A REMARK ON THE GRADIENT MAP

LEONARDO BILIOTTI, ALESSANDRO GHIGI, AND PETER HEINZNER

Abstract. For a Hamiltonian action of a compact group $U$ of isometries on a compact Kähler manifold $Z$ and a compatible subgroup $G$ of $U^\mathbb{C}$, we prove that for any closed $G$–invariant subset $Y \subset Z$ the image of the gradient map $\mu_p(Y)$ is independent of the choice of the invariant Kähler form $\omega$ in its cohomology class $[\omega]$.

1. Introduction

Let $(Z, \omega)$ be a compact Kähler manifold and let $U$ be a compact connected semisimple Lie group such that $U^\mathbb{C}$ acts holomorphically on $Z$, $U$ preserves $\omega$ and there is a momentum map $\mu : Z \to \mathfrak{u}^*$. Let $G \subset U^\mathbb{C}$ be a compatible subgroup. By this we mean a subgroup which is compatible with the Cartan involution $\Theta$ of $U^\mathbb{C}$ which defines $U$, i.e. if $\mathfrak{p} = \mathfrak{g} \cap i\mathfrak{u}$ and $K = U \cap G$, then $G = K \cdot \exp \mathfrak{p}$. Let $\mu_p : Z \to \mathfrak{p}$ be the associated gradient map (see [4, 5] or section 2).

In this note we prove the following.

Theorem 1. Let $Y \subset Z$ be a closed $G$-stable subset. Then up to translation the set $\mu_p(Y)$ is independent of the choice of the invariant Kähler form $\omega$ in the cohomology class $[\omega]$.

Since $Z$ is compact and $G$ is compatible there is a stratification of $Z$ analogous to the Kirwan stratification, see [4]. This gives a stratification of any closed $G$–invariant subset $Y$ of $Z$, by intersecting the strata in $Z$ with $Y$. It follows from Theorem 1 that when the momentum map is properly normalized (see Lemma 2) this stratification does not depend on the choice of $\omega$ in its cohomology class.

When $Z$ is a projective manifold and $\omega$ is the pull-back of a Fubini-Study form via an equivariant embedding of $Z$ in $\mathbb{P}^N$, Kirwan [6, §12]...

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proved that the stratification in terms of a properly normalized \( \mu \) can be defined purely in terms of algebraic geometry. In the present note we give a proof of this fact for a general compact Kähler manifold \( Z \) in the more general setting of gradient maps for actions of compatible subgroups on closed \( G \)-invariant subsets of \( Z \).

Another consequence of the above is the following. Assume that \( Z \) is a projective manifold and that \([\omega]\) is an integral class. Let \( Y \subset Z \) be a closed \( G \)-invariant real semi-algebraic subset whose real algebraic Zariski closure is irreducible. Let \( a \subset p \) be a maximal subalgebra and let \( a_+ \) be a closed Weyl chamber in \( a \). Then \( A(Y)_+ := \mu_p(Y) \cap a_+ \) is convex (see [2], which deals with the case when \( \omega \) is the restriction of a Fubini-Study metric).

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2. Background

Let \((Z, \omega)\) be a compact Kähler manifold and let \( U \) be a compact Lie group. Assume that \( U \) acts on \( Z \) by holomorphic Kähler isometries. Since \( Z \) is compact the \( U \)-action extends to a holomorphic action of the complexified group \( U^\mathbb{C} \). Assume also that there is a momentum map \( \mu : Z \to u^* \cong u \), where \( u^* \) is identified with \( u \) using a fixed \( U \)-invariant scalar product on \( u \) that we denote by \( \langle , \rangle \). We also denote by \( \langle , \rangle \) the scalar product on \( iu \) such that multiplication by \( i \) is an isometry of \( u \) onto \( iu \). If \( \xi \in u \) we denote by \( \xi_Z \) the fundamental vector field on \( Z \) and we let \( \mu^\xi \in C^\infty(Z) \) be the function \( \mu^\xi(z) := \langle \mu(z), \xi \rangle \). That \( \mu \) is the moment map means that it is \( U \)-equivariant and that \( d\mu^\xi = i\xi_Z \omega \).

