Maximum principle for stable operators

Florian Grube | Thorben Hensiek

Faculty of Mathematics, Bielefeld University, Bielefeld, Germany

Correspondence
Thorben Hensiek, Faculty of Mathematics, Bielefeld University, Bielefeld, Germany. Email: thensiek@math.uni-bielefeld.de

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Abstract
We prove a weak maximum principle for nonlocal symmetric stable operators including the fractional Laplacian. The main focus of this work is on minimal regularity assumptions of the functions under consideration.

KEYWORDS
maximum principle, nonlocal symmetric stable operator

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1 | INTRODUCTION

The study of maximum principles for harmonic functions can be traced back to the works [26] by Gauss and [41] by Riemann. They are a key tool in the theory of existence, particularly uniqueness, and regularity of solutions to linear second-order elliptic equations. Let Ω be a sufficiently regular, bounded domain. For the Laplacian, the following maximum principle is well known. If \( u \in L^1(\Omega) \) is distributionally subharmonic, that is,

\[
(u, -\Delta \eta)_{L^2(\Omega)} \leq 0 \quad \text{for all } \eta \in C^\infty_c(\Omega), \eta \geq 0,
\]

and satisfies the boundary assumption

\[
\lim_{\varepsilon \to 0^+} \varepsilon^{-1} \int_{\{x \in \Omega \mid \text{dist}(x, \partial \Omega) < \varepsilon\}} u^+(x) \, dx = 0,
\]

then \( u \leq 0 \) in \( \Omega \), see [40, Proposition 6.1, Proposition 20.2] by Ponce. Here, \( u^+ = \max\{u, 0\} \). The condition Equation (2) yields \( \int_{\partial \Omega} u^+ \, d\sigma = 0 \) for continuous functions \( u \in C(\overline{\Omega}) \), that is, \( u \leq 0 \) on \( \partial \Omega \). The goal of this paper is to prove a nonlocal version of this maximum principle.

In recent years, there has been an intense study of nonlocal operators. The most prominent and well-studied example is the fractional Laplacian. It is defined for \( s \in (0, 1) \) as

\[
(-\Delta)^s u(x) := \text{p.v.} \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x - y|^{d+2s}} \, dy.
\]
By the symmetry of the fractional Laplacian, we can define the operator for functions in \( L^1(\mathbb{R}^d, (1 + |x|)^{-d-2s} \, dx) \) via
\[
\langle (-\Delta)^s u, \eta \rangle := (u, (-\Delta)^s \eta)_{L^2(\mathbb{R}^d)} \text{ for } \eta \in C_c^\infty(\mathbb{R}^d).
\]
The space \( L^1(\mathbb{R}^d, (1 + |x|)^{-d-2s} \, dx) \) captures the decay of the Lévy measure \( \nu(dx) = |x|^{-d-2s} \, dx \) at infinity. It is typically called the tail space, for example, see the book [6] by Bogdan et al. and Bogdan and Byczkowski [4, Definition 3.1]. For the fractional Laplacian, our result, see Theorem 1.1, reads as follows. If \( u \in L^1(\mathbb{R}^d, (1 + |x|)^{-d-2s} \, dx) \) satisfies
\[
(u, (-\Delta)^s \eta)_{L^2(\mathbb{R}^d)} \leq 0 \text{ for all } \eta \in C_c^\infty(\Omega),
\]
then \( u \leq 0 \) a.e. in \( \Omega \). Instead of Equation (3) we could assume the stronger but more accessible assumption \( u^+ \in L^1(\Omega, \text{dist}(x, \partial\Omega)^{-s} \, dx) \).

### 1.1 Related literature

The following works contain weak maximum principles. Silvestre proved in [49, Proposition 2.17] the following: If \( u \in L^1(\mathbb{R}^d, (1 + |x|)^{-d-2s} \, dx) \) is upper-semicontinuous on \( \Omega \) and satisfies
\[
(u, (-\Delta)^s \eta)_{L^2(\mathbb{R}^d)} \leq 0 \text{ for all nonnegative } \eta \in C_c^\infty(\Omega),
\]
then \( u \leq 0 \) in \( \Omega \). We want to emphasize that the function \( u \) needs to be upper-semicontinuous up to the boundary of \( \Omega \). This result is analogous to the case of the Laplacian. It is well known that if \( u \in L^1_{\text{loc}}(\Omega) \) satisfies
\[
u \text{ is upper-semicontinuous in } \Omega \text{ and } u(x) \leq \frac{1}{|B(x)|} \int_{B(x)} u(y) \, dy \text{ for all balls } B(x) \subset \subset \Omega,
\]
then \( u \leq 0 \) in \( \Omega \). Note that the condition Equation (5) is equivalent to \( u \) being distributionally subharmonic. For both statements, we refer to the book [16, Chapter 27] by Donoghue. In [9, Theorem 5.2], Caffarelli and Silvestre extended this result to a larger class of operators. The condition Equation (3) is less restrictive than being upper-semicontinuous on \( \Omega \) since upper-semicontinuous functions attain their maximum on compact sets.

A nonlocal Green–Gauß formula motivates the following bilinear form associated to the fractional Laplacian. For \( u \in C^2_b(\mathbb{R}^d) \) and \( v \in C^1_c(\Omega) \),
\[
\int_\Omega (-\Delta)^s u(x)v(x) \, dx = \frac{1}{2} \iint_{(\Omega^c \times \Omega^c)^c} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+2s}} \, dy \, dx =: E_{(-\Delta)^s}(u, v).
\]
In this setup, the following weak maximum principle holds. Let \( u : \mathbb{R}^d \to \mathbb{R} \) be energy finite w.r.t. \( (-\Delta)^s \) in \( \Omega \), that is, it satisfies \( E_{(-\Delta)^s}(u, u) < \infty \). If \( u \leq 0 \) a.e. on \( \Omega^c \) and \( u \) is a weak subsolution in \( \Omega \), that is, \( E_{(-\Delta)^s}(u, v) \leq 0 \) for all \( v : \mathbb{R}^d \to [0, \infty) \) with \( E_{(-\Delta)^s}(v, v) < \infty \) and \( v = 0 \) a.e. on \( \Omega^c \), then \( u \leq 0 \) a.e.. This statement is a direct consequence of choosing \( v = u^+ = \max[u, 0] \), see Servadei and Valdinoci [48, Lemma 6]. The idea is often used in the proof of maximum principles for second-order elliptic operators, see Gilbarg and Trudinger [27]. This approach has been applied to a larger class of nonlocal operators, including those with nonsymmetric kernels, by Felsinger, Kassmann, and Voigt in [22, Theorem 4.1]. The assumption \( E_{(-\Delta)^s}(u, u) < \infty \) is stronger than Equation (3) as it requires regularity of \( u \) inside of \( \Omega \) and across the
boundary. For a detailed comparison to our result, we refer to Corollary 1.5 and Remark 1.6. Small-volume weak maximum principles, see Theorem 1.3 and Theorem 2.4, and a weak maximum principle on open, bounded sets, see Proposition 2.3, for a class of Schrödinger-type operators of the form $L + q$ are contained in the work [31] by Jarohs and Weth. In this setting, $Lu(x) := \text{p.v.} \int (u(x) - u(y))j(x - y) \, dy$ is a nonlocal operator with a symmetric kernel $j$. The authors formulated optimal conditions on the kernel $j$ and the function $q$ such that a weak maximum principle for weak subsolutions holds. Feulefack and Jarohs extended these results to integro-differential operators $L + q$ with kernels $j$ of small order and open sets $\Omega$ with Lipschitz boundary, which are not necessarily bounded, in their work [23, Proposition 1.2]. Their space of solutions is larger than that in [31] while still requiring their variational solutions to be energy finite in $\Omega$. In [24, Proposition 5.5], Feulefack, Jarohs, and Weth proved a modification of the small-volume weak maximum principle [31, Theorem 2.4] for unbounded domains. Chen and Weth studied the logarithmic Laplacian, related maximum principles, and necessary conditions on the domain, see [10, Theorem 1.8, Corollary 1.9, Corollary 1.10]. In [30, Proposition 3.5], Jarohs and Weth proved a weak maximum principle for antisymmetric, weak supersolutions in an open bounded set $U \subset H$ for operators $L + q$. Here, $L$ is a nonlocal operator and $H$ is an affine half space. The notion of antisymmetry is with respect to the half space $H$. They assumed mild symmetry and monotonicity assumptions on $L$. Chen, Li, and Li extended this result in the case of the fractional Laplacian to unbounded domains $U \subset H$ in [11]. Abatangelo proved a weak maximum principle for the fractional Laplacian in [1, Lemma 3.9]. He considered boundary blow-up solutions $(-\Delta)^s u = f$ in $\Omega$, $u(x) \to \infty$ as $\Omega \ni x \to z \in \partial \Omega$ and introduced a solution concept that can deal with the boundary blow-up and the prescribed data outside $\overline{\Omega}$, see [1, Definition 1.3]. In particular, he tests the equation with functions from the preimage of $C_c^\infty(\Omega)$ under the Green operator over $\Omega$. In his setup, the weak maximum principle [1, Lemma 3.9] follows directly from the choice of test functions. The question of regularity assumptions on a distributional subsolution such that the maximum principle holds is not discussed in his work.

