RADICALS AND IDEALS OF AFFINE NEAR-SEMIRINGS
OVER BRANDT SEMIGROUPS

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Abstract. This work obtains all the right ideals, radicals, congruences and
ideals of the affine near-semirings over Brandt semigroups.

1. Introduction

An algebraic structure \((\mathcal{N}, +, \cdot)\) with two binary operations \(+\) and \(\cdot\) is said to
be a near-semiring if \((\mathcal{N}, +)\) and \((\mathcal{N}, \cdot)\) are semigroups and \(\cdot\) is one-side, say left,
distributive over \(+\), i.e. \(a \cdot (b + c) = a \cdot b + a \cdot c\), for all \(a, b, c \in \mathcal{N}\). Typical examples
of near-semirings are of the form \(M(S)\), the set of all mappings on a semigroup \(S\). Near-semirings are not only the natural generalization of semirings and near-rings,
but also they have very prominent applications in computer science. To name a few:
process algebras by Bergstra and Klop [1], and domain axioms in near-semirings
by Struth and Desharnais [3].

Near-semirings were introduced by van Hoorn and van Rootselaar as a general-
ization of near-rings [11]. In [10], van Hoorn generalized the concept of Jacobson
radical of rings to zero-symmetric near-semirings. These radicals also generalize
the radicals of near-rings by Betsch [2]. In this context, van Hoorn introduced four-
teen radicals of zero-symmetric near-semiring and studied some relation between
them. The properties of these radicals are further investigated in the literature
(e.g. [5, 12]). Krishna and Chatterjee developed a radical (which is similar to the
Jacobson radical of rings) for a special class of near-semirings to test the minimality
of linear sequential machines in [6].

In this paper, we study the ideals and radicals of the zero-symmetric affine
near-semiring over a Brandt semigroup. First we present the necessary background
material in Section 2. For the near-semiring under consideration, we obtain the right
ideals in Section 3 and ascertain all radicals in Section 4. Further, we determine all
its congruences and consequently obtain its ideals in Section 5.

2. Preliminaries

In this section, we provide a necessary background material through two subsec-
tions. One is to present the notions of near-semirings, and their ideals and radicals.
In the second subsection, we recall the notion of the affine near-semiring over a
Brandt semigroup. We also utilize this section to fix our notations which used
throughout the work.

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2.1. **A near-semiring and its radicals.** In this subsection, we recall some necessary fundamentals of near-semirings from [5, 10, 11].

**Definition 2.1.** An algebraic structure \((N, +, \cdot)\) is said to be a **near-semiring** if

1. \((N, +)\) is a semigroup,
2. \((N, \cdot)\) is a semigroup, and
3. \(a \cdot (b + c) = a \cdot b + a \cdot c\), for all \(a, b, c \in N\).

Furthermore, if there is an element \(0 \in N\) such that

4. \(a + 0 = 0 + a = a\) for all \(a \in N\), and
5. \(a \cdot 0 = 0 \cdot a = 0\) for all \(a \in N\),

then \((N, +, \cdot)\) is called a **zero-symmetric near-semiring**.

**Example 2.2.** Let \((S, +)\) be a semigroup and \(M(S)\) be the set of all functions on \(S\). The algebraic structure \((M(S), +, \circ)\) is a near-semiring, where + is point-wise addition and \(\circ\) is composition of mappings, i.e., for \(x \in S\) and \(f, g \in M(S)\),

\[ x(f + g) = xf + xg \quad \text{and} \quad x(f \circ g) = (xf)g. \]

We write an argument of a function on its left, e.g. \(xf\) is the value of a function \(f\) at an argument \(x\). We always denote the composition \(f \circ g\) by \(fg\). The notions of homomorphism and subnear-semiring of a near-semiring can be defined in a routine way.

