ON THE SPECTRUM OF THE PERIODIC DIRAC OPERATOR

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ABSTRACT. The absolute continuity of the spectrum for the periodic Dirac operator
\[ \hat{D} = \sum_{j=1}^{n} \left( -i \frac{\partial}{\partial x_j} - A_j \right) \hat{\alpha}_j + \hat{V}^{(0)} + \hat{V}^{(1)}, \quad x \in \mathbb{R}^n, \ n \geq 3, \]
is proved given that either \( A \in C(\mathbb{R}^n; \mathbb{R}^n) \cap H^{2q}_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n), 2q > n - 2 \), or the Fourier series of the vector potential \( A : \mathbb{R}^n \to \mathbb{R}^n \) is absolutely convergent. Here, \( \hat{V}^{(s)} = (\hat{V}^{(s)})^* \) are continuous matrix functions and \( \hat{\alpha}_j, \hat{\alpha}_2^{n+1} = \hat{I}, s = 0, 1 \).

In [1], the absolute continuity of the spectrum for the periodic Dirac operator
\[ \hat{D} = \sum_{j=1}^{n} \left( -i \frac{\partial}{\partial x_j} - A_j \right) \hat{\alpha}_j + V \hat{I} + V_0 \hat{\alpha}_{n+1} \]
in \( \mathbb{R}^n, n \geq 2, \) was proved, where \( V, V_0 \in L^q_{\text{loc}}(\mathbb{R}^2; \mathbb{R}), A \in L^q_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2), q > 2, \) for \( n = 2 \) and \( V, V_0 \in C(\mathbb{R}^n; \mathbb{R}), A \in C^{2n+3}(\mathbb{R}^n; \mathbb{R}^n) \) for \( n \geq 3 \). Here, \( \hat{\alpha}_{n+1} \) is a Hermitian matrix anticommuting with the matrices \( \hat{\alpha}_j, j = 1, \ldots, n, \) and \( \hat{\alpha}_2^{n+1} = \hat{I}. \) For \( n = 2, \) the proof is based on the results in [2,3], where the two-dimensional periodic Schrödinger operator was considered. In [3], the absolute continuity of the spectrum for this operator was proved in the case of the scalar (electric) and the vector (magnetic) potentials \( V \) and \( A \) satisfying the conditions \( V \in L^q_{\text{loc}}(\mathbb{R}^2; \mathbb{R}) \) and \( A \in L^{2q}_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2), q > 1. \) For the periodic Dirac operator with \( n = 2, \) the same result as in [1] was independently obtained in [4]. However, it was assumed in [4] that \( V_0 \equiv m = \text{const}. \) But the functions \( V_0 \in L^q_{\text{loc}}(\mathbb{R}^2; \mathbb{R}), q > 2, \) can in fact be considered in this case as well without any significant changes. The proof in [4] used the method suggested in [5], where the absolute continuity of the spectrum was established for the two-dimensional Dirac operator with the periodic potential \( V \in L^q_{\text{loc}}(\mathbb{R}^2; \mathbb{R}), q > 2 \) (and \( A \equiv 0 \)). Sobolev’s results (see [6]) for the absolute continuity of the spectrum of the Schrödinger

\[ ^1 \text{In this version a few misprints have been corrected.} \]

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operator with the periodic vector potential \( A \in C^{2n+3}(\mathbb{R}^n; \mathbb{R}^n) \) were used in [1] for the case \( n \geq 3 \). Sobolev later replaced the last condition with the weaker condition \( A \in H^q_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n) \), \( 2q > 3n - 2 \), \( n \geq 3 \) (see the survey in [7]), which permitted changing the smoothness conditions on the vector potential \( A \) for the periodic Dirac operator [1,7] in an adequate manner. The absolute continuity of the spectrum for the Dirac operator in \( \mathbb{R}^n \), \( n \geq 3 \), with the periodic scalar potential \( V \) (for \( A \equiv 0 \)) was proved in [8–10] under various constraints on \( V \).

1. Let \( \mathcal{L}_M, M \in \mathbb{N} \), denote the linear space of complex \( M \times M \) matrices, let \( \mathcal{S}_M \) be the set of Hermitian matrices in \( \mathcal{L}_M \), and let the matrices \( \hat{\alpha}_j \in \mathcal{S}_M \), \( j = 1, \ldots, n \), satisfy the commutation relations \( \hat{\alpha}_j \hat{\alpha}_l + \hat{\alpha}_l \hat{\alpha}_j = 2 \delta_{jl} \hat{I} \), where \( \hat{I} \in \mathcal{L}_M \) is the identity matrix and \( \delta_{jl} \) is the Kronecker delta. We write
\[
\mathcal{L}_M^{(s)} = \{ \hat{L} \in \mathcal{L}_M : \hat{L} \hat{\alpha}_j = (-1)^s \hat{\alpha}_j \hat{L} \text{ for all } j = 1, \ldots, n \},
\]
\[
\mathcal{S}_M^{(s)} = \mathcal{L}_M^{(s)} \cap \mathcal{S}_M, \quad s = 0, 1.
\]

We consider the Dirac operator
\[
\hat{D} = \hat{D}_0 + \hat{V}^{(0)} + \hat{V}^{(1)} - \sum_{j=1}^{n} A_j \hat{\alpha}_j = \sum_{j=1}^{n} \left( -i \frac{\partial}{\partial x_j} - A_j \right) \hat{\alpha}_j + \hat{V}^{(0)} + \hat{V}^{(1)},
\]
where \( n \geq 3 \) (\( i^2 = -1 \)). The vector function \( A : \mathbb{R}^n \to \mathbb{R}^n \) and the matrix functions \( \hat{V}^{(s)} : \mathbb{R}^n \to \mathcal{S}_M^{(s)} \), \( s = 0, 1 \), are assumed to be periodic with a period lattice \( \Lambda \subset \mathbb{R}^n \). We set
\[
\hat{V} = \hat{V}^{(0)} + \hat{V}^{(1)} - \sum_{j=1}^{n} A_j \hat{\alpha}_j.
\]
The coordinates of the vectors in \( \mathbb{R}^n \) are set in an orthogonal basis \( \{ E_j \} \).

Here, \( E_j \) and \( E_j^* \) are the basis vectors in the lattice \( \Lambda \) and its reciprocal lattice \( \Lambda^* \), \( (E_j, E_l^*) = \delta_{jl} \) (\( |.| \) and (\( ., . \)) are the length and the inner product of vectors in \( \mathbb{R}^n \)),
\[
K = \left\{ x = \sum_{j=1}^{n} \xi_j E_j : 0 \leq \xi_j < 1, j = 1, \ldots, n \right\},
\]
\[
K^* = \left\{ y = \sum_{j=1}^{n} \eta_j E_j : 0 \leq \eta_j < 1, j = 1, \ldots, n \right\},
\]
and \( v(K) \) and \( v(K^*) \) are the volumes of the elementary cells \( K \) and \( K^* \).

The inner products and the norms in the spaces \( L^2(K; \mathbb{C}^M) \) and \( \mathbb{C}^M \) are introduced in the usual way with (as a rule) the usual notation (without
indicating the spaces themselves). The matrices in $L_M$ are identified with the operators on the space $\mathbb{C}^M$ (and their norm is defined as the norm of operators on $\mathbb{C}^M$). Let $H^q(\mathbb{R}^n; \mathbb{C}^d)$, $d \in \mathbb{N}$, be the Sobolev class of order $q \geq 0$, and let $\tilde{H}^q(K; \mathbb{C}^d)$ be the set of vector functions $\phi : K \to \mathbb{C}^d$ whose periodic extensions (with the period lattice $\Lambda$) belong to $H^q_{\text{loc}}(\mathbb{R}^n; \mathbb{C}^d)$. In what follows, the functions defined on the elementary cell $K$ are identified with their periodic extensions throughout the space $\mathbb{R}^n$.

We let

$$\chi_N = v^{-1}(K) \int_K \chi(x) e^{-2\pi i (N, x)} \, d^n x, \quad N \in \Lambda^*, \tag{1}$$

denote the Fourier coefficients of the functions $\chi \in L^1(K, U)$, where $U$ is the space $\mathbb{C}$ or $\mathbb{C}^M$ or $L_M$.

