Optimal estimates for the perfect conductivity problem with inclusions close to the boundary

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Abstract

When a convex perfectly conducting inclusion is closely spaced to the boundary of the matrix domain, a “bigger” convex domain containing the inclusion, the electric field can be arbitrary large. We establish both the pointwise upper bound and the lower bound of the gradient estimate for this perfect conductivity problem by using the energy method. These results give the optimal blow-up rates of electric field for conductors with arbitrary shape and in all dimensions. A particular case when a circular inclusion is close to the boundary of a circular matrix domain in dimension two is studied earlier by Ammari, Kang, Lee, Lee and Lim (2007). From the view of methodology, the technique we develop in this paper is significantly different from the previous one restricted to the circular case, which allows us further investigate the general elliptic equations with divergence form.

1 Introduction and main results

It is well known that in high-contrast fiber-reinforced composites high concentration of extreme electric field or mechanical loads will cause failure initiation in zones, which are created by extreme loads amplified by composite microstructure, including the narrow regions between two adjacent inclusions and the thin gaps between the inclusions and the matrix boundary. The main purpose of this paper is to study the blow-up estimate of \(|\nabla u|\) where the high concentration of electric field is created. Note that the anti-plane shear model is consistent with the two-dimensional conductivity model. Thus, the blow-up analysis for electric field have a valuable meaning in relation to in the failure analysis of composite material.

There have been many important works on the gradient estimates for the conductivity problem in the presence of inclusions. For two adjacent inclusions \(D_1\) and \(D_2\) with \(\varepsilon\) apart, Keller [22] was the first to use analysis to estimate the effective properties of

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particle reinforced composites. Bonnetier and Vogelius [13] and Li and Vogelius [25] proved the uniform boundedness of $|\nabla u|$ regardless of $\varepsilon$ provided that the conductivities stay away from 0 and $\infty$. Li and Nirenberg [24] extended the results in [25] to general divergence form second order elliptic systems including systems of elasticity. This in particular answered in the affirmative the question naturally led to by the numerical indication by Babuška, Andersson, Smith, and Levin [6] for the boundedness of the strain tensor as $\varepsilon$ tends to 0. On the other hand, in order to investigate the high-contrast conductivity problem, Ammari, Kang, and Lim [1] were the first to study the case of the close-to-touching regime of particles whose conductivities degenerate, a lower bound on $|\nabla u|$ was constructed there showing blow-up in both the perfectly conducting and insulating cases. This blow-up was proved to be of order $\varepsilon^{-1/2}$ in $\mathbb{R}^2$. In their subsequent work with H. Lee and J. Lee [4] they established upper and lower bounds on the electric field for the close-to-touching regime of two circular particles in $\mathbb{R}^2$ with degenerate conductivities. Another interesting case of a particle very close to the boundary is also considered and similar lower and upper bounds for $|\nabla u|$ are established. Subsequently, it has been proved by many mathematicians that for the two close-to-touching inclusions case the generic blow-up rate of $|\nabla u|$ blow-up is $\varepsilon^{-1/2}$ in two dimensions, $|\varepsilon \log \varepsilon|^{-1}$ in three dimensions, and $\varepsilon^{-1}$ in dimensions greater than four. See Bao, Li and Yin [7, 8], as well as Lim and Yun [26]. Further, more detailed, characterizations of the singular behavior of gradient of $u$ have been obtained by Ammari, Ciraolo, Kang, Lee and Yun [2], Ammari, Kang, Lee, Lim and Zribi [5], Bonnetier and Triki [11, 12], Gorb and Novikov [18] and Kang, Lim and Yun [20, 21]. For more related work on elliptic equations and systems from composites, see [9, 10, 14, 15, 16, 19, 23, 28, 31] and the references therein. However, for the second case of a particle close to the boundary, to the best of our knowledge, there has not been any further result after [4] on the investigation that how the boundary data effects the gradient of the solution until now.

Actually, in [4], Ammari, Kang, Lee, Lee, and Lim also studied the case that a small disk $B_r \subset \mathbb{R}^2$, with $\infty$ conductivity, is close to the boundary of a big disk $B_{\rho} (B_{\rho} \ni B_r)$, for which the blow-up rate $\varepsilon^{-1/2}$ is established. Essentially two-dimensional potential theory techniques for circular domain are used in [4], and the authors point out the importance of the three-dimensional case. In this paper we make use of energy method to establish the optimal gradient estimates in all dimensions when a general convex inclusion is very close to the boundary of a “bigger” convex domain which contains the inclusion.

Before stating our results, we first describe the nature of our domain. Let $D$ be a bounded open set in $\mathbb{R}^n (n \geq 2)$, $D_1$ be a strictly convex open subset of $D$, both being of class $C^{2,\alpha}$ ($0 < \alpha < 1$), and denote the distance $\text{dist}(D_1, \partial D) =: \varepsilon > 0$. We further assume that the $C^{2,\alpha}$ norms of $\partial D_1$ and $\partial D$ are bounded by some constant independent of $\varepsilon$.

Suppose that the conductivity of the inclusion $D_1$ degenerates to $\infty$; in other words,
the inclusion is a perfect conductor. We consider the following conductivity problem

\[
\begin{aligned}
\Delta u &= 0, & \text{in } D \setminus \bar{D}_1, \\
u &= C_1, & \text{on } \partial D, \\
\int_{\partial D_1} \frac{\partial u}{\partial n} &= 0, \\
u &= \varphi, & \text{on } \partial D,
\end{aligned}
\]

(1.1)

where \( \varphi \in C^2(\partial D) \), \( C_1 \) is some constant to be determined later, and

\[
\frac{\partial u}{\partial \nu} := \lim_{\tau \to 0} \frac{u(x + \nu \tau) - u(x)}{\tau}.
\]

Here and throughout this paper \( \nu \) is the outward unit normal to the domain and the subscript \( \pm \) indicates the limit from outside and inside the domain, respectively. The existence, uniqueness and regularity of solutions to equation (1.1) can be referred to the Appendix in [7], with a minor modification.

Throughout this paper, unless otherwise stated, \( C \) denotes a constant, whose value may vary from line to line, depending only on \( n, \kappa_0, \kappa_1 \) and an upper bound of the \( C^{2,\alpha} \) norms of \( \partial D_1 \) and \( \partial D \), but not on \( \varepsilon \). Also, we call a constant having such dependence a universal constant.

Let \( P \in \partial D \) be the nearest point to \( D_1 \). Let \( PP_1 \) denote the shortest line segment between \( \partial D \) and \( \partial D_1 \). Denote

\[
\rho_n(\varepsilon) = \begin{cases} 
\sqrt{\varepsilon}, & n = 2, \\
\frac{1}{|\ln \varepsilon|}, & n = 3, \\
1, & n \geq 4.
\end{cases}
\]

(1.2)

We have the following gradient estimates in all dimensions.

