha \) and \( \epsilon \), therefore the same bound hold after integration over \( u_1 \) and \( u \).

By Proposition 2.1, we can now shift our attention to the asymptotic behavior of \( \tilde{D}(\alpha, x; N) \), which has better convergence property as shown in Section 3.
3 Estimating the "tail" of the discrepancy function

This section is devoted to estimating the "tail" of \( \bar{D}(\alpha, x; N) \), note that \( \bar{D}(\alpha, x; N) \) is a sum of the products (where \( n \in \mathbb{Z}^{d+1} \backslash \{0\} \))

\[
f(n, x, \alpha) = i^{d+1} \frac{1 - e^{2\pi i n_1 x}}{2\pi n_1} \left( \frac{\sin 2\pi \left( \frac{n_1}{N^2} \right)}{2\pi \left( \frac{n_1}{N^2} \right)} \right)^2 \prod_{i=1}^{d} \left( \frac{1 - e^{-2\pi i N(n_1\alpha_i - n_{i+1})}}{2\pi (n_1\alpha_i - n_{i+1})} \right) \left( \frac{\sin 2\pi (n_1\alpha_i - n_{i+1})}{2\pi (n_1\alpha_i - n_{i+1})} \right)^2
\]

(3.1)

let:

\[
\bar{D}_4(\alpha, x; N) = \sum_{n \in U_4(\alpha; N)} f(n, x, \alpha),
\]

where

\[
U_4(\alpha; N) = \left\{ n \in \mathbb{Z}^{d+1} \mid \begin{array}{l}
|n_1| < N^2/4, \\
|n_1\alpha_i - n_{i+1}| = \|n_1\alpha_i\|, 1 \leq i \leq d \\
|n_1| \prod_{i=1}^{d} \|n_1\alpha_i\| > (\log N)^s
\end{array} \right\}
\]

(3.2)

\| \cdot \| \text{ denotes the distance to the closest integer, and } s \text{ is a large enough but fixed integer to be determined later (see Lemma 4.1).}

The main result of this section is the following:

**Proposition 3.1.** Let \( \varphi(n) \) be a positive increasing function such that \( \sum_{n=1}^{\infty} \frac{1}{\varphi(n)} < \infty \), then for almost every \( \alpha \in [0, 1]^d \), we have

\[
|\bar{D}_4(\alpha, x; N) - \bar{D}(\alpha, x, N)| \ll (\log N)^d \varphi^{\max\{3, d\}}(\log \log N)
\]

To prove Proposition 3.1, we need to control the components of the difference \( \bar{D}_4 - \bar{D} \) step by step.

3.1 Estimation for the sum when \( |n_1| \) is big.

First we will show that the sum of the terms when \( |n_1| \) is big does not contribute much. Define

\[
\bar{D}_1(\alpha, x; N) = \sum_{n \in U_1(N)} f(n, x, \alpha),
\]

(3.3)

where

\[
U_1(N) = \{ n \in \mathbb{Z}^{d+1} \backslash \{0\} : |n_1| < N^2(\log N)^2 \}
\]

We show the following:
Proposition 3.2. For almost every $\alpha \in \mathbb{R}^d$ we have

$$|\bar{D}_1(\alpha, x; N) - \bar{D}(\alpha, x; N)| = O(1),$$

where $O(1)$ represents an absolute bound which may depend on $\alpha$, but does not depend on $x$ or $N$.

Proof. First we decompose the sum into three parts:

$$|\bar{D}_1(\alpha, x; N) - \bar{D}(\alpha, x; N)| \ll \bar{D}_{1,0} + \sum_{j=1}^{d} \bar{D}_{1,j} \quad (3.4)$$

where

$$\bar{D}_{1,0} = \sum_{n \in U_{1,0}(\alpha; N)} \frac{N^4}{|n_1|^{d+1}} \prod_{i=1}^{d} \frac{1}{|n_1 \alpha_i - n_{i+1}|}$$

where

$$U_{1,0}(\alpha; N) = \left\{ n \in \mathbb{Z}^{d+1} : |n_1| > N^2 (\log N)^2, |n_1 \alpha_i - n_{i+1}| < 1/2, \ 1 \leq i \leq d \right\},$$

and

$$\bar{D}_{1,j} = \sum_{n \in U_{1,j}(\alpha; N)} \frac{N^4}{|n_1|^{d+1}} \prod_{i=1}^{d} \frac{1}{|n_1 \alpha_i - n_{i+1}|} \cdot \frac{1}{|n_1 \alpha_j - n_{j+1}|^{d+1}},$$

where

$$U_{1,j}(\alpha; N) = \left\{ n \in \mathbb{Z}^{d+1} : |n_1| > N^2 (\log N)^2, |n_1 \alpha_i - n_{i+1}| < 1/2, \ i \neq j, \ |n_1 \alpha_j - n_{j+1}| > \frac{1}{2} \right\}.$$

Define $r_{\alpha_i} = [\|n_1 \alpha_i - n_{i+1}\|], 1 \leq i \leq d$, we have that for $1 \leq j \leq d$,

$$D_{1,j} \ll \sum_{r_{\alpha_j} = 1}^{\infty} \sum_{n \in \mathbb{Z}^{d+1}, \begin{array}{l} |n_1| > N^2 (\log N)^2 \\ |n_1 \alpha_i - n_{i+1}| < 1/2 \\ i \neq j \end{array}} \frac{N^4}{|n_1|^{d+1}} \prod_{i=1}^{d} \frac{1}{|n_1 \alpha_i - n_{i+1}|} \frac{1}{r_{\alpha_j}^{d+1}} \ll \bar{D}_{1,0},$$

Therefore it suffices to show that $D_{1,0} = O(1)$, we will first prove a lemma:

Lemma 3.1. Let $\varphi(n)$ be a positive increasing function such that $\sum_{n=1}^{\infty} \frac{1}{\varphi(n)} < \infty$, then for almost every $\alpha \in \mathbb{R}^d$, the sum

$$\sum_{n=2}^{\infty} n \varphi(n) \prod_{i=1}^{d} (\|n \alpha_i\| \varphi(\|\log \|n \alpha_i\||))$$

converges.
Proof. Note that the integral
\[ J(n) = \int_{[0,1]^d} \frac{1}{\prod_{i=1}^d (\|n\alpha_i\| \varphi(\|\log \|n\alpha_i\||))} d\alpha \]
is finite and independent of \( n \), the series \( \sum_{n=2}^{\infty} \frac{1}{n \varphi(\log n)} \) also converges, so the integral
\[ \int_{[0,1]^d} \left( \sum_{n=2}^{\infty} \frac{1}{n \varphi(\log n)} \prod_{i=1}^d (\|n\alpha_i\| \varphi(\|\log \|n\alpha_i\||))) \right) d\alpha = \sum_{n=2}^{\infty} \frac{J(n)}{n \varphi(\log n)} \]is finite. Therefore we have that the series
\[ \sum_{n=2}^{\infty} \frac{1}{n \varphi(\log n)} \prod_{i=1}^d (\|n\alpha_i\| \varphi(\|\log \|n\alpha_i\||)) \]
converges for almost every \( \alpha \in [0,1]^d \) and ever \( \epsilon > 0 \), the periodicity of \( \| \cdot \| \) gives the result for almost every \( \alpha \in \mathbb{R}^d \).