Let $\mathcal{B}(\mathbb{R})$ be the set of Borel subsets $O \subseteq \mathbb{R}$, and let $\mathcal{M}_h$, $h > 0$, be the set of signed even Borel measures (charges) $\mu : \mathcal{B}(\mathbb{R}) \to \mathbb{R}$ such that

$$\hat{\mu}(p) = \int_{\mathbb{R}} e^{ipt} \, d\mu(t) = 1 \quad \text{for} \quad |p| \leq 2\pi h, \quad p \in \mathbb{R},$$

$$\|\mu\| = \sup_{O \in \mathcal{B}(\mathbb{R})} (|\mu(O)| + |\mu(\mathbb{R} \setminus O)|) < +\infty, \quad \mu \in \mathcal{M}_h. \tag{2}$$

For an arbitrary vector $\gamma \in \Lambda \setminus \{0\}$, an arbitrary measure $\mu \in \mathcal{M}_h$, $h > 0$, and any vector $\tilde{e} \in S_{n-2}(\gamma^{-1}\gamma) = \{e' \in S_{n-1} : (\gamma, e') = 0\}$, where $S_{n-1}$ is the unit sphere in $\mathbb{R}^n$, we write

$$\tilde{A}(\gamma, \mu, \tilde{e}; x) = \int_{\mathbb{R}} d\mu(t) \int_0^1 A(x - \xi \gamma - t\tilde{e}) \, d\xi, \quad x \in \mathbb{R}^n. \tag{3}$$

In this paper, we consider continuous (periodic) functions $A : \mathbb{R}^n \to \mathbb{R}^n$ and $\hat{V}^{(s)} : \mathbb{R}^n \to S_M^{(s)}$, $s = 0, 1$. In this case, $\hat{D} = \hat{D}_0 + \hat{V}$ is a self-adjoint operator on the Hilbert space $L^2(\mathbb{R}^n; \mathbb{C}^M)$ with the domain $D(\hat{D}) = D(\hat{D}_0) = H^1(\mathbb{R}^n; \mathbb{C}^M)$.

**Theorem 1.** Let $A : \mathbb{R}^n \to \mathbb{R}^n$ and $\hat{V}^{(s)} : \mathbb{R}^n \to S_M^{(s)}$, $s = 0, 1$, be continuous periodic functions with the period lattice $\Lambda \subset \mathbb{R}^n$, $n \geq 3$. If

$$\max_{\tilde{e} \in S_{n-2}(\gamma^{-1}\gamma)} \|\tilde{A}(\gamma, \mu, \tilde{e}; .) - A_0\|_{L^\infty(\mathbb{R}^n)} < \pi |\gamma|^{-1} \tag{4}$$

for some vector $\gamma \in \Lambda \setminus \{0\}$ and a measure $\mu \in \mathcal{M}_h$, $h > 0$, where

$$A_0 = v^{-1}(K) \int_K A(x) \, d^n x,$$
then the spectrum of operator (1) is absolutely continuous.

The operator \( \hat{D} \) is unitarily equivalent to the direct integral

\[
\int \bigoplus_{2\pi K^*} \hat{D}(k) \frac{d^n k}{(2\pi)^n v(K^*)},
\]

where

\[
\hat{D}(k) = \hat{D}_0(k) + \hat{V}, \quad \hat{D}_0(k) = \sum_{j=1}^{n} \left(-i \frac{\partial}{\partial x_j} + k_j\right) \hat{\alpha}_j, \quad k_j = (k, \mathcal{E}_j),
\]

\[
D(\hat{D}(k)) = D(\hat{D}_0(k)) = \tilde{H}^1(K; \mathbb{C}^M) \subset L^2(K; \mathbb{C}^M).
\]

The vector \( k \in \mathbb{R}^n \) is called a quasimomentum. The unitary equivalence is established using the Gel’fand transformation [11] (also see [9] for the case of the periodic Dirac operator). The self-adjoint operators \( \hat{D}(k) \) have compact resolvents and hence discrete spectra. Let \( E_\nu(k), \nu \in \mathbb{Z}, \) be the eigenvalues of the operators \( \hat{D}(k) \). We assume that they are arranged in an increasing order (counting multiplicities). The eigenvalues can be indexed for different \( k \) such that the functions \( \mathbb{R}^n \ni k \to E_\nu(k) \) are continuous.

Let \( e \in S_{n-1}. \) For \( k \in \mathbb{R}^n \) and \( \varkappa \geq 0, \) we write

\[
\hat{D}_0(k + i\varkappa e) = \hat{D}_0(k) + i\varkappa \sum_{j=1}^{n} e_j \hat{\alpha}_j, \quad e_j = (e, \mathcal{E}_j),
\]

\[
\hat{D}(k + i\varkappa e) = \hat{D}_0(k + i\varkappa e) + \hat{V}, \quad D(\hat{D}(k + i\varkappa e)) = D(\hat{D}_0(k + i\varkappa e)) = \tilde{H}^1(K; \mathbb{C}^M).
\]

**Proof of Theorem 1.** We use the Thomas method [12]. Because it is well known [4,9] (see [2,13] for the case of the periodic Schrödinger operator), we present only a brief scheme of the method. The decomposition of the operator \( \hat{D} \) into direct integral (3) and the piecewise analyticity of the functions \( \mathbb{R} \ni \xi \to E_\nu(k + \xi e), \nu \in \mathbb{Z}, k \in \mathbb{R}^n, \) imply (see Theorems XIII.85 and XIII.86 in [13]) that to prove the absolute continuity of the spectrum of operator (1), it suffices to show that the functions \( \xi \to E_\nu(k + \xi e) \) are not constant (for some unit vector \( e \)) on every interval \( (\xi_1, \xi_2) \subset \mathbb{R}. \) But if we suppose that \( E_\nu(k + \xi e) \equiv E \) for all \( \xi \in (\xi_1, \xi_2), \xi_1 < \xi_2, \) then it follows from the analytic Fredholm theorem that \( E \) is an eigenvalue of \( \hat{D}(k + (\xi + i\varkappa)e) \) for all \( \xi + i\varkappa \in \mathbb{C}. \) Consequently, it suffices to prove the invertibility of the operators \( \hat{D}(k + (\xi + i\varkappa)e) - E, k \in \mathbb{R}^n, E \in \mathbb{R}, \) for some \( \xi + i\varkappa \in \mathbb{C}. \) Theorem 1 is therefore a consequence of the following assertion.
Theorem 2. Let \( \gamma \in \Lambda \setminus \{0\} \), \( e = |\gamma|^{-1}\gamma \), \( \mu \in M_h \), \( h > 0 \). Let \( A : \mathbb{R}^n \to \mathbb{C}^n \) and \( \tilde{V}(s) : \mathbb{R}^n \to \mathcal{L}_M^{(s)} \), \( s = 0,1 \), be continuous periodic functions with the period lattice \( \Lambda \subset \mathbb{R}^n \), \( n \geq 3 \). If \( A_0 = 0 \) and

\[
\max_{\bar{\epsilon} \in S_{n-2}(|\gamma|^{-1}\gamma)} \| (\tilde{A}(\gamma, \mu, \bar{\epsilon};.), \bar{\epsilon}) + i(\tilde{A}(\gamma, \mu, \bar{\epsilon};.), e) \|_{L^\infty(\mathbb{R}^n)} = \tilde{\theta}_0|\gamma|^{-1},
\]

where \( \tilde{\theta} \in [0,1) \), then for any \( \theta \in (0,1-\tilde{\theta}) \), there exists a number \( \varkappa_0 = \varkappa_0(\gamma, h, \mu; \tilde{V}, \theta) > 0 \) such that the inequality

\[
\| \tilde{D}(k + i\varepsilon e)\phi \| \geq \theta\pi|\gamma|^{-1}\exp(-4C\|\mu\| \max\{|\gamma|, h^{-1}\} \|A\|_{L^\infty(\mathbb{R}^n;\mathbb{C}^n)} \|\phi\|
\]

holds for all \( k \in \mathbb{R}^n \) with \( (k, \gamma) = \pi \), all \( \varkappa \geq \varkappa_0 \), and all vector functions \( \phi \in \tilde{H}^1(K;\mathbb{C}^M) \), where \( C > 0 \) is a universal constant to be defined in Lemma 1.

Theorem 2 is proved in Section 3. The following theorem is a consequence of Theorem 1.

Theorem 3. Let \( A : \mathbb{R}^n \to \mathbb{R}^n \) and \( \tilde{V}(s) : \mathbb{R}^n \to S_M^{(s)} \), \( s = 0,1 \), be continuous periodic functions with the period lattice \( \Lambda \subset \mathbb{R}^n \), \( n \geq 3 \). If at least one of the conditions

1. \( A \in H^q_{loc}(\mathbb{R}^n;\mathbb{R}^n), \quad 2q > n - 2 \), or
2. \( \sum_{N \in \Lambda^*} \|A_N\|_{\mathbb{C}^n} < +\infty \)

holds, then the spectrum of operator (1) is absolutely continuous.

Theorem 4 is used to prove Theorem 3.

