LAGRANGIAN AND HAMILTONIAN FEYNMAN FORMULAE
FOR SOME FELLER SEMIGROUPS AND THEIR
PERTURBATIONS

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Abstract. A Feynman formula is a representation of a solution of an initial
(or initial-boundary) value problem for an evolution equation (or, equivalently,
a representation of the semigroup resolving the problem) by a limit of n-fold
iterated integrals of some elementary functions as \( n \to \infty \). In this note we ob-
tain some Feynman formulae for a class of semigroups associated with Feller
processes. Finite dimensional integrals in the Feynman formulae give approxi-
mations for functional integrals in some Feynman–Kac formulae corresponding
to the underlying processes. Hence, these Feynman formulae give an effective
tool to calculate functional integrals with respect to probability measures gen-
etrated by these Feller processes.

Keywords Feynman formulae; Feynman–Kac formulae; approximations of
functional integrals, approximations of transition densities.

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1. Introduction

In this note we consider a class of semigroups associated with Feller processes.
Feller processes are continuous-time Markov processes, which generalize the class of
stochastic processes with stationary and independent increment s, or Lévy processes.
Note that many diffusion processes belong to this class. Every Feller process \((ξ_t)_{t≥0}\)
in \(\mathbb{R}^d\) generates a strongly continuous positivity preserving contraction semigroup
\((T_t)_{t≥0}\) on the space \(C_∞(\mathbb{R}^d)\) of continuous functions vanishing at infinity:
\[ T_t f(x) = E_{x} \left[ f(ξ_t) \right] \] for any \( f ∈ C_∞(\mathbb{R}^d) \). Due to Courrèege it is known that (under a mild
richness condition on the domain) the infinitesimal generator \( A \) of a Feller semigroup
is a pseudo-differential operator (ΨDO, for short), i.e. an operator of the form
\[ Af(q) = -H(q,D)f(q) = -(2π)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(q-x)·p} H(q,p) f(x) \, dx \, dp, \quad f ∈ C_∞(\mathbb{R}^d). \]

The symbol of the operator, the function \(-H : \mathbb{R}^d × \mathbb{R}^d → \mathbb{C}, (q,p) ↦ -H(q,p),\)
is for fixed \( q \), given in terms of a Lévy-Khintchine representation
\[ H(q,p) = a(q) + iℓ(q) · p + p · Q(q)p + \int_{y ≠ 0} \left( 1 - e^{ip·y} + \frac{ip·y}{1 + |y|^2} \right) N(q,dy), \]
where, for each fixed \( q, ℓ(q) ∈ \mathbb{R}^d, Q(q) \) is a positive semidefinite symmetric matrix
and \( N(q,dy) \) is a measure kernel on \( \mathbb{R}^d \setminus \{0\} \) such that \( \int_{y ≠ 0} \frac{|y|^2}{1 + |y|^2} N(q,dy) < ∞ \). Note that these negative definite symbols do not belong to any of the classical
symbol classes of ΨDOs; consequently we do not have a Hörmander or Maslov
symbolic calculus at our disposal.

In a similar way, each operator \( T_t \) can be represented as a pseudo-differential op-
erator \( λ_t(·,D) \) with the symbol \( λ_t(q,p) = E_q [e^{i(ξ_t−q)·p}] \). It is known that \(-H(q,p) = \)
Feynman pseudomeasure on the set of paths in the configuration (or phase) space. Feynman path integrals with respect to a space of the system, described by the Schrödinger equation. In modern terminology, actually, these integrals range over Cartesian powers of the configuration (or phase) space. Feynman has defined this integral as a limit of some finite dimensional integrals; see [16], [17], to obtain the solution of the Schrödinger equation with a potential.

Brownian motion. This gives rise to several interesting problems: which negative definite symbols $H(q,p)$ lead to Feller processes and, if so, how can we represent or approximate the symbol $\lambda_t(q,p)$. The existence problem has been discussed at length in a series of papers, see [22, 4, 24] and the literature given there, and we would like now to investigate the problem how to represent the semigroup resp. its symbol if the (symbol of the) generator is known.

Consider an evolution equation $\frac{df}{dt}(t,q) = -H(q,D)f(t,q)$, where $-H(\cdot,D)$ is a generator of some Feller process $(\xi_t)_{t\geq 0}$. Following the terminology of mathematical physics, we call $H(\cdot,D)$ the Hamiltonian of the physical system, which is described by the above evolution equation. The solution of the Cauchy problem for this equation with initial data $f(0,q) = f_0(q)$ can be obtained by the Feynman-Kac formula $f(t,q) \equiv (T_tf_0)(q) = E^{(p)}f_0(\xi_t)$. Here the expectation $E^{(p)}f_0(\xi_t)$ is a functional integral (path integral) over the set of paths of the process $(\xi_t)_{t\geq 0}$ with respect to the measure generated by this process. If $(\xi_t)_{t\geq 0}$ is a diffusion process, then $E^{(p)}f_0(\xi_t) = \int_{\mathbb{R}^2} f_0(\xi_t) \mu(d\xi)$, where $\mu$ is a Gaussian measure, corresponding to this process; in particular, the Wiener measure corresponds to the process of Brownian motion.

The heuristic notion of a path integral has been introduced by R. Feynman, see [16], to obtain the solution of the Schrödinger equation with a potential. Feynman has defined this integral as a limit of some finite dimensional integrals; actually, these integrals range over Cartesian powers of the configuration (or phase) space of the system, described by the Schrödinger equation. In modern terminology this kind of path integrals are called Feynman path integrals with respect to a Feynman pseudomeasure on the set of paths in the configuration (or phase) space.

The classical Feynman-Kac formula, representing the solution of the Cauchy problem for the heat equation by a functional integral with respect to the Wiener measure can also be obtained applying Feynman’s construction. Here the functional integral is a limit of $n$-fold iterated integrals containing Gaussian exponents which are transition densities of a Brownian motion. This construction can be extended to a large class of Markov processes. However, in most cases the transition densities of Feller processes cannot be expressed by elementary functions and, hence, in order to compute functional integrals in Feynman-Kac formulae we need to approximate them. This gives rise to Feynman formulae.

A Feynman formula is a representation of the solution of an initial (or initial-boundary) value problem for an evolution equation (or, equivalently, a representation of the semigroup resolving the problem) as a limit of $n$-fold iterated integrals of some elementary functions, when $n \to \infty$. Obviously, the iterated integrals in a Feynman formula for some problem give approximations for a functional integral in the Feynman-Kac formula representing the solution of the problem. These approximations can be used for direct calculations and simulations.

The notion of a Feynman formula has been introduced in [33] and the method to obtain Feynman formulae for evolutionary equations has been developed in a series of papers [34–37]. Recently, this method has been successfully applied to obtain Feynman formulae for different classes of problems for evolutionary equations on different geometric structures, see, e.g. [5, 7], [26, 27, 32] and also to construct some surface measures on infinite dimensional manifolds (see [34–38]). This method is based on Chernoff’s theorem (see [12] and [33] for the version used
here), which is a generalization of the well-known Trotter formula. Trotter’s formula has been used to justify Feynman’s heuristic result for Schrödinger equations with a potential, e.g. [25], and to prove the classical Feynman-Kac formula mentioned earlier.

By Chernoff’s theorem a strongly continuous semigroup \((T_t)_{t \geq 0}\) on a Banach space can be represented as a strong limit: \(T_t = \lim_{n \to \infty} [F(t/n)]^n\) where \(F(t)\) is an operator-valued function satisfying certain conditions (see Theorem 2.4 for details). This equality is called a Feynman formula for the semigroup \((T_t)_{t \geq 0}\). We call this Feynman formula a Lagrangian Feynman formula, if the \(F(t)\), \(t > 0\), are integral operators with elementary kernels; if the \(F(t)\) are \(\Psi\)DOs, we speak of Hamiltonian Feynman formulæ. In particular, we obtain a Hamiltonian Feynman formula for a semigroup \(T_t \equiv e^{-tH(\cdot)}\) generated by a \(\Psi\)DO \(-H(\cdot, D)\) with the symbol \(-H(q, p)\) if

\[
e^{-tH(\cdot, D)} = \lim_{n \to \infty} \left[ e^{-\frac{t}{n}H(\cdot, D)} \right]^n,
\]

where \(e^{-\frac{t}{n}H(\cdot, D)}\) is the \(\Psi\)DO with the symbol \(e^{-\frac{1}{n}H(q, p)}\). Note that, in general \(e^{-\frac{t}{n}H(\cdot, D)}\) is not a semigroup and that \(\lambda_t(q, p) \neq e^{-tH(q, p)}\).

Our terminology is inspired by the fact that a Lagrangian Feynman formula gives approximations to a functional integral over a set of paths in the configuration space of a system (whose evolution is described by the semigroup \((T_t)_{t \geq 0}\)), while a Hamiltonian Feynman formula corresponds to a functional (Hamiltonian Feynman path) integral over a set of paths in the phase space of some system (cf. [8]). The corresponding Hamiltonian Feynman formula gives rise to a Hamiltonian Feynman path integral also in the case \(T_t = e^{itH(q, D)}\) (see [33]).

In this note we prove some Hamiltonian and Lagrangian Feynman formulæ for semigroups associated with Feller processes and for perturbations of such semigroups. Several results of the paper have been announced in [9]. The paper is organized as follows. Section 2 contains notation and some preliminaries; in particular, Chernoff’s theorem is formulated and the notion of Chernoff equivalence is introduced. In Section 3 we prove a Hamiltonian Feynman formula for a class of semigroups associated with Feller processes. In Section 4 we obtain a Lagrangian Feynman formula for a multiplicative perturbation of a Feller semigroup by a function \(a(\cdot)\) which is continuous, positive, bounded and bounded away from zero. Note, that analogous Lagrangian Feynman formulæ have been proved for some diffusion processes in [8] and have been presented for the Cauchy process in [9]. In Section 5 we consider gradient and bounded Schrödinger perturbations of Feller semigroups and obtain some Hamiltonian and Lagrangian Feynman formulæ for them.

