Quasinormal Modes, Reduced Phase Space and Area Spectrum of Black Holes

M.R. Setare *
Physics Dept. Inst. for Studies in Theo. Physics and Mathematics(IPM)
P. O. Box 19395-5531, Tehran, IRAN

Abstract
Motivated by the recent interest in quantization of black hole area spectrum, we consider the area spectrum of Schwarzschild, BTZ, extremal Reissner-Nordström, near extremal Schwarzschild-de Sitter, and Kerr black holes. Based on the proposal by Bekenstein and others that the black hole area spectrum is discrete and equally spaced, we implement Kunstatter’s method to derive the area spectrum for these black holes. We show that although as Schwarzschild black hole the spectrum is discrete, it is non equispaced in general. In the other hand the reduced phase space quantization is another technique which we discuss here. However there is a discrepancy between the result of the reduced phase space methodology and quasinormal modes approach for area spectrum of some black holes.

*E-mail: rezakord@ipm.ir
1 Introduction

Hawking was the first to observe that black holes are not completely black but they emit radiation [1]. This radiation is essentially thermal and the black hole emits field quanta of all frequencies. Now, may be one can ask this question: do quantum black holes have a discrete spectrum? Several authors have raised the possibility that Hawking radiation might in fact have a discrete spectrum. This idea was first proposed by Bekenstein in 1974 [2]. According to Bekenstein’s proposal, the eigenvalues of the black hole event horizon area are of the form

\[ A_n = \alpha l_p^2 n, \]  

where \( \alpha \) is a dimensionless constant, \( n \) ranges over positive integers and, by using gravitational units \( G = c = 1 \), \( l_p = \hbar^{1/2} \) is the Planck length. The black hole entropy-law \( (S = \frac{A}{4\hbar}) \) combined with a statistical interpretation of the entropy requires that \( \exp A\hbar \) is an integer. This, in turn, constrains the spacing constant parameter \( \alpha \) in Eq.(1) to take the form \( \alpha = 4 \ln b \) for some integer \( b \). Endorsement for Bekenstein’s proposal was provided by the observation that the area of the horizon \( A \) behaves, for a slowly changing black hole, as an adiabatic invariant [3]. It is significant that a classical adiabatic invariant corresponds to a quantum observable with a discrete spectrum, by virtue of Ehrenfest’s principle.

Since the original heuristic arguments of Bekenstein there has been a substantial amount of work in trying to derive the spectrum (1) by more rigorous means [2, 5]. Of particular relevance to the upcoming analysis is a program that was initiated by Barvinsky and Kunstatter [6]. Their methodology is based on expressing the black hole dynamics in terms of a reduced phase space and then applying an appropriate process of quantization. One vital assumption was required in this analysis; namely, the authors assumed that the conjugate to the mass is periodic over an interval of \( \frac{2\pi}{k} \) where the \( k \) is the surface gravity at the horizon. They did, however, justify this input by way of Euclidean considerations.

In the other hand, recently, the quantization of the black hole area has been considered [7, 8] as a result of the absorption of a quasi-normal mode excitation [9, 10]. Any non-dissipative systems has modes of vibrations, which forming a complete set, and called normal modes. Each mode having a given real frequency of oscillation and being independent of any other. The system once disturbed continues to vibrate in one or several of the normal modes. On the other hand, when one deals with open dissipative system, as a black hole, instead of normal modes, one considers quasi-normal modes for which the frequencies are no longer pure real, showing that the system is loosing energy. The possibility of a connection between the quasinormal frequencies of black holes and the quantum properties of the entropy spectrum was first observed by Bekenstein [11], and further developed by Hod [7]. In particular, Hod proposed that the real part of the quasi-
normal frequencies, in the infinite damping limit, might be related via the correspondence principle to the fundamental quanta of mass and angular momentum.

In this paper we would like to discuss on the discrepancy between the reduced phase space methodology and QNM approach for area spectrum of some black holes.

2 Quasinormal Modes and Area Spectrum

It can be shown that for any periodic system, there exists an adiabatic invariant, which can be calculated (up to a constant shift) as follows

\[ I \equiv \oint p\,dq \propto \int \frac{dE}{\omega(E)}, \]

where \((p, q)\) are its phase space variables, \(E\) is the energy of system and \(\omega(E)\) is vibrational frequency. Via Bohr-Sommerfeld quantization has an equally spaced spectrum in the semi-classical (large \(n\)) limit:

\[ I \approx n\hbar. \]

From this viewpoint, the main problem of black hole quantum mechanics has been to correctly identify the physically relevant period of vibrational frequency. Recently Hod [7] assumed an equally spaced area spectrum and used the apparent existence of a unique quasinormal mode frequency in the large damping limit to uniquely fix the spacing.

