We propose an ensemble technique for converting any classifier into a computationally secure classifier. We define a simpler security problem for random binary classifiers and prove a reduction from this model to the security of the overall ensemble classifier. We provide experimental evidence of the security of our random binary classifiers, as well as empirical results of the adversarial accuracy of the overall ensemble to black-box attacks. Our construction crucially leverages hidden randomness in the multiclass-to-binary reduction.

1 Introduction

Current machine learning models are vulnerable at test time to adversarial examples, which are data points that have been imperceptibly modified from legitimate data points but are misclassified with high confidence. This problem has attracted significant researcher interest [SZS+13][GSS15] in both explaining their existence and defending against an adversary who tries to compute them. Previous work has attempted to train models to be explicitly robust to attacks by incorporating robustness into the optimization problem [MMS+17][SRBB18], by input transformations and discretization to reduce model linearity [Jac18], or by injecting randomness at inference time [XWZ+17]. However these defenses have all been subsequently broken by changing the attack model slightly in terms of allowable perturbations [SC17] or by using more sophisticated attacks [ACW18].

Recent explanations suggest that the existence of adversarial examples is actually inevitable in high-dimensional spaces. [GMF+18][FGCC19] show that these examples exist for any linear classifier with nonzero error rate under additive Gaussian noise. This vulnerability is a simple geometrical fact when the dimension $d$ is large: because most of the mass of a Gaussian distribution is concentrated near the shell, the distance to the closest misclassified example is a factor $d^{1/2}$ closer than the distance to the shell. [IST+19] show that adversarial perturbations can actually be robust features for generalization, and thus their adversarial nature is just a misalignment with our natural human notions of robustness.

In light of the evidence for the inevitability of adversarial perturbations, one goal we can still hope to achieve is a computational separation between their existence and the computational complexity of computing them. We propose a cryptographic technique which uses hidden random binary codewords to prevent the adversary from easily computing these perturbations. Any instantiation of the random binary codewords produces an accurate classifier with high probability, so there is no danger of security through obscurity, because the model owner can sample his own fresh random bits. Furthermore, the space of all possible binary codewords is exponential in the number of output classes, so the adversary cannot simply try all of them.

A major show-stopper with black-box models which rely on hidden information is the phenomenon of transferability [PMG16a], where an adversarial perturbation computed for one model has a high chance of causing an independently trained model to simultaneously fail. The first model is called the substitute model, and the second model is the black-box oracle model. [PMG+16b] show that,
even if the adversary is only given black-box oracle access to predicted labels, existing machine learning models are vulnerable to transfer learning attacks executed by training substitute models. The transfer success rate is the probability that an adversarial example computed for the substitute model is also misclassified by the black-box oracle.

Thus, in order to hide the randomness in a single classifier, we must not allow the adversary to directly query it. We achieve this by a special ensemble scheme such that the adversary learns only the output of the overall ensemble without learning any of the intermediate representations. Previous ensemble techniques for increasing adversarial robustness only subsample or augment the training data within each class [TKP+17], whereas our ensemble samples random splits of the labels themselves within the overall multiclass classification setup. This means that the underlying classification problem is unknown to the adversary, and we argue that this randomness decreases the transfer success rate.

2 Preliminaries

Let $\mathcal{X} \subset \mathbb{R}^d$ be the feature space, and let $\mathcal{Y} = \{1, 2, \ldots, N\}$ be the set of classes. The learning problem is to construct a multiclass classifier that is allowed to abstain from making a prediction by returning the symbol $\omega$. We assume all classifier training is conducted using a fixed machine learning algorithm ML which is public knowledge. ML takes as input a set of binary-labeled data points $(x_i, z_i)_{i=1}^n$, where each $x_i \in \mathcal{X}$ and $z_i \in \{\pm 1\}$, and outputs a binary classifier $f : \mathcal{X} \rightarrow \{\pm 1\}$. Furthermore, we assume that $\text{ML}(\{(x_i, z_i)\}_{i=1}^n) = -\text{ML}(\{(x_i, -z_i)\}_{i=1}^n)$, which just means that the labels $-1$ and $+1$ have no intrinsic meaning. Lastly, we fix some space $\mathcal{P} \subset \mathcal{X}$ to be the set of allowable adversarial perturbations, for example $\{\rho \in \mathcal{X} \mid \|\rho\|_\infty < c\}$.

2.a Threat model

We consider the setting of a server hosting a fixed classifier $F : \mathcal{X} \rightarrow \{1, \ldots, N, \omega\}$ and users who interact with the server by presenting a query $q \in \mathcal{X}$ to the server and receiving the output label $F(q)$. We call $F$ a black-box classifier, because the user does not see any of the intermediate computation values of $F(q)$. Two types of users access the server: honest users who present queries drawn from a natural data distribution, and adversarial users who present adversarial examples designed to intentionally cause a misclassification. The desired property is to serve the honest users the true label while simultaneously preventing the adversarial users from causing a misclassification; the latter is accomplished by either continuing to return the true label on adversarial examples or by returning the abstain label $\omega$.

In order for this distinction to be well-defined, we need to separate natural misclassified examples from adversarial examples. We achieve this by fixing in advance a data point $x$ which is correctly classified by $F(x)$ and requiring the adversary to compute a perturbation $\rho \in \mathcal{P}$ for this specific $x$ such that $F(x + \rho) \notin \{F(x), \omega\}$. We think of $x$ as a parameter of the attack, for example the natural image of the face of an attacker who wishes to masquerade as someone else. The classifier $F$ is secure if for all $x$, the adversary cannot find a $\rho$ satisfying this.