2. Notations and preliminaries

Let \(C^\infty_c(\mathbb{R}^d)\) be a set of infinitely differentiable functions on \(\mathbb{R}^d\) with compact support and \(S(\mathbb{R}^d)\) be the Schwartz space of rapidly decreasing functions. Let us also consider a space \(C_\infty(\mathbb{R}^d)\) of all continuous functions vanishing at infinity. It is a Banach space with the norm \(\|f\|_\infty = \sup_{x \in \mathbb{R}^d} |f(x)|\). Write for the norm in \(C^k_c(\mathbb{R}^d)\), the space of \(k\) times continuously differentiable functions which vanish (with all their derivatives) at infinity,

\[
\|u\|_{(k)} := \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_\infty
\]

where \(\alpha \in \mathbb{N}^n_0\), \(\partial^\alpha = \partial^{\alpha_1}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}\), and \(|\alpha| = \alpha_1 + \ldots + \alpha_n\).

We use the following notations for the Fourier transform and its inverse:

\[
\hat{f}(p) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ipq} f(q) dq \quad \text{and} \quad \mathcal{F}^{-1}[f](q) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ipq} f(p) dp.
\]
2.1. Negative definite functions. Negative definite functions have been introduced by I.J. Schöhberg in connection with isometric embeddings of metric spaces into a Hilbert space. His original definition is the following.

Definition 2.1. A function \( \psi : \mathbb{R}^d \to \mathbb{C} \) is called negative definite if for any \( m \in \mathbb{N} \) and all \( p_1, \ldots, p_m \in \mathbb{R}^d \) the \( m \times m \) matrix \( \left( \psi(p_j) + \overline{\psi(p_k)} - \psi(p_j - p_k) \right)_{j,k=1,\ldots,m} \) is positive hermitian, i.e., if for all \( \lambda_1, \ldots, \lambda_m \in \mathbb{C} \)

\[
\sum_{j,k=1}^m \left( \psi(p_j) + \overline{\psi(p_k)} - \psi(p_j - p_k) \right) \lambda_j \overline{\lambda_k} \geq 0.
\]

A negative definite function is NOT the negative of a positive definite function. Recall that a function \( u : \mathbb{R}^d \to \mathbb{C} \) is called positive definite if for any choice of \( k \in \mathbb{N} \) and vectors \( p_1, \ldots, p_k \in \mathbb{R}^d \) the matrix \( (u(p_i - p_j))_{i,j=1,\ldots,k} \) is positive Hermitian, i.e. for all \( \lambda_1, \ldots, \lambda_k \in \mathbb{C} \) we have \( \sum_{i,j=1}^k u(p_i - p_j) \lambda_i \overline{\lambda_j} \geq 0 \).

Corollary 2.1. If \( u : \mathbb{R}^d \to \mathbb{C} \) is a positive definite function, then the function \( p \mapsto u(0) - u(p) \) is negative definite.

The deeper connection between positive definite and negative definite functions can be seen from the following Theorem 2.1 which also justifies the definition of continuous negative definite functions through the Lévy-Khintchine formula:

Definition 2.2. A function \( \psi : \mathbb{R}^d \to \mathbb{C} \) is called a continuous negative definite function if \( \psi \) is given by the Lévy-Khintchine formula

\[
\psi(p) = a + i\ell \cdot p + p \cdot Qp + \int_{y \neq 0} \left( 1 - e^{iy \cdot p} + \frac{iy \cdot p}{1 + |y|^2} \right) N(dy).
\]

The tuple \( (a, \ell, Q, N) \) consisting of \( a \in \mathbb{R}^+ \), \( \ell \in \mathbb{R}^d \), a positive semidefinite matrix \( Q \in \mathbb{R}^{d \times d} \) and a Radon measure \( N \) on \( \mathbb{R}^d \) is called Lévy characteristics (of \( \psi \)). The measure \( N \) is often called Lévy measure.

Obviously, the Lévy characteristics are uniquely determined by \( \psi \)—and vice versa.

Theorem 2.1. For \( \psi : \mathbb{R}^d \to \mathbb{C} \) the following properties are equivalent:

(a) \( \psi \) is continuous and negative definite in the sense of Definition 2.1;
(b) \( \psi \) is given by the Lévy-Khintchine formula (1);
(c) \( \psi(0) \geq 0 \) and \( e^{-t\psi} \) is for every \( t > 0 \) continuous and positive definite.

A proof of Theorem 2.1 can be found, e.g. in the monographs by Jacob [22] or by Berg and Forst [1](II.§7). All continuous positive definite functions are characterized by Bochner’s Theorem.

Theorem 2.2 (Bochner). A function \( \phi : \mathbb{R}^d \to \mathbb{C} \) is continuous and positive definite if, and only if, it is the Fourier transform of a bounded Radon measure \( \mu \in \mathcal{M}_b^+(\mathbb{R}^d) \), i.e.,

\[
\phi(p) = \hat{\mu}(p) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ip \cdot q} \mu(dq).
\]

From Definition 2.1 it is not hard to see that a negative definite function has positive real part \( \Re \psi \geq 0 \), satisfies \( \overline{\psi(p)} = \psi(-p) \) and that \( \sqrt{|\psi(\cdot)|} \) is subadditive, i.e.,

\[
\sqrt{|\psi(p_1 + p_2)|} \leq \sqrt{|\psi(p_1)|} + \sqrt{|\psi(p_2)|}, \quad p_1, p_2 \in \mathbb{R}^d.
\]
If $\psi$ is continuous, repeated applications of the subadditivity estimate yield the following growth bound
\[
|\psi(p)| \leq 2 \sup_{|\eta| \leq 1} |\psi(\eta)| \left(1 + |p|^2\right), \quad p \in \mathbb{R}^d.
\]

2.2. Feller and Lévy semigroups and their generators. A Feller process $(X_t)_{t \geq 0}$ with a state space $\mathbb{R}^d$ is a strong Markov process whose associated operator semigroup $(T_t)_{t \geq 0}$,
\[
T_t u(x) = \mathbf{E}^x [u(X_t)], \quad u \in C^\infty(\mathbb{R}^d), \quad t \geq 0, \quad x \in \mathbb{R}^d,
\]
enjoys the Feller property, i.e., it is a strongly continuous positivity preserving contraction semigroup on the space $C^\infty(\mathbb{R}^d)$. The semigroup $(T_t)_{t \geq 0}$ is said to be a Feller semigroup.

The (infinitesimal) generator $(A, D(A))$ of the semigroup or the process is given by the strong limit
\[
Au := \lim_{t \to 0} \frac{T_t u - u}{t}
\]
on the set $D(A) \subset C^\infty(\mathbb{R}^d)$ of those $u \in C^\infty(\mathbb{R}^d)$ for which the above limit exists w.r.t. the sup-norm. We will call $(A, D(A))$ a Feller generator for short.

Before we proceed with general Feller semigroups it is instructive to have a brief look at Lévy processes (and convolution semigroups) which are a particular subclass of Feller processes. Our standard reference for Lévy processes is the monograph by K. Sato [9]. A Lévy process $(Y_t)_{t \geq 0}$ is a stochastically continuous random process with stationary and independent increments. The Fourier transform of a Lévy process has a particularly simple structure,
\[
\mathbf{E}^x \left[ e^{ip(Y_t-x)} \right] = \mathbf{E}^0 \left[ e^{ipY_t} \right] = e^{-t \psi(p)},
\]
where $\psi : \mathbb{R}^d \to \mathbb{C}$ is the characteristic exponent which is a continuous negative definite function, i.e. $\psi$ is given by the Lévy-Khintchine formula [1]. Since $(Y_t)_{t \geq 0}$ is a Markov process both [9] and [11] characterize the finite dimensional distributions of $(Y_t)_{t \geq 0}$ and, hence, the process itself.

A Lévy process is spatially homogeneous. Therefore, the associated semigroup is of convolution type,
\[
S_t u(x) = \mathbf{E}^x [u(Y_t)] = \mathbf{E}^0 [u(Y_t + x)] = \int_{\mathbb{R}^d} u(x + y) \mathbf{P}^0(Y_t \in dy) = u \ast \tilde{\mu}_t(dy),
\]
\[
\tilde{\mu}_t(dy) = \mathbf{P}^0(Y_t \in -dy),
\]
and a short direct calculation shows that $(S_t)_{t \geq 0}$ is indeed a Feller semigroup with infinitesimal generator
\[
Bu(x) = -\psi(D) u(x) := -(2\pi)^{-n/2} \int_{\mathbb{R}^d} \psi(p) \hat{u}(p) e^{ix \cdot p} dp, \quad u \in C^\infty_c(\mathbb{R}^d).
\]

One can use the estimate (2) to show that integrals in (4) are convergent.

The operator $\psi(D)$ is a first example of a so-called pseudo differential operator with the symbol $\psi(p)$. Since $\psi$ does not depend on $x$ the operator has constant “coefficients”. Notice that the symbol $\psi$ is just the characteristic exponent of the process $(Y_t)_{t \geq 0}$. This shows that

every Lévy process is generated by a pseudo differential operator $-\psi(D)$ with the symbol $-\psi(p)$ where $\psi$ is the characteristic exponent of the process. Conversely, every pseudo differential operator $-\psi(D)$ with the symbol $-\psi(p)$, where $\psi$ is a continuous negative definite function, i.e., given by the Lévy-Khintchine formula (1), is the generator of a Lévy process.
Let us return to the general situation. It is not hard to see (cf. [15], p. 165, Theorem 2.2 (b)) that Feller generators satisfy the so-called positive maximum principle (PMP) if \( u \in D(A) \), \( \sup_{x \in \mathbb{R}^d} u(x) = u(x_0) \geq 0 \) then \( Au(x_0) \leq 0 \).

