On the Tree Augmentation Problem

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Abstract

In the Tree Augmentation problem we are given a tree \( T = (V, F) \) and a set \( E \subseteq V \times V \) of edges with positive integer costs \( \{c_e : e \in E\} \). The goal is to augment \( T \) by a minimum cost edge set \( J \subseteq E \) such that \( T \cup J \) is 2-edge-connected. We obtain the following results.

– Recently, Adjiashvili [SODA 17] introduced a novel LP for the problem and used it to break the 2-approximation barrier for instances when the maximum cost \( M \) of an edge in \( E \) is bounded by a constant; his algorithm computes a \( 1.96418 + \epsilon \) approximate solution in time \( n^{(M/\epsilon^2)\Omega(1)} \). Using a simpler LP, we achieve ratio \( \frac{12}{7} + \epsilon \) in time \( 2^{O(M/\epsilon^2)} \text{poly}(n) \). This gives ratio better than 2 for logarithmic costs, and not only for constant costs.

– One of the oldest open questions for the problem is whether for unit costs (when \( M = 1 \)) the standard LP-relaxation, so called Cut-LP, has integrality gap less than 2. We resolve this open question by proving that for unit costs the integrality gap of the Cut-LP is at most \( 28/15 = 2 - 2/15 \). In addition, we will prove that another natural LP-relaxation, that is much simpler than the ones in previous work, has integrality gap at most \( 7/4 \).

Keywords Tree augmentation · Approximation algorithm · Extreme points · Integrality gap

1 Introduction

We consider the following problem:

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**Tree Augmentation**

*Input:* A tree \( T = (V, F) \) and an additional set \( E \subseteq V \times V \) of edges with positive integer costs \( c = \{c_e : e \in E\} \).

*Output:* A minimum cost edge set \( J \subseteq E \) such that \( T \cup J \) is 2-edge-connected.

The problem was studied extensively, c.f. [5–7,9,12,13,15,21–23,25,27]. For a long time the best known ratio for the problem was 2 for arbitrary costs [15] and 1.5 for unit costs [12,23]; see also [13] for a simple 1.8-approximation algorithm. It is also known that the integrality gap of a standard LP-relaxation for the problem, so called Cut-LP, is at most 2 [15] and at least 1.5 [7]. Several other LP and SDP relaxations were introduced to show that the algorithm in [12,13,23] achieves ratio better than 2 w.r.t. to these relaxations, c.f. [5,22]. For additional algorithms with ratio better than 2 for restricted versions see [9,25].

Let \( M \) denote the maximum cost of an edge in \( E \). Recently Adjishvili [1] introduced a novel LP for the problem—so called the \( k \)-Bundle-LP, and used it to break the natural 2-approximation barrier for instances when \( M \) is bounded by a constant. To introduce this result we need some definitions.

The edges of \( T \) will be called \( T \)-edges to distinguish them from the edges in \( E \). **Tree Augmentation** can be formulated as a problem of covering the \( T \)-edges by paths. Let \( T_{uv} \) denote the unique \( uv \)-path in \( T \). We say that an edge \( uv \) covers a \( T \)-edge \( f \) if \( f \in T_{uv} \). Then \( T \cup J \) is 2-edge-connected if and only if \( J \) covers \( T \). For a set \( B \subseteq F \) of \( T \)-edges let \( \psi(B) \) denote the set of edges in \( E \) that cover some \( f \in B \), and \( \tau(B) \) the minimum cost of an edge set in \( E \) that covers \( B \). For \( J \subseteq E \) let \( x(J) = \sum_{e \in J} x_e \). The standard LP for the problem which we call the Cut-LP seeks to minimize \( c^T x = \sum_{e \in E} c_e x_e \) over the Cut-Polyhedron

\[
\Pi^\text{Cut} = \left\{ x \in \mathbb{R}^E : x(\psi(f)) \geq 1 \forall f \in F, x \geq 0 \right\}
\]

The \( k \)-Bundle-LP of [1] adds over the standard Cut-LP the constraints \( \sum_{e \in \psi(B)} c_e x_e \geq \tau(B) \) for any forest \( B \) in \( T \) that has at most \( k \) leaves, where \( k = \Theta(M/\epsilon^2) \). The algorithm of [1] computes a \( 1.96418+\epsilon \) approximate solution w.r.t. the \( k \)-Bundle-LP in time \( n^{k^{O(1)}} \). For unit costs, a modification of the algorithm achieves ratio \( 5/3 + \epsilon \).

Here we observe that it is sufficient to consider just certain subtrees of \( T \) instead of forests. Root \( T \) at some node \( r \). The choice of \( r \) defines an ancestor/descendant relation on \( V \). The leaves of \( T \) are the nodes in \( V \setminus \{r\} \) that have no descendants. For any subtree \( S \) of \( T \), the node \( s \) of \( S \) closest to \( r \) is the root of \( S \), and the pair \( S, s \) is called a rooted subtree of \( T, r \); we will not mention the roots of trees if they are clear from the context. We say that \( S \) is a complete rooted subtree if it contains all descendants of \( s \) in \( T \), and a full rooted subtree if for any non-leaf node \( v \) of \( S \) the children of \( v \) in \( S \) and \( T \) coincide; see Fig. 1a, b. A branch of \( S \), or a branch hanging on \( s \), is a rooted subtree \( B \) of \( S \) induced by the root \( s \) of \( S \) and the descendants in \( S \) of some child \( s' \) of \( s \); see Fig. 1c. We say that a subtree \( B \) of \( T \) is a branch if it is a branch of a full rooted subtree, or if it is a full rooted subtree with root \( r \). Equivalently, a branch is a union of a full rooted subtree and its parent \( T \)-edge.
Let $B_k$ denote the set of branches in $T$ with less than $k$ leaves. The $k$-Branch-LP seeks to minimize $c^T x = \sum_{e \in E} c_e x_e$ over the $k$-Branch-Polyhedron $\Pi_k^{Br} \subseteq \mathbb{R}^E$ defined by the constraints:

\[
\sum_{e \in \psi(f)} x_e \geq 1 \quad \forall f \in F
\]
\[
\sum_{e \in \psi(B)} c_e x_e \geq \tau(B) \quad \forall B \in B_k
\]
\[
x_e \geq 0 \quad \forall e \in E
\]

The set of constraints of the $k$-Branch-LP is a subset of constraints of the $k$-Bundle-LP of [1], hence the $k$-Branch-LP is both more compact and its optimal value is no larger than that of the $k$-Bundle-LP. The first main result in this paper is:

**Theorem 1** For any $1 \leq \lambda \leq k - 1$, Tree Augmentation admits a $4^k \cdot \text{poly}(n)$ time algorithm that computes a solution of cost at most $\rho + 8 \cdot \frac{\lambda M}{3k - \lambda M} + \frac{2}{\lambda}$ times the optimal value of the $k$-Branch-LP, where $\rho = \frac{12}{7}$ for arbitrary costs and $\rho = 1.6$ for unit costs.

For a given $\epsilon$, choosing properly $\lambda = \Theta(1/\epsilon)$ and $k = \Theta(M/\epsilon^2)$ gives ratio $\rho + \epsilon$ in time $2^{O(M/\epsilon^2)} \cdot \text{poly}(n)$.

In parallel to our work, Fiorini, Groß, Könemann, and Sanitá [14] augmented the $k$-Bundle-LP of [1] by additional constraints – $\{0, \frac{1}{2}\}$-Chvátal-Gomory Cuts, to achieve ratio $1.5 + \epsilon$ in $n^{(M/\epsilon^2)^{O(1)}}$ time, thus almost matching the best known ratio for unit costs [12,23]. Our result in Theorem 1, done independently, shows that already the $k$-Bundle-LP has integrality gap closer to 1.5 than to 2. Our version of the algorithm of [1] is also simpler than the one in [14]. In fact, combining our approach with [14] enables to achieve ratio $1.5 + \epsilon$ in $2^{O(M/\epsilon^2)} \cdot \text{poly}(n)$ time. Note that this allows to achieve this ratio for logarithmic costs, and not only for constant costs. We will provide an additional comparison of our results and those in [14] in Sect. 2.3.

Very recently the natural ratio 1.5 was improved to a smaller constant by Grandoni, Kalaitzis & Zenklusen [18]. Their approach also works for small integer costs and gives ratio that tends to 1.5 from below.
We note that while the running time of the combinatorial algorithm for unit costs of [23] is roughly the same as that of finding a maximum matching, the recent algorithms [14,18] that are based on the approach of Adjiashvili [1] have running time \(n^{(M/\epsilon^2)}O(1)\), with large constant hidden on the \(O(\cdot)\) term; this is very high even for unit costs and \(\epsilon = 0.1\). Our result in Theorem 1 substantially reduces the running time to \(2^{O(M/\epsilon^2)} \cdot poly(n)\).

