On the Treves theorem for the AKNS equation.

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Abstract. According to a theorem of Treves [11], the conserved functionals of the AKNS equation vanish on all pairs of formal Laurent series ($\tilde{q}$, $\tilde{r}$) of a specified form, both of them with a pole of the first order. We propose a new and very simple proof for this statement, based on the theory of Bäcklund transformations; using the same method, we prove that the AKNS conserved functionals vanish on other pairs of Laurent series. The spirit is the same of our previous paper [7] on the Treves theorem for the KdV [10], with some non-trivial technical differences.

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1 Introduction and preliminaries.

Some recent works of Treves (see [10] [11] and references therein) have introduced, in the words of Dickey [2], a "fresh idea" in the sector of integrable evolutionary PDEs (KdV, nonlinear Schrödinger, etc.). The discovery of Treves is that all the conserved functionals of these equations vanish when they are evaluated on certain formal Laurent series (intending the integrals which appear in the functionals as loop integrals in $\mathbb{C}$ around zero).

To be more specific, let us consider the KdV equation $q_t = q_{xxx} - 12qq_x$, and the functionals $h = h(q)$ which are integrals of polynomials in $q$ and its $x$-derivatives; for any such functional to be conserved by the KdV equation, it is necessary and sufficient [10] that $h(\tilde{q}) = 0$ for all formal Laurent series with complex coefficients of the form $\tilde{q} = 1/x^2 + \tilde{q}_0 + \sum_{k=2}^{+\infty} \tilde{q}_k x^k$. Now, let us pass to the coupled equations

$$q_t = \frac{1}{2} q_{xx} - q^2 r, \quad r_t = -\frac{1}{2} r_{xx} + qr^2$$ (1.1)

which have been considered in [11]; as well known, these become the nonlinear Schrödinger equation $i\bar{q}_r = (1/2)\bar{q}_{xx} - |q|^2 q$ if $r$ is the complex conjugate of $q$ and $\tau := it$; if $q, r$ are regarded as independent, the pair (1.1) is usually called the AKNS equation after Ablowitz, Kaup, Newell and Segur (see [8] and references therein), a terminology to which we also stick in this paper.

Let us consider the functionals $h(q, r)$ which are integrals of polynomials in $q, r$ and their $x$ derivatives; for any such functional to be conserved by Eq. (1.1), it is necessary [11] that $h(\tilde{q}, \tilde{r}) = 0$ for all pairs of Laurent series $\tilde{q} = e^{\varphi} (1/x + \alpha + \beta x + \sum_{k=2}^{+\infty} \chi_k x^k), \tilde{r} = e^{-\varphi} (1/x - \alpha + \beta x + \sum_{k=2}^{+\infty} \rho_k x^k)$ (with $\varphi, \alpha, \beta, \chi_k, \rho_k \in \mathbb{C}$); differently from the KdV case, the sufficiency of this condition has been conjectured but not proved in [11].

The proofs in the original papers of Treves are based on a long and highly technical analysis, where a central role is played by the recurrence relations for some standard basis of conserved functionals. A simple alternative proof of the necessary condition for the KdV, and an analogue of this result for the Boussinesq equation, were obtained by Dickey [2] using the dressing method for the Lax operator. Another proof of necessity for the KdV was proposed in a paper of ours [7], where we employed the invariance of the KdV conserved functionals under the (auto)-Bäcklund transformation. The relation
between [2] and [7] has been very recently discussed in [3], where the extension of the Bäcklund method to the Boussinesq and the other GD hierarchies has also been sketched.

In the present paper the Bäcklund invariance of the conserved functionals will be used in relation to the AKNS, to get a new proof of the Treves theorem for this equation. Using this idea we will show that the AKNS conserved functionals vanish on other nontrivial pairs of Laurent series, in particular for those of the form \( \tilde{q} = e^{q} (2/x^2 + \alpha + \beta x + \sum_{k=2}^{\infty} \chi_k x^k), \tilde{r} = e^{-q} (1 + \beta x^2 + \sum_{k=4}^{\infty} \delta_k x^k) \).

Both for the pairs \((\tilde{q}, \tilde{r})\) considered by Treves and for the ones in the above variant, our proof is very simple and conceptually similar to the argument of [7] for the KdV; in fact, we show that any \((\tilde{q}, \tilde{r})\) as before is the Bäcklund transform of a pair \((q, r)\) of holomorphic series (with no negative powers of \(x\)), on which every functional of polynomial type is zero for trivial reasons.

In comparison with our previous analysis of the KdV case, the AKNS is a bit more difficult regarding the precise definition of the Bäcklund transformation; in fact, even though this transformation for the AKNS is known in the literature [1] [9], its usual formulation involves rational maps that would cause some troubles in the present framework; for this reason, in this paper we give a different presentation which is more implicit but involves only polynomial mappings.

Let us describe the organization of the paper. In the rest of this Introduction, we generalize for our present needs the language of differential algebras and formal variational calculus already employed in [7]; the AKNS equation and the space of its conserved functionals are described formally within this framework. In Section 2 we state precisely the Treves theorem for the AKNS (Prop. 2.1) and our variant of it mentioned before (Prop. 2.2). In Section 3 we introduce the Bäcklund transformation and state the invariance under it of the AKNS conserved functionals, in a way suitable for our purposes; in Section 4 we use it to prove Prop.s 2.1 and 2.2.

Two Appendices have been added to review the matrix Lax formalism à la Drinfeld-Sokolov, and its relation to Bäcklund transformations, for certain classes of integrable systems and especially for the AKNS; the aim is, essentially, to justify the slightly non standard presentation of the Bäcklund machinery employed in this work.

All vector spaces considered in this paper are over \(\mathbb{C}\). As anticipated, hereafter we summarise some concepts from differential algebra and from the formal variational calculus of Gelfand-Dickey [4], including their applications to the AKNS theory.

**Differential algebras.** By a differential algebra, we mean an associative algebra (commutative or not) equipped with a derivation, i.e., with a linear map of the algebra into itself having the Leibnitz property w.r.t. the product. We do not require the algebra to possess a unity; if this exists, one easily proves that it is annihilated by the derivation. A morphism of differential algebras is an algebraic morphism respecting the derivations.

A differential algebra is typically written as \((\mathcal{Q}, \partial_x)\); the subscript \(x\) attached to the derivation is also used to denote its action on the elements \(q\) of the algebra, so \(\partial_x : \mathcal{Q} \to \mathcal{Q}, q \mapsto \partial_x q\).

We write \(\mathcal{Q}_x\) for the image of \(\partial_x\); if \(q, p \in \mathcal{Q}\) and \(p = q_x\), sometimes we say that \(q\) is a primitive of \(p\).

The quotient vector space

\[
\int \mathcal{Q} := \mathcal{Q}/\mathcal{Q}_x = \{ q + \mathcal{Q}_x \mid q \in \mathcal{Q} \}
\]

is called the space of integrals of \(\mathcal{Q}\). The corresponding quotient map is denoted with

\[
\int : \mathcal{Q} \to \int \mathcal{Q}, \quad q \mapsto \int q := q + \mathcal{Q}_x ,
\]

and we call \(\int q\) the integral of \(q\); of course \(\int q_x = 0\) for each \(q\). Let \(\mathcal{S}\) be any subset of \(\mathcal{Q}\); we write

\[
\int \mathcal{S} := \{ s + \mathcal{Q}_x \mid s \in \mathcal{S} \} \subset \int \mathcal{Q} ,
\]

and note that the restriction of \((1.3)\) is a map \(\mathcal{S} \to \int \mathcal{S}, s \mapsto \int s\).

Sometimes, it is necessary to specify the dependence from the differential algebra \(\mathcal{Q}\) of the previous operations of integration: in this case we write \(\int^\mathcal{Q}\) for the map \((1.3)\) or its restriction to \(\mathcal{S}\), and \(\int^\mathcal{Q} \mathcal{S}\) for the set \((1.4)\).
A differential subalgebra of a differential algebra \((Q, \partial_x)\) is a subalgebra \(P \subset Q\) closed under \(\partial_x\); of course, the differential subalgebra \(P\) with the restricted map \(\partial_x \mid P = \partial_x\) is itself a differential algebra. In this situation, one has to distinguish between two kinds of integration, the first one intrinsic for the differential algebra \((P, \partial_x)\), and the second one relative to \(Q\). In the first case, we introduce as usually the quotient space and the quotient map

\[
\int P := P/P_x; \quad \int : P \mapsto \int P, \quad p \mapsto \int p := p + P_x;
\]

in the second case, we consider the space and the map

\[
\int Q P := \{p + Q_x \mid p \in P\} \subset \int Q; \quad \int Q : P \mapsto \int Q p, \quad p \mapsto \int Q p := p + Q_x.
\]

One easily checks the existence of a unique map

\[
\psi : \int P \to \int Q P \quad \text{such that} \quad \psi(\int p) = \int Q p \quad \forall p \in P.
\]

The map \(\psi\) is linear and onto; furthermore, it is injective if and only if

\[
P \cap Q_x = P_x.
\]

If condition (1.5) holds, we will say that \(P\) is a strict differential subalgebra of \(Q\). (Of course, for any differential subalgebra \(P\) it is \(P \cap Q_x \subset P_x\); Eq. (1.5) means that any element of \(P\) with a primitive in \(Q\) also has a primitive in \(P\).)

