Large $\mathcal{N}$ phase transitions and the fate of small Schwarzschild-AdS black holes

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Abstract: Sufficiently small Schwarzschild-AdS black holes in asymptotically global AdS$_5 \times S^5$ spacetime are known to become dynamically unstable toward deformation of the internal $S^5$ geometry. The resulting evolution of such an unstable black hole is related, via holography, to the dynamics of supercooled plasma which has reached the limit of metastability in maximally supersymmetric large-$\mathcal{N}$ Yang-Mills theory on $\mathbb{R} \times S^3$. Puzzles related to the resulting dynamical evolution are discussed, with a key issue involving differences between the large $\mathcal{N}$ limit in the dual field theory and typical large volume thermodynamic limits.

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1 Introduction

AdS/CFT duality relates the dynamics of maximally supersymmetric SU($N$) Yang-Mills theory ($\mathcal{N} = 4$ SYM), in the limit of large $N$ and large ’t Hooft coupling $\lambda$, to classical supergravity on asymptotically AdS$_5 \times S^5$ spacetimes [1–3]. If the field theory is defined on a spatial three-sphere, then the relevant dual geometries are asymptotic to global AdS$_5$ (times $S^5$), whose conformal boundary may be taken to be $\mathbb{R} \times S^3$. One may understand the existence of a first order confinement/deconfinement transition in $\mathcal{N} = 4$ SYM, on $\mathbb{R} \times S^3$, as a transition between two different gravitational solutions both satisfying the required asymptotic behavior, namely the vacuum geometry AdS$_5$ and the Schwarzschild-AdS$_5$ black hole (both times $S^5$) [4]. Sufficiently large Schwarzschild-AdS black holes provide the dual description of thermal equilibrium states in $\mathcal{N} = 4$ SYM above the deconfinement transition.

Smaller Schwarzschild-AdS black holes (BH) should have a dual interpretation involving non-equilibrium states in $\mathcal{N} = 4$ SYM. In particular, it is known that Schwarzschild-AdS$_5 \times S^5$ black holes below a critical size become dynamically unstable [5–7]. The instability involves deformations of the internal $S^5$ geometry and is thought to lead to localization of the black hole on the internal space [5, 8–10].

The goal of this paper is to highlight a number of puzzling issues in this presumed evolution scenario. These include the inequivalence between canonical and microcanonical descriptions of equilibrium states, possible non-perturbative (finite $N$) instabilities in small but locally stable black holes, and the connections (or lack thereof) between the dynamics of unstable Schwarzschild-AdS black holes and spinodal decomposition in typical first order phase transitions. The possible validity of two alternative evolution scenarios is also discussed. One involves the potential existence of other stationary supergravity solutions to which the dynamical evolution of unstable Schwarzschild-AdS black holes might asymptote. A more radical possibility is that this dynamical evolution could fail to asymptote to any stationary solution. This would indicate a failure of the corresponding non-equilibrium initial states in $\mathcal{N} = 4$ SYM to thermalize. Such lack of thermalization, in a strongly coupled field theory, would be somewhat analogous to the phenomena of many body localization present in certain condensed matter systems [11–14].

A dissatisfying feature of the discussion in this paper is the lack of crisp answers to some of the puzzles and speculative possibilities considered. Despite this, it is hoped that there is value in bringing attention to the issues involved, on which future work may shed greater light. In an effort to make the presentation largely self-contained, section 2 summarizes key aspects of $\mathcal{N} = 4$ SYM thermodynamics and properties of Schwarzschild-AdS black holes. Section 3 reviews the behavior of typical systems with
a first order phase transition, focusing on cooling dynamics in which a system initially in equilibrium in the hot phase slowly loses energy, enters a metastable supercooled state, undergoes spinodal decomposition upon reaching the limit of metastability, and ultimately re-thermalizes. This material is relevant background for the subsequent discussion. Known results on the instability of small Schwarzschild-AdS black holes, other deformed and localized black hole solutions [7, 10], and the presumed dynamical scenario which could lead from an unstable Schwarzschild-AdS black hole to the known localized BH solutions is summarized in section 4. Section 5, discussing various issues in this evolution scenario, as well as possible alternatives, is the heart of this paper. As noted above, open questions significantly outnumber clear answers. The final section 6 contains concluding remarks.

2 \( \mathcal{N} = 4 \) SYM on \( \mathbb{R} \times S^3 \) and AdS black holes

We consider maximally supersymmetric \( SU(N) \) Yang-Mills theory defined on \( \mathbb{R} \times S^3 \) with the spatial three-sphere having radius \( L \), and focus on the limit of large \( N \). The finite spatial volume acts as an infrared cutoff and introduces the length \( L \) into the conformal field theory. The curvature of the three-sphere induces a non-vanishing vacuum Casimir energy, proportional to the number of degrees of freedom and scaling as \( 1/L \).

\[
\lim_{N \to \infty} \frac{E_{\text{vac}}}{N^2} = \frac{3}{16L}. \tag{2.1}
\]

The spectrum of the theory is discrete, and the density of states has a finite limit as \( N \to \infty \). The Boltzmann sum representation of the canonical partition function converges at sufficiently low temperatures, and the resulting equilibrium state may be regarded as a thermal gas of glueballs (gauge invariant excitations created, at large \( N \), by single trace operators). Since contributions to the pressure, or free energy, due to a thermal gas of excitations scales as \( O(N^0) \), whereas the vacuum energy grows quadratically with \( N \), the free energy in this “confined” phase is just

\[
F_{\text{conf}}(T) = E_{\text{vac}} + O(N^0). \tag{2.2}
\]

The entropy \( S \equiv -\partial F/\partial T \) and heat capacity \( C_V \equiv \partial E/\partial T \) both scale as \( O(N^0) \) in this phase, and the \( \mathbb{Z}_N \) center symmetry of \( \mathcal{N} = 4 \) SYM is not spontaneously broken.

\(^1\)As emphasized in Ref. [15], the Casimir energy on \( \mathbb{R} \times S^3 \) is inherently ambiguous. Adding the counterterm \( \int d^4x \sqrt{g} R^2 \) to the theory, with an arbitrary finite coefficient, shifts the vacuum energy on \( \mathbb{R} \times S^3 \). The value (2.1) results from calculations using \( \zeta \)-function regularization at zero coupling [16], as well as holographic renormalization [17] in the gravitational dual. See also Refs. [18, 19].
Above a transition temperature proportional to $1/L$,

$$T_c \equiv \frac{t_c(\lambda)}{L}, \quad (2.3)$$

the equilibrium state of the theory is a “deconfined” non-Abelian plasma in which the $\mathbb{Z}_N$ center symmetry is spontaneously broken (in the large $N$ limit), and both the entropy and heat capacity are $O(N^2)$. Quantitatively,

$$t_c(\lambda) = \begin{cases} 
1/\ln(7+4\sqrt{3}) \approx 0.380, & \lambda \to 0; \\
3/(2\pi) \approx 0.477, & \lambda \to \infty,
\end{cases} \quad (2.4)$$

with the weak-coupling value determined by a perturbative reduction to a solvable matrix model [20, 21] and the strong-coupling value from the holographic description reviewed below. The transition between the confined and deconfined phases is a genuine thermodynamic phase transition in the $N \to \infty$ limit, and is first order with a non-vanishing and $O(N^2)$ latent heat (or discontinuity in the internal energy),

$$\Delta E \equiv \lim_{T \to T^+} E(T) - \lim_{T \to T^-} E(T), \quad (2.5)$$

in both the weak coupling limit, $\lambda \equiv g^2N \to 0$, and at strong coupling, $\lambda \gg 1$.

