WEIGHT $q$-MULTIPLICITIES FOR REPRESENTATIONS OF THE EXCEPTIONAL LIE ALGEBRA $\mathfrak{g}_2$

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Abstract. Given a simple Lie algebra $\mathfrak{g}$, Kostant’s weight $q$-multiplicity formula is an alternating sum over the Weyl group whose terms involve the $q$-analog of Kostant’s partition function. For $\lambda$ (a weight of $\mathfrak{g}$), the $q$-analog of Kostant’s partition function is a polynomial-valued function defined by $\varphi_q(\xi) = \sum c_i q^i$ where $c_i$ is the number of ways $\xi$ can be written as a sum of $i$ positive roots of $\mathfrak{g}$. In this way, the evaluation of Kostant’s weight $q$-multiplicity formula at $q = 1$ recovers the multiplicity of a weight in a highest weight representation of $\mathfrak{g}$. In this paper, we give closed formulas for computing weight $q$-multiplicities in a highest weight representation of the exceptional Lie algebra $\mathfrak{g}_2$.

1. Introduction

We recall that the theorem of the highest weight asserts that a finite-dimensional complex irreducible representation of a simple Lie algebra $\mathfrak{g}$ is equivalent to $L(\lambda)$, a highest weight representation with dominant integral highest weight $\lambda$. The multiplicity of a weight $\mu$ in $L(\lambda)$, denoted by $m(\lambda, \mu)$, can be computed using Kostant’s weight multiplicity formula (as defined by Kostant in [15]):

$$m(\lambda, \mu) = \sum_{\sigma \in W} (-1)^{\ell(\sigma)} \varphi(\sigma(\lambda + \rho) - (\mu + \rho))$$

where $W$ is the Weyl group of $\mathfrak{g}$, $\ell(\sigma)$ denotes the length of $\sigma \in W$, and $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ with $\Phi^+$ being the set of positive roots of $\mathfrak{g}$, and where $\varphi$ denotes Kostant’s partition function, which counts the number of ways to express a weight as a nonnegative integral sum of positive roots.

In this paper, we consider the exceptional Lie algebra $\mathfrak{g}_2$ and study the $q$-analog of Kostant’s weight multiplicity formula, also known as Kostant’s weight $q$-multiplicity formula, which is a generalization of equation (1) defined by Lusztig in [10]:

$$m_q(\lambda, \mu) = \sum_{\sigma \in W} (-1)^{\ell(\sigma)} \varphi_q(\sigma(\lambda + \rho) - (\mu + \rho)).$$

In equation (2), $\varphi_q$ denotes the $q$-analog of Kostant’s partition function, which is a polynomial-valued function defined by

$$\varphi_q(\xi) = c_0 + c_1 q + c_2 q^2 + \cdots + c_n q^n,$$

where $c_i$ denotes the number of ways to express the weight $\xi$ as a sum of exactly $i$ positive roots. Note that equation (2) generalizes (1) since $\varphi_q(\xi)|_{q=1} = \varphi(\xi)$ for any weight $\xi$ and so $m_q(\lambda, \mu)|_{q=1} = m(\lambda, \mu)$. One important application of equation (2) is the celebrated result of Lusztig [10, Section 10, p. 226], which states that if $\mathfrak{g}$ is a finite-dimensional simple Lie algebra $\mathfrak{g}$ and $\tilde{\alpha}$ is the highest root, then $m_q(\tilde{\alpha}, 0) = q^{e_1} + q^{e_2} + \cdots + q^{e_r}$ where $e_1, e_2, \ldots, e_r$ are the exponents of $\mathfrak{g}$. In the case of the exceptional Lie algebra $\mathfrak{g}_2$, this implies that $m_q(\tilde{\alpha}, 0) = q + q^5$.

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Although formulas such as equation (1) and (2) exist, it is very difficult to give closed formulas for weight multiplicities for a Lie algebra of arbitrary rank. The difficulties in this work arise from both the lack of closed formulas for the partition functions involved, as well as the factorial growth of the Weyl group order as the rank of the Lie algebra increases. For some results related to computations of weight multiplicities in certain highest weight representations see [2-5, 8, 9, 11, 13]. In general, there has been some success in providing closed formulas for weight $q$-multiplicities for Lie algebras of low rank. This includes the work of Harris and Lauber [11] on weight $q$-multiplicities for the representations of $\mathfrak{sp}_4(\mathbb{C})$, which generalized the work of Refaghat and Shahryari [14], and the work of Garcia, Harris, Loving, Martinez, Melendez, Rennie, Rojas Kirby, and Tinoco [3] on weight $q$-multiplicities for $\mathfrak{sl}_4(\mathbb{C})$. Other work provides visualizations of the subsets of elements of the Weyl group which contribute non-trivially to the associated weight multiplicity, for examples see [11, 12]. Motivated by these works, we present a new formula for equation (2) giving weight $q$-multiplicities for representations of the exceptional Lie algebra $\mathfrak{g}_2$.

**Theorem 1.1.** Let $\varpi_1$ and $\varpi_2$ denote the fundamental weights of $\mathfrak{g}_2$. If $\lambda = m\varpi_1 + n\varpi_2$, $\mu = x\varpi_1 + y\varpi_2$, and $m, n, x, y \in \mathbb{N} := \{0, 1, 2, 3, \ldots\}$, then

$$m_q(\lambda, \mu) = \begin{cases} P - Q - R + S + T & \text{if and only if } a, b, c, d, e, f \in \mathbb{N}, \\ P - Q - R + S & \text{if and only if } a, b, c, d, e \in \mathbb{N}, f \notin \mathbb{N}, \\ P - Q - R + T & \text{if and only if } a, b, c, d, f \in \mathbb{N}, e \notin \mathbb{N}, \\ P - Q - R & \text{if and only if } a, b, c, d \in \mathbb{N}, e, f \notin \mathbb{N}, \\ P - Q & \text{if and only if } a, b, c \in \mathbb{N}, d, e, f \notin \mathbb{N}, \\ P & \text{if and only if } a, b \in \mathbb{N}, c, d, e, f \notin \mathbb{N}, \\ 0 & \text{otherwise} \end{cases}$$

where

$$P = \varphi_q((2m + 3n - 2x - 3y)\alpha_1 + (m + 2n - x - 2y)\alpha_2),$$

$$Q = \varphi_q((m + 3n - 2x - 3y - 1)\alpha_1 + (m + 2n - x - 2y)\alpha_2),$$

$$R = \varphi_q((2m + 3n - 2x - 3y)\alpha_1 + (m + n - x - 2y - 1)\alpha_2),$$

$$S = \varphi_q((m + 3n - 2x - 3y - 1)\alpha_1 + (n - x - 2y - 2)\alpha_2),$$

$$T = \varphi_q((m - 2x - 3y - 4)\alpha_1 + (m + n - x - 2y - 1)\alpha_2).$$

In general, using equation (2) to compute weight $q$-multiplicities for representations of $\mathfrak{g}_2$ requires the computation of Kostant’s partition function on 12 distinct inputs, as the Weyl group of $\mathfrak{g}_2$ is isomorphic to the dihedral group of order 12. However, Theorem 1.1 reduces all weight $q$-multiplicity computations to at most five such computations. Our second result, provides a formula for the $q$-analog of Kostant’s partition function for $\mathfrak{g}_2$, which can be used to compute each of the terms appearing in Theorem 1.1.

**Proposition 1.1.** If $m, n \in \mathbb{N}$, then the value of $\varphi_q(m\alpha_1 + n\alpha_2)$ is given by

$$\min\left(\left[\frac{m}{2}\right], \left[\frac{n}{2}\right]\right) \min\left(\left[\frac{m-n}{2}\right], \left[\frac{n-2i}{2}\right]\right) \min\left(\left[\frac{m-n}{2}\right], \left[\frac{n-2i-j}{2}\right]\right) \min(m-n-3j-2k, n-2i-j-k) \left(\sum_{l=0}^{\left[\min(m-3i-3j-2k, n-2i-j-k)\right]} q^l\right),$$

where $z = m + n - 4i - 3j - 2k - l$.

**Outline of the paper.** Section 2 provides the Lie theoretic background needed for the remainder of the manuscript. Section 3 contains the proof of Proposition 1.1. We prove Theorem 1.1 in
Section 4 and provide some detailed examples of how Theorem 1.1 can be used to compute weight $q$-multiplicities for representations of $g_2$. In Section 5 we provide a missing case in the proof of a formula of Harris and Lauber for the $q$-analog of Kostant’s partition function of the Lie algebra $\mathfrak{sp}_4(\mathbb{C})$ appearing in [10]. We end the manuscript with a section containing some open problems.

