Asymptotic security analysis of teleportation based quantum cryptography

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Abstract. We prove that the teleportation based quantum cryptography protocol presented in [Opt. Commun. 283, 184 (2010)], which is built using only orthogonal states encoding the classical bits that are teleported from Alice to Bob, is asymptotically secure against all types of individual and collective attacks. We then investigate modifications to that protocol leading to greater secret-key rates and to security against coherent attacks. In other words, we show an unconditional secure quantum key distribution protocol that does not need non-orthogonal quantum states to encode the bits of the secret key sent from Alice to Bob. We also revisit the security proof of the BB84 protocol by exploring the non-uniqueness of the Schmidt decomposition of its entanglement-based representation. This allows us to arrive at a secure transmission of the key for a slightly greater quantum bit error rate (quantum communication channel’s noise) when compared to its standard security analysis.

1 Introduction

In today’s world, where the amount of information produced, stored, and transmitted has increased to an unprecedented level, it is of utmost importance the construction of ways to store and transmit information in a secure way, where only authorized parties are able to access the content of the stored and transmitted data [1]. Cryptography is an interdisciplinary field of research whose
main goal is the construction of devices and protocols such that the transmission of information can be made in a secure way, preventing an eavesdropper (Eve) from deciphering the message sent from one party (Alice) to another one (Bob).

The standard solution currently employed by Alice to cipher a message, such that only Bob can decipher it, is based on public key protocols [2]. The security of those protocols is based on the assumption that there is no efficient algorithm to factor huge prime numbers. If this were the case, this algorithm could be adapted to break the security of all public key cryptography protocols. Although there is no classical algorithm that can efficiently factor prime numbers of any size, this is not the case quantumly [3]. The proof that a quantum computer can factor efficiently large prime numbers was the main trigger that boosted the field of quantum cryptography, which had already offered an alternative solution to secure communication [4] about 10 years before Peter Shor’s work [3].

The protocol presented by Bennett and Brassard in 1984 [4], later called the BB84 protocol, was the quantum solution to the key distribution problem. The key distribution problem is basically the classical impossibility to be 100% sure that only two parties, and no one else, agree on a random sequence of bits that is sent from one party to another. The transmission of the key from Alice to Bob via classical means can in principle be monitored by a clever and powerful enough Eve, who copies the random bits during their transmission without ever being detected by Alice and Bob. This is the scenario dictated by classical physics, in which copying or cloning of bits is always possible. However, if we use non-orthogonal quantum states to encode the bits to be transmitted from Alice to Bob, and if we believe in the correctness of quantum mechanics, no eavesdropper can tamper with the quantum key distribution without being discovered by Alice and Bob [4]. The quantum solution to the key distribution problem brought back to the table classical private key protocols, the one-time pad for example [1], which can now be made secure if the key distribution needed to its implementation is made quantumly using the BB84 protocol. Moreover, private key protocols, once a secure key is established between Alice and Bob, will not become insecure with the advent of a quantum computer. This is the main reason that led to the boom of research and development in the field of quantum cryptography in the last two decades [5,6,7], culminating with commercially viable quantum cryptography solutions [8,9,10].

The most important concept behind all standard quantum key distribution schemes is the use of non-orthogonal quantum states to encode the classical bits 0 and 1 [11]. Those bits are randomly generated by Alice and randomly encoded into non-orthogonal quantum states (qubits) that are sent to Bob. By properly preparing and measuring those qubits, Alice and Bob are able to share a secret random string of 0’s and 1’s, i.e., the secret key needed to the implementation of private key cryptography protocols [1]. And since non-

1. Alice and Bob must also share an authenticated classical channel, which can be totally insecure, to realize the key distribution.
orthogonal quantum states cannot be cloned \cite{12,13}, the transmission of a random sequence of bits using this strategy is secure according to the laws of quantum mechanics.

What happens if the qubits encoding the bits of the secret key are not physically sent but rather teleported from Alice to Bob \cite{14}? On the one hand we can think of this process as a substitution to the direct sending of the qubits from Alice to Bob through a physical channel. In this case, the teleportation of qubits is an alternative way to send to Bob the quantum information contained in those qubits, playing no direct role in the generation of the secret key. Also, no qualitative changes are made to the quantum key distribution protocols when we use the quantum teleportation protocol either as a substitute to the direct transmission of the qubits or to increase the physical distance between Alice and Bob in which they can still establish a secret key \cite{15}.

On the other hand, we can use the quantum teleportation protocol as the main ingredient to the construction of a quantum key distribution protocol, such that the teleportation protocol plays an active role in the generation of the key. This active role of the teleportation protocol in the generation of the key is the main feature of the quantum key distribution protocol originally presented in Ref. \cite{16}, from now on called the GR10 protocol. In addition to that, the GR10 protocol has two other interesting features. First, it works securely even when Alice and Bob use partially entangled states to implement the quantum teleportation protocol \cite{17,18,19,20,21,22,23,24,25}. Second, the classical bits of the secret key can be encoded into two orthogonal states that are subsequently teleported from Alice to Bob. This last feature is in contrast to standard quantum key distribution schemes, where the use of non-orthogonal quantum states to encode the classical bits of the key is mandatory. \cite{26} See Refs. \cite{28,29,30} for other quantum key distribution schemes where only orthogonal states are used in its execution.

Although so far no security loophole was found in the GR10 protocol, its complete security analysis is lacking. One of the main goals of this work is to fill this gap, proving that the GR10 protocol as originally conceived is secure against all types of individual and collective attacks. This is shown in Sec. \ref{sec:gr10} where we also review how the GR10 protocol works and give the main qualitative reasons of why it is indeed secure. We also discuss how the GR10 protocol can be modified in order to achieve asymptotic unconditional security, i.e., security against all types of attacks allowed by the laws of Physics. But before tackling the security analysis of the GR10 protocol, we revisit in greater detail the security analysis of the BB84 protocol in Sec. \ref{sec:bb84} presenting the main tools needed to handle the security analysis of the GR10 protocol. We also show that by using the non-uniqueness of the Schmidt decomposition, we can write the purification of the BB84 protocol in such a way that it is possible to prove its security for quantum bit error rates that are higher than those predicted by its standard security analysis. In Sec. \ref{sec:asymptotic} we modify the original GR10 protocol,

\footnote{We can also use maximally entangled states to create a secret key shared by Alice and Bob without using teleportation and without sending qubits encoding classical bits from Alice to Bob \cite{26,27}.}
transforming it into a deterministic protocol at the cost of a lower secret-key rate. And finally, in Sec. 5 we give our concluding remarks.

2 Revisiting the security analysis of the BB84 protocol

2.1 The BB84 protocol

Before we revisit the security analysis of the BB84 protocol [31, 32], we give a short description of how it basically works. It goes as follows:

(i) In the first place, Alice and Bob agreed beforehand that the encoding of the classical bits sent from Alice to Bob is randomly done using two non-orthogonal quantum states for each bit. For instance, the bit 0 is either encoded into the qubit $|0\rangle$ or $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and the bit 1 into $|1\rangle$ or $|\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$.

(ii) After such agreement, Alice randomly prepares her qubit in one of the four states described above and sends it to Bob. After receiving the qubit, Bob measures it randomly using either the z-basis, spanned by the states \{\(|0\rangle, |1\rangle\)}, or the x-basis, spanned by \{\(|+\rangle, |\rangle\)\}. Whenever Bob measurement results are $|0\rangle$ or $|+\rangle$, he assumes Alice sent the bit 0, while whenever his measurement results are $|1\rangle$ or $|\rangle$, he assigns the bit value 1. Step (ii) is repeated $N$ times.

(iii) After Bob has finished all his measurements on the $N$ qubits, Alice and Bob start a conversation over an authenticated public classical channel that can be fully insecure. In this step Alice reveals the basis used to prepare the qubits encoding each bit sent to Bob and he reveals the basis he used to implement his measurements. They discard the cases where different basis were used and keep the instances in which they used the same basis. Roughly $N/2$ cases are discarded and $N/2$ are kept. Those $N/2$ remaining shared bits are usually called the raw key and this process of selecting the useful part of the shared bits from what is not useful is called sifting. Note that the sifting step is crucial. According to the laws of quantum mechanics, and if there is no noise or eavesdropping, Alice and Bob will agree 100% of the time on the value of a given bit whenever Bob measures his qubit in the same basis used by Alice to prepare it. Had they insisted on keeping the cases where different basis were employed, even in principle they could not always agree on the values of the bits, making a mistake 50% of the time.

(iv) Alice and Bob then reveal a part of the bits of their raw key, a random sample of the $N/2$ remaining cases. This sample of bits is used to check for the presence of an eavesdropper and it usually consists of $N/4$ bits, half the size of the raw key [4]. If for every pair of bits in the disclosed sample Alice’s and Bob’s bits are the same, Alice and Bob assume no eavesdropping happened and the remaining $N/4$ bits are used as a secret key. If, as is always the case in practice, some of the bits disclosed by Alice and Bob are not the same, they use the disclosed information to estimate
the error rate of the quantum communication channel. This error rate is always assumed to be caused by Eve’s action, whether or not she really acted on the channel. Below a certain threshold for the error rate, Alice and Bob act classically on the remaining undisclosed bits implementing error correction and privacy amplification protocols. At the end, they share a reduced sequence of bits that can be considered secure and identical.

2.2 Unconditional security

One of the main problems of both classical and quantum cryptography is to provide a rigorous way to compute the error rate threshold below which a key distribution scheme is secure [5,6,7]. Specifically, we would like to have an upper bound on the error rate below which unconditional security is achieved, i.e., we want to know the error rate below which the key distribution protocol is secure against any possible attack that Eve might implement using the known laws of Physics.

The most general and powerful security attack on a quantum key distribution scheme is usually called a coherent attack. Using this terminology, the ultimate goal of quantum cryptography security analysis is thus to obtain the tightest upper bound on the error rate below which a given quantum key distribution protocol is secure against all coherent attacks.

To understand what a coherent attack is, it is important to discuss two other types of attacks [5,6,7]. The first one is called individual or incoherent attack, the least severe class of attacks. Individual attacks are such that Eve attacks each one of the \( N \) qubits traveling from Alice to Bob independently of the other qubits and always using the same strategy of attack. It is also assumed that Eve implements all her measurements on the probes (ancillas) she used to interact with the qubits sent by Alice before Alice and Bob start the classical post-processing of the data (error correction and privacy amplification).

The second type of attack, called collective attack, extends the possibilities of Eve in the sense that now she can do everything allowed by individual attacks and, in addition, she can decide to postpone the measurements on her ancillas until the moment she considers more convenient, storing the ancillas before any measurement in a quantum memory, for example. Also, her measurements can be joint measurements (collective measurements), in which she measures more than one ancilla simultaneously.

The coherent attack, the most powerful attack, is such that Eve can do everything allowed by individual and collective attacks, as well as implement any type of interaction (unitary operation) involving any number of her ancillas.

