ASSEMBLING LIE ALGEBRAS FROM LIEONS.

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Abstract. If a Lie algebra structures $\mathfrak{g}$ on a vector space is the sum of a family of mutually compatible Lie algebra structures $\mathfrak{g}_i$, we say that $\mathfrak{g}$ is simply assembled from $\mathfrak{g}_i$'s. By repeating this procedure several times one gets a family of Lie algebras assembled from $\mathfrak{g}_i$'s. The central result of this paper is that any finite dimensional Lie algebra over $\mathbb{R}$ or $\mathbb{C}$ can be assembled from two constituents, called $\hat{\odot}$- and $\hat{\diamond}$-lieons. A lieon is the direct sum of an abelian Lie algebra with a 2-dimensional nonabelian Lie algebra or with the 3-dimensional Heisenberg algebra.

Some techniques of disassembling Lie algebras are introduced and various results concerning assembling-disassembling procedures are obtained. In particular, it is shown how classical Lie algebras are assembled from lieons and is obtained the complete list of Lie algebras, which can be simply assembled from lieons.
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Introduction

According to the modern view, matter is of a compound nature. The constituents, elementary particles, are characterized by their symmetry properties. Since these properties are formalized in terms of Lie algebras, one may hypothesize that the compound nature of matter is somehow mirrored in the structure of symmetry algebras. This and some other similar considerations lead to a suspicion that Lie algebras possess, in a sense, compound structure. The study, some results of which are presented in this paper, was motivated by this question. The main result we have found is that finite dimensional Lie algebras over \( \mathbb{R} \) and \( \mathbb{C} \) are made from two "elementary particles", which we call lieons.

Obviously, prior to approaching "elementary particle theory" of "Lie matter" the exact meaning of "made from" in the context of Lie algebras should be established. A suggestion of how it could be done comes from Poisson geometry. Namely, from the one hand, a Lie algebra is naturally interpreted as a linear Poisson structure on its dual. On the other hand, it is natural to think that a Poisson structure/bivector \( \mathcal{P} \) is "made from" Poisson structures \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \), if \( \mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2 \). In such a case \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) are called compatible. So, by translating this idea into the language of Lie algebras we get the following.

Lie algebra structures \([\cdot, \cdot]_1\) and \([\cdot, \cdot]_2\) on a vector space \( V \) are compatible, if \([\cdot, \cdot]_1 + [\cdot, \cdot]_2\) is a Lie algebra structure as well. If a Lie algebra structure \([\cdot, \cdot]\) is presented in the form 
\[
[\cdot, \cdot] = [\cdot, \cdot]_1 + \cdots + [\cdot, \cdot]_m
\]
with mutually compatible structures \([\cdot, \cdot]_i\)'s, we speak of a disassembling of \([\cdot, \cdot]\), i.e., that \([\cdot, \cdot]\) is "made from" \([\cdot, \cdot]_i\)'s. Note that compatibility is not a transitive notion. Hence it has sense to go ahead by disassembling all compounds \([\cdot, \cdot]_i\). This way we get a 2-step disassembling of \([\cdot, \cdot]\), and so on. The central result of this paper states that any \( n \)-dimensional Lie algebras can be in this sense disassembled up to lieons \( J_n \) and \( \delta_n \), called \( J \) - and \( \delta \)-lieons, respectively. Here \( J_n = J \oplus \gamma_{n-2} \) and \( \delta_n = \delta \oplus \gamma_{n-3} \) with \( \gamma_m \) standing for the \( m \)-dimensional abelian Lie algebra, \( J \) for a non-abelian 2-dimensional algebra (all such algebras are isomorphic) and \( \delta \) for the 3-dimensional Heisenberg algebra. For instance, the algebra \( \mathfrak{u}(2) = \mathfrak{so}(3) \) can be disassembled into 3 pieces each of which is equivalent to \( \delta \). Speculatively, one
might interpret this fact by saying that $\mathfrak{u}(2)$, the symmetry algebra of nucleons, is composed of $3 \oplus$-lieons, each of them is the symmetry algebra of a hypothetical particle called, say, “quark”, etc. We, however, do not discuss eventual physical applications in this paper by concentrating only on purely mathematical questions.

The concept of compatible Poisson structures originates in F. Magri’s work [11] on bihamiltonian systems, and was subsequently developed and exploited by many authors in the context of integrable systems and Poisson geometry. However, as far as we know, it was not systematically studied in theory of Lie algebras. Also, it worth mentioning that translation of techniques and constructions of Poisson geometry into context of Lie algebras is very useful, and we exploit this possibility at full scale. This is why some parts of this paper are dedicated to necessary elements of Poisson geometry.

The contents, results and organization of the paper are as follows. Generalities concerning differential forms, multivector fields and Schouten bracket formalism we need are collected in section 2. Here we recall notions of compatibility of Poisson and Lie algebra structures and discuss their simplest properties.

Modularity properties of Poisson and Lie algebra structures are considered in section 3. Here we specify the general compatibility conditions for unimodular Poisson structures and as a result prove that the Lie rank of an unimodular Lie algebra is strictly lesser than its dimension. Then we show that a Poisson structure $P$ can be disassembled into unimodular and completely non-unimodular parts. The Poisson bivector corresponding to the non-unimodular part is $P_\nu \wedge \Xi = [P_\nu, \nu \Xi]$ where $\Xi$ is the modular vector field of $P$ and $\Xi(\nu) = 1$, $\nu \in C^\infty(M)$. This bivector is of rank two (if nontrivial) and characterized by the property that its unimodular part is trivial.

The second part of section 3 is dedicated to the matching problem: what are different (up to equivalence) ways to assemble Poisson structures from given ones. This problem in full generality seems to be very difficult. By this reason, we restrict ourself to a particular case of two completely non-unimodular structures. It turns out that even in this case equivalent classes of matchings are labeled by functional parameters (proposition 3.13).

The result and constructions concerning modular disassembling of Poisson structures are then adopted to Lie algebras in section 4. In particular, we call modular Lie algebras whose Poisson bivector is completely non-unimodular and show that by subtracting from a Lie algebra a suitable modular algebra one gets an unimodular algebra. The structure of modular Lie algebras is very simple. So, in this sense this result reduces the study of general Lie algebras to unimodular ones. Here we also discuss compatibility conditions for modular and unimodular Lie algebras, and, in particular, show that semisimple and modular Lie algebras are incompatible. In the second part of this section the matching problem for modular Lie algebras is solved. In contrast with general Poisson structures, this problem admits a complete solution. Essentially, matchings of modular Lie algebras are labeled by representations of 2-dimensional algebras (see theorem 4.1).

Section 5 is central in the paper. Here we prove that any finite dimensional Lie algebra over an algebraically closed ground field of zero characteristic, or over $\mathbb{R}$ can be assembled from lieons (theorems 5.1 and 5.2). The proof of this result naturally splits into “solvable” and “semisimple” parts, and we show that any solvable algebra over arbitrary ground field of characteristic zero can be assembled
from lieons (proposition 5.2). This part of the proof is rather simple. On the contrary, the “semisimple” one is more delicate. As a preliminary step, we reduce this part to the problem of disassembling abelian extensions of simple algebras.

The last problem is, essentially, a question on representations of simple Lie algebras, and as such could be analyzed on the basis of the well-known description of them. However, such an approach would be rather cumbersome and hardly instructive, if not to say “amoral”. Moreover, the fact of compound structure of Lie algebras seems to be more fundamental than classification of simple Lie algebras and their representations and, by this reason, must logically precede it. By all these reasons we have chosen another approach. It is based on the stripping procedure (see subsection 5.3), which reduces the problem to representations of simplest algebras, i.e., simple algebras without proper nonabelian subalgebras. Simplest algebras do not exist over an algebraically closed ground field of zero characteristic. The only simplest algebra over \( \mathbb{R} \) is \( \mathfrak{so}(3) \). This is, at the end, why the assembling-from-lieons theorem was proven in these two cases. In this connection it is worth mentioning that the reduction to representations of simplest algebras is based on representations of \( \mathfrak{sl}(2) \).

Besides the proof of these two main theorems, some useful disassembling techniques are also developed in section 5. Being mostly interested to some applications to differential geometry and theoretical physics we have been initially restricted in this paper to ground fields \( \mathbb{R} \) and \( \mathbb{C} \). But it turned out that many of developed here constructions and techniques work well for arbitrary ground fields too. In particular, they indicate possible approaches to the assembling-from-lieons problem for arbitrary fields. They are briefly discussed at the end of this section.

In section 6 we study first level Lie algebras, i.e., the algebras that can be assembled from lieons in one step. With this purpose, we analyze compatibility conditions of two lieons and show that these can be expressed in a purely geometrical manner, namely, in terms of the relative position of subspaces carrying centers and derived algebras of lieons in question. One of consequences of this fact is that the structure of first level algebras does not depend on the ground field in the sense that it is described exclusively in terms of the above-mentioned subspaces. The results of this sections are used in section 8 where we study the “chemistry” of a special class of Lie algebras, called coaxial.

How to assemble classical Lie algebras from \( \mathfrak{sl} \)-lieons over arbitrary ground fields of characteristic zero is shown in section 7. A remarkable fact is that this can be done in no more than 4 steps. More exactly, all simple 3-dimensional Lie algebras can be directly assembled from 3 \( \mathfrak{sl} \)-lieons, i.e., in one step. Simple algebras of higher dimensions require at least 2 steps. For instance, orthogonal algebras can be assembled from \( \mathfrak{sl} \)-lieons in 2 steps.

Sections 8 and 9 are dedicated to a natural question: what are all possible combinations of mutually compatible lieons. Informally speaking, we ask what are simple “molecules”, which can be synthesized from lieons. Essentially, this question is equivalent to the classification problem: what are maximal families of mutually compatible lieons. We solve a simplified version of this problem, when only coaxial, i.e., naturally related with a chosen base lieons are considered. This version is not only interested by itself but also gives useful hints toward the general “chemistry” of Lie algebras. The result we have obtained looks encouraging. Namely, it turned out that maximal families of mutually compatible coaxial lieons, called clusters, are
composed of structural groups surrounded by casings and connected by connectives like in the usual chemistry.

The distribution of the material in these two sections is such that in the first of them we introduce basic techniques and necessary terminology, and solve the problem for clusters composed only of \( \triangleleft \)-lions, or only of \( \triangledown \)-lions. In the second one we describe general clusters and on this basis describe the structure of coaxial Lie algebras. In particular, it turns out that the semisimple part of a coaxial algebra consists of 3-dimensional simple algebras, and the derived series of its solvable part is of length \( \leq 3 \). Here we also give some examples of infinite-dimensional Lie algebras assembled from lions.

In conclusive section 10 we briefly discuss some problems and perspectives of the theory we have started in this paper.

2. Preliminaries

In this section we collect necessary for the sequel facts concerning the calculus of multivectors and differential forms, Poisson geometry, compatibility of Poisson and Lie algebra structures, etc., and fix the notation. More details concerning material reported in this section the reader will find in [2, 15]. Everything in this article is assumed to be smooth.

2.1. Multivectors and differential forms. We use \( M \) for an \( n \)-dimensional manifold and

\[
(1) \quad D(M) = \bigoplus_{k \geq 0} D_k(M) \quad \text{for the exterior algebra of multivectors on } M,
\]

\[
D(M) = D_1(M) \quad \text{for the } C^\infty(M) \text{-module of vector fields on } M, \text{ and } " \wedge " \quad \text{for the wedge product in } D_1(M);
\]

\[
(2) \quad [\cdot, \cdot] \quad \text{for the Schouten bracket in } D_1(M);
\]

\[
(3) \quad \Lambda^\ast(M) = \bigoplus_{k \geq 0} \Lambda^k(M) \quad \text{for the exterior algebra of differential forms on } M \quad \text{and } " \wedge " \quad \text{for the wedge product in it.}
\]

If \( S \) is a \( \mathbb{Z} \)-graded object, say, a multivector, then we use \((-1)^{\deg S \cdot \deg \cdot} \) (resp., \((-1)^{\deg S \cdot \deg \cdot} \)) for \((-1)^{\deg S \cdot \deg \cdot} \) (resp., \((-1)^{\deg S \cdot \deg \cdot} \)). For instance, if \( P \in D_k(M) \) and \( Q \in D_l(M) \), then \((-1)^{PQ} = (-1)^{k(l-1)} \) and \((-1)^{P+Q} = (-1)^{k+l-1} \). This notation makes the formulas that involve signs of \( \mathbb{Z} \)-graded objects, more readable. In particular, graded anticommutativity and Jacobi identity for the Schouten bracket reads

\[
[[P,Q]] = -(-1)^{PQ} [Q,P]
\]

\[
(-1)^{PR}[P,[Q,R]] + (-1)^{QR}[R,[P,Q]] + (-1)^{QR}[Q,[R,P]] = 0
\]

Denote by \( \text{Hgr } \Lambda^\ast(M) \) the totality of graded \( \mathbb{R} \)-linear operators acting on the graded space \( \Lambda^\ast(M) \) and by \([\cdot, \cdot]^{gr} \) the graded commutator of such operators. An operator \( \Delta \in \text{Hgr } \Lambda^\ast(M) \) is a (graded) differential operator over \( \Lambda^\ast(M) \) if

\[
[\omega_0, [\omega_1, \ldots, [\omega_k, \Delta]^{gr}, \ldots]]^{gr} = 0, \quad \forall \omega_0, \omega_1, \ldots, \omega_k \in \Lambda^\ast(M),
\]

where \( \omega_i \)'s are understood to be left multiplication operators.

Insertion of a multivector \( Q \in D_k(M) \) into a differential form \( \omega \in \Lambda^l \) we denote by \( Q|\omega \in \Lambda^{l-k} \), and by \( i_Q : \Lambda^\ast(M) \to \Lambda^\ast(M) \) the operator \( \omega \mapsto Q|\omega \), i.e., \( i_Q(\omega) = Q|\omega \). Obviously,

\[
i_P \circ i_Q = i_{P \wedge Q} \quad \text{and} \quad [i_P, i_Q]^{gr} = 0, \quad P, Q \in D_1(M).
\]
The correspondence \( Q \leftrightarrow i_Q \) identifies the algebra \( D_*(M) \) with the algebra of \( C^\infty(M) \)-linear differential operators over \( \Lambda^*(M) \). More exactly, these operators of order \( k \) correspond to \( k \)-vectors. In terms of this identification the Schouten bracket is described by the formula

\[
i_{P,Q} = [i_P, d]^gr, i_Q]^gr = -(-1)^{\deg Q}[i_P, i_Q]^gr, \quad P, Q \in D_*(M). \tag{3}
\]

The \textit{Lie derivative operator} along a multivector \( Q \) is defined as

\[
L_Q = [i_Q, d]^gr : \Lambda^*(M) \to \Lambda^*(M) \tag{4}
\]

and (3) reads

\[
i_{[P,Q]} = [L_P, i_Q]^gr = -(-1)^Q[i_P, L_Q]^gr. \tag{5}
\]

Here the following useful formula should be mentioned:

\[
[i_Q, L_X]^gr = i_{L_X(Q)}, \quad X \in D(M), \ Q \in D_*(M), \tag{6}
\]

where \( L_X(Q) = [Q, X] \).

The \textit{liezation} operation \( L : Q \mapsto L_Q \) is a (graded right) derivation of the algebra \( D_*(M) \):

\[
L_{P \wedge Q} = i_P \circ L_Q + (-1)^Q L_P \circ i_Q. \tag{7}
\]

Another useful interpretation of the Schouten bracket is easily derived from (5) and (7):

\[
i_{[P,Q]} = (-1)^Q L_{P \wedge Q} - (-1)^Q i_P \circ L_Q - (-1)^P i_Q \circ L_P. \tag{8}
\]

A convenient coordinate-wise description of the above operations is as follows. Let \( x = (x_1, \ldots, x_n) \) be a local chart on \( M \). Instead of the standard local expression

\[
Q = \sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1, \ldots, i_k}(x) \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_k}}, \quad Q \in D_k(M),
\]

we shall use

\[
Q = \sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1, \ldots, i_k}(x) \xi_{i_1} \cdots \xi_{i_k} \tag{9}
\]

assuming that the variables \( \xi_i \)'s anticommute, i.e., \( \xi_i \xi_j = -\xi_j \xi_i \). This allows one to introduce “partial derivatives” \( \frac{\partial}{\partial x_i} \) and \( \frac{\partial}{\partial \xi_i} \) acting on skew-commutative polynomials (9) in \( \xi_i \)'s. Namely, the first of them just acts on coefficients \( a_{i_1, \ldots, i_k}(x) \), while the second is \( C^\infty(M) \)-linear and commutes with the multiplication by \( \xi_j \) operator by the rule \( \frac{\partial}{\partial \xi_i} \circ \xi_j + \xi_j \circ \frac{\partial}{\partial \xi_i} = \delta_{ij} \). In these terms the Schouten bracket reads

\[
[P, Q] = -\sum_i \left( \frac{\partial P}{\partial x_i} \frac{\partial Q}{\partial \xi_i} + (-1)^P \frac{\partial P}{\partial \xi_i} \frac{\partial Q}{\partial x_i} \right). \tag{10}
\]

In particular, by introducing the operator \( X_P : D_*(M) \to D_*(M) \), \( X_P(Q) = [P, Q] \), we have

\[
X_P = -\sum_i \left( \frac{\partial P}{\partial x_i} \frac{\partial}{\partial \xi_i} + (-1)^P \frac{\partial P}{\partial \xi_i} \frac{\partial}{\partial x_i} \right). \tag{11}
\]
2.2. Poisson manifolds. Recall that Poisson structure on a manifold $M$ is a Lie algebra structure on the $\mathbb{R}$–vector space $C^\infty(M)$

$$(f,g) \mapsto \{f,g\} \in C^\infty(M), \quad f,g \in C^\infty(M),$$

which at the same time is a biderivation, i.e.,

$$\{fg,h\} = f\{g,h\} + g\{f,h\} \quad \text{and} \quad \{f,gh\} = g\{f,h\} + h\{f,g\}.$$  

$P \in D_2(M)$ is a Poisson bivector if $[ [P,P] ] = 0$. The formula

$$\{f,g\} = P(df,dg), \quad f,g \in \mathcal{C}^\infty(M).$$

establishes one-to-one correspondence between Poisson bivectors and Poisson structures on $M$. The Poisson bracket associated this way with the Poisson bivector $P$ will be denoted by $\{\cdot,\cdot\}_P$.

A Poisson manifold is nondegenerate, if the corresponding Poisson bivector is nondegenerate, i.e., the correspondence

$$\gamma_P : \Lambda^1(M) \rightarrow \mathcal{D}(M), \quad \omega \mapsto P(\omega,\cdot),$$

is an isomorphism of $\mathcal{C}^\infty(M)$–modules. $\gamma_P$ naturally extends to an homomorphism of exterior algebras still denoted

$$\gamma_P : \Lambda^*(M) \rightarrow \mathcal{D}(M).$$

It is an isomorphism if $P$ is nondegenerate. In this case $\gamma_P(P) \in \Lambda^2(M)$ is a symplectic form on $M$. This way the class of nondegenerate Poisson manifolds is identified with the class of symplectic manifolds.

The Poisson differential

$$\partial_P : \mathcal{D}_\ast(M) \rightarrow \mathcal{D}_{\ast+1}(M), \quad \partial_P(Q) = [P,Q],$$

associated with a Poisson bivector $P$ supplies $\mathcal{D}_\ast(M)$ with a cochain complex structure. The vector field

$$P_f \overset{\text{def}}{=} \partial_P(f) = [P,f] = -\gamma_P(df) = -df \ f$$  

(12)

is called $P$–Hamiltonian corresponding to the Hamiltonian function $f$.

The following definition is central for this paper.

**Definition 2.1.** Poisson structures $P_1$ and $P_2$ on a manifold $M$ are called compatible if $P_1 + P_2$ is a Poisson structure as well.

**Proposition 2.1.** Poisson structures $P_1$ and $P_2$ are compatible if one of the following equivalent conditions holds:

1. $[P_1, P_2] = 0$;
2. $sP_1 + tP_2$, $s,t \in \mathbb{R}$, is a Poisson structure for all $s,t$;
3. the bracket $\{\cdot,\cdot\} = s\{\cdot,\cdot\}_{P_1} + t\{\cdot,\cdot\}_{P_2}$, $s,t \in \mathbb{R}$, is a Lie algebras structure on $C^\infty(M)$;
4. $\partial_{P_1} + \partial_{P_2}$ is a differential in $\mathcal{D}_\ast(M)$, or, equivalently, $\partial_{P_1}\partial_{P_2} + \partial_{P_2}\partial_{P_1} = 0$.

**Proof.** The first assertion directly follows from $[P_1 + P_2, P_1 + P_2] = 2[P_1, P_2]$, while (2) - (4) are obvious consequences of it. \qed
2.3. Lie algebras. In the literature the term "Lie algebra" is commonly used in two different meanings, namely, either as a concrete Lie algebra structure on a vector space, or as an isomorphism class of such structures. In various situations in this paper this distinction is essential and we will use "Lie algebra structure" instead of "Lie algebra" in an ambiguous in this sense context.

Lie algebra structures will be denoted by bold Fraktur characters, say, $\mathfrak{g}, \mathfrak{h}$, etc. The symbol $|\mathfrak{g}|$ refers to the supporting $\mathfrak{g}$ vector space. We use square brackets, if necessary with various indexes, for Lie product operations.

Let $\mathfrak{g}$ be a Lie algebra over a ground field $k$ and $V = |\mathfrak{g}|$. A Lie algebra structure is naturally defined in the algebra $k(V^*)$ of polynomials on $V^* = \text{Hom}_k(V, k)$. Namely, denoting by $f_v$ the linear function on $V^*$ corresponding to $v \in V$, we define the "Poisson bracket" $\{\cdot, \cdot\}$ on linear functions by putting

$$\{f_v, f_w\} = f_{[v, w]}, \quad v, w \in V,$$

and extend it onto the whole algebra as a biderivation. This construction remains valid for any larger algebra $A \supset k(V^*)$ with the property that any derivation of $k(V^*)$ uniquely extends to $A$. For instance, $C^\infty(V^*)$ is such an algebra if $k = \mathbb{R}$.

We shall refer to the so-defined Lie algebra as the Poisson structure on the dual to the Lie algebra $\mathfrak{g}$. The corresponding Poisson bivector on $V^*$ will be denoted $P_{\mathfrak{g}}$.

Let $\{e_i\}$ be a basis in $V$. Put $x_i = f_{e_i}$. Then

$$\{f, g\} = \sum_{i, j, k} c_{ij}^k x_k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}, \quad (13)$$

where $c_{ij}^k$ are structure constant of $\mathfrak{g}$ in the considered basis, and

$$P_{\mathfrak{g}} = \sum_{i, j, k} c_{ij}^k x_k \xi_i \xi_j, \quad (14)$$

Poisson structures of the form $P_{\mathfrak{g}}$ have linear coefficients in any cartesian chart on $V^*$ and vice versa. By this reason they are also called linear. If $Q \in D_\ast(W)$ is a linear, i.e., with linear in a cartesian chart coefficients, multivector on a vector space $W$, then

$$Q_\theta = [X_\theta, Q], \quad \theta \in W,$$

where $Q_\theta$ is the value of $Q$ at $\theta$ and $X_\theta$ is the corresponding to $\theta$ constant vector field on $W$. This observation is useful when dealing with linear Poisson structures.

Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebra structures on a $k$-vector space $V$ and $[\cdot, \cdot]_1, [\cdot, \cdot]_2$ the corresponding Lie products. The following is the analogue of definition 2.1 for Lie algebras.

**Definition 2.2.** Lie algebra structures $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are called compatible if $[\cdot, \cdot] \overset{\text{def}}{=} [\cdot, \cdot]_1 + [\cdot, \cdot]_2$ is a Lie product in $V$.

The Lie algebra structure on $V$ defined by the Lie product $[\cdot, \cdot]_1, +[\cdot, \cdot]_2$ will be denoted by $\mathfrak{g}_1 + \mathfrak{g}_2$. Obviously, we have

**Proposition 2.2.** Lie algebra structures $\mathfrak{g}_1$ and $\mathfrak{g}_2$ on $V$ are compatible if and only if the corresponding Poisson structures on $V^*$ are compatible. \[\square\]
2.4. Lie rank of Poisson manifolds and Lie Algebras. Recall that a bivector field \( Q \in D_2(M) \) generates a distribution (with singularities) on \( M \). This distribution is defined as a \( C^\infty(M) \)-submodule \( D_Q(M) \) of \( D(M) \) generated by vector fields \( Q_f = df | Q, f \in C^\infty(M) \).

Geometrically, \( D_Q(M) \) may be viewed as a family of vector spaces \( M \ni x \mapsto \triangle_Q(x) \subset T_xM \) on \( M \) where the subspace \( \triangle_Q(x) \subset T_xM \) is generated by vectors of the form \( Q_{f,x} \in T_xM, \ \forall f \in C^\infty(M) \). The function

\[
M \ni x \mapsto \text{rank}_Q(x) = \dim \triangle_Q(x)
\]

is, obviously, lower semicontinuous with values in even integers. In particular, \( \text{rank}_Q(x) \) is is locally constant except a thin closed subset in \( M \) and reaches its maximum value, say \( 2k \), in an open domain of \( M \).

If \( M \ni x \mapsto \triangle_Q(x) \subset T_xM \) on \( M \) where the subspace \( \triangle_Q(x) \subset T_xM \) is generated by vectors of the form \( Q_{f,x} \in T_xM, \ \forall f \in C^\infty(M) \). The function

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**Definition 2.3.** 1) A bivector field \( Q \) is said to be of rank \( 2k \) if

\[
Q^k \neq 0, \quad Q^{k+1} = 0
\]

2) A Lie algebra is said to be of Lie rank \( 2k \) if the associated linear Poisson bivector on its dual is of rank \( 2k \).

**Example 2.1.** If \( n = 2k + \epsilon, \quad \epsilon = 0, 1 \), then the direct sum of \( k \) copies of the non-commutative 2-dimensional Lie algebra and the 1-dimensional algebra, if \( \epsilon = 1 \), is an \( n \)-dimensional Lie algebra of maximal Lie rank \( 2k \).

If \( Q \) is a Poisson bivector, then \([Q_f, Q_g] = Q_{\{f,g\}}\), i.e., the distribution \( D_Q(M) \) is a Frobenius one. Locally maximal integral submanifolds of \( D_Q(M) \) constitute the canonical symplectic foliation of \( M \) associated with \( Q \) (see [14, 18]). If \( Q = P_g \), then the leaves of this foliation are orbits of the coadjoint representation of \( g \).

3. Modularity of Poisson and Lie algebra Structures

In this section we introduce and study a splitting of a Poisson or a Lie algebra structure into unimodular and non-unimodular parts. This splitting is canonical up to a gauge transformation and reduces, in a sense, the study of general Poisson or Lie algebras structures to unimodular ones. The central in this construction are notions of modular vector field and modular class introduced by J.-L. Koszul [8]. In survey [9] the reader will find an extensive bibliography about.

3.1. Unimodular Poisson structures. If \( \omega \in \Lambda^n(M) \) is a volume from, then the map

\[
Q \mapsto Q \rfloor \omega, \quad Q \in D_*(M),
\]

which we shall call \( \omega \)-duality, is an isomorphism between \( C^\infty(M) \)-modules \( D_*(M) \) and \( \Lambda^*(M) \). In particular, the \((n-2)\)-form \( \alpha = \alpha_{P,\omega} = P \rfloor \omega, \ \omega \)-dual to the Poisson bivector \( P \), completely characterizes this bivector.

**Proposition 3.1.** \( P \in D_2(M) \) is a Poisson bivector on \( M \) if and only if

\[
d(P \rfloor \alpha) = 2P \rfloor d\alpha \quad \text{with} \quad \alpha = P \rfloor \omega. \quad (15)
\]
Proof. Formula (5) for $P = P_1 = P_2$ can be rewritten as

$$L_P \wedge P - 2i_P \circ L_P = i_{[P,P]}$$

and hence

$$L_P(\omega) - 2P \cdot L_P(\omega) = [P,P](\omega)$$ (16)

On the other hand, by (4), we have

$$L_P(\omega) = -d(P \cdot \omega) = -d\alpha$$ (17)

and

$$L_P(\omega) = -d((i_{P\wedge P})(\omega)) = -d(P \cdot (P \cdot \omega)) = -d(P \cdot \alpha)$$

With these substitutions (16) takes the form

$$d(P \cdot \alpha) - 2P \cdot d\alpha = -[P,P](\omega).$$

Finally, observe that $Q = 0 \iff Q \cdot \omega = 0$ for any $Q \in D_*(M)$. \qed

Definition 3.1. A Poisson structure on $M$ is called unimodular with respect to $\omega$ (shortly, $\omega$-unimodular) if $L_P(\omega) = 0$.

Proposition 3.2. A Poisson structure $P$ is unimodular if and only if one of the following relations holds

1. $d\alpha = 0$ with $\alpha = P \cdot \omega$;
2. $L_P(\omega) = 0$, $\forall f \in C^\infty(M)$, i.e. Lie derivatives of $\omega$ along all $P$-hamiltonian fields vanish.

Proof. In view of (17) the first assertion is obvious. Then, by applying (4), we have

$$L_P(\omega) = d(P \cdot \omega) = -d((df \cdot P \cdot \omega)) = -d(df \wedge (P \cdot \omega)) = -df \wedge L_P(\omega)$$

i.e.

$$L_P(\omega) = -df \wedge L_P(\omega)$$ (18)

Since $df \wedge \rho = 0$, $\forall f \in C^\infty(M)$, implies $\rho = 0$ for $\rho \in \Lambda^k(M)$, $k < n$, we see that

$$L_P(\omega) = 0, \forall f \iff df \wedge L_P(\omega) = 0, \forall f \iff L_P(\omega) = 0. \qed$$

Remark 3.1. In fact, formula (18) shows that unimodularity of $P$ is guaranteed by $L_P(\omega) = 0$, $i = 1, \ldots, n$, for a local chart $(x_1, \ldots, x_n)$ on $M$.

Compatibility conditions for unimodular Poisson structures are simplified as follows.

Proposition 3.3. Let $P_1, P_2$ be $\omega$-unimodular Poisson structures on $M$. They are compatible if and only if $L_{P_1 \wedge P_2}(\omega) = 0$.

Proof. Since $L_{P_1 \wedge P_2}(\omega) = 0$ formula (8) applied to $\omega$ gives:

$$P_1 \cdot L_{P_2}(\omega) + P_2 \cdot L_{P_1}(\omega) + [P_1, P_2] \cdot \omega = 0$$

Due to unimodularity of $P_1$ and $P_2$ this formula reduces to $[P_1, P_2] \cdot \omega = 0$. which is equivalent to $[P_1, P_2] = 0$. \qed

Corollary 3.1. Two $\omega$-unimodular Poisson structures $P_1$ and $P_2$ are compatible if $P_1 \wedge P_2 = 0$. \qed
Corollary 3.2. Any two $\omega$--unimodular Poisson structures on a 3-dimensional $M$ are compatible.

Remark 3.2. Condition $L_{P_1 \wedge P_2} = 0$ implying, obviously, compatibility of $P_1$ and $P_2$ is not, in fact, weaker than the condition $P_1 \wedge P_2 = 0$, since $L_Q = 0$ is equivalent to $Q = 0$ for any $Q \in \mathcal{D}_s(M)$.

3.2. Lie rank of unimodular Lie algebras. The following example shows existence of unimodular odd-dimensional Lie algebras of maximal possible Lie rank, i.e., of rank $2k$ if $n = 2k + 1$.

Example 3.1. Let $(x_1, x_2, \ldots, x_{2k+1})$ be a cartesian chart on $V^*$, $\dim V = 2k + 1$. The bivector

$$P = x_{2k+1}(\xi_1 \wedge \xi_2 + \cdots + \xi_{2k-1} \wedge \xi_{2k})$$

is a Poisson one as well as bivectors $P_s = x_{2k+1}(\xi_{2s-1} \wedge \xi_{2s},$ $s = 1, \ldots, k,$ of rank one. Obviously, the rank of $P$ is $2k$, $P_s$’s are mutually compatible and $P = P_1 + \cdots + P_k$. So, Lie algebra structures $\mathfrak{g}, \mathfrak{g}_1, \ldots, \mathfrak{g}_k$ on $V$ corresponding $P, P_1, \ldots, P_k,$ respectively, are mutually compatible. $\mathfrak{g} = \mathfrak{g}_1 + \cdots + \mathfrak{g}_k$ and the Lie rank of $\mathfrak{g}$ is $2k$.

Now we will show that the Lie rank of an unimodular Lie algebra can not coincide with its dimension.

Proposition 3.4. If $Q$ is an $\omega$--modular Poisson structure on $M$ of rank $Q$ and $n = 2k$, then $Q^k \omega = \text{const} \neq 0$.

Proof. Since $Q$ is a Poisson structure, then, according to (8), $L_{Q^s} = 2i_Q \circ L_Q$. Inductively, one easily finds that

$$L_{Q^s} = si_{Q^{s-1}} \circ L_Q, \quad \forall s \geq 2$$

If $Q^k \neq 0$, then the function $f = Q^k \omega = (Q^k, \omega)$ is different from zero.

On the other hand, due to $\omega$-unimodularity of $Q$, $L_Q(\omega) = 0$ and hence

$$df = d(Q^k \omega) = L_{Q^k}(\omega) = kQ^{k-1} \omega L_Q(\omega) = 0. \quad \square$$

Corollary 3.3. The Lie rank of an unimodular Lie algebra is strictly lesser than its dimension.

Proof. Let $P$ be the associated linear Poisson bivector on $V$ and $\dim V = 2k$. If $k$ is the rank of $P$, then $f = P^k \omega = \text{const} \neq 0$ where $\omega$ is a cartesian volume form on $V$. This contradicts the fact that $f$ is a homogeneous polynomial of order $k$. \quad \square

3.3. Modular vector fields. Let $\omega$ be a volume form and $P$ a Poisson vector on $M$.

Definition 3.2. The vector field $\Xi = \Xi_{P, \omega}$ is uniquely defined by the relation

$$\Xi | \omega = d(P | \omega) \quad (19)$$

and is called the modular vector field of $P$ with respect to $\omega$ (shortly, $\omega$-modular).

It follows from (17) and (18) that

$$L_{P_f}(\omega) = df \wedge (\Xi | \omega)$$

On the other hand, $0 = \Xi | (df \wedge \omega) = \Xi(f) \omega - df \wedge (\Xi | \omega)$. So,

$$L_{P_f}(\omega) = \Xi(f) \omega \quad (20)$$
i.e. $\Xi(f) = \text{div}_\omega(P_f)$. In other words, $\Xi_{P,\omega}$ measures divergence of $P$-hamiltonian vector fields with respect to the volume from $\omega$. This property was the original definition of modular fields.

**Proposition 3.5.** The following relations hold for the modular field $\Xi = \Xi_{P,\omega}$ of a Poisson structure $P$ and its dual form $\alpha = \alpha_{P,\omega} = P|\omega$:

1. $L_{\Xi}(\alpha) = 0$
2. $L_{P}(d\alpha) = 0$
3. $L_{P}(\alpha) = -\Xi|\alpha$
4. $L_{\Xi}(\omega) = 0$
5. $L_{\Xi}(P) = [P,\Xi] = \partial_P(\Xi) = 0$
6. $P_f|\omega = -df \wedge \alpha$;
7. $[\Xi, P_f] = P_{\Xi(f)}$.

**Proof.** First, we have

$$L_{\Xi}(\alpha) = d(\Xi|\alpha) + \Xi|d\alpha.$$  

Then, by (15),

$$d(\Xi|\alpha) = d(P|d\alpha) = \frac{1}{2}d^2(P|\alpha) = 0$$

On the other hand, $\Xi|d\alpha = \Xi|(\Xi|\omega) = 0$ This proves (i).

Similarly,

$$L_{P}(d\alpha) = [i_P, d]^g_r(d\alpha) = -d(P|d\alpha) = 0$$

This proves (ii).

In its turn (iii) directly results from (15):

$$L_{P}(\alpha) = [i_P, d](\alpha) = P|d\alpha - d(P|\alpha) = -P|d\alpha = -\Xi|\alpha$$

Concerning (iv) we have:

$$L_{\Xi}(\omega) = d(\Xi|\alpha) = d^2\alpha = 0$$

To prove (v) it suffices to show that $L_{\Xi}(P)|\omega = 0$. But according to (i) and (6) we have

$$0 = L_{\Xi}(\alpha) = L_{\Xi}(P|\omega) = -L_{\Xi}(P)|\omega + P|L_{\Xi}(\omega) = -L_{\Xi}(P)|\omega.$$  

Now, a particular case of (3) is $i_{[f,P]} = [f, d]^g_r, i_P|g_r = -[df,i_P]$, and one gets (vi) by applying this relation to $\omega$.

Finally, recall the general formula

$$L_X(\rho \mid Q) = -L_X(\rho \mid Q + \rho \mid L_X(Q)$$

with $X \in D(M), \quad Q \in D_1(M), \quad \rho \in A^1(M)$. By specifying it to $X = \Xi, \quad Q = P$ and $\omega = d\nu$ one finds that

$$[\Xi, P_f] = -L_{\Xi}(P_f) = L_{\Xi}(df|P) = -L_{\Xi}(df)|P + df|L_{\Xi}(P).$$

Now the desired result follows from assertion (v) by observing that $L_{\Xi}(df) = d\Xi(f)$. □

Additivity is an important property of $\omega$–modular fields.

**Proposition 3.6.** If Poisson structure $P_1$ and $P_2$ are compatible and $\Xi_1, \Xi_2$ are their $\omega$–modular fields, respectively, then $\Xi_1 + \Xi_2$ is the $\omega$–modular field of $P_1 + P_2$.

**Proof.** Straightforwardly from (20). □
Corollary 3.4. If $P_i, \Xi_i, i = 1, 2$, are as above, then
\[ L\Xi_1(P_2) + L\Xi_2(P_1) = 0 \tag{21} \]

Proof. Directly from $L\Xi_1(P_1) = L\Xi_1(P_2) = L\Xi_2(P_1 + P_2) = 0$ (proposition 3.5,(v)). \qed

3.4. $\omega$–modular class. The following well-known fact describes dependence of the $\omega$–modular field on $\omega$. For completeness we report a short proof of it.

Proposition 3.7. If $\omega' = f \omega$ is another volume form on $M$, then
\[ \Xi_{P,f\omega} = \Xi_{P,\omega} - P \ln |f| \tag{22} \]

Proof. The vector field $\Xi_{P,f\omega}$ is determined as solution of
\[ \Xi_{P,f\omega} \cdot (f\omega) = d\alpha_{P,f\omega} \tag{23} \]
By putting $\Xi_{P,f\omega} = \Xi_{P,\omega} + Y$ and noticing that $\alpha_{P,f\omega} = P \cdot (f\omega) = f\alpha$ where $\alpha = \alpha_{P,\omega}$ one may rewrite (23) in the form
\[ fY \cdot \omega = df \wedge \alpha \iff Y \cdot \omega = d(\ln |f|) \wedge \alpha \]
(f is nowhere zero, since $f\omega$ is a volume form). Now proposition 3.5, (vi), shows that $Y = -P \ln |f|$. \qed

This result has the following cohomological interpretation. First, observe that (proposition 3.5,(v))
\[ \partial_P(\Xi) = \{ P; \Xi \} = L\Xi(P) = 0, \]
i.e., $\Xi$ is a 1-cocycle of the complex $\{ D_*(M), \partial_P \}$. Moreover, $P$–hamiltonian fields are coboundaries of this complex, namely, $P_g = \partial_P(g)$. Hence proposition 3.7 yields:

Corollary 3.5. The cohomology class of the $\omega$–modular field $\Xi_{P,\omega}$ in $\{ D_*(M), \partial_P \}$ does not depend on $\omega$ and, therefore, is well-defined by $P$. \qed

Definition 3.3. The $\partial_P$-cohomology class of the $\omega$–modular field is called the modular class of $P$.

Corollary 3.6. A Poisson structure $P$ is $\omega$–unimodular with respect to a volume form $\omega$ if and only if its modular class vanishes. \qed

If $P$ is nondegenerate, then $(M, \gamma_P(P))$ is a symplectic manifold. In this case the isomorphism $\gamma_P : D_*(M) \rightarrow \Lambda^*(M)$ is also an isomorphism of complexes $\{ \Lambda^*(M), d \}$ and $\{ D_*(M), d_P \}$. Therefore, if $H^1(M) = 0$, then any nondegenerate Poisson structure on $M$ is $\omega$–unimodular with respect to a suitable volume form $\omega$.

3.5. Modular disassembling of a Poisson structure. Now we shall show that the $\omega$–modular vector field of a Poisson structure $P$ allows one to disassemble this structure, at least, locally, into two parts, one of which is $\omega$–unimodular, while all “$\omega$–non– unimodularity” of $P$ is concentrated the second part.

Proposition 3.8. Let $\Xi$ be the $\omega$–modular vector field of a Poisson structure $P$ and $\nu \in C^\infty(M)$ be such that $\Xi(\nu) = 1$. Then
\begin{enumerate}
\item $\Xi \wedge P_\nu$ is a Poisson structure compatible with $P$;
\item $L_{\Xi \wedge P_\nu}(\omega) = -L_P(\omega)$;
\end{enumerate}
(3) \( P + \Xi \wedge P_\nu \) is an \( \omega \)-unimodular compatible with \( P \) Poisson structure and \( \nu \) is a Casimir function of it;
(4) \( P_\nu \wedge \Xi = [P, \nu \Xi] = \partial_\nu (\nu \Xi) \).

Proof. First, from proposition 3.5, (vii), we see that
\[
[\Xi, P_\nu] = [\Xi, P_\nu] = -L_\Xi (d\nu | P) = P_{\Xi(\nu)} = P_1 = 0.
\]
Since the Schouten bracket is a graded biderivation of the exterior algebra \( D_*(M) \), this implies
\[
[\Xi \wedge P_\nu, \Xi \wedge P_\nu] = 0
\]
i.e. \( \Xi \wedge P_\nu \) is a Poisson structure. By the same reason we have
\[
[P, \Xi \wedge P_\nu] = [P, \Xi] \wedge P_\nu - \Xi \wedge [P, P_\nu]
\]
The right hand side of this equality vanishes by proposition 3.5, (v), and the fact that \( P_\nu \) is a \( P \)-hamiltonian field. So, \( \Xi \wedge P_\nu \) is compatible with \( P \).

To prove the second assertion we specify (7) to \( P = \Xi, Q = P_\nu \) and then apply the result to \( \omega \):
\[
L_{\Xi \wedge P_\nu} (\omega) = \Xi | L_{P_\nu} (\omega) - L_{\Xi} (P_\nu | \omega)
\]
Similarly, formula (6), specified to \( X = \Xi, Q = P_\nu \) and then applied to \( \omega \), gives
\[
L_\Xi (P_\nu | \omega) = -L_\Xi (P_\nu) | \omega + P_\nu | L_\Xi (\omega)
\]
By proposition 3.5, (v), (vii), \( L_\Xi (\omega) = 0 \) and \( L_\Xi (P_\nu) = 0 \). Hence
\[
L_{\Xi \wedge P_\nu} (\omega) = \Xi | L_{P_\nu} (\omega)
\]
On the other hand, according to (20), \( L_{P_\nu} (\omega) = \Xi(\nu) \omega = \omega \) and hence
\[
L_{\Xi \wedge P_\nu} (\omega) = \Xi | \omega = d\alpha = -L_\nu (\omega).
\]

Next, \( \omega \)-unimodularity of \( P + \Xi \wedge P_\nu \) directly follows from assertion (2), while
\[
d\nu | (P + \Xi \wedge P_\nu) = d\nu | P + \Xi(\nu) P_\nu = 0
\]
proves that \( \nu \) is a Casimir function of this structure.

The fact that \( \partial_\nu \) is a graded derivation of the exterior algebra \( D_*(M) \) together with \( P_\nu = [P, \nu], [P, \Xi] = 0 \) (proposition 3.5, (v)) proves the last assertion.

Thus \( P \) is presented as the sum
\[
P = (P - P_\nu \wedge \Xi) + P_\nu \wedge \Xi
\]
(24) of two compatible Poisson structures, one of which is \( \omega \)-unimodular and another \( \omega \)-non-unimodular (if different from zero). \( P_\nu \wedge \Xi \) is an \( \omega \)-modular bivector associated with \( P \). Note that \( P_\nu \wedge \Xi \) is of rank 2 (if different from zero), i.e., is a smallest possible \( \omega \)-non-unimodular part of \( P \). This may be interpreted as a canonical disassembling of \( P \) into \( \omega \)-unimodular and \( \omega \)-non-unimodular parts.

Obviously, all \( \omega \)-modular bivectors associated with \( P \) form an affine subspace in \( D_2(M) \) modelled on the subspace \( \{ P_\nu \wedge \Xi | \Xi(f) = 0 \} \).

All \( \omega \)-modular bivectors associated with \( P \) are compatible each other. Indeed, by proposition 3.5, (viii), \( [P_\nu, \Xi] = 0 \) if \( \Xi(f) = \text{const} \). Since the Schouten bracket is a graded biderivation of \( D_*(M) \), this implies
\[
[P_\nu \wedge \Xi, P_\mu \wedge \Xi] = 0 \quad \text{if} \quad \Xi(\nu) = \Xi(\mu) = 1.
\]

It is remarkable that an \( \omega \)-modular bivector coincides with its \( \omega \)-non-unimodular part, i.e., is completely \( \omega \)-non-modular;
Proposition 3.9. If $P, \omega, \Xi$ and $\nu$ are as above, then

1. the $\omega$–modular field of the Poisson structure $P_\nu \wedge \Xi$ coincides with $\Xi$;
2. if $\Xi(\mu) = 1$, then $(P_\nu \wedge \Xi)_\mu \wedge \Xi = P_\nu \wedge \Xi$.

Proof. By definition (19), the first assertion is just an interpretation of proposition 3.8, (2). The second one follows from:

$$(P_\nu \wedge \Xi)_\mu = -d\mu \rfloor (P_\nu \wedge \Xi) = \Xi(\mu)P_\nu - \{\mu, \nu\} \Xi.$$

So, in contrast to general Poison structures, the non-unimodular part of an $\omega$–modular bivector is unique, i.e. does not depend on the choice of a normalizing function $\nu$.

Denote by $\{\cdot, \cdot\}_{\text{non}}$ (resp., $\{\cdot, \cdot\}_{\text{uni}}$) the Poisson bracket corresponding to the $\omega$-non-unimodular (resp., $\omega$-unimodular) part of $P$ according to (24). Then

$$\{f, g\}_{\text{non}} = \{f, \nu\} \Xi(g) - \{g, \nu\} \Xi(f).$$

If $P$–hamiltonian fields $P_f$ and $P_g$ are $\omega$–divergenceless i.e., $\Xi(f) = \Xi(g) = 0$ (see (20), then

$$\{f, g\}_{\text{non}} = 0 \iff \{f, g\}_{\text{uni}} = \{f, g\}.$$ 

This shows that restriction of the bracket $\{\cdot, \cdot\}_{\text{uni}}$ to the $\omega$–divergenceless part of the original Poisson structure does not depend on the choice of the normalizing function $\nu$ (see (24)).

An abstract description of $\omega$–modular bivectors is as follows.

Proposition 3.10. Let $X, \Xi \in D(M)$ and the volume form $\omega \in \Lambda^n(M)$ be such that

$$[X, \Xi] = 0, \quad L_\Xi(\omega) = 0, \quad L_X(\omega) = \omega \quad(25)$$

Then $P = X \wedge \Xi$ is a Poisson structure, which coincides with its $\omega$–non–unimodular part, and $\Xi$ is the $\omega$–modular vector field of it.

Proof. First, observe that conditions (25) implies independence of vector fields $X$ and $\Xi$. Since $[X \wedge \Xi, X \wedge \Xi] = 2[X, \Xi] \wedge X \wedge \Xi$, the condition $[X, \Xi] = 0$ implies that $P$ is a Poisson structure. On the other hand, in view of (6), (7) and (25) we have

$$L_P(\omega) = X \rfloor L_\Xi(\omega) - L_X(\Xi) \rfloor \omega = L_X(\Xi) \rfloor \omega = L_X(\Xi) = -\Xi \rfloor \omega$$

This shows (see (17) and (19)) that $\Xi$ is the $\omega$–modular vector field of $P$.

Since $X$ and $\Xi$ commute, there exists, at least, locally, a function $\nu$ such that $\Xi(\nu) = 1, \quad X(\nu) = 0$. For such a function $P_\nu = -d\nu \rfloor (X \wedge \Xi) = X$, i.e., locally, $P = P_\nu \wedge \Xi$ and, therefore, $P$ coincides with its $\omega$–non–unimodular part. \hfill $\Box$

Definition 3.4. A Poisson bivector described in proposition (3.10) is called an $\omega$–modular bivector.

In what follows we shall assume satisfied conditions of proposition 3.10 when referring to an $\omega$–modular bivector presented in the form $X \wedge \Xi$.

3.6. Compatibility of $\omega$–modular bivectors. Now we shall discuss compatibility conditions involving $\omega$–modular bivectors. First, we consider the inverse to the $\omega$–modular splitting procedure.
Proposition 3.11. An $\omega$–modular bivector $X \wedge \Xi$ and an unimodular structure $Q$ are compatible if and only if

$$L_\Xi(Q) = 0, \quad \Xi \wedge L_X(Q) = 0.$$ 

Proof. First, observe that $\Xi$ is the $\omega$–modular field of the Poisson structure $X \wedge \Xi + Q$ (proposition 3.6). Therefore, in view of proposition 3.5, (v), $0 = L_\Xi(X \wedge \Xi + Q) = L_\Xi(Q)$. On the other hand, the compatibility condition of $X \wedge \Xi$ and $Q$ is

$$0 = [X \wedge \Xi, Q] = X \wedge [\Xi, Q] - [X, Q] \wedge \Xi = L_\Xi(Q) \wedge X + L_X(Q) \wedge \Xi$$

Since $L_\Xi(Q) = 0$, this gives the desired result. $\square$

Remark 3.3. Generally, $X \wedge \Xi$ is not an $\omega$–modular bivector associated with $P = X \wedge \Xi + Q$. For example, if $M = \mathbb{R}^2$, $\omega = dx_1 \wedge dx_2$, $X = x_1 \xi_1$, $\Xi = \xi_2$ and $Q = \xi_1 \xi_2$, then $X \wedge \Xi$ is an $\omega$–modular bivector compatible with the unimodular Poisson bivector $Q$. But in this case $P$ is another $\omega$–modular bivector, i.e., the unimodular part of $P$ is trivial.