Throughout the paper, the term ideal is employed with the usual sense; a differential ideal of a differential algebra is an ideal closed under the derivation.

**Gelfand-Dickey differential algebra in any number \(\gamma\) of generators.** This is the commutative differential algebra

\[
\mathfrak{g} := \mathbb{C}[\xi_1, \ldots, \xi_2, \xi_{1,x}, \ldots, \xi_{\gamma,x}, \ldots]_0,
\]

made of complex polynomials in infinitely many indeterminates \(\xi_s, \xi_{s,x}, \xi_{s,xx}, \ldots\) \((s = 1, \ldots, \gamma)\), without free term (the absence of this is indicated by the subscript \(0\)); \(\mathfrak{g}\) is equipped with the unique derivation \(\partial_x \equiv _x\) such that

\[
(\xi_s)_x = \xi_{s,x}, \quad (\xi_{s,x})_x = \xi_{s,xx}, \quad \ldots.
\]

We write \(F, G\), etc. for the elements of \(\mathfrak{g}\). (For example: \(F := \xi_{1,x}^2\xi_{2,xx}, G := 3\xi_{1,x}\xi_2 \in \mathfrak{g}\); \(FG = 3\xi_{1,x}^3\xi_2\xi_{2,xx}\).)

For the elements of \(\int \mathfrak{g} \equiv \mathfrak{g}/\mathfrak{g}_x\), which have the form \(f = \int F\ (F \in \mathfrak{g})\), the general denomination of "integrals" is of course available; however, in this case the name functionals is more standard.

The Gelfand-Dickey algebra \(\mathfrak{g}\) can be represented in terms of transformations on any commutative differential algebra \((Q, \partial_x)\), in the following way. Let us consider the Cartesian product \(\times^\gamma Q \equiv Q^\gamma\); then, any \(F \in \mathfrak{g}\) induces a map

\[
F(\ ) : Q^\gamma \to Q, \quad (q_1, \ldots, q_\gamma) \mapsto F(q_1, \ldots, q_\gamma)
\]

where \(F(q_1, \ldots, q_\gamma)\) is obtained from the expression of the polynomial \(F\) replacing \(\xi_s\) with \(q_s;\ \xi_{s,x}\) with \(q_{s,x}\), etc. .

It is important to distinguish the elements of \(\mathfrak{g}\) from the maps on \(Q^\gamma\): this is the reason why the symbol of the map in (1.11) contains a bracket ( ). (In \([7]\), for the same reason we used bold symbols for the elements of \(\mathfrak{g}\), and non bold notations for the maps on \(Q\); in the present framework, the proliferation of bold symbols would be excessive). We note that

\[
F(\ ) \in Pol(Q^\gamma, Q),
\]

where \(Pol(X, Y)\) are the polynomial maps \([7]\) between any two vector spaces \(X, Y\) and \(Q^\gamma\) is regarded as a vector space with the product structure. \(Pol(Q^\gamma, Q)\) is a commutative algebra with the pointwise
product, and a differential algebra with the unique derivation \( \partial_x : P(\ ) \mapsto P_x(\ ) \) such that \( P_x(q) := P(q)_x \) for all \( P(\ ) \). The correspondence

\[
\mathfrak{g} \to \text{Pol}(Q^\gamma, Q), \quad F \mapsto F(\ )
\]

is a morphism of differential algebras. It also induces a linear map

\[
\int \mathfrak{g} \to \text{Pol}(Q^\gamma, \int Q), \quad f \mapsto f(\ )
\]

in the following way: if \( f = \int F \), then

\[
f(\ ) : Q^\gamma \to \int Q, \quad (q_1, ..., q_\gamma) \mapsto f(q_1, ..., q_\gamma) := \int F(q_1, ..., q_\gamma).
\]

The maps (1.11) and (1.15) will be called the representations on \( Q \) of \( F \) and \( f \), respectively.

**Vector fields and Lie derivatives.** We consider again the Gelfand-Dickey differential algebra (1.9). In this framework, by a vector field we simply mean a family \( X = (X_1, ..., X_\gamma) \in \mathfrak{g}^\gamma \) (this is represented as map \( X(\ ) = (X_1( ), ..., X_\gamma( )) \in \text{Pol}(Q^\gamma, Q^\gamma) \) on any commutative differential algebra \( Q \)). The Lie derivative on \( \mathfrak{g} \) induced by \( X \) is the unique derivation

\[
\mathcal{L}_X : \mathfrak{g} \to \mathfrak{g} \quad \text{such that} \quad \mathcal{L}_X \partial_x = \partial_x \mathcal{L}_X, \quad \mathcal{L}_X \xi_s = X_s \ (s = 1, ..., \gamma);
\]

the corresponding Lie derivative on \( \int \mathfrak{g} \) is the unique map

\[
\mathcal{L}_X : \int \mathfrak{g} \to \int \mathfrak{g} \quad \text{such that} \quad \mathcal{L}_X \int = \int \mathcal{L}_X;
\]

of course this map is linear. The set of conserved functionals of a vector field \( X \) is

\[
\mathfrak{z}_X := \{ h \in \int \mathfrak{g} \mid \mathcal{L}_X h = 0 \};
\]

this is a vector subspace of \( \int \mathfrak{g} \).

**The AKNS theory and its conserved functionals**. This is a theory in \( \gamma = 2 \) components. For our purposes, it is necessary to formulate it in the language of formal variational calculus; so, we introduce the Gelfand-Dickey algebra

\[
\mathfrak{g} := \mathbb{C}[\xi, \eta, \xi_x, \eta_x, ...]_0
\]

with generators \( \xi_1 \equiv \xi, \xi_2 \equiv \eta \). The AKNS vector field is

\[
X_{AKNS} \equiv X := (\frac{1}{2} \xi_{xx} - \xi^2 \eta, -\frac{1}{2} \eta_{xx} + \xi \eta^2) \in \mathfrak{g}^2.
\]

The space of conserved functionals

\[
\mathfrak{z}_{X_{AKNS}} \equiv \mathfrak{z}
\]

is known to be of infinite dimension, a remarkable property placing this vector field within the realm of integrable systems. In the Appendices \[1\, \[2\] we will review the Lax formalism to obtain the conserved functionals of this vector field and of similar systems. This approach gives a basis \( (h_i)_{i=1,2,...} \) for \( \mathfrak{z} \), derived from the "fundamental invariants" of the Lax operator; the first elements are

\[
h_1 := \frac{1}{2} \int \xi \eta, \quad h_2 := \frac{1}{4} \int \xi_x \eta, \quad h_3 := -\frac{1}{8} \int (\xi^2 \eta^2 + \xi_x \eta_x), \quad h_4 := \frac{1}{16} \int (-3 \eta^2 \xi_x + \xi_x \eta_{xx}).
\]
The Treves theorem (and some variant of it) for the AKNS.

As in the case of the KdV [10], this theorem concerns the representation of an integrable system on a peculiar differential algebra. This is the commutative differential algebra of formal Laurent series in one indeterminate \( x \) and complex coefficients, i.e.,

\[
Q := \{ q = \sum_{k=k_{\text{min}}}^{+\infty} q_k x^k \mid q_k \in \mathbb{C}, k_{\text{min}} = k_{\text{min}}(q) \in \mathbb{Z} \}; \quad (2.1)
\]

the product is the usual Cauchy product of series, and the derivation is

\[
\partial_x : Q \to Q, \quad q \mapsto q_x := \sum_{k=k_{\text{min}}}^{+\infty} k q_k x^{k-1}. \quad (2.2)
\]

Clearly, we have

\[
Q_x = \{ q \in Q \mid q_{-1} = 0 \} \quad (2.3)
\]

and the map \( \int Q \to \mathbb{C} \), \( \int q \mapsto q_{-1} \) is a linear isomorphism. For this reason, from now on we make the identifications

\[
\int Q \simeq \mathbb{C}; \quad \int q \simeq q_{-1} \quad \forall q \in Q \quad (2.4)
\]

(in [7], this was presented for simplicity as the very definition of \( \int \)). Of course, the above description of \( \int q \) as the "residue" \( q_{-1} \) suggests to interpret it as a loop integral in \( \mathbb{C} \) around zero.

