Multi-view graph structure learning using subspace merging on Grassmann manifold

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Abstract
Many successful learning algorithms have been recently developed to represent graph-structured data. For example, Graph Neural Networks (GNNs) have achieved considerable successes in various tasks such as node classification, graph classification, and link prediction. However, these methods are highly dependent on the quality of the input graph structure. One used approach to alleviate this problem is to learn the graph structure instead of relying on a manually designed graph. In this paper, we introduce a new graph structure learning approach using multi-view learning, named MV-GSL (Multi-View Graph Structure Learning), in which we aggregate different graph structure learning methods using subspace merging on Grassmann manifold to improve the quality of the learned graph structures. Extensive experiments are performed to evaluate the effectiveness of the proposed method on two benchmark datasets, Cora and Citeseer. Our experiments show that the proposed method has promising performance compared to single and other combined graph structure learning methods.

Keywords Graph convolutional network · Graph structure learning · Multi-view learning · Subspace merging · Grassmann manifold

1 Introduction
Graphs are known as a powerful tool for representing complex relationships between non-Euclidean data such as social networks, biological networks [45], e-commerce networks,
communication networks, sensor networks, co-word networks [37], and blockchain networks [34]. Conventional deep learning algorithms cannot be directly applied to graph data because this type of data is inherently complex and irregular. Thus, a new class of methods called Graph Neural Networks (GNNs) has been developed to apply deep learning methods to graph data [24].

GNNs use an information diffusion mechanism to refine the nodes’ representation. According to this mechanism, each node aggregates the information of its neighbors and combines the aggregated data with its feature vector. This process is repeated \( k \) times to obtain a new representation of each node according to the structural information of its \( k \)-hop neighbors [31, 51, 59, 60]. These methods utilize both node feature and network topology in an interactive learning process. Based on the used aggregation process, there are several variants of GNNs, such as Graph Convolutional Networks (GCN) [3, 6, 27], Graph Attention Networks (GAT) [48, 57], and Graph Autoencoder Networks (GAE) [26].

Recent studies show that the performance of GNNs is highly dependent on the quality of the used graph structure [5, 13, 19, 20, 43, 56, 61, 64]. An optimal graph provides better performance in the downstream task [4, 18]. For example, in node classification, the optimal graph leads to better node representation and subsequently better node classification. In addition, the optimal graph is robust against noise, i.e. the performance of the model does not degrade significantly when some edges are added or removed to/from the graph. However, in many applications, a sound and complete graph is not available. In some other applications, such as in natural language processing tasks, the data is not inherently graph-based. In these cases, the graph structure should be learned from the observed data. The simplest way to create a graph structure is to calculate dependencies between samples using a similarity function, and exploit a thresholding method, such as K-Nearest Neighbors (KNN). However, these methods face several important challenges, such as choosing an appropriate similarity function and selecting the value of the parameters (e.g., \( K \)), which can have a profound effect on the quality of the learned graph.

In recent years, many methods have been proposed for Graph Structure Learning (GSL) such as GLCN [18], ProGCN [19], and GRCN [55], to obtain more quality graphs. However, these methods deal with some challenges, such as:

- Most existing methods learn a single relationship between pairs of nodes. However, in many applications, there are several types of relationships between two nodes, such as the multi-relational structure of a knowledge graph; different protein-protein interactions in biological systems; and friendship, acquaintance, and business relations in social networks. Therefore, learning only a single graph cannot completely show the relationships between nodes.
- Existing single GSL methods utilize different techniques to discover similarities between nodes. As a result, each method explores a different aspect of the data that cannot be discovered by the other method.
- A single GSL method may have good performance on a dataset, while its performance is weak in another dataset. As our investigations show the GLCN method [18] cannot add edges to the graph, so if the graph has many missing edges, this method has poor performance.

To address these challenges, this paper introduces Multi-view Graph Structure Learning (MV-GSL) method that merges different learned graphs based on the Grassmann manifold [7] to exploit the multi-view knowledge. In fact, this paper combines different GSL methods. The fusion method provides more comprehensive knowledge of data and improves the
generalization capability of the model. In addition, the usage of multiple views can alleviate the negative impact of noise [49, 53]. There are different methods for combining graph structures. The easiest method is to average different graph structures. However, such approach mixes the local structure of each graph structure and thus leads to poor performance [39]. Therefore, researchers have proposed many methods for merging multiple views, such as co-training learning, multiple kernels learning, and subspace learning (described in Section 2.4).

Among these methods, we have selected the subspace merging based on the Grassmann manifold for several reasons. 1) Recent studies show that this method is flexible and has satisfactory performance [7, 25]. 2) Subspace learning methods assume that there is a common latent subspace between views that must be explored. In our proposed method, different graphs represent relationships among the same set of nodes from different aspects and features of the nodes are shared by all graphs. So, there is a shared informative graph that must be explored. 3) In the Grassmann merging, the aggregated graph is created using the projection distance between the basis vectors, instead of the point-to-point distance between the subspaces in the Euclidean space. Thus, the learned graph is more robust and stable against noise [39].

In summary, this work has the following contributions:

- We conduct an extensive study on single-graph structure learning methods.
- We propose a new graph structure learning approach that integrates multiple graphs using subspace merging. The adopted graphs are automatically constructed using the existing single-graph structure learning methods. The aggregated graph is used in a node classification task.
- Comprehensive experiments on two benchmark datasets show that our proposed method achieves better or highly competitive results compared to the competitors.

The rest of this paper is structured as follows. Section 2 describes the theoretical basic concepts. Related work is reviewed in Section 3. The key idea of the proposed method is introduced in Section 4. Section 5 presents experimental results and analysis. Finally, the paper is concluded in the Section 6.

2 Background

In this section, we introduce the related theoretical basis, including the graph convolutional network, Graph structure learning, and multi-view learning.

2.1 Graph

In this paper, a graph is represented as $G = (V, E)$, where $V \in R^n$ is a set of nodes and $E \subseteq \{(i, j)|i, j \in V\}$ is a set of relational edges. Each node $i$ has a set of features or a signal $x_i \in R^d$ and a label $y_i \in R^c$. In addition, each edge can have a set of features $e_i \in R^f$. Here, $n, c, d, f$ are the number of nodes, the number of classes, the size of feature vector of nodes, and the size of feature vector of edges, respectively. The structure of a graph is determined by the adjacency matrix $A \in R^{n \times n}$, where if there is an edge between two nodes, the corresponding element is greater than 0, and otherwise it is equal to 0. In addition to the adjacency matrix, the graph structure can be represented using the Laplacian matrix $L \in R^{n \times n}$, which is defined as follows:

$$L = D - A,$$
Where $D \in R^{n \times n}$ is a degree matrix whose diagonal elements are equal to the degree of each node and non-diagonal elements are 0. To create numerical stability in deep models, the symmetric normalized version of Laplacian matrix is usually used. It is defined as follows:

$$L = D^{-\frac{1}{2}} LD^{-\frac{1}{2}} = I - D^{-\frac{1}{2}} AD^{-\frac{1}{2}}. \quad (2)$$

The Laplacian matrix has different valuable and important properties. For example, when the adjacent matrix is non-negative and symmetric, its Laplacian matrix is a real symmetric positive semi-definite matrix and has a full set of orthonormal eigenvectors. Therefore, it can be rewritten as follows:

$$L = U \Lambda U^T, \quad (3)$$

where $U \in R^{n \times n}$ is the matrix of eigenvectors and $\Lambda$ is the diagonal matrix of eigenvalues. The Laplacian matrix can also generalize the concept of Fourier transform into graph signals through spectral decomposition [8].