Theorem 4. Let \( \Lambda \) be a lattice in \( \mathbb{R}^n \), \( n \geq 2 \). There are positive constants \( c_1 \) and \( c_2 \) depending on \( n \) and \( \Lambda \) such that for any nonnegative Borel measure \( \mu \) on the unit sphere \( S_{n-1} \subset \mathbb{R}^n \), any \( h > 0 \), and any \( R_0 \geq \min_{\gamma \in \Lambda \setminus \{0\}} |\gamma| \), there exists a vector \( \gamma \in \Lambda \setminus \{0\} \) such that

1. \( |\gamma| \leq R_0 \),
2. if \((\gamma, \gamma') = 0\) for some vector \( \gamma' \in \Lambda \setminus \{0\}\), then \( |\gamma'| > c_1 R_0^{1/(n-1)} \) (\( \Lambda^* \) is the reciprocal lattice of \( \Lambda \)),
3. \( \mu\{e' \in S_{n-1} : |(e', \gamma)| \leq h\} \leq c_2|\gamma|^{-1}\max\{h, R_0^{-1/(n-1)}\} \mu(S_{n-1}) \).

The proof of Theorem 4 for the lattice \( \Lambda = \mathbb{Z}^n \) and for \( h = c_3 R_0^{-1/(n-1)} \) (where \( c_3 = c_3(n) > 0 \)) is presented in [14] (see [15] for \( n = 3 \)). The proof in the general case follows the one suggested in [14] with some slight changes.

Proof of Theorem 3. It can be assumed that \( A_0 = 0 \). We write

\[
F(A; \gamma, \mu) = \max_{\bar{\epsilon} \in S_{n-2}(|\gamma|^{-1}\gamma)} |\gamma| \|\tilde{A}(\gamma, \mu, \bar{\epsilon};.)\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)}, \quad \gamma \in \Lambda \setminus \{0\}, \quad \mu \in M_h. \]
Let condition 1 hold. We define the measure
\[
\mu^{(1)}(.) = \sum_{N \in \Lambda^* \setminus \{0\}} |N|^{2q} \|A\|_{C^n}^2 \delta_{N/|N|}(.)
\]
on the unit sphere $S_{n-1}$, where $\delta_{e'}(.)$ is the Dirac measure concentrated at the point $e' \in S_{n-1}$. From Theorem 4 (applied to the measure $\mu^{(1)}$), it follows that for any $R_0 \geq \min_{\gamma \in \Lambda \setminus \{0\}} |\gamma|$ there is a vector $\gamma \in \Lambda \setminus \{0\}$ such that $|\gamma| \leq R_0$,
\[
\sum_{N \in \Pi(\gamma)} |N|^{2q} \|A_N\|_{C^n}^2 \leq c_2 |\gamma|^{-1} R_0^{-(n-1)} \sum_{N \in \Lambda^*} |N|^{2q} \|A_N\|_{C^n}^2,
\]
and $|\gamma'| > c_1 R_0^{1/(n-1)}$ for all $\gamma' \in \Pi(\gamma) \cap \{\gamma' \in \Lambda^* \setminus \{0\} : (\gamma, \gamma') = 0\}$. We take a measure $\mu \in \mathcal{M}_h$ (for some $h > 0$) such that $|\hat{\mu}(p)| \leq 1$ for all $p \in \mathbb{R}$ and $\hat{\mu}(p) = 0$ if $|p| \geq 2\pi h > 2\pi h$. For a vector $\tilde{e} \in S_{n-2}(|\gamma|^{-1} \gamma)$, we write $\Pi(\gamma, \tilde{e}) = \{\gamma' \in \Pi(\gamma) : |(\gamma', \tilde{e})| \leq h_1\}$. Because $2q > n - 2$, we have
\[
\sum_{N \in \Pi(\gamma, \tilde{e})} |N|^{-2q} \leq c_4 R_0^{-2q/(n-1)}
\]
for all $\tilde{e} \in S_{n-2}(|\gamma|^{-1} \gamma)$, where the constant $c_4 > 0$ depends on $n$, $\Lambda$, $q$, and $h_1$. Consequently,
\[
F(A ; \gamma, \mu) \leq \sup_{\tilde{e} \in S_{n-2}(|\gamma|^{-1} \gamma)} |\gamma| \sum_{N \in \Pi(\gamma, \tilde{e})} \|A_N\|_{C^n} \leq (4)
\]
\[
|\gamma| \left( \sup_{\tilde{e} \in S_{n-2}(|\gamma|^{-1} \gamma)} \sum_{N \in \Pi(\gamma, \tilde{e})} |N|^{-2q} \right)^{1/2} \left( \sum_{N \in \Pi(\gamma)} |N|^{2q} \|A_N\|_{C^n}^2 \right)^{1/2} \leq \sqrt{c_2 c_4 R_0^{(n-2-2q)/(2(n-1))} \left( \sum_{N \in \Lambda^*} |N|^{2q} \|A_N\|_{C^n}^2 \right)^{1/2}}.
\]
The right-hand side of (4) becomes arbitrarily small if a sufficiently large number $R_0$ is chosen (and inequality (2) consequently holds). Case 2, for which the Dirac measure $\mu = \delta$ is chosen, is considered in a similar (slightly simpler) way. Theorem 3 is proved.

2. We fix a vector $\gamma \in \Lambda \setminus \{0\}$ and a measure $\mu \in \mathcal{M}_h$, $h > 0$, $e = |\gamma|^{-1} \gamma$. In what follows, the constants we introduce can depend on $\gamma$, $h$, and $\mu$, but we do not indicate this dependence explicitly (until Theorem 8 below).

Let $\hat{P}^c$, where $C \subseteq \Lambda^*$, denote the orthogonal projection on $L^2(K; \mathbb{C}^M)$ that takes a vector function $\phi \in L^2(K; \mathbb{C}^M)$ to the vector function
\[
\hat{P}^c \phi = \phi^c = \sum_{N \in C} \phi_N e^{2\pi i (N,x)}
\]
and $\hat{\phi} \equiv 0$). We introduce the notation $\mathcal{H}(\mathcal{C}) = \{ \phi \in L^2(K; \mathbb{C}^M) : \phi_N = 0$ for $N \notin \mathcal{C} \}$.

Let $\mathcal{P}(e) = \{ \tau e : \tau \in \mathbb{R} \}$. For the vectors $x \in \mathbb{R}^n \setminus \mathcal{P}(e)$, we write

$$\widehat{e}(x) = (x - (x, e)e) |x - (x, e)e|^{-1} \in S_{n-2} (e),$$

where $S_{n-2} (e) = \{ \widehat{e} \in S_{n-1} : (e, \widehat{e}) = 0 \}$; we also write $\sigma_{n-2} = \text{mes} (S_{n-2})$, where $\text{mes} (\cdot)$ is the standard measure ('surface area') on the unit sphere $S_{n-2} = S_{n-2} (e)$. For $\beta > 0$ and $\varkappa > \beta$, we write

$$\mathcal{O}_\beta = \mathcal{O}_\beta (\varkappa) = \{ x \in \mathbb{R}^n : |(x, e)| < \beta \text{ and } |\varkappa - |x - (x, e)e|| < \beta \},$$

$$\mathcal{K}_\beta = \mathcal{K}_\beta (k; \varkappa) = \{ N \in \Lambda^* : k + 2\pi N \in \mathcal{O}_\beta \}, \quad k \in \mathbb{R}^n.$$

We set

$$\hat{P}^\pm_{\varepsilon} = \frac{1}{2} \left( \hat{T} \mp i \left( \sum_{j=1}^n e_j \hat{\alpha}_j \right) \left( \sum_{j=1}^n \tilde{e}_j \hat{\alpha}_j \right) \right),$$

for all $\tilde{e} \in S_{n-2} (e)$, where $\hat{P}^\pm_{\varepsilon}$ are orthogonal projections on $\mathbb{C}^M$.

For $k \in \mathbb{R}^n$, $\varkappa \geq 0$, and $N \in \Lambda^*$, we introduce the notation

$$\hat{D}_N (k; \varkappa) = \sum_{j=1}^n (k_j + 2\pi N_j + i \varkappa e_j) \hat{\alpha}_j,$$

$$G^\pm_N (k; \varkappa) = \left( (k + 2\pi N, e)^2 + (\varkappa \pm \sqrt{|k + 2\pi N|}^2 - (k + 2\pi N, e)^2) \right)^{1/2},$$

and $G_N (k; \varkappa) = G_N^- (k; \varkappa)$. The inequalities

$$G_N (k; \varkappa) \| u \| \leq \| \hat{D}_N (k; \varkappa) u \| \leq G_N^+ (k; \varkappa) \| u \|, \quad u \in \mathbb{C}^M,$$

hold. If $(k, \gamma) = \pi$, then $G_N (k; \varkappa) \geq |(k + 2\pi N, e)| \geq \pi |\gamma|^{-1}$. For all vector functions $\phi \in \tilde{H}^1 (K; \mathbb{C}^M)$,

$$\hat{D}_0 (k + i \varkappa e) \phi = \sum_{N \in \Lambda^*} \hat{D}_N (k; \varkappa) \phi_N e^{2\pi i (N,x)}.$$

In this case (for all $\varkappa \geq 0$ and $k + 2\pi N \notin \mathcal{P}(e)$), we have

$$\| \hat{D}_N (k; \varkappa) \hat{P}^\pm_{\varepsilon(k+2\pi N)} \phi_N \| = G_N^\pm (k; \varkappa) \| \hat{P}^\pm_{\varepsilon(k+2\pi N)} \phi_N \|,$$

and

$$\hat{P}^\pm_{\varepsilon(k+2\pi N)} \hat{D}_N (k; \varkappa) \hat{P}^\pm_{\varepsilon(k+2\pi N)} = \hat{O},$$

where $\hat{O} \in \mathcal{L}_M$ is the zero matrix.

