**ABSTRACT**

Radiative transfer is a notoriously difficult and computationally demanding problem. Yet, it is an indispensable ingredient in nearly all astrophysical and cosmological simulations. Choosing an appropriate discretization scheme is a crucial part of the simulation, since it not only determines the direct memory cost of the model but also largely determines the computational cost and the achievable accuracy. In this paper, we show how an appropriate choice of directional discretization scheme as well as spatial model mesh can help alleviate the computational cost, while largely retaining the accuracy. First, we discuss the adaptive ray-tracing scheme implemented in our 3D radiative transfer library \textsc{Magritte}, that adapts the rays to the spatial mesh and uses a hierarchical directional discretization based on HEALPix. Second, we demonstrate how the free and open-source software library Gmsh can be used to generate high quality meshes that can be easily tailored for \textsc{Magritte}. In particular, we show how the local element size distribution of the mesh can be used to optimise the sampling of both analytically and numerically defined models. Furthermore, we show that when using the output of hydrodynamics simulations as input for a radiative transfer simulation, the number of elements in the input model can often be reduced by an order of magnitude, without significant loss of accuracy in the radiation field. We demonstrate this for two models based on a hierarchical octree mesh resulting from adaptive mesh refinement (AMR), as well as two models based on smoothed-particle hydrodynamics (SPH) data.

**Key words:** radiative transfer, methods: numerical, software: development

\section*{1 INTRODUCTION}

Radiative transfer plays a critical role in various astrophysical and cosmological processes. Not only does it determine what we can and cannot observe, it also actively alters the physical and chemical conditions throughout the Universe through radiative pressure, heating and cooling, and via various photo-ionization and photo-dissociation reactions. Although it is computationally challenging, it is crucial to be able to accurately account for all relevant radiative processes in the wealth of modern astrophysical and cosmological simulations.

The first step in every computer simulation is finding a proper representation for the simulated objects in the model. Often, this comes down to finding an appropriate discretization for all physical quantities. This is an essential step, since the number of elements in the discretization will not only determine the direct memory cost of the model but also its computational cost and ultimately the maximal achievable accuracy of the simulation. Finding an appropriate discretization scheme is thus a question of optimizing the trade-off between accuracy and computational cost.

Well-established radiation transport solvers such as \textsc{OpenMC}\footnote{See also openmc.org/}.
In further contrast to the industrial radiative transfer applications, in astrophysics and cosmology we are almost never interested in a highly accurate solution of a specific model, but e.g. rather in understanding the more general driving mechanisms that govern a set of models. For instance, where in nuclear engineering it is crucial to be able to accurately describe the effect of adding a single fuel rod in a reactor, it is far less important in cosmology, for instance, to be able to describe the effect of one additional filament in the cosmic web. This allows us to optimize our models more aggressively, retaining only the essential features under investigation.

This is the second paper in a series on MAGRITTE\(^2\): a modern open-source software library for 3D radiative transfer (De Ceuster et al. 2019). MAGRITTE is a deterministic ray-tracer that uses a formal solver to compute the radiation field along a fixed set of rays (i.e. directions) through the model. In this paper, we discuss our implementation of an improved ray-tracing scheme, that adapts to the spatial discretization and uses a hierarchical directional discretization based on HEALPix\(^3\) (Gorski et al. 2005). Furthermore, we demonstrate how the free and open-source software library Gmsh\(^4\) (Geuzaine & Remacle 2009) can be used to efficiently generate high quality meshes that can be used to construct radiative transfer models in MAGRITTE. Finally, we present and demonstrate a simple algorithm that can be used to reduce the size (i.e. the number of elements) of an input model for radiative transfer simulations by an order of magnitude without a significant loss of accuracy.

The structure of this paper is as follows. In Section 2 we discuss the implementation of our adaptive ray-tracing scheme and explain how we use Gmsh to efficiently generate meshes for input models. Further, in Section 3, we demonstrate the mesh generation process with an analytic Archimedean spiral model and four snapshots of hydrodynamics simulations, two of which using AMR and two using SPH. Finally, in Section 4, we discuss our results and we present our conclusions in Section 5.