Let \(diam(T)\) denote the diameter of \(T\). TREE AUGMENTATION admits a polynomial time algorithm when \(diam(T) \leq 3\). If \(diam(T) = 2\) then \(T\) is a star and we get the EDGE-COVER problem, while the case \(diam(T) = 3\) is reduced to the case \(diam(T) = 2\) by “guessing” some optimal solution edge that covers the central \(T\)-edge. The problem becomes NP-hard when \(diam(T) = 4\) even for unit costs [15]. We prove that (without solving any LP) for arbitrary costs TREE AUGMENTATION with trees of diameter \(\leq 5\) admits ratio \(3/2\).

Our second main result resolves one of the oldest open questions concerning the problem—whether for unit costs the integrality gap of the CUT-LP is less than 2. This was conjectured in the 90’s by Cherian, Jordán & Ravi [6] for arbitrary costs, but so far there was no real evidence for this even for unit costs. Our second main result resolves this old open question.

**Theorem 2** For unit costs, the integrality gap of the CUT-LP is at most \(28/15 = 2 - 2/15\).

In addition, we will show that for unit costs, another natural simple LP-relaxation, has integrality gap at most 7/4.

## 2 Algorithm for Bounded Costs (Theorem 1)

The Theorem 1 algorithm is a modification of the algorithm of [1]. We emphasize some differences. We use the \(k\)-BRANCH-LP instead of the \(k\)-BUNDLE-LP of [1]. But, unlike [1], we do not solve our LP at the beginning. Instead, we combine binary search with the ellipsoid algorithm as follows. We start with lower and upper bounds \(p\) and \(q\) on the value of the \(k\)-BRANCH-LP, e.g., \(p = 0\) and \(q\) is the cost of some feasible solution to the problem. Given a “candidate” \(x\) with \(q \leq c^T x \leq p\), the outer iteration (see Algorithm 1) of the entire algorithm either returns a solution of cost at most \((\rho + \frac{8}{3} \frac{2^M}{k-2M} + \frac{2}{\rho})c^T x\) or a constraint of the \(k\)-BRANCH-LP violated by \(x\); we show that this can be done in time \(4^k \cdot poly(n)\), rather than in time \(n^{kO(1)}\) as in [1]. We set \(p \leftarrow \frac{p+q}{2}\) in the former case and \(q \leftarrow \frac{p+q}{2}\) in the latter case and continue to the next iteration, terminating when \(p - q\) is small enough. This essentially gives a \(4^k \cdot poly(n)\) time separation oracle for the \(k\)-BRANCH-LP (if a violated \(k\)-branch constraint is found). Since the ellipsoid algorithm uses a polynomial number of calls to the separation oracle, the running time is \(4^k \cdot poly(n)\). Note that checking whether \(x \in \Pi_{Cut}^T\) is trivial, hence for simplicity of exposition we will assume that the “candidate” \(x\) is in \(\Pi_{Cut}^T\).

For a set \(S\) of \(T\)-edges we denote by \(T/S\) the tree obtained from \(T\) by contracting every \(T\)-edge of \(S\). This defines a new TREE AUGMENTATION instance (that may have
loops and parallel edges), where contraction of a $T$-edge $uv$ leads to shrinking $u$, $v$ into a single node in the graph $(V, E)$ of edges. In the algorithm, we repeatedly take a certain complete rooted subtree $\hat{S}$, and either find a $k$-branch-constraint violated by some branch in $\hat{S}$, or a “cheap” cover $J_S$ of a subset $S$ of the $T$-edges of $\hat{S}$; in the latter case, we add $J_S$ to our partial solution $J$, contract $\hat{S}$, and iterate on the instance $T \leftarrow T/\hat{S}$. At the end of the loop, the edges that are still not covered by the partial solution $J$ are covered by a different procedure, by a total cost $2\lambda \cdot c_T x$, as follows.

We call a $T$-edge $f \in F$ $\lambda$-thin if $x(\psi(f)) \leq \lambda$, and $f$ is $\lambda$-thick otherwise. We need the following lemma from [1], for which we provide a proof for completeness of exposition.

**Lemma 1** ([1]) There exists a polynomial time algorithm that given $x \in \Pi_{\text{Cut}}$, $\lambda > 1$, and a set $F' \subseteq F$ of $\lambda$-thick $T$-edges computes a cover $J'$ of $F'$ of cost $\leq 2\lambda \cdot c_T x$.

**Proof** Since all $T$-edges in $F'$ are $\lambda$-thick, $x/\lambda$ is a feasible solution to the $\text{Cut- LP}$ for covering $F'$. Thus any polynomial time algorithm that computes a solution $J'$ of cost at most 2 times the optimal value of the $\text{Cut- LP}$ for covering $F'$ has the desired property. There are several such algorithms, see [15,16,20].

We say that a complete rooted subtree $S$ of $T$ is a $(k, \lambda)$-subtree if $S$ has at least $k$ leaves and if either the parent $T$-edge $f$ of $S$ is $\lambda$-thin or $s = r$. For $\lambda = \Theta(1/\epsilon)$ and $k = \Theta(M/\epsilon^2)$ we choose $\hat{S}$ to be an inclusionwise minimal $(k, \lambda)$-subtree. Let us focus on the problem of covering such $\hat{S}$. Let $S'$ be the set of $T$-edges of the inclusionwise maximal subtree of $\hat{S}$ that contains the root $s$ of $\hat{S}$ and has only $\lambda$-thick $T$-edges (possibly $S = \emptyset$); see Fig. 2a. We postpone covering the $T$-edges in $S'$ to the end of the algorithm, so we contract $S'$ into $s$ and consider the tree $\hat{S} \leftarrow \hat{S}/S'$; see Fig. 2b. In $S$, every branch $B$ hanging on $s$ has less than $k$ leaves, by the minimality of $S$, hence it has a corresponding constraint in the $k$-Branch- $\text{LP}$. We will show that for a $k$-branch $B$ an optimal set of edges that covers $B$ can be computed in time $4^k \cdot \text{poly}(n)$. If $\sum_{e \in \psi(B)} c_e x_e < \tau(B)$ for some branch $B$ hanging on $s$ in $S$, then we return the corresponding $k$-branch constraint violated by $x$; otherwise, we will show how to compute a “cheap” cover of $S$. More formally, in the next section we will prove:

**Lemma 2** Suppose that we are given a Tree Augmentation instance and $x \in \Pi_{\text{Cut}}$ such that any complete rooted proper subtree of the input tree has less than $k$ leaves. Then there exists a $4^k \cdot \text{poly}(n)$ time algorithm that either finds a $k$-branch constraint
violated by x, or computes a solution of cost \( \leq \rho \sum_{e \in E \setminus R} c_e x_e + \frac{4}{3} \sum_{e \in R} c_e x_e \), where \( \rho \) is as in Theorem 1 and \( R \) is the set of edges in \( E \) incident to the root.

To find a cheap cover of \( S \), we consider the Tree Augmentation instance obtained from \( T / S' \) by contacting into \( s \) all nodes not in \( S \). Note that every edge that was in \( \psi(S) \cap \psi(f) \) is now incident to the root. Thus since \( \rho \geq \frac{4}{3} \), Lemma 2 implies:

**Corollary 1** There exists a \( 4k \cdot \text{poly}(n) \) time algorithm that either finds a \( k \)-branch-constraint violated by \( x \), or a cover \( J_S \) of \( S \) of cost \( c(J_S) \leq \rho \sum_{e \in \gamma(S)} c_e x_e + \frac{4}{3} \sum_{e \in \psi(f)} c_e x_e \), where \( \rho \) is as in Theorem 1 and \( \gamma(S) \) denotes the set of edges with both endnodes in \( S \), and \( f \) is the parent \( T \)-edge of \( S \).

The outer iteration of the algorithm is as follows:

**Algorithm 1: Outer-Iteration** \((T = (V, F), E, x, c, k, r, \lambda)\)

1. \( J \leftarrow \emptyset, F' \leftarrow \emptyset \)
2. while \( T \) has at least 2 nodes do
   3. let \( \hat{S} \) be an inclusionwise minimal \((k, \lambda)\)-subtree of \( T \)
   4. let \( S' \) be the edge-set of the inclusionwise maximal subtree of \( \hat{S} \) that contains the root \( s \) of \( \hat{S} \) and has only \( \lambda \)-thick edges
   5. apply the algorithm from **Corollary 1** on \( S \leftarrow \hat{S} / S' \)
   6. if **Corollary 1** algorithm returns a cover \( J_S \) of \( S \) then do:
      7. \( F' \leftarrow F' \cup S', J \leftarrow J \cup J_S, T \leftarrow T / \hat{S} \)
   8. else, return a \( k \)-branch constraint violated by \( x \) and STOP
9. return \( J \cup J' \)

Note that at step 7 the \( T \)-edges in \( F' \) are all \( \lambda \)-thick and thus Lemma 1 applies. We will now analyze the performance of the algorithm assuming that no \( k \)-branch-constraint violated by \( x \) was found. Let \( \delta(S) \) denote the set of edges with exactly one endnode in \( S \) and \( \gamma(S) \) the set of edges with both endnodes in \( S \). Let \( f \) be the parent \( T \)-edge of \( S \). Since \( f \) is \( \lambda \)-thin

\[
\sum_{e \in \gamma(S)} c_e x_e \leq M \cdot x(\psi(f)) \leq M \lambda .
\]