**Theorem 1.1.** Let \( D_1 \subset D \subset \mathbb{R}^n (n \geq 2) \) be defined as above. Let \( u \in H^1(D) \cap C^1(D \setminus \bar{D}_1) \) be the solution to (1.1). Then for \( 0 < \varepsilon < 1/2 \), we have

\[
|\nabla u(x)| \leq \frac{C \rho_n(\varepsilon)}{\varepsilon + \text{dist}^2(x, PP_1)} + C \left( \frac{\text{dist}(x, PP_1)}{\varepsilon + \text{dist}^2(x, PP_1)} + 1 \right) \| \varphi \|_{C^2(\partial D)}, \quad x \in D \setminus D_1,
\]

(1.3)

and if \( |Q[\varphi]| \geq c^* \) for some universal constant \( c^* > 0 \), then

\[
|\nabla u(x)| \geq \frac{\rho_n(\varepsilon)|Q[\varphi]|}{C \varepsilon}, \quad x \in PP_1,
\]

(1.4)

where

\[
Q[\varphi] = \int_{\partial D_1} \frac{\partial v_0}{\partial \nu}
\]

(1.5)

is a bounded functional of \( \varphi \), and \( v_0 \in C^2(D \setminus \bar{D}_1) \) is uniquely determined by

\[
\begin{aligned}
\Delta v_0 &= 0, & \text{in } D \setminus \bar{D}_1, \\
v_0 &= 0, & \text{on } \partial D_1, \\
v_0 &= \varphi(x) - \varphi(P), & \text{on } \partial D.
\end{aligned}
\]

(1.6)
Remark 1.1. If \( \varphi = 0 \), then the solution of (1.1) is \( u \equiv 0 \). On the other hand, by (1.6), we have \( v_0 \equiv 0 \), so \( Q[\varphi] = 0 \). Thus, Theorem 1.1 is obvious. So we only need to prove it for \( \|\varphi\|_{C^2(\partial D)} = 1 \) by considering \( u/\|\varphi\|_{C^2(\partial D)} \). Our result do not really need \( D \) and \( D_1 \) to be strictly convex. In fact, our proof of Theorem 1.1 applies to more general situations where \( \partial D \) and \( \partial D_1 \) are relatively strictly convex in a neighborhood of \( \overline{PP_1} \). Even when they are not necessarily relatively convex near \( P \) and \( P_1 \), while the distance between them remains to be \( \varepsilon \), our method also can be applied; for more details, see discussions in Subsection 2.4.

Remark 1.2. The upper bound in (1.3) is a pointwise estimate, which provides more information than that in [7]. Moreover, the effect of the boundary data to the blow-up of \( |\nabla u| \) is captured by (1.3) and (1.4). In this sense, we can regards (1.3) and (1.4) as boundary estimates in relation to the interior estimates in [7], where two adjacent inclusions were considered. Furthermore, it turns out that the functional \( Q[\varphi] \) plays an important role in the blow-up analysis. It is interesting to know when \( |Q[\varphi]| \geq c^* \) for some positive universal constant \( c^* \). A sufficient condition for the existence of \( c^* \) is given in Subsection 2.3.

The approach developed in the proof of Theorem 1.1 can be extended to study general elliptic equations with a divergence form. Let \( n, D_1, D, \varepsilon \) and \( \varphi \) be the same as in Theorem 1.1 and let \( A_{ij}(x) \in C^2(D \setminus \overline{D}_1) \) be \( n \times n \) symmetric matrix functions and satisfy the uniform elliptic condition

\[
\lambda |\xi|^2 \leq A_{ij}(x)\xi_i\xi_j \leq \Lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \quad x \in D \setminus \overline{D}_1, \tag{1.7}
\]

where \( 0 < \lambda \leq \Lambda < +\infty \). We consider

\[
\begin{align*}
\partial_i \left( A_{ij}(x) \partial_j u \right) &= 0, & \text{in } D \setminus \overline{D}_1, \\
u &= C_1, & \text{on } \overline{D}_1, \\
\left| \int_{\partial D_1} (A_{ij}(x) \partial_j u)n_i \right|_+ &= 0, \\
u &= \varphi, & \text{on } \partial D.
\end{align*}
\tag{1.8}
\]

Then

**Theorem 1.2.** Let \( D_1 \subset D \subset \mathbb{R}^n \) \((n \geq 2)\) be defined as above. Let \( u \in H^1(D) \cap C^1(D \setminus \overline{D}_1) \) be the solution to (1.8). Then for \( 0 < \varepsilon < 1/2 \), we have (1.3) and (1.4) hold, where

\[
Q[\varphi] = \int_{\partial D_1} \frac{\partial V_0}{\partial y}
\]

where \( V_0 \) is uniquely determined by

\[
\begin{align*}
\partial_i \left( A_{ij}(x) \partial_j V_0 \right) &= 0, & \text{in } D \setminus \overline{D}_1, \\
V_0 &= 0, & \text{on } \partial D_1, \\
V_0 &= \varphi(x) - \varphi(P), & \text{on } \partial D.
\end{align*}
\tag{1.9}
\]

In order to prove Theorem 1.1, we decompose the solution \( u \) of (1.1) as follows

\[
u(x) = (C_1 - \varphi(P))v_1(x) + v_0(x) + \varphi(P), \quad x \in D \setminus \overline{D}_1, \tag{1.10}
\]
where \( v_1 \in C^2(D \setminus \overline{D}_1) \) satisfies

\[
\begin{cases}
\Delta v_1 = 0, & \text{in } D \setminus \overline{D}_1, \\
v_1 = 1, & \text{on } \partial D_1, \\
v_1 = 0, & \text{on } \partial D.
\end{cases}
\]

Then we have

\[
\nabla u(x) = (C_1 - \varphi(P))\nabla v_1(x) + \nabla v_0(x), \quad x \in D \setminus \overline{D}_1.
\]

Since \( u = C_1 \) on \( \partial D_1 \) and \( \|u\|_{H^1(\Omega)} \leq C \) (independent of \( \varepsilon \)), it follows from the trace embedding theorem that

\[
|C_1| \leq C.
\]

Thus, the proof of Theorem 1.1 is reduced to the estimate of \( |\nabla v_1| \) and \( |\nabla v_0| \).

Similarly, for Theorem 1.2 we define \( V_1 \in C^2(D \setminus \overline{D}_1) \) satisfying

\[
\begin{cases}
\partial_i \left(A_{ij}(x) \partial_j V_1 \right) = 0, & \text{in } D \setminus \overline{D}_1, \\
V_1 = 1, & \text{on } \partial D_1, \\
V_1 = 0, & \text{on } \partial D.
\end{cases}
\]

The rest of this paper is organized as follows. In section 2, we mainly estimate \( |\nabla v_1| \) and \( |\nabla v_0| \). By constructing an auxiliary function \( \bar{u} \), and proving the boundedness of \( |\nabla (v_1 - \bar{u})| \), we show that \( |\nabla \bar{u}| \) is actually the main term of \( |\nabla v_1| \). By the same way, we obtain the estimate of \( |\nabla v_0| \). Thus, the optimal gradient estimate is established for convex inclusions in all dimensions. In section 3 we give the main ingredients of the proof of Theorem 1.2. For general elliptic equations with a divergence form, we construct an auxiliary function \( \tilde{u} \), associated with the coefficients \( A_{ij} \), and then obtain the boundedness of \( |\nabla (V_1 - \tilde{u})| \) and the estimate of \( |\nabla V_0| \).

