A new type of the Gronwall-Bellman inequality and its application to fractional stochastic differential equations

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Abstract: This paper presents a new type of Gronwall-Bellman inequality, which arises from a class of integral equations with a mixture of nonsingular and singular integrals. The new idea is to use a binomial function to combine the known Gronwall-Bellman inequalities for integral equations having nonsingular integrals with those having singular integrals. Based on this new type of Gronwall-Bellman inequality, we investigate the existence and uniqueness of the solution to a fractional stochastic differential equation (SDE) with fractional order $0 < \alpha < 1$. Finally, the fractional type Fokker-Planck-Kolmogorov equation associated to the solution of the fractional SDE is derived using Itô's formula.

Subjects: Science; Mathematics & Statistics; Statistics & Probability; Probability; Probability Theory & Applications

Keywords: Gronwall-Bellman inequality; fractional stochastic differential equations (SDEs); existence and uniqueness; fractional Fokker-Planck equation

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PUBLIC INTEREST STATEMENT

Gronwall-Bellman inequality plays a significant role in mathematical modeling, particularly in applications of integral equations. For a mathematical model which arises from a class of integral equations with a mixture of nonsingular and singular integrals, there is lack of a powerful Gronwall-Bellman inequality to help researchers on this case. To derive such a Gronwall-Bellman inequality, the new idea is to use a binomial function to combine the known Gronwall-Bellman inequalities for integral equations having nonsingular integrals with those having singular integrals. Based on this new type of Gronwall-Bellman inequality, we investigate the existence and uniqueness of the solution to a fractional stochastic differential equation (SDE) with fractional order $0 < \alpha < 1$. This result generalizes the known existence and uniqueness theorem related to fractional order $\frac{1}{2} < \alpha < 1$. 
1. Introduction

It is well known that integral inequalities are instrumental in studying the qualitative analysis of solutions to differential and integral equations (Ames & Pachpatte, 1997). Among these inequalities, the distinguished Gronwall-Bellman type inequality from Bellman and Cooke (1963), and its associated extensions (Agarwal & Choi, 2016; Agarwal, Deng, & Zhang, 2005; Agarwal, Tariboon, & Ntouyas, 2016; Lipovan, 2000; Liu, Zhang, Agarwal, & Wang, 2016; Mao, 1989; Pachpatte, 1975; Wang, Agarwal, & Chand, 2014), are capable of affording explicit bounds on solutions of a class of linear differential equations with integer order. The following lemma concerns a standard Gronwall-Bellman inequality in Corduneanu (2008) for a differential equation with order one or equivalently an integral equation with nonsingular integrals.

**Lemma 1.1** Suppose $h(t), k(t),$ and $x(t)$ are continuous functions on $t_0 \leq t < T, \quad 0 < T \leq \infty$, with $k(t) \geq 0$. If $x(t)$ satisfies

$$ x(t) \leq h(t) + \int_{t_0}^{t} k(s)x(s) \, ds, $$

then

$$ x(t) \leq h(t) + \int_{t_0}^{t} h(s)k(s) \exp \left( \int_{s}^{t} k(u) \, du \right) \, ds. $$

Moreover, if $h(t)$ is nondecreasing, then

$$ x(t) \leq h(t) \exp \left( \int_{t_0}^{t} k(s) \, ds \right). $$

In order to investigate the qualitative properties of solutions to differential equations of fractional order, there are several generalizations of Gronwall-Bellman inequalities developed by many researchers (Atıcı & Eloe, 2012; Lazarević & Spasić, 2009; Ye, Gao, & Ding, 2007; Zheng, 2013). Let us recall the following generalized Gronwall-Bellman inequality proposed in Ye et al. (2007) for a fractional differential equation with order $\beta > 0$ or equivalently an integral equation with singular integrals.

**Lemma 1.2** Suppose $\beta > 0$, $a(t)$ is a nonnegative function which is locally integrable on $0 \leq t < T, \quad 0 < T \leq \infty$, and $g(t)$ is a nonnegative, nondecreasing continuous function defined on $0 \leq t < T$ with $g(t) \leq M$ (constant). If $u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with

$$ u(t) \leq a(t) + g(t) \left( t - s \right)^{\beta - 1} u(s) \, ds $$

on this interval, then

$$ u(t) \leq a(t) + \int_{t_0}^{t} \left( \sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)} (t - s)^{(n-1)\beta} a(s) \right) \, ds, $$

where $\Gamma(\cdot)$ is the gamma function. Furthermore, if $a(t)$ is nondecreasing on $0 \leq t < T$, then

$$ u(t) \leq a(t)E_{\beta}(g(t)\Gamma(\beta)t^{\beta}), $$

where $E_{\beta}(z)$ is the Mittag-Leffler function defined by $E_{\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta + 1)}$ for $z > 0$.

From many real applications, such as in physics, theoretical biology, and mathematical finance, there is substantial interest in a class of fractional SDEs (Jumarie, 2005a; Mandelbrot & Van Ness, 1968; Pedjeu & Ladde, 2012). The fractional SDEs take the form

$$ x(t) \leq h(t) + \int_{t_0}^{t} k(s)x(s) \, ds, $$

$$ x(t) \leq h(t) + \int_{t_0}^{t} h(s)k(s) \exp \left( \int_{s}^{t} k(u) \, du \right) \, ds. $$

Moreover, if $h(t)$ is nondecreasing, then

$$ x(t) \leq h(t) \exp \left( \int_{t_0}^{t} k(s) \, ds \right). $$
\[ dx(t) = b(t, x(t)) \, dt + \sigma_1(t, x(t)) \, dt^\alpha + \sigma_2(t, x(t)) \, dB_t, \]  

where the initial value is \( x(0) = x_0 \), \( 0 < \alpha < 1 \), and \( B_t \) is the standard Brownian motion. According to Jumarie (2005a) and Pedjeu and Ladde (2012), the integral equation corresponding to Equation (1) is

\[ x(t) = x_0 + \int_0^t b(s, x(s)) \, ds + \alpha \int_0^t (t-s)^{\alpha-1} \sigma_1(s, x(s)) \, ds + \int_0^t \sigma_2(s, x(s)) \, dB_s. \]  

Since \( 0 < \alpha < 1 \), there are nonsingular and singular integrals in the integral equation (Equation (2)). However, the above-mentioned types of Gronwall-Bellman inequalities, such as Lemmas 1.1 and 1.2, are not applicable to studying the qualitative properties of the solution to Equations (1) or (2).

