Further results on angular equivalence of norms

Eder Kikianty

Abstract

Angular equivalence of norms is introduced by Kikianty and Sinnamon (2017) and is a stronger notion than the usual topological equivalence. Given two angularly equivalent norms, if one norm has a certain geometrical property, e.g. uniform convexity, then the other norm also possesses such a property. In this paper, we show further results in this direction, namely angular equivalent norms share the property of uniform non-squareness, and that angular equivalence preserves the exposed points of the unit ball. A discussion on the (equivalence of the) dual norms of angularly equivalent norms is also given, giving a partial answer to an open problem as stated in the paper by Kikianty and Sinnamon (2017).

1 Introduction

In the paper [5], a new notion of norm equivalence, namely angular equivalence, is introduced. Two norms are angularly equivalent on a real vector space, if over all pairs of nonzero vectors, the angle of the pair with respect to one norm is comparable to the angle of the same pair with respect to the other norm. Any two norms that are angularly equivalent are also topologically equivalent. Angular equivalence preserves certain properties, e.g. uniform convexity, that the usual equivalence does not.

One needs a concept of angle in normed space to define such an equivalence. In a real normed space \((X, \|\cdot\|)\), the mapping \(g^\pm : X \times X \to \mathbb{R}\) given by

\[
  g^\pm(x, y) := \|x\| \lim_{t \to 0^\pm} \frac{1}{t} (\|x + ty\| - \|x\|)
\]

exists. The \(g\)-functional relative to \(\|\cdot\|\) is defined as the map \(g : X \times X \to \mathbb{R}\) given by

\[
  g(x, y) := \frac{1}{2}(g^+(x, y) + g^-(x, y)), \quad x, y \in X.
\]

We note that \(g\) is not symmetric in general. If \(x\) and \(y\) are non-zero vectors in \(X\), the norm angle from \(x\) to \(y\) is \(\theta = \theta(x, y)\), defined by \(0 \leq \theta \leq \pi\) and

\[
  \cos \theta(x, y) = \frac{g(x, y)}{\|x\|\|y\|}.
\]

With this norm angle, angular equivalence is defined as follows.
Definition 1 (Kikianty and Sinnamon [5]). Two norms $\| \cdot \|_1$ and $\| \cdot \|_2$, on a real vector space $X$ are angularly equivalent provided there exists a constant $C$ such that for all non-zero $x, y \in X$

$$\tan \left( \frac{\theta_2(x, y)}{2} \right) \leq C \tan \left( \frac{\theta_1(x, y)}{2} \right).$$

Here $\theta_1(x, y)$ and $\theta_2(x, y)$ are the norm angles from $x$ to $y$ relative to $\| \cdot \|_1$ and $\| \cdot \|_2$, respectively. Also $\tan(\pi/2)$ is taken to be $+\infty$.

It is straightforward to see that angular equivalence is both reflexive and transitive. Despite appearances, angular equivalence is a symmetric relation (cf. [5, p. 944]) and thus it is an equivalence relation. In what follows, we recall some results concerning angular equivalence, specifically the preservation of geometrical properties by this equivalence. For further results, we refer the readers to the paper [5].

Proposition 2 (Kikianty and Sinnamon [5]). Let $\| \cdot \|_1$ and $\| \cdot \|_2$ be two angularly equivalent norms on the real vector space $X$. Then, the following statements are true.

(AE1) Both norms $\| \cdot \|_1$ and $\| \cdot \|_2$ are topologically equivalent.

(AE2) The norm $\| \cdot \|_1$ is induced by an inner product if and only if $\| \cdot \|_2$ is induced by an inner product.

(AE3) For $0 \neq x \in X$, then $x/\|x\|_1$ is an extreme point of $B_{(X, \| \cdot \|_1)}$ if and only if $x/\|x\|_2$ is an extreme point of $B_{(X, \| \cdot \|_2)}$.

(AE4) The space $(X, \| \cdot \|_1)$ is strictly convex (uniformly convex), if and only if $(X, \| \cdot \|_2)$ is strictly convex (uniformly convex).

(AE5) If $p, q \in [1, \infty]$ and $n \in \mathbb{N}$ with $n \geq 2$, then the $\ell^p$ and $\ell^q$ norms on $\mathbb{R}^n$ are angularly equivalent, if and only if $p \neq q$.

In this paper, we further showcase how angular equivalent norms share other geometrical properties, similar to results (AE3) and (AE4) in Proposition 2. In Section 3, we see that angular equivalence also preserves uniform non-squareness, and in Section 4 we also show that angular equivalence preserve exposed points of a unit ball. In [5], a counter example is given to the following question: If $X$ is a real normed spaces with two angularly equivalent norms $\| \cdot \|_1$ and $\| \cdot \|_2$, are their dual norms $\| \cdot \|_1^*$ and $\| \cdot \|_2^*$ equivalent on $X^*$? In Section 5 extra conditions to the underlying space $X$ are given to obtain an affirmative answer, namely strict convexity, smoothness, and reflexivity.

2 Preliminary

Let $(X, \| \cdot \|)$ be a normed space. Throughout the paper, we use the standard notation of $S_X$ and $B_X$ for the unit sphere and unit ball, respectively, of the normed space $X$. Let $x_0 \in X$. The one-sided Gâteaux derivatives

$$G^\pm(x_0, y) = \lim_{t \to 0^\pm} \frac{1}{t} (\|x_0 + ty\| - \|x_0\|).$$
exist for all $y \in X$ \cite[Lemma 5.4.14]{Megginson}. Furthermore, Lemma 5.4.14 of Megginson \cite{Megginson} also gives the result that $G^\pm$ is sub-(super-)additive with respect to the second argument, as summarised in the following proposition.

