Identification of nonlinear heat conduction laws

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Abstract

We consider the identification of nonlinear heat conduction laws in stationary and instationary heat transfer problems. Only a single additional measurement of the temperature on a curve on the boundary is required to determine the unknown parameter function on the range of observed temperatures. We first present a new proof of Cannon’s uniqueness result for the stationary case, then derive a corresponding stability estimate, and finally extend our argument to instationary problems.

Keywords: parameter identification, nonlinear diffusion, quasilinear parabolic problems

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1 Introduction

This note is concerned with parameter identification problems in nonlinear heat transfer processes. Let us consider the quasilinear elliptic problem

\[- \text{div}(a(u)\nabla u) = f \quad \text{in } \Omega, \quad a(u)\partial_n u = j \quad \text{on } \partial \Omega.\]

Following [4], see also [13, 22], the parameter function \(a(u)\) can be uniquely determined from temperature measurements \(g = u|_{\gamma}\) on a boundary curve \(\gamma\). We present an alternative proof of this uniqueness result below which allows us to treat also perturbations in the data \(f, j,\) and in \(g\), and to obtain a stability result for the inverse problem. By using a proper experimental setup, we can then consider also parabolic problems of the form

\[u_t - \text{div}(a(u)\nabla u) = f \quad \text{on } \Omega \times (0,T),\]

with \(u = u_0\) on \(\Omega \times \{0\}\) and \(a(u)\partial_n u = j\) on \(\partial\Omega \times (0,T)\). In fact, the additional term \(u_t\) will be treated as a perturbation of the stationary equation. Our main result about identifiability in the parabolic case can be summarized as follows:

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For any \( \epsilon > 0 \) and any interval \([g_1, g_2]\), we can choose an experimental setup, i.e., data \( u_0, j, f \), and a time horizon \( T \), such that

\[
u = \tilde{u} \text{ on } \gamma \times (0, T) \implies |a(s) - \tilde{a}(s)| \leq \epsilon \text{ for all } s \in [g_1, g_2].\]

Here \( u, \tilde{u} \) denote the temperature distributions for parameters \( a, \tilde{a} \), respectively. It is thus possible to identify the coefficient function \( a(u) \) with any prescribed accuracy by a single measurement truly instationary experiment. Our proof of this result is based on the combination of an energy estimate and a perturbation argument. Similar energy estimates have been used recently also for parameter identification in linear elliptic equations [16].

Parameter identification in heat conduction has a long history [2, 3, 5]. To date, rigorous uniqueness results for quasilinear parabolic problems are however only available in one space dimension [6, 7, 8, 10, 11, 12]; but see [17] for multidimensional semilinear problems. To the best of our knowledge, the question of identifiability of a nonlinear heat conduction law in the multidimensional quasilinear parabolic problem has not been answered yet. Motivated by applications, several papers are also concerned with numerical methods for parameter estimation in nonlinear heat transfer, see e.g. [9, 21, 23]. For an overview about available uniqueness results and further references on parameter identification in the context of partial differential equations, let us refer to [18] and [19].

The remainder of this note is organized as follows: In Section 2 we introduce some basic assumptions and then present our new proof of the uniqueness result of [4] for the stationary case. In Section 3 we then derive the corresponding stability result for the inverse problem, which enables us to treat also quasilinear parabolic equations in Section 4. We conclude with a short discussion.

## 2 Uniqueness for the elliptic problem

Throughout the text, we will make some general assumptions that allow us to keep the presentation simple. Concerning the geometry, we assume that

(A1) \( \Omega \subset \mathbb{R}^d \) is a bounded domain with \( C^{1,1} \) boundary in \( d = 2, 3 \) space dimensions and \( \gamma : [0, 1] \to \partial \Omega \) is a \( C^1 \) curve on the boundary.

It should become clear from our analysis that the geometric regularity conditions can be further relaxed. Let us consider the following quasilinear elliptic problem

\[
- \text{div}(a(u)\nabla u) = 0 \quad \text{in } \Omega, \tag{1}
\]

\[
a(u)\partial_n u = j \quad \text{on } \partial \Omega. \tag{2}
\]

In order to ensure the well-posedness of this forward problem, we require some regularity and compatibility conditions for the parameter and the data, namely...
Via the transformation $A$, Theorem 1 concerning this particular problem. We thus obtain from standard results for linear elliptic equations [14]; see also [13] for details of solutions of the elliptic problem (1)–(2) for parameters $a < \infty$ by the additional temperature measurement $g$.

