I. INTRODUCTION

Our main intention in searching for the physical laws or formulating theories and models is for the adequate and accurate description of the available phenomenology. It does not make any sense to spend efforts in the building of a theoretical apparatus if a well-characterized phenomenology could not be put in accordance with the corresponding predictions. In the establishment of the present knowledge about the fundamental particles and interactions, many experimental data have been helpful. However, perhaps no other data are so special as the neutral electromagnetic pion decay. Through its phenomenology it was stated to be the most remarkable, subtle and intriguing aspect of the Quantum Field Theory (QFT); the triangle anomalies \[\pi\gamma\gamma\] whose theoretical implications go beyond the simple accurate description of the experimental data. It is fair to say that our present conception about the fundamental particles, two families of six elements, the quarks and the leptons, is a direct implication of the neutral electromagnetic pion decay phenomenology \[\pi\gamma\gamma\].

A renormalizable model is only achieved after the violations of the axial-vector Ward identity of the AVV (and AAA) one-loop triangle amplitude cancel, which requires a specific set of constituent particles; precisely those of the Standard Model \[\pi\gamma\gamma\]. The triangle anomalies were mentioned in the literature many years before their connection with the pion decay phenomenology was established, which was stated through the Sutherland-Veltman paradox \[\pi\gamma\gamma\]. The standard methods of current algebra, the LSZ formalism and PCAC hypothesis can be used to show that if all three Ward identities are required to be satisfied, when the AVV amplitude is evaluated, the low-energy limit of such an amplitude is not compatible with the predictions of a low-energy theorem which relates the AVV amplitude to the $\pi \rightarrow \gamma\gamma$ decay rate \[\pi\gamma\gamma\]. Only if an anomalous term is included in the AVV calculated amplitude, the divergence of the axial current will correspond to an amplitude with the correct low-energy limit. The fundamental nature of the anomaly phenomenon resides in the fact that no calculational method can avoid the occurrence of at least one symmetry relation violation to get the correct low-energy behaviour, which is a necessary requirement in order to put the theory in accordance with the phenomenology. The last sentence states that we cannot have a renormalizable theory with only one type of fermion due to the fact that if the theory is in agreement with the experimental data, it must violate a Ward identity, which eliminates the chance for the renormalizability. Conversely, if the theory is renormalizable, i.e., if all the Ward identities are preserved, the corresponding theory should not be useful because it does not describe the experimental data. All the general aspects previously mentioned about the anomaly phenomenon are very well-known. What remains a problem is the evaluation of the involved amplitudes and identification of the nature of the violating terms in the perturbative calculations, i.e., the justification of the anomalous term in the Ward identity connecting the AVV triangle amplitude to the $PVV$ one, through the axial Ward identity. Such a justification was proposed by Bell and Jackiw \[\pi\gamma\gamma\], before the introduction of the Dimensional Regularization (DR) \[\pi\gamma\gamma\], which remains even today as the accepted one. At least it can be found in almost all the QFT textbooks \[\pi\gamma\gamma\] in a closely related way to the point of view of the reference \[\pi\gamma\gamma\]. The main point of the argumentation resides in the divergent character, more precisely in the presence of linearly divergent Feynman integrals in the external momenta contracted expressions for the AVV amplitude. In consequence, it is assumed that the contracted expressions are ambiguous quantities since they are dependent on the internal lines momentum routing, which means that two different choices for the internal momentum labels lead to two different physical amplitudes. An expression can be led into any other one by a shift on the integrating momentum, but due to the linear divergence’s degree, a surface term must be added to the shifted amplitude, which will precisely constitute the difference between them \[\pi\gamma\gamma\]. The next step is to verify that there are no possible choices for the internal lines momenta such that all the Ward identities are simultaneously preserved. Given this impossibility, the vector Ward identities are chosen to be preserved and the axial one is assumed broken precisely by a quantity that corresponds to the required anomalous...
term, which is necessary for the correct low-energy limit of the AVV physical amplitude. In spite of this point of view having well succeeded in furnishment a justification for the anomalous term in the perturbative evaluation of the involved amplitudes, some doubts remain about the consistency of the arguments in a wider sense. In other words, if this procedure is adopted to treat other physical problems, should we expect a consistent description? This is due to the fact that from the point of view of the DR, our best reference concerning the consistency in perturbative calculations, all the momenta ambiguities are automatically eliminated. This means that the ambiguities only play a relevant role in the triangle anomaly justifications, where the DR cannot be applied. So, it is natural to ask ourselves if the violations are a consequence of the non-existence of an applicable consistent technique or if they represent a manifestation of a fundamental and unavoidable phenomenon of the nature. If it is the case, and it seems to be so, then we should expect that the source of the violations must not be related to specific ingredients of the perturbative calculations, due to the fact that if exact solutions were available, the divergences, and consequently the ambiguities, would certainly be absent while the anomalies should be still present in the problem. Another important aspect is the existence of identities at the traces level, which are valid before any calculations involving divergent structures have been made, which relate the n-point Green’s functions with an odd number of $\gamma_5$ Dirac matrix to those having an even number of such matrix. This means that, even in the case where the DR cannot be applied to the specific calculations involved, the mathematical structures to be evaluated can be connected with other ones, through exact relations, which can be evaluated within the scope of the DR. So, even in an indirect way, we can test the consistency of a procedure using the results of the DR.

The purpose of the present work is to make some clarification to the points mentioned above about the perturbative justifications of the triangle anomalies. Our main tool for the investigation is the adoption of a very general calculational strategy concerning the manipulations and calculations of divergent amplitudes. Within this strategy it is possible to avoid the explicit use of a regularization technique so that all the intrinsic arbitrariness of the perturbative calculations remain still present in the final expressions obtained this way. The mathematical objects which are crucially dependent on the specific regularization philosophy eventually applied are isolated from the independent ones. With this procedure it is easily possible to map the results into those which may be obtained in any traditional technique. This special property of our results will allow us to extract interesting, clean and sound conclusions about the perturbative justifications of the triangle anomalies. We will use these conclusions to explain the neutral pion decay through the anomalous Ward identity but without having recourse to internal momenta ambiguities.

The work is organized in the following way. In the section II we introduce the general aspects related to the discussion. In the section III we consider the lowest order expression for the involved three-point functions and their symmetry relations. The calculational strategy, to handle divergences, is introduced in the section IV. The traditional way to look at the problem is considered, from our results, in the section V. In the section VI we consider the explicit evaluation of the three-point functions involved and their symmetry relations verifications for, in the section VII, present the final remarks and conclusions.

II. THE NEUTRAL PION DECAY; GENERALITIES AND SUTHERLAND-VELTMAN PARADOX

Let us introduce in this section the generalities involved in the electromagnetic pion decay, which will be related to the discussion about the AVV anomaly. There are many textbooks where this issue can be found. We will follow in a closely related way the ref. in order to state the problem. The electromagnetic decay of a pion can be schematically represented by

$$\pi (q) \rightarrow \gamma (p) + \pi (p') .$$

The corresponding matrix element is given by

$$\langle \gamma (p, \varepsilon_1) , \gamma (p', \varepsilon_2) | \pi (q) \rangle = i (2\pi)^4 T_{\mu\nu} (q; p, p') \delta^4 (q - p - p') \varepsilon_1^\mu (p) \varepsilon_2^\nu (p') .$$

The tensor $T_{\mu\nu}$ is the amplitude connecting the external fields and needs to be specified by a theory or model. Its general structure, dictated by Lorentz invariance and CPT, can be written as

$$T_{\mu\nu} (q; p, p') = \varepsilon_{\mu\alpha\beta} p_\alpha p'_\beta \Gamma (q^2) .$$

In terms of the electromagnetic current and the pion field we write

$$T_{\mu\nu} (q; p, p') = e^2 \int d^4x d^4y e^{ipx + ip'y} \langle 0 | T (J_\mu (x) J_\nu (y)) | \pi (q) \rangle .$$
The PCAC hypothesis relates the pion field to the axial-vector current, \( \partial^\lambda A_\lambda^\pi = f_\pi m_\pi^2 \pi^a \), in such a way that we can write \( T_{\mu \nu} (q; p, p') \) only in terms of currents. Using the LSZ formalism it is possible to state such relation as

\[
T_{\mu \nu} (q; p, p') = i e^2 \frac{(q^2 - m_\pi^2)}{f_\pi m_\pi^2} \int d^4 x d^4 y e^{ip_\mu x + ip'_\nu y} \langle 0 \mid T (\partial^\lambda A_\lambda^\pi (x) J_\mu (x) J_\nu (y)) \mid 0 \rangle .
\]

(5)

The integral on the right hand side, on the other hand, can be related to another amplitude by

\[
q_\lambda T_{\lambda \mu \nu} (q; p, p') = (-i) \int d^4 x d^4 y e^{ip_\mu x + ip'_\nu y} \left\{ \langle 0 \mid T (\partial^\lambda A_\lambda^\pi (x) J_\mu (x) J_\nu (y)) \mid 0 \rangle + \langle 0 \mid T \left( [A_\lambda^0 (x) , J_\nu (y)] J_\mu (0) \right) \mid 0 \rangle \delta (x_0 - y_0) + \langle 0 \mid T \left( [A_\lambda^0 (x) , J_\mu (0)] J_\nu (y) \right) \mid 0 \rangle \delta (x_0) \right\} ,
\]

(6)

where

\[
T_{\lambda \mu \nu} (q; p, p') = \int d^4 x d^4 y e^{ip_\mu x + ip'_\nu y} \langle 0 \mid T (A_\lambda (x) J_\mu (y) J_\nu (0)) \mid 0 \rangle .
\]

(7)

Now we can analyze what the relation (6) implies for the pion decay. The first term can be associated with \( T_{\mu \nu} \) amplitude which should carry informations about the pion decay. The equation (6) states a prediction about the value of the \( \Gamma (q^2) \) in the kinematical limit \( q^2 \to 0 \). This is due to the fact that in the limit \( q_\lambda \to 0 \), the left hand side \( q_\lambda T_{\lambda \mu \nu} \), must vanish. The two commutators on the right hand side should also vanish by the current algebra properties. This means that \( T_{\mu \nu} \) should also vanish in this limit which implies that \( \Gamma (q^2) \to 0 \). Thus, if we identify \( T_{\mu \nu} \) as responsible for the pion decay description, and it obeys the predictions of the above discussed low-energy limits, the decay rate \( \pi^0 \to 2 \gamma \) vanishes for \( m_\pi^2 \to 0 \) and it is predicted a very small value at the physical \( m_\pi \). This is clearly in disagreement with the experimental data.

