A New Vector-Tensor Theory and Higher-Dimensional Cosmology

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Physica Scripta 43, No. 2, pp. 129–132 (1991)

Abstract
A new vector-tensor model of classical gravity, which contains coupling between the field strength of the vector field and the curvature tensors in six dimensions, is proposed. Cosmological solutions of the scale factors in this model with the compactified space are obtained. The generalization of the model is also considered.

1 INTRODUCTION
The theory of general relativity seems the most probable theory which describes a classical behavior of space-time geometry at a large scale. To seek for the other possibilities in the description of gravitation, various modifications of Einstein’s gravity have been investigated by many authors in the past several decades. In some of the theories gravity is coupled to other fields; for example, the well-known Brans-Dicke theory [1] has a crucial coupling between the scalar curvature of the space-time manifold and a scalar field. On the other hand, we are able to consider a theory which is described by an action that includes higher-derivative terms of the metric tensor, such as a curvature-squared term [2]. Moreover, from a relation to unified theories, gravity can be derived by an action which is not the pure Einstein–Hilbert action. These theories can be formulated in more than four-dimensions, such as Kaluza–Klein theory [3] and the string theory [4]. We must note that multidimensional theories need not involve the gravity in higher dimensions, since they have only to induce a theory which is consistent with Einstein’s general relativity in four dimensions [5]. True theory of gravity may have some of the above-mentioned structures, and then it is worth studying possible combinations of the ideas about the gravitation.
As another difficulty in the theory of gravity, there are many problems in a quantization of gravity; one of them is that renormalization has been impossible because the coupling constant of gravitons, i.e., Newton’s constant $G$, has dimensions. Although we will not consider the quantization of gravity in this letter, we will construct a model as a theory with a dimensionless coupling in higher dimensional space-time. And the theory induces a correct dimensionality of the effective coupling of gravity in four dimensions.

We start with some possible hypotheses. First, a coupling constant in our model is dimensionless. Second, there are higher-derivative terms of the metric but an action has first order of second derivatives of the metric. In this case, evolution equations can be obtained from the action as in the case with the Einstein–Hilbert action, and we also expect that the solution of the equation of motion could not have so many branches (cf. ref. [6]).

Furthermore, we require that our model can have more than four dimensions, and then it does not describe Einstein gravity in the higher dimensions (cf. ref. [5]). Of course, we require that our model consistently describe Einstein gravity after compactification of an extra space.

In this letter, we will show that our model has a good symmetry and well-behaved cosmological solution of homogeneous spaces.

To construct an action which satisfies the requirements above, we choose an action which is linear in curvature tensors. Further, we introduce a vector field coupled to the curvature nonminimally.

We deal with our vector-tensor model in six dimensions. The generalization to the other dimensions will be discussed later. Our action is analogous to four-dimensional Euler form [6, 7]. That is:

$$I = -\frac{\gamma}{16} \int d^6 x \sqrt{-g} \delta_{EFGH}^{ABCD} F_{AB} F^{EF} R_{CD}^{GH} + (\text{surface terms}),$$

where the generalized Kronecker’s symbol is defined by

$$\delta_{EFGH}^{ABCD} = 4! \delta_{[E}^{A} \delta_{F}^{B} \delta_{G}^{C} \delta_{H]}^{D},$$

Here $\gamma$ is a dimensionless coupling because our action is defined in six dimensions. Horndeski showed [8] that this form of the action in four dimensions leads to the conservation of the “electric charge” as well as the energy-momentum. The surface terms can be added to remove the second-derivative of the metric, as is the Gibbons–Hawking surface term in Einsteinian gravity [9]. We consider two dimensions compactified to $S^2$, and we put an ansatz on the field strength of the vector field on $S^2$ as: [10]

$$F_{ab} = q \epsilon_{ab},$$

where $q$ is a constant. This ansatz is used throughout this letter. This satisfies the equation of motion extracted from the action (1), at least when the background metric has a homogeneous structure as the case we will consider in this letter.
If we want to include the matter, we add the action of the matter fields in the form of the six-dimensional integral.

In order to show several cosmological solutions to the field equations obtained from our action, we will use the metric of the form:

\[ ds^2 = -N^2 dt^2 + a^2(t) \left( 1 + \frac{k}{4} r^2 \right)^{-2} \left\{ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right\} + b^2(t)(d\omega^2 + \sin^2 \omega d\chi^2). \]  

By using this metric and the "monopole" ansatz (3) for the vector field, the action reduces to:

\[ I = 3\gamma q^2 \int dt \left\{ \frac{1}{N} \left[ \frac{\dot{a}^2}{b^2} - 2\frac{a^2 \dot{a} \dot{b}}{b^3} \right] - Nk \frac{a}{b^2} \right\}. \]  

The field equations can be obtained by variation of the action (5) with respect to the functions \( a, b \) and \( N \): the variation with respect to \( N \) leads to the analogous equation to the time-time component of Einstein equations after taking \( N \to 1 \).

