SECOND FUNDAMENTAL FORM OF THE PRYM MAP IN THE RAMIFIED CASE

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Abstract. In this paper we study the second fundamental form of the Prym map $P_{g,r} : R_{g,r} \to A_{g-1+r}^\delta$ in the ramified case $r > 0$. We give an expression of it in terms of the second fundamental form of the Torelli map of the covering curves. We use this expression to give an upper bound for the dimension of a germ of a totally geodesic submanifold, and hence of a Shimura subvariety of $A_{g-1+r}^\delta$, contained in the Prym locus.

1. Introduction

Denote by $R_{g,r}$ the moduli space parametrising isomorphism classes of pairs $[(C, \alpha, R)]$ where $C$ is a smooth complex projective curve of genus $g$, $R$ is a reduced effective divisor of degree $2r$ on $C$ and $\alpha$ is a line bundle on $C$ such that $\alpha^2 = \mathcal{O}_C(R)$. To such data it is associated a double cover of $C$, $\pi : \tilde{C} \to C$ branched on $R$, with $\tilde{C} = \text{Spec}(\mathcal{O}_C \oplus \alpha^{-1})$.

The Prym variety associated to $[(C, \alpha, R)]$ is the connected component containing 0 of the kernel of the norm map $\text{Nm}_\pi : J\tilde{C} \to JC$. Notice that for $r > 0$, $\text{ker} \text{Nm}_\pi$ is connected. It is a polarized abelian variety of dimension $g - 1 + r$, denoted by $P(C, \alpha, R)$ or equivalently $P(\tilde{C}, C)$. The polarization $\Xi$ is induced by restricting the principal polarization on $J\tilde{C}$ and it is of type $\delta = (1, \ldots, 1, 2, \ldots, 2)$ for $r > 0$. For $r = 0, 1$ it is twice a principal polarization and we endow $P(\tilde{C}, C)$ with this principal polarization.

This defines the Prym map $P_{g,r} : R_{g,r} \to A_{g-1+r}^\delta$, $[(C, \alpha, R)] \mapsto [(P(C, \alpha, R), \Xi)]$, where $A_{g-1+r}^\delta$ is the moduli space of abelian varieties of dimension $g-1+r$ with a polarization of type $\delta$.

The codifferential of $P_{g,r}$ at a generic point $[(C, \alpha, R)]$ is given by the multiplication map $(dP_{g,r})^* : S^2 H^0(C, K_C \otimes \alpha) \to H^0(C, K_C^2(R))$ which is known to be surjective (see [11]), therefore $P_{g,r}$ is generically finite, if and only if

$$\dim R_{g,r} \leq \dim A_{g-1+r}^\delta.$$ 

This holds if: either $r \geq 3$ and $g \geq 1$, or $r = 2$ and $g \geq 3$, $r = 1$ and $g \geq 5$, $r = 0$ and $g \geq 6$.

If $r = 0$ the Prym map is generically injective for $g \geq 7$ ([9], [10]). If $r > 0$, Marcucci and Pirola [13], and later, for the missing cases, Marcucci and Naranjo [12] and Naranjo and Ortega [15] have proved the generic injectivity in all the cases except for $r = 2$, $g = 3$, which was previously studied by Nagaraj and Ramanan, and also by Bardelli, Ciliberto and Verra (see [14], [1]) and for which the degree of the Prym map is 3.

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In this paper we study the second fundamental form of the restriction of the Prym map to the open set \( R_{g,r}^0 \) where the Prym map is an immersion, with respect to the orbifold metric on \( A_{g-1+r}^0 \) induced by the symmetric metric on the Siegel space \( \mathcal{H}_{g-1+r} \).

In the unramified case \( r = 0 \), the second fundamental form of the Prym map was studied in [5] using the Hodge gaussian maps introduced in [8] and the flat structure on the degree 0 line bundle \( \alpha \) on \( C \). Here in Theorem 2.1 we give a description of the second fundamental form \( \rho_P \) for all \( r \geq 0 \) in terms of the second fundamental form \( \bar{\rho} \) of the Torelli map \( j : \mathcal{M}_{2g-1+r} \to A_{2g-1+r} \) described in [8], [3], [6]. At the point \([C, \alpha, R]\) we show that \( \rho_P \) is obtained from \( \bar{\rho} \) by first restricting to the kernel of \((dP_{g,r})^*\) and then projecting to \( S^2H^0(K_C^2(R)) \).

This also allows us to prove that the map \( \rho_P \) is a lifting of the second gaussian map of the line bundle \( K_C \otimes \alpha \) (see Proposition 2.3). This was already proved in the unramified case \( r = 0 \) in [5] and previously in the case of the Torelli map in [8].

In the second part of the paper we use this description of \( \rho_P \) to study totally geodesic submanifolds in the Prym loci.

We recall that a conjecture by Coleman and Oort says that for big enough genus, there should not exist Shimura subvarieties of \( \mathcal{A}_g \) generically contained in the Torelli locus, i.e. contained in the Torelli locus \( j(\mathcal{M}_g) \) and intersecting \( j(\mathcal{M}_g) \). Shimura subvarieties are totally geodesic, hence it is possible to approach the conjecture studying the second fundamental of the Torelli map. This viewpoint was used in [6], where an upper bound on the dimension of a germ of a totally geodesic submanifold contained in the Torelli locus, depending on \( g \), is given.

