Operator means deformed by a fixed point method

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Abstract
By means of a fixed point method we discuss the deformation of two-variable and multivariate operator means of positive definite matrices/operators. It is shown that the deformation of any operator mean in the Kubo–Ando sense becomes again an operator mean in the same sense. The operator means deformed by the weighted power means with two parameters are particularly examined.

Keywords Operator mean · Operator monotone function · Positive definite matrices · Fixed point · Thompson metric · Weighted geometric mean · Weighted power mean

Mathematics Subject Classification 47A64 · 47B65 · 47L07 · 58B20

1 Introduction
The notion of (two-variable) operator means of positive operators on a Hilbert space was introduced in an axiomatic way by Kubo and Ando [18]. The main theorem of [18] says that there is a one-to-one correspondence between the operator means \( \sigma \) and the positive operator monotone functions \( f \) on \([0, \infty)\) with \( f(1) = 1 \) in such a way that

\[
A \sigma B = A^{1/2} f(A^{-1/2} B A^{1/2}) A^{1/2}
\]

for positive invertible operators \( A, B \) on a Hilbert space \( \mathcal{H} \). The extension to general positive operators \( A, B \) is given as \( A \sigma B = \lim_{\varepsilon \searrow 0} (A + \varepsilon I) \sigma (B + \varepsilon I) \). The operator monotone function \( f \) on \([0, \infty)\) corresponding to \( \sigma \) is called the representing function.

Dedicated to Professor Rajendra Bhatia with admiration.

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of $\sigma$. Thus, most properties of operator means can be described in terms of their representing functions, so study of “operator” means in the Kubo–Ando sense can essentially be reduced to that of “numerical” operator monotone functions on $[0, \infty)$.

The geometric mean, first introduced by Pusz and Woronowicz [28] and then discussed by Ando [1] in detail, is probably an operator mean that has been paid the most attention. It was a long-standing problem to extend the notion of geometric mean to the case of more than two variables of matrices/operators. A breakthrough came when certain definitions of multivariate geometric means of positive definite matrices were found in an iteration method by Ando, Li and Mathias [2] and in a Riemannian geometry method by Moakher [25] and by Bhatia and Holbrook [5] (also [4]). In the latter method, the mean $G_w(A_1, \ldots, A_n)$ with a probability weight $w = (w_1, \ldots, w_n)$ is defined as the minimizer of the weighted sum of the squares $\sum_{j=1}^n w_j \delta^2(X, A_j)$, where $\delta(X, Y)$ is the Riemannian trace metric, and it is also characterized by the gradient zero equation $\sum_{j=1}^n w_j X^{-1/2} A_j X^{-1/2} = 0$, called the Karcher equation. Since then, Riemannian multivariate means have extensively been developed by many authors. Among others, the monotonicity property of $G_w(A_1, \ldots, A_n)$ was proved in [6,19]. In [22] the multivariate weighted power means $P_{w,r}(A_1, \ldots, A_n)$ for $r \in [-1, 1] \setminus \{0\}$ were introduced and the convergence lim$_{r \to 0} P_{w,r} = G_w$ was proved. Furthermore, the multivariate geometric and power means have recently been generalized to probability measures on the positive definite matrices based on the Wasserstein distance (see, e.g., [12,13,16]).

In recent studies of Riemannian multivariate means there are two significant features worth noting from the technical point of view. The one is that the positive definite matrices (of fixed size) form a space of nonpositive curvature (NPC) so that the general theory of NPC spaces [29] is of essential use. The other is that the fixed point method is often used in different places (see, e.g., [16,17,20–24,26,32]). For instance, the def-

The paper is organized as follows. In Sect. 2, $\tau$ and $\sigma$ are operator means in the Kubo–Ando sense for positive operators on $\mathcal{H}$ assumed infinite-dimensional, where $\sigma \neq I$ (I is the left trivial mean $X Y = X$). For any positive invertible operators $A, B$, as shown in [15] in the multivariate setting, the fixed point equation $X = (X \sigma A) \tau (X \sigma B)$ has a unique positive invertible solution, which is denoted by $A \tau_\sigma B$. Then we prove that $\tau_\sigma$ defines an operator mean in the Kubo–Ando sense, and call it the deformed operator mean. In Sect. 3 we present properties and examples of the deformed operator means $\tau_\sigma$. In particular, we examine the two-parameter deformations $\tau_{s,r} := \tau_{p_{s,r}}$ for $s \in (0, 1]$ and $r \in [-1, 1]$, where $p_{s,r}$ are the weighted power means. In Sect. 4 we consider an $n$-variable operator mean $M$ of positive invertible operators on $\mathcal{H}$ (of finite or infinite dimension), and assume that $M$ satisfies some basic properties such as joint monotonicity, etc. For any two-variable operator means $\sigma_1, \ldots, \sigma_n$ and positive invertible operators $A_1, \ldots, A_n$, we show that the equation
\( X = M(X\sigma_1 A_1, \ldots, X\sigma_n A_n) \) has a unique positive invertible solution, which defines the deformed operator mean \( M_{(\sigma_1, \ldots, \sigma_n)}(A_1, \ldots, A_n) \) satisfying properties inherited from \( M \). The notion of deformed operator means is considered as an extended version of generalized operator means in \([26]\). In particular, we examine the deformations of \( M \) by \( \sigma_1 = \cdots = \sigma_n = p_{s,r} \), from which we can produce a lot of multivariate operator means via deformation though their construction is not so concrete.

Since the two-variable operator mean is a special case of multivariate operator means, there is a bit redundancy between presentations of Sects. 2 and 3 and that of Sect. 4. But our main result is that \( \tau_\sigma \) becomes an operator mean in the Kubo–Ando sense again. So in the first part of the paper we argue the deformed operator means \( \tau_\sigma \) restricted to the two-variable case, by taking account of their representing functions.

The present article is a slightly shortened version of \([10]\) with the same title by removing details of some proofs which were already presented in \([15]\). A further extension of the fixed point method to the setting of probability measures on the positive invertible operators is carried out in \([14]\).

### 2 Two-variable operator means

Throughout this section we assume that \( \mathcal{H} \) is an infinite-dimensional Hilbert space and \( B(\mathcal{H}) \) is the Banach space of all bounded linear operators on \( \mathcal{H} \) with the operator norm \( \| \cdot \| \). An operator \( A \in B(\mathcal{H}) \) is positive if \( \langle \xi, A\xi \rangle \geq 0 \) for all \( \xi \in \mathcal{H} \). We denote by \( B(\mathcal{H})^+ \) the closed convex cone of positive operators in \( B(\mathcal{H}) \) and by \( B(\mathcal{H})^{++} \) the open convex cone of invertible positive operators in \( B(\mathcal{H}) \).

The Thompson metric on \( B(\mathcal{H})^{++} \) is defined by

\[
d_T(A, B) := \| \log A^{-1/2} BA^{-1/2} \|, \quad A, B \in B(\mathcal{H})^{++},
\]

which is also written as

\[
d_T(A, B) = \log \max \{ M(A/B), M(B/A) \},
\]

where \( M(A/B) := \inf \{ \alpha > 0 : A \leq \alpha B \} \). It is known in \([30]\) that \( d_T \) is a complete metric on \( B(\mathcal{H})^{++} \) and the topology on \( B(\mathcal{H})^{++} \) induced by \( d_T \) is the same as the operator norm topology.

A (two-variable) operator mean \( \sigma \) introduced by Kubo and Ando \([18]\) is a binary operation

\[
\sigma : B(\mathcal{H})^+ \times B(\mathcal{H})^+ \rightarrow B(\mathcal{H})^+
\]

satisfying the following four properties where \( A, A', B, B', C \in B(\mathcal{H})^+ \):

(I) Joint monotonicity: \( A \leq A' \) and \( B \leq B' \) imply \( A\sigma B \leq A'\sigma B' \).

(II) Transformer inequality: \( C(A\sigma B)C \leq (CAC)\sigma(CBC) \).

(III) Downward continuity: If \( A_k, B_k \in B(\mathcal{H})^+, k \in \mathbb{N}, A_k \nrightarrow A \) and \( B_k \nrightarrow B \), then \( A_k\sigma B_k \nrightarrow A\sigma B \), where \( A_k \nrightarrow A \) means that \( A_1 \geq A_2 \geq \cdots \) and \( A_k \rightarrow A \) in the strong operator topology.
Normalization: IσI = I, where I is the identity operator on $\mathcal{H}$.

Each operator mean $\sigma$ is associated with a non-negative operator monotone function $f_\sigma$ on $[0, \infty)$ with $f_\sigma(1) = 1$, called the representing function of $\sigma$, in such a way that

$$A \sigma B = A^{1/2}f_\sigma(A^{-1/2}BA^{-1/2})A^{1/2}, \quad A, B \in B(\mathcal{H})^{++},$$

(1)

which is extended to general $A, B \in B(\mathcal{H})^+$ as $A \sigma B := \lim_{\epsilon \searrow 0} (A + \epsilon I)\sigma (B + \epsilon I)$.

We write $l$ and $\tau$ to denote the two extreme left and right operator means, i.e., $A l := A$ and $A \tau B := B$ for every $A, B \in B(\mathcal{H})^+$. Throughout this and the next sections, we shall simply write “operator mean” for “two-variable operator mean in the Kubo–Ando sense.”

From now on we assume that $\tau$ and $\sigma$ are arbitrary operator means with a single restriction $\sigma \neq l$, and we will consider, given $A, B \in B(\mathcal{H})^{++}$, the equation

$$X = (X \sigma A)\tau (X \sigma B), \quad X \in B(\mathcal{H})^{++}.$$  

(2)

It might be instructive to start with a few typical examples of Eq. (2) and their solutions. Although the following examples are rather well-known, we discuss them in some detail for the reader’s convenience. Note in particular that the weighted power means were introduced in [22] by a fixed point method in the multivariable setting (see Example 6 below).

**Example 1** Let $\tau = \#_\alpha$, the weighted geometric mean with $0 \leq \alpha \leq 1$, i.e.,

$$A \#_\alpha B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}, \quad A, B \in B(\mathcal{H})^{++},$$

corresponding to the operator monotone function $t^\alpha, t \geq 0$. Let $\sigma = \#_r$ with $0 < r \leq 1$. Then a unique solution to (2) for any $A, B \in B(\mathcal{H})^{++}$ is $X = A \#_\alpha B$. Indeed, in this case, (2) is equivalent to $I = (X^{-1/2}AX^{-1/2})^{1/2}A \#_\alpha (X^{-1/2}BX^{-1/2})^{1/2}$. When $\alpha = 0$, this means that $X^{-1/2}AX^{-1/2} = I$ so that $X = A$. When $0 < \alpha \leq 1$, the identity above means that $X^{-1/2}BX^{-1/2} = (X^{-1/2}AX^{-1/2})^{1-\frac{1}{\alpha}}$, so that

$$B = X^{1/2}(X^{-1/2}AX^{-1/2})^{1-\frac{1}{\alpha}}X^{1/2} = A^{1/2}(A^{-1/2}XA^{-1/2})^{1-\frac{1}{\alpha}}A^{1/2},$$

which is solved as $X = A \#_\alpha B$.

**Example 2** Let $\tau = \nabla_\alpha$, the weighted arithmetic mean where $0 \leq \alpha \leq 1$, i.e., $A \nabla_\alpha B := (1 - \alpha)A + \alpha B$, and $\sigma = \#$, with $0 < r \leq 1$. With $Y := A^{-1/2}XA^{-1/2}$ and $C := A^{-1/2}BA^{-1/2}$, (2) in this case is equivalent to

$$Y = (1 - \alpha)(Y \tau I) + \alpha(Y \tau C) \quad \iff \quad I = (1 - \alpha)Y^{-r} + \alpha(Y^{-1/2}CY^{-1/2})^{r}$$

$$\iff \quad C = \left[\frac{Y^r - (1 - \alpha)I}{\alpha}\right]^{1/r}$$

$$\iff \quad Y = [(1 - \alpha)I + \alpha C^r]^{1/r}.$$
Therefore, a unique solution to (2) is \( X = A p_{\alpha,r}B \), where \( p_{\alpha,r} \) is the weighted power mean

\[
A p_{\alpha,r}B := A^{1/2} [(1 - \alpha)I + \alpha (A^{-1/2}BA^{-1/2})^r]^{1/r} A^{1/2}, \quad A, B \in B(\mathcal{H})^{++},
\]

(3)

corresponding to the operator monotone function \( f_{\alpha,r}(t) := (1 - \alpha + \alpha t^r)^{1/r}, t \geq 0 \).

**Example 3** Let \( \tau = \tau_{\alpha} \), the weighted harmonic mean where \( 0 \leq \alpha \leq 1 \), i.e., \( A \tau_{\alpha} B := [(1 - \alpha)A^{-1} + \alpha B^{-1}]^{-1} \), and \( \sigma = #_r \) with \( 0 < r \leq 1 \). In this case, (2) is equivalent to

\[
X^{-1} = (X^{-1} #_r A^{-1}) \gamma_{\alpha} (X^{-1} #_r B^{-1}),
\]

which has the unique solution \( X = (A^{-1} p_{\alpha,#r} B^{-1})^{-1} \). The last expression is indeed the weighted power mean

\[
A p_{\alpha,-r}B := A^{1/2} [(1 - \alpha)I + \alpha (A^{-1/2}BA^{-1/2})^{-r}]^{-1/r} A^{1/2}, \quad A, B \in B(\mathcal{H})^{++},
\]

(4)

for \( \alpha \in [0, 1] \) and \( -r \in [-1, 0] \), corresponding to the operator monotone function \( f_{\alpha,-r}(t) := (1 - \alpha + \alpha t^{-r})^{-1/r} \).

The main theorem (Theorem 1) of this section shows that Eq. (2) always has a unique solution and it indeed defines an operator mean again. Many assertions required to prove the theorem were shown in [15] in a more general setting of multivariate operator means (in [14] more generally for operator means of probability measures). So we first recall those assertions.

First of all, it is known by [15, Theorem 2.1] that for every \( A, B \in B(\mathcal{H})^{++} \) there exists a unique \( X_0 \in B(\mathcal{H})^{++} \) which satisfies (2). We write \( A \tau_{\sigma} B \) for this unique solution \( X_0 \in B(\mathcal{H})^{++} \) to (2). Hence we have a binary operation

\[
\tau_{\sigma} : B(\mathcal{H})^{++} \times B(\mathcal{H})^{++} \rightarrow B(\mathcal{H})^{++}.
\]

It was also shown in [15, Theorem 2.1] that the map \( \tau_{\sigma} \) satisfies the following properties:

(i) **Joint monotonicity:** If \( A \leq A' \) and \( B \leq B' \) in \( B(\mathcal{H})^{++} \), then \( A \tau_{\sigma} B \leq A' \tau_{\sigma} B' \).