We formalize this attack problem by the notion of a security challenge. The adversary is given all the information about $F$ except for any internal randomness used to initialize $F$. The adversary is then given the challenge point $(x, y)$ with $F(x) = y$ being the correct classification, and the adversary successfully solves the security challenge if he finds a $\rho$ such that $F(x + \rho) \notin \{\omega, F(x)\}$ with non-negligible probability. The solution to the security challenge is a successful attack.

The separation between existence of a solution and feasibility of finding it is given by resource constraints on the adversary, most commonly in the form of runtime. We say that a security challenge is hard if there does not exist an algorithm for finding a solution within these resource constraints. In addition to runtime, we also consider the constraint of how many times the adversary is allowed to interact with the classifier.

We make a distinction between these query points (denoted by $q$) and the challenge point (denoted by $x$), both of which are feature vectors in $\mathcal{X}$. Query points are arbitrarily chosen by the adversary for the purpose of learning more about the black-box $F$, and there is no notion of correctness for $F(q)$. The ability to obtain labels for arbitrary query points enables the adversary to mount more powerful black-box attacks, for example the substitute model training methods described by [PMG+16b].
This larger space of possible attacks is realistic but also makes direct empirical security analysis difficult. Cryptographic proofs of security provide an alternative to direct analysis.

2.b Security proofs in cryptography

Instead of directly trying to prove the security of \( F \), we define a simpler system \( f \) that is easier to empirically test and reason about. We then prove a reduction from the security challenge of \( F \) to the security challenge of \( f \), which shows that \( F \) is at least as hard to attack as \( f \). We define a security assumption that characterizes the hardness of attacking \( f \). This security assumption cannot be mathematically proven to be true, but nonetheless defining the right assumption makes the reduction is useful, because this assumption can be easier to empirically study. If the security assumption for the hardness of \( f \) is true, then \( F \) is secure.

The security assumption we define is the hardness of attacking a new type of randomized classifier without any query access to it. We give two reasons why this assumption is the right one to make. Firstly, the scope of attacks to analyze is greatly reduced when the attacker has no access to the classifier. The adversary can essentially only mount transfer learning attacks by training models on the public dataset. Secondly, we only require the probability of success of the adversary to be bounded below 1 by a constant, and the overall security of the ensemble can be boosted from this bound. We next describe this assumption detail.

2.c Random binary classifiers

In a multiclass classification problem with labels 1, \ldots, \( N \), suppose we have a binary classifier \( f : \mathcal{X} \to \{\pm 1\} \) for two particular classes \( y \) and \( t \), where class \( y \) is mapped to \(+1\) and class \( t \) is mapped to \(-1\). An adversary is given a data point \((x, y)\) with \( f(x) = +1 \), and the adversary wishes to attack this binary classifier by computing a perturbation \( \rho \) such that \( f(x + \rho) = -1 \). However, at training time \( f \) was not trained on just data points with original labels \( y \) or \( t \), but with all remaining \( N - 2 \) classes also having been randomly remapped to \( \pm 1 \) with equal probability. In other words, for each class \( k \not\in \{y, t\} \), we sample a Rademacher random variable \( z_k \sim \{\pm 1\} \) and assign every data point of original label \( k \) to the new binary label \( z_k \). This random assignment does not change the original \( y\text{-vs-}t \) classification task when all query data points are only of original class \( y \) or \( t \). The resulting \( f \) corresponding to training with the random binary labels \( z \sim \{\pm 1\}^N \) is a random binary classifier:

**Definition 1** (Random binary classifier). Let \( \mathcal{D} \) be a distribution over \( \{\pm 1\}^N \). The random binary classifier over \( \mathcal{D} \) is the distribution of \( f \) over \( z \sim \mathcal{D} \) where each training data point \( x_i \) is relabeled to \( \pm 1 \) by \( z_{y_i} \):

\[
f_z := \text{ML}\left(\left\{(x_i, z_{y_i})\right\}_{i=1}^n\right). \quad \square
\]

The security challenge for the random binary classifier is to compute a perturbation that changes its output with high probability over the sampling of \( z \).

**Definition 2** (Security challenge for random binary classifier). Let \( f_z := \text{ML}\left(\{(x_i, z_{y_i})\}_{i=1}^n\right). \) Let \( z \sim \{\pm 1\}^N \) be a Rademacher random vector, and let \( \mathcal{D}_{yt} \) be the distribution of \( z \) conditioned on \( z_y = +1, z_t = -1 \). The security challenge for a challenge data point \((x, y)\), failure rate \( \delta > 0 \), and target label \( t \neq y \) is to compute a perturbation \( \rho \in \mathcal{P} \) which changes the output of \( f_z(x) \) with failure rate no greater than \( \delta \):

\[
\Pr_{z \sim \mathcal{D}_{yt}}[f_z(x + \rho) \neq f_z(x)] > 1 - \delta.
\]

In particular, the adversary has no ability to obtain labels for query points from the random binary classifier. \( \square \)

Note that the adversary has knowledge of two of the bits of \( z \), corresponding to the original label \( y \) and some target label \( t \neq y \). Our security assumption is that for any \( \rho \in \mathcal{P} \), for all \( c > 0 \), there exists a constant \( N_0 > 0 \) such that
\[ \Pr_{z \sim P_{\mu}} \left[ f_z(x + \rho) \neq f(x) \right] \leq 1 - 1/N^\nu \]

whenever \( N \geq N_0 \).