We let $\hat{P}^\pm = \hat{P}^\pm (k)$, where $k \in \mathbb{R}^n$, denote the operators on $L^2 (K; \mathbb{C}^M)$ that take vector functions $\phi \in L^2 (K; \mathbb{C}^M)$ to the vector functions $\hat{P}^\pm \phi \in L^2 (K; \mathbb{C}^M)$.
$L^2(K; \mathbb{C}^M)$ with the Fourier coefficients $(\hat{P}^\pm \phi)_N = \hat{P}_{\epsilon(k+2\pi N)}^\pm \phi_N$ if $k + 2\pi N \notin \mathcal{P}(e)$ and $(\hat{P}^\pm \phi)_N = 0$ otherwise.

For the matrix function $\hat{V} = \hat{V}^{(0)} + \hat{V}^{(1)} - \sum_{j=1}^{n} A_j \hat{\alpha}_j$, where $\hat{V}^{(s)} : \mathbb{R}^n \rightarrow \mathcal{L}^{(s)}_M$, $s = 0, 1$, and $A : \mathbb{R}^n \rightarrow \mathbb{C}^n$ are continuous periodic functions with the period lattice $\Lambda$, we write
\[
W = W(\hat{V}) = n \|A\|_{L^\infty(\mathbb{R}^n; \mathbb{C}^n)} + \sum_{s=0,1} \|\hat{V}^{(s)}\|_{L^\infty(\mathbb{R}^n; \mathcal{L}_M)}.
\]

We set $c_5(A) = c_5(A; \gamma, h, \mu) = \exp \left( -4C \|\mu\| \max \{1, \theta^{-1}\} \|A\|_{L^\infty(\mathbb{R}^n; \mathbb{C}^n)} \right)$, where $C > 0$ is a universal constant to be defined in Lemma 1.

**Theorem 5.** Let $\tilde{\theta} \in [0, 1)$, $\theta \in (0, 1 - \tilde{\theta})$, $W_0 \geq 0$, $R \geq 1$, $\beta > 0$, and $a \in (0, 1]$. Also, let us fix a vector $\gamma \in \Lambda \setminus \{0\}$, a number $h > 0$, and a measure $\mu \in \mathcal{M}_2$; $e = |\gamma|^{-1} \gamma$. Then there are numbers $b = b(\tilde{\theta}, \theta, W_0; a) > 0$ and $\varkappa = \varkappa(\theta, \theta, W_0, R, \beta; a) > 4\beta + R$ such that the inequality
\[
\|\hat{P}^+(k) \hat{D}(k + i\varepsilon e) \phi\|^2 + a^2 \|\hat{P}^-(k) \hat{D}(k + i\varepsilon e) \phi\|^2 \geq c_5^2(A) \left( \left( \frac{\theta \pi}{|\gamma|} \right)^2 \|P^-(k) \phi\|^2 + \left( \frac{b \varkappa}{\beta + R} \right)^2 \|P^+(k) \phi\|^2 \right)
\]
holds for all vectors $k \in \mathbb{R}^n$ with $(k, \gamma) = \pi$, all $\varkappa \geq \varkappa_0$, all continuous periodic functions $\hat{V}^{(s)} : \mathbb{R}^n \rightarrow \mathcal{L}^{(s)}_M$, $s = 0, 1$, and $A : \mathbb{R}^n \rightarrow \mathbb{C}^n$ (with the period lattice $\Lambda \subset \mathbb{R}^n$, $n \geq 3$) such that $A_0 = 0$,
\[
W(\hat{V}) \leq W_0, \quad (5)
\]
\[
\max_{\varepsilon \in S_{n-2}(e)} \|((\tilde{A}(\gamma, \mu, \tilde{\varepsilon}; e)) + i(\tilde{A}(\gamma, \mu, \tilde{\varepsilon}; e), e))\|_{L^\infty(\mathbb{R}^n)} \leq \tilde{\theta} \pi |\gamma|^{-1}, \quad (6)
\]
\[
\hat{V}_N = 0 \text{ for } 2\pi |N| > R, \quad (7)
\]
and all vector functions $\phi \in \mathcal{H}(\mathcal{K}_\beta(k; \varkappa))$.

**Proof.** Without loss of generality we assume that the basis vector $\mathcal{E}_2$ coincides with $e$. We fix some numbers $\theta < \theta_4 < \theta_3 < \theta_2 < \theta_1 < 1 - \theta$ and write $\delta = 1 - \theta_4^2 \theta_3^2$ and $c_5' = \exp \left( -4C \|\mu\| \max \{|\gamma|, \theta^{-1}\} W_0 \right)$. We choose a number $\tilde{\varepsilon} \in (0, 1)$ proceeding from the condition $(c_5')^2 \left( (1 - \tilde{\varepsilon})^2 \delta \right)^2 \geq 2\delta^2 W_0^2 \tilde{\varepsilon}$. Lower bounds for the constant $\varkappa_0$ are specified in the course of the proof. We first suppose that $\varkappa_0 > 4\beta + R$. In this case, if $N \in \mathcal{K}_\beta(k; \varkappa)$, $k \in \mathbb{R}^n$, $\varkappa \geq \varkappa_0$, and $2\pi |N| \leq R$ (where $N' \in \Lambda^*$), then $|\tilde{e}(k + 2\pi (N + N')) - \tilde{e}(k + 2\pi N)| < 2\pi N|N|$. There is a number $c_6 = c_6(\tilde{\varepsilon}) > 0$ such that for all $\varkappa \geq \varkappa_0$, there are nonintersecting (nonempty)
We introduce the notation \( \rho = \tilde{\rho} + 2R/\varkappa, \rho' = \tilde{\rho} + 4R/\varkappa \). Let \( \Omega_\lambda = \left\{ \tilde{e} \in S_{n-2} : |\tilde{e} - \tilde{e}'| < \frac{2R}{\varkappa} \text{ for some } \tilde{e}' \in \tilde{\Omega}_\lambda \right\} \);

\( \tilde{\Omega}_\lambda \subset \Omega_\lambda \), and \( |\tilde{e}' - \tilde{e}''| > 4R/\varkappa \) for all \( \tilde{e}' \in \Omega_{\lambda_1}, \tilde{e}'' \in \Omega_{\lambda_2}, \lambda_1 \neq \lambda_2 \).

Property 3 implies that for any \( \phi \) any \( \varkappa = 1 \lambda = 1, \ldots, \lambda(R, \varkappa) \), such that

1. \( |\tilde{e} - E^\lambda| \leq \tilde{\rho} = c_6 R/\varkappa \) for all \( \tilde{e} \in \tilde{\Omega}_\lambda \);
2. \( |\tilde{e}' - \tilde{e}''| > 8R/\varkappa \) for all \( \tilde{e}' \in \tilde{\Omega}_{\lambda_1}, \tilde{e}'' \in \tilde{\Omega}_{\lambda_2}, \lambda_1 \neq \lambda_2 \);
3. \( \text{mes} \left( S_{n-2} \setminus \bigcup_{\lambda} \tilde{\Omega}_\lambda \right) < (1/2) \tilde{\varepsilon} \sigma_{n-2} \).