(ii) **Congruence invariance:** For every \( A, B, C \in B(\mathcal{H})^{++} \),

\[
C (A \tau_{\sigma} B) C = (C A C) \tau_{\sigma} (C B C).
\]

(iii) **Monotone continuity:** If \( A, B, A_k, B_k \in B(\mathcal{H})^{++}, k \in \mathbb{N}, A_k \searrow A \) and \( B_k \searrow B \) as \( k \to \infty \), then \( A_k \tau_{\sigma} B_k \to A \tau_{\sigma} B \) in the strong operator topology. The same holds if \( A_k \nearrow A \) and \( B_k \nearer B \) as \( k \to \infty \).

(iv) **Normalization:** \( I \tau_{\sigma} I = I \).

(v) For \( A, B, X \in B(\mathcal{H})^{++} \), if \( X \leq (X \sigma A) \tau (X \sigma B) \) (resp., \( X \geq (X \sigma A) \tau (X \sigma B) \)), then \( X \leq A \tau_{\sigma} B \) (resp., \( X \geq A \tau_{\sigma} B \)).
For every $A, B \in B(\mathcal{H})^+$, by joint monotonicity (i) we can define $A\tau_\sigma B$ in $B(\mathcal{H})^+$ by

$$A\tau_\sigma B := \lim_{\varepsilon \searrow 0} (A + \varepsilon I)\tau_\sigma (B + \varepsilon I). \quad (5)$$

By property (iii) we see that this binary operation $\tau_\sigma : B(\mathcal{H})^+ \times B(\mathcal{H})^+ \to B(\mathcal{H})^+$ extends the previous $\tau_\sigma : B(\mathcal{H})^+ \times B(\mathcal{H})^+ \to B(\mathcal{H})^+$. Then the extended $\tau_\sigma$ has the following properties.

**Lemma 1** The map $\tau_\sigma$ extended to $B(\mathcal{H})^+ \times B(\mathcal{H})^+$ satisfies joint monotonicity (I) and downward continuity (III).

**Proof** Joint monotonicity is obvious from (i) and definition (5). To prove downward continuity, assume that $A_k \searrow A$ and $B_k \searrow B$ in $B(\mathcal{H})^+$. For every $\xi \in \mathcal{H}$ one has

$$\langle \xi, (\lim_k A_k \tau_\sigma B_k)\xi \rangle = \inf_k \langle \xi, (A_k \tau_\sigma B_k)\xi \rangle$$

$$= \inf_{\varepsilon > 0} \inf_k \langle \xi, \{(A_k + \varepsilon I)\tau_\sigma (B_k + \varepsilon I)\}\xi \rangle$$

$$= \inf_{\varepsilon > 0} \inf_k \langle \xi, ((A + \varepsilon I)\tau_\sigma (B + \varepsilon I)\}\xi \rangle$$

$$= \inf_{\varepsilon > 0} \langle \xi, (A + \varepsilon I)\tau_\sigma (B + \varepsilon I)\rangle$$

$$= \langle \xi, (A \tau_\sigma B)\rangle,$$

where the second and the last equalities are by definition (5) and the fourth equality is due to (iii). Therefore, $A_k \tau_\sigma B_k \searrow A \tau_\sigma B$. \qed

A remaining requirement for $\tau_\sigma$ on $B(\mathcal{H})^+ \times B(\mathcal{H})^+$ to be an operator mean is transformer inequality (II). But it does not seem easy to directly prove inequality (II), so we take a detour by giving the following lemma that was also a key part of the proof of the main theorem of [18].

**Lemma 2** Let $A, B \in B(\mathcal{H})^{++}$ and $P \in B(\mathcal{H})$ be a projection commuting with $A, B$. Then $P$ commutes with $A\tau_\sigma B$ and

$$(AP)\tau_\sigma (BP) = (A\tau_\sigma B)P. \quad (6)$$

**Proof** Although we assumed that $\mathcal{H}$ is an infinite-dimensional Hilbert space, the whole discussions up to now are valid for any Hilbert space $\mathcal{H}$ (whichever finite-dimensional or infinite-dimensional). Let $\mathcal{H}_0 := P\mathcal{H}$ and $\mathcal{H}_1 := (I - P)\mathcal{H}$. One can then apply the unique existence of a solution to (2) to $AP, BP \in B(\mathcal{H}_0)^{++}$ and $A(I - P), B(I - P) \in B(\mathcal{H}_1)^{++}$ to have an $X_0 \in B(\mathcal{H}_0)^{++}$ such that

$$X_0 = \{X_0\sigma (AP)\} \tau \{X_0\sigma (BP)\}, \quad (7)$$

and an $X_1 \in B(\mathcal{H}_1)^{++}$ such that

$$X_1 = \{X_1\sigma (A(I - P))\} \tau \{X_1\sigma (B(I - P))\}.$$
For any $\varepsilon > 0$, multiplying this with $\varepsilon$ one furthermore has

$$\varepsilon X_1 = \{(\varepsilon X_1)\sigma(\varepsilon A(I - P))\} \tau\{(\varepsilon X_1)\sigma(\varepsilon B(I - P))\}. \quad (8)$$

Since the operator means $\sigma, \tau$ are computed component-wise for direct sum operators $Y_0 + Y_1$ with $Y_i \in B(H)^+$, $i = 0, 1$, we add (7) and (8) to have

$$X_0 + \varepsilon X_1 = \{(X_0 + \varepsilon X_1)\sigma(AP + \varepsilon A(I - P))\} \tau\{(X_0 + \varepsilon X_1)\sigma(BP + \varepsilon B(I - P))\},$$

which implies that

$$X_0 + \varepsilon X_1 = \{AP + \varepsilon A(I - P)\} \tau\sigma\{BP + \varepsilon B(I - P)\}, \quad \varepsilon > 0. \quad (9)$$

Letting $\varepsilon = 1$ in (9) gives

$$X_0 + X_1 = A\tau\sigma B. \quad (10)$$

From the downward continuity of $\tau\sigma$ (Lemma 1), letting $\varepsilon \downarrow 0$ in (9) gives

$$X_0 = (AP)\tau\sigma(BP). \quad (11)$$

From (10) and (11) we see that $P$ commutes with $A\tau\sigma B$ and (6) holds. \hfill \square

We are now in a position to present the main result of the section.

**Theorem 1** Let $\tau$ and $\sigma$ be operator means with $\sigma \neq 1$. Then the binary operation $\tau\sigma$ first defined by Eq. (2) for $A, B \in B(H)^+$ and then extended by (5) for $A, B \in B(H)^+$ is an operator mean.

Moreover, the operator monotone function $f_{\tau\sigma}$ corresponding to $\tau\sigma$ is determined in such a way that $x = f_{\tau\sigma}(t)$ for $t > 0$ is a unique solution to

$$(x\sigma 1)\tau(x\sigma t) = x, \quad x > 0, \quad (12)$$

that is,

$$f_{\sigma}(1/x)f_{\tau}\left(\frac{f_{\sigma}(t/x)}{f_{\sigma}(1/x)}\right) = 1, \quad x > 0. \quad (13)$$

**Proof** Since Lemma 2 implies that $I\tau\sigma(tI)$ for $t > 0$ commutes with all projections in $B(H)$, it follows that there is a function $f$ on $(0, \infty)$ such that

$$f(t)I = I\tau\sigma(tI), \quad t > 0.$$  

Here, it is immediate to see that $x = f(t)$ is determined by the numerical equation

$$x = (x\sigma 1)\tau(x\sigma t), \quad x > 0. \quad (14)$$
It is clear that \( f(1) = 1 \). From monotone continuity (iii) it follows that \( f \) is continuous on \((0, \infty)\). In the same way as in the proof of [18, Theorem 3.6] by use of Lemma 2, one can show that, for every \( A \in B(H)^{++} \),

\[
f(A) = I \tau_{\sigma} A.
\]  

This implies that if \( A, B \in B(H)^{++} \) and \( A \leq B \), then \( f(A) \leq f(B) \). Therefore, \( f \) is a positive operator monotone function on \((0, \infty)\), which can be extended to \([0, \infty)\) by \( f(0) := \lim_{t \downarrow 0} f(t) \).

Finally, by property (ii) and (15), one can write for every \( A, B \in B(H)^{++} \)

\[
A \tau_{\sigma} B = A^{1/2} (I \tau_{\sigma} (A^{-1/2} BA^{-1/2})) A^{1/2} = A^{1/2} f(A^{-1/2} BA^{-1/2}) A^{1/2}.
\]

Therefore, we have

\[
A \tau_{\sigma} B = m_f B, \quad A, B \in B(H)^{++},
\]

where \( m_f \) is the operator mean corresponding to \( f \). From the downward continuity of \( \tau_{\sigma} \) (Lemma 1) as well as \( m_f \) on \( B(H)^{+} \times B(H)^{+} \), the equality above extends to \( A, B \in B(H)^{+} \). Therefore, \( \tau_{\sigma} = m_f \), that is, \( \tau_{\sigma} \) is an operator mean with \( f_{\tau_{\sigma}} = f \). The determining Eq. (12) was already verified in (14), and (13) is a more explicit rewriting of (12).

We call the operator mean \( \tau_{\sigma} \) shown by Theorem 1 the **deformed operator mean** from \( \tau \) by \( \sigma \).

**Remark 1** More generally than (2) one may consider the equation

\[
X = (X \sigma_1 A) \tau (X \sigma_2 B), \quad X \in B(H)^{++},
\]

where \( \tau, \sigma_1, \sigma_2 \) are operator means with \( \sigma_1, \sigma_2 \neq I \). Then the whole arguments of this section can similarly be done with this generalized equation, so that one can define the operator mean \( \tau_{(\sigma_1, \sigma_2)} \) deformed from \( \tau \) by a pair \( (\sigma_1, \sigma_2) \). The corresponding operator monotone function \( f_{\tau_{(\sigma_1, \sigma_2)}} \) is determined in such a way that \( x = f_{\tau_{(\sigma_1, \sigma_2)}}(t) \) is a unique solution to \((x \sigma_1 t) \tau (x \sigma_2 t) = x \) or

\[
f_{\sigma_1}(1/x) f_{\tau} \left( \frac{f_{\sigma_2}(t/x)}{f_{\sigma_1}(1/x)} \right) = 1, \quad x > 0.
\]

This generalized setting will be adopted in Sect. 4 to discuss deformation of multi-variate operator means.

### 3 Properties and examples

In this section we will show some general properties of the deformed operator mean \( \tau_{\sigma} \) and examine \( \tau_{\sigma} \) when \( \sigma \) varies over the weighted power means with two parameters.
There are three important transformations on the operator means [18]. For an operator mean $\tau$, the transpose $\tau'$ of $\tau$ is defined as $A \tau' B := B \tau A$, whose representing operator monotone function is $f_{\tau'}(t) = t f_{\tau}(t^{-1})$. If $\tau = \tau'$, $\tau$ is said to be symmetric. The adjoint $\tau^*$ of $\tau$ is defined as $A \tau^* B := (A^{-1} \tau B^{-1})^{-1}$, whose representing function is $f_{\tau^*}(t) = f_{\tau}(t^{-1})^{-1}$. If $\tau = \tau^*$, $\tau$ is said to be self-adjoint. The dual $\tau^\perp$ of $\tau$ is defined as $\tau^\perp := (\tau^*)^* = (\tau^*)'$, whose representing function is $t / f_{\tau}(t)$.

In this section we assume as in Sect. 2 that $\tau, \sigma, \tau_1, \sigma_1$ are operator means with $\sigma, \sigma_1 \neq I$ (recall that $I$ and $\tau$ are the left and the right trivial means).

**Proposition 1**

(1) $\tau_{\tau} = \tau$ for $\sigma = \tau$.

(2) $I_\sigma = I$ and $\tau_{\sigma} = \tau$ for any $\sigma \neq I$.

(3) If $\tau \leq \tau_1$ and $\sigma \leq \sigma_1$, then $\tau_{\sigma} \leq (\tau_1)_{\sigma_1}$.

(4) $(\tau_{\sigma})' = (\tau')_{\sigma}$. Hence, if $\tau$ is symmetric, then so is $\tau_\sigma$.

(5) $(\tau_{\sigma})^* = (\tau^*)_{\sigma^*}$. Hence, if $\tau, \sigma$ are self-adjoint, then so is $\tau_{\sigma}$.

(6) $(\tau_{\sigma})^\perp = (\tau^\perp)_{\sigma^*}$. Hence, if $\tau = \tau^\perp$ and $\sigma$ is self-adjoint, then $(\tau_{\sigma})^\perp = \tau_\sigma$.

**Proof**

(1) is trivial.

(2) When $\tau = I$, Eq. (2) is $X = X \sigma A$, which has the solution $X = A$. Similarly, when $\tau = \tau$, (2) has the solution $X = B$.

(3) Let $A, B \in B(\mathcal{H})^{++}$ and set $X_0 := A \tau_{\sigma} B$ and $X_1 := A(\tau_1)_{\sigma_1} B$. Since

$$X_1 = (X_1 \sigma_1 A) \tau_1 (X_1 \sigma_1 B) \geq (X_1 \sigma A) \tau (X_1 \sigma B),$$

one has $X_1 \geq X_0$ thanks to property (v) in Sect. 2.

(4) is clear since

$$(X \sigma A) \tau' (X \sigma B) = (X \sigma B) \tau (X \sigma A).$$

(5) is clear since $X = (X \sigma^* A) \tau^* (X \sigma^* B)$ means that

$$X^{-1} = (X^{-1} \sigma A^{-1}) \tau (X^{-1} \sigma B^{-1}).$$

(6) immediately follows from (4) and (5). $\square$

The above properties (1), (3) and (5) are in [14, Proposition 4.1] for operator means of probability measures on the positive invertible operators.

**Proposition 2**

(1) The representing function of $(\sigma^*)_{\sigma}$ is determined in such a way that $x = f_{(\sigma^*)_{\sigma}}(t)$ is a solution of

$$x f_{\sigma}(1/x) = f_{\sigma}(t/x), \quad \text{i.e.,} \quad f_{\sigma'}(x) = f_{\sigma}(t/x), \quad x > 0.$$

Moreover, $(\sigma^*)_{\sigma} = \# if and only if $\sigma$ is symmetric.