In [7] a question similar to the one of Coleman and Oort for Pryms was asked in the case \( r = 0, 1 \), i.e. when the Prym variety is principally polarised and the Prym loci \( \overline{P}_{g,r}(R_{g,r}) \) contain the Torelli locus. More precisely, the question is about the existence, for big enough genus, of Shimura subvarieties generically contained in the Prym loci. We say that a subvariety \( Z \subset \mathcal{A}_{g-1+r} \) is generically contained in the Prym locus if \( Z \subset \overline{P}_{g,r}(R_{g,r}) \), \( Z \cap P_{g,r}(R_{g,r}) \neq \emptyset \) and \( Z \) intersects the locus of indecomposable polarised abelian varieties. In [7] examples in low dimension of Shimura curves generically contained in the Prym loci for \( r = 0, 1 \) are given, using Galois covers of \( \mathbb{P}^1 \).

In [4] we gave an upper bound on the dimension of a germ of a totally geodesic submanifold contained in the Prym locus when \( r = 0 \), depending only on \( g \), which is similar to the estimate in [6] for the Torelli locus, that we achieved via the second fundamental form.

Here we generalise the above question for any \( r \geq 0 \) and we find an upper bound for the dimension of a germ of a totally geodesic submanifold contained in the Prym loci which depends on \( g \) and \( r \) (Theorem 3.4). This is obtained as the generic case of a bound depending also on the gonality \( k \), given for a germ of a totally geodesic submanifold contained in the Prym loci passing through a point \([C, \alpha, R]\), with \( C \) a \( k \)-gonal curve (Theorem 3.2).

2. The 2nd fundamental form of the Prym map

Let \( C \) be a smooth complex projective curve of genus \( g \), \( R \) a reduced divisor of degree \( 2r \) on \( C \) and \( \alpha \) a line bundle on \( C \) such that \( \alpha^2 = \mathcal{O}_C(R) \). To such data corresponds a double cover \( \pi : \hat{C} \to C \) branched on \( R \). In the ramified case \( r > 0 \), the Prym variety \( P(C, \alpha, R) \) associated to this data is the polarised abelian variety given by the kernel of the norm map \( N_{\pi} : J\hat{C} \to JC \). For \( r = 0 \) it is its connected component containing the origin. For \( r > 1 \) the polarisation is given by \( \Xi := \Theta_{\hat{C}|P(C,\alpha,R)} \) where \( \Theta_{\hat{C}} \) is a theta divisor of \( J\hat{C} \). For \( r = 0, 1 \) the polarisation \( \Theta_{\hat{C}|P(C,\alpha,R)} \) is twice a principal polarisation \( \Xi \). Consider the Prym
map \( P_{g,r} : R_{g,r} \rightarrow A^g_{g-1+r} \), which associates to a point \([C, \alpha, R] \in R_{g,r}\) the isomorphism class of its Prym variety \( P(C, \alpha, R) \) with the polarisation \( \Xi \).

Denote by \( R^0_{g,r} \) the open subset of \( R_{g,r} \) where the Prym map \( P_{g,r} \) is an immersion.

Consider the orbifold tangent bundle exact sequence of the Prym map
\[
(2.1) \quad 0 \rightarrow T^0_{R^0_{g,r}} \rightarrow P^*_{g,r} T^0_{A^g_{g-1+r}} \rightarrow N^*_{R^0_{g,r}/A^g_{g-1+r}} \rightarrow 0
\]

On \( A^g_{g-1+r} \) we consider the orbifold metric induced by the symmetric metric on the Siegel space \( H_{g-1+r} \) and the associated second fundamental form with respect to the metric connection of the above exact sequence. Denote its dual by
\[
(2.2) \quad \rho_P : N^*_{R^0_{g,r}/A^g_{g-1+r}} \rightarrow S^2 \Omega^1_{R^0_{g,r}}.
\]

To describe this second fundamental form we study the second fundamental form of the Torelli map of the covering curves \( \tilde{C} \). Since our computations will be local, we will restrict to the open set \( U \) of \( R^0_{g,r} \) where there is a universal family \( \tilde{f} : \tilde{C} \rightarrow U \).

Denote by \( \tilde{H} := R^1 \tilde{f}_* \mathbb{C}, \tilde{H}_b = H^1(\tilde{C}_b, \mathbb{C}) \), by \( \tilde{\mathcal{F}} = \tilde{f}_* \omega^2_{\tilde{C}|U} \) the Hodge bundle, \( \tilde{\mathcal{F}}_b = H^0(\tilde{C}_b, K_{C_b}) \). The \( \mathbb{Z}/2\mathbb{Z} \) action corresponding to the \( 2 : 1 \) covering gives a decomposition \( \tilde{H} = \tilde{H}^+ \oplus \tilde{H}^- \) in \( \pm 1 \) eigenspaces and an analogous decomposition \( \tilde{\mathcal{F}} = \tilde{\mathcal{F}}^+ \oplus \tilde{\mathcal{F}}^- \). We have \( \tilde{\mathcal{F}}_b^+ \cong H^0(C_b, K_{C_b}), \tilde{\mathcal{F}}_b^- \cong H^0(C_b, K_{C_b} \otimes \alpha_b), \forall b \in U \).

