A Riemann–Hilbert approach to Painlevé IV

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The methods of [vdP-Sa, vdP1, vdP2] are applied to the fourth Painlevé equation. One obtains a Riemann–Hilbert correspondence between moduli spaces of rank two connections on \(\mathbb{P}^1\) and moduli spaces for the monodromy data. The moduli spaces for these connections are identified with Okamoto–Painlevé varieties and the Painlevé property follows. For an explicit computation of the full group of Bäcklund transformations, rank three connections on \(\mathbb{P}^1\) are introduced, inspired by the symmetric form for PIV, studied by M. Noumi and Y. Yamada.

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Introduction

In this paper we apply the methods of [vdP-Sa, vdP1, vdP2] to the fourth Painlevé equation. We refer only to a few items of the extensive literature on Okamoto–Painlevé varieties. More details on Stokes matrices and the analytic classification of singularities can be found in [vdP-Si].

The Riemann–Hilbert approach to the Painlevé equation PIV consists of the construction of a moduli space \(\mathcal{M}\) of connections on the projective line and a moduli space \(\mathcal{R}\) for the monodromy data. The Riemann–Hilbert morphism \(RH : \mathcal{M} \to \mathcal{R}\) assigns to a connection its monodromy data. The fibres of \(RH\), i.e., the isomonodromic families in \(\mathcal{M}\), are parametrized by \(t \in T = \mathbb{C}\). The explicit form of the fibres produces the solutions of PIV.

\[RH^{\text{ext}} : \mathcal{M}^+ (\theta_0, \theta_\infty) \to \mathcal{R}^+ (\theta_0, \theta_\infty) \times T\]

The extended Riemann–Hilbert morphism, is an analytic isomorphism between rather subtle moduli spaces \(\mathcal{M}^+ (\theta_0, \theta_\infty)\) and \(\mathcal{R}^+ (\theta_0, \theta_\infty) \times T\), depending on parameters \(\theta_0, \theta_\infty\) and provided with a level structure (or parabolic structure). The Painlevé Property for PIV with parameters \(\theta_0, \theta_\infty\) follows from this as well as the identification of \(\mathcal{M}^+ (\theta_0, \theta_\infty)\) with an Okamoto–Painlevé variety. Formulas for Bäcklund transformations, rational and Riccati solutions for PIV are derived.

The construction of \(\mathcal{M}\) involves the choice of a set \(S\) of differential modules over \(\mathbb{C}(z)\). In the first part of this paper the ‘classical’ choice for \(S\) is treated. The second choice for \(S\) is inspired by the symmetric form for PIV [No, No-Y], studied by M. Noumi and Y. Yamada. This leads to a different construction of \(\mathcal{M}, \mathcal{R}\) and Okamoto–Painlevé varieties, treated in §3.

1. The classical choice for \(S\) and \(\mathcal{R}\)

Let \(S\) be the set of the isomorphy classes of the differential modules \((M, \delta_M)\) over \(\mathbb{C}(z)\) (with \(\delta_M(fm) = \left(z \frac{d}{dz} f\right)m + f \delta_M(m)\)) having the properties:

- \(\dim M = 2\); \(\Lambda^2 M\) is trivial;
- \(0, \infty\) are the singular points and the Katz invariants are \(r(0) = 0, r(\infty) = 2\).
The variable $z$ is normalized such that the (generalized) eigenvalues at $\infty$ are $\pm(z^2 + \frac{1}{2}z)$. Finally, we exclude the case that $M$ is a direct sum of two proper submodules since this situation does not produce solutions for PIV.

The monodromy data at $\infty$ are given by the matrices

\[
\left( \begin{array}{cc}
\alpha & 0 \\
\frac{1}{\alpha} & 1
\end{array} \right), \left( \begin{array}{cc}
1 & 0 \\
a_1 & 1
\end{array} \right), \left( \begin{array}{cc}
1 & a_2 \\
0 & 1
\end{array} \right), \left( \begin{array}{cc}
1 & a_3 \\
a_3 & 1
\end{array} \right), \left( \begin{array}{cc}
1 & a_4 \\
0 & 1
\end{array} \right)
\]

with respect to a basis of the symbolic solution space $V(\infty)$ at $z = \infty$ corresponding to the direct sum expression $V(\infty) = V_{z^2 + \frac{1}{2}z} \oplus V_{-(z^2 + \frac{1}{2}z)} = \mathbb{C}e_1 \oplus\mathbb{C}e_2$. The first matrix is the formal monodromy and the others are the four Stokes matrices. The topological monodromy $top_\infty$ at $z = \infty$ (which equals the topological monodromy at $z = 0$) is the product of these matrices in this order. Further we exclude the case $a_1 = a_2 = a_3 = a_4 = 0$, since this corresponds to the direct sum situation. The monodromy data form a variety $\mathcal{A} := \mathbb{C}^* \times (\mathbb{C}^4 \setminus \{(0, 0, 0, 0)\})$.

The base change $e_1, e_2 \mapsto \lambda e_1, \lambda^{-1} e_2$ induces an action of $\mathbb{G}_m$ on $\mathcal{A}$. The monodromy space $\mathcal{R}$ is the quotient $\mathcal{A}/\mathbb{G}_m$. This quotient can be obtained by gluing the subspaces $\mathcal{R}_j, j = 1, \ldots, 4$ of $\mathcal{A}$, defined by $a_j = 1$.

We observe (see [vdP-Sa], Theorem 1.7) that the map $S \to \mathcal{R} \times T$, which maps a module in $S$ to its monodromy data and the value of $t \in T = \mathbb{C}$, is bijective.

The parameter space is $\mathcal{P} = \mathbb{C} \times \mathbb{C}^*$ and $\mathcal{R} \to \mathcal{P}$ maps an element of $\mathcal{R}$ to $(\text{trace}(top_\infty), \alpha)$. The fibre above $(s, \alpha)$ is denoted by $\mathcal{R}[s, \alpha]$. This fibre is a smooth, connected surface for $s \neq \pm 2$. The fibre $\mathcal{R}[2, \alpha]$ has one singular point and this point corresponds to $top_\infty = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)$. Similarly, $\mathcal{R}[-2, \alpha]$ has one singular point corresponding to $top_\infty = -\left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)$.

The singular points are the reason for introducing a level structure (or ‘parabolic structure’ in the terminology of [Bo, In, IIS1, IIS2, ISA]). For the monodromy data this is a line $L \subset V(\infty)$ which is invariant under $top_\infty$. The new monodromy space is denoted by $\mathcal{R}^+$. For a module $M$ in $S$ the level structure is a 1-dimensional submodule $N$ of $\mathbb{C}(\mathbb{C}(z)) \otimes M$. The submodule $N$ corresponds to an eigenvalue of the topological monodromy $top_0$ at $z = 0$ (which is equal to $top_\infty$). The new set is denoted by $S^+$. For the parameter space, the level structure is the introduction of an eigenvalue $\beta$ of $top_\infty$. The new parameter space $\mathcal{P}^+ = \mathbb{C}^* \times \mathbb{C}^*$ maps to $\mathcal{P}$ by $(\beta, \alpha) \mapsto (\beta + \beta^{-1}, \alpha)$.

The fibres of $\mathcal{R}^+ \to \mathcal{P}^+$ are denoted by $\mathcal{R}^+(\beta, \alpha)$. The morphism $\mathcal{R}^+((\beta, \alpha) \to \mathcal{R}[\beta + \beta^{-1}, \alpha]$ is an isomorphism for $\beta \neq \pm 1$. A computation shows that

**Lemma 1.1.** $\mathcal{R}^+((1, \alpha) \to \mathcal{R}[\pm 2, \alpha]$ is the minimal resolution.

The map $S^+ \to \mathcal{P}^+$ is defined by $\beta = e^{2\pi i \lambda}$ where $\delta_m n = 2n$ for a basis vector $n$ of $N \subset \mathbb{C}(\mathbb{C}(z)) \otimes M$ and $\alpha$ as before. The fibre is written as $S^+(\beta, \alpha)$.