(2) The representing function of $(\sigma^\perp)_{\sigma}$ is determined in such a way that $x = f_{(\sigma^\perp)_{\sigma}}(t)$ is a solution of

$$f_{\sigma}(1/x) = (x/t) f_{\sigma}(t/x), \quad \text{i.e.,} \quad f_{\sigma}(1/x) = f_{\sigma'}(x/t), \quad x > 0.$$

Moreover, $(\sigma^\perp)_{\sigma} = \# if and only if $\sigma$ is symmetric.
Proof  (1) When $\tau = \sigma^*$, Eq. (13) means that
\[
 f_\sigma(1/x) = f_\sigma\left(\frac{f_\sigma(1/x)}{f_\sigma(t/x)}\right), \quad x > 0.
\]
Since $\sigma \neq l$ implies that $f_\sigma$ is strictly increasing on $(0, \infty)$, the above equation is equivalent to
\[
 1/x = \frac{f_\sigma(1/x)}{f_\sigma(t/x)}, \quad \text{i.e.,} \quad x f_\sigma(1/x) = f_\sigma(t/x), \quad x > 0.
\]
Moreover, $(\sigma^*)_\sigma = #$ holds if and only if the above holds for any $t = x^2$, that is equivalent to $\sigma' = \sigma$.

(2) When $\tau = \sigma^\perp$, (13) becomes
\[
 \frac{f_\sigma(t/x)}{f_\sigma(1/x)} = 1, \quad x > 0,
\]
equivalently,
\[
 t/x = \frac{f_\sigma(t/x)}{f_\sigma(1/x)}, \quad \text{i.e.,} \quad f_\sigma(1/x) = (x/t) f_\sigma(t/x), \quad x > 0.
\]
Hence, $(\sigma^\perp)_\sigma = #$ if and only if the above holds for any $t = x^2$, that is equivalent to $\sigma' = \sigma$. \hfill \Box

The above proposition in particular says that
\[
 \nabla ! = !\nu = \#_{!} = #. \tag{16}
\]

In what follows we denote by $\text{OM}_{+,1}(0, \infty)$ the set of non-negative operator monotone functions $f$ on $[0, \infty)$ with $f(1) = 1$, i.e., the set of representing operator monotone functions of operator means. Recall [9, (2.3.2)] that if $f \in \text{OM}_{+,1}(0, \infty)$ and $f'(1) = \alpha$, then $\alpha \in [0, 1]$ and
\[
 \frac{t}{(1 - \alpha)t + \alpha} \leq f(t) \leq (1 - \alpha) + \alpha t, \quad t \in [0, \infty). \tag{17}
\]
When $f \in \text{OM}_{+,1}(0, \infty)$, we note by (17) that
\[
 \min\{1, t\} \leq f(t) \leq \max\{1, t\}, \quad t \in [0, \infty), \tag{18}
\]
and that if $f'(1) = 0$ then $f \equiv 1$.

The derivative $f'_\nu(1) \in [0, 1]$ provides a significant characteristic of an operator mean $\tau$. The next proposition says that this characteristic is preserved under taking the deformed operator mean $\tau_\sigma$ for any $\sigma \neq l$. 

\begin{flushright}
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Remark 2 The map \(\tau\rightarrow f_{\tau}(t)\) becomes \(f_{\tau}(1) = f_{\tau}^{\prime}(1)\) for every operator means \(\tau, \sigma\) with \(\sigma \neq 1\).

Proof First, assume that \(f_{\tau}^{\prime}(1) = 0\) and so \(f_{\tau} = 1\). Then \(x = 1\) is the solution to (13) for every \(t > 0\), so that \(f_{\tau}(f_{\sigma}(t)) = 1\) for all \(t > 0\). Since 1 is in the interior of the range of \(f_{\tau}\) thanks to \(\sigma \neq 1\), we have \(f_{\tau}^{\prime}(1) = 0 = f_{\tau}^{\prime}(1)\).

Next, assume that \(f_{\tau}^{\prime}(1) > 0\); then \(f_{\tau}\) is strictly increasing and the inverse function \(t = f_{\tau}^{-1}(x)\) exists in a neighborhood of 1. It follows from (13) that

\[
f_{\sigma}(1/x) f_{\tau} \left( \frac{f_{\sigma}(f_{\tau}^{-1}(x)/x)}{f_{\sigma}(1/x)} \right) = 1
\]

holds in a neighborhood of 1. Now, noting \(f_{\tau}(1) = f_{\sigma}(1) = f_{\tau}^{-1}(1) = 1\), we differentiate the above at \(x = 1\) to obtain

\[
-f_{\sigma}^{\prime}(1) + f_{\tau}^{\prime}(1) \left( f_{\sigma}^{\prime}(1)(f_{\tau}^{\prime}(1) - 1) + f_{\sigma}^{\prime}(1) \right) = 0,
\]

which, thanks to \(f_{\sigma}^{\prime}(1) > 0\), implies that \(f_{\tau}^{\prime}(1) = f_{\tau}^{\prime}(1)\). \(\square\)

Proposition 4 For every \(\sigma \neq 1\) the map \(\tau \mapsto \tau_{\sigma}\) is injective on the operator means.

Proof First, assume that \(\tau_{\sigma} = 1\) or \(f_{\tau_{\sigma}} = 1\). Then, as in the proof of the previous proposition, \(f_{\tau}(f_{\sigma}(t)) = 1\) for all \(t > 0\). Since \(f_{\sigma} \neq 1\), we have \(\tau = 1\).

Next, let \(\kappa := \tau_{\sigma}\) and assume \(\kappa \neq 1\); then the range of \(f_{\kappa}\) contains [\(a, b\)] with \(0 < a < 1 < b\). As in the proof of the previous proposition we have

\[
f_{\tau}(\phi(x)) = \frac{1}{f_{\tau}(1/x)}, \quad \text{where} \quad \phi(x) := \frac{f_{\sigma}(f_{\kappa}^{-1}(x)/x)}{f_{\sigma}(1/x)}, \quad x \in [a, b].
\]

Since \(f_{\kappa}^{-1}(a) < 1 < f_{\kappa}^{-1}(b)\),

\[
\phi(a) = \frac{f_{\sigma}(f_{\kappa}^{-1}(a)/a)}{f_{\sigma}(1/a)} < 1, \quad \phi(b) = \frac{f_{\sigma}(f_{\kappa}^{-1}(b)/b)}{f_{\sigma}(1/b)} > 1.
\]

Therefore, \(f_{\tau}(t)\) on [\(\phi(a), \phi(b)\)] is uniquely determined by \(f_{\sigma}\) and \(f_{\kappa}\). From the analyticity of \(f_{\tau}\) this implies that \(\tau\) is uniquely determined by \(\sigma\) and \(\kappa\). \(\square\)

Remark 2 The map \(\tau \mapsto \tau_{\sigma}\) is not surjective onto the operator means in general. For instance, when \(\kappa = \kappa\), (13) becomes \(f_{\tau}(t^{1/2}) = f_{\kappa}(t)^{1/2}\) or \(f_{\tau}(t) = f_{\kappa}(t^{1/2})^{1/2}\), \(t > 0\). If \(\kappa = \nabla\), then \(f_{\tau}(t) = (\frac{1+t}{2})^{1/2}\), which is not operator monotone. Hence there is no operator mean \(\tau\) satisfying \(\tau_{\#} = \nabla\). Also, assume that \(\tau_{\gamma} = \#\); then (13) becomes \(f_{\tau}(\frac{x+1}{x+1}) = \frac{2x}{x+1}\) for \(x = \frac{2t}{1+t}\). Therefore,

\[
f_{\tau} \left( \frac{x(3-x)}{(x+1)(2-x)} \right) = \frac{2x}{x+1}, \quad 0 < x < 2,
\]

from which we have \(\lim_{t \searrow 0} f_{\tau}(t)/t = 4/3\). On the other hand, since \(f_{\tau}^{\prime}(1) = 1/2\) by Proposition 3, it follows from (17) that \(\lim_{t \searrow 0} f_{\tau}(t)/t \geq 2\), a contradiction. Hence no
operator mean $\tau$ satisfies $\tau_{\nabla} = !$. See Example 4 below for more about the deformed operator means $\tau_{\sigma}$ by $\sigma = \#_{r}$ and $\sigma = \nabla_{r}$ with $0 < r \leq 1$.

On the other hand, for fixed $\tau$ the map $\sigma (\neq 1) \mapsto \tau_{\sigma}$ is not injective. A trivial example is the case of $\tau = l$ or $\tau = r$; see Proposition 1 (2). A non-trivial example appears from (28) below. Moreover, the map $\sigma \mapsto \tau_{\sigma}$ is not surjective onto the operator means $\kappa$ with $f'_{\kappa}(1) = \alpha$, where $\alpha := f'_{\tau}(1)$. For instance, let $\tau = \#$. Then (13) becomes $f_{\sigma}(1/x)f_{\sigma}(t/x) = 1$. If $\#_{r} = \nabla$, then $f_{\sigma}(\frac{2}{1+t})f_{\sigma}(\frac{2t}{1+t}) = 1$ and letting $t \searrow 0$ gives $f_{\sigma}(2)f_{\sigma}(0) = 1$. The concavity of $f_{\sigma}$ gives $(f_{\sigma}(0)+f_{\sigma}(2))/2 \leq f_{\sigma}(1) = 1$. Hence $f_{\sigma}(0) = f_{\sigma}(2) = 1$, which contradicts $\sigma \neq l$.

To show the continuous dependence of $\tau_{\sigma}$ on $\tau$ and $\sigma$, we prepare a basic fact on convergence of operator means or their representing operator monotone functions. For this we first give the next lemma.

**Lemma 3** For every $f \in \text{OM}_{+,1}(0, \infty)$,

$$f'(t) \leq \max\{1, (2t)^{-1}\}, \quad t \in (0, \infty).$$

Hence, for every $\delta \in (0, 1)$,

$$\sup\{f'(t) : t \geq \delta, \ f \in \text{OM}_{+,1}(0, \infty)\} \leq \delta^{-1}.$$

**Proof** It is well-known (see, e.g., [3,9]) that an operator monotone function on $[0, \infty)$ admits an integral expression

$$f(t) = a + bt + \int_{(0,\infty)} \frac{t(1+\lambda)}{t+\lambda} \, d\mu(\lambda), \quad t \in [0, \infty),$$

where $a \in \mathbb{R}$, $b \geq 0$ and $\mu$ is a positive finite measure on $(0, \infty)$. When $f \in \text{OM}_{+,1}(0, \infty)$, one has

$$a = f(0) \geq 0, \quad f(1) = a + b + \mu((0, \infty)) = 1. \quad (19)$$

Compute

$$\phi_{t}(\lambda) := \frac{\partial}{\partial t} \left[ \frac{t(1+\lambda)}{t+\lambda} \right] = \frac{\lambda(1+\lambda)}{(t+\lambda)^2}$$

and

$$\frac{d}{d\lambda} \phi_{t}(\lambda) = \frac{t + (2t - 1)\lambda}{(t + \lambda)^3}.$$

If $t \geq 1/2$, then $\phi_{t}(\lambda)$ is monotone increasing in $\lambda \in (0, \infty)$ and hence $\phi_{t}(\lambda) \leq \lim_{\lambda \to \infty} \phi_{t}(\lambda) = 1$. If $0 < t < 1/2$, then

$$\max_{\lambda \in (0, \infty)} \phi_{t}(\lambda) = \phi_{t}\left(\frac{t}{1-2t}\right) = \frac{1}{4t(1-t)} \leq \frac{1}{2t}.$$
These yield that $\phi_t(\lambda) \leq \max\{1, (2t)^{-1}\}$ for all $t, \lambda \in (0, \infty)$. Hence it follows from Lebesgue’s dominated convergence theorem that

$$f'(t) = b + \int_{(0, \infty)} \phi_t(\lambda) \, d\mu(\lambda) \leq b + \max\{1, (2t)^{-1}\} \mu((0, \infty)) \leq \max\{1, (2t)^{-1}\},$$

where the last inequality is due to (19).

The latter assertion is immediate since $\max\{1, (2t)^{-1}\} \leq \delta^{-1}$ for any $t \geq \delta$ where $\delta \in (0, 1)$.

**Lemma 4** For operator means $\tau$ and $\tau_k, k \in \mathbb{N}$, the following conditions are equivalent:

(a) $f_{\tau_k}(t) \to f_{\tau}(t)$ for any $t \in (0, \infty)$;

(b) $f_{\tau_k} \to f_{\tau}$ uniformly on $[\delta, \delta^{-1}]$ for any $\delta \in (0, 1)$;

(c) $A \tau_k B \to A \tau B$ in the norm for every $A, B \in B(\mathcal{H})^{++}$.

**Proof** For any $\delta \in (0, 1)$, since Lemma 3 shows that $f_{\tau_k}(t)$ is uniformly bounded for all $k$ and all $t \in [\delta, \delta^{-1}]$, it follows that $\{f_{\tau_k}\}$ is equicontinuous on $[\delta, \delta^{-1}]$. Hence the pointwise convergence of (a) yields the uniform convergence of $\{f_{\tau_k}\}$ on $[\delta, \delta^{-1}]$, so that we have (a) $\iff$ (b) since (b) $\implies$ (a) is trivial.

Next, note that, for every $X \in B(\mathcal{H})^{++}$ with the spectrum $\sigma(X)$,

$$\|f_{\tau_k}(X) - f_{\tau}(X)\| = \sup_{t \in \sigma(X)} |f_{\tau_k}(t) - f_{\tau}(t)|,$$

from which it is easy to see that (b) $\iff$ (c).

For $\tau, \tau_k$ as in Lemma 4 we say that $\tau_k$ properly converges to $\tau$ and write $\tau_k \to \tau$ properly, if the equivalent conditions of the lemma hold. Note that we do not take care of the convergence of $f_{\tau_k}(t)$ at $t = 0$. For example, when $\tau_k = (\#_{1/k} + \tau)/2$, we have $\tau_k \to \nabla = (1 + \tau)/2$ properly but $f_{\tau_k}(0) = 0 \not\to f_{\nabla}(0) = 1/2$.

**Proposition 5** Let $\tau, \sigma$ and $\tau_k, \sigma_k, k \in \mathbb{N}$, be operator means with $\sigma, \sigma_k \neq 1$, and assume that $\tau_k \to \tau$ and $\sigma_k \to \sigma$ properly. Then $(\tau_k)_{\sigma_k} \to \tau_{\sigma}$ properly. Hence for every $A, B \in B(\mathcal{H})^{++}$, $A(\tau_k)_{\sigma_k} B \to A\tau_{\sigma} B$ in the operator norm.

