B-spline Collocation with Domain Decomposition Method

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Abstract. A global B-spline collocation method has been previously developed and successfully implemented by the present authors for solving elliptic partial differential equations in arbitrary complex domains. However, the global B-spline approximation, which is simply reduced to Bezier approximation of any degree \( p \) with \( C^0 \) continuity, has led to the use of B-spline basis of high order in order to achieve high accuracy. The need for B-spline bases of high order in the global method would be more prominent in domains of large dimension. For the increased collocation points, it may also lead to the ill-conditioning problem. In this study, overlapping domain decomposition of multiplicative Schwarz algorithm is combined with the global method. Our objective is two-fold that improving the accuracy with the combination technique, and also investigating influence of the combination technique to the employed B-spline basis orders with respect to the obtained accuracy. It was shown that the combination method produced higher accuracy with the B-spline basis of much lower order than that needed in implementation of the initial method. Hence, the approximation stability of the B-spline collocation method was also increased.

Introduction

1.1. Overview

The method of collocation is attractive for solving differential equations due to its ease of implementation and efficiency. The method applies directly to the original differential equations that describe the physics of the problem. In the last decades, collocation methods using B-spline functions for the numerical solutions of various types of partial differential equations have been paid attention and becoming an active research area. The spline collocation methods are interesting due to their high accuracies. There are many variants of spline collocation methods, which are mainly related to the locations of the collocation points, such as nodal, orthogonal and modified collocation (see the survey of spline collocation methods by Fairweather and Meade [1], Bialecki and Fairweather [2]). Other variants of spline collocation methods have been investigated as well: Botella [3], Greville [4, 5] and
Moment [6] collocations. The accuracy of the spline collocation methods depends on the order of B-spline basis, regularity or continuity, and location of the collocation points [3-6]. Recently, B-spline collocation methods have been applied successfully in various areas including fluid dynamics [3, 4, 7-9], computational aero-acoustics [10], wave equation [11], non-linear analysis of laminated panels [12], vibration analysis [13, 14], and non-linear two-point boundary value problems [15].

1.2. Motivation

However, there are serious shortcomings in the implementation of the B-spline collocation methods. The fact that basis needed for approximation is a tensor product operation from one-dimensional B-spline basis functions has restricted the method extension to applications in higher dimensions. Moreover, the tensor product structure that is essentially rectangular in nature has also further limited the approximation to rectangular problem domain or other domains in which a rectangular structure can be envisaged. Obviously, this produces a quite severe limitation with respect to the flexibility and suitability of the method for applications in non-rectangular or irregular domains. In [16], it has been stated that the B-spline collocation methods have not yet reached the level of flexibility of the isoparametric formulation of the finite element method for arbitrary geometries.

1.3. Newer Strategy

A global B-spline collocation method has been previously developed and successfully implemented for solving elliptic partial differential equations in arbitrary complex domains. The implementation of global collocation has been inspired by successful application of radial basis functions (RBF) collocation in non-uniform grids [17]. However, the global B-spline approximation, which is simply reduced to Bezier approximation of any degree $p$ with $C^0$ continuity, has led to the use of B-spline basis of high order in order to achieve high accuracy. The need for B-spline bases of high order in the global method would be more prominent in domains of large dimension. For the increased collocation points, it may also lead to the ill-conditioning problem.

In this study, overlapping domain decomposition of multiplicative Schwarz algorithm [18] is combined with the global method. It is aimed that the obtained B-spline approximation can be more robust, but still preserving great flexibility of the initial method in dealing with arbitrary complex domains. Hence, our objective is two-fold that improving the accuracy with the combination technique, and also investigating influence of the combination technique to the employed B-spline basis orders with respect to the obtained accuracy.

1.4. Related Works

The present method, in the absence of triangulation, may be also related to the work of Hu, Han and Lai [19] in the sense that the constructed approximation rely upon the use of splines of various degrees. In the authors work, which is an extension of Awanou, Lai and Wensons spline method [20], the method of bivariate splines of various degrees constructed from Bernstein polynomials of degree $p$ over triangles was used for the numerical solution of Poisson and biharmonic equations in an adaptive manner. High order approximation is used to get a better approximation power, much similar also with $p$-adaptation in hp-finite element methods [21, 22]. In general, the method of bivariate splines falls into high order method.