**Lemma 1.2.** The map $S^+(\beta, \alpha) \to \mathcal{R}^+(\beta, \alpha) \times T$ is bijective.

The reducible locus of $\mathcal{R}^+$ (i.e., the monodromy data is reducible) is the disjoint union of the closed sets defined (in the notation of $\mathcal{A}$) by $a_2 = a_4 = 0$ and $a_1 = a_3 = 0$. The space $\mathcal{R}^+((\beta, \alpha)$ contains no reducible elements for $\beta \neq \alpha$. If $\beta = \alpha$, then the reducible locus of $\mathcal{R}^+((\beta, \alpha)$ consists of two non intersecting projective lines.
2. The moduli space \( \mathcal{M}(\theta_0, \theta_\infty) \)

Choose \( \theta_0 \) with \( \beta = e^{i\theta_0} \) and \( \theta_\infty \) with \( \alpha = e^{i\theta_\infty} \). The aim is to replace the set \( S^+ (\beta, \alpha) \) by a moduli space of connections \( \mathcal{M}(\theta_0, \theta_\infty) \) and to study the extended Riemann–Hilbert map \( RH^{ext} : \mathcal{M}(\theta_0, \theta_\infty) \to \mathfrak{g}^+ (\beta, \alpha) \times T \).

Let a module \( (M,N) \in S^+ (\beta, \alpha) \) be given. We define a connection \( (\nabla', \nabla) \) on the projective line with \( \nabla : \mathcal{V}' \to \Omega([0, 3]) \otimes \mathcal{V}' \), with generic fibre \( M \), by prescribing the connection \( D := \nabla' \).

This is equivalently to choosing ‘invariant lattices’ at \( z = 0 \) and \( z = \infty \). The invariant lattice at \( z = 0 \) is \( \mathbb{C}[z]g_1 + \mathbb{C}[z]g_2 \subset \mathbb{C}((z)) \otimes M \) with \( N = \mathbb{C}((z))g_1 \) and the matrix of \( \delta_M \) with respect to \( g_1, g_2 \) is \( \left( \begin{array}{cc} \frac{a_0}{z} & \frac{a_1}{z^2} \\ 0 & \frac{a_2}{z} \end{array} \right) \).

The invariant lattice at \( z = \infty \) is \( \mathbb{C}[\frac{1}{z}]h_1 + \mathbb{C}[\frac{1}{z}]h_2 \subset \mathbb{C}(\mathbb{C}^{(-1)}) \otimes M \) such that \( \delta_M h_1 = \omega h_1, \delta_M h_2 = -\omega h_2 \).

The second exterior power of \( \omega \) is \( d : O \to \Omega \). Thus \( \omega \) has degree 0 and type \( O(k) \oplus O(-k) \) with \( k \geq 0 \). If \( M \) is irreducible, then \( k \in \{0, 1\} \). The reducible modules are studied in Observations 2.2.

We consider the case \( k \in \{0, 1\} \). The connection \( (\nabla', \nabla) \), defined by replacing the invariant lattice \( \mathbb{C}[z]g_1 + \mathbb{C}[z]g_2 \) by \( \mathbb{C}[z]g_1 + \mathbb{C}[\frac{1}{z}]g_2 \), has type \( O \oplus O(-1) \). Further we identify \( \mathcal{V} \) with \( \mathcal{O}_1 \oplus O([-0])e_2 \).

2.1. The connections on \( \mathcal{V} := \mathcal{O}_1 + O([-0])e_2 \)

The connection \( D = \nabla' : \mathcal{V} \to O(2[\infty]) \otimes \mathcal{V} \), obtained from \( (M,N) \in S^+ (\beta, \alpha) \) and the prescribed invariant lattices, has, with respect to the basis \( e_1, e_2 \), the matrix \( \left( \begin{array}{cc} a & b \\ c & -d \end{array} \right) \) with \( a = a_0 + a_1z + a_2z^2 \), \( b = b_{-1}z^{-1} + \cdots + b_2z^2 \), \( c = c_1z + c_2z^2 \). The local data at \( z = \infty \) yields the equations

\[
a_0^2 + b_2c_2 = 1, \quad 2a_0a_2 + b_2c_1 + b_1c_2 = \tau, \quad 2a_0a_1 + a_1^2 + b_1c_1 + b_0c_2 = \theta_\infty + \frac{t^2}{4}.
\]

For \( z = 0 \) one obtains \( a_0(a_0 - 1) + b_{-1}c_1 = \frac{\theta_1}{2}(\frac{\theta_1}{2} - 1) \).

As a start, we forget the level structure \( N \) of the pair \( (M,N) \in S^+ (\beta, \alpha) \) and we assume that \( c_1z + c_2z^2 \neq 0 \). The above variables \( a_s, b_s, c_s, t \) and equations define a space \( \mathcal{G} \) of dimension 6. We have to divide by the group \( G \) of transformations \( e_1 \mapsto e_1, \quad e_2 \mapsto \lambda e_2 + (x_0 + x_1z^{-1})e_1 \) of \( \mathcal{V} \). The quotient \( \mathcal{G} / G \) is by definition the moduli space \( \mathcal{M}(\theta_0, \theta_\infty) \).

Proposition 2.1. The moduli space \( \mathcal{M}(\theta_0, \theta_\infty) \) is a good geometric quotient of \( \mathcal{G} \) in the sense that there exists a \( G \)-equivariant isomorphism \( G \times \mathcal{M}(\theta_0, \theta_\infty) \to \mathcal{G} \).

\( \mathcal{M}(\theta_0, \theta_\infty) \) is smooth for \( \theta_0 \neq 1 \). For a connection \( D \in \mathcal{M}(1, \theta_\infty) \), which is a singular point, there is a basis of \( \mathcal{V}_0 \) for which \( D \) has the form \( z \frac{d}{dz} + \left( \begin{array}{c} i \frac{t^2}{4} \\ 0 \end{array} \right) \).

Proof. The ‘first standard form’ \( ST_1 \) is the closed subset of \( \mathcal{G} \) defined by:

\[
z \frac{d}{dz} + \left( \begin{array}{c} a_2z^2 + b \\ a_1z^2 + c \end{array} \right) \quad \text{with} \quad b_2 = -c_2(\theta_\infty + \frac{t^2}{4} - b_0c_2) + t, \quad b_1 = \theta_\infty + \frac{t^2}{4} - b_0c_2, \quad b_{-1} = \frac{\theta_1}{2}(\frac{\theta_1}{2} - 1) \quad \text{and} \quad a_1^2 + a_2^2 - c_1^2(\theta_\infty + \frac{t^2}{4} - b_0c_2) - 1 = 0.
\]

The obvious morphism \( G \times ST_1 \to \{ (a_s, b_s, c_s) \in \mathcal{G} | c_1 \neq 0 \} \) is an isomorphism.
The coordinate ring of $ST_1$ is $\mathbb{C}[a_2, c_2, t, b_0]/(a_2^2 + c_2 t - c_2^2(\theta_0 + \frac{t}{4} - b_0 c_2) - 1)$ and $ST_1$ is non-singular.

The ‘second standard form’ $ST_2$ is the closed subset of $\mathcal{C}$ defined by:

$$z \frac{d}{dz} + \left( \begin{array}{c}
\frac{a_0}{c_1 z + z - \theta_0} \\
\frac{b}{c_1 z + z - \theta_0}
\end{array} \right)$$

with $b = z^2 + b_1 z + b_0 + b_{-1} z^{-1}$, $c = c_1 z + z^2$ and $c_1 + b_1 = t$, $b_1 c_1 + b_0 = \theta_0 + \frac{t}{4}$, $a_0(a_0 - 1) + b_{-1} c_1 = \frac{b_0}{2}(\frac{b_0}{2} - 1)$. The obvious morphism $G \times ST_2 \to \{(a_0, b_1, c_1) \in \mathcal{C} | c_2 \neq 0\}$ is an isomorphism.

The coordinate ring of $ST_2$ is $\mathbb{C}[a_0, c_1, t, b_{-1}]/(a_0(a_0 - 1) + b_{-1} c_1 - \frac{a_0}{2}(\frac{a_0}{2} - 1))$. For fixed $t$, one finds one singular point: $a_0 = 1/2$, $b_{-1} = c_1 = 0$, $\theta_0 = 1$.

In the above case, one easily verifies that $D$ has the form $z \frac{d}{dz} + (\frac{1}{4} 0 \frac{1}{2})$ w.r.t. a basis of $\mathcal{T}_0$. The quotient $\mathcal{C} / G$ is obtained by gluing the two ‘charts’ $ST_1$ and $ST_2$ in the obvious way.

Observations 2.1 The level structure for $\mathcal{M}(\theta_0, \theta_0)$.

For a connection $D \in \mathcal{M}(\theta_0, \theta_0)$, the level structure is a 1-dimensional submodule $N \subset \mathbb{C}((z)) \otimes \mathcal{T}_0$ with a generator $n$ such that $\delta n = e^{z \theta_0} n$. The space $\mathcal{M}^+(\theta_0, \theta_0)$ denotes the addition of this level structure to $\mathcal{M}(\theta_0, \theta_0)$.

If $top_0$, the topological monodromy at $z = 0$ of the connection $D$, is not $\pm (1 0 1)$, then the level structure $N$ is unique.