**Definition 2.3.** Let \(N\) be a zero-symmetric near-semiring. A semigroup \((S, +)\) with identity \(0_S\) is said to be a **\(N\)-semigroup** if there exists a composition \((s, a) \mapsto sa: S \times N \to S\) such that, for all \(a, b \in N\) and \(s \in S\),

1. \(s(a + b) = sa + sb\),
2. \(s(ab) = (sa)b\), and
3. \(s0 = 0_S\).

Note that the semigroup \((N, +)\) of a near-semiring \((N, +, \cdot)\) is an \(N\)-semigroup. We denote this \(N\)-semigroup by \(N^+\).

**Definition 2.4.** Let \(S\) be an \(N\)-semigroup. A semigroup congruence \(\sim_r\) of \(S\) is said to be a **congruence of \(N\)-semigroup** \(S\), if for all \(s, t \in S\) and \(a \in N\),

\[ s \sim_r t \implies sa \sim_r ta. \]

**Definition 2.5.** An **\(N\)-morphism** of an \(N\)-semigroup \(S\) is a semigroup homomorphism \(\phi\) of \(S\) into an \(N\)-semigroup \(S'\) such that

\[ (sa)\phi = (s\phi)a \]

for all \(a \in N\) and \(s \in S\). The kernel of an \(N\)-morphism is called an **\(N\)-kernel** of an \(N\)-semigroup \(S\). A subsemigroup \(T\) of an \(N\)-semigroup \(S\) is said to be an \(N\)-subsemigroup of \(S\) if and only if \(0_S \in T\) and \(TN \subseteq T\).

**Definition 2.6.** The kernel of a homomorphism of \(N\) is called an **ideal** of \(N\). The \(N\)-kernels of the \(N\)-semigroup \(N^+\) are called **right ideals** of \(N\).

One may refer to [10, 11] for a few other notions viz. strong ideal, modular right ideal and \(\lambda\)-modular right ideal, a special congruence \(r'_\Delta\) associated to a normal subsemigroup \(\Delta\) of a semigroup \(S\), and, for various \((\nu, \mu)\), the \(N\)-semigroups of type \((\nu, \mu)\). The homomorphism corresponding to \(r''_\Delta\) is denoted by \(\lambda_\Delta\).
**Definition 2.7.** Let $s$ be an element of an $\mathcal{N}$-semigroup $S$. The annihilator of $s$, denoted by $A(s)$, defined by the set $\{a \in \mathcal{N} : sa = 0_S\}$. Further, for a subset $T$ of $S$, the annihilator of $T$ is

$$A(T) = \bigcap_{s \in T} A(s) = \{a \in \mathcal{N} : Ta = 0_S\}.$$  

**Theorem 2.8** [5]. The annihilator $A(S)$ of an $\mathcal{N}$-semigroup $S$ is an ideal of $\mathcal{N}$.

We now recall the notions of various radicals in the following definition.

**Definition 2.9** [10]. Let $\mathcal{N}$ be a zero-symmetric near-semiring.

1. For $\nu = 0, 1$ with $\mu = 0, 1, 2, 3$ and $\nu = 2$ with $\mu = 0, 1$

   $$J_{(\nu, \mu)}(\mathcal{N}) = \bigcap_{S \text{ is of type } (\nu, \mu)} A(S).$$

2. $R_0(\mathcal{N})$ is the intersection of all maximal modular right ideals of $\mathcal{N}$.
3. $R_1(\mathcal{N})$ is the intersection of all modular maximal right ideals of $\mathcal{N}$.
4. $R_2(\mathcal{N})$ is the intersection of all maximal $\lambda$-modular right ideals of $\mathcal{N}$.
5. $R_3(\mathcal{N})$ is the intersection of all $\lambda$-modular maximal right ideals of $\mathcal{N}$.

In any case, the empty intersection of subsets of $\mathcal{N}$ is $\mathcal{N}$. The relations between these radicals are given in Figure 1 where $A \to B$ means $A \subset B$.

