Conservation laws and stability of higher derivative extended Chern-Simons

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Abstract. The higher derivative field theories are notorious for the stability problems both at classical and quantum level. Classical instability is connected with unboundedness of the canonical energy, while the unbounded energy spectrum leads to the quantum instability. For a wide class of higher derivative theories, including the extended Chern-Simons, other bounded conserved quantities which provide the stability can exist. The most general gauge invariant extended Chern-Simons theory of arbitrary finite order n admits (n − 1)-parameter series of conserved energy-momentum tensors. If the 00-component of the most general representative of this series is bounded, the theory is stable. The stability condition requires from the free extended Chern-Simons theory to describe the unitary reducible representation of the Poincaré group. The unstable theory corresponds to nonunitary representation.

1. Introduction

The higher derivative theories, whose Lagrangians involve second and higher time derivatives of the fields, constitute an important class of models of modern theoretical physics. The higher derivative modifications are known for a wide set of high energy physics models, including electrodynamics [1], Yang-Mills theory [2], conformal theories [3], and gravity [4, 5, 6, 7]. In many cases, the higher derivative theories have higher symmetry and better convergency properties at both the classical and quantum level compared to their counterparts without higher derivatives. In particular, the inclusion of higher derivatives is believed to be one of possible paths to construction of renormalizable quantum gravity [8], see also discussion in [9]. A notorious trouble of higher derivative models concerns instability of their dynamics. We mention the old work of Ostrogrdski [10] as a standard reference.

The instability problems are related to the specifics of construction of the canonical Hamiltonian formulation [10, 11], where all the higher derivatives of the fields are considered as new independent variables. It is a general fact, that the Hamiltonian becomes an unbounded function of the phase-space variables if the higher derivatives are included into the Lagrangian. At the quantum level, this corresponds to the ghost poles in the propagator and related problem of existence of the vacuum state with the lowest energy. The instability of dynamics is not a theorem for the singular theories, where the classical Hamiltonian can be bounded due to constraints [12, 13]. We mention the $f(R)$-gravity and its modifications [5, 6] as exceptional models of such type. For non-singular theories, the classical Hamiltonian is always unbounded, and the stability problem is still relevant [14, 15, 16].

In the recent papers [17, 18, 19], it has been observed that the alternative quantization scheme can be more adequate for a wide class of higher derivative models. Once such formalism exists,
it corresponds to the alternative Hamiltonian formulation with a bounded Hamiltonian at the classical level. For examples of alternative Hamiltonian formulations in the mechanical and field theories we refer to the articles [17, 18, 20, 21]. In all these models, the construction of alternative Hamiltonian formulation proceeds in two steps. First, the most general conservation law of the theory is considered as the Hamiltonian, and then the Poisson bracket such that brings the dynamics to the Hamiltonian form is found. Such special procedure of construction of Hamiltonian means that the structure of classical conserved quantities controls stability: the theory is stable if it admits the bounded integral of motion.

In the present paper, we study the stability of higher derivative theories from the viewpoint of existence of bounded conservation laws. We consider two particular higher derivative models: the toy theory of the Pais-Uhlenbeck oscillator [22] and gauge extended Chern-Simons model [23], which is a field theory of a current interest. We mostly follow the pattern of the paper [24]. We first find the series of higher symmetries that preserves the action functional, and then construct the associated set of conserved quantities by the Noether theorem. This set includes the unbounded canonical energy and the other bounded entries that can stabilize dynamics at both the classical and quantum level. The conditions of existence of bounded conserved quantities are explicitly identified in all the considered models. The proposed stability analysis is general, and it can be used in other physically relevant theories, for example, in the higher derivative gravities.

The rest of the paper is organized as follows. In Section 2, we consider a toy model of the Pais-Uhlenbeck oscillator. In Section 3, we analyze the conserved tensors and stability conditions for the extended Chern-Simons of arbitrary finite order. In Section 4, the general constructions of previous section are exemplified by the third-order extension of Chern-Simons theory. In Conclusion, we summarize the results.

2. Toy model
In this section, we illustrate how the idea of stabilizing dynamics by the conserved quantities works in the simplest higher derivative theory.