The first goal of this paper, presented in Section 2, is to derive a new type of Gronwall-Bellman inequality which is applicable to study the qualitative behaviors of the solution to the fractional SDE (Equation (1)) or the stochastic integral equation (Equation (2)). The second goal, accomplished in Section 3, is to apply the results from Section 2 to investigate existence and uniqueness of the solution to the fractional SDE (Equation (1)) of order \( 0 < \alpha < 1 \). Finally, in Section 4, a fractional type Fokker-Planck-Kolmogorov equation associated to the solution of the fractional SDE (Equation (1)) is derived.

### 2. Generalization of the Gronwall-Bellman inequality

In this section, we develop a new integral inequality, Equation (4) below, by verifying three claims. The first claim is established by using the method of induction and taking advantage of the binomial function; the second claim is verified by taking advantage of properties of the Gamma function; the third claim is verified by employing Gamma functions, Mittag-Leffler functions, and exponential functions. The established integral inequality is applicable to the fractional SDE (Equation (1)) or the stochastic integral equation (Equation (2)). Also, this new integral inequality can be considered as a generalization of the integral inequalities in Lemmas 1.1 and 1.2.

**Theorem 2.1** Let \( 0 < \alpha < 1 \) and consider the time interval \( I = [0, T) \), where \( T \leq \infty \). Suppose \( a(t) \) is a nonnegative function, which is locally integrable on \( I \) and \( b(t) \) and \( g(t) \) are nonnegative, nondecreasing continuous function defined on \( I \), with both bounded by a positive constant, \( M \). If \( u(t) \) is nonnegative, and locally integrable on \( I \) and satisfies

\[ u(t) \leq a(t) + b(t) \int_0^t u(s) \, ds + g(t) \int_0^t (t-s)^{\alpha-1} u(s) \, ds, \]  

then

\[ u(t) \leq a(t) + \sum_{n=1}^{\infty} \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) b^{n-i}(t) g^i(t) \frac{\Gamma(\alpha)}{\Gamma(i\alpha + n - i)} \int_0^t (t-s)^{ia-(i+1)\alpha} a(s) \, ds. \]  

**Proof** Let \( \phi \) be a locally integrable function and define an operator \( B \) on \( \phi \) as follows

\[ B\phi(t) = b(t) \int_0^t \phi(s) \, ds + g(t) \int_0^t (t-s)^{\alpha-1} \phi(s) \, ds, \quad t \geq 0. \]

From the inequality, Equation (3),

\[ u(t) \leq a(t) + Bu(t). \]
This implies

\[ u(t) \leq \sum_{k=0}^{n-1} b^k a(t) + B^n u(t). \]  

(5)

In order to get the desired inequality, Equation (4), from Equation (5), there are three claims to be verified.

The first claim provides a general bound on \( B^n(t) \):

\[ B^n u(t) \leq \sum_{i=0}^{n} \binom{n}{i} b^{n-i}(t) g'(s) \frac{[\Gamma(\alpha)]^i}{\Gamma(i\alpha + k - i)} \int_0^t \int_0^s (s-r)^{(i-\alpha+1-k)} u(r) \, dr \, ds. \]  

(6)

The method of induction will be used to verify the inequality in Equation (6). First let \( n = 1 \). Then the inequality in Equation (6) is true. Now, suppose that the inequality, Equation (6), holds for \( n = k \), and then compute \( B^{k+1} \) when \( n = k + 1 \).

\[ B^{k+1} u(t) = B(B^k u(t)) \leq b(t) \sum_{i=0}^{k} \binom{k}{i} b^{k-i}(s) g'(s) \frac{[\Gamma(\alpha)]^i}{\Gamma(i\alpha + k - i)} \int_0^t \int_0^s (s-r)^{(i-\alpha+1-k)} u(r) \, dr \, ds 
\]

\[ + g(t) \int_0^t (t-s)^{k+1} \sum_{i=0}^{k} \binom{k}{i} b^{k-i}(s) g'(s) \frac{[\Gamma(\alpha)]^i}{\Gamma(i\alpha + k - i)} \int_0^t \int_0^s (s-r)^{(i-\alpha+1-k)} u(r) \, dr \, ds. \]

Let

\[ C(t) = b(t) \sum_{i=0}^{k} \binom{k}{i} b^{k-i}(s) g'(s) \frac{[\Gamma(\alpha)]^i}{\Gamma(i\alpha + k - i)} \int_0^t \int_0^s (s-r)^{(i-\alpha+1-k)} u(r) \, dr \, ds, \]

and

\[ G(t) = g(t) \int_0^t (t-s)^{k+1} \sum_{i=0}^{k} \binom{k}{i} b^{k-i}(s) g'(s) \frac{[\Gamma(\alpha)]^i}{\Gamma(i\alpha + k - i)} \int_0^t \int_0^s (s-r)^{(i-\alpha+1-k)} u(r) \, dr \, ds. \]