2. Background

We use the same notation as appearing in [4], which the reader can look to for a more comprehensive treatment of some of the objects introduced here. We denote the simple roots of $g_2$ as $\alpha_1$ and $\alpha_2$, and the fundamental weights as $\varpi_1$ and $\varpi_2$. The positive roots of $g_2$ are given by

$$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}.$$  

Recall that $\varpi_1 = 2\alpha_1 + \alpha_2$, $\varpi_2 = 3\alpha_1 + 2\alpha_2$, and

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \varpi_1 + \varpi_2 = 5\alpha_1 + 3\alpha_2.$$  

We set $\lambda = (2m + 3n)\alpha_1 + (m + 2n)\alpha_2$ and $\mu = (2x + 3y)\alpha_1 + (x + 2y)\alpha_2$ where $m, n, x, y \in \mathbb{N}$. We make this choice to simplify our computations and we are able to do so since the fundamental weight lattice and the root lattice of $g_2$ are equal.

The Weyl group of $g_2$, denoted $W$, is generated by reflections about hyperplanes orthogonal to the simple roots. We denote the reflection through the hyperplane orthogonal to $\alpha_i$ by $s_i$ for $i = 1, 2$. In Figure 1 we illustrate the positive roots and in red we present the hyperplanes defining the reflections $s_1$ and $s_2$. The action of the generators of $W$ on the simple roots is given by

$$s_1(\alpha_1) = -\alpha_1, \quad s_1(\alpha_2) = 3\alpha_1 + \alpha_2,$$

$$s_2(\alpha_1) = \alpha_1 + \alpha_2, \quad s_2(\alpha_2) = -\alpha_2.$$  

Table 1 describes how the remaining elements of $W$ act on the simple roots.

![Figure 1. Positive root system for $g_2$ and the lines orthogonal to the simple roots which define $s_1$ and $s_2$.](image-url)
In this section, we provide a closed formula for the $q$-analog of Kostant’s partition function for the exceptional Lie algebra $g_2$, which was presented in equation (3). We restate the result below for ease of reference.

**Proposition 1.1.** If $m, n \in \mathbb{N}$, then the value of $\varphi_q (m \alpha_1 + n \alpha_2)$ is given by

$$
\sum_{i=0}^{\min(\lfloor\frac{m}{2}\rfloor, \lfloor\frac{n}{2}\rfloor)} \left( \sum_{j=0}^{\min(\lfloor\frac{m-3i}{2}\rfloor, n-2i)} \left( \sum_{k=0}^{\min(\lfloor\frac{m-3i-3j}{2}\rfloor, n-2i-j-k)} \left( \sum_{l=0}^{\min(m-3i-3j-2k, n-2i-j-k)} q^l \right) \right) \right),
$$

where $z = m + n - 4i - 3j - 2k - l$.

**Proof.** The number of ways we can write $m \alpha_1 + n \alpha_2$ as a nonnegative integral sum of positive roots is determined by the number of times each positive root in

$$
\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}
$$
is used.

If a partition includes $i$ multiples of the highest root $3\alpha_1 + 2\alpha_2$, then $0 \leq i \leq \min(\lfloor\frac{m}{2}\rfloor, \lfloor\frac{n}{2}\rfloor)$, so as to not exceed each coefficient of the weight $m \alpha_1 + n \alpha_2$ for $m$ and $n$. We are now left to partition $m \alpha_1 + n \alpha_2 - i(3\alpha_1 + 2\alpha_2) = (m-3i)\alpha_1 + (n-2i)\alpha_2$. If the partition of $(m-3i)\alpha_1 + (n-2i)\alpha_2$ includes $j$ multiples of the root $3\alpha_1 + \alpha_2$, then $0 \leq j \leq \min(\lfloor\frac{m-3i}{2}\rfloor, n-2i)$. In which case, we must partition $(m-3i)\alpha_1 + (n-2i-3j)\alpha_1 + (3\alpha_1 + \alpha_2) = (m-3i-3j)\alpha_1 + (n-2i-j)\alpha_2$ if the partition of $(m-3i-3j)\alpha_1 + (n-2i-j)\alpha_2$ includes $k$ multiples of the root $3\alpha_1 + \alpha_2$, then $0 \leq k \leq \min(\lfloor\frac{m-3i-3j}{2}\rfloor, n-2i-j)$. We must now partition $(m-3i-3j-k)\alpha_1 + (n-2i-j-k)\alpha_2 - k(2\alpha_1 + \alpha_2) = (m-3i-3j-2k)\alpha_1 + (n-2i-j-k)\alpha_2$. If the partition of $(m-3i-3j-2k)\alpha_1 + (n-2i-j-k)\alpha_2$ includes $l$ multiples of $\alpha_1 + \alpha_2$, then $0 \leq l \leq \min(m-3i-3j-2k, n-2i-j-k)$. We are left to partition $(m-3i-3j-2k-l)\alpha_1 + (n-2i-j-k-l)\alpha_2 - l(\alpha_1 + \alpha_2) = (m-3i-3j-2k-l)\alpha_1 + (n-2i-j-k-l)\alpha_2$. Finally, the coefficients of $\alpha_1$ or $\alpha_2$ in our partition are determined by our choice of $i, j, k, l$ and are $m-3i-3j-2k-l$ and $n-2i-j-k-l$, respectively.

It follows that the total number of roots used is given by $z = i + j + k + l + (m-3i-3j-2k-l) + (n-2i-j-k-l) = m + n - 4i - 3j - 2k - l$.

With the formula of Proposition 1.1 at hand, next we compute the values of $\sigma(\lambda + \mu) - (\mu + \rho)$ as they appear in (2) for each $\sigma \in W$. Recall that $\lambda = (2m+3n)\alpha_1 + (m+2n)\alpha_2$ and $\mu = (2x+3y)\alpha_1 + (x+2y)\alpha_2$, where $m, n, x, y \in \mathbb{N}$. To illustrate the computations, we consider the case when $\sigma = s_1$, and using equations (7), (8), and (9), we find that

$$
s_1(\lambda + \rho) - (\mu + \rho) = s_1((2m+3n)\alpha_1 + (m+2n)\alpha_2 + 5\alpha_1 + 3\alpha_2) - ((2x+3y)\alpha_1 + (x+2y)\alpha_2 + 5\alpha_1 + 3\alpha_2)
$$

$$
= ((2m + 3n + 5)(-\alpha_1) + (m + 2n + 3)(3\alpha_1 + \alpha_2) - (2x + 3y + 5)\alpha_1 - (x + 2y + 3)\alpha_2
$$
\[= (m + 3n - 2x - 3y - 1)\alpha_1 + (m + 2n - x - 2y)\alpha_2.\]

Repeating this process with every remaining Weyl group element yields the contents of Table 2.

| $\sigma$ | $\ell(\sigma)$ | $\sigma(\lambda + \rho) - (\mu + \rho)$ |
|----------|----------------|----------------------------------------|
| 1        | 0              | \((2m + 3n - 2x - 3y)\alpha_1 + (m + 2n - x - 2y)\alpha_2\) |
| $s_1$    | 1              | \((m + 3n - 2x - 3y - 1)\alpha_1 + (m + 2n - x - 2y)\alpha_2\) |
| $s_2$    | 1              | \((2m + 3n - 2x - 3y)\alpha_1 + (m + n - x - 2y - 1)\alpha_2\) |
| $s_2s_1$ | 2              | \((m + 3n - 2x - 3y - 1)\alpha_1 + (n - x - 2y - 2)\alpha_2\) |
| $s_1s_2$ | 2              | \((m - 2x - 3y - 4)\alpha_1 + (m + n - x - 2y - 1)\alpha_2\) |
| $s_1s_2s_1$ | 3            | \((-m - 2x - 3y - 6)\alpha_1 + (n - x - 2y - 2)\alpha_2\) |
| $s_2s_1s_2$ | 3           | \((m - 2x - 3y - 4)\alpha_1 + (-n - x - 2y - 4)\alpha_2\) |
| $s_1s_2s_1s_2$ | 4          | \((-m - 3n - 2x - 3y - 9)\alpha_1 + (-n - x - 2y - 4)\alpha_2\) |
| $s_2\{s_1s_2\}^2$ | 4        | \((-m - 2x - 3y - 6)\alpha_1 + (-m - n - x - 2y - 5)\alpha_2\) |
| $s_1\{s_2s_1\}^2$ | 5        | \((-2m - 3n - 2x - 3y - 10)\alpha_1 + (-m - n - x - 2y - 5)\alpha_2\) |
| $s_2\{s_1s_2\}^2$ | 5        | \((-m - 3n - 2x - 3y - 9)\alpha_1 + (-m - 2n - x - 2y - 6)\alpha_2\) |
| $\{s_1s_2\}^3$ | 6        | \((-2m - 3n - 2x - 3y - 10)\alpha_1 + (-m - 2n - x - 2y - 6)\alpha_2\) |

Table 2. Evaluations of $\sigma(\lambda + \rho) - (\mu + \rho)$ for $\sigma \in W$.