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3 If Eve uses different strategies, Alice and Bob can detect Eve by randomly choosing different samples of the raw key to check for security. This will lead to different error rates for different samples if Eve uses different attacks. Thus, different error rates mean that Eve tampered with the key distribution scheme. In this case Alice and Bob discard all the data and restart the key distribution protocol all over.
with any number of the qubits sent by Alice to Bob. Eve is also allowed to implement whatever type of measurements she likes at whatever time she believes is more convenient to her. In summary, Eve can do with the qubits sent from Alice to Bob whatever is allowed by the laws of quantum mechanics.

Here and in the rest of this work we restrict ourselves to one-way post-processing of the raw key. In the BB84 protocol, this is implemented in the step (iv) described above. One-way post-processing is optimally divided into an error correction (information reconciliation) first step and then a privacy amplification step [6]. These are classical processing protocols applied to the raw key and they are called one-way whenever only one party sends classical information during the implementation of those tasks. The other party acts according to previously established rules but never sends any feedback to the communicating party. Furthermore, if the communicating party is the one that sends the quantum states, we have direct reconciliation one-way post-processing. Using our terminology, Alice is the party who communicates classically with Bob during the post-processing stage and he is the one acting on his data without giving any feedback to Alice. If the size (number of bits) of the raw key is $R$, at the end of the classical post-processing step only a fraction $r$ of the original raw key will be secure and perfectly correlated. In other words, the shared secret key’s size is $K = rR$.

An important tool in the analysis of the security of a quantum key distribution scheme under collective attacks, one-way classical post-processing, and direct reconciliation is the Devetak-Winter bound [33],

$$ r = I(A : B) - \max_{Eve} \chi(A : E). $$

Here $r$ is the secret-key fraction as defined above and understood in the asymptotic case, namely, we are dealing with infinitely long raw keys ($N \to \infty$) [3]. The first term on the right hand side of Eq. (1), $I(A : B)$, is the mutual information between the classical data with Alice and Bob, i.e., the correlation between the string of bits with Alice and the string of bits with Bob after the sifting step. In an ideal scenario, with no noise and no Eve, $r = I(A : B) = 1$. This means that every single bit of Alice’s bit string is identical with the corresponding one of Bob’s bit string, i.e., the raw key is the secret shared key ($K = R$). The second term on the right hand side, $\chi(A : E)$, is the Holevo quantity [34]. It can be seen as a quantum generalization of the mutual information, quantifying Eve’s information about the raw key. Maximizing it over all possible collective attacks, that is what the notation $\max_{Eve}$ is telling us to do, we can get the secret-key rate of a quantum key distribution protocol by subtracting it from the mutual information of Alice and Bob. If $r > 0$, the protocol is secure, and if $r \leq 0$, it is insecure [33]. We will come back to a more detailed discussion about $I(A : B)$ and $\chi(A : E)$ later.

The subsequent understanding that a prepare-and-measure quantum key distribution protocol has an equivalent entanglement-based representation [5,6,17,27,35] led to practical ways to compute the secret-key fraction given by Eq. (1). Working directly with the entanglement-based version of the key distribution protocol, we can write the quantum state shared by Alice, Bob, and
Eve after the interference of Eve as a pure state (purification of the state with Alice and Bob). This is the best scenario for Eve [33] and using the purification we can estimate a lower bound for the secret-key rate $r$ that depends on the error rates of the protocol [33,34]. Note that the mapping of a prepare-and-measure protocol to its entanglement-based representation does not imply that the latter is equivalently easy to implement in practice as the former. This mapping just tells us that the security proof obtained for the entanglement-based protocol is as good as if we had worked directly with the prepare-and-measure protocol [5,6,7].

The next breakthrough in the security analysis of quantum key distribution protocols, generalizing the ideas given in Refs. [36,37], was the proof that unconditional security analysis can be carried out in the asymptotic regime by studying the security of a given protocol at the collective attack level [38,39]. With the help of the quantum de Finetti theorem, Renner [38], for discrete variable protocols, and Renner and Cirac [39], for continuous variable ones, showed in what sense collective and coherent attack security analysis are equivalent.

For a permutation invariant quantum key distribution protocol, the quantum de Finetti theorem guarantees that in the asymptotic limit Eve’s knowledge of the raw key, when she implements a coherent attack, is not crucially different from her knowledge of it if she had implemented a collective attack. In other words, security under collective attacks implies security under coherent attacks for any protocol whose entanglement-based representation is invariant under any permutation between the quantum states describing the $N$ pairs of particles shared by Alice and Bob. And if this permutation invariance is not already present due to the symmetry of the protocol, it can be enforced at a later stage by a suitable randomization of the classical data shared by Alice and Bob [6,7,38,39].

2.3 Mutual information and the Holevo quantity

Before we continue, it is important at this stage to give precise definitions of the mutual information [5,6,7,40] and of the Holevo quantity [5,6,7,34,40]. These two quantities are the main ingredients we need to compute the secret-key fraction, Eq. (1), and thus assess the security of any quantum key distribution protocol.

2.3.1 Mutual information

Let $A$ denote a random variable, $a$ its possible outcomes, and $p_A(a)$ the probability of $A$ having the value $a$. The Shannon entropy $H(A)$ associated to the random variable $A$ is [40]

$$H(A) = - \sum_a p_A(a) \log[p_A(a)],$$

(2)
where the base of the logarithm is 2 and $0 \log 0 = 0$. The Shannon entropy quantifies the information content of the random variable $A$. For a completely random variable whose sample space is composed of $n$ symbols, we have that $p_A(a) = 1/n$, any $a$. This leads to $H(A)$ attaining its maximum value possible: $\log n$. For the trivial case where only one $p_A(a) = 1$ and all the others are zero, we get $H(A) = 0$, the least amount of information possible. For any other random distribution of $n$ symbols, $0 \leq H(A) \leq \log n$.

If we now similarly introduce another random variable $B$, the probability of finding $B$ with the value $b$ given that $A$ has the value $a$ is $p_{B|A}(b|a)$. This is usually called the conditional probability. With the aid of the conditional probability we can define the conditional entropy as

$$H(B|A) = \sum_a p_A(a)H(B|A = a), \quad (3)$$

where

$$H(B|A = a) = -\sum_b p_{B|A}(b|a) \log [p_{B|A}(b|a)]. \quad (4)$$

The conditional entropy quantifies the average information contained on the several conditional “random variables $B|A$” that can be built for each outcome $a$ of the random variable $A$. In a certain sense, it measures the average information needed to completely characterize the random variable $B$ given our knowledge of $A$. If $A$ and $B$ are perfectly correlated, i.e., if the knowledge of $A$ completely determines $B$, we have that $H(B|A) = 0$. On the other hand, for completely independent variables $A$ and $B$, $H(B|A) = H(B)$.

Using the Shannon and the conditional entropies, the mutual information between the two random variables $A$ and $B$ reads,

$$I(A : B) = H(B) - H(B|A). \quad (5)$$

The mutual information quantifies the correlation between the random variables $A$ and $B$ and it is related to the statistical mutual dependence between these two random variables. For completely independent random variables $A$ and $B$, the mutual information acquires its minimal value possible, $I(A : B) = 0$, since $H(B|A) = H(B)$. For perfectly correlated variables $H(B|A) = 0$ and we thus get $I(A : B) = H(B)$.

### 2.3.2 The Holevo quantity

Let us assume that for each one of the possible outcomes $a$ of a random variable $A$ we associate the quantum state $\rho_{E|a}$. We can think of $\rho_{E|a}$ as the quantum state with Eve conditioned on Alice’s obtaining the state $|a\rangle$ after she implements, for instance, a projective measurement on her system. The probability $p_A(a)$ of getting the value $a$ from $A$ is such that it is equal to Alice’s chance of getting after her measurement the state $|a\rangle$. In this scenario,
the density matrix describing Eve’s system is given by an ensemble built on the states \( \rho_{E|a} \) with probability weight \( p_A(a) \),

\[
\rho_E = \sum_a p_A(a) \rho_{E|a}.
\] (6)

The state \( \rho_E \) can also be seen as Eve’s description of her system if she knows nothing about the measurement results obtained by Alice at a given run of a quantum key distribution protocol. This is the case in the BB84 and GR10 protocols, where the key is built out of the instances in which no information is revealed about Alice’s measurement results. Only the measurement basis is publicly revealed, which means that Eve has no information about which state \( |a\rangle \) Alice measured.

The state \( \rho_E \) is obtained by tracing out Alice’s system from the state below, which describes Alice and Eve’s joint state prior to any measurement,

\[
\rho_{AE} = \text{Tr}_A(\rho_{ABE}).
\] (7)

Using \( \rho_{AE} \) and the measurement postulate of quantum mechanics, if Alice projects her state onto \( |a\rangle \langle a| \) we have that

\[
p_A(a) = \text{Tr}[\rho_{AE}(|a\rangle \langle a| \otimes 1_E)],
\] (8)

where we now take the total trace and \( 1_E \) is the identity operator acting on Eve’s Hilbert space. The measurement postulate also implies that

\[
\rho_{E|a} = (|a\rangle \langle a| \otimes 1_E)\rho_{AE}(|a\rangle \otimes 1_E)/p_A(a).
\] (9)

The state \( \rho_{AE} \) is, in its turn, obtained from the partial trace with respect to Bob of \( \rho_{ABE} \), the total state describing Alice, Bob, and Eve,

\[
\rho_{AE} = \text{Tr}_B(\rho_{ABE}).
\] (10)

Using the previous notation, the Holevo quantity is given by

\[
\chi(A : E) = S(\rho_E) - \sum_a p_A(a) S(\rho_{E|a}),
\] (11)

where \( S(\rho) \) is the von Neumann entropy of the quantum state \( \rho \) [40].

\[
S(\rho) = -\text{Tr}(\rho \log \rho).
\] (12)

We can appreciate the meaning of the Holevo quantity \( \chi(A : E) \) noting that it bounds the communication capacity of a quantum channel, where we encoded the classical symbols \( a \), whose probability of occurrence is \( p_A(a) \), into the quantum states \( \rho_{E|a} \) [33,34]. Because of this property, the Holevo quantity is usually considered a quantum extension of the mutual information.
2.4 Entanglement-based representation of the BB84 protocol and its standard security analysis

The entanglement-based representation of the BB84 protocol is [6]

\[ |\Phi_1\rangle = |\Phi^+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle). \] (13)

This is one of the four Bell states and it reproduces exactly the statistics Alice and Bob obtain in the original prepare and measure scheme [4]. Indeed, if Alice and Bob measure their qubits in the z-basis, they will obtain with equal chances either the qubit |0⟩ or the qubit |1⟩. Also, noting that |\Phi^+\rangle = (|++\rangle + |--\rangle)/\sqrt{2}, a similar perfect correlation follows if they both measure their qubits in the x-basis.

The Bell state |\Phi^+\rangle together with the other three,

\[ |\Phi_2\rangle = |\Phi^-\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle), \] (14)

\[ |\Phi_3\rangle = |\Psi^+\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle), \] (15)

\[ |\Phi_4\rangle = |\Psi^-\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle), \] (16)

form a complete orthonormal basis that can be used to expand any pure two-qubit state.

