A Note on Operator-Theoretic Approach
to Classic Boundary Value Problems
for Harmonic and Analytic Functions
in the Complex Plane Domain

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August 30, 2009

Abstract
General spectral boundary value problems framework is utilized to restate Poincaré, Hilbert, and Riemann problems for harmonic and analytic functions in the abstract operator-theoretic setting.

Introduction
The last several years have witnessed increased interest revealed by the mathematical community to the abstract operator-theoretic methods in applications to spectral boundary value problems for differential operators and operator matrices. It is sufficient to point out numerous recently published works 1 2 3 4 5 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 along with their extensive bibliographies in order to appreciate the potential and vitality of emerging concepts and approaches. The general theory has been successfully applied to boundary value problems for general elliptic partial differential operators of even order in bounded Lipschitz domains, for nonselfadjoint (2 × 2)-block operator matrices acting in $L^2(0, 1) \times L^2(0, 1)$ known as Hain-Lüst operators, for additive perturbations of multiplication operators and some other cases inspired by the theory of elliptic partial differential operators.

The presented paper is an attempt to embrace the study of boundary value problems of complex analysis by the general operator theoretic framework. We follow the line of reasoning developed in 26 27 28 and hope to demonstrate utility of the abstract technique in formulating classic problems of Poincaré, Hilbert, and Riemann for harmonic and analytic functions in the bounded simply connected and sufficiently smooth domain of the complex plane. Keeping this goal in mind no attempt is made to report any function analytic results on solvability and properties of solutions of these problems. For the comprehensive treatment (at least in the classical settings) the interested reader is referred to the authoritative resources 6 14 22 23, where all the details can be found.

The paper consists of two sections. After recollecting relevant definitions and statements from 26 27 28 we apply obtained results to the Laplace operator on the plane domain. Then by appropriate choice of boundary conditions we arrive at the standard statements of three aforementioned problems of complex analysis.

As usual, $\mathbb{R}$, $\mathbb{C}$ are the sets of real and complex numbers. For two separable Hilbert spaces $H_1, H_2$ and linear operator $A$ from $H_1$ to $H_2$ the notation $A : H_1 \to H_2$ means that $A$ is defined everywhere in $H_1$ and bounded. Domain, range, and kernel of $A$ are $D(A)$, $R(A)$, and $\ker(A)$, respectively. The writing $A : f \mapsto g$ for $f \in D(A)$ is equivalent to $Af = g$. The symbol $\rho(A)$ is used for the resolvent set of $A$. If $A : H \to H$ and $\lambda \in \mathbb{C}$, then the inclusion $\lambda^{-1} \in \rho(A)$ means that the operator $I - \lambda A$ is boundedly invertible, i.e. the inverse $(I - \lambda A)^{-1}$ exists and is bounded in $H$. When discussing function theoretic concepts, the Lebesgue measure is assumed.
1 Spectral Boundary Value Problems

1.1 Spaces and operators

Let $H$ be a Hilbert space and $T : H \to H$ be a bounded linear operator. Assume $\ker(T) = \{0\}$ and denote $A_0$ the left inverse of $T$ so that

$$A_0 T f = f, \quad f \in H$$

Note that $A_0$ with domain $\mathcal{D}(A_0) = \mathcal{R}(T)$ need not be bounded, closed or even densely defined. Let $E$ be another Hilbert space and $\Pi : E \to H$ be a linear mapping with $\ker(\Pi) = \{0\}$ satisfying condition

$$\mathcal{R}(T) \cap \mathcal{R}(\Pi) = \{0\}$$

It follows that the linear set $\mathcal{R}(T) + \mathcal{R}(\Pi)$ is the direct sum $\mathcal{R}(T) \oplus \mathcal{R}(\Pi)$. Introduce linear operator $A$ in $H$ with the domain $\mathcal{D}(A) := \mathcal{R}(T) \oplus \mathcal{R}(\Pi)$ by

$$A : T f + \Pi \varphi \mapsto f, \quad f \in H, \varphi \in E$$

Obviously,

$$\ker(A) = \mathcal{R}(\Pi), \quad \mathcal{R}(A) = H, \quad A_0 = A|_{\mathcal{R}(T)}$$

Since $\ker(\Pi) = \{0\}$ there exists the left inverse $\gamma_0$ of $\Pi$ such that $\ker(\gamma_0) = \{0\}$ and $\gamma_0 \Pi \varphi = \varphi$, $\varphi \in E$

We extend the operator $\gamma_0$ from its domain $\mathcal{D}(\gamma_0) = \mathcal{R}(\Pi)$ to the linear map $\Gamma_0$ defined on $\mathcal{D}(A)$ by

$$\Gamma_0 : T f + \Pi \varphi \mapsto \varphi, \quad f \in H, \varphi \in E$$

It is clear that

$$\ker(\Gamma_0) = \mathcal{R}(T), \quad \mathcal{R}(\Gamma_0) = E, \quad \gamma_0 = \Gamma_0|_{\mathcal{R}(\Pi)}$$

1.2 Spectral boundary value problem

The spectral boundary value problem for unknown $u \in \mathcal{D}(A)$ is defined by the system of two equations

$$\begin{cases}
(A - \lambda I)u = f \\
\Gamma_0 u = \varphi
\end{cases} \tag{1.2.1}$$

where $f \in H$, $\varphi \in E$ and $\lambda \in \mathbb{C}$ is the spectral parameter. Since $T : H \to H$, the bounded inverse $(I - \lambda T)^{-1}$ exists for any $\lambda$ in a small neighborhood of $\lambda = 0$. To justify the terminology we note that in the applications below the first equation (1.2.1) is realized as the “main” equation for the operator $A$ defined in a bounded domain of the complex plane, whereas equality $\Gamma_0 u = \varphi$ plays the role of boundary condition. The operator $\Gamma_0$ is interpreted as a “boundary map” defined on $\mathcal{D}(A)$ with values in the “boundary space” $E$.