Observe that for $m, n, x, y \in \mathbb{N}$, the $q$-analogue of Kostant’s partition function evaluates to zero if the coefficient of either $\alpha_1$ or $\alpha_2$ is negative. Thus, given the computations appearing in Table 2, we note that the only elements of the Weyl group that contribute to Kostant’s weight $q$-multiplicity formula are $1, s_1, s_2, s_2s_1$, and $s_1s_2$. The remaining elements of $W$ never contribute and, hence, we disregard them moving forward. With these observations, we are now ready to prove Theorem 1.1 by evaluating $m_q(\lambda, \mu)$ as appearing in (2).

4. The $q$-analogue of Kostant’s Weight Multiplicity Formula

4.1. Evaluation of $m_q(\lambda, \mu)$. In the previous section, we established that $1, s_1, s_2, s_2s_1$, and $s_1s_2$ are the only Weyl group elements that contribute nontrivially to $m_q(\lambda, \mu)$ whenever $\lambda = m\varpi_1 + n\varpi_2 = (2m + 3n)\alpha_1 + (m + 2n)\alpha_2$ and $\mu \in \mathbb{N}$ is an integer. For the sake of simplicity, we make the following change of variables

\begin{align}
    a &= 2m + 3n - 2x - 3y, \\
    b &= m + 2n - x - 2y, \\
    c &= m + 3n - 2x - 3y - 1, \\
    d &= m + n - x - 2y - 1, \\
    e &= n - x - 2y - 2, \text{ and} \\
    f &= m - 2x - 3y - 4.
\end{align}

Utilizing this change of variables together with the evaluations in Table 2 for $\sigma = 1, s_1, s_2, s_2s_1$, and $s_1s_2$, we obtain

\begin{align}
P &= \varphi_q(1(\lambda + \rho) - (\mu + \rho)) = \varphi_q(aa\alpha_1 + bo\alpha_2), \\
Q &= \varphi_q(s_1(\lambda + \rho) - (\mu + \rho)) = \varphi_q(ca\alpha_1 + bo\alpha_2), \\
R &= \varphi_q(s_2(\lambda + \rho) - (\mu + \rho)) = \varphi_q(aa\alpha_1 + do\alpha_2), \\
S &= \varphi_q(s_2s_1(\lambda + \rho) - (\mu + \rho)) = \varphi_q(ca\alpha_1 + eo\alpha_2), \text{ and} \\
T &= \varphi_q(s_1s_2(\lambda + \rho) - (\mu + \rho)) = \varphi_q(fa\alpha_1 + do\alpha_2).
\end{align}
The expressions in equation (12) are precisely the expressions described in (5) and are the terms
needed to evaluate \( m_q(\lambda, \mu) \). However, there can be instances where certain values of \( m, n, x, y \in \mathbb{N} \) result in some of the expressions in (12) being zero, while others remain nonzero. When an expression is zero we say it contributes trivially to the \( q \)-multiplicity; if instead the expression is nonzero, then we say it contributes nontrivially to the \( q \)-multiplicity.

From (12), we know that there are at most five terms, namely \( P, Q, R, S, \) and \( T \) that can con-
tribute to \( m_q(\lambda, \mu) \) depending on the values of \( m, n, x, y \in \mathbb{N} \). This gives us at most \( 2^5 = 32 \) distinct
possible formulas for \( m_q(\lambda, \mu) \). In the work that follows, we will prove that of these 32 distinct
possible cases only 8 can occur.

As is standard, we let \( \lor \) denote the Boolean operator \( or \), and \( \land \) denote the Boolean operator \( and \).
Note that \( a, b, c, d, e, f \), as given in (11), are always integer quantities. Hence, when \( a, b, c, d, e, f \)
are nonnegative, then \( P, Q, R, S, \) and \( T \) contribute nontrivially to \( m_q(\lambda, \mu) \). To simplify notation,
we define the statements

\[
\begin{align*}
a_0 : & \quad a \geq 0, \quad a_1 : \quad a < 0, \quad b_0 : \quad b \geq 0, \quad b_1 : \quad b < 0, \quad c_0 : \quad c \geq 0, \quad c_1 : \quad c < 0, \\
d_0 : & \quad d \geq 0, \quad d_1 : \quad d < 0, \quad e_0 : \quad e \geq 0, \quad e_1 : \quad e < 0, \quad f_0 : \quad f \geq 0, \quad f_1 : \quad f < 0.
\end{align*}
\]

Thus, by definition of Kostant’s partition function we have that

\[
\begin{align*}
P & \text{ contributes nontrivially if and only if } a_0 \land b_0 \text{ holds true,} \\
Q & \text{ contributes nontrivially if and only if } c_0 \land b_0 \text{ holds true,} \\
R & \text{ contributes nontrivially if and only if } a_0 \land d_0 \text{ holds true,} \\
S & \text{ contributes nontrivially if and only if } c_0 \land e_0 \text{ holds true,} \\
T & \text{ contributes nontrivially if and only if } f_0 \land d_0 \text{ holds true.}
\end{align*}
\]

Hence,

\[
\begin{align*}
P & \text{ contributes trivially if and only if } a_1 \lor b_1 \text{ holds true,} \\
Q & \text{ contributes trivially if and only if } c_1 \lor b_1 \text{ holds true,} \\
R & \text{ contributes trivially if and only if } a_1 \lor d_1 \text{ holds true,} \\
S & \text{ contributes trivially if and only if } c_1 \lor e_1 \text{ holds true,} \\
T & \text{ contributes trivially if and only if } f_1 \lor d_1 \text{ holds true.}
\end{align*}
\]

We briefly illustrate our method of proof via an example. From the descriptions in (13) and
(14), we know that \( m_q(\lambda, \mu) = P - Q + T \) when \( P, Q, T \) contribute nontrivially and \( R, S \) contribute
trivially. This implies that the following necessary condition must be true:

\[(a_0 \land b_0) \land (c_0 \land b_0) \land (a_1 \lor d_1) \land (c_1 \lor e_1) \land (f_0 \land d_0).\]

However, we note that such a logical statement contains \((a_0 \land d_0) \land (a_1 \lor d_1)\), which can never be
true. This establishes that \( m_q(\lambda, \mu) \neq P - Q + T \) whenever \( m, n, x, y \in \mathbb{N} \). In this case, we would
state that \( P - Q + T \) is a forbidden \( q \)-multiplicity formula. We now give a general definition.

**Definition 1.** Fix \( \lambda = m\varpi_1 + n\varpi_2 \) and \( \mu = x\varpi_1 + y\varpi_2 \) with \( m, n, x, y \in \mathbb{N} \). Let \( P, Q, R, S, T \)
be as in (12), with \( \text{sgn}(P) = \text{sgn}(S) = \text{sgn}(T) = 1 \) and \( \text{sgn}(Q) = \text{sgn}(R) = -1 \). For any subset
\( X \subseteq \{P, Q, R, S, T\} \), if \( m_q(\lambda, \mu) \neq \sum_{x \in X} \text{sgn}(x)x \), then \( \sum_{x \in X} \text{sgn}(x)x \) is said to be a forbidden
\( q \)-multiplicity formula.

Using this new definition along with the technique illustrated above we establish the following.

**Lemma 4.1.** Let \( \lambda = m\varpi_1 + n\varpi_2 \) and \( \mu = x\varpi_1 + y\varpi_2 \) with \( m, n, x, y \in \mathbb{N} \). If \( P, Q, R, S, T \) are as
in (12), then the formulas \( \sum_{x \in X} \text{sgn}(x)x \), with \( X \subseteq \{P, Q, R, S, T\} \), listed in Table 3 are forbidden
\( q \)-multiplicity formulas.
Table 3. Forbidden q-multiplicity formulas for Lemma 4.1

| Case | Necessary Conditions |
|------|----------------------|
| 1    | \( P - R + T \)       |
| 2    | \( P - Q + S \)       |
| 3    | \( -Q + S + T \)      |
| 4    | \( -R + S + T \)      |
| 5    | \( -Q + S \)          |
| 6    | \( -Q + T \)          |
| 7    | \( -R + S \)          |
| 8    | \( -R + T \)          |
| 9    | \( S + T \)           |
| 10   | \( -Q \)              |
| 11   | \( -R \)              |
| 12   | \( S \)               |
| 13   | \( T \)               |

Table 4. Forbidden q-multiplicity formulas for Lemma 4.2

| Case | Necessary Conditions |
|------|----------------------|
| 1    | \( P - Q + S + T \)  |
| 2    | \( P - R + S + T \)  |
| 3    | \( -Q - R + S + T \) |
| 4    | \( P + S \)          |
| 5    | \( P + T \)          |
| 6    | \( -Q - R \)         |

Proof. Our work in the previous example has already established that \( P - Q + T \) is a forbidden q-multiplicity formula. Next, consider the case where \( m_q(\lambda, \mu) = P + S + T \). As a consequence of (13) and (14), the following statement must hold true:

\[(a_0 \wedge b_0) \wedge (c_0 \wedge e_0) \wedge (f_0 \wedge d_0) \wedge (c_1 \wedge b_1) \wedge (a_1 \wedge d_1).