Note that this implicitly assumes \( P \) does not contain any non-adversarial perturbations, such as those of the form \( x' - x \) where \( x' \) is a legitimate image of class \( t \). In Section 4.a, we experimentally justify this assumption by estimating the transfer success probability for all pairs of classes \((y, \ell)\) in the MNIST and CIFAR-10 datasets using the standard \( \ell_\infty \)-ball for \( P \).

**2.d Main construction**

Recall that our goal is to construct a multiclass classifier \( F : \mathcal{X} \rightarrow \{1, 2, \ldots, N, \omega\} \) which is allowed to abstain from making a prediction (as represented by the output \( \omega \)), and an adversarial perturbation \( \rho \) is only considered a successful attack if \( F(x + \rho) \notin \{F(x), \omega\} \).

Our ensemble construction is the error-correcting code approach for multiclass-to-binary reduction [DB94], except with completely random codes for security purposes.

**Construction 1** (Random ensemble classifier). Given a multiclass classification problem with labels \( \mathcal{Y} = \{1, \ldots, N\} \), a codelength \( M \), and a threshold parameter \( r \in (0, 1/3) \):

- Sample random matrix \( Z \in \{\pm 1\}^{N \times M} \), where each \( Z_{ij} \sim \{\pm 1\} \) independently and with equal probability
- For \( j = 1, \ldots, M \), construct the binary classifier \( f_j = \text{ML} (\{(x_i, Z_{y,j})\}_{i=1}^n) \)

Given a query data point \( x \), compute output \( F(x) \) by:

- Compute the predicted codeword vector \( C(x) := (f_1(x), \ldots, f_M(x)) \)
- Compute \((d^*, y^*) = \min_y \|Z_y - C(x)\|_H\) where \( y^* \) is the index and \( d^* \) is the Hamming distance to \( Z_{y^*} \).
- If \( d^* < Mr \), then output \( y^* \), else output \( \omega \)

In this construction, the codeword \( Z_y \in \{\pm 1\}^M \) acts as the identity of class \( y \), and thus the classification of a data point \( x \) is the class codeword which is closest to its predicted codeword \( C(x) \). We should think of the free parameters as \( M = \Omega(\text{poly}(N)) \) and \( r = O(1/N) \). \( M \) needs to be sufficiently large in order for the random ensemble classifier to be accurate on natural examples, and \( r \) needs to be sufficiently small for security purposes.

We give some intuition for why this construction has desirable security properties. In order to change the overall output of some test point \((x, y)\), he needs to change the output of sufficiently many binary classifiers \( f_j \) so that \( C(x + \rho) \) is close to some codeword \( Z_{t_i} \), \( t \neq y \). But the Hamming distance between \( Z_y \) and \( Z_t \) is \( M/2 \) on expectation, and \( x, x + \rho \) must be within distance \( Mr \) to \( Z_y, Z_t \) respectively. Since each \( f_i \) is constructed independently at random, the overall probability of success is exponentially decreasing in the probability of successfully changing the output of an individual classifier.

We proceed to define the security challenge for this construction. We will use the shorthand notation \( Z \sim \{\pm 1\}^{N \times M} \) to denote the distribution of \( Z \in \{\pm 1\}^{N \times M} \) where each entry is independently sampled from \( \{\pm 1\} \) with equal probability.

**Definition 3** (Security challenge for random ensemble). Let \( F_Z(\cdot) \) be the ensemble classifier constructed with random hidden code matrix \( Z \) as defined in Construction 1. The security challenge for a challenge data point \((x, y)\) and accuracy \( \varepsilon \in (0, 1) \) is a two-round protocol:

1. Provide \( Q \) nonadaptive queries to \( F_Z(\cdot) \) and receive answer labels, denoted by \( \{(q_k, a_k)\}_{k=1}^Q \). The queries cannot depend on \( Z \) but can depend on anything else, including the original training data set and the construction of \( F \).
2. Return a perturbation $\rho \in \mathcal{P}$ by some function of the query answers $\rho = \phi(\{a_k\}_{k=1}^Q)$ such that $\rho$ satisfies
\[
\Pr_{Z \sim \{\pm 1\}^{N \times M}} [F_Z(x + \rho) \not\in \{F_Z(x), \omega\}] > \varepsilon.
\]
An algorithm for solving the security challenge is determined by its query set $\{q_k\}_{k=1}^Q$ and the function $\phi$ for computing the final perturbation from the query answers.

For example, one possible attack captured by this model is substitute DNN training with a one epoch of data augmentation, which is a single epoch version of the black-box attack described by [PMG+16b]. The adversary obtains a pre-labeled dataset of arbitrary size (which could be the original training data set) and trains an initial substitute DNN on this dataset. The adversary then iteratively refines this initial DNN through substitute training epochs by using Jacobian data augmentation to construct new synthetic data points. These synthetic points are labeled using the black-box classifier and added to the labeled dataset using the classifier’s output as the label.