To arrive at this conclusion we can go by other track. We first note that the \( T_{\lambda \mu \nu} \) is a pseudo-tensor in such a way that its most general structure, by Lorentz, CPT and Bose’s statistics, is given by:

\[
T^{\lambda \mu \nu} = \varepsilon^{\mu \rho \omega \nu} p_\rho p'_\omega q^\lambda F_1 (q^2) + \varepsilon^{\lambda \mu \rho \nu} (p - p')_\rho F_2 (q^2) + \left[ \varepsilon^{\lambda \mu \omega \nu} p'^\omega - \varepsilon^{\lambda \nu \omega \mu} p'^\omega \right] p_\rho p'_\rho F_3 (q^2)
\]

(8)

To obtain the above structure we have also required the conservation of the vector current: \( p_\mu T_{\lambda \mu \nu} = p'_\nu T_{\lambda \mu \nu} = 0 \) and put the external vectors on the mass shell \( p^2 = p'^2 = 0 \) such that \( (p + p')^2 = q^2 = 2p \cdot p' \). The divergence of the axial current, or the contraction of the above equation with \( q_\lambda \), should give us the \( T_{\mu \nu} \) amplitude, which is

\[
q_\lambda T_{\lambda \mu \nu} = \varepsilon_{\lambda \mu \nu \rho} p_\rho [F_1 + F_3] .
\]

(9)

The left hand side vanishes in the limit \( q_\lambda \to 0 \), which implies that the right hand side needs to vanish too. Again the conclusion is that the ingredients used to state the result (8) lead to \( \Gamma (q^2) = 0 \) in the kinematical point \( q^2 = 0 \).

We can put what we have learned in these analyses as follows: The amplitude \( T_{\lambda \mu \nu} \) can be calculated from the point of view of any theory or model by using any calculational method. After this, four symmetry properties can be verified: three Ward identities and a low-energy limit. If all the Ward identities are obtained satisfied, then, necessarily, the low-energy behavior for such a \( T_{\lambda \mu \nu} \), which is related through the axial Ward identity to the pion decay amplitude, will give us a non-zero value, violating then the low-energy theorem obtained by the current algebra methods. If, however, we want to obtain the low-energy limit satisfied by constructing a \( T_{\mu \nu} \) amplitude, which vanishes at \( q^2 = 0 \) and simultaneously satisfies all the Ward identities, then the electromagnetic neutral pion decay rate is obtained in complete disagreement with the experimental data. Therefore, if we want a theoretical tool which adequately describes the phenomenology, it will be necessary to impose that the correct \( T_{\mu \nu} \) amplitude should have a non-zero low-energy limit. As a consequence, if we want to construct the \( T_{\lambda \mu \nu} \) such that in the limit \( q_\lambda \to 0 \)

\[
\begin{align*}
q_\lambda T_{\lambda \mu \nu} &= 0 \\
p_\mu T_{\lambda \mu \nu} &= 0 \\
p'_\nu T_{\lambda \mu \nu} &= 0,
\end{align*}
\]

(10)

we are forced to violate the axial Ward identity due to the fact that in order to obtain a vanishing value for the right hand side of the axial Ward identity, without to forbid the pion decay, it is necessary to assume the presence
of an extra term, i.e., $q_{\lambda} T_{\lambda \mu \nu} = T_{\mu \nu} + C_{\mu \nu}$, that means to break the Ward identity by the amount $C_{\mu \nu}$ which is an anomalous term. The value for the anomalous term is precisely the one of the low-energy value for the $T_{\mu \nu}$ amplitude which means that it is related to the experimental value for the electromagnetic neutral pion decay rate.

The ingredients of the preceding discussion are well-known and constitute the scope of the Sutherland-Veltman paradox. What remains as a question is the explicit evaluation of the involved amplitudes in the context of perturbative calculations due to the presence of divergences and, consequently, many types of arbitrariness. The main point is: if an anomalous term, whose value is fixed by the pion decay phenomenology, must be present in the calculated expressions, what is its source? In other words, how can we justify the origin of the anomalous term in perturbative calculations? Such a justification was given by Bell and Jackiw, that can be found in many QFT modern textbooks, and it is deeply founded in the divergences aspects involved, more precisely, in the linearly divergent character of the $T_{\lambda \mu \nu}$ perturbative amplitude. It was argued that there is an intrinsic arbitrariness due to the fact that a shift in the integrating momentum of the one-loop $T_{\lambda \mu \nu}$ amplitude gives raise to another expression which differs from the original one by a surface term whose coefficient is an undefined combination of the internal lines momentum, which is arbitrary. The justification of the anomalous term is then associated to the compulsory choices we must take, in order to give a definite value for the $T_{\lambda \mu \nu}$ perturbative amplitude. In spite of the adopted procedure being well-succeeded in the furnishment of a justification for the anomalous term in the one-loop perturbative calculations, some questions remained as interesting ones. First, the justification is founded in the ambiguities which is an exclusive ingredient of the perturbative calculation. However, the anomaly phenomenon is predicted for the exact amplitudes, in which case the divergences and their associated ambiguities are certainly absent. In view of these arguments we should expect that the origin of the anomalous term does not reside in the divergent aspects involved, which means that it cannot be associated to ambiguities. The second aspect, that invites us to think about the perturbative justification of triangle anomalies, is concerning the consistency in perturbative calculations. We would like to look at all physical amplitudes of all theories and models in the same way. This means to interpret the divergences following an unique recipe. Our main tool to handle divergences in QFT is undoubtedly the DR, which automatically eliminates all the ambiguities even that the degree of divergence is higher than the logarithmic one. This means that there is only one problem where the ambiguities are called to play a relevant role; in the pseudo-amplitudes where we find the triangle anomalies. So, the related question is: in a general method to handle the divergences, which is capable to map all the results of the DR, where this consistent technique can be applied, such that no restrictions of applicability are present for the treatment of amplitudes with an odd number of $\gamma_5$ matrix, can we explain the anomalies through ambiguities? The reason for this question is immediate: if a method maps the DR results it must automatically eliminate the ambiguities. So it is expected, also through this line of reasoning, that the perturbative justifications of the anomalous term cannot reside in the ambiguities.

After the preceding argumentation, we have stated our working lines; to investigate in this context the origin of the triangle anomaly involved in the pion decay phenomenology.

### III. $AVV$ and $PVV$ Amplitudes and Their Symmetry Relations

An explicit expression for $T_{\lambda \mu \nu}$ and $T_{\mu \nu}$, which have appeared in the preceding section is only achieved after the construction of a Lagrangian density or the involved currents. Given the hadronic character of the pion, we need to look at the decay in terms of intermediate states of quarks and anti-quarks. The lowest order Feynman diagrams connecting the external fields are triangle diagrams, with only one kind of quarks in the internal lines. Ignoring internal symmetries operators, the currents involved are

$$
\begin{align*}
V_\mu(x) &= \overline{\Psi}(x) \gamma_\mu \Psi(x) \\
A_\mu(x) &= \overline{\Psi}(x) i\gamma_\mu \gamma_5 \Psi(x) \\
P(x) &= \overline{\Psi}(x) \gamma_5 \Psi(x),
\end{align*}
$$

where $\Psi(x)$ is the massive quark field obeying the Dirac equation. As a consequence the above currents satisfy

$$
\begin{align*}
\partial^\mu V_\mu(x) &= 0 \\
\partial^\mu A_\mu(x) &= 2miP(x).
\end{align*}
$$

The corresponding expressions $AVV$ and $PVV$ amplitudes in the lowest order can be defined by

$$
\begin{align*}
T_{AVV}^{\lambda \mu \nu} &= \int \frac{d^4k}{(2\pi)^4} Tr \frac{1}{(k + k_3) - m} \gamma_5 \frac{1}{(k + k_1) - m} \gamma_\nu \frac{1}{(k + k_2) - m} \\
T_{PVV}^{\lambda \mu \nu} &= \int \frac{d^4k}{(2\pi)^4} Tr \gamma_5 \frac{1}{(k + k_3) - m} \gamma_\mu \frac{1}{(k + k_1) - m} \gamma_\nu \frac{1}{(k + k_2) - m}.
\end{align*}
$$
where \( k_1, k_2 \) and \( k_3 \) are the internal lines momenta, which are related to the external ones by their differences as follows (see fig.01)

\[
\begin{align*}
\left\{ \begin{array}{l}
k_3 - k_2 = q = p + p' \\
k_3 - k_1 = p \\
k_1 - k_2 = p'.
\end{array} \right.
\end{align*}
\]

\( (15) \)

Fig.01: Diagrammatic representation for the AVV and PVV three-point functions and for the AV two-point function, figs. (a), (b), and (c), respectively.