## 2 Proof of Theorem 1.1

After a possible translation and rotation if necessary, we may assume without loss of generality that \( D, D_1 \) are two strictly convex domains, which satisfy the following:

\[ D_1 \subset D, \quad P_1 = (0', \varepsilon) \in \partial D_1, \quad P = (0', 0) \in \partial D, \quad \text{and } \text{dist}(D_1, \partial D) = \varepsilon. \]

Denote \( \Omega := D \setminus \overline{D}_1 \). Near the origin, we assume that there exists a universal constant \( R_0 \), independent of \( \varepsilon \), such that \( \partial D_1 \) and \( \partial D \) can be represented by the graph of

\[ x_n = \varepsilon + h_1(x') \quad \text{and} \quad x_n = h(x'), \quad \text{for } |x'| \leq 2R_0, \]

respectively, where \( h_1 \) and \( h \) satisfy

\[
\begin{align*}
\varepsilon + h_1(x') &> h(x'), \quad \text{for } |x'| < 2R_0, \\
h_1(0') = h(0') = 0, \quad \nabla_{x'} h_1(0') = \nabla_{x'} h(0') = 0.
\end{align*}
\]
\[
\n\nabla^2_h h_1(0') \geq \kappa_0 I_{n-1}, \quad \nabla^2_h h(0') \geq 0, \quad \nabla^2_r (h_1-h)(0') \geq \kappa_1 I_{n-1} ,
\]
\[
\text{where } \kappa_0, \kappa_1 > 0, I_{n-1} \text{ is the } (n-1) \times (n-1) \text{ identity matrix, and}
\]
\[
\|h_1\|_{C^2(B'_{2R_0})} + \|h\|_{C^2(B'_{2R_0})} \leq C, 
\]
\[
\text{where } C \text{ is a universal constant. For } 0 < r \leq 2R_0, \text{ we denote}
\]
\[
\Omega_r : = \{(x', x_n) \in \mathbb{R}^n \mid h(x') < x_n < \varepsilon + h_1(x'), |x'| < r \},
\]
\[
\text{and}
\]
\[
\delta(x') : = \varepsilon + h_1(x') - h(x'), \quad \forall (x', x_n) \in \Omega_{R_0}. 
\]

### 2.1 Outline of the proof of Theorem 1.1

Most of the paper is devoted to these estimates. Now introduce a function \( \bar{u} \in C^2(\mathbb{R}^n) \), such that \( \bar{u} = 1 \) on \( \partial D_1 \), \( \bar{u} = 0 \) on \( \partial D \),
\[
\bar{u}(x) = \frac{x_n - h(x')}{\varepsilon + h_1(x') - h(x')}, \quad x \in \Omega_{R_0},
\]
and
\[
\|\bar{u}\|_{C^2(\Omega_{R_0})} \leq C. 
\]

Using the assumptions on \( h_1 \) and \( h \), (2.1)–(2.4), a direct calculation gives
\[
|\partial_{\eta_i} \bar{u}(x)| \leq \frac{C|x'|}{\varepsilon + |x'|^2}, \quad i = 1, \ldots, n-1, \quad \partial_{x_n} \bar{u}(x) = \frac{1}{\delta(x')}, \quad x \in \Omega_{R_0},
\]
where \( \delta(x') \) is defined by (2.5). Then we have

**Proposition 2.1.** Assume the above, let \( v_1, v_0 \in H^1(\Omega) \) be the weak solution of (1.11) and (1.6). Then
\[
\|\nabla (v_1 - \bar{u})\|_{L^\infty(\Omega)} \leq C. 
\]
and
\[
|\nabla v_0(x)| \leq C \left( \frac{|x'|}{\varepsilon + |x'|^2} + 1 \right) \|\varphi\|_{C^2(\partial D)}, \quad x \in \Omega. 
\]

Consequently,
\[
\frac{1}{C(\varepsilon + |x'|^2)} \leq |\nabla v_1(x)| \leq \frac{C}{\varepsilon + |x'|^2}, \quad x \in \Omega_{R_0},
\]
and
\[
|\nabla v_1(x)| \leq C, \quad x \in \Omega \setminus \Omega_{R_0}.
\]

**Remark 2.1.** Notice that (2.10) shows that \( |\nabla v_0| \) is bounded on the segment \( PP_1 \), because of the fact \( v_0(P_1) = v_0(P) = 0 \). However, for \( v_1(P_1) = 1 \) and \( v_1(P) = 0 \), \( |\nabla v_1| \sim \varepsilon^{-1} \) on \( PP_1 \). Actually, pointwise bound (2.11) is an improvement of its counterpart in [7], where the maximal principle is the main tool. In order to obtain (2.11), we make use of energy method and iteration technology, which is essentially different to that used in [7].
Proposition 2.1 is the main ingredient in the proof of Theorem 1.1. The proof will be given in the next subsection.

Define

\[ a_{11} := \int_{\Omega} |\nabla v_1|^2 \, dx. \]  

(2.12)

Then using (2.11),

\[ \frac{1}{C} \int_{\Omega} \frac{1}{(\varepsilon + |x'|^2)^2} \, dx \leq a_{11} \leq C \int_{\Omega} \frac{1}{(\varepsilon + |x'|^2)^2} \, dx. \]

By direct integration we obtain the following estimates for \( a_{11} \), which is essentially the same as lemmas 2.5–2.7 in [7].

**Lemma 2.2.** ([7]) For \( n \geq 2 \),

\[ \frac{1}{C \rho_n(\varepsilon)} \leq a_{11} \leq \frac{C}{\rho_n(\varepsilon)}, \]

where \( \rho_n(\varepsilon) \) is defined by (1.2).

**Proof of Theorem 1.1.** By the decomposition (1.10) and the third line of (1.1), we have

\[ (C_1 - \varphi(P)) \int_{\partial D_1} \frac{\partial v_1}{\partial \nu} + \int_{\partial D_1} \frac{\partial v_0}{\partial \nu} = 0. \]

(2.13)

Recalling the definition of \( v_1 \), we have

\[ \int_{\partial D_1} \frac{\partial v_1}{\partial \nu} = \int_{\partial D_1} \frac{\partial v_1}{\partial \nu} v_1 = -\int_{\Omega} |\nabla v_1|^2 = -a_{11}. \]

(2.14)

Hence

\[ C_1 - \varphi(P) = \frac{Q[\varphi]}{a_{11}}. \]

Thus

\[ \nabla u = (C_1 - \varphi(P)) \nabla v_1 + \nabla v_0 = \frac{Q[\varphi]}{a_{11}} \nabla v_1 + \nabla v_0. \]

Using the upper bounds of \( |\nabla v_1| \) and \( |\nabla v_0| \) in Proposition 2.1 and Lemma 2.2, we obtain (1.3). If \( |Q[\varphi]| \geq c^* \), then using the lower bound of \( |\nabla v_1| \) in (2.11) and boundless of \( |\nabla v_0| \) on the segment \( PP_1 \), we have (1.4) holds on the segment \( PP_1 \). Thus, Theorem 1.1 is proved. \( \square \)

### 2.2 Proof of Proposition 2.1

**Proof.** **STEP 1.** Proof of (2.9).

Denote

\[ w_1 := v_1 - \bar{u}. \]

(2.15)

By the definition of \( v_1 \), (1.11), and using (2.15), we have

\[ \begin{cases} -\Delta w_1 = \Delta \bar{u} & \text{in } \Omega, \\ w_1 = 0 & \text{on } \partial \Omega. \end{cases} \]

(2.16)
Since
\[ |\bar{u}| + |\nabla \bar{u}| + |\nabla^2 \bar{u}| \leq C, \quad \text{in} \quad \Omega \setminus \Omega_{R_0/2}, \quad (2.17) \]
by the standard elliptic theory, we know that
\[ |w_1| + |\nabla w_1| \leq C, \quad \text{in} \quad \Omega \setminus \Omega_{R_0}. \quad (2.18) \]
Therefore, in order to show (2.9), we only need to prove
\[ \|\nabla w_1\|_{L^\infty(\Omega_{R_0})} \leq C. \quad (2.19) \]
The rest proof of (2.19) is divided into three steps.

