Incentives in Resource Allocation under Dynamic Demands

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Abstract

Every computer system—from schedulers in public clouds (e.g., Amazon, Google, Microsoft etc.) to computer networks to hypervisors to operating systems—performs resource allocation across system users. The defacto allocation policy used in most of these systems, max-min fairness, guarantees many desirable properties like incentive compatibility and Pareto efficiency, assuming user demands are static (time-independent). However, in modern real-world production systems, user demands are dynamic, that is, vary over time. As a result, there is now a fundamental mismatch between the resource allocation goals of computer systems, and the properties enabled by classical resource allocation policies. This paper aims to bridge this mismatch.

We consider a natural generalization of the classical algorithm for max-min fair resource allocation for the case of dynamic demands: this algorithm guarantees Pareto optimality, while ensuring that resources allocated to users are as max-min fair as possible up to any time instant, given the allocation in previous periods. While this dynamic allocation scheme remains Pareto optimal, unfortunately, it is not incentive compatible. Our results show that the possible increase in utility by misreporting demand is minimal and, since this misreporting can lead to significant decrease in overall useful allocation, this suggests that it is not a useful strategy. Our main result is to show that when user demands are independent random variables, increasing the total amount of resource by a \((1+O(\varepsilon))\) factor compared to the sum of expected instantaneous user demands, makes the algorithm \((1+\varepsilon)\)-incentive compatible, that is, no user can improve her allocation by more than a factor of \((1+\varepsilon)\) by misreporting her demands, where \(\varepsilon \to 0\) as \(n \to \infty\) and \(n\) is the number of users. In the adversarial setting, we show that this algorithm is \(3/2\)-incentive compatible; we also show that this factor is nearly tight. We also present generalization of the above results for the case of multiple colluding users and multi-resource allocation.

1 Introduction

Resource allocation is a fundamental problem in computer systems. For example, companies like Google [Ver+15] and Microsoft [Gra+14] use schedulers in private clouds to allocate a limited and divisible amount of resources (e.g., CPU, memory, servers, etc.) among a number of selfish and strategic users that want to maximize their allocation; the goal of the scheduler is to maximize resource utilization while achieving fairness in resource allocation. The defacto allocation policy used in many of these systems is the classic max-min fairness policy. For instance, max-min fairness is used in most schedulers used in private clouds [Bou+14; Gho+11; Gra+14; Gra+16a; Gra+16b; Hin+11; SFS12; Vav+13; Ver+15]; max-min fairness is deeply entrenched in congestion

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control protocols like TCP and its variants [Ali+10; CJ89]; and max-min fairness is the default policy for resource allocation in most operating systems and hypervisors [Cha08; KVM16]. Such prevalence of max-min fairness policy is rooted in the properties it guarantees: Pareto-efficiency, envy-freeness, and incentive compatibility. To guarantee these properties, max-min fairness assumes that user demands do not change over time. This assumption does not hold in many scenarios—several recent studies in the systems community have shown that user demands have become highly dynamic, that is, vary significantly over time [CAD18; Rei+12; Vup+20b; YYR20]. For such dynamic user demands, naively using max-min fairness policy (e.g., to perform new instantaneously max-min fair allocation every unit of time) maintains Pareto-optimality but loses fairness—intuitively, since max-min fairness policy does not take past allocations into account, dynamic user demands can result in an increasingly large disparity between users’ allocations over time.

We study a natural generalization of max-min fair allocation for dynamic demands over divisible resources (referred to as dynamic max-min fairness, also see [Fre+18]). Every round, each user has a demand, which is the maximum amount of resources that is useful to her. Users want to maximize the sum of overall useful resources they get (over rounds). In every round, dynamic max-min fairness allocates resources so as to maintain Pareto-efficiency while being as fair as possible given the past allocations: first the minimum total allocation of any user is maximized, then the second minimum, etc. By construction, dynamic max-min fairness is Pareto-efficient: every round, it maintains the invariant that either all the resources are used or every user’s demand is satisfied. However, dynamic max-min fairness is not incentive compatible, i.e., it is possible that a user can misreport her demand on one round to increase her total useful allocation in the future by a small amount [Fre+18] (also see Theorem 3.1 for a stronger lower-bound). Nevertheless, studying dynamic max-min fairness is both important and interesting. First, similar to widely-used classical max-min fairness, it is simple and easy to understand; thus, it has the potential for real-world adoption (similar to many other non-incentive compatible mechanisms used in practice, e.g., non-incentive compatible auctions used by U.S. Treasury to sell treasury bills since 1929 and by the U.K to sell electricity [Har18; Kri09; Par18]). Second, our results show that dynamic max-min fairness is approximately incentive compatible, that is, strategic users can increase their allocation by misreporting their demands but this increase is bounded by a small factor and requires knowledge of future demands; moreover, misreporting demands in dynamic max-min fairness can lead to significant decrease in overall useful allocation, suggesting that misreporting unlikely to be a useful strategy for any user.

**Our Contribution.** Our goal is to show that dynamic max-min fair allocation is close to incentive compatible. A popular relaxation of incentive compatibility is $\gamma$-incentive compatibility [Arc+03; AB18; BSV19; DFP08; Düt+12; KPS03; MS14], which requires that the possible increase in utility by untruthful reporting must be bounded by a factor of $\gamma \geq 1$. Using this notion we show that users have very low incentive to be untruthful, especially when the number of users, $n$, becomes large and demands are independent random variables:

- Our main result in Section 3 is to show that when users’ demands are independent random variables, by increasing the total amount of resources by a factor of $(1 + O(\varepsilon))$ compared to the sum of expected instantaneous user demands, the algorithm is $(1 + \varepsilon)$-incentive compatible where $\varepsilon \to 0$ as $n \to \infty$. For example, when the expected demand of each user is identical and the standard deviation of the demands are at most proportional to their expectations, then we get the claimed result for $\varepsilon = \frac{1}{\sqrt{n}}$. We also extend this result to independent random demands with different means, but with a similar bound on the standard deviations.

For the case of adversarial demands, we show that dynamic max-min fairness is $3/2$-incentive compatible (Theorem 3.5), and give a lower bound of $4/3$ (Theorem 3.1) improving the lower bound of [Fre+18]. We also show that there is no incentive to over-report demand (Theorem 3.4), ensuring that all allocated resources are used.

- In Section 4 we study the generalization of max-min fairness to the case of multiple resources assuming that each user has a fixed ratio how it uses resources, and only the amount of the user’s need is variable. We study the natural extension of max-min fairness for this setting, dominant resource fairness [Gho+11], in repeated
settings. When users only need subsets of the resources, one cannot bound the incentive compatibility ratio: a user can over-report her demand, and gain added resources proportional to the number of users. In contrast, when all users need all the resources (even with very different ratios) we show results similar to the single resource case, but this time depending on a parameter \(\rho > 0\) measuring the similarity of the ratios of different users (\(\rho = 1\) when ratios are identical). Again when users’ demands are independent random variables with not too large variance, increasing the total amount of resources by a factor of \(1 + c/\sqrt{n\rho}\) makes dynamic max-min fairness \((1 + O(c^{-2}))\)-incentive compatible (Theorem 4.6). For adversarial demands dynamic max-min fairness is \(\frac{1+\rho}{\rho}\)-incentive compatible (Theorem 4.5), and over-reporting is not beneficial (Theorem 4.3).

- In Section 5 we study the effect of collusion on dynamic weighted max-min fairness (generalization of max-min fairness when users have different priorities). We again show that similar results are true: if users’ demands are random variables dynamic weighted max-min fairness even with collusion among the users tends to 1-incentive compatibility with similar rates as before (Theorem 5.3), in adversarial settings it is 2-incentive compatible (Theorem 5.2), and demand over-reporting does not increase utility (Theorem 5.1).

- Finally, in Section 6 we study how often can a user be allocated resources significantly above the share she would get if reporting truthfully. Assuming that the total resources allocated to users increase approximately linearly over time, we prove that the users cannot for long periods have a factor of \(\gamma\) more resources by misreporting, for any \(\gamma > 1\) (Theorem 6.1), and the time between the intervals when a user has \(\gamma\) more resources scales exponentially with \(\frac{2-\gamma}{3-2\gamma}\) (also Theorem 6.1).

**Related Work.** The simplest algorithm for resource allocation is *strict partitioning* [Ver+17; Vup+20a], that allocates a fixed amount of resources to each user independent of their demands. While incentive compatible, strict partitioning can have arbitrarily bad efficiency. Static max-min fairness [Fre+18; Gho+11; Gra+14; Gra+16b; SFS12] is Pareto-efficient, incentive compatible and fair but only when user demands are static. However, as shown in several recent results [Fre+18; Hos19; SCC21], naïvely using static max-min fairness in the case of dynamic demands can result in large disparity between resources allocated to users over time since past allocations are not taken into account.

[Fre+18; Hos19] study resource allocation for the case of dynamic demands, but the allocation model they consider is closer to max-min fairness separately in each epoch, and less aim to be fair overall. Under this model, they present two mechanisms that are incentive compatible but either offer only 0.5 times the utility of static allocation, or are efficient under strong assumptions: user demands being i.i.d random variables, and number of rounds growing large. [SCC21] presents minor improvements over static max-min fairness for dynamic demands. Their mechanism allocates resources in an incentive compatible way according to max-min fairness while marginally penalizing users with larger past allocations using a parameter \(\delta \in [0, 1]\). For both \(\delta = 0\) and \(\delta \rightarrow 1\), the penalty tends to 0 for every past allocation and the mechanism becomes identical to static max-min fairness; for other values of \(\delta\), the penalty is at most a \(\delta(1 - \delta) \leq 1/4\) fraction of past allocation surplus, and it reduces exponentially with time (users who were allocated large amount of resources further in the past receive even tinier penalty). Thus, for all values of \(\delta\) (and, in particular, for \(\delta = 0\) and \(\delta \rightarrow 1\)), their mechanism suffers from the same problems as static max-min fairness.

Several other papers study resource allocation where user demands can be dynamic, but with significantly different setting than ours. [AW19; ZP20] examine the setting where indivisible goods arrive over time and have to be allocated to users whose utilities are random; however [AW19] studies a much weaker version of incentive compatibility in which a mechanism is incentive compatible if misreporting cannot increase a user’s utility in the current round and [ZP20] does not consider strategic agents. In [Kan+20] it is assumed the users do not know their exact demands every round and need to provide feedback to the mechanism after each round of allocation to allow the mechanism to learn. The goal of the paper is to offer an (approximately) truthful and (approximately) efficient version of max-min fair allocation each iteration, despite the lack of information, but is not considering the dynamic notion of fairness that is the focus of our paper.
\(\gamma\)-incentive compatibility has seen a lot of recent applications. [Arc+03; Düt+12; KPS03; MS14] study combinatorial auctions that are almost incentive compatible. [DFP08] studies approximate incentive compatibility in machine learning, when users are asked to label data. [AB18] examines approximate incentive compatibility in large markets, where the number of users grows to infinity. [BSV19] develops algorithms that can estimate how incentive compatible mechanisms for buying, matching, and voting are. To the best of our knowledge we are the first to apply \(\gamma\)-incentive compatibility to the problem of resource sharing.

2 Notation

We use \([m]\) to denote the set \(\{1, 2, \ldots, m\}\) for any natural number \(m\). Additionally we define \(x^+ = \max(x, 0)\).

There are \(n\) users, where \(n \geq 2\). The set of users is denoted with \([n]\). The game is divided into epochs \(1, 2, \ldots, t, \ldots\). Every epoch there is a fixed amount of a resource shared amongst the users. We denote the total amount of resources with \(\mathcal{R}\), which w.l.o.g. we are usually going to assume that it is 1, unless stated otherwise.

We denote with \(r_i^t\) the allocation of user \(i\) in epoch \(t\). We also denote with \(R_i^t\) the cumulative allocation of user \(i\) up to round \(t\), i.e. \(R_i^t = \sum_{\tau=1}^t r_i^\tau\). By definition, \(R_i^0 = 0\).

Every epoch \(t\), each user \(i\) has a demand, denoted with \(d_i^t\). This represents the maximum allocation that is useful for that user, i.e. a user is indifferent between getting an allocation equal to her demand and an allocation higher than her demand, so the utility of user \(i\) on epoch \(t\) is \(u_i^t = \min(r_i^t, d_i^t)\). The total utility of user \(i\) after epoch \(t\) equals the sum of utilities up to that round, i.e. \(\hat{U}_i^t = \sum_{\tau=1}^t u_i^\tau = \sum_{\tau=1}^t \min(r_i^\tau, d_i^\tau)\).

Dynamic Max-min Fairness. In max-min fairness the resources are allocated such that the minimum amount of resources is maximized, then the second minimum is maximized, etc., as long as every user gains an amount of resources that does not exceed her demand. If for example we have \(\mathcal{R} = 1\) total resource and three users with demands \(1/4, 3/8,\) and \(1\), then the first user gets \(1/4\) resources and the other two get \(3/8\) resources each.

In dynamic max-min fairness, every epoch the max-min fairness algorithm is applied to the users’ cumulative allocations constrained by what they have already been allocated in previous iterations, i.e. given an epoch \(t\) and that every user \(i\) has cumulative allocation \(R_i^{t-1}\):

\[
\begin{align*}
&\text{choose } r_1^t, r_2^t, \ldots, r_n^t \\
&\text{applying max-min fairness on } R_1^{t-1} + r_1^t, R_2^{t-1} + r_2^t, \ldots, R_n^{t-1} + r_n^t \\
&\text{given the constraints } \sum_{i \in [n]} r_i^t \leq \mathcal{R}, \quad \forall i \in [n] : 0 \leq r_i^t \leq d_i^t
\end{align*}
\]

Incentives in Dynamic Max-min Fairness. It is known from [Gho+11] that applying max-min fairness when there is a single epoch is incentive compatible, i.e. users can never increase their allocation by misreporting their demand. In dynamic settings, however, this is not the case. As was shown by [Fre+18], a user can increase her allocation by misreporting. See also the improved lower bound (Theorem 3.1).

In this work we are interested in how far dynamic max-min fairness is from incentive compatible. W.l.o.g. we are usually going to study the possible deviations of user 1, i.e. how much user 1 can increase her allocation by lying about her demand. We use the symbols \(\hat{d}_i^t, \hat{r}_i^t, \hat{R}_i^t, \hat{u}_i^t, \hat{U}_i^t\) to denote the claimed demand and resulting outcome of some deviation of user 1. We want to prove that for some \(\gamma \geq 1\), dynamic max-min fairness is always \(\gamma\)-incentive compatible, i.e. for every users’ true demands \(\{d_i^t\}_{i,t}\), for every deviation of user 1 \(\{d_1^t\}_t\), and for every \(t\), to prove that

\[\hat{U}_1^t \leq \gamma U_1^t\]

\(\gamma\) is often referred to as the incentive compatibility ratio.
3 Approximate incentive compatibility ratio in single resource settings

**Bad Example.** As mentioned before, dynamic max-min fairness does not guarantee incentive compatibility. This is demonstrated in the following theorem, where user 1 can misreport her demand to increase her utility by a factor of almost 4/3.