The Gauss-Manin connection \( \nabla_{GM} \) on \( \tilde{H} \) induces a connection \( \nabla^{1,0} \) on the Hodge bundle \( \tilde{\mathcal{F}} \). Both connections are \( \mathbb{Z}/2\mathbb{Z} \) invariant, hence we have a connection \( \nabla^- \) on \( \tilde{\mathcal{F}}^- \), and an induced connection \( \nabla \) on \( S^2 \tilde{\mathcal{F}}^- \). With the identifications on \( U \): \( P^* \Omega^1_{A^g_{g-1+r}} \cong S^2 \tilde{\mathcal{F}}^- \), \( \Omega^1_{U} = (\tilde{f}_* \omega^2_{\tilde{C}|U})^+ \), the connection \( \nabla \) corresponds to the connection associated to the Siegel metric.

Consider the exact sequence
\[
(2.3) \quad 0 \rightarrow \mathcal{I}_2 \rightarrow S^2 \tilde{\mathcal{F}} \xrightarrow{m} \tilde{f}_* \omega^2_{\tilde{C}|U} \rightarrow 0
\]
where the map \( m \) is the multiplication map and it is the dual of the differential of the Torelli map of the curves \( \tilde{C} \) on \( U \).

The second fundamental form of the exact sequence (2.3) is a map
\[
(2.4) \quad \Psi : \mathcal{I}_2 \rightarrow \tilde{f}_* \omega^2_{\tilde{C}|U} \otimes \Omega^1_{U} \cong \tilde{f}_* \omega^2_{\tilde{C}|U} \otimes (\tilde{f}_* \omega^2_{\tilde{C}|U})^+
\]

Since the multiplication map \( m \) is \( \mathbb{Z}/2\mathbb{Z} \) equivariant, we also have the exact sequence:
\[
(2.5) \quad 0 \rightarrow \mathcal{I}_2^+ \rightarrow (S^2 \tilde{\mathcal{F}})^+ \xrightarrow{m} (\tilde{f}_* \omega^2_{\tilde{C}|U})^+ \rightarrow 0
\]

Clearly we have \((S^2 \tilde{\mathcal{F}})^+ \cong S^2 \tilde{\mathcal{F}}^+ \oplus S^2 \tilde{\mathcal{F}}^-\) and the restriction of the multiplication map \( m \) to \( S^2 \tilde{\mathcal{F}}^- \) is the dual of the differential of the Prym map \( P \). More precisely the dual of the exact sequence (2.1) on \( U \) can be written as
\[
(2.6) \quad 0 \rightarrow \mathcal{G} \rightarrow S^2 \tilde{\mathcal{F}}^- \xrightarrow{m} (\tilde{f}_* \omega^2_{\tilde{C}|U})^+ \rightarrow 0
\]
where \( \mathcal{G} = S^2 \tilde{\mathcal{F}}^- \cap \mathcal{I}_2^+ \). So the dual of the second fundamental form of the Prym map is a map
\[
(2.7) \quad \rho_P : \mathcal{G} \rightarrow (\tilde{f}_* \omega^2_{\tilde{C}|U})^+ \otimes (\tilde{f}_* \omega^2_{\tilde{C}|U})^+,
\]
which is symmetric. Clearly we have

\[(2.8)\quad \rho_P = p \circ \Psi|_g\]

where \( p : (\tilde{f}_*\omega^2_{C|U}) \otimes (\tilde{f}_*\omega^2_{C|U})^+ \to S^2(\tilde{f}_*\omega^2_{C|U})^+ \) is the natural projection.

Denote by \( \tilde{\gamma} := 2g - 1 + r \) and consider now the Torelli map \( \tilde{j} : M^0_g \to A_g \), where \( M^0_g \) is the complement of the hyperelliptic locus. Then \( \tilde{j} \) is an immersion and we denote by \( \tilde{\rho} \) the dual of the second fundamental form of \( \tilde{j} \). Since we are working in a local setting, we can assume that we have a modular map \( \mu : U \to V \) where \( V \) is an open subset of \( M^0_g \) on which there exists a universal family \( \varphi : \tilde{C}_V \to V \). The family \( \tilde{C} \) is the pullback of \( \tilde{C}_V \) via the map \( \mu \).

On \( V \) we have

\[(2.9)\quad 0 \to I_2(\omega_{\tilde{C}_V|V}) \to S^2(\varphi_*\omega_{\tilde{C}_V|V}) \xrightarrow{m} \varphi_*\omega^2_{\tilde{C}_V|V} \to 0\]

and the multiplication map \( m \) is the dual of the differential of the Torelli map.

On \( V \) the dual of the second fundamental form of the Torelli map is a map

\[(2.10)\quad \tilde{\rho} : I_2(\omega_{\tilde{C}_V|V}) \to \varphi_*\omega^2_{\tilde{C}_V|V} \otimes \varphi_*\omega^2_{\tilde{C}_V|V}\]

The pullback of the above exact sequence on \( U \) via \( \mu \) is the exact sequence \((2.3)\). Hence we have

\[(2.11)\quad \Psi = q \circ \mu^*\tilde{\rho}\]

where \( q : (\tilde{f}_*\omega^2_{C|U}) \otimes (\tilde{f}_*\omega^2_{C|U}) \to (\tilde{f}_*\omega^2_{C|U}) \otimes (\tilde{f}_*\omega^2_{C|U})^+ \) is the natural projection.

