The $K$-theory of Assemblers

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Abstract

In this paper we introduce the notion of an assembler, which formally encodes “cutting and pasting” data. An assembler has an associated $K$-theory spectrum, in which $\pi_0$ is the free abelian group of objects of the assembler modulo the cutting and pasting relations, and in which the higher homotopy groups encode further geometric invariants. The goal of this paper is to prove structural theorems about this $K$-theory spectrum, including analogs of Quillen’s localization and dévissage theorems. We demonstrate the uses of these theorems by analyzing the assembler associated to the Grothendieck ring of varieties and the assembler associated to scissors congruence groups of polytopes.

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1. Introduction

Scissors congruence groups appear in many different areas of mathematics, from classical geometry of Euclidean, spherical and hyperbolic space [4, 17, 3, 6] to the study of birational geometry and motivic integration [12, 13, 15] to questions about definable sets and logic [2, 9]. However, as the contexts in which these appear are so different there is no general theory for the study of such groups, and very little exploration of their peculiar blend of geometry and combinatorics. In this paper we propose a framework for the study of these groups by replacing them with $K$-theory spaces. This framework is optimized to work with the combinatorial nature of cutting and pasting sets together and does not rely on any further algebraic structure.

Our approach constructs an object called an assembler (see Definition 2.4), which encodes the data necessary to perform scissors congruence. The basic object underlying an assembler is a small Grothendieck site where all morphisms are monic; the topology then encodes exactly which objects “assemble” to form other objects. Two maps $A \to C$ and $B \to C$ are considered disjoint if $A \times_C B$ exists and is equal to the initial object. We prove the following theorem in Section 2.2.

**Theorem A.** There exists a functor $K : \text{Asm} \to \text{Sp}$ from the category of assemblers to the category of spectra such that for any assembler $C$, $\pi_0 K(C)$ is the free abelian group generated by objects of $C$ modulo the relations

$$[A] = \sum_{i \in I} [A_i] \quad \text{for any finite disjoint covering family } \{A_i \to A\}_{i \in I}. $$

The higher homotopy groups of assemblers encode further geometric information about the geometry of cutting and pasting. They are not generally trivial; for example, for the assembler whose underlying category is $\emptyset \to *$ and which has the trivial topology, the $K$-theory has the homotopy type of the sphere spectrum. In a sequel [23] we will construct generators for $\pi_1 K(C)$ and show that these are related to “self-scissors-congruences” of objects.
In this paper, however, we concern ourselves with more structural questions. It turns out that assemblers fall into a sweet spot in the definition of algebraic $K$-theory similar to the sweet spot found by Quillen [16]. When Waldhausen [20] developed a framework for the algebraic $K$-theory of spaces he had to discard many of the advantages of Quillen’s exact categories; in particular, while he had an analog of Quillen’s Localization Theorem ([16, Theorem 5], [20, Proposition 1.5.5]) he did not have an analog of Quillen’s Dévissage [16, Theorem 4]. This makes computations and analysis using Waldhausen’s approach much more difficult than Quillen’s. The approach used in this paper, while much more analogous to Waldhausen’s combinatorial approach than Quillen’s algebraic one, also has both localization and dévissage theorems.

**Theorem B** (Dévissage). Let $C$ be an assembler and $D$ a full subassembler. If for every object $A \in C$ there exists a finite disjoint covering family $\{D_i \to A\}_{i \in I}$ such that $D_i \in D$ for all $i \in I$ then the induced map $K(D) \to K(C)$ is an equivalence of spectra.

A subcategory $D$ of an assembler $C$ is called a subassembler if it is an assembler and the inclusion $D \to C$ is a morphism of assemblers. This theorem is proved in Section 4.1. For examples of applications of this theorem, see Sections 5.1 and 5.2.

We also have a localization theorem; unfortunately, the most general form of the theorem requires passing to simplicial assemblers. We define a simplicial assembler to be a functor $\Delta^{op} \to \text{Asm}$; for a simplicial assembler $C^\bullet$ we define $K(C^\bullet) = \operatorname{hocolim}_{[n] \in \Delta^{op}} K(C_n)$.

Any assembler $C$ can be considered a simplicial assembler $C$, by taking the constant functor at $C$. It turns out that inside the category of simplicial assemblers it is always possible to construct a nice model for the cofiber on $K$-theory of a morphism of assemblers.

**Theorem C** (Localization). Let $g: D \to C$ be a morphism of simplicial assemblers. There exists a simplicial assembler $(C/g)$, with a morphism of simplicial assemblers $\iota: C \to (C/g)$, such that the sequence

$$K(D) \xrightarrow{K(g)} K(C) \xrightarrow{K(\iota)} K((C/g)),$$

is a cofiber sequence.

This is proved in Section 6. This theorem gives a useful formal definition, but suffers from the same problem as many bar constructions: the homotopy type of a simplicial object can be difficult to identify. Even when $C$ and $D$ are constant simplicial assemblers, $(C/g)$, will not be; this generally makes using Theorem C difficult. However, in Section 7 we show that for certain types of morphisms of assemblers the cofiber turns out to be surprisingly simple. Let $C \setminus D$ be the full subcategory of $C$ containing the objects $((\text{ob} C) \setminus (\text{ob} D)) \cup \{\emptyset\}$;
for its assembler structure, see Definition 2.9. For an object $A$ of $C$, we say that $C$ has complements for $A$ if any morphism $A \to B$ is in a finite disjoint covering family of $B$.

**Theorem D.** Let $D$ be a subassembler of $C$ such that $D$ is a sieve in $C$ and such that $C$ has complements for all objects of $D$. Then

$$K(D) \to K(C) \to K(C \setminus D)$$

is a cofiber sequence.

As an application of this theory, in Section 5.2 we construct a spectral sequence relating the classical scissors congruence groups to Goodwillie’s total scissors congruence groups [7] and McMullen’s polytope algebra [14]. The differentials in this spectral sequence measure the difference between these groups; in particular, Goodwillie’s groups split as sums of classical groups if and only if certain differentials in the spectral sequences are zero.

**Theorem E.** Let $G_n$ be the assembler whose objects are polytopes of dimension $n$ in $E^n$, and let $\mathcal{G}_n$ be the assembler whose objects are polytopes of dimension at most $n$; then the $\pi_0K(G_n)$ are the scissors congruence groups of Dupont and Sah and the $\pi_0K(\mathcal{G}_n)$ are the scissors congruence groups of Goodwillie. There exists a spectral sequence

$E^1_{p,q} = \pi_pK(G_q) \to \pi_p(\mathcal{G}_n)$ for $q \leq n$.

A discussion of the differentials of this spectral sequence is given in [23, Section 5].

The theory developed in this paper will also be used in [24] to prove the following theorem:

**Theorem F.** Let $K_0[\mathcal{V}_k]$ be the Grothendieck ring of varieties. Any element $x$ in the kernel of multiplication by $[A^1]$ can be represented as $[X] - [Y]$ where $X$ and $Y$ are varieties such that $[X \times A^1] = [Y \times A^1]$ but $X \times A^1$ and $Y \times A^1$ are not piecewise isomorphic.

Some preliminary work including a computation of a spectral sequence similar to the one mentioned in Theorem 5.1 is proved in Section 5.1.

The theory of assemblers grew out of the theory of polytope complexes, developed in [21, 22]. Every polytope complex produces an assembler (see Example 3.6) but not vice versa, and assemblers are much simpler to work with and much more flexible than polytope complexes. For this paper familiarity with polytope complexes is unnecessary, although some of the approaches used here are similar to those used in [22].

This paper is organized as follows. In Section 2 we define assemblers and the $K$-theory functor and prove Theorem A. In Section 3 we discuss several examples of assemblers and their $K$-theories. Section 4 proves Theorem B and a reduction theorem. Section 5 discusses two applications of the theorems from Section 4 and proves Theorem C. Section 6 proves Theorem D.
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2. Abstract Scissors Congruence

In this section we introduce scissors congruence of abstract objects. We want to define a relation $A \simeq B$ on certain kinds of objects, where we say that $A \simeq B$ if $A$ can be “decomposed” into “disjoint” pieces $A_1, \ldots, A_n$, and $B$ can be “decomposed” into “disjoint” pieces $B_1, \ldots, B_n$ such that $A_i \cong B_i$ (for some definition of $\cong$). To be able to define this rigorously we introduce assemblers, which categorically codify this information in a natural way.

2.1. Assemblers

In this section we define the notion of an “assembler,” which is the fundamental object of study of this paper. For a discussion of several examples of assemblers, see Section 3.

Definition 2.1. In any category with an initial object $\varnothing$, we say that two morphisms $f: A \rightarrow C$ and $g: B \rightarrow C$ are disjoint if the pullback $A \times_C B$ exists and is equal to $\varnothing$. A family $\{f_i: A_i \rightarrow A\}_{i \in I}$ is a disjoint family if for $i \neq i'$ the morphisms $f_i$ and $f_i'$ are disjoint.

Although initial objects do not need to be unique, as all constructions in this paper depend only on the non-initial objects in an assembler we will assume that initial objects are unique.

Definition 2.2. Let $C$ be any category. A sieve in $C$ is a full subcategory $\mathcal{D}$ such that for all $A$ in $C$, if there exists a morphism $A \rightarrow B$ in $C$ with $B \in \mathcal{D}$, then $A$ is in $\mathcal{D}$. In other words, a full subcategory $\mathcal{D}$ is a sieve in $C$ if it is closed under precomposition with morphisms in $C$.

Note that any sieve in $C$ must be equal to its essential image; in other words, if $\mathcal{D}$ is a sieve in $C$ and $A \cong A'$ with $A' \in \mathcal{D}$ then $A \in \mathcal{D}$. Observe that if $\mathcal{S}$ is a sieve in the over category $C/C$ and $f: B \rightarrow C$ is a morphism in $C$, then the preimage of $\mathcal{S}$ under $f \circ : C/B \rightarrow C/C$ is also a sieve, generally called $f^* \mathcal{S}$.

Definition 2.3. A Grothendieck topology on a category $C$ is a collection $J(C)$ of sieves in $C/C$ for all objects $C \in C$. These collections must satisfy the following axioms:

(T1) If $\mathcal{S}$ is in $J(C)$ and $f: B \rightarrow C$ is a morphism in $C$ then $f^* \mathcal{S}$ is in $J(B)$.
(T2) Let \( S \) be in \( J(C) \) and \( T \) be any sieve in \( C/C \). If for every object \( f: B \rightarrow C \) in \( S \) the sieve \( f^*T \) is in \( J(B) \) then \( T \) is in \( J(C) \).

(T3) \( C/C \) is in \( J(C) \) for all \( C \in C \).

Given a family of morphisms \( \{f_i: A_i \rightarrow A\}_{i \in I} \) in \( C \), we say that it is a covering family if the full subcategory of \( C/A \) containing the objects

\[
\{g: X \rightarrow A \mid \exists i \in I, h: X \rightarrow A_i \text{ s.t. } f_ih = g\}
\]

is in \( J(A) \).

A category \( C \) with a Grothendieck topology is called a Grothendieck site.

**Definition 2.4.** An assembler \( C \) is a small Grothendieck site satisfying the following extra conditions:

(I) \( C \) has an initial object \( \emptyset \), and the empty family is a covering family of \( \emptyset \).

(R) For any \( A \), any two finite disjoint covering families of \( A \) have a common refinement which is itself a finite disjoint covering family.

(M) All morphisms in \( C \) are monomorphisms.

A large assembler is a Grothendieck site satisfying axioms (I), (R) and (M). An assembler is said to be closed if it has all pullbacks.

**Remark.** If a Grothendieck site is closed under pullbacks then (R) always holds.

The following lemma is direct from the definition of an assembler but is important enough that we wish to highlight it:

**Lemma 2.5.** If \( A \) is noninitial in \( C \) then \( C(A, \emptyset) = \emptyset \).

We can now rephrase the definition of scissors congruence in the following way:

**Definition 2.6.** Two objects \( A, B \) in an assembler \( C \) are scissors congruent, written \( A \simeq B \), if there exist finite disjoint covering families \( \{A_i \rightarrow A\}_{i=1}^n \) and \( \{B_i \rightarrow B\}_{i=1}^n \) such that \( A_i \) is isomorphic to \( B_i \) for \( 1 \leq i \leq n \).

**Definition 2.7.** Let \( C, D \) be two assemblers. A functor \( F: C \rightarrow D \) is a morphism of assemblers if it is continuous (in the sense of Grothendieck topologies) and preserves the initial object and disjointness. In other words, if \( f: A \rightarrow C \) and \( g: B \rightarrow C \) are disjoint in \( C \) then \( F(f) \) and \( F(g) \) are disjoint in \( D \). We denote the category of assemblers and morphisms of assemblers by \( \text{Asm} \). The subcategory of closed assemblers and pullback-preserving morphisms of assemblers is denoted by \( \text{cAsm} \).

For convenience, we denote the full subcategory of noninitial objects by \( C^o \). We have the following:

**Lemma 2.8.** \( \text{Asm} \) and \( \text{cAsm} \) have arbitrary products and coproducts.
Proof. Let $X$ be any set, and $\{C_x\}_{x \in X}$ an $X$-tuple of assemblers. We write $\bigvee_{x \in X} C_x$ for the assembler whose class of objects is $\emptyset \cup \prod_{x \in X} \text{ob} C_x$ and whose morphisms between noninitial objects come from the $C_x$. A morphism of assemblers $\bigvee_{x \in X} C_x \to D$ is then just a morphism $F_x: C_x \to D$ for each $x \in X$, so $\bigvee_{x \in X} C_x$ is the coproduct in $\text{Asm}$. We write $\prod_{x \in X} C_x$ for the assembler whose underlying category is $\prod C_x$ and whose topology is the product topology where a family is a covering family exactly when its projection to every coordinate is. Then a morphism of assemblers $D \to \prod_{x \in X} C_x$ is just an $X$-tuple of morphisms $D \to C_x$, so this is the categorical product in $\text{Asm}$.

When restricting to $c\text{Asm}$ the same proof applies.

We will need one other construction on assemblers.

**Definition 2.9.** Let $C$ be an assembler and $D$ a sieve in $C$. As a category, we define $C \setminus D$ to be the full subcategory of $C$ containing all objects not in $D$. Then $C \setminus D$ inherits an assembler structure from $C$, where a family $\{f_i: A_i \to A\}_{i \in I}$ in $C \setminus D$ is defined to be a covering family if there exists a family of morphisms $\{f_j: A_j \to A\}_{j \in J}$ such that each $A_j$ is in $D$ for all $j \in J$ and such that $\{f_i: A_i \to A\}_{i \in I \cup J}$ is a covering family in $C$.

In other words, a family in $C \setminus D$ is a covering family if it can be completed to a covering family in $C$ by morphisms whose domains are in $D$. There is a natural morphism of assemblers

$$c: C \to C \setminus D$$

sending each object in $(\text{ob} C) \setminus (\text{ob} D)$ to itself and all objects in $D$ to $\emptyset$. A morphism $A \to B$ with $A \notin D$ is sent to itself; a morphism $A \to B$ with $A \in D$ but $B \notin D$ is sent to the unique morphism $\emptyset \to B$.

The fundamental construction allowing us to to scissors congruence with assemblers is the category $\mathcal{W}(C)$, which has as its objects formal sums of objects of $C$, and as its morphisms “gluings” of finite disjoint covering families. As the morphisms of this category keep track of the different ways of “past- ing” objects together, its $K$-theory should contain important information about scissors congruence classes.