**Proposition 3.** Let $(X, \|\cdot\|)$ be a normed space. For any $x, y, z \in X$, we have

$$G^+(x, y + z) \leq G^+(x, z) + G^+(y, z)$$

and

$$G^-(x, y + z) \geq G^-(x, z) + G^-(y, z).$$

Let $(X, \|\cdot\|)$ be a real normed space. For any $x, y \in X$,

$$g^\pm(x, y) = \|x\| \lim_{t \to 0^+} \frac{1}{t} (\|x + ty\| - \|x\|) = \|x\| G^\pm(x, y).$$

We recall the following result (see \cite[Lemma 1]{Megginson}) which readily follows from the definition of the mapping $g^\pm$.

**Proposition 4.** Let $(X, \|\cdot\|)$ be a real normed space. For any $x, y \in X$, we have the following inequality

$$-\|x\| \|y\| \leq \|x\| (\|x\| - \|x - y\|) \leq g^-(x, y) \leq g^+(x, y) \leq \|x\| (\|x + y\| - \|x\|) \leq \|x\| \|y\|.$$
Proposition 7 (Dragomir [1], Proposition 4, p. 21). Let \((X, \|\cdot\|)\) be a normed space. Then \(X\) is smooth if and only if there exists a unique semi-inner product which generates \(\|\cdot\|\).

Let \((X, \|\cdot\|)\) be a real normed space. Recall that the \(g\)-functional relative to \(\|\cdot\|\) is the map \(g: X \times X \to \mathbb{R}\) given by
\[
g(x, y) = \frac{1}{2}(g^+(x, y) + g^-(x, y)), \quad x, y \in X.
\]
We are in a position to specify the construction of a (unique) semi-inner product, using the \(g\)-functional relative to \(\|\cdot\|\), which generates \(\|\cdot\|\).

Proposition 8. Let \((X, \|\cdot\|)\) be a real normed space. Define \([\cdot, \cdot]: X \times X \to \mathbb{R}\) by
\[
[y, x] := g(x, y), \quad \text{for all } x, y \in X.
\]
Then,
(i) \([\cdot, \cdot]\) satisfies properties (S2)-(S5) of Definition 5;
(ii) If \(X\) is smooth, then \([\cdot, \cdot]\) is the unique semi-inner product on \(X \times X\).

Proof. First we note that \(g(x, x) = \|x\|^2\) for all \(x \in X\). We omit the proof of (i), as the proof for (S2), (S3), and (S5) readily follows from the definition of \(g\), and (S4) follows from Proposition 4. We prove (ii). First, note that since \(X\) is assumed to be smooth, then \(g^+ \equiv g^-\), i.e.
\[
g(x, y) = \|x\| \lim_{t \to 0} \frac{1}{t}(\|x + ty\| - \|x\|), \quad \text{for all } x, y \in X.
\]
By Proposition 3, \(g^\pm\) is also sub-(super-)additive with respect to the second argument, and thus
\[
g^-(x, y) + g^-(x, z) \leq g^-(x, y + z) = g(x, y + z) = g^+(x, y + z) \leq g^+(x, y) + g^+(x, z)
\]
and since \(g^+ \equiv g^-\), we get equality, and therefore,
\[
[y + z, x] = g(x, y + z) = g(x, y) + g(x, z) = [y, x] + [z, x].
\]
This shows (S1) of Definition 5 and together with (i), we conclude that \([\cdot, \cdot]\) is a semi-inner product which generates \(\|\cdot\|\). Uniqueness follows from Proposition 7.

Example 9. [Miličić [8], p. 72] From Proposition 8 we note that the smoothness of the normed space implies the linearity of the \(g\)-functional (in the second argument). Let \(x = (x_i), y = (y_i) \in \ell^p\) with \(1 < p < \infty\). The functional
\[
[y, x]_{\ell^p} = g_{\ell^p}(x, y) = \begin{cases} \|x\|_{\ell^p}^{2-p} \sum_i |x_i|^{p-1} \text{sgn}(x_i) y_i, & x \neq 0; \\ 0, & x = 0; \end{cases}
\]
is the unique semi-inner product on \(\ell^p \times \ell^p\). We note that
\[
g_{\ell^1}(x, y) = \|x\|_{\ell^1} \sum_i \text{sgn}(x_i) y_i
\]
is linear in the second argument, and thus is a semi-inner product on \(\ell^1 \times \ell^1\), although the space is not smooth.
3 Uniform non-squareness

Let $(X, \|\cdot\|)$ be a normed space. Recall that $X$ is said to be uniformly convex if for all $\varepsilon \in (0, 2)$ there exists $\delta \in (0, 1)$ such that the following holds:

$$ \text{if } x, y \in S_X \text{ with } \|x - y\| \geq \varepsilon, \text{ then } \left\| \frac{x + y}{2} \right\| \leq 1 - \delta. $$

The notion of uniform non-squareness is introduced by James [3] as a weaker form of uniform convexity. In particular, James showed that a Banach space is reflexive provided that the unit ball is uniformly non-square and thus it gave a refinement to the implication of reflexivity by uniform convexity, that is,

Uniform convexity $\Rightarrow$ Uniform non squareness $\Rightarrow$ Reflexivity.

