STEENROD COALGEBRAS II. SIMPLICIAL COMPLEXES

JUSTIN R. SMITH

ABSTRACT. In this paper, we extend earlier work by showing that if \( X \) and \( Y \) are ordered simplicial complexes (i.e. simplicial sets whose simplices are determined by their vertices), a morphism \( g: N(X) \to N(Y) \) of Steenrod coalgebras (normalized chain-complexes equipped with extra structure) induces one of topological realizations \( \hat{g}: |X| \to |Y| \). If \( g \) is an isomorphism, then it induces an isomorphism between \( X \) and \( Y \), implying that \( |X| \) and \( |Y| \) are homeomorphic.

1. INTRODUCTION

It is well-known that the Alexander-Whitney coproduct is functorial with respect to simplicial maps. If \( X \) is a simplicial set, \( C(X) \) is the unnormalized chain-complex and \( R.S.2 \) is the bar-resolution of \( \mathbb{Z}_2 \) (see [9]), it is also well-known that there is a unique homotopy class of \( \mathbb{Z}_2 \)-equivariant maps (where \( \mathbb{Z}_2 \) transposes the factors of the target)

\[
\xi_X: R.S.2 \otimes C(X) \to C(X) \otimes C(X)
\]

cohomology, and that this extends the Alexander-Whitney diagonal. We will call such structures, Steenrod coalgebras and the map \( \xi_X \) the Steenrod diagonal. Done carefully (see section A), this Steenrod diagonal is functorial.

In [14], the author defined the functor \( \mathcal{C}(\ast) \) on simplicial sets — essentially the chain complex equipped with the structure of a coalgebra over an operad \( \mathcal{G} \). This coalgebra structure determined all Steenrod and other cohomology operations. Since these coalgebras are not nilpotent\(^1\) they have a kind of “transcendental” structure that contains much more information. In [13], the author showed that this transcendental structure even manifests in the sub-operad of \( \mathcal{G} \) generated by \( \mathcal{G}(2) = R.S.2 \) and proved

\[^1\text{In a nilpotent coalgebra, iterated coproducts of elements “peter out” after a finite number of steps. See [10, chapter 3] for the precise definition.}\]
Theorem. If $X$ and $Y$ are pointed reduced simplicial sets and 

$$f: C(X) \to C(Y)$$

is a morphism of Steenrod coalgebras — over unnormalized chain-complexes — then $f$ induces a commutative diagram

\[
\begin{array}{ccc}
X & \to & Y \\
\downarrow g_X & & \downarrow g_Y \\
\mathfrak{d} \circ \hat{f}(X) & \to & \mathfrak{d} \circ \hat{f}(Y) \\
\downarrow \phi_\mathfrak{d}(\hat{f}(X)) & & \downarrow \phi_\mathfrak{d}(\hat{f}(Y)) \\
\mathbb{Z}_\infty(\mathfrak{d} \circ \hat{f}(X)) & \xrightarrow{f_\infty} & \mathbb{Z}_\infty(\mathfrak{d} \circ \hat{f}(Y)) \\
\downarrow q_\mathfrak{d}(\hat{f}(X)) & & \downarrow q_\mathfrak{d}(\hat{f}(Y)) \\
\tilde{\mathbb{Z}}(\mathfrak{d} \circ \hat{f}(X)) & \xrightarrow{\tilde{f}} & \tilde{\mathbb{Z}}(\mathfrak{d} \circ \hat{f}(Y)) \\
\end{array}
\]

where $g_X$ and $g_Y$ are homotopy equivalences if $X$ and $Y$ are Kan complexes — and homotopy equivalences of their topological realizations otherwise. In particular, if $X$ and $Y$ are nilpotent and $f$ is an integral homology equivalence, then the topological realizations $|X|$ and $|Y|$ are homotopy equivalent.

Here, $\hat{f}$ and $\mathfrak{d}$ are functors defined in definition 2.2.

It follows that that the $C(\ast)$-functor determines a nilpotent space’s weak homotopy type. In the present paper, we complement the results of [13] by showing:

Corollary. 4.10. If $X$ and $Y$ are ordered simplicial complexes, any purely algebraic chain map of normalized chain complexes

$$f: N(X) \to N(Y)$$

that makes the diagram

\[
\begin{array}{ccc}
RS_2 \otimes N(X) & \xrightarrow{1 \otimes f} & RS_2 \otimes N(Y) \\
\downarrow \xi_X & & \downarrow \xi_Y \\
N(X) \otimes N(X) & \xrightarrow{f \otimes f} & N(Y) \otimes N(Y) \\
\end{array}
\]

commute induces a map of delta-complexes

$$\hat{f}: \mathfrak{d}(X) \to \mathfrak{d}(Y)$$
which are equipped with canonical inclusions

\[ \iota_X : X \to f \circ d(X) \]
\[ \iota_Y : Y \to f \circ d(Y) \]

that are homotopy equivalences of their topological realizations. If \( f \) is an isomorphism, then \( \hat{f}(\iota_X(X)) = \iota_Y(Y) \) and \( X \) and \( Y \) are isomorphic, hence homeomorphic.

Recall that an ordered simplicial complex is a simplicial set without degeneracies whose simplices are uniquely determined by their vertices (for instance, a piecewise linear manifold). The proof requires \( X \) and \( Y \) to be ordered simplicial complexes and is likely not true for arbitrary simplicial sets. Also note that we require diagram 1.1 to commute exactly, not merely up to a chain-homotopy (as is done when using it to compute Steenrod squares).

This and the main result in [13] imply that old mathematical structures like chain-complexes and Steenrod diagonals encapsulate vast amounts of information about a space — and that the traditional ways of studying them (taking cohomology, for example) throw most of this information away.

The author is indebted to Dennis Sullivan for several interesting discussions.

2. Definitions and Assumptions

Throughout this paper \( C(\ast) \) will denote the unnormalized chain complex and \( N(\ast) \) the normalized one.

We consider variations on the concept of simplicial set.

Definition 2.1. Let \( \Delta_+ \) be the ordinal number category whose morphisms are order-preserving monomorphisms between them. The objects of \( \Delta_+ \) are elements \( n = \{0 \to 1 \to \cdots \to n\} \) and a morphism

\[ \theta : m \to n \]

is a strict order-preserving map \( (i < k \implies \theta(i) < \theta(j)) \). Then the category of delta-complexes, \( D \), has objects that are contravariant functors

\[ \Delta_+ \to \text{Set} \]

to the category of sets. The chain complex of a delta-complex, \( X \), will be denoted \( N(X) \).

Remark. In other words, delta-complexes are just simplicial sets without degeneracies. Note that ordered simplicial complexes are particular types of delta-complexes.
A simplicial set gives rise to a delta-complex by “forgetting” its degeneracies — “promoting” its degenerate simplices to nondegenerate status. Conversely, a delta-complex can be converted into a simplicial set by equipping it with degenerate simplices in a mechanical fashion. These operations define functors:

**Definition 2.2.** The functor

\[ \hat{f} : S \to D \]

is defined to simply drop degeneracy operators (degenerate simplices become nondegenerate). The functor

\[ \hat{d} : D \to S \]

equips a delta complex, \( X \), with degenerate simplices and operators via

\[
\hat{d}(X)_m = \bigsqcup_{m-n} X_n
\]

for all \( m > n \geq 0 \).

**Remark.** The functors \( \hat{f} \) and \( \hat{d} \) were denoted \( F \) and \( G \), respectively, in [11]. Equation [2.1] simply states that we add all possible degeneracies of simplices in \( X \) subject only to the basic identities that face- and degeneracy-operators must satisfy.

Although \( \hat{f} \) promotes degenerate simplices to nondegenerate ones, these new nondegenerate simplices can be collapsed without changing the homotopy type of the complex: although the degeneracy operators are no longer built in to the delta-complex, they still define contracting homotopies.

The definition immediately implies that

**Proposition 2.3.** If \( X \) is a simplicial set and \( Y \) is a delta-complex, \( C(X) = N(\hat{f}(X)) \), \( N(\hat{d}(Y)) = N(Y) \), and \( C(X) = N(\hat{d} \circ \hat{f}(X)) \).