We write

\[ \tilde{e}^\lambda = \tilde{S}(k, \varkappa; \phi) E^\lambda, \]

\( \tilde{\mathcal{K}}^\lambda_\beta = \tilde{\mathcal{K}}^\lambda_\beta(k, \varkappa; \phi) = \{ N \in \mathcal{K}_{\beta}(k; \varkappa) : \tilde{e}(k + 2\pi N) \in \tilde{S}\tilde{\Omega}_\lambda \}, \)

\( \mathcal{K}^\lambda_\beta = \mathcal{K}^\lambda_\beta(k, \varkappa; \phi) = \{ N \in \mathcal{K}_{\beta}(k; \varkappa) : \tilde{e}(k + 2\pi N) \in \tilde{S} \Omega_\lambda \}, \quad \tilde{\mathcal{K}}^\lambda_\beta \subset \mathcal{K}^\lambda_\beta. \)

The choice of the orthogonal transformation \( \tilde{S} \) means that

\[ \left\| \left( \hat{P}^\pm \phi \right)^{\mathcal{K}^\lambda_\beta \cup \tilde{\mathcal{K}}^\lambda_\beta} \right\|^2 \leq \tilde{\varepsilon} \left\| \hat{P}^\pm \phi \right\|^2. \] (8)

For each index \( \lambda \) (and for all already chosen \( k, \varkappa, \) and \( \phi \)), we take an orthogonal system of vectors \( \mathcal{E}^{(\lambda)}_j \in S_{n-1}, j = 1, \ldots, n \), such that \( \mathcal{E}^{(\lambda)}_1 = \tilde{e}^\lambda \) and \( \mathcal{E}^{(\lambda)}_2 = \mathcal{E}_2 = e \). We let \( x^{(\lambda)}_j = (x, \mathcal{E}^{(\lambda)}_j) \) denote the coordinates of the vectors \( x = \sum_{j=1}^n x_j \mathcal{E}_j \in \mathbb{R}^n \) (and also of the vectors in \( \mathbb{C}^n \)). Let

\[ \mathcal{E}^{(\lambda)}_j = \sum_{l=1}^n T^{(\lambda)}_{lj} \mathcal{E}_l. \]

Then \( A^{(\lambda)}_j = \sum_{l=1}^n T^{(\lambda)}_{lj} A_l \) (where \( A_l = (A, \mathcal{E}_l) \) and \( A^{(\lambda)}_j = (A, \mathcal{E}^{(\lambda)}_j) \)), \( \tilde{A}^{(\lambda)}_j = \tilde{A}^{(\lambda)}_j(\gamma, \mu, \tilde{e}^\lambda; \cdot) = \sum_{l=1}^n T^{(\lambda)}_{lj} \tilde{A}_l \), and \( \tilde{A}_l = \tilde{A}_l(\gamma, \mu, \tilde{e}^\lambda; \cdot) \). We introduce the notation \( \tilde{\alpha}^{(\lambda)}_j = \sum_{l=1}^n T^{(\lambda)}_{lj} \tilde{\alpha}_l, j = 1, \ldots, n \). For the Fourier coefficients \( (\tilde{A}^{(\lambda)}_j)_N \) of the functions \( \tilde{A}^{(\lambda)}_j, j = 1, \ldots, n \), we have \( (\tilde{A}^{(\lambda)}_j)_N = \)
\[ \mu (2\pi N_1^{(\lambda)}) (A_j^{(\lambda)})_N \text{ if } N_2 = 0 \text{ and } (\tilde{A}_j^{(\lambda)})_N = 0 \text{ if } N_2 \neq 0. \] (Here, \((A_j^{(\lambda)})_N\) are the Fourier coefficients of \(A_j^{(\lambda)}\), \(N \in \Lambda^*\).)

Let \(\Phi^{(s,\lambda)} : \mathbb{R}^n \to \mathbb{C}, s = 1, 2\), be periodic trigonometric polynomials with the period lattice \(\Lambda\) and the Fourier coefficients \(\Phi_N^{(1,\lambda)} = \Phi_N^{(2,\lambda)} = 0\) if \(N_1^{(\lambda)} = 2\) and

\[ \begin{align*}
\Phi_N^{(1,\lambda)} &= (2\pi i ((N_1^{(\lambda)})^2 + N_2^2))^{-1} (N_1^{(\lambda)} (A_1^{(\lambda)} - \tilde{A}_1^{(\lambda)})_N + N_2 (A_2 - \tilde{A}_2)_N), \\
\Phi_N^{(2,\lambda)} &= - (2\pi i ((N_1^{(\lambda)})^2 + N_2^2))^{-1} (N_2 (A_1^{(\lambda)} - \tilde{A}_1^{(\lambda)})_N - N_1^{(\lambda)} (A_2 - \tilde{A}_2)_N)
\end{align*} \]

otherwise. We have

\[ \frac{\partial \Phi^{(1,\lambda)}}{\partial x_1^{(\lambda)}} - \frac{\partial \Phi^{(2,\lambda)}}{\partial x_2^{(\lambda)}} = A_1^{(\lambda)} - \tilde{A}_1^{(\lambda)}, \quad \frac{\partial \Phi^{(1,\lambda)}}{\partial x_2^{(\lambda)}} + \frac{\partial \Phi^{(2,\lambda)}}{\partial x_1^{(\lambda)}} = A_2 - \tilde{A}_2. \]

**Lemma 1.** There is a universal constant \(C > 0\) such that

\[ \|\Phi^{(s,\lambda)}\|_{L^\infty(\mathbb{R}^n)} \leq C \|\mu\| \max \{|\gamma|, h^{-1}\} \|A\|_{L^\infty(\mathbb{R}^n; \mathbb{C})}, \quad s = 1, 2. \]

**Proof.** Let \(\eta(.) \in C^\infty(\mathbb{R}; \mathbb{R}), \eta(\tau) = 0\) for \(\tau \leq \pi, 0 \leq \eta(\tau) \leq 1\) for \(\pi < \tau \leq 2\pi\), and \(\eta(\tau) = 1\) for \(\tau > 2\pi\). For \(x, y \in \mathbb{R}\) (and \(x^2 + y^2 > 0\)), we set

\[ G(x, y) = \frac{x}{x^2 + y^2} \int_0^\infty \frac{\partial \eta(\tau)}{\partial \tau} J_0(\tau \sqrt{x^2 + y^2}) \, d\tau, \]

where \(J_0(.)\) is the Bessel function of the first kind of order zero; \(G(., .) \in L^q(\mathbb{R}^2), q \in [1, 2]\). We write \(G_1(t; x, y) = t^{-1} G(t^{-1} x, t^{-1} y), t > 0, \) and \(G_2(t; x, y) = G_1(t; y, x); \|G_s(t; ., .)\|_{L^1(\mathbb{R}^2)} = t \|G(., .)\|_{L^1(\mathbb{R}^2)}, s = 1, 2\). For arbitrary continuous periodic functions \(\mathcal{F} : \mathbb{R}^n \to \mathbb{C}\) with the period lattice \(\Lambda\), we set

\[ (\mathcal{F} \ast_{\lambda} G_s(t; ., .))(x) = \iint_{\mathbb{R}^2} G_s(t; \xi_1, \xi_2) \mathcal{F}(x - \xi_1 e^\lambda - \xi_2 e) \, d\xi_1 d\xi_2, \quad x \in \mathbb{R}^n. \]

In this case, \((\mathcal{F} \ast_{\lambda} G_s(t; ., .))_N = 0\) if \(N_1^{(\lambda)} = N_2 = 0\) and

\[ (\mathcal{F} \ast_{\lambda} G_s(t; ., .))_N = - \frac{i N_s^{(\lambda)}}{(N_1^{(\lambda)})^2 + N_2^2} \eta \left(2\pi t \sqrt{(N_1^{(\lambda)})^2 + N_2^2}\right) \mathcal{F}_N \]

otherwise, \(s = 1, 2\). Let \(t = \max \{|\gamma|, h^{-1}\}\). Because \((A - \tilde{A})_N = 0\) for \(N_2 = 0, |N_1^{(\lambda)}| \leq h, \) and \(|N_2| = |\gamma|^{-1} |(N, \gamma)| \geq |\gamma|^{-1}\) for \(N_2 \neq 0\), we have

\[ 2\pi \Phi^{(1,\lambda)} = (A_1^{(\lambda)} - \tilde{A}_1^{(\lambda)}) \ast_{\lambda} G_1(t; ., .) + (A_2 - \tilde{A}_2) \ast_{\lambda} G_2(t; ., .), \]
\[2\pi \Phi^{(2,\lambda)} = -(A_1^{(\lambda)} - \tilde{A}_1^{(\lambda)}) *_\lambda G_2(t, \ldots) + (A_2 - \tilde{A}_2) *_\lambda G_1(t, \ldots).\]

Using the inequalities \(\|\tilde{A}\|_{L^\infty(\mathbb{R}^n; \mathbb{C}^n)} \leq \|\mu\| \|A\|_{L^\infty(\mathbb{R}^n; \mathbb{C}^n)}\) and \(\|\mu\| \geq 1\), and taking the constant \(C = 2\pi^{-1} \|G(\cdot, \cdot)\|_{L^1(\mathbb{R}^2)}\), we complete the proof of the lemma.

