DYNAMICAL QUENCHING OF THE $\alpha^2$ DYNAMO

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Received 2002 January 20; accepted 2002 February 12

ABSTRACT

We present a two-scale approximation for the dynamics of a nonlinear $\alpha^2$ dynamo. Solutions of the resulting nonlinear equations agree with the numerical simulations of Brandenburg and show that $\alpha$ is quenched by the buildup of magnetic helicity at the forcing scale $1/k_2$ as the $\alpha$ effect transfers it from the large scale $1/k_1$ ($>1/k_2$). For times $t > (k_1/k_2)Re_{M,2}$ in eddy turnover units (where $Re_{M,2}$ is the magnetic Reynolds number of the forcing scale), $\alpha$ is limited resistively in the form predicted for the steady state case. However, for $t \ll Re_{M,2}$, $\alpha$ takes on its kinematic value independent of $Re_{M,2}$, allowing the production of large-scale magnetic energy equal to $k_1/k_2$ times equipartition. Thus, the dynamic theory of $\alpha$ predicts substantial “fast” growth of a large-scale field despite being “slow” at large times.

Subject headings: galaxies: magnetic fields — ISM: magnetic fields — methods: numerical — MHD — stars: magnetic fields — turbulence

1. INTRODUCTION

Large-scale magnetic fields are often interpreted in terms of the equations of mean-field magnetohydrodynamics (MHD; Krause & Rädler 1980):

$$\partial_t \mathbf{B} = \alpha \mathbf{V} \times \mathbf{B} + (\beta + \lambda)\mathbf{V}^2 \mathbf{B},$$

where $\mathbf{B}$ is the mean (or large-scale) magnetic field,

$$\lambda = \frac{\eta c^2}{4\pi}$$

is the magnetic diffusivity in terms of the resistivity $\eta$, and $\alpha$ and $\beta$ are parameters of the underlying MHD turbulence. Steenbeck, Krause, & Rädler (1966) showed that if the turbulence is isotropic and incompressible and the back-reaction of $\mathbf{B}$ is neglected,

$$\alpha = -\frac{1}{2} \tau (\mathbf{V} \cdot \mathbf{V} \times \mathbf{V}),$$

$$\beta = \frac{1}{2} \tau (\mathbf{V}^2).$$

Here, $\tau$ is a typical correlation time of the flow $\mathbf{V}$, and $\frac{1}{2} (\mathbf{V} \cdot \mathbf{V} \times \mathbf{V})$ is its kinetic helicity, a measure of the net handedness of cyclonic motions (Parker 1955, 1979); $\beta$ represents turbulent diffusion of $\mathbf{B}$. In this paper, the brackets and overbar represent spatial averages.

As $\mathbf{B}$ grows, it exerts a back-reaction on the turbulent flow, and equations (3) and (4) must be modified to account for this. A number of attempts to describe the corresponding saturation of $\alpha$, or “$\alpha$ quenching,” have been made. As part of a general study of homogeneous, isotropic, helical MHD turbulence, Pouquet, Frisch, & Léorat (1976, hereafter PFL) used the eddy-damped quasi-normal Markov (EDQNM) approximation to derive evolution equations for the spectra of kinetic energy, magnetic energy, kinetic helicity, and magnetic helicity (defined as $\langle A \cdot \mathbf{V} \times A \rangle/2$, with $A$ the vector potential). They then solved a number of initial-value problems for these spectra and found an $\alpha$ effect like that predicted by Steenbeck et al. (1966). By expanding in terms of a small dimensionless expansion parameter $a \ll 1$, they found that $\alpha$ appropriate for a field having a large scale $k^{-1}$ is

$$\alpha(k) = -\frac{1}{2} \int_{k/a}^{\infty} \theta_{kq}(H_q^V - H_q^C) dq,$$

where $H_q^V$ is the spectrum of the small-scale kinetic helicity

$$H_q^V = \frac{1}{2} (\mathbf{V} \cdot \mathbf{V} \times \mathbf{V})$$

$H_q^C$ is the spectrum of the small-scale current helicity

$$H_q^C = \frac{1}{2} (\mathbf{b} \cdot \mathbf{V} \times \mathbf{b})$$

(where the small-scale field $b$, like other magnetic fields in this paper, is in velocity units), and $\theta_{kq}$ is the relaxation time for the interaction of two wavenumbers $q$ and $q' \sim q$ to excite $k \ll q$. Equation (5) is appropriate for the case that the lower limit of $q, k/a$, is much larger than $k$, the wavenumber of the large-scale field. If one replaces $\theta_{kq}$ by $\tau$, the first term in equation (5) agrees with equation (3). However, the second term in equation (5) is new, and its physical significance was discussed in PFL. It will play an important role in what follows.

