Vibration Analysis of Frame Structures Using Wavelet Finite Elements

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Abstract. The recently developed Wavelet Finite Element Method (WFEM) involves combining the versatile wavelet analysis with conventional Finite Element Method (FEM) by utilizing the wavelet and scaling functions as interpolating functions, providing an alternative to the conventional polynomial interpolation functions used in FEM. In this paper, the B-Spline Wavelet on bounded Interval (BSWI) wavelets are utilized in the construction of the wavelet based finite elements for the dynamic analysis of a Vierendeel frame structure subjected to a moving load. The numerical results for these WFEM formulations are compared with the classical Euler Bernoulli elements.

1. Introduction

Moving load dynamic problems have been studied in the last decades with various forms of interactions and varying loads as outlined by Ouyang [1]. The majority of the simple moving load problems may be analytically solved as described in Fryba’s monograph [2] and Yang et al. [3]. However, in the analysis of complicated moving load problems, particularly those with significant variations in the structural and loading properties as well as boundary conditions, conventional FEM is deemed to be the preferred solution method to obtain their approximate solutions. The use of this numerical analysis technique for problems with regions of the solution domain where the gradient of the field variables vary suddenly or rapidly, encounters difficulties. For this reason, research has been carried out in recent years by structural analysts to overcome these limitations and find more efficient and accurate tools to analyse these structural systems, particularly in moving load problems. Initially used mainly by mathematicians as a decomposition tool for data functions and operators, the application of wavelet analysis has vastly grown in various disciplines at an exponential rate due to the varying properties of the increasingly different wavelet families. Daubechies [4] and Chui [5] monographs outline detailed descriptions and formulations of general wavelets. In recent years, work has been carried out exploring the fusion of conventional FEM and wavelet analysis to obtain an approximate solution for various engineering problems, widely referred nowadays as the wavelet finite element method (WFEM). These wavelet based elements are constructed using the scaling functions of the wavelet families as the interpolating functions, contrary to the polynomial interpolating functions used in conventional FEM. Minimum compact support of wavelets is a valuable wavelet property in the formulation of the WFEM that leads to fewer degrees of freedom (DOFs). Another key property, multiresolution analysis (MRA) which stems from the “two-scale” relation, provides the opportunity for the analysis of systems globally and most importantly, locally, at different scales without alterations to the original modelling. An individual element or group of elements can be analysed at finer scales to obtain better accuracy once the meshing of the system has been carried out. Furthermore, MRA helps ensure the convergence condition is met. However, the properties of different wavelet families vary, and therefore the decision on which family, order and scale is the most adequate, becomes paramount to its application.
Ko et al [6] were among the first to propose the WFEM when they applied the Daubechies wavelet family to construct a wavelet based finite element (WFE) and analyse one dimensional and two dimensional Neumann problems. The development of orthogonal wavelets such as the Daubechies wavelet family, led to the development of a new finite element strategy, via Galerkin formulations. Zhou et al [7] applied the method to the bending of beams and plates. Ma et al [8] and Chen et al [9,10] progressed this research by using the Daubechies WFE to prove the method’s ability to solve static structural problems. Due to their minimum compact support, orthogonality and MRA properties, the Daubechies wavelets were initially preferred in the formulation of wavelet based finite elements (WFEs). However, their numerical implementation is challenging because of lacking a closed form, and because of the possibility of the orthogonality loss, when used in weak variational formulations for general boundary value problems. To overcome these problems, specialized numerical techniques are required [11], which increase the computational costs but maintain the approximation accuracy and orthogonality properties in the analysis. BSWI family of wavelets constructed by Chui and Quak [12], have also been used in some WFEM formulations [13-14]. This family of wavelets has, in addition to the multiresolution and compact support properties, explicit expressions and therefore there is no need to carry out additional calculations to obtain the integral of the products of the scaling function and/or their derivatives.

In this paper the WFEM is formulated using BSWI. The WFEs are used in the analysis of the dynamic response of a Vierendeel frame structure subjected to a moving load. The results are compared with analytical formulations and the conventional Euler Bernoulli beam elements in order to highlight the efficiency and general properties of such formulations. Section 2 of this paper highlights the general wavelet and MRA properties with a brief overview of BSWI formulations. Section 3 illustrates the wavelet Galerkin finite element formulations. Finally in section 4, simulated numerical examples for the dynamic response of a Vierendeel frame structure under a moving load are carried out to highlight the formulation efficiency as well as the difficulties encountered in the numerical implementation, followed by conclusions.

2. Wavelet Analysis
A brief overview of wavelet analysis and multiresolution analysis as well as their properties is carried out in this section. Multiresolution, representing an advantageous property of wavelets, refers to the simultaneous appearance of multiple scales in function decompositions in the Hilbert space \( L^2(\mathbb{R}) \) using a sequence of closed subspaces \( V_j \) which is represented mathematically as [5]:

\[
\ldots \subset V_{-2} \subset V_{-1} \subset V_0 \subset \ldots \subset V_j \subset V_{j+1} \subset \ldots
\]

These subspaces satisfy the following conditions:

1. \( V_j \subset V_{j+1} \)
2. \( \bigcap V_j = \{0\} \) and \( \bigcup V_j = L^2 \) (Completeness).