We come to the Treves theorem for the AKNS and for the differential algebra (2.1-2.2). In our notations, this reads:

2.1 Proposition [11]. Let \( h \in \mathfrak{z} \subset \mathfrak{f} \), and consider its representation \( h(\cdot) : Q^2 \to \mathbb{C} \). Then

\[
h(\tilde{q}, \tilde{r}) = 0 \quad \forall \quad \tilde{q} = e^{\varphi} \left( \frac{1}{x} + \alpha + \beta x + \sum_{k=2}^{+\infty} \tilde{\chi}_k x^k \right), \quad \tilde{r} = e^{-\varphi} \left( \frac{1}{x} - \alpha + \beta x + \sum_{k=2}^{+\infty} \tilde{\rho}_k x^k \right), \quad (2.5)
\]

\((\varphi, \alpha, \beta, \tilde{\chi}_k, \tilde{\rho}_k \in \mathbb{C})\). \quad \diamond

As anticipated, our aim in this paper is to give a new proof of this result, based on the Bäcklund transformations. This technique will allow us to prove the following variant of the previous result:

2.2 Proposition. Let \( h \in \mathfrak{z} \); then

\[
h(\tilde{q}, \tilde{r}) = 0 \quad \forall \quad \tilde{q} = e^{\varphi} \left( \frac{2}{x^2} + \alpha + \beta x + \sum_{k=2}^{+\infty} \tilde{\chi}_k x^k \right), \quad \tilde{r} = e^{-\varphi} \left( 1 + \beta x^2 + \sum_{k=4}^{+\infty} \tilde{\rho}_k x^k \right), \quad (2.6)
\]

\((\varphi, \alpha, \beta, \tilde{\chi}_k, \tilde{\rho}_k \in \mathbb{C})\). \quad \diamond

Both Props. 2.1 and 2.2 are proved in Section 4. Hereafter, as a preliminary step we discuss the AKNS Bäcklund transformation, in a formulation suitable for our purposes.

3 Bäcklund transformation for the AKNS theory.

Essentially, this is a transformation leaving invariant the AKNS conserved functionals. However, its description in the language of formal variational calculus requires some technical subtleties introduced hereafter. To this purpose, we consider besides \( \mathfrak{f} := \mathbb{C}[\xi, \eta, \xi_x, \eta_x, \ldots]_{0} \) a "copy" of it, say

\[
\tilde{\mathfrak{f}} := \mathbb{C}[\tilde{\xi}, \tilde{\eta}, \tilde{\xi}_x, \tilde{\eta}_x, \ldots]_{0}, \quad (3.1)
\]
with the derivation such that \( (\tilde{\xi})_x = \xi_x \), etc. Of course, there is a unique differential-algebraic isomorphism \( \tilde{\pi} : \tilde{\mathfrak{g}} \rightarrow \tilde{\mathfrak{f}} \), \( F \rightarrow \tilde{F} \) sending \( \xi, \eta \) into \( \tilde{\xi}, \tilde{\eta} \). This also induces a linear isomorphism
\[
\int \tilde{\mathfrak{g}} \rightarrow \int \tilde{\mathfrak{f}} \ , \quad f \mapsto \tilde{f} := \int \tilde{F} .
\]
(3.2)
We interpret \( \tilde{\mathfrak{g}}, \tilde{\mathfrak{f}} \) as describing the "initial" and "final variables" for the "transformation" to be introduced. The latter is in fact defined implicitly in terms of an "auxiliary variable" \( \nu \); its description mixes together the initial auxiliary and final variables, so we introduce a third differential algebra
\[
\mathfrak{m} := \mathbb{C}[\xi, \eta, \nu, \bar{\xi}, \bar{\nu}, \xi_x, \eta_x, \nu_x, \bar{\xi}_x, \bar{\nu}_x, \ldots]_0 ,
\]
writing again \( \partial_x \) for its derivation. Up to trivial identifications, we have
\[
\tilde{\mathfrak{g}}, \tilde{\mathfrak{f}} \subset \mathfrak{m} ;
\]
both \( \tilde{\mathfrak{g}} \) and \( \tilde{\mathfrak{f}} \) are strict differential subalgebras of \( \mathfrak{m} \) (in the sense of \( 1.8 \)), so
\[
\int \tilde{\mathfrak{g}} \simeq \int \mathfrak{m} \ , \quad \int \tilde{\mathfrak{f}} \simeq \int \mathfrak{m} \subset \mathfrak{m} .
\]
(3.5)

3.1 Definition. The AKNS Bäcklund ideal \( \mathcal{J}_{AKNS} \equiv \mathcal{J} \subset \mathfrak{m} \) is the ideal of \( \mathfrak{m} \) generated by the elements
\[
I_1 := \xi_x - \bar{\xi}_x + \nu(\xi + \bar{\xi}), \quad I_2 := \eta_x - \bar{\eta}_x + \nu(\eta + \bar{\eta}), \quad I_3 := \nu_x + \xi \eta - \bar{\xi} \bar{\eta} . \quad \bigcirc \quad (3.6)
\]
From standard commutative algebra,
\[
\mathcal{J} = \left\{ \frac{\sum_{j=1}^{3} F_j I_j}{F_j \in \mathfrak{m} \ \forall j} \right\} .
\]
(3.7)
In the language of formal variational calculus, the Bäcklund invariance of the AKNS conserved functionals can be expressed as follows.

3.2 Proposition. Let \( h \in \mathcal{J}_{AKNS} \equiv \mathcal{J} \subset \tilde{\mathfrak{g}} \), and define \( \tilde{h} \) following \( 1.2 \). Then
\[
\tilde{h} - h \in \int \mathcal{J} \quad (\int \equiv \int \mathfrak{m}) . \quad \bigcirc \quad (3.8)
\]
The above Proposition is essentially known in the literature, even though it is not usually formulated in the language of formal variational calculus. In any case, to make the paper self contained we propose a proof in the Appendix B.

In order to exemplify Eq. (3.8), let us consider the functionals \( h_i \) in Eq. (1.23) \((i = 1, 2, 3, 4)\) and their tilded images \( \tilde{h}_1 = \int \tilde{\xi} \tilde{\eta} \ldots \); it turns out that
\[
\tilde{h}_1 - h_1 = -\frac{1}{2} \int I_3 , \quad \tilde{h}_2 - h_2 = \frac{1}{4} \int (-\eta I_1 + \bar{\xi} I_2 + \nu I_3) ,
\]
(3.9)
\[
\tilde{h}_3 - h_3 = \frac{1}{8} \int ((-\bar{\eta} \nu + \bar{\eta}_x) I_1 + (\xi \nu + \xi_x) I_2 + (\xi \eta + \bar{\xi} \bar{\eta} - \nu^2) I_3) ,
\]
\[
\tilde{h}_4 - h_4 = \frac{1}{16} \int ((\xi \nu^2 - \bar{\xi} \bar{\eta} - \xi \eta \bar{\nu} + 2\bar{\xi} \bar{\eta} \bar{\nu} + \bar{\xi} \bar{\eta}^2 - \nu \bar{\nu}^2 + \nu \xi_x - \eta \xi_x) I_1 +
\]
\[
+(-\xi \bar{\xi} \eta + \bar{\xi}^2 \eta - 2\bar{\xi}^2 \bar{\eta} + \bar{\xi} \nu^2 + \nu \xi_x + \xi \nu_x + \bar{\xi} \nu_x + \xi \bar{\nu}_x) I_2 +
\]
\[
+(-\xi \nu \bar{\xi} \bar{\eta} + \bar{\xi} \bar{\eta} \nu - \bar{\xi} \bar{\eta} \nu + \nu \xi_x + \eta \xi_x - \xi \eta_x - \bar{\xi} \eta_x) I_3 ) ,
\]
We now present the consequences of the previous statements in terms of concrete differential algebras. From now on \((\mathcal{Q}, \partial_x)\) is a commutative differential algebra; so, the generators \(I_j\) in Eq. (3.10) induce maps

\[
I_j(\ ) : \mathcal{Q}^5 \to \mathcal{Q}, \quad (q, r, v, \tilde{q}, \tilde{r}) \mapsto I_j(q, r, v, \tilde{q}, \tilde{r}),
\]

\[
I_1(q, \ldots, \tilde{r}) := q_x - \tilde{q}_x + v(q + \tilde{q}), \quad I_2(q, \ldots, \tilde{r}) := r_x - \tilde{r}_x + v(r + \tilde{r}), \quad I_3(q, \ldots, \tilde{r}) := v_x + qr - q \tilde{r}.
\]

### 3.3 Definition

Let \(\mathcal{Q}^2 := \mathcal{Q} \times \mathcal{Q}\) and consider the set \(2\mathcal{Q}^2\) of all subsets of \(\mathcal{Q}^2\). The AKNS Bäcklund transformation for \(\mathcal{Q}\) is the map

\[
B_{\text{AKNS}}(\ ) \equiv B(\ ) : \mathcal{Q}^2 \to 2\mathcal{Q}^2, \quad (q, r) \mapsto B(q, r),
\]

\[
B(q, r) := \{(q, \tilde{r}) \in \mathcal{Q}^2 \mid \exists v \in \mathcal{Q} \text{ s.t. } I_j(q, r, v, \tilde{q}, \tilde{r}) = 0 \text{ for } j = 1, 2, 3 \}.
\]

### 3.4 Proposition

Let \(h \in \mathfrak{I}\) and consider its representation \(h(\ ) : \mathcal{Q}^2 \to \mathbb{C}\). For all \((q, r) \in \mathcal{Q}^2\) and \((\tilde{q}, \tilde{r}) \in B(q, r)\), it is

\[
h(\tilde{q}, \tilde{r}) = h(q, r).
\]