**Definition 2.4.** A simplicial set, \( X \), is defined to be *degeneracy-free* if

\[ X = \hat{d}(Y) \]

for some delta-complex, \( Y \).

**Remark.** Compare definition 1.10 in chapter VII of [5]. In a manner of speaking, \( X \) is freely generated by the degeneracy operators acting on a basis consisting of the simplices of \( Y \). Lemma 1.2 in chapter VII of [5] describes other properties of degeneracy-free simplicial sets (hence of the functor \( \hat{d} \)).

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2 Their definition has a typo, stating that \( \Delta_+ \) consists of *surjections* rather than *injections*. 
In [11], Rourke and Sanderson also showed that one could give a “somewhat more intrinsic” definition of degeneracy-freeness:

**Proposition 2.5.** If \( X \) is a simplicial set, let \( \text{Core}(X) \) consist of the nondegenerate simplices and their faces. This is a delta-complex and there exists a canonical map

\[
c: \partial(\text{Core}(X)) \to X
\]

sending simplices of \( \text{Core}(X) \) to themselves in \( X \) and degeneracies to suitable degeneracies of them. Then \( X \) is degeneracy-free if and only if \( c \) is an isomorphism.

Theorem 1.7 of [11] shows that there exists an adjunction:

\[
(2.2) \quad \partial: D \leftrightarrow S: f
\]

The composite (the counit of the adjunction)

\[
f \circ \partial: D \to D
\]

maps a delta complex into a much larger one — that has an infinite number of (degenerate) simplices added to it. There is a natural inclusion

\[
(2.3) \quad \iota_X: X \to f \circ \partial(X)
\]

and a natural map (the unit of the adjunction)

\[
(2.4) \quad g: \partial \circ f(X) \to X
\]

The functor \( g \) sends degenerate simplices of \( X \) that had been “promoted to nondegenerate status” by \( f \) to their degenerate originals — and the extra degenerates added by \( \partial \) to suitable degeneracies of the simplices of \( X \).

Rourke and Sanderson also prove:

**Proposition 2.6.** If \( X \) is a simplicial set and \( Y \) is a delta-complex then

1. \(|Y| \text{ and } \partial Y| \) are homeomorphic,
2. the map \(|g|: |\partial \circ f(X)| \to |X| \) is a homotopy equivalence, so that \(|\iota_Y|: |Y| \to |f \circ \partial(Y)| \) is a homotopy equivalence,
3. \( f: HS \to HD \) defines an equivalence of categories, where \( HS \) and \( HD \) are the homotopy categories, respectively, of \( S \) and \( D \).

The inverse is \( \partial: HD \to HS \).

**Remark.** Here, \(|*| \) denotes the topological realization functors for \( S \) and \( D \).

**Proof.** The first two statements are proposition 2.1 of [11]. \( \square \)
Definition 2.7. We will denote the category of \( \mathbb{Z} \)-free chain complexes by \( \text{Ch} \) and ones that are \textit{bounded from below} in dimension 0 by \( \text{Ch}_0 \).

We make extensive use of the Koszul Convention (see [6]) regarding signs in homological calculations:

Definition 2.8. If \( f : C_1 \to D_1, g : C_2 \to D_2 \) are maps, and \( a \otimes b \in C_1 \otimes C_2 \) (where \( a \) is a homogeneous element), then \( (f \otimes g)(a \otimes b) \) is defined to be \( (-1)^{\deg(g) \cdot \deg(a)} f(a) \otimes g(b) \).

Remark 2.9. If \( f_i, g_i \) are maps, it isn’t hard to verify that the Koszul convention implies that \( (f_1 \otimes g_1) \circ (f_2 \otimes g_2) = (-1)^{\deg(f_2) \cdot \deg(g_1)} (f_1 \circ f_2 \otimes g_1 \circ g_2) \).

The set of morphisms of chain-complexes is itself a chain complex:

Definition 2.10. Given chain-complexes \( A, B \in \text{Ch} \) define

\[
\text{Hom}_\mathbb{Z}(A, B)
\]

to be the chain-complex of graded \( \mathbb{Z} \)-morphisms where the degree of an element \( x \in \text{Hom}_\mathbb{Z}(A, B) \) is its degree as a map and with differential

\[
\partial f = f \circ \partial_A - (-1)^{\deg f} \partial_B \circ f
\]

As a \( \mathbb{Z} \)-module \( \text{Hom}_\mathbb{Z}(A, B)_k = \prod_j \text{Hom}_\mathbb{Z}(A_j, B_{j+k}) \).

Remark. Given \( A, B \in \text{Ch}^{S_n} \), we can define \( \text{Hom}_{\mathbb{Z},S_n}(A, B) \) in a corresponding way.

Definition 2.11. If \( G \) is a discrete group, let \( \text{Ch}_0^G \) denote the category of chain-complexes equipped with a right \( G \)-action. This is again a closed symmetric monoidal category and the forgetful functor \( \text{Ch}_0^G \to \text{Ch}_0 \) has a left adjoint, \( (-)[G] \). This applies to the symmetric groups, \( S_n \), where we regard \( S_1 \) and \( S_0 \) as the trivial group. The category of collections is defined to be the product

\[
\text{Coll}(\text{Ch}_0) = \prod_{n \geq 0} \text{Ch}_0^{S_n}
\]

Its objects are written \( \mathcal{V} = \{\mathcal{V}(n)\}_{n \geq 0} \). Each collection induces an endofunctor (also denoted \( \mathcal{V} \)) \( \mathcal{V} : \text{Ch}_0 \to \text{Ch}_0 \)

\[
\mathcal{V}(X) = \bigoplus_{n \geq 0} \mathcal{V}(n) \otimes_{\mathbb{Z}^{S_n}} X \otimes^n
\]

where \( X \otimes^n = X \otimes \cdots \otimes X \) and \( S_n \) acts on \( X \otimes^n \) by permuting factors. This endofunctor is a \textit{monad} if the defining collection has the structure of an \textit{operad}, which means that \( \mathcal{V} \) has a unit \( \eta : \mathbb{Z} \to \mathcal{V}(1) \) and
structure maps

\[ \gamma_{k_1, \ldots, k_n}: \mathcal{V}(n) \otimes \mathcal{V}(k_1) \otimes \cdots \otimes \mathcal{V}(k_n) \to \mathcal{V}(k_1 + \cdots + k_n) \]
satisfying well-known equivariance, associativity, and unit conditions — see [15], [7].

We will call the operad \( \mathcal{V} = \{ \mathcal{V}(n) \} \) \( \Sigma \)-cofibrant if \( \mathcal{V}(n) \) is \( \mathbb{Z}S_n \)-projective for all \( n \geq 0 \).

Remark. The operads we consider here correspond to symmetric operads in [15]. The term “unital operad” is used in different ways by different authors. We use it in the sense of Kriz and May in [7], meaning the operad has a 0-component that acts like an arity-lowering augmentation under compositions. Here \( \mathcal{V}(0) = \mathbb{Z} \).

The term \( \Sigma \)-cofibrant first appeared in [2].

We can also define operads in terms of compositions:

**Definition 2.12.** If \( \mathcal{V} \) is an operad with components \( \mathcal{V}(n) \) and \( \mathcal{V}(m) \), define the \( i \)th composition, with \( 1 \leq i \leq n \)

\[ o_i: \mathcal{V}(n) \otimes \mathcal{V}(m) \to \mathcal{V}(n + m - 1) \]

by

\[
\begin{array}{c}
\mathcal{V}(n) \otimes \mathcal{V}(m) \\
\mathcal{V}(n) \otimes \mathbb{Z}^{i-1} \otimes \mathcal{V}(m) \otimes \mathbb{Z}^{n-i} \\
\downarrow_{1 \otimes \eta^{i-1} \otimes 1 \otimes \eta^{n-i}} \\
\mathcal{V}(n) \otimes \mathcal{V}(1)^{i-1} \otimes \mathcal{V}(m) \otimes \mathcal{V}(1)^{n-i} \\
\downarrow_{\gamma} \\
\mathcal{V}(n + m - 1)
\end{array}
\]

Here \( \eta: \mathbb{Z} \to \mathcal{V}(1) \) is the unit.