Now we shall discuss compatibility of two $\omega$–modular bivectors. Assume $X_1, \Xi_i \in D(M), i = 1, 2$, to be as in proposition (3.10). By developing the compatibility condition $[P_1, P_2] = 0$ of $\omega$–modular bivectors $P_1 = X_1 \wedge \Xi_1$ and $P_2 = X_2 \wedge \Xi_2$ we obtain

$$[X_1, X_2] \wedge \Xi_1 \wedge \Xi_2 + [\Xi_1, \Xi_2] \wedge X_1 \wedge X_2 = [X_1, \Xi_2] \wedge \Xi_1 \wedge X_2 + [\Xi_1, X_2] \wedge X_1 \wedge \Xi_2. \quad (26)$$

However, formula (26) does not reflect modularity properties of $P_1$ and $P_2$, and these are to be added. Since (26) guarantees that $P = P_1 + P_2$ is a Poisson bivector with the modular field $\Xi = \Xi_1 + \Xi_2$, proposition 3.5 and its consequences are valid for $P, \Xi$.

For instance, formula (21) in the considered case can be rewritten as

$$[[X_2, \Xi_1] \wedge \Xi_2 + [X_1, \Xi_2] \wedge \Xi_1 + (X_1 - X_2) \wedge \Xi_1, \Xi_2] = 0 \quad (27)$$

Note that (27) is a formal consequence of (26) and modularity property of $P_i$'s. Moreover, by multiplying (27) by $X_2$, we can bring formula (26) to the form

$$[X_1, X_2] \wedge \Xi_1 \wedge \Xi_2 = [\Xi_1, X_2] \wedge (X_1 - X_2) \wedge \Xi_2 \quad (28)$$

or, similarly, to

$$[X_1, X_2] \wedge \Xi_1 \wedge \Xi_2 = [X_1, \Xi_2] \wedge (X_1 - X_2) \wedge \Xi_1 \quad (29)$$

Hence we have

Proposition 3.12. $\omega$–modular bivectors $X_1 \wedge \Xi_1$ and $X_2 \wedge \Xi_2$ are compatible if and only if (27) and one of formulae (28) or (29) holds. $\square$

Remark 3.4. Condition (21) is manifestly satisfied if $\Xi_1 = \lambda \Xi_2$, $0 \neq \lambda \in C^\infty(M)$, i.e., two $\omega$–modular bivectors are compatible if their $\omega$–modular fields are proportional.
3.7. **On complexity of the matching problem.** Let $P_i$ (resp., $G_i$), $i = 1, \ldots, m$, be diffeomorphism (resp., isomorphism) classes of Poisson (resp., Lie algebras) structures of the same dimension. The *matching problem* is to classify various realizations of $P_i$’s (resp., $G_i$’s) of these structures on the same manifold (resp., vector space) for which $P_i$ and $P_j$ (resp., $G_i$ and $G_j$) are compatible for all $i$ and $j$. Such a realization will be called a *matching*. An equivalence of two matchings is defined in an obvious manner, and the matching problem is: what are different, i.e., nonequivalent, matchings of given Poisson (resp., Lie algebra) structures?

This problem seems to be rather difficult. Below we shall discuss it for two $\omega$-modular bivectors in order to show its complexity. Namely, we shall solve compatibility conditions for $\omega$-modular bivectors $P_1 = X_1 \wedge \Xi_1$ and $P_2 = X_2 \wedge \Xi_2$ assuming that $\Xi_1$ and $\Xi_2$ are independent and $[\Xi_1, \Xi_2] = 0$. The last assumption is automatically satisfied for Lie algebras.

With these assumptions (27) becomes

$$[X_2, \Xi_1] \wedge \Xi_2 + [X_1, \Xi_2] \wedge \Xi_1 = 0,$$

or, equivalently,

$$[X_1, \Xi_2] = f_1 \Xi_1 + \lambda \Xi_2, \quad [X_2, \Xi_1] = \lambda \Xi_1 + f_2 \Xi_2$$

for some $f_1, f_2, \lambda \in C^\infty(M)$. Now each of formulae (28) and (29) can be brought to the form

$$([X_1, X_2] - \lambda (X_1 - X_2)) \wedge \Xi_1 \wedge \Xi_2 = 0$$

The last relation is equivalent to

$$[X_1, X_2] = \lambda (X_1 - X_2) + \mu_1 \Xi_1 - \mu_2 \Xi_2$$

for some $\mu_1, \mu_2 \in C^\infty(M)$.

**Lemma 3.1.** If $X_1, X_2, \Xi_1, \Xi_2$ are as above, then functions $f_1, f_2, \lambda, \mu_1, \mu_2$ occurring in (30) and (31) satisfy relations

$$\Xi_1(\lambda) = \Xi_2(\lambda) = \Xi_1(f_1) = \Xi_2(f_2) = 0$$

$$\Xi_1(\mu_2) = 2f_2 \lambda + X_1(f_2)$$

$$\Xi_2(\mu_1) = 2f_1 \lambda + X_2(f_1)$$

$$\lambda^2 + f_1 f_2 = -\Xi_1(\mu_1) - X_1(\lambda) = -\Xi_2(\mu_2) - X_2(\lambda).$$

**Proof.** These are consequences of relations (30), (31) and Jacobi identities involving vector fields $X_1, X_2, \Xi_1, \Xi_2$. For instance, Jacobi identity for $X_1, \Xi_1, \Xi_2$ gives $\Xi_1(\alpha) = \Xi_1(\lambda) = 0$.

The gauge $X_i \mapsto X_i + g_i \Xi_i$ with $\Xi_1(g_i) = 0$ annihilates $f_i, i = 1, 2$, if $\Xi_1(g_2) = -f_2, \Xi_2(g_1) = -f_1$. Due to relations $\Xi_1(\Xi_i) = 0$ from the above lemma such functions $g_i$ exist, at least, locally and relations (32) are simplified to

$$\Xi_1(\lambda) = \Xi_2(\lambda) = \Xi_1(\mu_2) = \Xi_2(\mu_1) = 0,$$

$$\Xi_1(\mu_1) + X_1(\lambda) = \Xi_2(\mu_2) + X_2(\lambda) = -\lambda^2.$$  

Note that the so-normalized vectors $X_i$’s are uniquely defined up to a transformation $X_i \mapsto X_i + \phi \Xi_i$ with $\Xi_j(\phi_i) = 0, i, j = 1, 2$. Such a transformation induces a transformation $\mu_i \mapsto \mu_i + \psi_i, i = 1, 2$, with $\Xi_j(\psi_i) = 0, i, j = 1, 2$. Here $\psi_1, \psi_2$ arbitrary satisfying the last conditions functions.
Now it is convenient to pass to vectors \( Z = X_1 - X_2 \) and \( W = X_1 + X_2 \) instead of \( X_1 \) and \( X_2 \) which will be assumed normalized as above. In these terms relations (30) and (31) become

\[
[Z, \Xi_1] = -\lambda \Xi_1, \quad [Z, \Xi_2] = -\lambda \Xi_2, \quad [W, \Xi_1] = \lambda \Xi_1, \quad [Z, W] = 2\lambda Z + 2\mu_1 \Xi_1 - 2\mu_2 \Xi_2
\]  
(34)

with (see (32))

\[
Z(\lambda) = \Xi_2(\mu_2) - \Xi_1(\mu_1), \quad W(\lambda) = -(2\lambda^2 + \Xi_1(\mu_1) + \Xi_2(\mu_2)).
\]  
(35)

Now suppose that the bidimensional foliation generated by \( \Xi_1 \) and \( \Xi_2 \) is a fibration \( \pi : M \to N \). This takes place, at least, locally. Then formula (34) tells that vector fields \( Z \) and \( W \) are \( \pi \)-projectable. Moreover, it follows from (34) and (35) that \( \Xi_i(\mu_i) = 0 \), \( i = 1, 2 \), and hence

\[
\lambda = \pi^*(z), \quad \Xi_i(\mu_i) = \pi^*(\nu_i), \quad i = 1, 2, \quad \text{for some } z, \nu_1, \nu_2 \in C^\infty(N).
\]

Let \( \bar{Z} = \pi(Z) \), \( \bar{W} = \pi(W) \), \( u = \nu_2 - \nu_1 \) and \( v = \nu_1 + \nu_2 \). Then

\[
\bar{Z}(z) = u, \quad \bar{W}(z) = -(2z^2 + u), \quad [\bar{Z}, \bar{W}] = 2z\bar{Z}.
\]  
(36)

We shall explicitly solve these relations assuming that functions \( z, u \) and \( v \) are functionally independent (the generic case). To this end note that there is a local chart \( (z, u, v, y_1, \ldots, y_{n-5}) \) on \( N \) such that \( \bar{Z}(y_i) = \bar{W}(y_i) = 0 \), \( \forall i \), and in this chart

\[
\bar{Z} = u\partial_z + \alpha\partial_u + \beta\partial_v, \quad \bar{W} = -(2z^2 + v)\partial_z + a\partial_u + b\partial_v
\]  
(37)

with \( \alpha, \beta, a \) and \( b \) being some functions of \( z, u \), and \( v \). By using first two relations in (36) and (37) one finds that \([\bar{Z}, \bar{W}](z) = -(4zu + a + \beta)\). On the other hand, the third relation in (36) gives \([\bar{Z}, \bar{W}](z) = 2zu \). Hence

\[
a = -(\beta + 6zu)
\]  
(38)

Next, by substituting fields (37) to \([\bar{Z}, \bar{W}] = 2z\bar{Z} \) and taking into account (38) one obtains

\[
\bar{Z}(a) - \bar{W}(a) = 2za, \quad \bar{Z}(b) - \bar{W}(b) = 2z\beta
\]  
(39)

The first of these equations can be resolved with respect to \( b \):

\[
b = \frac{1}{\alpha_v}\psi(z, u, v, \alpha, \beta, \alpha_z, \ldots, \beta_v)
\]  
(40)

with \( \psi \) being a polynomial of variables \( z, u, v \), functions \( \alpha \) and \( \beta \) and their first order derivatives. So, coefficients \( a \) and \( b \) of \( \bar{W} \) are completely determined by \( \alpha \) and \( \beta \).

Now, in view of (38) and (40), the second of equations (39) becomes a relation of the form

\[
\Phi(z, u, v, \alpha, \beta, \alpha_z, \ldots, \beta_v) = 0
\]  
(41)

with \( \Phi \) being a rational function of all involved arguments. The interested reader will easily find explicit expressions for \( \phi \) and \( \Phi \) which are not so instructive to be reported here.

**Proposition 3.13. Functions**

\[
\lambda = \pi^*(z), \quad \Xi_1(\mu_1) = -\frac{1}{2}\pi^*(u + v), \quad \Xi_2(\mu_2) = \frac{1}{2}\pi^*(u + v)
\]
are differential invariants of matchings of generic \( \omega \)-modular bivectors with respect to diffeomorphisms preserving the volume form \( \omega \) as well as functions \( \pi^*\alpha \) and \( \pi^*\beta \) of variables \( \pi^*(z), \pi^*(u), \pi^*(v) \) which are subject to differential relation (41).

Proof. The first assertion of this proposition is obvious from the above discussion. Also, vector fields \( \tilde{Z} \) and \( \tilde{W} \) are differential invariants of the problem. Hence their components \( \alpha \) and \( \beta \) in the invariant chart \( (z, u, v) \) are differential invariants as well assuming that all vector fields \( \tilde{Z}, \tilde{W} \) resolving relations (36) can be lifted to vector fields \( Z, W \) on \( M \) that resolve relations (34) and (35).

To prove the last assertion consider a local chart \( (x_1, x_2, z, u, v,...) \) in which \( \Xi_i = \partial_{x_i}, i = 1, 2 \). Then vector fields \( \tilde{Z}, \tilde{W} \) can be lifted to vector fields \( Z, W \) on \( M \) such that \( \tilde{Z}(x_i) = \tilde{W}(x_i) = 0, i = 1, 2 \). Vector fields \( Z, W \) on \( M \) of the form

\[
Z = \dot{Z} + \lambda(x_1 \partial_{x_1} - x_2 \partial_{x_2}) + \phi_1 \partial_{x_1} + \phi_2 \partial_{x_2}, \quad W = \dot{W} - \lambda(x_1 \partial_{x_1} + x_2 \partial_{x_2}) + \psi_1 \partial_{x_1} + \psi_2 \partial_{x_2}
\]

with \( \Xi_i(\phi_j) = \Xi_i(\psi_j) = 0, i, j = 1, 2 \). By using the gauge \( X_1 \mapsto X_1 + \phi_1 \Xi_1, X_2 \mapsto X_2 - \phi_2 \Xi_2 \) we eliminate functions \( \phi_1 \) and \( \phi_2 \). Now all relations in (34) are satisfied except the last one, which is satisfied if

\[
\tilde{Z}(\psi_1) - \lambda \psi_1 = \mu_1 - \Xi_1(\mu_1)x_1 \quad \text{and} \quad \tilde{Z}(\psi_2) + \lambda \psi_2 = \Xi_2(\mu_2)x_2 - \mu_2.
\]

Obviously, these equations admit (local) solutions in a neighborhood of a regular point of \( \tilde{Z} \). \( \square \)

Remark 3.5. Proposition 3.13 tells that matchings of two modular Poisson structures depend on functional parameters even when their modular vectors commute. If these do not commute the situation becomes much more complicated.

4. Modular structure of Lie algebras.

Now we shall specify results of the preceding section to linear Poisson structures, i.e., to Lie algebras. In this case \( M \) is replaced by the dual \( V^* \) of an \( n \)-dimensional vector space \( V \) over a ground field \( k \). Being algebraically formal the results of the preceding section remain valid in the differential calculus over the algebra \( k[V^*] \) of polynomials on \( V^* \).

4.1. Modular disassembling of Lie algebras. The cartesian volume from \( \omega = dx_1 \wedge \cdots \wedge dx_n \) associated with a standard cartesian chart \( (x_1, \ldots, x_n) \) on \( V^* \) is well-defined up to a scalar factor. Obviously, the concept of \( \omega \)-modularity does not change when passing from \( \omega \) to \( \lambda \omega, 0 \neq \lambda \in k \). So, the cartesian modularity, i.e., \( \omega \)-modularity with respect to a cartesian volume form \( \omega \), is well-defined on \( V^* \) and will be simply referred to as modularity. In this section we shall only deal with polynomial tensor fields on \( V^* \) and use adjectives constant, linear, etc, by referring to coefficients of these fields.

Below \( P \) stands for a linear Poisson structure on \( V^* \) which is identified with a Lie algebra structure on \( V \) (see n. 2.3). The differential form \( \alpha = \alpha_P = P|\omega \) is linear, while \( d\alpha \) is constant as well as the modular vector field \( \Xi = \Xi_P \). It is easy to see that \( \Xi \) does not depend on the choice of a cartesian volume form. Being constant the field \( \Xi \) is identified with a vector \( \theta = \theta_P \in V^* \) called the modular vector of \( P \) or of the corresponding Lie algebra.

Since \( \Xi \) is constant, a function \( \nu \) such that \( \Xi(\nu) = 1 \) can be chosen linear and, therefore, identified with a vector \( v \in V \) such that \( \theta(v) = 1 \). The Poisson bivector \( P_v \wedge \Xi \) is linear and hence corresponds to a Lie algebra structure on \( V \). Obviously,
it is well–defined by $P$. Therefore, the disassembling (24) defines a disassembling of the Lie algebra associated with $P$ into unimodular and non-unimodular parts. Namely,

$$g = g_{uni} + g_{non} \quad \text{with} \quad P = Pg, \quad P_{g_{uni}} = P + P_{\nu} \wedge \Xi, \quad P_{g_{non}} = P_{\nu} \wedge \Xi. \quad (42)$$

A direct description of this disassembling in terms of the Lie algebra $g$ is as follows. First, recall that with a linear vector field $X$ on $V^*$ a linear operator $A : V \rightarrow V$ is naturally associated. Namely, by identifying vectors of $V$ with linear functions on $V^*$, this becomes a tautology, namely, $A(u) = X(u)$. In particular, for $X = P_{\nu}$ we have

$$A(u) \overset{\text{def}}{=} [u, \nu] = P(du, d\nu), \quad u \in V^*. \quad \text{i.e., } A = -\text{ad}_u. \quad \text{Now the characteristic property (20) of } \Xi \text{ is translated as}$$

$$\theta(u) = -\text{tr}(\text{ad}_u). \quad (43)$$

This formula may be considered as a direct definition of $\theta$. It also tells that unimodular Lie algebras are those for which operators of the adjoint representation are traceless.

In these terms, the Lie algebra structure $g_{non}$ corresponding to $P_{\nu} \wedge \Xi$ reads

$$[u, v]_{non} = \theta(u)A(v) - \theta(v)A(u), \quad u, v \in V, \quad A = \text{ad}_\nu, \quad \text{(44)}$$

or, alternatively, $[u, v]_{non} = \theta(u)[\nu, v] - \theta(v)[\nu, u]$.

**Proposition 4.1.** The operator $A = \text{ad}_\nu$ and $\theta \in V^*$ satisfy relations:

$$A^*(\theta) = 0, \quad \text{tr } A = -1, \quad A(\nu) = 0, \quad \theta(\nu) = 1 \quad (45)$$

Conversely, if $\theta \in V^*, \nu \in V$ and $A : V \rightarrow V$ satisfy the above relations, then formula (44) defines a Lie algebra, which coincides with its non–unimodular part.

**Proof.** The relations to prove are just translations of relations (25) for $X = P_{\nu}$ and $P_{\nu}(\nu) = 0$ (see proposition 3.10).

If, conversely, $A^*(\theta) = 0$, i.e., $\theta(Au) = 0, \forall u \in V$, then (44) defines a Lie algebra structure $\mathfrak{h}$ on $V$. Relations $A(\nu) = 0, \theta(\nu) = 1$ imply that $A = \text{ad}_\nu$. Finally, definition (43) shows that the modular vector of $\mathfrak{h}$ coincides with $\theta$, since the trace of the operator $u \mapsto \theta(u)z$, $z \in V$, is equal to $\theta(z)$. □

Relations (45) except $\text{tr } A = -1$ mean that $V$ splits into the direct sum of two subspaces $W_0 = \ker \theta$ and $W_1 = \{ l\nu | l \in \mathbb{R} \}$ of dimensions $n - 1$ and 1, respectively. They are invariant with respect to $A$ and $W_1 \subset \ker A$. This shows that a Lie algebra defined by (44) with $A$ and $\theta$ satisfying (45) is uniquely up to isomorphism determined by the operator $A_0 = A|_{W_0} : W_0 \rightarrow W_0, \quad \text{tr } A_0 = 1$. Indeed, let $W_0$ and $W_1$ be vector spaces, $\dim W_0 = n - 1, \dim W_1 = 1$, and $0 \neq e \in W_1$. Then with any linear operator $A_0 : W_0 \rightarrow W_0$ one can associate a Lie algebra structure on $W = W_0 \oplus W_1$ given by the relations

$$[u, v] = 0 \quad \text{for } u, v \in W_0 \quad \text{and } [u, e] = A(u) \quad (46)$$

This structure is isomorphic to that given by (44) and (45) if $\text{tr } A_0 \neq 0$.

**Definition 4.1.** A Lie algebra defined by (46) with $\text{tr } A_0 \neq 0$ is called modular.

Observe that the product in a modular Lie algebra is of the form (44) with $\theta$ and $A$ satisfying relations (45) for some $\nu \in V$. Denote this algebra by $\mathfrak{t}_{A, \theta, \nu}$.

Thus any Lie algebra structure is the sum of a modular Lie algebra and a compatible with it unimodular one.
4.2. Compatibility of modular and unimodular Lie algebras. Now we shall discuss compatibility conditions of a modular Lie algebra and an unimodular one. A modular Lie algebra is of the form $\mathfrak{g} \ltimes \Xi$ where $X$ and $\Xi$ are commuting linear and constant vector fields, respectively, satisfying relations (25). The product in this algebra is given by (44) where $A : V \to V$ is corresponding to $X$ operator.

**Proposition 4.2.** An unimodular Lie algebra algebra $\mathfrak{g}$ is compatible with the algebra $\mathfrak{f}_{A,\theta,\nu}$ if and only if

$$\theta([\mathfrak{g}, \mathfrak{g}]) = 0 \quad \text{and} \quad \theta(u)([Av, w] + [v, Aw] - A[v, w]) + \text{cycle} = 0 \quad \forall u, v, w \in V, \quad (47)$$

where $[\cdot, \cdot]$ is the product in $\mathfrak{g}$.

**Proof.** Let $Y \in D(M), Q \in D_2(M), \omega, \rho \in \Lambda^1(M)$. Recall the general formula

$$L_Y(Q)(\omega, \rho) = Q(L_Y(\omega), \rho) + Q(\omega, L_Y(\rho)) - Y(Q(\omega, \rho))$$

By specifying it to $M = V^*, Y = \Xi, Q = P_{\mathfrak{g}}, \omega = du, \rho = dv$ we find

$$L_{P_{\mathfrak{g}}}(P_{\mathfrak{g}})(du, dv) = P_{\mathfrak{g}}(d(\theta(u)), du) + P_{\mathfrak{g}}(du, d(\theta(v))) - \theta([u, v])$$

since $\theta(u)$ and $\theta(v)$ are constant. Hence the condition $L_{\Xi}(P_{\mathfrak{g}}) = 0$ of proposition 3.5 specifies to $\theta([\mathfrak{g}, \mathfrak{g}]) = 0$. Similarly, for $Y = X$ we obtain

$$L_X(P_{\mathfrak{g}})(du, dv) = [Av, v] + [u, Av] - A([u, v])$$

Now it is easy to see that the second relation we have to prove is identical to the relation $\Xi \wedge L_X(P_{\mathfrak{g}}) = 0$ of proposition 3.11. \( \square \)

**Corollary 4.1.** If $\mathfrak{g}$ is unimodular and $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, then no modular Lie algebra is compatible with $\mathfrak{g}$. In particular, a semi-simple Lie algebra can not be compatible with a modular Lie algebra.

4.3. Matching modular Lie algebras. Let $X_1 \wedge \Xi_1, X_2 \wedge \Xi_2$ be linear Poisson bivectors on $V^*$ with $X_i$ and $\Xi_i$ as in sec.3.7. Being the modular field of $X_i \wedge \Xi_i$, $\Xi_i$ is a constant vector field on $V$ and, therefore, $[\Xi_1, \Xi_2] = 0$. By this reason $X_1, X_2, \Xi_1, \Xi_2$ are subject to relations (30) and (31) and as a consequence, to lemma (3.1). Since $X_1, X_2$ are linear vector fields on $V^*$ functions $f_1, f_2$ and $\lambda$ in (30) are constant while $\mu_1, \mu_2$ in (31) are linear. By a suitable gauge $X_i \mapsto X_i + g_i \Xi_i, i = 1, 2$, with linear $g_i$'s functions $f_1$ and $f_2$ can be eliminated. In this case relations in lemma (3.1) reduce to

$$\Xi_1(\mu_2) = \Xi_2(\mu_2) = 0, \quad \Xi_1(\mu_1) = \Xi_2(\mu_2) = -\lambda^2. \quad (48)$$

Let $\bar{V}^*$ be the quotient by the 2-dimensional subspace $\text{span}(\Xi_1, \Xi_2)$ space of $V^*$. Then relations (30) and (31) show that vector fields $X_1$ and $X_2$ project to some vector fields $\bar{X}_1, \bar{X}_2$ on $\bar{V}^*$, respectively, and $[\bar{X}_1, \bar{X}_2] = \lambda(\bar{X}_1 - \bar{X}_2)$. So, $\bar{X}_1$ and $\bar{X}_2$ generate a 2-dimensional Lie algebra on $\bar{V}^*$.

Let $V_i^*$ be a complement of $V_i^0 = \text{span}(\theta_1, \theta_2)$ in $V^*$ and $\pi_i : V^* \to V_i^*$, $i = 0, 1$, be the corresponding projections. A projectable on $\bar{V}^*$ vector field $X \in D(V^*)$ can be presented in the form

$$X = X_0 + a_1 \Xi_1 + a_2 \Xi_2, \quad a_1, a_2 \in C^\infty(V^*)$$

where $X_0$ is parallel to $V_i^*$. If $X$ is linear, then $X_0, a_1$ and $a_2$ are linear too. So, vector fields $Z = X_1 - X_2$ and $W = X_1 + X_2$ can be presented as

$$Z = Z_0 + \alpha_1 \Xi_1 + \alpha_2 \Xi_2, \quad W = W_0 + \beta_1 \Xi_1 + \beta_2 \Xi_2. \quad (49)$$
In these terms relations (34) are equivalent to
\begin{align}
-\Xi_1(\alpha_1) &= \Xi_1(\beta_1) = \Xi_2(\alpha_2) = \Xi_2(\beta_2) = -\lambda, \\
\Xi_1(\alpha_2) &= \Xi_1(\beta_2) = \Xi_2(\alpha_1) = \Xi_2(\beta_1) = 0. \\
Z_0(\beta_1) - W_0(\alpha_1) - \lambda(3\alpha_1 + \beta_1) &= 2\mu_1, \\
Z_0(\beta_2) - W_0(\alpha_2) + \lambda(\beta_2 - 3\alpha_2) &= -2\mu_2.
\end{align}  
(50)

Point out that linear functions \(\alpha_i\)'s and \(\beta_i\)'s depend on the choice of the complement \(V_1^*\).

To proceed on we need the following lemma

**Lemma 4.1.** Let \(\omega\) be a cartesian volume form on \(V_1^*\) and \(\omega = \pi_0^*(\omega_0) \wedge \pi_1^*(\omega_1)\) with \(\omega_i\) being a cartesian volume form on \(V_i^*\). Then
\[ L_X(\omega) = (\text{div}_{\omega_0} \bar{X}_0 + \Xi_1(\alpha_1) + \Xi_2(\alpha_2))\omega \]
where \(\bar{X}_0\) is the restriction of \(X_0\) to \(V_1^*\).

**Proof.** Since \(X_0|_{\pi_0^*(\omega_0)} = \Xi_0|_{\omega} = \Xi_1|_{\omega} = 0\), we have
\[ L_{X_0}(\pi_0^*(\omega_0)) = 0 \quad \text{and} \quad L_{\alpha_i\Xi_i}(\omega) = d\alpha \wedge (\Xi_i|_{\omega}) = \Xi_i(\alpha_i)\omega \]
It remains to note that
\[ L_{X_0}(\omega) = \pi_0^*(\omega_0) \wedge L_{X_0}(\pi_1^*(\omega_1)) = \pi_0^*(\omega_0) \wedge \pi_1^*(L_{\bar{X}_0}\omega_1) = \text{div}_{\omega_0} \bar{X}_0 \cdot \omega. \]

\[ \square \]

A linear function \(\varphi\) on \(V^*\) can be decomposed into the sum \(\varphi = \varphi^0 + \varphi^1\) where \(\varphi^i\) is linear and vanishes on \(V_i^*\), \(i = 1, 2\). Accordingly, relations (50) and (51) split into two parts. First of them explicitly describes functions \(\alpha_1, \beta_1, \ i = 1, 2\), while the second one, in view of (48), put no additional restrictions on these functions. We can assume that \(\alpha_0^0 = \alpha_2^0 = 0\) by making use of the gauge \(X_1 \rightarrow X_1 + \alpha_0^0\Xi_1, \ X_2 \rightarrow X_2 - \alpha_2^0\Xi_2\), which is still at our disposal. In this normalization relations (51) become
\[ Z_0(\beta_1^0) - \lambda\beta_1^0 = 2\mu_0^0, \quad Z_0(\beta_2^0) + \lambda\beta_2^0 = -2\mu_2^0. \]
(52)

Now we are ready to describe matchings of two modular Lie algebras. If \(\lambda \neq 0\) they are characterized by ordered quadruples \((V, A, B, \lambda)\) where \(V\) is a vector space, \(A, B : V \rightarrow V\) are linear operators such that \([A, B] = 2\lambda A\) and \(tr B = 2(1 + \lambda)\). If \(\lambda = 0\), then the matching is characterized by the quintuple \((V, A, B, \nu_1, \nu_2)\) with commuting \(A, B : V \rightarrow V\) such that \(tr A = 0, tr B = 2\), and \(\nu_1, \nu_2 \in \text{Ker} \ A\). Quadruples \((V, A, B, \lambda)\) and \((V', A', B', \lambda')\) are *equivalent* if \(\lambda = \lambda'\) and there exists an isomorphism \(\Phi : V \rightarrow V'\) such that \(A' = \Phi B \Phi^{-1}, B' = \Phi B\Phi^{-1}\). Similarly, quintuples \((V, A, B, \nu_1, \nu_2)\) and \((V', A', B', \nu'_1, \nu'_2)\) are *equivalent* if, additionally, \(\nu'_i = \Phi^{-1} \circ \nu_i\). In particular, the group \(gl(V)\) naturally acts on quadruples and quintuples defined on \(V\) and their equivalence classes are labeled by orbits of this action.

The quadruple (resp., quintuple) associated with a compatible pair \(X_1 \land \Xi_1, \ X_2 \land \Xi_2\) is constructed on \(V = V = \left(V^* / \text{span}(\theta_1, \theta_2)\right)\) with \(A\) and \(B\) being restrictions of \(\bar{Z} = X_1 - \bar{X}_2\) and \(\bar{W} = X_1 + \bar{X}_2\) to linear functions on \(V\) (resp., \(\nu = \beta_i^0\) for a suitable choice of \(V_i^*\)), respectively.

**Theorem 4.1.** Matchings of \(n\)-dimensional modular Lie algebras are classified by equivalence classes of quadruples \((V, A, B, \lambda)\) (resp., quintuples \((V, A, B, \nu_1, \nu_2)\), if \(\lambda \neq 0\) (resp., if \(\lambda = 0\)).
Proof. Choose the complement $V_1^*$ so that $\mu_1^0 = \mu_2^0 = 0$. It is not difficult to see that such $V_1^*$ exists and is unique. Then functions $\mu_i = \mu_i^1$ are uniquely defined by relations (48), and (52) simplifies to

$$A(\beta_1^0) - \lambda \beta_1^0 = 0, \quad A(\beta_2^0) + \lambda \beta_2^0 = 0. \quad (53)$$

The commutation relation $[Z, W] = 2\lambda \bar{Z}$ implies $[A, B] = 2\lambda A$. Hence $A$ is nilpotent if $\lambda \neq 0 \Rightarrow A \pm \lambda \text{id}$ is nondegenerate $\Rightarrow$ the only solution of (53) is $\beta_1^0 = \beta_2^0 = 0$. Moreover, Lemma 4.1 shows that $\text{tr} B = \text{div}_{\omega_0} \bar{X}_0 = 2(1 + \lambda)$. So, in this case the matching is completely characterized by the quadruple $(V_1^* \approx \bar{V}, A, B, \lambda)$. If $\lambda = 0$, then (53) just tells that $\beta_1^0, \beta_2^0 \in \text{Ker} A$ and we have no other restrictions on these functions. In this case the matching is completely characterized by the quintuple $(V_1^* \approx \bar{V}, A, B, \beta_1^0, \beta_2^0)$.

Conversely, by starting from an abstract quadruple (resp., quintuple) one can construct a pair of compatible modular Lie algebras, which is characterized by an equivalent to it quadruple (resp., quintuple). Indeed, let $\iota : \mathcal{V} \to \mathcal{V}^*$ be an imbedding. Put $V_1^* = \text{Im} \iota$ and choose a complement $V_0^*$ of $V_1^*$ together with two independent vectors $\theta_1, \theta_2 \in V_0^*$. Then $\Xi_i$ is defined as the corresponding to $\theta_i$ constant vector field.

Next, if $H : \mathcal{V} \to \mathcal{V}$ is an operator, then the operator $\bar{H} : V_1^* \to V_1^*$ is defined as the direct sum of the operator $\iota \circ H \circ \iota^{-1}$ on $V_1^*$ and the zero operator on $V_0^*$. Denote by $Y_H$ the corresponding to $H$ linear vector field on $\mathcal{V}^*$ and put $Z_0 = Y_A, W_0 = Y_B$.

Finally, define functions $\alpha_i, \beta_i$ and $\mu_i$ by putting

$$\alpha_i^0 = \mu_i^0 = 0, \quad \alpha_1^1 = \lambda \varphi_1, \quad \alpha_2^1 = -\lambda \varphi_2, \quad \beta_1^1 = -\lambda \varphi_1, \quad \mu_1^1 = -\lambda^2 \varphi_i$$

with $\varphi_i$ being the linear function on $V^*$ vanishing on $V_1^*$ and such that $\varphi_i(\theta_j) = \delta_{ij}$, and $\beta_1^0 = 0$ (resp., $\beta_2^0 = \pi_1^*(\nu_i)$ if $\lambda \neq 0$ (resp., if $\lambda = 0$).

Vector fields $Z$ and $W$ (and, therefore, $X_1 = \frac{1}{2}(Z + W), X_2 = \frac{1}{2}(W - Z)$) are defined by formula (49) with $\alpha_i$’s and $\beta_i$’s as above. Now a direct check shows that the so-constructed linear bivectors $X_i \wedge \Xi_i$ are compatible and modular with $X_i$’s and $\Xi_i$’s as in sec. 3.7. \qed

Remark 4.1. Since representations of bidimensional Lie algebras are well-known, theorem 4.1 gives an exhaustive description of matchings of modular Lie algebras. Also, it is worth stressing that for a given representation the trace relation $\text{tr} B = 2(1 + \lambda)$ describes the spectrum of the parameter $\lambda$.

Remark 4.2. It is easy see that matchings with proportional $\Xi_1$ and $\Xi_2$ are completely characterized by quadruples $(V, A_1, A_2, \nu)$ such that $\dim V = n - 1, \text{tr} A_1 = \text{tr} A_2 = 1$ and $0 \neq \nu \in \mathfrak{k}$. Namely, $V = V^*/\text{span}(\theta_1), \Xi_2 = \nu \Xi_1$ and $A_1$ is the restriction of the projected on $V$ vector field $X_1$ to linear functions on $V$.

5. The disassembling problem

This section is central in this paper. Here we discuss how a Lie algebra can be gradually disassembled into some other Lie algebras. Here we introduce some basic disassembling techniques and then prove (theorems 5.1 and 5.2) that any finite-dimensional Lie algebra over an algebraically closed field of zero characteristic or over $\mathbb{R}$ can be assembled in few steps from lions (see below).

In this section “Lie algebra” refers to a finite dimensional Lie algebra over a ground field $k$. We start with necessary terminology in order to properly state the **disassembling problem**.
5.1. Statement of the problem. Simple disassemblings and lieons.

Definition 5.1. A simple disassembling of a Lie algebra structure $g$ on a vector space $V$ is a representation of it as the sum

$$g = g_1 + \cdots + g_k$$

of mutually compatible Lie algebra structures $g_i$'s on $V$. Lie algebras $g_i$'s figuring in (54) are called primary constituents of $g$.

In such a situation we speak, slightly abusing the language, on a (simple) disassembling of the Lie algebra $g$ into algebras $g_1, \ldots, g_k$ or, alternatively, that $g$ is assembled from $g_1, \ldots, g_k$. Accordingly, we write

$$g_1 + \cdots + g_k = h_1 + \cdots + h_l,$$

(55)
in order to express one of the following two facts:

- a Lie algebra structure on a vector space $V$ admits two (different) disassemblings into Lie algebras structures $g_i$'s and $h_i$'s, respectively;
- Lie algebras assembled from Lie algebras $g_i$'s and $h_i$'s, respectively, are isomorphic.

Having disassembled a Lie algebra $g$ into constituents $g_1, \ldots, g_k$, it is natural to look for further disassembling of $g_i$'s and so on. This way one gets secondary, ternary, etc., constituents. The procedure, which is inverse to such a multi-step disassembling one, will be called an assembling procedure. A natural questions arising in this connection is:

**What are “finest” (“simplest”) constituents of which any Lie algebra over a given ground field can be assembled?**

It will be refereed to as the disassembling problem.

It is not difficult to come to the conclusion that the following Lie algebras must be in the list of these “finest” algebras:

- the 1-dimensional Lie algebra $\gamma$,
- the unique non-abelian 2-dimensional Lie algebra $\zeta$,
- the 3-dimensional Heisenberg (over $k$) algebra $\mathfrak{h}$.

In terms of generators algebras $\zeta$ and $\mathfrak{h}$ are described as follows:

$$\zeta = \{e_1, e_2 \mid [e_1, e_2] = e_2\}$$
$$\mathfrak{h} = \{\varepsilon_1, \varepsilon_2, \varepsilon_3 \mid [\varepsilon_1, \varepsilon_2] = \varepsilon_3, \ [\varepsilon_1, \varepsilon_3] = [\varepsilon_2, \varepsilon_3] = 0\}$$

They are “simplest” in any reasonable sense of this word. In particular, $\zeta$ is the “simplest” non-unimodular algebra, while $\mathfrak{h}$ is the “simplest” nontrivial unimodular one.

Denote by $\zeta_n, n \geq 2$, (resp., $\mathfrak{h}_n, n \geq 3$) the direct sum of $\zeta$ (resp., of $\mathfrak{h}$) and the $(n - 2)$-dimensional (resp., $(n - 3)$-dimensional) abelian Lie algebra. We will also use $\gamma_n$ for $n$-dimensional abelian Lie algebra.

Definition 5.2. Lie algebras $\mathfrak{h}_n$ and $\zeta_n$ are called $n$-dimensional $\mathfrak{h}$- and $\zeta$-lieons, respectively.

Solution of the disassembling problem for 3-dimensional Lie algebras is not difficult (see [12]) and is as follows.
Example 5.1. Any unimodular 3-dimensional Lie algebra can be simply assembled from \( l \) copies of \( \mathfrak{h} \), \( l \leq 3 \). Any non-unimodular 3-dimensional Lie algebra can be simply assembled from \( l \) copies of \( \mathfrak{h} \), \( l \leq 2 \), and one copy of \( \mathfrak{j} \). In this sense one can say that all 3-dimensional Lie algebras are assembled from \( \mathfrak{j} \)'s and \( \mathfrak{h} \)'s with help of \( \gamma \) (in order to construct \( \mathfrak{j}_3 \) from \( \mathfrak{j} \)).

In this connection see also [13] for an explicit description of the algebraic variety \( \text{Lie}(3) \) of all Lie algebra structures on a 3-dimensiona vector space.

Sometimes it is more expressive to use ”chemical” formulas like

\[
2 \mathfrak{h} = 2 \mathfrak{j} + 2\gamma. \tag{56}
\]

This formula, which will be proven below, is synonymous to \( \mathfrak{h} + \mathfrak{h} = \mathfrak{j} + \mathfrak{j} \). It should be stressed that formulas like (56) tell only that a Lie algebra can be in a way assembled both from algebras indicated in its left-hand side and from those in its right-hand side.

Fig. 1. An a-scheme of length 3.

Assemblage schemas.

Now we pass to a necessary bureaucracy. An assembling scheme (shortly, an a-scheme) \( \mathcal{S} \) is a finite graph, whose set of vertices \( \text{vert} \mathcal{S} \) is a disjoint union of nonempty subsets \( \text{vert}_s \mathcal{S} \), \( s = 0, \ldots, m \), called levels, such that

1. \( \text{vert}_0 \mathcal{S} \) consists of only one vertex \( 0_{\mathcal{S}} \), called the origin of \( \mathcal{S} \).
2. Edges of \( \mathcal{S} \) connect vertices of consecutive levels only. If \( v_0 \in \text{vert}_s \mathcal{S} \) and \( v_1 \in \text{vert}_{s+1} \mathcal{S} \) are ends of an edge, then they are called the origin and the end of this edge, respectively.
3. Any vertex \( v \in \text{vert}_s \mathcal{S} \), \( s > 0 \), is the end of only one edge.
4. None of vertices \( v \in \text{vert}_s \mathcal{S} \), \( s < m \), is the origin of only one edge. A vertex which is not the origin of an edge is called an end of \( \mathcal{S} \).

The number \( m \) is called the length of \( \mathcal{S} \) and denoted by \( |\mathcal{S}| \). Obviously, \( \mathcal{S} \) is a connected graph and there exists at most one edge connecting two given vertices of it. All vertices in \( \text{vert}_m \mathcal{S} \) are ends. A-schemes \( \mathcal{S} \) and \( \mathcal{S}' \) are equivalent if they are equivalent as graphs.

Multi-step disassemblies.

Definition 5.3. Let \( \mathfrak{g} \) be a Lie algebra structure on a vector space \( V \) and \( \mathcal{S} \) be an a-scheme. A system \( \{ \mathfrak{g}_v \}, v \in \text{vert} \mathcal{S} \), of Lie algebra structures on \( V \) is called an \( m \)-step (\( m = |\mathcal{S}| \)) disassembling of \( \mathfrak{g} \) if

1. \( \mathfrak{g} = \mathfrak{g}_{0_{\mathcal{S}}} \);
2. If \( v_1, \ldots, v_p \in \text{vert} \mathcal{S} \) are ends of edges having the common origin \( v \), then structures \( \mathfrak{g}_{v_1}, \ldots, \mathfrak{g}_{v_p} \) are mutually compatible and \( \mathfrak{g}_v = \mathfrak{g}_{v_1} + \cdots + \mathfrak{g}_{v_p} \).

\( \mathcal{S} \) is the scheme of this assembling and we shall speak on a \( \mathcal{S} \)-disassembling in order to stress an instance of that. The structure \( \mathfrak{g}_v, v \in \text{vert}_s \mathcal{S} \), is called an \( (s \text{-level}) \) term of the \( \mathcal{S} \)-disassembling It is an end term of it if \( v \) is an end point of \( \mathcal{S} \). We say that \( \mathfrak{g} \) is assembled from Lie algebras \( \mathfrak{g}_1, \ldots, \mathfrak{g}_r \) if these are in one-to-one correspondence with end terms of a disassembling of \( \mathfrak{g} \).

It is worth stressing that if \( v \) and \( w \) are not ends of two edges of a common origin, then \( \mathfrak{g}_v \) and \( \mathfrak{g}_w \) are not, generally, compatible.
**Definition 5.4.**

1. A disassembling of $\mathfrak{g}$ is called complete if all its end terms are isomorphic either to $\mathfrak{g}_n$, or to $\mathfrak{n}_n$.

2. Disassemblings of isomorphic Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are called equivalent if there exist an equivalence $\sigma : \text{vert } S \to \text{vert } S'$ of the corresponding $a$-schemes and an isomorphism $\varphi : \mathfrak{g} \to \mathfrak{h}$ which is also an isomorphism of $\mathfrak{g}_v$ onto $\mathfrak{h}_{\sigma(v)}$, $\forall v \in \text{vert } S$.

Obviously, nonequivalent disassemblings can have equivalent $a$-schemes. The above-mentioned formula $\mathfrak{n} + \mathfrak{n} = \mathfrak{g}_3 + \mathfrak{h}_3$ is a simple example of that.

The disassembling problem.

The disassembling problem is the question:

**Whether a given Lie algebra can be completely disassembled?**

Below we shall develop some disassembling techniques and prove that any Lie algebra over an algebraically closed field or over $\mathbb{R}$ can be completely disassembled. This result confirms that lieons are elementary constituents of which all Lie algebras are made. By mimicking physical terminology one may say that $\gamma$ creates the necessary “vacuum”, which makes possible interactions between constituents $\mathfrak{j}$ and $\mathfrak{n}$ of “Lie matter”.

In this connection it should be mentioned that the number of elementary constituents for Lie algebras can not be reduced to one. Indeed, according to proposition 3.6, by assembling unimodular Lie algebras one can only get unimodular ones. So, only unimodular Lie algebras can be assembled from $\mathfrak{n}$-lieons. On the other hand, it is not difficult to show (see [12]) that $\mathfrak{n}$ can not be assembled only from $\mathfrak{j}$-lieons (compare with (56) !). Hence the algebra $\mathfrak{n}$ can not be excluded from the list of “finest” Lie algebras.

Now we pass to some basic techniques and constructions that will be used in our analysis of the disassembling problem.

### 5.2. Reduction to Solvable and Semisimple Algebras

First of all, we shall show that the problem naturally splits into “solvable” and “semisimple” parts. The semidirect sum of a Lie algebra $\mathfrak{a}$ and of the abelian Lie algebra structure on a vector space $V$, which is defined by a representation $\rho : \mathfrak{a} \to \text{End } V$ will be denoted by $\mathfrak{a} \oplus \rho V$.

**Proposition 5.1.** Let algebra $\mathfrak{g}$ be the semidirect sum of an its subalgebra $\mathfrak{g}_0$ and an ideal $\mathfrak{h}$. Identifying $|\mathfrak{g}|$ and $|\mathfrak{g}_0| \oplus |\mathfrak{h}|$ we have the simple disassembling

$$\mathfrak{g} = (\mathfrak{g}_0 \oplus_{\rho} |\mathfrak{h}|) + (\gamma_m \oplus \mathfrak{h})$$

with $\rho$ being the canonical representation of $\mathfrak{g}_0$ in $|\mathfrak{h}|$ and $\gamma_m$, $m = \dim \mathfrak{g}_0$, the abelian structure on $|\mathfrak{g}_0|$.

**Proof.** By construction. \(\square\)

Now, apply proposition 5.1 to the Levi-Malcev decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r}$ of a Lie algebra $\mathfrak{g}$. Here $\mathfrak{r}$ is the radical of $\mathfrak{g}$ and $\mathfrak{h} \subset \mathfrak{g}$ a complementing $\mathfrak{r}$ semisimple subalgebra.

**Corollary 5.1.** The disassembling problem for Lie algebras over a field $k$ of characteristic zero reduces to that for solvable algebras (over $k$) and for abelian extensions of semisimple algebras, i.e., algebras of the form $\mathfrak{h} \oplus_{\rho} W$ with $\rho : \mathfrak{h} \to \text{End } W$ being a finite-dimensional representation of a semisimple algebra $\mathfrak{h}$ (over $k$).
Solvable Algebras.
The first of these two problems admits a simple solution.

**Proposition 5.2.** Any solvable Lie algebra over a field $k$ can be completely disassembled.

*Proof.* Let $g$ be a solvable algebra. Then any subspace of $\langle g, g \rangle$ containing the derived algebra $[g, g]$ is, obviously, an ideal of $g$. Consider such an ideal $s$ of codimension one and a complementing it one-dimensional subspace, which is automatically a subalgebra of $g$. This makes evident that $g$ is a semidirect product $\gamma \oplus_{\rho} s$. By applying to it proposition 5.1 we see that $g$ can be disassembled into two structures, one of which is $\gamma \oplus s$ with $s$ being a solvable algebra, while the other one is of the form $\gamma \oplus_{\rho} V$ with $\rho$ being a representation of $\gamma$ in the vector space $V = |s|$. Now obvious induction arguments reduce the problem to disassembling of algebras of the latter type. This can be done as follows.

Fix a base element $\nu \in \gamma$ and put $A = \rho(\nu) : V \to V$, $0 \neq \nu \in \gamma$. The product in the algebra $\gamma \oplus_{\rho} V$ is given by

$$[\nu, v] = Av, \quad [v_1, v_2] = 0, \quad \nu \in \gamma, \quad v_1, v_2 \in V,$$

i.e., is completely determined by the operator $A$. Denote the so-defined algebra by $\Gamma_A$. Since the operator $A$ in this construction is defined up to a scalar factor, algebras $\Gamma_A$ and $\Gamma_{\lambda A}$, $\lambda \in k$, are isomorphic. So, it remains to show that algebras $\Gamma_A$'s can be completely disassembled.

First, note that $\n = \Gamma_A$ if $A$ is the identity operator on an 1-dimensional vector space $V$ and $\hat{n} = \Gamma_A$ if $A$ is a nontrivial nilpotent operator on a 2-dimensional vector space $V$.

Second, if $\|a_{ij}\|$ is the matrix of $A$ in a basis $\{e_i\}$ of $V$, then $A = \sum a_{ij} E_{ij}$, where the operator $E_{ij} : V \to V$ is defined by $E_{ij}(e_i) = e_j$ and $E_{ij}(e_k) = 0$, $k \neq i$. Now it is easy to see that the structure $\Gamma_{E_{ij}}$ is isomorphic to $\hat{n}_n$, if $i \neq j$, and to $\n_n$, if $i = j$. Finally, since $\Gamma_{a_{ij}, E_{ij}} = \Gamma_{E_{ij}}$, if $a_{ij} \neq 0$, then

$$\Gamma_A = \sum_{i, j, a_{ij} \neq 0} \Gamma_{E_{ij}},$$

which is the desired disassembling. \hfill \Box

Disassembling (58) depends on the choice of a base in $V$. This fact can be used to illustrate nonuniqueness of complete disassemblies of a given algebra. For instance, let $\text{dim} V = 2$ and $A : V \to V$ be an operator, with eigenvalues $\pm 1$. Then in the basis of eigenvectors $e_1$ and $e_2$ disassembling (58) is $\Gamma_A = \Gamma_{E_{11}} + \Gamma_{E_{22}}$, i.e., symbolically, $\Gamma_A = 2 \n_3$. On the other hand, in the basis $\{e_1 + e_2, \ e_1 - e_2\}$ we have $\Gamma_A = \Gamma_{E_{12}} + \Gamma_{E_{21}} \Leftrightarrow \Gamma_A = 2 \hat{n}$. This proves formula (56).

The d-scheme of the complete disassembling procedure for a solvable algebra described above is as in fig. 2.

From Semisimple to Simple Algebras.
The disassembling problem for algebras $g \oplus_{\rho} V$ with semisimple $g$ is easily reduced to that for simple $g$. Indeed, observe that the direct sum of Lie algebras $g = g_1 \oplus \cdots \oplus g_k$ is a natural assemblage $g = g_1 + \cdots + g_k$, where the structure $g_i$, on $|g| = |g_1| \oplus \cdots |g_k|$ is the direct sum of abelian structures on $|g_j|$'s for $j \neq i$ and the structure $g_i$ on $|g_i|$. If $\rho$ is a representation of $g$ in $V$, then the representation
\[ \rho_i \text{ of } \mathfrak{g}_i \text{ in } V \text{ is defined to be trivial on } [\mathfrak{g}_j], \ j \neq i, \text{ and coinciding with } \rho \text{ on } \mathfrak{g}_i. \]

Then we have

**Proposition 5.3.**

\[ \mathfrak{g} \oplus \rho V = \mathfrak{g}_1 \oplus \rho_1 V + \cdots + \mathfrak{g}_k \oplus \rho_k V, \]  

(59)

is a simple disassembling of \( \mathfrak{g} \oplus \rho V \).

**Proof.** By construction. \( \square \)

**Corollary 5.2.** Let \( \mathfrak{g} \) be a semisimple Lie algebra and \( \mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k \) its decomposition into a sum of simple algebras. If \( \rho \) is a representation of \( \mathfrak{g} \) in \( V \), then

\[ \mathfrak{g} \oplus \rho V = (\mathfrak{g}_1 \oplus \gamma_{l_i}) \oplus \rho_i V + \cdots + (\mathfrak{g}_k \oplus \gamma_{l_k}) \oplus \rho_k V, \]  

(60)

where \( l_i = \dim \mathfrak{g} - \dim \mathfrak{g}_i, \ i = 1, \ldots, k, \) is a simple disassembling of \( (\mathfrak{g}_i \oplus \rho_i V) \).

