MULTIVARIABLE GENERALIZATIONS OF THE SCHUR CLASS: POSITIVE KERNEL CHARACTERIZATION AND TRANSFER FUNCTION REALIZATION

JOSEPH A. BALL, ANIMIKH BISWAS, QUANLEI FANG, AND SANNE TER HORST

Abstract. The operator-valued Schur-class is defined to be the set of holomorphic functions $S$ mapping the unit disk into the space of contraction operators between two Hilbert spaces. There are a number of alternate characterizations: the operator of multiplication by $S$ defines a contraction operator between two Hardy Hilbert spaces, $S$ satisfies a von Neumann inequality, a certain operator-valued kernel associated with $S$ is positive-definite, and $S$ can be realized as the transfer function of a dissipative (or even conservative) discrete-time linear input/state/output linear system. Various multivariable generalizations of this class have appeared recently, one of the most encompassing being that of Muhly and Solel where the unit disk is replaced by the strict unit ball of the elements of a dual correspondence $E^*$ associated with a $W^*$-correspondence $E$ over a $W^*$-algebra $A$ together with a $*$-representation $\sigma$ of $A$. The main new point which we add here is the introduction of the notion of reproducing kernel Hilbert correspondence and identification of the Muhly-Solel Hardy spaces as reproducing kernel Hilbert correspondences associated with a completely positive analogue of the classical Szegö kernel. In this way we are able to make the analogy between the Muhly-Solel Schur class and the classical Schur class more complete. We also illustrate the theory by specializing it to some well-studied special cases; in some instances there result new kinds of realization theorems.

1. Introduction

The classical Schur class $S$ (consisting of holomorphic functions mapping the unit disk $\mathbb{D}$ into the closed unit disk $\overline{\mathbb{D}}$) along with its operator-valued generalization has been an object of intensive study over the past century (see [45] for the original paper of Schur and [26] for a survey of some of the impact and applications in signal processing). To formulate the definition of the operator-valued version, we let $\mathcal{L}(\mathcal{U},\mathcal{Y})$ denote the space of bounded linear operators acting between Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$. We also let $H^2_\mathcal{U}(\mathbb{D})$ and $H^2_\mathcal{Y}(\mathbb{D})$ be the standard Hardy spaces of $\mathcal{U}$-valued (respectively $\mathcal{Y}$-valued) holomorphic functions on the unit disk $\mathbb{D}$. By the Schur class $S(\mathcal{U},\mathcal{Y})$ we mean the set of $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued functions holomorphic on the unit disk $\mathbb{D}$ with values $S(z)$ having norm at most 1 for each $z \in \mathbb{D}$. The class $S(\mathcal{U},\mathcal{Y})$ admits several remarkable characterizations. The following result is well known and is formulated as the prototype for the multivariable generalizations to follow.

Theorem 1.1. Let $S$ be an $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued function defined on $\mathbb{D}$. The following are equivalent:

1. $S \in S(\mathcal{U},\mathcal{Y})$, i.e., $S$ is analytic on $\mathbb{D}$ with contractive values in $\mathcal{L}(\mathcal{U},\mathcal{Y})$.

1’ The multiplication operator $M_S$: $f(z) \mapsto S(z) \cdot f(z)$ defines a contraction from $H^2_\mathcal{U}(\mathbb{D})$ into $H^2_\mathcal{Y}(\mathbb{D})$.

1” $S$ is analytic and satisfies the von Neumann’s inequality: if $T$ is any strictly contractive operator on a Hilbert space $\mathcal{K}$, i.e., $\|T\| < 1$, then $S(T)$ is a contraction operator ($\|S(T)\| \leq 1$), where $S(T)$ is the operator defined by

$$S(T) = \sum_{n=0}^{\infty} S_n \otimes T^n \in \mathcal{L}(\mathcal{U} \otimes \mathcal{K}, \mathcal{Y} \otimes \mathcal{K}) \quad \text{if} \quad S(z) = \sum_{n=0}^{\infty} S_n z^n.$$

2. The function $K_S$: $\mathbb{D} \times \mathbb{D} \rightarrow \mathcal{L}(\mathcal{Y})$ given by

$$K_S(z,w) = \frac{I_\mathcal{Y} - S(z)S(w)^*}{1 - zw}$$

is a positive kernel on $\mathbb{D} \times \mathbb{D}$.

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(3) There exists a Hilbert space $\mathcal{H}$ and a coisometric (or even unitary or contractive) connecting operator (or colligation) $U$ of the form

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ U \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H} \\ Y \end{bmatrix}$$

so that $S(z)$ can be realized in the form

$$S(z) = D + zC(I - zA)^{-1}B.$$  \hfill (1.1)

From the point of view of systems theory, the function (1.1) is the transfer function of the linear system

$$\Sigma = \Sigma(U) : \{ x(n+1) = Ax(n) + Bu(n) \\ y(n) = Cx(n) + Du(n) \}.$$  

The following well-known Proposition gives several equivalent definitions for the term “positive kernel” used in condition (2) in Theorem 1.1. The scalar case ($Y = \mathbb{C}$) of this result goes back to the paper of Aronszajn [7], but is also often attributed to E.H. Moore and Kolmogorov, while the vector-valued case has been well exploited in the function-theoretic operator theory literature over the years (see [17, 19]).

**Proposition 1.2.** Let $K : \Omega \times \Omega \rightarrow \mathcal{L}(Y)$ be a given function. Then the following conditions are equivalent:

1. For any finite collection of points $\omega_1, \ldots, \omega_N \in \Omega$ and of vectors $y_1, \ldots, y_N \in \mathcal{Y}$ ($N = 1, 2, \ldots$) it holds that

$$\sum_{i,j=1}^{N} \langle K(\omega_i, \omega_j)y_j, y_i \rangle \mathcal{Y} \geq 0.$$  \hfill (1.2)

2. There exists an operator-valued function $H : \Omega \rightarrow \mathcal{L}(\mathcal{H}, \mathcal{Y})$ for some auxiliary Hilbert space $\mathcal{H}$ so that

$$K(\omega', \omega) = H(\omega')H(\omega)^*.$$  \hfill (1.3)

3. There exists a Hilbert space $\mathcal{H}(K)$ of $\mathcal{Y}$-valued functions $f$ on $\Omega$ so that the function $K(\cdot, \omega)y$ is in $\mathcal{H}(K)$ for each $\omega \in \Omega$ and $y \in \mathcal{Y}$ and has the reproducing property

$$(f, K(\cdot, \omega)y)_{\mathcal{H}(K)} = (f(\omega), y)_{\mathcal{Y}}.$$  

When any (and hence all) of these equivalent conditions hold, we say that $K$ is a positive kernel on $\Omega \times \Omega$.

We provide a sketch of the proof of Theorem 1.1 as a model for how extensions to more general settings may proceed.

**Sketch of the proof of Theorem 1.1.** The easy part is (3) $\implies$ (2) $\implies$ (1$''$) $\implies$ (1$'$) $\implies$ (1):

(3) $\implies$ (2): Assume that $S(z)$ is as in (1.1) with $U$ unitary, and hence, in particular, coisometric. From the relations arising from the coisometric property of $U$:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

one can verify that

$$I - S(z)S(w)^* = I - [D + zC(I - zA)^{-1}B][D + wC(I - wA)^{-1}B]^*$$

$$= C(I - zA)^{-1}[(1 - z\omega)I_{\mathcal{H}}][I - \omega A^*]^{-1}C^*.$$  

This implies that $H(z) = C(I - zA)^{-1}$ satisfies (1.3).

(2) $\implies$ (1$'$): Due to $I - S(z)S(w)^* = H(z)(I - z\omega)I_{\mathcal{H}}H(\omega)^*$, we can see that for any $\|T\| < 1$

$$I - S(T)S(T)^* = H(T)(I - TT^*) \otimes I_{\mathcal{H}}H(T)^* \geq 0.$$  

(1$''$) $\implies$ (1$'$): Observe that $M_S = S(S) = s - \lim_{r \uparrow 1} S(rS)$ where $S$ is the shift operator $M_z$ on $H^2(\mathbb{D})$. Thus the fact that $\|S(rS)\|_\infty \leq 1$ for any $r \in (0, 1]$ implies $\|M_S\|_\infty \leq 1$.

(1$'$) $\implies$ (1): Note that since $S(z)u = M_S \cdot u$ for any $u \in \mathcal{U}$, we have $\|M_S\|_{op} = \|S\|_\infty$. So $\|M_S\| \leq 1$ implies that $S \in S(\mathcal{U}, \mathcal{Y})$.

The harder part is (1) $\implies$ (1$'$) $\implies$ (1$''$) $\implies$ (2) $\implies$ (3):

(1) $\implies$ (1$'$): We can view $H^2(\mathbb{D}) \subset L^2(\mathcal{T})$. Thus $\|M_Su\|_{L^2(\mathcal{T})} \leq \|S\|_\infty \cdot \|u\|_{L^2(\mathcal{T})}$.

(1$'$) $\implies$ (1$''$): According to the Sz.-Nagy dilation theorem, any contraction operator $T$ has a unitary dilation $U$. In the strictly contractive case $\|T\| < 1$, one can show that in fact the unitary dilation is the
bilateral shift with some multiplicity \( N \): \( U = S \otimes I_N \) (if \( N = \infty \), we interpret \( I_N \) as the identity operator on \( l^2 \)). We then have \( T^n = P_{S^*}(S \otimes I_N)^n \). Therefore \( \| S(T) \| = \| P_{S^*}(S \otimes I_N) \| \leq \| M_S \| \leq 1 \).

\((1') \implies (2)\): A direct proof of this implication can be done via a rather long, intricate argument using a Gelfand-Naimark-Segal construction in conjunction with a Hahn-Banach separation argument—we refer to this as a GNS/HB argument. For the polydisk setting, the argument originates in \([1]\); the version for a general semigroupoid setting in \([23]\) covers in particular the classical setting here.

Alternatively, one can avoid the GNS/HB argument via the following shortcut:

\((1') \implies (1) \implies (2)\): We have seen that \((1') \implies (1)\) is easy. For \((1) \implies (2)\), we assume \( \| M_S \| \leq 1 \). View \( H^2(\mathbb{D}) \) as the reproducing kernel Hilbert space \( H(k_{S_{\mathbb{D}}}) \), where \( k_{S_{\mathbb{D}}}(z, w) = \frac{1}{1 - \overline{z}w} \) is the Szegő kernel. Since \( M_\mathbb{D} k_{S_{\mathbb{D}}} (\cdot, w) y = k_{S_{\mathbb{D}}} (\cdot, w) S(w)^* y \), we see that

\[
\sum_{i,j=1,\ldots,N} \| (k_{S}(z_j, z_j)y_j) \|_Y = \| \sum_j k_{S_{\mathbb{D}}}(\cdot, z_j)y_j \|^2 - \| (M_S)^* \sum_j k_{S_{\mathbb{D}}}(\cdot, z_j)y_j \|^2 \geq 0
\]

and it follows (via criterion \((1)\)) that \( K_S \) is a positive kernel on \( \mathbb{D} \times \mathbb{D} \).

\((2) \implies (3)\): This implication can be done by the now standard lurking isometry argument—see \([8]\) where this coinage was introduced.

The purpose of this paper is to study recent extensions of the Schur class and the associated analogues of Theorem \(1.1\) to more general multivariable settings. In Section \(2\) we describe two such extensions: the Drury-Arveson space setting and the free-semigroup setting. We emphasize how all the ingredients of the proof of Theorem \(1.1\) sketched above have direct analogues in these two settings; hence the proof of the analogues of Theorem \(1.1\) for these two settings (see Theorem \(2.1\) and Theorem \(2.3\) below) directly parallel the proof of Theorem \(1.1\) as sketched above. A far more sophisticated generalized Schur class has been introduced by Muhly and Solel (see \([33, 36]\)). The main contribution of the present paper is to introduce the notion of reproducing kernel Hilbert correspondence and an analogue of the Fourier (or Z-) transform for the Muhly-Solel setting. The starting point for most of the constructions is a \( W^* \)-correspondence \( E \) over a \( W^* \)-algebra \( A \) together with a \( * \)-representation \( \sigma \) of \( A \). We show that the image, denoted in our notation as \( H^2(E, \sigma) \) which is an analogue of \( H^2 \), of a Muhly-Solel Fock space, denoted as \( F^2(E, \sigma) \) in our notation which is an analogue of \( \ell^2(\mathbb{Z}_+) \), under this \( Z \)-transform is a space of \( E \)-valued functions (\( E \) equal to a coefficient Hilbert space) on the Muhly-Solel generalized unit disk \( D((E^*)^+) \) and that an element \( S \) of the Muhly-Solel Schur class as introduced in \([36]\) induces a bounded multiplication operator on \( H^2(E, \sigma) \). We also obtain analogues of the other parts of Theorem \(1.1\) for this setting (see Theorem \(5.1\) in Section \(5\) below) and thus obtain a more complete analogy between the Muhly-Solel Schur class and the classical Schur class than that presented in \([36]\). Section \(3\) develops required preliminaries concerning general correspondences, including the notions of reproducing kernel correspondence and of reproducing kernel Hilbert correspondence; these are natural elaborations of the Kolmogorov decomposition for a completely positive kernel found in \([18]\). Section \(4\) introduces the spaces \( H^2(E, \sigma) \) and \( H^\infty(E, \sigma) \) which are the analogues of the Hardy spaces \( H^2 \) and \( H^\infty \) for this setting. The final section \(4\) applies the general theory to some familiar more concrete special cases. Specifically we make explicit how the classical case discussed above as well as the Drury-Arveson setting and the free-semigroup algebra setting discussed in Section \(2\) are particular cases of the Muhly-Solel setting. The general theory here leads to more structured versions of these well-studied settings and corresponding new types of realization theorems. We also discuss one of the main examples motivating the work in \([31, 33, 36]\), namely the setting of analytic crossed-product algebras. It is interesting to note that the realization theorem for a particular instance of this example amounts to the realization theorem for input-output maps of conservative time-varying linear systems obtained in \([4]\).

Another class of examples covered by the Muhly-Solel setting are graph algebras (also known as semigroupoid algebras \([30, 32, 27]\); we do not discuss these here. There are still other types of generalized Schur classes which are not subsumed under the Muhly-Solel Fock space/correspondence setup. We mention the Schur-Agler class for the polydisk (see \([1, 2, 13]\) and for more general domains \([5, 9]\)), the noncommutative Schur-Agler class (see \([12, 13]\), and higher-rank graph algebras (see \([28]\)). A differentiating feature of these variants of the Schur class is a more implicit version of condition \(2\) in Theorem \(1.1\) where the single positive

\[^3\]In nice cases, the general situation collapses to this statement; more correctly, the vector-valued functions are defined on \( D((E^*)^+) \times \sigma(A)' \) where \( \sigma(A)' \) is the commutant of the image \( \sigma(A) \) of \( \sigma \) in \( L(E) \).
kernel (the Szegö kernel $\frac{1}{1-z\bar{w}}$) is replaced by a whole family of positive kernels. An abstract framework using this feature as the point of departure is the semigroupoid approach of Dritschel-Marcantognini-McCullough [23] which incorporates all the aforementioned settings in [1, 2, 14, 12, 28]. However the theory in [23] does not appear to include the analytic crossed-product algebras included in the Muhly-Solel scheme since it does not allow for the action of a $W^*$-algebra $\mathcal{A}$ acting on the ambient Hilbert space. It is conceivable that some sort of synthesis of these two disparate approaches is possible; the recent work on product decompositions over general semigroups (see [46]) appears to be a start in this direction.

The notation is mostly standard but we mention here a few conventions for reference. For $\Omega$ any index set, $\ell^2(\Omega)$ denotes the space of complex-valued functions on $\Omega$ which are absolutely square summable:

$$\ell^2(\Omega) = \{\xi: \Omega \to \mathbb{C}: \sum_{\omega \in \Omega} |\xi(\omega)|^2 < \infty\}.$$  

Most often the choice $\Omega = \mathbb{Z}$ (the integers) or $\Omega = \mathbb{Z}_+$ (the nonnegative integers) appears. For $\mathcal{H}$ a Hilbert space, we use $\ell^2(\mathcal{H})$ as shorthand for $\ell^2(\Omega) \otimes \mathcal{H}$ (the space of $\mathcal{H}$-valued function on $\Omega$ square-summable in norm). More general versions where $\mathcal{H}$ may be a correspondence also come up from time to time.

2. Some multivariable Schur classes

In this section we introduce two multivariable settings (the Drury-Arveson space setting and the free semigroup algebra setting) for the Schur class and formulate the analogue of Theorem [13] for these two settings.

2.1. Drury-Arveson space. A multivariable generalization of the Szegö kernel $k(z, w) = (1 - z\bar{w})^{-1}$ much studied of late is the positive kernel

$$k_d(z, w) = \frac{1}{1 - \langle z, w \rangle} \text{ on } \mathbb{B}^d \times \mathbb{B}^d,$$

where $\mathbb{B}^d = \{z = (z_1, \ldots, z_d) \in \mathbb{C}^d : \langle z, z \rangle < 1\}$ is the unit ball of the $d$-dimensional Euclidean space $\mathbb{C}^d$. By $\langle z, w \rangle = \sum_{j=1}^d z_j \bar{w}_j$ we mean the standard inner product in $\mathbb{C}^d$. The reproducing kernel Hilbert space (RKHS) $\mathcal{H}(k_d)$ associated with $k_d$ via Aronszajn’s construction [7] is a natural multivariable analogue of the Hardy space $H^2$ of the unit disk and coincides with $H^2$ if $d = 1$.

For $\mathcal{Y}$ an auxiliary Hilbert space, we consider the tensor product Hilbert space $\mathcal{H}_\mathcal{Y}(k_d) := \mathcal{H}(k_d) \otimes \mathcal{Y}$ whose elements can be viewed as $\mathcal{Y}$-valued functions in $\mathcal{H}(k_d)$. Then $\mathcal{H}_\mathcal{Y}(k_d)$ can be characterized as follows:

$$\mathcal{H}_\mathcal{Y}(k_d) = \left\{ f(z) = \sum_{n \in \mathbb{Z}_+^d} f_n z^n : \|f\|^2 = \sum_{n \in \mathbb{Z}_+^d} \frac{|n|!}{|n|!} : \|f_n\|^2 < \infty \right\}.$$  

Here and in what follows, we use standard multivariable notations: for multi-integers $n = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d$ and points $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$ we set

$$|n| = n_1 + n_2 + \ldots + n_d, \quad |n|! = n_1! n_2! \ldots n_d!, \quad z^n = z_1^{n_1} z_2^{n_2} \ldots z_d^{n_d}.$$  

By $\mathcal{M}_d(\mathcal{U}, \mathcal{Y})$ we denote the space of all $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued analytic functions $S$ on $\mathbb{B}^d$ such that the induced multiplication operator

$$M_S : f(z) \mapsto S(z) \cdot f(z)$$

maps $\mathcal{H}_\mathcal{Y}(k_d)$ into $\mathcal{H}_\mathcal{Y}(k_d)$. It follows by the closed graph theorem that for every $S \in \mathcal{M}_d(\mathcal{U}, \mathcal{Y})$, the operator $M_S$ is bounded. We shall pay particular attention to the unit ball of $\mathcal{M}_d(\mathcal{U}, \mathcal{Y})$, denoted by

$$\mathcal{S}_d(\mathcal{U}, \mathcal{Y}) = \{ S \in \mathcal{M}_d(\mathcal{U}, \mathcal{Y}) : \|M_S\|_{\text{op}} \leq 1 \}.$$  

We refer to $\mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ as a generalized (d-variable) Schur class since $\mathcal{S}_1(\mathcal{U}, \mathcal{Y})$ collapses to the classical Schur class. Characterizations of $\mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ in terms of realizations originate in [3, 15, 23]. The following is the analogue of Theorem [13] for this setting; the result with condition (1") eliminated appeared e.g. in [15, 11].

**Theorem 2.1.** Let $S$ be an $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function defined on $\mathbb{B}^d$. The following are equivalent:

1. $S$ belongs to $\mathcal{S}_d(\mathcal{U}, \mathcal{Y})$, i.e., the multiplication operator $M_S : f(z) \mapsto S(z) f(z)$ defines a contraction from $\mathcal{H}_\mathcal{Y}(k_d)$ into $\mathcal{H}_\mathcal{Y}(k_d)$. 


(1′′) \( \| S(T) \| \leq 1 \) for any commutative row contraction \( T = (T_1, \ldots, T_d) \in \mathcal{L}(K)^d \), i.e., if \( S \) is given by 
\[
S(z) = \sum_{n \in \mathbb{Z}^d_+} S_n z^n \quad \text{and if (1), (2)} \text{ is any commuting } d\text{-tuple of bounded linear operators on a Hilbert space } K
\] such that the row matrix \([T_1 \cdots T_d]\) defines a strict contraction operator from \( K^d \) to \( K \), then the operator \( S(T) \in \mathcal{L}(U \otimes K, Y \otimes K) \) defined via the operator-norm limit of the series \( S(T) := \sum_n S_n \otimes T^n \) has \( \| S(T) \| \leq 1 \).

(2) The function \( K_S : B \times B \to \mathcal{L}(Y) \) given by 
\[
K_S(z, w) = \frac{I_Y - S(z)S(w)^*}{1 - \langle z, w \rangle}
\]
is a positive kernel (see Proposition 1.2).

(3) There exists a Hilbert space \( H \) and a unitary (or even coisometric or contractive) connecting operator (or colligation) \( U \) of the form 
\[
U = \begin{bmatrix} A & B \\
C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\
& \vdots & \ddots & \ddots \\
& & A_d & B_d \\
C & D \end{bmatrix} : \begin{bmatrix} H \\
U \end{bmatrix} \to \begin{bmatrix} H^d \\
Y \end{bmatrix}
\]
so that \( S(z) \) can be realized in the form 
\[
S(z) = D + C (I - z_I H - z_2 A_1 - \cdots - z_d A_d)^{-1} (z_1 B_1 + \cdots + z_d B_d)
\]
\[
= D + C (I - Z(z)A)^{-1} Z(z)B
\]
where we set
\[
Z(z) = [z_1 I_H \ldots z_d I_H], \quad A = \begin{bmatrix} A_1 \\
& \vdots \\
& & A_d \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\
& \vdots \\
& & B_d \end{bmatrix}.
\]

Remarks on the proof: \( (3) \implies (2) \implies (1′′) \implies (1′) \) follows in the same way as in the sketch of the proof of Theorem 1.1 above. For \( (1′) \implies (2) \), one can use the same reproducing kernel argument as the shortcut discussed in the proof Theorem 1.1 above. For \( (1′) \implies (1′′) \), one can follow the corresponding argument sketched above for Theorem 1.1 but with the Sz.-Nagy dilation theorem replaced with the Drury-Dilworth dilation theorem (see 24). The implication \( (2) \implies (3) \) follows exactly as in the classical case via the lurking isometry argument (see 15). Note that \( (1′′) \implies (2) \) also can be achieved directly by the GNS/HB argument in 23 specialized to the setting here, but this is not usually done since one has the alternative easier route \( (1′′) \implies (1′) \implies (2) \).

## 2.2. Free semigroup algebras

We now discuss the generalization of the Schur class associated with free semigroup algebras and models for row contractions (see 39 40 41 42 17). We follow the formalism and notation as used in 17 16.

Let \( z = (z_1, \ldots, z_d) \) and \( w = (w_1, \ldots, w_d) \) be two sets of noncommuting indeterminates. We let \( F_d \) denote the free semigroup generated by the \( d \) letters \( \{1, \ldots, d\} \). A generic element of \( F_d \) is a word \( w \) equal to a string of letters 
\[
\alpha = i_N \cdots i_1 \quad \text{where} \quad i_k \in \{1, \ldots, d\} \quad \text{for} \quad k = 1, \ldots, N.
\]

The product of two words is defined by the usual concatenation. The unit element of \( F_d \) is the empty word denoted by \( \emptyset \). For \( \alpha \) a word of the form \( (2.2) \), we let \( z^\alpha \) denote the monomial in noncommuting indeterminates \( z^\alpha = z_{i_N} \cdots z_{i_1} \) and we let \( z^\emptyset = 1 \). We extend this noncommutative functional calculus to a \( d \)-tuple of operators \( A = (A_1, \ldots, A_d) \) on a Hilbert space \( K \):

\[
A^v = A_{i_N} \cdots A_{i_1} \quad \text{if} \quad v = i_N \cdots i_1 \in F_d \setminus \{\emptyset\}; \quad A^\emptyset = I_K.
\]

We will also have need of the transpose operation on \( F_d \):

\[
\alpha^T = i_1 \cdots i_N \quad \text{if} \quad \alpha = i_N \cdots i_1.
\]
Given a coefficient Hilbert space $\mathcal{Y}$ we let $\mathcal{Y}(z)$ denote the set of all polynomials in $z = (z_1, \ldots, z_d)$ with coefficients in $\mathcal{Y}$:

$$\mathcal{Y}(z) = \left\{ p(z) = \sum_{\alpha \in \mathcal{F}_d} p_{\alpha} z^\alpha : p_{\alpha} \in \mathcal{Y} \text{ and } p_{\alpha} = 0 \text{ for all but finitely many } \alpha \right\},$$

while $\mathcal{Y}(\langle z \rangle)$ denotes the set of all formal power series in the indeterminates $z$ with coefficients in $\mathcal{Y}$:

$$\mathcal{Y}(\langle z \rangle) = \left\{ f(z) = \sum_{\alpha \in \mathcal{F}_d} f_{\alpha} z^\alpha : f_{\alpha} \in \mathcal{Y} \right\}.$$

Note that vectors in $\mathcal{Y}$ can be considered as Hilbert space operators between $\mathbb{C}$ and $\mathcal{Y}$. More generally, if $\mathcal{U}$ and $\mathcal{Y}$ are two Hilbert spaces, we let $\mathcal{L}(\mathcal{U}, \mathcal{Y})(z)$ and $\mathcal{L}(\mathcal{U}, \mathcal{Y})(\langle z \rangle)$ denote the space of polynomials (respectively, formal power series) in the noncommuting indeterminates $z = (z_1, \ldots, z_d)$ with coefficients in $\mathcal{L}(\mathcal{U}, \mathcal{Y})$. Given $S = \sum_{\alpha \in \mathcal{F}_d} s_{\alpha} z^\alpha \in \mathcal{L}(\mathcal{U}, \mathcal{Y})(\langle z \rangle)$ and $f = \sum_{\beta \in \mathcal{F}_d} f_{\beta} z^\beta \in \mathcal{U}(\langle z \rangle)$, the product $S(z) \cdot f(z) \in \mathcal{Y}(\langle z \rangle)$ is defined as an element of $\mathcal{Y}(\langle z \rangle)$ via the noncommutative convolution:

$$S(z) \cdot f(z) = \sum_{\alpha, \beta \in \mathcal{F}_d} s_{\alpha} f_{\beta} z^{\alpha + \beta} = \sum_{v \in \mathcal{F}_d} \left( \sum_{\alpha, \beta \in \mathcal{F}_d : \alpha \cdot \beta = v} s_{\alpha} f_{\beta} \right) z^v. \quad (2.3)$$

Note that the coefficient of $z^v$ in (2.3) is well defined since any given word $v \in \mathcal{F}_d$ can be decomposed as a product $v = \alpha \cdot \beta$ in only finitely many distinct ways.