Remark. Operads were originally called composition algebras and defined in terms of these operations — see [4].

It is well-known that the compositions and \( \gamma \) determine each other — see definition 2.12 and proposition 2.13 of [15]. It is also well-known (see lemma 2.14 of [15]) that:

**Lemma 2.13.** Compositions obey the identities

\[
(a \circ_i b) \circ_j c = \begin{cases} 
(-1)^{\dim b \cdot \dim c} (a \circ_{j-n+1} c) \circ_i b & \text{if } i + n - 1 \leq j \\
(a \circ_i (b \circ_{j-i+1} c)) & \text{if } i \leq j < i + n - 1 \\
(-1)^{\dim b \cdot \dim c} (a \circ_i c) \circ_{i+m-1} b & \text{if } 1 \leq j < i
\end{cases}
\]
where arity \( c = m \), arity \( a = n \), and

\[
a \circ_{\sigma(i)} (\sigma \cdot b) = T_{1, \ldots, n, \ldots, 1} (\sigma) \cdot (a \circ_i b)
\]

for \( \sigma \in S_n \), where \( T_{\alpha_1, \ldots, \alpha_n} (\sigma) \in S_{\sum \alpha_i} \) is a permutation that permutes the \( n \) blocks

\[
\{1, \ldots, \alpha_1\}, \{\alpha_1 + 1, \alpha_1 + \alpha_2\}, \ldots, \\
\{\alpha_1 + \cdots + \alpha_{n-1} + 1, \alpha_1 + \cdots + \alpha_n\}
\]

via \( \sigma \).

A simple example of an operad is:

**Example 2.14.** For each \( n \geq 0 \), \( \mathcal{G}_0(n) = \mathbb{Z}S_n \), with structure-map a \( \mathbb{Z} \)-linear extension of

\[
\gamma_{\alpha_1, \ldots, \alpha_n}: S_n \times S_{\alpha_1} \times \cdots \times S_{\alpha_n} \to S_{\alpha_1 + \cdots + \alpha_n}
\]

defined by

\[
\gamma_{\alpha_1, \ldots, \alpha_n}(\sigma \times \theta_1 \times \cdots \times \theta_n) = T_{\alpha_1, \ldots, \alpha_n}(\sigma) \circ (\theta_1 \oplus \cdots \oplus \theta_n)
\]

with \( \sigma \in S_n \) and \( \theta_i \in S_{\alpha_i} \), where \( T_{\alpha_1, \ldots, \alpha_n}(\sigma) \in S_{\sum \alpha_i} \) is defined above, in lemma 2.8. See \([14]\) for explicit formulas and computations. Another important operad is:

The operad, \( \mathcal{G} \), defined in \([14]\) is given by \( \mathcal{G}(n) = RS_n \) — the bar-resolution of \( \mathbb{Z} \) over \( \mathbb{Z}S_n \). This is well-known (like the closely-related Barrett-Eccles operad — see \([1]\)) to be a Hopf-operad, i.e. equipped with an operad morphism

\[
\delta: \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}
\]

and is important in topological applications. See \([14]\) for formulas for the structure maps.

For the purposes of this paper, the main example of an operad is

**Definition 2.15.** Given any \( C \in \text{Ch} \), the associated coendomorphism operad, \( \text{CoEnd}(C) \) is defined by

\[
\text{CoEnd}(C)(n) = \text{Hom}_{\mathbb{Z}}(C, C^{\otimes n})
\]

Its structure map

\[
\gamma_{\alpha_1, \ldots, \alpha_n}: \text{Hom}_{\mathbb{Z}}(C, C^{\otimes n}) \otimes \text{Hom}_{\mathbb{Z}}(C, C^{\otimes \alpha_1}) \otimes \cdots \otimes \text{Hom}_{\mathbb{Z}}(C, C^{\otimes \alpha_n}) \to \\
\text{Hom}_{\mathbb{Z}}(C, C^{\otimes \alpha_1 + \cdots + \alpha_n})
\]
simply composes a map in $\text{Hom}_\mathbb{Z}(C, C^\otimes n)$ with maps of each of the $n$ factors of $C$.

This is a non-unital operad, but if $C \in \text{Ch}$ has an augmentation map $\varepsilon: C \to \mathbb{Z}$ then we can regard $\varepsilon$ as the generator of $\text{CoEnd}(C)(0) = \mathbb{Z} \cdot \varepsilon \subset \text{Hom}_\mathbb{Z}(C, C^\otimes 0) = \text{Hom}_\mathbb{Z}(C, \mathbb{Z})$.

We use the coendomorphism operad to define the main object of this paper:

**Definition 2.16.** A coalgebra over an operad $\mathcal{V}$ is a chain-complex $C \in \text{Ch}$ with an operad morphism $\alpha: \mathcal{V} \to \text{CoEnd}(C)$, called its structure map. We will sometimes want to define coalgebras using the adjoint structure map,

$$\alpha: C \to \prod_{n \geq 0} \text{Hom}_{\mathbb{Z}S_n}(\mathcal{V}(n), C^\otimes n)$$

where $S_n$ acts on $C^\otimes n$ by permuting factors or the set of chain-maps $\alpha_n: C \to \text{Hom}_{\mathbb{Z}S_n}(\mathcal{V}(n), C^\otimes n)$ for all $n \geq 0$ or even $\beta_n: \mathcal{V}(n) \otimes C \to C^\otimes n$.

It is not hard to see how compositions (in definition $2.12$) relate to coalgebras.

**Proposition 2.17.** Let $\beta_n: \mathcal{V}(n) \otimes C \to C^\otimes n$ for all $n \geq 0$ define a coalgebra over an operad $\mathcal{V}$ and, for any $x \in \mathcal{V}(n)$ and any $n \geq 0$ define

$$\Delta_x = \beta_n(x \otimes \ast): C \to C^\otimes n$$

If $x \in \mathcal{V}(n)$ and $y \in \mathcal{V}(m)$, then

$$\Delta_{y \circ_i x} = 1 \otimes \cdots \otimes 1 \otimes \Delta_y \otimes 1 \otimes \cdots \otimes \Delta_x$$

with $i$th position.

**Proof.** Immediate, from definitions $2.12$ and $2.15$. \hfill \square

2.1. Types of coalgebras.

**Example 2.18.** Coassociative coalgebras are precisely the coalgebras over $\mathcal{S}_0$ (see $2.14$).

**Definition 2.19.** Commute is an operad defined to have one basis element $\{b_i\}$ for each integer $i \geq 0$. Here the arity of $b_i$ is $i$ and the degree is $0$ and the these elements satisfy the composition-law: $\gamma(b_n \otimes b_k \otimes \cdots \otimes b_{k_n}) = b_K$, where $K = \sum_{i=1}^{n} k_i$. The differential of this operad is identically zero. The symmetric-group actions are trivial.
Example 2.20. Coassociative, commutative coalgebras are the coalgebras over \textit{Commute}.

We can define a concept dual to that of a free algebra generated by a set:

**Definition 2.21.** Let \( D \) be a coalgebra over an operad \( \mathcal{V} \), equipped with a \( \text{Ch} \)-morphism \( \varepsilon: [D] \to E \), where \( E \in \text{Ch} \). Then \( D \) is called the \textit{cofree coalgebra} over \( \mathcal{V} \) cogenerated by \( \varepsilon \) if any morphism in \( \text{Ch} \)
\[
f: [C] \to E
\]
where \( C \in \mathcal{S}_0 \), induces a \textit{unique} morphism in \( \mathcal{S}_0 \)
\[
\alpha_f: C \to D
\]
that makes the diagram
\[
[C] \xrightarrow{[\alpha_f]} [D] \xrightarrow{\varepsilon} E
\]
Here \( \alpha_f \) is called the \textit{classifying map} of \( f \). If \( C \in \mathcal{S}_0 \) then
\[
\alpha_f: C \to L_{\mathcal{V}}[C]
\]
will be called the \textit{classifying map} of \( C \).

