An Optimal Algorithm for Bandit and Zero-Order Convex Optimization with Two-Point Feedback

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Abstract

We consider the closely related problems of bandit convex optimization with two-point feedback, and zero-order stochastic convex optimization with two function evaluations per round. We provide a simple algorithm and analysis which is optimal for convex Lipschitz functions. This improves on [4], which only provides an optimal result for smooth functions; Moreover, the algorithm and analysis are simpler, and readily extend to non-Euclidean problems. The algorithm is based on a small but surprisingly powerful modification of the gradient estimator.

1 Introduction

We consider the problem of bandit convex optimization with two-point feedback [1]. This problem can be defined as a repeated game between a learner and an adversary as follows: At each round \( t \), the adversary picks a convex function \( f_t \) on \( \mathbb{R}^d \), which is not revealed to the learner. The learner then chooses a point \( w_t \) from some known and closed convex set \( W \subseteq \mathbb{R}^d \), and suffers a loss \( f_t(w_t) \). As feedback, the learner may choose two points \( w'_t, w''_t \in W \) and receive \( f_t(w'_t), f_t(w''_t) \). The learner’s goal is to minimize average regret, defined as

\[
\frac{1}{T} \sum_{t=1}^{T} f_t(w_t) - \min_{w \in W} \frac{1}{T} \sum_{t=1}^{T} f_t(w).
\]

In this note, we focus on obtaining bounds on the expected average regret (with respect to the learner’s randomness).

A closely-related and easier setting is zero-order stochastic convex optimization. In this setting, our goal is to approximately solve \( F(w) = \min_{w \in W} \mathbb{E}_{\xi}[f(w; \xi)] \), given limited access to \( \{f(\cdot; \xi_t)\}_{t=1}^{T} \) where \( \xi_t \) are i.i.d. instantiations. Specifically, we assume that each \( f(\cdot; \xi_t) \) is not directly observed, but rather can be queried at two points. This models situations where computing gradients directly is complicated or infeasible. It is well-known [3] that given an algorithm with expected average regret \( R_T \) in the bandit optimization setting above, if we feed it with the functions \( f_t(w) = f(w; \xi_t) \), then the average \( \bar{w}_T = \frac{1}{T} \sum_{t=1}^{T} w_t \) of the points generated satisfies the following bound on the expected optimization error:

\[\mathbb{E}[F(\bar{w}_T)] - \min_{w \in W} F(w) \leq R_T.\]

\[\text{This is slightly different than the model of [1], where the learner only chooses } w'_t, w''_t \text{ and the loss is } \frac{1}{2} (f_t(w'_t) + f_t(w''_t)).\]

However, our results and analysis can be easily translated to their setting, and the model we discuss translates more directly to the zero-order stochastic optimization considered later.
Thus, an algorithm for bandit optimization can be converted to an algorithm for zero-order stochastic optimization with similar guarantees.

The bandit optimization setting with two-point feedback was proposed and studied in [1]. Independently, [8] and considered two-point methods for stochastic optimization. Both papers are based on randomized gradient estimates which are then fed into standard first-order algorithms (e.g. gradient descent, or more generally mirror descent). However, the regret/error guarantees in both papers were suboptimal in terms of the dependence on the dimension. Recently, [4] considered a similar approach for the stochastic optimization setting, attaining an optimal error guarantee when \( f(\cdot;\xi) \) is a smooth function (differential and with Lipschitz-continuous gradients). Related results in the smooth case were also obtained by [6]. However, to tackle the general case, where \( f(\cdot;\xi) \) may be non-smooth, [4] resorted to a non-trivial smoothing scheme and a significantly more involved analysis. The resulting bounds have additional factors (logarithmic in the dimension) compared to the guarantees in the smooth case. Moreover, an analysis is only provided for Euclidean problems (where the domain \( \mathcal{W} \) and Lipschitz parameter of \( f \) scale with the \( L_2 \) norm).