Extending earlier work of W. von Waldenfels [39, 40] Ph. Courrège showed in [13], see also [2], the following structure result for operators satisfying the positive maximum principle. We formulate his theorem only for Feller generators.

**Theorem 2.3** (Courrège). Let \((A, D(A))\) be a Feller generator such that \( C_c^\infty(\mathbb{R}^d) \subset D(A) \). Then \( A|_{C_c^\infty(\mathbb{R}^d)} \) is a pseudo differential operator,

\[
Au(q) = -H(q, D)u(q) = -(2\pi)^{-n/2} \int_{\mathbb{R}^d} H(q, p) \hat{u}(p) e^{ip \cdot q} \, dp, \quad u \in C_c^\infty(\mathbb{R}^d),
\]

with the symbol \( H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C} \) which is measurable, locally bounded in both variables \((q, p)\), and satisfies for fixed \( q \) the following Lévy-Khintchine representation

\[
H(q, p) = a(q) + i\ell(q) \cdot p + p \cdot Q(q) + \int_{\mathbb{R}^d \setminus \{0\}} \left( 1 - e^{iy \cdot p} + \frac{iy \cdot p}{1 + |y|^2} \right) N(q, dy),
\]

where \((a(q), \ell(q), Q(q), N(q, \cdot))\) is for each \( q \in \mathbb{R}^d \) the Lévy characteristics of \(-H(q, \cdot)\).

Observe that (5) automatically implies the continuity of \( p \mapsto H(q, p) \) for each \( q \in \mathbb{R}^d \).

Let \( H(q, D) \) be a pseudo differential operator with the symbol \( H(q, p) \) as in Theorem 2.3. Since \( H(q, p) \) is represented by the Lévy-Khintchine type formula (6) we can use Fourier inversion in (5) and find that the integro-differential operator

\[
A\varphi(q) = -a(q)\varphi(q) + \ell(q) \cdot \nabla \varphi(q) + \sum_{j,k=1}^d Q^{jk}(x) \partial_j \partial_k \varphi(q) \]

\[
+ \int_{\mathbb{R}^d \setminus \{0\}} \left( \varphi(q + y) - \varphi(q) - \frac{y \cdot \nabla \varphi(q)}{1 + |y|^2} \right) N(q, dy)
\]

extends \((-H(\cdot, D), C_c^\infty(\mathbb{R}^d))\) to the set \( C_c^\infty(\mathbb{R}^n) \). Note that the following Lemma 2.1, together with the integration properties of \( N(q, dy) \),

\[
\int_{y \neq 0} |y|^2/(1 + |y|^2) N(q, dy) < \infty,
\]

ensure that the integral in (7) converges. From now on we will use the pseudo differential representation (5) and the integro-differential representation (7) simultaneously.

**Lemma 2.1.** For all \( \varphi \in C_c^\infty(\mathbb{R}^d) \) we have

\[
|\varphi(q + y) - \varphi(q) - \frac{y \cdot \nabla \varphi(q)}{1 + |y|^2}| \leq 2 \frac{|y|^2}{1 + |y|^2} \|\varphi\|_2.
\]
Proof. By Taylor’s formula we get for all \( q, y \in \mathbb{R}^d \)

\[
\left| (1 + |y|^2) \left( \varphi(q + y) - \varphi(q) - \frac{y \cdot \nabla \varphi(q)}{1 + |y|^2} \right) \right| 
\leq |\varphi(q + y) - \varphi(q) - y \cdot \nabla \varphi(q)| + |y|^2 |\varphi(q + y) - \varphi(q)|
\leq \frac{1}{2} \sum_{j,k=1}^d y_j y_k \partial_j \partial_k \varphi(\xi_{q,y})
\leq 2|y|^2 \left( \|\varphi\|_\infty + \sqrt{\sum_{j,k=1}^d \|\partial_j \partial_k \varphi\|_\infty^2} \right)
\leq 2|y|^2 \|\varphi\|_{(2)}. \quad \Box
\]

In the sequel we will need also the following Lemma.

**Lemma 2.2.** We have

\[
\frac{|y|^2}{1 + |y|^2} = \int_{\mathbb{R}^d} \left( 1 - \cos(y \cdot p) \right) g(p) \, dp, \quad y \in \mathbb{R}^d,
\]

where \( g(p) = \frac{1}{2} \int_0^\infty (2\pi \lambda)^{-d/2} e^{-|p|^2/2\lambda} e^{-\lambda/2} d\lambda \) is integrable and has absolute moments of arbitrary order.

**Proof.** The Tonelli-Fubini Theorem and a change of variables show for \( k \in \mathbb{N}_0 \)

\[
\int_{\mathbb{R}^d} |p|^k g(p) \, dp = \frac{1}{2} \int_0^\infty (2\pi \lambda)^{-d/2} \int_{\mathbb{R}^d} |p|^k e^{-|p|^2/2\lambda} \, dp \, e^{-\lambda/2} d\lambda
\]

\[
= \frac{1}{2} \int_0^\infty (2\pi \lambda)^{-d/2} \int_{\mathbb{R}^d} \lambda^{k/2} |\eta|^k e^{-|\eta|^2/2\lambda} \, d\eta \, d\lambda
\]

\[
= \frac{1}{2} (2\pi)^{-d/2} \int_{\mathbb{R}^d} |\eta|^k e^{-|\eta|^2/2} \, d\eta \int_0^\infty \lambda^{k/2} e^{-\lambda/2} d\lambda,
\]

i.e., \( g \) has absolute moments of any order. Moreover, the elementary formula

\[
e^{-\lambda |y|^2/2} = (2\pi \lambda)^{-d/2} \int_{\mathbb{R}^d} e^{-|p|^2/2\lambda} e^{iy \cdot p} \, dp
\]

and Fubini’s Theorem yield

\[
\frac{|y|^2}{1 + |y|^2} = \frac{1}{2} \int_0^\infty \left( 1 - e^{-\lambda |y|^2/2} \right) e^{-\lambda/2} d\lambda
\]

\[
= \frac{1}{2} \int_0^\infty (2\pi \lambda)^{-n/2} \left( 1 - e^{iy \cdot p} \right) e^{-|p|^2/2\lambda} e^{-\lambda/2} d\lambda dp
\]

\[
= \int_0^\infty (1 - e^{iy \cdot p}) g(p) \, dp.
\]

The assertion follows since the left-hand side is real-valued. \( \Box \)
2.3. The Chernoff theorem. If $X, X_1, X_2$ are Banach spaces, then $L(X_1, X_2)$ denotes the space of continuous linear mappings from $X_1$ to $X_2$ equipped with the strong operator topology, $L(X) = L(X, X)$, $\| \cdot \|$ denotes the operator norm on $L(X)$ and $\text{Id}$ the identity operator in $X$. If $D(T) \subset X$ is a linear subspace and $T : D(T) \to X$ is a linear operator, then $D(T)$ denotes the domain of $T$.

The derivative at the origin of a function $F : [0, \varepsilon) \to L(X)$, $\varepsilon > 0$, is a linear mapping $F'(0) : D(F'(0)) \to X$ such that

$$F'(0)g := \lim_{t \searrow 0} \frac{F(t)g - F(0)g}{t},$$

where $D(F'(0))$ is the vector space of all elements $g \in X$ for which the above limit exists.

In the sequel we use the following version of Chernoff’s theorem (see [33]).

**Theorem 2.4** (Chernoff). Let $X$ be a Banach space, $F : [0, \infty) \to L(X)$ be a (strongly) continuous mapping such that $F(0) = \text{Id}$ and $\| F(t) \| \leq e^{\alpha t}$ for some $\alpha \in [0, \infty]$ and all $t \geq 0$. Let $D$ be a linear subspace of $D(F'(0))$ such that the restriction of the operator $F'(0)$ to this subspace is closable. Let $(A, D(A))$ be this closure. If $(A, D(A))$ is the generator of a strongly continuous semigroup $(T_t)_{t \geq 0}$, then for any $t_0 > 0$ the sequence $(F(t/n))_n \in N$ converges to $(T_{t/n})_n$ as $n \to \infty$ in the strong operator topology, uniformly with respect to $t \in [0, t_0]$, i.e., $T_t = \lim_{n \to \infty}(F(t/n))^n$.

A family of operators $(F(t))_{t \geq 0}$ is called Chernoff equivalent to the semigroup $(T_t)_{t \geq 0}$ if this family satisfies the assertions of Chernoff’s theorem; then, by Chernoff’s theorem we have in $L(X)$ locally uniformly with respect to $t$

$$T_t = \lim_{n \to \infty}(F(t/n))^n. \tag{9}$$

The equality (9) is called Feynman formula for the semigroup $(T_t)_{t \geq 0}$.

3. Hamiltonian Feynman formula for some Feller semigroups

Consider a function $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ which is measurable, locally bounded in both variables $(q, p)$, and satisfies for fixed $q$ the Lévy-Khintchine representation [4], i.e., $H(q, \cdot)$ is a continuous negative definite function for all $q \in \mathbb{R}^d$. Assume that

$$\sup_{q \in \mathbb{R}^d} |H(q, p)| \leq \kappa(1 + |p|^2) \quad \text{for all } p \in \mathbb{R}^d \quad \text{and some } \kappa > 0, \tag{10}$$

$$p \mapsto H(q, p) \quad \text{is uniformly (w.r.t. } q \in \mathbb{R}^d) \text{ continuous at } p = 0, \tag{11}$$

$$q \mapsto H(q, p) \quad \text{is continuous for all } p \in \mathbb{R}^d. \tag{12}$$

Consider a ΨDO $H(\cdot, D)$ with the symbol $H(q, p)$, i.e., for each $\varphi \in C_0^\infty(\mathbb{R}^d)$ we have

$$H(q, D)\varphi(q) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{iq \cdot p} H(q, p)\hat{\varphi}(p)dp. \tag{13}$$

Note that (due to the estimate (12)) the condition (10) actually means that a ΨDO $H(\cdot, D)$ is an operator with bounded “coefficients” $(a(q), \ell(q), Q(q), N(q, \cdot))$.