The following articles are closely related to our work. For the fractional Laplacian and the domain $\Omega = B_1(0)$, Li and Lü proved a weak maximum principle for distributional subsolutions in [35, Theorem 1.6]. The proof relies on the explicit representation of the Poisson kernel on a ball. The authors assumed $u \in L^{1/(1-s)}(B_1(0)) \cap L^1(\mathbb{R}^d, (1 + |x|)^{-d-2s} \, dx)$. The assumptions on $u$ in Theorem 1.1 are weaker. If $u \in L^{1/(1-s)}(B_1(0))$, then the Hölder inequality yields

$$\varepsilon^{-s} \int_{B_1 \setminus B_{1-\varepsilon}} u^s(x) \, dx \leq \varepsilon^{-s} \|u\|_{L^{1/(1-s)}(B_1 \setminus B_{1-\varepsilon})} (|B_1 \setminus B_{1-\varepsilon}|)^{\frac{s}{d}} \leq c \|u\|_{L^{1/(1-s)}(B_1 \setminus B_{1-\varepsilon})} \to 0 \text{ as } \varepsilon \to 0.$$ 

The article [37] by Liu and Zhuo contains our Theorem 1.1 in the special case of the fractional Laplacian. They also assumed the boundary condition Equation (16) on the function $u$. Note that their proof is rather different from ours since they use explicit upper Poisson kernel estimates. This is similar to the proof in [35, Theorem 1.6] and [34, Theorem 3.1] by Li and Liu. In contrast, our result is true for the larger class of stable, nondegenerate operators. Furthermore, our proof directly connects available boundary regularity of solutions to minimal assumptions for guaranteeing a maximum principle.

Now, we discuss results on strong maximum principles. Some of the aforementioned works for weak maximum principles contain results on strong maximum principles in the respective settings. We give a list without repeating the details. These are [31, Theorem 1.2, Theorem 1.2, Theorem 2.5, Theorem 2.6] by Jarohs and Weth, [10, Proposition 4.1] by Chen and Weth, [23, Proposition 1.5] by Jarohs and Feulefack, [30, Proposition 3.6] by Jarohs and Weth, [1, Theorem 2.3] by Chen, Li, and Li, as well as [35, Theorem 1.8, Theorem 1.9] by Li and Lü. Furthermore, Cabré and Sire proved a strong maximum principle for the fractional Laplacian, using a representation as a Dirichlet-to-Neumann map, see [8, Section 4.6]. In [5], Bogdan and Byczkowski used probabilistic methods to prove a strong maximum principle for supersolutions related to the Schrödinger operator $(-\Delta)^s + q$.

Lastly, we want to mention the work [2] by Abatangelo, Jarohs, and Saldaña in which the authors discussed the failure of maximum principles for higher order fractional Laplacians.

In this paper, we consider generators of symmetric, stable Lévy processes. These processes play a key role in the Generalized Central Limit Theorem, for example, see the book [46] by Samorodnitsky and Taqqu. Let $\sigma \in (0, 1)$, $(X_t)$ a symmetric, $2\sigma$-stable process, that is, for all $t > 0$

$$X_1 \overset{d}{=} t \frac{1}{2\sigma} X_t.$$
There exists a nonnegative, finite measure $\mu$ on the unit sphere $S^{d-1}$ such that its generator is $-A_s$, where

$$A_s u(x) := \text{p.v.} \int_{\mathbb{R}^d} (u(x) - u(x + h)) \nu(dh) \tag{7}$$

with the Lévy measure given in polar coordinates via

$$\nu(U) := \int_{\mathbb{R}} \int_{S^{d-1}} 1_{U}(r\theta) |r|^{-1-2s} \mu(d\theta) dr, \quad U \in \mathcal{B}(\mathbb{R}^d). \tag{8}$$

We integrate the radii over the whole real line $\mathbb{R}$ to ensure the symmetry of the measure $\nu$, that is, $\nu(-A) = \nu(A)$ for all measurable sets $A \subset \mathbb{R}$. This relation was established by Lévy in [33] and Khintchine in [32]. See also [47] by Sato for a proof. Additionally, we assume the nondegeneracy condition

$$0 < \inf_{\omega \in S^{d-1}} \int_{S^{d-1}} |\omega \cdot \theta|^{2s} \mu(d\theta). \tag{9}$$

The condition Equation (9) is satisfied as soon as $\mu$ is not supported on a hyperplane. It is rooted in the work [39] of Picard as an ellipticity assumption on the Lévy measure $\nu$. Another motivation to study operators like Equation (7) is Courrège’s theorem, which characterizes the operators satisfying a maximum principle, see the work [13, Theorem 1.4] by Courrège. We emphasize two examples in this class of operators. If we pick $\mu$ as a uniform distribution on the sphere, then the resulting operator is the aforementioned fractional Laplacian $(-\Delta)^s$, up to a constant. Another example is $\sum_{j=1}^{d} (-\Delta_j)^s$, which is a sum of one-dimensional fractional Laplacians in all coordinate directions. It is the generator of the process $(X_1^t, \ldots, X_d^t)$ where $X_i^t$ are independent, one-dimensional, symmetric, $2s$-stable processes. In this case, $\mu$ is a sum of Dirac measures $\delta_{e_i}$, where $e_i$ are basis vectors of $\mathbb{R}^d$.

The aim of this paper is to find the minimal regularity of a function $u : \mathbb{R}^d \to \mathbb{R}$ such that the maximum principle

$$A_s u \leq 0 \text{ in } \Omega, \ u \leq 0 \text{ on } \Omega^c \Rightarrow u \leq 0 \text{ in } \Omega \tag{10}$$

holds. We study the operator distributionally. Thereby,

$$\langle A_s u, \eta \rangle := (u, A_s \eta)_{L^2(\mathbb{R}^d)} \tag{11}$$

should be well defined and finite for all $\eta \in C_c^\infty(\Omega)$. We introduce the weighted $L^1$-space $L^1(\mathbb{R}^d, v^*(x) dx)$ with the weight

$$v^*(x) := \int_{\mathbb{R}} \int_{S^{d-1}} 1_{\Omega}(x + r\theta)(1 + |r|)^{-1-2s} \mu(d\theta) dr \tag{12}$$

We call it the tail space for $\nu$ and $\Omega$. The function $A_s \eta$ is bounded in $\Omega$ and decays at infinity like $v^*$ for $\eta \in C_c^\infty(\Omega)$. Therefore, $u \in L^1(\Omega) \cap L^1(\mathbb{R}^d, v^*(x) dx)$ is sufficient for Equation (11) to be well defined and finite, see Lemma 2.4. In the case of the fractional Laplacian, this tail space coincides with the aforementioned space $L^1(\mathbb{R}^d, (1 + |x|)^{-d-2s})$. This is proven in Lemma A.1. In our second example $\sum_{j=1}^{d} (-\Delta_j)^s$, the auxiliary measure $v^*(x) dx$ only measures sets close to the coordinate axes, dependent on $\Omega$. Here, functions in $L^1(\mathbb{R}^d, v^*(x) dx)$ are not necessarily integrable on $\Omega$. An example can be found in Example A.2. The weight $v^*$ captures the behavior of $\nu$ at infinity. Foghem and Kassmann introduced and discussed several possibilities of tail weights for a large class of Lévy measures in [25].

The integrability $u \in L^1(\Omega) \cap L^1(\mathbb{R}^d, v^*(x) dx)$ is not sufficient for Equation (10) to hold. The function

$$\begin{cases} u(x) := (1 - |x|^2)^{-1+s} & \text{for } x \in B_1(0), \\ u(x) := 0 & \text{for } x \notin B_1(0) \end{cases} \tag{13}$$
satisfies \((-Δ)u = 0\) in \(B_1(0)\) and \(u = 0\) on \(B_1(0)^c\), but it disobeys the maximum principle, see the works [29] by Hmissi, [3] by Bogdan, and [19] by Dyda for a proof. For a systematic approach to boundary blow-up solutions like Equation (13), we refer to the work [1] by Abatangelo.

Now we state the main result of this paper.

**Theorem 1.1.** Let \(Ω ⊂ \mathbb{R}^d\) be a bounded Lipschitz domain satisfying the uniform exterior ball condition, \(s ∈ (0, 1)\), and \(μ\) a nonnegative, finite measure on the unit sphere satisfying the nondegeneracy assumption Equation (9), \(A_s\) be as in Equation (7). If \(u ∈ L^1(Ω) ∩ L^1(\mathbb{R}^d, \nu^s(x)\,dx)\) satisfies

\[
(u, A_sη)_{L^2(\mathbb{R}^d)} ≤ 0 \text{ for all } η ∈ C_c^\infty(Ω), \eta ≥ 0,
\]

\[
u^s
u ≤ 0 \text{ a.e. on } Ω^c,
\]

\[
\lim_{\varepsilon → 0^+} \varepsilon^{-s} \int_{\{x ∈ Ω | \text{dist}(x, Ω^c) < \varepsilon\}} u^+(x)\,dx = 0,
\]

then \(u ≤ 0\) a.e. in \(Ω\).

**Remark 1.2.** Instead of Equation (16), we can assume the stronger but more accessible condition

\[
u^s_u^+ ∈ L^1(Ω, \text{dist}(x, Ω^c)^{-s}\,dx).
\]

This condition implies Equation (16) because if \(u^+ ∈ L^1(Ω, \text{dist}(x, Ω^c)^{-s}\,dx)\), then

\[
u^s_\varepsilon^{-s} \int_{\{x ∈ Ω | \text{dist}(x, Ω^c) < \varepsilon\}} u^+(x)\,dx ≤ \int_{\{x ∈ Ω | \text{dist}(x, Ω^c) < \varepsilon\}} u^+(x)\text{dist}(x, Ω^c)^{-s}\,dx → 0 \text{ as } \varepsilon → 0^+.
\]

**Remark 1.3.**

(i) By Equation (26), if \(u\) satisfies Equation (14), then \(u_{\supp(\nu^s)}\) satisfies Equation (14). Thus, we may replace Equation (15) with \(u ≤ 0\) a.e. on \(Ω^c ∩ \supp(\nu^s)\) in Theorem 1.1. This is particularly interesting if the Lévy measure \(ν\) does not have full support.

(ii) Theorem 1.1 is optimal in the following sense: The function \(u\) from Equation (13) disobeys the maximum principle and

\[
u^s_\varepsilon^{-s+δ} \int_{B_1 \setminus B_{1-\varepsilon}} (1 - |x|^2)^{-1+s}\,dx → 0 \text{ as } \varepsilon → 0^+
\]

for every \(δ > 0\) but not for \(δ = 0\).