For a closed subgroup \( G \subset U^\mathbb{C} \) let \( K := G \cap U \) and \( p := g \cap iu \). The group \( G \) is called compatible if \( G = K \cdot \exp p \) [4, 5]. In the following we fix a compatible subgroup \( G \subset U^\mathbb{C} \). If \( z \in Z \), let \( \mu_p(z) \in p \) denote \(-i\) times the component of \( \mu(z) \) in the direction of \( ip \). In other words we require that \( \langle \mu_p(z), \beta \rangle = -\langle \mu(z), i\beta \rangle \) for any \( \beta \in p \). The map

\[
\mu_p : Z \to p
\]

is called the gradient map (see [3]) or restricted momentum map. Let \( \mu^\beta_p \in C^\infty(Z) \) be the function \( \mu^\beta_p(z) = \langle \mu_p(z), \beta \rangle = \mu^{-i\beta}(z) \). Let \( \langle , \rangle \) be the Kähler metric associated to \( \omega \), i.e. \( \langle v, w \rangle = \omega(v, Jw) \). Then \( \beta_Z \) is the gradient of \( \mu^\beta_p \) with respect to \( \langle , \rangle \).
Example 1. (1) For any compact subgroup $K \subset U$, both $K$ and its complexification $G = K^\mathbb{C}$ are compatible. In particular $G = U^\mathbb{C}$ is a compatible subgroup. (2) If $G$ is a real form of $U^\mathbb{C}$, then $G$ is compatible. (3) For any $\xi \in iu$, the subgroup $G = \exp(\mathbb{R}\xi)$ is compatible.

Next we recall the Stratification Theorem for actions of compatible subgroups. Given a maximal subalgebra $a \subset p$ and a Weyl chamber $a^+ \subset a$ define

$$\eta_p : X \to \mathbb{R} \quad \eta_p(x) := \frac{1}{2}||\mu_p(x)||^2$$

$$C_p := \text{Crit}(\eta_p) \quad B_p := \mu_p(C_p) \quad B_p^+ := B_p \cap a^+$$

$$X(\mu) = \{x \in X : G \cdot x \cap \mu_p^{-1}(0) \neq \emptyset\}$$

where $X$ is a compact $G$-invariant subset of $Z$. Points lying in $X(\mu)$ are called semistable. Using semistability and the function $\eta_p$ one can define a stratification of $X$ in the following way, see [6] and [4]. For $\beta \in B_p^+$ set

$$X_{||\beta||^2} := \{x \in X : \exp(\mathbb{R}\beta) \cdot x \cap (\mu^\beta)^{-1}(||\beta||^2)\}$$

$$X^\beta := \{x \in X : \beta X(x) = 0\}$$

$$X_{||\beta||^2}^\beta := X^\beta \cap X_{||\beta||^2}$$

$$X_{||\beta||^2}^{\beta+} := \{x \in X_{||\beta||^2} : \lim_{t \to -\infty} \exp(t\beta) \cdot x \text{ exists and it lies in } X_{||\beta||^2}^\beta\}$$

$$G^{\beta+} := \{g \in G : \text{the limit } \lim_{t \to -\infty} \exp(t\beta)g \exp(-t\beta) \text{ exists in } G\}.$$ 

Set also

$$G^\beta = \{g \in G : \text{Ad} g(\beta) = \beta\} \quad p^\beta := \{\xi \in p : [\xi, \beta] = 0\}.$$ 

The group $G^\beta = K^\beta \cdot \exp(p^\beta)$ is a compatible subgroup of $U^\mathbb{C}$ and the set $X_{||\beta||^2}^{\beta+}$ is $G^{\beta+}$-invariant. Denote by $\mu_{p^\beta}$ the composition of $\mu_p$ with the orthogonal projection $p \to p^\beta$. Then $\mu_{p^\beta}$ is a gradient map for the $G^\beta$-action on $X_{||\beta||^2}^{\beta+}$. We set $\widehat{\mu_{p^\beta}} := \mu_{p^\beta} - \beta$. Since $\beta$ lies in the center of $g^\beta$ and since $G^\beta$ is a compatible subgroup of $(U^\beta)^\mathbb{C} = (U^\mathbb{C})^\beta$, it is a gradient map too. We let $S^{\beta+}$ denote the set of $G^\beta$-semistable points in $X_{||\beta||^2}^{\beta+}$ with respect to $\widehat{\mu_{p^\beta}}$, i.e.

$$S^{\beta+} := \{x \in X_{||\beta||^2}^{\beta+} : G^\beta \cdot x \cap \mu_{p^\beta}^{-1}(\beta) \neq \emptyset\}.$$ 

The set $S^{\beta+}$ coincides with the set of semistable points of the group $G^\beta$ in $X_{||\beta||^2}$ after shifting. By definition the $\beta$-stratum is given by $S_\beta := G \cdot S^{\beta+}$. 

Stratification Theorem. (See [4, Thm. 7.3]) Assume that \( X \) is a compact \( G \)-invariant subset of \( Z \). Then \( \mathcal{B}_p^+ \) is finite and
\[
X = \bigcup_{\beta \in \mathcal{B}_p^+} S_\beta.
\]
Moreover
\[
\overline{S_\beta} \subset S_\beta \cup \bigcup_{||\gamma||>||\beta||} S_\gamma.
\]

3. Proof of Theorem 1

For a \( U \)-invariant function \( f \) on \( Z \) we set
\[
\bar{\omega} := \omega + dd^c f
\]
where \( d^c f := -2J^* df \). Since \( Z \) is compact and \( U \) acts by holomorphic transformations, any \( U \)-invariant Kähler form \( \bar{\omega} \) in the Kähler class \([\omega]\) can be written in this way. Since pluriharmonic functions on \( Z \) are constant, the function \( f \) is unique up to a constant.