**Assumption A.**

(i) We assume that the function $H(q, p)$ is such that $-H(\cdot, D)$ is closable and the closure is the generator of a strongly continuous semigroup on $C_0^\infty(\mathbb{R}^d)$.

(ii) We assume also that the set $C_0^\infty(\mathbb{R}^d)$ of test functions is an operator core for this generator.
Remark 3.1. Conditions on the function $H(q,p)$ to fulfill Assumption A (i) can be found, for example, in Vol. 2 of [22] (Thms. 2.6.4, 2.6.9, 2.7.9, 2.7.16, 2.7.19, 2.8.1) or in [24]. For all these constructions $C_c^\infty(\mathbb{R}^d)$ is always an operator core. Note that Assumption A (ii) holds also for example for generators of Lévy processes, see [29] (Theo. 31.5).

Let $F(t)$ be a ΨDO with the symbol $e^{-tH(q,p)}$, i.e. for each $\varphi \in C_c^\infty(\mathbb{R}^d)$

$$F(t)\varphi(q) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ip\cdot q - tH(q,p)} \hat{\varphi}(p) dp.$$  

(14)

Lemma 3.1. For each $\varphi \in C_c^\infty(\mathbb{R}^d)$ the function $F(t)\varphi$ belongs to $C_\infty(\mathbb{R}^d)$.

Proof. The Fourier transform $\hat{\varphi}$ of a test function $\varphi \in C_c^\infty(\mathbb{R}^d)$ is in the Schwartz space $S(\mathbb{R}^d)$ of rapidly decreasing functions. Since $q \mapsto e^{-tH(q,p)}$ is continuous (by assumption [12]) and bounded ($\Re H \geq 0$ due to properties of continuous negative definite functions), Lebesgue’s Dominated Convergence theorem shows that $F(t)$ maps $C_c^\infty(\mathbb{R}^d)$ into $C(\mathbb{R}^d)$.

Let us prove, that $F(t)\varphi(q) \to 0$ when $|q| \to \infty$. Since $H(q,\cdot)$ is continuous negative definite for all $q \in \mathbb{R}^d$ then $e^{-tH(q,\cdot)}$ is continuous positive definite for all $q \in \mathbb{R}^d$ and all $t > 0$ due to Theorem [24]. Then the function

$$[p \mapsto h_t(q,p) := e^{-tH(q,0)} - e^{-tH(q,p)}]$$

is also continuous negative definite for all $q \in \mathbb{R}^d$ by Corollary [24]. Hence, $h_t(q,\cdot)$ satisfies a Lévy-Khintchine representation

$$h_t(q,p) = a_t(q) + i\ell_t(q) \cdot p + p \cdot Q_t(q)p + \int \left(1 - e^{iy\cdot p} + \frac{iy\cdot p}{1 + |y|^2}\right) N_t(q,dy),$$  

(15)  

where $(a_t(q),\ell_t(q),Q_t(q),N_t(q,\cdot))$ is for each $q \in \mathbb{R}^d$ the Lévy characteristics of $h_t(q,\cdot)$. Again we can consider a ΨDO $h_t(q,D)$ with the symbol $h_t(q,p)$, i.e. for each $\varphi \in C_c^\infty(\mathbb{R}^d)$

$$h_t(q,D)\varphi(q) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ip\cdot q} h_t(q,p) \hat{\varphi}(p) dp$$

$$= -a_t(q)\varphi(q) + \ell_t(q) \cdot \nabla \varphi(q) + \sum_{j,k=1}^d Q^{jk}_t(q) \partial_j \partial_k u(q)$$

$$+ \int_{y \neq 0} \left( \varphi(q+y) - \varphi(q) - \frac{y \cdot \nabla \varphi(q)}{1 + |y|^2} \right) N_t(q,dy),$$  

(16)

Note, that

$$F(t)\varphi(q) = (2\pi)^{-d/2} e^{-tH(q,0)} \int_{\mathbb{R}^d} e^{ip\cdot q} \hat{\varphi}(p) dp - (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ip\cdot q} h_t(q,p) \hat{\varphi}(p) dp.$$  

Since $\Re H \geq 0$ then $\sup_{q \in \mathbb{R}^d} |e^{-tH(q,0)}| \leq 1$, and the first integral in the above formula tends to zero as $|q| \to \infty$ by the Riemann–Lebesgue Theorem. Thus, we only need to show that

$$\left[q \mapsto (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ip\cdot q} h_t(q,p) \hat{\varphi}(p) dp \right] \in C_\infty(\mathbb{R}^d).$$
As \( \varphi \) has compact support, there is some \( R > 0 \) such that \( \text{supp} \varphi \subset B_R(0) \). For all \( |q| > 2R \) formula (16) becomes

\[
|h_t(q,D)\varphi(q)| = \left\| \int_{|y| > R} \varphi(q + y) N_t(q,dy) \right\| \leq 2 \int_{y \neq 0} \frac{|y/R|^2}{1 + |y/R|^2} N_t(q,dy) \cdot \|\varphi\|_\infty.
\]

The last line follows from the elementary inequality \( \frac{1}{2} \leq \frac{\eta}{1 + \eta^2} \) for \( |t| > 1 \) which applies if \( |y| > R \), and from \( \varphi(q + y) = 0 \) if \( |q| > 2R \) and \( |y| \leq R \). We can now use Lemma 2.2 the Lévy-Khintchine representation of \( h_t(q, \cdot) \) and the estimate (2) for a continuous negative definite function \( h_t(q, \cdot) \) to get

\[
|h_t(q,D)\varphi(q)| \leq 2 \int_{|y| > R} \left( 1 - \cos \frac{y \cdot \eta}{R} \right) \varphi(y) dy N_t(q,dy) \cdot \|\varphi\|_\infty
\]

\[
\leq 2 \int_{R^d} \text{Re} h_t \left( q, \frac{\eta}{R} \right) \varphi(y) dy \cdot \|\varphi\|_\infty
\]

\[
\leq 2 \int_{R^d} \left| h_t \left( q, \frac{\eta}{R} \right) \right| \varphi(y) dy \cdot \|\varphi\|_\infty
\]

\[
\leq 2 \sup_{|\xi| \leq 1} |h_t(q, \xi)| \int_{R^d} \left( 1 + |\eta|^2 \right) \varphi(y) dy \cdot \|\varphi\|_\infty.
\]

Since \( g(\eta) \) has absolute moments of any order, we see

\[
|h_t(q,D)\varphi(q)| \leq c_\varphi \sup_{q \in R^d} \sup_{|\xi| \leq 1} |h_t(q, \xi)| \cdot \|\varphi\|_\infty \quad \text{for all} \quad |q| > 2R.
\]

As \( h_t(q,0) = 0 \), the condition (11) tells us that \( \lim_{|q| \to \infty} h_t(q,D)\varphi(q) = 0 \). Therefore, \( h_t(q, \cdot)\varphi \in C^\infty_c(R^d) \).

\( \square \)

**Lemma 3.2.** For each \( t > 0 \) a mapping \( F(t) \) can be extended to a contraction \( F(t) : C^\infty_c(R^d) \to C^\infty_c(R^d) \).

**Proof.** Let us freeze the coefficients (see e.g. [23]). For each \( t > 0 \) and each \( q_0 \in R^d \) let us consider a PDO \( F^{q_0}(t) \) with the symbol \( e^{-tH(q_0,p)} \), i.e. for any \( \varphi \in C^\infty_c(R^d) \) we have

\[
F^{q_0}(t)\varphi(q) = (2\pi)^{-d/2} \int_{R^d} e^{ipq} e^{-tH(q_0,p)} \varphi(p) dp.
\]

Then \( F(t)\varphi(q) = F^{q}(t)\varphi(q) \) for any \( \varphi \in C^\infty_c(R^d) \) and any \( q \in R^d \). Since for each \( q_0 \in R^d \) the function \( e^{-tH(q_0,p)} \) is positive definite then there exists a convolution semigroup \( \mu^{q_0}_t \) such that \( F^{-1}[\mu^{q_0}_t] = (2\pi)^{-d/2} e^{-tH(q_0,p)} \) and \( F^{q_0}(t)\varphi(q) = \int_{R^d} \varphi(q - y) \mu^{q_0}_t(dy) \). Hence, for each \( q_0 \in R^d \) a family \( (F^{q_0}(t))_{t \geq 0} \) is a Feller semigroup, and for each \( q, q_0 \in R^d \) we have

\[
|F^{q_0}(t)\varphi(q)| = \left| \int_{R^d} \varphi(q - y) \mu^{q_0}_t(dy) \right| \leq \|\varphi\|_\infty.
\]
Let \( F(t) \) be measurable and locally bounded in both variables \((q,p)\). Assume that \( H(q,\cdot) \) is continuous and negative definite for all \( q \in \mathbb{R}^d \) and that conditions (10) and (11) hold. Under Assumption A the family \( \{F(t)\}_{t \geq 0} \) is Chernoff equivalent to a strongly continuous semigroup \( \{T_t\}_{t \geq 0} \) generated by the closure of the \( \Psi DO - H(\cdot, D) \) with the symbol \(-H(q,p)\), and the Hamiltonian Feynman formula \( T_t = \lim_{n \to \infty} \{F_t^n\} \) is valid in \( L(C_\infty(\mathbb{R}^d)) \) locally uniformly with respect to \( t \geq 0 \).