### 2.2 Graph neural network

Graph Convolutional Networks (GCNs) are one of the most popular types of GNNs. They generalize existing convolution operations for the Euclidean data (e.g., image, text, etc) to the non-Euclidean data [51]. Convolution operations in the Euclidean data learn new properties from adjacent pixels/words, similarly, convolution operations in graphs seek to learn new features from adjacent nodes (See Fig. 1).

In literature, two strategies have been used to define convolution filters. These strategies create two categories of graph convolution networks, including spectral or frequency-based methods and spatial-based methods.

#### 2.2.1 Spectral methods

The purpose of spectral GCN is to define graph convolution using Fourier transform. In the Fourier domain, the convolution operation is calculated by eigendecomposition of the Laplacian matrix [31] as follows:

$$h = x \ast g = \mathcal{F}^{-1}(\mathcal{F}(x) \odot \mathcal{F}(g)) = U(U^T g \odot U^T x), \quad (4)$$

where $x$ is the input feature vector (signal graph) and $h$ is the updated feature vector, $\mathcal{F}$ indicates Fourier function, and $U$ is eigenvectors matrix of the normalized Laplacian matrix.

![Fig. 1](image-url) Convolution operation in Euclidean data (image or text) (right) and non-Euclidean data (left)
By defining the spectral filter as $g_w = \text{diag}(U^T g)$, the graph convolution can be simplified as follows:

$$h = x \ast g_w = U g_w U^T x.$$  \hspace{1cm} (5)

Therefore, different types of spectral convolution networks can be formed depending on the filter $g_w$. For example, Bruna et al. [3] proposed the spectral CNN, which considers the filter as a diagonal matrix of learnable parameters $g_w = w_{ij}$. Therefore, the graph convolution operation is defined as follows:

$$
h_{:,j} = x_{:,j} \ast g_w = \sigma \left( \sum_{i=1}^{d_l-1} \sum Uw_{i,j} U^T x_{:,j} \right), \quad j = 1, 2, \ldots, d_l,
$$  \hspace{1cm} (6)

In this equation, $d_l$ is the number of output channels and $\sigma$ represent a non-linear activation function. However, this operation leads to high computational complexity, non-scalability for large graphs (due to the need to calculate the eigenvectors of the Laplacian matrix), and non-local filters (due to the use of non-parametric filters). To solve these challenges, Defferard et al. [6] proposed ChebNet, which approximates the convolution filter using $K$th order Chebyshev polynomials, i.e., $g_w = \sum_{k=0}^{K} w_k T_k(\hat{\Lambda})$ where $\hat{\Lambda} = \frac{2\lambda}{\lambda_{\text{max}}} - I_n$ and $\lambda_{\text{max}}$ is the largest eigenvalue. Then, the convolution operation is calculated as follows:

$$h = U \left( \sum_{i=0}^{k} w_i T_i(\hat{\Lambda}) \right) U^T x = \sum_{i=1}^{k} w_i T_i(\hat{\Lambda}) x,$$  \hspace{1cm} (7)

where $L = \frac{2L}{\lambda_{\text{max}}} - I_n$ and $T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x)$, with $T_0(x) = 1$, $T_1(x) = x$. This method solves the localization issue and reduces computational complexity. However, it can lead to the overfitting issue in local neighborhood structures for graphs with high degree distributions because it does not limit the polynomial order ($k$). To solve this problem, Kipf and Welling [27] have presented a simpler version of ChebNet, which limits the order of Chebyshev polynomials to 1 ($k = 1$) and approximates the largest eigenvalue ($\lambda_{\text{max}} = 2$). In the graph network literature, this network is known as the GCN. In this method, the convolution operation is simplified as follows:

$$h = \sum_{i=0}^{1} w_i T_i(\hat{\Lambda}) x = w_0 T_0(\hat{\Lambda}) x + w_1 T_1(\hat{\Lambda}) x = \sigma \left( w \left( I_n + \hat{D}^{-\frac{1}{2}} \hat{A} \hat{D}^{-\frac{1}{2}} \right) x \right),$$  \hspace{1cm} (8)

where $\hat{A} = A + I$ and subsequently, $\hat{D}_{ii} = \sum_j \hat{A}_{ij}$. The first-order approximation allows the convolution operation to update the representation of each node with its immediate neighbor information, i.e., it is spatially localized. In fact, it can be considered as a bridge between the spectral and spatial methods [58, 59]. Therefore, in order to use the information of the k-hop neighbors, several layers of convolution can be stacked.

### 2.2.2 Spatial methods

While spectral methods are suitable for stationary, simple, and small graphs, spatial methods can be used for dynamic and large graphs with rich feature information. Spatial methods perform filtering operations directly by defining the spatial structures of neighboring nodes in the graph [31, 51].

The main challenge of these methods is to define operations for different neighborhood sizes, maintain local stability and weight sharing [31]. To solve these challenges, Duvenaud et al. [11] have provided the Neural FPS method, which learns different weight matrices
for nodes with different degrees. Niepert, Ahmed, and Kutzkov [33] have extracted a fixed number of neighboring nodes. Hamilton, Ying, and Leskovec [16] have proposed the GraphSAGE method, which samples a constant-sized neighborhood for each node, then applies an aggregation function (mean or maximal properties of the sample neighbors) to the samples. However, these methods cannot fully utilize the capacity of all neighbors due to sampling from the neighbors.

### 2.3 Graph structure learning

The purpose of GSL is to learn the best representation of the observed data in the form of a graph [38, 46]. In other words, suppose $X \in \mathbb{R}^{M \times N}$ is the feature matrix, where $M$ is the number of samples and $N$ indicates the number of features, and there is a prior knowledge of data such as data distribution. In this case, relationships between samples can be represented in the form of a graph [9].

So far, various graph structure learning methods have been developed. Dong et al. [9] have been divided into three general approaches, including statistical or probability-based methods, physically-based methods, and signal processing-based methods.

- **Statistical or probability-based methods**: The observed data is obtained from a probability distribution. So, the probability distribution is used to model the relationships between the data. This means that there is a graph whose structure is based on the probability distribution of the observed data, such as LDS method [14].

- **Physically-based methods**: The observed data is the results of some physical phenomena. Thus, the aim is to infer the intrinsic structure of the graph from the physics of the observed data. For example, the information diffusion model on social networks.

- **Signal processing-based methods**: The observed data is represented based on its behavior in the field of graph spectrum. In these methods, the goal is to learn a graph that includes certain properties of the observed data, such as signal smoothness or sparse graph. In this category, smoothness-based methods are one of the simplest and most popular methods. (See [9, 46] for studying other methods.) Smoothness-based methods assume that signals (labels) change smoothly and slowly between adjacent nodes. For example, the temperature is identical in different places in a geographical area. The smoothness of the signals (labels) is usually measured using Dirichlet energy. Dirichlet energy is based on the quadratic form of the Laplacian matrix. It is calculated for each signal $x$ in the graph using the following equation:

$$tr(X^T LX) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} \| x_i - x_j \|^2_2.$$

(9)

Thus, in these methods, the goal is to find the matrix that minimizes the signal (label) variation on this matrix as follows:

$$loss_{gl} = \arg \min_{L > 0} tr(X^T LX) + \lambda f(L),$$

(10)

where $f(L)$ is the regularization term to guarantee the learning of the valid matrix (sparse matrix, low-rank matrix, or symmetric matrix).

