Navigating the Landscape of Games

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Abstract

Games are traditionally recognized as one of the key testbeds underlying progress in artificial intelligence (AI), aptly referred to as the “Drosophila of AI”. Traditionally, researchers have focused on using games to build strong AI agents that, e.g., achieve human-level performance. This progress, however, also requires a classification of how ‘interesting’ a game is for an artificial agent. Tackling this latter question not only facilitates an understanding of the characteristics of learnt AI agents in games, but also helps to determine what game an AI should address next as part of its training. Here, we show how network measures applied to so-called response graphs of large-scale games enable the creation of a useful landscape of games, quantifying the relationships between games of widely varying sizes, characteristics, and complexities. We illustrate our findings in various domains, ranging from well-studied canonical games to significantly more complex empirical games capturing the performance of trained AI agents pitted against one another. Our results culminate in a demonstration of how one can leverage this information to automatically generate new and interesting games, including mixtures of empirical games synthesized from real world games.

Introduction

Traditionally, games have played a pivotal role in artificial intelligence (AI) research and have been extensively investigated in machine learning, ranging from abstract benchmarks in game theory over popular board games such as Chess[182] and Go[98], to realtime strategy games such as StarCraft II[110] and Dota 2[4]. Chess and Go have been referred to as the Drosophila of AI research[99]. In recent years, other games such as Poker and StarCraft have reached a similar status in the AI community, witnessed by impressive results such as Libratus[12] and AlphaStar[110]. However, AI research has, so far, primarily sought to answer the question of how to build strong learning agents; we refer to this as the ‘Policy Problem’. In short, the ‘Policy Problem’ entails the search for (super) human-level AI behavior in the vast space of all possible strategies (also known as policies in the reinforcement learning literature) that can be learned by an artificial agent. In this work, we instead consider the inverse perspective also known as the ‘Problem Problem’, defined as “the engineering problem of generating large numbers of interesting adaptive environments to support research” by Leibo et al[66]. This definition raises a critical question: what makes a game ‘interesting’ enough for an AI agent to learn to play?

While the importance of games as a natural research platform for the development of learning algorithms and the measurement of progress in AI is well-established[84,102,121], the significance of the inverse problem remains largely underexposed but is expected to play a critical role for the algorithmic development of future AI entities general enough to solve complex tasks[44]. To enable progress towards solving the Problem Problem, we first need to investigate the underlying notion of what it means for a game to be ‘interesting’ in the first place, or more fundamentally, how to characterize the topological landscape of games.
Figure 1: A landscape of games revealed by the proposed response graph-based workflow. Notably, variations of games with related rules are well-clustered together, indicating strong similarity despite their widely-varying raw sizes. Instances of Blotto cluster together, despite their payoff table sizes ranging from $20 \times 20$ for Blotto(5,3) to $1000 \times 1000$ for Blotto(10,5). Games with strong transitive components (e.g., variations of Elo games, AlphaStar League, Random Game of Skill, and Normal Bernoulli Game) can be observed to be well separated from strongly cyclical games (Rock–Paper–Scissors and the Disc game). Closely-related real-world games (e.g., Hex, Tic-Tac-Toe, Connect Four and each of their respective Misère counterparts) are also well-clustered together.

For instance, one can consider simple characterizations of a game as quantified by cardinal measures, such as the number of strategies available, players involved, number of outcomes possible, or boolean measures, such as whether the game is symmetric, or whether it requires reasoning with imperfect information. One could also order the payouts to players, as done, for instance, in prior works exploring the space of $2 \times 2$ games. For more complex games, however, such measures are quite crude, failing to disambiguate differences in games with identical structures. In many instances, it seems more useful to consider measures that somehow characterize the strategic interactions in the game. As such, an alternative perspective is to consider the nuances in behavioral variations used in the game (e.g., under observed human play); unfortunately, such measures can be subjective or difficult to quantify. One may also seek to classify games from the standpoint of computational complexity. Games have long provided an interesting source of problems under this perspective, and the celebrated PPAD-completeness result shows that in general, computing equilibria for games is a hard problem. Despite this, a game that is computationally challenging to solve may not necessarily be interesting to play (e.g., consider a two-player game where one player chooses a string $S$, and the other needs to find a string $S'$ such that a cryptographic hash of $S'$ is $S$). Thus, it is evident that designation of a single measure quantifying the inherent topological landscape over games is a non-trivial task. As such, one of the central motivations of this paper is that, practically speaking, there is a rich hierarchy of complexity within the class of interesting real-world games.

To help narrow down the properties used to expose the topological landscape of games (and help identify or generate games of interest), we revisit the advances made in the context of the ‘Policy Problem’, which targets evaluation and training of capable agents. A number of these recent works have considered the problems of evaluating and training agents with complex behaviors or interactions. Many of these works focus on analyzing the interactions within populations of agents, relying on game-theoretic models capturing pairwise relations between them to enable novel training or evaluation schemes. Related models have considered the transitive-intransitive inter-agent relations to study games from a dynamical systems perspective. Moreover, the recent work of Czarnecki et al. investigated the geometric structure...
of symmetric zero-sum two-player games from an information-theoretic perspective, with particular emphasis again on transitive-intransitive relations. Fundamentally, the topological structure exposed by modeling pairwise agent interactions seems to be the key enabler of the powerful training and ranking techniques introduced in the above works.

In related literature, graph theory has been well-established as a useful framework for topological analysis of systems involving a large number of interacting entities; complexity analysis via graph-theoretic techniques span across the application domains of social networks, the webgraph, biological systems, econometrics, and linguistics; despite this, graph-theoretic techniques have been significantly less so explored for characterization of the topological structure of games. In this paper, we demonstrate that the combination of graph theory and game-theoretic models of pairwise agent interactions provides the appropriate tools for analyzing the structure of general-sum, many-player games.

From a graph-theoretic perspective, the interacting entities (nodes) of interest in our work are either strategies (in abstract games) or AI agents (in empirical games, where strategies correspond to learned or appropriately-sampled player policies). The interactions between these agents, as quantified by the game’s payoffs, constitute the structure of the graph under analysis. The specific transition structure studied here corresponds to the so-called $\alpha$-Rank response graph of the game, which has played a key role in recent works targeting evaluation and training in large-scale games. The response graph generated by $\alpha$-Rank yields useful insights into the inherent structure of the game, and we demonstrate that the complexity of this underlying graph is closely related to the complexity of solving the game itself.

The primary contribution of this work is a graph-based analytical toolkit that exposes the topological structure of a given collection of games. We use this toolkit to characterize a number of games, by first analyzing canonical and synthetically-designed games with well-defined structures, then extending to large-scale empirical games datasets including Poker, AlphaGo, and StarCraft II, yielding a landscape of games as visualized in Figure 1. We demonstrate correlation of the complexity of the graphs associated with games with the complexity of solving the game itself. This analysis culminates in a demonstration of how the topological structure over games can be used to tackle the ‘interestingness’ question of the ‘Problem Problem’, which seeks to automatically generate games or domains with characteristics interesting for learning agents. Importantly, curriculum learning has proven to be important for agents to learn complex tasks, starting from simpler ones, as part of the earlier mentioned ‘Policy Problem’. Devising such curricula is difficult and mostly done manually. Therefore, automated curricula hold great promise and become easier to achieve when being able to navigate the landscape of games, and as such providing means to automatically generate games of various levels of complexity will aid in solving the ‘Problem Problem’.

Results

We develop a foundational graph-theoretic toolkit that facilitates analysis of canonical and real-world large-scale games, providing insights into their related topological structure in terms of their high-level strategic interactions. We illustrate the toolkit in various games that are both popular with humans and wherein AI agents have reached human-level performance, including Go, MuJoCo Soccer, and StarCraft II. We also show how this toolkit can be used to automate generation of games (which can, for example, subsequently be used to train AI agents on).

We start this section with a simple motivating example to solidify intuitions and explain the workflow of our graph-theoretic toolkit. The prerequisite game theory background and technical details are provided in the Methods section, with full discussion of related works and additional details in the Supplementary Information.

Motivating example and workflow

We first motivate our graph-theoretic method by considering a class of games with an intuitive parametric structure in the player payoffs. Specifically, we consider games of three broad classes (generated as detailed in the Supplementary Information): games in which strategies have a clear transitive ordering (Figure 2.a); games in which strategies have a cyclical structure wherein all but the final strategy are transitive with respect to one another (Figure 2.d); and games with random (or no clear underlying) structure (Figure 2.g).
Figure 2: Motivating example of various classes of two-player, symmetric zero-sum games. (a), (d), and (g) respectively, visualize payoffs for instances of games with transitive, cyclical, and random structure. Despite the numerous payoff variations possible in each class of games illustrated, each shares the underlying payoff structure shown, respectively, in (b), (e), and (h). Moreover, variations in payoffs can notably impact the difficulty of solving (i.e., finding the Nash equilibrium) of these games, as visualized in (c) [f] [i].

We shall see that the core characteristics of games with shared underlying structure is recovered via the proposed analysis.

Each of these figures visualizes the payoffs corresponding to 4 instances of games of the respective class, with each game involving 10 strategies per player; more concretely, entry $M(s_i, s_j)$ of each matrix visualized in Figures 2.a, 2.d and 2.g quantifies the payoff received by the first player if the players, respectively, use strategies $s_i$ and $s_j$. Despite the variance in payoffs evident in the instances of games exemplified here, each essentially shares the payoff structure exposed by re-ordering their strategies, respectively, in Figures 2.b, 2.e and 2.h. In other words, the visual representation of the payoffs in this latter set of figures succinctly characterizes the ‘backbone’ of strategic interactions within these classes of games, despite not being immediately apparent in the individual instances visualized.

More importantly, the complexity of learning useful mixed strategies to play in each of these games is closely associated with this structural backbone. To exemplify this, consider the computational complexity of solving each of these games (i.e., finding a Nash equilibrium). Specifically, we visualize this computational complexity by using the Double Oracle algorithm which, starting from a sub-game consisting of a single randomly-selected strategy, iteratively expands the strategy space until discovery of the Nash equilibrium of the full underlying game. Figures 2.c, 2.f and 2.i visualize the distribution of Double Oracle iterations needed to solve the corresponding games, under random initializations. Note, in particular, that although the underlying payoff structure of the transitive and cyclical games respectively visualized in Figure 2.a and Figure 2.d is similar, the introduction of a cycle in the latter class of games has a substantial impact on the complexity of solving them (as evident in Figure 2.f). In particular, whereas the former class of games are solved using a low (and deterministic) number of iterations, the latter class requires additional iterations due to the presence of cycles increasing the number of strategies in the support of the Nash equilibrium.

Workflow Overall, characterization of the topological structure of games is an important and nuanced problem. To address this problem, we use graph theory to build an analytical toolkit automatically summa-
rizing the high-level strategic interactions within a game, and providing useful complexity measures thereof. Specifically, consider again our motivating transitive game, re-visualized using a collection of graph-based measures in Figure 3. Each of these measures provides a different viewpoint on the underlying game, collectively characterizing it. Specifically, given the game payoffs, Figure 3.b visualizes the so-called $\alpha$-Rank response graph of the game; here, each node corresponds to a strategy (for either player, as the game’s payoffs are symmetric). Transition probabilities between nodes are informed by a precise evolutionary model (detailed in Methods and Omidshafiei et al.); roughly speaking, a directed edge from one strategy to another indicates the players having a higher preference for the latter strategy, in comparison to the former. The response graph, thus, visualizes all preferential interactions between strategies in the game. Moreover, the color intensity of each node indicates its so-called $\alpha$-Rank, which measures the long-term preference of the players for that particular strategy, as dictated by the transition model mentioned above; specifically, darker colors here indicate more preferable strategies.