Note that we get uniqueness and a true a-priori estimate once the solution is fixed only on $\gamma$. Here and below, we write $(u, v) = \int_{\Omega} uv \, dx$ for the $L^2$ scalar product. In the last step, we used integration-by-parts and the identical Neumann data $\partial_n \tilde{A}(u) = a(u) \partial_n u = j = \tilde{a}(\tilde{u}) \partial_n \tilde{u} = \partial_n \tilde{A}(\tilde{u})$. Setting $\phi = A(u) - \tilde{A}(\tilde{u})$ now implies $\|\nabla A(u) - \nabla \tilde{A}(\tilde{u})\|_{L^2(\Omega)} = 0$, and hence $A(u) = \tilde{A}(\tilde{u}) + C$ with some constant $c \in \mathbb{R}$. Using the continuity of $u$ and $\tilde{u}$ up to the boundary and assuming identical temperature measurements $u|_{\gamma} = \tilde{u}|_{\gamma} = g$, we get

$$A(g) = \tilde{A}(g) + c \quad \text{on } \gamma.$$

By differentiation along the curve $\gamma$, we obtain Cannon’s uniqueness result [4].

**Theorem 2** Let (A1) hold and let $u, \tilde{u}$ denote the solutions of (1)–(2) for parameters $a, \tilde{a} \in A_{ad}$ with identical data $f, j$ satisfying (A3). Then the measurement $u|_{\gamma} = g = \tilde{u}|_{\gamma}$ on $\gamma$ implies that $a(s) = \tilde{a}(s)$ for $s \in \text{int}(g(\gamma(s)) : s \in [0, 1])$.

Note that the interval of identifiability is empty, if the temperature $g = u|_{\gamma}$ is constant on $\gamma$. In fact, no identification is possible in that case.
3 Stability for the inverse problem

As a second step of our analysis, we now investigate the stability of the identified parameter \( a \) with respect to perturbations in the data \( f, j \), and in the measurements \( g \). As above, let \( u, \tilde{u} \) denote the solutions of (1)–(2) for parameters \( a, \tilde{a} \in \mathcal{A}_{ad} \), and with data \( f, \tilde{f} \) and \( j, \tilde{j} \) satisfying assumption (A3). Proceeding as in the previous section, we then obtain the identity

\[
(f - \tilde{f}, \phi)_{\Omega} + (j - \tilde{j}, \phi)_{\partial\Omega} = (\nabla A(u) - \nabla \tilde{A}(\tilde{u}), \nabla \phi)_{\Omega}
\]

for all smooth functions \( \phi \). Choosing \( \phi = A(u) - \tilde{A}(\tilde{u}) \) as before and applying the Cauchy-Schwarz inequality to estimate the terms on the left hand side, we get

\[
\|\nabla A(u) - \nabla \tilde{A}(\tilde{u})\|_{L^2(\Omega)} \leq \|f - \tilde{f}\|_{L^2(\Omega)} \|A(u) - \tilde{A}(\tilde{u})\|_{L^2(\Omega)} + \|j - \tilde{j}\|_{L^2(\partial\Omega)} \|A(u) - \tilde{A}(\tilde{u})\|_{L^2(\partial\Omega)}.
\]

Without loss of generality, we can define the principals \( A \) and \( \tilde{A} \) in such a way that \( \int_{\Omega} A(u) - \tilde{A}(\tilde{u}) \, dx = 0 \). By means of the Poincaré inequality, we can then deduce that \( \|A(u) - \tilde{A}(\tilde{u})\|_{L^2(\Omega)} \leq C_P \|\nabla A(u) - \nabla \tilde{A}(\tilde{u})\|_{L^2(\Omega)} \) and similarly that \( \|A(u) - \tilde{A}(\tilde{u})\|_{L^2(\partial\Omega)} \leq C_P \|\nabla A(u) - \nabla \tilde{A}(\tilde{u})\|_{L^2(\Omega)} \). Thus we arrive at

\[
\|\nabla A(u) - \nabla \tilde{A}(\tilde{u})\|_{L^2(\Omega)} \leq C_P (\|f - \tilde{f}\|_{L^2(\Omega)} + \|j - \tilde{j}\|_{L^2(\partial\Omega)}).
\]