**Remark 2.22.** This universal property of cofree coalgebras implies that they are \textit{unique} up to isomorphism if they exist.

The paper [15] explicitly constructs cofree coalgebras for many operads:

- \( L_{\mathcal{V}}C \) is the \textit{general} cofree coalgebra over the operad \( \mathcal{V} \) — here, \( C \), is a chain-complex that is not necessarily concentrated in nonnegative dimensions. Then [15] constructs \( D = L_{\mathcal{V}}E \) as the maximal submodule of
\[
\prod_{n=1}^{\infty} \text{Hom}_{\mathbb{Z}S_n}(\mathcal{V}(n), E^{\otimes n})
\]
on which the dual of the structure-maps of \( \mathcal{V} \) define a coalgebra-structure: let \( i: D \to \prod_{n=1}^{\infty} \text{Hom}_{\mathbb{Z}S_n}(\mathcal{V}(n), E^{\otimes n}) \) be the inclusion of chain-complexes. In the notation of definition 2.21, an \( \mathcal{V} \)-coalgebra, \( C \), is defined by its \textit{structure map} (see equation 2.6)
\[
s: C \to \prod_{n=1}^{\infty} \text{Hom}_{\mathbb{Z}S_n}(\mathcal{V}(n), C^{\otimes n})
\]
and its classifying map $\alpha_f: D \to L_\gamma C$ is the coalgebra morphism defined by the diagram

$$
\begin{array}{ccc}
C & \xrightarrow{s} & \prod_{n=1}^{\infty} \text{Hom}_{\mathbb{Z}S_n}(V(n), C^\otimes n) \\
\downarrow{\alpha_f} & & \downarrow{\prod_{n=1}^{\infty} \text{Hom}_{\mathbb{Z}S_n}(1, f^\otimes n)} \\
D & \xrightarrow{f} & \prod_{n=1}^{\infty} \text{Hom}_{\mathbb{Z}S_n}(V(n), E^\otimes n)
\end{array}
$$

An inductive argument shows that this is the unique coalgebra morphism compatible with $f$.

3. STEENROD COALGEBRAS

We begin with:

**Definition 3.1.** A Steenrod coalgebra, $(C, \delta)$ is a chain-complex $C \in \text{Ch}$ equipped with a $\mathbb{Z}_2$-equivariant chain-map

$$
\delta: RS_2 \otimes C \rightarrow C \otimes C
$$

where $\mathbb{Z}_2$ acts on $C \otimes C$ by swapping factors and $RS_2$ is the bar-resolution of $\mathbb{Z}$ over $\mathbb{Z}S_2$. A morphism $f: (C, \delta_C) \to (D, \delta_D)$ is a chain-map $f: C \to D$ that makes the diagram

$$
\begin{array}{ccc}
C & \xrightarrow{f} & D \\
\downarrow{\delta_C} & & \downarrow{\delta_D} \\
C \otimes C & \xrightarrow{f \otimes f} & D \otimes D
\end{array}
$$

commute.

Steenrod coalgebras are very general — the underlying coalgebra need not even be coassociative. The category of Steenrod coalgebras is denoted $\mathcal{F}$.

**Definition 3.2.** Let, $\mathcal{F}$, denote the free operad generated by $RS_2$.

**Remark.** See sections 5.2 and 5.5 of [3] or section 5.8 of [2] for an explicit construction of $\mathcal{F}$. For instance

$$
\mathcal{F}(3) = RS_2 \otimes_{\mathbb{Z}S_2} \left( \underbrace{\mathbb{Z}S_3 \otimes_{\mathbb{Z}S_2} RS_2}_S \oplus \underbrace{\mathbb{Z}S_3 \otimes_{\mathbb{Z}S_2} RS_2}_S \right)
$$

where $S_2 = \mathbb{Z}_2$ swaps the summands and $\mathbb{Z}S_3$ acts on $\mathcal{F}(3)$ by acting on the factors $\mathbb{Z}S_3$ inside the parentheses.
Proposition 3.3. The identity map of $R S_2$ uniquely extends to an operad-morphism
\[ \xi: \mathcal{F} \to \mathcal{G} \]
and the kernel is an operadic ideal (see section 5.2.16 of [8]) denoted $\mathcal{R}$.

Remark. The image, $\xi(\mathcal{F}) \subset \mathcal{G}$, is the suboperad generated by $\mathcal{G}(2) = R S_2$.

Proof. All statements follow immediately from the defining property of free operads. \qed

Although the construction of $\mathcal{F}$ is fairly complex, it is easy to describe coalgebras over $\mathcal{F}$:

Proposition 3.4. The category of coalgebras over $\mathcal{F}$ is identical to that of Steenrod coalgebras.

Proof. If $C$ is an $\mathcal{F}$-coalgebra then there exists a $\mathbb{Z} S_2$-morphism
\[ \mathcal{F}(2) \otimes C = R S_2 \otimes C \to C \otimes C \]
so $C$ is a Steenrod coalgebra. If $C$ is a Steenrod coalgebra, it has an adjoint structure map
\[ R S_2 \to \text{Hom}_\mathbb{Z}(C, C \otimes C) = \text{CoEnd}(C)(2) \]
that uniquely extends to an operad-morphism
\[ \mathcal{F} \to \text{CoEnd}(C) \]
It is also clear that this correspondence respects morphisms. \qed

This has a number of interesting consequences:

Theorem 3.5. If $C$ is a chain-complex, there exists a universal Steenrod coalgebra $L_{\mathcal{F}} C$ — the cofree coalgebra over $\mathcal{F}$ cogenerated by $C$ — equipped with a chain-map
\[ \varepsilon: L_{\mathcal{F}} C \to C \]
with the property that, given any Steenrod coalgebra $D$ and any chain-map $f: D \to C$, there exists a unique morphism of Steenrod coalgebras
\[ \tilde{f}: D \to L_{\mathcal{F}} C \]
that makes the diagram
\[
\begin{array}{ccc}
D & \xrightarrow{\tilde{f}} & L_{\mathcal{F}} C \\
\downarrow f & & \downarrow \varepsilon \\
C & & C
\end{array}
\]
commute.

Proof. The conclusions are nothing but the defining properties of a cofree coalgebra over $\mathcal{F}$. So the result follows immediately from proposition 3.4.

□

4. Morphisms of Steenrod coalgebras

Proposition A.4 proves that if $e_n = [(1, 2)|\cdots|(1, 2)] \in RS_2$ and $x \in N(X)$ is the image of a $k$-simplex, then

$$\xi_X(e_k \otimes x) = \eta_k \cdot x \otimes x$$

where $\eta_k = (-1)^{k(k-1)/2}$ and

$$\xi_X: RS_2 \otimes N(X) \to N(X) \otimes N(X)$$

is the Steenrod diagonal (see definition 3.1).

Definition 4.1. If $k, m$ are positive integers, $C$ is a chain-complex, and $F_{2,m} = e_m$ and $F_{k,m} = e_m \circ_1 \cdots \circ_1 e_m \in \mathcal{F}(k)$ — compositions in the operad $\mathcal{F}$ — set

$$\rho_m = (\eta_m \cdot E_{2,m}, \eta_m^2 \cdot E_{3,m}, \eta_m^3 \cdot E_{4,m}, \ldots) \in \prod_{n=2}^{\infty} \mathcal{F}(n)$$

with $\eta_m = (-1)^{m(m-1)/2}$ and define

$$\gamma_m: \prod_{n=2}^{\infty} \text{Hom}_{\mathbb{Z}S_n}(\mathcal{F}(n), C_{\otimes n}) \to \prod_{n=2}^{\infty} C_{\otimes n}$$

via evaluation on $\rho_m$.