In this note, we present and analyze a simple algorithm with the following properties:

- For Euclidean problems, it is optimal up to constants for both smooth and non-smooth functions. This closes the gap between the smooth and non-smooth Euclidean problems in this setting.

- The algorithm and analysis are readily applicable to non-Euclidean problems. We give an example for the 1-norm, with the resulting bound optimal up to a \( \sqrt{\log(d)} \) factor.

- The algorithm and analysis are simpler than those proposed in [4]. They apply equally to the bandit and zero-order optimization setting, and can be readily extended using standard techniques (e.g. to strongly-convex functions, regret/error bounds holding with high-probability rather than just in expectation, and improved bounds if allowed \( k > 2 \) observations per round instead of just two).

Like previous algorithms, our algorithm is based on a random gradient estimator, which given a function \( f \) and point \( w \), queries \( f \) at two random locations close to \( w \), and computes a random vector whose expectation is a gradient of a smoothed version of \( f \). The papers [8, 4, 6] essentially use the estimator which queries at \( w \) and \( w + \delta u \) (where \( u \) is a random unit vector and \( \delta > 0 \) is a small parameter), and returns

\[
\frac{d}{\delta} (f(w + \delta u) - f(w)) \ u. \tag{1}
\]

The intuition is readily seen in the one-dimensional \((d = 1)\) case, where the expectation of this expression equals

\[
\frac{1}{2\delta} (f(w + \delta) - f(w - \delta)), \tag{2}
\]

which indeed approximates the derivative of \( f \) (assuming \( f \) is differentiable) at \( w \), if \( \delta \) is small enough.

In contrast, our algorithm uses a slightly different estimator (also used in [1]), which queries at \( w - \delta u, w + \delta u \), and returns

\[
\frac{d}{2\delta} (f(w + \delta u) - f(w - \delta u)) \ u. \tag{3}
\]

Again, the intuition is readily seen in the case \( d = 1 \), where the expectation of this expression also equals Eq. \( \ref{eq:smooth_approx} \).

When \( \delta \) is sufficiently small and \( f \) is differentiable at \( w \), both estimators compute a good approximation of the true gradient \( \nabla f(w) \). However, when \( f \) is not differentiable, the variance of the estimator in Eq. \( \ref{eq:smooth_approx} \)
can be quadratic in the dimension \(d\), as pointed out by [4]: For example, for \(f(w) = \|w\|_2\) and \(w = 0\), the second moment equals
\[
E \left[ \left\| \frac{d}{\delta} (f(\delta u) - f(0)) u \right\|^2 \right] = E \left[ d^2 \|u\|^2 \right] = d^2.
\]
Since the performance of the algorithm crucially depends on the second moment of the gradient estimate, this leads to a highly sub-optimal guarantee. In [4], this was handled by adding an additional random perturbation and using a more involved analysis. Surprisingly, it turns out that the slightly different estimator in Eq. (3) does not suffer from this problem, and its second moment is essentially linear in the dimension \(d\).

## 2 Algorithm and Main Results

We consider the algorithm described in Figure 1 which performs standard mirror descent using a randomized gradient estimator \(\tilde{g}_t\) of a (smoothed) version of \(f_t\) at point \(w_t\). We make the assumption that one can indeed query \(f_t\) at any point \(w_t + \delta_t u_t\) as specified in the algorithm.

**Algorithm 1 Two-Point Bandit Convex Optimization Algorithm**

Input: Step size \(\eta\), function \(r : \mathcal{W} \mapsto \mathbb{R}\), exploration parameters \(\delta_t > 0\)

Initialize \(\theta_1 = 0\).

for \(t = 1, \ldots, T - 1\) do

- Predict \(w_t = \arg \max_{w \in \mathcal{W}} \left\langle \theta_t, w \right\rangle - r(w)\)
- Sample \(u_t\) uniformly from the Euclidean unit sphere \(\{w : \|w\|_2 = 1\}\)
- Query \(f_t(w_t + \delta_t u_t)\) and \(f_t(w_t - \delta_t u_t)\)
- Set \(\tilde{g}_t = \frac{1}{2 \delta_t} (f_t(w_t + \delta_t u_t) - f_t(w_t - \delta_t u_t)) u_t\)
- Update \(\theta_{t+1} = \theta_t - \eta \tilde{g}_t\)

end for

The analysis of the algorithm is presented in the following theorem:

**Theorem 1.** Assume the following conditions hold:

1. \(r\) is 1-strongly convex with respect to a norm \(\| \cdot \|\), and \(\sup_{w \in \mathcal{W}} r(w) \leq R^2\) for some \(R < \infty\).
2. \(f_t\) is convex and \(G_2\)-Lipschitz with respect to the 2-norm \(\| \cdot \|_2\).
3. The dual norm \(\| \cdot \|_*\) of \(\| \cdot \|\) is such that \(\sqrt{E_{\|u_t\|_2^4}} \leq p_*\) for some \(p_* < \infty\).

If \(\eta = \frac{R}{p_* G_2 \sqrt{d} T}\), and \(\delta_t\) chosen such that \(\delta_t \leq p_* R \sqrt{\frac{d}{T}}\), then the sequence \(w_1, \ldots, w_T\) generated by the algorithm satisfies the following for any \(T\) and \(w^* \in \mathcal{W}^*\):
\[
E \left[ \frac{1}{T} \sum_{t=1}^{T} f_t(w_t) - \frac{1}{T} \sum_{t=1}^{T} f_t(w^*) \right] \leq c p_* G_2 R \sqrt{\frac{d}{T}},
\]
where \(c\) is some numerical constant.

\footnote{This may require us to query at a distance \(\delta_t\) outside \(\mathcal{W}\). If we must query within \(\mathcal{W}\), then one can simply run the algorithm on a slightly smaller set \((1 - \delta)\mathcal{W}\) where \(\delta \geq \delta_t\) for all \(t\), ensuring that we always query at \(\mathcal{W}\). Since the formal guarantee in Thm. 1 holds for arbitrarily small \(\delta_t\), and each \(f_t\) is Lipschitz, we can always take \(\delta\) and \(\delta_t\) small enough so that the additional regret/error incurred is negligible.}
We note that conditions 1 is standard in the analysis of the mirror-descent method (see the specific corollaries below), whereas conditions 2 and 3 are needed to ensure that the variance of our gradient estimator is controlled.

As mentioned earlier, the bound on the average regret which appears in Thm. 1 immediately implies a similar bound on the error in a stochastic optimization setting, for the average point $w_T = \frac{1}{T} \sum_{t=1}^{T} w_t$. We note that the result is robust to the choice of $\eta$, and is the same up to constants as long as $\eta = \Theta(R/p, G_2 \sqrt{dT})$. Also, the constant $c$, while always bounded above zero, shrinks as $\delta_t \to 0$ (see the proof for details).

As a first application, let us consider the case where $\| \cdot \|$ is the Euclidean $\| \cdot \|_2$. In this case, we can take $s(w) = \frac{1}{2} \| w \|_2^2$, and the algorithm reduces to a standard variant of online gradient descent, defined as $\theta_{t+1} = \theta_t - \delta_t$ and $w_t = \arg \min_{w \in \mathcal{W}} \| w - \theta_t \|_2$. In this case, we get the following corollary:

**Corollary 1.** Suppose $f_t$ for all $t$ is $G_2$-Lipschitz with respect to the Euclidean norm, and $\mathcal{W} \subseteq \{ w : \| w \|_2 \leq R \}$. Then using $\| \cdot \| = \| \cdot \|_2$ and $r(w) = \frac{1}{2} \| w \|_2^2$, it holds for some constant $c$ and any $w^* \in \mathcal{W}$ that

$$
\mathbb{E} \left[ \frac{1}{T} \sum_{t=1}^{T} f_t(w_t) - \frac{1}{T} \sum_{t=1}^{T} f_t(w^*) \right] \leq c G_2 R \sqrt{\frac{d}{T}},
$$

The proof is immediately obtained from Thm. 1 noting that $p_* = 1$ in our case. This bound matches (up to constants) the lower bound in [4], hence closing the gap between upper and lower bounds in this setting.

