SECOND ORDER ESTIMATES FOR HESSIAN EQUATIONS OF
PARABOLIC TYPE ON RIEMANNIAN MANIFOLDS

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ABSTRACT. In this paper, we establish the second order estimates of solutions to
the first initial-boundary value problem for general Hessian type fully nonlinear
parabolic equations on Riemannian manifolds. The techniques used in this article
can work for a wide range of fully nonlinear PDEs under very general conditions.

Keywords: Fully nonlinear parabolic equations, Riemannian manifolds, a priori
estimates, The first initial-boundary value problem.

1. Introduction

Let $(M^n, g)$ be a compact Riemannian manifold of dimension $n \geq 2$ with smooth
boundary $\partial M$ and $\bar{M} := M \cup \partial M$. We will study the equation

$$(1.1) \quad f(\lambda(\nabla^2 u + A[u])) - u_t = \psi(x, t, u, \nabla u)$$

in $M_T = M \times (0, T] \subset M \times \mathbb{R}$, where $f$ is a symmetric smooth function of $n$ variables,
$\nabla^2 u$ denotes the Hessian of $u(x, t)$ with respect to $x \in M$, $A[u] = A(x, t, \nabla u)$ is a
$(0, 2)$ tensor on $\bar{M}$ which may depend on $t \in [0, T]$ and $\nabla u$ and

$\lambda(\nabla^2 u + A[u]) = (\lambda_1, \ldots, \lambda_n)$

denotes the eigenvalues of $\nabla^2 u + A[u]$ with respect to the metric $g$.

In this paper we are mainly concerned with the a priori $C^2$ estimates for solutions
to (1.1) with boundary condition

$$(1.2) \quad u = \varphi \text{ on } \mathcal{P}M_T,$$

where $\varphi \in C^\infty(\mathcal{P}M_T)$ satisfying $\lambda(\nabla^2 \varphi(x, 0) + A[\varphi(x, 0)]) \in \Gamma$ for all $x \in \bar{M}$. Here
$\mathcal{P}M_T = BM_T \cup SM_T$ is the parabolic boundary of $M_T$ with $BM_T = M \times \{0\}$ and
$SM_T = \partial M \times [0, T]$.

The idea of this paper is mainly from Guan and Jiao [7] where the authors studied
the second order estimates for the elliptic counterpart of (1.1):

$$(1.3) \quad f(\lambda(\nabla^2 u + A(x, u, \nabla u))) = \psi(x, u, \nabla u).$$

Comparing with the elliptic case, the main difficulty in deriving the second order
estimates for the parabolic equation (1.1) is from its degeneracy which is overcome
by using the strict subsolution in this paper. Surprisingly, thanks to the strict subso-
lution, we are able to relax some restrictions to $f$. Again because of the degeneracy,
we do not get the higher estimates and the existence of classical solutions. It is useful to consider viscosity solutions to (1.1) which will be addressed in forthcoming papers.

The first initial-boundary value problem for equation of form (1.1) in $\mathbb{R}^n$ with $A \equiv 0$ and $\psi = \psi(x, t)$ was studied by Ivochkina and Ladyzhenskaya in [8] (when $f = \sigma_n^{1/n}$) and [9]. Jiao and Sui treated the case that $A \equiv \chi(x)$ and $\psi = \psi(x, t)$ on Riemannian manifolds using the techniques of [5] and [7]. For the elliptic Hessian equations on manifolds, we refer the readers to Li [11], Urbas [13], Guan [4, 5, 6], Guan and Jiao [7] and their references.

As in [2], in which the authors studied the equations (1.3) with $A \equiv 0$ and $\psi = \psi(x)$ in a bounded domain of $\mathbb{R}^n$, $f \in C^\infty(\Gamma) \cap C_0(\Gamma)$ is assumed to be defined on $\Gamma$, where $\Gamma$ is an open, convex, symmetric proper subcone of $\mathbb{R}^n$ with vertex at the origin and $\Gamma^+ = \{ \lambda \in \mathbb{R}^n : \text{each component } \lambda_i > 0 \} \subseteq \Gamma$.

and to satisfy the following structure conditions in this paper:

\begin{equation}
(1.4) \quad f_i \equiv \frac{\partial f}{\partial \lambda_i} > 0 \text{ in } \Gamma, \quad 1 \leq i \leq n,
\end{equation}

(1.5) $f$ is concave in $\Gamma$,

and

(1.6) $f > 0$ in $\Gamma$, $f = 0$ on $\partial \Gamma$.

Typical examples are given by $f = \sigma_1^{1/k}$ and $f = (\sigma_k/\sigma_j)^{1/(k-l)}$, $1 \leq l < k \leq n$, defined in the cone $\Gamma_k = \{ \lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, j = 1, \ldots, k \}$, where $\sigma_k(\lambda)$ are the elementary symmetric functions

$$
\sigma_k(\lambda) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}, \quad k = 1, \ldots, n.
$$

Another interesting example is $f = \log P_k$, where

$$
P_k(\lambda) := \prod_{i_1 < \cdots < i_k} (\lambda_{i_1} + \cdots + \lambda_{i_k}), \quad 1 \leq k \leq n,
$$

defined in the cone

$$
\mathcal{P}_k := \{ \lambda \in \mathbb{R}^n : \lambda_{i_1} + \cdots + \lambda_{i_k} > 0 \}.
$$

We call a function $u(x, t)$ admissible if $\lambda(\nabla^2 u + A[u]) \in \Gamma$ in $M \times [0, T]$. It is shown in [2] that (1.4) ensures that equation (1.1) is parabolic for admissible solutions. (1.5) means that the function $F$ defined by $F(A) = f(\lambda[A])$ is concave for $A \in \mathcal{S}^{n \times n}$ with $\lambda[A] \in \Gamma$, where $\mathcal{S}^{n \times n}$ is the set of $n \times n$ symmetric matrices.

Throughout the paper we assume $A[u]$ is smooth on $\bar{M}_T$ for $u \in C^\infty(\bar{M}_T)$, $\psi \in C^\infty(T^*\bar{M} \times [0, T] \times \mathbb{R})$ (for convenience we shall write $\psi = \psi(x, t, z, p)$ for $(x, p) \in T^*\bar{M}$, $t \in [0, T]$ and $z \in \mathbb{R}$ through). Note that for fixed $(x, t) \in \bar{M}_T$ and $p \in T^*_x M$,

$$
A(x, t, p) : T^*_x M \times T^*_x M \rightarrow \mathbb{R}
$$
is a symmetric bilinear map. We shall use the notation

\[ A^\xi \eta(x, \cdot, \cdot) := A(x, \cdot, \cdot)(\xi, \eta), \quad \xi, \eta \in T^*_x M \]

and, for a function \( v \in C^{2,1}_{x,t}(M_T) \), \( A[v] := A(x, t, \nabla v) \), \( A^\xi \eta[v] := A^\xi \eta(x, t, \nabla v) \) (see [7]).