**Proof.** Just to observe that in the considered case the algebra \( \mathfrak{g}_i \) in (59) is isomorphic to \( \mathfrak{g}_i \oplus \gamma_{l_i}, \ i = 1, \ldots, k \). \( \square \)

Thus, proposition 5.2 and corollary 5.2 reduce the disassembling problem to abelian extensions of of simple algebras, i.e., Lie algebras of the form \( \mathfrak{g} \oplus \rho V \) with simple \( \mathfrak{g} \). The *stripping procedure* we are passing to describe will be our main tool in disassembling algebras of this kind.

**5.3. The stripping procedure.** First, we shall introduce some special algebras which are used in this procedure.

**Dressing algebras.**

A *dressing algebra* is defined on the direct sum \( W_0 \oplus W \) of two vector spaces \( W_0 \) and \( W \) by means of a bilinear skew-symmetric \( W_0 \)-valued form \( \beta : W \times W \to W_0 \).

The product in this algebra is defined by formula

\[ [(w_0, w), (w'_0, w')] = (\beta(w, w'), 0), \quad w_0, w'_0 \in W_0, \ w, w' \in W. \]  

(61)

Denote the so-defined algebra by \( \mathfrak{a}_\beta \). If \( \dim W = 2, \ \dim W_0 = 1 \) and \( \beta \neq 0 \), then \( \mathfrak{a}_\beta \) is isomorphic to \( \mathfrak{n} \).

A Lie algebra \( \mathfrak{a} \) is isomorphic to a dressing one iff the derived subalgebra \( [\mathfrak{a}, \mathfrak{a}] \) belongs to its center. Indeed, in such a case one can take the center for \( W \) and any complementary to the center subspace for \( W_0 \).

**Proposition 5.4.** Let \( \beta \) and \( \beta' \) be \( W_0 \)-valued skew-symmetric bilinear forms on \( W \). Then Lie algebras \( \mathfrak{a}_\beta \) and \( \mathfrak{a}_{\beta'} \) are compatible. Moreover, any dressing algebra can be simply disassembled into a number of \( \mathfrak{n} \)-lieons.
The following evident fact is, nevertheless, one of most efficient in disassembling the stripping procedure techniques.

\[ \text{Example 5.3.} \]

The matrix transposition \( T : M \mapsto M^t, M \in \mathfrak{gl}(n, k) \) is an anti-automorphism of \( \mathfrak{gl}(n, k) \), i.e., \([M, N]^t = [N^t, M^t]\). So, \( t = -T \) is an involution of \( \mathfrak{gl}(n, k) \). The 1-eigenspace of \( t \) is formed by skew-symmetric matrices and is identified with the special orthogonal subalgebra \( \mathfrak{so}(n, k) \subset \mathfrak{gl}(n, k) \), while the \((-1)\)-eigenspace \( S(n, k) \) consists of symmetric matrices. So, \( (\mathfrak{so}(n, k), S(n, k)) \) is a \( d \)-pair in \( \mathfrak{gl}(n, k) \). The subalgebra \( \mathfrak{s}l(n, k) \subset \mathfrak{gl}(n, k) \) of traceless matrices is \( t \)-invariant. Hence \( t_0 = t|_{\mathfrak{s}l(n, k)} \) is an involution in \( \mathfrak{s}l(n, k) \) and \( (\mathfrak{so}(n, k), S_0(n, k)) \) with \( S_0(n, k) \)) being the space of symmetric traceless matrices is the corresponding \( d \)-pair.

The stripping procedure.

The following evident fact is, nevertheless, one of most efficient in disassembling techniques.

**Proof.** Obviously,
\[
\mathbf{a}_{\beta + \beta'} = \mathbf{a}_{\beta} + \mathbf{a}_{\beta'}.
\] (62)
Hence \( \mathbf{a}_{\beta} \) and \( \mathbf{a}_{\beta'} \) are compatible.

Choose a base \( e_1, \ldots, e_m \) in \( W_0 \) and a base \( e_1, \ldots, e_{n-m} \) in \( W \). Then
\[
\beta = \sum_{i,j,k} \beta_{ij}^k e_k
\]
for some \( k \)-valued skew-symmetric bilinear forms \( \beta_{ij}^k \) on \( W \) such that \( \beta_{ij}^k(e_p, e_q) = 0 \) if \((p, q)\) differs from \((i, j)\) and \((j, i)\). If \( \beta_{ij}^k \neq 0 \), then the algebra \( \mathbf{a}_{\beta_{ij}^k} \) with \( \beta_{ij}^k = \beta_{ij}^k e_k \) is isomorphic to \( \mathfrak{sl}_n \). Hence,
\[
\mathbf{a}_{\beta} = \sum_{i,j,k} \mathbf{a}_{\beta_{ij}^k},
\] (63)
where the summation \( \sum_{i,j,k} \) is extended on all triples \( i, j, k \) for which \( \beta_{ij}^k \neq 0 \). \( \Box \)

**D-pairs and involutions.**

Let now \( \mathfrak{g} \) be a Lie algebra, \( \mathfrak{s} \) an its subalgebra and \( W \) be a complement of \([\mathfrak{s}]\) in \([\mathfrak{g}]\). If \([W, W] \subset \mathfrak{s} \) and \([\mathfrak{s}, W] \subset W \) the pair \((\mathfrak{s}, W)\) is called a \( d \)-pair in \( \mathfrak{g} \). A \( d \)-pair \((\mathfrak{s}, W)\) is trivial if \( W \) is an abelian subalgebra.

The dressing algebra \( \mathbf{a}_{\beta} \) defined on \([\mathfrak{g}]\) with \( W_0 = [\mathfrak{s}] \) and \( \beta(w_1, w_2) = [w_1, w_2] \), \( w_1, w_2 \in W \) will be called associated with \((\mathfrak{s}, W)\).

**Example 5.2.** Let \( V \) be a vector space and \( V_1, V_2 \) be its subspaces complementary one to another. Consider the subalgebra \( \mathfrak{s} = \mathfrak{s}(V_1, V_2) \) of the Lie algebra \( \mathfrak{gl}(V) \) composed of operators, leaving \( V_1 \) and \( V_2 \) invariant. The linear subspace \( W = W(V_1, V_2) \) formed by operators, sending \( V_1 \) to \( V_2 \) and conversely, is a complement of \( \mathfrak{s}(V_1, V_2) \) in \( \mathfrak{gl}(V) \). Then \((\mathfrak{s}, W)\) is a \( d \)-pair in \( \mathfrak{gl}(V) \).

**Remark 5.1.** A \( d \)-pair \((\mathfrak{s}, W)\) in \( \mathfrak{g} \) supplies \( \mathfrak{g} \) with a structure of a graded \( \mathbb{F}_2 \)-algebra \((\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z})\) and vice versa. Namely, if \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \), then \( \mathfrak{s} = \mathfrak{g}_0, W = \mathfrak{g}_1 \).

The involution \( I : [\mathfrak{g}] \mapsto [\mathfrak{g}] \), \( I^2 = id|_{[\mathfrak{g}]} \) with \([\mathfrak{s}]\) and \( W \) being eigenspaces corresponding to eigenvalues 1 and \(-1\), respectively, is naturally associated with a \( d \)-pair \((\mathfrak{s}, W)\). Obviously, \( I \) is an automorphism of \( \mathfrak{g} \). Conversely, \( \pm 1 \)-eigenspaces of an involutive automorphism \( I \) of \( \mathfrak{g} \) form a \( d \)-pair in \( \mathfrak{g} \). So, there is a one-to-one correspondence between \( d \)-pairs and involutions of \( \mathfrak{g} \). It depends on the context, which of these points of view is more convenient.

**Example 5.3.**
**Lemma 5.1** (The Stripping Lemma). Let $\mathfrak{g}$ be a Lie algebra, $(\mathfrak{s}, \mathfrak{W})$ be a $d$-pair in it and $\mathfrak{a}_\beta$ be the associated dressing algebra. Then

$$\mathfrak{g} = (\mathfrak{s} \oplus_{\rho} \mathfrak{W}) + \mathfrak{a}_\beta,$$

with $\rho$ being the restriction of the adjoint representation of $\mathfrak{g}$ to $\mathfrak{s}$ is a simple disassembling of $\mathfrak{g}$.

**Proof.** By construction. □

**Remark 5.2.** The dressing algebra $\mathfrak{a}_\beta$ may be viewed as a “mantle” that covers “shoulders” $\mathfrak{s}$ of $\mathfrak{s} \oplus_{\rho} \mathfrak{W}$. This motivates the terminology. According to proposition 5.4, $\mathfrak{a}_\beta$ can be completely disassembled. This reduces the disassembling problem for $\mathfrak{g}$ to a simpler algebra, namely, $\mathfrak{s} \oplus_{\rho} \mathfrak{W}$.

The **stripping procedure** consists of consecutive applications of the Stripping Lemma which gradually simplify appearing in its course algebras. In order to give due rigor to the term ”simplification”, we define the **complexity** $l(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ as the dimension of its ”semi-simple part”, i.e., of an its Levi subalgebra.

Algebras of complexity zero are solvable and, according to proposition 5.2, can be completely disassembled. Therefore, we see that

*all Lie algebras over a given ground field $k$ can be completely disassembled if any algebra of the form $\mathfrak{g} \oplus_{\rho} \mathfrak{V}$ with simple $\mathfrak{g}$ admits a $d$-pair $(\mathfrak{s}, \mathfrak{W})$ such that $l(\mathfrak{s}) < l(\mathfrak{g})$.*

Indeed, the dressing algebra $\mathfrak{a}_\beta$ in the corresponding disassembling $\mathfrak{g} \oplus_{\rho} \mathfrak{V} = \mathfrak{s} \oplus_{\rho'} \mathfrak{W} + \mathfrak{a}_\beta$ (see (64)) can be completely disassembled (proposition (63)). On the other hand, $l(\mathfrak{s} \oplus_{\rho'} \mathfrak{W}) = l(\mathfrak{s}) < l(\mathfrak{g})$. So, by applying proposition 5.1 to the Levi-Malcev decomposition of the algebra $\mathfrak{s} \oplus_{\rho'} \mathfrak{W}$ we reduce the problem to an algebra of the form $\mathfrak{g} \oplus_{\rho} \mathfrak{V}$ with $\bar{\mathfrak{g}}$ being the semisimple part of $\mathfrak{s}$, since $l(\bar{\mathfrak{g}} \oplus_{\rho} \mathfrak{V}) = l(\bar{\mathfrak{g}}) = l(\mathfrak{s}) < l(\mathfrak{g})$. Finally, according to Proposition 5.3, the algebra $\bar{\mathfrak{g}} \oplus_{\rho} \mathfrak{V}$ disassembles into algebras of the form $\mathfrak{h} \oplus_{\tau} \mathfrak{U}$ with $l(\mathfrak{h}) \leq l(\bar{\mathfrak{g}})$ and simple $\mathfrak{h}$.

We shall call a $d$-pair $(\mathfrak{s}, \mathfrak{W})$ in a Lie algebra $\mathfrak{g}$ as well as the corresponding to it $d$-involution simplifying if the complexity of $\mathfrak{s}$ is lesser than that of $\mathfrak{g}$. In the rest of this section we shall concentrate on existence of simplifying $d$-pairs for Lie algebras of the form $\mathfrak{g} \oplus_{\rho} \mathfrak{V}$ with simple $\mathfrak{g}$.

**Multi-involution disassembling.**

Keeping in mind that the disassembling problem is reduced to abelian extensions of simple Lie algebras we shall adopt the Stripping Lemma to semidirect products $\mathfrak{g} = \mathfrak{h} \oplus_{\rho} \mathfrak{V}$. It is convenient to put the question in a more general context.

Let $P_1, \ldots, P_l$ be commuting involutions of a Lie algebra $\mathfrak{g}$. Denote by $\mathbb{F}_2^l$ the algebra of $\mathbb{F}_2$-valued $l$-vectors with coordinate-wise multiplication. Let $\varsigma = (\varsigma_1, \ldots, \varsigma_l) \in \mathbb{F}_2^l$. The common eigenspace of involutions $P_1, \ldots, P_l$, which correspond to their eigenvalues $\lambda_i = (-1)^{\varsigma_i}$, $i = 1, \ldots, l$, will be denoted by $|\mathfrak{g}|_{\varsigma}$. Then

$$|\mathfrak{g}| = \bigoplus_{\varsigma \in \mathbb{F}_2^l} |\mathfrak{g}|_{\varsigma}$$

(65)

Obviously, $|\mathfrak{g}|_{\varsigma} |\mathfrak{g}|_{\sigma} = |\mathfrak{g}|_{\varsigma + \sigma}$. Associate with any $\varsigma \in \mathbb{F}_2^l$ a skew-symmetric algebra structure denoted by $\mathfrak{g}_{\varsigma}$ on $|\mathfrak{g}|$ with the product $[\cdot, \cdot]_{\varsigma}$ defined on homogenous elements by the formula

$$[u, v]_{\varsigma} = [u, v], \quad \text{if} \quad \xi \cdot \tau = \varsigma, \quad u \in |\mathfrak{g}|_{\xi}, \quad v \in |\mathfrak{g}|_{\tau}, \quad \text{and zero otherwise.}$$

(66)
Proposition 5.5. \( g \) is a Lie algebra structure on \( |g| \).

Proof. First, we have to check the Jacobi identity for the bracket \([\cdot,\cdot]|_{\xi}\). As it directly follows from the definition, the double bracket \([u, [v, w]]|_{\xi}\) with \( u \in g_\mu, v \in g_\nu, w \in g_\xi \), can be different from zero only if \( \xi = 0 \) and \( \mu \nu = \nu \xi = \xi \mu = 0 \). In this case \([\cdot,\cdot]|_{\xi} = [u, [v, w]]|_{\xi} + \text{cycle} = [u, [v, w]] + \text{cycle} = 0 \). On the other hand, if \( \xi, \mu, \nu, \xi \) do not satisfy the above condition, all double commutators of elements \( u, v \) and \( w \) with respect to \([\cdot,\cdot]|_{\xi}\) vanish. \( \square \)

Generally, Lie algebras \( g\)'s are not mutually compatible. Nevertheless, some their combinations implicitly appear in the multi-involution disassembling procedure which is described below.

Let \( I_0 \) be the involution of \( g = h \oplus_\rho V \) whose proper subspaces corresponding to eigenvalues 1 and \(-1\) are \([h]\) and \( V \), respectively. An involution \( I \) (or the corresponding d-pair) of \( g \), which commutes with \( I_0 \), will be called adopted (to the semidirect sum structure of \( g \)). Obviously, both \([h]\) and \( V \) are \( I \)-invariant. Let \( h = h_0 \oplus h_1, V = V_0 \oplus V_1 \) be the splittings into proper subspaces corresponding to eigenvalues 1 and \(-1\) of \( I \), respectively. The associated with \( I \) d-pair is \((h_0 \oplus_\rho_0 V_0, [h_1] \oplus_\rho_1 V_1)\) where \( \rho_1 \) stands for the restriction of \( \rho \) to \( h_1 \) and \( V_1 \). By removing, according to the Stripping Lemma, the associated dressing algebra we get the Lie algebra

\[
(h_0 \oplus_\rho_0 V_0) \oplus_\varnothing ([h_1] \oplus V_1)
\]  
(67)

with the representation \( \varnothing \) defined by formulas

\[
g(h_0)(h_1) = [h_0, h_1], \quad g(h_0)(v_i) = \rho(h_0)(v_i), \quad g(v_0)(h_1) = -\rho(h_1)(v_0), \quad g(v_0)(v_1) = 0
\]

where \( h_i \in h_i, v_i \in V_i, i = 0, 1 \). On the other hand, algebra (67) may be viewed as the semidirect product of \( h_0 \) and the ideal \( I \) whose support is \( V_0 \oplus [h_1] \oplus V_1 \). The product \([\cdot,\cdot]'\) in this ideal is such that \([V_0, [h_1]]' \subset V_1, [V_0, V_1]' = [([h_1], V_1)'] = 0 \). So, \( I \) is nilpotent and as such can be completely disassembled. Since

\[
(h_0 \oplus_\rho_0 V_0) \oplus_\varnothing ([h_1] \oplus V_1) = h_0 \oplus_\varnothing_0 [I] + \gamma_m \oplus I, \quad m = \dim h_0,
\]

with \( \varnothing_0 \) being the direct sum of natural actions of \( h_0 \) on \([h_1], V_0 \) and \( V_1 \) the disassembling problem for the algebra (67) and, therefore, for \( g \) is reduced to that for \((h_0 \oplus_\varnothing_0 V_0) \oplus_\varnothing_0[I] \). The passage from \( h \oplus_\rho V \) to \((h_0 \oplus_\varnothing_0 V_0) \oplus_\varnothing_0[I] \) will be called the stripping of the semidirect product \( h \oplus_\rho V \) by \( I \).

Proposition 5.6. Let \( P_1, \ldots, P_t \) be commuting involutions of a Lie algebra \( g \). Then \( g \) can be assembled from lions and the algebra \( g_{(0,\ldots,0)} \oplus_\rho W \) where \( W = \oplus_{0 \neq \zeta} |g_\zeta| \) and \( \rho \) is the direct sum of natural actions of \( g_{(0,\ldots,0)} \) on \( g_\zeta \)’s.

Proof. This is an inductive procedure. First, we use the involution \( I_1 \) to show (by the Stripping Lemma) that \( g \) can be assembled from lions and the algebra \( g_0 \oplus_\varnothing_0 W_0, W_0 = [g_1] \), where \( g_0 \) and \( [g_1] \) are proper subspaces of \( I_1 \) corresponding to eigenvalues 1 and \(-1\), respectively, and \( \varnothing_0 \) is a natural action of \( g_0 \) on \([g_1] \). Since the involution \( I_2 \) commute with \( I_1 \), it leaves invariant both \( g_0 \) and \( [g_1] \) and, therefore, induces an adopted involution \( I \) of the algebra \( g_0 \oplus_\varnothing_0 W_0 \). Now, by stripping the semidirect product \( g_0 \oplus_\varnothing_0 W_0 \) by \( I \), we see that \( g \) is assembled from lions and \( g_{(0,0)} \oplus_\varnothing_0 W_1 \) where \( W_1 = \oplus_{0 \neq \zeta} |g_\zeta| \) with \( \zeta \in F_2^2 \). By continuing this process we get the desired result. \( \square \)
**Complete disassembling of classical Lie algebras.**

One simple application of proposition 5.6 is the following.

**Proposition 5.7.** Lie algebras $\mathfrak{sl}(n,k)$, $\mathfrak{o}(n,k)$, $\mathfrak{so}(n,k)$, $\mathfrak{u}$, $\mathfrak{su}$ can be completely disassembled.

**Proof.** Let $\{e_1, \ldots, e_n\}$ be a basis in a $k$-vector space $V$. We shall identify operators from $\text{End} V$ and their matrices in this basis. Consider the $d$-pair $(\mathfrak{s}_j, W_j)$ in $\mathfrak{g} = \mathfrak{gl}(V)$ associated, by the construction of example 5.2, with subspaces $V_1 = \text{span}\{e_j\}$ and $V_2 = \text{span}\{e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_n\}$, and denote by $I_j$ the corresponding to it involution of $\mathfrak{gl}(V)$. Then involutions $I_1, \ldots, I_n$ commute each other.

It is easy to see that the common proper subspace $|\mathfrak{g}_c|$, $\zeta \in \mathbb{F}_2^n$, of involutions $I_1, \ldots, I_n$, is different from zero iff the unit occurs in $\zeta$ either zero, or 2 times. In the first case the subspace $|\mathfrak{g}_c|_{(0,0,\ldots,0)}$ is composed of operators for which $e_1, \ldots, e_n$ are eigenvectors, i.e., it consists of diagonal matrices. In the second case, let $\zeta = (ij) \in \mathbb{F}_2^n$, $i \neq j$, be the $\mathbb{F}_2$-vector with two nonzero components on the $i$-th and $j$-th places. Then the subspace $|\mathfrak{g}_{ij}|$ consists of operators $\lambda e_{ij} + \mu e_{ji}$, $\lambda, \mu \in k$, with $e_{ij}$ being the operator sending $e_j$ to $e_i$ and annihilating $e_k$'s for $k \neq j$. So, in the considered situation the algebra $\mathfrak{g}_{(0,0,\ldots,0)}$ is abelian. Therefore, the algebra $\mathfrak{g}_{(0,0,\ldots,0)} \oplus W$, is solvable and as such can be completely disassembled. Now it directly follows from proposition 5.6 that the algebra $\mathfrak{g}(n,k) = \mathfrak{gl}(V)$ can be completely disassembled too.

Algebras $\mathfrak{sl}(n,k)$, $\mathfrak{o}(n,k)$, $\mathfrak{so}(n,k)$ are invariant with respect to the above constructed involutions $I_j$'s. Obviously, for each of them the subspace $|\mathfrak{g}_c|$ is a subspace of the corresponding subspace for the algebra $\mathfrak{gl}(n,k)$. In particular, this shows that the algebra $\mathfrak{g}_{(0,\ldots,0)} \oplus W$ is solvable, and proposition 5.6 gives the desired result.

In order to completely disassemble the symplectic algebra $\mathfrak{sp}(n,k)$ the preceding procedure must be slightly modified. Let $\sigma(u,v)$ be a symplectic form on $V$, dim $V = 2n$. We interpret the algebra $\mathfrak{sp}(n,k)$ as the algebra

$$\mathfrak{sp}(\sigma) = \{ A \in \text{End } V \mid \sigma(Aa, v) + \sigma(a, Av) = 0, \forall a, v \in V \}.$$ 

Let $V = V_1 \oplus \cdots \oplus V_n$, dim $V_i = 2$, $\forall i$, be a $\sigma$-orthogonal decomposition of $V$ and $P_i : V \to V$ the associated projector on $V_i$. Then $I_i = \text{id}_V - 2P_i$ is an involution of $\mathfrak{gl}(V)$. It is easy to see that involutions $I_i$'s commute and their common proper subspace, on which they all are the identity, is $\mathfrak{sp}(\sigma_1) \oplus \cdots \oplus \mathfrak{sp}(\sigma_n)$ with $\sigma_i = \sigma_{V_i}$ ($= \mathfrak{g}_{(0,\ldots,0)}$ in the notation of proposition 5.6). Note that $\mathfrak{sp}(\sigma_i)$ is isomorphic to $\mathfrak{sp}(2,k)$. So, in the considered case, in the contrast with the preceding case the algebra $\mathfrak{g}_{(0,\ldots,0)} \oplus W$ of proposition 5.6 is not solvable. So, it can not be completely disassembled on the basis of our previous results. This small difficulty can be resolved by introducing an additional involution.

Namely, let $J_i : V_i \to V_i$ be a complex structure on $V_i$ compatible with $\sigma_i$, i.e.,

$$J_i^2 = -\text{id}_{V_i} \quad \text{and} \quad \sigma_i(J_iu, v) + \sigma_i(u, J_iv) = 0,$$

and $J = J_1 \oplus \cdots \oplus J_n$. Then $\sigma(Ju, v) + \sigma(u, Jv) = 0$ and $J_0 : \text{End } V \to \text{End } V$, $A \mapsto -JAJ$ is an involution which leaves invariant the subalgebra $\mathfrak{sp}(\sigma)$. Moreover, $I_0$ commutes with involutions $I_1, \ldots, I_n$, and their common proper subspace on which they act as identity is the abelian subalgebra $\mathfrak{h}$ composed of elements $\lambda_1J_1 + \cdots + \lambda_nJ_n$, $\lambda_1, \ldots, \lambda_n \in k$. Now, by applying proposition 5.6 to involutions $I_0, I_1, \ldots, I_n$ and taking into account that in this case the algebra $\mathfrak{g}_{(0,\ldots,0)} \oplus W = \mathfrak{h} \oplus W$ is solvable, we see that $\mathfrak{sp}(\sigma)$ can be completely disassembled.
Similar arguments shows that $\mathfrak{u}(n)$ and $\mathfrak{su}(n)$ can be assembled from lions. To this end, interpret an $n$–dimensional vector space as a $2n$–dimensional $\mathbb{R}$–vector space $V$ supplied with a complex structure $J$, $J^2 = -\text{id}_V$ and split $V$ into a direct sum of 2-dimensional $J$-invariant subspaces. Then consider, as above, the corresponding involutions $I_1, \ldots, I_n$. In this case the algebra $\mathfrak{g}(0, \ldots, 0)$ is abelian. Namely, it consists, exactly as before, of elements $\lambda_1 J_1 \oplus \cdots \oplus \lambda_n J_n$, $\lambda_1, \ldots, \lambda_n \in k$, where $J_i = J|_{V_i}$. Hence proposition 5.6 gives the desired result.

This proof of proposition 5.6 is not very constructive in the sense that the corresponding a–scheme is rather complicated to efficiently work with. We reported it here with the aim to illustrate the Stripping Lemma at work. A short and constructive procedure of complete disassembling of classical Lie algebras will be described in section 7.

In fact, any simple Lie algebra $\mathfrak{g}$ over a ground field $k$ possesses a nontrivial involution. So, the next natural question is wether it can be extended to the algebra $\mathfrak{g} \oplus_s V$.

### 5.4. Extensions of d-pairs/involutions.

An extension of a d-pair in a Lie algebras $\mathfrak{g}$ to an adopted d-pair in $\mathfrak{g} \oplus_s V$ is described as follows. Let $\rho$ be a representation of $\mathfrak{g}$ in a vector space $V$ and $(\mathfrak{s}, W)$ a d-pair in $\mathfrak{g}$. A decomposition $V = V_0 \oplus V_1$ will be called an $\rho$-extension of $(\mathfrak{s}, W)$ if

1. $V_i$ is invariant with respect to operators $\rho(s), s \in \mathfrak{s}, i = 0, 1$;
2. $\rho(w)(V_0) \subset V_1, \rho(w)(V_1) \subset V_0$, if $w \in W$.

**Lemma 5.2.** If $V = V_0 \oplus V_1$ is a $\rho$–extension of $(\mathfrak{s}, W)$, then $(\mathfrak{s} \oplus_{\rho|_s} V_0, W \oplus V_1)$ is a d-pair in the Lie algebra $\mathfrak{g} \oplus_s V$.

**Proof.** Obviously, $(\mathfrak{s} \oplus_{\rho|_s} V_0)$ is a subalgebra of $\mathfrak{g} \oplus_s V$. Denoting by $[\cdot, \cdot]_\rho$ the Lie product in $\mathfrak{g} \oplus_s V$ we have

$$
[(s, v_0), (w, v_1)]_\rho = ([s, w], \rho(s)(v_1) - \rho(w)(v_0)) \in W \oplus V_1
$$

$$
[w, v_1], (w', v_1']_\rho = ([w, w'], \rho(w)(v_1') - \rho(w')(v_1)) \in \mathfrak{s} \oplus_{\rho|_s} V_0,
$$

i.e., that

$$
[(\mathfrak{s} \oplus_{\rho|_s} V_0, W \oplus V_1)_\rho \subset W \oplus V_1, \quad [W \oplus V_1, W \oplus V_1]_\rho \subset \mathfrak{s} \oplus_{\rho|_s} V_0.
$$

\[ \square \]

Let $(\mathfrak{s}, W)$ and $\rho$ be as above. A linear operator $A : V \to V$ is called splitting (with respect to $(\mathfrak{s}, W)$ and $\rho$) if

$$
\rho(s) \circ A - A \circ \rho(s) = 0, \quad \rho(w) \circ A + A \circ \rho(w) = 0, \quad s \in \mathfrak{s}, \quad w \in W. \quad (68)
$$

In particular, $A$ is an endomorphism of the $\mathfrak{s}$-module $(V, \rho|_s)$. A splitting operator $A$ is called a splitting involution if, in addition, $A^2 = \text{id}_V$. The splitting involution $id_{V_1} \oplus (-id_{V_2})$ is naturally associated with a $\rho$-extension $V = V_0 \oplus V_1$ and vice versa. Note also that splitting operators with respect to $(\mathfrak{s}, W)$ and $\rho$ form a vector space which will be denoted by $S_1 = S_1(\mathfrak{s}, W, \rho)$, and that the product of two splitting operators is an endomorphism of the $\mathfrak{g}$-module $(V, \rho)$. So, denoting by $S_0(\rho)$ the algebra of these endomorphisms we see that

$$
S(\mathfrak{s}, W, \rho) = S_0(\rho) \oplus S_1(\mathfrak{s}, W, \rho)
$$

is an associative $\mathbb{F}_2$-graded algebra.

Denote by $V(\lambda)$ the root space of $A$, corresponding to an eigenvalue $\lambda \in k$ of $A$. 

Since degenerate operators in fields is well-known, this proposition is of help when looking for infinite. Hence extensions of simple algebras over arbitrary ground fields.

**Corollary 5.3.** Assume that eigenvalues of a nondegenerate splitting operator $A$ belong to $k$ and divide them into two parts $\Lambda_0$ and $\Lambda_1$, in such a way that opposite eigenvalues $\lambda$ and $-\lambda$ do not belong to the same part. Then the pair $V_0 = \sum_{\lambda \in \Lambda_0} V_{(\lambda)}$, $V_1 = \sum_{\lambda \in \Lambda_1} V_{(\lambda)}$ is an extension of $(s, W)$. In particular, if $A$ is a splitting involution, then the pair $(V_{(1)}, V_{(-1)})$ is an extension of $(s, W)$.

**Proof.** Straightforwardly from the above lemma. □

**Remark 5.3.** Let $A, V_0$ and $V_1$ be as in the above corollary. Then one gets a splitting involution just by declaring $V_0$ and $V_1$ to be its proper subspaces corresponding to eigenvalues 1 and $-1$, respectively.

The following fact helps in searching for non-trivial splitting operators.

**Proposition 5.8.** Let $\bar{k}$ be an extension of the ground field $k$ and $\bar{s}, \bar{W}, \bar{\rho}$ be the corresponding extensions of $s, W$ and $\rho$, respectively. Then

1. $S_0(\bar{\rho}) = S_0(\rho) \otimes_k \bar{k}$;
2. $S_1(\bar{s}, \bar{W}, \bar{\rho}) = S_1(s, W, \rho) \otimes_k \bar{k}$;
3. If $S_1(\bar{s}, \bar{W}, \bar{\rho})$ contains a nondegenerate operator, then $S_1(s, W, \rho)$ also contains a such one.

**Proof.** Obviously, $S_1(\bar{s}, \bar{W}, \bar{\rho}) \otimes_k \bar{k}$ is the solution space of linear system (68) interpreted as a system over $\bar{k}$. So, its dimension over $\bar{k}$ coincides with that of the solution space of (68) over $k$, i.e., with $S_1(s, W, \rho)$. Hence $S_1(\bar{s}, \bar{W}, \bar{\rho}) \otimes_k \bar{k} = S_1(\bar{s}, \bar{W}, \bar{\rho})$. Similar arguments prove that $S_0(\bar{\rho}) = S_0(\rho) \otimes_k \bar{k}$.

To prove the second assertion consider the polynomial $P(t) = \det (t_i A_1 + \cdots + t_m A_m)$ in variables $t_i$’s with $A_1 \ldots A_m$ being a base of $S_1(s, W, \rho)$. Zeros $t = (t_1, \ldots, t_m)$ of $P(t)$ with $t_i \in \bar{k}$ correspond to degenerate operators in $S_1(\bar{s}, \bar{W}, \bar{\rho})$. Since $A_1 \ldots A_m$ is also a base of $S_0(\bar{s}, \bar{W}, \bar{\rho})$, zeros of $P(t)$ with $t_i$’s in $\bar{k}$ gives degenerate operators in $S_1(\bar{s}, \bar{W}, \bar{\rho})$. So, since $S_1(\bar{s}, \bar{W}, \bar{\rho})$ contains non-degenerate operators the polynomial $P(t)$ is nonzero. But being of zero characteristic $k$ is infinite. Hence $P(t)$ is a nonzero function on $k^n$.

Since the structure of representations of simple algebras over algebraically closed fields is well-known, this proposition is of help when looking for $d$-pairs for abelian extensions of simple algebras over arbitrary ground fields.
5.5. Some properties of the algebra $S(s,W,\rho)$. In this subsection we keep the notation of the previous one.

**Lemma 5.4.** Let $0 \neq A \in S_1(s,W,\rho)$ and $\rho$ is irreducible. If one of eigenvalues $\lambda$ of $A$ belongs to $k$, then

1. $V = \text{Ker}(A^2 - \lambda^2 I)$ and $\lambda \neq 0$;
2. $V_0 = \text{Ker}(A - \lambda I), V_1 = \text{Ker}(A + \lambda I)$ is a $\rho$-extension of $(s,W)$.

**Proof.** First, we have $0 \neq \text{Ker}(A - \lambda I) \subset \text{Ker}(A^2 - \lambda^2 I)$. On the other hand,

$$\text{Ker}(A - \lambda I) \overset{C}{\leftrightarrow} \text{Ker}(A + \lambda I), \quad \forall w \in W.$$

Therefore Ker$(A^2 - \lambda^2 I)$ is $\rho$-invariant, and $V = \text{Ker}(A^2 - \lambda^2)$, since $\rho$ is irreducible. If $\lambda = 0$, then, obviously, Ker$ A$ is $\rho$-invariant and hence Ker $ A = V$, i.e., $A = 0$ in contradiction with the assumption.

The second assertion is obvious in view of corollary 5.3. $\square$

An immediate consequence of this lemma is

**Corollary 5.4.** Let $k$ be algebraically closed and $\rho$ irreducible. If $S_1(s,W,\rho)$ is nontrivial, then there is a $\rho$-extension of $(s,W)$.

**Proposition 5.9.** Let $g$ be simple and $\rho$ irreducible. Then

1. $S_0(\rho)$ is a division algebra (over $k$).
2. If $S(s,W,\rho)$ is not a division algebra, then the d-pair $(s,W)$ admits a $\rho$-extension.

**Proof.** The first assertion is the classical Schur lemma. Next, let $A = A_0 + A_1 \in S(s,W,\rho)$ with $A_0 \in S_0(\rho)$, $A_1 \in S_1(s,W,\rho)$ be a degenerate operator. The first assertion of the proposition implies that $A_1 \neq 0$ and $A_0^{-1} \in S_0(\rho)$, if $A_0 \neq 0$. In this case the operator $B = A_0^{-1} = I + A_1 A_0^{-1}$ is degenerate too, and hence one of eigenvalues of $B_1 = A_1 A_0^{-1} \in S_1(s,W,\rho)$ is $-1$. Now lemma 5.4 proves the assertion. Moreover, this lemma shows that the assumption $A_0 \neq 0$ takes place.

Indeed, assuming the contrary we see that $A_1$ is degenerate and, therefore, one of its eigenvalues is $0$ in contradiction with the lemma. $\square$

Now we shall specify the above results to the case $k = \mathbb{R}$.

**Proposition 5.10.** Let the $g$ be a simple Lie algebra over $\mathbb{R}$, $\rho : g \to \text{End} V$ an irreducible representation of $g$, and $(s,W)$ a d-pair in $g$. If $S_1(s,W,\rho)$ is not trivial, then $(s,W)$ admits a $\rho$-extension except, possibly, the case when $S(s,W,\rho)$ is isomorphic to $\mathbb{C}$.

**Proof.** Proposition 5.9 allows us to restrict to the case when $S(s,W,\rho)$ is a division algebra. Since $S_1(s,W,\rho)$ is nontrivial, the dimension of this algebra is greater than $1$. So, by the classical Frobenius theorem, this algebra is isomorphic either to $\mathbb{C}$, or to $\mathbb{Q}$, and we have to analyze only the second alternative.

In this case, as it is easy to see, $S_0(\rho)$ is isomorphic to $\mathbb{C}$, and $V$ acquires a structure of a vector space over $\mathbb{C}$ by means of the operator $J \in \text{End} V$ that corresponds to $\sqrt{-1}$ via this isomorphism. Denote by $V_\mathbb{C}$ this complex vector space.

The representation $\rho$ naturally extends to a representation $\rho_\mathbb{C} : g_\mathbb{C} \to \text{End}_\mathbb{C} V_\mathbb{C}$ of the complexification $g_\mathbb{C} = g \otimes_\mathbb{R} \mathbb{C}$ of $g$ in $V_\mathbb{C}$:

$$\rho_\mathbb{C}(x \otimes \sqrt{-1}) \overset{\text{def}}{=} J(\rho(x)), \quad x \in g.$$
By corollary 5.4, the d-pair $\langle s \otimes_{\mathbb{R}} \mathbb{C}, W \otimes_{\mathbb{R}} \mathbb{C} \rangle$ in $g^\mathbb{C}$ admits a $\rho_C$-extension whose restriction to $\rho$ is, obviously, an $\rho$-extension of $\langle s, W \rangle$.

5.6. D-pairs associated with 3-dimensional simple subalgebras. Here we shall construct simplifying d-pairs for abelian extensions of Lie algebras possessing a simple 3-dimensional subalgebra.

First, we shall collect some necessary facts about simple 3-dimensional algebras (see, for instance, [3]). Let $h$ be a such one and $h \in h$ a regular element of it. Then there exists a base $(e_1, e_2, e_3 = h)$ such that $[e_1, e_2] = \alpha e_2$, $[h, e_2] = \beta e_1$, $\alpha, \beta \in k$ (the ground field), $\alpha \beta \neq 0$. Put $\kappa = \alpha \beta$. So, the characteristic polynomial of $ad h$ is $t(t^2 - \kappa)$. If $\kappa = \lambda^2$, $\lambda \in k$, then $h$ splits and there exists a base $(h', 2\lambda^{-1} h, x, y)$ of $h$ called a $sl_2$-triple, such that $[x, y] = 2x$, $[h', y] = -2y$, $[x, y] = h'$. If $\kappa$ is not a square in $k$, i.e., the polynomial $t^2 - \kappa$ is irreducible, consider the extension $\bar{k}$ of $k$ by adding to $k$ the roots of $t^2 - \kappa$. We still denote these roots by $\pm \lambda \in \bar{k}$. The extended algebra $h = h \otimes_k \bar{k}$ splits over $\bar{k}$ and, as before, one can find an $sl_2$-triple $(h' = 2\lambda^{-1} h, x, y)$ in it. Recall also, that if $\rho$ is a representation of $h$, or of $h$, then eigenvalues of $\rho(h')$ are integer and multiplicities of opposite eigenvalues are equal. So, eigenvalues of $h$ are of the form $\pm (m/2)\lambda$ with $\lambda^2 = -\kappa$, $m \in \mathbb{Z}$. Since the element $h$ is semisimple the operator $\rho(h)$ is semisimple as well (see [6]). Therefore the representation space $U$ of $\rho$ splits into a direct sum of 1-dimensional and 2-dimensional $\rho(h)$-invariant subspaces in such a way that 1-dimensional subspaces belong to $ker \rho(h)$ while each of 2-dimensional ones is annihilated by the operator $\rho(h)^2 - (1/4)m^2\kappa$ for a suitable integer $m \neq 0$. We shall call them eigenlines and $m$-eigenplanes, respectively. Obviously, if $h$ splits, then any eigenplane splits into two eigenlines generated by eigenvectors of eigenvalues $\pm (m/2)\lambda$. In the non-split case eigenplanes are irreducible with respect to $\rho(h)$.

Now we shall associate a d-pair with a simple 3-dimensional subalgebra $h$ of a Lie algebra $g$. First, we recall the following elementary fact. Let $a$ be a Lie algebra, $x \in a$ and $a_\mu$ the root space of the operator $ad x$ corresponding to the eigenvalue $\mu$. Then

$$[a_\mu, a_\nu] \subset a_{\mu+\nu}. \quad (69)$$

Let $h \in h$ be as above and $A = ad_h h$. Put

$$g_0 = ker A, \quad g_m = ker(A^2 - (m^2/4)\kappa id_g)$$

Then $g = \bigoplus_{m \geq 0} g_m$ and commutation relations

$$[g_k, g_l] \subset g_{k+l} \oplus g_{k-l}. \quad (70)$$

take place. If $h$ splits, this directly follows from (69). Indeed, in this case $g_m, m > 0$, splits into a direct sum of root spaces corresponding to eigenvalues $\pm (m/2)\lambda$ (notice that $g_k = g_{-k}$). If $h$ does not split one obtains the result by extending scalars from $k$ to $\bar{k}$. In fact, the extended subalgebra $\bar{h}$ of the extended algebra $\bar{g}$ splits and hence the extended analogues $\bar{g}_m$'s of subspaces $g_m$'s commute according to (70), while $g_m \subset \bar{g}_m$.

Relations (70) show that

$$s = \bigoplus_{m \geq 0} g_{2m}, \quad W = \bigoplus_{m \geq 0} g_{2m+1}. \quad (71)$$
is a d–pair in \( g \), which will be called the first d-pair associated with \( h \) if \( W \neq \{0\} \). Since \( h \subset s \), this d–pair is trivial iff \( g = s \oplus \rho W \). In particular, it is nontrivial and simplifying if \( g \) is semisimple.

If \( W = \{0\} \), i.e., \( g = \bigoplus_{m \geq 0} g_{2m} \), then
\[
  s = \bigoplus_{m \geq 0} g_{4m}, \quad W = \bigoplus_{m \geq 0} g_{4m+2}
\]
(72)
is the second d-pair associated with \( h \). Since \( h \in g_0 \subset s \) and \( x, y \in g_2 \subset W \), this d–pair is nontrivial and, obviously, simplifying if \( g \) is simple.

5.7. Solution of the disassembling problem for algebraically closed fields. D–pairs (71) and (72) allow us to solve the disassembling problem for Lie algebras over algebraically closed fields. In this subsection we keep the notation of the previous one and assume the ground field \( k \) to be algebraically closed.

**Proposition 5.11.** Let \( g \) be a Lie algebra possessing a simple 3-dimensional subalgebra and \( \rho \) a representation of \( g \) in \( V \). Then a nontrivial associated with \( h \) d–pair admits a \( \rho \)-extension.

**Proof.** We identify \( g \) (resp., \( V \)) with subalgebra \( g \oplus_\rho \{0\} \) (resp., \( \{0\} \oplus_\rho V \)) in the algebra \( g \oplus_\rho V \). Put
\[
  B = \rho(h), \quad V_0 = \ker B, \quad V_m = \ker(B^2 - (m^2/4)k \mathbb{I}_V).
\]
Obviously, \( V = \bigoplus_{m \geq 0} V_m \). Since \( g \) is simple, the first d–pair \( (s, W) \) associated with \( h \) is nontrivial if \( W \neq \{0\} \). Then
\[
  s_{\rho} \overset{\text{def}}{=} s \oplus \bigoplus_{m \geq 0} V_{2m}, \quad W_{\rho} \overset{\text{def}}{=} W \oplus \bigoplus_{m \geq 0} V_{2m+1}
\]
(73)
is the required extension. Indeed, this directly follows from commutation relations
\[
  [g_{k+1}, V_l] \subset V_{k+l} \oplus V_{k-l},
\]
(74)
which can be proved by the same arguments as for (70). Note that this part of the proof does not require algebraic closure of \( k \).

If the first d–pair associated with \( h \) is trivial, we consider finer decompositions of \( g \) and \( V \) using the fact that \( h \) splits if \( k \) is algebraically closed. Namely, put
\[
  g_m = \ker (A - \frac{m}{2} \lambda I_g), \quad V_m' = \ker (B - \frac{m}{2} \lambda I_V).
\]
(75)
Then, obviously, \( g_m = g_m' \oplus g_{-m}', V_m = V_m' \oplus V_{-m}' \), and (see 69)
\[
  [L_k, L_l] \subset L_{k+l},
\]
(76)
where \( L_s \) stands for one of subspaces \( g_s', V_s' \). Now it immediately follows from relations (76) that subspaces
\[
  V_0 = \bigoplus_{k \in \mathbb{Z}} (V_{4k} \oplus V_{4k+1}), \quad V_1 = \bigoplus_{k \in \mathbb{Z}} (V_{4k+2} \oplus V_{4k+3}).
\]
(77)
provides a \( \rho \)-extension of the second d–pair associated with \( h \).

An important consequence of proposition 5.11 is

**Corollary 5.5.** Let \( h \) be a simple 3-dimensional subalgebra of an algebra Lie \( g \) over an arbitrary ground field and \( (s, W) \) the associated with \( h \) d-pair. Then \( S_1(s, W, \rho) \) is nontrivial.

**Proof.** Immediately from proposition 5.8, (1).
Theorem 5.1. Any finite-dimensional Lie algebra over an algebraically closed field of characteristic zero can be completely disassembled.

Proof. By Morozov lemma, any simple Lie algebra \(\mathfrak{g}\) over an algebraically closed field \(k\) of characteristic zero possesses a 3-dimensional subalgebra \(\mathfrak{h}\) isomorphic to \(\mathfrak{sl}(2, k)\) (see [3], [6]). If the algebra \(\mathfrak{g}\) in proposition 5.11 is simple, then the \(\rho\)-extension of one of the \(d\)-pairs \((\mathfrak{s}, W)\) associated with \(\mathfrak{h}\) is simplifying. Indeed, the Stripping Lemma applied to this extended \(d\)-pair leads to an algebra of the form \(\mathfrak{s} \oplus \rho' \mathfrak{V}'\) (see the proof of proposition 5.11) whose semisimple part coincides with that of \(\mathfrak{s}\). Hence \(l(\mathfrak{s}) = l(\mathfrak{g})\), since \(\mathfrak{s}\) is a proper subalgebra of \(\mathfrak{g}\).

5.8. Simplest algebras. Simple Lie algebras, which can not be directly disassembled by the above methods, will be discussed in this section.

Definition 5.5. A simple Lie algebra is called simplest if all its proper subalgebras are abelian.

This definition is justified by the following

Proposition 5.12. A simple Lie algebra \(\mathfrak{g}\) over a field \(k\) of characteristic zero contains either a simplest subalgebra, or an subalgebra isomorphic to \(\mathfrak{sl}(2, k)\).

Proof. If \(\mathfrak{g}\) is not simplest, then it contains a proper non-abelian subalgebra \(\mathfrak{h}\). If the semisimple part \(\mathfrak{h}\) is nontrivial, then \(\mathfrak{h}\) contains a proper simple subalgebra, and one gets the desired result by obvious induction arguments. If, on the contrary, the semisimple part is trivial, then \(\mathfrak{h}\) is a nonabelian solvable algebra and, therefore, contains a nontrivial nilpotent element \(g\). Hence the endomorphism \(ad_g\) of \(|\mathfrak{g}|\) has a nontrivial nilpotent part. In other words, \(g\), considered as an element of \(\mathfrak{g}\), has a nontrivial nilpotent part \(g_n\) which, according to a well-known property of semisimple algebras, belongs to \(\mathfrak{g}\). Thus \(\mathfrak{g}\) possesses a nontrivial nilpotent element. By Morozov’s lemma, such an element is contained in a 3-dimensional subalgebra of \(\mathfrak{g}\) isomorphic to \(\mathfrak{sl}(2, k)\).

Corollary 5.6. All elements of a simplest algebra are semisimple.

Proof. Assume that \(\mathfrak{g}\) has a nonsemisimple element. Then the nilpotent part of such an element is nontrivial and, by a well-known property of semisimple Lie algebras, belongs to \(\mathfrak{g}\). Hence \(\mathfrak{g}\) has a nontrivial nilpotent element. In its turn this element is contained in a 3-dimensional subalgebra of \(\mathfrak{g}\) which is isomorphic to \(\mathfrak{sl}(2, k)\) (see the proof of the above proposition). But \(\mathfrak{sl}(2, k)\) and, therefore, \(\mathfrak{g}\) contains 2-dimensional nonabelian subalgebras in contradiction with the fact that \(\mathfrak{g}\) is a simplest algebra.

Existence and diversity of simplest algebras depend exclusively on arithmetic properties of the ground field \(k\). For instance, there are no simplest algebras over algebraically closed fields. Indeed, any simple algebra over such a field \(k\) contains a 3-dimensional simple subalgebra isomorphic to \(\mathfrak{sl}(2, k)\), which, in its turn, contain proper 2-dimensional non-abelian subalgebras. On the contrary, there is only one (up to isomorphism) simplest algebra over \(\mathbb{R}\), namely, \(\mathfrak{so}(3, \mathbb{R})\) (see proposition 5.15 below).

Now we shall collect some elementary properties of simplest algebras. Denote by \(C_x\) the centralizer of an element \(x \in \mathfrak{g}\).

Proposition 5.13. Let \(\mathfrak{g}\) be a simplest Lie algebra and \(0 \neq x \in \mathfrak{g}\). Then
We have to prove that \( g \) of \( z \) consider an element \( C \).

**Proof.**

(1) Since \( 0 \neq x \in C_x \), the center of \( C_x \) is nontrivial. But the center of \( g \) is trivial. Hence \( C_x \) does not coincide with \( g \), i.e., is a proper subalgebra of \( g \). As such it is abelian.

(2) Obviously, \( x \) and \( z \) belong to \( C_y \), which, by (1), is abelian.

(3) If \( 0 \neq y \in C_x \), then, according to (2), any \( z \in C_y \) belongs to \( C_x \).

(4) \( C_x \) is an ideal in its normalizer \( N_x \). Since \( g \) is simple \( N_x \) is a proper subalgebra of \( g \) and, as such, must be abelian. So, \( N_x \subset C_x \) and hence \( N_x = C_x \).

(5) Directly from (4).

(6) We have to prove that \( [y,C_x] \cap C_x = \{0\} \), \( \forall y \in g \). Assuming the contrary consider an element \( z \in C_x \) such that \( 0 \neq [y,z] \in C_x \). In view of (3) \( C_x = C_z \Rightarrow [y,z] \in C_z \). By (4), this implies that \( y \in C_z \) and, therefore, \( [y,z] = 0 \) in contradiction with the assumption.

To proceed on we need some information about operators of the adjoint representation of a simplest algebra.

### 5.9. On the adjoint representation of simplest algebras

First, we mention without proof the following elementary facts.

**Lemma 5.5.** Let \( V \) be a finite-dimensional vector space and \( b(\cdot, \cdot) \) a nondegenerate, symmetric bilinear form on \( V \). If \( A : V \to V \) is a linear, skew-symmetric with respect to \( b \) operator, i.e., \( b(Au,v) + b(u,Av) = 0 \), \( u,v \in V \), then the minimal polynomial of \( A \) is of the form \( t^r \varphi(t^2) \cdot \varphi(0) \neq 0, r \geq 0 \). If \( A \) is semisimple and \( \varphi = \varphi_1^{n_1} \cdot \cdots \cdot \varphi_m^{n_m} \) is the canonical factorization of the polynomial \( \varphi \) into irreducible and relatively prime factors, then \( r = 0, 1 \) and \( n_1 = \cdots = n_m = 1 \).

**Corollary 5.7.** The assertion of the above lemma is valid for operators of the adjoint representation of a simplest algebra and in this case \( r = 1 \).

