Bianchi Type IX Magnetized Stiff Fluid Cosmological Model

Rakeshwar Purohit 1, Anita Bagora (Menaria) 2

1 Department of Mathematics, M.L. Sukhadia University, Udaipur-313001, India
2 Department of Mathematics, Jaipur National University, Jaipur-302027, India

Email: anita_bagora@yahoo.com

Abstract. We have investigated homogeneous anisotropic tilted Bianchi type IX cosmological model for perfect fluid distribution with electro-magnetic field. The magnetic field is due to an electric current produced along x axis thus $F_{23}$ is the only non-vanishing component of electromagnetic field tensor. To get a determinate model, we have assumed the condition $a=b^2$ between metric potentials. The various physical and geometrical aspects of the model viz pressure, density, shear, expansion factor, electromagnetic field tensor and rotation are also discussed.

1. Introduction

Bianchi type IX cosmological models are interesting because these models allow not only expansion but also rotation and shear and in general these are anisotropic. Many relativists have taken keen interest in studying Bianchi type IX universe because familiar solutions like Robertson-Walker universe, the de-sitter universe, the Taub-Nut solutions etc. are of Bianchi type IX space-times. Bianchi type IX universe include closed FRW models. The homogeneous and isotropic FRW cosmological models, which are used to describe standard cosmological models are particular cases of Bianchi type I, V and IX space-times according to the constant curvature of the physical three-space, $t=constant$, is zero, negative or positive. The cosmological models with rotation and expansion have been discussed by various researchers viz. Novello and Reboucas[1], Reboucas and Lima[2-3], Rosquist[4], Bradley and Sviestins[5], Sviestins[6] and Patel et al.[7]. Also many authors have considered the behavior of individual Bianchi models that contain either a pure magnetic field or magnetic field plus fluid. The magnetic field has significant role in cosmological scale and is present in galactic and intergalactic spaces. Monaghan [8] has discussed the behaviour of magnetic field in stellar bodies. Lorenz [9] found Bianchi type cosmological models with electromagnetic field. Soares and Assad [10] have obtained anisotropic and spatially homogeneous Bianchi type VII and IX cosmologies in the presence of electromagnetic field. Vaidya and Patel [11] have studied spatially homogeneous space time of Bianchi type IX and they have outlined a general scheme for the derivation of exact solutions of Einstein’s field equations presence of perfect fluid and pure radiation fields. Krori et al. [12] and Chakraborty and Nandy [13] have derived cosmological models of Bianchi type II, VIII and IX. There are many other researchers viz. Uggla, and Zur-Muhlen...
[14], Burd, Buric and Ellis [15], Kind [16] and Paternoga et al. [17] have studied Bianchi type IX space-time in different context. Besides this, there has been a considerable interest in spatially homogeneous and anisotropic cosmological models in which the fluid flow is not normal to the hypersurface of homogeneity. These are called tilted universes. The tilted cosmological models in which the fluid vector is not normal to the hypersurface of homogeneity are more complicated than those of non-tilted one. Therefore, in recent years there has been a considerable interest in investigating such type of models. The general dynamics of these cosmological models have been studied in detail by King and Ellis [18], Ellis and King [19], Collins and Ellis [20], Ellis and Baldwin [21]. Dunn and Tupper [22] investigated that tilting universe is possible when electromagnetic field is present. Tilted Bianchi type I cosmological model for perfect fluid distribution in presence of magnetic field has been obtained by Bali and Sharma [23]. They have shown that tilted nature of the model is preserved due to magnetic field. As far as matter concern the solution with stiff fluid has great importance. Electromagnetic Bianchi type I model with stiff matter has been studied by Lorenz [24]. The relevance of the stiff equation of state \( \varepsilon = p \) to the matter content of the universe in its early stages has been discussed by Barrow [25]. Hajj Boutros [26] has obtained a stiff matter solution to Einstein equation which are spherically symmetric and with a perfect fluid distribution of matter satisfying equation of state \( \varepsilon = p \), where \( p \) is the pressure of fluid and \( \varepsilon \) is the density using the condition \( \varepsilon = np, n \geq 1 \). Bagora [27-28] have also investigated Bianchi type I, III tilted cosmological model in different context. Also, Bagora [29] obtained tilted dust Bianchi type IX bulk viscous cosmological model. Motivated by these studies, in this paper, we propose to find magnetized stiff tilted Bianchi type IX cosmological models. To get a determinate model, we have assumed the condition \( a=b^2 \) between metric potentials. The various physical and geometrical aspects of the model viz pressure, density, shear, expansion factor, electromagnetic field tensor and rotation are also discussed. The physical and geometrical aspects of the model together with singularities are discussed.