In this paper we assume that there exists an admissible function \( u \in C^2(M_T) \) satisfying

\[ f(\lambda(\nabla^2 u + A[u])) - u \geq \psi(x, t, u, \nabla u) + \delta_0 \quad \text{in} \quad M \times [0, T]. \]

for some positive constant \( \delta_0 \) with \( u = \varphi \) on \( \partial M \times [0, T] \) and \( u \leq \varphi \) in \( M \times \{0\} \).

We shall prove the following Theorem.

**Theorem 1.1.** Let \( u \in C^4(M_T) \) be an admissible solution of (1.7). Suppose (1.4)-(1.6) hold. Assume that

\[ -\psi(x, t, z, p) \] and \( A^\xi(x, t, p) \) are concave in \( p \), \( \forall \xi \in T_x M \),

and

\[ \psi_z \leq 0. \]

Then

\[ \max_{M_T} \mid \nabla^2 u \mid \leq C_1 \left( 1 + \max_{P_M T} \mid \nabla^2 u \mid \right) \]

where \( C_1 > 0 \) depends on \( |u|_{C^4(M_T)} \) and \( |u|_{C^2(M_T)} \). Suppose that \( u \) also satisfies the boundary condition (1.2) and, in addition, assume that

\[ \sum f_i(\lambda)\lambda_i \geq 0, \quad \forall \lambda \in \Gamma, \]

(1.11)

\[ f(\lambda(\nabla^2 \varphi(x, 0) + A[\varphi(x, 0)])) - \varphi_t(x, 0) = \psi[\varphi(x, 0)], \quad \forall x \in \bar{M}, \]

(1.12)

and

\[ \varphi_t(x, t) + \psi(x, t, z, p) > 0 \]

for each \( (x, t) \in SM_T, \ p \in T^*_x \bar{M} \) and \( z \in \mathbb{R} \). Then there exists \( C_2 > 0 \) depending on \( |u|_{C^4(M_T)}, |u|_{C^2(M_T)} \) and \( |\varphi|_{C^4(P_M T)} \) such that

\[ \max_{P_M T} \mid \nabla^2 u \mid \leq C_2. \]

(1.13)

Since \( u \) is admissible, we have, by (1.8),

\[ \triangle u + \text{tr} A_{pk}(x, t, 0) \nabla_k u + \text{tr} A(x, t, 0) \geq \triangle u + \text{tr} A(x, t, \nabla u) > 0 \]

and by the maximum principle it is easy to derive the estimate

\[ \max_{M_T} |u| + \max_{P_M T} |\nabla u| \leq C. \]

(1.14)

Combining with the gradient estimates (Theorem 5.1-5.3), we can prove the following theorem immediately.
Theorem 1.2. Let \( u \in C^4(\bar{M}_T) \) be an admissible solution of (1.7) in \( M_T \) with \( u \geq u \) in \( M_T \) and \( u = \varphi \) on \( \mathcal{P}M_T \). Suppose (1.4)-(1.6), (1.7)-(1.9), and (1.11)-(1.13) hold. Then we have

\[
(1.16) \quad |u|_{C^{2,1}(\bar{M}_T)} \leq C,
\]

where \( C > 0 \) depends on \( n, M \) and \( |u|_{C^2(\bar{M}_T)} \) under any of the following additional assumptions: (i) (5.1)-(5.3) hold for \( \gamma_1 < 4, \gamma_2 = 2 \) in (5.1); (ii) \((M^n, g)\) has nonnegative sectional curvature and (5.1) hold for \( \gamma_1, \gamma_2 < 2 \); (iii) (5.1), (5.16)-(5.20) hold for \( \gamma_1, \gamma_2 < 4 \) in (5.1) and \( \gamma < 2 \) in (5.18)-(5.20).

The rest of this paper is organized as follows. In Section 2, we introduce some preliminaries and present a brief review of some elementary formulas. In Section 3 and Section 4, we establish the global and boundary estimates for second order derivatives respectively. The gradient estimates are derived in Section 5.

2. Preliminaries

Throughout the paper \( \nabla \) denotes the Levi-Civita connection of \((M^n, g)\). The curvature tensor is defined by

\[
R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]}Z.
\]

Let \( e_1, \ldots, e_n \) be local frames on \( M^n \). We denote \( g_{ij} = g(e_i, e_j), \{g^{ij}\} = \{g_{ij}\}^{-1} \). Define the Christoffel symbols \( \Gamma^k_{ij} \) by \( \nabla_{e_i}e_j = \Gamma^k_{ij}e_k \) and the curvature coefficients

\[
R_{ijkl} = g(R(e_k, e_i)e_j, e_i), \quad R^k_{ijkl} = g^{im}R_{mjkl}.
\]

We shall use the notation \( \nabla_i = \nabla_{e_i}, \nabla_{ij} = \nabla_i \nabla_j - \Gamma^k_{ij} \nabla_k, \) etc.

For a differentiable function \( v \) defined on \( M^n \), we usually identity \( \nabla v \) with the gradient of \( v \), and use \( \nabla^2 v \) to denote the Hessian of \( v \) which is locally given by \( \nabla_{ij}v = \nabla_i(\nabla_jv) - \Gamma^k_{ij} \nabla_kv \). We recall that \( \nabla_{ij}v = \nabla_{ji}v \) and

\[
(2.1) \quad \nabla_{ijk}v - \nabla_{jik}v = R^l_{kij} \nabla_l v,
\]

\[
(2.2) \quad \nabla_{ijk}v - \nabla_{kij}v = R^m_{ijk} \nabla_{lm}v + \nabla_i R^m_{ijk} \nabla_m v + R^m_{lik} \nabla_{jm}v.
\]

Let \( u \in C^4(\bar{M}_T) \) be an admissible solution of equation (1.1). For simplicity we shall denote \( U := \nabla^2 u + A(x, t, \nabla u) \) and, under a local frame \( e_1, \ldots, e_n \),

\[
U_{ij} \equiv U(e_i, e_j) = \nabla_{ij}u + A^{ij}(x, t, \nabla u),
\]

\[
(2.3) \quad \nabla_k U_{ij} \equiv \nabla U(e_i, e_j, e_k) = \nabla_{kij}u + \nabla_k A^{ij}(x, t, \nabla u)
\]

\[
\equiv \nabla_{kij}u + A^{ij}_{zk}(x, t, \nabla u) + A^{ij}_{pl}(x, t, \nabla u) \nabla_{kl}u,
\]
such that is admissible satisfying (1.7), there exists a constant , and the following identities hold for

\[ u_t = (\nabla) + A_i^j(x,t,\nabla u) (\nabla u)_t \]

\[ \nabla_i u_t + A_i^j(x,t,\nabla u) (\nabla u)_t, \]

where \( A_i^j = A^i_{e_j} \) and \( A^i_{e_j} \) denotes the partial covariant derivative of \( A \) when viewed as depending on \( x \in M \) only, while the meanings of \( A_i^j \) and \( A^i_{p_k} \), etc are obvious. Similarly we can calculate \( \nabla_{kl} U_{ij} = \nabla_k \nabla_i U_{ij} - \Gamma_{kl}^m \nabla_m U_{ij} \), etc.