We have the following

**Theorem 2.1.** The dual of the second fundamental form of the Prym map on \( U \) is obtained as \( \rho_P = p' \circ (\mu^*\tilde{\rho})|_g \), where \( p' : S^2(\tilde{f}_*\omega^2_{C|U}) \to S^2((\tilde{f}_*\omega^2_{C|U})^+) \) is the natural projection.

**Proof.** By \((2.8)\) and \((2.11)\) we have \( \rho_P = p \circ \Psi|_g = p \circ q \circ (\mu^*\tilde{\rho})|_g = p' \circ (\mu^*\tilde{\rho})|_g \). \(\square\)

At the point \( b_0 := [(C, \alpha, R)] \in U \) corresponding to the \( 2 : 1 \) cover \( \pi : \tilde{C} \to C \), the space \( P^*g, \Omega^1_{g, 1+r, b_0} \) is isomorphic to \( S^2H^0(K_C \otimes \alpha) \), \( \Omega^1_{\mathcal{R}, g, r, b_0} \) is isomorphic to \( H^0(K_C^2(R)) \) and the dual of the exact sequence \((2.1)\) at the point \( b_0 \) becomes

\[0 \to I_2(K_C \otimes \alpha) \to S^2H^0(K_C \otimes \alpha) \xrightarrow{m} H^0(K_C^2(R)) \to 0.\]

The dual of the second fundamental form of the Prym map at the point \( b_0 \) is a map

\[(2.12)\quad \rho_P : I_2(K_C \otimes \alpha) \to S^2H^0(K_C^2(R))\]

Observe that \( \forall Q \in I_2(K_C \otimes \alpha) \xrightarrow{\pi^*} I_2(K_C^+) \), \( \forall v_1, v_2 \in H^1(T_C(-R)) \cong H^1(T_C)^+ \), by Theorem \((2.1)\) we have:

\[(2.13)\quad \rho_P(Q)(v_1 \otimes v_2) = \tilde{\rho}(\pi^*Q)(v_1 \otimes v_2).\]

Denote by

\[(2.14)\quad \tilde{\mu}_2 : I_2(K_C) \to H^0(K_C^4)\]

the second Gaussian map of the canonical bundle \( K_C \) and by
(2.15) \[
\mu_2 := \mu_{2,KC} : I_2(KC \otimes \alpha) \to H^0(K_C^2(R))
\]
the second Gaussian map of the line bundle \(KC \otimes \alpha\) (for the definition and a local expression of the second Gaussian maps see e.g. [2, section 2]). Notice that \(\mu_2\) is equivariant, hence it induces a map \(\tilde{\mu}_2 : I_2(K_C) \to H^0(K_C^4(2R))\).

We have the following

**Lemma 2.2.** For every \(Q \in I_2(KC \otimes \alpha)\), \(\mu_2(Q) = \tilde{\mu}_2(\pi^*Q)\) via the inclusion \(H^0(K_C^4(R)) \subset \tilde{\mu}_2(\pi^*Q)\).

**Proof.** We show the equality by a local computation. For a point \(P \not\in R\), take local coordinates \(z\) in a neighbourhood \(V\) of \(P\) and \(w\) in a neighbourhood \(U\) of a point \(T \in \pi^{-1}(P)\) such that the local expression of \(\pi : U \to V\) is \(w \mapsto w = z\). Since \(\alpha^2 = \mathcal{O}_C(R)\), \(\alpha^2|_U = \mathcal{O}_V\) and we choose a local frame \(a\) of \(\alpha\) on \(V\) such that \(a^2 = 1\) and \((\pi^*a)|_U = 1\). Fix a basis \(\{\omega_i\}\) of \(H^0(K_C \otimes \alpha)\), so locally \(\omega_i = f_i(z)dz \otimes a\). Then on \(U\) we have \(\pi^*(\omega_i) = f_i(w)dw\). Take a quadric \(Q = \sum_{i,j} a_{i,j}\omega_i \otimes \omega_j \in I_2(KC \otimes \alpha)\), hence locally \(\pi^*Q = \sum_{i,j} a_{i,j}f_i(w)dw \otimes f_j(w)dw\) and we have \(\tilde{\mu}_2(\pi^*Q) = -\sum_{i,j} a_{i,j}f_i'(w)f_j'(w)(dw)^4\). On the other hand on \(V\), \(\mu_2(Q) = -\sum_{i,j} a_{i,j}f_i'(z)f_j'(z)(dz)^4 a^2\), hence the statement follows, since \(a^2 = 1\).