This representation of a game as a graph enables a variety of useful insights into its underlying structure and complexity. For instance, consider the distribution of cycles in the graph, which play an important role in multiagent evaluation and training schemes and, as later shown, are correlated to the computational complexity of solving two-player zero-sum games (e.g., via Double Oracle). Figure 3.f makes evident the lack of cycles in the particular class of transitive games; while this is clearly apparent in the underlying (fully

Figure 3: Method workflow, with transitive game results accompanying.

Figure 4: Cyclical game results. (a) game payoffs, (b) response graph, (c) cycles histogram, (d) spectral response graph, (e) clustered response graph, (f) contracted response graph.
ordered) payoff visualization of Figure 3.a, it is less so in the unordered variants visualized in Figure 2.a. Even so, the high-level relational structure between the strategies becomes significantly evident by conducting a spectral analysis of the underlying game response graph. Full technical details of this procedure are provided in the Methods section. At a high level, the so-called Laplacian spectrum (i.e., eigenvalues) of a graph, along with associated eigenvectors, captures important information regarding it (e.g., number of spanning trees, algebraic connectivity, and numerous related properties). Reprojecting the response graph by using the top eigenvectors yields the spectral response graph visualized in Figure 3.c, wherein similar strategies are placed close to one another. Taking this one step further, one can cluster the spectral response graph, yielding the clustered response graph, which exposes three classes of strategies in Figure 3.d: a fully dominated strategy with only outgoing edges (a singleton cluster, on the bottom left of the graph), a transient cluster of strategies with both incoming and outgoing edges (top cluster), and a dominant strategy with all incoming edges (bottom right cluster). Finally, contracting the clustered graph by fusing nodes within each cluster yields the high-level characterization of transitive games shown in Figure 3.e.

We can also conduct this analysis for instances of our other motivating games, such as the cyclical game visualized in Figure 4.a. Note here the distinct differences with the earlier transitive game example; in the cyclical game, the $\alpha$-Rank distribution in the response graph (Figure 4.b) has higher entropy (indicating preference for many strategies, rather than one, due to the presence of cycles). Moreover, the spectral reprojection in Figure 4.d reveals a clear set of transitive nodes (left side of visualization) and a singleton cluster of a cycle-inducing node (right side). Contracting this response graph reveals the fundamentally cyclical nature of this game (Figure 4.f). Finally, we label each row and column of the original payoff table Figure 4.a based on this clustering analysis, thus clearly identifying the final strategy as the outlier enforcing the cyclical relationships in the game. Note that while there is no single graphical structure that summarizes the particular class of random games visualized earlier in Figure 2.h, we include this analysis for several instances of such games in the Supplementary Information.

Crucially, a key benefit of this analysis is that the game structure exposed is identical for all instances of the transitive and cyclical games visualized earlier in Figures 2.a and 2.d, making it significantly easier to characterize games with related structure, in contrast to raw analysis of payoffs. Our later case studies further exemplify this, exposing related underlying structures for several classes of more complex games.

Analysis of Canonical and Real-World Games

The insights afforded by our graph-theoretic approach apply to both small canonical games and larger empirical games (where strategies are synonymous with trained AI agents).

Canonical Games

Consider the canonical Rock–Paper–Scissors game, involving a cycle among the three strategies (wherein Rock loses to Paper, which loses to Scissors, which loses to Rock). Figure 5.I.a visualizes a variant of this game involving a redundant copy of the first strategy, Rock, which introduces a redundant cycle and thus affects the distribution of cycles in the game. Despite this, the spectral response graph (Figure 5.I.d) reveals that the redundant game topologically remains the same as the original Rock–Paper–Scissors game, thus reducing to the original game under spectral clustering.

This graph-based analysis also extends to general-sum games. As an example, consider the slightly more complex game of 11-20, wherein two players each request an integer amount of money between 11 to 20 units (inclusive). Each player receives the amount requested, though a bonus of 20 units is allotted to one player if they request exactly 1 unit less than the other player. The payoffs and response graph of this game are visualized, respectively, in Figures 5.II.a and 5.II.b, where strategies, from top-to-bottom and left-to-right in the payoff table, correspond to increasing units of money being requested. This game, first introduced by Arad and Rubinstein, is structurally designed to analyze so-called $k$-level reasoning, wherein a level-0 player is naive (i.e., here simply requests 20 units), and any level-$k$ player responds to an assumed level-$(k-1)$ opponent; e.g., a level-1 player best responds to an assumed level-0 opponent, thus requesting 19 units to ensure receiving the bonus units.

The spectral response graph here (Figure 5.II.d) reveals a more complex mix of transitive and intransitive relations between strategies. Notably, the contracted response graph (Figure 5.II.f) reveals 7 clusters of strategies. Referring back to the rows of payoffs in Figure 5.II.a, relabeled to match cluster colors, reveals that our technique effectively pinpoints the sets of strategies that define the rules of the game: weak strategies
Figure 5: Results for Redundant Rock–Paper–Scissors (RPS) and 11-20 game, in top and bottom rows, respectively. In Redundant RPS, the redundant copy of the first strategy (Rock) is clustered in the spectral response graph. In 11-20, seven clusters of strategies are revealed, exposing the cyclical nature of this game.

Figure 6: Results for empirical games of AlphaGo (top) and MuJoCo soccer dataset (bottom). Note that as these are empirical games, strategies here correspond to trained AI agents. In AlphaGo, the strong transitive relationship between agents is revealed via our analysis. In MuJoCo soccer, more complex relations between similarly-performing agents are revealed in the clusters produced.

(11 or 12 units, first two rows of the payoff table, and evident in the far-right of the clustered response graph), followed by a set of intermediate strategies with higher payoffs (clustered pairwise, near the lower-center of the clustered response graph), and finally the two key strategies that establish the cyclical relationship within the game through \( k \)-level reasoning (19 and 20 units, corresponding to level-0 and level-1 players, in the far-left of the clustered response graph).

This analysis extends to more complex instances of empirical games, which involve trained AI agents, as next exemplified.
AlphaGo Consider first the game of Go, as played by 7 AlphaGo variants: $AG(r)$, $AG(p)$, $AG(v)$, $AG(rv)$, $AG(rp)$, $AG(vp)$, and $AG(rvp)$, where each variant uses the specified combination of rollouts $r$, value networks $v$, and/or policy networks $p$. We analyze the empirical game where each strategy corresponds to one of these agents, and payoffs (Figure 6.I.a) correspond to the win rates of these agents when paired against each other (as detailed by Silver et al.\textsuperscript{93}, Table 9). The $\alpha$-Rank distribution indicated by the node (i.e., strategy) color intensities in Figure 6.I.b reveals $AG(rvp)$ as a dominant strategy, and the cycle distribution graph Figure 6.I.c reveals a lack of cycles here. The spectral response graph, however, goes further, revealing a fully transitive structure (Figures 6.I.d and 6.I.e), as in the motivating transitive games discussed earlier. The spectral analysis on this particular empirical game, therefore, reveals its simple underlying transitive structure (Figure 6.I.f).

MuJoCo Soccer Consider a more interesting empirical game, wherein agents are trained to play soccer in the continuous control domain of MuJoCo, exemplified in Figure 6 (second row). Each agent in this empirical game is generated using a distinct set of training parameters (e.g., feedforward vs. recurrent policies, reward shaping enabled and disabled, etc.), with full agent specifications and payoffs detailed by Liu et al.\textsuperscript{59}. The spectral response graph (Figure 6.II.e) reveals two outlier agents: a strictly dominated agent (node in the top-right), and a strong (yet not strictly dominant) agent (node in the top-left). Several agents here are clustered pairwise, revealing their closely-related interactions with respect to the other agents; such information could, for example, be used to discard or fuse such redundant agents during training to save computational costs.

AlphaStar League Consider next a significantly larger-scale empirical game, consisting of 888 StarCraft II agents from the AlphaStar Final league of Vinyals et al.\textsuperscript{110}. StarCraft II is a notable example, involving a choice of 3 races per player and realtime gameplay, making a wide array of behaviors possible in the game itself. The empirical game considered is visualized in Figure 7.I.a, and is representative of a large number of agents with varying skill levels. Despite its size, spectral analysis of this empirical game reveals that several key subsets of closely-performing agents exist here, illustrated in Figure 7.I.d. Closer inspection of the agents used to construct this empirical payoff table reveals the following insights, with agent types corresponding to those detailed in Vinyals et al.\textsuperscript{110}: i) the blue, orange, and green clusters are composed of
Figure 8: Results for Blotto(5,3), Blotto(10,3), and Go (board size=3). Despite the significant difference in sizes, both instances of Blotto yield a remarkably similar contracted response graph. Moreover, the contracted response graph for Go is notably different from AlphaGo results, due to the latter being an empirical game constructed from trained AI agents rather than a representative set of sampled policies.

agents in the initial phases of training, which are generally weakest (as observed in Figure 7.I.d, and also visible as the narrow band of low payoffs in the top of Figure 7.I.a); ii) the red cluster consists primarily of various, specialized 'exploiter' agents; iii) the purple and brown clusters are primarily composed of the 'league exploiters' and 'main agents', with the latter being generally higher strength than the former. To further ascertain the relationships between only the strongest agents, we remove the three clusters corresponding to the weakest agents, repeating the analysis in Figure 7 (bottom row). Here, we observe the presence of a series of progressively stronger agents (top nodes in Figure 7.II.d), as well as a single outlier agent which quite clearly bests several of these clusters (bottom node of Figure 7.II.d).

An important caveat, as this stage, is that the agents in AlphaGo, MuJoCo soccer, and AlphaStar above were trained to maximize performance, rather than to explicitly reveal insights into their respective underlying games of Go, soccer, and StarCraft II. Thus, this analysis focused on characterizing relationships between the agents from the ‘Policy Problem’ perspective, rather than the underlying games themselves, which provide insights into the interestingness of the game (‘Problem Problem’). This latter investigation would require a significantly larger population of agents, which cover the policy space of the underlying game effectively, as exemplified next.
**Blotto** Naturally, characterization of the underlying game can be achieved in games small enough where all possible policies can be explicitly compared against one another. For instance, consider Blotto($\tau, \rho)$, a zero-sum two-player game wherein each player has $\tau$ tokens that they can distribute amongst $\rho$ regions\textsuperscript{11}. In each region, each player with the most tokens wins (see Tuyls et al.\textsuperscript{10} for additional details). In the variant we analyze here, each player receives a payoff of $+1$, 0, and $-1$ per region respectively won, drawn, and lost. The permutations of each player’s allocated tokens, in turn, induce strong cyclical relations between the possible policies in the game. While the strategy space for this game is of size $\binom{\tau+\rho-1}{\rho-1}$, payoffs matrices can be fully specified for small instances, as shown for Blotto(5,3) and Blotto(10,3) in Figure 8 (first and second row, respectively). Despite the differences in strategy space sizes in these particular instances of Blotto, the contracted response graphs in Figures 8.I.d and 8.II.d capture the cyclical relations underlying both instances, revealing a remarkably similar structure.