**Proof** For simplicity write $f_k := f_{(\tau_k)_{\sigma_k}}$. By Lemma 4 it suffices to prove the pointwise convergence $f_k(t) \to f_{\tau}(t)$ for any $t \in (0, \infty)$. For each fixed $t \in (0, \infty)$ let $x_k := f_k(t), k \in \mathbb{N}$. Then by (18) there is a $\delta \in (0, 1)$ such that $x_k, x_k/t \in [\delta, \delta^{-1}], k \in \mathbb{N}$. Let $x_0$ be any limit point of $\{x_k\}$ so that $x_0$ is a limit of a subsequence $\{x_{k_j}\}$. By (13) for $\tau_k, \sigma_k$ one has

$$f_{\sigma_k}(1/x_k) f_{\tau_k}(\frac{f_{\sigma_k}(t/x_k)}{f_{\sigma_k}(1/x_k)}) = 1. \quad (20)$$
Since \( f_{\sigma k_j} \to f_{\sigma} \) properly, from (b) of Lemma 4 one can easily see that
\[
f_{\sigma k_j} \left( \frac{1}{x_{k_j}} \right) \to f_{\sigma} \left( \frac{1}{x_0} \right), \quad f_{\sigma k_j} \left( \frac{t}{x_{k_j}} \right) \to f_{\sigma} \left( \frac{t}{x_0} \right).
\]
Moreover, since \( f_{\tau k_j} \to f_{\tau} \) properly, one can similarly have
\[
f_{\tau k_j} \left( \frac{f_{\sigma k_j} \left( \frac{t}{x_{k_j}} \right)}{f_{\sigma k_j} \left( \frac{1}{x_{k_j}} \right)} \right) \to f_{\tau} \left( \frac{f_{\sigma} \left( \frac{t}{x_0} \right)}{f_{\sigma} \left( \frac{1}{x_0} \right)} \right).
\]
Hence, letting \( j \to \infty \) in (20) gives
\[
f_{\sigma} \left( \frac{1}{x_0} \right) f_{\tau} \left( \frac{f_{\sigma} \left( \frac{t}{x_0} \right)}{f_{\sigma} \left( \frac{1}{x_0} \right)} \right) = 1
\]
so that \( x_0 = f_{\tau \sigma} (t) \). Therefore, we find that \( f_{\tau \sigma} (t) \) is a unique limit point of \( \{ x_k \} \) so that \( x_k = f_k (t) \to f_{\tau \sigma} (t) \), as desired. \( \square \)

In the rest of this section we will discuss the deformed operator means from an arbitrary \( \tau \) by the weighted power means \( p_s, r \) with two parameters \( s \in (0, 1] \) and \( r \in [-1, 1] \). Recall that \( p_s, r \) for \( r \in [-1, 1] \setminus \{ 0 \} \) is the operator mean corresponding to the operator monotone function
\[
f_{s, r} (t) := (1 - s + st^r)^{1/r}, \quad t \in (0, \infty),
\]
see (3) and (4). Here we use the convention that
\[
p_{s, 0} := \#_s,
\]
that is justified as \( \lim_{r \to 0} f_{s, r} (t) = t^s \) for any \( t \in [0, \infty) \) so that \( p_{s, r} \to \#_s \) properly as \( r \to 0 \). Also we restrict \( s \) to \( (0, 1] \) in view of \( p_{0, r} = 1 \).

For each operator mean \( \tau \) we introduce the two-parameter deformations of \( \tau \) as
\[
\tau_{s, r} := \tau_{p_s, r}, \quad s \in (0, 1], \ r \in [-1, 1]. \quad (22)
\]
In particular, we have
\[
\tau_{s, -1} = \tau_{1_s}, \quad \tau_{s, 0} = \tau_{\#_s}, \quad \tau_{s, 1} = \tau_{\triangledown_s},
\]
the deformed operator means by the weighted harmonic, the weighted geometric and the weighted arithmetic means, respectively. Moreover,
\[
\tau_{1, r} = \tau_{\#_r} = \tau, \quad r \in [-1, 1].
\]
Note that if \( s_k \in (0, 1] \) and \( r_k \in [-1, 1], \ k \in \mathbb{N}, \) are such that \( s_k \to s \in (0, 1] \) and \( r_k \to r \in [-1, 1], \) then \( f_{s_k, r_k} (t) \to f_{s, r} (t) \) for any \( t \in (0, \infty) \) and hence Proposition 5
implies that \( \tau_{s_k, r_k} \rightarrow \tau_{s, r} \) properly. Thus, the two-parameter deformations \( \tau_{s, r} \) \((0 < s \leq 1, -1 \leq r \leq 1)\) of \( \tau \) form a continuous family of operator means bound with \( \tau \) (at \( s = 1 \)).

An interesting question here is to find which operator means appear as boundary values of \( \tau_{s, r} \) in the limit \( s \rightarrow 0 \), which we settle in the following theorem.

**Theorem 2** Let \( s_k \in (0, 1] \) and \( r_k \in [-1, 1], k \in \mathbb{N} \), be such that \( s_k \rightarrow 0 \) and \( r_k \rightarrow r \). Then \( \tau_{s_k, r_k} \rightarrow \rho_{\alpha, r} \) properly, where \( \alpha := f'_{\tau}(1) \in [0, 1] \). Thus, with

\[
\tau_{0.r} := \rho_{\alpha, r}, \quad r \in [-1, 1],
\]

the operator means \( \tau_{s, r} \) is continuous (in the sense of Lemma 4) in two parameters \( s \in [0, 1] \) and \( r \in [-1, 1] \).

**Proof** For simplicity write \( f_k := f_{s_k, r_k} \). By Lemma 4 it suffices to show the convergence \( f_k(t) \rightarrow f_{\alpha, r}(t) \) for any fixed \( r \in (0, \infty) \). Let \( x_k := f_k(t), k \in \mathbb{N} \); then as in the proof of Proposition 5, there is a \( \delta \in (0, 1) \) such that \( \delta \leq x_k \leq \delta^{-1}, k \in \mathbb{N} \). It remains to prove that \( f_{\alpha, r}(t) \) is a unique limit point of \( \{x_k\} \). For this, by replacing \( \{x_k\} \) with a subsequence, we may assume (for notational brevity) that \( \{x_k\} \) itself converges to some \( x_0 \). By (13) one has

\[
f_k(1/x_k) f_{\tau} \left( \frac{f_k(t/x_k)}{f_k(1/x_k)} \right) = 1. \tag{24}
\]

First, assume that \( r \neq 0 \). Since \( r_k \rightarrow r \), we may assume that \( r_k \in [r/2, 1], k \in \mathbb{N} \), for \( r > 0 \) and \( r_k \in [-1, r/2], k \in \mathbb{N} \), for \( r < 0 \). Since \( s_k \rightarrow 0 \), one has

\[
f_k(1/x_k) = \left( 1 - s_k + \frac{s_k}{x_k r_k} \right)^{1/r_k} = 1 + \frac{s_k}{r_k} \left( \frac{1}{x_k r_k} - 1 \right) + o(s_k) \quad \text{as } k \rightarrow \infty,
\]

where \( o(s_k)/s_k \rightarrow 0 \) as \( k \rightarrow \infty \). Similarly,

\[
f_k(t/x_k) = 1 + \frac{s_k}{r_k} \left( \frac{t r_k}{x_k^2} - 1 \right) + o(s_k) \quad \text{as } k \rightarrow \infty
\]

so that

\[
\frac{f_k(t/x_k)}{f_k(1/x_k)} = 1 + \frac{s_k}{r_k} \left( \frac{t r_k}{x_k^2} - \frac{1}{r_k x_k} \right) + o(s_k) \quad \text{as } k \rightarrow \infty.
\]

Since \( f_{\tau}(1 + x) = 1 + \alpha x + o(x) \) as \( x \rightarrow 0 \), one furthermore has

\[
f_{\tau} \left( \frac{f_k(t/x_k)}{f_k(1/x_k)} \right) = 1 + \frac{\alpha s_k}{r_k} \left( \frac{t r_k}{x_k^2} - \frac{1}{r_k x_k} \right) + o(s_k) \quad \text{as } k \rightarrow \infty.
\]
Therefore, by (24) we obtain
\[
\left[ 1 + \frac{s_k}{r_k} \left( \frac{1}{x_k} - 1 \right) + o(s_k) \right] \left[ 1 + \frac{\alpha s_k}{r_k} \left( \frac{t^r_k}{x_k} - \frac{1}{x_k^r} \right) + o(s_k) \right] = 1,
\]
that is,
\[
\frac{s_k}{r_k} \left( \frac{1}{x_k} - 1 \right) + \frac{\alpha s_k}{r_k} \left( \frac{t^r_k}{x_k} - \frac{1}{x_k^r} \right) = o(s_k) \quad \text{as } k \to \infty.
\]
This implies that
\[
\lim_{k \to \infty} \left[ \frac{1}{r_k} \left( \frac{1}{x_k} - 1 \right) + \frac{\alpha}{r_k} \left( \frac{t^r_k}{x_k} - \frac{1}{x_k^r} \right) \right] = 0.
\]
Therefore,
\[
\frac{1}{r} \left( \frac{1}{x_0} - 1 \right) + \frac{\alpha}{r} \left( \frac{t^r}{x_0} - \frac{1}{x_0^r} \right) = 0,
\]
that is, \(1 - x_0^r + \alpha(t^r - 1) = 0\), implying that \(x_0 = (1 - \alpha + \alpha t^r)^{1/r} = f_{\alpha,r}(t)\).

Next, assume that \(r = 0\), so \(s_k, r_k \to 0\). Since
\[
\frac{1}{x_k} - 1 = e^{-r_k \log x_k} - 1 = -r_k \log x_k + o(r_k) \quad \text{as } k \to \infty,
\]
one can compute
\[
\log f_k(1/x_k) = \frac{1}{r_k} \log \left[ 1 + s_k \left( \frac{1}{x_k} - 1 \right) \right]
\]
\[
= \frac{1}{r_k} \log \left[ 1 - s_k r_k \log x_k + s_k o(r_k) \right]
\]
\[
= \frac{1}{r_k} \left[ -s_k r_k \log x_k + s_k o(r_k) + o(s_k r_k) \right]
\]
\[
= -s_k \log x_k + o(s_k) \quad \text{as } k \to \infty,
\]
where the last equality follows from
\[
\frac{1}{s_k} \left[ s_k o(r_k) + o(s_k r_k) \right] = \frac{o(r_k)}{r_k} + \frac{o(s_k r_k)}{s_k r_k} \to 0 \quad \text{as } k \to \infty.
\]
Therefore,
\[
f_k(1/x_k) = \exp \left[ -s_k \log x_k + o(s_k) \right] = 1 - s_k \log x_k + o(s_k) \quad \text{as } k \to \infty,
\]
and similarly,

\[ f_k(t/x_k) = 1 - s_k \log \frac{x_k}{t} + o(s_k) \quad \text{as } k \to \infty. \]

Hence we obtain, as in the proof in the case \( r \neq 0 \),

\[
\left[ 1 - s_k \log x_k + o(s_k) \right] \left[ 1 + \alpha s_k \left( \log x_k - \log \frac{x_k}{t} \right) + o(s_k) \right] = 1,
\]

that is, \(-s_k \log x_k + \alpha s_k \log t = o(s_k)\) as \( k \to \infty \). Therefore,

\[
\lim_{k \to \infty} (-\log x_k + \alpha \log t) = 0,
\]

implying that \( x_0 = t^\alpha = f_{\alpha,0}(t) \).

The latter statement of the theorem is an immediate consequence of the convergence property just proved as well as the fact remarked just before the theorem.

The deformed operator means \( \tau_{s,r} \) \( (0 \leq s \leq 1, -1 \leq r \leq 1) \) constructed above are drawn in the following figure:

---

**Example 4** We here examine the representing operator monotone functions of the deformed operator means in the three lines \( \tau_{s,1} = \tau_{\#s}, \tau_{s,0} = \tau_{\flat s} \) and \( \tau_{s,-1} = \tau_{\uparrow s} \) with \( s \in [0, 1] \). We write \( f = f_r \) that can be an arbitrary element of \( \text{OM}_{+1}(0, \infty) \) with \( \alpha = f'(1) \).

(1) When \( \sigma = \#_r \) with \( 0 < r \leq 1 \), Eq. (13) is solved as \( x = f(r)^{1/r} \). It is well-known that \( f(r)^{1/r} \in \text{OM}_{+1}(0, \infty) \) for any \( r \in (0, 1] \), but it seems less well-known that \( f(r)^{1/r} \to r^\alpha \) properly as \( r \searrow 0 \), a particular case of Theorem 2. One may define a one-parameter continuous family of “generalized power means” \( p_{r,\alpha} \) for \( r \in [-1, 1] \) by

\[ p_{r,\alpha} = f(r)^{1/r}. \]
\[ p_{\tau,r} := \begin{cases} 
\tau_{#r} & (0 < r \leq 1), \\
#_{r} & (r = 0), \\
(\tau^*)_{#-r} & (-1 \leq r < 0), 
\end{cases} \]

joining \( \tau \) \((r = 1)\), \( #_{r} \) \((r = 0)\) and \( \tau^* \) \((r = -1)\), whose representing function is \( f(t^{r})^{1/r} \) for \( r \neq 0 \). A special case where \( \tau = \nabla_{\alpha} \) is the weighted power means \( p_{\alpha,r} \) for \( r \in [-1, 1] \) with \( \alpha \) fixed, dealt with in Examples 2 and 3.

(2) When \( \sigma = \nabla_{s} \) with \( 0 < s \leq 1 \), Eq. (13) becomes

\[ f\left(\frac{(1-r)x + rt}{(1-s)x + st}\right) = \frac{x}{(1-s)x + s}, \quad x > 0, \] (25)

whose solution is \( x = f_{\tau_{\nabla_{s}}}(t) \). For instance, when \( \tau = !_{\alpha} \), the above equation means that

\[ \frac{(1-s)x + s(1-\alpha)t + s\alpha}{(1-s)x + st} = \frac{(1-s)x + s}{x}. \]

Solving this we find that the representing function of \((!_{\alpha})_{\nabla_{s}}\) (where \( 0 < s < 1 \)) is

\[ f_{(!_{\alpha})_{\nabla_{s}}}(t) = \frac{1 - \alpha - s + (\alpha-s)t + \sqrt{(1-\alpha-s + (\alpha-s)t)^2 + 4s(1-s)t}}{2(1-s)}. \] (26)

When \( s = 0 \), the above right-hand side reduces to \( f_{\nabla_{s}}(t) \), which is compatible with the fact that \((!_{\alpha})_{\nabla_{s}}\) approaches to \( \nabla_{\alpha} \) as \( s \searrow 0 \), a particular case of Theorem 2. In particular, when \( \sigma = \nabla_{s} \) and \( \tau = # \), (25) means that

\[ [(1-s)x + s][(1-s)x + st] = x^2. \]

Solving this shows that the representing function of \( #_{\nabla_{s}} \) is

\[ f_{#_{\nabla_{s}}}(t) = \frac{(1-s)(1+t) + \sqrt{(1-s)^2(1+t)^2 + 4s(2-s)t}}{2(2-s)}. \] (27)

Note that when \( s = 0 \) the above reduces to \( f_{\nabla}(t) = (1 + t)/2 \), which is compatible with the fact that \( #_{\nabla_{s}} \) approaches to \( \nabla \) as \( s \searrow 0 \).

