Electroweak corrections to $e^+e^- \rightarrow \gamma\gamma$ as a luminosity process at FCC-ee

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Abstract
We consider large-angle two photon production in $e^+e^-$ annihilation as a possible process to monitor the luminosity of a future $e^+e^-$ circular collider (FCC-ee). We review and assess the status of the theoretical accuracy by performing a detailed phenomenological study of next-to-leading order electroweak corrections and leading logarithmic QED contributions due to multiple photon radiation. We also estimate the impact of photonic and fermion-loop corrections at next-to-next-to-leading order and the uncertainty induced by the hadronic contribution to the vacuum polarization. Possible perspectives to address the target theoretical accuracy are briefly discussed.

Keywords: electron-positron colliders, luminosity, two photon production, QED, electroweak corrections, theoretical accuracy

1. Introduction

FCC-ee is a proposed high-luminosity $e^+e^-$ circular collider under consideration at CERN as one of the accelerators for next-generation particle physics experiments [1,2]. With a centre-of-mass energy (c.m.) between 90 and 365 GeV, it will provide the opportunity to test the Standard Model with unprecedented accuracy and perform indirect searches for New Physics through precision measurements. In particular, it could enable detailed investigations of the mechanism of electroweak symmetry breaking and high-precision studies of the properties of the $Z$, $W$, Higgs and top particles, as well as of the strong interaction.

The accomplishment of the above goals depends crucially on a number of critical factors, among which a precise knowledge of the collider luminosity. The ambitious FCC-ee target is a luminosity measurement with a total relative error

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of the order of $10^{-4}$ (or even better), not to spoil the expected statistical un-
certainty of the main processes of interest. This precision exceeds that obtained 
at LEP and calls for a major effort by both the experimental and theoretical 
community.

At FCC-ee, the standard luminosity process is expected to be \( e^+e^- \rightarrow e^+e^- \) 
(Bhabha) scattering, as measured by means of dedicated calorimeters put in 
the very forward region close to the beams, likewise at LEP [4]. In this respect, 
available precision calculations and related tools for small-angle Bhabha scat-
ering have been reviewed in Ref. [5] and a possible path to 0.01\% theoretical 
luminosity accuracy has been outlined in Ref. [6].

However, also the process of large-angle photon-pair production, i.e. \( e^+e^- \rightarrow \gamma\gamma \), has been recently proposed as a possible alternative normalization process 
for the FCC-ee physics program [7–9]. It was already used to monitor the 
luminosity at \( e^+e^- \) flavor factories with c.m. energy at the GeV scale [10,11] and 
is still employed by BESIII Collaboration to cross-check the luminosity obtained 
by analyzing large-angle Bhabha scattering events [12]. Actually, \( e^+e^- \rightarrow \gamma\gamma \) 
turns out to be a particularly adequate normalization process from a theoretical 
point of view. It is a purely QED process at leading order (LO) at any energy, 
it receives QED corrections from the initial state only and does not contain at 
order $\alpha$ the contribution due to vacuum polarization (in particular, hadronic 
loops), which enters at next-to-next-to-leading order (NNLO) only. On the 
other hand, the process is affected by a large background due to large-angle 
Bhabha scattering, which is huge on the $Z$ peak but more manageable at higher 
energies.

In spite of the above limitation, the possibility of using two photon produc-
tion as a luminosity process at FCC-ee is an interesting option to be pursued. 
Contrarily to Bhabha scattering, that received a lot of attention over the past 
decades, there is a rather poor theoretical literature about \( e^+e^- \rightarrow \gamma\gamma \) anni-
hilation and the most recent phenomenological results refer to \( e^+e^- \) colliders 
of moderate energies [13–16]. Also the available Monte Carlo (MC) generators [14,16], which are necessary for experimental simulations and feasibility 
studies, are tailored for low-energy accelerators and need to be improved for the 
high-energy, high-precision requirements of FCC-ee.