If $top_0 = \pm (1 0 1)$, then $\theta_0 \in \mathbb{Z}$. Further $\mathcal{T}_0$ has a basis $v_1, v_2$ for which $D$ obtains the matrix form

$$z \frac{d}{dz} + \left( \begin{array}{c}
\frac{a_0}{c_1 z + z - \theta_0} \\
\frac{b}{c_1 z + z - \theta_0}
\end{array} \right)$$

If $\frac{b_0}{2} \neq \frac{\theta_0}{2}$, the basis $v_1, v_2$ is unique up to multiplication by constants. Then one defines the level structure $N$ by $N = \mathbb{C}((z)) v_1$.

In the final case $top_0 = -(1 0 1)$ and $\theta_0 = 1$, the connection $D$ does not prescribe a level structure. We replace $\mathcal{M}(1, \theta_0)$ by $\mathcal{M}^+(1, \theta_0)$ defined as the closed subspace of $\mathcal{M}(1, \theta_0) \times \mathbb{P}^1$ consisting of the equivalence classes of the tuples $(D, L)$ with $D \in \mathcal{M}(1, \theta_0)$ and $L$ a line in $\mathcal{T}_0$ at $z = 0$, invariant under $D$. We will verify that (for fixed $t$) $\mathcal{M}^+(1, \theta_0) \to \mathcal{M}(1, \theta_0)$ is the minimal resolution.

Verifications. The chart $ST_2$ of $\mathcal{M}(1, \theta_0)$ consists of the differential operators $z \frac{d}{dz} + \left( \begin{array}{c}
\frac{a_0}{c_1 z + z - \theta_0} \\
\frac{b}{c_1 z + z - \theta_0}
\end{array} \right)$ with

$$b = z^2 + b_1 z + b_0 + b_{-1} z^{-1}, c_1 + b_1 = t, b_1 c_1 + b_0 = \theta_0 + \frac{t}{4}, a_0(a_0 - 1) + b_{-1} c_1 = -\frac{1}{4}.$$ 

The line $L = \mathbb{C}((c_1))$ is generated by a non-zero element $(c_1) \in \mathbb{C}[[z]]^2$ satisfying the equation

$$\{z \frac{d}{dz} + \left( \begin{array}{c}
\frac{a_0}{c_1 z + z - \theta_0} \\
\frac{b}{c_1 z + z - \theta_0}
\end{array} \right) \} (x_1) = \frac{1}{2} (x_1).$$

In the case $a_0 = \frac{1}{2}$, $b_{-1} = c_1 = 0$, the operator $z \frac{d}{dz} + \left( \begin{array}{c}
\frac{a_0}{c_1 z + z - \theta_0} \\
\frac{b}{c_1 z + z - \theta_0}
\end{array} \right)$ is equivalent over $\mathbb{C}[[z]]$ to $z \frac{d}{dz} + (\frac{1}{2} 0 \frac{1}{4})$. Thus the possible lines $L$ form a projective line. In the opposite case, the operator is equivalent over $\mathbb{C}[[z]]$ to $z \frac{d}{dz} + (\frac{1}{2} \frac{1}{2})$ and there is only one $L$.

Observations 2.2 The reducible locus of $\mathcal{M}(\theta_0, \theta_0)$.

Put $\omega = \omega^2 + \frac{1}{2} z + \frac{\theta_0}{2}$ and let $c \in \mathbb{C}[z^{-1}, z]$. If a reducible connection is present in $\mathcal{M}(\theta_0, \theta_0)$, then $\frac{\theta_0}{2} \in \mathbb{Z} + \mathbb{Z}$. There are two types of reducible modules in $S^+$. Type (1) is represented by $z \frac{d}{dz} + \left( \begin{array}{c}
\omega \\
\omega
\end{array} \right)$ and Type (2) is represented by $z \frac{d}{dz} + \left( \begin{array}{c}
\omega \\
- \omega
\end{array} \right)$.
For a given reducible module $M$, say of type (1), one defines (as before) the connection $(\mathcal{V}, D)$ with generic fibre $M$ by the local operators $z \frac{d}{dz} + \left( a \frac{1}{z} + b \right)$ at $z = 0$ and $z \frac{d}{dz} + \left( a \frac{1}{z} - b \right)$ at $z = \infty$. Assume that type (1) is not present in $\mathcal{M}(\theta_0, \theta_\infty)$. Then $\mathcal{V} \cong O(\kappa) \oplus O(-k - 1)$ with $k \geq 1$ and one identifies $\mathcal{V}$ with $O(k[0])e_1 \oplus O(-(k + 1)[0])e_2$. A computation of $D$ in this case leads to two possible relations, namely $\frac{\theta}{a} = \frac{\theta}{c} = k$ or $1 - \frac{\theta}{a} = \frac{\theta}{c} - k$. Thus one finds the list for type (1). The list for type (2) is found in a similar way.

Type (1) is present in precisely the following cases:

$\theta_0 \geq \theta_\infty$ for $\frac{\theta}{a} \in \frac{\theta}{a} + \mathbb{Z}$ and $\frac{\theta}{a} \not\in -\frac{\theta}{a} + \mathbb{Z}$

$\theta_0 \leq -\theta_\infty + 2$ for $\frac{\theta}{a} \not\in -\frac{\theta}{a} + \mathbb{Z}$ and $\frac{\theta}{a} \in -\frac{\theta}{a} + \mathbb{Z}$

$\theta_0 \geq \theta_\infty$ or $\theta_0 \leq -\theta_\infty + 2$ for $\frac{\theta}{a} \in -\frac{\theta}{a} + \mathbb{Z}$ and $\frac{\theta}{a} \not\in -\frac{\theta}{a} + \mathbb{Z}$

Type (2) is present in precisely the following cases:

$\theta_0 \leq \theta_\infty + 2$ for $\frac{\theta}{a} \not\in \frac{\theta}{a} + \mathbb{Z}$ and $\frac{\theta}{a} \not\in -\frac{\theta}{a} + \mathbb{Z}$

$\theta_0 \geq -\theta_\infty$ for $\frac{\theta}{a} \not\in \frac{\theta}{a} + \mathbb{Z}$ and $\frac{\theta}{a} \not\in \frac{\theta}{a} + \mathbb{Z}$

$\theta_0 \leq \theta_\infty + 2$ or $\theta_0 \geq -\theta_\infty$ for $\frac{\theta}{a} \not\in \frac{\theta}{a} + \mathbb{Z}$ and $\frac{\theta}{a} \not\in -\frac{\theta}{a} + \mathbb{Z}$

Examples: We use the notation $z \frac{d}{dz} + \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$ of the space $\mathcal{V}$. Suppose $b = 0$.

Then $a_1^2 = 1$, $2a_1a_2 = t$, $2a_0a_2 + a_1^2 = \theta_\infty + \frac{\theta}{a}^2$, $(a_0 - \frac{1}{2})^2 = \left( \frac{\theta}{a} - \frac{1}{2} \right)^2$ and the non zero element $e_1z^2 + e_2z^2$ is unique up to multiplication by a non zero constant. One finds in general four reducible families (with some overlap for $\theta_0 = 1$ and/or $\theta_\infty = \pm 1$):

$b = 0$, $a_2 = 1$, $a_1 = \frac{1}{2}$, $a_0 = \frac{\theta}{a}$, $\frac{\theta}{a} = \frac{1}{2} \pm \left( \frac{\theta}{a} - \frac{1}{2} \right)$ and

$b = 0$, $a_2 = -1$, $a_1 = -\frac{1}{2}$, $a_0 = -\frac{\theta}{a}$, $-\frac{\theta}{a} = \frac{1}{2} \pm \left( \frac{\theta}{a} - \frac{1}{2} \right)$.

We observe that these examples are precisely the cases of an equality sign in the lists for type (1) and type (2).

**Proposition 2.2.** Put $\beta = e^{\pi i a_0}, \alpha = e^{\pi i a_0}$. Let $F : \mathcal{M}^+(\theta_0, \theta_\infty) \to S^+(\beta, \alpha)$ be the map that sends a tuple $(D, L)$ to $(M, N)$ where $M$ is the generic fibre of $D$ and $N = \mathbb{C}(\mathbb{Z}) \otimes L$. The map $F$ is injective and its image contains the ‘irreducible locus’ of $S^+(\beta, \alpha)$. A component of the ‘irreducible locus’ lies in the image of $F$ if and only if $\theta_0, \theta_\infty$ satisfy the corresponding inequality of Observations 2.2.