Continuing with the proof for Proposition 3.2 since \( \sum_{1}^{\infty} \frac{1}{n^p} = O(1) \), by Borel-Cantelli Lemma, we have for almost every \( \alpha \in \mathbb{R}^d \),
\[ \|n\alpha_i\| \geq \frac{1}{n^2}, \quad 1 \leq i \leq d, \]
for all but finitely many \( n \in \mathbb{N} \), or equivalently,
\[ |\log \|n\alpha_i\|| \leq 2 \log n, \quad 1 \leq i \leq d, \]for all but finitely many \( n \in \mathbb{N}^+ \).
Take \( \varphi(n) = n^{1+\epsilon} \) where \( \epsilon > 0 \) is small, then \( \tilde{D}_{1,0} \) can be estimated as follows:
\[ \tilde{D}_{1,0} = \sum_{n \geq N^2(\log N)^2} \frac{1}{n \prod_{i=1}^d \|n\alpha_i\|} \leq \sum_{n \geq N^2(\log N)^2} \frac{1}{(\log n)^{d+3\epsilon} \prod_{i=1}^d \|n\alpha_i\|} \]
\[ \leq \sum_{n \geq N^2(\log N)^2} \frac{1}{\log n} \prod_{i=1}^d (\|n\alpha_i\| \varphi(\|\log \|n\alpha_i\||)) \]
\[ \leq \sum_{n=2}^{\infty} \frac{1}{\log n} \prod_{i=1}^d (\|n\alpha_i\| \varphi(\|\log \|n\alpha_i\||)) \]
\[ = O(1), \]
finishing the proof.
3.2 Estimation for the sum when $|n_1\alpha_i|$ is bigger than $\frac{1}{2}$ for one of $1 \leq i \leq d$.

Let

$$D_2(\alpha, x; N) = \sum_{n \in U_2(\alpha; N)} f(n, x, \alpha)$$

where $n = (n_1, \ldots, n_{d+1})$ satisfies

$$U_2(\alpha; N) = \left\{ n \in \mathbb{Z}^{d+1} \setminus \{0\} \mid \begin{array}{l}
|n_1| < N^2(\log N)^2, \\
|n_1\alpha_i - n_i + 1| = \|n_1\alpha_i\|, \ 1 \leq i \leq d
\end{array} \right\}$$

(3.7)

We show that $\bar{D}_1$ can be replaced by $\bar{D}_2$:

**Proposition 3.3.** Let $\varphi(n)$ be a positive increasing function such that $\sum_{n=1}^{\infty} \frac{1}{\varphi(n)} < \infty$, then for almost every $\alpha \in [0, 1]^d$, we have

$$|\bar{D}_2(\alpha, x; N) - \bar{D}_1(\alpha, x; N)| \ll (\log N)^d \varphi^d(\log \log N)$$

**Proof.** First we have the following inequality:

$$|\bar{D}_2(\alpha, x; N) - \bar{D}_1(\alpha, x; N)| \ll \sum_{j=1}^{d} \sum_{n \in U_{2,j}} \frac{1}{|n_1| \prod_{i \neq j} |n_1\alpha_i - n_i + 1|} \frac{1}{|n_1\alpha_j - n_j + 1|^d}$$

$$\ll \sum_{j=1}^{d} \sum_{n=1}^{N^2(\log N)^2} \frac{1}{n \prod_{i \neq j} \|n\alpha_i\|},$$

(3.8)

where

$$U_{2,j} = \left\{ n_1 \leq N^2(\log N)^2, \ |n_1\alpha_i - n_i + 1| < \frac{1}{2}, \ i \neq j, \ |n_1\alpha_j - n_j + 1| \geq \frac{1}{2} \right\}$$

The proposition follows from the following lemma:

**Lemma 3.2.** Let $\varphi(n)$ be an arbitrary positive increasing function of $n$ with $\sum_{n=1}^{\infty} \frac{1}{\varphi(n)} < \infty$, then for almost every $\alpha \in \mathbb{R}^d$ and every $1 \leq j \leq d$, we have

$$\sum_{n=1}^{N^2(\log N)^2} \frac{1}{n \prod_{i \neq j} \|n\alpha_i\|} \ll (\log N)^d \varphi^d(\log \log N),$$

**Proof.** It suffices to prove for every $1 \leq j \leq d$ the bound above holds, we employ the same technique as in Lemma 3.1. Denote $\bar{d}\alpha_j = \bar{d}\alpha_1 \cdots \bar{d}\alpha_j \cdots \bar{d}\alpha_d$, where $\bar{d}\alpha_j$ is omitted. Note that the integral

$$J(n) = \int_{[0,1]^{d-1}} \frac{1}{\prod_{i \neq j} (\|n\alpha_i\| \log \|n\alpha_i\| \varphi(\|\log \log \|n\alpha_i\|)))^d} \alpha_1 \cdots \alpha_j \cdots \alpha_d$$

is finite and independent of \( n \), the series \( \sum_{n=2}^{\infty} \frac{1}{n \log n} \varphi(\log \log n) \) also converges, so the integral

\[
\int_{[0,1]^{d-1}} \left( \sum_{n=2}^{\infty} \frac{1}{n \log n} \prod_{i \neq j} \left( ||n\alpha_i|| \log ||n\alpha_i|| \varphi(\log \log ||n\alpha_i||) \right) \right) d\alpha_j
\]

\[
= \sum_{n=2}^{\infty} \frac{J(n)}{n \log n} \varphi(\log \log n)
\]

is finite. Therefore we have that the series

\[
\sum_{n=2}^{\infty} \frac{1}{n \log n} \varphi(\log \log n) \prod_{i \neq j} \left( ||n\alpha_i|| \log ||n\alpha_i|| \varphi(\log \log ||n\alpha_i||) \right)
\]

converges for almost every \( \alpha \in [0,1]^d \) and ever \( \varepsilon > 0 \), the periodicity of \( \| \cdot \| \) gives the convergence for almost every \( \alpha \in \mathbb{R}^d \). Using the inequality (3.9), we have

\[
N^2(\log N)^2 \sum_{n=1}^{\infty} \frac{1}{n \prod_{i \neq j} ||n\alpha_i||} \ll \left( \sum_{n=2}^{\infty} \frac{(\log N)^d \varphi(\log \log N)^d}{n \log n} \prod_{i \neq j} \left( ||n\alpha_i|| \log ||n\alpha_i|| \varphi(\log \log ||n\alpha_i||) \right) \right) \ll (\log N)^d \varphi(\log \log N)^d.
\]

Since the sum in (3.8) is a finite sum over \( 1 \leq j \leq d \), the proof for the proposition is completed.