**Theorem 3.1.** There is an instance with \( n + 2 \) users, in which a user can misreport her demand to increase her utility by a factor of \( \frac{4}{3} - \frac{1}{3} \cdot 2^{-n} \).

We defer the proof of the theorem to Appendix A. To provide intuition about how a user can increase her utility by misreporting, we include here the example of [Fre+18], where user 1 can increase her utility by a factor of 10/9.

**Example 1 ([Fre+18]).** There are 3 users and 3 epochs. The real demands of the users are shown in Table 1, as well as their allocations when user 1 is truthful and when she misreports.

| epoch | 1  | 2  | 3  |
|-------|----|----|----|
| user 1 | 1/2 | 1/4 | 1/2 |
| user 2 | 1/2 | 1  | 0  |
| user 3 | 0  | 0  | 1/2 |
|       |    | 3/4 | 1/2 |

Table 1: The black numbers are the users’ demands, the blue ones are the users’ allocation when user 1 is truthful, and the red ones are the allocations when user 1 misreports her demand on epoch 1 by demanding 0.

Because user 1 under-reports her demand on the first epoch, on the second epoch she manages to “steal” some of user 3’s resources. Then, on the third epoch the allocation mechanism equalizes the total allocations of users 1 and 2, making user 1 get back some of the resources she lost in epoch 1. This results in user 1 having 5/4 total resources instead of 9/8, i.e. her allocation increases by a factor of 10/9. ■

Both in Theorem 3.1 and Example 1, it is important to note that user 1 can increase her utility only by a small constant factor. Additionally, this is done by user 1 under-reporting her demand, not over-reporting; this is important because it implies that the resources allocated are always used by the users. As we will show next, both of these facts are true in general.

**Bounding allocations while one user deviates.** To prove the above, first we show a lemma offering a simple condition on which pair of users can gain overall allocations from one another. When users’ demands are not satisfied, for a user to get more resources someone else needs to get less. The lemma will allow us to reason about how a deviation by user 1 can lead to a user \( i \) (possibly \( i = 1 \)) getting more resources and another user \( j \) getting less.

**Lemma 3.2.** Fix an epoch \( t \) and let \( i, j \) be two different users. If the following conditions hold

- \( r^t_i < \hat{r}^t_i \) and \( r^t_i < d^t_i \), i.e. user \( i \) gets more resources on epoch \( t \) when user 1 deviates and user \( i \) could have gotten more resources when user 1 does not deviate.
- \( r^t_j > \hat{r}^t_j \) and \( r^t_j < d^t_j \), i.e. user \( j \) gets less resources on epoch \( t \) when user 1 deviates and user \( j \) could have gotten more resources when user 1 deviates.

then \( R^t_i \geq R^t_j \) and \( \hat{R}^t_i \leq \hat{R}^t_j \), implying

\[
\hat{R}^t_i - R^t_i \leq \hat{R}^t_j - R^t_j \tag{1}
\]

It should be noted that the conditions for \( i \) (similarly for \( j \)) can be simplified if \( i \) has the same demand in both outcomes (which is trivially true if \( i \neq 1 \)): if \( r^t_i < \hat{r}^t_i \) the other inequality is implied as \( d^t_i = d^t_i \) and \( r^t_i \leq d^t_i \).
Proof. Because of the conditions, we notice that \( r_i^t < d_i^t \) and \( r_j^t > 0 \), which implies that it would have been feasible to increase \( r_i^t \) by decreasing \( r_j^t \). This implies that \( \hat{R}_i^t \geq R_i^t \); otherwise it would have been more fair to give some of the resources user \( j \) got to user \( i \). With the analogous inverse argument (we can increase \( r_i^t \) by decreasing \( r_j^t \)) we can prove that \( \hat{R}_i^t \leq R_i^t \). This completes the proof. \[\blacksquare\]

The main technical tool in our work is the following lemma bounding the total amount all the users have “won” because of user 1 deviating, i.e. \( \sum_k (\hat{R}_k^t - R_k^t)^+ \). So rather than bounding the deviating user 1’s gain directly, it is better to consider the overall increase in all users combined. More specifically, the lemma upper bounds the increase of that amount after any epoch, given that user 1 does not over-report her demand (which as we are going to show later in Theorem 3.4 users have no benefit in doing). The bound on the total over-allocation \( \sum_k (\hat{R}_k^t - R_k^t)^+ \) then follows by summing over the time periods. The bound on the increase of \( \sum_k (\hat{R}_k^t - R_k^t)^+ \) after any epoch is different according to three different cases:

- If all users’ demands are satisfied, then the increase is at most 0.
- If user 1 is truthful the increase is again at most 0 so in these steps over-allocation can move between users but cannot increase. This is the reason working with the total over-allocation is so helpful.
- If user 1 under-reports the increase is bounded by the amount of resources she receives when she is truthful.

**Lemma 3.3.** Fix any \( t \geq 1 \). Let \( \{R_i^{t-1}\}_{i \in [n]} \) and \( \{\hat{R}_i^{t-1}\}_{i \in [n]} \) be the cumulative allocations up to epoch \( t - 1 \). Assume that \( \{d_i^t\}_{i \in [n]} \) are some users’ demands and that \( \{\hat{d}_i^t\}_{i \in [n]} \) are the same demands except user 1’s, who deviates but does not over-report, i.e. \( \hat{d}_1^t \leq d_1^t \). Then it holds that

\[
\sum_{k \in [n]} (\hat{R}_k^t - R_k^t)^+ - \sum_{k \in [n]} (\hat{R}_{k-1}^t - R_{k-1}^t)^+ \leq r_1^t \left[ \sum_k d_k^t > R \text{ and } \hat{d}_1^t < d_1^t \right]
\]  

(2)

When all demands are satisfied, user 1 clearly cannot change other users’ allocation by under-reporting. We will use Lemma 3.2 to show that if user 1 is truthful on epoch \( t \), then the l.h.s. of (2) is at most 0; as max-min fairness allocates resources such that the large \( \hat{R}_i^t - R_i^t \) are decreased and the small \( \hat{R}_i^t - R_i^t \) are increased. Finally, if user 1 under-reports her demand then the (at most) \( r_1^t \) resources user 1 does not get might increase the total over-allocation by the same amount.

Proof. We first focus on the case when the users’ demands are satisfied. In this case, because user 1 can only under-report her demand, she cannot alter other users’ allocations and she can only decrease her allocation. This entails that \( \hat{r}_k^t \leq r_k^t \) for all \( k \), so \( \hat{R}_k^t - R_k^t \leq \hat{R}_{k-1}^t - R_{k-1}^t \), proving this case of the lemma.

To prove the other case of the lemma, define \( P^t = \{i \in [n] : \hat{R}_i^t \geq R_i^t\} \) for all \( t \). Suppose by contradiction:

\[
\sum_{k \in P^t} (\hat{R}_k^t - R_k^t) - \sum_{k \in P^{t-1}} (\hat{R}_{k-1}^t - R_{k-1}^t) > r_1^t \left[ \hat{d}_1^t < d_1^t \right]
\]

Because \( \sum_{k \in P^t} (\hat{R}_k^t - R_k^t) \leq \sum_{k \in P^{t-1}} (\hat{R}_{k-1}^t - R_{k-1}^t) \), the above inequality implies

\[
\sum_{k \in P^t} (\hat{r}_k^t - r_k^t) > r_1^t \left[ \hat{d}_1^t < d_1^t \right]
\]

(3)

Because user 1 does not over-report her demand, it holds that \( \sum_k r_k^t \geq \sum_k \hat{r}_k^t \), i.e. the total resources allocated to the users does not increase when user 1 deviates. Combining this fact with (3) we get that

\[
\sum_{k \notin P^t} (r_k^t - \hat{r}_k^t) > r_1^t \left[ d_1^t < \hat{d}_1^t \right]
\]

(4)

We notice that because of (3), there exists a user \( i \in P^t \) for whom \( \hat{r}_i^t > r_i^t \); because of (4), there exists a user \( j \notin P^t \) for whom \( r_j^t > \hat{r}_j^t \). Additionally for that \( j \) we can assume that \( d_j^t = \hat{d}_j^t \) because:
proves that this is no longer the case with multiple

3.4

by summing (4) implies \( \sum_{k \in P^t, k \neq 1} (r_k^1 - r_k^j) > 0 \), i.e. \( j \neq 1 \) and we assumed that only user 1 deviates.

Thus we have \( d_i^t \leq d_i^j \) (since no user over-reports), \( \hat{d}_j^t = d_j^t, \hat{r}_j^t > r_j^t \), and \( \hat{r}_j^t < r_j^t \). Now Lemma 3.2 proves that \( \hat{R}_i^t - R_i^t \leq \hat{R}_j^t - R_j^t \). This leads to a contradiction, because \( i \in P^t \) and \( j \notin P^t \), i.e. \( \hat{R}_i^t - R_i^t \geq 0 > \hat{R}_j^t - R_j^t \). ■

Adversarial Demands. In this section we will prove upper bounds on the incentive compatibility ratio when users’ demands are picked adversarially. We will prove that users have no incentive to over-report their demand. The immediate effect of over-reporting is allocating resources to user 1 in excess of her demand, which do not contribute to her utility. Intuitively, this suggest that user 1 is put into a disadvantageous position: other users get less resources which makes them be favored by the allocation algorithm in the future, while user 1 becomes less favored. However, a small change in the users’ resources causes a cascading change in future allocations making the proof of this theorem hard. We will see in Section 4 that this is no longer the case with multiple resources when users only use a subset of them. We defer the proof the theorem to the end of this section.

Theorem 3.4. Users have nothing to gain by declaring a demand higher than their actual demand.

First we show that using this theorem and Lemma 3.3 allows us to bound the incentive to deviate. A bound of 2 is easy to get by summing (2) for all \( t \) up to some certain epoch. We give an incentive compatibility bound of 3/2 by using the same lemma, but arguing that some other user \( j \) must also share the same increased allocation of resources using Lemma 3.2.

Theorem 3.5. No user can misreport her demand to increase her utility by a factor larger than \( \frac{3}{2} \), i.e. for any user \( i \) and deviation user \( i \) makes, for all \( t \), \( \hat{U}_i^t \leq \frac{3}{2} U_i^t \).

Proof. Theorem 3.4 implies that it is no loss of generality to assume that user 1 does not over-report her demand, since any benefit gained can be gained by changing every over-report to a truthful one. This means that instead of \( \hat{U}_i^t \leq \frac{3}{2} U_i^t \) we can show \( R_i^t \leq \frac{3}{2} R_i^t \). Towards a contradiction, let \( t \) be the first epoch when user 1 gets more than \( 3/2 \) more resources by some deviation of demands, i.e. \( R_i^t > \frac{3}{2} R_i^t \) and \( \hat{R}_i^t \leq \frac{3}{2} R_i^t \). This implies that \( \hat{r}_i^t > r_i^t \), which in turn entails that there exists a user \( j \) for whom \( \hat{r}_j^t < r_j^t \), since the total resources allocated when user 1 is under-reporting cannot be less than those when 1 is truthful. Because \( \hat{r}_i^t > r_i^t, \hat{r}_j^t < r_j^t, d_i^t \leq d_i^j \), and \( d_j^t = d_j^j \), we can use Lemma 3.2 and get \( R_i^t - R_i^t \leq \hat{R}_j^t - R_j^t \). This inequality, \( \hat{R}_i^t - R_i^t \geq 0 \), and Lemma 3.3 by summing (2) for every epoch up to \( t \), implies

\[
2 (\hat{R}_i^t - R_i^t) \leq (\hat{R}_i^t - R_i^t) + (\hat{R}_j^t - R_j^t) \leq \sum_{k} (\hat{R}_k^t - R_k^t) \leq \sum_{t=1}^{T} r_i^t = R_i^t
\]

The above inequality leads to \( \hat{R}_i^t \leq \frac{3}{2} R_i^t \), a contradiction. ■

Next we prove Theorem 3.4, that users have no incentive to over-report.

Proof of Theorem 3.4. Fix an epoch \( T \) and let \( \{\hat{d}_i^t\}_{i,t} \) be any demands (that possibly involve user 1 both over and under-reporting). We are going to show that if user 1 changes every over-report to a truthful one, then her utility on epoch \( T \) is not going to decrease. Let \( T_0 \leq T \) be the last epoch where user 1 over-reported. For all users \( i \) and epochs \( t \in [1, T] \), let \( d_i^t = \hat{d}_i^t \), except for \( d_i^{T_0} \) which is 1’s actual demand (note that \( d_i^{T_0} \neq \hat{d}^{T_0}_i \)). Let \( \bar{r}, \bar{R}, \bar{u}, \bar{U} \) be the result of demands \( \bar{d} \). We will show that \( U_T^{\bar{d}} \geq U_T^{d} \), i.e. user 1 does not prefer the demand sequence \( \{\hat{d}_i^t\}_{i,t} \) over the demand sequence \( \{d_i^t\}_{i,t} \). If we apply this inductively for every epoch before \( T \) where user 1 over-reports, we are going to get that over-reporting is not a desirable strategy.
Up to epoch $T_0 - 1$ all users’ demands in $\hat{d}$ and $\tilde{d}$ are the same and thus so are the allocations and utilities: for all $i$, $\hat{R}_{i_{T_0 - 1}} = \hat{R}_{i_{T_0 - 1}}$ and $\tilde{U}_{i_{T_0 - 1}} = \tilde{U}_{i_{T_0 - 1}}$. Because $\hat{d}^{T_0} > \hat{d}_{1}^{T_0}$ and for $i \neq 1$, $\hat{d}^{T_0} = \tilde{d}_{i}^{T_0}$, user 1 may earn some additional resources on $T_0$, i.e. $\hat{r}_{1}^{T_0} - \hat{r}_{1}^{T_0} = \hat{R}_{1}^{T_0} - \hat{R}_{1}^{T_0} = x$, for some $x \geq 0$, while other users $i \neq 1$ get less resources: $\hat{r}_{i}^{T_0} - \hat{r}_{i}^{T_0} = \hat{R}_{i}^{T_0} - \hat{R}_{i}^{T_0} \leq 0$. We first note that the $x$ additional resources that user 1 gets are in excess of 1’s true demand, meaning they do not contribute towards 1’s utility:

$$\hat{U}_{1}^{T_0} - \hat{U}_{1}^{T_0} = \hat{R}_{1}^{T_0} - \hat{R}_{1}^{T_0} - x = 0$$  

(5)

Additionally, because user 1 does not over-report $\hat{d}$ or $\tilde{d}$ in epochs $T_0 + 1$ to $T$ (by assumption $T_0$ is the last epoch before $T$ where user 1 over-reports), it holds that for $t \in [T_0 + 1, T]$, user 1’s utility is the same as the resources she receives: $\hat{u}_{1}^{t} = \hat{r}_{1}^{t}$ and $\tilde{u}_{1}^{t} = \tilde{r}_{1}^{t}$. This fact, combined with (5) proves that

$$\forall t \in [T_0, T], \hat{U}_{1}^{t} - \hat{U}_{1}^{t} = \hat{R}_{1}^{t} - \hat{R}_{1}^{t} - x$$  

(6)

Thus, in order for this over-reporting to be a strictly better strategy, it must hold that $\hat{R}_{1}^{T} - \hat{R}_{1}^{T} > x$. We will complete the proof by proving that the opposite holds. Since there is no over-reporting in periods $t \in [T_0 + 1, T]$ we can use Lemma 3.3, where we use $\tilde{d}_{1}^{t} \leq \hat{d}_{1}^{t}$ in place of $\hat{d}_{1}^{t} \leq d_{1}^{t}$ and summing (2) for all $t \in [T_0 + 1, T]$ and noticing that $\tilde{d}_{1}^{t} = \hat{d}_{1}^{t}$ we get

$$\sum_{k} \left( \hat{R}_{k}^{T} - \tilde{R}_{k}^{T} \right)^{+} - \sum_{k} \left( \hat{R}_{k}^{T_0} - \tilde{R}_{k}^{T_0} \right)^{+} \leq 0$$

The above, because $(\hat{R}_{k}^{T} - \tilde{R}_{k}^{T})^{+} \geq 0, \hat{R}_{k}^{T_0} - \tilde{R}_{k}^{T_0} \leq 0$ if $k \neq 1$, and $\hat{R}_{1}^{T_0} - \tilde{R}_{1}^{T_0} = x \geq 0$, proves that $\hat{R}_{1}^{T} - \tilde{R}_{1}^{T} \leq x$. This completes the proof.  