Let us explain the main ideas in the proof of Theorem 1.1. In an ideal situation, we would use the Green function in place of \(η\) in Equation (14). This is not permitted in our setup. Thereby, we approximate the Green function. We consider a sequence \(D_ε \subset⊂ Ω\) of subdomains exhausting \(Ω\). We fix \(ψ ∈ C_c^\infty(Ω)\) and solve the Dirichlet problem \(A_sφ_ε = ψ\) in \(D_ε\) and \(φ_ε = 0\) on \(D_ε^c\). The solutions \(φ_ε\) approximate the Green function if we pick an approximation of the identity in place of \(ψ\). Regularity results on the solution \(φ_ε\) are crucial in our proof.

**Remark 1.4.** The assumptions on the regularity of the boundary of the domain \(Ω\) in Theorem 1.1 seem unnatural. They are only needed for boundary regularity of solutions to the Dirichlet problem in Proposition 2.8, see Ros-Oton and Serra [43, Proposition 4.5]. The restriction to the class of stable operators is also due to the interior and boundary regularity estimates, see [43, Theorem 1.1, Proposition 4.5]. If regularity estimates are available, then our proof generalizes to nonlocal operators with a symmetric kernel \(k\), which is comparable to a kernel of a \(2s\)-stable operator and satisfies \(k(x, x + h) = k(x, x - h)\) for any \(x ∈ \mathbb{R}^d, h ∈ B_1(0)\). Additionally, the methods in our proof of Theorem 1.1 can be applied to the case of the Laplacian on sufficiently regular domains to prove the classical maximum principle, see Equation (1) and Equation (2).
Nonlocal operators and related Dirichlet problems are often studied in a Hilbert space setting. Following the works [22] by Felsinger, Kassmann, and Voigt and [48] by Servadei and Valdinoci, we define a bilinear form associated to $A_s$,

$$
\mathcal{E}_A(u, v) := \frac{1}{2} \int_{\mathbb{R}^d} \int_{S^{d-1}} 1_{(\Omega^c \times \Omega^c)^c}(x, x + r \theta) (u(x) - u(x + r \theta))(v(x) - v(x + r \theta)) \frac{1}{|r|^{1+2s}} \mu(d\theta) \, dr \, dx.
$$

This is motivated by a nonlocal Green–Gauß formula, see Du et al. [17] for bounded kernels, Dipierro, Ros-Oton, and Valdinoci [15, Lemma 3.3] for the fractional Laplacian, and Foghem and Kassmann [25] for more general Lévy measures.

**Corollary 1.5.** Let $\Omega$ and $\mu$ satisfy the assumptions from Theorem 1.1 and $A_s$ be as in Equation (7). If $u \in H^s(\Omega) \cap L^1(\mathbb{R}^d, \nu^*(x) \, dx)$ satisfies

$$
\mathcal{E}_A(u, \eta) \leq 0 \quad \text{for all} \quad \eta \in C^\infty_0(\Omega), \eta \geq 0,
$$

$$
u \leq 0 \quad \text{a.e. on} \quad \Omega^c,
$$

then $u \leq 0$ a.e. in $\Omega$.

**Remark 1.6.**

(i) The assumption $u \in H^s(\Omega) \cap L^1(\mathbb{R}^d, \nu^*(x) \, dx)$ is sufficient for $\mathcal{E}_A(u, \eta)$ to be well defined and finite, see Lemma 2.5.

(ii) The proof of Corollary 1.5 uses $u \in H^{s/2}(\Omega)$ and the fractional Hardy inequality to deduce the integrability $u \in L^1(\Omega, \text{dist}(x, \partial \Omega)^{-s} \, dx)$. But $u \in H^{s/2}(\Omega) \cap L^1(\mathbb{R}^d, \nu^*(x) \, dx)$ is not sufficient for $\mathcal{E}_A(u, \eta)$ to be well defined and finite for all $s \in (0, 1)$.

### 1.2 Outline

In Section 2, we prove basic properties of the operator $A_s$ and introduce the concept of distributional solutions. Additionally, we state a boundary regularity result of solutions and provide technical ingredients for the proof of Theorem 1.1. Lemma 2.13 connects the Hölder regularity of solutions with the assumption Equation (16). In Section 3, we prove Theorem 1.1 and Corollary 1.5. In the Appendix A we compare tail spaces.

### 2 PRELIMINARIES

We set $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$, $a^+ := \max\{a, 0\}$, and $|a| = \max\{z \in \mathbb{Z} \mid z \leq a\}$ for $a, b \in \mathbb{R}$. We denote the set of Hölder continuous functions on $\Omega \subset \mathbb{R}^d$ by $C^\alpha(\Omega) = C^{[\alpha]}(\mathbb{R}^d)$ for all $\alpha > 0$. In the case that $\Omega$ is bounded, we equip the space $C^\alpha(\Omega)$ with the usual norm

$$
\|u\|_{C^\alpha(\Omega)} := \sum_{|\beta| \leq \alpha} \sup_{x \in \Omega} |\partial^\beta u(x)| + \sum_{|\beta| = \alpha} [\partial^\beta u]_{C^{\alpha-|\beta|}(\Omega)}
$$

with the Hölder seminorm for $0 < s < 1$

$$
[u]_{C^s(\Omega)} = \sup_{x, y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^s}.
$$

We denote by $H^s(\Omega)$ the standard $L^2$-based Sobolev–Slobodeckij space for $s \in (0, 1)$. We also define the distance function

$$
\text{dist}(x, \Omega) := \inf\{|x - y| \mid y \in \Omega\}, \quad \text{dist}(\Omega, \Omega') := \inf\{\text{dist}(x, \Omega') \mid x \in \Omega\},
$$
where $\Omega, \Omega' \subset \mathbb{R}^d$ are open sets and $x \in \mathbb{R}^d$. For an open set $\Omega \subset \mathbb{R}^d$ and $\varepsilon > 0$, we introduce the thinned and thickened sets

$$
\Omega_\varepsilon := \{x \in \Omega \mid \text{dist}(x, \partial \Omega) > \varepsilon\}, \quad \Omega^\varepsilon := \{x \in \mathbb{R}^d \mid \text{dist}(x, \Omega) < \varepsilon\}.
$$

(19)

We say that a domain $\Omega \subset \mathbb{R}^d$ satisfies the uniform exterior ball condition if there exists a radius $\rho > 0$ such that for every $x \in \partial \Omega$, there exists a ball $B \subset \overline{\Omega}^c$ of radius $\rho$ such that $B \cap \overline{\Omega} = \{x\}$.

The following lemma ensures the existence of a sequence of $C^\infty$ subdomains exhausting a bounded Lipschitz domain satisfying the uniform exterior ball condition. This type of domain exhaustion is classical. We are particularly interested in a uniform bound of the exterior ball radius. The result is taken from the work by Mitrea, see [38, Lemma 6.4].

**Lemma 2.1** ([38, Lemma 6.4]). Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain satisfying the uniform exterior ball condition. There exists a sequence of $C^\infty$-domains $\{D_\varepsilon\}_{\varepsilon > 0}$ such that

(i) $D_\varepsilon \subset D_{\varepsilon'} \subset \subset \Omega$ for all $0 < \varepsilon' < \varepsilon < \varepsilon_0$, $\bigcup_{0<\varepsilon<\varepsilon_0} D_\varepsilon = \Omega$,

(ii) $D_\varepsilon$ satisfies the uniform exterior ball condition with a radius independent of $\varepsilon$,

(iii) there exists a constant $\lambda > 1$ such that

$$
\varepsilon \leq \text{dist}(\partial D_\varepsilon, \partial \Omega), \quad \sup \{|\text{dist}(x, \partial \Omega)| \mid x \in \partial D_\varepsilon\} \leq \lambda \varepsilon
$$

for all $0 < \varepsilon < \varepsilon_0$.

**Proof.** The existence of the sequence $\{D_\varepsilon\}_{\varepsilon > 0}$ satisfying the properties (i) and (ii) follows from [38, Lemma 6.4]. Property (iii) can be ensured by choosing the constants in the construction of $D_\varepsilon$ in the proof of [38, Lemma 6.4] accordingly. □

The symmetry of the Lévy measure $\nu$ allows us to rewrite $A_s$ to a double difference.

**Proposition 2.2.** Fix an open set $\Omega \subset \mathbb{R}^d$ and $\alpha > 0$. Let $u \in C(\mathbb{R}^d) \cap C^{2s+\alpha}(\Omega) \cap L^1(\mathbb{R}^d, \nu^*(x) \, dx)$. The term $A_s u(x)$ is well defined, finite, and

$$
A_s u(x) = \frac{1}{2} \int_{\mathbb{R}} \int_{S^{d-1}} \frac{2u(x) - u(x - r \theta) - u(x + r \theta)}{|r|^{1+2s}} \mu(d\theta) \, dr
$$

for any $x \in \Omega$. Furthermore, the function $A_s u$ is continuous in $\Omega$.

This result is standard, for example, see Bogdan and Byczkowski [4, Lemma 3.5], Silvestre [49, Proposition 2.4], and Di Nezza, Palatucci, and Valdinoci [14, Lemma 3.2].