Gruzinov & Diamond (1994, hereafter GD, 1995, 1996) and Bhattacharjee & Yuan (1995, hereafter BY) recognized that the current helicity term in equations (5) and (7) is related to magnetic helicity, a conserved quantity in ideal MHD, and exploited this fact to find how $\alpha$ is quenched for a closed system when it has reached a steady state. In this paper, we also link the current helicity contribution to $\alpha$ with the equation for magnetic helicity evolution, but, in addition to considering a steady state, we solve the time-dependent problem. As we discuss, the results lead ultimately to different conclusions than those of GD and BY.

There is an important assumption built into our approach: we assume that the PFL current helicity con-
conclude in Appendix B). We implemented by Appendix A.) In § 2, we discuss the model of PFL and produce a two-scale simplification of their equations. (This is supplemented by Appendix A.) In § 3, we solve the resulting time-dependent equations for large-scale field growth and show that the results agree well the numerical simulations of Brandenburg (2001, hereafter B01). In § 4, we compare our results to the implications of previous α-quenching models (and supplement this in Appendix B). We conclude in § 5.

2. USING PFL IN A TWO-SCALE APPROXIMATION

PFL studied the spectra of kinetic energy

$$E^V = \frac{1}{2} \langle \mathbf{v}^2 \rangle,$$  

(8)

magnetic energy

$$E^M = \frac{1}{2} \langle \mathbf{B}^2 \rangle,$$  

(9)

kinetic helicity $H^V$ (eq. [6]), and magnetic helicity

$$H^M = \frac{1}{2} \langle \mathbf{A} \cdot \nabla \times \mathbf{A} \rangle = \frac{1}{2} \langle \mathbf{A} \cdot \mathbf{B} \rangle.$$  

(10)

(Note the factor of $\frac{1}{2}$ in eq. [10].) It is easy to show that the spectrum of current helicity (eq. [7]) is related to that of $H^M$ by

$$H_C^M = k^2 H_k^M.$$  

(11)

Therefore, the evolution of $H_C^M$, needed for evaluating $\alpha$ according to equation (5), is tied to that of $H_k^M$.

Magnetic helicity conservation (e.g., Woltjer 1958) will play a role in what follows. Following Moffatt (1978), we can use the induction equation ($c = 1$) in the form

$$\partial_t \mathbf{B} = -\nabla \times \mathbf{E} = \nabla \times (\mathbf{v} \times \mathbf{B} - \lambda \mathbf{V} \times \mathbf{B})$$  

(12)

to show that

$$\partial_t \left( \frac{1}{2} \mathbf{A} \cdot \mathbf{B} \right) = -\frac{1}{2} \nabla \cdot \left[ \mathbf{B} (\phi - \mathbf{v} \cdot \mathbf{A}) + \mathbf{v} (\mathbf{A} \cdot \mathbf{B}) \right] - \lambda \mathbf{B} \cdot \nabla \times \mathbf{B}.$$  

(13)

With appropriate boundary conditions on $\partial V$, the average of the divergence over the volume $V$ vanishes, so

$$\partial_t H^M = -2\lambda H^C,$$  

(14)

showing that if $\lambda = 0$, then $H^M$ is conserved. As shown by Moffatt (1978), this expresses the fact that the linkage between magnetic lines of force cannot change if they are frozen in the fluid. As we shall see, even though the total magnetic helicity is conserved if $\lambda = 0$, $\alpha$ causes it to flow from small scales to large scales.

Equations (3.2) and (3.4) of PFL are

$$\begin{align*}
\partial_t + 2\lambda k^2 E^M_k &= k \Gamma \left( E^V_k - E^M_k \right) \\
+ 2\alpha(k) H^C_k - 2\beta(k) k^2 E^M_k,
\end{align*}$$  

(15)

$$\begin{align*}
\partial_t + 2\lambda k^2 H^M_k &= \frac{\Gamma}{k} \left( H^C_k - H^M_k \right) \\
+ 2\alpha(k) E^M_k - 2\beta(k) H^C_k.
\end{align*}$$  

(16)

We have restored the Ohmic dissipation term $2\lambda k^2$, which appears in PFL’s Table 1 but is omitted in PFL’s equations (3.2) and (3.4). The quantity $\beta(k)$ is a turbulent diffusion transport coefficient, which we will need to prescribe later. We also have omitted the quantity $\Gamma_k$ of PFL, which is a first-order correction to $\Gamma_k$.

