NONHOMOGENEOUS SYSTEMS INVOLVING CRITICAL OR SUBCRITICAL NONLINEARITIES

MOUSOMI BHAKTA\textsuperscript{1}, SOUPTIK CHAKRABORTY\textsuperscript{1}, AND PATRIZIA PUCCI\textsuperscript{2}

Abstract. This paper deals with existence of a nontrivial positive solution to systems of equations involving nontrivial nonhomogeneous terms and critical or subcritical nonlinearities. Via a minimization argument we prove existence of a positive solution whose energy is negative provided that the nonhomogeneous terms are small enough in the dual norm.

\textbf{2010 MSC:} 35J50, 35R11, 35J47, 35A15, 35B33, 35J60.

1. Introduction

In this paper we consider the following system of equations

\begin{align*}
(-\Delta)^s u + \gamma u &= \frac{\alpha}{\alpha + \beta} |u|^{\alpha-2} u |v|^{\beta} + f(x) \text{ in } \mathbb{R}^N, \\
(-\Delta)^s v + \gamma v &= \frac{\beta}{\alpha + \beta} |v|^{\beta-2} v |u|^{\alpha} + g(x) \text{ in } \mathbb{R}^N,
\end{align*}

where $N > 2s$, $\alpha, \beta > 1$, $\alpha + \beta \leq 2s^*$, $2s^* := 2N/(N-2s)$, $f, g$ are nontrivial nonnegative functionals in the dual space of $\dot{H}^s(\mathbb{R}^N)$ if $\alpha + \beta = 2s^*$ and of $H^s(\mathbb{R}^N)$ if $\alpha + \beta < 2s^*$, while $\gamma = 0$ if $\alpha + \beta = 2s^*$ and $\gamma = 1$ if $\alpha + \beta < 2s^*$. Here $(-\Delta)^s$ denotes the fractional Laplace operator which can be defined for the Schwartz class functions $\varphi$ as follows

\[-\Delta^s \varphi(x) := c_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s}} dy, \quad c_{N,s} = \frac{4^s \Gamma(N/2 + s)}{\pi^{N/2} \Gamma(-s)}.
\]

Let

\[\dot{H}^s(\mathbb{R}^N) := \left\{ u \in L^{2s}(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dx dy < \infty \right\},\]

be the homogeneous fractional Sobolev space, endowed with the inner product $\langle \cdot, \cdot \rangle_{H^s}$ and corresponding Gagliardo norm

\[\|u\|_{H^s} := \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dx dy \right)^{1/2}.
\]

\textsuperscript{1}{\textsc{Department of Mathematics, Indian Institute of Science Education and Research, Dr. Homi Bhaba Road, Pune-411008, India}}

\textsuperscript{2}{\textsc{Dipartimento di Matematica e Informatica, Università degli Studi di Perugia Via Vanvitelli 1, I-06123 Perugia, Italy}}

\textit{E-mail addresses:} mousomi@iiserpune.ac.in, soupchak9492@gmail.com, patrizia.pucci@unipg.it

\textit{Key words and phrases.} Nonlocal system of equations, fractional Laplacian, positive solutions, nontrivial solution, local minimum.
While $H^s(\mathbb{R}^N)$ is the standard fractional Sobolev Hilbert space with inner product $\langle \cdot, \cdot \rangle_{H^s(\mathbb{R}^N)}$ and corresponding norm
\[ \|u\|_{H^s} := \left( \|u\|_2^2 + \|u\|_{H^s}^2 \right)^{1/2}, \]
where in general $\| \cdot \|_p$ is the standard norm on the Lebesgue space $L^p(\mathbb{R}^N)$, when $p \geq 1$.

In the vectorial case, the natural solution space for (S) is the Hilbert space $\dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$, equipped with the inner product
\[ \langle (u, v), (\phi, \psi) \rangle_{\dot{H}^s \times \dot{H}^s} := \langle u, \phi \rangle_{\dot{H}^s} + \langle v, \psi \rangle_{\dot{H}^s}, \]
and the norm
\[ \|(u, v)\|_{\dot{H}^s \times \dot{H}^s} := \left( \|u\|_{\dot{H}^s}^2 + \|v\|_{\dot{H}^s}^2 \right)^{1/2}, \]
when $\alpha + \beta = 2^*_s$, while is $H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$ equipped with the inner product
\[ \langle (u, v), (\phi, \psi) \rangle_{H^s \times H^s} := \langle u, \phi \rangle_{H^s} + \langle v, \psi \rangle_{H^s} + \langle u, \phi \rangle_{L^2} + \langle v, \psi \rangle_{L^2}, \]
and the norm
\[ \|(u, v)\|_{H^s \times H^s} := \left( \|u\|_{H^s}^2 + \|v\|_{H^s}^2 \right)^{1/2}, \]
if $\alpha + \beta < 2^*_s$.

When $\alpha + \beta = 2^*_s$, we say $(u, v) \in \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$ is a solution of (S) if $u, v > 0$ in $\mathbb{R}^N$ and
\[ \langle (u, v), (\phi, \psi) \rangle_{\dot{H}^s \times \dot{H}^s} = \frac{\alpha}{2^*_s} \int_{\mathbb{R}^N} |u|^{2^*_s-2} u |\phi| \, dx + \frac{\beta}{2^*_s} \int_{\mathbb{R}^N} |v|^{2^*_s-2} v |\psi| \, dx + \langle \dot{u}, \phi \rangle_{\dot{H}^s} + \langle \dot{v}, \psi \rangle_{\dot{H}^s} \]
holds for every $(\phi, \psi) \in \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$, while if $\alpha + \beta < 2^*_s$ a couple $(u, v) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$ is said to be a solution of (S) if $u, v > 0$ in $\mathbb{R}^N$ and
\[ \langle (u, v), (\phi, \psi) \rangle_{H^s \times H^s} = \frac{\alpha}{\alpha + \beta} \int_{\mathbb{R}^N} |u|^{2^*_s-2} u |\phi| \, dx + \frac{\beta}{\alpha + \beta} \int_{\mathbb{R}^N} |v|^{2^*_s-2} v |\psi| \, dx + \langle \dot{u}, \phi \rangle_{H^s} + \langle \dot{v}, \psi \rangle_{H^s} \]
holds for every $(\phi, \psi) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$.