As a second application, let us consider the case where $\| \cdot \|$ is the 1-norm, $\| \cdot \|_1$, the domain $\mathcal{W}$ is the simplex in $\mathbb{R}^d$, $d > 1$ (although our result easily extends to any subset of the 1-norm unit ball), and we use a standard entropic regularizer:

**Corollary 2.** Suppose $f_t$ for all $t$ is $G_1$-Lipschitz with respect to the $L_1$ norm. Then using $\| \cdot \| = \| \cdot \|_1$ and $r(w) = \sum_{i=1}^{d} w_i \log (dw_i)$, it holds for some constant $c$ and any $w^* \in \mathcal{W}$ that

$$
\mathbb{E} \left[ \frac{1}{T} \sum_{t=1}^{T} f_t(w_t) - \frac{1}{T} \sum_{t=1}^{T} f_t(w^*) \right] \leq c G_1 \sqrt{\frac{d \log^2(d)}{T}}.
$$

This bound matches (this time up to a logarithmic factor) the lower bound in [4] for this setting.

**Proof.** The function $r$ is 1-strongly convex with respect to the 1-norm (see for instance [9], Example 2.5), and has value at most $\log(d)$ on the simplex. Also, if $f_t$ is $G_1$-Lipschitz with respect to the 1-norm, then it must be $\sqrt{d} G_1$-Lipschitz with respect to the Euclidean norm. Finally, to satisfy condition 3 in Thm. 1, we upper bound $\sqrt{d} G_1$ using the following lemma, whose proof is given in the appendix:

**Lemma 1.** If $u$ is uniformly distributed on the unit sphere in $\mathbb{R}^d$, $d > 1$, then $\sqrt{d} \| u \|_2 \leq c \sqrt{\frac{\log(d)}{d}}$ where $c$ is a positive numerical constant independent of $d$.

Plugging these observations into Thm. 1 leads to the desired result.

\[\Box\]

### 3 Proof of Theorem 1

As discussed in the introduction, the key to getting improved results compared to previous papers is the use of a slightly different random gradient estimator, which turns out to have significantly less variance. The formal proof relies on a few simple lemmas listed below. The key lemma is Lemma 3 which establishes the improved variance behavior.
Lemma 2. For any $w^* \in \mathcal{W}$, it holds that

$$
\frac{1}{T} \sum_{t=1}^{T} (\hat{g}_t, w_t - w^*) \leq \frac{1}{\eta} R^2 + \eta \sum_{t=1}^{T} \|\hat{g}_t\|_2^2.
$$

This lemma is the canonical result on the convergence of online mirror descent, and the proof is standard (see e.g. [9]).

Lemma 3. Define the function

$$
\hat{f}_t(w) = \mathbb{E}_{u_t} [f_t(w + \delta_t u_t)],
$$

over $\mathcal{W}$, where $u_t$ is a vector picked uniformly at random from the Euclidean unit sphere. Then the function is convex, Lipschitz with constant $G_2$, satisfies

$$
\sup_{w \in \mathcal{W}} |\hat{f}_t(w) - f_t(w)| \leq \delta_t G_2,
$$

and is differentiable with the following gradient:

$$
\nabla \hat{f}_t(w) = \mathbb{E}_{u_t} \left[ \frac{d}{\delta_t} f_t(w + \delta_t u_t) u_t \right].
$$

Proof. The fact that the function is convex and Lipschitz is immediate from its definition and the assumptions in the theorem. The inequality follows from $u_t$ being a unit vector and that $f_t$ is assumed to be $G_2$-Lipschitz with respect to the 2-norm. The differentiability property follows from Lemma 2.1 in [5].

Lemma 4. For any function $g$ which is $L$-Lipschitz with respect to the 2-norm, it holds that if $u$ is uniformly distributed on the Euclidean unit sphere, then

$$
\sqrt{\mathbb{E} \left[ \left( g(u) - \mathbb{E}[g(u)] \right)^4 \right]} \leq c \frac{L^2}{d},
$$

for some numerical constant $c$.