The numerical results of PFL show that if helical MHD turbulence is excited predominantly at a single wavenumber $k_2$ (the outer scale of the turbulence), a pulse of excitation moves toward smaller values (larger scales) in what is sometimes called an inverse cascade. B01 has shown that this inverse cascade is nonlocal in the sense that the excitation jumps from $k_2$ to $k_1 \ll k_2$, where $k_1$ is limited from below by the wavenumber associated with the box scale. Maron & Blackman (2002) showed that the presence of a two-peaked spectrum (with peaks at the forcing scale and the box scale) requires sufficient input kinetic helicity.

We are interested in applying equation (16) to the case $k = k_1$. Because, according to equation (5), $\alpha(k_1)$ is based on helicity at $k > k_1$ and $E^M(k_1)$ can be significant, the $\alpha$ effect pumps magnetic helicity from $k_2$ to $k_1$. As we will see later, if $\lambda$ is small, then magnetic-helicity conservation requires that an equal and opposite amount of helicity must be established at $k_2$. Because of equation (11), the last term of equation (16), $2\beta(k) H^C_k$, has the same qualitative effect as $2\lambda k^2 H^M_k$. The first term on the right is proportional to

$$\Gamma_k = \frac{1}{2} \int_0^k \theta_{k^2 q} E^M_q \, dq,$$  

(17)

which is essentially the square root of the magnetic energy in velocity units for $k < k_1$. In the numerical results of PFL, B01, and Maron & Blackman (2002), there is a peak at $k_1$, with little energy at $k < k_1$. Hence, we assume that the $\Gamma_k$ term is negligible. Note that $\alpha(k) = \frac{1}{2} \alpha^c k^3$ and $\beta(k) = \nu_k^V$ of PFL.

We may integrate equations (15) and (16) over $k$ and approximate the results by

$$\begin{align*}
\partial_t E^M_1 &= 2\alpha k^1 H^M_1 - 2\lambda \beta k^1 E^M_1, \\
\partial_t H^M_1 &= 2\alpha E^M_1 - 2\lambda \beta k^1 H^M_1,
\end{align*}$$  

(18)

where

$$E^M_1 = \int_{-k}^k E^M_q \, dq,$$  

(20)

eq., in effect setting $\alpha(k_1)$ and $\beta(k_1)$ equal to their values derived from contributions at $k = k_2$. (Note that $\alpha$ is dimensionally a speed and $\beta$ a diffusivity. For magnetic and kinetic spectra approximately Kolmogorov, the dominant contribution to both $\alpha$ and $\beta$ comes from the forcing scale.) This two-scale approach, wherein the forcing scale is equal to the scale at which the small-scale field is peaked, is justified only when the forcing is sufficiently helical (Maron & Blackman 2002).
It is reassuring that equations (18) and (19) are exactly the equations one gets from two-scale theory applied to the small-scale $k_2$ and the large-scale $k_1$ (Appendix A). In what follows, we will often use $E^M = B^2_1/2$, where $B_1$ is the field at scale $k_1^{-1}$. Note that $B_1$ can be used interchangeably with $B$.

In the same spirit, we can replace $\theta_{eq0}$ in equation (5) by a typical value $\tau$ related to the peak at $k_2$ to obtain

$$\alpha = -\frac{3}{2} \tau (H^V_1 - H^C_2).$$

(21)

3. DYNAMICAL QUENCHING AND COMPARISON TO NUMERICAL SIMULATIONS

Here, we show that the solutions of the equations in the two-scale formalism of the previous section agree well with the numerical results of B01. B01 studied the dynamo effect in a nearly incompressible conducting fluid with periodic boundary conditions; $B_1$ is allowed to grow at various wavenumbers $k$, consistent with the boundary conditions, as a result of the $\alpha$ effect, thus simulating a nonlinear $\alpha^2$ dynamo. The results of B01 are qualitatively similar to the numerical results of PFL, in that a pulse of excitation propagates to large scales. As B01 kept $H^C_2$ approximately constant by driving the MHD turbulence with a helical force, in the light of equation (21), equations (18) and (19) become

$$\partial_t E^M = 2(\alpha_0 + \frac{2}{3} \tau k^2_1 H^M_1) k^2_1 H^M - 2(\lambda + \beta) k^2_1 E^M,$$

$$\partial_t H^M_1 = 2(\alpha_0 + \frac{2}{3} \tau k^2_1 H^M_2) E^M - 2(\lambda + \beta) k^2_1 H^M_1,$$

where

$$\alpha_0 = -\frac{3}{2} \tau H^V_1 = const.$$

(24)

