DISTRIBUTED COLORING OF GRAPHS
WITH AN OPTIMAL NUMBER OF COLORS

ÉTIENNE BAMAS AND LOUIS ESPERET

Abstract. This paper studies sufficient conditions to obtain efficient distributed algorithms coloring graphs optimally (i.e. with the minimum number of colors) in the LOCAL model of computation. Most of the work on distributed vertex coloring so far has focused on coloring graphs of maximum degree \( \Delta \) with at most \( \Delta + 1 \) colors (or \( \Delta \) colors when some simple obstructions are forbidden). When \( \Delta \) is a sufficiently large and \( k \geq \Delta - k \Delta + 1 \), for some integer \( k \Delta \approx \sqrt{\Delta} - 2 \), we give a distributed algorithm that given a \( k \)-colorable graph \( G \) of maximum degree \( \Delta \), finds a \( k \)-coloring of \( G \) in \( \min \{ O(\Delta^{\lambda} \log n), 2^{O(\log \Delta + \sqrt{\log \log n})} \} \) rounds w.h.p., for any \( \lambda > 0 \). The lower bound \( \Delta - k \Delta + 1 \) is best possible in the sense that for infinitely many values of \( \Delta \), we prove that when \( \chi(G) \leq \Delta - k \Delta \), finding an optimal coloring of \( G \) requires \( \Omega(n) \) rounds. Our proof is a light adaptation of a remarkable result of Molloy and Reed, who proved that for \( \Delta \) large enough, for any \( k \geq \Delta - k \Delta \) deciding whether \( \chi(G) \leq k \) is in \( \mathbb{P} \), while Embden-Weinert et al. proved that for \( k \leq \Delta - k \Delta - 1 \), the same problem is \( \mathbb{NP} \)-complete. Note that the sequential and distributed thresholds differ by one.

Our first result covers the case where the chromatic number of the graph ranges between \( \Delta - \sqrt{\Delta} \) and \( \Delta + 1 \). Our second result covers a larger range, but gives a weaker bound on the number of colors: For any sufficiently large \( \Delta \), and \( \Omega(\log \Delta) \leq d \leq \Delta/100 \), we prove that every graph of maximum degree \( \Delta \) and clique number at most \( \Delta - d \) can be efficiently colored with at most \( \Delta - \varepsilon d \) colors, for some absolute constant \( \varepsilon > 0 \), with a randomized algorithm running w.h.p. in \( \min \{ O(\log \Delta \log n), 2^{O(\log \Delta + \sqrt{\log \log n})} \} \) rounds.

1. Introduction

The graph coloring problem plays an important role in distributed computing, since it is used as a subroutine in distributed algorithms for a large variety of problems (see the recent survey book of Barenboim and Elkin [1] for more details and further references). The central problem in distributed coloring is the \((\Delta + 1)\)-coloring problem, where a graph of maximum degree at most \( \Delta \) has to be colored with at most \( \Delta + 1 \) colors (see [9] and [4] for the fastest deterministic and randomized algorithms to date). The bound \( \Delta + 1 \) on the number of colors is best possible in general, but it follows from Brooks’ Theorem that any connected graph of maximum degree \( \Delta \) which is neither an odd cycle nor a complete graph can indeed be colored with \( \Delta \) colors, instead of \( \Delta + 1 \), and there has been some work to find fast distributed algorithms coloring such graphs with \( \Delta \) colors. The problem was first considered by Panconesi and Srinivasan [17], and it was recently proved in [11] that the \( \Delta \)-coloring problem can be solved with a randomized algorithm running in \( O(\log \Delta) + 2^{O(\sqrt{\log \log n})} \) rounds when \( \Delta \geq 4 \), or \( O((\log \log n)^2) \) rounds when \( \Delta \) is a constant. On the other hand, it was proved in [3] that a randomized algorithm solving the \( \Delta \)-coloring problem needs \( \Omega(\log \log n) \) rounds. These results, as well as all the other algorithms mentioned in this paper, are proved in the LOCAL model of computation (see below for more details).

The main idea of \( \Delta \)-coloring is that by forbidding some simple obstructions (complete graphs and odd cycles), we can save one color (compared with the easier \((\Delta + 1)\)-coloring problem) while

Partially supported by ANR Project GATO (anr-16-ce40-0009-01) and LabEx PERSYVAL-Lab (anr-11-labx-0025).
still having a fast algorithm, whether sequential or distributed. A natural question is: can we go further? Is there some small set of obstructions (that can be easily recognized locally, at least when $\Delta$ is sufficiently large), such that if we forbid these obstructions we can find fast distributed algorithms coloring graphs of maximum degree $\Delta$ with $\Delta - 1$ colors? Or $\Delta - 2$ colors? Or $\Delta - k$ colors, for some constant $k$?

The sequential version of this question turned out to have a very precise answer. For any $\Delta$, let $k_\Delta$ be the maximum integer $k$ such that $(k + 1)(k + 2) \leq \Delta$. It can be checked that $k_\Delta = \lfloor \sqrt{\Delta + 1/4} - 3/2 \rfloor$ and thus $\sqrt{\Delta} - 3 < k_\Delta < \sqrt{\Delta} - 1$. The following was proved by Embden-Weinert et al. [7].

**Theorem 1.1** ([7]). For $3 \leq c \leq \Delta - k_\Delta - 1$, we cannot test for $c$-colorability of graphs with maximum degree $\Delta$ in polynomial time unless $P = NP$.

The following strong converse was then proved by Molloy and Reed [16].

**Theorem 1.2** ([16]). For sufficiently large (but constant) $\Delta$, and every $c \geq \Delta - k_\Delta$, there is a linear time deterministic algorithm to test whether graphs of maximum degree $\Delta$ are $c$-colorable. Furthermore, there is a polynomial time deterministic algorithm that will produce a $c$-coloring whenever one exists.

Our main result will be to prove that a similar dichotomy occurs in the LOCAL model, with a slightly larger tractability threshold ($\Delta - k_\Delta + 1$ instead of $\Delta - k_\Delta$).

**Theorem 1.3.** For sufficiently large $\Delta$, and any $\lambda > 0$, there is a distributed randomized algorithm running w.h.p. in $\min\{O(\Delta^\lambda \log n), 2^O(\log \Delta + \sqrt{\log \log n})\}$ rounds, that takes a graph $G$ with maximum degree $\Delta$ in input, and outputs, for any $c \geq \Delta - k_\Delta + 1$, either a certificate that $G$ is not $c$-colorable, or a $c$-coloring of $G$.

Here, w.h.p. (with high probability) means with probability at least $1 - O(n^{-\alpha})$, for any fixed $\alpha > 0$. We will prove that the value of $\Delta - k_\Delta + 1$ is sharp, in the following sense.

**Theorem 1.4.** When $c \leq \Delta - k_\Delta - 1$ (for any value of $\Delta$), and when $c = \Delta - k_\Delta$ (for infinitely many values of $\Delta$), there exist arbitrarily large graphs $G$ of maximum degree $\Delta$ for which $\chi(G) = c$, and such that any distributed algorithm coloring $G$ with $c$ colors takes $\Omega(n/\Delta)$ rounds.

In the LOCAL model of computation, if the algorithm runs in $r$ rounds, the color assigned to a vertex $v$ is based only on the (subgraph induced by the) vertices at distance at most $r$ from $v$. The fact that when $c \geq \Delta - k_\Delta + 1$, it can be decided whether $G$ is $c$-colorable by only looking at each neighborhood was already proved by Molloy and Reed [16] (see Theorem 4.1). In this paper, we are mostly interested in producing such a coloring in a distributed way, and it is a priori unclear that it can be done in a small number rounds. For instance, in the LOCAL model it can be decided in a single round whether a graph has maximum degree at most two (and is therefore 3-colorable), but finding a 3-coloring of a path takes an unbounded number of rounds [14].

An interesting difference between Theorems 1.3 and 1.2 (besides the fact that the sequential and distributed thresholds are not the same), is that in the sequential result it is important that $\Delta$ is a constant. If $\Delta$ depends on $n$, then Molloy and Reed [16] proved that the tractability threshold is around $\Delta - \Theta(\log \Delta)$ colors. On the other hand, in the distributed setting there is no requirement on $\Delta$ a priori, except that a gap appears between the lower and upper bounds on the round complexity when $\Delta \leq n^{1-\epsilon}$, for some arbitrary $\epsilon > 0$.

It should be mentioned that efficient algorithms in distributed computing involving solutions that are best possible from an existential point of view (for instance coloring results involving the
chromatic number) are extremely rare. The single example we know of is the following result of Schneider and Wattenhofer [19]: when $\Delta = \Omega(\log^{1+1/\log^* n} n)$ and $\chi = O(\Delta/\log^{1+1/\log^* n} n)$, they find a randomized distributing algorithm coloring graphs of maximum degree $\Delta$ and chromatic number $\chi$ with at most $(1 - 1/O(\chi))\Delta$ colors w.h.p., and running w.h.p. in $O(\log \chi + \log^* n)$ rounds. Two significant differences with our result are the requirement on $\Delta$ and the fact that the number of colors is not best possible.

Theorem 1.3 covers the situation where $\chi(G) \geq \Delta - \sqrt{\Delta} + 1$ (and in this case, gives an efficient algorithm to obtain an optimal coloring of the graph). Recall that Brooks’ theorem (and its algorithmic variants) colors graphs of maximum degree $\Delta \geq 3$ distinct from $K_{\Delta+1}$ (or equivalently, with clique number at most $\Delta$) with at most $\Delta$ colors. Our next result generalizes the algorithmic versions of Brooks’ theorem in the following direction.

