Employment of hyper-cycloidal oscillatory motion for finding the coefficient of rolling friction. Part 1: Theoretical model

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Abstract. The oscillation motions of physical pendula constructed with contacts between ball and cylindrical external surfaces for materializing the theoretical frictionless pivot point were applied for determining the coefficient of rolling friction (occurring as a characteristic of the relative motion between the contacting surfaces). The condition imposed for estimation of the coefficient of rolling friction consists in the requirement that the theoretical curve and the experimental data describing the decrease of angular amplitude should superpose each other. To be mentioned that specialized software for dynamic analysis characterizes the rolling friction by the internal damping of the contacting materials instead of applying the coefficient of rolling friction between the materials.

1. Introduction

The mechanisms met in applications can be categorized into three main classes [1]: linkage, cam and gear mechanisms. One of the major advantages brought by the presence of a higher pair is the significant decrease of the number of structural elements. To illustrate this affirmation, the problem of transmission of motion between two shafts with arbitrary directions is considered. This goal may be accomplished by designing a RCCC mechanism, as example [2] that contains cylindrical pairs and rotation and/or translation pairs. When for finding the solution, only fifth class pairs are preferred, the simplest structural solution includes seven binary elements joined by seven pairs of fifth class. From structural point of view, the problem is solved simpler if the higher pairs are accepted in the structure of the transmission mechanism. The rotation motion can be transmitted between two shafts of random direction by means of a higher pair of first class, as it is exposed in [3-4]. Another structural solution for motion transmission between the two shafts consists in using a first class pair of surface-surface type, met in the case of cylindrical gear mechanisms. The relative motion in a higher pair is complex and its effect is the presence of a friction torsor that has all components nonzero. For plane kinematic chains, the components of the friction torsor are: the friction force oriented along the tangent in the contact point and the moment of rolling friction, with the direction normal to the plane of motion.

Even for the plane kinematic chains, the motion between the generatrix curves of the contacting surfaces is extremely complex, depending directly on the nature of the contacting curves. As principle, the curves with best technological constraints are the circle and the straight line. In [5] is presented a technique for replacing the uphill and downhill profiles of a rotating cam with circular arcs and the effect if this substitution upon the law of motion of a translating knife-edge follower. The use only of
circles and lines as generatrix curves for the contacting surfaces substantially diminishes the possible relative motions between these. Practically, when one of the curves is maintained immobile, the points from the other one may describe five types of curves, resumed in [6]. One of the feasible situations is the outer contact between a fixed circle and a mobile circle that rolls without sliding over the first. The points attached to the mobile circle describe a family of hyper-cycloids whose shapes depend on the relative position of the point with respect to the mobile circle. In a series of recent works, [7-9] different types of pendula for which the contacting surfaces are combinations of cylindrical and/or plane surfaces were studied, with the objective of finding the value of the rolling friction torque. All these models are based on the principle of equality between the experimental damping and the theoretical decrease of amplitude.

2. The hyper-cycloidal pendulum. Characteristics and equation of motion

The hyper-cycloidal pendulum is presented in figure 1. A mobile circle (r) of r radius is brought into contact with a fixed circle (R) of R radius, on the outer side.

![Figure 1. The hyper-cycloidal pendulum.](image1)

![Figure 2. Family of hyper-cycloids.](image2)

The position of the mobile circle is characterized using a Cartesian coordinate system, attached to the circle, with the origin in the center of it. The O'x' axis passes through the point C - the center of mass of the pendulum, positioned at a distance ξ with respect to the center of the mobile circle. For different ξ values, a family of hyper-cycloids can be generated, as in figure 2. The position of the mobile circle is described by two parameters: the angle θ of the vector radius of the center of the mobile circle with respect to the fixed frame and the angle φ made by the axes of the mobile system with the axes of the fixed frame. To conclude, for the general case, the pendulum has two degrees of freedom. Concerning the external forces, the single force is the own weight G. The torsor of the reactions in the point C contains the normal reaction N oriented along the radius of the contact point, the friction force T, tangent to the two circles in the contact point and the moment of rolling friction, normal to the plane of motion and oriented as to oppose to the relative rotation between the two circles. The versors of the mobile frame are denoted by i', j' and the versor of the vector radius of the contact point is u, as seen in figure 1. In order to establish the motion of the pendulum, there are applied: the theorem of motion of the center of mass:

\[ \vec{M}_G = \vec{G} + \vec{N} + \vec{T} \]  \hspace{1cm} (1)
and the moment of momentum theorem with respect to the center of mass:

$$J_z\ddot{\theta}k = \frac{GC}{x(N + T)} + \tilde{M}_r k$$  \hspace{1cm} (2)