In this section we use the observation about quasinormal mode frequency to drive the general form in the semi-classical limit of the Bekenstein-Hawking entropy spectrum for Schwarzschild, BTZ, extremal Reissner-Nordström, near extremal Schwarzschild-de Sitter, Kerr black holes and near extremal black branes [17, 21–25].

We start from the observation that for a Schwarzschild black hole of mass \(M\) and radius \(R\), the real part of the quasinormal mode frequency approaches a fixed non-zero value in the large damping limit as [15, 16]

\[ \omega_R = \frac{\text{ln} \, 3}{4\pi R}. \]

Following [7, 8], Kunstatter in his recent interesting paper [17] assume that this classical frequency plays an important role in the dynamics of the black hole and is relevant to its quantum properties. He consider \(\omega_R\) to be a fundamental vibrational frequency for a black hole of energy \(E = M\). Using Eq. (2), we are lead to the following adiabatic invariant:

\[ I = 4\pi \int \frac{dER}{\text{ln} \, 3} = \frac{\hbar S_{BH}}{\text{ln} \, 3} + c, \]

where we have used the fact that \(R = 2M = 2E\) and the definition of the Bekenstein-Hawking entropy \(S_{BH} = \frac{A}{4\hbar} = \frac{\pi R^2}{4\hbar}\). Eq. (3) then implies that the entropy spectrum is equally spaced:

\[ S_{BH} = n \text{ln} \, 3 \]
Then, the degeneracy of quantum states is given by:

$$\Omega(E) = \exp(S_{BH}) = 3^n$$  \hspace{1cm} (7)

In this case also we obtain

$$\Delta A = 4\hbar \ln 3$$  \hspace{1cm} (8)

then the spacing constant parameter $\alpha$ in Eq.(1) fine as

$$\alpha = 4 \ln 3$$  \hspace{1cm} (9)

It has been suggested that the presence of $\ln 3$ implies a change in the loop quantum gravity gauge group from $SU(2)$ to $SO(3)$ [8]. Now we would like to obtain the area spectrum of non-rotating BTZ black hole. The non rotated BTZ black hole line element is as as following [19, 20]

$$ds^2 = -\left(-M + \frac{r^2}{l^2}\right)dt^2 + \frac{dr^2}{\left(-M + \frac{r^2}{l^2}\right)} + r^2d\theta^2,$$  \hspace{1cm} (10)

which has an horizon at

$$r_+ = \sqrt{\frac{M}{\Lambda}},$$  \hspace{1cm} (11)

and is similar to Schwarzschild black hole with the important difference that it is not asymptotically flat but it has constant negative curvature. The quasi-normal frequencies for non-rotating BTZ black hole have been obtained by Cardoso and Lemos in [18]

$$\omega = \pm m - 2iM^{1/2}(n + 1) \quad n = 0, 1, 2, ...$$  \hspace{1cm} (12)

where $m$ is the angular quantum number. Let $\omega = \omega_R - i\omega_I$, now by taking $\omega_R$ as previous example, we have

$$I = \int \frac{dE}{\omega_R} = \int \frac{1}{\pm m}dM = \frac{M}{\pm m} + c,$$  \hspace{1cm} (13)

where $c$ is a constant, therefore the mass spectrum is equally spaced

$$M = mn\hbar, \quad m \geq 0$$

$$M = -mn\hbar, \quad m < 0.$$  \hspace{1cm} (14)

Then for mass spacing we have

$$\Delta M = \pm m\hbar,$$  \hspace{1cm} (15)

which is the fundamental quanta of black hole mass. As one can see the mass spectrum is equally spaced only for a fixed $m$. For different $m$ there are multiplets with different values of spacing which is given by Eq.(28). On the other hand, the black hole horizon area is given by

$$A = 2\pi r_+.$$  \hspace{1cm} (16)
Using Eq. (11) we obtain

\[ A = 2\pi \sqrt{\frac{M}{\Lambda}}. \]  

(17)

The Bohr-Sommerfeld quantization law and Eq. (10) then implies that the area spectrum is as following [21],

\[ A_n = 2\pi \sqrt{\frac{nm\hbar}{\Lambda}}. \]  

(18)

As one can see although the mass spectrum is equally spaced but the area spectrum is not equally spaced.