As discussed by Witten [4] and many other authors, the above features of $\mathcal{N} = 4$ SYM thermodynamics, in the strong coupling and large $N$ limits, have a simple dual gravitational description. The geometry dual to the $\mathcal{N} = 4$ SYM vacuum state is $\text{AdS}_5 \times S^5$. A convenient form for the metric on this space is

$$ds^2 = -(1+\rho^2)dt^2 + L^2 \left[ \frac{d\rho^2}{1+\rho^2} + \rho^2 d\Omega_3^2 + d\Omega_5^2 \right], \quad (2.6)$$

where the radial coordinate $\rho$ runs from 0 to $\infty$ at the AdS boundary. In the gravitational description, the length $L$ is both the AdS curvature scale and the radius of the five-sphere. The total energy, extracted from the boundary stress tensor [17], is

$$E_{\text{vac}} = \frac{3\pi L^2}{32 G_5} = \frac{3N^2}{16L}, \quad (2.7)$$

where the last form uses the holographic relations

$$N^2 = \frac{\pi L^3}{2 G_5} = \frac{\pi^4 L^8}{2 G_{10}}, \quad (2.8)$$

\footnote{Whether the deconfinement transition remains first order at small but non-zero coupling in $\mathcal{N} = 4$ SYM remains an open question [21–23].}
connecting the 5D and 10D Newton’s constants \( G_5 \) and \( G_{10} \) to the rank of the \( SU(N) \) gauge group of the dual SYM theory. Henceforth, these relations will be used routinely to eliminate \( G_5 \) and \( G_{10} \) and express results in terms of \( N \). The Euclidean signature version of this geometry,

\[
ds^2 = (1+\rho^2) \, dt^2 + L^2 \left[ \frac{d\rho^2}{1+\rho^2} + \rho^2 \, d\Omega_3^2 + d\Omega_5^2 \right], \tag{2.9}
\]

with the periodic identification \( t = t + \beta \), provides the gravitational description of the thermal equilibrium state at temperature \( T \equiv \beta^{-1} \) in the confined phase of \( \mathcal{N} = 4 \) SYM theory. This geometry is referred to as “thermal AdS”. The boundary stress-energy associated with this geometry gives the leading \( O(N^2) \) contributions to the field theory stress-energy tensor, which is completely temperature independent and equals the vacuum Casimir stress-energy. On the gravitational side, this temperature independence reflects the fact that the periodic identification introducing temperature has no effect whatsoever on local aspects of the geometry. To see subleading \( O(N^0) \) thermal contributions to the stress-energy, or related thermodynamic observables, one must include effects of quantum fluctuations in the geometry.

The geometry (2.6) has no event horizon and hence vanishing gravitational entropy. Consequently, the free energy \( F \equiv E - TS \) coincides with the vacuum energy,

\[
F_{\text{thermal-AdS}} = \frac{3N^2}{16L}, \tag{2.10}
\]

up to subleading \( O(N^0) \) corrections. This is in accord with the field theory entropy (in the confined phase) scaling as \( O(N^0) \), and the free energy having the form (2.2).

Equilibrium states in the deconfined phase of \( \mathcal{N} = 4 \) SYM have a dual gravitational description in terms of Schwarzschild black holes in asymptotically \( \text{AdS}_5 \times S^5 \) spacetimes [4]. These “AdS-BH” geometries may be described by the metric

\[
ds^2 = -f(\rho) \, dt^2 + L^2 \left[ \frac{d\rho^2}{f(\rho)} + \rho^2 \, d\Omega_3^2 + d\Omega_5^2 \right], \tag{2.11}
\]

where

\[
f(\rho) \equiv 1 + \rho^2 - (1 + \rho_h^2) \frac{\rho_h^2}{\rho^2}. \tag{2.12}
\]

The dimensionless parameter \( \rho_h \) controls the black hole size. The black hole horizon is located at \( \rho = \rho_h \) and the geometry reduces to that of global \( \text{AdS}_5 \times S^5 \) at \( \rho_h = 0 \). The horizon area \( A = 2\pi^5 L^8 \rho_h^3 \), and the associated black hole entropy is

\[
S \equiv \frac{A}{4G_{10}} = \pi N^2 \rho_h^3. \tag{2.13}
\]
The horizon temperature

\[ T \equiv \frac{\kappa}{2\pi} = \rho_h^{-1} + 2\rho_h \frac{\rho h}{2\pi L}, \]  

(2.14)

where \( \kappa \equiv \frac{1}{2} f'(\rho_h)/L \) is the surface gravity at the horizon. The temperature (2.14) has a minimum value

\[ T_{\text{min}} = \frac{\sqrt{2}}{\pi L}. \]  

(2.15)

at \( \rho_h = 1/\sqrt{2} \), and diverges in the limit of both large and small \( \rho_h \). Inverting the relation (2.14) gives the black hole size as a double-valued function of temperature,

\[ \rho_h = \frac{1}{2} \left[ \pi LT \pm \sqrt{(\pi LT)^2 - 2} \right]. \]  

(2.16)

The + branch is referred to as describing “large” AdS black holes, while the – branch describes “small” AdS black holes.

The energy of the Schwarzschild-AdS solution (2.11), extracted from the boundary stress-energy tensor [17], is

\[ E = \frac{3N^2}{16L} \left( 1 + 2\rho_h^2 \right)^2, \]  

(2.17)

and the corresponding free energy

\[ F \equiv E - TS = \frac{N^2}{4L} \left[ \frac{3}{4} + \rho_h^2(1 - \rho_h^2) \right]. \]  

(2.18)

The left panel of Figure 1 shows a plot of the free energy (2.18) together with the constant value (2.10) of (the \( O(N^2) \) part of) the thermal AdS free energy. The right panel shows the internal energy (2.17) of the AdS black hole, along with the constant value of the thermal AdS energy (2.7), as a function of temperature.

The pressure \( p \) coincides with \( E/(3V) = E/(6\pi^2 L^3) \), as required for a conformal theory with a traceless stress-energy tensor (on \( \mathbb{R} \times S^3 \)). This agrees, as it must, with the thermodynamic definitions \( p = -\frac{dE}{dV} \bigg|_S = -\frac{dE}{dV} \bigg|_T \). For \( \rho_h \gg 1 \), the pressure approaches that of strongly coupled \( \mathcal{N} = 4 \) SYM plasma in flat space, \( p \sim \frac{N^2}{8\pi^2} \rho_h^4 L^{-4} = \frac{\pi^2}{8} N^2 T^4 \).

In a canonical description of thermodynamics, genuine equilibrium states are global minima of the free energy. The black hole free energy (2.18) falls below the thermal AdS free energy (2.10) only when \( \rho_h > 1 \). The point \( \rho_h = 1 \) lies on the large BH branch and corresponds to a transition temperature

\[ T_c = \frac{3}{2\pi L}, \]  

(2.19)

where the nature of the equilibrium state switches between confined and deconfined phases. The transition is first order with two distinct equilibrium states, thermal AdS
Figure 1. Left panel: Free energy of asymptotically AdS$_5 \times$ S$^5$ Schwarzschild black holes, in units of $N^2/L$, as a function of the black hole size $\rho_h$. The solid black portion of the curve with $\rho_h > 1$ represents stable equilibrium states of deconfined $\mathcal{N} = 4$ SYM plasma. The long dashed blue portion of the curve with $\rho_* < \rho_h < 1$ represents locally stable states of supercooled plasma, while the short dashed purple portion with $\rho_h < \rho_*$ corresponds to locally unstable states. The dotted horizontal line shows the free energy of thermal-AdS. Right panel: Energy of AdS-Schwarzschild black holes, and thermal-AdS, in units of $N^2/L$ and plotted as a function of temperature. The thin vertical line marks the transition temperature. Different markings on the curves have the same meanings as in the left panel.