**Definition 2.10.** Let $C$ be an assembler. We define the category $\mathcal{W}(C)$ to have objects $\{A_i\}_{i \in I}$, where $I$ is a finite set and $A_i$ is a noninitial object of $C$ for all $i \in I$. A morphism $f: \{A_i\}_{i \in I} \to \{B_j\}_{j \in J}$ in $\mathcal{W}(C)$ is a map of sets $f: I \to J$ together with a tuple of morphism $f_i: A_i \to B_{f(i)}$ for all $i \in I$ such that for all $j \in J$ the family $\{f_i: A_i \to B_j\}_{i \in f^{-1}(j)}$ is a finite disjoint covering family. Note that $\mathcal{W}$ is a functor $\text{Asm} \to \text{Cat}$.

**Proposition 2.11.** Let $C$ be an assembler.

(1) All morphisms in $\mathcal{W}(C)$ are monomorphisms.
(2) Any diagram $A \rightarrow C \leftarrow B$ in $W(C)$ can be completed to a commutative square. If $C$ is closed then $W(C)$ has all pullbacks.

(3) For any family of assemblers $\{C_x\}_{x \in X}$, let $\bigoplus W(C_x)$ be the full subcategory of $\prod W(C_x)$ where all but finitely many of the objects are the object indexed by the empty set. The functor

$$P: W(\bigvee_{x \in X} C_x) \longrightarrow \prod_{x \in X} W(C_x)$$

induced by the morphisms of assemblers $F_x: \bigvee_{x \in X} C_x \longrightarrow C_x$ which sends all $C_y$ for $y \neq x$ to the initial object, and sends $C_x$ to itself via the identity induces an equivalence of categories

$$W(\bigvee_{x \in X} C_x) \longrightarrow \bigoplus_{x \in X} W(C_x).$$

Proof. We prove these in turn. Proof of (1): Let $f: \{A_i\}_{i \in I} \rightarrow \{B_j\}_{j \in J}$ be a morphism of $W(C)$, and consider two morphisms $g, h: \{C_k\}_{k \in K} \rightarrow \{A_i\}_{i \in I}$ such that $fg = fh$; we show that $g = h$. Consider any $k \in K$. We have $fg(k) = fh(k)$, which implies that the square

$$\begin{array}{ccc}
C_k & \xrightarrow{g_k} & A_{g(k)} \\
\downarrow^{h_k} & & \downarrow^{f_{g(k)}} \\
A_{h(k)} & \xrightarrow{f_h(k)} & B_{f_{g(k)}}
\end{array}$$

commutes. If $g(k) \neq h(k)$ then $f_{g(k)}$ and $f_{h(k)}$ are disjoint, and thus $C_k = \emptyset$, a contradiction. Thus $g(k) = h(k)$. But as each morphism in $C$ is a monomorphism, this means that we must have $g_k = h_k$ as well.

Proof of (2): This follows directly from axiom (R); the second part follows from the definition of $W$ if $C$ has pullbacks.

Proof of (3): As each object of $W(\bigvee_{x \in X} C_x)$ is indexed by a finite set, $P$ is actually a functor $W(\bigvee_{x \in X} C_x) \longrightarrow \bigoplus_{x \in X} W(C_x)$. To see that $P$ is an equivalence, note that it is full and faithful and hits all objects indexed by disjoint indexing sets. Essential surjectivity follows because for any finite tuple of finite sets there exists an isomorphic one where the sets are disjoint.

2.2. The $K$-theory of an assembler

We are now ready to define the $K$-theory of an assembler.

Recall that the topological analog of the tensor product is the smash product $\wedge$, defined for pointed simplicial sets $X$ and $Y$ by

$$(X \wedge Y)_n \overset{\text{def}}{=} X_n \times Y_n / ((X_n \times \{\ast\}) \cup (\{\ast\} \times Y_n)).$$
Let $S^1$ be the pointed simplicial set $\Delta^1/\partial \Delta^1$, whose set of $n$-simplices is $\{*, 1, \ldots, n\}$. Let $S^k = (S^1)^\wedge k$. We have an action of $\Sigma_k$ on $S^k$ which permutes the $S^1$ factors.

We write $N : \mathbf{Cat} \to \mathbf{sSets}$ for the nerve of a category.

**Definition 2.12.** For a pointed set $X$ and an assembler $\mathcal{C}$, write $X \wedge \mathcal{C}$ for the assembler $\bigvee_{x \in X \setminus \{\ast\}} \mathcal{C}$. This gives a tensoring of $\mathbf{Asm}$ over $\mathbf{FinSet}$, for the definition of a tensoring see for example [11, Section 3.7]). For any pointed set $X$ write $X^\circ = X \setminus \{\ast\}$. Then we have an induced map

$$X \wedge N W(\mathcal{C}) \to N W(X \wedge \mathcal{C})$$

given by

$$X \wedge N W(\mathcal{C}) \cong \bigvee_{X^\circ} N W(\mathcal{C}) \xrightarrow{\bigoplus} N W(X \wedge \mathcal{C}).$$

This map is natural in $X$.

The spectrum $K(\mathcal{C})$ is defined to be the symmetric spectrum of simplicial sets where the $k$-th space is given by the diagonal of the bisimplicial set

$$[n] \mapsto N W((S^k)_n \wedge \mathcal{C}),$$

with the $\Sigma_k$-action induced from the action on $S^k$. Considering $S^1$ to be a bisimplicial set constant in one direction, we have a map of simplicial sets

$$\varphi_n : (S^1 \wedge N W(S^k \wedge \mathcal{C}))_n \cong (S^1)_n \wedge N W((S^k)_n \wedge \mathcal{C})$$

$$\xrightarrow{} N W((S^1)_n \wedge (S^k)_n \wedge \mathcal{C}) \cong N W((S^k+1)_n \wedge \mathcal{C}).$$

The spectral structure map $S^1 \wedge N W(S^k \wedge \mathcal{C}) \to N W(S^k+1 \wedge \mathcal{C})$ is the diagonal of the map of bisimplicial sets $\varphi$.

We write $K_i(\mathcal{C})$ for $\pi_i K(\mathcal{C})$.

Theorem 2.13 now follows from the definition of $K$ and the following theorem:

**Theorem 2.13.** For any assembler $\mathcal{C}$, $\pi_0 K(\mathcal{C})$ is the free abelian group generated by objects of $\mathcal{C}$ modulo the relations

$$[A] = \sum_{i \in I} [A_i] \quad \text{for any finite disjoint covering family } \{A_i \to A\}_{i \in I}.$$

**Proof.** To find generators and relations on $K_0(\mathcal{C})$ we will use the theory of $\Gamma$-spaces; for background on $\Gamma$-spaces, see for example [18] or [1]. To every $\Gamma$-space $X$ there is associated a symmetric spectrum $\mathbf{BX}$, which has $(\mathbf{BX})_n = [X(S^n)]$. The functor $\mathbf{X} : \mathbf{n} \mapsto |NW(n \wedge \mathcal{C})|$ is a special $\Gamma$-space by Proposition 2.11[3], and the spectrum $\mathbf{BX}$ is exactly equal to $K(\mathcal{C})$. Thus to find $\pi_0 K(\mathcal{C})$ it suffices to find $\pi_0 \mathbf{BX}$.

Since $X$ is special, $\pi_0 X(1)$ is a monoid with operation induced by

$$\pi_0 X(1) \times \pi_0 X(1) \equiv \pi_0 X(2) \xrightarrow{X(2 \to 1)} \pi_0 X(1).$$
Here, the first bijection is induced by the functor \( P \) from Proposition 2.11(3). Rewriting this in terms of \( W \) it is simply the functor

\[
W(C) \oplus W(C) \xrightarrow{P^{-1}} W(C \lor C) \xrightarrow{\mu} W(C),
\]

where \( P^{-1} \) is any inverse equivalence to \( P \), for example the one taking a pair of objects \( \{A_i\}_{i \in I}, \{B_j\}_{j \in J} \) to the object \( \{C_k\}_{k \in I \cup J} \), where \( C_k = A_k \) in the left copy of \( C \) if \( k \in I \) and \( C_k = B_j \) in the right copy of \( C \) if \( k \in J \); the functor \( \mu \) is then just the functor induced by the fold map of assemblers. This operation is therefore the operation which sums objects by taking the disjoint unions of their indexing sets. By \([18\text{ Section 4}]\) \( \pi_0B_X \) is the group completion of this monoid, so \( \pi_0B_X \) is a quotient of the free abelian group generated by the noninitial objects of \( C \).

The relations on the group are induced by morphisms of \( W(C) \). A morphism \( f: \{A_i\}_{i \in I} \to \{B_j\}_{j \in J} \) can be written as \( \bigsqcup_{j \in J} \{A_i\}_{i \in f^{-1}(j)} \to \{B_j\} \), which is a \( J \)-fold formal sum of the component morphisms. Each component is a finite disjoint covering family, and gives the relation

\[
[B_j] = \sum_{i \in f^{-1}(j)} [A_i].
\]

In addition, any finite disjoint covering family \( \{A_i \to A\}_{i \in I} \) gives a morphism \( \{A_i\}_{i \in I} \to \{A\} \). Thus the relations on \( \pi_0(X) \) given by the morphisms exactly correspond to the finite disjoint covering families of noninitial objects in \( C \).

It remains to check that the group in the statement of the theorem gives the same group as the description of \( \pi_0K(C) \) above. The only difference between these two descriptions is the presence of the initial object. However, the initial object has an empty covering family (which is tautologically finite and disjoint), so by the relation in the statement of the theorem \( [\emptyset] = 0 \). Thus its presence in the description does not affect the group.

We introduce the notion of a simplicial assembler and its \( K \)-theory.

**Definition 2.14.** A simplicial assembler is a functor \( \Delta^{op} \to \text{Asm} \). A morphism of simplicial assemblers is a natural transformation of functors. We define the \( K \)-theory spectrum of a simplicial assembler \( C \) by

\[
K(C) = \text{hocolim}_{[n] \in \Delta^{op}} K(C_n).
\]

We write \( s\text{Asm} \) for the category of simplicial assemblers.

**Remark.** Recall that homotopy colimits of spectra can be computed levelwise. A diagram \( \Delta^{op} \to s\text{Sets} \) is a bisimplicial set; the diagonal of the bisimplicial set is a model for the homotopy colimit of the diagram. We can thus give an explicit model for \( K(C) \) in an analogous way to Definition 2.12 as follows.

For any simplicial set \( X \), and any simplicial assembler \( C \), we define a simplicial assembler \( X \land C \), by \( (X \land C)_n = X_n \land C_n \). We define \( K(C)_k = \lvert \mathcal{W}(S^k \land C) \rvert \). The
spectral structure maps are constructed analogously to those in Definition 2.12.

In particular, if we consider an assembler to be a constant simplicial assembler, then $K(C)_k = |W(S^k \land C)|$.

3. Examples

In this section we examine several examples of assemblers and their $K$-theories.

Example 3.1. Let $*$ be the assembler whose underlying category is trivial. This assembler has no noninitial objects, so $W(*)$ is the trivial category, and thus $K(*) \cong *$.

Example 3.2. Fix a discrete group $G$, and let $S_G$ be the assembler with two objects, $\emptyset$ and $*$, one non-invertible morphism $\emptyset \to *$ and with $\text{Aut}(*) = G$. Then $W(S_G)$ has as its objects the finite sets and its morphisms $I \to J$ are isomorphisms $I \to J$ together with an element $g_i$ for all $i \in I$. By the Barratt–Priddy–Quillen–Segal Theorem (see [18]) $K(S_G)$ is stably equivalent to $\Sigma_+^{\infty} BG$. A homomorphism of groups $\varphi: H \to G$ gives a morphism of assemblers $S_H \to S_G$, which we also denote by $\varphi$.

In the special case when $G$ is the trivial group, $K(S_G) \cong \mathbb{S}$; we write $\mathbb{S}$ instead of $\mathbb{S}_1$.

Example 3.3. Let $C$ be the partial order of all open subsets of a topological space $X$, with the usual Grothendieck topology. Then $C$ is an assembler. However, the only way that we can have a finite disjoint covering family $\{U \to U\}_{i \in I}$ is if the $U_i$ are disjoint connected components of $U$. Thus for any connected open subset $U$ of $X$ there are no nontrivial finite disjoint covering families of $U$. In this case $K(C)$ is a wedge of spheres, one for each connected open subset of $X$, and is therefore completely uninteresting.

Here is an example relating assemblers to logic.

Example 3.4. For any small category $I$ and any (large) assembler $C$, let $[I, C]$ be the category of functors $I \to C$ and natural transformations between them. We define a topology on $[I, C]$ by saying that $\{F_\alpha \to F\}_{\alpha \in A}$ is a covering family if for all objects $X$ in $C$, the family $\{F_\alpha(X) \to F(X)\}_{\alpha \in A}$ is a covering family. Then $[I, C]$ is a (large) assembler.

As a special case, let $C = \text{Sets}_i$ be the large assembler of sets and injective functions, and let $I$ be the category of models and elementary inclusions of a logical theory $T$ (for an introduction to model theory, see for example [8]). Since any subcategory of an assembler containing the initial object and satisfying (R) is also an assembler, we can conclude that the category of definable sets of a theory is also an assembler. $K_0$ of this assembler is the abelian group underlying the Grothendieck ring of definable sets.

Remark. Section 5.1 and Examples 3.2 and 3.4 mention Grothendieck rings but we have not yet introduced any structures on assemblers that can lead to ring structures on $K_0$. In fact, it turns out that the category of assemblers is a symmetric monoidal category and $K$ is a symmetric monoidal functor, which
allows us to construct $E_\infty$ ring structures on the $K$-theory spectra of these assemblers. This will be discussed in more detail in future work.

Using the fact that $K(S) \simeq S$, we can construct homotopy types of all suspension spectra.

**Example 3.5.** For any pointed simplicial set $X$, $X \wedge S$ is a simplicial assembler. By definition,

$$K(X, \wedge S) \simeq \hocolim_n K(X_n \wedge S) \simeq \hocolim_n \bigvee_{X_n \setminus \{\ast\}} S \simeq \Sigma^\infty X.$$ 

Thus all homotopy types of suspension spectra are in the image of the functor $K : s\text{Asm} \to \text{Sp}$.

The original theory of [21, 22] led to the theory of assemblers; the following example shows that assemblers are a strict generalization of the ideas in those papers.

**Example 3.6.** The definition of a polytope complex and the notation $\cdots$ can be found in [21, Definition 3.1]. Given a polytope complex $C$ we define an assembler $C_a$ in the following manner. We set $\text{ob} C_a = \text{ob} C$, and a morphism $A \to B$ in $C_a$ is given by an object $A'$ and a diagram

$$A \stackrel{\sigma}{\leftarrow} A' \to B \in C.$$

The composite of $A \stackrel{\sigma}{\leftarrow} A' \to B$ and $B \stackrel{\tau}{\leftarrow} B' \to C$ is given by

$$A \stackrel{\sigma \tau}{\leftarrow} A' \to \tau^* A' \to \sigma^p C.$$

A family $\{A_i \leftarrow A'_i \to \cdots A\}_{i \in I}$ is a covering family exactly when $\{A'_i \to \cdots A\}_{i \in I}$ is a covering family in $C$. These definitions make $C_a$ into an assembler, and $K(C_a) \simeq K(C)$.