When the domain is a open bounded subset of $\mathbb{R}^N$, in a pioneering work, Tarantello [28] proves existence of two positive solutions for the nonhomogeneous problem
\[ \begin{align*}
- \Delta u &= |u|^{2^*_s-2} u + f \quad \text{in } \Omega, \\
0 &= u \quad \text{on } \partial \Omega, \\
2^*_s &= \frac{2N}{N-2},
\end{align*} \tag{1.1} \]
where $0 \leq f \in H^{-1}(\Omega)$ satisfies suitable condition. In [10, 25] the authors study existence of sign changing solutions of (1.1). In [3], the first and third author of the current paper treat the scalar version of (S) with the critical nonlinearity, namely the equation:
\[ \begin{cases}
(-\Delta)^s u = a(x)|u|^{2^*_s-2} u + f(x) & \text{in } \mathbb{R}^N, \\
u > 0 & \text{in } \mathbb{R}^N, \\
u \in \dot{H}^s(\mathbb{R}^N),
\end{cases} \]
where $0 < a \in L^\infty(\mathbb{R}^N)$, $a(x) \to 1$ as $|x| \to \infty$ and $f \in \dot{H}^s(\mathbb{R}^N)'$ and prove existence of at least two positive solutions when $\|f\|_{(\dot{H}^s)'}$ is small enough. For the scalar version of (S), with subcritical nonlinearities, we refer to [11, 21, 29] in the local case and to [4] in the nonlocal case. In all these papers existence of at least two positive solutions is actually proved.
Elliptic systems arise in biological applications (e.g., population dynamics) or physical applications (e.g., models of a nuclear reactor) and have been drawn a lot of attention (see [2, 11, 22, 24, 26, 27] and references therein). For systems in bounded domains with nonhomogeneous terms we refer to [6]. In the case of vector valued solutions for Schrödinger systems of equations in \( \mathbb{R}^3 \) with nonhomogeneous perturbation, we refer to [23], where the authors have applied Lyapunov–Schmidt reduction scheme to construct multiple solutions.

In the nonlocal case, there are not so many papers, in which weakly coupled systems of equations have been studied. To quote a few, we refer to [3, 9, 12, 15, 19, 20]. Actually all these papers deal with Dirichlet systems of equations in bounded domains. For the nonlocal systems of equations in the entire space \( \mathbb{R}^N \), we cite [16, 17] and the references therein. To the best of our knowledge, so far there have been no papers in the literature, where existence of nontrivial solutions to system of equations, with fractional Laplacian and the critical or subcritical exponents in \( \mathbb{R}^N \), have been established in the nontrivial nonhomogeneous case.

The main result in the paper is new even in the local case \( s = 1 \) and is stated below, where \( \ker(f) \) denotes the kernel of \( f \).

**Theorem 1.1.** (i) If \( \alpha + \beta = 2s^* \), and \( f, g \) are nontrivial nonnegative functionals in the dual space \( \hat{H}^s(\mathbb{R}^N) \) of \( H^s(\mathbb{R}^N) \) such that \( \ker(f) = \ker(g) \), then system (S) admits a nontrivial solution \((\bar{u}, \bar{v})\) such that \( \bar{u} > 0 \) and \( \bar{v} > 0 \), provided that \( 0 < \max\{\|f\|_{(\hat{H}^s)^*}, \|g\|_{(\hat{H}^s)^*}\} \leq d \) for some \( d > 0 \) sufficiently small.

(ii) If \( \alpha + \beta < 2s^* \), and \( f, g \) are nontrivial nonnegative functionals in the dual space \( H^{-s}(\mathbb{R}^N) \) of \( H^s(\mathbb{R}^N) \) such that \( \ker(f) = \ker(g) \), then (S) admits a nontrivial solution \((\bar{u}, \bar{v})\) such that \( \bar{u} > 0 \) and \( \bar{v} > 0 \), provided that \( 0 < \max\{\|f\|_{H^{-s}}, \|g\|_{H^{-s}}\} \leq d \) for some \( d > 0 \) sufficiently small.

Furthermore, in both the cases (i) and (ii) if \( f \equiv g \), then the solution \((\bar{u}, \bar{v})\) has the property that \( \bar{u} \neq \bar{v} \), whenever \( \alpha \neq \beta \). Finally, if \( \alpha = \beta \) but \( f \neq g \), then \( \bar{u} \neq \bar{v} \).

Let us emphasize that here we introduce suitable assumptions under which system (S) admits a positive solution, with different components, while papers devoted to systems seem not to address this question at all. Therefore, we actually solve system (S) when it does not reduce into a single equation.

To the best of our knowledge, the question of finding at least two nontrivial solutions to (S) remains open in the vectorial case. In the scalar case we are able to exhibit existence of two different solutions in the recent papers [4, 5].