In general, given a coefficient Hilbert space $\mathcal{C}$, we use the $\mathcal{C}$ inner product to generate a pairing

$$\langle \cdot, \cdot \rangle_{\mathcal{C} \times \mathcal{C}(\langle w \rangle)} : \mathcal{C} \times \mathcal{C}(\langle w \rangle) \rightarrow \mathbb{C}(\langle w \rangle)$$

via

$$\left\langle c, \sum_{\beta \in \mathcal{F}_d} f_{\beta} w^\beta \right\rangle_{\mathcal{C} \times \mathcal{C}(\langle w \rangle)} = \sum_{\beta \in \mathcal{F}_d} \left\langle c, f_{\beta} \right\rangle_{\mathcal{C}} w^\beta \in \mathbb{C}(\langle w \rangle).$$

Similarly we can consider $\langle \sum_{\alpha \in \mathcal{F}_d} f_{\alpha} w^\alpha, c \rangle_{\mathcal{C}(\langle w \rangle) \times \mathcal{C}}$ or the more general pairing

$$\left\langle \sum_{\alpha \in \mathcal{F}_d} f_{\alpha} w^\alpha, \sum_{\beta \in \mathcal{F}_d} g_{\beta} w^\beta \right\rangle_{\mathcal{C}(\langle w \rangle) \times \mathcal{C}(\langle w \rangle)} = \sum_{\alpha, \beta \in \mathcal{F}_d} \left\langle f_{\alpha}, g_{\beta} \right\rangle_{\mathcal{C}} w^\beta w^\alpha.$$

Suppose that $\mathcal{H}$ is a Hilbert space whose elements are formal power series in $\mathcal{Y}(\langle z \rangle)$ and that $K(z, w) = \sum_{\alpha, \beta \in \mathcal{F}_d} K_{\alpha, \beta} z^\alpha w^\beta$ is a formal power series in the two sets of $d$ noncommuting indeterminates $z = (z_1, \ldots, z_d)$ and $w = (w_1, \ldots, w_d)$. We say that $\mathcal{H}$ is a NFRKHS (noncommutative formal reproducing kernel Hilbert space) if for each $\alpha \in \mathcal{F}_d$, the linear operator $\Phi_{\alpha} : \mathcal{H} \rightarrow \mathcal{Y}$ defined by $f(z) = \sum_{\beta \in \mathcal{F}_d} f_{\beta} z^\beta \mapsto f_{\alpha}$ is continuous. In this case there must be a formal power series $k_{\alpha}(z) \in \mathcal{L}(\mathcal{Y})(\langle z \rangle)$ so that $k_{\alpha}(\cdot y) \in \mathcal{H}$ for each $\alpha \in \mathcal{F}_d$ and $y \in \mathcal{Y}$ and

$$\langle f, k_{\alpha} y \rangle_{\mathcal{H}} = \langle f_{\alpha}, y \rangle_{\mathcal{Y}}.$$

If we set $K(z, w) = \sum_{\beta \in \mathcal{F}_d} k_{\beta}(z) w^{\beta\top}$, then we have the reproducing property

$$\langle f, K(\cdot, w) y \rangle_{\mathcal{H} \times \mathcal{H}(\langle w \rangle)} = \langle f(w), y \rangle_{\mathcal{Y}(\langle w \rangle) \times \mathcal{Y}}.$$

In this case we say that $K(z, w)$ is the reproducing kernel for the NFRKHS $\mathcal{H}$. As explained in detail in [16], we have the following equivalent characterizations for such kernels which parallel the statements of Proposition 1.2 for the classical case.

**Proposition 2.2.** Let $K(z, w) \in \mathcal{L}(\mathcal{Y})(\langle z, w \rangle)$ be a formal power series in two sets of noncommuting indeterminates with coefficients $K_{\alpha, \beta}$ equal to bounded operators on the Hilbert space $\mathcal{Y}$. Then the following conditions are equivalent:

1. For all finitely supported $\mathcal{Y}$-valued functions $\alpha \mapsto y_{\alpha}$ it holds that
$$\sum_{\alpha, \alpha' \in \mathcal{F}_d} \langle K_{\alpha, \alpha'} y_{\alpha'}, y_{\alpha} \rangle \geq 0,$$
Theorem 2.3. The following is the noncommutative analogue of Theorem 1.1 for this setting. We refer to [39, 40] for details. Thus we have in general (2.3) defines a bounded operator from $H^2_\mathcal{Y}(\mathcal{F}_d)$ to $H^2_\mathcal{Y}(\mathcal{F}_d)$. The noncommutative Schur class $S_{nc,d}(\mathcal{U},\mathcal{Y})$ is defined to consist of such multipliers $S$ for which $M_S$ has operator norm at most 1:

$$S_{nc,d}(\mathcal{U},\mathcal{Y}) = \{ S \in \mathcal{L}(\mathcal{U},\mathcal{Y})(\langle z \rangle) : M_S : H^2_\mathcal{Y}(\mathcal{F}_d) \to H^2_\mathcal{Y}(\mathcal{F}_d) \text{ with } \| M_S \|_{op} \leq 1 \}.$$ 

The following is the noncommutative analogue of Theorem 1.1 for this setting. We refer to [39, 40] for details.

**Theorem 2.3.** Let $S(z) \in \mathcal{L}(\mathcal{U},\mathcal{Y})(\langle z \rangle)$ be a formal power series in $z = (z_1, \ldots, z_d)$ with coefficients in $\mathcal{L}(\mathcal{U},\mathcal{Y})$. Then the following are equivalent:

1. $S \in S_{nc,d}(\mathcal{U},\mathcal{Y})$, i.e., $M_S : \mathcal{U}(z) \to \mathcal{Y}(z)$ given by $M_S : p(z) \to S(z)p(z)$ extends to define a contraction operator from $H^2_\mathcal{Y}(\mathcal{F}_d)$ into $H^2_\mathcal{Y}(\mathcal{F}_d)$.
2. For each strict row contraction $(T_1, \ldots, T_d)$, i.e., a $d$-tuple $(T_1, \ldots, T_d)$ of operators on a Hilbert space $\mathcal{K}$ (commutative or not) such that the row matrix $[T_1 \cdots T_d]$ defines a strict contraction operator from $\mathcal{K}^d$ to $\mathcal{K}$, we have

$$\| S(T) \| \leq 1,$$

where

$$S(T) = \sum_{\alpha \in \mathcal{F}_d} s_\alpha \otimes T^\alpha \in \mathcal{L}(\mathcal{U} \otimes \mathcal{K}), \mathcal{Y} \otimes \mathcal{K}) \quad \text{if} \quad S(z) = \sum_{\alpha \in \mathcal{F}_d} s_\alpha z^\alpha$$

and where we set

$$T^\alpha = T_{i_N} \cdots T_{i_1} \quad \text{if} \quad \alpha = i_N \cdots i_1 \in \mathcal{F}_d.$$
(2) The formal power series given by

\[ K_S(z, w) := k_{Sz, nc}(z, w) - S(z)k_{Sz, nc}(z, w)S(w) \]

is a noncommutative positive kernel (see Proposition 2.2).

(3) There exists a Hilbert space \( \mathcal{H} \) and a unitary connection operator \( U \) of the form

\[
U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{bmatrix} : \begin{bmatrix} X \\ U \end{bmatrix} \rightarrow \begin{bmatrix} X \\ Y \end{bmatrix}
\]

so that \( S(z) \) can be realized as a formal power series in the form

\[
S(z) = D + \sum_{j=1}^{d} \sum_{v \in \mathcal{F}_d} C^* A_j z^j \cdot z_j = D + C(I - Z(z)A)^{-1} Z(z)B
\]

where \( Z(z) \), \( A \) and \( B \) are as in (2.1) but where now \( z_1, \ldots, z_d \) are noncommuting indeterminates rather than commuting variables.

**Sketch of the proof of Theorem 2.3.** The proof of (3) \( \implies \) (2) \( \implies \) (1′′) \( \implies \) (1) formally goes through in the same was as the classical case. Let us just note that (1′′) \( \implies \) (1') involves viewing \( M_S : H^2_d(\mathcal{F}_d) \rightarrow H^2_d(k_d) \) as \( M_S = S(S) \) where \( S = (S_1, \ldots, S_d) \) are the left creation operators of multiplication by \( z_j \) on the left on the Fock space \( H^2_d(\mathcal{F}_d) \). From the assumption (1''), we know that \( \|S(rS)\| \leq 1 \) for each \( r < 1 \) and hence \( \|M_S\| = \lim_{r \rightarrow 1} \|S(rS)\| \leq 1 \) as well.

We discuss the harder direction (1') \( \implies \) (1'') \( \implies \) (2) \( \implies \) (3).

(1'') \( \implies \) (2): This implication again can be done via an appropriate version of the GNS/HB argument; see [12] for a slightly more general version and [23] for an even more general version.

Alternatively, one can follow the route (1'') \( \implies \) (1') \( \implies \) (2): As we have already discussed (1'') \( \implies \) (1'), it suffices to discuss (1') \( \implies \) (2). This can be done by an adaptation of the argument for the classical case to the present setting of formal, noncommutative reproducing kernel Hilbert spaces—see [16] Theorem 3.15.

(2) \( \implies \) (3): The lurking isometry argument works in this context as well—see [16] Theorem 3.16.

3. Reproducing kernel \((\mathcal{A}, \mathcal{B})\)-correspondences

The notion of a vector-valued reproducing kernel Hilbert space based on an operator-valued positive kernel has been a standard tool in operator theory as well as in other applications for some time now. Recently, Barreto, Bhat, Liescher and Skeide [18] introduced a finer notion of positive kernel (completely positive kernel) and gave several equivalent characterizations, but did not develop the connections with reproducing kernel Hilbert spaces. The purpose of this section is to fill in this gap, as it is the natural tool for the discussion to follow.

Let \( \mathcal{B} \) be a \( C^* \)-algebra and \( E \) a linear space. For some of the discussion to follow, it will be convenient to assume that \( \mathcal{B} \) has a unit. However, any \( C^* \)-algebra has an approximate identity (see [23] Theorem 1.4.8); by making use of such an approximate identity, most arguments using a unit element \( 1_\mathcal{B} \) can be adapted to an approximation argument yielding the desired result for the general case where \( \mathcal{B} \) is not assumed to possess a unit. In the sequel we usually leave the details of this adaptation to the reader.

We say that \( E \) is a (right) pre-Hilbert \( C^* \)-module over \( \mathcal{B} \) if \( E \) is a right module over \( \mathcal{B} \) and is endowed with a \( \mathcal{B} \)-valued inner product \( \langle \cdot , \cdot \rangle_E \) satisfying the following axioms for any \( \lambda , \mu \in \mathbb{C} , e, f, g \in E \) and \( b \in \mathcal{B} \):

1. \( \langle \lambda e + \mu f, g \rangle_E = \lambda \langle e, g \rangle_E + \mu \langle f, g \rangle_E \);
2. \( \langle e \cdot b, f \rangle_E = \langle e, f \rangle_E b \);
3. \( \langle e, f \rangle_E^* = \langle f, e \rangle_E \);
4. \( \langle e, e \rangle_E \geq 0 \) (as an element of \( \mathcal{B} \));
5. \( \langle e, e \rangle_E = 0 \) implies that \( e = 0 \)
We also impose that \((\lambda e) \cdot b = e \cdot (\lambda b)\) for all \(e \in E, b \in B\) and \(\lambda \in \mathbb{C}\). Note that if \(B\) has a unit, this last condition is automatic from the axioms for the identification \(\lambda \mapsto \lambda \cdot 1_B\) and the axioms for \(E\) being a module over \(B\). (Unlike some other authors, we take the \(B\)-valued inner-product to be linear in the first variable and conjugate-linear in the second variable as is usually done in the Hilbert-space setting \((B = \mathbb{C})\) rather than the reverse.) Note that it then follows that

\[
\langle e, f \cdot b \rangle_E = b^* \langle e, f \rangle_E.
\]

When the inner product is clear, we drop the subscript \(E\) and write simply \(\langle e, f \rangle\) for the \(B\)-valued inner product. If \(E\) is a pre-Hilbert module over \(B\), then \(E\) is a normed linear space with norm given by

\[
\|e\|_E = \|\langle e, e \rangle\|^{1/2}_B.
\]

(3.1)

Here \(\|\|_B\) denotes the norm associated with the \(C^*\)-algebra \(B\). One can always complete \(E\) to a Banach space in the norm (3.1) to get what we shall call a Hilbert \(C^*\)-module over \(B\). Moreover, \(E\) has additional structure, namely \(E\) carries the structure of an operator space, i.e., \(E\) is the upper right corner of a subalgebra of operators acting on a Hilbert space with a representation as \(2 \times 2\) block operator matrices (the linking algebra)—see [31] or [43].

Given two Hilbert \(C^*\)-modules \(E\) and \(F\) over the same \(C^*\)-algebra \(B\), it is natural to consider the space \(\mathcal{L}(E, F)\) of bounded linear operators \(T : E \to F\) between the Banach spaces \(E\) and \(F\). Unlike the Hilbert space case, for a linear map \(T\) from \(E\) to \(F\) it may or may not happen that there is an adjoint operator \(T^* \in \mathcal{L}(F, E)\) so that

\[
\langle Te, f \rangle_F = \langle e, T^* f \rangle_E\quad\text{for all } e \in E \text{ and } f \in F.
\]

In case there exists an operator \(T^* \in \mathcal{L}(F, E)\) with this property we say that \(T\) is adjointable and we denote the set of all adjointable linear operators between \(E\) and \(F\) as \(\mathcal{L}^a(E, F)\) (with the usual abbreviation \(\mathcal{L}^a\) in case \(E = F\)). When the mapping \(T : E \to F\) is adjointable in this sense, necessarily \(T \in \mathcal{L}(E, F)\) with the additional property that \(T\) is a \(B\)-module map:

\[
T(e \cdot b) = T(e) \cdot b\quad\text{for all } e \in E \text{ and } b \in B.\tag{3.2}
\]

However, this additional property (3.2) alone is not sufficient for admission of \(T\) in the class \(\mathcal{L}^a(E, F)\) of adjointable maps (see [13] Example 2.19).

Following [31, 33] (see also the books [29, 43] for more comprehensive treatments), we now introduce the notion of an \((\mathcal{A}, B)\)-correspondence. If \(E\) is a right Hilbert \(C^*\)-module over \(B\) and \(\mathcal{A}\) is another \(C^*\)-algebra, we say that \(E\) is a \((\mathcal{A}, B)\)-correspondence if \(E\) is also a left module over \(\mathcal{A}\) which makes \(E\) an \((\mathcal{A}, B)\)-bimodule:

\[
(a \cdot e) \cdot b = a \cdot (e \cdot b)\quad\text{for all } a \in \mathcal{A}, e \in E \text{ and } b \in B
\]

with the additional compatibility condition

\[
\langle a \cdot e, f \rangle_E = \langle e, a^* \cdot f \rangle_E.\tag{3.3}
\]

The compatibility condition in (3.3) is equivalent to requiring that each of the left multiplication operators \(\varphi(a) : e \mapsto a \cdot e\) on \(E\) is a bounded linear operator on \(E\) for each \(a \in \mathcal{A}\) and \(\varphi\) is a \(C^*\)-homomorphism from \(\mathcal{A}\) into the \(C^*\)-algebra \(\mathcal{L}^a(E)\) of bounded adjointable operators on \(E\): thus \(\varphi(a)\) is adjointable for each \(a \in \mathcal{A}\) with \(\varphi(a)^* = \varphi(a^*)\). We shall occasionally write \(\varphi(a)e\) rather than \(a \cdot e\).

Note the lack of symmetry in the roles of \(\mathcal{A}\) and \(B\): the identities \(\langle e \cdot b, f \rangle = \langle e, f \rangle b\) together with \(\langle e, f \cdot b^* \rangle = b^* \langle e, f \rangle\) preclude the validity in general of the identity \(\langle e \cdot b, f \rangle = \langle e, f \cdot b^* \rangle\) (the would-be \(B\) analogue of (3.3)) unless \(B\) is commutative.

If both \(\mathcal{A}\) and \(B\) have units, we also demand that the scalar multiplication on \(E\) is compatible with both the identification \(\lambda \mapsto \lambda 1_{\mathcal{A}}\) of \(\mathbb{C}\) as a subalgebra of \(\mathcal{A}\) and the identification \(\lambda \mapsto \lambda 1_B\) of \(\mathbb{C}\) as a subalgebra of \(B\). This is consistent with demanding the additional axioms

\[
(\lambda a) \cdot e = a \cdot (\lambda e),\quad (\lambda e) \cdot b = e \cdot (\lambda b)
\]

for the general case.
The classical case is the one where $E$ is a Hilbert space $\mathcal{E}$, $\mathcal{B} = \mathbb{C}$ and $\mathcal{A} = \mathcal{L}(\mathcal{E})$ with the operations given by

$$a \cdot e = ae \quad \text{(the operator $a$ acting on the vector $e$)}$$
$$e \cdot b = be \quad \text{(scalar multiplication in $\mathcal{E}$)},$$

$$\langle e, f \rangle \quad \text{(the $\mathcal{E}$ Hilbert-space inner product)}.$$

Another easy example is to take $E = A = B$ all equal to a $C^*$-algebra with

$$a \cdot e = ae, \quad e \cdot b = eb, \quad \langle e, f \rangle_E = f^* e. $$

We encourage the reader to peruse Section 6 for a variety of additional examples and references for more complete details.

We will have need of various constructions for making new correspondences out of given correspondences. We give formal definitions as follows.

**Definition 3.1.**

1. **Direct sum:** Suppose that $E$ and $F$ are two $(\mathcal{A}, \mathcal{B})$-correspondences. Then the direct-sum correspondence $E \oplus F$ is defined to be the direct sum vector space $E \oplus F$ together with the diagonal left-$\mathcal{A}$ action and right-$\mathcal{B}$ action and the direct-sum $\mathcal{B}$-valued inner product:

$$a \cdot (e \oplus f) = (a \cdot e) \oplus (a \cdot f), \quad (e \oplus f) \cdot b = (e \cdot b) \oplus (f \cdot b),$$

$$\langle e \oplus f, e' \oplus f' \rangle_{E \oplus F} = \langle e, e' \rangle_E + \langle f, f' \rangle_F.$$

2. **Tensor product:** Suppose that we are given three $C^*$-algebras $A, B$ and $C$ together with an $(\mathcal{A}, \mathcal{B})$-correspondence $E$ and a $(\mathcal{B}, C)$-correspondence $F$. Then we define the tensor product correspondence $E \otimes_B F$ (sometimes abbreviated to $E \otimes F$) to be the completion of the linear span of all tensors $e \otimes f$ (with $e \in E$ and $f \in F$) subject to the identification

$$\langle e \otimes f, e' \otimes f' \rangle_{E \otimes F} = \langle \langle e, e' \rangle_E \cdot f, f' \rangle_F.$$

It is a straightforward exercise to verify that the balanced tensor-product construction is well-defined. For example the computation

$$\langle (e \otimes f, (e' \otimes b') \otimes f') = \langle b^* \cdot (e, e') \cdot b \cdot f, f' \rangle$$

$$\quad = \langle (e, e') \cdot b, b' \cdot f' \rangle$$

$$\quad = \langle e \otimes (b \cdot f), e' \otimes (b' \cdot f') \rangle$$

shows that the $E \otimes F$-inner product is well-defined.

**Remark 3.2.** Bounded linear operators between direct sum correspondences admit operator matrix decompositions in precisely the same way as in the Hilbert space case ($\mathcal{B} = \mathbb{C}$), while adjointability of such an operator corresponds to the operators in the decomposition being adjointable. For bounded linear operators between tensor-product correspondences the situation is slightly more complicated. We give an example how operators can be constructed. Let $E$ and $E'$ be $(\mathcal{A}, \mathcal{B})$-correspondences and $F$ and $F'$ $(\mathcal{B}, C)$-correspondences, for $C^*$-algebras $A, B$ and $C$. Furthermore, let $X \in \mathcal{L}(E, E')$ and $Y \in \mathcal{L}(F, F')$ be $\mathcal{B}$-module maps. Then we write $X \otimes Y$ for the operator in $\mathcal{L}(E \otimes_B F, E' \otimes_B F')$ which is determined by

$$X \otimes Y(e \otimes f) = (X e) \otimes (Y f) \quad \text{for each } e \otimes f \in E \otimes_B F. \quad (3.5)$$

The $\mathcal{B}$-module map properties are needed to guarantee that for each $e \otimes f \in E \otimes_B F$ and all $b \in \mathcal{B}$ we have

$$X \otimes Y(eb \otimes f) = (X(eb)) \otimes (Yf) = (Xe)b \otimes (Yf) = (Xe) \otimes (bYf) = (Xe) \otimes (Y(bf))$$

$$= X \otimes Y(e \otimes bf).$$
Thus the balancing in the tensor product (see (3.4)) is respected by the operator $X \otimes Y$. Moreover, $X \otimes Y$ is adjointable in case $X$ and $Y$ are adjointable operators, with $(X \otimes Y)^* = X^* \otimes Y^*$. Indeed, this is the case since for $f \otimes g \in E \otimes F$ and $e' \otimes f' \in E' \otimes F'$ we have

$$
\langle (X \otimes Y)(e \otimes f), e' \otimes f' \rangle_{E' \otimes F'} = (X e \otimes Y f, e' \otimes f')_{E' \otimes F'}
$$

In particular, the left action on $E \otimes F$ can now be written as $a \mapsto \varphi(a) \otimes I_F \in \mathcal{L}(E \otimes F, E \otimes F)$, where $I_F \in \mathcal{L}(F, F)$ is the identity operator on $F$. We will have occasions to use operators constructed in this way in the sequel.

We now introduce the notion of reproducing kernel $(A, B)$-correspondence.

**Definition 3.3.** Let $A$ and $B$ be $C^*$-algebras. By an $(A, B)$-reproducing kernel correspondence on a set $\Omega$, we mean an $(A, B)$-correspondence $E$ whose elements are $B$-valued functions $f : (\omega, a) \mapsto f(\omega, a) \in B$ on $\Omega \times A$, which is a vector space with respect to the usual point-wise vector-space operations and such that for each $\omega \in \Omega$ there is a kernel element $k_\omega \in E$ with

$$f(\omega, a) = \langle a \cdot f, k_\omega \rangle_E. \quad (3.6)$$

When this is the case we say that the function $K : \Omega \times \Omega \rightarrow \mathcal{L}(A, B)$ given by

$$K(\omega, \omega')[a] = k_{\omega'}(\omega, a) \quad (3.7)$$

is the reproducing kernel for the reproducing kernel correspondence $E$.

From the inner product characterization in (3.6) of the point evaluation for elements in an $(A, B)$-reproducing kernel correspondence $E$ on $\Omega$ one easily deduces that the left $A$-action and the right $B$-action are given by

$$(a \cdot f)(\omega', a') = f(\omega', a'a) \quad \text{and} \quad (f \cdot b)(\omega', a') = f(\omega', a'b). \quad (3.8)$$

It is implicit in Definition 3.3 that the map $a \mapsto k_\omega(\omega, a) \in B$ is linear in $a \in A$ for each $\omega, \omega' \in \Omega$. In fact the mapping from $A$ to $B$ given by $a \mapsto f(\omega, a)$ is $A$-linear for each fixed $f \in E$ and $\omega \in \Omega$. If $A$ has a unit $1_A$, this follows from the general identity $f(\omega, a) = (a \cdot f)(\omega, 1_A) \quad (a \text{ consequence of } \ref{3.8})$ together with the linearity of the point-evaluation map $f \mapsto f(\omega, 1_A)$ from $E$ to $B$ for each fixed $\omega \in \Omega$ which in turn is an easy consequence of (3.6). The general case follows by adapting this argument to the setting where one has only an approximate identity. Note also that we recover the element $k_{\omega'} \in B$ from $K$ by using formula (3.7) to define $k_{\omega'}$ as a function of $(\omega, a)$ for each $\omega' \in \Omega$.

The next proposition gives some elementary observations concerning the structure of reproducing kernel correspondences.

**Proposition 3.4.** If $E$ is a reproducing kernel $(A, B)$-correspondence with kernel elements $k_\omega$ for $\omega \in \Omega$, then the bounded evaluation map $e_{\omega, a}$ from $E$ to $B$ given by $e_{\omega, a} : f \mapsto f(\omega, a)$ is adjointable for each fixed $(\omega, a) \in \Omega \times A$ and we have

$$a^* k_\omega b = e_{\omega, a}^* b \quad \text{for each } \omega \in \Omega, \ a \in A, \text{ and } b \in B. \quad (3.9)$$

Conversely, suppose that $E$ is an $(A, B)$-correspondence of $B$-valued functions on the set $\Omega \times A$ satisfying (3.8) and such that the evaluation map $e_{\omega, a} : f \mapsto f(\omega, a)$ is a bounded and adjointable map from $E$ to $B$ for each $\omega \in \Omega$ and $a \in A$. Then $E$ is a reproducing kernel $(A, B)$-correspondence with reproducing kernel elements determined by (3.9).

Moreover, in either case, for each fixed $(\omega, a)$ the point-evaluation map $e_{\omega, a} : E \rightarrow B$ is a $B$-module map:

$$(f \cdot b)(\omega, a) = f(\omega, a)b \quad \text{for all } b \in B.$$
Proof. Suppose $E$ is a reproducing kernel $(\mathcal{A}, \mathcal{B})$-correspondence with kernel elements $k_\omega$ for $\omega \in \Omega$. If $e_{\omega,a}$ denotes the evaluation map from $E$ to $\mathcal{B}$ given by $e_{\omega,a} : f \mapsto f(\omega,a)$, we have
\[
\langle e_{\omega,a} f, b \rangle_{\mathcal{B}} = b^* f(\omega,a) = b^* \langle a \cdot f, k_\omega \rangle_E = \langle f, a^* k_\omega \cdot b \rangle.
\]
So $e_{\omega,a}$ is adjointable with $e_{\omega,a}^* b = a^* k_\omega b$ for any $b \in \mathcal{B}$.

On the other hand, if the evaluation map
\[
e_{\omega,a} : f \mapsto f(\omega,a)
\]
is a bounded and adjointable map from $E$ to $\mathcal{B}$ for each $\omega \in \Omega$ and $a \in \mathcal{A}$, then there exists an $e_{\omega,a}^*$ so that
\[
b^* (e_{\omega,a} f) = \langle e_{\omega,a} f, b \rangle_{\mathcal{B}} = \langle f, e_{\omega,a}^* b \rangle_{E}.
\]
If $\mathcal{A}$ and $\mathcal{B}$ have identities $1_\mathcal{A}$ and $1_\mathcal{B}$ respectively, we set $k_\omega = e_{\omega,1_\mathcal{A}}^* (1_\mathcal{B})$. Using the first identity in (3.8) it follows from a computation similar to that above, that $a^* k_\omega = e_{\omega,a}^* (1_\mathcal{B})$. We readily see that
\[
f(\omega,a) = e_{\omega,a} f = \langle f, a^* k_\omega \rangle_E = \langle a \cdot f, k_\omega \rangle_E.
\]

If $\mathcal{A}$ and/or $\mathcal{B}$ does not have a unit, one can do an approximate version of the above argument using an approximate identity for $\mathcal{A}$ and/or $\mathcal{B}$. In any case, it follows that $E$ is a reproducing kernel $(\mathcal{A}, \mathcal{B})$-correspondence with reproducing kernel elements determined by (3.9).

The last part follows from the definition of the right $\mathcal{B}$-action given by (3.9). $\square$

Given a reproducing kernel $(\mathcal{A}, \mathcal{B})$-correspondence as in Definition 3.3 one can show that the associated reproducing kernel function $\mathbb{K} : \Omega \times \Omega \rightarrow \mathcal{L}(\mathcal{A}, \mathcal{B})$ defined by (3.7) is a completely positive kernel in the sense of [18], i.e., the function
\[
((\omega, a), (\omega', a')) \mapsto \mathbb{K}(\omega, \omega') [a^* a']
\]
is a positive kernel in the classical sense of Aronszajn [2] (extended to the $C^*$-algebra-valued case), that is, $\sum_{i,j=1}^{N} b_i^* \mathbb{K}(\omega_i, \omega_j) [a_i^* a_j] b_j$ is a positive element of $\mathcal{B}$ for each choice of finitely many $(\omega_1, a_1), \ldots, (\omega_N, a_N)$ in $\Omega \times \mathcal{A}$ and $b_1, \ldots, b_N$ in $\mathcal{B}$. In fact, by the axioms of an $(\mathcal{A}, \mathcal{B})$-correspondence combined with the reproducing property of the kernel elements $k_\omega$, we have
\[
\sum_{i,j=1}^{N} b_i^* \mathbb{K}(\omega_i, \omega_j) [a_i^* a_j] b_j = \sum_{i,j=1}^{N} b_i^* \langle a_i^* a_j k_{\omega_i} \rangle_{E} b_j = \sum_{i,j=1}^{N} \langle a_j k_{\omega_i} b_j, a_i k_{\omega_i} b_i \rangle_{E} = \left\langle \sum_{i=1}^{N} a_j k_{\omega_i} b_j, \sum_{i=1}^{N} a_i k_{\omega_i} b_i \right\rangle_{E} \geq 0.
\]

Actually, we have the following equivalent statements.