For larger games, the cardinality of the pure policy space typically makes it infeasible to fully enumerate policies and construct a complete empirical payoff table. In these instances, we rely on sampling policies in a manner that captures a set of representative policies, with varying transitive and intransitive relationships. We use the policy sampling procedure proposed by Czarnecki et al.\textsuperscript{20}, which also seeks a set of representative policies for a given game. The specifics of this procedure are detailed in the Supplementary Information, and at a high level involve three phases: i) using a combination of tree search algorithms, Alpha-Beta\textsuperscript{18} and Monte Carlo Tree Search\textsuperscript{15}, with varying tree depth limits for the former and varying number of simulations allotted to the latter, thus yielding policies of varying transitive strengths; ii) using a range of random seeds in each instantiation of the above algorithms, thus producing a range of policies for each level of transitive strength; iii) repeating the same procedure with negated game payoffs, thus also covering the space of policies that actively seek to lose the original game. While this sampling procedure is a heuristic, it produces a representative set of policies with varying degrees of transitive and intransitive relations, and thus provides an approximation of the underlying game that can be feasibly analyzed.

**Go** Let us revisit the example of Go, constructing our empirical game using the above policy sampling scheme, rather than the AlphaGo agents used earlier. We analyze a variant of the game with board size $3 \times 3$, as shown in Figure 8 (third row). Notably, the contracted response graph (Figure 8.III.d) reveals the presence of a strongly-cyclical structure in the underlying game, in contrast to the AlphaGo empirical game (Figure 6). Moreover, the presence of a reasonably strong agent (visible in the top of the contracted response graph) becomes evident here, though this agent also shares cyclical relations with several sets of other agents. Overall, this analysis exemplifies the distinction between analyzing an underlying game (e.g., Go) versus analyzing the agent training process (e.g., AlphaGo). Investigation of links between these two lines of analysis, we believe, makes for an interesting avenue for future work.

**Linking response graph complexity to computational complexity** A question that naturally arises is whether certain measures over response graphs are correlated with the computational complexity of solving their associated games. We investigate this in Figure 9, which compares several response graph complexity measures against the number of iterations needed to solve a large collection of games using the double oracle algorithm\textsuperscript{28}. The results here consider specifically the $\alpha$-Rank entropy, number of 3-cycles, and mean in-degree (with details in Methods and results for additional measures included in the Supplementary Information). As in earlier experiments, solution of small-scale games is conducted using payoffs over full enumeration of pure policies, whereas that of larger games is done using the empirical games over sampled policies. Each graph complexity measure reported is normalized with respect to the maximum measure possible in a graph of the same size, and the number of iterations to solve is normalized with respect to the number of strategies in the respective game. Thus, for each game, the normalized number of iterations to solve provides a measure of its relative computational complexity compared to games with the same strategy space size.

Several trends are of note in these results. First, the entropy of the $\alpha$-Rank distribution associated with each game correlates well with its computational complexity (see Spearman’s correlation coefficient $\rho_s$ in the top-right Figure 9.a). This matches intuition, as higher entropy $\alpha$-Rank distributions indicate a larger support over the strategy space (i.e., strong strategies, with non-zero $\alpha$-Rank mass), thus requiring additional iterations to solve. Moreover, the number of 3-cycles in the response graph also correlates well with computational complexity, again matching intuition as the intransitivities introduced by cycles typically
make it more difficult to traverse the strategy space. Finally, the mean in-degree over all response graph nodes correlates less so with computational complexity (though degree-based measures still serve a useful role in characterizing and distinguishing graphs of differing sizes). Overall, these results indicate that response graph complexity provides a useful means of quantifying the computational complexity of games, bearing potential for more formal future investigation.

The Landscape of Games

The results, thus far, have demonstrated that graph-theoretic analysis can simplify games (via spectral clustering), uncover their topological structure (e.g., transitive structure of the AlphaGo empirical game), and yield measures correlated to the computational complexity of solving these games. Overall, it is evident that the perspective offered by graph theory yields a useful characterization of games across multiple fronts. Given this insight, we next consider whether this characterization can be used to compare a widely-diverse set of games.

To achieve this, we construct empirical payoff tables for a suite of games, using the policy sampling scheme described earlier for the larger instances (also see Supplementary Information for full details, including description of the games considered). For each game, we compute the response graphs and several associated local and global complexity measures (e.g., α-Rank distribution entropy, number of 3-cycles, node-wise in- and out-degree statistics, and several other measures detailed in the Methods section), which constitute a feature vector capturing properties of interest. Finally, a principal component analysis of these features yields the low-dimensional visualization of the landscape of games considered, shown in Figure 1.

We make several key insights given this empirical landscape of games. Notably, variations of games with related rules are well-clustered together, indicating strong similarity despite the widely-varying sizes of their policy spaces and empirical games used to construct them; specifically, all considered instances of Blotto cluster together, with empirical game sizes ranging from $20 \times 20$ for Blotto(5,3) to $1000 \times 1000$ for Blotto(10,5). Moreover, games with strong transitive components (e.g., variations of Elo games, AlphaStar League, Random Game of Skill, and Normal Bernoulli Game) are also notably separated from strongly cyclical games (Rock–Paper–Scissors, Disc game, and Blotto variations). Closely-related real-world games
(e.g., Hex, Tic-Tac-Toe, Connect Four, and each of their respective Misère counterparts wherein players seek to lose) are also well-clustered. Crucially, the strong alignment of this analysis with intuitions of the similarity of certain classes of games serves as an important validation of the graph-based analysis technique proposed in this work. Additionally, the analysis and corresponding landscape of games make clear that several games of interest for AI seem well-clustered together, which also holds for less interesting games (e.g., Transitive and Elo games). As such, we believe this type of investigation can be considered a method to taxonomize which future games may be interesting, and which ones less so to train AI agents on.

The Problem Problem Revisited: Procedural Game Generation

Having now established various graph-theoretic tools for characterizing games of interest, we revisit the so-called Problem Problem, which targets automatic generation of ‘interesting’ environments. At a high level, we establish the feedback loop visualized in Figure 3, enabling automatic generation of games as driven by our graph-based analytical workflow.

Full details of the game generation procedure are provided in the Methods Section. At a high level, given a parameterization of a generated game that specifies an associated payoff tensor, we synthesize its response graph and associated measures of interest (as done when generating the earlier landscape of games); we use the multidimensional Elo parameterization for generating payoffs, due to its inherent ability to specify complex transitive and intransitive games. We then specify an objective function of interest to optimize over these graph-based measures. As only the evaluations of such graph-based measures (rather than their gradients) are typically available, we use a gradient-free approach to iteratively generate games optimizing these measures (CMA-ES is used in our experiments).

Naturally, we can maximize any individual game complexity measures, or a combination thereof, directly (e.g., entropy of the α-Rank distribution, number of 3-cycles, etc.). More interestingly, however, we can leverage our low-dimensional landscape of games to directly drive the generation of new games towards existing ones with properties of interest. Consider the instance of game generation shown in Figure 10.a, which shows an overview of the above pipeline generating a 5 × 5 game minimizing Euclidean distance within the low-dimensional complexity landscape to the standard 3 × 3 Rock–Paper–Scissors game. Each point on this plot corresponds to a generated game instance. The payoffs visualized, from left to right, respectively correspond to the initial procedural game parameters (which specify a game with constant payoffs), intermediate parameters, and final optimized parameters; projections of the corresponding games within the games landscape are also indicated, with the targeted game of interest (Rock–Paper–Scissors here) highlighted in green. Notably, the final optimized game exactly captures the underlying rules that specify a general-size Rock–Paper–Scissors game, in that each strategy beats as many other strategies as it loses to. In Figure 10.b, we consider a larger 13 × 13 generated game, which seeks to minimize distance to a 1000 × 1000 Elo game (which is transitive in structure, as in our earlier motivating example in Figure 2.a). Once again, the generated game captures the transitive structure associated with Elo games.

Next, we consider generation of games that exhibit properties of mixtures of several target games. For example, consider what happens if 3 × 3 Rock–Paper–Scissors were to be combined with the 1000 × 1000 Elo game above; one might expect a mixture of transitive and cyclical properties in the payoffs, though the means of generating such mixed payoffs directly is not obvious due to the inherent differences in sizes of the targeted games. Using our workflow, which conducts this optimization in the low-dimensional graph-based landscape, we demonstrate a sequence of generated games targeting exactly this mixture in Figure 10.c. Here, the game generation objective is to minimize Euclidean distance to the mixed principal components of the two target games (weighted equally). The payoffs of the final generated game exhibit exactly the properties intuited above, with predominantly positive (blue) upper-triangle of payoff entries establishing a transitive structure, and the more sporadic positive entries in the lower-triangle establishing cycles.

Naturally, this approach opens the door to an important avenue for further investigation, targeting generation of yet more interesting combinations of games of different sizes and rulesets (e.g., as in Figure 10.d, which generates games targeting a mixture of Go (board size=3) and the AlphaStar League), and subsequent training of AI agents using such a curriculum of generated games. Overall, these examples illustrate a key benefit of the proposed graph-theoretic measures in that it captures the underlying structure of various classes of games. The characterization of games enabled by our approach directly enables the navigation of the associated games landscape to generate never-before-seen instances of games with fundamentally related
Figure 10: Visualization of procedural game generation projected in the games landscape. Each figure visualizes the generation of a game of specified size, which targets a pre-defined game (or mixture of games) of a different size. The three payoffs in each respective figure, from left to right, correspond to the initial procedural game parameters, intermediate parameters, and final optimized parameters. Strategies are sorted by mean payoffs in (b) and (c) to more easily identify transitive structures expected from an Elo game.
Discussion

In 1965, mathematician Alexander Kronrod stated that “chess is the Drosophila of artificial intelligence”, referring to the genus of flies used extensively for genetics research. This parallel drawn to biology invites the question of whether a family, order, or, more concretely, shared structures linking various games can be identified. Our work demonstrated a means of revealing this topological structure, extending beyond related works investigating this question for small classes of games (e.g., $2 \times 2$ games\cite{19,79,80}). We believe that such a topological landscape of games can help to identify and generate related games of interest for AI agents to tackle, as targeted by the Problem Problem, hopefully significantly extending the reach of AI system capabilities. As such, this paper presented a comprehensive study of games under the lens of graph theory and empirical game theory, operating on the response graph of any game of interest. The proposed approach applies to general-sum, many-player games, enabling richer understanding of the inherent relationships between strategies (or agents), contraction to a representative (and smaller) underlying game, and identification of a game’s inherent topology. We highlighted insights offered by this approach when applied to a large suite of games, including canonical games, empirical games consisting of trained agent policies, and real-world games consisting of representative sampled policies, extending well beyond typical characterizations of games using raw payoff visualizations, cardinal measures such as strategy or game tree sizes, or strategy rankings. We demonstrated that complexity measures associated with the response graphs analyzed correlate well to the computational complexity of solving these games, and importantly enable the visualization of the landscape of games in relation to one another (as in Figure 1).