Then, compute \( C(t) \) and \( G(t) \) term by term to reach the desired inequality (Equation (6)). Since \( b(t) \) and \( g(t) \) are nonnegative and nondecreasing functions,

\[ C(t) = \sum_{i=0}^{k} b^{k-i}(t) g'(s) \frac{[\Gamma(\alpha)]^i}{\Gamma(i\alpha + k - i)} \int_0^t \int_0^s (s-r)^{(i-\alpha+1-k)} u(r) \, dr \, ds 
\]

\[ = \sum_{i=0}^{k} b^{k-i}(t) g'(s) \frac{[\Gamma(\alpha)]^i}{\Gamma(i\alpha + k - i)} \int_0^t \int_0^s (s-r)^{(i-\alpha+1-k)} u(r) \, ds \, dr 
\]

\[ = \sum_{i=0}^{k} b^{k-i}(t) g'(s) \frac{[\Gamma(\alpha)]^i}{\Gamma(i\alpha + k - i + 1)} (t-r)^{(i-\alpha+1-k)} u(r) \, dr 
\]

\[ = b^{k+1}(t) g'(s) \frac{1}{\Gamma(k+1)} \int_0^t (t-r)^{k} u(r) \, dr 
\]

Similarly, compute \( G(t) \).
Note that for the purpose of notation simplification during the proof of the second claim, define

\[ G(t) \leq \sum_{i=0}^{k} b^{k-i}(t) g^{i+1}(t) \left( \begin{array}{c} k \\ i \end{array} \right) \frac{[\Gamma(\alpha)]^i}{\Gamma(i \alpha + k + i)} \int_{0}^{t} (s-u)^{i-1} \left( s-\tau \right)^{(a-i+k-1)} u(\tau) \, d\tau \, ds \]

\[ = \sum_{i=0}^{k} b^{k-i}(t) g^{i+1}(t) \left( \begin{array}{c} k \\ i \end{array} \right) \frac{[\Gamma(\alpha)]^i}{\Gamma(i \alpha + k + i)} \int_{0}^{t} (s-u)^{i-1} (s-\tau)^{(a-i+k-1)} u(\tau) \, d\tau \, ds \]

\[ = \sum_{i=0}^{k} b^{k-i}(t) g^{i+1}(t) \left( \begin{array}{c} k \\ i \end{array} \right) \frac{[\Gamma(\alpha)]^i}{\Gamma(i \alpha + k + i)} \int_{0}^{t} (s-u)^{i-1} (s-\tau)^{(a-i+k-1)} u(\tau) \, d\tau \, ds \]

Combining Equations (7) and (8) yield

\[ B^{k+1} u(t) = C(t) + G(t) \leq b^{k+1}(t) \left( \begin{array}{c} k \\ 0 \end{array} \right) \frac{1}{\Gamma(k+1)} \int_{0}^{t} (t-\tau)^0 u(\tau) \, d\tau \]

\[ + b(t) \sum_{i=1}^{k} \left[ \left( \begin{array}{c} k \\ i \end{array} \right) g^{i+1}(t) \right] b^{k-i}(t) g^{i+1}(t) \left( \begin{array}{c} k \\ i \end{array} \right) \frac{[\Gamma(\alpha)]^i}{\Gamma(i \alpha + k + i)} \int_{0}^{t} (s-u)^{i-1} (s-\tau)^{(a-i+k-1)} u(\tau) \, d\tau \]

\[ + g^{k+1}(t) \left( \begin{array}{c} k \\ k \end{array} \right) \frac{[\Gamma(\alpha)]^{k+1}}{\Gamma((k+1) \alpha)} \int_{0}^{t} (t-\tau)^{(k+1)a-1} u(\tau) \, d\tau \]

\[ = \sum_{i=0}^{k+1} \left( \begin{array}{c} k+1 \\ i \end{array} \right) b^{k+1-i}(t) g^{i+1}(t) \frac{[\Gamma(\alpha)]^i}{\Gamma(i \alpha + k + i)} \int_{0}^{t} (t-u)^{i-1} (t-\tau)^{(a-i+k-1)} u(\tau) \, d\tau. \]

This implies that for any \( n \in \mathbb{N}^* \), the first claim, Equation (6), holds.

The second claim shows that \( B^n u(t) \) vanishes as \( n \) increases. For each \( t \) in \([0, T]\),

\[ B^n u(t) \to 0, \quad \text{as } n \to \infty. \]

For the purpose of notation simplification during the proof of the second claim, define

\[ H_{n}(t) := \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) b^{n-i}(t) g^{i+1}(t) \frac{[\Gamma(\alpha)]^i}{\Gamma((n+1-\alpha) \alpha + i)} \int_{0}^{t} (s-u)^{i-1} (s-\tau)^{(a-i+n-\alpha)} u(\tau) \, d\tau. \]

Note that \( \Gamma(\chi) \) is positive and decreasing on \((0, 1]\) but positive and increasing on \([2, \infty)\). Let \( x_i = i \alpha + n - i \). Then, the sequence \( x_i \) is decreasing over \([0, n]\) since \( x_{i+1} - x_i = \alpha - 1 < 0 \) when \( i \) is an integer and \( i \in [0, n] \). This means \( x_i^{\min} = \alpha \) and \( x_i^{\max} = n \). Furthermore, for a fixed \( \alpha \), there exists a large enough \( n_0 \) such that for any \( n > n_0 \), there is \( n_0 > \frac{\alpha}{\alpha - 1} \). So the sequence satisfies \( x_i \geq 2 \) for any integer \( i \) in \([0, n]\) if \( n \) is large enough. Thus, for any \( i \in [0, n] \) and \( \Gamma(x_i^{\min}) < \Gamma(x_i) \), and

\[ H_{n}(t) \leq \frac{1}{\Gamma(n \alpha)} \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) b^{n-i}(t) g^{i+1}(t) [\Gamma(\alpha)]^i \int_{0}^{t} (s-u)^{i-1} (s-\tau)^{(a-i+n-\alpha)} u(\tau) \, d\tau, \quad n > n_0. \]