Lemma 1.1. Suppose $\lambda^{-1} \in \rho(T)$. Then

$$\ker(A - \lambda I) = \mathcal{R}((I - \lambda T)^{-1}\Pi)$$

Proof. Let $u \in \mathcal{D}(A)$. Since $u \in \mathcal{D}(A) = \mathcal{R}(T) \oplus \mathcal{R}(\Pi)$ there exist $f \in H$ and $\varphi \in E$ such that $u = T f + \Pi \varphi$. Then

$$(A - \lambda I)u = (A - \lambda I)(T f + \Pi \varphi) = f - \lambda(T f + \Pi \varphi) = (I - \lambda T)f - \lambda \Pi \varphi$$

Assuming $(A - \lambda I)u = 0$ and $\lambda^{-1} \in \rho(T)$ we obtain $f = \lambda(I - \lambda T)^{-1}\Pi \varphi$. Substitution into $u = T f + \Pi \varphi$ yields

$$u = \lambda T(I - \lambda T)^{-1}\Pi \varphi + \Pi \varphi = [I + \lambda T(I - \lambda T)^{-1}]\Pi \varphi = (I - \lambda T)^{-1}\Pi \varphi$$
To prove the inverse, put \( v = (I - \lambda T)^{-1} \Pi \varphi \) with some \( \varphi \in E \) and observe that

\[
(A - \lambda I)(I - \lambda T)^{-1} = (A - \lambda I)(I + \lambda T(I - \lambda T)^{-1})
\]
\[
= A - \lambda I + \lambda(A - \lambda I)T(I - \lambda T)^{-1}
\]
\[
= A - \lambda I + \lambda(I - \lambda T)(I - \lambda T)^{-1} = A
\]

Since \( \text{Ker}(A) = \mathcal{R}(\Pi) \),

\[
(A - \lambda I)v = (A - \lambda I)(I - \lambda T)^{-1} \Pi \varphi = \lambda \Pi \varphi = 0
\]

which completes the proof.

The following theorem describes solutions of (1.2.1) when \( \lambda^{-1} \in \rho(T) \).

**Theorem 1.2.** If \( \lambda^{-1} \in \rho(T) \), then the problem (1.2.1) is uniquely solvable for any \( f \in H \), \( \varphi \in E \) with the solution

\[ u_\lambda^{f,\varphi} = T(I - \lambda T)^{-1} f + (I - \lambda T)^{-1} \Pi \varphi \]  

(1.2.2)

**Proof.** Uniqueness of solution follows from the standard arguments. Namely, if \( u_1, u_2 \in \mathcal{D}(A) \) are two solutions, then for their difference \( u_0 = u_1 - u_2 = A f_0 + \Pi \varphi_0 \) with some \( f_0 \in \mathcal{H}, \varphi_0 \in E \) we have \( (A - \lambda I)u_0 = 0 \) and \( \Gamma_0 u_0 = 0 \). Since \( \text{Ker}(\Gamma_0) = \mathcal{R}(T) \) and \( \Gamma_0 \Pi = I \), the second identity gives \( \varphi_0 = 0 \). Then the first identity yields \( 0 = (A - \lambda I)T f_0 = (I - \lambda T)f_0 \) and the equality \( f_0 = 0 \) follows from the assumption \( \lambda^{-1} \in \rho(T) \).

Let us prove the representation (1.2.2). Due to Lemma 1.1 the term \( (I - \lambda T)^{-1} \Pi \varphi \) belongs to \( \text{Ker}(A - \lambda I) \). Thus we have

\[
(A - \lambda I)u_\lambda^{f,\varphi} = (A - \lambda I)(I - \lambda T)^{-1} f = (A - \lambda)(I - \lambda T)^{-1} T f = AT f = f
\]

The condition \( \Gamma_0 u = 0 \) is fulfilled for \( u_\lambda^{f,\varphi} \) defined by (1.2.2) due to obvious calculations

\[
\Gamma_0 u_\lambda^{f,\varphi} = \Gamma_0 (I - \lambda T)^{-1} \Pi \varphi = \Gamma_0 [I + \lambda T(I - \lambda T)^{-1}] \Pi \varphi = \Gamma_0 \Pi \varphi = \varphi
\]

where we used equality \( \text{Ker}(\Gamma_0) = \mathcal{R}(T) \). The proof is complete.