Lemma 2. If \( \mu : Z \to \mathfrak{u} \) is a momentum map for the \( U \)-action on \( Z \) with respect to \( \omega \), then the function \( \bar{\mu} : Z \to \mathfrak{u} \) defined by
\[
\bar{\mu}^\xi := \mu^\xi - d^c f(\xi_Z)
\]
is a momentum map for the \( U \)-action on \( Z \) with respect to \( \bar{\omega} \).

Proof. That \( \bar{\mu} \) is a momentum map follows from Cartan formula using that \( L_{\xi_Z} d^c f = d^c L_{\xi_Z} f = 0 \). This in turn follows from the assumption that the action of \( U \) is holomorphic and \( f \) is \( U \)-invariant. \( \square \)

A more precise version of Theorem 1 is the following.

**Theorem 4.** For any closed \( G \)-stable subset \( Y \subset Z \) we have \( \mu_p(Y) = \bar{\mu}_p(Y) \).

**Proof.** Let \( \mathfrak{a} \subset \mathfrak{p} \) be a maximal subalgebra and set \( A := \exp \mathfrak{a} \). The group \( A \) is a compatible subgroup. Let \( \mu_a : Z \to \mathfrak{a} \) be the restricted gradient map. Any connected subgroup \( B \subset A \) is compatible. Given such a \( B \), set \( Z^{(B)} := \{ z \in Z : A_z = B \} \). A connected component \( S \) of \( Z^{(B)} \) will be called an \( A \)-stratum of type \( \mathfrak{b} \). For a given \( S \) let \( C \) denote the connected component of \( Z^B \) containing \( S \). Then \( C \) is a complex submanifold of \( Z \) and the Slice Theorem (see Theorem 14.10 and 14.21 in [3] or Theorem 2.2 in [2]) applied to the \( A \)-action on \( C \) shows that \( S \) is open and dense in \( C \).

Let \( A^c \) be the Zariski closure of \( A \) in \( U^C \). The group \( A^c \) is a compatible subgroup of \( U^C \), \( A^c \cap U = T \) is a torus and \( A^c = T \exp(it) \), where
t denotes the Lie algebra of T. Moreover $\overline{S}$ is $A^c$-stable [2, Lemma 3.3 (1)]. Denote by $\mu_t : Z \rightarrow t$ the momentum map obtained by projecting $\mu : Z \rightarrow u$ to $t$, and denote by $\Pi : it \rightarrow a$ the orthogonal projection. Then $\mu_a = \Pi \circ i\mu_t$ and $\mu_a(S) = \Pi(i\mu_t(S))$. By the convexity theorem of Atiyah-Guillemin-Sternberg $\mu_t(S)$ is a convex polytope and its vertices are images of points fixed by $A^c$. It follows that $\mu_a(S)$ is a convex polytope as well. Since $\Pi$ is linear, any vertex of $\mu_a(S)$ is the projection of at least one vertex of $i\mu_t(S)$. Therefore $\mu_a(S)$ is the convex hull of $\mu_a(S^A)$. Now we use Lemma [2] if $x \in \overline{S^A}$, then $\xi_Z(x) = 0$, so $\tilde{\mu}\xi(x) = \mu\xi(x)$, for any $\xi \in a$. Therefore $\tilde{\mu}_a(x) = \mu_a(x)$ for every $A$-fixed point $x$. It follows that both $\mu_a(S)$ and the affine subspace spanned by $\mu_a(S)$ do not depend on the choice of the Kähler form $\omega$.

Let $\Sigma$ be the collection of affine hyperplanes of $a$ that are affine hulls of $\mu_a(S)$ for some $A$-stratum $S$. Set $P := \mu_a(Z)$ and $P_0 := P - \bigcup_{H \in \Sigma} P \cap H.$ (See [2]). The set $P_0$ is an open subset of $a$. Let $C(P_0)$ denote the set of its connected components. This is a finite set. For $\gamma \in C(P_0)$ let $P(\gamma)$ be the closure of the connected component $\gamma$. Then $P(\gamma)$ is a convex polytope. Since both $P$ and the hyperplanes $H$ are independent of $\omega$, also the polytopes $P(\gamma)$ do not depend on $\omega$. By [2, Corollary 5.8]

$$\mu_p(Y) \cap a = \bigcup_{\gamma \in F(\omega)} P(\gamma),$$

where $F(\omega) \subset \Gamma$ is some subset of $C(P_0)$. One can join $\omega$ to $\tilde{\omega}$ continuously, e.g. by $\omega_t := \omega + tdd^c f$. Then $\tilde{\mu}_t := \mu - tdd^c f(\cdot , z)$ also depends continuously on $t$. So $P(\gamma) \subset \mu_p(Y) \cap a$ if and only if $P(\gamma) \subset \mu_{t,p}(Y) \cap a$. Therefore $F(\omega_t)$ is constant and the same is true of $\mu_p(Y) \cap a$. This implies $\mu_p(Y) = K(\mu_p(Y) \cap a).$ Hence $\tilde{\mu}(Y) = \tilde{\mu}_p(Y).$ □