The synthetic data points are the queries $q_1, \ldots, q_Q$, and thus our proof shows that a single epoch of data augmentation is not sufficient to construct a successful attack (assuming the security assumption is true). The actual implementation of this attack in [PFC+16] uses a constant number of substitute training epochs, and our proof does not apply directly to this implementation, because the second round of queries can depend on the answers in the first round. Nonetheless, we show empirically in Section 4.b that our construction is still secure against this attack involving a constant number of rounds of queries.

3 Security analysis

The main theoretical result is a reduction from solving the random classifier challenge to solving the random ensemble challenge. In our reduction, we make the simplifying assumption that the space of allowable perturbations $\mathcal{P}$ is the same in both security challenges. This allows us to get away with not explicitly defining which perturbations are adversarial and which are legitimate, because a perturbation which makes $x + \rho$ a legitimate image of the class $t$ would solve both security challenges simultaneously. We also assume without loss of generality that $r$ is chosen such that $Mr \in \mathbb{Z}$, because Hamming distance is an integer.

**Theorem 4.** Suppose there exists an algorithm $A$ that can solve the security challenge for the random ensemble with any threshold $r \in (0, 1/3)$ such that $Mr \in \mathbb{Z}$ using $Q$ queries and with accuracy $\varepsilon \in (0, 1)$. Then there is an algorithm that can compute a perturbation $\rho$ which solves the security challenge for a random binary classifier with failure rate
\[
\delta < 2 \left( r + \sqrt{\frac{\log(1/\varepsilon)}{2M}} \right).
\]

The algorithm succeeds in computing this perturbation with probability (over $Z$) at least
\[
1 - 4NQ \sqrt{\frac{M}{2\pi} \cdot \frac{1 - r}{r} \cdot \left( \frac{3}{2^{5/3}} \right)^M}.
\]

The theorem shows that if such an algorithm $A$ exists, $r = O(1/N)$, and $M = \Omega(\text{poly}(N))$, then the failure rate decreases as $O(1/N^c)$ for some constant $c$, which contradicts the security assumption (Assumption 1). Conversely, if the security assumption is true, then an adversary cannot solve the security challenge for the random ensemble with $O(\text{poly}(N))$ nonadaptive queries to the ensemble classifier.

We give a brief proof sketch here, deferring the full proof to Section A. Given a single random classifier $f_z$, we can simulate the entire ensemble classifier $F_Z$ by constructing the remaining $M-1$ random classifiers using the public data set and ML. However, we cannot apply $A$ to $F_Z$ directly, because in Definition 2 there is no query access to $f_z$. Thus we first show in Lemma A.1 that we can simulate the output of the entire ensemble using only $M-1$ classifiers with high probability.
Then, applying the algorithm $A$ the ensemble of $M - 1$ classifiers produces an attack perturbation $\rho$ which also applies to the entire ensemble of $M$ classifiers. Now we want to compute the probability of the output of each individual classifier in the ensemble being changed, but the $Q$ queries could potentially leak information about some column $z^j$. We use Lemma A.1 for each column $j$ to show that this is not the case; i.e. that the query answers are completely determined by the remaining $M - 1$ columns with high probability and thus independent of column $j$ itself. Then we show in Lemma A.2 that an overall success probability of $\varepsilon$ gives an upper bound on $\delta$ for each individual classifier.

4 Empirical results

We provide empirical analysis on both the security assumption (Assumption 1) and the adversarial test accuracy for the MNIST [LCB98] and CIFAR-10 [Kri09] datasets. We use code from the CleverHans adversarial examples library [PFC+16] and from the MadryLab CIFAR10 adversarial examples challenge [Mgd17] for the base classifier architecture, training, and attacks. The only modification to the base classifier architecture was to change the output layer from dimension 10 to dimension 2 for a binary output; no further architecture tuning was performed to optimize natural accuracy.

4.a Analysis of random binary classifiers

First, we empirically estimate the transfer success rate for all pairs of classes. We train a sample size of 30 random binary classifiers and then compute an adversarial perturbation for each test data point and each target class. The perturbation is computed by using a pre-trained standard model for the respective dataset with all $N$ output dimensions. We then compute whether each random binary classifier makes a different prediction on the original test data point versus the perturbed test data point. Finally, for each pair $(y, t)$, we empirically estimate the probability of the output of $f_z(\cdot)$ being changed conditioned on $z_y \neq z_t$ and plot this. The goal of this analysis is to show that this probability is bounded below 1 by a constant.

4.a.1 MNIST

We use the Fast Gradient Sign Method applied to a simple convolutional neural network which achieves 99.3% test accuracy and 61.6% black-box adversarial test accuracy as the substitute model, as implemented in CleverHans [PFC+16]. The perturbation space is an $\ell_\infty$ ball with radius $\varepsilon$ (note that this $\varepsilon$ is standard notation for the step size of the attack in the literature; we no longer refer to $\varepsilon$ in the main theorem). The parameter setting $\varepsilon = 0.3$ is chosen by [PMG+16b] as being optimal in the sense that increasing $\varepsilon$ does not increase the attacker’s power. We also show results for $\varepsilon = 1$ to illustrate the robustness of our assumption.

Figure 1 shows the success probabilities over all pairs of classes $(y, t)$ averaged over all of the test data points. The vertical axis corresponds to the original label $y$, while the horizontal axis corresponds to the target label $t$. The color scheme is the viridis palette, which scales uniformly from 0 (black) to 1 (yellow). The warmest coordinate $(1, 7)$ for $\varepsilon = 0.3$ corresponds to a probability of 0.14.