2. The Metric and Field Equations

We consider the homogeneous anisotropic Bianchi type IX metric in the form
\[
ds^2 = -dt^2 + a^2(t) \ dx^2 + b^2(t) \ dy^2 + (b^2 \sin^2 y + a^2 \cos^2 y) \ dz^2 - 2a^2 \cos y \ dx \ dz,
\]
where \( a \) and \( b \) are functions of \( t \) alone.

The energy-momentum tensor for perfect fluid distribution with heat conduction given by Ellis [30] is given by
\[
T^i_j = (\varepsilon + p) v^i v^j + p g^i_j + q_i v^j + q_j v^i + E^i_j,
\]
(2)

To get a determinate model, we have assumed the condition \( a=b^2 \) between metric potentials. The various physical and geometrical aspects of the model viz pressure, density, shear, expansion factor, electromagnetic field tensor and rotation are also discussed. The physical and geometrical aspects of the model together with singularities are discussed.

Here \( E^i_j \) is the electromagnetic field given by Lichnerowicz [31] as
\[
E^i_j = \mp \left[ h^i_k \left( v^j_k + \frac{1}{2} g^j_k \right) - h^j_k h^i_k \right],
\]
(6)

where \( \mp \) is magnetic permeability and \( h_i \) is the magnetic flow vector defined by
\[
h_i = \sqrt{-g} \ e_{ijk} F^{jk},
\]
(7)

Here \( F_{ij} \) is the electromagnetic field tensor and \( e_{ijk} \) the Levi-Civita tensor density.

From (7) we find that \( h_1=0, h_2=0, h_3=0 \). This leads to \( F_{12} = 0 = F_{13} \) by virtue of (7). We also find that \( F_{14} = 0 = F_{24} \) due to the assumption of infinite conductivity of the fluid. We take the incident magnetic field to be in the direction of x-axis so that the only non-vanishing component of \( F_{ij} \) is \( F_{23} \). The first set of Maxwell’s equation
\[ F_{ijk} + F_{jki} + F_{kij} = 0 \].

leads to \[ F_{23} = Hsiny \text{ (say)} \).

Now \[ h_1 = \frac{H}{\mu_0} \cosh \lambda \], \[ h_4 = \frac{H}{\mu_0} \sinh \lambda \].

Since

\[
\begin{align*}
| b |^2 &= b_i h_i' \\
&= h_1 h^1 + h_2 h^2 \\
&= g_{ij} (h_i')^2 + g_{44} (h_4)^2 \\
&= \frac{H^2 \cosh^2 \lambda}{\mu^2 b^2} - \frac{H^2 \sinh^2 \lambda}{\mu^2 b^2} \\
&= \frac{H^2}{\mu^2 b^2}.
\end{align*}
\]

Equation (6) leads to

\[
E_1 = -\frac{H^2}{2b^2} = -E_2 = -E_3 = E_4. \tag{8}
\]

Here \( p \) is the isotropic pressure, \( \varepsilon \) the matter density and \( q_i \) the heat conduction vector orthogonal to \( v^i \). The fluid flow vector \( v^i \) has the components \( \left( \frac{\sinh \lambda}{a}, 0, 0, \cosh \lambda \right) \) satisfying the condition (3) and \( \lambda \) is the tilt angle.