We have

Corollary 4.2. If $X$ is an ordered simplicial complex and $c \in C(X)$ is an element generated by an $n$-simplex, then the image of $c$ under the composite

$$N(X)_n \xrightarrow{\alpha_{N(X)}} L_\mathcal{F}C(X) \xrightarrow{\prod_{k=1}^\infty \text{Hom}_{\mathbb{Z}S_k}(\mathcal{F}(k), N(X)^{\otimes k})} \prod_{k=1}^\infty N(X)^{\otimes k}$$

is

$$e(c) = (c, c \otimes c, \ldots)$$

Here

$$\alpha_{N(X)}: N(X) \to L_\mathcal{F}C(X)$$

is the classifying map to a cofree coalgebra defined in definition 2.21.

Here, $N(X)$ is the (normalized) chain complex of $X$. 
Proof. This follows immediately from proposition A.4 and proposition 2.17. □

Lemma B.1 implies that:

**Corollary 4.3.** Let $X$ be a simplicial set and suppose

$$f: N^n = N(\Delta^n) \to N(X)$$

is a nontrivial Steenrod coalgebra morphism. Then the image of the generator $\Delta^n \in N(\Delta^n)$ is a generator of $N(X)$ defined by an $n$-simplex of $X$.

**Remark.** As the statement implies, we do not need $X$ to be an ordered simplicial complex in this result.

**Proof.** Suppose

$$f(\Delta^n) = \sum_{k=1}^{t} c_k \cdot \sigma^n_k \in N(X)$$

where the $\sigma^n_k$ are images of $n$-simplices of $X$ and the $c_k \in \mathbb{Z}$. If $f(\Delta^n)$ is not equal to one of the $\sigma^n_k$ (i.e. if more than one of the $c_k$ is nonzero, or if only one is nonzero but not equal to 1), lemma [B.1] implies that its image under $\gamma_n \circ \alpha_{N(X)} \circ f$ in corollary [4.2] is linearly independent of the images of the $\sigma^n_k$, a contradiction. □

We also conclude that:

**Corollary 4.4.** If $f: N(\Delta^n) \to N(\Delta^n)$ is

1. an isomorphism of Steenrod coalgebras in dimension $n$ and
2. an endomorphism of Steenrod coalgebras in lower dimensions

then $f$ is the identity map.

**Proof.** Corollary [4.3] implies that $f$ maps every sub-simplex of $\Delta^n$ to one of the same dimension. We may identify a $k$ dimensional sub-simplex of $\Delta^n$ with a set of $k+1$ vertices $\{i_0, \ldots, i_k\}$ with $i_0 < \cdots < i_k$.

We are given that $f$ is an isomorphism in dimension $n$ — i.e. it is bijective. We use downward induction on dimension to show that it is bijective in lower dimensions.

If $f$ is bijective in dimension $k$, every set of $k+1$ vertices $\{j_0, \ldots, i_k\}$ occurs exactly once as $f(\Delta^k_\ell)$ for some $\ell$. Given any $k$-simplex, $\Delta^k$, with $f(\Delta^k) = \{j_0, \ldots, i_k\}$, the boundary $\partial f(\Delta^k)$ is a linear combination of $k + 1$ distinct faces — namely all $k$-element subsets of $\{j_0, \ldots, i_k\}$. Since $f$ is a chain-map $f(\partial \Delta^k)$ must be a linear combination of all $k$-element subsets of $\{j_0, \ldots, i_k\}$. It follows that every $k$-element subset of every $k+1$-element set occurs in $f(\Delta^k_{\ell-1})$ for some
t = 1, \ldots, \binom{n+1}{k}$. The Pigeonhole Principle implies that each such $k$-element subset occurs exactly once in the image of $f$, so that $f$ is bijective on $k-1$-simplices.

We conclude that $f$ is an automorphism of $N(\Delta^n)$. Now we show that $f$ is the identity map:

In dimension 0, let $f$ be a permutation, $\pi: \{0, \ldots, n\} \to \{0, \ldots, n\}$ of vertices. If $s = (i_1, i_2)$ with $i_1 < i_2$ is a 1-simplex, $f(s) = (j_1, j_2)$ with $j_1 < j_2$ is a 1-simplex, and

$$f(\partial s) = f(i_1) - f(i_2) = \partial f(s) = (j_1) - (j_2) = (\pi i_1) - (\pi i_2)$$

Given the signs of the terms in the boundary, we conclude that $i_1 < i_2 \implies \pi i_1 < \pi i_2$ for all $0 \leq i_1 < i_2 \leq n$ (in other words, $\pi$ cannot swap the ends of a 1-simplex). This forces $\pi$ to be the identity permutation. It follows that $f$ is the identity map on 1-simplices.

If $k > 1$, $w = (i_0, \ldots, i_k)$ is any $k$-simplex in $\Delta^n$, and

$$\delta_k = (1 \otimes \cdots \otimes \delta) \circ \cdots \circ \delta: N(\Delta^n) \to N(\Delta^n) \otimes^k$$

where $\delta: N(\Delta^n) \to N(\Delta^n) \otimes N(\Delta^n)$ is the Alexander-Whitney diagonal, then the image of $\delta_k(w)$ in

$$N(\Delta^n) \otimes^k / (N(\Delta^n) \otimes^k)_0$$

is

$$Z = (i_0, i_1) \otimes (i_1, i_2) \otimes \cdots \otimes (i_{k-1}, i_k) \in N(\Delta^n)_{1} \otimes^k$$

where each edge, $(i_t, i_{t+1})$, is the result of a sequence, $F_0 \cdots F_{t-1} F_{t+1} \cdots F_n$, of face-operations applied to $w$. Since these edges are mapped via the identity map (by the argument above) $f \otimes^k(Z) = Z \in N(\Delta^n)_{1} \otimes^k$, which implies that $f(w)$ has the same vertices as $w$ so $f(w) = w$. It follows that $f$ is the identity map in all dimensions. \(\square\)

A similar line of reasoning implies that:

**Corollary 4.5.** Let $X$ be an ordered simplicial complex and let

$$f: N(\Delta^n) \to N(X)$$

map $\Delta^n$ to an $n$-simplex $\sigma \in N(X)$ defined by the inclusion $\iota: \Delta^n \to X$. Then

$$f(N(\Delta^n)) \subset N(\iota)(N(\Delta^n))$$

so that $f = N(\iota)$. 
Proof. Since $X$ is an ordered simplicial complex, the map $\imath$ is an inclusion.

Suppose $\Delta^k \subset \Delta^n$ and $f(N(\Delta^k))_k \subset N(\Delta^k)_k$. Since the boundary of $\Delta^k$ is an alternating sum of $k+1$ faces, and since they must map to $k-1$-dimensional simplices of $N(f(\Delta^k))$ with the same signs (so no cancellations can take place) we must have $f(F(\Delta^k)) \subset N(f(\Delta^k))$ and the conclusion follows by downward induction on dimension. The final statements follow immediately from corollary 4.4. □

Next, we consider degeneracies:

**Proposition 4.6.** If $n > m$, then the Steenrod-coalgebra morphisms

$$f: N(\Delta^n) \to N(\Delta^m)$$

are in a 1-1 correspondence with surjective morphisms

$$f: n \twoheadrightarrow m$$

of ordered sets, where $n = 0 < \cdots < n$ and $m = 0 < \cdots < m$.

Proof. Certainly any Steenrod-coalgebra morphism, $f$, defines a surjective morphism of vertices: $f = \alpha(f)$. Given $f$, corollary 4.3 implies that the $m$-dimensional sub-simplices of $\Delta^n$ can either map to $\Delta^m$ (in a unique way, by corollary 4.4) or 0. The sets

$$f^{-1}(0), \ldots, f^{-1}(m)$$

represent sub-simplices of $\Delta^n$, which we can imagine that $f$ collapses to points — defining a morphism of ordered simplicial complexes and a chain-map. Each possible selection $i_0 \in f^{-1}(0), \ldots, i_m \in f^{-1}(m)$ defines a unique $m$-simplex $\Delta^m_{i_0, \ldots, i_m} \subset \Delta^n$ for which there is a unique Steenrod coalgebra morphism (by corollary 4.4)

(4.1)

$$f_{i_0, \ldots, i_m}: N(\Delta^m_{i_0, \ldots, i_m}) \to N(\Delta^m)$$

We can define a Steenrod coalgebra morphism

$$f: N(\Delta^n) \to N(\Delta^m)$$

that sends each of these to $N(\Delta^m)$ and all other sub-simplices of $\Delta^n$ to 0. We will call this morphism $\beta(f)$.

