Interpolation on Jets

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1 Introduction

We work over a field $k$ of characteristic zero.

A jet in $n$-dimensional projective space $\mathbb{P}_k^n$ over $k$ will be any divisor on a one dimensional linear subspace (i.e. a line in $\mathbb{P}_k^n$) with support a point. The length of a jet will be its degree as a divisor and we will say $r$-jet for a jet of length $r$. When $1 < r$, an $r$-jet is contained in a unique line, which we call its axis. As for $r = 1$, we will distinguish 1-jets, which are furnished with an axis (namely a line containing them) from (free) points. Finally, by convention, a 0-jet is the empty subscheme.

We say that a closed sub-scheme $Y \hookrightarrow \mathbb{P}_k^n$ has maximal rank in degree $d \geq 0$ if the canonical map $H^0(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(d)) \rightarrow H^0(Y, \mathcal{O}_Y(d))$ has maximal rank as a linear map.

Our theorem 1.2 slightly refines a result proved by A. Eastwood in [E1,2] characterizing those generic unions of jets having maximal rank. The proof in [1, 2] used complex specialisation arguments and his proof runs to some sixty pages. Our proof is independent of this earlier one and much simpler. It works by an induction argument based on the simple proposition 2.1, which allows us to reduce to the extremal case where the sequence of lengths is maximal (among the allowed ones) with respect to the lexicographical order. This case can then be treated by elementary techniques when $n = 2$ and for $n \geq 3$, using, as was done in [1, 2], using an old theorem of Hartshorne and the second author [3]. This theorem says that generic unions of lines have maximal rank in any degree.

Definition. 1.1 We will say that a set $S$ of lines is in general position if for any subset of $S$, the corresponding union has maximal rank in any degree.

Observe that in $\mathbb{P}^2$, any finite set of lines is in general position, while in $\mathbb{P}^n$, it follows from the above mentioned result that any generic finite set of lines is in general position.

Theorem. 1.2 Let $L = (L_1, \ldots, L_m)$ be a sequence of $m$ lines in general position in $\mathbb{P}^n_k$, $r = (r_1, \ldots, r_m)$ be a sequence of positive integers in non-increasing order and $d$ an integer such that the sum $r_1 + \cdots + r_m$ is at most \( \binom{n+d}{d} \). Then the union $J_1 \cup \ldots \cup J_m$ where $J_i$ is the generic $r_i$-jet on $L_i$ has maximal rank in degree $d$ if and only if the following (necessary) numerical condition $C(n, d)$ holds: $r_1 \leq d + 1$ and, if $n = 2$, then for any $1 \leq s \leq d + 1$, $r_1 + \cdots + r_s \leq d.s + 1 - \binom{s-1}{2}$. 

requiring a contact of order $r_i$ at the generic point $x_i$ of $L_i$ for $i = 1, \ldots, m$, are linearly independent precisely under the expected numerical conditions given in the theorem.

Before passing on to the proof, we introduce the following notation.

**Definition. 1.3** For a fixed $n$, we say that a sequence of non-negative integers $(\chi, r_1, \ldots, r_m)$ is $d$-admissible if

1. $\chi + r_1 \cdots + r_m = \binom{n+d}{d}$
2. $r_1 \geq \cdots \geq r_m \geq 1$
3. if $n = 2$ then $r_1 + \cdots + r_s \leq ds + 1 - (s-1)(s-2)/2$

for $s = 1, \ldots, d+1$.

and we denote by $S_d$ the set of all $d$-admissible sequences, which we give the total lexicographical order (i.e. $(\alpha_i)_i > (\beta_i)_i$ if the first non-zero term in the sequence $(\alpha_i - \beta_i)_i$ is strictly positive).

**Definition. 1.4** Given a $d$-admissible sequence $r = (\chi, r_1, \ldots, r_m)$ and a sequence of lines in general position $L = (L_1, \ldots, L_m)$ it is clear that the union $J_1 \cup \cdots \cup J_m$ of the generic $r_i$-jets $J_i$ in $L_i$, has maximal rank in degree $d$ if and only if the union $J := J_{L,r} = J_1 \cup \cdots \cup J_m \cup R$ has maximal rank, where $R$ is the union of $\chi$ generic (free) points. We call such a union $J$ a $d$-admissible union of jets and we say that

$$r = (\chi, r_1, \ldots, r_m)$$

is the weight of $J$.