Let \( F \) be the function defined by

\[ F(h) = f(\lambda(h)) \]

for a \((0,2)\) tensor \( h \) on \( M \).

Following the literature we denote throughout this paper

\[ F^{ij} = \frac{\partial F}{\partial h_{ij}}(U), \quad F^{ij,kl} = \frac{\partial^2 F}{\partial h_{ij} \partial h_{kl}}(U) \]

under an orthonormal local frame \( e_1, \ldots, e_n \). The matrix \( \{F^{ij}\} \) has eigenvalues \( f_1, \ldots, f_n \) and is positive definite by assumption (1.4), while (1.5) implies that \( F \) is a concave function of \( U_{ij} \) (see [1]). Moreover, when \( \{U_{ij}\} \) is diagonal so is \( \{F^{ij}\} \), and the following identities hold

\[ F^{ij} U_{ij} = \sum f_i \lambda_i, \quad F^{ij} U_{ik} U_{kj} = \sum f_i \lambda_i^2, \quad \lambda(U) = (\lambda_1, \ldots, \lambda_n). \]

Define the linear operator \( L \) locally by

\[ Lv = F^{ij} \nabla_{ij} v + (F^{ij} A^j_{p_k} - \psi_{p_k}) \nabla_k v - v_t, \]

for \( v \in C^{2,1}_{x,t}(M_T) \). We can prove

**Theorem 2.1.** Let \( u \) be an admissible solution to (1.1) with \( u \geq \bar{u} \) in \( M_T \). Assume that (1.4), (1.5), (1.8) and (1.9) hold. Then there exists a constant \( \theta > 0 \) depending only on \( \delta_0 \) and \( \bar{u} \) such that

\[ L(u - \bar{u}) \geq \theta(1 + \sum F^{ii}) \]

**Proof.** Since \( u \) is admissible satisfying (1.7), there exists a constant \( \varepsilon_0 > 0 \) such that \( \{x \in M_T : \lambda(\nabla^2 u + A[u] - \varepsilon_0 g)\} \) is a compact subset of \( \Gamma \) and

\[ f(\lambda(\nabla^2 u + A[u] - \varepsilon_0 g)) - \bar{u} \geq \psi(\bar{u}) + \frac{\delta_0}{2} \text{ in } M_T. \]

Let \( \theta = \min\{\frac{f_u}{2}, \varepsilon_0\} \). For each \((x,t) \in M_T\), we may assume \( \{U_{ij}\} = \{\nabla_{ij} u + A^{ij}\} \) is diagonal at \((x,t)\). From (1.8), (1.9) and the concavity of \( F \), we see, at \((x,t)\),

\[ F^{ii}(U_{ii} - \varepsilon_0 g_{ii} - U_{ii}) - (u - \bar{u})_t \geq \psi(x,t,u,\nabla u) - \psi(x,t,u,\nabla u) + \frac{\delta_0}{2} \]

\[ \geq \psi(x,t,u,\nabla u) - \psi(x,t,u,\nabla u) + \frac{\delta_0}{2} \]

\[ \geq \psi_{p_k} \nabla_k (u - \bar{u}) + \frac{\delta_0}{2}, \]

for a \((0,2)\) tensor \( h \) on \( M \).
By (1.8) again, we have
\[ F^{ii}(U_{ii} - U_{ii}) = F^{ii}\nabla_{ii}(u - u) + F^{ii}(A^{ii}(x, t, \nabla u) - A^{ii}(x, t, \nabla u)) \geq F^{ii}\nabla_{ii}(u - u) + F^{ii}A_{pi}^{ii}\nabla_{k}(u - u). \]

Combining (2.6) and (2.7), we get
\[ \mathcal{L}(u - u) \geq \varepsilon_0 \sum F^{ii} + \delta_0 \geq \theta(1 + \sum F^{ii}). \]

3. Global estimates for second derivatives

In this section, we prove (1.10) in Theorem 1.1 for which we set
\[ W = \max_{(x, t) \in \bar{M}_T} \max_{\xi \in T_x \bar{M}, |\xi| = 1} (\nabla \xi \xi u + A \xi \xi (x, u, \nabla u)e^\phi), \]
as in [7], where \( \phi \) is a function to be determined. It suffices to estimate \( W \). We may assume \( W \) is achieved at \( (x_0, t_0) \in \bar{M}_T - P M_T \). Choose a smooth orthonormal local frame \( e_1, \ldots, e_n \) about \( x_0 \) such that \( \nabla_i e_j = 0 \), and \( U \) is diagonal at \( (x_0, t_0) \). We assume \( U_{11}(x_0, t_0) \geq \ldots \geq U_{nn}(x_0, t_0) \). We have \( W = U_{11}(x_0, t_0)e^{\phi(x_0, t_0)} \).

At the point \( (x_0, t_0) \) where the function \( \log U_{11} + \phi \) attains its maximum, we have
\[ \frac{\nabla_i U_{11}}{U_{11}} + \nabla_i \phi = 0 \text{ for each } i = 1, \ldots, n, \]
\[ \frac{(U_{11})_t}{U_{11}} + \phi_t \geq 0, \]
and
\[ 0 \geq \sum_i F^{ii} \left\{ \frac{\nabla_i U_{11}}{U_{11}} - \left( \frac{\nabla_i U_{11}}{U_{11}} \right)^2 + \nabla_{ii} \phi \right\}. \]

Differentiating equation (1.11) twice, we find
\[ F^{ii} \nabla_k U_{ii} - \nabla_k u_t = \psi_{x_k} + \psi_{u} \nabla_k u + \psi_{p_i} \nabla_{kj} u, \text{ for all } k, \]
and
\[ F^{ii} \nabla_{11} U_{ii} + F^{ij,kl} \nabla_{1} U_{ij} \nabla_{1} U_{kl} - \nabla_{11} u_t \geq \psi_{p_i} \nabla_{11} u + \psi_{p_{p_k}} \nabla_{1k} u \nabla_{1i} u - C U_{11} \]
\[ \geq \psi_{p_i} \nabla_{j} U_{11} + \psi_{p_{p_l}} U_{11}^2 - C U_{11} \]
\[ = - U_{11} \psi_{p_j} \nabla_{j} \phi + \psi_{p_{p_l}} U_{11}^2 - C U_{11}. \]
Next, by (3.1) and (3.4),
\begin{equation}
F^{ii}(\nabla_{ii} A^{11} - \nabla_{11} A^{ii}) \geq F^{ii}(A^{11}_{i} \nabla_{ij} u - A^{ii}_{j} \nabla_{jj} u) \\
+ F^{ii}(A^{11}_{p} U^{2}_{ii} - A^{ii}_{p} U^{2}_{ii}) - CU_{11} \sum F^{ii}
\end{equation}
\begin{equation}
\geq U_{11} F^{ii} A^{11}_{p} \nabla_{j} \phi + A^{11}_{p} \nabla_{j} u t - CU_{11} \sum F^{ii} - CU_{11} \\
- C \sum_{i \geq 2} F^{ii} U^{2}_{ii} - U^{2}_{ii} \sum F^{ii} A^{ii}_{pj}.
\end{equation}