**Random Demands.** In this section we study the case where the users’ demands are random variables. Note that Theorem 3.4 still holds, i.e. it is no loss of generality to assume no user over-reports her demands. Using Lemma 3.3 we are going to show that by increasing the amount of total available resources, $\mathcal{R}$, by a little, we can lower the additional expected amount user 1 can gain by deviating. The setup is the following:

1. $d_{i}^{t}$ is the demand of user $i$ at round $t$, drawn by some distribution. Note that we do not require $d_{i}^{t}$ and $d_{i}^{t}'$ to be independent or to come from the same distribution.
2. For $i \neq j$, $d_{i}^{t}$ and $d_{j}^{t}$ are distributed independently.
3. We assume that the expected sum of demands does not increase over time. More specifically, w.l.o.g. we assume that for every $t$, $\sum_{i} E [d_{i}^{t}] \leq 1$.
4. For every $t$, $\sum_{i} \text{Var} (d_{i}^{t}) = O(1/n)$.
5. For all $t$ and for all $i$, $\max (d_{i}^{t}) = O(1/\sqrt{n})$.

Recall that by Lemma 3.3, the maximum amount of resources user 1 can get by misreporting in a single round is $r_{1}^{t} \mathbb{1} \left[ \sum_{i} d_{i}^{t} > \mathcal{R} \right]$. For adversarial demands this quantity can always be $r_{1}^{t}$. However, when demands are randomized their sum is larger that their expectation (which is at most 1) with small probability. This allows us to bound the expectation of $r_{1}^{t} \mathbb{1} \left[ \sum_{i} d_{i}^{t} > \mathcal{R} \right]$, which will imply that the expected benefit of misreporting is a small fraction of $r_{1}^{t}$. In the next lemma we bound the expectation of $r_{1}^{t} \mathbb{1} \left[ \sum_{i} d_{i}^{t} > \mathcal{R} \right]$ by a quantity that can be then bounded using any concentration inequality.

**Lemma 3.6.** If conditions 1-5 are true, then for every $t$ and for every allocation $\left\{ R_{i}^{t-1} \right\}_{i \in [n]}$ it holds that

$$E \left[ r_{1}^{t} \mathbb{1} \left[ \sum_{i} d_{i}^{t} > \mathcal{R} \right] \right] \leq E \left[ r_{1}^{t} \right] \mathbb{P} \left[ \left| \sum_{i \neq 1} d_{i}^{t} - \sum_{i \neq 1} E \left[ d_{i}^{t} \right] \right| > \mathcal{R} - 1 - \max (d_{1}^{t}) \right]$$  

(7)

where the expectations and the probabilities are taken over $\left\{ d_{i}^{t} \right\}_{i \in [n]}$. 

8
Proof. The allocation \( r^t_i \) is a complicated function of the variables \( \{ R_i^{t-1} \}_{i \in [n]} \) and the random variables \( \{ d_i^t \}_{i \in [n]} \). This means that \( r^t_i \) and \( 1 \left[ \sum_i d_i^t > R \right] \) are not independent, so we cannot bound the expectation of their product by the product of their expectations, which makes the the proof more involved. For this reason we need to bound them by quantities that are independent. We first note that the following holds for any realization of the random variables.

\[
\begin{align*}
    r^t_i 1 \left[ \sum_i d_i^t \leq R \right] &\geq r^t_i 1 \left[ \max(d_i^t) + \sum_{i \neq 1} d_i^t \leq R \right], \\

\end{align*}
\]

This makes the two terms on the right hand side “less dependant” but they are still not independent: the realization of \( \{ d_i^t \}_{i \in [n]} \) affects both terms. We then take the expectation of the above

\[
\begin{align*}
    \mathbb{E} \left[ r^t_i 1 \left[ \sum_i d_i^t \leq R \right] \right] &\geq \mathbb{E} \left[ r^t_i 1 \left[ \max(d_i^t) + \sum_{i \neq 1} d_i^t \leq R \right] \right], \\

\end{align*}
\]

\[
\begin{align*}
    \left( \text{law of total expectation} \right) \quad \mathbb{E} \left[ r^t_i \right] \mathbb{P} \left[ \max(d_i^t) + \sum_{i \neq 1} d_i^t \leq R \right] &\geq \mathbb{E} \left[ r^t_i \right] \mathbb{P} \left[ \max(d_i^t) + \sum_{i \neq 1} d_i^t \leq R \right], \\

\end{align*}
\]

\[
\begin{align*}
    \left( \text{demands are satisfied: } r^t_i \right) \quad \mathbb{E} \left[ r^t_i \right] \mathbb{P} \left[ \max(d_i^t) + \sum_{i \neq 1} d_i^t \leq R \right] &\geq \mathbb{E} \left[ r^t_i \right] \mathbb{P} \left[ \max(d_i^t) + \sum_{i \neq 1} d_i^t \leq R \right], \\

\end{align*}
\]

We can express \( \mathbb{E} \left[ r^t_i \right] = \mathbb{E} \left[ r^t_i 1 \left[ \sum_i d_i^t > R \right] \right] + \mathbb{E} \left[ r^t_i 1 \left[ \sum_i d_i^t \leq R \right] \right] \) which makes the above inequality

\[
\begin{align*}
    \mathbb{E} \left[ r^t_i 1 \left[ \sum_i d_i^t > R \right] \right] &\leq \mathbb{E} \left[ r^t_i \right] \mathbb{P} \left[ \max(d_i^t) + \sum_{i \neq 1} d_i^t > R \right], \\

\end{align*}
\]

\[
\begin{align*}
    &\leq \mathbb{E} \left[ r^t_i \right] \mathbb{P} \left[ \sum_{i \neq 1} d_i^t - \sum_{i \neq 1} \mathbb{E} \left[ d_i^t \right] > R - 1 - \max(d_i^t) \right], \\

\end{align*}
\]

where in the last inequality we used the fact that \( \sum_{i \neq 1} \mathbb{E} \left[ d_i^t \right] \leq \sum_i \mathbb{E} \left[ d_i^t \right] \leq 1 \). 

Using the above lemma and Lemma 3.3 we can use Chebyshev’s inequality to show that as we make \( R \) larger than 1, the additional amount of resources that user 1 can get by deviating diminishes.

\textbf{Theorem 3.7.} If conditions 1-5 are true and for the total resources it holds that \( R \geq 1 + c/\sqrt{n} \) for some \( c \in \Omega(1) \), then for all \( t \) and any deviation of user \( i \) it holds that

\[
\mathbb{E} [\hat{U}^t_i] \leq \mathbb{E} [U^t_i] \left( 1 + O \left( 1/c^2 \right) \right)
\]

With the assumptions in the theorem, we can use Chebyshev’s inequality to upper bound the probability in (7) by \( O(1/c^2) \). If we combine this with Lemma 3.3, we get that every epoch the expected additional resources user 1 can get by deviating is at most \( \mathbb{E} [r^t_1 O(1/c^2)] \). The proof is included in Appendix A.

If we use \( c = \sqrt{n} \) in Theorem 3.7 we get the following corollary, in which both the incentive compatibility ratio and the total resources tend to 1 as \( n \to \infty \):

\textbf{Corollary 3.8.} If the total available resources are \( R \geq 1 + \frac{1}{\sqrt{n}} \), then \( \mathbb{E} [\hat{U}^t_i] \leq \mathbb{E} [U^t_i] \left( 1 + O \left( \frac{1}{\sqrt{n}} \right) \right) \).
Remark 3.9. We extend our results to dynamic weighted max-min fairness, where every user $i$ has a weight $w_i$ and the mechanism applies max-min fairness on $\{R_t^i/w_i\}_{i \in [n]}$ instead of $\{R_t^i\}_{i \in [n]}$. The same holds for the case where a group of users $I \subset [n]$ forms a coalition to increase their total allocation $\sum_{i \in I} R_t^i$. In Section 5 we show that dynamic weighted max-min fairness with coalitions satisfies a 2-incentive compatibility upper bound (Theorem 5.2) and a result similar to Theorem 3.7 for randomized demands (Theorem 5.3).

4 Multiple Resources

In this section we are going to explore the generalization of dynamic max-min fairness for multiple resources, dynamic dominant resource fairness. There are $m \geq 1$ different resources, denoted with $1, 2, \ldots, m$. W.l.o.g., we assume that for every resource the amount available is $R_i$, and $R = 1$ unless stated otherwise.

Every user $i$ has $m$ non-negative values, $a_{i1}, a_{i2}, \ldots, a_{im}$ called ratios, the ratios of the different resources that the user needs for the application running, i.e. every epoch, for some $x \geq 0$ user $i$ uses $x a_{iq}$ of every resource $q \in [m]$. W.l.o.g. we assume that $\max_{i,q} a_{iq} = 1$.

We assume that the users’ ratios do not change over time. The ratios depend on the type of application running by the user (e.g., Spark tasks [Zah+10] or web caches [Ber+20]), so we will assume they are fixed given the type of application, and also publicly known as the application running is public. The total amount of resources that the user needs for each epoch changes, depending on the current traffic. We denote with $r_t^i$ the allocation of user $i$ in epoch $t$, i.e. on epoch $t$ user $i$ receives $r_t^i a_{iq}$ of every resource $q \in [m]$. Let $R_t^i$ be the cumulative allocation of user $i$ up to epoch $t$, i.e. $R_t^i = \sum_{\tau=1}^t r_t^i$. By definition, for every $i$, $R_0^i = 0$. As in the single resource case we assume $i$’s utility on epoch $t$ is $U_t^i = \min(r_t^i, d_t^i)$ and her total utility up to epoch $t$ is $U_t^i = \sum_{\tau=1}^t u_t^i$.

Dynamic Dominant Resource Fairness. Dominant resource fairness (DRF) is the generalization of max-min fairness for the case of multiple resources, where the fairness criteria is applied to the allocations $r_t^i$. Using our notation, dynamic DRF is easy to describe. For a given epoch $t$, assuming that every user $i$ has cumulative allocation $R_t^{i-1}$: choose $r_1^t, r_2^t, \ldots, r_n^t$

applying max-min fairness on $R_t^{i-1} + r_1^t, R_t^{i-1} + r_2^t, \ldots, R_t^{i-1} + r_n^t$

given the constraints $\forall i \in [n]: 0 \leq r_t^i \leq d_t^i$, $\forall q \in [m]: \sum_{i \in [n]} r_t^i a_{iq} \leq 1$

It is nice to observe a dissimilarity: if for every user $i$ it holds that $\max_q a_{iq} = 1$, then every user is treated similarly by the allocation algorithm. In contrast, if for some user $i$, $\max_q a_{iq} < 1$, then user $i$ is less favored: if for example $\max_q a_{iq} = 1/2$, then user $i$ needs double the allocation to get the same amount of her dominant resource. This means that the allocation algorithm that we use is the dynamic version of a generalization of DRF, weighted DRF, which uses the values $\max_q a_{iq}$ as weights to favor some users more than others.

Potential Value of Over-reporting. We first show that when some resources are only used by a subset of users, a user can over-report her demand to increase her allocation by an arbitrary amount. Next we will show that this is no longer possible once all resources are used by all users, even if the ratios are very different.

Theorem 4.1. There is an instance with $n$ users where a user can over-report her demand to increase her utility by a factor of $\Omega(n)$.

We sketch the idea of the example here, with details presented in Appendix B, see Table 3. Consider an example with two resources. User 1 has $a_{11} = 1$ and $a_{12} = 0$, some users $i \in A \subset [n]$ have $a_{i1} = 1$ and $a_{i2} = 1 - 1/|A|$,
and the rest of the users \( j \in B \subseteq [n] \) have \( a_{j1} = 0 \) and \( a_{j2} = 1/|B| \). If the demands of users in \( A \) and \( B \) are 1, and the demand of user 1 is 0, then for \( i \in A \) and \( j \in B \), \( r_i = r_j = 1/|A| \).

However, if user 1 over-reports her demand from 0 to 1, then for \( i \in A \), \( \hat{r}_i = \hat{r}_1 = 1/(|A|+1) \) and for \( j \in B \), \( \hat{r}_j = 2/(|A|+1) \). This does put user 1 in a disadvantage, because she has earned \( 1/(|A|+1) \) useless resources, but has also increased the allocation of every user in \( B \) by close to \( 1/(|A|+1) \), for a total increase of almost \( |B|/(|A|+1) \); this will allow user 1 to increase her future allocation because users in \( B \) also have a larger allocation.

**Approximate Incentive Compatibility of Dynamic Resource Fairness.** We will see that the benefit of over-reporting can only happen if users use only a subset of the resources. The main results of the rest of this section are extending the results of Section 3 assuming users use every resource: over-reporting is no longer incentive compatible (Theorem 4.3), and the approximate incentive compatibility ratio is bounded (Theorem 4.5 and Theorem 4.6) with the bound depending on the ratio \( \rho \) of the values \( a_{iq} \): \( \rho = \min_{i,j,q} \frac{a_{iq}}{a_{jq}} \).

Throughout the remainder of this section, we will assume that all users use each resource, that is \( a_{iq} > 0 \) for all \( i, q \). This is often the case in practice when resource sharing is applied in computer systems: most tasks run by a user require non-zero amount of each resource (e.g. CPU, memory, and storage).

**Bounding allocations while one user deviates.** We start by pointing out why the example of the incentive to over-report in the previous section no longer works if \( a_{j1} > 0 \) for \( j \in B \) and \( a_{12} > 0 \): in order for the resulting allocation to be fair, when user 1 over-reports, all the users’ allocation must be the same. This is true because decreasing the allocation of any user always makes feasible the increase of other users’ allocation (up to their demand). This boils down to the fact that if every \( a_{iq} \) is positive then if users’ demands are not met it is because of a single resource being saturated. In contrast, when some users’ ratios are zero, different users are limited by different saturated resources.