(3) When \( \sigma = !_{s} \) with \( 0 < s \leq 1 \), (13) becomes

\[ f\left(\frac{(1-s)t + stx}{(1-s)t + sx}\right) = 1 - s + sx, \quad x > 0, \]

whose solution is \( x = f_{\tau_{s}}(t) \). Note that \( x = f_{\tau_{s}}(t) \) is equivalent to \( x^{-1} = f_{(\tau^*)_{\nabla_{s}}}(t^{-1}) \), as seen from Proposition 1 (5). For instance we have \( f_{#_{s}}(t) = \ldots \)
means. Let \( x \in \mathbb{R} \) and \( r \) be such a regular continuous family, then so is the deformed family if and only if there exists an \( f_{\psi}(t) = f^{(\psi)}(t) \), whose explicit forms can be computed from (27) and (26).

We end the section with some discussions on one-parameter families of operator means. Let \( \{m_{\alpha}\}_{\alpha \in [0, 1]} \) be a one-parameter continuous (in the sense of Lemma 4) family of operator means. We say that such a family is regular if \( f_{m_{\alpha}}(1) = \alpha \) (equivalently, \( !_{\alpha} \leq m_{\alpha} \leq \nabla_{\alpha} \), as noted in (17)) for all \( \alpha \in [0, 1] \). Note that the condition in particular contains \( m_{0} = 1 \) and \( m_{1} = \tau \). In [8] Fujii and Kamei constructed, given a symmetric operator mean \( \sigma \), a regular continuous family of operator means \( \{m_{\alpha}\}_{\alpha \in [0, 1]} \) in the following way:

\[
m_{0} := 1, \quad m_{1/2} := \sigma, \quad m_{1} := \tau,
\]

and inductively

\[
Am_{2^{k+1}}B := (Am_{2^{k}}B)\sigma(Am_{2^{k+1}}B)
\]

for \( n, k \in \mathbb{N} \) with \( 2k + 1 < 2^{n+1} \). The construction was extended by Pálfia and Petz [27] to an arbitrary (not necessarily symmetric) operator mean \( \sigma \) \((\neq 1, \tau)\) in such a way that \( m_{s} = \sigma \) when \( s = f_{\alpha}(1) \); see [31] for the equivalence between the two constructions for a symmetric \( \sigma \). By Propositions 3 and 5 note also that if \( \{m_{\alpha}\}_{\alpha \in [0, 1]} \) is such a regular continuous family, then so is the deformed \( \{m_{\alpha}\}_{\alpha \in [0, 1]} \) by any \( \sigma \neq 1 \); for instance, \( \{m_{s}\}_{\alpha \in [0, 1]} \) for any \( s \in (0, 1] \) given in (26).

Here, of our particular concern is a family of operator means having the interpolation property, introduced in [7,8] as follows: A continuous family of operator means \( \{m_{\alpha}\}_{\alpha \in [0, 1]} \) is called an interpolation family if

\[
(am_{\alpha}b)m_{\delta}(am_{\beta}b) = am_{(1-\delta)\alpha+\delta\beta}b, \quad a, b \in (0, \infty),
\]

for all \( \alpha, \beta, \delta \in [0, 1] \). In this case, for every \( \alpha, \beta \in [0, 1] \) one has

\[
\{(am_{\alpha}b)m_{\beta}a\}(am_{\alpha}b)m_{\beta}b\} = \{(am_{\alpha}b)m_{\beta}(am_{0}b)\}m_{\alpha}\{(am_{\alpha}b)m_{\beta}(am_{1}b)\}
\]

\[
= \{(am_{(1-\beta)\alpha}(am_{(1-\beta)\alpha}\alpha+\beta)b)
\]

\[
= am_{(1-\alpha)(1-\beta)\alpha+\alpha((1-\beta)\alpha+\beta)}b
\]

\[
= am_{\alpha}b,
\]

which implies that \( x = 1m_{\alpha}t = f_{m_{\alpha}}(t) \) is a solution to Eq. (12) for \( \tau = m_{\alpha} \) and \( \sigma = m_{\beta} \), that is, \( (m_{\alpha})m_{\beta} = m_{\alpha} \) for all \( \alpha \in [0, 1] \) and \( \beta \in (0, 1] \). It was proved in [31] that if \( \{m_{\alpha}\}_{\alpha \in [0, 1]} \) is a regular continuous family of operator means, then it is an interpolation family if and only if there exists an \( r \in [-1, 1] \) such that \( m_{\alpha} = p_{\alpha,r} \) for all \( \alpha \in [0, 1] \). Therefore, we have, for every \( \alpha \in [0, 1], \beta \in (0, 1] \) and \( r \in [-1, 1] \),

\[
(p_{\alpha,r})p_{\beta,r} = p_{\alpha,r}, \quad (28)
\]
whose direct proof is also easy. In particular, for every $\alpha \in [0, 1]$ and $\beta \in (0, 1]$,\[ (\nabla_\alpha) \nabla_\beta = \nabla_\alpha, \quad (\#_\alpha) \#_\beta = \#_\alpha, \quad (!_\alpha)_!_\beta = !_\alpha. \]

The following proposition may be worth giving while it is not essentially new.

**Proposition 6** For every symmetric operator mean $\sigma$ consider the following conditions:

(i) for some $r \in [-1, 1]$, $\sigma = p_{1/2,r}$, i.e., $f_\sigma(x) = \left(1 + \frac{x^r}{2}\right)^{1/r}$ (meant as $f_\sigma(x) = x^{1/2}$ for $r = 0$),

(ii) $(a \sigma b)\sigma(c \sigma d) = (a \sigma c)\sigma(b \sigma d)$ for all $a, b, c, d \in (0, \infty)$,

(iii) $f_\sigma(x)\sigma f_\sigma(y) = f_\sigma(x \sigma y)$ for all $x, y \in (0, \infty)$,

(iv) $\sigma_\sigma = \sigma$.

Then the following implications hold: (i) $\iff$ (ii) $\implies$ (iii) $\implies$ (iv).

**Proof** (i) $\implies$ (ii) is immediately seen.

(ii) $\implies$ (i). Let $\{m_\alpha\}_{\alpha \in [0,1]}$ be the regular continuous family constructed in [8] (as mentioned above) from $\sigma$ so that $m_{1/2} = \sigma$. It was shown in [7, Theorem 1] that $\sigma$ satisfies condition (ii) if and only if $\{m_\alpha\}$ is an interpolation family. Then it follows from [31, Theorem 6] that $\{m_\alpha\} = \{p_{\alpha,r}\}_\alpha$ for some $r \in [-1, 1]$. Hence $\sigma = m_{1/2} = p_{1/2,r}$.

(ii) $\implies$ (iii) is obvious by taking $a = c = 1$ in (ii), as mentioned in [7].

(iii) $\implies$ (iv). For every $t > 0$ let $x := f_\sigma(t)$; then one has by (iii)

$$f_\sigma(1/x)f_\sigma\left(\frac{f_\sigma(t/x)}{f_\sigma(1/x)}\right) = f_\sigma(1/x)\sigma f_\sigma(t/x) = f_\sigma((1/x)\sigma(t/x))$$

$$= f_\sigma((1/x)f_\sigma(t)) = f_\sigma(1) = 1.$$

This implies by Theorem 1 that $\sigma = \sigma_\sigma$.

**Problem 1** Is there an operator mean $\sigma (\neq 1)$, apart from the weighted power means $p_{\alpha,r}$, such that $\sigma_\sigma = \sigma$? The condition $\sigma_\sigma = \sigma$ seems quite hard to hold unless $\sigma$ is in the family $p_{\alpha,r}$. For example, for $-1 \leq q \leq 2$ let $\sigma_q$ be a power difference mean with the representing function

$$f_q(t) = \frac{q - 1}{q} \cdot \frac{1 - t^q}{1 - t^{q-1}};$$

see [11, Proposition 4.2] for the fact that $f_q \in OM_{+,1}(0, \infty)$ when (and only when) $-1 \leq q \leq 2$. A numerical computation verifies the failure of $\sigma_\sigma = \sigma$ for $\sigma = \sigma_q$ except when $q = -1, 1/2, 2$ (the cases of the harmonic, geometric and arithmetic means).
4 Multivariate operator means

The aim of this section is to generalize our previous discussions on deformation of two-variable operator means (in the Kubo–Ando sense) to multivariate operator means. We here assume, unless otherwise stated, that $H$ is a general Hilbert space whichever finite-dimensional or infinite-dimensional. In what follows, we consider an $n$-variable ($n \geq 2$) operator mean

$$M : (B(H)^{++})^n \rightarrow B(H)^{++}$$

having the following properties:

(A) Joint monotonicity: If $A_j, B_j \in B(H)^{++}$ and $A_j \leq B_j$ for $1 \leq j \leq n$, then $M(A_1, \ldots, A_n) \leq M(B_1, \ldots, B_n)$.

(B) Homogeneity: $M(\alpha A_1, \ldots, \alpha A_n) = \alpha M(A_1, \ldots, A_n)$ for every $A_j \in B(H)^{++}$ and $\alpha > 0$.

(C) Monotone continuity: If $A_{j,k} \in B(H)^{++}$, $k \in \mathbb{N}$, and $A_{j,k} \searrow A_j \in B(H)^{++}$ as $k \to \infty$ for $1 \leq j \leq n$, then $M(A_{1,k}, \ldots, A_{n,k}) \rightarrow M(A_1, \ldots, A_n)$ in the strong operator topology. The same holds if $A_{j,k} \nearrow A_j \in B(H)^{++}$ as $k \to \infty$ for $1 \leq j \leq n$.

(D) Normalization: $M(I, \ldots, I) = I$.

We note that the above (A)–(D) are more or less similar to (I)–(IV), respectively, given in Sect. 2 for Kubo-Ando’s definition of two-variable operator means, so properties (A)–(D) might potentially serve as an abstract definition of $n$-variable operator means.

The above properties will always be assumed as minimal requirements for our arguments below on deformation of $M$ by two-variable operator means. Unlike two-variable operator means treated in Sects. 2 and 3, we fix the domain of multivariate means to $B(H)^{++}$ and do not consider to extend them to $B(H)^+$. In the sequel we will thus use a simpler notation $\mathbb{P}$ in place of $B(H)^{++}$. We denote by $w = (w_1, \ldots, w_n)$ a probability weight vector, i.e., $w_j \geq 0$ with $\sum_{j=1}^n w_j = 1$. The most familiar examples of multivariate operator means are the weighted arithmetic and harmonic means

$$\mathcal{A}_w(A_1, \ldots, A_n) := \sum_{j=1}^n w_j A_j, \quad \mathcal{H}_w(A_1, \ldots, A_n) := \left( \sum_{j=1}^n w_j A_j^{-1} \right)^{-1}.$$ 

More substantial and recently the most studied examples are the multivariate extensions of the weighted geometric mean and the weighted power means, as briefly described below (also see the descriptions in [15, Example 2.4]).

Example 5 The weighted multivariate geometric mean $G_w(A_1, \ldots, A_n)$, variously called the Riemannian mean, the Karcher mean, the Cartan mean, etc. was introduced for positive definite matrices by Moakher [25] and by Bhatia and Holbrook [5] in a Riemannian geometry approach, whose monotonicity property (A) was proved by
Lawson and Lim [19]. A significant feature of $G_w$ is that it is determined as a unique solution to the Karcher equation

$$\sum_{j=1}^{n} w_j \log X^{-1/2} A_j X^{-1/2} = 0, \quad X \in \mathbb{P}. \quad (29)$$

This equation is also used to define $G_w(A_1, \ldots, A_n)$ for infinite-dimensional positive operators, see [20,21]. Note that $G_w$ is indeed the extension of the weighted geometric mean $\#_\alpha$ as $G_{(1-\alpha,\alpha)}(A, B) = A\#_\alpha B$.

Although it is not explicitly mentioned in [21], one can easily see from the arguments there that $G_w$ satisfies monotone continuity (C). (In the finite-dimensional case, this is obvious from the continuity of $G_w$ in the operator norm shown in [21].) Indeed, assume that $A_{j,k} \searrow A_j \in \mathbb{P}$ for $1 \leq j \leq n$, and let $X_k := G_w(A_{1,k}, \ldots, A_{n,k})$. One can choose a $\delta \in (0, 1)$ so that $\delta I \leq A_{j,k} \leq \delta^{-1}I$ for all $j, k$. Then $\delta I \leq X_k \leq \delta^{-1}I$ as well for all $k$ and the monotonicity property implies that $X_k \searrow X_0$ for some $X_0 \in \mathbb{P}$. Since $(A, Y) \mapsto \log Y^{-1/2} A Y^{-1/2}$ is continuous in the strong operator topology on $\delta I \leq A, Y \leq \delta^{-1}I$ (see [21]), it follows that $X_0$ satisfies the Karcher equation in (29) for $A_j$, so $X_0 = G_w(A_1, \ldots, A_n)$. The proof in the case $A_{j,k} \nearrow A_j$ is similar.

From the viewpoint of the fixed point method, it is worth noting that, for every $A_j \in \mathbb{P}$ and $0 < r \leq 1$, $X = G_w(A_1, \ldots, A_n)$ is a unique solution to the equation

$$X = G_w(X_\#_r A_1, \ldots, X_\#_r A_n), \quad X \in \mathbb{P}. \quad (30)$$

This was shown in [13, Theorem 3.4] in the setting of probability measures on the positive definite matrices, but the same proof is valid in the infinite-dimensional case as well.