3.3 Estimation for the sum when \( |n_1| \prod_{i=1}^{d} ||n_i\alpha_i|| \) is small.

Define:

\[
\tilde{D}_3(\alpha; x; N) = \sum_{n \in U_3(\alpha; N)} f(n, x, \alpha)
\]

where

\[
U_3(\alpha; N) = \left\{ n \in \mathbb{Z}^{d+1} \setminus \{0\} \mid \begin{array}{l}
1 \leq |n_1| \leq N^2(\log N)^2, \\
|n_1| \prod_{i=1}^{d} ||n_1\alpha_i|| > (\log N)^s, \\
|n_1\alpha_i - n_{i+1}| = ||n_1\alpha_i||, \quad 1 \leq i \leq d
\end{array} \right\}.
\]

Note that in (3.11), \( \{n_{i+1}\}_{1 \leq i \leq d} \) are respectively the closest integers to \( \{n_1\alpha_i\}_{1 \leq i \leq d} \), therefore in the sequel we only need to discuss \( n_1 \) instead of \( n \in \mathbb{Z}^{d+1} \). The main result of this step is the following:
Proposition 3.4. Let $\varphi(n)$ be an arbitrary positive increasing function of $n$ with $\sum_{n=1}^{\infty} \frac{1}{\varphi(n)} < \infty$, then for almost every $\alpha$, we have

$$|\bar{D}_3(\alpha, x; N) - \bar{D}_2(\alpha, x; N)| \ll (\log N)^d \varphi^{d+1}(\log \log N).$$

First we need some preparation lemmas about the lower bound of the small divisors.

Lemma 3.3. Let $\varphi(n)$ be an arbitrary positive increasing function of $n$ with $\sum_{n=1}^{\infty} \frac{1}{\varphi(n)} < \infty$, then for almost every $\alpha \in [0,1]^d$, we have for every $|n| > 1$,

$$|n| \prod_{i=1}^{d} ||n\alpha_i|| \gg \frac{1}{(\log |n|)^d \varphi(\log \log |n|)}$$

and

$$||n\alpha_i|| \gg \frac{1}{|n|\varphi(\log |n|)}; \quad 1 \leq i \leq d,$$

with a constant that depends on $\alpha \in [0,1]^d$.

Proof. For $\epsilon > 0$ fixed, define

$$E_n := \{ \alpha \in [0,1]^d : |n| \prod_{i=1}^{d} ||n\alpha_i|| < 1/(\log |n|)^d \varphi(\log \log |n|) \},$$

by direct calculation, we have:

$$\text{Leb}(E_n) \leq \frac{2}{n(\log n)\varphi(\log \log |n|)}$$

therefore $\sum_{n \in \mathbb{Z}\setminus\{0\}} \text{Leb}(E_n) < \infty$, by Borel-Cantelli Lemma, we have that for for almost every $\alpha$,

$$|n| \prod_{i=1}^{d} ||n\alpha_i|| \gg 1/(\log |n|)^d \varphi(\log \log |n|)$$

with a constant that depends on $\alpha \in [0,1]^d$. The proof for $|n||na|$ is similar. \qed

Proposition 3.4 follows easily from the following lemma:

Lemma 3.4. Let $\varphi(n)$ be an arbitrary positive increasing function of $n$ with $\sum_{n=1}^{\infty} \frac{1}{\varphi(n)} < \infty$, then for almost every $\alpha$, and $s \in \mathbb{N}$ fixed, we have the following estimation:

$$\sum_{1 \leq n \leq N^2(\log N)^2} \frac{1}{n \prod_{i=1}^{d} ||n\alpha_i||} \ll (\log N)^d \varphi^{d+1}(\log \log N) \quad (3.12)$$

where the inequality holds for a constant that depends on $s, \alpha$.
Proof. Divide \( n \) in discussion into the following sets:

\[
S^\alpha(p, v) = \left\{ e^{e^p - 1} \leq n < e^p \left| \frac{2^{v-1}}{(\log n)^d \varphi(\log \log |n|)} \leq n \prod_{i=1}^d \| n\alpha_i \| < \frac{2^v}{(\log n)^d \varphi(\log \log |n|)} \right. \right\}
\]

Where \( 0 \leq p \leq \log \log N + 1, \ v \ll \log \log N \). By Lemma 3.3, there exists a constant \( K = K(\alpha, \varphi) > 0, \) such that \( v \geq -K \). We prove a sublemma for the upper bound of the number of elements inside each \( S^\alpha(p, v) \):

**Sublemma 3.4.1.** Let \( \varphi(n) \) be an arbitrary positive increasing function of \( n \) with \( \sum_{n=1}^{\infty} \frac{1}{\varphi(n)} < \infty \), then for almost every \( \alpha \), and every \( p > 1, v \geq -K \),

\[
\#S^\alpha(p, v) \ll 2^v \varphi(v),
\]

with a constant that depends only on \( \alpha, \varphi \).

**Proof.** Define the corresponding set

\[
C(n, v) = \left\{ \alpha \in [0, 1]^d \left| \frac{2^{v-1}}{(\log n)^d \varphi(\log \log |n|)} \leq n \prod_{i=1}^d \| n\alpha_i \| < \frac{2^v}{(\log n)^d \varphi(\log \log |n|)} \right. \right\},
\]

we have:

\[
\text{Leb}(C(n, v)) \ll \frac{2^v}{n(\log n) \varphi(\log \log n)},
\]

and

\[
\#S^\alpha(p, v) = \sum_{e^{e^p - 1} \leq n < e^p} 1_{C(n, v)}(\alpha).
\]

We have:

\[
\mathbb{E}(\#S^\alpha(p, v)) = \sum_{e^{e^p - 1} \leq n < e^p} \text{Leb}(C(n, v)) \ll \frac{2^v}{\varphi(p)}
\]

and

\[
\text{Leb}\{\#S^\alpha(p, v) \geq 2^v \varphi(v)\} \leq \mathbb{E}(\#S^\alpha(p, v)) \ll \frac{1}{\varphi(p) \varphi(v)}
\]

So

\[
\sum_{p > 1 \atop v \geq -K} \text{Leb}\{\#S^\alpha(p, v) \geq 2^v \varphi(v)\} < \infty.
\]

By Borel-Cantelli Lemma, we have that for almost every \( \alpha \),

\[
\#S^\alpha(p, v) \ll 2^v \varphi(v).
\]

\[\square\]
Continuing with the proof for Lemma 3.4, we have the following estimation:

\[
\sum_{1 \leq n \leq N^2(\log N)^2} \frac{1}{n \prod_{i=1}^{d} \|n\alpha_i\|} \ll \sum_{p=1}^{\log \log N+1} -K \leq v \leq \log \log N \left( \frac{\varphi(p)}{2^p} \cdot 2^n \varphi(v) \right) \ll (\log N)^{d} \varphi(\log \log N) \cdot (\varphi(\log \log N) \log \log N) \ll (\log N)^{d} \varphi^{d+1}(\log \log N).
\]

Therefore Lemma 3.4 is proven. \qed

**Proof for Proposition 3.3**

Note that

\[
|\tilde{D}_3(\alpha, x; N) - \tilde{D}_2(\alpha, x; N)| \ll \sum_{1 \leq n \leq N^2(\log N)^2} \frac{1}{n \prod_{i=1}^{d} \|n\alpha_i\|}.
\]

Proposition 3.4 follows from Lemma 3.4.