In this language, theorem 1.2 simply says that every $d$-admissible union of jets has maximal rank in degree $d$.

Without further comment, we will freely use the fact that “maximal rank in degree $d$ ”is stable by generisation. We express our gratitude to the referee for having suggested changes which have improved the presentation of the proof.

## 2 The proof

**Proposition. 2.1** Fix a closed subscheme $Y \subset \mathbb{P}^n_k$ and let $D$ be a line not contained in $Y$. Suppose that for some $d, v > 0$ the union of $Y$ with the generic jet of length $v + 1$ (resp $v - 1$) in $D$ has maximal rank in degree $d$, then the union of $Y$ with the generic $v$-jet in $D$ has maximal rank in degree $d$.

**Proof.** We denote by $Y_r$ the union of $Y$ with the generic $r$-jet; denoted $J_r$; in $D$ and by $m_r$ the canonical map

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(\mathbb{P}^n, \mathcal{O}_Y(d)) \times H^0(\mathbb{P}^n, \mathcal{O}_J(d)) = H^0(\mathbb{P}^n, \mathcal{O}_{Y_r}(d))$$

...
If \( m_{v+1} \) is surjective, then so is \( m_v \), and if \( m_{v-1} \) is injective, then so is \( m_v \). In the remaining case, \( m_{v+1} \) is injective and not surjective, and \( m_{v-1} \) is surjective and not injective and we have to prove that \( m_v \) is bijective.

What is clear is that \( h^0(\mathbb{P}^n, I_Y(d)) = v \), where \( I_Y \) is the ideal sheaf of \( Y \) as a subscheme of \( \mathbb{P}^n \), and that the canonical map
\[
H^0(\mathbb{P}^n, I_Y(d)) \hookrightarrow H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))
\]
is injective. It follows that the map
\[
H^0(\mathbb{P}^n, I_Y(d)) \hookrightarrow H^0(D, \mathcal{O}_D(d))
\]
is injective. If \( W \) is the image subspace of this latter map we need only show that the generic \( v \)-jet in \( D \) imposes linearly independent conditions on \( W \), but this is true for any \( v \)-dimensional subspace of \( H^0(D, \mathcal{O}_D(d)) \). This follows from the well known fact that a family of polynomials \( f_1, \ldots, f_v \in k[t] \) is linearly independent if and only if the \( v \times v \) Wronskian
\[
W(f_1, \ldots, f_v) = \begin{bmatrix} \partial^i f_j \\ \partial t^i \end{bmatrix}
\]
has rank \( v \).

2.1 Proof of theorem 1.2 for \( \mathbb{P}^2 \)

As indicated after definition 1.4, we will prove that every \( d \)-admissible union of jets has maximal rank in degree \( d \).

We will write a \( d \)-admissible weight \( r = (\chi, \sigma_1, \ldots, \sigma_m) \) in the form
\[
r = (\chi, m_1, \ldots, m_p, r_1, \ldots, r_q)
\]
where \( m_1, \ldots, m_p \) is the extremal sequence \( d+1, \ldots, d+2-p \) given by the condition 1.3 (iii) and \( r_1 < d+1-p \).

Fix a \( d \)-admissible union of jets \( J \) of weight
\[
r = (\chi, m_1, \ldots, m_p, r_1, \ldots, r_q)
\]
and suppose that every \( d \)-admissible union of jets \( J' \) of weight \( r' > r \) has maximal rank in degree \( d \). We will show that this implies that \( J \) has maximal rank in degree \( d \) completing the proof.

If \( q \leq 1 \) (i.e. \( r_2 = 0 \)) then by specialising the \( \chi \) free points, we can specialise \( J \) to the \( d \)-admissible union of jets \( J_d \) of weight \( (0, d+1, \ldots, 1) \). A simple induction then shows that any homogeneous form of degree \( d \) on \( \mathbb{P}^2 \) which vanishes on \( J_d \) vanishes on all \( d+1 \) axes and is thus identically zero. This shows that \( J \) has maximal rank in degree \( d \) if \( r_2 = 0 \).