The name “polytope complex” was inspired by the ideas of classical scissors congruence; the new theory of assemblers allows us to make new connections in those applications as well.

### 4. Equivalences between assemblers

In this section we explore two types of morphisms between assemblers which produce equivalences on the level of $K$-theory. The two main results, Theorem B and Theorem 4.8, are useful for identifying assemblers that come up in various contexts. For concretes applications of these results, see Sections 5.1 and 5.2.

We begin with a result which is the basis of all of our assembler calculations. It is the “inductive step” which allows us to show that if we have an equivalence of $K$-theory spectra at level 0, then we must have an equivalence of $K$-theory spectra at all levels. Although the proof is straightforward from previous results, we present it in its entirety here as it is the key point in many analyses of assemblers.

By an abuse of notation, for a $k$-simplicial category $\mathcal{E}$... we will write $|\mathcal{E}|$ for the simplicial set whose $n$-simplices are $N_n \mathcal{E}_{n \to n}$.  

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Lemma 4.1. Suppose that \( p : \mathcal{C} \to \mathcal{D} \) is a morphism of simplicial assemblers such that the induced map of simplicial sets
\[
|W(p)| : |W(\mathcal{C})| \to |W(\mathcal{D})|
\]
is a weak equivalence of simplicial sets. Then \( K(p) \) is an equivalence of spectra.

Proof. It suffices to show that \( K(p)_k \) is an equivalence for all \( k \). Thus we want to show that \(|W(S^k \wedge \mathcal{C})| \to |W(S^k \wedge \mathcal{D})|\) is a weak equivalence. Let \( S^k \wedge \mathcal{C} \) be the bisimplicial assembler whose \((n,m)\)-th entry is \( S^k \wedge C_m \). Applying \( W \) pointwise and applying \(|·|\), we see that
\[
|W(S^k \wedge \mathcal{C})| \cong |W(S^k \wedge \mathcal{C})|.
\]
Thus it remains to show that \( p \) induces a weak equivalence
\[
|W(S^k \wedge \mathcal{C})| \to |W(S^k \wedge \mathcal{D})|.
\]
It suffices to show that for all \( n \), \(|W(S^k_n \wedge \mathcal{C})| \to |W(S^k_n \wedge \mathcal{D})|\) is a weak equivalence. We have the following commutative diagram:
\[
\begin{array}{ccc}
W(S^k_n \wedge \mathcal{C}) & \xrightarrow{W(S^k \wedge p)} & W(S^k_n \wedge \mathcal{D}) \\
\simeq & & \simeq \\
W(\mathcal{C})^n_k & \xrightarrow{p^n_k} & W(\mathcal{D})^n_k
\end{array}
\]
The vertical morphisms are level equivalences of simplicial categories by Proposition 2.11(3), and the bottom morphism is an equivalence after applying \(|·|\), as it is just an \( n^k \)-fold product of \( p \) with itself. By two-of-three we can therefore conclude that \( W(S^k \wedge p) \) is a weak equivalence after geometric realization.

Corollary 4.2. If \( F : \mathcal{C} \to \mathcal{D} \) is a morphism of assemblers which is an equivalence on the underlying categories, then \( K(F) \) is a weak equivalence of spectra.

Proof. By Lemma 4.1 it suffices to check that \( NW(F) \) is a weak equivalence. However, since \( F \) is an equivalence of categories it is straightforward that \( W(F) \) is, as well; since equivalences of categories are homotopy equivalences on nerves, we are done.

We use Quillen’s Theorem A to show that morphisms become equivalences after applying \(|·|\).

Quillen’s Theorem A. Suppose that \( F : \mathcal{C} \to \mathcal{D} \) is a functor between small categories such that \( N(F/Y) \) is contractible for all objects \( Y \) in \( \mathcal{D} \). Then \( NF : NC \to ND \) is a homotopy equivalence.
We say a preorder $C$ is cofiltered if for any two objects $A, B$ in $C$ there exists a diagram

$$A \leftarrow X \rightarrow B$$

in $C$. Quillen’s Theorem A has the following corollary:

**Lemma 4.3.** Any cofiltered preorder is contractible.

This lemma is the dual of the specialization to preorders of [16, Corollary 2, Section 1].

### 4.1. Restriction to subcomplexes

Recall that a subcategory $D$ of an assembler $C$ is called a subassembler if it is an assembler and the inclusion $D \rightarrow C$ is a morphism of assemblers. In this section we will prove Theorem B.

**Theorem B.** Let $C$ be an assembler and $D$ a full subassembler. If for every object $A$ there exists a finite disjoint covering family $\{D_i \rightarrow A\}_{i \in I}$ such that $D_i$ is in $D$ for all $i \in I$ then the induced map $K(D) \rightarrow K(C)$ is an equivalence of spectra.

**Proof.** We begin with a couple of observations. Since $D$ is a full subcategory of $C$, $W(D)$ is a full subcategory of $W(C)$. In addition, for every object $A$ in $W(C)$ there exists an object $B$ in $W(D)$ and a morphism $B \rightarrow A$ in $W(C)$. To see this, write $A = \{A_i\}_{i \in I}$, and choose finite disjoint covering families $\{B_j^{(i)} \rightarrow A_i\}_{j \in J_i}$ with $B_j^{(i)}$ in $D$ for all $i \in I$ and $j \in J_i$. Then setting $B = \{B_j^{(i)}\}_{(i,j) \in \coprod_{i \in I} J_i}$ gives us the desired object, and the covering families define the morphism $B \rightarrow A$.

By Proposition 2.11(2), for any two morphisms $A \rightarrow Y$ and $B \rightarrow Y$ there exists a commutative square

$$\begin{array}{ccc}
C & \rightarrow & A \\
\downarrow & & \downarrow \\
B & \rightarrow & Y
\end{array}$$

in $W(C)$; thus $W(C)$ is cofiltered.

By Lemma 4.1 it suffices to show that $i: W(D) \rightarrow W(C)$ is an equivalence after geometric realization. By Quillen’s Theorem A it suffices to show that the category $i/Y$ is contractible for all $Y$ in $W(C)$. As $W(C)/Y$ is a preorder and $W(D)$ is a subcategory of $W(C)$, $i/Y$ is also a preorder. Thus by Lemma 4.3 it suffices to show that it is cofiltered. To find an object above $A \rightarrow Y$ and $B \rightarrow Y$ in $i/Y$ we complete

$$\begin{array}{ccc}
A & \rightarrow & Y \\
\downarrow & & \downarrow \\
B & \leftarrow &
\end{array}$$

to a square in $W(C)$ with apex $C$, and then choose a morphism $C' \rightarrow C$ with $C'$ in $W(D)$. As $W(D)$ is a full subcategory, the morphisms $C' \rightarrow A$ and $C' \rightarrow B$ are morphisms in $W(D)$, and therefore $i/Y$ is cofiltered. □
As levelwise weak equivalences in simplicial spectra map to weak equivalences under realization, we immediately have the following:

**Corollary 4.4.** Let $D \to C$ be a morphism of simplicial assemblers such that for each $n$, $D_n \to C_n$ is an inclusion of a subassembler with sufficiently many covers. Then the map $K(D) \to K(C)$ is a weak equivalence of spectra.

4.2. Epimorphic assemblers with sinks

**Definition 4.5.** Let $C$ be an assembler. We say that $C$ is an epimorphic assembler with a sink if it satisfies the following three conditions:

(S) $C$ contains an object $S$, called a sink, such that for all other objects $A$, $C(A,S) \neq \emptyset$.

(Ep) All morphisms with noninitial domain in $C$ are epimorphisms, and all families $\{A \to B\}$ consisting of single morphisms with $A \neq \emptyset$ are covering families.

(D) If $A, B \neq \emptyset$ no two morphisms $A \to C$ and $B \to C$ are disjoint.

Morally speaking, in this case $C$ behaves like the assembler $S_G$, which has a single noninitial object $S$ with a group of automorphisms acting on it. If for a noninitial $A$ we let $C_A$ be the full subassembler of $C$ containing all objects $B$ for which there exists a morphism $B \to A$, then by Theorem B the inclusion $C_A \to C$ induces an equivalence on $K$-theory. Thus $C$ is “homogeneous” in a certain sense. By axiom (Ep) every object in $W(C)$ has a morphism to an object each of whose components is equal to $S$, and by axiom (D) all morphisms preserve the cardinality of the indexing set. Thus the higher homotopical structure of $K(C)$ must come from “partially defined automorphisms” of $S$, which are given by zigzags of morphisms in $W(C)$. In this section we show that this intuition is correct by showing that there exists a group $G$ and a morphism of assemblers $C \to S_G$ which induces an equivalence on $K$-theory.

**Definition 4.6.** Let $C$ be an epimorphic assembler with a sink. The group $G$ associated to $C$ is defined as follows. The elements of $G$ are equivalence classes of diagrams $S \xleftarrow{A}$ in $C$, where $A$ is any noninitial object. We define

$$
\left( \begin{array}{c}
S \\
A
\end{array} \right) \sim \left( \begin{array}{c}
S \\
B
\end{array} \right)
$$

if there exists an object $C$ that fits into a diagram

$$
\begin{array}{c}
S \\
A
\end{array} \xleftarrow{f_1} \xrightarrow{f_2} A
$$

$$
\begin{array}{c}
B \\
C
\end{array} \xleftarrow{g_1} \xrightarrow{g_2} S
$$
which commutes. The multiplication on \( G \) of \( \left[ \begin{array}{c} S \\ f_1 \\
 \end{array} \right] \boxtimes \left[ \begin{array}{c} A \\ f_2 \\
 \end{array} \right] \) and \( \left[ \begin{array}{c} S \\ g_1 \\
 \end{array} \right] \boxtimes \left[ \begin{array}{c} B \\ g_2 \\
 \end{array} \right] \) is represented by the composition down the left and the right in the diagram

\[
\begin{array}{ccc}
g'_1 & X & f'_2 \\
f_1 & A & g_1 \\
 & S & B \\
g_2 & S & \\
\end{array}
\]

where the middle square is any completion of \( f_2 \) and \( g_1 \) to a commutative square where \( X \) is noninitial (which exists by axiom (D)). The identity in \( G \) is represented by two copies of the identity morphism \( 1_S \); the inverse of \( \left[ \begin{array}{c} S \\ f \\
 \end{array} \right] \boxtimes \left[ \begin{array}{c} A \\
 \end{array} \right] \) is \( \left[ \begin{array}{c} S \\ g \\
 \end{array} \right] \boxtimes \left[ \begin{array}{c} A \\
 \end{array} \right] \).

To see that the product in \( G_C \) is well-defined, first suppose that there exists a diagram

\[
\begin{array}{ccc}
g'_1 & X & Y \\
& f'_2 & g'_2 \\
A & & B \\
g_2 & S & \\
\end{array}
\]

which gives us two different completions. Let \( Z \) be noninitial with morphisms \( \alpha: Z \rightarrow X \) and \( \beta: Z \rightarrow Y \) which complete \( f'_2 \) and \( f'_2'' \) to a commutative square. Then we have

\[
(Z \xrightarrow{\alpha} X \xrightarrow{g'_1} A \xrightarrow{f_2} S) = (Z \xrightarrow{\alpha} X \xrightarrow{f'_2} B \xrightarrow{g_1} S) = (Z \xrightarrow{\beta} Y \xrightarrow{f'_2''} B \xrightarrow{g_1} S) = (Z \xrightarrow{\beta} Y \xrightarrow{g'_2''} A \xrightarrow{f_2} S).
\]

As all morphisms in \( C \) are monic, this means that \( \alpha \) and \( \beta \) also complete \( g'_1 \) and \( g'_2'' \) to a commutative square, so we have a diagram
and thus the two different representatives represent the same class and the product in $G$ is well-defined. Associativity follows analogously.

**Definition 4.7.** Let $C$ be an epimorphic assembler with a sink, and let $G$ be the group associated to $C$. Let $F$ be a choice of a morphism $f_A : A \to S$ for each object $A$ in $C$, with the assumption that $f_S = 1_S$. Then we can define a morphism of assemblers $\pi_F : C \to S_G$ by

$$\pi_F(A) = *$$

and

$$\pi_F(g : A \to B) = \left[ \begin{array}{c} S \\ \hline \end{array} \right]_{f_A g}.$$

**Theorem 4.8.** Let $C$ be an epimorphic assembler with a sink, and let $G$ be the group associated to $C$. Then for every choice of family $F = \{ f_A : A \to S \}$ the morphism of assemblers $\pi_F : C \to S_G$ induces an equivalence on $K$-theory.

**Proof.** Let $\pi = \pi_F$. The functor $\pi$ is continuous by definition. Thus to see that this is a valid morphism of assemblers we need to check that it preserves disjointness, or, equivalently, that no two morphisms between noninitial objects in $C$ are disjoint. This is true by property (D).

We now need to check that this induces an equivalence on $K$-theory. By Lemma 4.1 it suffices to show that the map $W(\mathcal{C}) \to W(S_G)$ induces a homotopy equivalence on geometric realization.

We use Quillen’s Theorem A. We need to show that for all $Y$ in $W(S_G)$, $W(\pi)/Y$ is contractible. But $Y = \{ * \}_{i \in I}$ and $W(\pi)/Y \simeq (W(\pi)/\{ * \})^{[I]}$, so it suffices to show that $D = W(\pi)/\{ * \}$ is contractible. An object of $D$ is a pair $(A, g)$ with $A$ in $C$ and $g \in G$. We claim that $D$ is a preorder. Indeed, suppose that we have two objects $(A, g)$ and $(B, h)$. A morphism $\{ A \}_{\{ * \}} \to \{ B \}_{\{ * \}}$ in $W(C)$ is a morphism $f : A \to B$ in $C$; thus a morphism $(A, g) \to (B, h)$ is an $f$ in $C$ such that $\pi(f) = h^{-1} g$. Therefore $D$ is a preorder exactly when $\pi$ is faithful. To show that $\pi$ is faithful we need to show that if $r, s \in C(A, B)$ are such that there exists a diagram

$$
\begin{array}{c}
\begin{array}{c}
A \\
\hline
r \\
\hline
S
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
C \\
\hline
f_B \\
\hline
B
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
B \\
\hline
S
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A \\
\hline
f_A \\
\hline
S
\end{array}
\end{array}
\end{array}
$$

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then \( r = s \). As \( f_A \) is monic, it follows that \( h_1 = h_2 \). As \( h_1 \) is epic, it follows that \( r = s \), so we see that \( \pi \) is faithful.

As \( D \) is a preorder, to show that it is contractible by Lemma 4.3 it suffices to show that it is cofiltered. Let \((A, g)\) and \((B, h)\) be two objects in \( D \). An object \((C, k)\) above them is a pair of morphisms \( f: C \to A \) and \( f': C \to B \) in \( C \), such that \( \pi(f) = g k^{-1} \) and \( \pi(f') = h k^{-1} \); thus we want to find an object \( C \) in \( C \) and a pair of morphisms \( f: C \to A \) and \( f': C \to B \) such that \( g^{-1} \pi(f) = h^{-1} \pi(f') \).

Pick a representative \( S \subseteq X \) for \( g \) and a representative \( S \subseteq Y \) for \( h \).