2 METHODS

2.1 Adaptive ray-tracing

Various methods have been devised to compute the radiation field in a given medium by solving the radiative transfer equation. MAGRITTE uses a long-characteristic formal solver and thus solves the radiative transfer equation along a set of predefined rays through the model. The ray-tracer follows a straight line through the model and gathers the emissivities and opacities along the ray that are then used to solve the radiative transfer equation. By stepping from one point to the next it indirectly determines the optical depth increments, i.e. the step size, in the discretized transfer equation used by the solver. The first version of MAGRITTE employed a second-order radiative transfer solver (Feautrier 1964; De Ceuster et al. 2019), but the current version also provides the more accurate (Hermitian) fourth-order scheme by Auer (1976), with second and third-order boundary conditions respectively (Auer 1967). In the following sections, we describe three ways in which the rays are adapted to further improve the accuracy of the solver.

2.1.1 Adaptive velocity sampling

Since Doppler shifts can cause significant variations in emissivity and opacity within a frequency bin, it is crucial to carefully sample the velocities encountered along a ray. This becomes even more important for line radiative transfer, where significant changes occur in particularly narrow frequency ranges. In De Ceuster et al. (2019), it was already discussed how MAGRITTE accounts for this by interpolating the velocity along a ray between two points if its change is too large. In our new version of the solver, we extended this by employing a similar method to the optical depth increments encountered along a ray.

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\(^2\) See also www.cea.fr/nucleaire/tripoli-4/.
\(^3\) See also laws.lanl.gov/vhosts/mcnp.lanl.gov/index.shtml.
\(^4\) See also github.com/Magritte-code.
\(^6\) See also healpix.sourceforge.net.
\(^5\) See also gmsh.info.
2.1.2 Adaptive optical depth increments

In MAGRITTE, the radiative transfer equation is solved in its second-order or Feautrier form (Feautrier 1964),

\[
1 - \frac{d^2}{dt^2} u(x, \hat{n}) = S(x, \hat{n}),
\]

where \( u(x, \hat{n}) \) is the mean intensity along direction \( \hat{n} \), and \( S(x, \hat{n}) \) and \( \tau(x, \hat{n}) \) are respectively the effective source function and effective optical depth, which can be derived from the local emissivities and opacities (see De Ceuster et al. 2019, for more details). The optical depth is thus the relevant dependent variable in the differential equation. Assuming a proper sampling of the emissivity and opacity, the discretization error will thus be determined mainly by the size of the optical depth increments. This means that a model mesh can perfectly sample the relevant optical data, but nevertheless produce a large discretization error. To resolve this, we adapt the optical depth increments as they are computed. In particular, we divide the interval on the ray between the (projected) points \( n \) and \( n + 1 \) in \( n_{\text{inter}} \) equal parts and linearly interpolate the emissivities and opacities on the sub-intervals. By defining the number of interpolations as

\[
n_{\text{inter}} = \left\lceil \frac{\Delta \chi}{\Delta \tau_{\text{max}}} \right\rceil,
\]

where \( \chi \) denotes the opacity and \( \Delta \tau \) the distance increment along the ray, we can ensure that the optical depth increments along the ray are always smaller than a predefined value \( \tau_{\text{max}} \).

It is important to note that the adaptive velocity sampling is still a separate process that happens before the adaptive optical depth increments are computed and that the former cannot be included in or replaced by the latter. For example, a large change in velocity along a ray which Doppler shifts a line from one wing to the other will not result in a particularly large optical depth increment, but will leave the line unaccounted for, increasing the error on the computed radiation field.

2.1.3 Adaptive directional discretization

Many applications of radiative transfer computations require directional integrals over the radiation field to compute, for instance, the radiative heating or cooling, or to compute the radiative pressure. In those cases also the directions of the rays need to be discretized. Given a function \( f(x, \hat{n}) \), the directional integral is discretized as

\[
\int d\Omega f(x, \hat{n}) \rightarrow \sum_{\hat{n} \in R_i} w_{i,r} f_{i,r},
\]

where the \( i \) and \( r \) indicate the point and ray index respectively, and \( R_i \) is the set of rays originating from point \( i \). The main difficulty is to determine this set of rays \( (R_i) \) for each point. Once these are known, the corresponding weights \( w_{i,r} \) can readily be computed.