The Einstein’s field equation

\[
\begin{align*}
R_j^i - \frac{1}{2} R g_j^i &= -8\pi T_j^i. \tag{A=0} \\
( & c = G = 1)
\end{align*}
\]

The field equation for the line element (1) leads to

\[
\begin{align*}
\frac{2b}{b} + \frac{b^2}{4b} - \frac{3a^2}{4b^2} + \frac{1}{b^2} &= -8\pi \left[ (\varepsilon + p) \sinh^2 \lambda + p + 2q_i \frac{\sinh \lambda}{a} - \frac{H^2}{2\mu b^2} \right], \tag{9}
\end{align*}
\]

\[
\begin{align*}
\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{ab}{4b} + \frac{a^2}{b^2} &= -8\pi \left[ p + \frac{H^2}{2\mu b^2} \right], \tag{10}
\end{align*}
\]

\[
\begin{align*}
\frac{2\dot{a}}{ab} \frac{\dot{b}}{b} + \frac{a^2}{b^2} + \frac{1}{b^2} - \frac{4a}{b^2} &= 8\pi \left[ (\varepsilon + p) \cosh^2 \lambda - p + 2q_i \frac{\sinh \lambda}{a} + \frac{H^2}{2\mu b^2} \right], \tag{11}
\end{align*}
\]

\[
(\varepsilon + p) a \sinh \lambda \cosh \lambda + q_i \cosh \lambda + q_i \frac{\sinh^2 \lambda}{\cosh \lambda} = 0. \tag{12}
\]

Here \( (.) \) denotes the ordinary differentiation with respect to cosmic time 't'.

3. Solution of the Field Equations

Equations from (9) to (12) are four equations in six unknown \( a, b, \varepsilon, p, \lambda \) and \( q_i \). For the complete determination of these quantities we need two extra conditions.

Firstly, we assume that the model is filled with stiff fluid distribution which leads to \( \varepsilon = p \).

Secondly, we assumed relation between metric potentials as
\[ a = b^2. \]  

Equations (9) and (11) lead to
\[ \frac{2b}{b} + \frac{2b^3}{a^2} + 2\frac{ab}{b^2} + \frac{2}{b^3} a^2 = 8\pi (\varepsilon - p) + \frac{K}{b^3}. \]  

Using stiff fluid condition from (13) in equation (15), we have
\[ \frac{2b}{b} + \frac{2b^3}{a^2} + 2\frac{ab}{b^2} + \frac{2}{b^3} a^2 = \frac{K}{b^3}, \]  

where \( K = \frac{8\pi H^2}{\mu}. \)

Again using (14) in equation (16), we have
\[ \frac{2b}{b} + \frac{6b^3}{a^2} + \frac{2}{b^3} = \frac{K}{b^3}. \]  

Let \( b = f(b) \) and \( \dot{b} = ff'. \)

Using (18) in equation (17), we have
\[ ff' + \frac{6b^2}{b} = \frac{K + b^2 - 2b^2}{b^3}. \]  

Equation (19) leads to
\[ b = \frac{1}{2b^7} \left[ 6Kb^3 + 3b^6 - 8b^5 + 24N \right], \]  

\[ \dot{b} = \frac{b^5 - 2Kb^4 - 24N}{4b^7}, \]  

where \( 'N' \) is constant of integration.

The metric (1) reduces to the form
\[ ds^2 = -\left( \frac{dt}{db} \right)^2 db^2 + b^5 (b^3 dx^2 + dy^2) + b^7 (\sin^2 y + b^5 \cos^2 y) dz^2 - 2b^4 \cos y dx dz. \]  

Using (20) in the metric (22), we have
\[ dS^2 = \frac{-24T^6dT^2}{[6KT^4 + 3T^8 - 8T^6 + 24N]} + T^2 (T^2dx^2 + dY^2) + T^2 (\sin^2 Y + T^2 \cos^2 Y) dZ^2 - 2T^4 \cos YdXdZ, \]  

where \( b = T, x = X, y = Y \) and \( z = Z. \)

### 4. Some Physical and Geometrical Features

The matter density and isotropic pressure for the model (23) are given by
\[ 8\pi \varepsilon = 8\pi p = \frac{120N + 32T^6 - 27T^8 - 18KT^4}{24T^8}. \]  