### 1.3 M-operator

Let \( \Lambda \) be a linear operator in \( E \) defined on the domain \( \mathcal{D}(\Lambda) \subseteq E \). Introduce the linear map \( \Gamma_1 \) on \( \mathcal{D}(\Gamma_1) = \mathcal{R}(\mathcal{D}) + \Pi \mathcal{D}(\Lambda) \subseteq \mathcal{D}(A) \) with the range \( \mathcal{R}(\Gamma_1) \subseteq E \) by

\[
\Gamma_1 : T f + \Pi \varphi \mapsto \Pi^* f + \Lambda \varphi, \quad f \in \mathcal{H}, \varphi \in \mathcal{D}(\Lambda)
\]

Obviously,

\[
\Gamma_1 T = \Pi^*, \quad \Gamma_1 \Pi = \Lambda
\]  

(1.3.1)

Note that \( \Gamma_1 T \) is bounded as an adjoint to the bounded operator. In applications below \( \Gamma_1 \) is realized as the “second boundary operator” complementary to \( \Gamma_0 \).

**Definition 1.3.** The M-operator is an operator-function \( M(\lambda) \) of the spectral parameter \( \lambda \) defined on \( \mathcal{D}(\Lambda) \) for \( \lambda^{-1} \in \rho(T) \) by the equality

\[
M(\lambda) \Gamma_0 u_\lambda = \Gamma_1 u_\lambda, \quad u_\lambda \in \text{Ker}(A - \lambda I) \cap \mathcal{D}(\Gamma_1)
\]

To check correctness of this definition assume \( u_\lambda \in \text{Ker}(A - \lambda I) \cap \mathcal{D}(\Gamma_1) \) and \( \lambda^{-1} \in \rho(T) \). Then according to Lemma 1.1 \( u_\lambda = (I - \lambda T)^{-1} \Pi \varphi \) where \( \Gamma_0 u_\lambda = \varphi \) with some \( \varphi \in \mathcal{D}(\Lambda) \). Therefore \( \Gamma_0 u_\lambda = 0 \) means \( \varphi = 0 \), which in turn implies \( u_\lambda = 0 \) and \( \Gamma_1 u_\lambda = 0 \).

**Theorem 1.4.** For \( \lambda^{-1} \in \rho(T) \)

\[
M(\lambda) = \Gamma_1 (I - \lambda T)^{-1} \Pi = \Lambda + \lambda \Pi^* (I - \lambda T)^{-1} \Pi, \quad \mathcal{D}(M(\lambda)) = \mathcal{D}(\Lambda)
\]
Proof. By Lemma 1.1 any \( u_\lambda \in \text{Ker}(A - \lambda I) \) has the form \( u_\lambda = (I - \lambda T)^{-1}\Pi \varphi \) with some \( \varphi \in E \). Assuming \( \varphi \in \mathcal{D}(\lambda) \) we have \( u_\lambda \in \mathcal{D}(\Gamma_1) \) and
\[
\Gamma_1 u_\lambda = \Gamma_1(I - \lambda T)^{-1}\Pi \varphi = \Gamma_1[I + \lambda T(I - \lambda T)^{-1}]\Pi \varphi = [\Gamma_1\Pi + \lambda\Gamma_1 T(I - \lambda T)^{-1}]\Pi \varphi
\]
The statement follows from equalities \( \Gamma_0 u_\lambda = \varphi, \Gamma_1 T = \Pi^*, \) and \( \Gamma_1 \Pi = \Lambda \).

**Corollary 1.5.** Values of the analytic operator-function \( M(\lambda) - M(0), \lambda^{-1} \in \rho(T) \) are bounded operators in \( E \).

### 1.4 Boundary conditions

Let \( \beta_0, \beta_1 \) be two linear operators, \( \mathcal{D}(\beta_0) \supset \mathcal{D}(\lambda) \) and \( \beta_1 : E \to E \). Consider spectral boundary value problem for unknown \( u \in \mathcal{D}(\Gamma_1) \subset \mathcal{D}(A) \) defined by
\[
\begin{aligned}
(A - \lambda I)u &= f \\
(\beta_0 \Gamma_0 + \beta_1 \Gamma_1)u &= \varphi
\end{aligned}
\tag{1.4.1}
\]
where \( f \in H, \varphi \in E \) and \( \lambda \in \mathbb{C} \) is the spectral parameter.

**Theorem 1.6.** Assume \( \lambda^{-1} \in \rho(T) \) is such that the equation
\[
[\beta_0 + \beta_1 M(\lambda)] \psi = g
\tag{1.4.2}
\]
with unknown \( \psi \in E \) is uniquely solvable for any \( g \in E \). Then the boundary value problem (1.4.1) has unique solution \( u_{\lambda}^{f,\varphi} \in \mathcal{D}(A) \) given by
\[
u_{\lambda}^{f,\varphi} = T(I - \lambda T)^{-1}f + (I - \lambda T)^{-1}\Pi \Psi_{\lambda}^{f,\varphi}
\tag{1.4.3}
\]
where \( \Psi_{\lambda}^{f,\varphi} \in E \) solves (1.4.2) with
\[
g = \varphi - \beta_1 \Pi^*(I - \lambda T)^{-1}f
\tag{1.4.4}
\]

**Remark 1.7.** Formally the left hand side of (1.4.2) is meaningful only for \( \psi \in \mathcal{D}(M(\lambda)) = \mathcal{D}(\lambda) \). However, the domain of \( \beta_0 + \beta_1 M(\lambda) \) can be wider than \( \mathcal{D}(\lambda) \), for example if the operator sum \( \beta_0 + \beta_1 \Lambda \) is bounded. Taking such possibilities into consideration the general solution to (1.4.2) is sought in the whole space \( E \).