With these motivations in mind, we take a first step towards a critical assess-
ment of the current status of the theoretical accuracy for large-angle photon-pair 
production at FCC-ee energies. For this purpose, we make use of the MC pro-
gram BabaYaga@nlo [14,17–20] and improve it by including purely weak 
corrections due to heavy boson exchange. We examine in detail the effects 
of next-to-leading (NLO) electroweak corrections and higher-order QED contrib-
utions for both integrated and differential cross sections and according to 
different event selection criteria. We also explore the rôle played by photonic and 
fermion-loop corrections at NNLO and the uncertainty driven by the hadronic

\[ \text{1 It was extensively used and is still a reference code for luminosity measurements at flavor factories.} \]
contribution to the vacuum polarization. QED corrections to $e^+e^- \rightarrow \gamma\gamma$ at order $\alpha$ were previously computed in Refs. [21,23] and NLO electroweak corrections were obtained in Refs. [24,26]. A generator based on Ref. [23] was used at LEP for the analysis of photon-pair production at energies above the $Z$ mass [27]. It is worth noting that the full set of electroweak corrections is unavailable in any modern MC generator but the version of BABAYAGA@nlo developed for this study, which also provides the exponentiation of QED leading logarithmic (LL) contributions due to multiple photon radiation.

With respect to our contribution to the workshop proceedings as in Ref. [28], the present paper provides theoretical details and further numerical results relevant for two photon production as a luminosity process at FCC-ee.

The structure of the article is as follows. In Sect. 2 we sketch the theoretical formulation of the QED radiation inherent to BABAYAGA@nlo and the computation of one-loop weak corrections. The results of our numerical study are shown in Sect. 3. In Sect. 4 we draw the main conclusions of our analysis and discuss possible ways to achieve the target theoretical accuracy of FCC-ee.

2. Electroweak corrections

The photonic and weak corrections to $e^+e^- \rightarrow \gamma\gamma$ form two gauge-invariant subsets and can be treated separately.

According to the theoretical formulation implemented in BABAYAGA@nlo, the photonic corrections are computed by using a fully-exclusive QED Parton Shower (PS) matched to QED contributions at NLO. The master formula for the cross section calculation reads as follows:

$$d\sigma = F_{SV} \Pi^2 (Q^2, \epsilon) \sum_{n=0}^{\infty} \frac{1}{n!} \left( \prod_{i=0}^{n} F_{H,i} \right) |M_{n,LL}|^2 d\Phi_n$$

(1)

In Eq. (1), $\Pi (Q^2, \epsilon)$ is the Sudakov form factor, which accounts for the exponentiation of LL contributions due to soft and virtual corrections, $Q^2$ and $\epsilon$ being the hard scale of the process and a soft-hard photon separator, respectively. We set $Q^2 = s$, where $s$ is the squared c.m. energy, in such a way that the big collinear logarithms $L = \ln(s/m_e^2)$ due to initial-state radiation are resummed to all orders. $|M_{n,LL}|^2$ is the squared matrix element describing the emission in LL approximation of $n$ hard photons (i.e. with energy larger than $\epsilon$) on top of the two LO ones. The factor $d\Phi_n$ is the exact phase space element of the process $2\gamma$ plus $n$ additional radiated photons.

In Eq. (1) the matching of the above PS ingredients with the NLO QED corrections is realized by the finite-order factors $F_{SV}$ and $F_H$, whose definitions can be found in Refs. [14,20]. They are infrared/collinear safe correction factors that account for those $O(\alpha)$ non-logarithmic terms entering the NLO calculation and absent in the PS approach. For the soft+virtual factor $F_{SV}$ we use the results of Ref. [23], while the matrix element of the radiative process $e^+e^- \rightarrow \gamma\gamma\gamma$, which is needed for the calculation of the hard bremsstrahlung factor $F_H$, is computed by means of the symbolic manipulation program FORM [29,30].
By construction, the matching procedure as in Eq. (1) is such that the $O(\alpha)$ expansion of Eq. (1) reproduces the NLO cross section and exponentiation of LL contribution is preserved as in a pure PS algorithm. Moreover, as a by-product of its factorized structure, the bulk of the photonic sub-leading contributions at NNLO, i.e those of the order of $\alpha^2 L$, is automatically included by means of terms of the type $F_{SV} \mid H \otimes LL$ corrections [31].