The games landscape exposed here was then leveraged to procedurally generate games, providing a principled means of better understanding and facilitating the solution of the so-called Problem Problem. While the classes of games generated in this paper were restricted to the normal-form (e.g., generalized variants of Rock–Paper–Scissors), they served as an important validation of the proposed approach. Specifically, this work provides a foundational layer for generating games that are of interest in a richer context of domains. As such, an important line of future work will involve generalization of this approach to generation of more complex games, e.g., by optimizing parameters associated with richer underlying games (e.g., goal sizes or opponent speed variations for training drills in MuJoCo soccer, deck variations in Poker, map variations in games such as Capture-the-Flag\cite{48}, etc.) and synthesizing associated empirical payoff tensors to better navigate the game landscape. Moreover, as the principle contribution of this paper was to establish a graph-theoretic approach for investigating the landscape of games, we focused our investigation on empirical analysis of a large suite of games. As such, for larger games, our analysis relied on sampling of a representative set of policies to characterize them. Thus, a limitation of this approach is that such a policy sampling scheme can be inherently expensive for extremely large games, making it important to further investigate alternative sampling schemes. Consideration of an expanded set of such policies (e.g., those that balance the odds for players by ensuring a near-equal win probability) and correlations between the empirical game complexity and the complexity of the underlying policy representations (e.g., deep versus shallow neural networks or, whenever possible, Boolean measures of strategic complexity\cite{29,87}) also seem interesting to investigate. Finally, our approach is general enough to be applicable to other areas in social and life sciences, which can be modeled either as response graphs over games; applications of these game-theoretic techniques to characterize the complex ecologies in these related fields would be an interesting application area.

Overall, we hope that this work paves the way for related investigations of theoretical properties of graph-based games analysis, and further links to related works investigating the geometry and structure of games\cite{19,79,80}.

Methods

We present a detailed overview of the underlying methodology and techniques used for conducting the graph theoretic analysis in this work.
Prerequisites

Games Our work applies to $K$-player, general-sum games, wherein each player $k \in [K]$ has a finite set $S^k$ of pure strategies. The space of pure strategy profiles is denoted $S = \prod_k S^k$, where a specific pure strategy profile instance is denoted $s = (s^1, \ldots, s^K) \in S$. For a given profile $s \in S$, the payoffs vector is denoted $M(s) = (M^1(s), \ldots, M^K(s)) \in \mathbb{R}^K$, where $M^k(s)$ is the payoff for each player $k \in [k]$. We denote by $s^{-k}$ the profile of strategies used by all but the $k$-th player. A game is said to be zero-sum if $\sum_k M^k(s) = 0$ for all $s \in S$. A game is said to be symmetric if all players have the same strategy set, and $M^k(s^1, \ldots, s^K) = M^\rho(k)(s^{\rho(1)}, \ldots, s^{\rho(K)})$ for all permutations $\rho$, strategy profiles $s$, and player indices $k \in [K]$.

Empirical games For the real-world games considered (e.g., Go, Tic–Tac–Toe, etc.), we conduct our analysis using an empirical game theoretic approach. Specifically, rather than consider the space of all pure strategies in the game (which can be enormous, even in the case of, e.g., Tic–Tac–Toe), we construct an empirical game over meta-strategies, which can be considered higher-level strategies over atomic actions. In empirical games, a meta-strategy $s^k$ for each player $k$ corresponds to a sampled policy (e.g., in the case of our real-world games examples), or an AI agent (e.g., in our study of AlphaGo, where each meta-strategy was a specific variant of AlphaGo). Empirical game payoffs are calculated according to the win/loss ratio of these meta-strategies against one another, over many trials of the underlying games. From a practical perspective, game-theoretic analysis applies to empirical games (over agents) in the same manner as standard games (over strategies); thus, we consider strategies and agents as synonymous in this work. Overall, empirical games provide a useful abstraction of the underlying game that enables the study of significantly larger and more complex interactions.

Finite population models and $\alpha$-Rank In game theory, one often seeks algorithms or models for evaluating and training strategies with respect to one another (i.e., models that produce a score or ranking over strategy profiles, or an equilibrium over them). As a specific example, the Double Oracle algorithm, which is used to quantify the computational complexity of solving games in some of our experiments, converges to Nash equilibria, albeit only in two-player zero-sum games. More recently, a line of research has introduced and applied the $\alpha$-Rank algorithm for evaluation of strategies in general-sum, $n$-player many-strategy games. $\alpha$-Rank leverages notions from stochastic evolutionary dynamics in finite populations, in the limit of rare mutations, which are subsequently analyzed to produce these scalar ratings (one per strategy or agent). At a high level, $\alpha$-Rank models the probability of a population transitioning from a given strategy to a new strategy, by considering the additional payoff the population would receive via such a deviation. These evolutionary relations are considered between all strategies in the game, and are summarized in its so-called response graph. $\alpha$-Rank then uses the stationary distribution over this response graph to quantify the long-term propensity of playing each of the strategies, assigning a scalar score to each.

Overall, $\alpha$-Rank yields a useful representation of the limiting behaviors of the players, providing a summary of the characteristics of the underlying game—a 1-dimensional one (a scalar rating per strategy profile). In our work, we exploit the higher-dimensional structural properties of the $\alpha$-Rank response graph, to make more informed characterizations of the underlying game, rather than compute scalar rankings.

Response graphs The $\alpha$-Rank response graph provides the mathematical model that underpins our analysis. It constitutes an analogue (yet, not equivalent) model of the invasion graphs used to describe the evolution dynamics in finite populations in the limit when mutations are rare (see, e.g., Fudenberg and Imhof, Hanert et al., Segbroeck, Vasconcelos et al.). In this small-mutation approximation, directed edges stand for the fixation probability of a single mutant in a monomorphic population of resident individuals (the vertices), such that all transitions are computed through a process involving only two strategies at a time. Here, we use a similar approach. Let us consider a pure strategy profile $s = (s^1, \ldots, s^K)$. Consider a unilateral deviation (corresponding to a mutation) of player $k$ from playing $s^k \in S^k$ to a new strategy $s^k \in S^k$, thus resulting in a new profile $s = (s^k, s^{-k})$. The response graph associated with the game considers all such deviations, defining transition probabilities between all pairs of strategy profiles involving a unilateral deviation. Specifically, let $E_{s,\sigma}$ denote the transition probability from
$s$ to $\sigma$ (where the latter involves a unilateral deviation), defined as

$$E_{s,\sigma} = \begin{cases} 
\eta \frac{1 - \exp(-\alpha(M^k(\sigma) - M^k(s)))}{1 - \exp(-\alpha m(M^k(\sigma) - M^k(s)))} & \text{if } M^k(\sigma) \neq M^k(s) \\
\frac{1}{m} & \text{otherwise},
\end{cases}$$

(1)

where $\eta$ is a normalizing factor denoting the reciprocal of the total number of unilateral deviations from a given strategy profile, i.e., $\eta = (\sum_{k=1}^{K} (|S^k| - 1))^{-1}$. Furthermore, $\alpha \geq 0$ and $m \in \mathbb{N}$ are parameters of the underlying evolutionary model considered and denote, respectively, to the so-called selection pressure and population size.

To further simplify the model and avoid sweeps over these parameters, we consider here the limit of infinite-$\alpha$ introduced by Omidshafiei et al.\textsuperscript{72}, which specifies transitions from lower-payoff profiles to higher-payoff ones with probability $\eta(1 - \varepsilon)$, the reverse transition with probability $\eta\varepsilon$, and transition between strategies of equal payoff with probability $\eta/2$, where $0 < \varepsilon \ll 1$ is a small perturbation factor. We use $\varepsilon = 1e-10$ in our experiments, and found low sensitivity of results to this choice given a sufficiently small value. For further theoretical exposition of $\alpha$-Rank under this infinite-$\alpha$ regime, see Rowland et al.\textsuperscript{82}. Given the pairwise strategy transitions defined as such, the self-transition probability of $s$ is subsequently defined as,

$$E_{s,s} = 1 - \sum_{k \in |K| \backslash \{s\}} E_{s,\sigma}.$$  

(2)

As mentioned earlier, if two strategy profiles $s$ and $\sigma$ do not correspond to a unilateral deviation (i.e., differ in more than one player’s strategy), no transition occurs between them under this model (i.e., $E_{s,\sigma} = 0$).

The transition structure above is informed by particular models in evolutionary dynamics as explained in detail in Omidshafiei et al.\textsuperscript{72}. The introduction of the perturbation term $\varepsilon$ effectively ensures the ergodicity of the associated Markov chain with row-stochastic transition matrix $E$. This transition structure then enables definition of the $\alpha$-Rank response graph of a game.

**Definition 1 (Response graph)** The response graph of a game is a weighted directed graph (digraph) $G = (S, E)$ where each node corresponds to a pure strategy profile $s \in S$, and each weighted edge $E_{s,\sigma}$ quantifies the probability of transitioning from profile $s$ to $\sigma$.

For example, the response graph associated with a transitive game is visualized in Figure 3.b, where each node corresponds to a strategy $s$, and directed edges indicate transition probabilities between nodes. Omidshafiei et al.\textsuperscript{72} define $\alpha$-Rank $\pi \in \Delta^{|S| - 1}$ as a probability distribution over the strategy profiles $S$, by ordering the masses of the stationary distribution of $E$ (i.e., solution of the eigenvalue problem $\pi^T E = \pi^T$). Effectively, the $\alpha$-Rank distribution quantifies the average amount of time spent by the players in each profile $s \in S$ under the associated discrete-time evolutionary population model\textsuperscript{81}. Our proposed methodology uses the $\alpha$-Rank response graphs in a more refined manner, quantifying the structural properties defining the underlying game, as detailed in the workflow outlined in Figure 3 and, in more detail, below.

**Workflow: Spectral, clustered, and contracted response graphs**

This section details the workflow used to perform spectral analysis of games’ response graphs (i.e., the steps visualized in Figure 3.c to Figure 3.e). Response graphs are processed in two stages: i) symmetrization (i.e., transformation of the directed response graphs to an associated undirected graph), and ii) subsequent spectral analysis. This two-phase approach is a standard technique for analysis of directed graphs, which has proved effective in a large body of prior works (see Malliaros and Vazirgiannis\textsuperscript{61}, Van Lierde\textsuperscript{109} for comprehensive surveys). Additionally, spectral analysis of the response graph is closely-associated with the eigenvalue analysis required when solving for the $\alpha$-Rank distribution, establishing a shared formalism of our techniques with those of prior works.

**Symmetrization** Let $A$ denote the adjacency matrix of the response graph $G$, where $A = E$ as $G$ is a directed weighted graph. We seek a transformation such that response graph strategies with similar
relationships to neighboring strategies tend to have higher adjacency with one another. Bibliometric symmetrization provides a useful means to do so in application to directed graphs, whereby the symmetrized adjacency matrix is defined \( A = AA^T + A^T A \). Intuitively, in the first term, \( AA^T \), the \((s, \sigma)\)-th entry captures the weighted number of other strategies that both \( s \) and \( \sigma \) would deviate to in the response graph \( G \); the same entry in the second term, \( A^T A \), captures the weighted number of other strategies that would deviate to both \( s \) and \( \sigma \). Hence, this symmetrization captures the relationship of each pair of response graph nodes \((s, \sigma)\) with respect to all other nodes, ensuring high values of weighted adjacency when these strategies have similar relational roles with respect to all other strategies in the game. More intuitively, this ensures that in games such as Redundant Rock–Paper–Scissors (see Figure 5, first row), sets of redundant strategies are considered to be highly adjacent to each other.