**Proof.** Due to Lemma 1.1 the second term in (1.4.3) belongs to \( \text{Ker}(A - \lambda I) \). Therefore
\[
(A - \lambda I)u_{\lambda}^{f,\varphi} = (A - \lambda I)T(I - \lambda T)^{-1}f = f
\]
Thus the element (1.4.3) solves the first equation in (1.4.1). Let us verify fulfillment of the second equation in (1.4.1). To that end we need to calculate \( (\beta_0 \Gamma_0 + \beta_1 \Gamma_1)u_{\lambda}^{f,\varphi} \) where \( u_{\lambda}^{f,\varphi} \) is defined by (1.4.3). Assuming for the moment that \( \Psi_{\lambda}^{f,\varphi} \in \mathcal{D}(A) \) so that \( u_{\lambda}^{f,\varphi} \in \mathcal{D}(\Gamma_1) \), we have according to properties of \( \Gamma_0, \Gamma_1 \) and Theorem 1.1
\[
\begin{aligned}
\Gamma_0 u_{\lambda}^{f,\varphi} &= \Gamma_0[T(I - \lambda T)^{-1}f + (I - \lambda T)^{-1}\Pi \Psi_{\lambda}^{f,\varphi}] = \Psi_{\lambda}^{f,\varphi} \\
\Gamma_1 u_{\lambda}^{f,\varphi} &= \Gamma_1[T(I - \lambda T)^{-1}f + (I - \lambda T)^{-1}\Pi \Psi_{\lambda}^{f,\varphi}] = \Pi^*(I - \lambda T)^{-1}f + M(\lambda)\Psi_{\lambda}^{f,\varphi}
\end{aligned}
\]
Since \( \Psi_{\lambda}^{f,\varphi} \) solves (1.4.2), (1.4.4), we have
\[
(\beta_0 \Gamma_0 + \beta_1 \Gamma_1)u_{\lambda}^{f,\varphi} = (\beta_0 \Psi_{\lambda}^{f,\varphi} + \beta_1 [\Pi^*(I - \lambda T)^{-1}f + M(\lambda)\Psi_{\lambda}^{f,\varphi}]) = (\beta_0 + \beta_1 M(\lambda))\Psi_{\lambda}^{f,\varphi} + \beta_1 \Pi^*(I - \lambda T)^{-1}f
\]
\[
= \varphi - \beta_1 \Pi^*(I - \lambda T)^{-1}f + \beta_1 \Pi^*(I - \lambda T)^{-1}f = \varphi
\]
Now the condition \( u_{\lambda}^{f,\varphi} \in \mathcal{D}(\Gamma_1) \) can be relaxed by treating the expression \( \beta_0 \Gamma_0 + \beta_1 \Gamma_1 \) as an operator sum initially defined on \( \mathcal{D}(\Gamma_1) \) and then extended to its maximal domain in \( \mathcal{D}(A) \subset E \).

Calculations above show that \( u_{\lambda}^{f,\varphi} \) solves the system \((A - \lambda I)u = f, \Gamma_0 u = \Psi_{\lambda}^{f,\varphi}) \). According to the uniqueness part of Theorem 1.2 this solution is unique if equalities \( f = 0 \) and \( \Psi_{\lambda}^{f,\varphi} = 0 \) imply \( \varphi = 0 \). In turn, this implication follows from the unique solvability of (1.4.2). The proof is complete. \( \blacksquare \)
1.5 Operator node

In this subsection we discuss connections of the spectral boundary value problems \[\text{[24, 31]}\] to the theory of open systems thereby translating the setting of previous sections into alternative, in some sense more intuitive, terms. We refer the reader to the books \[\text{[12, 13, 24, 29]}\] for background information on the linear systems theory.

The collection \(\{T, \Pi, \Lambda; H, E\}\) of two Hilbert spaces and three operators introduced above defines the block operator matrix acting in the space \(H \oplus E\) and often called the operator node

\[
\mathbf{M} = \begin{pmatrix} T & \Pi \\ \Pi^* & \Lambda \end{pmatrix}
\]  

(1.5.1)

The node \(\mathbf{M}\) is associated with an open system \(\widehat{\mathbf{M}}\) defined as follows. The state and the input-output spaces of the system \(\widehat{\mathbf{M}}\) are identified with \(H, E\) respectively. The inner states of \(\widehat{\mathbf{M}}\) are realized as elements of \(H\) and are governed by the equation \((A - \lambda I)u = 0\). Elements of \(E\) represent external control and observation data sent to the input and read from the output of the system \(\widehat{\mathbf{M}}\) by the external control and observation processes. For \(\lambda^{-1} \in \rho(T)\) and \(\varphi \in E\) the control process is given as the input-state mapping \(\varphi \mapsto u_\lambda^\varphi = (I - \lambda T)^{-1}\Pi \varphi\). The state-output mapping representing the observation process is defined as \(u_\lambda^\varphi \mapsto \Gamma u_\lambda^\varphi\) assuming \(u_\lambda^\varphi \in D(\Gamma_1)\), or equivalently, \(\varphi \in D(\Lambda)\). In this model the transfer function that maps inputs into outputs coincide with the \(M\)-operator \(M(\lambda) : \varphi \mapsto \Gamma u_\lambda^\varphi\). The map \(\Lambda\) is called the feedthrough operator. The role of \(\Lambda\) becomes clear if we note that for \(\lambda = 0\) the input-output mapping reduces to the correspondence \(\varphi \mapsto \Lambda \varphi\).