Assuming that the points in the mesh properly sample the relevant distributions in the model, for each point in the mesh, the directions of the other points with respect to that point will properly sample the relevant directions for that point. Therefore, ideally, the discretization of the directions for a point in the mesh would follow the distribution of the directions of the other points with respect to that point. Furthermore, since this discretization has to be generated for every point in the mesh, the procedure cannot be too computationally demanding.

Therefore, we opted for a structured adaptively refining scheme based on HEALPix (Gorski et al. 2005). Given a level of refinement, \( l \), HEALPix provides a discretization of the unit sphere in \( 12 \times 4^l \) uniformly distributed pixels of equal area. By stitching together parts of these uniform HEALPix discretizations with different levels of refinement, we can obtain a locally refined directional discretization adapted to the point density in the mesh.

We start with a HEALPix discretization with a minimal level of refinement, \( l_{\text{min}} \), and refine the pixels according to the distribution of the directions of the other points, until a certain maximal level of refinement, \( l_{\text{max}} \), is reached. Figure 1 shows a Cartesian projection of an example of the resulting discretization of the unit sphere and the end points of the corresponding direction vectors. Since MAGRITTE considers pairs of antipodal rays, there is an antipodal symmetry in the directional discretisation.

To obtain the distribution of the directions of the other points with respect to the point under consideration, we draw a uniformly distributed sample of 10000 points from the mesh and record the HEALPix pixels they belong to for each level of refinement from \( l_{\text{min}} \) up to and including \( l_{\text{max}} \). In practice, we only need to compute the distribution at the highest level of refinement. The results for the other levels can be obtained by downgrading the resulting map, leveraging HEALPix’ nested ordering scheme (Gorski et al. 2005).

To decide which pixels to refine, we order them according to the number of directions of other points belonging to them, and only refine the top half. To ensure that the same number of rays is traced for each point we fix the number of pixels that is refined at each level \( l \) to be \( 6 \times 2^l \). This implies that at each level (except \( l_{\text{max}} \)), half of the pixels is further refined.

Since HEALPix partitions the unit sphere in pixels of equal area, the corresponding weight for a pixel obtained with a level of refinement \( l(i, r) \) is the inverse of the number of pixels at that level

\[
w_{i,r} = \frac{1}{12 \times 4^{l(i, r)}},
\]

where the level of refinement depends on the originating point \( i \) and the direction of the ray \( r \). This adaptive refinement scheme allows to sample the directions better with fewer rays.

2.2 Mesh constructing and reduction

Over the years, many different algorithms have been devised to partition a given volume for use in a computer, see for instance the
free and open-source meshing library called software libraries. For all models in this paper we have used the of these algorithms have furthermore been implemented in various

Delaunay tetrahedralization. Tousethese meshes in M/a.pc/g.pc/r.pc/i.pc/t.pc/t.pc/e.pc

count by George & Frey (2008), and the references therein. Many

t Gestures that are difficult to describe with analytic models, it is

Although many astrophysical objects are characterised by irregular

2.2.1 Meshing analytic models

Although many astrophysical objects are characterised by irregular

Figure 2. Ray traced through the Voronoi mesh (top) and its topologically
dual Delaunay mesh (bottom). Note that every distance increment along the
ray is the projection of the edge of the Delaunay triangle connecting the

by Thompson et al. (1998), the more recent account by George & Frey (2008), and the references therein. Many

It requires a point cloud and nearest neighbour information), the De-

M/a.pc/g.pc/r.pc/i.pc/t.pc/t.pc/e.pc

once an appropriate background mesh is created with the de-

$$G f(x) = \max_{\hat{n} \in S^2} \left\{ \hat{n} \cdot \nabla f(x) / f(x) \right\} = \| \nabla \log f(x) \|. \quad (5)$$

This can be used in a map to obtain the desired element size dis-

2.2.2 Re-meshing existing models

Since radiative transfer simulations are often only a component in a

$$G f_i = \max \left\{ \frac{f_i - f_n}{f_i + f_n}, \text{ for each neighbour } n \text{ of point } i \right\}. \quad (6)$$