In [26] Birmingham et al have shown that the quantum mechanics of the rotating BTZ black hole is characterized by a Virasoro algebra at infinity. Identifying the real part of the quasi-normal frequencies with the fundamental quanta of black hole mass and angular momentum, they found that an elementary excitation corresponds exactly to a correctly quantized shift of the Virasoro generator \( L_0 \) or \( \bar{L}_0 \) in this algebra. Similar to [26] we have not found a quantization of horizon area. The result for the area of event horizon is not equally spaced, in contrast with area spectrum of Schwarzschild black hole.

Now we would like to obtain the area and entropy spectrum of extremal Reissner-Nordström (RN) black holes. The RN black hole’s (event and inner) horizons are given in terms of the black hole parameters by

\[ r_{\pm} = M \pm \sqrt{M^2 - Q^2}, \]  

(19)

where \( M \) and \( Q \) are respectively mass and charge of black hole. In the extreme case these two horizons are coincides

\[ r_{\pm} = M, \quad M = Q. \]  

(20)

According a very interesting conclusion follows [39] (see also more recent paper [40]) , the real part of the quasinormal frequency for extremal RN black holes coincides with the Schwarzschild value

\[ \omega_{RN}^R = \frac{\ln 3}{4\pi R_H}, \]  

(21)

where

\[ R_H = 2M. \]  

(22)

Now by taking \( \omega_{RN}^R \) in this context, we have

\[ I = \int \frac{dE}{\omega_{RN}^R} = \int \frac{4\pi R_H}{\ln 3} dM = \frac{4\pi}{\ln 3} \int 2MdM = \frac{4\pi}{\ln 3} M^2 + c, \]  

(23)

where \( c \) is a constant. In the other hand, the black hole horizon area is given by

\[ A = 4\pi r_{\pm}^2. \]  

(24)
Which using Eq.(20) in extremal case is as following

\[ A = 4\pi M^2. \]  

(25)

The Bohr-Sommerfeld quantization law and Eq.(23) then implies that the area spectrum is equally spaced,

\[ A_n = n\hbar\ln 3. \]  

(26)

By another method we can obtain above result. From Eq.(25) we get

\[ \Delta A = 8\pi M \Delta M = 8\pi M \hbar \omega_{RN} \]  

(27)

where we have associated the energy spacing with a frequency through \( \Delta M = \Delta E = \hbar \omega_{RN} \). Now using Eqs.(21, 22) we have

\[ \Delta A = \hbar \ln 3, \]  

(28)

therefore the extremal RN black hole have a discrete spectrum as

\[ A_n = n\hbar\ln 3. \]  

(29)

Which is exactly the result of Eq.(26). Using the definition of the Bekenstein-Hawking entropy we have

\[ S = \frac{A_n}{4\hbar} = \frac{n \ln 3}{4}. \]  

(30)

Now our aim is to obtain the area and entropy spectrum of near extremal Schwarzschild-de Sitter black holes. The metric of the Schwarzschild-de Sitter spacetime is as following

\[ ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2, \]  

(31)

where

\[ f(r) = 1 - \frac{2M}{r} - \frac{r^2}{a^2}, \]  

(32)

where \( M \) is the mass of black hole, and \( a \) denoting the de Sitter curvature radius, which related to the cosmological constant by \( a^2 = \frac{3}{\Lambda} \). The spacetime possesses two horizons, the black hole horizon is at \( r = r_b \), and the cosmological horizon is at \( r = r_c \), with \( r_b < r_c \). The third zero of \( f(r) \) locates at \( r_0 = -(r_b + r_c) \). It is useful to express \( M \) and \( a^2 \) as function of \( r_b, r_c \)

\[ a^2 = r_b^2 + r_br_c + r_c^2 \]  

(33)

and

\[ 2Ma^2 = r_br_c(r_b + r_c) \]  