Observe that the equality can also be checked locally around a critical point \(T\) over a point \(P \in R\). So we can assume that the map \(\pi : U \to V\) is of the form \(w \mapsto w^2 = z\). Now \((\pi^*\alpha)|_U = \mathcal{O}_U(T)\) so we choose a local frame \(a\) of \(\alpha\) on \(V\) such that \(a^2 = \frac{1}{z}\) and \((\pi^*a)|_U = \frac{1}{w}\). Now locally \(\pi^*(\omega_i) = 2f_i(w^2)dw\) and \(\pi^*(Q) = 4\sum_{i,j} a_{i,j}f_i'(w^2)dw \otimes f_j'(w^2)dw\). So we have

\[
\tilde{\mu}_2(\pi^*Q) = -4 \sum_{i,j} a_{i,j}f_i'(w^2)f_j'(w^2)(w^2)(dw)^4 = -\sum_{i,j} a_{i,j}f_i'(z)f_j'(z)(dz)^4 \frac{1}{z}.
\]

On the other hand locally \(\mu_2(Q) = -\sum_{i,j} a_{i,j}f_i'(z)f_j'(z)(dz)^4 a^2 = \tilde{\mu}_2(\pi^*Q)\). \(\square\)

We recall the definition of a Schiffer variation of a line bundle \(L\) on a curve \(C\) at a point \(p \in C\). Consider the evaluation map \(v : H^0(K_C \otimes L) \otimes \mathcal{O}_C \to K_C \otimes L\) and its dual

\[\xi_L : T_C \otimes L^{-1} \to H^1(L^{-1}) \otimes \mathcal{O}_C.\]

A Schiffer variation \(\xi_{P,L}\) is a generator of the image of the map

\[(T_C \otimes L^{-1})_P \to H^1(L^{-1}).\]

Note that this map can be seen as the coboundary map of the exact sequence

\[0 \to L^{-1} \to L^{-1}(P) \to L^{-1}(P)_P \to 0,\]

with the identification \(T_C|_P = \mathcal{O}_C(P)|_P\).

If we fix a local coordinate \(z\) centred in \(P\) a choice of a Dolbeault representative of \(\xi_{P,L}\) is \(\theta_{P,L} = \frac{\partial b_P}{\partial l^{-1}}\), where \(b_P\) is a bump function at \(P\) and \(l\) is a local frame of \(L\).

**Proposition 2.3.** We have the following commutative diagram

\[
\begin{array}{ccc}
I_2(KC \otimes \alpha) & \xrightarrow{\rho_P} & S^2(H^0(K_C^2(R))) \\
\downarrow^{4\pi i \mu_2} & & \downarrow^{m} \\
H^0(K_C^4(R)) & \longrightarrow & H^0(K_C^4(2R))
\end{array}
\]
Proof. We recall that we have a similar statement for \( \hat{\rho} \) and \( \hat{\mu}_2 \), namely \( m \circ \hat{\rho} = -2\pi i \hat{\mu}_2 \) (see [8, Thm. 3.1], [6, Thm. 2.2]). Denote as usual by \( \pi : \tilde{C} \to C \) the double cover and by \( \sigma \) the covering involution on \( \tilde{C} \), take a point \( T \in \tilde{C} \) with \( T \neq \sigma(T) \) and fix a local coordinate on \( \tilde{C} \) around \( T \) (and correspondingly around \( \sigma(T) \)) and on \( C \) around \( P := \pi(T) \). For a point \( S \in \tilde{C} \), denote by \( \xi_S := \xi_{S,KC} \).

Then by [8, Thm. 3.1], [6, Thm. 2.2] we have:

\[
(2.16)\quad \hat{\rho}(\pi^*Q)(\xi_T \circ \xi_{\sigma(T)}) = -4\pi i(\pi^*Q)(T, \sigma(T)) \cdot \hat{\eta}_T(\sigma(T)) = -4\pi iQ(P, P) \cdot \hat{\eta}_T(\sigma(T)) = 0,
\]

since \( Q(P, P) = 0 \) and \( \hat{\eta}_T \in H^0(K^2_{\tilde{C}}(2T)) \) has only one double pole in \( T \), so it is holomorphic around \( \sigma(T) \). Set \( v := \xi_T + \xi_{\sigma(T)} \in H^1(T_{\tilde{C}})^+ \). By (2.16) and by the \( \mathbb{Z}/2\mathbb{Z} \) equivariance of \( \hat{\rho} \) we have:

\[
(2.17)\quad 2\hat{\rho}(\pi^*Q)(\xi_T \circ \xi_T) = \hat{\rho}(\pi^*Q)(v \circ v) = \rho_P(Q)(v \circ v)
\]

where the last equality is (2.13).

By [8, Thm. 3.1], [6, Thm. 2.2] we have

\[
(2.18)\quad \hat{\rho}(\pi^*Q)(\xi_T \circ \xi_T) = (m(\pi^*(Q)))(T) = -2\pi i\hat{\mu}_2(Q)(T) = -2\pi i\mu_2(Q)(P),
\]

by Lemma (2.2).

Consider the line bundle \( L = K_C(R) \) and a Schiffer variation \( \xi_{P,L} \in H^1(T_C(-R)) \). Then with the identification of \( H^1(T_C(-R)) \) with \( H^1(T_{\tilde{C}})^+ \), the Schiffer variation \( \xi_{P,L} \) corresponds to \( v = \xi_T + \xi_{\sigma(T)} \). This can be checked as follows: if we take the exact sequence

\[
0 \to T_{\tilde{C}} \to T_{\tilde{C}}(T + \sigma(T)) \to T_{\tilde{C}}(T + \sigma(T))_{|T+\sigma(T)} \to 0,
\]

apply \( \pi_* \) and take the invariant part, we get the exact sequence

\[
0 \to T_C(-R) \to T_C(-R)(P) \to T_C(-R)(P)_{|P} \to 0,
\]

hence \( v \) can be identified with \( \xi_{P,L} \).

