Regulating atomic imbalance in double-well lattices

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Abstract

An insulating optical lattice with double-well sites is considered. In the case of the unity filling factor, an effective Hamiltonian in the pseudospin representation is derived. A method is suggested for manipulating the properties of the system by varying the shape of the double-well potential. In particular, it is shown that the atomic imbalance can be varied at will and a kind of the Morse-alphabet sequences can be created.

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1 Introduction

Physics of cold atomic gases, Bose [1–8] and Fermi [9,10], is a fastly developing field of research, both theoretically and experimentally. Cold atoms in optical lattices provide a versatile tool for various applications [11–14]. Recently, a novel type of optical lattices has been realized in experiments [15–17], the so-called double-well lattices. Each site of such a lattice is formed by a double-well potential.

Generally, the theoretical description of the double-well lattices is essentially more complicated than that of the usual lattices. Therefore, in order to describe the properties of such lattices, it is useful to consider some particular types of them.

In the present paper, we consider a particular kind of the double-well lattice, which is characterized by the following main features: atoms in the lattice are in an insulating state, and the atomic filling factor is equal to unity. Such a type of a lattice is of principal importance providing a convenient setup for realizing quantum information processing and quantum computing [11]. Another important property of the lattices, we shall be considering, is the existence of atomic interactions between different lattice sites. The principal goal of the present paper is to demonstrate that the atomic imbalance between the wells of each double well can be manipulated and that arbitrary sequences of the order-disorder transition can be generated.

2 Insulating double-well lattice

We start with the standard form of the energy Hamiltonian, expressed through the field operators in the Heisenberg representation. Since the system is assumed to be in the insulating state, the field operators can be expanded over localized orbitals, which yields the Hamiltonian

\[ \hat{H} = \sum_{nj} E_n c_{nj}^\dagger c_{nj} + \frac{1}{2} \sum_{\{j\}} \sum_{\{n\}} \phi_{n_jn_2n_3n_4} c_{n_1j_1}^\dagger c_{n_2j_2}^\dagger c_{n_3j_3} c_{n_4j_4}, \]

in which \( c_{nj} \) is a field operator labelled by the quantum index \( n \) and the site number \( j \), related to a lattice vector \( a_j \). The atoms can be either bosons or fermions, with the operator commutation relations

\[ [c_{nj}, c_{n_j}^\dagger] = \delta_{nm} \delta_{ij}, \quad [c_{mi}, c_{nj}] = 0, \]

where the commutator is assumed for bosons and anticommutator for fermions. The value \( E_n \) represents the energy levels of an atom in a double-well located at a site of the lattice. The quantity \( \phi_{n_jn_2n_3j_4} \) is a matrix element of the interaction potential with respect to the localized orbitals labelled by the indices \( n \) and \( j \). We assume that the interaction potential is sufficiently strong, so that the interactions of atoms between at least nearest-neighbor sites cannot be neglected. This can be easily achieved, for instance, with the long-range potentials, such that exist between polar molecules and between Rydberg atoms [18] or between the atoms with large magnetic moments [19]. A known example of the latter atoms is Cr that can be cooled to ultracold temperatures [20]. In the case of ions, long-range forces would be due to the Coulomb interaction.
The assumption that each lattice site contains just a single atom can be formalized by means of the unipolarity condition

$$\sum_n c_{nj}^\dagger c_{nj} = 1, \quad c_{nj} c_{nj} = 0.$$  \hspace{1cm} (2)

This is the known condition, used earlier by Bogolubov [21] for treating ferromagnets. Keeping in mind low temperatures, we can consider only two lowest energy levels of a double-well potential, enumerated by \( n = 1, 2 \), so that \( E_1 < E_2 \). As is known [22], the ground-state orbital, of an atom in a double-well potential, is symmetric with respect to spatial inversion, while that for the first excited state is antisymmetric.