- **Other**: Some research studies use other methods to learn graph. For example, CoGl [44] that uses reconstruction error to learn graphs. In this method, the updated feature representation must be close to the original feature representation.
In addition to the mentioned categories, structure learning methods can be categorized into two classes, 1) *task-independent learning methods* and 2) *task-driven learning methods*.

The first class learns the optimal graph from the observed data, then use the learned graph in the downstream task (node, graph, or edge classification), such as [8, 12, 22]. However, these methods only produce graphs based on structural information independently downstream tasks. This can reduce performance in combination with graph neural networks [54]. In addition, these approaches may create subgraphs with suboptimal structural features because they do not obtain any feedback from the downstream task. The second class follows this key idea that learning a better graph structure depends on better classification of the node (graph or edge), and vice versa. Thus, to achieve the optimal solution, they simultaneously learn the graph structure and the node (graph or edge) classification such as [14, 18, 19, 28, 36, 54, 55].

Also, according to another view, structure learning methods can be categorized into two major groups: 1) *Full graph parameterization* and 2) *Similarity-metric learning*.

The first group of methods uses learnable parameters to model the weight of edges, such as ProGCN [19], and GLNN [15]. These methods are flexibility designed, but have high memory overhead and time cost, and are not scalable for large graphs due to the over-parameterized problem [14, 28, 55]. To solve these problems, the second group learns a similarity metric between the pairs of nodes embedding to model the weight of edges [62]. This increases the learning speed and reduces the number of learnable parameters. Some examples of this group are the GAT [48] and GLCN [18] methods, which use the attention-based similarity measure to learn the weight of the edges, or the RGLN [47] and AGCN [28] methods, which use the Mahalanobis distance measure.

### 2.4 Multi-view learning

Multi-view learning is a rapidly growing research field in machine learning. It integrates and learns from multiple views, including multiple relation or various types of features, to improve the generalization performance [49, 52, 53].

Views can be integrated in three levels, including *feature, relation (structure)* and *results*. In recent years, researchers have proposed many different algorithms based on feature and relation levels. For example, co-training learning, multiple kernels learning, and subspace learning are among the first approaches for multi-view learning at feature and relation levels [52, 53]. Co-training approaches execute an iterative process to maximize mutual agreement between learners on different views. Multi-kernel approaches identify different kernels corresponding to different views and combine them linearly or nonlinearly. Subspace approaches identify common latent subspaces between different views. Dong et al. [7] and Khan and Blumenstock [25] believe that subspace learning approaches are better than the other two approaches due to feasibility and satisfactory performance. Therefore, we focus on this approach in this paper.

In addition to these approaches, the emergence of deep learning and graph-based neural networks has led to the formation of another class based on these networks called multi-view GNN [53], such as MR-GCN [17], Relational Graph Convolutional Network (RGCN) [41], and mGCN [32].

Finally, ensemble learning approaches are among the most popular methods for combining results. Ensemble approaches create a model on each view and then combine their results using different methods such as voting, averaging, and meta learning [40]. For example, Keramatfar, Amirkhani, and Bidgoly [23] have proposed a stacking model based on
multiple GCNs to leverage the knowledge embedded in different graphs. However, these methods ignore the correlation between views [17].

3 Related work

In this section, we review recent related works about graph structure learning and multi-view learning based on GCN.

3.1 Graph structure learning

Graph Attention Network (GAT) is one of the first methods that simultaneously learn graph and prediction task [48]. GAT does not explicitly generate graphs, but only learns the weight of the relation between each node and its neighbors using the attention-based similarity measure. Therefore, this method cannot add a new edge, but only reweights the edges with a local view.

Inspired by the GAT, Jiang et al. [18] have proposed the GLCN method based on weight calculation between each node and all graph nodes. In fact, this method utilizes a global view of the graph to learn its general structure. In addition, they have added graph learning loss to the classification loss to ensure that the matrix is valid (sparse and smooth). However, this method also reweights existing edges and cannot add edges.

To solve the edge addition challenge, Yu et al. [55] have developed the GRCN method, which includes a revision module for predicting missing edges and reweighting existing edges. Like the previous two methods, this approach uses similarity between pairs of nodes to identify relationships between nodes. However, there is one difference, GRCN utilizes the updated representation of nodes and the dot product function to calculate the similarity. Then, The obtained similarity matrix is added to the original adjacency matrix to insert, delete, and reweight edges. They also select only edges with high confidence to reduce computational costs and prevent excessive noise. However, this method can only improve the existing graph and cannot be used for cases where the graph does not exist.

Chen et al. [4] have transformed the graph structure learning problem into an iterative process of similarity metric learning. They used the multi-head version of the weighted cosine similarity function to calculate the similarity between pairs of nodes and construct the graph. Also, to reduce computational costs and memory consumption, they have developed an anchor-based method that learns the node-anchor affinity matrix instead of learning the similarity matrix between all pairs of nodes.

Similar to previous methods Tang et al. [47] have provided similarities metric learning-based method under a low-rank assumption. They used the Mahalanobis distance measure to identify relationships between nodes. Then, it is optimized by minimizing a similarity-preserving loss. Also, this method considers a low-rank model between features to decompose the similarity matrix into a low-dimensional matrix for implementing and reducing computational costs effectively.

Jin et al. [19] have developed a robust graph neural network method based on three characteristics of real graphs, including feature smoothness, low-rank, and sparsity graph to clear the perturbed graph against adversarial attacks.

Like the previous method, Gao et al. [15] have suggested a method to learn the parametrized graph based on the properties of real graphs, including feature smoothness, sparsity graph, loopless and symmetric graph.
Zhan et al. [56] mention that point-to-point relationships cannot show complex relations. So, they use hypergraph with complex multivariate relations to establish the initial graph. Then, they learn a new parameterized graph based on smoothness of labels and closeness of the new graph to the initial graph.

Lin, Kang, Liao, Zhao, and Chen [30] have combined GAT and GLCN methods to present a new method called DGL. In this method, the similarity graph is learned from the feature matrix based on the GLCN similarity measure. Then, the learned graph and the feature matrix are fed to the GAT method to update the feature matrix. After applying the GLCN and GAT layers several times, three local and global representations are created for each node. Finally, the weighted sum of the features is used to classify nodes.

Shi et al. [44] believe that existing methods seek to adapt the content network and original network structure, while node content and network structure are two distinct but highly correlated data sources. Therefore, they are different in terms of representation learning. As a result, instead of creating compatibility between the two networks, they have proposed a new co-alignment method that models the incompatibility of the two networks for optimal node representation. This method uses a co-training manner so that the content network learns a good node representation for best network reconstruction, and the origin network structure learns an optimal node representation. Like the GAT and GLCN methods, they have used the attention-based similarity measure to calculate the similarity between pairs of nodes.

Peng, Kong, Liu, and Kuang [35] have used the Gaussian kernel to calculate the weight of the edges. They have combined the learned graph with the original graph to use the information in the initial graph to solve the add edge problem of GLCN.

Yang et al. [54] believe that adjacent nodes tend to share the same label, so they have modeled the graph structure learning problem as a label propagation process (label smoothness). They have used the dot product of predicted labels to calculate the similarity between pairs of nodes.