Proof. A standard result on the concentration of Lipschitz functions on the Euclidean unit sphere implies that

$$
\Pr(|g(u) - \mathbb{E}[g(u)]| > t) \leq 2 \exp \left(-c' dt^2 / L^2 \right)
$$

for some numerical constant $c' > 0$ (see the proof of Proposition 2.10 and Corollary 2.6 in [7]). Therefore,

$$
\sqrt{\mathbb{E} \left[ \left( g(u) - \mathbb{E}[g(u)] \right)^4 \right]} = \sqrt{\int_{t=0}^{\infty} \Pr \left( |g(u) - \mathbb{E}[g(u)]| > t \right) dt}
$$

$$
= \sqrt{\int_{t=0}^{\infty} \Pr \left( |g(u) - \mathbb{E}[g(u)]| > \sqrt{t} \right) dt \leq \sqrt{\int_{t=0}^{\infty} 2 \exp \left(-c' \frac{d \sqrt{t}}{L^2} \right) dt = \sqrt{\frac{L^4}{(c'd)^2}},
$$

which equals $cL^2 / d$ for some numerical constant $c$. □

Lemma 5. It holds that $\mathbb{E} [\hat{g}_t | w_t] = \nabla \hat{f}_t(w_t)$ (where $\hat{f}_t(\cdot)$ is as defined in Lemma 3), and $\mathbb{E} [\|\hat{g}_t\|_2^2 | w_t] \leq c d p^2 G^2_2$ for some numerical constant $c$. 

5
Proof. For simplicity of notation, we drop the $t$ subscript. Since $u$ has a symmetric distribution around the origin,

$$
\mathbb{E}[\tilde{g}|w] = \mathbb{E}_u \left[ \frac{d}{2\delta} (f(w + \delta u) - f(w - \delta u)) u \right]
$$

$$
= \mathbb{E}_u \left[ \frac{d}{2\delta} (f(w + \delta u)) u \right] + \mathbb{E}_u \left[ \frac{d}{2\delta} f(w - \delta u)(-u) \right]
$$

$$
= \mathbb{E}_u \left[ \frac{d}{2\delta} (f(w + \delta u)) u \right] + \mathbb{E}_u \left[ \frac{d}{2\delta} f(w + \delta u)(u) \right]
$$

which equals $\nabla \tilde{f}(w)$ by Lemma[3].

As to the second part of the lemma, we have the following, where $\alpha$ is an arbitrary parameter and where we use the elementary inequality $(a - b)^2 \leq 2(a^2 + b^2)$.

$$
\mathbb{E}[\|\tilde{g}\|_w^2 | w] = \mathbb{E}_u \left[ \| \frac{d}{2\delta} (f(w + \delta u) - f(w - \delta u)) u \|_w^2 \right]
$$

$$
= \frac{d^2}{4\delta^2} \mathbb{E}_u \left[ \| u \|_w^2 (f(w + \delta u) - f(w - \delta u))^2 \right]
$$

$$
= \frac{d^2}{4\delta^2} \mathbb{E}_u \left[ \| u \|_w^2 ((f(w + \delta u) - \alpha) - (f(w - \delta u) - \alpha))^2 \right]
$$

$$
\leq \frac{d^2}{2\delta^2} \mathbb{E}_u \left[ \| u \|_w^2 ((f(w + \delta u) - \alpha)^2 + (f(w - \delta u) - \alpha)^2) \right]
$$

Again using the symmetrical distribution of $u$, this equals

$$
\frac{d^2}{2\delta^2} \mathbb{E}_u \left[ \| u \|_w^2 (f(w + \delta u) - \alpha)^2 \right] + \mathbb{E}_u \left[ \| u \|_w^2 (f(w - \delta u) - \alpha)^2 \right].
$$