In the presence of the matter Lagrangian, the field equations are:

\[ \left( \frac{\dot{a}}{a} \right)^2 + 2\frac{\ddot{a}}{a} + 2\frac{\dot{a} \dot{B}}{a B} + \frac{\ddot{B}}{B} + \frac{k}{a^2} = -\frac{1}{\gamma q^2 B^2} (p)_{\text{matter}}, \]  

\[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{\ddot{a}}{a} + \frac{k}{a^2} = \frac{1}{3\gamma q^2 B^2} (p')_{\text{matter}}, \]  

\[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{\dot{a} \dot{B}}{a B} + \frac{k}{a^2} = \frac{1}{3\gamma q^2 B^2} (\rho)_{\text{matter}}, \]

where \( B \equiv b^{-2} \), and \( (\rho, p, p')_{\text{matter}} \) represent (the energy density, pressure in the three-dimensional space, pressure in the extra two-dimensional space) respectively.

At first in order to inspect whether our model has a well-behaved cosmological solution, we solve a vacuum solution as the simplest case. The field equations can be solved exactly,

\[ a(t) = (c_1 t - kt^2)^{1/2}, \quad (c_1: \text{const.}) \]  

and

\[ b(t) = c_2 \left| \frac{t(c_1 - kt)}{(c_1 - 2kt)^2} \right|^{1/4}, \quad (c_2: \text{const.}). \]

Although the radius of the compactified inner space diverges at \( t = c_1/2 \), when \( k = 1 \), we can consider that we have not reached the time yet; the early evolution in this vacuum solution is relevant for the present universe. The behavior of the scale factor of our space \( (a) \) corresponds to a solution with radiation in the four-dimensional Friedmann-Robertson-Walker (FRW) model.

The behavior of \( a \) is determined by eq. (7).
Next we consider solutions in the presence of matters. Here we deal with some typical examples of matters, such as the quadratic of the field strength, dust matter, four-dimentional radiation and six-dimensional radiation. Some of the field equations with matter can be exactly solved and we first treat this kind of solutions.

We will show a solution with an extra contribution of the field strength. Namely, we add the usual Maxwell-type action of the quadratic term in the field strength to our action. That is to say, we incorporate the term \[ \int d^6x \sqrt{-g} \frac{\beta}{4} F_{MN} F^{MN} \]
into the action. We substitute the “monopole” form of the field strength in the extra space as the previous case. The equation of motion of the scale factor of three-dimensional space is solvable, and the solution is given, in the particular case where \( k = 0 \), as:

\[ a(t) = \left[ c_1 \sinh(2\alpha'/t) \right]^{1/2}, \quad (c_1 : \text{const.}), \]  

where \( \alpha' \equiv \beta/6\gamma \).

In this case, the scale of the extra space is:

\[ b(t) = c_2 \left[ \tanh(2\alpha'/t) \right]^{1/4} \left[ \cosh(2\alpha'/t) \right]^{-1}, \quad (c_2 : \text{const.}). \]  

The field equations can be solved exactly even if \( k \neq 0 \), after some manipulations. But we omit the exhibition of the solutions.

Just as the vacuum solution, the behavior of the scale factor of our space corresponds to the solution in the presence of radiation as well as a cosmological constant in four-dimensional FRW model.

Next we consider dust and four-dimensions radiation as a matter. which are analogue of matter and radiation dominated era in four-dimensional FRW model. The field equation of the three-dimensional scale factor is the same as a vacuum case:

\[ a(t) = (c_1 t - kt^2)^{1/2}. \]  

And the scale factor of the compactified extra space in each case is

For dust matter:

\[ b_d = \left[ \frac{C'}{3k\gamma q^2(c_1 t - kt^2)^{1/2}} + c_2 \frac{c_1 - 2kt}{(c_1 t - kt^2)^{1/2}} \right]^{-1/2}, \quad (k \neq 0) \]  

\[ b_d = \left[ \frac{2C'}{3\gamma q^2 c_1^{3/2}} + c_2 \frac{c_1}{t^{1/2}} \right]^{-1/2}, \quad (k = 0) \]

For four-dimensional radiation:

\[ b_r = \left[ \frac{4C}{3\gamma q^2} + c_2 \frac{c_1 - 2kt}{(c_1 t - kt^2)^{1/2}} \right]^{-1/2}, \]  

where \( b_d \) and \( b_r \) are the solution in the presence of a dust matter and the one in the presence of four-dimensional radiation, respectively. \( c_1 \) and \( c_2 \) are
integration constants. Here the amount of the matter is represented as \( \rho_{\text{dust}} = C'/\left(a^3 b^2\right) \) and \( \rho_{\text{4-rad.}} = C/\left(a^4 b^2\right) \) in each case.

Finally we show the solution of the universe dominated by six-dimensional radiation. Since in this case the field equations cannot be solved in the analytic form, we show the leading behaviors of the scale factors at the early stage of the evolution. For six-dimensional radiation, its energy density is written as

\[ \rho = \rho_0 (a^3 b^2)^{-6/5} \]  

Here we take \( k = 0 \), because we need a solution which corresponds to the early universe. We assume that solutions of the scale factors in this case are proportional to the power of cosmic time, thus we can write \( a = t^\alpha \) and \( b = t^\beta \), respectively. Substituting these in eqs.(6,7,8), we find a set of solutions for \( \alpha \) and \( \beta \):

\[ \alpha = \frac{5}{9} \quad \text{and} \quad \beta = 0 \]  

These behaviors of the scale factors are very close to the vacuum case (\( \alpha = 1/2 \)). And this fact indicates that more moderate change of the scale of the extra space is permitted in the presence of the six-dimensional radiation.