**Theorem 1.5.** There exists $\Delta_0 > 0$ such that for every $\Delta \geq \Delta_0$ and $2^{59} \log \Delta \leq d \leq \frac{\Delta}{100}$, there exists a randomized distributed algorithm that given an $n$-vertex graph of maximum degree $\Delta$, either outputs a clique of size more than $\Delta - d$ if such a clique exists, or a coloring with at most $\Delta - 2^{-23}d$ colors. The round complexity is $\min\{O(\log \Delta n), 2^{O(\log \Delta + \sqrt{\log \log n})}\}$ rounds w.h.p., and in particular it is w.h.p. $O(\log n/\log \log n)$.

We note that in the setting of Theorems 1.2 and 1.1, the chromatic number is an additive factor away from the maximum degree, while the result of Schneider and Wattenhofer [19] mentioned above asks for a much larger (multiplicative) gap between $\chi$ and $\Delta$.

We start with some preliminaries on distributed computing, probability, and graph theory in Section 2. We then prove Theorem 1.5 in Section 3. It turns out that the proof of Theorem 1.5 contains several ingredients that will be reused in the proof of Theorem 1.3. In Section 4 we prove Theorem 1.4 and explain how to adapt the proof of Theorem 1.2 in [16] to prove Theorem 1.3. We conclude with some remarks in Section 5.

2. Preliminaries

2.1. **Distributed computing.** We consider the classical LOCAL model of computation, which is a distributed model in which the network corresponds to the graph under consideration, i.e. each vertex of the graph corresponds to a processor, with infinite computational power, and vertices can communicate with their neighbors in synchronous rounds (in this model there is no restriction on the size of the messages exchanged by two neighboring vertices during each round of communication). Each vertex knows the number $n$ of vertices and its own id (a distinct integer between 1 and $n$). In this paper, the vertices also know the maximum degree $\Delta$ of the graph, and some number $c$ of colors. Once the communication between the nodes is over, each vertex outputs a value (in our case, an integer between 1 and $c$ corresponding to its color in a proper coloring of the graph, or some subset of its neighbors which cannot be colored with $c$ colors). The complexity of the algorithm is the number of rounds of communication.

2.2. **Vertex coloring.** A $k$-coloring of a graph $G$ is an assignment of integers from $\{1, \ldots, k\}$ to the vertices of $G$ such that any two adjacent vertices receive distinct colors. The chromatic number $\chi(G)$ of $G$ is the least $k$ such that $G$ has a $k$-coloring.

In this paper it will be convenient to consider a slightly more general scenario, in which the colors available for each vertex are not necessarily the same. A list-assignment $L$ for $G$ is a collection of lists $L(v)$ of colors, one for each vertex $v$ of $G$. Given a list-assignment $L$, an $L$-list-coloring of $G$ is a coloring of $G$ (i.e. any two adjacent vertices receive distinct colors, as before), with the additional constraint that each vertex $v$ is colored with a color from its own list $L(v)$. A simple
greedy algorithm shows that if for each vertex \( v \), \( |L(v)| \geq d(v) + 1 \) (where \( d(v) \) denotes the degree of \( v \) in \( G \)), then \( G \) has an \( L \)-list-coloring. This is a very useful generalisation of the fact that any graph of maximum degree \( \Delta \) is \((\Delta + 1)\)-colorable.

In this paper we will repeatedly use the following two important algorithmic results on list-coloring. The first result was proved in [2].

**Theorem 2.1** ([2]). Let \( G \) be a graph of maximum degree \( \Delta \) and let \( L \) be a list-assignment such that for any vertex \( v \), \( |L(v)| \geq d(v) + 1 \). Then an \( L \)-list-coloring of \( G \) can be found w.h.p. in \( O(\log \Delta) + 2^{O(\sqrt{\log \log n})} \) rounds.

The following stronger result was then proved in [6].

**Theorem 2.2** ([6]). Let \( G \) be a graph of maximum degree \( \Delta \) and let \( L \) be a list-assignment such that for any vertex \( v \), \( |L(v)| \geq d(v) + \epsilon \Delta \), for some \( \epsilon > 0 \). Then an \( L \)-list-coloring of \( G \) can be found w.h.p. in \( O(\log(1/\epsilon)) + 2^{O(\sqrt{\log \log n})} \) rounds.

Note that Theorem 2.1 can be deduced from Theorem 2.2 by simply setting \( \epsilon = 1/\Delta \).

The setting in which these two results will be applied is the following. Let \( G \) be a graph of maximum degree \( \Delta \) with a subset \( S \) of vertices that are colored with at most \( k \) colors. We want to extend the \( k \)-coloring of \( S \) to a \( k \)-coloring of \( G \) (i.e. find a \( k \)-coloring of \( G \) that agrees with the original coloring on \( S \)).

Let \( U = V(G) - S \) be the set of uncolored vertices, and for each vertex \( u \in U \), let \( L(u) \) be the subset of colors from \( 1, \ldots, k \) that do not appear among the neighbors of \( u \) in \( S \). Note that extending the \( k \)-coloring of \( S \) to a \( k \)-coloring of \( G \) is the same as finding an \( L \)-list-coloring of \( G[U] \), the subgraph of \( G \) induced by \( U \). The following observations will be particularly useful in combination with Theorems 2.1 and 2.2.

**Observation 2.3.** If \( u \in U \) has degree at most \( \delta \) in \( G \), then \( |L(u)| \geq d_U(u) + k - \delta \), where \( d_U(u) \) denotes the number of neighbors of \( u \) in \( U \), or equivalently the degree of \( u \) in \( G[U] \).

To see this, consider first that all neighbors of \( u \) are uncolored; at this point we have \( |L(u)| = k \). Every time we color one neighbor, \( |L(u)| \) decreases by at most 1 and since at most \( \delta - d_U(u) \) vertices in the neighborhood of \( u \) are colored, the result holds.

**Observation 2.4.** If \( u \in U \) has at least \( \ell \) repeated colors in its neighborhood, then \( |L(u)| \geq d_U(u) + k - \Delta + \ell \).

To see this, for every color that is repeated, uncolor one neighbor of \( u \) that has this color. We now have a new coloring and therefore a new set \( U' \) such that \( d_{U'}(u) \geq d_U(u) + \ell \) but the list of colors that do not appear in the neighborhood of \( u \) has not changed. Now apply Observation 2.3 with \( \delta = \Delta \).

2.3. **Probabilistic tools.** Consider a set \( X \) of independent random variables, and a set \( B = B_1, \ldots, B_n \) of (typically bad) events, each depending on a subset of the variables from \( X \). Consider the graph \( H \) with vertex-set \( B \), with an edge between two events if the set of variables they depend on intersect. The graph \( H \) is called the **event dependency graph**. Let \( d \geq 2 \) be the maximum degree of \( H \), and let \( p \) be the maximum probability of an event from \( B \).

We will use the following algorithmic versions of the Lovász Local Lemma [3, 10].

**Theorem 2.5** ([5]). If \( epd^2 < 1 \), then there is a distributed randomized algorithm, running in \( H \) w.h.p. in \( O(\log_1/epd(n)) \) rounds, that finds a value assignment to the variables of \( X \) such that no event from \( B \) holds.
Theorem 2.6 ([10]). If \( 2^{15}pd^8 < 1 \), then there is a distributed randomized algorithm, running in \( H \) w.h.p. in \( 2^{O(\log d + \sqrt{\log \log n})} \) rounds, that finds a value assignment to the variables of \( X \) such that no event from \( B \) holds.

It should be noted that in each subsequent application of Theorem 2.5 or 2.6, the event dependency graph \( H \) will only be considered implicitly. The reason is that the variables of \( X \) will be associated to the vertices of some other graph \( G \), and the events from \( B \) will correspond to connected subgraphs of \( G \) of constant radius. Thus, the outcomes of Theorems 2.5 and 2.6 will be computed in \( G \) directly (the round complexity is then simply multiplied by a constant, which does not change the asymptotic complexity).

We shall also use the following version of Talagrand’s inequality (see the appendix in [16]).

Theorem 2.7 (Talagrand’s Inequality). Let \( X \) be a non-negative random variable whose value is determined by \( n \) independent trials \( T_1,...,T_n \) and satisfying the following for some \( c, r \geq 0 \):

- changing the outcome of any one trial changes the value of \( X \) by at most \( c \).
- for any \( s \), if \( X \geq s \) then there is a set of at most \( rs \) trials whose outcome certify \( X \geq s \).

Then for any \( t \geq 0 \),

\[
\mathbb{P} \left( |X - \mathbb{E}(X)| > t + 20c\sqrt{r\mathbb{E}(X)} + 64c^2r \right) \leq 4 \cdot \exp \left( \frac{t^2}{8c^2r(\mathbb{E}(X) + t)} \right)
\]

2.4. The dense decomposition. The graph decomposition described in this section is due to Reed [18] (see also [15, 16]). A somewhat similar (although not completely equivalent) decomposition was recently used by Harris et al. [13] (see also [4]) in the context of distributed \((\Delta+1)\)-coloring algorithms.

Consider a graph \( G = (V,E) \) of maximum degree \( \Delta \). We call a vertex \( d \)-dense if its neighborhood has more than \( \left( \frac{\Delta}{2} \right) - d\Delta \) edges (note that \( d \) might depend on \( \Delta \)). A vertex \( v \) that is not \( d \)-dense is said to be \( d \)-sparse.

We say that \( S, X_1, X_2, \ldots, X_t \) is a \( d \)-dense decomposition of \( G \) if:

1. \( S, X_1, X_2, \ldots, X_t \) partition \( V \).
2. every \( X_i \) has between \( \Delta - 8d \) and \( \Delta + 4d \) vertices.
3. there are at most \( 8d\Delta \) edges between \( X_i \) and \( V - X_i \).
4. a vertex is adjacent to at least \( \frac{3\Delta}{4} \) vertices of \( X_i \) if and only if it is in \( X_i \).
5. every vertex in \( S \) is \( d \)-sparse.