Retaining two lowest levels allows us to invoke the pseudospin representation by introducing the pseudospin operators

$$S^x_j = \frac{1}{2} \left( c_{1j}^\dagger c_{2j} - c_{2j}^\dagger c_{1j} \right), \quad S^y_j = \frac{i}{2} \left( c_{1j}^\dagger c_{2j} - c_{2j}^\dagger c_{1j} \right), \quad S^z_j = \frac{1}{2} \left( c_{1j}^\dagger c_{2j} + c_{2j}^\dagger c_{1j} \right).$$  \hspace{1cm} (3)

To understand the meaning of these operators, one can introduce the left and right location operators, respectively, as

$$c_{jL} = \frac{1}{\sqrt{2}} (c_{1j} + c_{2j}), \quad c_{jR} = \frac{1}{\sqrt{2}} (c_{1j} - c_{2j}).$$  \hspace{1cm} (4)

Then operators (3) become

$$S^x_j = \frac{1}{2} \left( c_{jL}^\dagger c_{jR} + c_{jR}^\dagger c_{jL} \right), \quad S^y_j = -\frac{i}{2} \left( c_{jL}^\dagger c_{jR} - c_{jR}^\dagger c_{jL} \right), \quad S^z_j = \frac{1}{2} \left( c_{jL}^\dagger c_{jL} - c_{jR}^\dagger c_{jR} \right).$$  \hspace{1cm} (5)

Note that this representation is valid for both statistics, so that atoms can be either bosons or fermions. The physical meaning of these operators is as follows: \( S^x_j \) characterizes the tunneling intensity between the wells of a double-well potential, \( S^y_j \) describes the Josephson current between the wells, and \( S^z_j \) defines the atomic imbalance between the wells. It is worth emphasizing that for a double well it is necessary to take into account at least two lowest energy levels, but not just one ground state, since only with two levels there exists tunneling between the wells. Actually, it is exactly because of the tunneling that the energy levels split into pairs [22].

Let us use the notation

$$E_0 \equiv \frac{E_1 + E_2}{2},$$

and

$$V_{ij}^{mm'm'} \equiv \Phi_{ijj'i}^{mm'm'} \pm \Phi_{ijij}^{mm'm'},$$

where the sign plus or minus is for bosons or fermions, respectively. Also, let us denote the matrix elements

$$A_{ij} \equiv \frac{1}{4} \left( V_{ij}^{1111} + V_{ij}^{2222} + 2V_{ij}^{1221} \right), \quad B_{ij} \equiv \frac{1}{2} \left( V_{ij}^{1111} + V_{ij}^{2222} - 2V_{ij}^{1221} \right).$$
\[ C_{ij} \equiv \frac{1}{2} \left( V_{ij}^{2222} - V_{ij}^{1111} \right), \quad I_{ij} \equiv -2V_{ij}^{1122}. \]  \hfill (6)

An important quantity is the tunneling frequency
\[ \Omega \equiv E_2 - E_1 + \sum_{j(\neq i)} C_{ij} \]  \hfill (7)

characterizing the atomic tunneling between the wells of a double-well potential.

Using the above conditions and notations, we reduce Hamiltonian (1) to the pseudospin form
\[ \hat{H} = E_0 N + \frac{1}{2} \sum_{i \neq j} A_{ij} - \Omega \sum_j S_j^z + \sum_{i \neq j} B_{ij} S_i^x S_j^x - \sum_{i \neq j} I_{ij} S_i^z S_j^z. \]  \hfill (8)

The first two terms do not contain operators, so do not play role in what follows. The magnitude of \( B_{ij} \) can be comparable with \( \Omega \), hence, it cannot be neglected. The value of the tunneling frequency \( \Omega \) can be varied in a wide range, depending on the shape of the double-well potential. To illustrate this, let us take the latter in the form
\[ V(r) = V_0 \left( \frac{r_x}{r_0} \right)^2 \left[ \left( \frac{r_x}{r_0} \right)^2 - 2 \right] + V_{yz}, \]