Pick commutative squares

\[
\begin{array}{cccc}
X' & \xrightarrow{g_2'} & A & \quad \text{and} \\
\downarrow f'_A & & \downarrow f_A & \\
X & \xrightarrow{g_2} & S & \quad \text{and} \\
& & \downarrow S & \\
Y' & \xrightarrow{h_2'} & B & \quad \text{and} \\
\downarrow f'_B & & \downarrow f_B & \\
Y & \xrightarrow{h_2} & S & \quad \text{and} \\
& & \downarrow S & \\
\end{array}
\]

with noninitial \( X' \) and \( Y' \); these exist because no two morphisms in \( C \) are disjoint. Let \( C \) be any completion of

\[
\begin{array}{cccc}
X' & \xrightarrow{f'_A} & X & \xrightarrow{g_1} & S & \xrightarrow{h_1} & Y & \xleftarrow{f'_B} & Y' \\
& & \downarrow S & \quad \text{and} \\
& & \downarrow S & \quad \text{and} \\
\end{array}
\]

to a square, with morphisms \( \alpha: C \to X' \) and \( \beta: C \to Y' \). Define \( f \) and \( f' \) by

\[
\begin{array}{cccc}
C & \xrightarrow{\alpha} & X' & \xrightarrow{g_2'} & A & \quad \text{and} \\
& & \downarrow & \quad \text{and} \\
& & \downarrow & \quad \text{and} \\
C & \xrightarrow{\beta} & \quad Y' & \xrightarrow{h_2'} & B.
\end{array}
\]

We want to show that \( f \) and \( f' \) satisfy the desired conditions. We have the following diagram, where the starred square may not commute:
Then \( g = \left[ \begin{array}{c} S \xleftarrow{g_2} X \end{array} \right] = \left[ \begin{array}{c} S \xleftarrow{g_1 f_A \alpha} C \end{array} \right] = \left[ \begin{array}{c} S \xleftarrow{f_C} C \end{array} \right] = \left[ \begin{array}{c} S \xleftarrow{f_A \alpha} C \end{array} \right] \), and thus

\[
g^{-1} \pi(f) = \left[ \begin{array}{c} S \xleftarrow{f_C} C \end{array} \right] = \left[ \begin{array}{c} f_C \end{array} \right] = \left[ \begin{array}{c} S \xleftarrow{g_1 f_A \alpha} C \end{array} \right] = \left[ \begin{array}{c} f_C \end{array} \right] = \left[ \begin{array}{c} S \xleftarrow{f_A \alpha} C \end{array} \right].
\]

Analogously,

\[
h^{-1} \pi(f') = \left[ \begin{array}{c} S \xleftarrow{f_C} C \end{array} \right] = \left[ \begin{array}{c} f_C \end{array} \right] = \left[ \begin{array}{c} h_1 f_B \beta \end{array} \right] = \left[ \begin{array}{c} S \xleftarrow{g_1 f_A \alpha} C \end{array} \right] = \left[ \begin{array}{c} f_C \end{array} \right] = \left[ \begin{array}{c} S \xleftarrow{f_A \alpha} C \end{array} \right].
\]

But since the outside of the diagram commutes, \( h_1 f_B \beta = g_1 f_A \alpha \), and \( g^{-1} \pi(f) = h^{-1} \pi(g') \). Thus \( D \) is cofiltered and by Quillen’s Theorem A we have that \(|\mathcal{W}(C)| \rightarrow |\mathcal{W}(S_G)|\) is a weak equivalence.

One important thing to observe is that while \( K(\pi_F) \) is an equivalence this equivalence is not canonical: we may get a different equivalence for each choice of morphisms to the sink. We cannot always find a relation between two different choices of \( F \), but in nice geometric cases we can:

**Proposition 4.9.** Suppose that \( F \) and \( F' \) are two choices of families as in Definition 4.7, and let \( \psi: C \rightarrow C \) be an automorphism such that \( \psi(S) = S \). If there exists a family of isomorphisms \( \{ \phi_A: A \rightarrow \psi(A) \mid A \in C \} \) such that for all \( A \neq S \) the diagrams

\[
\begin{array}{ccc}
A & \xrightarrow{\phi_A} & \psi(A) \\
\downarrow f'_A & & \downarrow f_A \\
S & \xrightleftharpoons{\psi} & S
\end{array}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{\phi_A} & \psi(A) \\
\downarrow f'_A & & \downarrow f_A \\
S & \xrightleftharpoons{\psi} & S
\end{array}
\]

commute, then

\( \pi_{F'} = \Phi \pi_F \),

where \( \Phi: S_G \rightarrow S_G \) is the morphism of assemblers induced by conjugation by

\[
\left[ \begin{array}{c} S \xleftarrow{\phi_S} S \end{array} \right] \text{ in } G.
\]

**Proof.** The desired formula holds tautologically on objects, so we simply need
to check that it holds on morphisms. We have

\[
\pi_{F'}(g: A \rightarrow B) = \begin{bmatrix} f_A' & f_{A}\end{bmatrix} = \begin{bmatrix} S \circ A & S \circ A \end{bmatrix} = \begin{bmatrix} \phi S f_A & \phi S f_A \end{bmatrix}
\]

\[
= \begin{bmatrix} S \circ f_A & \phi S \circ f_A \end{bmatrix} = \begin{bmatrix} \phi S \circ f_A & \phi S \circ f_A \end{bmatrix}.
\]

Proposition 4.10. Let \( \mathcal{C} \) be an epimorphic assembler with sinks and let \( U \) be a noninitial object of \( \mathcal{C} \). Let \( \mathcal{C}_U \) be the full subcategory of \( \mathcal{C} \) of those objects \( A \) such that \( \text{Hom}_\mathcal{C}(A, U) \neq \emptyset \); this is again an epimorphic assembler with sink \( U \). Let \( \mathcal{F} = \{ f_A: A \rightarrow S \} \) be a family of morphisms to the sink \( S \) of \( \mathcal{C} \), and let \( \mathcal{F}' = \{ f_A': A \rightarrow U \} \) be a family of morphisms to \( U \) in \( \mathcal{C}_U \) which satisfy \( f_A = f_U f_A' \) for all \( A \) in \( \mathcal{C}_U \). Let \( G \) and \( G' \) be the groups associated to \( \mathcal{C} \) and \( \mathcal{C}_U \), respectively, and let \( \phi: G \rightarrow G' \) be defined by

\[
\phi \begin{bmatrix} U \circ A \end{bmatrix} = \begin{bmatrix} f_U f & f_U \end{bmatrix}.
\]

Then \( \phi \) is an isomorphism of groups and the diagram

\[
\begin{array}{ccc}
\mathcal{C}_U & \xrightarrow{\pi_{F'}} & \mathcal{C} \\
\downarrow{S_{G'}} & & \downarrow{S_{F}} \\
S_{G'} & \xrightarrow{\phi} & S_{G}
\end{array}
\]

commutes.

Proof. The fact that the diagram commutes follows directly from the definitions. To check that \( \phi \) is an isomorphism, note that it is injective because \( \mathcal{C}_U \) contains all objects above \( U \), and it is surjective because any representing pair can be modified to factor through \( f_U \).

5. Applications of the main theorems

In this section we give two applications of the theorems in Section 3 and of Theorem D. Although Theorem D has not yet been proved, the proof is technical, long and not terribly illuminating, so we prefer to defer the proof until after some interesting applications of the theorem are illustrated. For details of the proof, see Sections 6 and 7.

For ease of reading, we reproduce the theorem here. Recall that for an object \( A \) of \( \mathcal{C} \), we say that \( \mathcal{C} \) has complements for \( A \) if any morphism \( A \rightarrow B \) is in a finite disjoint covering family of \( B \).
**Theorem D.** Let $D$ be a subassembler of $C$ such that $D$ is a sieve in $C$ and such that $C$ has complements for all objects of $D$. Then

$$K(D) \to K(C) \to K(C \setminus D)$$

is a cofiber sequence.

### 5.1. The Grothendieck ring of varieties

Let $k$ be a field. We define $V_k$ to be the category whose objects are $k$-varieties (by which we mean reduced separated schemes of finite type over $k$), and whose morphisms are finite composites of open embeddings and closed embeddings. The topology on $V_k$ is generated by the coverage $\{Y \to X, X \setminus Y \to X\}$ for closed embeddings $Y \to X$. (For background on coverages, see for example [10].) Then $V_k$ is an assembler. $K_0(V_k)$ is the abelian group underlying the Grothendieck ring of varieties.

**Example 5.1.** Let $k$ be a finite field and let $F$ be the assembler of finite sets and injections. Let $L$ be a finite algebraic extension of $k$. Then we have a morphism of assemblers $c_L: V_k \to F$ given by $c_L(X) = X(L)$. Thus point counting is an example of a morphism of assemblers, and taking $K$-theory gives us a "derived" notion of point counting: a map $K(V_k) \to \mathbb{S}$. (The $K$-theory of $F$ is equivalent to the sphere spectrum by Theorem [B] for the inclusion $\mathbb{S} \to F$ taking $\ast$ to a singleton set.)

Let $V_k^{(n)}$ be the full subassembler of $V_k$ consisting of all varieties of dimension at most $n$. The goal of this section is to prove the following theorem:

**Theorem 5.2.** Let $B_n$ be the set of birational isomorphism classes of varieties of dimension $n$. For any variety $X$ over $k$, let $k(X)$ be its function field. Then

$$\text{hocofib} (K(V_k^{(n-1)}) \to K(V_k^{(n)})) \simeq \bigvee_{[X] \in B_n} \Sigma^\infty_+ B\text{Aut}_k(X).$$

This theorem gives us a spectral sequence

$$E_1^{p,q} = \bigoplus_{[X] \in B_n} \pi^p B\text{Aut}_k(X) \to K_p K(V_k)$$

that is a key tool in the proof of Theorem [D].

Note that as a subassembler of $V_k^{(n)}$, the assembler $V_k^{(n-1)}$ satisfies the conditions of Theorem [D]. Thus by Theorem [D]

$$\text{hocofib} (K(V_k^{(n-1)}) \to K(V_k^{(n)})) \simeq K(V_k^{(n)} \setminus V_k^{(n-1)}).$$

The rest of this section is dedicated to analyzing the homotopy type of $K(V_k^{(n)} \setminus V_k^{(n-1)})$.

For a fixed irreducible variety $X$ of dimension $n$, let $C_X$ be the full assembler of $V_k^{(n)} \setminus V_k^{(n-1)}$ with objects varieties $Y$ of dimension $n$ for which there exists
a morphism $Y \to X$ (although we do not include this morphism in the data). For a nonempty variety $Y$ the only finite disjoint covering families are of the form $\{Y' \to Y\}$ for a noninitial $Y'$; any morphism between noninitial objects gives a finite disjoint covering family.

Let $B_n$ be as above, and pick a representative $X_\alpha$ for each $\alpha \in B_n$. Let $\mathcal{C} = \bigvee_{\alpha \in B_n} \mathcal{C}_{X_\alpha}$. We claim that $\mathcal{C}$ is a full subassembler of $\mathcal{V}_k^{(n)} \setminus \mathcal{V}_k^{(n-1)}$ which satisfies the conditions of Theorem [3] and thus $K(\mathcal{V}_k^{(n)} \setminus \mathcal{V}_k^{(n-1)}) \simeq K(\mathcal{C})$. The subassemblers $\mathcal{C}_{X_\alpha}$ do not intersect inside $\mathcal{V}_k^{(n)} \setminus \mathcal{V}_k^{(n-1)}$, since if some noninitial $U$ were inside two of them then there would be morphisms $U \to X_\alpha$ and $U \to X_\beta$, which gives a birational isomorphism between $X_\alpha$ and $X_\beta$, which cannot happen when $\alpha \neq \beta$. Therefore $\mathcal{C}$ is a subassembler of $\mathcal{V}_k^{(n)} \setminus \mathcal{V}_k^{(n-1)}$, and it remains to check that it is full. Suppose it were not full, so that there existed some morphism $U_\alpha \to V_\beta$ with $U_\alpha \in \mathcal{C}_{X_\alpha}$ and $V_\beta \in \mathcal{C}_{X_\beta}$. Then again there would exist a morphism $U_\alpha \to V_\beta \to X_\beta$, and $X_\alpha$ would be birational to $X_\beta$, a contradiction.

Now we claim that $\mathcal{C}$ satisfies the conditions of Theorem [3] inside $\mathcal{V}_k^{(n)} \setminus \mathcal{V}_k^{(n-1)}$. Suppose that $Y$ is any variety of dimension $n$. It can be written as $Y = \bigcup_{i=1}^\ell Y_i$, where the $Y_i$ are the irreducible components of $Y$; we also assume without loss of generality that there exists $1 \leq m \leq \ell$ such that $\dim Y_i = n$ if $i \leq m$ and $\dim Y_i < n$ otherwise. Since $B_n$ is all birational isomorphism classes of irreducible varieties, $[Y_i] = [X_\alpha]$ for some $\alpha \in B_n$. Thus there is some open subset $U_i$ of $Y_i$ which is disjoint from all other $Y_j$ and such that there exists a morphism $U_i \to X_\alpha$. Therefore $U_i \in \mathcal{C}$. The family $\{U_i \to Y\}$ is a covering family in $\mathcal{V}_k^{(n)} \setminus \mathcal{V}_k^{(n-1)}$, since it can be completed to the finite disjoint covering family

$$\left\{ U_i \to Y, \bigcup_{i=1}^m (Y_i \setminus U_i) \cup \bigcup_{i=m+1}^\ell Y_i \to Y \right\}$$

in $\mathcal{V}_k^{(n)}$.

From this we can conclude that

$$K(\mathcal{V}_k^{(n)} \setminus \mathcal{V}_k^{(n-1)}) \simeq K(\mathcal{C}) \simeq \bigvee_{[X_\alpha] \in B_n} K(\mathcal{C}_{X_\alpha}).$$

Thus we can now restrict our attention to determining the homotopy type of $K(\mathcal{C}_X)$ for any variety $X$.

In the following analysis, we will fix the model of $\mathcal{C}_X$: we think of the objects of $\mathcal{C}_X$ as subsets of $X(\overline{k})$; then for every $Y$ in $\mathcal{C}_X$ there is a preferred morphism $\iota_Y : Y \to X$ which is an inclusion of points. With the choice of this model we have chosen “coordinates” for all objects of $\mathcal{C}_X$ simultaneously; by Corollary [4.2] this choice does not affect the homotopy type of the $K$-theory of the assembler.

We prove a slightly more general theorem than is needed to complete the proof of Theorem [5.2] as it will be necessary for the proof of Theorem [F].
**Theorem 5.3.** Let $X$ be an irreducible variety of dimension $n$, and let $\mathcal{C}_X$ be the assembler whose objects are subvarieties of $X$ of dimension $n$, modeled as algebraic subsets of $X(\bar{k})$ defined over $k$.

(1) $K(\mathcal{C}_X) \simeq \Sigma^\infty_+ B\text{Aut}_k(X)$.