The sets \( X_i \) are called the dense components and \( S \) is called the sparse component. Note that a simple consequence of (4) and (2) is that each dense component has diameter at most 2, provided that \( d \leq \frac{\Delta}{8} \).

Lemma 2.8. A \( d \)-dense decomposition of \( G \) can be constructed in \( O(1) \) rounds for every \( d \leq \frac{\Delta}{100} \).

Proof. Each \( d \)-dense vertex \( v \) applies the following procedure in parallel in order to build a cluster \( D_v \):

Phase 1.

1. \( D_v = v \cup N(v) \)
2. while there is some vertex \( u \) in \( D_v \) with \( |N(u) \cap D_v| < \frac{3\Delta}{4} \), remove \( u \) from \( D_v \).
3. while there is some vertex \( u \) outside \( D_v \) with \( |N(u) \cap D_v| \geq \frac{3\Delta}{4} \), add \( u \) to \( D_v \).
Note that only vertices at distance at most two from \( v \) are added or removed from \( D_v \), so this 3-step procedure can indeed be performed in \( O(1) \) rounds. It follows from Lemma 15.2 in [15] that (i) \( v \in D_v \), (ii) every vertex \( x \) is in \( D_v \) if and only if \( |N(x) \cap D_v| \geq 3\Delta \), (iii) \( \Delta - 8d \leq |D_v| \leq \Delta + 4d \), (iv) there are at most \( 8d\Delta \) edges between \( D_v \) and \( V - D_v \), and (v) if \( x \) and \( y \) are two \( d \)-dense vertices and \( D_x \cap D_y \neq \emptyset \) then \( x \in D_y \) and \( y \in D_x \).

**Phase 2.** Now, every \( d \)-sparse vertex that is not in any cluster \( D_v \) joins the set \( S \), and every other (sparse or dense) vertex \( v \) considers the \( d \)-dense vertex \( u \) with smallest id such that \( v \in D_u \), and joins \( D_u \) (while leaving all the other sets \( D_w \) it was part of). After this step, each sparse vertex \( v \) checks whether the set \( D_u \) it joined during Phase 2 is such that \( u \) also joined \( D_u \) during Phase 2. If this is the case \( v \) remains in \( D_u \), and if not \( v \) joins \( S \).

We now prove that \( S \) together with the resulting non-empty clusters \( D_v \) form the desired \( d \)-dense decomposition of \( G \). We first note that these sets partition \( V \), as each vertex joining a cluster \( D_u \) also leaves all the other clusters it was part of. Observe now that property (v) above implies that if some \( d \)-dense vertex \( v \) does not join \( D_v \) during Phase 2, then no \( d \)-dense vertex of \( D_v \) joins \( D_v \), and since the \( d \)-sparse vertices of \( D_v \) join \( S \), then \( D_v \) is empty after Phase 2. On the other hand, property (v) implies that if \( v \) joined \( D_v \) during Phase 2, then all vertices of \( D_v \) (sparse or dense) also join \( D_v \) during this phase. Using properties (i)-(v) above, this concludes the proof of Lemma 2.8.

### 3. Graphs with small clique number

In this section we prove Theorem [1.5]. The proof is a simple combination of ideas developed in the proofs of Lemmas 10 and 16 in [16] (see also Section 10.3 in [15]). The proofs there are given for a slightly different range of parameters, so we decided to include the full proof here instead of simply pointing to appropriate parts of their results. More specifically, we will need the following two results.

**Lemma 3.1.** Let \( G \) be a graph of (sufficiently large) maximum degree \( \Delta \) and let \( \ell \geq 2^{54}\log \Delta \). Then we can find a partial coloring of \( G \) with \( \Delta/2 \) colors in \( \min\{O(\log \Delta \cdot n), 2^{O(\log \Delta + \sqrt{\log \log n})}\} \) rounds w.h.p., such that for each uncolored vertex \( v \) with at least \( \ell \Delta \) pairs of non-adjacent vertices in \( N(v) \), there are more than \( 2^{-18\ell} \) repeated colors in \( N(v) \).

**Lemma 3.2.** Let \( S, X_1, \ldots, X_t \) be a \( 2^{-4}\Delta \)-dense decomposition of a graph \( G \) of maximum degree \( \Delta \geq 30d \) and clique number at most \( \Delta - d \). Then any \( k \)-coloring of \( S \) with \( k \geq \Delta - d/48 \) colors can be extended to a \( k \)-coloring of \( G \) w.h.p. in \( O(\log_d \Delta) + 2^{O(\sqrt{\log \log n})} \) rounds.

We now explain how these two results can be combined to provide a proof of Theorem 1.5. It should be mentioned that we have made no significant effort to optimize the various constants appearing throughout the proof, and have chosen instead to focus on making the proof as simple as possible. Lemmas 3.1 and 3.2 will be proved at the end of the section.

**Proof of Theorem 1.5.** If \( G \) contains a clique on more than \( \Delta - d \) vertices, it can be found in \( O(1) \) rounds so we may assume in the remainder that \( G \) has clique number at most \( \Delta - d \).

We start by using Lemma 2.8 to compute a \( 2^{-4}\Delta \)-dense decomposition \( S, X_1, X_2, \ldots, X_t \) of \( G \). Let \( T \) be the vertices of \( S \) with degree at least \( \Delta - 2^{-5}d \) in \( S \). Since each vertex of \( v \in T \) is \( 2^{-4}d \)-sparse, \( N(v) \) contains at least

\[
\left( \Delta - 2^{-5}d \right) / 2 - \left( \Delta / 2 \right) + 2^{-4}d\Delta \geq 2^{-5}d\Delta
\]
pairs of non-adjacent vertices in $S$.

Using Lemma 3.1 with $\ell = 2^{-5}d$, we then obtain a partial coloring of $S$ with at most $\Delta/2 \leq \Delta - 2^{-24}d$ colors in $\min\{O(\log_\Delta n), 2^{O(\log_\Delta + \sqrt{\log \log n})}\}$ rounds w.h.p., such that each uncolored vertex of $T$ has more than $2^{-23}d$ repeated colors in its neighborhood. Let $U$ be the set of uncolored vertices of $S$, and for each vertex of $v \in U$, let $L(v)$ be the set of colors from $1,\ldots, \Delta - 2^{-24}d$ that do not appear in the neighborhood of $v$. We claim that

$$\text{for each } v \in U, |L(v)| \geq d_U(v) + 2^{-24}d,$$

(1)

where $d_U(v)$ denotes the number of neighbors of $v$ in $U$, or equivalently the degree of $v$ in $G[U]$.

To see why (1) holds, consider first the case $v \in U - T$. Observe that in this case $v$ has degree at most $\Delta - 2^{-5}d$ in $S$, and thus (1) follows directly from Observation 2.3 with $k = \Delta - 2^{-24}d$ and $\delta = \Delta - 2^{-5}d$ (which implies $k - \delta = \Delta - 2^{-24}d - \Delta + 2^{-5}d \geq 2^{-24}d$).

Assume now that $v \in U \cap T$. Since each uncolored vertex of $T$ has more than $2^{-23}d$ repeated colors in its neighborhood, (1) follows directly from Observation 2.4 with $k = \Delta - 2^{-24}d$ and $\ell = 2^{-23}d$ (which implies $k - \Delta + \ell = \Delta - 2^{-24}d - \Delta + 2^{-23}d = 2^{-24}d$). This concludes the proof of (1).

It follows from (1) that we can use Theorem 2.2 with $\epsilon = 2^{-24}d/\Delta$ to extend the partial coloring of $S$ to all the vertices of $S$ in $O(\log(\Delta/d)) + 2^{O(\sqrt{\log \log n})}$ rounds (w.h.p.).

It remains to extend the coloring of $S$ to the dense components $X_1,\ldots,X_\ell$. Using Lemma 3.2, the coloring of $S$ can then be extended to $X_1,\ldots,X_\ell$ in $O(\log(\Delta/d)) + 2^{O(\sqrt{\log \log n})}$ rounds (w.h.p.). It follows that the overall round complexity is the minimum of $O(\log_\Delta n)$ and $2^{O(\log_\Delta + \sqrt{\log \log n})}$.

In particular, it is w.h.p. $O(\log n/\log \log n)$, for any value of $\Delta$, which concludes the proof of Theorem 1.5.

We now turn to the proof of Lemma 3.1 which is a classical application of the probabilistic method, see Lemma 10 in [16], or Section 10.3 in [15] (which considered a slightly smaller range of values for the parameter $d$, namely $d = \Omega(\log^3 \Delta)$ instead of $d = \Omega(\log \Delta)$).

Proof of Lemma 3.1 Let $C = \frac{\Delta}{2}$. We apply the following simple randomized procedure in two steps (see Section 3.2 in [16] or Section 10.3 in [15]).

1. Each vertex $v$ with at least $\ell \Delta$ pairs of non-adjacent vertices in $N(v)$ chooses a color uniformly at random from $\{1,\ldots,C\}$, independently of the other vertices.

2. If the color $c$ chosen by $v$ is also chosen by a neighbor of $v$ at Step 1, then $v$ uncolors itself.

Note that two adjacent vertices that received the same color at Step 1 will both be uncolored at Step 2 (it is the reason why this procedure is sometimes called the wasteful coloring procedure).