**Proof.** Without loss of generality, we assume $2s + \alpha < 2$ and $2s + \alpha < 1$ for $s < \frac{1}{2}$. Take $x \in \Omega$ and any ball $B_\varepsilon(x) \subset \subset \Omega$. We define for $\varepsilon > 0$,

$$
A_\varepsilon^s u(x) := \int_{\{|r| \geq \varepsilon\}} \int_{S^{d-1}} \frac{u(x) - u(x + r \theta)}{|r|^{1+2s}} \mu(d\theta) \, dr.
$$

(20)

This term is well defined and finite by the assumption $u \in L^1(\mathbb{R}^d, \nu^*(x) \, dx)$. Now we write $A_\varepsilon^s u(x) = \frac{1}{2} A_\varepsilon^s u(x) + \frac{1}{2} A_\varepsilon^s u(x)$ and use the transformation $r \mapsto -r$ in its second occurrence. This yields

$$
A_\varepsilon^s u(x) = \frac{1}{2} \int_{\{|r| \geq \varepsilon\}} \int_{S^{d-1}} \frac{u(x) - u(x + r \theta)}{|r|^{1+2s}} \mu(d\theta) \, dr + \frac{1}{2} \int_{\{|r| \geq \varepsilon\}} \int_{S^{d-1}} \frac{u(x) - u(x - r \theta)}{|r|^{1+2s}} \mu(d\theta) \, dr
$$

$$
= \frac{1}{2} \int_{\{|r| \geq \varepsilon\}} \int_{S^{d-1}} \frac{2u(x) - u(x - r \theta) - u(x + r \theta)}{|r|^{1+2s}} \mu(d\theta) \, dr.
$$

(21)
For $s < \frac{1}{2}$, any $\theta \in S^{d-1}$, and $|r| < \delta$,
\[
\frac{|2u(x) - u(x - r\theta) - u(x + r\theta)|}{2|r|^{1 + 2s}} \leq |u|_{C^{2s+\alpha}(B_\delta(x))}|r|^{\alpha-1}.
\]
(22)

For $s \geq \frac{1}{2}$, $\theta \in S^{d-1}$, and $|r| < \delta$, the fundamental theorem of calculus yields
\[
|2u(x) - u(x - r\theta) - u(x + r\theta)| = \left| \int_0^1 (\nabla u(x - tr\theta) - \nabla u(x + tr\theta)) \cdot r\theta \, dr \right|
\leq \frac{(2|r|)^{2s+\alpha}}{2s + \alpha} |u|_{C^{2s+\alpha}(B_\delta(x))}.
\]
(23)

Lastly, for all $s \in (0,1)$, $\theta \in S^{d-1}$, and $|r| \geq \delta$,
\[
|2u(x) - u(x - r\theta) - u(x + r\theta)| \leq \left( 1 + \frac{1}{\delta} \right)^{1+2s} \frac{|u(x)| + |u(x + r\theta)| + |u(x + r\theta)|}{(1 + |r|)^{1+2s}}.
\]
(24)

Additionally, we know using a change of variables and Fubini's theorem that $u \in L^1(\mathbb{R}^d, \nu(x) \, dx)$ if
\[
\int_{\Omega} \int_{\mathbb{R}^d} \int_{S^{d-1}} |u(y + r\theta)|(1 + |r|)^{-1-2s} \mu(d\theta) \, dy \, dr < \infty.
\]

Therefore, the right-hand side of Equation (21) is finite for a.e. $x \in \Omega$. Since $u$ is continuous on $\mathbb{R}^d$, this property holds for every $x \in \Omega$. By the dominated convergence theorem with the previous bounds, the limit $\kappa \to 0$ in Equation (21) exists.

Lastly, we show that $A_s u$ is continuous in $\Omega$. We fix $x \in \Omega$ and write $A_s u(x)$ as
\[
A_s u(x) = \frac{1}{2} \left( \int_{\{|r| \geq \delta\}} + \int_{\{|r| < \delta\}} \right) \int_{S^{d-1}} \frac{2u(x) - u(x - r\theta) - u(x + r\theta)}{|r|^{1+2s}} \mu(d\theta) \, dr.
\]

The dominated convergence theorem, the estimates Equation (22), Equation (23), and Equation (24), and the continuity of $u$ yield the continuity of $A_s u$ at $x$. \hfill \Box

**Proposition 2.3.** Let $\Omega \subset \mathbb{R}^d$ be open and bounded, $s \in (0,1)$ and $\alpha > 0$. There exists a constant $C > 0$ such that
\[
\|A_s \phi\|_{L^\infty(\Omega)} \leq C \|\phi\|_{C^{2s+\alpha}(\Omega)}
\]
(25)

for all $\phi \in C^c_c(\Omega)$.

**Proof.** The claim follows by using the arguments in Proposition 2.2 uniformly for any $x \in \Omega$. Instead of picking a small ball $B_\delta(x)$ in the beginning of the proof of Proposition 2.2, we choose a ball $B \supset \supset \Omega$. \hfill \Box

**Lemma 2.4.** Let $\Omega \subset \mathbb{R}^d$ be open and bounded. If $u \in L^1(\Omega) \cap L^1(\mathbb{R}^d, \nu(x) \, dx)$ and $\eta \in C^\infty_0(\Omega)$, then $(u, A_s \eta)_{L^2(\mathbb{R}^d)}$ is well defined and finite.

**Proof.** Proposition 2.3 yields $A_s \eta \in L^\infty(\Omega)$. This and $u \in L^1(\Omega)$ imply that $(u, A_s \eta)_{L^2(\mathbb{R}^d)}$ is well defined and finite. Fix $\delta = \text{dist}(\text{supp}\, \eta, \Omega^c) > 0$. For any $x \in \Omega^c$,
\[
A_s \eta(x) = -\frac{1}{2} \int_{\{|r| > \delta\}} \int_{S^{d-1}} \frac{\eta(x - r\theta) + \eta(x + r\theta)}{|r|^{1+2s}} \mu(d\theta) \, dr = -\int_{\{|r| > \delta\}} \int_{S^{d-1}} \frac{\eta(x + r\theta)}{|r|^{1+2s}} 1_{\Omega}(x + r\theta) \mu(d\theta) \, dr.
\]
Recall the definition of $\nu^*$ in Equation (12). Thereby, for $x \in \Omega^c$,

$$|A_x\eta(x)| \leq \|\eta\|_{L^\infty} \left(1 + \frac{1}{s}\right)^{1+2s} \nu^*(x).$$  \hspace{1cm} (26)

The claim follows from $u \in L^1(\mathbb{R}^d, \nu^*(x) \, dx)$.

**Lemma 2.5.** Let $\Omega$ be a bounded Lipschitz domain. For any $u \in H^s(\Omega) \cap L^1(\mathbb{R}^d, \nu^*(x) \, dx)$ and $\eta \in C^\infty_c(\Omega)$, the bilinear form $E_A(u, \eta)$ is well defined and finite.

**Proof.** We split the energy into $\Omega \times \Omega, \Omega \times \Omega^c$ and $\Omega^c \times \Omega$. The proof for $\Omega \times \Omega^c$ and $\Omega^c \times \Omega$ coincide. We begin with $\Omega^c \times \Omega$. Define $\delta := \text{dist}(\text{supp} \, \eta, \Omega^c) > 0$, $C := (1 + 1/\delta)^{1+2s}$ and notice $C \, |r|^1 \geq (1 + |r|)^{1+2s}$ for any $r > \delta$. Thus,

$$\left| \int_{\Omega^c} \int_{\mathbb{R}^d} \int_{S^{d-1}} 1_{\Omega}(x + r \theta) (u(x) - u(x + r \theta)) (0 - \eta(x + r \theta)) |r|^{1+2s} \, \mu(d\theta) \, dr \, dx \right|$$

$$\leq C \|\eta\|_{L^\infty} \int_{\Omega^c} \int_{\mathbb{R}^d} \int_{S^{d-1}} 1_{\Omega}(x + r \theta) \frac{|u(x) - u(x + r \theta)|}{(1 + |r|)^{1+2s}} \, \mu(d\theta) \, dr \, dx$$

$$\leq C \|\eta\|_{L^\infty} \left( \|u\|_{L^1(\mathbb{R}^d, \nu^*(x) \, dx)} + \frac{1}{\delta^{2s}} \|u\|_{L^1(\Omega)} \right).$$

For $\Omega \times \Omega$, we want to apply [20, Proposition 6.1] by Dyda and Kassmann. This is possible since the Lévy measure satisfies

$$\int_{\mathbb{R}^d} (t \wedge |z|)^2 \nu(dz) = \mu(S^{d-1}) t^{2-2s} \left( \frac{1}{1-s} + \frac{1}{5} \right)$$

for all $t > 0$ and we assumed $\Omega$ to be bounded with Lipschitz boundary. Therefore, [20, Proposition 6.1] and Hölder inequality yield

$$\left| \int_{\Omega} \int_{\mathbb{R}^d} (u(x) - u(x + h))(\eta(x) - \eta(x + h)) 1_{\Omega}(x + h) \nu(dh) \, dx \right| \leq c \|u\|_{H^s(\Omega)} \|\eta\|_{H^s(\Omega)}.$$

**Definition 2.6** (distributional solution). Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $\psi \in L^\infty(\Omega)$. We say that $\phi$ is a distributional solution to the problem

$$A_x\phi = \psi \quad \text{in} \, \Omega,$$

$$\phi = 0 \quad \text{on} \, \Omega^c,$$  \hspace{1cm} (27)

if $\phi \in L^1_{\text{loc}}(\mathbb{R}^d), \phi = 0$ on $\Omega^c$, and

$$\int_{\mathbb{R}^d} \phi(x) A_x \eta(x) \, dx = \int_{\Omega} \psi(x) \eta(x) \, dx$$

for all $\eta \in C^\infty_c(\Omega)$.

There is a rich theory on existence and uniqueness of distributional solutions. We refer the reader to Bogdan et al. [7], Felsinger, Kassmann, and Voigt [22], Grzywny, Kassmann, and Ležaj [28], and Rutkowski [45]. In the following proposition, we use the existence theorem from [45] to deduce the existence of solutions in the sense of Definition 2.6.

