A note on a Sung-Wang’s paper

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Abstract

The purpose of this note is to study the connectedness at infinity of manifold by using the theory of $p$-harmonic functions. We show that if the first eigenvalue $\lambda_{1,p}$ for the $p$-Laplacian achieves its maximal value on a Kähler manifold or a quaternionic Kähler manifold then such a manifold must be connected at infinity unless it is a topological cylinder with an explicit warped product metric.

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1. Introduction

It is well-known that the theory of $L^2$ harmonic functions/forms has close relation to geometry of manifolds, in particular, geometric structure at infinity. We refer the reader to [4, 5, 6, 7, 8] for further details of this topic. From a variational point of view, $p$-harmonic functions are natural extensions of harmonic functions. Therefore, it is very natural to study $p$-harmonic functions/forms on submanifolds and ask what is the relationship between geometry of such these submanifolds and the space of $p$-harmonic functions/forms. We emphasize that compared with the theory for harmonic functions, the study of $p$-harmonic functions is generally harder, even though elliptic, is degenerate and the regularity results are far weaker. In [2], Buckley and Koskela gave volume estimate of $p$-parabolic ends, $p$-nonparabolic ends in term of the first eigenvalue of the $p$-Laplacian. Then, in [2], Batista, Cavalcante and Santos used $p$-harmonic function to introduce a definition of $p$-parabolic ends (also see [9]). They proved that if $E$ is an end of a complete Riemannian manifold and satisfies a Sobolev-type inequality then $E$ must either have finite volume or to be $p$-nonparabolic. A characterization of $p$-nonparabolic ends in the context of submanifold is also verified. Recently, Chang, Chen and Wei (see [1]) studied $p$-harmonic maps with finite $q$-energy and prove a Liouville type theorem. As an application, they extended this theorem to some $p$-harmonic maps such as $p$-harmonic morphisms and conformal maps between Riemannian manifolds.

The main purpose of this note is to understand manifolds whose principal eigenvalue $\lambda_{1,p}$ for the $p$-Laplacian achieves its maximal. When $p = 2$, this problem has
been studied by Li and Wang in [6, 7, 8]. In [4], Kong, Li and Zhou proved a splitting theorem on quaternionic Kähler manifolds. In the general case of \( p \geq 2 \), Sung and Wang generalized Li-Wang’s results on a Riemannian manifold with \( \lambda_{1,p} \) obtaining its maximal value. They showed that such a manifold must be connected at infinity unless it is a topological cylinder endowed with an explicit warped product metric. Motivated by these results, we consider the same problem on Kähler manifolds and obtain following two theorems.

**Theorem 1.1.** Let \( M^{2m} \) be a complete Kähler manifold of complex dimension \( m \geq 1 \) with holomorphic bisectional curvature bounded by

\[
BK_M \geq -1.
\]

If \( \lambda_{1,p} \geq \left( \frac{2m}{p} \right)^p \), then either

1. \( M \) has no \( p \)-parabolic end; or
2. \( M \) splits as a warped product \( M = \mathbb{R} \times N \) where \( N \) is a compact manifold. Moreover, the metric is given by

\[
ds_M^2 = dt^2 + e^{-4t} \omega_2^2 + e^{-2t} \sum_{\alpha=3}^{2m} \omega_\alpha^2,
\]

where \( \{\omega_2, \ldots, \omega_{2m}\} \) are orthonormal coframes for \( N \).

Similarly, we obtain a splitting theorem on quaternionic Kähler manifolds, under a weaker assumption on the scalar curvature

**Theorem 1.2.** Let \( M^{4m} \) be a complete noncompact quaternionic Kähler manifold of real dimension \( 4m \) with the scalar curvature of \( M \) bounded by

\[
S_M \geq -16m(m+2).
\]

If \( \lambda_{1,p} \geq \left( \frac{2(2m+1)}{p} \right)^p \), then either

1. \( M \) has no \( p \)-parabolic end; or
2. \( M \) splits as a warped product \( M = \mathbb{R} \times N \) where \( N \) is a compact manifold. Moreover, the metric is given by

\[
ds_M^2 = dt^2 + e^{4t} \sum_{p=2}^{4} \omega_p^2 + e^{2t} \sum_{\alpha=5}^{4m} \omega_\alpha^2,
\]

where \( \{\omega_2, \ldots, \omega_{4m}\} \) are orthonormal coframes for \( N \).