(Confined) and the AdS-BH (deconfined), co-existing at $T_c$. The internal energy is discontinuous, with

$$E_c^- \equiv \lim_{T \to T_c^-} E(T) = \frac{3N^2}{16L}, \quad E_c^+ \equiv \lim_{T \to T_c^+} E(T) = \frac{27N^2}{16L},$$

and latent heat $\Delta E = \frac{3N^2}{2L}$. Examination of the expectation value of the (dual description of the) Polyakov loop confirms that the $\mathbb{Z}_N$ center symmetry is unbroken in the thermal AdS geometry, but is spontaneously broken in the AdS-BH geometry [4], in complete accord with the interpretation of the transition between these geometries as a confinement/deconfinement phase transition. Finally, AdS black holes on the large BH branch are dynamically stable; all quasinormal mode frequencies lie in the lower half plane, corresponding to exponentially damped behavior. To summarize, the interpretation of large AdS-Schwarzschild black holes with $\rho_h \geq 1$ as the holographic duals of equilibrium states of deconfined non-Abelian $\mathcal{N} = 4$ SYM plasma (inside a three-sphere, in the large $N$ and strong coupling limits) is well established.

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3In this finite volume theory, the Polyakov loop expectation value $\langle \frac{1}{N} \operatorname{tr} \Omega \rangle$, in the large $N$ limit, may be defined via the large $N$ factorization relation, $\lim_{N \to \infty} \langle | \frac{1}{N} \operatorname{tr} \Omega |^2 \rangle = | \langle \frac{1}{N} \operatorname{tr} \Omega \rangle |^2$. For equivalent alternative definitions and related discussion see, for example, Refs. [4, 21].
For $\rho_h < 1$, the AdS-BH is no longer the minimum of the free energy [24]. The heat capacity, given by
\[ C_V \equiv \frac{\partial E}{\partial T} = 3\pi N^2 \rho_h^3 \frac{2\rho_h^2 + 1}{2\rho_h^2 - 1}, \tag{2.21} \]
diverges to $+\infty$ as $\rho_h \to 1/\sqrt{2}$ from above, corresponding to $T \to T_{\text{min}}$, and becomes negative on the small-BH branch with $\rho_h < 1/\sqrt{2}$. As the black hole size decreases below $\rho_h = 1/\sqrt{2}$, the global AdS black hole ceases to be locally dynamically stable when $\rho_h$ reaches a critical value [5–7],
\[ \rho_* \approx 0.4402373. \tag{2.22} \]

At $\rho_h = \rho_*$, one quasinormal mode frequency crosses the real axis and, for $\rho_h < \rho_*$, moves into the upper half plane, indicating an exponentially growing instability. The unstable mode involves a deformation of the geometry involving $\ell = 1$ harmonics on the internal $S^5$. Further instabilities, involving higher harmonics on the $S^5$, appear at a series of progressively smaller values of $\rho_h$ [5]. At each instability threshold there is a zero mode in the static fluctuation spectrum, signaling a bifurcation in the space of static solutions. The new branches of static solutions which appear at the first two such bifurcations were studied numerically in Ref. [7]. These new solutions have deformations which break the $SO(6)$ symmetry of the internal $S^5$ down to $SO(5)$, and have been termed “lumpy” AdS black holes.

Schwarzschild-AdS black hole geometries with $\rho_* < \rho_h < 1$ must be the gravitational dual of some states in the dual field theory. The non-minimal free energy indicates that this geometry does not represent a true equilibrium state in $\mathcal{N} = 4$ SYM. Nevertheless, the geometry is the smooth continuation of the thermodynamically stable black hole to lower values of energy and, as noted above, it remains locally dynamically stable. It should be clear that this geometry provides the dual description of deconfined $\mathcal{N} = 4$ SYM plasma which is supercooled below the confinement/deconfinement transition. The point $\rho_h = \rho_*$ represents the limit of local metastability of the supercooled system.\(^4\)

3 Cooling through first order phase transitions

Before examining possible scenarios for the dynamical evolution of small AdS black holes which reach the limit of metastability, we first review cooling dynamics at typ-

\(^4\)Supercooled phases, before they reach the limit of metastability, typically have non-perturbative instabilities involving nucleation and growth [25]. In $SU(N)$ gauge theories, however, such nucleation probabilities vanish exponentially as $N \to \infty$ and may largely be ignored when considering the dynamics of SYM plasma at large $N$. Section 5 contains further discussion of this issue.
Figure 2. Schematic form energy density $\epsilon$ as a function of temperature $T$ in a typical system with a first order phase transition at some temperature $T_c$. The energy density jumps discontinuously from $\epsilon_c^-$ to $\epsilon_c^+$. Dashed curves show metastable supercooled states with energy densities and temperatures varying from $(\epsilon_c^+, T_c)$ down to the limit of metastability at $(\epsilon_*, T_*)$, and corresponding metastable superheated states extending upward from $(\epsilon_c^-, T_c)$.

ical first order phase transitions. This will be useful background for the subsequent discussion.

Consider a system with a first order phase transition in the usual thermodynamic limit of spatial volume $V$ tending to infinity. Assume, for simplicity, that this is not a symmetry-breaking phase transition but instead something like a gas-to-liquid transition for which there is a unique equilibrium state on either side of the transition. Figure 2 sketches the typical behavior of the energy density $\epsilon$ as a function of temperature, showing a monotonically increasing function with a discontinuous jump at the transition temperature $T_c$. Let $\epsilon_c^\pm \equiv \lim_{T \to T_c^\pm} \epsilon(T)$ denote the energy densities at the transition, with limits taken from the indicated side; $\Delta \epsilon \equiv \epsilon_c^+ - \epsilon_c^-$ is the latent heat per unit volume. The high temperature phase has a metastable supercooled continuation below $T_c$, and likewise the low temperature phase has a metastable superheated continuation above $T_c$, both indicated by dashed lines in the figure. The limit of metastability of the supercooled system lies at some temperature $T_*$ and energy density $\epsilon_*$.

Given the assumed absence of symmetry breaking, at temperatures other than $T_c$ there is a unique equilibrium state. Uniqueness of the equilibrium state implies, for example, that local (or compactly supported) observables have no sensitivity to the choice of boundary conditions placed on the spatial boundary of the theory before the volume $V$ is sent to infinity.

Precisely at $T_c$ there are multiple equilibrium states. These include the two "pure phases" with energy densities $\epsilon_c^\pm$ which are the limits of the unique equilibrium states
on either side of the transition. Let $\rho_{\pm}$ denote the statistical density matrices of these equilibrium states. Possible equilibrium states also include statistical mixtures of the two pure phases,$^5$

$$\rho = x \rho_+ + (1-x) \rho_- , \quad (3.1)$$

with $x \in (0,1)$. Such “mixed states” form the interior of a convex domain whose extremal points are the pure phases.$^6$ A key point is that equilibrium states at $T_c$ exist with any desired energy density in between the extremal values, $\epsilon \in [\epsilon^-, \epsilon^+]$.