**Proof.** By Lemma 3.2 each \( F(t) \) is a contraction operator on \( C_\infty(\mathbb{R}^d) \), thus we only need to prove that for all \( F(t) \in C_\infty(\mathbb{R}^d) \) we have \( \lim_{t \to 0} \|F(t)\varphi - \varphi\|_\infty = 0 \) and \( F'(0) = -H(\cdot, D) \) on a core of \(-H(\cdot, D)\).

For any \( F \in C_\infty(\mathbb{R}^d) \) we have, due to the estimate (10),

\[
\lim_{t \to 0} \|F(t)\varphi - \varphi\|_\infty = \lim_{t \to 0} \sup_{q \in \mathbb{R}^d} \left| \frac{2\pi}{2d/2} \int_{\mathbb{R}^d} e^{ip \cdot q} \widehat{\varphi}(p) \left( e^{-tH(q,p)} - 1 \right) dp \right|
\leq \lim_{t \to 0} \left( \frac{2\pi}{2d/2} \right) \sup_{q \in \mathbb{R}^d} \left\{ \frac{e^{-tH(q,p)} - 1}{-tH(q,p)} \right\} \|H(q,p)\| dp
\leq \lim_{t \to 0} \left( \frac{2\pi}{2d/2} \right) \int_{\mathbb{R}^d} t\kappa(1 + |p|^2) |\widehat{\varphi}(p)| dp
= 0,
\]

since \( \widehat{\varphi} \in S(\mathbb{R}^d) \). Hence, \( \lim_{t \to 0} \|F(t)\varphi - \varphi\|_\infty = 0 \) for all \( \varphi \in C_\infty(\mathbb{R}^d) \). As \( \|F(t)\| \leq 1 \), then the last equality is true for all \( \varphi \in C_\infty(\mathbb{R}^d) \) by a \( \varepsilon \)-argument.

In a similar way, for any \( \varphi \in C_\infty(\mathbb{R}^d) \) we have

\[
\lim_{t \to 0} \left\| \frac{F(t)\varphi - \varphi}{t} + H(\cdot, D)\varphi \right\|_\infty
= \lim_{t \to 0} \sup_{q \in \mathbb{R}^d} \left| \frac{2\pi}{2d/2} \int_{\mathbb{R}^d} e^{ip \cdot q} \widehat{\varphi}(p) \left( \frac{e^{-tH(q,p)} - 1}{t} + (H(q,p)) \right) dp \right|
\leq \lim_{t \to 0} \left( \frac{2\pi}{2d/2} \right) \int_{\mathbb{R}^d} |\widehat{\varphi}(p)| t\kappa^2(1 + |p|^2)^2 dp
= 0.
\]

Thus, all assumptions of Chernoff’s theorem are fulfilled, and the family \( F(t) \) is Chernoff equivalent to the semigroup \( T_t \) generated by \(-H(\cdot, D)\). □

**Remark 3.2.** (i) Let us assume additionally that \( H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C} \) satisfies the following condition:

\[
\exists C > 0 \text{ such that } \|\partial_q \partial_p e^{itH}\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)} \leq C,
\]

where \( \alpha, \beta \in \mathbb{N}_0 \), \( \alpha = 0 \) or 1, \( \beta = 0 \) or 1, \( \partial_q \partial_p \) are derivatives in the distributional sense. Note, that this condition is fulfilled, e.g. if \( H : |H(q,p)| \geq c|p|^r \) for \( |p| \gg 1 \), some \( c > 0 \) and some \( r \in (0, 2) \). Then by Theorem 2 of Ref. [19] we have \( F(t) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \). In this case the Hamiltonian Feynman formula obtained
in Theorem 3.1 has the following form:

\[(T(t)\varphi)(q_0)\]

\[
= \lim_{n \to \infty} \frac{1}{(2\pi)^{dn}} \int_{\mathbb{R}^{dn}} e^{i \sum_{k=1}^{n} p_k \cdot (q_{k-1} - q_k)} e^{-\frac{1}{4} \sum_{k=1}^{n} H(q_{k-1}, q_k)} \varphi(q_n) dq_1 dp_1 \cdots dq_n dp_n,
\]

where the equality holds in \(L_2\)-sense (i.e. the integrals in the right hand side must be considered in a regularized sense). We refer to [10] for further conditions on \(H(q, p)\) ensuring \(F(t) : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)\).

(ii) If the function \(H\) satisfies sufficient conditions for \(F(t)\varphi\) to be in \(S(\mathbb{R}^d)\) for each \(\varphi \in S(\mathbb{R}^d)\) then for any \(\varphi \in S(\mathbb{R}^d)\) the equality in the Hamiltonian Feynman formula (17) holds in each point \(q_0 \in \mathbb{R}^d\). Such conditions can be found in the following lemma.

**Lemma 3.3.** Let \(H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}\) be a continuous function such that for each \(q \in \mathbb{R}^d\) a mapping \(p \mapsto H(q, p)\) is negative definite and \(H(\cdot, \cdot) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)\). Assume that for each \(p \in \mathbb{R}^d\) and for each \(\alpha, \beta \in \mathbb{N}_0^d\), the following estimates hold:

\[
\sup_{q \in \mathbb{R}^d} |\partial_q^\alpha \partial_p^\beta H(q, p)| \leq f_{\alpha, \beta}(p),
\]

where all functions \(f_{\alpha, \beta}\) are continuous on \(\mathbb{R}^d\) and have at most polynomial growth at infinity. Then \(F(t)\varphi \in S(\mathbb{R}^d)\) for each \(\varphi \in S(\mathbb{R}^d)\).

**Proof.** By Lemma 3.2 we have \(F(t)\varphi \in C_\infty(\mathbb{R}^d)\). Let us show that for all \(\alpha, \beta \in \mathbb{N}_0^d\) the norm

\[
\|F(t)\varphi\|_{\alpha, \beta} = \sup_{q \in \mathbb{R}^d} |q^\alpha \partial_q^\beta [F(t)\varphi](q)|
\]

is finite. Note that for any \(\beta \in \mathbb{N}_0^d\) the function \(\partial_q^\beta e^{-tH(q, p)} q^p\) is continuous. By (13) it is also majorized (uniformly for all \(q\)) by some continuous function of \(p\) which has at most polynomial growth at infinity. Hence, by Lebesgue’s dominated convergence theorem, we have

\[
q^\alpha \partial_q^\beta [F(t)\varphi](q) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} q^\alpha \partial_q^\beta e^{-tH(q, p)} q^p \varphi(p) dp
\]

\[
= (2\pi)^{-d/2} \sum_{0 \leq \gamma \leq \beta} \int_{\mathbb{R}^d} q^\alpha \partial_q^\beta (e^{ip \cdot q}) \partial_q^{\beta-\gamma} (e^{-tH(q, p)}) \varphi(p) dp.
\]

Since \(\partial_q^\gamma (e^{ip \cdot q}) = e^{ip \cdot q} R_\gamma(p)\), where \(R_\gamma\) is a polynomial of \(p\), we can use integration by parts and get

\[
q^\alpha \partial_q^\beta [F(t)\varphi](q) = (2\pi)^{-d/2} \sum_{0 \leq \gamma \leq \beta} \int_{\mathbb{R}^d} q^\alpha e^{ip \cdot q} [R_\gamma(p) \partial_q^{\beta-\gamma} (e^{-tH(q, p)}) \varphi(p)] dp
\]

\[
= (2\pi)^{-d/2} \sum_{0 \leq \gamma \leq \beta} (-i)^{|\alpha|} \int_{\mathbb{R}^d} \partial_p^\alpha e^{ip \cdot q} [R_\gamma(p) \partial_q^{\beta-\gamma} (e^{-tH(q, p)}) \varphi(p)] dp
\]

Since \(\partial_p^\alpha [R_\gamma(p) \partial_q^{\beta-\gamma} (e^{-tH(q, p)}) \varphi(p)]\) is bounded by an \(L_1\)-function which is independent of \(q\), we can use (13) to see that the expression in the last line is finite. Hence, the norm \(\|F(t)\varphi\|_{\alpha, \beta}\) is finite. \(\square\)
Remark 3.3. If in Assumption A (i) we require the existence of not just a strongly continuous but a Feller semigroup, we obtain a Hamiltonian Feynman formula for the corresponding Feller process $\xi_t$. Besides, for each fixed $n$ the operator $[F(t/n)]^n$ in the Hamiltonian Feynman formula corresponds to the approximation of the process $\xi_t$ by a Markov chain $\{Y^{t/n}(k)\}_{k=0}^n$ with Lévy increments. This Markov chain is obtained by splitting the time interval $[0, t]$ onto $n$ equal steps and “freezing” the coefficient $q$ in the transition probabilities of $\xi_{kt}$ at each step (cf. [1]). Moreover, the transition kernels $\mu_{q,t/n}$ of this Markov chain correspond to the transition operator $W_{t/n}$, $W_{t/n}\varphi(q) = \int_{\mathbb{R}^d} \varphi(y)\mu_{q,t/n}(dy)$. Hence, $[F(t/n)]^n = [W_{t/n}]^n$. This allows us to transform the obtained Hamiltonian Feynman formula for the Feller semigroup $T_t$ associated with the process $\xi_t$ into a Lagrangian Feynman formula $T_t\varphi(q) = \lim_{n \to \infty} [W_{t/n}]^n \varphi(q)$.