Once an appropriate background mesh is created with the de-

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3.1 Meshing analytic models

As an example of an analytically defined model, we consider a stellar wind described by an Archimedean spiral with generic parameters, following Homan et al. (2015). The full details of the model together with a notebook implementation can be found online. Figure 3 shows two slices through the model, showing the density distribution as well as the underlying mesh generated with Gmsh using the method described in Section 2.2.1. In this example, the density was used as a tracer function to determine the desired local element size distribution, with a minimum desired element size of ℓ_{\text{min}} = 15 AU and a maximum desired element size of ℓ_{\text{max}} = 50 AU. The regular Cartesian background mesh is defined in a cubic box of size (1200 AU)^3 and a resolution of 100^3 elements. The resulting mesh consists of 49 347 points, a modest number considering the relatively complex morphology. One can easily obtain even sparser meshes, either by increasing the minimum or maximum desired element sizes, or by applying the method presented in Section 2.2.2. The latter technique will be demonstrated in the next section.

3.2 Re-meshing existing models

In the following we will consider four specific examples of how hydrodynamics models can be reduced or coarsened before they are used as input for a radiative transfer solver. We consider two models based on an hierarchical octree mesh resulting from adaptive mesh refinement (AMR), and two models based on smoothed-particle hydrodynamics (SPH) simulation data.

3.2.1 AMR models

The idea of an octree discretization of a 3D space is to locally subdivide an initial cubic cell into eight sub-cells until the desired local mesh size is achieved. Hydrodynamics solvers using adaptive mesh refinement (AMR) often use an octree as underlying geometric data structure. We consider late snapshots of two hydrodynamics models of the intricate stellar outflow produced by a mass-losing asymptotic giant branch (AGB) star as it is perturbed by a companion, modelled...
using MPI-AMRVAC\textsuperscript{8} (Xia et al. 2018). We used the code with a Cartesian mesh and allowed for 8 levels of adaptive refinement.

The first example, shown in Figure 4, contains a relatively regular spiral outflow. Reducing the model using GMSH and the algorithm described in Section 2.2.2, the resulting reduced mesh contains about 10.2 times fewer points than the original one. This results in a speedup of 18.3 for the computation of the radiation field. In the second example, shown in Figure 5, we consider a more erratic spiral outflow. This leads to a reduced mesh containing 8.4 times fewer points than the original one, which results in a 12.6x speedup. The parameters of the reduction algorithm can be found in Table 1 and properties of the original and reduced meshes are summarised in Table 2.

In both examples, one can see in the reduced meshes some artefacts of the levels of refinement in the original meshes. This is due to the fact that the desired element sizes in the algorithm are determined by the original element sizes.

To quantify the quality of the reduced meshes we compute the radiation field for both the original and reduced models using MAGRITTE and calculate the absolute relative difference between the results. The solution on the reduced mesh can be mapped to the original by barycentric interpolation on the reduced mesh to the results. The solution on the reduced mesh can be mapped to the radiation field for both the original and reduced models using determined by the original element sizes.

due to the fact that the desired element sizes in the algorithm are summarized in Table 2.

In Table 1 and properties of the original and reduced meshes are summarised in Table 2.

The first example, shown in Figure 4, contains a relatively regular spiral outflow. Reducing the model using GMSH and the algorithm described in Section 2.2.2, the resulting reduced mesh contains 11.1 times fewer points than the original. This results in a speedup of 35.0 for the computation of the radiation field. In the second example, shown in Figure 11, we consider a much more erratic spiral outflow. Despite the complex morphology, the reduced mesh still contains 10.7 times fewer points than the original, resulting in a 34.1x speedup. The parameters of the reduction algorithm can be found in Table 1 and the properties of the original and reduced meshes are summarised in Table 2.