**Proof.** Recall that operators of the adjoint representation of a Lie algebra \( g \) are skew-symmetric with respect to the Killing form, which is nondegenerate for a semisimple \( g \). Moreover, according to corollary 5.6, these operators are semisimple. So, it suffices to take the Killing form for \( b \) and to observe that the kernel of an adjoint representation operator is nontrivial.

It is not difficult to see that if \( g(t) \) is an irreducible polynomial, then the polynomial \( h(t) = g(t^2) \) is either irreducible, or \( h(t) = \psi(t)\psi(-t) \) with irreducible and relatively prime \( \psi(t) \) and \( \psi(-t) \). Hence, by lemma 5.6, the minimal polynomial \( f(t) \) of a semisimple skew-adjoint operator \( A \) is of the form

\[
F(t) = t^\epsilon f_1(t^2) \cdot \cdots \cdot f_k(t^2)\psi_1(t)\psi_1(-t) \cdots \psi_i(t)\psi_i(-t), \quad \epsilon = 0, 1,
\]

with relatively prime and irreducible factors. Under the hypothesis of lemma 5.6 with \( \epsilon = 1 \) we have the following direct sum decomposition

\[
V = \ker A \oplus \ker f_1(A^2) \oplus \cdots \oplus \ker f_k(A^2) \oplus \ker g_1(A^2) \oplus \cdots \oplus \ker g_l(A^2)
\]

with \( g_i(t^2) = \psi_i(t)\psi_i(-t) \).
Lemma 5.6. Subspaces in decomposition (79) are mutually orthogonal with respect to the form $b$.

Proof. This is a direct consequence of the fact that $b(\varphi(A^2), \ker \phi(A)) = 0$, if polynomials $\varphi(t^2)$ and $\phi(t)$ are relatively prime. To prove this assertion, consider the identity

$$\varphi(t^2)\alpha(t) + \phi(t)\beta(t) = 1$$

with $\alpha(t)$, $\beta(t)$ being some polynomials. Let $u \in \ker \varphi(A^2)$, $v \in \ker \phi(A)$ and $\alpha(t) = \alpha_0(t^2) + t\alpha_1(t^2)$. Then

$$0 = b(\varphi(t^2)\alpha_0(A^2), \varphi(A^2))u, v = b(u, \varphi(t^2)\alpha_1(A^2))v = b(u, \varphi(A^2))v = b(u, v - \phi(A)\beta(A)v) = b(u, v).$$

□

Finally, we shall fix some properties of operators of the adjoint representation of a simplest Lie algebra $g$ (over $\mathbb{k}$). Below $(\cdot, \cdot)$ stands for the Killing form on $g$ and $k_f$ for the splitting field of a polynomial $f \in \mathbb{k}[t]$.

Proposition 5.14. Let $g$ be a simplest Lie algebra, $0 \neq x \in g$ and $A = ad\ x$. Then

1. the minimal polynomial $F(t)$ of $A$ has the form $(78)$ with $\epsilon = 1$ and decomposition (79) with Killing orthogonal summands for $V = |g|$ holds;
2. $C_x = \ker A$;
3. nonzero roots of $F(t)$ do not belong to $k$;
4. the lattice (in $k_F$) generated by roots of $F(t)$ coincides with that of $f_i(t)$ and with that of $\varphi_j(t)$, $i = 1, \ldots, k$, $j = 1, \ldots, l$;
5. $k_F = k_{f_i} = k_{\varphi_j}$, $i = 1, \ldots, k$, $j = 1, \ldots, l$.

Proof. (1) Directly from corollary 5.7 and lemma 5.6.

(2) Directly from proposition 5.13.

(3) Assuming the contrary we observe that $x$ and an eigenvector $y \in g$ corresponding to a nonzero eigenvalue $\lambda$ of $A$ span a 2-dimensional nonabelian subalgebra of $g$. Being simplest $g$ must coincide with this subalgebra in contradiction with simplicity of $g$.

(4) Let $h(t)$ be one of polynomials $f_i$, $\varphi_j$'s and $\lambda_1, \ldots, \lambda_m \in k_h$ be its roots. Consider the subalgebra $\mathfrak{h}_h$ of $g$ generated by $W = \ker h(A)$ and $C_x$. Obviously, $\mathfrak{h}$ is nonabelian and hence $g = \mathfrak{h}_h$. On the other hand, in view of (69) applied to the $k_F$-extension of $g$, eigenvalues of $A \mid_{\mathfrak{h}}$ belong to the lattice $L_h$ in $k_F$ generated by $\lambda_1, \ldots, \lambda_m$. Lattices $L_{f_i}$'s and $L_{\varphi_j}$'s must coincide, since, otherwise, one of the subalgebras would be proper in $g$.

(5) Directly from (4). □

5.10. Complete disassembling of real Lie algebras. Now we are ready to prove that any Lie algebra over $\mathbb{R}$ can be completely disassembled. The stripping procedure in this case is based on the following fact.

Proposition 5.15. Simplest Lie algebras over $\mathbb{R}$ are isomorphic to $\mathfrak{so}(3, \mathbb{R})$.

Proof. Let $g$ be a simplest real Lie algebra. Note that the minimal polynomial $F(t)$ of $A = ad\ x$, $x \in g$, is of the form $F(t) = t(t^2 + \lambda_1^2) \ldots (t^2 + \lambda_k^2)$, $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$, $k \geq 1$. Indeed, according to proposition 5.14 (3), all nonzero roots of $A$ do not belong to $\mathbb{R}$. Let $C$ be a Cartan subalgebra of $g$. Then, according to proposition 5.13, $\{ad_y\}_{y \in C}$ is a family of commuting semisimple operators. Recall that such a
family possesses a primitive element, i.e., a such one that any its invariant subspace is also invariant with respect to all operators of the family. Let \( A = \text{ad}_x, \ x \in C \) be primitive for the family \( \{\text{ad}_y\} \) and \( f(t) = t^2 + \lambda^2 \) be one of irreducible factors of its minimal polynomial \( F(t) \). If \( 0 \neq y \in \ker f(A), \) then \( P = \text{span} < y, Ay > \) is, obviously, \( A \)-invariant and hence \( \text{ad}_z \)-invariant for any \( z \in C \). Equivalently, \( [C, P] \subset P \). Obviously, \( \dim P = 2 \). Moreover, \( A([y, Ay]) = [Ay, Ay] + [y, A^2y] = 0, \) since \( A^2y = -\lambda^2y \). But, according to proposition 5.13, \( C = C_x = \ker A \). This shows that \( [y, Ay] \in C \iff [P, P] \subset C \). Hence \( \mathfrak{h} = C + P \) is a nonabelian subalgebra of \( \mathfrak{g} \).

Observe now that \( [P, P] \neq 0 \). Indeed, otherwise, elements \( x, y \) and \( Ay \) would span a 3-dimensional subalgebra of \( \mathfrak{g} \), which is nonabelian, since \( 0 \neq Ay = [x, y] \), and solvable. But being simplest \( \mathfrak{g} \) must coincide with this subalgebra in contradiction with simplicity of \( \mathfrak{g} \). Moreover, \( \dim [P, P] = 1 \), since \( [P, P] = \text{span} [y, Ay] \neq 0 \). Therefore \( [P, P] \) and \( P \) span a nonabelian 3-dimensional subalgebra \( \mathfrak{h}_0 \) of \( \mathfrak{h} \). So, \( \mathfrak{g} = \mathfrak{h}_0 \), since \( \mathfrak{g} \) is simplest. But any 3-dimensional simple Lie algebra over \( \mathbb{R} \) is isomorphic either to \( \mathfrak{so}(3, \mathbb{R}) \), or to \( \mathfrak{sl}(2, \mathbb{R}) \) (Bianchi’s classification), and the latter is not simplest one.

**Theorem 5.2.** Any finite-dimensional Lie algebra over \( \mathbb{R} \) can be completely disassembled.

**Proof.** As we have seen earlier the problem reduces to existence of simplifying d-pairs for algebras \( \mathfrak{g} \oplus_{\rho} V \) with a simple \( \mathfrak{g} \) and an irreducible \( \rho : \mathfrak{g} \to \text{End} V \). We shall construct such a pair with help of a simple 3-dimensional subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \). By propositions 5.12 and 5.15, \( \mathfrak{g} \) contains either a subalgebra isomorphic to \( \mathfrak{sl}(2, \mathbb{R}) \) or a subalgebra isomorphic to \( \mathfrak{so}(3, \mathbb{R}) \). In the first case proposition 5.11 proves existence of a simplifying d-pair. Hence we have to analyze the situation when \( \mathfrak{h} \) is isomorphic to \( \mathfrak{so}(3, \mathbb{R}) \). If in this case the first d-pair associated with \( \mathfrak{h} \) is nontrivial, then d-pair (73) in the proof of proposition 5.11 solves the problem.

So, we shall assume that the d-pair \( \{\mathfrak{h}, W\} \) associated with \( \mathfrak{h} \) is of the second type. In particular, in the notation of proposition 5.11, we have \( \mathfrak{g} = \bigoplus_{k \geq 0} \mathfrak{g}_{2k} \). Put also

\[
V_{\text{even}} = \bigoplus_{k \geq 0} V_{2k} \quad \text{and} \quad V_{\text{odd}} = \bigoplus_{k \geq 0} V_{2k+1}.
\]

Commutation relations (74) show that subspaces \( V_{\text{even}} \) and \( V_{\text{odd}} \) are \( \rho \)-invariant. Since \( \rho \) is irreducible, one of these subspaces is trivial.

First, assume that \( V_{\text{odd}} \) is trivial. Once again, relations (74) show that

\[
V_0 = \bigoplus_{k \geq 0} V_{4k}, \quad V_1 = \bigoplus_{k \geq 0} V_{4k+2}
\]

is a \( \rho \)-extension of \( \{\mathfrak{s}, W\} \).

Finally, assume that \( V_{\text{even}} \) is trivial. First of all, we note that, by corollary 5.5, \( S_1(\mathfrak{s}, W, \rho) \) is nontrivial. Moreover, proposition 5.10 reduces the problem to the case when the \( \mathbb{F}_2 \)-graded algebra \( S(\mathfrak{s}, W, \rho) \) is isomorphic to \( \mathbb{C} \). In this case \( S_1(\mathfrak{s}, W, \rho) \) is 1-dimensional and contains an operator \( J \) such that \( J^2 = -1 \). So, \( J \) supplies \( V \) with a \( \mathbb{C} \)-vector space structure which will be denoted by \( V_\mathbb{C} \).

Below we shall adopt the notation used in the proof of proposition 5.11. By construction, operators \( \rho(z), \ z \in \mathfrak{s} \), commute with \( J \), i.e., they are \( \mathbb{C} \)-linear in \( V_\mathbb{C} \). In particular, \( B = \rho(h) \) is such an operator, since \( h \in \mathfrak{s} \). Obviously, eigenvalues of
$B$ are $\frac{m}{2}\sqrt{-1}$, $m \in \mathbb{Z}$, and

$$V_C = \bigoplus_{m \in \mathbb{Z}} V_m, \quad V_m = \ker(B - \frac{m}{2}\sqrt{-1}I)(v) = \ker(B - \frac{m}{2}J)$$

$(V_m$’s are subspaces of $V_C$). Also, observe that in the considered case $V_m$ may be nontrivial only for odd $m$ and consider subspaces

$$W_1 = \bigoplus_{m \in \mathbb{Z}} V_{4m+1} \quad \text{and} \quad W_2 = \bigoplus_{m \in \mathbb{Z}} V_{4m-1}$$
of $V_C$. Then $V_C = W_1 \oplus W_2$ and $\dim W_1 = \dim W_2$, since $V_m = V_{m} \oplus V_{-m}$. We shall prove that $W_1$ and $W_2$ are $\rho$-invariant. If so, one of these subspaces must be trivial, since $\rho$ is irreducible. But $\dim W_1 = \dim W_2$ and hence the other subspace must be trivial too. This, however, is impossible, since $V$ is nontrivial.

Let $v \in V_m$ and $w \in g_{4k+2} \subset W$. Then

$$Bv = \frac{m}{2}\sqrt{-1}v = \frac{m}{2}Jv, \quad [h,w] \in W, \quad [h,[h,w]] = -(2k+1)^2w,$$

and, therefore,

$$\rho(w)J + J\rho(w) = 0 = \rho([h,w])J + J\rho([h,w]).$$

Now we have

$$B(\rho(w)v) = \rho(w)(Bv) + [B,\rho(w)]v = \frac{m}{2}\rho(w)(Jv) + [\rho(h),\rho(w)]v =$$

$$-\frac{m}{2}J(\rho(w)v) + \rho([h,w])v = -\frac{m}{2}\sqrt{-1}\rho(w)v + \rho([h,w])v$$

and

$$B(\rho([h,w])v) = \rho([h,w])(Bv) + [B,\rho([h,w])]v = \frac{m}{2}\rho([h,w])(Jv) +$$

$$+ [\rho(h),\rho([h,w])]v = -\frac{m}{2}J(\rho([h,w])v) + \rho([h,[h,w]])v =$$

$$-\frac{m}{2}\sqrt{-1}\rho([h,w])v - (2k+1)^2\rho(w)v.$$

It follows from (81) and (82) that vectors $\rho(w)v$ and $\rho([h,w])v$ span a 2-dimensional subspace $\Pi$ in $V_C$, which is invariant with respect to $B$. As it is easy to see, eigenvalues of $B|_\Pi$ are $-\frac{m \pm (4k+2)}{2}\sqrt{-1}$. This shows that $\Pi$ is spanned by eigenvectors of $B|_\Pi$ corresponding to these eigenvalues, i.e., that vectors $\rho(w)v$ and $\rho([h,w])v$ belong to $V_{-m+4k+2} \oplus V_{-(m+4k+2)}$. Since $m = 4s \pm 1$, the residue of $m$ mod 4 coincides with that of $-m \pm (4k+2)$. Therefore, vectors $\rho(w)v$ and $\rho([h,w])v$ belong to the same subspace $W_i$ as $v$.

If $z \in g_{4k} \subset s$, then $[h,z] \in s, [h,[h,z]] = -4k^2z$ and operators $\rho(z)$ and $\rho([h,z])$ commute with $J$. The same arguments as above show that vectors $\rho(z)v$ and $\rho([h,z])v$ belong to $V_{m+4k} \oplus V_{-(m-4k)}$, i.e., to the same subspace $W_i$ as $v$. This proves that subspaces $W_i$’s are $\rho$-invariant.

\begin{remark}
\textbf{Remark 5.4.} The final part of the proof of proposition 5.2 shows that $S(s,W,\rho)$ cannot be a division algebra, if $S_0(\rho)$ is isomorphic to $\mathbb{R}$ and $V_{\text{even}}$ is trivial.
\end{remark}

On the disassembling problem for arbitrary fields.
It seems rather plausible that any finite-dimensional Lie algebra over a field of characteristic zero can be completely disassembled. In view of proposition 5.12 the disassembling problem is reduced to simplest algebras and their finite-dimensional representations. This approach presumes a description of simplest algebras over
arbitrary ground fields $k$ and involutions of them. This problem doesn’t appear to be extremely challenging not to attempt to resolve it. Probably, the circle of ideas that one can find in the last chapter of [3] would be sufficient for the solution.

Derived algebras $[A_{\text{Lie}}, A_{\text{Lie}}]$ of Lie algebras $A_{\text{Lie}}$ associated with division algebras $A$ over $k$ give examples of simplest algebras. In this connection it should be also stressed that simplest Lie algebras and their representations have all merits to be studied in its own right. For instance, they appear to be natural substitutes for $\mathfrak{sl}_2$-triples in the perspective of developing analogues of root space decompositions for simple Lie algebras over arbitrary fields.

An alternative approach to the disassembling problem could be a direct description of suitable $d$-pairs in simple algebras over a given field based in its turn on a description of these algebras as, for instance, it is done in [3]. However, in order to become practical such a description must be duly extended to representations of these algebras. Moreover, a serious deficiency of this approach is that it, apart of being rather boring, does not reveal the true nature of the phenomenon.

The conjecture that all Lie algebras can be assembled from lions and the fact that lions are, in fact, Lie algebras over $k$ give examples of simplest algebras. In this connection it should be also stressed that simplest Lie algebras and their representations have all merits to be studied in its own right. For instance, they appear to be natural substitutes for $\mathfrak{sl}_2$-triples in the perspective of developing analogues of root space decompositions for simple Lie algebras over arbitrary fields.

An alternative approach to the disassembling problem could be a direct description of suitable $d$-pairs in simple algebras over a given field based in its turn on a description of these algebras as, for instance, it is done in [3]. However, in order to become practical such a description must be duly extended to representations of these algebras. Moreover, a serious deficiency of this approach is that it, apart of being rather boring, does not reveal the true nature of the phenomenon.

The conjecture that all Lie algebras can be assembled from lions and the fact that lions are, in fact, Lie algebras over $k$ lead to suspect that Lie algebras over a field $k$ are obtained as specifications to $k$ of some universal assemblage schemes. In the rest of this paper some facts supporting and clarifying this idea will be given.

6. Matching lieons and first level Lie algebras

A Lie algebra is of the first level if it can be completely disassembled in one step. In other words, first level Lie algebras are ones that can be assembled from a number of mutually compatible lieons and, on this basis, construct examples of first level Lie algebras.

6.1. Compatible $\cap$-lieons. Let $g$ be a an $n$-dimensional $\cap$-lieon and $V = |g|$. Denote by $C = C_g$ the center of $g$ and put $l = l_g = \{g, g\}$. Then $\dim C = n - 2$, $\dim l = 1$ and $l \subset C$. A basis $e_1, \ldots, e_n$ of $V$ such that $e_3 \in l$ and $e_i \in C$, if $i > 2$, will be called normal for $g$. The only nontrivial product in this basis is $[e_1, e_2] = \alpha e_3$, $\alpha \neq 0$. The associated Poisson bivector on $V^*$ in the corresponding coordinates is $P = P_g = \alpha x_3 x_1 x_2$. This shows that, up to proportionality, $g$ is uniquely defined by the pair $(C, l)$.

Consider now two $\cap$-lieons $g_1$ and $g_2$ on $V$, i.e., $|g_1| = |g_2| = V$, and put $C_i = C_{g_i}$, $l_i = l_{g_i}$, $P_i = P_{g_i}$, $i = 1, 2$, $C_1 \cap C_2$. Obviously, $n - 4 \leq \dim C_1 \leq n - 2$.

Below we use formula (10) for computations of the occurring Schouten brackets.

Lemma 6.1. If $\dim C_1 = n - 4$, then $g_1$ and $g_2$ are compatible iff $l_i \subset C_i$, $i = 1, 2$.

Proof. We have to examine the following four cases.

$A_1$: $l_i$ does not belong to $C_i$, $i = 1, 2$. In this case a basis $e_5, \ldots, e_n$ in $C_12$ can be completed by some vectors $e_1 \in l_1, e_2 \in C_1, e_3 \in l_2, e_4 \in C_2$ up to a basis in $V$. The only nonzero product $[e_1, e_2]$ in $g_1$ is $[e_3, e_4] = \alpha_1 e_1$, and $[e_1, e_2]_2 = \alpha_2 e_3$ in $g_2$ with some $\alpha_1, \alpha_2 \in k$. Hence $P_1 = \alpha_1 x_3 x_1 x_2$ and $P_2 = \alpha_2 x_3 x_4 x_2$ and a direct computation by using formula shows that $[P_1, P_2] \neq 0$, i.e., that $g_1$ and $g_2$ are not compatible.

$A_2$: One of subspaces $l_i$’s, say, $l_1$, belongs to $C_12$ and the other, $l_2$, does not. In this case we complete a basis $e_5 \in l_1, e_6, \ldots, e_n$ in $C_12$ by some vectors $e_1, e_2 \in
$C_1, e_3 \in l_2, e_4 \in C_2$ up to a basis in $V$. By similar reasons as above, $P_1 = \alpha_1 x_5 \xi_4, P_2 = \alpha_2 x_5 \xi_4$ in the corresponding coordinates. Since $[P_1, P_2] \neq 0$, $g_1$ and $g_2$ are not compatible.

$A_3$: $l_1 \subset C_{12}, i = 1, 2$, and $l_1 \neq l_2$. In this case we consider a basis $e_5 \in l_1, e_6 \in l_2, \ldots, e_n$ in $C_{12}$ and complete it by independent mod $C_{12}$ vectors $e_1, e_2 \in C_1, e_3, e_4 \in C_2$ up to a basis in $V$. In such a basis $P_1 = \alpha_1 x_5 \xi_3 \xi_4, P_2 = \alpha_2 x_6 \xi_1 \xi_2$ and $[P_1, P_2] = 0$. So, $g_1$ and $g_2$ are compatible and it is easy to see that $g_1 + g_2$ is isomorphic to $\cap \oplus \cap \oplus \gamma_{n-6}$.

$A_4$: $l_1 = l_2 \subset C_{12}$. A basis $e_5 \in l_1 = l_2, e_6, \ldots, e_n$ in $C_{12}$ can be completed by independent mod $C_{12}$ vectors $e_1, e_2 \in C_1, e_3, e_4 \in C_2$ up to a basis in $V$. Then $P_1 = \alpha_1 x_5 \xi_4, P_2 = \alpha_2 x_5 \xi_1 \xi_2$ and $[P_1, P_2] = 0$, i.e., $g_1$ and $g_2$ are compatible. The Poisson bivector corresponding to $g_1 + g_2$ is proportional to $x_5(\xi_2 + \xi_3 \xi_4)$.

**Lemma 6.2**. If dim $C_{12} = n - 3$, then $g_1$ and $g_2$ are compatible.

**Proof.** Put $C = C_1 + C_2$. Then dim $C = n - 1$. We have four qualitatively different situations as before.

$B_1$: $l_1$ and $l_2$ do not belong to $C_{12} \leftrightarrow C = l_1 \oplus l_2 \oplus C_{12}$. Consider a basis $e_1, \ldots, e_n$ in $V$ with $e_i \in l_i, i = 1, 2, e_i \in C_{12}, i > 3$. Then, in the corresponding coordinates, $P_1 = \alpha_1 x_5 \xi_4, P_2 = \alpha_2 x_6 \xi_3 $.

$B_2$: *One of subspaces $l_i$'s, say, $l_1$, belongs to $C_{12}$ and the other, $l_2$, does not.* Then $V = \langle v \rangle \oplus C$, if $v \in V \setminus C$, and $C = \langle w \rangle \oplus C_{12}$, if $w \in C \setminus C_{12}$. So, vectors $e_1 = w, 0 \leq e_2 \in l_1, 0 \leq e_3, e_4 \in l_2, e_5, \ldots, e_n$ form a basis in $V$ assuming that $e_2, e_5, \ldots, e_n$ is a basis in $C_{12}$. In the corresponding coordinates we have $P_1 = \alpha_1 x_5 \xi_4, P_2 = \alpha_2 x_6 \xi_3 $.

$B_3$: $l_i \subset C_{12}, i = 1, 2$, and $l_1 \neq l_2$. In this case there is a basis $e_1, \ldots, e_n$ in $V$ such that $e_i \in C_i \setminus C_{12}, i = 1, 2, e_3 \in V \setminus C$ and $e_i \in C_{12}$ for $i > 5$. Then, in the corresponding coordinates, $P_1 = \alpha_1 x_5 \xi_3 \xi_4, P_2 = \alpha_2 x_5 \xi_1 \xi_3 $.

$B_4$: $l_i \subset C_{12}, i = 1, 2$, and $l_1 = l_2$. Similarly to the preceding case there is a basis $e_1, \ldots, e_n$ in $V$ such that $e_i \in C_i \setminus C_{12}, i = 1, 2, e_3 \in l_1 = l_2, e_4 \in V \setminus C$ and $e_i \in C_{12}$ for $i > 4$. Then, in the corresponding coordinates, $P_1 = \alpha_1 x_5 \xi_3 \xi_4, P_2 = \alpha_2 x_5 \xi_1 \xi_4 $.

Now a simple computation shows that $[P_1, P_2] = 0$ in any of these cases.

**Lemma 6.3**. If dim $C_{12} = n - 2$, then $g_1$ and $g_2$ are compatible.

**Proof.** In this case $C_1 = C_2 = C_{12}$ and $V = \langle v_1 \rangle \oplus \langle v_2 \rangle \oplus C_{12}$ for any independent mod $C_{12}$ vectors $v_1, v_2$. Here we have two possibilities:

$D_1$: $l_1 \neq l_2$. Consider a basis $e_1, \ldots, e_n$ in $V$ with $e_i \in l_i, e_{i+2} = v_i, i = 1, 2, e_j \in C_{12}$ for $j > 4$. Then, as above, $P_1 = \alpha_1 x_5 \xi_3 \xi_4, P_2 = \alpha_2 x_5 \xi_1 \xi_2$ and $[P_1, P_2] = 0$.

$D_2$: $l_1 = l_2$. In this case lies $g_1$ and $g_2$ are proportional.

Using lemmas 6.1, 6.2 and 6.3 it is not difficult to construct various families of mutually compatible $\mathfrak{h}_n$-structures on a vector space $V$. Two such constructions are described below.

A (finite) family $\{C_i\}$ of $(n - 2)$-dimensional subspaces of $V$ will be called *tight* if dim $C_i \cap C_j > n - 4$. Tight families are easily described.

**Lemma 6.4.** A family $\{C_i\}$ of $(n - 2)$-dimensional subspaces of $V$ is tight if either all $C_i$’s are contained in a common $(n - 1)$-dimensional subspace ("co-pencil"), or all $C_i$’s have a common $(n - 3)$-dimensional subspace ("pencil").
A finite family \( \{ \mathfrak{g}_i \} \) of \( \mathfrak{h} \)-lieons on \( V \) will be called \emph{tight} if the family \( \{ C_i \} \) of their centers is tight. It follows from lemmas 6.2 and 6.3 that \( \mathfrak{h} \)-lieons belonging to a tight family are mutually compatible. This observation together with lemma 6.4 prove

**Proposition 6.1.** Let \( \{ C_i \} \) be a co-pencil (resp., pencil) of \((n - 2)\)-dimensional subspaces of \( V \). Assign to each \( C_i \) a 1-dimensional subspace \( l_i \subset C_i \). Then \( \mathfrak{h} \)-lieons characterized by pairs \((C_i, l_i)\) are mutually compatible so that their linear combinations are first level Lie algebras. \( \square \)

Also we have,

**Proposition 6.2.** Let \( \{ \mathfrak{g}_1, \ldots, \mathfrak{g}_m \} \) be a family of \( \mathfrak{h} \)-lieons characterized by pairs \((C_1, l_1), \ldots, (C_m, l_m)\). If span \((l_1, \ldots, l_m) \subset \bigcap_{i=1}^m C_i \), then \( \mathfrak{g}_i \)'s are mutually compatible so that their linear combinations are first level Lie algebras. \( \square \)

**Proof.** If \( \dim C_i \cap C_j > n - 4 \), then \( \mathfrak{g}_i \) and \( \mathfrak{g}_j \) are compatible by lemmas 6.2 and 6.3. Otherwise, they are compatible by lemma 6.1. \( \square \)

These constructions illustrate the diversity of combinations of \( \mathfrak{h} \)-lieons that produce first level Lie algebras.

6.2. **Compatible \( \mathfrak{h} \)- and \( \mathfrak{j} \)-lieons.** A \( \mathfrak{j} \)-lieon \( \mathfrak{g} \) on \( V \) is up to proportionality characterized by its center \( C = C_\mathfrak{g} \) and the derived algebra \( \Delta = \Delta_\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \). Since both \( \Delta \) and \( C \) are abelian, we identify them with the supporting them subspaces of \( V \). Obviously, \( \dim C = n - 2 \), \( \dim \Delta = 1 \), \( C \cap \Delta = \{0\} \). So, up to a scalar factor \( \mathfrak{g} \) is completely determined by the pair \((C, \Delta)\) of subspaces of \( V \) and vice-versa.

Consider now a \( \mathfrak{h} \)-lieon \( \mathfrak{g}_0 \) and a \( \mathfrak{j} \)-lieon \( \mathfrak{g}_\mathfrak{h} \) on \( V \) and the characterizing them pairs \((C_0, \Delta)\) and \((C_\mathfrak{h}, l)\). Put \( C_{12} = C_0 \cap C_\mathfrak{h} \). Then \( n - 4 \leq \dim C_{12} \leq n - 2 \).

**Lemma 6.5.** If \( \dim C_{12} = n - 4 \), then \( \mathfrak{g}_0 \) and \( \mathfrak{g}_\mathfrak{h} \) are incompatible.

**Proof.** First, note that \( \mathfrak{g}_0 \) and \( \mathfrak{g}_\mathfrak{h} \) are compatible iff the factorized lieons \( \mathfrak{g}_0/C_{12} \) and \( \mathfrak{g}_\mathfrak{h}/C_{12} \) are compatible. So, we can assume that \( n = 4 \) and \( C_{12} = 0 \Leftrightarrow V = C_0 \oplus C_\mathfrak{h} \) with \( \dim C_0 = \dim C_\mathfrak{h} = 2 \). Projections defined by this splitting of \( V \) send the line \( \Delta \) to subspaces \( \Delta^0 \subset C_0 \) and \( \Delta^\mathfrak{h} \subset C_\mathfrak{h} \), respectively. Since \( \Delta \) does not belong to \( C_0 \), \( \dim \Delta^0 = 1 \). There may occur one of the following three cases.

1. \( I_1 : \dim \Delta^0 = 1 \) and \( C_\mathfrak{h} = l \oplus \Delta^\mathfrak{h} \). Let \( e_1 \in \Delta^0 \), \( e_2 \in \Delta^\mathfrak{h} \) be such that \( e_1 + e_2 \) generates \( \Delta \). If \( e_3 \) generates \( l \) and \( e_1, e_4 \) generate \( C_0 \), then \( e_1, \ldots, e_4 \) is a basis in \( V \) and, in the corresponding coordinates, Poisson bivectors associated with \( \mathfrak{g}_0 \) and \( \mathfrak{g}_\mathfrak{h} \) are proportional to \( P_1 = (x_1 + x_2)\xi_3\xi_4 \) and \( P_2 = x_3\xi_1\xi_4 \), respectively. Now a computation shows that \( [P_1, P_2] \neq 0 \).

2. \( I_2 : \dim \Delta^0 = 0 \) and \( C_\mathfrak{h} = l \oplus \Delta^\mathfrak{h} \). Consider a basis \( e_1, \ldots, e_4 \) in \( V \) with \( e_1, e_2 \in C_0, e_3 \in l, e_4 \in \Delta^\mathfrak{h} \). In the corresponding coordinates \( P_1 \) and \( P_2 \) are proportional to \( x_4\xi_3\xi_4 \) and \( x_3\xi_1\xi_2 \), respectively, and one finds that \( [P_1, P_2] \neq 0 \).

3. \( I_3 : \dim \Delta^0 = 0 \) and \( l = \Delta^\mathfrak{h} \). Similarly, in a basis \( e_1, \ldots, e_4 \) in \( V \) with \( e_1, e_2 \in C_0, e_3 \in l, e_4 \in C_\mathfrak{h} \), \( P_1 \) and \( P_2 \) are proportional to \( x_3\xi_3\xi_4 \) and \( x_3\xi_1\xi_2 \), respectively, and a computation show that \( [P_1, P_2] \neq 0 \). \( \square \)

Put \( C = C_0 + C_\mathfrak{h} \) and note that \( \dim C = n - 1 \) iff \( \dim C_{12} = n - 3 \). In this case there are two possibilities: \( \Delta \cap C = \{0\} \) and \( \Delta \subset C \).

**Lemma 6.6.** If \( \dim C_{12} = n - 3 \) and \( \Delta \cap C = \{0\} \), then \( \mathfrak{g}_0 \) and \( \mathfrak{g}_\mathfrak{h} \) are incompatible.
Proof. In this case $V = \Delta \oplus C$. If $l$ does not belong to $C_{12}$, then, as in the preceding lemma, the factorization by $C_{12}$ reduces the problem to $n = 3$. If $n = 3$, then $\dim C_0^l = \dim C_0^\eta = 1$, $l = C_0^\eta$ and $V = \Delta \oplus C_0^\eta \oplus l$. In this case, $e_1, e_2, e_3$ of $V$ with $e_1 \in l$, $e_2 \in C_0^\eta$, $e_3 \in \Delta$ we have $P_1 \sim x_3 \xi_1 \xi_3$, $P_2 \sim x_1 \xi_2 \xi_3$ and find that $\{P_1, P_2\} \neq 0$.

If $l \subset C_{12}$, then $C_{12} = l \oplus C'$, $\dim C' = n - 4$, and the factorization by $C'$ reduces the situation to $n = 4$. In this particular case $\dim C_0^l = \dim C_0^\eta = 2$ and $C_0^\eta \cap C_0^\eta = l$. In a basis $e_1, \ldots, e_4$ such that $e_1 \in C_0^\eta, e_2 \in C_0^\eta, e_3 \in \Delta, e_4 \in l$ we have $P_1 \sim x_3 \xi_1 \xi_3, P_2 \sim x_4 \xi_2 \xi_3$ with $\{P_1, P_2\} \neq 0$.

Lemma 6.7. If $\dim C_{12} = n - 3$ and $\Delta \subset C$, then $g_0$ and $g_\eta$ are compatible.

Proof. In this case $C = \Delta \oplus C_0^\eta$ and $V = W \oplus C$ for a 1-dimensional subspace $W$ of $V$. If $l$ does not belong to $C_{12}$, the factorization of $V$ by $C_{12}$ reduces the situation to a 3-dimensional one in which $C = C_0^\eta \oplus C_0^\eta$, $\dim C_0^l = \dim C_0^\eta = 1$, $l = C_0^\eta$ and $V = C_0^\eta \oplus C_0^\eta \oplus W$. Consider a basis $e_1, e_2, e_3$ in $V$ with $e_1 \in C_0^\eta, e_2 \in C_0^\eta, e_3 \in W$. Since $\Delta \subset C_0^\eta \oplus C_0^\eta$ and $\Delta \cap C_0 = \{0\}, \Delta$ is generated by a vector of the form $e_1 + \lambda e_2, \lambda \in k$. In the corresponding to such a basis coordinates we have $P_1 \sim x_1 \xi_2 \xi_3, P_2 \sim (x_1 + \lambda x_2) \xi_1 \xi_3$ with $\{P_1, P_2\} = 0$.

If $l \subset C_{12}$, then $C_{12} = l \oplus C'$, $\dim C' = n - 4$, and the factorization by $C'$ reduces the situation to $n = 4$. In this case $\dim C_0^l = \dim C_0^\eta = 2$ and $\dim C_0^\eta \cap \dim C_0^\eta = l$. Consider a basis $e_1, \ldots, e_4$ in $V$ such that $e_1 \in C_0^\eta, e_2 \in C_0^\eta, e_3 \in l, e_4 \in W$. Since $\Delta \cap C_0 = \{0\}, \Delta \subset C = C_0^\eta \oplus C_0^\eta$ is generated by a vector of the form $e_1 + \lambda e_2 + \mu e_3$. In the corresponding coordinates, we have $P_1 \sim x_3 \xi_2 \xi_4, P_2 \sim (x_1 + \lambda x_2 + \mu x_3) \xi_1 \xi_4$ and $\{P_1, P_2\} = 0$.

Lemma 6.8. If $\dim C_{12} = n - 2$, then $g_0$ and $g_\eta$ are compatible.

Proof. In this case $C_0^\eta = C_\eta = C$ and $V = \Delta \oplus W \oplus C$, $\dim W = 1$. In a basis $e_1, \ldots, e_n$ in $V$ such that $e_1 \in \Delta, e_2 \in W, e_3 \in l \subset C, e_4, \ldots, e_n \in C$, we have $P_1 \sim x_3 \xi_2 \xi_4, P_2 \sim x_1 \xi_1 \xi_2$ and see that $\{P_1, P_2\} = 0$.

A summary of the above lemmas is

Proposition 6.3. A $\mathfrak{g}$-lieon and a $\mathfrak{h}$-lieon on $V$ are compatible iff the intersection of their centers is not generic, i.e., of dimension grater than $n - 4$.

The following assertion immediately results from propositions 6.1 and 6.3

Corollary 6.1. Let $g_1, \ldots, g_m$ be $\mathfrak{h}$-lieons and $g$ a $\mathfrak{g}$-lieon. If centers of all these lieons are contained in a common hyperplane in $V$, then $g + \alpha_1 g_1 + \cdots + \alpha_m g_m, \alpha_1, \ldots, \alpha_m \in k$, is a non-unimodular first level Lie algebra.

6.3. Compatible $\mathfrak{g}$-lieons. Consider two $\mathfrak{g}$-lieons $g_i$, $i = 1, 2$, on a vector space $V$ and the characterizing them pairs $(\Delta_i, C_i), i = 1, 2$. Recall that $\dim \Delta_i = 1, \dim C_i = n - 2$ and $\Delta_i \cap C_i = \{0\}$. Put $C_{12} = C_1 \cap C_2$. Obviously, $n - 4 \leq \dim C_{12} \leq n - 2$ and compatibility of $g_i$’s is equivalent to that of factorized $\mathfrak{g}$-lieons $g_i / C_{12}$’s.

Lemma 6.9. If $\dim C_{12} = n - 4$, then $g_1$ and $g_2$ are compatible iff $\Delta_1 \subset C_2$ and $\Delta_2 \subset C_1$. 


Proof. Passing to the factorized structures $\mathfrak{g}_i/\mathfrak{C}_{12}$’s we can assume that dim $V = 4$. In this particular case dim $C_1 = $ dim $C_2 = 2$ and $V = C_1 \oplus C_2$. Let $p_1 : V \to C_1$ be a natural projection and $L = \langle \Delta_1, \Delta_2 \rangle$ be the span of $\Delta_1$ and $\Delta_2$. Examine now various situations occurring in this context.

$K_1$ : dim $L = 2$ and $L \cap C_i = \{0\}$, $i = 1, 2$. Then $p_1|_L : L \to C_i$ is an isomorphism, $i = 1, 2$, and hence there is a basis $e_1, \ldots, e_4$ in $V$ such that $e_1, e_2 \in C_1$, $e_3, e_4 \in C_2$ and $e_1 + e_2 \in \Delta_1$, $e_2 + e_4 \in \Delta_2$. In the corresponding coordinates we have $P_1 \sim (x_1 + x_3)\xi_3\xi_4$, $P_2 \sim (x_2 + x_4)\xi_1\xi_2$ and a computation shows that $[P_1, P_2] \neq 0$.

$K_2$ : dim $L = 2$, dim $L \cap C_1 = 1$ and $L \cap C_2 = \{0\}$. Then $p_1|_L$ is an isomorphism. So, if $0 \neq e_i \in \Delta_i$, $i = 1, 2$, then $e_1 = p_1(e_1)$, $e_2 = p_1(e_2)$ is a basis in $C_1$. Also $e_3 = p_2(e_1) \neq 0$, since $\Delta_1 \cap C_1 = \{0\}$, and $p_2(e_1)$ and $p_2(e_2)$ are proportional. If $e_4 \in C_2$ is not proportional to $e_3$, then $e_1, \ldots, e_4$ is a basis in $V$. By construction $e_1 = e_1 + e_3$ and $e_2 = e_2 + \lambda e_3$. So, in the corresponding coordinates, $P_1 \sim (x_1 + x_3)\xi_3\xi_4$, $P_2 \sim (x_2 + \lambda x_3)\xi_1\xi_2$ and we can see that $[P_1, P_2] \neq 0$.

$K_3$ : dim $L = 2$, dim $L \cap C_i = 1$, $i = 1, 2$. If $e_1$, $e_2$ are as above, then $e_3 = p_2(e_1) \neq 0$, $e_1 = p_1(e_2) \neq 0$ and $p_2(e_2) = \lambda e_3$, $p_1(e_1) = \mu e_1$ for some $\lambda, \mu \in k$. By construction $e_1 = \mu e_1 + e_3$, $e_2 = e_1 + \lambda e_3$. Complete vectors $e_1, e_3$ to a basis in $V$ by vectors $e_2 \in C_1$, $e_4 \in C_2$. Then $P_1 \sim (\mu x_1 + x_3)\xi_3\xi_4$, $P_2 \sim (x_1 + \lambda x_3)\xi_1\xi_2$, $e_2 = \xi_1\xi_2$ in the corresponding coordinates, and $[P_1, P_2] \sim -\lambda (\mu x_1 + x_3)\xi_1\xi_2 - \mu (x_1 + \lambda x_3)\xi_2\xi_4$. Now a computation shows that $[P_1, P_2] = 0$ iff $\mu = \lambda = 0$. Geometrically, this condition tells that $\Delta_1 \subset C_2$, $\Delta_2 \subset C_1$, or, equivalently, that $\mathfrak{g}_1 + \mathfrak{g}_2$ is isomorphic to $\xi \oplus \xi$ for $n = 4$ and to $\xi \oplus \xi \oplus \gamma_{n-4}$ in the general case.

$K_4$ : dim $L = 1 \Leftrightarrow \Delta_1 = \Delta_2$. In this case one easily constructs a basis $e_1, \ldots, e_4$ in $V$ with $e_1, e_2 \in C_1$ and $e_3, e_4 \in C_2$ and $e_1 + e_3 \in \Delta_1 = \Delta_2$. As earlier we see that $P_1 \sim (x_1 + x_3)\xi_3\xi_4$, $P_2 \sim (x_1 + x_3)\xi_1\xi_2$ and $[P_1, P_2] \neq 0$. \hfill $\square$

Lemma 6.10. If dim $C_{12} = n - 3$, then $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are compatible either if $\Delta_1 = \Delta_2 \mod C_{12}$, or if $\Delta_1 \subset C_1 + C_2$, $i = 1, 2$.

Proof. The factorization mod $C_{12}$ reduces the problem, as above, to $n = 3$. In this case dim $C_1 = $ dim $C_2 = 1$ and $C_1 \cap C_2 = \{0\}$. Equivalently, if $C = C_1 + C_2$, then dim $C = 2$. Here two possibilities occur:

$J_1$ : One of $\Delta_i$’s, say, $\Delta_1$, does not belong to $C$. In a basis $e_1, e_2, e_3$ in $V$ with $e_i \in C_i$, $e_3 \in \Delta_1$ we have $P_1 \sim x_3\xi_2\xi_3$, $P_2 \sim \sum_{i=1}^3 \alpha_i x_i \xi_i\xi_3$ and $[P_1, P_2] \sim (\alpha_1 x_1 + \alpha_2 x_2)\xi_1\xi_2\xi_3$. Now a computation shows that $[P_1, P_2] = 0$ iff $\alpha_1 = \alpha_2 = 0 \Leftrightarrow \Delta_1 \subset \Delta_2$. For arbitrary $n$ the last condition means that $\Delta_1 \subset \Delta_2 \cap C_{12}$.

$J_2$ : $\Delta_1 \subset C$, $i = 1, 2$. Let $0 \neq e_i \in C_i$, $i = 1, 2$. Then $e_1 + \lambda e_2 \in \Delta_2$ and $\mu e_1 + e_2 \in \Delta_1$ for some $\lambda, \mu \in k$. Complete $e_1, e_2$ to a basis in $V$ by a vector $e_3$. Then $P_2 \sim (x_1 + \lambda x_2)\xi_1\xi_3$, $P_1 \sim (\mu x_1 + x_2)\xi_2\xi_3$ and $[P_1, P_2] = 0$. \hfill $\square$

Remark 6.1. The results of this section show that compatible configurations of lieons can be described in a manner which does not refer explicitly to a concrete ground field $k$. Namely, this description operates with the characterizing pairs and the relative position of composing them elements. Concreteness of $k$ is exclusively confined to coefficients of linear combinations of basic “abstract” lions from which first level lie algebras over $k$ are made.

It is not difficult to extract from the proof of theorems 5.1 and 5.2 that there is a number $\nu(n)$ such that any $n$-dimensional Lie algebra can be assembled from more than $\nu(n)$ lieons. On the other hand, the results of this section show that even first level lie algebras can be assembled from an unlimited number of $\sigma$- and
∅-lieons intertwined one another in a chaotic manner. This makes nontrivial the problem of recognizing isomorphic Lie algebras on the basis of their α-schemes. By this reason more regular assembling procedures are of interest. One of them, in a sense simplest, will be discussed the sections dedicated to coaxial algebras.

7. Canonical disassemblings of classical Lie algebras

In this section we shall describe canonical, in a sense, complete disassemblings of classical Lie algebras. This will be done in a way which simultaneously covers Lie algebras over ℝ and C. More exactly, classical Lie algebras are among symmetry algebras of some bilinear and volume forms, and this interpretation make it possible to completely disassemble them over an arbitrary ground field k of characteristic zero. The techniques we use here are mainly based on the Schouten bracket formalism and the stripping procedure.

7.1. Disassembling of g-orthogonal algebras. Let \( g = \sum_{i=1}^{n} a_i x_i^2 \), \( 0 \neq a_i \in k \), be a nondegenerate quadratic form on a k-vector space V. The Lie algebra so\((g)\) of (infinitesimal) symmetries of g is composed of linear vector fields X on V such that \( X(g) = 0 \). Obviously, \( e_{ij} = a_i x_i \partial_j - a_j x_j \partial_i \in so(g) \) and \([e_{ij}, e_{jk}] = a_j e_{ik}\). Moreover, fields \( \{e_{ij}\}_{i<j} \) form a base of so\((g)\). For instance, this is a standard base of so\((p,q)\), \( q = n - p \), if \( k = ℝ \) and \( a_1 = \cdots a_p = 1 \), \( a_{p+1} = \cdots = a_n = -1 \).

Let \( x_{ij} \) be the linear function on \( so(g) \) corresponding to \( e_{ij} \). Obviously, \( x_{ij} = -x_{ji} \) and \( \{x_{ij}\}_{i<j} \) is a cartesian chart on \( so(g)^* \). The only nonzero Poisson brackets, which involve functions \( x_{ij} \)'s, are \( \{x_{ij}, x_{jk}\} = a_j x_{ik} \). The Poisson bivector

\[
P = \sum_{i<j,\alpha} a_{\alpha} x_{ij} \xi_{i\alpha} \wedge \xi_{j\alpha} \quad \text{with} \quad \xi_{ij} = \frac{\partial}{\partial x_{ij}}
\]

represents the associated with \( so(g) \) Poisson structure on \( so(g)^* \) in terms of coordinates \( \{x_{ij}\} \). Observe that

\[
P = \sum_{\alpha} a_{\alpha} P_{\alpha} \quad \text{with} \quad P_{\alpha} = \sum_{i<j} x_{ij} \xi_{i\alpha} \wedge \xi_{j\alpha}
\]

Since P is a Poisson bivector for arbitrary \( a_{\alpha} \)'s, this shows that \( P_1, \ldots, P_n \) are mutually compatible Poisson bivectors.

The same result may be obtained by observing that

\[
2P_{\alpha} = [P, X_{\alpha}] = [P_{\alpha}, X_{\alpha}] \quad \text{with} \quad X_{\alpha} = \sum_s x_{as} \xi_{as}, \quad \text{and} \quad [P_{\alpha}, X_{\beta}] = 0, \quad \forall \alpha \neq \beta.
\]

Indeed, \([P, P_{\alpha}] = \frac{1}{2} \partial^2_{P_{\alpha}}(X_{\alpha}) = 0\) and

\[
[P_{\alpha}, P_{\beta}] = \frac{1}{2} [P_{\alpha}, [P, X_{\beta}]] = \pm \frac{1}{2} \sum [P_{\alpha}, P] \cdot [P, [P_{\alpha}, X_{\beta}]] = 0.
\]

So, finally, we have

\[
P_{\alpha} = \sum_{i<j} P_{\alpha,ij} \quad \text{with} \quad P_{\alpha,ij} = x_{ij} \xi_{i\alpha} \wedge \xi_{j\alpha}.
\]

For a fixed \( \alpha \) Poisson bivectors \( P_{\alpha,ij} \)'s are, obviously, compatible each other. Each of them is associated with an algebra isomorphic to \( \bigotimes m, m = n(n-1)/2 \). This shows that the algebra so\((g)\) can be assembled in two steps from \( n(n-1)(n-2)/2 \) ∅-lieons.
By translating the above results in terms of Lie brackets one easily finds that

$$[\cdot, \cdot] = [\cdot, \cdot]_1 + \cdots + [\cdot, \cdot]_n$$

where $[\cdot, \cdot]$ stands for the Lie bracket in $\mathfrak{so}(g)$ and the structure $[\cdot, \cdot]_\alpha$ is defined by relations

$$[e_{\alpha i}, e_{\alpha j}]_\alpha = \alpha_{ij} e_i e_j \quad \text{and} \quad [e_{ij}, e_{kl}]_\alpha = 0, \text{ if } \alpha \notin \{i,j\} \cap \{k,l\}.$$

In its turn, $[\cdot, \cdot]_\alpha = \sum_{i<j} [\cdot, \cdot]_{\alpha, ij}$ where the only nontrivial product $[\cdot, \cdot]_{\alpha, ij}$ involving base vectors $e_{kl}$’s is $[e_{\alpha i}, e_{\alpha j}]_{\alpha, ij} = \alpha_{ij} e_i e_j$.

**Remark 7.1.** The Poisson bivectors $P_{\alpha} = \sum_{i<j} x_{ij} \xi_{\alpha i} \wedge \xi_{\alpha j}$ may be interpreted as bivectors over the ring $\mathbb{Z}[x_{ij}]_{1 \leq i < j \leq n}$. Any formal linear combination of these bivectors with coefficients in a field $k$ is naturally interpreted as a linear bivector over the polynomial algebra $k[x_{ij}]_{1 \leq i < j \leq n}$, i.e., as a Lie algebra over $k$. In this sense $P_{\alpha}$’s are universal building blocks for $g$-orthogonal algebras. For instance, if $k = \mathbb{R}$, then

$$P_1 + \cdots + P_s = P_{s+1} - \cdots - P_n, \quad s = n - r.$$

is the Poisson bivector associated with $\mathfrak{so}(r,s)$.

### 7.2. Disassembling of symplectic Lie algebras $\mathfrak{sp}(2n)$.

Let $\beta(v, w)$ be a non-degenerate skew-symmetric form on a $k$–vector space $V$. Then the dimension of $V$ is even, say, $2n$, and there exists a (canonical) basis $\{e_1, \ldots, e_n, e'_1, \ldots, e'_n\}$ in $V$ such that

$$\beta(e_i, e'_j) = \delta_{ij}, \quad \beta(e_i, e_j) = \beta(e'_i, e'_j) = 0, \quad i, j = 1, \ldots, n.$$  

The symplectic Lie algebra $\mathfrak{sp}(\beta)$ consists of operators $A \in \text{End } V$ such that

$$\beta(Av, w) + \beta(v, Aw) = 0, \quad v, w \in V.$$

The algebra $\mathfrak{sp}(\beta)$ can be completely disassembled essentially by the same method as for orthogonal algebras. It will be described below in a form, which is better adapted to the symplectic situation.