**Spectral response graphs** Following bibliometric symmetrization of the response graph, clustering proceeds as follows. Specifically, for any partitioning of the strategy profiles \( S \) into sets \( S_1 \subset S \) and \( \bar{S}_1 = S \setminus S_1 \), define \( w(S_1, \bar{S}_1) = \sum_{s \in S_1, \sigma \in \bar{S}_1} E_{s, \sigma} \). Let the sets of disjoint strategy profiles \( \{S_k\}_{k \in [K]} \) partition \( S \) (i.e., \( \bigcup_{k \in [K]} S_k = S \)). Define the \( K \)-cut of graph \( G \) under partitions \( \{S_k\}_{k \in [K]} \) as

\[
\text{cut}(\{S_k\}) = \sum_k w(S_k, \bar{S}_k),
\]

which, roughly speaking, measures the connectedness of points in each cluster; i.e., a low \( \text{cut} \) indicates that points across distinct clusters are not well-connected. A standard technique for cluster analysis of graphs is to choose the set of \( K \) partitions, \( \{S_k\}_{k \in [K]} \), which minimizes (3). In certain situations, balanced clusters (i.e., clusters with similar numbers of nodes) may be desirable; here, a more suitable metric is the so-called normalized \( K \)-cut, or \( \text{Ncut} \), of graph \( G \) under partitions \( \{S_k\}_{k \in [K]} \),

\[
\text{Ncut}(\{S_k\}) = \sum_k \frac{w(S_k, \bar{S}_k)}{w(S_k, S)}.
\]

Unfortunately, the minimization problem associated with (4) is NP-hard even when \( K = 2 \) (see Shi and Malik). A typical approach is to consider a spectral relaxation of this minimization problem, which corresponds to a generalized eigenvalue problem (i.e., efficiently solved via standard linear algebra); interested readers are referred to Shi and Malik, Van Lierde, for further exposition. Define the Laplacian matrix \( L = D - A \) (respectively, \( L = I - D^{-1/2} A D^{-1/2} \)), where degree matrix \( D \) has diagonal entries \( D_{i,i} = \sum_j \tilde{A}_{i,j} \), and zeroes elsewhere. Then the eigenvectors associated with the lowest nonzero eigenvalues of \( L \) provide the desired spectral projection of the datapoints (i.e., spectral response graph), with the desired number of projection dimensions corresponding to the number of eigenvectors kept. We found that using the unnormalized graph Laplacian \( L = D - A \) yielded intuitive projections in our experiments, which we visualize 2-dimensionally in our results (e.g., see Figure 3.c).

**Clustered and contracted response graphs** The relaxed clustering problem detailed above is subsequently solved by application of a standard clustering algorithm to the spectral-projected graph nodes. Specifically, we use agglomerative average-linkage clustering in our experiments (see Rokach and Maimon Chapter 15 for details). For determining the appropriate number of clusters, we use the approach introduced by Pham et al., which we found to yield more intuitive clusterings than the gap statistic for the games considered.

Following computation of clustered response graphs (e.g., Figure 3.d), we contract clustered nodes (summing edge probabilities accordingly), as in Figure 3.e. Note that for clarity, our visualizations only show edges corresponding to transitions from lower-payoff to higher-payoff strategies in the standard, spectral, and clustered response graphs, as these bear the majority of transition mass between nodes; reverse edges (from higher- to lower-payoff nodes) and self-transitions are not visually indicated, despite being used in the underlying spectral clustering. The exception is for contracted response graphs, where we do visualize weighted edges from higher- to lower-payoff nodes; this is due to the node contraction process potentially yielding edges with non-negligible weight in both directions.
Workflow: Low-dimensional landscape & game generation

We next detail the approach used to compute the games landscape and to procedurally generate games, as respectively visualized in Figures 1 and 10.

Low-dimensional games landscape To compute the low-dimensional games landscape, we use principal component analysis (PCA) of key features associated with the games’ response graphs. Specifically, we use a collection of features we found to correlate well with the underlying computational complexity of solving these games (as detailed in the Results section).

Specifically, using the response graph \( G \) of each game, we compute in- and out-degrees for all nodes, the entropy of the \( \alpha \)-Rank distribution \( \pi \), and the total number of 3-cycles. We normalize each of these measures as follows: dividing node-wise in- and out-degrees via the maximum possible degrees for a response graph of the same size; dividing the \( \alpha \)-Rank distribution entropy via the entropy of the uniform distribution of the same size; finally, dividing the number of 3-cycles via the same measure for a fully connected directed graph.

We subsequently construct a feature vector consisting of the normalized \( \alpha \)-Rank distribution entropy, the normalized number of 3-cycles, and statistics related to normalized in- and out-degrees. Specifically, we consider the mean, median, standard deviation, skew, and kurtosis of the in- and out-degrees across all response graph nodes, similar to the NetSimile\(^5\) approach, which characterized undirected graphs. This yields a feature vector of fixed size for all games. We subsequently conduct a PCA analysis of the resulting feature vectors, visualizing the landscape in Figure 1 via projection of the feature vectors onto the top two principal components, yielding a low-dimensional embedding \( v_g \) for each game \( g \).

Multidimensional Elo The games generated in Figures 2 and 10 use the multidimensional Elo (mElo) parametric structure\(^4\), an extension of the classical Elo\(^2\) rating system used in Chess and other games. In mElo games, each strategy \( i \) is characterized by two sets of parameters: i) a scalar rating \( r_i \in \mathbb{R} \), capturing the strategy’s transitive strength, and ii) a \( 2k \)-dimensional vector \( c_i \), capturing the strategy’s intransitive relations to other strategies. The payoff a strategy \( i \) receives when played against a strategy \( j \) in a mElo game is defined by \( M(i, j) = \sigma(r_i - r_j + c_i^T \Omega c_j) \) where \( \sigma(z) = (1 + \exp(-z))^{-1} \), \( \Omega = \sum_{i=1}^{k} (e_{2i-1}e_{2i}^T - e_{2i}e_{2i-1}^T) \), and \( e_i \) is the unit vector with coordinate \( i \) equal to 1. This parametric structure is particularly useful as it enables definition of a wide array of games, ranging from those with fully transitive strategic interactions (e.g., those with a single dominant strategy, as visualized in Figure 2.a), to intransitive interactions (e.g., those with cyclical relations, as visualized in Figure 2.d), to a mix thereof.

Procedural game generation The procedural game generation visualized in Figure 10 is conducted as follows. First, we compute the low-dimensional game embeddings for the collection of games of interest, as detailed above. Next, for an initially randomly-generated mElo game of the specified size and rank \( k \), we concatenate the associated mElo parameters \( r_i \) and \( c_i \) for all strategies, yielding a vector of length \( |S|(1+2k) \) fully parameterizing the mElo game, and constituting the decision variables of the optimization problem used to generate new games. We used a rank 5 mElo parameterization for all game generation experiments. For any such setting of mElo parameters, we compute the associated mElo payoff matrix \( M \), then the associated response graph and features, and finally project these features onto the principal components previously computed for the collection of games of interest, yielding the projected mElo components \( v_{mElo} \).

Subsequently, given a game \( g \) of interest that we would like to structurally mimic via our generated mElo game, we use a gradient-free optimizer, CMA-ES\(^3\) to minimize \( ||v_{mElo} - v_g||^2 \) by appropriately setting the mElo parameters. For targeting mixtures of games (e.g., as in Figures 10.c and 10.d), we simply use a weighted mixture of their principal components \( v_g \) (with equal weights used in our experiments). We found the opens-source implementation of CMA-ES\(^5\) to converge to suitable parameters within 20 iterations for all experiments, with the exception of the larger game generation results visualized in Figure 10.d, which required 40 iterations.

Statistics

To generate the distributions of Double Oracle iterations needed to solve the motivating examples (Figure 2), we used 20 generated games per class (transitive, cyclical, random), showing four examples of each in
Figures 2.a, 2.d and 2.g. For each of these 20 games, we used 10 random initializations of the Double Oracle algorithm, reporting the full distribution of iterations. To generate the complexity results in Figure 9, we likewise used 10 random initializations of Double Oracle per game, with standard deviations shown in the scatter plots (which may require zooming in). For the Spearman correlation coefficients shown in each of Figures 9.a to 9.c, the reported p-value is two-sided.

Data Availability

We use OpenSpiel\textsuperscript{54} as the backend providing many of the games and associated payoff datasets studied here (see Supplementary Information for details). Payoff datasets for empirical games in the literature are referenced in the main text.

Code Availability

We use OpenSpiel\textsuperscript{54} for the implementation of $\alpha$-Rank and Double Oracle.

References

[1] Krzysztof R Apt and Sunil Simon. A classification of weakly acyclic games. Theory and Decision, 78(4):501–524, 2015.
[2] Ayala Arad and Ariel Rubinstein. The 11-20 money request game: A level-k reasoning study. American Economic Review, 102(7):3561–73, 2012.
[3] David Balduzzi, Sebastien Racaniere, James Martens, Jakob Foerster, Karl Tuyls, and Thore Graepel. The mechanics of $n$-player differentiable games. In International Conference on Machine Learning (ICML), 2018.
[4] David Balduzzi, Karl Tuyls, Julien Perolat, and Thore Graepel. Re-evaluating evaluation. In Advances in Neural Information Processing Systems (NeurIPS), 2018.
[5] David Balduzzi, Marta Garnelo, Yoram Bachrach, Wojciech M Czarnecki, Julien Perolat, Max Jaderberg, and Thore Graepel. Open-ended learning in symmetric zero-sum games. In International Conference on Machine Learning (ICML), 2019.
[6] Marc G. Bellemare, Yavar Naddaf, Joel Veness, and Michael Bowling. The Arcade Learning Environment: An evaluation platform for general agents. J. Artif. Intell. Res., 47:253–279, 2013.
[7] Yoshua Bengio, Jérôme Louradour, Ronan Collobert, and Jason Weston. Curriculum learning. In International Conference on Machine Learning (ICML), 2009.
[8] Michele Berlingerio, Danai Koutra, Tina Eliassi-Rad, and Christos Faloutsos. NetSimile: A scalable approach to size-independent network similarity. arXiv preprint arXiv:1209.2684, 2012.
[9] Stefano Boccaletti, Vito Latora, Yamir Moreno, Martin Chavez, and D-U Hwang. Complex networks: Structure and dynamics. Physics reports, 424(4-5):175–308, 2006.
[10] Danail D Bonchev and Dennis Rouvray. Complexity in chemistry, biology, and ecology. Springer Science & Business Media, 2007.
[11] Emile Borel. La théorie du jeu et les équations intégrales noyau symétrique. Comptes rendus de l’Académie des Sciences, 173(1304-1308):58, 1921.
[12] Noam Brown and Tuomas Sandholm. Superhuman AI for heads-up no-limit poker: Libratus beats top professionals. Science, 359(6374):418–424, 2018.
[13] Murray Campbell, A Joseph Hoane Jr, and Feng-hsiung Hsu. Deep Blue. Artificial intelligence, 134(1-2):57–83, 2002.
[14] Xi Chen, Xiaotie Deng, and Shang-Hua Teng. Settling the complexity of computing two-player Nash equilibria. *Journal of the ACM (JACM)*, 56(3):1–57, 2009.