So far we obtain some cosmological solutions of our six-dimensional vector-tensor model including the exact solutions. For the vacuum solution and the solution in the presence of the extra Maxwell term, there are relations to other cosmological models in other dimensions.

Changing variables in our model as \( 1/b^2 = B \) reduces our action (1) to the five-dimensional Kaluza-Klein action with taking the metric, \( ds^2 = -dt^2 + a^2 d\Omega^2 + b^2 dy^2 \). Accordingly, the vacuum solution in our model corresponds to the vacuum solution in the five-dimensional model, up to the substitution of \( 1/b^2 = B \). In the same way, the Maxwell term \( \beta F^2/4 \), which is equivalent to \( \beta(2q^2)/4b^4 \) by use of the monopole ansatz, becomes the form of \( \Lambda'B^2 \) by changing variable as \( 1/b^2 = B \) and \( \Lambda' = \beta q^2/2 \). This term corresponds to the cosmological term in that of five-dimensional Kaluza-Klein theory. It is known [11] that the five-dimensional Kaluza-Klein cosmological solution with a cosmological constant (including vanishing cosmological constant) can be related to the universe dominated by four dimensional radiation with an appropriately normalized cosmological constant.

As an extension of our vector-tensor model, we would like to consider a model with more than six dimensions, that is, \( 4 + N \) dimensions. A natural extension can be discovered in (\( 4 + N \)) dimensions is described by an action which contains an antisymmetric tensor field. The possible action is

\[ I = -\frac{\gamma_N}{4(N!)^2} \int d^{4+N}x \sqrt{-g} \theta_{B_1 \cdots B_{N+2}} H_{A_1 \cdots A_N} H^{B_1 \cdots B_N} R^{B_{N+1} B_{N+2}}_{A_{N+1} A_{N+2}} + \text{(surface terms)}, \]  

where Kronecker’s delta is generalized to have \( 2(N+2) \) indices. Here \( \gamma_N \) is not dimensionless, regretfully. But the dimension of \( \gamma_N \) is (mass)\(^2-N\); then it is not so bad in the sense of powercounting, though we never consider the quantization here. If we assume the “expectation value” of the form \( H_{\cdots} = q\varepsilon_{\cdots} \) on the extra
$N$-manifold, the vacuum solution in this model can be solved exactly. At early stage of the evolution of our three-space behaves like $t^{1/2}$, whereas the scale factor of the extra space behaves like

$$b(t) \approx t^{1/2}. \quad (19)$$

Therefore, if $N$ is sufficiently large, time evolution of the coupling “constant” like the gravitational coupling and gauge couplings (if we add), which are concerned with $b$, can be so small as to be consistent with the observations [12].

In summary, we obtain the cosmological solutions in our vector-tensor model in six dimensions with compactification in the presence of various matter fields. The relation to the other dimensional cosmological model makes the derivation of the solutions simple. Note that the extra space in our model admits a non-Abelian isometry. The generalization of our model to higher-dimensions is straightforward, and the rapid change in the coupling constant can be avoided if we take the model with sufficiently large number of total dimensions.

Spherically symmetric vacuum solution can be obtained analytically because of the correspondence with five-dimensional “black hole” solution by the similar substitution as in this letter; in particular, Schwarzschild solution with extra space of constant radius is permitted. But if we consider “charged” object, the solution in our vector-tensor model differs from the Reissner-Nordstrom solution in general relativity. The numerical study on the solution of this type is in progress and will be reported.

Another interesting topic is the making of the whole universe as a topological defect [13]. We start with flat six dimensions. We introduce a complex scalar minimally coupled to the vector field in order to construct a Nielsen-Olesen vortex [14]. Since the field strength is concentrated at the core of the vortex, the gravity “exists” only there. Now we obtain a “thin” universe with spatial three dimensions. Further study about this scenario including examination of the “confinement” in the universe and the back reaction is a future problem.

We have treated our models as a classical theory. Although the coupling constants have favorable dimensionality, quantization of our models is not straightforward. This is typically shown by the fact that the kinetic term of the vector field has “wrong” sign in the compactified background. We think that this is not a great drawback, because the component with negative sign always exists in the Einstein gravity and the minimal corrections of the theory. We must study the quantization of our model as well as the gravity theories in general.

**Acknowledgements**

The authors would like to thank A. Nakamula for some comments.

This work is supported in part by the Grant-in-Aid for Encouragement of Young Scientist from the Ministry of Education, Science and Culture (# 63790150).

One of the authors (KS) is grateful to the Japan Society for the Promotion of Science for the fellowship. He also thanks Iwanami Fujukai for financial aid.
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