Let $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ be coordinates in $V$ with respect to the above basis and $\omega = \sum_i dp_i \wedge dq_i$. Then the algebra $\mathfrak{sp}(\beta)$ may be interpreted as the algebra of linear vector fields $X$ on $V$ such that $L_X(\omega) = 0$. They are hamiltonian (with respect to $\omega$) fields $X_f$ corresponding to quadratic in $p$’s and $q$’s hamiltonians $f = f(p, q)$. So, in this interpretation hamiltonian fields corresponding to monomials $p_i p_j, q_i q_j, p_i q_j, i, j = 1, \ldots, n$, form a base of $\mathfrak{sp}(\beta)$ and the Lie product in $\mathfrak{sp}(\beta)$ is interpreted as commutator of vector fields. Alternatively, the identification $f \leftrightarrow X_f$, $\{f, g\} \leftrightarrow [X_f, X_g] = X_{\{f, g\}}$ allows us to interpret $\mathfrak{sp}(\beta)$ as the Lie algebra of quadratic polynomials $k_2[p, q] = k_2[p_1, \ldots, p_n, q_1, \ldots, q_n]$ in $p$’s and $q$’s with respect to the Poisson bracket $\{\cdot, \cdot\}$ determined by the Poisson bivector $\Pi = \sum_i \partial_{p_i} \wedge \partial_{q_i}$. In other words, we model the algebra $\mathfrak{sp}(2n, k)$ as the vector space $k_2[p, q]$ supplied with the bracket $[\cdot, \cdot] = \{\cdot, \cdot\}_{k_2[p, q]}$. Observe that $\Pi = \Pi_1 + \cdots + \Pi_n$ with $\Pi_i = \partial_{p_i} \wedge \partial_{q_i}$ and denote by $\{\cdot, \cdot\}_i$ the bracket associated with the Poisson bivector $\Pi_i$. Then $[\cdot, \cdot] = [\cdot, \cdot]_1 + \cdots + [\cdot, \cdot]_n$ with $[\cdot, \cdot]_i = \{\cdot, \cdot\}_i^{k_2[p, q]}$. Since bivectors $\Pi_i$’s are compatible each other, the brackets $[\cdot, \cdot]_i$’s are mutually compatible as too and we get the disassembling

$$(k_2[p, q], [\cdot, \cdot]) = (k_2[p, q], [\cdot, \cdot]_1) + \cdots + (k_2[p, q], [\cdot, \cdot]_n).$$
Obviously, Lie algebras \( \mathfrak{sp}_1(2n, k) = (k_2[p, q], [\cdot, \cdot]_k), i = 1, \ldots, n \), are isomorphic one to another. So, it suffices to completely disassemble one of them, say, \( \mathfrak{sp}_1(2n, k) \). To this end, observe that the Levi-Malcev decomposition of \( \mathfrak{sp}_1(2n, k) \) is

\[
\mathfrak{sp}_1(2n, k) = \langle p_1^2, p_1q_1, q_1^2 \rangle \oplus \langle p_1p_2, p_1q_2, q_1p_2, p_1q_2, q_2 q_2 \rangle_{1 < i, j \leq n}
\]

where \( \langle a, \ldots, b \rangle \) denotes the subalgebra of \( \mathfrak{sp}_1(2n, k) \) spanned by \( a, \ldots, b \).

The semisimple part \( s = \langle p_1^2, p_1q_1, q_1^2 \rangle \) of \( \mathfrak{sp}_1(2n, k) \) is isomorphic to \( \mathfrak{sl}(2, k) \). The radical \( r \) of it is

\[
r = \langle p_1p_2, p_1q_2, q_1p_2, q_1q_2 \rangle_{1 < i, j \leq n}
\]

and

\[
c = \langle p_1p_2, p_1q_2, q_1q_2 \rangle_{1 < i, j \leq n}
\]

is the center of \( r \). Note that \( [r, r] \subset c \).

According to proposition 5.1, the algebra \( \mathfrak{sp}_1(2n, k) \) is assembled from \( s \oplus_p \langle r \rangle \) and \( \gamma_m \oplus r \) for a suitable \( m \) where \( \rho \) stands for a natural representation of \( s \) in the vector space \( \langle r \rangle \) supporting the ideal \( r \). So, it remains to disassemble each of these two algebras.

The algebra \( r \) contains the following Heisenberg subalgebras:

\[
\begin{align*}
\mathfrak{h}^{pp}_{ij} &= \langle p_1p_i, p_1p_j, p_1p_j \rangle, & \mathfrak{h}^{pp}_{ij} &= \langle p_1p_i, q_1q_j, p_1p_j \rangle & (84) \\
\mathfrak{h}^{pp}_{ij} &= \langle p_1q_i, q_1p_j, q_1p_j \rangle, & \mathfrak{h}^{pp}_{ij} &= \langle p_1q_i, q_1q_j, q_1q_j \rangle & (85)
\end{align*}
\]

Each subalgebra \( \mathfrak{h}^{ab}_{ij} \) from this list naturally extends to the unique \( \mathfrak{h} \)-linear \( r \)-dimensional \( \mathfrak{h}^{ab}_{ij} \) on \( \langle r \rangle \), \( r = \dim r \), whose center contains all quadratic monomials, which do not figure in the definition of \( \mathfrak{h}^{ab}_{ij} \). It is easy to check (see subsection 6.1) that \( \mathfrak{h}^{ab}_{ij} \)'s are compatible each other and hence completely disassemble \( r \).

Aiming to disassemble the algebra \( s \oplus_p \langle r \rangle \) consider the following \( d \)-pair in it:

\[
(\langle p_1q_1, V_p \rangle, \langle p_1^2, q_1^2, V_q, c \rangle) \quad \text{with} \quad V_p = \langle p_1p_1, p_1q_1 \rangle_{1 < i, j \leq n}, V_q = \langle q_1p_1, q_1q_1 \rangle_{1 < i, j \leq n}
\]

(86)

Easily verified inclusions

\[
[V_p, V_p]_1 = [V_q, V_q]_1 = 0, \quad [V_p, V_q]_1 \subset V_p, \quad [[p_1^2], V_p]_1 = [[q_1^2], V_q]_1 = 0, \quad [[p_1^2], V_p]_1 \subset V_q, \quad [[[p_1^2], q_1^2], V_p]_1 \subset \langle p_1q_1 \rangle, \quad [[[p_1^2], q_1^2], q_1^2]_1 \subset \langle p_1q_1 \rangle, \quad [[[p_1^2], q_1^2], V_q]_1 \subset V_p
\]

prove that (86) is a \( d \)-pair.

Nontrivial relations among quadratic monomials in the dressing algebra of this \( d \)-pair are \( [p_1^2, q_1^2]_1 = 4p_1q_1, [p_1^2, q_1p_1]_1 = 2p_1q_1, [p_1^2, q_1q_1]_1 = 2p_1q_1, 1 < i \leq n \). So, the triples \( (p_1^2, q_1^2, p_1q_1), (p_1^2, q_1p_1, p_1q_1), (p_1^2, q_1q_1, p_1q_1), 1 < i \leq n \), span subalgebras of the dressing algebra, which are isomorphic to \( \mathfrak{h} \). As in the case of subalgebras \( \mathfrak{h}^{ab}_{ij} \) these subalgebras naturally extend to some \( \mathfrak{h} \)-lieons. These extensions are mutually compatible and, therefore, disassemble the dressing algebra.

Now it remains to disassemble the algebra

\[
(\langle p_1q_1, V_p \rangle \oplus_p \langle p_1^2, q_1^2, V_q, c \rangle)
\]

where \( p \) is a natural representation of the subalgebra \( \langle p_1q_1, V_p \rangle \) in the vector space \( \langle p_1^2, q_1^2, V_q, c \rangle \). This algebras is, in fact, the semidirect product

\[
a = \langle p_1q_1 \rangle \oplus_c (V_p \oplus_p \langle p_1^2, q_1^2, V_q, c \rangle)
\]
where the action \( \varsigma \) of \( \langle p_1 q_1 \rangle \) on \( V_p \) and \( \langle p_1^2, q_1^2, V_q, c \rangle \) is induced by the bracket \([\cdot, \cdot]_1\).

It is easy to see that nontrivial relations among elements of the basis of the ideal \( i = V_p \oplus \langle p_1^2, q_1^2, V_q, c \rangle \subset a \) are

\[
\langle q_1^2, p_1 r \rangle_1 = 2q_1 r \in V_q, \ \langle p_1 r, q_1 s \rangle_1 = rs \in c \quad \text{with} \quad r, s = p_i, q_j, \ 0 < i, j \leq n.
\]

Now we see that the triples \( \langle q_1^2, p_1 r, 2q_1 r \rangle, \ \langle p_1 r, q_1 s, rs \rangle \) span Heisenberg subalgebras in \( i \). By the same arguments as before, their natural extensions disassemble the algebra \( i \) into a number of \( \mathfrak{n} \)-lieons.

The final step is to disassemble the algebra

\[
\langle p_1 q_1 \rangle \oplus \theta |i| = \langle p_1 q_1 \rangle \oplus \langle p_1^2, q_1^2, V_p, V_q, c \rangle.
\]

Observe that \( |i| \) is the direct sum of \( \theta \)-invariant subspaces

\[
\langle p_1^2, q_1^2 \rangle, \ \langle p_1 r, q_1 r \rangle \text{ with } r = p_i, q_i, \ 0 < i \leq n, \quad \text{and } |c|.
\]

The action \( \theta \) on \( |c| \) is trivial and each of subalgebras \( \langle p_1 q_1 \rangle \oplus \theta (\langle p_1^2, q_1^2 \rangle \oplus |c|) \), \( \langle p_1 q_1 \rangle \oplus \theta (\langle p_1 r, q_1 r \rangle \oplus |c|) \) is isomorphic to \( 2 \mathfrak{o}_{2m} = 2 \mathfrak{n}_m \) for a suitable \( m \) (see formula (56) and subsection 5.2). This disassembles the algebra \( \langle p_1 q_1 \rangle \oplus \theta |i| \) into \( 2n - 1 \) \( \mathfrak{n} \)-lieons.

7.3. Disassembling of \( \mathfrak{gl}(n, k), \mathfrak{sl}(n, k), \mathfrak{u}(n) \) and \( \mathfrak{su}(n) \). First, we shall construct a 3-step disassembling of \( \mathfrak{gl}(n, k) \). It is convenient to interpret this algebra as the algebra of linear vector fields on an \( n \)-dimensional vector space \( V \). Vector fields \( e_{ij} = x_i \partial_{x_j} \), \( 1 \leq i, j \leq n \), form a natural basis of it. The nontrivial Lie products in this algebra are

\[
[e_{i\alpha}, e_{\alpha j}] = e_{ij}, \quad \text{if} \quad i \neq j, \quad \text{and} \quad [e_{i\alpha}, e_{\alpha i}] = e_{ii} - e_{\alpha i}, \quad \text{if} \quad i \neq \alpha.
\]

Let \( z_{ij} \)'s be the corresponding to \( e_{ij} \)'s coordinates on \( \mathfrak{gl}(n, k)^* \). Then the associated with \( \mathfrak{gl}(n, k) \) Poisson bivector is

\[
P = \sum_{1 \leq i, j, \alpha \leq n} z_{ij} \xi_{i\alpha} \xi_{\alpha j} \quad \text{with} \quad \xi_{ij} = \frac{\partial}{\partial x_{ij}}. \tag{87}
\]

In the basis

\[
\{e_{ij}^t = t_i t_j e_{ij} \}, \quad 0 \neq t_i \in k, \quad i = 1, \ldots, n,
\]

we, obviously, have \( P = \sum \xi_{i\alpha} z_{ij}^t \xi_{i\alpha} t_i \) where \( z_{ij}^t \) and \( \xi_{i\alpha} \) stand for coordinates and partial derivatives with respect to this basis. The isomorphism identifying the second basis with the first one transforms \( P \) into the Poisson bivector

\[
P_t = \sum_{1 \leq \alpha \leq n} t_i^2 \xi_{i\alpha} \quad \text{with} \quad P_\alpha = \sum_{(i,j) \neq (\alpha, \alpha)} z_{ij} \xi_{i\alpha} \xi_{\alpha j}, \quad t = (t_1, \ldots, t_n). \tag{88}
\]

This implies that \( P_t \)'s are mutually compatible Poisson bivectors, and, in particular, that \( P = P_1 + \cdots + P_n \). In its turn, \( P_\alpha \) disassembles as

\[
P_\alpha = \sum_{i, i \neq \alpha} (z_{i\alpha} \xi_{i\alpha} \xi_{\alpha i} + z_{i\alpha} \xi_{\alpha i} \xi_{i\alpha}) + \sum_{i, j, i \neq \alpha, j \neq \alpha} (z_{ij} \xi_{i\alpha} \xi_{\alpha j}) \tag{89}
\]

The first of Poisson bivectors in the left hand side of (89) corresponds to an algebra of the form \( \Gamma_A \), while the second to a dressing algebra. They both can be simply disassembled into a number of \( n^2 \)-dimensional \( \mathfrak{n} \)-lieons (see (58)).

However, Poisson bivectors \( P_\alpha \)'s in (88) do not restrict to the subalgebra \( \mathfrak{sl}(n, k) \) of \( \mathfrak{gl}(n, k) \) and hence can not be used to disassemble it. With this purpose we consider another basis in \( \mathfrak{gl}(n, k) \):

\[
e_{ij}^0 = e_{ij} - e_{ji}, \quad 1 \leq i < j \leq n, \quad e_{ij}^1 = e_{ij} + e_{ji}, \quad 1 \leq i \leq j \leq n, \tag{90}
\]
which respects the matrix transposition. We also put $e^0_{ij} = -e^0_{ji}$, $e^1_{ij} = e^1_{ji}$. The nontrivial Lie products of elements of basis (90) are

\[ [e^0_{ij}, e^0_{kl}] = e^{0+0}_{ij}, \text{ if } i \neq k, \ i \neq j, \ i \neq k; \]
\[ [e^0_{ij}, e^0_{kl}] = 2(e^1_{ij} - e^1_{kl}), \text{ if } i \neq j, \ \sigma \neq \tau; \]
\[ [e^0_{ij}, e^1_{kl}] = 2e^{0+1}_{ij}, \text{ if } i \neq j. \quad (91) \]

Here we interpret upper indices of vectors $e^i_{ij}$ as elements of $\mathbb{F}_2$. The corresponding d–pair in $\mathfrak{gl}(k, n)$ is $(s = \langle e^0_{ij} \rangle, W = \langle e^1_{ij} \rangle)$. Obviously, $s$ is isomorphic to $\mathfrak{so}(g)$ for $g = \sum_{i=1}^n x_i^2$, $s \subset \mathfrak{s}l(n, k) \subset \mathfrak{gl}(n, k)$ and

\[ W_0 \overset{\text{def}}{=} W \cap \mathfrak{s}l(n, k) = \langle e^1_{ij}, e^1_{ii} - e^1_{jj} \rangle_{1 \leq i \neq j \leq n}. \]

In particular, $(s, W_0)$ is a d–pair in $\mathfrak{s}l(n, k)$. Denote by $a_\beta$ (resp., $a_{\beta_0}$) the dressing algebra (see subsection 5.3) corresponding to the d–pair $(s, W)$ (resp., in $(s, W_0)$), and by $\rho$ (resp., $\rho_0$) the corresponding representation of $s$ in $W$ (resp., $W_0$). So, by construction, we have

\[ \mathfrak{gl}(n, k) = s \oplus W + a_\beta, \quad \mathfrak{s}l(n, k) = s \oplus_{\rho_0} W_0 + a_{\beta_0}. \quad (92) \]

Moreover, we have

**Lemma 7.1.** For $k = \mathbb{R}$ the algebras

\[ u(n, k) = s \oplus_{\rho} W + a_\beta, \quad s\mathfrak{u}(n, k) = s \oplus_{\rho_0} W_0 + a_{-\beta_0}. \]

are isomorphic to the unitary and special unitary Lie algebras, respectively.

**Proof.** A direct check on the basis of relations (91). \(\square\)

**Remark 7.2.** The isomorphism class of algebras $\mathfrak{gl}_\lambda(n, k) \overset{\text{def}}{=} s \oplus_{\rho} W + a_{\lambda\beta}$ and $\mathfrak{s}l_\lambda(n, k) \overset{\text{def}}{=} s \oplus_{\rho_0} W_0 + a_{\lambda\beta_0}$ depend on the quadratic residue of $\lambda$. Namely, $\mathfrak{gl}_\lambda$ and $\mathfrak{gl}_{\lambda'}$ (resp., $\mathfrak{s}l_\lambda$ and $\mathfrak{s}l_{\lambda'}$) are isomorphic iff $\lambda' = \lambda \mu^2$, $\mu \in k$.

Since a dressing algebra can be simply disassembled into a number of $\mathfrak{n}$–lieons, we shall focus on the algebras $s \oplus_{\rho} W$ and $s \oplus_{\rho_0} W_0$. In virtue of (92) and lemma 7.1 a complete disassembling of this algebra automatically gives complete disassemblings of algebras $\mathfrak{gl}(n, k)$, $\mathfrak{s}l(n, k)$, $\mathfrak{u}(n, k)$ and $\mathfrak{s}u(n, k)$.

Denote by $z_{ij}$ (resp., $w_{pq}$) linear functions on $s$ (resp., $|W|$) corresponding to $e^0_{ij}$ (resp., $e^1_{pq}$). They together form a cartesian chart on $|s \oplus_{\rho} W| = |s| \oplus |W|$. In this chart the Poisson bivector of the algebra $s \oplus_{\rho} W$ reads

\[ Q = \sum_{\alpha, \lambda < j} z_{ij} \xi_{\alpha \lambda} + \sum_{p \neq \alpha \neq q} w_{pq} \xi_{\alpha q} + 2 \sum_{p \neq q} w_{pq} \xi_{pq} \eta_{pq} \quad (93) \]

with $\xi_{ij} = \frac{\partial}{\partial z_{ij}}, \eta_{pq} = \frac{\partial}{\partial w_{pq}}$. By the same arguments as in subsection 7.1, we see that if

\[ Q_\alpha = \sum_{i < j} z_{ij} \xi_{\alpha \lambda} + \sum_{p \neq q \neq \alpha} w_{pq} \xi_{\alpha q} + 2 \sum_{p} w_{pq} \xi_{\alpha q} \eta_{\alpha q}, \]

then $Q = Q_1 + \cdots + Q_n$ is a simple disassembling of $Q$.

In order to disassemble $Q_\alpha$ note that single summand in the left hand side of (93) are $n^2$–dimensional $\mathfrak{n}$–lieons, and the only incompatible pairs of the are

\[ w_{pq} \xi_{\alpha q} \eta_{\alpha q}, \quad p \neq q, \ p \neq \alpha, \ q \neq \alpha. \]
This shows that
\[ Q_\alpha^1 = \sum_{i<j} z_{ij} \xi_{ij} \zeta_{ij}, \quad Q_\alpha^2 = \sum_{p,q \neq \alpha} w_{pq} \zeta_{pq} \eta_{pq}, \quad Q_\alpha^3 = \sum_p w_{p\alpha} \zeta_{p\alpha} \eta_{p\alpha} \] (94)
are Poisson bivectors and \([Q_\alpha^1, Q_\alpha^2] = [Q_\alpha^1, Q_\alpha^3] = 0\). Moreover, by using formula (10) we easily find that \([Q_\alpha^2, Q_\alpha^3]\) is a simple disassembling of \(Q_\alpha\). Finally, it follows from formula (94) that Poisson bivectors \(Q_\alpha^i\) are assembled from mutually compatible \(\mathfrak{n}\)-leons. This way we get a complete common disassembling of \(\mathfrak{g}(n, k)\) and \(u(n, k)\) in 4 steps.

In order to apply a similar approach to the algebra \(s \oplus_{p_0} W_0\) we have to choose a base in \(W_0\). A such one is
\[ e_i^j, \quad 1 \leq i < j \leq n, \quad e_i = \frac{1}{2} (e_i - e_{11}) = x_i \partial_i - x_1 \partial_1, \quad 1 < i \leq n. \]
Denote by \(w_{ij}\) and \(w_i\) the corresponding linear functions on \(W_0^*\), respectively. Together with functions \(z_{ij}\)'s they form a cartesian chart on \([s \oplus_{p_0} W_0]^* = [s]^* \oplus [W_0]^*\).

Also, put \(\eta_{ij} = \frac{\partial}{\partial \vartheta_{ij}}, \quad \eta_i = \frac{\partial}{\partial \vartheta_i}\). As it follows from (91), in this chart the Poisson bivector of \(s \oplus_{p_0} W_0\) reads
\[ Q^0 = \sum_{j,i<k} z_{ik} \xi_{ik} \zeta_{ik} + \sum_{i,j,k \neq i} w_{ik} \xi_{ij} \eta_{jk} + 2 \sum_{1<i<j} (w_i - w_j) \xi_{ij} \eta_{ji} - 2 \sum_{1<i<j} w_i \xi_{ii} \eta_{ii} + \sum_{1<i<j,i \neq j} w_{ij} \xi_{ij} \eta_{ij} + 2 \sum_{1<i} w_{1i} \xi_{i1} \eta_i + \sum_{1<i,j,i \neq j} w_{ij} \xi_{ij} \eta_{ij} \] (95)
Now apply the trick we have used in subsection 7.1 to \(Q^0\).

The expression of \(Q^0\) in the basis
\[ t_i t_j e_i^0, \quad 1 \leq i < j \leq n, \quad t_i^2 e_i, \quad 1 < i \leq n. \]
of \(s \oplus_{p_0} W_0\) is of the form \(Q^0 = t_1^1 Q_1^0 + \cdots + t_n^1 Q_n^0\) with \(Q_1^0\) non depending on \(t_i\)'s.

This proves that \(Q^0\)'s are mutually compatible Poisson bivectors. In particular, \(Q^0 = Q_1^0 + \cdots + Q_n^0\) is a simple disassembling of \(Q^0\). Exact expressions of bivectors \(Q_j^0\)'s are easily obtained from (95). Namely, we have
\[ Q_1^0 = \sum_{i<k} z_{ik} \xi_{1i} \zeta_{ik} + \sum_{i\neq 1,k \neq 1} w_{ik} \xi_{i1} \eta_{ik} - 2 \sum_{1<i} w_i \xi_{i1} \eta_{i1} \] (96)
and for \(j > 1\)
\[ Q_j^0 = \sum_{i<k,j \notin \{i,k\}} z_{ik} \xi_{ij} \zeta_{jk} + \sum_{i \neq k,j \notin \{i,k\}} w_{ik} \xi_{ij} \eta_{jk} + 2 \sum_{i,j} w_{ij} \xi_{ij} \eta_{ji} + \sum_{i,j} w_{ij} \xi_{ij} \eta_{ij} + 2 \sum_{i,j} w_{ij} \xi_{ij} \eta_{ij} + \sum_{i,j} w_{ij} \xi_{ij} \eta_{ij}. \] (97)
Each single term of summations (96) and (97) is an \((n^2 - 1)\)-dimensional \(\mathfrak{n}\)-lieon. It is easy to verify by a direct check or by using lemmas 6.1, 6.2 and 6.3 that
- all \(\mathfrak{n}\)-leons in (96) are mutually compatible;
- incompatible pairs of \(\mathfrak{n}\)-leons in (97) are
\[ z_{1k} \xi_{1j} \zeta_{kj}, \quad k > 1; \quad w_{1k} \xi_{1j} \eta_{jk}, \quad k > 1; \quad w_{1k} \xi_{kj} \eta_{j1}, \quad k > 1; \quad w_{1k} \xi_{kj} \eta_{j1}, \quad k > 1; \]
This shows that
• $Q_0^0$ is simply disassembled into a number of $\mathfrak{n}$–lieons;
• $Q_j^j$, $j > 1$, is simply disassembled into a number of $\mathfrak{n}$–lieons and structures

\[
Q_j^j = \sum_{k>1,k\neq j} z_{1k}\xi_{1j}\xi_{jk} + \sum_{k>1,k\neq j} w_{1k}\xi_{1j}\eta_{jk} + 2w_{1j}\xi_{1j}\eta_j
\]  \hspace{1cm} (99)

\[
Q_j'' = \sum_{k>1,k\neq j} w_{1k}\xi_{1k}\eta_j + \sum_{k>1,k\neq j} w_{kj}\xi_{kj}\eta_j + \sum_{k>1,k\neq j} w_{1k}\xi_{kj}\eta_{j1}
\]  \hspace{1cm} (100)

Indeed, by (98), $Q_j'$ and $Q_j''$ are composed of mutually compatible $\mathfrak{n}$–lieons and, therefore, are Poisson bivectors. Moreover, a direct computation by using formula (10) proves that $[Q_j', Q_j''] = 0$.

Finally, by the above said both $Q_j'$ and $Q_j''$ can be simply disassembled into a number of $\mathfrak{n}$–lieons. Thus we have explicitly described a 4-step complete disassembling of considered simple Lie algebras.

**Remark 7.3.** Identify the algebra $\mathfrak{so}(n, k)$ with $\mathfrak{so}(g)$, $g$ being the standard ‘scalar’ product on $V = k^n$. Then the above discussed representation $\rho$ of the algebra $\mathfrak{so}(n, k)$ is identified with the canonical representation of this algebra in $S^2V^\ast$. It is easy to see that the disassembling procedure discussed in this subsection can be applied to all canonical representations of $\mathfrak{so}(n, k)$ in tensor powers of $V$.

By concluding this section we note that the discussed in it simple algebras belonging to the same series are assembled from the same ‘universal elements’ with coefficients belonging to a given ground field (see remarks 7.1 and 7.2). The same can be said about their representations (see remark 7.3). These observations lead to a natural conjecture. To state it we, first, recall that the type of a central simple Lie algebra $\mathfrak{g}$ over a field is the type of a splitting simple algebra obtained from $\mathfrak{g}$ by an extension of scalars. Second, we say that two Lie algebras are assembled from the same elements if the corresponding a–schemas are equivalent and the corresponding terms of them are proportional.

**Conjecture.** All central simple Lie algebras of the same type can be assembled from the same “universal” elements which are Lie algebra structures over $\mathbb{Z}$.

An approach, which appear more boring than difficult, is to apply the techniques of this section to the known description of simple Lie algebras over arbitrary fields of characteristic zero (see, for instance, chapter X of Jacobson’s book [3]).

8. **Coaxial Lie algebras**

Coaxial Lie algebras form a natural subclass of first level Lie algebras, which is, in a sense, attached to a chosen basis of the supporting vector space $V$ (see below). Informally speaking, these are “molecules” which can be directly, i.e., in one step, “synthesized” from lions which play the role of ”atoms” in this context. Description of these “molecules” is a combinatorial problem which is solved in this and the subsequent sections. Coaxial algebras provide us with necessary ingredients for synthesis of more complicated “molecules”. For instance, such ones have already appeared in the procedure of disassembling of classical Lie algebras in the previous section.

The central point in this section is a description of some “maximal” families of mutually compatible lions, called clusters. It is rather instructive to see how lions in clusters are self-organized in structural groups, surrounded by casings and tied
Coaxial algebras: definitions. Fix a basis $B = \{ e_1, \ldots, e_m \}$ in $V$. Suppose that numbers $1 \leq i, j, k \leq n$ differ each other and denote by $[i,j|k]$ (resp., by $[i|j]$) the $n$–dimensional $\cap$–lieon (resp., the $\|$–lieon) for which $[e_i,e_j] = -[e_j,e_i] = e_k$ (resp., $[e_i,e_j] = -[e_j,e_i] = e_j$) are the only nontrivial of Lie products involving base vectors. Call them $B$–base, or, simply, base $\cap$– and $\|$lieons, respectively. We also shall use the notation like $[A,B|C]$ or $[A|B]$ instead of $[i,j|k]$ and $[i|j]$, respectively, if $A = e_i, B = e_j$ and $C = e_k$.

Definition 8.1. A linear combination (over $k$) of some mutually compatible $B$–base lieons is called a $B$-coaxial, or, simply, coaxial (Lie algebra) structure. Such a structure will be called $\cap$-coaxial (resp., $\|$-coaxial) if it is composed only of base $\cap$–lieons (resp., of base $\|$–lieons).

A coaxial Lie algebra $g$ may be presented in the form

$$g = \sum \alpha_{(i,j|k)} [i,j|k] + \sum \beta_{(p|q)} [p|q], \quad \alpha_{(i,j|k)}, \beta_{(p|q)} \in k,$$

(101)

where figuring in it base structures with nonzero coefficients are compatible each other. Vectors $e_i,e_j$ and $e_k$ (resp., $e_i,e_j$) are called vertices of $[i,j|k]$ (resp., of $[i|j]$).

Fig. 3. Base lieons.

Vectors $e_i,e_j$ are ends of $[i,j|k]$, while $e_k$ is its center. The origin and the end of $[i,j|k]$ are $e_i$ and $e_j$, respectively. In the sequel we do not distinguish between $[i,j|k]$ and $[i,j|k] = -[j,i|k]$, since they have identical compatibility properties.

Poisson bivectors on $V^*$ corresponding to $[i,j|k]$ and $[i|j]$ are $\langle i,j|k \rangle = x_k \xi_i \xi_j$ and $\langle i|j \rangle = x_j \xi_i \xi_j$, respectively. They will be called base bivectors. Obviously, the coordinate expression of the Poisson bivector $P_g$ associated with a Lie algebra structure $g$ on $V$ is a linear combination of base bivectors:

$$P_g = \sum \alpha_{(i,j|k)} \langle i,j|k \rangle + \sum \beta_{(p|q)} \langle p|q \rangle, \quad \alpha_{(i,j|k)}, \beta_{(p|q)} \in k.$$

(102)

These bivectors are not generally mutually compatible. For instance, such are base bivectors occurring in coordinate expressions of Poisson bivectors of classical Lie algebras which were discussed in section 7. The Poisson bivector associated with a $\cap$–, $\|$–coaxial algebra will be also called $\cap$–, $\|$–coaxial, respectively.

Base lieons are conveniently, up to proportionality, represented as diagrams in Fig. 3. In the sequel such diagrams will be systematically used in construction of families of mutually compatible base lieons.
Compatibility of base structures.
We shall say that two base lieons are trivially compatible if they either coincide up to the sign, or have no common vertices at all.

Proposition 8.1. 1) Two base $\cap$-lieons are nontrivially compatible if and only if they have either a common center vertex, or a common end vertex at least.
2) Two base $\cup$-lieons are incompatible if and only if the origin of one of them is the end of the other one and they have no other common vertices.
3) A $\cap$-lieon is nontrivially compatible with a $\cup$-lieon if and only if its origin coincides with one of the ends of this $\cap$-lieon.

Proof. A direct consequence of lemmas 6.1-6.10. □

A graphical interpretation of proposition 8.1 is given in Fig. 4.

Example 8.1. Poisson bivectors $P_{\alpha}$'s that disassemble the algebra $\mathfrak{so}(g)$ (see subsection 7.1) are coaxial with respect to the base $\{e_{ij}\}$. The corresponding to $P_{\alpha}$ coaxial Lie algebra is isomorphic to the direct sum of $(n - 1)(n - 2)/2$ copies of the Heisenberg algebra and an abelian one. This algebra is, obviously, coaxial. So, $\mathfrak{so}(g)$ is assembled from $n$ coaxial Lie algebras of this kind. Poisson bivectors $Q'_j$ and $Q''_j$ which appear in the disassembling procedure in subsection 7.3 are also $\cap$-coaxial.

8.2. Clusters. Lie algebra structures (resp., Poisson bivectors), which are compatible with two proportional ones, are, obviously, the same. So, it is convenient to work with classes of proportional base lieons or corresponding Poisson bivectors. A class of proportional base $\cap$-lieons (resp., $\cup$-lieons) will be called a tee (resp., a dee). We shall use keep the notation $\lfloor i,j \rfloor_k$ (resp., $\lfloor i \rfloor_j$) for the corresponding tee (resp., dee). The ends and center of a tee are those of the corresponding base $\cap$-lieon, and similarly, for the end and origin of a dee.

A family of tees (resp., dees) will be called a tee (resp., dee) family. Union of a tee and of a dee families will be called a base family $\Phi$. A graph $\Upsilon_\Phi$ is naturally associated with a base family $\Phi$. Namely, let $S(\Phi) = \{e_{i_1}, \ldots, e_{i_m}\} \subset \{e_1, \ldots, e_n\}$ be the totality of vertices of all tees and dees composing $\Phi$ and $I(\Phi) = \{i_1, \ldots, i_m\}$. Vertices $v_1, \ldots, v_m$ of $\Upsilon_\Phi$ are in one-to-one correspondence with base vectors $e_{i_1}, \ldots, e_{i_m}$. Vertices $v_k$ and $v_l$ of $\Upsilon_\Phi$ are connected by an unique edge iff $e_{i_k}, e_{i_l}$ are either vertices of a tee, or the center and one of the ends of a tee belonging to $\Phi$. For example, the graph corresponding to $\Phi = ([i,j][k], [i][k], [l][j])$ has four vertices and three edges. Base vectors belonging to $S(\Phi)$ will be called vertices of $\Phi$. Obviously, $\Phi = \Phi'_1 \cup \Phi''_1$ where $\Phi'_1$ (resp., $\Phi''_1$) is the set of all dees (resp., tees) belonging to $\Phi$.

Base families $\Phi$ and $\Phi'$ are equivalent if there exists a one-to-one correspondence $S(\Phi) \leftrightarrow S(\Phi')$ which induces a one-to-one correspondence between tees and dees belonging to $\Phi$ and those belonging to $\Phi'$.

A base family will be called compatible if composing it lieons are mutually compatible. A compatible family, denoted by $\Phi_g$, is naturally associated with a coaxial
Lie algebra $\mathfrak{g}$. Namely, it consists of tees and dees corresponding to nonzero terms in expression (101) for $\mathfrak{g}$. Denote by $\Phi^e_\mathfrak{g}$ (resp., $\Phi^d_\mathfrak{g}$) the $\mathfrak{g}$-family (resp., $\mathfrak{g}$–family) composed of dees (resp., tees) belonging to $\Phi_\mathfrak{g}$. Finally, two compatible families will be called compatible if composing them tees and dees are mutually compatible.

A Lie algebra $\mathfrak{g}$ will be called associated with a base family $\Phi$ if $\Phi = \Phi^g_\mathfrak{g}$.

The associated with $\Phi^g_\mathfrak{g}$ graph will be denoted by $\Upsilon_\mathfrak{g}$, i.e., $\Upsilon_\mathfrak{g} = \Upsilon^g_\Phi$. Let $\Upsilon$ be a connected component of $\Upsilon_\mathfrak{g}$, $\{e_{i_1}, \ldots, e_{i_s}\}$ the vertices of $\Upsilon$ and $I = \{i_1, \ldots, i_s\}$.

A Lie algebra structure on the subspace of $V$ generated by $e_{i_1}, \ldots, e_{i_s}$ is defined as the linear combination

$$\mathfrak{g}_I = \sum_{i,j,k \in I} \alpha_{(i,j | k)} [i,j][k] + \sum_{p,q \in I} \beta_{(p|q)} [p|q], \quad \alpha_{(i,j | k)}, \beta_{(p|q)} \in \mathfrak{k};$$

assuming that $\mathfrak{g}$ is given by (101). Obviously, $\mathfrak{g} = \mathfrak{g}_{I_1} \oplus \cdots \oplus \mathfrak{g}_{I_m} \oplus \gamma_I$, $l = n - (\dim \mathfrak{g}_{I_1} + \cdots + \dim \mathfrak{g}_{I_m})$. (104)

where $I_j$’s are sets of indices corresponding to connected components of $\Upsilon_\mathfrak{g}$.

A compatible base family is maximal if it is not contained in a larger compatible one, which have the same set of vertices. A maximal compatible family will be called a cluster if the corresponding to it graph is connected. The number of vertices of the graph associated with a cluster is called the dimension of it. Obviously, a compatible family with the connected graph is contained in a cluster. So, in view of decomposition (104), the problem of description of coaxial algebras is reduced to description of clusters. Similarly are defined maximal compatible tee– and dee–families and, accordingly, tee-clusters and dee-clusters. It should be stressed that a tee-cluster, or a dee-cluster is not necessarily a cluster (see below).

According to the above said we see that

the problem of description of coaxial Lie algebras
is reduced to description of clusters.

By this reason in the sequel we shall concentrate on study of clusters.

The following terminology will be useful in our further analysis of the structure of clusters. We shall say that a tee/dee $\vartheta$ blocks (alternatively, is blocking) a tee/dee $\vartheta$ if it is incompatible with $\vartheta$. So, we have the following

Blocking rule: Let $\Phi$ be a (tee/dee-)cluster and vertices of $\vartheta$ belong to $S(\Phi)$. Then $\vartheta$ belongs to $\Phi$, if $\Phi$ does not contain tees/dees that block $\vartheta$.

Base families are conveniently represented by means of their diagrams (see, for instance, Fig. 4). The use of such diagrams makes much more clear proofs of various assertions about compatibility of base families appearing in the forthcoming analysis of the structure of clusters. It turned out impossible to accompany these proofs, due to their numerosity, by the corresponding pictures. So, the reader is strongly suggested to do that on his own.

8.3. Structure of dee-clusters. Dee-clusters are easily classified. With this purpose we shall introduce the following compatible dee-families.

**Double.** This is a compatible dee-family of the form $\{[p|q], [q|p]\}$.

**Basic property of doubles:** If a base lieon is compatible with a double $D$, then it is a $\bigtriangledown$–lieon whose ends are vertices of $D$ (proposition 8.1).

In particular, if $D$ belongs to a compatible dee-family $\Phi$, then a dee $\vartheta \in \Phi \setminus D$ has no common vertices with $D$. Vertices of $D$ will be called double vertices of $\Phi$. 
Therefore if $\nu$ is not a double vertex of $\Phi$, then $\nu$ is either the common origin, or the common end of all dees $\vartheta \in \Phi$ that have $\nu$ as one of its vertices. Accordingly, vertices of a compatible dee-family are subdivided into double, initial and end vertices, respectively.

**Spider.** Let $I_0 = \{i_1, \ldots, i_k\}$ and $I_1 = \{j_1, \ldots, j_l\}$ be nonempty subsets of $\{1, \ldots, n\}$ such that $I_0 \cap I_1 = \emptyset$. The dee-family

$$Sp(I_0, I_1) \overset{\text{def}}{=} \{ |p|q| \mid p \in I_0, q \in I_1 \}$$

will be called a $(k, l)$-spider. Vertices $e_i, \ldots, e_k$ (resp., $e_{j_1}, \ldots, e_{j_l}$) will be called initial (resp., end) vertices of the spider $Sp(I_0, I_1)$. Obviously, all $(k, l)$-spiders are equivalent. We shall use the notation $Sp_k^0$ when referring to a $(k, l)$-spider.

**Proposition 8.2.**

1. If a double $D$ is contained in a dee-cluster $\Phi$, then $n = 2$ and $D = \Phi$.
2. If $n > 2$, then a dee-cluster is a $(m, n-m)$-spider, $1 \leq m < n$.

**Proof.**

1. If a dee $\delta$ has a common vertex with a double $D$ and $\delta \notin D$, then, according to proposition 8.1, 2), $\delta$ is incompatible with $D$.
2. Let $\Phi$ be a dee-cluster and $\{p_i|q_i\} \in \Phi, i = 1, 2$. By assertion (1), $\Phi$ does not contain doubles. So, by proposition 8.1, 2), $p_i$ (resp., $q_i$) is not the end (resp., the origin) of a dee belonging to $\Phi$. Therefore the dees $\{p_1|q_2\}$ and $\{p_2|q_1\}$ are compatible with all dees from $\Phi$, and, by maximality property of $\Phi$, they must belong to $\Phi$.

Put

$$C_{m,n}^0 = Sp(I_0, I_1) \quad \text{with} \quad I_0 = \{1, \ldots, m\}, I_1 = \{m + 1, \ldots, n\}, 1 \leq m < n.$$

**Proposition 8.3.**

1. $C_{1,2}^0 = \{[1|2], [2|1]\}$ is the unique 2-dimensional dee-cluster and, at the same time, the unique 2-dimensional cluster.
2. For $n > 2$ there exists an unique $n$-dimensional cluster $C_{m,n}$ containing $C_{m,n}^0$. Namely,

$$C_{1,n} = C_{1,n}^0 \cup \{ |1,k|l \mid 2 \leq k, l \leq n \};
C_{2,n} = C_{2,n}^0 \cup \{ |1,2|k|l \mid 3 \leq k \leq n \};
C_{m,n} = C_{m,n}^0, \quad m \geq 3.$$

**Proof.** The first assertion is obvious.

If $n > 2$, then, as it follows from proposition 8.1, 2), $C_{m,n}$ contains all tees that are compatible with $C_{m,n}^0$. In particular, there are no such ones, if $m \geq 3$. Moreover, by proposition 8.1, 1), these tees are mutually compatible.

As it is easy to see, a $\delta$-coaxial Lie algebra $g$, which is associated with $C_{m,n}^0$, $n > 2$, is of the form $g = a \oplus_{\rho} W$ with $\rho$ being a representation of an $m$-dimensional abelian algebra $a$ in $W$ whose operators have a common diagonalizing basis. The corresponding Poisson bivector is

$$P_{m,n}^0(\alpha) = \sum_{1 \leq i \leq m, m < j \leq n} \alpha_{ij} x_j \xi_i \xi_j, \quad \alpha = \{\alpha_{ij}\}, \; \alpha_{ij} \in k. \quad (105)$$
Poisson bivectors representing coaxial Lie algebras associated with clusters \( C_{m,n} \) are

\[
P_{1,n}(\alpha, \beta) = P^\updownarrow_{1,n}(\alpha) + \sum_{2 \leq k, l \leq n} \beta_{kl} x_k \xi_l \xi_k, \quad \beta = \{ \beta_{kl} \}_{2 \leq k, l \leq n}; \quad (106)
\]

\[
P_{2,n}(\alpha, \tau) = P^\updownarrow_{2,n}(\alpha) + \sum_{3 \leq k \leq n} \tau_k x_k \xi_1 \xi_2 \xi_k, \quad \tau = \{ \tau_k \}_{3 \leq k \leq n}; \quad (107)
\]

\[
P_{m,n}(\alpha) = P^\updownarrow_{m,n}(\alpha), \quad m \geq 3. \quad (108)
\]

The Lie algebra corresponding to Poisson bivector (106) is isomorphic to an algebra \( \Gamma_A \), \( A : W \rightarrow W, \ A = \text{ad} e_1 \) (see subsection 5.2). Here the operator \( A \) can be arbitrary.

The Lie algebra \( g \) corresponding to Poisson bivector (107) has an abelian ideal \( I, |I| = \langle e_3, \ldots, e_n \rangle \), of codimension 2 such that \( [g, g] \subseteq I \). In particular, \( g \) is solvable and its derived series consists of two nontrivial terms.

8.4. Structural groups of tee-clusters. Tee–clusters, unlike dee–clusters, are much more diversified and have a quite complex structure. Their description is a specific combinatorial problem whose solution requires, like in chemistry, determination of basic structural elements.

First of all, it is useful to distinguish vertices of a compatible tee-family \( \Phi \). Namely, a vertex \( \nu \in S(\Phi) \) will be called an end (resp., a center) vertex of \( \Phi \) if it is an end (resp., the center) vertex of any tee \( \vartheta \in \Phi \) such that \( \nu \) is one of vertices of \( \vartheta \). Otherwise, \( \nu \) will be called a mixing vertex. Vertices of the graph \( \Upsilon_\Phi \) will be called accordingly.

The role of various type vertices is illustrated by the following proposition.

Proposition 8.4. Let \( g \) be a coaxial Lie algebra such that \( \Phi_\cdot g = \Phi_\updownarrow_\cdot \). Then

1. The subspace of \( |g| \) generated by all center vertices of \( \Phi_\cdot g \) supports a central ideal \( I \) of \( g \).
2. Let \( g = \sum_{\vartheta \in \Phi_\cdot} a_\vartheta \vartheta \) and \( \Psi \subseteq \Phi_\cdot g \) be the family formed by the tees \( \vartheta \in \Phi_\cdot \) whose centers are not centers of \( \Phi_\cdot g \). Then for a suitable \( m \) the algebra \( g/I \oplus \gamma_m \) is isomorphic to the algebra \( \sum_{\vartheta \in \Psi} a_\vartheta \vartheta \).
3. \( ||g, g|| \) belongs to the subspace generated by all mixing and center vertices of \( \Phi \).

Proof. Straightforwardly from proposition 8.1 and the definitions. \( \square \)

Now we pass to describe structural groups, which are some special compatible tee-families. They are building blocks of which tee-clusters are made. This description is accompanied by basic properties of structural groups. They are direct consequences of proposition 8.1 and by this reason the proofs are omitted.

Triangle. A compatible tee-family of the form

\[
\Delta_{ijk} = \{ [i, j|k], [j, k|i], [k, l|i] \}, \quad 1 \leq i, k, l \leq n.
\]

will be called a triangle. It is easy to see (proposition 8.1) that triangles are 3-dimensional clusters and vice versa. Lie algebras associated with a triangle are of the form \( \mathfrak{so}(g) \) with \( g \) being a 3-dimensional nondegenerate quadratic form, and hence are simple.
Basic property of triangles: The ends of a tee nontrivially compatible with a triangle are vertices of this triangle. All such tees structures are compatible each other.

Hedgehog. Consider a disjoint, i.e., without common vertices, family of triangles $\triangle_1, \ldots, \triangle_p$ and base vectors $\varepsilon_1 = e_i(1), \ldots, \varepsilon_q = e_i(q)$ which are not vertices of these triangles. Enlarge the family $\triangle_1 \cup \cdots \cup \triangle_p$ by adding to it all tees whose centers are in $(\varepsilon_1, \ldots, \varepsilon_q)$ and ends in one of triangles $\triangle_i$'s. The so-obtained family is compatible and is called a hedgehog of type $(p, q)$, or a $(p, q)$-hedgehog. Vertices $\varepsilon_i$'s of the hedgehog are its thorns and $\triangle_i$'s are its base triangles. A triangle can be viewed as a $(1, 0)$-hedgehog.

A $(p, q)$-hedgehog is a $(3p + q)$-dimensional cluster. A Lie algebras associated with a $(p, q)$–hedgehog is a central extension of the direct sum of $p$, 3-dimensional simple algebras (see proposition 8.4).

Basic property of hedgehogs: If a tee $\vartheta \notin \Phi$ is nontrivially compatible with a hedgehog $\Phi$, then either its center is one of thorns of $\Phi$, while its ends do not belong to $S(\Phi)$, or its ends belong to one of triangles $\triangle_i$'s, while its center is not a thorn of $\Phi$. In particular, if $\Phi$ is contained in a compatible tee family $\Psi$, then the thorns of $\Phi$ are center points of $\Psi$.

Twain. A family of the form $\wedge_{i,j}^k = \{[i, k|j], [j, k|i]\}$ will be called a twain. The vertex $e_k$ is the top of $\wedge_{i,j}^k$, while $e_i$ and $e_j$ form its bottom. The unique triangle containing $\wedge_{i,j}^k$ is $\Delta_{i,j}$.

Basic property of twains: If a tee $\vartheta$ is compatible with a twain $\wedge$ and the center of $\vartheta$ is the top of $\wedge$, then the ends of $\vartheta$ belong to the bottom of $\wedge$, i.e., $\wedge \cup \{\vartheta\}$ is a triangle. Consequently, if a twain $\wedge$ belongs to a compatible family $\Phi$, then either its top is an end vertex of $\Phi$, or $\wedge \subset \Delta \subset \Phi$ where $\Delta$ is the unique triangle which contains $\wedge$.

An associated with a twain Lie algebra is the algebra of infinitesimal symmetries of a plain “metric” $adx^2 + bdy^2$, $a, b, \in k$, or, in other words, the Killing algebra of this metric.

Trey. An $m$-trey is a family equivalent to
$$\mathcal{T}_m = \{\wedge_{2,3}^1, [2, 3|4], \ldots, [2, 3|m + 4]\}, \ m > 0.$$ Note that $\wedge_{2,3}^1$ is the unique twain contained in $\mathcal{T}_m$ and $e_4, \ldots, e_{m+4}$ are center vertices of $\mathcal{T}_m$. Bottom vertices of this twain will be called side vertices of this $m$-trey. Center vertices of an $m$-trey are a center vertices of any containing it compatible tee-family. A 1-trey will be called simply a trey.

Basic property of treys: If a tee $\vartheta$ is compatible with an $m$-trey $\mathcal{T}_m$ and the center of $\vartheta$ is a side vertex of $\mathcal{T}_m$, then $\vartheta$ belongs to the twain contained in $\mathcal{T}_m$.

A Lie algebra associated with an $m$-trey is an $m$-dimensional central extension of a Lie algebra associated with the contained in it twain.

Pyramid. An $m$-dimensional pyramid, or, shortly, $m$-pyramid, is a tee-family equivalent to
$$\nabla_{1}^m = \bigcup_{2 \leq i,j \leq m+1} \wedge_{i,j}^1.$$ The common end of all composing a pyramid tees is the top of it, while other vertices form its bottom. A 2-pyramid is a twain.
Notice that a pyramid is a tee-cluster if \( m > 2 \), but not a cluster. Indeed, \( \{ [1 \mid i], 2 \leq i \leq m + 1 \} \) are all compatible with \( \nabla_1^m \) dees. They are also mutually compatible. Hence
\[
C_1^m = \nabla_1^m \bigcup C_{1, m+1}^0
\]
is the unique containing \( \nabla_1^m \) cluster of the same dimension called a dressed pyramid.

**Multiplex.** An \( m \)-plex is a family composed of \( m \) tees, which have one common end and one common center vertex. For instance,
\[
\neg m = \bigcup_{3 \leq i \leq m+2} [2, i][1].
\]
is a such one. The common end (resp., center) of these tees will be called the origin (resp., center) of the \( m \)-plex. All other vertices of it are its ends.

**Basic property of \( m \)-plexes:** If a family \( \Phi \) contains an \( m \)-plex, \( m \geq 3 \), then the origin of this \( m \)-plex is an end vertex of \( \Phi \).

**Multiped.** An \( (p,q) \)-ped, \( p \geq 2 \), is a family equivalent to
\[
\nabla_p^q = \bigcup_{1 \leq i, j \leq p, p+1 \leq k \leq p+q} [i, j][k]. \tag{109}
\]
A \( (p,q) \)-multiped has \( p+q \) vertices \( q \) of which are center and \( p \) end vertices of it. Multipeds with one center and \( m \) ends will be called \( m \)-peds. Among them tripods (=3-peds) are of a special interest. A \( (p,q) \)-multiped is the union of \( q \) \( p \)-peds which have common ends. All tees composing an \( m \)-ped that have a common end vertex form an \( (m-1) \)-plex. Multipeds with more than three ends are clusters.