Now suppose that \( r_2 > 0 \). We will apply 2.1 with
\[
Y = J_1 \cup \cdots \cup J_p \cup J''_1 \cup \cdots \cup J''_q \cup R
\]
where \( J_i \) is the \( m_i \)-jet, \( J''_i \) is the \( r_i \)-jet and \( R \) is the union of the \( \chi \) free points. We...
On the one hand, \( J_{(-)} \) is contained in the union \( \tilde{J}_{(-)} \) obtained by adding a generic free point to \( J_{(-)} \). Now \( \tilde{J}_{(-)} \) is clearly a \( d \)-admissible union of jets of weight \( \tilde{r}_{(-)} > r \) so that \( J_{(-)} \) has maximal rank in degree \( d \) by the induction hypothesis.

On the other hand, \( J_{(+)} \) contains the union of jets \( \tilde{J}_{(+)} \) obtained by contracting \( J'_2 \) to length \( r_2 - 1 \) and it will suffice to show that the weight

\[
\tilde{r}_{(+)} = (X, m_1, \ldots, m_p, r_1 + 1, r_2, \ldots, r_{t-1}, r_t - 1, r_{t+1}, \ldots, r_\delta)
\]

of \( \tilde{J}_{(+)} \) is \( d \)-admissible, where \( r_2 = \cdots = r_t > r_{t+1} \). This is equivalent to showing that the series of inequalities

\[
r_1 + (s - 1)r_2 \leq (d - p)s + 1 - (s - 1)(s - 2)/2
\]

for \( 1 \leq s \leq \min(t, d - p + 1) \) implies strict inequality for \( 1 \leq s \leq \min(t - 1, d - p + 1) \). Since the associated quadratic form

\[
Q(s) = s^2 - (3 + 2(d - p - r_2))s + 2(r_1 - r_2)
\]

vanishes on the interval \([0, 1]\), there is only one root \( \geq 1 \). This completes the proof if \( t \leq d - p + 1 \) or if \( Q \) has no integer roots \( \leq d - p + 1 \). We will now finish by showing that if \( t > d - p + 1 \) then \( Q \) does not have integer roots. In fact if \( Q \) has integer roots, they must be 0 and \( s_1 = 3 + 2(d - p - r_2) \leq d - p + 1 \). In this case \( r_2 \geq (d - p + 2)/2 \). However we have

\[
tr_2 \leq r_1 + (t - 1)r_2 \leq r_1 + \ldots + r_t \leq (d - p + 2)(d - p + 1)/2
\]

showing that \( t \leq d - p + 1 \). □

### 2.2 Proof of theorem 1 for \( \mathbb{P}^n, n \geq 3 \).

The proof is similar to the previous one. We write the weight \( r \) of a \( d \)-admissible union of jets \( J \) in the form

\[
r = (X, m_1, \ldots, m_p, r_1, \ldots, r_\delta)
\]

where \( m_i = d + 1 \) and \( r_1 \leq d \).

Now let \( J \) be a \( d \)-admissible union of jets of weight \( r \) and suppose that every \( d \)-admissible union of jets \( J' \) of weight \( r' > r \) has maximal rank in degree \( d \). We will show that this implies that \( J \) has maximal rank in degree \( d \).

If \( r_2 \neq 0 \) one easily applies 2.1 as before. So suppose that \( r_2 = 0 \). By specialising the \( \chi \) free points firstly to the \( r_1 \)-jet then to further lines in general position we can specialise \( J \) to a \( d \)-admissible union of jets of weight \( r = (0, m_1, \ldots, m_p, \delta) \) where \( 0 \leq \delta \leq d \). This is equivalent to the corresponding problem obtained by replacing the \( p, (d + 1) \)-jets by their corresponding axes. That is to say, we must show that if \( D_1, \ldots, D_{p+1} \) are \( p + 1 \) lines in general position, then the union of \( D_1, \ldots, D_p \) with the generic \( \delta \)-jet in \( D_{p+1} \) has maximal rank in degree \( d \). The general position hypothesis implies that \( h^0(\mathbb{P}^n, I_{D_1 \cup \cdots \cup D_p}(d)) = \delta \) and that the canonical map

\[
H^0(\mathbb{P}^n, I_{D_1 \cup \cdots \cup D_p}(d)) \to H^0(D_{p+1}, O_{D_{p+1}}(d))
\]

is injective. □
Note: our method would work equally well for any union of curves: from a maximal rank statement concerning a union of curves, we derive a maximal rank statement for unions of generic jets on these curves, under natural necessary numerical conditions.

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