Long period sequences generated by the logistic map over finite fields with control parameter four

Kazuyoshi Tsuchiya  † Yasuyuki Nogami  ‡

Abstract

Pseudorandom number generators have been widely used in Monte Carlo methods, communication systems, cryptography and so on. For cryptographic applications, pseudorandom number generators are required to generate sequences which have good statistical properties, long period and unpredictability. A Dickson generator is a nonlinear congruential generator whose recurrence function is the Dickson polynomial. Aly and Winterhof obtained a lower bound on the linear complexity profile of a Dickson generator. Moreover Vasiga and Shallit studied the state diagram given by the Dickson polynomial of degree two. However, they do not specify sets of initial values which generate a long period sequence.

In this paper, we show conditions for parameters and initial values to generate long period sequences, and asymptotic properties for periods by numerical experiments. We specify sets of initial values which generate a long period sequence. For suitable parameters, every element of this set occurs exactly once as a component of generating sequence in one period.

In order to obtain sets of initial values, we consider a logistic generator proposed by Miyazaki, Araki, Uehara and Nogami, which is obtained from a Dickson generator of degree two with a linear transformation. Moreover, we remark on the linear complexity profile of the logistic generator. The sets of initial values are described by values of the Legendre symbol. The main idea is to introduce a structure of a hyperbola to the sets of initial values. Our results ensure that generating sequences of Dickson generator of degree two have long period. As a consequence, the Dickson generator of degree two has some good properties for cryptographic applications.

1 Introduction

Pseudorandom number generators have been widely used in Monte Carlo methods, communication systems, cryptography and so on. For cryptographic applications, pseudorandom number generators are required to generate sequences which have good statistical properties, long period and unpredictability. Therefore nonlinearity of a state transition function is important.

Linear complexity profile of a sequence is a measure of nonlinearity. Gutierrez, Shparlinski and Winterhof [4] gave a lower bound on the linear complexity profile of a general nonlinear congruential generator. For some special recurrence
functions, much better results were shown, namely, the inverse functions \([4]\), the power functions \([15, 3]\), the Dickson polynomials \([1]\) and the Rédéi functions \([5]\) (See \([21]\) and \([11, 10.4.4.2]\)).

A Dickson generator is a nonlinear congruential generator whose recurrence function is the Dickson polynomial. A lower bound on the linear complexity profile of a Dickson generator was given by Aly and Winterhof \([1]\). Moreover the state diagram was studied by Vasiga and Shallit \([18]\) in the case of degree 2. Vasiga and Shallit obtained a period of a sequence for any initial value, and showed the structure of the state diagram. However, they do not specify sets of initial values which generate a long period sequence.

In this paper, we show conditions for parameters and initial values to generate long period sequences, and asymptotic properties for periods by numerical experiments. We specify sets of initial values which generate a long period sequence. For suitable parameters, every element of this set occurs exactly once as a component of generating sequence in one period. In order to obtain sets of initial values, we consider a logistic generator with control parameter four, which can be obtained from a Dickson generator of degree 2 with a linear transformation. A logistic generator was proposed by Miyazaki, Araki, Uehara and Nogami. They showed properties of the generator in \([6, 7, 8, 9, 10]\). The sets of initial values are described by values of the Legendre symbol. The main idea is to introduce a structure of a hyperbola to the sets of initial values. Although we use a structure of a hyperbola to give conditions for initial values and parameters to have a long period, we can use this structure to improve a lower bound on the linear complexity profile of a Dickson generator of degree 2.

This paper is organized as follows: In Sect.2, we introduce the definition of the Dickson generator and known results for the Dickson generator of degree 2. In Sect.3, we introduce the definition of the logistic map over finite fields and the logistic generator. In Sect.4, we consider periods of logistic generator sequences with control parameter four. In particular, we show conditions for parameters to be maximal on certain sets of initial values, and asymptotic properties for periods by numerical experiments. In Sect.5, we remark on the linear complexity profile of the logistic generator. In Sect.6, we consider a possibility of generalization. Finally, we describe some conclusions in Sect.7.

We give some notations. For a prime number \(p\), \(\mathbb{F}_p\) denotes the finite field with \(p\) elements. For a prime number \(p\) and an integer \(a\), \((a/p)\) denotes the Legendre symbol. \(D_0\) and \(D_1\) denote the set of non-zero quadratic residues modulo \(p\) and of quadratic non-residues modulo \(p\), respectively. For an integer \(b\), put \(D_i - b = \{c \in \mathbb{F}_p \mid c + b \in D_i\}, \quad i = 0, 1\).

For a finite set \(A\), \(#A\) denotes the number of elements in \(A\). For a finite field \(\mathbb{F}_p\) and the quadratic extension \(\mathbb{F}_{p^2}\) of \(\mathbb{F}_p\), \(N_{\mathbb{F}_{p^2}/\mathbb{F}_p}: \mathbb{F}_{p^2} \to \mathbb{F}_p\) denotes the norm map, namely, \(N_{\mathbb{F}_{p^2}/\mathbb{F}_p}(\alpha) = \alpha \alpha^p\) for \(\alpha \in \mathbb{F}_{p^2}\). For a finite group \(G\) and an element \(g \in G\), \(\text{ord}_G g\) denotes the order of \(g\) in \(G\). In particular, if \(G\) is the multiplicative group \((\mathbb{Z}/N\mathbb{Z})^\times\) of the quotient ring \(\mathbb{Z}/N\mathbb{Z}\) of integers modulo \(N\), then we write \(\text{ord}_N g\). Let \(S = (s_n)_{n \geq 0}\) be an eventually periodic sequence. Then there exists the least positive integer \(r = r(S)\) such that \(s_r \in \{s_0, \ldots, s_{r-1}\}\). Let \(t = t(S)\) be the least non-negative integer such that \(s_t = s_0\). If \(t > 0\), we call \((s_0, \ldots, s_{t-1})\) the tail of \(S\). We call \((s_t, \ldots, s_{r-1})\) the cycle of \(S\). Let \(c(S) = r - t\). We call \(c(S)\) the period of \(S\).
A preliminary version [17] of this paper was presented at the seventh International Workshop on Signal Design and its Applications in Communications (IWSDA 2015).

2 The Dickson generator of degree 2

In this section, we introduce known results for the Dickson generator of degree 2 by Vasiga and Shallit [18].