**STEP 1.1.** Proof of boundedness of the energy in \( \Omega \), that is,
\[ \int_{\Omega} |\nabla w_1|^2 \leq C. \quad (2.20) \]
Using the maximum principle, we have \( 0 < v_1 < 1 \) in \( \Omega \), so that
\[ \|w_1\|_{L^\infty(\Omega)} \leq C. \quad (2.21) \]
By a direct computation,
\[ |\Delta \bar{u}(x)| \leq \frac{C}{\varepsilon + |x'|^2}, \quad x \in \Omega_{R_0}. \quad (2.22) \]
Multiplying the equation in (2.16) by \( w_1 \) and integrating by parts, it follows from (2.17) and (2.22) that
\[ \int_{\Omega} |\nabla w_1|^2 = \int_{\Omega} w_1 (\Delta \bar{u}) \leq \|w_1\|_{L^\infty(\Omega)} \left( \int_{\Omega_{R_0}} |\Delta \bar{u}| + C \right) \leq C. \]
So (2.20) is proved.

**STEP 1.2.** Proof of
\[ \int_{\widetilde{\Omega}_s(z')} |\nabla w_1|^2 dx \leq \begin{cases} C\varepsilon^n, & \text{if } |z'| \leq \sqrt{\varepsilon}, \\ C|z'|^{2n}, & \text{if } \sqrt{\varepsilon} < |z'| \leq R_0, \end{cases} \quad (2.23) \]
where
\[ \widetilde{\Omega}_s(z') := \{ x \in \mathbb{R}^n \mid h(x') < x_n < \varepsilon + h_1(x'), |x' - z'| < s \}. \]
The iteration scheme here we use is similar in spirit to that used in [9, 23]. For \( 0 < t < s < R_0 \), let \( \eta \) be a smooth cutoff function satisfying \( \eta(x') = 1 \) if \( |x' - z'| < t \), \( \eta(x') = 0 \) if \( |x' - z'| > s \), \( 0 \leq \eta(x') \leq 1 \) if \( t \leq |x' - z'| \leq s \), and \( |\nabla \eta(x')| \leq \frac{1}{s-t} \). Multiplying the equation in (2.16) by \( w_1 \eta^2 \) and integrating by parts leads to
\[ \int_{\widetilde{\Omega}_s(z')} |\nabla w_1|^2 \leq \frac{C}{(s-t)^2} \int_{\widetilde{\Omega}_s(z')} |w_1|^2 + (s-t)^2 \int_{\widetilde{\Omega}_s(z')} |\Delta \bar{u}|^2. \quad (2.24) \]

**Case 1.** For \( |z'| \leq \sqrt{\varepsilon} \). For \( 0 < s < \sqrt{\varepsilon} \), using (2.22), we have
\[
\int_{\Omega_{t}(c')} |\Delta \bar{u}|^2 \leq \frac{Cs^{\alpha-1}}{\varepsilon}, \quad \text{if } 0 < s < \sqrt{\varepsilon}. \tag{2.25}
\]

Note that
\[
\int_{\Omega_{t}(c')} |w_1|^2 \leq C\varepsilon^2 \int_{\Omega_{t}(c')} |\nabla w_1|^2, \quad \text{if } 0 < s < \sqrt{\varepsilon}. \tag{2.26}
\]

Denote
\[F(t) := \int_{\Omega_{t}(c')} |\nabla w_1|^2.\]

It follows from (2.24), (2.25) and (2.26) that
\[
F(t) \leq \left( \frac{c_1 \varepsilon}{s - t} \right)^2 F(s) + C(s - t)^2 \frac{s^{n-1}}{\varepsilon}, \quad \forall \ 0 < t < s < \sqrt{\varepsilon}, \tag{2.27}
\]

where \(c_1\) is a universal constant.

Let \(k = \left[ \frac{1}{4c_1 \sqrt{\varepsilon}} \right] \) and \(t_i = \delta + 2c_1 i \varepsilon, \ i = 0, 1, 2, \cdots, k.\) Then by (2.27) with \(s = t_{i+1}\) and \(t = t_i,\) we have
\[
F(t_i) \leq \frac{1}{4} F(t_{i+1}) + C(i + 1)^{n-1} \varepsilon^n.
\]

After \(k\) iterations, using (2.20), we have
\[
F(t_0) \leq \left( \frac{1}{4} \right)^k F(t_k) + C \varepsilon^{n} \sum_{i=0}^{k-1} \left( \frac{1}{4} \right)^i (i + 1)^{n-1} \leq C \varepsilon^n.
\]

Therefore
\[
\int_{\Omega_{t}(c')} |\nabla w_1|^2 \leq C \varepsilon^n.
\]

**Case 2.** For \(\sqrt{\varepsilon} \leq |z'| \leq R_0,\) Estimate (2.25) becomes
\[
\int_{\Omega_{t}(c')} |\Delta \bar{u}|^2 \leq \frac{Cs^{\alpha-1}}{|z'|^2}, \quad \text{if } 0 < s < |z'|. \tag{2.28}
\]

Estimate (2.26) becomes
\[
\int_{\Omega_{t}(c')} |w_1|^2 \leq C|z'|^4 \int_{\Omega_{t}(c')} |\nabla w_1|^2, \quad \text{if } 0 < s < |z'|. \tag{2.29}
\]

Estimate (2.27) becomes, in view of (2.24),
\[
F(t) \leq \left( \frac{c_2 |z'|^2}{s - t} \right)^2 F(s) + C(s - t)^2 \frac{s^{n-1}}{|z'|^2}, \quad \forall \ 0 < t < s < |z'|, \tag{2.30}
\]

where \(c_2\) is another universal constant.

Let \(k = \left[ \frac{1}{4c_2 |z'|} \right] \) and \(t_i = \delta + 2c_2 i |z'|^2, \ i = 0, 1, 2, \cdots, k.\) Then applying (2.30) with \(s = t_{i+1}\) and \(t = t_i,\) we have
\[
F(t_i) \leq \frac{1}{4} F(t_{i+1}) + \frac{C(t_{i+1} - t_i)^2 t_{i+1}^{n-1}}{|z'|^2} \leq \frac{1}{4} F(t_{i+1}) + C(i + 1)^{n-1} |z'|^{2n},
\]
After $k$ iterations, using (2.20) we have

$$F(t_0) \leq \left( \frac{1}{4} \right)^k F(t_k) + C|z'|^{2n} \sum_{i=0}^{k-1} \frac{1}{4} (i + 1)^{n-1} \leq C|z'|^{2n}.$$  

This implies that

$$\int_{\tilde{\Omega}_d(z')} |\nabla w_1|^2 \leq C|z'|^{2n}.$$  

(2.23) is proved.

