THE CIRCULAR SPATIAL RESTRICTED 3-BODY PROBLEM

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ABSTRACT. We show that the hypersurface of the regularized spatial circular restricted three body problem is of contact type whenever the energy level is below the first critical value (energy level of first Lagrange point) or if energy level is slightly above it. A dynamical consequence is in this energy range there is no blue sky catastrophe.

1. Introduction

In this paper we will discuss the restricted 3-body problem. This means a version of the 3-body problem with a massless third object which we call the satellite, and heavy primary which rotate around each other in a circular orbit. We call the primaries Earth and Moon.

If we assume the all objects move in a plane then the Hamiltonian is given by

\[ H(q,p) = \frac{1}{2}|p|^2 - \frac{\mu}{|q-M|} - \frac{1-\mu}{|q-E|} + p_1q_2 - p_2q_1 \]

where \( E = (\mu, 0) \) and \( M = (-1 + \mu, 0) \). This Hamiltonian comes from a time dependent transformation of time dependent Hamiltonian which is given by the sum of the potential and kinetic energy.

In (1.1) \( q \) and \( p \) represent position and momentum of the satellite, respectively, and \( \mu \in [0,1] \) is the mass ratio of Earth and Moon. Especially we call the case \( \mu = 0 \) or \( \mu = 1 \) as rotating Kepler problem. To derive the equation (1.1) we refer to [1].

Albers et.al prove in [4] that a component of a level set \( H^{-1}(c) \) of this planar Hamiltonian which projection to the \( q \)-plane is bounded admits a contact structure for energy level, where \( c \) value are below or slightly above the first critical value (this is critical value of the effective potential). In this paper, we will consider the spatial restricted 3-body problem i.e. the positions of Earth and Moon are given by \( E = (\mu, 0, 0) \) and \( M = (-1 + \mu, 0, 0) \) where the two additional coordinates \( q_3 \) and \( p_3 \) represent the additional coordinates and its momentum. The satellite does not need to remain in the plane anymore, so we call it the spatial restricted 3-body problem. We will show that in the spatial case, a component of hypersurface \( H^{-1}(c) \) which projection to the \( q \)-plane is bounded is of contact type.

The energy hypersurface \( H^{-1}(c) \) is non-compact, because of collisions of the satellite with Moon or Earth. These are singular points of Hamiltonian. To remove these singularities, we will consider the regularization introduced by Moser [12]. He introduced a new Hamiltonian \( K \) which is given by

\[ K = (H-c)|q-q^0|. \]

The new Hamiltonian \( K \) is explicitly given by

\[ K(q,p) = \left( \frac{1}{2}|p|^2 - \frac{\mu}{|q-q^0|} - \frac{1-\mu}{|q-E|} + p_1q_2 - p_2q_1 - c \right)|q-q^0|. \]

This \( K \) is called the regularized Hamiltonian and we will discuss \( K \) in Section in 6 more detail. At \( K = 0 \) the dynamics are a reparametrization of the dynamics of \( H = c \). If the energy value \( c \) is lower than the first critical value (denote this critical value as \( H(L_1) \)), the energy hypersurface consists of three connected components. Two of these have projection to \( q \)-plane is bounded and contain Earth and Moon, respectively, and the other have projection to \( q \)-plane is unbounded. We denote
the bounded component containing the Earth by $\Sigma_c^E$ and the bounded component containing the Moon by $\Sigma_c^M$. Each such bounded component of the energy hypersurface $\Sigma_c^E$ is regularized to $\Sigma_c^E$ by Moser’s method. This regularized hypersurface is diffeomorphic to $ST^*S^3 \approx S^2 \times S^1$ and if $c$ is slightly above the first critical value, the energy hypersurface consists of two connected components. One is bounded and the other is unbounded. Because the bounded component contains the Earth and Moon, we will denote this component by $\Sigma_c^{E,M}$ and its regularization by $\Sigma_c^{E,M}$. The latter is diffeomorphic to $ST^*S^3 \# ST^*S^3$.

**Theorem 1.1.** For every mass ratio $\mu \in (0, 1)$, the bounded components $\Sigma_c^E$ and $\Sigma_c^M$ with $c < H(L_1)$ are of contact type. Furthermore there exists $\epsilon > 0$ such that the bounded component $\Sigma_c^{E,M}$ is also of contact type for $H(L_1) < c < H(L_1) + \epsilon$.

**Remark 1.1.** In the planar case, there are other regularizations of restricted 3-body problem such Levi-Civita regularization. For this see [6].

After regularization we get results about so-called blue sky catastrophes. A blue sky catastrophe is a phenomenon of a 1-parameter following periodic orbits on a compact manifold family whose period goes to infinity as the parameter goes to some finite value. Combined with Theorem 1.1 we have the Corollary.

**Corollary 1.1.** For $\mu \in (0, 1)$, there is no blue sky catastrophe on $\Sigma_c^E$, $\Sigma_c^M$ for $c < H(L_1)$ and $\Sigma_c^{E,M}$ for $H(L_1) < c < H(L_1) + \epsilon$.

We will give proof of Corollary in Section 7.

2. Construction of restricted spatial 3-body problem

First we consider three bodies called Earth, Moon and satellite. The Earth and Moon are denoted by $E$ and $M$. The non-negative real number $m_E$, $m_M$ indicate mass of Earth and mass of Moon respectively. We assume the satellite is massless, so we only care about $\mu$ which the ratio of $m_E$ and $m_M$

$$\mu = \frac{m_M}{m_E + m_M} \in [0, 1].$$

From now on we consider $\mu$ as the mass of Moon and $1 - \mu$ as the mass of Earth. Also we assume that the Earth and Moon rotate according to Newton’s law of gravitation in circles around their center of mass which contained in the plane. The position of Earth and Moon is given by

$$E(t) = (\mu \cos t, \mu \sin t, 0), \quad M(t) = (-(1 - \mu) \cos t, -(1 - \mu) \sin t, 0).$$

Then the time-dependent Hamiltonian of the satellite is $H_t : (\mathbb{R}^3 \setminus \{M(t), E(t)\}) \times \mathbb{R}^3 \rightarrow \mathbb{R}$, given by

$$H_t(q, p) = \frac{1}{2} |p|^2 - \frac{\mu}{|q - M(t)|} - \frac{1 - \mu}{|q - E(t)|}.$$ 

We change to a synodical coordinate system and get an autonomous Hamiltonian. The positions of Earth and Moon are then given by

$$E = (\mu, 0, 0), \quad M = (-(1 - \mu), 0, 0).$$

In the new coordinate system, the Hamiltonian becomes autonomous and is given by $H : (\mathbb{R}^3 \setminus \{M, E\}) \times \mathbb{R}^3 \rightarrow \mathbb{R}$

$$H(q, p) = \frac{1}{2} |p|^2 - \frac{\mu}{|q|} - \frac{1 - \mu}{|q - e|} + p_1 q_2 - p_2 q_1.$$ 

By centering the Moon at the origin, the Hamiltonian is changed as follows

$$H(q, p) = \frac{1}{2} |p|^2 - \frac{\mu}{|q|} - \frac{1 - \mu}{|q - e|} + p_1 q_2 - p_2 (q_1 - 1 + \mu)$$ 

(2.2)
where \( e = (1, 0, 0) \). Also we can rewrite the Hamiltonian \( H \) as

\[
H = \frac{1}{2} ((p_1 + q_2)^2 + (p_2 - q_1 + 1 - \mu)^2 + p_3^2) - \frac{\mu}{|q|} - \frac{1-\mu}{|q-e|} - \frac{1}{2}((q_1 - 1 + \mu)^2 + q_2^2). \tag{2.3}
\]

From this equation, we can define the effective potential \( U \) as

\[
U = -\frac{\mu}{|q|} - \frac{1-\mu}{|q-e|} - \frac{1}{2}((q_1 - 1 + \mu)^2 + q_2^2). \tag{2.4}
\]

The effective potential \( U \) has five critical points \( l_1, l_2, l_3, l_4, l_5 \) which are called Lagrange points. These Lagrange points can be ordered by the critical values of the effective potential \( U \). There are three critical points on axis between the Earth and Moon. And we regard \( l_1 \) as taking minimal value of effective potential in five Lagrange points. Then the first Lagrange point \( l_1 \) lies between Moon and Earth.