However, this also implies that \((a_0 \wedge d_0) \wedge (a_1 \wedge d_1)\), which is a contradiction. Therefore, \( P + S + T \) is a forbidden q-multiplicity formula.

In Table 3 we give a total of eleven cases (including the two considered above) which give rise to forbidden q-multiplicity formulas. Note that for each case, we specify both the necessary condition that must be true in order for that formula to hold, as well as the contradiction that arises from such a case.

Our next result establishes 13 additional forbidden q-multiplicity formulas.

Lemma 4.2. Let \( \lambda = m \varpi_1 + n \varpi_2 \) and \( \mu = x \varpi_1 + y \varpi_2 \) with \( m, n, x, y \in \mathbb{N} \). If \( P, Q, R, S, T \) are as in (12), then the formulas \( \sum_{x \in X} \text{sgn}(x)x \), with \( X \subseteq \{P, Q, R, S, T\} \), listed in Table 4 are forbidden q-multiplicity formulas.
Proof. We begin by describing a set of statements that give rise to contradictions. These cases will allow us to establish that the $q$-multiplicities listed in Table 4 are forbidden.

Case A: Assume the statement $e_0 \land d_1$ holds true. If $d = m + n - x - 2y - 1 < 0$, then $m + n - 2y - 1 < x$. Also, if $e = n - x - 2y - 2 \geq 0$, then $n - 2y - 2 \geq 0$. Thus, $n - 2y - 2 > m + n - 2y - 1$. Solving for $m$ explicitly yields $m < -1$, implying that whenever $e_0 \land d_1$ holds true the corresponding system of inequalities does not have a nonnegative integer solution.

Case B: Assume the statement $f_0 \land c_1$ holds true. If $c = m + 3n - 2x - 3y - 1 < 0$, then $m + 3n - 3y - 1 < 2x$. Also, if $f = m - 2x - 3y - 4 \geq 0$, then $m - 3y - 4 \geq 2x$. Hence, $m - 3y - 4 > m + 3n - 3y - 1$. Solving for $n$ explicitly yields $n < -1$, implying that this corresponding system of inequalities does not have a nonnegative integer solution.

Case C: Assume the statement $c_0 \land a_1$ holds true. We observe that if $a = 2m + 3n - 2x - 3y < 0$, then $2m + 3n - 3y < 2x$. Also, if $c = m + 3n - 2x - 3y - 1 \geq 0$, then $m + 3n - 3y - 1 \geq 2x$. We join these two inequalities to obtain $m + 3n - 3y - 1 > 2m + 3n - 3y$. If we solve for $m$ explicitly, we obtain that $m < -1$, implying that this system has no solutions.

Case D: Assume the statement $d_0 \land b_1$ holds true. We observe that if $b = m + 2n - x - 2y < 0$, then $m + 2n - 2y < x$. Also, if $d = m + n - x - 2y - 1 \geq 0$, then $m + n - 2y - 1 \geq x$. We join these two inequalities to obtain $m + n - 2y - 1 > m + 2n - 2y$. If we solve for $n$ explicitly, we obtain that $n < -1$, implying that such a system has no solutions.

Utilizing the cases above, we are now ready to consider each $q$-multiplicity listed in Table 4 and show each is forbidden.

Case 1: The necessary condition for $m_q(\lambda, \mu) = P - R + T$ is given by

$$(a_0 \land b_0) \land (c_1 \lor b_1) \land (a_0 \land d_0) \land (c_1 \lor e_1) \land (f_0 \land d_0).$$

Since the logical statement must hold true and it contains $a_0 \land b_0 \land d_0 \land f_0$, it must be that $(c_1 \lor b_1) \land (c_1 \lor e_1)$ reduces to $c_1$ or $c_1 \land e_1$. Otherwise, it would contain the contradiction $b_0 \land b_1$. We list all the possible ways in which the necessary condition for this case can be true and describe a contradiction arising from each possibility.

| Possible Logical Conditions | Contradiction |
|-----------------------------|--------------|
| $(a_0 \land b_0 \land d_0 \land f_0) \land c_1$ | $f_0 \land c_1$ (Case B) |
| $(a_0 \land b_0 \land d_0 \land f_0) \land (c_1 \land e_1)$ | $f_0 \land c_1$ (Case B) |

Case 2: The necessary condition for $m_q(\lambda, \mu) = P - Q + S$ is given by

$$(a_0 \land b_0) \land (c_0 \land b_0) \land (a_1 \lor d_1) \land (c_0 \land e_0) \land (f_1 \lor d_1).$$

Since the logical statement must hold true and it contains $a_0 \land b_0 \land c_0 \land e_0$, it must be that $(a_1 \lor d_1) \land (f_1 \lor d_1)$ reduces to $d_1$ or $d_1 \land f_1$. Otherwise, it would contain the contradiction $a_0 \land a_1$. We list all the possible ways in which the necessary condition for this case can be true and describe a contradiction arising from each possibility.

| Possible Logical Conditions | Contradiction |
|-----------------------------|--------------|
| $(a_0 \land b_0 \land c_0 \land e_0) \land d_1$ | $e_0 \land d_1$ (Case A) |
| $(a_0 \land b_0 \land c_0 \land e_0) \land (d_1 \land f_1)$ | $e_0 \land d_1$ (Case A) |

Case 3: The necessary condition for $m_q(\lambda, \mu) = -Q + S + T$ is given by

$$(a_1 \lor b_1) \land (c_0 \land b_0) \land (a_1 \lor d_1) \land (c_0 \land e_0) \land (f_0 \land d_0).$$

Since the logical statement must hold true and it contains $b_0 \land c_0 \land d_0 \land e_0 \land f_0$, it must be that $(a_1 \lor b_1) \land (a_1 \lor d_1)$ reduces to $a_1$. Otherwise, it would contain the contradiction $b_0 \land b_1$ or $d_0 \land d_1$. Thus, the only possible way in which the necessary condition for this case can be true is if $(b_0 \land c_0 \land d_0 \land e_0 \land f_0) \land a_1$ is true. However, this case contains the contradiction $c_0 \land a_1$ as seen in Case C.
Case 4: The necessary condition for $m_q(\lambda, \mu) = -R + S + T$ is given by

$$(a_1 \lor b_1) \land (c_1 \lor b_1) \land (a_0 \land d_0) \land (c_0 \land e_0) \land (f_0 \land d_0).$$

Since the logical statement must hold true and it contains $a_0 \land c_0 \land d_0 \land e_0 \land f_0$, it must be that $(a_1 \lor b_1) \land (c_1 \lor b_1)$ reduces to $b_1$. Otherwise, it would contain the contradiction $a_0 \land a_1$ or $c_0 \land c_1$. Thus, the only possible way in which the necessary condition for this case can be true is if $(a_0 \land c_0 \land d_0 \land c_0 \land e_0) \land (f_0 \land b_1)$ is true. However, this case contains the contradiction $d_0 \land b_1$ as seen in Case D.

Case 5: The necessary condition for $m_q(\lambda, \mu) = -Q + S$ is given by

$$(a_1 \lor b_1) \land (c_0 \land b_0) \land (a_1 \lor d_1) \land (c_0 \land e_0) \land (f_1 \lor d_1).$$

Since the logical statement must hold true and it contains $b_0 \land c_0 \land e_0$, it must be that $(a_1 \lor b_1) \land (a_1 \lor d_1) \land (f_1 \lor d_1)$ reduces to $a_1 \land d_1, a_1 \land f_1$, or $a_1 \land d_1 \land f_1$. Otherwise, it would contain the contradiction $b_0 \land b_1$. Thus, there are three possible ways in which the necessary condition for this case can be true. Next, we list all the possible ways in which the necessary condition for this case can be true and describe a contradiction arising from each possibility.

| Possible Logical Conditions | Contradiction |
|----------------------------|---------------|
| $(b_0 \land c_0 \land d_0) \land (a_1 \land d_1)$ | $e_0 \land d_1$ (Case A) |
| $(b_0 \land c_0 \land e_0) \land (a_1 \land f_1)$ | $e_0 \land a_1$ (Case C) |
| $(b_0 \land c_0 \land e_0) \land (a_1 \land d_1 \land f_1)$ | $c_0 \land a_1$ (Case C) |

Case 6: The necessary condition for $m_q(\lambda, \mu) = -Q + T$ is given by

$$(a_1 \lor b_1) \land (c_0 \land b_0) \land (a_1 \lor d_1) \land (c_1 \land e_1) \land (f_0 \land d_0).$$

Since the logical statement must hold true and it contains $b_0 \land c_0 \land e_0$, it must be that $(a_1 \lor b_1) \land (a_1 \lor d_1) \land (c_1 \lor e_1)$ reduces to $a_1 \land e_1$. Otherwise, it would contain the contradiction $b_0 \land b_1, c_0 \land c_1$, or $d_0 \land d_1$. Thus, the only possible way in which the necessary condition for this case can be true is if $(b_0 \land c_0 \land d_0 \land f_0) \land (a_1 \land e_1)$ is true. However, this case contains the contradiction $c_0 \land a_1$ as seen in Case C.