**Proof.** From Prop. 3.2 we know that \(\tilde{h} - h = \int I\) for some \(I\) in the Bäcklund ideal \(\mathfrak{I}\). So, using the representation \(I(\ ) : \mathcal{Q}^5 \to \mathcal{Q}\) we find

\[
h(\tilde{q}, \tilde{r}) - h(q, r) = \int I(q, r, v, \tilde{q}, \tilde{r}) \quad \forall(q, r, v, \tilde{q}, \tilde{r}) \in \mathcal{Q}^5.
\]

On the other hand, by comparison with (3.7) we have

\[
I(\ ) = \sum_{j=1}^{3} F_j( ) I_j( )
\]

where \(F_j( ) : \mathcal{Q}^5 \to \mathcal{Q}\) are certain polynomial maps. In particular, let \(\tilde{q}, \tilde{r} \in B(q, r)\); if \(v\) is as in (3.11), we have \(I_j(q, r, v, \tilde{q}, \tilde{r}) = 0\) implying \(I(q, r, v, \tilde{q}, \tilde{r}) = 0\), and Eq. (3.13) gives \(h(\tilde{q}, \tilde{r}) - h(q, r) = 0\).\^\hfill\(\Box\)

### 4 Proofs of Propositions 2.1, 2.2

From now on, \((\mathcal{Q}, \cdot, \int)\) is the differential algebra of formal Laurent series described in Section 2. For convenience, we consider therein the differential subalgebra of "holomorphic series"

\[
\mathcal{Z} := \{q \in \mathcal{Q} \mid q = \sum_{k=0}^{+\infty} q_k x^k \}.
\]

Trivially, we have

### 4.1 Lemma

Consider any \(h \in \int \mathfrak{I}\) and its representation \(h(\ ) : \mathcal{Q}^2 \to \mathbb{C};\) then

\[
h(\ ) \mid \mathcal{Z}^2 = 0.
\]

**Proof.** Write \(h = \int H\). If \(q, r \in \mathcal{Z}\), it is also \(H(q, r) \in \mathcal{Z}^2\), because this is a polynomial in \(q, r\) and their derivatives. Thus, the residue \(h(q, r) = \int H(q, r)\) is zero.\^\hfill\(\Box\)

We consider the AKNS Bäcklund transformation \(B(\ ) : \mathcal{Q}^2 \to 2\mathcal{Q}^2\) (see Def. 3.3); then, combining the previous Lemma with the Bäcklund invariance of all the AKNS conserved functionals (Prop. 3.4) we get
4.2 Proposition. Let \( h \in \mathfrak{A}_{KNS} \equiv 3 \); then
\[
h(\ ) \mid B(\mathbb{Z}^2) = 0 , \quad B(\mathbb{Z}^2) := \cup_{(q,r) \in \mathbb{Z}^2} B(q,r) . \tag{4.3}
\]
We now fix the attention on the set of Laurent series appearing in the Treves theorem \([24]\) i.e.,
\[
T := \left\{ \tilde{q}, \tilde{r} \in \mathbb{Q}^2 \mid \tilde{q} = e^\varphi \left( \frac{1}{x} + \alpha + \beta x + \sum_{k=2}^{+\infty} \tilde{\chi}_k x^k \right) , \quad \tilde{r} = e^{-\varphi} \left( \frac{1}{x} - \alpha + \beta x + \sum_{k=2}^{+\infty} \tilde{\rho}_k x^k \right) , \quad (\varphi, \alpha, \beta, \tilde{\chi}_k, \tilde{\rho}_k \in \mathbb{C}) \right\} . \tag{4.4}
\]

4.3 Lemma. i) Let \( \tilde{q}, \tilde{r} \) be as in Eq.\,(4.4). Then, there are uniquely determined series of the form
\[
q = e^\varphi \left( -\alpha + \sum_{k=2}^{+\infty} \chi_k x^k \right) , \quad r = e^{-\varphi} \left( \alpha + \sum_{k=2}^{+\infty} \rho_k x^k \right) , \quad v = -\frac{1}{x} + 2\beta x + \sum_{k=2}^{+\infty} v_k x^k \tag{4.5}
\]
such that
\[
I_j(q, r, v, \tilde{q}, \tilde{r}) = 0 \quad (j = 1, 2, 3) . \tag{4.6}
\]
ii) Statement i) implies
\[
T \subset B(\mathbb{Z}^2) . \tag{4.7}
\]
Proof. i) Let \( q, \ldots, \tilde{r} \) be as in Eqs.\,(4.6), and put for brevity \( I_j \equiv I_j(q, r, v, \tilde{q}, \tilde{r}) \). Direct computation gives:
\[
I_1 = e^\varphi \sum_{k=1}^{+\infty} J_{1k} x^k , \quad I_2 = e^{-\varphi} \sum_{k=1}^{+\infty} J_{2k} x^k , \quad I_3 = \sum_{k=1}^{+\infty} I_{3k} x^k ; \tag{4.8}
\]
\[
J_{11} := \chi_2 - 3\tilde{\chi}_2 + v_2 , \quad J_{12} := 2\chi_3 - 4\tilde{\chi}_3 + 2\beta^2 + v_3 , \quad J_{13} := 3\chi_4 - 5\tilde{\chi}_4 + \beta(2\chi_2 + 2\tilde{\chi}_2 + v_2) + v_4 , \nonumber
\]
\[
J_{1k} := k\chi_{k+1} - (k+2)\tilde{\chi}_{k+1} + \beta(2\chi_{k-1} + 2\tilde{\chi}_{k-1} + v_{k-1} + v_{k+1} + \sum_{\ell=2}^{k-2} v_{k-\ell}(\chi_\ell + \tilde{\chi}_\ell)) \quad (k \geq 4) ; \tag{4.9}
\]
\[
J_{21} := \rho_2 - 3\tilde{\rho}_2 + v_2 , \quad J_{22} := 2\rho_3 - 4\tilde{\rho}_3 + 2\beta^2 + v_3 , \quad J_{23} := 3\rho_4 - 5\tilde{\rho}_4 + \beta(2\rho_2 + 2\tilde{\rho}_2 + v_2) + v_4 , \nonumber
\]
\[
J_{2k} := k\rho_{k+1} - (k+2)\tilde{\rho}_{k+1} + \beta(2\rho_{k-1} + 2\tilde{\rho}_{k-1} + v_{k-1} + v_{k+1} + \sum_{\ell=2}^{k-2} v_{k-\ell}(\rho_\ell + \tilde{\rho}_\ell)) \quad (k \geq 4) ; \tag{4.10}
\]
\[
I_{31} := 2v_2 - \tilde{\rho}_2 - \tilde{\chi}_2 , \quad I_{32} := 3v_3 - \alpha(\rho_2 - \chi_2 + \tilde{\rho}_2 - \tilde{\chi}_2) - \tilde{\rho}_3 - \tilde{\chi}_3 - \beta^2 , \nonumber
\]
\[
I_{33} := 4v_4 - \alpha(\rho_3 - \chi_3 + \tilde{\rho}_3 - \tilde{\chi}_3) - \tilde{\rho}_4 - \tilde{\chi}_4 - \beta(\rho_2 + \tilde{\rho}_2) , \tag{4.11}
\]
\[
I_{3k} := (k+1)v_{k+1} - \alpha(\rho_k - \chi_k + \tilde{\rho}_k - \tilde{\chi}_k) - \tilde{\rho}_{k+1} - \tilde{\chi}_{k+1} - \beta(\rho_{k-1} + \tilde{\rho}_{k-1} + \chi_{k-1} + \tilde{\chi}_{k-1}) + \sum_{\ell=2}^{k-2} (\chi_{k-\ell} + \rho_{k-\ell} + \tilde{\chi}_{k-\ell} - \tilde{\rho}_{k-\ell}) , \quad (k \geq 4) . \nonumber
\]
We must show that the equations \( J_{1k} = 0 , J_{2k} = 0 , I_{3k} = 0 \) for all \( k \geq 1 \) have uniquely determined solutions for the coefficients \( \chi_k, \rho_k, v_k \) \( (k \geq 2) \).
The proof is recursive: for \( k = 1, 2, 3, \ldots \), the equation \( I_{3k} = 0 \) determines \( v_{k+1} \), and inserting the result into \( J_{2k} = 0 , J_{1k} = 0 \) one determines, respectively, \( \rho_{k+1} \) and \( \chi_{k+1} \).

ii) Let \( (\tilde{q}, \tilde{r}) \in T \), and \( q, r, v \) as in i). It is clear that \( (q, r) \in \mathbb{Z}^2 \); Eq.\,(4.3) means \( (\tilde{q}, \tilde{r}) \in B(q, r) . \) \hfill □

Proof of the Treves theorem (Prop.\,2.1). Put together Eqs.\,(4.3) and \,(4.7). \hfill □
In a similar way we now prove Prop. 2.2 that concerns the set

\[ S := \{ \tilde{q}, \tilde{r} \in \mathbb{Q}^2 \mid \tilde{q} = e^\varphi \left( \frac{2}{x^2} + \alpha + \beta x + \sum_{k=2}^{+\infty} \chi_k x^k \right), \]  

\[ \tilde{r} = e^{-\varphi} \left( 1 + \beta x^3 + \sum_{k=4}^{+\infty} \tilde{\rho}_k x^k \right), \quad (\varphi, \alpha, \beta, \chi_k, \tilde{\rho}_k \in \mathbb{C}) \}; \]

everything relies on the following

**4.4 Lemma.** i) Let \( \tilde{q}, \tilde{r} \) be as in Eq. (4.12). Then, there are uniquely determined series of the form

\[ q = e^\varphi \sum_{k=2}^{+\infty} \chi_k x^k, \quad r = e^{-\varphi} \left( -1 + \sum_{k=3}^{+\infty} \rho_k x^k \right), \quad v = -\frac{2}{x} + \alpha x + \sum_{k=2}^{+\infty} \nu_k x^k \]  