**STEP 1.3. Rescaling and $L^\infty$ estimates.** Denote $\delta := \delta(z')$. Making a change of variables

\begin{align*}
\begin{cases}
  x' - z' = \delta y', \\
  x_n = \delta y_n,
\end{cases}
\end{align*}

(2.31)

then $\tilde{\Omega}_d(z')$ becomes $Q'_1$, where

$$Q'_r = \left\{ y \in \mathbb{R}^n \left| \frac{1}{\delta} h(\delta y' + z') < y_n < \frac{\varepsilon}{\delta} + \frac{1}{\delta} h_1(\delta y' + z'), |y'| < r \right. \right\}, \quad \text{for } r \leq 1,$$

and the top and bottom boundaries become

$$y_n = \hat{h}_1(y') := \frac{1}{\delta} (\varepsilon + h_1(\delta y' + z')), \quad |y'| < 1,$$

and

$$y_n = \hat{h}(y') := \frac{1}{\delta} h(\delta y' + z'), \quad |y'| < 1.$$  

Then

$$\hat{h}_1(0') - \hat{h}(0') := \frac{\varepsilon + h_1(z') - h(z')}{\delta} = 1,$$  

and by (2.1)–(2.4),

$$|\nabla_x \hat{h}_1(0')| + |\nabla_x \hat{h}(0')| \leq C|z'|, \quad |\nabla^2_x \hat{h}_1(0')| + |\nabla^2_x \hat{h}(0')| \leq C\delta.$$  

Since $R_0$ is small, $\|\hat{h}_1\|_{C^{1,1}((-1,1)^{n-1})}$ and $\|\hat{h}\|_{C^{1,1}((-1,1)^{n-1})}$ are small and $Q'_1$ is essentially a unit square (or a unit cylinder) as far as applications of Sobolev embedding theorems and classical $L^p$ estimates for elliptic equations are concerned. Let

$$U(y', y_n) := \tilde{u}(\delta y' + z', \delta y_n), \quad W(y', y_n) := w_1(\delta y' + z', \delta y_n), \quad y \in Q'_1,$$

(2.32)

then by the equation in (2.16),

$$-\Delta W = \Delta U, \quad y \in Q'_1.$$  

(2.33)

where

$$|\Delta U| = \delta^2 |\Delta \tilde{u}|.$$  

Since $W = 0$ on the top and bottom boundaries of $Q'_1$, it follows from the Poincaré inequality that

$$\|W\|_{H^1(Q'_1)} \leq C \|\nabla W\|_{L^2(Q'_1)}.$$
By \( W^{2,p} \) estimates for elliptic equations and Sobolev embedding theorems, for \( p > n \),
\[
\| \nabla W \|_{L^p(Q_{1/2})} \leq C \| W \|_{W^{2,p}(Q_{1/2})} \leq C \left( \| \nabla W \|_{L^2(Q_{1})} + \| \Delta U \|_{L^p(Q_{1})} \right).
\]
Therefore
\[
\| \nabla W_1 \|_{L^p(\Omega;\mathbb{C}_z')} \leq \frac{C}{\delta} \left( \delta^{1-\frac{z}{n}} \| \nabla W_1 \|_{L^2(\Omega;\mathbb{C}_z')} + \delta^2 \| \Delta \tilde{u} \|_{L^p(\Omega;\mathbb{C}_z')} \right). \tag{2.34}
\]

**Case 1.** For \( |z'| \leq \sqrt{\varepsilon} \). Using (2.22) and (2.5),
\[
\delta^2 |\Delta \tilde{u}| \leq \frac{C \delta^2}{\varepsilon} \leq C \varepsilon, \quad \text{in } \tilde{\Omega}_0(z').
\]
It follows from (2.34) and (2.23) that
\[
|\nabla W_1(z', x_n)| \leq \frac{C (\delta^{1-\frac{z}{n}} \varepsilon + \varepsilon)}{\delta} \leq C, \quad \forall \ h(z') < x_n < \varepsilon + h_1(z').
\]

**Case 2.** For \( \sqrt{\varepsilon} \leq |z'| \leq R_0 \). Using (2.22) and (2.5),
\[
\delta^2 |\Delta \tilde{u}| \leq \frac{C \delta^2}{|z'|^2} \leq C|z'|^2, \quad \text{in } \tilde{\Omega}_0(z').
\]
We deduce from (2.34) and (2.23) that
\[
|\nabla W_1(z', x_n)| \leq \frac{C (\delta^{1-\frac{z}{n}} |z'|^n + |z'|^2)}{\delta} \leq C, \quad \forall \ h(z') < x_n < \varepsilon + h_1(z').
\]

Estimate (2.9) is established.

**STEP 2.** Proof of (2.10).

Similar to the proof of (2.9), we introduce a function \( \hat{u} \in C^2(\mathbb{R}^n) \), such that \( \hat{u} = 0 \) on \( \partial D_1 \), \( \hat{u} = \varphi(x) - \varphi(0) \) on \( \partial D \),
\[
\hat{u}(x) = (1 - \bar{u}(x))(\varphi(x', h(x')) - \varphi(0)), \quad \text{in } \Omega_{R_0}, \tag{2.35}
\]
and
\[
||\hat{u}||_{C^2(\mathbb{R}^n \setminus \Omega_{R_0})} \leq C. \tag{2.36}
\]
Denote
\[
w_0 := v_0 - \hat{u}.
\]
Then by the definitions of \( v_0 \), (1.6),
\[
\begin{align*}
-\Delta w_0 &= \Delta \hat{u}, & \text{in } \Omega, \\
w_0 &= 0, & \text{on } \partial \Omega.
\end{align*} \tag{2.37}
\]
Similarly as (2.18) and (2.21), we have
\[
||w_0||_{L^p(\Omega)} + ||\nabla w_0||_{L^p(\Omega_{R_0/2})} \leq C. \tag{2.38}
\]
Thus, in order to prove (2.10), we only need to prove
\[ \| \nabla w_0 \|_{L^\infty(\Omega_{R_0})} \leq C. \]

By a direct calculation, we have for \( x \in \Omega_{R_0}, \)
\[
\partial_{x_i} \tilde{u} = (1 - \bar{u}) \left( \partial_{x_i} \varphi + \partial_{x_i} \varphi \partial_{x_i} h \right) - \partial_{x_i} \tilde{u} \left( \varphi(x', h(x')) - \varphi(0) \right), \quad i = 1, 2, \ldots, n - 1,
\]
\[
\partial_{x_n} \tilde{u} = -\partial_{x_n} \tilde{u} \left( \varphi(x', h(x')) - \varphi(0) \right).
\]

Using the assumption on \( \varphi, \) we have
\[
\left| \varphi(x', h(x')) - \varphi(0) \right| \leq C \| \varphi \|_{C^1(\partial D)} |x'|, \quad \text{in} \ \Omega_{R_0},
\]
(2.39)
then in view of (2.1)–(2.4),
\[
|\nabla \tilde{u}| \leq C \left( \frac{|x'|}{\varepsilon + |x'|^2} + 1 \right) \| \varphi \|_{C^1(\partial D)}, \quad \text{in} \ \Omega_{R_0}.
\]
(2.40)