### 3.4 Control the sum when \( n_1 \) is between \( N^2/4 \) and \( N^2(\log N)^2 \).

The goal of this step is to prove the following:

**Proposition 3.5.** For almost every \( \alpha \), we have

\[
|\tilde{D}_4(\alpha, x; N) - \tilde{D}_3(\alpha, x; N)| \ll (\log N)^2 \log N.
\]

With the proposition above, we could limit the range of \( (2\pi n_1/N^2) \) in \( f(n, x, \alpha) \) to \((-\pi/2, \pi/2)\), which makes \( f \) better-behaved for later estimations.

The range for \( n_1 \) in the difference \( D_4 - D_3 \) can be decomposed as follows:

\[
T_\alpha(I; N) = \begin{cases} 
  n_1 \in \mathbb{Z} & \left( \frac{2^{l_1}}{2^{l_1+1}}, \frac{2^{-l_{i+1}}}{2^{-l_{i+1}+1}} \right), \\
  2^{-l_{i+1}} \leq \|n_1 \alpha_i\| < 2^{-l_{i+1}+1}, & 1 \leq i \leq d-1 \\
  2\sum_{i=1}^{d} l_{i+1} - l_i \leq \|n_1 \alpha_d\| < 2\sum_{i=1}^{d} l_{i+1} - l_i + 1
\end{cases}
\]

where \( l_1 \) defines the range for \( n_1 \), \( l_{i+1} \) defines the range for \( \|n_1 \alpha_i\| \), \( 1 \leq i \leq d-1 \), and \( l_{d+1} \) defines the range for \( \|n_1 \prod_{i=1}^{d} \|n_1 \alpha_i\| \). By Lemma 3.3, the range for \( I = (l_1, \ldots, l_{d+1}) \in L_1(N) \) is defined by:

\[
L_1(N) := \begin{cases}
  1 \in \mathbb{Z}^{d+1} & \left( \frac{2 \log_2 N - 2 \leq l_1 \leq 2 \log_2 N + 2 \log_2 \log N}, \\
  2 \leq l_{i+1} \ll \log N, & 1 \leq i \leq d-1, \\
  \log \log N \ll l_{d+1} \leq l_1 \ll \log N,
\end{cases}
\]

(3.15)
We will first prove a uniform upper bound for $\#T_{\alpha}(l; N)$:

**Lemma 3.5.** If $s > 3$, then for almost every $\alpha \in \mathbb{R}^d$ and all the $1$ in $L_1(N)$, we have

$$\#T_{\alpha}(l; N) \ll 2^{d+1}$$

where the constant is uniform for $l$ in $L_1(N)$.

**Proof.** The number of elements inside $T_{\alpha}(l; N)$ corresponds to the number of $n_1 \in [2^{l_1}, 2^{l_1+1})$, such that vector $n_1 \alpha \mod 1 = (n_1 \alpha_1 \mod 1, \ldots, n_1 \alpha_d \mod 1)$ visit one of the $2^d$ target boxes:

$$d \cdot \prod_{i=1}^{d-1} \pm [2^{-l_{i+1}}, 2^{-l_{i+1}+1}) \times \pm [2^{\sum_{i=1}^{d} l_{i+1} - l_i}, 2^{\sum_{i=1}^{d} l_{i+1} - l_i + 1}),$$

where the negative signs deal with the case when $\frac{1}{2} \leq \{n_1 \alpha_i\} < 1$. Therefore the expected value for $\#T_{\alpha}(l; N)$ is $O(2^{d+1})$, from Theorem 1 we know that the error term for the cardinality is $O((\log 2^{l_1})^{2+\epsilon})$ for any target box, we have that

$$\#T_{\alpha}(l; N) = 2^{d+1} + O(l_1^{2+\epsilon}).$$

Note that $l_1^{2+\epsilon} \ll (\log N)^{2+\epsilon} < (\log N)^s < 2^{d+1}$, the claim follows. \(\square\)

Now we can estimate the contribution of the terms for which $N^2/4 \leq n_1 \leq N^2(\log N)^2$.

**Proof of Proposition 3.5**

**Proof.** Inside each $T_{\alpha}(l; N)$, the divisors $n_1 \prod_{i=1}^{d} \|n_i \alpha_i\|$ is between $2^{d+1}$ and $2^{d+1+d+1}$, therefore

$$|D_4 - D_3| \leq \sum_{l \in L_1(N)} \sum_{T_{\alpha}(l; N)} \frac{1}{|n_1 \prod_{i=1}^{d} \|n_1 \alpha_i\|} \leq \sum_{l \in L_1(N)} \sum_{T_{\alpha}(l; N)} \frac{1}{2^{d+1}} \ll \sum_{l \in L_1(N)} \frac{1}{2^{d+1}} \#T_{\alpha}(l; N) \ll \sum_{l \in L_1(N)} 1 \ll (\log N)^d \log \log N.$$

The last inequality follows from the range of $l_i$’s in $L_1(N)$ (see (3.15)), which gives the number of possible $l_i$’s is $O(\log \log N)$, and the number of possible $l_i$’s are $O(\log N)$ for $2 \leq i \leq d + 1$. \(\square\)

**Proof of Proposition 3.1**
Proof. Combining Propositions 3.2-3.5, we have

\[ |\bar{D}_4 - D| \ll O(1) + (\log N)^d \varphi^d (\log \log N) + (\log N)^d \varphi^{d+1} (\log \log N) + (\log N)^d \log \log N \]
\[ \ll (\log N)^d \varphi^{\max\{d,3\}} (\log \log N). \]

\[ \square \]

4 Cancellation of the main terms.

To estimate the contribution of the remaining terms, we first decompose the product
\[ f(n, x, \alpha) \]
into 2 distinct parts,
\[ \bar{D}_4 = i^{d+1} \left( \sum_{n \in U_4(\alpha,N)} \frac{1}{n_1 \prod_{i=1}^{d} (n_1 \alpha_i - n_{i+1})} \cdot g(n, \alpha; N) \right) \]
\[ + \sum_{s} \sum_{n \in U_4(\alpha;N)} e^{2\pi i s \cdot \mathcal{L}_s(n)} \cdot g(n, \alpha; N) \] \hfill (4.1)

Now \( g(n, \alpha; N) \) is the product below (observe that \(|g(n, \alpha; N)| \leq 1\):
\[ \left( \frac{\sin 2\pi \left( \frac{n_1}{N^2} \right)}{2\pi \left( \frac{n_1}{N^2} \right)} \right)^2 \prod_{i=1}^{d} \left( \frac{\sin 2\pi (n_1 \alpha_i - n_{i+1})}{2\pi (n_1 \alpha_i - n_{i+1})} \right)^2 \] \hfill (4.2)

and finally, \( \mathcal{L}_s = \mathcal{L}_{s,x,N,\alpha} \) is one of the \( 2^{d+1} - 1 \) linear forms of \( d + 1 \) variables:
\[ \mathcal{L}_s(n) = \mathcal{L}_s(n_1, \ldots, n_{d+1}) = \delta_1 n_1 x - \sum_{i=1}^{d} \delta_i N(n_1 \alpha_i - n_{i+1}) \] \hfill (4.3)

where \( s = (\delta_1, \ldots, \delta_{d+1}) \in \{0,1\}^{d+1} \) and \( s \neq 0 \).