**Proposition 2.7.** Let $\Omega \subset \mathbb{R}^d$ be an open and bounded set and $\psi \in L^\infty(\Omega)$. There exists a solution $\phi \in L^\infty(\Omega)$ to Equation (27).
Proof. We introduce the function space $V_{\nu,0}(\Omega | \mathbb{R}^d) = \{ u : \mathbb{R}^d \to \mathbb{R} | \mathcal{E}_A(u,u) < \infty, u = 0 \text{ on } \Omega^c \}$ endowed with the inner product $(u,v)_{V_{\nu,0}} := (u,v)_{L^2(\Omega)} + \mathcal{E}_A(u,v)$. In [45, Theorem 4.2], Rutkowski proved, via an application of the Lax–Milgram lemma, the existence of a unique $u \in V_{\nu,0}(\Omega | \mathbb{R}^d)$ satisfying

$$
\mathcal{E}_A(u, v) = \int_\Omega \psi(x) u(x) \, dx
$$

for all $v \in V_{\nu,0}(\Omega | \mathbb{R}^d)$. The function $u$ is in $L^\infty(\Omega)$ by [45, Lemma 5.7]. It remains to show that $u$ is a solution in the sense of Definition 2.6. Take any $\eta \in C_c^\infty(\Omega) \subset V_{\nu,0}(\Omega | \mathbb{R}^d)$. A small calculation yields

$$
\int_{\mathbb{R}^d} u(x) A^s_r \eta(x) \, dx = \frac{1}{2} \int_{\mathbb{R}^d} \int_{|r| > s} \int_{S^{d-1}} \frac{(u(x) - u(x + r \theta))(\eta(x) - \eta(x + r \theta))}{|r|^{1+2s}} \mu(d\theta) \, dr \, dx.
$$

The left-hand side of Equation (29) converges to $\int_\Omega u(x) A^s_r \eta(x) \, dx$ by dominated convergence since $u \in L^1(\Omega)$, supp($u) \subset \bar{\Omega}$, and $A^s_r \eta \in L^\infty(\Omega)$ by Equation (25). The right-hand side of Equation (29) converges to $\mathcal{E}_A(u, \eta)$ by dominated convergence. Thus, the solution $u$ satisfies $\int u A^s_r \eta = \int \psi \eta$ for all $\eta \in C_c^\infty(\Omega)$.

Hölder regularity up to the boundary of solutions to Dirichlet problems was proven by Ros-Oton and Serra in [43]. The following proposition is taken from their work. For the fractional Laplacian, see also Ros-Oton and Serra [42].

**Proposition 2.8** [43, Proposition 4.5]. Let $\Omega$ be a bounded Lipschitz domain satisfying the uniform exterior ball condition. Suppose $s \in (0, 1)$ and let $\mu$ be any bounded measure on $S^{d-1}$ satisfying the nondegeneracy assumptions Equation (9). Let $\psi \in L^\infty(\Omega)$ and $\phi$ be a bounded distributional solution of Equation (27). Then, $\phi \in C^s(\bar{\Omega})$ and there exists a constant $C_{d,\Omega,s,\mu} > 0$ such that

$$
\|\phi\|_{C^s(\Omega)} \leq C_{d,\Omega,s,\mu} \|\psi\|_{L^\infty(\Omega)}.
$$

**Remark 2.9.** The constant $C_{d,\Omega,s,\mu}$ depends on $d$, $s$, the nondegeneracy constant of $\mu$, see Equation (9), the diameter of $\Omega$, the Lipschitz constant, and the uniform exterior ball radius of $\Omega$, see [43, Proposition 4.5] and [42, Proposition 1.1, Lemma 2.7, Lemma 2.9].

Now, we combine interior regularity results of Ros-Oton and Serra for solutions to Equation (27) with Proposition 2.8 and the existence theorem, see Proposition 2.7, to prove the following existence and uniqueness of classical solutions.

**Proposition 2.10.** Let $\Omega$ be a bounded Lipschitz domain satisfying the uniform exterior ball condition, $s \in (0, 1)$, and $\mu$ a bounded measure on $S^{d-1}$ satisfying Equation (9). We fix $\psi \in C_c^\infty(\Omega)$. There exists a classical solution $\phi \in C^{2s+\alpha}(\Omega) \cap C^s(\bar{\Omega})$ to

$$
A^s_r \phi = \psi \text{ in } \Omega,
$$

$$
\phi = 0 \text{ on } \Omega^c.
$$

**Proof.** By Proposition 2.7, there exists a bounded distributional solution $\phi \in L^\infty(\Omega)$ to Equation (27) in the sense of Definition 2.6. The solution satisfies $\phi \in C^s(\mathbb{R}^d) \cap C^{2s+\alpha}(\Omega)$ by Proposition 2.8 and the interior regularity theory, see [43, Theorem 1.1] by Ros-Oton and Serra. Here, $\alpha > 0$ such that $2s + \alpha$ is not an integer. By Proposition 2.2, $A^s_r \phi$ is continuous in $\Omega$. We fix $\eta \in C_c^\infty(\Omega)$. Since $A^s_r \phi$ is continuous in $\Omega$, $A^s_r \phi \in L^\infty(\text{supp}(\eta))$. The choice of $\phi$, see Equation (28), yields

$$
\int_\Omega \psi(x) \eta(x) \, dx = \mathcal{E}_A(\phi, \eta).
$$

The dominated convergence theorem, $A^s_r \phi \in L^\infty(\text{supp}(\eta))$, $\eta \in C_c^\infty(\Omega)$, and Equation (29) yield

$$
\int_\Omega A^s_r \phi(x) \eta(x) \, dx = \mathcal{E}_A(\phi, \eta) = \int_\Omega \psi(x) \eta(x) \, dx.
$$

This identity holds for all $\eta \in C_c^\infty(\Omega)$. Since $A^s_r \phi$ is continuous in $\Omega$, we conclude $A^s_r \phi(x) = \psi(x)$ for all $x \in \Omega$. 

□
The following lemma shows how $A_s u$ changes under mollification of $u$. This result is standard and well known for translation invariant operators, for example, see [4, Theorem 3.12] for the fractional Laplacian. We prove the statement for the convenience of the reader.

**Lemma 2.11.** Let $u \in L^1(\Omega) \cap L^1(\mathbb{R}^d, \nu^*(x)\,dx)$ satisfy Equation (14) and Equation (15). Fix a radial bump function $\eta \in C_\infty^\circ(\mathbb{R}^d)$ with $\text{supp}\,\eta = B_1(0)$, $\eta \geq 0$, and $\|\eta\|_{L^1(B_1(0))} = 1$. We define an approximation of the identity $\eta_\varepsilon := \varepsilon^{-d} \eta \left( \frac{\cdot}{\varepsilon} \right)$. The mollification $u_\varepsilon = u * \eta_\varepsilon$ satisfies

\[
A_s u_\varepsilon \leq 0 \quad \text{in} \quad \Omega_\varepsilon,
\]

\[
u_\varepsilon \leq 0 \quad \text{on} \quad (\Omega_\varepsilon)^c
\]

pointwise. Here, $\Omega_\varepsilon$ and $\Omega^c$ are as in Equation (19).

**Proof.** The second claim follows from Equation (15) and $\text{supp}(\eta_\varepsilon) = B_\varepsilon(0)$.

Since $u \in L^1(\Omega) \cap L^1(\mathbb{R}^d, \nu^*(x)\,dx)$ and $\eta \in C_\infty^\circ(\mathbb{R}^d)$, the convolved function satisfies $u_\varepsilon \in C_\infty^\circ(\mathbb{R}^d) \cap L^1(\mathbb{R}^d, \nu^*(x)\,dx)$. By Proposition 2.2, $A_s u_\varepsilon$ is continuous in $\Omega$. For the first property, we fix $x \in \Omega_\varepsilon$. Now we use Fubini's theorem, the radiality of $\eta$, and a change of variables to conclude

\[
A_s u_\varepsilon(x) = \int_{\mathbb{R}^d} u(y) \text{ p.v.} \int_{S^{d-1}} \frac{\eta_\varepsilon(x - y) - \eta_\varepsilon(x + r\theta - y)}{|r|^{1+2s}} \mu(d\theta) \, dr \, dy \leq 0.
\]

The first equality is true by Equation (25) and dominated convergence. The last inequality is true because $\eta_\varepsilon(\cdot - x) \in C_\infty^\circ(\Omega)$ for any $x \in \Omega_\varepsilon$.

The next lemma shows the following: If $u$ is a subsolution, then $u^+ = \max\{u, 0\}$ is again a subsolution. Silvestre proved this result for the fractional Laplacian in [49, Lemma 2.18]. We follow the technique in [36, Lemma 2] by Li, Wu, and Xu as it generalizes more easily to our class of operators.

**Lemma 2.12.** If $u \in L^1(\Omega) \cap L^1(\mathbb{R}^d, \nu^*(x)\,dx)$ satisfies Equation (14) and Equation (15), then $u^+ = \max\{u, 0\}$ satisfies Equation (14) and $u^+ = 0$ on $\Omega^c$.

**Proof.** The proof of the second claim is immediate. We divide the proof into two steps.

**Step 1.** Suppose $u \in L^1(\Omega) \cap L^1(\mathbb{R}^d, \nu^*(x)\,dx) \cap C_\infty^\circ(\mathbb{R}^d)$ satisfies Equation (14). Let $\eta \in C_\infty^\circ(\Omega)$. Equation (29) and dominated convergence yield

\[
\int_{\mathbb{R}^d} A_s u(x) \eta(x) \, dx = \mathcal{E}_{A_s}(u, \eta) = \int_{\mathbb{R}^d} u A_s \eta(x) \, dx.
\]

By Proposition 2.2, $A_s u$ is continuous in $\Omega$ and, thus, $A_s u \in L^\infty(\text{supp}\,\eta)$. Therefore, the function $x \mapsto A_s u(x)$ is integrable over $\Omega$. The term $\mathcal{E}_{A_s}(u, \eta)$ is well defined and finite by Lemma 2.5. By Lemma 2.4 and $u \in L^1(\Omega) \cap L^1(\mathbb{R}^d, \nu^*(x)\,dx)$, the last term $\int_{\mathbb{R}^d} u(x) A_s \eta(x) \, dx$ is finite. Therefore, the application of dominated convergence is justified. Since $\eta \in C_\infty^\circ(\Omega)$ was arbitrary, we conclude $A_s u(x) \leq 0$ for all $x \in \Omega$.