**Theorem 3.5.** Given a function $\mathbb{K} : \Omega \times \Omega \rightarrow \mathcal{L}(\mathcal{A}, \mathcal{B})$, the following are equivalent:

1. $\mathbb{K}$ is a completely positive kernel in the sense that the function from $(\Omega \times \mathcal{A}) \times (\Omega \times \mathcal{A}) \rightarrow \mathcal{L}(\mathcal{A}, \mathcal{B})$ given by

   $$(\omega, a)(\omega', a') \mapsto \mathbb{K}(\omega', \omega)[a^* a']$$

   is a positive kernel in the sense of Aronszajn:

   $$\sum_{i,j=1}^{N} b_i^* \mathbb{K}(\omega_i, \omega_j)[a_i^* a_j] b_j \geq 0 \text{ in } \mathcal{B} \text{ for all } (\omega_1, a_1), \ldots, (\omega_N, a_N) \in \Omega \times \mathcal{A} \text{ and } b_1, \ldots, b_N \in \mathcal{B}.$$ 

2. $\mathbb{K}$ has a Kolmogorov decomposition in the sense of [18], i.e., there exists an $(\mathcal{A}, \mathcal{B})$-correspondence $E$ and a mapping $\omega \mapsto k_\omega$ from $\Omega$ into $E$ such that

   $$\mathbb{K}(\omega', \omega)[a] = \langle a \cdot k_{\omega}, k_{\omega'} \rangle_{E} \text{ for all } a \in \mathcal{A}.$$
(3) $\mathbb{K}$ is the reproducing kernel for an $(\mathcal{A}, \mathcal{B})$-reproducing kernel correspondence $E = E(\mathbb{K})$, i.e., there is an $(\mathcal{A}, \mathcal{B})$-correspondence $E = E(\mathbb{K})$ whose elements are $\mathcal{B}$-valued functions on $\Omega \times \mathcal{A}$ such that the function $k_\omega : (\omega', a') \mapsto \mathbb{K}(\omega', \omega)[a']$ is in $E(\mathbb{K})$ for each $\omega \in \Omega$ and has the reproducing property

$$
(a \cdot f, k_\omega)_{\mathbb{K}(\mathbb{K})} = (f, a^* \cdot k_\omega)_{\mathbb{K}(\mathbb{K})} = f(\omega, a) \text{ for all } \omega \in \Omega \text{ and } a \in \mathcal{A}
$$

where $a^* \cdot k_\omega$ is given by

$$
(a^* \cdot k_\omega)(\omega', a') = \mathbb{K}(\omega', \omega)[a'a^*].
$$

Proof. For the equivalence of (1) and (2), we refer to Theorem 3.2.3 in [18]. The argument in the paragraph preceding the statement of the theorem shows that (3) $\implies$ (1). To see that (2) $\implies$ (3), assume that $E$ is an $(\mathcal{A}, \mathcal{B})$-correspondence as in (2). Without loss of generality we may assume that $E = \text{span}\{a \cdot k_\omega b : a \in \mathcal{A}, \omega \in \Omega, b \in \mathcal{B}\}$. (3.11)

We view elements $f$ of $E$ as $\mathcal{B}$-valued functions on $\Omega \times \mathcal{A}$ by defining

$$
f(\omega, a) = (a \cdot f, k_\omega)_{E} \text{ for each } \omega \in \Omega \text{ and } a \in \mathcal{A}.
$$

The nondegeneracy assumption (3.11) says that

$$
f(\omega, a)b = 0 \text{ for all } a \in \mathcal{A}, b \in \mathcal{B}, \omega \in \Omega \implies f = 0 \in E.
$$

Hence the map $f \mapsto f(\cdot, \cdot)$ is injective. Finally (3.11) holds by definition. \qed

We now tailor this general theorem to the case where $\mathcal{B} = \mathcal{L}(\mathcal{E})$ for a Hilbert space $\mathcal{E}$. Note that $\mathcal{E}$ is a $(\mathcal{L}(\mathcal{E}), \mathbb{C})$-correspondence, i.e., a Hilbert space with a $*$-representation $b \mapsto \varphi(b) \in \mathcal{L}(\mathcal{E})$ (namely, the identity representation). Hence, given that $E$ is an $(\mathcal{A}, \mathcal{L}(\mathcal{E}))$-correspondence, we may form the tensor product $E \otimes_{\mathcal{L}(\mathcal{E})} \mathcal{E}$ to obtain an $(\mathcal{A}, \mathbb{C})$-correspondence, i.e., a Hilbert space which we will denote by $\mathcal{H}$ equipped with an $\mathcal{L}(\mathcal{H})$-valued $*$-representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ of $\mathcal{A}$. Similarly, if we view $\mathcal{B} = \mathcal{L}(\mathcal{E})$ as a $(\mathcal{L}(\mathcal{E}), \mathcal{L}(\mathcal{E}))$-correspondence, we may form the tensor product $\mathcal{L}(\mathcal{E}) \otimes \mathcal{E}$ to arrive at the Hilbert space $\mathcal{E}$, via the balancing [3, 4], viewed as a $(\mathcal{L}(\mathcal{E}), \mathbb{C})$-correspondence. Let us suppose also that $E$ is a reproducing kernel correspondence. Then via the formula $f \otimes e \in E \otimes \mathcal{E} \mapsto f(\omega, a) \otimes e \in \mathcal{L}(\mathcal{E}) \otimes \mathcal{E}$ for each $\omega \in \Omega$ and $a \in \mathcal{A}$ extended via linearity and continuity to the whole space $E \otimes \mathcal{E}$, we may view each $f \in \mathcal{H} = E \otimes \mathcal{E}$ as a $\mathcal{E}$-valued function on $\Omega \times \mathcal{A}$ such that point-evaluation $f \mapsto f(\omega, a)$ is continuous, i.e., $\mathcal{H}$ is a reproducing kernel Hilbert space of vector-valued functions on $\Omega \times \mathcal{A}$, but with the additional wrinkle that there is also a representation $a \mapsto \pi(a)$ of $\mathcal{A}$ on $\mathcal{H}$ with $\pi(a)(f \otimes e) = (a \cdot f) \otimes e$ such that

$$
(\pi(a)(f \otimes e))(\omega', a') = f(\omega', a' a) \otimes e
$$

with reproducing kernel (in the sense of a vector-valued reproducing kernel Hilbert space) $K(\cdot, \cdot)$ of the special form

$$
K((\omega', a'), (\omega, a)) = \mathbb{K}(\omega', \omega)[a'a']
$$

for a completely positive kernel $\mathbb{K} : \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{E}))$: for $f \in \mathcal{H}^*(\mathbb{K})$, $e \in \mathcal{E}$ and $(\omega, a) \in \Omega \times \mathcal{A},$

$$
(f, \mathbb{K}(\cdot, \omega)[a]e)_{\mathcal{H}} = (f(\omega, a), e)_{\mathcal{E}}
$$

where $\mathbb{K}$ is completely positive. This leads us to an alternative reproducing-kernel interpretation of a completely positive kernel $\mathbb{K} : \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{B})$ for the case where $\mathcal{B} = \mathcal{L}(\mathcal{E})$ for a Hilbert space $\mathcal{E}$.

Theorem 3.6. Suppose that $\mathcal{A}$ is a $C^*$-algebra, $\mathcal{E}$ is a Hilbert space and that a function $\mathbb{K} : \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{E}))$ is given. The following conditions are equivalent.

(1) The function $\mathbb{K}$ is a completely positive kernel in the sense that

$$
\sum_{i,j=1}^{N} \langle \mathbb{K}(\omega_i, \omega_j)[a_i^* a_j]e_j, e_i \rangle \geq 0
$$

for all finite collections $\omega_1, \ldots, \omega_N \in \Omega$, $a_1, \ldots, a_N \in \mathcal{A}$ and $e_1, \ldots, e_N \in \mathcal{E}$ for $N = 1, 2, \ldots$. (2) The kernel $\mathbb{K}$ has a Kolmogorov decomposition: there is a Hilbert space $\mathcal{H}$ together with a $*$-representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ of $\mathcal{A}$ and a mapping $H : \Omega \to \mathcal{L}(\mathcal{H}, \mathcal{E})$ so that

$$
\mathbb{K}(\omega', \omega)[a] = H(\omega')\pi(a)H(\omega)^*.
$$
When (3.12) does occur and if also $A$ kernel $K$ In this case it may or may not happen that point evaluation is a $n$-algebra $E$-valued functions on $\Omega \times A$ such that:

(a) The $*$-representation $\pi$ is given by

$$
\langle \pi(a)f, e \rangle = f(a')a'.
$$

(b) The function $k_\omega: \Omega \times A \to \mathcal{L}(E)$ given by

$$
k_\omega(a', a') = e \mapsto K(a', \omega)[a']e
$$

is such that $k_\omega e \in \mathcal{H}(K)$ for each $\omega \in \Omega$ and $e \in E$ and has the reproducing kernel property:

$$
\langle f, \pi(a)*k_\omega e \rangle_{\mathcal{H}(K)} = \langle f, \pi(a) \rangle_{\mathcal{H}(K)}.
$$

Combining (3.14) and (3.15) gives (3.13) as claimed.

Let us say that the object described in part (3) of Theorem 3.6 a reproducing kernel Hilbert correspondence (over the $C^*$-algebra $A$ with values in the coefficient space $E$).

**Remark 3.7.** If $\mathcal{H}(K)$ is a reproducing kernel Hilbert correspondence space as in part (3) of Theorem 3.6, a special situation occurs if the coefficient space $E$ is also equipped with a $*$-representation $\pi_E: A \to \mathcal{L}(E)$. In this case it may or may not happen that point evaluation is an $A$-module map, i.e., that

$$
(a \cdot f)(\omega', a') = a \cdot f(\omega', a') \quad \text{or equivalently} \quad (\pi(a)f)(\omega', a') = \pi_E(a)f(\omega', a'). \quad (3.12)
$$

When (3.12) does occur and if also $A$ has a unit $1_A$, one can show that the associated completely positive kernel $K(\omega, \omega'[a]$ has the special property

$$
K(\omega, \omega'[a^*a'] = \pi_E(a)^*K(\omega, \omega')[-1_A][\pi_E(a') \quad (3.13)
$$

and hence complete positivity of $K$ reduces to standard (Aronszajn) positivity for the kernel $K_0: \Omega \times \Omega \to \mathcal{L}(E)$ given by

$$
K_0(\omega, \omega') = K(\omega, \omega')[-1_A].
$$

Indeed, the computation

$$
\langle K(\omega, \omega')[a^*a']e', e \rangle_E = \langle a^*a' \cdot k_\omega e', k_\omega e \rangle_{\mathcal{H}(K)}
$$

shows that

$$
K(\omega, \omega')[a^*a'] = a^*K(\omega, \omega')[a']. \quad (3.14)
$$

On the other hand, the positive-kernel property of the kernel

$$
((\omega, a), (\omega', a')) \mapsto K((\omega, a), (\omega', a')) := K(\omega, \omega'[a^*a']
$$

implies that $K$ is Hermitian, i.e., $K((\omega, a), (\omega', a')) = K((\omega', a'), (\omega, a))^*$, i.e.,

$$
K(\omega, \omega')[a^*a'] = (K(\omega', \omega)[a^*a])^*.
$$

In particular,

$$
K(\omega, \omega')[a'] = (K(\omega', \omega)[a^*])^* = (a^*K(\omega', \omega)[-1_A])^* \quad \text{(by (3.11))}
$$

and hence also

$$
K(\omega, \omega')[a'] = K(\omega, \omega')[-1_A][a']. \quad (3.15)
$$

Combining (3.14) and (3.15) gives (3.13) as claimed.
4. Function-theoretic operator theory associated with a correspondence $E$

In this section we obtain the analogues of Hardy spaces, Toeplitz operators, $Z$-transform and Schur class attached to a $\mathcal{A}$-$W^*$-correspondence $E$ together with a $*$-representation $\sigma$ of $\mathcal{A}$. These results flesh out more fully the function-theoretic aspects of the work of Muhly-Solel \cite{Muhly-Solel2002, Muhly-Solel2004, Muhly-Solel2008}.

4.1. Hardy Hilbert spaces associated with a correspondence $E$. In this section we shall consider the situation where $\mathcal{A} = B$; we abbreviate the term $(\mathcal{A}, \mathcal{A})$-correspondence to simply $\mathcal{A}$-correspondence. We also now restrict our attention to the case where $\mathcal{A}$ is a von Neumann algebra and let $E$ be a $\mathcal{A}$-$W^*$-correspondence. This means that $E$ is a $\mathcal{A}$-correspondence which is also self-dual in the sense that any right $\mathcal{A}$-module map $\rho: E \to \mathcal{A}$ is given by taking the inner product against some element $e_\rho$ of $E$: $\rho(e) = \langle e, e_\rho \rangle_E \in \mathcal{A}$. Moreover, the space $\mathcal{L}^n(E)$ of adjointable operators on the $W^*$-correspondence $E$ is in fact a $\mathcal{A}$-$\mathcal{A}$-algebra, i.e., is the abstract version of a von Neumann algebra with an ultra-weak topology (see \cite{Sz.-Nagy1961}).

Since $E$ is a $\mathcal{A}$-correspondence, we may use Definition \ref{def:Correspondence} to define the self-tensor product $E^{\otimes 2} = E \otimes_\mathcal{A} E$ which is again an $\mathcal{A}$-correspondence, and, inductively, an $\mathcal{A}$-correspondence $E^{\otimes n} = E \otimes_\mathcal{A} (E \otimes_\mathcal{A} (E \otimes_\mathcal{A} (\cdots)))$ for each $n = 1, 2, \ldots$. If we use $a \mapsto \varphi(a)$ to denote the left $\mathcal{A}$-action $\varphi(a)e = a \cdot e$ on $E$, we denote the left $\mathcal{A}$-action on $E^{\otimes n}$ by $\varphi^{(n)}$:

$\varphi^{(n)}(a): \xi_n \otimes \xi_{n-1} \otimes \cdots \otimes \xi_1 \mapsto (\varphi(a)\xi_n) \otimes \xi_{n-1} \otimes \cdots \otimes \xi_1$.

Note that, using the notation in \cite{Muhly-Solel2002, Muhly-Solel2004, Muhly-Solel2008}, we may write $\varphi^{(n)}(a) = \varphi(a) \otimes I_{E^{\otimes n-1}}$. We formally set $E^{\otimes 0} = \mathcal{A}$. Then the Fock space $\mathcal{F}^2(E)$ is defined to be

$$\mathcal{F}^2(E) = \bigoplus_{n=0}^{\infty} E^{\otimes n}$$

and is also an $\mathcal{A}$-correspondence. We denote the left $\mathcal{A}$-action on $\mathcal{F}(E)$ by $\varphi_{\infty}$:

$$\varphi_{\infty}(a): \bigoplus_{n=0}^{\infty} \xi^{(n)} \mapsto \bigoplus_{n=0}^{\infty} (\varphi^{(n)}(a)\xi^{(n)}) \text{ for } \bigoplus_{n=0}^{\infty} \xi^{(n)} \in \bigoplus_{n=0}^{\infty} E^{\otimes n},$$

or, more succinctly,

$$\varphi_{\infty}(a) = \text{diag}(a, \varphi^{(1)}(a), \varphi^{(2)}(a), \ldots).$$

In addition to the von Neumann algebra $\mathcal{A}$ and the $\mathcal{A}$-correspondence $E$, suppose that we are also given an auxiliary Hilbert space $\mathcal{E}$ and a nondegenerate $*$-homomorphism $\sigma: \mathcal{A} \to \mathcal{L}(\mathcal{E})$; as this will be the setting for much of the analysis to follow, we refer to such a pair $(E, \sigma)$ as a correspondence-representation pair. Then the Hilbert space $\mathcal{E}$ equipped with $\sigma$ becomes an $(\mathcal{A}, \mathcal{C})$-correspondence with left $\mathcal{A}$-action given by $\sigma$:

$$a \cdot y = \sigma(a)y \text{ for all } a \in \mathcal{A} \text{ and } y \in \mathcal{E}.$$ 

We let $E \otimes_\sigma \mathcal{E}$ be the associated tensor-product $(\mathcal{A}, \mathcal{C})$-correspondence $E \otimes_\mathcal{A} \mathcal{E}$ as in Definition \ref{def:Correspondence} As $\mathcal{F}^2(E)$ is also an $\mathcal{A}$-correspondence, we may also form the $(\mathcal{A}, \mathcal{C})$-correspondence

$$\mathcal{F}^2(E, \sigma) := \mathcal{F}^2(E) \otimes_\sigma \mathcal{E} = \bigoplus_{n=0}^{\infty} (E^{\otimes n} \otimes_\sigma \mathcal{E}),$$

with left $\mathcal{A}$-action given by the $*$-representation

$$\varphi_{\infty, \sigma}(a) = \varphi_{\infty}(a) \otimes I_{\mathcal{E}}.$$ 

It turns out that $\mathcal{F}^2(E, \sigma)$ is also a $(\sigma(\mathcal{A}'), \mathcal{C})$-correspondence, where $\sigma(\mathcal{A}') \subset \mathcal{L}(\mathcal{E})$ denotes the commutant of $\sigma(\mathcal{A})$:

$$\sigma(\mathcal{A})' = \{ b \in \mathcal{L}(\mathcal{E}) : b \sigma(a) = \sigma(a) b \text{ for all } a \in \mathcal{A} \},$$

and the left $\sigma(\mathcal{A})'$-action is given by the $*$-representation $\iota_{\infty, \sigma}$ of $\sigma(\mathcal{A})'$ on $\mathcal{F}(E, \sigma)$:

$$\iota_{\infty, \sigma}(b) = I_{\mathcal{F}^2(E)} \otimes b \text{ for each } b \in \sigma(\mathcal{A})',$$

using the notation in \cite{Muhly-Solel2002, Muhly-Solel2004, Muhly-Solel2008}. Note that $b \in \mathcal{L}(\mathcal{E})$ is in $\sigma(\mathcal{A})'$ precisely when $b$ is an $\mathcal{A}$-module map, so that $I_{\mathcal{F}^2(E)} \otimes b$ is a well defined operator on $\mathcal{F}(E, \sigma)$. Moreover, $\varphi_{\infty, \sigma}(a)$ commutes with $\iota_{\infty, \sigma}(b)$ for each $a \in \mathcal{A}$ and $b \in \sigma(\mathcal{A})'$ since

$$\varphi_{\infty, \sigma}(a) \iota_{\infty, \sigma}(b) = \varphi_{\infty}(a) \otimes b = \iota_{\infty, \sigma}(b) \varphi_{\infty, \sigma}(a).$$

Thus $\iota_{\infty, \sigma}(b)$ is a $\mathcal{A}$-module map for each $b \in \sigma(\mathcal{A})'$ and $\varphi_{\infty, \sigma}(a)$ is a $\sigma(\mathcal{A})'$-module map for each $a \in \mathcal{A}$. 
We denote by $E^\sigma$ the set of all bounded linear operators $\mu : \mathcal{E} \to E \otimes_\sigma \mathcal{E}$ which are also $\mathcal{A}$-module maps:

$$E^\sigma = \{ \mu : \mathcal{E} \to E \otimes_\sigma \mathcal{E} : \mu \sigma(a) = (\varphi(a) \otimes I_\mathcal{E}) \mu \},$$

and $(E^\sigma)^*$ for the set of adjoints (which are also $\mathcal{A}$-module maps):

$$(E^\sigma)^* = \{ \eta : E \otimes_\sigma \mathcal{E} \to \mathcal{E} : \eta^* \in E^\sigma \}. \tag{4.6}$$

More generally, for a given $\eta \in (E^\sigma)^*$, we may define operators $\eta^n : E^\otimes_n \otimes_\sigma \mathcal{E} \to \mathcal{E}$ (generalized powers) by

$$\eta^n = \eta(I_E \otimes \eta) \cdots (I_{E^\otimes_n} \otimes \eta)$$

where we use the identification

$$E^\otimes_n \otimes_\sigma \mathcal{E} = E^{\otimes n-1} \otimes_\mathcal{A} (E \otimes_\sigma \mathcal{E})$$

in these definitions. We also set $\eta^0 = I_E \in \mathcal{L}(\mathcal{E})$. Again the fact that $\eta$ is an $\mathcal{A}$-module map ensures that $I_{E^\otimes k} \otimes \eta$ is a well-defined operator in $\mathcal{L}(E^\otimes k+1 \otimes_\sigma \mathcal{E}, E^\otimes k \otimes_\sigma \mathcal{E})$. The defining $\mathcal{A}$-module property of $\eta$ in (4.6) then extends to the generalized powers $\eta^n$ in the form

$$\eta^n(\varphi^{(m)}(a) \otimes I_E) = \sigma(a)\eta^n,$$  \tag{4.7}

i.e., $\eta^n$ is also an $\mathcal{A}$-module map.

Denote by $\mathbb{D}(E^\sigma)^*$ the set of strictly contractive elements of $(E^\sigma)^*$:

$$\mathbb{D}(E^\sigma)^* = \{ \eta \in (E^\sigma)^* : \|\eta\| < 1 \}. \tag{4.8}$$

Then, for $\eta \in \mathbb{D}(E^\sigma)^*$ and $b \in \sigma(\mathcal{A})'$, we may define a bounded operator $f \mapsto f^\wedge(\eta, b)$ from $\mathcal{F}^2(E, \sigma)$ into $\mathcal{E}$ by

$$f^\wedge(\eta, b) = \sum_{n=0}^{\infty} \eta^n(\iota_{\sigma, \sigma}(b)f)_{n} = \sum_{n=0}^{\infty} \eta^n(I_{E^\otimes_n} \otimes b)f_n \text{ if } f = \oplus_{n=0}^{\infty} f_n. \tag{4.8}$$

Note that the fact that $\|\eta\| < 1$ guarantees that the series in (4.8) converges. The $\mathcal{A}$-module properties of $\iota_{\sigma, \sigma}(b)$ and each generalized power $\eta^n$ (see (4.7)) for given $b \in \sigma(\mathcal{A})'$ and $\eta \in \mathbb{D}(E^\sigma)^*$ imply that the point-evaluation $f \mapsto f^\wedge(\eta, b)$ is also an $\mathcal{A}$-module map:

$$(\varphi_{\infty, \sigma}(a)f)^\wedge(\eta, b) = \sigma(a)f^\wedge(\eta, b).$$

However, the point-evaluation $f \mapsto f^\wedge(\eta, b)$ is not a $\sigma(\mathcal{A})'$-module map, i.e., there is no guarantee for the general validity of the identity $(\iota_{\sigma, \sigma}(b)f)^\wedge(\eta', b') = b f^\wedge(\eta', b')$, but rather we have the property

$$(\iota_{\sigma, \sigma}(b)f)^\wedge(\eta', b') = f^\wedge(\eta', b'b).$$

We denote the space of all $\mathcal{E}$-valued functions on $\mathbb{D}(E^\sigma)^* \times \sigma(\mathcal{A})'$ of the form $(\eta, b) \mapsto f^\wedge(\eta, b)$ for some $f \in \mathcal{F}^2(E, \sigma)$ by $H^2(E, \sigma)$ with norm $\|f^\wedge\|_{H^2(E, \sigma)}$ chosen so as to make the map $f \mapsto f^\wedge$ a coisometry from $\mathcal{F}^2(E, \sigma)$ to $H^2(E, \sigma)$:

$$H^2(E, \sigma) = \{ f^\wedge : f \in \mathcal{F}^2(E, \sigma) \} \text{ with } \|f^\wedge\|_{H^2(E, \sigma)} = \|P_{(\ker \Phi)^\perp} f \|_{\mathcal{F}^2(E, \sigma)}$$

where we denote by $\Phi$ (the generalized Fourier or Z-transform for this setting) the transformation from $\mathcal{F}^2(e, \sigma)$ into $H^2(E, \sigma)$ given by

$$\Phi : f \mapsto f^\wedge. \tag{4.9}$$

Then we have the following result.

**Theorem 4.1.** The space $H^2(E, \sigma)$ is a reproducing kernel Hilbert correspondence $\mathcal{H}(\mathbb{K})$ (as in part (3) of Theorem 3.6) over $\sigma(\mathcal{A})'$ consisting of $\mathcal{E}$-valued functions on $\mathbb{D}(E^\sigma)^* \times \sigma(\mathcal{A})'$ with the $*$-representation of $\sigma(\mathcal{A})'$ given by

$$(b \cdot f^\wedge)(\eta', b') = (\iota_{\sigma, \sigma}(b)f)^\wedge(\eta', b') \text{ for } b \in \sigma(\mathcal{A})'. \tag{4.10}$$

The completely positive kernel $\mathbb{K}$ associated with $H^2(E, \sigma)$ as in Theorem 3.6

$$\mathbb{K}_{E, \sigma} : \mathbb{D}(E^\sigma)^* \times \mathbb{D}(E^\sigma)^* \to \mathcal{L}(\sigma(\mathcal{A})', \mathcal{E})$$

is the Szegö kernel for our setting given by

$$\mathbb{K}_{E, \sigma}(\eta, \zeta | b) = \sum_{n=0}^{\infty} \eta^n(I_{E^\otimes_n} \otimes b)(\zeta^n)^* \text{ for } b \in \sigma(\mathcal{A})'. \tag{4.11}$$
Proof. Define \( \Phi : F^2(E, \sigma) \to H^2(E, \sigma) \) as in (4.9). By the definition of the norm on \( H^2(E, \sigma) \), \( \Phi \) is a coisometry. For each \( b \in \sigma(A)' \) and \( \eta \in \mathbb{D}(\langle (E^\sigma)^* \rangle) \), define an associated controllability operator \( \mathcal{C}_{b, \eta} : F^2(E, \sigma) \to \mathcal{E} \) by

\[
\mathcal{C}_{b, \eta} : f \mapsto f^\wedge(\eta, b) \text{ if } f \in F^2(E, \sigma).
\]

By definition,

\[
\ker \Phi = \bigcap_{b \in \sigma(A)' \setminus \mathbb{D}(\langle (E^\sigma)^* \rangle)} \ker \mathcal{C}_{b, \eta}.
\]

The initial space of the coisometry \( \Phi \) is the orthogonal complement of its kernel, namely

\[
\ker \Phi^\perp = \overline{\text{span}}\{\text{Ran} \mathcal{C}_{b, \eta} : b \in \sigma(A)', \eta \in \mathbb{D}(\langle (E^\sigma)^* \rangle)\},
\]

where the observability operator \( \mathcal{C}_{b, \eta}^* \) is given by

\[
\mathcal{C}_{b, \eta}^* : e \mapsto \oplus_{n=0}^\infty (I_{E \otimes m} \otimes b^*) (\eta^n)^* e \in F^2(E) \otimes_\sigma \mathcal{E}.
\]

We compute

\[
(f^\wedge(\zeta, b), e)_\mathcal{E} = (\mathcal{C}_{b, \eta} f, e)_\mathcal{E} = (f, \mathcal{C}_{b, \eta}^* e)_{F^2(E, \sigma)} = (f^\wedge, \Phi(\mathcal{C}_{b, \eta}^* e))_{H^2(E, \sigma)} = (f^\wedge, b^* \cdot \Phi(\mathcal{C}_{b, \eta}^* e))_{H^2(E, \sigma)},
\]

where we use the fact seen above that \( \mathcal{C}_{b, \eta}^* e \) is in the initial space of \( \Phi \) and that \( \Phi((I_{F^2(E)} \otimes b)f) = b(\Phi f) \) for each \( b \in \sigma(A)' \) and \( f \in F^2(\sigma, \eta) \). Hence the operator

\[
k_{E, \sigma, \zeta} : = \Phi \mathcal{C}_{b, \eta}^* \mathcal{E} \to H^2(E, \sigma)
\]

has the reproducing property for \( H^2(E, \sigma) \); see part (3.b) in Theorem 3.4. Since \( \Phi \) is a coisometry and \( I_E \in \sigma(A)' \), we obtain that the reproducing kernel \( K_{E, \sigma}(\eta, \zeta) \) is necessarily given by

\[
K_{E, \sigma}(\eta, \zeta) = C_{b, \eta} \Phi^* \mathcal{C}_{b, \eta}^* = \sum_{n=0}^\infty \eta^n (I_{E \otimes m} \otimes b)(\zeta^n)^*.
\]

In agreement with (4.11).