Figure 1: Success probabilities for targeted attacks on MNIST random binary classifiers
Next we plot the success probabilities of each individual test data point for the highest misclassified pairs. Recall that our total sample size of random binary classifiers is 30, but \( \Pr[z_y \neq z_t] = 1/2 \), so the expected sample size for each data point and each pair \((y, t)\) is 15 samples. Figure 2 shows the distribution for the two highest probabilities in the \( \varepsilon = 0.3 \) plot. We can see that even the worst-case test data points have probabilities bounded far away from 1.

![Success probabilities for individual MNIST test data points, \( \varepsilon = 0.3 \)](image)

**Figure 2:** Distribution of success probabilities for individual MNIST test data points, \( \varepsilon = 0.3 \)

### 4.a.2 CIFAR10

We use Projected Gradient Descent on the cross-entropy loss with an \( \ell_\infty \) norm bound of \( \varepsilon = 8 \), as implemented in the MadryLab CIFAR10 Adversarial Examples Challenge [Mad17]. The pre-trained substitute is a w28-10 wide residual network [ZK16], and the random binary classifiers are the same ResNet architecture but with two output dimensions instead of ten. Figure 3 shows the empirical success probabilities over the CIFAR-10 data set for all pairs of classes.

![Success probabilities for targeted attacks on CIFAR-10 random binary classifiers](image)

**Figure 3:** Success probabilities for targeted attacks on CIFAR-10 random binary classifiers

We see that attacks with target label \( y = 6 \) (frog) have particularly high success rate on random binary classifiers when the source class is another animal. However for the majority of pairs, the security assumption is valid. We plot in Figure 4 the individual test data point distributions for the \((y, t)\) pairs \((4, 5)\) and \((5, 6)\). We see that our security assumption actually fails when transforming cats, deer, and dogs into frogs, but the failure of the security assumption for these cases is at least interpretable in the sense that the easily confused classes are also close to each other by human perception.

### 4.b Analysis of black-box adversarial accuracy

Next, we empirically analyze the robustness of our random ensemble construction to black-box transfer learning attacks. We use the CleverHans attack library [PFC+16] as a standard benchmark. The attack algorithm trains a two-layer fully connected substitute model iteratively augmenting its training data set via queries to the random ensemble scheme and then uses the Fast Gradient Sign Method attack on the substitute model.

Because the attack library is not designed for querying classifier which abstains, we perform substitute model training with a non-abstaining random ensemble (i.e. \( r = 1/2 \)). We consider the
threshold $r$ at the end when analyzing the final true and adversarial test accuracies. In order to incorporate the abstain label, we use the following definitions of accuracy for our experiments. The true test accuracy requires the classifier to make the correct, non-abstaining prediction. However when computing adversarial accuracy, we also consider it a success if the classifier outputs $\omega$.

**Definition 5** (True and adversarial test accuracy). Given a multiclass classifier $F : \mathcal{X} \rightarrow \{1, 2, \cdots, N, \omega\}$ which is allowed to abstain from making a prediction (as represented by the output $\omega$), the relevant accuracy benchmarks are

$$
\text{True accuracy} := \mathbb{E}_{(x,y)} \left[ I[F(x) = y] \right]
$$

$$
\text{Adversarial accuracy} := \mathbb{E}_{(\hat{x},y)} \left[ I[F(\hat{x}) \in \{y, \omega\}] \right],
$$

where $x$ is the original data point and $\hat{x}$ is an adversarial perturbation of $x$.

All random binary classifiers used in these experiments are the same architecture as the random binary classifiers in Section 4.a. Figure 5 shows that the ensemble enjoys good adversarial accuracy in the low-$r$ regime, although there is a tradeoff with the true test accuracy.

![Figure 5: Accuracy versus Hamming distance ratio ($r$)](image)

5 Conclusion

We proposed a novel approach to provable robustness at test time in the adversarial setting. We formalized a smaller attack problem which is easier to study and which we conjecture to be hard. We also show that our overall ensemble construction enjoys high adversarial accuracy against black-box attacks with standard measures of perturbation size while being completely agnostic of these parameters. Our formal proof framework introduces some techniques in analysis of cryptographic constructions to the adversarial learning problem, and we hope it can lead to more principled empirical and theoretical work in this area.
References

[ACW18] Anish Athalye, Nicholas Carlini, and David Wagner. Obfuscated Gradients Give a False Sense of Security: Circumventing Defenses to Adversarial Examples. In Icml, 2018. 1

[DB94] T. G. Dietterich and G. Bakiri. Solving Multiclass Learning Problems via Error-Correcting Output Codes. Journal of Artificial Intelligence Research, 2, 1994. 2.d

[FGCC19] Nic Ford, Justin Gilmer, Nicolas Carlini, and Dogus Cubuk. Adversarial Examples Are a Natural Consequence of Test Error in Noise. 2019. 1

[GMF+18] Justin Gilmer, Luke Metz, Fartash Faghri, Samuel S. Schoenholz, Maithra Raghu, Martin Wattenberg, and Ian Goodfellow. Adversarial Spheres. 2018. 1

[GSS15] Ian J. Goodfellow, Jonathon Shlens, and Christian Szegedy. Explaining and Harnessing Adversarial Examples. International Conference on Learning Representations, pages 1–11, 2015. 1