**Definition 10.** Let $(X, \|\cdot\|)$ be a normed space. The space $X$ is said to be uniformly non-square if there exists $\delta \in (0, 1)$ such that

$$ \text{if } x, y \in S_X \text{ with } \left\| \frac{x - y}{2} \right\| \geq 1 - \delta, \text{ then } \left\| \frac{x + y}{2} \right\| \leq 1 - \delta. $$

**Remark 11.**

1. Definition 10 is rewritten from its original definition in [3].
2. In $\mathbb{R}^2$, if $1 < \lambda < \sqrt{2}$, then the norm $\|\cdot\|_\lambda$ defined by

$$ \| (x, y) \|_\lambda := \max \left\{ (x^2 + y^2)^{\frac{1}{2}}, \lambda \max \{|x|, |y|\} \right\}, \ (x, y) \in \mathbb{R}^2, $$

is uniformly non-square but not strictly convex (hence, not uniformly convex). This example is due to Kato and Takahashi [4, p. 1058].

Our aim is to show that uniform non-squareness is shared by angularly equivalent norms. We start with two lemmas which provide characterisations of uniform non-squareness using norm angles. We follow the main idea of the proof of Theorem 2.6 of [5].

**Lemma 12.** Let $(X, \|\cdot\|)$ be a normed space. Then $X$ is uniformly non-square if and only if there exists $\delta \in (0, 1)$ such that the following holds:

$$ \text{if } x, y \in S_X \text{ with } \left\| \frac{x - y}{2} \right\| \geq 1 - \delta, \text{ then } \tan \left( \frac{\theta(x, y)}{2} \right) \geq \sqrt{\delta}. $$

**Proof.** Assume that $X$ is uniformly non-square, i.e. there exists $\eta \in (0, 1)$ such that

$$ \text{if } x, y \in S_X \text{ with } \left\| \frac{x - y}{2} \right\| \geq 1 - \eta, \text{ then } \left\| \frac{x + y}{2} \right\| \leq 1 - \eta. $$

Set $\delta := \eta$. Let $x, y \in S_X$ with $\left\| \frac{x - y}{2} \right\| \geq 1 - \delta = 1 - \eta$. Since $x, y \in S_X$, we have the following inequality

$$ -1 \leq 1 - \|x - y\| \leq g(x, y) \leq \|x + y\| - 1 \leq 1,$$
from Proposition 4. Therefore, we have $1 + g(x, y) \leq 2$ and $1 - g(x, y) \geq 2 - \|x + y\|$. Then,

$$
\tan \left( \frac{\theta(x, y)}{2} \right) \geq \sqrt{\frac{1 - g(x, y)}{1 + g(x, y)}} \\
\geq \sqrt{\frac{1 - g(x, y)}{2}} \geq \sqrt{1 - \|x + y\|/2} \geq \sqrt{\eta} = \sqrt{\delta}.
$$

Conversely, assume there exists $\eta \in (0, 1)$ such that

if $x, y \in S_X$ with $\left\| \frac{x - y}{2} \right\| \geq 1 - \eta$, then $\tan \left( \frac{\theta(x, y)}{2} \right) \geq \sqrt{\eta}.$

Choose $\delta := \min\{\frac{\eta}{2}, \frac{\eta}{1+\eta}\} > 0$. Let $x, y \in S_X$ with $\left\| \frac{x - y}{2} \right\| \geq 1 - \delta \geq 1 - \eta$, since $\delta \leq \frac{\eta}{2} < \eta$. If $\|x + y\| = 0$, then $\left\| \frac{x + y}{2} \right\| = 0 \leq 1 - \delta$. We consider the case $\|x + y\| \neq 0$. Now,

$$
\left\| (2 - \|x + y\|)x - \|x + y\| \left( \frac{x + y}{\|x + y\|} - x \right) \right\| \\
= \|2x - \|x + y\| + \|x + y\| x - x + y\| = \|x - y\| \geq 2(1 - \delta).
$$

Thus, either

$$
\| (2 - \|x + y\|)x \| \geq 2\delta
$$

or

$$
\left\| \|x + y\| \left( \frac{x + y}{\|x + y\|} - x \right) \right\| \geq 2(1 - \delta) - 2\delta = 2 - 4\delta,
$$

which follows from the triangle inequality. In the first case, we have

$$
2 - \|x + y\| = \| (2 - \|x + y\|)x \| \geq 2\delta
$$

that is

$$
\left\| \frac{x + y}{2} \right\| \leq 1 - \delta,
$$

and we are done. In the second case, we have

$$
\left\| \frac{x + y}{\|x + y\|} - x \right\| \geq \frac{2 - 4\delta}{\|x + y\|} \geq 1 - 2\delta \geq 1 - \eta
$$

by our choice of $\delta \leq \frac{\eta}{2}$. Therefore, by our assumption,

$$
\sqrt{\eta} \leq \tan \left( \frac{\theta(x, y)}{2} \right) = \sqrt{\frac{1 - g\left( \frac{x + y}{\|x + y\|}, x \right)}{1 + g\left( \frac{x + y}{\|x + y\|}, x \right)}},
$$

and by rearranging we obtain

$$
g \left( \frac{x + y}{\|x + y\|}, x \right) \leq \frac{1 - \eta}{1 + \eta}.$$
By Proposition 4 with $x + y$ and $x$, we have

$$
\|x + y\| - 1 \leq \frac{g(x + y, x)}{\|x + y\|}
$$

and thus

$$
\left\| \frac{x + y}{2} \right\| \leq \frac{1}{2} \left( 1 + \frac{g(x + y, x)}{\|x + y\|} \right) \leq \frac{1}{2} \left( 1 + \frac{1 - \eta}{1 + \eta} \right) = \frac{1}{1 + \eta} = 1 - \frac{\eta}{1 + \eta} \leq 1 - \delta
$$

as we choose $\delta \leq \frac{\eta}{1 + \eta}$. This completes the proof. □

**Lemma 13.** Let $(X, \|\cdot\|)$ be a normed space. Then the following are equivalent.

(i) $X$ is uniformly nonsquare.