Li et al. [28] have proposed the AGCN method, which is the first study for learning graph for each sample. This method learns a residual Laplacian matrix and add it to the original Laplacian matrix. This residual matrix is obtained from the learned adjacency matrix by calculating the Gaussian normalization of the Mahalanobis distance between pairs of nodes. Unlike these methods, which can only learn one type of relationship between nodes, our proposed method seeks to learn several types of relationships between nodes. Table 1 summarizes the reviewed papers.

3.2 Multi-view learning

Zhuang and Ma [63] proposed DualGCN method. This method uses adjacent matrix and positive pointwise mutual information (PPMI) matrix to embed local consistency-based knowledge and global consistency-based knowledge, respectively. They use a new regularizer function to control the different convolutional results for better label prediction. Note that in this method, the PPMI matrix only helps to better represent the features of adjacent nodes in the adjacency matrix, and no combinations are made at the feature level or graph or results. Also, unlike our method, the graphs are fixed in DualGCN.

Schlichtkrull et al. [41] have proposed the RGCN method for multi-relational graphs by integrating features. In this method, after applying a GCN on each relation and updating the features, the average of the updated features on each node is used to predict the label of nodes.
| Method       | Learning method | Similarity-based | Similarity measure      | Task                        |
|--------------|-----------------|------------------|-------------------------|-----------------------------|
| GAT [48]     | –               | Similarity-based | Attention-based         | Node classification         |
| GLCN [18]    | Smoothness-based| Similarity-based | Attention-based         | Node classification         |
| GRCN [55]    | Smoothness-based| Similarity-based | Dot product             | Node classification         |
| IDGL [4]     | Smoothness-based| Similarity-based | Multi-head weighted cosine similarity | Node and graph classification |
| RGLN [47]    | Smoothness-based| Similarity-based | Mahalanobis distance    | Node classification         |
| ProGCN [19]  | Smoothness-based| Full graph       |                         | Node classification         |
| GLNN [15]    | Smoothness-based| Full graph       |                         | Node classification         |
| [56]         | Smoothness-based| Full graph       |                         | Node classification         |
| DGL [30]     | Smoothness-based| Similarity-based | Attention-based         | Node classification         |
| CoGL [44]    | Reconstruction error | Similarity-based | Attention-based         | Node classification         |
| DGCN [35]    | Smoothness-based| Similarity-based | Gaussian kernel         | Node classification         |
| To-GCN [54]  | Smoothness-based| Similarity-based | Dot Product             | Node classification         |
| AGCN [28]    | –               | Similarity-based | Mahalanobis distance    | Graph classification        |
Ma et al. [32] have suggested the mGCN method at the feature level. They believe that each node has two types of relationships. First, the relationship between each node and its neighbors in each dimension (within-dimension interactions). Second, the relationship between each node and its copy in the other dimensions (across-dimension interactions). Thus, they create a specific-dimension representation for each node by combining the within- and across-dimension representations. Finally, they concatenate the specific-dimension representations to create a general representation for each node. This representation can be used to predict the label of nodes.

Khan and Blumenstock [25] have proposed a Multi-GCN method for multi-view learning. They have used Grassmann learning to merge different graphs. However, the used graphs in this method are static.

Lin, Wang, Liao, Zhao, and Chen [29] have proposed the SF-GCN method, which fusions different graphs by exploring the common and specific properties of structures. Unlike the previous method, SF-GCN uses the weighted sum of the graphs and considers the importance of each structure in the fusion process. It uses Grassmann learning for obtaining the weight of the importance of each graph.

Wang et al. [50] argue that the GCN method cannot adaptively integrate the information contained in the feature space and structure space. Therefore, they have proposed a multi-channel method called AM-GCN for combining this information. In addition to the topology graph, this method creates a feature graph using the cosine similarity based on the features of the nodes. Then, two GCN networks are created on the feature graph and the topology graph, and a shared GCN network is created using two graphs. The importance of the features obtained from these networks is identified using the attention mechanism. Finally, the extracted features are combined based on the attention weight for the classification task. This method also uses static graphs, unlike our proposed method.

Huang et al. [17] have proposed a special convolutional operation for multi-relation graphs based on the eigen-decomposition of a Laplacian tensor, which takes into account the correlations across the relations. The eigen-decomposition is formulated with a generalized tensor product, which can correspond to any unitary transform instead of limited merely to Fourier transform. This method also used fixed graphs during the training.

Yu et al. [55] theoretically show that the GRCN method can also be used for multiple graphs, due to the use of the addition operator. Therefore, an integrated matrix can be created according to the following steps. Firstly, GCN is applied on each graph; then, similarity matrices are calculated using the dot product of the new representation related to each node; finally, similarity matrices and initial matrices are summed together. This method also uses only one similarity measure to create multiple dynamic graphs. In addition, it uses only the classification loss function, which may not guarantee a valid graph.

Peng et al. [35] have presented the DGCN method, which combines both the features and the graph. Firstly, DGCN calculates the similarity graph of each view using the Gaussian kernel. Then, in order to benefit from the comprehensive information of all views, the weighted average of the learned graph corresponding to each view and the initial graphs of all views are fed into $M$ GCN networks. Finally, the updated representations of each node are combined for node classification. Although this method uses the dynamic graph, unlike our method, only one graph is learned from a view, and the learned graph is combined with the existing original graphs.

Adaloglou, Vretos, and Daras [1] have presented the MV-AGC method for multi-relational graphs. MV-AGC is an extended version of AGCN [28]. In this method, first a few similarity graphs are learned using the Mahalanobis distance measure. Then, in order to maintain the original structure of the graph, the initial Laplacian matrix is added to the
learned Laplacian matrix. Then, multiple GCNs are used to create multiple new representations for each node. Finally, the maximum of normalized representations of each node is used as the final representation to predict the label. Similar to our method, this method learns different relationships between pairs of nodes. However, it differs from our proposed method in several aspects. First, it is used to classify graphs, while we focus on node classification. Second, it utilizes a single learning method (Mahalanobis similarity measure) to create multiple graphs, which may not guarantee the different types of relationships. Third, since it just uses the classification loss function, it may not guarantee valid graph learning; while we include sparseness and smoothness of the learned graphs in the proposed method.

In summary, unlike our proposed method, most reviewed methods use static graph, while the static graph is not enough optimal for the downstream task. Some methods learn a single graph from data. While as stated earlier, a single graph may not be able to show comprehensive knowledge of data and is not robust against noise. Some methods also learn multiple graphs using a similarity measure that may not be able to learn more diverse relationships. Most methods use simple fusion methods such as sum and average to fuse structures or features. While our proposed method uses the Grassmann merging, which provides a better fusion graph and is more robust against noise because it uses the projection distance between the basic vectors. Table 2 summarizes the reviewed articles.