Cluster decomposition provides a diagnostic indicating whether a given equilibrium state is pure or mixed. For later purposes, note that the usual statement of cluster decomposition,

$$\lim_{|x-y| \to \infty} \left[ \langle (\mathcal{O}(x)\mathcal{O}(y)) \rangle - \langle \mathcal{O}(x) \rangle \langle \mathcal{O}(y) \rangle \right] = 0 \quad (3.2)$$

for some local observable $\mathcal{O}(x)$ (in infinite volume), implies the integrated form,

$$\lim_{|\mathcal{R}| \to \infty} \left[ \left\langle \left( \frac{1}{|\mathcal{R}|} \int_\mathcal{R} \mathcal{O}(x) \right)^2 \right\rangle - \left\langle \frac{1}{|\mathcal{R}|} \int_\mathcal{R} \mathcal{O}(x) \right\rangle^2 \right] = 0 , \quad (3.3)$$

where the region $\mathcal{R}$ is, for example, a ball of volume $|\mathcal{R}|$. In other words, cluster decomposition implies vanishing variance of intensive observables spatially averaged over increasingly large volumes in the thermodynamic limit. Equilibrium states which satisfy cluster decomposition are called “extremal” or “pure” states,$^7$ while equilibrium states which violate cluster decomposition are mixed states of the form (3.1), decomposable into a positively weighted average of extremal states.

One might expect further distinct “phase-separated” equilibrium states to also exist at $T_c$, in which an interface (or domain wall) is present on some planar surface, with properties on one side approaching those of the pure phase $\rho_+$ while properties on the other side approach those of $\rho_-$. This is the case in four or more spatial dimensions, where bounded transverse fluctuations of an interface allow non-translationally invariant extremal equilibrium states (satisfying cluster decomposition) to exist.$^8$ Such

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$^5$In infinite volume, an “equilibrium state” means a probability measure which satisfies the Dobrushin-Lanford-Ruelle conditions (see, e.g., Ref. [26]), namely that when restricted to any finite subvolume with degrees of freedom outside the subvolume fixed, the measure reduces to the canonical Gibbs measure conditioned on the fixed degrees of freedom.

$^6$More generally if some discrete symmetry, say $\mathbb{Z}_p$, is unbroken on one side of the transition and spontaneously broken on the other side, then there can be $p+1$ pure phases at $T_c$, with the most general mixed phase an arbitrary convex linear combination of these $p+1$ pure phases.

$^7$This use of “pure states” in statistical physics, as a synonym of “extremal”, is distinct from pure states in quantum mechanics.

$^8$For lattice theories in three space dimensions, non-translationally invariant equilibrium states exist below the interface roughening temperature [27–29]. The lower limit of four spatial dimensions
states may be viewed as the limit of finite volume equilibrium states in which one fixes degrees of freedom on the boundary of the volume in a non-uniform manner which pins the interface location on the boundary and selects the desired volume fraction $x$ [29]. But in three or fewer spatial dimensions, which the following discussion assumes, the transverse fluctuations in interface position diverge as $V \to \infty$. As a result, it becomes completely indeterminate whether some given spatial region $\mathcal{R}$, of finite but arbitrarily large extent, lies on one side of the interface or the other. The probability that the interface runs through any given region $\mathcal{R}$ vanishes as $V \to \infty$. Consequently, as probed by any local (or compactly supported) observable, a phase-separated equilibrium state, in three or fewer space dimensions, is a mixed phase of the form (3.1), and does not satisfy cluster decomposition.\(^9\)

Dynamically, in the infinite volume limit, an interface surface will continually fluctuate, undergoing unbounded stochastic motion, and never settle down to a well-defined position. Interpreting the mixed ensemble (3.1) as a phase-separated equilibrium state with a completely uncertain interface location makes clear that these non-extremal equilibrium states are also present in a microcanonical description of thermodynamics. Fixing the total energy, or rather energy density, to lie in between the extremal values, $\epsilon_c^- < \epsilon < \epsilon_c^+$, determines the volume fraction $x$ but leaves the location of the separating interface completely undetermined.

Given this understanding of possible equilibrium states, consider a system which is initially prepared and fully equilibrated at some temperature $T > T_c$, and then slowly cooled (for example, by adiabatic expansion in a fluid system). The cooling removes energy from the system and lowers its temperature through $T_c$, causing the system to enter the metastable supercooled regime. Within this regime, assume that the cooling rate is small compared to microscopic relaxation rates but is large compared to the rate for bubble nucleation (within some finite region of interest), so that the system remains in the supercooled state up to the limit of metastability, whereupon it becomes unstable. What happens next?

The subsequent evolution is termed \textit{spinodal decomposition}. Typically, perturbations with a range of wavenumbers become unstable in the homogeneous supercooled state when $T < T_\ast$, leading to growth of structure with intricate spatial patterns. In, for example, solids with first order compositional phase transitions, if a material undergoes to continuum theories with continuous rotation and translation invariance, which the following discussion assumes.

\(^9\)Note, however, that adding a non-translationally invariant perturbation to the system, such as a tiny gravity gradient in a fluid system, can serve to localize an interface separating pure phases, thereby producing a non-translationally invariant phase-separated equilibrium state which does satisfy cluster decomposition.
going spinodal decomposition is cooled sufficiently rapidly then finely dispersed spatial microstructure can become frozen in place, significantly changing material properties in technologically useful fashions. But suppose, instead, that the external cooling ceases at the onset of spinodal decomposition, so that the system subsequently evolves as an isolated system. Although the details of the dynamics may be highly complex, in any normal thermodynamic system the endpoint of the evolution is easy to describe: the system re-thermalizes. The energy density of the system is $\epsilon_*$ at the onset of spinodal decomposition and therefore, since energy is conserved, the system must end up in an equilibrium state with the same energy density $\epsilon_*$.  

There are two characteristic possibilities, illustrated in Fig. 3. If $\epsilon_* < \epsilon_c^-$, then the system can equilibrate to an equilibrium state on the low temperature side of the phase transition, with a final temperature $T_f$ which is below $T_c$. Alternatively, if $\epsilon_* > \epsilon_c^-$, then the system must equilibrate to a mixed phase with energy density $\epsilon_*$ and final temperature $T_f = T_c$. Given that $\epsilon_* < \epsilon_c^+$, it is impossible for the system to equilibrate with a final temperature $T_f > T_c$. The basic point to be emphasized is that, if $\epsilon_* > \epsilon_c^-$, then equilibration after spinodal decomposition must lead to a final state at temperature $T_c$.

4 Supercooled $\mathcal{N}=4$ plasma and small black holes

We now return to consideration of supercooled $\mathcal{N}=4$ SYM plasma, or small AdS black holes. The basic scenario is the same as discussed above: $\mathcal{N}=4$ plasma is confined to a three-sphere of radius $L$ and initially in equilibrium at some temperature $T > T_c$.
The three-sphere is slowly expanded. As the radius \( L \) increases, the positive pressure of the plasma implies that the plasma does work and loses energy. When the energy diminishes below \( E^+_c \equiv 27N^2/(16L) \) the system passes into the supercooled regime. As noted earlier (footnote 4), homogeneous nucleation within the supercooled regime is exponentially suppressed at large \( N \) and may be ignored. The expansion continues until the plasma crosses the limit of metastability at an energy of

\[
E_* \equiv \frac{3N^2}{16L} (1 + 2\rho^2) \approx \frac{0.361028 N^2}{L},
\]

and then ceases. The peculiar form (2.21) of the heat capacity shows that the temperature is non-monotonic during the expansion, initially decreasing and passing through \( T_c = 1.5/(\pi L) \), reaching \( T_{\text{min}} \approx 1.414/(\pi L) \) and then increasing to \( T_* \approx 1.576/(\pi L) \). Nothing dynamically significant happens at \( T_{\text{min}} \); the pressure remains positive and energy is continually extracted as the volume expands. A local instability of the supercooled plasma arises when the energy drops below \( E_*(4.1) \), and the question we wish to consider is what can be said about the endpoint of the subsequent evolution. This is the same as asking what, in the dual gravitational description, is the fate of a small Schwarzschild-AdS black hole after it becomes unstable at \( \rho < \rho_c \)?