**Example 6** The multivariate weighted power mean $P_{w,r}(A_1, \ldots, A_n)$ for $r \in [-1, 1] \setminus \{0\}$ was introduced for positive definite matrices by Lim and Pálfia [22], which was extended to infinite-dimensional operators in [20,21]. For $A_j \in \mathbb{P}$, $P_{w,r}(A_1, \ldots, A_n)$ is defined as a unique solution $X \in \mathbb{P}$ to either of the equations

$$X = \mathcal{A}_w(X_\#_r A_1, \ldots, X_\#_r A_n) \quad \text{for} \quad 0 < r \leq 1, \quad (31)$$

$$X = \mathcal{H}_w(X_\#_{-r} A_1, \ldots, X_\#_{-r} A_n) \quad \text{for} \quad -1 \leq r < 0, \quad (32)$$

which are the extension of $p_{\alpha,r}$ given in Examples 2 and 3 as $P_{(1-\alpha,\alpha),r}(A, B) = A p_{\alpha,r} B$. Clearly, $P_{w,1} = \mathcal{A}_w$ and $P_{w,-1} = \mathcal{H}_w$. An important fact proved in [20–22] is that

$$\lim_{r \to 0} P_{w,r}(A_1, \ldots, A_n) = G_w(A_1, \ldots, A_n)$$

in the strong operator topology.

Joint monotonicity (A) of $P_{w,r}$ was shown in [20–22] by a fixed point method. By an argument similar to that in Example 5 one can show that $P_{w,r}$ satisfies monotone continuity (C). Indeed, assume that $A_{j,k} \searrow A_j \in \mathbb{P}$ for $1 \leq j \leq n$, and choose a
\( \delta \in (0, 1) \) such that \( \delta I \leq A_{j,k} \leq \delta^{-1} I \) for all \( j, k \). Let \( X_k := P_{w,r}(A_{1,k}, \ldots, A_{n,k}) \); then \( X_k \searrow X_0 \) for some \( X_0 \in \mathbb{P} \). It then follows that

\[
\mathcal{A}_w(X_k\#_r A_{1,k}, \ldots, X_k\#_r A_{n,k}) \longrightarrow \mathcal{A}_w(X_0\#_r A_{1}, \ldots, X_0\#_r A_{n}), \\
\mathcal{H}_w(X_k\#_{-r} A_{1,k}, \ldots, X_k\#_{-r} A_{n,k}) \longrightarrow \mathcal{H}_w(X_0\#_{-r} A_{1}, \ldots, X_0\#_{-r} A_{n})
\]

in the strong operator topology for \( 0 < r \leq 1 \) and \(-1 \leq r < 0\), respectively. Hence \( X_0 = P_{w,r}(A_{1}, \ldots, A_{n}) \). The proof when \( A_{j,k} \searrow A_j \) is similar.

All of the multivariate means \( \mathcal{A}_w, \mathcal{H}_w, G_w \) and \( P_{w,r} \) given so far satisfy all properties (A)–(D). Additionally they satisfy, among others, the following properties (see [20,21,33] and references therein):

(E) *Congruence invariance:* For every \( A_j \in \mathbb{P} \) and any invertible \( C \in \mathcal{B}(\mathcal{H}) \),

\[
CM(A_1, \ldots, A_n)C^* = M(CA_1C^*, \ldots, CA_nC^*).
\]

(F) *Joint concavity:* For every \( A_j, B_j \in \mathbb{P} \) and \( 0 < \lambda < 1 \),

\[
M(\lambda A_1 + (1 - \lambda)B_1, \ldots, \lambda A_n + (1 - \lambda)B_n) \\
\geq \lambda M(A_1, \ldots, A_n) + (1 - \lambda)M(B_1, \ldots, B_n).
\]

From homogeneity (B) this is equivalent to

\[
M(A_1 + B_1, \ldots, A_n + B_n) \geq M(A_1, \ldots, A_n) + M(B_1, \ldots, B_n).
\]

(G) \( \mathcal{A}_w, \mathcal{H}_w \) *weighted mean inequalities:* With some weight vector \( w \), for every \( A_j \in \mathbb{P} \),

\[
\mathcal{H}_w(A_1, \ldots, A_n) \leq M(A_1, \ldots, A_n) \leq \mathcal{A}_w(A_1, \ldots, A_n).
\]

It is indeed known [20–22] that

\[
\mathcal{H}_w \leq P_{w,-s} \leq P_{w,-r} \leq G_w \leq P_{w,r} \leq P_{w,s} \leq \mathcal{A}_w \text{ if } 0 < r < s < 1.
\]

From now on, assume that \( M : \mathbb{P}^n \rightarrow \mathbb{P} \) is an \( n\)-variable operator mean satisfying (A)–(D) stated at the beginning of the section, and let \( \sigma_1, \ldots, \sigma_n \) be two-variable operator means (in the Kubo and Ando sense) such that \( \sigma_j \neq I \) for any \( j \). For given \( A_1, \ldots, A_n \in \mathbb{P} \) we consider the equation

\[
X = M(X\sigma_1 A_1, \ldots, X\sigma_n A_n), \quad X \in \mathbb{P}, \tag{33}
\]

which generalizes (30)–(32).
For every $A_j, B_j \in P$ the following $d_T$-inequality is well-known for $G_w$ and $P_{w,r}$ (see [21]). The inequality for $M$ easily follows from (A) and (B) as shown in [15, Lemma 2.2]:

$$d_T(M(A_1, \ldots, A_n), M(B_1, \ldots, B_n)) \leq \max_{1 \leq j \leq k} d_T(A_j, B_j).$$

(34)

In particular, this implies that $M(A_1, \ldots, A_n)$ is continuous on $P^n$ in the operator norm.

In a similar way to the proof of [15, Theorem 2.1] treating the case where $\sigma_1 = \cdots = \sigma_n = \sigma$, we can show that for every $A_j \in P$ there exists a unique $X_0 \in P$ which satisfies (33). We write $M(\sigma_1, \ldots, \sigma_n)(A_1, \ldots, A_n)$ for this unique solution $X_0 \in P$ to (33), and so we have a map

$$M(\sigma_1, \ldots, \sigma_n) : P^n \rightarrow P.$$  

(35)

It can be also shown as in [15, Theorem 2.1] that the map $M(\sigma_1, \ldots, \sigma_n)$ satisfies (A)–(D) as $M$ does the same and furthermore the following holds.

**Lemma 5** Let $X \in P$.

1. If $X \geq M(X\sigma_1 A_1, \ldots, X\sigma_n A_n)$, then $X \geq M(\sigma_1, \ldots, \sigma_n)(A_1, \ldots, A_n)$.
2. If $X \leq M(X\sigma_1 A_1, \ldots, X\sigma_n A_n)$, then $X \leq M(\sigma_1, \ldots, \sigma_n)(A_1, \ldots, A_n)$.

**Remark 3** When $\mathcal{H}$ is finite-dimensional, we can prove the unique existence of the solution to Eq. (33) under (A), (B) and (D) without (C). To see this, choose a $\delta \in (0, 1)$ such that $\delta I \leq A_j \leq \delta^{-1} I$ for $1 \leq j \leq n$, and let $\Sigma_\delta := \{X \in P : \delta I \leq X \leq \delta^{-1} I\}$ which is a compact convex subset of the $d^2$-dimensional Euclidean space $B(\mathcal{H})^{sa}$, the space of self-adjoint $X \in B(\mathcal{H})$, where $d := \dim \mathcal{H}$. Define the map $F : P \rightarrow P$ by

$$F(X) := M(X\sigma_1 A_1, \ldots, X\sigma_n X_n), \quad X \in P.$$  

(36)

Then it follows from (A), (B) and (D) of $M$ that $F$ maps $\Sigma_\delta$ into itself. Since $F$ is continuous in the operator norm, the existence of the solution follows from Brouwer’s fixed point theorem, and its uniqueness is shown as in [15, Theorem 2.1], where assumption (C) is unnecessary. However, under this situation without (C) we are not able to show that the resulting map (35) satisfies (A) and (B).

We call $M(\sigma_1, \ldots, \sigma_n)$ the **deformed operator mean** from $M$ by $(\sigma_1, \ldots, \sigma_n)$. When $\sigma_1 = \cdots = \sigma_n = \sigma$, we simply write $M_{\sigma}$ and call it the deformed operator mean from $M$ by $\sigma$. The deformed operator mean $\tau_\sigma$ discussed in Sects. 2 and 3 is the special case where $M = \tau$ is a two-variable operator mean and $\sigma_1 = \sigma_2 = \sigma$. The multivariate weighted power means $P_{w,r}$ in Example 6 are typical examples of deformed operator means as

$$P_{w,r} = (\mathcal{A}_w)^{\#_r}, \quad P_{w,-r} = (\mathcal{H}_w)^{\#_r} \quad \text{for} \quad 0 < r \leq 1.$$
From Example 5 note also that

\[ G_w = (G_w)_r, \quad \text{for} \quad 0 < r \leq 1. \]

**Remark 4** In [26] Pálfia introduced a generalized notion of operator means of probability measures on \( \mathbb{P} \) determined by the generalized Karcher equation. Restricted to the \( n \)-variable situation, the equation is given as

\[ \sum_{j=1}^{n} w_j g(X^{-1/2} A_j X^{-1/2}) = 0, \quad (37) \]

where \( w = (w_1, \ldots, w_n) \) is a weight vector and \( g \) is an operator monotone function on \((0, \infty)\) with \( g(1) = 0 \) and \( g'(1) = 1 \). For a two-variable operator mean \( \sigma \neq l \) let \( g_\sigma(x) := (f_\sigma(x) - 1)/f_\sigma'(1); \) then \( g_\sigma \) is operator monotone on \((0, \infty)\) satisfying \( g_\sigma(1) = 0 \) and \( g_\sigma'(1) = 1 \), and Eq. (33) for \( M = \mathcal{A}_w \) and \( \sigma_1 = \cdots = \sigma_n = \sigma \) is equivalent to (37) with \( g = g_\sigma \). Hence, in the special case \( M = \mathcal{A}_w \), a deformed operator mean \( (\mathcal{A}_w)_\sigma \) is a generalized operator mean determined by (37). The unique existence of the solution of (37) in [26] is based on the Banach contraction principle with respect to the Thompson metric \( d_T \), while our proof (see [15]) is based on the monotone continuity of \( M \).

The *adjoint* operator mean \( M^* \) of \( M \) is defined by

\[ M^*(A_1, \ldots, A_n) := M(A_1^{-1}, \ldots, A_n^{-1})^{-1}, \quad A_j \in \mathbb{P}. \]

It is immediate to verify that \( M^* \) satisfies (A)–(D) again. The operator mean \( M \) is said to be *self-adjoint* if \( M = M^* \). Note that for multivariate operator means the term “dual” is rather used for \( M^* \) as in [20–22], but we prefer to use the term “adjoint” in accordance with the case of operator means in the Kubo–Ando sense.

The next proposition in particular shows the preservation of properties (E)–(F) under taking deformed means.

**Proposition 7**

1. \( M_{\bar{t}} = M \).
2. Let \( \hat{M} \) be an \( n \)-variable operator mean with (A)–(D) and \( \hat{\sigma}_j \), \( 1 \leq j \leq n \), be two-variable operator means. If \( M \leq \hat{M} \) and \( \sigma_j \leq \hat{\sigma}_j \) for \( 1 \leq j \leq n \), then \( M(\sigma_1, \ldots, \sigma_n) \leq \hat{M}(\hat{\sigma}_1, \ldots, \hat{\sigma}_n) \).
3. \( (M(\sigma_1, \ldots, \sigma_n))^* = (M^*)(\sigma_1^*, \ldots, \sigma_n^*) \). Hence, if \( M \) and \( \sigma_j \) are self-adjoint, then so is \( M(\sigma_1, \ldots, \sigma_n) \).
4. For any permutation \( \pi \) on \( \{1, \ldots, n\} \) define

\[ M^\pi(A_1, \ldots, A_n) := M(A_{\pi^{-1}(1)}, \ldots, A_{\pi^{-1}(k)}). \]

Then \( M(\sigma_1, \ldots, \sigma_n)^\pi = (M^\pi)(\sigma_{\pi(1)}, \ldots, \sigma_{\pi(k)}) \). Hence, if \( M \) is permutation invariant, then so is \( M_{\sigma} \) for any two-variable operator mean \( \sigma \).
5. If \( M \) satisfies (E), then \( M(\sigma_1, \ldots, \sigma_n) \) does the same.
6. If \( M \) satisfies (F), then \( M(\sigma_1, \ldots, \sigma_n) \) does the same.
(7) Assume that $M$ satisfies (G) with a weight vector $\mathbf{w}$, and let

$$\hat{\mathbf{w}} := \left( \sum_{j=1}^{n} w_j \alpha_j \right)^{-1} (w_1 \alpha_1, \ldots, w_n \alpha_n),$$

where $\alpha_j := f'_\sigma(1)$, $1 \leq j \leq n$. Then $M(\sigma_1, \ldots, \sigma_n)$ satisfies (G) with the weight vector $\hat{\mathbf{w}}$. In particular, for any two-variable operator mean $\sigma$, $M_\sigma$ satisfies (G) with the same $\mathbf{w}$.

Proof (1) is obvious. The proofs of (2) and (3) are similar to those of Proposition 1 (3) and (4), respectively.

(4) immediately follows from

$$M^\pi (X \sigma_\pi(1) A_1, \ldots, X \sigma_\pi(n) A_n) = M(X \sigma_1 A_{\pi^{-1}(1)}, \ldots, X \sigma_n A_{\pi^{-1}(n)}).$$

(5) For $X_0 := M(\sigma_1, \ldots, \sigma_n)(A_1, \ldots, A_n)$ and invertible $C$ one has

$$CX_0 C^* = M((CX_0 C^*) \sigma_1 (CA_1 C^*), \ldots, (CX_0 C^*) \sigma_n (CA_n C^*))$$

so that $CX_0 C^* = M(\sigma_1, \ldots, \sigma_n)(CA_1 C^*, \ldots, CA_n C^*)$.

(6) Let $X_0 := M(\sigma_1, \ldots, \sigma_n)(A_1, \ldots, A_n)$ and $Y_0 := M(\sigma_1, \ldots, \sigma_n)(B_1, \ldots, B_n)$. Since, by [18, Theorem 3.5],

$$(X_0 + Y_0) \sigma_j (A_j + B_j) \geq (X_0 \sigma_j B_j) + (Y_0 \sigma_j B_j),$$

we have, by (A) and (F) of $M$,

$$M((X_0 + Y_0) \sigma_1 (A_1 + B_1), \ldots, (X_0 + Y_0) \sigma_n (A_n + B_n))$$

$$\geq M(X_0 \sigma_1 A_1, \ldots, X_0 \sigma_n A_n) + M(Y_0 \sigma_1 B_1, \ldots, Y_0 \sigma_n B_n) = X_0 + Y_0,$$

which implies by Lemma 5 (2) that

$$M(\sigma_1, \ldots, \sigma_n)(A_1 + B_1, \ldots, A_n + B_n) \geq X_0 + Y_0,$$

as required.