Using the uniform boundedness of \( A(u) \) and \( \tilde{A}(\tilde{u}) \) in \( W^{1,p}(\Omega) \) with \( p \) arbitrarily large, interpolation between \( L^2(\Omega) \) and \( W^{1,p}(\Omega) \) [15], embedding of \( W^{0,q}(\Omega) \) into \( C(\Omega) \) [1], and moving to the boundary, we get

\[
\|\nabla A(g) - \nabla \tilde{A}(\tilde{g})\|_{L^\infty(\gamma)} \leq C_\beta (\|f - \tilde{f}\|_{L^2(\Omega)} + \|j - \tilde{j}\|_{L^2(\partial\Omega)})^\beta
\]

for all \( 0 \leq \beta < 3/5 \) and \( C_\beta \) depending only on \( \beta \), on the domain, and the bounds for the coefficients and the data. For the choice \( \beta = 1/2 \), we obtain via the triangle inequality, the assumption on the set of admissible parameters, and by selecting only a tangential component of the gradient

\[
\|\partial_\tau A(g) - \partial_\tau \tilde{A}(\tilde{g})\|_{L^\infty(\gamma)} \leq C_\beta (\|f - \tilde{f}\|_{L^2(\Omega)} + \|j - \tilde{j}\|_{L^2(\partial\Omega)})^{1/2} + \tilde{a} \|g - \tilde{g}\|_{W^{1,\infty}(\gamma)}.
\]

Here \( \partial_\tau \) denotes the derivative along \( \gamma \). To proceed further, let us assume that

(A4) \( [g_1, g_2] \subset \{g(\gamma(s)) : s \in [0, 1]\} \) and \( |\partial_\gamma g| \geq c > 0 \).

Note that this is only a technical condition that can easily be satisfied by a proper experimental setup. By combining the previous estimates, we then obtain
To ensure the unique solvability and uniform a-priori estimates, we assume that tended to parabolic problems of the form

We will now demonstrate how the argument of the previous section can be extended to parabolic problems of the form

The constant $c$ in this theorem only depends on the geometry and the bounds for the coefficients and the data. If $f = f$ and $j = j$, we obtain Lipschitz continuity of the parameter $a$ with respect to perturbations in the measurements $g$; compare also with the stability result proven in [13].

4 Identification in the parabolic case

We will now demonstrate how the argument of the previous section can be extended to parabolic problems of the form

\begin{align*}
   u_t - \text{div}(a(u)\nabla u) &= f & \text{on } \Omega \times (0, T), \\
   a(u)\partial_n u &= j & \text{on } \partial\Omega \times (0, T).
\end{align*}

To ensure the unique solvability and uniform a-priori estimates, we assume that

(A5) $u(x, 0) = 0, j \in W^{1,\infty}(\partial\Omega \times (0, T))$ with $j(x, 0) = 0$, and $f \in L^{\infty}(\Omega \times (0, T))$.

Note that more general initial conditions could be incorporated easily and the regularity requirements for the $f$ and $j$ could be further relaxed. By standard solvability results for quasilinear parabolic problems [20], we obtain

**Theorem 4** Let the assumptions (A1)–(A2) and (A5) hold. Then (3)–(4) has a unique solution $u \in L^2(0, T; W^{2,p}(\Omega))$ for all $p < \infty$ that satisfies

$$
\|u\|_{L^\infty(0, T; W^{1,p}(\Omega))} + \|u\|_{L^2(0, T; W^{2,p}(\Omega))} + \|u_t\|_{L^2(0, T; L^p(\Omega))} 
\leq C_p(\|f\|_{L^2(0, T; L^{\infty}(\Omega))} + \|j\|_{W^{1,\infty}(\Omega \times (0, T))} + \|u\|_{L^\infty(\gamma \times (0, T))}).
$$

The constant $C_p$ in the estimate depends only on $p$, on the geometry, and on the bounds for the coefficients and the data used in the assumptions.

By embedding theorems, $u$ and $\nabla u$ are even Hölder continuous on $\overline{\Omega} \times [0, T]$.