Since \( x(\delta(v)) \geq 1 \) for every leaf \( v \) of \( S \), \( c_e \geq 1 \) for every \( e \in E \), and since \( S \) is a \((k, \lambda)\)-subtree

\[
2 \sum_{e \in \gamma(S)} c_e x_e = \sum_{v \in S} \sum_{e \in \delta(v)} c_e x_e - \sum_{e \in \gamma(f)} c_e x_e \geq \sum_{v \in \hat{S} \setminus \{s\}} x(\delta(v)) - \lambda M \geq k - \lambda M .
\]

Consider a single iteration in the while-loop. Let \( \Delta(c^T x) \) denote the decrease in the LP-solution value as a result of contracting \( \hat{S} \). Then

\[
\Delta(c^T x) = \sum_{e \in \gamma(S)} c_e x_e \geq \frac{k - \lambda M}{2} .
\]
On the other hand, by Lemma 2, the partial solution cost increases by at most
\[
c(J_S) \leq \rho \sum_{e \in \gamma(S)} c_e x_e + \frac{4}{3} \sum_{e \in \gamma(f)} c_e x_e \leq \rho \sum_{e \in \gamma(\hat{S})} c_e x_e + \frac{4}{3} \lambda M.
\]
Thus
\[
\frac{c(J_S)}{\Delta(c^T x)} \leq \rho + \frac{8}{3} \frac{\lambda M}{k - \lambda M}.
\]
The while-loop terminates when the LP-solution value becomes 0, hence by a standard local-ratio/induction argument we get that at the end of the while-loop \(c(J) \leq \left(\rho + \frac{8}{3} \frac{\lambda M}{k - \lambda M}\right) c^T x\). At step 7 we add an edge set of cost \(\leq 2 \lambda c^T x\), and Theorem 1 follows. It only remains to prove Lemma 2, which we will do in the subsequent sections.

### 2.1 Proof of Lemma 2

Assume that we are given an instance \(T = (V, E), c\) of Tree Augmentation with root \(r\) and \(x\) as in Lemma 2. It is known that Tree Augmentation instances when \(T\) is a path can be solved in polynomial time. Thus by a standard “metric completion” type argument we may assume that the graph \((V, E)\) is a complete graph and that \(c_{uv} = \tau(T_{uv})\) for all \(u, v \in V\). Indeed, then for each \(u, v \in V\) corresponds a set \(P_{uv}\) of edges of cost \(c_{uv} = \tau(T_{uv})\) that covers \(T_{uv}\), and whenever and edge \(uv\) is chosen to the solution, it can be replaced by \(P_{uv}\) without increasing the cost. Note that we use this assumption only in the proof of Lemma 2, where the running time does not depend on the maximum cost \(M\) of an edge in \(E\). Let us say that an edge \(uv \in E\) is:

- **A cross-edge** if \(r\) is an internal node of \(T_{uv}\);
- **An in-edge** if \(r\) does not belong to \(T_{uv}\);
- **An \(r\)-edge** if \(r = u\) or \(r = v\);
- **An up-edge** if one of \(u, v\) is an ancestor of the other.

For a subset \(E' \subseteq E\) of edges the E'-up vector of \(x\) is obtained from \(x\) as follows: for every non-up edge \(e = uv \in E'\) increase \(x_{ua}\) and \(x_{ea}\) by \(x_e\) and then reset \(x_e\) to 0, where \(a\) is the least common ancestor of \(u\) and \(v\). The **fractional cost** of a set \(J\) of edges w.r.t. \(c\) and \(x\) is defined by \(\sum_{e \in J} c_e x_e\). Let \(C^i, C^c, C^r\) denote the fractional cost of in-edges, cross-edges, and \(r\)-edges, respectively, w.r.t. \(c\) and \(x\). We fix some \(x^* \in \Pi^{Cut}\) and denote by \(C^i, C^c, C^r\) the fractional cost of in-edges, cross-edges, and \(r\)-edges, respectively, w.r.t. \(c\) and \(x^*\). We give two rounding procedures, given in Lemmas 3 and 4. The rounding procedure in Lemma 3 is similar to that of [1], but we show that it can be implemented in time \(4^k \cdot poly(n)\) instead of \(n^{\Omega(1)}\).

**Lemma 3** There exists a \(4^k \cdot poly(n)\) time algorithm that either finds a \(k\)-branch inequality violated by \(x^*\), or returns an integral solution of cost at most \(C^i + 2C^c + C^r\).
Proof Let $B$ be the set of branches hanging on $r$. For every $B \in B$ compute an optimal solution $J_B$. If for some $B \in B$ we have $\tau(B) > \sum_{e \in \psi(B)} c_e x^*_e$ then a $k$-branch inequality violated by $x^*$ is found. Else, the algorithm returns the union $J = \bigcup_{B \in B} J_B$ of the computed edge sets. As every cross-edge has its endnodes in two distinct branches, while every in-edge or $r$-edge has its both endnodes in the same branch, we get

$$c(J) \leq \sum_{B \in B} \tau(B) \leq \sum_{B \in B} \sum_{e \in \psi(B)} c_e x^*_e = \sum_{B \in B} \left( \sum_{e \in \delta(B)} c_e x^*_e + \sum_{e \in \gamma(B)} c_e x^*_e \right) = 2C^{cr} + C^{in} + C^r.$$

It remains to show that an optimal solution in each branch of $r$ can be computed in time $4^k \cdot \text{poly}(n)$. More generally, we will show that TREE AUGMENTATION instances with $k$ leaves can be solved optimally within this time bound. Recall that we may assume that the graph $(V, E)$ is a complete graph and that $c_{uv} = \tau(T_{uv})$ for all $u, v \in V$. We claim that then $T$ has no node $v$ with $\deg_T(v) = 2$. This is a well known reduction (e.g. see [26]). In more details, we show that any solution $J$ can be converted into a solution of no greater cost that has no edge incident to $v$, and thus $v$ can be “shortcut”. If $J$ has edges $uv, vw$ then it is easy to see that $J \setminus \{uv, vw\} \cup \{uw\}$ is also a feasible solution, of cost at most $c(J)$, since $c_{uw} \leq c_{uv} + c_{vw}$. Applying this operation repeatedly we may assume that $\deg_J(v) \leq 1$. If $\deg_J(v) = 0$, we are done. Suppose that $J$ has a unique edge $e = uv$ incident to $v$. Let $vu$ and $vu'$ be the two $T$-edges incident to $v$, where assume that $vu'$ is not covered by $e$. Then there is an edge $e' \in J$ that covers $vu'$. Since $e'$ is not incident to $v$, it must be that $e'$ covers $vu$. Replacing $e$ by the edge $vu$ gives a feasible solution without increasing the cost.

Consequently, we reduce our instance to an equivalent instance with at most $2k - 1$ tree edges. Now recall that TREE AUGMENTATION is a particular case of the MIN-COST SET-COVER problem, where the set $F$ of $T$-edges are the elements and $\{T_e : e \in E\}$ are the sets. The MIN-COST SET-COVER problem can be solved in $2^n \cdot \text{poly}(n)$ time via dynamic programming, where $n$ is the number of elements; such an algorithm is described in [11][Sect. 6.1] for unit costs, but the proof extends to arbitrary costs [10]. Thus our reduced TREE AUGMENTATION instance can be solved in $2^{2k-1} \cdot \text{poly}(n) \leq 4^k \cdot \text{poly}(n)$ time.

For the second rounding procedure [1] proved that for any $\lambda > 1$ one can compute in polynomial time an integral solution of cost at most $2\lambda C^{in} + \frac{4}{3} \frac{\lambda}{\lambda - 1} C^{cr}$. We prove:

**Lemma 4** There exists a polynomial time algorithm that computes a solution of cost $\frac{4}{3}(2C^{in} + C^{cr} + C^r)$, and a solution of size $2C^{in} + \frac{4}{3} C^{cr} + C^r$ in the case of unit costs.

Consider the case of arbitrary bounded costs. If $C^{in} \geq \frac{2}{5} C^{cr}$ we use the rounding procedure from Lemma 3 and the rounding procedure from Lemma 4 otherwise. In both cases we get $c(J) \leq \frac{12}{7}(C^{in} + C^{cr}) + \frac{4}{3} C^r$. In the case of unit costs, if $C^{in} \geq \frac{2}{5} C^{cr}$ we use the rounding procedure from Lemma 3, and the procedure from Lemma 4 otherwise. In both cases we get $c(J) \leq 1.6(C^{in} + C^{cr}) + C^r$. 

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Lemma 4 is proved in the next section. The proof relies on properties of extreme points of the \textsc{Cut-Polyhedron} $\Pi^{Cut}$ that are of independent interest.