2. Proof of Theorem 1.1

Before proving the main Theorem 1.1 let us present some useful notation and auxiliary results. Define

\[
S = \inf_{u \in H^s(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|_{H^s}^2}{\|u\|_{2s}^2}, \quad S_{\alpha + \beta} = \inf_{u \in H^s(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|_{H^s}^2}{\|u\|_{\alpha + \beta}^2},
\]
and

\[
S_{(\alpha, \beta)} = \begin{cases} 
\inf_{(u, v) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) \setminus \{(0,0)\}} \left( \frac{\|u\|_{H^s}^2 + \|v\|_{H^s}^2}{\left( \int_{\mathbb{R}^N} |u|^{\alpha} |v|^{\beta} \, dx \right)^{2/\alpha \beta}} \right), & \text{if } \alpha + \beta = 2^*_s \\
\inf_{(u, v) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) \setminus \{(0,0)\}} \left( \frac{\|u\|_{H^s}^2 + \|v\|_{H^s}^2}{\left( \int_{\mathbb{R}^N} |u|^{\alpha} |v|^{\beta} \, dx \right)^{2/(\alpha + \beta)}} \right), & \text{if } \alpha + \beta < 2^*_s.
\end{cases}
\]

In the celebrated paper [8] Chen, Li and Ou prove that when $\alpha + \beta = 2^*_s$ the Sobolev constant $S_{\alpha + \beta} = S$ is achieved by $w$, where $w$ is the unique positive solution (up to translations and dilations) of

\[ (-\Delta)^s w = w^{2^*_s - 1} \quad \text{in } \mathbb{R}^N, \quad w \in H^s(\mathbb{R}^N). \]

Indeed, any positive solution of the above equation is radially symmetric, with respect to some point $x_0 \in \mathbb{R}^N$, strictly decreasing in $r = |x - x_0|$, of class $C^\infty(\mathbb{R}^N)$ and so of the explicit parametric form

\[ w(x) = c_{N, s} \left( \frac{\lambda}{\lambda^2 + |x - x_0|^2} \right)^{\frac{N - 2s}{2}}, \]

for some $\lambda > 0$. On the other hand, when $2 < \alpha + \beta < 2^*_s$, Frank, Lenzmann and Silvestre in their celebrated paper [18] prove that $S_{\alpha + \beta}$ is achieved by unique (up to a translation) positive ground state solution $w$ of

\[ (-\Delta)^s w + w = w^{\alpha + \beta - 1} \quad \text{in } \mathbb{R}^N, \quad w \in H^s(\mathbb{R}^N). \]

Furthermore, $w$ is radially symmetric, symmetric decreasing $C^\infty(\mathbb{R}^N)$ function which satisfies the following decay property in $\mathbb{R}^N$

\[ \frac{C^{-1}}{1 + |x|^{N+2s}} \leq w(x) \leq \frac{C}{1 + |x|^{N+2s}}, \]

with some constant $C > 0$ depending on $N$, $\alpha + \beta$, $s$.

**Lemma 2.1.** There exists a positive constant $C = C(\alpha, \beta, s, N)$ such that when $\alpha + \beta = 2^*_s$

\[ \left( \int_{\mathbb{R}^N} |u|^{\alpha} |v|^{\beta} \, dx \right)^{1/2^*_s} \leq C \|(u, v)\|_{H^s \times H^s}, \]

for all $(u, v) \in \hat{H}^s(\mathbb{R}^N) \times \hat{H}^s(\mathbb{R}^N)$, while if $\alpha + \beta < 2^*_s$

\[ \left( \int_{\mathbb{R}^N} |u|^{\alpha} |v|^{\beta} \, dx \right)^{1/(\alpha + \beta)} \leq C \|(u, v)\|_{H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)}, \]

for all $(u, v) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$.

**Proof.** It easily follows from the definition of $S_{\alpha + \beta}$ and the inequality

\[ |t|^{\alpha} |\tau|^{\beta} \leq |t|^{\alpha + \beta} + |\tau|^{\alpha + \beta} \]

for all $(t, \tau) \in \mathbb{R}^2$. \qed

Next, we recall a result from [15] which states the relation between $S_{(\alpha, \beta)}$ and $S_{\alpha + \beta}$.
Lemma 2.2. \([15] \text{Lemma 5.1}\) In all cases \(\alpha > 1, \beta > 1,\) with \(\alpha + \beta \leq 2_s^*,\) it results

\[ S_{(\alpha, \beta)} = \left[ \left( \frac{\alpha}{\beta} \right)^{\frac{\alpha}{\alpha+\beta}} + \left( \frac{\alpha}{\beta} \right)^{\frac{\alpha}{\alpha+\beta}} \right] S_{\alpha+\beta}. \]

Moreover, if \(w\) achieves \(S_{\alpha+\beta}\) then \((Bw, Cw)\) achieves \(S_{(\alpha, \beta)}\) for all positive constants \(B\) and \(C\) such that \(B/C = \sqrt{\alpha/\beta}.\)

Finally we prove a short useful result

**Lemma 2.3.** In all cases \(\alpha > 1, \beta > 1,\) with \(\alpha + \beta \leq 2_s^*,\)

\[ S_{(\alpha, \beta)} > S_{\alpha+\beta} \]

holds true.

**Proof.** If \(\alpha > \beta,\) then using Lemma [15],

\[ \frac{S_{(\alpha, \beta)}}{S_{\alpha+\beta}} = \left( \frac{\alpha}{\beta} \right)^{\frac{\alpha}{\alpha+\beta}} + \left( \frac{\alpha}{\beta} \right)^{\frac{\alpha}{\alpha+\beta}} = \left( \frac{\alpha}{\beta} \right)^{\frac{\alpha}{\alpha+\beta}} \frac{\alpha + \beta}{\alpha} > 1. \]

Similarly, if \(\alpha < \beta\) then

\[ \frac{S_{(\alpha, \beta)}}{S} = \left( \frac{\alpha}{\beta} \right)^{\frac{\alpha}{\alpha+\beta}} + \left( \frac{\alpha}{\beta} \right)^{\frac{\alpha}{\alpha+\beta}} = \left( \frac{\alpha}{\beta} \right)^{\frac{\alpha}{\alpha+\beta}} \frac{\alpha + \beta}{\beta} > 1. \]

Further, \(S_{(\alpha, \beta)} > 2S_{\alpha+\beta}\) for \(\alpha = \beta.\) \(\square\)

We are finally in a position to prove the main result and we simply say that a couple \((u, v)\) is positive if both components are positive.