The condition \((\beta_0 \Gamma_0 + \beta_1 \Gamma_1)u = \varphi\) can be interpreted as a description of the system obtained from \(\widehat{\mathbf{M}}\) by \text{“mixing”} its inputs and outputs into a new input defined by the operator sum \(\beta_0 \Gamma_0 + \beta_1 \Gamma_1\). The second term represents a feedback procedure that sends the original output \(\Gamma_1 u_\lambda^\varphi\), modified along the way by the operator \(\beta_1\), back to the system’s input. In a similar way, with a suitable choice of operators \(\alpha_0, \alpha_1\), the output can be redefined as the sum \((\alpha_0 \Gamma_0 + \alpha_1 \Gamma_1)u_\lambda^\varphi\), where \((A - \lambda)u_\lambda^\varphi = 0\) and \(\Gamma_0 u_\lambda^\varphi = \varphi\) is the input of system \(\widehat{\mathbf{M}}\). Combination of these two \text{“mixing”} operations leads to the system with the output \((\alpha_0 \Gamma_0 + \alpha_1 \Gamma_1)u_\lambda^\varphi\) where \(u_\lambda^\varphi \in \text{Ker}(A - \lambda I)\) is the state satisfying condition \((\beta_0 \Gamma_0 + \beta_1 \Gamma_1)u = \varphi\), and \(\varphi\) is considered as the input. The resulting system \(\widehat{\mathbf{N}}\) is called the fractional linear transformation of \(\widehat{\mathbf{M}}\). It is not difficult to see that the mapping

\[N(\lambda) : (\beta_0 + \beta_1 M(\lambda)) \varphi \mapsto (\alpha_0 + \alpha_1 M(\lambda)) \varphi\]

is the transfer function of \(\widehat{\mathbf{N}}\). Here \(\varphi \in D(\Lambda)\) is regarded as a parameter. If \((\beta_0 + \beta_1 M(\lambda))\) is boundedly invertible, then \(N(\lambda)\) can be written in the form of linear operator

\[N(\lambda) = (\alpha_0 + \alpha_1 M(\lambda))(\beta_0 + \beta_1 M(\lambda))^{-1}\]

In general case when \((\beta_0 + \beta_1 M(\lambda))\) is not invertible \(N(\lambda)\) is a multi-valued map, or in other terminology, a linear relation on the Hilbert space \(E \oplus E\). Trivial inputs satisfying \((\beta_0 + \beta_1 M(\lambda)) \varphi = 0\) correspond to the inner states that always exist and produce non-trivial output regardless of the input applied to the system.

Expression for the feedthrough operator \(\Theta\) of system \(\widehat{\mathbf{N}}\) is obtained by setting \(\lambda = 0\),

\[\Theta : (\beta_0 + \beta_1 \Lambda) \varphi \mapsto (\alpha_0 + \alpha_1 \Lambda) \varphi\]

Assuming \((\beta_0 + \beta_1 \Lambda)\) is invertible, \(\Theta = (\alpha_0 + \alpha_1 \Lambda)(\beta_0 + \beta_1 \Lambda)^{-1}\). Existence of both factors as well as existence of their product here and in the formula for \(N(\lambda)\) above requires further justification, especially in cases where participating operators are unbounded. The detailed discussion of relevant issues in the setting of abstract boundary value problems can be found in [28]. A brief illustration of these concepts is given below for the case of Hilbert boundary value problem for analytic functions.

2 Applications

Let \(D \subset \mathbb{C}\) be a bounded simply connected domain of the complex plane \(\mathbb{C}\) with smooth boundary \(\partial D\). Let us define the main and boundary Hilbert spaces as \(H = L^2(D)\), \(E = L^2(\partial D)\). It is well known that the
inhomogeneous boundary value problem for the Dirichlet Laplacian in $H$

$$\Delta u = f, \quad u|_{\partial D} = 0 \quad (2.0.2)$$

is uniquely solvable for any $f \in H$. Let $T : H \to H$ be the corresponding solution operator $T : f \mapsto u$ acting in $L^2(D)$. The range $R(T)$ consists of all functions from the Sobolev class $W^2_2(D)$ vanishing on the boundary [7]. Therefore $R(T)$ is dense in $L^2(D)$. Following the general schema, we define $\Pi : L^2(\partial D) \to L^2(D)$ to be the solution operator for the problem

$$\Delta u = 0, \quad u|_{\partial D} = \varphi \quad (2.0.3)$$

where $\varphi \in L^2(\partial D)$. Clearly, $\Pi$ is the operator of harmonic continuation of functions defined on $\partial D$ into the domain $D$. It is an integral operator with the kernel expressed in terms of Green’s function of the domain $D$. If $u^\varphi$ is a solution to (2.0.3) corresponding to $\varphi \in L^2(\partial D)$, then the element $\varphi$ is uniquely (in sense of $L^2(\partial D)$) recovered from $u^\varphi$ by the boundary trace mapping $\gamma_0 : u \mapsto u|_{\partial D}$. Thus $\gamma_0 \Pi = I_E$.