Note that the sign \( \pm \) in the second part of (4.1) is in fact \( \pm = (-1)^{\sum_{i=1}^{d+1} \delta_i} \), and so it is independent of \( n \in \mathbb{Z}^{d+1} \).

We begin with the first part of \( \bar{D}_4 \):

Let
\[ \bar{D}_5(\alpha, x; N) = \sum_{n \in U_4(\alpha;N)} \frac{g(n, \alpha; N)}{n_1 \prod_{i=1}^{d} (n_1 \alpha_i - n_{i+1})}. \] \hfill (4.4)

The goal of this section is to prove the following:

**Proposition 4.1.** For almost every \( \alpha \in [0,1]^d \), we have
\[ |\bar{D}_5(\alpha, x; N)| \ll \log N \]
Let $\delta_N = 1/(\log N)^d$, $\epsilon = \{\epsilon_1, \ldots, \epsilon_{d+1}\} \in \{\pm 1\}^{d+1}$, and $l = \{l_1, \ldots, l_{d+1}\} \in \mathbb{Z}^{d+1}$, and define the sets:

$$S_\alpha(l, \epsilon; N) = \left\{ n \in \mathbb{Z}^{d+1} : (1 + \delta_N)^{l_i} \leq \epsilon_i n_i < (1 + \delta_N)^{l_i+1}, (1 + \delta_N)^{-l_{i+1}} \leq \epsilon_{i+1} (n_1 \alpha_1 - n_{i+1}) < (1 + \delta_N)^{-l_{i+1}+1}, 1 \leq i \leq d - 1 \right\},$$

$$\left\{ (1 + \delta_N)^\sum_{i=1}^{d} l_{i+1-l_i} \leq \epsilon_{d+1} (n_1 \alpha_d - n_{d+1}) \leq (1 + \delta_N)^\sum_{i=1}^{d} l_{i+1-l_i+1}, n_{i+1}'s \text{ are respectively the closest integer to } n_i \alpha_i, 1 \leq i \leq d \right\},$$

where $l_i$’s are positive integers, and the range of $l = (l_1, \ldots, l_{d+1})$ is as the following:

$$L_2(N) = \left\{ l \in \mathbb{N}^{d+1} \left| \begin{array}{c}
(\log N)^s \leq (1 + \delta_N)^{l_{d+1}} \leq (1 + \delta_N)^{l_1} \leq N^2/4, \\
(1 + \delta_N)^{l_{i+1} - l_i} \geq (\log N)^s, 1 \leq i \leq d - 1.
\end{array} \right. \right\}$$

which gives

$$\log \log N/\delta_N \ll l_{d+1} \leq l_1 \ll \log N/\delta_N, \quad l_1 - l_{i+1} \gg \log \log N/\delta_N, 1 \leq i \leq d - 1. \quad (4.6)$$

By integration over $\alpha \in [0, 1)$, the expected value for $\#S_\alpha(l, \epsilon; N)$ is:

$$E(l, \epsilon; N) = \int_{\alpha \in [0, 1]^d} \#S_\alpha(l, \epsilon; N) \, d\alpha = \delta_N^{d+1}(1 + \delta_N)^{l_{d+1}}.$$

A more precise description for the number of elements in $\#S_\alpha(l, \epsilon; N)$ is the following:

**Lemma 4.1.** If $s > (d + 2)d + 3$, then for almost every $\alpha \in \mathbb{R}^d$, and every $l \in L_2(N)$, we have

$$|\#S_\alpha(l, \epsilon; N) - E(l, \epsilon; N)| \ll \delta_N E(l, \epsilon; N)$$

where the constant is uniform for $l \in L_2(N)$ and $N$.

**Proof.** With the same reasoning as in Lemma 3.5, the number of elements inside $S_\alpha(l, \epsilon; N)$ corresponds to the number of

$$n_1 \in [(1 + \delta_N)^{l_1}, (1 + \delta_N)^{l_1+1}),$$

such that vector $n_1 \alpha$ lies inside the target box

$$\prod_{i=1}^{d-1} [(1 + \delta_N)^{-l_{i+1}}, (1 + \delta_N)^{-l_{i+1}+1}] \times [(1 + \delta_N)^\sum_{i=1}^{d} l_{i+1-l_i}, (1 + \delta_N)^\sum_{i=1}^{d} l_{i+1-l_i+1}].$$

therefore the expected value for $\#S_\alpha(l, \epsilon; N)$ is $\delta_N^{d+1}(1 + \delta_N)^{l_{d+1}}$, from Theorem 3.3, we know that the error term is $O((\log(1 + \delta_N)^l)^{2+\epsilon})$ for any box, i.e.

$$\#S_\alpha(l, \epsilon; N) = \delta_N^{d+1}(1 + \delta_N)^{l_{d+1}} + O((\log(1 + \delta_N)^l)^{2+\epsilon}).$$

Note that $(\log(1 + \delta_N)^l)^{2+\epsilon} \ll (\log N)^{2+\epsilon} < \delta_N^{d+2}(\log N)^s \leq \delta_N^{d+2}(1 + \delta_N)^{l_{d+1}} = \delta_N E(l, \epsilon; N)$, the claim follows. \qed
For the sake of simplicity, we abbreviate $S_\alpha(1, \epsilon; N)$ and note it by $\mathcal{S}(1, \epsilon; N)$ in later discussions. Using the Lemma above, we can estimate the sum $\bar{\mathcal{D}}_5$ by cancelling out the main terms.

**Proof of Proposition 4.3**

Proof. Let $\epsilon^+$ and $\epsilon^-$ be two vectors in $\{-1, +1\}^{d+1}$ such that one and only one coordinate is different, and the sign of the divisor is $+$ for $\mathcal{S}(1, \epsilon^+; N)$, and $-$ for $\mathcal{S}(1, \epsilon^-; N)$.

$$\bar{\mathcal{D}}_5 = \sum_{n \in L_2(N) \setminus \text{pairs of } \epsilon^\pm} \left( \sum_{n \in S(1, \epsilon^+; N)} \frac{g(n, \alpha; N)}{n_1 \prod_{i=1}^d (n_1 \alpha_i - n_{i+1})} + \sum_{n \in S(1, \epsilon^-; N)} \frac{g(n, \alpha; N)}{n_1 \prod_{i=1}^d (n_1 \alpha_i - n_{i+1})} \right).$$