We can define a projection \( \pi \) as

\[
\pi : (\mathbb{R}^3 \setminus \{E, M\}) \times \mathbb{R}^3 \to \mathbb{R}^3 \setminus \{E, M\}. \tag{2.5}
\]

There is a one to one correspondence between critical point of effective potential and of the Hamiltonian (2.3).

\[
\pi^{-1} : \text{crit}(U) \to \text{crit}(H) \quad (q_1, q_2, 0) \mapsto (q_1, q_2, 0, -q_2, q_1, 0) \tag{2.6}
\]

We denote a level set by

\[
\Sigma_c := H^{-1}(c) \quad \text{for} \quad c \in \mathbb{R}. \tag{2.7}
\]

Then the Hill’s region is defined as

\[
K_c = \pi(\Sigma_c) \in \mathbb{R}^3 \setminus \{E, M\}. \tag{2.8}
\]

If \( c < H(L_1) \), then \( K_c \) contains three connected components which are one unbounded region and two bounded regions. The closure of the two bounded regions contain the Moon and Earth, respectively. So we denote these two components as \( K_c^M, K_c^E \) such that

\[
M \in K_c^M, \quad E \in K_c^E. \]

We denote the bounded components of the hypersurface \( H^{-1}(c) \) by

\[
\Sigma_c^M = \pi^{-1}(K_c^M) \cap \Sigma_c, \quad \Sigma_c^E = \pi^{-1}(K_c^E) \cap \Sigma_c.
\]

3. Below the first critical level in spatial case

In this section we will consider spherical coordinates

\[
q_1 = \rho \cos \theta \sin \varphi, \quad q_2 = \rho \sin \theta \sin \varphi, \quad q_3 = \rho \cos \varphi.
\]

\[
0 \leq \theta < 2\pi, \quad 0 \leq \varphi \leq \pi.
\]

Since we only consider the case that the energy level is below the first critical value, the radius \( \rho \) must be smaller than the distance from moon to the first Lagrange point which is clearly less than 1.

The effective potential can be rewritten as

\[
U(\rho, \varphi, \theta) = -\frac{\mu}{\rho} - \frac{1-\mu}{\sqrt{\rho^2 - 2\rho \cos \theta \sin \varphi + 1}} - \frac{1}{2}((\rho \cos \theta \sin \varphi - 1 + \mu)^2 + \rho^2 \sin^2 \theta \sin^2 \varphi).
\]

\[
\tag{3.1}
\]
For fixed $\rho$, we will find the minimal value of $U$ on this sphere, which we do by differentiating $U$ with respect to $\theta$.

**Lemma 3.1.** For fixed $\rho$ and $\varphi$, the function $U$ has minimal value on $\theta = 0$ or $\theta = \pi$.

**Proof.** The first derivative of $U$ with respect to $\theta$ is given by

$$\frac{\partial U}{\partial \theta} = (1 - \mu)\rho \sin \theta \sin \varphi \left( \frac{1}{(\rho^2 - 2\rho \cos \theta \sin \varphi + 1)^{3/2}} - 1 \right)$$

The solutions of $\frac{\partial U}{\partial \theta} = 0$ are $\sin \theta = 0$ or $\sin \varphi = 0$ or $\left( \frac{1}{(\rho^2 - 2\rho \cos \theta \sin \varphi + 1)^{3/2}} - 1 \right) = 0$. Define $A = \rho^2 - 2\rho \cos \theta \sin \varphi$. Then

$$\frac{1}{(\rho^2 - 2\rho \cos \theta \sin \varphi + 1)^{3/2}} - 1 = \frac{1 - (A + 1)^3}{(A + 1)^{3/2}(1 + (A + 1)^{3/2})} = -A^3 - 3A^2 - 3A$$

We can show that $A^2 + 3A + 3 \neq 0$, so $A$ must be zero and this means

$$\rho(\rho - 2\cos \theta \sin \varphi) = 0 \iff \rho - 2\cos \theta \sin \varphi = 0.$$

Therefore the solutions of $\frac{\partial U}{\partial \theta} = 0$ are as follows

(3.2) $\rho - 2\cos \theta \sin \varphi = 0$

(3.3) $\theta = 0, \pi$

(3.4) $\varphi = 0, \pi$.

First, we consider the case $\varphi \neq 0, \pi$ (Because $\varphi = 0, \pi$ are North Pole and South Pole respectively. And we will consider (3.4) later.) Then (3.2) can be rewritten as

$$\cos \theta = -\frac{\rho}{2 \sin \varphi}.$$

So there are three cases

(3.6) $\frac{\rho}{2 \sin \varphi} > 1, \quad \frac{\rho}{2 \sin \varphi} = 1, \quad 0 < \frac{\rho}{2 \sin \varphi} < 1.$

This means (3.5) has no real solution in the first case of (3.6). For the second case of (3.6), $\theta = 0$ is solution of (3.5). We will consider the case $\theta = 0$ later. For the third case of (3.6) there are two $\theta$ solutions which are not equal to 0, $\pi$.

For fixed $\varphi$, each domain is a circle which is parallel to the $\{q_3 = 0\}$ plane. So we have to determine whether these critical points are minima, maxima or saddles on each circle. For this, consider the second derivative

$$\frac{\partial^2 U}{\partial \theta^2} = (1 - \mu)\rho \sin \varphi \left( \cos \theta \left( \frac{1}{(\rho^2 - 2\rho \cos \theta \sin \varphi + 1)^{3/2}} - 1 \right) + \sin \theta \left( -\frac{3\rho \sin \theta \sin \varphi}{(\rho^2 - 2\rho \cos \theta \sin \varphi + 1)^{3/2}} \right) \right).$$

**Claim.** The second derivative of the effective potential at $\theta = \pi$ is positive except $\varphi = 0, \pi$. i.e. $\frac{\partial^2 U}{\partial \theta^2}(\rho, \varphi, \pi) > 0$ for all $\rho$ and $\varphi \neq 0, \pi$

**Proof.** The second derivative of effective potential with respect to $\theta$ at $\theta = \pi$ is given by

$$\frac{\partial^2 U}{\partial \theta^2}(\rho, \varphi, \pi) = -(1 - \mu)\rho \sin \varphi \left( \frac{1}{(\rho^2 + 2\rho \sin \varphi + 1)^{3/2}} - 1 \right)$$
and it is clear that \( \rho^2 + 2\rho \sin \varphi + 1 > 1 \), so \( \frac{1}{(\rho^2+2\rho \sin \varphi+1)^{1/2}} - 1 < 0 \) and we get \( \frac{\partial^2 U}{\partial \rho^2}(\rho, \varphi, \pi) > 0 \).