Also for \( \alpha \in (0, 1), \Gamma(\alpha) > 1 \). Therefore,


\[ H_n(t) \leq \left[ \frac{\Gamma(\alpha)}{\Gamma(n\alpha)} \right] \sum_{i=0}^{n} \left( \frac{n}{i} \right) b^{n-i}(t) g'(t) \int_{0}^{t} \frac{1}{(s-t)^{\alpha(i+i-1)-n}} u(s) \, ds. \]

Let \( y_i = ia + n - i - 1 \). Similar to the sequence \( x_i \), there is \( y_i^{\text{min}} = n\alpha - 1 \geq 1 \) for a large enough \( n \) and \( y_i^{\text{max}} = n - 1 \). Since \( t \in [0, T) \), split the interval \([0, T]\) into two subintervals \([0, 1]\) and \([1, \infty)\). For \( t \in [0, 1] \), \( (t-s)^\alpha \leq t^{\alpha\min} = t^{\alpha-1} \) while if \( t \in [1, T) \), \( (t-s)^\alpha \leq t^{\alpha \max} = t^\alpha \). Thus,

\[ H_n(t) \leq \frac{\left[ \frac{\Gamma(\alpha)}{\Gamma(n\alpha)} \right]}{\left[ \frac{\Gamma(\alpha)}{\Gamma(n\alpha)} \right]} \sum_{i=0}^{n} \left( \frac{n}{i} \right) b^{n-i}(t) g'(t) \int_{0}^{t} \frac{1}{(s-t)^{\alpha(i+i-1)-n}} u(s) \, ds. \]

Notice that \( b(t) \) and \( g(t) \) are both bounded by a positive constant \( M \), i.e. \( b(t) \leq M \) and \( g(t) \leq M \), and \( u(s) \) is locally integrable over \( 0 \leq t < T \). This means that from Equation (10), \( H_n(t) \to 0 \) as \( n \to \infty \) because the Gamma function, \( \Gamma(n\alpha) \), is growing faster than a power function. Therefore, the second claim, Equation (9), is verified since \( B^* u(t) \leq H_n(t) \) for any \( n \in \mathbb{N}^+ \).

The third claim establishes that the right-hand side (RHS) of Equation (4) exists on \( 0 \leq t < T \). In order to show this statement, we first prove that for \( 0 \leq t < T \), the following infinite sum of sequences denoted by \( L(t;r) \) is convergent.

\[
L(t;r) := \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left( \frac{n}{i} \right) b^{i+j}(t) g'(t) \frac{\Gamma(\alpha)}{\Gamma(i\alpha+j+1)} r^{i+j+n-j-1}
\]

\[
= \sum_{i=0}^{\infty} g'(t) \frac{\Gamma(\alpha)}{\Gamma(i\alpha+j+1)} r^{i} \sum_{j=0}^{\infty} \left( \frac{n}{i} \right) b^{i+j}(t) \frac{1}{\Gamma(i\alpha+j+1)} r^{n-j} \tag{11}
\]

where \((i + \alpha - n) \ldots (i + 1)\) is a product and it takes one if \((ia + n - i) < ia + 1\). Let \( k = n - i \), then compute

\[
\left( \frac{n}{i} \right) \frac{1}{(i+\alpha-n-1)(i+1)} = \frac{(k+i)!}{i!k!} \frac{1}{(i+\alpha)(i+1)} = \frac{(k+i)(i+1)}{k! (i+\alpha)(i+1)} \leq \frac{1}{a^k k!}.
\]

Substituting \( k = n - i \) and Equation (12) into Equation (11) gives

\[
L(t;r) \leq \sum_{i=0}^{\infty} \frac{g'(t)\Gamma(\alpha)}{\Gamma(i\alpha+j+1)} \sum_{k=0}^{\infty} \frac{1}{a^k k!} r^k = E_x(g(t)\Gamma(\alpha)r^x) \exp\left( \frac{1}{a} b(t)r \right),
\]

which is finite for \( 0 \leq t < T \). Furthermore, since \( b(t) \leq M \) and \( g(t) \leq M \), define

\[
L(M;r) := \sum_{i=0}^{\infty} \frac{M^i \Gamma(\alpha)}{\Gamma(i\alpha+j+1)} \sum_{k=0}^{\infty} \frac{M^k r^k}{a^k k!} = E_x(M\Gamma(\alpha)r^x) \exp\left( \frac{1}{a} M r \right),
\]

which means \( L(M;r) \) is finite and \( L(t;r) \leq L(M;r) \). Then, compute the RHS of Equation (4)
RHS = \( a(t) + \sum_{n=1}^{\infty} \sum_{i=0}^{n} \binom{n}{i} b^{n-i}(t) g(t) \frac{[\Gamma(a)]^i}{\Gamma(i a + n + 1)} \int_0^t (t-s)^{i(a+n-i)} a(s) \, ds \)
\leq \( a(t) + \sum_{n=1}^{\infty} \sum_{i=0}^{n} \binom{n}{i} M^{n-i} M^i \frac{[\Gamma(a)]^i}{\Gamma(i a + n + 1)} \int_0^t (t-s)^{i(a+n-i)} a(s) \, ds \)
= \( a(t) + \int_0^t dL(Mt-s) \, a(s) \, ds \).