### 2.2 Properties of Extreme Points of the \textsc{Cut-Polyhedron} (Lemma 4)

W.l.o.g., we augment the \textsc{Cut-LP} by the constraints $x_e \leq 1$ for all $e \in E$, while using the same notation as before. The (modified) \textsc{Cut-LP} always has an optimal solution $x$ that is an extreme point or a basic feasible solution of $\Pi^{Cut}$. Geometrically, this means that $x$ is not a convex combination of other points in $\Pi^{Cut}$; algebraically this means that there exists a set of $|E|$ inequalities in the system defining $\Pi^{Cut}$ such that $x$ is the unique solution for the corresponding linear equations system. These definitions are known to be equivalent and we will use both of them, c.f. [24].

A set family $L$ is laminar if any two sets in the family are either disjoint or one contains the other. Note that TREE AUGMENTATION is equivalent to the problem of covering the laminar family of the node sets of the complete rooted proper subtrees of $T$, where an edge covers a node set $S$ if it has exactly one endnode in $S$. In particular, note that the constraint $x(\psi(f)) \geq 1$ is equivalent to the constraint $x(\delta(S)) \geq 1$ where $S$ is the node set of the complete rooted subtree with parent $T$-edge $f$. Let $\mathbb{N}_0$ denote the set of non-negative integers.

**Lemma 5** Let $(V, E)$ be a graph, $L$ a laminar family on $V$, and $b \in \mathbb{N}_0^{|L|}$. Suppose that for every $S \in L$ there is no edge between two distinct children of $S$ and that the equation system $\{x(\delta(S)) = b_S : S \in L\}$ has a unique solution $0 < x^* < 1$. Then $x_e^* = 1/2$ for all $e \in E$. Furthermore, each endnode of every $e \in E$ belongs to some $S \in L$.

**Proof**

For every $uv \in E$ put one token at $u$ and one token at $v$. The total number of tokens is $2|E|$. For $S \in L$ let $t(S)$ be the number of tokens placed at nodes in $S$ that belong to no child of $S$. Since $L$ is laminar, every token is placed in at most one set in $L$, and thus $\sum_{S \in L} t(S) \leq 2|E|$. Let $S \in L$ and let $C(S)$ be the set of children of $S$ in $L$. Let $E_S$ be the set of edges in $\delta(S)$ that cover no child of $S$, and $E_{C(S)}$ the set of edges not in $\delta(S)$ that cover some child of $S$. By the assumption of the lemma, no $e \in E_{C(S)}$ connects two children of $S$. Observe that $b_S^* = x^*(E_S) - x^*(E_{C(S)})$ is an integer since

$$x^*(E_S) - x^*(E_{C(S)}) = x^*(\delta(S)) - \sum_{C \in C(S)} x^*(\delta(C)) = b_S - \sum_{C \in C(A)} b_C \equiv b_S^*.$$

If $|E_S| = |E_{C(S)}| = 0$ then $\delta(S) = \bigcup_{C \in C(S)} \delta(C)$, contradicting linear independence. If $|E_S| + |E_{C(S)}| = 1$ then $E_S \cup E_{C(S)}$ has a unique edge that has an integer $x^*$-value, contradicting that $0 < x^* < 1$. Thus $|E_S| + |E_{C(S)}| \geq 2$. Since no $e \in E$ goes between children of $S$, $t(S) \geq |E_S| + |E_{C(S)}|$. Consequently, since $\sum_{S \in L} t(S) \leq 2|E|$, we get: $t(S) = |E_S| + |E_{C(S)}| = 2 \forall S \in L$. Moreover, if an endnode of some $e \in E$ belongs to no $S \in L$, then we get the contradiction $\sum_{S \in L} t(S) \geq 2|E| + 1$. Now we replace our equation system by an equivalent one $\{x(E_S) - x(E_{C(S)}) = b_S^* : S \in L\}$ obtained by elementary operations on the rows of the coefficients matrix. Note that $x^*$

\[ Springer \]
is also a unique solution to this new equation system. Moreover, this equation system has exactly two variables in each equation and all its coefficients are integral. By [19], the solution of such equation systems is always half-integral.

Let us say that TREE AUGMENTATION instance is spider-shaped if every in-edge in $E$ is an up-edge. By a standard “iterative rounding” argument (c.f. [24]), and using the correspondence between rooted trees and laminar families, we get from Lemma 5:

**Corollary 2** Suppose that we are given a spider-shaped TREE AUGMENTATION instance and $b \in \mathbb{N}_0^F$. Let $x$ be an extreme point of the polytope $\{x \in \mathbb{R}^E : x(\psi(f)) \geq b_f \forall f \in F, 0 \leq x \leq 1\}$. Then $x$ is half-integral (namely, $x_e \in \{0, \frac{1}{2}, 1\}$ for all $e \in E$) and $x_e \in \{0, 1\}$ for every $e \in \delta(r)$.

**Proof** Note that the assumptions of the lemma remain valid after each one of the following two operations:

- If $x_e = 0$ for some $e \in E$ then remove $e$.
- If $x_e = 1$ for some $e \in E$ then set $b_f \leftarrow \max\{b_f - 1, 0\}$ for every $f \in T_e$, and remove $e$.

We thus can argue by induction. The base case $|E| = 1$ is obvious. If there is $e \in E$ with $x_e \in \{0, 1\}$ then apply an appropriate operation above and the induction hypothesis. Otherwise, $0 < x < 1$. Let $\mathcal{L}_T$ be the laminar family of $T$. Since $x$ is an extreme point, there exists a laminar family $\mathcal{L} \subseteq \mathcal{L}_T$ such that $x$ is the unique solution to the equation system $\{x(\delta(S)) = b_S : S \in \mathcal{L}\}$. Note that since our TREE AUGMENTATION instance is spider-shaped, there is no edge that connects two children of some $S \in \mathcal{L}$. Thus Lemma 5 implies $x^* = 1/2$ for all $e \in E$, and the proof is complete.

The algorithm that computes an integral solution of cost $\frac{4}{3}(2C^{in} + C^{cr} + C^r)$ is as follows. We obtain a spider-shaped instance by removing all non-up in-edges and compute an optimal extreme point solution $x$ to the CUT-LP. By Corollary 2, $x$ is half-integral and $x_e \in \{0, 1\}$ for every $e \in \delta(r)$. We take into our solution every edge $e$ with $x_e = 1$ and round the remaining $1/2$ entries using the algorithm of Cheriyan, Jordán & Ravi [6], that showed how to round a half-integral solution to the CUT-LP to integral solution within a factor of $4/3$. Thus we can compute a solution $J$ of cost at most $c(J) \leq \frac{4}{3}c^T x \leq \frac{4}{3}c^T x^*$. We claim that $c^T x \leq 2C^{in} + C^{cr} + C^r$. To see this let $E^{in}$ be the set of in-edges and let $x'$ be the $E^{in}$-up vector of $x^*$. Then $x'$ is a feasible solution to the CUT-LP of value $2C^{in} + C^{cr} + C^r$, in the obtained TREE AUGMENTATION instance with all non-up in-edges removed. But since $x$ is an optimal solution to the same LP, we have $c^T x \leq c^T x' = 2C^{in} + C^{cr} + C^r$. This concludes the proof of Lemma 4 for the case of arbitrary costs.

In the rest of this section we consider the case of unit costs.

**Lemma 6** Let $a, b \geq 0$ and let $x$ be an extreme point of the polytope

$$\Pi = \{x \in \Pi^{Cut} : C^{in}_x = a, C^{cr}_x = b\}$$

such that $x_e > 0$ for every cross-edge $e$. Then the graph $(V, E^{cr})$ of cross-edges has no even cycle and each one of its connected components has at most one cycle.
Proof Let $Q$ be a cycle in $E^{cr}$ and let $\epsilon = \min_{e \in Q} x_e$. Since $x_e > 0$ for all $e \in E^{cr}$, $\epsilon > 0$. If $|Q|$ is even, let $Q', Q''$ be a partition of $Q$ into two perfect matchings. Let $z$ be a vector defined by $z_e = \epsilon$ if $e \in Q'$, $z_e = -\epsilon$ if $e \in Q''$, and $z_e = 0$ otherwise. By the choice of $\epsilon$, $x+z, x-z$ are non-negative, and it is not hard to verify that $x+z, x-z \in \Pi$. However, $x = \frac{1}{2}(x+z) + \frac{1}{2}(x-z)$, contradicting that $x$ is an extreme point.

Suppose that $|Q|$ is odd. Let $u, v$ be nodes on $Q$, possibly $u = v$. We claim that $(V, E^{cr} \setminus Q)$ has no $uv$-path; this also implies that any two odd cycles in $(V, E^{cr})$ are node disjoint. Suppose to the contrary that $(V, E^{cr} \setminus Q)$ has a $uv$-path $P$. Let $P'$ and $P''$ be the two internally disjoint $uv$-paths in $Q$ where $|P'|$ is odd and $|P''|$ is even. Then one of $P \cup P'$ and $P \cup P''$ is an even cycle, contradicting that $(V, E^{cr})$ has no even cycles.