The necessary crossed diagram, to construct the physical process, can be obtained by changing \( \mu \) and \( \nu \) and changing the internal lines arbitrary momenta \( k \)'s to \( l \)'s, satisfying then

\[
\begin{align*}
l_3 - l_2 &= q = p + p' \\
l_3 - l_1 &= p' \\
l_1 - l_2 &= p.
\end{align*}
\]

\( (16) \)

The explicit evaluation of the above structures should allow us to verify what we have announced in the context of the Sutherland-Veltman paradox. This is precisely the step that generates the difficulties we have in understanding the involved ingredients. Both structures are, by power counting, superficially linearly divergent. In practice, after the traces evaluation, \( T_{\mu\nu}^{PVV} \) exhibits a finite character while \( T_{\lambda\mu\nu}^{AVV} \) remains linearly divergent. The last sentence implies that, in order to verify the four symmetry properties previously discussed, we need to handle, in some way, divergent integrals. Such calculations involve, as it is well-known, many types of arbitrariness. Consequently only the adoption of a consistent strategy to handle the divergences can avoid the transformation of the arbitrariness into ambiguities. By ambiguities, we mean all the dependence on the final results of the specific choices required by the involved arbitrariness. The worse aspect related to such ambiguities is the invariable breaking of some symmetries associated to them. In what follows, the aspects arbitrariness, ambiguities and symmetry violations, will play the most important role in our discussions. Before the explicit evaluation, let us verify what we can expect to find from general grounds for the amplitudes.

The three-point functions \( T_{\mu\nu}^{PVV} \) and \( T_{\lambda\mu\nu}^{AVV} \) should exhibit the symmetry properties of the amplitudes \( T_{\mu\nu} \) and \( T_{\lambda\mu\nu} \) which we have presented in the section II. The verification of such properties is made by taking the contractions of the Green’s functions with the external momenta associated with the respective Lorentz index. Such operation can be used in the level of traces to state conditions to be fulfilled in terms of the Green’s functions with a lower number of points. In principle, a divergent Green’s function with a lower number of points presents a higher degree of divergence. This means, for example, that by successive contractions we can associate a logarithmic divergent Green’s function with those linear and quadratic ones. This type of association may allow us to test the choices we need to make, in the calculations involving divergent structures, in a broader sense. For our present problem, this means to generate relations between the one-time contracted three-point functions with other three and two-point functions.

To construct such relations we may consider identities like \( (13) \)

\[
\begin{align*}
(k_3 - k_2) \left\{ \gamma_\nu \frac{1}{(k + k_2) - m} + i\gamma_\mu \gamma_5 \frac{1}{(k + k_3) - m} \right\} \\
+ \left\{ \gamma_\nu \frac{1}{(k + k_2) - m} - i\gamma_\mu \gamma_5 \frac{1}{(k + k_3) - m} \right\} = -2mi \left\{ \gamma_\nu \frac{1}{(k + k_2) - m} + i\gamma_\mu \gamma_5 \frac{1}{(k + k_3) - m} \right\}.
\end{align*}
\]

\( (17) \)

Taking the traces and integrating in the momentum \( k \) in both sides, this means that (see fig.02)

\[
(k_3 - k_2) \lambda T_{\lambda\mu\nu}^{AVV} = -2mi T_{\mu\nu}^{PVV} + T_{\mu\nu}^{AVV} (k_1, k_2; m) - T_{\nu\mu}^{AVV} (k_3, k_1; m).
\]

\( (18) \)
Following this procedure, we also note the identity structures. So, in the next section we will define our calculational strategy to handle divergences.

Also, in a similar way, we get (see fig03)

\[(k_3 - k_1)_{\mu} T^{AV}_{\lambda \nu} = T^{AV}_{\lambda \nu}(k_1, k_2; m) - T^{AV}_{\lambda \nu}(k_3, k_2; m).\]  \hfill (21)

Also, in a similar way, we get (see fig04)

\[(k_1 - k_2)_{\nu} T^{AV}_{\lambda \mu} = T^{AV}_{\lambda \mu}(k_3, k_2; m) - T^{AV}_{\lambda \mu}(k_3, k_1; m).\]  \hfill (22)

At this point we can ask ourselves: what does it mean the equations (18), (21), and (22)? Clearly, when the explicit calculations of all the involved amplitudes are performed, from the point of view of any chosen calculational strategy or regularization technique, independent of the value to be attributed by the adopted method for the involved structures, the relations should be maintained. Note that the value for the two-point functions involved seems to play a role of conditions for the corresponding Ward identity preservation (the crossed channel leads to similar relations). This does not mean that the calculation of the two-point function structures guarantees that the symmetry properties of the three-point functions are automatically verified, even when the desirable value is obtained for the right hand side of the equations. It only means that the arbitrariness involved in the explicit evaluation of the three-point functions are the same as those we find in the evaluation of the corresponding two-point functions. In what follows we expect to clarify all these points. In order to give some progress in our discussions, we need to explicitly calculate the involved structures. So, in the next section we will define our calculational strategy to handle divergences.

**IV. THE CALCULATIONAL METHOD TO HANDLE DIVERGENT INTEGRALS**

As we have discussed in the previous section, if the explicit evaluation of perturbative (divergent) amplitudes is in order, we need to specify a philosophy to handle the mathematical indefinities involved. Usually the calculations
become reliable only after the adoption of a regularization technique. After this, in the intermediary steps, we
invariably assume some specific consequences for the results, intrinsically associated to the properties attributed
to the divergent integrals resulting from the (arbitrariness) choices for the mathematical indefinities, implied by the
adopted regularization. In the final form this way obtained for the amplitudes in general, it is not possible to specify
in a clear way what are the particular effects of the adopted regularization for the result or, in other words, to evaluate
in what sense the expression is dependent on the adopted technique. In order to perform, in a way as safe as possible,
an analysis of the properties of the divergent amplitudes, including their symmetry relations and the question of
ambiguities related to the arbitrariness involved in the routing of the loop internal momenta, we need to avoid as
much as possible specific choices in intermediary steps in such a way that all possibilities remain still contained in
our final results. If it is possible, we can change the usual focus of the analysis from the verification by testing the
consistency of the regularization technique, to the identification of eventual properties that a technique should have
to be consistent. The implication of the preceding argument, that will become clear in what follows, play a central
role in the discussions we want to make.

To explicitly evaluate the divergent integrals involved, we adopted an alternative strategy to handle divergences
\[.\] Rather than the specification of some regularization, to justify all the necessary manipulations, we will assume
the presence of a regulating distribution only in an implicit way. Schematically

\[
\int \frac{d^4k}{(2\pi)^4} f(k) \rightarrow \int \frac{d^4k}{(2\pi)^4} f(k) \left\{ \lim_{\Lambda_i^2 \rightarrow \infty} G_{\Lambda_i}(k, \Lambda_i^2) \right\} = \int \frac{d^4k}{(2\pi)^4} f(k). \tag{23}
\]

Here \(\Lambda_i's\) are parameters of the generic distribution \(G(\Lambda_i^2, k)\) that, in addition to the obvious finiteness character of
the modified integral, must have two other very general properties; it must be even in the integrating momentum \(k,
\) due to Lorentz invariance maintenance, and as well as a well-defined connection limit must exist, i.e.,

\[
\lim_{\Lambda_i^2 \rightarrow \infty} G_{\Lambda_i}(k^2, \Lambda_i^2) = 1. \tag{24}
\]

The first property implies that all odd integrals vanish. The second one guarantees, in particular, that the value of
the finite integrals in the amplitudes will not be modified. Having this in mind we manipulate the integrand of the
divergent integrals to generate a mathematical expression where all the divergences are located in internal momenta
independent structures. This goal can be achieved by using an adequate identity like

\[
\frac{1}{[(k + k_i)^2 - m^2]} = \sum_{j=0}^{N} \frac{(-1)^j (k_i^2 + 2k_i \cdot k)^j}{(k^2 - m^2)^{j+1}} + \frac{(-1)^{N+1} (k_i^2 + 2k_i \cdot k)^{N+1}}{(k^2 - m^2)^{N+1}} \left( k + k_i \right)^2 - m^2, \tag{25}
\]

where \(k_i\) is (in principle) an arbitrary choice for the routing of a loop internal line momentum. The value for \(N\)
should be adequately chosen. The minor value should be the one that leads the last term in the above expression to be
present in a finite integral, and therefore, by virtue of the well-defined connection limit assumptions, the corresponding
integration can be performed without restrictions and free from specific effects of the eventual regularization. All the
remaining structures become independent of the internal lines momenta. We then eliminate all the integrals with
odd integrand, as a trivial consequence of the even character of the regulating implicit distribution. In the divergent
structures obtained this way, no additional assumptions are taken. They are organized in five objects, namely

\[
\begin{align*}
\mathbb{D}_{\alpha\beta\mu\nu} &= \int \frac{d^4k}{(2\pi)^4} \frac{24k_\mu k_\nu k_\alpha k_\beta}{(k^2 - m^2)^4} - g_{\alpha\beta} \int \frac{d^4k}{(2\pi)^4} \frac{4k_\mu k_\nu}{(k^2 - m^2)^3} \\
g_{\alpha\nu} \int \frac{d^4k}{(2\pi)^4} \frac{4k_\beta k_\mu}{(k^2 - m^2)^3} - g_{\alpha\mu} \int \frac{d^4k}{(2\pi)^4} \frac{4k_\beta k_\nu}{(k^2 - m^2)^3} \tag{26}
\end{align*}
\]

\[
\begin{align*}
\Delta_{\mu\nu} &= \int \frac{d^4k}{(2\pi)^4} \frac{4k_\mu k_\nu}{(k^2 - m^2)^3} - \int \frac{d^4k}{(2\pi)^4} \frac{g_{\mu\nu}}{(k^2 - m^2)^2} \tag{27}
\end{align*}
\]

\[
\begin{align*}
\nabla_{\mu\nu} &= \int \frac{d^4k}{(2\pi)^4} \frac{2k_\nu k_\mu}{(k^2 - m^2)^2} - \int \frac{d^4k}{(2\pi)^4} \frac{g_{\mu\nu}}{(k^2 - m^2)} \tag{28}
\end{align*}
\]

\[
\begin{align*}
I_{\text{log}}(m^2) &= \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2)^2} \tag{29}
\end{align*}
\]

\[
\begin{align*}
I_{\text{quad}}(m^2) &= \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2)^2}. \tag{30}
\end{align*}
\]
This systematization is sufficient for discussions in fundamental theories at the one-loop level \cite{11}. In non-renormalizable ones, new objects can be defined following this philosophy \cite{10,12}. In the two (or more) loop level of calculations new basic divergent structures can be equally defined in a completely analogous way. The main point is to avoid the explicit evaluation of such divergent structures, in which case a regulating distribution needs to be specified.