Furthermore,
\[
\Delta \tilde{u} = (1 - \bar{u}) \left( \Delta \varphi + 2 \nabla \varphi \cdot \nabla h + (\partial_{x_n} \varphi) \Delta h + \partial_{x_n} \varphi (\nabla h)^2 \right)
\]
\[
- 2 \nabla \tilde{u} \cdot \left( \nabla \varphi \right) (\partial_{x_n} \varphi) \nabla h \right) - \Delta \tilde{u} \left( \varphi(x', h(x')) - \varphi(0) \right),
\]
and using (2.39) and (2.1)–(2.4) again,
\[
|\Delta \tilde{u}| \leq C \left( \frac{|x'|}{\varepsilon + |x'|^2} + 1 \right) \| \varphi \|_{C^2(\partial D)}, \quad \text{in} \ \Omega_{R_0},
\]
which is better than (2.22). Therefore, the rest of the proof is completely the same as step 1.1–1.3 above. (2.10) is proved. The proof of Proposition 2.1 is completed. \( \square \)

### 2.3 Estimates of \( Q[\varphi] \)

In order to identify the lower bound (1.4), we estimate \( |Q[\varphi]| \) in this subsection. Let
\( D_1^* := D_1 - (0', \varepsilon) \) and \( \Omega^* := D \setminus \overline{D_1^*} \) and define
\[
\begin{cases}
\Delta v_1^* = 0, & \text{in} \ \Omega^*, \\
v_1^* = 1, & \text{on} \ \partial D_1^* \setminus \{0\}, \quad \text{and} \quad \Delta v_0^* = 0, & \text{in} \ \Omega^*, \\
v_0^* = 0, & \text{on} \ \partial D \setminus \{0\}, \quad \text{and} \quad v_0^* = \varphi(x) - \varphi(0), & \text{on} \ \partial D.
\end{cases}
\]
(2.41)

**Lemma 2.3.** There exists a unique \( v_i^* \in L^\infty(\Omega^*) \cap C^0(\overline{\Omega^*} \setminus \{0\}) \cap C^2(\Omega^*), \) \( i = 0, 1, \) which solve (2.41). Moreover, \( v_i^* \in C^1(\overline{\Omega^*} \setminus \{0\}). \)

**Proof.** The existence of solutions of (2.41) can easily be obtained by Perron’s method, see theorem 2.12 and lemma 2.13 in [17]. For the readers’ convenience, we give a simple
proof of the uniqueness for \( n \geq 3 \). The case \( n = 2 \) is similar. We only need to prove that 0 is the only solution in \( L^\infty(\Omega^*) \cap C^0(\Omega^* \setminus \{0\}) \cap C^2(\Omega^*) \) of the following equation

\[
\begin{cases}
\Delta w = 0, & \text{in } \Omega^*, \\
w = 0, & \text{on } \partial \Omega^* \setminus \{0\}.
\end{cases}
\]

Indeed, noticing that \( w = 0 \) on \( \partial \Omega^* \setminus \{0\} \), it follows that for any \( \epsilon > 0 \),

\[
|w(x)| \leq \frac{\epsilon^{n-2} \|w\|_{L^\infty(\Omega^*)}}{|x|^{n-2}}, \quad \text{on } \partial(\Omega^* \setminus B_\epsilon(0)).
\]

Using the maximum principle, we have

\[
|w(x)| \leq \frac{\epsilon^{n-2} \|w\|_{L^\infty(\Omega^*)}}{|x|^{n-2}}, \quad \text{in } \Omega^* \setminus B_\epsilon(0).
\]

Thus, \( w = 0 \) in \( \Omega^* \). The additional regularity follows from standard elliptic estimates and the smoothness of \( \partial D_1 \) and \( \partial D \). \( \square \)

**Lemma 2.4.** For \( i = 0, 1 \),

\[
v_i \to v_i^*, \quad \text{in } C^2_{loc}(\Omega^*), \quad \text{as } \epsilon \to 0,
\]

and

\[
\int_{\partial D_1} \frac{\partial v_0}{\partial v} \to \int_{\partial D_1} \frac{\partial v_0^*}{\partial v}, \quad \text{as } \epsilon \to 0.
\]

**Proof.** By the maximum principle, \( \|v_i\|_{L^\infty} \) is bounded by a constant independent of \( \epsilon \). By the uniqueness part of Lemma 2.3 we obtain (2.42) using standard elliptic estimates. It follows from the definition of \( v_1 \) and the Green’s formula that

\[
\int_{\partial D_1} \frac{\partial v_0}{\partial v} = \int_{\partial D_1} \frac{\partial v_0}{\partial v} v_1 = \int_{\partial D} \frac{\partial v_0}{\partial v} = \int_{\partial D} \frac{\partial v_1}{\partial v} (\varphi(x) - \varphi(0)).
\]

Similarly,

\[
\int_{\partial D_1} \frac{\partial v_0^*}{\partial v} = \int_{\partial D_1} \frac{\partial v_0^*}{\partial v} v_1^* = \int_{\partial D} \frac{\partial v_1^*}{\partial v} (\varphi(x) - \varphi(0)).
\]

(2.43) follows from (2.42). \( \square \)

Define

\[
Q^*[\varphi] = \int_{\partial D_1} \frac{\partial v_0^*}{\partial v}.
\]

By Lemma 2.4

\[
Q[\varphi] \to Q^*[\varphi], \quad \text{as } \epsilon \to 0.
\]

**Corollary 2.5.** If \( \varphi \in C^2(\partial D) \) satisfies \( Q^*[\varphi] \neq 0 \), then \( |Q[\varphi]| \geq c^* \), for some positive universal constant \( c^* \) which is independent of \( \epsilon \).

**Remark 2.2.** It follows from the definition of \( Q^*[\varphi] \) and (2.44) that \( Q^*[\varphi] = \int_{\partial D} \frac{\partial v_0^*}{\partial v} (\varphi(x) - \varphi(0)) \).

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2.4 More general $D_1$ and $D$

As mentioned in Remark 1.1, the strict convexity assumption of $D_1$ and $D$ can be weakened. In fact, our proof of Proposition 2.1 applies, with minor modification, to more general situations: In $\mathbb{R}^n$, $n \geq 2$, under the same assumptions in the beginning of Section 2 except the strict convexity assumptions (2.3). We assume that

$$\lambda_0 |x'|^m \leq h_1(x') - h(x') \leq \lambda_1 |x'|^m, \quad \text{for } |x'| < 2R_0,$$

and

$$|\nabla_x h_1(x')|, |\nabla^2_x h(x')| \leq C |x'|^{m-1}, \quad |\nabla^2_x h_1(x')|, |\nabla^2_x h(x')| \leq C |x'|^{m-2}, \quad \text{for } |x'| < 2R_0,$$

for some $\varepsilon$-independent constants $0 < \lambda_0 < \lambda_1$, and $m \geq 2$. Clearly,

$$\frac{1}{C}(\varepsilon + |x'|^m) \leq \delta(x') \leq C(\varepsilon + |x'|^m).$$