Remarkably, the nonlinear differential equations (22) and (23) have a force-free solution for $B_1$ in which

$$E^M_1 = k_1 H^M_1.\tag{25}$$

Thus, equations (23) and (24) reduce to one equation that $H_1$ must satisfy, namely,

$$\partial_t H^M_1 = 2k_1(\alpha_0 + \frac{2}{3} \tau k^2_1 H^M_2) H^M_1 - 2(\lambda + \beta) k^2_1 H^M_1.\tag{26}$$

To solve equation (26), we need to express $H^M_2$ in terms of $H^M_1$. To do this, we use the conservation of magnetic helicity (eq. (14)) in the form

$$\partial_t H^M_1 + \partial_t H^M_2 = -2\lambda (k^2_1 H^M_1 + k^2_2 H^M_2).\tag{27}$$

Equations (26) and (27) are the coupled equations in $H^M_1$ and $H^M_2$ that need to be solved. If $H^M_2$ is small, $H^M_1$, and hence the large-scale field $B_1$, grows exponentially, driven by the first term on the right-hand side of equation (26). The magnetic helicity conservation equation (27) shows that for small $\lambda$, growth of $H^M_2$ is not free but comes at the expense of growing $H^M_1$ with the opposite sign. This decreases the value of $\alpha$ in equation (26). This $\alpha$ quenching slows the growth of $H^M_2$, leading to a steady state when the right-hand side of equation (26) vanishes.

To solve equations (26) and (27), we rewrite them in dimensionless form. We define the dimensionless magnetic helicities $h_1 \equiv 2H^M_1 k_2 v_2^2$ and $h_2 \equiv 2H^M_2 k_2 v_2^2$ and write time in units of $1/k_2 v_2$. We also define $Re_M \equiv (v_2/k_1)/\lambda$. [Note that this definition of $Re_M$ is based on the forcing-scale rms velocity but on the large scale, $k_1^{-1}$. We will later employ a second magnetic Reynolds number $Re_M \equiv Re_M(k_1/k_2)$.] We also need a prescription for $\alpha_0$ and for $\beta$. We assume that the kinetic helicity is forced maximally and take $\tau = 2/k_2 v_2$, implying that $\alpha_0 = 2v_2/3$. Unfortunately, a rigorous prescription for $\beta$ in three dimensions is lacking, but as in B01, we will consider two cases, $\beta = \beta_0 \alpha/\alpha_0$ and $\beta = \beta_0 \equiv v_2/k_2$.

Using the above scalings, we can replace equations (26) and (27) with dimensionless equations given by

$$\partial_t h_1 = \frac{4}{3} \left(\frac{k_1}{k_2}\right)^2 h_1(1 + h_2)
- 2h_1 \left[ \frac{k_1}{k_2 Re_M} + \frac{k_2}{k_1^2} (1 + q_2 h_2) \right],$$

$$\partial_t h_2 = -\frac{4}{3} \left(\frac{k_1}{k_2}\right)^2 h_1(1 + h_2) + 2h_1 \frac{k_2^2}{k_1^2} (1 + q_2 h_2)
- \frac{2}{\text{Re}_M} \frac{h_2 k_2}{k_1},\tag{29}$$

where $q_2 = 0$ in the above equations corresponds to $\beta(t) = \beta_0$ = constant and $q_2 = 1$ corresponds to $\beta(t) = \alpha(t) = 0$. Solutions of these coupled equations are shown in Figures 1–4. The key parameters are $k_2/k_1$, $Re_M$, and $q_2$. In the figures, we have also compared these results to the empirical fits of numerical simulations in B01. We have taken $h_1(t = 0) = 10^{-3}$, but the sensitivity to $h_1(0)$ is only logarithmic (see eq. [35] below). In Figure 1, we have used $k_2/k_1 = 5$, following B01, and in Figure 2, we have used $k_2/k_1 = 20$.