**Basic property of multipeds:** If a compatible tee-family \( \Phi \) contains a \( (p,q) \)-ped \( \nabla \), \( p \geq 3 \), then any center vertex of \( \nabla \) is a center vertex of \( \Phi \).

**Hybrid.** A \( (p,q \mid r) \)-hybrid is union of a \( (p,r) \)-multiped and a \( (q,r) \)-hedgehog, which have common center vertices and no other common ones. Here \( p \geq 2 \), \( q \geq 1 \), \( r \geq 1 \). A \( (p,q \mid r) \)-hybrid is a cluster and hence a tee-cluster if \( p > 3 \). This directly follows from the basic property of hedgehogs and proposition 8.1, 3).

**Cross.** Two tees with the common center and mutually different end vertices form a cross. For instance, \( ([2,3][1], [4,5][1]) \) is a cross. The common center of these tees is called the center of the cross.

**Basic property of crosses:** The center of a cross belonging to a compatible tee-family \( \Phi \) is a center vertex of \( \Phi \).

**Catena.** An open \( m \)-catena is a family equivalent to
\[
l_m = \bigcup_{2 \leq l \leq m+1} [1, l+1][l], \quad m \geq 2. \tag{110}
\]
An open \( m \)-catena has two end and one center vertices. The common end of tees belonging to a catena is the initial vertex of it (\( e_1 \) for catena (110)). The other end vertex of it is its final vertex (\( e_{m+1} \) for catena (110)). There is only one tee belonging to an open catena whose end vertices coincide with end vertices of this catena.
A closed \((m,k)\)-catena is a tee-family equivalent to
\[
\iota^k_m = \iota_{m-1} \cup [1, m \mid k - 1], \quad m \geq 2, \quad 3 \leq k \leq m.
\] (111)

For instance, \(\iota^1_2\) is a twain. If \(k \geq 2\) a closed catena has only one end vertex and one center vertex. For catena (111) these vertices are \(e_1\) and \(e_2\), respectively.

Diagrams of described above structural groups together with their icons are presented in Fig. 5.

8.5. Types of vertices of a compatible tee-family and casings. The following terminology will be useful in our further analysis of tee-clusters. We shall say that a tee/dee \(\vartheta\) blocks (alternatively, is blocking) a tee/dee \(\theta\) if it is incompatible with \(\theta\). So, we have the following

Blocking rule: If \(\Phi\) is a (tee/dee-)cluster and vertices of \(\theta\) belong to \(S(\Phi)\), then \(\theta\) belongs to \(\Phi\), if there are no blocking \(\theta\) elements in \(\Phi\).
Tees of a compatible tee-family $\Phi$ are naturally subdivided into six classes:

$$ece-,\ ee-,\ ec-,\ e-,\ c-\ \text{and\ } 0-\text{tees}$$

according to their vertices which are mixing (in $\Phi$). For instance, a tee $\theta \in \Phi$ is an $ec$-tee if one of its end vertices and the center vertex are mixing, while the remaining third one is another end vertex. End vertices of an $ee$–tee are mixing but the center is not. All vertices of a $0$-tee are not mixing, etc.

In what follows we shall determine the structure of tee-clusters by analyzing "neighborhoods" of tees of each of these types. Schematically, a tee-cluster consists of "neighborhoods", called casings, of the above described structural groups which are tied together by means of tees called connectives. Exact meaning of these terms is explained below.

**Hedgehogs and $ece$-tees.**

First, we have

**Lemma 8.1.** Let $\Phi$ be a compatible tee-family. Then any $ece$-structure in $\Phi$ belongs to an unique contained in $\Phi$ triangle.

**Proof.** Let $\theta = [i,j,k] \in \Phi$ be an $ece$-structure. Since $e_i$ (resp., $e_j$) is mixing, there is a tee $\theta_1 \in \Phi$ (resp., $\theta_2 \in \Phi$) whose center vertex is $e_i$ (resp., $e_j$). Since $\theta_1$ (resp., $\theta_2$) is compatible with $\theta$, it must be of the form $\theta_1 = [j,l|i]$ (resp., $\theta_2 = [i,m|j]$). Moreover, compatibility of $\theta_1$ and $\theta_2$ implies $l = m$. If $l = m \neq k$, then $\theta_1, \theta_2$ and $\theta$ form a trey contained in $\Phi$. In this case the center vertex $e_k$ of this trey is a center vertex of $\Phi$ by one of basic properties of treys in contradiction with the hypothesis of the lemma. Hence $l = m = k$ and $\theta_1, \theta_2$ together with $\theta$ form a triangle. □

**Corollary 8.1.** If all vertices of a compatible tee-family $\Phi$ are mixing, then $\Phi$ is a disjoint, i.e., without common vertices, union of triangles. In particular, if all vertices of a tee-cluster are mixing, then it is a triangle.

**Proof.** If all vertices of a compatible tee-family $\Phi$ are mixing, then all belonging to it tees are $ece$–tees. Therefore, $\Phi$ is a union of triangles. On the other hand, if two triangles have one or two common vertices, then, obviously, they contain incompatible tees. □

Let $\Delta \subset \Phi$ be a triangle. It is easy to see that any nontrivially compatible with $\Delta$ tee, whose center is a center vertex of $\Phi$, is also compatible with all tees belonging to $\Phi$. Moreover, all such tees are, obviously, compatible each other. They all form the casing of $\Delta$ (in $\Phi$). So, if $\Phi$ is a tee-cluster this casing belongs to $\Phi$. Hence all triangles in a tee-cluster $\Phi$ together with their casings form a hedgehog contained in $\Phi$. Denote it by $\Phi_h$. Emphasize that the thorns of $\Phi_h$ are center vertices of $\Phi$ and vice versa, and the tees forming this casing are $ee$-tees.

By summing up these observations and lemma 8.1 we get

**Proposition 8.5.** Let $\Phi$ be a tee-cluster. Then $\Phi_h$ is not empty if and only if $\Phi$ has at least one center vertex and at least one $ece$-tee.

**Pendent twains and $ee$-tees.**

Let $\Phi$ be a compatible tee-family. A twain $\wedge \subset \Phi$ will be called pendent, or, shortly, $p$-twain, if it belongs to a trey $\top \subset \Phi$ and the top of $\wedge$ is an end vertex of $\Phi$. 

Lemma 8.2. (1) The ends of an ee-tee are either side vertices of a p-twain, or vertices of a triangle belonging to Φ. The center of such a tee is a center vertex of Φ.

(2) If the ends of a tee θ are bottom vertices of a pendent twain ∧ ⊂ Φ and the center of θ is a center vertex of Φ, then θ is compatible with Φ.

(3) Let T₁ ≠ T₂ be two teys such that S(T₁) ∩ S(T₂) ≠ ∅. If belonging to them tees are compatible each other, then either T₁ ∩ T₂ is a twain, or T₁ ∩ T₂ = ∅. In the last case T₁ and T₂ have a common center, or a common end vertex, or both.

Proof. (1) The proof literally repeats that of lemma 8.1 with the exception that in this case k may differ from l = m. If l = m ≠ k, then tees θ, θ₁ and θ₂ (see the proof of lemma 8.1) form a contained in Φ trey. The second alternative takes place if the triangle containing the twain {θ₁, θ₂} belongs to Φ.

(2) A tee ϑ ∈ Φ might be incompatible with θ only if its center is one of end vertices of θ. Therefore, the center of ϑ is a side point of a trey ⊤ ⊂ Φ, which contains ∧. Then, by basic property of treys, ϑ ∈ ∧. Hence ϑ and θ have a common end vertex and as such are compatible.

(3) A direct check by paying attention to the basic property of treys. □

All tees described in assertion (2) of the above lemma constitute the ee-casing of the pendent twain ∧. By this assertion, this casing belongs to Φ if Φ is a tee-cluster.

Now consider a tee such that its end vertices are end and bottom vertices of a p-twain ∧ ⊂ Φ, while its center is a center vertex of Φ. All such tees form the e-casing of ∧. We shall see below (lemma 8.9) that the e-casing of a p-twain ∧ is compatible with Φ and hence belongs to Φ if Φ is a tee-cluster. Obviously, all tees composing an e-casing (resp., an ee-casing) are e-tees (resp., ee-tees). A trey together with its e-casing will be called completed. A completed trey is obtained from an (1,q)-hedgehog by removing one tee from its base triangle. So, it is not a tee-cluster.

Example 8.2. 1-trey cluster. Let E, C, B₁, B₂ be the end, center and side vertices of a trey ⊤, respectively. Add to them a new vertex D and consider tees [E, Bᵢ | D], [E, Bᵢ | C], i = 1, 2, and [E, D | C]. These 5 tees together with ⊤ form a tee-cluster, called 1-trey cluster and denoted 1⊤. By adding to it the dee [E | D] one gets a cluster.

Standing twains, ee-tees and pyramids.

A twain ∧ ⊂ Φ which does belong neither to a triangle, nor to a trey contained in Φ will be called standing, or, shortly, an s-twain. Obviously, the top of an s-twain is an end vertex of Φ. Standing twains can be characterized as those which have no common vertices with ee-tees. They naturally appear in connection with ee-tees.

To proceed on we need the following terminology. A tee θ ∈ Φ (resp., a twain, or a pyramid belonging to Φ) one end of which (resp., the top) is an end vertex E of Φ will be called rooted at E. Let ∧ ⊂ Φ be a twain. A tee θ ∈ Φ, θ ∉ ∧, which is rooted at the same vertex as ∧, will be called a side (resp., lateral) tee for ∧ if the center (resp., the second end) of θ is one of bottom vertices of ∧.

Lemma 8.3. Let Φ be a tee-cluster and ∧ ⊂ Φ be rooted at E s-twain. Then

(1) if Φ possesses a center vertex, then at least one of side tees of ∧ belongs to Φ;
(2) all nontrivially compatible with∧tees are rooted at E and hence are compatible each other;

(3) if θ ∈ Φ is a lateral tee of∧and the center of θ is not a center vertex of Φ, then θ belongs to an s-twain∧′⊂Φ.

(4) the set of bottom vertices of all rooted at E s-twains form the bottom of a rooted at E pyramid P ⊂ Φ. The pyramid P contains any belonging to Φ pyramid, which is rooted at E, and any lateral tee of it does not belong to Φ;

(5) two compatible pyramids rooted at different vertices have no common vertices.

Proof. (1) Let B₁, B₂ be bottom vertices of∧and C a center vertex of Φ. The tee θ = [B₁,B₂|C] does not belong to Φ, since∧is an s-twain. On the other hand, since Φ is a tee-cluster, there is a tee θ ∈ Φ which is incompatible with θ. Since C is a center vertex of Φ, the center of θ is either B₁, or B₂. But being compatible with∧the tee θ is rooted at E and hence is a side tee for∧.

(2) It can be easily checked that a tee which is nontrivially compatible with a twain∧and not rooted at its top vertex is of the form θ = [B₁,B₂|C] with B₁,B₂ being bottom vertices of∧. Obviously, this is impossible for an s-twain.

(3) Let B₁,B₂ ∈ S(Φ) be bottom vertices of∧and θ = [E,B₂|C]. It suffices to show that θ = [E,C|B₂] belongs to Φ, i.e. there is no incompatible with θ tee ρ ∈ Φ. Such a tee must have either its center at C, or one of its ends at B₂. In the first case one of the ends of ρ must be either B₂, or E. Indeed, otherwise, ρ and θ would form a cross whose center C will be, by the basic property of crosses, a center vertex of Φ. But a tee of the form ρ = [D,B₂|C], C ≠ E, is compatible with θ′ = [E,B₁|B₂] ∈∧iff D = B₁. This is impossible, since ρ = [B₁,B₂|C] together with∧form a trey in contradiction with the hypothesis that∧is an s-twain. On the other hand, ρ can not have an end at E, since any rooted at E tee is compatible with θ.

Finally, if one end of ρ is B₂, then B₁ is its second end, since ρ is compatible with∧but not compatible with θ. But we have already seen that this is impossible.

(4) Let ∧₁,…,∧ₘ be rooted at E s-twains that belong to Φ and P the mentioned in the statement pyramid. If θ ∈ P, then θ is either a lateral tee for one of twains∧’s, or belongs to one of them. In the first case θ belongs to Φ in virtue of assertion (3). This proves that P ⊂ Φ.

The remaining part of (4) also directly follows from assertion (3).

(5) Directly from proposition 8.1. □

The pyramid figuring in assertion (4) of the above lemma will be denoted by P_E = P_E(Φ). Its casing is composed of all tees of the form [E,B|C] with B being a bottom vertex of P_E and C a center vertex of Φ. It will be shown that the casing of P_E belongs to Φ (see lemma 8.9 below).

As in the case of twains we shall call a rooted at E tee a side (resp., lateral) tee of P_E if its center (resp., second end) belongs to the bottom of P_E, while the remaining third vertex of it is not a vertex of P_E.

8.6. Connectives and nec-tees. Now we shall describe the situations when ec-tees appear in a non-subordinate manner, i.e., not as elements of p-, or s-twains. Such a tee will be called a nec-tee.
Lemma 8.4. Let $\Phi$ be a tee-cluster and $\theta = [E, A|C] \in \Phi$ a rooted at $E$ nec-tee. Then it holds:

1. $\theta$ is included into a 3-catenae of the form
   \[ l = \{ \theta_{or} = [E, C|B], \theta, \theta_{end} = [E, D|A] \} \text{ with } A \neq B \text{ and } C \neq D; \]
2. $D \neq B$;
3. If $P_E \neq \emptyset$, then any $q = [E, C|B']$ with $B' \in S(P_E)$ belongs to $\Phi$. In particular, any such a tee can be taken for $\theta_{or}$.
4. If $P_E \neq \emptyset$, then $\theta$ is a side tee of $P_E$.
5. $\Phi$ contains a tee of the form $q = [A, D|Q]$ with $Q \neq C$, i.e., $\theta$ is contained in the compatible family $\Psi(\theta) = \{q\} \cup \tau \subset \Phi$. If $Q = B$, then $B$ is a center vertex.
6. If $D$ is a mixing vertex, then $A$ is a bottom vertex of a p-twain, and, therefore, $\theta$ is a lateral tee of it.
7. If $P_E = \emptyset$, then the center $B$ of $\theta_{or}$ is a center vertex of $\Phi$.
8. If $C$ is a vertex of a tee $\vartheta$, which is compatible with $l$, then $\vartheta$ is rooted at $E$.
9. Let $C'$ be the center of a nec-tee $\vartheta$, which is rooted at $E$. If $C \neq C'$, then
   \[ \{[E, C|C'], [E, C'|C]\} \text{ is an s-twain in } \Phi. \]
   In particular, $P_E \neq \emptyset$.
10. If $A$ is the center vertex of a tee, which is compatible with $\Psi(\theta)$ (see assertion (5)), then this tee coincides with $\theta_{end}$.

Proof. (1) Since $C$ is a mixing vertex, there is a compatible with $\theta$ tee $\vartheta$ one of whose ends is $C$. Another end vertex of such a $\vartheta$ must be either $A$, or $E$. The first of these alternative is impossible. Indeed, since $A$ is a mixing vertex there is a tee $q \in \Phi$ with the center at $A$. The only such tee, which is compatible with $\theta$ and $\vartheta$, is $q = [E, C|A]$. But $\{\theta, q\} \subset \Phi$ is a containing $\theta$ twain in contradiction with the hypothesis that $\theta$ is a nec-tee. So, we may take $\vartheta = [E, C|A]$ for $\theta_{or}$.

Next, since $C$ is a mixing vertex, there is a compatible with $\theta$ tee $\vartheta'$ whose center is $A$. Any such tee is of the form $[E, D|A]$. Finally, any of equalities $A = B$, or $C = D$ implies that $\theta$ belongs to a twain contained in $\Phi$. But this is impossible, since $\vartheta'$ is a nec-tee.

(2) Assume that $B = D$. In this case any tee, which is nontrivially compatible with the family $\{\theta_{or}, \theta, \theta_{end}\} \subset \Phi$, is rooted at $E$. By this reason, such a tee and, in particular, any $q \in \Phi$ is compatible with the tee $\vartheta = [E, C|A]$. This tee forms a twain together with $\theta$ and belongs to $\Phi$, since $\Phi$ is a cluster. But this contradicts the hypothesis that $\theta$ is a nec-tee.

(3) A tee $q$, which is incompatible with $\vartheta = [E, C|B']$, must have either its center at $C$, or one of its ends at $B'$. If, additionally, $q \in \Phi$, then $q$ is compatible with $P_E \cap l$. But all such tees are rooted at $E$ and hence are compatible with $\vartheta$. This proves that all tees belonging to $\Phi$ are compatible with $\theta$ and hence $\vartheta \in \Phi$.

(4) According to (3) we can assume that $B$ is a bottom vertex of $P_E$. It follows that a tee, which has the center (resp., one of its ends) at $B$ (resp., at $C$) is compatible with $l$ and $P_E$, is rooted at $E$. As previously, this shows that $\vartheta = [E, B|C]$ is compatible with $\Phi$ and hence belongs to $\Phi$. So, the twain $\{[E, C|B'], [E, B|C]\}$ belongs to $\Phi$ and, therefore, to $P_E$ (Lemma 8.3, (3)). This proves that $C$ is a bottom vertex of $P_E$.

(5) Since $\theta$ is a nec-tee, the tee $\vartheta = [E, C|A]$, which forms a twain with $\theta$, does not belong to $\Phi$. Therefore, $\Phi$, being a tee-cluster, contains a tee $q$, which is
incompatible with \( \vartheta \). But any tee, which is incompatible with \( \vartheta \) and at the same time is compatible with the catena \( \ell \subset \Phi \), is of the form \([A,D|Q]\). Moreover, \( Q \neq C \), since, otherwise, \( \varrho \) and \( \vartheta_{or} \) would be incompatible. Finally, if \( Q = B \), then, in view of assertion (1), \( \varrho \) and \( \vartheta_{or} \) form a cross. So, \( B \) is a center vertex of \( \Phi \) by the basic property of crosses.

(6) In this case the tee \([A,D|Q]\) from (5) is an ee-tee. So, by lemma 8.2, \( A \) and \( D \) are bottom vertices of a rooted at \( E \) p-twain.

(7) The tee \( \vartheta = [E,C|B'] \) does not belong to \( \Phi \). Indeed, otherwise, the twain \( \{[E,C|B'],[E,B|C']\} \) would belong to \( \Phi \) in contradiction with the hypothesis that \( P_E = \emptyset \). Hence there is a tee \( \vartheta \in \Phi \) which is incompatible with \( \vartheta \). On the other hand, \( \vartheta \), as a tee from \( \Phi \), is compatible with \( \ell \). It remain to observe that any tee which satisfies these conditions has its center at \( B \) and form a cross with \( \vartheta_{or} \). Hence \( B \) is a center vertex of \( \Phi \) by the basic property of crosses.

(8) Immediately from assertion (1).

(9) A tee from \( \Phi \), which is incompatible with one of tees \([E,C|C'],[E,C'|C']\), has one of its vertices either at \( C \), or at \( C' \). But this is impossible, since, according to assertion (8), any such tee is rooted at \( E \).

(10) Obvious. \( \square \)

The fact that any nec-structure \( \vartheta \) is included in a family of the form \( \Psi(\vartheta) \) (assertion (5) of the preceding lemma) will be often used in the sequel. It is worth noticing that \( \Psi(\vartheta) \) is not unique and tees composing \( \Psi(\vartheta) \) have different center vertices except, possibly, \( B \) and \( Q \) which may coincide.

**Pendent nec-tees**

Below we shall keep the notation of lemma 8.4 for vertices of \( \Psi(\vartheta) \). A nec-tee will be called pendent if it is not a side tee of a contained in \( \Phi \) pyramid.

**Corollary 8.2.** Let \( \Phi \) be a tee-cluster, \( E \in S(\Phi) \) an end vertex of it and \( \vartheta \) a rooted at \( E \) nec-tee. Then

1. \( \vartheta \) is pendent if and only if \( P_E = \emptyset \).
2. \( \vartheta \) is a side tee of \( P_E \), if \( P_E \neq \emptyset \) or a pendent one, if \( P_E = \emptyset \). In both cases it may at the same time be a lateral tee of a p-twain.
3. If a side tee of \( P_E \) is also lateral for a p-twain in \( \Phi \), then it belongs to \( \Phi \).
4. If \( \vartheta \) is pendent, then the center of any other rooted at \( E \) pendent tee coincides with the center of \( \vartheta \).
5. Any tee \([E,C|B']\), which is lateral for a p-twain in \( \Phi \), belongs to \( \Phi \).
6. If the center vertex of \( \vartheta \) is a vertex of \( \vartheta \in \Phi \), then \( \vartheta \) is rooted at \( E \).

**Proof.** (1) Directly from the definition and lemma 8.4, (4).

(2) Obviously from assertion (1) and lemma 8.4, (6).

(3) Let \( \vartheta = [E,Q|R] \) be lateral for a p-twain \( \wedge \subset \Phi \). It could be blocked either by a tee with the center at \( Q \), or by a tee with an end at \( R \). In the first case \( Q \) is a side vertex of the trey, which contains \( \wedge \). But, by the basic property of treys, \( \vartheta \in \wedge \) and hence does not block \( \vartheta \). In the second case \( R \in S(P_E) \). Since any nontrivially compatible with \( P_E \) tee is rooted at \( E \), it does not block \( \vartheta \). So, by the blocking rule, \( \vartheta \) belongs to \( \Phi \).

(4) Let \( \vartheta = [E,A|C] \) and \( \vartheta' = [E,A'|C'] \) be pendent tees and \( C \neq C' \). If one of vertices of a tee \( \vartheta \) is the center \( C \) of \( \vartheta \) and \( \vartheta \) is compatible with \( \Psi(\vartheta) \), then, as it is easy to see, \( \vartheta \) is rooted at \( E \) and similarly for \( \vartheta' \). By this reason, tees forming the rooted at \( E \) twain \( \wedge \) with bottom vertices \( C \) and \( C' \) can not be blocked by a tee.
from \( \Phi \). So, by the blocking rule \( \land \subset \Phi \). Obviously, \( \land \) is an s-twain. This shows that \( P_E \neq \emptyset \) in contradiction with the made assumption.

(5) To prove this assertion it suffices to substitute \( \Psi(\theta) \) for \( P_E \) in the proof of assertion (3).

(6) Immediately from assertions (1) and (8) of lemma 8.4. \( \square \)

According to assertion (4) of this corollary, all pendent nec-tees rooted at \( E \) have the common center vertex. Denote it by \( C_{pn}^E \) and consider the multiplex \( \perp_{pt}^E \) with the origin \( E \) and the center \( C_{pn}^E \) whose ends are bottom vertices of all rooted at \( E \) p-twains. It will be called the twain multiplex at \( E \). Assertions (4) and (5) of the corollary 8.2 show that \( \perp_{pt}^E \) belongs to \( \Phi \). When \( \Phi \) does not have rooted at \( E \) s-twains, i.e., \( P_E = \emptyset \), then the "rod" with ends \( E \) and \( C_{pn}^E \) may be viewed as the "collapsed" \( P_E \). The casing of \( \perp_{pt}^E \) is composed of all tees with ends at \( E \) and \( C_{pn}^E \) and whose centers are center vertices of \( \Phi \). If \( \Phi \) is a tee-cluster, then this casing belongs to \( \Phi \) (see lemma 8.9 below).

A nec-tee will be called a pyt-connective (resp., pt-connective) at \( E \) if it is lateral for a rooted at \( E \) p-twain and simultaneously a side tee for \( P_E \) (resp., a rooted at \( E \) nec–pendent tee if \( P_E = \emptyset \)). Informally speaking, these connectives "consolidate" the system of rooted at \( E \) p-twains around the "central pillar" \( P_E \) (resp., the "rod" \( EC_{pn}^E \)) into a "rigid structure". Both pyt- and pt-connectives are internal in the sense that they join p- and s-twains rooted at the same vertex. External connectives will be discussed below.

8.7. C- and e-tees. Let \( \Phi \) be a compatible tee-family. Denote by \( \Phi_E \) the set of all rooted at \( E \) ec-tees. A c-tee with end vertices \( E \) and \( D \) connects families \( \Phi_E \) and \( \Phi_D \). The following lemma allow us to subdivide c-tees into two classes.

**Lemma 8.5.** Let \( C, D \in S(\Phi) \) be end vertices of \( \Phi \) and

\[ \iota_2 = \{ [E,D|C], [E,C|B] \} \subset \Phi. \]

Then it holds:

1. \( \theta = [E,D|C] \in \iota_2 \) is a c-tee and, conversely, any c-tee is contained in a 2-catena \( \iota_2 \subset \Phi \).
2. If \( C \) is a vertex of a tee \( \varrho \) which is nontrivially compatible with \( \iota_2 \), then \( \varrho \) is of one of the following types: \( [E,A|C] \) (type I at \( E \)), or \( [D,C|Q] \) (type II at \( E \)), or \( [E,C|B'] \) (type III at \( E \)). Tees of type I and II at \( E \) are incompatible.
3. If \( \Phi \) is a tee-cluster and \( \varrho \in \Phi \) is of type I at \( E \), then any \( \theta' = [E,E'|C] \) with \( E' \) being an end vertex belongs to \( \Phi \).
4. Let \( \varrho \in \Phi \) be of type I at \( E \) and \( C \) be a vertex of a tee \( \vartheta \in \Phi \). Then \( \vartheta \) is rooted at \( E \).

**Proof.** (1)-(2) Obviously.

(3) A tee \( \vartheta \in \Phi \) which could block \( \theta' \) must have one of its ends at \( C \). But being compatible with \( \iota_2 \), \( \vartheta \) is either of type II, or of type III at \( E \). The first alternative is impossible, since tees of types I and II are incompatible. On the other hand, a tee of type III does not block \( \theta' \).

(4) An obvious consequence of compatibility of \( \vartheta \) with \( (\iota_2 \cup \{ \varrho \}) \). \( \square \)
Let \( \theta \in (I_2 \cup \{\varnothing\}) \subset \Phi \) be as in lemma 8.5. The e-tee \( \theta \) will be called a hook at \( E \) (resp., a bridge) if \( \varnothing \) is of type I at \( E \) (resp., of type II). We shall say that the bridge \( [E, D] \) connects vertices \( E \) and \( D \).

**Hooks.**

Now we shall describe “environments” of hooks.

**Lemma 8.6.** Let \( \Phi \) be a tee-cluster. Then it holds:

1. Let \( \theta = [E, D, C] \in \Phi \) be a hook at \( E \) and \( C \) a vertex of \( \emptyset \in \Phi \). Then \( \emptyset \) is rooted at \( E \).
2. Let \( \theta_i = [E, E_i, C_i] \in \Phi \), \( i = 1, 2 \), be hooks at \( E \). Then either \( C_1 = C_2 \), or \( \emptyset = \{[E, C_1|C_2]\}, \{[E, C_2|C_1]\} \) is an s-twin.
3. Let \( \emptyset_1 = [E, E_1|C_1] \) be a hook at \( E \) and \( \emptyset_2 = [E, E_2|C_2] \) be a pending nec–tee at \( E \). Then either \( C_1 = C_2 \) or \( \emptyset = \{[E, C_1|C_2]\}, \{[E, C_2|C_1]\} \) is an s-twin.
4. Let \( E \) and \( E' \) be end vertices and \( C \) be a bottom vertex of \( P_E \), or the center vertex of a hook.

Then \( \emptyset = [E, E'|C] \) is a hook at \( E \).
5. Let \( [E, E'|C] \) be a hook at \( E \) and \( S_1, S_2 \) be bottom vertices of a rooted at \( E \) p-twin \( \emptyset \). Then \( \emptyset_i = [E, S_i|C] \in \Phi \), \( i = 1, 2 \).

**Proof.**

1. A particular case of lemma 8.5, (4).
2. Assume that \( C_1 \neq C_2 \). If \( C_i \), \( i = 1, 2 \), is a vertex of a tee \( \emptyset \), then, in virtue of assertion (1), \( \emptyset \) is rooted at \( E_i \) and, therefore, does not block tees that compose \( \emptyset \). Moreover, tees of the form \( [C_1, C_2|Q] \) are incompatible with \( \emptyset_1 \) and \( \emptyset_2 \) and hence do not belong to \( \emptyset \), i.e., \( \emptyset \) belongs to \( \Phi \). Moreover, tees of the form \( [C_1, C_2|Q] \) are incompatible with \( \emptyset_1 \) and \( \emptyset_2 \) and hence do not block them.
3. Observe that one of vertices of a tee \( \emptyset \in \Phi \) which could block a belonging to \( \emptyset \) tee must be either \( C_1 \) or \( C_2 \). But any such structure is rooted at \( E \) (corollary 8.2, (6), and lemma 8.5, (4)) and hence can not block them.
4. Observe that if a tee \( \emptyset \in \Phi \) blocks \( \emptyset \), then \( C \) must be one of its vertices. But all such tees are rooted at \( E \) (corollary 8.2, (6), and lemma 8.5, (4)) and hence can not block \( \emptyset \).
5. By the basic property of treys and lemma 8.5, (4), \( \emptyset \) does not contain blocking \( \emptyset_i \) tees. \( \square \)

According to assertion (2) of lemma 8.6 all hooks at \( E \) have a common center vertex if \( P_E = \emptyset \). Denote it by \( C_{E}^{hk} \). Denote also by \( \perp_{E}^{hk} \) the multiplex constituted by all tees \( [E, E'|C_{E}^{hk}] \) with \( E' \) running all different from \( E \) end vertices of \( \Phi \). It will be called the hook multiplex at \( E \).

**Corollary 8.3.** Let \( \Phi \) be a tee-cluster and \( E \) an end vertex of it. Then

1. all hooks at \( E \) are side tees of \( P_E \) if \( P_E \neq \emptyset \).
2. If \( \Phi \) contains at least one pendent tee and one hook at \( E \), then \( C_{E}^{pn} = C_{E}^{hk} \).
3. \( \perp_{E}^{hk} \subset \Phi \).

**Proof.** Assertion (1) directly follows from assertions (2) and (4) of lemma 8.6, and assertion (2) from corollary 8.2, (1) and lemma 8.6 (2), (3). Assertion (3) is a particular case of assertion (4) of this lemma. \( \square \)

In view of corollary 8.3, (2), there would be no confusion to use the common notation \( C_{E} \) for centers \( C_{E}^{hk} \) and \( C_{E}^{pn} \).
The casing of \( \perp_{\Phi} \) is composed of all tees \( \{E, C_{E}^{\Phi}\} \) with \( C \) running all center vertices of \( \Phi \). Obviously, it coincides with the casing of \( \perp_{\Phi}^{\Psi} \) assuming that \( \perp_{\Phi}^{\Psi} \neq \emptyset \).

**Bridges.**

Bridges join different families \( \Phi_{E} \)'s and in this sense are external connectives. On the other hand, they are naturally combined with internal connectives as will be shown below. First, we need the following

**Lemma 8.7.** Let \( \Phi \) be a tee-cluster and \( \theta = [E, A|C] \in \Phi \) be a nec-tee rooted at \( E \). Then \( A \) is either a bottom vertex of a rooted at \( E \) \( p \)-twain, or the center vertex of a bridge one end vertex of which is \( E \).

**Proof.** Recall that \( \theta \in \Psi(\theta) \) (see lemma 8.4, (5)). If, in the notation of lemma 8.4, the vertex \( D \) is mixing, then the first alternative occurs. If \( D \) is an end vertex, then the c-tee \( [E, D|A] \in \Psi(\theta) \) is a bridge, since the unique tee with the center at \( A \) which is compatible with \( \Psi(\theta) \) is, obviously, \( \theta \). \( \square \)

The following lemma shows the role of bridges in the structure of tee-clusters.

**Lemma 8.8.** Let \( \Phi \) be a tee-cluster and \( \varrho = [E, D|Z] \in \Phi \) be a bridge. Then it holds:

1. If \( \varrho \in \Phi, \varrho \neq \emptyset \), and \( Z \) is a vertex of \( \varrho \), then \( Z \) is an end vertex of \( \varrho \).
2. If one of end vertices of a side structure \( \vartheta \) of \( \Phi \) is \( Z \), then \( \vartheta \) belongs to \( \Phi \).
3. The tee \( \theta = [E, Z|C_{E}] \) belongs to \( \Phi \).

**Proof.** (1) Directly from the definition of a bridge.

(2)-(3). Let \( \vartheta = [E, Z|B] \) where \( B \) is either a bottom vertex of \( \Phi \), or \( B = C_{E} \). \( \varrho \) can be blocked by a tee \( \vartheta \in \Phi \), which either has the center at \( Z \) or one of its ends at \( B \). Assertion (1) of this lemma excludes the first of these possibilities. Next, \( B \) is either the center of a \( nec \)-structure, or the center of a hook rooted at \( E \). In each of these cases a tee one of whose vertices is \( B \) is rooted at \( E \) (corollary 8.2, (6), and lemma 8.6, (1)). But such a tee does not block \( \theta \). \( \square \)

A \( pyt \)-connective (resp., \( pb \)-connective) at an end vertex \( E \) is a side structure of \( \Phi \) (resp., \( [E, Z|C_{E}] \)), one of whose ends is the center \( Z \) of a bridge connecting \( E \) with another end vertex. All \( pb \)-connectives at \( E \) form a multiplex, denoted by \( \perp_{E}^{\psi} \). If \( \Phi \) is a tee-cluster, then, by lemma 8.8,(3), this multiplex belongs to \( \Phi \). Also, denote by \( \perp_{E}^{\psi} \) (resp., \( \perp_{E}^{\psi} \)) the family of all \( pyt \)-connectives (resp., \( pyt \)-connectives) at \( E \). If \( \Phi \) is a tee-cluster, then both \( \perp_{E}^{\psi} \) and \( \perp_{E}^{\psi} \) belong to \( \Phi \). Each of these families is the union of multiplexes with common origin and ends whose centers run bottom vertices of \( \Phi \). The casing of a bridge \( \{E_{i}, Z|C_{i}\}, i = 1, 2 \), with \( C \) being a center vertex of \( \Phi \). In contrast to all previously introduced casings, it is naturally subdivided into two parts that are composed of rooted at \( E_{1} \) and of rooted at \( E_{2} \) tees, respectively. Lemma 8.9 below tells that this casing belongs to \( \Phi \) as well.

**Casings and e-tees.**

Now we shall describe e-structures.

**Lemma 8.9.** Let \( \Phi \) be a tee-cluster and \( E \) an end vertex of it. Then \( \theta = [E, Z|C] \in \Phi \) is a (rooted at \( E \)) e-tee if and only if \( Z \) is the center either of an ec-, or of an c-tee rooted at \( E \).
Proof. First, note that if $\theta$ is an $e$-tee, then $Z$ is the center vertex of a tee $\varrho \in \Phi$. Being compatible with $\theta$ the tee $\varrho$ is rooted at $E$. Moreover, $\theta$ and $\varrho$ form an open 2-catena $\gamma_2$ whose initial vertex is $E$. This shows that $\varrho$ is either an $ee$-, or an $c$-tee rooted at $E$.

Conversely, assume that $Z$ is the center vertex of an $ee$-, or of an $c$-tee $\varrho \in \Phi$ which is rooted at $E$. Since $C$ is a center vertex of $\Phi$, $\theta$ can be blocked only by a tee $\vartheta \in \Phi$ whose center vertex is $Z$. But from the previous description of rooted at $E$ $ee$-, and $c$-tees we see that all such tees are rooted at $E$ too. Hence they can not block $\theta$. \qed

Now we can state that if $\Phi$ is a tee-cluster, then all previously considered casings belong to $\Phi$ and any $e$-tee belongs to one of these casings.

$0$-tees.
These are tees whose end and center vertices are end and center vertices of $\Phi$, respectively. If $\Phi$ is tee-cluster, then they, obviously, belongs to it and constitute a multiped $\Phi_{mp} \subset \Phi$. If $\Phi$ has at least two ends vertices and one $ee$-structure, then both $\Phi_h$ and $\Phi_{mp}$ are not empty. In this case they form the hybrid $\Phi_{hyb} = (\Phi_h \cup \Phi_{mp}) \subset \Phi$.

A graphical summary of various kinds of casings and connectives is given in Fig. 6.
8.8. The card of a tee-cluster. The above analysis revealed basic structural units of which all tee-clusters are made. On this ground we can now describe all tee-clusters.

First of all, observe that a tee-cluster $\Phi$ is naturally divided into two parts, $\Phi_h$ and $\Phi_{end}$. The family $\Phi_{end}$ is composed of all tees $\theta \in \Phi$ one of whose vertices is an end vertex of $\Phi$. It is easy to see that $\Phi_{end} = \Phi \setminus \Phi_h$ and $\Phi_{mp} \subset \Phi_{end}$. So, $\Phi = \Phi_h \cup \Phi_{end}$, $\Phi_h \cap \Phi_{end} = \emptyset$ and $S(\Phi_h) \cap S(\Phi_{end})$ consists of all center vertices of $\Phi$. As motivated by proposition 8.8 below, $\Phi_h$ (resp., $\Phi_{end}$) will be called the semi-simple (resp., solvable) part of $\Phi$.

The data characterizing a compatible tee-family $\Phi$ are: $n_c=$(the number of center vertices), $n_e=$(the number of end vertices), $n_{tr}=$(the number of triangles), $t_E=$(the number of p-twains rooted at the end vertex $E$), $p_E=$(the dimension of $P_E$) and $b_{E,D}=$(the number of bridges connecting end vertices $E$ and $D$). The numbers $t_E$ and $p_E$ will be called the twain and pyramid numbers at $E$, respectively, and $b_{E,D}$ the bridge number at $(E, D)$. Each of these numbers is a nonnegative integer. Since the dimension of a true pyramid is greater than 1, the value $p_E = 1$ requires a comment. Namely, it is interpreted as existence of the vertex $C_E \in S(\Phi)$, the common center of all connectives and hooks rooted at $E$. Informally speaking, $p_E = 1$ refers to the ”collapsed” pyramid $P_E$, that is the “rod” $ECE$.

All above numbers forms the card of $\Phi$, which will be denoted by $C(\Phi)$. More precisely, numerate end vertices of $\Phi$ and put $t_i = t_{E_i}$, $p_i = p_E$ if $E$ is the $i$-th end vertex and, similarly, $b_{ij} = b_{E_i,D_j}$ if $D$ is the $j$-th end vertex. Thereby we have the twain vector $t = (t_1, ..., t_{n_e})$, the pyramid vector $p = (p_1, ..., p_n)$ and the bridge matrix $B = \|b_{ij}\|$. A renumbering of end vertices corresponds to a simultaneous permutation of components of these vectors and the matrix. So, the triple $(t, p, B)$ will be considered as a representative of the corresponding equivalence class $[t, p, B]$ modulo these permutations. Thus

$$C(\Phi) = (n_c, n_e, n_{tr}, [t, p, B]).$$

(112)

Proposition 8.6. Two tee-clusters are equivalent if and only if their cards are equal.

Proof. First, observe that if the layout of center and end vertices, triangles, p-twains, pyramids and bridges of a tee-cluster $\Phi$ is known, then $\Phi$ is automatically and uniquely restored just by adding to these data all possible connectives, hooks and the casing.

So, it suffices to show that if $\Phi$ and $\Phi'$ are tee-clusters with equal cards, then there exists a one-to-one correspondence $\zeta : S(\Phi) \rightarrow S(\Phi')$ which identifies center and end vertices of these clusters as well as their triangles, p-twains, pyramids (including ”collapsed”) and bridges. We shall construct such a map gradually by starting from a map $\zeta_1 : S(\Phi_h) \rightarrow S(\Phi'_h)$ which establishes an equivalence of $(n_{tr}, n_c)$-hedgehogs $\Phi_h$ and $\Phi'_h$. After that we shall extend $\zeta_1$ to a biunique correspondence $\zeta_2$ of end vertices of $\Phi$ and $\Phi'$ in such a way that $t_E(\Phi) = t_{G(E)}(\Phi')$, $p_E(\Phi) = p_{G(E)}(\Phi')$ and $b_{E,D}(\Phi) = B_{G(E),\zeta_2(D)}(\Phi')$ for all end vertices of $\Phi$. This is, obviously, possible. Since $p_E = p_{G(E)}$, there is a bijection between bottom vertices of $P_E$ and $P_{G(E)}$. This way $\zeta_2$ is extended to pyramid, and we shall proceed on similarly for p-twains and bridges. \qed

Abstract cards.
Proposition 8.6 reduces the classification of tee-clusters to description of their cards. Namely, an abstract card is an ordered set of the form \((k, l, m, [t, p, B])\) where \(k, l, m \in \mathbb{N}_0\), \(t, p \in \mathbb{N}_0^l\) and \(B\) is a symmetric \(l \times l\)-matrix with entries in \(\mathbb{N}_0\) and zero diagonal elements. As earlier, \([t, p, B]\) stands for the orbit of the triple \((t, p, B)\) under a natural action of the symmetric group \(S_l\). The number

\[
\text{card } \mathcal{J} = k + l + 3m + \sum_{i=1}^{l} p_i + 2 \sum_{i=1}^{l} t_i + \sum_{1 \leq i < j \leq l} b_{ij}
\]

will be called the dimension of \(\mathcal{J}\). If \(\mathcal{J} = C(\Phi)\), then \(\text{card } \mathcal{J} = \dim \Phi\).

Obviously, the card of a tee-cluster is an abstract card but the converse is not true. So, the problem is to find exact conditions that distinguish cards of tee-clusters among other abstract cards. To this end the notion of a realization of an abstract card will be useful. Namely, choose among base vectors the following disjoint groups:

\[
\{C_1, \ldots, C_k\}, \quad \{E_1, \ldots, E_t\}, \quad \{T_{i1}, T_{i2}, T_{i3}\}, \quad 1 \leq i \leq m,
\]

\[
\{W_{j1}^s, W_{j2}^s\}, \quad 1 \leq i \leq l, \quad 1 \leq j \leq t_i, \quad \{P_{ir}\}, \quad 1 \leq i \leq l, \quad 1 \leq r \leq p_i, \quad (113)
\]

\[
\{C_{ij}^s\}, \quad 1 \leq i < j \leq l, \quad 1 \leq s \leq b_{ij}
\]

Vectors \(C_i\)’s (resp., \(E_i\)’s) will be called declared center (resp., end) vertices. Similarly, \(T_{iq}\), \(1 \leq q \leq 3\), are vertices of the \(i\)-th declared triangle, \(\{W_{j1}^s, W_{j2}^s\}\) are bottom vertices of the \(j\)-th declared p-twain rooted at \(E_i\), \(P_{ir}\), \(1 \leq r \leq p_i\), are bottom vertices of the declared pyramid rooted at \(E_i\) and \(C_{ij}^s\) is the center vertex of the \(s\)-th declared bridge connecting \(E_i\) and \(E_j\). Then we shall consider the corresponding declared triangles, p-twains, pyramids and bridges by adding to them all possible declared connectives and the casing. For instance, an abstract card of tee-cluster is of the form \([E_i, W_{j1}^s|P_{ir}]\), etc. The so-constructed family, which is, obviously, compatible, will be called a realization of \(\mathcal{J}\) and denoted \(\Phi_{\mathcal{J}}\). Two realizations of a given abstract card are, obviously, equivalent. Also, if \(\mathcal{J} = C(\Phi)\), then, as it is easy to see, \(\Phi_{\mathcal{J}} = \Phi\). Hence the above problem can be reformulated as:

**what are abstract cards \(\mathcal{J}\) such that \(\Phi_{\mathcal{J}}\) a tee-cluster.**

The following necessary conditions are on the surface. First, the graph of \(\Phi_{\mathcal{J}}\) must be connected and, second, \(\mathcal{J}\) must be equal to \(C(\Phi_{\mathcal{J}})\), i.e., that the declared parameters must coincide with actual ones. For instance, if the declared parameters are \(k = 0, l = m = p_1 = t_1 = 1\), i.e., \(\mathcal{J} = (0, 1, 1, [[(1), (1), (0)]])\), then \(\Phi_{\mathcal{J}}\) consists of two rooted at the same vertex twains and one disjoint from them triangle. So, the graph of \(\Phi_{\mathcal{J}}\) is disconnected and, moreover, none of these two twains can be distinguished as a p-twain, i.e., the declared value \(t_1 = 1\) differs from the actual. If the declared parameters are \(k = 0, l = m = 1, p_1 = t_1 = 0\), then \(\Phi_{\mathcal{J}}\) is a triangle. So, in this case, the realization does not have the declared end vertex \(E_1\).

These necessary conditions (resp., an satisfying them abstract card) will be called consistency conditions (resp., a consistent card).

In subsections 8.5 - 8.7 we have established the role of various kinds of tees in the construction of a tee-cluster. Now it is convenient to bring together the obtained results in order to ease further discussion of consistency conditions.
DISTRIBUTION OF TEES IN A TEE-CLUSTER

| type | description |
|------|-------------|
| eee- | belongs to a triangle |
| ee-  | belongs to a p-, or s-twain, or is a connective |
| ee-  | belongs to a hedgehog but not to a triangle, or to the ee-casing of a p-twain |
| c-   | is a bridge |
| e-   | belongs to a casing different from ee-type |
| 0 -  | belongs to a multiped |

This table will be referred as the DT-table.

Join operations.
In order to explicitly describe consistent cards we need the following four operations with tee-clusters. Below Φ stands for a tee-cluster.

Joining a triangle. Assumption: \( n_c \neq 0 \). Include the \((n_r, n_c)\)-hedgehog Φ\(_h\) into an \((n_r + 1, n_c)\)-hedgehog \(\bar{\Phi}_h\) by adding three new vertices to \(S(\Phi)\). The new tee-cluster is \(\Phi_h \cup \Phi_{\text{end}}\).

Joining a p-twain. Assumption: \( n_c \neq 0, n_e \neq 0 \). Let \( E \) be an end vertex of Φ and \( B_1, B_2 \notin S(\Phi) \). First, add to Φ the twain \( \Lambda = \{[E, B_1|B_2], [E, D_2|B_1]\} \) and all tees of the form \([B_1, B_2|C]\) with \( C \) being a center vertex of Φ. Then add all the connectives and casings to the so-obtained family. The resulting compatible tee-family will be denoted by Φ\(_\Lambda, E\), or, simply, Φ\(_\Lambda\).

Joining a pyramid. Assumption: \( p_E \neq 0 \). Let \( E \) be an end vertex of Φ and \( B_1 \ldots, B_r \) base vectors not belonging to \( S(\Phi) \). We assume that \( r \geq 1 \), if at least one of families \( P_E, \perp_{\text{en}}^p, \perp_{\text{pt}}^p \) is nonempty (equivalently, \( p_E > 0 \)), and \( r > 1 \) otherwise. Consider the pyramid \( \nabla \) whose top vertex is \( E \) and bottom vertices are that of \( P_E \), if \( P_E \neq \emptyset \), or \( C_E \), if \( P_E = \emptyset \) and \( C_E \) exists, and \( B_1 \ldots, B_r \). Now we get a new compatible tee-family by adding all new connectives, hooks and casings to \( \Phi \cup \nabla \).

If \( p_E = 0 \), then the “collapsed pyramid”, i.e., a new vertex interpreted as \( C_E \), can be created, assuming that \( n_e > 0 \) and Φ contains either or both a rooted at \( E \) trey and a bridge with one end at \( E \). Namely, first, we add the tees \([E, A|C_E]\) and \([E, C_E|E']\) to Φ where \( A \) is the center of a bridge, or a side vertex of a rooted at \( R \) trey, and \( E' \) is a center vertex of Φ. Then we complete the so-obtained tee-family by adding to it all possible new connectives, hooks and casings.

Joining a bridge. Assumption: \( n_e \geq 2 \). First, add to Φ the tee \( \theta = [E_1, E_2|C] \) where \( E_1 \) and \( E_2 \) are end vertices of Φ and \( C \notin S(\Phi) \). Then add to \( \Phi \cup \{\theta\} \) all tees of the form \([E_1, C|A]\) where \( A \) is a bottom vertex of \( P_{E_1} \), or \( C_{E_1} \), or a center vertex of Φ.

Observe that these join operations commute, preserve both end and center vertices of the original tee-cluster and do not create new ones.
An end (resp., center) vertex of a compatible tee-family $\Psi$ remaining be such in any containing $\Psi$ compatible tee-family will be called stable. The following assertion is obvious.

**Lemma 8.10.** We have:

1. An end vertex $E$ of $\Psi$ is stable if there are at least three rooted at $E$ tees with mutually different second ends.
2. The center of a cross or a tripod belonging to $\Psi$ is a stable center vertex of $\Psi$. □

**Proposition 8.7.** If the ends of a tee-cluster $\Phi$ are stable, then the result of any of the above joining procedures is a tee-cluster.

**Proof.** For triangles the assertion is obvious. In order to prove that $\Phi$ is a tee-cluster we have to show that any compatible with $\Phi$ tee $\theta$ at least one of whose vertices is $B_i$, $i = 1, 2$, and others are in $S(\Phi_A)$ belongs to $\Phi_A$. But such a tee, due to stability of end vertices of $\Phi$, is either of the form $[B_1, B_2, C]$ with $C$ being a center vertex of $\Phi$ or of the form $[E, B_i, Z]$ where $E$ is the top of $\wedge$ and $Z$ is a center/mixing vertex of $\Phi$. In the first case $\theta$ belongs to $\Phi_A$ by construction as well as in the case when $Z$ is a center vertex $\Phi$. If $Z$ is mixing, then it may be a bottom vertex of a p-twain, or of a pyramid $P_D$, or $C_D$, or the center of a bridge in $\Phi$ as it follows from the description of mixing vertices of a tee-cluster. The first and the fourth of these possibilities are manifestly impossible. For the rest, compatibility with $\Phi$ conditions imply that $\theta$ must be rooted at $D$ and, therefore, that $E = D$. In other words, $Z$ is a bottom vertex of $P_E$, or $C_E$ and hence, by construction, $[E, B_i, Z] \in \Phi$.

Similar arguments together with DT-table prove the remaining two assertions. □

**Corollary 8.4.** Let $J = (k, l, m, [t, p, B])$ be an abstract card. If $k \geq 1$ and $l \geq 4$, then $\Phi_J$ is a tee-cluster.