[15] Jeff Clune. AI-GAs: AI-generating algorithms, an alternate paradigm for producing general artificial intelligence. *arXiv preprint arXiv:1905.10985*, 2019.

[16] Karl Cobbe, Oleg Klimov, Chris Hesse, Taehoon Kim, and John Schulman. Quantifying generalization in reinforcement learning. In *International Conference on Machine Learning (ICML)*, 2019.

[17] Luciano da F Costa, Francisco A Rodrigues, Gonzalo Travieso, and Paulino Ribeiro Villas Boas. Characterization of complex networks: A survey of measurements. *Advances in physics*, 56(1):167–242, 2007.

[18] Rémi Coulom. Efficient selectivity and backup operators in Monte-Carlo tree search. In *International Conference on Computers and Games (ICCG)*, 2006.

[19] Jacob W Crandall, Mayada Oudah, Fatimah Ishowo-Oloko, Sherief Abdallah, Jean-François Bonnefon, Manuel Cebrian, Azim Shariff, Michael A Goodrich, and Iyad Rahwan. Cooperating with machines. *Nature communications*, 9(1):1–12, 2018.

[20] Wojciech Marian Czarnecki, Gauthier Gidel, Brendan Tracey, Karl Tuyls, Shayeegom Omidshafiei, David Balduzzi, and Max Jaderberg. Real world games look like spinning tops. *arXiv preprint arXiv:2004.09468*, 2020.

[21] Constantinos Daskalakis. On the complexity of approximating a Nash equilibrium. *ACM Transactions on Algorithms (TALG)*, 9(3):1–35, 2013.

[22] Constantinos Daskalakis, Paul W Goldberg, and Christos H Papadimitriou. The complexity of computing a Nash equilibrium. *SIAM Journal on Computing*, 39(1):195–259, 2009.

[23] Robyn M Dawes. Social dilemmas. *Annual review of psychology*, 31(1):169–193, 1980.

[24] Matthias Dehmer. *Structural analysis of complex networks*. Springer Science & Business Media, 2010.

[25] Debora Donato, Luigi Laura, Stefano Leonardi, and Stefano Millozzi. Large scale properties of the webgraph. *The European Physical Journal B*, 38(2):239–243, 2004.

[26] Jeffrey L Elman. Learning and development in neural networks: The importance of starting small. *Cognition*, 48(1):71–99, 1993.

[27] Arpad Elo. *The Rating of Chess players, Past and Present*. Ishi Press International, 1978.

[28] Richard Everett, Adam Cobb, Andrew Markham, and Stephen Roberts. Optimising worlds to evaluate and influence reinforcement learning agents. In *International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, 2019.

[29] Jacob Feldman. Minimization of boolean complexity in human concept learning. *Nature*, 407(6804):630–633, 2000.

[30] Carlos Florensa, David Held, Markus Wulfmeier, Michael Zhang, and Pieter Abbeel. Reverse curriculum generation for reinforcement learning. In *Conference on Robot Learning (CoRL)*, 2017.

[31] Drew Fudenberg and Lorenz A Imhof. Imitation processes with small mutations. *Journal of Economic Theory*, 131(1):251–262, 2006.

[32] Bertrand Georgeot, Olivier Giraud, and Dima L Shepelyansky. Spectral properties of the Google matrix of the world wide web and other directed networks. *Physical Review E*, 81(5):056109, 2010.

[33] Herbert Gintis. *Game Theory Evolving*. Princeton University Press, 2nd edition, 2009.
[34] Alex Graves, Marc G. Bellemare, Jacob Menick, Remi Munos, and Koray Kavukcuoglu. Automated curriculum learning for neural networks. In *International Conference on Machine Learning (ICML)*, 2017.

[35] Guy Hacohen and Daphna Weinshall. On the power of curriculum learning in training deep networks. In *International Conference on Machine Learning (ICML)*, 2019.

[36] Nikolaus Hansen. The CMA evolution strategy: A comparing review. In *Towards a new evolutionary computation*, pages 75–102. Springer, 2006.

[37] Nikolaus Hansen, Youhei Akimoto, and Petr Baudis. CMA-ES/pycma on Github. Zenodo, DOI:10.5281/zenodo.2559634, February 2019. URL https://doi.org/10.5281/zenodo.2559634

[38] Garrett Hardin. The tragedy of the commons. *Science*, 162(3859):1243–1248, 1968.

[39] Christoph Hauert, Arne Traulsen, Hannelore Brandt, Martin A Nowak, and Karl Sigmund. Via freedom to coercion: The emergence of costly punishment. *Science*, 316(5833):1905–1907, 2007.

[40] Ricardo Hausmann, César A Hidalgo, Sebastián Bustos, Michele Coscia, Alexander Simoes, and Muhammed A Yildirim. The atlas of economic complexity: Mapping paths to prosperity. MIT Press, 2014.

[41] Johannes Heinrich, Marc Lanctot, and David Silver. Fictitious self-play in extensive-form games. In *International Conference on Machine Learning (ICML)*, 2015.

[42] José Hernández-Orallo. *The Measure of All Minds: Evaluating Natural and Artificial Intelligence*. Cambridge University Press, 2017.

[43] José Hernández-Orallo. Evaluation in artificial intelligence: From task-oriented to ability-oriented measurement. *Artif. Intell. Rev.*, 48(3):397–447, 2017.

[44] José Hernández-Orallo, Peter A. Flach, and César Ferri. A unified view of performance metrics: translating threshold choice into expected classification loss. *J. Mach. Learn. Res.*, 13:2813–2869, 2012.

[45] Álvaro Francisco Huertas-Rosero. A cartography for 2x2 symmetric games. In *III Colombian Congress and the I International Andean conference on Operations Research (CCIO)*, 2004.

[46] Lorens A Imhof, Drew Fudenberg, and Martin A Nowak. Evolutionary cycles of cooperation and defection. *Proc Natl Acad Sci USA*, 102(31):10797–10800, 2005.

[47] J. J. Hofbauer and K. Sigmund. Evolutionary games and population dynamics. *Cambridge University Press*, 1998.

[48] Max Jaderberg, Wojciech M Czarnecki, Iain Dunning, Luke Marris, Guy Lever, Antonio Garcia Castaneda, Charles Beattie, Neil C Rabinowitz, Ari S Morcos, Avraham Ruderman, et al. Human-level performance in 3D multiplayer games with population-based reinforcement learning. *Science*, 364 (6443):859–865, 2019.

[49] Niels Justesen, Ruben Rodriguez Torrado, Philip Bontrager, Ahmed Khalifa, Julian Togelius, and Sebastian Risi. Illuminating generalization in deep reinforcement learning through procedural level generation. In *NeurIPS Deep Reinforcement Learning Workshop*, 2018.

[50] Jongkwang Kim and Thomas Wilhelm. What is a complex graph? *Physica A: Statistical Mechanics and its Applications*, 387(11):2637–2652, 2008.

[51] Danai Koutra, Joshua T Vogelstein, and Christos Faloutsos. Deltacon: A principled massive-graph similarity function. In *International Conference on Data Mining*, 2013.

[52] Kai A Krueger and Peter Dayan. Flexible shaping: How learning in small steps helps. *Cognition*, 110 (3):380–394, 2009.
[53] Marc Lanctot, Vinicius Zambaldi, Audrunas Gruslys, Angeliki Lazaridou, Karl Tuyls, Julien Pérolat, David Silver, and Thore Graepel. A unified game-theoretic approach to multiagent reinforcement learning. In *Advances in Neural Information Processing Systems (NeurIPS)*, 2017.

[54] Marc Lanctot, Edward Lockhart, Jean-Baptiste Lespiau, Vinicius Zambaldi, Satyaki Upadhyay, Julien Pérolat, Sriram Srinivasan, Finbarr Timbers, Karl Tuyls, Shayegan Omidshafiei, Daniel Hennes, Dustin Morrill, Paul Muller, Timo Ewalds, Ryan Faulkner, János Kramár, Bart De Vylder, Brennan Saeta, James Bradbury, David Ding, Sebastian Borgeaud, Matthew Lai, Julian Schrittwieser, Thomas Anthony, Edward Hughes, Ivo Danihelka, and Jonah Ryan-Davis. OpenSpiel: A framework for reinforcement learning in games. *arXiv preprint arXiv:1908.09453*, 2019.

[55] Joel Z Leibo, Vinicius Zambaldi, Marc Lanctot, Janusz Marecki, and Thore Graepel. Multi-agent reinforcement learning in sequential social dilemmas. In *International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, 2017.

[56] Joel Z Leibo, Edward Hughes, Marc Lanctot, and Thore Graepel. Autocurricula and the emergence of innovation from social interaction: A manifesto for multi-agent intelligence research. Technical report, DeepMind, 2019.

[57] Annick Lesne. Complex networks: From graph theory to biology. *Letters in Mathematical Physics*, 78(3):235–262, 2006.

[58] Wim BG Liebrand. A classification of social dilemma games. *Simulation & Games*, 14(2):123–138, 1983.

[59] Siqi Liu, Guy Lever, Nicholas Heess, Josh Merel, Saran Tunyasuvunakool, and Thore Graepel. Emergent coordination through competition. In *International Conference on Learning Representations (ICLR)*, 2019.

[60] Marlos C. Machado, Marc G. Bellemare, Erik Talvitie, Joel Veness, Matthew J. Hausknecht, and Michael Bowling. Revisiting the Arcade Learning Environment: Evaluation protocols and open problems for general agents. *J. Artif. Intell. Res.*, 61:523–562, 2018.

[61] Fragkiskos D Malliaros and Michalis Vazirgiannis. Clustering and community detection in directed networks: A survey. *Physics Reports*, 533(4):95–142, 2013.

[62] John McCarthy. AI as sport. *Science*, 276(5318):1518–1519, 1997.

[63] H Brendan McMahan, Geoffrey J Gordon, and Avrim Blum. Planning in the presence of cost functions controlled by an adversary. In *International Conference on Machine Learning (ICML)*, 2003.

[64] Vahab S Mirrokni and Adrian Vetta. Convergence issues in competitive games. In *Approximation, randomization, and combinatorial optimization. algorithms and techniques*, pages 183–194. Springer, 2004.

[65] Bojan Mohar. The Laplacian spectrum of graphs. *Graph theory, combinatorics, and applications*, 2 (871-898):12, 1991.

[66] Paul Muller, Shayegan Omidshafiei, Mark Rowland, Karl Tuyls, Julien Perolat, Siqi Liu, Daniel Hennes, Luke Marris, Marc Lanctot, Edward Hughes, Zhe Wang, Guy Lever, Nicolas Heess, Thore Graepel, and Remi Munos. A generalized training approach for multiagent learning. In *International Conference on Learning Representations (ICLR)*, 2020.

[67] John F Nash. Equilibrium points in $n$-person games. *Proc Natl Acad Sci U S A*, 36(1):48–49, 1950.

[68] Allen Newell and Herbert A Simon. Computer science as empirical inquiry: Symbols and search. *Communications of the ACM*, 1976.