In the ideal scenario, after each run of the entanglement-based version of the BB84 protocol, Alice and Bob share the state |\Phi^+\rangle. After \(N\) runs they share the state |\Phi^+\rangle^{\otimes N}. However, in the presence of Eve and at the collective attack level, Eve attacks each one of the qubits traveling to Bob individually and using the same strategy [4]. This means that the state shared by Alice, Bob, and Eve after \(N\) runs of the protocol can be written as

\[ |\Psi\rangle^{\otimes N}_{ABE} = |\Psi\rangle_{ABE} \otimes \cdots \otimes |\Psi\rangle_{ABE}, \] (17)

where

\[ |\Psi\rangle_{ABE} = \sum_{j=1}^{4} \sqrt{\lambda_j} |\Phi_j\rangle_{AB} |\epsilon_j\rangle_E. \] (18)

Equation (18) is a purification of the state describing Alice, Bob, and Eve and we can always use it to describe the global state of our parties due to the Schmidt decomposition theorem [40]. For a large enough Hilbert space describing Eve’s states, we can always write |\Psi\rangle_{ABE} as given above, with all \(\lambda_j\) non negative, \(\sum_j \lambda_j = 1\), and |\epsilon_j⟩ forming an orthonormal basis.

Since after \(N\) runs of the protocol the global state shared by all parties is |\Psi\rangle^{\otimes N}_{ABE}, we have an ensemble of \(N\) replicas of the state |\Psi\rangle_{ABE}. Thus, all

\[ 4 \] Although we will be dealing with collective attacks, the following analysis remains valid for the more general class of coherent attacks due to the results of Refs. [36,37,38,39].
correlation between $AB$ (Alice and Bob) and $E$ (Eve) can be analyzed working directly with $|\Psi\rangle_{ABE}$. Also, note that without Eve or noise we have $\lambda_1 = 1$ and $\lambda_j = 0$, $j = 2, 3, 4$. When Eve is present, however, we have in general that $\lambda_j \neq 0$, for all $j$.

Using Eq. (18) we are able to determine in which way the presence of Eve affects the BB84 protocol or, equivalently, we can study how the presence of noise changes the operation of the BB84 protocol from its expected noiseless case. When Eve is present and Alice and Bob use the same basis to prepare and measure their qubits, the probability of agreement on the value of the bit sent from Alice is no longer one.

Considering the instances in which Alice and Bob employed the $z$-basis, the probability of agreement on the value of the bit sent from Alice to Bob is $1 - \varepsilon_z = p_A(0)p_{B|A}(0|0) + p_A(1)p_{B|A}(1|1)$, while the probability of Bob making a mistake is $\varepsilon_z = p_A(0)p_{B|A}(1|0) + p_A(1)p_{B|A}(0|1)$. However, a standard result from the theory of probabilities states that $p_{AB}(a, b) = p_A(a)p_{B|A}(b|a)$, where $p_{AB}(a, b)$ is the joint probability of Alice and Bob seeing simultaneously the values $a$ and $b$, respectively. This leads to

$$\varepsilon_z = p_{AB}(0, 1) + p_{AB}(1, 0)$$

and to a similar expression for $1 - \varepsilon_z$.

Applying the measurement postulate of quantum mechanics we get

$$p_{AB}(a, b) = \text{Tr}(P_{ab}\rho_{ABE}),$$

where

$$\rho_{ABE} = (|\Psi\rangle_{ABE}\langle\Psi|),$$

with $|\Psi\rangle_{ABE}$ given by Eq. (18), and

$$P_{ab} = |a\rangle_A \langle a| \otimes |b\rangle_B \langle b| \otimes 1_E.$$  

Inserting Eqs. (21) and (22) into (20) we obtain

$$p_{AB}(0, 1) = p_{AB}(1, 0) = (\lambda_3 + \lambda_4)/2$$

and consequently

$$\varepsilon_z = \lambda_3 + \lambda_4.$$  

If we now consider the cases where Alice and Bob employed the $x$-basis to prepare and measure their qubits, a similar calculation leads to the following probability of Bob making a mistake on the value of the bit sent by Alice,

$$\varepsilon_x = \lambda_2 + \lambda_4.$$  

We are now ready to compute the two quantities needed to calculate the secret-key fraction of the BB84 protocol (see Eq. (1)). We begin with the mutual information and then we move on to the Holevo quantity.
2.4.1 The mutual information of the BB84 protocol

In order to compute the mutual information between Alice and Bob, we consider the cases in which they have employed the z-basis to prepare and measure their qubits. The cases in which they both use the x-basis will only be employed to estimate the error $\varepsilon_x$ needed to the computation of the Holevo quantity [6].

Due to the symmetry of the state $\rho_{AB} = \text{Tr}_E(\rho_{ABE})$ under the permutation of the qubit with Alice with the one with Bob, we have that $p_A(a) = p_B(a)$.

Thus, inserting Eqs. (21) and (10) into (8) we get

$$p_A(0) = p_A(1) = p_B(0) = p_B(1) = \frac{1}{2}. \quad (26)$$

Using Eq. (26), it is not difficult to see that Eq. (2) becomes

$$H(A) = H(B) = 1. \quad (27)$$

Now, using that $p_{AB}(a, b) = p_A(a)p_B(b|a)$ and that $\sum_b p_{AB}(a, b) = p_A(a)$, it is possible to compute Eq. (3) with the aid of Eqs. (23) and (26). This leads to

$$H(B|A) = h(\varepsilon_z), \quad (28)$$

with

$$h(x) = -x \log x - (1 - x) \log(1 - x) \quad (29)$$

being the binary entropy, and where we have used that $p_{AB}(0, 0) = p_{AB}(1, 1) = (\lambda_1 + \lambda_2)/2$ to write Eq. (28) as shown above.

Combining Eqs. (27) and (28), we finally get the mutual information between Alice and Bob (see Eq. (5)),

$$I(A : B) = 1 - h(\varepsilon_z). \quad (30)$$

2.4.2 The Holevo quantity of the BB84 protocol

Looking at Eq. (11), we see that to compute the Holevo quantity we need $\rho_E = \text{Tr}_{AB}(\rho_{ABE})$, Eve’s quantum state after she interacts her probes with the qubit sent by Alice, and $\rho_{E|a}$, Eve’s description of her physical system if she knows that Alice projected her qubit onto the state $|a\rangle$.

Tracing out Alice and Bob from $\rho_{ABE}$, Eq. (21), we get

$$\rho_E = \sum_{j=1}^{4} \lambda_j |\epsilon_j\rangle_E \langle \epsilon_j|, \quad (31)$$

which, after inserted into Eq. (12), gives

$$S(\rho_E) = -\sum_{j=1}^{4} \lambda_j \log \lambda_j. \quad (32)$$
Furthermore, a direct calculation using Eqs. (9), (10), (21), and (26) leads to
\[
\rho_{E|0} = \sum_{j=1}^{4} \lambda_j |\epsilon_j\rangle_{E} \langle \epsilon_j | + \sqrt{\lambda_1 \lambda_2} (|\epsilon_1\rangle_{E} \langle \epsilon_2 | + h.c.),
\]
\[
\rho_{E|1} = \sum_{j=1}^{4} \lambda_j |\epsilon_j\rangle_{E} \langle \epsilon_j | - \sqrt{\lambda_1 \lambda_2} (|\epsilon_1\rangle_{E} \langle \epsilon_2 | + h.c.),
\]
where \( h.c. \) denotes the Hermitian conjugate of the operator appearing before it. Noting that the eigenvalues of both \( \rho_{E|0} \) and \( \rho_{E|1} \) are 0, 0, \( \lambda_1 + \lambda_2 \), and \( \lambda_3 + \lambda_4 \), the von Neumann entropy, Eq. (12), for those states becomes
\[
S(\rho_{E|0}) = S(\rho_{E|1}) = h(\varepsilon_z),
\]
where we have used Eq. (24) to eliminate the \( \lambda \)'s in favor of the error rate \( \varepsilon_z \).

Inserting Eqs. (32) and (35) into (11), and using that \( p_A(a) = 1/2 \), the Holevo quantity reads
\[
\chi(A : E) = -\sum_{j=1}^{4} \lambda_j \log \lambda_j - h(\varepsilon_z).
\]

### 2.4.3 The secret-key fraction of the BB84 protocol

If we insert Eqs. (30) and (36) into (11), we get the following expression for the secret-key fraction of the BB84 protocol,
\[
r = 1 - h(\varepsilon_z) - \max_{\text{Eve}} \left\{ -\sum_{j=1}^{4} \lambda_j \log \lambda_j - h(\varepsilon_z) \right\}
\]
\[
= 1 - h(\varepsilon_z) + h(\varepsilon_z) - \max_{\text{Eve}} \left\{ -\sum_{j=1}^{4} \lambda_j \log \lambda_j \right\}
\]
\[
= 1 + \min_{\text{Eve}} \left\{ \sum_{j=1}^{4} \lambda_j \log \lambda_j \right\}.
\]

Note that to arrive at the last equality we used that
\[
- \max_x [f(x)] = \min_x [-f(x)],
\]
i.e., the negative of the maximum value of a function \( f \) over its domain is equal to the minimum of \(-f\) over the same domain.
For the original BB84 protocol, we have shown that the $\lambda$’s satisfy the following three equations,

$$\lambda_3 + \lambda_4 = \varepsilon_z,$$  \hspace{1cm} (38)

$$\lambda_2 + \lambda_4 = \varepsilon_x,$$ \hspace{1cm} (39)

$$\sum_{i=1}^{4} \lambda_i = 1.$$ \hspace{1cm} (40)

Since we have four $\lambda$’s, we cannot uniquely express them in terms of the experimentally determined error rates $\varepsilon_z$ and $\varepsilon_x$. This means that we will need to minimize the expression inside the curly brackets in Eq. (37) in order to get a lower bound of the secret-key fraction as a function of those error rates. This is most easily done following the prescription given in Ref. [6].

Using Eq. (38) we see that $\lambda_4 = \varepsilon_z - \lambda_3$ and thus that $0 \leq \lambda_3 \leq \varepsilon_z$ since no $\lambda_j$ can be negative. This means that we can write

$$\lambda_3 = v\varepsilon_z,$$ \hspace{1cm} (41)

$$\lambda_4 = (1 - v)\varepsilon_z,$$ \hspace{1cm} (42)

where $v \in [0,1]$.

Now, Eq. (40) can be written as $\lambda_2 = 1 - \lambda_1 - \lambda_3 - \lambda_4 = (1 - \varepsilon_z) - \lambda_1$, where we used Eq. (38) to arrive at the last equality. Since $\lambda_2$ cannot be negative we have $0 \leq \lambda_1 \leq 1 - \varepsilon_z$ and, as before, we can write

$$\lambda_1 = u(1 - \varepsilon_z),$$ \hspace{1cm} (43)

$$\lambda_2 = (1 - u)(1 - \varepsilon_z),$$ \hspace{1cm} (44)

with $u \in [0,1]$.