It is not hard to see that $f = \alpha \circ \beta(f)$. That $f = \beta \circ \alpha(f)$ follows from the uniqueness of the morphisms $\{f_{i_0, \ldots, i_m}\}$ in equations 4.1. It follows that $\alpha$ and $\beta$ define inverse one-to-one correspondences. □

We define a complement to the $N(*)$-functor:
Definition 4.7. Define a functor
\[ \text{hom}_S(\star, \ast) : \mathcal{S} \to \mathcal{D} \]
to the category of delta-complexes, as follows:
If \( C \in \mathcal{S} \), define the \( n \)-simplices of \( \text{hom}_S(\star, C) \) to be the Steenrod coalgebra morphisms
\[ N^n \to C \]
where \( N^n = N(\Delta^n) \) is the normalized chain-complex of the standard \( n \)-simplex, equipped with the Steenrod coalgebra structure defined in theorem A.2.
Face-operations are duals of coface-operations
\[ d_i : [0, \ldots, i - 1, i + 1, \ldots n] \to [0, \ldots, n] \]
with \( i = 0, \ldots, n \) and vertex \( i \) in the target is not in the image of \( d_i \).

Proposition 4.8. If \( X \) is an ordered simplicial complex there exists a natural inclusion
\[ u_X : X \to \text{hom}_S(\star, N(X)) \]

Remark. This is also true if \( X \) is an arbitrary simplicial set and we replace \( N(X) \) with the functor \( C(X) \) defined in [13]. A priori, it is not clear that \( \text{hom}_S(\star, N(X)) \) is an ordered simplicial complex.

Proof. To prove the first statement, note that any simplex \( \Delta^k \) in \( X \) comes equipped with a canonical inclusion
\[ \iota : \Delta^k \to X \]
The corresponding order-preserving map of vertices induces an Steenrod-coalgebra morphism
\[ N(\iota) : N(\Delta^k) = N^k \to N(X) \]
so \( u_X \) is defined by
\[ \Delta^k \mapsto N(\iota) \]
It is not hard to see that this operation respects face-operations. \( \square \)

So, \( \text{hom}_S(\star, N(X)) \) naturally contains a copy of \( X \). The interesting question is how much more it contains:

Theorem 4.9. If \( X \in \text{SC} \) is an ordered simplicial complex, then
\[ \text{hom}_S(\star, N(X)) = \mathcal{A} \circ \mathcal{A}(X) \]
and the canonical inclusion
\[ \iota_X : X \to \text{hom}_S(\star, N(X)) \]
defined in proposition 4.8 is the unit of the adjunction in equation 2.3 and a homotopy equivalence of topological realizations.
Proof. This follows immediately from corollary 4.3 which implies that simplices map to simplices and corollary 4.5 which implies that these maps are unique. Proposition 4.6 implies that

$$\text{hom}_S(\star, N(X)) = \bigsqcup_{m-n} X_n = f \circ \partial(X)$$

since morphisms of simplices are uniquely determined by their corresponding vertex-set maps. These added degenerate simplices are only subject to the basic identities between face- and degeneracy-operators. The conclusion follows from proposition 2.6. □

Corollary 4.10. If X and Y are ordered simplicial complexes, any morphism of Steenrod coalgebras

$$g: N(X) \to N(Y)$$

induces a map

$$\hat{g}: f \circ \partial(X) \to f \circ \partial(Y)$$

of delta-complexes and a map of topological realizations

$$|X| \xrightarrow{\iota_X} |f \circ \partial(X)| \xrightarrow{\hat{g}} |f \circ \partial(Y)| \xrightarrow{|\iota_Y|^{-1}} |Y|$$

where $\iota_X$ and $\iota_Y$ are defined in equation 2.3, $|\iota_Y|^{-1}$ is a homotopy inverse, and $|\star|$ is topological realization.

If $g$ is an isomorphism then $\hat{g}$ is an isomorphism and $\hat{g}(\iota_X(X)) = \iota_Y(Y)$ so that $\hat{g}$ induces an isomorphism

$$\iota_Y^{-1} \circ \hat{g} \circ \iota_X: X \to Y$$

Remark. Roughly speaking, the final statement follows from the fact that Steenrod coalgebra morphisms are very well-behaved with respect to maps between simplices of the same dimension.

Proof. A morphism $g: N(X) \to N(Y)$ induces a morphism of delta-complexes

$$\text{hom}_S(\star, g): \text{hom}_S(\star, N(X)) \to \text{hom}_S(\star, N(Y))$$

which is an isomorphism (and homeomorphism) if $g$ is an isomorphism. The conclusion follows from theorem 4.9 which implies that the canonical inclusions

$$|\iota_X|: |X| \to |\text{hom}_S(\star, N(X))|$$

$$|\iota_Y|: |Y| \to |\text{hom}_S(\star, N(Y))|$$

are homotopy equivalences.

If $w: \Delta^n \to X$ is a simplex and $g$ is an isomorphism, corollary 4.5 implies that $g(N(w)): N(\Delta^n) \to N(Y)$ must equal $N(q): N(\Delta^n) \to$
\[ N(Y), \text{where } q: \Delta^n \to Y \text{ is the inclusion of a simplex — moreover this must respect face-operators. It follows that } \hat{g}(\iota_X(X)) = \iota_Y(Y). \]

**Appendix A. The functor \( N(\ast) \)**

We begin by constructing a contracting cochain on the normalized chain-complex of a standard simplex:

**Definition A.1.** Let \( \Delta^k \) be a standard \( k \)-simplex with vertices \([0], \ldots, [k]\) and \( j \)-faces \([i_0, \ldots, i_j]\) with \( i_0 < \cdots < i_j \) and let \( s^k \) denote its normalized chain-complex with boundary map \( \partial \). This is equipped with an augmentation

\[ \epsilon: s^k \to \mathbb{Z} \]

that maps all vertices to \( 1 \in \mathbb{Z} \) and all other simplices to 0. Let

\[ \iota_k: \mathbb{Z} \to s^k \]

denote the map sending \( 1 \in \mathbb{Z} \) to the image of the vertex \([n]\). Then we have a contracting cochain

\[
\varphi_k([i_0, \ldots, i_t]) = \begin{cases} 
(-1)^{t+1}[i_0, \ldots, i_t, k] & \text{if } i_t \neq k \\
0 & \text{if } i_t = k 
\end{cases}
\]

and \( 1 - \iota_k \circ \epsilon = \partial \circ \varphi_k + \varphi_k \circ \partial \).

**Theorem A.2.** The normalized chain-complex of \([i_0, \ldots, i_k] = \Delta^k\) has a Steenrod coalgebra structure that is natural with respect to order-preserving mappings of vertex-sets

\[ [i_0, \ldots, i_k] \to [j_0, \ldots, j_t] \]

with \( j_0 \leq \cdots \leq j_t \) and \( \ell \geq k \). This Steenrod coalgebra is denoted \( N^k \).

If \( X \) is an ordered simplicial complex, then the normalized chain-complex of \( X \) has a natural Steenrod coalgebra structure

\[ N(X) = \lim_{\Delta^n \downarrow X} N^k \]

for \( \Delta^n \in \Delta \downarrow X — \text{the simplex category of } X, \) with Steenrod diagonal

\[ \xi: RS_2 \otimes N(X) \to N(X) \otimes N(X) \]

**Remark.** The author has a Common LISP program for computing \( \xi(x \otimes C(\Delta^k)) — \text{the number of terms grows exponentially as the dimension of } x \) increases.

Compare this with the functor \( C(\ast) \) defined in [14] and [12].
Proof. If \( C = s^k = N(\Delta^k) \) — the normalized chain complex — we can define a corresponding contracting homotopy on \( C \otimes C \) via
\[
\Phi = 1 \otimes \varphi_k + \varphi_k \otimes t_k \circ \epsilon
\]
where \( \varphi_k, t_k, \) and \( \epsilon \) are as in definition A.1. Above dimension 0, \( \Phi \) is effectively equal to \( 1 \otimes \varphi_k \). Now set \( M_2 = C \otimes C \) and \( N_2 = \text{im}(\Phi) \).