The tilt angle \( \lambda \) is given by
\[ \cosh \lambda = \sqrt{\frac{120N + 8T^6 - 9T^8}{120N - 16T^6 + 9T^8 + 18KT^4}}, \]  

\[ \sinh \lambda = T^2 \sqrt{\frac{6(4T^2 - 3T^2 - 3K)}{120N - 16T^6 + 9T^8 + 18KT^4}}. \]

The scalar expansion \( \theta \) calculated for the flow vector \( v_i \) is given by
\[ \theta = \frac{4\psi_4}{T^3} \sqrt{\frac{6KT^4 + 3T^8 - 8T^6 + 24N}{6\psi_1\psi_3^3}}. \]
The flow vectors $v^i$ and heat conduction vectors $q^i$ for the model (20) are given by

$$v^1 = \frac{6\psi_2}{\psi_3}, \quad (28)$$

$$v^4 = \frac{\psi_1}{\psi_3}, \quad (29)$$

$$q_1 = -\frac{\psi_1}{24\pi T^4} \sqrt{\frac{6\psi_2}{\psi_3}}, \quad (30)$$

$$q_4 = \frac{6\psi_2}{24\pi T^4} \sqrt{\frac{\psi_3}{\psi_3}}. \quad (31)$$

The non-vanishing components of shear tensor $(\sigma_{ij})$ and rotation tensor $(\omega_{ij})$ are given by

$$\sigma_{11} = 8\psi_5 \sqrt{\frac{\psi_1(6KT^4 + 3T^8 - 8T^6 + 24N)}{6\psi_3}}, \quad (32)$$

$$\sigma_{22} = -2\psi_5 \sqrt{\frac{6KT^4 + 3T^8 - 8T^6 + 24N}{3T^2}} \frac{1}{6\psi_1\psi_3}, \quad (33)$$

$$\sigma_{33} = -2(\psi_6 T^2 \cos^2 y - \psi_5 \sin^2 y) \sqrt{\frac{6KT^4 + 3T^8 - 8T^6 + 24N}{3T^2}} \frac{1}{6\psi_1\psi_3}, \quad (34)$$

$$\sigma_{44} = 8\psi_2\psi_5 \sqrt{\frac{6KT^4 + 3T^8 - 8T^6 + 24N}{6\psi_1\psi_3}}, \quad (35)$$

$$\sigma_{14} = -4\psi_5 \sqrt{\frac{\psi_1(6KT^4 + 3T^8 - 8T^6 + 24N)}{3T^2}} \frac{1}{\psi_3}, \quad (36)$$

$$\omega_{14} = \sqrt{\frac{\psi_1(6KT^4 + 3T^8 - 8T^6 + 24N)}{\psi_3}}. \quad (37)$$

In the absence of magnetic field i.e. $K=0$, the above quantities lead to

$$8\pi \varepsilon = \frac{8\pi p}{\epsilon} = \frac{120N + 32T^6 - 27T^8}{24T^8}. \quad (38)$$

$$\theta = \frac{4\psi_7}{T^4} \sqrt{\frac{3T^8 - 8T^6 + 24N}{6(120N + 8T^6 - 9T^8)(120N - 16T^6 + 9T^8)^3}}, \quad (39)$$

$$\cosh \lambda = \frac{120N + 8T^6 - 9T^8}{120N - 16T^6 + 9T^8}, \quad (40)$$

$$\sinh \lambda = T^3 \sqrt{\frac{6(4 - 3T^2)}{120N - 16T^6 + 9T^8}}, \quad (41)$$

$$\sigma_{11} = 8\psi_8 \sqrt{\frac{(3T^8 - 8T^6 + 24N)(120N + 8T^6 - 9T^8)}{6(120N - 16T^6 + 9T^8)^5}}, \quad (42)$$