Case 7: The necessary condition for $m_q(\lambda, \mu) = -R + S$ is given by

$$(a_1 \lor b_1) \land (c_1 \lor b_1) \land (a_0 \land d_0) \land (c_0 \land e_0) \land (f_1 \lor d_1).$$

Since the logical statement must hold true and it contains $a_0 \land c_0 \land d_0 \land e_0$, it must be that $(a_1 \lor b_1) \land (c_1 \lor b_1) \land (f_1 \lor d_1)$ reduces to $b_1 \land f_1$. Otherwise, it would contain the contradiction $a_0 \land a_1, c_0 \land c_1$ or $d_0 \land d_1$. Thus, there is only one possible way in which the necessary condition for this case can be true, namely, if $a_0 \land c_0 \land d_0 \land e_0 \land b_1 \land f_1$ is true. However, this gives rise to the contradiction $d_0 \land b_1$ as seen in Case D.

Case 8: The necessary condition for $m_q(\lambda, \mu) = -R + T$ is given by

$$(a_1 \lor b_1) \land (c_1 \lor b_1) \land (a_0 \land d_0) \land (c_1 \lor e_1) \land (f_0 \land d_0).$$

Since the logical statement must hold true and it contains $a_0 \land d_0 \land f_0$, it must be that $(a_1 \lor b_1) \land (c_1 \lor b_1) \land (c_1 \lor e_1)$ reduces to $b_1 \land c_1, b_1 \land e_1$, or $b_1 \land c_1 \land e_1$. Otherwise, it would contain the contradiction $a_0 \land a_1$. Thus, there are three possible ways in which the necessary condition for this case can be true. We list the three possible ways in which the necessary condition can be true and describe a contradiction arising from each possibility.

| Possible Logical Conditions | Contradiction |
|----------------------------|---------------|
| $(a_0 \land d_0 \land f_0) \land (b_1 \land c_1)$ | $f_0 \land c_1$ (Case B) |
| $(a_0 \land d_0 \land f_0) \land (b_1 \land e_1)$ | $d_0 \land b_1$ (Case D) |
| $(a_0 \land d_0 \land f_0) \land (b_1 \land c_1 \land e_1)$ | $f_0 \land c_1$ (Case B) |
Case 9: The necessary condition for \( m_q(\lambda, \mu) = S + T \) is given by

\[
(a_1 \lor b_1) \land (c_1 \lor b_1) \land (a_1 \lor d_1) \land (c_0 \land e_0) \land (f_0 \land d_0).
\]

Since the logical statement must hold true and it contains \( c_0 \land d_0 \land e_0 \land f_0 \), it must be that \((a_1 \lor b_1) \land (c_1 \lor b_1) \land (a_1 \lor d_1) \rightarrow a_1 \land b_1\). Otherwise, it would contain the contradiction \( c_0 \land c_1 \) or \( d_0 \land d_1 \). Thus, there is only one possible way in which the necessary condition for this case can be true, namely, if \( c_0 \land d_0 \land e_0 \land f_0 \land a_1 \land b_1 \) is true. However, this gives rise to the contradiction \( c_0 \land a_1 \) as seen in Case C.

Case 10: The necessary condition for \( m_q(\lambda, \mu) = -Q \) is given by

\[
(a_1 \lor b_1) \land (c_0 \land b_0) \land (a_1 \lor d_1) \land (c_1 \lor e_1) \land (f_1 \lor d_1).
\]

Since the logical statement must hold true and it contains \( b_0 \land c_0 \), it must be that \((a_1 \lor b_1) \land (a_1 \lor d_1) \land (f_1 \lor d_1) \land (c_1 \lor e_1) \rightarrow b_1 \land a_1 \land e_1 \land d_1 \land f_1\). Otherwise, it would contain the contradiction \( b_0 \land b_1 \lor c_0 \land c_1 \). Thus, there are three possible ways in which the necessary condition for this case can be true. We list these possibilities and describe a contradiction arising from each possibility.

| Possible Logical Conditions | Contradiction |
|-----------------------------|--------------|
| \((b_0 \land c_0) \land (a_1 \land e_1) \land f_1)\) | \(c_0 \land a_1\) (Case C) |
| \((b_0 \land c_0) \land (a_1 \land e_1) \land d_1)\) | \(c_0 \land a_1\) (Case C) |
| \((b_0 \land c_0) \land (a_1 \land e_1 \land d_1 \land f_1)\) | \(c_0 \land a_1\) (Case C) |

Case 11: The necessary condition for \( m_q(\lambda, \mu) = -R \) is given by

\[
(a_1 \lor b_1) \land (c_1 \lor b_1) \land (a_0 \land d_0) \land (c_1 \lor e_1) \land (f_1 \lor d_1).
\]

Since the logical statement must hold true and it contains \( a_0 \land d_0 \), it must be that \((a_1 \lor b_1) \land (c_1 \lor b_1) \land (f_1 \lor d_1) \land (c_1 \lor e_1) \rightarrow b_1 \land f_1 \land c_1 \land e_1\) or \(b_1 \land f_1 \land c_1 \land e_1\). Otherwise, it would contain the contradiction \( a_0 \land a_1 \) or \( d_0 \land d_1 \). Thus, there are only three possible ways in which the necessary condition for this case can be true. We list these possibilities and describe a contradiction arising from each possibility.

| Possible Logical Conditions | Contradiction |
|-----------------------------|--------------|
| \((a_0 \land d_0) \land (b_1 \land f_1 \land c_1)\) | \(d_0 \land b_1\) (Case D) |
| \((a_0 \land d_0) \land (b_1 \land f_1 \land e_1)\) | \(d_0 \land b_1\) (Case D) |
| \((a_0 \land d_0) \land (b_1 \land f_1 \land c_1 \land e_1)\) | \(d_0 \land b_1\) (Case D) |

Case 12: The necessary condition for \( m_q(\lambda, \mu) = S \) is given by

\[
(a_1 \lor b_1) \land (c_1 \lor b_1) \land (a_1 \lor d_1) \land (c_0 \land e_0) \land (f_1 \lor d_1).
\]

Since the logical statement must hold true and it contains \( c_0 \land e_0 \), it must be that \((a_1 \lor b_1) \land (c_1 \lor b_1) \land (a_1 \lor d_1) \land (f_1 \lor d_1) \rightarrow a_1 \land b_1 \land d_1 \land f_1, a_1 \land b_1 \land d_1, a_1 \land b_1 \land f_1, b_1 \land d_1 \land f_1, \) or \(b_1 \land d_1 \). Otherwise, it would contain the contradiction \( c_0 \land c_1 \). Thus, there are five possible ways in which the necessary condition for this case can be true. We list these possibilities and describe a contradiction arising from each possibility.

| Possible Logical Conditions | Contradiction |
|-----------------------------|--------------|
| \((c_0 \land e_0) \land (a_1 \land b_1 \land d_1 \land f_1)\) | \(e_0 \land d_1\) (Case A) |
| \((c_0 \land e_0) \land (a_1 \land b_1 \land d_1)\) | \(e_0 \land d_1\) (Case A) |
| \((c_0 \land e_0) \land (a_1 \land b_1 \land f_1)\) | \(c_0 \land a_1\) (Case C) |
| \((c_0 \land e_0) \land (b_1 \land d_1 \land f_1)\) | \(e_0 \land d_1\) (Case A) |
| \((c_0 \land e_0) \land (b_1 \land d_1)\) | \(e_0 \land d_1\) (Case A) |