2.1 The Dickson generator

We recall the definition of the Dickson polynomials. For details, see [11, 9.6]. Let \( p \) be a prime number. For \( a \in \mathbb{F}_p \), the family of Dickson polynomials \( D_e(X, a) \in \mathbb{F}_p[X] \) is defined as the recurrence relation \( D_e(X, a) = XD_{e-1}(X, a) - aD_{e-2}(X, a) \), \( e \geq 2 \) with initial values \( D_0(X, a) = 2 \) and \( D_1(X, a) = X \). Then the degree of \( D_e(X, a) \) is \( e \) for any \( e \geq 0 \). Note that \( F_0(X) = X^2 - uX + 1 \in \mathbb{F}_p[X] \) is the characteristic polynomial of the family \( D_e(X, a) \), \( e \geq 0 \) for \( u \in \mathbb{F}_p \).

For \( s_0 \in \mathbb{F}_p \), the sequence \( S = (s_n)_{n \geq 0} \) defined as the recurrence relation

\[
 s_{n+1} = D_e(s_n, 1), \quad n \geq 0
\]

is called a Dickson generator sequence of degree \( e \).

2.2 The Dickson generator of degree 2

By the definition, \( D_2(X, 1) = X^2 - 2 \). Vasiga and Shallit [18] studied the state diagram of the Dickson generator of degree 2. They showed a period of a sequence for any initial value. For an odd integer \( m \), define \( \text{ord}_m^2 \) to be the least positive integer \( k \) such that \( 2^k \equiv \pm 1 \mod m \).

**Theorem 1** ([18] Theorem 12). Let \( S = (s_n)_{n \geq 0} \) be a Dickson generator sequence of degree 2. Let \( \alpha \) and \( \beta \) be the roots of \( F_0(X) = X^2 - s_0X + 1 \). Let \( \text{ord}_G \alpha = 2^e \cdot m \), where \( G = \mathbb{F}_p^\times \) and \( m \) is odd. Then the length of the tail of \( S \) is \( e \), and the period of \( S \) is \( \text{ord}_m^2 \).

**Theorem 2** ([18] Corollary 15). Let \( p > 2 \) be a prime number. Suppose that \( p - 1 = 2^f \cdot m \), \( p + 1 = 2^{f'} \cdot m' \), where \( m \) and \( m' \) are odd integers. For any divisor \( d \neq 1 \) of \( m \), the state diagram given by \( D_2(X, 1) = X^2 - 2 \) consists of \( \varphi(d)/(2\text{ord}_d^2) \) cycles of period \( \text{ord}_d^2 \), where \( \varphi \) is Euler’s totient function. A complete binary tree of height \( f - 1 \) is attached to each element in these cycles. Similarly, for any divisor \( d' \neq 1 \) of \( m' \), the state diagram given by \( D_2(X, 1) = X^2 - 2 \) consists of \( \varphi(d')/(2\text{ord}_{d'}^2) \) cycles of period \( \text{ord}_{d'}^2 \). A complete binary tree of height \( f' - 1 \) is attached to each element in these cycles. Finally, the element 0 is the root of a complete binary tree of height \( f - 2 \) (resp. \( f' - 2 \)) if \( p \equiv 1 \mod 4 \) (resp. \( p \equiv 3 \mod 4 \)). The state diagram consists of the directed edges \( (0, -2), (-2, 2), (2, 2) \).

Although they showed the structure of the state diagram given by \( D_2(X, 1) = X^2 - 2 \), they do not specify sets of initial values which generate a long period sequence. In order to obtain these sets, we introduce the logistic generator proposed by Miyazaki, Araki, Uehara and Nogami.
3 The logistic generator

In this section, we introduce the logistic map over finite fields and the logistic generator.

Let $p$ be a prime number and $\mu_p \in \mathbb{F}_p - \{0\}$. The logistic map $LM_{\mathbb{F}_p}[\mu_p] : \mathbb{F}_p \rightarrow \mathbb{F}_p$ over $\mathbb{F}_p$ with control parameter $\mu_p$ is defined as $LM_{\mathbb{F}_p}[\mu_p](a) = \mu_p a(a + 1)$ for any $a \in \mathbb{F}_p$ (See [7] or [8]). If $p > 3$ and $\mu_p = 4$, it is simply referred to as $LM_{\mathbb{F}_p}$.

Assume that $p > 3$. For $s_0 \in \mathbb{F}_p$, the sequence $S = (s_n)_{n \geq 0}$ defined as the recurrence relation

$$s_{n+1} = LM_{\mathbb{F}_p}(s_n), \quad n \geq 0 \quad (2)$$

is called a logistic generator sequence. Note that a logistic generator is a kind of quadratic congruential generator modulo $p$. For any $n \geq 0$, put $s'_n = 4s_n + 2$. Then $S' = (s'_n)_{n \geq 0}$ is a Dickson generator sequence of degree 2.

4 Periods of logistic generator sequences

In this section, we consider periods of logistic generator sequences with control parameter four. In particular, we show conditions for parameters to be maximal on certain sets of initial values, and asymptotic properties for periods by numerical experiments.

Throughout this section, let $p > 3$ be a prime number.

4.1 The sets of initial values and a structure of the hyperbola

In order to obtain conditions for sets of initial values, we observe examples of state diagrams given by $LM_{\mathbb{F}_p}$.

Example 3. Figure 1 (resp. Figure 2) describes the state diagram given by $LM_{\mathbb{F}_p}$ in the case of $p = 17$ (resp. $p = 23$). Here, an integer $a$ in a circle means that $(a/p) = 1$, an integer $a$ in a rectangle means that $(a/p) = -1$ and an integer $a$ in a triangle means that $a \equiv 0 \mod p$.

It is observed that $\#\{LM_{\mathbb{F}_p}^{-1}(a)\} = 0$ if and only if $(a/p) \neq (LM_{\mathbb{F}_p}(a)/p)$ for $a \in \mathbb{F}_p - \{-1\}$. In the case of $p = 17$ (resp. $p = 23$), an element $a \in \mathbb{F}_p$ such
that \((a/p) = (\text{LM}_{F_p}(a)/p) = -1\) (resp. \((a/p) = (\text{LM}_{F_p}(a)/p) = 1\)) generates a sequence of long period. Note that \(17 \equiv 1 \pmod{4}\) and \(23 \equiv 3 \pmod{4}\).

We begin to show properties of logistic generator sequences.

**Lemma 4.** Let \(a \in F_p\).

1. \(#\{\text{LM}^{-1}_{F_p}(a)\} = 2\) if and only if \((a/p) = (\text{LM}_{F_p}(a)/p)\).

2. \(#\{\text{LM}^{-1}_{F_p}(a)\} = 1\) if and only if \(a = -1\).