Example 3.1. Let us consider the symbol $H_1(q,p) = a(q)|p|^\alpha$, where $\alpha \in (0, 2]$ and $a(\cdot) \in C^\infty(\mathbb{R}^d)$ is a strictly positive and bounded function. Then $-H_1(\cdot, D)$ generates a Feller semigroup $(T_t^1)_{t \geq 0}$ (see [17]). If $\alpha = 2$ this semigroup corresponds to the process of diffusion with variable diffusion coefficient. All conditions of the Theorem 3.1 are fulfilled and by the Hamiltonian Feynman formula (17) for any $\varphi \in C_c^\infty(\mathbb{R}^d)$ and any $q_0 \in \mathbb{R}^d$ we have:

$$(T^1_t \varphi)(q_0) = \lim_{n \to \infty} \frac{1}{(2\pi)^dn} \int_{\mathbb{R}^{2dn}} e^{i \sum_{k=1}^n p_k (q_{k-1} - q_k)} e^{-\frac{1}{2} \sum_{k=1}^n a(q_{k-1})|p_k|^\alpha} \varphi(q_n)dp_1 \cdots dq_n dp_n.$$ 

Example 3.2. Let us consider the symbol $H_2(q,p) = \sqrt{|p|^\alpha + m^2(q) - m(q)}$, where $m(\cdot) \in C^\infty(\mathbb{R}^d)$ is a strictly positive and bounded function on $\mathbb{R}^d$, $\alpha \in (0, 2]$. If additionally the function $m(\cdot)$ is such that the Assumption A holds (e.g. if $m \equiv \text{const}$), then the following Hamiltonian Feynman formula is valid for the corresponding semigroup $(T^2_t)_{t \geq 0}$:

$$(T^2_t \varphi)(q_0) = \lim_{n \to \infty} \frac{1}{(2\pi)^dn} \int_{\mathbb{R}^{2dn}} e^{i \sum_{k=1}^n p_k (q_{k-1} - q_k)} e^{-\frac{1}{2} \sum_{k=1}^n |p_k|^\alpha + m^2(q_{k-1}) - m(q_{k-1})} \times \varphi(q_n)dp_1 \cdots dq_n dp_n,$$

where $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $q_0 \in \mathbb{R}^d$. In the case $\alpha = 2$ the operator $H_2(\cdot, D)$ can be considered as a Hamiltonian of a free relativistic (quasi-)particle with variable mass (cf. [15], [20]).

4. Lagrangian Feynman formula for multiplicative perturbations of Feller semigroups

Let $H(\cdot, D)$ be a $\Psi$DO with the symbol $H(q,p)$. Assume that $-H(\cdot, D)$ generates a Feller semigroup $(T_t)_{t \geq 0}$,

$$T_t\varphi(q) = \int_{\mathbb{R}^d} \varphi(y)p_t(q,dy),$$

where $p_t(q,dy) = P^t(X_t \in dy)$ is the transition probability of the underlying Feller process $X_t$, $\varphi \in C_c(\mathbb{R}^d)$. Let $a(\cdot) : \mathbb{R}^d \to [c, 1/c]$, $c > 0$, be a continuous function. Then (due to [13]) a $\Psi$DO $-\tilde{H}(\cdot, D)$ with the symbol $-\tilde{H}(q,p) = -a(q)H(q,p)$ generates a strongly continuous semigroup $(\tilde{T}_t)_{t \geq 0}$ on the space $C_c(\mathbb{R}^d)$. 

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Consider now a family \((\tilde{F}(t))_{t\geq 0}\) of operators on \(C_\infty(\mathbb{R}^d)\) defined by the formula:

\[
\tilde{F}(t)\varphi(q) = (T_{a(q)t}\varphi)(q) = \int_{\mathbb{R}^d} \varphi(y)p_{a(q)t}(q,dy)
\]

\[ (19) \]

**Theorem 4.1.** The family \((\tilde{F}(t))_{t\geq 0}\) given by the formula \((19)\) is Chernoff equivalent to the semigroup \((T_t)_{t\geq 0}\) generated by a \(\Psi DO\) \(-\hat{H}(\cdot, D)\) with the symbol \(-\hat{H}(q,p) = -a(q)H(q,p)\) and, hence, the Lagrangian Feynman formula

\[
\tilde{T}_t\varphi(q_0) = \lim_{n\to\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \varphi(q_0)p_{a_1(q_0)t/n}(q_1, dq_1)p_{a_2(q_1)t/n}(q_2, dq_2) \cdots p_{a_n(q_n)t/n}(q_{n-1}, dq_{n-1}) \cdot \cdot \cdot p_{a_n(q_n)t}(q_{n-1}, dq_n)
\]

\[ (20) \]

is valid in \(L(C_\infty(\mathbb{R}^d))\) locally uniformly with respect to \(t \geq 0\).

**Remark 4.1.** The transformation of a Feller process under which a symbol \(H(q,p)\) transfers into a symbol \(a(q)H(q,p)\) may be understood as a position-dependent time re-scaling of the process: \(t \sim a(q)t\). Indeed, for each random variable \(X_{a(q)}\), we have:

\[
\mathbb{E}[\exp(a(X_{a(q)} - q) - p)] - 1 = a(q)\mathbb{E}[\exp(a(q) - p)] - 1 \rightarrow -a(q)H(q,p), \quad t \to 0.
\]

Note, that \(p(t,x,dy) = p_{a(x)}(x,dy)\) is NOT the transition function of the re-scaled process.

If the explicit form of the transition density of the original Feller process is known then the Lagrangian Feynman formula \((20)\) for the time re-scaled process contains only explicit—usually elementary—functions.

**Proof.** First, let us prove that the family \((\tilde{F}(t))_{t\geq 0}\) acts in the space \(C_\infty(\mathbb{R}^d)\). For any fixed \(\varphi \in C_\infty(\mathbb{R}^d)\) we have

\[
\lim_{q \to q_0} |\tilde{F}(t)\varphi(q) - \tilde{F}(t)\varphi(q_0)| = \lim_{q \to q_0} |(T_{a(q)t}\varphi)(q) - (T_{a(q_0)t}\varphi)(q_0)| \\
\leq \lim_{q \to q_0} \left( |(T_{a(q)t}\varphi)(q) - (T_{a(q_0)t}\varphi)(q)| + |(T_{a(q_0)t}\varphi)(q) - (T_{a(q_0)t}\varphi)(q_0)| \right) \\
\leq \lim_{q \to q_0} \|T_{a(q)t} - T_{a(q_0)t}\|_\infty + \lim_{q \to q_0} \|T_{a(q_0)t}\varphi(q) - (T_{a(q_0)t}\varphi)(q_0)| = 0.
\]

Therefore, the function \(\tilde{F}(t)\varphi\) is continuous. Since

\[
\lim_{|q| \to \infty} \tilde{F}(t)\varphi(q) = \lim_{|q| \to \infty} |(T_{a(q)t}\varphi)(q)| \\
\leq \lim_{|q| \to \infty} \sup_{q_0 \in \mathbb{R}^d} |T_{a(q_0)t}\varphi(q)| \\
= \lim_{|q| \to \infty} \sup_{s \in [ct, t/c]} |T_s\varphi(q)|,
\]

we get that \(\tilde{F}(t) : C_\infty(\mathbb{R}^d) \to C_\infty(\mathbb{R}^d)\).

Since the semigroup \((T_t)_{t\geq 0}\) acts in \(C_\infty(\mathbb{R}^d)\), then for any \(t \geq 0\) and \(\varepsilon > 0\) there exists \(R_\varepsilon > 0\) such that for any \(q \in \mathbb{R}^d : |q| > R_\varepsilon\) the inequality \(|T_t\varphi(q)| < \frac{1}{4}\varepsilon\) holds. Due to the strong continuity of \((T_t)_{t\geq 0}\) there exists \(\delta_\varepsilon > 0\) such that for all \(\tau, \tau' \in [ct, t/c]\) with \(|\tau - \tau'| < \delta_\varepsilon\) the inequality \(\|T_{\tau'}\varphi - T_{\tau}\varphi\|_\infty < \frac{1}{4}\varepsilon\) holds. Let us fix \(\varepsilon > 0\). Consider a partition \(\tau_0 = ct < \tau_1 < \ldots < \tau_N = t/c\) of a segment \([ct, t/c]\) such that \(\max_{1 \leq k \leq N} |\tau_k - \tau_{k-1}| < \delta_\varepsilon\). Then for any \(\tau \in [ct, t/c]\) there exists \(\tau_k\)
with $|\tau - \tau_k| < \delta$. Let now $R_\varepsilon = \max_{0 \leq k \leq N} R_{\varepsilon, \tau_k}$. Then for any $q \in \mathbb{R}^d : |q| > R_\varepsilon$ and any $\tau \in [ct, t/c]$ we have

$$|T_\tau \varphi(q)| \leq |T_\tau \varphi(q) - T_{\tau_k} \varphi(q)| + |T_{\tau_k} \varphi(q)|$$

$$\leq \|T_\tau \varphi - T_{\tau_k} \varphi\|_\infty + |T_{\tau_k} \varphi(q)|$$

$$\leq \frac{1}{2} \varepsilon + \frac{1}{2} \varepsilon = \varepsilon.$$

Therefore, $\lim_{|q| \to \infty} \sup_{\tau \in [ct, t/c]} |T_\tau \varphi(q)| = 0$ and, hence, the function $\tilde{F}(t)\varphi$ vanishes at infinity.