where \( V_{yz} \) is, say, a harmonic potential in the \( y \)- and \( z \)-directions. Then the tunneling frequency strongly depends on the parameter
\[ \alpha \equiv \frac{1}{\sqrt{2m r_0^2 V_0^2}} = \frac{a}{\pi r_0} \sqrt{\frac{E_R}{V_0}}, \]  \hfill (9)

in which \( a \) is the lattice spacing in the \( x \)-direction and \( E_R \equiv \pi^2/2ma^2 \) is the recoil energy. By varying the interwell distance \( r_0 \) or the potential depth \( V_0 \), the parameter \( \alpha \) can be varied in a wide range between \( \alpha \ll 1 \) and \( \alpha \gg 1 \). Direct calculations [22] show that the tunneling frequency \( \Omega \) can also be made either small or large, such that
\[ \Omega \simeq 6V_0 \exp \left( -2\pi \frac{r_0}{a} \frac{r_0}{a} \sqrt{\frac{V_0}{E_R}} \right) \quad (\alpha \ll 1), \]
\[ \Omega \simeq 9V_0 \left( \frac{a}{\pi r_0} \right)^{2/3} \left( \frac{E_R}{V_0} \right)^{1/3} \quad (\alpha \gg 1). \]  \hfill (10)

That is, \( \Omega \) can be regulated by governing the shape of the double-well potential. And varying \( \Omega \) it is possible to regulate the system properties. 

The important system characteristics are the average tunneling intensity
\[ x \equiv \frac{2}{N_L} \sum_j \langle S_j^x \rangle, \]  \hfill (11)

average Josephson current
\[ y \equiv \frac{2}{N_L} \sum_j \langle S_j^y \rangle, \]  \hfill (12)
and the average well imbalance

\[ z \equiv \frac{2}{N_L} \sum_j < S_j^z > , \tag{13} \]

where \( N_L \) is the number of lattice sites and \(< \cdot >\) implies statistical averaging. These values depend on the system parameters, such as the dimensionless transverse interaction

\[ b \equiv \frac{B}{I+B} , \tag{14} \]

in which

\[ B \equiv \frac{1}{N_L} \sum_{i \neq j} B_{ij} , \quad I \equiv \frac{1}{N_L} \sum_{i \neq j} I_{ij} , \]

and the dimensionless tunneling frequency

\[ \omega \equiv \frac{\Omega}{I+B} . \tag{15} \]

3 Manipulating atomic imbalance

The equations of motion for the average quantities (11) to (13) can be obtained by averaging the Heisenberg equations for the spin operators \( S_j^\alpha \). In this procedure, we assume that the system is at zero temperature, we employ the local-field approximation [23], taking into account particle interactions occurring in the local field of other particles. Measuring time in units of \( 1/(I+B) \), we obtain

\[ \frac{dx}{dt} = (1-b)yz - \gamma_2(x-x_t) , \quad \frac{dy}{dt} = (\omega - x)z - \gamma_2(y-y_t) , \quad \frac{dz}{dt} = (bx - \omega)y - \gamma_1(z-z_t) , \tag{16} \]

where \( \gamma_1 \) and \( \gamma_2 \) are the attenuation parameters that are expressed through atomic interactions \( I_{ij} \) and \( B_{ij} \) similarly to the corresponding damping parameters in spin systems [24], and where the local fields are

\[ x_t = \frac{\omega - bx}{h} , \quad y_t = 0 , \quad z_t = \frac{1-b}{h} \quad z , \tag{17} \]

with

\[ h = \sqrt{(\omega - bx)^2 + (1-b)^2z^2} . \]

By their form, the local fields (17) correspond to the equilibrium solutions for averages (11) to (13). However, since here \( x = x(t) \) and \( z = z(t) \) depend on time, the local fields (17) describe the locally equilibrium state.

Accomplishing the Lyapunov stability analysis for the system of equations (16), we find two fixed points. One is given by the equations

\[ x_1^* = \omega , \quad y_1^* = 0 , \quad z_1^* = \sqrt{1-\omega^2} \quad (\omega < 1) , \tag{18} \]
this fixed point being stable for \( \omega < 1 \), but unstable for \( \omega > 1 \). And the other fixed point is

\[
x_2^* = 1 , \quad y_2^* = 0 , \quad z_2^* = 0 \quad (\omega > 1) ,
\]

which is stable for \( \omega > 1 \), but unstable for \( \omega < 1 \).