The assumption that all users are using each resource allows us to prove a lemma analogous to Lemma 3.2, as with this assumption, we can increase any user’s allocation by decreasing another allocation, allowing the proof to be identical to the proof of Lemma 3.2. For example, if \( a_{iq} = 0 \) and \( a_{jq} > 0 \), then decreasing the allocation of user \( i \) does not free any amount of resource \( q \) for user \( j \). Proving this lemma will lead to results similar to Theorems 3.4, 3.5, and 3.7.

**Lemma 4.2.** Fix an epoch \( t \) and assume that for all users \( i \in [n] \) and resources \( q \in [m] \) it holds that \( a_{iq} > 0 \). Let \( i, j \) be two different users. If the following conditions hold

- For \( i \), \( r_i < \hat{r}_i \) and \( r_i < d_i \).
- For \( j \), \( r_j < \hat{r}_j \) and \( r_j < d_j \),

then \( R_i \geq \hat{R}_i \) and \( \hat{R}_j \leq R_j \), implying

\[
\hat{R}_i - R_i \leq \hat{R}_j - R_j
\]

Similarly to Theorem 3.4, we can now prove that there is no benefit to over-reporting. As mentioned previously, this is a very important property because it guarantees that every resource allocated is utilized.

**Theorem 4.3.** Assume that for all users \( i \in [n] \) and resources \( q \in [m] \) it holds that \( a_{iq} > 0 \). Then the users have nothing to gain by declaring a demand higher than their actual demand.

Because of Lemma 4.2 the proof of this theorem is quite similar to the one in Theorem 5.1: if user 1 over-reports her demand to get \( x \) more not useful resources, we can prove that \( \hat{R}_1 - R_1 \leq x \), i.e. user 1 will not get additional useful resources. The full proof can be found in Appendix B.

Next we present an auxiliary lemma, similar to Lemma 3.3, but this time we bound the increase of any user’s allocation when user 1 deviates, i.e. \( \hat{R}_k - R_k \) for each \( k \). Unfortunately, it is a weaker version of Lemma 3.3,
involving the parameter $\rho_1 = \min_{k,q} \frac{a_{kq}}{a_{iq}}$. If the users ratios are the same, i.e. $a_{iq} = a_{jq}$ for every $q$ and $i \neq j$, then $\rho_1 = 1$ and the following lemma would allow us to prove a 2-incentive compatibility ratio upper bound. In general however, the incentive compatibility ratio depends on $\rho_1$ and it becomes larger the smaller $\rho_1$ is.

**Lemma 4.4.** Assume that for all users $i \in [n]$ and resources $q \in [m]$ it holds that $a_{iq} > 0$. Then, for every user $i$ and every epoch $t$, if user 1 does not over-report her demand, it holds that

$$\rho_1 \left( \hat{R}_i^t - R_i^t \right) \leq \sum_{t=1}^{\tau} \sum_{q \in [m]} a_{kq} d_k^t > R$$

The lemma’s proof is similar to Lemma 3.3. If the users’ demands are satisfied then user 1 cannot decrease the allocation of other users. If users’ demands are not satisfied, then user 1 can free at most $r_1^t a_{1q}$ from every resource $q$ which can increase user $k$’s allocation by at most $r_1^t \frac{a_{1q}}{a_{kq}}$. The full proof can be found in Appendix B.

**Adversarial Demands.** Using Lemma 4.4 it is easy to derive an upper bound on how much user 1 can increase her allocation when deviating.

**Theorem 4.5.** Assume that for all users $i \in [n]$ and resources $q \in [m]$ it holds that $a_{iq} > 0$. Then for any $i \in [n]$, user $i$ cannot misreport her demand to increase her utility by a factor larger than $1 + \rho_i$, where $\rho_i = \min_{k,q} \frac{a_{kq}}{a_{iq}}$.

**Proof.** W.l.o.g. we are going to prove the theorem for $i = 1$. Because of Theorem 4.3 we can assume that user 1 does not over-report her demand and thus we can bound $\hat{R}_1^t$ instead of $\hat{U}_1^t$. Lemma 4.4 implies that for any $t$, $\rho_1 \left( \hat{R}_1^t - R_1^t \right) \leq R_1^t$, which proves the theorem. \qed

**Random Demands.** Next we present a result similar to Theorem 3.7 when users’ demands are independent random variables. By raising the total amount of resources by a little, we can make the expectation of the r.h.s. of (8) arbitrarily small, using Chebyshev’s inequality. The full proof of this theorem can be found in Appendix B.

**Theorem 4.6.** Assume that the following conditions hold

- For every epoch $t$ and every users $i \neq j$, $d_i^t$ and $d_j^t$ are distributed independently.
- For every epoch $t$, $\sum_i \mathbb{E} \left[ d_i^t \right] \leq 1$.
- For every epoch $t$, $\sum_i \text{Var} \left( d_i^t \right) = O(1/n)$.
- For every epoch $t$ and every user $i$, $\max(d_i^t) = O(1/\sqrt{n})$.

Fix a user $i$ and let $\rho_i = \min_{k,q} \frac{a_{kq}}{a_{iq}}$. If for the total resources it holds that $R \geq 1 + c/\sqrt{n}$ for some $c = \Omega(1)$, then for all epochs $t$ and any deviation of user $i$ it holds that

$$\mathbb{E} [\hat{U}_i^t] \leq \mathbb{E} [U_i^t] \left( 1 + O \left( 1/c^2 \right) \right)$$

**Remark 4.7.** The result of this section extend to time varying ratios as long as the ratios are public knowledge. Assume on epoch $t$ the ratios of user $i$ are $\{a_{i,q}^t\}_{q \in [m]}$. If we define $\rho_i = \sup_{k,q,t} \frac{a_{kq}}{a_{iq}}$ then Theorems 4.5 and 4.6 still hold.

## 5 Coalitions and weighted max-min fairness

In this section we present a generalized version of our previous results: we extend the results of Section 3 to the case of weighted max-min fairness and when users collude to increase their overall resources.
In weighted max-min fairness, the allocation algorithm applies max-min fairness on the the weighted allocations of the users, i.e. every epoch \( t \) the following problem is solved

\[
\text{choose } r^t_1, r^t_2, \ldots, r^t_n \text{ applying max-min fairness on } \frac{R^{t-1}_1 + r^t_1}{w_1}, \frac{R^{t-1}_2 + r^t_2}{w_2}, \ldots, \frac{R^{t-1}_n + r^t_n}{w_n}
\]

given the constraints

\[
\sum_{i \in [n]} r^t_i \leq R, \quad \forall i \in [n] : 0 \leq r^t_i \leq d^t_i
\]

where \( w_1, w_2, \ldots, w_n \) are positive numbers. Assuming that \( \sum_i w_i = 1 \), the numbers \( w_1, w_2, \ldots, w_n \) usually correspond to the entitlement of the users, i.e. \( w_i \) is the fraction of resources user \( i \) is expected to use every round or reserved.

When users form coalitions they try to increase the sum of their utilities by each member of the coalition deviating. We bound that increase, i.e. if the set \( I \subseteq [n] \) of users forms a coalition and demands \( \{d^t_i\}_{i \in I, t} \) instead of \( \{d^t_i\}_{i \in I, t} \), then for some \( \gamma \geq 1 \) and for all \( t \) we want to prove that \( \sum_{i \in I} \hat{U}^t_i \leq \gamma \sum_{i \in I} U^t_i \). Proofs of the results in this section are included in Appendix C.

Analogously to Theorem 3.4 we have that users have nothing to gain by over-reporting.

**Theorem 5.1.** Let \( I \subseteq [n] \) be a set of users that form a coalition and \( w_1, \ldots, w_n \) be any weights, according to which weighted max-min fairness allocates resources. Then the users in \( I \) have nothing to gain by declaring a demand higher than their actual demand.

Next we show a generalized version of Theorem 3.5. More specifically, that users in the coalition cannot misreport their demands to increase their total utility by a factor more than 2 and if there is no coalition (\(| I | = 1 \)) no user \( i \) can increase their utility by a factor larger than \( 1 + \max_{j \neq i} \frac{w_j}{w_i + w_j} \) (which is strictly less than 2).

**Theorem 5.2.** Let \( I \subseteq [n] \) be a set of users that form a coalition and \( w_1, \ldots, w_n \) be any weights, according to which weighted max-min fairness allocates resources. Then for any deviation of the users in \( I \) and any epoch \( t \) it holds that

\[
\sum_{i \in I} \hat{U}^t_i \leq 2 \sum_{i \in I} U^t_i
\]

Additionally, when \( I = \{i\} \) for any user \( i \), then

\[
\hat{U}^t_i \leq \left( 1 + \max_{j \neq i} \frac{w_j}{w_i + w_j} \right) U^t_i
\]

To prove the above theorem we prove Lemma C.2, a generalization of Lemma 3.3 bounding the additional allocation of all users, where the \( r^t_1 \) in the right hand side of the inequality is replaced with \( \sum_{i \in I} r^t_i \). The same lemma improves the incentive compatibility ratio when the users’ demands are random variables, like in Theorem 3.7.

**Theorem 5.3.** Let \( I \subseteq [n] \) be a set of users that form a coalition, \( w_1, \ldots, w_n \) be any weights according to which weighted max-min fairness allocates resources and assume that the following conditions hold

- For every \( t \) and every \( i \notin I \) and \( j \notin I \) such that \( i \neq j \), \( d^t_i \) and \( d^t_j \) are distributed independently.
- For every \( t \), \( \sum_{i \notin I} \mathbb{E} \left[ d^t_i \right] \leq 1 \).
- For every \( t \), \( \sum_{i \notin I} \text{Var} \left( d^t_i \right) = O(1/n) \).
- For all \( t \) and for all \( i \in I \), \( \max(d^t_i) = O(1/\sqrt{n}) \).

If for the total resources it holds that \( R \geq 1 + c/\sqrt{n} \) for some \( c = \Omega(|I|) \), then for all epochs \( t \) and any deviations of the users in \( I \) it holds that

\[
\mathbb{E} \left[ \sum_{i \in I} \hat{U}^t_i \right] \leq \mathbb{E} \left[ \sum_{i \in I} U^t_i \right] \left( 1 + O \left( \frac{1}{c^2} \right) \right)
\]
6 Getting more utility multiple times

Finally, in this section we study what happens when user 1 deviates and gets more utility over an extended time period, or multiple times. More specifically, assuming that there are alternating intervals where either \( \tilde{R}_1 > R_1 \) or \( \tilde{R}_1 < R_1 \) we study the length of those intervals and the duration between between them. We first make the following definitions:

- For \( \ell = 0, 1, 2, \ldots \) let \( s_\ell \) be distinct and ordered times (i.e. \( s_{\ell-1} < s_\ell \)) when user 1 begins having more resources by misreporting, i.e. \( \tilde{R}_{1s_\ell-1}^s \leq R_{1s_{\ell-1}}^s \) and \( \tilde{R}_{1s_\ell}^s > R_{1s_\ell}^s \).
- For \( \ell = 0, 1, 2, \ldots \) let \( e_\ell \) be the first time after epoch \( s_\ell \) when user 1 begins having less resources by misreporting, i.e. \( \tilde{R}_{1e_\ell-1}^e \geq R_{1e_{\ell-1}}^e \) and \( \tilde{R}_{1e_\ell}^e < R_{1e_\ell}^e \).

Note that \( 0 < s_0 < e_0 < s_1 < e_1 < \ldots \) by definition. Using the above notation we prove that if during every interval \([s_\ell, e_\ell]\) user 1 got a factor of \( \gamma \) more resources on epoch \( t_\ell \) for some \( t_\ell \in [s_\ell, e_\ell] \) by misreporting, then \( t_\ell \) cannot be much larger than \( s_\ell \) and that also \( t_\ell \) scales exponentially with \( \ell \).

**Theorem 6.1.** Let there be only one resource. Assume that for every \( t, R_t^1 \in \Theta(t) \) and for every \( \ell = 0, 1, \ldots \) there exists an epoch \( t_\ell \in [s_\ell, e_\ell] \) for which \( \tilde{R}_{1t_\ell}^1 \geq \gamma R_{1t_\ell}^1 \), for some \( \gamma > 1 \). Then for any \( \ell = 0, 1, \ldots \) and any \( t_\ell \in [s_\ell, e_\ell] \) such that \( \tilde{R}_{1t_\ell}^1 \geq \gamma R_{1t_\ell}^1 \), it holds that

\[
t_\ell = O(s_\ell) \quad \text{and} \quad t_\ell = \left( \frac{2 - \gamma}{3 - 2 \gamma} \right)^{\ell} \Omega(t_0)
\]

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A Deferred Proofs of Section 3

Theorem 3.1. There is an instance with \( n + 2 \) users, in which a user can misreport her demand to increase her utility by a factor of \( \frac{4}{3} - \frac{1}{3 \cdot 2^n} \).

Proof. The example in which a user (here called Alice) can deviate to increase her allocation by a factor of close to 4/3 is the following.

- There is a total of 1 resource: \( R = 1 \).
- There are \( n + 2 \) users: Alice and users 0, 1, \ldots, \( n \).
- There are \( n + 2 \) epochs, numbered \(-1\) to \( n + 1\):
  - On epoch \(-1\), only user 0 demands 1 resource.
  - On epoch \( i \), for \( i = 0, 1, \ldots, n \), only Alice and user \( i \) demand 1 resource each.
  - On epoch \( n + 1 \), only Alice and user 0 demand 1 resource each.

It turns out the in the above setting, if Alice is truthful she ends up with \( \frac{3}{2} \) resources, but if she declares a demand of 0 on epoch 0 she can earn \( 2 - \frac{1}{2^n} \) resources instead. The details that lead to this are in Table 2.

| Users | Epoch \(-1\) | Epoch 0 | Epoch 1 | Epoch 2 | ... | Epoch \( n \) | Epoch \( n + 1 \) |
|-------|------------|---------|---------|---------|------|--------------|--------------|
| Alice | 0          | 1       | 1 0     | 1 0     | 1/2 | 1 0 1/4      | 1 0 1/2n 1 1/2 1 |
| 0     | 1 1 1     | 0 1 0   | 0       | 0       | ... | 0            | 0            |
| 1     | 0         | 0 0     | 0 1 1   | 1 1/2   | 0   | ... 0       | 0            |
| 2     | ...       | ...     | ...     | ...     | ... | ...         | ...          |
| n     | 0         | 0 0     | 0 0     | 0       | 0   | 1 1 (2\(^n\)-1)/2\(^n\) | 0            |

Table 2: The black numbers denote the users’ demands, the blue numbers are the users’ allocations when Alice is truthful, and the red numbers are the allocations when Alice misreports her demand on epoch 0 by demanding 0 instead of 1.

This entails that Alice increases her resources by a factor of \( \frac{4}{3} - \frac{1}{3 \cdot 2^n} \), as desired. ■

We now present the missing proof of Theorem 3.7.