**Remark 2.10** [2, 4, 9]. If $\mathcal{N}$ is a near-ring, then $J_{(0, \mu)}(\mathcal{N})$, $\mu = 0, 1, 2, 3$ are the radical $J_0(\mathcal{N})$; $J_{(1, \mu)}(\mathcal{N})$, $\mu = 0, 1, 2, 3$ are the radical $J_1(\mathcal{N})$; $J_{(2, \mu)}(\mathcal{N})$, $\mu = 0, 1$, are the radical $J_2(\mathcal{N})$; and $R_0(\mathcal{N})$, $\nu = 0, 1, 2, 3$ are the radical $D(\mathcal{N})$ of Betsch. Further, if $\mathcal{N}$ is a ring, then all the fourteen radicals are the radical of Jacobson.

**Definition 2.11.** A zero-symmetric near-semiring $\mathcal{N}$ is called $(\nu, \mu)$-primitive if $\mathcal{N}$ has an $\mathcal{N}$-semigroup $S$ of type $(\nu, \mu)$ with $A(S) = \{0\}$. 

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**Figure 1.** Relation between various radicals of a near-semiring

![Diagram illustrating the relations between various radicals of a near-semiring](image-url)
2.2. An affine near-semiring over a Brandt semigroup. In this subsection, we present the necessary fundamentals of affine near-semirings over Brandt semigroups. For more detail one may refer to [7, 8].

Let \((S, +)\) be a semigroup. An element \(f \in M(S)\) is said to be an affine map if \(f = g + h\), for some endomorphism \(g\) and a constant map \(h\) on \(S\). The set of all affine mappings over \(S\), denoted by \(\text{Aff}(S)\), need not be a subnear-semiring of \(M(S)\). The affine near-semiring, denoted by \(A^+(S)\), is the subnear-semiring generated by \(\text{Aff}(S)\) in \(M(S)\). Indeed, the subsemigroup of \((M(S), +)\) generated by \(\text{Aff}(S)\) equals \((A^+(S), +)\) (cf. [8 Corollary 1]). If \((S, +)\) is commutative, then \(\text{Aff}(S)\) is a subnear-semiring of \(M(S)\) so that \(\text{Aff}(S) = A^+(S)\).

Definition 2.12. For any integer \(n \geq 1\), let \([n] = \{1, 2, \ldots, n\}\). The semigroup \((B_n, +)\), where \(B_n = ([n] \times [n]) \cup \{\vartheta\}\) and the operation \(+\) is given by

\[
(i, j) + (k, l) = \begin{cases} 
(i, l) & \text{if } j = k; \\
\vartheta & \text{if } j \neq k
\end{cases}
\]

and, for all \(\alpha \in B_n\), \(\alpha + \vartheta = \vartheta + \alpha = \vartheta\), is known as Brandt semigroup. Note that \(\vartheta\) is the (two sided) zero element in \(B_n\).

Let \(\vartheta\) be the zero element of the semigroup \((S, +)\). For \(f \in M(S)\), the support of \(f\), denoted by \(\text{supp}(f)\), is defined by the set

\[
\text{supp}(f) = \{\alpha \in S \mid \alpha f \neq \vartheta\}.
\]

A function \(f \in M(S)\) is said to be of \(k\)-support if the cardinality of \(\text{supp}(f)\) is \(k\), i.e. \(|\text{supp}(f)| = k\). If \(k = |S|\) (or \(k = 1\)), then \(f\) is said to be of full support (or singleton support, respectively). For \(X \subseteq M(S)\), we write \(X_k\) to denote the set of all mappings of \(k\)-support in \(X\), i.e.

\[
X_k = \{f \in X \mid f \text{ is of } k\text{-support}\}.
\]

For ease of reference, we continue to use the following notations for the elements of \(M(B_n)\), as given in [8].

Notation 2.13.