To meet the high-energy, high-precision requirements of FCC-ee, we improved the theoretical content of BABAYAGA@nlo by calculating the one-loop weak corrections due to $W$, $Z$ and Higgs exchange. We computed them using the computer program Recola [32, 33], which internally adopts the Collier [34] library for the evaluation of one-loop scalar and tensor integrals. The calculation has been performed in the on-shell renormalization scheme, with complex mass values for the heavy boson masses [35–37].

The fermion-loop corrections to the photon self energy, which are needed for the estimate of the two loop contributions addressed in the next Section, are taken into account using the following expression for the vacuum polarization correction
\[
\Delta \alpha(s) = \Delta \alpha_{\text{lep}}(s) + \Delta \alpha_{\text{had}}(s) + \Delta \alpha_{\text{top}}(s)
\] (2)

For the leptonic correction $\Delta \alpha_{\text{lep}}(s)$ and the top-quark contribution $\Delta \alpha_{\text{top}}(s)$ we use the well known results in one-loop approximation. The hadronic (light-quark) correction is accounted for according to a dispersive approach based on time-like data for the process $e^+ e^- \rightarrow \text{hadrons}$, as implemented in the latest version of the *hadr5n16.f* routine [38].

3. Numerical results

For the presentation of the results of our numerical study, we use the following set of input parameters:

\[
\begin{align*}
\alpha(0) &= 1/137.0359895000034580626 \\
M_Z &= 91.15348 \text{ GeV} \quad \Gamma_Z = 2.49427 \text{ GeV} \\
M_W &= 80.35797 \text{ GeV} \quad \Gamma_W = 2.08430 \text{ GeV} \\
M_H &= 125 \text{ GeV} \\
m_e &= 0.51099 \text{ MeV} \\
m_\mu &= 0.10566 \text{ GeV} \\
m_\tau &= 1.777 \text{ GeV} \\
m_{\text{top}} &= 173.2 \text{ GeV}
\end{align*}
\] (3)

We consider four c.m. energy values, which are representative of the expected FCC-ee operation program ($Z$-pole, $WW$, $ZH$ and $t\bar{t}$ thresholds)
\[
\sqrt{s} = 91, \ 160, \ 240, \ 365 \text{ GeV}
\] (4)

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2At NNLO, there is a contribution to fermion-loop corrections arising from light-by-light scattering, which is not considered in the present study and whose evaluation is left to future work.

3Available at [http://www-com.physik.hu-berlin.de/~fjeger/software.html](http://www-com.physik.hu-berlin.de/~fjeger/software.html)
For those c.m. energies, the collinear logarithm $L$ varies in the range between 24 and 27, and the QED expansion parameter $\beta = 2 \alpha/\pi(L - 1)$ is of the order of 0.1.

To study the dependence of the QED corrections on the applied cuts, we consider two different simulation setup

[a] Full phase space, i.e. no cuts

[b] Acceptance cuts, i.e. at least two photons with: $20^\circ < \theta_\gamma < 160^\circ$ and $E_\gamma \geq 0.25 \times \sqrt{s}$

In Tab. 1 and Tab. 2, we examine the impact of the QED radiative corrections on the integrated cross sections, when considering setup [a] (Tab. 1) and setup [b] (Tab. 2).

| $\sqrt{s}$ (GeV) | LO (pb)   | NLO (pb) | w h.o. (pb) |
|------------------|-----------|----------|-------------|
| 91               | 364.68    | 447.27 [+23%] | 445.6(9) [-0.46%] |
| 160              | 123.71    | 154.37 [+25%] | 153.2(2) [-0.95%] |
| 240              | 56.816    | 71.809 [+26%] | 71.07(6) [-1.30%] |
| 365              | 25.385    | 32.515 [+28%] | 32.09(2) [-1.67%] |

Table 1: The two photon production cross section at LO, NLO QED and according to Eq. (1), for four FCC-ee c.m. energies and according to setup [a]. The numbers in parenthesis are the relative contributions due to NLO and higher-order LL corrections.