This question, and closely related issues, have been considered previously. The conventional expectation is that the horizon of the AdS black hole, with \( S^3 \times S^5 \) topology, becomes increasingly distorted on the \( S^5 \) in a manner analogous to the Gregory-Laflamme instability of black strings in asymptotically flat space [30]. A cascade of instabilities on different scales leads to the development of fractal structure [31, 32]. The classical gravity description breaks down when the length of minimal cycles around the horizon reach the string scale \( \ell_s \). At this point, topology changing transitions which break thin “necks” in the horizon become possible. The presumption has been that this process will lead to one or more black holes with \( S^8 \) horizon topology, which eventually merge and settle down to form a single stationary black hole, localized on the \( S^5 \), with a geometry which is invariant under the \( SO(4) \) symmetry of the spatial \( S^3 \) and at most an \( SO(5) \) subgroup of the \( SO(6) \) symmetry of the asymptotic \( S^5 \).

Much of this picture is implicit in early 1998 discussions of AdS black holes and associated thermodynamics in a microcanonical perspective by Banks, Douglas, Horowitz, and Martinec [8] and by Peet and Ross [9]. The first explicit calculation of instability thresholds of Schwarzschild-AdS\(_5 \times S^5 \) black holes was performed by Hubeny and Rangamani [5], who emphasized that from a gauge theory perspective these instabilities should be interpreted as indicating the existence of phase transitions in the microcanonical ensemble at which the \( SO(6)_R \) symmetry of \( \mathcal{N} = 4 \) SYM is spontaneously broken. As mentioned earlier, “lumpy” static black hole branches which emerge from the first
two bifurcations were constructed numerically by Dias, Santos, and Way [7]. For the leading \((\ell = 1)\) bifurcation they found that the entropy decreases as the deformation increases, so these deformed black holes are even less thermodynamically relevant than undeformed Schwarzschild-AdS black holes at the same energy. More recently [10], these authors succeeded in finding “localized” black hole solutions with \(S^8\) horizon topology and \(SO(4) \times SO(5)\) symmetry. The localized black holes have higher entropy

\[\delta S / N^2 \]

\[\delta E L / N^2 \]

\[\delta F L / N^2 \]

**Figure 4.** Left: Entropy difference \(\delta S \equiv S - S_{\text{Schw-AdS}}\) of localized black holes relative to the corresponding Schwarzschild-AdS \(S^5 \times S^5\) black hole, as a function of energy excess \(\delta E \equiv E - E_{\text{vac}}\). Individual black data points show the additional entropy \(\delta S\) of localized \(S^8\) horizon topology black holes; the dashed purple curve through these points is a fit to the data. The horizontal dashed red line at \(\delta S = 0\) represents Schwarzschild-AdS \(S^5 \times S^5\) black holes. The green diamond and pink square mark the \(\ell = 1\) and \(\ell = 2\) bifurcations, respectively, of the AdS \(S^5 \times S^5\) black hole. The thick blue curve emerging from the \(\ell = 1\) bifurcation shows lumpy \(S^3 \times S^5\) horizon topology black holes. The thin dashed vertical line identifies the localized black hole with the same energy as the Schwarzschild-AdS black hole at the onset of instability. Right: Free energy excess \(\delta F \equiv F - E_{\text{vac}}\) as a function of temperature. Different curves and symbols have the same meanings as in the left plot. The upper and lower red dotted curves show the free energy excess of small and large Schwarzschild-AdS black holes, respectively. The single black dot on the lower branch marks the phase transition point where the black hole free energy falls below that of thermal-AdS. The inset plot magnifies the region near the \(\ell = 1\) bifurcation. Localized \(S^8\) black holes (black data points with purple fit to the data) have lower free energy than small Schwarzschild-AdS black holes for \(T \gtrsim 0.585/L\). Plots courtesy of O. Dias and reproduced from Ref. [10].
than Schwarzschild-AdS$_5 \times S^5$ black holes for energies below [10]

$$E_R \approx 0.413 N^2/L.$$  \hfill (4.2)

The localized BH with energy $E_*$ has a temperature

$$T_{*}\text{loc} \approx 1.7098/(\pi L),$$ \hfill (4.3)

higher than both $T_c$ and $T_*$. These localized solutions may eventually meet the branch of lumpy black holes emerging from the $\ell=1$ bifurcation at a cusp-like topological transition point lying at yet higher energy.

The left panel of Figure 4 shows the entropy difference $\delta S \equiv S - S_{\text{Schw-AdS}}$ of these localized black holes relative to that of an undeformed Schwarzschild-AdS black hole, as a function of energy excess $\delta E \equiv E - E_{\text{vac}}$. The black data points with purple dashed fit to the data represent localized black holes. The horizontal red line at $\delta S = 0$ corresponds to Schwarzschild-AdS black holes. On this line, the green diamond at $\delta E_* \equiv E_* - E_{\text{vac}} \approx 0.1735 N^2/L$ indicates the $\ell = 1$ bifurcation (i.e., the metastability limit), while the pink square shows the next $\ell = 2$ bifurcation. The blue curve emerging from the $\ell = 1$ bifurcation represents lumpy black hole solutions. The intersection of the localized BH curve with the horizontal red line at $\delta E_R = E_R - E_{\text{vac}} \approx 0.225 N^2/L$ represents the point where the (extrapolated) entropy of the localized solutions falls below that of a Schwarzschild-AdS black hole. (Plots courtesy of O. Dias and reproduced from Ref. [10].)

The right panel of Fig. 4 shows the free energy excess $\delta F \equiv F - E_{\text{vac}}$ of these various solutions as a function of temperature. Different curves have the same meaning as in the left panel. The upper and lower red dotted curves show the free energy excess of small and large Schwarzschild-AdS black holes, respectively. The black dot at $\delta F = 0$ indicates the phase transition point. The inset plot magnifies the region around the $\ell = 1$ bifurcation. One sees from this figure that the localized black hole free energy is a convex function of temperature (as is the small Schwarzschild-AdS black hole branch). Since $C_V = -T \partial^2 F/\partial T^2$, this implies that the localized $S^8$ horizon topology black holes have negative heat capacity, just like undeformed small Schwarzschild-AdS black holes.

Dias, Santos, and Way interpret the energy $E_R$ as a transition point, within the microcanonical ensemble, to a phase of spontaneously broken $R$-symmetry with properties characterized by their localized black hole solutions. Dynamically, they suggest that Schwarzschild-AdS$_5 \times S^5$ black holes which reach the $\ell=1$ instability threshold (4.1) should subsequently evolve toward these static localized $S^8$ horizon topology solutions in the manner described above.
5 Evolution scenarios

The above scenario, with evolution asymptoting to the localized black hole solutions of Dias et al. [10] at a temperature $T^\text{loc}_*$ greater than $T_c$, is consistent with currently known information. But there are alternative possibilities which also warrant consideration. There may be other possible final states, not currently known, with yet higher entropy. Or, the presumption of asymptotic stationarity may be false, with dynamical evolution never settling down to a well-defined equilibrium state. In this section, we examine various open questions and issues associated with each of these three logical possibilities.