Now we inductively define
\[
\xi: \text{RS}_2 \otimes C \rightarrow C \otimes C
\]
In dimension 0, we define \( \xi \) for all \( n \) via:
\[
\xi(A \otimes [0]) = \begin{cases} 
[0] \otimes [0] & \text{if } A = [0] \\
0 & \text{if } \dim A > 0
\end{cases}
\]
This clearly makes \( s^0 \) a Steenrod coalgebra.

Suppose that \( \xi \) is defined below dimension \( k \). Then \( N(\partial \Delta^k) \) is well-defined and satisfies the conclusions of this theorem. We define \( f(a[a_1| \ldots |a_j] \otimes [0, \ldots, k]) \) by induction on \( j \),
\[
(\text{A.2}) \quad \xi(A \otimes s^k) = \Phi \circ \xi(\partial A \otimes s^k) + (-1)^{\dim A} \Phi \circ \xi(A \otimes \partial s^k)
\]
where \( A \in A(S_2, 1) \subset \text{RS}_2 \) and the term \( \xi(A \otimes \partial s^k) \) refers to the coalgebra structure of \( N(\partial \Delta^k) \). The term \( \xi(A \otimes \partial s^k) \) is defined by induction on \( k \). The term \( \xi(\partial A \otimes s^k) \) is defined by induction on the dimension of \( A \). We ultimately get an expression for \( \xi(x \otimes [0, \ldots, k]) \) as a sum of tensor-products of sub-simplices of \([0, \ldots, k]\) — given as ordered lists of vertices.

We claim that this Steenrod coalgebra structure is natural with respect to ordered mappings of vertices. This follows from the fact that the only significant property that the vertex \( k \) has in equation A.1 and equation A.2 is that it is the highest numbered vertex. \( \square \)

We conclude this section some computations of higher coproducts:

Example A.3. If \([0, 1, 2] = \Delta^2\) is a 2-simplex, then
\[
(\text{A.3}) \quad \xi([0] \otimes \Delta^2) = \Delta^2 \otimes F_0 F_1 \Delta^2 + F_2 \Delta^2 \otimes F_0 \Delta^2 + F_1 F_2 \Delta^2 \otimes \Delta^2
\]
— the standard (Alexander-Whitney) coproduct — and
\[
\xi([(1, 2)] \otimes \Delta^2) = [0, 1, 2] \otimes [1, 2] - [0, 2] \otimes [0, 1, 2] - [0, 1, 2] \otimes [0, 1]
\]
or, in face-operations
Proof. If we write $\Delta^2 = [0, 1, 2]$, we get
\[
\xi([\cdot] \otimes \Delta^2) = [0, 1, 2] \otimes [2] + [1, 2] \otimes [0] + [0] \otimes [0, 1, 2]
\]
To compute $\xi((1, 2) \otimes \Delta^2)$ we have a version of equation A.4:
\[
\xi(e_1 \otimes \partial \Delta^2) = \Phi_2(\xi(e_1 \otimes \Delta^2) - \Phi_2(\xi(e_1 \otimes \partial \Delta^2)
= -\Phi_2(\xi((1, 2) \cdot [\cdot] \otimes \Delta^2) + \Phi_2(\xi([\cdot] \otimes \Delta^2) - \Phi_2(\xi(e_1 \otimes \partial \Delta^2)
\]
Now
\[
\Phi_2 \cdot (\xi((1, 2) \cdot [\cdot] \otimes \Delta^2) = (1 \otimes \varphi_2)([2] \otimes [0, 1, 2] - [1, 2] \otimes [0, 1]
+ [0, 1, 2] \otimes [0])
+ (\varphi_2 \otimes \iota_2 \circ \epsilon)([2] \otimes [0, 1, 2]
- [1, 2] \otimes [0, 1] + [0, 1, 2] \otimes [0, 1])
= [1, 2] \otimes [0, 1, 2] - [0, 1, 2] \otimes [0, 2]
\]
where the $+\,$ sign on the term $[1, 2] \otimes [0, 1, 2]$ is due to the Koszul convention and definition. We also get
\[
\Phi_2(\xi([\cdot] \otimes \Delta^2) = (1 \otimes \varphi_2)([0, 1, 2] \otimes [2] + [0, 1] \otimes [1, 2]
+ [0] \otimes [0, 1, 2])
= 0
\]
In addition, proposition A.4 implies that
\[
\xi(e_1 \otimes \partial \Delta^2) = -[1, 2] \otimes [1, 2] + [0, 2] \otimes [0, 2]
- [0, 1] \otimes [0, 1]
\]
so that
\[
\Phi_2\xi(e_1 \otimes \partial \Delta^2) = -[0, 1] \otimes [0, 1, 2]
\]
We conclude that
\[
\xi([(1, 2)] \otimes \Delta^2) = -[1, 2] \otimes [0, 1, 2] - [0, 1, 2] \otimes [0, 2]
+ [0, 1] \otimes [0, 1, 2]
\]
which implies equation A.4. \qed

With this in mind, note that images of simplices in $N(*)$ have an interesting property:
Proposition A.4. Let $X$ be a simplicial set with $C = N(X)$ and with coalgebra structure
$$\xi: RS_2 \otimes N(X) \to N(X) \otimes N(X)$$
and suppose $RS_2$ is generated in dimension $n$ by $e_n = [(1, 2) \cdots (1, 2)]$.

If $x \in C$ is the image of a $k$-simplex, then
$$\xi(e_k \otimes x) = \eta_k \cdot x \otimes x$$
where $\eta_k = (-1)^{k(k+1)/2}$.

Remark. This is just a chain-level statement that the Steenrod operation $Sq^0$ acts trivially on mod-$2$ cohomology. A weaker form of this result appeared in [3].

Proof. Recall that $(RS_2)_n = \mathbb{Z}[\mathbb{Z}_2]$ generated by $e_n = [(1, 2) \cdots (1, 2)]$.

Let $T$ be the generator of $\mathbb{Z}_2$ — acting on $C \otimes C$ by swapping the copies of $C$.

Since the normalized chain-complex, $N(\Delta^k)$, has the property that $N(\Delta^k)_j = 0$ for $j > k$
$$j > k \implies \xi(e_j \otimes N(\Delta^k)) = 0$$

As in section 4 of [14], if $e_0 = [] \in RS_2$ is the 0-dimensional generator, we define
$$\xi: RS_2 \otimes C \to C \otimes C$$
inductively by
$$\xi(e_0 \otimes [i]) = [i] \otimes [i]$$
$$\xi(e_0 \otimes [0, \ldots, k]) = \sum_{i=0}^{k} [0, \ldots, i] \otimes [i, \ldots, k]$$

Let $\sigma = \Delta^k$ and inductively define
$$\xi(e_k \otimes \sigma) = \Phi_k(\xi(\partial e_k \otimes \sigma)) + (-1)^k \Phi_k \xi(e_k \otimes \partial \sigma)$$
$$= \Phi_k(\xi(\partial e_k \otimes \sigma))$$

because of equation A.5.