**B-spline Collocation**

Consider the following boundary value problem:

\[ Lu(x) = f(x) \quad \forall x \in \Omega \]
\[ B^h u(x) = h(x) \quad \forall x \in \partial \Omega^h \]
\[ B^s u(x) = g(x) \quad \forall x \in \partial \Omega^s \]  \hspace{1cm} (1)

\[ Lu(x) = f(x) \quad \forall x \in \Omega \]
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\[ B^s u(x) = g(x) \quad \forall x \in \partial \Omega^s \]  \hspace{1cm} (1)
where $\Omega$ is the problem domain, $\partial \Omega^h$ is the Dirichlet boundary, $\partial \Omega^g$ is the Neumann boundary, $\partial \Omega = \partial \Omega^h \cup \partial \Omega^g$ is the problem domain boundary, $L$ is the Laplace operator $\left( \nabla^2 \right)$ in $\Omega$, $B^h$ is the differential operator on $\partial \Omega^h$, $B^g$ is the differential operator on $\partial \Omega^g$, $\mathbf{x} \in \mathbb{R}^d$.

Let also $\{CP_i = (\xi_i)\}_{i=1}^{NC}$ be $NC$ collocation points in $\Omega$, of which $\{(\xi_i)\}_{i=1}^{NI}$ are interior points, $\{(\xi_i)\}_{i=1}^{N_h}$ are boundary points on $\partial \Omega^h$ and $\{(\xi_i)\}_{i=1}^{N_g}$ are boundary points on $\partial \Omega^g$, hence $NC = NI + N_h + N_g$.

The approximate solution $\hat{u}(\mathbf{x})$ for the problem (1) is to be obtained with the collocation using B-splines. B-spline basis construction is briefly described in what follows.

Consider one-dimensional B-spline basis function construction given by the Cox-de Boor recursion formula [23, 24]:

$$N_{i,k}(t) = \begin{cases} 1 & \text{if } \tau_i < t < \tau_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$N_{i,k}(t) = \frac{t - \tau_i}{\tau_{i+k+1} - \tau_i} N_{i,k-1}(t) + \frac{\tau_{i+k+1} - t}{\tau_{i+k+1} - \tau_{i+1}} N_{i+1,k-1}(t) \quad \text{for} \quad 2 \leq k \leq n+1$$

In the formula, the convention $0/0 = 0$ is used for the division calculation. Thus, $b = \{N_0,k, ..., N_{n,k}\}$ are the univariate B-spline bases for the space $C$ of polynomial functions of degree less than or equal to $k-1$ on interval $I = [a_1, b_1] = [\tau_0, \tau_{n+k}]$. It is also clear from (3) that for $k > 1$ the $N_{i,k}(t)$ are a linear combination of two $(k-1)$ order basis functions.

The relationship between the number of knots in the knot vector $(\tau_i)_{i=0}^{n+k}$ ($mk$), control points $(n+1)$ and B-spline order $k$ is also given as follows:

$$mk = n + k + 1$$

For higher dimensional problems, the basis functions are constructed by taking the product of the one-dimensional B-spline basis functions. For approximation in 2D, the basis functions are given by:

$$B := \left( N_{i,k} \otimes M_{j,l} \right)_{i=0,j=0}^{n,m}$$

where $B$ are the tensor product B-spline bases for the space $D$ of tensor product polynomial splines on the tensor product domain $[a_1, b_1] \times [a_2, b_2] = [\tau_0, \tau_{n+k}] \times [\xi_0, \xi_{m+l}]$. $N_{i,k}$ and $M_{j,l}$ are the B-spline basis functions of orders $k$ and $l$.

Note that the tensor product B-spline bases have the partition of unity property:

$$\sum_{i=0}^{n} N_{i,k}(t) M_{j,l}(s) = 1$$

In addition, continuity of a B-spline curve depends generally on its order. A B-spline curve of order $k$ has in general $C^{k-2}$ continuity. There is however a continuity constraint with respect to the
multiplicity \((mp)\) of a knot in the knot vector. It is called as knot continuity, where increasing the multiplicity \((mp)\) of a knot reduces the continuity of the curve to \(C^{k-mp}\) at that knot. It also means only \(p-mp\) derivatives exist for a knot of multiplicity \(mp\), where \(p\) is the B-spline degree [25].