### 4 Proposed method

While learning multiple relationships between nodes can provide more comprehensive knowledge of the data and improve model performance, most existing methods learn a single relationship between pairs of nodes. In this section, we present the details of our multi-view graph structure learning approach, MV-GSL, which exploits several kinds of relationships between pairs of nodes.

| Method       | Graph type | Aggregation method | Aggregation level | Weighted or unweighted aggregation | Weighted method     |
|--------------|------------|--------------------|-------------------|------------------------------------|---------------------|
| DualGCN [63] | Static     | –                  | –                 | –                                  | –                   |
| RGCN [41]    | Static     | Average            | Feature           | Unweighted                         | –                   |
| mGCN [32]    | Static     | Concatenate        | Feature           | Weighted Full connected layer       |                     |
| Multi-GCN [25]| Static     | Grassmann learning | Structure         | Unweighted                         | –                   |
| SF-GCN [29]  | Static     | Sum and multiple   | Structure         | Weighted Grassmann learning         | –                   |
| AM-GCN [50]  | Static     | Sum                | Feature           | Weighted Attention                 |                     |
| MR-GCN [17]  | Static     | Multi-relational convolution operator | Feature | Unweighted – |
| GRCN [55]    | Dynamic    | Sum                | Structure         | Unweighted –                       |
| DGCN [35]    | Dynamic    | Average and sum    | Structure and feature | Weighted Attention – |
| MV-AGC [1]   | Dynamic    | Sum and max        | Structure and feature | Unweighted – |

Table 2 Summery of multi-view learning methods
Fig. 2 Framework of the proposed MV-GSL method

As shown in Fig. 2, the proposed framework consists of three modules: learning, merging, and classification. The learning module utilizes some of the single-graph structure learning methods to obtain multiple graph structures. The merging module unifies the learned graphs by subspace merging using Grassmann manifold. Finally, the classification module utilizes the unified graph to classify nodes. The details of each module are described in the following.

4.1 Learning

The purpose of this module is to create multiple graphs to be used as multiple views. For this purpose, different single-graph structure learning methods are used to learn $m$ different graphs $\tilde{A}_i$, $1 \leq i \leq m$. Each method can use two knowledge sources, the feature matrix $X$ and the graph topology matrix $A$. For effective merging, the base graphs should be diverse and complementary. In this paper, we exploit five methods: GAT [48], GLCN [18], RGLN [47], GRCN [55], and ProGCN [19].

The GAT method learns the graph structure with attention-based similarity metric with a local view as

$$
\tilde{A}_{ij} = \frac{\exp(\text{leakyReLU}(a^T (Wx_i \| Wx_j)))}{\sum_{k \in N_i} \exp(\text{leakyReLU}(a^T (Wx_i \| Wx_k)))},
$$

where $\|$ is the concatenation operator, $W$ is the parameter matrix for the shared linear transform, $a$ is the weighted vector of the shared attention, $N_i$ is the set of neighbors of node $i$, and LeakyRelu is the nonlinear activation function.

The GCLN method learns the graph structure with attention-based similarity metric similar to GAT but with a global view as

$$
\tilde{A}_{ij} = \frac{A_{ij} \exp(\text{ReLU}(a^T (Wx_i - Wx_j)))}{\sum_{k=1}^{n} A_{ik} \exp(\text{ReLU}(a^T (Wx_i - Wx_k)))},
$$

where $A_{ij}$ is initial graph. It also minimizes the following loss function along with the classification loss to ensure learning of a valid graph:

$$
\text{loss}_{gl} = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} \|x_i - x_j\|_2^2 + \alpha \|\tilde{A}\|_F^2,
$$

where the first term ensures the smoothness of adjacent signals and the second term controls the sparsity of the learned graph.
The RGLN method learns the graph structure with Mahalanobis distance metric as

\[
\tilde{A}_{ij} = A_{ij} + \exp(-\|R^T(x_i - x_j)\|^2),
\]

where \( R \in \mathbb{R}^{d \times s} \) is a low-rank weight matrix, \( s \ll d \) which significantly reduces the number of the learnable parameters due to low-rank property, and \( A_{ij} \) is initial graph. It also ensures learning of a valid graph by minimizing the smoothness constraint as

\[
\text{loss}_{gl} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} \|x_i - x_j\|^2_2.
\]

The GRCN method modifies the original graph with the dot product of the updated node representations after using GCN as follow:

\[
\tilde{A} = A + \text{dot}(Z, Z).
\]

The ProGCN method learns a full parametric matrix by optimizing the following loss function:

\[
\text{loss}_{gl} = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} \|x_i - x_j\|^2_2 + \alpha \|\tilde{A}\|_1 + \beta \|\tilde{A}\|_* + \gamma \|A - \tilde{A}\|^2_2,
\]

where the terms control smoothness, sparsity, low-rankness, and closeness of the learned graph to the original graph, respectively. Also \( \alpha, \beta, \) and \( \gamma \) are hyperparameters that determine the contributions of the constraints.

4.2 Merging

The output of the learning module is \( m \) matrices \( \{\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_m\} \), where each matrix \( \tilde{A}_i \in \mathbb{R}^{n \times n} \) is a learned graph. We use the Grassmann learning to merge the learned graphs to obtain an informative combination of the base graphs [7]. The usage of Grassmann learning retains the specific structural properties of each learned graph and creates a common structure between graphs rather than mixing them.

The Grassmann manifold is a special manifold related to Euclidean space [2, 7]. Mathematically, the Grassmann manifold \( Gr(n, p) \) is the space of \( n \)-by-\( p \) matrices (e.g., \( Y \)) with orthonormal columns, where \( 0 \leq p \leq n \), i.e.,

\[
Gr(n, p) = \{Y \mid Y \in \mathbb{R}^{n \times p}, Y^TY = I\}.
\]

According to Grassmann theory, each orthonormal matrix forms a unique subspace, so it can be mapped to a unique point in the Grassmann manifold [29]. Since the eigenvector matrix of the normalized Laplacian matrix (\( U \in \mathbb{R}^{n \times p} \)), which contains the first \( p \) eigenvectors, is orthonormal [51], it also forms a unique subspace that can map a single point on the Grassmann manifold.

On the other hand, each row of the eigenvector matrix represents the spectral embedding of each node in the \( p \)-dimensional space. So, the two adjacent nodes have close embedding vectors. This subspace representation, which summarizes graph information, can be used for a variety of tasks, such as clustering, classification, and merging graphs [7].
We use subspace representation for merging graphs. The integrated subspace $U$ should have the shortest distance to all subspaces $\{U_i\}_{i=1}^m$, while preserving the connections between the nodes in each individual subspace as much as possible. This is obtained using the following objective function:

$$\min_{U \in \mathbb{R}^{n \times p}} \sum_{i=1}^m \text{tr}(U^T L_i U) + \alpha (p \times m - \sum_{i=1}^m \text{tr}(U U^T U_i U_i^T)), \quad \text{s.t.} \quad U^T U = I, \quad (19)$$

where $m$ is the number of graphs and $\alpha$ is a hyperparameter to control the relative importance of two losses. In this equation, the first term controls the node connectivity based on spectral embedding, and the second term controls the distance between the merged subspace and the individual subspaces. The second term is based on the projection distance between the principal angles of subspaces $\theta_{ij}$ as follows:

$$\sum_{i=1}^m d^2(U, U_i) = \sum_{i=1}^m \sum_{j=1}^p \sin^2 \theta_{ij}$$

$$= \sum_{i=1}^m (p - \sum_{j=1}^p \cos^2 \theta_{ij})$$

$$= \sum_{i=1}^m (p - \text{tr}(U U^T U_i U_i^T))$$

$$= p \times m - \sum_{i=1}^m \text{tr}(U U^T U_i U_i^T). \quad (20)$$

Ignoring the constant value $p \times m$ and solving this optimization problem using the Rayleigh-Ritz theorem, the new Laplacian matrix is obtained as

$$L_{new} = \sum_{i=1}^M L_i - \alpha \sum_{i=1}^M U_i U_i^T. \quad (21)$$

After calculating the eigenvectors matrix $U_i$ corresponding to the Laplacian matrix $L_i$ of each graph $\tilde{A}_i$ and calculating the aggregated Laplacian matrix, the aggregated adjacency matrix is obtained as follows:

$$\hat{A} = D - L_{new}. \quad (22)$$

Aggregated graphs may have negative values due to the presence of a correlation between eigenvectors, while real graphs have normally positive edge weights. Therefore, the ReLU function is used to remove the negative weights while keeping the positive weights.