We can say that this procedure furnishes an universal point of view for the calculated amplitudes, once it becomes possible to map the final expressions obtained this way into the corresponding results of other techniques. All the steps followed and all the assumptions are perfectly valid in the reasonable regularization prescriptions, including the DR. All we need, to extract from our results those of a specific technique, is to evaluate the divergent structures remaining at the final expression according to the specific chosen prescription. Another important fact we call the attention is that no shifts or expansions are used in intermediary steps. We assume the most general as possible routing for all amplitudes. The potential ambiguous terms are still present in the final results. Consequently it is possible to make contact with those corresponding to the explicit evaluation of surface term involved when shifts in the internal momenta are performed. This is an important aspect of our analysis because we want to make contact with the traditional approach used to justify the triangle anomalies.

In order to clarify the above described method, to handle divergences, let us apply the calculational strategy in the treatment of some divergent integrals. For this purpose we take two of them that will play an important role in our analysis. They are two-point functions structures defined as follows

\[ (I_2; I_2^\mu) = \int \frac{d^4k}{(2\pi)^4} \frac{(1; k^\mu)}{[(k + k_1)^2 - m^2][(k + k_2)^2 - m^2]}. \]  

(31)

The first indicated above, the \( I_2 \) integral is a logarithmically divergent structure while \( (I_2)^\mu \) is a linearly one. In these structures \( k_1 \) and \( k_2 \) represent, in principle, arbitrary choices for the internal lines momenta. Therefore we can expect a dependence on \( k_1 \) and \( k_2 \) other than the difference between them only for the \( (I_2)^\mu \) integral.

Taken first the \( I_2 \) integral we choose, in the identity (25), \( N = 1 \) to rewrite both denominators. Then we get

\[ I_2 = \int \frac{d^4k}{2\pi^4} \frac{1}{(k^2 - m^2)^2} \frac{1}{[(k + k_1)^2 - m^2][(k + k_2)^2 - m^2]} \]

\[ - \int \frac{d^4k}{2\pi^4} \frac{(k_1^2 + 2k_1 \cdot k)^2}{[(k + k_1)^2 - m^2][(k + k_2)^2 - m^2]} \]

\[ - \int \frac{d^4k}{2\pi^4} \frac{(k_2^2 + 2k_2 \cdot k)^2}{[(k + k_1)^2 - m^2][(k + k_2)^2 - m^2]} \]

\[ + \int \frac{d^4k}{2\pi^4} \frac{(k_1^2 + 2k_1 \cdot k)(k_2^2 + 2k_2 \cdot k)}{[(k + k_1)^2 - m^2][(k + k_2)^2 - m^2][(k + k_1)^2 - m^2][(k + k_2)^2 - m^2]]}. \]

(32)

The right hand side exhibits the desirable form. The divergent term is located in a structure which is independent of internal momenta and which we can identify as \( I_{log} (m^2) \), defined in equation (29). The remaining structures are finite ones and we use what we call connection limit existence to drop the \( \Lambda \) subscript on the integral, or equivalently to remove the eventual regulating distribution under the argumentation that the integration and the connection limit can be perfectly interchanged. The thus obtained finite Feynman integrals can be solved without any problem. The answer can be written as

\[ I_2 = I_{log} (m^2) - \left( \frac{i}{4\pi^2} \right) Z_0((k_1 - k_2)^2; m^2), \]  

(33)

where we have introduced (in shorthand notation) the two-point function structures

\[ Z_k(\lambda_1^2, \lambda_2^2, q^2; \lambda^2) = \int_0^1 dz z^k \ln \left( \frac{q^2 z (1 - z) + (\lambda_1^2 - \lambda_2^2) z - \lambda_1^2}{(-\lambda^2)} \right). \]  

(34)

Analytical expressions can be easily obtained \cite{13} but for our present purposes this is not necessary.

Following the procedure we can also evaluate the \( I_2^\mu \) integral. The first step is the same: the use of the identity (25) to rewrite the integrand, now to the form

\[ (I_2)^\mu = \frac{1}{2} (k_1 + k_2) \int \frac{d^4k}{(2\pi)^4} \frac{4k_\alpha k_\mu}{(k^2 - m^2)^3} \]

8
Following the same prescription, for future use, we can calculate three new integrals; namely, 
\[ \int d^4k \frac{(k^2 + 2k_1 \cdot k)^2 k_\mu}{(2\pi)^4 (k^2 - m^2)^3[(k + k_1)^2 - m^2]} \]
\[ + \int d^4k \frac{(k^2 + 2k_2 \cdot k)^2 k_\mu}{(2\pi)^4 (k^2 - m^2)^3[(k + k_2)^2 - m^2]} \]
\[ + \int d^4k \frac{(k^2 + 2k_1 \cdot k)(k_3^2 + 2k_2 \cdot k)k_\mu}{(2\pi)^4 (k^2 - m^2)^2[(k + k_1)^2 - m^2][(k + k_2)^2 - m^2]} \]  
\[ (35) \]

In the above expression, we have dropped two odd \( k \) integrals, by virtue of the even character of the implicit regulating distribution as well as the \( \Lambda \) subscript in the last three terms due to the finite character. After the integration of the finite terms, we are led to the result

\[ (I_\mu)_\alpha = -\frac{1}{2}(k_1 + k_2)_\alpha (\Delta_{\alpha\mu}) \frac{1}{2}(k_1 + k_2)_\mu \left\{ I_{\log}(m^2) - \left( \frac{i}{(4\pi)^2} \right) Z_0((k_1 - k_2)^2; m^2) \right\} \]
\[ = -\frac{1}{2}(k_1 + k_2)_\alpha (\Delta_{\alpha\mu}) - \frac{1}{2}(k_1 + k_2)_\mu (I_\mu). \]  
\[ (36) \]

Following the same prescription, for future use, we can calculate three new integrals;

\[ (I_3; I_3^\mu; I_3^{\mu\nu}) = \int d^4k \frac{(1; k^\mu; k^\nu)}{(2\pi)^4 [(k + k_1)^2 - m^2][(k + k_2)^2 - m^2][(k + k_3)^2 - m^2]} \]  
\[ (37) \]

This is a very easy job because only one of them is a divergent structure. We write the results as

\[ \bullet I_3 = \left( \frac{i}{(4\pi)^2} \right) \xi_{00} \]  
\[ (38) \]
\[ \bullet (I_3)_\mu = \left( \frac{i}{(4\pi)^2} \right) \left\{ (k_1 - k_2)_\mu \xi_{01} - (k_3 - k_1)_\mu \xi_{10} \right\} - k_{1\mu} I_3 \]  
\[ (39) \]
\[ \bullet (I_3)_{\mu\nu} = \left( \frac{i}{(4\pi)^2} \right) \left\{ -\frac{g_{\mu\nu}}{2} \eta_{00} + (k_1 - k_2)_\mu (k_1 - k_2)_\nu \xi_{02} + (k_3 - k_1)_\mu (k_3 - k_1)_\nu \xi_{20} \right\} \]
\[ - (k_1 - k_2)_\mu (k_3 - k_1)_\nu \xi_{11} - (k_1 - k_2)_\nu (k_3 - k_1)_\mu \xi_{11} \right\} + \frac{g_{\mu\nu}}{4} [I_{\log}(m^2)] + \frac{\Delta_{\mu\nu}}{4} - k_{1\mu} (I_3)_{\nu} - k_{1\nu} (I_3)_\mu + k_{1\mu} k_{1\nu} I_3, \]  
\[ (40) \]

where we have introduced the three-point function structures \( \xi_{nm} \), defined as

\[ \xi_{nm}(k_3 - k_1, k_1 - k_2; m) = \int_0^1 dz \int_0^{1-z} dy \frac{z^n y^m}{Q(y, z)}, \]  
\[ (41) \]

where \( Q(y, z) = (k_1 - k_2)^2 y(1-y) + (k_3 - k_1)^2 z(1-z) + 2(k_1 - k_2) \cdot (k_3 - k_1) yz - m^2 \), and

\[ \eta_{00} = \frac{1}{2} Z_0((k_3 - k_2)^2; m^2) - \left( \frac{1}{2} + m^2 \xi_{00} \right) + \frac{1}{2}(k_3 - k_1)^2 \xi_{10} + \frac{1}{2}(k_1 - k_2)^2 \xi_{01}. \]  
\[ (42) \]

At this point it is important to emphasize the general aspects of the method. No shifts have been performed and, in fact, no divergent integrals have been calculated. All final results produced by this approach can be mapped into those of any specific technique. The finite parts are the same as they should be by physical reasons. The divergent parts can be easily obtained. All we need is to evaluate the remaining divergent structures. By virtue of this general character, the present strategy can be simply used to systematize the procedures, even if one wants to use traditional techniques. Those parts that depend on the specific regularization method are naturally separated allowing us to analyze such dependence in a particular problem, which is very interesting separately. Let us now use the above obtained result to calculate physical amplitudes.