Since the Mittag-Leffler function \( E_i(t^a) \) is an entire function in \( t^a \), see Gorenflo, Loutchko, Luchko, and Mainardi (2002), the exponential function \( \exp(t) \) is uniformly continuous in \( t \), and both \( t^{a-1} \) and \( a(t) \) are locally integrable over \( 0 \leq t < T \), the integral \( \int_0^t \frac{dL(Mt-s)}{dt} a(s) \, ds \) is finite. This implies that the RHS of Equation (4) is finite. So the last claim is also verified, thereby completing the proof.

\[ \text{Corollary 2.1} \]
Suppose the conditions in Theorem 2.1 are satisfied and furthermore, \( a(t) \) is nondecreasing on \( 0 \leq t < T \). Then
\[ u(t) \leq a(t) E_i g(t) \Gamma(a) t^a \exp \left( \frac{1}{a} b(t) t \right). \]

**Proof** From the proof of Theorem 2.1,
\[ u(t) \leq a(t) + \sum_{n=1}^{\infty} \sum_{i=0}^{n} \binom{n}{i} b^{n-i}(t) g(t) \frac{[\Gamma(a)]^i}{\Gamma(i a + n + 1)} \int_0^t (t-s)^{i(a+n-i)} a(s) \, ds. \]

Since \( a(t) \) is nondecreasing,
\[ u(t) \leq a(t) \sum_{n=1}^{\infty} \sum_{i=0}^{n} \binom{n}{i} b^{n-i}(t) g(t) \frac{[\Gamma(a)]^i}{\Gamma(i a + n + 1)} \int_0^t (t-s)^{i(a+n-i)} \, ds \]
\leq a(t) \sum_{n=1}^{\infty} \sum_{i=0}^{n} \binom{n}{i} b^{n-i}(t) g(t) \frac{[\Gamma(a)]^i}{\Gamma(i a + n + 1)} t^{i(a+n-i)}
\leq a(t) E_i g(t) \Gamma(a) t^a \exp \left( \frac{1}{a} b(t) t \right). \]
This completes the proof.

**Remark 2.1** From Theorem 2.1 and Corollary 2.1, we see that if \( a = 1 \), Theorem 2.1 and Corollary 2.1 are the same as Lemma 1.1; while if \( b(t) \equiv 0 \), Theorem 2.1 and Corollary 2.1 become Lemma 1.2.

### 3. Existence and uniqueness of the solution to fractional SDEs

In this section, using the main results from Section 2, we investigate the existence and uniqueness of the solution to the fractional SDE (Equation (1)) with fractional order \( 0 < a < 1 \). By application of the classical Picard-Lindelöf successive approximation scheme and the standard Gronwall-Bellman inequality, existence and uniqueness of the solution to Equation (1) with fractional order \( \frac{1}{2} < a < 1 \) is discussed in Pedjeu and Ladde (2012). However, the case with \( 0 < a \leq \frac{1}{2} \) remains to be investigated. We can apply the generalized Gronwall-Bellman inequality developed in Section 2 to derive existence and uniqueness of the solution to Equation (1) when \( 0 < a < 1 \).

**Theorem 3.1** Let \( 0 < a < 1 \), \( T > 0 \), and \( \mathbb{B} \) be a \( m \)-dimensional Brownian motion on a complete probability space \( \Omega \equiv (\Omega, \mathcal{F}, \mathbb{P}) \). Assume that \( b(\cdot, \cdot), \sigma_1(\cdot, \cdot), \sigma_2(\cdot, \cdot) : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \sigma_2(\cdot, \cdot) : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{m} \) are measurable functions satisfying the linear growth condition,
\[ |b(t, x)|^2 + |\sigma_1(t, x)|^2 + |\sigma_2(t, x)|^2 \leq K^2(1 + |x|^2), \]  

(13)

for some constant \( K > 0 \) and the Lipschitz condition,

\[ |b(t, x) - b(t, y)| + |\sigma_1(t, x) - \sigma_1(t, y)| + |\sigma_2(t, x) - \sigma_2(t, y)| \leq L|x - y|, \]  

(14)

for some constant \( L > 0 \). Let \( x_0 \) be a random variable, which is independent of the \( \sigma \)-algebra \( \mathcal{F}_t \subset \mathcal{F}_\infty \) generated by \( \{B_s, t \geq 0\} \) and satisfies \( \mathbb{E}[|x_0|^2] < \infty \). Then, the fractional stochastic differential equation (Equation (1)) has a unique \( t \)-continuous solution \( x(t, \omega) \) with the property that \( x(t, \omega) \) is adapted to the filtration \( \mathcal{F}_t \) generated by \( x_0 \) and \( \{B_s, t \geq 0\} \), and

\[ \mathbb{E}\left[ \int_0^T |x(t)|^2 \, dt \right] < \infty. \]

**Proof (Existence)** From Equation (2), the corresponding equivalent stochastic integral equation of the fractional stochastic differential equation (Equation (1)) is rewritten as

\[ x(t) = x_0 + \int_0^t b(s, x(s)) \, ds + \alpha \int_0^t (t - s)^{\alpha - 1} \sigma_1(s, x(s)) \, ds + \int_0^t \sigma_2(s, x(s)) \, dB_s, \]

where \( 0 \leq t < T \) and \( 0 < \alpha < 1 \). For more details about this equivalence between Equation (1) and Equation (2), we refer to Jumarie (2005a, 2005b, 2006). By the method of Picard-Lindelöf successive approximations, define \( x^0(t) = x_0 \) and \( x^k(t) = x^k(t, \omega) \) inductively as follows

\[ x^{k+1}(t) = x_0 + \int_0^t b(s, x^k(s)) \, ds + \alpha \int_0^t (t - s)^{\alpha - 1} \sigma_1(s, x^k(s)) \, ds + \int_0^t \sigma_2(s, x^k(s)) \, dB_s, \]  

(15)