From the equations (1) and (2) it results two and one projection equations, respectively and thus, three scalar equations are available. The unknowns of the problem are the angles $\theta$ and $\varphi$, necessary for the characterization of motion and the values of the components of the reaction torsor in the contact point $C$, denoted $N$, $T$ and $M_r$. So, two supplementary scalar equations are needed. One of these equations is get assuming that the relation between the moment of rolling friction and the normal reaction $N$ is known. Though recent researches, based on the theory of elasticity [10] demonstrate a power law dependency, for the simplicity of the calculus, a linear dependence is accepted:

$$M_r = s_r N$$  \hspace{1cm} (3)

The above relation is accepted in most of the cases [11-13]. Concerning the last equation necessary for compatibility reasons, it depends on the hypothesis assumed for the relative motion between the two circles, in the point $C$. Consequently, if $v_{relc} = 0$ is considered, situation characteristic to pure rolling, there is a relation between the angles $\theta$ and $\varphi$, and the system has one degree of freedom. The force $T$ is unknown as value and the pure rolling condition must be fulfilled:

$$\frac{T}{N} < \mu_d$$  \hspace{1cm} (4)

where $\mu_d$ is the dynamic coefficient of sliding friction. On contrary case, when sliding exists between the two surfaces, the friction force attains the limit value:

$$T = \mu_d N$$  \hspace{1cm} (5)

and there is no connection between the two angles $\theta$ and $\varphi$ any more.

3. **Deducing the equation of motion of the pendulum for the pure rolling case**

With the circle $\Gamma$ considered immobile, the condition of pure rolling between the two circles is expressed by cancelling the velocity of the point of contact $C$ from the mobile circle:

$$v_{C'} = \dot{k} \times (R + r)u + \varphi k \times (-ru) = 0$$  \hspace{1cm} (6)

From equation (6) it results:

$$\dot{\theta} = \dot{\varphi} \{ r/(R + r) \}$$  \hspace{1cm} (7)

The equation (7) is integrated with the hypothesis that at the initial moment the orientations of the two axes were identical and it results:

$$\theta(\varphi) = r\varphi/(R + r),$$  \hspace{1cm} (8)

that is the equation for the relationship between the two angles. The parametric equations of the trajectories of the center of mass of the pendulum are expressed by:

$$\begin{cases} x = (R + r)\cos \theta(\varphi) + \xi \cos \theta \\ y = (R + r)\sin \theta(\varphi) + \xi \sin \theta \end{cases}$$  \hspace{1cm} (9)

For different values of the parameter $\xi$ the equations (9) represent a family of hyper-cycloids. By differentiating twice with respect to time the relations (9) the components of the acceleration of the center of mass with respect to the fixed system are obtained:
\[ a_{Gx} = \left[ \sin \theta + \frac{\xi}{r} \sin \left( \frac{r+R}{r} \theta \right) \right] \ddot{\theta}(R+r) - \left[ \cos \theta + \frac{\xi}{r} \cos \left( \frac{r+R}{r} \theta \right) \right] \dot{\theta}^2(R+r) \] (10a)

\[ a_{Gy} = \left[ \cos \theta + \frac{\xi}{r} \cos \left( \frac{r+R}{r} \theta \right) \right] \ddot{\theta}(R+r) - \left[ \sin \theta + \frac{\xi}{r} \sin \left( \frac{r+R}{r} \theta \right) \right] \dot{\theta}^2(R+r) \] (10b)

In order to apply the equation (1), the vectors from the left member of the vector equation (1) are expressed via the projections on the axes of the fixed frame:

\[
G + N + T = \begin{bmatrix}
-Mg & \cos \theta & \sin \theta \\
0 & \sin \theta & -\cos(\theta) \\
0 & 0 & 0
\end{bmatrix}
\] (11)

So as to apply the moment of momentum theorem (2), in addition to vector expressions of the normal force \( N \) and friction force \( T \), the expression of the vector \( \overline{GC} \) that specifies the position of the contact point \( C \) with respect to the centre of mass of the pendulum \( G \), is necessary:

\[
\overline{GC} = r_C - r_G = \begin{bmatrix}
R \cos \theta \\
(R+r) \cos \theta + \xi \cos[(R+r)\theta / r] \\
R \sin \theta \\
(R+r) \sin \theta + \xi \sin[(R+r)\theta / r] \\
0 \\
0
\end{bmatrix}
\] (12)

together with the expression of the vector of rolling friction torque:

\[
M_r = -k \cdot s \cdot N \cdot \text{sgn}(\dot{\theta})
\] (13)

where \( \text{sgn}(x) \) represents the signum function. The equation (13) shows that the moment of friction is proportional to the value of the normal reaction \( N \) and always has opposite sense to the vector of angular velocity. The above relations permit finding the differential equation of motion of the pendulum:

\[
\ddot{\theta} = \frac{\left( J_z + Mr^2 \frac{\xi}{r} s \cdot \text{sgn}(\dot{\theta}) \right) + \frac{\xi}{r} R \sin \theta}{\left( J_z + Mr^2 \frac{\xi}{r} s \cdot \text{sgn}(\dot{\theta}) \right) + \frac{\xi}{r} R \sin \theta - s \cdot \text{sgn}(\dot{\theta}) \cos \theta} \cdot \frac{R \theta}{r} + \frac{\xi}{r} R \sin \theta \right) \dot{\theta}^2 + \frac{\xi}{r} \left( R + r \right) \dot{\theta}^2 \\
+ \frac{\left( J_z + Mr^2 \frac{\xi}{r} s \cdot \text{sgn}(\dot{\theta}) \right) + \frac{\xi}{r} R \sin \theta}{\left( J_z + Mr^2 \frac{\xi}{r} s \cdot \text{sgn}(\dot{\theta}) \right) + \frac{\xi}{r} R \sin \theta - s \cdot \text{sgn}(\dot{\theta}) \cos \theta} \cdot \frac{R \theta}{r} + \frac{\xi}{r} R \sin \theta \right) \dot{\theta}^2 + \frac{\xi}{r} \left( R + r \right) \dot{\theta}^2 \\

Mg
\] (14)

The equation (14) is a nonlinear differential equation. After the numerical integration of the equation for stipulated initial conditions, the friction force \( T \) and the normal reaction \( N \) must be found, since they are necessary for confirming the pure rolling condition (4).

\[
T = -M(R+r) \left[ 1 + \frac{\xi}{r} \cos \left( \frac{R \theta}{r} \right) \right] \dot{\theta} + M(R+r) \frac{\xi}{r^2} \dot{\theta}^2 \sin \left( \frac{R \theta}{r} \right) + Mg \sin \theta
\] (15)
In [14-15] it is underlined that for the case of damped oscillations under dry friction condition, a condition necessary for validation of the obtained equation is the linear decrease of amplitude. To test the fulfillment of this stipulation, the equation (14) was integrated numerically using the Runge-Kutta IV, considering that the pendulum is launched from resting at an initial angular amplitude. For the parameters occurring in equation (14) a set of values of the parameters were considered:

\[ R = 0.5 \text{ m}; r = 0.1 \text{ m}; \xi = -0.033 \text{ m}; M = 36.757 \text{ Kg}; J_2 = 0.461 \text{ Kg} \cdot \text{m}^2; s_r = 5 \mu \text{m}. \]

In figures 3 and 4 there are presented the variation of the angular elongation and the variation of the coefficient of static friction for two values of initial amplitude. From these figures it can be observed that for small oscillations, the ratio \( T / N \) takes values lower than 0.2 that is the limit value of the coefficient of sliding friction and as consequence, the angular amplitude presents linear decrease. When the launching amplitude has appreciable values, in the initial phase of the motion the pure rolling condition is disobeyed and as result, the angular amplitude presents a nonlinear diminishment. An interesting aspect is that towards the end of the motion, the pure rolling condition is satisfied and therefore the decrease of amplitude becomes linear. This aspect could be used in estimating the limit value of the dynamic coefficient of sliding friction.

**Figure 3.** Variation of angular elongation (a) and variation of the coefficient of static friction (b) for the initial amplitude \( \theta_0 = 5^\circ \).

**Figure 4.** Variation of angular elongation (a) and of the coefficient of static friction (b) for \( \theta_0 = 18^\circ \).

4. Conclusions

The paper presents a method and the related device used for establishing the coefficient of rolling friction. The instrument is a pendulum with external cylindrical surface as contact surface. It contacts a fixed cylindrical surface and thus the two cylinders make contact over a generatrix. When there is no sliding between the two cylindrical bodies, the points from the mobile one describe a family of hyperr-
cycloids and from hereby the name of the technique. The fundamental theorems of vector mechanics are applied under the assumption of pure rolling conditions and the differential characteristic equation of motion of the pendulum is obtained. Once this law established, the characteristics of the reaction torsor in the contact between the two cylinders can be found. The normal reaction force and the tangential one permit testing if the condition of pure rolling between the two cylinders is valid. For a set of initial parameters, the equation of motion was numerically integrated and a linear damping of the amplitude of the pendulum was observed. The condition that the theoretical model and the experimental pendulum should have identical decreases of angular amplitude is the base of finding the coefficient of rolling friction. The comparison between the results of the theoretical model and the experimental data are the subject of the second part of the paper.

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