1. For \(c \in B_n\), the constant map that sends all the elements of \(B_n\) to \(c\) is denoted by \(\xi_c\). The set of all constant maps over \(B_n\) is denoted by \(CB_n\).
2. For \(k, l, p, q \in [n]\), the singleton support map that send \((k, l)\) to \((p, q)\) is denoted by \((k, l)_{(p, q)}\).
3. For \(p, q \in [n]\), the \(n\)-support map which sends \((i, p)\) (where \(1 \leq i \leq n\)) to \((i, q)\) using a permutation \(\sigma \in S_n\) is denoted by \((p, q; \sigma)\). We denote the identity permutation on \([n]\) by \(id\).

Note that \(A^+(B_1) = \{(1, 1; id)\} \cup CB_1\). For \(n \geq 2\), the elements of \(A^+(B_n)\) are given by the following theorem.

Theorem 2.14 [8]. For \(n \geq 2\), \(A^+(B_n)\) precisely contains \((n! + 1)n^2 + n^4 + 1\) elements with the following breakup.

1. All the \(n^2 + 1\) constant maps.
2. All the \(n^4\) singleton support maps.
3. The remaining \((n!)n^2\) elements are the \(n\)-support maps of the form \((p, q; \sigma)\), where \(p, q \in [n]\) and \(\sigma \in S_n\).
We are ready to investigate the radicals and ideals of \( A^+(B_n) \) – the affine near-semiring over a Brandt semigroup. Since the radicals are defined in the context of zero-symmetric near-semirings, we extend the semigroup reduct \( (A^+(B_n), +) \) to monoid by adjoining 0 and make the resultant near-semiring zero-symmetric. In what follows, the zero-symmetric affine near-semiring \( A^+(B_n) \cup \{0\} \) is denoted by \( \mathcal{N} \), i.e.

1. \( \mathcal{N}, + \) is a monoid with identity element 0,
2. \( \mathcal{N}, \circ \) is a semigroup,
3. \( 0f = f0 = 0 \), for all \( f \in \mathcal{N} \), and
4. \( f(g + h) = fg + fh \), for all \( f, g, h \in \mathcal{N} \).

In this work, a nontrivial congruence of an algebraic structure is meant to be a congruence which is neither equality nor universal relation.

3. Right ideals

In this section, we obtain all the right ideals of the affine near-semiring \( \mathcal{N} \) by ascertaining the concerning congruences of \( \mathcal{N} \)-semigroups. We begin with the following lemma.

**Lemma 3.1.** Let \( \sim \) be a nontrivial congruence over the semigroup \( (\mathcal{N}, +) \) and \( f \in A^+(B_n)_{n^+} \). If \( f \sim \xi_\vartheta \), then \( \sim = (A^+(B_n) \times A^+(B_n)) \cup \{(0,0)\} \).

**Proof.** First note that \( (A^+(B_n) \times A^+(B_n)) \cup \{(0,0)\} \) is a congruence relation of the semigroup \( (\mathcal{N}, +) \). Let \( f = \xi_{(p_0,q_0)} \) and \( \xi_{(p,q)} \) be an arbitrary full support map. Since

\[
\xi_{(p,q)} = \xi_{(p_0,p_0)} + \xi_{(q_0,q_0)} + \xi_{(p_0,q_0)} + \xi_\vartheta + \xi_{(q_0,q)} = \xi_\vartheta,
\]

we have \( \xi_{(p,q)} \sim \xi_\vartheta \) for all \( p, q \in [n] \). Further, given an arbitrary \( n \)-support map \( (k,l;\sigma) \), since \( \xi_{(p,l)} \sim \xi_\vartheta \), we have

\[
(k,l;\sigma) = (k,p;\sigma) + \xi_{(p,l)} \sim (k,p;\sigma) + \xi_\vartheta = \xi_\vartheta.
\]

Thus, all \( n \)-support maps are related to \( \xi_\vartheta \) under \( \sim \). Similarly, given an arbitrary \( \xi_{(p,q)} \in A^+(B_n)_1 \), since \( \xi_{(p,q)} \sim \xi_\vartheta \), for \( \sigma \in S_n \) such that \( k\sigma = q \), we have

\[
(k,l)\xi_{(p,q)} = (l,q;\sigma) \sim (l,q;\sigma) + \xi_\vartheta = \xi_\vartheta.
\]

Hence, all elements of \( A^+(B_n) \) are related to each other under \( \sim \). \( \square \)

Now, using Lemma 3.1, we determine the right ideals of \( \mathcal{N} \) in the following theorem.