| $\sqrt{s}$ (GeV) | LO (pb)   | NLO (pb) | w h.o. (pb) | Bhabha LO (pb) |
|------------------|-----------|----------|-------------|----------------|
| 91               | 39.821    | 41.043 [+3.07%] | 40.870(4) [-0.43%] | 2625.9 |
| 160              | 12.881    | 13.291 [+3.18%] | 13.228(1) [-0.49%] | 259.98 |
| 240              | 5.7250    | 5.9120 [+3.27%] | 5.8812(6) [-0.54%] | 115.77 |
| 365              | 2.4752    | 2.5581 [+3.35%] | 2.5438(3) [-0.58%] | 50.373 |

Table 2: The same as in Tab. 1 according to the cuts of setup [b]. In the last column, the LO large-angle Bhabha cross section in the same setup is shown for the sake of comparison.

The photon-pair production cross section is shown according to different accuracy levels, i.e. at LO, NLO QED and according to Eq. (1), that includes the LL contributions due to multiple photon radiation. The numbers in parenthesis are the relative contributions due to NLO and higher-order LL corrections to LO, respectively. It can be noticed that the NLO corrections are particularly relevant in the inclusive setup [a], varying in the range between 20% and 30%, whereas they get largely reduced, to a few percent level, in setup [b]. The same trend is observed for the higher-order contributions, which range between 0.5% and 2% in setup [a], while they always amount to about five per mille in setup [b], independently of the c.m. energy. The difference between the results of the two configurations can be ascribed to the presence in setup [a] of
Figure 1: Upper panel: the LO angular distributions of $e^+e^- \rightarrow \gamma\gamma$ and large-angle Bhabha scattering, for four FCC-ee c.m. energies and according to setup [b]. Lower panel: ratio of the differential cross sections of the two processes.

logarithmic enhancements due to radiation of photons emitted collinear to the beams, that are largely removed by the angular cuts of setup [b]. This suggests that the overall contribution of QED corrections can be made sufficiently small by means of appropriate event selections, keeping at the same time an acceptable statistical uncertainty thanks to the high FCC-ee luminosity.

The higher-order effects discussed above are dominated by $O(\alpha^2 L^2)$ LL corrections but also include sub-leading contributions beyond NLO, among which $O(\alpha^2 L)$ photonic corrections. As already remarked, they originate as a by-product of the factorized form of Eq. (1) and their contribution can be estimated according to the following chain formula [14, 20]

$$\delta_{\alpha^2 L} = \frac{\sigma - \sigma_{\text{NLO}}} {\sigma_{\text{LO}}} - \sigma_{\text{PS}} + \sigma_{\text{PS}}^\alpha.$$

In Eq. (5), $\sigma$ is the full factorized cross section as in Eq. (1), $\sigma_{\text{NLO}}$ the exact NLO cross section, $\sigma_{\text{PS}}$ and $\sigma_{\text{PS}}^\alpha$ are the pure PS cross sections with all-order and $O(\alpha)$ corrections, respectively. An estimate of $O(\alpha^2 L)$ contributions is relevant because it allows to probe the size of the most relevant higher-order corrections beyond the LL approximation [6, 31]. By using Eq. (5), we studied the impact of $O(\alpha^2 L)$ photonic corrections as a function of the selection criteria, also considering an additional acollinearity cut of 10° that tends to single out elastic events. We obtained that $\delta_{\alpha^2 L}$ varies in setup [b] in the few per mille range as a function of $\sqrt{s}$ and gets reduced below 0.1% when adding the acollinearity cut to the conditions of setup [b].