The quality of the meshes can again be quantified by comparing the results of a radiative transfer computation using MAGRITTE between the original and reduced meshes. The results of the reduced mesh can be mapped to the original one in the same way as with the AMR models. Figure 6 shows the cumulative distribution of the relative errors between the original and reduced meshes. The SPH erratic model shows a relative error below 10% for about 90% of its points. This can be attributed the fact that the original SPH model already had a higher sampling that follows the morphology more closely. Since the desired element sizes are based on this sampling, this will result in higher quality meshes.

The point cloud structure of an SPH data set maps naturally to MAGRITTE’s internal geometric data structure. However, it should be noted that in this way we do not account for the smoothing kernels.

Figures 7 and 8 show a comparison between a dense regular directional discretization containing $12 \times 2^8 = 3072$ rays and our adaptive scheme containing 552 rays for a point half way along the $\hat{z}$-axis looking down on the $xy$-plane of the AMR models. The adaptive scheme allows for 3 levels of refinement from $l_{\text{min}} = 1$ up until $l_{\text{max}} = 4$. Although the result on the coarser adaptive discretization clearly shows some differences with the finer regular one, the overall relative errors are limited, as can be seen from the cumulative distribution of the relative errors between the dense regular and adaptive models shown in Figure 9. More than 60% of all the rays originating from all the points show a relative error below 10%, and more than 80% show a relative error below 20%. Note also here that about 20% of the points have a relative error below $10^{-3}$, which can be attributed to the fact that the maximally refined directions in the adaptive scheme have the same order and hence exactly the same rays as in the dense regular scheme, yielding exactly the same results and a negligible error.

\textsuperscript{8} See also amrvac.org.

\textsuperscript{9} See also phantomsph.bitbucket.io.

3.2.2 SPH models

In smoothed-particle hydrodynamics (SPH) simulations, rather than defining the physical quantities on a mesh, the model is described by a number of particles with definite properties and smoothing kernels describing their proliferation. We again consider late snapshots of two hydrodynamics models of the intricate stellar outflow produced by a mass-losing asymptotic giant branch (AGB) star as it is perturbed by a companion, this time modelled using the smoothed-particle hydrodynamics code PHANTOM\textsuperscript{9} (Price et al. 2018).

The first example, shown in Figure 10, describes a very regular spiral outflow. Reducing the model using GMSH and the algorithm described in Section 2.2.2, the resulting reduced mesh contains 11.1 times fewer points than the original. This results in a speedup of 35.0 for the computation of the radiation field. In the second example, shown in Figure 11, we consider a much more erratic spiral outflow. Despite the complex morphology, the reduced mesh still contains 10.7 times fewer points than the original, resulting in a 34.1x speedup. The parameters of the reduction algorithm can be found in Table 1 and the properties of the original and reduced meshes are summarised in Table 2.

The quality of the meshes can again be quantified by comparing the results of a radiative transfer computation using MAGRITTE between the original and reduced meshes. The results of the reduced mesh can be mapped to the original one in the same way as with the AMR models. Figure 6 shows the cumulative distribution of the relative errors between the original and reduced meshes. The SPH erratic model shows a relative error below 10% for about 90% of its points. This can be attributed the fact that the original SPH model already had a higher sampling that follows the morphology more closely. Since the desired element sizes are based on this sampling, this will result in higher quality meshes.

The point cloud structure of an SPH data set maps naturally to MAGRITTE’s internal geometric data structure. However, it should be noted that in this way we do not account for the smoothing kernels.

Figures 12 and 13 show a comparison between a dense regular directional discretization containing $12 \times 2^8 = 3072$ rays and our adaptive scheme containing 552 rays for a point half way along the $\hat{z}$-axis looking down on the $xy$-plane of the SPH models. The adaptive scheme allows for 3 levels of refinement from $l_{\text{min}} = 1$ up until $l_{\text{max}} = 4$. In accordance with the results for the AMR models, the result for the SPH models on the coarser adaptive discretization clearly shows some differences with the finer regular one, while the overall relative errors are again limited, as can be seen from the cumulative distribution of the relative errors between the dense regular and adaptive models shown in Figure 9. More than 60% of all the rays originating from all the points show a relative error below 10%, and more than 80% show a relative error below 20%. Note also here that about 20% of the points have a relative error below $10^{-3}$, which can be attributed to the fact that the maximally refined directions in the adaptive scheme have the same order and hence exactly the same rays as in the dense regular scheme, yielding exactly the same results and a negligible error.
Figure 4. Comparison between the original and reduced model mesh for the octree version of the regular spiral model. The properties of the meshes can be found in Table 2. The top row shows slices through the centre along the $x y$-plane and the bottom row shows slices along the $x z$-plane. The relative error in the rightmost column is computed as the average over all directions and frequency bins of the absolute relative difference between the radiation field computed on the original mesh and the radiation field computed on the reduced mesh when interpolated to the original.