**Theorem 3.2.** \( \mathcal{N} \) and \( \{0\} \) are only the right ideals of \( \mathcal{N} \).

**Proof.** Let \( I \neq \{0\} \) be a right ideal of \( \mathcal{N} \) so that \( I = \ker \varphi \), where \( \varphi : \mathcal{N}^+ \rightarrow S \) is an \( \mathcal{N} \)-morphism. Note that \( I = [0]_\sim \), where \( \sim \) is the congruence over the \( \mathcal{N} \)-semigroup \( \mathcal{N}^+ \) defined by \( a \sim b \) if and only if \( a\varphi = b\varphi \), i.e. the relation \( \sim \), on \( \mathcal{N} \) is compatible with respect to \( + \) and if \( a \sim b \) then \( ac \sim bc \) for all \( c \in \mathcal{N} \).

Let \( f \) be a nonzero element of \( \mathcal{N} \) such that \( f \sim 0 \). First note that

\[
\xi_\vartheta = f\xi_\vartheta \sim 0 \xi_\vartheta = 0.
\]

Further, for any full support map \( \xi_{(p,q)} \), we have

\[
\xi_{(p,q)} = f\xi_{(p,q)} \sim 0 \xi_{(p,q)} = 0
\]

so that, by transitivity, \( \xi_{(p,q)} \sim \xi_\vartheta \). Hence, by Lemma 3.1 \( \sim = \mathcal{N} \times \mathcal{N} \) so that \( I = \mathcal{N} \). \( \square \)
Remark 3.3. The ideal \{0\} is the maximal right ideal of \(\mathcal{N}\).

4. Radicals

In order to obtain the radicals of the affine near-semiring \(\mathcal{N}\), in this section, we first identify an \(\mathcal{N}\)-semigroup which satisfies the criteria of all types of \(\mathcal{N}\)-semigroups by van Hoorn. Using the \(\mathcal{N}\)-semigroup, we ascertain the radicals of \(\mathcal{N}\). Further, we observe that the near-semiring \(\mathcal{N}\) is \((\nu, \mu)^\text{-primitive}\) (cf. Theorem 4.3).

Consider the subsemigroup \(\mathcal{C} = C_{B_n} \cup \{0\}\) of \((\mathcal{N}, +)\) and observe that \(\mathcal{C}\) is an \(\mathcal{N}\)-semigroup with respect to the multiplication in \(\mathcal{N}\). The following properties of the \(\mathcal{N}\)-semigroup \(\mathcal{C}\) are useful.

Lemma 4.1.

1. Every nonzero element of \(\mathcal{C}\) is a generator. Moreover, the \(\mathcal{N}\)-semigroup \(\mathcal{C}\) is strongly monogenic and \(A(g) = \{0\}\) for all \(g \in \mathcal{C} \setminus \{0\}\).
2. The subsemigroup \(\{0\}\) is the maximal \(\mathcal{N}\)-subsemigroup of \(\mathcal{C}\).
3. The \(\mathcal{N}\)-semigroup \(\mathcal{C}\) is irreducible.

Proof. (1) Let \(g \in C_{B_n}\). Note that \(g\mathcal{N} \subseteq \mathcal{C}\) because the product of a constant map with any map is a constant map. Conversely, for \(f \in \mathcal{C}\), since \(gf = f\), we have \(g\mathcal{N} = \mathcal{C}\) for all \(g \in \mathcal{C} \setminus \{0\}\). Further, since \(0\mathcal{N} = \{0\}\) and \(C\mathcal{N} = \mathcal{C} \neq \{0\}\). Hence, \(\mathcal{C}\) is strongly monogenic.