**Proof.** The injectivity of $F$ follows from the construction of $\mathcal{M}^+(\theta_0, \theta_\infty)$. If $M$ is irreducible then the vector bundle $\mathcal{V}$, introduced in the beginning of this section, has type $O(k) \oplus O(-k)$ for some $k \in \{0, 1\}$. Therefore the subbundle $\mathcal{V}$ has type $O \oplus O(-1)$ and can be identified with $Oe_1 \oplus O(-[0])e_2$. Thus the image of $F$ contains the ‘irreducible locus’ of $S^+(\beta, \alpha)$. The final statement follows from Observations 2.2. \qed

Define $S^+(\theta_0, \theta_\infty) \subset S^+(\beta, \alpha)$ (for $\beta = e^{\pi i a_0}, \alpha = e^{\pi i a_0}$) to be the image of $F$ and let $R^+(\theta_0, \theta_\infty) \subset R^+(\beta, \alpha)$ be the corresponding open subset.

**Corollary 2.1.** $RH^{ext} : \mathcal{M}^+(\theta_0, \theta_\infty) \to R^+(\theta_0, \theta_\infty) \times T$, the extended Riemann–Hilbert map, is an analytic isomorphism.

**Proof.** $RH^{ext}$ is bijective since $S^+(\theta_0, \theta_\infty) \to R^+(\theta_0, \theta_\infty) \times T$ is bijective. The two spaces $\mathcal{M}^+(\theta_0, \theta_\infty)$ and $R^+(\theta_0, \theta_\infty) \times T$ are smooth and so $RH^{ext}$ is an analytic isomorphism (see [vdPl, vdP2]). \qed
2.2. Isomonodromic families, Okamoto–Painlevé spaces

An isomonodromic family above the chart ST$_2$ of $\mathcal{M}^+(\theta_0, \theta_\infty)$ has the form $z \frac{d}{dz} + A$ with $A = (a_0 \ - b) \ _{c \ - d}$ with $c = z^2 - qz$, $b = \frac{1}{2}z^2 + b_1 z + b_0 + b_{-1}z^{-1}$, $b = \frac{1}{2}z^2 + (t + q)z + q(t + q) + \theta_\infty + \frac{t^2}{4} + (\frac{8 - a_0 - \frac{1}{2})^2}{q} - (\frac{8 - a_0 - \frac{1}{2})^2}{q}$ and that the entries of $M$ are holomorphic and have no zeros. Consider the operator $Q = \frac{1}{2}z^2 + b_1 z + b_0 + b_{-1}z^{-1}$, where $a_0$ and $q$ are functions of $t$. Isomonodromy is equivalent to the existence of an operator $\frac{d}{dt} + B$, commuting with $z \frac{d}{dz} + A$. In other terminology $\frac{d}{dt} + A$, $\frac{d}{dt} + B$ is a Lax pair. This is equivalent to the equation $\frac{d}{dt}(A) = \frac{d}{dz}(B) + [A, B]$. One observes that $B$ has trace zero and that the entries of $B$ have the form $d_{-1}(t)z + d_0(t) + d_1(t)z$. Using MAPLE one obtains the isomonodromic family of operators $M$ appearing in the formula for the chart ST$_2$. Then $Q$ is a Riccati solution of PIV with $d = \theta_\infty$ and $d = 1 \pm (\theta_0 - 1)$.

Isomonodromy for reducible families. An isomonodromic family of operators $z \frac{d}{dz} + (\omega \ _{0 \ - \omega})$, with $\omega = \frac{1}{2}z^2 + \frac{1}{2}z + \frac{1}{2}$, commutes with an operator of the form $\frac{d}{dt} + (\tau \ _{\theta \ - \tau})$. One computes that $\tau = \frac{z^2 + 2z + t}{4}$ and $q' = q^2 + \frac{1}{2}q + \frac{d}{2}$. Then $q$ is a Riccati solution of PIV with $d = \theta_\infty$ and $d = -1 \pm (\theta_0 - 1)$.

Observations 2.3 The solutions $q_r$ with $r \in \mathcal{R}^+(\theta_0, \theta_\infty)$. The fibre of $\mathcal{M}^+(\theta_0, \theta_\infty)$ is, by Corollary 2.1, isomorphic to $T$. Write $q_r$ for the function $q$ appearing in the formula for the chart ST$_2$. Then $q_r$ is a meromorphic solution of PIV, defined on all of $T$.

Theorem 2.1. The fourth Painlevé equation has the Painlevé property. The moduli space $\mathcal{M}^+(\theta_0, \theta_\infty)$ is analytically isomorphic to the Okamoto–Painlevé space for PIV with parameters $\theta_0, \theta_\infty$.

Proof. Let a local solution $Q$ of PIV with parameters $\theta_0, \theta_\infty$ be given. Let $U$ be an open disk, where $Q$ is holomorphic and has no zeros. Consider the operator $z \frac{d}{dz} + (\omega \ _{0 \ - \omega})$ with $\omega = \frac{1}{2}z^2 + \frac{1}{2}z + \frac{1}{2}$. This defines an analytic map $U \to \mathcal{M}^+(\theta_0, \theta_\infty)$. Since $Q$ is a local solution of PIV, the map $U \to \mathcal{M}^+(\theta_0, \theta_\infty) \to \mathcal{R}^+(\theta_0, \theta_\infty)$ is constant. Let $r$ be its image. Then $Q$ coincides with $q_r$ on $U$. Thus $Q$ extends to a global solution of PIV and this equation has the Painlevé property.

The bundle $\mathcal{M}^+(\theta_0, \theta_\infty) \to T$, with its foliation defined by the fibres of the morphism $\mathcal{M}^+(\theta_0, \theta_\infty) \to \mathcal{R}^+(\theta_0, \theta_\infty)$, is the Okamoto–Painlevé variety according to the isomorphism of Corollary 2.1.

We note that $\mathcal{R}^+(\theta_0, \theta_\infty)$ is the space of initial conditions.
2.3. \( \text{Aut}(\mathbb{S}^+) \) and Bäcklund transformations

Natural automorphisms of \( \mathbb{S}^+ \) are:

(1). \( \sigma_1 : (M,N) \mapsto (M,N^*) \) where \( N^* \) is a submodule of \( \mathbb{C}(z) \otimes M \) such that \( N \oplus N^* = \mathbb{C}(z) \otimes M \).

This is well defined for \( \beta \neq \pm 1 \). For \( \beta = \pm 1 \), the module \( N^* \) might not exist or might not be unique. It seems correct to define \( N^* := N \) for \( \beta = \pm 1 \).

(2). \( \sigma_2 : (M,N) \mapsto (M \otimes A,N \otimes A) \), where \( A = \mathbb{C}(z) a \) and \( \delta a = \frac{1}{2} a \).

(3). \( \sigma_3 : (M,N) \mapsto \mathbb{C}(z) \otimes \phi (M,N) \), where \( \phi \) is the automorphism of \( \mathbb{C}(z) \) which is the identity on \( \mathbb{C} \) and maps \( z \) to \( iz \). Let \( \text{Aut}(\mathbb{S}^+) \) denote the group generated by \( \sigma_j, j = 1,2,3 \).

| \( \beta \) | \( \alpha \) | \( t \) | \( z \) |
|---|---|---|---|
| \( \sigma_1 \) | \( \beta^{-1} \) | \( \alpha \) | \( t \) | \( z \) |
| \( \sigma_2 \) | \( -\beta \) | \( -\alpha \) | \( t \) | \( z \) |
| \( \sigma_3 \) | \( \beta \) | \( \alpha^{-1} \) | \( it \) | \( iz \) |

The above group is commutative and has order 16. The following table is a choice of lifting the generators to actions on \( \theta_0, \theta_\infty, t, z \).

| \( \theta_0 \) | \( \theta_\infty \) | \( t \) | \( z \) |
|---|---|---|---|
| \( \sigma_1 \) | \( 2 - \theta_0 \) | \( \theta_\infty \) | \( t \) | \( z \) |
| \( \sigma_2 \) | \( \theta_0 + 1 \) | \( \theta_\infty + 1 \) | \( t \) | \( z \) |
| \( \sigma_3 \) | \( \theta_0 \) | \( -\theta_\infty \) | \( it \) | \( iz \) |

The induced morphisms \( \tilde{\sigma}_1 : \mathcal{M}^+ (\theta_0, \theta_\infty) \to \mathcal{M}^+ (-\theta_0 + 2, \theta_\infty) \) and \( \tilde{\sigma}_3 : \mathcal{M}^+ (\theta_0, \theta_\infty) \to \mathcal{M}^+ (\theta_0 + 1, \theta_\infty + 1) \) are evident from the standard operators representing the points of \( \mathcal{M}^+ (\theta_0, \theta_\infty) \). A MAPLE computation yields the explicit morphisms \( \tilde{\sigma}_2 : \mathcal{M}^+ (\theta_0, \theta_\infty) \to \mathcal{M}^+ (\theta_0 - 1, \theta_\infty - 1) \).