We introduce the notation

\[\hat{D}_0^{(\lambda)} = \left(-i \frac{\partial}{\partial x_1^{(\lambda)}} + k_1^{(\lambda)}\right) \hat{\alpha}_1^{(\lambda)} + \left(-i \frac{\partial}{\partial x_2^{(\lambda)}} + k_2 + i\chi\right) \hat{\alpha}_2^{(\lambda)} ,\]

\[\hat{D}^{(\lambda)} = \hat{D}_0^{(\lambda)} - \tilde{A}_1^{(\lambda)} \hat{\alpha}_1^{(\lambda)} - \tilde{A}_2 \hat{\alpha}_2^{(\lambda)} ,\]

\[\hat{D}^{(\lambda)}(k + i\chi) = e^{-i\hat{\alpha}_1^{(\lambda)} \hat{\Phi}^{(2,\lambda)} e^{i\hat{\alpha}_1^{(\lambda)} \hat{\Phi}^{(1,\lambda)}} e^{-i\hat{\alpha}_1^{(\lambda)} \hat{\Phi}^{(2,\lambda)^*} \hat{\Phi}^{(1,\lambda)^*}} ,\]

\[\hat{\nabla}^{(\lambda)} = \hat{\nabla}^{(0)} + \hat{\nabla}^{(1)} + \sum_{j=3}^{n} \left(-i \frac{\partial}{\partial x_j^{(\lambda)}} + k_j^{(\lambda)} - A_j^{(\lambda)}\right) \hat{\alpha}_j^{(\lambda)} ,\]

\[\hat{D}(k + i\chi) = \hat{D}^{(\lambda)}(k + i\chi) + \hat{\nabla}^{(\lambda)} .\]

If \(N \in K^\lambda_{\beta}\), then \(|\bar{c}(k + 2\pi N) - \bar{e}^{\lambda}| < \rho\) and therefore

\[|k + 2\pi N - (k_2 + 2\pi N_2) + \chi| < \beta + \rho \chi .\]

It follows that

\[\sum_{j=3}^{n} \left(k_j^{(\lambda)} + 2\pi N_j^{(\lambda)}\right) E_j^{(\lambda)} < \beta + \rho \chi , \quad |k_1^{(\lambda)} + 2\pi N_1^{(\lambda)} - \chi| < \beta + \rho \chi , \quad (9)\]

and

\[\|\hat{\nabla}^{(\lambda)} \phi K^\lambda_{\beta}\| \leq \left(\beta + (c_6 + 2)R + W\right) \|\phi K^\lambda_{\beta}\| .\]

We use the brief notation \(\hat{P}_\lambda^{\pm} = \hat{P}_\mp = (1/2)(\hat{I} \pm i\hat{\alpha}_1^{(\lambda)} \hat{\alpha}_2^{(\lambda)}\). We set \(\chi^{(\lambda)} = e^{-i\hat{\Phi}^{(1,\lambda)} \hat{\Phi}^{(2,\lambda)} \phi K^\lambda_{\beta}}\). The relation

\[\hat{D}_0^{(\lambda)} \hat{P}_\lambda^{\pm} \chi^{(\lambda)} = \sum_{N \in \Lambda^*} \left(k_2 + 2\pi N_2 + i(k_1^{(\lambda)} + 2\pi N_1^{(\lambda)}\right) \hat{\alpha}_2^{(\lambda)} \hat{P}_\lambda^{\pm} \chi^{(\lambda)} e^{2\pi i(Nx)} \quad (10)\]

holds.

We write \(O^{(\lambda)}(\tau) = \{N \in \Lambda^* : |k_1^{(\lambda)} + 2\pi N_1^{(\lambda)} - \chi| < 2\tau\}, \tau > 0\).

Inequalities (9) imply that there is a constant

\[c_7 = c_7(\tilde{\theta}, \theta, W_0, R, \beta) > \frac{1}{2} \left(\beta + (c_6 + 2)R\right)\]

such that (for all \(\lambda\))

\[\left\|\sum_{N \in \Lambda^* \setminus O^{(\lambda)}(c_7)} \hat{P}_\lambda^{\pm} \chi^{(\lambda)} e^{2\pi i(Nx)}\right\| \leq \frac{1}{2} \left\|\hat{P}_\lambda^{\pm} \chi^{(\lambda)}\right\| . \quad (11)\]
In what follows, we assume that \( \kappa_0 \geq c_7 \). As a consequence of (10) and (11), we obtain

\[
\| \hat{D}_0^{(\lambda)} \hat{P}_\lambda^+ \chi^{(\lambda)} \| \geq v^{1/2}(K) \left( \sum_{N \in \mathcal{O}(\lambda)} \| \kappa + (k_1^{(\lambda)} + 2\pi N_1^{(\lambda)}) \| \| \hat{P}_\lambda^+ \chi_N^{(\lambda)} \| \right)^{1/2} \geq \\
2 (\kappa - c_7) \left\| \sum_{N \in \mathcal{O}(\lambda)} \hat{P}_\lambda^+ \chi_N^{(\lambda)} e^{2\pi i (N, x)} \right\| \geq (\kappa - c_7) \| \hat{P}_\lambda^+ \chi^{(\lambda)} \| .
\]

On the other hand, we have \(|k_2 + 2\pi N_2| \geq \pi|\gamma|^{-1}\). Condition (6) implies that

\[
\| \tilde{A}^{(\lambda)} \tilde{\alpha}_1^{(\lambda)} + \tilde{A}_2 \tilde{\alpha}_2 \|_{L^\infty(\mathbb{R}^n; \mathcal{L}_M)} \leq \tilde{\theta}_1 \pi|\gamma|^{-1},
\]

and therefore (see (10))

\[
\| \hat{D}^{(\lambda)} \hat{P}_\lambda^- \chi^{(\lambda)} \| \geq \| \hat{D}_0^{(\lambda)} \hat{P}_\lambda^- \chi^{(\lambda)} \| - \tilde{\theta}_1 \pi|\gamma|^{-1} \| \hat{P}_\lambda^- \chi^{(\lambda)} \| \geq (1 - \tilde{\theta}_1) \pi|\gamma|^{-1} \| \hat{P}_\lambda^- \chi^{(\lambda)} \| .
\]

The operators \( \hat{P}_\lambda^\pm \) commute with the operators \( e^{\pm i\Phi^{(1, \lambda)}} \), \( e^{-i\tilde{\alpha}_1^{(\lambda)} \tilde{\alpha}_2 \Phi^{(2, \lambda)}} \), and \( \hat{V}^{(\lambda)} \), and we have \( \hat{P}_\lambda^\pm \hat{D}^{(\lambda)} = \hat{D}^{(\lambda)} \hat{P}_\lambda^\pm \). Consequently,

\[
\hat{P}_\lambda^\pm \hat{D}(k + i\kappa e) = \hat{D}^{(\lambda)}(k + i\kappa e) \hat{P}_\lambda^\mp + \hat{V}^{(\lambda)} \hat{P}_\lambda^\pm.
\]

Using the above estimates and also the inequality

\[
\| e^{\pm i\Phi^{(1, \lambda)}} e^{i\tilde{\alpha}_1^{(\lambda)} \tilde{\alpha}_2 \Phi^{(2, \lambda)}} \|_{L^\infty(\mathbb{R}^n; \mathcal{L}_M)} \leq c_5^{-1/2} (A),
\]

we derive

\[
\| \hat{P}_\lambda^+ \hat{D}(k + i\kappa e) \phi^\kappa_\beta \| \geq (1 - \tilde{\theta}_1) \pi|\gamma|^{-1} c_5 (A) \| \hat{P}_\lambda^- \phi^\kappa_\beta \| - \| \hat{V}^{(\lambda)} \hat{P}_\lambda^+ \phi^\kappa_\beta \| ,
\]

\[
(\kappa - c_7 - \pi|\gamma|^{-1}) c_5 (A) \| \hat{P}_\lambda^+ \phi^\kappa_\beta \| - \| \hat{V}^{(\lambda)} \hat{P}_\lambda^- \phi^\kappa_\beta \| .
\]

Let

\[
\sigma = \theta_2^2 \theta_3^{-2} - 1,
\]

\[
\tilde{a} = \min \{1, \sqrt{\sigma} a, (1 - \tilde{\theta} - \theta_1) \pi|\gamma|^{-1} c_5' (\beta + (c_6 + 2)R + W_0)^{-1} \},
\]