Further we use a freezing-in technique. For each $q_0 \in \mathbb{R}^d$ consider a family of operators $(F^{q_0}(t))_{t \geq 0}$ on $C_\infty(\mathbb{R}^d)$ such that

$$F^{q_0}(t)\varphi(q) = \int_{\mathbb{R}^d} \varphi(y) p_{a(q_0), t}(q, dy) \equiv (T_{a(q_0)t}) \varphi(q).$$

Then $\tilde{F}(t)\varphi(q) = F^q(t)\varphi(q)$ for all $\varphi \in C_\infty(\mathbb{R}^d)$, $q \in \mathbb{R}^d$. Moreover, $\tilde{F}(0) = T_0 = \text{Id}$ and, as $(T_t)_{t \geq 0}$ is a contraction semigroup, we have

$$\|\tilde{F}(t)\varphi\|_\infty \leq \sup_{q_0 \in \mathbb{R}^d} \sup_{q \in \mathbb{R}^d} |F^{q_0}(t)\varphi(q)|$$

$$\leq \sup_{q_0 \in \mathbb{R}^d} \|T_{a(q_0)t}\varphi\|_\infty$$

$$\leq \|\varphi\|_\infty.$$

The family $(\tilde{F}(t))_{t \geq 0}$ is strongly continuous since

$$\lim_{t \to 0} \|\tilde{F}(t)\varphi - \varphi\|_\infty = \lim_{t \to 0} \sup_{q \in \mathbb{R}^d} |F^q(t)\varphi(q) - \varphi(q)|$$

$$\leq \lim_{t \to 0} \sup_{q_0 \in \mathbb{R}^d, q \in \mathbb{R}^d} |F^{q_0}(t)\varphi(q) - \varphi(q)|$$

$$= \lim_{t \to 0} \sup_{q_0 \in \mathbb{R}^d} \|T_{a(q_0)t}\varphi - \varphi\|_\infty$$

$$\leq \lim_{t \to 0} \sup_{a \in [c, 1/c]} \|T_{at}\varphi - \varphi\|_\infty$$

$$= 0.$$

And for all $\varphi \in D(-H(\cdot, D))$ we have

$$\left\| \frac{\tilde{F}(t)\varphi - \varphi}{t} + \tilde{H}(\cdot, D)\varphi \right\|_\infty$$

$$= \sup_{q \in \mathbb{R}^d} \left| \frac{F^q(t)\varphi(q) - \varphi(q)}{t} + a(q)H(q, D)\varphi(q) \right|$$

$$\leq \sup_{q_0 \in \mathbb{R}^d} \sup_{q \in \mathbb{R}^d} \left| \frac{F^{q_0}(t)\varphi(q) - \varphi(q)}{t} + a(q_0)H(q, D)\varphi(q) \right|$$

$$= \sup_{q_0 \in \mathbb{R}^d} \sup_{q \in \mathbb{R}^d} \left| \frac{-1}{a(q_0)t} \int_0^{a(q_0)t} a(q_0)H(q, D)(T_\tau \varphi(q) - \varphi(q))d\tau \right|$$

$$\leq \frac{1}{t} \int_0^{t/c} \|H(q, D)(T_\tau \varphi(q) - \varphi(q))\|_\infty d\tau$$

$$\rightarrow 0, \quad t \to 0.$$
Therefore, all assumptions of Chernoff’s theorem are fulfilled and, hence, the family $(\tilde{F}(t))_{t\geq 0}$ is Chernoff equivalent to the semigroup $(\tilde{T}_t)_{t\geq 0}$ generated by a $\Psi DO$ $-\tilde{H}(\cdot, D) = -a(\cdot)H(\cdot, D)$. □

**Remark 4.2.** One can show that (the appropriate modification of) Theorem 4.1 remains true for multiplicative perturbations of not necessary Feller but just strongly continuous semigroups on $C_\infty(\mathbb{R}^d)$.

**Remark 4.3.** If the symbol $-\tilde{H}(q, p)$ of a $\Psi DO$ $-\tilde{H}(\cdot, D)$ generating a Feller semigroup $(\tilde{T}_t)_{t\geq 0}$ satisfies the assumptions of Theorem 3.1 then the symbol $-\tilde{H}(q, p) = -\frac{1}{2}a(q)|p|^2$ (with continuous $a(\cdot) : \mathbb{R}^d \to [c, 1/c]$, $c > 0$) also satisfies the assumptions of this Theorem. Hence, the Hamiltonian Feynman formula obtained in Theorem 3.1 remains valid for the perturbed semigroup $(\tilde{T}_t)_{t\geq 0}$ as well.

**Example 4.1** (diffusion with variable diffusion coefficient). Let $\psi(p) = \frac{1}{2}|p|^2$ be the characteristic exponent of a Brownian motion in $\mathbb{R}^d$. The generator of Brownian motion is $-\psi(D) = \frac{1}{2}\Delta$. The transition density is given by Gaussian density

$$p_t^{BM}(x) = (2\pi t)^{-d/2} \exp \left\{ -\frac{|x|^2}{2t} \right\}.$$ 

Consider the semigroup $(\tilde{T}_t)_{t\geq 0}$, generated by a $\Psi DO$ $-\tilde{H}(\cdot, D)$ with the symbol $-\tilde{H}(q, p) = -\frac{1}{2}a(q)|p|^2$, where $a(\cdot)$ is as before. Then by Theorem 4.3 for each $\varphi \in C_\infty(\mathbb{R}^d)$ we have (cf. [4], [5]):

$$\tilde{T}_t\varphi(q_0) = \lim_{n \to \infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} (2\pi a(q_0)t/n)^{-d/2} \exp \left\{ -\frac{|q_0 - q_1|^2}{2a(q_0)t/n} \right\} \cdots$$

$$\cdots (2\pi a(q_{n-1})t/n)^{-d/2} \exp \left\{ -\frac{|q_{n-1} - q_n|^2}{2a(q_{n-1})t/n} \right\} \varphi(q_n) dq_1 \cdots dq_n.$$

**Example 4.2** (Cauchy type process with variable coefficient). Let $\psi(p) = |p|$ be the characteristic exponent of the Cauchy process in $\mathbb{R}^d$. The generator of the Cauchy process is $-\psi(D) = -\sqrt{-\Delta}$. The transition density is given by the formula

$$p_t(x) = \Gamma\left(\frac{d}{2} + \frac{1}{2}\right) \frac{t}{\pi |x|^2 + t^2(d+1)/2},$$

where $\Gamma(\cdot)$ is Euler’s Gamma function.

Consider the semigroup $(\tilde{T}_t)_{t\geq 0}$, generated by a $\Psi DO$ $-\tilde{H}(\cdot, D)$ with the symbol $-\tilde{H}(q, p) = -a(q)|p|$, where $a(\cdot)$ is as before. Then by Theorem 4.1 for each $\varphi \in C_\infty(\mathbb{R}^d)$ we have (cf. [5]):

$$\tilde{T}_t\varphi(q_0) = \lim_{n \to \infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \left[ \Gamma\left(\frac{d}{2} + \frac{1}{2}\right) \right]^n \frac{a(q_0)t/n}{[(a(q_0)t/n)^2 + (\pi|q_0 - q_1|^2(d+1)/2)]} \cdots$$

$$\cdots \frac{a(q_{n-1})t/n}{[(a(q_{n-1})t/n)^2 + (\pi|q_{n-1} - q_n|^2(d+1)/2)]} \varphi(q_n) dq_1 \cdots dq_n.$$

5. **Feynman formulae for additive perturbations of semigroups**

**Theorem 5.1.** Let $X$ be a Banach space with a norm $\| \cdot \|_X$. Let $(T_k(t))_{t\geq 0}$, $k = 1, \ldots, m$, be strongly continuous semigroups on $X$ with generators $(A_k, D(A_k))$ respectively. Assume that $A = A_1 + \cdots + A_m$ with domain $D(A) = \cap_{k=1}^m D(A_k)$ is closable and that the closure is the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on $X$. Let $(F_k(t))_{t\geq 0}$, $k = 1, \ldots, m$, be families of operators in $X$ which are Chernoff equivalent to the semigroups $(T_k(t))_{t\geq 0}$ respectively, i.e. for each $k \in \{1, \ldots, m\}$ we have $F_k(0) = 1d$, $\|F_k(t)\| \leq e^{a_k t}$ for some $a_k > 0$ and there is a
Let \( D_k \subset D(\mathcal{A}_k) \), which is a core for \( \mathcal{A}_k \), such that \( \lim_{t \to 0} \| F_m(t) \varphi - A_t \varphi \|_X = 0 \) for each \( \varphi \in D_k \). Assume that there exists a set \( D \subset \cap_{k=1}^m D_k \) which is a core for \( A \). Then the family \( (F(t))_{t \geq 0} \), where \( F(t) = F_1(t) \circ \cdots \circ F_m(t) \) is Chernoff equivalent to the semigroup \( (T(t))_{t \geq 0} \) and, hence, the Feynman formula

\[
T_t = \lim_{n \to \infty} \left[ F(t/n) \right]^n
\]
is valid in the strong operator topology locally uniformly with respect to \( t \geq 0 \).

**Proof.** Obviously, the family \( (F(t))_{t \geq 0} \) is strongly continuous, \( F(0) = \text{Id} \) and

\[
\| F(t) \| \leq \| F_1(t) \| \cdots \| F_m(t) \| \leq e^{(a_1+\cdots+a_m)t}.
\]

Let \( D \subset \cap_{k=1}^m D_k \) be a core for \( A \). Then for each \( \varphi \in D \) we have

\[
\lim_{t \to 0} \left\| \frac{F(t) \varphi - \varphi}{t} - A_t \varphi \right\|_X = \lim_{t \to 0} \left\| \frac{F_1(t) \circ \cdots \circ F_m(t) \varphi - \varphi}{t} - A_1 \varphi - \cdots - A_m \varphi \right\|_X
\]

\[
= \lim_{t \to 0} \left\| \frac{F_1(t) \circ \cdots \circ F_m(t) \varphi - \varphi}{t} \right\|_X
\]

\[
\leq \cdots \leq \lim_{t \to 0} \left\| \frac{F_1(t) \varphi - \varphi}{t} - A_1 \varphi \right\|_X
\]

\[
= 0.
\]

Note, that if some of the \( (T_k(t))_{t \geq 0} \) are known explicitly and if \( \| T_k(t) \| \leq e^{a_k t} \) for some \( a_k \geq 0 \) then we can take \( \tilde{F}_k(t) \equiv T_k(t) \) in the corresponding Feynman formulae.