Inside each $S(1, \epsilon; N)$, denote by $g(n, \alpha; N)_{\text{max/min}}$ the maximal/minimal value of $g(n, \alpha; N)$ for $n \in S(1, \epsilon; N)$, we have:

$$|g(n, \alpha; N)_{\text{max}} - g(n, \alpha; N)_{\text{min}}| \ll \delta_N g(n, \alpha; N)_{\text{min}}$$

and

$$0 < g(n, \alpha; N)_{\text{max}} \leq 1.$$

For each pair, the sum would cancel out as follows:

$$\sum_{n \in S(1, \epsilon^+; N)} \frac{g(n, \alpha; N)}{n_1 \prod_{i=1}^d (n_1 \alpha_i - n_{i+1})} + \sum_{n \in S(1, \epsilon^-; N)} \frac{g(n, \alpha; N)}{n_1 \prod_{i=1}^d (n_1 \alpha_i - n_{i+1})} \leq \frac{g(n, \alpha; N)_{\text{max}}}{(1 + \delta_N)^d} (1 + \delta_N) \mathcal{E}(1, \epsilon; N) - \frac{g(n, \alpha; N)_{\text{min}}}{(1 + \delta_N)^d} (1 - \delta_N) \mathcal{E}(1, \epsilon; N)$$

$$\ll (g(n, \alpha; N)_{\text{max}} + g(n, \alpha; N)_{\text{min}})(1 + \delta_N)^{d+1} \mathcal{E}(1, \epsilon; N) + \frac{g(n, \alpha; N)_{\text{max}}}{(1 + \delta_N)^d} \delta_N \mathcal{E}(1, \epsilon; N)$$

$$\ll \delta_N^{-1} \mathcal{E}(1, \epsilon; N) \ll \delta_N^{-d+2}.$$

The other direction is the same:

$$\sum_{n \in S(1, \epsilon^+; N)} \frac{g(n, \alpha; N)}{n_1 \prod_{i=1}^d (n_1 \alpha_i - n_{i+1})} + \sum_{n \in S(1, \epsilon^-; N)} \frac{g(n, \alpha; N)}{n_1 \prod_{i=1}^d (n_1 \alpha_i - n_{i+1})} \gg -\delta_N^{d+2}.$$

Summing over all the possible pairs of $\epsilon^\pm$ and $l_1, \ldots, l_{d+1}$, note that from (4.6), $0 \leq l_{i+1} \leq l_i$, $1 \leq i \leq d$, we have:

$$|\bar{\mathcal{D}}_5| \ll \sum_{l \in L_2(N)} \delta_N^{d+2} \ll \sum_{\frac{\log N}{\delta_N} \leq l_1 \leq \frac{\log N}{\delta_N}} \delta_N^{d+2} l_1 \ll \delta_N^{d+2} \cdot \frac{(\log N)^{d+1}}{\delta_N^{d+1}} = \log N.$$

\[\square\]
5 Estimation of small exponentials.

Finally we study the contribution of the linear forms \( \mathcal{L}_d \) in \( (4.1) \).

Let

\[
D_6 = D_6^{(s)} = \sum_{n \in U^d(\alpha; N)} \frac{e^{2\pi i \mathcal{L}_s(n)}}{n_1 \prod_{i=1}^{d} (n_1 \alpha_i - n_{i+1})} \cdot g(n, \alpha; N)
\]

\[
= \sum_{l \in L_2(N)} \left( \sum_{n \in S(1, \epsilon^+; N)} \tilde{f}(n, x, \alpha) + \sum_{n \in S(1, \epsilon^-; N)} \tilde{f}(n, x, \alpha) \right),
\]

where

\[
\tilde{f}(n, x, \alpha) = \frac{e^{2\pi i \mathcal{L}_s(n)}}{n_1 \prod_{i=1}^{d} (n_1 \alpha_i - n_{i+1})} \cdot g(n, \alpha; N)
\]

and \( \mathcal{L} \) is one of the \( 2^{d+1} - 1 \) linear forms \( \mathcal{L}_{d,x,N,\alpha} \) defined in \( (4.3) \). Following the same line of reasoning as in J. Beck\(^1\), we prove a version of Key Lemma as in \( (4.3) \) that can be adopted in our case. We shall emphasize the key ingredient in the proof is that arithmetic progressions will contribute like a single term inside the progression. Using Lemma \( 5.1 \) we can estimate the number of \( \epsilon \)-big vectors defined as follows, which can be used to cancel the extra log \( N \) in the crude estimation by bounding every numerator by 1.

**Definition 5.1.** We say \( l = (l_1, \ldots, l_{d+1}) \) is "\( \epsilon \)-big" vector if

\[
\frac{|S(1, \epsilon^+; N)| + |S(1, \epsilon^-; N)|}{\log N} \leq \left| \sum_{n \in S(1, \epsilon^+; N)} e^{2\pi i \mathcal{L}_s(n)} - \sum_{n \in S(1, \epsilon^-; N)} e^{2\pi i \mathcal{L}_s(n)} \right|,
\]

where \( |S| = \#S \) denotes the number of elements inside the set \( S \), and as in the proof of Proposition \( (4.1) \), \( \epsilon^+ \) and \( \epsilon^- \) are a pair of vectors in \( \{-1, +1\}^{d+1} \) such that one and only one coordinate is different, and the sign of the divisor is + for \( S(1, \epsilon^+; N) \), and − for \( S(N, 1, \epsilon^-) \).

For convenience denote

\[
S(1, \epsilon^\pm; N) = S(1, \epsilon^+; N) \cup S(1, \epsilon^-; N).
\]

**Definition 5.2.** Two integral vectors \( l = (l_1, \ldots, l_{d+1}) \) and \( h = (h_1, \ldots, h_{d+1}) \) satisfying \( (4.6) \) are called "neighbors" if

\[
(1 + \delta_N)^{h_1-l_1} = (\log N)^9,
\]

\[
(1 + \delta_N)^{h_{i+1}-l_{i+1}} = (\log N)^9, \quad 1 \leq i \leq d - 1,
\]

\[
(1 + \delta_N)^{h_{d+1}-l_{d+1}} = (\log N)^9(d+1).
\]
The notation $l \rightarrow h$ means that the ordered pair $(l, h)$ of vectors satisfies $(5.9)$. Note that by slightly modifying the value of $\delta_N \approx (\log N)^{-d}$, we can make sure that the above definitions are met for integer vectors $l$ and $h$.

**Definition 5.3.** A sequence $H = (h^{(1)}, h^{(2)}, h^{(3)}, \ldots)$ of vectors satisfying $(1.10)$ is called a "special line" if $h^{(1)} \rightarrow h^{(2)} \rightarrow h^{(3)} \rightarrow \ldots$, that is, any two consecutive vectors in $H$ are neighbors.

**Lemma 5.1.** For almost every $\alpha$ every special line contains at most one $\epsilon$-big vector.

**Proof.** Let $H = (h^{(1)}, h^{(2)}, h^{(3)}, \ldots)$ be a special line with two $\epsilon$-big vectors $h^{(p)}$ and $h^{(q)}$, $1 \leq p < q$. If

$$||\mathcal{L}(n)|| \leq (\log N)^{-2} \text{ for every } n \in S(h^{(p)}, \epsilon^{\pm}; N), \quad (5.7)$$

then

$$|1 - e^{2\pi i \mathcal{L}(n)}| \ll (\log N)^{-2} \text{ for every } n \in S(h^{(p)}, \epsilon^{\pm}; N). \quad (5.8)$$