From the proof of Theorem 4.1 we see that we have the identification

\[
b^* \cdot k_{E, \sigma, \eta} e = \mathcal{C}_{b, \eta}^* e = \oplus_{n=0}^\infty (I_{E \otimes m} \otimes b^*) \eta^n e
\]

and the initial space for the coisometry \( \Phi : F(E) \otimes_\sigma \mathcal{E} \to H^2(E, \sigma) \) can be identified as

\[
[F(E) \otimes_\sigma \mathcal{E}]_{\text{initial}} = \overline{\text{span}}\{b \cdot k_{E, \sigma, \eta} e : b \in \sigma(A)', \eta \in \mathbb{D}(\langle (E^\sigma)^* \rangle), e \in \mathcal{E}\}.
\]

4.2. Analytic Toeplitz algebras associated with a correspondence \( E \). Given an \( A \)-W*-correspondence \( E \), we let \( F^2(E) \) be the associated Fock space as in (4.1). We have already defined the \( * \)-representation of \( A \) to \( \mathcal{L}^\infty(F^2(E)) \) given by \( a \mapsto \varphi_\infty(a) \) as in (4.2). If we view operators on \( F^2(E) \) as matrices induced by the decomposition \( F^2(E) = \bigoplus_{n=0}^\infty E^\otimes n \) of \( F^2(E) \), we see that each \( \varphi_\infty(a) \) has a diagonal representation \( \varphi_\infty(a) = \text{diag}_{n=0,1, \ldots} \varphi(n)(a) \). In addition to the operators \( \varphi_\infty(a) \in \mathcal{L}^\infty(F^2(E)) \), we introduce the so-called creation operators on \( F^2(E) \) given, for each \( \xi \in E \), by the subdiagonal (or shift) block matrix

\[
T_\xi = \begin{bmatrix}
0 & 0 & 0 & \cdots \\
T^{(0)}_\xi & 0 & 0 & \cdots \\
0 & T^{(1)}_\xi & 0 & \cdots \\
\ddots & \ddots & \ddots & \ddots
\end{bmatrix}
\]

2The terminology is motivated by connections with system theory; for a systematic account for the Drury-Arveson and free-semigroup algebra settings, we refer to [10].
where the block entry $T_{\xi}^{(n)}: E^{\otimes n} \to E^{\otimes n+1}$ is given by

$$T_{\xi}^{(n)}: \xi_n \otimes \cdots \otimes \xi_1 \mapsto \xi \otimes \xi_n \otimes \cdots \otimes \xi_1.$$ 

The operator $T_{\xi}$ is also in $\mathcal{L}^a(\mathcal{F}^2(E))$. In summary, both $T_{\xi}$ and $\varphi_{\infty}(a)$ are $\mathcal{A}$-module maps with respect to the right $\mathcal{A}$-action on $\mathcal{F}^2(E)$ for each $\xi \in E$ and $a \in \mathcal{A}$. Moreover, one easily checks that

$$\varphi_{\infty}(a)T_{\xi} = T_{\varphi(a)\xi} \quad \text{and} \quad T_{\xi}\varphi_{\infty}(a) = T_{\xi a} \quad \text{for each } a \in \mathcal{A} \text{ and } \xi \in E.$$

We let $\mathcal{F}^\infty(E)$ denote the weak-* closed algebra generated by the collection of operators

$$\{\varphi_{\infty}(a), T_{\xi}: a \in \mathcal{A} \text{ and } \xi \in E\}$$

in the $W^*$-algebra $\mathcal{L}^a(\mathcal{F}(E))$—we prefer this notation over the notation $H^\infty(E)$ used for this object in [31,33].

Suppose now that we are also given a $*$-representation $\sigma$ of $\mathcal{A}$ on a Hilbert space $\mathcal{E}$. Rather than the algebra $\mathcal{F}^\infty(E)$ of adjointable operators on the $\mathcal{A}$-correspondence $\mathcal{F}^2(E)$, our main focus of interest will be on the algebra $\mathcal{F}^\infty(E) \otimes I_{\mathcal{E}}$ of all operators on the Hilbert space $\mathcal{F}^2(E, \sigma)$ of the form $R = T \otimes I_{\mathcal{E}}$ with $T \in \mathcal{F}^\infty(E)$ acting on the Hilbert space $\mathcal{F}^2(E, \sigma)$. Note that the operator $R = T \otimes I_{\mathcal{E}}$ is properly defined since $R$ is an $\mathcal{A}$-module map with respect to the right $\mathcal{A}$-action on $\mathcal{F}^2(E)$. For convenience we shall use the abbreviated notation

$$\mathcal{F}^\infty(E, \sigma) = \mathcal{F}^\infty(E) \otimes I_{\mathcal{E}},$$

and

$$\varphi_{\infty, \sigma}(a) = \varphi_{\infty}(a) \otimes I_{\mathcal{E}} \quad \text{and} \quad T_{\xi, \sigma} = T_{\xi} \otimes I_{\mathcal{E}} \quad \text{for all } a \in \mathcal{A} \text{ and } \xi \in E.$$

The algebra $\mathcal{F}^\infty(E, \sigma)$ can also be described as the weak-* closed algebra generated by the collection of operators

$$\{\varphi_{\infty, \sigma}(a), T_{\xi, \sigma}: a \in \mathcal{A}, \xi \in E\}. \quad (4.13)$$

The following alternative characterization of $\mathcal{F}^\infty(E, \sigma)$ will be useful. Here we define $E^\sigma$ and $\sigma(\mathcal{A})'$ as in (4.10) and (4.14). Note that each element $\mu$ of $E^\sigma$ induces a dual creation operator $T_{\mu, \sigma}^{d}$ in $\mathcal{L}^a(\mathcal{F}^2(E, \sigma))$ given by

$$T_{\mu, \sigma}^{d} = \begin{bmatrix}
0 & 0 & 0 & \cdots \\
T_{\mu, \sigma}^{d,(0)} & 0 & 0 & \cdots \\
0 & T_{\mu, \sigma}^{d,(1)} & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots
\end{bmatrix}$$

where $T_{\mu, \sigma}^{d,(n)}: E^{\otimes n} \otimes \sigma \mathcal{E} \to E^{\otimes n+1} \otimes \sigma \mathcal{E}$ is given by

$$T_{\mu, \sigma}^{d,(n)}: \xi_n \otimes \cdots \otimes \xi_1 \otimes e \mapsto \xi_n \otimes \cdots \otimes \xi_1 \otimes \mu e$$

where as usual we make the identification

$$E^{\otimes n} \otimes (\sigma \mathcal{E}) = E^{\otimes n+1} \otimes \sigma \mathcal{E}.$$

Using the notation in (4.5) we can write $T_{\mu, \sigma}^{d} = I_{\mathcal{F}^2(E)} \otimes \mu$, where we identify $\mathcal{F}^2(E) \otimes \sigma \mathcal{E}$ with $\mathcal{F}^2(E)$, which makes sense because $\mu$ is an $\mathcal{A}$-module map. Also recall that $t_{\infty, \sigma}$ in (4.4) defines a $*$-representation of $\sigma(\mathcal{A})'$ on $\mathcal{F}^2(E, \sigma)$.

**Proposition 4.2.** An operator $R \in \mathcal{L}(\mathcal{F}^2(E, \sigma))$ is in $\mathcal{F}^\infty(E, \sigma)$ if and only if $R$ commutes with each of the operators $I_{\mathcal{F}(E)} \otimes \sigma b$ and $T_{\mu, \sigma}^{d}$ for $b \in \sigma(\mathcal{A})'$ and $\mu \in E^\sigma$. Consequently, the operator $R \in \mathcal{L}(\mathcal{F}^2(E, \sigma))$ with infinite block-matrix representation

$$R = [R_{i,j}]_{i,j=0,1,2,\ldots} \quad \text{where } R_{i,j}: E^{\otimes j} \otimes \sigma \mathcal{E} \to E^{\otimes i} \otimes \sigma \mathcal{E}$$

is in $\mathcal{F}^\infty(E, \sigma)$ if and only if $R$ is lower triangular ($R_{i,j} = 0$ for $i < j$) and for $i \geq j$ $R_{i,j}$ satisfies the following compatibility (Toeplitz-like) conditions:

$$R_{i,j}(I_{E^{\otimes j}} \otimes b) = (I_{E^{\otimes i}} \otimes b)R_{i,j} \quad \text{for all } b \in \sigma(\mathcal{A})', \quad (4.14)$$

$$R_{i+1,j+1}(I_{E^{\otimes j}} \otimes \mu) = (I_{E^{\otimes i}} \otimes \mu)R_{i,j} \quad \text{for all } \mu \in E^\sigma. \quad (4.15)$$

and hence, inductively,

$$R_{i,j}^{\mu} = (I_{E^{\otimes j-i}} \otimes \mu^j)R_{i,j,0}. \quad (4.16)$$
Proof. The first part follows from Theorem 3.9 of [33]. The second part is then a straightforward translation of these commutativity conditions to expressions involving the block entries. \( \square \)

Taking the cue from Proposition 4.2, we view elements \( R \) of \( \mathcal{F}^\infty(E, \sigma) \) as the analytic Toeplitz operators for this Fock-space/correspondence setting.

While it is in general not the case that \( R(T_{\mu, \sigma}^d)^* = (T_{\mu, \sigma}^d)^*R \) for \( R \in \mathcal{F}^\infty(E, \sigma) \) and \( \mu \in E^\sigma \), this is almost the case as is made precise in the following proposition.

**Proposition 4.3.** For \( R \in \mathcal{F}^\infty(E, \sigma) \) and \( \mu \in E^\sigma \), we have

\[
R(T_{\mu, \sigma}^d)^*|_{\oplus_{n=1}^\infty E^\otimes n \otimes E} = (T_{\mu, \sigma}^d)^*R|_{\oplus_{n=1}^\infty E^\otimes n \otimes E},
\]

or, in terms of matrix entries, we have inductively

\[
R_{i,j}(I_{E^\otimes i} \otimes \eta) = (I_{E^\otimes i} \otimes \eta)R_{i+1,j+1} \text{ for all } i, j = 0, 1, \ldots.
\]

for \( \eta = \mu^* \in (E^\sigma)^* \).

Proof. To prove that \((4.17)\) holds for all \( R \in \mathcal{F}^\infty(E, \sigma) \), it suffices to show that it holds for each \( R \) in the generating set \((4.13)\). We are thus reduced to showing that \((4.18)\) holds for all \( R \) of the special form \( \varphi_\infty(a) \) for an \( a \in \mathcal{A} \) and \( T_{\xi}^f \) for a \( \xi \in E \). This in turn is a routine calculation which we leave to the reader. \( \square \)

Suppose that we are given \( R \in \mathcal{F}^\infty(E, \sigma) \). We regard \( \mathcal{E} \) as a subspace of \( \mathcal{F}^2(E, \sigma) \) via the identification \( y \cong y \oplus 0 \oplus 0 \oplus \cdots \). Then the restriction of \( R \) to \( \mathcal{E} \) defines an operator from \( \mathcal{E} \) into \( \mathcal{F}^2(E, \sigma) \) where we have a point evaluation in \( \mathbb{D}((E^\sigma)^* \times \sigma(A)') \) defined in \((4.15)\). We may then define an operator \( R^\wedge(\eta) \in \mathcal{L}(\mathcal{E}) \) by

\[
R^\wedge(\eta)e = (Re)^\wedge(\eta, I_{\mathcal{E}}).
\]

Explicitly, we have

\[
R^\wedge(\eta) = \sum_{n=0}^\infty \eta^n R_{n,0} \in \mathcal{L}(\mathcal{E}).
\]

Note that, as a consequence of Proposition 4.2, the full function \( f(\eta, b) = (Re)^\wedge(\eta, b) \) is then determined from \( R^\wedge(\eta) \) and \( e \in \mathcal{E} \) according to

\[
(Re)^\wedge(\eta, b) = (\iota_{\infty, \sigma}(b)Re)(\eta, I_{\mathcal{E}}) = (R\iota_{\infty, \sigma}(b)e)(\eta, I_{\mathcal{E}}) = (Rbe)(\eta) = R^\wedge(\eta)(be)
\]

for \( \eta \in \mathbb{D}((E^\sigma)^*) \) and \( b \in \sigma(A)' \). This implies that if we would extend the point evaluation to \( \mathbb{D}((E^\sigma)^* \times \sigma(A)') \) by \( R^\wedge(\eta, b)e = (Re)^\wedge(\eta, b) \), the result would just give \( R^\wedge(\eta, b) = R^\wedge(\eta)b \).

It is of interest that this transform \( R \rightarrow R^\wedge(\cdot) \) is multiplicative.

**Proposition 4.4.**

1. Suppose that \( R \) and \( S \) are two elements of \( \mathcal{F}^\infty(E, \sigma) \). Then

\[
(RS)^\wedge(\eta) = R^\wedge(\eta)S^\wedge(\eta)
\]

for all \( \eta \in \mathbb{D}((E^\sigma)^*) \).

2. Suppose that \( R \) is an operator in \( \mathcal{F}^\infty(E, \sigma) \) and that \( f \) is an element of \( \mathcal{F}^2(E, \sigma) \). Then

\[
(Rf)^\wedge(\eta, b) = R^\wedge(\eta)f^\wedge(\eta, b)
\]

for all \( \eta \in \mathbb{D}((E^\sigma)^*) \) and \( b \in \sigma(A)' \).

Proof. Suppose that \( R = [R_{i,j}]_{i,j=0,1,\ldots} \) is an operator in \( \mathcal{F}^\infty(E, \sigma) \) and that \( f = \oplus_{j=0}^\infty f_j \) is an element of \( \mathcal{F}^2(E, \sigma) \). We first note that a special case of \((4.18)\) is

\[
R_{\ell, 0} \eta = (I_{E^\otimes \ell} \otimes \eta)R_{\ell+1, 1}.
\]

Iteration of \((4.18)\) in turn leads to

\[
R_{\ell, 0} \eta^j = I_{E^\otimes \ell} \otimes \eta^j R_{\ell+j, 1} : E^\otimes j \otimes \sigma \mathcal{E} \rightarrow E^\otimes \ell \otimes \sigma \mathcal{E}.
\]
Then we compute for \( \eta \in \mathbb{D}((E^\sigma)^*) \) and \( b \in \sigma(A)' \) that
\[
R^\wedge(\eta)f^\wedge(\eta, b) = \left( \sum_{\ell=0}^{\infty} \eta^\ell R_{\ell,0} \right) \left( \sum_{j=0}^{\infty} \eta^j (I_{E^\otimes j} \otimes b)f_j \right)
\]
\[
= \sum_{\ell,j=0}^{\infty} \eta^l R_{\ell,0} \eta^j (I_{E^\otimes j} \otimes b)f_j
\]
\[
= \sum_{\ell,j=0}^{\infty} \eta^{\ell+j} R_{\ell+j,j} (I_{E^\otimes j} \otimes b)f_j \tag{4.20}
\]
\[
= \sum_{\ell,j=0}^{\infty} \eta^{\ell+j} (I_{E^\otimes j} \otimes b)R_{\ell+j,j} f_j \tag{by 4.14}
\]
\[
= \sum_{n=0}^{\infty} \eta^n (I_{E^\otimes n} \otimes b) \left( \sum_{j=0}^{n} R_{n,j} f_j \right)
\]
\[
= \sum_{n=0}^{\infty} \eta^n (I_{E^\otimes n} \otimes b)[Rf]_n = (Rf)^\wedge(\eta, b)
\]

and part (2) of the Proposition follows. Part (1) follows as the special case where \( b = I_E \) and \( f = Se \) for arbitrary \( e \in \mathcal{E} \).

**Remark 4.5.** We note that a consequence of the formula \(4.19\) is that the operator \( M_{R^\wedge} \) of multiplication by \( R^\wedge \) on \( H^2(E, \sigma) \)
\[
M_{R^\wedge} : f^\wedge(\eta, b) \mapsto R^\wedge(\eta)f^\wedge(\eta, b)
\]
commutes with the \( \sigma(A)' \)-left action on \( H^2(E, \sigma) \):
\[
M_{R^\wedge}(b \cdot f^\wedge) = b \cdot M_{R^\wedge} f^\wedge \quad \text{where} \quad (b \cdot f)^\wedge(\eta', b') = f^\wedge(\eta', b') \quad \text{for all} \quad b, b' \in \sigma(A)' \quad \text{and} \quad \eta' \in \mathbb{D}((E^\sigma)^*) .
\]
This can also be seen as a consequence of applying the \( Z \)-transform to the identity
\[
R_{t, \infty, \sigma}(b) = t_{\infty, \sigma}(b)R \quad \text{for all} \quad b \in \sigma(A)'.
\]

Proposition 4.4 leads immediately to the following corollary.

**Corollary 4.6.**

1. The kernel of the Fourier transform \( \Phi : f \rightarrow f^\wedge \) in \( \mathcal{F}^2(E, \sigma) \)
\[
\text{Ker } \Phi = \{ f \in \mathcal{F}^2(E, \sigma) : f^\wedge(\eta, b) = 0 \quad \text{for all} \quad \eta \in \mathbb{D}((E^\sigma)^*) \quad \text{and} \quad b \in \sigma(A)' \}
\]
is invariant under the analytic Toeplitz operators:
\[
f^\wedge(\eta, b) = 0 \quad \text{for all} \quad \eta \in \mathbb{D}((E^\sigma)^*) \quad \text{and} \quad b \in \sigma(A)', \quad R \in \mathcal{F}^\infty(E, \sigma)
\]
\[
\implies (Rf)^\wedge(\eta, b) = 0 \quad \text{for all} \quad \eta \in \mathbb{D}((E^\sigma)^*) \quad \text{and} \quad b \in \sigma(A)' .
\]

2. The initial space \( [\mathcal{F}^2(E, \sigma)]_{\text{initial}} \) of the Fourier transform \( \Phi \) is invariant under the adjoints of the analytic Toeplitz operators:
\[
f \in [\mathcal{F}^2(E, \sigma)]_{\text{initial}}, \quad R \in \mathcal{F}^\infty(E, \sigma) \implies R^* f \in [\mathcal{F}^2(E, \sigma)]_{\text{initial}} .
\]

Explicitly, the action of \( R^* \) on a generic vector in the spanning set \(4.12\) for \( [\mathcal{F}^2(E) \otimes \mathcal{E}]_{\text{initial}} \) is given by
\[
R^*(b^* \cdot k_{E, \sigma, \eta})e = b^* \cdot k_{E, \sigma, \eta} R^\wedge(\eta)^* e .
\]

**Proof.** If \( f^\wedge(\eta, b) = 0 \) for all \( \eta \) and \( b \), then, by (4.19) we see immediately that
\[
(Rf)^\wedge(\eta, b) = R^\wedge(\eta)f^\wedge(\eta, b) = 0
\]
for all \( \eta \) and \( b \) as well as for any \( R \in \mathcal{F}^\infty(E, \sigma) \). The first part of the second statement then follows by simply taking adjoints.
To verify the second part of the second statement, it suffices to verify on the generators $R = T_\xi$ and $R = \varphi(a)$ for $\xi \in E$ and $a \in A$; this in turn is straightforward.

\[ \square \]

**Remark 4.7.** We note that the definition of $R^\circ(\eta)$ involves only the first column of $R$. From the relations (4.16) and (4.14) one can see that the first column of $R$ already uniquely determines the action of $R$ on all of $[\mathcal{F}(E, \sigma)]_{\text{initial}}$.

**Remark 4.8.** Let $\mu \in E^\sigma$ and $\eta \in (E^\sigma)^*$ and $b \in (\sigma(A))'$. Then an easy verification using the relations $\mu \sigma(a) = \varphi(a) * I_E \mu$ and $\sigma(a) \eta = \eta \varphi(a) * I_E$ shows that

\[
\eta(I_E \otimes b) \mu \in \sigma(A)'.
\]

This observation has several consequences.

1. Given $\mu \in E^\sigma$ and $\eta \in (E^\sigma)^*$ we may define a mapping $\theta_{\eta, \mu}$ on $\sigma(A)'$ by

\[
\theta_{\eta, \mu}(b) = \eta(I_E \otimes b) \mu.
\]

Iteration of this map gives

\[
\theta_{\eta, \mu}^2(b) = \eta(I_E \otimes \eta(I_E \otimes b) \mu) = \eta^2(I_{E^\otimes 2} \otimes b) \mu^2
\]

and more generally

\[
\theta_{\eta, \mu}^n(b) = \eta^n(I_{E^\otimes n} \otimes b) \mu^n
\]

where we make use of the generalized power $\eta^n$ for an element $\eta$ of $(E^\sigma)^*$ (and set $\mu^n = ((\mu)^*)^n : E \to E^\otimes n \otimes_\sigma E$). For $\eta, \zeta \in \mathbb{D}((E^\sigma)^*)$, we may take $\mu = \zeta^*$ and then we have $\|\theta_{\eta, \zeta^*}\| < 1$. Then we may use the geometric series to compute the inverse of $I - \theta_{\eta, \zeta^*}$ to get

\[
(I - \theta_{\eta, \zeta^*})^{-1}(b) = \sum_{n=0}^\infty (\theta_{\eta, \zeta^*})^n(b) = \sum_{n=0}^\infty \eta^n(I_{E^\otimes n} \otimes b)(\zeta^n)^*.
\]

We conclude that the Szegö kernel (4.11) can also be written as

\[
\mathbb{K}_E,\sigma(\eta, \zeta)[b] = (I - \theta_{\eta, \zeta^*})^{-1}(b).
\]

This is the form of the Szegö kernel used in [33, 36].

2. Suppose that we are given two elements $\eta, \zeta \in E^\sigma$. The special case of (4.21) with $b = I_E$ and $\eta = \mu^*$ for a $\mu' \in E^\sigma$ enables us to define a $\sigma(A)'$-valued inner product on $E^\sigma$:

\[
(\mu, \mu')_{E^\sigma} = \mu^\sigma \mu \in \sigma(A)' \text{ for } \mu, \mu' \in E^\sigma.
\]

Moreover one can check that $E^\sigma$ has a well-defined right $\sigma(A)'$-action

\[
(\mu \cdot b)(e) = \mu(be)
\]

and a well-defined left $\sigma(A)'$-action

\[
(b \cdot \mu)(e) = (I_E \otimes b) \mu(e).
\]

It is then straightforward to check that $E^\sigma$ is a $\sigma(A)'$-correspondence. This observation plays a key role in the duality theory in [33] (see also Proposition 4.22 above).

Next we introduce the space

\[
H^\infty(E, \sigma) = \{ R^\wedge : R \in \mathcal{F}^\infty(E, \sigma) \},
\]

where we interpret $R^\wedge$ as a function mapping $\mathbb{D}((E^\sigma)^*)$ into $\mathcal{L}(E)$. Then $H^\infty(E, \sigma)$ is closed under addition ($(R_1 + R_2)^\wedge = R_1^\wedge + R_2^\wedge$), scalar multiplication ($(\lambda R)^\wedge = \lambda R^\wedge$) and pointwise multiplication (Proposition 4.1(1)). Moreover, part (2) of Proposition 4.4 implies that a function $S \in H^\infty(E, \sigma)$ defines a multiplication operator $M_S$ on $H^2(E, \sigma)$ by

\[
(M_S f^\wedge)(\eta, b) = S(\eta)f^\wedge(\eta, b) \text{ for each } \eta \in \mathbb{D}((E^\sigma)^*), b \in \sigma(A)' , f^\wedge \in H^2(E, \sigma).
\]

In fact, we have the following result.
Proposition 4.9. A function $S : \mathbb{D}((E^\sigma)^*) \to \mathcal{L}(\mathcal{E})$ is in $H^\infty(E, \sigma)$ if and only if $S$ defines a multiplication operator $M_S$ on $H^2(E, \sigma)$ by $(4.22)$. In case $S \in H^\infty(E, \sigma)$, we have $\|M_S\| \leq \|R\|$ for each $R \in F^\infty(E, \sigma)$ with $S = R^\wedge$ and there exists a $R \in F^\infty(E, \sigma)$ with $S = R^\wedge$ such that $\|M_S\| = \|R\|$. Moreover, if $S \in H^\infty(E, \sigma)$, then $M_S$ is a $\sigma(A)$-module map that in addition commutes with the operators

$$\Phi(I_{F(E)} \otimes \mu)\Phi^*$$

for each $\mu \in \sigma(E)$.

Here $\Phi$ is the coisometry from $F^2(E, \sigma)$ into $H^2(E, \sigma)$ given by $\Phi : f \mapsto f^\wedge$.

Proof. We already observed that $S \in H^\infty(E, \sigma)$ guarantees that $M_S$ in $(4.22)$ defines a multiplication operator on $H^2(E, \sigma)$. Moreover, for $R \in F^\infty(E, \sigma)$ with $S = R^\wedge$ we have

$$(M_S\Phi f)(\eta, b) = (M_Sf^\wedge)(\eta, b) = S(\eta)f^\wedge(\eta, b) = R^\wedge(\eta)f(\eta, b) = (RF)(\eta, b) = \Phi(Rf)(\eta, b)$$

for each $f \in F^2(E, \sigma)$, $\eta \in \mathbb{D}((E^\sigma)^*)$ and $b \in \sigma(A)'$. Hence

$$M_S\Phi = \Phi R.$$

In particular we have $M_S = \Phi R \Phi^*$ and thus $\|M_S\| \leq \|R\|$ since $\Phi$ is a coisometry.

Now assume that $S$ defines a multiplication operator $M_S$ on $H^2(E, \sigma)$ by $(4.22)$. The definition of $M_S$ and of the left action on $H^2(E, \sigma)$ in $(4.10)$ shows that, for $b, b' \in \sigma(A)'$ and $\eta \in \mathbb{D}((E^\sigma)^*)$, we have

$$(M_Sb'f^\wedge)(\eta, b) = S(\eta)f^\wedge(\eta, bb') = (b'M_Sf^\wedge)(\eta, b) \quad \text{for each } f^\wedge \in H^2(E, \sigma).$$

Hence $M_S$ is a $\sigma(A)'$-module map.

We now show that there exists $R \in F^\infty(E, \sigma)$ with $R^\wedge = S$. We first note that

$$((I_{F^2(E)} \otimes \mu)f)^\wedge(\eta, b) = \sum_{n=1}^\infty \eta^n(I_{E^\otimes n} \otimes b)(I_{E^\otimes n-1} \otimes \mu)f_{n-1}$$

$$= \sum_{n=1}^\infty \eta^{n-1}(I_{E^\otimes n-1} \otimes \eta(I_E \otimes b)\mu)f_{n-1}$$

$$= f^\wedge(\eta, \eta(I_E \otimes b)\mu)$$

(4.23)

and where we use the observation from Remark 4.8 that $\eta(I_E \otimes b)\mu$ is in $\sigma(A)'$.

From $(4.23)$, it readily follows that $I_{F^2(E)} \otimes \mu$ on $F^2(E, \sigma)$ leaves $\text{Ker } \Phi$ invariant. The same holds for the operator $I_{F^2(E)} \otimes b$. Consequently, denoting by $P(= \Phi^*\Phi)$ the projection on $\mathcal{G} = (\text{Ker } \Phi)^\perp$, we note that

$$PX = PXP \quad \text{for } X = I_{F^2(E)} \otimes \mu, I_{F^2(E)} \otimes b.$$

We show that the operator $\Phi^*M_S\Phi$ commutes with $I_{F^2(E)} \otimes b'$ for all $b' \in \sigma(A)'$. To see this, let $f \in F^2(E, \sigma)$ and $\Phi^*M_S\Phi(I_{F^2(E)} \otimes b')f = g$. Due to $(4.24)$, we have $g^\wedge(\eta, b) = S(\eta, bb')$. Now if we let $\Phi^*M_S\Phi f = h$, it follows that $h^\wedge(\eta, b) = S(\eta)f^\wedge(\eta, b)$ and consequently

$$((I_{F^2(E)} \otimes b')\Phi^*M_S\Phi f)^\wedge(\eta, b) = S(\eta)f^\wedge(\eta, bb')$$

and the claim follows. A similar computation using $(4.24)$ shows that $P(I_{F^2(E)} \otimes \mu)A = AP(I_{F^2(E)} \otimes \mu)\mathcal{G}$ for all $\mu \in E^\sigma$, where $A = \Phi^*M_S\Phi(\mathcal{G})$.

We recall now that the maps $\mu \in E^\sigma, b \in \sigma(A)'$ form an isometric covariant representation of the $\sigma(A)'$-correspondence $E^\sigma$ (see pages 369-370 in [33]—the precise definition is covariant representation is given in the text surrounding formulas $(1.20)$–$(1.28)$ below). We may now apply the commutant lifting theorem for covariant representations of a correspondence due to Muhly-Solel (see Theorem 4.4, [31]) to obtain an operator that commutes with the operators $I_{F^2(E)} \otimes \mu$ and $I_{F^2(E)} \otimes b$ (which implies $R \in F^\infty(E)$ by Proposition 4.2) which moreover satisfies $PR = AP$. This immediately implies that $R^\wedge = S$. Furthermore, we can choose $R$ such that $\|R\| = \|M_S\|$.