It is worth emphasizing that the fixed points (18) and (19) are stable, for the corresponding values of \( \omega \), only when \( \gamma_1 \) and \( \gamma_2 \) are not zero. When the latter parameters are zero, the dynamical system (16) is structurally unstable. This stresses the necessity of taking into account the attenuation effects, without which there would be no correct description of dynamics.

The stationary solution (18) characterizes the phase of the system with a nonzero well imbalance \( z_1^* > 0 \). Hence, this can be called the ordered phase. The stationary solution (19) describes a disordered phase, where the well imbalance is zero, \( z_2^* = 0 \). In an equilibrium system, the transition between the ordered and disordered phases would correspond to a quantum phase transition, with the well imbalance playing the role of an order parameter. For a nonequilibrium system, the value \( \omega = 1 \) is a bifurcation point, where a dynamical phase transition takes place.

As has been discussed above, the magnitude of the tunneling frequency can be varied in a wide range by changing the shape of the double well. This means that we have a straightforward opportunity of regulating the state of the double-well lattice by reswitching the value of \( \omega \) between that one corresponding to the ordered state and another related to the disordered state. That is, we can manipulate with the well imbalance, reswitching it, according to our will, between the zero and nonzero values.

As an illustration of this remarkable feasibility of regulating the well imbalance, we solve numerically Eqs. (16), with a time-modulated tunneling frequency \( \omega = \omega(t) \), which is varied according to the rule

\[
\omega(t) = \begin{cases} 
\omega_1, & 0 \leq t < \Delta t_1 \\
\omega_2, & \Delta t_1 \leq t < \Delta t_1 + \Delta t_2 \\
\omega_1, & \Delta t_1 + \Delta t_2 \leq t < \Delta t_1 + \Delta t_2 + \Delta t_3 , \\
\ldots 
\end{cases}
\]

where \( \omega_1 < 1 \) and \( \omega_2 > 1 \). The time intervals \( \Delta t_n \) can be chosen arbitrarily. For instance, we can take all of them being equal, as in Fig. 1, where \( \Delta t_1 = \Delta t_2 = \ldots = \Delta t_n \). Or we can organize a periodic sequence with two unequal time intervals \( \Delta t_1 = \Delta t_3 = \ldots = \Delta t_{2n+1} \) and \( \Delta t_2 = \Delta t_4 = \ldots = \Delta t_{2n} \), as in Fig. 2. Finally, we can take arbitrary time intervals \( \Delta t_n \), as in Fig. 3, realizing the sequences, according to our will, in which information, as in the Morse alphabet, could be encoded. Such Morse-alphabet sequences for the pseudospin system, representing a double-well lattice, are analogous to the punctuated sequences for superradiating spin systems [25].

In conclusion, we have derived a pseudospin representation for insulating double-well lattices, with the unity filling factor. Such a system can be either in an ordered or in a disordered state, which are characterized by a nonzero or zero well imbalance, respectively. For correctly describing the system dynamics, one has to take account of relaxation effects, without which the related dynamical system is structurally unstable. By a temporal modulation
of the lattice parameters, e.g., of the tunneling frequency, it is possible to regulate the well imbalance, organizing the Morse-alphabet sequences, which can be employed for quantum information processing. The possibility of regulating the well imbalance in a double-well lattice, by organizing arbitrary temporal sequences of ordered and disordered states, is the main result of the present paper.