3. Assume that \(a \neq -1\). Then \(#\{\text{LM}^{-1}_{F_p}(a)\} = 0\) if and only if \((a/p) \neq (\text{LM}_{F_p}(a)/p)\).

**Proof.** For \(a \in F_p\), the discriminant of the polynomial \(4X(X + 1) - a = D := 16(a + 1)\). Hence the statement follows from \((\text{LM}_{F_p}(a)/p) = (a/p)(D/p)\).

By Lemma 4, every element in \(D_0 \cap (D_0 - 1)\) and \(D_1 \cap (D_0 - 1)\) has two inverse images of \(\text{LM}_{F_p}\). That is to say, these elements are candidates for being in cycles.

**Lemma 5.** Let \(a \in D_0 \cap (D_0 - 1)\) or \(a \in D_1 \cap (D_0 - 1)\). Put \(\{c_1, c_2\} = \text{LM}^{-1}_{F_p}(a)\).

1. If \(p \equiv 3 \pmod{4}\) and \(a \in D_0 \cap (D_0 - 1)\), then \((c_1/p) \neq (c_2/p)\).

2. If \(p \equiv 1 \pmod{4}\) and \(a \in D_1 \cap (D_0 - 1)\), then \((c_1/p) \neq (c_2/p)\).

**Proof.** Since \(4X(X + 1) - a = 4(X - c_1)(X - c_2), -a = 4c_1c_2\). Hence we have

\[
\left(\frac{c_1}{p}\right) \left(\frac{c_2}{p}\right) = \left(\frac{-1}{p}\right) \left(\frac{a}{p}\right).
\]

The statement follows from the first supplement to quadratic reciprocity.

By Lemma 5, if \(p \equiv 3 \pmod{4}\) (resp. \(p \equiv 1 \pmod{4}\)), then every element in \(D_0 \cap (D_0 - 1)\) (resp. \(D_1 \cap (D_0 - 1)\)) is in a cycle. Therefore one can expect that an element which satisfies the conditions in Lemma 5 generates a long period sequence.
In order to analyze $D_0 \cap (D_0 - 1)$ and $D_1 \cap (D_0 - 1)$, we introduce a structure of a hyperbola. Let $C$ be the hyperbola over $\mathbb{F}_p$ defined by the equation $x^2 - y^2 = 1$. For an extension field $K/\mathbb{F}_p$, $C(K)$ denotes the set of $K$-rational points on $C$. Then we have a bijective map $\psi_K : K \setminus \{0\} \to C(K)$ defined as $\psi_K(t) = (2^{-1}(t + t^{-1}), 2^{-1}(t - t^{-1}))$ for any $t \in K \setminus \{0\}$ (See Silverman [16, I.1.3.1]).

Assume that $p \equiv 3 \mod 4$. Put $\text{Param}_3 = \mathbb{F}_p - \{0, \pm 1\}$. We define a map $\Phi_3 : \text{Param}_3 \to D_0 \cap (D_0 - 1)$ as $\Phi_3(t) = (2^{-1}(t - t^{-1}))^2, t \in \text{Param}_3$.

Assume that $p \equiv 1 \mod 4$. Put
\[
\text{Param}_1 = \{t \in \mathbb{F}_p^\times | N_{\mathbb{F}_p/\mathbb{F}_p}(t) = 1\} - \{\pm 1\}.
\]

We define a map $\Phi_1 : \text{Param}_1 \to D_1 \cap (D_0 - 1)$ as $\Phi_1(t) = (2^{-1}(t - t^{-1}))^2, t \in \text{Param}_1$.

**Proposition 6.** Let $p > 3$ be a prime number. If $p \equiv 3 \mod 4$ (resp. $p \equiv 1 \mod 4$), then $\Phi_3$ (resp. $\Phi_1$) is four-to-one map.

**Proof.** First we consider the case of $p \equiv 3 \mod 4$. Let $a \in D_0 \cap (D_0 - 1)$. Then there are $b, c \in \mathbb{F}_p$ such that $a = b^2$ and $a + 1 = c^2$. Hence $(c, b) \in C(\mathbb{F}_p)$. Since $a \neq 0, b \neq 0$. Hence we have four-to-one map $\Phi_3$.

Next we consider the case of $p \equiv 1 \mod 4$. Let $a \in D_1 \cap (D_0 - 1)$. Then there are $\beta \in \mathbb{F}_{p^2} - \mathbb{F}_p$ and $c \in \mathbb{F}_p$ such that $a = \beta^2$ and $a + 1 = c^2$. Hence $(c, \beta) \in \{(u, v) \in C(\mathbb{F}_{p^2}) | u \in \mathbb{F}_p, v \in \mathbb{F}_{p^2} - \mathbb{F}_p\}$. Now, we have
\[
\{t \in \mathbb{F}_{p^2} | t + t^{-1} \in \mathbb{F}_p, t - t^{-1} \in \mathbb{F}_{p^2} - \mathbb{F}_p\} = \{t \in \mathbb{F}_{p^2} - \mathbb{F}_p | N_{\mathbb{F}_{p^2}/\mathbb{F}_p}(t) = 1\}.
\]

Note that $t \in \mathbb{F}_p$ and $N_{\mathbb{F}_{p^2}/\mathbb{F}_p}(t) = 1$ if and only if $t = \pm 1$. Hence we have four-to-one map $\Phi_1$. \qed

**Remark 7.** For $j \in \{1, 3\}$, we have
\[
\Phi_j(t) = \Phi_j(-t) = \Phi_j(t^{-1}) = \Phi_j(-t^{-1}), \quad t \in \text{Param}_j.
\]

**Remark 8.** Let $\mathbb{G}_m$ be the multiplicative group scheme and $T_2 = \ker[N_{\mathbb{F}_{p^2}/\mathbb{F}_p}]$.

Res_{\mathbb{F}_{p^2}/\mathbb{F}_p} \mathbb{G}_m \to \mathbb{G}_m$ the norm one torus (For details, see [20], [19] and [14]).

Then we have $\text{Param}_1 = \mathbb{G}_m(\mathbb{F}_p) - \{\pm 1\}$. Since $\text{Param}_1(T_2(\mathbb{F}_p) - \{\pm 1\})$. Thus, they have a common structure of a set of $\mathbb{F}_p$-rational points on an algebraic torus of dimension one except elements of order 1 and 2.

**Corollary 9.** Let $p > 3$ be a prime number. If $p \equiv 3 \mod 4$ (resp. $p \equiv 1 \mod 4$), then $\# \{D_0 \cap (D_0 - 1)\} = (p - 3)/4$ (resp. $\# \{D_1 \cap (D_0 - 1)\} = (p - 1)/4$).