In this study a special class of B-spline approximation obtained by using an open-uniform knot vector with full multiplicity of its end knots (no interior knots) is used for the collocation. Such a B-spline approximation is chosen because it will produce global approximation in order to deal with arbitrary complex domains with arbitrary distribution of knot points. It is noted here the B-spline approximation is simply reduced into Bezier approximation of any degree \(p\) with \(C^0\) continuity [26].

Thus, we have here for the collocation (1) the following two-dimensional B-spline approximation:

\[
\hat{u}(x,y) = \sum_{i,j=0}^{n,m} N_{i,j}(x)M_{j,i}(y)\alpha_{i,j}
\]  

(7)

The B-spline approximation can be simplified to be much similar with shape functions in mesh-less and finite element methods [27] as:

\[
\hat{u}(x,y) = \sum_{n=1}^{N_b} N^n(x,y)\alpha_n
\]  

(8)

where \(N^n(x,y) = N_{i,k}(x)M_{j,l}(y)\), \(\alpha_n\) are the coefficients of approximation related to \(N^n(x,y)\) and \(N_b\) is the number of B-spline bases used in approximation.

**Overlapping Domain Decomposition Method**

By overlapping domain decomposition method, a given computational domain is partitioned into smaller overlapping sub-domains [18, 28, 29]. A typical option to accomplish such a decomposition task is by partitioning the computational domain \(\Omega\) into \(K\) non-overlapping sub-domains and then extending from the shared boundaries of neighbouring sub-domains \(\Omega_i\) and \(\Omega_j\) an amount of overlapping portions \(\gamma_{ij}\) to a larger domain \(\Omega_i\). The artificial interior boundaries are hence introduced here. This schematic is illustrated in Figure 1.

**Figure 1.** Schematic of domain decomposition of \(\Omega\) into smaller
overlapping sub-domains $\Omega_i$. The artificial boundaries $\Gamma_i$ are part of the boundary of $\Omega_i$ that is interior of $\Omega$.

Denote $\Omega_i$, $\partial\Omega_i \setminus \Gamma_i$ and $\Gamma_i$ the extended sub-domain, the natural boundary and the artificial interior boundary overlapped with other neighbouring sub-domains, respectively. Let also $\Omega_i = \Omega_i \cup \partial\Omega_i \setminus \Gamma_i \cup \Gamma_i$ denote the closed sub-domain, $S$ denote the operator of artificial interior boundary conditions and $A_i$ be the artificial interior boundary value of sub-domain $\Omega_i$ extracted from the neighbouring sub-domains. The original global problem is now reformulated as a series of sub-domain problems:

\begin{align}
L u(x) &= f_i & \text{in } \Omega_i \\
B u(x) &= b_i & \text{on } \partial\Omega_i \setminus \Gamma_i \\
S u(x) &= A_i & \text{on } \Gamma_i
\end{align}

$B, S$ specify Dirichlet, Neumann or mixed (Robin) boundary condition operator and $i = 1, 2, \ldots, K$.

Communication and transmission of information among the sub-domains is carried out iteratively in the overlapping regions $\sigma_j$ i.e. through the artificial interior boundaries $\Gamma_i$, which is described in the following. For each sub-domain the B-spline approximation is obtained as follows:

\begin{equation}
\begin{bmatrix} A_{\bar{\Omega} \setminus \Gamma_i} \\ A_{\Gamma_i} \end{bmatrix} \begin{bmatrix} \mathbf{a}_i \\ \mathbf{F}_{\Gamma_i} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{\bar{\Omega} \setminus \Gamma_i} \\ \mathbf{F}_{\Gamma_i} \end{bmatrix}
\end{equation}

where: $A_{\bar{\Omega} \setminus \Gamma_i}$ are matrices representing the operators acting on $\bar{\Omega}_i$ excluding $\Gamma_i$, $\mathbf{F}_{\bar{\Omega} \setminus \Gamma_i}$ are the corresponding right-hand side vectors, in which the natural boundary conditions are imposed. The sub-matrices $A_{\Gamma_i}$ and vectors $\mathbf{F}_{\Gamma_i}$ are produced from the artificial interior boundary conditions.