The solution of homogeneous problem $\Delta u = 0$ with condition $u|_{\partial D} = 0$ is trivial and therefore the equality $R(T) \cap R(\Pi) = \{0\}$ holds. Define the operator $A$ as the Laplacian with the dense domain $D(A) = R(T) + R(\Pi)$ and introduce $\Gamma_0$ on $D(\Gamma_0) = D(A)$ as the trace operator $\gamma_0$ extended as the null mapping to set $R(T) = D(A) \setminus R(\Pi)$. Denote $A_0$ the restriction of $A$ to $R(T)$. Trace properties of functions from the Sobolev class $W^2_2(D)$ imply that $A_0$ is in fact the Dirichlet Laplacian on $D(A_0) = R(T)$ and $A_0 T = I$.

Let $\Gamma_1 : u \mapsto \frac{\partial u}{\partial n}|_{\partial D}$ be the trace of the outer normal derivative of $u \in D(A)$ defined on the dense set of sufficiently smooth functions in the closure of $D$. The integral representation for $T = A_0^{-1}$ and application of the Fubini theorem show that $\Gamma_1 T = \Pi' : H \to E$, as prescribed in (1.3.1), see [28]. All components of the operator node $\Psi$ from (1.5.1) are now completely determined except for the parameter $\Lambda$ defined on domain $D(\Lambda) \subset L^2(\partial D)$. Below we give three definitions of $\Lambda$ resulting in three boundary value problems for harmonic and analytic functions in $D$. We are concerned with the equation (2.0.3) and for simplicity only the case $\lambda = 0$ of system (1.4.1) is discussed. Results for the spectral problem with any $\lambda \in \mathbb{C}$ easily follow from the abstract considerations of Section 1.

### 2.1 Poincaré problem

Definitions of operators $\Gamma_0$ and $\Gamma_1$ given above suggest the “natural” choice of $\Lambda$. Since $\Gamma_1$ maps a smooth function defined in $\overline{D}$ to the trace of its normal derivative on $\partial D$, and $\Pi$ is the operator of harmonic continuation, we have for smooth $\varphi$

$$\Gamma_1 \Pi : \varphi \mapsto \left. \frac{\partial u^\varphi}{\partial n} \right|_{\partial D}$$

where $u^\varphi$ is the solution to $Au = 0$ satisfying boundary condition $u|_{\partial D} = \varphi$. Operator $\Omega := \Gamma_1 \Pi$ is called the Dirichlet-to-Neumann map for the Laplacian $\Delta$ in $D$. It is known that $\Omega$ defined on the Sobolev class $W^1_2(\partial D)$ is selfadjoint in $L^2(\partial D)$. Let $\Lambda = \Omega$ with the domain $D(\Lambda) = W^1_2(\partial D)$.

According to Theorem 1.3 for two mappings $\beta_0 : W^1_2(\partial D) \to L^2(\partial D)$ and $\beta_1 : L^2(\partial D) \to L^2(\partial D)$, and $g \in L^2(\partial D)$ the solvability of system

$$Au = 0, \quad (\beta_0 \Gamma_0 + \beta_1 \Gamma_1) u = g \quad (2.1.1)$$

is equivalent to the solvability of

$$(\beta_0 + \beta_1 \Omega) \varphi = g. \quad (2.1.2)$$

Let $\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\gamma}$ be complex valued measurable functions on $\partial D$. Define operators $\beta_0$ and $\beta_1$ by

$$\beta_0 : \varphi \mapsto \tilde{\beta}_0 \frac{d\varphi}{ds} + \tilde{\gamma} \varphi \quad \beta_1 : \varphi \mapsto \tilde{\beta}_1 \varphi \quad (2.1.3)$$

where $\frac{d}{ds}$ is the operator of (generalized) differentiation in $L^2(\partial D)$. For sufficiently smooth $\varphi$ the harmonic function $u^\varphi = \Pi \varphi$ is continuously differentiable in the closure $\overline{D}$ and the trace of its tangential derivative $\frac{\partial u^\varphi}{\partial \tau}$ on the boundary $\partial D$ satisfies

$$\left. \frac{\partial u^\varphi}{\partial \tau} \right|_{\partial D} = \frac{d\varphi}{ds}$$
Thus the boundary condition in (2.1.1) is meaningful at least for harmonic functions \( u \in L^2(D) \) with boundary values from \( W^1_2(\partial D) \). Solvability of (2.1.1) with the choice (2.1.3) therefore is determined by the solvability of

\[
\left( \frac{d}{ds} + \tilde{\beta}_0 \Omega + \tilde{\gamma} \right) \varphi = g, \quad g \in L^2(\partial D)
\]

for unknown \( \varphi \in W^1_2(\partial D) \). Since \( \Omega \varphi = \frac{\partial u}{\partial \eta} \big|_{\partial D} \), this condition can be rendered as

\[
\tilde{\beta}_0 \frac{\partial u}{\partial \eta} \big|_{\partial D} + \tilde{\beta}_1 \frac{\partial u}{\partial n} \big|_{\partial D} + \tilde{\gamma} u \big|_{\partial D} = g, \quad g \in L^2(\partial D)
\]

(2.1.4)

for the unknown \( u \) harmonic in \( D \). When \( \tilde{\beta}_0, \tilde{\beta}_1, \tilde{\gamma} \), and \( g \) are sufficiently regular and real valued, the problem (2.1.4) reduces to the classical Poincaré’s problem for harmonic functions [23].