In the figures, the solid curves represent our numerical solutions to equations (28) and (29), whereas the dotted curves represent the formula given in B01, which is an empirical fit to simulation data assuming that $\alpha$ and $\beta$ are prescribed according to equations (32) and (33) below. More explicitly, B01 found that the growth of $B^2$ was well described by the formula

$$\frac{B^2_1}{B^2_{1,0}} \left(1 - \frac{B^2_1}{B^2_{1,\text{sat}}}\right)^{1 - \left(\nu_0 k_1 - k_1^2/\lambda\right)} = e^{2(\nu_0 k_1 - k_1^2/\lambda)},$$

(30)

where $B_{1,0} = B_1(t = 0)$. This can be rewritten using the notation above as a dimensionless equation for $t$ in units of

![Figure 1](image-url)

FIG. 1.—Solution for $h_1(t), f_1 = 1, \text{and } q_2 = 1$. Here, $k_2/k_1 = 5$, and the three curves from left to right have $Re_M = 100, 250$, and 500, respectively. The dotted curves result from using eqs. (32) and (33) to quasi-empirically fit the simulations of B01 at late times, as discussed in the text.
The three curves from left to right have \( \text{Re}_M = 10^2, 10^4, \) and \( 10^6 \), respectively. The dotted curves result from using eqs. (32) and (33) to quasi-empirically fit the simulations of B01 at late times, as discussed in the text.

\[
t = \frac{k_2 \ln((h_1/h_{1,0})(1 - h_1k_2^2/k_2^3\text{Re}_M(k_1/k_2 - 2)/3))}{2k_1},
\]

where \( h_{1,0} \) is \( h_1(t=0) \). Note that equations (30) and (31) correspond to \( \alpha \) and \( \beta \) quenching of the form

\[
\alpha = \frac{\alpha_0}{1 + sB_1/v_2},
\]

\[
\beta = \frac{\beta_0}{1 + sB_1/v_2},
\]

where \( sB \sim \text{Re}_M(k_1/k_2)(2/3 - k_1/k_2) = \text{Re}_M(2/3 - k_1/k_2) \) and \( \text{Re}_M(2/3) = v_1/k_2 \).

Equations (32) and (33) are derived from those in B01 by rescaling equation (55) of B01 with our notation. It can also be shown directly that, up to terms of the order of \( 1/\text{Re}_M \), equation (31) is consistent with that derived by substituting equations (32) and (33) into equation (28) and solving for \( t \).

Note that in contrast to the suggestion of B01, it is actually the forcing-scale magnetic Reynolds number \( \text{Re}_{M^2} \) that plays a prominent role in these formulæ.

The solutions of equations (28) and (29) are subtle and interesting. Some insight can be gained by their sum,

\[
\beta_1h_1 + \beta_2h_2 = -\frac{2}{\text{Re}_M(1/k_2^2 + k_2/k_1^2)},
\]

which corresponds to equation (27), the conservation of total magnetic helicity. If we make the astrophysically relevant assumption that \( \text{Re}_M \gg 1 \), then for a significant period beginning from \( t = 0 \), the right-hand side of equation (27) is small for all \( h_1 \) and \( h_2 \). It follows that \( \beta_1(h_1 + h_2) \approx 0 \) and for \( h(t=0) = 0 \), this implies \( h_2 \approx -h_1 \). In this period, we can self-consistently ignore \( 1/\text{Re}_M \) in equation (28). If \( q_2 = 1 \), this phase ends when \( h_2 \to -1 \), so that \( h_1 \approx 1 \). This is manifested in Figure 3.

This kinematic phase precedes the asymptotic saturation of the dynamo investigated by other authors, in which all time derivatives vanish exactly. For this to happen, the right-hand side of equation (34) must vanish, which is equivalent to demanding that \( h_2 = -(k_2/k_1)^2h_1 \). Since the right-hand sides of equations (28) and (29) are proportional to \( 1 + h_2 \) when terms of the order of \( 1/\text{Re}_M \) are neglected, their vanishing requires that \( h_2 = -1 \) and therefore that \( h_1 = (k_2/k_1)^2 \). This is observed in Figures 1 and 2. The asymptotic saturation (when the field growth ceases) takes a time of the order of \( t_{\text{sat}} \sim \text{Re}_M/k_2k_1 \), which in astrophysics is often huge. Thus, although in principle it is correct that \( \alpha \) is resistively limited (as seen from our solutions in Figs. 4 and 5), as suggested by BY, GD, Vainshtein & Cattaneo (1992), and Cattaneo & Hughes (1996), this is less important for the field saturation strength than the fact that for a time \( t_{\text{kin}} \), the kinematic value of \( \alpha \) applies. The timescale \( t_{\text{kin}} \) is given by a few kinematic growth timescales for the \( \alpha^2 \) dynamo; more specifically,

\[
t_{\text{kin}} \sim \ln \left[ \frac{1}{h_1(0)} \left( \frac{k_2}{k_1} \right) \left( \frac{4}{3} \frac{2k_1}{k_2} \right) \right].
\]

For \( h_1(0) = 0.001 \), \( k_2/k_1 = 5 \) and \( t_{\text{kin}} \sim 37 \), as seen in Figure 3.