V. TWO-POINT FUNCTIONS EVALUATION AND THE TRADITIONAL WAY TO LOOK AT ANOMALIES

With the calculational strategy presented in the previous section, the evaluation of physical amplitudes becomes a direct and simple job. First we perform the involved traces in order to write the amplitudes as a combination of...
Feynman integral even if some of them are divergent mathematical structures. After the identification of the integrals, we simply bring the results previously obtained. Let us take, as a first example, the two-point function structure obtained in the right hand side of the equations (18), (21), and (22). After the traces have been performed we write,

\[ T^{AV}_{\mu \nu}(k_1, k_2; m) = 2 \varepsilon_{\mu \rho \alpha \beta} (k_1 - k_2)_\beta \{ (k_1 + k_2)_\alpha I_2 + 2 (I_2)_\alpha \}. \]  (43)

The two integrals we need have already been calculated. Substituting the expressions (33) and (36) we get:

\[ T^{AV}_{\mu \nu} = 2 \varepsilon_{\mu \rho \alpha \beta} (k_1 - k_2)_\alpha (k_1 + k_2) \xi [\Delta_{\xi \beta}]. \]  (44)

The above expression represents the last step we can perform without assuming any arbitrary choices, in order to get a definite result for the physical amplitude. First, we need to adopt some regularization recipe or equivalent philosophy to attribute some significance to the object \( \Delta \), once it is an undefined mathematical structure. Such a choice is arbitrary because it is not present in the Feynman rules of a theory or model that has generated the AV amplitude. If the result attributed to the \( \Delta \) object presents a dependence on the specific regularization adopted, then the arbitrariness becomes an ambiguity.

Another arbitrariness, which plays an important role in such type of discussions, is that related to the choices for the routing of the internal lines momenta. In our present problem the choices are taken as the most general ones. Only the difference \( k_1 - k_2 \) is a definite physical quantity, in such a way the result (44) presents a potentially ambiguous character. For this purpose it is sufficient that the object \( \Delta \) assumes a nonzero value as a consequence of the adopted regularization. This is materialized by the dependence on \( k_1 + k_2 \). Again, we are free to choose \( k_1 \) and \( k_2 \), this is arbitrary, but if our final result presents a dependence on this choice, which seems to be the case, then the arbitrariness will become an ambiguity.

Let us now return to the question of the symmetry relations for the AVV amplitude. Inserting the results for the AV term we obtain for the equations (18), (21), and (22) respectively

\[ (k_3 - k_2) \lambda X^{AV}_{\lambda \mu \nu} = -2mi[T^{PVV}_{\mu \nu}] + 2 \varepsilon_{\mu \rho \alpha \beta} [(k_1 - k_3)_\beta (k_1 + k_3) \xi + (k_2 - k_1)_\beta (k_1 + k_2) \xi] \Delta_{\xi \alpha} \]  (45)

\[ (k_3 - k_1) \mu X^{AV}_{\lambda \mu \nu} = 2 \varepsilon_{\lambda \rho \alpha \beta} [(k_2 - k_1)_\beta (k_1 + k_2) \xi + (k_3 - k_2)_\beta (k_2 + k_3) \xi] \Delta_{\xi \alpha} \]  (46)

\[ (k_1 - k_2) \nu X^{AV}_{\lambda \mu \nu} = 2 \varepsilon_{\lambda \mu \rho \beta} [(k_3 - k_1)_\beta (k_1 + k_3) \xi + (k_2 - k_3)_\beta (k_2 + k_3) \xi] \Delta_{\xi \alpha}. \]  (47)

Before the analysis we want to make from our proposed point of view, let us show how the general result put above can furnish what we call the traditional way to look at the anomalies. The referred point of view assumes that the object \( \Delta \) presents a dependence on the specific regularization adopted, then the arbitrariness becomes an ambiguity.

The left hand side is a total derivative which, due to the Gauss theorem, is a surface term. The value can be immediately obtained as

\[ \Delta_{\mu \nu} = - \left( \frac{i}{32 \pi^2} \right) g_{\mu \nu}, \]  (49)

which is then identified as the surface term evaluated in the traditional approach. In the next step, once we have a dependence on arbitrary momenta given this value for \( \Delta \), we parameterize the internal momenta in terms of the external ones. We adopt then

\[
\begin{align*}
  k_1 &= ap' + bp \\
  k_2 &= bp + (a-1)p' \\
  k_3 &= ap' + (b+1)p,
\end{align*}
\]  (50)

where \( a \) and \( b \) are constants. Notice that: \( k_1 - k_2 = p' \), \( k_3 - k_1 = p \) and \( k_3 - k_2 = p' + p = q \), where \( q \) is obviously the momentum of the axial vector. After these substitutions we get
we will obtain the contribution of the crossed diagram whose parameterization for the internal lines momenta can be assumed as

\[
\begin{align*}
(k_3 - k_2)_{\lambda} T_{A\mu}^{AVV} &= -2miT_{\mu\nu}^{PVV} - \frac{(a - b)}{8\pi^2} i\varepsilon_{\mu\nu\alpha\beta} p^\alpha p'^\beta \\
(k_3 - k_1)_{\mu} T_{A\mu}^{AVV} &= - \frac{(1 - a)}{8\pi^2} i\varepsilon_{\lambda\nu\alpha\beta} p^\alpha p'^\beta \\
(k_1 - k_2)_{\nu} T_{A\mu}^{AVV} &= - \frac{(1 + b)}{8\pi^2} i\varepsilon_{\lambda\mu\alpha\beta} p^\alpha p'^\beta.
\end{align*}
\]

In the above expressions the arbitrariness related to the routing of internal lines now present in the parameters \(a\) and \(b\) remains. In addition we note that there are no values for \(a\) and \(b\) in such a way that all the expected relations among the involved Green’s functions are simultaneously satisfied. If we follow this line of reasoning and include the contribution of the crossed diagram whose parameterization for the internal lines momenta can be assumed as

\[
\begin{align*}
l_1 &= cp + dp' \\
l_2 &= dp' + (c - 1)p \\
l_3 &= cp + (d + 1)p',
\end{align*}
\]

we will obtain

\[
\begin{align*}
(l_3 - l_2)_{\lambda} T_{A\mu}^{AVV} &= -2miT_{\mu\nu}^{PVV} - \frac{(c - d)}{8\pi^2} i\varepsilon_{\mu\nu\alpha\beta} p_\alpha p_\beta' \\
(l_1 - l_2)_{\mu} T_{A\mu}^{AVV} &= - \frac{(d + 1)}{8\pi^2} i\varepsilon_{\lambda\nu\alpha\beta} p_\alpha p_\beta' \\
(l_3 - l_1)_{\nu} T_{A\mu}^{AVV} &= - \frac{(c - 1)}{8\pi^2} i\varepsilon_{\lambda\mu\alpha\beta} p_\alpha p_\beta'.
\end{align*}
\]

The addition of the two contributions gives us

\[
\begin{align*}
q_\lambda T_{A\mu}^{AVV} &= -2miT_{\mu\nu}^{PVV} - \frac{(a - b + c - d)}{8\pi^2} i\varepsilon_{\mu\nu\alpha\beta} p_\alpha p_\beta' \\
p_\mu T_{A\mu}^{AVV} &= - \frac{(d - a + 2)}{8\pi^2} i\varepsilon_{\lambda\nu\alpha\beta} p_\alpha p_\beta' \\
p'_{\mu} T_{A\mu}^{AVV} &= - \frac{(c - b - 2)}{8\pi^2} i\varepsilon_{\lambda\mu\alpha\beta} p_\alpha p_\beta'.
\end{align*}
\]

A closer contact with the usual results can be obtained if it is assumed the same significance for the arbitrary internal momenta, i.e., \(a = c\) and \(b = d\) in the equations (51)-(53), and (55)-(57). We get then

\[
\begin{align*}
q_\lambda T_{A\mu}^{AVV} &= -2miT_{\mu\nu}^{PVV} - \frac{(a - b)}{4\pi^2} i\varepsilon_{\mu\nu\alpha\beta} p_\alpha p_\beta' \\
p_\mu T_{A\mu}^{AVV} &= - \frac{(b - a + 2)}{8\pi^2} i\varepsilon_{\lambda\nu\alpha\beta} p_\alpha p_\beta' \\
p'_{\mu} T_{A\mu}^{AVV} &= - \frac{(c - b + 2)}{8\pi^2} i\varepsilon_{\lambda\mu\alpha\beta} p_\alpha p_\beta'.
\end{align*}
\]

Finally, we choose the value \(a = 1\) in the above expression to get

\[
\begin{align*}
q_\lambda T_{A\mu}^{AVV} &= -2miT_{\mu\nu}^{PVV} - \frac{(1 - b)}{4\pi^2} i\varepsilon_{\mu\nu\alpha\beta} p_\alpha p_\beta' \\
p_\mu T_{A\mu}^{AVV} &= - \frac{(1 + b)}{8\pi^2} i\varepsilon_{\lambda\nu\alpha\beta} p_\alpha p_\beta' \\
p'_{\mu} T_{A\mu}^{AVV} &= - \frac{(1 + b)}{8\pi^2} i\varepsilon_{\lambda\mu\alpha\beta} p_\alpha p_\beta'.
\end{align*}
\]

The result this way obtained, can be immediately recognized as the traditional one \[\square\] \[\square\] \[\square\]. It is now clear that there is no value for the \(b\) parameter in order to preserve all Ward identities. Following the usual arguments and choosing the value \(b = -1\) the \(U(1)\) gauge symmetry is maintained, the axial one is violated. However, this analysis cannot reveal us anything about the fourth symmetry property of the \(AVV\) physical amplitude, which is certainly the most important aspect related to the fundamental nature of the anomaly phenomenon. The low-energy limit of the \(AVV\) amplitude is modified because of the presence of the anomalous term and the pion decay rate is obtained

\[\square\]
in agreement with the experimental data. This result is compatible with the statements of the Sutherland-Veltman paradox: two symmetry relations and the low energy theorem are chosen to be satisfied. One symmetry relation is assumed violated. If we stop our discussion at this point, nothing new is added to the status of the problem. All our presentations can be considered as an alternative way to perform the calculations, perhaps more simple and direct but nothing more than this.