[IST+19] Andrew Ilyas, Shibani Santurkar, Dimitris Tsipras, Logan Engstrom, Brandon Tran, and Aleksander Madry. Adversarial Examples Are Not Bugs, They Are Features. 2019. 1

[Jac18] Ian Goodfellow Jacob Buckman, Aurko Roy, Colin Raffell. Thermometer Encoding: One Hot Way To Resist Adversarial Examples. Iclr, 19(1):92–97, 2018. 1

[Kri09] Alex Krizhevsky. Learning Multiple Layers of Features from Tiny Images. arXiv 2009, 2009. 4

[LCB98] Y LeCun, C Cortes, and C J C Burges. The MNIST dataset of handwritten digits. http://yann.lecun.com/exdb/mnist/, 1998. 4

[Mad17] Aleksander Madry. MadryLab CIFAR10 Adversarial Examples Challenge. https://github.com/MadryLab/cifar10_challenge, 2017. 4, 4.a.2

[MMS+17] Aleksander Madry, Aleksandar Makelov, Ludwig Schmidt, Dimitris Tsipras, and Adrian Vladu. Towards Deep Learning Models Resistant to Adversarial Attacks. pages 1–27, 2017. 1

[MS78] F J MacWilliams and N J. A. Sloane. The Theory of Error-Correcting Codes. 1978. A

[PFC+16] Nicolas Papernot, Fartash Faghri, Nicholas Carlini, Ian Goodfellow, Reuben Feinman, Alexey Kurakin, Cihang Xie, Yash Sharma, Tom Brown, Aurko Roy, Alexander Matyasko, Vahid Behzadan, Karen Hambardzumyan, Zhishuai Zhang, Yi-Lin Jiang, Zhi Li, Ryan Sheatsley, Abhiebav Garg, Jonathan Uesato, Willi Gierke, Yinpeng Dong, David Berthelot, Paul Hindricks, Jonas Rauber, Rujun Long, and Patrick McDaniel. Technical Report on the CleverHans v2.1.0 Adversarial Examples Library. pages 1–12, 2016. 2.d, 4, 4.a.1, 4.b

[PMG16a] Nicolas Papernot, Patrick McDaniel, and Ian Goodfellow. Transferability in Machine Learning: from Phenomena to Black-Box Attacks using Adversarial Samples. 2016. 1

[PMG+16b] Nicolas Papernot, Patrick McDaniel, Ian Goodfellow, Somesh Jha, Z. Berkay Celik, and Ananthram Swami. Practical Black-Box Attacks against Machine Learning. 2016. 1, 2.a, 2.d, 4.a.1

[SC17] Yash Sharma and Pin-Yu Chen. Attacking the Madry Defense Model with SL_1S-based Adversarial Examples. pages 1–9, 2017. 1

[SRBB18] Lukas Schott, Jonas Rauber, Matthias Bethge, and Wieland Brendel. Towards the first adversarially robust neural network model on MNIST. 3:1–16, 2018. 1

[SZS+13] Christian Szegedy, Wojciech Zaremba, Ilya Sutskever, Joan Bruna, Dumitru Erhan, Ian Goodfellow, and Rob Fergus. Intriguing properties of neural networks. pages 1–10, 2013. 1

[TKP+17] Florian Tramèr, Alexey Kurakin, Nicolas Papernot, Ian Goodfellow, Dan Boneh, and Patrick McDaniel. Ensemble Adversarial Training: Attacks and Defenses. pages 1–20, 2017. 1
[XWZ⁺17] Cihang Xie, Jianyu Wang, Zhishuai Zhang, Zhou Ren, and Alan Yuille. Mitigating Adversarial Effects Through Randomization. pages 1–16, 2017.

[ZK16] Sergey Zagoruyko and Nikos Komodakis. Wide Residual Networks. 2016.
A Proofs

Lemma A.1. Fix any query point \( q \) and threshold \( r < 1/3 \) such that \( M r \in \mathbb{Z} \). Given a random ensemble function \( F_Z : \mathcal{X} \rightarrow \{1, \ldots, N\} \) with \( M \) independently and identically generated random classifiers and threshold \( r < 1/3 \), fix some \( j \in \{1, \ldots, M\} \) and let \( F_{Z^{-j}} \) denote the modified ensemble which ignores the \( j \)th random classifier and takes the vote only over the remaining \( M - 1 \) classifiers. Then

\[
\Pr_{\mathbb{Z}^{-j} \sim \{\pm 1\}^{N \times (M-1)}} \left[ F_Z(q) \neq F_{Z^{-j}}(q) \right] \leq 4N \sqrt{\frac{1-r}{2\pi Mr}} \left( \frac{3}{2^{5/3}} \right)^M,
\]

where the probability is taken only over the matrix \( \mathbb{Z}^{-j} \) and is independent of the column \( Z^j \).

The lemma shows that for any \( j \), with high probability over \( \mathbb{Z}^{-j} \) the query answer \( F_Z(q) \) is independent of \( Z^j \), so that no information is revealed by the queries about column \( j \). In the following proofs we will use the shorthand \( f_j := f_{Z^j} \), i.e. the random classifier constructed from the \( j \)th column of \( Z \).