Solving further equation (10), it is obtained:

\begin{equation}
\mathbf{a}_i = A_i^{-1} \mathbf{F}_i
\end{equation}

where: $A_i = \begin{bmatrix} A_{\bar{\Omega} \setminus \Gamma_i} & A_{\Gamma_i} \end{bmatrix}$, $\mathbf{F}_i = \begin{bmatrix} \mathbf{F}_{\bar{\Omega} \setminus \Gamma_i} \\ \mathbf{F}_{\Gamma_i} \end{bmatrix}$ and $\mathbf{a}_i$ are the vector of coefficients of approximation of each sub-domain. Note that $\mathbf{F}_i$, containing the artificial interior boundary conditions, will continuously change with the updating results of its neighbouring sub-domains.

The approximated values of the unknown function in sub-domain $\bar{\Omega}_i$ are:

\begin{equation}
\phi_i(x) = G_i(x) \mathbf{a}_i
\end{equation}
where: \( G_i(x) = \left[ N^\alpha(x_1), N^\alpha(x_2), \ldots, N^\alpha(x_{N_i}) \right]^T \) and \( N_i \) is the total number of collocation points in sub-domain \( \Omega_i \).

It can be seen that the approximated values of the unknown function in sub-domain \( \Omega_i \) are involving update in the artificial interior boundary conditions. In addition, depending on how artificial interior boundary conditions \( \mathbf{F}_i \) are updated, there are two types of overlapping domain decomposition methods i.e. additive Schwarz and multiplicative Schwarz methods. For clarity, both these overlapping domain decomposition techniques are described here.

In the overlapping additive Schwarz method, the artificial interior boundary values of each sub-domain problem at the \((n+1)\)th step are updated from the results of the \(n\)th step. This is illustrated as follows: Let \( \Omega \) be partitioned into two sub-domains \( \Omega_1 \) and \( \Omega_2 \), where \( \Omega_1 \cap \Omega_2 \neq \emptyset \). The additive Schwarz domain decomposition can be written as:

\[
\begin{align*}
L \phi_1^* &= f & \text{in } \Omega_1 \\
B \phi_1^* &= b & \text{on } \partial \Omega_1 \setminus \Gamma_1 \\
\phi_1^* &= \phi_2^{n-1} & \text{on } \Gamma_1 \\
L \phi_2^* &= f & \text{in } \Omega_2 \\
B \phi_2^* &= b & \text{on } \partial \Omega_2 \setminus \Gamma_2 \\
\phi_2^* &= \phi_1^{n-1} & \text{on } \Gamma_2
\end{align*}
\]  

(13)

On the other hand, in the overlapping multiplicative Schwarz method, each sub-domain problem is solved sequentially, where each sub-domain uses the most recent updates from its neighbouring sub-domains. The multiplicative Schwarz domain decomposition is then illustrated as:

\[
\begin{align*}
L \phi_1^* &= f & \text{in } \Omega_1 \\
B \phi_1^* &= b & \text{on } \partial \Omega_1 \setminus \Gamma_1 \\
\phi_1^* &= \phi_2^{n-1} & \text{on } \Gamma_1 \\
L \phi_2^* &= f & \text{in } \Omega_2 \\
B \phi_2^* &= b & \text{on } \partial \Omega_2 \setminus \Gamma_2 \\
\phi_2^* &= \phi_1^{n} & \text{on } \Gamma_2
\end{align*}
\]  

(14)

Both the multiplicative and additive Schwarz domain decomposition methods can be expressed in terms of the Dirichlet artificial interior boundary conditions as follows [28]:

**Algorithm (Overlapping Domain Decomposition Method)**

\( \phi \leftarrow \) Guess initial solution on the artificial boundaries  
For \( i = 1, 2, \ldots \), until convergence  
For \( j = 1, 2, \ldots, K \) Do:  
\[ \text{Solve equation (17), get } \mathbf{a}_i \]
\[ \phi \leftarrow G, \alpha, \]