The classical analysis of the procedure is as follows. For $v$ with at least $\ell \Delta$ pairs of non-adjacent vertices in $N(v)$, we define $B_v$ as the event that there are at most $2^{-18\ell}$ repeated colors in $N(v)$. We will prove that $\mathbb{P}(B_v) \leq \Delta^{-2^6}$. Since each event $B_v$ only depends of the colors of the vertices at distance at most two from $v$, the maximum degree of the dependency graph in Theorems 2.5 and 2.6 is at most $\Delta^4$ and it follows from these theorems that we can find a partial color assignment avoiding all events $B_v$ in $\min\{O(\log_\Delta n), 2^{O(\log_\Delta + \sqrt{\log \log n})}\}$ rounds w.h.p., as $2^{15} \Delta^{-2^6} (\Delta^4)^8 \leq \frac{1}{\Delta} < 1$ for sufficiently large $\Delta$.

Let $P_v$ be number of pairs of non-adjacent vertices $u,w$ in $N(v)$ such that (1) $u$ and $w$ were assigned the same color and (2) no other neighbor of $u$, $v$, or $w$ was assigned the same color. Note that the number of repeated colors in $N(v)$ after Step 2 is at least $P_v$. Since $v$ has at least $\ell \Delta$ pairs of non-adjacent vertices in $N(v)$ and at most $3\Delta$ vertices are neighbors of $u$, $v$, or $w$, we have
Lemma 16 in [16] (which only considered the special case $d = 0$), we certify this, and if $P_v$ differs from its expectation by at least $2^{-18}\ell$, then $Y_v$ or $Z_v$ differs from its expectation by at least $2^{-19}\ell$.

Note first that $\mathbb{E}(Z_v) \leq (\ell\Delta \cdot \frac{1}{C}) + 2\ell$. Observe also that any change on the color of a single vertex affects the values of $Y_v$ and $Z_v$ by at most 1 (removing the old color can only decrease the variables, by at most 1, and adding the new color can only increase the variables, also by at most 1). Moreover, if $Y_v \geq s$ there is a set of $2s$ color assignments to the vertices of $N(v)$ that certify this, and if $Z_v \geq s$ there is a set of $3s$ color assignments to the vertices at distance at most two from $v$ that certify this. We can thus apply Theorem 2.7 to the variable $Y_v$ with $c = 1$ and $r = 2$, and to the variable $Z_v$ with $c = 1$ and $r = 3$. We obtain

$$
\mathbb{P}\left( |Y_v - \mathbb{E}(Y_v)| > t + 20\sqrt{2\mathbb{E}(Y_v)} + 2^7 \right) \leq 4 \cdot \exp\left( -\frac{t^2}{24(\mathbb{E}(Y_v) + t)} \right)
$$

Take $t = 2^{-19}\ell - 20\sqrt{2\mathbb{E}(Y_v)} + 2^7$ and note that $t \geq 2^{-19}\ell - 2^6\sqrt{\ell} - 2^7 \geq 2^{-20}\ell$ for sufficiently large $\ell$ (recall that $\ell = \Omega(\log \Delta)$ and $\Delta$ is assumed to be sufficiently large). Note also that $\mathbb{E}(Y_v) + t \leq 2\ell + 2^{-19}\ell \leq 4\ell$. As a consequence,

$$
\mathbb{P}\left( |Y_v - \mathbb{E}(Y_v)| > 2^{-19}\ell \right) \leq 4 \cdot \exp\left( -\frac{2^{-40}\ell^2}{2^6\ell} \right) \leq 4 \cdot \exp\left( -2^{-46}\ell \right).
$$

For $Z_v$ we obtain similarly:

$$
\mathbb{P}\left( |Z_v - \mathbb{E}(Z_v)| > t + 20\sqrt{3\mathbb{E}(Z_v)} + 3 \cdot 2^6 \right) \leq 4 \cdot \exp\left( -\frac{t^2}{24(\mathbb{E}(Z_v) + t)} \right)
$$

By taking $t = 2^{-19}\ell - 20\sqrt{3\mathbb{E}(Z_v)} + 3 \cdot 2^6$, and noting that for sufficiently large $\Delta$, we have $2^{-20}\ell \leq t \leq 2^{-19}\ell$ and $\mathbb{E}(Z_v) + t \leq 4\ell$, we obtain:

$$
\mathbb{P}\left( |Z_v - \mathbb{E}(Z_v)| > 2^{-19}\ell \right) \leq 4 \cdot \exp\left( -\frac{2^{-40}\ell^2}{2^7\ell} \right) \leq 4 \cdot \exp\left( -2^{-47}\ell \right).
$$

Note that since $\ell \geq 2^{24}\log \Delta$, we have $4 \cdot \exp\left( -2^{-47}\ell \right) \leq \frac{1}{2}\Delta^{-2^6}$ for sufficiently large $\Delta$. It follows that the probability that $P_v$ differs from its expectation by at least $2^{-18}\ell$ is at most $\Delta^{-2^6}$, as desired. This concludes the proof of Lemma 3.1.

We conclude this section with the proof of Lemma 3.2, which is an extension of the proof of Lemma 16 in [16] (which only considered the special case $d = \sqrt{\Delta}$). A significant difference is that in [16], the coloring is extended to each dense set sequentially, while here we color all the dense sets $X_i$ at once.

**Proof of Lemma 3.2.** By the definition of a $2^4d$-dense decomposition, recall that

(1) $X_i$ has between $\Delta - d/2$ and $\Delta + d/4$ vertices.
(2) There are at most $d\Delta/2$ edges between $X_i$ and $V - X_i$.
(3) A vertex is adjacent to at least $\frac{3\Delta}{4}$ vertices of $X_i$ if and only if it is in $X_i$.
Consider a maximal matching in the complement of $X_i$ (the graph with vertex-set $X_i$ in which two vertices are adjacent if and only if they are non-adjacent in $G$). Note that the set $C$ of vertices of $X_i$ not covered by the matching forms a clique (of size at most $\Delta - d$), and since $X_i$ has size at least $\Delta - d/2$, the matching has size at least \(\frac{1}{4}(\Delta - d/2 - |C|) \geq \frac{1}{4}(\Delta - d/2 - \Delta + d) \geq d/4\).

Let $M_i$ be a set of precisely $d/4$ pairs of distinct vertices $(u_1, v_1), \ldots, (u_{d/4}, v_{d/4})$ of $X_i$, that are pairwise disjoint, and such that for any $1 \leq j \leq d/4$, $u_j$ is non-adjacent to $v_j$ in $G$. Let $U_i$ be the set of vertices of $X_i$ not covered by $M_i$, and note that $\Delta - d \leq |U_i| \leq \Delta - \frac{d}{4}$. We say that a vertex $w \in U_i$ dominates a pair $(u_j, v_j)$ of $M_i$ if $w$ is adjacent to both $u_j$ and $v_j$. Fix a pair $(u_j, v_j)$ in $M_i$, and observe that by property (3) above, the number of vertices of $U_i$ that dominate $(u_j, v_j)$ is at least $3\Delta/2 - 4|M_i| - |U_i| \geq 3\Delta/2 - d - \Delta + \frac{d}{4} \geq \Delta/2 - \frac{3d}{4} \geq |U_i|/3$ (the final inequality follows from the fact that $d \leq \Delta/30 \leq 2\Delta/9$). A simple double counting argument then shows that the number of vertices of $U_i$ that dominate at least $|M_i|/6$ pairs of $M_i$ is at least $|U_i|/5$. Let $Z_i$ be the set of such vertices of $U_i$, and note that $|Z_i| \geq |U_i|/5 \geq \Delta/5 - d/5 \geq \Delta/6$ (since $d \leq \Delta/30 \leq \Delta/6$).

We now divide $U_i - Z_i$ into 3 parts: the set $W_{i}^{0}$ of vertices of $U_i - Z_i$ of degree at most $\Delta - d$ in $G$, the set $W_{i}^{+}$ of vertices of $U_i - (Z_i \cup W_{i}^{0})$ with at least $|Z_i|/4$ neighbors in $Z_i$, and the set $W_{i}^{-}$ of vertices of $U_i - (Z_i \cup W_{i}^{0})$ with less than $|Z_i|/4$ neighbors in $Z_i$. Note that the vertices of $W_{i}^{-}$ have degree at least $\Delta - d - \Delta/2$ in $G$ (since they are not in $W_{i}^{0}$) and thus they have at least $\Delta - d - (|X_i| - 3|Z_i|/4) \geq \Delta - d - \Delta - d/4 + \Delta/8 \geq \Delta/8 - 5d/4 \geq \Delta/12$ neighbors outside of $X_i$ (in the final inequality follows from the fact that $d \leq \Delta/30$). Since there are at most $d\Delta/2$ edges between $X_i$ and $v - X_i$, we have $|W_{i}^{-}| \cdot \Delta/12 \leq d\Delta/2$ and thus $|W_{i}^{-}| \leq 6d$.

We are now ready to extend the coloring of $S$ to the sets $X_i$. We proceed in the following order.

1. We start by coloring the vertices covered by the $M_i$’s. Consider the graph $H_1$ obtained from $G$ by identifying the vertex $u_j$ with the vertex $v_j$, for each pair $(u_j, v_j)$ of each set $M_i$. The coloring of $S$ in $G$ corresponds to a coloring of $S$ in $H_1$, and we want to extend this coloring of $S$ in $H_1$ to the newly created vertices (in $G$, this will correspond to an extension of the coloring of $S$ to all the vertices covered by the $M_i$’s, such that in any pair $(u_j, v_j)$ of some $M_i$, the two vertices $u_j$ and $v_j$ are assigned the same color).

