Article

Study on Date–Jimbo–Kashiwara–Miwa Equation with Conformable Derivative Dependent on Time Parameter to Find the Exact Dynamic Wave Solutions

Md Ashik Iqbal 1, Ye Wang 2,3,*, Md Mamun Miah 4 and Mohamed S. Osman 5,*

1 Department of Mathematics, Bangladesh Army University of Engineering and Technology, Natore 6431, Bangladesh; ashikiqbalmath@gmail.com
2 Department of Mathematics, Huzhou University, Huzhou 313000, China
3 Institute for Advanced Study Honoring Chen Jian Gong, Hangzhou Normal University, Hangzhou 311121, China
4 Department of Mathematics, Khulna University of Engineering & Technology, Khulna 9203, Bangladesh; mamun0954@gmail.com
5 Department of Mathematics, Faculty of Science, Cairo University, Giza 12613, Egypt
* Correspondence: 03019@zjhu.edu.cn (Y.W.); mofatzi@.cu.edu.eg or mofatzi@sci.cu.edu.eg (M.S.O.)

Abstract: In this article, we construct the exact dynamical wave solutions to the Date–Jimbo–Kashiwara–Miwa equation with conformable derivative by using an efficient and well-established approach, namely: the two-variable ($G'/G$, $1/G$)-expansion method. The solutions of the Date–Jimbo–Kashiwara–Miwa equation with conformable derivative play a vital role in many scientific occurrences. The regular dynamical wave solutions of the abovementioned equation imply three different fundamental functions, which are the trigonometric function, the hyperbolic function, and the rational function. These solutions are classified graphically into three categories, such as singular periodic solitary, kink soliton, and anti-kink soliton wave solutions. Furthermore, the effect of the fractional parameter on these solutions is discussed through 2D plots.

Keywords: two-variable ($G'/G$, $1/G$)-expansion method; conformable Date–Jimbo–Kashiwara–Miwa equation; exact dynamical wave solutions

1. Introduction

Many phenomena are represented by nonlinear partial differential equations known as quasi linear or nonlinear evolution equations. Almost all of the occurrences in our natural world, especially in the science and engineering sectors, can be simulated by this complicated mathematical equation, which are space- and time-related, nonlinear evolution equations (NLEEs). To explain the natural incidents occurred in many scientific fields—such as the field of engineering, solid-state physics, plasma physics, applied physics, applied mathematics, plasma waves, fluid dynamics, quantum physics, electrodynamics, magneto hydrodynamics and turbulence, biology, fiber optics, chemistry, chemical physics, astrophysics, general theory of relativity, cosmology, medical science, and so on [1–48]—we need to look for the exact solutions of the NLEEs. Many researchers have built up various types of methods to obtain the exact solutions for the NLEEs, including the following: the homogeneous balance method [1,2], the first integral method [3,4], the Jacobi elliptic function expansion method [5,6], the exponential function method [7,8], the tanh–function method [9,10], the auxiliary ordinary differential equation method [11], the homotopy analysis method [12], the tanh–coth method [13], the sine–Gordon expansion method [14], the $(G'/G^2)$-expansion method [15], the unified method [16], the variational iteration method [17], the extended direct algebraic method [18], the exp($(-\phi(\xi))$)-expansion method [19,20], Lie group method [21], the Hirota’s bilinear method [22,23], the generalized Kudryshov method [24–26], the Backlund transform method [27], the Cole–Hopf
transformation method [28], the Riccati equation method [29], the \((G'/G)\)-expansion method [30,31], the new generalized \((G'/G)\)expansion method [32], the modified \((G'/G)\)-expansion method [33], and so on.