The Pais-Uhlenbeck oscillator of fourth order is a theory of a single dynamical variable $x(t)$ with the action functional

$$\mathcal{S}[x(t)] = \int L(x, \dot{x}, \ddot{x}) dt, \quad L(x, \dot{x}, \ddot{x}) = \frac{1}{2} (-\dddot{x}^2 + (\omega_1^2 + \omega_2^2)\dot{x}^2 - \omega_1^2 \omega_2^2 x^2).$$

Here, the parameters $\omega_i, i = 1, 2$, are the frequencies of oscillations. We assume that the frequencies are different and nonzero,

$$\omega_1 \neq \omega_2, \quad \omega_1^2 + \omega_2^2 \neq 0.$$  \hfill (2)

The Euler-Lagrange equation for the model (1) has the form,

$$\frac{\delta \mathcal{S}}{\delta x} \equiv x^{(4)} + (\omega_1^2 + \omega_2^2)\ddot{x} + \omega_1^2 \omega_2^2 x^2 = 0.$$ \hfill (3)

The solution to this equation is the bi-harmonic oscillation,

$$x(t) = A_1 \sin(\omega_1 t + \varphi_1) + A_2 \sin(\omega_2 t + \varphi_2),$$ \hfill (4)

where the amplitudes $A_i$, and initial phases $\varphi_i, i = 1, 2$, are integration constants. The motion is finite,

$$|x(t)| \leq |A_1| + |A_2|.$$ \hfill (5)
Thus, the Pais-Uhlenbeck oscillator is a stable model. Let us explain the stability of the Pais-Uhlenbeck theory from the viewpoint of the conserved quantities of the model. There are two-parameter series of symmetries of the action functional (1),

\[ \delta x = \beta_1 \dot{x} + \beta_2 \ddot{x}, \]

where \( \beta_1, \beta_2 \) are infinitesimal transformation parameters, being constants. The first symmetry in the set is time translation. The second transformation in (6) is the higher symmetry. The two-parameter series of conserved quantities is associated with these symmetries by the Noether theorem:

\[ J = \beta_1 E^{\text{can}} + \beta_2 J^2, \]

where \( E^{\text{can}} \) is the canonical energy of the model, and \( J^2 \) is another independent integral of motion,

\[ E^{\text{can}} = \left( \dot{x}^2 + \frac{1}{2} \ddot{x}^2 \right) + \frac{1}{2} \left( (\omega_1^2 + \omega_2^2) \dot{x}^2 + \omega_1^2 \omega_2^2 \ddot{x}^2 \right), \]

\[ J^2 = \frac{1}{2} \left( \dot{x}^2 + (\omega_1^2 + \omega_2^2) \ddot{x}^2 \right) + \omega_1^2 \omega_2^2 \left( x \ddot{x} - \frac{1}{2} \dddot{x}^2 \right). \]

Both of these quantities are unbounded quadratic forms of initial data \( \dot{x}, \ddot{x}, \ldots x \), but they can be joined in two bounded combinations,

\[ J_1 = J^2 + \omega_2^2 E^{\text{can}}, \quad J_2 = J^2 + \omega_1^2 E^{\text{can}}, \]

\[ J_i = \frac{1}{2} \left( (x^{(3)} + \omega_i^2 \dot{x})^2 + (\omega_1^2 + \omega_2^2 - \omega_i^2) (\ddot{x} + \omega_i^2 x)^2 \right), \quad i = 1, 2. \]

Any bounded combination of these bounded quantities with positive coefficients is a positive-definite quadratic form of initial data.

The positive definite conserved quantity, being constructed from the integrals of motion (9), selects the stationary bounded surface in the phase-space of the theory. This bounded conserved quantity stabilizes the dynamics of the Pais-Uhlenbeck theory.

3. Extended Chern-Simons

Consider 3d Minkowski space with the local coordinates \( x^\mu, \mu = 0, 1, 2 \), and the metric

\[ \eta_{\mu\nu} = \text{diag}(+1, -1, -1). \]

The extended Chern-Simons is a gauge theory of vector field \( A = A_\mu(x) dx^\mu \) with the action functional

\[ S[A(x)] = m^2 \sum_{p=1}^{n} \alpha_p \int A_\mu(x) F^{(p)}(x) d^3x. \]

Here, \( m \) is a parameter with dimension of mass, and dimensionless real constants \( \alpha_1, \ldots, \alpha_n \) are parameters of the model. Without loss of generality we assume that \( \alpha_n \neq 0 \), and the notation is used:

\[ F^{(p)} = m^{-p} \varepsilon_{\mu_1 \mu_2} \partial^{\rho_1} F^{(p-1)}(x^{\mu_1}), \quad F^{(0)} \equiv A_\mu, \quad r = 1, \ldots, n, \]

where \( \varepsilon \) denotes the 3d Levi-Civita symbol with \( \varepsilon_{012} = 1 \). The Euler-Lagrange equations for the action functional (11) have the form

\[ \frac{\delta S}{\delta A} = m^2 \sum_{p=1}^{n} \alpha_p F^{(p)} = 0. \]
These equations involve the \( n \)-th time derivatives of \( A \).