(2) We show that the semigroups \(\mathcal{C}\) and \(\{0\}\) are the only \(\mathcal{N}\)-subsemigroups of \(\mathcal{C}\). Let \(T\) be an \(\mathcal{N}\)-subsemigroup of \(\mathcal{C}\) such that \(\{0\} \neq T \subseteq \mathcal{C}\). Then there exist \(f \neq 0 \in T\) and \(g \in \mathcal{C} \setminus T\). Since \(fg = g \notin T\), we have \(TN \nsubseteq T\); a contradiction to \(T\) is an \(\mathcal{N}\)-subsemigroup. Hence, the result.

(3) By Lemma 4.1(1), the \(\mathcal{N}\)-semigroup \(\mathcal{C}\) is monogenic with any nonzero element \(g\) as generator such that \(A(g) = \{0\}\); thus, \(A(g)\) is maximal right ideal in \(\mathcal{N}\) (cf. Remark 3.3). Hence, by [10, Theorem 8], \(\mathcal{C}\) is irreducible.

Remark 4.2. Since a strongly monogenic \(\mathcal{N}\)-semigroup is monogenic we have, for \(\mu = 0, 1, 2, 3\) and \(\nu = 2\) with \(\mu = 0, 1\), we have the following.

Theorem 4.3. For \(\nu = 0, 1\) with \(\mu = 0, 1, 2, 3\) and \(\nu = 2\) with \(\mu = 0, 1\), we have the following.

1. The \(\mathcal{N}\)-semigroup \(\mathcal{C}\) is of type \((\nu, \mu)\) with \(A(\mathcal{C}) = 0\).
2. The near-semiring \(\mathcal{N}\) is \((\nu, \mu)^\text{-primitive}\) for all \(\nu\) and \(\mu\).
3. \(J_{(\nu, \mu)}(\mathcal{N}) = \{0\}\) for all \(\nu\) and \(\mu\).

Proof. In view of Remark 4.2, we prove (1) in the following cases.

Type \((1, \mu)\): Note that, by Lemma 4.1(1), the \(\mathcal{N}\)-semigroup \(\mathcal{C}\) is strongly monogenic.

(i) By Lemma 4.1(3), we have \(\mathcal{C}\) is irreducible. Hence, \(\mathcal{C}\) is of type \((1, 0)\).

(ii) By Lemma 4.1(1) and Remark 3.3, for any generator \(g\), \(A(g)\) is a maximal right ideal. Hence, \(\mathcal{C}\) is of type \((1, 1)\).

(iii) Note that the ideal \(\{0\}\) is strong right ideal so that for any generator \(g\), \(A(g)\) is a strong maximal right ideal (see ii above). Further, note that \(A(g)\) is a maximal strong right ideal (cf. Remark 3.3). Hence, \(\mathcal{C}\) is of type \((1, 2)\) and \((1, 3)\).
Type (2,μ): Since $C$ is monogenic and, for any generator $g$ of $C$, $A(g)$ is a maximal $N$-subsemigroup of $C$ (cf. Lemma 4.1(1) and Lemma 4.1(2)). Thus, $C$ is of type (2,1). By [10, Theorem 9], every $N$-semigroup of type (2,1) will be of type (2,0). Hence, $C$ is of type (2,0).

Proofs for (2) and (3) follow from (1).

Theorem 4.4. For $\nu = 0, 1$, we have $R_\nu(N) = \{0\}$.

Proof. In view of Figure 1 we prove that result by showing that the right ideal $\{0\}$ is a modular maximal right ideal. By Lemma 4.1(1), the $N$-semigroup $C$ is monogenic and has a generator $g$ such that $A(g) = \{0\}$. Hence, the right ideal $\{0\}$ is modular (cf. [10, Theorem 7]). Further, since $\{0\}$ is a maximal right ideal (cf. Remark 3.3), we have $\nu = 0, 1$ is a modular maximal right ideal.