Now consider $P = \{x \in \mathbb{R}^d \mid u(x) > 0\}$. For $x > 0$ and $\eta \in C_\infty^\circ(\Omega)$ with $\eta \geq 0$, we define $A_s^\chi \eta$ as in Equation (20). We calculate

\[
\int_{\mathbb{R}^d} u^+(x) A_s^\chi \eta(x) \, dx = \int_{\mathbb{R}^d} u(x) \int_{\{r > x\} \times S^{d-1}} \frac{\eta(x) - \eta(x + r\theta)}{|r|^{1+2s}} \mu(d\theta) \, dr \, dx.
\]
\[
\begin{align*}
&= \int_{P} \eta(x) \int_{\{\|r\| \geq \varepsilon\} \mathbb{S}^{d-1}} \frac{u(x) - u(x + r\vartheta)}{|r|^{1+2s}} \mu(d\vartheta) \, dr \, dx \\
&\quad + \int_{P} \int_{\{\|r\| \geq \varepsilon\} \mathbb{S}^{d-1}} u(x + r\vartheta) \eta(x) - u(x) \eta(x + r\vartheta) \frac{|r|^{1+2s}}{|r|^{1+2s}} \mu(d\vartheta) \, dr \\
&\quad + \int_{P} \int_{\{\|r\| \geq \varepsilon\} \mathbb{S}^{d-1}} u(x + r\vartheta) \eta(x) \eta(x + r\vartheta) \frac{|r|^{1+2s}}{|r|^{1+2s}} \mu(d\vartheta) \, dr \\
&=: (I) + (II) + (III).
\end{align*}
\]

These integrals are well defined and finite since \(\eta\) has compact support and \(u \in L^1(\mathbb{R}^d, \nu^*(x) \, dx)\). By symmetry, (II) is zero and (III) is nonpositive by the definition of \(P\). We conclude
\[
\int_{\mathbb{R}^d} u^+(x) A_x^s \eta(x) \, dx \leq \int_{P} \eta(x) A_x^s u(x) \, dx.
\]

By Proposition 2.2 and dominated convergence, we conclude
\[
\int_{\mathbb{R}^d} u^+(x) A_x^s \eta(x) \, dx \leq 0.
\]

**Step 2.** Now suppose \(u \in L^1(\Omega) \cap L^1(\mathbb{R}^d, \nu^*(x) \, dx)\) satisfies Equation (14) and Equation (15). By Lemma 2.11 and Step 1,

\[
\left( (u_\varepsilon^+), A_x \phi \right)_{L^2(\mathbb{R}^d)} \leq 0 \text{ for any nonnegative } \phi \in C_c^\infty(\Omega_\varepsilon), \quad (u_\varepsilon^+) = 0 \text{ on } (\Omega_\varepsilon)^c.
\]

Notice \(u^+ = 0\) on \(\Omega^c\) by assumption Equation (15). Now consider any \(\phi \in C_c^\infty(\Omega)\) with \(\phi \geq 0\). Fix \(\varepsilon_0 > 0\) such that supp \(\phi \subset \Omega_{\varepsilon_0}\). Then, (\(u_\varepsilon^+, A_x \phi\)) \(\leq 0\) follows for any \(0 < \varepsilon < \varepsilon_0\). Since \(|a^+ - b^+| \leq |a - b|\) holds for any two real numbers \(a, b\), we notice for any \(0 < \varepsilon < \varepsilon_0\)
\[
\begin{align*}
\|(u_\varepsilon^+) - u^+\|_{L^1(\Omega_\varepsilon)} &= \int_{\Omega_\varepsilon} |(u_\varepsilon^+) - u^+(x)| \, dx \\
&= \int_{\Omega_\varepsilon} |(u 1_{\Omega^c} \ast \eta_\varepsilon(x) - u(x) 1_{\Omega(x)}| \, dx \\
&\leq \int_{\Omega_\varepsilon} ||u 1_{\Omega^c} \ast \eta_\varepsilon(x) - u(x) 1_{\Omega(x)}||_{L^1(\mathbb{R}^d)}. 
\end{align*}
\]

This term converges to zero for \(\varepsilon \to 0\) since \(u 1_{\Omega^c} \in L^1(\mathbb{R}^d)\). Lastly,
\[
\begin{align*}
\|(u_\varepsilon^+) - u^+\|_{L^1(\Omega_\varepsilon^c)} &= \int_{\Omega_\varepsilon^c} |(u_\varepsilon^+) - u^+(x)| \, dx \\
&\leq \int_{\Omega_\varepsilon^c} ||u_\varepsilon^+(x) + u^+(x)|| \, dx \\
&\leq \int_{\Omega_\varepsilon^c} ||u_\varepsilon^+(x)| + |u^+(x)|| \, dx \\
&\leq \int_{\Omega_\varepsilon^c} ||(u_\varepsilon^+) e(x) - u^+(x)| + 2 |u^+(x)|| \, dx \\
&\leq \|(u_\varepsilon^+) - u^+\|_{L^1(\mathbb{R}^d)} + 2 \|u^+\|_{L^1(\mathbb{R}^d)^c}.
\end{align*}
\]

The function \(u^+\) is in \(L^1(\mathbb{R}^d)\), because of Equation (15) and \(u \in L^1(\Omega)\). Therefore, the right-hand side of the previous inequality converges to zero as \(\varepsilon \to 0\). Since \(\phi \in C_c^\infty(\Omega)\), we know \(\|A_x \phi\|_{L^\infty(\Omega^0)} < \infty\) by Equation (25). Thus, we conclude
The following lemma is the key technical estimate in the proof of Theorem 1.1. It connects the regularity of distributional solutions, see Proposition 2.8, with the assumption Equation (16).

**Lemma 2.13.** Let \( s \in (0,1) \) and \( \Omega \subset \mathbb{R}^d \) be a bounded Lipschitz domain satisfying the uniform exterior ball condition. Let \( \{D_\varepsilon\}_{\varepsilon > 0} \) be \( C^\infty \)-domains exhausting \( \Omega \) from Lemma 2.1 with \( \lambda > 1 \). Additionally, we fix a nonnegative, \( L^1 \)-normalized bump function \( \eta \in C_\infty(\mathbb{R}^d) \) with \( \text{supp} \eta = B_1(0) \) and an approximation of the identity \( \eta_\varepsilon := \varepsilon^{-d} \eta(\varepsilon \cdot) \in C_\infty(\mathbb{R}^d) \). There exists a constant \( C > 0 \) such that

\[
\sup\{\text{dist}(\cdot, \partial D_\varepsilon)^{-s} * \eta_\varepsilon(x) \mid x \in \Omega, \ \text{dist}(x, \partial \Omega) < (1 + \lambda)\varepsilon\} \leq C \varepsilon^{-s}
\]

for all \( 0 < \varepsilon < \varepsilon_0 \).

**Proof.** Fix any \( 0 < \varepsilon < \varepsilon_0 \) and any \( x \in \Omega \) satisfying \( \text{dist}(x, \partial \Omega) < (1 + \lambda)\varepsilon \). Now we estimate the convolved distance function:

\[
\text{dist}(\cdot, \partial D_\varepsilon)^{-s} * \eta_\varepsilon(x) = \int_{\mathbb{R}^d} \text{dist}(y, \partial D_\varepsilon)^{-s}\eta_\varepsilon(x-y) \, dy \leq \varepsilon^{-d} \|\eta\|_{L^\infty} \int_{B_\varepsilon(x)} \text{dist}(y, \partial D_\varepsilon)^{-s} \, dy.
\]

For any \( y \in B_\varepsilon(x) \),

\[
\text{dist}(y, \partial D_\varepsilon) \leq |y - x| + \text{dist}(x, \partial \Omega) + \sup\{\text{dist}(z, \partial \Omega) \mid z \in \partial D_\varepsilon\} \leq 2(1 + \lambda)\varepsilon.
\]

Now we apply the coarea formula, see Federer [21], to the right-hand side of Equation (30) with the Lipschitz continuous function \( \text{dist}(\cdot, \partial D_\varepsilon) \), which satisfies \( |\nabla \text{dist}(\cdot, \partial D_\varepsilon)| = 1 \) a.e.. This yields

\[
\text{dist}(\cdot, \partial D_\varepsilon)^{-s} * \eta_\varepsilon(x) \leq \varepsilon^{-d} \|\eta\|_{L^\infty} \int_0^{2(1 + \lambda)\varepsilon} t^{-s} \sigma(\{y \in B_\varepsilon(x) \mid \text{dist}(y, \partial D_\varepsilon) = t\}) \, dt.
\]

Here, \( \sigma \) is the surface measure. The surface measure of a ball \( B_r \) intersecting a hyperplane scales like \( r^{d-1} \). Thus, there exists a constant \( c > 0 \) such that

\[
\sigma(\{y \in B_\varepsilon(x) \mid \text{dist}(y, \partial D_\varepsilon) = t\}) \leq ce^{d-1},
\]

see, for example, Ros-Oton and Valdinoci [44, Lemma A.4 (A.19)]. The constant \( c \) does not depend on \( \varepsilon \). Now we combine this estimate and Equation (31) to conclude

\[
\text{dist}(\cdot, \partial \Omega_\varepsilon)^{-s} * \eta_\varepsilon(x) \leq c \|\eta\|_{L^\infty} \varepsilon^{-1} \int_0^{2(1 + \lambda)\varepsilon} t^{-s} \, dt = 2^{1-s} c \|\eta\|_{L^\infty} (1 + \lambda)^{1-s} \varepsilon^{-s}.
\]

\[\Box\]

### 3 PROOF OF THE MAXIMUM PRINCIPLES

In this section, we provide the proofs of Theorem 1.1 and Corollary 1.5.