**Corollary 5.** Assume that $Z$ is connected and let $\omega$ and $\tilde{\omega}$ be two cohomologous Kähler forms with momentum maps $\mu$ and $\tilde{\mu}$ respectively as in Lemma [2]. Then $\tilde{\mu}$ is the unique momentum map such that $\mu(Z) = \tilde{\mu}(Z)$.

**Proof.** Since two momentum maps with respect to $\tilde{\omega}$ differ by addition of an element of the center of $u$, it is clear that there is at most one such map with the image equal to $\mu(Z)$. To complete the proof it is therefore enough to check that $\tilde{\mu}(Z) = \mu(Z)$. This is a special case of the previous theorem. □
**Theorem 6.** Let $\omega$ and $\tilde{\omega}$ be two cohomologous Kähler forms on $Z$, with momentum maps $\mu$ and $\tilde{\mu}$ respectively as in Lemma 2. Then the set $B^+_0$ is the same for both momentum maps and the two stratifications of $X$ coincide.

**Proof.** By \[4, Corollary 7.6\]

$$B_p = \{ \beta \in \mathfrak{p} : \text{there exists } x \in X : \frac{||\beta||^2}{2} = \inf_{G \cdot x} \eta_{p} \text{ and } \beta \in \mu_p(G \cdot x) \}. \tag{7}$$

Moreover for $\beta \in B_p$

$$S_\beta = \{ x \in X : \frac{||\beta||^2}{2} = \inf_{G \cdot x} \eta_{p} \text{ and } \beta \in \mu_p(G \cdot x) \}. \tag{8}$$

For any point $x \in X$, the set $G \cdot x$ is closed and $G$-invariant. Hence by Theorem 4 $\mu_p(G \cdot x) = \tilde{\mu}_p(G \cdot x)$. From this it follows that $\inf_{G \cdot x} \eta_{p} = \inf_{G \cdot x} \tilde{\eta}_p$, where $\tilde{\eta}_p := ||\mu_p||^2/2$. The result follows from (7) and (8). □

From the above we obtain the following generalization.

**Corollary 9.** If $Z$ is a complex projective manifold, $U$ is a compact connected semisimple Lie group acting on $Z$, $\omega$ is a $U$-invariant Hodge metric and $Y \subset Z$ is a closed $G$-invariant real semi-algebraic subset whose real algebraic Zariski closure is irreducible, then $A(Y)_+$ is convex. Moreover if $G$ is semisimple, then $X(\mu)$ is dense (if it is nonempty).

**Proof.** By assumption there is a very ample line bundle $L \to Z$ such that $[\omega] = 2\pi c_1(L)/m$ for an integer $m > 0$. Let $\omega_{FS}$ be a $U$-invariant Fubini-Study metric on $\mathbb{P}(H^0(Z, L)^*)$. Let $\mu_{FS}$ be the moment map with respect to $\omega_{FS}|_Z$. In [2] the convexity theorem has been proved for $\mu_{FS}$. A rescaling in the symplectic form yields a corresponding rescaling in the momentum map. Therefore the convexity theorem also holds for the momentum map $\tilde{\mu}$ relative to the symplectic form $\tilde{\omega} := \omega_{FS}/m$. So it holds also for $\mu$, since $\mu_p(Y) = \tilde{\mu}_p(Y)$ by Theorem 4. The proof of the last statement is similar: see [2] and Corollary 5. □

**Corollary 10.** Under the same assumptions, any local minimum of $|\mu_p|^2$ is a global minimum.

**Proof.** This follows since $|\mu_p|^2$ is $K$-invariant and $\mu(Z)_+$ is a convex subset of $a_+$. □

**Corollary 11.** If $\omega$ and $\omega'$ are cohomologous Kähler forms on $Z$ with momentum maps $\mu$ and $\tilde{\mu}$ as in Lemma 2 then $X(\mu) = X(\tilde{\mu})$. 
Proof. It is enough to observe that $X(\mu) = S_0$. 

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Università di Parma
E-mail address: leonardo.biliotti@unipr.it

Università di Milano Bicocca
E-mail address: alessandro.ghigi@unimib.it

Ruhr Universität Bochum
E-mail address: peter.heinzner@rub.de