The integrated graph may be very dense, causing high computational complexity for the classification module and low accuracy due to noisy edges. To resolve this issue, we use KNN sparsification, where just the $k$ edges with the highest value are kept for each node. Therefore, the integrated graph is updated as:

$$\tilde{A}_{ij} = \begin{cases} \hat{A}_{ij}, & \text{if } \hat{A}_{ij} \in \text{Top } k \\ 0, & \text{otherwise.} \end{cases} \quad (23)$$
Finally, the final graph is made symmetric using the following equation:

\[
\hat{A} = \frac{(\tilde{A} + \tilde{A}^T)}{2}.
\]  

(24)

### 4.3 Classification

For the classification module, we adopt a two-layered GCN [27] as follows:

\[
Z = \text{Softmax}(\tilde{A}\text{ReLU}(\tilde{A}XW^0)W^1),
\]

(25)

where \(\tilde{A} = I_n + \hat{D}^{-\frac{1}{2}} \hat{A} \hat{D}^{-\frac{1}{2}}\), and \(W^0\) and \(W^1\) are weight matrices. Softmax and ReLU are activation functions in the hidden and output layers, respectively. In this equation, the first layer creates a new representation of the original raw features \(X\), while the second layer turns the created intermediate representation into the final representation \(Z\). The optimal weight matrices are obtained by minimizing the cross-entropy loss function as follows:

\[
\text{loss}_{CE} = \sum_{i=1}^{n} \sum_{j=1}^{c} y_{ij} \ln z_{ij}.
\]

(26)

### 5 Experiments

We conduct extensive experiments to compare the proposed method with previous state-of-the-art methods in terms of classification accuracy. For base methods, the results are obtained using publicly available codes. All the reported results are averaged over five runs. We compare the proposed MV-GSL method with both single-graph and multi-graph structure learning methods. We also investigate different methods based on their matrix representations.

#### 5.1 Setting

The node classification task is used to evaluate performances. In this regard, we use two popular paper citation network datasets, Cora [42] and Citeseer [42]. In these datasets, the nodes represent the papers that were published in a journal, and the edges show citations. Each publication is described by a sparse one-hot feature vector, which indicates the absence or presence of the corresponding word from a learned dictionary.

We also perform the experiments on the Digits dataset [10], where the graph structure is not available. This dataset contains 1797 8 × 8 images. Each image is a hand-written digit. In order to create the image network, the images were considered as graph nodes described by a vector of size 64 (8 × 8). The initial graph is also created using the KNN method so that neighbor size is equal to 16 and the cosine similarity measure is considered as the distance metric.

---

1. The datasets are available at https://github.com/kimiyoung/planetoid/tree/master/data
2. The dataset is available at https://scikit-learn.org/stable/auto_examples/datasets/plot_digits_last_image.html
Table 3 summarizes the statistics of the datasets. As shown in the table, all have low label rates making them appropriate semi-supervised datasets which are commonly used. We use a transductive setting, which assumes all unlabeled data are available at the training time.

We work on five popular state-of-the-art single-graph structure learning method: GLCN [18], GAT [48], RGLN [47], GRCN [55], and ProGCN [19]. To satisfy the requirements of input matrices in the Grassmann method, we make the matrices symmetric and remove non-zero diagonal elements.

In the experiments, we use the train-test split used in Kipf and Welling [27] for citation datasets. For the methods that already used this split, including GAT,3 RGLN,4 and GRCN,5 we use the same hyperparameter values presented in their papers. For methods that used a different train-test split, including GLCN6 and ProGCN,7 we optimize the hyperparameter values to make a fair comparison. For image dataset, we use the train-test split used in [4] and optimize the hyperparameter values to get best result.

We use the random search method on the validation dataset for tuning hyperparameters. In the merging module, the size of subspace is set to 10 times number of classes and the regularization parameter \( \alpha \) is chosen from \{0.1, 0.2, ..., 0.9\}. In the classification module, we choose the learning rate from \{0.01, 0.03, 0.1, 0.2, 0.3, 0.5\}, number of hidden units from \{16, 32, 64, 128, 256, 512\}, dropout rate from \{0.5, 0.6, 0.7, 0.8, 0.9\}, and weight decay from \{5e^{-3}, 5e^{-4}, 5e^{-5}\}. The number of epochs is 5000, and the learning process stops when the training loss is higher than the average loss in the last 10 epochs.

### 5.2 Results and discussion

Table 4 shows the results of comparing the proposed MV-GSL method with the five single-graph learning methods as well as four multi-graph methods. The used multi-graph methods are

- **Average** which uses the average of the symmetric normalized adjacency matrices

\[
\left( \frac{1}{m} \sum_{i=1}^{m} \tilde{D}^{-\frac{1}{2}} \tilde{A} \tilde{D}^{-\frac{1}{2}} \right),
\]

- **Ensemble** which aggregates the outputs of different GCN models, using either average or maximum probability, instead of merging the graphs,

- **RGCN** [41] which uses the average of features.

As shown in this table, the MV-GSL method outperforms all single and multi-graph methods on all datasets.

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3https://github.com/PetarV/-GAT.git  
4https://github.com/ashawkey/JLGCN.git  
5https://github.com/PlusRoss/GRCN.git  
6https://github.com/jiangboahu/GLCN-tf.git  
7https://github.com/ChandlerBang/Pro-GNN.git
Table 4 The accuracy of different single and multi-graph methods on the datasets Cora, Citeseer, and Digits

| Method            | Cora    | Citeseer | Digits   |
|-------------------|---------|----------|----------|
| GAT [48]          | 83.5 ± 0.5 | 73.0 ± 0.5 | 93.7 ± 0.2 |
| GLCN [18]         | 82.0 ± 0.3 | 71.8 ± 1.3 | 94.0 ± 0.1 |
| GRCN [55]         | 84.1 ± 0.3 | 73.0 ± 0.5 | 93.5 ± 0.6 |
| RGLN [47]         | 81.5 ± 0.7 | 73.5 ± 1.7 | 93.5 ± 0.9 |
| ProGCN [19]       | 81.0 ± 0.3 | 68.4 ± 0.6 | 93.2 ± 1.9 |
| Average           | 83.2 ± 0.3 | 73.4 ± 0.2 | 93.4 ± 0.9 |
| Ensemble (Average Prob) | 83.5 ± 0.3 | 71.8 ± 0.4 | 93.9 ± 0.4 |
| Ensemble (Max Prob) | 83.2 ± 0.7 | 71.1 ± 1.1 | 80.0 ± 4.1 |
| RGCN [41]         | 82.7 ± 1.1 | 72.7 ± 0.6 | 93.7 ± 0.6 |
| MV-GSL (proposed method) | **84.9 ± 0.6** | **74.0 ± 0.4** | **94.1 ± 0.5** |

Best results are in **bold**

To visualize the learned graphs, we show the spy plot of the adjacency matrices in Fig. 3. It visualizes the non-zero values \((nz)\) of the matrices, ordered such that the samples of each class are next to each other. It is clear that, GAT and GLCN just reweight the edges, as previously stated. RGLN create a fully connected matrix, although most edges are light weight. RGLN and GRCN establishing good intra-class relationships. However, they has added a lot of inter-class relationships, which increases noise. ProGCN has identified more inter-class relationships in both datasets than other methods, which has caused a reduction in performance. On the other hand, our method provides better results than the base methods in both datasets by reducing inter-class relations and reducing false edges.