Applying Cauchy-Schwartz and using the condition $\sqrt{\mathbb{E}_u[\|u\|^4]} \leq p_*$ stated in the theorem, we get the upper bound

$$
\frac{d^2}{\delta^2} \sqrt{\mathbb{E}_u[\|u\|^4]} \sqrt{\mathbb{E}_u \left[ (f(w + \delta u) - \alpha)^4 \right]} = \frac{p_2^2 d^2}{\delta^2} \sqrt{\mathbb{E}_u \left[ (f(w + \delta u) - \alpha)^4 \right]}.
$$

In particular, taking $\alpha = \mathbb{E}_u[f(w + \delta u)]$ and using Lemma[4](noting that $f(w + \delta u)$ is $G_2\delta$-Lipschitz w.r.t. $u$ in terms of the 2-norm), this is at most $\frac{p_2^2 d^2}{\delta^2} \frac{(G_2\delta)^2}{\delta^2} = cd\delta^2 G_2^2$ as required. \hfill \Box

We are now ready to prove the theorem. Taking expectations on both sides of the inequality in Lemma[4] we have

$$
\mathbb{E} \left[ \sum_{t=1}^T \langle \tilde{g}_t, w_t - w^* \rangle \right] \leq \frac{1}{\eta} R^2 + \eta \sum_{t=1}^T \mathbb{E} \left[ \| \tilde{g}_t \|^2 \right] = \frac{1}{\eta} R^2 + \eta \sum_{t=1}^T \mathbb{E} \left[ \mathbb{E} \left[ \| \tilde{g}_t \|^2 | w_t \right] \right]. \tag{4}
$$
Using Lemma 5, the right hand side is at most
\[ \frac{1}{\eta} R^2 + \eta c d \rho s G^2_2 T \]
The left hand side of Eq. (4), by Lemma 5 and convexity of \( \hat{f}_t \), equals
\[ \mathbb{E} \left[ \sum_{t=1}^{T} (\mathbb{E}[\tilde{g}_t | w_t], w_t - w^*) \right] = \mathbb{E} \left[ \sum_{t=1}^{T} (\nabla \hat{f}_t(w_t), w_t - w^*) \right] \geq \mathbb{E} \left[ \sum_{t=1}^{T} (\hat{f}_t(w_t) - \hat{f}_t(w^*)) \right]. \]
By Lemma 3 this is at least
\[ \mathbb{E} \left[ \sum_{t=1}^{T} (f_t(w_t) - f_t(w^*)) \right] - G_2 \sum_{t=1}^{T} \delta_t. \]
Combining these inequalities and plugging back into Eq. (4), we get
\[ \mathbb{E} \left[ \sum_{t=1}^{T} (f_t(w_t) - f_t(w^*)) \right] \leq G_2 \sum_{t=1}^{T} \delta_t + \frac{1}{\eta} R^2 + c d \rho s G^2_2 \eta T. \]
Choosing \( \eta = R / (p_s G_2 \sqrt{d/T}) \), and any \( \delta_t \leq p_s R \sqrt{d/T} \), we get
\[ \mathbb{E} \left[ \sum_{t=1}^{T} (f_t(w_t) - f_t(w^*)) \right] \leq (c + 2) p_s G_2 R \sqrt{d/T}. \]
Dividing both sides by \( T \), the result follows.

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A Proof of Lemma 1

We note that the distribution of $u$ is equivalent to that of $\frac{\|n\|_4}{\|n\|_2}$, where $n \sim \mathcal{N}(0, I_d)$ is a standard Gaussian random vector. Moreover, by a standard concentration bound on the norm of Gaussian random vectors (e.g. Corollary 2.3 in [2], with $\epsilon = 1/2$):

$$\max \left\{ \Pr \left( \|n\|_2 \leq \sqrt{\frac{d}{2}} \right), \Pr \left( \|n\|_2 \geq \sqrt{2d} \right) \right\} \leq \exp \left( -\frac{d}{16} \right).$$