Proof. The only way the additional vote of \( f_j(q) \) can influence the vote of \( F_{Z^{-j}}(q) \) is if the predicted codeword of length \( M - 1 \) is on the decision boundary between some class \( i \) and the abstaining space corresponding to \( \omega \). In the boolean hypercube \( \{\pm 1\}^{M-1} \), the number of points that are at a distance of exactly \( k \) to any fixed point is \( \binom{M-1}{k} \). Because we want our probability bound to hold true regardless of the value of \( f_j \), we have to consider the possibility of \( f_j(q) \) influencing the points on either side of the decision boundary. To account for this, we multiply the number by 2. Then over all \( N \) classes, the number of possible points on the decision boundary is at most \( 2^N \binom{M-1}{M-r} \) by a union bound.

Recall that we assumed the machine learning oracle is symmetric; that is, \( \text{ML}\{(x_i, y_i)_{i=1}^n\} = -\text{ML}\{(x_i, -y_i)_{i=1}^n\} \). Thus given any query data point \( q \), \( \Pr_{z}[f_z(q) = 1] = 1/2 \) when \( z \sim \{\pm 1\}^M \) uniformly, because the probabilities of sampling \( z \) and \( -z \) are identical. Then over sampling of \( Z^{-j} \), the predicted codeword vector has independently distributed Rademacher entries, which means the probability mass on any point in \( \{\pm 1\}^{M-1} \) is \( 1/2^{M-1} \). Thus the probability of \( q \) being on the decision boundary is at most

\[
\frac{2N}{2^{M-1}} \binom{M-1}{M r} = \frac{4N(1-r)}{2^M} \binom{M}{M r}.
\]

We now apply the binomial coefficient upper bound from [MS78], stated in Appendix B.1, to obtain

\[
\binom{M}{M r} \leq \frac{1}{\sqrt{2\pi Mr(1-r)}} 2^{MH_2(r)},
\]

where \( H_2(r) = -r \log_2 r - (1-r) \log_2(1-r) \) is the negative entropy function. Note that \( H_2(r) \) is monotonically increasing in \( r \in (0, 1/2) \) and reaches \( \log_2 r \) at \( r = 1/2 \), so when \( r < 1/3 \) then \( H_2(r) \leq \log_2 \frac{3}{2e^r} \approx 0.6365 \). Thus the probability in (1) can be bounded by

\[
\frac{4N(1-r)}{2^M} \frac{1}{\sqrt{2\pi Mr(1-r)}} 2^{MH_2(r)} \leq 4N \sqrt{\frac{1-r}{2\pi Mr}} \left( \frac{3}{2^{5/3}} \right)^M.
\]

Since \( \frac{3}{2^{5/3}} \approx 0.945 \), this gives an exponentially decaying probability bound in \( M \).

The next lemma is a concentration result that holds when no information is revealed by the queries about any individual column.
Lemma A.2. Suppose that the event $f_j(x + \rho) \neq f_j(x)$ is independent and identical for each column $j$. Fix a data point $(x, y)$. Given a perturbation $\rho$ which solves the security challenge for the random ensemble with target probability $\varepsilon > 0$, then for every random classifier in the ensemble, $\rho$ solves the security challenge for it with failure rate $\delta < 2(r + \sqrt{\log(1/\varepsilon)/2M})$.

Proof. Recall that the adversary is said to have solved the security challenge for the random ensemble if the vector of code bits $C_Z(x + \rho) := (f_1(x + \rho), \ldots, f_M(x + \rho))$ has Hamming distance less than $Mr$ to any other codeword $Z_i$, where $i \neq y$. Since each entry of the code matrix is sampled independently, we can consider the probability of this event bit-by-bit.

Let $E_{ij}$ be the event where $f_j(x + \rho) = Z_{ij}$. Let $\mathcal{E}_t$ be the probability of the event where $||C_Z(x + \rho') - \tilde{Z}_i||_1 \leq Mr$, meaning the codeword for class $t$ is the closest. By the independence assumption, we have $\Pr[\mathcal{E}_t] = \Pr[X > M(1 - r)]$ where $X \sim \text{Binom}(M, \Pr[\mathcal{E}_{ij}])$, or equivalently,

$$\Pr[\mathcal{E}_t] = \Pr[X < Mr \mid X \sim \text{Binom}(M, 1 - \Pr[\mathcal{E}_{ij}])].$$

The probability of changing $F(x)$ from $y$ to any other class can be bounded by applying the union bound to all $t \neq y$. We obtain

$$\Pr[F_Z(x + \rho) \neq F_Z(x)] \leq (N - 1) \Pr[\mathcal{E}_t],$$

and by the assumption of the lemma we know the left-hand side probability is $\delta > 0$. Thus we just need to compute $\Pr[\mathcal{E}_{ij}]$ and apply a tail inequality for the binomial distribution.

Fix one underlying code bit $j$ and some other class $t \neq y$. Each bit $Z_{ij}$ differs from the corresponding bit of $C_{ij}$ with probability $1/2$ under the random code sampling scheme. Without loss of generality, we’ll let $Z_{yj} = +1$. We analyze the probability of the event $f_j(x + \rho) = Z_{ij}$ by conditioning on $Z_{ij}$, obtaining

$$\Pr[\mathcal{E}_{ij}] = \Pr[Z_{ij} = -1] \Pr[f_j(x + \rho) = -1 \mid Z_{ij} = -1, Z_{yj} = 1] + \Pr[Z_{ij} = +1] \Pr[f_j(x + \rho) = +1 \mid Z_{ij} = +1, Z_{yj} = +1].$$