5.1 Known localized black hole as final state?

One reason to expect the scenario of evolution toward the known localized black hole solutions to be correct is the obvious: the successful construction [10] of these solutions for energies up to and a little beyond $E_*$, with the demonstration that these localized black holes have higher entropy than the corresponding Schwarzschild-AdS black holes. There are, however, multiple curious or puzzling features implied by this scenario. One may wonder how this evolution can be consistent with conservation of $R$-charge. A final temperature $T_f > T_c$ is at odds with the conventional picture of re-thermalization after spinodal decomposition discussed in section 3. Why the difference, and should this have been expected? This scenario implies a microcanonical description of equilibrium states which is fundamentally different from the canonical description of equilibrium states, despite the fact that the large $N$ limit can be viewed as a thermodynamic limit [33]. Again, why? Finally, for finite values of $N$, what does this scenario predict for the fate of supercooled plasma with energy between $E_*$ and $E^+_c$ (in the absence of further cooling)?

We begin with the disconnect between the description of $N = 4$ SYM equilibrium states in the canonical and microcanonical ensembles. In the putative microcanonical picture, if the total energy $E$ is less than $E_R$ (and greater than $E_{\text{vac}}$ by an $O(N^2)$ amount), then in equilibrium the $SO(6)$ $R$-symmetry is spontaneously broken down to $SO(5)$ and the heat capacity $C_V$ is negative. In contrast, in the canonical description when the total energy $\hat{E}$ is less than $E^+_c$ (and greater than $E_{\text{vac}}$), then the equilibrium state is a statistical mixture of the confined and deconfined pure phases at $T_c$, and the heat capacity $C_V = \partial\hat{E}/\partial T = \langle (E - \langle E \rangle)^2 \rangle / T^2$ is necessarily positive. Nowhere in the canonical description of equilibrium states is there any sign of spontaneous breaking of $R$-symmetry.

Substantial differences between microcanonical and canonical descriptions are to be expected in small systems where fluctuations in thermodynamic quantities can be substantial and the presence or absence of energy transfer to some external environment...
may play a significant role. But in the usual thermodynamic limit, the infinite system can itself serve as a heat bath for any subsystem of interest. Equivalence of ensembles is a basic result directly linked to the vanishing variance (3.3) of spatially averaged observables.

We are considering a finite volume system, $\mathcal{N} = 4$ SYM on a spatial three-sphere, in the large $N$ limit. As $N \to \infty$, the number of degrees of freedom diverges and thermodynamic functions can develop non-analyticities (i.e., phase transitions). Large $N$ factorization [34–36],

$$
\lim_{N \to \infty} \left[ \langle O^2 \rangle - \langle O \rangle^2 \right] = 0,
$$

may be viewed as the direct analog of the integrated form (3.3) of cluster decomposition. Here, $O$ is any “classical” large $N$ observable [36], such as a fixed product of single trace operators normalized to have $O(N^0)$ expectation values. Just as with cluster decomposition, large $N$ factorization does not hold automatically for all states in the large $N$ limit, but rather is a diagnostic for whether a given state, as probed by “classical” operators, is indistinguishable from a statistical mixture of states which do satisfy large $N$ factorization [36, 37]. In theories with a holographic dual, only those states satisfying large $N$ factorization will have a dual description involving a single classical geometry. States defined by extremization (e.g., minimizing the free energy $F[\rho]$ or maximizing entropy $S[\rho]$) will satisfy large $N$ factorization whenever the extremum is unique.

The mismatch between canonical and microcanonical descriptions of $\mathcal{N} = 4$ SYM equilibrium states arises from the apparent absence of any analog of phase-separated equilibrium states, coexisting with the pure phases at $T_c$. In our large $N$ limit, such (hypothetical) “phase-separated” equilibrium states would be states with energy intermediate between $E_c^+$ and $E_c^-$, negligible energy fluctuations, a horizon temperature equal to $T_c$, and (in a Euclidean description) expectation values of Polyakov loops, possibly multiply wound, indicative of an eigenvalue distribution for the gauge field holonomy in the time direction which is neither gapped nor $\mathbb{Z}_N$ invariant [21]. If such a state is a co-existing equilibrium state at $T_c$, then (the $O(N^2)$ part of) its free energy must equal $E_{\text{vac}}$, implying that the entropy of such a state is directly related to its energy, $S = (E - E_{\text{vac}})/T_c = \frac{2\pi L}{3} (E - E_{\text{vac}})$. This entropy would be a substantially larger than that of the localized black holes shown in Fig. 4.

In the usual large volume thermodynamic limit, it is spatial locality (i.e., sufficiently rapid decrease of interaction strength with distance) which guarantees the existence of phase-separated equilibrium states in both canonical and microcanonical descriptions. A domain wall separating regions resembling pure phases will have a free energy excess proportional to its area, but this excess makes a vanishing contribution to the volume.
average of the free energy density (or internal energy density) in the infinite volume limit. Sufficiently far from the domain wall, on either side, expectation values of local observables will be indistinguishable from those in the corresponding pure phase.

In large $N$ matrix models (or gauge theories), correlations and fluctuations in the space of eigenvalues should be regarded as the analog of spatial correlations and fluctuations in a typical statistical theory. But there is no direct analog of spatial locality as seen, for example, in the logarithm of the Vandermonde determinant which appears in every matrix model—every eigenvalue interacts with every other eigenvalue. So features of large volume thermodynamic limits which inextricably rely on spatial locality may simply have no analog in large $N$ thermodynamic limits.

A negative heat capacity, in a microcanonical ensemble, is closely related to the absence of phase-separated equilibrium states. Recall that first and second derivatives of the entropy $S(E)$ determine the temperature $T$ and heat capacity $C_V$ via

$$\frac{\partial S}{\partial E} = \frac{1}{T}, \quad \frac{\partial^2 S}{\partial E^2} = -\frac{1}{T^2} \frac{\partial T}{\partial E} = -\frac{C_V^{-1}}{T^2}. \quad (5.2)$$

A negative heat capacity implies that the entropy violates the usual concavity relation (i.e., $\partial^2 S/\partial E^2 < 0$), and is instead convex. If the microcanonical entropy $S(E)$ is convex for some range of energies, say $E_1 < E < E_2$, then $S(E)$ lies below its concave hull—which in this interval is the straight line $S_{\text{concave-hull}}(E) = (1-x) S(E_1) + x S(E_2)$ with $x \equiv (E-E_1)/(E_2-E_1)$. So convexity of the microcanonical entropy in this interval amounts to an assertion that it is impossible to construct any state of the system in which some fraction $x$ of the degrees of freedom behave like a low energy equilibrium state with entropy/energy ratio $S(E_1)/E_1$ while the complementary fraction $1-x$ of the degrees of freedom behave like a high energy equilibrium state with entropy/energy ratio $S(E_2)/E_2$, with negligible interaction between the two subsets. Such a state, if it exists, would have an entropy which lies on the above straight line connecting $S(E_1)$ and $S(E_2)$. In ordinary large volume thermodynamic limits, phase-separated equilibrium states illustrate exactly this sort of partitioning of degrees of freedom. They play an essential role in ensuring concavity of the entropy.