Let $S = (s_n)_{n \geq 0}$ be a sequence defined as the recurrence relation (2) with $s_0 \in D_0 \cap (D_0 - 1)$ (resp. $s_0 \in D_1 \cap (D_0 - 1)$) if $p \equiv 3 \mod 4$ (resp. $p \equiv 1 \mod 4$). By Corollary 9, the period of $S$ is upper bounded by $(p - 3)/4$ (resp. $(p - 1)/4$) if $p \equiv 3 \mod 4$ (resp. $p \equiv 1 \mod 4$).

**Example 10.** Let $p = 23$. Then we have $D_0 \cap (D_0 - 1) = \{1, 2, 3, 8, 12\}$ (See Fig. 2). Table 1 shows the correspondence between $\text{Param}_3$ and $D_0 \cap (D_0 - 1)$.
Table 1: The correspondence between $\text{Param}_3$ and $D_0 \cap (D_0 - 1)$ in the case of $p = 23$.

| $t \in \text{Param}_3$ | $\Phi_3(t) \in D_0 \cap (D_0 - 1)$ |
|-------------------------|-----------------------------------|
| 4, 6, 17, 19            | 1                                 |
| 2, 11, 12, 21           | 2                                 |
| 5, 9, 14, 18            | 3                                 |
| 7, 10, 13, 16           | 8                                 |
| 3, 8, 15, 20            | 12                                |

Table 2: The correspondence between $\text{Param}_1$ and $D_1 \cap (D_0 - 1)$ in the case of $p = 17$.

| $t \in \text{Param}_1$ | $\Phi_1(t) \in D_1 \cap (D_0 - 1)$ |
|-------------------------|-----------------------------------|
| $2 + \alpha, 15 + 16\alpha, 2 + 16\alpha, 15 + \alpha$ | 3 |
| $5 + 5\alpha, 12 + 12\alpha, 5 + 12\alpha, 12 + 5\alpha$ | 7 |
| $8 + 2\alpha, 9 + 15\alpha, 8 + 15\alpha, 9 + 2\alpha$ | 12 |
| $7 + 4\alpha, 10 + 13\alpha, 7 + 13\alpha, 10 + 4\alpha$ | 14 |

Example 11. Let $p = 17$. Then we have $D_1 \cap (D_0 - 1) = \{3, 7, 12, 14\}$ (See Fig. 1). Let $\alpha$ be a root of $X^2 - 3 \in \mathbb{F}_p[X]$. Then $\mathbb{F}_p(\alpha)$ is a quadratic extension of $\mathbb{F}_p$. Table 2 shows the correspondence between $\text{Param}_1$ and $D_1 \cap (D_0 - 1)$.

For understanding periods, we relate the logistic map on the sets of initial values and the square map on the parameter spaces of the hyperbola.

Lemma 12. If $p \equiv 3 \text{ mod } 4$ (resp. $p \equiv 1 \text{ mod } 4$), then $\text{LM}_{\mathbb{F}_p}(\Phi_3(t)) = \Phi_3(t^2)$, $t \in \text{Param}_3$ (resp. $\text{LM}_{\mathbb{F}_p}(\Phi_1(t)) = \Phi_1(t^2)$, $t \in \text{Param}_1$).

Proof. Assume that $p \equiv 3 \text{ mod } 4$. For $t \in \text{Param}_3$,

$$\text{LM}_{\mathbb{F}_p}(\Phi_3(t)) = 4 \times \{2^{-1}(t - t^{-1})\}^2 \times \{2^{-1}(t + t^{-1})\}^2$$

$$= \{2^{-1}(t^2 - t^{-2})\}^2$$

$$= \Phi_3(t^2).$$

For the case of $p \equiv 1 \text{ mod } 4$, one can show the statement similarly.

Lemma 12 follows that periods of logistic generator sequences on the sets of initial values are induced by those of sequences generated by the square map on the parameter spaces of the hyperbola. The latter has been studied by Rogers [13] and Vasiga and Shallit [18].

4.2 Periods of sequences

The period of a logistic generator sequence and the state diagram given by the logistic map are obtained by Theorem 1 and Theorem 2. However, we revisit
similar ones because we give conditions to be maximal on the sets of initial values below.

**Theorem 13.** Let \( p > 3 \) be a prime number. Let \( t \in \text{Param}_3 \) (resp. \( t \in \text{Param}_1 \)) and \( S = (s_n)_{n \geq 0} \) a sequence defined as the recurrence relation (2) with \( s_0 = \Phi_3(t) \in D_0 \cap (D_0 - 1) \) (resp. \( s_0 = \Phi_1(t) \in D_1 \cap (D_0 - 1) \)) if \( p \equiv 3 \mod 4 \) (resp. \( p \equiv 1 \mod 4 \)). Let \( \text{ord}_G t = 2^e \cdot m \), where \( G = \mathbb{F}_{p^2}^\times \) and \( m \) is odd. Then the length of the tail of \( S \) is

\[
t(S) = \begin{cases} 
0 & (e = 0) \\
 e - 1 & (e > 0) 
\end{cases}, \tag{4}
\]

and the period of \( S \) is \( c(S) = \text{ord}_m' 2 \) (For the definition of \( \text{ord}_m' 2 \), see Subsect. 2.2).

**Proof.** Note that the sequence \( (t^e)_{n \geq 0} \) has period \( \text{ord}_m 2 \) and tail length \( e \) (Vasiga and Shallit [18, Theorem 1]). By (3), we have \( c(S) = \text{ord}_m' 2 \). On the other hand, \( \text{ord}_G t = m \) for some odd integer \( m \) if and only if \( \text{ord}_G (-t) = 2m \) for some odd integer \( m \). Hence we have (4). \( \square \)

**Theorem 14.** Let \( p > 3 \) be a prime number. If \( p \equiv 3 \mod 4 \) (resp. \( p \equiv 1 \mod 4 \)), suppose that \( p - 1 = 2m \) (resp. \( p + 1 = 2m \)), where \( m \) is an odd integer. If \( p \equiv 3 \mod 4 \) (resp. \( p \equiv 1 \mod 4 \)), for any divisor \( d \neq 1 \) of \( m \) the state diagram given by \( \text{LM}_{p^2} \) on \( D_0 \cap (D_0 - 1) \) (resp. \( D_1 \cap (D_0 - 1) \)) consists of \( n_d := \varphi(d)/(2\text{ord}_d' 2) \) cycles of period \( c_d := \text{ord}_d' 2 \), where \( \varphi \) is Euler’s totient function.