It means \( \theta = \pi \) is a local minimum, for any fixed \( \rho \) and \( \varphi \). Next, we will consider the \( \theta = 0 \) case.

\[
(3.7) \quad \frac{\partial^2 U}{\partial \theta^2}(\rho, \varphi, 0) = (1 - \mu)\rho \sin \varphi \left( \frac{1}{(\rho^2 - 2\rho \sin \varphi + 1)^{3/2}} - 1 \right)
\]

The sign of the second derivative at \( \theta = 0 \) depends on the sign of \( \rho^2 - 2\rho \sin \varphi \) and we know that if \( \frac{\rho}{2 \sin \varphi} > 1 \), the \( 3.3 \) has no solution. The inequality is equivalent to \( \rho^2 - 2\rho \sin \varphi > 0 \). Therefore we get \( \frac{1}{(\rho^2 - 2\rho \sin \varphi + 1)^{1/2}} - 1 < 0 \). This means that there are two critical points \( \theta = 0, \theta = \pi \) and \( \theta = 0 \) is a maximum on a circle and \( \theta = \pi \) is a minimum on each circle. So we can show that

If \( \frac{\rho}{2 \sin \varphi} > 1 \), then \( \theta = 0 \) is a maximum and there are two critical points on each circle.

Also we know that if \( \frac{\rho}{2 \sin \varphi} < 1 \), then \( 3.3 \) has two additional \( \theta \) solutions which are not equal to \( \theta = 0, \pi \). So in this case, there are four critical points and \( \theta = 0 \) is a local minimum on each circle. In the same way we can show that

If \( \frac{\rho}{2 \sin \varphi} < 1 \), then \( \theta = 0 \) is a local minimum and there are four critical points on each circle.

Next consider the case \( \frac{\rho}{2 \sin \varphi} = 1 \). We notice that \( \theta = 0 \) is a solution of \( 3.3 \). This means there are two critical points (\( \theta = 0, \theta = \pi \)) and we already have shown that \( \theta = \pi \) is local minimum for any \( \rho, \varphi \). Also the domain is a circle which is compact without boundary. Therefore \( \theta = 0 \) must be a maximum on a circle (\( \theta = \pi \) is minimum on a circle).

To summarize, for fixed \( \rho \) and \( \varphi \), each circle attains a minimum of \( U \) on \( \theta = 0 \) or \( \theta = \pi \). So the global minimum of \( U \) on each sphere appears on \( \theta = 0 \) or \( \theta = \pi \). We didn’t consider the case \( 3.4 \).

But it is enough if we investigate the critical points of the great circle which contains North Pole and South Pole with \( \theta = 0 \) and \( \theta = \pi \) on each circle.

If we fix the radius of the sphere, the effective potential \( U \) has global minimum on \( \theta = 0 \) or \( \theta = \pi \). To find the global minimum of \( U \) on each sphere, we differentiate \( U \) with respect to \( \varphi \).

**Theorem 3.1.** For fixed \( \rho \in (0, 1) \), the function \( U \) attains its minimum at \( \theta = 0, \varphi = \frac{\pi}{2} \).

**Proof.** Because of Lemma 3.1 we know that \( U \) has a minimum at \( \theta = 0 \) or \( \theta = \pi \). In other words, \( U \) attains its minimum on the great circle which can be parametrized by \( (\rho \cos \phi, 0, \rho \sin \phi) \) where \( \phi \in [0, 2\pi) \). Now we can consider the restriction of \( U \) to the great circle. Then the restriction of \( U \) is given by

\[
U|_{(\rho \cos \phi, 0, \rho \sin \phi)} = -\frac{\mu}{\rho} \frac{1 - \mu}{(\rho^2 - 2\rho \cos \phi + 1)^{1/2}} - \frac{1}{2} (\rho \cos \phi - 1 + \mu)^2.
\]

The derivative of \( U \) with respect to \( \phi \) is given by

\[
\frac{\partial U}{\partial \phi}|_{(\rho \cos \phi, 0, \rho \sin \phi)} = \frac{(1 - \mu)\rho \sin \phi}{(\rho^2 - 2\rho \cos \phi + 1)^{3/2}} + (\rho \cos \phi - 1 + \mu)\rho \sin \phi
= \rho \sin \phi \left( \frac{1 - \mu}{(\rho^2 - 2\rho \cos \phi + 1)^{3/2}} + \rho \cos \phi - 1 + \mu \right).
\]

To find the critical points, we have to look for the solutions of \( \frac{\partial U}{\partial \phi}|_{(\rho \cos \phi, 0, \rho \sin \phi)} = 0 \). We can see that \( \phi = 0, \phi = \pi \) are solutions of the given equation. We will show that there are two more solutions. For this, we use the substitution \( \cos \phi = t \) where \( t \in [-1, 1] \). Then the second term of \( \frac{\partial U}{\partial \phi}|_{(\rho \cos \phi, 0, \rho \sin \phi)} \) can be rewritten as

\[
f(t) = \frac{1 - \mu}{(\rho^2 - 2\rho t + 1)^{3/2}} + \rho t - 1 + \mu.
\]
If we denote this function by \( f(t) \), we can see that
\[
f(-1) = \frac{1 - \mu}{(\rho^2 + 2\rho + 1)^{3/2}} - \rho - (1 - \mu) < 0
\]
\[
f(1) = \frac{1 - \mu}{(\rho^2 - 2\rho + 1)^{3/2}} + \rho - (1 - \mu) > 0
\]
and \( f'(t) \) is given by
\[
f'(t) = \frac{3\rho(1 - \mu)}{(\rho^2 - 2\rho t + 1)^{5/2}} + \rho > 0
\]
This means there is exactly one solution on \( t \in (-1, 1) \) where \( t = \cos \phi \), so there are four critical points on this great circle. The solutions of the second term are not equal to \( \phi = 0 \) or \( \phi = \pi \). This means \( \frac{\partial U}{\partial \phi} \) have four distinct solutions.

Next we will determine whether given critical points are a minimum, a maximum or a saddle. To do this, consider the second derivative.
\[
\frac{\partial^2 U}{\partial \phi^2} = \rho \cos \phi \left( \frac{1 - \mu}{(\rho^2 - 2\rho \cos \phi + 1)^{3/2}} + \rho \cos \phi - 1 + \mu \right) + g(\phi)\rho \sin \phi
\]

Here \( g(\phi) \) represents the derivative of second term of \( \frac{\partial U}{\partial \phi} \). Then we can see the type of the critical points at \( \phi = 0 \) and \( \phi = \pi \).

\[
\left. \frac{\partial^2 U}{\partial \phi^2} \right|_{\phi=0} = \rho \left( \frac{1 - \mu}{(\rho^2 - 2\rho + 1)^{3/2}} + \rho - 1 + \mu \right) > 0
\]
\[
\left. \frac{\partial^2 U}{\partial \phi^2} \right|_{\phi=\pi} = -\rho \left( \frac{1 - \mu}{(\rho^2 + 2\rho + 1)^{3/2}} - \rho - 1 + \mu \right) > 0
\]
This means the critical points \( \phi = 0 \) and \( \phi = \pi \) are both local minima, and because of compactness of \( S^1 \), the other two critical points are local maxima. Therefore, \( U \) attains its global minimum at \( \phi = 0 \) or \( \phi = \pi \) and because of the Lemma 5.2. of [4], we see that the global minimum is attained at \( \phi = 0 \). In spherical coordinates it can be written as
\[
\varphi = \frac{\pi}{2}, \theta = 0
\]