Finally, we show that no two cycles in $(V, E^{cr})$ are connected by a path. Suppose to the contrary that $(V, E^{cr})$ has a $uv$-path $P$ that connects two distinct cycles $Q_u$ and $Q_v$, see Fig. 3. Let $z$ be defined as in Fig. 3. By the choice of $\epsilon$, each one of the vectors $x+z$ and $x-z$ is non-negative, and they are both in $\Pi$. However, $x = \frac{1}{2}(x+z) + \frac{1}{2}(x-z)$, contradicting that $x$ is an extreme point. \hfill \qed

Note that Lemma 6 implies that extreme points of $\Pi^{Cut}$ have the property given in the lemma. From Lemma 6 we also get:

Corollary 3 In the case of unit costs there exists a polynomial time algorithm that computes $x \in \Pi$ such that the graph $(V, E^{cr})$ of cross edges of positive $x$-value is a forest and such that $C^x_{cin} = C^x_{cin}$, $C^x_{cr} = C^x_{cr}$, and $C^x_{cr} \leq \frac{4}{3} C^{cr}$.

Proof Let $\Pi$ be as in Lemma 6 where $a = C^{cin}$ and $b = C^{cr}$ and let $x$ be an optimal extreme point solution to the LP $\min(\sum_{e \in E} x_e : x \in \Pi)$. Let $Q$ be a cycle of cross-edges and $e$ the minimum $x$-value edge in $Q$. We update $x$ by adding $x_e$ to each of $x_{e^e}, x_{e''}$ and setting $x_e = 0$. The increase in the value of $x$ is at most $\frac{1}{3} \sum_{e \in Q} x_e$, and it is not hard to verify that $x$ remains a feasible solution. In this way we can eliminate all cycles, ending with $x \in \Pi$ as required. \hfill \qed

Remark Corollary 3 holds also for arbitrary costs, but in this case the proof is much more involved. Specifically, we use the following statement, which we do not prove here, since it currently has no application: Let $q \geq 3$ and let $c_i, x_i \geq 0$ be reals, $i = 0, \ldots, q - 1$. Denote $a_i = c_{i-1} - c_i + c_{i+1}$ where the indices are modulo $k$. Then $\sum_{i=0}^{k-1} c_i x_i \geq 3 \cdot \min_{0 \leq i \leq k-1} a_i x_i$.

Let $x$ be as in Corollary 3 and let $x'$ be an $E^{cin}$-up vector of $x$. Note that $x' \in \Pi^{Cut}$, since $x \in \Pi^{Cut}$. We will show how to compute a solution $J$ of size $c(J) \leq x'(E) \leq
inequalities of the form $Ax \geq b$ for every $A$ such that $A$ is a given integral matrix, and let $\Pi_I(b)$ be convex hull of the integral points in $\Pi(b)$. The $0, 1, 1/2$-Chvátal-Gomory cuts (see [4,8,17]) are inequalities of the form $\lambda^T A + \mu^T x \geq \lceil \lambda^T b \rceil$, for vectors $\lambda$, $\mu$ with entries in $\{0, 1/2\}$ such that $\lambda^T A + \mu^T$ is an integral vector.

A matrix $A$ is 2-regular if each of its non-singular square submatrices is half-integral. It is known that $A$ is 2-regular if and only if the extreme points of $\Pi(b)$ are half-integral for any integral vector $b$, and that if $A$ is 2-regular then $P_I(b)$ is described by the $0, 1/2$-Chvátal-Gomory cuts [2]. Thus in matrix terms our Corollary 2 implies the following:

**Corollary 4** In spider-shaped Tree Augmentation instances, the incidence matrix $A$ of the $T$-edges and the paths $\{T_e : e \in E\}$ is 2-regular.

Note that 2-regularity of $A$ does not imply that the corresponding integer program $\min_{x \in \Pi_I(b)} \{c^T x : x \in \Pi_I(b)\}$ is in P, since we have no guarantee that the separation problem for $0, 1/2$-Chvátal-Gomory cuts is in P. However, a particular class of 2-regular matrices has this nice property. A matrix $A$ is a binet matrix if there exists a square non-singular integer matrix $R$ such that $\|z\|_1 \leq 2$ for any column $z$ of $R$ or of $RA$, where $\|z\|_1 = \sum_j |z_j|$ is the $L_1$-norm of $z$. It is known that any binet matrix is 2-regular, but binet matrices have the advantage that the separation problem for $0, 1/2$-Chvátal-Gomory cuts is in P [3]. All in all, we have that if $A$ is binet then the integer program $\min_{x \in \Pi_I(b)} \{c^T x : x \in \Pi_I(b)\}$ can be solved efficiently, by a combinatorial algorithm [3]. The following result, that is stronger than our Corollary 4, was proved by Fiorini, Groß, Könenmann & Sanitá [14] in parallel to our work; for completeness of exposition we provide a proof-sketch.
Lemma 7 ([14]) In spider-shaped Tree Augmentation instances, the incidence matrix $A$ of the $T$-edges and the paths $\{T_e : e \in E\}$ is binet.

Proof For $f \in F$ let $ch(f)$ denote the set of child $T$-edges of $f$ in $T$. Define a square matrix $R \in \{−1, 0, 1\}^{F \times F}$ as follows: $R_{f,g} = 1$ if $g \in ch(f)$, and the other entries of $R$ are 0. Let $z$ be the column in $R$ of $g \in F$. Then $z_g = 1$ and if $g$ has a parent $T$-edge $f$ then $z_f = −1$; other entries of $z$ are 0. Thus $\|z\|_1 \leq 2$. We prove by induction on $|F|$ that $R$ is non-singular. The case $|F| = 1$ is trivial. If $|F| \geq 2$, let $f$ be a leaf $T$-edge. The row of $f$ in $R$ has a unique non-zero entry $R_{f,f} = 1$. Let $T'$ be obtained from $T$ by removing $f$ and the leaf of $f$. The matrix $R'$ that corresponds to $T'$ is obtained from $R$ by removing the row of $f$ and the column of $f$. By the induction hypothesis, det$(R') \neq 0$. Thus $\det(R) = \det(R') \neq 0$, implying that $R$ is non-singular.

We now describe the entries of the matrix $RA$. Let $y$ be the row in $R$ of $f \in F$. Then $y_f = 1$ and $y_g = −1$ for $g \in ch(f)$; other entries of $y$ are 0. Column $e$ in $A$ encodes the path $T_e$, namely, has 1 for each $T_e$-edge; other entries are 0. Thus

$$(RA)_{f,e} = |f \cap T_e| − |ch(f) \cap T_e|. \tag{1}$$

In particular, if $z$ is the column in $RA$ of $e \in E$ then:

- If $f \in T_e$ then $z_f = 1$ if $|ch(f) \cap T_e| = 0$ and $z_f = 0$ otherwise.
- If $f \notin T_e$ then $z_f = −|ch(f) \cap T_e|$.\n
Now let $e = uv$ and let $a$ be the least common ancestor of $u, v$. Consider two cases, in which we indicate only non-zero entries of $z$. If $a \in \{u, v\}$ ($e$ is an up-edge), say $a = v$, then $z_f = 1$ if $f$ is the parent $T$-edge of $u$ and $z_f = −1$ if $a \neq v$ and $f$ is the parent $T$-edge of $v$. If $a \notin \{u, v\}$ then $z_f = 1$ if $f$ is the parent $T$-edge of $u$ or of $v$, and $z_f = −2$ if $f$ is the parent $T$-edge of $a$; however, in a spider-shaped Tree Augmentation instance we cannot have $z_f = −2$, since if $e$ is a cross edge then $a = r$ and thus $a$ has no parent $T$-edge. Consequently, in both cases $\|z\|_1 \leq 2$.\n
By a result of [3] (an integer program $\min \{c^T x : x \in \Pi_1(b)\}$ is in P if $A$ is binet), Lemma 7 immediately implies:

Corollary 5 ([14]) Spider-shaped Tree Augmentation instances admit a polynomial time algorithm.

In [14] it is also provided a direct simple proof that the problem of separating the $\{0, 1/2\}$-Chvátal-Gomory cuts of the CUT LP is in P. Combining this with our Corollary 4 and a result of [2] ($P_1(b)$ is described by the $\{0, 1/2\}$-Chvátal-Gomory cuts if $A$ is 2-regular), also enables to deduce Corollary 5.

Fiorini et al. [14] considered the ODD-CUT $k$-BUNDLE LP obtained by adding to the $k$-BUNDLE LP of [1] the “odd-cuts” (the $\{0, 1/2\}$-Chvátal-Gomory cuts). They showed that this LP is compatible with the decomposition of [1], namely, that if $x$ a feasible solution to this LP, then for any $k$-bundle $B$ the restriction of $x$ to $\psi(B)$ is a feasible solution to this LP on $B$. Since each $k$-branch is a $k$-bundle, the more compact ODD-CUT $k$-BRANCH LP is also compatible with [1] decomposition. As was mentioned in the Introduction, combining the [14] and our paper techniques gives:
For any $1 \leq \lambda \leq k - 1$, Tree Augmentation admits a $4^k \cdot \text{poly}(n)$ time algorithm that computes a solution of cost at most $\frac{3}{2} + \frac{2kM}{k-\lambda M} + \frac{2}{\lambda}$ times the optimal value of the Odd-Cut $k$-Branch LP.