Note that
\begin{equation}
\nabla_{ii} U_{11} \geq \nabla_{11} U_{ii} + \nabla_{ii} A^{11} - \nabla_{11} A^{ii} - CU_{11}.
\end{equation}
Thus, by (3.5), (3.6) and (3.2), we have, at \((x_{0}, t_{0})\),
\begin{equation}
F^{ii} \nabla_{ii} U_{11} \geq F^{ii} \nabla_{11} U_{ii} - CU_{11}(1 + \sum F^{ii}) + A^{11}_{i} \nabla_{j} u t \\
- C \sum_{i \geq 2} F^{ii} U^{2}_{ii} - U^{2}_{ii} \sum F^{ii} A^{ii}_{pj} + U_{11} F^{ii} A^{ii}_{pj} \nabla_{j} \phi \\
\geq U_{11} \mathcal{L} \phi - U_{11} F^{ii} \nabla_{ii} \phi - F^{ij,kl} \nabla_{1i} U_{ij} \nabla_{1k} U_{kl} + \psi_{p1p1} U^{2}_{11} \\
- CU_{11}(1 + \sum F^{ii}) - CF^{ii} U^{2}_{ii} - U^{2}_{ii} \sum F^{ii} A^{ii}_{p1p1}.
\end{equation}

It follows that, by (3.3),
\begin{equation}
\mathcal{L} \phi \leq U_{11} \sum_{i \geq 2} F^{ii} A^{ii}_{p1p1} - \psi_{p1p1} U_{11} + C(1 + \sum F^{ii}) + \frac{C}{U_{11}} F^{ii} U^{2}_{ii} + E,
\end{equation}
where
\begin{equation}
E = \frac{1}{U^{2}_{11}} F^{ii}(\nabla_{i} U_{11})^{2} + \frac{1}{U_{11}} F^{ij,kl} \nabla_{1i} U_{ij} \nabla_{1k} U_{kl}.
\end{equation}

Let
\begin{equation}
\phi = \frac{\delta |\nabla u|^{2}}{2} + b \eta,
\end{equation}
where \(b, \delta\) are undetermined constants, \(0 < \delta < 1 \leq b\), and \(\eta\) is a \(C^{2}\) function which may depend on \(u\) but not on its derivatives. We calculate, at \((x_{0}, t_{0})\),
\begin{equation}
\nabla_{i} \phi = \delta \nabla_{j} u \nabla_{ij} u + b \nabla_{i} \eta = \delta \nabla_{i} u U_{ii} - \delta \nabla_{j} u A^{ij} + b \nabla_{i} \eta
\end{equation}
\begin{equation}
\phi_{t} = \delta \nabla_{j} u (\nabla_{j} u)_{t} + b \eta_{t}
\end{equation}
\begin{equation}

\nabla_{ii} \phi \geq \frac{\delta}{2} U^{2}_{ii} - C \delta + \delta \nabla_{j} u \nabla_{ij} u + b \nabla_{ii} \eta.
\end{equation}
From (2.1) and (3.4), we derive

\[
F^{ii} \nabla_j u \nabla_{ijj} u \geq F^{ii} \nabla_j u (\nabla_j U^{ii} - \nabla_j A^{ii}) - C |\nabla u|^2 \sum F^{ii} \\
\geq (\psi_{pi} - F^{ii} A^{ii}) \nabla_j u \nabla_{ijj} u + \nabla_j u \nabla_j (u_t) - C(1 + \sum F^{ii}).
\]

(3.13)

Therefore,

\[
L \phi \geq b L \eta + \frac{\delta}{2} F^{ii} U_{ii}^2 - C \sum F^{ii} - C.
\]

(3.14)

Let \( \eta = \bar{u} - u \). We get from (3.10) that

\[
(\nabla_i \phi)^2 \leq C \delta^2 (1 + U_{ii}^2) + 2b^2 (\nabla_i \eta)^2 \leq C \delta^2 U_{ii}^2 + C b^2.
\]

(3.15)

For fixed \( 0 < s \leq 1/3 \) let

\[
J = \{ i : U_{ii} \leq -s U_{11} \}, \quad K = \{ i : U_{ii} > -s U_{11} \}.
\]

Using a result of Andrews \[1\] and Gerhardt \[3\] as in \[5\] and \[7\] (see \[13\] also), we have

\[
E \leq C b^2 \sum_{i \in J} F^{ii} + C \delta^2 \sum_{i \in J} F^{ii} U_{ii}^2 + C \sum F^{ii} + C (\delta^2 U_{11}^2 + b^2) F^{11}.
\]

(3.16)

Therefore, by (3.9), (3.14) and (3.16), we have

\[
b L \eta \leq \left( C \delta^2 - \frac{\delta}{2} + \frac{C}{U_{11}} \right) F^{ii} U_{ii}^2 + C b^2 \sum_{i \in J} F^{ii} + C \sum F^{ii} \\
+ C(\delta^2 U_{11}^2 + b^2) F^{11} + C
\]

\[
\leq \left( C \delta^2 - \frac{\delta}{2} + \frac{C}{U_{11}} \right) F^{ii} U_{ii}^2 + C b^2 \sum_{i \in J} F^{ii} + C \sum F^{ii} \\
+ C b^2 F^{11} + C.
\]

(3.17)

Choose \( \delta \) sufficiently small such that \( C \delta^2 - \frac{\delta}{2} \) is negative and let

\[
c_1 := -\frac{1}{2} \left( C \delta^2 - \frac{\delta}{2} \right) > 0.
\]

We may assume

\[
C \delta^2 - \frac{\delta}{2} + \frac{C}{U_{11}} \leq -c_1
\]

for otherwise we have \( U_{11} \leq \frac{C}{c_1} \) and we are done. Thus, by (2.5), choosing \( b \) sufficiently large, we derive from (3.17) that

\[
c_1 F^{ii} U_{ii}^2 - C b^2 F^{11} - C b^2 \sum_{i \in J} F^{ii} \leq 0.
\]

Then we can get a bound \( U_{11}(x_0, t_0) \leq C \) since \( |U_{ii}| \geq s U_{11} \) for \( i \in J \). The proof of (1.10) is completed.
4. Boundary estimates for second derivatives

In this section, we consider the estimates of second order derivatives on parabolic boundary $\mathcal{P}M_T$. We may assume $\varphi \in C^4(M_T)$.