The formulas are given with respect to the coordinates \( a = a_0, q \) of an open subset (namely \( q \neq 0 \) in the chart \( ST_2 \)) of the first space and \( \tilde{a} = \tilde{a}_0, \tilde{q} \) of an open subset of the second space. The assumption that the operator \( z \frac{d}{dz} + A(a,q,\theta_0,\theta_\infty,z) \), belonging to \( \mathcal{M}^+ (\theta_0, \theta_\infty) \), is equivalent, by a transformation of the type \( U \frac{d}{dz} + U_0 + U_1 z + U_2 z^2 \), to the operator \( \frac{d}{dt} + A(\tilde{a}, \tilde{q}, \theta_0 + 1, \theta_\infty + 1, z) \), belonging to \( \mathcal{M}^+ (\theta_0 + 1, \theta_\infty + 1) \), leads to the following formulas

\[
\tilde{q} = \frac{-4q^2 \theta_\infty + 4a^2 - 4q^2 t - q^2 t^2 - 4q^4 - 4q^2 \theta_0 + \theta_0^2 - 4a \theta_0}{4q(t - \theta_0 + 2a + 2q^2)}
\]

\[
\tilde{a} = \frac{long}{16q^2(t - \theta_0 + 2a + 2q^2)^2}.
\]

The substitution \( a = q' + \frac{1}{2} \) in the first formula produces \( \tilde{q} \) in terms of \( q, q' \) and the parameters \( \theta_0, \theta_\infty \), this is the Bäcklund transformation in terms of solutions. The second formula is obtained by substitution \( \tilde{a} = \tilde{q'} + \frac{1}{4} \) and an expression for \( \tilde{q}' \) coming from the first formula and the equation for \( q'' \).

The term \( qt - \theta_0 + 2a + 2q^2 \) in the denominator of the formulas indicates that the morphism \( \tilde{\sigma}_3 \) is in general a rational equivalence and is not defined on leaves of the foliation with \( a = q' + \frac{1}{2} \) and \( q' + q^2 + \frac{1}{2} q + \frac{-\theta_0 + 1}{2} = 0 \). This occurs precisely when \( \theta_0 = -\theta_\infty \) and the reducible locus of \( \mathcal{M}^+ (\theta_0, \theta_\infty) \) is not present in the corresponding \( \mathcal{M}^+ (1 + \theta_0, 1 + \theta_\infty) \) (compare Observations 2.2 and the Riccati equations for reducible families).

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We note that the group $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$ contains the two shifts $\theta_0 \mapsto \theta_0 + 2$, $\theta_\pi \mapsto \theta_\pi$ and $\theta_0 \mapsto \theta_0$, $\theta_\pi \mapsto \theta_\pi + 2$. One observes that, in comparison with the book of Gromak et al. [Gr] and Okamoto’s paper [O3], there is still a missing generator for the group of all Bäcklund transformations of PIV. This generator does not seem to come from a ‘natural’ automorphism of $S^+$ (i.e., constructions of linear algebra for differential modules and operations with the differential field $C(z)$). In the final section we will investigate another set of differential modules $S$, inspired by Noumi’s symmetric form of PIV ([No, No-Y]). As is shown by Noumi, this will easily produce all Bäcklund transformations and moreover all rational and Riccati solutions.

3. The Noumi–Yamada family

M. Noumi and Y. Yamada produced a $3 \times 3$-Lax pair, arising from the Lax formalism of the modified KP hierarchy, for the symmetric form of PIV, namely

$$
\frac{dz}{dt} + \begin{pmatrix}
\epsilon_1 f_1 & 1 \\
\epsilon_2 f_2 & f_0z \\
f_0z & \epsilon_3
\end{pmatrix}, \quad \frac{dz}{dt} + \begin{pmatrix}
-q_1 & 1 & 0 \\
0 & -q_2 & 1 \\
0 & 0 & -q_3
\end{pmatrix}, \quad \text{leading to equations}
$$

$$
\epsilon'_1 = \epsilon'_2 = \epsilon'_3 = 0; \quad f_1 - f_2 = -q_1 + q_3; \quad f_2 - f_0 = q_1 - q_2; \quad f_0 - f_1 = q_2 - q_3,
$$

$$
f'_0 = f_0(f_1 - f_2) + (1 - \epsilon_1 - \epsilon_3); \quad f'_1 = f_1(f_2 - f_0) + (\epsilon_1 - \epsilon_2); \quad f'_2 = f_2(f_0 - f_1) + (\epsilon_2 - \epsilon_3).
$$

Since the local exponents $\epsilon_\pi$ at $z = 0$ are constants in an isomonodromic family, we can and will suppose $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$. Then we may and will also suppose that $q_1 + q_2 + q_3 = 0$. Further it is assumed that $t = f_0 + f_1 + f_2$.

Then $f_1$ satisfies the fourth Painlevé equation $y'' = \frac{(y')^2}{2y} + \frac{3}{2}y^3 + 2y^2 + (\frac{1}{2} + \theta_\pi)y - \frac{(\theta_\pi - 1)^{2}}{25}$ with $\theta_0 = 1 + \epsilon_1 - \epsilon_2$, and $\theta_\pi = 1 + \epsilon_1 - \epsilon_3$. After rescaling $t \mapsto \frac{t}{\sqrt{2}}, \quad y \mapsto -\sqrt{2}y$ one obtains ‘our’ equation $y'' = \frac{(y')^2}{3} + \frac{3}{3}y^3 + y^2 + (t^2 + 4\theta_\pi)^{\frac{3}{8}} - \frac{(\theta_\pi - 1)^2}{8y}$.

Using this symmetric form one finds the extended Weyl group of $A_2$ as group of Bäcklund transformations. For example $\pi: (f_0, f_1, f_2) \mapsto (f_1, f_2, f_0)$ translates into the ‘missing’ Bäcklund transformation of §2.3, namely

$$
\theta_0 \mapsto -\frac{1}{2}\theta_0 + \frac{1}{2}\theta_\pi + 2, \quad \theta_\pi \mapsto -\frac{3}{2}\theta_0 - \frac{1}{2}\theta_\pi + 2.
$$

This is the inspiration for the new class $S$ of differential modules $M$ over $C(z)$, defined by:

$\dim M = 3$; $\Lambda^3 M$ is trivial; the only singular points are $0, \infty$; $0$ is regular singular and the Katz invariant of $\infty$ is $\frac{1}{2}$. After scaling the variable $z$ the generalized eigenvalues at $\infty$ are:

$q_0 = \zeta^{2/3} + \frac{1}{3}\zeta^{1/3}, \quad q_1 = \zeta^{2/3} + \frac{1}{3}\zeta^{1/3}, \quad q_2 = \zeta^{2/3} + \frac{1}{3}\zeta^{1/3}$

where $\zeta = e^{2\pi i/3}$.

Invariance at $z = \infty$. For $M \in S$ the operator $D = \nabla_{\frac{1}{\sqrt{2}}}$ has at $z = \infty$ has the form $\frac{d}{dz} + \text{diag}(q_0, q_1, q_2)$ with respect to a basis $e_0, e_1, e_2$. A lattice $\Lambda$ at $z = \infty$ is called invariant if $z^{-1}D(\Lambda) \subset \Lambda$. We respect to the basis $h_0 = e_0 + e_1 + e_2$, $h_1 = z^{1/3}(e_0 + \zeta e_1 + \zeta^2 e_2)$, and $h_2 = z^{-1/3}(e_0 + \zeta^2 e_1 + \zeta e_2)$, $D$ has the form $\frac{d}{dz} + \begin{pmatrix}
0 & \frac{1}{3} & 1 \\
\frac{1}{3} & 0 & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & 0
\end{pmatrix}$. Thus $\Lambda_0 := \langle h_0, h_1, h_2 \rangle >$ is an invariant lattice. $\Lambda_1 := \langle h_0, z^{-1}h_1, h_2 \rangle$ is the only invariant lattice of codimension 1 in $\Lambda_0$.  

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and the operator $D$ has with respect to this basis the form $z \frac{d}{dz} + \begin{pmatrix} 0 & 1 & \frac{\ell}{3} \\ \frac{\ell}{3} & -\frac{2}{3} & z \\ z & \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$. The invariant lattice $\Lambda_2 := \langle z^{-1}h_0, z^{-1}h_1, h_2 \rangle$ has codimension 2 in $\Lambda_0$. All invariant lattices at $z = \infty$ are $\{z^n \Lambda_0 | n \in \mathbb{Z}, i = 0, 1, 2\}$.