Expanding $\Phi_k$, we get
$$\xi(e_k \otimes \sigma) = (1 \otimes \varphi_k)(\xi(\partial e_k \otimes \sigma)) + (\varphi_k \otimes i_k \circ e)\xi(\partial e_k \otimes \sigma)$$
$$= (1 \otimes \varphi_k)(\xi(\partial e_k \otimes \sigma))$$

because $\varphi_k^2 = 0$ and $\varphi_k \circ i_k \circ e = 0$. 
Noting that \( \partial e_k = (1 + (-1)^k T)e_{k-1} \in RS_2 \), we get

\[
\xi(e_k \otimes \sigma) = (1 \otimes \varphi_k)(\xi(e_{k-1} \otimes \sigma) + (-1)^k (1 \otimes \varphi_k) \cdot T \cdot \xi(e_{k-1} \otimes \sigma)
\]

\[
= (-1)^k (1 \otimes \varphi_k)) \cdot T \cdot \xi(e_{k-1} \otimes \sigma)
\]

again, because \( \varphi_k^2 = 0 \) and \( \varphi_k \circ i_k \circ \epsilon = 0 \). We continue, using equation [A.8] to compute \( \xi(e_{k-1} \otimes \sigma) \):

\[
\xi(e_k \otimes \sigma) = (-1)^k (1 \otimes \varphi_k)) \cdot T \cdot \xi(e_{k-1} \otimes \sigma)
\]

\[
= (-1)^k (1 \otimes \varphi_k)) \cdot T \cdot (1 \otimes \varphi_k)) \left( \xi(\partial e_{k-1} \otimes \sigma) \right)
\]

\[
= (-1)^{k+1} \varphi_k \otimes \varphi_k \cdot T \cdot \left( \xi(\partial e_{k-1} \otimes \sigma) \right)
\]

— where the factor of \( (-1)^{k+1} \) is the result of applying the Koszul Convention — \( (1 \otimes \varphi_k) \circ (\varphi_k \otimes 1) = -\varphi_k \otimes \varphi_k \).

If \( k-1 = 0 \), then the left term vanishes. If \( k-1 = 1 \) so \( \partial e_{k-1} \) is 0-dimensional then equation [A.6] gives \( \xi(\partial e_1 \otimes \sigma) \) and this vanishes when plugged into \( \varphi_k \otimes \varphi_k \). If \( k-1 > 1 \), then \( \xi(\partial e_{k-1} \otimes \sigma) \) is in the image of \( \varphi_k \), so it vanishes when plugged into \( \varphi_k \otimes \varphi_k \).

In all cases, we can write

\[
\xi(e_k \otimes \sigma) = (-1)^{k+1} \varphi_k \otimes \varphi_k \cdot T \cdot (-1)^{k-1} \xi(e_{k-1} \otimes \partial \sigma)
\]

\[
= \varphi_k \otimes \varphi_k \cdot T \cdot \xi(e_{k-1} \otimes \partial \sigma)
\]

If \( \xi(e_{k-1} \otimes \Delta^{k-1}) = \eta_{k-1} \Delta^{k-1} \otimes \Delta^{k-1} \) (the inductive hypothesis), then

\[
\xi(e_{k-1} \otimes \partial \sigma) = \sum_{i=0}^{k} \eta_{k-1} \cdot (-1)^i [0, \ldots, i-1, i+1, \ldots k] \otimes [0, \ldots, i-1, i+1, \ldots k]
\]

and the only term that does not get annihilated by \( \varphi_k \otimes \varphi_k \) is

\[
(-1)^k [0, \ldots, k-1] \otimes [0, \ldots, k-1]
\]
(see equation \[A.1\]). We get
\[
\xi(e_k \otimes \sigma) = \eta_{k-1} \cdot \varphi_k \otimes \varphi_k \cdot T \cdot (-1)^k [0, \ldots, k-1] \otimes [0, \ldots, k-1]
\]
\[
= \eta_{k-1} \cdot \varphi_k \otimes \varphi_k (-1)^{(k-1)^2 +k} [0, \ldots, k-1] \otimes [0, \ldots, k-1]
\]
\[
= \eta_{k-1} \cdot (-1)^{(k-1)^2 +2k-1} \varphi_k [0, \ldots, k-1] \otimes \varphi_k [0, \ldots, k-1]
\]
\[
= \eta_{k-1} \cdot (-1)^k [0, \ldots, k] \otimes [0, \ldots, k]
\]
\[
= \eta_k \cdot [0, \ldots, k] \otimes [0, \ldots, k]
\]
where the sign-changes are due to the Koszul Convention. We conclude that \( \eta_k = (-1)^k \eta_{k-1} \). \qed

**APPENDIX B. LEMMA B.1**

**Lemma B.1.** Let \( C \) be a free abelian group, let
\[
\hat{C} = \mathbb{Z} \oplus \prod_{i=1}^{\infty} C^{\otimes i}
\]
Let \( e: C \to \hat{C} \) be the function that sends \( c \in C \) to
\[
(1, c, c \otimes c, c \otimes c \otimes c, \ldots) \in \hat{C}
\]
For any integer \( t > 1 \) and any set \( \{c_1, \ldots, c_t\} \in C \) of distinct, nonzero elements, the elements
\[
\{e(c_1), \ldots, e(c_t)\} \in \mathbb{Q} \otimes \mathbb{Z} \hat{C}
\]
are linearly independent over \( \mathbb{Q} \). It follows that \( e \) defines an injective function
\[
\bar{e}: \mathbb{Z}[C] \to \hat{C}
\]

**Proof.** We will construct a vector-space morphism
\[
(B.1) \quad f: \mathbb{Q} \otimes \mathbb{Z} \hat{C} \to V
\]
such that the images, \( \{f(e(c_i))\} \), are linearly independent. We begin with the “truncation morphism”
\[
r_t: \hat{C} \to \mathbb{Z} \oplus \bigoplus_{i=1}^{t-1} C^{\otimes i} = \hat{C}_{t-1}
\]
which maps \( C^{\otimes 1} \) isomorphically. If \( \{b_i\} \) is a \( \mathbb{Z} \)-basis for \( C \), we define a vector-space morphism
\[
g: \hat{C}_{t-1} \otimes \mathbb{Z} \mathbb{Q} \to \mathbb{Q}[X_1, X_2, \ldots]
\]
by setting
\[
g(c) = \sum_{\alpha} z_{\alpha} X_{\alpha}
\]
where \( c = \sum_{\alpha} z_{\alpha} b_{\alpha} \in C \otimes_{\mathbb{Z}} \mathbb{Q} \), and extend this to \( \hat{C}_{t-1} \otimes_{\mathbb{Z}} \mathbb{Q} \) via
\[
g(c_1 \otimes \cdots \otimes c_j) = g(c_1) \cdots g(c_j) \in \mathbb{Q}[X_1, X_2, \ldots]
\]
The map in equation (B.1) is just the composite
\[
\hat{C} \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{r_{r-1} \otimes 1} \hat{C}_{t-1} \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \mathbb{Q}[X_1, X_2, \ldots]
\]
It is not hard to see that
\[
p_i = f(e(c_i)) = 1 + f(c_i) + \cdots + f(c_i)^{t-1} \in \mathbb{Q}[X_1, X_2, \ldots]
\]
for \( i = 1, \ldots, t \). Since the \( f(c_i) \) are linear in the indeterminates \( X_i \), the degree-\( j \) component (in the indeterminates) of \( f(e(c_i)) \) is precisely \( f(c_i)^j \). It follows that a linear dependence-relation
\[
\sum_{i=1}^{t} \alpha_i \cdot p_i = 0
\]
with \( \alpha_i \in \mathbb{Q} \), holds if and only if
\[
\sum_{i=1}^{t} \alpha_i \cdot f(c_i)^j = 0
\]
for all \( j = 0, \ldots, t-1 \). This is equivalent to \( \det M = 0 \), where
\[
M = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
f(c_1) & f(c_2) & \cdots & f(c_t) \\
\vdots & \vdots & \ddots & \vdots \\
f(c_1)^{t-1} & f(c_2)^{t-1} & \cdots & f(c_t)^{t-1}
\end{bmatrix}
\]
Since \( M \) is the transpose of the Vandermonde matrix, we get
\[
\det M = \prod_{1 \leq i < j \leq t} (f(c_i) - f(c_j))
\]
Since \( f|C \otimes_{\mathbb{Z}} \mathbb{Q} \subset \hat{C} \otimes_{\mathbb{Z}} \mathbb{Q} \) is injective, it follows that this only vanishes if there exist \( i \) and \( j \) with \( i \neq j \) and \( c_i = c_j \). The second conclusion follows. \( \square \)

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Current address: Department of Mathematics, Drexel University, Philadelphia, PA 19104

E-mail address: jsmith@drexel.edu

URL: http://vorpal.math.drexel.edu