**Proof.** For any divisor \( d \) of \( m \), the state diagram given by the square map consists of \( \varphi(d)/\text{ord}_d' 2 \) cycles of period \( \text{ord}_d 2 \) on \( \text{Param}_3 \) (resp. \( \text{Param}_1 \)) if \( p \equiv 3 \mod 4 \) (resp. \( p \equiv 1 \mod 4 \)). For details, see Rogers [13, Theorem] or Vasiga and Shallit [18, Corollary 3].

Let \( d \neq 1 \) be a divisor of \( m \).

Suppose that there is no integer \( k \in \mathbb{Z} \) such that \( 2^k \equiv -1 \mod d \). Then a cycle of period \( c_d \) on \( D_0 \cap (D_0 - 1) \) (resp. \( D_1 \cap (D_0 - 1) \)) corresponds to two cycles of period \( \text{ord}_d 2 \) on \( \text{Param}_3 \) (resp. \( \text{Param}_1 \)). Hence the state diagram consists of \( \varphi(d)/(2\text{ord}_d' 2) \) cycles of period \( \text{ord}_d 2 \).

Suppose that there is an integer \( k \in \mathbb{Z} \) such that \( 2^k \equiv -1 \mod d \). Then a cycle of period \( c_d \) on \( D_0 \cap (D_0 - 1) \) (resp. \( D_1 \cap (D_0 - 1) \)) corresponds to a cycle of period \( \text{ord}_d 2 \) on \( \text{Param}_3 \) (resp. \( \text{Param}_1 \)). Hence the state diagram consists of \( \varphi(d)/\text{ord}_d' 2 \) cycles of period \( \text{ord}_d 2/2 \). \( \square \)

**Example 15.** Let \( p = 23 \). Put \( m = 11 \), then \( p = 2m + 1 \). Table 3 shows the structure of the state diagram given by \( \text{LM}_{p^2} \) on \( D_0 \cap (D_0 - 1) = \{1, 2, 3, 8, 12\} \). In fact, the state diagram consists of the cycle \( (1, 8, 12, 3, 2) \) (See Fig. 2).

**Example 16.** Let \( p = 17 \). Put \( m = 9 \), then \( p = 2m - 1 \). Table 4 shows the structure of the state diagram given by \( \text{LM}_{p^2} \) on \( D_1 \cap (D_0 - 1) = \{3, 7, 12, 14\} \). In fact, the state diagram consists of the cycles \( \{12\} \) and \( \{3, 14, 7\} \) (See Fig. 1).

### 4.3 Long period sequences

In order to apply the logistic map over finite fields to a pseudorandom number generator, a generating sequence is required to have a long period. Now, we show the conditions for parameters to be maximal on the sets of initial values.
Table 3: The structure of the state diagram given by $LM_{p_{23}}$ on $D_0 \cap (D_0 - 1)$.

| $d$ | ord$_d$ | $\varphi(d)$ | $n_d$ | $c_d$ |
|-----|---------|---------------|------|------|
| 11  | 10      | 10            | 1    | 5    |

Table 4: The structure of the state diagram given by $LM_{p_{17}}$ on $D_1 \cap (D_0 - 1)$.

| $d$ | ord$_d$ | $\varphi(d)$ | $n_d$ | $c_d$ |
|-----|---------|---------------|------|------|
| 3   | 2       | 2             | 1    | 1    |
| 9   | 6       | 6             | 1    | 3    |

**Corollary 17.** Let $p > 3$ be a prime number. If $p \equiv 3 \mod 4$ (resp. $p \equiv 1 \mod 4$), let $S = (s_n)_{n \geq 0}$ be a sequence defined as the recurrence relation (2) with $s_0 \in D_0 \cap (D_0 - 1)$ (resp. $s_0 \in D_1 \cap (D_0 - 1)$). Then $S$ attains a maximal period if and only if there exists a prime number $p_1$ such that

$$
\begin{cases}
  p = 2p_1 + 1 & \text{if } p \equiv 3 \mod 4 \\
  p = 2p_1 - 1 & \text{if } p \equiv 1 \mod 4
\end{cases}
$$

and

$$
\text{ord}_{p_1} 2 = p_1 - 1 \text{ or } \begin{cases}
  \text{ord}_{p_1} 2 = (p_1 - 1)/2 \\
  (p_1 - 1)/2 \text{ is an odd integer}
\end{cases}
$$

**Proof.** Suppose that $S$ attains a maximal period. Assume that $p \equiv 3 \mod 4$. Since the state diagram given by $LM_{p_{23}}$ on $D_0 \cap (D_0 - 1)$ has one cycle, $p_1 := (p - 1)/2$ is a prime number and (6) holds by Theorem 14. Similarly, if $p \equiv 1 \mod 4$, then $p_1 := (p + 1)/2$ is a prime number and (6) holds.

Conversely, suppose that there exists a prime number $p_1$ such that (5) and (6) hold. Then $S$ has period $(p_1 - 1)/2$ by Theorem 14.

Assume that $p \equiv 3 \mod 4$. $p$ is called a maximal prime on $D_0 \cap (D_0 - 1)$ if there exists a prime number $p_1$ such that (5) and (6) hold. Assume that $p \equiv 1 \mod 4$. $p$ is called a maximal prime on $D_1 \cap (D_0 - 1)$ if there exists a prime number $p_1$ such that (5) and (6) hold.

**Remark 18.** For a prime number $p$, $p$ is a safe prime if there is a prime number $p_1$ such that $p = 2p_1 + 1$. For a safe prime $p = 2p_1 + 1$, $p$ is a 2-safe prime if $p_1$ is also a safe prime. If $p$ is a 2-safe prime, then $p$ is a maximal prime on $D_0 \cap (D_0 - 1)$. Peinado, Montoya, Muñoz and Yuste [12, Remark 2] and Miyazaki, Araki, Uehara and Nogami [9] considered in this situation. We show small 2-safe primes as follows:

$$
11, 23, 47, 167, 359, 719, 1439, 2039, 2879, 4079.
$$
Remark 19. Let $p > 3$ be a prime number such that $p \equiv 1 \mod 4$. Assume that there is a prime number $p_1$ such that $p = 2p_1 - 1$ and $p_1$ is a safe prime ($p$ is analogous to a $2$-safe prime). Then $p$ is a maximal prime on $D_1 \cap (D_0 - 1)$. However, a prime number that satisfies these conditions is only $13$. In fact, we may assume that $p_1 > 3$. Since $p \not\equiv 0 \mod 3$, $p_1 \equiv 1 \mod 3$. Put $p_1 = 2p_2 + 1$ for some prime number $p_2$. Since $p_1 \equiv 1 \mod 3$, $p_2 \equiv 0 \mod 3$, that is $p_2 = 3$. Hence $p = 4p_2 + 1 = 13$.