The parameters $u$ and $v$ are not independent since Eq. (39) implies that $(1 - u)(1 - \varepsilon_z) + (1 - v)\varepsilon_z = \varepsilon_x$. Solving for $u$ we get

$$u = (1 - \varepsilon_x - v\varepsilon_z)/(1 - \varepsilon_z),$$ \hspace{1cm} (45)

leaving us with only $v$ as a free parameter. Inserting Eq. (45) into (41) and (42) and using Eqs. (41)-(43), we can write the secret-key fraction (37) as

$$r = 1 + \min_{\text{Eve}} \{ \theta(1 - \varepsilon_x - \varepsilon_z v) + \theta(\varepsilon_x - \varepsilon_z (1 - v)) + \theta(\varepsilon_z - \varepsilon_z v) + \theta(\varepsilon_z v) \},$$ \hspace{1cm} (46)

where

$$\theta(x) = x \log x.$$ \hspace{1cm} (47)

Minimizing $r$ as a function of $v$, i.e., solving $dr/dv = 0$ gives

$$v = 1 - \varepsilon_x,$$ \hspace{1cm} (48)

which is indeed a minimum for $r$ since a direct calculation leads to $d^2r/dv^2 > 0$ when $v = 1 - \varepsilon_x$. 
Using Eqs. (49) and (52) we can express the λ’s giving the lower bound for the secret-key fraction as

\[ \lambda_1 = (1 - \varepsilon_x)(1 - \varepsilon_z), \]
\[ \lambda_2 = \varepsilon_x(1 - \varepsilon_z), \]
\[ \lambda_3 = \varepsilon_z(1 - \varepsilon_x), \]
\[ \lambda_4 = \varepsilon_x \varepsilon_z. \]

Inserting Eqs. (49)-(52) into (37) we obtain the lower bound for the secret-key fraction of the BB84 protocol in terms of the measurable quantities \( \varepsilon_z \) and \( \varepsilon_x \),

\[ r = 1 - h(\varepsilon_x) - h(\varepsilon_z). \]

The quantities \( \varepsilon_x \) and \( \varepsilon_z \) are the probability of Bob making a mistake (error rate) about the value of a bit sent by Alice when they both use the same basis, either the x- or z-basis, to prepare and measure the qubit encoding that bit.

If we assume that \( \varepsilon_x = \varepsilon_z = \varepsilon \), the secret-key fraction can be written as

\[ r = 1 - 2h(\varepsilon), \]

where it is not difficult to see that \( r > 0 \) whenever

\[ \varepsilon \lesssim 11\%. \]

In other words, for symmetrical error rates below 11% the BB84 protocol can be considered secure [6].

2.5 Non-uniqueness of the Schmidt decomposition and the security of the BB84 protocol

Instead of Eq. (18), a perfectly legitimate Schmidt decomposition representing the state of Alice, Bob, and Eve is

\[ |\tilde{\Psi}\rangle_{ABE} = \sum_{j=1}^{4} \sqrt{\lambda_j} |\tilde{\Phi}_j\rangle_{AB} |\tilde{\epsilon}_j\rangle_E, \]

where

\[ |\tilde{\Phi}_1\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle), \]
\[ |\tilde{\Phi}_2\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle), \]
\[ |\tilde{\Phi}_3\rangle = |01\rangle, \]
\[ |\tilde{\Phi}_4\rangle = |10\rangle, \]

span a complete orthonormal basis that can be used to describe any two-qubit pure state. Note that \( |\Phi_1\rangle = |\tilde{\Phi}_1\rangle \) and \( |\Phi_2\rangle = |\tilde{\Phi}_1\rangle \), where \( |\Phi_1\rangle \) and \( |\Phi_2\rangle \) are
given by Eqs. (13) and (14). Moreover, when Eve is absent \( \lambda_1 = 1 \), with the other \( \lambda_j \)'s being zero, recovering the entanglement-based representation of the BB84 protocol in the ideal scenario.

Equation (56) is another purification describing the state of Alice, Bob, and Eve after Eve has coupled her ancillas with the qubit sent from Alice to Bob. The same arguments that led us to write Eq. (18) apply here and, as before, \( 0 \leq \lambda_j \leq 1 \), \( \sum_j \lambda_j = 1 \), and \( |\tilde{\epsilon}_j\rangle, j = 1, \ldots, 4 \), span an orthonormal basis.

The purifications (18) and (56) are connected by the following relation

\[
|\tilde{\Psi}\rangle_{ABE} = (U_{AB} \otimes U_E)|\Psi\rangle_{ABE},
\]

where

\[
U_{AB} = \sum_{j=1}^{2} |\Phi_j\rangle_{AB AB}\langle\Phi_j| + |01\rangle_{AB AB}\langle\Phi_3| + |10\rangle_{AB AB}\langle\Phi_4|,
\]

\[
U_E = \sum_{j=1}^{4} |\tilde{\epsilon}_j\rangle_E \langle\epsilon_j|.
\]

It is worth mentioning that the unitary operation connecting the two purifications are local with respect to the partition \( AB \) and \( E \).

Since this unitary operation is local with respect to Eve, we can alternatively go from (18) to (56) by identifying \( |\tilde{\epsilon}_j\rangle \) with \( |\epsilon_j\rangle \), \( \lambda_j \) with \( \tilde{\lambda}_j \), and by simply applying the following unitary operation,

\[
U_{AB} \otimes 1_E,
\]

where \( 1_E \) is the identity operator acting on Eve’s Hilbert space. Since Eve’s unitary operation is given by the identity operator, it is clear that no changes are made on the states describing her system. Also, whether we think of \( |\tilde{\Psi}\rangle_{ABE} \) as given by \( U_{AB} \otimes U_E|\Psi\rangle_{ABE} \) or by \( U_{AB} \otimes 1_E|\Psi\rangle_{ABE} \), we will get the same lower bound for the secret-key fraction since we need to compute the Holevo quantity maximizing it over all possible strategies that Eve might employ. As such, and in order to simplify notation, we will write \( |\epsilon_j\rangle \) and \( \lambda_j \) instead of \( |\tilde{\epsilon}_j\rangle \) and \( \tilde{\lambda}_j \).

However, the purification (56) is not equivalent to (18) in the following sense. As we will show below, the way we write the purification (56) forces us to introduce another constraint among the \( \lambda_j \)'s that is absent from (18). This will allow us to get a tighter lower bound for the secret-key fraction, which ultimately leads to an increase of the error rates below which the BB84 protocol continues to operate securely.

Let us now present this extra constraint. If we repeat the steps that led to Eq. (26) using Eq. (56) instead of (18) we get,

\[
p_A(0) = (\lambda_1 + \lambda_2)/2 + \lambda_3,
\]

\[
p_A(1) = (\lambda_1 + \lambda_2)/2 + \lambda_4.
\]
But whether or not Eve interferes, the BB84 protocol is such that we always have
\[ p_A(0) = p_A(1) = 1/2, \]  
(67)
since Alice randomly chooses with equal chances whether she sends Bob the bit 0 or 1. Enforcing the condition (67) onto Eqs. (65) and (66), we see that we must have
\[ \lambda_3 = \lambda_4 = \lambda. \]  
(68)
Using the constraint (68) it is not difficult to see that a direct calculation, similar to what we did in Sec. 2.4, leads to
\[ p_A(0) = p_A(1) = p_B(0) = p_B(1) = 1/2 \]  
(69)
and to
\[ p_{AB}(0,0) = p_{AB}(1,1) = (\lambda_1 + \lambda_2)/2, \]  
(70)
\[ p_{AB}(0,1) = p_{AB}(1,0) = \lambda. \]  
(71)
Similarly, we also obtain for the mutual information between Alice and Bob,
\[ I(A:B) = 1 - h(\varepsilon_z), \]  
(72)
where
\[ \varepsilon_z = 2\lambda, \]  
(73)
\[ \varepsilon_x = \lambda_2 + \lambda. \]  
(74)
We now move on to the calculation of the Holevo quantity, Eq. (11), using the new purification (56). Using Eq. (56) and the constraint (68) we get
\[ \rho_E = \text{Tr}_{AB}(\rho_{ABE}) = \sum_{j=1}^{2} \lambda_j |\varepsilon_j\rangle_E E \langle \varepsilon_j| + \lambda |\varepsilon_3\rangle_E E \langle \varepsilon_3| + |\varepsilon_4\rangle_E E \langle \varepsilon_4| \]  
and thus
\[ S(\rho_E) = -\lambda_1 \log \lambda_1 - \lambda_2 \log \lambda_2 - 2\lambda \log \lambda. \]  
(75)
Now, employing Eq. (56) instead of (18) we get after Eqs. (69), (70), (71), and (72),
\[ \rho_{E|0} = \lambda_1 |\varepsilon_1\rangle_E E \langle \varepsilon_1| + \lambda_2 |\varepsilon_2\rangle_E E \langle \varepsilon_2| + 2\lambda |\varepsilon_3\rangle_E E \langle \varepsilon_3| + \sqrt{\lambda_1 \lambda_2} |\varepsilon_2\rangle_E E \langle \varepsilon_2| + h.c. \]  
(76)
\[ \rho_{E|1} = \lambda_1 |\varepsilon_1\rangle_E E \langle \varepsilon_1| + \lambda_2 |\varepsilon_2\rangle_E E \langle \varepsilon_2| + 2\lambda |\varepsilon_4\rangle_E E \langle \varepsilon_4| - \sqrt{\lambda_1 \lambda_2} |\varepsilon_2\rangle_E E \langle \varepsilon_2| + h.c. \]  
(77)
The eigenvalues of both \( \rho_{E|0} \) and \( \rho_{E|1} \) are 0, 0, \( \lambda_1 + \lambda_2 \), and 2\( \lambda \), leading to the following value for the von Neumann entropy,
\[ S(\rho_{E|0}) = S(\rho_{E|1}) = h(\varepsilon_z), \]  
(78)
The Holevo quantity is obtained when we insert Eqs. (69), (75), and (78) into (11), giving
\[
\chi(A : E) = -\lambda_1 \log \lambda_1 - \lambda_2 \log \lambda_2 - 2\lambda \log \lambda - h(\varepsilon_z).
\] (79)

Finally, the secret-key fraction, Eq. (1), comes from subtracting Eq. (79) from (72),
\[
r = 1 + \lambda_1 \log \lambda_1 + \lambda_2 \log \lambda_2 + 2\lambda \log \lambda.
\] (80)

It is worth mentioning that we do not need this time to minimize \(r\) over all strategies that Eve might employ because all quantities are already fixed by the experimental data, i.e., all we need to compute \(r\) is \(\varepsilon_x\) and \(\varepsilon_z\). This comes about due to the extra constraint, Eq. (68), which allows us to uniquely determine the \(\lambda\)'s as functions of the error rates \(\varepsilon_x\) and \(\varepsilon_z\). Using Eqs. (73), (74), and the normalization condition of the \(\lambda\)'s, namely, \(\lambda_1 + \lambda_2 + 2\lambda = 1\), we have
\[
\lambda_1 = 1 - \varepsilon_x - \varepsilon_z / 2, \quad \lambda_2 = \varepsilon_x - \varepsilon_z / 2, \quad \lambda = \varepsilon_z / 2.
\] (81-83)

The secret-key fraction is obtained inserting Eqs. (81)-(83) into (80), which can be written after some algebra as follows,
\[
r = \varepsilon_z \log \varepsilon_z + (1 - \varepsilon_x - \varepsilon_z / 2) \log(2 - 2\varepsilon_x - \varepsilon_z) + (\varepsilon_x - \varepsilon_z / 2) \log(2\varepsilon_x - \varepsilon_z).
\] (84)

Assuming, as before, we deal with a depolarizing channel \((\varepsilon_x = \varepsilon_z = \varepsilon)\) the secret-key fraction becomes
\[
r = (3\varepsilon / 2) \log \varepsilon + (1 - 3\varepsilon / 2) \log(2 - 3\varepsilon).
\] (85)

Searching for the maximal \(\varepsilon\) giving \(r > 0\) we get
\[
\varepsilon \lesssim 12.61\%.
\] (86)

The above error rate threshold is the same one obtains working with the six-state protocol \([41, 42, 43, 32]\), where in addition to the z- and x-basis we have the bits encoded in the y-basis. This means that both the BB84 and the six-state protocols are secure under the same level of noise. It is worth stressing once more that we could only achieve the previous result due to the exploration of the non-uniqueness of the purification (Schmidt decomposition) describing the state of Alice, Bob, and Eve.