Note that \( t_{\text{kin}} \) is sensitive to \( k_2/k_1 \) and independent of \( \text{Re}_M \). Figure 3 shows that there is significant disagreement
in this regime with equation (32), but this formula was used in B01 only to model the regime $t > \text{Re}_M$, so the result is not unexpected. We can see from the solution for $\alpha$ itself that indeed our solutions do match equation (32) for $t > \text{Re}_M$ (Figs. 4 and 5). Figure 4 shows the difference in $\alpha$ along with equation (33) for the two values $\text{Re}_M = 10^2$ and $10^3$. Notice again the disagreement with the formula (eq. [32]) until $t = \text{Re}_M$ and agreement afterward. The emergence of this timescale is expected, as it marks the time at which the resistive term on the right-hand side of equation (28) becomes competitive with the terms involving $(1 + h_2)$. Asymptotic saturation does not occur until $t \sim t_{\text{sat}} = \text{Re}_M k_2/k_1$ as described above.

Finally, note that $q_2 = 0$ corresponds to $\beta = \beta_0$. In general, this leads to a lower value of $h_1$ in the asymptotic saturation phase because this enforces zero saturation of $\beta$, whereas there is still some saturation of $\alpha$ in this limit. (Note that $q_2 = 0$ corresponds to the case of GD discussed further in Appendix B.) For large $k_2/k_1$, the solutions of equations (28) and (29) are insensitive to $q_2 = 0$ or $q_2 = 1$. This is because the larger the $k_2/k_1$, the smaller the influence of the $q_2$ terms in equations (28) and (29). This is highlighted in Figure 6, where the result for $q_2 = 0$ is plotted with the B01 fit. This suggests that for large-scale separation, the magnetic-energy saturation is insensitive to the form of $\beta$ quenching. In real dynamos, however, magnetic flux and not just magnetic energy may be needed, so the insensitivity can be misleading because $\beta$ is needed to remove flux of the opposite sign. From the low $k_2/k_1$ cases, it is clear that $q_2 = 1$ is a better fit to the simulations of B01.

**4. IMPLICATIONS AND COMPARISON TO PREVIOUS WORK**

The physical picture of the quenching process just described is this: helical turbulence is forced at $k_2 (= 5$ in B01) and kept approximately constant by forcing. Hence, $\alpha_0 = -2\pi H_2^M/3 = \text{constant}$. If $H_2^M$, the magnetic helicity at $k_1$ (which reaches 1 here as a result of boundary conditions), is initially small, so that $2k_1^2 H_2^M/3 \ll |\alpha_0|$, equation (26) (or eq. [28]) shows that it will be exponentially amplified provided that the damping due to $\beta + \lambda$ does not overcome the $\alpha$ effect.

Initially, $\alpha = \alpha_0$, acting like a pump that moves magnetic helicity from $k_2$ to $k_1$ and driving the dynamo. This kinematic phase lasts until $t_{\text{kin}}$, as given by equation (35). Eventually, the growing $H_2^M$ results in a growing $H_2^M$ of opposite sign, which reduces $\alpha$ through $H_2^M$. The $\text{Re}_M$-dependent quenching kicks in at $t = t_{\text{kin}}$, but it is not until $t = \text{Re}_M$ that the asymptotic equations (32) and (33) are appropriate. Asymptotic saturation, defined by the time at which $B_1$ approaches its maximum possible value of $(k_2/k_1)^{1/2}v_2$, occurs at $t = t_{\text{sat}} = \text{Re}_M k_2/k_1$. (Note that $B_1$ approaches equipartition at a time $\sim \text{Re}_M/2$. This can be derived approximately from eqs. [28] and [29] by setting $h_1 = k_2/k_1$, $\partial h_1/\partial t \sim h_1/t$, and solving for $h_2$ and $t$.) For $t \geq \text{Re}_M$, our numerical solution, like the full numerical simulations of B01, is well fitted by the $\alpha$ in equation (32) with a corresponding $\beta$ of equation (33). Our two-scale approach is also consistent with the simulations of B01, where magnetic helicity was found to jump directly from $k_2$ to $k_1$ without filling in the intermediate wavenumbers.

The emergence of the timescale $t_{\text{kin}}$ is interesting because it shows how one can misinterpret the implications of the asymptotic quenching formula (32) and (33). These formulae are appropriate only for $t > \text{Re}_M$. The large-scale field actually grows kinematically up to a value $B_1 = (k_1/k_2)^{1/2}v_2$ by $t = t_{\text{kin}}$ and ultimately up to $B_1 \sim (k_2/k_1)^{1/2}v_2$ by $t = t_{\text{sat}}$. For large $\text{Re}_M$, these values of $B_1$ are both much larger than the quantity $v_2/\text{Re}_M^{1/2}$, which would have been inferred to be the saturation value if one assumed equations (32) and (33) were valid at all times.