Inc et al. [34] proposed the two-variable \((G'/G, 1/G)\)-expansion method to solve the Kaup–Kupershmidt equation, while Miah et al. [35] used this technique to investigate different exact traveling wave solutions of some NLEEs. In [36–38], many researchers have used this method to solve many other NLEEs. In this paper, we investigate the conformable Date–Jimbo–Kashiwara–Miwa (CDJKM) equation, and we solve it by using the two-variable \((G'/G, 1/G)\)-expansion method. The CDJKM equation moved to \((2 + 1)\)-dimensional Date–Jimbo–Kashiwara–Miwa equation when the fractional derivative \(\theta = 1\). Many researchers investigated the \((2 + 1)\)-dimensional Date–Jimbo–Kashiwara–Miwa equation [39–42], but no one used our desired method to solve the CDIKM equation. For the first time, Guo and Lin [44] introduced the CDJKM equation and Kumar et al. [45] successfully investigated the solutions for this equation via the sine–Gordon expansion and the improved Bernoulli sub equation methods. In [45], only one real solution is presented, and the rest of the solutions are complex. Therefore, our purpose here is to present new real wave solutions for the CDJKM equation by using the two-variable \((G'/G, 1/G)\)-expansion method and explain the dynamical behavior of the obtained solutions.

The CDJKM equation has the following form [43,44]:

\[
\begin{align*}
&u_{xxxx} + 4u_{xxy}u_x + 2u_{xxy}u_y + 6u_{xxy}u_{xx} - \alpha u_{yyy} - 2\beta \frac{\partial}{\partial x} \left( \frac{\partial \theta}{\partial x} \left( \frac{\partial^2 u}{\partial \rho^2} \right) \right) = 0 \\
&\text{where } u = u(x, y, t), \beta \text{ is a constant, and } 0 < \theta \leq 1.
\end{align*}
\]

where \(L = \frac{G'('}{G(\eta)}\), \(H = \frac{1}{G(\eta)}\)

By using Equations (2) and (3), we obtain two new relations, as follows:

\[
L' = -L^2 + \mu H - \lambda, \quad H' = -LH.
\]

Equation (2) has three types of general solutions, depending on \(\lambda\), which can be classified as follows:

**Condition I.** For \(\lambda > 0\), Equation (2) has a general solution in the following form:

\[
G(\eta) = A \sin \left( \sqrt{\lambda} \eta \right) + B \cos \left( \sqrt{\lambda} \eta \right) + \frac{\mu}{\lambda}
\]

Consequently, we have the following:

\[
H^2 = \frac{\lambda(L^2 - 2\mu H + \lambda)}{\lambda^2 \rho_1 - \mu^2}
\]
where \( \rho_1 = A^2 + B^2 \) and \( A, B \) are two arbitrary constants.

**Condition II.** For \( \lambda < 0 \), Equation (2) has a general solution in the following:

\[
G(\eta) = A \sinh \left( \sqrt{-\lambda} \eta \right) + B \cosh \left( \sqrt{-\lambda} \eta \right) + \frac{\mu}{\lambda} 
\]

(7)

Therefore, we have the following:

\[
H^2 = \frac{-\lambda \left( L^2 - 2\mu H + \lambda \right)}{\lambda^2 \rho_2 + \mu^2} 
\]

(8)

where \( \rho_2 = A^2 - B^2 \).

**Condition III.** For \( \lambda = 0 \), Equation (2) has a general solution in the following form:

\[
G(\eta) = \frac{\mu}{2} \eta^2 + A\eta + B 
\]

(9)

Thus, we have the following:

\[
H^2 = \frac{(L^2 - 2\mu H)}{A^2 - 2\mu B} 
\]

(10)

Now, consider the following nonlinear partial differential equation:

\[
R \left( u, u_x, u_t, u_y, u_{tt}, u_{yy}, u_{xx}, u_{yy}, u_{tt}, u^2, \ldots \right) = 0 
\]

(11)

where \( R \) is a polynomial in \( u(x, y, t) \) and it is partial derivatives.

**Action 1.** Convert the function \( u \) of the three independents variables \( x, y, \) and \( t \) into a function of single parameter by the following conventional wave transformation rule, as follows:

\[
u(x, y, t) = u(\eta), \quad \eta = lx + py - ct \theta \]

(12)

where \( l \) (coefficient of spatial variable \( x \)), \( p \) (coefficient of spatial variable \( y \)), and \( c \) (velocity of wave) are nonzero arbitrary constants. Using Equation (12) in Equation (11), we obtain the following converted nonlinear ordinary equation:

\[
Q \left( u, u', u'' , u^2, \ldots \right) = 0 
\]

(13)

where \( Q \) is a polynomial in its arguments.