Theorem 3.7. If conditions 1-5 are true and for the total resources it holds that \( R \geq 1 + c/\sqrt{n} \) for some \( c \in \Omega(1) \), then for all \( t \) and any deviation of user \( i \) it holds that

\[
\mathbb{E}[U_i^t] \leq \mathbb{E}[U_i^t] \left(1 + O\left(\frac{1}{c^2}\right)\right)
\]

Proof. Fix a \( t \). Because of Theorem 3.4 it is no loss of generality to assume that user 1 does not over-report her demand and thus \( \hat{U}_1^t = \hat{R}_1^t \) and \( U_1^t = R_1^t \). By combining Lemmas 3.3 and 3.6 we get that

\[
\mathbb{E} \left[ \hat{R}_1^t - R_1^t \right] \leq \sum_{\tau=1}^{t} \mathbb{E} \left[ r_1^{\tau} \right] P \left[ \sum_{i \neq 1} d_i^{\tau} - \sum_{i \neq 1} \mathbb{E} [d_i^{\tau}] \right] > R - 1 - \max(d_1^t) \right] \tag{9}
\]

Using Chebyshev’s inequality and the independence of \( \{d_i^t\}_{i \neq 1} \), for every \( \tau \) we have that

\[
P \left[ \sum_{i \neq 1} d_i^{\tau} - \sum_{i \neq 1} \mathbb{E} [d_i^{\tau}] \right] > R - 1 - \max(d_1^t) \right] \leq \frac{\sum_{i \neq 1} \text{Var} (d_i^t)}{(R - 1 - \max(d_1^t))^2} \leq \frac{1}{c^2} \tag{10}
\]
where in the last inequality we assumed that there exists a $C > 0$ such that for all $\tau$

$$\mathcal{R} \geq 1 + \max(d_i^T) + C \sqrt{\sum_{i \neq 1} \text{Var}(d_i^T)}$$

(11)

Note that as long as $C \in \Omega(1)$, (11) is the same as the constraint $\mathcal{R} \geq 1 + c/\sqrt{n}$, since $\max(d_i^T) \in O(1/\sqrt{n})$ and $\sum_i \text{Var}(d_i^T) \in O(1/n)$. Combining (9) and (10) proves the theorem. \hfill \blacksquare

## B  Deferred Proofs of Section 4

In this section we prove the theorems of Section 4.

### Theorem 4.1. There is an instance with $n$ users where a user can over-report her demand to increase her utility by a factor of $\Omega(n)$. 

**Proof.** Table 3 presents the example graphically. In detail, the example has $x + y + 1$ users and $y + 2$ resources:

- Alice wants every resource with a ratio of 1, except for the second resource which she does not want.
- There are $x$ copies of a user, Bob, where each wants the first resource with a ratio of 1, the second one with a ratio of $x^3/x^4$, and does not want any other resource.
- User $i$, for all $i \in [y]$, wants the second resource with a ratio of $1/x^y$, the $(i + 2)$-th one with a ratio of 1, and does not want any other resources.
- On epoch 1 the total available amount of every resource is 1. On all other epochs the total available amount for each resource is $1/x^2$.

| Users | Ratios | Epoch 1 | Epoch 2 | Epoch 3 | Epoch 4 | Epoch 5 | ... |
|-------|--------|---------|---------|---------|---------|---------|-----|
| Alice | 1 0 1 1 1 ... 1 | 0 0 $x^2+1$ $x^2$ $x^2$ | $x^2$ $x^2$ $x^2$ | $x^2$ | $x^2$ | $x^2$ | ... |
| x Bobs | $x^3-1$ $x^4$ | 0 0 0 ... 0 | 1 $1/x^2$ $x^2+1$ $x^2$ | $x^2$ | $x^2$ | $x^2$ | $x^2$ | $x^2$ | ... |
| 1 0 | $1/x^y$ 1 0 0 ... 0 | 1 $1/x^2$ $x^2+1$ $x^2$ | $x^2$ | $x^2$ | $x^2$ | $x^2$ | ... |
| 2 0 | $1/x^y$ 0 1 0 ... 0 | 1 $1/x^2$ $x^2+1$ $x^2$ | $x^2$ | $x^2$ | $x^2$ | $x^2$ | ... |
| 3 0 | $1/x^y$ 0 0 1 ... 0 | 1 $1/x^2$ $x^2+1$ $x^2$ | $x^2$ | $x^2$ | $x^2$ | $x^2$ | ... |
| ... | ... | ... | ... | ... | ... | ... | ... | ... |
| y 0 | $1/x^y$ 0 0 0 ... 1 | 1 $1/x^2$ $x^2+1$ | $x^2$ | $x^2$ | $x^2$ | $x^2$ | ... |

Table 3: For every epoch the black numbers denote the users’ demands, the blue numbers are the users’ allocation when Alice is truthful, and the red numbers are the allocations when Alice over-reports her demand on epoch 1 by asking for 1 resource. The available amount of every resource on epoch 1 is 1, while on the other epochs it is $1/x^2$.

If Alice lies on the first epoch and requests resources instead of requesting zero, she can take resources from every Bob, which increases the allocation of the other users by a factor of $\Theta(x)$. This entails that in epochs 2, 3, ..., $y+1$, instead of Alice getting allocated $1/x^2$, $1/2x^2$, ..., $1/2^{y-1}x^2$, respectively, she gets $1/x^2$ every round. Note that this is true if the following inequality is true, which guarantees that when Alice is untruthful the resulting allocation in rounds 2 to $y+1$ is max-min fair:

$$\frac{1}{x^2 + 1} + \frac{y}{x^2} \leq \frac{x + 1}{x^2 + 1}$$

(12)

We can make that assumption w.l.o.g. because we can split the first epoch where there is 1 amount of every resource into $x^2$ epochs where there is $x^{-2}$ amount every time.
which is true if \( y \leq \frac{x^3}{x^2 + 1} \). This allows us to set \( y = \Theta(x) \). In total, Alice gets an allocation of \( \frac{2}{x^2} \left( 1 - \frac{1}{2x^2} \right) \leq \frac{2}{x^2} \) if she is truthful and a (useful) allocation of \( \frac{x}{2} \) if she is untruthful, i.e. she increases her utility by a factor of at least \( y/2 \). Noticing that \( y = \Theta(x) \) and \( n = x + y + 1 = \Theta(y) \) proves the theorem. \( \square \)

Next we turn to the proof of Theorem 4.3. First we restate Lemma 4.2 for completeness (which was proven in Section 4) and then we prove that over-reporting is not beneficial.

**Lemma 4.2.** Fix an epoch \( t \) and assume that for all users \( i \in [n] \) and resources \( q \in [m] \) it holds that \( a_{iq} > 0 \). Let \( i, j \) be two different users. If the following conditions hold

- For \( i \), \( r_i^t < r_i^t \) and \( r_i^t < d_i^t \).
- For \( j \), \( r_j^t < r_j^t \) and \( r_j^t < d_j^t \).

then \( R_i^t \geq R_j^t \) and \( \hat{R_i}^t \leq \hat{R_j}^t \), implying

\[
\hat{R_i}^t - R_i^t \leq \hat{R_j}^t - R_j^t
\]

**Theorem 4.3.** Assume that for all users \( i \in [n] \) and resources \( q \in [m] \) it holds that \( a_{iq} > 0 \). Then the users have nothing to gain by declaring a demand higher than their actual demand.

**Proof.** Let \( \{d_{i,t}\}_{i,t} \) be any demands and fix an epoch \( T \). We are going to show that user 1 has nothing to lose by epoch \( T \) by reporting her demand truthfully instead of over-reporting on epochs 1 to \( T \). Let \( T_0 \leq T \) be the last epoch where user 1 over-reported. For all users \( i \) and epochs \( t \), let \( \hat{d}_{i,t} = d_{i,t} \), except for \( \hat{d}_{1,T_0} \) which is 1’s actual demand (note that \( \hat{d}_{1,T_0} < d_{1,T_0} \)). We will show that \( \hat{U}_1^{T_0} \geq \hat{U}_1^T \), i.e. on epoch \( T \), user 1 does not prefer the demand sequence \( \{\hat{d}_{i,t}\}_{i,t} \) over the demand sequence \( \{d_{i,t}\}_{i,t} \). If we apply this inductively for every epoch before \( T \) where user 1 over-reports, we are going to get that over-reporting is not a desirable strategy.

Up to epoch \( T_0 - 1 \) the users’ demands are the same and thus so are the allocations and utilities: for all \( i \), \( R_i^{T_0 - 1} = R_i^{T_0 - 1} \) and \( U_i^{T_0 - 1} = U_i^{T_0 - 1} \). Because \( \hat{d}_{1,T_0} > d_{1,T_0} \), user 1 earns some additional resources on \( T_0 \), i.e. \( \hat{r}_1^{T_0} - r_1^{T_0} = \hat{R}_1^{T_0} - R_1^{T_0} = x \), for some fixed \( x \geq 0 \). We first note that these \( x \) resources are in excess of 1’s true demand, meaning they do not contribute towards 1’s utility, i.e.

\[
\hat{U}_1^{T_0} - \hat{U}_1^T = \hat{R}_1^{T_0} - R_1^{T_0} - x = 0
\]

Additionally, because user 1 does not over-report in epochs \( T_0 + 1 \) to \( T \), it holds that for \( t \in [T_0 + 1, T] \), \( \hat{a}_1^t = r_1^t \) and \( \hat{a}_1^t = \hat{r}_1^t \). This fact, combined with (12) proves that

\[
\forall t \in [T_0, T], \quad \hat{U}_1^t - \hat{U}_1^t = \hat{R}_1^t - R_1^t - x
\]

Thus, in order for this over-reporting to be a strictly better strategy, it must hold that \( \hat{R}_1^T - \hat{R}_1^T > x \). To prove that this is not the case, we are going to prove, using induction on \( t \), that

\[
\forall k, \forall t \in [T_0, T], \quad \hat{R}_k^t - \hat{R}_k^t \leq x
\]

For \( t = T_0 \), the induction trivially holds for \( k = 1 \), as \( \hat{R}_1^{T_0} = \hat{R}_1^{T_0} + x \). For other users it holds that \( \hat{R}_k^{T_0} \leq \hat{R}_k^{T_0} \), to prove this assume a contradiction, i.e. there exists a user \( i \neq 1 \) such that \( \hat{R}_i^T > \hat{R}_i^T \). Because \( \hat{R}_i^{T - 1} = \hat{R}_i^{T - 1} \), this implies that \( \hat{r}_i^T > \hat{r}_i^T \). Additionally, it holds that there must exist a user \( j \neq 1 \), such that \( \hat{r}_j^T > \hat{r}_j^T \); otherwise we would have been able to strictly increase i’s allocation without decreasing the allocation of other users. Because \( \hat{d}_{i,T_0} = d_{i,T_0} \) and \( \hat{d}_{j,T_0} = d_{j,T_0} \) we can use Lemma 4.2 to get that

\[
\hat{R}_i^T - \hat{R}_i^T \leq \hat{R}_j^T - \hat{R}_j^T
\]

which, because \( \hat{R}_i^{T - 1} = \hat{R}_i^{T - 1} \) and \( \hat{R}_j^{T - 1} = \hat{R}_j^{T - 1} \), implies that \( \hat{r}_i^T - \hat{r}_i^T \leq \hat{r}_j^T - \hat{r}_j^T \), which is a contradiction because of \( \hat{r}_i^T > \hat{r}_i^T \) and \( \hat{r}_j^T > \hat{r}_j^T \).
For any \( t \in [T_0 + 1, T] \), we again assume a contradiction: there exists a user \( i \) such that
\[
\hat{R}_i^t - R_i^t > x
\]  
(13)

Because of the inductive hypothesis, \( \hat{R}_i^{t-1} - R_i^{t-1} \leq x \), the above implies that \( \hat{r}_i^t > r_i^t \). Because we cannot increase a user’s allocation without decreasing the allocation of someone else (which is true because for all \( k \), \( d_k^t \equiv d_k^t \)), there must exist a user \( j \) such that \( \hat{r}_j^t < r_j^t \). The last two facts, combined with Lemma 4.2 imply
\[
\hat{R}_i^t - R_i^t \leq \hat{R}_j^t - R_j^t
\]

This leads to a contradiction because of (13) and \( \hat{R}_j^t - R_j^t \leq \hat{R}_j^{t-1} - R_j^{t-1} \leq x \) (this inequality is implied by \( \hat{r}_j^t < r_j^t \) and the inductive hypothesis).

Now we prove the lemma that is similar to Lemma 3.3 bounding each user’s additional allocated resource due to user 1’s misreports, but includes the parameter \( \rho_1 = \min_{k,q} \frac{a_{iq}}{a_{iq}} \).

**Lemma 4.4.** Assume that for all users \( i \in [n] \) and resources \( q \in [m] \) it holds that \( a_{iq} > 0 \). Then, for every user \( i \) and every epoch \( t \), if user 1 does not over-report her demand, it holds that
\[
\rho_1 \left( \hat{R}_i^t - R_i^t \right) \leq \sum_{\tau=1}^{t} r_{i}^\tau 1 \left[ \exists q \in [m], \sum_{k \in [n]} a_{kq} d_k^\tau > \mathcal{R} \right]
\]  
(8)

**Proof.** We will prove the lemma using induction on \( t \). For \( t = 0 \), the lemma trivially holds.

For every \( \tau \), let \( f^\tau = r_{i}^\tau 1 \left[ \exists q \in [m], \sum_{k \in [n]} a_{kq} d_k^\tau > \mathcal{R} \right] \). Assume that (8) holds for some \( t \geq 0 \) and for all users \( i \). For \( t + 1 \), assume by contradiction that there exists a user \( i \), such that
\[
\rho_1 \left( \hat{R}_i^{t+1} - R_i^{t+1} \right) > \sum_{\tau=1}^{t+1} f^\tau
\]  
(14)

Because of the inductive hypothesis, \( \rho_1 \left( \hat{R}_i^t - R_i^t \right) \leq \sum_{\tau=1}^{t} f^\tau \), this entails that
\[
\rho_1 \left( \hat{r}_i^{t+1} - r_i^{t+1} \right) > f^{t+1} = r_{i}^{t+1} 1 \left[ \exists q \in [m], \sum_{k \in [n]} a_{kq} d_k^{t+1} > \mathcal{R} \right]
\]  
(15)

(15) leads to a contradiction for the case when the users’ true demands are met in epoch \( t + 1 \) (in which case \( f^{t+1} = 0 \)): because user 1 does not over-report, for every \( k \neq 1 \) it holds that \( \hat{r}_k^t = r_k^t = d_k^t \) and for user 1 it holds that \( \hat{r}_1^t \leq r_1^t = d_1^t \).