The Noumi–Yamada Lax pair has one additional feature, namely: there exists $U \in \text{GL}(3, \mathbb{C}[[z]])$ with $U = 1 + U_1z + U_2z^2 + \ldots$ such that

$$U^{-1}\left\{z \frac{d}{dz} + \begin{pmatrix} \epsilon_1 & f_1 & 1 \\ z & \epsilon_2 & f_2 \\ f_0z & z & \epsilon_3 \end{pmatrix}\right\}U = z \frac{d}{dz} + \begin{pmatrix} \epsilon_1 & * & * \\ 0 & \epsilon_2 & * \\ 0 & 0 & \epsilon_3 \end{pmatrix}, \text{ with all } * \in \mathbb{C}.$$

**Level structure for $S$.** This leads to a ‘level structure’ or ‘parabolic structure’ for the elements $M \in S$ consisting of differential submodules $M_1 \subset M_2 \subset \mathbb{C}((z)) \otimes M$ of dimensions 1 and 2 over $\mathbb{C}((z))$. Let $S^+$ denote the set of the differential modules in $S$, provided with a level structure.

The moduli space $\mathcal{R}$ for the analytic data.

The singular directions $d$ for $q_k - q_\ell$ are computed as follows. $z \frac{d}{dz}(y) = (q_k - q_\ell)y$ has solution $\exp(\frac{3}{2}(\zeta^{2k} - \zeta^{2\ell})z^{2/3} + 3(\zeta^k - \zeta^\ell)z^{1/3})$. Write $z = e^{id}$ and $\zeta^{2k} - \zeta^{2\ell} = |\zeta^{2k} - \zeta^{2\ell}| e^{i\phi(k, \ell)}$. Then $|y(re^{id})|$ has maximal descent for $r \to \infty$ if and only if $\frac{3}{2}d + \phi(k, \ell) = \pi + \mathbb{Z}2\pi$ (equivalently, $d = \frac{3}{2}\pi - \frac{3}{2}\phi(k, \ell) + \mathbb{Z}3\pi$).

| $k$ | $\ell$ | $\phi$ | $d$ |
|-----|-------|-------|-----|
| 0   | 1     | $\frac{\pi}{6}$ | $\frac{\pi}{3}$ |
| 1   | 0     | $\frac{\pi}{6}$ | $\frac{\pi}{3}$ |
| 0   | 2     | $\frac{\pi}{3}$ | $\frac{\pi}{3}$ |
| 2   | 0     | $\frac{\pi}{2}$ | $\frac{\pi}{3}$ |
| 1   | 2     | $\frac{\pi}{2}$ | $\frac{\pi}{3}$ |
| 2   | 1     | $\frac{\pi}{2}$ | $\frac{\pi}{3}$ |

The analytic data consists of the formal monodromy and the six Stokes matrices at $z = \infty$. The product of the formal monodromy and the Stokes matrices for the singular directions $d \in [0, 2\pi)$

$$
\begin{pmatrix} 0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
x_4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\
x_3 & 1 & 0 \\
0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & x_1 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{pmatrix}
$$

is equal to the topological monodromy $M(x_s) := \begin{pmatrix} x_4 & 0 & x_1x_4 + 1 \\
1 & 0 & x_1 \\
x_3 & 1 & x_1x_3 + x_2 \end{pmatrix}$. Its characteristic polynomial is $\lambda^3 - (x_2 + x_4 + x_1x_3)\lambda^2 + (-x_1 - x_3 + x_2x_4)\lambda - 1$.

The moduli space $\mathcal{R}$ for the analytic data consists of the tuples $x_s = (x_1, \ldots, x_4)$ since the other two Stokes matrices can be expressed in the Stokes matrices for the singular directions in $[0, 2\pi)$. Thus $\mathcal{R} \cong \mathbb{A}^4$. The elements of the parameter space $\mathcal{P}$ are the sets of eigenvalues of the topological monodromy. Thus $\mathcal{P} = \{\lambda^3 - e_1\lambda^2 + e_2\lambda - 1 | e_1, e_2 \in \mathbb{C}\}$. Let $\mathcal{R}(P)$ be the fibre above $P \in \mathcal{P}$. If $P$ has three distinct roots, then $\mathcal{R}(P)$ is a smooth surface. If $P$ has roots
a, a, a^{-2} with a \neq a^{-2}, then the point \( x_1 \in \mathcal{R}(P) \) with \( x_1 \neq (-a^{-1}, a, -a^{-1}, a) \) is regular and \( M(x_1) \) has two Jordan blocks. The point \((-a^{-1}, a, -a^{-1}, a) \in \mathcal{R}(P)\) is singular and has type \( A_1 \). Further \( M(-a^{-1}, a, -a^{-1}, a) \) has three Jordan blocks. If \( P \) has roots \( a, a, a \) (and thus \( a^3 = 1 \)), then the point \( x_2 \in \mathcal{R}(P) \) with \( x_2 \neq (-a^{-1}, a, -a^{-1}, a) \) is regular and \( M(x_2) \) has one Jordan block. The point \( x_3 = (-a^{-1}, a, -a^{-1}, a) \) is singular and has type \( A_2 \). The matrix \( M(-a^{-1}, a, -a^{-1}, a) \) has two Jordan blocks.

**Level structure for \( \mathcal{R} \) and \( \mathcal{P} \).** For an element of \( \mathcal{R} \) we introduce a ‘level structure’ which consists of subspaces \( L_1 \subset L_2 \subset \mathbb{C}^3 \) of dimensions 1 and 2 which are invariant under the topological monodromy (at \( z = \infty \) or, equivalently, at \( z = 0 \)). The corresponding space is denoted by \( \mathcal{R}^+ \). The level structure for a \( P \in \mathcal{P} \) consists of a tuple \((\mu_1, \mu_2, \mu_3)\) with \( \mu_1 \mu_2 \mu_3 = 1 \) and \( P = \prod_{j=1}^3 (\lambda - \mu_j) \). The corresponding space is denoted by \( \mathcal{P}^+ \). The morphism \( \text{par} : \mathcal{R}^+ \to \mathcal{P}^+ \) is defined by \((x_1, L_1, L_2) \mapsto (\mu_1, \mu_2, \mu_3)\), where \( \mu_1 \) is the eigenvalue of \( M(x_1) \) on \( L_1 \) and \( \mu_2 \) is that of \( M(x_1) \) on \( L_2/L_1 \).

One observes that \( \mathcal{R}^+ \) is the closed subspace of \( \mathbb{C}^4_x \times \mathbb{P}^2_y \times (\mathbb{P}^2)^*_x \times \mathcal{P}^+ \), where \( \mathcal{P}^+ \) equals \( \{(\mu_1, \mu_2, \mu_3) \in \mathbb{C}^3 | \mu_1 \mu_2 \mu_3 = 1 \} \), given by the equations:

\[
M(x_1)y = \mu_1y, \quad y := \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad zM(x_1) = \mu_3z, \quad z := (z_1, z_2, z_3), \quad \Sigma y_j z_j = 0.
\]

Indeed, \( \mathbb{C}_y \) and the kernel of \( z \in (\mathbb{C}^3)^* \) are the \( M(x_1) \)-invariant spaces \( L_1 \subset L_2 \subset \mathbb{C}^3 \). Further \( \text{par} : \mathcal{R}^+ \to \mathcal{P}^+ \) is the projection onto the last factor. The fibre \( \mathcal{R}^+(\mu_1) \) of \( \text{par} \) above the point \((\mu_1, \mu_2, \mu_3) \in \mathcal{P}^+ \) maps to the fibre \( \mathcal{R}(P) \) of \( \mathcal{R} \to \mathcal{P} \) above the point \( P = (\lambda - \mu_1)(\lambda - \mu_2)(\lambda - \mu_3) \).

**Proposition 3.1.** \( \text{res} : \mathcal{R}^+(\mu_1) \to \mathcal{R}(P) \) is the minimal resolution of \( \mathcal{R}(P) \).

A straightforward computation proves this statement. In particular, the fibre of \( \text{res} \) above a non singular point is just one point since there is only one level structure possible. If two of the \( \mu_1 \) are equal, then the preimage under \( \text{res} \) of the singular point is a \( \mathbb{P}^1 \), consisting of the lines \( \mathbb{C}_y \) in the two-dimensional eigenspace for \( a \) and the kernel of \( z \) is this two-dimensional eigenspace. If the three \( \mu_1 \) are equal, then the preimage under \( \text{res} \) of the singular point is a pair of intersecting projective lines. In this case the Jordan form of \( M(x_1) \) has two blocks, \( \mathbb{C}_y \) is a line in the two-dimensional eigenspace of \( \mathbb{C}^3 \), \( \mathbb{C}_z \) is a line in the two-dimensional eigenspace of the dual \( (\mathbb{C}^3)^* \) and \( \Sigma y_j z_j = 0 \).