The action (11) is Poincaré-invariant. The space-time translations are symmetries of the action functional (11),

\[
\delta \xi A_\mu = \xi^\nu \partial_\nu A_\mu ,
\]

where \( \xi \) is the transformation parameter. The canonical energy-momentum is associated with this symmetry,

\[
T_{\mu\nu}^{\text{can}}(\alpha) = \frac{m^2}{2} \sum_{p=1}^{n} \sum_{r+s=p} \alpha_p \left( F^{(r)} \mu F^{(s)} \nu + F^{(r)} \nu F^{(s)} \mu - \eta_{\mu\nu}\eta^{\rho\sigma} F^{(r)} \rho F^{(s)} \sigma \right).
\]

The 00-component of the energy-momentum tensor has the form

\[
T_{00}^{\text{can}}(\alpha) = \frac{m^2}{2} \sum_{p=1}^{n} \sum_{r+s=p} \alpha_p F^{(r)} \mu F^{(s)} \mu .
\]

This quantity is linear in \( F^{(n-1)} \) for \( n > 2 \),

\[
T_{00}^{\text{can}}(\alpha) = \frac{m^2}{2} \alpha_n F^{(n-1)} \mu F^{(1)} \mu + \ldots,
\]

the dots denote the terms that do not include \( F^{(n-1)} \). Thus, the energy in the extended Chern-Simons theory is unbounded whenever the higher derivatives are included in the Lagrangian.

The series of higher symmetries, which generalize (14) of the action functional (11) has the form

\[
\delta \xi A_\mu = \sum_{q=1}^{n-1} \beta_q \xi^\nu \partial_\nu F^{(q-1)} \mu ,
\]

where \( \beta_q, q = 1, \ldots, n - 1 \), are the parameters of symmetry series, being real constants. The \((n-1)\)-parameter series of conserved tensors is connected with this symmetry by the Noether theorem,

\[
T_{\mu\nu}(\alpha, \beta) = \sum_{r,s=1}^{n-1} C_{r,s}(\alpha, \beta) \left( F^{(r)} \mu F^{(s)} \nu + F^{(r)} \nu F^{(s)} \mu - \eta_{\mu\nu}\eta^{\rho\sigma} F^{(r)} \rho F^{(s)} \sigma \right).
\]

The Bezout matrix \( C_{r,s}(\alpha, \beta) \) of two polynomials is defined by the generating relation

\[
C_{r,s}(\alpha, \beta) = \left[ \frac{\partial^{r+s}}{\partial z^r \partial z'^s} \left( \frac{M(z)N(z') - M(z')N(z)}{z - z'} \right) \right]_{z = z' = 0},
\]

where \( z \) and \( z' \) are two independent variables, and

\[
M(z) = \sum_{p=1}^{n} \alpha_p z^p , \quad N(z) = \sum_{q=1}^{n-1} \beta_q z^q .
\]

The representatives of the series (19) are defined by the formula

\[
T_{\mu\nu}^{(q)}(\alpha) = \frac{\partial T_{\mu\nu}(\alpha, \beta)}{\partial \beta_q} , \quad q = 1, \ldots, n - 1 .
\]
By construction, \( T^{(1)}_{\mu\nu} = T^{\text{can}}_{\mu\nu} \), and other conserved tensors are independent.

The 00-component of general conserved tensor (19) has the form

\[
T_{00}(\alpha,\beta) = \frac{m^2}{2} \sum_{\mu=0}^{n-1} \sum_{r,s=1}^{n-1} C_{r,s}(\alpha,\beta) F^{(r)}_{\mu} F^{(s)}_{\mu}.
\]  

This quantity is a quadratic form of the variables \( F^{(r)}_{\mu} \). So, it is bounded if

\[
C_{r,s}(\alpha,\beta) \text{ is a positive definite matrix}.
\]  

This condition is a restriction on the parameters \( \beta \) in the series of energy-momentum tensors (19). It is consistent, iff the polynomial

\[
M'(z) = \sum_{q=0}^{n-1} \alpha_{q+1} z^q
\]  

has simple and real roots. From the viewpoint of the representation theory, it means that the stability condition requires from the free extended Chern-Simons theory to describe the unitary reducible representation of the Poincaré group. If the roots of \( M'(z) \) are multiple or complex, there is no bounded integral of motion that can stabilize the dynamics.