**Example 5.1** (bounded Schrödinger perturbations). Let \( X = C_{\infty}(\mathbb{R}^d) \) with the supremum norm, \( (T_t)_{t \geq 0} \) be a strongly continuous semigroup with a generator \( (A, D(A)) \), \( (F(t))_{t \geq 0} \) be Chernoff equivalent to \( (T_t)_{t \geq 0} \). Let \( V(\cdot) : \mathbb{R}^d \to \mathbb{R} \) be a bounded continuous function. Then an operator \( A + V \), such that \( D(A + V) = D(A) \) and \( (A + V) \varphi(q) = A \varphi(q) + V(q) \varphi(q) \) for all \( \varphi \in D(A + V) \), generates a strongly continuous semigroup \( (T_{t}^{A+V})_{t \geq 0} \) on \( C_{\infty}(\mathbb{R}^d) \). By Theorem 5.1, the Feynman formula \( T_{t}^{A+V} = \lim_{n \to \infty} \left[ e^{V(t/n)} \circ T(t/n) \right]^n \) is valid. In particular, if \( \| T_t \| \leq e^{a t} \) for some \( a \in [0, +\infty) \) and all \( t \geq 0 \), then we have \( T_{t}^{A+V} = \lim_{n \to \infty} \left[ e^{V(t/n)} \circ T(t/n) \right]^n \). In both formulae the operator \( e^{tV} \) is an operator of multiplication with the function \( e^{tV} \).

**Example 5.2** (gradient perturbations). Let again \( X = C_{\infty}(\mathbb{R}^d) \) with the supremum norm. Let \( b(\cdot) : \mathbb{R}^d \to \mathbb{R}^d \) be a bounded continuous vector field and \( \nabla = \left( \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^d} \right) \). Consider an operator \( \nabla \), such that \( b \nabla \varphi(q) = b(q) \nabla \varphi(q) \) for all \( \varphi \in D(b \nabla) \). Consider a family \( (S(t))_{t \geq 0} \) of operators in \( C_{\infty}(\mathbb{R}^d) \) defined by the formula:

\[
S(t) \varphi(q) = \varphi(q + tb(q)).
\]
Then \((S(t))_{t \geq 0}\) is Chernoff equivalent to the semigroup \(e^{tb\nabla}\) generated by \(b\nabla\). Indeed, \(S(0) = \text{Id}, \|S(t)\| = 1\) and for all \(\varphi \in D(b\nabla)\)

\[
\lim_{t \to 0} \left\| \frac{S(t)\varphi - \varphi}{t} - b\nabla \varphi \right\|_{\infty} = \lim_{t \to 0} \sup_{q \in \mathbb{R}^d} \left| \frac{\varphi(q + tb(q)) - \varphi(q) - b(q)\nabla \varphi(q)}{t} \right| \\
\leq \lim_{t \to 0} t \sup_{q \in \mathbb{R}^d, s \in [0,t]} \left| b(q) \cdot \text{Hess} \varphi(q + sb(q))b(q) \right| \\
= 0,
\]

where \(\text{Hess} \varphi\) is a Hessian of \(\varphi\). Hence, \(\frac{d}{dt}S(t)|_{t=0}\varphi = b\nabla \varphi\) for each \(\varphi \in D(b\nabla)\).

Let now \((T_t)_{t \geq 0}\) be a strongly continuous semigroup with a generator \((A, D(A))\) and a family \((F(t))_{t \geq 0}\) be Chernoff equivalent to \((T_t)_{t \geq 0}\). Consider an operator \(A + b\nabla\) such that \((A + b\nabla)\varphi(q) = A\varphi(q) + b(q)\nabla \varphi(q)\) for all \(\varphi \in D(A + b\nabla) = D(A) \cap D(b\nabla)\). Assume that \(A + b\nabla\) generates a strongly continuous semigroup \((T_t ^{A+b\nabla})_{t \geq 0}\) on \(C_\infty (\mathbb{R}^d)\). Note, that the assumption holds, e.g. if the operator \(b\nabla\) is \(A\)-bounded, i.e. \(D(A) \subset D(b\nabla)\) and for all \(\varphi \in D(A), \lambda \in [0,1)\) and \(\gamma \geq 0\) the estimate

\[
\|b\nabla \varphi\|_X \leq \lambda \|A \varphi\|_X + \gamma \|\varphi\|_X
\]

holds. In particular, \(b\nabla\) is \(A\)-bounded for \(\alpha = -(-\Delta)^{n/2}, \alpha \in (1,2]\).

By Theorem 5.1 the family \((S(t) \circ F(t))_{t \geq 0}\) is Chernoff equivalent to the semigroup \((T_t ^{A+b\nabla})_{t \geq 0}\) and the Feynman formula \(T_t ^{A+b\nabla} = \lim_{n \to \infty} [S(t/n) \circ F(t/n)]^n\) is valid.

**Corollary 5.1** (Hamiltonian Feynman formula for gradient and bounded Schrödinger perturbations of Feller semigroups). Let \(b()\), \(V()\) be as in the above examples. Let a function \(H(q,p)\) be as in Theorem 5.1. \(H(\cdot, D)\) be a \(\Psi DO\) with the symbol \(H(q,p)\) (see formula (13)) and \(F(t)\) be given by the formula (14). We assume that the function \(H(q,p)\) satisfies sufficient conditions for \(\tilde{-H}(\cdot, D)\) and \(A := \tilde{-H}(\cdot, D) + b\nabla + V\) to be closable and the closures to generate strongly continuous semigroups on \(C_\infty (\mathbb{R}^d)\). Then by Theorems 5.1, 5.2 and due to the above examples the following Hamiltonian Feynman formula is valid for the semigroup \((T_t ^A)_{t \geq 0}\), generated by \(A = -\tilde{-H}(\cdot, D) + b\nabla + V\):

\[
(T_t ^A) \varphi(q_0) = \lim_{n \to \infty} \left[ e^{\frac{H}{n} V} \circ S(t/n) \circ F(t/n) \right]^n \varphi(q_0) \\
= \lim_{n \to \infty} (2\pi)^{-dn} \int_{(\mathbb{R}^d)^{2n}} \frac{e^{\frac{1}{n} \sum_{k=1}^n V(q_{k-1}) e^\sum_{k=1}^n p_k (q_{k-1} - q_k + b(q_{k-1})) \varphi(q_{n} )dq_{1} dp_{1} \cdots dq_{n} dp_{n} }{e^{\frac{1}{n} \sum_{k=1}^n H(q_{k-1} + \frac{b(q_{k-1})}{n})}}.
\]

where again the integrals in the formula must be understood in a proper sense (see Remark 5.2).

**Corollary 5.2** (Lagrangian Feynman formula for a mixture of multiplicative and additive perturbations of Feller semigroups). Let \(b()\), \(V()\) be as in the above examples, \(\tilde{H}(\cdot, D)\) be as in Section 4, \(\tilde{F}(t)\) — as in (19). We assume that the function \(\tilde{H}(q,p)\) satisfies sufficient conditions for \(\tilde{-H}(\cdot, D)\) and \(B := -\tilde{-H}(\cdot, D) + b\nabla + V\) to be closable and the closures to generate strongly continuous semigroups on \(C_\infty (\mathbb{R}^d)\). Then by Theorems 5.1, 5.2 and due to the above examples the following Lagrangian Feynman formula is valid for the semigroup \((T_t ^B)_{t \geq 0}\), generated by \(B = -\tilde{-H}(\cdot, D) + b\nabla + V\):
the heat semigroup ($T_t$)

Example 5.3 (Lagrangian Feynman formula for perturbations of the heat semigroup)

Let again $b(\cdot)$, $V(\cdot)$ be as before. Consider $A = \frac{1}{2}\Delta + b\nabla + V$:

$$T^B_t \varphi(q_0) = \lim_{n \to \infty} \left[ e^{\frac{t}{n}V} \circ S(\frac{q_n}{n}) \circ \tilde{F}(\frac{q_0}{n}) \right]^n \varphi(q_0)$$

$$= \lim_{n \to \infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} e^{\frac{t}{n} \sum_{k=1}^n V(q_{k-1})} \varphi(q_n) p_a^{(q_n+b(q_n)\frac{q_0}{n})} \frac{q_0 + b(q_0)\frac{q_0}{n}}{n} \cdot dq_1 \times \cdots \times p_{a+1+b(q_{n-1})\frac{q_0}{n}} \frac{q_{n-1} + b(q_{n-1})\frac{q_0}{n}}{n} \cdot dq_n.$$ 

Since $|x + b(x) - y|^2 = |x - y|^2 + 2tb(x)(x - y) + t^2|b(x)|^2$ then

$$T^{b\nabla + V}_t \varphi(q_0) = \lim_{n \to \infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} e^{\frac{t}{n} \sum_{k=1}^n V(q_{k-1})} e^{-\sum_{k=1}^n b(q_{k-1})(q_{k-1} - q_k)} \times \varphi(q_n) dq_1 \cdots dq_n,$$

where $p^{BM}_t(x) = (2\pi t)^{(-d/2)} \exp \left\{ -\frac{|x|^2}{2t} \right\}$ is the transition density of Brownian motion. Therefore, one can show, that the limit in the right hand side of the last formula coincides with the functional integral

$$E^\varnothing \left[ e^{\int_0^T V(\xi) \, d\xi} e^{\int_0^T b(\xi) \, d\xi} e^{-\frac{1}{2} \int_0^T |b(\xi)|^2 \, d\xi} \int f(\xi_t) \right]$$

with respect to Wiener measure concentrated on the paths starting at $q_0$. Hence, the machinery of Feynman–Kac formula (21) and Girsanov’s formula but also to extend them for the case of variable diffusion coefficients (cf. [8]).

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