   Note that each newly created vertex $x$ in some $M_i$ has at most $\frac{\Delta}{4} + \frac{d}{4} = \Delta/2$ neighbors outside $X_i$ and at most $|M_i| \leq d/4$ neighbors among the newly created vertices of $M_i$, thus $x$ has degree at most $\Delta/2 + d/4$ in $H_1$. If $L(x)$ denotes the list of colors available for $x$ in $H_1$ (i.e. the colors that do not appear among the neighbors of $x$ in $S$), then it follows from Observation 2.3 with $k = \Delta - d/48$ and $\delta = \Delta/2 + d/4$ that $|L(x)|$ exceeds the number of neighbors of $x$ in $H_1 - S$ by at least $k - \delta = \Delta - d/48 - \Delta/2 - d/4 \geq \Delta/48$. We can thus extend the coloring of $S$ to the newly created vertices of $H_1$ in $2^{O(\sqrt{\log \log n})}$ rounds, w.h.p., using Theorem 2.2 with $\epsilon = 1/48$. In $G$, this corresponds to a coloring of the vertices covered by the $M_i$’s extending the coloring of $S$, such that for any pair $(u, v)$ in any $M_i$, $u$ and $v$ have the same color.

2. We then color $W^{-} = \bigcup_i W_{i}^{-}$. To do this, observe that since each set $W_{i}^{-}$ has size at most $6d$, and each corresponding set $M_i$ has size $d/4$, it follows that each vertex $v \in W_{i}^{-}$ has at most $\Delta/4 + 6d + 2 \cdot d/4$ neighbors that are either in $W^{-}$ or already colored. Combining Observation 2.3 (with $k = \Delta - d/48$ and $\delta = \Delta/4 + 6d + 2 \cdot d/4$) and Theorem 2.2, we can then extend the current coloring to $W^{-}$ w.h.p. in $2^{O(\sqrt{\log \log n})}$ rounds.

3. We then color $W^{+} = \bigcup_i W_{i}^{+}$. These vertices have at least $|Z_i|/4 \geq \Delta/24$ neighbors in the corresponding set $Z_i$, which are all uncolored at this point (they will be colored at the next step), thus each vertex of $W^{+}$ has at most $23\Delta/24$ neighbors that are either in $W^{+}$
or already colored. Combining Observation 2.3 (with \( k = \Delta - d/48 \) and \( \delta = 23\Delta/24 \)) and Theorem 2.2, we can then extend the current coloring to \( W^+ \) w.h.p. in \( 2^{O(\sqrt{\log \log n})} \) rounds.

(4) We now color \( Z = \bigcup_i Z_i \). Since each vertex of some \( Z_i \) is adjacent to both members of at least \( \lceil \frac{M_i}{6} \rceil \) pairs of \( M_i \), it has at least \( \frac{M_i}{6} = d/24 \) repeated colors in its neighborhood. Combining Observation 2.4 (with \( k = \Delta - d/48 \) and \( \ell = d/24 \), and thus \( k - \Delta + \ell = d/48 \)) and Theorem 2.2, we can then extend the current coloring to \( W^- \) w.h.p. in \( O(\log(\Delta/d)) + 2^{O(\sqrt{\log \log n})} \) rounds.

(5) We now color \( W^0 = \bigcup_i W_i^0 \). Each vertex in this set has degree at most \( \Delta - d \) in \( G \) and can thus be colored by combining Observation 2.3 and Theorem 2.2, we can then extend the current coloring to \( W^+ \) w.h.p. in \( O(\log(\Delta/d)) + 2^{O(\sqrt{\log \log n})} \) rounds.

This concludes the proof of Lemma 3.2.

\[ \square \]

4. Graphs with chromatic number close to the maximum degree

In this section, we prove the main result of this paper.

We start with the (fairly simple) proof of Theorem 1.4 and then prove Theorem 1.3 or rather explain how it can be deduced from appropriate parts of the proof of Theorem 1.2 in [16]. It should be noted that our assumption that \( c \geq \Delta - k \Delta + 1 \) makes the proof of Theorem 1.3 significantly easier than the proof of Theorem 1.2 in [16], where the main difficulty comes from the case \( c = \Delta - k \Delta \).

4.1. Reducers. A \( c \)-reducer in a graph \( G \) is a subset \( D \) of vertices consisting of a clique \( C \) with \( c - 1 \) vertices and a disjoint stable set \( S \) such that every vertex of \( C \) is adjacent to all of \( S \) but none of \( V(G) - D \) (see Figure 1, right). Given a graph \( G \) with a \( c \)-reducer \( D = (C, S) \), the graph \( H \) obtained from \( G \) by removing \( C \) and identifying all the vertices of \( S \) into a single vertex is called the reduction of \( G \) with respect to \( D \) (see Figure 1, left). Note that \( G \) is \( c \)-colorable if and only if \( H \) is \( c \)-colorable, and thus \( c \)-reductions preserve \( c \)-colorability and non-\( c \)-colorability.

Proof of Theorem 1.4. Let \( \Delta \) be an integer, and assume that either

- \( c \leq \Delta - k \Delta - 1 \), or
- \( c = \Delta - k \Delta \) and \( \Delta = (k + 1)(k \Delta + 2) \).

For \( i \geq 1 \), we define a graph \( G_i \) of maximum degree \( \Delta \) and a subset \( C_i \) of \( G_i \) inductively as follows. \( G_1 \) is the complete graph on \( c + 1 \) vertices, and \( C_1 \) is the set of vertices of \( G_1 \). For any \( i \geq 2 \), \( G_i \) is obtained from \( G_{i-1} \) by removing an arbitrary vertex \( v_{i-1} \) of \( G_{i-1} \), adding a stable set \( S_i \) of size \( \Delta - c + 2 \) and a \( (c - 1) \)-clique \( C_i \) such that (1) each neighbor of \( v_{i-1} \) in \( G_{i-1} \) is adjacent to exactly one vertex of \( S_i \), and (2) each vertex of \( S_i \) is adjacent to all the vertices of \( C_i \). The construction of \( G_i \) from \( G_{i-1} \) is depicted in Figure 1.

In order to make sure that the maximum degree of \( G_i \) is at most \( \Delta \), while performing (1) we split as evenly as possible the degree of \( v_{i-1} \) between the vertices of \( S_i \) (each edge between \( v_{i-1} \) and some neighbor \( u \) in \( G_{i-1} \) becomes an edge joining \( u \) and some vertex of \( S_i \) in \( G_i \), and we want the degrees of the vertices of \( S_i \) to be as balanced as possible). Since \( |S_i| = \Delta - c + 2 \), each vertex of \( C_i \) has degree \( \Delta \) in \( G_i \). Each vertex of \( S_i \) must also have degree at most \( \Delta \) so it can have up to \( \Delta - c + 1 \) neighbors in \( G_{i-1} \). Since \( v_{i-1} \) has degree at most \( \Delta \), and \( (\Delta - c + 2)(\Delta - c + 1) \geq \Delta \), the edges incident to \( v_{i-1} \) in \( G_{i-1} \) can be split among the vertices of \( S_i \) in such way that each vertex of \( S_i \) has degree at most \( \Delta \) in \( G_i \).

We now make a couple of remarks on \( G_i \). It can be observed that \( G_{i-1} \) is the reduction of \( G_i \) with respect to some \( c \)-reducer, and since \( G_1 \) is a clique on \( c + 1 \) vertices and reductions preserve \( c \)-non-colorability, \( G_i \) is not \( c \)-colorable. It is also easy to see that any proper subgraph of \( G_i \) has chromatic number at most \( c \) (see Observation 3 in [16]). Note that \( G_i \) consists of \( i \) layers, each
being the union of a clique of size at most \( c \leq \Delta \) and a stable set of size at most \( \Delta - c + 2 \leq \Delta \) (see Figure 2), and thus \( G_i \) has diameter at least \( \frac{n}{\Delta} \), where \( n \) denotes the number of vertices of \( G_i \). Let \( G \) be the graph obtained from \( G_i \) by deleting a single edge between a vertex of layer \( i/2 \) (i.e., a vertex that was added at step \( i/2 \)) and a vertex of layer \( i/2 + 1 \). As a proper subgraph of \( G_i \), \( G \) has maximum degree at most \( \Delta \) and chromatic number at most \( c \), and it can be checked that any ball of radius less than \( \frac{n}{\Delta} \) in \( G_i \) is isomorphic to a ball of the same radius in \( G \). Since \( G_i \) is not \( c \)-colorable, it follows from a classical observation\(^1\) of Linial [14], that \( G \) cannot be colored optimally (i.e., with \( c \) colors) in less than \( \frac{n}{\Delta} \) rounds. This concludes the proof of Theorem 1.4. \( \square \)

### 4.2. Overview of the proof of Theorem 1.3

We start by considering the first part of the statement of Theorem 1.3 if \( G \) is not \( c \)-colorable, then we are supposed to output a certificate that \( G \) is not \( c \)-colorable. In order to do so, we will use the following result of Molloy and Reed (Theorem 5 in [16]).

\(^1\)This observation is not explicitly stated in [14], but is the essence of the proof of Theorem 3.1 in that paper. Namely, if two graphs \( G, H \) are such that \( |V(H)| \leq |V(G)| \) and any ball of radius \( t + 1 \) in \( H \) is isomorphic to some ball of radius \( t + 1 \) in \( G \), then the \( t \)-neighborhood graph \( N_t(H) \) of \( H \) is a subgraph of the \( t \)-neighborhood graph \( N_t(G) \) of \( G \). Since \( H \) is a subgraph of \( N_t(H) \), it is also a subgraph of \( N_t(G) \). It follows from [14] Proposition 2.3(1)] that a graph \( G \) can be \( c \)-colored in \( t \) rounds if and only if its \( t \)-neighborhood graph \( \chi_t(G) \) is \( c \)-colorable. This implies that \( G \) cannot be colored with less than \( \chi(H) \) colors in \( t \) rounds in the LOCAL model.
**Theorem 4.1.** For sufficiently large $\Delta$, and for $c \geq \Delta - k_\Delta + 1$, if $G$ has maximum degree at most $\Delta$, and $\chi(G) > c$, then there is some vertex $v$ in $G$ such that the subgraph induced by $\{v\} \cup N(v)$ is not $c$-colorable.