(4.13)

such that \( I_j(q, r, v, \tilde{q}, \tilde{r}) = 0 \) for \( j = 1, 2, 3 \).

ii) Statement i) implies

\[ S \subset B(\mathbb{Z}^2) . \]  

(4.14)

**Proof.** i) Let \( q, v, \tilde{r} \) be as in Eqs. (4.11, 4.2), and \( I_j \equiv I_j(q, r, v, \tilde{q}, \tilde{r}) \). Then

\[ I_1 = e^\varphi \sum_{k=0}^{+\infty} J_{1k} x^k, \quad I_2 = e^{-\varphi} \sum_{k=2}^{+\infty} J_{2k} x^k, \quad I_3 = x \sum_{k=0}^{+\infty} J_{3k} x^k ; \]  

(4.15)

\[ J_{10} := 2v_2 - 3\beta , \quad J_{11} := 2v_3 - 4\tilde{\chi}_2 + \alpha^2 , \quad J_{12} := \chi_3 + 2v_4 + \alpha v_2 - 5\tilde{\chi}_3 + \alpha \beta , \]  

\[ J_{13} := 2\chi_4 + 2v_5 + \alpha v_3 + \beta v_2 - 6\tilde{\chi}_4 + \alpha (\chi_2 + \tilde{\chi}_2) , \]  

\[ J_{1k} := (k-1)\chi_{k+1} + 2v_{k+2} - (k+3)\tilde{\chi}_{k+1} + \alpha v_k + \beta v_{k-1} + \alpha (\chi_{k-1} + \tilde{\chi}_{k-1}) + \sum_{\ell=2}^{k-2} v_k - \ell (\chi_\ell + \tilde{\chi}_\ell) \quad (k \geq 4) ; \]  

(4.16)

\[ J_{22} := \rho_3 - 5\beta , \quad J_{23} := 2\rho_4 - 6\tilde{\rho}_4 , \quad J_{24} := 3\rho_5 + \alpha (\rho_3 + \beta) - 7\tilde{\rho}_5 , \]  

\[ J_{25} := 4\rho_6 + v_2 \rho_3 + \beta v_2 - 8\tilde{\rho}_6 + \alpha (\rho_4 + \tilde{\rho}_4) , \]  

(4.17)

\[ J_{2k} := (k-1)\rho_{k+1} - (k+3)\tilde{\rho}_{k+1} + \beta v_{k-3} + \alpha (\rho_{k-1} + \tilde{\rho}_{k-1}) + \sum_{\ell=3}^{k-2} v_k - \ell \rho_\ell + \sum_{\ell=4}^{k-2} v_k - \ell \tilde{\rho}_\ell \quad (k \geq 6) ; \]  

\[ J_{30} := 2v_2 - 3\beta , \quad J_{31} := 3v_3 - \chi_2 - \tilde{\chi}_2 - 2\tilde{\rho}_4 , \quad J_{32} := 4v_4 - \chi_3 + \tilde{\chi}_3 - 2\tilde{\rho}_5 - \alpha \beta , \]  

\[ J_{33} := 5v_5 - \chi_4 + \tilde{\chi}_4 - 2\tilde{\rho}_6 - \alpha \tilde{\rho}_4 - \beta^2 , \quad J_{34} := 6v_6 - \chi_5 + \tilde{\chi}_5 - 2\tilde{\rho}_7 - \alpha \tilde{\rho}_5 - \beta \tilde{\rho}_4 + \chi_2 \rho_3 - \tilde{\chi}_2 \beta , \]  

\[ J_{3k} := (k+2) v_{k+2} - \chi_{k+1} - \tilde{\chi}_{k+1} - 2\tilde{\rho}_{k+3} + \alpha \tilde{\rho}_{k+1} - \beta \tilde{\rho}_k + \sum_{\ell=3}^{k-1} \chi_{k+1} - \ell \rho_\ell - \sum_{\ell=4}^{k-1} \chi_{k+1} - \ell \tilde{\rho}_\ell , \quad (k \geq 5) . \]  

Again, we must show that the equations \( J_{1k} = 0, J_{2k} = 0, J_{3k} = 0 \) for all \( k \) have uniquely determined solutions for the coefficients \( \chi_k, \rho_k, \tilde{\rho}_k \).

In fact: from \( J_{10} = 0 \) and \( J_{11} = 0 \) one uniquely determines \( v_2, v_3 \); now, the equation \( J_{30} = 0 \) is automatically fulfilled, and \( J_{31} = 0 \) gives \( \chi_2 \). At this point, we must determine \( v_{k+2} \), \( \rho_{k+1} \) and \( \chi_{k+1} \) for \( k \geq 2 \), which is performed recursively in the following way. From \( J_{1k} = 0 \) and \( J_{3k} = 0 \) one computes \( v_{k+2} \) and \( \chi_{k+1} \); to find them, one must solve a linear system whose matrix \( \begin{pmatrix} 2 & k-1 \\ k+2 & -1 \end{pmatrix} \) has determinant \( -k(k+1) \neq 0 \). Finally, from \( J_{2k} = 0 \) one gets \( \rho_{k+1} \).

ii) Let \( (\tilde{q}, \tilde{r}) \in S \), and \( q, r, v \) as in i). Then \( (q, r) \in \mathbb{Z}^2 \), and statement i) means \( (\tilde{q}, \tilde{r}) \in B(q, r) \).
Proof of Prop. 2.2. Put together Eq.s (4.3) and (4.14).

Remark. In any case, the following holds:

\[(\tilde{q}, \tilde{r}) \in B(Z^2), \tilde{q} = \sum_{k=a}^{+\infty} \tilde{q}_k x^k, \tilde{q}_a \neq 0, \quad \tilde{r} = \sum_{k=b}^{+\infty} \tilde{r}_k x^k, \tilde{r}_b \neq 0, \quad \min(a,b) < 0 \quad (4.19)\]

\[\Rightarrow a + b = -2, \quad \tilde{q}_a \tilde{r}_b = -\min(a,b).\]

In fact, we are assuming \(I_j \equiv I_j(q,r,v,\tilde{q},\tilde{r}) = 0\) for some holomorphic \(q,r \in Z\) and \(v \in Q\). To fix the ideas, let us assume \(\min(a,b) = a\); then, from \(I_1 = 0\) we easily infer \(v = \sum_{k=-1}^{+\infty} v_k x^k\) with \(v_{-1} = a\). Inserting these facts into \(I_3 = 0\), we obtain \(a + b = -2\) and \(\tilde{q}_a \tilde{r}_b = -a\). If \(\min(a,b) = b\) we proceed similarly, using the equations \(I_2 = 0\) and \(I_3 = 0\).

Of course, if \((\tilde{q}, \tilde{r})\) are in the subsets \(T\) or \(S\) we have, respectively, \(a = b = -1\) or \(a = -2, b = 0\).
A Appendix. Matrix Lax operators: the Drinfeld-Sokolov formulation.

Some more algebra. Consider any differential algebra \((\mathcal{U}, \partial_x \equiv \cdot_x)\), possibly non commutative. Then, the algebra of differential operators with coefficients in \(\mathcal{U}\) is the associative (and non commutative) algebra \(\text{Diff}(\mathcal{U})\) with generators

\[ \partial_x, \quad U \quad (U \in \mathcal{U}) \]  \hspace{1cm} \text{(A.1)}

and defining relations:

\[ UV = \underbrace{UV}_{\text{product in } \text{Diff}(\mathcal{U})} \quad ; \quad \partial_x U = U \partial_x + U_x \quad (U, V \in \mathcal{U}). \]  \hspace{1cm} \text{(A.2)}

Any \(D \in \text{Diff}(\mathcal{U})\) has a representation

\[ D = \sum_{k=0}^{d} D_k \partial_x^k \quad (d \in \mathbb{N}, D_k \in \mathcal{U} \forall k), \]  \hspace{1cm} \text{(A.3)}

which is unique under the condition \(D_d \neq 0\) if \(d \neq 0\); the unique integer \(d\) determined in this way is called the order of the differential operator \(D\). \(\mathcal{U}\) can be identified with the subalgebra of \(\text{Diff}(\mathcal{U})\) made of zero order operators; if \(\mathcal{U}\) has unity 1, this is also the unity of \(\text{Diff}(\mathcal{U})\) (so, the invertible zero order operators are just the invertible elements of \(\mathcal{U}\)).