**Proof of Theorem 1.1 – Part (i).** Let \(\alpha + \beta = 2_s^*.\) We note that system \((\mathbf{S})\) is variational and the underlying functional is

\[ I_{f,g}(u, v) := \frac{1}{2} \|(u, v)\|_{\dot{H}^s \times \dot{H}^s}^2 - \frac{1}{2} \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \, dx - \langle \dot{H}^s, (f, u) \rangle_{\dot{H}^s} - \langle \dot{H}^s, (g, v) \rangle_{\dot{H}^s}, \]

which is well defined in \(\dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)\) and of class \(C^1(\dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)).\) Moreover, if \((u, v)\) is a solution of \((\mathbf{S}),\) then \((u, v)\) is a positive critical point of \(I_{f,g}\) and vice versa.

Let us now introduce the auxiliary functional

\[ J_{f,g}(u, v) := \frac{1}{2} \|(u, v)\|_{\dot{H}^s \times \dot{H}^s}^2 - \frac{1}{2} \int_{\mathbb{R}^N} u_+^\alpha v_+^\beta \, dx - \langle \dot{H}^s, (f, u) \rangle_{\dot{H}^s} - \langle \dot{H}^s, (g, v) \rangle_{\dot{H}^s}, \]

which is well defined in \(\dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)\) and of class \(C^1(\dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)),\) with second derivative. Indeed, for all \((u, v), (\phi, \psi) \in \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N))\)

\[ J''_{f,g}(u, v)((\phi, \psi), (\phi, \psi)) = \|(\phi, \psi)\|_{\dot{H}^s \times \dot{H}^s}^2 - \frac{\alpha(\alpha - 1)}{2_s} \int_{\mathbb{R}^N} u_+^{\alpha-2} v_+^\beta \phi^2 \, dx \]

\[ - \frac{\beta(\beta - 1)}{2_s} \int_{\mathbb{R}^N} u_+^\alpha v_+^{\beta-2} \psi^2 \, dx - \frac{2\alpha \beta}{2_s} \int_{\mathbb{R}^N} u_+^{\alpha-1} v_+^{\beta-1} \phi \psi \, dx. \]

(2.1)
Using Hölder’s and Sobolev’s inequalities, we estimate the second term on the RHS as follows

\[
\int_{\mathbb{R}^N} u_+^{\alpha-1} v_+^{\beta-1} |\phi|^2 \, dx \leq \left( \int_{\mathbb{R}^N} |\phi|^2 \, dx \right)^{\frac{2}{2^*}} \left( \int_{\mathbb{R}^N} |u|^2 \, dx \right)^{\frac{\alpha-2}{2}} \left( \int_{\mathbb{R}^N} |v|^2 \, dx \right)^{\frac{\beta}{2}}
\]

\[
\leq S^{-1 - \frac{\alpha-2-\beta}{2^*}} \|u\|_{H^s}^{\alpha-2} \|v\|_{H^s}^\beta \|\phi\|_{H^s}^2 \\
\leq S^{-\frac{2}{2^*}} \|u,v\|_{H^s}^{\alpha-2} \|\phi,\psi\|_{H^s}^2.
\]

In the inequality we have used the fact that $\|u\|_{\dot{H}^s} \leq \|(u,v)\|_{H^s \times \dot{H}^s}$ and $\alpha + \beta = 2s^*$. Similarly,

\[
\int_{\mathbb{R}^N} u_+^{\alpha} v_+^{\beta} |\phi|^2 \, dx \leq S^{-\frac{2}{2^*}} \|u,v\|_{H^s \times \dot{H}^s}^2 \|\phi,\psi\|_{H^s}^2.
\]

Furthermore,

\[
\int_{\mathbb{R}^N} u_+^{\alpha-1} v_+^{\beta-1} |\phi|^2 \, dx \leq \left( \int_{\mathbb{R}^N} |\phi|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |\phi|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |u|^2 \, dx \right)^{\frac{\alpha-1}{\alpha}} \left( \int_{\mathbb{R}^N} |v|^2 \, dx \right)^{\frac{\beta}{\beta}}
\]

\[
\leq S^{-\frac{1}{2}} \left( \frac{\alpha-1}{2} \right) \|\phi\|_{H^s} \|\psi\|_{\dot{H}^s} \|u\|_{H^s}^{\alpha-1} \|v\|_{H^s}^\beta
\]

\[
\leq S^{-\frac{2}{2^*}} \|\phi,\psi\|_{H^s \times \dot{H}^s} \|\phi,\psi\|_{H^s \times \dot{H}^s}^2 \|u,v\|_{H^s}^{2s^*-2} \|\phi,\psi\|_{H^s \times \dot{H}^s}^2.
\]

Thus, substituting the above three estimates in (12.1), we obtain

\[
J_{f,g}''(u,v) \geq \left( 1 - S^{-\frac{2}{2s^*}} \|u,v\|_{H^s \times \dot{H}^s}^{2s^*-2} \right) \left[ \alpha(\alpha-1) + \beta(\beta-1) + \alpha \beta \right] \times \|\phi,\psi\|_{H^s \times \dot{H}^s}^2.
\]