Figure 5. Comparison between the original and reduced model mesh for the octree version of the erratic spiral model. The properties of the meshes can be found in Table 2. The top row shows slices through the centre along the $x y$-plane and the bottom row shows slices along the $x z$-plane. The relative error in the rightmost column is computed as the average over all directions and frequency bins of the absolute relative difference between the radiation field computed on the original mesh and the radiation field computed on the reduced mesh when interpolated to the original.
DISCUSSION

4.1 Adaptive ray-tracing

We should point out that our approach is quite different from Abel & Wandelt (2002), who originally coined the term “adaptive ray-tracing” in the context of radiative transfer in cosmological simulations. Their idea, to split rays as they reach further away from their origin to obtain a more constant volume coverage, is ideal for direct solvers or short-characteristics methods but would be difficult to implement in our second-order solver. Our approach is more related to adaptive mesh refinement (AMR) methods, but applied on-the-fly along the ray and in the directional discretization.

4.2 Interpolating between meshes

The choice of interpolation method between coarser and finer meshes is crucial in the determination of the error caused by the mesh reduction. In this paper we used a barycentric scheme implemented in scipy’s LinearNDInterpolator to interpolate between coarser and finer meshes (Virtanen et al. 2020). However, using for instance a more primitive mapping to nearest neighbours yielded significantly worse results, increasing the relative errors by about a factor 4. Higher order schemes might provide better results but would incur a higher computational cost. Since the barycentric scheme is already quite expensive (it requires roughly about 10% of the time to generate a mesh reduction) it appears to be the most suitable approach for now.

4.3 Model symmetries, 1D and 2D ray-tracers

When a model possesses a symmetry, it is most advantageous to use a solver that can leverage that symmetry. For instance, a spherically symmetric model can be solved most efficiently using a solver that is effectively 1D. Similarly, models that possess a cylindrical symmetry, such as disks and bi-polar outflows, can be solved most efficiently using a solver that is effectively 2D. Nevertheless, we tried to generate 3D meshes for models possessing a symmetry (spherical or cylindrical), to see if we could use our methods to generate relatively sparse 3D meshes for these effectively lower dimensional models. However, since Magritte internally uses a Cartesian co-ordinate system, it turned out that each of these models required an objectionable amount of points to properly represent these models in 3D. To accommodate that, we equipped Magritte with a dedicated 1D (spherical symmetric) and 2D (cylindrical symmetric) ray-tracer.

The methods outlined in this paper can readily be applied to generate lower dimensional (1D or 2D) models that can now also be processed with Magritte.

4.4 Future work

Now we have a set of methods at our disposal that allow us to easily generate meshes tailored to radiative transfer calculations, we can leverage these techniques, for instance, to develop multi-physics multi-grid methods for the iterative solvers in Magritte that aim to find self-consistent solutions for the different physical processes in the model. A first step in this direction will be the implementation of a multi-grid radiative transfer solver in Magritte.

Currently, mesh generation is a separate pre-processing step to the radiative transfer simulation. However, in order to effectively leverage the methods presented here, for instance in coupled radiation-hydrodynamics simulations, mesh generation should happen on-the-fly and preferably in a way that is aware of the evolution of the mesh.