Let us briefly describe the modifications needed for this combined result.

- Lemma 1 is used in the same way as before, namely, just to cover by cost $\frac{2}{\lambda} c^T x$ the $\lambda$-thick edges uncovered by the main algorithm.
- Recall that [14] showed that separating the odd-cuts is in P. The new Lemma 3 would state that given $x^* \in \mathbb{R}^E$, there exists a $4^k \cdot \text{poly}(n)$ time algorithm that either finds a $k$-branch inequality or an odd-cut inequality violated by $x^*$, or returns an integral solution of cost at most $C^{in} + 2C^{cr} + C^r$.
- Lemma 4 will be replaced by a result of [14] that a solution of cost $2C^{in} + C^{cr} + C^r$ can be computed in polynomial time.
- In an improved version of Corollary 1 one gets that if no violated inequality is found then $c(J_S) \leq \sum_{e \in \gamma(S)} c_e x_e + \sum_{e \in \psi(f)} c_e x_e$. And then, the same calculations as after Algorithm 1 give $\frac{c(J_S)}{\Delta(c^T x)} \leq \frac{3}{2} + \frac{2kM}{k-\lambda M}$.

Let us now illustrate another application of Corollary 5.

**Lemma 8** Tree Augmentation admits ratio $3/2$ for trees of diameter $\leq 5$.

**Proof** The case $\text{diam}(T) = 5$ is reduced to the case $\text{diam}(T) \leq 4$ by “guessing” some optimal solution edge that covers the central $T$-edge. So assume that $\text{diam}(T) \leq 4$. Let $r$ be a center of $T$. Let $r$ be a center of $T$. Fix some optimal solution and let $C^{in}$ and $C^{cr}$ denote the fractional cost of in-edges and cross-edges in this solution. As before, apply the following two procedures.

1. Each branch $B$ hanging on $r$ is a tree of diameter $\leq 3$, hence an optimal cover $J_B$ of $B$ can be computed in polynomial time. The union of the edge sets $J_B$ gives a solution of cost at most $C^{in} + 2C^{cr}$.
2. Compute an optimal solution of the spider-shaped instance obtained by removing all non-up in-edges using Lemma 7; the cost of this solution is $2C^{in} + C^{cr}$.

Choosing the better among the two computed solutions gives a solution of cost at most $\min\{C^{in} + 2C^{cr}, 2C^{in} + C^{cr}\}$, while the optimal solution cost is $C^{in} + C^{cr}$. It is easy to see that the approximation ratio is bounded by $3/2$; if $C^{in} \leq C^{cr}$ then $C^{in} + 2C^{cr} \leq \frac{3}{2} (C^{in} + C^{cr})$, while if $C^{in} > C^{cr}$ then $2C^{in} + C^{cr} < \frac{3}{2} (C^{in} + C^{cr})$. \hfill $\square$

Lemma 8 can be used further to obtain ratio $9/5$ for trees of diameter $\leq 7$. As before, we can reduce the case $\text{diam}(T) = 7$ to the case $\text{diam}(T) \leq 6$ by guessing some optimal solution edge that covers the central $T$-edge. We compute a $3/2$-approximate cover of each branch, which gives a solution of cost at most $\frac{3}{2} (C^{in} + 2C^{cr})$. We also compute a solution of cost at most $2C^{in} + C^{cr}$ as before, using Corollary 5. The worse case is when these two bounds are equal, namely, when $C^{in} = 4C^{cr}$. In this case we get that $\frac{2C^{in} + C^{cr}}{C^{in} + C^{cr}} = 1 + \frac{C^{in}}{C^{in} + C^{cr}} = 1 + \frac{4}{4+1} = \frac{9}{5}$. In a similar way, one can further obtain ratio better than 2 when $\text{diam}(T) \leq 9$, and so on, but the ratio approaches 2 when the diameter becomes higher.
We note that the effort in proving Lemma 7 of [14] and our Corollary 4 is roughly the same. However, the result in Lemma 7 of [14] is more general and thus enables to obtain easily results for related problems, as we illustrate below. Note however that our result not only substantially simplifies and reduces the time complexity of algorithms based on the approach of Adjiashvili [1], but also qualitatively extends the range of costs for which a ratio better than 2 can be achieved. Moreover, the proof idea of Corollary 2 might be useful for half-integral network design problems for which the corresponding matrix is not binet.

It is known that if $A$ is binet then also the problem of minimizing $c^T x$ over $\{ x \in \Pi_I(b) : p \leq x \leq q \}$ can be solved in polynomial time for any integer vectors $p$ and $q$ [2,3]. Now consider the following generalization of Tree Augmentation, which we call the Generalized Tree Augmentation problem. Here we are also given demands $\{ b_f : f \in F \}$ on the $T$-edges, and require that at least $b_f$ edges will cover every $T$-edge $f \in F$; we also require that for every edge $e \in E$ at most $q$ copies of $e$ are selected. Then from Lemma 7 of [2,3,14] one can deduce the following (the proof is omitted):

**Corollary 6** Spider-shaped Generalized Tree Augmentation instances admit a polynomial time algorithm.

### 3 Bound on the Integrality Gap of the Cut-LP (Theorem 2)

Let us write the (unit costs) Cut-LP as well as its dual LP explicitly:

$$
\begin{align*}
\min & \quad \sum_{e \in E} x_e \\
\text{s.t.} & \quad \sum_{e \in \psi(f)} x_e \geq 1 \forall f \in F \\
& \quad x_e \geq 0 \forall e \in E
\end{align*}
$$

$$
\begin{align*}
\max & \quad \sum_{f \in F} y_f \\
\text{s.t.} & \quad \sum_{\psi(f) \ni e} y_f \leq 1 \forall e \in E \\
& \quad y_f \geq 0 \forall f \in F
\end{align*}
$$

To prove that the integrality gap of the Cut-LP is at most $28/15$ we will show that a simplified version from [22] of the algorithm of [13] has the desired performance. For the analysis, we will use the dual fitting method. We will show how to construct a (possibly infeasible) dual solution $y \in \mathbb{R}_+^F$, that has the following two properties:

**Property 1** $y$ fully pays for the constructed solution $J$, namely, $|J| \leq \sum_{f \in F} y_f$.

**Property 2** $y$ may violate the dual constraints by a factor of at most $\rho = 28/15$.

From the second property we get that $y/\rho$ is a feasible dual solution, hence by weak duality the value of $y$ is at most $\rho$ times the optimal value of the Cut-LP. Combining with the first property we get that $|J|$ is at most $\rho$ times the optimal value of the Cut-LP.

The algorithm iteratively finds a pair $T', J'$ where $T'$ is a subtree of the current tree and $J'$ covers $T'$, contracts $T'$, and adds $J'$ to $J$. We refer to nodes created by contractions as compound nodes and denote by $C$ the set of non-leaf compound
nodes of the current tree. Non-compound nodes are referred to as original nodes. For technical reasons, the root $r$ is considered as a compound node. Whenever $T'$ contains the root of $T$, the new compound node becomes the root of the new tree.

To identify a pair $T', J'$ as above, the algorithm maintains a matching $M$ on the original leaves. We denote by $U$ the leaves of the current tree unmatched by $M$. A subtree $T'$ of $T$ is $M$-compatible if for any $bb' \in M$ either both $b, b'$ belong to $T'$ or none of $b, b'$ belongs to $T'$; in this case we will also say that a contraction of $T'$ is $M$-compatible. Assuming all compound nodes were created by $M$-compatible contractions, then the following type of contractions is also $M$-compatible.

**Definition 1** (greedy contraction) Adding to the partial solution $J$ an edge $e$ with both endnodes in $U$ and contracting $T_e$ is called a greedy contraction.

Given a complete rooted $M$-compatible subtree $T'$ of $T$ we use the notation:

- $M' = M(T')$ is the set of edges in $M$ with both endnodes in $T'$.
- $U' = U(T')$ is the set of unmatched leaves of $T'$.
- $C' = C(T')$ is the set of non-leaf compound nodes of $T'$.

**Definition 2** (semi-closed tree) Let $T'$ be a complete rooted subtree of $T$. For a subset $A$ of nodes of $T'$ we say that $T'$ of is $A$-closed if there is no edge from $A$ to a node outside $T'$, and $T'$ is $A$-open otherwise. Given a matching $M$ on the leaves of $T$, we say that $T'$ is semi-closed if it is $M$-compatible and $U'$-closed.

The following definition characterizes semi-closed subtrees that we want to avoid. We will say that $T'$ with 3 leaves is of type (i) if it has two nodes with exactly two children each (see the node $w$ and its parent in Fig. 4(i)) and $T'$ is of type (ii) otherwise (see Fig. 4(ii)).