Therefore, $J_{f,g}''(u,v)$ is positive definite for $(u,v)$ in the ball centered at 0 and of radius $r$ in $\dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$, where

\[
r = \left( \frac{2s^*}{\alpha^2 + \beta^2 + \alpha \beta - 2s^*} \right)^{\frac{1}{2s^*-2}} S^{\frac{2}{2^*}}.
\]

Hence $J_{f,g}$ is strictly convex in $B_r$. For $(u,v) \in \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$, with $(u,v)_{H^s \times \dot{H}^s} = r$,

\[
J_{f,g}(u,v) = \frac{1}{2} \|u,v\|_{H^s \times \dot{H}^s}^2 - \frac{1}{2} \int_{\mathbb{R}^N} u_+ v_+ |\phi|^2 c dx - (\dot{H}^s)^\prime \langle f, u \rangle_{H^s} - (\dot{H}^s)^\prime \langle g, v \rangle_{\dot{H}^s}
\]

\[
\geq \left( \frac{1}{2} - \frac{1}{2s^*} S_{(\alpha,\beta)} \right) r^{2s^*-2} \left( \|f\|_{(H^s)^\prime} \|u\|_{H^s} + \|g\|_{(\dot{H}^s)^\prime} \|v\|_{\dot{H}^s} \right)
\]

\[
\geq \left( \frac{1}{2} - \frac{1}{2s^*} S_{(\alpha,\beta)} \right) r^{2s^*-2} \left( \|f\|_{(H^s)^\prime} + \|g\|_{(\dot{H}^s)^\prime} \right) r.
\]

As $r^{2s^*-2} = \frac{2s^*}{\alpha^2 + \beta^2 + \alpha \beta - 2s^*} S^{\frac{2}{2^*}}$, we obtain

\[
J_{f,g}(u,v) \geq \left[ \frac{1}{2} - \frac{1}{\alpha^2 + \beta^2 + \alpha \beta - 2s^*} S \left( \frac{S}{S_{(\alpha,\beta)}} \right)^{\frac{2}{2^*}} \right] r^{2} - r \left( \|f\|_{(H^s)^\prime} + \|g\|_{(\dot{H}^s)^\prime} \right).
\]
We claim that

\[(\alpha^2 + \beta^2 + \alpha \beta - 2s^*) \left( \frac{S(\alpha, \beta)}{S} \right)^{2s^*/2} > 2. \tag{2.3} \]

By Lemma 2.2 and \(\alpha + \beta = 2s^*\), we have

\[(\alpha^2 + \beta^2 + \alpha \beta - 2s^*) \left( \frac{S(\alpha, \beta)}{S} \right)^{2s^*/2} > (\alpha^2 + \beta^2 + \alpha \beta - 2s^*) \frac{S(\alpha, \beta)}{S} = [2s^*(2s^* - 1) - \alpha \beta] \left( \frac{\alpha}{\beta} \right) \frac{\beta}{2s^*}. \]

Since \(2s^* > 2\), to prove (2.3) it is enough to show that

\[\left[ \frac{1}{2} (\alpha^2 + \beta^2 - 1) - \frac{1}{2} \right] \left( \frac{\alpha}{\beta} \right) \frac{\beta}{2s^*} > 1. \]

Now,

\[\left[ \frac{1}{2} (\alpha^2 + \beta^2 - 1) - \frac{1}{2} \right] \left( \frac{\alpha}{\beta} \right) \frac{\beta}{2s^*} \geq 1 \iff \frac{\alpha^2 + \beta^2 - 1}{\alpha^2 + \beta^2} > \frac{1}{\alpha} \iff \frac{\alpha^2 + \beta^2 - 1}{\alpha^2 + \beta^2} > \frac{1}{\alpha}. \]

Since, \(\alpha, \beta > 1\) and \(\alpha + \beta = 2s^*\), we have

\[\alpha \left[ 1 + \frac{1}{\alpha^{2s^*/2} \beta^{2s^*/2}} \right] < 2 \alpha \leq \frac{(\alpha + \beta)^2}{2} = \frac{(2s^*)^2}{2} < \frac{1}{2}. \]

Hence the claim (2.3) follows.

Now, by (2.2) and (2.3) there exists a number \(d > 0\) such that

\[\inf_{\|u,v\|_{H^s \times H^s} = r} J_{f,g}(u,v) > 0, \quad \text{provided that } 0 < \max\{\|f\|_{\dot{H}^{s}}\|, \|g\|_{\dot{H}^{s}}\} \leq d. \]

Furthermore, for \((u,v) \in \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)\), with \(u > 0\) and \(v \geq 0\),

\[J_{f,g}(tu, tv) \begin{cases} < 0 & \text{for } t > 0 \text{ small enough} \\ > 0 & \text{for } t < 0 \text{ small enough,} \end{cases} \tag{2.4} \]

since \(f\) and \(g\) are nontrivial. Combining this along with the fact that \(J_{f,g}\) is strictly convex in \(B_r\) and

\[\inf_{\|u,v\|_{H^s \times H^s} = r} J_{f,g}(u,v) = 0 = J_{f,g}(0,0), \]

we conclude that there exists a unique critical point \((\bar{u}, \bar{v})\) of \(J_{f,g}\) in \(B_r\) such that

\[J_{f,g}(\bar{u}, \bar{v}) = \inf_{\|u,v\|_{H^s \times H^s} < r} J_{f,g}(u,v) < J_{f,g}(0,0) = 0. \]