We also point out that we have carried out the security analysis for the BB84 protocol using the Schmidt decomposition (56) without imposing the constraint (68). In this case we had to minimize \(r\) over all possible strategies of Eve because there still remained a free \(\lambda\) in the expression for \(r\). At the end of the minimization process we found out that the constraint (68) naturally emerged as the scenario that is best to Eve. Since the constraint (68) is related to the fact that \(p_A(0) = p_A(1) = 1/2\), it is possible that a non-symmetrical protocol, such that \(p_A(0) \neq p_A(1)\), may lead to an increase in the level of noise in which it still operates securely.
3 Asymptotic security analysis of the GR10 protocol

To better appreciate all the steps involved in the operation of the GR10 protocol and to get a qualitative feeling of why it is secure, it is crucial to understand how the probabilistic teleportation protocol works [17,18,19,22,23]. A brief review of the probabilistic teleportation protocol, gauged to the purposes of this work, is the subject of Sec. 3.1. In Sec. 3.2 we present the GR10 protocol using the notation employed in this work. In Sec. 3.3 we start by highlighting the main qualitative features of the GR10 protocol that makes it secure against an eavesdropper attack and then we provide its full and rigorous security analysis, similar in spirit to the ones previously shown for the BB84 protocol.

3.1 The probabilistic teleportation protocol

Alice’s goal is to teleport to Bob the state $|\phi\rangle_A = \alpha |0\rangle_A + \beta |1\rangle_A$ (87) through the following entangled state shared with him,$$
|\Phi^1_{AB}\rangle = \frac{|00\rangle_{AB} + n|11\rangle_{AB}}{\sqrt{1 + n^2}}. \tag{88}
$$Here $\alpha, \beta$ are complex numbers, $|\alpha|^2 + |\beta|^2 = 1$ is the normalization condition, and $0 \leq n \leq 1$. Note that Eq. (88) is not a maximally entangled state whenever $n < 1$. Equation (88) together with the generalized Bell states [17,18,19]

$$
|\Phi^2_{AB}\rangle = \frac{n|00\rangle - |11\rangle}{\sqrt{1 + n^2}}, \tag{89}
$$

$$
|\Phi^3_{AB}\rangle = \frac{|01\rangle + n|10\rangle}{\sqrt{1 + n^2}}, \tag{90}
$$

$$
|\Phi^4_{AB}\rangle = \frac{n|01\rangle - |10\rangle}{\sqrt{1 + n^2}}, \tag{91}
$$

form a complete set of orthonormal vectors spanning a two-qubit Hilbert space.

All the steps of the probabilistic teleportation protocol are similar to the deterministic one [14]. The two differences are related to the use of a non-maximally entangled state connecting Alice and Bob, $|\Phi^0_{AB}\rangle$, and to the basis onto which Alice’s projects her qubits. Instead of projecting her qubits onto maximally entangled Bell states ($n = 1$), she projects them onto generalized Bell states $|\Phi^m_{AA}\rangle$, where we assume for the moment that $m \neq n$.

The probability $p_j$ of Alice measuring a particular generalized Bell state $|\Phi^m_j\rangle$ is

$$
p_1 = |f_1(\alpha, \beta)|^2, \quad p_2 = |f_2(\alpha, \beta)|^2, \quad p_3 = |f_2(\beta, \alpha)|^2, \quad p_4 = |f_1(\beta, \alpha)|^2, \tag{92}
$$
where

\[
\begin{align*}
\left| f_1(\alpha, \beta) \right| &= \sqrt{\frac{|\alpha|^2 + m^2n^2|\beta|^2}{(1 + m^2)(1 + n^2)}} , \\
\left| f_2(\alpha, \beta) \right| &= \sqrt{\frac{m^2|\alpha|^2 + n^2|\beta|^2}{(1 + m^2)(1 + n^2)}} .
\end{align*}
\]  
(93)

After implementing her measurement, Alice tells Bob via a classical communication channel the result she obtained. If she measured the state \(|\Phi_m^m\rangle\), Bob’s state collapses to \(U_j^*|\phi_j\rangle_B\), with \(|\phi_j\rangle_B\) given by Eqs. (94)-(97). To finish the teleportation protocol, Bob uses the information received from Alice to apply the corresponding unitary operation \(U_j\) on his qubit. Here \(U_1 = 1\) is the identity matrix and \(U_2 = \sigma_z, U_3 = \sigma_x,\) and \(U_4 = \sigma_z\sigma_x\) are given by the standard Pauli matrices. After applying \(U_j\) his qubit is given by one of these four possibilities,

\[
\begin{align*}
|\phi_1\rangle_B &= \frac{\alpha|0\rangle_B + mn\beta|1\rangle_B}{\sqrt{|\alpha|^2 + m^2n^2|\beta|^2}} , \\
|\phi_2\rangle_B &= \frac{\alpha|0\rangle_B + n\beta|1\rangle_B}{\sqrt{|\alpha|^2 + n^2|\beta|^2}} , \\
|\phi_3\rangle_B &= \frac{\alpha|0\rangle_B + m\beta|1\rangle_B}{\sqrt{|\alpha|^2 + m^2|\beta|^2}} , \\
|\phi_4\rangle_B &= \frac{mn\alpha|0\rangle_B + \beta|1\rangle_B}{\sqrt{m^2n^2|\alpha|^2 + |\beta|^2}} .
\end{align*}
\]  
(94)-(97)

In other words, if Alice obtained the state \(|\Phi_m^m\rangle\) after projecting her two qubits onto the generalized Bell states, Bob’s qubit at the end of the teleportation protocol is correspondingly described by the state \(|\phi_j\rangle_B\). It is worth mentioning that for \(m = n = 1\) we get back to the original teleportation protocol [14], where \(p_j = 1/4\) and \(|\phi_j\rangle_B = \alpha|0\rangle_B + \beta|1\rangle_B\) for any \(j\).

Looking at Eqs. (94)-(97), we realize that Bob gets a perfect replica of Alice’s teleported qubit if she chooses \(m = n\). This condition for the values of \(m\) and \(n\) is often called the matching condition [16][18][19][22][23] and it is clear that only two out of the four possible measurement results of Alice lead to a perfect teleportation. This happens whenever she obtains either \(|\Phi_m^m\rangle\) or \(|\Phi_m^n\rangle\), with \(m = n\). In this scenario, \(|\phi_2\rangle_B\) and \(|\phi_3\rangle_B\) are given by \(\alpha|0\rangle_B + \beta|1\rangle_B\) and the teleportation protocol is considered successful. The probability of success is

\[
p_{\text{succ}}(n) = p_2 + p_3 = \frac{2n^2}{(1 + n^2)^2} .
\]  
(98)

3.2 The GR10 protocol

Once the probabilistic teleportation protocol is understood, we are ready to show how the GR10 protocol works. In what follows we will briefly review all the steps in the execution of the GR10 protocol, similarly to the way Ref. [15] originally introduced it.
(i) Alice and Bob agree beforehand on two points. First, the encoding of the classical bits will be made using only one orthonormal basis. This basis is chosen such that its base vectors are non-orthogonal to the base vectors used to express the partially entangled Bell states employed in the probabilistic teleportation protocol. For instance, if the entangled resource shared between Alice and Bob is given by $|\Phi_n^{11}\rangle_{AB} = (|00\rangle_{AB} + n|11\rangle_{AB})/\sqrt{1 + n^2}$, they agree that the qubit $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ encodes the bit 0 and $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$ the bit 1. In other words, they use the x-basis to encode the bits and the z-basis to write the Bell states. Second, they also agree on the possible values of $n$. In the original version of the GR10 protocol, only two values for $n$ are used, $n_1$ and $n_2$.

(ii) Alice randomly prepares a qubit in one of the two states described above, namely, $|+\rangle$ or $|-\rangle$. Bob, in its turn, randomly generates either the partially entangled two-qubit state $|\Phi_n^{11}\rangle_{AB}$ or $|\Phi_n^{12}\rangle_{AB}$, sending one of the qubits to Alice and keeping the other with him.

(iii) After receiving her share of the partially entangled state, Alice initiates the probabilistic teleportation protocol as previously described. Specifically, she projects the qubit encoding the classical bit and the qubit received from Bob onto the generalized Bell state $|\Phi_m^{ij}\rangle_{AA}$, where $m$ is randomly chosen between $n_1$ or $n_2$. Note that at this stage of the protocol, neither Bob tells Alice the value of $n$ he chose to prepare the entangled state nor Alice tells Bob the value of $m$ she chose in order to fix the generalized Bell basis used to project her qubits. Alice only tells Bob the generalized Bell state $|\Phi_m^{ij}\rangle_{AA}$ she got, $j = 1, \ldots, 4$, but not the value of $m$.

(iv) After receiving the news from Alice (the value of $j$ but not of $m$), Bob implements the corresponding unitary operation to correct his qubit as described in Sec. 3.1. His qubit, after that, is in one of the four possible states listed in Eqs. (94)-(97), where $\alpha = \beta = 1/2$. He then projects his qubit onto the x-basis. If his measurement result is $|+\rangle$, he assumes Alice sent the bit 0, and if it yields the state $|-\rangle$, he assigns the bit value 1. This whole process, steps (i) to (iv), is repeated $N$ times.

(v) After Bob has finished all his measurements on the $N$ qubits, Alice and Bob, similarly to what is done in the BB84 protocol, start a conversation over an authenticated public classical channel that can be fully insecure. In this public discussion, they reveal to each other the following pieces of information. He tells Alice the values of $n$ ($n_1$ or $n_2$) he employed to prepare each one of the partially entangled states and Alice tells him the values of $m$ ($n_1$ or $n_2$) used in each one of the $N$ generalized Bell measurements she made. They discard the instances in which $m \neq n$, keeping only the cases where the matching condition occurred ($m = n$). About $N/2$ cases are discarded and $N/2$ are kept. Of the remaining $N/2$ cases, another sifting

\footnote{We can always use more than one orthonormal basis to encode the classical bits, with non-orthogonal states encoding the same bit. If we use two such basis, we can see the present protocol as an additional security layer to the BB84 protocol. But the whole point of the GR10 protocol, which will be made clearer when we present its security analysis, is that it is secure even if we use only one orthogonal basis to encode the classical bits.}
process is implemented. Due to the particular operation of the probabilistic teleportation protocol, the teleported qubit reaching Bob is an exact replica of Alice’s qubit if \( m = n \) and her measurement outcome is either \( |\Phi^+_n\rangle \) or \( |\Phi^+_n\rangle \). Therefore, they discard the cases in which Alice’s measurement resulted in the states \( |\Phi^+_n\rangle \) and \( |\Phi^+_n\rangle \). The remaining bits are the raw key which, after Eq. (98), is given by

\[
R = \left( \frac{p_{\text{suc}}(n_1)}{2} + \frac{p_{\text{suc}}(n_2)}{2} \right) \frac{N}{2}.
\]

(vi) Finally, Alice and Bob disclose part of the bits of the raw key, which is used to detect the presence of an eavesdropper. The publicly revealed information is employed to estimate the error rate of the quantum communication channel, allowing them to act accordingly to increase the security of the shared key via classical reconciliation and privacy amplification protocols.