**Proof.** Consider the contained in $\Phi_J$ $(m, l|k)$-hybrid $\Psi$. It is a tee-cluster. Since $k \geq 1$, any center vertex of $\Psi$ contains a tripod and hence is stable. Also, since $l \geq 4$, at least three 0-tees rooted at an end vertex of $\Psi$ have different second end vertices. By this reason end vertices of $\Psi$ are stable too. Now it remains to observe that $\Phi_J$ is obtained from $\Psi$ by a series of join operations and apply proposition 8.7 □

**8.9. Exceptional cards.** Corollary 8.4 shows that nontrivial consistency conditions may occur only if $k = 0$ (case I), or if $k > 0, l < 4$ (case II). Consider them separately by anticipating the following evident facts (see subsection 8.7 for the notation):

**Lemma 8.11.** Let $\Psi$ be a compatible tee-family, $E, E'$ end vertices of $\Psi$ and $\theta \in \Psi$. Then

1. if the center of $\theta$ is in $S(\Psi_E)$, then $\theta$ is rooted in $E$;
2. if the ends of $\theta$ are in $S(\Psi_E)$ and $S(\Psi_{E'})$, respectively, then they coincide with $E$ and $E'$. 
Case I: \( k = 0 \). If \( \Phi \) is a tee-cluster, then \( n_e = 0 \) implies that \( n_{tr} \leq 1 \), \( t = 0 \), \( p_i \neq 1 \), \( \forall i \). Indeed, all triangles of \( \Phi \) belong to the hedgehog \( \Phi_h \) which has at least one thorn, if \( n_{tr} \geq 2 \). Also, existence of p-twains and “collapsed pyramids” in \( \Phi \) presumes (see lemma 8.4, (6), (7) existence of center vertices in \( \Phi \). So, the consistency conditions in this case are: \( k = 0 \Rightarrow m \leq 1 \), \( t = 0 \), \( p_i \neq 1 \), \( \forall i \).

If, moreover, \( n_{tr} = 1 \), then \( \Phi_{\text{end}} = \emptyset \), since \( S(\Phi_h) \cap S(\Phi_{\text{end}}) \) consists of center vertices of \( \Phi \). In other words, \( n_e = 0 \Rightarrow n_e = 0 \) and the corresponding consistency condition is: \( k = 0 \), \( m = 1 \Rightarrow t = 0 \) and hence \( t = p = 0 \), \( B = 0 \), i.e., \( J = (0,0,1,0,0,0) \) and \( \Phi_J \) is a triangle.

If, on the contrary, \( n_{tr} = 0 \), then \( n_e \neq 0 \) (see DT-table). To analyze this case, denote by \( n_{e,0} \) (resp., \( l_0 \)) the number of end vertices \( E \) of \( \Phi \) for which \( P_E = \emptyset \) (resp., the number of components \( p_i \) of \( p \) in an abstract card, which are equal to zero). Consider cases \( n_{e,0} = 0 \) and \( n_{e,0} \neq 0 \) separately.

If \( n_{e,0} = 0 \) and \( n_e = 1 \), then, obviously, \( \Phi = P_E \) where \( E \) is the unique end vertex of \( \Phi \). But \( P_E \) is a tee-cluster if its dimension is greater than 2. So, the corresponding card is \((0,1,0,[(0),(p),0])\) with \( p \geq 3 \).

If \( l > 1 \), then the family \( S_{\text{r}} \) is a tee-cluster (resp., \( l_0 \)) the family \( S_{\text{r}} \) consists of mutually disjoint s-twains \( \wedge_1, \ldots, \wedge_l \) rooted at some vertices \( E_1, \ldots, E_l \) and all hooks of the form \( [E_i,E_j|B_i] \in \Phi, i \neq j \), with \( B_i \) being a bottom vertex of \( \wedge_i \) (see also DT-table). \( E_1, \ldots, E_l \) are end vertices of \( S_{\text{r}} \). By lemma 8.10, they are stable if \( l > 1 \). So, if a tee \( \theta \) is compatible with \( S_{\text{r}} \) and its vertices belong to \( S(S_{\text{r}}) \), then the center of \( \theta \) is a bottom vertex a twain \( \wedge_i \). It implies that \( \theta \) is a rooted at \( E_i \) hook and hence belong to \( S_{\text{r}} \).

Now, by appropriately joining pyramids and bridges to the tee-cluster \( S_{\text{r}} \), we can construct a tee-cluster with arbitrary card of the form \((0,l,0,[(0),(p_1,\ldots,p_l),B])\), \( l > 1 \), \( p_i \geq 2 \), \( \forall i \) (see proposition 8.7).

Assume now that \( l_0 \neq 0 \), \( J = (0,l,0,[(0),(p_1,\ldots,p_l),B]) \) and the end vertices \( E_1, \ldots, E_l \) of \( \Phi_J \) are numbered in such a way that \( p_i = 0 \), if \( i \leq l_0 \), and \( p_i \geq 2 \), if \( i > l_0 \). If \( l - l_0 = 1 \), then \( \theta = [B,E_{l_0},E_1] \) with \( B \) being a bottom vertex of \( P_{E_{l_0}} \) is compatible with \( \Phi_J \) but does not belong to \( \Phi_J \). So, \( \Phi_J \) is not a tee-cluster. If \( l - l_0 = 1 \), then \( l \geq 3 \) and \( [E_{l-1},E_1|E_1] \notin \Phi_J \) is compatible with \( \Phi_J \). So, \( \Phi_J \) is not a tee-cluster in this case too. On the contrary, all end vertices of \( \Phi_J \) are stable if \( l - l_0 \geq 3 \). Moreover, similar arguments as above show that in this case \( \Phi_J \) a tee-cluster. In other words, \( l - l_0 \geq 3 \) is the consistency condition in the case when \( k = m = 0, l_0 \neq 0 \).

Thus cards of tee-clusters without center vertices are:

\[
\begin{align*}
(0,0,1,[(0),(0),0]) & \quad \text{(triangle),} \\
(0,1,0,[(0),(p),0]) & \quad \text{if } p \geq 3 \text{ (p - pyramid)} \\
(0,l,0,[(0),(p_1,\ldots,p_l),B]), & \quad \text{if } l > 1, \quad p_i \geq 2, \quad 1 \leq i \leq l; \quad (114) \\
(0,l,0,[(0),(0,\ldots,0,p_{l_0+1},\ldots,p_l),B]), & \quad \text{if } l - l_0 \geq 3, \quad p_i \geq 2, \quad i \geq l_0.
\end{align*}
\]

Denote by \( O_{s}S_{\text{r}} \), a tee-cluster whose card is \((0,r+s,0,[(0),(0,\ldots,0,2,\ldots,2),0])\) with \( l = r + s \) and \( l_0 = s \) and by \( P_{r+s} \) the \( k \)-dimensional pyramid. Then all tee-clusters from the above list, except the triangle, are obtained from \( P_{r+s} \), \( S_{\text{r}} \), \( l > 1 \), and \( O_{s}S_{\text{r}}, s > 0, r > 2 \), by joining to them pyramids and bridges.

Case II: \( k > 0, 0 \leq l \leq 3 \). This case is subdivided into four subcases, \( \mathbb{I}_0, \ldots, \mathbb{I}_3 \), according to the value of \( l \).
II₀. If \( l = 0 \Leftrightarrow \Phi_{\text{end}} = \emptyset \), then \( \Phi = \Phi_h \) is an \((m, k)\)-hedgehog.

II₁. If \( \mathcal{J} = (k, 1, 0, [(1), (1), 0]) \), then \( \Phi_\mathcal{J} \) is a tee-cluster. This is easily verified by a direct check using basic properties of treys. Denote the class of equivalent to it tee-clusters by \( \text{Pr}_1 \mathcal{P} \). End and center vertices of such a cluster are stable. Realizations of cards \((k, 1, 0, [(r), (s), 0]) \), \( r \geq 1, s \geq 1 \), are obtained by joining p-twains and pyramids to \( \text{Pr}_1 \mathcal{P} \). By proposition 8.7 they are tee-clusters. So, within the considered case it remains to check consistence of abstract cards with \((t) = (0)\) and \((p) = (0)\).

In the first of these cases \( \mathcal{J} = (k, 1, m, [(0), (p), 0]) \) and \((\Phi_\mathcal{J})_{\text{end}} \) is the \( p \)-pyramid together with tees of the form \([E, B[C_i]], i = 1, \ldots, k\), where \( E = E_1 \) and \( B \) is a bottom vertex of \( P_E \) (see (113)). If \( m = 0 \), then \( \Phi_\mathcal{J} = (\Phi_\mathcal{J})_{\text{end}} \) belongs to the rooted at \( E \) pyramid whose bottom vertices are those of \( P_E \) and also declared center vertices \( C_i \). Hence it is not a tee-cluster. Also, \( \Phi_\mathcal{J} \) is not a cluster if \( m > 0, p = 2 \).

Indeed, in this case the tee \( |B_1, B_2|E| \not\in \Phi_\mathcal{J} \) with \( B_1 \)'s being the bottom vertices of the twin \( P_E \) is compatible with \( \Phi_\mathcal{J} \). On the contrary, if \( m > 0, p \geq 2 \), then end and center vertices of \( \Phi_\mathcal{J} \) are stable and it is a tee-cluster.

In the second case \( \mathcal{J} = (k, 1, m, [(t), (0), 0]) \) and \( \Phi_\mathcal{J} \) is a tee-cluster iff \( t \geq 2 \). It is easily follows from the basic property of treys.

Thus the list of cards of tee-clusters in the considered case is:

\[
\begin{align*}
(k, 1, m, [(t), (p), 0]) & \quad k > 0, \quad m \geq 0, \quad t \geq 1, \quad p \geq 1; \\
(k, 1, m, [(0), (p), 0]) & \quad k > 0, \quad m > 0, \quad p \geq 3; \\
(k, 1, m, [(t), (0), 0]) & \quad k > 0, \quad m \geq 0, \quad t \geq 2.
\end{align*}
\]

II₂. Let \( \mathcal{J} = (k, 2, 0, [(0), (0), B]) \) with the \( 2 \times 2 \) bridge matrix \( B = \| b_{ij} \|, b_{12} = 2 \). Put \( \text{Br}_k = \Phi_\mathcal{J} \). By definition, \( \text{Br}_k \) contains a \((2, k)\)-multiped. The casing of each of two bridges that are contained in \( \text{Br}_k \) consists of \( 2k \) tees. By using lemmas 8.11 and 8.10, it is not difficult to verify that \( \text{Br}_k \) is a tee-cluster with stable end and center vertices. Now, by using the join operations and proposition 8.7, we see that realizatations of abstract cards of the form \((k, 2, m, [(t_1, t_2), (p_1, p_2), B]) \) with \( b_{12} \geq 2 \) are tee-clusters.

Similarly, a direct check shows that a realization of \((k, 2, 0, [(1), (0), 0]) \) is a tee-cluster with stable end and center vertices. Hence, by proposition 8.7, reallizations of abstract cards of the form \((k, 2, m, [(t_1, t_2), (p_1, p_2), B]) \) such that \( t_1 t_2 \geq 1 \) are tee-clusters. So, within the considered case we have to analyze abstract cards with \( b_{12} \leq 1, t_1 t_2 = 0 \).

First, assume that \( t = (t_1, 0), t_1 \geq 1 \). If \( \mathcal{J} = (k, 2, 0, [(t_1, 0), (p_1, p_2), B]) \) with \( p_2 < 2, b_{12} \leq 1 \), then \( \Phi_\mathcal{J} \) is not a tee-cluster. Indeed, observe that in this case \( C_{E_2} \) does not exist so that \( p_2 \neq 1 \) and hence \( p_2 = 0 \). If \( b_{12} = 0 \), then the tee \( |E_1, B[E_2]| \) with \( B \) being a bottom vertex of a rooted at \( E_1 \) p-twain in \( \Phi_\mathcal{J} \) is compatible with \( \Phi_\mathcal{J} \) but does not belong to it. If \( b_{12} = 1 \), then such is the tee \( |E_1, C[E_2]| \) with \( C \) being the center of the unique connecting \( E_1 \) and \( E_2 \) bridge in \( \Phi_\mathcal{J} \). This proves that \( p_2 \geq 2 \).

On the other hand, if \( \mathcal{J} = (k, 2, 0, [(1), (0), (0, 2), B]), b_{12} \leq 1 \), then \( \Phi_\mathcal{J} \) is a tee-cluster with stable end and center vertices. Now proposition 8.7 shows that realizatations of abstract cards \((k, 2, m, [(t_1, 0), (p_1, p_2), B]) \) with \( k \geq 1, t_1 \geq 1, p_2 \geq 2 \) are tee-clusters.
Now it remains to examine abstract cards with \( t = (0,0), b_{12} \leq 1 \). First of all, observe that a realization of the abstract card \((k, 2, 0, [(0,0), (2,2), 0])\) is a tee-cluster with stable end and center vertices. By the same arguments as earlier this implies that realizations of cards \((k, 2, m, [(0,0), (p_1, p_2), B])\) with \( p_i \geq 2, i = 1, 2 \), are tee-clusters too. So, the next step is to examine realization of abstract cards \( \mathcal{J} = (k, 2, m, [(0,0), (p_1, p_2), B])\) with \( p_1 \geq 2, p_2 \leq 1 \). Since \( t_2 = 0 \), equality \( p_2 = 1 \), i.e., existence of the vertex \( C_{E_2} \) in \( \Phi_{\mathcal{J}} \), is possible, iff \( b_{12} = 1 \). But, as earlier, the tee \([E_1, C|E_2]\) \( \notin \Phi_{\mathcal{J}} \) with \( C \) being the center of the unique bridge in \( \Phi_{\mathcal{J}} \) is compatible with \( \Phi_{\mathcal{J}} \). So, in this case \( \Phi_{\mathcal{J}} \) is not a tee-cluster. If \( p_2 = 0 \), then the tee \([E_1, B|E_2]\) \( \notin \Phi_{\mathcal{J}} \) with \( B \) being a bottom vertex of \( PE_i \) is compatible with \( \Phi_{\mathcal{J}} \), so that \( \Phi_{\mathcal{J}} \) is not a tee-cluster in this case too.

The same arguments show that realizations of abstract cards with \( p_i \leq 1, i = 1, 2 \), and \( b_{12} = 1 \) are not tee-clusters. If \( b_{12} = 0 \), then \( p_1 = p_2 = 0 \), as we have observed earlier. In this case \( m = 0 \) is, obviously, impossible, while realization of the abstract card \((k, 2, m, [(0,0), (0,0), 0])\), \( m > 0 \), is a \((2,k|m)\)-hybrid. A peculiarity of this cluster is that its ends are not stable.

Thus cards of tee-clusters in the considered case are:

\[
\begin{align*}
(k, 2, m, [(t_1, t_2), (p_1, p_2), B]), & \quad b_{12} \geq 2; \\
(k, 2, m, [(t_1, t_2), (p_1, p_2), B]), & \quad t_1 t_2 \neq 0; \\
(k, 2, m, [(t_1, 0), (p_1, p_2), B]), & \quad t_1 \geq 1, p_2 \geq 2, b_{12} \leq 1; \\
(k, 2, m, [(0, 0), (p_1, p_2), B]), & \quad p_1 \geq 2, p_2 \geq 2, b_{12} \leq 1; \\
(k, 2, m, [(0, 0), (0, 0), 0]), & \quad m > 0.
\end{align*}
\]

\( \mathbb{I}_3 \). The \((k,3)\)-ped \( \Phi_{mp} \) contained in a tee-cluster \( \Phi \) with \( n_e = k > 0, n_c = 3 \) is not a tee-cluster. Center vertices of \( \Phi_{mp} \) are, obviously, stable, while the end ones are not. Indeed, if \( E_i, i = 1, 2, 3 \), are end vertices of \( \Phi \) and, therefore, of \( \Phi_{mp} \), then tees \( \theta_{rs|t} = [E_r, E_s|E_t], \{r,s,t\} = \{1,2,3\} \), being compatible with \( \Phi_{mp} \) do not belong to it. Moreover, they do not belong to \( \Phi \), since, otherwise, \( E_i \)'s the would not be end vertices. So, \( \Phi \) must contain tees that block \( \theta_{rs|t} \)'s. But a tee blocking \( \theta_{rs|t} \) must be rooted at \( E_i \) and have a mixing second end. According to DT-table this happens if \( \Phi \) contains at least one of the following structure group rooted at \( E_i \): a pyramid (possibly "collapsed"), a p-twain, a bridge. Moreover, a bridge connecting \( E_s \) and \( E_t \) simultaneously blocks \( \theta_{rs|t} \) and \( \theta_{rs|s} \). The following abstract cards describe all minimal combinations of these groups, which simultaneously block all \( \theta_{rs|t} \)'s:

\[
\begin{align*}
(k, 3, 0, [(0, 0), (0, 0), 0], B), & \quad b_{12} = b_{13} = 1, b_{23} = 0; \\
(k, 3, 0, [(\varepsilon, 0, 0), (1 - \varepsilon, 0, 0), B]), & \quad b_{12} = b_{13} = 0, b_{23} = 1, \varepsilon = 0, 1; \\
(k, 3, 0, [(\varepsilon_1, \varepsilon_2, \varepsilon_3), (1 - \varepsilon_1, 1 - \varepsilon_2, 1 - \varepsilon_3), 0]), & \quad \varepsilon_1, \varepsilon_2, \varepsilon_3 = 0, 1.
\end{align*}
\]

A simple direct check shows that realizations of these abstract cards are, in fact, tee-clusters with stable center and end vertices. Now proposition 8.7 allows to obtain the full list of tee-clusters in the considered case in which is assumed that \( t = (t_1, t_2, t_3), p = (p_1, p_2, p_3) \) and \( B = [b_{ij}], 1 \leq i, j \leq 3 \).

\[
(k, 3, m, [(t, p, B)]) \quad \text{with} \quad \begin{cases}
(1) & b_{12} > 0, b_{13} > 0, b_{23} = 0. \\
(2) & b_{23} > 0, t_1 \geq \varepsilon, p_1 \geq 1 - \varepsilon, \varepsilon = 0, 1. \\
(3) & t_i \geq \varepsilon, p_i \geq 1 - \varepsilon, \varepsilon_i = 0, 1.
\end{cases}
\]
8.10. **Classification of tee-clusters. Generators.** By summing up the results of subsequences 8.8 (corollary 8.4) and 8.9 we get the following description of tee-clusters.

**Theorem 8.1.** Tee-clusters are in one-to-one correspondence with their cards. An abstract card \( (k, l, m, [t, p, B]) \) is the card of a tee-cluster if and only if \( k > 0, l > 3, \) or it belongs to one of lists (115), (116), (117).

An alternative way to describe tee clusters is as follows. Let \( \Phi \) be a tee-cluster, \( C(\Phi) = (k, l, m, [t, p, B]) \). Denote by \( \langle \Phi \rangle \), or, equivalently, by \( \langle k, l, m | t, p, B \rangle \) the set of equivalence classes of tee-clusters that are obtained by successively applying to \( \Phi \) join operations. We shall say that they are equivalence classes of tee-clusters generated by \( \Phi \). The totality \( \mathcal{T}_h \) of equivalence classes of all tee-clusters can be described by indicating a base of it, i.e., a system of tee-clusters \( \Phi_\alpha \) such that \( \mathcal{T}_h = \cup \langle \Phi_\alpha \rangle \). One of such bases, which is easily extracted from the above description of tee-clusters (see lists (115), (116), (117)), is the following. In the list below is assumed that \( k > 0, m > 0, r > 1 \).

\[
\begin{align*}
\text{Tr} : & (0, 0, 1, [0, 0, 0], 0, 0, 0), & \text{(triangle);} \\
\text{Pr}_3 : & (0, 1, 0, [0, 0], 0, 0), & \text{(3 - pyramid);} \\
\text{St}_r : & (0, r, 0, [0, 0, 0], 0, 0), & \text{r} \geq 3, \text{s} > 0 (\text{rooted} r - \text{twain}); \\
\text{O}_{r+s} \text{St}_r : & (0, r + s, 0, [0, 0, 0], (0, 0, 0), (2, 2, 2), 0), & \text{r} \geq 3, \text{s} > 0 (\text{rooted} r - \text{twain}); \\
\text{Hg}_1^k : & (k, 0, 1, [0, 0, 0]), & ((k, 1) - \text{hedgehog}); \\
\text{Pr}_1 \text{Pr}_3^k : & (k, 1, 0, [0, 0], (0), (3), 0), & (k - \text{trey cluster}); \\
\text{S}^k \text{Pr}_3 : & (k, 1, 1, [0, 0], (0), (3), 0), & (k - \text{trey cluster}); \\
\text{Pt}_3^k : & (k, 1, 1, [0, 0], (0), (0), (3)), & (k - \text{trey cluster}); \\
\text{Br}_k : & (k, 2, 0, [0, 0], (0, 0), (B)), & \text{b}_{12} = 2, \\
(k, 2, 0, [0, 1], (0, 0), (0), (B)), & \text{b}_{12} = 2, \text{b}_{13} = 1, \text{b}_{23} = 0; \\
(k, 2, 0, [1, 0], (0, 0), (0), (B)), & \text{b}_{12} = 2, \text{b}_{13} = 0, \text{b}_{23} = 1, \varepsilon = 0, 1; \\
(k, 2, 0, [0, 0], (2, 2), (0), (B)), & \varepsilon_1, \varepsilon_2, \varepsilon_3 = 0, 1; \\
(k, 2, 0, [0, 0], (0, 0), (0), (B)), & \text{b}_{12} = 2, \text{b}_{13} = 1, \text{b}_{23} = 0; \\
(k, 3, 0, [0, 0, 0], (0, 0, 0), (B)), & \text{b}_{12} = 2, \text{b}_{13} = 0, \text{b}_{23} = 1, \varepsilon = 0, 1; \\
(k, 3, 0, [(\varepsilon, 0, 0), (1 - \varepsilon, 0, 0)], (B)), & \varepsilon_1, \varepsilon_2, \varepsilon_3 = 0, 1; \\
(k, 3, 0, [(\varepsilon_1, \varepsilon_2, \varepsilon_3), (1 - \varepsilon_1, 1 - \varepsilon_2, 1 - \varepsilon_3), (0)], & \varepsilon_1, \varepsilon_2, \varepsilon_3 = 0, 1; \\
\text{Mp}_k^l : & (k, l, 0, [0, 0], (0, 0), (0, 0), (0)), & l > 3. 
\end{align*}
\]

**Example 8.3.** (4- and 5-dimensional tee-clusters.) The following lists of 4- and 5-dimensional tee-clusters are easily extracted from the above description.
4-dimensional tree-clusters: $\text{Hg}_1^1$ ((1,1)-hedgehog), $\text{Pr}_3$ (3-pyramid).

5-dimensional tree-clusters: $\text{Hg}_2^1$ ((2,1)-hedgehog), $\text{Pr}_4$ (4-pyramid), $\text{Pt}_1^1\text{O}$ (1-trey cluster), $\text{Br}_2$ (2-bridge cluster), $\text{Mp}_4^1$ (4-ped).

8.11. Coaxial Lie algebras associated with tee-clusters. Now we are ready to answer the question: what are coaxial Lie algebra associated with tee-clusters. The following observation is useful to this end.

**Lemma 8.12.** Let $\mathfrak{g}$ be a Lie algebra associated with a compatible tee-family $\Phi$, $S_i \subset S(\Phi)$, $i=1,2$, and $V_i$ the subspace of $|\mathfrak{g}|$ spanned by $S_i$. Then

1. The subspace $[V_1, V_2] = \text{Span}\{[v_1, v_2] | v_i \in V_i, i=1,2\}$ of $|\mathfrak{g}|$ is spanned by center vertices of all tees $[E_1, E_2 | C] \in \Phi$ such that $E_i \in S_i$.
2. The subspace $V$ of $|\mathfrak{g}|$ spanned by a subset $S \subset S(\Phi)$ is a subalgebra (resp., an ideal) of $\Phi$, if the center vertex of any tee $\theta \in \Phi$ with ends in $S$ (resp., with one end in $S$) also belongs to $S$.
3. $|[\mathfrak{g}, \mathfrak{g}]|$ belongs to the subspace spanned by center and mixing vertices of $\Phi$.

**Proof.** The first assertion is obvious, while the others are immediate consequences of it.

**Proposition 8.8.** Let $\mathfrak{g}$ be a coaxial Lie algebra such that $\Phi = \Phi_\mathfrak{g}$ is a tee-cluster. Then

1. The subspace of $|\mathfrak{g}|$ spanned by $S(\Phi_h)$ (resp., by $S(\Phi_{\text{end}})$) supports an ideal $\mathfrak{h}$ (resp., $\mathfrak{r}$) of $\mathfrak{g}$.
2. $[\mathfrak{h}, \mathfrak{r}] = 0$;
3. $\mathfrak{u} = \mathfrak{h} \cap \mathfrak{r}$ is a central ideal of $\mathfrak{g}$ whose support is the subspace spanned by center vertices of $\Phi$;
4. The quotient algebra $\mathfrak{g}/\mathfrak{r} = \mathfrak{h}/\mathfrak{u}$ is the direct sum of 3-dimensional simple Lie algebras, associated with triangles contained in $\Phi$;
5. The second derived ideal $\mathfrak{r}^{(2)} = [\mathfrak{r}^{(1)}, \mathfrak{r}^{(1)}]$ of $\mathfrak{r}$ is abelian; $\mathfrak{r}^{(1)} = [\mathfrak{r}, \mathfrak{r}]$ is abelian if $\Phi$ does not contain p-twains.
6. $\mathfrak{r}$ is the radical of $\mathfrak{g}$.

**Proof.** Assertions (1)-(4) directly follow from lemma 8.12. By the third assertion of this lemma, $|[\mathfrak{r}, \mathfrak{r}]|$ belongs to the subspace spanned by center and mixing vertices of $\Phi_{\text{end}}$. Therefore, $|[\mathfrak{r}^{(1)}, \mathfrak{r}^{(1)}]|$ belongs to the subspace spanned by centers of ee tees contained in $\Phi_{\text{end}}$. But, according to DT-table, ee tees form the ee casing of p-twains contained in $\Phi$, and hence their centers are center vertices of $\Phi$. This shows that $[\mathfrak{r}^{(1)}, \mathfrak{r}^{(1)}] \subset \mathfrak{u}$. In particular, $[\mathfrak{r}^{(1)}, \mathfrak{r}^{(1)}] = \{0\}$, if $\Phi$ does not contain p-twains. This proves assertion (5). Finally, the last assertion directly follows from (4) and (5) ones.

**Corollary 8.5.** Let $\mathfrak{g}$ be a coaxial Lie algebra and $\mathfrak{r}$ the radical of it. Then $\mathfrak{r}^{(3)} = 0$ and the semisimple part of $\mathfrak{g}$ is isomorphic to a direct sum of 3-dimensional simple Lie algebras.

**Proof.** This is a direct consequence of proposition 8.8 and the fact that $\Phi_\mathfrak{g}$ is contained in a tee-cluster.
9. Generic clusters

Let \( \Phi \) be a compatible family of base structures. Then \( \Phi = \Phi_0 \cup \Phi_1 \) where \( \Phi_0 \) is a tee-family and \( \Phi_1 \) is a dee-family. A cluster will be called generic, if \( \Phi_0 \neq \emptyset, \Phi_1 \neq \emptyset \). The previous analysis of dee- and tee-clusters naturally extends to generic ones whose structure will be described in this section.

9.1. Framed tee-clusters. A natural question is: what are generic clusters one can construct by adding some dees to a tee-cluster. To answer it we need the following

**Lemma 9.1.** Let \( \Phi \) be a compatible family and \( \varrho \in \Phi_1 \). Then

1. If the origin of \( \varrho \) belongs to \( S(\Phi_0) \), then it is an end vertex of \( \Phi_0 \);
2. If the end of \( \varrho \) belongs to \( S(\Phi_0) \), then the origin of \( \varrho \) also belongs to \( S(\Phi_0) \);
3. If \( \Phi_0 \) is a tee-cluster, then the end of \( \varrho \) can not be a center vertex of \( \Phi_0 \);
4. If \( \Phi_0 \) is a tee-cluster, then the end of \( \varrho \) is neither a bottom vertex of a \( p \)-twain in \( \Phi_0 \), nor a center vertex of a bridge.

**Proof.** First two assertions are direct consequences of proposition 8.1 and definitions. The third one follows from the fact that a dee whose end is the center of a cross (resp., a tripod) is not compatible with this cross (resp., the tripod). Indeed, if \( C \) is a center vertex of a tee-cluster \( \Psi \), then \( \Psi \) contains a cross, or a tripod with the center at \( C \) as one can see from the description of tee-clusters. Concerning the last assertion, observe that a dee whose end is a side vertex of a trey is not compatible with this trey. But a \( p \)-twain is contained in a trey belonging to \( \Phi_0 \). Finally, the center of a bridge is manifestly impossible by proposition 8.1. \( \square \)

Diagrams of doubles, rotators and spiders are illustrated in Fig. 6.

**Fig. 7.** Doubles, rotators and spiders.

Let \( \Psi \) be a tee-cluster. Add to it all dees of the form \([E|B]\) with \( E \) being an end vertex of \( \Psi \) and \( B \) a bottom vertex of \( P_E \) \( (B = C_E, \text{ if } P_E = \emptyset) \). The so-obtained family, denoted by \( \tilde{\Psi} \), is, obviously, compatible. It will be called a framed tee-cluster. It will be shown (proposition 9.3) that a framed tee-cluster, which is different from a \((k,2|m)\)-hybrid, is also a cluster.

We stress that the framed twain is a 3-dimensional cluster. So, a framed \( k \)-pyramid is a cluster, if \( k \geq 2 \). The framing of a framed \( k \)-pyramid \( \tilde{P} \) is the \((1,k)\)-spider, which is formed by all dees \([E|B]\) where \( E \) and \( B \) are top and bottom vertices of \( P \), respectively.

Framed tee-clusters do not exhaust generic ones as the following two examples show.

**Rotator.** An \( m \)-rotator, \( m \geq 0 \), is a family equivalent to

\[ \{[e_1|e_2], [e_2|e_1], [e_1, e_2|e_3], \ldots, [e_1, e_2|e_{m+2}] \}. \]
It is easily verified that an $m$-rotator is a generic cluster if $m > 0$. The double $\{[e_1|e_2], [e_2|e_1]\}$ is its axis, $e_1, e_2$ are its ends and $e_i$’s, $i = 3, \ldots, m + 2$, are its thorns.

**Basic properties of rotators.** No dee is nontrivially compatible with a rotator. A tee $\theta = [E_1, E_2|C]$ is nontrivially compatible with a rotator $r$ if either $E_1, E_2$ are ends of $r$, or $C$ is a thorn of $r$, which is the unique common vertex of $\theta$ and $r$.

**D-bridge.** This is a family equivalent to $\{[e_1|e_3], [e_2|e_3], [e_1, e_2|e_3]\}$. It is, obviously, a (generic) cluster. We shall refer to $e_1, e_2$ as the ends of this d-bridge, to $e_3$ as its center and to $[e_1, e_2|e_3]$ as its axis. We shall say also that a d-bridge connects its ends. $\text{Bd}$ will be standard notation of a d-bridge.

**Basic properties of d-bridges.** If one vertex of a tee/dee, which is compatible with a d-bridge $\text{Bd}$, is the center vertex of $\text{Bd}$, then this tee/dee belongs to $\text{Bd}$.

**Raft.** Let $\mathcal{R} = \{\text{Bd}_1, \ldots, \text{Bd}_k\}$ be a compatible family of d-bridges whose end vertices belong to a subset $\{E_1, \ldots, E_l\}$ of base vectors. Obviously, $2 \leq l \leq 2k$.

The family $\mathcal{R}$ is compatible iff center vertices of d-bridges $\text{Bd}_i$’s differ each other.

The union $\text{Bd}_1 \cup \cdots \cup \text{Bd}_k$ is called a $(k,l)$-raft. If $l$ differs from 2 and $2k$, then there are more than one nonequivalent $(k,l)$-rafts. By basic property of d-bridges, a raft is a cluster if its graph is connected. Such a cluster will be called a raft cluster.

Let $\mathcal{R}$ be as above and $\{C_1, \ldots, C_m\} \cap S(\mathcal{R}) = \emptyset$. A suspended from $\{C_1, \ldots, C_m\}$ raft (shortly, s-raft) is the union of $\mathcal{R}$ and the $(l,m)$-ped whose center vertices are $C_i$’s and the end vertices are those of $\mathcal{R}$. As it is easy to see, a suspended raft is a cluster, if $l > 2$. Diagrams of d-bridges and rafts are shown in Fig. 8.

Fig. 8. D-bridges and rafts.

9.2. Types of dees in a generic cluster. Now we pass to a systematic study of generic clusters by starting with the necessary terminology. Let $\Phi$ be a compatible family. A vertex $e_i \in S(\Phi)$ will be called an end vertex of $\Phi$, if $e_i$ is either or both the origin of a dee $\varrho \in \Phi_1$ and an end vertex of $\Phi_0$. Assertion (1) of Lemma 9.1 guarantees correctness of this terminology. A center vertex of $\Phi_0$, which is not a vertex of $S(\Phi_1)$, will be called a $t$-center vertex of $\Phi$. Similarly, the end of a dee in $\Phi_1$, which does not belong to $S(\Phi_0)$, will be called a $d$-center vertex of $\Phi$.

**Lemma 9.2.** Let $\Phi$ be a generic cluster. Then it holds:

1. If a double belongs to $\Phi_1$, then it is the axis of the belonging to $\Phi$ rotator whose thorns are $t$-center vertices of $\Phi$.
2. If $E$ and $D$ are an end and an $d$-center vertices of $\Phi$, respectively, then the $\text{dee } [E|D]$ belongs to $\Phi$.
3. If $E_1, E_2$ are end vertices of $\Phi$, while $C$ is an $t$-center of it, then the $\text{tee } [E_1, E_2|C]$ belongs to $\Phi$.

**Proof.** A direct consequence of proposition 8.1. \hfill $\square$

Denote by $\Phi^{db}_1$ (resp., $\Phi^{br}_1$) the union of all doubles (resp., dees in d-bridges) that belong to $\Phi$. A dee $\varrho \in \Phi$ will be called immersed if its vertices belong to $S(\Phi_0)$ and
A spike (resp., a poker) is a family equivalent to \{[e_i, e_j], [e_i, e_j, e_k]\} (resp., \{[e_i, e_j, e_k, e_l]\}). Obviously, an immersed dee belongs to at least one spike or poker contained in \(\Phi\). Denote by \(\Phi^{sp}\) the set of all immersed dees in \(\Phi\) and put \(\Phi^p = \Phi \setminus (\Phi^{im} \cup \Phi^{db} \cup \Phi^{lr})\). So, \(\Phi^p\) is the disjoint union
\[
\Phi^p = \Phi^{im} \cup \Phi^{db} \cup \Phi^{sp} \cup \Phi^{lr}.
\]

Obviously, the end of a dee in \(\Phi^p\) is a d-center vertex, and, as it follows from lemma 9.2 (2), \(\Phi^{sp}\) is a \((k,l)\)-spider (see subsection 8.3) where \(k\) (resp., \(l\)) is the number of end (resp., d-center) vertices of \(\Phi\).

**Lemma 9.3.** Let \(\Phi\) be a generic cluster with \(k\) end vertices. Then \(k \geq 3\), if \(\Phi\) has at least one d-center vertex.

**Proof.** First, assume that \(k = 2\). Let \(E_1, E_2\) be end vertices of \(\Phi\). If \(D\) is a d-vertex of \(\Phi\), then \([E_i[D], i = 1, 2\), are the only dees in \(\Phi\) that have \(D\) as one of their vertices. On the other hand, \(\theta = [E_1, E_2][D]\) is the only tee, which is compatible with these dees and has \(D\) as one of its vertices. This shows that \(\theta\) is compatible with \(\Phi\) and, therefore, belongs to \(\Phi\) in contradiction with the assumption that \(D\) is a d-center vertex.

Now assume that \(k = 1\). Let \(E\) be the end vertex and \(D\) one of d-center vertices of \(\Phi\). So, by lemma 9.1, \([E][D]\) is the unique dee in \(\Phi\) with the end at \(D\). If \(C\) is an \(t\)-center vertex of \(\Phi\), then, as it is easy to check, the tee \([E, D[C]]\) is compatible with \(\Phi\) and hence belongs to \(\Phi\). This is, however, impossible, since \(D\) is a d-center vertex. So, \(\Phi\) has no \(t\)-center vertices. Consider now a rooted at \(E\) tee \(\theta = [E, P][Q] \in \Phi\). Then the tee \(\vartheta = [E, D][Q]\) is compatible with \(\Phi\). Indeed, \(\vartheta\) can not be blocked by a dee \(\sigma\) in \(\Phi\), since its origin is at \(E\) by the assumption \(k = 1\). On the other hand, \(\vartheta\) can be blocked only by a tee \(\varrho \in \Phi\) one whose ends is at \(Q\) and which is not rooted at \(E\), i.e., \(\varrho = [P, Q][R] \in \Phi\), \(R \in S(\Phi)\). Since \(k = 1\), \(P\) is not the end vertex of a dee in \(\Phi\) (lemma 9.1). Moreover, \(P\) is not an end vertex of \(\Phi\), since \(P \neq E\). This implies that there is a tee \(\rho \in \Phi\) with the center at \(P\). But the unique tee, which is compatible with \(\theta\) and \(\varrho\), is \(\rho = [E, Q][P]\). Since \(\{\theta, \varrho, \rho\}\) is a trey, the center vertex \(R\) of it is an \(t\)-center vertex of \(\Phi\). But, as it was already proved, this is impossible.

**Corollary 9.1.** Let \(\Phi\) be a cluster. Then it holds:

1. If \(\Phi^{sp} \neq \emptyset\), then the number \(k\) of end vertices of \(\Phi\) is greater than 2 and \(\Phi^{sp}\) is a \((k,l)\)-spider with \(l\) being the number of d-center vertices of \(\Phi\).
2. If the center of a \((2,1)\)-spider \(\Sigma \subset \Phi^1\) is not a d-center vertex of \(\Phi\), then \(\Sigma\) belongs to an unique \(d\)-bridge contained in \(\Phi\).

**Proof.** The first assertion is a direct consequence of lemma 9.2 (2) and lemma 9.3. Next, the center of \(\Sigma\) is a vertex of a tee \(\theta \in \Phi\). But being compatible with \(\Sigma\) the tee \(\theta\) has common ends with \(\Sigma\).

The following lemma clarifies the status of immersed dees.

**Lemma 9.4.** Let \(\Phi\) be a cluster, \(E\) be an end vertex of \(\Phi\) and \(\varrho = [E][D] \in \Phi^{im}\). Then

1. \(\varrho\) belongs to a contained in \(\Phi\) poker.
2. If \(A \in S(\Phi), A \neq E\), is an end vertex of \(\Phi\), or a bottom vertex of a rooted at \(E\) twain in \(\Phi_0\), or the center of a bridge in \(\Phi_0\) with one end at \(E\), then \([E, A][D] \in \Phi\).
(3) If \( \theta_i = [E|D_i] \in \Phi_1^{im} \), \( i = 1, \ldots, m \), then \( D_1, \ldots, D_m \) are bottom vertices of the (possibly, "collapsed") pyramid \( P_E \subset \Phi_0 \).
(4) If \( \rho \) is the unique tee in \( \Phi_1^{im} \), which is rooted at \( E \), then \( D = C_E \) and vice versa.

Proof. (1). Since a tee in \( \Phi_1^{im} \) belongs either to a spike, or to a poker, only the former case is to be examined. Let \( \theta = [E, D|B] \in \Phi \) complements \( \rho \) to a spike. Since \( \rho \notin \Phi_1^{db} \), the tee \( \rho' = [D|E] \) is blocked. As it is easy to see, \( \rho' \) can be blocked either by a tee forming a poker with \( \rho \), or by a rooted at \( E \) tee whose second end \( A \) is different from \( D \), or by a tee with the origin at \( E \). In the second of these cases let \( \theta = [E, A|C] \in \Phi \) be a blocking \( \rho' \) tee. Then a direct check shows that the tee \( [E, A|D] \), which forms a poker with \( \rho \), is compatible with any tee/dee, which is compatible with tees \( \rho, \theta \) and \( \theta \). Hence \( [E, A|D] \) belongs to \( \Phi \). In the third case let \( \sigma = [E|D'] \in \Phi \) be a tee, which blocks \( \rho' \), and \( \rho = [E, D'|D] \). Observe now two facts. First, any tee whose vertices are \( E, D, D' \), which is incompatible with the family \( \Sigma = \{ \rho, \theta, \sigma \} \), is rooted at \( E \) and hence does not block \( \rho \). Second, a tee, which is compatible with \( \Sigma \) but not compatible with \( \rho \), is of the form \( \sigma' = [E'|D'], E \neq E' \). So, the tee \( \rho \), which forms a poker with \( \rho \), can be blocked only by such a tee belonging to \( \Phi \). This proves the assertion assuming that \( \Phi \) does not contain tees of this kind. If, on the contrary, \( \{ \sigma, \sigma' \} \subset \Phi \), then, according to corollary 9.1, either \( D \) is a d-center vertex or \( \{ \sigma, \sigma' \} \) belongs to a d-bridge in \( \Phi \). In each of these cases \( \Phi \) has more than one end vertices (see lemma 9.3). If \( E' \neq E \) is a such one, then the tee \( [E, E'|D] \), which forms a poker with \( \rho \), is not blocked and hence belongs to \( \Phi \).

(2) By using basic property of twains and lemma 9.3, (2), one easily verifies that \( \Phi \) does not contain tees/dees that block \([E, A|D]\).

(3) If \([E|D], [E|D'] \in \Phi_1^{im}\), then, by the same reasons as previously, is easy to verify, the tee \([E, D'|D]\) is not blocked and hence belongs to \( \Phi \). This shows that the rooted at \( E \) pyramid with bottom vertices \( D_1, \ldots, D_m \) belongs to \( \Phi \). Moreover, if \( D \) is a bottom vertex of a rooted at \( E \) s-twain in \( \Phi_0 \), then, obviously, \([E|D] \in \Phi \). These two facts proves the assertion.

(4) A direct check by using proposition 8.1 and assertion (1).

The tee described in lemma 9.4, (4), will be called a single (rooted at \( E \)). A single may be thought as the framing of a “collapsed pyramid”. According to lemma 9.4 a rooted at \( E \) single belongs to a nonzero number of rooted at \( E \) pokers, which belong to \( \Phi \).

Corollary 9.2. Let \( \Phi \) be a cluster and \( E_1, \ldots, E_m \) end vertices of it. All s-twains \( \wedge \in \Phi_0 \) rooted at \( E_i \) form a (possibly empty) pyramid \( P_{E_i} \). The framing \( \Phi_1^{im} \) of \( P_{E_i} \) belongs to \( \Phi \). \( \Phi_1^{im} \) is disjoint union of framings \( \Phi_1^{im} \) and singles that correspond to end vertices \( E_j \) such that \( p_{E_j} = 1 \).

The following lemma describes how a poker \( \Pi \in \Phi \) is attached to the rest of \( \Phi \).

Lemma 9.5. Let \( \Pi = [E|B], [E, C|B] \in \Phi \) and \([E|B] \in \Phi_1^{im} \). Then one of the following possibilities takes place:

(1) \( \Pi \subset \overline{P}_E \).
(2) \( C \) is an end vertex of \( \Phi \).
(3) \( C \) is a side vertex of a rooted at \( E \) trey.
(4) \( C \) is the center vertex of a bridge \( [E, E'|C] \in \Phi \). If \( |C, E'|B'| \in \Phi \), then either \( |E'|B'| \in \Phi \), or \( B' \) is a center vertex of \( \Phi \).

**Proof.** If \( \Pi \notin \hat{P}_E \), then the dee \( g = [E|C] \) is blocked. But all other dees, which are compatible with \( \Pi \), are of the form \( [C|B'] \). So, \( C \) is an end vertex of \( \Phi \) if \( g \) is blocked by such a dee. Let now \( \theta \) be a compatible with \( \Pi \) tee, which blocks \( g \). A simple prove by exhaustion shows that \( \theta \) must be of the form \( [C, E'|B'] \) with vertices \( E', B' \) not belonging to \( \{E, B, C\} \). The vertex \( C \) is either an end or a mixing vertex of \( \Phi_0 \). In the first case \( C \) is an end vertex of \( \Phi \) too. Indeed, otherwise, \( C \) would be the end vertex of a dee belonging to \( \Phi \). But any such dee is incompatible with \( \Pi \cup \{\theta\} \). So, it remains to analize the second of these possibilities only.

In this case \( \Phi_0 \) contains a tee \( \rho \) whose center vertex is \( C \). The only such tee, which is compatible with \( \Pi \cup \{\theta\} \), is \( \rho = [E, E'|C] \). Note that a tee is incompatible with \( \Pi \cup \{\theta, \rho\} \), if \( E' \) is the end vertex of it. By this reason, \( E' \) is an end vertex of \( \Phi \), if it is an end vertex of \( \Phi_0 \). So, in this case \( \rho \) is the bridge from assertion (4). Moreover, the second part of this assertion follows the fact that the dee \( [E'|B'] \) can be blocked either by the tee \( [E, C|B'] \) or by a tee which forms a cross with \( [E', C|B'] \).

Finally, if \( E' \) is a mixing vertex of \( \Phi_0 \), then there is a tee \( \sigma \in \Phi_0 \) with the center at \( E' \). But the only such one, which is compatible with \( \Pi \cup \{\theta, \rho\} \), is \( [E, C|E'] \). This tee completes \( \{[E, C|B'], [C, E'|B']\} \) up to a trey. \( \square \)

**Corollary 9.3.** A poker in a cluster belongs to a framed pyramid or to a d-bridge, or the belonging to it tee is a connective.

**Proof.** If in the notation of lemma 9.5 \( C \) is an end vertex, then \( \Phi \) may possess a dee of the form \( [C|B'] \). If \( B = B' \), then \( \Pi \) belongs to a d-bridge. Otherwise, the considered poker is as in lemma 9.5. \( \square \)

### 9.3. Join operations and cards of clusters.

The join operations defined earlier for tee-clusters can be applied also to arbitrary clusters. Their definitions remain literally the same except the pyramid join operation where pyramids we must be replaced by framed pyramids. Additionally, we have three new join operations we are passing to describe. Below \( \Phi \) stands for a cluster.

**Joining a rotator.** Add two new vertices \( A_1 \) and \( A_2 \) to \( S(\Phi) \) and consider the rotator \( r \) whose axis is \( [A_1|A_2] \) and thorns are t-center vertices of \( \Phi \). Then, by the basic property of rotators, \( \Phi \cup r \) is a cluster.

**Joining a d-bridge.** There are two versions of this procedure.

1) Consider the d-bridge \( \sim = \{[E_1|C], [E_2|C], [E_1, E_2|C]\} \) with \( E_1, E_2 \) being some end vertices of \( \Phi \) and \( C \notin S(\Phi) \). Then, by the basic property of d-bridges, \( \Phi \cup \sim \) is a cluster.

2) Let \( \sim \) be as above but \( S(\Phi) \cap S(\sim) = \{E_1\} \) with \( E_1 \) being an end vertex of \( \Phi \). By adding to \( \Phi \cup \sim \) new connects and all dees \( \{[E_2|C]\} \) with \( C \) running through d-center vertices of \( \Phi \), we get a cluster.

**Joining a spider.** Add a new vertex \( C \) to \( S(\Phi) \) and consider the spider \( S \) composed of all dees \( \{E|C\} \) with \( E \) running through end vertices of \( \Phi \). Then \( \Phi \cup S \) is a cluster and \( C \) is a d-center vertex of it, if the number of end vertices of \( \Phi \) is not less than 3.

These new join operations preserve original t-center and end vertices and do not create new ones except the second join bridge operation. In this case one new end
vertex comes out. These operations also commute each other as well as with old join operations. Moreover, we have

**Proposition 9.1.** Let $\Phi$ be a cluster such that $\Phi_0$ is a tee-cluster. Then $\Phi$ is obtained from the framed tee-cluster $\Phi_0$ by applying to it rotator, d-bridge and spider join operations.

**Proof.** First of all, observe that $\Phi_0 = \Phi_0 \cup \Phi_1^{im}$. So, $\Phi$ is obtained from the framed tee-cluster $\Phi_0$ by adding to it the spider $\Phi_1^{ip}$, all d-bridges corresponding to the forming $\Phi_1^{br}$ (2,1)-spiders (corollary 9.1, (2)) and all rotators whose axes are the composing $\Phi_1^{tr}$ doubles and thorns are t-center vertices of $\Phi$. $\square$

By summing up the previous results of this section and that of the preceding one we see that clusters are made of basic groups, their casings and connectives. Namely, denote by $n_t, n_d$ and $n_e$ numbers of t-center, d-center and end vertices of $\Phi$, respectively, and by $n_{tr}, n_r$ numbers of triangles and doubles in $\Phi$. The list of basic groups is as follows:

- $(n_{tr}, n_r)$-hedgehog $\Phi_h$,
- $(n_e, n_d)$-multipeded $\Phi_{mp}$,
- $(n_r, n_t)$-rotator $\Phi_{rt}$,
- $(n_e, n_d)$-spider $\Phi_{sp}$,
- $n_t$-treys,
- framed pyramids,
- bridges,
- d-bridges.

Here $\Phi_h = (\Phi_0)_h$. We also emphasize that the end and center vertices of $\Phi_{mp}$ are those of $\Phi$, and, similarly, for $\Phi_{rt}$ and $\Phi_{sp}$. Also, associate with $\Phi$ the s-raft $\Phi_{sf} \subset \Phi$ composed of all d-bridges belonging to $\Phi$, which are suspended from center vertices of $\Phi$.

$\Phi_{FW} = \Phi_h \cup \Phi_{mp} \cup \Phi_{rt} \cup \Phi_{sp}$ is the framework of $\Phi$. This part of $\Phi$ is well defined by numbers $n_t, n_d, n_e, n_{tr}$ and $n_r$, while basic groups in the right column of this list are attached to end vertices of $\Phi$ and their numbers may vary almost arbitrarily when the end vertices of $\Phi$ remain fixed. Let $E_1, \ldots, E_{n_e}$ be end vertices of $\Phi$. Put $t = (t_1, \ldots, t_{n_e})$ (resp., $p = (p_1, \ldots, p_{n_e})$) with $t_i$ (resp., $p_i$) being the number of $n_t$-treys rooted at $E_i$ (resp., the dimension of $P_{E_i}$). Here $P_{E_i}$ is the framed pyramid $P_{E_i} \in \Phi_0$. Denote by $b_{ij}$ (resp., $d_{ij}$) the number of bridges (resp., d-bridges) connecting $E_i$ and $E_j$. By putting $b_{ii} = d_{ii} = 0$ we have symmetric matrices $B = [b_{ij}]$ and $D = [d_{ij}]$. The card of a cluster $\Phi$ is

$$\text{Card}(\Phi) = (n_t, n_e, n_d, n_{tr}, n_r, [t, p, B, D]).$$

Here the meaning of the bracket $[\ldots]$ is as for tee-clusters (see subsection 8.8). The tee-part (resp., dee-part) of Card($\Phi$) is $C_t(\Phi) = (n_t, n_e, n_{tr}, [t, p, B])$ (resp., $C_d(\Phi) = (n_t, n_e, n_d, n_r, [p, B])$). Obviously, $C_t(\Phi) = C(\Phi_0)$ and

$$\dim \Phi = n_t + n_e + n_d + 3n_{tr} + 2n_r + \sum_{i=1}^{n_e} (p_i + 2t_i) + \sum_{1 \leq i < j \leq n_e} (b_{ij} + d_{ij}).$$

**Proposition 9.2.** Two clusters are equivalent if and only if their cards are equal.

**Proof.** If all basic groups composing a cluster $\Phi$ are known, then completing it casings and connectives are uniquely restored. The proof is essentially the same as that of proposition 8.6, and we omit the details. $\square$
Lemma 9.6. \( k \) assertion. The remaining ones are by a simple direct check.