4. Transversality

In this section we will prove the main theorem. We have to compute the derivative of \( H \) given by (2.2). The exterior derivative of \( H \) is given by
\[
(4.8) \quad dH = p\, dp + \frac{\mu}{|q|^2} q\, dq + \frac{1 - \mu}{|q - 1|^3} (q - 1)\, dq + p_1\, dq_2 + q_2\, dp_1 - p_2\, dq_1 - (q_1 - 1 + \mu)\, dp_2.
\]

If we put the Moon at the origin, the Liouville vector field is given by
\[
X = q \frac{\partial}{\partial q}.
\]

Insert the Liouville vector field into (4.8), then \( dH(X) \) is given by
\[
(4.9) \quad dH(X) = X(H) = \frac{\mu}{|q|} + (1 - \mu)\frac{q \cdot (q - 1)}{|q - 1|^3} + p_1 q_2 - p_2 q_1.
\]

**Theorem 4.1.** The Liouville vector field \( X \) intersects \( \Sigma_c^M \) transversely. i.e \( X(H)|_{\Sigma_c^M} \) is positive for \( c < H(L^1) \).
To prove this theorem, differentiate the effective potential \( U \) with respect to \( \rho \),

\[
\frac{\partial U}{\partial \rho} = \frac{\mu}{\rho^2} + \frac{(1 - \mu)(\rho - \cos \theta \sin \varphi)}{(\rho^2 - 2\rho \cos \theta \sin \varphi + 1)^{3/2}} - \rho \cos^2 \theta \sin^2 \varphi + \cos \theta \sin \varphi(1 - \mu) - \rho \sin^2 \theta \sin^2 \varphi.
\]

Let \( d := |M - l^1| \), where \( M \) is the position of the moon and \( l^1 \) is the first Lagrange point. Also define the ball \( B \) as

\[
B = \{ q \in \mathbb{R}^3 : |q - M| \leq d \}
\]

then we get the Lemma.

**Lemma 4.1.** The derivative of \( U \) with respect to \( \rho \) is positive if \( q \in B \setminus \{ M, l^1 \} \).

**Proof.** There always exists \( \theta' \) such that

\[
\cos \theta' = \cos \theta \sin \varphi,
\]

so (4.10) reduces to

\[
\frac{\partial U}{\partial \rho} = \frac{\mu}{\rho^2} + \frac{(1 - \mu)(\rho - \cos \theta')}{(\rho^2 - 2\rho \cos \theta' + 1)^{3/2}} + \cos \theta'(1 - \mu) - \rho \sin^2 \varphi.
\]

By Lemma 5.4 of [4], we can show that this term is positive. \( \square \)

The second derivative of \( U \) with respect to \( \rho \) is given by

\[
\frac{\partial^2 U}{\partial \rho^2} = -\frac{2\mu}{\rho^3} - \frac{(1 - \mu)(2\rho^2 - 4\rho \cos \theta \sin \varphi + 3 \cos^2 \varphi - 1)}{(\rho^2 - 2\rho \cos \theta \sin \varphi + 1)^{5/2}} - \sin^2 \varphi.
\]

Now we can apply equation (4.11) to (4.12). Then the second derivative simplified as follows

\[
\frac{\partial^2 U}{\partial \rho^2} = -\frac{2\mu}{\rho^3} - \frac{(1 - \mu)(2\rho^2 - 4\rho \cos \theta' + 3 \cos^2 \theta' - 1)}{(\rho^2 - 2\rho \cos \theta' + 1)^{5/2}} - \sin^2 \varphi.
\]

We can prove the following lemma.

**Lemma 4.2.** For every \( q \in B \setminus \{ M \} \), it holds that \( \frac{\partial^2 U}{\partial \rho^2} \leq -\sin^2 \varphi \).

**Proof.** Define the function

\[
W(\rho, \theta') := -\frac{2\mu}{\rho^3} - \frac{(1 - \mu)(2\rho^2 - 4\rho \cos \theta' + 3 \cos^2 \theta' - 1)}{(\rho^2 - 2\rho \cos \theta' + 1)^{5/2}}.
\]

Then \( W \) is non-positive by Lemma 5.5 of [4] and by (4.13). \( \square \)

Now we can prove Theorem 4.1 by using Lemma 4.1 and Lemma 4.2.

**Proof of Theorem 4.1.** We change to spherical coordinates with the Liouville vector field \( X = \rho \frac{\partial}{\partial \rho} \).

We now compute

\[
X(H) = \rho \frac{\partial U}{\partial \rho} + \rho \sin \theta \sin \varphi(p_1 + \rho \sin \theta \sin \varphi) - \rho \cos \theta \sin \varphi(p_2 - \rho \cos \theta \sin \varphi + 1 - \mu)
\]

\[
\geq \rho \frac{\partial U}{\partial \rho} - \rho \sin \varphi \sqrt{(p_1 + \rho \sin \theta \sin \varphi)^2 + (p_2 - \rho \cos \theta \sin \varphi + 1 - \mu)^2}
\]

\[
= \rho \frac{\partial U}{\partial \rho} - \rho \sin \varphi \sqrt{2(H - U) - p_1^2}
\]

\[
\geq \rho \frac{\partial U}{\partial \rho} - \rho \sin \varphi \sqrt{2(H - U)}.
\]

In the second step, we have used the Cauchy-Schwarz inequality.
Since \((q, p) \in \Sigma^M_c\), we have \(H(q, p) = c\). Also \(U\) depends only \(q\), so it is enough to show the following

\[
\rho \left( \frac{\partial U}{\partial \rho} - \sin \varphi \sqrt{2(c - U)} \right) |_{\kappa^M_c} > 0.
\]

For this we consider the following inequality.

\[
U(d, \theta, \varphi) \geq U(d, 0, \frac{\pi}{2}) = U(1^1) = H(L^1) > c.
\]

The first inequality of (4.15) is satisfied by Theorem 3.1, also, \(U(q) = U(\rho, \varphi, \theta) \leq c\), so there exists \(\tau \in [0, d - \rho]\) such that \(U(\rho + \tau, \varphi, \theta) = c\). Hence we get following result

\[
\begin{align*}
\therefore \left( \frac{\partial U(\rho, \varphi, \theta)}{\partial \rho} \right)^2 &= \left( \frac{\partial U(\rho + \tau, \varphi, \theta)}{\partial \rho} \right)^2 - \int_0^\tau \frac{d}{dt} \left( \frac{\partial U(\rho + \tau, \varphi, \theta)}{\partial \rho} \right)^2 dt \\
&> -2 \int_0^\tau \frac{\partial U(\rho + \tau, \varphi, \theta)}{\partial \rho} \frac{\partial^2 U(\rho + \tau, \varphi, \theta)}{\partial \rho^2} dt \\
&\geq 2 \int_0^\tau \sin^2 \varphi \frac{\partial U(\rho + \tau, \varphi, \theta)}{\partial \rho} dt \\
&= 2 \sin^2 \varphi (c - U(\rho, \varphi, \theta)).
\end{align*}
\]

In this inequality, we use Lemma 4.1 and Lemma 4.2 That means (4.14) holds. Therefore we finish the proof the Theorem 4.1. \(\square\)

**Remark 4.1.** In Theorem 4.1, we consider the transversality of the Liouville vector field only for \(E < H(L_1)\) but we can slightly extend the range of \(E\). To this end, first take a small ball \(B_\delta(L_1)\) around the Lagrange point \(L_1\) with radius \(\delta\). Then the set \(H^{-1}(E) \setminus B_\delta(L_1)\) consists two components. One component contains the moon and the other contains the earth. So we will call the components the moon component and the earth component, respectively. If we choose a sufficiently small \(\epsilon\), the set \(H^{-1}(E) \setminus B_\delta(L_1)\) is still divided into two components by \(B_\delta(L_1)\) for \(E < H(L_1) + \epsilon\). By Theorem 3.1, the Hills region of the moon \(\Sigma^M_{H(L_1)}\) has maximal length along the ray \(\varphi = \frac{\pi}{2}, \theta = 0\). So the moon component of \(H^{-1}(E)\) for \(E < H(L_1) + \epsilon\) has maximal distance for \(\theta\) and \(\varphi\) close to 0 and \(\frac{\pi}{2}\) respectively. The maximal distance is smaller than \(d\) (where \(d = |M - l_1|\)), for sufficiently small \(\epsilon, \delta\). That means the moon component is contained in \(B_\delta(M)\).