**Definition 3** (dangerous semi-closed tree) A semi-closed subtree $T'$ of $T$ is dangerous if it is as in Fig. 4. Namely, $|M'| = 1$, $|U'| = 1$, $|C'| = 0$, and if $a$ is the leaf of $T'$ unmatched by $M$ then: $T'$ is $a$-closed and there exists an ordering $b, b'$ of the matched leaves of $T'$ such that $ab' \in E$, the contraction of $ab'$ does not create a new leaf, and $T'$ is $b$-open.
**Definition 4** *(twin-edge, stem)* Let \( L \) denote the set of leaves of \( T \). An edge on \( L \) is a **twin-edge** if its contraction results in a new leaf. The least common ancestor of the endnodes of a twin-edge is a **stem**.

In [13] the following is proved:

**Lemma 9** ([13]) Suppose that \( M \) has no twin-edges and that the current tree \( T \) was obtained from the initial tree by sequentially applying a greedy contraction or a semi-closed tree contraction, and that \( T \) has no greedy contraction. Then there exists a polynomial time algorithm that finds a non-dangerous semi-closed subtree \( T' \) of \( T \) and a cover \( J' \) of \( T' \) of size \( |J'| = |M'| + |U'| \).

Let \( L(M) \) denote the set of leaves matched by \( M \). The algorithm is as follows:

**Algorithm 2: ITERATIVE-CONTRACTION** \((T = (V, F), E)\)

1. **initialize:** \( M \leftarrow \) maximal matching on \( L \) among non twin-edges
2. contract every link in \( J \)
3. **while** \( T \) has at least 2 nodes **do**
4. **exhaust greedy contractions**
5. **if** \( T \) has at least 2 nodes **then** for \( T' \), \( J' \) as in Lemma 9 do:
6. return \( J \)

We now describe how to construct \( y \) satisfying Properties 1 and 2 as above. For simplicity of exposition let us use the notation \( y_v \) to denote the dual variable of the parent \( T \)-edge of \( v \). With this notation, Algorithm 3 incorporates into Algorithm 2 the steps of the construction of the dual (possibly infeasible) solution \( y \).

**Algorithm 3: DUAL-CONSTRUCTION** \((T = (V, F), E)\)

1. **initialize:** \( M \leftarrow \) maximal matching on \( L \) among non twin-edges
2. contract every link in \( J \)
3. **while** \( T \) has at least 2 nodes **do**
4. **exhaust greedy contractions**
5. **if** \( T \) has at least 2 nodes then for \( T' \), \( J' \) as in Lemma 9 do:
6. return \( J \)
We now define certain quantities that will help us to prove that at the end of the algorithm \(|J| \leq \sum_{f \in F} y_f\) and that \(y\) violates the dual constraints by a factor of at most 28/15.

**Definition 5** (load of an edge) Given \(y \in \mathbb{R}^F_+\) and an edge \(e \in E\), the **load** \(\sigma(e)\) of \(e\) is the sum of the dual variables in the constraint of \(e\) in the dual LP, namely \(\sigma(e) = \sum_{\psi(f) \ni e} y_f\).

**Definition 6** (credit of a node) Consider a constructed dual solution \(y\) and a node \(c\) of \(T\) during the algorithm, where \(c\) is obtained by contracting the (possibly trivial) subtree \(S\) of \(T\). The **credit** \(\pi(c)\) is defined as follows. Let \(\pi'(c)\) be the sum of the dual variables \(y\) of the edges of \(S\) and the parent edge of \(c\) minus the number of edges used by the algorithm to contract \(S\) into \(c\). Then \(\pi(c) = \pi'(c) + 1\) if \(r \in S\) and \(\pi(c) = \pi'(c)\) otherwise.

Our goal is to prove that at the end of the algorithm \(\sigma(e) \leq 28/15\) for all \(e \in E\), and that the unique node of \(T\) has credit at least 1. For an edge \(e\) that connects nodes \(u, v\) of the current tree \(T\) the **level** \(\ell(e)\) of \(e\) (w.r.t. the current tree \(T\)) is the number of compound nodes and original leaves (of the current tree \(T\)) in \(\{u, v\}\). Clearly, \(\ell(e) \in \{0, 1, 2\}\) and note that if both endnodes of \(e\) lie in the same compound node then \(e\) is a loop and \(\ell(e) = 2\).

**Lemma 10** At the end of step 2 of Algorithm 3, and then at the end of every iteration in the “while” loop, the following holds.

(i) \(\pi(c) \geq 1\) if \(c\) is an unmatched leaf or a compound node of \(T\).

(ii) For any edge \(e\):

\[
\begin{align*}
\bullet \ & \sigma(e) \leq 28/15 \quad \text{if } \ell(e) = 2. \\
\bullet \ & \sigma(e) \leq 16/15 \quad \text{if } \ell(e) = 1. \\
\bullet \ & \sigma(e) = 0 \quad \text{if } \ell(e) = 0.
\end{align*}
\]

**Proof** It is easy to see that the statement holds at the end of step 2, see Fig. 5. We will prove by induction that the statement continues to hold after each contraction step of the while-loop. Let us consider such contraction step that resulted in a new compound node \(c\) and denote by \(\sigma', \ell', \pi'\) the new values of \(\sigma, \ell, \pi\) after the contraction. By the induction hypothesis \(\sigma, \ell, \pi\) satisfy properties (i) and (ii) above, and we prove that \(\sigma', \ell', \pi'\) satisfy (i) and (ii) as well.

For (i) it is sufficient to prove that \(\pi'(c) \geq 1\), as \(\pi' = \pi\) for other nodes. Consider a greedy contraction with an edge \(e\) connecting two unmatched leaves \(u\) and \(v\). By the induction hypothesis, \(\pi(u), \pi(v) \geq 1\). Thus \(\pi'(c) \geq \pi(u) + \pi(v) - 1 \geq 1\). Now suppose that a semi-closed tree \(T'\) was contracted into \(c\). Let \(\Delta(y)\) denote the increase in the value of \(y\) during the contraction step and note that

\[
\pi'(c) \geq \left(\pi(C') + \frac{8}{5}|M'| + |U'|\right) - (|M'| + |U'|) + \Delta(y) \geq |C'| + \frac{3}{5}|M'| + \Delta(y).
\]

If \(|C'| \geq 1\) or \(|M'| \geq 2\) then \(\pi'(c) \geq |C'| + \frac{3}{5}|M'| \geq 1\). If \(|M'| = 0\) (Fig. 6a) then \(\pi'(c) \geq \Delta(y) \geq \frac{1}{2}(|U'| + 1) \geq 1\). If \(|M'| = 1\) then \(\Delta(y) = \frac{2}{5}\), since in all cases in
Figs. 5 and 6, the number of “+” signs is larger by one than the number of “−” signs; thus \( \pi'(c) \geq \frac{3}{5} |M'| + \frac{2}{5} \geq 1 \). In all cases \( \pi'(c) \geq 1 \), as required.

We now show that property (ii) holds. Note that if \( \sigma'(e) = \sigma(e) \) then (ii) continues to hold for \( e \), since contractions can only increase the edge level and since the bounds in (ii) are increasing with the level. Thus we only need to consider the cases when we change the dual variables, namely, when a semi-closed tree \( T' \) was contracted into \( c \); these are the cases given in Figs. 6 and 7.

It is sufficient to consider edges with at least one endnode in \( T' \), as \( \sigma' = \sigma \) and \( \ell' = \ell \) holds for other edges. Let \( e \) be an edge that has an endnode in \( T' \). Let \( q(e) \) denote the number of “+” signs minus the number of “−” signs in Figs. 6 and 7 along the path \( T_e \); we have \( \sigma'(e) - \sigma(e) = \frac{1}{2}q(e) \) in Fig. 6a and \( \sigma'(e) - \sigma(e) = \frac{3}{2}q(e) \) in all other cases. One can verify that \( q(e) \leq 0 \) if \( e \) connects a leaf of \( T' \) to another leaf of \( T' \) or to a node outside \( T' \). Thus it remains to consider the case when \( e \) is incident to a non-leaf node of \( T' \). Then \( \ell'(e) > \ell(e) \), since \( |C'| = 0 \). One can verify that \( q(e) \leq 1 \), except one case – \( q(e) = 2 \) if in Fig. 7a \( e \) connects the leaf \( a \) to a node \( v \) in \( T' \) that is an ancestor of \( w \); this tight case is the one that determined our initial assignment of dual variables. In all cases we have \( \sigma'(e) - \sigma(e) \leq \frac{4}{5} \), which equals the minimum difference \( \frac{28}{15} - \frac{16}{15} \) in the bounds in (ii) due to an increase of an edge level. This concludes the proof of (ii) and of the lemma. □
Fig. 7 Non-dangerous trees with \(|M'| = |U'| = 1| and duals updates in Case 2 of Algorithm 3. Here “+” means increasing the dual variable by 2/5 and “−” means decreasing the dual variable by 2/5. All trees are \(a\)-closed. The trees in \(a, b\) are non-dangerous trees of type (i), and the trees in \(c, d, e\) are non-dangerous trees of type (ii). In (a) the edge \(ab'\) is missing and in (b) \(ab'\) is present and \(T'\) is \(b\)-closed. In (c) both edges \(ab\) and \(ab'\) are present, hence to be non-dangerous the tree must be both \(b'\)-closed and \(b\)-closed. In (d) \(ab'\) is present hence the tree must be \(b\)-closed; the case when \(ab\) present and the tree is \(b'\)-closed is identical. In (e) both \(ab\) and \(ab'\) are missing.