[69] Martin A. Nowak and Karl Sigmund. Evolutionary dynamics of biological games. *Science*, 303(5659):793–799, 2004.
[70] Martin A Nowak, Akira Sasaki, Christine Taylor, and Drew Fudenberg. Emergence of cooperation and evolutionary stability in finite populations. *Nature*, 428(6983):646–650, 2004.

[71] Mancur Olson. *The Logic of Collective Action*. Harvard University Press, 1974.

[72] Shayegan Omidshafiei, Christos Papadimitriou, Georgios Piliouras, Karl Tuyls, Mark Rowland, Jean-Baptiste Lespiau, Wojciech M Czarnecki, Marc Lanctot, Julien Perolat, and Rémi Munos. α-Rank: Multi-agent evaluation by evolution. *Scientific Reports*, 9, 2019.

[73] OpenAI, Christopher Berner, Greg Brockman, Brooke Chan, Vicki Cheung, Przemyslaw Dębiak, Christy Dennison, David Farhi, Quirin Fischer, Shariq Hashme, Chris Hesse, Rafał Józefowicz, Scott Gray, Catherine Olsson, Jakub Pachocki, Michael Petrov, Henrique Pondé de Oliveira Pinto, Jonathan Raiman, Tim Salimans, Jeremy Schlatter, Jonas Schneider, Szymon Sidor, Ilya Sutskever, Jie Tang, Filip Wolski, and Susan Zhang. Dota 2 with large scale deep reinforcement learning. *arXiv preprint arXiv:1912.06680*, 2019.

[74] Lawrence Page, Sergey Brin, Rajeev Motwani, and Terry Winograd. The PageRank citation ranking: Bringing order to the web. Technical report, Stanford InfoLab, 1999.

[75] Georgios A Pavlopoulos, Maria Secrier, Charalampos M Moschopoulos, Theodoros G Soldatos, Sophia Kossida, Jan Aerts, Reinhard Schneider, and Pantelis G Bagos. Using graph theory to analyze biological networks. *BioData mining*, 4(1):10, 2011.

[76] Duc Truong Pham, Stefan S Dimov, and Chi D Nguyen. Selection of K in K-means clustering. *Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science*, 219(1):103–119, 2005.

[77] Steve Phelps, Simon Parsons, and Peter McBurney. An evolutionary game-theoretic comparison of two double-auction market designs. In *AAMAS Workshop on Agent-Mediated Electronic Commerce: Theories for and Engineering of Distributed Mechanisms and Systems*, 2004.

[78] John Platt. Social traps. *American psychologist*, 28(8):641, 1973.

[79] Anatol Rapoport and Melvin Guyer. A taxonomy of 2×2 games. *General systems*, 11:203–214, 1966.

[80] David Robinson and David Goforth. *The topology of the 2x2 games: A new periodic table*, volume 3. Psychology Press, 2005.

[81] Lior Rokach and Oded Maimon. Clustering methods. In *Data mining and knowledge discovery handbook*, pages 321–352. Springer, 2005.

[82] Mark Rowland, Shayegan Omidshafiei, Karl Tuyls, Julien Perolat, Michal Valko, Georgios Piliouras, and Remi Munos. Multiagent evaluation under incomplete information. In *Advances in Neural Information Processing Systems (NeurIPS)*, 2019.

[83] Arthur L Samuel. Programming computers to play games. In *Advances in Computers*, volume 1, pages 165–192. Elsevier, 1960.

[84] W.H. Sandholm. *Population Games and Evolutionary Dynamics*. Economic Learning and Social Evolution. MIT Press, 2010.

[85] A Sanfeliu and King-Sun Fu. A distance measure between attributed relational graphs for pattern recognition. *IEEE Transactions on Systems, Man, and Cybernetics, SMC-13(3)*, 1983.

[86] Terence D Sanger. Neural network learning control of robot manipulators using gradually increasing task difficulty. *IEEE transactions on Robotics and Automation*, 10(3):323–333, 1994.

[87] Fernando P Santos, Francisco C Santos, and Jorge M Pacheco. Social norm complexity and past reputations in the evolution of cooperation. *Nature*, 555(7695):242–245, 2018.
[88] Venu Satuluri and Srinivasan Parthasarathy. Symmetrizations for clustering directed graphs. In *International Conference on Extending Database Technology*. ACM, 2011.

[89] Jonathan Schaeffer. A gamut of games. *AI Magazine*, 22(3):29–29, 2001.

[90] John Scott. Social network analysis. *Sociology*, 22(1):109–127, 1988.

[91] Sven Van Segbroeck, Jorge M. Pacheco, Tom Lenaerts, and Francisco C. Santos. Emergence of fairness in repeated group interactions. *Physical Review Letters*, 108:158104, 2012.

[92] Jianbo Shi and Jitendra Malik. Normalized cuts and image segmentation. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 22(8):888–905, 2000.

[93] David Silver, Aja Huang, Chris J. Maddison, Arthur Guez, Laurent Sifre, George van den Driessche, Julian Schrittwieser, Ioannis Antonoglou, Veda Panneershelvam, Marc Lanctot, Sander Dieleman, Dominik Grewe, John Nham, Nal Kalchbrenner, Ilya Sutskever, Timothy Lillicrap, Madeleine Leach, Koray Kavukcuoglu, Thore Graepel, and Demis Hassabis. Mastering the game of Go with deep neural networks and tree search. *Nature*, 529(7587):484–489, 2016.

[94] David Silver, Thomas Hubert, Julian Schrittwieser, Ioannis Antonoglou, Matthew Lai, Arthur Guez, Marc Lanctot, Laurent Sifre, Dharshan Kumaran, Thore Graepel, Timothy Lillicrap, Karen Simonyan, and Demis Hassabis. A general reinforcement learning algorithm that masters chess, shogi, and Go through self-play. *Science*, 362(6419):1140–1144, 2018.

[95] Satinder P Singh, Michael J Kearns, and Yishay Mansour. Nash convergence of gradient dynamics in general-sum games. In *Conference on Uncertainty in Artificial Intelligence (UAI)*, 2000.

[96] Andrea Tacchella, Matthieu Cristelli, Guido Caldarelli, Andrea Gabrielli, and Luciano Pietronero. Economic complexity: Conceptual grounding of a new metrics for global competitiveness. *Journal of Economic Dynamics and Control*, 37(8):1683–1691, 2013.

[97] Christine Taylor, Drew Fudenberg, Akira Sasaki, and Martin A Nowak. Evolutionary game dynamics in finite populations. *Bulletin of mathematical biology*, 66(6):1621–1644, 2004.

[98] Robert Tibshirani, Guenther Walther, and Trevor Hastie. Estimating the number of clusters in a data set via the gap statistic. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 63(2):411–423, 2001.

[99] Arne Traulsen, Martin A. Nowak, and Jorge M. Pacheco. Stochastic dynamics of invasion and fixation. *Phys. Rev. E*, 74:011909, 2006.

[100] Arne Traulsen, Jorge M. Pacheco, and Loren A. Imhof. Stochasticity and evolutionary stability. *Phys. Rev. E*, 74:021905, 2006.

[101] Anton Tsitsulin, Davide Mottin, Panagiotis Karras, Alexander Bronstein, and Emmanuel Müller. NetLSD: Hearing the shape of a graph. In *International Conference on Knowledge Discovery & Data Mining*, 2018.

[102] Alan M Turing. Digital computers applied to games. In *Faster than thought: A symposium on digital computing machines*. Pitman Publishing, 1953.

[103] Karl Tuyls, Julien Perolat, Marc Lanctot, Joel Z Leibo, and Thore Graepel. A generalised method for empirical game theoretic analysis. In *International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, 2018.

[104] Karl Tuyls, Julien Pérolat, Marc Lanctot, Edward Hughes, Richard Everett, Joel Z. Leibo, Csaba Szepesvári, and Thore Graepel. Bounds and dynamics for empirical game theoretic analysis. *Auton. Agents Multi Agent Syst.*, 34(1):7, 2020.
[105] Valeriu Ungureanu and Ana Botnari. Nash equilibria sets in mixed extended $2 \times 3$ games. *Computer Science Journal of Moldova*, 13(2):38, 2005.

[106] Hadrien Van Lierde. Spectral clustering algorithms for directed graphs. Master's thesis, Universite Catholique de Louvain., 2015.

[107] Maarten Van Steen. Graph theory and complex networks: An introduction. 2010.

[108] Vítor V Vasconcelos, Fernando P Santos, Francisco C Santos, and Jorge M Pacheco. Stochastic dynamics through hierarchically embedded Markov chains. *Physical Review Letters*, 118(5):058301, 2017.

[109] Carl Veller and Laura K Hayward. Finite-population evolution with rare mutations in asymmetric games. *Journal of Economic Theory*, 162:93–113, 2016.

[110] Oriol Vinyals, Igor Babuschkin, Wojciech M. Czarnecki, Michaël Mathieu, Andrew Dudzik, Junyoung Chung, David H. Choi, Richard Powell, Timo Ewalds, Petko Georgiev, Junhyuk Oh, Dan Horgan, Manuel Kroiss, Ivo Danihelka, Aja Huang, Laurent Sifre, Trevor Cai, John P. Agapiou, Max Jaderberg, Alexander S. Vezhnevets, Rémi Leblond, Tobias Pohlen, Valentin Dalilard, David Budden, Yury Sulsky, James Molloy, Tom L. Paine, Caglar Gulcehre, Ziyu Wang, Tobias Pfaff, Yuhuai Wu, Roman Ring, Dani Yogatama, Dario Wünsch, Katrina McKinney, Oliver Smith, Tom Schaul, Timothy Lillicrap, Koray Kavukcuoglu, Demis Hassabis, Chris Apps, and David Silver. Grandmaster level in StarCraft II using multi-agent reinforcement learning. *Nature*, 575(7782):350–354, 2019.

[111] Michael S Vitevitch. What can graph theory tell us about word learning and lexical retrieval? *Journal of Speech, Language, and Hearing Research*, 2008.

[112] W. E. Walsh, R. Das, G. Tesauro, and J.O. Kephart. Analyzing complex strategic interactions in multi-agent games. In *AAAI Workshop on Game Theoretic and Decision Theoretic Agents*, 2002.

[113] Rui Wang, Joel Lehman, Jeff Clune, and Kenneth O Stanley. Paired open-ended trailblazer (POET): Endlessly generating increasingly complex and diverse learning environments and their solutions. *arXiv preprint arXiv:1901.01753*, 2019.

[114] Xiaofeng Wang and Tuomas Sandholm. Reinforcement learning to play an optimal Nash equilibrium in team Markov games. In *Advances in Neural Information Processing Systems (NeurIPS)*, 2003.

[115] Stanley Wasserman and Katherine Faust. *Social network analysis: Methods and applications*. Cambridge University Press, 1994.

[116] J.W. Weibull. *Evolutionary Game Theory*. MIT Press, 1995.

[117] Daphna Weinshall, Gad Cohen, and Dan Amir. Curriculum learning by transfer learning: Theory and experiments with deep networks. In *International Conference on Machine Learning (ICML)*, 2018.

[118] Michael P Wellman. Methods for empirical game-theoretic analysis. In *AAAI Conference on Artificial Intelligence*, 2006.