Therefore, \((\bar{u}, \bar{v})\) is a nontrivial solution of

\[
\begin{align*}
(-\Delta)^s u &= \frac{\alpha}{2s} u_+^{s-1} v_+^{s} + f(x) \quad &\text{in } \mathbb{R}^N, \\
(-\Delta)^s v &= \frac{\beta}{2s} u_+^{s-1} v_+^{s} + g(x) \quad &\text{in } \mathbb{R}^N, \\
u, v &\in \dot{H}^s(\mathbb{R}^N).
\end{align*} \tag{2.5}
\]
Since, \( f \) and \( g \) are nonnegative functionals, then taking \( (\phi, \psi) = (\bar{u}_-, \bar{v}_-) \) as a test function in \([23]\), we obtain
\[
-\|\bar{u}_-\|_{H^s}^2 - \iint_{\mathbb{R}^{2N}} \frac{[\bar{u}_-(y)\bar{u}_-(x) + \bar{u}_-(x)\bar{u}_-(y)]}{|x - y|^{N+2s}} \, dx \, dy = -\|\bar{v}_-\|_{H^s}^2.
\]
This in turn implies \( \bar{u}_- = 0 \) and \( \bar{v}_- = 0 \), i.e., \( \bar{u} \geq 0 \) and \( \bar{v} \geq 0 \). Therefore, \( (\bar{u}, \bar{v}) \) is a nontrivial nonnegative solution of \( (S) \).

Next we assert that \( (\bar{u}, \bar{v}) \neq (0, 0) \) implies \( \bar{u} \neq 0 \) and \( \bar{v} \neq 0 \). Suppose not, that is assume for instance that \( \bar{u} \neq 0 \) but \( \bar{v} = 0 \). Then taking the test function \( (\phi, \psi) = (\bar{u}, 0) \) we get
\[
\|\bar{u}\|_{H^s}^2 = (\bar{H}\phi, \bar{u})_{H^s} = 0.
\]
Hence, \( \|\bar{u}\|_{H^s} = 0 \), since \( \ker(f) = \ker(g) \) by assumption. This contradicts the fact that \( (\bar{u}, \bar{v}) \neq (0, 0) \). Similarly, we can show that if \( \bar{u} = 0 \) then \( \bar{v} = 0 \) too. Hence the assertion follows.

Let us claim that \( \bar{u} > 0 \) and \( \bar{v} > 0 \) in \( \mathbb{R}^N \). To prove the claim, first we note that taking the test function \( (\phi, \psi) = (\phi, 0) \), where \( \phi \in H^s(\mathbb{R}^N) \) with \( \phi \geq 0 \), we obtain
\[
\langle \bar{u}, \phi \rangle_{H^s} = \frac{\alpha}{2^s} \int_{\mathbb{R}^N} \bar{u}^{\alpha-1} \bar{v}^\beta \phi \, dx + (\bar{H}\phi, \bar{f})_{H^s} \geq 0,
\]
as \( f \) is a nonnegative functional and \( \bar{u}, \bar{v} \geq 0 \). This implies \( \bar{u} \) is a weak supersolution to
\[
(-\Delta)^s u = 0.
\]
Therefore, applying the maximum principle [13] Theorem 1.2 (ii)], with \( c \equiv 0 \) and \( p = 2 \) there, it follows that \( \bar{u} > 0 \) in \( \mathbb{R}^N \). Similarly, taking the test function \( (\phi, \psi) = (0, \psi) \), with \( \psi \in \bar{H}^s(\mathbb{R}^N) \) and \( \psi \geq 0 \), yields \( \bar{v} > 0 \) in \( \mathbb{R}^N \). This proves the claim.

The final assertion will be shown below by the method of contradiction. Therefore, let us suppose \( \bar{u} \equiv \bar{v} \) and divide the proof in the two cases covered by the theorem.

First, we assume \( f \equiv g \) but \( \alpha \neq \beta \). Then, taking the test function \( (\phi, \psi) = (\bar{u}, -\bar{u}) \) yields
\[
\frac{1}{2^s} (\alpha - \beta) \int_{\mathbb{R}^N} \bar{u}^{\alpha+\beta} \, dx = 0.
\]
This is impossible since \( \bar{u} \) is positive in \( \mathbb{R}^N \).

In the remaining case, we assume \( \alpha = \beta \) but \( f \not\equiv g \) and \( \ker(f) = \ker(g) \). Then taking the test function \( (\phi, \psi) = (\phi, -\phi) \), where \( \phi \in C_0^\infty(\mathbb{R}^N) \), we obtain
\[
(\bar{H}\phi, \bar{f} - g, \phi)_{H^s} = 0.
\]
This in turn implies \( f \equiv g \) as \( \phi \in C_0^\infty(\mathbb{R}^N) \) is arbitrary. This contradiction completes the proof of Part (i).

Part (ii). The proof follows along the same lines as in Part (i), therefore we just mention only the differences. It is easy to see that the associated functional corresponding to \( (S) \) is now
\[
\bar{I}_{f,g}(u, v) := \frac{1}{2} \|(u, v)\|_{H^s \times H^s}^2 - \frac{1}{\alpha + \beta} \int_{\mathbb{R}^N} |u|^{\alpha} |v|^{\beta} \, dx = -\|\bar{v}_-\|_{H^s}^2 - H^{-s}(f, u)_{H^s} - H^{-s}(g, v)_{H^s}.
\]
Let us introduce the auxiliary functional as

\[ \tilde{J}_{f,g}(u,v) := \frac{1}{2} \| (u,v) \|^2_{H^s \times H^s} - \frac{1}{\alpha + \beta} \int_{\mathbb{R}^N} u^\alpha+ v^\beta \, dx - H^{-1}(f,u)_{H^s} - H^{-1}(g,v)_{H^s}, \]