For any vector space \(\mathcal{V}\), we introduce the vector space of \(n \times n\) matrices

\[ \text{Mat}_n(\mathcal{V}) = \{ V = (V_{ab})_{a,b=1,...,n} \mid V_{ab} \in \mathcal{V} \forall a,b \}; \]  \hspace{1cm} \text{(A.4)}

we often consider therein the supplementary subspaces \(\text{Diag}_n(\mathcal{V}), \text{Off}_n(\mathcal{V})\) made, respectively, by the diagonal and off-diagonal matrices.

If \((\mathcal{V}, \partial_x)\) is a differential algebra, \(\text{Mat}_n(\mathcal{V})\) is a differential algebra when equipped with the usual "row by column" product and with the "term by term" derivation \(\mathcal{V} \mapsto \mathcal{V}_x := (V_{ab,x})\). Up to trivial identifications, we have \(\text{fMat}_n(\mathcal{V}) = \text{Mat}_n(\text{fV})\) and \(\text{fV} = (\text{fV}_{ab})\). Of course, \(\text{Diag}_n(\mathcal{V})\) is a differential subalgebra of \(\text{Mat}_n(\mathcal{V})\). If \(\mathcal{I}\) is an ideal, or a differential (\(\partial_x\)-closed) ideal in \(\mathcal{V}\), the same occurs for \(\text{Mat}_n(\mathcal{I})\) in \(\text{Mat}_n(\mathcal{V})\).

Consider again a vector space, now denoted for convenience with \(\mathcal{W}\). We can associate to it the vector space of formal series in one indeterminate \(\lambda\)

\[ \mathcal{W}(\lambda) := \{ W = \sum_{i=i_{\text{min}}}^{+\infty} W_i \lambda^{-i} \mid i_{\text{min}} = i_{\text{min}}(W) \in \mathbb{Z}, W_i \in \mathcal{W} \forall i \}; \]  \hspace{1cm} \text{(A.5)}

we will often fix the attention on the subspaces

\[ \mathcal{W}(\lambda)_\leq := \{ \sum_{i=0}^{+\infty} W_i \lambda^{-i} \}, \quad \mathcal{W}(\lambda)_< := \{ \sum_{i=1}^{+\infty} W_i \lambda^{-i} \}. \]  \hspace{1cm} \text{(A.6)}

If \(\mathcal{W}\) is a differential algebra with derivation \(\partial_x \equiv \cdot_x\), then \(\mathcal{W}(\lambda)\) is a differential algebra, with the usual Cauchy product of series and the derivation \(\partial_x : W \mapsto W_x := \sum_i W_i \lambda^{-i}\). Of course, \(\mathcal{W}\) can be identified with the differential subalgebra of \(\mathcal{W}(\lambda)\) made of series \(W\) with \(W_i = 0\) for \(i \neq 0\); \(\mathcal{W}(\lambda)_\leq\), \(\mathcal{W}(\lambda)_<\) are also differential subalgebras. If \(\mathcal{W}\) has unity 1, this is also the unity of \(\mathcal{W}(\lambda)\); the subset

\[ 1 + \mathcal{W}(\lambda)_< = \{ 1 + \sum_{i=1}^{+\infty} W_i \lambda^{-i} \}\]  \hspace{1cm} \text{(A.7)}
is a group with respect to the Cauchy product.
To conclude these preliminaries we point out that the notation $[\cdot,\cdot]$, employed in the sequel for matrices or differential operators, stands for the usual commutator.

Drinfeld-Sokolov theory. Let us consider a commutative differential algebra $(\mathcal{F},\partial_x)$; to fix the ideas, on can think $\mathcal{F}$ to be the algebra of differential operators with coefficients in the algebra $(A.9)$. Moreover, the matrix functionals of $(A.8)$ are made of differential operators with coefficients in the algebra $(A.9)$. In the sequel, elements of $\mathcal{F}$ and $Mat_n(\mathcal{F})$ will be called, respectively, the scalar and the $n \times n$ matrix integrals, or functionals of $\mathcal{F}$.
For technical reasons appearing in the sequel, we need the direct sum of this algebra and the algebra of integrals, or functionals $F$ of matrices with complex entries, i.e.,

$$\text{Mat}_n(\mathcal{F}) \oplus \text{Mat}_n(\mathbb{C}) \quad \text{(A.8)}$$

which is in an obvious way an associative algebra (the product between elements of $\text{Mat}_n(\mathcal{F})$ and $\text{Mat}_n(\mathbb{C})$ is defined again row by column); this algebra contains $\text{Mat}_n(\mathcal{F})$ as an ideal. The derivation $\partial_x$ of $\text{Mat}_n(\mathcal{F})$ is extended to the previous direct sum, prescribing that it annihilates $\text{Mat}_n(\mathbb{C})$; in this way, $(A.8)$ is a differential algebra and $\text{Mat}_n(\mathcal{F})$ a differential ideal of it. Of course the unity of $(A.8)$ is $1 := \text{diag}_n(1,\ldots,1)$.

The next step is to form the differential algebra of formal series

$$\left(\text{Mat}_n(\mathcal{F}) \oplus \text{Mat}_n(\mathbb{C})\right)(\lambda) := \{N = \sum_{i_{\text{min}}}^{\infty} N_i \lambda^{-i} \mid i_{\text{min}} \in \mathbb{Z}, N_i \in \text{Mat}_n(\mathcal{F}) \oplus \text{Mat}_n(\mathbb{C}) \forall i\} \quad \text{(A.9)}$$

(which of course contains the differential ideal $\text{Mat}_n(\mathcal{F})(\lambda)$, made of series as above with coefficients $N_i \in \text{Mat}_n(\mathcal{F})$). The ultimate step is the algebra

$$\mathcal{D}_n(\mathcal{F}) := \text{Diff}\left(\left(\text{Mat}_n(\mathcal{F}) \oplus \text{Mat}_n(\mathbb{C})\right)(\lambda)\right) \quad \text{(A.10)}$$

made of differential operators with coefficients in the algebra $(A.9)$.

A.1 Definition. A first order, $n \times n$ Lax operator is a differential operator of the form

$$L \equiv L_{A,S} := \partial_x - \lambda A - S \in \mathcal{D}_n(\mathcal{F}) \quad \text{(A.11)}$$

$S \in \text{Mat}_n(\mathcal{F})$, $A = \text{diag}(a_1,\ldots,a_n) \in \text{Diag}_n(\mathbb{C})$, $a_i \neq 0$ for all $i$, $a_i \neq a_j$ for $i \neq j$.
We will write $\Sigma_n(\mathcal{F})$ for the set of these operators.

The first result in the Drinfeld-Sokolov theory is a diagonalization theorem for these operators.

A.2 Proposition. Consider an operator $L = L_{A,S} \in \Sigma_n(\mathcal{F})$. Then, there is a pair of objects

$$U = 1 + \sum_{i=1}^{\infty} U_i \lambda^{-i} \in 1 + \text{Mat}_n(\mathcal{F})(\lambda)_\prec \quad \text{(A.12)}$$

$$H = \sum_{i=0}^{\infty} H_i \lambda^{-i} \in \text{Diag}_n(\mathcal{F})(\lambda)_\preceq \quad \text{(A.13)}$$

such that

$$L = U(\partial_x - \lambda A - H) U^{-1} \quad \text{(A.14)}$$

moreover, the matrix functionals

$$h_{L,i} \equiv h_i := \int H_i \quad (i = 0, 1, 2, \ldots) \quad \text{(A.15)}$$

are uniquely determined by $L$. 

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Proof. See Section 1 of [5]. The main point is that Eq. (A.14) is equivalent to $LU = U (\partial_x - \lambda A - H)$, and that the expansion in powers of $\lambda$ of both sides in this equality gives rise to recursion equations for the sequences $(H_i)$, $(U_i)$.

A.3 Definition. Any pair $(U, H)$ as in Prop. A.2 will be called a diagonalizing pair for $L$. The matrix functionals $h_{L,i}$ will be called the fundamental invariants of $L$.

Remark. Suppose $(\mathcal{F}, \partial_x)$ is a strict differential subalgebra of a commutative differential algebra $(\mathcal{M}, \partial_x)$ (Eq. (13); recall that $\int \mathcal{F} \cong \int \mathcal{M} \subset \int \mathcal{M}$). Then, by the uniqueness statement of the previous Proposition, the diagonalizations of $L$ as an element of $\mathcal{L}_n(\mathcal{F})$, or as an element of $\mathcal{L}_n(\mathcal{M})$, give rise to the same fundamental invariants which belong in any case to $\text{Diag}_n(\int \mathcal{F})$.

The forthcoming Proposition considers a situation of this kind; the result stated therein clarifies the origin of the term "invariant" for the functionals $h_i$, $\tilde{h}_i$.