For the sake of comparison, we also show in Tab. 2 the LO cross section of the large-angle Bhabha process, which turns out to be the main background to the
two photon signature. The Bhabha scattering cross section is evaluated using the same cuts of event selection \[b\] applied to \(e^+e^- \rightarrow \gamma\gamma\). As can be seen, Bhabha scattering provides at the \(Z\) resonance an overwhelming background \((\sigma_{\text{Bhabha}} \simeq 66 \times \sigma_{\gamma\gamma})\), which remains sizeable but less important for the other energy points \((\sigma_{\text{Bhabha}} \simeq 20 \times \sigma_{\gamma\gamma})\). To get further insight on the interplay between the two processes, we show in Fig. 1 a comparison between the LO angular distributions of \(e^+e^- \rightarrow \gamma\gamma\) and large-angle Bhabha scattering. The distributions refer to the four c.m. energies and to the cuts of setup \[b\] for both processes. As usual, the scattering angles are defined with respect to the direction of the incoming electron. In the upper panel of Fig. 1 one can notice the strongly asymmetric behavior of Bhabha scattering, which is dominated by \(t\)-channel photon exchange, in comparison with the symmetric shape of \(e^+e^- \rightarrow \gamma\gamma\). As can be seen, the cross section of two photon production is significantly larger than that of Bhabha scattering in the backward \(\theta_{e^-}\) hemisphere (large electron scattering angles) for all energies, except at the \(Z\) resonance. Obviously, the same holds for the ratio of the two distributions as a function of \(\theta_{e^+}\) in the forward \(\theta_{e^-}\) hemisphere, i.e., at small positron scattering angles. This simple analysis seems to suggest that, at least well above the \(Z\) pole, the angular distributions can provide a handle to control the Bhabha background, provided a sufficiently good discrimination of the electron/positron charge is performed in the experiment.

A representative example of the effects due to QED corrections on the differential cross sections is given in Fig. 2 which shows the angular distribution of the most energetic photon for the four energy points, according to setup \[b\]. As can be noticed, the NLO corrections are particularly important in the central region, where they are large and negative, reaching the 20-30\% level, as

![Figure 2: Upper panel: the angular distribution of the most energetic photon, for four FCC-ee c.m. energies and according to setup \[b\]. Lower panel: relative contributions of NLO and higher-order LL corrections.](image-url)
mainly due to soft-photon radiation. This effect is partially compensated by higher-order corrections, that amount to some percents in the same region.

In Fig. 3 we show the contribution of weak corrections to the integrated cross section as a function of the c.m. energy in the case of setup [b] and to the photon angular distribution at the four canonical energy points. As expected, the correction to the integrated cross section is of increasing importance as the energy increases, varying from a few per mille to one per cent. It amounts to about 0.5% around the $W$-pair production threshold, it passes through zero around the $ZH$ threshold and becomes more and more negative from $ZH$ to the $t\bar{t}$ production thresholds. Concerning the angular distribution, the contribution of weak corrections is practically negligible at the $Z$ resonance, at the per cent level for the other energy points and more pronounced in the central region for any energy, where it is of the same order as higher-order QED contributions at high energies.

Our estimate of the fermion-loop correction to the integrated cross section is given in Tab. 3, using for definitiveness the setup [b] that includes acceptance cuts. The numerical results of Tab. 3 are obtained by factorization of the NLO photonic correction with the vacuum polarization contribution according to the following formula

$$\sigma_{\Delta\alpha}^{NNLO} \pm \delta\sigma_{\text{had}} \simeq (\sigma_{\text{QED}}^{\text{NLO}} - \sigma_{\text{LO}}^{\text{NLO}}) \times [\Delta\alpha(s) \pm \delta\Delta\alpha_{\text{had}}]$$  \hspace{1cm} (6)

where $\delta\Delta\alpha_{\text{had}}$ is the data-driven uncertainty due to the hadronic contribution to $\Delta\alpha$, as returned by the hadr5n16.f routine. The factorized approach as in Eq. (6) gives rise to corrections dominated by $O(\alpha^2L^2)$ contributions and was proved in Ref. [39] to be an excellent approximation of the perturbative result.
based on an exact NNLO calculation. As can be seen, the vacuum polarization correction due to both leptonic(+top) and hadronic loops amounts to about 0.1% at all c.m. energies. In particular, we checked that top-quark contribution is always completely negligible. Note that the parametric uncertainty induced by the hadronic contribution to \( \Delta \alpha \) is much smaller than the target accuracy and therefore is not a limiting factor for the theoretical predictions for \( e^+e^- \rightarrow \gamma\gamma \). This is a strength of two photon production and is in contrast to small-angle Bhabha scattering, where the same uncertainty presently contributes at the \( 10^{-4} \) level \[6\]. It must be also noticed that a sound assessment of this class of corrections requires an explicit two-loop computation, as well as the combination of loop effects with the same-order contribution of real pair emission, which partially cancels the fermion-loop correction, as shown in past precision calculations for Bhabha scattering \[39–42\].