(2) For any variety $X$ there is a morphism of assemblers $\mathcal{C}_X \to \mathcal{C}_X \times \mathbb{A}^1$ which takes a variety $Y$ to $Y \times \mathbb{A}^1$ and a morphism $f: Y \to f \times 1_{\mathbb{A}^1}$. Write $\text{Aut}_k(X \times \mathbb{A}^1)$ as $\text{Aut}_k(X)(t)$ for a transcendental $t$; let $\varphi: \text{Aut}_k(X) \to \text{Aut}_k(X)(t)$ be the homomorphism that includes $\text{Aut}_k(X)$ as those automorphisms which fix $t$. Then the diagram

\[
\begin{array}{ccc}
\mathcal{C}_X & \longrightarrow & \mathcal{C}_X \times \mathbb{A}^1 \\
\downarrow & & \downarrow \\
\text{S}_{\text{Aut} k(X)} & \varphi & \text{S}_{\text{Aut} k(X \times \mathbb{A}^1)}
\end{array}
\]

commutes.

**Proof.** This proof relies on Theorem 4.8. For any variety $Z$ we define the family $\mathcal{F}_Z = \{ \iota_A: A \to Z \mid A \in \mathcal{C}_Z \}$.

Proof of (1): The assembler $\mathcal{C}_X$ satisfies the conditions of Theorem 4.8 so we get a morphism of assemblers $\pi_{\mathcal{F}_X}: \mathcal{C}_X \to \text{S}_G$ which induces an equivalence after applying $K$-theory. In this case $G$ is the group $\text{Aut}_k(X)$, the birational automorphisms of $X$.

Proof of (2): With our definition of $\mathcal{F}_{X \times \mathbb{A}^1}$, for any subvariety $Y$ of $X$ we have $\iota_{Y \times \mathbb{A}^1} = \iota_Y \times 1_{\mathbb{A}^1}$. Thus if we define

\[
\varphi \left[ X \xrightarrow{f} A \right] \overset{\text{def}}{=} \left[ X \times \mathbb{A}^1 \xrightarrow{f \times 1_{\mathbb{A}^1}} A \times \mathbb{A}^1 \right]
\]

the given diagram commutes. This $\varphi$ is injective and hits all partial automorphisms of $X \times \mathbb{A}^1$ which are defined “fiberwise” on some $U \times \mathbb{A}^1$, which gives the desired algebraic description. \qed

### 5.2. Classical scissors congruence

There are two standard approaches to scissors congruence for polytopes. The first focuses on one dimension at a time by saying that two $n$-polytopes are “interior disjoint” if their intersection contains no $n$-polytopes. We can then define scissors congruence of $n$-polytopes in $\mathbb{R}^\infty$ in the following manner. An $n$-simplex is defined to be the convex hull of $n+1$ points in general position; an $n$-polytope is a finite union of $n$-simplices. We then say that two $n$-polytopes $P$ and $Q$ are scissors congruent if
we can write \( P = \bigcup_{i=1}^{m} P_i \) and \( Q = \bigcup_{i=1}^{m} Q_i \) such that \( P_i \cong Q_i \), and

\[
P_i \cap P_j \text{ and } Q_i \cap Q_j \text{ contain no } n\text{-simplices for } i \neq j.
\]

This is the approach considered in the work of Dupont and Sah (see [17], [4] and [3]). We write the scissors congruence group of \( n \)-polytopes defined through this approach as \( P_n \).

An alternate approach considers all intersections of all dimensions. Here we want to be able to decompose polytopes into completely disjoint sets. The simplest way to write this down is to say that an \( n \)-simplex is the interior (in the \( n \)-space spanned by the points) of the convex hull of \( n + 1 \) points, and that an \( n \)-polytope is a finite union of simplices \( X_i, \ldots, X_k \), where \( \dim X_i \leq n \) and for some \( i_0 \), \( \dim X_{i_0} = n \). Thus for example, a closed interval is a 1-polytope because a point is a 0-simplex, an open interval is a 1-simplex, and a closed interval is the union of an open interval and two points. We then say that two polytopes \( P \) and \( Q \) are scissors congruent if

- we can write \( P = \bigcup_{i=1}^{m} P_i \) and \( Q = \bigcup_{i=1}^{m} Q_i \) such that \( P_i \cong Q_i \), and
- \( P_i \cap P_j = Q_i \cap Q_j = \emptyset \).

This is an approach which is analogous to the scissors congruence of definable sets (considered in, for example, [2]). When the only isometries allowed are translations these groups were analyzed by McMullen in [14]; the more general case was studied by Goodwillie in [7]. We denote the scissors congruence group of such polytopes by \( \mathcal{P} \).

As dimension is a scissors-congruence invariant, we have the following observation:

\textbf{Observation 5.4.} The group \( \mathcal{P} \) is filtered by dimension. More formally, if we define \( \mathcal{P}_n \) to be the scissors congruence group of \( m \)-polytopes with \( m \leq n \) then we have a sequence of homomorphisms

\[
0 = \mathcal{P}_{-1} \rightarrow \mathcal{P}_0 \leftarrow \mathcal{P}_1 \rightarrow \cdots \rightarrow \mathcal{P}.
\]

We have a filtration on \( \mathcal{P} \) where the \( n \)-th filtered piece is the image of \( \mathcal{P}_n \) inside \( \mathcal{P} \).

However, it is unclear whether or not higher dimensional polytopes can induce relations between lower-dimensional ones, so we do not know whether \( \mathcal{P}_n \) is the \( n \)-th graded piece of \( \mathcal{P} \). We want to compute the associated graded spectrum of this filtration, and use it to learn about the structure of \( \mathcal{P} \).

First, we consider the analog of Dupont and Sah’s approach. We define an assembler \( G_n \) whose objects are pairs \( (U, P) \) where \( P \) is an \( n \)-polytope in \( \mathbb{R}^\infty \) and \( U \) is the smallest affine subspace containing \( P \). A morphism \( (U, P) \rightarrow (V, Q) \) is an isometry \( \varphi: U \rightarrow V \) such that \( \varphi(P) \subseteq Q \). The Grothendieck topology is defined by defining a family

\[
\{ \varphi_\alpha: (U_\alpha, P_\alpha) \rightarrow (U, P) \}_{\alpha \in A}
\]
to be a covering family if \( P = \bigcup_{\alpha \in A} \varphi_\alpha(P_\alpha) \).

Now we construct the analog of McMullen and Goodwillie’s approach. We define an assembler \( \mathcal{G} \) to have as objects pairs \((U, P)\), where \( P \) is an \( n \)-polytope in \( \mathbb{R}^\infty \) and \( U = \text{span} \ P \). Morphisms \((U, P) \to (V, Q)\) of pairs consist of an isometry \( \varphi: U \to V \) such that \( \varphi(P) \subseteq Q \). Note that these are almost the same definitions as in the Dupont and Sah approach; the difference is that here we take all \( n \), and that our polytopes have a slightly different definition, in that they do not necessarily contain their boundaries. The Grothendieck topology is defined analogously to the Grothendieck topology on \( \mathcal{G}_n \).

Remark. Both of these constructions can be done using only simplices, without considering general polytopes. By Theorem 13 this approach gives an equivalent \( K \)-theory. However, as the traditional approach uses polytopes, we do as well.

Morphisms in \( \mathcal{G} \) are defined in exactly the same way as the morphisms in \( \mathcal{G}_n \), so that we have an inclusion functor \( \mathcal{G}_n \hookrightarrow \mathcal{G} \). From the definition of the topologies this is a continuous functor. However, this is not a morphism of assemblers as it does not preserve disjointness: given any two \( n \)-polytopes with a nonempty measure-0 intersection, their pullback (over their union) is \( \emptyset \) in \( \mathcal{G}_n \) but not in \( \mathcal{G} \).

Naively we might think that \( \mathcal{G} \) is a product of \( \mathcal{G}_n \)’s for different \( n \), but the above observation about the functor \( \mathcal{G}_n \hookrightarrow \mathcal{G} \) means that this is not the case. Even playing around with small values of \( n \) shows that something is different, as in \( \mathcal{G} \) we have an extra invariant on polytopes: the Euler characteristic. (For more on this, see [2].)

**Proposition 5.5.** Let \( \mathcal{G}_n \) be the full subcategory of \( \mathcal{G} \) containing all pairs \((U, P)\) where \( P \) is an \( i \)-polytope with \( i \leq m \). This is a subassembler, and we get a sequence

\[ * = G_{-1} \xrightarrow{i_0} G_0 \xrightarrow{i_1} G_1 \xrightarrow{i_2} \cdots \rightarrow G, \]

where \( * \) is the assembler containing only the initial object. We have

\[ \text{hocofib} K(i_n) \simeq K(\mathcal{G}_n). \]

Thus \( \bigvee_{n=0}^{\infty} K(\mathcal{G}_n) \) is the associated graded spectrum of \( K(\mathcal{G}) \).

**Proof.** Since \( \mathcal{G}_{n-1} \) satisfies the conditions of Theorem D it follows that

\[ \text{hocofib} K(i_n) \simeq K(\mathcal{G}_n \setminus \mathcal{G}_{n-1}). \]

Therefore it suffices to construct a morphism of assemblers \( F: \mathcal{G}_n \to \mathcal{G}_n \setminus \mathcal{G}_{n-1} \) which induces an equivalence on \( K \)-theory. Let \( F': \mathcal{G}_n \to \mathcal{G}_n \) take each pair \((U, P)\) to the pair \((U, \bar{P})\), where \( \bar{P} \) is the open interior of \( P \). This functor preserves pullbacks but is not continuous. We claim that after mapping to \( \mathcal{G}_n \setminus \mathcal{G}_{n-1} \) this functor induces an equivalence of \( K \)-theories.

Let \( F \) be the composite

\[ \mathcal{G}_n \xrightarrow{F'} \mathcal{G}_n \xrightarrow{\pi} \mathcal{G}_n \setminus \mathcal{G}_{n-1}, \]
where \( \pi \) is the canonical morphism \( \mathcal{G}_n \xrightarrow{\pi} \mathcal{G}_n \setminus \mathcal{G}_{n-1} \). As both \( \pi \) and \( F' \) preserve disjointness and the initial object, so does \( F \). Thus to show that \( F \) is a valid morphism of assemblers it suffices to show that it preserves covering families. A covering family in \( \mathcal{G}_n \) is a family \( \{ f_\alpha : P_\alpha \to P \}_{\alpha \in A} \) such that \( \bigcup_{\alpha \in A} f_\alpha(P_\alpha) = P \). For each \( \alpha \) we can write \( P_\alpha = \partial P_\alpha \cup \partial P_\alpha \), where \( \partial P_\alpha \) is the boundary of \( P_\alpha \). Note that \( \partial P_\alpha \subseteq \mathcal{G}_n \setminus \mathcal{G}_{n-1} \). We then have a covering family \( \{ f_\alpha : \partial P_\alpha \to P \}_{\alpha \in A} \) in \( \mathcal{G}_n \). Each source in the second half is in \( \mathcal{G}_n \setminus \mathcal{G}_{n-1} \), so it is killed by \( \pi \); thus by definition of the topology on \( \mathcal{G}_n \setminus \mathcal{G}_{n-1} \) \( \{ f_\alpha : \partial P_\alpha \to P \}_{\alpha \in A} \) is a covering family, and we see that \( F \) preserves covering families.

Note that \( F \) includes \( \mathcal{G}_n \) as a subassembler of \( \mathcal{G}_n \setminus \mathcal{G}_{n-1} \). The objects of \( \mathcal{G}_n \setminus \mathcal{G}_{n-1} \) which are hit by \( F \) are exactly those objects \((U, P)\) such that \( \partial P = P \).

Any object \((U, P)\) has a disjoint covering family
\[
\{ (\text{span}(\partial P), \partial P) \to (U, P), (\text{span} \hat{P}, \hat{P}) \to (U, P) \},
\]
so the family \( \{ (\text{span} \hat{P}, \hat{P}) \to (U, P) \} \) is a covering family of \((U, P)\) in \( \mathcal{G}_n \setminus \mathcal{G}_{n-1} \). The second one of these is in the image of \( F \) and we see that \( F(\mathcal{G}_n) \) satisfies the condition of Theorem \ref{thm:cofiber}. Thus \( K(\mathcal{G}_n) \simeq K(\mathcal{G}_n \setminus \mathcal{G}_{n-1}) \).

As a corollary we have:

**Theorem 5.6.** There is a spectral sequence
\[
E_{p,q}^1 = K_p(\mathcal{G}_q) \longrightarrow K_p(\mathcal{G}).
\]

If we filter \( \mathcal{G}_n \) instead of \( \mathcal{G} \) we get the spectral sequence in Theorem \ref{thm:cofiber}.

**Proof.** These are the spectral sequences associated to the filtered spectra
\[
K(\mathcal{G}_0) \longrightarrow K(\mathcal{G}_1) \longrightarrow \cdots \longrightarrow K(\mathcal{G})
\]
and
\[
K(\mathcal{G}_0) \longrightarrow K(\mathcal{G}_1) \longrightarrow \cdots \longrightarrow K(\mathcal{G}_n).
\]

\( \square \)

6. The cofiber theorem

Our goal in this section is to prove Theorem \ref{thm:cofiber}.

**Theorem \ref{thm:cofiber}.** Let \( g : \mathcal{D} \longrightarrow \mathcal{C} \) be a morphism of simplicial assemblers. There exists a simplicial assembler \( (\mathcal{C}/g) \), with a morphism of simplicial assemblers \( \iota : \mathcal{C} \longrightarrow (\mathcal{C}/g) \), such that the sequence
\[
K(\mathcal{D}) \xrightarrow{K(g)} K(\mathcal{C}) \xrightarrow{K(\iota)} K((\mathcal{C}/g)).
\]
is a cofiber sequence.
Definition 6.1. Suppose that we are given a morphism of simplicial assemblers \( g : D \to C \). We define the simplicial assembler \((C/g)\) by

\[(C/g)_n = C_n \vee ((S^1)_n \wedge D_n).\]

As \( \vee \) is the coproduct in \( \text{Asm} \), in order to define the face and degeneracy maps it suffices to construct them on each component separately. On \( C_n \), the face and degeneracy maps are induced by the simplicial structure on \( C \); on \((S^1)_n \wedge D_n\) all structure maps other than \( d_0 \) are induced by the simplicial structure maps of \( S^1 \) and \( D \). Recall that \( n = \{0, 1, \ldots, n\} \). Note that \((S^1)_n \cong n \cong 1 \vee (n-1)\).

We define \( d_0 \) to be the composite

\[C_n \vee (S^1)_n \wedge D_n \xrightarrow{\cong} C_n \vee (1 \vee (S^1)_{n-1}) \wedge C_n \vee D_n \xrightarrow{\cong} C_n \vee D_n \vee ((S^1)_{n-1} \wedge D_n) \xrightarrow{g \circ (1 \wedge d_0)} C_{n-1} \vee ((S^1)_{n-1} \wedge D_{n-1}).\]

The simplicial assembler \((C/g)\) comes with a natural inclusion \( i : C \to (C/g) \) and a natural projection \( \pi_D : (C/g) \to S^1 \vee D \). Theorem \( C \) states that

\[K(D) \xrightarrow{K(g)} K(C) \xrightarrow{K(i)} K((C/g)).\]

is a cofiber sequence.

We prove this using an approach analogous to the one used by Waldhausen for \([20\text{, Propositions 1.5.5, 1.5.6}]\). However, as the structure of our spectra is much simpler than his, the proof turns out to be much more basic. The key point is the additivity theorem: all we need is the two-term case of Proposition \( 2.11(3) \) to show that \( W(C \vee C) \) is equivalent to \( W(C)^2 \).