\[
b'' = \min \begin{cases} 
1, \\
\sqrt{\sigma} a, \\
(1 - \tilde{\theta} - \theta_1) \pi|\gamma|^{-1} c_5' (c_6 + 2)^{-1} (1 + W_0)^{-1}, \\
2 (\theta_1 - \theta_2) \pi|\gamma|^{-1} (c_6 + 2)^{-1}.
\end{cases}
\]
Since \((\beta + R)^{-1} b'' < \tilde{a}\), we can pick a number \(\tilde{a}'\) such that \((\beta + R)^{-1} b'' \leq \tilde{a}' < \tilde{a}\). For an adequate choice of the number \(\kappa_0\) (and for \(\kappa \geq \kappa_0\)), inequalities (12) and (13) imply the estimate
\[
\| \hat{P}_\lambda^+ \hat{D}(k + i\varepsilon\varepsilon) \phi K_\beta^+ \| + \tilde{a} \| \hat{P}_\lambda^- \hat{D}(k + i\varepsilon\varepsilon) \phi K_\beta^- \| \geq \]
\[
c_5(A) \left( \theta_1 \pi |\gamma|^{-1} \| \hat{P}_\lambda^- \phi K_\beta^+ \| + \tilde{a}' \| \hat{P}_\lambda^+ \phi K_\beta^- \| \right).
\]
For all \(\tilde{e} \in \tilde{S}_\Omega \subset S_{n-2}(e)\), we have
\[
\| (\hat{P}_\lambda^\pm - \hat{P}_\lambda^\pm) \phi K_\beta^\pm \| \leq \frac{1}{2} |\tilde{e} - \tilde{e}^\lambda| \| \phi K_\beta^\pm \| \leq \frac{\rho}{2} \| \phi K_\beta^\pm \|. \tag{14}
\]
If \((\hat{D}(k + i\varepsilon\varepsilon) \phi K_\beta^\pm)_N \neq 0\) for some \(N \in \Lambda^*\), then \(k + 2\pi N \notin \mathcal{P}(e)\) and \(|\tilde{e}(k + 2\pi N) - \tilde{e}^\lambda| < \rho + 2R/\kappa = \rho'\). Therefore,
\[
\| (\hat{P}_\lambda^\pm - \hat{P}_\lambda^\pm) \hat{D}(k + i\varepsilon\varepsilon) \phi K_\beta^\pm \| \leq \frac{\rho'}{2} \| \hat{D}(k + i\varepsilon\varepsilon) \phi K_\beta^\pm \|.
\]
Consequently,
\[
\| \hat{P}_\lambda^+ \hat{D}(k + i\varepsilon\varepsilon) \phi K_\beta^+ \| + \tilde{a} \| \hat{P}_\lambda^- \hat{D}(k + i\varepsilon\varepsilon) \phi K_\beta^- \| \leq \]
\[
(1 + \rho' \tilde{a}^{-1}) \left( \| \hat{P}_\lambda^+ \hat{D}(k + i\varepsilon\varepsilon) \phi K_\beta^+ \| + \tilde{a} \| \hat{P}_\lambda^- \hat{D}(k + i\varepsilon\varepsilon) \phi K_\beta^- \| \right). \tag{15}
\]
Since \((\beta + R)^{-1} b'' \leq \tilde{a}'\) and \((c_6 + 2) b'' \leq 2 (\theta_1 - \theta_2) \pi |\gamma|^{-1}\), for an adequately chosen number \(\kappa_0\) (and for \(\kappa \geq \kappa_0\)) inequality (14) implies that
\[
\theta_1 \frac{\pi}{|\gamma|} \| \hat{P}_\lambda^- \phi K_\beta^+ \| + \frac{b'' \kappa}{\beta + R} \| \hat{P}_\lambda^+ \phi K_\beta^- \| \geq \]
\[
(1 + \rho' \tilde{a}^{-1}) \left( \theta_2 \frac{\pi}{|\gamma|} \| \hat{P}_\lambda^- \phi K_\beta^+ \| + \frac{b'' \kappa}{2(\beta + R)} \| \hat{P}_\lambda^+ \phi K_\beta^- \| \right). \tag{16}
\]
From (15) and (16), it follows that
\[
\| \hat{P}_\lambda^+ \hat{D}(k + i\varepsilon\varepsilon) \phi K_\beta^+ \| + \tilde{a} \| \hat{P}_\lambda^- \hat{D}(k + i\varepsilon\varepsilon) \phi K_\beta^- \| \geq \]
\[
c_5(A) \left( \theta_2 \frac{\pi}{|\gamma|} \| \hat{P}_\lambda^- \phi K_\beta^+ \| + \frac{b'' \kappa}{2(\beta + R)} \| \hat{P}_\lambda^+ \phi K_\beta^- \| \right).
\]
We write \(b' = (1/2)(1 + \sigma)^{-1/2} b''\). Then
\[
\| \hat{P}_\lambda^+ \hat{D}(k + i\varepsilon\varepsilon) \phi K_\beta^+ \|^2 + a^2 \| \hat{P}_\lambda^- \hat{D}(k + i\varepsilon\varepsilon) \phi K_\beta^- \|^2 \geq \]
\[
(1 + \sigma)^{-1} \left( \| \hat{P}_\lambda^+ \hat{D}(k + i\varepsilon\varepsilon) \phi K_\beta^+ \| + \tilde{a} \| \hat{P}_\lambda^- \hat{D}(k + i\varepsilon\varepsilon) \phi K_\beta^- \| \right)^2 \geq \]
\[
c_3^2(A) \left( \left( \theta_3 \frac{\pi}{|\gamma|} \right)^2 \| \hat{P}_\lambda^- \phi K_\beta^+ \|^2 + \left( \frac{b' \kappa}{2(\beta + R)} \right)^2 \| \hat{P}_\lambda^+ \phi K_\beta^- \|^2 \right). \tag{17}
\]
If $N \in \Lambda^*$ and $\lambda_1 \neq \lambda_2$, then either $2\pi|N-N'| > R$ for all $N' \in K^\lambda_{\beta}$ or $2\pi|N-N''| > R$ for all $N'' \in K^\lambda_{\beta}$. Therefore,

$$\widehat{V}_\phi \bigcup_{\lambda} K^\lambda_{\beta} = \sum_{\lambda} \widehat{V}_\phi K^\lambda_{\beta}, \quad \widehat{D}(k + i\varepsilon) \bigcup_{\lambda} K^\lambda_{\beta} = \sum_{\lambda} \widehat{D}(k + i\varepsilon) K^\lambda_{\beta}.$$

If $N \in \bigcup_{\lambda} K^\lambda_{\beta}$, then

$$(\widehat{D}(k + i\varepsilon) \bigcup_{\lambda} K^\lambda_{\beta})_N = \left(\widehat{D}(k + i\varepsilon) \bigcup_{\lambda} K^\lambda_{\beta}\right)_N + \left(\widehat{V}_\phi K^\lambda_{\beta} \bigcup_{\lambda} K^\lambda_{\beta}\right)_N.$$

If $N \in \Lambda^* \bigsetminus \bigcup_{\lambda} K^\lambda_{\beta}$, then

$$\left(\widehat{D}(k + i\varepsilon) \bigcup_{\lambda} K^\lambda_{\beta}\right)_N = \left(\widehat{V}_\phi K^\lambda_{\beta} \bigcup_{\lambda} K^\lambda_{\beta}\right)_N.$$

These relations (for each of the signs) imply the estimates

$$\|\widehat{P}^{\pm} \widehat{D}(k + i\varepsilon)\| \geq \sum_{N \in \bigcup_{\lambda} K^\lambda_{\beta}} v(K) \left\|\left(\widehat{P}^{\pm} \widehat{D}(k + i\varepsilon) \bigcup_{\lambda} K^\lambda_{\beta}\right)_N + \left(\widehat{V}_\phi K^\lambda_{\beta} \bigcup_{\lambda} K^\lambda_{\beta}\right)_N\right\| \geq$$

$$(1-\delta) \left\|\widehat{P}^{\pm} \widehat{D}(k + i\varepsilon) \bigcup_{\lambda} K^\lambda_{\beta}\right\|^2 - (1-\delta) \left\|\widehat{P}^{\pm} \widehat{V}_\phi K^\lambda_{\beta} \bigcup_{\lambda} K^\lambda_{\beta}\right\|^2 - (1-\delta) \delta^{-1} \left\|\widehat{P}^{\pm} \widehat{V}_\phi K^\lambda_{\beta} \bigcup_{\lambda} K^\lambda_{\beta}\right\|^2 \geq$$

$$(1-\delta) \left\|\widehat{D}(k + i\varepsilon) \bigcup_{\lambda} K^\lambda_{\beta}\right\|^2 - (1-\delta) \delta^{-1} \left\|\widehat{D}(k + i\varepsilon) \bigcup_{\lambda} K^\lambda_{\beta}\right\|^2 - \delta^{-1} W^2 \left\|\widehat{V}_\phi K^\lambda_{\beta} \bigcup_{\lambda} K^\lambda_{\beta}\right\|^2.$$
\[ \frac{2}{\delta} W^2 \bar{\varepsilon} \| \phi \|^2 \geq (1 - \bar{\varepsilon}) c^2_\delta(A) \left( \left( \frac{\theta_4}{|\gamma|} \right)^2 \| \hat{P}^\pm \phi \|^2 + (1 - \delta) \left( \frac{b' \kappa}{2(\beta + R)} \right)^2 \| \hat{P}^\pm \phi \|^2 \right) - \frac{2}{\delta} W^2 \bar{\varepsilon} (\| \hat{P}^- \phi \|^2 + \| \hat{P}^+ \phi \|^2) \geq c^2_\delta(A) \left( \left( \frac{\theta}{|\gamma|} \right)^2 \| \hat{P}^- \phi \|^2 + (1 - \delta) \left( \frac{b \kappa}{\beta + R} \right)^2 \| \hat{P}^+ \phi \|^2 \right). \]

Theorem 5 is proved.