Dynamical quenching or time-dependent approaches recognizing the current helicity as a contributor to $\alpha$ have been discussed elsewhere (Zeldovich, Ruzmaikin, & Sokoloff 1983; Kleeorin et al. 2000; Kleeorin, Rogachevskii, & Ruzmaikin 1995; Kleeorin & Rogachevskii 1999; see also Ji 1999; Ji & Prager 2001), but here we have specifically linked the PFL $\alpha$ correction to the helicity conservation in a simple two-scale approach. Other quenching studies for closed systems such as those by Cattaneo & Hughes (1996) and BY advocated values of $\alpha$ that are resistively limited and of a form in agreement with equation (32) but with the assumption of a steady $B_1$. Assuming equation (21) and using equations (26) and (27) in the steady state, their formulae can be derived easily. However, one must also have a prescription for $\beta$. If $\beta$ is proportional to $\alpha$, then formulae such as equations (32) and (33) emerge. If $\beta(t) \simeq \beta_0$, as in GD, then a formula for $\alpha$ without resistively limited quenching emerges (this requires a reinterpretation of their formulae; see
Appendix B). Additional work is needed to determine $\beta$ dynamically. On the other hand, we have shown that for large $k_2/k_1$, the dynamo quenching is largely insensitive to $\beta$.

An important point to reemphasize is that even when resistive-quenching formulae are found from steady state analyses, this does not necessarily reflect the saturation value of $B_1$. The fact that there exists a kinematic regime up until $t_{\text{fin}}$ means that by the time formulae such as equations (32) and (33) are valid, the field may have already grown substantially, as we have shown. That being said, all of our analysis here is for the growth of magnetic energy to saturation; all of our solutions agree with the numerical simulations of B01 for the regime of $t > Re_M$, where B01 showed that equation (32) fits the data.

5. CONCLUSION

We have shown that the evolution equations of PFL, together with their formula for $\alpha$, leads to dynamical $\alpha$ quenching from the $\alpha$-induced flow of magnetic helicity from small to large scales; the associated buildup of small-scale current helicity of the opposite sign eventually suppresses $\alpha$. This simple $\alpha^2$ dynamo process can be modeled using a two-scale formalism. We have identified a timescale $t_{\text{fin}}$ up to which the dynamo in a periodic box operates independently of $Re_M$ and grows to large values of the order of $(k_1/k_2)^{1/2} v_2$. At later times, the dynamo becomes slow. The dynamo coefficients become resistively limited, depending strongly on $Re_M$. Our solutions agree with the numerical simulations of B01.

We thank the ITP at University of California, Santa Barbara, and its participants for stimulating interactions during the spring 2000 workshop on astrophysical turbulence. In particular, we thank A. Brandenburg, S. Cowley, R. Kulsrud, J. Maron, and P. Diamond for discussions. We also warmly thank the participants of the memorable Virgin Gorda MHD turbulence Workshop of December 2001, namely, A. Brandenburg, R. Kulsrud, J. Maron, B. Mathaeus, and A. Pouquet, for the intense and extended discussions that have directly influenced the revised version of this paper. E. B. acknowledges support from DOE grant DE-FG02-00ER54600.

APPENDIX A

THE EQUIVALENCE OF PFL AND TWO-SCALE THEORY

Here, we show that equations (18) and (19) also follow from two-scale theory. Multiplying equation (1) by $B$ gives

$$\partial_t \left( \frac{1}{2} B^2 \right) = \alpha B \cdot \nabla \times B - (\lambda + \beta) B \cdot \nabla^2 B,$$

$$= \alpha B \cdot \nabla \times B + (\lambda + \beta) B \cdot \nabla \times \nabla \times B,$$

$$= \alpha B \cdot \nabla \times B - (\lambda + \beta) (\nabla \times B)^2,$$  \hspace{1cm} (A1)

where $\doteq$ means equal to within a divergence; from equation (13), it is the true equality for the indicated boundary conditions. Now, let $k^{-1}_1$ be the scale of $B$. Then,

$$B \cdot \nabla \times B = B_1 \cdot \nabla \times B_1$$

$$= 2H_1^C = 2k_1^2 H_1^M,$$

$$(\nabla \times B)^2 = 2k_1^2 E_1^M.$$  \hspace{1cm} (A2)

Thus, equation (A1) becomes

$$\partial_t E_1^M = 2\alpha k_1^2 H_1^M - 2(\lambda + \beta) k_1^2 E_1^M$$  \hspace{1cm} (A3)

in agreement with equation (18).