(7) Since $1_{\alpha_j} \leq \sigma_j \leq \nabla_{\alpha_j}$, it follows from the assertion (2) above that

$$(\mathcal{A}_\mathbf{w})(1_{\alpha_1}, \ldots, 1_{\alpha_n}) \leq M(\sigma_1, \ldots, \sigma_n) \leq (\mathcal{A}_\mathbf{w})(\nabla_{\alpha_1}, \ldots, \nabla_{\alpha_n}).$$

The equation

$$X = \mathcal{A}_\mathbf{w}(X \nabla_{\alpha_1} A_1, \ldots, X \nabla_{\alpha_n} A_n)$$
Proof Let \( \Delta \), while we assume here that 
\[ \sum_{j=1}^{n} w_j \alpha_j \] 
is easily solved as 
\[ X = \left( \sum_{j=1}^{n} w_j \alpha_j \right)^{-1} \sum_{j=1}^{n} w_j \alpha_j A_j = \mathcal{H}_w(A_1, \ldots, A_n), \]
while the equation 
\[ X = \mathcal{H}_w(X !_{\alpha_1} A_1, \ldots, X !_{\alpha_n} A_n) \]
is solved as 
\[ X = \mathcal{H}_w(A_1, \ldots, A_n). \] (The latter also follows from the former and the assertion (3)). Therefore, \( M(\sigma_1, \ldots, \sigma_n) \) satisfies (G) with the weight vector \( \hat{w} \). \( \square \)

**Proposition 8** For each \( j = 1, \ldots, n \), let \( \sigma_j \) and \( \sigma_{j,k}, k \in \mathbb{N} \), be two-variable operator means with \( \sigma_j, \sigma_{j,k} \neq 1 \), and assume that \( \sigma_{j,k} \rightarrow \sigma_j \), that is, \( f_{\sigma_{j,k}}(t) \rightarrow f_{\sigma_j}(t) \) for any \( t \in (0, \infty) \) as \( k \rightarrow \infty \). Then for every \( A_j \in \mathbb{P} \),
\[ M(\sigma_{1,k}, \ldots, \sigma_{n,k}) (A_1, \ldots, A_n) \rightarrow M(\sigma_1, \ldots, \sigma_n) (A_1, \ldots, A_n) \] as \( k \rightarrow \infty \)
in the strong operator topology. The upward convergence holds similarly when \( \sigma_{j,k} \nrightarrow \sigma_j \) as \( k \rightarrow \infty \) for each \( j \).

**Proof** Let \( X_k := M(\sigma_{1,k}, \ldots, \sigma_{n,k}) (A_1, \ldots, A_n) \) for each \( k \in \mathbb{N} \). It follows from Proposition 7 (2) that \( X_1 \geq X_2 \geq \cdots \). Choose a \( \delta > 0 \) such that \( A_j \geq \delta I \) for \( 1 \leq j \leq n \). Since \( \delta I \leq M((\delta I) \sigma_{1,k} A_1, \ldots, (\delta I) \sigma_{n,k} A_n) \), Lemma 5 (2) gives \( X_k \geq \delta I \) for all \( k \). Therefore, \( X_k \nrightarrow X_0 \) for some \( X_0 \in \mathbb{P} \). We may now prove that \( X_0 = M(\sigma_1, \ldots, \sigma_n) (A_1, \ldots, A_n) \), that is, \( X_0 = M(X_0 \sigma_1 A_1, \ldots, X_0 \sigma_n A_n) \). From (C) it suffices to show that \( X_k \sigma_{j,k} A_j \nrightarrow X_0 \sigma_j A_j \) as \( k \rightarrow \infty \) for every \( j = 1, \ldots, n \). But this is easy to verify since \( X_k \nrightarrow X_0 \) and \( \sigma_{j,k} \nrightarrow \sigma_j \). The proof is similar when \( \sigma_{j,k} \nrightarrow \sigma_j \). \( \square \)

In the following we present the multivariate versions of Proposition 5 and Theorem 2, while we assume here that \( \mathcal{H} \) is finite-dimensional.

**Proposition 9** Assume that \( \mathcal{H} \) is finite-dimensional. For each \( j = 1, \ldots, n \) let \( \sigma_j, \sigma_{j,k}, k \in \mathbb{N} \), be two-variable operator means with \( \sigma_j, \sigma_{j,k} \neq 1 \), and assume that \( \sigma_{j,k} \rightarrow \sigma_j \) properly as \( k \rightarrow \infty \). Then for every \( A_j \in \mathbb{P} \),
\[ M(\sigma_{1,k}, \ldots, \sigma_{n,k}) (A_1, \ldots, A_n) \rightarrow M(\sigma_1, \ldots, \sigma_n) (A_1, \ldots, A_n) \]
in the operator norm.

**Proof** Choose a \( \delta \in (0, 1) \) such that \( \delta I \leq A_j \leq \delta^{-1} I \) for \( 1 \leq j \leq n \). Let \( X_0 := M(\sigma_1, \ldots, \sigma_n) (A_1, \ldots, A_n) \) and \( X_k := M(\sigma_{1,k}, \ldots, \sigma_{n,k}) (A_1, \ldots, A_n) \); then by Lemma 5 (1) and (2), \( \delta I \leq X_k \leq \delta^{-1} I \) as well for all \( k \). From the finite-dimensionality assumption, note that \( \{ X \in B(\mathcal{H})^+ : \delta I \leq X \leq \delta^{-1} I \} \) is compact in the operator norm. Hence, to prove that \( X_k \rightarrow X_0 \) in the operator norm, it suffices to show that \( X_0 \) is a unique
limit point of \( \{X_k\} \). By replacing \( \{X_k\} \) by a subsequence we may assume (for notational brevity) that \( \{X_k\} \) itself converges to a \( Y_0 \in \mathbb{P} \). For each \( j = 1, \ldots, n \) we have

\[
\|X_k \sigma_{j,k} A_j - Y_0 \sigma_j A_j\| = \|A_j \sigma_{j,k} X_k - A_j \sigma_j Y_0\|
\]

\[
\leq \|A_j\| \|f_{\sigma_{j,k}}(A_j^{-1/2} X_k A_j^{-1/2}) - f_{\sigma_j}(A_j^{-1/2} Y_0 A_j^{-1/2})\|
\]

\[
\leq \|A_j\| \|f_{\sigma_{j,k}}(A_j^{-1/2} X_k A_j^{-1/2}) - f_{\sigma_j}(A_j^{-1/2} X_k A_j^{-1/2})\|
\]

\[
+ \|A_j\| \|f_{\sigma_j}(A_j^{-1/2} X_k A_j^{-1/2}) - f_{\sigma_j}(A_j^{-1/2} Y_0 A_j^{-1/2})\|
\]

where \( \sigma_{j,j,k} \) are the transposes of \( \sigma_j, \sigma_{j,k} \). Since \( f_{\sigma_{j,k}}(t) = tf_{\sigma_{j,k}}(t^{-1}) \rightarrow t f_{\sigma_j}(t^{-1}) = f_{\sigma_j}(t) \) for any \( t \in (0, \infty) \) and \( \delta A_j^{-1} \leq A_j^{-1/2} X_k A_j^{-1/2} \leq \delta^{-1} A_j^{-1} \) for all \( k \), it follows from Lemma 4 that

\[
\|f_{\sigma_{j,k}}(A_j^{-1/2} X_k A_j^{-1/2}) - f_{\sigma_j}(A_j^{-1/2} X_k A_j^{-1/2})\| \rightarrow 0 \quad \text{as } k \rightarrow \infty
\]

so that \( \|X_k \sigma_{j,k} A_j - Y_0 \sigma_j A_j\| \rightarrow 0 \). Therefore, by (34),

\[
X_k = M(X_k \sigma_{1,k} A_1, \ldots, X_k \sigma_{n,k} A_n) \longrightarrow M(Y_0 \sigma_1 A_1, \ldots, Y_0 \sigma_n A_n)
\]

in the operator norm. This implies that \( Y_0 = M(Y_0 \sigma_1 A_1, \ldots, Y_0 \sigma_n A_n) \), and hence \( Y_0 = X_0 \) follows.

\[\square\]

**Theorem 3** Assume that \( H \) is finite-dimensional and \( M \) satisfies, in addition to (A)–(D), (G) with a weight vector \( w \). Then for every \( A_j \in \mathbb{P}, \)

\[
M_{(p_{x,r_1}, \ldots, p_{x,r_n})}(A_1, \ldots, A_n)
\]

\[
\longrightarrow \begin{cases} (\mathcal{A}_w)(#_{r_1}, \ldots, #_{r_n}) (A_1, \ldots, A_n) & \text{if } r_1, \ldots, r_n \in (0, 1), \\ (\mathcal{H}_w)(#_{r_1}, \ldots, #_{r_n}) (A_1, \ldots, A_n) & \text{if } r_1, \ldots, r_n \in [-1, 0), \\ G_w (A_1, \ldots, A_n) & \text{if } r_1 = \cdots = r_n = 0, \end{cases}
\]

in the operator norm as \( s \searrow 0 \) with \( s \in (0, 1] \), where

\[
\hat{w} := \begin{cases} \left( \sum_{j=1}^{n} \frac{w_j}{r_j} \right)^{-1} \left( \frac{w_1}{r_1}, \ldots, \frac{w_n}{r_n} \right) & \text{if } r_1, \ldots, r_n \in (0, 1), \\ \left( \sum_{j=1}^{n} \frac{w_j}{-r_j} \right)^{-1} \left( \frac{w_1}{-r_1}, \ldots, \frac{w_n}{-r_n} \right) & \text{if } r_1, \ldots, r_n \in [-1, 0). \end{cases}
\]

**Proof** In view of property (G) and Proposition 7 (2) we may prove the result for \( M = \mathcal{A}_w \) and for \( M = \mathcal{H}_w \). Choose a \( \delta \in (0, 1) \) such that \( \delta I \leq A_j \leq \delta^{-1} I \) for \( 1 \leq j \leq n \). First, let us prove the case \( M = \mathcal{A}_w \), and let either \( r_1, \ldots, r_n \in (0, 1) \) or \( r_1, \ldots, r_n \in [-1, 0) \). For \( s \in (0, 1] \) let \( X_s := (\mathcal{A}_w)(p_{x,r_1}, \ldots, p_{x,r_n}) (A_1, \ldots, A_n) \). Since

\[
I = (\mathcal{A}_w)(p_{x,r_1}, \ldots, p_{x,r_n}) (X_s^{-1/2} A_1 X_s^{-1/2}, \ldots, X_s^{-1/2} A_n X_s^{-1/2})
\]
by Proposition 7 (5), we have

$$I = \sum_{j=1}^{n} w_j \left[ (1-s)I + s(X_{s}^{-1/2} A_j X_{s}^{-1/2})^{1/r_j} \right].$$

(38)

By taking account of $\{X_s\}_{s \in (0,1]}$ being in the compact set $\{X \in \mathbb{P} : \delta I \leq X \leq \delta^{-1} I\}$ (due to the finite-dimensionality of $\mathcal{H}$), let $Y_0$ be any limit point of $X_s$ as $s \searrow 0$, so $X_k := X_{s_k} \rightarrow Y_0$ for some sequence $s_k \in (0,1], s_k \searrow 0$. By (38) one can write

$$I = \sum_{j=1}^{n} w_j \left[ I + \frac{s_k}{r_j} \left( (X_k^{-1/2} A_j X_k^{-1/2})^{1/r_j} - I \right) + o(s_k) \right]$$

so that

$$0 = \sum_{j=1}^{n} \frac{w_j}{r_j} \left( (X_k^{-1/2} A_j X_k^{-1/2})^{1/r_j} - I \right) + \frac{o(s_k)}{s_k} \quad \text{as } k \to \infty.$$

(39)

When $r_1, \ldots, r_n \in (0, 1]$, this means that

$$X_k = \sum_{j=1}^{n} \hat{w}_j (X_k^{-1} A_j) + \frac{o(s_k)}{s_k} \quad \text{as } k \to \infty.$$

Letting $k \to \infty$ gives $Y_0 = \sum_{j=1}^{n} \hat{w}_j (Y_0^{-1} A_j)$, and hence $Y_0 = X_0$ follows, where $X_0 := (\mathcal{X}(A_1, \ldots, A_n))$. Since $\{X_s\}$ has a unique limit point $X_0$, we find that $X_s \rightarrow X_0$ as $s \searrow 0$.

On the other hand, when $r_1, \ldots, r_n \in [-1, 0)$, (39) is rewritten as

$$0 = \sum_{j=1}^{n} \frac{w_j}{-r_j} \left( (X_k^{1/2} A_j^{-1} X_k^{1/2})^{-r_j} - I \right) + \frac{o(s_k)}{s_k},$$

that is,

$$X_k^{-1} = \sum_{j=1}^{n} \hat{w}_j (X_k^{-1} A_j^{-1}) + \frac{o(s_k)}{s_k} \quad \text{as } k \to \infty.$$

Letting $k \to \infty$ gives

$$Y_0^{-1} = \sum_{j=1}^{n} \hat{w}_j (Y_0^{-1} A_j^{-1}) = \sum_{j=1}^{n} \hat{w}_j (Y_0^{-1} A_j^{-1}).$$
which means that \( Y_0 = X_0 \), where \( X_0 := (\mathcal{H}_w(#_{-r_1}, \ldots, #_{-r_n}))(A_1, \ldots, A_n) \). Therefore, \( X_s \to X_0 \) as \( s \searrow 0 \).