By repeating the argument of the cancellation of the main term with the above equation, we obtain (Err means error for the number of elements in the set $S$)

$$\left| \sum_{n \in S(h^{(p)}, \epsilon^{+}; N)} e^{2\pi i \mathcal{L}(n)} - \sum_{n \in S(h^{(p)}, \epsilon^{-}; N)} e^{2\pi i \mathcal{L}(n)} \right| \ll (1 + (\log N)^{-2})(E(h^{(p)}, \epsilon; N)) + |\text{Err}| - (1 - (\log N)^{-2})(E(h^{(p)}, \epsilon; N)) - |\text{Err}|$$

$$\ll |\text{Err}| + (\log N)^{-2} E(h^{(p)}, \epsilon; N)$$

$$\ll \delta_N (E(h^{(p)}, \epsilon; N))$$

$$\ll \frac{|S(h^{(p)}, \epsilon^{\pm}; N)|}{(\log N)^2}.$$ 

But this contradicts the assumption that $h^{(p)}$ is $\epsilon$-big, see $(5.3)$. So there is an $n^* \in S(h^{(p)}, \epsilon^{\pm}; N)$ such that

$$||\mathcal{L}(n)|| > (\log N)^{-2}. \quad (5.9)$$

For every $m \in S(h^{(q)}, \epsilon^{\pm}; N)$ (another $\epsilon$-big vector), consider the "arithmetic progression" with difference $n^*$:

$$m + r \cdot n^* = (m_1 + r \cdot n_1^*, \ldots, m_{d+1} + r \cdot n_{d+1}^*), \quad r = 0, \pm 1, \pm 2, \ldots$$

We will estimate how many consecutive members $m + r \cdot n^*$ are contained in $S(N, h^{(q)}, \epsilon^{\pm})$. Since $n^* \in S(h^{(p)}, \epsilon^{\pm}; N)$, the definition for $S(h^{(q)}, \epsilon^{\pm}; N)$ (see $(1.10)$) gives the following:

$$(1 + \delta_N)h_i^{(p)} \leq \epsilon_1 n_i^* < (1 + \delta_N)h_i^{(p)} + 1, \quad (5.10)$$

$$(1 + \delta_N)h_{i+1}^{(p)} - h_i^{(p)} \leq \epsilon_{i+1} n_i^* - n_{i+1}^* < (1 + \delta_N)h_{i+1}^{(p)} - h_i^{(p)} + 1, \quad (5.11)$$

$$(1 + \delta_N)\sum_{i=1}^d h_{i+1}^{(p)} - h_i^{(p)} \leq \epsilon_{d+1} n_i^* - n_{d+1}^* < (1 + \delta_N)\sum_{i=1}^d h_{i+1}^{(p)} - h_i^{(p)} + 1. \quad (5.12)$$
Definition 5.4. An \( \mathbf{m} \in S(\mathbf{h}^q, \varepsilon^\pm; N) \) is called an inner point if

\[
(1 + \delta_N)^{h_1^{(q)}} (1 + \frac{\delta_N}{(\log N)^2}) \leq \epsilon_1 m_1 < (1 - \frac{\delta_N}{(\log N)^2})(1 + \delta_N)^{h_1^{(q)}+1},
\]

(5.13)

\[
(1+\delta_N)^{-h_{i+1}^{(q)}} (1 + \frac{\delta_N}{(\log N)^2}) \leq \epsilon_{i+1}(m_1 \alpha_i - m_{i+1}) < (1 - \frac{\delta_N}{(\log N)^2})(1+\delta_N)^{-h_{i+1}^{(q)}+1},
\]

(5.14)

\[
(1+\delta_N)^{\sum_{i=1}^{d} h_{i+1}^{(q)} - h_1^{(q)}} (1 + \frac{\delta_N}{(\log N)^2}) \leq \epsilon_{d+1}(m_1 \alpha_{d+1} - m_{d+1}) < (1 - \frac{\delta_N}{(\log N)^2})(1+\delta_N)^{\sum_{i=1}^{d} h_{i+1}^{(q)} - h_1^{(q)}+1}.
\]

(5.15)

The rest of the points in \( S(\mathbf{h}^q, \varepsilon^\pm; N) \) are called border points.

For every inner point \( \mathbf{m} \in S(\mathbf{h}^q, \varepsilon^\pm; N) \), and for every \( |r| \leq (\log N)^4 \), it follows from (5.14), (5.10) and (5.13) that,

\[
(1 + \delta_N)^{h_1^{(q)} < (1 + \delta_N)^{h_1^{(q)}} (1 + \frac{\delta_N}{(\log N)^2}) - (\log N)^4 (1 + \delta_N)^{h_1^{(q)}+1}
\]

\[
\leq \epsilon_1 (m_1 + r \cdot n^*_1)
\]

\[
< (1 - \frac{\delta_N}{(\log N)^2})(1 + \delta_N)^{h_1^{(q)}+1} + (\log N)^4 (1 + \delta_N)^{h_1^{(q)}+1} < (1 + \delta_N)^{h_1^{(q)}+1}
\]

(5.16)

Similarly, from (5.5), (5.11) and (5.17), and from (5.6), (5.12) and (5.15), we obtain the following, for \( 1 \leq i \leq d - 1 \),

\[
(1 + \delta_N)^{-h_{i+1}^{(q)}} < \epsilon_{i+1} ((m_1 + r n^*_1) \alpha_{i+1} - (m_{i+1} + r \cdot n^*_1)) < (1 + \delta_N)^{-h_{i+1}^{(q)}+1},
\]

(5.17)

and

\[
(1+\delta_N)^{\sum_{i=1}^{d} h_{i+1}^{(q)} - h_1^{(q)}} < \epsilon_{d+1} ((m_1 + r n^*_1) \alpha_{d+1} - (m_{d+1} + r \cdot n^*_1)) < (1+\delta_N)^{\sum_{i=1}^{d} h_{i+1}^{(q)} - h_1^{(q)}+1}
\]

(5.18)

In view of (5.16)-(5.18), for any inner point \( \mathbf{m} \in S(\mathbf{h}^q, \varepsilon^\pm; N) \), at least \( (\log N)^4 \) consecutive members in the progression \( \mathbf{m} + r \cdot \mathbf{n}^* \) are contained in \( S(\mathbf{h}^q, \varepsilon^\pm; N) \). Therefore, we can decompose \( S(\mathbf{h}^q, \varepsilon^\pm; N) \) into three parts:

\[
S(\mathbf{h}^q, \varepsilon^\pm; N) = AP^+ \cup AP^- \cup BP
\]

(5.19)

where \( AP^\pm \) denotes the family of arithmetic progressions \( \{ \mathbf{m} + r \cdot \mathbf{n}^* : 0 \leq r \leq l - 1 \} \) in \( S(\mathbf{h}^q, \varepsilon^\pm; N) \) and \( S(\mathbf{h}^q, \varepsilon^-; N) \) respectively, where \( l = l(\mathbf{m}) \) is the length of the progression starting from \( \mathbf{m} \), and \( l \geq (\log N)^4 \). \( BP \) denotes a set of border points of \( S(\mathbf{h}^q, \varepsilon^\pm; N) \) that are not included in any arithmetic progressions. Using \( ||\mathcal{L}(\mathbf{m})|| > 22 \)
(\log N)^{-2} \) (see (5.9)), the linearity of \( \mathcal{L} \), we obtain

\[
\left| \sum_{\text{AP}^+} e^{2\pi i \mathcal{L}(n)} \right| \leq \sum_{\text{arithmetic progressions}} \left| \sum_{r=0}^{l-1} e^{2\pi i \mathcal{L}(m+rn^*)} \right|
\]

\[
= \sum_{\text{arithmetic progressions}} \left| \sum_{r=0}^{l-1} e^{2\pi i \mathcal{L}(m)+r \mathcal{L}(n^*)} \right|
\]

\[
\ll \sum_{\text{arithmetic progressions}} \frac{1}{\|\mathcal{L}(n)\|} \cdot \sum_{\text{arithmetic progressions}} (\log N)^2
\]

since each length \( \geq (\log N)^4 \). Similarly,

\[
\left| \sum_{\text{AP}^-} e^{2\pi i \mathcal{L}(n)} \right| \ll \frac{|S(h^q, \varepsilon^\pm; N)|}{(\log N)^2} \quad (5.21)
\]