4. Conclusions

We have studied large-angle two photon production in \( e^+e^- \) annihilation as a possible process to measure the luminosity at FCC-ee experiments. We have assessed the status of the theoretical accuracy by performing a thorough phenomenological study of the radiative corrections to \( e^+e^- \rightarrow \gamma\gamma \) annihilation for all the relevant c.m. energies. To that purpose, we have upgraded the theoretical content of the code \textsc{BabaYaga@nlo}, that includes exact NLO QED corrections matched to PS, by computing the weak corrections due to the presence of heavy bosons in the internal loops.

We have shown that in a realistic setup including acceptance cuts the NLO QED corrections are fairly small, being at the level of a few percents for all the relevant c.m. energies. In the same conditions, the effects due to multiple photon emission, which are dominated by \( O(\alpha^2 L^2) \) contributions, amount to about 0.5%. The one-loop weak corrections to the integrated cross section and angular distribution are below 1% and at a few percent level, respectively. As a whole, these results point out that NLO electroweak and higher-order QED corrections to photon-pair production at FCC-ee are moderate but strictly necessary for precision luminosity monitoring.

We have also probed the size of some radiative corrections entering at NNLO accuracy. We have shown that sub-leading \( O(\alpha^2L) \) photonic contributions vary

| \( \sqrt{s} \) (GeV) | \( \sigma_{\Delta \alpha \text{lep+top}}^{\text{NNLO}}/\sigma_{LO} \) | \( \sigma_{\Delta \alpha \text{had}}^{\text{NNLO}}/\sigma_{LO} \) | \( \delta \sigma_{\text{had}}/\sigma_{LO} \) |
|-----------------|-----------------|-----------------|-----------------|
| 91              | 0.096%          | 0.085%          | 3.7 \( \cdot \) 10\(^{-6} \) |
| 160             | 0.108%          | 0.098%          | 3.8 \( \cdot \) 10\(^{-6} \) |
| 240             | 0.115%          | 0.108%          | 3.9 \( \cdot \) 10\(^{-6} \) |
| 365             | 0.119%          | 0.120%          | 4.0 \( \cdot \) 10\(^{-6} \) |

Table 3: Relative contribution of the NNLO leptonic(+top) and hadronic vacuum polarization correction to the cross section in setup \([b]\) and for four FCC-ee c.m. energies. In the last column, the uncertainty due to the hadronic contribution is shown.
from 0.01% to a few 0.1%, depending on the applied cuts and the considered c.m. energy. We have also estimated the vacuum polarization correction due to leptonic and hadronic loops, to conclude that it provides a contribution at the one per mille level for all the c.m. energies of interest. Moreover, we have shown that the uncertainty induced by the hadronic loops is below $10^{-5}$ and therefore does not provide a limitation for the theoretical predictions, contrarily to small-angle Bhabha scattering. Note that at NNLO accuracy fermion-loop corrections also involve the contribution from light-by-light scattering. The evaluation of this effect and of its uncertainty is left to future work.

As far as QED corrections are concerned, the accuracy of the present calculation can be estimated to be at the 0.1% level or slightly better [1,15,20,39]. However, the above conclusions about photonic and vacuum polarization corrections at NNLO suggest that the theoretical formulation implemented in our code, if supplemented by the contribution of $O(\alpha^2)$ fermion-loop and real pair corrections, should be sufficient to get close to an accuracy at the $10^{-4}$ level. Previous results about Bhabha scattering as luminosity process at LEP [6,42] and flavor factories [15,20,39] support this expectation. Beyond a 0.01% accuracy, a full calculation of NNLO QED corrections and, eventually, of two-loop weak contributions will be ultimately needed to reach the challenging frontier of the 10 ppm theoretical accuracy.

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