### 3.3 Security analysis of the GR10 protocol

The simplest way to qualitatively understand the security of the GR10 protocol is to push as far as possible the analogy between the GR10 and the BB84 protocols [16]. To do this, we first observe that in the GR10 protocol Alice and Bob discard all the instances in which her generalized Bell measurements yield \( |\Phi^+_n\rangle \) and \( |\Phi^+_n\rangle \). Of the remaining two possibilities, four different cases emerge. Alice can either teleport the state \( |+\rangle \) and obtain \( |\Phi^+_n\rangle \) for her Bell measurement or she can teleport \( |+\rangle \) and obtain \( |\Phi^+_n\rangle \). Similarly, she can choose to teleport to Bob the state \( |--\rangle \) and either obtain \( |\Phi^+_n\rangle \) or \( |\Phi^+_n\rangle \) after implementing the generalized Bell measurement. The quantum states describing Bob’s qubit for each one these four possibilities are respectively (see Eqs. (95) and (96)),

\[
|\tilde{0}\rangle_B = \frac{m|0\rangle + n|1\rangle}{\sqrt{n^2 + m^2}} = \frac{(m + n)|+\rangle + (m - n)|-\rangle}{\sqrt{2(n^2 + m^2)}},
\]

\[
|\tilde{1}\rangle_B = \frac{m|0\rangle - n|1\rangle}{\sqrt{n^2 + m^2}} = \frac{(m - n)|+\rangle + (m + n)|-\rangle}{\sqrt{2(n^2 + m^2)}},
\]

where we have dropped the subindex \( B \) at the right hand side of the above expressions for simplicity. It is not difficult to see that the set \( \{|\tilde{0}\rangle_B, |\tilde{1}\rangle_B\} \) defines an orthonormal basis as well as the set \( \{|\tilde{+}\rangle_B, |\tilde{-}\rangle_B\} \). Let us call them the \( \tilde{z} \)-basis and \( \tilde{x} \)-basis, respectively.

The analogy with the BB84 protocol is now clear. Whenever Alice teleports the state \( |+\rangle \), Bob randomly gets either the state \( |\tilde{0}\rangle \) or \( |\tilde{+}\rangle \), depending on which Bell measurement she obtained. If the matching condition is achieved
(n = m), we have that <0, 0> = +\rangle = +\rangle and Bob will correctly assign the bit value 0 after measuring his qubit in the x-basis. Similarly, if Alice teleports the state |−\rangle, Bob’s qubit is either |\bar{1}\rangle or |\bar{−}\rangle. For n = m we have |\bar{1}\rangle = |\bar{−}\rangle = |−\rangle and Bob will get the correct bit value 1 after finishing his measurement in the x-basis.

On the other hand, if the matching condition is not fulfilled (n ≠ m), the probability of Bob making a mistake is not zero and it is given by \((m - n)^2/(2(n^2 + m^2))\). Moreover, the fact that m ≠ n breaks the degeneracy of Bob’s possible states after the teleportation, i.e., |0\rangle ≠ |\bar{+}\rangle and |\bar{1}\rangle ≠ |\bar{−}\rangle. This induces an encoding into non-orthogonal states for the bit values sent by Alice, exactly as it happens in the BB84 protocol. Indeed, in this scenario the bit value 0 is either associated to |0\rangle or |\bar{+}\rangle, where \langle 0|\bar{+}\rangle = 2nm/(n^2 + m^2), and the bit value 1 is either related to |\bar{1}\rangle or |\bar{−}\rangle, where again \langle \bar{1}|\bar{−}\rangle = \langle 0|\bar{+}\rangle.

Putting it another way, although in the GR10 protocol we employ only one set of orthonormal states to encode the key, the fact that its operation is based on the probabilistic teleportation protocol leads to an effective non-orthogonal encoding whenever m ≠ n. And since Eve cannot always guess the correct values of m and n in her eavesdropping before Alice and Bob finish all their measurements, she will necessarily be caught tampering with the execution of the GR10 protocol.

Let us now move to the quantitative security analysis of the GR10 protocol. First we note that, according to the discussion in the previous paragraphs, whenever the matching condition is fulfilled (m = n) Bob gets an exact replica of the teleported state. Thus, the entanglement-based representation of the post-selected successful cases of the GR10 protocol is

\[ |\bar{Φ}_1\rangle = \frac{1}{\sqrt{2}}(|++\rangle + |−−\rangle). \] (104)

Equation (104) is the Bell state (13) written in the x-basis.

After Eve’s eavesdropping, a possible purification representing the state of Alice, Bob, and Eve is

\[ |Ψ\rangle_{ABE} = \sum_{j=1}^{4} \sqrt{λ_j}|\bar{Φ}_j\rangle_{AB}|\bar{ε}_j\rangle_E, \] (105)

where |\bar{Φ}_1\rangle was just defined and

\[ |\bar{Φ}_2\rangle = \frac{1}{\sqrt{2}}(|−\rangle + |−\rangle), \] \(106\)

\[ |\bar{Φ}_3\rangle = |01\rangle, \] \(107\)

\[ |\bar{Φ}_4\rangle = |10\rangle. \] \(108\)

Equation (106) is the standard Bell state (14) rewritten in the x-basis.

The entanglement-based representation (105) is the one we employed when we reassessed the security analysis of the BB84 protocol using a different purification, namely, Eq. (56). Here, however, we deal with the x-basis alone,
while in Sec. 2.5 both the z- and x-basis were possible preparation and measurement basis for Alice and Bob. The fact that we now have no knowledge of $\varepsilon_z$ will reflect in one of the $\lambda$’s being undetermined. This forces us to carry out the maximization of Eve’s accessible information over all possible strategies that she might eventually use. When compared to the calculations of Sec. 2.5, this will lead to a lower value for the error rate $\varepsilon_x$ below which we have a positive secret-key fraction and therefore security.

The states teleported by Alice are either $|+\rangle$ or $|-\rangle$. Since they are chosen with equal chances, Alice’s ensemble of qubits are given by $(|+\rangle\langle+| + |-\rangle\langle-|)/2$ or, equivalently, by $(|0\rangle\langle0| + |1\rangle\langle1|)/2$. Hence, as in Sec. 2.5, we must have

$$p_A(0) = p_A(1) = \frac{1}{2},$$

which leads to the constraint (68),

$$\lambda_3 = \lambda_4 = \lambda.$$ (110)

Carrying out similar calculations to the ones detailed in Sec. 2, we get the following expression for the mutual information between Alice and Bob,

$$I(A : B) = 1 - h(\varepsilon_x).$$ (111)

The error rate $\varepsilon_x$ is the probability of Bob assigning the wrong bit value to the bit teleported by Alice and is given by

$$\varepsilon_x = \lambda_2 + \lambda = (1 + \lambda_2 - \lambda_1)/2,$$ (112)

where we used the normalization condition

$$\lambda_1 + \lambda_2 + 2\lambda = 1$$ (113)

to arrive at the last equality.

By the same token we get

$$S(\rho_E) = -\lambda_1 \log \lambda_1 - \lambda_2 \log \lambda_2 - 2\lambda \log \lambda,$$ (114)

$$\rho_{E|+} = \sum_{j=1}^{4} \lambda_j |\epsilon_j\rangle_E \langle \epsilon_j| + \sqrt{\lambda_1 \lambda_2 \lambda_3 \lambda_4} (|\epsilon_1\rangle_E \langle \epsilon_3| + h.c.)$$

$$+ \sqrt{\lambda_1 \lambda_4 \lambda_3 \lambda_2} (|\epsilon_1\rangle_E \langle \epsilon_4| + h.c.) - \sqrt{\lambda_2 \lambda_3 \lambda_1 \lambda_4} (|\epsilon_2\rangle_E \langle \epsilon_3| + h.c.)$$

$$+ \sqrt{\lambda_2 \lambda_4 \lambda_3 \lambda_1} (|\epsilon_2\rangle_E \langle \epsilon_4| + h.c.),$$ (115)

$$\rho_{E|-} = \sum_{j=1}^{4} \lambda_j |\epsilon_j\rangle_E \langle \epsilon_j| - \sqrt{\lambda_1 \lambda_2 \lambda_3 \lambda_4} (|\epsilon_1\rangle_E \langle \epsilon_3| + h.c.)$$

$$- \sqrt{\lambda_1 \lambda_4 \lambda_3 \lambda_2} (|\epsilon_1\rangle_E \langle \epsilon_4| + h.c.) + \sqrt{\lambda_2 \lambda_3 \lambda_1 \lambda_4} (|\epsilon_2\rangle_E \langle \epsilon_3| + h.c.)$$

$$- \sqrt{\lambda_2 \lambda_4 \lambda_3 \lambda_1} (|\epsilon_2\rangle_E \langle \epsilon_4| + h.c.).$$ (116)
The two non-null eigenvalues of either $\rho_{E|+}$ or $\rho_{E|\pm}$ are $(1 + \lambda_1 - \lambda_2)/2 = 1 - \varepsilon_x$ and $(1 + \lambda_2 - \lambda_1)/2 = \varepsilon_x$, where we used that $\lambda_3 = \lambda_4$ and Eq. (112) to write the eigenvalues as shown. We thus get

$$S(\rho_{E|+}) = S(\rho_{E|\pm}) = h(\varepsilon_x)$$

(117)

and with the aid of Eq. (114) we can write the Holevo quantity as follows,

$$\chi(A:E) = -\lambda_1 \log \lambda_1 - \lambda_2 \log \lambda_2 - 2\lambda \log \lambda - h(\varepsilon_x).$$

(118)

Finally, the secret-key fraction (11) becomes

$$r = 1 + \min_{\varepsilon_x} \left\{ \lambda_1 \log \lambda_1 + \lambda_2 \log \lambda_2 + 2\lambda \log \lambda \right\}$$

$$= 1 + \min_{\lambda} \left\{ (1 - \varepsilon_x - \lambda) \log(1 - \varepsilon_x - \lambda) + (\varepsilon_x - \lambda) \log(\varepsilon_x - \lambda) + 2\lambda \log \lambda \right\}.$$  

(119)

To obtain the last equality we used Eqs. (112) and (113) to express $\lambda_1$ and $\lambda_2$ in terms of $\varepsilon_x$ and $\lambda$,

$$\lambda_1 = 1 - \varepsilon_x - \lambda, \quad \lambda_2 = \varepsilon_x - \lambda.$$  

(120)

Solving

$$\frac{dr}{d\lambda} = 0$$

(121)

for $\lambda$ we get

$$\lambda_{\min} = \varepsilon_x (1 - \varepsilon_x)$$

(122)

and that

$$\frac{d^2r(\lambda_{\min})}{d\lambda^2} > 0,$$  

(123)

proving that we indeed have found the minimum of $r(\lambda)$.