For the rest of the proof we assume that the users’ true demands are not met in epoch \( t + 1 \). Because the demands \( \{d_k^{t+1}\}_{k \in [n]} \) are not met, there must exist a resource \( q \) which was saturated, entailing that
\[
\sum_{k \in [n]} a_{kq} r_k^{t+1} \geq \sum_{k \in [n]} a_{kq} \hat{r}_k^{t+1}
\]  
(16)

We now multiply (15) with \( a_{1q} \) and add it with (16) and get
\[
a_{1q} \rho_1 \left( \hat{r}_i^{t+1} - r_i^{t+1} \right) + a_{1q} \left( r_i^{t+1} - \hat{r}_i^{t+1} \right) + \sum_{k \neq i} a_{kq} \left( r_k^{t+1} - \hat{r}_k^{t+1} \right) > a_{1q} r_1^{t+1}
\]  
(17)
We now use the facts that $a_{1q} \rho_1 \leq a_{iq}$ (due to the definition of $\rho_1$), $\hat{r}_i^{t+1} > r_i^{t+1}$ (from (15)), and $\hat{r}_i^{t+1} \geq 0$ to rewrite (17):
\[
\sum_{k \neq i, 1} a_{kq} (r_k^{t+1} - \hat{r}_k^{t+1}) > 0
\]  
(18) implies that there must exists a user $j \neq 1$ such that $r_j^{t+1} > \hat{r}_j^{t+1}$. Combining this with $\hat{r}_i^{t+1} > r_i^{t+1}$ and the fact that user 1 does not over-report her demand, we can use Lemma 4.2 and get that
\[
\hat{R}_i^{t+1} - R_i^{t+1} \leq \hat{R}_j^{t+1} - R_j^{t+1}
\]
This leads to a contradiction because of (14) and
\[
\rho_1 \left( \hat{R}_j^{t+1} - R_j^{t+1} \right) < \rho_1 \left( \hat{R}_j^{t} - R_j^{t} \right) \leq \sum_{\tau=1}^{t+1} f^\tau
\]
where the above inequalities are implied by $\hat{r}_j^t < r_j^t$ and the inductive hypothesis.

Now we will focus on the case where the users’ demands are random variables.

**Theorem 4.6.** Assume that the following conditions hold

- For every epoch $t$ and every users $i \neq j$, $d_i^t$ and $d_j^t$ are distributed independently.
- For every epoch $t$, $\sum_i \mathbb{E}[d_i^t] \leq 1$.
- For every epoch $t$, $\sum_i \text{Var}(d_i^t) = O(1/n)$.
- For every epoch $t$ and every user $i$, $\max(d_i^t) = O(1/\sqrt{n})$.

Fix a user $i$ and let $\rho_i = \min_{k,q} \frac{a_{kq}}{a_{iq}}$. If for the total resources it holds that $R \geq 1 + c/\sqrt{\rho_i} n$ for some $c = \Omega(1)$, then for all epochs $t$ and any deviation of user $i$ it holds that
\[
\mathbb{E}[\hat{U}_i^t] \leq \mathbb{E}[U_i^t] \left( 1 + O\left(1/e^2 \right) \right)
\]
We first prove a lemma that will help bound the expectation of $r_i^t \mathbbm{1} \left[ r_i^t \mathbbm{1} \left[ \exists q \in [m] : \sum_i a_{iq} d_i^t > R \right] \right]$ using any concentration inequality.

**Lemma B.1.** If the conditions of Theorem 4.6 are true then for every $t$ it holds that
\[
\mathbb{E} \left[ r_i^t \mathbbm{1} \left[ \exists q \in [m] : \sum_i a_{iq} d_i^t > R \right] \right] \leq \mathbb{E} \left[ r_i^t \mathbbm{1} \left[ \sum_{i \neq 1} d_i^t - \sum_{i \neq 1} \mathbb{E}(d_i^t) \right] > R - 1 - \max(d_i^t) \right]
\]
**Proof.** The allocation $r_i^t$ is a complicated function of the variables $\{R_i^{t-1}\}_i$ and the random variables $\{d_i^t\}_i$. This means that $r_i^t$ and $\mathbbm{1} \left[ \exists q \in [m] : \sum_i a_{iq} d_i^t > R \right]$ are not independent, so we cannot bound the expectation of their product by the product of their expectations, which makes the the proof more involved. For this reason we need to bound them by quantities that are independent. We first note that the following holds for any realization of the random variables and because for all $i, q, a_{iq} \leq 1$:
\[
r_i^t \mathbbm{1} \left[ \exists q \in [m] : \sum_i a_{iq} d_i^t \leq R \right] \geq r_i^t \mathbbm{1} \left[ \sum_i d_i^t \leq R \right] \geq r_i^t \mathbbm{1} \left[ \max(d_i^t) + \sum_{i \neq 1} d_i^t \leq R \right]
\]
This makes the two terms on the right hand side “less dependent” but they are still not independent: the realization of \( \{d_i^t\}_t \) affects both terms. We then take the expectation of the above

\[
E \left[ r_1^t \mathbb{1} \left( \exists q \in [m] : \sum_i a_{iq} d_i^t \leq \mathcal{R} \right) \right] \geq E \left[ r_1^t \mathbb{1} \left( \max(d_1^t) + \sum_{i \neq 1} d_i^t \leq \mathcal{R} \right) \right]
\]

(using the law of total expectation)

\[= E \left[ r_1^t \mathbb{1} \left( \max(d_1^t) + \sum_{i \neq 1} d_i^t \leq \mathcal{R} \right) \right] \mathbb{P} \left[ \max(d_1^t) + \sum_{i \neq 1} d_i^t \leq \mathcal{R} \right]
\]

(\( \text{demands are satisfied: } r_1^t \text{ does not depend on } \{d_i^t\}_{i \neq 1} \))

We can express \( E \left[ r_1^t \right] \) as

\[E \left[ r_1^t \right] = E \left[ r_1^t \mathbb{1} \left( \exists q \in [m] : \sum_i a_{iq} d_i^t > \mathcal{R} \right) \right] + E \left[ r_1^t \mathbb{1} \left( \exists q \in [m] : \sum_i a_{iq} d_i^t \leq \mathcal{R} \right) \right]
\]

using this fact the above inequality implies

\[E \left[ r_1^t \mathbb{1} \left( \exists q \in [m] : \sum_i a_{iq} d_i^t > \mathcal{R} \right) \right] \leq E \left[ r_1^t \right] \mathbb{P} \left[ \max(d_1^t) + \sum_{i \neq 1} d_i^t > \mathcal{R} \right]
\]

\[\leq E \left[ r_1^t \right] \mathbb{P} \left[ \sum_{i \neq 1} d_i^t - \sum_{i \neq 1} E[d_i^t] > \mathcal{R} - 1 - \max(d_1^t) \right]
\]

where in the last inequality we used the fact that \( \sum_{i \neq 1} E[d_i^t] \leq \sum_i E[d_i^t] \leq 1. \)

**Proof of Theorem 4.6.** Fix a \( t \) and w.l.o.g. assume that \( i = 1 \). Because of Theorem 4.3 we can assume that user 1 does not over-report her demand, which implies that \( \hat{U}_1^t = \hat{R}_1^t \). By combining Lemmas 4.4 and B.1 we get that

\[\rho_1 E \left[ \hat{R}_1^t - R_1^t \right] \leq \sum_{\tau = 1}^t E \left[ r_1^\tau \right] \mathbb{P} \left[ \left| \sum_{k \neq 1} d_k^\tau - \sum_{k \neq 1} E[d_k^\tau] \right| > \mathcal{R} - 1 - \max(d_1^\tau) \right] \]

(19)

Using Chebyshev’s inequality and the independence of \( \{d_k^\tau\}_{k \in [n]} \), for every \( \tau \) we have that

\[\mathbb{P} \left[ \left| \sum_{k \neq 1} d_k^\tau - \sum_{k \neq 1} E[d_k^\tau] \right| > \mathcal{R} - 1 - \max(d_1^\tau) \right] \leq \frac{\sum_{k \neq 1} \text{Var}(d_k^\tau)}{(\mathcal{R} - 1 - \max(d_1^\tau))^2} \leq \frac{\rho_1}{C^2}
\]

(20)

where in the last inequality we assumed that for all \( t \) and for some \( C \in \Omega(1) \)

\[\mathcal{R} \geq 1 + \max(d_1^\tau) + \frac{C}{\sqrt{\rho_1}} \sqrt{\sum_{k \neq 1} \text{Var}(d_k^\tau)}
\]

Note that (21) is the same as the constraint \( \mathcal{R} \geq 1 + c/\sqrt{n} \), since \( \rho_1 \) is a constant, \( \max(d_1^\tau) \in O(1/\sqrt{n}) \), and \( \sum_k \text{Var}(d_k^\tau) \in O(1/n) \). Combining (19) and (20) proves the theorem:

\[E \left[ \hat{R}_1^t \right] \leq E \left[ R_1^t \right] \left( 1 + \frac{1}{C^2} \right)
\]

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C  Deferred Proofs of Section 5

First we generalize Lemma 3.2.

Lemma C.1. Fix an epoch $t$ and let $i, j$ be two different users. If the following conditions hold

- For $i$, $r_i^t < \hat{r}_i^t$ and $r_i^t < d_i^t$.
- For $j$, $r_j^t > \hat{r}_j^t$ and $\hat{r}_j^t < d_j^t$.

then

$$\frac{R_i^t}{w_i} \geq \frac{\hat{R}_i^t}{w_j} \quad \text{and} \quad \frac{\hat{R}_i^t}{w_i} \leq \frac{\hat{R}_j^t}{w_j},$$

and consequentially

$$\frac{\hat{R}_i^t - R_i^t}{w_i} \leq \frac{\hat{R}_j^t - R_j^t}{w_j} \quad (22)$$

Proof. Because of the conditions, we notice that $r_i^t < d_i^t$ and $r_j^t > 0$, which implies that it would have been feasible to increase $r_i^t$ by decreasing $r_j^t$. This implies that $R_i^t/w_i \geq R_j^t/w_j$; otherwise it would have been true that $R_i^t/w_i > R_j^t/w_j$, entailing that taking resources from $j$ and giving them to $i$ would have been favored by max-min fairness. With the inverse argument (we can increase $r_j^t$ by decreasing $r_i^t$) we can prove that $\hat{R}_i^t/w_i \leq \hat{R}_j^t/w_j$. This completes the proof. \qed

Next we generalize Lemma 3.3.

Lemma C.2. Fix any $t \geq 1$. Let $\{R_i^{t-1}\}_{i \in [n]}$ and $\{\hat{R}_i^{t-1}\}_{i \in [n]}$ be the cumulative allocations up to epoch $t - 1$. Assume that $\{d_i^t\}_{i \in [n]}$ are some users’ demands and that $\{\hat{d}_i^t\}_{i \in [n]}$ are the same demands except users’ in $I$, who deviate but do not over-report, i.e. for every user $i \in I$, $\hat{d}_i^t \leq d_i^t$. Then it holds that

$$\sum_{k \in [n]} \left(\hat{R}_k^t - R_k^t\right)^+ - \sum_{k \in [n]} \left(\hat{R}_k^{t-1} - R_k^{t-1}\right)^+ \leq 1 \left[ \sum_k d_k^t > R \quad \text{and} \quad \exists k \in I : d_k^t < \hat{d}_k^t \right] \sum_{k \in I} r_k^t \quad (23)$$

Proof. When users’ demands are satisfied, because users in $I$ can only under-report their demand, we have that for $k \in I$, $\hat{r}_k^t \leq r_k^t = d_k^t$ and for users $k \not\in I$, $\hat{r}_k^t = r_k^t = d_k^t$. This entails that for all users $k \in [n]$

$$\hat{R}_k^t - R_k^t \leq \hat{R}_k^{t-1} - R_k^{t-1}$$

The above inequality proves that $\sum_k (\hat{R}_k^t - R_k^t)^+ \leq \sum_k (\hat{R}_k^{t-1} - R_k^{t-1})^+$, which completes the proof when users’ demands are satisfied.

To prove the other case of the lemma we assume that demands are not met and for every epoch $t$ we define $P_t = \{k \in [n] : R_k^t \geq R_k^t \}$. We now assume a contradiction:

$$\sum_{k \in P_t} \left(\hat{R}_k^t - R_k^t\right) - \sum_{k \in P_t^{t-1}} \left(\hat{R}_k^{t-1} - R_k^{t-1}\right) > 1 \left[ \exists k \in I : d_k^t < \hat{d}_k^t \right] \sum_{k \in I} r_k^t$$

Because $\sum_{k \in P_t} (\hat{R}_k^{t-1} - R_k^{t-1}) \leq \sum_{k \in P_t^{t-1}} (\hat{R}_k^{t-1} - R_k^{t-1})$, the above inequality implies

$$\sum_{k \in P_t} (\hat{r}_k^t - r_k^t) > 1 \left[ \exists k \in I : \hat{d}_k^t < d_k^t \right] \sum_{k \in I} r_k^t \quad (24)$$
Because users do not over-report their demands, it holds that $\sum_k r_i^t \geq \sum_k \hat{r}_i^t$, i.e. the total resources allocated to the users does not increase when users in $I$ deviate. Combining this fact with (24) we get that

$$\sum_{k \notin P^t} (r_k^t - \hat{r}_k^t) > 1 \left[ \exists k \in I: \hat{d}_k^t < d_k^t \right] \sum_{k \in I} r_k^t$$  \hspace{1cm} (25)$$

We notice that because of (24), there exists a user $i \in P^t$ for whom $\hat{r}_i^t > r_i^t$, because of (25), there exists a user $j \notin P^t$ for whom $\hat{r}_j^t < r_j^t$. Additionally for that $j$ we can assume that $\hat{d}_j^t = d_j^t$ because:

- If for all users $k \in I$, $\hat{d}_k^t = d_k^t$, then because only users in $I$ deviate, for all users $k \in [n]$, $\hat{d}_k^t = d_k^t$.
- If for some user $k \in I$, $\hat{d}_k^t < d_k^t$, then (25) implies $\sum_{k \notin P^t, k \notin I} (r_k^t - \hat{r}_k^t) > 0$, i.e. $j \notin I$ and we assumed that only users in $I$ deviate.

Thus we have that $\hat{d}_i^t \leq d_i^t$ (since no user over-reports), $\hat{d}_j^t = d_j^t$, $r_i^t > r_j^t$, and $\hat{r}_j^t < r_j^t$, meaning we can apply Lemma C.1 to prove that

$$\frac{\hat{R}_i^t - R_i^t}{w_i} \leq \frac{\hat{R}_j^t - R_j^t}{w_j}$$

The above leads to a contradiction, because $i \in P^t$ and $j \notin P^t$, i.e. $\hat{R}_i^t - R_i^t \geq 0$ and $\hat{R}_j^t - R_j^t < 0$. \hfill \Box

**Adversarial Demands.** Now we prove that users do not want to over-report their demand.

**Theorem 5.1.** Let $I \subseteq [n]$ be a set of users that form a coalition and $w_1, \ldots, w_n$ be any weights, according to which weighted max-min fairness allocates resources. Then the users in $I$ have nothing to gain by declaring a demand higher than their actual demand.