### 2.2 Hilbert problem

Hilbert problem in the domain \( D \) consists in seeking an analytic function \( w = u + iv \) defined in \( D \) with the real and imaginary parts \( u, v \) satisfying following condition on the boundary \( \partial D \)

\[
a(s)u(s) + b(s)v(s) = g(s),
\]

(2.2.1)

with real valued functions \( a, b, \) and \( g \). For simplicity we consider the case when \( D \) is the unit disc \( \mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \} \) in the complex plane with the boundary \( \mathbb{T} = \{ z \in \mathbb{C} \mid |z| = 1 \} \). In order to apply the general schema we need to recall some properties of Hilbert transform \( H \) acting in \( L^2(\mathbb{T}) \), see [15] [21]. The operator \( H \) is bounded in \( L^2(\mathbb{T}) \) and for real \( \varphi \in L^2(\mathbb{T}) \) the function \( \varphi + iH\varphi \) is boundary value of the function \( w = u + iv \) analytic in \( \mathbb{D} \). In other words, if \( w = u + iv \) is analytic in \( \mathbb{D} \) with real valued harmonic functions \( u, v \) and such that the trace \( \varphi = u|_\mathbb{T} \) is in \( L^2(\mathbb{T}) \), then \( \tilde{\varphi} = v|_\mathbb{T} \) is also in \( L^2(\mathbb{T}) \) and functions \( \varphi, \tilde{\varphi} \) are related by equality \( \tilde{\varphi} = H\varphi \). The function \( \tilde{\varphi} \) is called the harmonic conjugate of \( \varphi \).

Define \( \Lambda \) to be the Hilbert transform, \( \Lambda = H \). Then the boundary condition (1.4.1) results in the equation

\[
(\beta_0 + \beta_1 H)\varphi = g
\]

(2.2.2)

Let \( \beta_0 : \varphi \mapsto a\varphi, \beta_0 : \varphi \mapsto b\varphi \) be two multiplication operators by measurable functions \( a, b \) on \( \mathbb{T} \). Under additional assumption that \( a, b, \varphi, \) and \( g \) are real valued, the condition (2.2.2) corresponds to the Hilbert problem (2.2.1) for unknown function \( w = u + iv \) analytic in \( \mathbb{D} \). If \( \varphi \in L^2(\mathbb{T}) \) solves the equation

\[
a(s)\varphi(s) + b(s)(\mathcal{H}\varphi)(s) = g(s),
\]

(2.2.3)

for almost all \( s \in \mathbb{T} \) then the solution to (2.2.1) is \( w = u + iv \) with real and imaginary parts \( u = \Pi \varphi \) and \( v = \Pi H\varphi \).

In the language of open systems theory the equation (2.2.1) can be treated as redefined input of the system \( \mathfrak{R} \) corresponding to the operator node (1.5.1) with \( \Lambda = H \). As an example, consider the left hand side of (2.2.2) with \( \beta_0 = 1, \beta_1 = i \) as the input of the new system \( \mathfrak{R} \) and with \( \beta_0 = 1, \beta_1 = -i \) as the output of \( \mathfrak{R} \). Then the feedthrough operator of \( \mathfrak{R} \) is the map

\[
\Theta : (I + i\mathcal{H})\varphi \mapsto (I - i\mathcal{H})\varphi, \quad \varphi \in L^2(\mathbb{T})
\]

which can not be written in the form \( \Theta = (I - i\mathcal{H})(I + i\mathcal{H})^{-1} \) because \( I + i\mathcal{H} \) is not boundedly invertible. Property \( \mathcal{H}^2 = -I \) of the Hilbert transform yields \( (I + i\mathcal{H})(I - i\mathcal{H}) = 0 \) and therefore \( \text{Ker}(I + i\mathcal{H}) \) is not trivial. In fact, \( \text{Ker}(I + i\mathcal{H}) = \mathcal{R}(I - i\mathcal{H}) \). Thus the mapping \( \Theta \) is the linear relation on the space \( L^2(\mathbb{T}) \oplus L^2(\mathbb{T}) \). However, its restriction to the set \( (I + i\mathcal{H})\mathcal{R}(L^2(\mathbb{T})) \oplus \{ 0 \} \) where \( \mathcal{R}(L^2(\mathbb{T})) \) is the set of all real valued functions from \( L^2(\mathbb{T}) \), defines an operator \( \theta = (I - i\mathcal{H})(I + i\mathcal{H})^{-1} \). It maps boundary values of functions \( w = u + iv \) analytic in \( \mathbb{D} \) with \( u|\mathbb{T} \in \mathcal{R}(L^2(\mathbb{T})) \) to the boundary values of complex conjugate function \( \bar{w} = u - iv \). Note that the operator \( \theta \) is not linear over the field of complex numbers because \( a\theta w \neq \theta aw \) for \( w \in \mathcal{D}(\theta) \) and \( a \in \mathbb{C} \) unless \( a \) is a real number.
2.3 Riemann problem