Applying the inequality \( |x + y + z|^2 \leq 3|x|^2 + 3|y|^2 + 3|z|^2 \) leads to

\[ \mathbb{E}[|x^{k+1}(t) - x^k(t)|^2] \leq 3\mathbb{E}\left[ \int_0^t \left( b(s, x^k(s)) - b(s, x^{k-1}(s)) \right)^2 \, ds \right] 
\quad + 3\mathbb{E}\left[ \alpha \int_0^t (t - s)^{\alpha - 1} \left( \sigma_1(s, x^k(s)) - \sigma_1(s, x^{k-1}(s)) \right)^2 \, ds \right] 
\quad + 3\mathbb{E}\left[ \left( \sigma_2(s, x^k(s)) - \sigma_2(s, x^{k-1}(s)) \right)^2 \, ds \right] 
\quad := I_1 + I_2 + I_3. \]

Using the Cauchy–Schwarz inequality on the first two terms, \( I_1 \) and \( I_2 \), plus Itô's Isometry, see in Oksendal (2013), in the third term, \( I_3 \) produces

\[ \mathbb{E}[|x^{k+1}(t) - x^k(t)|^2] \leq 3\mathbb{E}\left[ \left( b(s, x^k(s)) - b(s, x^{k-1}(s)) \right)^2 \, ds \right] 
\quad + 3\alpha^2 \int_0^t (t - s)^{2\alpha - 1} \, ds \mathbb{E}\left[ (t - s)^{\alpha - 1} \left( \sigma_1(s, x^k(s)) - \sigma_1(s, x^{k-1}(s)) \right)^2 \, ds \right] 
\quad \quad + 3\mathbb{E}\left[ \left( \sigma_2(s, x^k(s)) - \sigma_2(s, x^{k-1}(s)) \right)^2 \, ds \right] 
\quad := J_1 + J_2 + J_3. \]
Finally, using the Lipschitz condition (Equation (14)) on all terms, $J_1$, $J_2$, $J_3$, evaluating the first integral in the second term, $J_2$, and combining the first and third terms, $J_1$ and $J_3$ yields

$$E|\xi^{k+1}(t) - \xi^k(t)|^2 \leq 3L^2(1 + T) \int_0^t E|\xi^{k}(s) - \xi^{k-1}(s)|^2 \, ds$$

$$+ 3L^2(1 + T) \int_0^t (t - s)^{-1}E|\xi^k(s) - \xi^{k-1}(s)|^2 \, ds.$$  

(16)

Thus, for locally integrable function $\psi(t)$, define an operator $B$ as follows

$$B\psi(t) = 3L^2(1 + T) \left\{ \int_0^t \psi(s) \, ds + \int_0^t (t - s)^{-1} \psi(s) \, ds \right\}.$$  

Then, iterating Equation (16) yields

$$E|\xi^{k+1}(t) - \xi^k(t)|^2 \leq B(E|\xi^k(t) - \xi^{k-1}(t)|^2) \leq \cdots \leq B^m(E|\xi^1(t) - \xi^0(t)|^2).$$  

Since $0 < \alpha < 1$ and $E|\xi^1(t) - \xi^0(t)|^2$ is nonnegative and locally integrable, from the first claim, Equation (6), and the Equation (10) in the proof of the second claim in Section 2, we know that

$$E|\xi^{k+1}(t) - \xi^k(t)|^2 \leq B^m(E|\xi^1(t) - \xi^0(t)|^2)$$

$$\leq \frac{[\Gamma(\alpha)]^m \max\{T^{k-1}, T\}^k}{\Gamma(k\alpha)} [6L^2(1 + T)^4] t \int_0^t E|\xi^1(s) - \xi^0(s)|^2 \, ds.$$  

Similarly, apply the Cauchy–Schwartz inequality, the Itō’s Isometry, and the linear growth condition, Equation (13), instead of Lipschitz condition, Equation (14), to compute

$$E|\xi^1(t) - \xi^0(t)|^2 \leq 3(1 + T)K^2(1 + E|x_0|^2)(t + t^*).$$

This implies

$$\sup_{0 \leq t \leq T} E|\xi^{k+1}(t) - \xi^k(t)|^2 \leq M_0 \frac{[\Gamma(\alpha)]^m \max\{T^{k-1}, T\}^k}{\Gamma(k\alpha)} [6L^2(1 + T)^4],$$

(17)

where $M_0 = 3(1 + T)K^2(1 + E|x_0|^2)(\frac{T^k}{k} + \frac{T^{k+1}}{k+1})$ is independent of $k$ and $t$. Thus, for any $m > n > 0$,

$$\|\xi^m(t) - \xi^n(t)\|_{L^2_\psi}^2 \leq \sum_{k=m}^n \|\xi^{k+1}(t) - \xi^k(t)\|_{L^2_\psi}^2 = \sum_{k=m}^n \int_0^t E|\xi^{k+1}(t) - \xi^k(t)|^2 \, dt$$

$$\leq M_1 \sum_{k=m}^n \frac{[\Gamma(\alpha)]^k \max\{T^{k-1}, T\}^k}{\Gamma(k\alpha)} [6L^2(1 + T)^4]$$

$$\rightarrow 0, \text{ as } m, n \rightarrow \infty,$$

where $M_1 = 3(1 + T)K^2(1 + E|x_0|^2)(\frac{T^k}{k} + \frac{T^{k+1}}{k+1})$ is independent of $k$ and $t$. This means the successive approximations $(\xi^k(t))$ are mean-square convergent uniformly on $[0, T]$. It remains now to show that the sequence of successive approximations $(\xi^k(t))$ is almost surely convergent. First, apply Chebyshev’s inequality to yield

$$\sum_{k=1}^\infty P \left\{ \sup_{0 \leq s \leq T} |\xi^{k+1}(t) - \xi^k(t)| > \frac{1}{k} \right\} \leq \sum_{k=1}^\infty k^4 E \left( \sup_{0 \leq s \leq T} |\xi^{k+1}(t) - \xi^k(t)| \right)^2$$

$$= \sum_{k=1}^\infty k^4 E \left( \sup_{0 \leq s \leq T} |\xi^{k+1}(t) - \xi^k(t)|^2 \right).$$
By computations similar to those leading to Equation (17) and Doob’s Maximal Inequality for martingales,

$$
\sum_{k=1}^{\infty} \mathbb{P} \left( \sup_{0 \leq t \leq T} |x^{k+1}(t) - x^k(t)| > \frac{1}{k^2} \right) \leq M_0 \sum_{k=1}^{\infty} \frac{\Gamma(\alpha)k^k}{\Gamma(k+1)} \left( 6L^2(1+T)^{k+4}k^4 \right),
$$

which is finite. Then, applying the Borel-Cantelli lemma yields,

$$
\mathbb{P} \left( \sup_{0 \leq t \leq T} |x^{k+1}(t) - x^k(t)| > \frac{1}{k^2} \text{ for infinitely many } k \right) = 0.
$$