Therefore we can prove transversality of \(X\) and the moon component of \(H^{-1}(E)\) by the same argument of the proof of theorem 4.1. The same holds for the Earth component. This remark will be used in next section.

5. CONNECTED SUM

So far, we have seen that \(H^{-1}(c)\) admits a contact structure whenever \(c < H(L_1)\). Now we will show that if \(H(L_1) < c < H(L_1) + \epsilon\) for sufficiently small \(\epsilon\), then \(H^{-1}(c)\) also admits a contact structure. This result is already proved for planar case in section 7 of [4]. In the spatial case, we apply the same technique and use the same notation. First we will use Conley’s work on [7] and consider the Hamiltonian \(H\) and effective potential \(V\) which are given by

\[
H(q, p) = \frac{1}{2}((p_1 + q_2)^2 + (p_2 - q_1)^2 + p_3^2)) + V(q)
\]

\[
V(q) = -\frac{1}{2}(q_1^2 + q_2^2) - \frac{\mu}{|q - q'|} - \frac{1 - \mu}{|q - q'|}
\]
where $q^0 = (-1 - \mu, 0, 0)$ and $q^1 = (\mu, 0, 0)$. Then we can rewrite $V(q)$ by using the Taylor expansion at a critical point $q^L$.

\begin{equation}
V(q) = \tilde{Q}(q - q^L) + R(q - q^L)
\end{equation}

In (5.16), $\tilde{Q}$ represents the quadratic part of $V(q)$ and $R$ represents the remainder term (higher order than 2). Then we can expand the Taylor series at the first Lagrange point $(q^L, p^L) = (q_1^L, q_2^L, 0, -q_2^L, q_1^L, 0)$. Now we obtain

\[
H(q, p) = \frac{1}{2}((p_1 + q_2)^2 + (p_2 - q_1)^2 + p_3^2) + \tilde{Q}(q - q^L) + R(q - q^L).
\]

We can notice that the first term of $H(q, p)$ is a second order term so we combine the first term of $H(q, p)$ with $\tilde{Q}$, we obtain the quadratic part of $H$ which we write by $Q$. Then $H(q, p)$ is given by

\[
H(q, p) = Q(q - q^L, p + (q_2^L, -q_1^L, 0)) + R(q - q^L)
\]

where the $6 \times 6$ matrix $Q$ which represented by ordered basis $(q_1, q_2, p_1, p_2, q_3, p_3)$ is given by

\[
Q = \frac{1}{2}
\begin{pmatrix}
-2\rho & 0 & 0 & -1 & 0 & 0 \\
0 & \rho & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \rho & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

In the matrix $Q$, the letter $\rho$ means

\[
\rho = \frac{\mu}{|q^L - q^0|^3} + \frac{1 - \mu}{|q^L - q^1|^3}.
\]

For convenience we will consider translated terms and write $Q(q, p), R(q)$, so $H$ is given by

\[
H(q, p) = Q(q, p) + R(q)
\]

then for $q, p$ near 0, we get the vector field

\[
Y_{a,b,\gamma} = (q_1, q_2, p_1, p_2, q_3, p_3)
\begin{pmatrix}
a & 0 & 0 & 0 & 0 & 0 \\
b & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & a & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -b & 0 \\
0 & 0 & 0 & 0 & \gamma & 0 \\
0 & 0 & 0 & 0 & 0 & 1 - \gamma
\end{pmatrix}
\begin{pmatrix}
\partial_{q_1} \\
\partial_{q_2} \\
\partial_{p_1} \\
\partial_{p_2} \\
\partial_{q_3} \\
\partial_{p_3}
\end{pmatrix}
\]

Note that the symplectic form is $\omega = dp \wedge dq$. The Lie derivative $\mathcal{L}_Y \omega$ is given by

\[
\mathcal{L}_Y \omega = d(-aq_1 dp_1 + (1 - a)p_1 dq_1 - bq_2 dp_2 + (1 - b)p_2 dq_2 - \gamma q_3 dp_3 + (1 - \gamma)p_3 dq_3) = \omega
\]

Therefore the vector field $Y_{a,b,\gamma}$ is Liouville and we will show transversality of $Y_{a,b,\gamma}$.

**Lemma 5.1.** For $a < 0, b > 0$ and $0 < \gamma < 1$, the vector field $Y_{a,b,\gamma}$ transverse to $H^{-1}(c)$ in $B_r(l_1)$ where the energy $c$ is contained in $[H(L_1) - \epsilon, H(L_1) + \epsilon] - \{H(L_1)\}$.

**Proof.** Note that $Y(Q)$ is a quadratic form in $(q, p)$. For proper value of $a, b$ and $\gamma$, we can directly check that $Y(Q)$ is positive definite. And matrix $Q$ decomposes into two block matrices.

\[
Q = \frac{1}{2}
\begin{pmatrix}
-2\rho & 0 & 0 & -1 & 0 & 0 \\
0 & \rho & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \rho & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
In the matrix $Q$, the upper block is same as the planar case matrix. And lower block gives $\gamma \rho q_3^2 + (1-\gamma)p_3^2$ in $Y(Q)$. For proper value $\gamma$ this part is clearly positive definite and has positive eigenvalues. So $Y(Q)$ has positive eigenvalues. On the other hand, the remainder term $Y(R)$ has higher order than 2. For a sufficiently small neighborhood of first Lagrange point $(q_1^L, q_2^L, 0, q_2^L, -q_1^L, 0)$ we can estimate $Y(H)$ by

$$Y(H) = Y(Q + R) = Y(Q) + Y(R) \geq Y(Q) - 1/2|Y(Q)| > 0.$$ 

\[\square\]

The induced Liouville form from the Liouville vectorfield $Y$ is given by

$$\alpha = -(q_1 - q_1^L)dp_1 - b_2dp_2 - \gamma q_3dp_3 + (1-a)p_1dq_1 + (1-b)(p_2 - q_1^L)dq_2 + (1-\gamma)p_3dq_3.$$ 

Because of Remark 4.1, we already know that for energy $c$ which contained in $[H(L_1), H(L_1) + \epsilon]$ the vector field $X = (q - q_1)\partial_q$ is Liouville and transverse to hypersurface $H^{-1}(c)$ in $B_3(L_1)$. Its contact form $\alpha_0$ is given by

$$\alpha_0 = (q_1 - q_1^L)dp_1 - q_2dp_2 - q_3dp_3.$$ 

The difference of $\alpha_0$ and $\alpha_1$ is given by

$$\alpha_1 - \alpha_0 = (1-a)((q_1 - q_1^L)dp_1 + p_1dq_1) + (q_1^L - q_1^L)dp_1 + (1-b)(q_2dp_2 + (p_2 - q_1^L) dq_2) + (1-\gamma)(q_3dp_3 + p_3dq_3)$$