By defining an orthogonal complement \( W_j \) to the subspace \( V_j \), the next subspace \( V_{j+1} \) is completed

\[
V_j \oplus W_j = V_{j+1}
\]

and the space \( L^2(\mathbb{R}) \) can be represented as a direct sum:

\[
L^2(\mathbb{R}) = \bigoplus W_j
\]

2.1. B-Spline Wavelets on the Interval \([0,1]\)
Given a knot vector, \( t^j \) comprising of a sequence of knots \( t_i^j \) on \([0,1]\), via joining the piecewise polynomials between the knots, one is able to construct B-Splines at a scale \( j \in \mathbb{N}_0[12,15] \).
The \( m \)th order B-Splines are defined as:

\[
t_i^m := \{t_i^j\}
\]

The \( m \)th order B-Splines are defined as:

\[
B_m^j(x) := \{t_i^j + m - t_i^j\}[t_i^j, t_i^{j+1}, ..., t_i^{j+m}]_t(t - x)^{m-1}
\]

Where \([t_i^j, t_i^{j+1}, ..., t_i^{j+m}]_t\), is the \( m \)th divided difference of the truncated power function \((t - x)^{m-1}\) with respect to variable \( t \).

\[
\psi_{m,i}^j(x) = \psi_m(2^j x - k) = \frac{1}{2^{m-1}} \sum_{k=0}^{2^m-2} (-1)^k N_{2m}(k + 1) B_m^{j+1}
\]

\[
supp\psi_{m,i}^j = \left[\frac{i}{2^j}, \frac{i + 2m - 1}{2^j}\right]
\]

With values of \( 0 \leq i \leq 2^j - 2m + 1 \), \( supp \psi_{m,i}^j \subseteq [0,1] \). In order to have one inner wavelet on the interval [0,1], the following condition must be met [15].

\[
2^{j_0} \geq 2m - 1
\]

Where \( j_0 \) is the smallest scale \( j \) of order \( m \) that will ensure that there is at least one inner wavelet on the interval [0,1]. Thus, the scaling functions, \( \phi_{m,k}^j(x) \), and corresponding semi-orthogonal wavelet functions, \( \psi_{m,k}^j(x) \) of BSWI of order \( m \) at scale \( j \geq j_0 \) (BSWI\( mj \)) are evaluated as follows:

3. Wavelet Galerkin Finite Element Formulation

For a wavelet finite element formulation, the scaling functions specific for each wavelet formulation are used to construct the element displacement interpolation functions, instead of the conventional polynomial interpolating functions, with the variable field approximated using the Riesz basis of the fundamental space \( V_0 \):

\[
\tilde{\omega}(\xi) = \sum a_{mk}^j \phi_{m,k}^j(\xi)
\]

With \( \{a\} \) composed of projection coefficients in the wavelet space \( a_k \) and \( [N] \) the shape functions matrix composed of the wavelet basis functions. For the differential operator \( L \) associated with a specific analysis, it follows that:

\[
[B] = L[N]
\]

In the case of structural dynamics, using equation (12), the elemental stiffness, mass matrices and force vector are given as

\[
\mathbb{K}_e = \int_0^1 [B]^T [B] \, d\xi, \quad \mathbb{M}_e = \int_0^1 [N]^T [N] \, d\xi, \quad \mathbb{F}_e = \int_0^1 q(x) \, [N]^T \, d\xi
\]

The matrices and the load vector are in wavelet space and thus \( a_k \) are the corresponding DOF in this wavelet space. It is necessary to transform the vector and matrices into physical space so as to satisfy the boundary conditions of the beam as well as the connectivity of the neighbouring elements that are
linked by the respective nodes. Thus a transformation matrix $T$ is calculated from the scaling functions from equations (10) and (12). The DOFs transformation expression is given as:

$$w = [T] \alpha$$

(15)

Where the vector $w = [u_1 \ w_1 \ \theta_1 \ u_2 \ w_2 \ \theta_2 \ \ldots \ u_{n_{et} + 1} \ w_{n_{et} + 1} \ \theta_{n_{et} + 1}]^T$, consists of axial deformations and vertical displacements along the element and two rotations at the element ends to ensure effective connection between adjacent elements. Subsequent, the transformation from the wavelet space into the physical space for the stiffness and mass matrices as well as load vectors are given by:

$$K_e = a [T^{-1}]^T \bar{K}_e [T^{-1}]$$

$$M_e = \rho c L [T^{-1}]^T \bar{M}_e [T^{-1}]$$

$$R_e = [T^{-1}]^T \bar{R}_e$$

(16)