We note that any $R \in F^\infty(E, \sigma)$ is of the form $\widetilde{R} \otimes I_E$ for a $\widetilde{R} \in F^\infty(E)$. Moreover, the map $\widetilde{R} \mapsto R = \widetilde{R} \otimes I_E$ is an $\mathcal{L}(F^2(E, \sigma))$-valued representation of $F^\infty(E)$ which actually extends to a $\ast$-representation $T \mapsto T \otimes I_E$ of all of $\mathcal{L}^\infty(F^2(E))$—the restriction of $T \mapsto T \otimes I_E$ to $T \in F^\infty(E)$ is called the induced representation of $F^\infty(E)$ in the terminology of [31] [33]. The content of Proposition 4.4 is that, for each
\( \eta \in \mathbb{D}(\mathcal{E}^\sigma) \), the map \( R \mapsto R^\wedge(\eta) \) is an \( \mathcal{L}(\mathcal{E}) \)-valued representation of \( \mathcal{F}^\infty(E, \sigma) \). It follows that the composition

\[
\pi_\eta(R) = (\hat{R} \otimes I_\mathcal{E})^\wedge(\eta)
\]

is an (even completely contractive) representation of \( \mathcal{F}^\infty(E) \) (see [33]). For some \( \eta \in (\mathcal{E}^\sigma)^* \) of norm equal to 1, \( \pi_\eta \) still defines a representation of \( \mathcal{F}(E) \). It is the case that every \( \eta \) in the closed unit ball of \( (\mathcal{E}^\sigma)^* \) gives rise to a completely contractive representation of \( \mathcal{T}_\eta(E) \) (the norm-closure of the span of left multipliers \( \varphi_\infty(a) \) \((a \in \mathcal{A}) \) and creation operators \( T_\xi(\xi \in E) \) in \( \mathcal{L}^\infty(F^2(E)) \)), while it is not clear for which \( \eta \) the representation can be extended to \( \mathcal{F}^\infty(E) \)—this is one of the open problems in the theory (see [33]). It is the case that each completely contractive representation \( \pi \) of \( \mathcal{F}^\infty(E) \) comes from an \( \eta \in \overline{\mathbb{D}}((\mathcal{E}^\sigma)^*) \) for some weak-\( \ast \) continuous \( \ast \)-representation \( \sigma : \mathcal{A} \to \mathcal{L}(\mathcal{E}) \). Indeed, given a completely contractive representation \( \pi : \mathcal{F}^\infty(E) \to \mathcal{L}(\mathcal{E}) \), one can construct \( \sigma \) and \( \eta \) as follows. Define \( \sigma : \mathcal{A} \to \mathcal{L}(\mathcal{E}) \) by

\[
\sigma(a) = \pi(\varphi_\infty(a)).
\]

Then define \( \eta : E \to \mathcal{L}(\mathcal{E}) \) by

\[
\eta(\xi) = \pi(T_\xi).
\]

We wish to verify that \( \eta(\varphi(a)\xi \cdot a') = \sigma(a)\eta(\xi)\sigma(a') \), i.e., that the pair \((\eta, \sigma)\) is a covariant representation of \( E \) in the terminology of Muhly-Sole [31, 33]. As a first step for the verification of (4.28), one can easily check that

\[
T_{\varphi(a)\xi}a' = \varphi_\infty(a)T_\xi\varphi_\infty(a') \quad \text{for} \quad a \in \mathcal{A}, \xi \in E.
\]

We then compute

\[
\eta(\varphi(a)\xi \cdot a') = \pi(T_{\varphi(a)\xi}a') = \pi(\varphi_\infty(a)T_\xi\varphi_\infty(a')) = \pi(\varphi_\infty(a))\pi(T_\xi)\pi(\varphi_\infty(a')) = \sigma(a)\eta(\xi)\sigma(a')
\]

and (4.28) follows. As in [31], a covariant representation \((\eta, \sigma)\) of \( E \) determines an element \( \eta : E \otimes_\sigma \mathcal{E} \to \mathcal{E} \) of \( (\mathcal{E}^\sigma)^* \) according to the formula

\[
\eta(\xi \otimes e) = \eta(\xi)e.
\]

Here note that the property \( \eta(\varphi(a)\xi) = \sigma(a)\eta(\xi) \) is what is needed to verify that (4.24) is well-defined while the property \( \eta(\varphi(a)\xi) = \sigma(a)\eta(\xi) \) is what is needed to verify that \( \eta \) is in \( (\mathcal{E}^\sigma)^* \), i.e., that \( \eta \) has the \( \mathcal{A} \)-module-map property

\[
\eta(\varphi(a) \otimes I_\mathcal{E}) = \sigma(a)\eta.
\]

There is a converse: given an element \( \eta \in \mathbb{D}((\mathcal{E}^\sigma)^*) \), we may use (4.29) to define \( \eta \) so that \((\eta, \sigma)\) is a completely contractive covariant representation of \( E \). The mapping \( \pi \) given in (4.26)—(4.27) then extends to a representation of \( \mathcal{F}^\infty(E) \) (see [31]). For our situation here where \((\eta, \sigma)\) is given in terms of a representation \( \pi \) via (4.26)—(4.27), we see that a representation \( \pi \) of \( \mathcal{F}^\infty(E) \) determines a representation \( \sigma = \sigma_\pi \) of \( \mathcal{A} \) according to (4.26) along with an element \( \eta_\pi \) of \( (\mathcal{E}^\sigma)^* \) according to the formula

\[
\eta_\pi(\xi \otimes e) = \pi(T_\xi)e.
\]

It is then straightforward to check that the formula

\[
\pi(\hat{R}) = (\hat{R} \otimes I_\mathcal{E})^\wedge(\eta_\pi)
\]

holds for the cases where

\[
\hat{R} = \varphi_\infty(a) \quad \text{for some} \quad a \in \mathcal{A}, \quad \hat{R} = T_\xi \quad \text{for some} \quad \xi \in E.
\]

Under the assumption that \( \pi \) is continuous with respect to the weak-\( \ast \) topologies on \( \mathcal{F}^\infty(E) \) and \( \mathcal{L}(\mathcal{E}) \), it then follows that (4.30) holds for all \( \hat{R} \in \mathcal{F}^\infty(E) \), i.e., we recover \( \pi \) as \( \pi = \pi_\eta \) where in general \( \pi_\eta \) is given by (4.25).

It is of interest to apply this construction to the induced representation

\[
\pi_{\text{ind}} : \tilde{R} \mapsto \tilde{R} \otimes I_\mathcal{E}
\]
of \( \mathcal{F}^\infty(E) \) into \( \mathcal{L}(\mathcal{F}^2(E, \sigma)) \). We collect this result in the following Proposition.

**Proposition 4.10.** Suppose that we are given an \( \mathcal{A} \)-correspondence \( E \) together with a representation \( \sigma: \mathcal{A} \to \mathcal{L}(\mathcal{E}) \) for a Hilbert space \( \mathcal{E} \) and let \( \pi_{\text{ind}}: \mathcal{F}^\infty(\mathcal{E}) \to \mathcal{L}(\mathcal{F}^2(E, \sigma)) \) be the induced representation as in (4.31). Define \( \eta_{\text{ind}}: E \to \mathcal{L}(\mathcal{F}^2(E, \sigma)) \) and \( \sigma_{\text{ind}}: \mathcal{A} \to \mathcal{L}(\mathcal{F}^2(E, \sigma)) \) by

\[
\eta_{\text{ind}}(\xi) = T_\xi, \quad \sigma_{\text{ind}}(a) = \varphi_{\infty, \sigma}(a).
\]

Then \( (\eta_{\text{ind}}, \sigma_{\text{ind}}) \) is an (isometric) covariant representation of \( E \) with \( \eta_{\text{ind}}: E \otimes \mathcal{F}^2(E, \sigma) \to \mathcal{F}^2(E, \sigma) \) of \( (\mathcal{F}^\infty)^* \) associated with \( (\eta_{\text{ind}}, \sigma_{\text{ind}}) \) as in (4.29) given by

\[
\eta_{\text{ind}}: \xi \otimes \left[ \sum_{n=0}^{\infty} \xi(n) \otimes e_n \right] \to 0 \oplus \left[ \sum_{n=1}^{\infty} \xi \otimes \xi(n-1) \otimes e_{n-1} \right].
\]

Moreover, we recover \( \mathcal{R} = \mathcal{R} \otimes I_\mathcal{E} \in \mathcal{F}^\infty(E, \sigma) \) via the point evaluation

\[
\mathcal{R} \otimes I_\mathcal{E} = (\mathcal{R} \otimes I_{\mathcal{F}^2(E, \sigma)})^\wedge(\eta_{\text{ind}}).
\]

**Proof.** The proof is a simple specialization of the general construction sketched in the paragraph preceding the statement of the proposition. \( \square \)

It will be convenient to work also with the analytic Toeplitz operators acting between \( H^2(E, \sigma) \)-spaces of different multiplicity. For this purpose, we suppose that \( \mathcal{U} \) and \( \mathcal{Y} \) are two additional auxiliary Hilbert spaces (to be thought of as an input space and output space respectively). We consider higher multiplicity versions of \( H^2(E, \sigma) \) by tensoring with an auxiliary Hilbert space (which is to be thought of as adding multiplicity):

\[
H^2_0(E, \sigma) := H^2(E, \sigma) \otimes \mathcal{U}, \quad H^2_\infty(E, \sigma) := H^2(E, \sigma) \otimes \mathcal{Y}.
\]

Here we view \( \mathcal{U} \) and \( \mathcal{Y} \) as \((\mathbb{C}, \mathbb{C})\)-correspondences and apply the tensor-product construction of Definition 3.1 (2). The space \( H^2_0(E, \sigma) \) then is a reproducing kernel \( (\sigma(A)' \otimes \mathcal{U}, \mathcal{L}(\mathcal{E} \otimes \mathcal{U})) \)-correspondence on \( \mathbb{D}(E^\infty) \) where the point evaluation at a point \((\eta, b) \in \mathbb{D}(E^\infty) \times \sigma(A)\)' of a function \( f^\wedge \otimes u \in H^2_0(E, \sigma) \) (with \( f^\wedge \in H^2(E, \sigma) \) and \( u \in \mathcal{U} \)) is given by \((f^\wedge \otimes u)(\eta, b) = f^\wedge(\eta, b) \otimes u \in \mathcal{E} \otimes \mathcal{U} \). Moreover, note that the left \( \sigma(A)' \)-action is given by \( b \mapsto b \circ I_\mathcal{U} \). The completely positive kernel \( K_{(E, \sigma) \otimes \mathcal{U}} \) associated with it as in Theorem 4.9 is given by

\[
K_{(E, \sigma) \otimes \mathcal{U}}(\eta, \zeta)[b] = K_{E, \sigma}(\eta, \zeta)[b] \otimes I_\mathcal{U},
\]

where \( K_{E, \sigma} \) denotes the kernel for \( H^2(E, \sigma) \) defined in Theorem 4.1. Similar statements hold for \( H^2_\infty(E, \sigma) \), where the analogous kernel is denoted by \( K_{(E, \sigma) \otimes \mathcal{Y}} \).

We now define a higher-multiplicity version of the algebra of analytic Toeplitz operators \( H^\infty(E, \sigma) \) to be the linear space

\[
H^\infty_0(\mathcal{L}(\mathcal{U}, \mathcal{Y}), (E, \sigma)) := H^\infty(E, \sigma) \otimes \mathcal{L}(\mathcal{U}, \mathcal{Y}).
\]

This space consists of \( \mathcal{L}(\mathcal{E} \otimes \mathcal{U}, \mathcal{Y}) \)-valued functions on \( \mathbb{D}(E^\infty) \) with point evaluation of an element \( \mathcal{S} \otimes \mathcal{N} \in H^\infty_0(\mathcal{L}(\mathcal{U}, \mathcal{Y}), (E, \sigma)) \) at \( \eta \in \mathbb{D}(E^\infty) \) given by \((\mathcal{S} \otimes \mathcal{N})(\eta) = \mathcal{S}(\eta) \otimes \mathcal{N} \). Moreover, the functions in \( H^\infty_0(\mathcal{L}(\mathcal{U}, \mathcal{Y}), (E, \sigma)) \) define multiplication operators in \( \mathcal{L}(H^2_0(E, \sigma), H^2_\infty(E, \sigma)) \), in the same way as \( H^\infty(E, \sigma) \). For \( \mathcal{S} \otimes \mathcal{N} \in H^\infty_0(\mathcal{L}(\mathcal{U}, \mathcal{Y}), (E, \sigma)) \), \( \mathcal{S} \in H^\infty(E, \sigma) \) and \( \mathcal{N} \in \mathcal{L}(\mathcal{U}, \mathcal{Y}) \), the multiplication operator \( M_{\mathcal{S} \otimes \mathcal{N}} \) becomes \( M_{\mathcal{S} \otimes \mathcal{N}} = M_{\mathcal{S}} \otimes \mathcal{N} \).

In addition there are Fock space versions of all these spaces, namely

\[
\mathcal{F}_0^2(E, \sigma) := \mathcal{F}^2(E, \sigma) \otimes \mathcal{U}, \quad \mathcal{F}_\infty^2(E, \sigma) := \mathcal{F}^2(E, \sigma) \otimes \mathcal{Y}, \quad \mathcal{F}_0^\infty(E, \sigma) := \mathcal{F}^\infty(E, \sigma) \otimes \mathcal{U} \otimes \mathcal{Y}.
\]

Point evaluation for elements in \( \mathcal{F}_0^2(E, \sigma) \) and points in \( \mathbb{D}(E^\infty) \) is determined by attaching to \( f \otimes u \in \mathcal{F}_0^2(E, \sigma) \), \( f \in \mathcal{F}^2(E, \sigma) \) and \( u \in \mathcal{U} \), and \( (\eta, b) \in \mathbb{D}(E^\infty) \times \sigma(A)' \) the value \( (f \otimes u)(\eta, b) = f^\wedge(\eta, b) \otimes u \), so that the map

\[
\Phi_{\mathcal{U}}: u \mapsto f^\wedge \otimes u \quad \text{for} \quad f^\wedge \in \mathcal{F}_0^2(E, \sigma)
\]

defines a coisometry from \( \mathcal{F}_0^2(E, \sigma) \) onto \( H_0^2(E, \sigma) \). The analogous coisometry for \( \mathcal{F}_\infty^2(E, \sigma) \) is denoted by \( \Phi_\mathcal{Y} \).

Similarly we determine point evaluation for elements in \( \mathcal{F}_0^\infty(E, \sigma) \) and points in \( \mathbb{D}(E^\infty) \) by attaching to \( R \otimes X \in \mathcal{F}_0^\infty(E, \sigma) \), \( R \in \mathcal{F}^\infty(E, \sigma) \) and \( X \in \mathcal{L}(\mathcal{U}, \mathcal{Y}) \), and \( \eta \in \mathbb{D}(E^\infty) \) the value

\[
(R \otimes X)(\eta)(\xi) := R(\eta)(\xi) \otimes X \in \mathcal{L}(\mathcal{E} \otimes \mathcal{U}, \mathcal{E} \otimes \mathcal{Y}).
\]

(4.32)
Then $H^\infty_{\mathcal{L}(U,Y)}(E,\sigma)$ is recovered as

$$\{R^\wedge : R \in \mathcal{F}^\infty_{\mathcal{L}(U,Y)}(E,\sigma)\},$$

where $R^\wedge$ should be interpreted as a function mapping $\mathbb{D}((E^*)^*)$ into $\mathcal{L}((E \otimes U, E \otimes Y))$, while the space of multiplication operators in $\mathcal{L}(H^\infty_{\mathcal{L}(U,Y)}(E,\sigma), H^\infty_{\mathcal{L}(U,Y)}(E,\sigma))$ defined by $H^\infty_{\mathcal{L}(U,Y)}(E,\sigma)$ is given by

$$\{\Phi_Y R \Phi_{Y*}^*: R \in \mathcal{F}^\infty_{\mathcal{L}(U,Y)}(E,\sigma)\}.$$

In fact, it is easy to check that Proposition 4.9 guarantees that for $S \in H^\infty_{\mathcal{L}(U,Y)}(E,\sigma)$ we have $M_S = \Phi_Y R \Phi_{Y*}^*$ whenever $R \in \mathcal{F}^\infty_{\mathcal{L}(U,Y)}(E,\sigma)$ satisfies $S = R^\wedge$, so that $\|M_S\| \leq \|R\|$, and that there exists a $R \in \mathcal{F}^\infty_{\mathcal{L}(U,Y)}(E,\sigma)$ with $S = R^\wedge$ and $\|M_S\| = \|R\|$.

Alternatively, $\mathcal{F}^\infty_{\mathcal{L}(U,Y)}(E,\sigma)$ can be characterized as bounded operators from $\mathcal{F}^2_1(E,\sigma)$ to $\mathcal{F}^2_2(E,\sigma)$ with block-matrix representation

$$R = [R_{i,j}]_{i,j=0,1,...} \quad \text{with} \quad R_{i,j} : E^{\otimes j} \otimes \sigma E \otimes U \to E^{\otimes i} \otimes \sigma E \otimes Y$$

subject to

$$R_{i,j}(I_{E^{\otimes i}} \otimes b \otimes I_U) = (I_{E^{\otimes j}} \otimes b \otimes I_Y)R_{i,j} \text{ for } b \in \sigma(A)^\prime,$$

$$R_{i+1,j+1}(I_{E^{\otimes i}} \otimes \eta^* \otimes I_U) = (I_{E^{\otimes j}} \otimes \eta^* \otimes I_Y)R_{i,j} \text{ for } \eta^* \in E^\sigma.$$

For such $R \in \mathcal{F}^\infty_{\mathcal{L}(U,Y)}(E,\sigma)$ point evaluation in $\eta \in \mathbb{D}((E^*)^*)$ can be written as

$$R^\wedge(\eta) = \sum_{n=0}^\infty (\eta^\wedge \otimes I_Y)R_{n,0}.$$

In addition it is routine to see that part (1) of Proposition 4.3 can be extended to the following statement: if $S \in \mathcal{F}^\infty_{\mathcal{L}(U,Y)}(E,\sigma)$ and $R \in \mathcal{F}^\infty_{\mathcal{L}(Y,Z)}(E,\sigma)$, then $RS \in \mathcal{F}^\infty_{\mathcal{L}(U,Z)}(E,\sigma)$ and

$$(RS)^\wedge(\eta) = R^\wedge(\eta)S^\wedge(\eta).$$

**Remark 4.11.** Suppose that $R \in \mathcal{F}^\infty_{\mathcal{L}(U,Y)}(E,\sigma)$ has the form

$$R = \tilde{R} \otimes_\sigma I_E \otimes X$$

(4.34)

where $\tilde{R} \in \mathcal{F}^\infty(E)$ and $X \in \mathcal{L}(U,Y)$. In particular the point evaluation (4.32) defines $R^\wedge(\eta) \in \mathcal{L}(E \otimes U, E \otimes Y)$ for each $\eta \in \mathbb{D}((E^*)^*)$. Suppose now that $\sigma' : A \to \mathcal{L}(E')$ is another $*$-representation of $\mathcal{A}$ and $\eta' \in \mathbb{D}((E^*)^*)$. Then we may define a related function $\eta'' \mapsto R^{\wedge'}(\eta'') \in \mathcal{L}(E' \otimes U, E' \otimes Y)$ by

$$R^{\wedge'}(\eta'') = (\tilde{R} \otimes_\sigma I_{E'} \otimes X)^\wedge(\eta'').$$

While not all elements $R$ of $\mathcal{F}^\infty_{\mathcal{L}(U,Y)}(E,\sigma)$ are of the special form (4.34), finite linear combinations of elements of the special form (4.34) are weak-$*$ dense in $\mathcal{F}^\infty_{\mathcal{L}(U,Y)}(E,\sigma)$. By using linearity and a limiting process, one can then make sense of $R^{\wedge'}(\eta'') \in \mathcal{L}(E' \otimes U, E' \otimes Y)$ for any $\eta'' \in \mathbb{D}((E^*)^*)$. This fact will be useful for the formulation of condition (1'\textsuperscript{\textdagger}) in Theorem 5.1 below.

5. The Schur class associated with $(E,\sigma)$

Given a correspondence-representation pair $(E,\sigma)$ (where $\sigma : \mathcal{A} \to \mathcal{L}(E)$) along with auxiliary Hilbert spaces $U$ and $Y$, we define the associated Schur class $S_{E,\sigma}(U,Y)$ by

$$S_{E,\sigma}(U,Y) = \{S : \mathbb{D}((E^*)^*) \to \mathcal{L}(E \otimes U, E \otimes Y) : S(\eta) = R^\wedge(\eta) \text{ for all } \eta \in \mathbb{D}((E^*)^*)$$

for some $R \in \mathcal{F}^\infty_{\mathcal{L}(U,Y)}(E,\sigma)$ with $\|R\| \leq 1\}.$

(5.1)

We have the following characterization of the Schur class $S_{E,\sigma}(U,Y)$ analogous to the characterization of the classical Schur class given in Theorem 1.1 and to the multivariable extensions in Theorems 2.1 and 2.3. When specialized to the classical case (see Section 6.1 below), (5.1) gives the classical Schur class as defined in the Introduction, but from a different point of view. Rather than simply holomorphic, contractive,
Then the following conditions are equivalent:

\[ (3) \]

We know that \( \eta \) is completely positive, or more explicitly, there exists an auxiliary Hilbert space \( U \) which defines a contraction operator from \( \ell^2_n(\mathbb{Z}_+) \) to \( \ell^2_n(\mathbb{Z}_+) \). Thus the label (1) in Theorem \[3.3\] when specialized to the classical case, corresponds to a somewhat different statement than (1) in Theorem \[1.1\]. The other labels (1'), (1''), (2) and (3) in Theorem \[5.1\] correspond exactly to the corresponding statements in Theorems \[1.1\] 2.1 and 2.3.

**Theorem 5.1.** Suppose that we are given a correspondence-representation pair \((E, \sigma)\) (where \( \sigma: A \to \mathcal{L}(E) \)) along with auxiliary Hilbert spaces \( U \) and \( Y \) and an operator-valued function \( S: \mathbb{D}((E^\sigma)^*) \to \mathcal{L}(E \otimes U, E \otimes Y) \). Then the following conditions are equivalent:

1. \( S \in \mathcal{E}_{E \otimes U}(U, Y) \), i.e., there exists an \( R \in \mathcal{F}_{E \otimes U}(E, \sigma) \) with \( \|R\| \leq 1 \) such that \( S(\eta) = R^\wedge(\eta) \) for all \( \eta \in \mathbb{D}((E^\sigma)^*) \).

1' The multiplication operator \( M_S : f(\eta, b) \mapsto S(\eta)f(\eta, b) \) maps \( H^2(E, \sigma) \) contractively into \( H^2(E, \sigma) \).

1'' \( S \) is such that \( S(\eta) = R^\wedge(\eta) \) for all \( \eta \in \mathbb{D}((E^\sigma)^*) \) for an \( R \in \mathcal{F}_{E \otimes U}(E, \sigma) \) with the additional property: for each representation \( \sigma': A \to \mathcal{L}(E') \) and \( \eta' \in \mathbb{D}((E')^*) \) it happens that

\[ \|R^\wedge(\eta')\| \leq 1, \]

where \( R^\wedge(\eta') \) is defined as in Remark \[4.11\].

2 The function \( K_S : \mathbb{D}((E^\sigma)^*) \to \mathbb{D}((E^\sigma)^*) \) defined by

\[ K_S(\eta, \zeta)[b] := K_{(E, \sigma) \otimes Y}(\eta, \zeta)[b] - S(\eta)K_{(E, \sigma) \otimes U}(\eta, \zeta)[b]S(\zeta)^* \]

is completely positive, or more explicitly, there exists an auxiliary Hilbert space \( H \), an operator-valued function \( H: \mathbb{D}((E^\sigma)^*) \to \mathcal{L}(H, E \otimes Y) \) and a \( * \)-representation \( \pi \) of \( \sigma(A)' \) on \( H \) so that

\[ \left( \sum_{n=0}^\infty \eta^n(I_{E^\otimes n} \otimes b)(\zeta^n)^* \right) \otimes I_Y - S(\eta) \left[ \sum_{n=0}^\infty \eta^n(I_{E^\otimes n} \otimes b)(\zeta^n)^* \right] \otimes I_U \] \[ S(\zeta)^* = H(\eta)\pi(b)H(\zeta)^* \] (5.2)

for all \( \eta, \zeta \in \mathbb{D}((E^\sigma)^*) \) and \( b \in \sigma(A)' \).

3 There exists an auxiliary Hilbert space \( H \), a \( * \)-representation \( \pi: \sigma(A)' \to \mathcal{L}(H) \), and a coisometric colligation

\[ U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} H' \end{bmatrix} \to \begin{bmatrix} E^\sigma \otimes H' \\ E \otimes Y \end{bmatrix} \] (5.3)

which is a \( \sigma(A)' \)-module map, i.e.,

\[ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \pi(b)h \\ (b \otimes I_U)u \end{bmatrix} = \begin{bmatrix} (I_E \otimes b) \otimes I_H & 0 \\ 0 & b \otimes I_Y \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix} \] (5.4)

for \( h \in H \) and \( u \in E \otimes U \), so that \( S \) can be realized as

\[ S(\eta) = D + C(I - L_\eta^* A)^{-1}L_\eta^* B. \] (5.5)

Here \( L_\eta: H \to E^\sigma \otimes H \) is given by

\[ L_\eta h = \eta^* \otimes h \text{ for each } h \in H. \]

**Proof.** Both (1) \( \implies \) (1') and (1') \( \implies \) (1) follow immediately after extending Proposition \[4.9\] to the case with the added multiplicity as mentioned at the end of Section \[4\].

(1) \( \implies \) (1''): Given \( \eta' \in \mathbb{D}((E')^*) \) (so \( \|\eta'\| < 1 \)), by the dilation result in \[33\] Theorem 2.13 (see also \[22\]) we know that \( \eta' \) has a dilation to an induced representation \( \eta_{ind}: \mathcal{F}^\infty(E) \to \mathcal{L}(\mathcal{F}^2(E, \sigma_{ind})) \) associated.
with a representation \( \sigma_{\text{ind}} : A \rightarrow \mathcal{L}(\mathcal{E}_{\text{ind}}) \). As \( R \) is contractive by assumption, it then follows that \( R^\wedge_{\text{ind}}(\eta_{\text{ind}}) \) is also contractive. Since \( \eta_{\text{ind}} \) is a dilation of \( \eta \), we then also have

\[
\|R^\wedge(\eta')\| = \|P_{E \otimes Y}R^\wedge_{\text{ind}}(\eta_{\text{ind}})|_{E \otimes Ud}\| \\
\leq \|R^\wedge_{\text{ind}}(\eta_{\text{ind}})\| = \|R\| \leq 1
\]

and (1″) follows.

(1″) \( \implies \) (2): This implication requires an adaptation of the GNS/HB construction to the setting of completely positive (rather than classical positive) kernels. If \( \mathbb{K}(\omega', \omega)[a] \) is a completely positive kernel, then

\[
K((\omega', a'), (\omega, a)) = \mathbb{K}(\omega, \omega')|a'^*a'|
\]

is a positive kernel in the classical sense on \( \Omega \times \mathcal{A} \). In this way one can reduce to the classical setting and adapt the GNS/HB construction in [23] to the situation here. We leave complete details for another occasion.

(1″) \( \implies \) (1′): Assume that \( R \in \mathcal{F}_{\mathcal{E}(\mathcal{U}, \mathcal{Y})}(E, \sigma) \) and that \( S = R^\wedge \). From Proposition [4.10] extended to the higher multiplicity setting, we see that we recover \( R \) via the point-evaluation calculus as

\[
R = R^\wedge(\eta_{\text{ind}}).
\]

Hence we also recover \( R \) as the strong limit

\[
R = \lim_{r \downarrow 1} R^\wedge(\eta_{\text{ind}}).
\]

The assumption (1″) tells us that

\[
\|R^\wedge(\eta_{\text{ind}})\| \leq 1
\]

for each \( r < 1 \). Hence \( \|R\| \leq 1 \).

(1′) \( \implies \) (2) Assume that \( M_S \) is as in \( (1') \). From the definitions we see that

\[
(b \cdot M_S f^\wedge)(\eta', b') = S(\eta')f^\wedge(\eta', b'b) = (M_S(b \cdot f^\wedge))(\eta', b')
\]

and hence any multiplication operator \( M_S \) is a \( \sigma(A)' \)-module map. The computation

\[
\langle M_S, b \cdot (k_{E, \sigma, \zeta} \otimes I_Y)(e \otimes y) \rangle_{H^2(E, \sigma)} = \langle b \cdot M_S f, (k_{E, \sigma, \zeta} \otimes I_Y)(e \otimes y) \rangle_{H^2(E, \sigma)}
\]

\[
= \langle M_S(b \cdot f), (k_{E, \sigma, \zeta} \otimes I_Y)(e \otimes y) \rangle_{H^2(E, \sigma)}
\]

\[
= \langle S(\zeta)(b \cdot f)(\zeta), e \otimes y \rangle_{E \otimes Y}
\]

\[
= \langle b \cdot f, (k_{E, \sigma, \zeta} \otimes I_U)S(\zeta)^* (e \otimes y) \rangle_{H^2(E, \sigma)}
\]

\[
= \langle f, b \cdot (k_{E, \sigma, \zeta} \otimes I_U)S(\zeta)^* (e \otimes y) \rangle_{H^2(U, E, \sigma)}
\]

shows that

\[
M_S^* : b \cdot (k_{E, \sigma, \zeta} \otimes I_Y)(e \otimes y) \mapsto b \cdot (k_{E, \sigma, \zeta} \otimes I_U)S(\zeta)^* (e \otimes y).
\]

Since \( \|M_S\| \leq 1 \) by assumption, for any finite collection of \( b_j \in \sigma(A) \), \( \zeta_j \in \mathbb{D}((E^\sigma)^*) \) and \( e_j \otimes y_j \in \mathcal{E} \otimes \mathcal{Y} \) \((j = 1, \ldots, N)\), we have

\[
\left\| \sum_{j=1}^n b_j \cdot (k_{E, \sigma, \zeta_j} \otimes I_Y)(e_j \otimes y_j) \right\|^2 - \left\| M_S^* \sum_{j=1}^n b_j \cdot (k_{E, \sigma, \zeta_j} \otimes I_Y)(e_j \otimes y_j) \right\|^2 \geq 0.
\]

Expanding out inner products and using [5.6] and the basic general identities

\[
(b' \cdot (k_{E, \sigma, \zeta} \otimes I_Y)(e' \otimes y'), b \cdot (k_{E, \sigma, \eta} \otimes I_Y)(e \otimes y))_{H^2(E, \sigma)} =
\]

\[
\langle \mathbb{K}_{(E, \sigma) \otimes Y}(\eta, \zeta)|b^*b'|(e' \otimes y'), e \otimes y\rangle_{E \otimes Y},
\]

\[
(b' \cdot (k_{E, \sigma, \zeta} \otimes I_U)S(\zeta)^* (e' \otimes y'), b \cdot (k_{E, \sigma, \eta} \otimes I_U)S(\eta)^* (e \otimes y))_{H^2(E)} =
\]

\[
\langle S(\eta)\mathbb{K}_{(E, \sigma) \otimes Y}(\eta, \zeta)|b^*b'|(\zeta)^* (e' \otimes y'), e \otimes y\rangle_{E \otimes Y}
\]

we see that the left hand side of [5.7] is equal to

\[
\sum_{i,j=1}^N \langle \mathbb{K}_E(\zeta_i, \zeta_j)|b_i^*b_j|(e_i \otimes y_j), e_i \otimes y_i\rangle_{E \otimes Y}
\]
and we conclude that $K_\mathcal{E}$ is a completely positive kernel as wanted. The characterization given in (5.2) follows from part (2) of Theorem 3.6.