Most of the work of the proof is contained in the following lemma:

Lemma 6.2. For all \( k \geq 1 \), the sequence

\[K(C)_k \to K((C/g))_k \to K(S^1 \wedge D)_k\]

is a homotopy fiber sequence.

Proof. Each term in this sequence is the geometric realization of a \( k \)-simplicial category. As the geometric realization is a diagonal, we can turn this into a sequence of bisimplicial sets by taking a partial diagonal. By choosing this partial diagonal appropriately, we get a sequence of bisimplicial sets which at level \((m, n)\) the sequence is equal to

\[W((S^{k-1})_n \wedge (S^1)_m \wedge C_n) \to W((S^{k-1})_n \wedge ((S^1)_m \wedge (C_n \vee ((S^1)_n \wedge D_n)))) \to W((S^{k-1})_n \wedge (S^1)_m \wedge ((S^1)_n \wedge D_n)).\]

By generalizing \([19\text{, Lemma 5.2}]\) to trisimplicial sets, to prove the lemma it suffices to show that:

(1) the composition is constant,
(2) for all $n$ and all pointed sets $X$, $[m] \mapsto W(X \land (S^1)_m \land (S^1)_n \land D_n)$ is connected, and

(3) for all $n$ and all pointed sets $X$,

$$[m] \mapsto W(X \land (S^1)_m \land C_n)$$

$$\longrightarrow [m] \mapsto W(X \land (S^1)_m \land (C_n \lor ((S^1)_n \land D_n)))$$

$$\longrightarrow [m] \mapsto W(X \land (S^1)_m \land (S^1)_n \land D_n)$$

is a homotopy fiber sequence.

In (2) and (3), we really only care about $X = (S^{k-1})_n$. (1) is clear because the map of simplicial assemblers composes to a constant map. (2) follows because the 0-simplices arise from the trivial category. Write $C' = X \land C_n$ and $D' = X \land (S^1)_n \land D_n$. There exists a functor

$$W((S^1)_m \land (C' \lor D')) \longrightarrow W((S^1)_m \land C') \times W((S^1)_m \land D')$$

which takes an object $\{A_i\}_{i \in I}$ to the pair

$$\left( \{A_i\}_{i \in I \mid A_i \in (S^1)_m \land C'}, \{A_i\}_{i \in I \mid A_i \in (S^1)_m \land D'} \right);$$

this is an equivalence of categories by Proposition 2.11. This functor fits into a commutative diagram

$$\begin{array}{ccc}
W((S^1)_m \land C') & \xrightarrow{W(\iota)} & W((S^1)_m \land (C' \lor D')) \\
\downarrow & \cong & \downarrow \\
W((S^1)_m \land C') & \xrightarrow{W(\pi_D)} & W((S^1)_m \land D')
\end{array}$$

where the bottom row is a fiber sequence. The morphisms in the diagram also commute with the simplicial structure maps, and thus assemble into a diagram of simplicial categories. Note that it is very important in this context that we are fixing $n$, as if $n$ varies then the middle morphism does not commute with $d_0$. Upon taking geometric realization, the middle map turns into a weak equivalence of simplicial spaces. As the bottom row stays a fiber sequence even after geometric realization, the top row is a homotopy fiber sequence.

We are now ready to prove Theorem C.

**Proof of Theorem C.** Consider the commutative diagram

$$\begin{array}{ccc}
D \longrightarrow (D/1), & \xrightarrow{\pi_D} & S^1 \land D, \\
\downarrow g & & \downarrow \\
C \longrightarrow (C/g), & \xrightarrow{\pi_D} & S^1 \land D.
\end{array}$$

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After applying $K$, the top and bottom are levelwise homotopy fiber sequences by Lemma 6.2 and thus homotopy fiber sequences of spectra. The right-hand map is the identity, so the left-hand square is a homotopy pullback square. $K((\mathcal{D}/1)^{•})$ is obtained from a simplicial shift of $K(S^1 \cap \mathcal{D})$, so it is contractible. Thus the composition around the bottom is a homotopy fiber sequence.

This has the following corollary:

**Corollary 6.3.** $K(\mathcal{C})$ is an $\Omega$-spectrum above level 0.

*Proof.* Let $*$ be the assembler with no non-initial objects. Consider the functor $p: \mathcal{C} \rightarrow *$ which sends all objects of every level of $\mathcal{C}$ to the initial object. Then $(*/p)$, has as its $n$-th level $(S^1)_n \cap \mathcal{C}_n$ and so $K((*/p))_k \simeq K(\mathcal{C})_{k+1}$. Therefore by Lemma 6.2 we get a homotopy fiber sequence $K(\mathcal{C})_k \rightarrow * \rightarrow K(\mathcal{C})_{k+1}$.

Thus we get an equivalence $K(\mathcal{C})_k \simeq \Omega K(\mathcal{C})_{k+1}$. □

### 7. Proof of Theorem D

Recall Theorem D:

**Theorem D.** For an object $A$ of $\mathcal{C}$, we say that $\mathcal{C}$ has complements for $A$ if any morphism $A \rightarrow B$ is in a finite disjoint covering family of $B$. Let $\mathcal{D}$ be a subassembler of $\mathcal{C}$ such that $\mathcal{D}$ is a sieve in $\mathcal{C}$ and such that $\mathcal{C}$ has complements for all objects of $\mathcal{D}$. Then

$$K(\mathcal{D}) \rightarrow K(\mathcal{C}) \rightarrow K(\mathcal{C} \setminus \mathcal{D})$$

is a cofiber sequence.

We begin by fixing some notation. Let $i: \mathcal{D} \rightarrow \mathcal{C}$ be the inclusion of assemblers, and $c: \mathcal{C} \rightarrow \mathcal{C} \setminus \mathcal{D}$ be the canonical projection. There is a morphism of simplicial assemblers $p: (\mathcal{C}/i)^{•} \rightarrow \mathcal{C} \setminus \mathcal{D}$ given by projecting all copies of $\mathcal{D}$ in $(\mathcal{C}/i)^n$ to the initial object, and using the canonical morphism $c: \mathcal{C} \rightarrow \mathcal{C} \setminus \mathcal{D}$.

Consider the following commutative diagram of simplicial assemblers:

$$
\begin{array}{ccc}
\mathcal{D} & \rightarrow & (\mathcal{C}/i)^{•} \\
\downarrow & & \downarrow p \\
\mathcal{C} \setminus \mathcal{D}
\end{array}
$$

After applying $K$-theory, by Theorem D the composition along the top becomes a cofiber sequence. Thus it suffices to show that $p$ induces an equivalence of spectra to ensure that the composition along the bottom is also a cofiber sequence.

By Lemma 4.1 it suffices to show that $W(p): W((\mathcal{C}/i)^{•}) \rightarrow W(\mathcal{C} \setminus \mathcal{D})$ is an equivalence after geometric realization, which is the statement of the following proposition.

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Proposition 7.1. Suppose that the conditions of Theorem D hold, and let 
\( i: \mathcal{D} \to \mathcal{C} \) be the inclusion of \( \mathcal{D} \) into \( \mathcal{C} \). Then
\[
|W(p)|: |W((\mathcal{C}/i).)| \longrightarrow |W(\mathcal{C} \setminus \mathcal{D})|
\]
is a weak equivalence.

The rest of this section is dedicated to the proof of this proposition. As the 
proof is quite long, we start with the setup and an outline of the proof, and
finish the section with a series of lemmas that fill in the details.

Let \( P \) be the map \( NW((\mathcal{C}/i).) \to NW(\mathcal{C} \setminus \mathcal{D}) \) of bisimplicial sets; we want 
to show that it is a weak equivalence. We assume that the nerve-direction is hori-

tzonal, and the internal \( (\mathcal{C}/i). \)-direction is vertical. Thus \( NW(\mathcal{C} \setminus \mathcal{D}) \) is constant 
in the vertical direction. For conciseness we define
\[
\kappa = W(c): W(\mathcal{C}) \longrightarrow W(\mathcal{C} \setminus \mathcal{D}).
\]

For any subcategory \( \mathcal{C}' \subseteq \mathcal{C} \) and any object \( A = \{ A_i \}_{i \in I} \) in \( W(\mathcal{C}) \) we define
\[
A_{\mathcal{C}'} = \{ A_i \}_{i \in I, A_i \in \mathcal{C}'}.
\]

Observe that while \( \mathcal{C} \setminus \mathcal{D} \) is not a 
subassembler of \( \mathcal{C} \), it is a subcategory of \( \mathcal{C} \),
and we will often be treating it as such. More generally, for any diagram
\[
D = (A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n)
\]
in \( W(\mathcal{C}) \) we write
\[
D_{\mathcal{C}'} = (A'_0 \xrightarrow{} A'_1 \xrightarrow{} \cdots \xrightarrow{} A'_n),
\]
where \( A'_n = (A_n)_{\mathcal{C}'} \) and \( A'_i \) is the subobject of \( A_i \) whose image is in \( A'_n \). More
formally, if we write \( A_i = \{ A_{ij} \}_{j \in J_i} \) then we define \( A'_i = \{ A'_{ij} \}_{j \in J'_i} \), with
\( J'_n = \{ j \in J_n | A_{nj} \in \mathcal{C}' \} \) and \( J'_i = f_i^{-1}(J'_{i+1}) \). Note that \( D_{\mathcal{C}'} \) is still a diagram
in \( W(\mathcal{C}) \), not \( W(\mathcal{C}') \), since the \( A_i \) for \( i < n \) may have components which are not
in \( \mathcal{C}' \). Thus for objects \( A, \kappa(A) = A_{\mathcal{C} \setminus \mathcal{D}} \), but this is not true for morphisms.

We define
\[
Y_n = \{(A_0 \longrightarrow \cdots \longrightarrow A_n) \in N_n W(\mathcal{C}) | A_n \in W(\mathcal{C} \setminus \mathcal{D}) \}.
\]

We want to make \( Y \) into a simplicial set. We define \( d_n \) on an \( n \)-simplex
\[
D = (A_0 \longrightarrow A_1 \longrightarrow \cdots \longrightarrow A_n)
\]
to be \((A_0 \longrightarrow \cdots \longrightarrow A_{n-1})_{\mathcal{C} \setminus \mathcal{D}} \); we let the other face maps and all degeneracies
be inherited from \( NW(\mathcal{C}) \). This is a well-defined simplicial set because \( \mathcal{D} \) is a 
sieve in \( \mathcal{C} \).

Define the category \( W(\mathcal{C}, \mathcal{D}) \) to have \( \text{ob} W(\mathcal{C}, \mathcal{D}) = \text{ob} W(\mathcal{C} \setminus \mathcal{D}) \). Define a 
morphism
\[
f: \{ A_i \}_{i \in I} \longrightarrow \{ B_j \}_{j \in J} \in W(\mathcal{C}, \mathcal{D})
\]
to be a map of sets $f: I \rightarrow J$ together with morphisms $f_i: A_i \rightarrow B_{f(i)}$ for $i \in I$ such that for all $j \in J$ there exists a family $\{D_k \rightarrow B_j\}_{k \in K}$ of morphisms in $C$ with $D_k \in D$ such that

$$\{f_i: A_i \rightarrow B_j\}_{i \in f^{-1}(j)} \cup \{D_k \rightarrow B_k\}_{k \in K}$$

is a finite disjoint covering family in $C$. Then $W(C, D)$ is a subcategory of $W(C \setminus D)$ with inclusion functor

$$I: W(C, D) \rightarrow W(C \setminus D).$$

**Example 7.2.** Let $C$ be the assembler whose objects are open, half-open and closed segments in $\mathbb{R}$, together with the points of $\mathbb{R}$, and whose morphisms are compositions of translations and inclusions. Let $D$ be the subassembler containing only points. Then $C \setminus D$ has as its objects all segments of nonzero length, and the covering families in $W(C \setminus D)$ are families $\{f_i: A_i \rightarrow B\}$ such that $f_i(A_i) \cap f_j(A_j)$ is a finite set. On the other hand, morphisms of $W(C, D)$ consist of those covering families whose endpoints do not intersect at all. Even though these categories are different, we will prove in Section 7.3 that they are homotopy equivalent.

**Example 7.3.** Suppose that $C$ is the following preorder

![Diagram](https://via.placeholder.com/150)

with the only nontrivial covering family being $\{B \rightarrow D, C \rightarrow D\}$. Suppose that $D$ is the full subcategory with $\text{ob } D = \{\emptyset, A\}$. Then $W(C \setminus D)$ has objects triples of integers $(b, c, d)$ which count how many copies of $B$, $C$ and $D$ (respectively) the object contains. There exist morphisms $(b, c, d) \rightarrow (b-k, c-k, d+k)$ for all $k \leq \min(b, c)$. However, no covering family with objects in $C \setminus D$ has a completion to a finite disjoint covering family in $C$, so $W(C, D)$ is the maximal subgroupoid of $W(C \setminus D)$. In this case, $I$ is not a homotopy equivalence.

Recall that $P = NW(p)$. By considering $Y$, $NW(C, D)$ and $NW(C \setminus D)$ as bisimplicial sets which are constant in the vertical direction, we can factor $P$ as

$$NW((C/i)_0) \xrightarrow{p_1} Y \xrightarrow{\kappa} NW(C, D) \xrightarrow{N} NW(C \setminus D),$$

where $p_1$ is defined on a simplex in $N_m W((C/i)_0)$ by

$$p_1(A_0 \rightarrow \cdots \rightarrow A_m) = (A_0 \rightarrow \cdots \rightarrow A_m)_{C \setminus D},$$

and defined on simplices in $N_m W((C/i)_0)$ by $p_1 \circ d^n_0$. Here, the use of $\kappa$ is a slight abuse of notation, as $\kappa$ is a functor and not a morphism of simplicial sets;
however, as the map we want applies $\kappa$ to each $n$-simplex, we use it for clarity. Thus it suffices to show that $p_1$, $\kappa$ and $I$ are weak equivalences after geometric realization; this is shown in Sections [7.1, 7.2] and [7.3] respectively.

The[4]orem [4] has two conditions: that $D$ is a sieve in $C$ and that $C$ has all complements for all objects in $D$. It turns out that the second condition is only used to prove that $NI$ is a homotopy equivalence, so we see that the map

$$|W((C/i),)| \rightarrow |W(C, D)|$$

is always a homotopy equivalence when $D$ is a sieve in $C$.

### 7.1. The map $p_1$

First we consider $p_1$. We begin with some technical results.

**Lemma 7.4.** Write

$$C = (A_0 \rightarrow \cdots \rightarrow A_n) \in N_nW(C),$$

$$D = (B_0 \rightarrow \cdots \rightarrow B_n) \in N_nW(D),$$

and assume that for all $i$, the indexing sets of $A_i$ and $B_i$ are disjoint. We write

$$C \cup D$$

for the diagram

$$A_0 \cup B_0 \rightarrow A_1 \cup B_1 \rightarrow \cdots \rightarrow A_n \cup B_n,$$

where we use $\cup$ instead of $\sqcup$ to emphasize that we only need to take simple unions of the indexing sets to form the coproduct. Then we have

$$C = C_{C \cup D} \cup C_D \quad (C \cup D)_{C \setminus D} = C_{C \setminus D} \quad (C \cup D)_D = C_D \cup D.$$

**Proof.** This follows directly from the definitions. The reason we need the assumption on indexing sets is that if the indexing sets are not disjoint then we may end up taking different versions of the disjoint union on the two sides of the third equality. $\square$

Since $W(C \cup D'^k)$ is a full subcategory of $W(C) \times W(D)^k$ consisting of those tuples with disjoint indexing sets, we can consider a diagram $A_0 \rightarrow \cdots \rightarrow A_m$ in $W(C \cup D'^k)$ as a $k+1$-tuple of diagrams $(C, D_1, \ldots, D_k)$ with $C$ in $N_mW(C)$ and $D_i$ in $N_mW(D)$ for all $i$. Note that, by definition, for each $j = 0, \ldots, k$ the $j$-th objects in all diagrams have disjoint indexing sets. In the subsequent proofs we will often be using this identification.