(ii) there exists $\delta \in (0, 1)$ such that the following holds:

if $x, y \in S_X$ with $\left\| \frac{x - y}{2} \right\| \geq 1 - \delta$, then $\tan \left( \frac{\theta(x, y)}{2} \right) \geq \sqrt{\delta}$.

(iii) there exists $\varepsilon \in (0, 2)$ and $\delta \in (0, 1)$ such that the following holds:

if $x, y \in S_X$ with $\|x - y\| \geq \varepsilon$, then $\tan \left( \frac{\theta(x, y)}{2} \right) \geq \delta$.

**Proof.** The equivalence of (i) and (ii) follows from Lemma 12. We show that (ii) and (iii) are equivalent. Assume that there exists $\eta \in (0, 1)$ such that the following holds:

if $x, y \in S_X$ with $\left\| \frac{x - y}{2} \right\| \geq 1 - \eta$, then $\tan \left( \frac{\theta(x, y)}{2} \right) \geq \sqrt{\eta}$.

Set $\varepsilon := 2(1 - \eta) > 0$ and $\delta := \sqrt{\eta} > 0$. Let $x, y \in S_X$ be such that $\|x - y\| \geq \varepsilon$. Thus, $\|x - y\| \geq 2(1 - \eta)$, that is $\left\| \frac{x - y}{2} \right\| \geq 1 - \eta$. By assumption,

$$
\tan \left( \frac{\theta(x, y)}{2} \right) \geq \sqrt{\eta} = \delta.
$$

Now we assume that there exists $\varepsilon \in (0, 2)$ and $\eta \in (0, 1)$ such that the following holds:

if $x, y \in S_X$ with $\|x - y\| \geq \varepsilon$, then $\tan \left( \frac{\theta(x, y)}{2} \right) \geq \eta$.

Set $\delta := \min\{1 - \frac{\varepsilon}{2}, \eta^2\} > 0$. Let $x, y \in S_X$ be such that $\left\| \frac{x - y}{2} \right\| \geq 1 - \delta$. Thus, by our choice of $\delta \leq 1 - \frac{\varepsilon}{2}$, we have

$$
\left\| \frac{x - y}{2} \right\| \geq 1 - \delta \geq \frac{\varepsilon}{2}, \text{ and so } \|x - y\| \geq \varepsilon.
$$

By assumption, we have $\tan \left( \frac{\theta(x, y)}{2} \right) \geq \eta \geq \sqrt{\delta}$, by our choice of $\delta \leq \eta^2$. □
Now we prove our main result of the section.

**Theorem 14.** Let $X$ be a real normed space with two angularly equivalent norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Then $X$ is uniformly non-square with respect to $\|\cdot\|_1$ if and only if $X$ is uniformly non-square with respect to $\|\cdot\|_2$.

**Proof.** We need to only prove one side of the implication, as the other side follows by reversing the roles of $\|\cdot\|_1$ and $\|\cdot\|_2$. Let $C > 1$ be such that

$$\tan \left( \frac{\theta_1(x, y)}{2} \right) \leq C \tan \left( \frac{\theta_2(x, y)}{2} \right)$$

for all $x, y \in X$, where $\theta_1(x, y)$ is the norm angle from $x$ to $y$ with respect to $\|\cdot\|_1$. Since angular equivalence implies norm equivalence, let $M, m > 0$ be such that

$$m \|x\|_1 \leq \|x\|_2 \leq M \|x\|_1,$$

for all $x \in X$. Let $X$ be uniformly non-square with respect to $\|\cdot\|_1$. By Lemma 13 part (iii) there exist $\nu, \eta > 0$ such that

if $x, y \in S_{(X, \|\cdot\|_1)}$ with $\|x - y\|_1 \geq \nu$, then $\tan \left( \frac{\theta_1(x, y)}{2} \right) \geq \eta.$

Set $\varepsilon := 2 \frac{M \nu}{m} > 0$ and $\delta := \frac{\eta}{C} > 0$. Let $x, y \in S_{(X, \|\cdot\|_2)}$ with $\|x - y\|_2 \geq \varepsilon$. Let $\hat{x} = \frac{x}{\|x\|_2}$ and $\hat{y} = \frac{y}{\|y\|_2}$. Note that $\|\hat{x}\|_1 = 1 = \|\hat{y}\|_1$. Also, since $x \in S_{(X, \|\cdot\|_2)}$, we have $\|\hat{x}\|_2 = \frac{\|x\|_2}{\|x\|_1} = \frac{1}{\|x\|_1}$, and thus

$$x = \|x\|_1 \hat{x} = \frac{\hat{x}}{\|\hat{x}\|_2},$$

and similarly, $y = \frac{\hat{y}}{\|\hat{y}\|_2}$.