Then by the same procedure in the proof of Proposition 2.1 we have

**Proposition 2.6.** Assume the above, under the assumptions (2.45) and (2.46), instead of (2.3). Let $v_1, v_0 \in H^1(\Omega)$ be the weak solution of (1.11) and (1.6). Then

$$\|\nabla(v_1 - \bar{u})\|_{L^\infty(\Omega)} \leq C.$$  \hfill (2.47)

and

$$|\nabla v_0(x)| \leq C \left( \frac{|x'|^{m-1}}{\varepsilon + |x'|^m} + 1 \right) \|\phi\|_{L^2(\partial D)}, \quad x \in \Omega.$$  \hfill (2.48)

Consequently,

$$\frac{1}{C(\varepsilon + |x'|^m)} \leq |\nabla v_1(x)| \leq \frac{C}{\varepsilon + |x'|^m}, \quad x \in \Omega_{R_0},$$  \hfill (2.49)

and

$$|\nabla v_1(x)| \leq C, \quad x \in \Omega \setminus \Omega_{R_0}.$$

Thus, using (2.49), we have

$$\frac{1}{C} \int_{\Omega} \frac{1}{(\varepsilon + |x'|^m)^2} dx \leq a_{11} \leq C \int_{\Omega} \frac{1}{(\varepsilon + |x'|^m)^2} dx.$$

By direct integration we obtain the following estimates for $a_{11}$, instead of Lemma 2.2.

**Lemma 2.7.** For $n \geq 2$ and $m \geq 2$,

$$\frac{1}{C \rho_n^m(\varepsilon)} \leq a_{11} \leq \frac{C}{\rho_n^m(\varepsilon)}, \quad \text{where } \rho_n^m(\varepsilon) = \begin{cases} e^{\varepsilon \frac{n-1}{m}}, & \text{if } n - 1 < m, \\
\frac{1}{n^{m}}, & \text{if } n - 1 = m, \\
1, & \text{if } n - 1 > m. \end{cases}$$

Hence, we have the following more general theorem.
Theorem 2.8. Let $D_1 \subset D \subset \mathbb{R}^n$ ($n \geq 2$) be defined as above, under the assumptions (2.45) and (2.46), instead of (2.3). Let $u \in H^1(D) \cap C^1(D \setminus \overline{D_1})$ be the solution to (1.1). Then for $0 < \varepsilon < 1/2$, we have

$$|\nabla u(x)| \leq \frac{C \rho_n^m(\varepsilon) |Q[\varphi]|}{\varepsilon + \mathrm{dist}^m(x, PP_1)} + C \left(\frac{\mathrm{dist}^{m-1}(x, PP_1)}{\varepsilon + \mathrm{dist}^m(x, PP_1)} + 1\right)\|\varphi\|_{C^2(\partial D)}, \quad x \in D \setminus D_1,$$

(2.50)

and if $|Q[\varphi]| \geq c^*$ for some universal constant $c^* > 0$, then

$$|\nabla u(x)| \geq \frac{\rho_n^m(\varepsilon) |Q[\varphi]|}{C \varepsilon}, \quad x \in PP_1,$$

(2.51)

where $Q[\varphi]$ is defined by (1.5).

3 Proof of Theorem 1.2

Following the approach developed in the proof of Theorem 1.1, we construct an auxiliary function $\tilde{u} \in C^2(\mathbb{R}^n)$, such that $\tilde{u} = 1$ on $\partial D_1$, $\tilde{u} = 0$ on $\partial D$,

$$\tilde{u}(x) = \bar{u}(x) + \frac{\sum_{i=1}^{n-1} A_{ii}(x) \partial_i (h_1 - h)(x')}{4A_{mm}(x)} \left(\frac{2x_n - (\varepsilon + h_1(x') + h(x'))}{\varepsilon + h_1(x') - h(x')}\right)^2 - 1, \quad \text{in} \quad \Omega_{R_0},$$

(3.1)

and

$$\|	ilde{u}\|_{C^2(\Omega_{R_0})} \leq C.$$  

(3.2)

Using the assumptions on $h_1$ and $h$, (2.1)–(2.4), a direct calculation still gives

$$\frac{1}{C(\varepsilon + |x'|^2)} \leq |\nabla \tilde{u}(x)| \leq \frac{C}{\varepsilon + |x'|^2}, \quad x \in \Omega_{R_0}.$$  

(3.3)

More importantly, thanks to the corrector term in (3.1), we obtain the following bound

$$|\partial_i(A_{ij}(x) \partial_j \tilde{u}(x))| \leq \frac{C}{\varepsilon + |x'|^2}, \quad x \in \Omega_{R_0},$$

(3.4)

the same as (2.22). This is the point, which plays an important role in the proof of the following Proposition.

Proposition 3.1. Assume the above, let $V_0, V_1 \in H^1(\Omega)$ be the weak solution of (1.9) and (1.14), respectively. Then

$$\|\nabla (V_1 - \tilde{u})\|_{L^\infty(\Omega)} \leq C,$$

(3.5)

and

$$|\nabla V_0(x)| \leq C \left(\frac{|x'|}{\varepsilon + |x'|^2} + 1\right)\|\varphi\|_{C^2(\partial D)}, \quad x \in \Omega.$$  

(3.6)

Consequently,

$$\frac{1}{C(\varepsilon + |x'|^2)} \leq |\nabla V_1(x)| \leq \frac{C}{\varepsilon + |x'|^2}, \quad x \in \Omega_{R_0},$$

(3.7)

and

$$|\nabla V_1(x)| \leq C, \quad x \in \Omega \setminus \Omega_{R_0}.$$
Proof of Proposition \[\text{STEP 1. Proof of } (3.5).\]

Let

$$\tilde{w}_1 = V_1 - \tilde{u}.$$ 

Similarly, instead of (2.16), we have

$$\begin{cases}
\{-\partial_j(A_{ij}(x)\partial_j\tilde{w}_1) = \partial_j(A_{ij}(x)\partial_j\tilde{u}) =: \tilde{f}, & \text{in } \Omega, \\
\tilde{w}_1 = 0, & \text{on } \partial\Omega.
\end{cases} \quad (3.8)$$ 

By the standard elliptic theory,

$$|\tilde{w}_1| + |\nabla \tilde{w}_1| \leq C, \quad \text{in } \Omega \setminus \Omega_{\varepsilon}.$$ 

(3.9)

On the other hand, by the maximum principle, we have

$$\|\tilde{w}_1\|_{L^\infty(\Omega)} \leq C. \quad (3.10)$$

**STEP 1.1. Boundedness of the energy.** Multiplying the equation in (3.8) by \(\tilde{w}_1\), integrating by parts, using (1.7), (3.10) and (3.4), we have

$$\lambda \int_{\Omega} |\nabla \tilde{w}_1|^2 \, dx \leq \int_{\Omega} A_{ij}\partial_j \tilde{w}_1 \partial_j \tilde{w}_1 \, dx = \int_{\Omega} \tilde{f} \tilde{w}_1 \, dx \leq \|\tilde{w}_1\|_{L^\infty(\Omega)} \left(\int_{\Omega_{\varepsilon}} |\tilde{f}| \, dx + C\right) \leq C.$$ 