In the next experiment, we investigate the effect of changing the value of trade-off parameter \(\alpha\) in (21). As mentioned in Section 4.2, the merging module combines the sum of Laplacian matrices with a ratio of the new correlation matrix to create the merged matrix. This correlation matrix is created based on spectral embedding of the nodes. When \(\alpha\) is 0, the merged matrix is equivalent to the sum of Laplacian matrices. Figure 4 shows the results of changing this parameter. According to this figure, the parameter \(\alpha\) plays an important role in the classification accuracy, which shows that the correlation matrix has a significant effect on the merged matrix.

Finally, we examine the effect of the sparsification parameter \((k)\). As shown in Fig. 5, this parameter has a significant effect on the performance of MV-GSL method. As mentioned earlier, applying the sparsification not only improves the performance of the model but also drastically reduces the computational cost.

6 Conclusion

In this paper, we proposed a multi-relational graph structure learning method. The proposed framework learns multiple relationships and performs node classification simultaneously. It obtains higher accuracies by creating different relationships between nodes. It also can be
Fig. 3  Spy plots of the adjacency matrices. RGLN matrix is displayed with the precision of 0.0001
used for applications that need to learn several kinds of relationships between nodes. To verify the effectiveness of the proposed method, extensive experiments were performed on two citation datasets, Cora and Citeseer, and an image dataset, Digits. The experiments showed that the proposed method obtains better results compared to single-relational methods as well as other integrating methods.

The proposed method has also some limitations which can be investigated in future work. First, it is modular which can lead to suboptimal results. In other words, it learns each graph in isolation and keeps the learned graphs fixed during integration. An end-to-end approach which improves the learned graphs during merging can obtain better results. Second, different relationships between nodes can have different effects on the final result. So, weighting the relationships to consider their importance can lead to further improvements. Third, the use of different structure learning methods has led to an increase in the number of parameters of the proposed model. The use of parameters redundancy reduction methods can reduce the computational cost and improve the performance of the model [21].
Data Availability The citation datasets analysed during the current study are available in the Relational Dataset repository, https://relational.fit.cvut.cz/search. The image dataset analysed during the current study is available in the Scikit Learn library, https://scikit-learn.org/stable/modules/classes.html?highlight=datasets#module-sklearn.datasets. These datasets were derived from the following public domain resources: https://github.com/kimiyoung/planetoid/tree/master/data; https://linqs.org/datasets/; https://scikit-learn.org/stable/auto_examples/datasets/plot_digits_last_image.html

Declarations

Competing interests The authors declare no competing financial interests.