If (Multiplicative method) Update \( F_i \) for each \( \Omega_i \) part of \( \Omega \) from neighbouring domains
End Do

If (Additive method) Update all \( \Omega_i \) from previous steps
End Do

**Numerical Results and Discussion**

We consider the following elliptic PDE:

\[
\begin{align*}
\nabla^2 u &= -\frac{75}{144} \sin \left( \frac{\pi x}{6} \right) \sin \left( \frac{\pi y}{4} \right) \sin \left( \frac{3\pi x}{4} \right) \sin \left( \frac{5\pi x}{4} \right) - \frac{7\pi^2}{12} \cos \left( \frac{\pi x}{6} \right) \cos \left( \frac{\pi y}{4} \right) \sin \left( \frac{3\pi x}{4} \right) \sin \left( \frac{5\pi x}{4} \right) \\
&+ \frac{15\pi^2}{8} \sin \left( \frac{\pi x}{6} \right) \sin \left( \frac{7\pi x}{4} \right) \cos \left( \frac{3\pi x}{4} \right) \cos \left( \frac{5\pi x}{4} \right)
\end{align*}
\]

(17)

which is applied on the L-shaped domain as shown in Figure 2 and has the exact solution as follows:

\[
u(x, y) = \sin \left( \frac{\pi x}{6} \right) \sin \left( \frac{7\pi x}{4} \right) \sin \left( \frac{3\pi x}{4} \right) \sin \left( \frac{5\pi x}{4} \right)
\]

(18)

For clarity, the PDE solution on the L-shaped domain is further depicted in Figure 3. As can be seen, the solution shows wave characteristic on the domain with peak values near the corner point (2, 2).

B-spline approximations in both the global domain and domain decomposition application are compared here. In implementation, evaluation of B-spline basis functions can be either carried out symbolically or directly. In the symbolic manner, B-spline basis functions (including their derivatives) are first constructed and expressed symbolically, then followed by substitution of collocation points. Here, to clearly assess the performance of DD technique the symbolic manner is used. All numerical computations were carried out in MATLAB 2006 environment and run on Toshiba Satellite with OS Windows Vista Basic, processor of Intel Pentium Dual-Core and RAM of 1 GB.

For comparison of numerical accuracies, the following \( L^2 \) error is used:

\[
E_n = \sqrt{\frac{\sum_{i=1}^{NC} [u(x_i) - \bar{u}(x_i)]^2}{\sum_{i=1}^{NC} u(x_i)^2}}
\]

(19)

where \( NC \) is the number of collocation points, \( u(x_i) \) and \( \bar{u}(x_i) \) represent the exact and numerical solution, respectively. In addition, a fixed error tolerance \( 1 \times 10^{-7} \) and zero initial guess on the artificial interior boundaries are used in all numerical simulations. Moreover, equation (11) is solved by using QR factorization technique [30].

Note that the number of sub-domains of 3 and 4 are respectively used in the L-shaped domain for the implementations of domain decomposition method. Several numerical simulations are carried out to examine the influence of the number of sub-domains as well as that of the amount of overlapping.

Table 1 first describes the B-spline approximation on the one large domain. In Table 1, it is shown that while the global collocation effectively solves the PDE problem with arbitrary knot points, it leads to
the use of B-spline bases of high orders. This may be explained from the general continuity posed by
the global B-spline approximation which is \( C^0 \) only.

![Diagram](image)

**Figure 2.** L-shaped domain with: (a) 3 sub-domains, and (b) 4 sub-domains, used in the overlapping domain decomposition
method.

![Graph](image)

**Figure 3.** Exact solution of the PDE problem on L-shaped domain.

**Table 1.** Accuracies of B-spline collocation in global domain for different number of collocation points and B-spline order.