Finally, for border points, at least one of the inequalities in definition 5.4 is violated, thus the range for at least one components is shrunk with ratio \( \frac{\delta_N (\log N)^2}{\delta_N (\log N)^2} \). Using the same reasoning as in Lemma 5.5 and 4.1 and the cardinality of the set \( BP \) can be controlled by the total number of border points of \( S(h^q, \varepsilon^\pm; N) \), we have for almost every \( \alpha \),

\[
|BP| \ll \frac{|S(h^q, \varepsilon^\pm; N)|}{(\log N)^2} \quad (5.22)
\]

Combining (5.19)-(5.22), for almost every \( \alpha \), we obtain

\[
\left| \sum_{n \in S(h^{(p)}, \varepsilon^+; N)} e^{2\pi i \mathcal{L}(n)} - \sum_{n \in S(h^{(p)}, \varepsilon^-; N)} e^{2\pi i \mathcal{L}(n)} \right| \ll \frac{|S(h^{(p)}, \varepsilon^\pm; N)|}{(\log N)^2} \quad (5.23)
\]

which contradicts the assumption that \( h^{(q)} \) is \( \varepsilon \)-big (see (5.3)), therefore for almost every \( \alpha \), every special line contains at most one \( \varepsilon \)-big vector, which proves the lemma.

\textbf{Corollary 5.1.1.} \textit{The number of \( \varepsilon \)-big vectors is} \( \ll (\log \log N)(\log N)^d \delta_{\varepsilon}^{-d+1} \).

\textit{Proof.} First we estimate the number of maximal special lines. Let \( H = \langle h^{(1)}, h^{(2)}, h^{(3)}, \ldots \rangle \) be a "special line", here \( h^{(1)} \) is the first element of \( H \), that is, if \( h^{(0)} \rightarrow h^{(1)} \) holds for some \( h^{(0)} \), then at least one of the following inequalities is violated:

\[
(1 + \delta_N)^{h^{(0)}} \geq (\log N)^s, \quad (1 + \delta_N)^{h^{(0)}} - h^{(0)} \geq (\log N)^s, \quad 1 \leq i \leq d-1,
\]

\[
(1 + \delta_N)^{h^{(0)}} - h^{(i)} \geq (\log N)^s, \quad 1 \leq i \leq d-1,
\]

\[
(1 + \delta_N)^{h^{(0)}} - h^{(d-1)} \geq (\log N)^s, \quad (1 + \delta_N)^{h^{(0)}} - h^{(d)} \geq (\log N)^s.
\]
Thus, by (5.4) and (5.6), one of the inequalities below holds:

\[ (\log N)^s \leq (1 + \delta_N) h_{1+1}^{(1)} \leq (\log N)^{s+9}, \]

\[ (\log N)^s \leq (1 + \delta_N) h_{d+1}^{(1)} - h_{i+1}^{(1)} \leq (\log N)^{s+18}, \quad 1 \leq i \leq d-1, \]

\[ (\log N)^s \leq (1 + \delta_N) h_{d+1}^{(1)} \leq (\log N)^{s+9(d+1)}, \]

So at least one coordinate of \( h_{1+1}^{(1)}, h_{d+1}^{(1)}, \) or \( h_{1+1}^{(1)} - h_{i+1}^{(1)} \) of the first element \( \mathbf{h}^{(1)} \) of \( H \) is restricted to a short interval of length \( \text{const} \cdot \log \log N \cdot \delta - 1 \), the rest, by the condition (4.6), are restricted to an interval of length \( \text{const} \cdot \log N \cdot \delta^{-1} \). Since the starting vector determines the whole special line, the number of special lines is

\[ \ll (\log \log N) \cdot (\log N)^d \cdot \delta^{- (d+1)}. \]

By Lemma 5.1, the total number of \( \epsilon \)-big vectors is also

\[ \ll (\log \log N) \cdot (\log N)^d \cdot \delta^{- (d+1)}. \]

With the help of the Lemma 5.1, we can estimate the contribution of the exponential terms, we have the following claim:

**Proposition 5.1.** For almost every \( \alpha, \beta \) we have

\[ |D_6| \ll (\log N)^2 (\log \log N) \]

**Proof.**

\[ D_6 = \sum_{\text{small}} + \sum_{\text{big}} \]

where

\[ \sum_{\text{small}} = \sum_{\epsilon \in \text{not } \epsilon \text{-big}} \sum_{n \in S(N,1,\epsilon)} f(n,x,\alpha) \quad (5.23) \]

\[ \sum_{\text{big}} = \sum_{\epsilon \in \text{is } \epsilon \text{-big}} \sum_{n \in S(N,1,\epsilon)} f(n,x,\alpha) \quad (5.24) \]

By Lemma 4.1, range for 1, (4.6), and (5.3),

\[ \sum_{\text{small}} \ll \sum_{\epsilon \in \text{is not } \epsilon \text{-big}} (1 + \delta_N)^{-1} \epsilon \cdot \frac{E(1,\epsilon;N)}{\log N} \]

\[ \ll \sum_{\epsilon \in \mathcal{L}(N)} \frac{\delta^{d+1}}{\log N} \]

\[ \ll \frac{\delta^{d+1}}{\log N} \cdot \frac{(\log N)^{d+1}}{\delta^{d+1}} \]

\[ \ll (\log N)^d \]
By Corollary 5.1.1,

\[ \sum_{\text{big}} | \sum_{l \text{ is } \varepsilon \text{-big}} (1 + \delta_N)^{-l_{d+1}} \cdot E(l, \varepsilon; N) \ll \left( \log \log N \right)^d \log \log N \]

Combining Proposition 2.1, 3.1, 4.1, 5.1, we finally proved Theorem 1.2.
References

[1] József Beck. Probabilistic diophantine approximation, i. kronecker sequences. *Annals of Mathematics*, pages 449–502, 1994.

[2] Michael Drmota and Robert F Tichy. *Sequences, discrepancies and applications*. Springer, 2006.

[3] A Khintchine. Ein satz über kettenbrüche, mit arithmetischen anwendungen. *Mathematische Zeitschrift*, 18(1):289–306, 1923.

[4] Wolfgang M Schmidt. Metrical theorems on fractional parts of sequences. *Transactions of the American Mathematical Society*, 110(3):493–518, 1964.