Inserting Eq. (122) into (119) we obtain the following lower bound for the secret-key fraction of the GR10 protocol,

$$r = 1 - 2h(\varepsilon_x).$$  

(124)

Searching for the root of Eq. (124) we get that $r > 0$ if

$$\varepsilon_x \lesssim 11\%.$$  

(125)

In other words, for error rates less than 11% the GR10 protocol operates securely. In Fig. 1 we show Eq. (124) as a function of $\varepsilon_x$.

Similarly to what we did when presenting the alternative security proof of the BB84 protocol, we have worked out the security analysis for the GR10 protocol without imposing constraint (68). Again, we obtained Eq. (68) at the end of the minimization process as the optimal scenario for Eve.

Moreover, the purification (105) is crucial to attain a tight lower bound for the secret-key fraction $r$. Had we worked with the purification (18), as usually
done in the security analysis of the BB84 protocol, we would get a too strong lower bound, one that yields \( r < 0 \) for any value of \( \varepsilon_x > 0 \).

Finally, it is important to note that the security analysis presented here and in the next section can be directly extended to coherent attacks, leading to the unconditional security of the GR10 protocol and its modified version given in Sec. [1]. This is true due to the works of Renner [38] and Renner and Cirac [39], built on the previous works of Refs. [36,37]. The key result of Ref. [38] was the proof that in the asymptotic limit security against collective attacks are essentially equivalent to security against coherent ones.

### 4 Modifications to the GR10 protocol and their security analysis

We now modify the operation of the GR10 protocol in the following sense. Originally, as given in Ref. [16], all instances in which Alice’s measurement results yielded the generalized Bell states \(|\Phi_{m}^{n_1}\rangle\) and \(|\Phi_{m}^{n_2}\rangle\) were discarded as well as the cases where the matching condition did not occur \((m \neq n)\). We now want to consider a protocol where all cases are included as a valid outcome. In this way we increase the size of the raw key \( R \) at the cost of decreasing the value of the secret-key fraction \( r \). Whether the secret key’s size \( K = rR \) will be greater in one or the other way of implementing the GR10 protocol will depend of the value of the error rate and on the values of \( n_1 \) and \( n_2 \) adopted in its operation.

When we consider all instances, i.e., all Alice’s possible measurement outcomes and whether or not \( n = m \), Bob will inevitably fail sometimes in the correct identification of the bit value teleported by Alice, even in the noiseless
case. Therefore, the entanglement-based representation of the modified GR10 protocol will inherit this property and it is given by

$$|\xi_1\rangle = \sqrt{p_{AB}(\cdot, +)} |++\rangle + \sqrt{p_{AB}(\cdot, -)} |-\rangle$$

$$+ \sqrt{p_{AB}(\cdot, \cdot)} |-\rangle + \sqrt{p_{AB}(\cdot, \cdot)} |+-\rangle .$$

(126)

Here $p_{AB}(\cdot, +) + p_{AB}(\cdot, -)$ is the probability of Alice and Bob obtaining a perfect correlation between their bit values and $p_{AB}(\cdot, \cdot) + p_{AB}(\cdot, \cdot)$ is the probability of them disagreeing about those values.

According to the functioning of the probabilistic teleportation protocol, a direct calculation gives

$$p_{AB}(\cdot, +) = p_{AB}(\cdot, -) = p,$$

(127)

$$p_{AB}(\cdot, \cdot) = p_{AB}(\cdot, \cdot) = \frac{1}{2} - p,$$

(128)

where

$$p = \frac{1}{4} + \frac{(n_1 + n_2)^2 (1 + n_1 n_2)^2}{4(1 + n_1^2)(1 + n_2^2)^2}.$$ 

(129)

It is worth noting that $p$ is symmetric if we change the value of $n_1$ with that of $n_2$. Moreover, $p$ is a monotonically increasing function of either $n_1$ or $n_2$ and whenever $n_1 \neq 0$ or $n_2 \neq 0$, $p > 1/4$. Also, for $n_1 = n_2 = 1$ we have $p = 1/2$.

Since we must have some entanglement shared between Alice and Bob in the operation of the GR10 protocol, in what follows $1/4 < p \leq 1/2$.6

Thus, using Eqs. (127) and (128) the entanglement-based representation (126) becomes

$$|\xi_1\rangle = \sqrt{p}(|++\rangle + |-\rangle) + \sqrt{1/2 - p}(|-\rangle + |+-\rangle)$$

$$= \sqrt{2p}\Phi_1 + \sqrt{1-2p}\Phi_2,$$

(130)

where $|\Phi_1\rangle$ and $|\Phi_2\rangle$ are the Bell states (13) and (14). Note that we have suppressed for simplicity the subscripts $AB$ when writing the above kets.

6 Incidentally, it is worth mentioning that if in the BB84 protocol we accept all instances as a valid outcome, even when Alice and Bob use different preparation and measurement basis, we get an error rate of 25% in the ideal case (no Eve or noise). When the matching condition is not satisfied, the outcomes of Bob’s measurements are completely uncorrelated to the bit values encoded by Alice in the qubits sent to him. On the other hand, the modified GR10 protocol’s ideal error rate, $1 - 2p$, depends on the entanglement of the quantum states shared between Alice and Bob (the values of $n_1$ and $n_2$, cf. Eq. (129)). As such, for the modified GR10 protocol we can tune this error rate as we wish and whenever $p \geq 3/8 \approx 0.375$ the error rate $1 - 2p$ is lower than 25%, approaching zero as $n_1$ and $n_2$ tend to one. Furthermore, due to the GR10 protocol’s teleportation based operation, Bob’s measurement outcomes are not completely independent of Alice’s teleported qubits, including the instances in which Alice and Bob assign different bit values at a given run of the protocol. It is this entanglement-dependent ubiquitous correlation that allows the modified GR10 protocol to operate securely for not too low levels of entanglement.
The state describing Alice, Bob, and Eve after Eve tampered with the key transmission can be represented by the following purification,

\[ |\Psi\rangle_{ABE} = \sum_{j=1}^{4} \sqrt{\lambda_j} |\xi_j\rangle_{AB} |\epsilon_j\rangle_E. \]  

(131)

Here \(|\xi_1\rangle\) is given by Eq. (130) and

\[ |\xi_2\rangle = \sqrt{2p} |\Phi_2\rangle - \sqrt{1-2p} |\Phi_1\rangle, \]  

(132)

\[ |\xi_3\rangle = |\Phi_3\rangle, \]  

(133)

\[ |\xi_4\rangle = |\Phi_4\rangle, \]  

(134)

where \(|\Phi_3\rangle\) and \(|\Phi_4\rangle\) are the Bell states (15) and (16).

As before, this particular way of writing the purification is crucial in our quest for a tight lower bound for the secret-key fraction \(r\). This is true since we will get an additional constraint on the possible values of the \(\lambda\)’s due to the specific operation of the GR10 protocol. This constraint is obtained noting that since Alice always teleports with equal chances the states \(|+\rangle\) or \(|-\rangle\), we must necessarily have,

\[ p_A(0) = p_A(1) = 1/2. \]  

(135)

Note that \(p_A(+) = p_A(-) = 1/2\) is trivially satisfied.

A direct calculation gives

\[ p_A(0) = 1/2 + (\lambda_1 - \lambda_2)\sqrt{2p(1-2p)}, \]  

(136)

\[ p_A(1) = 1/2 - (\lambda_1 - \lambda_2)\sqrt{2p(1-2p)}, \]  

(137)

and Eq. (135) is satisfied for \(p \neq 1/2\) if, and only if,

\[ \lambda_1 = \lambda_2 = \lambda. \]  

(138)

For \(p = 1/2\) this restriction is not needed since we automatically have \(p_A(0) = p_A(1) = 1/2\). However, employing arguments of continuity for the value of the secret-key fraction as a function of \(p\) we can set \(\lambda_1 = \lambda_2\) for the whole range of \(p\) without getting into any physical or mathematical inconsistency.\(^7\)

Before we proceed it is important to define in the present context what we mean by the “error” made by Bob when measuring his qubits due to the presence of Eve. Since now even without Eve it is possible that Bob gets the wrong bit value sent by Alice, it is more appropriate to talk about “deviation” from the expected results in the ideal case (no Eve). Therefore, we define the deviation \(\delta_x\) as follows,

\[ \delta_x = p_{AB}(+,-) + p_{AB}(-,+) - p_{AB}^0(+,-) - p_{AB}^0(-,+), \]  

(139)

\(^7\) We have also employed a different entanglement-based representation designed to handle the \(p = 1/2\) case alone. The secret-key fraction we obtained was the same as the one we got for \(p = 1/2\) using the present entanglement-based representation with the constraint \(\lambda_1 = \lambda_2\).
where \( p_{AB}^1(a, b) \) is the joint probability of Alice and Bob obtaining, respectively, the values \( a \) and \( b \) when Eve does not interfere. Note that \( p_{AB}^1(\cdot, -) + p_{AB}^1(-, \cdot) \) is the probability of Alice and Bob disagreeing about the bit values in the absence of Eve while \( p_{AB}(\cdot, -) + p_{AB}(-, \cdot) \) is the probability of they disagreeing when Eve is present.

We also define the relative deviation of agreement as follows,

\[
\Delta_x = \frac{p_{AB}^0(+, +) + p_{AB}^0(-, -) - p_{AB}(+, +) - p_{AB}(-, -)}{p_{AB}^0(+, +) + p_{AB}^0(-, -)}, \tag{140}
\]

where the last equality comes from the fact that \( \sum_{a,b} p_{AB}(a, b) = 1 \) and from Eq. (139). Note that \( \delta_x = \Delta_x = \varepsilon_x \) whenever the probability of making a mistake in the ideal scenario is zero, as it happens in the BB84 and in the original GR10 protocols (cf. Eq. (19)).