Let us now turn to the parameter estimation problem: As before, we denote by $u$ and $\tilde{u}$ the solutions of (3)–(4) for parameters $a$ and $\tilde{a}$ with identical data $f$ and $j$. Proceeding like in the elliptic case, we obtain for every $0 < t \leq T$

$$
(u_t - \tilde{u}_t, \phi)_\Omega = -(\Delta A(u) - \Delta \tilde{A}(\tilde{u}), \phi)_\Omega = (\nabla A(u) - \nabla \tilde{A}(\tilde{u}), \nabla \phi)_\Omega.
$$

Choosing $\phi = A(u) - \tilde{A}(\tilde{u})$ and applying the Cauchy-Schwarz inequality in order to estimate the left hand side, we further get

$$
\|u_t - \tilde{u}_t\|_{L^2(\Omega)} \|A(u) - \tilde{A}(\tilde{u})\|_{L^2(\Omega)} \geq \|\nabla A(u) - \nabla \tilde{A}(\tilde{u})\|^2_{L^2(\Omega)}.
$$
We can define the principals $A$ and $\tilde{A}$ in such a way, maybe differently for every point in time, such that $\int_\Omega A(u) - \tilde{A}(\tilde{u})dx = 0$. By the Poincaré inequality, we then have $\| A(u) - \tilde{A}(\tilde{u}) \|_\Omega \leq C_P \| \nabla A(u) - \nabla \tilde{A}(\tilde{u}) \|_\Omega$. Using this in the previous estimates, we arrive at

$$\| \nabla A(u) - \nabla \tilde{A}(\tilde{u}) \|_\Omega \leq C_P^{-1} \| u_t - \tilde{u}_t \|_\Omega$$

for any $0 < t \leq T$.

Note that this inequality is local in time and independent of the choice of the principals $A$ and $\tilde{A}$. By the parabolic nature of the problem, the temperature distribution converges exponentially fast to that of the corresponding stationary problem, if we keep the data $j$ and $f$ constant over some time and assume that they satisfy the compatibility condition (A3). A slow variation of $f$ and $j$ over time also implies a slow variation of the temperature distribution. By a careful design of the experiment, we may therefore always assume that

(A6) $j$, $f$, and $T$ are chosen such that for all $0 < t \leq T$ we have $\| u_t \|_{L^2(\Omega)} \leq c_1 \epsilon^2$, $|\partial_t g| \geq 1/c_1$, and in addition $[g_1, g_2] \subset \{ u(\gamma(s), t) : s \in [0, 1], t \in [t_1, t_2] \}$ for some $0 < t_1 \leq t_2 \leq T$.

Using this condition, the uniform a-priori estimates for the solution, an interpolation argument, and moving to the boundary, we conclude

**Theorem 5** Let (A1) and (A5) hold and denote by $u$, $\tilde{u}$ the solutions of problem (3)–(4) with parameters $a$, $\tilde{a}$ satisfying (A2). Moreover, assume that the experiment is designed such that (A6) is valid with $c_1, c_2$ sufficiently small. Then from measurements $u(x, t) = \tilde{u}(x, t) = g(x, t)$ on $\gamma \times [t_1, t_2]$, we may conclude that $|a(s) - \tilde{a}(s)| \leq \epsilon$ for all $s \in [g_1, g_2]$.

Note that the constants $c_1, c_2$ do only depend on the domain and the bounds for the coefficients and the data. It should become clear from the derivation above that the statement of the theorem can be localized in time, i.e., we may identify $a(s)$ on $[g_1, g_2]$ from measurements $g = u(\gamma, t^*)$ at a single point in time on the corresponding range of temperatures. Similar as in the elliptic case, perturbations in the measurements and the data could be incorporated as well.

## 5 Discussion

In this note, we investigated the identification of nonlinear heat conduction laws in stationary and instationary heat transfer processes. We presented a new proof for Cannon’s uniqueness result for the quasilinear elliptic problem which allowed us to derive a corresponding stability result with respect to perturbations in the data. Using this stability result, the parabolic problem could then be treated as a perturbation of the elliptic case. We finally could obtain the approximate identification of the unknown parameter function with arbitrary accuracy by a
single experiment. Let us mention that, in principle, one could also apply a time-
independent temperature flux, wait until the system reaches equilibrium, and
then apply the results for the perturbed elliptic problem. By the parabolic nature,
the system will reach the stationary equilibrium exponentially fast. In contrast
to such an approach, the setting considered in Section 4 is truly instationary,
but close to the stationary equilibrium for all times.