A.4 Proposition. Let $(\mathcal{M}, \partial_x)$ be a commutative differential algebra containing $\mathcal{F}$ as a strict differential subalgebra; further, let $\bar{\mathcal{F}}$ denote another strict differential subalgebra of $\mathcal{M}$. Consider two operators

$$L = L_{\mathcal{A}, \mathcal{S}} \in \mathcal{L}_n(\mathcal{F}), \quad \bar{L} = \bar{L}_{\mathcal{A}, \mathcal{S}} \in \mathcal{L}_n(\bar{\mathcal{F}})$$

(with the same $\mathcal{A}$) and their fundamental invariants $h_{L,i} \equiv h_i \in \text{Diag}_n(\int \mathcal{F})$, $\bar{h}_{\bar{L},i} \equiv \bar{h}_i \in \text{Diag}_n(\int \bar{\mathcal{F}})$. Let $\mathcal{I}$ denote an ideal of $\mathcal{M}$, and assume there is

$$V = 1 + \sum_{i=1}^{+\infty} V_i \lambda^{-i} \in 1 + \text{Mat}_n(\mathcal{M})(\lambda)_{<}$$

such that

$$\bar{L} - V L V^{-1} \in \text{Mat}_n(\mathcal{I})(\lambda)_{<}.$$  

(A.18)

Then

$$\bar{h}_i - h_i \in \text{Diag}_n(\int \mathcal{I}) \quad (i = 0, 1, 2, \ldots)$$

(A.19)

(\text{where, in the last equation, } \int \equiv \int \mathcal{M}).

In a few words: if $L$, $\bar{L}$ are similar up to a series with coefficients in the ideal $\text{Mat}_n(\mathcal{I})$, their fundamental invariants coincide up to elements of $\text{Diag}_n(\int \mathcal{I})$. This result is essential for our purposes; since our language is slightly different from the one of [5], it is convenient to report the Proof of Prop. A.4 We choose a diagonalizing pair $(U, H)$ for $L$, so as to fulfill Eqs. (A.12), (A.14), and proceed in three steps.

Step 1. There are

$$W \in 1 + \text{Mat}_n(\mathcal{M})(\lambda)_{<}, \quad J \in \text{Mat}_n(\mathcal{I})(\lambda)_{<} \quad \text{such that} \quad \bar{L} = W (\partial_x - \lambda A - H - J) W^{-1}.$$  

(A.20)

In fact, assumption (A.18) means $\bar{L} = V L V^{-1} + F$, where $F \in \text{Mat}_n(\mathcal{I})(\lambda)_{<}$; from here and (A.14) we get

$$\bar{L} = V U (\partial_x - \lambda A - H) U^{-1} V^{-1} + F = V U (\partial_x - \lambda A - H + U^{-1} V^{-1} F V U) U^{-1} V^{-1};$$

(A.21)

this gives the thesis (A.20), with $W := V U$ and $J := - U^{-1} V^{-1} F V U$ (the relations $W \in 1 + \text{Mat}_n(\mathcal{M})(\lambda)_{<}$, $J \in \text{Mat}_n(\mathcal{I})(\lambda)_{<}$ are easily checked from the previous definitions recalling that $1 + \text{Mat}_n(\mathcal{M})(\lambda)_{<}$ is a subalgebra and $\text{Mat}_n(\mathcal{I})$ an ideal).
Step 2. Consider the differential ideal $\mathcal{G}$ of $\mathcal{M}$ generated by $\mathcal{I}$ (i.e., the smallest differential ideal containing $\mathcal{I}$). Then $\tilde{L}$, as an element of $\mathcal{L}_n(\mathcal{M})$, admits a diagonalizing pair $(\tilde{U}, \tilde{H})$ with $\tilde{H}$ of the form

$$\tilde{H} = H + G, \quad G \in \text{Diag}_n(\mathcal{G})(\lambda) \leq .$$

(A.22)

To prove this, we write

$$\tilde{U} = WZ, \quad W \text{ as Step 1,} \quad Z = 1 + \sum_{i=1}^{+\infty} Z_i \lambda^{-i} \in 1 + \text{Off}_n(\mathcal{G})(\lambda) \leq \text{ to be found,}$$

(A.23)

$$\tilde{H} = H + G = \sum_{i=0}^{+\infty} (H_i + G_i) \lambda^{-i}, \quad G_i \in \text{Diag}_n(\mathcal{G}) \text{ to be found}$$

(A.24)

(recall that $\text{Off}_n$ stands for the the off-diagonal $n \times n$ matrices). Due to these representations and to Eq (A.20) for $L$, the diagonalizing condition $L = \tilde{U}(\partial_x - \lambda A - H)\tilde{U}^{-1}$ is fulfilled if

$$(\partial_x - \lambda A - H)Z = Z(\partial_x - \lambda A - H - G).$$

(A.25)

Recalling that $\partial_x Z = Z\partial_x + Zx$ and expanding the last equation in powers of $\lambda$, we see that (A.25) is fulfilled if

$$-G_i + [A, Z_{i+1}] = Z_{i,x} - J_i + \sum_{k=1}^{i} \left( [Z_k, H_{i-k}] + Z_k G_{i-k} - J_{i-k} Z_k \right) \quad (i = 0, 1, 2, ...$$

(A.26)

(intending $Z_{0,x} := (1)_x = 0$ and $\sum_{k=1}^{0} := 0$). We will show that the system (A.26) can be solved recursively. To this purpose, we introduce the projections $\mathcal{D}_n, \mathcal{O}_n$ of $Mat_n(\mathcal{M})$ onto the supplementary subspaces $\text{Diag}_n(\mathcal{M}), \text{Off}_n(\mathcal{M})$; furthermore, recalling that $A$ is diagonal with non zero and all different eigenvalues, we infer that $\text{ad}_A := [A, \cdot] : \text{Off}_n(\mathcal{M}) \to \text{Off}_n(\mathcal{M})$ is a linear isomorphism. These remarks yield for (A.26) the solution

$$G_i = \mathcal{D}_n \left( J_i - \sum_{k=1}^{i} (Z_k G_{i-k} - J_{i-k} Z_k) \right),$$

(A.27)

$$Z_{i+1} = \text{ad}_A^{-1} \left( Z_{i,x} - \mathcal{O}_n(J_i) + \sum_{k=1}^{i} [Z_k, H_{i-k}] + \sum_{k=1}^{i} \mathcal{O}_n(Z_k G_{i-k} - J_{i-k} Z_k) \right) \quad (i = 0, 1, 2, ...$$

(A.28)

(note that $[Z_k, H_{i-k}]$ is purely off-diagonal). From these equations, with $i = 0$, one gets $G_0 = \mathcal{D}_n(J_0)$, $Z_1 = -\text{ad}_A^{-1} \mathcal{O}_n(J_0)$; subsequently, one uses recursively Eqs (A.27) to determine at each step $G_i, Z_{i+1}$. The fact that $Mat_n(\mathcal{G})$ is a differential ideal and the structure of the above equations also make evident that $G_i, Z_{i+1}$ belong to $Mat_n(\mathcal{G})$ for all $i$; by construction these matrices are, respectively, diagonal and off-diagonal as required.

Step 3. Conclusion of the proof. Let us intend $\int \equiv \int^\mathcal{M}$ (recalling the remark on $\int \mathcal{F}$ just before the statement of this Proposition, and the analogous one for $\int \mathcal{F}$). Eq (A.22) implies

$$\bar{h}_i - h_i = \int \bar{H}_i - \int H_i = \int G_i \in \text{Diag}_n(\mathcal{G})$$

(A.28)

the thesis (A.10) follows from here, and from the remark that

$$\int \mathcal{G} = \int \mathcal{I}.$$  

(A.29)

To prove Eq. (A.20), we note that any element of the differential ideal $\mathcal{G}$ has the form

$$G = \sum_{\ell \in \Lambda} F_\ell I^{(\sigma_\ell)} \quad (\text{A a finite set, } F_\ell \in \mathcal{M}, \ I_\ell \in \mathcal{I}, \ \sigma_\ell \in \mathbb{N} \ \forall \ell \in \Lambda),$$

(A.30)
where \( (\sigma_t) \) indicates the \( \sigma_t \)-th power of the derivation \( \partial_x \). Now, standard integration by parts gives

\[
\int G = \int I , \quad I := \sum_{\ell \in \Lambda} (-1)^{\sigma_t} F_\ell^{(\sigma_t)} I_\ell \in \mathcal{I} .
\]  

(A.31)

We now review some known relations between Lax operators, their diagonalization and evolutionary problems. From now on, we work with the Gelfand-Dickey differential algebra

\[
\mathfrak{g} = \mathbb{C}[\xi_1, \ldots, \xi_\gamma, \xi_{1,x}, \ldots, \xi_{\gamma,x}, \ldots]_0
\]  

(A.32)

(see Eq. (1.9)). Also, we are given a vector field \( X = (X_1, \ldots, X_\gamma) \in \mathfrak{g}^3 \); as explained in Section I there are Lie derivative operators \( \mathcal{L}_X : \mathfrak{g} \to \mathfrak{g} \) and \( \mathcal{L}_X : \mathfrak{g} \to \mathfrak{g} \) (see Eqs. (1.17-1.18)); these induce “componentwisely” maps \( \mathcal{L}_X \) of \( \text{Mat}_n(\mathfrak{g}) \) or \( \text{Mat}_n(\mathfrak{g}) \) into itself. Trivially, \( \mathcal{L}_X \) can be extended to a map of \( \text{Mat}_n(\mathfrak{g}) \oplus \text{Mat}_n(\mathfrak{C}) \) into itself, defining it to be zero on \( \text{Mat}_n(\mathfrak{C}) \).