Let us now analyze the equations (45), (46), and (47), which are completely arbitrary, but now following our arguments. We call the attention to the fact that the AV amplitude, which usually is looked only as an integral, is in fact a physical amplitude. There are, therefore, physical consequences implied by the assumed value for such amplitude. All the possibilities for the choices of the undefined quantities should be considered in light of the consistency requirements imposed by physical reasons. So, we then ask for the physical constraints that the AV is submitted. First, it is a two-point function with two internal propagators. Due to unitarity reasons (Cutkosky’s rules) it should exhibit a complex threshold at the external momentum \((k_1 - k_2)^2 = 4m^2\). Looking at the expression (44) such possibility is not available by the arbitrariness still present. So, if it is not vanishing, it is not compatible with the unitarity. For the second, if a nonzero value is attributed, the amplitude may connect an axial particle to a vector one. CPT is clearly violated. We can add other symmetry reasons stated by Ward identities. The AV amplitude possesses two Lorentz indexes and therefore there are two Ward identities. The contractions with the external momentum reveal

\[(k_1 - k_2)_\mu T^{AV}_{\mu\nu} = (k_1 - k_2)_\nu T^{AV}_{\mu\nu} = 0. \quad (67)\]

If \(T^{AV}_{\mu\nu}\) is not identically zero it carries two conserved currents, which is clearly inconsistent. With the above argumentation it is clear that there are many reasons to constrain the AV to the identically zero value. Guided by these reasons we ask for the ways that we have to make choices in order to obtain this value. There are two very different options. The first is to choose a regularization such that \(\Delta_{\mu\nu}^{reg} = 0\). With this choice we automatically eliminate the potentially ambiguous terms. The second option is to choose the ambiguous dependence on internal momenta in such a way \(k_1 + k_2 = 0\). In this case it is immaterial the value for \(\Delta_{\mu\nu}\). There is a price to be paid in adopting such a choice. In all the amplitudes in all theories or models the procedure should be the same, if we want a method. If one assumes the case-by-case analysis, i.e., in one problem a zero value for the \(\Delta_{\mu\nu}\) is assumed by some reason and in another a different value may be accepted, then our worries do not make sense. The advantage of the choice \(\Delta_{\mu\nu}^{reg} = 0\) is that is a property for divergent integrals, not for the amplitudes. In the DR technique such a value is automatically zero where this method can be applied. If we choose the value for the arbitrary momenta, then we are assuming that the physical amplitudes are intrinsically ambiguous quantities. The predictive power of the QFT in perturbative solutions is completely destroyed. We can use it only for adjustments of well-known phenomenologies. Predictions cannot be made in a definite way. In addition, we are assuming that the space-time homogeneity is lost in the calculations. This is evidently unacceptable as a final form of knowledge. So, it seems very reasonable to assume that the only consistent possibility is the choice \(\Delta_{\mu\nu}^{reg} = 0\) and we ask then for the consequences. First we note that the symmetry relations (45), (46) and (47) will lead us to

\[\begin{align*}
  p_\mu T_{\lambda\mu\nu}^{\lambda\nu} &= p_\mu T_{\lambda\mu\nu}^{\lambda\nu} = 0 \\
  q_\lambda T_{\lambda\mu\nu}^{\lambda\nu} &= -2miT_{\mu\nu}^{\mu\nu}. 
\end{align*}\]

(68)  

(69)  

At first sight we can learn that this means that no symmetry relation can be violated. It is not necessarily true but if it would be the case this would not be inconsistent with the Sutherland-Veltman paradox, because such an expression for the \(T_{\lambda\mu\nu}^{AV\nu}\) should violate the low energy theorem. The point is that, until this point, we have not explicitly calculated the three-point functions yet. So, in particular, we have not accessed the low energy behavior in order to verify such property. To clarify these statements, let us consider in the next section the explicit calculation for the AVV and PVV amplitudes and verify the Ward identities by using these expressions.

**VI. EXPLICIT CALCULATIONS FOR THE AVV AND PVV AMPLITUDES**

Let us follow the procedure we have described in the section III to explicitly calculate the AVV and PVV amplitudes. First we consider the expression (13) and after performing the Dirac traces we write

\[T_{\lambda\mu\nu}^{AVV} = -4 \left\{ -F_{\lambda\mu\nu} + N_{\lambda\mu\nu} + M_{\lambda\mu\nu} + P_{\lambda\mu\nu} \right\}, \quad (70)\]

by reasons that we will see in what follows. In the above expression we have adopted the definitions
\( P_{\mu \nu} = g_{\mu \nu} \epsilon_{\alpha \beta \lambda \xi} \int \frac{d^4k}{(2\pi)^4} \frac{(k + k_1)_{\alpha} (k + k_2)_{\beta} (k + k_3)_{\xi}}{[(k + k_1)^2 - m^2][(k + k_2)^2 - m^2][(k + k_3)^2 - m^2]} \) 

\( F_{\mu \nu} = \int \frac{d^4k}{(2\pi)^4} \left\{ \epsilon_{\nu \beta \lambda \xi} (k + k_1)_{\mu} (k + k_2)_{\beta} (k + k_3)_{\xi} 
+ \epsilon_{\mu \beta \lambda \xi} (k + k_1)_{\nu} (k + k_2)_{\beta} (k + k_3)_{\xi} 
+ \epsilon_{\mu \alpha \beta \lambda} (k + k_1)_{\alpha} (k + k_2)_{\beta} (k + k_3)_{\lambda} 
+ \epsilon_{\mu \nu \beta \lambda} (k + k_1)_{\alpha} (k + k_3)_{\xi} (k + k_2)_{\lambda} \right\} \times \left\{ \frac{1}{[(k + k_1)^2 - m^2][(k + k_2)^2 - m^2][(k + k_3)^2 - m^2]} \right\} \) 

\( N_{\mu \nu} = \frac{\epsilon_{\mu \nu \alpha \lambda}}{2} \left\{ \int \frac{d^4k}{(2\pi)^4} \frac{(k + k_1)_{\alpha}}{[(k + k_1)^2 - m^2][(k + k_2)^2 - m^2][(k + k_3)^2 - m^2]} 
+ \int \frac{d^4k}{(2\pi)^4} \frac{(k + k_1)_{\alpha}}{[(k + k_1)^2 - m^2][(k + k_2)^2 - m^2][(k + k_3)^2 - m^2]} 
+ [2m^2 - (k_2 - k_3)^2] \times \int \frac{d^4k}{(2\pi)^4} \frac{(k + k_1)_{\alpha}}{[(k + k_1)^2 - m^2][(k + k_2)^2 - m^2][(k + k_3)^2 - m^2]} \right\} \) 

\( M_{\mu \nu} = m^2 \epsilon_{\mu \nu \alpha \lambda} \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{(k + k_2)_{\alpha} - (k + k_1)_{\alpha} + (k + k_3)_{\alpha}}{[(k + k_1)^2 - m^2][(k + k_2)^2 - m^2][(k + k_3)^2 - m^2]} \right\} \) 

To arrive at these results only the traces were evaluated and the identity 

\[(k + k_1) \cdot (k + k_2) = \frac{1}{2} [(k + k_1)^2 - m^2] + \frac{1}{2} [(k + k_2)^2 - m^2] + \frac{1}{2} [2m^2 - (k_1 - k_2)^2], \]

was used in the \( N_{\mu \nu} \) term. Note that \( F_{\mu \nu} \) is logarithmically divergent, \( N_{\mu \nu} \) is linearly divergent and \( M_{\mu \nu} \) is finite. All we need is to substitute the results for the integrals (33), (36), (38), (39), and (40) in the expressions for the structures we have defined. We get then

\( P_{\mu \nu} = 0 \) 

\( M_{\mu \nu} = \left( \frac{i}{(4\pi)^2} \right) \epsilon_{\mu \nu \alpha \lambda} m^2 \left\{ (k_1 - k_2)_{\alpha} (\xi_{00} - \xi_{01}) - (k_3 - k_1)_{\alpha} (\xi_{00} - \xi_{10}) \right\} \) 