which is well defined in \( H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) \) and of class \( C^1(\mathbb{H}^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)) \), with second derivative. Arguing as before, we obtain for all \((u,v), (\phi, \psi) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)\)

\[ \tilde{J}''_{f,g}(u,v)((\phi, \psi), (\phi, \psi)) = \| (\phi, \psi) \|^2_{H^s \times H^s} - \frac{\alpha(\alpha - 1)}{\alpha + \beta} \int_{\mathbb{R}^N} u^\alpha-2 v^\beta \, dx \]

\[ - \frac{\beta(\beta - 1)}{\alpha + \beta} \int_{\mathbb{R}^N} u^\alpha v^\beta - 2 \frac{\alpha \beta}{\alpha + \beta} \int_{\mathbb{R}^N} u^\alpha-1 v^\beta \, dx, \]

\[ \geq \left( 1 - \frac{S_{\alpha+\beta} - \alpha \beta}{\alpha + \beta} \right) \| (u,v) \|^2_{H^s \times H^s} \left[ \alpha(\alpha - 1) + \beta(\beta - 1) + \alpha \beta \right] \times \| (\phi, \psi) \|^2_{H^s \times H^s}. \]

Therefore, \( \tilde{J}''_{f,g}(u,v) \) is positive definite for \((u,v) \) in the ball centered at 0 and of radius \( r \) in \( H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) \), where

\[ r = \left( \frac{\alpha + \beta}{\alpha^2 + \beta^2 + \alpha \beta - (\alpha + \beta)} \right)^{\alpha+\beta-2} S_{\alpha+\beta}^{\alpha+\beta-2}. \]

Hence \( \tilde{J}_{f,g} \) is strictly convex in \( B_r \). Furthermore, for all \((u,v) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) \), with \( \| (u,v) \|_{H^s \times H^s} = r \),

\[ \tilde{J}_{f,g}(u,v) \geq \left[ \frac{1}{2} - \frac{1}{\alpha^2 + \beta^2 + \alpha \beta - (\alpha + \beta)} \left( \frac{S_{\alpha+\beta}}{S_{\alpha,\beta}} \right)^{\alpha+\beta/2} \right] r^2 - r(\| f \|_{H^{-1}} + \| g \|_{H^{-1}}). \quad (2.6) \]

Since \( S_{\alpha,\beta} > S_{\alpha+\beta} \) by Lemma 2.3, we have

\[ (\alpha^2 + \beta^2 + \alpha \beta - (\alpha + \beta)) \left( \frac{S_{\alpha+\beta}}{S_{\alpha,\beta}} \right)^{(\alpha+\beta)/2} \geq \left( \frac{\alpha^2 + \beta^2 + \alpha \beta - (\alpha + \beta)}{\lambda_{\alpha+\beta}} \right) S_{\alpha+\beta}^{\alpha+\beta/2} \frac{\lambda_{\alpha+\beta}}{\alpha} \]

\[ = [(\alpha + \beta)(\alpha + \beta - 1) - \alpha \beta] \left( \frac{\alpha}{\beta} \right)^{\alpha+\beta} \frac{\alpha + \beta}{\alpha}. \]

Therefore, to prove

\[ [(\alpha^2 + \beta^2 + \alpha \beta - (\alpha + \beta)) \left( \frac{S_{\alpha+\beta}}{S_{\alpha,\beta}} \right)^{(\alpha+\beta)/2} > 2, \]

it is enough to show that

\[ [(\alpha + \beta)(\alpha + \beta - 1) - \alpha \beta] \left( \frac{\alpha}{\beta} \right)^{\alpha+\beta} \frac{1}{\alpha} > 1, \]

since \( \alpha + \beta > 2 \). Actually, the above expression is equivalent to

\[ (\alpha + \beta)(\alpha + \beta - 1) > \alpha \beta \left[ 1 + \frac{1}{\alpha^{\alpha+\beta} \beta^{\alpha+\beta}} \right]. \]
As $\alpha, \beta > 1$, a straightforward computation yields
\[
\alpha \beta \left[ 1 + \frac{1}{\alpha^{\alpha} \beta^{\beta}} \right] < 2\alpha \beta \leq \frac{(\alpha + \beta)^2}{2} < (\alpha + \beta)(\alpha + \beta - 1).
\]
Therefore, (2.6) implies the existence of a number $d > 0$ such that
\[
\inf_{\|(u,v)\|_{H^s \times H^s} = r} \tilde{J}_{f,g}(u,v) > 0,
\]
provided that $0 < \max\{\|f\|_{H^{-s}}, \|g\|_{H^{-s}}\} \leq d$.

From here on, proceeding as in the proof of Part (i), with obvious changes, we get the assertion.

Acknowledgement: M. Bhakta wishes to express her sincere gratitude to the Dipartimento di Matematica e Informatica of Università degli Studi di Perugia, where part of this work started during a visit of her to that institution. The research of M. Bhakta is partially supported by the SERB MATRICS grant (MTR/2017/000168). S. Chakraborty is supported by NBHM grant 0203/11/2017/RD-II.

P. Pucci is member of the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). P. Pucci was also partly supported by of the Fondo Ricerca di Base di Ateneo – Esercizio 2017–2019 of the University of Perugia, named PDEs and Nonlinear Analysis.

References

[1] Adachi, S.; Tanaka, K. Four positive solutions for the semilinear elliptic equation: $-\Delta u + u = a(x)u^p + f(x)$ in $\mathbb{R}^N$, Calc. Var. Partial Differential Equations 11 (2000), no. 1, 63–95.