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References

[1] R. A. Adams. *Sobolev Spaces*. Pure and Applied Mathematics. Academic
Press, New York, 1975.

[2] O. M. Alifanov. *Inverse Heat Transfer Problems*. Springer, New York, 1994.

[3] J. V. Beck, B. Blackwell, and C. R. StClair. *Inverse Heat Conduction: Ill-
Posed Problems*. Wiley Interscience, New York, 1985.

[4] J. R. Cannon. Determination of the unknown coefficient $k(u)$ in the equation
$\nabla \cdot k(u)\nabla u = 0$ from overspecified boundary data. *J. Math. Anal. Appl.*, 18:
112–114, 1967.

[5] J. R. Cannon. *The One-Dimensional Heat Equation*. Addison-Wesley, Cam-
bridge, 1984.

[6] J. R. Cannon and P. Duchateau. Determining unknown coefficients in a
nonlinear heat conduction problem. *SIAM J. Appl. Math.*, 24:298–314, 1973.

[7] J. R. Cannon and P. Duchateau. An inverse problem for a nonlinear diffusion
equation. *SIAM J. Appl. Math.*, 39(2):272–289, 1980.

[8] J. R. Cannon and P. Duchateau. Design of an experiment for the determina-
tion of an unknown coefficient in a nonlinear conduction-diffusion equation.
*International Journal of Engineering Science*, 25(8):1067 – 1078, 1987.

[9] G. Chavent and P. Lemonnier. Identification de la non-linéarité d’une
équation parabolique quasilineaire. *Appl. Math. Optim.*, 1(2):121–162,
1974/75.
[10] C. Cortázar and M. Elgueta. A monotonicity result related to a parabolic inverse problem. *Inverse Problems*, 6(4):515–521, 1990.

[11] P. Duchateau. Monotonicity and uniqueness results in identifying an unknown coefficient in a nonlinear diffusion equation. *SIAM J. Appl. Math.*, 41(2):310–323, 1981.

[12] P. Duchateau, R. Thelwell, and G. Butters. Analysis of an adjoint problem approach to the identification of an unknown diffusion coefficient. *Inverse Problems*, 20:601–625, 2004.

[13] H. Egger, J.-F. Pietschmann, and M. Schlottbom. Numerical identification of a nonlinear diffusion law via regularization in Hilbert scales. *Inverse Problems*, 30:025004, 2014.

[14] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*, volume 224 of *Grundlehren der mathematischen Wissenschaften*. Springer, Berlin, 2001.

[15] J. A. Griepentrog, K. Gröger, H. C. Kaiser, and J. Rehberg. Interpolation for function spaces related to mixed boundary value problems. *Math. Nachr.*, 241:110–120, 2002.

[16] B. Harrach and J. K. Seo. Exact shape-reconstruction by one-step linearization in electrical impedance tomography. *SIAM J. Math. Anal.*, 42:1505–1518, 2010.

[17] V. Isakov. On uniqueness in inverse problems for semilinear parabolic equations. *Archive for Rational Mechanics and Analysis*, 124:1–12, 1993.

[18] V. Isakov. *Inverse Problems for Partial Differential Equations*, volume 127 of *Applied Mathematical Sciences*. Springer Science+Business Media, 2006.

[19] M. V. Klibanov and A. Timonov. *Carleman estimates for coefficient inverse problems and numerical applications*. Inverse and Ill-posed Problems Series. VSP, Utrecht, 2004.

[20] O. A. Ladyzenskaja, V. A. Solonnikov, and N. N. Ural’ceva. *Linear and Quasi-Linear Equations of Parabolic Type*. American Mathematical Society, 1968.

[21] A. Lorenzi. An inverse problem for a quasilinear parabolic equation. *Ann. Mat. Pura Appl. (4)*, 142:145–169 (1986), 1985.

[22] M. Pilant and W. Rundell. A uniqueness theorem for determining conductivity from overspecified boundary data. *J. Math. Anal. Appl.*, 136:20–28, 1988.
[23] M. Pilant and W. Rundell. Multiple undetermined coefficient problems for quasi-linear parabolic equations. *Numer. Methods Partial Differential Equations*, 5(4):297–311, 1989.