Using Dunkl-Williams inequality, we get

$$\varepsilon \leq \|x - y\|_2 \leq \left\| \hat{x} - \hat{y} \right\|_2 \leq \frac{4 \|\hat{x} - \hat{y}\|_2}{\|\hat{x}\|_2 + \|\hat{y}\|_2} \leq \frac{4M \|\hat{x} - \hat{y}\|_1}{m \|\hat{x}\|_1 + m \|\hat{y}\|_1} = \frac{2M}{m} \|\hat{x} - \hat{y}\|_1$$

Thus,

$$\|\hat{x} - \hat{y}\|_1 \geq \frac{m \varepsilon}{2M} = \nu.$$

Therefore,

$$\eta \leq \tan \left( \frac{\theta_1(\hat{x}, \hat{y})}{2} \right) = \tan \left( \frac{\theta_1(x, y)}{2} \right) \leq C \tan \left( \frac{\theta_2(x, y)}{2} \right),$$

that is,

$$\tan \left( \frac{\theta_2(x, y)}{2} \right) \geq \frac{\eta}{C} = \delta,$$

and this completes the proof. \qed
4 Exposed points

Our aim in this section is to prove a similar result to that of Proposition 2 part (AE3), by considering exposed points instead of extreme points. First we recall the following definitions.

Definition 15. Let \((X, \|\cdot\|)\) be a real normed space and \(A\) be a subset of \(X\). A nonzero \(f \in X^*\) is a support functional for \(A\) if there is an \(x_0 \in A\) such that \(f(x_0) = \sup\{f(x) : x \in A\}\), in which case \(x_0\) is a support point of \(A\), the set \(\{x : x \in X, f(x) = f(x_0)\}\) is a support hyperplane for \(A\) and the functional \(f\) and the support hyperplane are both said to support \(A\) at \(x_0\).

Remark 16. Note that as a consequence of the Hahn-Banach theorem, for any \(x \in X\) there exists \(f \in S_{X^*}\) such that \(f(x) = \|x\|\). Also, \(f \in S_{X^*}\) supports \(B_X\) at \(x_0 \in S_X\) if and only if \(f(x_0) = 1\).

Definition 17. Let \((X, \|\cdot\|)\) be a real normed space and \(C\) be a nonempty closed convex subset of \(X\). A point \(x \in C\) is said to be an exposed point of \(C\) if there is \(f \in X^*\) such that \(f\) is bounded from above on \(C\) and attains its supremum on \(C\) at \(x\) and only at \(x\). In this case we call \(f\) an exposing functional of \(C\) and exposing \(C\) at \(x\).

Remark 18. If \(x_0\) is an exposed point of a nonempty closed convex subset \(C\) of \(X\), then it is also an extreme point. The converse is not true. For instance, the point \(A\) in Figure 1 is an extreme point that is not an exposed point of the bounded region.

![Figure 1: An extreme point that is not an exposed point](image)

We recall the following result and refer the readers to Lemma 5.4.16 from Megginson [7, p. 486] for its proof. We reformulate this for any real normed space.

Proposition 19. Let \(X\) be a real normed space, \(x_0 \in S_X\) and \(f \in S_{X^*}\). Then \(f\) supports \(B_X\) at \(x_0\) if and only if

\[
\lim_{t \to 0^-} \frac{\|x_0 + ty\| - \|x_0\|}{t} = G_-(x_0, y) \leq f(y) \leq G_+(x_0, y) = \lim_{t \to 0^+} \frac{\|x_0 + ty\| - \|x_0\|}{t}
\]

for all \(y \in X\).
Lemma 20. Let \((X, \|\cdot\|)\) be a real normed space. Then \(x_0 \in S_X\) is an exposed point of \(B_X\) if and only if \(\{y \in S_X : g(x_0, y) = 1\} = \{x_0\}\).

Proof. Let \(x_0 \in S_X\) be an exposed point of \(B_X\) with exposing functional \(f\). Thus, \(f(x_0) = 1\) and \(f(x_0) > f(y)\) for all \(y \in S_X\). By Proposition 19 we have the following inequality

\[
g^-(x_0, y) = G^-(x_0, y) \leq f(y) \leq G^+(x_0, y) = g^+(x_0, y),
\]

for all \(y \in X\). Suppose that there exists \(y_0 \in S_X\) with \(y_0 \neq x_0\) such that \(g(x_0, y_0) = 1\), i.e., \(g^-(x_0, y_0) + g^+(x_0, y_0) = 2\). Since \(f(y_0) < 1\), by assumption, we have \(g^-(x_0, y_0) \leq f(y_0) < 1\) and thus

\[
g^+(x_0, y_0) = 2 - g^-(x_0, y_0) > 1,
\]

contradicting Proposition 4. Conversely, assume that \(\{y \in S_X : g(x_0, y) = 1\} = \{x_0\}\) and suppose that \(x_0 \in S_X\) is not an exposed point of \(B_X\). Thus, if \(f \in S_X^*\) with \(f(x_0) = \|x_0\| = 1\), there exists \(y_0 \in S_X\) distinct from \(x_0\) such that \(f(y_0) = \|y_0\| = 1\). Note that for any \(t \in [0, 1]\), we have

\[
f((1-t)x_0 + ty_0) = tf(x_0) + (1-t)f(y_0) = 1.
\]

Since \(f \in S_X^*\), we have \(\|(1-t)x_0 + ty_0\| = 1\) for all \(t \in [0, 1]\). Now,

\[
g^+(x_0, y_0) = \lim_{t \to 0^+} \frac{1}{t} (\|(1-t)x_0 + ty_0\| - 1)
 = \lim_{s \to 0^+} \frac{1-s}{s} \left(\|x_0 + sy_0\| - 1\right)
 = \lim_{s \to 0^+} \frac{1}{s} \left(\|(1-s)x_0 + sy_0\| - 1 + s\right) = 1.
\]

Thus, \(g(x_0, y_0) = 1\) which contradicts the assumption. Therefore, \(x_0 \in S_X\) must be an exposed point of \(B_X\).