In Corollary 17, we state the necessary and sufficient condition for initial values and parameters to be maximal. Now, we estimate density for maximal sequences by numerical experiments.

We show the graphs of the percentage of $N$-bit maximal primes on $D_0 \cap (D_0 - 1)$ in $N$-bit primes congruent to $3$ modulo $4$ and of the percentage of $N$-bit maximal primes on $D_1 \cap (D_0 - 1)$ in $N$-bit primes congruent to $1$ modulo $4$ in Fig. 3 for $3 \leq N \leq 32$. From Fig. 3, it is observed that the both percentages are slowly decreasing as $N$ grows. Moreover, the percentage of $N$-bit maximal primes on $D_0 \cap (D_0 - 1)$ in $N$-bit primes congruent to $3$ modulo $4$ is larger than that of $N$-bit maximal primes on $D_1 \cap (D_0 - 1)$ in $N$-bit primes congruent to $1$ modulo $4$ for $N \geq 12$. The cause of this phenomenon is not clear, however Remark 18 and Remark 19 provide pieces of evidence.

This observation shows that maximal primes are rare. Therefore we estimate the number of cycles and their periods on the sets of initial values from Theorem 14. We show the graphs of the average of the number of cycles in Fig. 4 and the graphs of the average of periods in Fig. 5 for $17 \leq N \leq 32$. From Fig. 4 and Fig. 5, it is observed that the both averages of the number of cycles and of periods are increasing rapidly as $N$ grows. As in Fig. 3, the average of periods in the case of $N$-bit primes congruent to $3$ modulo $4$ is larger than that in the case of $N$-bit primes congruent to $1$ modulo $4$. On the other hand, the average of the number of cycles in the case of $N$-bit primes congruent to $3$ modulo $4$ is almost the same as that in the case of $N$-bit primes congruent to $1$ modulo $4$.
5 Remarks on the linear complexity profile of the logistic generator

In this section, we remark on the linear complexity profile of the logistic generator. In particular, we show two lower bounds on the linear complexity profile and compare the two lower bounds.

5.1 Definitions

We recall the definition of linear complexity profile. For details, see [11, 10.4].

Let \( q \) be a prime power. Let \( S = (s_n)_{n \geq 0} \) be a sequence over \( \mathbb{F}_q \). The linear complexity \( L(S) \) of \( S \) is the length \( L \) of the shortest linear recurrence relation

\[
s_{n+L} = a_{L-1}s_{n+L-1} + \cdots + a_0s_n, \quad n \geq 0
\]

for some \( a_0, \ldots, a_{L-1} \in \mathbb{F}_q \). For \( N \geq 1 \in \mathbb{Z} \), the \( N \)-th linear complexity \( L(S, N) \) of \( S \) is the length \( L \) of the shortest linear recurrence relation

\[
s_{n+L} = a_{L-1}s_{n+L-1} + \cdots + a_0s_n, \quad 0 \leq n \leq N - L - 1
\]
for some \(a_0, \ldots, a_{L-1} \in \mathbb{F}_q\). The linear complexity profile of \(S\) is the sequence \((L(S, N))_{N \geq 1}\) of integers.

Assume that \(S\) is periodic of length \(T\). Put \(s^T(X) = s_0 + s_1X + \cdots + s_{T-1}X^{T-1} \in \mathbb{F}_q[X]\). Then \(L(S) = L(S, 2T) \leq T\) and \(L(S) = T - \deg \text{GCD}(X^T - 1, s^T(X))\) (See [11, Theorem 10.4.27]).

### 5.2 Lower bounds on the linear complexity profile

Aly and Winterhof obtained a lower bound on the linear complexity profile of a Dickson generator as follows:

**Theorem 20** ([1] Theorem 1). Let \(p > 2\) be a prime number. Let \(S = (s_n)_{n \geq 0}\) be a sequence defined as the recurrence relation (1). Assume that \(S\) is periodic of length \(T\). Then the lower bound

\[
L(S, N) \geq \frac{\min\{N^2, 4T^2\}}{16(p+1)} - (p+1)^{1/2}, \quad N \geq 1
\]

holds.

Let \(p > 3\) be a prime number. Let \(S = (s_n)_{n \geq 0}\) be a sequence defined as the recurrence relation (2) over \(\mathbb{F}_p\). Since \(S\) is obtained from a Dickson generator sequence of degree 2 with a linear transformation, the linear complexity profile of \(S\) satisfies the lower bound (7). However, we obtain a little refined lower bound on the linear complexity profile of \(S\).

**Theorem 21.** Let \(m > 2\) be an odd integer such that \(p := 2m + 1\) (resp. \(p := 2m - 1\)) is a prime number and \(S = (s_n)_{n \geq 0}\) a sequence defined as the recurrence relation (2) with \(s_0 \in D_0 \cap (D_0 - 1)\) (resp. \(s_0 \in D_1 \cap (D_0 - 1)\)) over \(\mathbb{F}_p\). Then the lower bound

\[
L(S, N) \geq \frac{\min\{N^2, 4T^2\}}{16m} - m^{1/2}, \quad N \geq 1
\]

holds, where \(T\) is the period of \(S\).

**Proof.** Exactly like the proof of [1, Theorem 1]. See A.

Since the lower bound (8) is larger than the lower bound (7), we can ensure a higher security level for a Dickson generator sequence of degree 2.

In the case of arbitrary control parameter, we obtain another lower bound on the linear complexity profile by the same approach as the proof in Griffin and Shparlinski [3, Theorem 7].

**Theorem 22.** Let \(p > 2\) be a prime number and \(\mu_p \in \mathbb{F}_p - \{0\}\). Let \(S = (s_n)_{n \geq 0}\) be a sequence defined as the recurrence relation \(s_{n+1} = LM_{\mu_p}(s_n), n \geq 0\). Assume that \(S\) is periodic. Then the lower bound

\[
L(S, N) \geq \min \left\{ (2N)^{1/2} - 3, L(S) \right\}, \quad N \geq 1
\]

holds.

**Proof.** Exactly like the proof of [3, Theorem 7]. See B.
Figure 6: The graphs of the lower bound (8) and the lower bound (9) in the case of \( p = 6599 \). If an estimated value is a negative integer, then we plot zero instead of the value.