Case 13: The necessary condition for \( m_q(\lambda, \mu) = T \) is given by

\[
(a_1 \lor b_1) \land (c_1 \lor b_1) \land (a_1 \lor d_1) \land (c_1 \lor e_1) \land (f_0 \land d_0).
\]
Since the logical statement must hold true and it contains \(d_0 \land f_0\), it must be that \((a_1 \lor b_1) \land (c_1 \lor d_1) \land (a_1 \lor d_1)\) reduces to \(a_1 \land b_1 \land c_1 \land e_1\), \(a_1 \land b_1 \land e_1\), \(a_1 \land c_1 \land e_1\), or \(a_1 \land c_1\). Otherwise, it would contain the contradiction \(d_0 \land d_1\). Thus, there are five possible ways in which the necessary condition for this case can be true. We list these possibilities and describe a contradiction arising from each possibility.

| Possible Logical Conditions | Contradiction |
|-----------------------------|---------------|
| \((d_0 \land f_0) \land (a_1 \land b_1 \land c_1 \land e_1)\) | \(f_0 \land c_1\) (Case B) |
| \((d_0 \land f_0) \land (a_1 \land b_1 \land c_1)\) | \(f_0 \land c_1\) (Case B) |
| \((d_0 \land f_0) \land (a_1 \land b_1 \land e_1)\) | \(d_0 \land b_1\) (Case D) |
| \((d_0 \land f_0) \land (a_1 \land c_1 \land e_1)\) | \(f_0 \land c_1\) (Case B) |
| \((d_0 \land f_0) \land (a_1 \land c_1)\) | \(f_0 \land c_1\) (Case B) |

With the proof of Lemma 4.2 concluded, we are now prepared to give the proof of our main result.

**Proof of Theorem 1.1.** Note that after applying Lemma 4.1 and Lemma 4.2 it suffices to demonstrate the existence of the remaining eight cases that are listed in the statement of Theorem 1.1. Table 5 provides examples of these cases.

| Evaluations | Necessary Conditions | \((m, n, x, y)\) | \((a, b, c, d, e, f)\) |
|-------------|----------------------|----------------|-------------------|
| \(P - Q - R + S + T\) | \(a_0 \land b_0 \land c_0 \land d_0 \land e_0 \land f_0\) | \((5, 6, 0, 0)\) | \((28, 17, 22, 10, 4, 1)\) |
| \(P - Q - R + S\) | \(a_0 \land b_0 \land c_0 \land d_0 \land e_0 \land f_1\) | \((0, 4, 0, 0)\) | \((12, 8, 11, 3, 2, -4)\) |
| \(P - Q - R + T\) | \(a_0 \land b_0 \land c_0 \land d_0 \land e_1 \land f_0\) | \((5, 0, 0, 0)\) | \((10, 4, 4, -2, 1)\) |
| \(P - Q - R\) | \(a_0 \land b_0 \land c_0 \land d_0 \land e_1 \land f_1\) | \((5, 4, 0, 4)\) | \((10, 5, 4, 0, -6, -11)\) |
| \(P - Q\) | \(a_0 \land b_0 \land c_0 \land d_1 \land e_1 \land f_1\) | \((0, 50, 51, 0)\) | \((48, 49, 47, -2, -3, -106)\) |
| \(P - R\) | \(a_0 \land b_0 \land c_1 \land d_0 \land e_1 \land f_1\) | \((2, 0, 1, 0)\) | \((2, 1, -1, 0, -3, -4)\) |
| \(P\) | \(a_0 \land b_0 \land c_1 \land d_1 \land e_1 \land f_1\) | \((0, 0, 0, 0)\) | \((0, 0, -1, -1, -2, -4)\) |
| \(0\) | \((a_1 \lor b_1) \land (c_1 \lor b_1) \land (a_1 \lor d_1) \land (f_1 \lor d_1) \land (c_1 \lor e_1)\) | \((0, 0, 8, 0)\) | \((-16, -8, -17, -9, -10, -20)\) |

**Table 5. Examples establishing the existence of certain \(q\)-multiplicity formulas.**

With the existence of these evaluations established, we now show that each evaluation implies the corresponding statement given in Theorem 1.1. We first establish additional statements that give rise to contradictions. Our methods are similar to those employed in the proof of Lemma 4.2.

**Case E:** Assume the statement \(a_0 \land f_0 \land d_1\) holds true. We observe that if \(d = m + n - x - 2y - 1 < 0\), then \(2m + 2n - 4y - 2 < 2x\). Also, if \(f = m - 2x - 3y - 4 \geq 0\), then \(-m - 3y - 4 \geq 2x\). Finally, if \(a = 2m + 3n - 2x - 3y \geq 0\), then \(2m + 3n - 3y \geq 2x \geq 0\), implying that \(2m + 3n \geq 3y\). We join the first two inequalities to obtain \(3y > 6n + 3m + 6\). We then join the inequality just obtained and the third inequality to see that \(-6 > 3n + m\). This is impossible since \(n, m\) are non-negative, so such a system has no solution.

**Case F:** Assume the statement \(c_0 \land c_1\) holds true. We observe that if \(c = m + 3n - 2x - 3y - 1 < 0\), then \(m + 3n - 3y - 1 < 2x\). Also, if \(e = 2n - 2x - 4y - 4 \geq 0\), then \(2n - 4y - 4 \geq 2x\). We join these two inequalities to obtain \(2n - 4y - 4 > m + 3n - 3y - 1\). If we solve for \(m\) explicitly, we obtain that \(-n - y - 3 > m\), implying that such a system has no solutions.

Utilizing these cases, we consider each \(q\)-multiplicity listed in Theorem 1.1.
Case I: The necessary condition for \( m_q(\lambda, \mu) = P - Q - R + S + T \) is given by
\[
(a_0 \land b_0) \land (c_0 \land b_0) \land (a_0 \land d_0) \land (c_0 \land e_0) \land (f_0 \land d_0).
\]
This reduces to
\[
a_0 \land b_0 \land c_0 \land d_0 \land e_0 \land f_0,
\]
and so \( m_q(\lambda, \mu) = P - Q - R + S + T \) implies \( a_0 \land b_0 \land c_0 \land d_0 \land e_0 \land f_0 \).

Case II: The necessary condition for \( m_q(\lambda, \mu) = P - Q - R + S \) is given by
\[
(a_0 \land b_0) \land (c_0 \land b_0) \land (a_0 \land d_0) \land (c_0 \land e_0) \land (f_1 \lor d_1).
\]
Since the logical statement must hold true and it contains \( a_0 \land b_0 \land c_0 \land d_0 \land e_0 \), it must be that \( (f_1 \lor d_1) \) reduces to \( f_1 \). Otherwise, it would contain the contradiction \( d_0 \land d_1 \). Thus, there is only one possible way for the necessary condition for this case to be true. Therefore, \( m_q(\lambda, \mu) = P - Q - R + S \) implies \( a_0 \land b_0 \land c_0 \land d_0 \land e_0 \land f_1 \).

Case III: The necessary condition for \( m_q(\lambda, \mu) = P - Q - R + T \) is given by
\[
(a_0 \land b_0) \land (c_0 \land b_0) \land (a_0 \land d_0) \land (c_1 \lor e_1) \land (f_0 \land d_0).
\]
Since the logical statement must hold true and it contains \( a_0 \land b_0 \land c_0 \land d_0 \land f_0 \), it must be that \( (c_1 \lor e_1) \) reduces to \( e_1 \). Otherwise, it would contain the contradiction \( c_0 \land c_1 \). Thus, there is only one possible way for the necessary condition for this case to be true. Therefore, \( m_q(\lambda, \mu) = P - Q - R + T \) implies \( a_0 \land b_0 \land c_0 \land d_0 \land e_1 \land f_1 \).

Case IV: The necessary condition for \( m_q(\lambda, \mu) = P - Q - R \) is given by
\[
(a_0 \land b_0) \land (c_0 \land b_0) \land (a_0 \land d_0) \land (c_1 \lor e_1) \land (f_1 \lor d_1).
\]
Since the logical statement must hold true and it contains \( a_0 \land b_0 \land c_0 \land d_0 \), it must be that \( (c_1 \lor e_1) \lor (f_1 \lor d_1) \) reduces to \( e_1 \lor f_1 \). Otherwise, it would contain the contradiction \( d_0 \lor d_1 \). Thus, there is only one possible way for the necessary condition for this case to be true. Therefore, \( m_q(\lambda, \mu) = P - Q - R \) implies \( a_0 \land b_0 \land c_0 \land d_0 \land e_1 \land f_1 \).

Case V: The necessary condition for \( m_q(\lambda, \mu) = P - Q \) is given by
\[
(a_0 \land b_0) \land (c_0 \land b_0) \land (a_1 \lor d_1) \land (c_1 \lor e_1) \land (f_0 \land d_0).
\]
Since the logical statement must hold true and it contains \( a_0 \land b_0 \land c_0 \land a_1 \), it must be that \( (a_1 \lor d_1) \lor (c_1 \lor e_1) \lor (f_1 \lor d_1) \) reduces to \( d_0 \lor d_1 \). Thus, there are two possible ways in which the necessary condition for this case can be true. However, if we consider the statement \( a_0 \land b_0 \land c_0 \land d_1 \lor e_1 \land f_0 \), it contains the statement \( a_0 \land f_0 \lor d_1 \), a contradiction given by Case E. Therefore, \( m_q(\lambda, \mu) = P - Q \) implies \( a_0 \land b_0 \land c_0 \land d_0 \land e_1 \land f_1 \).