**Proof of Theorem 1.1.** Let \( u, \mu, \Omega \) be as in Theorem 1.1, \( s \in (0,1) \). It is sufficient to prove the result for \( u^+ \) by Lemma 2.12. We fix a nonnegative, radial bump function \( \eta \in C_\infty(\mathbb{R}^d) \) with \( \text{supp} \eta = B_1(0) \) as well as \( \|\eta\|_{L^1(\mathbb{R}^d)} = 1 \) and construct an approximation of the identity by \( \eta_\varepsilon := \varepsilon^{-d} \eta(\varepsilon^{-1} \cdot) \). Additionally, we fix the sequence of \( C^\infty \) subdomains \( \{D_\varepsilon\}_{0 < \varepsilon < \varepsilon_0} \) from Lemma 2.1 with \( \lambda > 1 \) as given therein. Recall the definition of \( \Omega_\varepsilon, \Omega^2 \) from Equation (19). Notice that \( \Omega_{\lambda \varepsilon} \subset D_\varepsilon \subset \Omega_\varepsilon \). Now fix an arbitrary nonnegative \( \psi \in C_\infty(\Omega) \) and \( 0 < \varepsilon_1 < \varepsilon_0 \) such that \( \text{supp} \psi \subset \subset \Omega_{\lambda \varepsilon} \).
For any $0 < \varepsilon < \varepsilon_1$, we define the function $v_{\varepsilon} = u^\ast \eta_{\varepsilon}$. Lemma 2.11 and Lemma 2.12 yield

\begin{align}
A_{\varepsilon}v_{\varepsilon} &\leq 0 \text{ in } \Omega_{\varepsilon}, \\
v_{\varepsilon} &= 0 \text{ on } (\Omega_{\varepsilon})^c.
\end{align}

Since $u^\ast \in L^1(\mathbb{R}^d)$, we know that $v_{\varepsilon} \in C^\infty_c(\mathbb{R}^d)$. For any $0 < \varepsilon < \varepsilon_1$ let $\phi_{\varepsilon} \in C^c(\mathbb{R}^d) \cap C^{2s+\alpha}(D_{\varepsilon})$ be the pointwise solution to

\begin{align}
A_{\varepsilon}\phi_{\varepsilon} &= \psi \text{ in } D_{\varepsilon}, \\
\phi_{\varepsilon} &= 0 \text{ on } D_{\varepsilon}^c.
\end{align}

from Proposition 2.10. Here, $0 < \alpha \leq s$ is such that $2s + \alpha$ is not an integer.

Now we define $A_{\varepsilon}^\kappa$ for any $\kappa > 0$ as in Equation (20). For any $0 < \varepsilon < \varepsilon_1$ and $\kappa > 0$,

\begin{align}
\int_{D_{\varepsilon}} A_{\varepsilon}^\kappa v_{\varepsilon}(x) \phi_{\varepsilon}(x) - v_{\varepsilon}(x) A_{\varepsilon}^\kappa \phi_{\varepsilon}(x) \, dx
= \int_{D_{\varepsilon}} \int_{S^{d-1}} \int_{\{|r| \geq \kappa\}} \frac{v_{\varepsilon}(x) \phi_{\varepsilon}(x + r\theta) - v_{\varepsilon}(x + r\theta) \phi_{\varepsilon}(x)}{|r|^{1+2s}} \, dr \, d\theta \, dx
= \int_{D_{\varepsilon}} \int_{S^{d-1}} \int_{\{|r| \geq \kappa\}} 1_{D_{\varepsilon}}(x + r\theta) \frac{v_{\varepsilon}(x + r\theta) \phi_{\varepsilon}(x) - v_{\varepsilon}(x + r\theta) \phi_{\varepsilon}(x)}{|r|^{1+2s}} \, dr \, d\theta \, dx
- \int_{D_{\varepsilon}} \int_{S^{d-1}} \int_{\{|r| \geq \kappa\}} 1_{\Omega \setminus D_{\varepsilon}}(x + r\theta) \frac{v_{\varepsilon}(x + r\theta) \phi_{\varepsilon}(x)}{|r|^{1+2s}} \, dr \, d\theta \, dx
= : R_{\varepsilon,\kappa}.
\end{align}

The first term in $R_{\varepsilon,\kappa}$ is zero by symmetry. Our goal is to show that the remainder $R_{\varepsilon,\kappa}$ converges to zero as $\varepsilon \to 0+$ uniformly in $\kappa$. A change of variables yields

\begin{align}
|R_{\varepsilon,\kappa}| \leq \int_{\Omega \setminus D_{\varepsilon}} v_{\varepsilon}(x) \int_{S^{d-1}} \int_{\{|r| \geq \kappa\}} \frac{\phi_{\varepsilon}(x + r\theta)}{|r|^{1+2s}} \, dr \, d\theta \, dx.
\end{align}

For any $x \in \Omega \setminus D_{\varepsilon}$, we know $\phi_{\varepsilon}(x) = 0$. Therefore, Proposition 2.8 yields

\begin{align}
\int_{S^{d-1}} \int_{\{|r| > \text{dist}(x, \partial D_{\varepsilon})\}} \frac{\phi_{\varepsilon}(x + r\theta)}{|r|^{1+2s}} \, dr \, d\theta \leq \|\phi_{\varepsilon}\|_{C^c(\mathbb{R}^d)} \int_{S^{d-1}} \int_{\{|r| > \text{dist}(x, \partial D_{\varepsilon})\}} \frac{1}{|r|^{1+s}} \, dr \, d\theta
\leq C_{d,\varepsilon,\delta,\mu}(\mu(S^{d-1})) \|\psi\|_{L^\infty(\mathbb{R}^d)} \text{dist}(x, \partial D_{\varepsilon})^{-s}.
\end{align}

There exists a constant $C_1 > 0$ such that $C_{d,\varepsilon,\delta,\mu}(\mu(S^{d-1})) \leq C_1$ by Remark 2.9. This constant $C_1$ is independent of $\varepsilon$ because the exterior ball radius of $D_{\varepsilon}$ is independent of $\varepsilon$, see Lemma 2.1. Notice that

\begin{align}
\text{dist}(y, \partial \Omega) \leq |x - y| + \text{dist}(x, \partial \Omega) \leq (1 + \lambda)\varepsilon
\end{align}

for any $x \in \Omega \setminus D_{\varepsilon}$ and $y \in B_{\varepsilon}(x) \cap \Omega$. The previous calculation and Equation (34) yield the following bound on the remainder $R_{\varepsilon,\kappa}$:

\begin{align}
|R_{\varepsilon,\kappa}| \leq C_1 \|\psi\|_{L^\infty(\mathbb{R}^d)} \int_{\Omega \setminus D_{\varepsilon}} u_{\varepsilon}(x) \text{dist}(x, \partial D_{\varepsilon})^{-s} \, dx
= C_1 \|\psi\|_{L^\infty(\mathbb{R}^d)} \int_{\Omega \setminus (1 + \lambda)\varepsilon} u^+(y) \int_{\Omega \setminus D_{\varepsilon}} \eta_{\varepsilon}(x - y) \text{dist}(x, \partial D_{\varepsilon})^{-s} \, dx \, dy
\end{align}
\[
\frac{|R_{\varepsilon,\kappa}|}{C_1 \|\psi\|_{L^\infty(\mathbb{R}^d)}} \leq \int_{\Omega \setminus (\Omega_{1+\lambda})_{\varepsilon}} u^+(y) (\text{dist}(\cdot, \partial D) - s \ast \eta_{\varepsilon})(y) \, dy.
\]

By the previous estimate and Lemma 2.13, there exists a constant \(C > 0\) such that
\[
|R_{\varepsilon,\kappa}| \leq C C_1 \|\psi\|_{L^\infty(\mathbb{R}^d)} (1 + \lambda) \varepsilon^{-s} \int_{\Omega \setminus (\Omega_{1+\lambda})_{\varepsilon}} u^+(x) \, dx.
\]  

(35)

This converges to zero as \(\varepsilon \to 0^+\) by assumption Equation (16).

Now we finish the proof. The choice of \(\phi_{\varepsilon}\) as well as Equation (32) and Equation (33) yield
\[
0 \geq \int_{D_\varepsilon} A_s u_\varepsilon(x) \phi_{\varepsilon}(x) \, dx = \lim_{\kappa \to 0^+} \int_{D_\varepsilon} A_s^\kappa u_\varepsilon(x) \phi_{\varepsilon}(x) \, dx
\]
\[
= \lim_{\kappa \to 0^+} \left[ \int_{D_\varepsilon} u_\varepsilon(x) A_s^\kappa \phi_{\varepsilon}(x) \, dx - \int_{1/\varepsilon \setminus D_\varepsilon} \phi_{\varepsilon}(x) \, dx \right]
\]
\[
= \int_{D_\varepsilon} (u^+ \ast \eta_{\varepsilon})(x) \psi(x) \, dx + \lim_{\kappa \to 0^+} R_{\varepsilon,\kappa}.
\]  

(36)

In the previous calculation, we used dominated convergence and Proposition 2.2. Equation (35) implies that \(R_{\varepsilon,\kappa}\) converges to zero as \(\varepsilon \to 0^+\) uniformly in \(\kappa\). We take the limit \(\varepsilon \to 0^+\) in Equation (36). Thereby,
\[
0 \geq \int_{\Omega} u^+(x) \psi(x) \, dx
\]

because \(u^+ \in L^1(\mathbb{R}^d)\), \(\{\eta_{\varepsilon}\}_\varepsilon\) is an approximation of the identity, and \(\text{supp} \psi \subset \Omega \subset \Omega \subset D_\varepsilon \subset D_\varepsilon \subset \Omega\). We conclude the claim as the nonnegative function \(\psi \in C_c^\infty(\Omega)\) was chosen arbitrarily.