This differential form is exact as $d\alpha_1 - d\alpha_0 = \omega - \omega = 0$ and has a primitive given by

$$G := (1-a)(q_1 - q_1^L)p_1 + (q_1^L - q_1^L)p_1 + (1-b)(p_2 - q_1^L)q_2 + (1-\gamma)p_3q_3.$$ 

Now we will consider the parallel transformation given by $(q_1^L, q_2^L, 0, -q_2^L, q_1^L, 0) \rightarrow (0, 0, 0, 0, 0)$ and denote its transformed coordinate by $(q, p)$. Then $G$ is rewritten by

$$G = (1-a)q_1p_1 + (q_1^L - q_1^L)p_1 + (1-b)p_2q_2 + (1-\gamma)p_3q_3.$$ 

We denote the two Liouville vector fields by $X_0, X_1 (X_0$ correspond to $\alpha_0$ and $X_1$ correspond to $\alpha_1$). Now we define the Hamiltonian vector field $X_G$ of $G$ by

$$i_{X_G}\omega = dG,$$

then we have

$$X_1 = X_0 + X_G.$$ 

Now we want to make a global vector field transverse to $H^{-1}(c)$ for $H(L_1) < c < H(L_1) + \epsilon$. For this, choose a cut-off function $f$ such that depending on $q_1 + \frac{1}{p}p_2$ and satisfy $X_1 = X_0 + X_{fG}$ for large $q_1$ and $X_0 = X_0 + X_{fG}$ for 0 close $q_1$. In other words, $f = 1$ when close to the first Lagrange point and $f = 0$ in outside of small neighborhood of the first Lagrange point. Then we define vector field $X$ as

$$X = X_0 + X_{fG}.$$ 

We are going to show that the energy hypersurface $H^{-1}(L_1)$ is separated into two connected component by the set $\{q_1 + \frac{1}{p}p_2 = 0\}$

To show this, we have to notice that the singular energy hypersurface $\{H = H(L_1)\}$ corresponds to $\{(Q = 0)\}$. If we show that the hyperplane $\{q_1 + \frac{1}{p}p_2 = \delta\}$ intersects the quadric $\{Q = 0\}$ in 4-sphere when $\delta \neq 0$ and intersect point when $\delta = 0$, then we can prove that hyperplane $\{q_1 + \frac{1}{p}p_2 = 0\}$ divides $H^{-1}(L_1)$ into two components.

For this, we can rewrite hyperplane equation by $p_2 = \rho (\delta - q_1)$ and put it into $\{(Q = 0)\}$, then the equation is given by

$$\frac{1}{2}
(q_1 + q_2)^2 + (p_2 - q_1)^2 - (2\rho + 1)q_1^2 + (\rho - 1)q_2^2 + \rho q_3^2 + p_3^2 = 0$$

and this equation can be rewritten by

$$(p_1 + q_2)^2 + (\rho q_1 - (\rho + 1)\delta)^2 + (\rho - 1)q_2^2 + \rho q_3^2 + p_3^2 = (2\rho + 1)\delta^2$$
It means when $\delta = 0$, the solution for this equation is just a point. So we have divided the hypersurface into two connected components. If one component contains the Earth, then we call it as the Earth component and the other the Moon component.

Now we want to show that the Liouville vector field $X$ is transverse to level sets of $H^{-1}(c)$ where $c \in (H(L_1), H(L_1) + \epsilon)$. To show this, we bring a cut-off function $f$ which is already defined. Also we have already checked that $X_0(H) > 0$ away from the Lagrange point in Remark 4.1. Then the only remaining thing is interpolating part of $f$. We will use the Hamiltonian given by

$$
X(H) = X_0(H) + X_{fG}(H) = dH(X_0) + \{H, fG\} = dH(X_0) - \{fG, H\}
$$

$$
= (1 - f)dH(X_0) + f dH(X_0) - f\{G, H\} - G\{f, H\}
$$

$$
= (1 - f)dH(X_0) + f dH(X_0 + X_G) + GdH(X_f).
$$

The first term is non-negative near the Lagrange point because $f = 0$ near the Lagrange point. Also the second term is non-negative near the Lagrange point because $X_0 + X_G = Y$. For the last term, first we compute $X_f$ given by

$$
X_f = f' \left( \frac{\partial}{\partial p_1} - \frac{1}{\rho} \frac{\partial}{\partial p_2} \right)
$$

Then we can compute $dH(X_f)$

$$
dH(X_f) = f' \left( \left(1 - \frac{1}{\rho}\right) p_1 - \frac{q_2}{\rho} \left( \frac{\mu}{\sqrt{|q-q^0|^3}} + \frac{1 - \mu}{|q-q^1|^2} - \rho \right) \right).
$$

Note that the second term vanishes at $q^2$ because of $\rho = \frac{\mu}{|q-q^0|^2} + \frac{1 - \mu}{|q-q^1|^2}$. In other words, we can estimate $dH(X_f) \sim f'(1 - \frac{1}{\rho})p_1$.

Next we consider the leading order term of $G$. Then we note that the only 1-degree term of $G$ is $(q_1^1 - q_1^0)p_1$ and leading order term of last term of $X(H)$ is given by $(q_1^1 - q_1^0)p_1 \cdot f'(1 - \frac{1}{\rho})p_1$. It means the leading order term is

$$
\rho' (q_1^1 - q_1^0)^2 (1 - \frac{1}{\rho}) p_1^2
$$

also one can notice that $f'$ and $(q_1^1 - q_1^0)$ is negative, $p_1^2$ and $1 - \frac{1}{\rho}$ is positive. So the last term of $X(H)$ is non-negative

**Lemma 5.2.** There exists $\epsilon > 0$ and a Liouville vector field $\hat{X}$ such that $\hat{X}$ is transverse to level sets $H = E$ for all $H(L_1) < E < H(L_1) + \epsilon$.

**Proof.** Near the Lagrange point, the two sets $\{Q = 0\}$ and $\{H = E\}$ are almost the same. $(H = Q + R$ and $R$ has order more than 2.) So, $f$, which depends on $q_1 + \frac{1}{\rho}p_2$, divides the level set $\{H = E\}$ into a Moon and Earth component.

Then we can choose $\epsilon_1, \epsilon_2$ such that $dH(X_1) > 0$ where $B_{\epsilon_2}(L_1)$ and $\epsilon_1 < \epsilon_2$, then first term of $(1 - f)dH(X_0)$ is positive on $[\epsilon_1, 0)$ and $f dH(X_0 + X_G)$ is positive on $[0, \epsilon_2)$ and $GdH(X_f)$ is non-negative. So we defined a Liouville vector field that is transverse to the entire level set $\{H = E\}$. \( \square \)

6. Moser regularization of level sets of $H$

In this section we follow Moser’s work to regularize our hypersurface $H^{-1}(c)$. J. Moser’s work is given in [12]. We will use the Hamiltonian given by

$$
H(p, q) = \frac{1}{2} |p|^2 - \frac{\mu}{|q-q^0|} - \frac{1 - \mu}{|q-q^1|} + p_1 q_2 - p_2 q_1
$$

where $q_1 = M$ and $q_2 = E$. This Hamiltonian has a singularity at $q = q^0$ and $q = q^1$. The purpose of Moser regularization is for remove singularities. So we should consider the cases $|q - q^0| < \epsilon$ or $|q - q^1| < \epsilon$. Without loss of generality, We can only consider the case $|q - q^0| < \epsilon$. 