From equation (13),

$$\partial_t \left( \frac{1}{2} A \cdot B \right) \doteq - E \cdot B.$$  \hspace{1cm} (A4)

Since, from equations (1) and (12),

$$E = -\alpha B + (\lambda + \beta) \nabla \times B,$$

$$\partial_t \left( \frac{1}{2} A \cdot B \right) \doteq (\lambda + \beta) B \cdot \nabla \times B,$$  \hspace{1cm} (A5)

or

$$\partial_t H_1^M \doteq 2\alpha E_1^M - 2(\lambda + \beta) k_1^2 H_1^M,$$  \hspace{1cm} (A6)

in agreement with equation (19). We conclude that if the Alfvén effect is omitted and the field is concentrated at $k_1$ and $k_2$, equations (2.32) and (2.34) of PFL are equivalent to the two-scale approximation.
APPENDIX B

REINTERPRETATION OF THE GD QUenching FORMULA

GD were the first to use the conservation of magnetic helicity to obtain a formula for $\alpha$ in a closed system in a steady state. Their conclusion that $\alpha$ saturates when $B_1$ is of the order of $\text{Re}_M^{-1/2}v_2$ stimulated the present investigation, for if correct, it would imply that the $\alpha$ effect in the Galaxy would be useless in explaining any fields larger than $10^{-16}$ G, as $\text{Re}_M \approx 10^{20}$ in the interstellar medium. Taking their formula for $\alpha$, we show that their result was misinterpreted.

GD did not assume that $B_1$ is constant in space, so we can use equation (19) with $k_1 \neq 0$. Because they assumed a steady state, we put $\partial_t = 0$, so that

$$\alpha E_1^M = \lambda H_1^C + \beta H_1^C,$$

which agrees with equation (9) in GD (except for a sign error in the latter, which is not propagated in the rest of their paper).

From equations (21) and (24), we have

$$\alpha = \alpha_0 + \frac{2}{3} \tau H_2^C,$$

so

$$H_2^C = \frac{3}{2\tau} (\alpha - \alpha_0).$$

If $t > t_{\text{sat}}$, then

$$H_1^C = -H_2^C = -\frac{3}{2\tau} (\alpha - \alpha_0).$$

If we substitute equation (B4) into the first term on the right-hand side of equation (B1), we get

$$\alpha = \frac{\alpha_0 + \beta_0 R \cdot \nabla \times R}{1 + \text{Re}_M^2},$$

where

$$R = \left( \frac{\tau}{3\lambda} \right)^{1/2} B_1 = \frac{\sqrt{2} \text{Re}_M^{1/2}}{v_2} B_1,$$

and we have used $\tau = 2/k_2v_2$ and $\lambda = \beta_0/\text{Re}_M = v_2/3k_2\text{Re}_M$. Following GD, we have put $\beta = \beta_0$, where $\beta_0$ is a constant.

Equation (B5) is the same as equation (4) of GD, so their work is consistent with this paper for $t > t_{\text{sat}}$. However, GD went on to conclude that $\alpha$ saturates when $B_1 \sim \text{Re}_M^{-1/2}v_2$, apparently assuming from equation (B5) that the criterion for saturation is $R \sim 1$.

However, one must be careful about the second term in equation (B5). Recall that it is proportional to $H_1^C$, which is constrained by equation (B4). When one substitutes $H_1^C$ from equation (B4) into the second term in equation (B1), one finds that

$$\alpha = \frac{\alpha_0}{1 + R^2/(1 + \text{Re}_M^2)} = \frac{\alpha_0}{1 + [\text{Re}_M/(1 + \text{Re}_M)] (B_1/v_2)^2},$$

so that $\alpha$ saturates not at $B_1 \sim \text{Re}_M^{-1/2}v_2$, but at

$$B_{1,\text{sat}} \sim \left( \frac{1 + \text{Re}_M^2}{\text{Re}_M} \right)^{1/2} v_2,$$

or, as $\text{Re}_M$ gets large

$$B_{1,\text{sat}} \sim v_2,$$

rather than being resistively limited as GD suggest.

However, if instead of $\beta = \beta_0$ we employ $\beta = \beta_0/\alpha_0$ (or $q_2 = 1$ in eqs. [28] and [29]), then it can be shown analytically that the resistively limited asymptotic equations (32) and (33) are correct.

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