In the **LOCAL** model of computation, testing the $c$-colorability of all closed neighborhoods (i.e. all the balls of radius 1) in $G$ can be done in a constant number of rounds, and any vertex finding a non-$c$-colorable subgraph in its closed neighborhood can simply output this subgraph as a certificate of non-$c$-colorability of $G$. It might be worth pointing that we heavily use the unbounded computational power of the nodes in the **LOCAL** model here when $\Delta \gg \log n$. However, when $\Delta = O(\log n)$, all the closed neighborhoods have logarithmic size, so testing their $c$-colorability takes polynomial time (in $n$) in any classical model of computation. Moreover, when $\Delta = O(1)$ the same task can be performed in constant time in any classical model of computation.

We can now assume that $G$ is $c$-colorable, and the goal is to find a $c$-coloring of $G$ in $\min\{O(\Delta^3 \log n), 2^{O(\log \Delta + \log \log n)}\}$ rounds w.h.p., for any $\lambda > 0$. The high-level description of the proof is as follows: we set $d = 10^6 \sqrt{\Delta}$ and start by computing a $d$-dense decomposition $S, X_1, \ldots, X_t$ of $G$. We then delete all the sets $X_i$ that are $c$-reducers or such that $G[X_i]$ has a matching of size at least $100\sqrt{\Delta}$. These sets will be colored at the very end, once the rest of the graph will be colored, using a proof very similar to that of Lemma 3.2 in $O(\log \Delta) + 2^{O(\log \log n)}$ additional rounds (Lemmas 4.2 and 4.3). So we can assume that no set $X_i$ is a $c$-.reducer or has a large antimatching. Using this assumption, we then find a specific $c$-coloring in each set $X_i$, independently of the other sets $X_j$, with desirable properties (Lemma 4.4). Using this coloring of each set $X_i$, we will construct a new graph $F$ from $G$ by contracting the color classes from the dense sets into single vertices, and adding suitable edges at strategic places in the graph (Lemma 4.5). All these contractions and edge additions can be easily simulated in $G$, since they involve pairs of vertices at distance at most 4 apart. The final part will consist in coloring $F$ with $c$ colors, and from this coloring it will be easy to deduce a $c$-coloring of $G$. Note that because of the edge additions and contraction, the maximum degree of $F$ is not bounded by $\Delta$ anymore, but it remains $O(\Delta)$. The coloring of $F$ is then obtained by a very intricate semi-random process. Fortunately, for us it boils down to repeated applications of the Lovász Local Lemma (more precisely, $O(\Delta^3)$ successive applications, for any $\lambda > 0$), and we just need to make sure that Theorems 2.5 and 2.6 can be substituted everywhere in the proof (Lemma 4.6). With this high-level view in mind, we now proceed with the proof.

4.3. **Proof of Theorem 1.3.** Let $d = 10^6 \sqrt{\Delta}$. We first compute a $d$-dense decomposition $S, X_1, \ldots, X_t$ of $G$ in $O(1)$ rounds using Lemma 2.8

A $c$-reducer $D = (C, S')$ is said to be deletable if there are fewer than $c$ vertices in $G - D$ with a neighbor in $S$. Observe that if $D = (C, S')$ is a deletable $c$-reducer in $G$, then any $c$-coloring of $G - D$ can be extended to $D$ (since there is a color which does not appear in the neighborhood of $S'$ in $G - D$). It was observed in [16] Observation 8] that when $c \geq \Delta - k_\Delta + 1$, any $c$-reducer is deletable. It has the following consequence.

**Lemma 4.2.** Let $X^r$ be the union of all the $c$-reducers $X_i$. Then any $c$-coloring of $G - X^r$ can be extended to $G$ w.h.p. in $O(\log \Delta) + 2^{O(\log \log n)}$ rounds.

**Proof.** For each $c$-reducer $X_i = (C_i, S_i)$, perform the reduction of $G$ with respect to $X_i$ (i.e. delete the clique $C_i$, and identify all the vertices of $S_i$ into a single vertex $v_i$). Let $R$ be the resulting graph, and let $N$ be the set of newly created vertices in $R$. Note that the $c$-coloring of $G - X^r$ corresponds to a $c$-coloring of $R - N$, and our goal is simply to extend this coloring to $R$ (once this
is done, we only have to assign the color of $v_i$ to all the vertices of the stable set $S_i$ in $G$, and to color $C_i$ with the $c - 1$ colors distinct from that of $v_i$, which can clearly be done in $O(1)$ rounds.

Note since each $X_i$ we consider here is deletable, each vertex $v_i \in N$ has degree at most $c - 1$ in $R$. It follows from Observation 2.3 and Theorem 2.1 that the $c$-coloring of $R - N$ can be extended to $R$ in $O(\log \Delta) + 2^{O(\sqrt{\log \log n})}$ rounds w.h.p., as desired. \hfill \Box

We say that a dense set $X_i$ is if hollow if $\overline{G[X_i]}$ (the complement of $G[X_i]$) contains a matching of size at least $100\sqrt{\Delta}$. We now rephrase Lemma 16 from [16] for our convenience (the proof of Lemma 4.3 follows the same lines as that of Lemma 3.2).

Lemma 4.3. Let $X^h$ be the union of the all the hollow sets $X_i$. Then any $c$-coloring of $G - X^h$ can be extended to $G$ w.h.p. in $O(\log \Delta) + 2^{O(\sqrt{\log \log n})}$ rounds.

We temporarily delete from $G$ all the $X_i$ that are $c$-reducers or hollow. These sets of vertices will be colored at the very end using Lemmas 4.2 and 4.3. Let $H$ be the graph obtained from $G$ by removing the dense components from Lemmas 4.2 and 4.3. Note that the restriction of the decomposition $S, X_1, \ldots, X_t$ to $H$ is still a $d$-dense decomposition of $H$, and for convenience we keep denoting it in this way (even if some sets $X_i$ have disappeared). It follows from our construction that no dense set $X_i$ in $H$ is a $c$-.reducer or is such that $\overline{H[X_i]}$ contains a matching of size at least $100\sqrt{\Delta}$.

Given a subset $Y$ of vertices from some dense component $X_i$, an external neighbor of $Y$ is a vertex outside of $X_i$ with a neighbor in $Y$. Given a $c$-coloring of $X_i$, we define $C_i$ as the set of vertices of $X_i$ whose color class is a singleton. We say that a $c$-coloring of $X_i$ is nice if:

1. $C_i$ is a clique of size at least $\Delta - 2 \cdot 10^6 \sqrt{\Delta}$,
2. each vertex from any color class of size at least $3$ is adjacent to all the vertices of $C_i$, and
3. if $\{x, y\}$ is a color class of size $2$, then either there is $z \in C_i$ such that $x, y$ are both adjacent to all the vertices of $C_i - \{z\}$, or one of $x, y$ is adjacent to all the vertices of $C_i$ and the other is adjacent to all but at most $\frac{\Delta}{2} + 10^7 \sqrt{\Delta}$ vertices of $C_i$.

Note that the unique $c$-coloring of a $c$- reducer is nice. Lemma 4.3 (or more specifically the fact that no set $X_i$ in $H$ contains a large antimalting) now allows us to use the following result of [16].

Lemma 4.4 (Lemmas 19, 20, 21, and 25 in [16]). Each dense set $X_i$ of $H$ has a nice $c$-coloring such that:

a) If a color class is not the unique largest color class in $X_i$, then it has at most $\frac{\Delta}{2} + 10\sqrt{\Delta}$ external neighbors.

b) Every color class of $X_i$ has at most $c - \sqrt{\Delta} + 3$ external neighbors.

c) If there is a color class of $X_i$ with more than $c - 10^8 \sqrt{\Delta}$ external neighbor, then $|C_i| \geq c - 2 \cdot 10^8$ and each vertex of $C_i$ has at most $3 \cdot 10^8$ external neighbors.

d) If there is a color class of $X_i$ with more than $c - 2\sqrt{\Delta} + 3$ external neighbors then $|C_i| = c - 1$ and each vertex of $C_i$ has at most 5 external neighbors.

e) If there is a color class of $X_i$ with more than $c - 2\Delta^{3/4}$ external neighbors then $|C_i| \geq c - 5\Delta^{1/4}$ and each vertex of $C_i$ has at most $8\Delta^{1/4}$ external neighbors.

We stress that the union of the $c$-colorings of each of the dense components $X_i$ is not necessarily a $c$-coloring of the union of the dense components: there might be some edges between vertices of different sets $X_i$ having the same color. It should be noted that parts (b)–(e) of this result, as stated here, look a bit different from their counterparts from Lemma 25 in [16]. Indeed, each of properties (b)–(e) in Lemma 25 from [16] starts by the precondition “If $X_i$ is not a reducer of a near-reducer”.
We assumed earlier that $X_i$ is not a $c$-reducer, so this part of the precondition can certainly be omitted in our case. A $c$-near-reducer is a subgraph $D$ which is the union of a clique $C$ of size $c - 1$ and a stable set $S'$ of size $\Delta - c + 1$, such that each vertex of $C$ is adjacent to every vertex of $S'$ (in particular each vertex of $C$ has at most one neighbor outside $D$). Note that each vertex of $S'$ has at most $\Delta - c + 1$ neighbors outside $D$, and thus $S'$ has at most $(\Delta - c + 1)^2$ neighbors outside $D$. Since $c \geq \Delta - k_\Delta + 1$ and $\sqrt{\Delta} - 3 < k_\Delta = \left[\sqrt{\Delta + 1/4} - 3/2\right] \leq \sqrt{\Delta + 1/4} - 3/2$, $S'$ has at most 

$$(\Delta - c + 1)^2 \leq k_\Delta^2 \leq \Delta - 3k_\Delta - 2 \leq c - 2k_\Delta - 3 \leq c - 2\sqrt{\Delta} + 3$$

neighbors outside $D$. In particular, in our case (i.e. when $c \geq \Delta - k_\Delta + 1$), any dense set $X_i$ which is a $c$-near-reducer satisfies Lemma 4.4(a)–(e), so we can indeed remove the preconditions from Lemma 25 in [16]. Note also that since each dense set $X_i$ has diameter at most 2, a nice coloring of each $X_i$ with the additional properties of Lemma 4.4 can be found in $O(1)$ rounds.