Now, we compare the lower bound (8) with the lower bound (9). Since the order of the lower bound (8) in terms of \( N \) is larger than that of the lower bound (9), the graph of the lower bound (8) is upper than that of the lower bound (9) for large \( N \). In the case of maximal primes, if the size of \( p \) is large enough, the lower bound (8) is close to the period asymptotically as \( N \) grows. On the other hand, the lower bound (8) does not make sense for small \( N \) because the estimated values are negative integers by the term \(-m^{1/2}\) of (8). Thus the two lower bounds are complementary to each other. Figure 6 describes the graphs of the two lower bounds in the case of \( p = 6599 \).

6 A possibility of generalization

In this section, we consider a possibility of generalization.

Let \( p > 2 \) be a prime number and \( \mu_p \in \mathbb{F}_p - \{0\} \). Let \( a \in \mathbb{F}_p \). Then the discriminant of the polynomial \( \mu_p X(X + 1) - a \) is \( D_{\mu_p} := \mu_p^2 + 4 \mu_p a \). If \( \mu_p = 4 \), then \( (LM_{\mu_p}(a)/p) \) is described by \( (D_{\mu_p}/p) \) as in the proof of Lemma 4. On the other hand, if \( \mu_p \neq 4 \), then \( (LM_{\mu_p}(a)/p) \) is not described by \( (D_{\mu_p}/p) \). Therefore, it is difficult to apply our methods to the case of \( \mu_p \neq 4 \). However, note that a logistic generator with \( \mu_p = 4 \) is linearly transformed to a logistic generator with \( \mu_p = p - 2 \) by [8, Theorem 1]. We do not know whether one can apply our methods to a general quadratic congruential generator. Note that Peinado, Montoya, Muñoz and Yuste [12] showed theoretical results about upper bounds on a period of the quadratic congruential generator sequence defined by the quadratic polynomial \( X^2 - c \in \mathbb{F}_p[X] \).

From another point of view, one can consider a generalization of our methods to the Dickson generator of degree \( n \geq 2 \). Unfortunately, how to apply our methods to the case of degree \( n \geq 3 \) is not clear.
7 Conclusion

In this paper, we investigate periods of logistic generator sequences with control parameter four. In particular, we show the conditions for initial values and parameters to be maximal, and estimate the percentage of maximal primes, the number of cycles and their periods on the sets of initial values. Therefore, we can ensure that generating sequences of Dickson generator of degree 2 have long period. Moreover, we obtain a little refined lower bound on the linear complexity profile of these sequences. As a consequence, the Dickson generator of degree 2 has some good properties for cryptographic applications.

Acknowledgments

The authors would like to thank Satoshi Uehara, Shunsuke Araki and Takeru Miyazaki for useful discussion. In particular, the authors would like to thank Satoshi Uehara for his valuable comments. This research was supported by JSPS KAKENHI Grant-in-Aid for Scientific Research (A) Number 16H01723.

References

[1] H. Aly and A. Winterhof. On the linear complexity profile of nonlinear congruential pseudorandom number generators with Dickson polynomials. Designs, Codes and Cryptography, 39(2):155–162, 2006.

[2] D. Gomez-Perez, J. Gutierrez, and I. E. Shparlinski. Exponential sums with Dickson polynomials. Finite Fields and Their Applications, 12(1):16–25, 2006.

[3] F. Griffin and I. E. Shparlinski. On the linear complexity profile of the power generator. IEEE Trans. Inf. Theory, 46(6):2159–2162, 2000.

[4] J. Gutierrez, I. E. Shparlinski, and A. Winterhof. On the linear and nonlinear complexity profile of nonlinear pseudorandom number generators. IEEE Trans. Inf. Theory, 49(1):60–64, 2003.

[5] W. Meidl and A. Winterhof. On the linear complexity profile of nonlinear congruential pseudorandom number generators with Rédei functions. Finite Fields and Their Applications, 13(3):628–634, 2007.

[6] T. Miyazaki, S. Araki, S. Uehara, and Y. Nogami. A study of the logistic map over prime fields with the safe prime. In 2013 Annual Meeting of the Japan Society for Industrial and Applied Mathematics, Fukuoka, 2013. (in Japanese).

[7] T. Miyazaki, S. Araki, S. Uehara, and Y. Nogami. A study on the pseudorandom number generator for the logistic map over prime fields. In The 30th Symposium on Cryptography and Information Security, Kyoto, 2013. (in Japanese).

[8] T. Miyazaki, S. Araki, S. Uehara, and Y. Nogami. A study of an automorphism on the logistic maps over prime fields. In The International
[9] T. Miyazaki, S. Araki, S. Uehara, and Y. Nogami. A study of averages of periods for sequences generated by the logistic map over prime fields with the doubly safe prime. In The 31st Symposium on Cryptography and Information Security, Kagoshima, 2014. (in Japanese).

[10] T. Miyazaki, S. Araki, S. Uehara, and Y. Nogami. Distribution of correlations for sequences generated by the logistic map over prime field. In The 32nd Symposium on Cryptography and Information Security, Kokura, 2015. (in Japanese).

[11] G. L. Mullen and D. Panario. Handbook of finite fields. Discrete Mathematics and Its Applications. CRC Press, 2013.

[12] A. Peinado, F. Montoya, J. Muñoz, and A. J. Yuste. Maximal periods of $x^2 + c$ in $\mathbb{F}_q$. In S. Boztas and I. E. Shparlinski, editors, Applied Algebra, Algebraic Algorithms and Error-Correcting Codes, volume 2227 of Lecture Notes in Computer Science, pages 219–228. Springer, Heidelberg, 2001.

[13] T. D. Rogers. The graph of the square mapping on the prime fields. Discrete Math., 148(1–3):317–324, 1996.

[14] K. Rubin and A. Silverberg. Torus-based cryptography. In D. Boneh, editor, Advances in cryptology – CRYPTO 2003, volume 2729 of Lecture Notes in Computer Science, pages 349–365. Springer, Heidelberg, 2003.

[15] I. E. Shparlinski. On the linear complexity of the power generator. Designs, Codes and Cryptography, 23(1):5–10, 2001.

[16] J. H. Silverman. The arithmetic of elliptic curves, volume 106 of Graduate Texts in Mathematics. Springer-Verlag New York, New York, NY, 2nd edition, 2009.

[17] K. Tsuchiya and Y. Nogami. Periods of sequences generated by the logistic map over finite fields with control parameter four. In The seventh International Workshop on Signal Design and its Applications in Communications, pages 155–159, Bengaluru, 2015.

[18] T. Vasiga and J. Shallit. On the iteration of certain quadratic maps over $GF(p)$. Discrete Math., 277(1–3):219–240, 2004.