\( N_{\mu \nu} = \frac{\epsilon_{\mu \nu \alpha \lambda}}{4} (k_1 - k_2)_{\alpha} \left\{ I_{\log} (m^2) - \left( \frac{i}{(4\pi)^2} \right) Z_0 \left( (k_1 - k_2)^2 ; m^2 \right) \right. 
+ \left( \frac{i}{(4\pi)^2} \right) \left[ 2m^2 - (k_3 - k_2)^2 \right] \left( 2\xi_{01} \right) \right\} 
- \frac{\epsilon_{\mu \nu \alpha \lambda}}{4} (k_3 - k_1)_{\alpha} \left\{ I_{\log} (m^2) - \left( \frac{i}{(4\pi)^2} \right) Z_0 \left( (k_1 - k_3)^2 ; m^2 \right) \right. 
+ \left( \frac{i}{(4\pi)^2} \right) \left[ 2m^2 - (k_3 - k_2)^2 \right] \left( 2\xi_{10} \right) \right\} 
- \frac{\epsilon_{\mu \nu \alpha \lambda}}{4} \left[ (k_1 + k_2)_{\beta} + (k_3 + k_1)_{\beta} \right] \Delta_{\alpha \beta} \) 

\( F_{\mu \nu} = \left( \frac{i}{(4\pi)^2} \right) (k_3 - k_1)_{\xi} (k_1 - k_2)_{\beta} \left\{ \epsilon_{\nu \beta \lambda \xi} [(k_1 - k_2)_{\mu} (\xi_{02} + \xi_{11} - \xi_{10}) 
- (k_3 - k_1)_{\mu} (\xi_{20} + \xi_{11} - \xi_{10})] 
+ \epsilon_{\mu \beta \lambda \xi} [(k_1 - k_2)_{\nu} (\xi_{02} + \xi_{11} - \xi_{01}) \right\} \)
The relations between the structure functions $\xi^\lambda_{\mu\nu}$ calculations a reasonable algebraic effort is involved. Substantial simplifications can be achieved if we note some arbitrariness involved is located in the last four terms. Only the last one is potentially ambiguous. Following a similar procedure we can also calculate the triangle amplitude. After taking Dirac traces and using the results (38) and (39) we get

\begin{equation}
T^{AVV}_{\lambda\mu\nu} = \left( \frac{i}{4\pi^2} \right) \frac{4}{2} (k_3 - k_1)_{(k_1 - k_2)} \beta \left\{ \varepsilon_{\nu\lambda\beta} [(k_3 - k_1)_{(k_2 - k_3)}] \right\}
\end{equation}

Putting all results together we write the full expression for the AVV triangle amplitude

\begin{equation}
T^{AVV}_{\lambda\mu\nu} = \left( \frac{i}{4\pi^2} \right) 4 (k_3 - k_1)_{(k_1 - k_2)} \beta \left\{ \varepsilon_{\nu\lambda\beta} [(k_3 - k_1)_{(k_2 - k_3)}] \right\}
\end{equation}

The above expression for the AVV triangle is the most general one concerning the divergences aspects. All the arbitrariness involved is located in the last four terms. Only the last one is potentially ambiguous. Following a similar procedure we can also calculate the $T^{PVV}_{\mu\nu}$ amplitude. After taking Dirac traces and using the results (38) and (39) we get

\begin{equation}
T^{PVV}_{\mu\nu} = \left( \frac{1}{4\pi^2} \right) m \varepsilon_{\mu\nu\alpha\beta} (k_1 - k_2)_{(k_3 - k_1)} \alpha (\xi_{00}).
\end{equation}

Let us verify the identities (18), (21), and (22) before any assumptions about the arbitrariness. For the necessary calculations a reasonable algebraic effort is involved. Substantial simplifications can be achieved if we note some relations between the structure functions $\xi_{nm}(q, p; m)$ and $Z_k (m^2; p^2)$. They are

\begin{equation}
q^2 (\xi_{11}) - (p \cdot q) (\xi_{02}) = \frac{1}{2} \left\{ - \frac{1}{2} Z_0 (m^2; p^2) + \frac{1}{2} Z_0 (m^2; p^2) + q^2 (\xi_{01}) \right\}
\end{equation}

\begin{equation}
q^2 (\xi_{20}) - (p \cdot q) (\xi_{11}) = \frac{1}{2} \left\{ - \frac{1}{2} + m^2 (\xi_{00}) + \frac{p^2}{2} (\xi_{01}) + \frac{3q^2}{2} (\xi_{10}) \right\}
\end{equation}
The above expressions can be rewritten in terms of other structures if we observe the equation (81) and (44). As an
the respective Ward identities, obtained by the addition of the crossed diagrams, will be violated.

As a consequence
and in consequence the equation (90) becomes identical to the equation (18).

\begin{align}
\bullet (k_3 - k_1)_\mu T_{\lambda\mu\nu}^{AVV} &= (k_3 - k_1)_\mu \Gamma_{\lambda\mu\nu}^{AVV} \\
&\quad + \left( \frac{i}{8\pi^2} \right) \varepsilon_{\nu\beta\lambda\xi} (k_3 - k_1)_\xi (k_1 - k_2)_\beta \\
\bullet (k_1 - k_2)_\mu T_{\lambda\mu\nu}^{AVV} &= (k_1 - k_2)_\mu \Gamma_{\lambda\mu\nu}^{AVV} \\
&\quad - \left( \frac{i}{8\pi^2} \right) \varepsilon_{\mu\delta\lambda\xi} (k_3 - k_1)_\xi (k_1 - k_2)_\beta \\
\bullet (k_3 - k_2)_\lambda T_{\lambda\mu\nu}^{AVV} &= (k_3 - k_2)_\lambda \Gamma_{\lambda\mu\nu}^{AVV} \\
&\quad + \left( \frac{i}{4\pi^2} \right) \varepsilon_{\mu\xi\nu\beta} (k_3 - k_1)_\xi (k_1 - k_2)_\beta \left[ 2m^2 \xi_{00} \right],
\end{align}

where we have defined

\[ \Gamma_{\lambda\mu\nu}^{AVV} = \varepsilon_{\mu\nu\beta\xi} \left[ (k_2 - k_1)_\beta + (k_3 - k_1)_\beta \right] \Delta_{\lambda\xi} + \varepsilon_{\nu\lambda\xi\beta} (k_3 - k_2)_\beta \Delta_{\mu\xi} + \varepsilon_{\mu\lambda\beta\xi} (k_3 - k_2)_\beta \Delta_{\nu\xi} + \varepsilon_{\mu\nu\lambda\alpha} \left[ (k_1 + k_2)_\beta + (k_3 + k_1)_\beta \right] \Delta_{\alpha\beta}. \]

The above expressions can be rewritten in terms of other structures if we observe the equation (81) and (44). As an example, note that

\[ (k_3 - k_2)_\lambda \Gamma_{\lambda\mu\nu}^{AVV} = 2\varepsilon_{\mu\nu\alpha\beta} \left[ (k_1 - k_3)_\beta (k_1 + k_3)_\xi + (k_2 - k_1)_\beta (k_2 + k_1)_\xi \right] \Delta_{\xi\alpha}, \]

which means that

\[ (k_3 - k_2)_\lambda \Gamma_{\lambda\mu\nu}^{AVV} = T_{\mu\nu}^{AVV}(k_1, k_2; m) - T_{\nu\mu}^{AVV}(k_3, k_1; m), \]

and in consequence the equation (90) becomes identical to the equation (18).

When we compare the results (88)-(90) with the expected identities, which are the equations (18), (21), and (22), it is immediate to see that the first one is satisfied but the two remaining ones present a disagreement. As a consequence the respective Ward identities, obtained by the addition of the crossed diagrams, will be violated.

Now if the consistent value for the \( AV \) amplitude is assumed, i.e., adopting \( \Delta_{\xi\alpha} = 0 \), the equations (88)-(90) become

\begin{align}
\bullet (k_3 - k_1)_\mu T_{\lambda\mu\nu}^{AVV} &= \left( \frac{i}{8\pi^2} \right) \varepsilon_{\nu\beta\lambda\xi} (k_3 - k_1)_\xi (k_1 - k_2)_\beta \\
\bullet (k_1 - k_2)_\mu T_{\lambda\mu\nu}^{AVV} &= -\left( \frac{i}{8\pi^2} \right) \varepsilon_{\mu\beta\lambda\xi} (k_3 - k_1)_\xi (k_1 - k_2)_\beta \\
\bullet (k_3 - k_2)_\lambda T_{\lambda\mu\nu}^{AVV} &= -2mi \left( T_{\mu\nu}^{PVV} \right),
\end{align}

The contribution of the crossed diagram gives us

\begin{align}
\bullet (l_1 - l_2)_\mu T_{\lambda\mu\nu}^{AVV} &= -\left( \frac{i}{8\pi^2} \right) \varepsilon_{\nu\beta\lambda\xi} (l_3 - l_1)_\xi (l_1 - l_2)_\beta \\
\bullet (l_3 - l_1)_\mu T_{\lambda\mu\nu}^{AVV} &= \left( \frac{i}{8\pi^2} \right) \varepsilon_{\mu\beta\lambda\xi} (l_3 - l_1)_\xi (l_1 - l_2)_\beta \\
\bullet (l_3 - l_2)_\lambda T_{\lambda\mu\nu}^{AVV} &= -2mi \left( T_{\nu\mu}^{PVV} \right),
\end{align}
so that we get

\[ p_\mu T^{A\to VV}_{\lambda\mu} = \left( \frac{i}{4\pi^2} \right) \varepsilon_{\nu\beta\lambda\xi} p_\xi p_\beta \]  

(100)

\[ p'_\nu T^{A\to VV}_{\lambda\nu} = -\left( \frac{i}{4\pi^2} \right) \varepsilon_{\mu\beta\lambda\xi} p_\xi p_\beta \]  

(101)

\[ q_\lambda T^{A\to VV}_{\lambda\mu} = -2mi \left\{ T_{\mu\nu}^{P\to VV} \right\} . \]  

(102)

These above results are perfectly compatible with what states the Sutherland-Veltman paradox: two Ward identities and the low-energy theorem have been obtained violated. Only the axial vector symmetry relation is satisfied. At this point the most important aspect is the fact that, in order to verify the violations, no fundamental symmetries were broken. The anomalies are not conditioned or associated to the typical arbitrariness present in the perturbative calculations. No particular treatment has been used. The violations have emerged in a natural way within a general calculational strategy that treats all the physical amplitudes of all theories and models in the same way. As a result of this treatment, i.e., by using the same expressions for the integrals we have used here, it can be shown that in all other three-point functions the symmetries are preserved (except obviously for the AAA which is anomalous too). In spite of such verification has been in fact performed we can convince ourselves by a simple argument. All the results of the DR can be immediately mapped by the choices \( \Delta = V = 0 \). Then if the DR is accepted as a consistent treatment, concerning the avoidance of ambiguities and symmetry preservation, it is almost immaterial such explicit verification. After these discussions we can return to the pion decay problem.