For a rod element, $a = E A L^{-1}$ and for a beam element $a = E I L^{-3}$. $E$ – Young’s Modulus, $I$ – moment of inertia, $A$ – cross sectional area and $L$ – Length. Equation (16) is formulated in local coordinates and therefore, transformation into global coordinates is carried out thereafter. Using equation (16), the system’s dynamics for a moving load problem is then described by:

$$M \ddot{q}(t) + C \dot{q}(t) + Kq(t) = R \delta(x - ct)$$

(17)

where $\dot{q}, \ddot{q}$ and $q$ are the accelerations, velocities and displacements at the nodal degrees of freedom in the physical space, $M$, $C$, $K$ are the mass, damping and stiffness matrices and $R$ represents the load vector.

4. Numerical Analysis

The Vierendeel frame as illustrated in figure 1, is subjected to a moving point load of magnitude $P = 2000$N moving along elements 1 - 6. The normalised deflection at point A is presented in addition to the first 10 angular frequencies. The material properties of the beams are $E = 207 \text{ Nm}^{-2}$, $A = 8.06 \times 10^{-5} \text{ m}^2$, $\rho = 7.81 \times 10^3 \text{ kg m}^{-3}$, $I = 2.71 \times 10^{-10} \text{ m}^4$ and $L = 0.305 \text{ m}$. The load travels at a velocity $c \text{ ms}^{-1}$ corresponding to the speed parameter $\alpha$ of values $10^{-5}$, 1 and 2 where $c = 6aL\omega_1/\pi (\omega_1 \text{ – First angular frequency})$. The absolute value deflection $\delta_0$ at point A is obtained from static loading of the frame at the same point A with load $P$. The results using 8 BSWI WFEs at scale $j = 3$ to model the Vierendeel frame are compared with a similar analysis using 48 Euler Bernoulli elements.

![Figure 1. Layout of Vierendeel frame structure.](image)

![Figure 2. Deformation, $\delta_0$ of Vierendeel frame subjected to static load $P$ applied at Point A.](image)
Results from figure 2 indicate that the deformation of the frame under a static load $P$ using 8 BSWI WFEs gives accurate results, similar to 48 conventional finite elements. This is further validated in table 1, where the first 10 angular frequencies of the frame are compared with that from [16].

| Mode No. | Jara-Almonte’s Exact Element [16] | 48 Euler Bernoulli Elements | 8 BSWI WFEs |
|----------|-----------------------------------|-----------------------------|-------------|
| 1        | 107                               | 107                         | 107         |
| 2        | 377                               | 337                         | 377         |
| 3        | 397                               | 397                         | 397         |
| 4        | 476                               | 475                         | 475         |
| 5        | 1099                              | 1098                        | 1098        |
| 6        | 1316                              | 1315                        | 1315        |
| 7        | 1504                              | 1503                        | 1503        |
| 8        | 1912                              | 1910                        | 1910        |
| 9        | 2061                              | 2060                        | 2060        |
| 10       | 2248                              | 2246                        | 2246        |

Defining $\delta$ as the deflection at Point A as the load $P$ moves across the beam at $c$ ms$^{-1}$, the plots of the normalised dynamic deflection $\delta / \delta_0$ are plotted in figures 3, 4 and 5 for values of $\alpha = 10^{-5}$, 1 and 2 respectively. The plots show that using 8 BSWI elements gives similar results to 48 Euler Bernoulli finite elements. Using the WFEM requires significantly fewer elements in comparison to the conventional FEM with high levels of accuracy.

Figure 3. Dynamic Deflection $\delta / \delta_0$ at point A as load moves over the frame when $\alpha = 10^{-5}$

Figure 4. Dynamic Deflection $\delta / \delta_0$ at point A as load moves over the frame when $\alpha = 1$

Conclusion

The use of B-Spline Wavelets on bounded Interval (BSWI) in the formulation of wavelet finite elements (WFEs) and its application to vibration analysis of frame structures subjected to a moving load has been highlighted in this paper. The numerical example of the Vierendeel frame structure subjected to a moving load illustrates high levels of accuracy attained using WFEM with fewer elements in comparison to conventional FEM. The natural frequencies of the frame structure obtained using WFEM, particularly at higher modes, are also of better accuracy than the conventional FEM approach. However, the kind, order and scale of the wavelet family is dependent on the application and level of accuracy required, therefore an increase in either scale or order leads to an increase in computational complexity. Furthermore, with the property of multiresolution, WFEM allows for an
individual element or group of elements to be analysed at finer scales to obtain better accuracy in regions of the solution domain where the gradient of the field variables vary suddenly or rapidly. Hence, the potential of WFEM for more complex moving load problems.

![Figure 5. Dynamic Deflection $\delta / \delta_0$ at point A as load moves over the frame when $\alpha = 2$.]

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