(34)
The surface gravity $k_b$ associated with the black hole horizon, as defined by the relation 

$$k_b = \frac{1}{2} \frac{df}{dr}|_{r=r_b}.$$ 

Easily one can show 

$$k_b = \frac{(r_c - r_b)(r_b - r_0)}{2a^2 r_b}. \quad (35)$$

Let us now specialize to the near extremal SdS black hole, which is defined as the spacetime for which the cosmological horizon is very close to the black hole horizon. According a recent study [27](based on identifying the relevant scattering potential with that of the Poschl-Teller model [28,29]) the quasinormal spectrum of SdS space depends strongly on the orbital angular momentum of the perturbation field. Moreover, in the near extremal SdS case, the real part of the frequency, goes almost linearly with orbital angular momentum. In a more recent paper [30], the quasinormal mode spectrum was calculated for SdS space by way of the monodromy method [31]. When this form of the spectrum is then subjected to the near extremal limit, as was done explicitly in [30], there is absolutely no orbital angular momentum dependence in evidence. In recent paper by Medved and Martin [32] a possible resolution for this discrepancy have provided. According to this paper, the monodromy-based calculation, although perfectly valid in the non-degenerate regime, can not necessarily be extrapolated up to the point of horizon coincidence.

The analytical quasinormal mode spectrum for the near extremal SdS black hole has been derived by Cardoso and Lemos [33] as following

$$\omega = k_b \left[ -(n + 1/2)i + \sqrt{\frac{v_0}{k_b^2} - 1/4 + o[\Delta]} \right], \quad n = 0, 1, 2, ... \quad (36)$$

where

$$v_0 = k_b^2 l(l + 1), \quad (37)$$

for scalar and electromagnetic perturbations, and

$$v_0 = k_b^2 (l + 2)(l - 1), \quad (38)$$

for gravitational perturbations, $l$ is the angular quantum number. In Eq.(39) $\Delta$ is squeezing parameter [32] which is as following

$$\Delta = \frac{r_c - r_b}{r_b} \ll 1. \quad (39)$$

Now by taking $\omega_R$ in this context, we have

$$I = \int \frac{dE}{\omega_R} = \int \frac{dM}{k_b \sqrt{\frac{v_0}{k_b^2} - 1/4}} = \frac{M}{k_b \sqrt{\frac{v_0}{k_b^2} - 1/4}} + c, \quad (40)$$

where $c$ is a constant. Boher-Sommerfeld quantization then implies that the mass spectrum is equally spaced,

$$M = n\hbar k_b \sqrt{\frac{v_0}{k_b^2} - 1/4}. \quad (41)$$
The use of Eqs. (34, 52) leads us to
\[
\delta M = \frac{r_b \Delta r \delta r_b}{2a^2} = \hbar k_b \sqrt{\frac{v_0}{k_b^2}} - 1/4.
\]
(42)

In the other hand, the black hole horizon area is given by
\[
A_b = 4\pi r_b^2,
\]
(43)
by the variation of the black hole horizon and use of Eq. (42) we have
\[
\delta A_b = 8\pi r_b \delta r_b = 8\pi \frac{2a^2 \hbar k_b \sqrt{\frac{v_0}{k_b^2}} - 1/4}{\Delta r}.
\]
(44)

Now by using Eq. (56), one can obtain
\[
\delta A_b = 24\pi \hbar \sqrt{\frac{v_0}{k_b^2}} - 1/4.
\]
(45)

Similar result for \( \delta A_b \) by another method have been obtained in [34]. Then we can obtain the quantization of a near extremal SdS black hole area as
\[
A_b = 24\pi n \hbar \sqrt{\frac{v_0}{k_b^2}} - 1/4.
\]
(46)

Using the definition of the Bekenstein-Hawking entropy we have
\[
S = \frac{A_b}{4\hbar} = 6\pi n \sqrt{\frac{v_0}{k_b^2}} - 1/4.
\]
(47)