Fix a point $(x_0, t_0) \in SM_T$. We shall choose smooth orthonormal local frames $e_1, \ldots, e_n$ around $x_0$ such that when restricted to $\partial M$, $e_n$ is normal to $\partial M$. Since $u - \underline{u} = 0$ on $SM_T$ we have

\begin{equation}
\nabla_{\alpha\beta}(u - \underline{u}) = -\nabla_n(u - \underline{u})\Pi(e_{\alpha}, e_{\beta}), \quad \forall \ 1 \leq \alpha, \beta < n \text{ on } SM_T,
\end{equation}

where $\Pi$ denotes the second fundamental form of $\partial M$. Therefore,

\begin{equation}
|\nabla_{\alpha\beta}u| \leq C, \quad \forall \ 1 \leq \alpha, \beta < n \text{ on } SM_T.
\end{equation}

Let $\rho(x)$ denote the distance from $x \in M$ to $x_0$,

$$
\rho(x) \equiv \text{dist}_{M^n}(x, x_0),
$$

and set

$$
M_\delta = \{X = (x, t) \in M \times (0, T] : \rho(x) < \delta, t \leq t_0 + \delta\}.
$$

For the mixed tangential-normal and pure normal second derivatives at $(x_0, t_0)$, we shall use the following barrier function as in [5],

\begin{equation}
\Psi = A_1v + A_2\rho^2 - A_3\sum_{1 < l<n} |\nabla_l(u - \varphi)|^2
\end{equation}

where $v = u - \underline{u}$. By differentiating the equation (1.1) and straightforward calculation, we obtain

\begin{equation}
\mathcal{L}(\nabla_k(u - \varphi)) \leq C\left(1 + \sum f_i|\lambda_i| + \sum f_i\right), \quad \forall \ 1 \leq k \leq n.
\end{equation}

Similar to [5] (see [7] also), using Proposition 2.19 and Corollary 2.21 of [5] and Theorem 2.1, we can prove that there exist uniform positive constants $\delta$ sufficiently small, and $A_1, A_2, A_3$ sufficiently large such that

\begin{equation}
\mathcal{L}(\Psi \pm \nabla_{\alpha}(u - \varphi)) \leq 0 \text{ in } M_\delta
\end{equation}

and $\Psi \pm \nabla_{\alpha}(u - \varphi) \geq 0$ on $\mathcal{P}M_\delta$. Thus, by the maximum principle, we see $\Psi \pm \nabla_{\alpha}(u - \varphi) \geq 0$ in $M_\delta$. Then we get

\begin{equation}
|\nabla_{n\alpha}u(x_0, t_0)| \leq \nabla_n\Psi(x_0, t_0) \leq C, \quad \forall \ \alpha < n.
\end{equation}

It remains to derive

\begin{equation}
\nabla_{n\alpha}u(x_0, t_0) \leq C
\end{equation}

since $\Delta u \geq -C$. We shall use an idea of Trudinger [12] as [5] and [7] to prove that there exist uniform positive constants $c_0$, $R_0$ such that for all $R > R_0$, $(\lambda'[U], R) \in \Gamma$ and

\begin{equation}
f(\lambda'[U], R) - u_t \geq \psi[u] + c_0 \text{ on } SM_T
\end{equation}
which implies (4.7) by Lemma 1.2 in [2], where \( \lambda[U] = (\lambda'_1, \ldots, \lambda'_{n-1}) \) denote the eigenvalues of the \((n - 1) \times (n - 1)\) matrix \( \{U_{\alpha\beta}\}_{1 \leq \alpha, \beta \leq (n-1)} \) and \( \psi[u] = \psi(\cdot, \cdot, u, \nabla u) \).

For \( R > 0 \) and a symmetric \((n-1)^2\) matrix \( \{r_{\alpha\beta}\} \) with \((\lambda'\{r_{\alpha\beta}\}, R) \in \Gamma\), define

\[
G[r_{\alpha\beta}] \equiv f(\lambda'[\{r_{\alpha\beta}\}], R)
\]

and consider

\[
m_R \equiv \min_{(x, t) \in \overline{SM_T}} G[U_{\alpha\beta}(x, t)] - u_t(x, t) - \psi[u].
\]

Note that \( G \) is concave and \( m_R \) is increasing in \( R \) by (1.4), and that

\[
c_R \equiv \inf_{\overline{SM_T}} (G[U_{\alpha\beta}] - u - \psi[u]) \geq \inf_{\overline{SM_T}} (G[U_{\alpha\beta}] - F[U_{ij}]) > 0
\]

when \( R \) is sufficiently large.

We wish to show \( m_R > 0 \) for \( R \) sufficiently large. Without loss of generality we assume \( m_R < c_R/2 \) (otherwise we are done) and suppose \( m_R \) is achieved at a point \((x_0, t_0) \in \overline{SM_T}\). Choose local orthonormal frames around \( x_0 \) as before and assume \( \nabla_{nn} u(x_0, t_0) \geq \nabla_{nn} u(x_0, t_0) \). Let \( \sigma_{\alpha\beta} = \langle \nabla_\alpha e_\beta, e_n \rangle \) and

\[
G'_{0\alpha\beta} = \frac{\partial G}{\partial r_{\alpha\beta}}[U_{\alpha\beta}(x_0, t_0)].
\]

Note that \( \sigma_{\alpha\beta} = \Pi(e_\alpha, e_\beta) \) on \( \partial M \) and that

\[
G'_{0\alpha\beta}(r_{\alpha\beta} - U_{\alpha\beta}(x_0, t_0)) \geq G[r_{\alpha\beta}] - G[U_{\alpha\beta}(x_0, t_0)]
\]

for any symmetric matrix \( \{r_{\alpha\beta}\} \) with \((\lambda'[\{r_{\alpha\beta}\}], R) \in \Gamma\) by the concavity of \( G \).

In particular, since \( u_t = \overline{u}_t = \varphi_t \) on \( \overline{SM_T} \), we have

\[
G'_{0\alpha\beta} U_{\alpha\beta} - \psi[u] - \varphi_t - G'_{0\alpha\beta} U_{\alpha\beta}(x_0, t_0) + \psi[u](x_0, t_0) + u_t(x_0, t_0)
\]

\[
\geq G[U_{\alpha\beta}] - \psi[u] - u_t - m_R \geq 0
\]

on \( \overline{SM_T} \).

From (4.11) we see that

\[
U_{\alpha\beta} = U_{\alpha\beta} - \nabla_n (u - \overline{u}) \sigma_{\alpha\beta} + A_{\alpha\beta}[u] - A_{\alpha\beta}[\overline{u}] \text{ on } \overline{SM_T}.
\]
Note that at \((x_0, t_0)\), we have

\[
\nabla_n (u - \bar{u}) G_0^{\alpha \beta} \sigma_{\alpha \beta} = G_0^{\alpha \beta} (U_{\alpha \beta} - U_{\alpha \beta}) + G_0^{\alpha \beta} (A^{\alpha \beta}[u] - A^{\alpha \beta}[\bar{u}])
\geq G[U_{\alpha \beta}] - G[U_{\alpha \beta}] + G_0^{\alpha \beta} (A^{\alpha \beta}[u] - A^{\alpha \beta}[\bar{u}])
\geq c_R - m_R + \psi[u] - u_t - \psi[u] - u_t
\geq c R - m R + \psi[u] - u_t
\geq c R - m R + \psi[u] - u_t
\geq c R - m R + \psi[u] - u_t
\geq \frac{c R}{2} + H[u] - H[u]
\]

(4.12)

where \(H[u] = G_0^{\alpha \beta} A^{\alpha \beta}[u] - \psi[u]\).