Proposition 9.3. Let \( \Phi \) be a tee-generated cluster. Then either \( \Phi = \overline{\Phi_0} \cup \Phi_{sp} \), or \( \Phi_0 \) is a \( (n_t,2|n_r) \)-hybrid and \( \Phi = \Phi_0 \cup \{ \theta \} \) where \( \theta \) is the double whose ends are end vertices of \( \Phi_0 \).

Proof. First, note that \( \Phi \) does not contain d-bridges. Indeed, the axis of a d-bridge \( \text{Bd} \in \Phi \) belongs to \( \Phi_0 \). On the other hand, by the basic property of d-bridges, the center \( C \) of this axis, which is also the center of \( \text{Bd} \), is a center vertex of \( \Phi_0 \). The axis is the unique tee in \( \Phi_0 \) that passes through \( C \). But there are no tee-clusters with such a center vertex (see, for instance, the list of tee-clusters in subsection 8.10).

Second, if \( \Phi \) contains rotators \( r_1, \ldots, r_l \), then the ends \( E_{i1}, E_{i2} \) of the axis of \( r_i, i = 1, \ldots, l \), are end vertices of \( \Phi_0 \). If \( l > 1 \), then the tee \( |E_{11}, E_{12}|C| \) with \( C \) being a center vertex of \( \Phi_0 \) and \( i \neq j \) is compatible with \( \Phi_0 \) but does not belong to it. This proves that \( \Phi_0 \) is not a tee-cluster if \( l > 1 \). If \( l = 1 \), then the tee \( |E_{11}, E|C| \) with \( C \) being a center vertex of \( \Phi_0 \) is compatible with \( \Phi_0 \) but does not belong to it if \( E \) is an end vertex of \( \Phi_0 \), which is different from \( E_{11} \) and \( E_{12} \). So, \( \Phi_0 \) is not a tee-cluster, if \( n_e(\Phi_0) > 2 \). The last condition is, obviously, equivalent to \( n_e(\Phi) > 0 \). On the contrary, if \( n_r(\Phi) = 0 \) and \( l = n_r(\Phi) = 1 \), \( \Phi_0 \) is a tee cluster, namely, a \((n_e,2|n_r)\)-hybrid. In this case \( \Phi_{sp} = \Phi_1^{im} = \emptyset \).

Thus if \( n_r = 0 \), then \( \Phi_1 = \Phi_1^{im} \cup \Phi_1^{sp} \Leftrightarrow \Phi = \Phi_0 \cup \Phi_1^{im} \cup \Phi_1^{sp} = \overline{\Phi_0} \cup \Phi_{sp} \). To conclude the proof it remains to note that \( \overline{\Phi_0} \) is a cluster, if \( \Phi_0 \) is a tee-cluster and \( n_r = 0 \).

A direct consequence of the above proposition is:

Corollary 9.4. A tee-cluster \( \Psi \) is also a cluster if and only if it is not a \((k,2|m)\)-hybrid and \( p(\Psi) = 0 \).

If \( \Phi \) is a framed tee-cluster, then \( \Phi = \Phi_0 \cup \Phi_1^{im} \). The converse is not true as the following examples show.

Consider the compatible family \( \Psi_{k,l} \) composed of dees \( |e_i|e_{k+i}| \), \( 1 \leq i \leq k \), and tees \( |e_i,e_j|e_{k+i}| \) where \( 1 \leq i < j \), and \( j \in \{1, \ldots, k\} \cup \{2k+1, \ldots, 2k+l\} \). We also assume that \( k \geq 1 \) and \( l \geq 1 \), if \( k = 1 \).

Lemma 9.6. If \((k,l) \neq (1,1)\), then the family \( \Psi_{k,l} \) is included in a unique cluster, denoted \( \Psi_{k,l} \) such that \( S(\Psi_{k,l}) = S(\Psi_{k,l}) \). More exactly, we have:

1. \( \Psi_{k,l} = \Psi_{k,l} \), i.e., \( \Psi_{k,l} \) is a cluster, if \( k \geq 3 \).
2. \( \Psi_{1,l} = (l+1)\)-dimensional framed pyramid, and \( e_1 \) is its top vertex.
3. \( \Psi_{2,0} \setminus \Psi_{2,0} = \{ e_1, e_2 | e_3 \}, i.e., \Psi_{2,0} \) is composed of two d-bridges, which have common end vertices \( e_1 \) and \( e_2 \).
4. If \( l \geq 1 \), then \( \Psi_{2,l} \setminus \Psi_{2,l} = \{ e_1, e_2 | e_{k+l} \}, i.e., \Psi_{2,l} \) is the system of l bridges connecting \( e_1 \) and \( e_2 \), which are “suspended” on the corresponding connectives to singles \( |e_1, e_3| \) and \( |e_2, e_4| \).

Proof. If \( k \geq 3 \), then all end vertices of \( \Psi_{k,l} \) are stable. This proves the first assertion. The remaining ones are by a simple direct check.

It follows from this lemma that \( \Phi = \Psi_{k,l} \) is a cluster of the form \( \Phi = \Phi_0 \cup \Phi_1^{im} \). If \( k \geq 2 \) and \((k,l) \neq (2,0)\), then \( \Phi_0 \) is not a tee-cluster,
In the sequel we shall use the notation \( \Psi_{k,l} \) also for a cluster, which is equivalent to the above described model. Let \( n_i \geq 1, i = 1, \ldots, k, \) and \( p_i = (p_1, \ldots, p_k) \). The cluster denoted by \( \Psi_{k,l}(p') = \Psi_{k,l}(p_1, \ldots, p_k) \) is obtained from \( \Psi_{k,l} \) by applying to the latter \( (p_i - 1) \)-times the pyramid join operation at \( e_i \) for all \( i = 1, \ldots, k \).

The “suspended” version of \( \Psi_{k,l}(p') \) is obtained by adding to it \( m \) center vertices and the corresponding casings. It will be denoted by \( \hat{\Psi}_{k,l} \), or simply \( \Psi_{k,l}(m) \), if \( p_i = (1, \ldots, 1) \). For instance, \( \Psi_{1,1}(m), l > 1 \), is a suspended framed pyramid.

9.5. **Description of clusters.** As in the case of tee-clusters proposition 9.2 reduces classification of generic clusters to description of their cards. We subdivide this problem into four separate cases:

1. clusters \( \Phi \) with \( n_d > 0 \) \( \iff \) \( \Phi_{sp} \neq \emptyset \).
2. clusters \( \Phi \) with \( n_d = 0, n_{tr} + n_r > 0 \) \( \iff \) \( \Phi_{sp} = \emptyset, \Phi_h \cup \Phi_{rt} \neq \emptyset \).
3. clusters \( \Phi \) with \( n_d = n_{tr} = n_r = 0, n_t > 0 \) \( \iff \) \( \Phi_{sp} \cup \Phi_h \cup \Phi_{rt} = \emptyset \).
4. clusters \( \Phi \) with \( n_d = n_{tr} = n_r = n_t = 0 \) \( \iff \) \( \Phi_{FW} = \emptyset \).

According to this subdivision, clusters will be called of types I,...,IV, respectively.

**Type I.** Recall that \( n_e \geq 3 \), if \( n_d > 0 \) (lemma 9.3), i.e., \( \Phi_{sp} \) has at least 3 end vertices, which, at the same time, are end vertices of \( \Phi \). If \( n_t > 0 \), then \( \Phi_{mp} \neq \emptyset \). Any center vertex of the multipid \( \Phi_{mp} \) is the center vertex of a contained in \( \Phi_{mp} \) tripod and, therefore, is stable. By this reason, \( \Phi = \Phi_{mp} \cup \Phi_{sp} \) is a cluster, and \( \Phi \) is obtained from \( \Phi \) by means of suitable join operations. So, in this case, \( n_{tr}, n_r \) and \( m, p, B, D \) may be arbitrary.

If \( n_t = 0 \), then, obviously, \( n_{tr} = n_r = 0, t = 0 \). In this case, \( \Phi \) is obtained from \( \Phi = \Phi_{sp} \) by means of bridge, d-bridge and (framed) pyramid join operations. Since \( n_t = 0 \), a bridge connecting end points \( E_i \) and \( E_j \) may exist only if \( p_i \) and \( p_j \) are nonzero. Since \( p_i, p_j, b_{ij} \) are nonnegative, this condition is equivalent to \( p_ip_jb_{ij} \geq b_{ij} \). Thus cards of clusters of type I are completely described by the following relations:

**Type I:** \( n_d > 0 \).

\[
\begin{align*}
I_+: & \ n_t > 0, n_e \geq 3. \\
I_0: & \ n_t = n_{tr} = n_r = 0, n_e \geq 3, p_ip_jb_{ij} \geq b_{ij}, 1 \leq i, j \leq n_e. \\
\end{align*}
\]

**Type II.** In this case \( n_t > 0 \) if \( \dim \Phi > 3 \). If \( n_e \geq 3 \), then \( \Phi_{mp} \) is a cluster with stable ends as well as \( \Phi = \Phi_h \cup \Phi_{rt} \cup \Phi_{mp} \). Moreover, \( \Phi \) and \( \hat{\Phi} \) have the same center and end vertices. This shows that \( \Phi \) is obtained from \( \hat{\Phi} \) by means of p-twain, (framed) pyramid, bridge and d-bridge join operations, i.e., in this case \( [t, p, B, D] \) may be arbitrary. So, it remains to analyze the case \( n_e \leq 3 \). We subdivide it into three subcases according to the number \( n_{rf} \) of end vertices of \( \Phi_{rf} \). Obviously, in the considered case \( n_{rf} = 0, 2, 3 \).

If \( n_{rf} = 3 \), then the s-raft \( \Phi_{rf} \) contains two d-bridges, which have one common end vertex, say, \( E_1 \). In other words, ends of \( \Phi_{rf} \) coincide with ends of \( \Phi \), and, moreover, \( \Psi = \Phi_h \cup \Phi_{rt} \cup \Phi_{rf} \subset \Phi \) is a cluster with the same center and end vertices as \( \Phi \). By this reason \( \Phi \) is obtained from \( \Psi \) by means of p-twain, (framed) pyramid and bridge join operations. This shows that in the considered case \( [t, p, B] \) may be arbitrary.

In the case \( n_{rf} = 2, n_e = 3 \) numerate end vertices \( E_1, E_2, E_3 \) of \( \Phi \), so that \( E_1, E_2 \) be common ends of the belonging to \( \Phi \) d-bridges. So, end vertices \( E_1, E_2 \) are
automatically stable, while \( E_3 \) is stable if and only if there is either a tee belonging to \( \Phi \) with the origin at \( E_3 \), or a rooted at \( E_3 \) tee \( \theta \in \Phi \) whose second end differs from \( E_1, E_2 \). A direct item-by-item examination shows that this occurs only if one of the following three conditions holds:

(1) \( p_3 > 0 \) \iff the framing of the pyramid \( p_{E_3} \) is nonempty.

(2) \( p_3 = 0, \; t_3 > 0 \).

(3) \( p_3 = t_3 = 0, \; B \neq 0 \).

By using suitable join operations we easily find that there are no more limitations on \( \text{Card}(\Phi) \).

Let \( n_{rf} = 2, n_e = 2 \) and \( B_r \) be the union of d-bridges in \( \Phi \). In this case \( \Phi = B_r \cup \Phi_{mp} \cup \Phi_h \cup \Phi_{rt} \) is a cluster such that \( S(\Phi) = S(\hat{\Phi}) \). Hence \( \Phi \) is obtained from \( \Phi \) by means of suitable join operations. This shows that \( \text{Card}(\Phi) \) is arbitrary in the considered case.

If \( n_{rf} = 0 \), then \( \Phi = \Phi_1^{mp} \cup \Phi_h \cup \Phi_{rt} \). Denote by \( n^+_e \) the number of end vertices of \( \Phi \) for which \( p_E \neq 0 \) and by \( \mathbf{p}^+ \) the arithmetic vector obtained from the vector \( \mathbf{p} \) by canceling all its zero components. In the considered case \( 0 \leq n^+_e \leq 3 \). We shall label the occurring subcases by couple \((n_e, n^+_e)\). Below \( E_1, E_2, E_3 \) stand for end vertices of \( \Phi \).

\((n_e, n^+_e) = (3,3)\): In this case \( \Phi \) contains the cluster \( \hat{\Psi}_{3,0}(\mathbf{p}^+; n_1) \) (see subsection 9.4). Then \( \Phi = \hat{\Psi}_{3,0}(\mathbf{p}^+; n_1) \cup \Phi_h \cup \Phi_{rt} \) is a cluster as well and \( S(\Phi) = S(\hat{\Phi}) \).

So, \( \Phi \) is obtained from \( \hat{\Phi} \) by means of s-twain and bridge join operations. Hence in this case \( p_1 > 0, \; i = 1, 2, 3 \), and \([t, B] \) is arbitrary.

\((n_e, n^+_e) = (3,2)\): Assume that the end vertices \( E_1, E_2, E_3 \) are enumerated in such a way that \( p_3 = 0 \) and consequently \( p_1p_2 > 0 \). Then \( E_1, E_2 \) are stable end vertices, while \( E_3 \) is such one if either \( t_3 > 0 \), or \( \Phi \) contains a bridge with one of its ends at \( E_3 \) (equivalently, \( b_{13} + b_{23} > 0 \)). Indeed, otherwise \([E_1, E_2][E_3] \notin \Phi \) is compatible with \( \Phi \). Moreover, any of this conditions implies that \( \Phi \) is a cluster.

\((n_e, n^+_e) = (3,1)\): Similarly, assume that \( p_1 > 0, \; p_2 = p_3 = 0 \). Tees \([E_1, E_2][E_3] \) and \([E_1, E_3][E_2] \) deprive \( E_2 \) and \( E_3 \) of their status of end vertices of \( \Phi \). Therefore, they are blocked. It is easy to see that in the considered situation the tees are among tees belonging to treys and bridges of \( \Phi \). An item-by-item examination shows that it occurs only in one of the following situations:

(1) \( tzt_3 > 0 \).

(2) \( t_2 > 0, \; t_3 = 0, \; b_{13} > 0 \).

(3) \( t_2 = t_3 = 0, \; b_{12}b_{13} > 0 \) \iff there are bridges in \( \Phi \) connecting \( E_1 \) with \( E_2 \) and \( E_3 \).

(4) \( t_2 = t_3 = b_{12} = b_{13} = 0, \; b_{23} > 0 \) \iff there is a bridge in \( \Phi \) connecting \( E_2 \) and \( E_3 \).

Alternatively, this list is equivalent to the following list of inequalities

\[(1) \; tzt_3 > 0, \quad (2) \; t_2b_{13} > 0, \quad (3) \; b_{12}b_{13} > 0, \quad (4) \; b_{23} > 0.\]

Moreover, these conditions, as it is easily verified, guarantee that \( \Phi \) is a cluster.

\((n_e, n^+_e) = (3,0)\): Since \( n_e = 3 \), the multiped \( \Phi_{mp} \) has stable center vertices. By this reason, \( \hat{\Phi} = \Phi \setminus (\Phi_h \cup \Phi_{rt}) \) is a cluster as well. On the other hand, \( \Phi \) does not contain dees and, therefore, is a tee-cluster. The clusters that simultaneously are tee-clusters (“double clusters”) are described in corollary 9.4, and their cards are easily extracted from the lists in section 8. So, in the considered case all clusters
are obtained from “double clusters” with \( n_c = 3 \) by means of double and triangle join operations.

\((n_e, n^+_e) = (2, 2)\): Let \( E_1, E_2 \) be end vertices of \( \Phi \) and \( \Xi \) a tee-family composed of tees of the form \([E_i, B_i|C]\), \(i=1, 2\), with \( B_i \) running through bottom vertices of \( P_{E_i} \) and \( C \) trough center vertices of \( \Phi \). Obviously, \( Xi \) belongs to the union of casings of \( P_{E_i}'s \). Then \( \Psi_{2,0} \cup \Xi \cup \Phi_{mp} \subset \Phi \) is a cluster and \( \Phi \) is obtained from it by means of double, p-twain, bridge and triangle join operations. Hence in the considered case \( n_r, n_{tr}, t, B \) are arbitrary, while \( p \) is subjected by the condition \( p_1 p_2 > 0 \).

\((n_e, n^+_e) = (2, 1)\): Let \( E_1, E_2 \) be end vertices of \( \Phi \). We may assume that \( p_1 > 0, p_2 = 0 \). Obviously, tees \([E_1, B|E_2]\) with \( B \) running bottom vertices of \( P_{E_2} \) do not belong to \( \Phi \) and, therefore, are blocked. As earlier, the blocking base structures in the considered context are among tees belonging to treys and bridges of \( \Phi \). A simple check shows that the blocking tees may come either from a rooted at \( E_2 \) trey, or from two bridges connecting \( E_1 \) and \( E_2 \). This implies that, in the considered case \( \Phi \) is a cluster iff either or both of the inequalities \( t_2 > 0 \) and \( b_{12} \geq 2 \) holds.

\((n_e, n^+_e) = (2, 0)\): In this case \( \Phi^i \cup \Phi^m \subset \Phi \) contains only onedee, say, \( \theta = [E|B] \) and, obviously, \( B = 0 \). This dee belongs to a poker \( \Pi = \{\theta, [E, C|B]\} \subset \Phi \). In the considered context \( C \) may be either a bottom vertex of \( P_{E_i} \) or a side vertex of a trey in \( \Phi \). As previously, this proves that in the considered case \( \Phi \) is a cluster iff one of the inequalities \( p = p_1 \geq 2 \) or \( t = t_1 > 0 \) holds.

\((n_e, n^+_e) = (1, 0)\): The family \( \Phi \setminus \Phi^h \cup \Phi^t \) consists of a number of multi-treys, which are rooted at the unique end vertex, and their casings. This number is positive. Indeed, otherwise the graph \( \Upsilon_\Phi \) would non be connected. Since \( \Phi \) is a cluster, this end vertex is stable. This is so iff \( t = t_1 \geq 2 \). Moreover, this condition guarantees that \( \Phi \) is a cluster.

Ultimately, we get the following list of clusters of type II where \( D = ||d_{ij}||, B = ||b_{ij}|| \). In this and other lists that follow the alternatives which are “embraced” by square bracket do not exclude one another.

**Type II:** \( n_t > 0, n_d = 0, n_{tr} + n_r > 0 \).
$\Pi_+ : n_e > 3$.  
$\Pi_{32} : n_e = 3, \ d_{12}d_{13} > 0$.  
$\Pi_{31} : n_e = 3, \ d_{12} > 0 \Rightarrow \left\{ \begin{array}{l} p_3 > 0. \\ p_3 = 0, \ t_3 > 0. \\ p_3 = t_3 = 0, \ B \neq 0. \\ p_1p_2p_3 > 0. \end{array} \right.$  
$\Pi_{30} : n_e = 3, \ D = 0 \Rightarrow \left\{ \begin{array}{l} p_1 > 0, \ p_2 > 0, \ p_3 = 0 \Rightarrow \left\{ \begin{array}{l} t_3 > 0. \\ b_{13} + b_{23} > 0. \end{array} \right. \\ p_1 > 0, \ p_2 = p_3 = 0 \Rightarrow \left\{ \begin{array}{l} t_3 > 0. \\ b_{12}b_{13} > 0. \end{array} \right. \end{array} \right.$  
$\Pi_{300} : n_e = 3, \ D = 0, \ p = 0 \Rightarrow \left\{ \begin{array}{l} t_1t_2t_3 > 0. \\ t_1t_2 > 0, \ t_3 = 0, \ b_{13} + b_{23} > 0. \\ t_1 > 0, \ t_2 = t_3 = 0 \Rightarrow \left\{ \begin{array}{l} b_{23} > 0. \\ b_{12}b_{13} > 0. \end{array} \right. \end{array} \right.$  
(119)

$\Pi_2 : n_e = 2, \ D \neq 0$.  
$\Pi_{22} : n_e = 2, \ p_1p_2 > 0, \ D = 0$.  
$\Pi_{21} : n_e = 2, \ p_1 > 0, \ p_2 = 0, \ D = 0 \Rightarrow \left\{ \begin{array}{l} t_2 > 0. \\ b_{12} \geq 2. \end{array} \right.$  
(120)

$\Pi_{20} : n_e = 2, \ p_1 = p_2 = 0, \ D = 0$.  
$\Pi_{11} : n_e = 1 \Rightarrow \left\{ \begin{array}{l} p_1t_1 > 0. \\ p_1 \geq 2, \ t_1 = 0. \end{array} \right.$  
$\Pi_{10} : n_e = 1, \ p_1 = 0, \ t_1 \geq 2$.  

**Type III.** First, note that joining to a cluster of type III some rotators and triangles we get a cluster of type II. So, clusters of type III are among families of the form $\Phi_{cstr} = \Phi \setminus (\Phi_h \cup \Phi_{rt})$ with $\Phi$ being a cluster of type II. So, we shall get a description of clusters of type III just by running through lists (119) and (120) of clusters of type II and singling out those of them for which families $\Phi_{cstr}$ are clusters. It should be stressed that the end and center vertices of $\Phi$ are those of $\Phi_{cstr}$.

It is easy to see that a compatible family $\Phi_{cstr}$ is a cluster iff its center vertices are stable. This is so, if $n_e \geq 3$. Indeed, in this case the multiped $\Phi_{mp} \subset \Phi_{cstr}$ has stable center vertices, and these are center vertices of $\Phi_{cstr}$. Hence relations of list (119) describe clusters also in the case when $\Phi_h \cup \Phi_{rt} = \emptyset$ also.

Let $n_e = 2$ and $E_1, E_2$ be end vertices of $\Phi$. In this case center vertices of $\Phi_{cstr}$ are stable, if $\Phi$ (equivalently, $\Phi_{cstr}$) contains a trey or a bridge or two nonempty pyramids $P_{E_1}$ and $P_{E_2}$. Indeed, $\Phi_{pm}$ together with the casing of any of these families contains a tripod or a cross and, at the same time, is contained in $\Phi_{cstr}$.

In other words, $\Phi_{cstr}$ may not be a cluster only if relations

$$p_1p_2 = 0, \quad t = 0, \quad B = 0.$$

(121) holds. So, it remains to describe clusters whose cards satisfy relations (120) and (121). We shall examine cases $\Pi_2, \ldots, \Pi_{20}$ (see 120) one after another.

$\Pi_2$. Assume that $P_{E_1} \neq \emptyset, P_{E_2} = \emptyset$, i.e., that $p_1 > 0, p_2 = 0$, and let $C$ be a center vertex of $\Phi$. Then $\theta = [E_1, E_2\mid C]$ is the only tee in $\Phi_{cstr}$ with the center at
C. Since $t = 0$, $B = 0$, this shows that all tees from $\Phi_{cstr}$ are rooted at $E_1$. Let $B$ be a bottom vertex of $P_E$, or $C_E$, if $p_1 = 1$. Then $\rho = |E_1, C|B| \notin \Phi_{cstr}$. But being rooted at $E_1$, $\rho$ is compatible with $\Phi_{cstr}$. Hence in the considered case $\Phi_{cstr}$ is not a cluster, if one of pyramids $P_E$ is nonempty.

In the remaining subcase $p = 0$ assume that $C_1, C_2$ are center vertices of $\Phi_{cstr}$. Then $\theta = [E_1, C_1, C_2] \notin \Phi$. But by the same reasons as above, $\theta$ is compatible with $\Phi_{cstr}$. So, $\Phi_{cstr}$ is not a cluster, if $n_t \geq 2$. On the contrary, this is so, if $n_t = 1$. The corresponding cluster is an $s$-raft composed of one $d$-bridge and having one center vertex. Thus under conditions $\Pi_2$ and (121) $\Phi_{cstr}$ is a cluster iff $p = 0$, $n_t = 1$.

$\Pi_22 - \Pi_21$. The corresponding relations in (120) are, obviously, inconsistent with (121). So, in the considered case $\Phi_{cstr}$ is a cluster.

$\Pi_{20}$. It follows from (121) that in this case $\Phi_{cstr}$ consists of only one tee and hence is not a cluster.

Finally, consider the situation when $n_e = 1$. In the case $\Pi_{11}$ center vertices of $\Phi_{cstr}$ are stable only if $p_1 t_1 > 0$. Indeed, the dee $|E|C$ where $E = E_1$ is the end vertex of $\Phi$ and $C$ is one of its center vertices is compatible with $\Phi_{cstr}$ but does not belong to it. The condition $t = t_1 \geq 2$ guarantees stability of $\Phi_{cstr}$ in the case $\Pi_{10}$. So, if $n_e = 1$, then $\Phi_{cstr}$ is a cluster either or both $p_1 t_1 > 0$ and $t_1 \geq 2$.

The results concerning type III are synthesized in the following list:

**Type III:** $n_t > 0$, $n_d = n_{tr} = n_e = 0$.

III$_1$: the same relations as in (119).

III$_2$: $n_e = 2$, $p_1 p_2 > 0$.

III$_{2d}$: $n_e = 2$, $n_t = 1$, $p = t = 0$, $B = 0$, $D \neq 0$.

III$_{21}$: $n_e = 2$, $p_1 > 0$, $p_2 = 0$, $D = 0 \Rightarrow \begin{cases} t_2 > 0, \\ b_{12} \geq 2. \end{cases}$ (122)

III$_{11}$: $n_e = 1$, $p_1 t_1 > 0$.

III$_{10}$: $n_e = 1$, $p_1 = 0$, $t_1 \geq 2$.

**Type IV.** Clusters of this type are made of framed pyramids (including singles), bridges, $d$-bridges and the corresponding connectives. These connectives are of the form $|E, A|B|$ where $E$ is an end vertex of $\Phi$, $B$ is a bottom vertex of $P_E$ and $A$ is either an end vertex of $\Phi$ or the center of a bridge contained in $\Phi$. They will be called $e$-connectives and $b$-connectives, respectively. Denote by $\Psi_\Phi$ the family composed of all framings of $P_E$’s and all $e$-connectives. Also, recall that $n_+^e$ stands for the number of end vertices $E$ of $\Phi$ such that $p_E \neq 0$ and $p^+$ for the vector obtained from $p$ by canceling its zero components. End vertices $E$ of $\Phi$ for which $p_E = 0$ will be referred as free. We shall classify clusters of type IV according to the value of $n_+^e$.

$n_+^e \geq 3$: Since $n_+^e \geq 3$, it follows from lemma 9.6, (1), that $\Psi_\Phi$ is equivalent to $\bar{\Psi}_k, l(p^+)$ with $k = n_+^e$ and $l = n_e - n_+^e$ (see subsection 9.4) and hence is a cluster. Since $\Psi_\Phi$ and of $\Phi$ have common end vertices, $\Phi$ is obtained from $\Psi_\Phi$ by means of bridge and $d$-bridge join operations. Hence $[B, D]$ may be arbitrary in this case.

$n_+^e = 2$: Let $E_1, E_2$ be non-free end vertices of $\Phi$, i.e., $p_1 p_2 > 0$ and $p_1 = 0$ if $i > 2$. A cluster of the considered type is completely characterized by the following two tautological properties:
(1) all its free end vertices are stable;
(2) dees \(|E_1|B_2|\) and \(|E_2|B_1|\) with \(B_i\) being a bottom vertex of \(P_{E_i}, i = 1, 2,\) are blocked.

Examine them separately.

(1) First, note that free end vertices of \(\Phi\) are not stable as end vertices of \(\Psi_\Phi\) (see lemma 9.6). Since such a free end vertex is not a vertex of a b-connective, it is stable only if it is an end of a d-bridge belonging to \(\Phi\). In other words, in the considered context free end vertices of a cluster \(\Phi\) are among endvertices of the raft \(R_\Phi\) formed by all d-bridges contained in \(\Phi\). This may be expressed algebraically in terms of the following inequalities:

\[
\alpha_i(p, D) \overset{\text{def}}{=} p_i + \sum_{j=1}^{n_e} d_{ij} > 0, \quad i = 1, \ldots, n_e. \tag{123}
\]

(2) The dees in question are blocked if any bottom vertex \(B_i\) of \(P_{E_i}\) is the center of at least two pokers, which belong to \(\Phi\) rooted at \(E_i, i = 1, 2\). This manifestly takes place if one of the following conditions is fulfilled: (a) \(P_{E_i}\) is not a single \(p_i > 1\) for \(i = 1, 2;\) (b) there is at least one bridge in \(\Phi\) \(\iff B \neq 0;\) (c) \(\Phi\) possesses at least one free end vertex \(\iff n_e > 2\). On the other hand, one easily sees that \(\Phi\) is not a cluster, if none of this conditions is satisfied.

Thus in the considered case an abstract card is the card of a cluster if inequalities (123) and one of conditions (a) - (c) hold.

\(n_e^+ = 1\): First, note that \(B = 0\) in this case. Assume that \(p_1 > 0\) and \(p_i = 0\) if \(i > 1\). The same arguments as previously show that free end vertices of \(\Phi\) are end vertices of the raft \(R_\Phi, i.e., inequalities (123) hold in the considered situation too. Moreover, it is easily verified that this is an unique restriction on Card(\(\Phi\)), if \(\Phi\) has at least two free end vertices, i.e., if \(n_e \geq 3\). If \(n_e = 2\), then, obviously, \(\Phi\) is a cluster iff \(d_{12} > 0 \iff D \neq 0\) and \(p_1 \geq 2\). Finally, a framed pyramid is the only cluster if \(n_e = 1\), i.e., if \(p_1 \geq 2\).

\(n_e^+ = 0\): In this case \(\Phi = R_\Phi\) is a raft cluster. Its card is characterized by two obvious requirements: \(n_e \geq 2\) and the matrix \(D\) is “connected". The last means that the associated with \(D\) graph \(\Upsilon_D\) whose \(i\)-th and \(j\)-the vertices are connected by \(d_{ij}\) edges, \(1 \leq i, j, \leq n_e,\) is \(\text{connected}\). Indeed, in the considered context connectedness of \(\Upsilon_D\) is equivalent to that of \(\Upsilon_\Phi\).

**Type IV:** \(n_t = n_d = n_tr = n_r = 0\).

\(\text{IV}_+ : p_1p_2p_3 > 0.\)

\(\text{IV}_2 : p_1p_2 > 0, p_i = \alpha_i(p, D) = 0\) if \(i > 2 \Rightarrow \begin{cases} p_1 > 1, & p_2 > 1. \\ B \neq 0. \\ n_e > 2. \end{cases} \tag{124}\)

\(\text{IV}_1 : p_1 > 0, p_i = 0\) if \(i > 1 \Rightarrow \begin{cases} n_e \geq 3. \\ n_e = 2, & p_1 \geq 2, & D \neq 0. \\ n_e = 1, & p_1 \geq 2. \end{cases} \)

Thus we have proven

**Theorem 9.1.** Equivalence classes of clusters are in one-to-one correspondence with equivalence classes of cards listed in (118), (119), (120), (122), (124).
9.6. **Low dimensional clusters.** The use of join operations simplifies much classification of clusters. This, however, leads to a certain loss of control of dimensions of the so-constructed clusters. In particular, the only way to describe \( n \)-dimensional clusters for a given \( n \) is to extract a list of them from lists (118), (119), (120), (122) and (124). This, however, would be hardly instructive. Moreover, the expected result would be too cumbersome to report it here. Below we shall list clusters of dimensions \( \leq 5 \) to illustrate the situation. To this end we have to introduce three special clusters before.

- **A single-center-single cluster** has one center and two end vertices and is composed of two singles and the corresponding connectives and casings.

- **single-twain cluster** has 2 end vertices in which are rooted a framed twain and a single together with necessary external connectives.

- **bridge-d-bridge cluster** consists of one bridge and one d-bridge with common end vertices, which are suspended from one t-center vertex.

A \( d \)-multiplex is a multiplex to which is added the dee \( [O|C] \) where \( O \) (resp., \( C \) is the origin (resp., center) of the multiplex.

Below we use the notation \( \Psi = \Psi' \Join J \), which tells that the family \( \Psi \) is obtained from the family \( \Psi' \) by means of the join operation \( J \).

- \( n=2 \): double.
- \( n=3 \): triangle, (1,1)-rotator, framed twain, d-bridge.
- \( n=4 \): (3,1)-spider, (1,1)-hedgehog, (1,2)-rotator, (2,2)-raft, framed 3-pyramid.
- \( n=5 \): (4,1)- and (3,2)-spiders, (3,1)-spider\( \Join \)d-bridge, tripod \( \Join (3,1) \)-spider, (3,1)-spider \( \Join \)single, (1,2)-hedgehog, (1,3)- and (2,1)-rotators, (4,1)-multiped, 2-bridge cluster, single-center-single cluster, bridge-d-bridge cluster, suspended (2,2)-raft, trey \( \Join \)single, d-2-plex \( \Join \)bridge, (2,3)-raft, single-twain cluster, (d-bridge\( \Join \)single)\( \Join \)single, d-bridge\( \Join \)framed twain, (3,2)-raft, framed 4-pyramid.

One can see from these lists that their irregularity is due to a small number of end and center vertices. In that case their stability is not guaranteed on a common basis. By this reason, it is natural to call a cluster \( \Phi \) **stable**, if \( n_t \geq 1 \) and \( n_e \geq 4 \). Indeed, in this case \( \Phi_{mp} \) is a cluster with stable end and center vertices, so that the numbers of structural groups (triangles, doubles, etc) composing \( \Phi \) are no longer constrained by some conditions.

9.7. **On the structure of coaxial Lie algebras.** Basic structure elements of a coaxial algebra \( g \) can be directly read from \( \Phi_g \). The key is the following simple lemma, an analogue of lemma 8.12.

**Lemma 9.7.** Let \( g \) be a Lie algebra associated with a compatible family \( \Phi \), \( S_i \subset S(\Phi), i=1,2 \), and \( V_i \) the subset of \( |g| \) spanned by \( S_i \). Then

1. the subspace \( [V_1, V_2] = \text{Span}\{[v_1, v_2] | v_i \in V_i, i = 1, 2\} \) of \( |g| \) is spanned by center vertices of all tees \( [e_i, e_j | e_k] \in \Phi \) and end vertices of all dees \( [e_p | e_q] \in \Phi \) such that \( e_i, e_j \) and \( e_p, e_q \) belong to different subsets \( S_i \), respectively.
(2) the subspace of $|\mathfrak{g}|$ spanned by a subset $S \subset S(\Phi)$ is a subalgebra of $\mathfrak{g}$, if center vertices of tees $\theta \in \Phi$ with ends in $S$ also belong to $S$.

(3) the subspace of $|\mathfrak{g}|$ spanned by a subset $S \subset S(\Phi)$ is an ideal of $\mathfrak{g}$, if (1) the center vertex of any tee in $\Phi$ with one end in $S$, and (2) the end vertex of any dee in $\Phi$ with the origin in $S$ also belong to $S$.

(4) $\langle [\mathfrak{g}, \mathfrak{g}] \rangle$ belongs to the subspace spanned by center and mixing vertices of $\Phi_0$ and end vertices of dees $g \in \Phi_1$.

(5) The modular vector of $\mathfrak{g}$ is a linear combination of end vertices of dees belonging to $\Phi$.

Proof. The first assertion is obvious, while 2) - 4) are its immediate consequences. The last assertion follows from proposition 3.6 and the fact that the modular vector of a $\hat{\theta}$-lieon which is proportional to $|e_p|e_q$ is proportional to $e_q$. □

Now, let $\mathfrak{g}$ be a coaxial Lie algebra such that $\Phi = \Phi_0$ is a cluster. Lemma 9.7 allows us to single out a some basic ideals in $\mathfrak{g}$. For a subset $S \subset S(\Phi)t$ we shall use the notation $S \Rightarrow \mathfrak{g}_{(an \ \text{index})} \subset \mathfrak{g}$ to express the fact that the subspace $\text{Span}(S) \subset |\mathfrak{g}|$ supports the ideal $\mathfrak{g}_{(an \ \text{index})}$ of $\mathfrak{g}$.

\[
\begin{align*}
\text{Span}\{\text{center vertices}\} & \Rightarrow \mathfrak{g}_c, \quad \text{Span}\{\text{d-center vertices}\} \Rightarrow \mathfrak{g}_{cd}, \\
S(\Phi_h) & \Rightarrow \mathfrak{g}_h, \quad S(\Phi_{rt}) \Rightarrow \mathfrak{g}_{rt}, \\
S(\Phi \setminus (\Phi_h \cup \Phi_{rt})) & \Rightarrow \mathfrak{g}_{sr}, \quad S(\Phi \setminus \Phi_h) \Rightarrow \mathfrak{g}_{cd}, \\
S(\Phi \setminus \Phi_{rt}) & \Rightarrow \mathfrak{g}_{sr}, \\
S(\Phi) & \Rightarrow \mathfrak{g}_c.
\end{align*}
\]

(125)

The ideal $\mathfrak{g}_c$ belongs to the center of $\mathfrak{g}$. Even more, it coincides with it if the linear combination of base lieons of $\Phi$ that defines $\mathfrak{g}$ is generic. In particular, $\mathfrak{g}_c$ commute with all ideals (125). Nontrivial commutation relations involving these ideals are

\[
\begin{align*}
\mathfrak{g}_{cd} : \mathfrak{g}_{cd} & \subset \mathfrak{g}_{cd}, & \mathfrak{g}_{cd} : \mathfrak{g}_{sr} & \subset \mathfrak{g}_{cd}, & \mathfrak{g}_{cd} : \mathfrak{g}_{0} & \subset \mathfrak{g}_{cd}, \\
\mathfrak{g}_{rad} : \mathfrak{g}_{sr} & \subset \mathfrak{g}_{sr}, & \mathfrak{g}_{rad} : \mathfrak{g}_{0} & \subset \mathfrak{g}_{sr}, & \mathfrak{g}_{sr} : \mathfrak{g}_{0} & \subset \mathfrak{g}_{sr}.
\end{align*}
\]

(126)

Obviously, the quotient algebra $\mathfrak{g}_{\text{smpl}} = \mathfrak{g}_h/\mathfrak{g}_c = \mathfrak{g}/\mathfrak{g}_{rad}$ is associated with the family $\Phi_0^{\text{mix}}$ of all triangles contained in $\Phi$. Hence it is isomorphic to the direct sum of 3-dimensional simple Lie algebras associated with these triangles. In its turn this shows that the ideal $\mathfrak{g}_h$ is an abelian extension of $\mathfrak{g}_{\text{smpl}}$, i.e., that $\mathfrak{g}_h = \mathfrak{g}_{\text{smpl}} \oplus_{\rho} [\mathfrak{g}_c]$ for a suitable representation $\rho$.

On the other hand, $\mathfrak{g}_{rad}$ is the radical of $\mathfrak{g}$. To prove this it is sufficient to show that $\mathfrak{g}_{rad}$ is solvable, since $\mathfrak{g}_{\text{smpl}} = \mathfrak{g}/\mathfrak{g}_{rad}$ is semisimple. Obviously, solvability of $\mathfrak{g}_{rad}$ is equivalent to solvability of $\mathfrak{g}_{rad}/\mathfrak{g}_c$. To this end note that the algebra $\mathfrak{g}_{rad}'$ is associated with the family composed of $\Phi_1$ and the family $\Phi_0^{\text{mix}}$ of all tees with mixing (in $\Phi$) center vertices, which are rooted at end vertices of $\Phi$. It is easy to see that $\mathfrak{g}_{rad}'$ splits into a direct sum of an algebra associated with $\Phi_1^{db}$ and an algebra $\mathfrak{g}_{rad}''$ associated with $(\Phi_1 \setminus \Phi_1^{db}) \setminus \Phi_0^{\text{mix}}$. The first of them is isomorphic to the direct sum of $n_{\Phi_0}$ copies of $\hat{\theta}$ and an abelian algebra $\mathfrak{g}_c$, and, therefore, is solvable. On the other hand, $[\mathfrak{g}_{rad}'', \mathfrak{g}_{rad}'] \subset \mathfrak{g}_{rad}''$ where $\mathfrak{g}_{rad}'' \subset \mathfrak{g}_{rad}$ is an abelian subalgebra generated by mixing vertices of $\Phi_0$, center vertices of d-bridges and d-center vertices of $\Phi$. This shows that the derived series of $\mathfrak{g}_{rad}$ is of length 2. Consequently, the derived series of $\mathfrak{g}_{rad}$ is of length 3. Finally, we see that the image of a natural imbedding $\mathfrak{g}_{\text{smpl}} \rightarrow \mathfrak{g}_{\text{smpl}} \oplus_{\rho} [\mathfrak{g}_c] = \mathfrak{g}_h \subset \mathfrak{g}$ is the Levi subalgebra of $\mathfrak{g}$.

The algebra $\mathfrak{g}_0 = \mathfrak{g}_t/\mathfrak{g}_c$ is associated with $\Phi_1^{db}$ and $\mathfrak{g}_t$, is isomorphic to $\mathfrak{g}_0 \oplus_{\rho} [\mathfrak{g}_c]$ for a suitable representation $\rho$. There is a certain similitude between ideals $\mathfrak{g}_h$, $\mathfrak{g}_{sr}$, $\mathfrak{g}_{rt}$, $\mathfrak{g}_{rad}$, which is associated with triangles, which are simplest tee-clusters.
Similarly, the ideal \( g_2 \) is an abelian extension of a direct sum of nonabelian 2-dimensional Lie algebras, which are associated with doubles, i.e., with simplest dee-clusters. Since the semisimple part of \( g \) intrinsically defined, the number \( n_{tr} \) does not depend on the way \( g \) is assembled from basic lieons. Similarly, it can be shown that the number \( n_{tr} \) is intrinsically defined for “generic” in a sense coaxial algebras. This simple example illustrates the fact that \( \Phi_g \) reveal some finer details of the structure of \( g \), which go unnoticed in the standard approach.

9.8. Infinite-dimensional “disassemblable” Lie algebras. There are natural infinite-dimensional analogues of coaxial Lie algebras. Namely, let \( \{e_i\}_{1 \leq i < \infty} \) be a base of a numerable vector space \( V \) over a field \( k \). Recall that elements of \( V \) are, by definition, linear combinations \( \sum_{i<\infty} \lambda_i e_i, \lambda_i \in k \), with finitely many nonzero coefficients. Then a formal combination

\[
\sum_{1 \leq i,j,k<\infty} \alpha_{ijk} [e_i, e_j] e_k + \sum_{1 \leq p,q<\infty} \beta_{pq} [e_p] e_q, \quad \alpha_{ijk}, \beta_{pq} \in k, \tag{127}
\]

defines a Lie algebra structure in \( V \), if the figuring in (127) base lieons with nonzero coefficients are mutually compatible. We shall call countable this kind of coaxial algebras.

The pro-finite version of this construction is as follows. Let \( W \) be a pro-finite vector space and \( \{e_i\}_{1 \leq i < \infty} \) a (pro-finite) base of it. Elements of \( W \) are formal linear combinations \( \sum_{i<\infty} \lambda_i e_i, \lambda_i \in k \). Assume that formal combination (127) is locally finite. This means that for a given \( k \) (resp., \( q \)) only a finite number of coefficients \( \alpha_{ijk} \) (resp., \( \beta_{pq} \)) is different from zero. Then a locally finite combination correctly defines a Lie algebra structure in \( W \), if the occurring base structures with nonzero coefficients are mutually compatible.

Example 9.1. Let \( \{e_{ij}\}_{-\infty < i,j < \infty} \) be a base in a pro-finite vector space \( W \). The tee-family \( \Theta_{ij}, i,j \in \mathbb{Z} \), composed of four tee-structures \( [e_{i\pm1,j}, e_{i,j\pm1}]e_{ij} \), is compatible. It is easy to see that \( \Theta_{ij} \) and \( \Theta_{kl} \) are compatible iff \( i + j \equiv k + l \mod 2 \). So, each of families

\[
\Theta_{\text{even}} = \bigcup_{i+j \text{ is even}} \Theta_{ij}, \quad \Theta_{\text{odd}} = \bigcup_{i+j \text{ is odd}} \Theta_{ij}
\]

is compatible but they are not compatible one another. Similarly, the dee-family \( D_{ij} \) composed of four dee-structures \( [e_{ij}]e_{i\pm1,j} \) and \( [e_{ij}]e_{i,j\pm1} \) is compatible as well as families

\[
D_{\text{even}} = \bigcup_{i+j \text{ is even}} D_{ij}, \quad D_{\text{odd}} = \bigcup_{i+j \text{ is odd}} D_{ij},
\]

which are not compatible one another. Also, a family \( \Theta \) and a family \( D \) are compatible iff they are of different “parities”. In particular, any formal linear combination of base structures belonging to \( \Theta_{\text{even}} \cup D_{\text{odd}} \) defines a Lie algebra structure in \( W \).

An interesting property of families \( \Theta_{\text{even}} \) and \( \Theta_{\text{odd}} \) is that they are absolutely in incompatible. This means that any Lie algebras associated with \( \Theta_{\text{even}} \) is incompatible with any algebra associated with \( \Theta_{\text{odd}} \).

The above example is naturally related with a 2-dimensional lattice and is easily generalized to lattices of higher dimensions.

By combining compatible countable/pro-finite coaxial Lie algebras we can construct infinite-dimensional “disassemblable” Lie algebras of second level, etc. Infinite analogues of classical Lie algebras are examples of this kind algebras. For
instance, the countable special orthogonal algebra $\mathfrak{so}_\infty$ is given by the formal combination

$$
\mathfrak{so}_\infty(\lambda_1, \lambda_2, \ldots) = \sum_{k=1}^{\infty} \lambda_k P_k \quad \text{with} \quad P_k = \sum_{1 \leq i, j < \infty} [e_{ik}, e_{kl} e_{ij}], \quad \lambda_k \in k,
$$

assuming that $e_{ij} = -e_{ji}$. Countable coaxial algebras corresponding to $P_k$'s are compatible each other and hence disassemble algebra $\mathfrak{so}_\infty(\lambda_1, \lambda_2, \ldots)$.

This way we get a new class of infinite-dimensional Lie algebras, which merits to be studied in its own right (see [7] in this connection).

10. SOME PROBLEMS AND PERSPECTIVES

In connection with topics discussed in this paper a series of natural questions arise together with various perspective applications. Below we shall mention some of them, which, at present, look most interesting.

**Complete disassembling for arbitrary ground fields.** It is rather plausible that the complete disassembling theorem takes place for arbitrary ground fields of characteristic zero (see a more detailed discussion at the end of section 5). But what about nonzero characteristic?

**Algebraic variety of Lie algebra structures.** The algebraic variety $\mathcal{L}ie(V)$ of all Lie algebra structures is an intersection of quadrics in $\mathcal{A}(V) = \text{Hom}_k(V \otimes V, V)$. The subspace of $\mathcal{A}(V)$ spanned by a family of mutually compatible structures belongs to $\mathcal{L}ie(V)$. In this sense $\mathcal{L}ie(V)$ is "woven" of such subspaces. This suggests to use this "web structures" in describing $\mathcal{L}ie(V)$. An instructive example of this kind is given in [13]. Also, a more deep understanding of the structure of $\mathcal{L}ie(V)$ for algebraically closed ground field $k$ could shed some light on the general disassembling problem.

**Deformations.** On the basis of a disassembling of a Lie algebra $\mathfrak{g}$ one can construct some deformations of it by substituting $\lambda_n \mathfrak{g}_n$, $\lambda_n \in k$, for $\mathfrak{g}_n$ in the corresponding a-scheme (see subsection 5.1). Here factors $\lambda_n$ are constrained by some relations, which are absent in the case of first level algebras. The conjecture that all essential deformations of a given Lie algebra are of this kind is, at present, rather plausible.

**Length of disassemblies.** The procedure we have used in the proof of the complete disassembling theorem allows to estimate the minimal number of necessary for this steps as $n + \text{const}$. On the other hand, classical Lie algebras can be completely disassembled in $\leq 4$ steps independently of their dimensions (see section 7). So, a natural question is: whether there is an universal constant $N$ such that any Lie algebra can be completely disassembled in no more than $N$ steps. In the case of positive answer the problem of describing the variety $\mathcal{L}ie(V)$ passes to a more constructive basis.

**Lie algebras of second level.** It seems to be still possible to explicitly describe second level Lie algebras, i.e., Lie algebras, which can be completely disassembled in two steps, as it has been done for coaxial algebras in section 8. This could shed, among other things, a new light on problems mentioned before.

**Invariants of Lie algebras.** A natural question is: how to construct invariants of a Lie algebra if a disassembling of it is known. This question has an evident homological flavour and, probably, leads to constructive theory of Lie algebras,
i.e., a theory, which from the very beginning considers Lie algebras as compound structures.

**Cohomological aspects.** Poisson (resp., Lie algebra) structures, which are compatible with a given one, are closed 2-forms in the associated Lichnerowicz-Poisson (resp., Chevalley-Eilenberg) complex. This fact was not explicitly exploited in this paper. Nevertheless, a more deep understanding of topics we have discussed here is related with this cohomological aspect. In this connection we mention that one of methods to completely disassemble a classical Lie algebra \( g \) is to represent the corresponding Poisson bivector \( P_g \) in the form \( P_g = \sum_i [X_i, P_g] \) for suitable vector fields \( X_i \) on \( g \). Here the terms \([X_i, P_g]\) are exact 2-cohains in the associated with \( P_g \) Lichnerowicz-Poisson complex.

It is naturally to think that the cohomology a Lie algebra assembled from some other ones is, in a sense, “assembled” from their cohomology. An exact formalization of this idea requires some special techniques of homological algebra and will be discussed in our subsequent paper.

**Generalizations.** Assemblage techniques and some results of this paper can be directly extended in many directions. First of all, there should be mentioned graded and multiple Lie algebras (see [4, 5, 12, 16, 17]). An interesting point here is that natural compatibility of hereditary structures associated with an n-ary Lie algebra links the disassembling problems for Lie algebra of different multiplicities together. More generally, compatibility problems for any kind of Poisson structures, say, algebroids and their n-ary analogues (see [10]), are tightly intertwined.

**Physical applications.** The concept of two compatible Poisson structures was introduced by F. Magri at 1977 (see [11]) and since that was studied and widely exploited by numerous authors in the context integrable systems. We do not discuss here these well-known aspects. The fundamental question arising in connection with compound nature of Lie algebras is:

> Let \( S \) be a physical system and \( g \) a Lie algebra of its infinitesimal symmetries. What one can say about intrinsic structure of \( S \), if a disassembling \( g \) is known?

This question is, of course, too general to allow an universal answer. It should be duly specified each time according to the nature of the system in question.

Interpretation of compounds of the symmetry algebra as symmetry waves is suggested itself. Since these waves are nonlinear, it leads to the conclusion that characterizing them quantities need not be additive. Probably, this kind of considerations could be useful in the theory of quarks.

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