So we have

\[
(2.19)\quad \rho_P(v \circ v) = \rho_P(\xi_{P,L} \circ \xi_{P,L}) = (m(\rho_P(Q)))(P),
\]

by the definition of \( \xi_{P,L} \), and putting together (2.17), (2.18), (2.19) we get

\[
(2.20)\quad (m(\rho_P(Q)))(P) = -4\pi i\mu_2(Q)(P)
\]

for all \( P \) which is not a critical value of \( \pi \). Hence \( m \circ \rho_P \) and \(-4\pi i\mu_2 \) are two sections of \( H^0(K^4_C(R)) \) that coincide in the complement of a finite set in \( C \), so they coincide.

\[\square\]

3. Totally geodesic submanifolds

In this section, following the ideas of [6] and [4], we give an upper bound for the dimension of a germ of a totally geodesic submanifold of \( \mathcal{A}^g_{g-1+r} \) contained in the Prym locus.

**Proposition 3.1.** Assume that \( [(C, \alpha, R)] \in \mathcal{R}^0_{g,r} \) is such that \( C \) is a \( k \)-gonal curve of genus \( g \), with \( g + r \geq k + 3 \) and \( \alpha^2 = \mathcal{O}_C(R) \).

1. If \( r > k + 1 \), then there exists a quadric \( Q \in I_2(K_C \otimes \alpha) \) such that \( \text{rank} \rho(Q) \geq 2g - 2 - k + r \).
2. If \( r \leq k + 1 \), then there exists a quadric \( Q \in I_2(K_C \otimes \alpha) \) such that \( \text{rank} \rho(Q) \geq 2g - 2k - 4 + 2r \).
Proof. Let \( F \) be a line bundle on \( C \) such that \(|F|\) is a \( g_1^1 \) and choose a basis \( \{x, y\} \) of \( H^0(F) \). Set \( M = K_C \otimes \alpha \otimes F^{-1} \) and denote by \( B \) the base locus of \(|M|\). By Riemann Roch
\[
(3.1) \quad h^0(M) = h^0(M(-B)) = h^0(F \otimes \alpha^{-1}) + g - 1 - k + r \geq g - k + r - 1 \geq 2
\]
by assumption.

Note that in case (1) \( B = \emptyset \), since \( \text{deg}(M) = 2g - 2 + r - k > 2g - 1 \).

Take a pencil \((t_1, t_2)\) in \( H^0(M) \). If \( B \neq \emptyset \), write \( t_i = t_i' s \) for a section \( s \in H^0(C, \mathcal{O}_C(B)) \) with \( \text{div}(s) = B \). Then \((t_1', t_2')\) is a base point free pencil in \(|M(-B)|\). Let \( \psi : C \to \mathbb{P}^1 \) be the morphism induced by this pencil and \( \tilde{\psi} = \psi \circ \pi : \tilde{C} \to \mathbb{P}^1 \) and set \( d := \text{deg}(\psi) = \text{deg}(M(-B)) \). Denote by \( \varphi \) the morphism induced by the pencil \(|F|\) and and \( \tilde{\varphi} = \varphi \circ \pi : \tilde{C} \to \mathbb{P}^1 \).

Consider the rank 4 quadric \( Q := xt_1 \odot yt_2 - xt_2 \odot yt_1 \). Clearly \( Q \in I_2(K_C \otimes \alpha) \). We want to show that \( rk\tilde{\varphi}(\pi^*Q) \geq d \).

Consider the set \( E := \psi(R \cup \text{Crit}(\varphi) \cup \text{Crit}(\psi) \cup B) \) where \( \text{Crit}(\varphi) \) (resp. \( \text{Crit}(\psi) \)) denote the set of critical points of \( \varphi \) (resp. \( \psi \)). Let \( z \in \mathbb{P}^1 \setminus E \) and let \( \{P_1, \ldots, P_d\} \) be the fibre of \( \psi \) over \( z \). By changing coordinates on \( \mathbb{P}^1 \) we can assume \( z = [0,1] \), i.e. \( t_i'(P_i) = 0 \) for \( i = 1, \ldots, d \). Then clearly \( t_j(P_i) = 0 \), so \( Q(P_i, P_j) = 0 \) for all \( i, j \). Set \( \{T_i, \sigma(T_i)\} = \pi^{-1}(P_i) \), so \( \pi^*(Q(T_i, T_j)) = \pi^*Q(T_i, \sigma(T_j)) = Q(P_i, P_j) = 0 \).

Let us fix a local coordinate at the relevant points and write \( \xi_T := \xi_{T, K_C} \) for a Schiffer variation of \( \tilde{C} \) at \( T \). Set \( v_i := \xi_{T_i} + \xi_{\sigma(T_i)} \). Clearly \( v_i \in H^1(T_{\tilde{C}})^\perp \cong H^1(T_C(-R)) \), so by (2.13) we have \( \rho_P(Q)(v_i \odot v_j) = \tilde{\rho}(\pi^*Q)(v_i \odot v_j) \).