In large $N$ matrix models, the non-locality of interactions in “eigenvalue space” may preclude any analogous partitioning of degrees of freedom into nearly independent subsets. Nevertheless, it is the case that eigenvalues which are farther apart interact more weakly than eigenvalues which are closer together. Asplund and Berenstein [38] and more recently Hanada and Maltz [39] have argued that one should view a small AdS black hole, localized on the $S^5$, as representing states of $\mathcal{N} = 4$ SYM in which a subset of the eigenvalues of the scalar fields are clumped together, while the complementary set of eigenvalues are widely dispersed. These papers are largely qualitative, but Ref. [39]
successfully reproduces the scaling properties of localized $S^8$ horizon topology BHs in the limit of very small energy. These arguments rely on an approximate notion of locality in the $\mathbb{R}^6$ eigenvalue space (of the six SYM scalar fields) replacing ordinary spatial locality. Except for the substitution of eigenvalue space for ordinary space, the overall picture is highly reminiscent of the construction of phase-separated states.

This suggests that it might, to some degree, make sense to view small black holes localized on the $S^5$ as analogs of phase-separated states associated with a first order transition. However, both the temperature and the free energy of these localized black hole states are higher than those of the pure phases at $T_c$. This difference could be viewed as indicating that the approximate degree of eigenvalue locality is insufficient to make the free energy excess associated with an “interface” in eigenvalue space subdominant, in the large $N$ limit, compared to the pure phase free energies. Moreover, it is not clear, at least to this author, whether this suggestion that localized BHs are analogs of phase-separated states (albeit with “extensive” interface free energy) is compatible with the above-mentioned characteristic of putative phase-separated states: that Polyakov loop expectation values interpolate between confined and deconfined phase expectation values in a manner indicative of a $\mathbb{Z}_N$ non-invariant, non-gapped eigenvalue distribution for the temporal holonomy.

Turning to other issues, it should be noted that the initial state of supercooled plasma at the metastability limit is invariant under the $SO(6)_R$ symmetry. So, in the absence of any $R$-symmetry violating perturbations, the final state must also be symmetric under $SO(6)_R$ and hence cannot literally be described by a single geometry with a black hole localized on the $S^5$. This, however, is not a real problem; it just means that the putative final state is a statistical mixture of localized black hole states averaged over all positions on the five-sphere. It may seem strange that a state described by a single classical geometry (in the large $N$ limit) could evolve into a state whose dual description involves a statistical mixture of geometries. In particular, this means that an initial state satisfying large $N$ factorization (5.1) evolves into a final state which violates large $N$ factorization. However, this is a typical feature of quantum dynamics whenever a system undergoes spontaneous symmetry breaking; negligible quantum fluctuations can become amplified and produce a superposition of classically distinct states which are indistinguishable from a statistical mixture.

A more significant and puzzling feature of this scenario concerns the fate, at large but finite $N$, of supercooled plasma with energy above the metastability limit at $E_*$. As discussed above, Schwarzschild-AdS black holes with energy between $E_*$ and $E_c^+$ provide the dual description of such supercooled states. These states are locally stable (for $E > E_*$) but one would expect non-perturbative tunneling or nucleation instabilities to exist with decay rates which vanish exponentially as $N \to \infty$. (After all, such
an instability, however slow, is what distinguishes the supercooled state from true equilibrium plasma at $T \geq T_c$. Viewed as a dynamical process at fixed energy, what is the endpoint of non-perturbative decay of supercooled plasma with energy $E > E^*$?

In a normal large volume thermodynamic limit, a locally stable supercooled phase can decay via nucleation and growth of critical size bubbles [25]. Once one or more critical bubbles of the low temperature phase are formed, they expand via conversion of latent heat into kinetic energy of expanding bubble walls, with subsequent bubble wall collisions leading to re-thermalization. Bubble nucleation rates depend on the amount of supercooling but are independent of the overall spatial volume (in the large volume limit), again reflecting the spatial locality of interactions. The endpoint of bubble nucleation events in the supercooled phase will be a phase-separated state with the same energy but greater entropy than the initial supercooled state (assuming the initial energy $E > E^*_c$).

In contrast, in large $N$ SYM, the rate for a single Polyakov loop eigenvalue to pass through a free energy barrier via tunneling or thermal activation is $O(N)$, since every eigenvalue interacts with every other one.\textsuperscript{10} Hence, unlike the situation in large volume thermodynamic limits, the decay rate of a locally stable supercooled phase (in finite spatial volume, at large $N$) should vanish exponentially as $N \to \infty$. Nevertheless, it is interesting to consider dynamics of the system at large but finite values of $N$. In the gravity-side description, a Schwarzschild-AdS black hole will Hawking radiate, on an $O(N^2)$ time scale, until it is in equilibrium with (super)gravitational radiation at the horizon temperature. Since the black hole heat capacity is $O(N^2)$, while that of the radiation is $O(1)$, the black hole retains all but an $O(N^{-2})$ fraction of the total energy. In the dual QFT, this process looks like formation of dilute gas of glueballs mixed in (and weakly interacting) with the deconfined plasma, rather analogous to Coulomb plasmas in which there is no sharp distinction between “bound” atoms and “free” ions.

Given that all radiation is trapped by the asymptotic AdS boundary conditions, the net effect of including Hawking radiation is merely to produce relative $O(N^{-2})$ corrections to thermodynamic quantities. But what happens to this state, whose gravitational description is a small black hole in equilibrium with its radiation, on exponentially long time scales? Given the existence of a free energy barrier separating this state from confined phase states with much lower free energy, one expects the presence of some non-perturbative instability, but what could be the endpoint of such an instability?

As shown in Fig. 4, the entropy of the localized BHs of Ref. [10] appears to cross that of Schwarzschild-AdS BHs at an energy $E_R$ (4.2) which is well below $E^*_c$. \textsuperscript{10}The $O(N)$ scaling of the free energy barrier for a single eigenvalue may be seen explicitly in the weak-coupling results of Refs. [21, 22].
is based on an extrapolation of available numerical data but, examining the quality of the fit and its extrapolation in Fig. 4, it is very hard to believe that this branch of solutions has an entropy which remains above that of the Schwarzschild-AdS BH all the way up to $E_c^+$. Assuming that the extrapolation shown in Fig. 4 is qualitatively correct, this scenario lacks any description of a possible endpoint for non-perturbative decays of supercooled plasma with energy in the range $E_R \leq E < E_c^+$.

5.2 Alternative final states?

Might there be other final states, not currently known, with higher entropy than the known localized BH solutions? This is a conceivable possibility, but at this point it is pure speculation. However, it is worth noting that stability of the localized black holes of Ref. [10] has not been fully explored. In particular, perturbations involving IIB supergravity fields other than the metric and the self-dual five-form are compatible with the symmetries of the localized BH solutions but have not yet been studied. So, for example, localized “fuzzy” black holes with non-zero values of the three-form fields may well exist. It is conceivable that such solutions could have horizon temperatures precisely equal to $T_c$, clearly implying an interpretation as phase-separated equilibrium states, although it is hard to see what would pick out this particular value.

Perhaps the strongest reason for suspecting that higher entropy solutions remain to be identified is the last puzzle mentioned above: the lack of any identified endpoint toward which non-perturbative decays of Schwarzschild-AdS black holes with energy between $E_R$ and $E_c^+$ could asymptote.