4 Integrality Gap of the 3-BUNCH-LP

The following simple LP-relaxation was suggested by the author several years before [1] and [14]. Let us call an odd size set \(B\) of edges of \(T\) a bunch if no 3 edges in \(B\) lie on the same path in \(T\). Let \(\mathcal{B}\) denote the set of bunches in \(T\). For every \(B \in \mathcal{B}\) at least \(w_B := (|B| + 1)/2\) edges are needed to cover \(B\). The corresponding BUNCH-LP and its dual LP are:

\[
\begin{align*}
\text{min} & \sum_{e \in E} x_e \\
\text{s.t.} & \sum_{e \in \psi(B)} x_e \geq w_B \quad \forall B \in \mathcal{B} \\
& x_e \geq 0 \quad \forall e \in E
\end{align*}
\]

\[
\begin{align*}
\text{max} & \sum_{B \in \mathcal{B}} w_B y_B \\
\text{s.t.} & \sum_{(B) \ni e} y_B \leq 1 \quad \forall e \in E \\
& y_B \geq 0 \quad \forall B \in \mathcal{B}
\end{align*}
\]

A \(k\)-bunch is a bunch of size \(k\). Let \(k\)-BUNCH-LP be the restriction of the BUNCH-LP to bunches of size \(\leq k\). Note that \(1\)-BUNCH-LP is just the CUT-LP, and that Theorem 1 says that the integrality gap of the \(1\)-BUNCH-LP is at most 28/15. We can easily prove a better bound for the \(3\)-BUNCH-LP.

Theorem 3 For unit costs, the integrality gap of the 3-BUNCH-LP is at most 7/4.

Proof We use the same algorithm as before, but define the dual variables differently. In the initialization step we set (see Fig. 8):

- \(y_v \leftarrow 1\) if \(v \in L \setminus L(M \cup J)\)
Fig. 8 Initial duals of the 3-Bunch-LP and the initial loads. The 3-bunch of the edges incident to the stem is shown by a light gray circle.

- $y_v \leftarrow \frac{3}{4}$ if $v \in L(M)$
- $y_v \leftarrow \frac{1}{2}$ if $v \in L(J)$
- $y_B \leftarrow \frac{1}{2}$ if $B$ is the 3-bunch of a stem of an edge in $J$

In the updates of the dual variables in Figs. 6 and 7, “+” and “−” means increasing and decreasing the dual variable by $\frac{1}{2}$, respectively, with one exception: in Fig. 7(a) the updates are $y_b \leftarrow y_b - \frac{1}{2}$ and $y_B \leftarrow \frac{1}{2}$, where $B$ is the 3-bunch formed by the parent $T$-edges of $a$, $b$, $w$. Similarly to Lemma 10 we prove that after step 2 the following holds:

(i) $\pi(c) \geq 1$ if $c$ is an unmatched leaf or a compound node of $T$.
(ii) For any edge $e$: $\sigma(e) \leq \frac{7}{4}$ if $\ell(e) = 2$, $\sigma(e) \leq 1$ if $\ell(e) = 1$, and $\sigma(e) = 0$ if $\ell(e) = 0$.

It is easy to see that the statement holds at the end of step 2, see Fig. 8; note that after step 2 the edge with load $\frac{5}{4}$ has level 2. As in Lemma 10 we continue by induction while using the same notation, but focus only on the arguments that are different from the ones in Lemma 10.

Suppose that a semi-closed tree $T'$ was contracted into a compound node $c$. Then

$$\pi'(c) \geq \left( \pi(C') + \frac{3}{2}|M'| + |U'| \right) - |M'| + |U'| + \Delta(y) \geq |C'| + \frac{1}{2}|M'| + \Delta(y).$$

If $|C'| \geq 1$ or $|M'| \geq 2$ then $\pi'(c) \geq |C'| + \frac{1}{2}|M'| \geq 1$. If $|M'| = 0$ (Fig. 6a) then $\pi'(c) \geq \Delta(y) \geq \frac{1}{2}(|U'| + 1) \geq 1$. If $|M'| = 1$ then $\Delta(y) = \frac{1}{2}$ and thus $\pi'(c) \geq \frac{1}{2}|M'| + \Delta(y) \geq 1$; this is since in each one of the cases in Figs. 6b and 7b, $c$, $d$, $e$ the number of “+” signs is larger by one than the number of “−” signs, while in the case in Fig. 7a we gain $2 \cdot \frac{1}{2} = 1$ when increasing by $\frac{1}{2}$ the dual variable of a 3-bunch, and loose just $\frac{1}{2}$ by decreasing $y_b$ by $\frac{1}{2}$. In all cases we have $\pi'(c) \geq 1$, as required.

We now show that property (ii) holds. Consider a semi-closed tree $T'$ was contracted into $c$ and an edge $e$ with at least one endnode in $T'$. Note that now $\frac{3}{4}$ is the minimum difference in the bounds in (ii) due to an increase of an edge level.

Let us consider the case in Fig. 7a. If $e$ is incident to $b$ or if $e = vb'$ for some $v \in T'$ then $\sigma'(e) \leq \sigma(e)$. In all the other cases we have $\ell'(e) > \ell(e)$ and $\sigma'(e) - \sigma(e) \leq \frac{1}{2} < \frac{3}{4}$. Hence the induction step holds in this case.
For the other cases, as before, let \( q(e) \) denote the number of “+” signs minus the number of “−” signs in Figs. 6 and 7b, c, d, e along the path \( T_e \); we have \( \sigma'(e) - \sigma(e) = \frac{1}{2}q(e) \) in all cases. One can verify that \( q(e) \leq 0 \) if \( e \) connects a leaf of \( T' \) to another leaf of \( T' \) or to a node outside \( T' \). If \( e \) is incident to a non-leaf node of \( T' \) then \( \ell'(e) > \ell(e) \) and \( q(e) \leq 1 \), which implies \( \sigma'(e) - \sigma(e) \leq \frac{1}{2} \). This concludes the proof of (ii) and of the lemma.

5 Conclusions

In this paper we presented an improved algorithm for Tree Augmentation, based on the idea of Adjiashvili [1]. A minor improvement is that the algorithm is simpler, as it avoids a technical discussion on so called “early compound nodes”, see [1] and [14]. A more important improvement is in the running time—\( 4^k \text{poly}(n) \) instead of \( n^{kO(1)} \), where \( k = \Theta(M/\epsilon^2) \). This allows ratio better than 2 also for logarithmic costs, and not only costs bounded by a constant. These two improvements are based, among others, on a more compact and simpler LP for the problem. Another important improvement is in the ratio—\( \frac{12}{7} + \epsilon \) instead of \( 1.96418 + \epsilon \) in [1]. This algorithm is based on a combinatorial result for spider-shaped Tree Augmentation instances. We showed that for spider-shaped instances, the extreme points of the Cut-Polyhedron are half-integral, and thus Tree Augmentation on such instances can be approximated within 4/3. As was mentioned, a related recent result of [14] shows that for spider-shaped instances, augmenting the Cut-LP by \{0, 1/2\}-Chvátal-Gomory Cuts gives an integral polyhedron and that such instances can be solved optimally in polynomial time. Overall we get that spider-shaped instances behave as “star-instances”—when \( T \) is a star (this is essentially the Edge-Cover problem): the extreme points of the Cut-LP are half-integral, while augmenting it by \{0, 1/2\}-Chvátal-Gomory Cuts gives an integral polyhedron. The description of the \{0, 1/2\}-Chvátal-Gomory Cuts in [14] is somewhat complicated, and a natural question is whether using the simpler Bunch-LP gives the same result. This is so when \( T \) is a star, c.f. [28] where an equivalent Edge-Cover problem is considered.

Our second main result is that in the case of unit costs the integrality gap of the Cut-LP is less than 2, which resolves a long standing open problem. Our goal here was just to present the simplest verifiable proof for this fact, and we believe that our bound \( 2 - 2/15 \) can be improved by a slightly more complex algorithm and analysis. As was mentioned, several LP and SDP relaxations, more complex than the Cut-LP, were shown to have integrality gap less than 2 for particular cases (e.g., the \( k \)-Branch-LP with logarithmic costs). The hope was that this may lead to ratio better than 2 for the general case. Our result suggests that already the simplest Cut-LP, combined with the dual fitting method, may be the right one to study to achieve this goal. More complex LP’s (e.g., the Bunch-LP or the Odd-Cut LP) may be used to improve the ratio.

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