We note that the term $\Pr[f_j(x + \rho) = -1 \mid Z_{ij} = -1, Z_{yj} = +1]$ is exactly the the probability $1 - \delta$ in Definition 2. Then $\Pr[\mathcal{E}_{ij}]$ can be bounded by

$$\Pr[\mathcal{E}_{ij}] \leq \frac{1}{2}(1 - \delta) + \frac{1}{2}(1) = 1 - \frac{\delta}{2}.$$

Then the probability in (2) can be bounded by using Hoeffding’s inequality B.2:

$$\Pr[\mathcal{E}_t \leq Mr] \leq \exp\left(-2M \left(r - \frac{\delta}{2}\right)^2\right).$$

Thus we have

$$\log(1/\varepsilon) > 2M \left(\frac{\delta}{2} - r\right)^2,$$

which is equivalent to

$$\delta < 2\left(r + \sqrt{\frac{\log(1/\varepsilon)}{2M}}\right).$$

$\square$
Proof of Theorem 4. We are given an instance of the security challenge for a random binary classifier (Definition 2). Let \( f_\star \) be the random binary classifier, where \( \mathcal{Z} \sim \{\pm 1\}^N \) is uniformly sampled. We can simulate the entire random ensemble by constructing \( M - 1 \) additional random classifiers in the same way that \( f_\star \) is sampled, so that \( f_1 = f_\star \) and \( f_2, \ldots, f_M \) are freshly sampled. Let \( Z^{-j} \) denote the matrix \( Z \) without the \( j \)th column, so that \( F_{Z^{-j}} : \mathcal{X} \to \{1, \ldots, N\} \) denotes the output of the random ensemble ignoring \( f_j \).

By the definition of the security challenge, the adversary cannot query \( f_1 \); however since \( F_{Z^{-j}} \) is simulated by the adversary, he can make queries to \( F_{Z^{-j}} \) and run \( \mathcal{A} \) to produce a perturbation \( \rho \) attacking \( F_{Z^{-j}} \). But if \( F_{Z^{-j}}(q_i) = F_Z(q_i) \) for each query \( q_i \), then \( \mathcal{A} \) would have produced the same perturbation \( \rho \) attacking \( F_Z \).

By Lemma A.1 and a union bound over the number of queries, the hypothetical query answers \( a_1, \ldots, a_Q \) to the entire ensemble \( F_Z \) depend only on \( F_{Z^{-j}} \) with probability at least

\[
1 - \Pr_{Z^{-j}} \left[ \exists i \text{ } F_{Z^{-j}}(q_i) \neq F_Z(q_i) \right] \geq 1 - 4NQ \sqrt{\frac{1 - r}{2\pi Mr}} \left( \frac{3}{2^{5/3}} \right)^M .
\]  

(3)

Now in order to apply Lemma A.2 to bound \( \varepsilon \) as a function of \( \delta \), we want to show for each \( j \) that the event \( f_j(x + \rho) \neq f_j(x) \) is independent of the query answers \( a_1, \ldots, a_Q \). This can be done by applying Lemma A.1 again to each column \( j \) to show that with high probability, the query answers only depend on the random sampling of \( Z^{-j} \). Since \( \rho = \phi(\{a_k\}_{k=1}^Q) \) is a function of the query answers, then this means that the adversary’s chosen \( \rho \) also only depends on \( Z^{-j} \). We obtain

\[
\Pr_{Z^{-j}} \left[ f_j(x + \rho) \neq f_j(x) \mid a_1, \ldots, a_Q \right] = \Pr_{Z^{-j}} \left[ f_j(x + \rho) \neq f_j(x) \mid Z^{-j} \right]
\]

\[
= \Pr_{Z^{-j}} \left[ f_j(x + \rho) \neq f_j(x) \right],
\]

and we see that this probability has no dependence on the actual column \( j \) since \( Z_j \) is independent and identical for each \( j \). We incur a factor \( M \) in the probability of failure by applying a union bound of the failure probability in (3) over all \( j = 1, \ldots, M \). Thus the event \( f_j(x + \rho) \neq f_j(x) \) is independent and identical for each column \( j \) with probability at least

\[
1 - 4MNQ \sqrt{\frac{1 - r}{2\pi Mr}} \left( \frac{3}{2^{5/3}} \right)^M .
\]

Then by Lemma A.2, the probability of \( \rho \) changing the output of \( f_\star \) is at least

\[
1 - 2 \left( r + \sqrt{\frac{\log(1/\varepsilon)}{2M}} \right).
\]

\[ \square \]

B Probability inequalities

Lemma B.1. Suppose \( \lambda n \) is an integer, where \( 0 < \lambda < 1 \). Then

\[
\binom{n}{\lambda n} \leq \frac{1}{\sqrt{2\pi n}(1 - \lambda)^{2nH_2(\lambda)}},
\]

where \( H_2(\lambda) = -\lambda \log_2 \lambda - (1 - \lambda) \log_2 (1 - \lambda) \) is the negative entropy function.

Lemma B.2. [Hoeffding’s inequality] Suppose \( X \sim \text{Binom}(n, p) \). Then for any \( \alpha > 0 \),

\[
\Pr \left[ X \leq (p - \alpha)n \right] \leq \exp \left( -2\alpha^2 n \right).
\]
C Link to code for experiments

https://www.dropbox.com/sh/l5p242guwh1by8t/AAAJBk1iw4YRHXRZ5nH0_dv-a?dl=0