Define

\[
\Phi = -\eta \nabla_n (u - \bar{u}) + H[u] - \varphi_t + Q
\]

where \(\eta = G_0^{\alpha \beta} \sigma_{\alpha \beta}\) and

\[
Q \equiv G_0^{\alpha \beta} \nabla_{\alpha \beta} \bar{u} - G_0^{\alpha \beta} U_{\alpha \beta}(x_0, t_0) + \psi[u](x_0, t_0) + u_t(x_0, t_0).
\]

By virtue of \((4.10)\) and \((4.11)\) we see that \(\Phi \geq 0\) on \(\overline{S M_T}\) and \(\Phi(x_0, t_0) = 0\). Next, by \((4.4)\) and \((1.8)\),

\[
\mathcal{L} H \leq H_z[u] \mathcal{L} u + H_{pk}[u] \mathcal{L} \nabla_k u + F^{ij} H_{p k l}[u] \nabla_{k l} u \nabla_{i j} u
+ C \left( \sum F^{i i} + \sum f_i | \lambda_i | + 1 \right)
\leq C \left( \sum F^{i i} + \sum f_i | \lambda_i | + 1 \right) + H_z[u] \mathcal{L} u.
\]

Since \(H_z[u] \geq 0\), by Theorem 2.1 we have

\[
\mathcal{L} u = \mathcal{L}(u - \bar{u}) + \mathcal{L} u \leq C (1 + \sum F^{i i}).
\]

It follows that

\[
\mathcal{L} H \leq C \left( \sum F^{i i} + \sum f_i | \lambda_i | + 1 \right).
\]

Therefore,

(4.13)

\[
\mathcal{L} \Phi \leq C \left( \sum F^{i i} + \sum f_i | \lambda_i | + 1 \right).
\]

By the compatibility condition \((1.12)\), we find that

\[
c'_R \equiv \inf_{x \in M} G(\nabla_{\alpha \beta} \varphi + A[\varphi])(x, 0) - \psi[\varphi](x, 0) - \varphi_t(x, 0) > 0
\]
when $R$ is sufficiently large. We may assume $m_R < \frac{c_R'}{2}$ (otherwise we are done). For $x \in \bar{M}$, by the concavity of $G$ again, we have

$$\Phi(x, 0) = G_0^{\alpha\beta}(U_{\alpha\beta}(x, 0) - U_{\alpha\beta}(x_0, t_0))$$
$$\quad - \psi[u](x, 0) - \varphi_t(x, 0) + \psi[u](x_0, t_0) + u_t(x_0, t_0)$$
$$= G_0^{\alpha\beta}(A_{\alpha\beta}\varphi + A[\varphi](x, 0) - U_{\alpha\beta}(x_0, t_0))$$
$$\quad - \varphi(x, 0) + u_t(x_0, t_0) + \psi[u](x_0, t_0) - \psi[\varphi](x_0)$$
$$\geq G(\nabla_{\alpha\beta}\varphi + A[\varphi](x, 0) - G(U_{\alpha\beta}(x_0, t_0))$$
$$\quad - \varphi(x, 0) + u_t(x_0, t_0) + \psi[u](x_0, t_0) - \psi[\varphi](x_0)$$
$$\geq c_R' - m_R > \frac{c_R'}{2}.$$  

It means that $\Phi > 0$ on $BM_T$. Thus, we get $\Phi \geq 0$ on $PM_\delta$.

Consider the function $\Psi$ defined in (4.3) as before. Similarly, there exist another group of constants $A_1 \gg A_2 \gg A_3 \gg 1$ such that

$$\begin{cases}
\mathcal{L}(\Psi + \Phi) \leq 0 & \text{in } M_\delta, \\
\Psi + \Phi \geq 0 & \text{on } PM_\delta.
\end{cases}$$

By the maximum principle we find $\Psi + \Phi \geq 0$ in $M_\delta$. It follows that $\nabla_n \Phi(x_0, t_0) \geq -\nabla_n \Psi(x_0, t_0) \geq -C$.

Following [7], we write $u^s = su + (1 - s)\underline{u}$ and

$$H[u^s] = G_0^{\alpha\beta}A^{\alpha\beta}[u^s] - \psi[u^s].$$

We have

$$H[u] - H[u] = \int_0^1 \frac{dH[u^s]}{dt} ds$$
$$= (u - \underline{u}) \int_0^1 H_z[u^s] ds + \sum \nabla_k(u - \underline{u}) \int_0^1 H_{pk}[u^s] ds.$$ 

Therefore, at $(x_0, t_0)$,

$$H[u] - H[u] = \nabla_n(u - \underline{u}) \int_0^1 H_{pn}[u^s] ds$$

(4.15)
and
\[
\nabla_n H[u] = \nabla_n H[u] + \sum \nabla_{kn}(u - u) \int_0^1 H_p[u^s]ds \\
+ \nabla_n(u - u) \int_0^1 (H_z[u^s] + H_{x_n}[u^s] + H_{x_n}[u]\nabla_n u^s)ds \\
(4.16)
+ \nabla_n(u - u) \sum \int_0^1 H_p[u^s]\nabla_{ln} u^s ds \\
\leq \nabla_{nn}(u - u) \int_0^1 (H_p[u^s] + sH_p[u^s]\nabla_n(u - u))ds + C \\
\leq \nabla_{nn}(u - u) \int_0^1 H_p[u^s]ds + C
\]
since \( H_{p,n} \leq 0, \nabla_{nn}(u - u) \geq 0 \) and \( \nabla_n(u - u) \geq 0 \). It follows that
\[
\nabla_n\Phi(x, t, 0) \leq -\eta(x, t, 0)\nabla_{nn}(x, t, 0) + \nabla_n H[u](x, t, 0) + C \\
(4.17)
\]
\[
\leq \left( -\eta(x, t, 0) + \int_0^1 H_p[u^s](x, t, 0)ds \right)\nabla_{nn} u(x, t, 0) + C.
\]
By (4.12) and (4.15),
\[
\eta(x, t, 0) - \int_0^1 H_p[u^s](x, t, 0)ds \geq \frac{c_R}{2\nabla_n(u - u)(x, t, 0)} \geq \epsilon_1 c_R > 0
\]
for some uniform \( \epsilon_1 > 0 \) independent of \( R \). This gives
\[
\nabla_{nn} u(x, t, 0) \leq \frac{C}{\epsilon_1 c_R}. \quad (4.19)
\]
So we have an \textit{a priori} upper bound for all eigenvalues of \( \{U_{ij}(x, t, 0)\} \). Now by (1.13), there exists a constant \( \nu_0 > 0 \) such that
\[
\inf_{(x, t) \in \mathbb{S}^M_T} \varphi_t(x, t) + \psi(x, t, u, \nabla u) \geq \nu_0.
\]
It follows that \( \lambda[\{U_{ij}(x, t, 0)\}] \) is contained in a compact subset of \( \Gamma \) by (1.6), and therefore
\[
m_R = G[U_{0, \beta}(x, t, 0)] - u_t(x, t, 0) - \psi[u](x, t, 0) > 0
\]
when \( R \) is sufficiently large. Then (1.8) is valid and the proof of (1.14) is completed.