So there exists a random variable $x(t)$, which is the limit of the following sequence

$$
x^1(t) = x^0(t) + \sum_{n=0}^{k-1} (x^{n+1}(t) - x^n(t)) \rightarrow x(t) \text{ a.s.,}
$$

uniformly on $[0, T]$. Also $x(t)$ is $t$-continuous since $x^k(t)$ is $t$-continuous for all $k$. Therefore, taking the limit on both sides of Equation (15) as $k \to \infty$, there is a stochastic process $x(t)$ satisfying Equation (2).

(Uniqueness) The uniqueness is due to the Itô Isometry and the Lipschitz condition, Equation (14). Let $x_1(t) = x_1(t, \omega)$ and $x_2(t) = x_2(t, \omega)$ be solutions of Equation (2), which have the initial values, $x_1(0) = y_1$ and $x_2(0) = y_2$, respectively. Similarly, apply the Cauchy–Schwartz inequality, the Itô Isometry, and the Lipschitz condition (Equation (14)) to compute

$$
E|\frac{d}{dt}|x_1(t) - x_2(t)|^2 \leq 4E|y_1 - y_2|^2 + 4L^2(1+T) \int_0^t E|\frac{d}{dt}|x_1(s) - x_2(s)|^2 \, ds
$$

$$
+ 4\alpha L^2 T \int_0^t (t-s)^{\alpha - 1} E|x_1(s) - x_2(s)|^2 \, ds.
$$

By application of the generalized Gronwall–Bellman inequality in Corollary 2.1, we have

$$
E|\frac{d}{dt}|x_1(t) - x_2(t)|^2 \leq 4E|y_1 - y_2|^2 E_s(4\alpha L^2 T^\alpha \Gamma(\alpha) t^\alpha) \exp \left( \frac{1}{\alpha} 4L^2(1+T)t \right).
$$

Since $x_1(t)$ and $x_2(t)$ both satisfy the stochastic integral equation (Equation (2)), the initial values $y_1$ and $y_2$ are both equal to $x_0$. This means $E|\frac{d}{dt}|x_1(t) - x_2(t)|^2 = 0$ for all $t > 0$. Furthermore,

$$
\mathbb{P} \{ |x_1(t) - x_2(t)| = 0 \text{ for all } 0 \leq t \leq T \} = 1.
$$

Therefore, the uniqueness of the solution to Equation (2) is proved. \hfill \square

4. Fractional Fokker-Planck-Kolmogorov equation

Based on the existence and uniqueness Theorem 3.1 developed in Section 3, we derive the fractional Fokker-Planck-Kolmogorov equation associated to the unique solution of the fractional SDE, Equation (1). Before deriving the fractional Fokker-Planck-Kolmogorov equation, we first introduce an Itô formula from Pedjeu and Ladde (2012) to the following Itô process

$$
x(t) = x_0 + \int_0^t b(s, x(s)) \, ds + \int_0^t \sigma_1(s, x(s)) \, ds + \int_0^t \sigma_2(s, x(s)) \, dB_s,
$$

where $0 < \alpha < 1, B_t$ is the $m$-dimensional standard Brownian motion, and functions $b, \sigma_1, \sigma_2$ satisfy the conditions in Theorem 3.1.

**Lemma 4.1** Let $X(t)$ satisfy the Equation (18) and furthermore, let $V \in C(R^+ \times R^+, R^{m\times d})$ and assume that $V_t, V_x, V_{xx}$ exist and continuous for $(t,x) \in R^+ \times R^d$, where $V_t(t,x)$ is an $m \times n$ Jacobian matrix of $V(t,x)$ and $V_{xx}(t,x)$ is an $m \times n$ Hessian matrix. Then,
\[
dV(t, X(t)) = L_1 V(t, X(t)) \, dt + L_2 V(t, X(t)) \, dt^\ast + L_3 V(t, X(t)) \, dB_t,
\]

where

\[
L_1 V(t, x) = V_1(t, x) + V_2(t, x)b(t, x) + \frac{1}{2} \sigma_2(t, x)^T V_{xx}(t, x) \sigma_2(t, x)
\]

and

\[
L_2 V(t, x) = V_3(t, x) \sigma_1(t, x), \quad L_3 V(t, x) = V_4(t, x) \sigma_2(t, x).
\]

By applying the existence and uniqueness Theorem 3.1 and Itô’s formula, Lemma 4.1, the following fractional Fokker-Planck-Kolmogorov equation is established.