VII. PION DECAY AND ANOMALIES; FINAL REMARKS AND CONCLUSIONS

Given the obtained results in the previous section and considering all the argumentation put in the remaining ones, what is the situation for the pion decay relative to the \( AVV \) triangle anomaly? From general aspects, this specific phenomenology is only a particular consequence of a theory or model where the involved final states, the pseudo-scalar and the photon fields, can be connected through the coupling to the quark-antiquark fields. However, in a theory or model where we also have an axial-vector, there are two different paths to construct the amplitude which must describe the pion decay in two photons. The first is the direct evaluation of the \( PVV \) amplitude, equation (81), which in fact produces a low-energy limit compatible with the pion decay rate. The second is to evaluate the \( AVV \) amplitude and after this the \( PVV \) one through the identity (see equation (18))

\[ T^{PVV}_{\mu\nu} = \frac{i}{2m} \left\{ (k_3 - k_2)_{\lambda} T^{AVV}_{\lambda\mu} + T^{AV}_{\mu\nu} (k_3, k_1; m) - T^{AV}_{\mu\nu} (k_1, k_2; m) \right\} . \]  

(103)

The same \( AVV \) amplitude must also satisfy (see equation (21) and (22))

\[ (k_3 - k_1)_{\mu} T^{AVV}_{\lambda\mu} = T^{AV}_{\lambda\nu} (k_1, k_2; m) - T^{AV}_{\lambda\nu} (k_3, k_2; m) \]  

(104)

\[ (k_1 - k_2)_{\nu} T^{AVV}_{\lambda\mu} = T^{AV}_{\lambda\mu} (k_3, k_2; m) - T^{AV}_{\lambda\mu} (k_3, k_1; m), \]  

(105)

and then the things do not seem to work as expected.

This is due to the fact that by using the \( PCAC \) hypothesis, the \( LSZ \) formalism and the standard procedures of the current algebra, it is possible to derive a low-energy theorem which states that

\[ \lim_{q_\lambda \to 0} q_\lambda T^{A\to VV}_{\lambda\mu} = 0, \]  

(106)

and in addition from the current algebra methods

\[ p_\mu T^{A\to VV}_{\lambda\mu} = p'_\nu T^{A\to VV}_{\lambda\nu} = 0 \]  

(107)

\[ q_\lambda T^{A\to VV}_{\lambda\mu} = -2mi T^{P\to VV}_{\mu\nu} , \]  

(108)

which creates a paradoxical situation involving the pion decay. If the low-energy limit (106) is satisfied by the \( AVV \) amplitude, due to the axial Ward identity (108), the \( PVV \) amplitude should also vanish at this limit. Consequently, the pion decay rate is not obtained as the experimental one. So, if the \( AVV \) and \( PVV \) satisfy the four equations above, the pion does not decay. What can we make in order to reconcile the theory with the phenomenology? There are two different options:
i) The traditional approach:

The usual procedure is the admittance of an anomalous term on the right hand side of the equation (108) which must have precisely the value that corresponds to the one necessary for the correct pion decay rate. In order to justify the presence of such term, it is looked at the right hand side of the identity (103) and it is attributed to the AV amplitude the value

\[ T_{\mu\nu}^{AV}(k_i, k_j; m) = i\varepsilon_{\mu\nu\alpha\beta} (k_i - k_j)_{\alpha} (k_j + k_i)_{\beta} \left( \frac{1}{32\pi^2} \right), \]

where \( k_i - k_j \) are the external momenta. The combination \( k_j + k_i \) are undefined quantities. After the substitution of this expression in the equations (103), (104), and (105) the values for the undefined quantities are chosen in order to get the desirable situation. The justification for the existence of an anomaly in the problem resides in the fact that it is not possible to obtain the three summations of AV amplitudes involved in the equations (103), (104), and (105) simultaneously vanishing. Given this impossibility, which implies in the violation of at least one Ward identity, we are authorized to perform the most adequate choice for the undefined quantities \( k_1 + k_2, k_3 + k_2 \) and \( k_1 + k_3 \). The two vector Ward identities are chosen to be preserved and the axial one is assumed violated precisely by the amount necessary to the low-energy limit required by the pion decay phenomenology.

ii) The proposed interpretation:

In view of the argumentations we have presented in this contribution, the above described interpretation for the situation involving the pion decay phenomenology, in spite of being well succeeded in the furnishment of a justification for the needed anomalous term, is founded in ingredients which can be questionable. The main point is the attribution of a nonzero value for the AV two-point function amplitude. We have argued that this means to violate the unitarity, CPT and the axial Ward identity, once

\[ (k_j - k_i)_{\mu} T_{\mu\nu}^{AV} = 0, \]

if it is non-vanishing. This procedure makes use of the potentially ambiguous character of perturbative physical amplitudes. This means to assume that we cannot make definite predictions within the perturbative solutions of QFT in general. Only adjustments for well-known phenomenologies, like that of the pion decay, can be obtained due to the fact that, in all the calculations, completely undefined pieces can be present in a calculated amplitude. The justification through the infinities or ambiguities is also in contradiction with the logical expectations: the anomaly involved in the pion decay through the AV amplitude is predicted for the exact physical amplitudes, in which case we certainly do not have infinities and ambiguities. So it does not seem to be reasonable any justifications, even for perturbative evaluations of the involved amplitudes, based on ingredients which are exclusive of perturbative calculations. In addition, now specifically referring to the perturbative aspects, the adopted procedure is in contradiction with the interpretations for the divergences included in the DR technique, which represents our best reference concerning the consistency in perturbative calculations. The interpretations of the traditional approach, to justify the AV anomaly, cannot be mapped in the ones of DR, in a problem where both procedures can be applied.

In the proposed interpretation, which maps the DR results in all places where it is possible to apply this technique, the properties for the divergent integrals are taken as universal ones. The AV two-point function is obtained identically zero and the ambiguities are automatically eliminated. The Ward identities for the AV amplitude are verified over the explicit expression. The AV structures, which are expected to be present in contracted expressions, are in fact identified but the anomalous term, present in the calculation, is not associated to such potentially ambiguous structures. The violating term emerges as a particular property of the AV amplitude, which is independent of the infinities or ambiguities involved, as should be expected. All other two- and three-point functions, which are constructed by the same set of Feynman integrals, have their Ward identities preserved (mapping with DR).

After these important remarks we can conclude our proposed interpretation. In the construction of theories where this amplitude is present, we have no option than to choose, in an arbitrary way, what is the symmetry content we want to have. All the options are in principle equivalent, due to the fundamental nature of the anomalies. If we want to have a theory consistent with the pion decay phenomenology, which seems to be the most important ingredient, we need to redefine the AV amplitude in order to impose the correct behavior to the limit of the vanishing axial vertex momentum

\[ (T_{A\toVV}^{AV}(p, p'))_{phy} = T_{A\toVV}^{AV}(p, p') - T_{A\toVV}^{AV}(0), \]

where

\[ T_{A\toVV}^{AV}(0) = -\left( \frac{i}{4\pi^2} \right) \varepsilon_{\mu\nu\lambda\xi} [p_\xi - p'_\xi], \]
is, as it should be, the required anomalous term. Consequently, we get for the $AVV$ physical amplitude

$$\bullet \ p'_\nu \left(T_{A^\mu VV}^{\lambda\mu\nu}\right)_{phy} = 0$$ (113)
$$\bullet \ p_\mu \left(T_{A^\mu VV}^{\lambda\mu\nu}\right)_{phy} = 0$$ (114)
$$\bullet \ q_\lambda \left(T_{A^\mu VV}^{\lambda\mu\nu}\right)_{phy} = -2mi \left\{T^{\mu\nu\rightarrow VV}_{\mu\nu} \right\} - \left(\frac{i}{2\pi^2}\right) \varepsilon_{\mu\nu\alpha\beta} p_\alpha p'_\beta.$$ (115)

There is certainly an arbitrariness in the imposition of such redefinition, but it is intrinsic to the problem and is in agreement with the renormalization procedures. The degree of arbitrariness certainly imposes a price to be paid that is more acceptable because it does not plague other amplitudes or problems. Anyway, in the traditional approach to justify the anomaly, an arbitrary choice is always necessary at the end. The difference is that such a choice is taken over an undefined quantity. Here the physical amplitudes do not become ambiguous quantities and the most fundamental space-time symmetry, that is the space-time homogeneity, need not be assumed broken neither CPT or unitarity. The redefined amplitude, allows the neutral pion decay in agreement with the experimental data, as it should be.

**Acknowledgements:** G. Dallabona acknowledges a grant from CNPq/Brazil.

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