Again trivially, we extend the Lie derivative to a map \( \mathcal{L}_X \) of \( \text{Mat}_n(\mathfrak{g}) \oplus \text{Mat}_n(\mathfrak{C}) \) into itself, setting \( \mathcal{L}_X \left( \sum_i N_i \lambda^{-i} \right) := \sum_i (\mathcal{L}_X N_i) \lambda^{-i} \); we finally define \( \mathcal{L}_X : \mathfrak{g}^n \to \mathfrak{g}^n \) by \( \mathcal{L}_X \left( \sum_k D_k \partial_k^k \right) := \sum_k (\mathcal{L}_X D_k) \partial_k^k \).

A.5 Definition. \( X \) is said to admit an \( n \times n \) Lax formulation if there are an operator \( L = L_{A,S} \in \mathcal{L}_n(\mathfrak{g}) \subset \mathcal{D}_n(\mathfrak{g}) \) and a zero order operator \( C \in \text{Mat}_n(\mathfrak{g})(\lambda) \subset \mathcal{D}_n(\mathfrak{g}) \) such that

\[
\mathcal{L}_X L = [L, C] .
\]  

(A.33)

A.6 Proposition. If \( X \) admits a Lax formulation as above, the fundamental invariants \( h_{L,i} = h_i \) \((i = 0, 1, 2, \ldots)\) are conserved matrix functionals for \( X \):

\[
\mathcal{L}_X h_i = 0 .
\]  

(A.34)

Proof. [5], Section 1.

Of course, from here we get conserved scalar functionals taking all matrix elements \( h_{i,ab} \in \int \mathfrak{g} \).

B Appendix. Lax formalism and Bäcklund transformations for the AKNS theory.

As in Section II we consider the differential algebra

\[
\mathfrak{g} = \mathbb{C}[\xi, \eta, \xi_x, \eta_x, \ldots]
\]  

(B.1)

with two generators \( \xi, \eta \), and the vector field \( X_{\text{AKNS}} = X \). This is known to admit a Lax formulation of the type \( \mathcal{A.36} \), with \( n = 2 \); the matrices \( S, A \) of \( L \), and the matrix \( C \) are given by

\[
S := \begin{pmatrix} 0 & \xi \\ \eta & 0 \end{pmatrix} , \quad A := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad C := \frac{1}{2} \begin{pmatrix} -\xi \eta & \xi_x \\ -\eta \xi_x & \xi \eta \end{pmatrix} + \lambda S + \lambda^2 A .
\]  

(B.2)

This operator has a diagonalizing pair \( U = 1 + U_1/\lambda + U_2/\lambda^2 + U_3/\lambda^3 + \ldots \), \( H = H_0 + H_1/\lambda + H_2/\lambda^2 + H_3/\lambda^3 + \ldots \), where

\[
U_1 = \frac{1}{2} \begin{pmatrix} 0 & -\xi \\ \eta & 0 \end{pmatrix} , \quad U_2 = \frac{1}{4} \begin{pmatrix} 0 & -\xi_x \\ -\eta_x & 0 \end{pmatrix} , \quad U_3 = \frac{1}{8} \begin{pmatrix} 0 & -\xi_{xx} + \xi^2 \eta \\ \eta_{xx} - \xi \eta^2 & 0 \end{pmatrix} ,
\]  

(B.3)
\[H_0 = 0, \quad H_1 = \frac{1}{2} \begin{pmatrix} \xi \eta & 0 \\ 0 & -\xi \eta \end{pmatrix}, \quad H_2 = -\frac{1}{4} \begin{pmatrix} \xi \eta_x & 0 \\ 0 & \eta \xi_x \end{pmatrix}, \quad H_3 = \frac{1}{8} \begin{pmatrix} \xi \eta_{xx} - \xi^2 \eta^2 & 0 \\ 0 & -\eta \xi_{xx} + \xi^2 \eta^2 \end{pmatrix}, \quad H_4 := \frac{1}{16} \begin{pmatrix} -\xi \eta_{xxx} + \xi \xi_x \eta^2 + 4 \xi^2 \eta \eta_x & 0 \\ 0 & -\eta \xi_{xxx} + \eta \xi_x \xi^2 + 4 \eta^2 \xi \xi_x \end{pmatrix}, \ldots \] (B.4)

The fundamental invariants \(h_i := \int h_i\) are conserved functionals for \(X\); for all \(i\), the diagonal elements of \(H_i\) are opposite up to total derivatives, so that
\[h_i = h_{L,i} = \begin{pmatrix} h_i & 0 \\ 0 & -h_i \end{pmatrix}. \quad \text{(B.5)}\]

For \(i = 1, 2, 3, 4\), the conserved scalar functionals \(h_i \in \mathfrak{H}\) are the ones appearing in Eq. (B.25).

Up to a rescaling of each element by a suitable constant, the sequence \((h_i)_{i=1,2,\ldots}\) can be identified with the basis of \(3_{\text{AKNS}} \equiv \mathfrak{H}\) considered in [11]. (The basis in [11] is not defined via the Lax formalism but, rather, by a precise formulation of the known biHamiltonian recursion scheme for the AKNS [6]: the fact that the functionals \((h_i)\) fulfill this recursion scheme reflect a general feature of the Drinfeld-Sokolov approach: see again [5], Section 1 where a general construction is given for the biHamiltonian structure of the evolution equations arising from matrix first order Lax operators).

We come to the Bäcklund transformation. Let us recall the formalism of Section 3 about it: this involves the "initial variables" \(\xi, \eta\) generating \(\mathfrak{g}\), the "final variables" \(\bar{\xi}, \bar{\eta}\) generating \(\bar{\mathfrak{g}}\), and the "auxiliary variable" \(\nu\) generating with all the previous ones the differential algebra \(\mathfrak{M}\) (see Eq. (3.3)).

Of course: the "tilde" map \(\tilde{\mathfrak{g}} \rightarrow \bar{\mathfrak{g}}\) induces componentwisely a tilde map \(\tilde{\nu} : Mat_n(\mathfrak{g}) \rightarrow Mat_n(\bar{\mathfrak{g}})\); the inclusions \(\mathfrak{g}, \bar{\mathfrak{g}} \subset \mathfrak{M}\) induce inclusions of the corresponding spaces of matrices, formal series in \(\lambda\) and differential operators. In particular, we fix the attention on the Lax operators
\[
L_{A,\mathfrak{S}} = L \in \mathfrak{D}_2(\mathfrak{g}) \subset \mathfrak{D}_2(\mathfrak{M}), \quad L_{A,\bar{\mathfrak{S}}} = \bar{L} \in \mathfrak{D}_2(\bar{\mathfrak{g}}) \subset \mathfrak{D}_2(\mathfrak{M}) \quad \text{ (B.6)}
\]

(A, S as in Eq. (3.2); as stipulated before, \(\bar{\mathfrak{S}}\) means \(\begin{pmatrix} 0 & \bar{\xi} \\ \bar{\eta} & 0 \end{pmatrix}\).

**B.1 Lemma.** Let
\[V := 1 + \frac{1}{2\lambda} \begin{pmatrix} \nu & \xi - \bar{\xi} \\ \bar{\eta} - \eta & -\nu \end{pmatrix}. \quad \text{(B.7)}\]

Then
\[
\bar{L} - VLV^{-1} \in Mat_2(\mathfrak{J})(\lambda)_{<}, \quad \text{(B.8)}
\]

where \(\mathfrak{J}\) is the Bäcklund ideal of Def. [5].

**Proof.** One finds by direct computation that
\[
\bar{L}V - VL = V_x - \bar{S}V + VS - \lambda[A,V] = 1 , \quad I := \frac{1}{2\lambda} \begin{pmatrix} I_3 & I_1 \\ -I_2 & -I_3 \end{pmatrix}, \quad \text{(B.9)}
\]

where \(I_j (j = 1, 2, 3)\) are the generators [3.6] of the Bäcklund ideal. This implies \(\bar{L} - VLV^{-1} = F\), where \(F := IV^{-1} \in Mat_2(\mathfrak{J})(\lambda)_{<}\).

The previous Lemma allows to give the

**Proof of Prop. 3.2.** The vector space \(\mathfrak{J}\) is generated by the sequence \((h_i)\), so it suffices to prove that
\[
\bar{h}_i - h_i \in \mathfrak{J} \quad (i = 1, 2, 3, \ldots). \quad \text{(B.10)}
\]

From Eq. (B.26) and its tilded analogue, we know that
\[
\begin{pmatrix} h_i & 0 \\ 0 & -h_i \end{pmatrix}, \quad \begin{pmatrix} \bar{h}_i & 0 \\ 0 & -\bar{h}_i \end{pmatrix} \quad \text{(B.11)}
\]
are the invariant matrix functionals of $L$ and $\tilde{L}$, respectively; by Prop. A.4 and Lemma B.1 these differ by elements of $\text{Diag}_2(f)$, yielding the thesis.

As an example, in Eq. (3.9) we have given explicit representations of $\tilde{h}_i - h_i$ as integrals of elements of $\mathcal{J}$, for $i = 1, 2, 3, 4$; these representations have been computed specializing to this case the general argument employed to prove Prop. A.4. In particular, this has required the application of the recursion relations (A.27) up to $i = 4$.

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