6.1. Transforming the Hamiltonian from $T^*\mathbb{R}^3$ to $T^*\mathbb{S}^3$. In this section we will follow Moser’s work to regularize the hypersurface $\Sigma^M_c$. First consider the new time parameter and new Hamiltonian

$$s = \int \frac{dt}{|q - q^0|}, \quad K = (H - c)|q - q^0|.$$ 

Here $K$ is a new Hamiltonian and $c$ is the energy level. Then we can express $K$ in $(q, p)$ coordinates. It is given by

$$K(q, p) = \left( \frac{1}{2}|p|^2 - \frac{\mu}{|q - q^0|} - \frac{1 - \mu}{|q - q^1|} + p_1q_2 - p_2q_1 - c \right)|q - q^0|.$$ 

We apply the canonical transformation given by $p = -x, \ y = q - q^0$ to exchange the role of position and momentum. The transformed Hamiltonian $\tilde{K}$ is given by

$$\tilde{K}(x, y) = \frac{1}{2}|x|^2|y - \mu - \frac{(1 - \mu)|y|}{|y + q^0 - q^1|} - x_1y_2|y| + x_2(y_1 + q^0_1)|y| - c|y|$$

$$= \frac{1}{2}(|x|^2 + 1)|y| - \mu - (1 - \mu)\frac{|y|}{|y + q^0 - q^1|} + (x_2y_1 - x_1y_2)|y| + x_2q^0_1|y| - (c + \frac{1}{2})|y|.$$ 

We can apply the symplectic transformation between $T^*\mathbb{R}^3$ and $T^*\mathbb{S}^3$ induced by stereographic projection. This is given by

$$x_k = \frac{\xi_k}{1 - \xi_0}, \quad k = 1, 2, 3,$$
$$y_k = \eta_k(1 - \xi_0) + \xi_k\eta_0, \quad k = 1, 2, 3.$$ 

Here $(x, y)$ represents the coordinates of $T^*\mathbb{R}^3$ and $(\xi, \eta)$ represents the coordinate of $T^*\mathbb{S}^3$, and we also have the relations

$$\xi_0 = \frac{|x|^2 - 1}{|x|^2 + 1}, \quad \xi_k = \frac{2x_k}{|x|^2 + 1} \quad \text{for} \ k = 1, 2, 3,$$
$$\eta_0 = \langle x, y \rangle, \quad \eta_k = \frac{|x|^2 + 1}{2}y_k - \langle x, y \rangle x_k \quad \text{for} \ k = 1, 2, 3,$$
$$|\eta| = \frac{(|x|^2 + 1)|y|}{2} = \frac{|y|}{1 - \xi_0}.$$ 

By using the above equations we get the transformed Hamiltonian $F$ on $T^*\mathbb{S}^3$ and $F$ is given by

$$F(\xi, \eta) = |\eta|\left(1 - \frac{(1 - \mu)(1 - \xi_0)}{|\eta|(1 - \xi_0) + \xi_0\eta_0 + q^0 - q^1|} + (1 - \xi_0)(\xi_2\eta_1 - \xi_1\eta_2) + \xi_2q^0_1 - (c + \frac{1}{2})(1 - \xi_0) \right) - \mu.$$ 

For convenience let us define $f(\xi, \eta)$ by

$$f(\xi, \eta) = 1 - \frac{(1 - \mu)(1 - \xi_0)}{|\eta|(1 - \xi_0) + \xi_0\eta_0 + q^0 - q^1|} + (1 - \xi_0)(\xi_2\eta_1 - \xi_1\eta_2) + \xi_2q^0_1 - (c + \frac{1}{2})(1 - \xi_0)$$

and consider the new Hamiltonian $Q(\xi, \eta)$ given by

$$Q(\xi, \eta) = \frac{1}{2}|\eta|^2f(\xi, \eta)^2.$$ 

Then we note that the level set $\{ H = k \} = \{ K = 0 \} = \{ Q = \frac{1}{2}\mu^2 \}$. So we will consider the level set $\{ Q = \frac{1}{2}\mu^2 \}$ instead of level set $\{ H = k \}$.

6.2. Transversality near the moon. In this section, we want to show that the level set $\{ Q = \frac{1}{2}\mu^2 \}$ admits a contact structure. For this, we will show that the Liouville vector field $X = \eta\partial_\eta$ is transverse to the level set $\{ Q = \frac{1}{2}\mu^2 \}$ near the moon. i.e. points $(\xi, \eta)$ which satisfy $|\eta|(1 - \xi_0) < \epsilon$. 
Of course there are two connected component but we consider the moon component. Then for $\mu > 0$ $X(Q)$ is given by

$$X(Q) = |\eta|^2 f(\xi, \eta)^2 + |\eta|^2 f(\xi, \eta)\eta \partial_\eta f(\xi, \eta)$$

$$= 2Q + |\eta|^2 f(\xi, \eta) \left( -\eta \partial_\eta \frac{(1 - \mu)(1 - \xi_0)}{|\eta(1 - \xi_0) + \xi_0^1 + q^0 - q^1|} + (1 - \xi_0)(\xi_2 \eta_1 - \xi_1 \eta_2) \right)$$

The first term is trivially positive. To estimate the second term, first we will estimate a lower bound of $|f(\xi, \eta)|$. We can get a upper bound of $|\eta|$ from the level set condition $\{Q = \frac{1}{2} \mu^2\}$. $|f(\xi, \eta)|$ can be bounded by following inequality

$$1 - \frac{(1 - \mu)(1 - \xi_0)}{|\eta(1 - \xi_0) + \xi_0^1 + q^0 - q^1|} + (1 - \xi_0)(\xi_2 \eta_1 - \xi_1 \eta_2) + \xi_2 q_0^0 - (c + \frac{1}{2})(1 - \xi_0) \geq 0$$

$$1 + (1 - \xi_0) \left( |c| - \frac{1 - \mu}{|\eta(1 - \xi_0) + \xi_0^1 + q^0 - q^1|} \right) - (1 - \xi_0)|\eta||\xi| - |\xi||q^0|$$

In this estimate we use the negativity of $c$. Because we are interesting in the level set under the $H(L_1)$. Also we know that $|q^0| = 1 - \mu, |\xi| = 1$ and $(1 - \xi_0)|\eta| < \epsilon$, then

$$|f(\xi, \eta)| \geq 1 + (1 - \xi_0) \left( |c| - \frac{1 - \mu}{|\eta(1 - \xi_0) + \xi_0^1 + q^0 - q^1|} \right) - \epsilon - (1 - \mu) \geq \frac{\mu}{2}$$

We can show the last inequality because the first three terms give $\mu - \epsilon$ and we know that $(1 - \xi_0)$ is always non-negative. Then second term is rewritten by

$$(6.19) \quad (1 - \xi_0) \left( |c| - \frac{1 - \mu}{|q - q^1|} \right)$$

then by the triangle inequality we get

$$|q - q^0| + |q - q^1| \geq |q^0 - q^1| = 1, \quad |q - q^1| \geq 1 - \epsilon.$$ 