Now we extend directly the Kunstatter’s approach [17] to determine mass and area spectrum of Kerr and extreme Kerr black holes. The metric of a four-dimensional Kerr black hole given in Boyer-Lindquist coordinates is
\[
d s^2 = -(1 - \frac{2Mr}{\Sigma}) d t^2 - \frac{4Mar \sin^2 \theta}{\Sigma} d t d \phi + \frac{\Sigma}{\Delta} d r^2 + \Sigma d \theta^2 + (r^2 + a^2 + 2Ma^2 r \sin^2 \theta) \sin^2 \theta d \phi^2
\]
(48)
where
\[
\Delta = r^2 - 2Mr + a^2
\]
(49)
\[
\Sigma = r^2 + a^2 \cos \theta
\]
(50)

and \( M \) is the mass of black hole. The roots of \( \Delta \) are given by
\[
r_\pm = M \pm \sqrt{M^2 - a^2}
\]
(51)
where \( r_+ \) is the radius of the event (outer) black hole horizon and \( r_- \) is the radius of the inner black hole horizon. In addition, we have defined the specific angular momentum as
\[
a = \frac{J}{M}
\]
(52)
where \( J \) is the angular momentum of the black hole. The Kerr black hole is rotating with angular velocity

\[
\Omega = \frac{a}{r^2 + a^2} \quad \text{(53)}
\]

\[
= \frac{J}{2M \left(M^2 + \sqrt{M^4 - J^2}\right)} \quad \text{(54)}
\]

which has been evaluated on the event black hole horizon. In gravitational units, the Kerr black hole horizon area and its Hawking temperature are given, respectively, by

\[
A = 4\pi (r_+^2 + a^2) \quad \text{(55)}
\]

\[
= 8\pi \left(M^2 + \sqrt{M^4 - J^2}\right) \quad \text{(56)}
\]

and

\[
T_H = \frac{\sqrt{M^4 - J^2}}{4\pi M \left(M^2 + \sqrt{M^4 - J^2}\right)} \quad \text{(57)}
\]

By applying Bohr’s correspondence principle, Hod \cite{35} conjectured that the real part of the asymptotic quasinormal frequencies of Kerr black hole is given by the formula

\[
\omega_R = T_H \ln 3 + m\Omega \quad \text{(59)}
\]

where \( m \) is the azimuthal eigenvalue of the oscillation. There was compelling evidence that the conjectured formula \text{(59)} must be wrong.

A systematic exploration of the behavior of the first few overtones, i.e., small values of the principal quantum number \( n \), was first accomplished by Onozawa \cite{36}. Onozawa used the Leaver’s continued fraction method to carry out the numerical calculations. Berti and Kokkotas \cite{37} confirmed Onozawa’s results and extended them to higher overtones, i.e., highly damped QNMs. They found that the formula conjectured by Hod, i.e., equation \text{(59)}, does not seem to provide a good fit to the asymptotic modes. Furthermore, Berti and Kokkotas showed that, as the mode order increases, modes having their orbital angular momentum eigenvalue \( l \) and azimuthal eigenvalue \( m \) to satisfy \( l = m = 2 \), are fitted extremely well by the relation

\[
\omega = 2\Omega + i2\pi T_H n \quad \text{(60)}
\]

\(^1\)A more sophisticated study of highly damped Kerr QNMs is performed in \cite{41}. In this work the authors provide complementing and clarifying results that were presented in previous works \cite{36,37}.
where the term $2\pi T_H$ that appears in the imaginary part of the mode frequencies is the surface gravity of the event horizon of the Kerr black hole. Therefore, the real part of the asymptotic frequencies having $l = m = 2$ is given by the expression

$$\omega_R = 2\Omega.$$  \hspace{1cm} (61)

The first law of black hole thermodynamics now takes the form

$$dM = \frac{1}{4}T_H dA + \Omega dJ$$  \hspace{1cm} (62)

and obviously the corresponding expression for adiabatic invariant is now given by the expression

$$I = \int \frac{dM - \Omega dJ}{\omega_R}.$$  \hspace{1cm} (63)

The real part ($\omega_R$) of the asymptotic (highly damped) quasinormal frequencies of the Kerr black hole is given by equation (71) and the angular velocity is given by equation (54). Therefore, the adiabatically invariant integral (63) is written as

$$I = \frac{2}{mJ} \int M \left(M^2 + \sqrt{M^4 - J^2}\right) dM - \frac{1}{m} \int dJ$$  \hspace{1cm} (64)

and after integration, we get

$$I = \frac{1}{2mJ} \left[M^2 \left(M^2 + \sqrt{M^4 - J^2}\right) - J^2 \ln \left(M^2 + \sqrt{M^4 - J^2}\right)\right] - \frac{1}{m} J .$$  \hspace{1cm} (65)