Theorem 21. Let \(X\) be a real normed space with two angularly equivalent \(\|\cdot\|_1\) and \(\|\cdot\|_2\). Then, \(x/\|x\|_1\) is an exposed point of \(B_{(X,\|\cdot\|_1)}\) if and only if \(x/\|x\|_2\) is an exposed point of \(B_{(X,\|\cdot\|_2)}\).

Proof. Let \(C > 0\) such that

\[
\frac{1 - \cos \theta_1(x, y)}{1 + \cos \theta_1(x, y)} \leq C \frac{1 - \cos \theta_2(x, y)}{1 + \cos \theta_2(x, y)}
\]

for all \(x, y \in X\). It is sufficient to prove one side of the implication as the reverse implication follows from swapping the roles of \(\|\cdot\|_1\) and \(\|\cdot\|_2\). We argue the contrapositive. Assume that \(x_0 \in S_{(X,\|\cdot\|_2)}\) is not an exposed point of \(B_{(X,\|\cdot\|_2)}\). By Lemma 20 there exists \(y_0 \in S_{(X,\|\cdot\|_2)}\) distinct from \(x_0\) such that \(g_2(x_0, y_0) = 1\), i.e. \(\cos \theta_2(x_0, y_0) = 1\) since \(x_0, y_0 \in S_{(X,\|\cdot\|_2)}\). Thus, by angular equivalence,

\[
\cos \theta_1(x_0, y_0) = 1
\]

that is,

\[
g_1 \left( \frac{x_0}{\|x_0\|_1}, \frac{y_0}{\|y_0\|_1} \right) = 1.
\]

By Lemma 20 again, since \(\frac{x_0}{\|x_0\|_1} \neq \frac{y_0}{\|y_0\|_1}, \frac{x_0}{\|x_0\|_1}\) is not an exposed point of \(B_{(X,\|\cdot\|_1)}\).
5 Dual norms

The following theorem is due to Giles [2, Theorem 6].

**Theorem 22** (Giles, 1967). Let \((X, \|\cdot\|)\) be a smooth and uniformly convex Banach space and \([\cdot, \cdot]\) be a semi-inner product which generates \(\|\cdot\|\). Then for all \(f \in X^*\), there exists a unique \(x \in X\) such that \(f(y) = [y, x]\) for all \(y \in X\).

One of the tools that is used in proving Theorem 22 is that every closed convex subset in a uniformly convex space is a Chebyshev set. Recall that a non-empty subset \(A\) of a metric space \((M, d)\) is a Chebyshev set if for every element \(x \in M\), there exists exactly one element \(y \in A\) such that

\[
d(x, y) = d(x, A) := \inf_{z \in A} d(x, z).
\]

However, the assumption of uniform convexity may be replaced by a weaker assumption. This result is due to MM Day (cf. [7, Corollary 5.1.19]):

**Lemma 23** (Day, 1941). If a normed space is strictly convex and reflexive, then each of its nonempty closed convex subsets is a Chebyshev set.

Recall that uniform convexity implies strict convexity and reflexivity. We prove a version of Theorem 22 by replacing uniform convexity with strict convexity and reflexivity and reformulate it in terms of the \(g\)-functional. We first state some results from [1] and [2] which are reformulated in terms of the \(g\)-functional, with the aid of Proposition 8. Recall that, from Proposition 8 when \(X\) is a smooth normed space, then the \(g\)-functional gives rise to a unique semi-inner product given by

\[
[x, y] = g(y, x), \quad x, y \in X.
\]

In a normed space \((X, \|\cdot\|)\) over the field \(\mathbb{K}\), \(x \in X\) is said to be \(B\)-orthogonal to \(y \in X\) if \(\|x + \lambda y\| \geq \|x\|\) for all \(\lambda \in \mathbb{K}\). In the usual manner, we say that \(x \in X\) is \(B\)-orthogonal to a subset \(Y \subseteq X\), if \(x\) is \(B\)-orthogonal to every \(y \in Y\). We restate the following results from [2], in terms of the \(g\)-functional, instead of a semi-inner product (via Proposition 8).

**Lemma 24** (Giles [2], Theorem 2). If \((X, \|\cdot\|)\) is smooth normed space over \(\mathbb{K}\), then \(g(x, y) = 0\) if and only if \(x\) is \(B\)-orthogonal to \(y\).

**Lemma 25** (Giles [2], Lemma 5). Let \((X, \|\cdot\|)\) be a smooth normed space over reals. Then \(X\) is strictly convex if and only if for any nonzero \(x, y \in X\), if \(g(x, y) = \|x\| \|y\|\), then \(y = \lambda x\) for some real number \(\lambda > 0\).

We now restate Theorem 6 of Giles [2] (Theorem 22 above) with a weaker assumption of strict convexity and reflexivity in place of uniform convexity.

**Theorem 26.** Let \((X, \|\cdot\|)\) be a smooth, strictly convex, and reflexive space. Then for all \(f \in X^*\), there exists a unique \(x \in X\) such that \(f(y) = g(x, y)\) for all \(y \in X\). Furthermore, \(\|f\| = \|x\|\).
Lemma 24, we get \( \|a\| = \|z\| \) for any \( z \neq 0 \).

(1) If \( z \in N \), then \( f(z) = 0 = g(x, z_0) \), for any \( x = \alpha x_0 \) with \( \alpha \in \mathbb{R} \).