$$\sigma_{22} = -\frac{2\psi_8}{3T^2} \sqrt{\frac{3T^8 - 8T^6 + 24N}{6(120N + 8T^6 - 9T^8)(120N - 16T^6 + 9T^8)^3}}, \quad (43)$$
\[\sigma_{33} = -\frac{2(\psi_3 T^2 \cos^2 y - \psi_8 \sin^2 y) \sqrt{(3T^8 - 8T^6 + 24N)}}{3T^2} \]
\[\sigma_{44} = 8T^2 (4 - 3T^2) \psi_8 \sqrt{\frac{3T^8 - 8T^6 + 24N}{6(120N + 8T^6 - 9T^8)(120N - 16T^6 + 9T^8)^3}}, \]  
\[\sigma_{14} = -\frac{4\psi_8}{3T} \sqrt{\frac{(4-3T^2)(3T^8 - 8T^6 + 24N)}{(120N - 16T^6 + 9T^8)^3}}, \]
\[\omega_{14} = \frac{(4 - 3T^2)(3T^8 - 8T^6 + 24N)}{(120N - 16T^6 + 9T^8)^3}, \]
\[v^1 = T \frac{6(4 - 3T^2)}{120N - 16T^6 + 9T^8}, \]
\[v^4 = \frac{120N + 8T^6 - 9T^8}{120N - 16T^6 + 9T^8}, \]
\[q_1 = -\frac{120N + 8T^6 - 9T^8}{24\pi T^3} \sqrt{\frac{6(4 - 3T^2)}{120N - 16T^6 + 9T^8}}, \]
\[q_4 = \frac{(4 - 3T^2)}{4\pi} \frac{120N + 8T^6 - 9T^8}{120N - 16T^6 + 9T^8}. \]

Here
\[\psi_1 = 120N + 8T^6 - 9T^8, \]
\[\psi_2 = 4T^2 - 3T^4 - 3K, \]
\[\psi_3 = 120N - 16T^6 + 9T^8 + 18KT^4, \]
\[\psi_4 = 7200N^2 - 600N^6 - 1080N^8 + 540KNT^4 - 104.5KT^{12} + 90K^{10} + 117T^{14} - 64T^{12} - 40.5T^{16}, \]
\[\psi_5 = 3600N^2 + 1920N^6 - 2160N^8 - 540KNT^4 - 121.5KT^{12} + 72K^{10} + 72T^{14} - 32T^{12} - 20.25T^{16}, \]
\[\psi_6 = 3600N^2 - 2400N^6 + 2160N^8 + 1620KNT^4 + 40.5KT^{12} + 36T^{14} - 32T^{12} - 20.25T^{16}, \]
\[\psi_7 = 7200N^2 - 600N^6 - 1080N^8 + 117T^{14} - 64T^{12} - 40.5T^{16}, \]
\[\psi_8 = 3600N^2 + 1920N^6 - 2160N^8 + 72T^{14} - 32T^{12} - 20.25T^{16}, \]
\[\psi_9 = 3600N^2 - 2400N^6 + 2160N^8 + 36T^{14} - 32T^{12} - 20.25T^{16}. \]

The rates of expansion \(H_i\) in the direction of x, y and z axes are given by
\[H_1 = \frac{1}{T^4} \sqrt{\frac{(6KT^4 + 3T^8 - 8T^6 + 24N)}{6}}, \]
\[H_2 = H_3 = \frac{1}{2T^4} \sqrt{\frac{(6KT^4 + 3T^8 - 8T^6 + 24N)}{6}}. \]

5. Conclusion

The model (21) represents a tilted model. The model starts with a big-bang at \(T=0\) and the expansion in the model decreases as time increases. The model has cigar type singularity at \(T=0\).
MacCallum, [32]). At the initial stage energy density $\epsilon \rightarrow \infty$ but it vanishes asymptotically. Tiltness of the model increases when magnetic field increases. The velocity components vanish at initial stage and tend to infinity on later stage. The heat conduction vectors tend to infinity at initial stage but vanish asymptotically. The model represents expanding, shearing, rotating and tilted universe in general. Since $T \rightarrow \infty \frac{\sigma}{\theta} \neq 0$, the model does not approach isotropy for large value of $T$. The Hubble components become infinitely large at initial stage whereas vanish asymptotically. The model completely follows the restrictions imposed by Ellis [30] and Dunn and Tupper [22] for tilted models. Therefore, this model is a realistic model.

In the absence of magnetic field, density is the decreasing function of time. The tilt angle is zero at $T=0$ so that model is non-tilted. Also, the rotation is throughout uniform In general, the model represents an anisotropic universe.

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