The note has two sections. In the section 2, we give a unified proof of Theorems 1.1 and 1.2.
2. Structure theorems on Kähler manifolds with maximal $\lambda_{1,p}$

In this section, we provide a unified proof of Theorems 1.1 and 1.2. Our argument is close to the proof of Theorem 3.1 in [10]. First, recall that a smooth function $u$ is said to be $p$-harmonic if

$$\Delta_p u := div(|\nabla u|^{p-2}\nabla u) = 0.$$ 

Proof of theorems 1.1 and 1.2. Note that by theorem 5.1 and 5.2 in [2], we have that $\lambda_{1,p}$ achieves its maximal value. Suppose that $M$ has a $p$-parabolic end $E$. Let $\beta$ be the Busemann function associated with a geodesic ray $\gamma$ contained in $E$, namely,

$$\beta(q) = \lim_{t \to \infty} (t - \text{dist}(q, \gamma(t))).$$

The Laplacian comparison theorems in [8] and [4] imply

$$\Delta \beta \geq -a,$$

where $a = 2m$ in theorem 1.1 (see [8]) and $a = 2(2m + 1)$ in theorem 1.2 (see [4]). Hence, for $b = \frac{a}{p}$, we have

$$\Delta_p (e^{b\beta}) = div \left( b^{p-2} e^{b(p-2)\beta} \nabla (e^{b\beta}) \right)$$

$$= b^{p-1} div \left( e^{b(p-1)\beta} \nabla \beta \right)$$

$$\geq b^{p-1} e^{b(p-1)\beta} (-bp + b(p-1)) = -b p e^{b\beta}.$$

Let $g = e^{b\beta}$, we obtain

$$\Delta_p (g) \geq -\lambda_{1,p} g^{p-1}.$$ 

The variational characterization of $\lambda_{1,p}$ gives

$$\lambda_{1,p} \int_M (\phi g)^p \leq \int_M |\nabla (\phi g)|^p,$$

for any nonnegative compactly supported smooth function $\phi$ on $M$. Integration by parts implies

$$\int_M \phi^p g \Delta_p g = -\int_M \phi^p |\nabla g|^p - p \int_M \phi^{p-1} g \langle \nabla \phi, \nabla g \rangle |\nabla g|^{p-2}.$$ 

We note that

$$|\nabla (\phi g)|^p = (|\nabla \phi|^2 g^2 + 2\phi g \langle \nabla \phi, \nabla g \rangle + \phi^2 |\nabla g|^2)^{\frac{p}{2}}$$

$$\leq \phi^p |\nabla g|^p + p\phi g \langle \nabla \phi, \nabla g \rangle \phi^{p-2} |\nabla g|^{p-2} + c |\nabla \phi|^2 g^p.$$
for some constant $c$ depending on $p$. Therefore, we have

$$\int_M \phi^p g(\Delta_p g + \lambda_{1,p} g^{p-1})$$

$$= \lambda_{1,p} \int_M (\phi g)^p - \int_M (\phi^p |\nabla g|^p - p \int_M \phi^{p-1} g \langle \nabla \phi, \nabla h \rangle |\nabla h|^{p-2}$$

$$\leq \int_M |\nabla (\phi g)|^p - \int_M |\phi^p |\nabla g|^p - p \int_M \phi^{p-1} g \langle \nabla \phi, \nabla h \rangle |\nabla h|^{p-2}$$

$$\leq c \int_M |\nabla \phi|^2 g^p. \quad (2.1)$$

Now, for $R > 0$, we choose the test function $0 \leq \phi \leq 1$ such that

$$\phi = \begin{cases} 1, & \text{on } B(R) \\ 0, & \text{on } M \setminus B(2R) \end{cases}$$

and $|\nabla \phi| \leq \frac{2}{R}$. It turns out that

$$\int_M |\nabla \phi|^2 g^p = \int_M |\nabla \phi|^2 e^{a\beta} \leq \frac{4}{R^2} \int_{B(2R) \setminus B(R)} e^{a\beta}$$

$$\leq \frac{4}{R^2} \int_{E \cap (B(2R) \setminus B(R))} e^{a\beta} + \frac{4}{R^2} \int_{(M \setminus E) \cap (B(2R) \setminus B(R))} e^{a\beta}. \quad (2.2)$$

Since $\lambda_{1,p} = b^p = \left(\frac{a}{p}\right)^p$, the theorem 0.1 in [3] implies $V(E \setminus B(R)) \leq ce^{-aR}$. Therefore, the first term of (2.2) tends to 0 when $R \to \infty$. Note that Li and Wang (see [8]) showed that

$$\beta(q) \leq -r(q) + c$$

on $M \setminus E$. Moreover, by volume comparison theorem in [4, 8], we have $V(B(R)) \leq ce^{aR}$. This implies that the second term of (2.2) converges to 0 as $R$ goes to infinity. Hence, by (2.1) we have

$$\Delta_p g + \lambda_{1,p} g^{p-1} \equiv 0.$$

Thus,

$$\Delta \beta = -a.$$

The conclusion is followed by using the argument in [8] (in theorem 1.1) and [4] (in theorem 1.2). The proof is complete. \qed

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