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References

[1] L. Pitaevskii, S. Stringari, Bose-Einstein Condensation, Clarendon, Oxford, 2003.
[2] V.I. Yukalov, Laser Phys. Lett. 1 (2004) 435.
[3] J.O. Anderson, Rev. Mod. Phys. 76 (2004) 599.
[4] K. Bongs, K. Sengstock, Rep. Prog. Phys. 67 (2004) 907.
[5] V.I. Yukalov, M.D. Girardeau, Laser Phys. Lett. 2 (2005) 375.
[6] A. Posazhennikova, Rev. Mod. Phys. 78 (2006) 1111.
[7] V.I. Yukalov, Laser Phys. Lett. 4 (2007) 632.
[8] N.P. Proukakis, B. Jackson, J. Phys. B 41 (2008) 203002.
[9] S. Giorgini, L.P. Pitaevskii, S. Stringari, Rev. Mod. Phys. 80 (2008) 1215.
[10] W. Ketterle, M.W. Zwierlein, Riv. Nuovo Cimento 31 (2008) 247.
[11] D. Jaksch, P. Zoller, Ann. Phys. (N.Y.) 315 (2005) 52.
[12] O. Morsch, M. Oberthaler, Rev. Mod. Phys. 78 (2006) 179.
[13] C. Moseley, O. Fialko, K. Ziegler, arXiv:0707.1979 (2007).
[14] I. Bloch, J. Dalibard, W. Zwerger, Rev. Mod. Phys. 80 (2008) 885.
[15] J. Sebby-Straley, M. Anderlini, P.S. Jessen, J.V. Porto, Phys. Rev. A 73 (2006) 033605.
[16] J. Sebby-Straley, B.L. Brown, M. Anderlini, P.J. Lee, W.D. Phillips, J.V. Porto, Phys. Rev. Lett. 98 (2007) 200405.
[17] P.J. Lee, M. Anderlini, B.L. Brown, J. Sebby-Straley, W.D. Phillips, J.V. Porto, Phys. Rev. Lett. 99 (2007) 020402.
[18] T.F. Gallagher, Rydberg Atoms, Cambridge University, Cambridge, 1994.
[19] A. Griesmaier, J. Phys. B 40 (2007) 91.
[20] J. Stuhler, A. Griesmaier, S. Giovanazzi, P. Pedri, L. Santos, Phys. Rev. Lett. 95 (2005) 150406.
[21] N.N. Bogolubov, Lectures on Quantum Statistics, Vol. 2, Gordon and Breach, New York, 1970.
[22] V.I. Yukalov, E.P. Yukalova, J. Phys. A 29 (1996) 6429.
[23] R.K. Wangness, Phys. Rev. 98 (1955) 927.
[24] A. Abragam and M. Goldman, Nuclear Magnetism, Clarendon, Oxford, 1982.
[25] V.I. Yukalov, E.P. Yukalova, Phys. Rev. Lett. 88 (2002) 257601.
Figure Captions

Fig. 1. Population imbalance as a function of dimensionless time for the periodic reswitching of the tunneling between $\omega_1 = 0.1$ and $\omega_2 = 5$, with equal time intervals $\Delta t_1 = \Delta t_2 = 10$. The system parameters are $\gamma_1 = \gamma_2 = 1$ and $b = 0.5$. Initial conditions are $x_0 = 0.33$, $y_0 = 0.8$, $z_0 = 0.5$.

Fig. 2. Population imbalance as a function of dimensionless time for the periodic reswitching of the tunneling between $\omega_1 = 0.1$ and $\omega_2 = 1.1$, with unequal time intervals $\Delta t_1 = 10$ and $\Delta t_2 = 25$. Other parameters are as in Fig. 1.

Fig. 3. Population imbalance as a function of dimensionless time for a nonperiodic reswitching of the tunneling between $\omega_1 = 0.1$ and $\omega_2 = 1.1$, with varying time intervals $\Delta t_j$. Other parameters are as in Fig. 1.
Figure 1: Population imbalance as a function of dimensionless time for the periodic reswitching of the tunneling between $\omega_1 = 0.1$ and $\omega_2 = 5$, with equal time intervals $\Delta t_1 = \Delta t_2 = 10$. The system parameters are $\gamma_1 = \gamma_2 = 1$ and $b = 0.5$. Initial conditions are $x_0 = 0.33$, $y_0 = 0.8$, $z_0 = 0.5$. 
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