5 CONCLUSION

Radiative transfer models are a crucial but very computationally demanding component of almost all astrophysical and cosmological simulations. In this paper, we demonstrate how the choice of both the directional and spatial discretization scheme can help alleviate the computational cost. First, we presented the improved adaptive ray-tracing scheme implemented in the 3D radiative transfer library Magritte, that uses an adaptive hierarchical scheme to discretize directions based on HEALPix (Gorski et al. 2005). Second, we demonstrated how the free and open-source software library Gmsi (Geuzaine & Remacle 2009) can be used to generate sparse meshes, even for morphologically complex models, that are ideally suited for radiative transfer simulations. Furthermore, we proposed two simple algorithms, one for analytically and one for numerically defined models, which can extract a desired mesh element size distribution from a model that will result in a sparse mesh, while preserving a proper sampling of key model features. Since typically the output of hydrodynamics models is used as input for radiative transfer simulations, we applied these algorithms to snapshots of several hydrodynamic models and showed that the number of elements can be reduced by an order of magnitude, without a significant loss of accuracy in the computation of the radiation field. As a result, the radiation field on the reduced meshes can be computed more than an order of magnitude faster. This reduced computational and memory cost can either be used to speedup the computation or it can be invested to increase the accuracy by refining the mesh in critical locations. The examples included both models based on an hierarchical octree mesh resulting from adaptive mesh refinements, as well as smoothed-particle hydrodynamics data. We conclude that carefully constructing an appropriate directional and spatial discretization scheme, using the methods described above, can significantly decrease the computational cost of radiative transfer simulations and make feasible simulations that would otherwise be intractable.

Figure 6. Cumulative distribution function of the relative errors with respect to the original meshes for the different models, computed for 1000 bins. The properties of the different meshes can be found in Table 2.
Figure 7. Comparison between a regular and an adaptive discretization of the directions for a point located at \((x, y, z) = (0, 0, 42)\) AU in the regular spiral AMR model. The point and rotation are chosen such that the viewing angle resembles the slice in the top row of Figure 4. Since the mean intensity along a ray \((u)\) is symmetric, each plot shows only half of a Cartesian projection.

Figure 8. Comparison between a regular and an adaptive discretization of the directions for a point located at \((x, y, z) = (0, 0, 6)\) AU in the erratic spiral AMR model. The point and rotation are chosen such that the viewing angle resembles the slice in the top row of Figure 5. Since the mean intensity along a ray \((u)\) is symmetric, each plot shows only half of a Cartesian projection.

Figure 9. Cumulative distribution function of the relative errors of the adaptively ray-traced models with respect to regular ones, computed for 1000 bins. The reduced meshes (see Table 2), were used as spatial discretization.

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Figure 10. Comparison between the original and reduced model mesh for the SPH version of the regular spiral model. The properties of the meshes can be found in Table 2. The top row shows slices through the centre along the $x,y$-plane and the bottom row shows slices along the $x,z$-plane. The relative error in the rightmost column is computed as the average over all directions and frequency bins of the absolute relative difference between the radiation field computed on the original mesh and the radiation field computed on the reduced mesh when interpolated to the original.

Figure 11. Comparison between the original and reduced model mesh for the SPH version of the erratic spiral model. The properties of the meshes can be found in Table 2. The top row shows slices through the centre along the $x,y$-plane and the bottom row shows slices along the $x,z$-plane. The relative error in the rightmost column is computed as the average over all directions and frequency bins of the absolute relative difference between the radiation field computed on the original mesh and the radiation field computed on the reduced mesh when interpolated to the original.
Figure 12. Comparison between a regular and an adaptive discretization of the directions for a point located at \((x, y, z) = (0, 0, 81)\) AU in the regular spiral SPH model. The point and rotation are chosen such that the viewing angle resembles the slice in the top row of Figure 10. Since the mean intensity along a ray \((u)\) is symmetric, each plot shows only half of a Cartesian projection.

Figure 13. Comparison between a regular and an adaptive discretization of the directions for a point located at \((x, y, z) = (0, 0, 18)\) AU in the erratic spiral SPH model. The point and rotation are chosen such that the viewing angle resembles the slice in the top row of Figure 11. Since the mean intensity along a ray \((u)\) is symmetric, each plot shows only half of a Cartesian projection.

DATA AVAILABILITY

The models underlying this article and the corresponding scripts to process them can be found in the online repository [github.com/Magritte-code/Examples](http://github.com/Magritte-code/Examples).

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