The Riemann problem for analytic functions is another case that can be studied by means of the general theory of Section 1. Let \( D \) be the simply connected bounded domain \( D \subset \mathbb{C} \) with regular boundary \( \partial D \) and \( B, g \) be measurable complex valued functions on \( \partial D \). A pair of functions \( \Phi^\pm \) is a solution to the corresponding Riemann problem if \( \Phi^+ \) is analytic in \( D \), \( \Phi^- \) is analytic in \( \mathbb{C} \setminus \overline{D} \), non-tangential boundary values of \( \Phi^\pm \) on the contour \( \partial D \) exist almost everywhere, and

\[
\Phi^+(s) - B(s)\Phi^-(s) = g(s), \quad a.e. \ s \in \partial D \tag{2.3.1}
\]

Note that all considerations carried out in the beginning of this section for the Laplacian remain fully applicable, as \( \Phi^\pm \) are linear combinations of harmonic functions defined in their corresponding domains.

Let \( S \) be the Cauchy singular integral operator on the contour \( \partial D \) defined for \( \varphi \in L^2(\partial D) \) by

\[
S : \varphi \mapsto \Phi(s) = \frac{1}{\pi i} \int_{\partial D} \frac{\varphi(t)dt}{t-s} = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi i} \int_{|t-s|>\varepsilon} \frac{\varphi(t)dt}{t-s}
\]

Operator \( S \) is bounded in \( L^2(\partial D) \). For notational convenience denote \( D^+ = D \) and \( D^- = \mathbb{C} \setminus \overline{D} \). Two functions

\[
\Phi^\pm(z) = \frac{1}{\pi i} \int_{\partial D} \frac{\varphi(t)dt}{t-z}, \quad z \in D^\pm
\]

where \( \varphi \in L^2(\partial D) \) are analytic and possess non-tangential boundary values almost everywhere on the contour \( \partial D \)

\[
\lim_{z \to s, \ s \in D^\pm} \Phi^\pm(z) = \Phi^\pm(s), \quad a.e. \ s \in \partial D
\]

The Sokhotski-Plemelj formulae \[6 \ [14 \ 23\]

\[
\Phi^+(s) = \varphi(s) + \Phi(s), \quad \Phi^-(s) = -\varphi(s) + \Phi(s) \quad a.e. \ s \in \partial D \tag{2.3.2}
\]

and boundedness of \( S \) show that \( \Phi^\pm \in L^2(\partial D) \).

Introduce two multiplication operators \( \beta_0 : \varphi \mapsto a(s)\varphi(s) \), \( \beta_1 : \varphi \mapsto b(s)\varphi(s) \), where \( a, b \) are measurable functions of \( s \in \partial D \). Then the choice \( \Lambda : \varphi \mapsto S\varphi \) and the boundary condition from \(1.4.1\) leads to the equation for unknown \( \varphi \in L^2(\partial D) \)

\[
a(s)\varphi(s) + b(s)(S\varphi)(s) = g(s) \tag{2.3.3}
\]

Put \( a = A + B \), \( b = A - B \) with some measurable functions \( A, B \) defined on \( \partial D \). Then for \( \Phi = S\varphi \) the equation \(2.3.3\) takes the form

\[
a\varphi + bS\varphi = (A + B)\varphi + (A - B)S\varphi = A(\varphi + \Phi) - B(-\varphi + \Phi)
\]

Therefore due to \(2.3.2\) the equation \(2.3.3\) becomes

\[
A(s)\Phi^+(s) - B(s)\Phi^-(s) = g(s), \quad s \in \partial D
\]

For \( A(s) = 1 \) we arrive at the Riemann boundary value problem \(2.3.4\).

Other types of boundary value problems can be described by the equation \(2.3.3\) if we continue to treat \( a \) and \( b \) as linear operators. For example, let \( \tau : \varphi(s) \mapsto \varphi(\alpha(s)) \), \( s \in \partial D \) be the composition operator where \( \alpha(s) \) is an arbitrary one-to-one mapping of the contour \( \partial D \) onto itself with continuous derivative \( \alpha'(s) \neq 0 \). The choice \( a = Ar + B, b = Ar - B \) where two multiplication operators \( A, B \) are as above, results in the so-called shifted Riemann boundary value problem with, see \[14\] for details.

\[
A(s)\Phi^+[\alpha(s)] - B(s)\Phi^-(s) = g(s)
\]

Note in conclusion that the case \( \lambda \neq 0 \) of the general spectral problem \(1.4.1\) appears to be irrelevant for the study of analytic functions in the paper’s context. However, the spectral theory approach may prove beneficial in the study of boundary value problems for the first-order differential operators of complex analysis, most notably, Cauchy-Riemann and Beltrami operators on domains (see for example \[4\] for their definitions).
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