**Proposition 3.2.** The natural maps \( \mathbb{S} \to \mathbb{R} \times \mathbb{T} \) and \( \mathbb{S}^+ \to \mathbb{R}^+ \times \mathbb{T} \) (with \( \mathbb{T} = \mathbb{C} \)) are bijections.

**Proof.** In the first case one applies [vdP-Saito], Theorem 1.7. The second case follows from the observation that the level structure \( M_1 \subset M_2 \subset \mathbb{C}((z)) \otimes M \) induces subspaces \( L_1 \subset L_2 \subset \mathbb{C}^3 \) of dimensions 1 and 2, invariant under the topological monodromy, and visa versa. \( \square \)

The Noumi–Yamada moduli space \( \mathcal{N}^+(\varepsilon_s) \)

\( \varepsilon_s \) denotes a triple \((\varepsilon_1, \varepsilon_2, \varepsilon_3)\) with \( \Sigma \varepsilon_j = 0 \). The set \( \mathbb{S}^+(\varepsilon_s) \) consists of the tuples \((M, M_1 \subset M_2) \in \mathbb{S}^+ \) such that \( M_1 = \mathbb{C}((z)) b_1 \) with \( \delta_M(b_1) = \varepsilon_1 b_1 \) and \( M_2/M_1 = \mathbb{C}((z)) b_2 \) with \( \delta_M(b_2) = \varepsilon_2 b_2 \). Let \( \mathbb{S}^\prime \) denote the free bundle on \( \mathbb{P}^1 \) of rank 3.

The points of the moduli space \( \mathcal{N}^+(\varepsilon_s) \) correspond to the isomorphism classes of connections \( D := \nabla_{\varepsilon_s} : \mathcal{Y} \to O(\{\infty\}) \otimes \mathcal{Y}^\prime \) with a level structure which consists of the \( D \)-invariant submodules \( V_j \subset V_2 \subset V_0 \) of rank 1 and 2 such that \( V_0/V_j, \ j = 1, 2 \) have no torsion and such that there is a tuple.
$$(M, M_1 \subset M_2) \in S^+(\varepsilon_i)$$ with $M$ is the generic fibre of $\mathcal{V}$, $\mathbb{C}((z)) \otimes V_j = M_j$, $j = 1, 2$ and $\hat{\mathcal{V}}_0$ is the lattice $\Lambda_0 \subset \mathbb{C}((z^{-1})) \otimes M$.

**Proposition 3.3.** $\mathcal{N}^+(\varepsilon_i)$ is the affine space $\mathbb{A}_3^3$ with coordinates $f_0, f_1, f_2$. $t = f_0 + f_1 + f_2$ and the connection is represented by $z \frac{d}{dz} + \begin{pmatrix} \varepsilon_1 & f_1 & 1 \\ \varepsilon_2 & f_2 & f_0 \varepsilon_3 & z & z \end{pmatrix}$.

**Proof.** The level structure provides $H^0(\mathcal{V})$ with a basis $e_1, e_2, e_3$ such that $D = z \frac{d}{dz} + A_0 + A_1 z$ with traceless constant matrices $A_0, A_1$ and $A_0 = \begin{pmatrix} \varepsilon_1 & * & * \\ 0 & \varepsilon_2 & * \\ 0 & 0 & \varepsilon_3 \end{pmatrix}$. This is unique up to the action of $B = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$. The lattice condition at $z = \infty$ is equivalent to $U \{ z \frac{d}{dz} + A_0 + A_1 z \} U^{-1} = z \frac{d}{dz} + \begin{pmatrix} 0 & z & 1 \\ 0 & 1 & 1 \\ \frac{1}{z} & \frac{1}{z} & -1 \end{pmatrix}$ for some $U = U_0(1 + U_{-1} z^{-1} + \ldots) \in \text{GL}_3(\mathbb{C}[[z^{-1}])]$. This is equivalent to the equations $A_1 = U_0^{-1} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & \frac{1}{z} & 0 \end{pmatrix} U_0$ and $A_0 = U_0^{-1} \begin{pmatrix} 0 & \frac{1}{z} & 1 \\ 0 & 1 & -1 \end{pmatrix} U_0 + [A_1, U_{-1}]$. A MAPLE computation produces matrices $U_0$ and $U_{-1}$ such that $A_0 = \begin{pmatrix} \varepsilon_1 & * & * \\ 0 & \varepsilon_2 & * \\ 0 & 0 & \varepsilon_3 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \varepsilon_3 \\ * & 1 & 0 \end{pmatrix}$. Moreover $U_0$ is unique up to multiplication by a non zero constant and $U_{-1}$ is unique up to adding a matrix $V$ with $[A_1, V] = 0$. Thus we found a representation of the connection in the ‘Noumi–Yamada form’ and this form is unique with respect to the action of the Borel group $B$ on $\mathcal{V}$. Therefore the Noumi–Yamada form represents the moduli space $\mathcal{N}^+(\varepsilon_i)$. 

The map $\mathcal{N}(\varepsilon_i) \to S^+(\varepsilon_i)$ is injective and not bijective. This is due to the choice of a free vector bundle $\mathcal{V}$ in the construction of $\mathcal{N}(\varepsilon_i)$. The aim is to avoid this choice and to construct a smooth partial completion $\mathcal{N}(\varepsilon_i)$ such that $\mathcal{N}(\varepsilon_i) \to S^+(\varepsilon_i)$ is bijective. As in $\S 2$, this will imply that the extended Riemann–Hilbert map $\mathcal{N}(\varepsilon_i) \to \mathbb{R}^+_{+}(\mu_s) \times T$ (with $\mu_j = e^{2\pi i \varepsilon_j}$ for $j = 1, 2, 3$) is an analytic isomorphism. Moreover $\mathcal{N}(\varepsilon_i)$ is the Okamoto–Painlevé space and $\mathbb{R}^+_{+}(\mu_s)$ is the space of the initial conditions.

**Construction of $\mathcal{N}(\varepsilon_i)$.**

The points of $\mathcal{N}(\varepsilon_i)$ correspond to (the isomorphism classes of) the tuples $(\mathcal{V}, D, L_1, L_2)$ with $D = \nabla_{\frac{d}{dz}} : \mathcal{V} \to \mathcal{O}(\infty) \otimes \mathcal{V}$ is a connection on a vector bundle $\mathcal{V}$ of rank 3.

We require the following: The connection $\hat{\mathcal{V}}_0$ is isomorphic to $\Lambda_1$. In other terms $\hat{\mathcal{V}}_0$ has a basis over $\mathbb{C}[[z^{-1}]]$ for which $D$ has the form $z \frac{d}{dz} + \begin{pmatrix} 0 & 1 & \frac{1}{z} \\ \frac{1}{z} & -\frac{2}{3} & \frac{3}{z} \\ \frac{1}{z} & \frac{1}{z} & -1 \end{pmatrix}$. Further $L_1 = \mathbb{C}[[z]] Y$ is a subconnection of $\hat{\mathcal{V}}_0$ such that $\hat{\mathcal{V}}_0/L_1$ has no torsion and $D Y = \varepsilon_1 Y$. Further $L_2 = \mathbb{C}[[z]] Z$ is a subconnection of $\hat{\mathcal{V}}_0$, the dual of $\hat{\mathcal{V}}_0$, such that $\hat{\mathcal{V}}_0^* / L_2$ has no torsion and $D Z = \varepsilon_2 Z$. Moreover $\Lambda^3(\hat{\mathcal{V}}_0)$ is trivial and
\( L_2(L_1) = 0. \)

The map \( F: \tilde{\mathcal{N}}(\varepsilon_\ast) \to S^+(\varepsilon_\ast) \), sends \((\mathcal{V}, D, L_1, L_2)\) to its generic fibre \( M \) together with the level structure on \( \mathbb{C}(z) \otimes M \) obtained from \( L_1, L_2 \). Conversely, for a given element \((M, M_1 \subset M_2) \in S^+(\varepsilon_\ast)\) one defines the connection \((\mathcal{V}, D)\) with generic fibre \( M \), by prescribing \( \hat{\mathcal{Y}}_0 \cong \Lambda_1 \). The additional data \( L_1, L_2 \) imply that \( \hat{\mathcal{Y}}_0 \) is represented by \( z^d \left[ \begin{array}{ccc} \varepsilon_1 & \ast & \ast \\ 0 & \varepsilon_2 & \ast \\ 0 & 0 & \varepsilon_3 \end{array} \right] \) with all \( \ast \in \mathbb{C}[z] \). This implies that \( F \) is bijective in the following cases:

\[ \begin{align*}
\mu_1, \mu_2, \mu_3 & \text{ are distinct;} \\
\mu_1 = \mu_2 & \neq \mu_3 \text{ and } \varepsilon_2 - \varepsilon_1 \geq 0; \\
\mu_1 = \mu_3 & \neq \mu_2 \text{ and } \varepsilon_3 - \varepsilon_1 \geq 0; \\
\mu_1 & \neq \mu_2 = \mu_3 \text{ and } \varepsilon_3 - \varepsilon_2 \geq 0; \\
\mu_1 = \mu_2 = \mu_3 & \text{ and } \varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_2 \geq 0.
\end{align*} \]

In the sequel we will only consider these cases.