5.3 No stationary final state?

A final scenario to consider is the possibility that supercooled $\mathcal{N} = 4$ SYM plasma, upon reaching the limit of metastability and becoming locally unstable, fails to re-equilibrate.\(^{11}\) In other words, the possibility that a Schwarzschild-AdS black hole with energy $E < E_\ast$, when infinitesimally perturbed, becomes a time dependent solution which fails to ever settle down. Although unexpected in a strongly coupled relativistic field theory, such behavior would be reminiscent of many body localization, a phenomena of current interest in condensed matter physics in which systems of interacting particles develop unusual correlations which prevent thermalization [11–14].

In this hypothetical scenario, the evolution of an unstable small BH, slightly perturbed, surely begins as described earlier in section 4: the unstable mode grows and the horizon becomes increasingly distorted on the $S^5$. It is quite plausible that a cascade of subsequent instabilities will lead to the development of horizon structure in a

\(^{11}\)More precisely, fails to re-equilibrate on a time scale which remains bounded as $N \to \infty$. 

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manner similar to Gregory-Laflamme dynamics in asymptotically flat space \[31, 32\]. But does this process reach the string scale in finite time (as seen from the boundary)? Does the subsequent string theory dynamics then exit the stringy domain with different horizon topology on an $O(1)$ time scale? At the moment, these portions of the “widely expected” scenario are conjectural. It is certainly conceivable that these expectations are incorrect.

A critical difference between asymptotically flat and asymptotically AdS gravitational dynamics is the impossibility, in the asymptotic AdS case, for a time-dependent black hole to lose energy (or angular momentum) to radiation which propagates ever-outward with negligible subsequent effect on the horizon dynamics. As is well known (and fundamental to AdS/CFT duality), the asymptotic AdS boundary acts like a box; outward propagating gravitational waves reflect off the boundary and return to the interior of the spacetime. How this affects cascading horizon instabilities is far from clear. Perhaps the horizon develops chaotic or fractal structure which never reaches the string scale (in $O(1)$ time).\(^\text{12}\) Or perhaps the geometry does reach the string scale, but the subsequent dynamics remains both stringy and time-dependent (for all $O(1)$ times), and fails to return to the classical GR domain.

Based on experience with other examples of horizon dynamics it would be very surprising if this scenario turned out to be correct, as gravitational dynamics, when supplemented with a boundary condition of regularity on a future event horizon, is effectively dissipative. But this intuition is based on studies of gravitational dynamics with smooth geometries which are always far from the string scale. Moreover, the specific problem under consideration is inherently highly non-generic since the initial state is very specific: supercooled $\mathcal{N} = 4$ SYM plasma. Hence, the possibility of some non-generic long time behavior is at least worth considering.

From both analytic and numerical studies of perturbations of AdS spacetime, there is strong evidence that there are regions of initial conditions with non-zero measure for which the evolution does not lead to horizon formation, and complementary regions, also of non-zero measure, for which horizons do form. (See the review \[41\] and references therein.) Much of this work considers initial conditions for which the total energy $E = E_{\text{vac}}(1 + \epsilon)$, with $\epsilon$ infinitesimal. In other words, the excitation energy $E - E_{\text{vac}}$ is $O(N^2)$ but tiny compared to the vacuum energy. Most research in this area has been restricted to spherical symmetry and work to date has focused on asymptotically AdS spacetimes without additional compact dimensions. Consequently, how this partitioning of phase space into stable and unstable regions deforms as the exci-

\(^{12}\)Although the recent paper \[40\] examining similar dynamical issues in an asymptotically AdS model of exotic hairy black branes does support the expectation that string-scale structure will appear in finite time.
tation energy increases, particularly in the presence of additional compact dimensions, is not well understood. Nevertheless, it is quite plausible that in the asymptotically $\text{AdS}_5 \times S^5$ case both collapsing and non-collapsing regions continue to be present (with non-zero measure) for energies $E < E^+_c$. Assuming so, consider approaching the separatrix between collapsing and non-collapsing regions of initial data from the collapsing side. Continuity, at finite times, with respect to small changes in initial conditions implies that the horizon ring-down time (i.e., the inverse of the lowest quasinormal mode frequency) must diverge in this limit. Hence, solutions with initial data approaching this separatrix would be examples of dynamical black holes which fail to equilibrate on any $O(1)$ time scale. It is conceivable that infinitesimal perturbations of unstable Schwarzschild-AdS black holes correspond to just such initial data.

However improbable this scenario appears, it offers a possible answer to the apparent non-existence of any stationary solution which could represent the endpoint of non-perturbative decays of supercooled but locally stable plasma (i.e., Schwarzschild-AdS black holes with $E_\ast < E < E^+_c$).

6 Concluding remarks

At the very least, the above discussion of multiple possible scenarios for the dynamical evolution of unstable small Schwarzschild-AdS black holes should make clear that interesting open questions remain concerning the dynamics of supercooled SYM plasma beyond the limit of metastability. It is surely worthwhile to investigate the possible existence of further stationary supergravity solutions, particularly solutions which go beyond the metric plus self-dual five-form ansatz which suffices for the lumpy and localized BH solutions of Dias et al. [7, 10]. Efforts to study numerically the time evolution of (slightly perturbed) unstable Schwarzschild-AdS black holes have, so far, proven frustratingly difficult [42]. Even with improved numerical methods it will, at best, be possible to follow the evolution for only a limited period of time if fractal-like horizon structure develops.

Many of the conceptual issues discussed above also arise if one considers the dynamics of supercooled states in large $N$ pure Yang-Mills theory on $\mathbb{R} \times S^3$. If the radius of the three-sphere is small compared to the inverse strong scale $\Lambda^{-1}$, then a weak coupling analysis is possible [20, 21]. For small but non-zero effective coupling, a calculable free energy barrier separates the confined and deconfined pure phases at

\footnote{Given that solutions in the non-collapsing region have no horizon, one might expect finite time continuity with respect to changes in initial data to also imply vanishing of the horizon area (and hence entropy) of solutions in the collapsing region as the separatrix is approached. But this is fallacious, as it ignores the teleological nature of event horizons.}
The deconfined phase continues as a local minimum in the free energy for temperatures below \( T_c \) down to \( T_* = T_c - (T_H - T_c) \), where \( T_H \) is the Hagedorn temperature (or the limit of metastability of the superheated low temperature phase). There is no sign of any additional phase-separated equilibrium states coexisting at \( T_c \); and there is no internal global symmetry analogous to the \( SU(6)_R \) symmetry of SYM which could spontaneously break.\(^{14}\) Hence, there is no known equilibrium state with the same energy as the supercooled plasma at the limit of metastability, to which the unstable supercooled system might subsequently equilibrate.

It would be very surprising if the qualitative fate of supercooled large \( N \) non-Abelian plasma, upon reaching the limit of metastability, is profoundly different depending on whether the plasma is pure glue or maximally supersymmetric. Largely because of this, if the author were a betting person, he would wager on the last scenario discussed above: failure to re-thermalize with no stationary final state.

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\(^{14}\)This assumes that the key result of Ref. [22] is correct. The author has no reason to doubt the validity of this work, but is unaware of any independent confirmation.

\(^{15}\)Moreover, for both pure YM on a small \((L \ll \Lambda^{-1})\) spatial three-sphere and conformal \( \mathcal{N} = 4 \) SYM on any size \( S^3 \), spontaneous breaking of the spatial \( SO(4) \) symmetry seems exceedingly implausible as \( L \) is the only relevant spatial scale. There is no evident shorter spatial scale which might play a distinct role as a characteristic de Broglie wavelength of excitations or a domain wall width.
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