Hence, by [6] Thm. 2.2, for \( i \neq j \)
\[
(3.2) \quad \rho_P(Q)(v_i \odot v_j) = 2\tilde{\rho}(\pi^*Q)(v_i \odot v_j) = 2\tilde{\rho}(\pi^*Q)(\xi_{T_i} \odot \xi_{T_j}) + 2\tilde{\rho}(\pi^*Q)(\xi_{\sigma(T_i)} \odot \xi_{T_j}) = \nonumber
\]
\[= -8\pi i(\pi^*Q)(T_i, T_j)\tilde{\eta}_{T_j}(T_i) - 8\pi i(\pi^*Q)(\sigma(T_i), T_j)\tilde{\eta}_{T_j}(\sigma(T_i)) = 0 \]
and
\[
(3.3) \quad \tilde{\rho}(\pi^*Q)(v_i \odot v_i) = 2\tilde{\rho}(\pi^*Q)(\xi_{T_i} \odot \xi_{T_i}) + 2\tilde{\rho}(\pi^*Q)(\xi_{\sigma(T_i)} \odot \xi_{T_i}) = \nonumber
\]
\[= -4\pi i\mu_2(\pi^*Q)(T_i) - 8\pi i(\pi^*Q)(\sigma(T_i), T_i)\tilde{\eta}_{T_i}(\sigma(T_i)) = -4\pi i\mu_2(\pi^*Q)(T_i) = -4\pi i\mu_2(Q)(P_i). \]

For a rank 4 quadric the second Gaussian map can be computed as follows: \( \mu_2(Q) = \mu_{1,F}(x \wedge y)\mu_{1,M}(t_1 \wedge t_2) \), where \( \mu_{1,F} \) and \( \mu_{1,M} \) are the first Gaussian maps of the line bundles \( F \) and \( M \) (see [2] Lemma 2.2)). Now \( \mu_{1,F}(x \wedge y)(P_i) \neq 0 \), because \( P_i \notin \text{Crit}(\varphi) \) by the choice of \( z \). Moreover \( P_i \) is not in the base locus \( B \). On \( C \setminus B \) the morphism \( \psi \) coincides with the map associated to \((t_1, t_2)\). Since \( P_i \notin \text{Crit}(\psi) \), it is not a critical point for the latter map. Therefore also \( \mu_{1,M}(t_1 \wedge t_2)(P_i) \neq 0 \). Thus \( \mu_2(Q)(P_i) = \mu_{1,F}(x \wedge y)(P_i)\mu_{1,M}(t_1 \wedge t_2)(P_i) \neq 0 \) for every \( i = 1, \ldots, d \).

We claim that the vectors \( \{v_1, \ldots, v_d\} \) are linearly independent in \( H^1(T_{\tilde{C}})^\perp \cong H^1(T_C(-R)) \).

In fact we show that the subspace \( W := \langle \xi_{T_1}, \xi_{\sigma(T_1)}, \ldots, \xi_{T_d}, \xi_{\sigma(T_d)} \rangle \) of \( H^1(C, T_{\tilde{C}}) \) has dimension \( 2d \). This is equivalent to say that the annihilator \( \text{Ann}(W) \) of \( W \) in \( H^0(\tilde{C}, K_{\tilde{C}}^2) \) has codimension \( 2d \). Observe that \( \text{Ann}(W) = H^0(\tilde{C}, K_{\tilde{C}}^2(-D)) \), where \( D = T_1 + \sigma(T_1) + \ldots + T_d + \sigma(T_d) \). Then by Riemann Roch we have \( h^0(K_{\tilde{C}}^2(-D)) = h^0(T_C(D)) + 4(\tilde{g} - 1) - 2d - (\tilde{g} - 1) = 3\tilde{g} - 3 - 2d \), since \( \text{deg}(T_{\tilde{C}}(D)) = -2(\tilde{g} - 1) + 2d = -4g - 2r + 4 + 2d \leq -4g - 2r + 4 + 2(2g - 2 + r - k - \text{deg}(B)) = -2k - 2\text{deg}(B) < 0 \).

This shows that \( \{\xi_{T_1}, \xi_{\sigma(T_1)}, \ldots, \xi_{T_d}, \xi_{\sigma(T_d)} \} \) are linearly independent in \( H^1(\tilde{C}, T_{\tilde{C}}) \) and hence also \( \{v_1, \ldots, v_d\} \) are linearly independent in \( H^1(T_{\tilde{C}})^\perp \).
By (3.2), (3.3) one immediately obtains that the restriction of $\tilde{\rho}(Q)$ to the subspace $W' := \langle v_1, \ldots, v_d \rangle$ is represented in the basis $\{v_1, \ldots, v_d\}$ by a diagonal matrix with entries

$$-4\pi i \mu_2(\pi^*Q)(T_i) = -4\pi i \mu_2(Q)(P_i) \neq 0$$

on the diagonal. So $\rho_P(Q)$ has rank at least $d$.