Finally, for any value of $n$, we always have $\|n\|_\infty \|n\|_2 \leq 1$, since the Euclidean norm is always larger than the infinity norm. Combining these observations, and using $1_A$ for the indicator function of the event $A$, we have

$$\mathbb{E}[\|u\|_\infty^4] = \mathbb{E} \left[ \frac{\|n\|_4}{\|n\|_2} \right] = \Pr \left( \|n\|_2 \leq \sqrt{\frac{d}{2}} \right) \mathbb{E} \left[ \frac{\|n\|_4}{\|n\|_2} \left| \|n\|_2 \leq \sqrt{\frac{d}{2}} \right. \right] + \Pr \left( \|n\|_2 > \sqrt{\frac{d}{2}} \right) \mathbb{E} \left[ \frac{\|n\|_4}{\|n\|_2} \left| \|n\|_2 > \sqrt{\frac{d}{2}} \right. \right]$$

$$\leq \exp \left( -\frac{d}{16} \right) + \Pr \left( \|n\|_2 > \sqrt{\frac{d}{2}} \right) \mathbb{E} \left[ \frac{\|n\|_4}{\sqrt{d/2}} \left| \|n\|_2 > \sqrt{\frac{d}{2}} \right. \right]$$

$$= \exp \left( -\frac{d}{16} \right) + \left( \frac{2}{d} \right)^2 \mathbb{E} \left[ \frac{\|n\|_4}{\sqrt{d/2}} 1_{\|n\|_2 > \sqrt{d/2}} \right]$$

$$\leq \exp \left( -\frac{d}{16} \right) + \frac{4}{d^2} \mathbb{E} \left[ \|n\|_\infty^4 \right].$$

Thus, it remains to upper bound $\mathbb{E} \left[ \|n\|_\infty^4 \right]$ where $n$ is a standard Gaussian random variable. Letting $n = (n_1, \ldots, n_d)$, and noting that $n_1, \ldots, n_d$ are independent and identically distributed standard Gaussian random variables, we have for any scalar $z \geq 1$ that

$$\Pr (\|n\|_\infty \leq z) = \prod_{i=1}^{n} \Pr (|n_i| \leq z) = (\Pr (|n_1| \leq z))^d$$

$$= (1 - \Pr (|n_1| > z))^d \geq 1 - d \Pr (|n_1| > z)$$

$$= 1 - 2d \Pr (n_1 > z) \geq 1 - d \exp(-z^2/2),$$

where (1) is Bernoulli’s inequality, and (2) is using a standard tail bound for a Gaussian random variable. In particular, the above implies that

$$\Pr (\|n\|_\infty > z) \leq d \exp(-z^2/2).$$
Therefore, for an arbitrary positive scalar \( r \geq 1 \),

\[
\mathbb{E} \left[ \| n \|_\infty^4 \right] = \int_{z=0}^{\infty} \Pr \left( \| n \|_\infty^4 > z \right) dz \\
\leq \int_{z=0}^{r} 1dz + \int_{z=r}^{\infty} \Pr \left( \| n \|_\infty > \sqrt[4]{z} \right) dz \\
\leq r + \int_{z=r}^{\infty} d \exp \left( -\frac{\sqrt{z}}{2} \right) dz \\
= r + 4d(2 + \sqrt{r}) \exp \left( -\frac{\sqrt{r}}{2} \right).
\]

In particular, plugging \( r = 4 \log^2(d) \) (which is larger than 1, since we assume \( d > 1 \)), we get \( 4(2 + 2 \log(d) + \log^2(d)) \). Plugging this back into Eq. (5), we get that

\[
\mathbb{E}[\| u \|_\infty^4] \leq \exp \left( -\frac{d}{16} \right) + 16 \frac{2 + 2 \log(d) + \log^2(d)}{d^2},
\]

which can be shown to be at most \( c' \left( \frac{\log(d)}{d} \right)^2 \) for all \( d > 1 \), where \( c' < 150 \) is a numerical constant. In particular, this means that \( \sqrt[4]{\mathbb{E}[\| u \|_\infty^4]} \leq 4c' \sqrt[4]{\frac{\log(d)}{d}} \) as required.