(2) Observe that

\[
\begin{align*}
\|z\| & = \inf_{z \in N} \|y - z\| \\
& = \inf_{z \in N} \|y - z_0 + z - z_0\| \\
& = \inf_{z \in N} \|y - z_0\| + \|z - z_0\| \\
& = \inf_{z \in N} \|y - z_0\| + \|z - z_0\| \\
& = \inf_{z \in N} \sup_{\alpha \neq 0} \|\alpha x_0 - z\| \\
& = \inf_{z \in N} \sup_{\alpha \neq 0} \|\alpha x_0 - z\|.
\end{align*}
\]

Thus, any \( y \in X \) can be written as \( y = z_0 + x_0 \), where \( z_0 \in N \), and \( 0 \neq x_0 \in X \) is such that \( g(x_0, z) = 0 \) for all \( z \in N \). Set \( x = \frac{f(x_0)}{\|x_0\|} \). Since \( z_0 \in N \), observation (1) gives us \( f(z_0) = g(x, z_0) \) and (2) give us \( f(x_0) = g(x, x_0) \). Therefore,

\[
\begin{align*}
f(y) &= f(z_0 + x_0) \\
&= f(z_0) + f(x_0) \\
&= g(x, z_0) + g(x, x_0) = g(x, z_0 + x_0) = g(x, y).
\end{align*}
\]

To prove uniqueness, let \( x, x' \in X, x \neq x' \) such that \( f(y) = g(x, y) \) and \( f(y) = g(x', y) \) for all \( y \in X \). Then,

\[
\|x\| = \|g(x, x)\| = \|g(x', x)\| \leq \|x'\| \|x\|
\]

so \( \|x\| \leq \|x'\| \) and

\[
\|x'\| = \|g(x', x')\| = \|g(x, x')\| \leq \|x\| \|x'\|
\]

so \( \|x'\| \leq \|x\| \). Thus, \( \|x'\| = \|x\| \), and

\[
\|x\|^2 = g(x', x)
\]

gives us

\[
\|x\| \|x'\| = g(x', x)
\]

and so by Lemma 23, we conclude that \( x = \lambda x' \). Combining this with \( \|x'\| = \|x\| \), we conclude that \( x = x' \). Finally,

\[
|f(y)| = |g(x, y)| \leq \|x\| \|y\|
\]

and so

\[
\|f\| = \sup_{0 \neq y \in X} \frac{|f(y)|}{\|y\|} \leq \|x\|
\]

and

\[
\|x\|^2 = |g(x, x)| = |f(x)| \leq \|f\| \|x\|
\]

so \( \|x\| \leq \|f\| \). This completes the proof. \( \square \)
We now restate Theorem 7 of Giles [2] in terms of the \( g \)-functional.

**Corollary 27.** Let \((X, \|\|)\) be a normed space. Assume that \(X\) is smooth, strictly convex, and reflexive. Then, the dual space \(X^*\) is smooth, strictly convex, and reflexive; and the \( g \)-functional on \(X^*\), is given by

\[
g(\phi, \psi) = g(x_\phi, x_\psi), \quad \text{for any } \phi, \psi \in X^*,
\]

where \(x_\phi\) and \(x_\psi\) in \(X\) are associated to \(\phi\) and \(\psi\), respectively, as given in Theorem 26.

**Proof.** By reflexivity of \(X\), it follows that \(X^*\) is reflexive, and since \(X\) is smooth and strictly convex, \(X^*\) is smooth and strictly convex. Let \(\phi, \psi \in X^*\). By Theorem 26, there exist \(x_\phi, x_\psi \in X\) such that

\[
\phi(z) = g(x_\phi, z) \quad \text{and} \quad \psi(z) = g(x_\psi, z), \quad \text{for all } z \in X,
\]

with \(\|\phi\| = \|x_\phi\|\) and \(\|\psi\| = \|x_\psi\|\). Define \([\cdot, \cdot] : X \times X \to \mathbb{R}\) by

\[
[\phi, \psi] := g(x_\phi, x_\psi), \quad \text{for any } \phi, \psi \in X^*.
\]

It is sufficient to show that \([\cdot, \cdot]\) is a semi-inner product on \(X^*\), since smoothness of \(X^*\), implies that \([\cdot, \cdot]\) is the unique semi-inner product on \(X^*\) which in turn implies that the \( g \)-functional in \(X^*\) is given by

\[
g(\psi, \phi) = [\phi, \psi] = g(x_\phi, x_\psi), \quad \text{for any } \phi, \psi \in X^*,
\]

as desired. Let \(\phi, \psi, \tau \in X^*\) and \(\alpha, \beta \in \mathbb{R}\). Firstly we note the following,

\[
[\phi, \psi] = g(x_\phi, x_\psi) = \phi(x_\psi).
\]

Now we show that \([\cdot, \cdot]\) satisfies the properties of semi-inner product. We have

\[
[\phi + \psi, \tau] = (\phi + \psi)(x_\tau) = \phi(x_\tau) + \psi(x_\tau) = g(x_\phi, x_\tau) + g(x_\psi, x_\tau) = [\phi, \tau] + [\psi, \tau].
\]

Next, we note that for all \(z \in X\),

\[
(\alpha \phi)(z) = \alpha \phi(z) = \alpha g(x_\phi, z) = g(\alpha x_\phi, z),
\]

that is, a one-to-one correspondence between \(\alpha \phi \in X^*\) with \(\alpha x_\phi \in X\). Thus

\[
[\alpha \phi, \beta \psi] = g(\alpha x_\phi, \beta x_\psi) = \alpha \beta g(x_\phi, x_\psi) = \alpha \beta [\phi, \psi].
\]