**Theorem 4.1** Let \( B(t) \) be the \( m \)-dimensional standard Brownian motion. Suppose that \( X(t) \) is the solution to the fractional SDE (Equation (1)) whose coefficient functions \( b \), \( \sigma_1 \) and \( \sigma_2 \) satisfy the conditions in Theorem 3.1. Then the transition probabilities \( P^x(t, x) = P^x_1(t, x_0, x_0) \) of \( X(t) \) satisfy the following fractional type differential equation

\[
dP^x(t, x) = A^x_1 P^x(t, x) \, dt + B^x_1 P^x(t, x) \, dt^\ast
\]

with initial condition \( P^x_0(0, x) = \delta_\nu(x) \), the Dirac delta function with mass on \( x_0 \) and \( A^x_1, B^x_1 \) are spatial operators defined, respectively, by

\[
A^x_1 h(x) = -\sum_{i=1}^n \frac{\partial}{\partial x_i} [b_i(t, x) h(x)] + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} \left[ \sum_{k=1}^m \sigma_{ik} \delta_2^j (t, x) h(x) \right]
\]

and

\[
B^x_1 = -\sum_{i=1}^n \frac{\partial}{\partial x_i} [\delta_1^i h(x)],
\]

where \( b = (b_1, \ldots, b_m)^T \), \( \delta_1 = (\delta_{1,1}, \ldots, \delta_{1,m})^T \), and \( \delta_2 \) is an \( n \times m \) matrix with elements \( \delta_{2,1} = \delta_{2,2} \).

**Proof** Let \( f \in C^\infty_c(R^n) \), i.e. \( f \) is an infinitely differential function on \( R^n \) with compact support. Since \( X(t) \) is the solution of the stochastic fractional differential equation (Equation (1)), this means \( X(t) \) satisfies the stochastic integral equation (Equation (18)). So apply Itô formula Lemma 4.1 on \( f(X(t)) \) to yield

\[
f(X(t)) - f(x_0) = \int_0^t \left( f_x(X(s)) b(s, X(s)) + \frac{1}{2} \sigma_2^2(s, X(s)) f_{xx}(X(s)) \sigma_2(s, X(s)) \right) \, ds
\]

\[
+ \int_0^t f_x(X(s)) \sigma_1(s, X(s)) \, ds^\ast + \int_0^t f_x(X(s)) \sigma_2(s, X(s)) \, dB_t.
\]

Notice the fact that

\[
\int_0^t f_x(X(s)) \sigma_1(s, X(s)) \, ds^\ast = a \left[ (t - s)^{\alpha - 1} f_x(X(s)) \sigma_1(s, X(s)) \right] \, ds,
\]

and more details on this equality can be found in Jumarie (2005a) and Pedjeu and Ladde (2012). Thus Equation (20) can be written as
\[
\begin{align*}
\quad f(X(t)) - f(x_0) &= \int_0^t \left( f_x(X(s))b(s, X(s)) + \frac{1}{2}\sigma_x^2(s, X(s))f_{xx}(X(s))\sigma_x(s, X(s)) \right) \, ds \\
&\quad + \alpha \int_0^t (t-s)^{\alpha-1}f_x(X(s))\sigma_x(s, X(s)) \, ds + \int_0^t f_x(X(s))\sigma_x(s, X(s)) \, dB_s.
\end{align*}
\] (21)

Since the integral \( \int_0^t f_x(X(s))\sigma_x(s, X(s)) \, dB_s \) is a martingale with respect to the filtration \( \mathcal{F}_t \), take conditional expectations on both sides of Equation (21) to obtain

\[
E[f(X(t)|X(0) = x_0) - f(x_0)] = E \left[ \int_0^t f_x(X(s))b(s, X(s)) \, ds | X(0) = x_0 \right]
+ E \left[ \frac{1}{2} \int_0^t \sigma_x^2(s, X(s))f_{xx}(X(s))\sigma_x(s, X(s)) \, ds | X(0) = x_0 \right]
+ \alpha E \left[ \int_0^t (t-s)^{\alpha-1}f_x(X(s))\sigma_x(s, X(s)) \, ds | X(0) = x_0 \right].
\] (22)

By Fubini’s Theorem and integration by parts, the above Equation (22) can be rewritten as

\[
\int_{\mathbb{R}^n} f(x)P^x(t, x) \, dx - f(x_0) = \int_{\mathbb{R}^n} f_x(x)b(s, x)P^x(s, x) \, dx \, ds \\
+ \frac{1}{2} \int_{\mathbb{R}^n} \sigma_x^2(s, x)f_{xx}(x)\sigma_x(s, x)P^x(s, x) \, dx \, ds \\
+ \alpha \int_{\mathbb{R}^n} (t-s)^{\alpha-1}f_x(s, x)P^x(s, x) \, dx \, ds
= \int_{\mathbb{R}^n} f(x)A^x_1P^x(s, x) \, dx \\
+ \int_{\mathbb{R}^n} f(x)\alpha (t-s)^{\alpha-1}A^x_2P^x(s, x) \, dx \, ds.
\]

Since \( f \in C_+^{\alpha}(\mathbb{R}^n) \) is arbitrary and \( C_+^{\alpha}(\mathbb{R}^n) \) is dense in \( L^2(\mathbb{R}^n) \),

\[
P^x(t, x) - \delta_{x_0}(x) = \int_0^t A^x_1P^x(s, x) \, ds + \alpha \int_0^t (t-s)^{\alpha-1}A^x_2P^x(s, x) \, ds,
\] (23)

where \( \delta_{x_0}(x) \) is a generalized function taking value \( \delta_{x_0}(x) = P^x(0, x_0) \). Finally, take the derivative with respect to time \( t \) on both sides of Equation (23) to yield the desired result (Equation (19)). \( \square \)

5. Conclusion

In this paper, a new type of Gronwall-Bellman inequality is established for a class of integral equations with a mixture of nonsingular and singular integrals. This new type of Gronwall-Bellman inequality can be considered as a generalization of known Gronwall-Bellman inequalities dealing with an integral equation having nonsingular or singular integrals, separately. With this new type of Gronwall-Bellman inequality, existence and uniqueness of the solution to a fractional SDE with fractional order \( 0 < \alpha < 1 \) is investigated. Furthermore, based on the existence and uniqueness result, a fractional type Fokker-Planck-Kolmogorov equation associated to the solution of a fractional SDE is derived.
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