By equating expressions (3) and (65), we get

$$M^2 \bar{A} - J^2 \ln \bar{A} - c = \left(2mJl_p^2\right) n$$  \hspace{1cm} (66)

where the parameter $c$ is equal to $2J^2$, the quantity $\bar{A}$ is the reduced horizon area

$$\bar{A} = \frac{A}{8\pi}$$  \hspace{1cm} (67)

and the area $A$ of the Kerr black hole horizon is given by equation (56). The solution to equation (66) is the principal branch of Lambert W-function and thus the area of the Kerr black hole is written

$$A = 8\pi \left(-\frac{J^2}{M^2}\right) W_0[z]$$  \hspace{1cm} (68)

where the argument of Lambert W-function is given by

$$z = \left(-\frac{M^2}{J^2}\right) e^{-2\left(1 + \frac{2m}{\bar{A}}\right)}.$$  \hspace{1cm} (69)

Since we are only interested in highly damped quasinormal frequencies, i.e. $n \to \infty$, we keep only the first term of the series expansion of the Lambert W-function and we get

$$A = 8\pi e^{-2\left(1 + \frac{2m}{\bar{A}}\right)}.$$  \hspace{1cm} (70)
It is obvious that the area spectrum, although discrete, is not equivalently spaced even to first order.

Hod studied again analytically the QNMs of Kerr black hole [38] and he concluded that the asymptotic quasinormal frequencies of Kerr black hole are given by the simpler expression

$$\omega_R = m\Omega$$

which is obviously in agreement with the aforesaid numerical results of Berti and Kokkotas. This classical frequency plays an important role in the dynamics of the black hole and is relevant to its quantum properties [7, 8].

3 Reduced Phase Space and Area Spectrum

We now consider a specific reduced phase space model of charged black hole [6] (see also [12]). The starting point of this model is the result of [13] that the dynamics of static spherically symmetric charged configuration in any classical theory of gravity in \(d\)-spacetime dimensions is governed by an effective action of the form

$$\Gamma = \int dt (P_M \dot{M} c^2 + P_Q \dot{Q} - H(M, Q)),$$

where \(M\) and \(Q\) are the mass and the charge respectively and \(P_M, P_Q\) the corresponding conjugate momenta. The momentum \(P_M\) has the interpretation of asymptotic time difference between the left and right wedges of a Kruskal diagram. Note that \(H\) is independent of \(P_M\) and \(P_Q\), such that from Hamilton’s equations, \(M\) and \(Q\) are constant of motion. Now, to incorporate thermodynamics of black holes, one assumes that the conjugate momentum \(P_M\) is periodic with period equal to inverse Hawking temperature times \(\hbar\). That is,

$$P_M \sim P_M + \frac{\hbar}{k_B T_H(M, Q)}.$$  \hspace{1cm} (73)

Similar assumptions were made in the past using different arguments [14]. Note that the above identification implies that the \((M, P_M)\) phase subspace has a wedge removed from it, which makes it difficult, if not possible to quantize on the full phase-space. Thus, one can make a canonical transformation \((M, Q, P_M, P_Q) \rightarrow (X, Q, \Pi_X, \Pi_Q)\), which on the one hand opens up the phase space, and on the other hand, naturally incorporate the periodicity Eq.(73) [13]

$$X = \sqrt{\frac{\hbar(S_{BH} - S_0(Q))}{\pi k_B}} \cos(2\pi P_M k_B T_H / \hbar)$$ \hspace{1cm} (74)

$$\Pi_X = \sqrt{\frac{\hbar(S_{BH} - S_0(Q))}{\pi k_B}} \sin(2\pi P_M k_B T_H / \hbar)$$ \hspace{1cm} (75)
\[ Q = Q \]  
\[ \Pi_Q = P_Q + \phi P_M + S'_M T_H/k_B \]

(76)

(77)

where the entropy at extremality is given by \( S_0(Q) = \frac{n k_B Q^2}{\hbar} \), \( \equiv \frac{d}{dQ} \) and \( \phi \) is the electrostatic potential at the horizon. The new phase space is \( R^4 \), on which a rigorous quantization can be performed in a straightforward fashion. Moreover, as shown in [6,12] it is straightforward to calculate the adiabatic invariant for charged black holes in this parameterization. In particular, Eqs.(47,26) generalize to the following invariant:

\[ I \equiv \oint \Pi_X dX = \frac{\hbar(S_{BH} - S_0(Q))}{2\pi k_B} = (2n + 1)\hbar \]

(78)

Quantization yields the following spectra for entropy and charge of the four dimensional quantum black hole (for more details refer to the [12]):

\[ S_{BH} = (2n + p + 1)\pi k_B, \quad n = 0, 1, 2, ... \]

\[ \frac{Q}{e} = m, \quad m = 0, \pm 1, \pm 2, ... \]

\[ p \equiv \frac{Q^2}{\hbar c} = m^2 \frac{e^2}{\hbar c} \]

(79)

(80)

(81)

where consistency requires \( p \) to be a non-negative integer.