So that

$$\int_{\Omega} |\nabla \tilde{w}_1|^2 \, dx \leq C. \quad (3.11)$$

**STEP 1.2. Local energy estimates.** Multiplying the equation in (3.8) by \(\eta^2 \tilde{w}_1\), where \(\eta\) is the same cutoff function defined before, and integrating by parts, we deduce

$$\int_{\tilde{\Omega}_\varepsilon(\varepsilon')} A_{ij}\partial_j \tilde{w}_1 \partial_j (\eta^2 \tilde{w}_1) \, dx = \int_{\tilde{\Omega}_\varepsilon(\varepsilon')} \tilde{f} \eta^2 \tilde{w}_1 \, dx.$$ 

Then

$$\int_{\tilde{\Omega}_\varepsilon(\varepsilon')} A_{ij}(\eta \partial_j \tilde{w}_1)(\eta \partial_j \tilde{w}_1) \, dx = \int_{\tilde{\Omega}_\varepsilon(\varepsilon')} \left[ -2A_{ij}(\eta \partial_j \tilde{w}_1)\tilde{w}_1 \partial_j \eta + \tilde{f} \eta^2 \tilde{w}_1 \right] \, dx.$$ 

By (1.7) and the Cauchy inequality,

$$\lambda \int_{\tilde{\Omega}_\varepsilon(\varepsilon')} |\eta \nabla \tilde{w}_1|^2 \, dx \leq \frac{\lambda}{4} \int_{\tilde{\Omega}_\varepsilon(\varepsilon')} |\eta \nabla \tilde{w}_1|^2 \, dx + \frac{C}{(s-t)^2} \int_{\tilde{\Omega}_\varepsilon(\varepsilon')} \tilde{w}_1^2 \, dx + C(s-t)^2 \int_{\tilde{\Omega}_\varepsilon(\varepsilon')} |\tilde{f}|^2 \, dx.$$ 

Thus

$$\int_{\tilde{\Omega}_\varepsilon(\varepsilon')} |\nabla \tilde{w}_1|^2 \, dx \leq \frac{C}{(s-t)^2} \int_{\tilde{\Omega}_\varepsilon(\varepsilon')} |\tilde{w}_1|^2 \, dx + C(s-t)^2 \int_{\tilde{\Omega}_\varepsilon(\varepsilon')} |\tilde{f}|^2 \, dx. \quad (3.12)$$

Then using estimate (3.4), instead of (2.22), we have

$$\int_{\tilde{\Omega}_\varepsilon(\varepsilon')} |\tilde{f}|^2 \, dx \leq \begin{cases}
\frac{C^{r_{g-1}}}{|\varepsilon'|}, & \text{if } |\varepsilon'| \leq \sqrt{\varepsilon}, \\
\frac{C^{r_{g-1}}}{|\varepsilon'|}, & \text{if } \sqrt{\varepsilon} < |\varepsilon'| \leq R_0.
\end{cases}$$
Using the iteration argument, similar as step 1.2 in the proof of Proposition [2.1], we have \( \tilde{w}_1 \) also satisfies (2.23), that is,

\[
\int_{\tilde{\Omega}(z')} |\nabla \tilde{w}_1|^2 \, dx \leq \begin{cases} 
C \varepsilon, & \text{if } |z'| \leq \sqrt{\varepsilon}, \\
C |z'| \varepsilon^n, & \text{if } \sqrt{\varepsilon} < |z'| \leq R_0.
\end{cases}
\]

Thus, similar as step 1.3 in the proof of Proposition [2.1] (3.5) is established.

STEP 2. Proof of (3.6).

Using \( \tilde{u} \) instead of \( \bar{u} \), we define a function \( \hat{u}_2 \in C^2(\mathbb{R}^n) \), such that \( \hat{u}_2 = 0 \) on \( \partial D_1 \), \( \hat{u}_2 = \varphi(x) - \varphi(0) \) on \( \partial D \),

\[
\hat{u}_2(x) = (1 - \tilde{u}(x)) (\varphi(x', h(x')) - \varphi(0)), \quad \text{in } \Omega_{R_0},
\]

and

\[
\| \hat{u}_2 \|_{C^2(\mathbb{R}^n \setminus \Omega_{R_0})} \leq C.
\]

Denote

\[
\tilde{w}_0 := V_0 - \hat{u}_2.
\]

Instead of (2.37), we have

\[
\begin{aligned}
-\d_i (A_{ij}(x) \d_j \tilde{w}_0) &= \d_i (A_{ij}(x) \d_j \tilde{u}_2) =: \tilde{f}, \quad \text{in } \Omega, \\
\tilde{w}_0 &= 0, \quad \text{on } \partial \Omega.
\end{aligned}
\]

STEP 2.1. The boundedness of the energy is the same as step 1.1. By a direct computation, we have

\[
|\tilde{f}| \leq C \left( \frac{|x'|}{\varepsilon + |x'|^2} + 1 \right) \| \varphi \|_{C^2(\partial D)}, \quad x \in \Omega_{R_0},
\]

and

\[
\int_{\tilde{\Omega}(z')} |\tilde{f}|^2 \, dx \leq C \| \varphi \|_{C^1(\partial D)}^2 \left( \frac{\varepsilon^{-1}}{|z'|} \right), \quad \text{if } |z'| \leq \sqrt{\varepsilon}, \\
\int_{\tilde{\Omega}(z')} |\tilde{f}|^2 \, dx \leq C \| \varphi \|_{C^1(\partial D)}^2 \left( \frac{\varepsilon^{-1}}{|z'|} \right), \quad \text{if } \sqrt{\varepsilon} < |z'| \leq R_0,
\]

similar as step 1.2 in the proof of Proposition [2.1], \( \tilde{w}_0 \) also satisfies (2.23). The rest is the same. Proposition 3.1 is established.

Proof of Theorem 1.2. Similarly as in the proof of Theorem 1.1, we decompose the solution \( u \) of (1.8) as

\[
u(x) = (C_1 - \varphi(P)) V_1(x) + V_0(x) + \varphi(P), \quad \text{in } D \setminus \overline{D_1}.
\]

Define

\[
a_{11} := -\int_{\partial D_1} A_{ij}(x) \d_j V_1 \, v_i.
\]

By integrating by parts,

\[
0 = \int_{\Omega} \d_i (A_{ij}(x) \d_j V_1) \cdot V_1 = -\int_{\Omega} A_{ij}(x) \d_j V_1 \d_i V_1 - \int_{\partial D_1} A_{ij}(x) \d_j V_1 \nu_i \cdot 1
\]

\[
= -\int_{\Omega} A_{ij}(x) \d_i V_1 \d_j V_1 + a_{11}.
\]
That is,

\[ a_{11} = \int_{\Omega} A_{ij}(x) \partial_i V_1 \partial_j V_1. \]

By the uniform elliptic condition (1.7), and (3.7),

\[ \frac{1}{C} \int_{\Omega} \frac{1}{(\varepsilon + |x'|^2)^2} \leq \lambda \int_{\Omega} |\nabla V_1|^2 \leq a_{11} \leq \Lambda \int_{\Omega} |\nabla V_1|^2 \leq C \int_{\Omega} \frac{1}{(\varepsilon + |x'|^2)^2}. \]

So that Lemma 2.2 holds still. Then, combining with Proposition 3.1, the proof of Theorem 1.2 is completed. \(\square\)

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