3. The following theorems are a consequence of Theorem 5. The proof of Theorem 6 is based on applying the relation
\[ \hat{P}^\pm(k) \hat{D}_0(k + i\kappa \varepsilon) = \hat{D}_0(k + i\kappa \varepsilon) \hat{P}^\pm(k) \]
and on selecting an arbitrarily small number \( a \in (0, 1] \). The proof of Theorem 7 essentially uses the arbitrariness in the choice of the number \( \beta > 0 \) (see below). Theorem 6 is used to prove the absolute continuity of the spectrum of a periodic Schrödinger operator.

**Theorem 6.** Let \( \tilde{\theta} \in [0, 1) \), \( W_0 \geq 0 \), \( R \geq 1 \), and \( \beta > 0 \) (for a fixed vector \( \gamma \in \Lambda \setminus \{0\} \) and a fixed measure \( \mu \in \mathcal{M}_h \), \( h > 0 \); \( \varepsilon = |\gamma|^{-1} \gamma \)). Then there are numbers \( c_8 = c_8(\tilde{\theta}, W_0) > 0 \) and \( \kappa_0 = \kappa_0(\tilde{\theta}, W_0, R, \beta) > 4\beta + 5R \) such that for all vectors \( k \in \mathbb{R}^n \) with \( (k, \gamma) = \pi \), all \( \kappa \geq \kappa_0 \), all continuous periodic functions \( \tilde{V}^{(s)} : \mathbb{R}^n \rightarrow \mathcal{L}^{(s)}_M \), \( s = 0, 1 \), and \( A : \mathbb{R}^n \rightarrow \mathbb{C}^n \) (with the period lattice \( \Lambda \subset \mathbb{R}^n \), \( n \geq 3 \)) for which \( A_0 = 0 \) and conditions (5) – (7) are satisfied, and all vector functions \( \phi \in \mathcal{H}(K; \kappa; \nu) \), the inequality
\[ \| \hat{D}^2(k + i\kappa \varepsilon) \phi \| \geq \frac{c_8 \kappa}{\beta + R} \| \phi \| \]
holds.

**Theorem 7.** Let \( \tilde{\theta} \in [0, 1) \), \( \theta \in (0, 1 - \tilde{\theta}) \), \( W_0 \geq 0 \), \( R \geq 1 \), and \( \delta \in (0, 1] \) (for a fixed vector \( \gamma \in \Lambda \setminus \{0\} \) and a fixed measure \( \mu \in \mathcal{M}_h \), \( h > 0 \); \( \varepsilon = |\gamma|^{-1} \gamma \)). Then there are numbers \( \mathcal{D} = \mathcal{D}(\theta, W_0, \delta) \geq 1 \) and \( \kappa_0 = \kappa_0(\tilde{\theta}, \theta, W_0, R, \delta) > (4\mathcal{D} + 1)R \) such that for all vectors \( k \in \mathbb{R}^n \) with \( (k, \gamma) = \pi \), all \( \kappa \geq \kappa_0 \), all continuous periodic functions \( \tilde{V}^{(s)} : \mathbb{R}^n \rightarrow \mathcal{L}^{(s)}_M \), \( s = 0, 1 \), and \( A : \mathbb{R}^n \rightarrow \mathbb{C}^n \) (with the period lattice \( \Lambda \subset \mathbb{R}^n \), \( n \geq 3 \)) for which \( A_0 = 0 \) and conditions (5) – (7) are satisfied, and all vector functions \( \phi \in \tilde{H}^1(K; \mathbb{C}^M) \), the inequality
\[ \| \hat{D}(k + i\kappa \varepsilon) \phi \|^2 \geq \]
\[(1 - \delta) \left( c_2^2(A_\alpha \left( \theta \frac{\pi}{|\gamma|} \right)^2 \| \mathcal{K}_{\alpha} \|^2 + v(K) \sum_{N \in \Lambda^\ast \setminus \mathcal{K}_{\beta}} G_2^2(k; \mathcal{X}) \| \phi_N \|^2 \right) \]

holds.

**Proof of Theorem 2.** Let \( \theta < \theta' < 1 - \tilde{\theta} \), \( \tilde{V}_{\nu}^{(s)} : \mathbb{R}^n \to \mathcal{L}_M^{(s)}, s = 0, 1 \), and \( A_\nu : \mathbb{R}^n \to \mathbb{C}^n, \nu \in \mathbb{N} \), be sequences of trigonometric polynomials with the period lattice \( \Lambda \) that uniformly converge as \( \nu \to +\infty \) to the functions \( \tilde{V}^{(s)} \) and \( A \), let \( (A_\nu)_0 = 0 \) for all \( \nu \in \mathbb{N} \), and let \( \tilde{V}_{\nu} = \tilde{V}_{\nu}^{(0)} + \tilde{V}_{\nu}^{(1)} - \sum_{j=1}^{n} (A_\nu)_j \alpha_j \).

From Theorem 7 (because \( G_N(k; \mathcal{X}) \geq \pi |\gamma|^{-1}, N \in \Lambda^\ast \)) it follows that for all sufficiently large \( \nu \), there are numbers \( \mathcal{X}_0^{(\nu)} > 0 \) such that for all \( k \in \mathbb{R}^n \) with \( (k, \gamma) = \pi \), all \( \mathcal{X} \geq \mathcal{X}_0^{(\nu)} \), and all vector functions \( \phi \in \tilde{H}^1(K; \mathbb{C}^M) \), the inequality
\[
\| \tilde{D}_0(k + i \mathcal{X} e) + \tilde{V}_{\nu} \| \phi \| \geq c_5 (A_\nu) \theta' \pi |\gamma|^{-1} \| \phi \|
\]
is valid. For a sufficiently large index \( \nu \) (and for \( \mathcal{X} \geq \mathcal{X}_0^{(\nu)} \)), it follows that the desired inequality holds. Theorem 2 is proved.

**Theorem 8.** Let \( \tilde{V}^{(s)} : \mathbb{R}^n \to \mathcal{L}_M^{(s)}, s = 0, 1 \), and \( A : \mathbb{R}^n \to \mathbb{C}^n \) be continuous periodic functions with the period lattice \( \Lambda \subset \mathbb{R}^n \), \( n \geq 3 \). If \( A_0 = 0 \) and condition (6) with \( \tilde{\theta} \in [0, 1) \) is satisfied for a vector \( \gamma \in \Lambda \setminus \{0\} \) \( (e = |\gamma|^{-1} \gamma) \) and a measure \( \mu \in \mathcal{M}_h, h > 0 \), then for any \( \delta \in (0, 1) \), there are numbers \( \beta = \beta (\gamma, h, \mu; \tilde{V}, \delta) > 0 \) and \( \mathcal{X}_0 = \mathcal{X}_0 (\gamma, h, \mu; \tilde{V}, \delta) > 0 \) such that for all \( k \in \mathbb{R}^n \) with \( (k, \gamma) = \pi \), all \( \mathcal{X} \geq \mathcal{X}_0 \), and all vector functions \( \phi \in \tilde{H}^1(K; \mathbb{C}^M) \), the inequality
\[
\| \tilde{D}(k + i \mathcal{X} e) \phi \|^2 \geq (1 - \delta) \left( c_5^2(A; \gamma, h, \mu) (1 - \tilde{\theta})^2 \left( \frac{\pi}{|\gamma|} \right) \| \phi_{\mathcal{K}_\beta} \|^2 + v(K) \sum_{N \in \Lambda^\ast \setminus \mathcal{K}_{\beta}} G_2^2(k; \mathcal{X}) \| \phi_N \|^2 \right)
\]
holds.

Theorem 8 also follows from Theorem 7 in view of the uniform approximation of the functions \( \tilde{V}^{(s)} \) and \( A \) by trigonometric polynomials with the period lattice \( \Lambda \).

**Corollary.** Let \( \tilde{V}^{(s)} : \mathbb{R}^n \to \mathcal{L}_M^{(s)}, s = 0, 1 \), and \( A : \mathbb{R}^n \to \mathbb{C}^n \) be continuous periodic functions with the period lattice \( \Lambda \subset \mathbb{R}^n \), \( n \geq 3 \), let \( A_0 = 0 \), and let condition (6) with \( \tilde{\theta} \in [0, 1) \) hold for some vector \( \gamma \in \Lambda \setminus \{0\} \) \( (e = |\gamma|^{-1} \gamma) \) and a measure \( \mu \in \mathcal{M}_h, h > 0 \). Then there are numbers \( c_9 = c_9 (\gamma, h, \mu; \tilde{V}) > 0 \) and \( \mathcal{X}_0 = \mathcal{X}_0 (\gamma, h, \mu; \tilde{V}) > 0 \) such that for all \( k \in \mathbb{R}^n \)
with \((k, \gamma) = \pi\), all \(\kappa \geq \kappa_0\), and all vector functions \(\phi \in \tilde{H}^1(K; \mathbb{C}^M)\), the inequality
\[
\| \hat{D}(k + i\kappa e)\phi \|^2 \geq c_9 v(K) \sum_{N \in \Lambda^*} G_N^2(k; \kappa) \|\phi_N\|^2
\]
is fulfilled.

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