Case VI: The necessary condition for \( m_q(\lambda, \mu) = P - R \) is given by
\[
(a_0 \land b_0) \land (c_1 \lor b_1) \land (a_0 \land d_0) \land (c_1 \lor e_1) \land (f_1 \lor d_1).
\]
Since the logical statement must hold true and it contains \( a_0 \land b_0 \land d_0 \), it must be that \( (c_1 \lor b_1) \lor (c_1 \lor e_1) \lor (f_1 \lor d_1) \) reduces to \( c_1 \land f_1 \) or \( c_1 \land e_1 \land f_1 \). Otherwise, it would contain the contradiction \( b_0 \land b_1 \) or \( d_0 \land d_1 \). Thus, there are two possible ways in which the necessary condition for this case can be true. However, if we consider the statement \( a_0 \land b_0 \land c_1 \land d_0 \land e_0 \land f_1 \), it contains the statement \( e_0 \land c_1 \), a contradiction given by Case F. Therefore, \( m_q(\lambda, \mu) = P - R \) implies \( a_0 \land b_0 \land c_1 \land d_0 \land e_1 \land f_1 \).

Case VII: The necessary condition for \( m_q(\lambda, \mu) = P \) is given by
\[
(a_0 \land b_0) \land (c_1 \lor b_1) \land (a_1 \lor d_1) \land (c_1 \lor e_1) \land (f_1 \lor d_1).
\]
Since the logical statement must hold true and it contains \( a_0 \land b_0 \), it must be that \( (c_1 \lor b_1) \lor (a_1 \lor d_1) \lor (c_1 \lor e_1) \lor (f_1 \lor d_1) \) reduces to \( c_1 \lor d_1 \lor e_1 \lor f_1 \). Otherwise, it would contain the contradiction \( a_0 \lor a_1 \) or \( b_0 \lor b_1 \). Thus, there are four
possible ways in which the necessary condition for this case can be true. We list three of
these possibilities and describe a contradiction arising from each possibility.

| Possible Logical Conditions | Contradiction |
|-----------------------------|--------------|
| \((a_0 \land b_0 \land e_0 \land f_0) \land (c_1 \land d_1)\) | \(a_0 \land d_1 \land f_0\) (Case E) |
| \((a_0 \land b_0 \land f_0) \land (c_1 \land d_1 \land e_1)\) | \(a_0 \land d_1 \land f_0\) (Case E) |
| \((a_0 \land b_0 \land e_0) \land (c_1 \land d_1 \land f_1)\) | \(e_0 \land c_1\) (Case F) |

Therefore, \(m_q(\lambda, \mu) = P\) implies \(a_0 \land b_0 \land c_1 \land d_1 \land e_1 \land f_1\).

Case VIII: Thus, we are left with the final case in which \(m_q(\lambda, \mu) = 0\). \(\square\)

We now present some examples of computing weight \(q\)-multiplicities using our formulas.

**Example 1.** If \(\lambda\) is the highest root of \(\mathfrak{g}_2\), i.e. \(\lambda = 3\alpha_1 + 2\alpha_2 = \varpi_2\), and \(\mu = 0\), then by Theorem 1.1 we have that \(m = x = y = 0\) and \(n = 1\) and, hence, \(a = 3, b = c = 2, d = 0, e = -1,\) and \(f = -4\). This implies that

\[
m_q(\lambda, \mu) = \varphi_q(3\alpha_1 + 2\alpha_2) - \varphi_q(2\alpha_1 + 2\alpha_2) - \varphi_q(3\alpha_1).\]

By Proposition 1.1 we note that

\[
\varphi_q(3\alpha_1 + 2\alpha_2) = q(1 + 2q + 2q^2 + q^3 + q^4),
\]

\[
\varphi_q(2\alpha_1 + 2\alpha_2) = q^2(2 + q + q^2),
\]

and \(\varphi_q(3\alpha_1) = q^3\).

Therefore \(m_q(\lambda, \mu) = q + q^5\), which recovers a known result of Lusztig which shows that \(m_q(\lambda, 0) = \sum_{i=1}^r q^{e_i}\), where \(\lambda\) is the highest root and \(e_1, \ldots, e_r\) are the exponents of the corresponding simple Lie algebra of rank \(r\) \([10]\). In addition, note that \(m(\lambda, \mu) = 2\).

**Example 2.** If \(\lambda = 3\varpi_2\) and \(\mu = \varpi_1 + 2\varpi_2\), then by Theorem 1.1 we have that \(m = 0, n = 3,\) \(x = 1, y = 2\) and, hence, \(a = 1, b = c = 0, d = -3, e = -4,\) and \(f = -12\). This implies that

\[
m_q(\lambda, \mu) = \varphi_q(\alpha_1 + \alpha_2) - \varphi_q(\alpha_2).\]

By Proposition 1.1 we note that

\[
\varphi_q(\alpha_1 + \alpha_2) = q(1 + q) \quad \text{and} \quad \varphi_q(\alpha_2) = q.
\]

Therefore \(m_q(\lambda, \mu) = q^2\) and \(m(\lambda, \mu) = 1\). This recovers a special case of \([13]\, \text{Theorem 6}\).

We recall the following formulas for the value of Kostant’s partition function for the exceptional Lie algebra \(\mathfrak{g}_2\) given by Tarski.

**Lemma 4.3** (Tarski p. 9-10 \([17]\)). Let \(m, n \in \mathbb{N}\).

(1) If \(m \leq n\), then \(\varphi(m\alpha_1 + n\alpha_2) = g(m)\)

(2) If \(n \leq m \leq \frac{3}{2}n\), then \(\varphi(m\alpha_1 + n\alpha_2) = g(m) - h(m - n - 1)\)

(3) If \(\frac{3}{2}n \leq m \leq 2n\), then \(\varphi(m\alpha_1 + n\alpha_2) = h(n) - g(3n - m - 1) + h(2n - m - 2)\)

(4) If \(2n \leq m \leq 3n\), then \(\varphi(m\alpha_1 + n\alpha_2) = h(m) - g(3n - m - 1)\)

(5) If \(3n \leq m\), then \(\varphi(m\alpha_1 + n\alpha_2) = h(n)\)

where for \(k \geq -2\),

\[
g(k) = \begin{cases} 
\frac{1}{32}(k + 6)(k^3 + 14k^2 + 54k + 72) & \text{for } k \equiv 0 \mod 6 \\
\frac{1}{32}(k + 5)^2(k^2 + 10k + 13) & \text{for } k \equiv 1 \mod 6 \\
\frac{1}{32}(k + 4)(k^3 + 16k^2 + 74k + 68) & \text{for } k \equiv 2 \mod 6 \\
\frac{1}{32}(k + 3)^2(k + 5)(k + 9) & \text{for } k \equiv 3 \mod 6 \\
\frac{1}{32}(k + 2)(k + 8)(k^2 + 10k + 22) & \text{for } k \equiv 4 \mod 6 \\
\frac{1}{32}(k + 1)(k + 5)(k + 7)^2 & \text{for } k \equiv 5 \mod 6 
\end{cases}
\]

and

\[
h(k) = \begin{cases} 
\frac{1}{18}(k + 2)(k + 4)(k^2 + 6k + 6) & \text{for } k \text{ even} \\
\frac{1}{18}(k + 1)(k + 3)^2(k + 5) & \text{for } k \text{ odd.}
\end{cases}
\]
We remark that one could instead use Lemma 4.3 along with Theorem 1.1 to compute weight multiplicities rather than setting \( q = 1 \) in Proposition 1.1 as we did in the above examples. We provide the details of these computations using our previous examples.

**Example 3.** Following Example 1 we let \( \lambda = 3\alpha_1 + 2\alpha_2 = \varpi_2, \mu = 0 \), and by Theorem 1.1 we know \( m(\lambda, \mu) = \varphi(3\alpha_1 + 2\alpha_2) - \varphi(2\alpha_1 + 2\alpha_2) - \varphi(3\alpha_1) \). Using Lemma 4.3 parts (b), (a), and (e), respectively, we note that

\[
\varphi(3\alpha_1 + 2\alpha_2) = g(3) - h(3 - 2) = \frac{1}{432} (6)^2 (8)(12) - \frac{1}{48} (2)(4)(6) = 7,
\]

\[
\varphi_q(2\alpha_1 + 2\alpha_2) = g(2) = \frac{1}{432} (6)(2^3 + 16(2)^2 + 74(2) + 68) = 4, \quad \text{and}
\]

\[
\varphi_q(3\alpha_1) = h(0) = 1.
\]

Therefore, \( m(\lambda, \mu) = 7 - 4 - 1 = 2 \), as previously computed.