References

1. Adaloglou N, Vretos N, Daras P (2020) Multi-view adaptive graph convolutions for graph classification. In: European conference on computer vision, pp 398–414
2. Bendokat T, Zimmermann R, Absil P-A (2020) A grassmann manifold handbook: basic geometry and computational aspects. arXiv:2011.13699
3. Bruna J, Zaremba W, Szlam A, LeCun Y (2014) Spectral networks and deep locally connected networks on graphs. In: 2nd international conference on learning representations, iclr 2014
4. Chen Y, Wu L, Zaki M (2020) Iterative deep graph learning for graph neural networks: better and robust node embeddings. Adv Neural Inf Process Syst 33
5. Dai H, Li H, Tian T, Huang X, Wang L, Zhu J, Song L (2018) Adversarial attack on graph structured data. In: International conference on machine learning, pp 1115–1124
6. Defferrard M, Bresson X, Vandergheynst P (2016) Convolutional neural networks on graphs with fast localized spectral filtering. Adv Neural Inf Process Syst 29:3844–3852
7. Dong X, Frossard P, Vandergheynst P, Nefedov N (2013) Clustering on multi-layer graphs via subspace analysis on grassmann manifolds. IEEE Trans Signal Process 62(4):905–918
8. Dong X, Thanou D, Frossard P, Vandergheynst P (2016) Learning laplacian matrix in smooth graph signal representations. IEEE Trans Signal Process 64(23):6160–6173
9. Dong X, Thanou D, Rabbat M, Frossard P (2019) Learning graphs from data: a signal representation perspective. IEEE Signal Process Mag 36(3):44–63
10. Dua D, Graff C (2017) Uci machine learning repository. University of California, Irvine. Sch Inform Comput Sci
11. Duvenaud DK, Maclaurin D, Iparraguirre J, Bombarell R, Hirzel T, Aspuru-Guzik A, Adams RP (2015) Convolutional networks on graphs for learning molecular fingerprints. In: Cortes C, Lawrence N, Lee D, Sugiyama M, Garnett R (eds) Advances in neural information processing systems, vol 28. Curran Associates, Inc
12. Egilmez HE, Pavez E, Ortega A (2017) Graph learning from data under laplacian and structural constraints. IEEE J Sel Top Signal Process 11(6):825–841
13. Fox JS, Rajamanickam S (2020) How robust are graph neural networks to structural noise? (Tech. Rep.). Sandia National Lab. (SNL-NM), Albuquerque, NM (United States)
14. Franceschi L, Niepert M, Pontil M, He X (2019) Learning discrete structures for graph neural networks. In: International conference on machine learning, pp 1972–1982
15. Gao X, Hu W, Guo Z (2020) Exploring structure-adaptive graph learning for robust semi-supervised classification. In: 2020 ieee international conference on multimedia and expo (icme), pp 1–6
16. Hamilton WL, Ying R, Leskovec J (2017) Inductive representation learning on large graphs. In: Proceedings of the 31st international conference on neural information processing systems, pp 1025–1035
17. Huang Z, Li X, Ye Y, Ng MK (2020) Mr-gcn: multi-relational graph convolutional networks based on generalized tensor product. In: Ijcai, pp 1258–1264
18. Jiang B, Zhang Z, Lin D, Tang J, Luo B (2019) Semi-supervised learning with graph learning-convolutional networks. In: Proceedings of the ieee/cvf conference on computer vision and pattern recognition, pp 11313–11320
19. Jin W, Ma Y, Liu X, Tang X, Wang S, Tang J (2020) Graph structure learning for robust graph neural networks. In: Proceedings of the 26th acm sigkdd international conference on knowledge discovery & data mining, pp 66–74
20. Jin W, Li Y, Xu H, Wang Y, Ji S, Aggarwal C, Tang J (2021) Adversarial attacks and defenses on graphs. ACM SIGKDD Explorations Newsl 22(2):19–34
21. Joudar N-E, Ettaouil M et al (2022) Krr-cnn: kernels redundancy reduction in convolutional neural networks. Neural Comput Appl 34(3):2443–2454
22. Kalofolias V (2016) How to learn a graph from smooth signals. In: Artificial intelligence and statistics, pp 920–929
23. Keramatfar A, Amirkhani H, Bidgoly AJ (2022) Modeling tweet dependencies with graph convolutional networks for sentiment analysis. Cogn Comput:1–12
24. Keramatfar A, Rafiee M, Amirkhani H (2022) Graph neural networks: a bibliometrics overview. arXiv:2201.01188
25. Khan MR, Blumenstock JE (2019) Multi-gcn: graph convolutional networks for multi-view networks, with applications to global poverty. In: Proceedings of the aaai conference on artificial intelligence, vol 33, pp 609–613
26. Kipf TN, Welling M (2016) Variational graph auto-encoders. NIPS Workshop on Bayesian Deep Learning
27. Kipf TN, Welling M (2017) Semi-supervised classification with graph convolutional networks. In: International conference on learning representations (iclr)
28. Li R, Wang S, Zhu F, Huang J (2018) Adaptive graph convolutional neural networks. In: Proceedings of the aaai conference on artificial intelligence, vol 32
29. Lin G, Wang J, Liao K, Zhao F, Chen W (2020) Structure fusion based on graph convolutional networks for node classification in citation networks. Electronics 9(3):432
30. Lin G, Kang X, Liao K, Zhao F, Chen Y (2021) Deep graph learning for semi-supervised classification. Pattern Recogn 118:108039
31. Liu Z, Zhou J (2020) Introduction to graph neural networks. Synth Lect Artif Intel Mach Learn 14(2):1–127
32. Ma Y, Wang S, Aggarwal CC, Yin D, Tang J (2019) Multi-dimensional graph convolutional networks. In: Proceedings of the 2019 siam international conference on data mining, pp 657–665
33. Niepert M, Ahmed M, Kutzkov K (2016) Learning convolutional neural networks for graphs. In: International conference on machine learning. pp 2014–2023
34. Pass R, Seeman L, Shelat A (2017) Analysis of the blockchain protocol in asynchronous networks. In: Annual international conference on the theory and applications of cryptographic techniques, pp 643–673
35. Peng L, Kong F, Liu C, Kuang P (2021) Robust and dynamic graph convolutional network for multi-view data classification. Comput J
36. Pilco DS, Rivera AR (2019) Graph learning network: a structure learning algorithm. arXiv:1905.12665
37. Pourhatami A, Kaviyani-Charati M, Kargar B, Baziyad H, Kargar M, Olmeda-Gómez C (2021) Mapping the intellectual structure of the coronavirus field (2000–2020): a co-word analysis. Scientometrics 126(8):6625–6657
38. Pu X, Chau SL, Dong X, Sejdinovic D (2021) Kernel-based graph learning from smooth signals: a functional viewpoint. IEEE Trans Signal Inf Process over Netw 7:192–207
39. Rong W, Zhuo E, Peng H, Chen J, Wang H, Han C, Cai H (2021) Learning a consensus affinity matrix for multi-view clustering via subspaces merging on grassmann manifold. Inf Sci 547:68–87
40. Safari O, Rokach L (2018) Ensemble learning: a survey. Wiley Interdiscip Rev: Data Min Knowl Discov 8(4):e1249
41. Schlichtkrull M, Kipf TN, Bloem P, Van Den Berg R, Titov I, Welling M (2018) Modeling relational data with graph convolutional networks. In: European semantic web conference, pp 593–607
42. Sen P, Namata G, Bilgic M, Getoor L, Galligher B, Eliassi-Rad T (2008) Collective classification in network data. AI Mag 29(3):93–93
43. Shanthamallu US, Thiagarajan JJ, Spanias A (2020) A regularized attention mechanism for graph attention networks. In: ICASSP 2020-2020 ieee international conference on acoustics, speech and signal processing (icassp), pp 3372–3376
44. Shi M, Tang Y, Zhu X (2021) Topology and content co-alignment graph convolutional learning. IEEE Trans Neural Netw Learn Syst
45. Shirazi S, Baziyad H, Ahmadi N, Albadvi A (2019) A new application of louvain algorithm for identifying disease fields using big data techniques. J Biostat Epidemiol 5(3):183–193
46. Subbareddy B, Reddy PC, Siripuram A, Zhang J (2019) A survey of signal processing based graph learning techniques. In: 2019 1st international conference on industrial artificial intelligence (iai), pp 1–6
47. Tang J, Gao X, Hu W (2021) Rgln: robust residual graph learning networks via similarity-preserving mapping on graphs. In: Icassp 2021-2021 ieee international conference on acoustics, speech and signal processing (icassp), pp 2940–2944
48. Vešićković P, Cucurull G, Casanova A, Romero A, Liò P, Bengio Y (2018) Graph attention networks. International Conference on Learning Representations. https://openreview.net/forum?id=rJXMpikCZ (accepted as poster). Accessed 31 Oct 2020
49. Wan L, Dong C, Pei X (2022) Self-paced learning-based multi-graphs semi-supervised learning. Multimedia Tools Appl:1–22
50. Wang X, Zhu M, Bo D, Cui P, Shi C, Pei J (2020) Am-gcn: adaptive multi-channel graph convolutional networks. In: Proceedings of the 26th acm sigkdd international conference on knowledge discovery & data mining, pp 1243–1253
51. Wu Z, Pan S, Chen F, Long G, Zhang C, Philip SY (2020) A comprehensive survey on graph neural networks. IEEE Trans on Neural Netw Learn Syst 32(1):4–24
52. Xu C, Tao D, Xu C (2013) A survey on multi-view learning. arXiv:1304.5634
53. Yan X, Hu S, Mao Y, Ye Y, Yu H (2021) Deep multi-view learning methods: a review. Neurocomputing 448:106–129
54. Yang L, Kang Z, Cao X, Jin D, Yang B, Guo Y (2019) Topology optimization based graph convolutional network. In: Ijcai, pp 4054–4061
55. Yu D, Zhang R, Jiang Z, Wu Y, Yang Y (2020) Graph-revised convolutional network. In: Joint European conference on machine learning and knowledge discovery in databases, pp 378–393
56. Zhan M, Gan J, Lu G, Wan Y (2020) Graph convolutional networks of reconstructed graph structure with constrained laplacian rank. Multimed Tools Appl:1–12
57. Zhang J, Shi X, Xie J, Ma H, King I, Yeung D-Y (2018) Gaan: gated attention networks for learning on large and spatiotemporal graphs. In: Proceedings of the thirty-fourth conference on uncertainty in artificial intelligence, pp 339–349
58. Zhang S, Tong H, Xu J, Maciejewski R (2019) Graph convolutional networks: a comprehensive review. Comput Soc Netw 6(1):1–23
59. Zhang Z, Cui P, Zhu W (2020) Deep learning on graphs: a survey. IEEE Trans Knowl Data Eng
60. Zhou J, Cui G, Hu S, Zhang Z, Yang C, Liu Z, Wang L, Li C, Sun M (2020) Graph neural networks: a review of methods and applications. AI Open 1:57–81
61. Zhu D, Zhang Z, Cui P, Zhu W (2019) Robust graph convolutional networks against adversarial attacks. In: Proceedings of the 25th acm sigkdd international conference on knowledge discovery & data mining, pp 1399–1407
62. Zhu Y, Xu W, Zhang J, Liu Q, Wu S, Wang L (2021) Deep graph structure learning for robust representations: a survey. arXiv:2103.03036
63. Zhuang C, Ma Q (2018) Dual graph convolutional networks for graph-based semi-supervised classification. In: Proceedings of the 2018 world wide web conference, pp 499–508
64. Züger D, Akbarnejad A, Günemann S (2018) Adversarial attacks on neural networks for graph data. In: Proceedings of the 24th ACM SIGKDD international conference on knowledge discovery & data mining, pp 2847–2856

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