Also for every $\mu \in [0, 1]$ we know that $H(L_1) \leq -\frac{3}{2}$. So we can get a lower bound of the term (6.19) by

$$(1 - \xi_0) \left( |c| - \frac{1 - \mu}{|q - q^1|} \right) \geq (1 - \xi_0) \left( \frac{3}{2} - \frac{1 - \mu}{1 - \epsilon} \right)$$

if we send $\epsilon \to 0$, the right hand side goes to $(1 - \xi_0)\mu > 0$ and it is clear that the right hand side is increasing as $\epsilon \to 0$. It follows that there is some $\epsilon'$ such that if $\epsilon < \epsilon'$ then, the right hand side is bigger than 0. Now we get

$$|f(\xi, \eta)| \geq \mu - \epsilon - 0 \geq \frac{\mu}{2}.$$ 

We now find

$$\frac{1}{2} \mu^2 = Q = \frac{1}{2} |\eta|^2 |f(\xi, \eta)|^2 \geq \frac{1}{2} |\eta|^2 \frac{\mu^2}{4}.$$ 

Therefore we get

$$|\eta| \leq 2.$$ 

By using this result, we can show the positivity of $X(Q)$

$$X(Q) \geq 2Q - |\eta|^2 |f(\xi, \eta)| \left( -\eta \partial_\eta \frac{(1 - \mu)(1 - \xi_0)}{|\eta(1 - \xi_0) + \xi_0^1 + q^0 - q^1|} + (1 - \xi_0)(\xi_2 \eta_1 - \xi_1 \eta_2) \right).$$
Note that $|\eta| \leq 2$, $|\eta| |f(\xi, \eta)| = \sqrt{2Q} = \mu$. Then we can apply the triangle inequality

$$X(Q) \geq 2Q - 2\mu \left( |\eta| \partial_\epsilon (1 - \mu)(1 - \xi_0) \over |\eta|(1 - \xi_0) + \xi_0(1 - q_0) + q^0 - q^1| + |1 - \xi_0)(\xi_2 - \xi_1\xi_2)| \right).$$

The term $\partial_\epsilon f(\xi, \eta)$ has maximum in our region. Because the function has a singularity at $q_1$ but our region is away from the $q_1$ and compact. So we has finite upper bound $C$ for this term. Then we can estimate $X(Q)$ like this

$$X(Q) \geq \mu^2 - 2\mu(1 + (1 - \mu)C).$$

and $\epsilon$ is small positive number. So we show that $X(Q)$ is positive.

In short, $X(Q) > 0$ for any $\mu$ (We consider the case of the moon so $\mu = 0$ is excluded). This result means that we show that the Liouville vector field $X = \eta \partial_\eta$ is transverse to our regularized hypersurface. In other words, the regularized hypersurface is of contact type and diffeomorphic to $ST^*S^3$ below the first critical value. Because $S^4$ is parallelizable, this space is diffeomorphic to $S^2 \times S^3$.

7. Blue sky catastrophes

In the previous section, we see that the level set $H^{-1}(c)$ admits a contact structure and also that it is regularized to $ST^*S^3$ or to $ST^*S^3 \# ST^*S^3$, which are both compact. Then we can say something about blue sky catastrophes.

**Definition 7.1.** A one-parameter family of vector fields $X_\epsilon$ where $\epsilon \in [0, 1]$ on a compact manifold has a blue sky catastrophe if for $\epsilon \in [0, 1)$ there are corresponding periodic orbits which are continuously depend on the parameter and its period goes to infinity as the parameter $\epsilon$ goes to 1. Also the distance from the orbit to the singular points of the field has a positive lower bound.

We can see that there is no blue sky catastrophe on the regularized surface $ST^*S^3$. For this we consider the action functional on an exact symplectic manifold where $\omega = d\lambda$.

$$\mathcal{A}^H(\gamma, \tau) = \int_{S^1} \gamma^*\lambda - \tau \int H(\gamma(t))dt.$$

where $\gamma \in C^\infty(S^1, M)$ and $\tau \in (0, \infty)$. Then we can show that if $\gamma, \tau$ satisfy

$$\partial_\gamma \gamma = \tau X_H(\gamma), \quad H(\gamma) = 0$$

these two conditions, they are critical points of the functional $\mathcal{A}^H$. These critical points are periodic orbits of $X_H$ with period $\tau$. To show there is no blue sky catastrophe we prove the following theorem, see also chapter 7.6 of [8].

**Theorem 7.1.** Suppose that $(r, \tau_r)$ for $r \in [0, 1)$ is a smooth family of periodic Reeb orbits $\gamma_r$ with period $\tau_r$. If there is one parameter family of contact form $\lambda|_{H^{-1}(0)}$ for every $r \in [0, 1]$, then there exist $\tau_1 \in (0, \infty)$ such that $\tau_r$ converges to $\tau_1$.

We conclude that there is no blue sky catastrophe on the regular energy hypersurface $H^{-1}(c)$ if $c < H(L_1)$ or $c \in (H(L_1), H(L_1) + e)$

**Proof.** First we can notice that for fixed $r$, $H^{-1}(0)$ the energy level and has a contact form for all $r \in [0, 1]$. So we know that the Hamiltonian flow is given by a reparameterization of Reeb flow of contact form $\lambda|_{H^{-1}}$. This means their images are same and Reeb vector field $R_r$ of $\lambda|_{H^{-1}}$ is parallel to Hamiltonian vector field $X_{H_1}|_{H^{-1}(0)}$. It means $R_r$ is Reeb vector field of $\lambda|_{H^{-1}(0)}$. Note that Hamiltonian vector field $X_{H_1}|_{H^{-1}(0)}$ is parallel to $R_r$. So first we just assume that

$$R_r = X_{H_1}|_{H^{-1}(0)}.$$

Now consider the family of functionals

$$\mathcal{A}^{H_r} : C^\infty(S^1, M) \times (0, \infty) \to \mathbb{R}$$
We compute the action $A^{H_r}$ at the critical point $(\gamma, \tau)$

$$A^{H_r}(\gamma, \tau) = \int_0^1 \lambda(\tau X_{H_r}(\gamma)) \, dt = \tau \int_0^1 \lambda(R_r) \, dt = \tau$$

Now let $(\gamma_r, \tau_r)$ for $r \in [0, 1)$ be a smooth family of periodic orbits. This is because we deal with Hamiltonian flow of each $H_r$ and they are critical points of action functional. We want to show that $\tau_r$ remains bounded. For this we compute $\partial_r \tau_r$,

$$\partial_r \tau_r = \frac{d}{dr} (A^{H_r}(\gamma_r, \tau_r)) = (\partial_r A^{H_r})(\gamma_r, \tau_r) = -\tau_r \int_{S^1} (\partial_r H_r)(\gamma_r) \, dt$$

in second equation we use the fact that $(\gamma_r, \tau_r)$ is a critical point of $A^{H_r}$. And our hypersurface $H_r^{-1}(0)$ is compact for all $r \in [0, 1]$, so there exists $k > 0$ such that

$$\left| \partial_r H_r |_{H_r^{-1}(0)} \right| < k.$$ 

This means

$$|\partial_r \tau_r| = \left| -\tau_r \int_{S^1} (\partial_r H_r)(\gamma_r) \, dt \right| \leq \tau_r \int_{S^1} |(\partial_r H_r)(\gamma_r)| \, dt < k \tau_r, \quad \text{where} \quad k > 0.$$

Especially, if $0 \leq r_1 < r_2 < 1$, then this implies

$$e^{-k(r_2-r_1)} \tau_{r_1} < \tau_{r_2} < e^{k(r_2-r_1)} \tau_{r_1}.$$ 

So we prove that $\tau_r$ remains bounded, when $r$ goes to 1.

One can notice that reparameterization of flow is not affect to convergence of period. Hence in our situation, there is no blue sky catastrophes and we prove the theorem.

\[ \square \]

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