Theorem 4.5. For $\nu = 2, 3$, we have $R_\nu(N) = N$.

Proof. In view of Figure 1 and Theorem 4.4 we prove that the homomorphism $\lambda_{(0)}$ is not modular. Note that the congruence relation $r''_{(0)}$ is the equality relation on $(N, +)$, where $r'_{(0)}$ is the transitive closure of the two sided stable reflexive and symmetric relation relation $r_{(0)}$ associated with a normal subsemigroup $\{0\}$ of the semigroup $(N, +)$. Consequently, the semigroup homomorphism $\lambda_{(0)}$ is an identity map on $(N, +)$. If the morphism $\lambda_{(0)}$ is modular, then there is an element $u \in N$ such that $x = ux$ for all $x \in N$, but there is no left identity element in $N$. Consequently, $\lambda_{(0)}$ is not modular. Thus, there is no maximal $\lambda$-modular right ideal. Hence, for $\nu = 2, 3$, we have $R_\nu(N) = N$.

5. Ideals

In this section, we prove that there is only one nontrivial congruence relation on $N$ (cf. Theorem 4.1). Consequently, all the ideals of $N$ are determined.

Theorem 5.1. The near-semiring $N$ has precisely the following congruences.

1. Equality relation
2. $N \times N$
3. $(A^+(B_n) \times A^+(B_n)) \cup \{(0,0)\}$

Hence, $N$ and $\{0\}$ are the only ideals of the near-semiring $N$.

Proof. In the sequel, we prove the theorem through the following claims.

Claim 1: Let $\sim$ be a nontrivial congruence over the near-semiring $N$ and $f \in N \setminus \{0, \xi_0\}$. If $f \sim \xi_0$, then $\sim = (A^+(B_n) \times A^+(B_n)) \cup \{(0,0)\}$.

Proof: First note that $(A^+(B_n) \times A^+(B_n)) \cup \{(0,0)\}$ is a congruence relation of the near-semiring $N$. If $f \in A^+(B_n)_{n^2+1}$, since $\sim$ is a congruence of the semigroup $(N, +)$, by Lemma 3.1 we have the result. Otherwise, we reduce the problem to Lemma 3.1 in the following cases.

Case 1.1: $f$ is of singleton support. Let $f = (k,l)\xi_{(p,q)}$. Since $(k,l)\xi_{(p,q)} \sim \xi_0$ we have

$$
\xi_{(k,l)}(k,l)\xi_{(p,q)} \sim \xi_{(k,l)}\xi_0
$$

so that $\xi_{(p,q)} \sim \xi_0$.

Case 1.2: $f$ is of $n$-support. Let $f = (p,q;\sigma)$. Since $(p,q;\sigma) \sim \xi_0$ we have

$$
\xi_{(k,p)}(p,q;\sigma) \sim \xi_{(k,p)}\xi_0
$$

so that $\xi_{(k,p,q)} \sim \xi_0$. 

Claim 2: If two nonzero elements are in one class under a nontrivial congruence over $N$, then the congruence is $(A^+(B_n) \times A^+(B_n)) \cup \{(0,0)\}$.

Proof: Let $f, g \in N \setminus \{0\}$ such that $f \sim g$ under a congruence $\sim$ over $N$. If $f$ or $g$ is equal to $\xi_\varnothing$, then by Claim 1, we have the result. Otherwise, we consider the following six cases classified by the supports of $f$ and $g$. In each case, we show that there is an element $h \in A^+(B_n) \setminus \{\xi_\varnothing\}$ such that $h \sim \xi_\varnothing$ so that the result follows from Claim 1.