In case (1) $B$ is empty, hence $d = 2g - 2 + r - k$. In case (2), by Clifford Theorem we have:

$$2(h^0(M(-B)) - 1) \leq \deg(M(-B)) = 2g - 2 + r - k - \deg(B),$$

hence

$$\deg(B) \leq 2g - 2 + r - k - 2h^0(M(-B)) + 2 \leq 2g + r - k - 2g + k + r - 1 = k - r + 2$$

and $d = 2g - 2 + r - k - \deg(B) \geq 2g + 2r - 2k - 4$.

\[\square\]

**Theorem 3.2.** Assume that $[(C, \alpha, R)] \in R^0_{g,r}$, where $C$ is a $k$-gonal curve of genus $g$ with $g + r \geq k + 3$. Let $Y$ be a germ of a totally geodesic submanifold of $A^6_{g-1+r}$, which is contained in the Prym locus and passes through $P(C, \alpha, R)$. Then

1. If $r > k + 1$, then $\dim Y \leq 2g - 2 + \frac{3r + k}{2}$.
2. If $r \leq k + 1$, then $\dim Y \leq 2g + r + k - 1$.

\[\text{Proof.}\] Since $Y$ is totally geodesic, for any $v \in T_{[(C, \alpha, R)]}Y$ we must have that $\rho(Q)(v \circ v) = 0$ for any $Q$ in $I_2(K_C \otimes \alpha)$. Hence if a quadric $Q$ is such that the rank of $\rho(Q)$ is at least $m$

$$\dim T_{[C]}Y \leq (3g - 3 + 2r) - \frac{m}{2},$$

The result then follows by the existence of a quadric $Q \in I_2(K_C \otimes \alpha)$ shown in Theorem 3.1 with $\text{rank}(\rho(Q)) \geq m$, where $m = 2g - 2 - k + r$ in case (1) and $m = 2g - 2k - 4 + 2r$ in case (2).

**Remark 3.3.** In case (2) of Theorem 3.1 if $|M|$ is base point free we have the same estimate as in case (1), namely $\text{rank}(\rho(Q)) \geq 2g - 2 - k + r$. So in this case the bound on the dimension of a germ of a totally geodesic submanifold $Y$ contained in the Prym locus and passing through $P(C, \alpha, R)$ with $C$ a $k$-gonal curve such that $|M|$ is base point free becomes: $\dim Y \leq 2g - 2 + \frac{3r + k}{2}$.

**Theorem 3.4.** Let $Y$ be a germ of a totally geodesic submanifold of $A^6_{g-1+r}$ which is contained in the Prym locus.

1. If $g < 2r - 5$, then $\dim Y \leq \frac{9}{2}g + \frac{3}{2}r - \frac{5}{4}$.
2. If $g \geq 2r - 5$, then $\dim Y \leq \frac{9}{2}g + r + \frac{1}{2}$.

\[\text{Proof.}\] This immediately follows from Theorem 3.2 since the gonality of a genus $g$ curve is at most $[(g + 3)/2]$.

**Remark 3.5.** In [7] examples of Shimura curves (hence totally geodesic) of $A_{g-1+r}$ contained in $R_{g,r}$ when $r = 0, 1$ have been constructed using families of Galois covers of $\mathbb{P}^1$. The examples when $r = 0$ are all contained in $A_{g-1}$ with $g \leq 13$, while the ones with $r = 1$ are all contained in $A_g$ with $g \leq 8$.

**References**

[1] Bardelli, F., Ciliberto, C., Verra, A., Curves of minimal genus on a general abelian variety, Compos. Math. 96 (1995), no. 2, 115–147.

[2] Colombo, E., Frediani, P., Some results on the second Gaussian map for curves. Michigan Math. J. Vol. 58, 3 (2009), 745-758.

[3] Colombo, E., Frediani, P., Siegel metric and curvature of the moduli space of curves. Transactions of the Amer. Math. Soc. 362 (2010), no. 3, 1231-1246.
Colombo, E., Frediani, P., A bound on the dimension of a totally geodesic submanifold in the Prym locus. Collectanea Mathematica. DOI: 10.1007/s13348-018-0215-0.

Colombo, E., Frediani, P., Prym map and second Gaussian map for Prym-canonical line bundles. Adv. Math. 239 (2013), 47-71.

Colombo, E., Frediani, P., Ghigi, A., On totally geodesic submanifolds in the Jacobian locus. Internat. J. Math. 26 (2015), no. 1, 1550005, 21 pp.

Colombo, E., Frediani, P., Ghigi, A., Penegini M., Shimura curves in the Prym locus. Communications in Contemporary Mathematics. DOI: 10.1142/S0219199718500098.

Colombo, E., Pirola, G.P., Tortora, A., Hodge-Gaussian maps, Ann. Scuola Normale Sup. Pisa Cl. Sci. (4) 30 (2001), no. 1, 125-146.

Friedman, Robert; Smith, Roy, The generic Torelli theorem for the Prym map. Invent. Math. 67 (1982), no. 3, 473-490.

Kanev, V. I., A global Torelli theorem for Prym varieties at a general point. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 46 (1982), no. 2, 244-268, 431.

Lange, H., Ortega, A., Prym varieties of cyclic coverings, Geom. Dedicata 150 (2011), 391-403.

Marcucci, V, Naranjo, J.C., Prym varieties of double coverings of elliptic curves, Int. Math. Res. Notices 6 (2014), 1689-1698.

Marcucci, V, Pirola, G.P., Generic Torelli for Prym varieties of ramified coverings, Compos. Math. 148 (2012), 1147-1170.

Nagaraj, D.S., Ramanan, S., Polarizations of type \((1, 2, \ldots, 2)\) on abelian varieties, Duke Math. J. 80 (1995), 157-194.

Naranjo, J.C., Ortega, A., Verra, A., Generic injectivity of the Prym map for double ramified coverings. Trans of AMS. DOI: 10.1090/tran/7459.

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