THREE TOPOLOGICAL PROBLEMS ABOUT INTEGRAL FUNCTIONALS ON SOBOLEV SPACES

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ABSTRACT. In this paper, I propose some problems, of topological nature, on the energy functional associated to the Dirichlet problem $-\Delta u = f(x, u)$ in $\Omega$, $u_{|\partial \Omega} = 0$. Positive answers to these problems would produce innovative multiplicity results on this Dirichlet problem.

In the present very short paper, I wish to propose some problems, of topological nature, on the energy functional associated to the Dirichlet problem $(P_f)$

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and explain their motivations as well.

So, let $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) be an open bounded set. Put $X = W^{1,2}_0(\Omega)$. For $q > 0$, denote by $A_q$ the class of all Carathéodory functions $f: \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$\sup_{(x,\xi) \in \Omega \times \mathbb{R}} \frac{|f(x,\xi)|}{1 + |\xi|^q} < +\infty.$$ 

For $0 < q \leq \frac{n+2}{n+2}$ and $f \in A_q$, put

$$\Phi_f(u) = \int_{\Omega} \left( \int_0^{u(x)} f(x, \xi) d\xi \right) dx$$

and

$$J_f(u) = \frac{1}{2} \int_{\Omega} |\nabla u(x)|^2 dx - \Phi_f(u)$$

for all $u \in X$.

So, the functional $J_f$ is continuously Gâteaux differentiable on $X$ and one has

$$J'_f(u)(v) = \int_{\Omega} \nabla u(x) \nabla v(x) dx - \int_{\Omega} f(x, u(x)) v(x) dx$$

for all $u, v \in X$. Hence, the critical points of $J_f$ in $X$ are exactly the weak solutions of problem $(P_f)$.

If $q < \frac{n+2}{n-2}$, the functional $\Phi_f$ is sequentially weakly continuous, by Rellich-Kondrachov theorem. However, $\Phi_f$ may be discontinuous with respect to the weak topology. In this connection, consider the following.

**Example 1.** If $f(x,\xi) = |\xi|^{q-1}\xi$ with $0 < q \leq \frac{n+2}{n-2}$, then $\Phi_f$ is discontinuous with respect to the weak topology.

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In fact, if \( V \) is any neighbourhood of 0 in the weak topology of \( X \), then \( V \) does contain an infinite-dimensional linear subspace \( F \) of \( X \). Consequently, if we choose \( u \in F \setminus \{0\} \), we have \( \lambda u \in V \) for all \( \lambda \in \mathbb{R} \) as well as
\[
\lim_{\lambda \to +\infty} \Phi_f(\lambda u) = \lim_{\lambda \to +\infty} \int_{\Omega} |u(x)|^q + 1 dx \lambda^q + 1 = +\infty,
\]
and so \( \Phi_f \) is weakly discontinuous at 0.

On the other hand, when \( f \) does not depend on \( \xi \), the functional \( \Phi_f \) is weakly continuous being linear and continuous. The above remarks then lead to the following natural question:

**Problem 1.** Is there some \( f \in A_q \), with \( q < \frac{n+2}{n-2} \), which is not of the form \( f(x, \xi) = a(x) \), such that the functional \( \Phi_f \) is continuous with respect to the weak topology of \( X \)?

To formulate the next problem, denote by \( \tau_s \) the topology on \( X \) whose members are the sequentially weakly open subsets of \( X \). That is, a set \( A \subseteq X \) belongs to \( \tau_s \) if and only if for each \( u \in A \) and each sequence \( \{u_n\} \) in \( X \) weakly convergent to \( u \), one has \( u_n \in A \) for all \( n \) large enough.

**Problem 2.** Is there some \( f \in A_q \), with \( q < \frac{n+2}{n-2} \), such that, for each \( \lambda > 0 \) and \( r \in \mathbb{R} \), the functional \( J_{\lambda f} \) is unbounded below and the set \( J_{\lambda f}^{-1}(r) \) has no isolated points with respect to the topology \( \tau_s \)?

The interest for the study of Problem 2 comes essentially from the following result:

**Theorem 1** ([3, Theorem 3]). Let \( f \in A_q \) with \( q < \frac{n+2}{n-2} \). Then, there exists some \( \lambda^* > 0 \) such that the functional \( J_{\lambda^* f} \) has local minimum with respect to the topology \( \tau_s \).

If \( \Phi_f \) is weakly continuous, then the conclusion of Theorem 1 becomes stronger: the topology \( \tau_s \) can be replaced by the weak topology. This remark is a further motivation for the study of Problem 1.

In the light of Theorem 1, the relevance of Problem 2 is clear. Actually, if \( f \) was answering Problem 2 in the affirmative, then, by Theorem 1, for some \( \lambda^* > 0 \), the functional \( J_{\lambda^* f} \) would have infinitely many local minima in the topology \( \tau_s \). Consequently, problem \((P_{\lambda^* f})\) would have infinitely many weak solutions.

It is also worth noticing that if \( f \in A_q \) with \( q < \frac{n+2}{n-2} \) and \( \lim_{\|u\| \to +\infty} J_f(u) = +\infty \), then the local minima of \( J_f \) in the strong and in the weak topology of \( X \) do coincide ([3, Theorem 1]). On the other hand, if \( f(x, \xi) = |\xi|^{q-1} \xi \) with \( 1 < q < \frac{n+2}{n-2} \), then, for some constant \( \lambda > 0 \), it turns out that 0 is a local minimum of \( J_{\lambda f} \) in the strong topology but not in the weak one ([3, Example 2]). However, I do not know any example of \( f \) for which \( J_f \) has a local minimum in the strong topology but not in \( \tau_s \).

To introduce the third problem (the most difficult, in my opinion), let me recall that in any vector space there is the strongest vector topology of the space ([1, p. 42]).

**Problem 3.** Denote by \( \tau \) the strongest vector topology of \( X \). Is there some \( f \in A_{\frac{n+2}{n-2}} \) such that the set \( \{ (u, v) \in X \times X : J_f'(u)(v) = 1 \} \) is disconnected in \((X, \tau) \times (X, \tau)\)?
The motivation for the study of Problem 3 comes from the following result:

**Theorem 2** ([5], Theorem 1.2). Let $S$ be a topological space, $Y$ a real topological vector space (with topological dual $Y^*$), and $A: S \to Y^*$ a weakly-star continuous operator. Then, the following assertions are equivalent:

(i) The set $\{(s, y) \in S \times Y : A(s)(y) = 1\}$ is disconnected.

(ii) The set $S \setminus A^{-1}(0)$ is disconnected.

Assume that $f \in A_{n+2}$ have the property required in Problem 3. Since $J_f \in C^1(X)$, clearly the operator $J'_f: X \to X^*$ is $\tau$-weakly-star-continuous. Hence, by Theorem 2, the set $X \setminus (J'_f)^{-1}(0)$ is $\tau$-disconnected. Then, this implies, in particular, that the set $(J'_f)^{-1}(0)$ is not $\tau$-relatively compact ([4], Proposition 3)), and hence is infinite. So, for such an $f$, problem $(P_f)$ would have infinitely many weak solutions.

Of course, to recognize the disconnectedness of the set $\{(u, v) \in X \times X : J'_f(u)(v) = 1\}$ in $(X, \tau) \times (X, \tau)$, it is enough to check that this set is disconnected in $(X, \tau_1) \times (X, \tau_1)$, where $\tau_1$ is any vector topology on $X$ (which, to be meaningful in view of Theorem 2, should also be stronger than the norm topology).

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