Next, we have

\[
[\phi, \phi] = g(x_\phi, x_\phi) = \|x_\phi\|^2 = \|\phi\|^2.
\]

Thus, \([\phi, \phi] = \|\phi\|^2 \geq 0\) and \([\phi, \phi] = 0\) implies \(\|\phi\|^2 = 0\), so \(\|\phi\| = 0\). Finally,

\[
|[\phi, \psi]| = |g(x_\phi, x_\psi)| \leq \|x_\phi\| \|x_\psi\| = \|\phi\| \|\psi\|.
\]

This completes the proof. \(\square\)
Theorem 28. Let $X$ be a normed space with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ that are both strictly convex, smooth, and reflexive, and that both norms are angularly equivalent. Then, the dual norms $\|\cdot\|_1^*$ and $\|\cdot\|_2^*$ are also angularly equivalent.

Proof. Denote by $g_1$ and $g_2$, the g-functional associated to the norm $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. By the assumption of angular equivalence, there exists $C > 0$ such that

$$\frac{1 - g_2(x, y)}{1 + g_2(x, y)} \leq C \frac{1 - g_1(x, y)}{1 + g_1(x, y)},$$

for any $x, y \in X$. Take two elements $\phi$ and $\psi$ of the dual space $X^*$. By Theorem 26 there exists $x_\phi$ and $x_\psi$ in $X$ such that

$$\phi(y) = g_1(x_\phi, y) \quad \text{and} \quad \psi(y) = g_2(x_\psi, y), \quad \text{for all} \ y \in X.$$

Thus, we have

$$\frac{1 - g_2(x_\psi, x_\phi)}{1 + g_2(x_\psi, x_\phi)} \leq C \frac{1 - g_1(x_\psi, x_\phi)}{1 + g_1(x_\psi, x_\phi)}, \quad (3)$$

By Corollary 27, we have the g-functionals on $(X^*, \|\cdot\|_1^*)$ and $(X^*, \|\cdot\|_2^*)$, denoted by $g_1^*$ and $g_2^*$, are given by

$$g_i^*(\phi, \psi) = g_i(x_\psi, x_\phi), \quad i = 1, 2.$$

Consequently, (3) becomes

$$\frac{1 - g_2^*(\phi, \psi)}{1 + g_2^*(\phi, \psi)} \leq C \frac{1 - g_1^*(\phi, \psi)}{1 + g_1^*(\phi, \psi)},$$

which shows that the dual norms $\|\cdot\|_1^*$ and $\|\cdot\|_2^*$ are also angularly equivalent. \qed

6 Discussion

The assumptions of Theorem 28 are as follows.

(A1) A real vector space $X$ with two angularly equivalent norms $\|\cdot\|_1$ and $\|\cdot\|_2$.

(A2) Both $\|\cdot\|_1$ and $\|\cdot\|_2$ are strictly convex.

(A3) Both $\|\cdot\|_1$ and $\|\cdot\|_2$ are smooth.

(A4) Both $\|\cdot\|_1$ and $\|\cdot\|_2$ are reflexive.

Corollary 2.2 of [5] states that angular equivalence preserves strict convexity and thus (A2) may be weakened to only requiring one of the norms to be strictly convex. This led to the following questions:

(Q1) Does angular equivalence preserves smoothness?

(Q2) Does angular equivalence preserves reflexivity?
Note also that the statement of Theorem 28 remains true, when the assumptions (A2)-(A4) are changed to the following.

(A2*) Both $\| \cdot \|_1$ and $\| \cdot \|_2$ are uniformly convex.

(A3*) Both $\| \cdot \|_1$ and $\| \cdot \|_2$ are uniformly smooth.

By Corollary 2.7 of [5], since angular equivalence preserves uniform convexity, (A1*) may be weakened to only requiring that one of the norms to be uniformly convex. This led to the question:

(Q3) Does angular equivalence preserves uniform smoothness?

An affirmative answer to (Q1)-(Q3) will strengthen the result of Theorem 28.

References

[1] Dragomir, SS. Semi-inner products and applications. Nova Science Publishers, Inc., Hauppauge, NY, 2004.
[2] Giles, JR. Classes of semi-inner-product spaces. Trans. Amer. Math. Soc. 129 (1967), 436–446.
[3] James, RC. Uniformly non-square Banach spaces. Ann. of Math. (2) 80 (1964), 542–550.
[4] Kato, M and Takahashi, Y. On the von Neumann-Jordan constant for Banach spaces. Proc. Amer. Math. Soc. 125 (1997), no. 4, 1055–1062.
[5] Kikianty, E and Sinnamon, G. Angular equivalence of normed spaces. J. Math. Anal. Appl. 454 (2017), no. 2, 942–960.
[6] Lumer, G. Semi-inner-product spaces. Trans. Amer. Math. Soc. 100 (1961), 29–43.
[7] Megginson, RE. An introduction to Banach space theory, Graduate Texts in Mathematics 183, Springer-Verlag, New York, 1998.
[8] Miličić, PM. A generalization of the parallelogram equality in normed spaces. J. Math. Kyoto Univ. 38 (1998), no. 1, 71–75.
[9] Miličić, PM. Characterizations of convexities of normed spaces by means of g-angles. Mat. Vesnik 54 (2002), no. 1-2, 37–44.
[10] Miličić, PM. On duality mapping and canonical isometry of a normed space. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. 15 (2004), 87–91.