Based on the nice $c$-coloring of each of the dense components $X_i$ resulting from Lemma 4.4, we now construct (locally) a new graph $F$ from $H$, which will be easier to color with a semi-random procedure, and such that any $c$-coloring of $F$ can be turned (locally and efficiently) into a $c$-coloring of $H$.

**Lemma 4.5** (Lemma 12 in [16]). We can construct locally in $H$ in $O(1)$ rounds a graph $F$ of maximum degree at most $10^9\Delta$ (such that a $c$-coloring of $H$ can be deduced from any $c$-coloring of $F$ in $O(1)$ rounds) and find a partition of the vertices of $F$ into $S, B, A_1, \ldots, A_t$ such that:

(a) Every $A_i$ is a clique with $c - 10^8\sqrt{\Delta} \leq |A_i| \leq c$.

(b) Every vertex of $A_i$ has at most $10^8\sqrt{\Delta}$ neighbors in $F - A_i$.

(c) There is a set $A_{i,j} \subseteq B$ of $c - |A_i|$ vertices which are adjacent to all of $A_i$. Every other vertex of $F - A_i$ is adjacent to at most $\frac{3}{4}\Delta + 10^8\sqrt{\Delta}$ vertices of $A_i$.

(d) Every vertex of $S$ either has fewer than $\Delta - 3\sqrt{\Delta}$ neighbors in $S$ or has at least $900\Delta^{3/2}$ non-adjacent pairs of neighbors within $S$.

(e) Every vertex of $B$ has fewer than $c - \sqrt{\Delta} + 9$ neighbors in $F - \bigcup_j A_j$.

(f) If a vertex $v \in B$ has at least $c - \Delta^{3/4}$ neighbors in $F - \bigcup_j A_j$, then there is some $i$ such that:

- $v$ has at most $c - \sqrt{\Delta} + 9$ neighbors in $F - A_i$ and every vertex of $A_i$ has at most $30\Delta^{1/4}$ neighbors in $F - A_i$.

(g) For every $A_i$, every two vertices outside of $A_i \cup \bigcup_j A_j$ which have at least $2\Delta^{9/10}$ neighbors in $A_i$ are joined by an edge of $F$.

We explain briefly how the graph $F$ is constructed in [16] (to stress that the construction can indeed be performed locally in $H$ (and then in $G$).

The construction starts by doing the following for each dense component $X_i$: each color class of size at least 2 is contracted into a single vertex, and vertices and edges are added inside $X_i$ to make it into a clique $D_i$ of size precisely $c$. It can be proved using Lemma 4.4 that the maximum degree does not increase too much and that each clique $D_i$ is not much larger than $C_i$ (see Lemma 29 in [16]).

A significant issue when trying to find a $c$-coloring of $H$ (or rather the current modification of $H$) is that given a clique $C_i$, there might be vertices outside $D_i$ that have many neighbors (say more than $\frac{3\Delta}{4}$) in $C_i$. Each such vertex must be in $D_j - C_j$, for some $j \neq i$. Consider such a vertex $v \in D_j - C_j$, with many neighbors in $C_i$. We need to make sure that the color of $v$ will be used by one of the few non-neighbors of $v$ in $C_i$, and one way to do it is, for each vertex $w \in D_i - C_i$, to construct a set $R_w$ of vertices with many neighbors in $C_i$ such that $\{w\} \cup R_w$ is a stable set.
and every vertex with many neighbors in some $C_i$ lies in such a set $R_w$. We then contract each set \{w\} $\cup$ $R_w$ into a single vertex (this will force that all these vertices have the same color at the end), and denote by $A_i$ the set $C_i$ after the removal of the vertices $w$ for which some set $R_w$ was defined. We also set $All_i = D_i$ – $A_i$. Again it can be proved that the maximum degree does not increase too much and each $A_i$ is not too small compared to $C_i$ (see Lemma 30 in [16]).

A second issue (related to the issue described above) is that we need to prevent that many different external neighbors of $A_i$ are all colored with the same color, and their neighborhoods cover $A_i$ (this would prevent this color from being used in $A_i$). The way it is solved in [16] is by adding an edge between every pair of external neighbors of $A_i$ having at least $\Delta^{9/10}$ neighbors in $A_i$. It is proved (see Lemma 31 in [16]) that it does not increase the maximum degree too much and is enough to deduce properties properties (a)–(g) of Lemma 4.5 (the issue raised in this paragraph is in particular related to property (g)).

To sum up, $F$ has been obtained from $H$ by identifying (or adding edges between) pairs of vertices at distance at most 4, since each dense component has diameter at most 2 and any two vertices that have been identified or joined by an edge have a neighbor in the same dense component. Moreover, each modification has been carried out independently by each dense set $X_i$ (even if the modifications had some impact outside of $X_i$), so $F$ can be simulated by $H$ (and then by $G$) with at most a small multiplicative loss on the round complexity. It is also clear that a $c$-coloring of $H$ can be obtained from any $c$-coloring of $F$ in $O(1)$ rounds.

It remains to show how to efficiently color $F$ with $c$ colors.

**Lemma 4.6.** The graph $F$ described in Lemma 4.5 can be colored with $c$ colors w.h.p. in $\min\{O(\Delta^3 \log n), 2^{O(\log \Delta + \sqrt{\log \log n})}\}$ rounds, for any $\lambda > 0$.

We will be rather brief here (the proof of the corresponding sequential statement, Lemma 13 in [16], takes 20 pages). Consider some $1 \leq i \leq t$. Since $All_i \cup A_i$ forms a clique of size $c$, we need to make sure that the colors that do not appear in $All_i$ do not appear either on too many external neighbors of $A_i$. A key property of the construction of $F$ (see properties (c) and (g) in Lemma 4.5) is that for any color $x$, there is at most one vertex $v \not\in All_i \cup A_i$ having at least $2\Delta^{9/10}$ neighbors in $A_i$ that is colored $x$, and moreover $v$ has at most $\frac{3}{4}\Delta + o(\Delta)$ neighbors in $A_i$. The goal will be to maintain this property throughout the whole process, namely that all of the time, at most $\frac{3}{4}\Delta + o(\Delta)$ vertices of $A_i$ have a neighbor colored $x$ outside of $All_i \cup A_i$ (let us call this event $E(i,x)$).

The starting point will be to color $S$ (the $d$-sparse vertices, see property (d) of Lemma 4.5) randomly as in the proof of Lemma 3.1 i.e. with the property that many colors are repeated in the neighborhoods of the high degree vertices, but also with the additional property that $E(i,x)$ still holds for any $i, x$ after the coloring.

We then proceed to extend the coloring to $B$. Recall that by property (e) of Lemma 4.5 each vertex of $B$ has at most $c - \Omega(\sqrt{\Delta})$ neighbors in $F - \bigcup_j A_j$. It turns out that it is a bit too high to extend randomly the coloring of $S$ to $B$ while maintaining property $E(i,x)$, so instead we color the remaining vertices in this order:

1. We first color the set $B_H$ of vertices of $B$ with at most $c - \Delta^{3/4}$ neighbors in $F - \bigcup_j A_j$ (coloring these vertices will preserve $E(i,x)$).
2. We then color the sets $A_i$ such that each vertex of $A_i$ has at most $30\Delta^{1/4}$ neighbors outside of $All_i \cup A_i$.
3. We color $B_L = B - B_H$, using property (f) of Lemma 4.5 (which implies that property $E(i,x)$ can now be preserved while coloring these vertices).
Finally we color the sets $A_i$ that have not been colored yet. 

The proofs that desirable properties are maintained during the coloring of the vertices of $S$ and $B$ and the $A_i$ are fairly similar to the proof of Lemma 3.1 in the sense that they boil down to the estimation of the expectation of some random variables, the proof that these random variables are highly concentrated, and then some application of the Lovász Local Lemma.

We should note two important differences, though.

- The first is that instead of a single random partial coloring, followed by a greedy procedure completing the coloring, the process for coloring $S$, $B$, and $A_i$ here involves multiple rounds (more specifically, at most $O(\Delta^\lambda)$ rounds, for any $\lambda > 0$, w.h.p.) of random partial coloring and a careful study of all the random variables throughout the process.

- The second is that while coloring the $A_i$, the partial random coloring procedure is a bit different than in the proof of Lemma 3.1. Recall that each $A_i$ is a clique, so assigning each vertex a color uniformly at random, and then uncoloring pairs of vertices with the same color would be extremely unpractical. Instead, each $A_i$ is colored with a permutation of the $|A_i|$ colors not appearing on All$_i$, taken uniformly at random among all the possible permutations. A consequence is that instead of using Talagrand’s Inequality to prove the concentration of random variables around their expectation, McDiarmid’s Inequality has to be used instead (see [16]), but the resulting bounds are of a similar order of magnitude.