5. Gradient estimates

In this section we establish the gradient estimates to prove Theorem 5.1-5.3 below. Throughout the section, we assume (1.4)-(1.5), (1.8) and the following growth conditions hold
\[
\begin{align*}
p \cdot \nabla_x A^{\xi}(x, t, z, p) \leq \psi_1(x, t, z)|\xi|^2(1 + |p|^{\gamma_1}) \\
p \cdot \nabla_x \psi(x, t, z, p) + |p|^2\psi_x(x, t, z, p) \geq -\psi_2(x, t, z)(1 + |p|^{\gamma_2})
\end{align*}
(5.1)
\]
for some functions $\tilde{\psi}_1, \tilde{\psi}_2 \geq 0$ and constants $\gamma_1, \gamma_2 > 0$.

Since the proofs of Theorem 5.1-5.3 are similar to those of Theorem 6.1-6.3 in [7], we only provide a sketch here. For more details we refer the reader to [7] where the elliptic Hessian equations are treated.

**Theorem 5.1.** Let $u \in C^3(\bar{M}_T)$ be an admissible solution of (1.1). Assume, in addition, that

$$\lim_{\sigma \to \infty} f(\sigma 1) = +\infty$$

where $1 = (1, \ldots, 1) \in \mathbb{R}^n$ and there exists a constant $c_0 > 0$ such that

$$A_{x^i x^j}(x, t, p) \eta_i \eta_j \leq -c_0|\xi|^2|\eta|^2 + c_0|g(\xi, \eta)|^2, \quad \forall \xi, \eta \in T_x M.$$  

Suppose that $\gamma_1 < 4, \gamma_2 = 2$ in (5.1), and that there is an admissible function $u \in C^2(\bar{M}_T)$. Then

$$\max_{\bar{M}_T} |\nabla u| \leq C_3 \left(1 + \max_{\bar{P}_M} |\nabla u|\right)$$

where $C_3$ is a positive constant depending on $|u|_{C^0(\bar{M}_T)}$ and $|u|_{C^2(\bar{M}_T)}$.

**Proof.** Let $w = |\nabla u|$ and $\phi$ a positive function to be determined. Suppose the function $w \phi^{-a}$ achieves a positive maximum at an interior point $(x_0, t_0) \in M_T - \bar{P}_M$ where $a < 1$ is a positive constant. Choose a smooth orthonormal local frame $e_1, \ldots, e_n$ about $x_0$ such that $\nabla_i e_j = 0$ at $x_0$ and $\{U_{ij}(x, t_0)\}$ is diagonal.

The function $\log w - a \log \phi$ attains its maximum at $(x_0, t_0)$ where for $i = 1, \ldots, n$,

$$\frac{\nabla_i w}{w} - a \frac{\nabla_i \phi}{\phi} = 0,$$

$$\frac{w_t}{w} - a \frac{\phi_t}{\phi} \geq 0$$

and

$$\frac{\nabla_i w}{w} + \frac{(a - a^2)|\nabla_i \phi|^2}{\phi^2} - a \frac{\nabla_i \phi}{\phi} \leq 0.$$  

Note that

$$w \nabla_i w = \nabla_i u \nabla_i u, \quad w w_t = \nabla_i u (\nabla_i u)_t.$$

By (2.1), (5.5) and (3.4),

$$w \nabla_i w = \nabla_i u \nabla_i u + \nabla_i u \nabla_i u - \nabla_i w \nabla_i w = \left(\nabla_i u + R_{iil}^k \nabla_k u\right) \nabla_i u + \left(\delta_{kl} - \frac{\nabla_k u \nabla_i u}{w^2}\right) \nabla_k u \nabla_l u$$

$$\geq \left(\nabla_i U_{ij} - A_{x^i x^j} A_{x^i x^j} \nabla_i u - C|\nabla u|^2\right) \nabla_i u - \frac{aw^2}{\phi} A_{x^i x^j} \nabla_k \phi - \nabla_i u A_{x^i} - Cw^2.$$  

$$w \nabla_i w = \nabla_i u \nabla_i u + \nabla_i u \nabla_i u - \nabla_i w \nabla_i w$$

$$\geq \left(\nabla_i U_{ij} - A_{x^i x^j} A_{x^i x^j} \nabla_i u - C|\nabla u|^2\right) \nabla_i u - \frac{aw^2}{\phi} A_{x^i x^j} \nabla_k \phi - \nabla_i u A_{x^i} - Cw^2.$$
By (3.4), (5.5) and (5.6),
\begin{equation}
F^{ii} \nabla_t u \nabla_t U_{ii} = \nabla_t u \psi_{x_i} + \psi_u |\nabla u|^2 + \psi_{p_k} \nabla_t u \nabla_{lk} u + \nabla_t u \nabla_t u_t \\
\geq \nabla_t u \psi_{x_i} + \psi_u |\nabla u|^2 + \frac{aw^2}{\phi} \psi_{p_k} \nabla_k \phi + \frac{aw^2}{\phi} \phi_t.
\end{equation}

Let \( \phi = (u - u) + b > 0 \), where \( b = 1 + \sup_{\bar{M}_T} (u - u) \).

By (5.3) we have
\begin{equation}
-A^{ii}_{p_k} \nabla_k \phi = A^{ii}_{p_k}(x, t, \nabla u) \nabla_k (u - u) \\
\geq A^{ii}(x, t, \nabla u) - A^{ii}(x, t, \nabla u) + \frac{c_0}{2} (|\nabla \phi|^2 - |\nabla_i \phi|^2).
\end{equation}

We may assume that \( c_0 \) is sufficiently small and that
\[
\frac{2a - 2a^2 - c_0 a \phi}{2\phi^2} > 0
\]
by choosing \( a \) sufficiently small.

Thus, by (5.7), (5.8), (5.9) and (5.10), we find
\begin{equation}
0 \geq a \frac{\phi}{F^{ii}} (U_{ii} - U_{i}) + \frac{ac_0 |\nabla \phi|^2}{2\phi} \sum F^{ii} + \frac{2a - 2a^2 - c_0 a \phi}{2\phi^2} F^{ii} |\nabla_i \phi|^2 \\
- \frac{1}{w^2} F^{ii}_{x_i} \nabla_l u + \frac{1}{w^2} \psi_{x_i} \nabla_l u + \psi_u + \frac{a}{\phi} \psi_{p_k} \nabla_k \phi + \frac{a}{\phi} \phi_t - C \sum F^{ii} \\
\geq a \frac{\phi}{F^{ii}} (U_{ii} - U_{i}) + \frac{ac_0 |\nabla \phi|^2}{2\phi} \sum F^{ii} - C \sum F^{ii} \\
+ \frac{a}{\phi} (\psi(x, t, u, \nabla u) - \psi(x, t, u, \nabla u)) \\
- \frac{1}{w^2} F^{ii}_{x_i} \nabla_l u + \frac{1}{w^2} \psi_{x_i} \nabla_l u + \psi_u + \frac{a}{\phi} (u - u)_t.
\end{equation}

Choose \( B > 0 \) sufficiently large such that (see [7])
\[
F(2Bg + U) \geq F(Bg) \text{ in } \bar{M}_T.
\]