Case 2.1: $f, g \in A^+(B_n)_1$. Let $f = (i,j) \xi_{(k,l)}$ and $g = (s,t) \xi_{(u,v)}$. If $(i,j) \neq (s,t)$, we have

$$\xi_{\varnothing} = (i,j) \xi_{(k,l)} + (s,t) \xi_{(u,v)} = (s,t) \xi_{(u,v)} + (i,j) \xi_{(k,l)} = \xi_{(s,t)} \xi_{(u,v)}.$$ 

Otherwise, $(i,j) = (s,t)$ so that $(k,l) \neq (u,v)$. Now, if $k \neq u$, then we have

$$(i,j) \xi_{(k,l)} = (i,j) \xi_{(k,k)} + (i,j) \xi_{(k,l)} \sim (i,j) \xi_{(k,k)} + (i,j) \xi_{(u,v)} = \xi_{\varnothing}.$$ 

Similarly, if $l \neq v$, we have

$$\xi_{\varnothing} = (i,j) \xi_{(k,l)} + (i,j) \xi_{(v,v)} \sim (i,j) \xi_{(u,v)} + (i,j) \xi_{(v,v)} = (i,j) \xi_{(u,v)}.$$ 

Case 2.2: $f, g \in A^+(B_n)_{n^2+1}$. Let $f = \xi_{(k,l)}$ and $g = \xi_{(u,v)}$. By considering full support maps whose images are the same as in various subcases of Case 1, we can show that there is an element in $A^+(B_n) \setminus \{\xi_\varnothing\}$ that is related to $\xi_\varnothing$ under $\sim$.

Case 2.3: $f, g \in A^+(B_n)_n$. Let $f = (i,j) \sigma$ and $g = (k,l) \rho$. If $l \neq j$, then

$$(i,j) \sigma = (i,j) \sigma + \xi_{(j,j)} \sim (k,l) \rho + \xi_{(j,j)} = \xi_{\varnothing}.$$ 

Otherwise, we have $(i,j) \sigma \sim (k,j) \rho$. Now, if $i \neq k$, then

$$\xi_{\varnothing} = (k,k; \id)(i,j) \sigma \sim (k,k; \id)(k,j) \rho = (k,j) \rho.$$ 

In case $i = k$, we have $\sigma \neq \rho$. Thus, there exists $t \in [n]$ such that $t \sigma \neq t \rho$. Now, $(i,j) \sigma \sim (i,j) \rho$ implies $\xi_{(k,i)}(i,j) \sigma \sim \xi_{(k,i)}(i,j) \rho$, i.e. $\xi_{(k,i)} \sim \xi_{(k,i)}$. Consequently,

$$\xi_{(k,i)} = \xi_{(k,i; \sigma)} + \xi_{(k,s,i)} \sim \xi_{(k,i; \sigma)} + \xi_{(k,p,i)} = \xi_{\varnothing}.$$ 

Case 2.4: $f \in A^+(B_n)_{1}, g \in A^+(B_n)_{n^2+1}$. Let $f = (k,l) \xi_{(p,q)}$ and $g = \xi_{(i,j)}$. Now, for $(s,t) \neq (k,l)$, we have

$$\xi_{\varnothing} = \xi_{(s,t)} f \sim \xi_{(s,t)} g = \xi_{(i,j)}.$$ 

Case 2.5: $f \in A^+(B_n)_{n^2+1}, g \in A^+(B_n)_n$. Let $f = \xi_{(p,q)}$ and $g = (i,j) \sigma$. Now, for $l \neq i$, we have

$$(k,l) \xi_{(p,q)} = (k,l) \xi_{(p,p)} + f \sim (k,l) \xi_{(p,p)} + g = \xi_{\varnothing}.$$ 

Case 2.6: $f \in A^+(B_n)_1, g \in A^+(B_n)_n$. Let $f = (k,l) \xi_{(p,q)}$ and $g = (i,j) \sigma$. Now, for $l \neq i$, we have

$$\xi_{\varnothing} = \xi_{(i,i)} f \sim \xi_{(i,i)} g = \xi_{(i,j)}.$$ 

$\square$
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