**Proof of Corollary 1.5.** We want to apply Theorem 1.1. Let \(u \in H^s(\Omega) \cap L^1(\mathbb{R}^d, \nu^*(x) \, dx)\) satisfy Equation (17) and Equation (18). Fix any nonnegative \(\eta \in C_c^\infty(\Omega)\). The dominated convergence theorem and Equation (29) yield
\[
\mathcal{E}_{A_s}(u, \eta) = \int_{\mathbb{R}^d} u A_s \eta(x) \, dx.
\]

The bilinear form \(\mathcal{E}_{A_s}(u, \eta)\) is well defined and finite by Lemma 2.5. By Lemma 2.4 and \(u \in L^1(\Omega) \cap L^1(\mathbb{R}^d, \nu^*(x) \, dx)\), the last term \(\int_{\mathbb{R}^d} u(x) A_s \eta(x) \, dx\) is well defined and finite. Thus, the application of dominated convergence was justified. Therefore, Equation (14) is satisfied. Since \(C_c^\infty(\Omega)\) is dense in \(H^{s/2}(\Omega)\), the Hölder inequality and the fractional Hardy inequality, see Chen and Song [12, Corollary 2.4] or Dyda [18, eq. (17)], imply
\[
\int_{\Omega} \frac{|u(x)|}{\text{dist}(x, \partial \Omega)^{s/2}} \, dx \leq \left\| \text{dist}(\cdot, \partial \Omega)^{-s/2} \right\|_{L^2(\Omega)} \left( \int_{\Omega} \frac{u(x)^2}{\text{dist}(x, \partial \Omega)^2} \, dx \right)^{1/2} \leq C \|u\|_{H^{s/2}(\Omega)}.
\]

An application of Theorem 1.1 finishes the proof, see Remark 1.2. □

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**CONFLICT OF INTEREST STATEMENT**

The authors declare no potential conflict of interests.
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APPENDIX A: DISCUSSION OF TAIL SPACES

The following lemma compares the tail space $L^1(\mathbb{R}^d, (1 + |x|)^{-d-2s} \, dx)$ with $L^1(\mathbb{R}^d, \nu^*(x) \, dx)$.

**Lemma A.1.** Let $s \in (0, 1)$, $\Omega \subset \mathbb{R}^d$ open and bounded, and $\nu$ be as in Equation (8). Suppose the measure $\mu$ on $S^{d-1}$ has a density with respect to the surface measure $\sigma$. Additionally, we assume that the density is bounded from below and above by positive constants. The corresponding tail space $L^1(\mathbb{R}^d, \nu^*(x) \, dx)$, see Equation (12), coincides with the space $L^1(\mathbb{R}^d, (1 + |x|)^{-d-2s} \, dx)$.

Out of convenience, we will use the notation $\simeq, \lesssim, \gtrsim$, and, respectively, $\lesssim$ and $\gtrsim$ when we say $f(x) \lesssim g(x)$ for real-valued functions if there exists a constant $C > 0$ such that $f(x) \leq Cg(x)$ for all $x$. The definition of $\gtrsim$ is now self-explanatory and $f(x) \simeq g(x)$ is defined as $f(x) \lesssim g(x)$ and $f(x) \gtrsim g(x)$.

**Proof.** It is sufficient to show that the weights are comparable, that is,

$$\nu^*(x) \simeq \frac{1}{(1 + |x|)^{d+2s}}.$$
We assume that $\mu \simeq \sigma$. Therefore,

$$
\nu^*(x) = \int_{\mathbb{R}^d} \int_{S^{d-1}} 1_\Omega(x + r\vartheta)(1 + |r|)^{-1-2s}\mu(d\vartheta) \, dr \simeq \int_{\mathbb{R}^d} \int_{S^{d-1}} 1_\Omega(x + r\vartheta)(1 + |r|)^{-1-2s}\sigma(d\vartheta) \, dr
$$

$$
\simeq \int_{\mathbb{R}^d} 1_\Omega(x + y)(1 + |y|)^{-1-2s}|y|^{1-d} \, dy.
$$

We begin proving the upper bound on $\nu^*$. We choose a ball $B_R(0)$ that contains $\Omega \subset B_R(0)$ and $\text{dist}(\Omega, \partial B_R(0)) = 2 \text{diam}(\Omega)$. We split the integration domain into $B_R(0)$ and its complement. Notice $(1 + R)^{-1-2s} \leq (1 + |y|)^{-1-2s} \leq 1$ for $y \in B_R(0)$. Therefore,

$$
\int_{B_R(0)} 1_\Omega(x+y)(1 + |y|)^{-1-2s}|y|^{1-d} \, dy \simeq \int_{B_R(0)} 1_\Omega(x+y)|y|^{1-d} \, dy.
$$

By the choice of $R$, we know the following: If $y \in B_R(0)$ and $x + y \in \Omega \subset B_R(0)$, then $x \in B_{2R}(0)$. Therefore,

$$
\int_{B_R(0)} 1_\Omega(x+y)|y|^{1-d} \, dy \leq \sigma(S^{d-1})R \, 1_{B_{2R}(0)}(x) \lesssim \frac{1}{(1 + |x|)^{d+2s}} \, 1_{B_{2R}(0)}(x).
$$

Now we prove an appropriate estimate for the integral over $B_R(0)^c$. First, notice

$$
|y| \geq |x| - |x + y| \geq |x| - R \geq \frac{1}{2}|x|
$$

for any $x \in B_{2R}(0)^c$ and $x + y \in \Omega \subset B_R(0)$. Therefore, for any $x \in B_{2R}(0)^c$,

$$
\int_{B_R(0)^c} 1_\Omega(x+y)(1 + |y|)^{-1-2s}|y|^{1-d} \, dy \leq |\Omega| \, \left(\frac{1}{2}|x|\right)^{-d-2s} \, 1_{B_{2R}(0)}(x) \lesssim \frac{1}{(1 + |x|)^{d+2s}} \, 1_{B_{2R}(0)}(x).
$$

(A1)

For any $x \in B_{2R}(0)$,

$$
\int_{B_R(0)^c} 1_\Omega(x+y)(1 + |y|)^{-1-2s}|y|^{1-d} \, dy \leq |\Omega| \, R^{1-d} \, 1_{B_{2R}(0)}(x) \lesssim \frac{1}{(1 + |x|)^{d+2s}} \, 1_{B_{2R}(0)}(x).
$$

(A2)

The estimates Equation (A1) and Equation (A2) yield the upper bound

$$
\nu^*(x) \lesssim \frac{1}{(1 + |x|)^{d+2s}}.
$$

Now, we prove a lower bound on $\nu^*$. For any $x, y \in \mathbb{R}^d$ such that $x + y \in \Omega$, the estimate $|y| \leq |x| + \max_{z \in \Omega} |z|$ holds. Therefore,

$$
(1 + |y|)^{-1-2s} \geq (1 + |x| + \max_{z \in \Omega} |z|)^{-1-2s} \geq (1 + |x|)^{-1-2s},
$$

$$
|y|^{1-d} \geq (\max_{z \in \Omega} |z| + |x|)^{1-d} \geq |x|^{1-d} \geq (1 + |x|)^{1-d}.
$$

Thus, for $x \in \mathbb{R}^d$,

$$
\int_{\mathbb{R}^d} 1_\Omega(x+y)(1 + |y|)^{-1-2s}|y|^{1-d} \, dy \gtrsim (1 + |x|)^{-d-2s} \int_{\mathbb{R}^d} 1_\Omega(x+y) \, dy \gtrsim \frac{1}{(1 + |x|)^{d+2s}}.
$$

We conclude the desired bounds

$$
\frac{1}{(1 + |x|)^{d+2s}} \lesssim \nu^*(x) \lesssim \frac{1}{(1 + |x|)^{d+2s}}.
$$

$\square$
Now, we give an example of $\nu$, see Equation (8), and a specific $\Omega$ such that functions in the tail space $L^1(\mathbb{R}^d, \nu^*(x) \, dx)$ are not necessarily integrable on $\Omega$.

**Example A.2.** We consider the domain

$$
\Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 < x, y < 1, x < 2y < 3x\} \subset \mathbb{R}^2,
$$

$e_1 = (1, 0), e_2 = (0, 1)$, and the operator $(-\delta_{e_1})^s + (-\delta_{e_2})^s$, which corresponds to $\mu = \delta_{e_1} + \delta_{e_2}$ in Equation (8). We define the function $u : \mathbb{R}^2 \to \mathbb{R}, u(x, y) := (xy)^{-1}$ for $(x, y) \in \Omega$ and $u(x, y) = 0$ else. The singularity in the origin is not integrable and thus $u \notin L^1(\Omega)$. This also implies $u \notin L^1(\mathbb{R}^d, (1 + |x|)^{-d-2s} \, dx)$. On the other hand, recall the definition of the tail space $L^1(\mathbb{R}^d, \nu^*(x) \, dx)$. In this case, we estimate the decay weight by

$$
\nu^*(x, y) \leq \int_{\mathbb{R}^1\Omega} (x + r, y) \, dr + \int_{\mathbb{R}^1\Omega} (x, y + r) \, dr.
$$

Notice for any $(x, y) \in \Omega$,

$$
\int_{\mathbb{R}^1\Omega} (x + r, y) \, dr \leq \frac{4y}{3}, \quad \int_{\mathbb{R}^1\Omega} (x, y + r) \, dr \leq x.
$$

And by

$$
\infty > \frac{4}{3} \ln(3) + \ln(3) \geq \frac{4}{3} \int_{\Omega} x^{-1} \, d(x, y) + \int_{\Omega} y^{-1} \, d(x, y)
$$

$$
\geq \int_{\Omega} (xy)^{-1} \int_{\mathbb{R}} 1_{\Omega}(x + r, y) \, dr \, d(x, y) + \int_{\Omega} (xy)^{-1} \int_{\mathbb{R}} 1_{\Omega}(x, y + r) \, dr \, d(x, y)
$$

$$
\geq \int_{\mathbb{R}^2} |u(x, y)| \nu^*(x, y) \, d(x, y),
$$

we conclude $u \in L^1(\mathbb{R}^d, \nu^*(x) \, dx)$. 