**Example 4.** Following Example 2 we let \( \lambda = 3\varpi_2, \mu = \varpi_1 + 2\varpi_2 \), and by Theorem 1.1 we know \( m(\lambda, \mu) = \varphi(\alpha_1 + \alpha_2) - \varphi(\alpha_2) \). Using Lemma 4.3 part (a) we note that

\[
\varphi(\alpha_1 + \alpha_2) = g(1) = \frac{1}{432} (6)^2 (1^2 + 10(1) + 13) = 2 \quad \text{and} \quad \varphi_q(\alpha_2) = g(0) = \frac{1}{432} (6)(72) = 1.
\]

Therefore, \( m(\lambda, \mu) = 2 - 1 = 1 \), as previously computed.

5. Revision of the \( q \)-analog of Kostant’s weight multiplicity for \( \mathfrak{sp}_4(\mathbb{C}) \)

Harris and Lauber considered the Lie algebra \( \mathfrak{sp}_4(\mathbb{C}) \) and gave a closed formula for the \( q \)-multiplicity formula. However, their partition function formula omitted an edge case, which resulted in a missing case in their work. The formula for \( \varphi_q(m\alpha_1 + n\alpha_2) \) given in [10, Proposition 1.2] is correct, and we restate it here

\[
\varphi_q(m\alpha_1 + n\alpha_2) = \sum_{i=0}^{\min\left(\left\lfloor \frac{m}{2} \right\rfloor , n\right)} \sum_{j=\max\left(\left(\frac{m}{2}\right) - n, 0\right)}^{m+n-2i} q^j,
\]

where \( m \) and \( n \) are integers, \( \alpha_1 \) and \( \alpha_2 \) are the simple roots, and \( \Phi^+ = \{ \alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2 \} \) are the positive roots of the Lie algebra \( \mathfrak{sp}_4(\mathbb{C}) \). The mistake occurs in Corollary 3.3 of [10]. We provide the corrected statement and its proof below.

**Corollary 5.1** (Corrected Corollary 3.3 [10]). If \( g = \mathfrak{sp}_4(\mathbb{C}) \) and \( m, n \in \mathbb{N} \), then

\[
\varphi(m\alpha_1 + n\alpha_2) = \begin{cases} 
\left( \left\lfloor \frac{m}{2} \right\rfloor + 1 \right) \left( m - \left\lfloor \frac{m}{2} \right\rfloor + 1 \right) & \text{if } n \geq m \\
\frac{2mn-m^2-m^2+m+n}{2} + \left\lfloor \frac{m}{2} \right\rfloor \left( m - \left\lfloor \frac{m}{2} \right\rfloor \right) + 1 & \text{if } 2n - 1 > m > n \\
\left( m + 1 \right) \left( n - \frac{1}{2} \left\lfloor \frac{m}{2} \right\rfloor + 1 \right) & \text{if } 2n > m \geq 2n - 1 > n \\
\frac{1}{2} n(n+2) & \text{if } m \geq 2n.
\end{cases}
\]

**Proof.** Setting \( q = 1 \) into equation (4) we find that

\[
\varphi(m\alpha_1 + n\alpha_2) = \sum_{i=0}^{\min\left(\left\lfloor \frac{m}{2} \right\rfloor , n\right)} \min(m-i, n) - \frac{1}{2} \min\left(\left\lfloor \frac{m}{2} \right\rfloor , n\right) \left( \min\left(\left\lfloor \frac{m}{2} \right\rfloor , n\right) + 1 \right) + \min\left(\left\lfloor \frac{m}{2} \right\rfloor , n\right) + 1.
\]

(17)
We now consider each case individually. If \( n \geq m \), then equation (17) simplifies to
\[
\left( \left\lfloor \frac{m}{2} \right\rfloor + 1 \right) \left( m - \left\lfloor \frac{m}{2} \right\rfloor + 1 \right).
\]
If \( m \geq 2n \), then equation (17) simplifies to \( (n+1)(n+2) \). If \( 2n - 1 > m > n \), then equation (17) yields
\[
(18) \quad \left( \sum_{i=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \min(m - i, n) \right) - \frac{\left\lfloor \frac{m}{2} \right\rfloor (\left\lfloor \frac{m}{2} \right\rfloor + 1)}{2} + \frac{\left\lfloor \frac{m}{2} \right\rfloor + 1}{2}.
\]
Let us consider the first term of expression (18). Since \( c = \left\lfloor \frac{m}{2} \right\rfloor \), we have that \( \min(m - i, n) \leq m - i \) and hence \( \min(m - i, n) = n \). If \( i > m - n \), then \( n > m - i \) and hence \( \min(m - i, n) = m - i \). Thus,
\[
(19) \quad \sum_{i=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \min(m - i, n) = \frac{2mn - m^2 - n^2 + m + n}{2} + m \frac{\left\lfloor \frac{m}{2} \right\rfloor - \left\lfloor \frac{m}{2} \right\rfloor + 1}{2}.
\]
Substituting equation (19) into equation (18) yields the desired result. If \( 2n - 1 > m > n \), then \( 2n - m \leq 1 \) which implies that \( \left\lfloor \frac{m - 2n + 2n}{2} \right\rfloor = \left\lfloor \frac{m}{2} \right\rfloor - m + n \leq 0 \), so \( \left\lfloor \frac{m}{2} \right\rfloor \leq m - n \). Thus, for all \( i \) it holds that \( m - n \geq i \), implying that \( m - i \geq n \) and we obtain that
\[
(20) \quad \sum_{i=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \min(m - i, n) = n \left( \left\lfloor \frac{m}{2} \right\rfloor + 1 \right).
\]
Substituting equation (20) into equation (18) yields the desired result. \( \square \)

As a consequence of this correction to Corollary 3.3 of [10], the following result replaces Corollary 4.1 in [10].

**Corollary 5.2** (Corrected Corollary 4.1 [10]). Let \( \lambda = m\varpi_1 + n\varpi_2 \) and \( \mu = x\varpi_1 + y\varpi_2 \) with \( m, n, x, y \in \mathbb{N} := \{0, 1, 2, \ldots\} \) be weights of \( \mathfrak{sp}_4(\mathbb{C}) \) and define \( a = m + n - x - y \), \( b = n - y + \frac{m-x}{2} \), \( c = n - x - y - 1 \), and \( d = -y - 1 + \frac{m-x}{2} \). Then
\[
(21) \quad m(\lambda, \mu) = \begin{cases} 
P - Q - R & \text{if } a, b, c, d \in \mathbb{N} \\
P - Q & \text{if } a, b, c \in \mathbb{N} \text{ and } d \notin \mathbb{N} \\
P - R & \text{if } a, b, d \in \mathbb{N} \text{ and } c \notin \mathbb{N} \\
P & \text{if } a, b \in \mathbb{N} \text{ and } c, d \notin \mathbb{N} \\
0 & \text{otherwise}
\end{cases}
\]
where
\[
P = \begin{cases} 
\left( \left\lfloor \frac{a}{2} \right\rfloor + 1 \right) \left( a - \left\lfloor \frac{a}{2} \right\rfloor + 1 \right) & \text{if } b \geq a \\
\frac{2ab - a^2 - b^2 + a + b}{2} + \left( \left\lfloor \frac{a}{2} \right\rfloor + 1 \right) \left( a - \left\lfloor \frac{a}{2} \right\rfloor \right) + 1 & \text{if } 2b - 1 > a > b \\
\left( \left\lfloor \frac{b}{2} \right\rfloor + 1 \right) \left( b - \left\lfloor \frac{b}{2} \right\rfloor + 1 \right) & \text{if } 2b > a \geq 2b - 1 > b,
\end{cases}
\]
\[
Q = \left( \frac{c + 2}{2} \right)^2,
\]
\[
R = \frac{(d + 1)(d + 2)}{2}.
\]
Finding formulas for Kostant’s partition has recently been connected to counting multiplex juggling sequences [1, 7]. These bijections have been considered for all classical Lie algebras, but extending them to the exceptional Lie algebras, such as $\mathfrak{g}_2$, remains an open problem. For a second direction of research, we remark that one could consider giving explicit formulas for the $q$-analog of Kostant’s partition function for $\mathfrak{g}_2$. This would require working through the expansion of Proposition 1.1 using the coefficient constraints given by Tarski in Lemma 4.3. We omitted such a computation because of its tedious and technical nature.

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