| Number of collocation points | Number of domain \( k \) | B-spline order | L2 error \( (\text{sec}) \) | Elapsed time \( (\text{sec}) \) |
|-----------------------------|--------------------------|----------------|----------------|----------------|
| 721                         | 1                        | 23             | 5.086e 2       | 84             |
|                             |                          | 25             | 2.759e 2       | 116            |
|                             |                          | 27             | 4.634e 3       | 154            |
| 1155                        | 1                        | 23             | 3.813e 3       | 89             |
|                             |                          | 25             | 5.568e 4       | 119            |
|                             |                          | 27             | 3.093e 4       | 163            |
| 1791                        | 1                        | 23             | 2.250e 4       | 98             |
|                             |                          | 25             | 1.971e 5       | 126            |
|                             |                          | 27             | 1.757e 6       | 180            |
Table 2. Influence of the amount of overlapping in B-spline collocation with domain decomposition.

| Number of collocation points | Number of sub-domains | Overlapping ratio $o/H_t$ | B-spline order $k$ | L2 error | Iteration steps | Elapsed time (sec) |
|-----------------------------|-----------------------|---------------------------|-------------------|----------|-----------------|-------------------|
| 1155                        | 3                     | 6.3%                      | 16                | 6.031e4  | 30              | 48                |
|                             |                       |                           | 18                | 2.917e5  | 27              | 66                |
|                             |                       |                           | 20                | 8.797e6  | 34              | 99                |
|                             |                       | 12%                       | 16                | 6.802e4  | 12              | 46                |
|                             |                       |                           | 18                | 5.272e5  | 11              | 63                |
|                             |                       |                           | 20                | 7.674e6  | 12              | 93                |
| 1791                        | 3                     | 6.3%                      | 16                | 9.484e4  | 30              | 57                |
|                             |                       |                           | 18                | 1.917e5  | 27              | 76                |
|                             |                       |                           | 20                | 1.943e6  | 33              | 113               |
|                             |                       | 12%                       | 16                | 7.400e4  | 12              | 51                |
|                             |                       |                           | 18                | 4.130e5  | 12              | 72                |
|                             |                       |                           | 20                | 2.330e6  | 14              | 100               |

Table 3. Influence of the number of sub-domains in B-spline collocation with domain decomposition.

| Number of collocation points | Number of sub-domains | Overlapping ratio $o/H_t$ | B-spline order $k$ | L2 error | Iteration steps | Elapsed time (sec) |
|-----------------------------|-----------------------|---------------------------|-------------------|----------|-----------------|-------------------|
| 1791                        | 3                     | 12%                       | 16                | 7.400e4  | 12              | 51                |
|                             |                       |                           | 18                | 4.130e5  | 12              | 72                |
|                             |                       |                           | 20                | 2.330e6  | 14              | 100               |
| 4                           | 12%                   | 16                        | 1.928e4           | 23       | 63              |                   |
|                             |                       |                           | 18                | 1.740e5  | 27              | 93                |
|                             |                       |                           | 20                | 5.283e6  | 33              | 132               |
| 4                           | 18%                   | 16                        | 3.450e4           | 15       | 61              |                   |
|                             |                       |                           | 18                | 1.753e5  | 18              | 86                |
|                             |                       |                           | 20                | 9.352e7  | 19              | 129               |

From the results, important features of the combination technique can be described as follows:

(1) As the amount of overlapping increases, the number of iteration steps needed in the multiplicative Schwarz method decreases. This may be understood that more overlapping requires more information exchange, thus allowing faster convergence.

(2) It can be seen that, compared with the global collocation, B-spline order is greatly reduced in the combination technique, thus avoiding ill-conditioning problem and producing increased accuracies. A decrease of B-spline order up to 7 was observed with the combination technique. For example: B-spline collocation with domain decomposition for the number of collocation points 1155, overlapping ratio 6.3% and 3 sub-domains employed the B-spline order of 16, in comparison to the global B-spline collocation of order 23.

(3) The use of more collocation points in the B-spline collocation with domain decomposition can increase the accuracy, as it is also observed in the application of global B-spline collocation. Stable approximation is thus produced with the combined techniques.
Conclusions
In this study, overlapping domain decomposition method of multiplicative Schwarz type has been combined with the global B-spline collocation method. It was shown that the combination method produced higher accuracy with the B-spline basis of much lower order than that needed in implementation of the initial method. In addition, as the approximation stability was also increased, the combined techniques are very promising for advanced problems and applications.

More variations in number of sub-domains/amount of overlapping can be implemented. Other domain decomposition methods are also to be investigated, which are subjects of further publications.

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