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A new irreducible component of the moduli space of stable Godeaux surfaces

Received: 17 January 2015 / Accepted 8 July 2015
Published online: 28 July 2015

Abstract. We construct from a general del Pezzo surface of degree 1 a Gorenstein stable surface $X$ with $K_X^2 = 1$ and $p_g(X) = q(X) = 0$. These surfaces are not smoothable but give an open subset of an irreducible component of the moduli space of stable Godeaux surfaces. In a particular example we also compute the canonical ring explicitly and discuss the behaviour of pluricanonical maps.

1. Introduction

One of the most vexing problems in the classification of surfaces of general type is the fact that we are not yet able to classify surfaces with the smallest possible invariants $K_X^2 = 1$ and $p_g(X) = q(X) = 0$. Such surfaces are usually called (numerical) Godeaux surfaces [1].

Nowadays the moduli space of surfaces of general type comes with a modular compactification, the moduli space of stable surfaces [10, 12]. The stable surfaces occurring in the compactification can be used to gain insight in the classical moduli space but are also interesting in their own right.

This article grew out of two motivations: first of all we hoped to give a new construction of some Godeaux surfaces by smoothing stable Godeaux surfaces in the spirit of [13] but starting with a Gorenstein surface. Secondly, in our study of pluricanonical maps of stable surfaces [14] we were looking for examples of Gorenstein stable surfaces with $|5K_X|$ not very ample; so it was natural to study some examples with small invariants.

Both motivations lead us to the construction described in Sect. 2 where from any general smooth del Pezzo surface of degree 1 we construct a (Gorenstein) stable Godeaux surface, that is a Gorenstein stable surface $X$ with $K_X^2 = 1$ and $p_g(X) = q(X) = 0$. We also show that $X$ is topologically simply connected and compute its integral homology.

It turns out that our first hope was futile: the surfaces we construct are far away from any smooth surfaces.
Theorem A. Let $X$ be a stable Godeaux surface of the type constructed in Sect. 2. Then the Kuranishi space of $X$ is smooth of dimension 8 and all deformations are equisingular. Such surfaces form an irreducible open subset of the moduli space $\overline{\mathcal{M}}_{1,1}$ of stable surface with $K_X^2 = \chi(O_X) = 1$. This subset of the moduli space has at most finite quotient singularities.

The proof of Theorem A will be given in Sect. 5. The key point is the explicit control over infinitesimal deformations in Sect. 4, where we have to overcome the annoying fact that we are handling local normal crossing surfaces which do not have global normal crossings.

In Sect. 6 we will concentrate on one explicit example and compute, with the help of a computer algebra system, its canonical ring. The result is a rather complicated ring, Gorenstein in codimension ten, but it is generated in degree up to 5 and, in particular, $|5K_X|$ is very ample. In fact, one can show directly that the latter property is true for all surfaces in our family [7].

In principle, it would be possible to study degenerations of our examples via the del Pezzo surfaces, however it is unclear if in this way one would be able to connect to a known component of the moduli space of Godeaux surfaces.

Starting from the results in [6] one can actually classify Gorenstein stable Godeaux surfaces with worse than canonical singularities. These will be studied in a forthcoming joint paper with Marco Franciosi and Rita Pardini.

2. The surface

Construction 2.1. Let $\tilde{X}$ be a smooth del Pezzo surface of degree 1, that is, a smooth projective surface with $-K_{\tilde{X}}$ ample and $K_{\tilde{X}}^2 = 1$. Assume that there are two different curves $\tilde{D}_1, \tilde{D}_2 \in |- K_{\tilde{X}}|$ such that $\tilde{D}_i$ is isomorphic to a plane nodal cubic. We will see below that this condition is satisfied for a general choice of $\tilde{X}$. Let $\tilde{D} = \tilde{D}_1 \cup \tilde{D}_2$ and let $\tilde{\nu} : \tilde{D}^\nu \to \tilde{D}$ be the normalisation map.

We specify an involution $\tau$ on $\tilde{D}^\nu$, which exchanges the two components and preserves the preimages of the nodes, and construct a surface $X$ as the pushout in the diagram

$$
\begin{array}{ccc}
\tilde{X} & \leftarrow \tilde{D} & \leftarrow \tilde{D}^\nu \\
\downarrow \pi & & \downarrow \pi \\
X & \leftarrow D & \leftarrow D^\nu \\
\end{array}
$$

Since $\tilde{D}^\nu$ is the disjoint union of two copies of $\mathbb{P}^1$, each with three marked points that are preimages of nodes of $\tilde{D}$, the involution $\tau$ is uniquely determined by its action on these points. This and some further notation is given in Fig. 1.

Note that $\pi^* K_X = K_{\tilde{X}} + \tilde{D} = -K_{\tilde{X}}$, which is ample. By [11, Thm. 5.13] the surface $X$ exists as a projective scheme, has semi-log-canonical singularities and $K_X$ is ample. In fact, in our case it has the singularities of a normal crossing divisor: a degenerate cusp singularity at $P$, locally isomorphic to the origin of