Carrying out the calculations of the joint probabilities we get for Eqs. (139) and (140),

\[
\delta_x = \lambda + \lambda_4 - 1 + 2p, \tag{141}
\]
\[
\Delta_x = \delta_x/(2p). \tag{142}
\]

Using Eqs. (138) and (141), together with the normalization condition \( \sum_{j=1}^4 \lambda_j = 1 \), we can express the four \( \lambda \)’s as follows,

\[
\lambda_1 = \lambda_2 = \lambda, \tag{143}
\]
\[
\lambda_3 = 2p - \delta_x - \lambda = \lambda_+ - \lambda, \tag{144}
\]
\[
\lambda_4 = 1 - 2p + \delta_x - \lambda = \lambda_- - \lambda. \tag{145}
\]

This allows us to write the mutual information between Alice and Bob as

\[
I(A : B) = 1 - h(2p - \delta_x). \tag{146}
\]

The relevant quantities needed to the calculation of the Holevo quantity are obtained in the same fashion already discussed for the BB84 and the original GR10 protocols:

\[
S(\rho_E) = -2\lambda \log \lambda - \lambda_3 \log \lambda_3 - \lambda_4 \log \lambda_4, \tag{147}
\]

\[
\rho_{E|+} = \sum_{j=1}^4 \lambda_j |\psi_j\rangle_E \langle \psi_j| + \sqrt{2p\lambda_1 \lambda_3} (|\psi_1\rangle_E \langle \psi_3| + \text{h.c.})
    - \sqrt{(1 - 2p)\lambda_1 \lambda_4} (|\psi_1\rangle_E \langle \psi_4| + \text{h.c.}) - \sqrt{(1 - 2p)\lambda_2 \lambda_3} (|\psi_2\rangle_E \langle \psi_3| + \text{h.c.})
    - \sqrt{2p\lambda_2 \lambda_4} (|\psi_2\rangle_E \langle \psi_4| + \text{h.c.}), \tag{148}
\]

\[
\rho_{E|-} = \sum_{j=1}^4 \lambda_j |\psi_j\rangle_E \langle \psi_j| - \sqrt{2p\lambda_1 \lambda_3} (|\psi_1\rangle_E \langle \psi_3| + \text{h.c.})
    + \sqrt{(1 - 2p)\lambda_1 \lambda_4} (|\psi_1\rangle_E \langle \psi_4| + \text{h.c.}) + \sqrt{(1 - 2p)\lambda_2 \lambda_3} (|\psi_2\rangle_E \langle \psi_3| + \text{h.c.})
    + \sqrt{2p\lambda_2 \lambda_4} (|\psi_2\rangle_E \langle \psi_4| + \text{h.c.}). \tag{149}
\]
The eigenvalues of $\rho_{E|+}$ and $\rho_{E|-}$ are the same. The non-zero ones can be written as $\lambda_+$ and $\lambda_-$ if we use Eqs. (143)-(145). This allows us to write

$$S(\rho_{E|+}) = S(\rho_{E|-}) = h(\lambda_+) = h(2p - \delta_x).$$

(150)

The Holevo quantity then becomes

$$\chi(A : E) = -2\lambda \log \lambda - (\lambda_+ - \lambda) \log(\lambda_+ - \lambda) - (\lambda_- - \lambda) \log(\lambda_- - \lambda) - h(\lambda_+),$$

leading to the following secret-key fraction

$$r = 1 + \min_{\text{Eve}} \{2\lambda \log \lambda + (\lambda_+ - \lambda) \log(\lambda_+ - \lambda) + (\lambda_- - \lambda) \log(\lambda_- - \lambda)\}.$$  

(151)

Remembering that

$$\lambda_+ = 2p - \delta_x, \quad \lambda_- = 1 - 2p + \delta_x,$$

(152)

we can solve

$$\frac{dr}{d\lambda} = 0$$

(153)

for $\lambda$ and get

$$\lambda_{\text{min}} = \lambda_+ + \lambda_- = (2p - \delta_x)(1 - 2p + \delta_x).$$

(154)

A direct calculation shows that

$$\frac{d^2r(\lambda_{\text{min}})}{d\lambda^2} > 0,$$

(155)

which proves that we got the minimum value of $r(\lambda)$.

Therefore, using Eqs. (153) and (155), we can write the lower bound for the secret-key fraction (152) as follows,

$$r = 1 - 2h(\lambda_+) = 1 - 2h(2p - \delta_x) = 1 - 2h(2p(1 - \Delta_x)],$$

(156)

where the last equality is obtained using Eq. (142).

In the main panel of Fig. 2 we show $r$ as a function of $\Delta_x$ for several values of $p$. We see that the greater the value of $p$ the greater $r$ and thus the greater the secret-key fraction for a given value of $\Delta_x$. For values of $p$ close to its maximal value 1/2 we can guarantee security for the modified GR10 protocol for a deviation rate as great as 11%. As we start decreasing $p$, which is related to less entanglement shared between Alice and Bob, we no longer have $r > 0$ whenever $p \lesssim 0.45$. In other words, whenever the level of entanglement shared between Alice and Bob is such that $p < 0.45$, we can only achieve security using the original GR10 protocol, which works for any value of $p > 1/4$ ($n_1 \neq 0$ and $n_2 \neq 0$). However, the price to pay as we decrease $p$ in the original GR10 protocol is the corresponding reduction of the size of the raw key $R$.

In the inset of Fig. 2 we show how the lower bound to the secret-key fraction responds to a reconciliation protocol (classical error correction step) whose efficiency is not optimal. We model a non-ideal reconciliation protocol
by including an effective reduction of the mutual information between Alice and Bob in the expression for $r$ [5,6,7]. In this scenario it changes to

$$r = \beta I(A:B) - \max_{\text{Eve}} \chi(A:E),$$

(158)

where $0 \leq \beta \leq 1$. Repeating all the steps of the previous calculation when we had $\beta = 1$, we get for the lower bound of the secret-key fraction,

$$r = \beta [1 - h(\lambda_\perp)] - h(\lambda_\perp).$$

(159)

![Fig. 2](image)

**Fig. 2** Main panel: Lower bound of $r$, the secret-key fraction for the modified GR10 protocol as given by Eq. (157), as a function of $\Delta_x$, the relative deviation from the expected probability of Alice and Bob getting the same bit values with and without the presence of Eve. Inset: The same as in the main panel but now assuming a reconciliation efficiency of $\beta = 0.8$.

For discrete variable quantum key distribution schemes, the reconciliation protocols have $\beta \approx 1$. In any case, we have tested how the modified GR10 protocol responds to a reconciliation protocol with $\beta = 0.8$, a very conservative value. As we can see in the inset of Fig. 2, it still keeps working securely for $p \geq 0.46$ when we have deviation rates $\Delta_x$ lower or equal to 1% while for $p = 0.49$ we have security for $\Delta_x \lesssim 7\%$.

It is worth mentioning that the lower bound for the secret-key fraction computed above implies that the modified GR10 protocol works for $n_1 = n_2$ and, in particular, for $n_1 = n_2 = 1$. The latter case corresponds to using the standard and deterministic probabilistic teleportation protocol to establish a secret-key between Alice and Bob, where Alice teleports only the states $|+\rangle$ and $|-\rangle$. In other words, this means that by using the teleportation protocol as originally presented [14], we can establish a secret key between Alice and
Bob employing only orthogonal states to encode bit values teleported from Alice to Bob. There is no need to use non-orthogonal states as in the BB84 or B92 protocols [11].

This “counterintuitive” behavior is readily understood by noting that the teleportation protocol has a built-in probabilistic aspect that cannot be overcome by any “superpowerful” quantum Eve. Indeed, if quantum mechanics is the correct description of nature, no one, not even Eve, has control over what will be the Bell state measured by Alice. This inherently probabilistic aspect of the quantum teleportation protocol precludes Eve from tampering with the teleportation protocol without being detected by Alice and Bob. This is so because the unitary operations that Bob needs to implement on his qubits at the end of the protocol are dictated by what Alice informs him of the results of her Bell state measurements and by the entangled resource shared between them. In this way, if Eve probes too deep trying to figure out the state teleported to Bob, she will cause changes to the Bell state shared between Alice and Bob. As such, the correct unitary operation that Bob must implement to correct his qubit will change too. Bob, being unaware of this change, will implement the unitary operation associated to the original Bell state shared with Alice. The use of the wrong unitary operation will make Bob assign the wrong value for the bit teleported from Alice and will lead to Alice and Bob detecting the presence of Eve when they compare the bit values of a sample of the shared string of bits. Mathematically, this intuitive fact will eventually reflect itself in a positive secret-key rate \( r \) even when \( n_1 = n_2 \).

We end this section by pointing to two other ways in which we can modify the GR10 protocol. The first one corresponds to increasing the number of quantum states employed to encode the bits teleported from Alice to Bob. We can, similarly to the BB84 protocol, use the states \( |0\rangle \) and \( |1\rangle \) together with the states \( |+\rangle \) and \( |--\rangle \) to encode the bit values. In this case we will be using non-orthogonal states to encode the bits and we can think of the GR10 protocol as an additional layer of security to the BB84 protocol. Second, we can choose to work with different post-selected subensembles. For example, instead of considering all runs of the teleportation protocol as we did above, we can select only the cases in which Bob sent the partially entangled state with \( n = n_1 \), discarding the \( n = n_2 \) cases. Or we can work with the cases in which Alice implemented the generalized Bell measurements with \( m = n_1 \), discarding the \( m = n_2 \) cases. This will reduce the size of the raw key \( R \), but if \( n_1 > n_2 \) we can show that by working with these subsets we can get a positive secret-key fraction (security) for a wider range of the deviation \( \Delta_x \) when compared to the case where all runs of the teleportation protocol is accepted as a valid outcome.

\footnote{The entanglement-based representation and all the calculations leading to the secret-key fraction \( r \) when we deal with these subensembles are equal to the ones shown in Sec. 4. The only change is in the value of \( p \), Eq. \( 156 \). It still depends on \( n_1 \) and \( n_2 \) but has a different functional form.}
5 Conclusion

In this work we presented a rigorous security proof for the GR10 quantum key distribution protocol, whose operation is based on the probabilistic teleportation protocol and on the use of only orthogonal quantum states to encode the bits of the secret key [15]. Being more specific, we have carried out the asymptotic security analysis of the GR10 protocol against all types of individual and collective attacks, determining the error rates below which we guarantee a secure operation of the key distribution scheme. Moreover, applying the results of Refs. [36,37,38,39] we argued that the present security analysis is easily extended to coherent attacks, leading to the unconditional security of the GR10 protocol.

Furthermore, we revisited the security analysis of the BB84 protocol [4] by exploring the non-uniqueness of the Schmidt decomposition (purification) that describes the quantum state of Alice, Bob, and the eavesdropper Eve after the latter has tampered with the key transmission. This allowed us to show that the BB84 protocol is secure for greater values of error rates than the standard security analysis predicted [6]. This non-uniqueness of the purification was the key ingredient allowing us to obtain tight lower bounds for the secret-key fraction of the GR10 protocol and its modified version.

We also showed a modified version of the GR10 protocol that operates deterministically, providing its full security analysis. We showed that this version of the GR10 protocol is still secure even if we use only orthogonal states to encode the bit values teleported from Alice to Bob. We showed that the price to pay when going from the probabilistic to the deterministic protocol is a reduction of the value of the secret-key fraction, specially when the degree of entanglement shared between Alice and Bob is small.

Finally, we would like to point out two possible extensions of the ideas here presented that we believe is worthy of investigation. First, it would be interesting to study how to devise a GR10-like protocol that operates with qudits instead of qubits. The main quest here is to obtain a scenario where working with qudits leads to an increase of the error rate below which the protocol is secure. The second extension is a bit more difficult and it lies in how to reshape the GR10 protocol in order to make it work with continuous variable systems.

Acknowledgments

DL thanks CAPES (Brazilian Agency for the Improvement of Personnel of Higher Education) for funding and GR thanks the Brazilian agencies CNPq (National Council for Scientific and Technological Development) and CNPq and FAPERJ (State of Rio de Janeiro Research Foundation) for financial support through the National Institute of Science and Technology for Quantum Information.
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