Recently the above formalism was extended to rotating (uncharged) black holes in 4 spacetime dimensions [41]. The authors showed that the corresponding area spectrum is also discrete and equispaced:

\[ A = 8\hbar\pi(n + m + 1/2) \]

(82)

where \( n \) and \( m \) are non-negative integers, while \( n \) signifies the departure of the black hole from extremality, \( m \) measures the classical angular momentum of the black hole. More recently, Das et al [42], extend their formalism to include BTZ black holes as well as 5 and higher dimensional Myers-Perry type rotating black holes with multiple angular momenta parameters. They show that while the area spectrum is discrete in each case in general it is not equispaced.

4 Conclusions

Bekenestein’s idea for quantizing a black hole is based on the fact that its horizon area, in the nonextremal case, behaves as a classical adiabatic invariant. Discrete spectra arise in quantum mechanics in the presence of a periodicity in the classical system which in

\[ \text{In recent paper [43] the authors have extended the results obtained for a charged black hole in an asymptotically flat spacetime to the scenario with non vanishing negative cosmological constant.} \]
turn leads to the existence of an adiabatic invariant or action variable. Bohr-Sommerfeld quantization implies that this adiabatic invariant has an equally spaced spectrum in the semi-classical limit. In this article we have reviewed some recent study in black hole area spectrum, including Kunstatter’s approach, and the reduced phase space methodology. We have extended the Kunstatter’s approach for Schwarzschild black hole to another black holes including, BTZ, extremal Reissner-Nordström, near extremal Schwarzschild-de Sitter, and Kerr black holes, and have shown that although as Schwarzschild black hole the spectrum is discrete, it is non equispaced in general. We have shown that while the area spectrum is discrete in each case in general it is not equispaced. The result for the area of event horizon in BTZ black hole case is as: $A_n = 2\pi \sqrt{\frac{nm\hbar}{\Lambda}}$ which is not equally spaced. Then we have found that the quantum area for extremal RN black hole in four dimensional spacetime, is as: $\Delta A = \hbar \ln 3$. Although the real parts of highly damped quasi-normal modes for schwarzschild and extremal RN black hole is equal [40] $\omega_R = \frac{\ln 3}{8\pi M}$, as one can see for example in [17, 40, 44], $\Delta A = 4\hbar \ln 3$ for schwarzschild black hole. Therefore in contrast with claim of [40], $\Delta A$ is not universal for all black holes. Also we have found that the quantum area for near extremal SdS black hole in four dimensional spacetime is as: $\Delta A = 24\pi \hbar \sqrt{\frac{m}{b^3} - 1/4}$. Also Abdalla et al [34] have shown that the results for spacing of the area spectrum for near extreme Kerr black holes differ from that for schwarzschild, as well as for non-extreme Kerr black holes. Such a difference for problem under consideration in this letter also in [34] as the authors have been mentioned may be justified due to the quite different nature of the asymptotic quasi-normal mode spectrum of the near extreme black hole. Finally we have shown that the area spectrum for Kerr black hole is as: $A = 8\pi e^{-2(1+\frac{m\hbar}{n})}$, it is obvious that the area spectrum, although discrete, is not equivalently spaced.

In the other hand the reduced phase space quantization is another technique which we have discussed here. However there is a discrepancy between the result of the reduced phase space methodology and QNM approach for area spectrum of some black holes. As for why there is an apparent discrepancy between the QNM calculations and that of reduced phase space methodology, there is no easy answer. It is not clear what role (if any) quasinormal spectrum plays in quantum theory of gravity (which as of now is nonexistent). I see only two useful applications of quasinormal spectrum: classical general relativity and AdS/CFT correspondence [45], where QNM appear as poles of the retarded correlations functions in the quantum field theory dual to a particular gravitational background [46].

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