In order to give \( \tilde{\mathcal{N}}(\varepsilon_\ast) \) the structure of an algebraic variety we observe that \( \mathcal{V} \) has degree -1 and type \( O \oplus O \oplus O(-1) \) since \((\mathcal{V}, D)\) is irreducible. We identify \( \mathcal{V} \) with \( Oe_1 \oplus Oe_2 \oplus O(-[\infty])e_3 \). The matrix of \( D \) with respect to the basis \( e_1, e_2, e_3 \) has trace zero and is denoted by \( \left( \begin{array}{ccc} 0 & e_1 & e_2 \\ e_3 & 0 & e_4 \\ e_5 & e_6 & 0 \end{array} \right) \). The generator \( Y = \sum_{n \geq 0} Y_n z^n \) of \( L_1 \) with \( Y_n = Y_n(1)e_1 + Y_n(2)e_2 + Y_n(3)e_3 \) is unique up to multiplication by a constant. The generator \( Z = \sum_{n \geq 0} Z_n z^n \) of \( L_2 \) with \( Z_n = Z_n(1)e_1^3 + Z_n(2)e_2^3 + Z_n(3)e_3^3 \) is unique up to multiplication by a constant. The \( Y_\ast(\ast), Z_\ast(\ast) \) are regarded as homogeneous coordinates.

The space \( \mathcal{A} \) is defined by the indeterminates \( a_\ast, Y_\ast(\ast), Z_\ast(\ast) \) and the relations induced by the above requirements. We note that for given \( \varepsilon_\ast \), such that the above restrictions are satisfied, the \( Y_n(\ast), Z_n(\ast) \) are for \( n \geq 1 \) eliminated by the relations. Thus \( \mathcal{A} \) is an algebraic variety.

The group \( G \) of the automorphisms of \( \mathcal{V} \) act upon \( \mathcal{A} \). By construction, the set theoretic quotient \( \mathcal{A}(\mathbb{C})/G \) coincides with \( S^+(\varepsilon_\ast) \). Thus the analytic map \( R: \mathcal{A} \to \mathcal{A}^+(\mu_\ast) \times T \), where \( \mu_j = e^{2\pi i \xi_j} \), for \( j = 1, 2, 3 \), is surjective and \( R(\xi_1) = R(\xi_2) \) if and only if there is a \( g \in G \) with \( g^\xi_1 = \xi_2 \). A long MAPLE session verifies: \( \mathcal{A} \) has a smooth geometric quotient by \( G \). This quotient is by definition \( \tilde{\mathcal{N}}(\varepsilon_\ast) \) and the extended Riemann–Hilbert map \( \tilde{\mathcal{N}}(\varepsilon_\ast) \to \mathcal{A}^+(\mu_\ast) \times T \) is an analytic isomorphism.

References

[Bo] P. Boalch, Symplectic Manifolds and Isomonodromic Deformations, Adv. in Math. 163 (2001) 137–205.
[Gr] V.I. Gromak, I. Laine, S. Shimomura, Painlevé differential equations in the complex plane. de Gruyter Studies in Mathematics, 28. Walter de Gruyter & Co., Berlin, 2002.
[In] M. Inaba, Moduli of parabolic connections on a curve and Riemann-Hilbert correspondence, Preprint, 2006, arXiv:math/0602004.
[IIS1] M. Inaba, K. Iwasaki, and M.-H. Saito, Moduli of stable parabolic connections, Riemann-Hilbert correspondence and geometry of Painlevé equation of type VI. I, Publ. Res. Inst. Math. Sci. 42 (2006) no. 4 987–1089.
[IIS2] M. Inaba, K. Iwasaki, and M.-H. Saito, Moduli of stable parabolic connections, Riemann-Hilbert correspondence and geometry of Painlevé equation of type VI. II, Moduli spaces and arithmetic geometry (Tokyo), Adv. Stud. Pure Math., vol. 45, Math. Soc. Japan, Tokyo, 2006, pp. 387–432.
[IISA] M. Inaba, K. Iwasaki and M.-H. Saito, Dynamics of the sixth Painlevé Equations, Théories Asymptotiques et Équations de Painlevé, Angers, Juin, 2004, "Séminaires et Congrès" of the Société Mathématique de France (SMF)14, 2006, 103–167.

[JMU] M. Jimbo, T. Miwa and K. Ueno, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients, I. General theory and τ-function, Physica D 2 (1981) 306–352.

[JM] M. Jimbo and T. Miwa, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients, II, Physica D 2 (1981) 407–448.

[No] M. Noumi, Painlevé equations through symmetry. Translated from the 2000 Japanese original by the author. Translations of Mathematical Monographs, 223. American Mathematical Society, Providence, RI, 2004.

[No-Y] M. Noumi and Y. Yamada, Symmetries in the fourth Painlevé equation and Okamoto polynomials. Nagoya Math. J. 153 (1999) 53–86.

[OO] Y. Ohyama, S. Okumura, A coalescent diagram of the Painlevé equations from the viewpoint of isomonodromic deformations. J. Phys. A 39 (2006) no. 39 12129–12151.

[O1] K. Okamoto, Sur les feuilletages associés aux équations du second ordre à points critiques fixes de P. Painlevé, Espaces des conditions initiales, Japan. J. Math. 5 (1979) 1–79.

[O2] K. Okamoto, Isomonodromic deformation and Painlevé equations and the Garnier system, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math. 33 (1986) 575–618.

[O3] K. Okamoto, Studies on the Painlevé Equations III Second and Fourth Painlevé Equations, PII and PIV, Math. Ann. 275 (1986) 221–255.

[O4] K. Okamoto, The Hamiltonians associated to the Painlevé equations. The Painlevé property, 735–787, CRM Ser. Math. Phys. Springer, New York, 1999.

[vdP1] M. van der Put, Families of linear differential equations and the Painlevé equations, SMF, Séminaires & Congrès 27 (2012) 203–220.

[vdP2] M. van der Put, Families of linear differential equations related to the second Painlevé equation in: Algebraic Methods in Dynamical Systems, Banach Center Publications, Volume 94, 2011, 247–262.

[vdP-Sa] M. van der Put and M-H. Saito, Moduli spaces for linear differential equations and the Painlevé equations., Ann. Inst. Fourier 59 (2009) no. 7 2611–2667.

[vdP-Si] M. van der Put, M.F. Singer, Galois Theory of Linear Differential Equations, Grundlehren der mathematischen Wissenschaften, Volume 328, Springer Verlag 2003.

[STT] M.-H. Saito, T. Takebe, H. Terajima, Deformation of Okamoto-Painlevé pairs and Painlevé equations, J. Algebraic Geom. 11 (2002) no. 2 311–362.

[S-Ta] Saito, M-H., Takebe, T., Classification of Okamoto-Painlevé pairs, Kobe J. Math. 19 (2002) no. 1 21–50.

[SU] M.-H. Saito, H. Umemura, Painlevé equations and deformations of rational surfaces with rational double points. Physics and combinatorics 1999 (Nagoya), 320–365, World Sci. Publishing, River Edge, NJ, 2001.

[STe] M.-H. Saito, H. Terajima, Nodal curves and Riccati solutions of Painlevé equations, J. Math. Kyoto Univ. 44 (2004) no. 3 529–568.

[Sakai] H. Sakai, Rational surfaces associated with affine root systems and geometry of the Painlevé equations, Comm. Math. Phys. 220 (2001) 165–229.

[T] H. Terajima, Families of Okamoto-Painlevé pairs and Painlevé equations, Ann. Mat. Pura Appl. (4), 186 (2007) no. 1 99–146.