(2) $\implies$ (3): The argument here is an adaptation of the proof of Theorem 3.5 in [36] to our setting. Assume that (2) holds. By Remark 4.8, the equality (5.2) can be rewritten as

$$
(I - \theta_{\eta,\zeta})^{-1}(b \otimes I_Y - S(\eta))(I - \theta_{\eta,\zeta})^{-1}(b \otimes I_d)S(\zeta)^* = H(\eta)\pi(b)H(\zeta)^*.
$$

Replace $b$ by $[I - \theta_{\eta,\zeta}](b) = b - \theta_{\eta,\zeta}(b)$ to rewrite this last expression as an Agler decomposition (see [1])

$$
b \otimes I_Y - S(\eta)(b \otimes I_d)S(\zeta)^* = H(\eta)\pi(b - \eta(I_E \otimes b)\zeta^*)H(\zeta)^*. 
$$

(5.8)

Rearranging and conjugating by two generic vectors $y$ and $y'$ in $\mathcal{E} \otimes \mathcal{Y}$ then gives us

$$
y^*H(\eta)\pi(b)H(\zeta)^*y' + y^*S(\eta)(b \otimes I_d)S(\zeta)^*y' = y^*H(\eta)\pi(\eta(I_E \otimes b)\zeta^*)H(\zeta)^*y' + y^*(b \otimes I_Y)y'.
$$

(5.9)

From Remark 4.8 we know that $E^\sigma$ is a $\sigma(A)'$-correspondence. We may also view the Hilbert space $\mathcal{H}$ as a $(\sigma(A)', \mathbb{C})$-correspondence with the left $\sigma(A)'$-action given by the representation $\pi$. We may then form the tensor-product $(\sigma(A)', \mathbb{C})$-correspondence $E^\sigma \otimes \mathcal{H}$ as in Definition 3.4. Explicitly, the $\mathbb{C}$-valued inner product on $E^\sigma \otimes \mathcal{H}$ is given by

$$
\langle \mu \otimes h, \mu' \otimes h' \rangle_{E^\sigma \otimes \mathcal{H}} = \langle \pi(\mu^* \mu)h, h' \rangle_{\mathcal{H}} = h'^*\pi(\mu^* \mu)h.
$$

It follows that the first term on the right-hand side of (5.9) can be written as

$$
y^*H(\eta)\pi(\eta(I_E \otimes b)\zeta^*)H(\zeta)^*y' = \langle (I_E \otimes b)\zeta^* \otimes H(\zeta)^*y', \eta^* \otimes H(\eta)\pi(\eta(I_E \otimes b)\zeta^*)H(\zeta)^*y' \rangle_{E^\sigma \otimes \mathcal{H}}.
$$

(5.10)

If we replace $b$ with $b'b'$ (where $b, b'$ are two elements of $\sigma(A)'$), use (5.10) and do some rearranging, we see that the equality (5.9) can be expressed in terms of inner products

$$
\langle \pi(b')H(\zeta)^*y', \pi(b\eta(I_E \otimes b)\zeta^*)H(\zeta)^*y' \rangle_{E^\sigma \otimes \mathcal{H}} + \langle (b' \otimes I_d)S(\zeta)^*y', (b \otimes I_d)S(\eta)^*y \rangle_{E^\sigma \otimes \mathcal{H}}
$$

$$
= \langle (I_E \otimes b')\zeta^* \otimes H(\zeta)^*y', (I_E \otimes b)\eta^* \otimes H(\eta)\pi(\eta(I_E \otimes b)\zeta^*)H(\zeta)^*y' \rangle_{E^\sigma \otimes \mathcal{H}} + \langle (b' \otimes I_Y)y', (b \otimes I_Y)y \rangle_{E^\sigma \otimes \mathcal{H}}.
$$

(5.11)

Introduce subspaces

$$
\mathcal{D}_V = \text{span} \left\{ \left[ (I_E \otimes b)\eta^* \otimes H(\eta)^*y \right], y \in \mathcal{E} \otimes \mathcal{Y}, \eta \in \mathbb{D}((E^\sigma)^*), b \in \sigma(A)' \right\} \subseteq \left[ E^\sigma \otimes \mathcal{H} \right]_{\mathcal{E} \otimes \mathcal{Y}}.
$$

$$
\mathcal{R}_V = \text{span} \left\{ \left[ \pi(b)H(\zeta)^*y \right], y \in \mathcal{E} \otimes \mathcal{Y}, \eta \in \mathbb{D}((E^\sigma)^*), b \in \sigma(A)' \right\} \subseteq \left[ \mathcal{H} \right]_{\mathcal{E} \otimes \mathcal{U}}.
$$

Note that both $\mathcal{D}_V$ and $\mathcal{R}_V$ are invariant under the left action of $\sigma(A)'$ on $(E^\sigma \otimes \mathcal{H}) \oplus (\mathcal{E} \otimes \mathcal{Y})$ and on $\mathcal{H} \oplus (\mathcal{E} \otimes \mathcal{U})$ respectively, i.e. $\mathcal{D}_V$ and $\mathcal{R}_V$ are $\sigma(A)'$-submodules of $(E^\sigma \otimes \mathcal{H}) \oplus (\mathcal{E} \otimes \mathcal{Y})$ and $\mathcal{H} \oplus (\mathcal{E} \otimes \mathcal{U})$ respectively. The import of (5.11) is that the formula

$$
V : \left[ (I_E \otimes b)\eta^* \otimes H(\eta)^*y \right] \mapsto \left[ \pi(b)H(\zeta)^*y \right]
$$

(5.12)

extends by linearity and continuity to a well-defined unitary operator from $\mathcal{D}_V$ onto $\mathcal{R}_V$. One easily checks that

$$
V(b \cdot d) = b \cdot Vd
$$

for $b \in \sigma(A)'$ and $d \in \mathcal{D}_V$.

By restricting in (5.12) to $b = I_E \in \sigma(A)'$ and $\eta = 0 \in \mathbb{D}((E^\sigma)^*)$ we see that $\{0\} \oplus (\mathcal{E} \otimes \mathcal{Y}) \subset \mathcal{D}_V$. In particular

$$
\mathcal{X} := ((E^\sigma \otimes \mathcal{H}) \oplus (\mathcal{E} \otimes \mathcal{Y})) \subset \mathcal{D}_V \subset (E^\sigma \otimes \mathcal{H}) \oplus \{0\}.
$$

Moreover, because $\mathcal{D}_V$ is invariant under the left $\sigma(A)'$-action we can see $\mathcal{X}$ as a $(\sigma(A)', \mathbb{C})$-correspondence, where the left action is obtained by restricting the left action on $E^\sigma \otimes \mathcal{H}$ to $\mathcal{X}$. Hence we can form the $(\sigma(A)', \mathbb{C})$-correspondence $K = \mathcal{H} \oplus (F^2(E^\sigma) \otimes \mathcal{X})$. Note that

$$
E^\sigma \otimes K = (E^\sigma \otimes \mathcal{H}) \oplus (E^\sigma \otimes F^2(E^\sigma) \otimes \mathcal{X}) = (E^\sigma \otimes \mathcal{H}) \oplus (E^\sigma \otimes F^2(E^\sigma) \otimes \mathcal{X}).
$$
So we can define an operator \( U \) from \( \mathcal{K} \oplus (\mathcal{E} \otimes \mathcal{U}) \) to \((E^\sigma \otimes \mathcal{K}) \oplus (\mathcal{E} \otimes \mathcal{Y})\) via

\[
\begin{bmatrix}
V P_{D_Y} & 0 & \cdots \\
0 & I_{E^\sigma \otimes \mathcal{K}} & \\
0 & 0 & \ddots
\end{bmatrix}
\begin{bmatrix}
(E^\sigma \otimes \mathcal{H}) \oplus (\mathcal{E} \otimes \mathcal{Y}) \\
E^\sigma \otimes \mathcal{X} \\
(E^\sigma)^{\otimes 2} \otimes \mathcal{X}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\mathcal{H} \oplus (\mathcal{E} \otimes \mathcal{U}) \\
E^\sigma \otimes \mathcal{X} \\
(E^\sigma)^{\otimes 2} \otimes \mathcal{X}
\end{bmatrix}.
\tag{5.13}
\]

Here \( P_{D_Y} \) and \( P_X \) stand for the projections onto \( D_Y \) and \( \mathcal{X} \) respectively. One easily checks that \( U^* \) is an isometric \( \sigma(A)' \)-module map. In other words, \( U \) is a coisometry, and a \( \sigma(A)' \)-module map. The construction in (5.13) is closely related to the dilation result in [31]; see also Section 3 in [34] for more details.

Next we decompose \( U \) as follows:

\[
U = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} : \begin{bmatrix}
\mathcal{H} \\
\mathcal{E} \otimes \mathcal{U}
\end{bmatrix} \rightarrow \begin{bmatrix}
E^\sigma \otimes \mathcal{H} \\
\mathcal{E} \otimes \mathcal{Y}
\end{bmatrix}.
\]

The definition of \( V \) and the construction of \( U \) imply that

\[
\begin{bmatrix}
A^* & C^* \\
B^* & D^*
\end{bmatrix}
\begin{bmatrix}
(I_X \otimes \eta^*) \circ H(\eta)^*y \\
(b \otimes I_Y)y
\end{bmatrix}
= \begin{bmatrix}
\pi(b) H(\eta)^*y \\
(b \otimes I_\mathcal{U}) S(\eta)^* y
\end{bmatrix}.
\]

By specifying this for \( b = I_X \) and observing that

\[
\eta^* \circ H(\eta)^* y = L_{\eta^*} H(\eta)^* y,
\]

we get

\[
A^* L_{\eta^*} H(\eta)^* + C^* = H(\eta)^* \quad \text{and} \quad B^* L_{\eta^*} H(\eta)^* + D^* = S(\eta)^*.
\tag{5.14}
\]

Moreover, for \( h \in \mathcal{H} \) we have

\[
\|L_{\eta^*} h\|^2 = \|\eta^* \otimes h\|^2 = \langle \eta^* \otimes h, \eta^* \otimes h \rangle = \langle \pi(\eta^*) \otimes h, h \rangle = \|\pi(\eta^*)\| \cdot \|h\|^2 \\
\leq \|\pi(\eta^*)\|^2 \cdot \|h\|^2 \cdot \|\eta\|^2 \leq \|\eta\|^2 \cdot \|h\|^2.
\]

This proves that \( \|L_{\eta^*}\| = \|\eta\| < 1 \). Hence \( I - A^* L_{\eta^*} \) is invertible and (5.14) shows that

\[
H(\eta)^* = (I_\mathcal{K} - A^* L_{\eta^*})^{-1} C^*.
\]

and thus,

\[
S(\eta)^* = D^* + B^* L_{\eta^*} (I_\mathcal{K} - A^* L_{\eta^*})^{-1} C^*.
\]

By taking adjoints we arrive at (5.5).

(3) \(\Rightarrow\) (2): Assume that (3) holds. We prove that \( \mathbb{K}_S \) admits an Agler decomposition as in (6.8) with \( H(\eta) := C(I - L_{\eta^*} A)^{-1} \). That this is equivalent to the complete positivity of the kernel \( \mathbb{K}_S \) can be seen via the change of variable used in the derivation of (5.8). The fact that \( U \) is a coisometric \( \sigma(A)' \)-module map can also be written as

\[
D(b \otimes I_Y) D^* + C \pi(b) C^* = b \otimes I_Y, \quad B(b \otimes I_\mathcal{U}) D^* = -A \pi(b) C^*, \\
A \pi(b) A^* + B(b \otimes I_\mathcal{U}) B^* = (I_E \otimes b) \otimes I_\mathcal{K}, \quad D(b \otimes I_\mathcal{U}) B^* = -C \pi(b) A^*.
\]

Note that

\[
H(\eta) = C(I - L_{\eta^*} A)^{-1} = C + C(I - L_{\eta^*} A)^{-1} L_{\eta^*} A = C + H(\eta) L_{\eta^*} A,
\]

and

\[
S(\eta) = D + H(\eta) L_{\eta^*} B.
\]
Hence
\[ H(\eta)\pi(b)H(\zeta)^* = (C + H(\eta)L_\eta^*)\pi(b)(C^* + A^*L_\zeta\cdot H(\zeta)^*) \]
\[ = C\pi(b)C^* + C\pi(b)A^*L_\zeta\cdot H(\zeta)^* + H(\eta)L_\eta^*A\pi(b)C^* \]
\[ + H(\eta)L_\eta^*A\pi(b)A^*L_\zeta\cdot H(\zeta)^* \]
\[ = b \otimes I^* - D(b \otimes I^*)D^* - (b \otimes I^*)B^*L_\zeta\cdot H(\zeta)^* \]
\[ - H(\eta)L_\eta^*B(b \otimes I^*)D^* - H(\eta)L_\eta^*B(b \otimes I^*)B^*L_\zeta\cdot H(\zeta)^* \]
\[ + H(\eta)L_\eta^*((I_E \otimes b) \otimes I_\mathcal{H})L_\zeta\cdot H(\zeta)^* \]
\[ = b \otimes I^* - S(\eta)(b \otimes I^*)S(\zeta)^* + H(\eta)\pi(\eta(I_E \otimes b)\zeta^*)H(\zeta)^*. \]

In this way we have proved that (5.18) holds.

(2) \implies (1'): Assume that (2) holds. Consider the formula
\[ (M_S)^*: b^* \cdot (k_{E,\sigma,\zeta} \otimes I_Y)Y \mapsto b^* \cdot (k_{E,\sigma,\zeta} \otimes I_U)S(\zeta)^*Y \quad (5.15) \]
for \( b \in \sigma(A)', \zeta \in \mathbb{D}((E^\sigma)^*) \) and \( Y \in \mathcal{E} \otimes \mathcal{Y} \). Then the complete positivity of the kernel \( K_S \) is exactly what is needed to see that the formula (5.15) can be extended by linearity and continuity to define a contraction operator \((M_S)^*\) from \( H^2_2(E, \sigma) \) into \( H^2_2(E, \sigma) \) which is also a \( \sigma(A)' \)-module map:
\[ b^* \cdot (M_S^* f) = M_S^*(b^* \cdot f) \quad \text{for all } b \in \sigma(A)' \text{ and } f \in H^2_2(E, \sigma). \]

Here we are using that the span of the collection of kernel functions
\[ \{ b^* \cdot (k_{E,\sigma,\zeta} \otimes I_Y)Y : b \in \sigma(A)', \zeta \in \mathbb{D}((E^\sigma)^*), Y \in \mathcal{E} \otimes \mathcal{Y} \} \]
is dense in \( H^2_2(E, \sigma) \). Then the computation
\[ \langle (M_S f)(\zeta, b), Y \rangle_{\mathcal{E} \otimes \mathcal{Y}} = \langle (M_S f, b^* \cdot (k_{E,\sigma,\zeta} \otimes I_Y)Y \rangle_{H^2_2(E, \sigma)} \]
\[ = \langle f, M_S^*(k_{E,\sigma,\zeta} \otimes I_Y)Y \rangle_{H^2_2(E, \sigma)} \]
\[ = \langle f, b^* \cdot (k_{E,\sigma,\zeta} \otimes I_Y)S(\zeta)^*Y \rangle_{H^2_2(E, \sigma)} \]
\[ = \langle f(\zeta, b), S(\zeta)^*Y \rangle_{\mathcal{E} \otimes \mathcal{U}} \]
\[ = \langle S(\zeta)f(\zeta, b), Y \rangle_{\mathcal{E} \otimes \mathcal{Y}} \]
shows that \( M_S \) is indeed the operator of multiplication by \( S \).

6. Examples

In this section we illustrate the general theory for some more concrete special cases. For simplicity we consider here only examples of the theory developed in Sections 3, 4 and 5 with \( \mathcal{U} = \mathcal{Y} = \mathbb{C} \). Unlike what one might expect, this does not lead to scalar versions of the results discussed in Sections 1 and 2, but rather to square versions, that is, we regain Theorems 1.1, 2.1 and 2.3 for the case \( \mathcal{U} = \mathcal{Y} \), but not necessarily equal to \( \mathbb{C} \).

6.1 The classical case. In this example, we take \( A = \mathcal{L}(\mathcal{G}) \) for a given Hilbert space \( \mathcal{G} \). Let \( E = \mathcal{L}(\mathcal{G}) \) viewed a correspondence over itself in the standard way:
\[ a \cdot \xi = a\xi, \quad \xi \cdot a' = \xi a' \quad \text{(the operator multiplication in } \mathcal{L}(\mathcal{G})) \text{ for } a, a' \in A, \xi \in E, \]
\[ \langle \xi', \xi \rangle = \xi^*\xi' \quad \text{(the } \mathcal{L}(\mathcal{G})\text{-inner product when considered as a correspondence over itself) for } \xi', \xi \in E. \]
Since
\[ \xi_n \otimes \xi_{n-1} \otimes \cdots \otimes \xi_1 = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \xi_n \cdots \xi_1, \]
we can identify \( E^\otimes n \) with \( E = \mathcal{L}(\mathcal{G}) \) and then the Fock space \( \mathcal{F}^2(E) \) has the form
\[ \mathcal{F}^2(E) = \oplus_{n=0}^{\infty} E^\otimes n = \ell^2_{\mathcal{L}(\mathcal{G})}(\mathbb{Z}_+). \]
The abstract analytic Toeplitz algebra \( \mathcal{F}^\infty(E) \) is the collection of all lower triangular Toeplitz matrices with \( \mathcal{L}(\mathcal{G}) \)-block entries acting as bounded operators on \( \ell^2_{\mathcal{L}(\mathcal{G})}(\mathbb{Z}_+). \)
Now suppose we are given a Hilbert space $E_0$, let $E = G \otimes E_0$ and $\sigma$ be the representation of $A = \mathcal{L}(G)$ on $\mathcal{L}(E)$ given by $\sigma(a) = a \otimes I_{E_0}$. Then

$$\sigma(A) = \{ b \in \mathcal{L}(E) : b \sigma(a) = \sigma(a) b \text{ for all } a \in A \}$$

$$= \{ b \in \mathcal{L}(E) : b(a^0 \otimes I_{E_0}) = (a^0 \otimes I_{E_0}) b \text{ for all } a^0 \in \mathcal{L}(G) \}$$

$$= \{ I_G \otimes b^0 : b^0 \in \mathcal{L}(E_0) \}$$

and hence $\sigma(A)'$ can be identified with $\mathcal{L}(E_0)$.

We next note that

$$\mathcal{F}^2(E, \sigma) = \mathcal{F}^2(E) \otimes_{\sigma} \mathcal{E} = \mathcal{L}(G) Z_+ \otimes_{\sigma} (G \otimes E_0) = \mathcal{L}(E_0) = \mathcal{L}(E) \mathcal{L}(G).$$

The representations $\varphi_{\infty, \sigma} : A = \mathcal{L}(G) \to \mathcal{L}(\ell_2^0(Z_+))$ and $\iota_{\infty, \sigma} : \sigma(A) \to \mathcal{L}(E_0) = \mathcal{L}(E) \otimes \mathcal{L}(G)$ are given by

$$\varphi_{\infty, \sigma}(a) = I_{\ell_2^0(Z_+)} \otimes a \otimes I_{E_0}, \quad \iota_{\infty, \sigma}(b^0) = I_{\ell_2^0(Z_+)} \otimes I_G \otimes b^0.$$

We next compute

$$(E^\sigma)^* = \{ \eta : E \otimes \sigma E \to E : \eta(a \otimes I_E) = \sigma(a) \eta, a \in \mathcal{L}(G) \}$$

$$= \{ \eta : \mathcal{L}(G) \otimes \sigma G \otimes E_0 \to \mathcal{G} \otimes E_0 : \eta(a \otimes I_E) = (a \otimes I_{E_0}) \eta, a \in \mathcal{L}(G) \}$$

$$= \{ \eta : \mathcal{G} \otimes E_0 \to \mathcal{G} \otimes E_0 : \eta(a \otimes I_{E_0}) = (a \otimes I_{E_0}) \eta, a \in \mathcal{L}(G) \}$$

$$= \{ I_G \otimes \eta^0 : \eta^0 \in \mathcal{L}(E_0) \}.$$

We conclude that $(E^\sigma)^*$ can be identified with $\mathcal{L}(E_0)$.

The creation operators and dual creation operators then have the form

$$T_{\xi, \sigma} = S \otimes \xi \otimes I_{E_0} \text{ for } \xi \in A = \mathcal{L}(G),$$

$$T_{\mu^0, \sigma} = S \otimes I_G \otimes \mu^0 \text{ for } \mu^0 \in \mathcal{L}(E_0) \cong E^\sigma$$

where $S$ is the standard shift operator on $\ell_2^0(Z_+)$:

$$S : \{ c_n \}_{n \in \mathbb{Z}_+} \mapsto \{ c'_n \}_{n \in \mathbb{Z}_+} \text{ where } c'_0 = 0, c'_n = c_{n-1} \text{ for } n \geq 1.$$

Note that the commutativity properties laid out in Proposition 4.2 are now transparent for this example.

Then, for $f = \oplus_{n=0}^\infty f_n \in \mathcal{F}^2(E, \sigma)$, the Fourier transform $\Phi f = f^\wedge$ is given by

$$f^\wedge(\eta^0, b^0) = \sum_{n=0}^\infty (I_G \otimes (\eta^0)^n b^0) f_n \in \mathcal{E}$$

for $\eta \in \mathcal{B}(\mathcal{L}(E_0))$ (the open unit ball of $\mathcal{L}(E_0)$) and $b^0 \in \mathcal{L}(E_0)$. One can check that $\Phi$ is injective. It follows that $\Phi$ is a unitary transformation from $\ell_2^0(Z_+)$ onto a Hilbert space $H^2(E, \sigma)$ of $E$-valued functions on $\mathcal{B}(\mathcal{L}(E_0)) \times \mathcal{L}(E_0)$ carrying a $\mathcal{L}(E_0)$-representation:

$$\pi_{H^2(E, \sigma)}(b^0) : f^\wedge(\eta^0, b^0) \mapsto f^\wedge(\eta^0, b^0 b^0).$$

In fact $f^\wedge(\eta^0, I_{E_0}) = 0$ for all $\eta^0 \in \mathcal{B}(\mathcal{L}(E_0))$ already forces $f$ to be zero in $\ell_2^0(Z_+)$ so the function $f^\wedge$ is determined completely by its single-variable restriction $f^{\wedge 1} := f^\wedge(\cdot, I_{E_0})$ and one can work with the space $\tilde{H}^2(E, \sigma) = \{ f^{\wedge 1} : f \in \ell_2^0 \}$ instead. One can identify $\tilde{H}^2(E, \sigma)$ with functions of the form

$$g(\eta^0) = \sum_{n=0}^\infty (I_G \otimes (\eta^0)^n) g_n \text{ where } \oplus_{n=0}^\infty g_n \in \ell_2^0(Z_+) \text{ with } \| g \|_{\tilde{H}^2(E, \sigma)} = \| \oplus_{n=0}^\infty g_n \|_{\ell_2^0(Z_+)}$$

and with the $\sigma(A)' \cong \mathcal{L}(E_0)$-left action given by

$$(b^0 \cdot g)(\eta) = \sum_{n=0}^\infty (I_G \otimes (\eta^0)^n)(I \otimes b^0) g_n \text{ if } g(\eta^0) = \sum_{n=0}^\infty (I_G \otimes (\eta^0)^n) g_n.$$

An element $S$ of $\mathcal{F}^{\infty}(E, \sigma)$ is an operator on $\ell_2^0(Z_+)$ having a lower-triangular Toeplitz matrix representation

$$R = [R_{i-j}]_{i,j=0,1,...}.$$
where each $R_n$ is an operator on $\mathcal{E}$ of the form $R_n = R_n^0 \otimes I_{\mathcal{E}_n}$ for an operator $R_n^0 \in \mathcal{L}(\mathcal{G})$ with $R_n^0 = 0$ for $n < 0$. Given such an $R$, the associated $\mathcal{L}(\mathcal{E})$-valued function $R^\lambda : \mathcal{L}(\mathcal{E}_0) \to \mathcal{L}(\mathcal{E})$ is then given by

$$R^\lambda(\eta^0) = \sum_{n=0}^{\infty} R_n^0 \otimes (\eta^0)^n.$$ 

The Schur class $S(E, \sigma)$ for this case can be identified with the set of functions $S : \mathcal{B}(\mathcal{L}(\mathcal{E}_0)) \to \mathcal{L}(\mathcal{E})$ with a presentation of the form

$$S(\eta^0) = \sum_{n=0}^{\infty} S_n^0 \otimes (\eta^0)^n$$

for which the associated Toeplitz matrix $[S_n^0]_{i,j=0,1,\ldots}$ defines a contraction operator on $\ell^2(\mathbb{Z}_+)$. If we use the single-variable version $\tilde{H}^2(E, \sigma)$ of the Hardy space, the condition in part (2) of Theorem 5.1 means not only that $M_S : f^{\lambda_1}(\eta^0) \mapsto S(\eta^0)f^{\lambda_1}(\eta^0)$ maps $\tilde{H}^2(E, \sigma)$ contractively into $\tilde{H}^2(E, \sigma)$, but also that $M_S$ is a $\mathcal{L}(\mathcal{E}_0)$-module map:

$$M_S(b \cdot f^{\lambda_1}) = b \cdot M_S f^{\lambda_1}.$$ 

The realization formula (6.4) and (6.5) from part (3) of Theorem 5.1 tells us that such functions $S$ are characterized by having a realization of the form

$$S(\eta^0) = D + C(I - \pi(\eta^0)A)^{-1}\pi(\eta^0)B$$

where

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{H}_E \to \mathcal{H}_E$$

is a unitary operator and $\pi$ is a *-representation of $\mathcal{L}(\mathcal{E}_0)$ to $\mathcal{L}(\mathcal{H})$ which is also a $\mathcal{L}(\mathcal{E}_0)$-module map:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \pi(b^0) & 0 \\ 0 & I_{\mathcal{G}} \otimes b^0 \end{bmatrix} = \begin{bmatrix} \pi(b^0) & 0 \\ 0 & I_{\mathcal{G}} \otimes b^0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$  

(6.3)

Here we use that $E^\sigma \otimes_\sigma \mathcal{H}$ can be identified with $\mathcal{H}$ since $(I_{\mathcal{G}} \otimes (\eta^0)^*) \otimes h = I_{\mathcal{E}} \otimes \pi((\eta^0)^*)h$.

We note that it is easy to see that a realization as in (6.2) implies that $S$ has a presentation of the form (6.1). Indeed, if $U$ is unitary and satisfies (6.3), since $A$ commutes with $\pi(\eta^0)$ we see that $(\pi(\eta^0)A)^n = A^n \pi(\eta^0)^n$. Hence expansion of the inverse in (6.2) as a geometric series and repeated usage of (6.3) gives

$$S(\eta^0) = \sum_{n=0}^{\infty} S_n(I \otimes (\eta^0)^n)$$

where

$$S_0 = D, \quad S_n = CA^{n-1}B$$

for $n \geq 1$.