4. Extended Chern-Simons of order 3

Let us demonstrate the general construction in the case \( n = 3 \). The action functional of the model reads

\[
S[A(x)] = \frac{1}{2} \int \left( \alpha_3 m^{-1} \varepsilon_{\mu\nu\rho} \partial^\rho G^\mu + \alpha_2 G^\mu + \alpha_1 m F^\mu \right), \quad \alpha_3 \neq 0,
\]  

where the notation is used

\[
G^\mu \equiv F^{(2)}_{\mu} = m^{-2} \left( \partial_\mu \partial^\nu A^\nu - \partial_\nu \partial^\nu A_\mu \right), \quad F^\mu \equiv F^{(1)}_{\mu} = m^{-1} \varepsilon_{\mu\nu\rho} \partial^\rho A^\nu.
\]  

The model (26) is invariant under the following two-parameter series of symmetries:

\[
\delta_\xi A_\mu = \beta_1 \xi^\nu \partial_\nu A_\mu + \beta_2 \xi^\nu \partial_\nu F^\mu,
\]  

The corresponding two-parameter series of conserved tensors reads

\[
T^{\text{can}}_{\mu\nu}(\alpha,\beta) = \beta_1 T^{\text{can}}_{\mu\nu}(\alpha) + \beta_2 T^{(2)}_{\mu\nu}(\alpha),
\]  

where

\[
T^{\text{can}}_{\mu\nu}(\alpha) = \frac{m^2}{2} \left( \alpha_3 \left( G^\mu F^\nu + G^\nu F^\mu - \eta_{\mu\nu} G^\rho F^\rho \right) + \alpha_2 \left( 2F^\mu F^\nu - \eta_{\mu\nu} F^\rho F^\rho \right) \right)
\]  

is the canonical energy-momentum, and

\[
T^{(2)}_{\mu\nu}(\alpha) = m^2 \left( \alpha_3 \left( G^\mu G^\nu - \frac{1}{2} \eta_{\mu\nu} G^\rho G^\rho \right) - \alpha_1 \left( F^\mu F^\nu - \frac{1}{2} \eta_{\mu\nu} F^\rho F^\rho \right) \right)
\]
is another independent conserved tensor. The canonical energy is linear in $G_\mu$ and unbounded, 

$$ T_{\text{can}}^{00}(\alpha) = m^2 \sum_{\mu=0}^{2} \left( \alpha_5 G_\mu F_\mu + \frac{1}{2} \alpha_2 F_\mu F_\mu \right). \quad (32) $$

The 00-component of the general conserved tensor (29) has the form 

$$ T_{00}(\alpha, \beta) = m^2 \sum_{\mu=0}^{2} \left( \beta_2 \alpha_5 G_\mu G_\mu + 2 \beta_1 \alpha_5 G_\mu F_\mu + (\beta_1 \alpha_2 - \beta_2 \alpha_1) F_\mu F_\mu \right). \quad (33) $$

It can be bounded. The boundedness condition for the quadratic form reads 

$$ \alpha_3 \beta_2 > 0, \quad -\alpha_3 \beta_2^2 + \alpha_2 \beta_1 \beta_2 - \alpha_1 \beta_2^2 > 0. \quad (34) $$

It is consistent if parameters of the model (26) satisfy the condition 

$$ \alpha_2^2 - 4 \alpha_1 \alpha_3 > 0. \quad (35) $$

According to the representation theory, the stable theory corresponds to one of the two cases: theory of two self-dual massive spin 1 with different masses, or theory of massless spin 1 and massive spin 1 subject to a self-duality condition [25, 26].

5. Conclusion
We considered a class of vector field models whose wave operator is a polynomial in the Chern-Simons operator. We demonstrated that the gauge theory of order $n$ admits $(n-1)$-parameter series of conserved tensors, whose 00-component can be bounded, while the canonical energy is always unbounded for $n > 2$. The bounded conservation laws ensure the stability of dynamics at classical level. At quantum level the stability is provided by bounded Hamiltonian. The constrained Hamiltonian formulations with bounded Hamiltonian of the extended Chern-Simons were constructed in [21, 27].

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