It can be checked that in all the applications of the Lovász Local Lemma in [16], bad events correspond to subgraphs of $H$ of bounded radius, and the probabilities of the bad events are smaller than any fixed polynomial function of the maximum degree of the event dependency graph (these probabilities are typically of order $\exp(-d^\alpha)$ or $\exp(-\beta \log^2 d)$, where $\alpha, \beta > 0$ and $d$ is the maximum degree of the event dependency graph), so in particular Theorem 2.5 and 2.6 can be substituted everywhere in the proof, and since the semi-random process involves at most $O(\Delta^\lambda)$ successive applications of the Lovász Local Lemma (for any $\lambda > 0$), the $c$-coloring of $F$ can be obtained w.h.p. in $\min\{O(\Delta^\lambda \log n), 2^{O(\log \Delta + \sqrt{\log \log n})}\}$ rounds, for any $\lambda > 0$.

We find it necessary to insist on a technical (but important) detail here. Theorems 2.5 and 2.6 use the so-called variable setting of the Local Lemma, which covers most applications of the original Local Lemma but not all of them. In particular we have to be careful here since the coloring of the $A_i$ involved random permutations of colors assigned to a given set of vertices, instead of colors chosen uniformly at random for each vertex, and it is not clear at first sight whether the former can be handled in the variable setting. It turns out that it can, since in the proof of Lemmas 39 and 40 in [16] the graph under consideration has one vertex for each uncolored $A_i$, and an edge between two vertices if the corresponding sets $A_i$ are adjacent in $H$ (since each set $A_i$ is a clique, this graph can be simulated within $H$). The variable associated to each vertex is the random permutation of colors assigned to the corresponding set $A_i$, so this is indeed an instance of the variable setting of the Local Lemma, and we can use Theorems 2.5 and 2.6.

Now that $F$ has been colored with $c$ colors, we obtain a $c$-coloring of $H$ in $O(1)$ rounds using Lemma 4.5, and it remains to color the dense components $X_i$ that are $c$-reducers, or such that $G[X_i]$ contains a matching of size at least $100\sqrt{\Delta}$ (recall that these dense components had been removed from the graph at the beginning of the procedure). It follows from Lemmas 4.2 and 4.3 that the $c$-coloring of $H$ can be extended to the remaining dense components of $G$ w.h.p. in $O(\log \Delta) + 2^{O(\sqrt{\log \log n})}$ rounds, which concludes the proof of Theorem 1.3. \hfill \Box

4.4. Summary of our contributions. We now make a brief summary of our contributions (to make clear what we added and subtracted from the proof of Molloy and Reed [16]).
In [16], c-reducers are dealt with slightly differently: some are simply removed as we do here, but some are reduced as in the definition of c-reduction of Section 4.1 (i.e. by removing the clique and contracting the stable set into a single vertex). This operation can create new c-reducers, and thus c-reducers have to be reduced sequentially until no c-reducer appears in the graph (the fact that it has to be done sequentially is essentially the proof of Theorem 1.4). For c-near-reducers, the situation is slightly more complicated (see Lemma 26 in [16]) but again inherently sequential. It is fortunate that in our case (i.e. when \(c \geq \Delta - k\Delta + 1\), we do not need to worry about these cases. So our contribution is simply to have checked that the initial \(d\)-dense decomposition can be computed locally (see Lemma 2.8), that the construction of \(F\) can be performed locally, that all the applications of the Local Lemma can be also carried out locally in the phase where the c-coloring of \(F\) is obtained, and that the resulting coloring of \(H\) can be extended to \(G\) locally and efficiently (see Lemmas 4.2 and 4.3).

5. Concluding remarks

Note that using recent results of Ghaffari et al. [10], the randomized algorithms in Theorem 1.5 and 1.3 can be replaced by deterministic algorithms with a round complexity of \(2^{O(\log \Delta + \sqrt{\log \log n})}\). An interesting question is whether the dependency in \(\Delta\) can be significantly reduced here (the same question can be asked for Theorem 1.8 in [10] or the ad-hoc techniques from [8], do not work well in our case.

When the maximum degree \(\Delta\) is a constant, the list-coloring problem where every vertex \(v\) has a list of at least \(d(v) + 1\) colors can be solved in \(O(\log^* n)\) rounds [12, 14], which is much faster than the round complexity of Theorems 2.1 and 2.2. In this case it is interesting to use a slightly faster version of Theorem 2.6 from [10], with round complexity \(\exp(\exp(\log^{(i+1)} n))\), \(\exp(\exp(\log^{(i+1)} n))\), or \(\exp(\exp(\log^{(i+1)} n))\) for any \(1 \leq i \leq \log^* n - 2\log^* \log^* n\). It is not difficult to see that in this case this round complexity dominates the other parts of the algorithms used in this paper. It follows that the round complexity in Theorem 1.5 and 1.3 in the bounded degree case can be replaced by \(\exp^{(i)}(\log^{(i+1)} n)\) for any \(1 \leq i \leq \log^* n - 2\log^* \log^* n\).

We have proved that the threshold between efficient tractability and intractability of finding an optimal coloring of a graph of (sufficiently large) maximum degree in the LOCAL model occurs at \(c = \Delta - k\Delta + 1\) colors (for all values of \(\Delta\)), or when \(c = \Delta - k\Delta\) and \((k\Delta + 1)(k\Delta + 2) = \Delta\). So a natural question is the status of the round complexity of obtaining an optimal coloring when \(c = \Delta - k\Delta\) and \((k\Delta + 1)(k\Delta + 2) < \Delta\). We have no clear idea of what the right answer should be, but in this case we can at least decide if the chromatic number is at most \(c\) in \(O(\Delta^{5/2})\) rounds (deterministically), using Corollary 7c(ii) in [10], which says that in this case we only need to check the \(c\)-colorability of connected subgraphs of size \(O(\Delta^{5/2})\), which can be done in \(O(\Delta^{5/2})\) rounds in the LOCAL model of computation.

Acknowledgments. We thank David Harris for pointing out the updated version of [10] and for his kind remarks on earlier versions of the paper.

References

[1] L. Barenboim and M. Elkin, Distributed graph coloring: Fundamentals and recent developments, Synthesis Lectures on Distributed Computing Theory 4(1) (2013), 1–171.
[2] L. Barenboim, M. Elkin, S. Pettie, and J. Schneider, The locality of distributed symmetry breaking, In Proc. of the 53rd Annual Symposium on Foundations of Computer Science (FOCS) 2012, 321–330.

[3] S. Brandt, O. Fischer, J. Hirvonen, B. Keller, T. Lempíainen, J. Rybicki, J. Suomela, and J. Uitto, A lower bound for the distributed Lovász local lemma, In Proc. of the ACM Symposium on Theory of Computing (STOC) 2016, 479–488.

[4] Y.-J. Chang, W. Li, and S. Pettie, An optimal distributed (Δ + 1)-coloring algorithm?, In Proc. of the 50th ACM Symposium on Theory of Computing (STOC) 2018.

[5] K.-M. Chung, S. Pettie, and H.-H. Su, Distributed algorithms for the Lovász Local Lemma and graph coloring, In Proc. of the Symposium on Principles of Distributed Computing (PODC) 2014, 134–143.

[6] M. Elkin, S. Pettie, and H.-H. Su, (2Δ – 1)-edge-coloring is much easier than maximal matching in the distributed setting, In Proc. of the ACM-SIAM Symposium on Discrete Algorithms (SODA) 2015, 355–370.

[7] S. Embden-Weinert, S. Hougardy and B. Kreuter, Uniquely Colourable Graphs and the Hardness of Colouring Graphs of Large Girth, Combin. Prob. Comput. 7 (1998), 375–386.

[8] M. Fischer and M. Ghaffari, Sublogarithmic distributed algorithms for Lovász local lemma with implications on complexity hierarchies, In Proc. 31st Symp. on Distributed Computing (DISC), 2017.

[9] P. Fraigniaud, M. Heinrich, and A. Kosowsky, Local conflict coloring, In Proc. 57th IEEE Symposium on Foundations of Computer Science (FOCS) 2016, pages 625–634.

[10] M. Ghaffari, D.G. Harris, F. Kuhn, On Derandomizing Local Distributed Algorithms, In Proc. of the IEEE Symposium on Foundations of Computer Science (FOCS) 2018.

[11] M. Ghaffari, J. Hirvonen, F. Kuhn, and Y. Maus, Improved Distributed Δ-Coloring, In Proc. of the Symposium on Principles of Distributed Computing (PODC) 2018.

[12] A. Goldberg, S. Plotkin, and G. Shannon, Parallel symmetry-breaking in sparse graphs, SIAM J. Discrete Math. 1(4) (1988), 434–446.

[13] D. Harris, J. Schneider, and H.-H. Su, Distributed (Δ + 1)-coloring in sublogarithmic rounds, In Proc. of the 48th ACM Symposium on Theory of Computing (STOC) 2016, 465–478.

[14] N. Linial, Locality in distributed graph algorithms, SIAM J. Comput. 21 (1992), 193–201.

[15] M. Molloy and B. Reed, Graph Colouring and the Probabilistic Method, Springer, 2002.

[16] M. Molloy and B. Reed, Colouring graphs when the number of colours is almost the maximum degree, J. Combin. Theory Ser. B 109 (2014), 134–195.

[17] A. Panconesi, and A. Srinivasan, The local nature of Δ-coloring and its algorithmic applications, Combinatorica 15 (1995), 255–280.

[18] B. Reed, ω, Δ, and χ, J. Graph Theory 27 (1998), 177–212.

[19] J. Schneider and R. Wattenhofer, Distributed Coloring Depending on the Chromatic Number or the Neighborhood Growth, In 18th International Colloquium on Structural Information and Communication Complexity (SIROCCO) 2011.

School of Computer and Communication Sciences, École Polytechnique Fédérale de Lausanne, Switzerland
E-mail address: etienne.bamas@epfl.ch

Laboratoire G-SCOP (CNRS, Univ. Grenoble Alpes), Grenoble, France
E-mail address: louis.esperet@grenoble-inp.fr