Additional usage of (6.3) gives us

$$S_n(I \otimes \eta^0) = (I \otimes \eta^0)S_n$$

for all $\eta^0 \in \mathcal{L}(\mathcal{E}_0)$, from which we conclude that $S_n$ has the form $S_n = S_n^0 \otimes I_{\mathcal{E}_n}$ for operators $S_n^0$ acting on $\mathcal{G}$, and hence $S(\eta^0)$ has the form as in (6.1).

Conversely, if $S : \mathcal{B}(\mathcal{L}(\mathcal{E}_0)) \to \mathcal{L}(\mathcal{E})$ is of the form (6.1), it follows that $S(\lambda) = \sum_{n=0}^{\infty} S_n^0 \lambda^n$ is in the classical Schur class $S(\mathcal{G}, \mathcal{G})$. By the classical realization theorem we can write

$$S^0(\lambda) = D^0 + \lambda C^0(I - \lambda A^0)^{-1}B^0$$

where

$$U^0 = \begin{bmatrix} A^0 & B^0 \\ C^0 & D^0 \end{bmatrix} : \mathcal{H}^0_{\mathcal{G}} \to \mathcal{H}^0_{\mathcal{G}}$$

is coisometric (or even unitary). Then

$$U = U^0 \otimes I_{\mathcal{E}_0} = \begin{bmatrix} A^0 \otimes I_{\mathcal{E}_0} & B \otimes I_{\mathcal{E}_0} \\ C^0 \otimes I_{\mathcal{E}_0} & D^0 \otimes I_{\mathcal{E}_0} \end{bmatrix} : \mathcal{H}_{\mathcal{E}} \to \mathcal{H}_{\mathcal{E}}$$
(where we set $\mathcal{H} = H^0 \otimes E_0$) with
\[
\pi(b) = I_{H^0} \otimes b \in \mathcal{L}(\mathcal{H}) \text{ for } b \in \mathcal{L}(E^0)
\]
provides a realization for $S$ as in (6.2). Thus the general theory provides a new kind of realization result, but one can easily derive this result directly from the classical realization theorem.

Two special cases of the above analysis are of interest.

(1) If we take $G = \mathcal{E}, E_0 = \mathbb{C}$ in the example, we have $\mathcal{F}^2(E) = l^2_{\mathcal{E}(\mathbb{Z}_+)}$ with $\mathcal{F}^\infty(\mathcal{E})$ equal to the collection of all lower triangular Toeplitz matrices with $\mathcal{L}(\mathcal{E})$-block entries acting on $l^2_{\mathcal{E}(\mathbb{Z}_+)}$. In this case $\sigma(A)' = \mathcal{C}I_{\mathcal{E}}$, and $(E^\sigma)^* = I_{\mathcal{E}} \otimes \mathbb{C}$ is isomorphic to $\mathbb{C}$; thus $\mathbb{D}((E^\sigma)^*)$ may be identified with the open unit disk $\mathbb{D}$ of $\mathbb{C}$. Moreover $\mathcal{F}(E) \otimes_\sigma \mathcal{E} = l^2_{\mathcal{E}}(\mathbb{Z}_+)$ and for a given $\lambda \in \mathbb{D}$, we have the bounded point-evaluation:
\[
f = \oplus_{n=0}^{\infty} f_n \in \mathcal{F}(E) \otimes_\sigma \mathcal{E} = l^2_{\mathcal{E}}(\mathbb{Z}_+) \rightarrow \hat{f}(\lambda) = \sum_{n=0}^{\infty} f_n \lambda^n \in H^2_{\mathcal{E}}(\mathbb{D}) .
\]
Then
\[
H^2(E, \sigma) = H^2_{\mathcal{E}}(\mathbb{D})
\]
and
\[
H^\infty(E, \sigma) = H^\infty_{\mathcal{E}}(\mathbb{D}) \otimes I_{\mathcal{E}} = H^\infty_{\mathcal{E}}(\mathbb{D}) .
\]
Hence $S \in H^\infty(E, \sigma)$ means, for $\lambda \in \mathbb{D}$, that $S(\lambda) = \sum_{n=0}^{\infty} S_n \lambda^n$ with $S_n \in \mathcal{L}(\mathcal{E})$. The operators in $H^\infty(\mathcal{E}, \sigma)$ with norm at most equal to 1 form the classical Schur class. If we apply the general theorem 5.1 for this case, we simply recover Theorem 1.1 (where $\mathcal{U} = \mathbb{Y} = \mathcal{G}$).

(2) If we take $G = \mathbb{C}, E_0 = \mathcal{E}$, then $\mathcal{F}^2(E) = \ell^2(\mathbb{Z}_+), \mathcal{F}^\infty(\mathcal{E})$ is the collection of all lower triangular Toeplitz matrices acting on $\ell^2(\mathbb{Z}_+)$. In this case $\sigma(A)' = \mathcal{E}$ and
\[
(E^\sigma)^* = \{ \eta: \mathbb{C} \otimes_\sigma \mathcal{E} \rightarrow \mathcal{E}: \eta(\sigma \otimes I_{\mathcal{E}}) = a \eta; a \in \mathbb{C} \} .
\]
Since $\mathbb{C} \otimes_\sigma \mathcal{E}$ can be identified with $\mathcal{E}$ in the obvious way, $(E^\sigma)^*$ amounts to $\mathcal{L}(\mathcal{E})$. We also have $\mathcal{F}^2(E) \otimes_\sigma \mathcal{E} = l^2_{\mathcal{E}}(\mathbb{Z}_+)$ for a given $\eta \in \mathbb{D}((E^\sigma)^*) = \mathbb{B}(\mathcal{L}(\mathcal{E}))$, i.e., $\eta \in \mathcal{L}(\mathcal{E})$ with $\| \eta \| < 1$, we have the bounded point evaluation:
\[
f = \oplus_{n=0}^{\infty} f_n \in \mathcal{F}(E) \otimes_\sigma \mathcal{E} = \ell^2_{\mathcal{E}}(\mathbb{Z}_+) \rightarrow \left( (\eta, b) \in \mathbb{B}(\mathcal{L}(\mathcal{E})) \times \mathcal{L}(\mathcal{E}) \rightarrow f^\wedge(\eta, b) = \sum_{n=0}^{\infty} \eta^n b f_n \right) .
\]
We may view this $H^2(E, \sigma)$ simply as functions of the form $\eta \mapsto \sum_{n=0}^{\infty} \eta^n f_n$ with $\oplus_{n=0}^{\infty} f_n \in \ell^2_{\mathcal{E}}$ which also carries an $\mathcal{L}(\mathcal{E})$-action:
\[
f(\eta) \mapsto (b \cdot f)(\eta) = \sum_{n=0}^{\infty} \eta^n b f_n \text{ if } f(\eta) = \sum_{n=0}^{\infty} \eta^n f_n
\]
or
\[
b \cdot f = \sum_{n=0}^{\infty} S^n b P_{\mathcal{E}} S^{*n} f
\]
where $P_{\mathcal{E}}$ is the projection onto the constant functions and $S$ is the operator-argument shift operator $(Sf)(\eta) = \eta \cdot f(\eta)$. If we identify $S = S^0 \otimes I_{\mathcal{E}} \in H^\infty(E, \sigma) = H^\infty \otimes I_{\mathcal{E}}$ with the scalar-valued function $S^0 \in H^\infty(\mathbb{D})$, then the associated function with operator argument
\[
\eta \mapsto S^0(\eta) = \sum_{n=0}^{\infty} \eta^n (S_n^0 \otimes I_{\mathcal{E}}) \in \mathcal{L}(\mathcal{E})
\]
corresponds to the functional calculus for scalar holomorphic functions with operator argument usually defined via the holomorphic functional calculus (see e.g. [48]). The positivity of the kernel $K_S(\eta, \zeta) = (I - \eta \zeta^*)^{-1} - S(\zeta)(I - \eta \zeta^*)^{-1} S(\eta)^*$ guarantees that the multiplication operator
\[
M_S: f(\eta) \mapsto S(\eta)f(\eta)
\]
is contractive on $H^2(E, \sigma)$ while complete positivity of the enlarged kernel

$$\mathbb{K}_S(\eta, \zeta)[b] = \sum_{n=0}^{\infty} \eta^n b \zeta^{*n} - S(\eta) \left( \sum_{n=0}^{\infty} \eta^n b \zeta^{*n} \right) S(\zeta)^*$$

guarantees in addition that $S$ has the form \([6,4]\) and that the associated multiplication operator $M_S$ commutes with the $\ell(\xi)$-action:

$$b \cdot (M_S f) = M_S (b \cdot f) \text{ for } b \in \ell(E), f \in H^2(e, \sigma).$$

The realization result (the equivalence of (1) and (3) in Theorem \([5,1]\)) follows from the classical realization result for scalar-valued Schur-class functions in the same way as was explained above for the general case of this example.

6.2. Free semigroup algebras. In this example, we take $A = \ell(G)$ for a given Hilbert space $G$ and $E$ to be the $d$-fold column space $\col_d^d \ell(G)$ over $\ell(G)$ viewed as a correspondence over $\ell(G)$ in the standard way (see \([31,43]\)):

$$a \cdot \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_d \end{bmatrix} = \begin{bmatrix} a\xi_1 \\ \vdots \\ a\xi_d \end{bmatrix}, \quad \xi_{\alpha} = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_d \end{bmatrix}, \quad a = \begin{bmatrix} a_{1,1} \\ \vdots \\ a_{d,d} \end{bmatrix}, \quad \begin{bmatrix} \xi_{\alpha} \\ \vdots \\ \xi_{\beta} \end{bmatrix} = \sum_{j=1}^{d} \xi_j \xi_j^*$$

for $\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_d \end{bmatrix}, \xi' = \begin{bmatrix} \xi_1' \\ \vdots \\ \xi_d' \end{bmatrix} \in E$ and $a \in \ell(G)$. (6.5)

One can then identify $E^\otimes n$ with the column space $\oplus_{\alpha: |\alpha| = n} \ell(G)$ where $\alpha = i_n \cdots i_1$ is in the free semigroup $F_d$ with notation as in Subsection \([22]\). Then the associated Fock space is

$$\mathcal{F}^2(E) = \oplus_{n=0}^{\infty} E^\otimes n = \oplus_{n=0}^{\infty} \left[ \oplus_{\alpha \in F_d: |\alpha| = n} \ell(G) \right]$$

can equally well be viewed as

$$\mathcal{F}^2(E) = \mathcal{L}_d \otimes \ell(G),$$

Then the analytic Toeplitz algebra $\mathcal{F}^\infty(E)$ can be identified as

$$\mathcal{F}^\infty(E) = \mathcal{L}_d \otimes \ell(G),$$

where $\mathcal{L}_d$ is the free semigroup algebra discussed by Davidson and Pitts in \([21]\) and is also the ultraweak closure of Popescu’s noncommutative disk algebra (see \([38]\)).

Just as done in the discussion of the classical case above, we now let $E_0$ be another Hilbert space and take $E$ to be $E = G \otimes E_0$. We consider the $*$-representation $\sigma$ of $A = \ell(G)$ to $\ell(E)$ given by

$$\sigma(a) = a \otimes I_{E_0} \text{ for } a \in \ell(G).$$

We compute $\sigma(A)'$ as follows:

$$\sigma(A)' = \{ b \in \ell(E) : b \sigma(a) = \sigma(a) e \text{ for all } a \in A \}$$

$$= \{ b \in \ell(E) : b(a \otimes I_{E_0}) = (a \otimes I_{E_0}) b \text{ for all } a \in \ell(G) \}$$

$$= \{ I_G \otimes b^0 : b^0 \in \ell(E_0) \}.$$ 

Hence $\sigma(A)'$ can be identified with $\ell(E_0)$.

We next note that

$$E^\otimes n \otimes \sigma E = \col_{|\alpha| = n} \ell(G) \otimes \sigma E \cong \col_{|\alpha| = n} E$$

and hence we identify $\mathcal{F}^2(E, \sigma) = \mathcal{F}^2(E) \otimes \sigma E$ as

$$\mathcal{F}^2(E, \sigma) = \oplus_{\alpha \in F_d} E_0 = \ell_2^2(F_d).$$

The representations $\varphi_{\infty, \sigma} : \ell(G) \to \ell_2^2(F_d)$ and $\iota_{\infty, \sigma} : \ell(E_0) \to \sigma(A)'$ can be seen to be given by

$$\varphi_{\infty, \sigma}(a) = I_{\ell_2^2(F_d)} \otimes a \otimes I_{E_0} \text{ for } a \in \ell(G) = A,$$

$$\iota_{\infty, \sigma}(b^0) = I_{\ell_2^2(F_d)} \otimes I_G \otimes b^0 \text{ for } b^0 \in \ell(E_0) \cong \sigma(A)'.$$
We now observe that $E \otimes_{\sigma} \mathcal{E}$ can be identified with $\mathcal{E}^d$ (the $d$-fold direct sum of $\mathcal{E}$ with itself) under the identification

\[
\begin{bmatrix}
\xi_1 \\
\vdots \\
\xi_d
\end{bmatrix} \otimes e \equiv \begin{bmatrix}
(\xi_1 \otimes I_{\mathcal{E}_0})e \\
\vdots \\
(\xi_d \otimes I_{\mathcal{E}_0})e
\end{bmatrix} \quad \text{for } \xi_1, \ldots, \xi_d \in \mathcal{L}(\mathcal{G}) \text{ and } e \in \mathcal{E}.
\]

Then $\eta \in (E^{\sigma})^*$ means that $\eta$ is a block row-matrix $\eta = [\eta_1 \cdots \eta_d]$ mapping $E \otimes_{\sigma} \mathcal{E} \cong \mathcal{E}^d$ into $\mathcal{E}$ with the additional property that

\[
[\eta_1 \cdots \eta_d] \operatorname{diag}(a \otimes I_{\mathcal{E}_0}) = (a \otimes I_{\mathcal{E}_0}) [\eta_1 \cdots \eta_d] \quad \text{for all } a \in \mathcal{L}(\mathcal{G}).
\]

It follows that $\eta_j(a \otimes I_{\mathcal{E}_0}) = (a \otimes I_{\mathcal{E}_0}) \eta_j$ and hence that $\eta_j = I_{\mathcal{E}} \otimes \eta_j^0$ for some $\eta_j^0 \in \mathcal{L}(\mathcal{E}_0)$ for $j = 1, \ldots, d$ and we have an identification

\[ (E^{\sigma})^* \cong \mathcal{L}(\mathcal{E}_0, \mathcal{E}_0). \]

One can check that the creation and dual creation operators are given by

\[
T_{\xi, \sigma} = \sum_{j=1}^d S_j \otimes \xi_j \otimes I_{\mathcal{E}_0} \quad \text{for } \xi = \begin{bmatrix}
\xi_1 \\
\vdots \\
\xi_d
\end{bmatrix} \in \mathcal{L}(\mathcal{G})^d = E,
\]

\[
T_{\mu^0, \sigma} = \sum_{j=1}^d S_j \otimes I_{\mathcal{E}} \otimes \mu_j^0 \quad \text{for } \mu^0 = \begin{bmatrix}
\mu_1^0 \\
\vdots \\
\mu_d^0
\end{bmatrix} \in \mathcal{L}(\mathcal{E}_0, \mathcal{E}_0^d) \cong E^\sigma.
\]

For $\eta^0 = [\eta_1^0 \cdots \eta_d^0] \in \mathcal{B}(\mathcal{E}_0^d, \mathcal{E}_0) \cong \mathcal{B}(E^\sigma)$ and $b^0 \in \mathcal{L}(\mathcal{E}_0) \cong \operatorname{sgn}(\mathcal{A})'$, we have the bounded point-evaluation operator on $\mathcal{F}^2(E, \sigma) = \ell_2^d(\mathcal{F}_d)$:

\[
f = \oplus_{n \in \mathcal{F}_d} f_n \mapsto f^\wedge(n^0, b^0) := \sum_{n \in \mathcal{F}_d} (I_{\mathcal{E}} \otimes (n^0)^* b^0) f_n \tag{6.6}
\]

where $(n^0)^\alpha = n_i^0_\alpha \cdots n_i^0_1$ for $\alpha = i_N \cdots i_1 \in \mathcal{F}_d$. To continue a detailed analysis, we now consider in turn two divergent special cases.

**Case 1:** $\mathcal{E}_0 = \mathbb{C}$ so $\mathcal{E} = \mathcal{G}$: In this case we identify $\mathcal{D}(E^\sigma)^* \cong \mathcal{B}(\mathcal{E}_0^d, \mathcal{E}_0)$ with the unit ball in $\mathbb{C}^d$

\[
\mathbb{B}^d := \left\{ \lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d : \sum_{j=1}^d |\lambda_j|^2 < 1 \right\}.
\]

The point-evaluation map

\[
f^\wedge(\lambda, b) = \sum_{n \in \mathbb{Z}_+^d} \left[ \sum_{|\alpha| = n} b f_\alpha \right] \lambda^n = b \cdot f^\wedge(\lambda, I_{\mathcal{E}}) \tag{6.7}
\]

where we use the standard $(\mathcal{L}(\mathcal{E}), \mathbb{C})$-correspondence structure on $\mathcal{E}$. Here also we use the standard commutative multivariable notation

\[
\lambda^n = \lambda_1^{n_1} \cdots \lambda_d^{n_d} \quad \text{if } n = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d.
\]

From [6.7] we see that we are in the situation of Remark 3.7 and completely positivity of the kernel

\[
\mathbb{K}(\lambda, \lambda') [b^* b'] = \sum_{n \in \mathbb{Z}_+^d} (\lambda^n I_{\mathcal{E}})(b^* b')(\lambda')^n I_{\mathcal{E}}
\]

associated with $H^2(E, \sigma)$ for this case reduces to classical Aronszajn positivity for the Drury-Arveson kernel

\[
k(\lambda, \lambda') = \sum_{n \in \mathbb{Z}_+^d} \lambda^n (\lambda')^n = \frac{1}{1 - \langle \lambda, \lambda' \rangle}.
\]
In this case the Fourier transform map \( \Phi: f \mapsto f^\wedge \) has a kernel with the cokernel given by the symmetric Fock space spanned by symmetric tensors

\[
\left\{ \sum_{\alpha \in \mathcal{F}_d: |\alpha| = n} \{ \delta_{\alpha,\alpha'} \}_{\alpha' \in \mathcal{F}_d} e: e \in \mathcal{E} \right\}
\]

where \( \delta_{\alpha,\alpha'} \) is the standard Kronecker delta

\[
\delta_{\alpha,\alpha'} = \begin{cases} 
1 & \text{if } \alpha = \alpha', \\
0 & \text{otherwise.}
\end{cases}
\]

Then it is known (see \[22, 6, 10\]) that the image of \( \Phi \) in this case, i.e., the space \( H^2(E, \sigma) \) of all functions on the ball of the form \( f^\wedge \) for an \( f \in \ell_2^d(\mathcal{F}_d) \), is exactly the Arveson-Drury space and the associated space \( H^\infty(E, \sigma) \) is exactly the multiplier space \( \mathcal{M}(\mathcal{E}) \) of the Arveson space. When we specialize the general Theorem \[5,1\] to this case we simply recover Theorem \[2,1\] for the case \( \mathcal{U} = \mathcal{Y} = \mathcal{E} \).

**Case 2:** \( \mathcal{G} = \mathbb{C} \) and \( \mathcal{E} = \mathcal{E}_0 \) is a separable, infinite-dimensional Hilbert space: In this case the generalized unit disk \( \mathbb{D}((E^*\sigma)^*) = \mathbb{B}(\mathcal{L}(E^d, \mathcal{E})) \) consists of row contractions

\[
\eta = [\eta_1 \ \cdots \ \eta_d]: \mathcal{E}^d \to \mathcal{E}.
\]

The Fock correspondence \( \mathcal{F}^2(\mathcal{E}) = \ell_2^d(\mathcal{F}_d) \) has scalar coefficients while the Hilbert Fock space \( \mathcal{F}^2(E, \sigma) = \ell_2^d(\mathcal{F}_d) \) has \( \mathcal{E} \)-valued coefficients. The point-evaluation map (6.6) has the form

\[
f = \{ f_\alpha \}_{\alpha \in \mathcal{F}_d} \mapsto f^\wedge(\eta, b) = \sum_{\alpha \in \mathcal{F}_d} \eta^\alpha b f_\alpha.
\]

The completely positive kernel associated with \( H^2(E, \sigma) \) for this case is

\[
\mathcal{K}_{E,\sigma}(\eta, \zeta)[b] = \sum_{\alpha \in \mathcal{F}_d} \eta^\alpha [b] \zeta^{\alpha^*}.
\]

where \( b \in \sigma(\mathcal{A}') = \mathcal{L}(\mathcal{E}) \).

The analytic Toeplitz algebra \( \mathcal{F}^\infty(E) \) is the free semigroup algebra \( \mathcal{L}_d \) acting on \( \mathcal{F}^2(\mathcal{E}) = \ell_2^d(\mathcal{F}_d) \) having noncommutative Toeplitz matrix representation

\[
R = [R_{\alpha,\beta}]_{\alpha,\beta \in \mathcal{F}_d} \text{ where } R_{\alpha,\beta} = R_{\alpha \beta^{-1}} \emptyset.
\]

where the matrix entries \( R_{\alpha,\beta} \) are scalars. Here \( \emptyset \) refers to the empty word in \( \mathcal{F}_d \) (the unit element for the semigroup \( \mathcal{F}_d \)) and we use the convention

\[
\alpha \beta^{-1} = \begin{cases} 
\alpha' & \text{if } \alpha = \alpha' \beta, \\
\text{undefined} & \text{otherwise,}
\end{cases}
\]

\[
R_{\text{undefined}} = 0.
\]

Then it is easily seen that \( R \otimes_{\sigma} I_{\mathcal{E}} \in \mathcal{L}(\ell^2_\mathcal{F}(\mathcal{F}_d)) \) is simply the infinite-multiplicity inflation of \( R \):

\[
R \otimes I_{\mathcal{E}} = [R \otimes I_{\mathcal{E}}]_{\alpha,\beta} \text{ where } [R \otimes I_{\mathcal{E}}]_{\alpha,\beta} = R_{\alpha \beta^{-1}} \otimes I_{\mathcal{E}}.
\]

The point-evaluation \( \eta \mapsto (R \otimes_{\sigma} I_{\mathcal{E}})^\wedge(\eta) \) for \( R \otimes_{\sigma} I_{\mathcal{E}} \in \mathcal{F}^\infty(E, \sigma) \) and \( \eta = [\eta_1 \ \cdots \ \eta_d] \in \mathbb{B}(\mathcal{L}(E^d, \mathcal{E})) \) is given by

\[
(R \otimes_{\sigma} I_{\mathcal{E}})^\wedge(\eta) = \sum_{\alpha \in \mathcal{F}_d} \eta^\alpha (R_{\alpha} \otimes I_{\mathcal{E}}).
\]

Viewing the operator \( R_{\alpha} \otimes_{\sigma} I_{\mathcal{E}} \) as simply multiplication by the scalar \( R_{\alpha} \), we can rewrite this as

\[
(R \otimes_{\sigma} I_{\mathcal{E}})^\wedge(\eta) = \sum_{\alpha \in \mathcal{F}_d} R_{\alpha} \eta^\alpha. \tag{6.8}
\]

As a consequence of the fact that there are no polynomial identities valid for matrices of all sizes (see \[4, pp. 22-23\]), it follows that the point-evaluation map

\[
R \in \mathcal{F}^\infty(E) = \mathcal{L}_d \mapsto (\eta \in \mathbb{B}(\mathcal{L}(E^d, \mathcal{E}) \mapsto (R \otimes_{\sigma} I_{\mathcal{E}})^\wedge(\eta) \in \mathcal{L}(\mathcal{E}))
\]

is injective.
For $\pi$ a $*$-representation of $\mathcal{L}(E)$ into $\mathcal{L}(H)$ for a Hilbert space $H$, one can check that
\[
\begin{bmatrix}
\mu_1 \\
\vdots \\
\mu_d
\end{bmatrix} \otimes h \cong \begin{bmatrix}
\pi(\mu_1)h \\
\vdots \\
\pi(\mu_d)h
\end{bmatrix}
\] (6.9)
gives an identification of $E^* \otimes H$ with $H^d$. For a colligation $U$ to be of the form (5.5) and to satisfy (5.4) means that there is a Hilbert space $H$ together with a $*$-representation $\pi: \mathcal{L}(E) \to \mathcal{L}(H)$ such that, after the identification of $E^* \otimes \pi H$ with $H^d$ via (6.9),
\[
U = \begin{bmatrix}
A_1 & B_1 \\
\vdots & \vdots \\
A_d & B_d
\end{bmatrix} : [H] \to [H]
\] (6.10)
subject to
\[
\begin{bmatrix}
\pi(b) \\
\vdots \\
\pi(b)
\end{bmatrix} \begin{bmatrix}
A_1 & B_1 \\
\vdots & \vdots \\
A_d & B_d
\end{bmatrix} = \begin{bmatrix}
\pi(b) & 0 \\
\vdots & \vdots \\
0 & b
\end{bmatrix},
\]
or, equivalently,
\[
A_j \pi(b) = \pi(b)A_j, \quad B_j b = \pi(b)B_j \text{ for } j = 1, \ldots, d,
\]
\[
C \pi(b) = bC, \quad Db = bD,
\] (6.11)
for all $b \in \mathcal{L}(E)$. For $\eta = [\eta_1 \cdots \eta_d] \in \mathbb{D}((E^*)^\ast) = \mathbb{B}(\mathcal{L}(E^d, E))$, one can check that the operator $L_\eta^*: H \to E^* \otimes \pi H$ given by $L_\eta^*: h \mapsto \eta^* \otimes h$, after the identification (6.9), is simply the column contraction
\[
L_\eta^*: h \mapsto \begin{bmatrix}
\eta_1 \\
\vdots \\
\eta_d
\end{bmatrix}
\]
with adjoint equal to
\[
L_\eta^* = [\eta_1 \cdots \eta_d] : \mathcal{H}^d \to \mathcal{H}.
\]
Suppose that $S \in H^\infty(E, \sigma)$ for this example of $(E, \sigma)$. Then the realization formula for $S \in \mathcal{F}^\infty(E, \sigma)$ given by (5.3) for this case becomes
\[
S(\eta) = D + C(I - \eta A)^{-1}B \text{ for } \eta = [\eta_1 \cdots \eta_d] \in \mathbb{B}(\mathcal{L}(E^d, E))
\] (6.12)
where the coisometric $U$ is as in (6.10). Using the relations (6.11) and using the expansion
\[
(I - \eta A)^{-1} = \sum_{n=0}^{\infty} (\eta A)^n,
\]
we see that (6.12) can be rewritten as
\[
S(\eta) = D + \sum_{\alpha \in \mathcal{F}_d} \sum_{j=1}^{d} CA^\alpha B \eta^\alpha \eta_j.
\]
Moreover, again from the relations (6.11) we see that
\[
(CA^\alpha B_j)b = b(CA^\alpha B_j) \text{ and } Db = bD \text{ for all } b \in \mathcal{L}(E),
\]
i.e., $CA^\alpha B_j =: s_{\alpha j}$ for all $\alpha \in \mathcal{F}_d$ and $j = 1, \ldots, d$ as well as $D =: s_0^0$ are all scalar operators:
\[
s_\alpha = s_0^\alpha I_E \text{ where } s_0^\alpha \in \mathbb{C}.
\]
From the complete positive kernel condition in Theorem 5.1 it is easily seen that $S(\eta) = \sum_{\alpha \in \mathcal{F}_d} s_{\alpha} \eta^{\alpha}$ is contractive for each row contraction $\eta = [\eta_1 \cdots \eta_{d}] \in \mathcal{B}(\mathcal{L}(E, E))$. Thus the formal power series

$$S^0(z) = \sum_{\alpha \in \mathcal{F}_d} s_{\alpha}^{0} z^{\alpha}$$

is in the formal noncommutative Schur class with scalar coefficients $S_{nc,d}(\mathbb{C}, \mathbb{C})$ introduced in Section 2.2.