Next, assume that \( r_1 = \cdots = r_n = 0 \), and let \( X_s := (\mathcal{H}_w)_{#_s}(A_1, \ldots, A_n) \) (recall the convention \( p_{s,0} = #_s \) in (21)). Then

\[
I = \sum_{j=1}^{n} w_j (I #_{s} X_s^{-1/2} A_j X_s^{-1/2}) = \sum_{j=1}^{n} w_j (X_s^{-1/2} A_j X_s^{-1/2})^s.
\]

Let \( Y_0 \) be any limit point of \( X_s \) as \( s \searrow 0 \), so \( X_k : = X_{s_k} \to Y_0 \) for some sequence \( s_k \in (0, 1] \), \( s_k \searrow 0 \). Since

\[
I = \sum_{j=1}^{n} w_j \exp(s_k \log X_k^{-1/2} A_j X_k^{-1/2})
\]

\[
= \sum_{j=1}^{n} w_j \left[ I + s_k \log X_k^{-1/2} A_j X_k^{-1/2} + o(s_k) \right]
\]

\[
= I + s_k \sum_{j=1}^{n} w_j \log X_k^{-1/2} A_j X_k^{-1/2} + o(s_k) \quad \text{as} \quad k \to \infty,
\]

one has

\[
\sum_{j=1}^{n} w_j \log X_k^{-1/2} A_j X_k^{-1/2} + \frac{o(s_k)}{s_k} = 0 \quad \text{as} \quad k \to \infty.
\]

Letting \( k \to \infty \) gives the Karcher equation \( \sum_{j=1}^{n} w_j \log Y_0^{-1/2} A_j Y_0^{-1/2} = 0 \), implying that \( Y_0 = G_w(A_1, \ldots, A_n) \). Therefore, \( G_w(A_1, \ldots, A_n) \) is a unique limit point of \( X_s \), so \( X_s \to G_w(A_1, \ldots, A_n) \) as \( s \searrow 0 \).

Second, let us treat the case \( M = \mathcal{H}_w \). The proof in this case is similar to the above case. When either \( r_1, \ldots, r_n \in (0, 1] \) or \( r_1, \ldots, r_n \in [-1, 0) \), for \( s \in (0, 1] \) let \( X_s := (\mathcal{H}_w)(p_{s,r_1}, \ldots, p_{s,r_n})(A_1, \ldots, A_n) \). Then (38) is replaced with

\[
I = \sum_{j=1}^{n} w_j \left[(1-s)I + s(X_s^{-1/2} A_j X_s^{-1/2})^{-1/r_j} \right],
\]

which yields the same equation as (39). Hence the remaining proof is the same as before. When \( r_1 = \cdots = r_n = 0 \), the proof is similar to the above case as well.

Similarly to the two-parameter deformations \( \tau_{s,r} \) for a two-variable operator mean \( \tau \) defined in (22) and (23), for \( M \) satisfying (G) with a weight vector \( w \), we now define the family \( M_{s,r} \) of \( n \)-variable operator means with two parameters \( s \in [0, 1] \) and \( r \in [-1, 1] \) by

\[ Birkhäuser \]
\[ M_{s,r} := \begin{cases} M_{p,r}, & s \in (0, 1], \, r \in [-1, 1], \\ P_{w,r}, & s = 0, \, r \in [-1, 1] \setminus \{0\}, \\ G_w, & s = 0, \, r = 0. \end{cases} \]

Then the next result holds.

**Theorem 4** Assume that \( \mathcal{H} \) is finite-dimensional and \( M \) satisfies, in addition to (A)–(D), (G) with a weight vector \( w \). Then the family \( M_{s,r} \) is continuous (in the sense of pointwise operator norm convergence) in \( s \in [0, 1] \) and \( r \in [-1, 1] \).

**Proof** The continuity of \( M_{s,r} \) on \((s, r) \in (0, 1] \times [-1, 1]\) is a consequence of Proposition 9. Hence we may show the continuity at \((0, r)\) where \( r \in [-1, 1] \). Theorem 3 in particular says that \( M_{s_k, r} \to M_{0, r} \) as \( s_k \to 0 \) with \( s_k \in (0, 1] \) for any fixed \( r \in [-1, 1] \). But, the proof of Theorem 3 can slightly be modified to show that
\[ M_{s_k, r_k} \to M_{0, r} = P_{w,r} \text{ as } s_k \to 0 \text{ and } r_k \to r \neq 0 \text{ where } s_k \in (0, 1] \text{ and } r_k \in [-1, 1]. \]
Moreover, it is known [20–22] that \( M_{0, r} = P_{w,r} \to M_{0, 0} = G_w \) as \( r \to 0 \). Thus it remains to show that \( M_{s_k, r_k} \to M_{0, 0} = G_w \) as \( s_k \to 0 \) and \( r_k \to 0 \) with \( s_k \in (0, 1] \) and \( r_k \in [-1, 1] \setminus \{0\} \). For this, as in the proof of Theorem 3 we may prove the convergence for \( M = \mathcal{A}_w \) and for \( M = \mathcal{H}_w \). For \( M = \mathcal{A}_w \) let \( X_k := (\mathcal{A}_w)_{p_{s_k, r_k}} (A_1, \ldots, A_n) \); then
\[
I = \sum_{j=1}^n w_j \left[ (1 - s_k)I + s_k (X_k^{-1/2} A_j X_k^{-1/2}) r_k \right]^{1/r_k}.
\]

As in the proof (the part of \( s_k, r_k \to 0 \)) of Theorem 2 by replacing \( 1/x_k \) with the operator \( X_k^{-1/2} A_j A_k^{-1/2} \), one can prove that
\[
\left[ I + s_k \left( X_k^{-1/2} A_j X_k^{-1/2} r_k - I \right) \right]^{1/r_k} = 1 + s_k \log X_k^{-1/2} A_j X_k^{-1/2} + o(s_k) \quad \text{as } k \to \infty.
\]

Inserting this into (40) gives
\[
\sum_{j=1}^n w_j \log X_k^{-1/2} A_j X_k^{-1/2} + \frac{o(s_k)}{s_k} = 0 \quad \text{as } k \to \infty,
\]
from which \( X_k \to G_w(A_1, \ldots, A_n) \) follows as in the proof (the part of \( r_1 = \cdots = r_n = 0 \)) of Theorem 3. The proof for the case \( M = \mathcal{H}_w \) is similar. \( \square \)

**Problem 2** Properties of two-variable operator means reduce to those of operator monotone functions on \([0, \infty)\) due to Kubo and Ando [18], however this is not the case for multivariate operator means. Therefore, in the proofs of Proposition 9 and Theorems 3 and 4 we have used the compactness argument, the reason why \( \mathcal{H} \) is assumed finite-dimensional. It seems that we have to find a new technique to prove those results in the infinite-dimensional setting, while they are likely to hold.
Example 7 From Proposition 7 and Theorem 4 one has a lot of one-parameter continuous families of $n$-variable operator means satisfying all of (A)–(G) (at least $\mathcal{H}$ is finite-dimensional). For example, when $M = \mathcal{A}_w$ or $G_w$ or $\mathcal{H}_w$, $\{M_{s,r}\}_{-1 \leq r \leq 1}$ is a continuous family of operator means interpolating $M_s$ ($r = -1$), $M_r$ ($r = 0$) and $M_s$ ($r = 1$) for each $s \in (0, 1)$, and $\{M_{s,r}\}_{0 \leq s \leq 1}$ is a continuous family of operator means joining $P_{w,r}$ ($s = 0$) and $M$ ($s = 1$) for each $r \in [-1, 1] \setminus \{0\}$ (also joining $G_w$ and $M$ when $r = 0$). In particular, for $\mathcal{A}$ and $\mathcal{H}$ with weight $w = (1/n, \ldots, 1/n)$, $X = \mathcal{A}_s(A_1, \ldots, A_n)$ and $X = \mathcal{H}_s(A_1, \ldots, A_n)$ are respectively the solutions to

$$X = \frac{1}{n} \sum_{j=1}^{n} X! A_j, \quad X^{-1} = \frac{1}{n} \sum_{j=1}^{n} X^{-1}! A^{-1}_j, \quad X \in \mathbb{P}.$$  

Note that unlike $\triangledown_1 = ! \triangledown = \#$ in (16), $\mathcal{A}_1$ and $\mathcal{H}_1 = (\mathcal{A}_1)\ast$ are not $G$ ( = $G_w$ with $w = (1/n, \ldots, 1/n)$). In fact, even for scalars $a_1, a_2, a_3 > 0$, $x = \mathcal{H}_s(a_1, a_2, a_3)$ is a solution to

$$x^3 + \frac{a_1 + a_2 + a_3}{3} x^2 - \frac{a_1 a_2 + a_2 a_3 + a_3 a_1}{3} x - a_1 a_2 a_3 = 0, \quad x > 0,$$

which is not equal to $G(a_1, a_2, a_3) = (a_1 a_2 a_3)^{1/3}$.

Example 8 Since Eq. (30) determines $X = G_w(A_1, \ldots, A_n)$, we have

$$(G_w)_{\#\alpha} = G_w, \quad \alpha \in (0, 1].$$

Extending (28) we furthermore have

$$(P_{w,r})_{p_{\alpha,r}} = P_{w,r}$$

for every weight vector $w$, $\alpha \in (0, 1]$ and $r \in [-1, 1] \setminus \{0\}$. To see this, let $X_0 := P_{w,r}(A_1, \ldots, A_n)$, which is determined by the equation

$$I = \sum_{j=1}^{n} w_j(X_0^{-1/2} A_j X_0^{-1/2})^r.$$

For every $\alpha \in (0, 1]$ this is equivalent to

$$I = \sum_{j=1}^{n} w_j [(1 - \alpha) I + \alpha (X_0^{-1/2} A_j X_0^{-1/2})^r]$$

$$= \sum_{j=1}^{n} w_j (I p_{\alpha,r}(X_0^{-1/2} A_j X_0^{-1/2}))^r,$$

which means that

$$I = P_{w,r}(I p_{\alpha,r}(X_0^{-1/2} A_1 X_0^{-1/2}), \ldots, I p_{\alpha,r}(X_0^{-1/2} A_n X_0^{-1/2})).$$
Therefore, $X_0 = P_{w,r}(X_0 \rho_{\alpha,r}A_1, \ldots, X_0 \rho_{\alpha,r}A_n)$, that is, $X_0 = (P_{w,r})\rho_{\alpha,r}(A_1, \ldots, A_n)$.

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References

1. Ando, T.: Concavity of certain maps on positive definite matrices and applications to Hadamard Products. Linear Algebra Appl. 26, 203–241 (1979)
2. Ando, T., Li, C.-K., Mathias, R.: Geometric means. Linear Algebra Appl. 385, 305–334 (2004)
3. Bhatia, R.: Matrix Analysis. Springer-Verlag, New York (1996)
4. Bhatia, R.: Positive Definite Matrices. Princeton University Press, Princeton (2007)
5. Bhatia, R., Holbrook, J.: Riemannian geometry and matrix geometric means. Linear Algebra Appl. 413, 594–618 (2006)
6. Bhatia, R., Karandikar, R.L.: Monotonicity of the matrix geometric mean. Math. Ann. 353, 1453–1467 (2012)
7. Fujii, J.I.: Interpolation for symmetric operator means. Sci. Math. Jpn. 75, 267–274 (2012)
8. Fujii, J.I., Kamei, E.: Uhlmann’s interpolational method for operator means. Math. Japon. 34, 541–547 (1989)
9. Hiai, F.: Matrix analysis: matrix monotone functions, matrix means, and majorization. Interdiscip Inf Sci 16, 139–248 (2010)
10. Hiai, F.: Operator means deformed by a fixed point method, arXiv:1711.10170 [math.FA]
11. Hiai, F., Kosaki, H.: Means for matrices and comparison of their norms. Indiana Univ. Math. J. 48, 899–936 (1999)
12. Hiai, F., Lim, Y.: Log-majorization and Lie-Trotter formula for the Cartan barycenter on probability measure spaces. J. Math. Anal. Appl. 453, 195–211 (2017)
13. Hiai, F., Lim, Y.: Geometric mean flows and the Cartan barycenter on the Wasserstein space over positive definite matrices. Linear Algebra Appl. 533, 118–131 (2017)
14. Hiai, F., Lim, Y.: Operator means of probability measures, Preprint (2019); arXiv:1901.03858 [math.FA]
15. Hiai, F., Seo, Y., Wada, S.: Ando-Hiai type inequalities for multivariate operator means. Linear Multilinear Algebra 67, 2253–2281 (2019)
16. Kim, S., Lee, H.: The power mean and the least squares mean of probability measures on the space of positive definite matrices. Linear Algebra Appl. 465, 325–346 (2015)
17. Kim, S., Lee, H., Lim, Y.: A fixed point mean approximation to the Cartan barycenter of positive definite matrices. Linear Algebra Appl. 496, 420–437 (2016)
18. Kubo, F., Ando, T.: Means of positive linear operators. Math. Ann. 246, 205–224 (1980)
19. Lawson, J., Lim, Y.: Monotonic properties of the least squares mean. Math. Ann. 351, 267–279 (2011)
20. Lawson, J., Lim, Y.: Weighted means and Karcher equations of positive operators. Proc. Natl. Acad. Sci. USA 110, 15626–15632 (2013)
21. Lawson, J., Lim, Y.: Karcher means and Karcher equations of positive definite operators. Trans. Amer. Math. Soc. Ser. B 1, 1–22 (2014)
22. Lim, Y., Pálfia, M.: Matrix power means and the Karcher mean. J. Funct. Anal. 262, 1498–1514 (2012)
23. Lim, Y., Pálfia, M.: Approximations to the Karcher mean on Hadamard spaces via geometric power means. Forum Math. 27, 2609–2635 (2015)
24. Lim, Y., Pálfia, M.: Existence and uniqueness of the $L^1$-Karcher mean, Preprint (2017); arXiv:1703.04292 [math.FA]
25. Moakher, M.: A differential geometric approach to the geometric mean of symmetric positive-definite matrices. SIAM J. Matrix Anal. Appl. 26, 735–747 (2005)
26. Pálfia, M.: Operator means of probability measures and generalized Karcher equations. Adv. Math. 289, 951–1007 (2016)
27. Pálfia, M., Petz, D.: Weighted multivariable operator means of positive definite operators. Linear Algebra Appl. 463, 134–153 (2014)
28. Pusz, W., Woronowicz, S.L.: Functional calculus for sesquilinear forms and the purification map. Rep. Math. Phys. 8, 159–170 (1975)
29. Sturm, K.-T.: Probability measures on metric spaces of nonpositive curvature, in: Heat Kernels and Analysis on Manifolds, Graphs, and Metric Spaces (Paris, 2002), pp. 357–390, Contemp. Math., 338, Amer. Math. Soc., Providence, RI (2003)
30. Thompson, A.C.: On certain contraction mappings in a partially ordered vector space. Proc. Amer. Math. Soc. 14, 438–443 (1963)
31. Udagawa, Y., Yamazaki, T., Yanagida, M.: Some properties of weighted operator means and characterizations of interpolational means. Linear Algebra Appl. 517, 217–234 (2017)
32. Yamazaki, T.: The Riemannian mean and matrix inequalities related to the Ando-Hiai inequality and chaotic order. Oper. Matrices 6, 577–588 (2012)
33. Yamazaki, T.: An elementary proof of arithmetic-geometric mean inequality of the weighted Riemannian mean of positive definite matrices. Linear Algebra Appl. 438, 1564–1569 (2013)