Therefore, by the concavity of \( F \),
\begin{equation}
F^{ii}(U_{ii} - U_{i}) \geq F(2Bg + U) - F(U) - 2B \sum F^{ii} \\
\geq F(Bg) - 2B \sum F^{ii} - \psi(x, t, u, \nabla u) - u_t.
\end{equation}
It follows from (5.1), (5.2), (5.11) and (5.12) that

$$0 \geq \frac{a}{\phi} F(Bg) - C - (C + 2B) \sum F^{ii} + \frac{ac_0|\nabla \phi|^2}{2\phi} \sum F^{ii}$$

(5.13)

\[-\frac{1}{u^2} F^{ii} A^{ii}_{x_i} \nabla_l u + \frac{1}{u^2} \psi x_i \nabla_l u + \psi u \]

\[\geq \left( \frac{ac_0|\nabla \phi|^2}{2\phi} - 3B - C|\nabla u|^{\gamma_1 - 2} \right) \sum F^{ii}\]

provided \(B\) is chosen sufficiently large. Thus, we get a bound \(|\nabla u(x_0, t_0)| \leq C\) and so the proof of Theorem 5.1 is completed. \(\square\)

**Theorem 5.2.** Let \(u \in C^3(M_T)\) be an admissible solution of (1.1) with \(u \geq u\) in \(M_T\). Assume, in addition, that (1.7), (1.9) and (5.1) hold for \(\gamma_1, \gamma_2 < 2\) in (5.1) and that \((M^n, g)\) has nonnegative sectional curvature. Then (5.4) holds.

**Proof.** Since \((M^n, g)\) has nonnegative sectional curvature, in orthonormal local frame,

\[R^{ii}_{x_i} \nabla_l u \nabla_l u \geq 0.\]

In the proof of Theorem 5.1, similar to (5.8), we have

(5.14)

\[w \nabla_l \nabla_l u \geq \nabla_l u \nabla_l u_i - \frac{aw^2}{\phi} A^{ii}_{x_i} \nabla_k \phi - \nabla_l u A^{ii}_{x_i}.\]

It follows from (2.5), (5.1), (5.7), (5.9) and (5.14) that

\[0 \geq \frac{a}{\phi} \mathcal{L}(u - u) + \frac{1}{u^2} \nabla_l u \psi x_i + \psi u - \frac{\nabla_l u}{u^2} F^{ii} A^{ii}_{x_i} + \frac{a - a^2}{\phi^2} F^{ii} |\nabla_l \phi|^2\]

(5.15)

\[\geq \frac{a}{\phi} \theta(1 + \sum F^{ii}) - C|\nabla u|^{\gamma_1 - 2} \sum F^{ii} - C|\nabla u|^{\gamma_2 - 2} + \frac{a - a^2}{\phi^2} F^{ii} |\nabla_l \phi|^2\]

provided \(|\nabla u|\) is sufficiently large. Choosing \(a\) sufficiently small, we can obtain a bound \(|\nabla u(x_0, t_0)| \leq C\) and (5.4) holds. \(\square\)

**Theorem 5.3.** Let \(u \in C^3(M_T)\) be an admissible solution of (1.1) in \(M_T\). Assume, in addition, that (5.1) hold for \(\gamma_1, \gamma_2 < 4\),

(5.16)

\[f \text{ is homogeneous of degree one,}\]

(5.17)

\[f_j(\lambda) \geq \nu_i \left( 1 + \sum f_i(\lambda) \right) \text{ for any } \lambda \in \Gamma \text{ with } \lambda_j < 0,\]

where \(\nu_i\) is a uniform positive constant and there exist a continuous function \(\bar{\psi} \geq 0\) and a positive constant \(\gamma < 2\) such that when \(|p|\) is sufficiently large,

(5.18)

\[p \cdot D_p \bar{\psi}(x, t, z, p), -p \cdot D_p A^{\xi}(x, t, z, p)/|\xi|^2 \leq \bar{\psi}(x, t, z)(1 + |p|^\gamma),\]

(5.19)

\[-\psi(x, t, z, p) \leq \bar{\psi}(x, t, z)(1 + |p|^\gamma),\]

(5.20)

\[|A^{\xi}(x, t, z, p)| \leq \bar{\psi}(x, t, z)|\xi||\eta|(1 + |p|^\gamma), \forall \xi, \eta \in T_x \bar{M}; \xi \perp \eta.\]
Then (5.4) holds.

Proof. In the proof of Theorem 5.1, we take \( \phi = -u + \sup_{M_T} u + 1 \). By the concavity of \( A^{ii} \) with respect to \( p \),

\[
A^{ii} = A^{ii}(x, t, \nabla u) \leq A^{ii}(x, t, 0) + A_{pk}^{ii}(x, t, 0) \nabla_k u
\]

Thus, from (5.16), (5.19) and (5.21), we find

\[
-F^{ii} \nabla^{ii} \phi = F^{ii} \nabla^{ii} u = F^{ii} U^{ii} - F^{ii} A^{ii} = u_t + \psi - F^{ii} A^{ii}
\]

By virtue of (5.7), (5.8), (5.9), (5.1), (5.18) and (5.22), we see that for \( a < 1 \),

\[
0 \geq \frac{(a - a^2)}{\phi^2} F^{ii} |\nabla_i u|^2 + \frac{\nabla_i u \psi x_i}{w^2} + \psi_u - \frac{a}{\phi} \psi_{pk} \nabla_k u - \frac{a}{\phi} u_t
\]

\[
+ \frac{a}{\phi} F^{ii} A_{pk}^{ii} \nabla_k u - F^{ii} \nabla_i u A_{x_i}^{ii} \frac{\nabla_i u}{w^2} + \frac{a}{\phi} u_t
\]

\[
- C |\nabla u|^{\gamma} - C (1 + |\nabla u|) \sum F^{ii}
\]

\[
\geq c_1 F^{ii} |\nabla_i u|^2 - C (|\nabla u|^{\gamma - 2} + |\nabla u|^{\gamma})
\]

\[
- C (1 + |\nabla u| + |\nabla u|^{\gamma - 2} + |\nabla u|^{\gamma}) \sum F^{ii}
\]

provided \( |\nabla u| \) is sufficiently large.

Without loss of generality we assume \( \nabla_1 u(x_0, t_0) \geq \frac{1}{n} |\nabla u(x_0, t_0)| > 0 \). Recall that \( U_{ij}(x_0, t_0) \) is diagonal. By (5.5), (5.21) and (5.20), we have

\[
U_{11} = -\frac{a}{\phi} |\nabla u|^2 + A^{11} + \frac{1}{\nabla_1 u} \sum_{k \geq 2} \nabla_k u A^{1k}
\]

\[
\leq -\frac{a}{\phi} |\nabla u|^2 + C (1 + |\nabla u| + |\nabla u|^{\gamma - 2}) < 0
\]

provided \( |\nabla u| \) is sufficiently large. Therefore, by (5.16),

\[
f_1 \geq \nu_0 \left( 1 + \sum_{i=1}^{n} f_i \right)
\]

and a bound \( |\nabla u(x_0, t_0)| \leq C \) follows from (5.23).

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