Conversely, if $S^0(z) = \sum_{\alpha \in \mathcal{F}_d} s_{\alpha}^{0} z^{\alpha}$ is in the formal noncommutative Schur class $S_{nc,d}(\mathbb{C}, \mathbb{C})$, then part (3) of Theorem 2.3 assures us that $S(z)$ has a realization of the form

$$S^0(z) = D^0 + C^0(I - Z(z)A^0)^{-1}Z(z)B^0$$

(6.13) for a coisometric (even unitary) colligation

$$U^0 = \begin{bmatrix} A^0_1 & B^0_1 \\ \vdots & \vdots \\ A^0_d & B^0_d \\ C^0 & D^0 \end{bmatrix} : \mathcal{H}^0 \to \mathcal{H}.$$ 

Let us form a new tensored colligation $U$ of the form

$$U = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{bmatrix} : \mathcal{H} \to \mathcal{H}^0.$$ 

where we set

$$\mathcal{H} = \mathcal{H}^0 \otimes \mathcal{E}, \quad A_j = A^0_j \otimes I_\mathcal{E}, \quad B_j = B^0_j \otimes I_\mathcal{E}, \quad C = C^0 \otimes I_\mathcal{E}, \quad D = D^0 \otimes I_\mathcal{E}$$

where $j = 1, \ldots, d$. We may define a $*$-representation $\pi : \mathcal{L}(\mathcal{E}) \to \mathcal{L}(\mathcal{H})$ by

$$\pi(b) = I_{\mathcal{H}^0} \otimes b.$$ 

Then it is easily seen that this $U$ satisfies (6.10) and (6.11). Moreover, from these relations and the realization (6.13) for the formal noncommutative Schur-class function $S^0(z)$, we see that we have a realization for the associated function $\eta \mapsto S^0(\eta)$ of the form (6.12):

$$S(\eta) := \sum_{\alpha \in \mathcal{F}_d} s_{\alpha}^{0} \eta^{\alpha} = D + C(I - L_\eta^{*} A)^{-1}L_\eta^{*} B.$$ 

We conclude: there is a one-to-one correspondence between formal power series $S^0(z) = \sum_{\alpha \in \mathcal{F}_d} s_{\alpha}^{0} z^{\alpha}$ in the noncommutative scalar-coefficient Schur class $S_{nc,d}(\mathbb{C}, \mathbb{C})$ and functions $\eta \mapsto S(\eta)$ in the Muhy-Solel class $H^\infty(E, \sigma)$ for the particular choice of $(E, \sigma)$ (described in (5.5) with $\mathcal{G} = \mathbb{C}$ and $\mathcal{E}_0 = \mathcal{E}$ infinite-dimensional), given by

$$S^0(z) = \sum_{\alpha \in \mathcal{F}_d} s_{\alpha}^{0} z^{\alpha} \mapsto \left( \eta \mapsto S(\eta) := \sum_{\alpha \in \mathcal{F}_d} \eta^{\alpha}(s_{\alpha}^{0} I_\mathcal{E}) \right).$$

Here we have made explicit the correspondence between condition (3) in Theorem 2.3 for $S^0(z)$ versus condition (3) in Theorem 5.1 for $S(\eta)$. An amusing exercise would be to understand directly the equivalence between any of the other conditions in Theorem 2.3 for $S^0(z)$ and the corresponding condition for $S(\eta)$ in Theorem 5.1

6.3. Analytic crossed-product algebras. We discuss here a particular case of analytic crossed-product algebras (see Example 2.6 in 31 as well as the references there). This particular case has strong connections with time-varying system theory and was discussed in connection with point-evaluation and generalized Nevanlinna-Pick interpolation in 35 (see Examples 2.5, 2.6 and 2.25 there). Here we wish to draw out the connections between the realizaton theorem (the equivalence of (1) and (3) in Theorem 6.1 for this case) and a result from 41 that any lower-triangular contractive operator on $l^2(\mathbb{Z})$ can be realized as the input-output map of a linear time-varying input/state/output system. For simplicity we discuss in detail only the multiplicity-free case ($\mathcal{U} = \mathcal{Y} = \mathbb{C}$).
We take the algebra $\mathcal{A}$ to be the algebra $\ell^\infty(\mathbb{Z})$ with coordinate-wise multiplication with correspondence $E$ equal to $A = \ell^\infty(\mathbb{Z})$ as a set. Let $\alpha$ be the automorphisms $\alpha(a)(k) = a(k-1)$ ($k \in \mathbb{Z}$) for $a : \mathbb{Z} \rightarrow \mathbb{C}$ in $A$. We consider $E$ as a correspondence over $A$ with left and right action given by

$$(a \cdot \xi)(k) = (\alpha(a))(k) = a(k-1)\xi(k), \quad (\xi \cdot a)(k) = (\xi a)(k) = \xi(k)a(k)$$

and with the $\mathcal{A}$-valued inner product

$$\langle \xi', \xi \rangle_E(k) = \overline{\xi(k)}\xi'(k)$$

for $d \in \mathbb{Z}$, $a \in A = \ell^\infty(\mathbb{Z})$ and $\xi', \xi \in E = \ell^\infty(\mathbb{Z})$. Then it is easily seen that $E^\otimes n$ is the correspondence over $A$ identified again with $E = \ell^\infty(\mathbb{Z})$ as a set with $\mathcal{A}$-valued inner product as in (6.14) but with left and right $A$-action given by

$$a \cdot (\otimes_{n=0}^\infty \xi^{(n)})(k) = \oplus_{n=0}^\infty a^n(\xi^{(n)})(k), \quad (\otimes_{n=0}^\infty \xi^{(n)})(k) \cdot a = \oplus_{n=0}^\infty \xi^{(n)}(k)a$$

for $k \in \mathbb{Z}$, $\xi^{(n)} \in E^\otimes n = \ell^\infty(\mathbb{Z})$ and $a \in A = \ell^\infty(\mathbb{Z})$. The Fock space $F^2(E)$ is then the correspondence $\oplus_{n=0}^\infty \ell^\infty(\mathbb{Z})$ with left and right $A$-action given by

$$a \cdot (\oplus_{n=0}^\infty \xi^{(n)})(k) = \oplus_{n=0}^\infty a^n(\xi^{(n)})(k), \quad (\oplus_{n=0}^\infty \xi^{(n)})(k) \cdot a = \oplus_{n=0}^\infty \xi^{(n)}(k)a.$$

More generally, when $\mathcal{A}$ is a general von Neumann algebra and $\alpha$ is an automorphism of $\mathcal{A}$, this construction gives rise to analytic crossed-product algebras which have been studied by a number of authors over the past several decades (see [31] [33] [35] and the references therein).

An appealing alternative representation of the correspondence, as explained in Example 2.6 in [35], is as follows. View $\mathcal{A}$ as the algebra $D$ of all diagonal operators acting on $\ell^2(\mathbb{Z})$, let $U$ be the bilateral shift operator $Ue_k = e_{k+1}$ (where $e_k(k')$ is the Kronecker delta function) and let $E = UD \subset \mathcal{L}(\ell^2(\mathbb{Z}))$. Then define the left and right actions of $\mathcal{A} = D$ on $E = UD$ simply by left and right operator multiplications with the inner product given by

$$\langle UD_1, UD_2 \rangle_E = D_2^*D_1 \in D.$$

One can check that this $A$-correspondence $E$ is unitarily equivalent to the $A'$-correspondence $E'$ given in the previous paragraph with the obvious identifications:

$$\{d(k)\}_{k \in \mathbb{Z}} \in A' = \ell^\infty(\mathbb{Z}) \mapsto \text{diag}_{k \in \mathbb{Z}}d(k) \in D = A,$$

$$\{\xi(k)\}_{k \in \mathbb{Z}} \in A' = \ell^\infty(\mathbb{Z}) \mapsto U : \text{diag}_{k \in \mathbb{Z}}\xi(k) \in UD = E.$$

One can easily check that the identification map

$$UD_2 \otimes UD_1 \in UD \otimes UD \mapsto UD_2UD_1 = U^2(U^*D_2U)D_1 \in U^2D.$$

is unitary from $E \otimes E$ to $U^2D$. After this identification, the left and right $D$-action on $E^\otimes 2 \cong U^2D$ is again given by left and right operator multiplication. More generally, we view $E^\otimes n$ as $U^nD$ with left and right $D$-action given by operator multiplication and with inner product inherited from $\mathcal{L}(\ell^2(\mathbb{Z}))$:

$$\langle U^nD_1, U^nD_2 \rangle_{E^\otimes n} = D_2^*U^nU^*D_1 = D_2^*D_1 \in D.$$

The Fock space $F^2(E)$ can then be identified with lower triangular matrices $T$ with diagonal expansion $T = \sum_{n=0}^\infty U^nD_n$ ($D_n \in D$) such that

$$\sum_{n=0}^N D_n^*D_n \text{ is bounded above in } D.$$  

(6.15)

The Toeplitz algebra $F^\infty(E)$ consists of all lower triangular matrices $R$ which give rise to bounded operators on $\ell^2(\mathbb{Z})$. As elements of $F^\infty(E)$, they act on $F^2(E)$ (lower triangular matrices satisfying (6.15)) via multiplication on the left. We can view this algebra as generated by a single creation operator $T_I$ (the creation operator associated with the identity matrix $I \in D$, namely the bilateral shift operator $U$), together with the diagonal operators $D$. Note that $U$ is really a unilateral shift operator since it is restricted to the space $F^2(E)$ of lower triangular matrices (with action equal to a shifting of the subdiagonals).

We now set $E = \ell^2(\mathbb{Z})$ and let $\sigma$ be the identity representation of $D$ on $E = \ell^2(\mathbb{Z})$. Then $E^\otimes n \otimes_\sigma E$ can be identified with $E = \ell^2(\mathbb{Z})$ in the natural way

$$\iota : U^nD \otimes e \mapsto De.$$
When this is done the left action of $\mathcal{A} = \mathcal{D}$ becomes
\[
d \cdot e = U^{n*} d U^n e
\]
since
\[
i(d \cdot (U^n D \otimes e)) = i(U^n D \otimes e) = i(U^n (U^{n*} d U^n) D \otimes e) = U^{n*} d U^n D e = U^{n*} d U^n i(U^n D \otimes e).
\]
Hence we identify $\mathcal{F}^2(E, \sigma) = \mathcal{F}^2(E) \otimes_\sigma \mathcal{E}$ with
\[
\mathcal{F}^2(E, \sigma) = \ell^2_{\mathbb{Z}_+}(\mathbb{Z}_+)
\]
with left action by $\mathcal{A} = \mathcal{D}$ given by
\[
b \cdot \{e_n\}_{n \in \mathbb{Z}_+} = \{U^{n*} b U^n e_n\}_{n \in \mathbb{Z}_+}.
\]
One can see that the image of the generating creation operator $T_d = U \otimes I_\mathcal{E}$ after these identifications is the unilateral shift operator $S \otimes I_{\ell^2(\mathbb{Z})}$ acting on $\mathcal{F}^2(E, \sigma) = \ell^2_{\mathbb{Z}_+}(\mathbb{Z}_+)$:
\[
T_d = [\ell_{i,j}]_{i,j \in \mathbb{Z}_+} \text{ where } \ell_{i,j} = \begin{cases} I_{\ell^2(\mathbb{Z})} & \text{if } i = j + 1, \\ 0 & \text{otherwise.} \end{cases}
\]
The elements $R$ of $\mathcal{F}^\infty(E, \sigma) \subset \mathcal{L}(\ell^2_{\mathbb{Z}_+}(\mathbb{Z}_+))$ can then be identified as the following algebra of sparse matrices: there is a sequence $\{d_n\}_{n \in \mathbb{Z}_+} \subset \mathcal{D}$ of diagonal operators on $\ell^2(\mathbb{Z})$ so that $R$ has the form
\[
R = [R_{i,j}]_{i,j \in \mathbb{Z}_+} \text{ where } R_{i,j} = \begin{cases} U U^* d_{i-j} U^* U & \text{for } i \geq j, \\ 0 & \text{otherwise,} \end{cases}
\]
or, in block-matrix form,
\[
R = \begin{bmatrix} d_0 & 0 & 0 & \ldots \\ d_1 & U^* d_0 U & 0 & \ldots \\ d_2 & U^* d_1 U & U^* U^2 d_0 U & \ldots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}.
\]
We identify $(E^n)^*$ for this example as follows. The space $(E^n)^*$ consists of operators $\eta: E \otimes_\sigma \mathcal{E} \to \mathcal{E}$ such that $\eta(\varphi(a) \otimes I_\mathcal{E}) = \sigma(a) \eta$. For the present situation, both $E \otimes_\sigma \mathcal{E}$ and $\mathcal{E}$ are identified with $\ell^2(\mathbb{Z})$ but the left action by an element $d \in \mathcal{A} = \mathcal{D}$ is given by multiplication by $U^* d U$ in the first case and by multiplication by $d$ in the second. Thus the operator $\eta \in \mathcal{L}(\ell^2(\mathbb{Z}))$ is required to satisfy
\[
\eta U^* D U = D \eta
\]
which means that $\eta U^*$ is diagonal, so $(E^n)^*$ is identified with weighted shift operators
\[
(E^n)^* \cong \{ \eta = D_{\eta} U = U (U^* D_{\eta} U) \in \mathcal{L}(\ell^2(\mathbb{Z})): D_{\eta} \in \mathcal{D} \} = U D.
\]
Recall that there is a representation of $\mathcal{F}^\infty(E^n)$ on $\mathcal{F}^2(E, \sigma)$ (where $E^n$ is viewed as a $\sigma(A)'$-correspondence). For our situation here, $\sigma(A)' = \mathcal{A} = \mathcal{D}$ considered as acting on $\mathcal{E} = \ell^2(\mathbb{Z})$ and the representation of $\sigma(A)'$ on $\mathcal{F}(E, \sigma) = \ell^2_{\mathbb{Z}_+}(\mathbb{Z}_+)$ turns out to be the diagonal action:
\[
b \cdot (\oplus_{n=0}^\infty e_n) = \oplus_{n=0}^\infty b e_n \quad \text{for } b \in \mathcal{D}, \oplus_{n=0}^\infty e_n \in \ell^2_{\mathbb{Z}_+}(\mathbb{Z}_+).
\]
For purposes of getting a generating set for $\mathcal{F}^\infty(E^n)$, it suffices to consider the single creation operator associated with $\eta^* = U^*$: the associated action on $\mathcal{F}^2(E, \sigma)$ turns out to be
\[
T_d^{ij} \cdot \ell_{li,j} = \begin{cases} U^* & \text{if } j = i+1, \\ 0 & \text{otherwise.} \end{cases}
\]
According to the duality result from [33], an operator $R$ on $\ell^2_{\mathbb{Z}_+}(\mathbb{Z}_+)$ is of the form (6.16) if and only if $R$ commutes with the scalar-diagonal operators (6.18) and the $E^n$-creation operator (6.19); an amusing exercise for the reader is to verify this fact directly for this example.

We now identify the $Z$-transform and compute the function spaces $H^2(E, \sigma)$ and $H^\infty(E, \sigma)$ as follows. By (6.17) we have an identification of $(E^n)^*$ with the space of weighted shift operators $UD$ in $\mathcal{L}(\ell^2(\mathbb{Z}))$. After carrying out the identifications $E^{\otimes n} \otimes_\sigma \mathcal{E} = \ell^2(\mathbb{Z})$, one can check that the generalized power $\eta^n: E^{\otimes n} \otimes_\sigma \mathcal{E} \to \mathcal{E}$
of an \( \eta \in (E^\gamma)^* = UD \) coincides with the usual power \( \eta^n \) as an element of the operator algebra \( \mathcal{L}(l^2(\mathbb{Z})) \). Therefore, for \( f = \{f_n\}_{n \in \mathbb{Z}_+} \in \ell^2_\mathcal{E}(\mathbb{Z}_+) \) and \( \eta = D_\eta U \in \mathcal{D}((E^\sigma)^*) \) (with \( D_\eta \in \mathbb{D} \)), we have

\[
f^\wedge(\eta, b) = \sum_{n=0}^\infty \eta^n b f_n = \sum_{n=0}^\infty (D_\eta U)^n b f_n.
\]

If we restrict the second variable \( b \in \sigma(\mathcal{A})' = \mathcal{D} \) to be \( b = I_{\ell^2(\mathbb{Z})} \), we have the restricted Fourier transform

\[
\Phi^1: f = \{f_n\}_{n \in \mathbb{Z}_+} \mapsto f^\wedge = \sum_{n=0}^\infty \eta^n f_n = \sum_{n=0}^\infty (D_\eta U)^n f_n.
\]

We assert that the restricted \( Z \)-transform \( \Phi^1: f \mapsto f^\wedge = f^\wedge(\cdot, I_{\ell^2(\mathbb{Z})}) \) is injective. Indeed, if \( f^\wedge(\eta) = 0 \) for all \( \eta \), evaluating at \( \eta = 0 \) gives that \( f_0 = 0 \) and hence \( f^\wedge(\eta) = \eta \cdot \sum_{n=0}^\infty f_{n+1} \eta^n = 0 \). Choosing \( \eta \) invertible and premultiplying by \( \eta^{-1} \) then gives that

\[
\sum_{n=0}^\infty f_{n+1} \eta^n = 0
\]

for all invertible \( \eta \). By approximating a noninvertible \( \eta \) by invertible \( \eta \)'s, we see that \( (6.20) \) actually holds for all \( \eta \in \mathcal{D}((E^\sigma)^*) \). Iteration of the same argument now gives that \( f_n = 0 \) for all \( n \in \mathbb{Z}_+ \), i.e., \( f = \{f_n\}_{n \in \mathbb{Z}_+} \) is the zero element of \( \ell^2_\mathcal{E}(\mathbb{Z}_+) \), and the assertion follows. Note that the \( \sigma(\mathcal{A})' = \mathcal{D} \)-action on \( F^2(E, \sigma) \) is given by

\[
d \cdot \{f_n\}_{n \in \mathbb{Z}_+} = \{df_n\}_{n \in \mathbb{Z}_+} \text{ for } d \in \mathcal{D}.
\]

The completely positive kernel \( \mathbb{K} \) associated with the reproducing kernel Hilbert correspondence \( H^2(E, \sigma) = \Phi(\ell^2_\mathcal{E}(\mathbb{Z}_+)) \) is

\[
\mathbb{K}(\eta, \zeta)[b] = \sum_{n=0}^\infty \eta^n b \zeta^* e^n \text{ for } \eta, \zeta \in \mathcal{D}((E^\sigma)^*) = \mathbb{B}(UD) \text{ and } b \in \mathcal{D}.
\]

Note that \( \Phi^* \mathbb{K}(\cdot, \zeta)[b]e = b b \zeta^* e \) where

\[
bb \zeta^* e = \{b \zeta^* e_n\}_{n \in \mathbb{Z}_+} \in \ell^2_\mathcal{E}(\mathbb{Z}_+).
\]

We conclude that the subcollection

\[
\{bb \zeta^* e: b \in \mathcal{D}, \zeta \in \mathbb{B}(UD), e \in \ell^2(\mathbb{Z})\}
\]

has dense span in \( \ell^2_\mathcal{E}(\mathbb{Z}_+) \).

An element \( R \) of \( F^\infty(E) \) is identified with a lower triangular matrix representing a bounded operator on \( \ell^2(\mathbb{Z}) \); it is convenient to represent such a matrix via a generalized Fourier series along subdiagonals:

\[
R \sim \sum_{n=0}^\infty U^n d_n \text{ where } d_n \in \mathcal{D}.
\]

(The Caesaro averages of the partial sums of the series converges to \( R \) in the weak-* topology but we shall not need this.) Then \( R \otimes I_{\ell^2(\mathbb{Z})} \), after the identification of \( F^2(E) \otimes_\sigma \ell^2(\mathbb{Z}) \) with \( \ell^2_\mathcal{E}(\mathbb{Z}_+) \), is identified with the operator acting on \( \ell^2_\mathcal{E}(\mathbb{Z}_+) \) with the sparse matrix representation \( (6.15) \). For \( \eta \in \mathcal{D}((E^\sigma)^*) = \mathbb{B}(UD) \), the associated point evaluation of \( R \otimes I_{\mathcal{E}} \) is then given by

\[
(R \otimes I_{\ell^2(\mathbb{Z})})^\wedge(\eta) = \sum_{n=0}^\infty \eta^n R_{n,0} = \sum_{n=0}^\infty \eta^n d_n
\]

if \( R \) is given by \( (6.21) \). In particular, formally we recover \( R \) from \( (R \otimes I_{\ell^2(\mathbb{Z})})^\wedge \) as

\[
R = (R \otimes I_{\ell^2(\mathbb{Z})})^\wedge(U).
\]
More precisely, we interpret the right-hand side of (6.23) as

\[(R \otimes I_{\ell^2(\mathbb{Z})})^\wedge(U) = \lim_{r \to 1}(R \otimes I_{\ell^2(\mathbb{Z})})^\wedge(\pi(U)).\]

The realization theorem (the equivalence of (1) and (3) in Theorem 5.1) assures us that any function of the form \((R \otimes I_{\ell^2(\mathbb{Z})})^\wedge\) with \(|R| \leq 1\) can be realized as follows. Suppose first that \(\mathcal{H}\) is a \((\sigma(A)' = D, \mathbb{C})\)-correspondence, i.e., \(\mathcal{H}\) is a Hilbert space and there is a *-representation \(\pi\) of \(\sigma(A)' = D\) with values in \(\mathcal{L}(\mathcal{H})\). Noting that

\[U^* d \otimes \pi h = U^* \otimes \pi(d)h \equiv \pi(d)h\]

for \(\mu = U^* d \in E^\sigma = U^* \mathcal{D}\) (so \(d \in \mathcal{D} = \sigma(A)\)' and \(h \in \mathcal{H}\), we see that \(E^\sigma \otimes \pi \mathcal{H}\) can be identified with \(\mathcal{H}\), but at the price that the left \((\sigma(A)' = D)\)-action on \(\mathcal{H}\) is given by \(\pi(1); b \mapsto \pi(UBU^*)\) rather than by \(\pi\). With this identification, we see that the unitary colligation \(U\) in (5.3) and (5.4) for this case has the form

\[
U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{H} \rightarrow \ell^2(\mathbb{Z})
\]

subject to

\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \pi(b) & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} \pi(1)(b) & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\]

for all \(b \in \sigma(A)' = D\), or, equivalently,

\[
A\pi(b) = \pi(UBU^*)A, \quad Bb = \pi(UBU^*)B, \quad C\pi(b) = bC, \quad Db = bD \quad \text{for all } b \in \mathcal{D}. \quad (6.24)
\]

The realization theorem then tells us that any \((R \otimes I_{\ell^2(\mathbb{Z})})^\wedge\) (where \(R \in \mathcal{L}(\ell^2(\mathbb{Z}))\) is lower-triangular and contractive) can be realized as

\[(R \otimes I_{\ell^2(\mathbb{Z})})^\wedge(\eta) = D + C(I - \pi(\eta U^*)A)^{-1}\pi(\eta U^*)B. \quad (6.25)\]

Let us now consider a time-varying input/state/output linear system of the form

\[
\Sigma : \begin{cases} x(n+1) = A(n)x(n) + B(n)u(n) \\ y(n) = C(n)x(n) + D(n)u(n) \end{cases}
\]

(6.26)

determined by the time-varying system matrix

\[
U(n) = \begin{bmatrix} A(n) & B(n) \\ C(n) & D(n) \end{bmatrix} : \mathcal{H}(n) \rightarrow \ell^2(\mathbb{Z}).
\]

We say that the system is conservative (respectively, dissipative) if each \(U(n)\) is unitary (respectively, contractive). Let us assume that we have a dissipative time-varying linear system with time-varying system matrix \(U(n) = \begin{bmatrix} A(n) & B(n) \\ C(n) & D(n) \end{bmatrix}\). Then it can be shown that, given an input string \(\{u(n)\}_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z})\), there is a unique system trajectory \((u(n), x(n), y(n))\), i.e., solution of the system equations (6.26), such that \(\lim_{n \to -\infty} x(n) = 0\) with the resulting output string \(\{y(n)\}_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z})\). In this way there is defined an input-output map \(T_{\Sigma}\) on \(\ell^2(\mathbb{Z})\) such that \(T_{\Sigma} : \{u(n)\}_{n \in \mathbb{Z}} \mapsto \{y(n)\}_{n \in \mathbb{Z}}\).

Let us introduce an aggregate state space

\[\mathcal{H} = \bigoplus_{n \in \mathbb{Z}} \mathcal{H}(n)\]

(6.27)

and an aggregate system matrix

\[
U = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})
\]

(6.28)

with \(\mathcal{A}, \mathcal{B}, \mathcal{C}\) and \(\mathcal{D}\) specified by block-matrix entries

\[
\begin{bmatrix} \mathcal{A} \end{bmatrix}_{i,j} = A(j)\delta_{i,j+1}, \quad \begin{bmatrix} \mathcal{B} \end{bmatrix}_{i,j} = B(j)\delta_{i,j+1}, \\
\begin{bmatrix} \mathcal{C} \end{bmatrix}_{i,j} = C(j)\delta_{i,j}, \quad \begin{bmatrix} \mathcal{D} \end{bmatrix}_{i,j} = D(j)\delta_{i,j}.
\]

(6.29)

If the operator \(\mathcal{A}\) has spectral radius strictly less than 1 as an operator on \(\mathcal{H}\), then one can compute that \(T_{\Sigma}\) is given by

\[T_{\Sigma} = \mathcal{D} + \mathcal{C}(I - \mathcal{A})^{-1}\mathcal{B} \in \mathcal{L}(\ell^2(\mathbb{Z})).\]

(6.30)
Even if $\mathcal{A}$ does not have spectral radius strictly less than 1, there are various ways whereby one can still make sense of the formula (6.30); one such is via a limit

$$T_\Sigma = \lim_{r \to 1} \mathcal{D} + C(I - r\mathcal{A})^{-1}(r\mathcal{B}).$$

From the representation (6.30) for $T_\Sigma$, one can compute that $T_\Sigma$ has the diagonal decomposition

$$T_\Sigma = \sum_{n=0}^{\infty} U^n d_n$$

where $d_0 = \mathcal{D}$ and $d_n = U^{*n} \mathcal{C} \mathcal{A}^{n-1} \mathcal{B}$ for $n \geq 1$.

Hence an application of (6.22) gives us

$$(T_\Sigma \otimes I_{L^2(\mathbb{Z})})^\wedge(\eta) = \mathcal{D} + \sum_{n=0}^{\infty} \eta^n U^{*n} \mathcal{C} \mathcal{A}^{n-1} \mathcal{B}.$$  \hspace{1cm} (6.31)

Given $\mathcal{H}$ in the form (6.27), we may define a representation $\pi$ of $\mathcal{D}$ by

$$\pi(b) : \oplus_{n \in \mathbb{Z}} h(n) \mapsto \oplus_{n \in \mathbb{Z}} b(n) h(n)$$

for $b = \text{diag}_{n \in \mathbb{Z}} \{b(n)\} \in \mathcal{D}$.

Note that if $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are as in (6.29), then $\mathcal{U}$ as in (6.28) satisfies the $\mathcal{D}$-module property (6.24) (with $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ in place of $A, B, C, D$). By a careful induction argument making use of these relations, one can show that

$$\mathcal{C}(\pi(\eta U^*)) \mathcal{A}^{k-1} \pi(\eta U^*) = \eta^k U^{*k} \mathcal{C} \mathcal{A}^{k-1}$$

for $k = 1, 2, \ldots$.

One can then show that

$$\mathcal{D} + \mathcal{C}(I - \pi(\eta U^*))^{-1} \pi(\eta U^*) \mathcal{B} = \mathcal{D} + \sum_{n=1}^{\infty} \mathcal{C}(\pi(\eta U^*) \mathcal{A})^{n-1} \pi(\eta U^*) \mathcal{B}$$

$$= \mathcal{D} + \sum_{n=1}^{\infty} \eta^n U^{*n} \mathcal{C} \mathcal{A}^{n-1} \mathcal{B}$$

$$= (T_\Sigma \otimes I_{L^2(\mathbb{Z})})^\wedge(\eta),$$

i.e., the aggregate colligation $\mathcal{U} = [\mathcal{A} \mathcal{B} \mathcal{C} \mathcal{D}]$ arising from the realization of $T_\Sigma$ as the input-output map for the time-varying linear system (6.20) gives rise to a realization of the form (6.25) for the function $(T_\Sigma \otimes I_{L^2(\mathbb{Z})})^\wedge$ in the Muhly-Solel Schur class for this special setting.

This suggests a different approach to the realization theorem (the equivalence of (1) and (3) in Theorem 5.1) for this particular case. Given a contractive lower-triangular operator $\overline{R}$ on $L^2(\mathbb{Z})$, it is known (see [4, Theorem 6.2]) that one can realize $\overline{R}$ as the input-output map $R = T_\Sigma$ of a conservative time-varying input/state/output linear system as in (6.20); the solution in [4] is given via a time-varying analogue of the Pavlov functional model, or, alternatively, via a time-varying analogue of the Sz.-Nagy-Foias or de Branges-Rovnyak functional model. Once we have realized $R$ as $R = T_\Sigma$ with $\Sigma$ as in (6.20), we get $(R \otimes I_{L^2(\mathbb{Z})})^\wedge$ realized in the form (6.25) and hence we have recovered the implication $(1) \implies (3)$ of Theorem 5.1. We conclude that the Muhly-Solel realization theorem for this case, after some translation, has essentially the same content as the conservative realization theorem for linear time-varying systems in [4].

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