**Lemma 7.5.** The map $p_1: NW((C/i),.) \rightarrow Y$ is a map of bisimplicial sets.

**Proof.** The fact that $p_1$ commutes with the horizontal simplicial structure maps follows directly from the definitions. It remains to check the vertical simplicial direction. As all vertical simplicial structure maps on $Y$ are trivial, we just need to check that applying vertical simplicial structure maps does not affect the image under $p_1$. Consider $A_0 \rightarrow \cdots \rightarrow A_m$ in $N_mW(C \cup D'^k)$ as a $k+1$-tuple of diagrams $(C, D_1, \ldots, D_k)$ By Lemma 7.4, $(C, D_1, \ldots, D_k)_{C \setminus D} = C_{C \setminus D},$
so we can conclude that $p_1$ commutes with all of the vertical simplicial structure maps other than $d_0$. But (again by Lemma [7.4])

$$p_1 d_0(C, D_1, \ldots, D_k) = p_1 (C \cup D_1, \ldots, D_k) = (C \cup D_1)_{C \setminus \mathcal{D}}$$

$$= C_{C \setminus \mathcal{D}} = p_1 (C, D_1, \ldots, D_k),$$

so $p_1$ commutes with $d_0$ as well. \qed

We are now ready to prove that $p_1$ is an equivalence after geometric realization. We do this by showing that it is a levelwise homotopy equivalence.

**Lemma 7.6.** For all $n \geq 0$, the simplicial set $X_n = N_n \mathcal{W}((C/i)_n)$ is homotopically discrete, and $\pi_0 X_n = Y_n$. When restricted to $X$, we have $p_1 = \pi_0$.

**Proof.** Write $X_{-1} = Y_n$, and consider $p_1 : X_0 \rightarrow Y_n$ as an augmentation of $X$. An augmented simplicial set with an extra degeneracy contracts onto its augmentation [5, Lemma III.5.1], so it remains for us only to construct $s_{-1} : X_m \rightarrow X_{m+1}$ for $m \geq -1$.

We define the map $s_{-1} : X_m \rightarrow X_0$ to be the inclusion. For $n \geq 0$, we define the map $s_{-1} : X_m \rightarrow X_{m+1}$ in the following manner. Consider a diagram $D = (A_0 \rightarrow \cdots \rightarrow A_n)$ in $\mathcal{W}((C/i)_m)$. Write this diagram as an $m+1$-tuple of diagrams $(C, D_1, \ldots, D_m)$ and define $s_{-1}(D) = (C_{C \setminus \mathcal{D}}, C_{\mathcal{D}}, D_1, \ldots, D_m)$.

Checking that $s_{-1}$ satisfies the conditions to be an extra degeneracy and that $p_1$ is an augmentation is a straightforward application of Lemma [7.4]. \qed

7.2. The map $\kappa$

We now move on to $\kappa$. We begin with several technical results. When the proofs are straightforward from definitions we omit them. Recall that we write $\kappa = \mathcal{W}(c)$.

**Lemma 7.7.** Let $C$ be a closed assembler. Suppose that $f : A \rightarrow C$ and $g : B \rightarrow C$ in $\mathcal{W}(C)$ are two morphisms such that there exists $h : \kappa(A) \rightarrow \kappa(B)$ in $\mathcal{W}(C \setminus \mathcal{D})$ making the diagram

$$\begin{array}{ccc}
\kappa(A) & \xrightarrow{\kappa(f)} & \kappa(C) \\
\downarrow{h} & & \downarrow{\kappa(g)} \\
\kappa(B) & & \kappa(B)
\end{array}$$

commute. Then for any pullback square

$$\begin{array}{ccc}
D & \xrightarrow{g'} & B \\
\downarrow{f'} & & \downarrow{f} \\
A & \xrightarrow{g} & C
\end{array}$$

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we have $\kappa(D) = A$, $\kappa(f') = 1_{\kappa(A)}$ and $\kappa(g') = h$.

**Lemma 7.8.** If $f: A \rightarrow B$ in $\mathcal{W}(C)$ has $B_D = \emptyset$ and $\kappa(f) = 1_B$ then $f = 1_B$.

**Lemma 7.9.** Suppose that we have a square

\[
\begin{array}{ccc}
\{A_i\}_{i \in I} & \xrightarrow{f} & \{B_j\}_{j \in J} \\
\downarrow{f'} & & \downarrow{g} \\
\{C_k\}_{k \in K} & \xrightarrow{g'} & \{D_l\}_{l \in L}
\end{array}
\]

in $\mathcal{W}(C)$ such that $\kappa(g) = \kappa(g')$. Then $f_{C \smallsetminus D} = f'_{C \smallsetminus D}$.

**Proof.** Let $\tilde{J} = \{ j \in J \mid B_j \in C \smallsetminus D \}$, and define $\tilde{K}$ analogously. As $\kappa(g) = \kappa(g')$ it follows that $\tilde{J} = \tilde{K}$. We first show that $f^{-1}(\tilde{J}) = (f')^{-1}(\tilde{K})$. Suppose that $i \in I$ is such that $f(i) \in \tilde{J}$. Then we have the following square in $\mathcal{C}$:

\[
\begin{array}{ccc}
A_i & \xrightarrow{f_i} & B_{f(i)} \\
\downarrow{f'_i} & & \downarrow{g_{f(i)}} \\
C_{f'(i)} & \xrightarrow{g'_{f(i)}} & D_{gf(i)}
\end{array}
\]

Thus the morphisms $g_{f(i)}$ and $g'_{f'(i)}$ are not disjoint; since both appear in $\kappa(g)$ we must have $f(i) = f'(i)$, and thus $f'(i)$ is in $\tilde{K}$. Thus $f^{-1}(\tilde{J}) \subseteq (f')^{-1}(\tilde{K})$, and by symmetry these sets are equal. But we also know that

$$g_{f(i)} = g'_{f(i)} = g'_{f'(i)}$$

since all morphisms in $\mathcal{C}$ are monic, we can conclude that $f_i = f'_i$. Thus $f$ and $f'$ are equal when restricted to $\{A_i\}_{i \in f^{-1}(\tilde{J})}$, which implies that $f_{C \smallsetminus D} = f'_{C \smallsetminus D}$. \qed

**Lemma 7.10.** Let $X_\bullet$ be a simplicial set. Suppose that for every $m$-cycle $\epsilon = \sum_{i=1}^k c_i x_i$ with $x_i \in X_m$ there exist simplices $\tilde{x}_i \in X_{m+1}$ such that $d_0 \tilde{x}_i = x_i$ and whenever $d_j x_i = d_j' x_i'$ it is also the case that $d_{j+1} \tilde{x}_i = d_{j+1} \tilde{x}_i'$. Then $H_m X = 0$.

**Proof.** Observe that for all $i$ and $j$, the terms $d_j x_i$ are elements of a basis for a free abelian group, so that the only way that $\partial \epsilon = \sum_{i=1}^k c_i \sum_{j=0}^m (-1)^j d_j x_i$ can be zero is if the coefficients of each basis element are 0. Therefore the $d_j x_i$ match up in some way which ensures that the coefficients of the simplices in the end are zero.
Let $\beta = \sum_{i=1}^{k} c_i \tilde{x}_i$. Then
\[
\partial \beta = \partial \sum_{i=1}^{k} c_i \tilde{x}_i = \sum_{i=1}^{k} \left( \sum_{j=1}^{m+1} (-1)^j c_i d_j \tilde{x}_i + c_i d_0 \tilde{x}_i \right) = \sum_{i=1}^{k} c_i \sum_{j=1}^{m+1} (-1)^j d_j \tilde{x}_i + \epsilon = \epsilon.
\]
The last step follows because the construction of $\tilde{x}_i$ ensures that the terms in the sum cancel the same way that they did for $\epsilon$.

To show that $\kappa$ is a weak equivalence, we use the simplicial version of Quillen’s Theorem A (see [20, Proposition 1.4.A]), which states that the following lemma implies that $\kappa$ is a homotopy equivalence.

**Lemma 7.11.** Let $\alpha: \Delta^n \to NW(C \setminus D)$ be any $n$-simplex of $NW(C, D)$. Define $\kappa/(n, \alpha)$ to be the pullback of the diagram
\[
\Delta^n \xrightarrow{\alpha} NW(C, D) \xleftarrow{\kappa} Y.
\]
Then $\kappa/(n, \alpha)$ is contractible for all $n$ and $\alpha$.

**Proof.** We prove this by showing three things:

(a) $\kappa/(n, \alpha)$ is connected,

(b) $\pi_1(\kappa/(n, \alpha)) = 0$, and

(c) for all $i > 0$, $H_i(\kappa/(n, \alpha)) = 0$.

Together, these imply that all homotopy groups of $\kappa/(n, \alpha)$ are trivial, showing that it is contractible. Write
\[
\alpha = A_0 \longrightarrow A_1 \longrightarrow \cdots \longrightarrow A_n \in N_n W(C, D),
\]
and consider $\Delta^n$ to be the nerve of $0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow n$. Fix a lift
\[
\tilde{\alpha} = A_0 \sqcup D_0 \longrightarrow A_1 \sqcup D_1 \longrightarrow \cdots \longrightarrow A_n \in Y_n
\]
of $\alpha$ along $\kappa$, where $D_i$ is in $W(D)$; such a lift exists by the definition of $W(C, D)$. Note that there is no $D_n$ at the end of the diagram, as all elements of $Y_n$ have last term in $W(C \setminus D)$. We prove (a)-(c) in order.

Proof of (a): The vertices of $\kappa/(n, \alpha)$ are pairs $(i, A)$ for $0 \leq i \leq n$ and $A$ in $Y_0$ such that $\alpha(i) = \kappa(A)$. By definition, the elements of $Y_0$ are exactly objects in $W(C \setminus D)$, and $\kappa$ is the identity of these. Thus the vertices of $\kappa/(n, \alpha)$ are pairs $(i, A_i)$. The vertices of $\tilde{\alpha}$ are $A_0, A_1, \ldots, A_n$. Thus there exists a simplex of $\kappa/(n, \alpha)$ which contains all of its vertices; since a simplex is connected $\kappa/(n, \alpha)$ is, as well.
Proof of (b): An element of $\pi_1(\kappa/(n, \alpha))$ is represented by a zigzag

$$i_0 \xrightarrow{\beta_0} i_1 \xleftarrow{\beta_1} i_2 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_m} i_m \in \Delta^n,$$

with $i_0 = i_m$, together with morphisms $f_j : A_{i_j} \sqcup E_j \to A_{i_{j'}}$ in $Y_1$, where $\beta_j : i_j \to i_{j'}$ and $E_j$ is in $\mathcal{W}(\mathcal{D})$. We show that any such zigzag is null-homotopic. In this, we use the usual approach for replacing zigzags in categories closed under pullbacks: we show that we can replace any two arrows going in the same direction by a “composition”, and then we show that we can replace arrows with the same codomain by arrows with the same domain using the pullback. These two steps allow us to reduce the problem to zigzags of length 1 or 2. Finally, we analyze these two cases. We include all details of this approach because the unusual simplicial structure on $Y$ introduces several subtleties.

First, suppose that two successive $\beta_j$’s go in the same direction; without loss of generality, we assume that $\beta_j : i_j \to i_{j+1}$ and $\beta_{j+1} : i_{j+1} \to i_{j+2}$. Then we can construct a 2-simplex represented by

$$A_{i_j} \sqcup E_j \sqcup E_{j+1} \xrightarrow{f_j \sqcup 1_{E_{j+1}}} A_{i_{j+1}} \sqcup E_{j+1} \xrightarrow{f_{j+1}} A_{i_{j+2}},$$

which shows that this zigzag is homotopic to the one where $\beta_j$ and $\beta_{j+1}$ are replaced by $i_j \to i_{j+2}$ with morphism $A_{i_j} \sqcup (E_j \sqcup E_{j+1}) \to A_{i_{j+2}}$.

Now suppose that $\beta_j : i_j \to i_{j+1}$ and $\beta_{j+1} : i_{j+2} \to i_{j+1}$ and assume without loss of generality that $i_j \leq i_{j+2}$. Take the pullback of $f_j$ and $f_{j+1}$ to form a square

$$
\begin{array}{ccc}
A_{i_j} \sqcup E & \xrightarrow{f_j'} & A_{i_{j+2}} \sqcup E_{j+2} \\
\downarrow f_{j+1} & & \downarrow f_{j+1} \\
A_{i_j} \sqcup E_j & \xrightarrow{f_j} & A_{i_{j+1}} \\
\end{array}
$$

The pullback is of the form $A_{i_j} \sqcup E$ by Lemma 7.7 and because $i_j \leq i_{j+2}$. Thus we can replace these two simplices in the zigzag by the two simplices represented by

$$(i_j \to i_j, (A_{i_j} \sqcup E \to A_{i_j} \sqcup E_j)c \setminus d)$$

and

$$(i_j \to i_{j+1}, (A_{i_j} \sqcup E \to A_{i_{j+1}} \sqcup E_{j+1})c \setminus d),$$

thus reversing the directions of the two arrows in the zigzag. (By Lemma 7.8 the first of these is trivial, but that does not matter for the current analysis.)

We now only need to consider zigzags of length 1 or 2. A zigzag of length 1 is a morphism $A_i \sqcup E \to A_i$ which projects down to the identity morphism in $\mathcal{W}(\mathcal{C} \setminus \mathcal{D})$. By Lemma 7.8 this morphism must be the identity morphism, and thus such a zigzag is represented by a degenerate simplex, and is therefore
contractible. Now consider a zigzag of length 2. Such a zigzag is represented by
a pair of morphisms

\[ A_i \xrightarrow{f_1} A_j \sqcup E \xrightarrow{f_2} A_i. \]

We thus have a pullback square

\[
\begin{array}{ccc}
A_j \sqcup E'' & \xrightarrow{f_2'} & A_j \sqcup E' \\
\downarrow{h} & & \downarrow{f_2} \\
A_j \sqcup E & \xrightarrow{f_1} & A_i
\end{array}
\]

where the pullback is \( A_j \sqcup E'' \) and \( (f_1')_C \times D = (f_2')_C \times D = 1_{A_j} \) by Lemma 7.7.

We can replace this zigzag by

\[ (j \to i, h), (j \to j, (f_1')_C \times D), (j \to j, (f_2')_C \times D), (j \to i, h). \]

As \( (f_1')_C \times D = (f_2')_C \times D = 1_{A_j} \), the middle two simplices are degenerate and can be removed. The outside two simplices are just the same simplex repeated twice, so we see that this zigzag is also null-homotopic.