**Proof.** Let $\{\hat{d}_i^t\}_{i,t}$ be any demands and fix an epoch $T$. We are going to show that the total utility of users in $I$ does not decrease on epoch $T$ by reporting their demand truthfully instead of over-reporting on epochs $1$ to $T$. Let $T_0 \leq T$ be the last epoch where any user in $I$ over-reported. Specifically, let $I' \subseteq I$ be the users in $I$ who over-reported in $T_0$. For all users $j$ and epochs $t$, define $\hat{d}_j^t = d_j^t$, except for every $i \in I'$, where $\hat{d}_i^{T_0}$ is $i$’s actual demand (note that $\hat{d}_i^{T_0} < d_i^{T_0}$). We will show that $\sum_{i \in I} \hat{U}_i^t \geq \sum_{i \in I} \hat{U}_i^{T_0}$, i.e. on epoch $T$, $\{\hat{d}_i^t\}_{i,t}$ are not preferable over $\{d_i^t\}_{i,t}$ for users in $I$. If we apply this inductively for every epoch before $T$ where a user in $I$ over-reports, we are going to get that over-reporting is not a desirable strategy.

Up to epoch $T_0$ the users’ allocations and utilities are the same, i.e. for all users $j$, $\hat{R}_j^{T_0-1} = R_j^{T_0-1}$ and $\hat{U}_j^{T_0-1} = U_j^{T_0-1}$. On epoch $T_0$, users’ allocations change. More specifically it holds that for any user who got more resources, that user must be in $I'$ and that additional resources must be in excess of their true demand:

$$\text{if } \hat{r}_i^{T_0} - r_i^{T_0} = R_i^{T_0} - R_i^{T_0} > 0 \text{ then } \hat{r}_i^t = d_i^{T_0} \text{ and } i \in I'$$ \hspace{1cm} (26)$$

To prove (26) we study two cases:

- If $\hat{r}_i^{T_0} - r_i^{T_0} = R_i^{T_0} - R_i^{T_0} > 0$ and $\hat{r}_i^{T_0} < d_i^{T_0}$, then $\hat{r}_i^{T_0} < d_i^{T_0}$. This means that demands are not satisfied both in $\hat{d}_i^{T_0}$ and $\hat{d}_i^{T_0}$, which implies that since $\hat{r}_i^{T_0} > r_i^{T_0}$, for some $j$, $r_j^{T_0} - r_j^{T_0} = R_j^{T_0} - R_j^{T_0} < 0$, which because of Lemma C.1 and $\hat{d}_j^t \leq d_j^t$ entails that $\hat{R}_i^{T_0} - R_i^{T_0} \leq \hat{R}_j^{T_0} - R_j^{T_0}$, a contradiction.

- If $\hat{r}_i^{T_0} - r_i^{T_0} = R_i^{T_0} - R_i^{T_0} > 0$ and $i \notin I'$, then, because user $i$ got more resources in $\hat{d}_i^{T_0}$ with the same demand, for some user $j$, $r_j^{T_0} - r_j^{T_0} = \hat{R}_j^{T_0} - R_j^{T_0} < 0$, which because of Lemma C.1, $\hat{d}_i^{T_0} = d_i^{T_0}$, and $\hat{d}_j^{T_0} \leq d_j^{T_0}$ entails that $\hat{R}_j^{T_0} - R_j^{T_0} \leq \hat{R}_j^{T_0} - R_j^{T_0}$, a contradiction.

Because of (26) we see that any additional resources that users in $I$ get in $T_0$ when they over-report are in excess of their demand (because for $i \in I'$, $\hat{d}_i^{T_0}$ is the true demand). This means that $\sum_{i \in I} (\hat{r}_i^{T_0} - r_i^{T_0})^+ = x \geq 0$ is
in excess of their demand and therefore
\[
\sum_{i \in I} (\hat{U}_T^0 - \hat{U}_T^0) = \sum_{i \in I} (\hat{R}_T^0 - \hat{R}_T^0) - x \leq 0 \tag{27}
\]

Now we notice that because in epochs \([T_0 + 1, T]\) users in \(I\) do not over-report their demand, it holds \(\hat{r}_i^t = \hat{u}_i^t\) and \(\hat{r}_i^t = \hat{u}_i^t\) for every \(t\) and \(i \in I\); combining this with (27) implies \(\sum_{i \in I} (\hat{U}_T^t - \hat{U}_T^t) = \sum_{i \in I} (\hat{R}_k^t - \hat{R}_k^t) - x\) for every \(t \in [T_0, T]\). Thus to show that over-reporting in \(T_0\) is not profitable we can show that \(\sum_{i \in I} (\hat{R}_i^T - \hat{R}_i^T) \leq x\).

We will complete the proof by proving this, using Lemma C.2: by summing (23) for all \(t \in [T_0 + 1, T]\) and noticing that for all \(k \in I\), \(d_k^t = d_k^t\) we get
\[
\sum_k \left( \hat{R}_k^T - \hat{R}_k^T \right) - \sum_k \left( \hat{R}_k^T - \hat{R}_k^T \right) \leq 0
\]

The above, because \((\hat{R}_k^T - \hat{R}_k^T)^+ \geq 0\), \(\hat{R}_k^0 - \hat{R}_k^T \leq 0\) if \(k \notin I\), and \(\sum_{i \in I} (\hat{R}_i^T - \hat{R}_i^T)^+ = x\), proves that \(\sum_{i \in I} (\hat{R}_i^T - \hat{R}_i^T) \leq x\). This completes the proof. ■

Now we prove the inventive compatibility bound for adversarial demands.

**Theorem 5.2.** Let \(I \subseteq [n]\) be a set of users that form a coalition and \(w_1, \ldots, w_n\) be any weights, according to which weighted max-min fairness allocates resources. Then for any deviation of the users in \(I\) and any epoch \(t\) it holds that
\[
\sum_{i \in I} \hat{U}_i^t \leq 2 \sum_{i \in I} U_i^t
\]

Additionally, when \(I = \{i\}\) for any user \(i\), then
\[
\hat{U}_i^t \leq \left(1 + \max_{j \neq i} \frac{w_j}{w_i + w_j}\right) U_i^t
\]

**Proof.** Because of Theorem 5.1 we assume without loss of generality that users do not over-report their demand. This entails that we can focus on the users’ allocations instead of their utilities.

Fix an epoch \(t\) and let \(T \leq t\) be the last time where \(\sum_{i \in I} \hat{r}_i^t > \sum_{i \in I} r_i^t\). We notice that
\[
\sum_{i \in I} (\hat{R}_i^t - R_i^t) = \sum_{i \in I} (\hat{R}_i^t - R_i^t) + \sum_{t = T + 1}^t \sum_{i \in I} (\hat{r}_i^t - r_i^t) \leq \sum_{i \in I} (\hat{R}_i^t - R_i^t) \tag{28}
\]

Because \(\sum_{k \in I} r_k^T > \sum_{k \in I} r_k^T\) and \(\sum_{k \in [n]} r_k^T \leq \sum_{k \in I} r_k^T\) (since users do not over-report) there must exist a \(i \in I\) such that \(r_i^T > r_i^T\) and a \(j \notin I\) such that \(r_j^T < r_j^T\). Because \(d_j^t \leq d_j^T\) and \(d_j^t = d_j^T\) we can use Lemma C.2 to get
\[
\frac{w_i}{w_j} (\hat{R}_i^T - R_i^T) \leq \hat{R}_j^T - R_j^T \tag{29}
\]

If we use Lemma C.2 and sum (23) for epochs up to \(T\) we get
\[
\sum_{k \in [n]} (\hat{R}_k^T - R_k^T)^+ \leq \sum_{k \in I} R_k^T
\]

which combined with (29) gives
\[
\min_{j \notin I} \frac{w_i}{w_j} (\hat{R}_i^T - R_i^T)^+ + \sum_{k \in I} (\hat{R}_k^T - R_k^T)^+ \leq \sum_{k \in I} R_k^{t \geq T} \leq \sum_{k \in I} R_k^t \tag{30}
\]
(28) and (30) prove that \( \sum_{k \in I} \hat{R}_k^t \leq 2 \sum_{k \in I} R_k^t \), which proves the first part of the lemma.

If \( |I| = 1 \), then \( I = \{i\} \) where \( i \) is the user appearing in (30). If \( \hat{R}_i^T - R_i^T < 0 \) the desired bound is true; otherwise, (28) and (30) prove that

\[
\min_{j \neq i} \frac{w_i}{w_j} (\hat{R}_i^t - R_i^t) + \hat{R}_i^t - R_i^t \leq \hat{R}_i^t
\]

which proves the desired bound: \( \hat{R}_i^t \leq \left( 1 + \max_{j \neq i} \frac{w_i}{w_i + w_j} \right) R_i^t \).

**Random Demands.** We will prove the following theorem similarly to how we proved Theorem 3.7.

**Theorem 5.3.** Let \( I \subseteq [n] \) be a set of users that form a coalition, \( w_1, \ldots, w_n \) be any weights according to which weighted max-min fairness allocates resources and assume that the following conditions hold

- For every \( t \) and every \( i \notin I \) and \( j \notin I \) such that \( i \neq j \), \( d_i^t \) and \( d_j^t \) are distributed independently.
- For every \( t \), \( \sum_{i \notin I} \mathbb{E}[d_i^t] \leq 1 \).
- For every \( t \), \( \sum_{i \notin I} \text{Var}(d_i^t) = O(1/n) \).
- For all \( t \) and for all \( i \in I \), \( \max(d_i^t) = O(1/\sqrt{n}) \).

If for the total resources it holds that \( R \geq 1 + c/\sqrt{n} \) for some \( c = \Omega(|I|) \), then for all epochs \( t \) and any deviations of the users in \( I \) it holds that

\[
\mathbb{E} \left[ \sum_{i \in I} \hat{U}_i^t \right] \leq \mathbb{E} \left[ \sum_{i \in I} U_i^t \right] \left( 1 + O \left( 1/c^2 \right) \right)
\]

Again we first prove a lemma that will help us apply Chebyshev’s inequality.

**Lemma C.3.** If the conditions of Theorem 5.3 are true, then for every \( t \) it holds that

\[
\mathbb{E} \left[ \sum_{i \in I} \left( \sum_{k \in [n]} d_k^t \right) \mathbbm{1} \left[ \sum_{k \in I} d_k^t > R \right] \right] \leq \mathbb{E} \left[ \sum_{i \in I} r_i^t \right] \mathbb{P} \left[ \left| \sum_{k \notin I} d_k^t - \sum_{k \notin I} \mathbb{E}[d_k^t] \right| > R - 1 - \sum_{k \in I} \max(d_k^t) \right]
\]

**Proof.** The allocation \( \{r_i^t\}_{i \in I} \) is a complicated function of the variables \( \{R_i^{t-1}\}_i \) and the random variables \( \{d_i^t\}_i \). This means that \( \{r_i^t\}_{i \in I} \) and \( \mathbbm{1} \left[ \sum_{k \in I} d_k^t > R \right] \) are not independent, so we cannot bound the expectation of their product by the product of their expectations, which makes the proof more involved. For this reason we need to bound them by quantities that are independent. We first note that the following holds for any realization of the random variables.

\[
\sum_{i \in I} r_i^t \mathbbm{1} \left[ \sum_{k \in [n]} d_k^t \leq R \right] \geq \sum_{i \in I} r_i^t \mathbbm{1} \left[ \sum_{k \in I} \max(d_k^t) + \sum_{k \notin I} d_k^t \leq R \right]
\]

This makes the two terms on the right hand side “less dependant” but they are still not independent: the real-
ization of \( \{d^t_k\}_i \) affects both terms. We then take the expectation of the above

\[
E \left[ \sum_{i \in I} r^t_i \mathbb{1} \left( \sum_{k \in [n]} d^t_k \leq R \right) \right] \geq E \left[ \sum_{i \in I} r^t_i \mathbb{1} \left( \sum_{k \in I} \max(d^t_k) + \sum_{k \notin I} d^t_k \leq R \right) \right]
\]

(\text{total law of expectation})

\[
= E \left[ \sum_{i \in I} r^t_i \right] \sum_{k \in I} \max(d^t_k) + \sum_{k \notin I} d^t_k \leq R \right] \mathbb{P} \left[ \sum_{k \in I} \max(d^t_k) + \sum_{k \notin I} d^t_k \leq R \right]
\]

\[
\text{demands are satisfied: } \sum_{i \in I} r^t_i \right] \mathbb{P} \left[ \sum_{k \in I} \max(d^t_k) + \sum_{k \notin I} d^t_k \leq R \right]
\]

We can express \( E \left[ \sum_{i \in I} r^t_i \right] \) as

\[
E \left[ \sum_{i \in I} r^t_i \right] = E \left[ \sum_{i \in I} r^t_i \mathbb{1} \left( \sum_{k \in [n]} d^t_k > R \right) \right] + E \left[ \sum_{i \in I} r^t_i \mathbb{1} \left( \sum_{k \in [n]} d^t_k \leq R \right) \right]
\]

using the fact the above inequality implies

\[
E \left[ \sum_{i \in I} r^t_i \mathbb{1} \left( \sum_{k \in [n]} d^t_k > R \right) \right] \leq E \left[ \sum_{i \in I} r^t_i \right] \mathbb{P} \left[ \sum_{k \in I} \max(d^t_k) + \sum_{k \notin I} d^t_k > R \right]
\]

\[
\leq E \left[ \sum_{i \in I} r^t_i \right] \mathbb{P} \left[ \sum_{k \notin I} d^t_k - \sum_{k \notin I} d^t_k \right] \geq R - 1 - \sum_{k \in I} \max(d^t_k) \right]
\]

where in the last inequality we used the fact that \( \sum_{k \notin I} E \left[ d^t_k \right] \leq 1 \).

Now we can prove the theorem.

**Proof of Theorem 5.3.** Fix a \( t \). Because of Theorem 5.1 we can assume that users in \( I \) do not over-report their demand and thus we can focus on allocations instead of demands. By combining Lemmas C.2 and C.3 we get that

\[
E \left[ \sum_{i \in I} \left( \hat{R}^t_i - R^t_i \right) \right] \leq \sum_{\tau = 1}^t \left( E \left[ \sum_{i \in I} r^t_i \right] \mathbb{P} \left[ \sum_{k \notin I} d^t_k - \sum_{k \notin I} E \left[ d^t_k \right] \right] > R - 1 - \sum_{k \in I} \max(d^t_k) \right) \right]
\]

(32)

Using Chebyshev’s inequality and the independence of \( \{d^t_k\}_{k \notin I} \), for every \( \tau \) we have that

\[
\mathbb{P} \left[ \left| \sum_{k \notin I} d^t_k - \sum_{k \notin I} E \left[ d^t_k \right] \right| > R - 1 - \sum_{k \in I} \max(d^t_k) \right] \leq \frac{\sum_{k \notin I} \text{Var}(d^t_k)}{(R - 1 - \sum_{k \in I} \max(d^t_k))^2} \leq \frac{1}{C^2}
\]

(33)

where in the last inequality we assumed that there exists a \( C \in \Omega(|I|) \) such that for all \( \tau \)

\[
R \geq 1 + \sum_{k \in I} \max(d^t_k) + C \sqrt{\sum_{k \notin I} \text{Var}(d^t_k)}
\]

(34)

Note that (34) is the same as the constraint \( R \geq 1 + c/\sqrt{n} \), since for \( k \in I \), \( \max(d^t_k) \in O(1/\sqrt{n}) \) and \( \sum_{k \notin I} \text{Var}(d^t_k) \in O(1/n) \). Combining (32) and (33) proves the theorem:

\[
E \left[ \sum_{i \in I} \hat{R}^t_i \right] \leq E \left[ \sum_{i \in I} R^t_i \right] \left( 1 + \frac{1}{C^2} \right)
\]

\[\blacksquare\]
D Deferred Proof of Section 6

We first prove a more general version of Lemma 3.3.