[2] Alves, C.O.; de Morais Filho, D.C.; Souto, M.A.S. On systems of elliptic equations involving subcritical or critical Sobolev exponents, Nonlinear Anal. 42 (2000), 771–787.

[3] Bhakta, M.; Nguyen, P.–T. On the existence and multiplicity of solutions to fractional Lane–Emden elliptic systems involving measures, to appear in Adv. Nonlinear Anal., arXiv:1809.07909.

[4] Bhakta, M.; Chakraborty, S; Ganguly, D. Existence and multiplicity of positive solutions of certain nonlocal scalar field equations, preprint, arXiv:1910.07919.

[5] Bhakta, M.; Pucci, P. On multiplicity of positive solutions for nonlocal equations with critical nonlinearity, preprint, arXiv:2003.02665.

[6] Bonheure, D.; Ramos, M. Multiple critical points of perturbed symmetric strongly indefinite functionals. Ann. Inst. H. Poincaré Anal. Non Linéaire 26, 675–688 (2009).

[7] Cao, D.M.; Zhou, H.S. Multiple positive solutions of nonhomogeneous semilinear elliptic equations in $\mathbb{R}^N$, Proc. Roy. Soc. Edinburgh 126 A (1996), 443–463.

[8] Chen, W.; Li, C.; Ou, B. Classification of solutions for an integral equation, Comm. Pure Appl. Math. 59 (2006), 330–343.

[9] Chen, W.; Squassina, M. Critical nonlocal systems with concave–convex powers, Adv. Nonlinear Stud. 16 (2016), 821–842.

[10] Clapp, M.; del Pino, M.; Musso, M. Multiple solutions for a non–homogeneous elliptic equation at the critical exponent, Proc. Roy. Soc. Edinburgh Sect. A 134 (2004), 69–87.

[11] Clément, Ph.; Fleckinger, J.; Mitidieri, E.; de Thélin, F. Existence of positive solutions for a nonvariational quasilinear system, J. Differential Equations 166 (2000), 455–477.

[12] Costa, D.; Miyagaki, O.H.; Squassina M.; Yang J. Asymptotics of ground states for Hénon type systems with fractional diffusion, Contributions to Nonlinear Elliptic Equations and Systems, Progress in Nonlinear Differential Equations and Their Applications, A Tribute to Djairo Guedes de Figueiredo on the Occasion of his 80th Birthday, 86 (2015), 133–161.

[13] Del Pezzo, L.M.; Quaas, A. A Hopf’s lemma and a strong minimum principle for the fractional $p$-Laplacian, J. Differential Equations 263 (2017), 765–778.

[14] Del Pino, M.; Felmer, P.L. Local mountain passes for semilinear elliptic problems in unbounded domains, Calc. Var. Partial Differential Equations 4 (1996), 121–137.
REFERENCES

[15] Faria, L.F.O.; Miyagaki, O.H.; Pereira, F.R.; Squassina, M.; Zhang, C. The Brézis–Nirenberg problem for nonlocal systems, *Adv. Nonlinear Anal.* 5 (2016), 85–103.

[16] Fiscella, A.; Pucci, P.; Saldi, S. Existence of entire solutions for Schrödinger–Hardy systems involving two fractional operators, *Nonlinear Anal.* 158 (2017), 109–131.

[17] Fiscella, A.; Pucci, P.; Zhang, B. $p$–fractional Hardy–Schrödinger–Kirchhoff systems with critical nonlinearities, *Adv. Nonlinear Anal.* 8 (2019), 1111–1131.

[18] Frank, R.L; Lenzmann, E.; Silvestre, L. Uniqueness of radial solutions for the fractional Laplacian, *Comm. Pure Appl. Math.* 19 (2016), 1671–1726.

[19] Giacomoni, J.; Mukherjee, T; Sreenadh, K. Doubly nonlocal system with Hardy–Littlewood–Sobolev critical nonlinearity, *J. Math. Anal. Appl.* 467 (2018), 638–672.

[20] He, X.; Squassina, M.; Zou, W. The Nehari manifold for fractional systems involving critical nonlinearities, *Commun. Pure Applied Anal.* 15 (2016), 1285–1308.

[21] Jeanjean, L. Two positive solutions for a class of nonhomogeneous elliptic equations, *Differential Integral Equations* 10 (1997), 609–624.

[22] Lin, T., Wei, J. Ground state of $N$ coupled nonlinear Schrödinger equations in $\mathbb{R}^n$, $n \geq 3$. *Commun. Math. Phys.* 255, 629–653 (2005).

[23] Long, W.; Peng, S. Positive vector solutions for a Schrödinger system with external source terms. *NoDEA Nonlinear Differential Equations Appl.* 27 (2020), no. 1, Art. 5, 36 pp.

[24] Mitidieri, E. Nonexistence of positive solutions of semilinear elliptic systems in $\mathbb{R}^N$, *Differential Integral Equations* 9 (1996), 465–479.

[25] Musso, M. Sign–changing blowing–up solutions for a non–homogeneous elliptic equation at the critical exponent, *J. Fixed Point Theory Appl.* 19 (2017), 345–361.

[26] Quittner, P.; Souplet, Ph. A priori estimates and existence for elliptic systems via bootstrap in weighted Lebesgue spaces, *Arch. Rational Mech. Anal.* 174 (2004), 49–81.

[27] Reichel W; Zou, H., Non–existence results for semilinear cooperative elliptic systems via moving spheres, *J. Differential Equations* 161 (2000), 219–243.

[28] Tarantello, G. On nonhomogeneous elliptic equations involving critical Sobolev exponent, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 9 (1992), 281–304.

[29] Zhu, X.P. A perturbation result on positive entire solutions of a semilinear elliptic equation, *J. Differential Equations* 92 (1991), 163–178.