Proof of (c): Let \( m \geq 2 \), and let \( \epsilon = \sum_{i=1}^{k} c_i(\alpha_i, x_i) \) be an \( m \)-cycle in \( \kappa/(n, \alpha) \).

We use Lemma 7.10 to prove that \( \epsilon = \partial \beta \) for some \( \beta \).

We write \( x_i \in Y_m \) as \( X_{i0} \to X_{i1} \to \cdots \to X_{im} \),

where \( X_{im} = A_{\alpha_i(m)} \). For each simplex \((\alpha_i, x_i)\) we have a diagram

\[ X_{i0} \sqcup D_{\alpha_i(m)} \to \cdots \to X_{im} \sqcup D_{\alpha_i(m)} \to A_n \in \mathcal{W}(\mathcal{C}). \]

Let \( \mathcal{I} \) be the category with

\[ \text{ob} \mathcal{I} = \{1, \ldots, k\} \times \{0, \ldots, m+1\}/((i, m+1) \sim (i', m+1)) \]

for all \( 1 \leq i, i' \leq k \), and

\[ \mathcal{I}((i, j), (i', j')) = \begin{cases} * & \text{if } i = i' \text{ and } j \leq j', \\ * & \text{if } j' = m+1 \\ \emptyset & \text{otherwise.} \end{cases} \]

Let \( \chi: \mathcal{I} \to \mathcal{W}(\mathcal{C}) \) be the functor taking the pair \((i, j)\) to \( X_{ij} \sqcup D_{\alpha_i(m)} \) and the morphisms to the morphisms defined by the simplices above. We define \( \chi(i, m+1) = A_n \). Let \( \tilde{X} \) be the limit of this diagram, which exists because \( \mathcal{W}(\mathcal{C}) \) has pullbacks, and write \( f_i: \tilde{X} \to X_{i0} \). Let \( \mu = \min_{i} \alpha_i(0); \) note that by repeated application of Lemma 7.7 we have \( \tilde{X}_{C \times D} = A_{\mu} \). Let \( \tilde{\alpha}_i: [m+1] \to [n] \)
be defined by $\tilde{\alpha}_i(j) = \alpha_i(j - 1)$ for $j > 0$, and $\tilde{\alpha}_i(0) = \mu$, and let $\tilde{x}_i$ be the simplex represented by the diagram

$$(\tilde{X} \xrightarrow{f_i} X_{i0} \sqcup D_{\alpha_i(m)} \xrightarrow{f} X_{i1} \sqcup D_{\alpha_i(m)} \rightarrow \cdots \rightarrow X_{im} \sqcup D_{\alpha_i(m)})_{\mathcal{C} \prec \mathcal{D}},$$

which we write

$$\tilde{X}_i \xrightarrow{h_i} X_{i0} \xrightarrow{f} X_{i1} \rightarrow \cdots \rightarrow X_{im}.$$

By definition, $d_0(\tilde{\alpha}_i, \tilde{x}_i) = (\alpha_i, x_i)$, so to apply Lemma 7.10 it remains to check that if $d_j(\alpha_i, x_i) = d_j'(\alpha_{i'}, x_{i'})$ then $d_{j+1}(\tilde{\alpha}_i, \tilde{x}_i) = d_{j+1}(\tilde{\alpha}_{i'}, \tilde{x}_{i'})$.

Let $0 \leq j, j' \leq m$. We want to show that if $d_j(\alpha_i, x_i) = d_j'(\alpha_{i'}, x_{i'})$ then $d_{j+1}(\tilde{\alpha}_i, \tilde{x}_i) = d_{j+1}(\tilde{\alpha}_{i'}, \tilde{x}_{i'})$. We need to show that the two diagrams

$$D = (\tilde{X} \rightarrow X_{i0} \sqcup D_{\alpha_i(m)} \rightarrow \cdots \rightarrow \tilde{X}_{ij} \sqcup D_{\alpha_i(m)} \rightarrow \cdots \rightarrow X_{im} \sqcup D_{\alpha_i(m)})_{\mathcal{C} \prec \mathcal{D}}$$

and

$$D' = (\tilde{X} \rightarrow X_{i'0} \sqcup D_{\alpha_{i'}(m)} \rightarrow \cdots \rightarrow \tilde{X}_{i'j'} \sqcup D_{\alpha_{i'}(m)} \rightarrow \cdots \rightarrow X_{i'm} \sqcup D_{\alpha_{i'}(m)})_{\mathcal{C} \prec \mathcal{D}}$$

are equal. Note that

$$D = Y \rightarrow X_{i0} \rightarrow \cdots \rightarrow \tilde{X}_{ij} \rightarrow \cdots \rightarrow X_{im}$$

for some $Y$, and analogously for $D'$. The assumption that $d_j x_i = d_{j'} x_{i'}$ implies that the composition of all but the first morphisms are the same in both diagrams, so (as all morphisms in $W(\mathcal{C})$ are monic by Proposition 2.11(1)) it suffices to show that the total composition of $D$ is the same as the total composition of $D'$.

For the following discussion, we assume that $0 < j, j' < m$; however, as this is assumed only for ease of notation, it does not affect the proof. We have the following diagram:

```
   X_{i0} \sqcup D_{\alpha_i(m)} \xrightarrow{f \sqcup 1} X_{im} \sqcup D_{\alpha_i(m)}

\tilde{X} \xrightarrow{f_i} \tilde{X}_{i0} \sqcup D_{\alpha_i(m)} \xrightarrow{f' \sqcup 1} \tilde{X}_{i'm} \sqcup D_{\alpha_{i'}(m)}
```

As $d_j \alpha_i = d_{j'} \alpha_{i'}$, we know that $\alpha_i(m) = \alpha_{i'}(m)$, and in particular $\kappa(g) = \kappa(g')$. Thus Lemma 7.9 applies, and

$$h_i f = (f_i(f \sqcup 1))_{\mathcal{C} \prec \mathcal{D}} = (f'_{i'}((f' \sqcup 1)))_{\mathcal{C} \prec \mathcal{D}} = h_{i'} f'.'
$$

(When $j$ and $j'$ are either 0 or $m$ this just affects the indices in the above diagram; we define $f$ and $f'$ to be the full compositions in $d_j x_i$ and $d_{j'} x_{i'}$, respectively.)
7.3. The functor \( I \)

Lastly we consider \( I \). First, we need the following technical result, which allows us to remove “errors” in extending weak equivalences in \( \mathcal{W}(\mathcal{C} \setminus \mathcal{D}) \) to weak equivalences in \( \mathcal{W}(\mathcal{C}) \).

**Lemma 7.12.** Suppose that \( \mathcal{C} \) is a closed assembler and has complements for all objects in \( \mathcal{D} \). Suppose we are given a finite collection of morphisms \( f_i: A_i \to B \) in \( \mathcal{C} \) such that for \( i \neq i' \), \( A_i \times_B A_{i'} \) is in \( \mathcal{D} \). Then there exists a morphism \( g: \{Z_k\}_{k \in K} \to \{A_i\}_{i \in I} \) in \( \mathcal{W}(\mathcal{C}) \) such that for all \( k \neq k' \in K \) the morphisms \( f_{g(k)}g_k \) and \( f_{g(k')}g_{k'} \) are either equal or disjoint. If they are equal then \( g(k) \neq g(k') \).

**Proof.** We prove this by induction on \( |I| \). If \( |I| \leq 1 \) then the lemma holds tautologically, so we only need to check the inductive step.

Pick \( i_0 \in I \), and consider the collection \( \{f_i\}_{i \neq i_0} \). The conditions of the lemma apply to this collection, and thus there exists \( g: \{Z_k\}_{k \in K} \to \{A_i\}_{i \neq i_0} \) satisfying the conditions of the lemma. As \( \mathcal{C} \) has complements for all objects of \( \mathcal{D} \) and \( \mathcal{D} \) is a sieve in \( \mathcal{C} \), \( \mathcal{C} \) has complements for \( Z_k \times_B A_{i_0} \) (which sits above \( A_{g(k)} \times_B A_{i_0} \) for all \( k \in K \). Let \( \mathcal{G}_k \) be the finite disjoint covering family of \( Z_k \) which includes the morphism \( Z_k \to Z_{k'} \) where any diagram \( f \) of Quillen’s Theorem A.

A morphism \( \{W_i\}_{i \in I} \to \{A_i\}_{i \in I} \) is a collection of finite disjoint covering families, one for each \( i \in I \). Thus in order to construct the desired morphism, it suffices to give a covering family for each \( i \in I \). For \( i_0 \), we take \( \mathcal{G} \). For \( i \neq i_0 \), take a refinement of \( \{g_k: Z_k \to A_i\}_{k \in g^{-1}(i)} \) by the relevant \( \mathcal{G}_k \). This then is the desired morphism.

The last condition in the lemma follows because if \( g(k) = g(k') \) then (as all morphisms are monic) \( g_k = g_{k'} \) and the covering family

\[ \{g_i: Z_i \to A_{g(k)}\}_{i \in g^{-1}(g(k))} \]

is not a finite disjoint covering family; hence, \( g \) is not a valid morphism of \( \mathcal{W}(\mathcal{C}) \), which is a contradiction. Thus \( g(k) \neq g(k') \) if the two morphisms \( f_{g(k)}g_k \) and \( f_{g(k')}g_{k'} \) are equal.

In order to prove that \( |I| \) is a homotopy equivalence we need a special case of Quillen’s Theorem A.

**Lemma 7.13.** Let \( \mathcal{C} \) be a category where all morphisms are monomorphisms, and where any diagram \( B \to A \leftarrow C \) can be completed to a commutative square. Suppose that \( \mathcal{D} \) is a subcategory which contains all objects and has the property that for any morphism \( f: A \to B \) in \( \mathcal{C} \) there exists a morphism \( g: Z \to A \) in \( \mathcal{D} \) such that \( fg \) is in \( \mathcal{D} \). Then the inclusion \( F: \mathcal{D} \to \mathcal{C} \) is a homotopy equivalence.
Proof. We use Quillen’s Theorem A to show that this functor is a homotopy equivalence. As all morphisms are monomorphisms in \( C \), \( C/A \) is a preorder for all \( A \). Thus \( F/A \) is also a preorder, and by Lemma 4.3 it suffices to show that it is cofiltered to show that it is contractible. Suppose that we are given two objects \( f: B \rightarrow A \) and \( g: C \rightarrow A \) in \( F/A \). We then have the following diagram:

\[
\begin{array}{ccc}
Z & \xrightarrow{\beta} & Y \\
\downarrow{g'} & & \downarrow{g} \\
B & \xrightarrow{f} & A
\end{array}
\]

Here, \( f' \) and \( g' \) complete \( f \) and \( g \) to a commutative square, as we assumed was possible in \( C \). Let \( \alpha \) be a morphism \( Y \rightarrow X \) in \( \mathcal{D} \) such that \( g'\alpha \) is in \( \mathcal{D} \); there exists such an \( \alpha \) by the property in the lemma. Similarly, let \( \beta \) be a morphism \( Z \rightarrow Y \) in \( \mathcal{D} \) such that \( f'\alpha\beta \) is in \( \mathcal{D} \). Thus the morphism \( Z \rightarrow C \) is in \( \mathcal{D} \), and the morphism \( Z \rightarrow B \) is in \( \mathcal{D} \), so the object \( g\beta \): \( Z \rightarrow A \) is an object of \( F/A \) over both \( f \) and \( g \).

\[
|I| \text{ is a homotopy equivalence.}
\]

Lemma 7.14. \( |I| \) is a homotopy equivalence.

Proof. We show that \( |I| \) is a homotopy equivalence using Lemma 7.13. In \( \mathcal{W}(\mathcal{C} \setminus \mathcal{D}) \) all morphisms are monic and any diagram \( B \rightarrow A \leftarrow C \) can be completed to a commutative square. The only property that remains to be checked is that for any morphism \( f: A \rightarrow B \) in \( \mathcal{W}(\mathcal{C} \setminus \mathcal{D}) \), there exists a morphism \( g: Z \rightarrow A \) in \( \mathcal{W}(\mathcal{C}, \mathcal{D}) \) such that \( fg \) is in \( \mathcal{W}(\mathcal{C}, \mathcal{D}) \). As any morphism in \( \mathcal{W}(\mathcal{C} \setminus \mathcal{D}) \) can be written as a coproduct of morphisms \( f: \{A_i\}_{i \in I} \rightarrow \{B\} \) it suffices to check that \( g \) exists for such morphisms.

Consider any morphism \( f: \{A_i\}_{i \in I} \rightarrow \{B\} \in \mathcal{W}(\mathcal{C} \setminus \mathcal{D}) \). The family \( \{f_i: A_i \rightarrow B\}_{i \in I} \) can be completed to a covering family in \( \mathcal{C} \) by morphisms with domains in \( \mathcal{D} \); write this whole family \( \{f_i: A_i \rightarrow B\}_{i \in I} \). It satisfies the condition of Lemma 7.12 and thus we have a

\[
\tilde{g}: Z = \{Z_k\}_{k \in K} \rightarrow \{A_i\}_{i \in I} \in \mathcal{W}(\mathcal{C})
\]

such that for all \( k \neq k' \in K \), \( f_{g(k)}g_k \) and \( f_{g(k')}g_{k'} \) are either disjoint or equal. Thus the family \( \mathcal{F} = \{f_{g(k)}g_k: Z_k \rightarrow B\}_{k \in K} \) is a finite covering family, but it may not be disjoint because for some \( k \neq k' \) \( f_{g(k)}g_k \) might equal \( f_{g(k')}g_{k'} \). However, in this case, \( Z_k \) can complete \( f_{g(k)} \) and \( f_{g(k')} \) to a square. By Lemma 7.12 in this case \( g(k) \neq g(k') \), which means that \( Z_k \) is in \( \mathcal{D} \). Thus the only obstructions to \( \mathcal{F} \) defining a morphism of \( \mathcal{W}(\mathcal{C}) \) is duplication of some morphisms with domains in \( \mathcal{D} \). Consequently, the morphism defined by the family

\[
\{f_{g(k)}g_k: Z_k \rightarrow B\}_{k \in K},
\]
where $\tilde{K} = \{ k \in K \mid Z_k \in C \setminus D \}$, is a valid morphism of $W(C, D)$. We therefore define $g: \{ Z_k \}_{k \in \tilde{K}} \rightarrow \{ A_i \}_{i \in I}$ to be the restriction of $\hat{g}$ to $\tilde{K}$; this also gives a valid morphism of $W(C, D)$.

To check that $g$ satisfies the conditions of Lemma 7.13 it remains to check that the composition $\tilde{f}g$ is in $W(C, D)$. $\tilde{f}g$ is the morphism $\{ Z_k \}_{k \in \tilde{K}} \rightarrow \{ B \}$; we need to check that this covering family can be completed to a finite disjoint covering family in $C$. Consider the set $\{ f_{\tilde{g}(k)} \} \} | k \in K \}$. These are a covering family of $B$ in $C$, as they are the refinement of a covering family $\{ f_i: A_i \rightarrow B | i \in I \}$ by the covering families $\{ Z_k \rightarrow A_i | k \in \tilde{g}^{-1}(i) \}$. From the construction of $\tilde{g}$, all distinct elements in the set are disjoint, so this is a finite disjoint covering family which is the completion of the family we are considering.

This wraps up the proof of Theorem D.

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