[119] M.P. Wellman, T.H. Kim, and Q. Duong. Analyzing incentives for protocol compliance in complex domains: A case study of introduction-based routing. In *Proceedings of the 12th Workshop on the Economics of Information Security*, 2013.

[120] James R Wright and Kevin Leyton-Brown. A formal separation between strategic and nonstrategic behavior. *arXiv preprint arXiv:1812.11571*, 2018.

[121] Georgios N Yannakakis and Julian Togelius. *Artificial intelligence and games*. Springer, 2018.
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Author contributions

SO, KT, and FCS conceptualized the graph-based framework used in the paper. SO implemented and generated the experimental results (spectral analysis of response graphs, clusterings, landscape of games, procedural games generation). WMC devised and implemented the policy sampling scheme for real world games, and generated the payoff tables for the games listed in Figure 1. All authors contributed actively to discussions, analysis, writing, and review of the paper.

Competing interests

The authors declare no competing interests.

Additional information

Materials & correspondence Material requests and correspondence should be requested to Shayegan Omidshafiei (somidshafiei@google.com).
Related Work

This paper largely focuses on what kinds of interactions and environments present interesting challenges for artificial agents. As such, this work sits at the intersection of several different research disciplines, including game theory, machine learning, and network science.

The central question revolving around the interestingness of environments for artificial learning agents has a long history in machine learning. Initial work concerned the use of curricula of tasks in training neural networks via supervised learning. This continues to be an active field of research to this day, with many new methods for curriculum learning being developed to match the increasingly intricate architectures used within deep learning, and the increasingly wide range of applications for deep neural networks, such as deep reinforcement learning and procedural environment generation to the forefront of curriculum learning.

Recently, open-ended learning via multiagent interaction and procedural environment generation has brought open-ended learning to the forefront of curriculum learning. This work contributes to this line of inquiry by framing the intrinsic interestingness of games as a driving factor in open-ended multiagent learning. While equilibrium computation in many classes of games is PPAD-complete (thus generally considered intractable), and while the computational complexity results in this work focus on the two-player zero-sum case, it is worth mentioning that our methods also directly apply to many-player general-sum settings (as exemplified by the analysis of the general-sum 11-20 game in the main text, and of many-player variants of Kuhn Poker in Figure 12).

Several prior works have analyzed game-theoretic response graphs from the perspective of agent evaluation and more generally there is a wide literature concerning analysis of directed graphs across a range of domains and disciplines. Additionally, there is an established line of work in game theory that seeks a discrete classification of games based on payoff tables. The space of 2x2 normal-form games have been classified into groups with restrictions, such as ordering the payout through topological and graph-theoretic techniques. More recent work has introduced a framework for binary classification of agent behaviors (e.g., as strategic and non-strategic) in simultaneous-move normal form games.

Social dilemma games (such as Prisoners Dilemma, Chicken, or Hoarding), categorized by Dawes, Liebrand, are a subset of games encapsulating the trade-offs found in social dilemmas. Social dilemmas such as the tragedy of the commons show the potential conflict between individual and group self-interest. Early work in social dilemmas showed how the dilemmas can arise from reinforced behavior. More recently, reinforcement learning (RL) has been applied to sequential social dilemmas where strategic decisions to cooperate or defect were shown to coincide with the learning of policies. Agent behavior was also shown to depend on factors such as the relative difficulties of learning policies and in the case of cooperation, possibly non-occurring coordination sub-problems. RL has also been used to train agents which cooperate with humans across a variety of two-player repeated games.

In addition to analysis of individual graphs, there has been much work on the analysis and clustering of collections of graphs in a variety of contexts. The graph-based analysis conducted herein draws on these works (e.g., using several local and global graph features and distributions to characterize our games, as in Berlingerio et al.). However, our approach focuses specifically on games, exploiting several measures related to the complexity of games, enabling navigation of their underlying landscape and subsequent generation of new games of interest.

A wide-spread strategy comparison method in machine learning is the Elo rating which ranks agents against one another in a purely transitive manner. Balduzzi et al. showed the importance of intransitive strategies when comparing agents, leading to the so called multidimensional Elo model, modeling more complex relations between agents. Finally, in a line of work related to training of agents, Policy-Space Response Oracles (PSRO) and Rectified PSRO lead to learning a collection of possibly intransitive strategies, but not to directly analyze them, as done in this paper.
Methods: Additional details

This section provides additional discussions and exposition of methods.

Policy sampling scheme

The policy sampling scheme we use for large, real-world games follows the policy space coverage procedure first specified by Czarnecki et al.\textsuperscript{20}, and is detailed as follows:

1. We use a tree search algorithm, Alpha-Beta\textsuperscript{68} with varying tree depth limits \(d\), and seeds \(s\). This involves running the Alpha-Beta algorithm to depth \(d\), and using random actions with seed \(s\) thereafter (i.e., if the game does not terminate). This yields policies of varying transitive strengths (controlled by depth \(d\)), with a range of related policies per depth (controlled by seed \(s\)). This also covers the case of a purely random policy when \(d\) is set to 0 (with seed \(s\) controlling the randomness).

2. We repeat the same procedure with negated game payoffs, thus also covering the space of policies that seek to lose the original game.

3. To further expand the policy space, we also define an augmented Alpha-Beta search, which assumes that branches of the game tree with depth beyond \(d\) have a value of 0. We likewise run this variant with negated payoffs as well.

4. To cover the policy space for more difficult games, we further augment this sampling strategy by using MCTS\textsuperscript{18} on each game, with \(k\) simulations and varying seeds \(s\).

For each variant of Alpha-Beta detailed above, we use depth parameters \(d \in \{1, \ldots, 9\}\). For MCTS, we use \(k \in \{10, 100, 1000\}\) simulations. For all algorithms, we also sweep over seeds \(s \in \{1, \ldots, 50\}\). While this sampling procedure is a heuristic, it produces a range of policies with varying degrees of transitive and intransitive relations, and thus provide a useful approximation of the underlying game.

Description of games analyzed

We provide an overview of the games analyzed (noting that we omit descriptions of 11-20, AlphaGo, MuJoCo soccer, Blotto, and AlphaStar League, as they are detailed in the main text). As our methods operate on \(\alpha\)-Rank response graphs, they apply to many-player general-sum games. For the specific instances of two-player zero-sum games analyzed here, we symmetrize payoffs and standardize them such that \(M \in [-1,1]\).

**Redundant Rock–Paper–Scissors** We modify the standard Rock–Paper–Scissors payoffs, \(M_{RPS}\), by duplicating the first strategy (Rock), yielding the payoffs for the redundant variant, \(M_{RRPS}\).

\[
M_{RPS} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \quad M_{RRPS} = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix}.
\] (5)

**Disc game** The Disc game is a cyclical game, defined as a differentiable generalization of the standard game of Rock–Paper–Scissors\textsuperscript{8}. We construct the Disc game payoffs as in Czarnecki et al.\textsuperscript{20}, by first uniformly sampling 1000 points, \(\{S_i\}_{i=1000}\), in the unit circle, subsequently defining payoffs,

\[
M(i,j) = S_i^T \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} S_j.
\] (6)

**Elo games and noisy variants** The variety of Elo games essentially correspond to the multidimensional Elo model detailed in the main paper, with the intransitive components removed, and payoffs rescaled. Noisy variants of these Elo games are generated via specifying a noise parameter \(\sigma^2\), adding zero-mean normally distributed noise of the specified variance to the non-noisy Elo payoff table, thus yielding noisy payoffs \(M_{\sigma^2}\). Finally, we symmetrize these payoffs, yielding final payoffs \(\tilde{M} = M_{\sigma^2} - M_{\sigma^2}^T\).
Games in motivating examples The variants of transitively-structured games in the motivating examples are Elo games, as specified above. The cyclical games are $N \times N$ mElo games of rank $k = 1$, with all transitive $r$ set to 0, and intransitive components specified as follows: $c_{0,N-2,0} = [0, 1, \ldots, N - 2]$, $c_{0,N-2,1} = [N - 2, N - 1, \ldots, 0]$, and $c_{N-1,2} = [-1, 1]$ The random-structured games are mElo games with rank $k = 3$, and transitive and intransitive parameters i.i.d. sampled from $N(0, 1)$.

Transitive game The Transitive Game payoffs are simply set to +1 for the upper-triangle, −1 for the lower-triangle, and 0 across the diagonal.

Random Game of Skill The Random Game of Skill is a $1000 \times 1000$ game, generated as defined by Czarnecki et al. Specifically, payoffs are $M(i,j) = 0.5(W_{ij} - W_{ji}) + S_i - S_j$, where $W_{ij}$, $W_{ji}$, $S_i$, and $S_j$ are sampled from $N(0, 1)$. In the Normal Bernoulli Game, parameters $S_i$ and $S_j$ are sampled likewise, while $W_{ij}$ and $W_{ji}$ are sampled from $U(0, 1)$ and payoffs are specified as $M(i,j) = W_{ij} - W_{ji} + S_i - S_j$.

3-move parity game The parity game is generated per the definition in Czarnecki et al. 20.

Real-world games We use the OpenSpiel implementations of the following games: Tic–Tac–Toe, Hex (board size=3), Quoridor (board size=3), Quoridor (board size=4), Go (board size=3), Go (board size=4), Connect Four, Kuhn Poker. For Go, we use a Komi (first-move advantage) of 6.5 points. With the exception of Kuhn Poker (which has only 12 information states, and fully enumerable policy space), we use the policy sampling scheme detailed above for this collection of games.

Additional results

This section provides additional results, including response graphs and experiments conducted for generating the landscapes in the main text, which could not be included there due to figure limits.

Motivating examples: results for randomly-generated games

Figure 11 illustrates a sample of response graphs for generated games of random structure, as discussed in the Motivating Examples section of the main text.

Additional response graph analysis

For completeness, Figures 12 to 17 present the response graph-based analysis for the additional games considered in the main text. Figure 12 is of particular note here, as it exemplifies the application of the methodology to asymmetric, many-player games. Specifically, the empirical games constructed here correspond to agents trained via extensive-form fictitious play (XFP) in 2-, 3-, and 4-player variants of Kuhn Poker.

Additional complexity results

Figure 18 provides an overview of additional response graph-based measures, in comparison to the normalized number of iterations required to solve each of the games considered in the main text.

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Figure 11: Results for randomly-generated games.
Figure 12: Additional response graph analysis results I. Specifically, these results pertain to Kuhn Poker agents trained via extensive-form fictitious play (XFP)\textsuperscript{41} with empirical games constructed as detailed in Omidshafiei et al.\textsuperscript{72}
Figure 13: Additional response graph analysis results II.
Figure 14: Additional response graph analysis results III.
Figure 15: Additional response graph analysis results IV.
Figure 16: Additional response graph analysis results V.
Figure 17: Additional response graph analysis results VI.
Figure 18: Response graph complexity vs. computational complexity of solving associated games. Each figure plots a respective measure of graph complexity against the normalized number of iterations needed to solve the associated game via the double oracle algorithm (with normalization done with respect to the total number of strategies in each underlying game). Note that mean node-wise in- and out-degrees (in (c) and (h) respectively) match across the games here due to the degree sum formula; other distributional statistics, however, do not necessarily match across in- and out-degrees, as evident above.