**Lemma D.1.** Fix any $t \geq 1$. Let $\{R_{i}^{t-1}\}_{i \in [n]}$ and $\{\hat{R}_{i}^{t-1}\}_{i \in [n]}$ be the cumulative allocations up to epoch $t-1$. Assume that $\{d_{i}^{t}\}_{i \in [n]}$ are some users’ demands and that $\{\hat{d}_{i}^{t}\}_{i \in [n]}$ are the same demands except user 1’s, who deviates but does not over-report, i.e. $\hat{d}_{1}^{t} \leq d_{1}^{t}$. Then it holds that

$$\sum_{k \in [n]} (\hat{R}_{k}^{t} - R_{k}^{t})^+ - \sum_{k \in [n]} (\hat{R}_{k}^{t-1} - R_{k}^{t-1})^+ \leq g^t$$

where $g^t = \min \left( (r_{1}^{t} - \hat{r}_{1}^{t})^+, (R_{1}^{t} - \hat{R}_{1}^{t})^+ \right)$.

**Proof.** Let $P^t = \{k : \hat{R}_{k}^{t} \geq R_{k}^{t}\}$ and suppose by contradiction that

$$\sum_{k \in P^t} (\hat{R}_{k}^{t} - R_{k}^{t})^+ - \sum_{k \in P^{t-1}} (\hat{R}_{k}^{t-1} - R_{k}^{t-1})^+ > g^t$$

(36)

In order to get a contradiction we are going to show that the following two conditions are implied by (36):

(I) There exists a user $i \in P^t$, such that $\hat{r}_{i}^{t} > r_{i}^{t}$.

(II) There exists a user $j \notin P^t$, such that $j \neq 1$ and $\hat{r}_{j}^{t} < r_{j}^{t}$.

The reason conditions (I) and (II) lead to a contradiction is the following: users $i$ and $j$ satisfy the conditions of Lemma 3.2, meaning that $\hat{R}_{i}^{t} - R_{i}^{t} \leq \hat{R}_{j}^{t} - R_{j}^{t}$. However this is a contradiction due to the facts that $i \in P^t$ and $j \notin P^t$.

To prove that (36) implies conditions (I) and (II), we will now distinguish two cases; these are shown in Propositions D.2 and D.3.

**Proposition D.2.** If $\hat{r}_{1}^{t} \geq r_{1}^{t}$ or $\hat{R}_{1}^{t} \geq R_{1}^{t}$, (36) implies conditions (I) and (II).

**Proof of Proposition D.2.** In this case we have that $g^t = 0$. Because of the definition of $P^t$, we can re-write (36)

$$\sum_{k \in P^t} \hat{r}_{k}^{t} - r_{k}^{t} > 0$$

(37)

(37) implies (I). (37) and the fact that $\sum_{k} r_{k}^{t} \geq \sum_{k} \hat{r}_{k}^{t}$ (which comes from user 1 not over-reporting her demand) proves that

$$\sum_{k \notin P^t} r_{k}^{t} - \hat{r}_{k}^{t} > 0$$

The above implies condition (II): there exists a $j \notin P^t$ such that $\hat{r}_{j}^{t} < r_{j}^{t}$. The reason that the aforementioned $j$ cannot be user 1 is because of our assumptions: either $\hat{r}_{1}^{t} \geq r_{1}^{t}$ or $\hat{R}_{1}^{t} \geq R_{1}^{t}$.

**Proposition D.3.** If $\hat{r}_{1}^{t} < r_{1}^{t}$ and $\hat{R}_{1}^{t} < R_{1}^{t}$, (36) implies conditions (I) and (II).

**Proof of Proposition D.3.** Because $\hat{R}_{1}^{t} < R_{1}^{t}$, it holds that $1 \notin P^t$. Consider two cases:

- If $\hat{R}_{1}^{t-1} < R_{1}^{t-1}$, then $g^t = r_{1}^{t} - \hat{r}_{1}^{t}$. In this case (36) implies

$$\sum_{k \in P^t} \hat{r}_{k}^{t} - r_{k}^{t} > r_{1}^{t} - \hat{r}_{1}^{t} > 0$$
• If \( \hat{R}_1^{t-1} \geq R_1^{t-1} \), then \( g^t = R_1^t - \hat{R}_1^t \) and \( 1 \in P^{t-1} \). In this case (36) implies

\[
\sum_{k \in P^t} (\hat{R}_k^t - R_k^t) - \sum_{k \in P^{t-1}} (\hat{R}_k^{t-1} - R_k^{t-1}) > R_1^t - \hat{R}_1^t
\]

\[
\sum_{k \in P^t} (\hat{R}_k^t - R_k^t) - \sum_{k \in P^{t-1} \setminus \{1\}} (\hat{R}_k^{t-1} - R_k^{t-1}) > r_1^t - \hat{r}_1^t
\]

\[
\sum_{k \in P^t} \hat{r}_k^t - r_k^t > r_1^t - \hat{r}_1^t > 0
\]

where to get the last inequality we use the fact that 1 \( \notin P^t \).

Thus in both cases the following inequality holds:

\[
\sum_{k \in P^t} \hat{r}_k^t - r_k^t > r_1^t - \hat{r}_1^t > 0 \tag{38}
\]

(38) implies (I), (38) and \( \sum_k r_k^t \geq \sum_k \hat{r}_k^t \) (which is true because user 1 does not over-report) prove that

\[
\sum_{k \notin P^t} r_k^t - \hat{r}_k^t > r_1^t - \hat{r}_1^t
\]

\[
\sum_{k \notin P^t \atop k \neq 1} r_k^t - \hat{r}_k^t > 0
\]

The above implies condition (II), which completes the proposition’s proof.

Due to Propositions D.2 and D.3 we have proven that (36) always leads to a contradiction. This proves the lemma.

We now use the above lemma to prove a corollary that directly bounds \( \hat{R}_1^t - R_1^t \).

**Corollary D.4.** Let \( g^t = \min \left( (r_1^t - \hat{r}_1^t)^+, (R_1^t - \hat{R}_1^t)^+ \right) \) and assume that user 1 does not over-report her demand. Then for every epoch \( t \)

\[
2 (\hat{R}_1^t - R_1^t) \leq \sum_{\tau=1}^t g^\tau \tag{39}
\]

**Proof.** Fix an epoch \( t \) and let \( t' \leq t \) be the last epoch before \( t \) where \( \hat{r}_1^{t'} > r_1^{t'} \). If no such epoch exists, then \( \hat{R}_1^t \leq R_1^t \), in which case the lemma holds. We notice that \( \hat{R}_1^t - R_1^t \leq \hat{R}_1^{t'} - R_1^{t'} \).

Using Lemma D.1 and summing (35) for \( t \) from 1 to \( t' \) we get

\[
\sum_{k} \left( \hat{R}_k^{t'} - R_k^{t'} \right)^+ \leq \sum_{\tau=1}^{t'} g^\tau
\]

Because \( \hat{r}_1^t > r_1^t \) and \( \sum_i r_i^{t'} \geq \sum_i \hat{r}_1^{t'} \) (since user 1 does not over-report) it holds that for some user \( j \neq 1 \), \( \hat{r}_j^{t'} < r_j^{t'} \). This means we can use Lemma 3.3 to prove that \( \hat{R}_1^t - R_1^t \leq \hat{R}_j^{t'} - R_j^{t'} \), which makes the above inequality

\[
2(\hat{R}_1^t - R_1^t) \leq 2(\hat{R}_1^{t'} - R_1^{t'}) \leq \hat{R}_j^{t'} - R_j^{t'} + \hat{R}_j^{t'} - R_j^{t'} \leq \sum_{\tau=1}^{t'} g^\tau \leq \sum_{\tau=1}^{t} g^\tau
\]

This completes the proof.
Now we prove a series of lemmas with the notation introduced in Section 6, in order to prove Theorem 6.1, which we restate for completeness.

**Theorem 6.1.** Let there be only one resource. Assume that for every $t$, $R_1^t \in \Theta(t)$ and for every $\ell = 0, 1, \ldots$ there exists an epoch $t_\ell \in [s_\ell, e_\ell)$ for which $\hat{R}_1^{t_\ell} \geq \gamma R_1^{t_\ell}$, for some $\gamma > 1$. Then for any $\ell = 0, 1, \ldots$ and any $t_\ell \in [s_\ell, e_\ell)$ such that $\hat{R}_1^{t_\ell} \geq \gamma R_1^{t_\ell}$, it holds that

$$t_\ell = O(s_\ell) \text{ and } t_\ell = \left(2 - \gamma \frac{1}{2 - 2\gamma}\right)^\ell \Omega(t_0)$$

**Lemma D.5.** If user 1 does not over-report her demand, for any $\ell = 0, 1, 2, \ldots$ and any $t_\ell \in [s_\ell, e_\ell)$ it holds that

$$2 \left(\hat{R}_1^{t_\ell} - R_1^{t_\ell}\right) \leq R_1^{s_\ell - 1} - \sum_{k=0}^{\ell-1} \left(\hat{R}_1^{e_k} - R_1^{s_k - 1}\right)$$

**Proof.** We are going to use Corollary D.4: for every $k \in [0, \ell - 1]$ and $t \in [s_k, e_k)$ it holds that the r.h.s. of (39) is $g^t = 0$, because $\hat{R}_1^t \geq R_1^t$. Fix a $t_\ell \in [s_\ell, e_\ell)$ and we notice that

$$2 \left(\hat{R}_1^{t_\ell} - R_1^{t_\ell}\right) \leq \sum_{\tau=1}^{t_\ell} g^\tau = \sum_{\tau=1}^{s_0 - 1} g^\tau + \sum_{\tau=s_0}^{s_1 - 1} g^\tau + \ldots + \sum_{\tau=s_{\ell-1}}^{s_\ell - 1} g^\tau$$

$$g^\tau \leq R_1^{s_\ell - 1} - \sum_{k=0}^{\ell-1} \left(\hat{R}_1^{e_k} - R_1^{s_k - 1}\right)$$

Now all that is left to complete the proof is to prove that for every $k$, $R_1^{e_k} - g^e_k \geq \hat{R}_1^{e_k}$. This actually holds with an equality: due to the definition of $e_k$, $\hat{R}_1^{e_k} = R_1^{e_k}$ and thus $\hat{R}_1^{e_k} = R_1^{e_k}$. ■

**Lemma D.6.** Assume that for every $\ell = 0, 1, \ldots$ there exists a $t_\ell \in [s_\ell, e_\ell)$ such that $\hat{R}_1^{t_\ell} \geq \gamma R_1^{t_\ell}$ for some $\gamma \in [1, 3/2)$. Then for any such $\{t_\ell\}_\ell$ and any $\ell \geq 1$:

$$R_1^{t_\ell} \geq \frac{\gamma - 1}{3 - 2\gamma} \sum_{k=0}^{\ell-1} R_1^{t_\ell}$$

**Proof.** Fix an $\ell \geq 1$. We use Lemma D.5 and get that

$$2 \left(\hat{R}_1^{t_\ell} - R_1^{t_\ell}\right) \leq R_1^{s_\ell - 1} - \sum_{k=0}^{\ell-1} \left(\hat{R}_1^{e_k} - R_1^{s_k - 1}\right) \leq R_1^{t_\ell} - \sum_{k=0}^{\ell-1} \left(\hat{R}_1^{t_\ell} - R_1^{t_\ell}\right)$$

Using the fact that for every $k = 0, \ldots, \ell$, $\hat{R}_1^{t_\ell} \geq \gamma R_1^{t_\ell}$, the above inequality becomes

$$(2\gamma - 3)R_1^{t_\ell} \leq -(\gamma - 1) \sum_{k=0}^{\ell-1} R_1^{t_\ell}$$

which proves the lemma. ■

**Corollary D.7.** If the conditions of Lemma D.6 hold, then for all $\ell \geq 1$,

$$R_1^{t_\ell} \geq \frac{\gamma - 1}{2 - \gamma} \left(\frac{2 - \gamma}{3 - 2\gamma}\right)^\ell R_1^{t_0}$$  \(40\)
Proof. We will prove the corollary with induction on $\ell$. For $\ell = 1$, (40) follows from Lemma D.6. Assume that for some $L$, (40) holds for all $\ell = 1, 2, \ldots, L$. Using Lemma D.6 we have that

$$R_1^{t_{L+1}} \geq \frac{\gamma - 1}{3 - 2\gamma} \sum_{\ell=0}^{L} R_1^{t_{\ell}} \geq R_1^{t_0} \frac{\gamma - 1}{3 - 2\gamma} \left( 1 + \sum_{\ell=1}^{L} \frac{\gamma - 1}{2 - \gamma} \left( \frac{2 - \gamma}{3 - 2\gamma} \right)^\ell \right) = R_1^{t_0} \frac{\gamma - 1}{2 - \gamma} \left( \frac{2 - \gamma}{3 - 2\gamma} \right)^{L+1}$$

This corollary proves the second part of Theorem 6.1. Now we are going to prove the first part.

Corollary D.8. If the conditions of Lemma D.6 hold and for all $t$, $R_1^t \in \Theta(t)$, then for all $\ell \geq 1$

$$t_\ell = \left( \frac{2 - \gamma}{3 - 2\gamma} \right)^\ell \Omega(t_0)$$

The lemma follows from the facts that $R_1^t = \Theta(t_\ell)$ and $R_1^{s_{\ell} - 1} = \Theta(s_\ell)$.

Lemma D.9. Fix an $\ell \in \{0, 1, \ldots\}$ and assume that for some $t_\ell \in [s_\ell, e_\ell)$ it holds that $\hat{R}_1^{t_\ell} \geq \gamma R_1^{t_\ell}$, for some constant $\gamma > 1$. If $R_1^t \in \Theta(t)$ for all $t$, then for all $\ell \in \{0, 1, \ldots\}$

$$t_\ell \in O(s_\ell)$$

Proof. Using Lemma D.5 and $\hat{R}_1^{t_\ell} \geq \gamma R_1^{t_\ell}$ we can easily prove that

$$2(\gamma - 1)R_1^{t_\ell} \leq R_1^{s_{\ell} - 1}$$

The lemma follows from the facts that $R_1^{t_\ell} = \Theta(t_\ell)$ and $R_1^{s_{\ell} - 1} = \Theta(s_\ell)$. 

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