[19] V. E. Voskresenski. Algebraic groups and their birational invariants, volume 179 of Translations of mathematical monographs. American Mathematical Society, New York, NY, 1998.

[20] W. C. Waterhouse. Introduction to affine group schemes, volume 66 of Graduate Texts in Mathematics. Springer-Verlag New York, New York, NY, 1979.

[21] A. Winterhof. Recent results on recursive nonlinear pseudorandom number generators (Invited Paper). In C. Carlet and A. Pott, editors, Sequences and Their Applications – SETA 2010, volume 6338 of Lecture Notes in Computer Science, pages 113–124. Springer, Heidelberg, 2010.
A Proof of Theorem 21

We may assume that \( N \leq 2T \). Put

\[
t = m, \quad H = h = \left\lfloor \frac{N + 1}{2} \right\rfloor.
\]

Then we have

\[
t = m \geq \text{ord}_d 2 \geq T \geq \left\lfloor \frac{N + 1}{2} \right\rfloor = H = h
\]

for some divisor \( d \neq 1 \) of \( m \) (See Theorem 14). By [3, Lemma 1], we see that the number \( U \) of solutions of the congruence

\[
2^x \equiv y \mod t, \quad 0 \leq x \leq H - 1, \quad 0 \leq y \leq h - 1
\]

satisfies

\[
U \geq \frac{Hb}{4m} - m^{1/2} \geq \frac{N^2}{16m} - m^{1/2}.
\]  \hspace{1cm} (10)

Let \((j_1, k_1), \ldots, (j_U, k_U)\) be the corresponding solutions.

Assume that \( L(S,N) \leq U - 1 \). Let \( W = (w_n)_{n \geq 0} \) be a sequence of linear complexity \( L(W) = L(S,N) \) with \( w_n = s_n \) for \( 0 \leq n \leq N - 1 \). By [3, Lemma 2], there exist \( c_1, \ldots, c_U \in \mathbb{F}_p \), not all equal to zero, such that

\[
\sum_{i=1}^{U} c_i w_{n+j_i} = 0, \quad n \geq 0.
\]

For any \( i \geq 0 \), we have

\[
s_i = \{2^{-1}(t^{2^i} - t^{-2^i})\}^2 = 4^{-1} \left( t^{2i+1} + t^{-2i+1} - 2 \right)
\]

for some \( t \in \text{Param}_3 \) (resp. \( t \in \text{Param}_1 \)) if \( p = 2m + 1 \) and \( s_0 \in D_0 \cap (D_0 - 1) \)

(resp. \( p = 2m - 1 \) and \( s_0 \in D_1 \cap (D_0 - 1) \)). Put \( b_i = t^{2i+1} + t^{-2i+1} \in \mathbb{F}_p \) for \( i \geq 0 \).

Since the discriminant \( D_i \) of the polynomial \( F_{b_i}(X) = X^2 - b_iX + 1 \in \mathbb{F}_p[X] \) is

\[
D_i = b_i^2 - 4 = 16s_i(s_i + 1),
\]

we have \((D_i/p) = (16s_i(s_i + 1)/p) = (LM_{\mathbb{F}_p}(s_i)/p) = 1\).

If \( p = 2m + 1 \) and \( s_0 \in D_0 \cap (D_0 - 1) \), then \((D_i/p) = (LM_{\mathbb{F}_p}(s_i)/p) = 1\). Hence \( F_{b_i}(X) \) has two roots \( t^{2i+1}, t^{-2i+1} \in \mathbb{F}_p \) and \((t^{2i+1})^m = (t^{2i})^{p-1} = 1\). Thus \( D_{b_i}(b_i, 1) = D_f(b_i, 1) \) if \( e \equiv f \mod m \). If \( p = 2m - 1 \) and \( s_0 \in D_1 \cap (D_0 - 1) \), then \((D_i/p) = (LM_{\mathbb{F}_p}(s_i)/p) = -1\). Hence \( F_{b_i}(X) \) has two roots \( t^{2i+1}, t^{-2i+1} \in \mathbb{F}_{p^2} - \mathbb{F}_p \) and \((t^{2i+1})^m = (t^{2i})^{p+1} = 1\). Thus \( D_{b_i}(b_i, 1) = D_f(b_i, 1) \) if \( e \equiv f \mod m \) (See [2, Lemma 6]).

We have

\[
w_{n+j_i} = s_{n+j_i} = 4^{-1} \left( t^{2n+1+j_i} + t^{-2n+1+j_i} - 2 \right)
\]

\[
= 4^{-1} \left( D_{2i}(t^{2n+1} + t^{-2n+1}, 1) - 2 \right)
\]

\[
= 4^{-1} \left( D_{b_i}(b_n, 1) - 2 \right)
\]

for \( 0 \leq n \leq N - 1 - j_i, 1 \leq i \leq U \). Put

\[
f(X) = \sum_{i=1}^{U} 4^{-1} c_i (D_{b_i}(X, 1) - 2) \in \mathbb{F}_p[X].
\]
Then $f(X)$ is non-zero polynomial of degree

$$\deg f(X) \leq \max_{1 \leq i \leq U} k_i \leq h - 1. \quad (11)$$

On the other hand, $f(X)$ has at least

$$\min \left\{ T, N - \max_{1 \leq i \leq U} j_i \right\} \geq H = h$$

distinct zeros $b_n$, $0 \leq n \leq \min\{T, N - \max_{1 \leq i \leq U} j_i\} - 1$. It contradicts (11). Hence we have $L(S, N) \geq U$ and (8) follows from (10).

\section*{B Proof of Theorem 22}

Let $W = (w_n)_{n \geq 0}$ be a sequence of the linear complexity $L(W) = L(S, N)$ with $w_n = s_n$ for $0 \leq n \leq N - 1$. Let $U = (u_n)_{n \geq 0}$ be the sequence defined as $u_n = w_{n+1} - \text{LMF}_{p_{[\mu]}}(w_n)$, $n \geq 0$. Then we have

$$L(U) \leq L((w_{n+1})_{n \geq 0}) + \frac{L(W)(L(W) + 1)}{2} + L(W)$$

$$= 2L(S, N) + \frac{L(S, N)(L(S, N) + 1)}{2}.$$ 

Unless $U$ is the all-zeros sequence,

$$2N \leq 4L(S, N) + L(S, N)(L(S, N) + 1) \leq (L(S, N) + 3)^2.$$ 

Noting that $L(S, N) = L(S)$ if $U$ is the all-zeros sequence, we have (9).