**Action 2.** Assume the solution of Equation (13), which can be written in terms of \( L(\eta) \) and \( H(\eta) \), has the following form:

\[
u(\eta) = \sum_{j=0}^{M} a_j L^j + \sum_{j=1}^{M} b_j L^{j-1} H 
\]

(14)

where \( a_j (j = 0, 1, 2, \ldots, M) \), \( b_j (j = 1, 2, \ldots, M) \), \( c \), \( \lambda \), and \( \mu \) are constants to be determined later. By applying the homogeneous balance rule between the highest order derivative and the highest nonlinear term, the number \( M \) can be evaluated.

**Action 3.** Insert the value of \( M \) in Equation (14) and substitute Equation (14) into Equation (13) through Equations (4) and (6), we convert the left side of Equation (13) in terms of \( L \) and \( H \), where the degree of \( L \) is taken from zero to any positive integer number but the degree of \( H \) cannot be greater than one.

**Action 4.** Comparing the like powers of the polynomial from both sides then we have a system of algebraic equations in \( a_j, b_j, c, \lambda, \mu, A, \) and \( B \). By using any computation software, like Mathematica, we can solve the obtained algebraic system. By inserting these values into Equation (14), we obtain the solution of Equation (13), and after setting the
value of \( \eta = l x + p y - c \theta \) in Equation (11), we reach the desirable traveling wave solution of Equation (11). For more details about the proposed method, see \([35–38]\).

3. Regular Traveling Wave Solutions of the CDJKM Equation

In this section, the two-variable \((G'/G, \ 1/G)\)-expansion method is applied to solve the CDJKM equation. By using the wave transformation rule given by Equation (11), we convert Equation (1) into a nonlinear ODE, as follows:

\[
l^4 p u''' + 3 l^3 p (u')^2 - (a p^3 - 2 l^2 c^2) u' = 0
\]  

(15)

Now after setting \( u' = U \), we obtain the following:

\[
l^4 p U''' + 3 l^3 p U^2 - (a p^3 - 2 l^2 c^2) U = 0
\]  

(16)

where the prime indicates the derivative with respect to \( \eta \).

By applying the homogeneous balance rule between \( U'' \) and \( U^2 \) in Equation (16), we obtain \( M = 2 \). Thus, the general solution takes the following form:

\[
U(\eta) = a_0 + a_1 L(\eta) + a_2 L^2(\eta) + b_1 H(\eta) + b_2 L(\eta) H(\eta)
\]  

(17)

where \( a_j \) \((j = 0, 1, 2, \ldots, M)\) and \( b_j \) \((j = 1, 2, \ldots, M)\) are constants and the new functions \( L(\eta) \) and \( H(\eta) \) are defined in Equation (4).

Now, Equation (16) have three types of solutions which have the following cases:

**Condition I.** For \( \lambda > 0 \). Differentiate Equation (17) two times, and by using Equations (4) and (6), Equation (16) is converted in terms of \( L(\eta) \) and \( H(\eta) \). By comparing the coefficients of the required polynomial from both sides, we obtain a group of algebraic equations in \( a_0, \ a_1, a_2, b_1, b_2, \lambda \ and \ c \). After solving this algebraic system, we obtain two different sets of solutions.

**Set 1:**

\[
a_0 = -l \lambda, \ a_1 = 0, \ a_2 = -l, \ b_1 = l \mu, \ b_2 = \pm \frac{l \sqrt{\lambda^2 \rho_1 - \mu^2}}{\sqrt{\lambda}} \text{ and } c = \pm \frac{p \ (p^2 a + l^4 \lambda)}{2 l^2 \beta}
\]  

(18)

**Set 2:**

\[
a_0 = -\frac{2 l \lambda}{3}, \ a_1 = 0, \ a_2 = -l, \ b_1 = l \mu, \ b_2 = \pm \frac{l \sqrt{\lambda^2 \rho_1 - \mu^2}}{\sqrt{\lambda}} \text{ and } c = \pm \frac{p \ (p^2 a + l^4 \lambda)}{2 l^2 \beta}
\]  

(19)

Now, after putting these values into Equation (17), we obtain the solutions of Equation (16). Firstly, inserting the values from Equation (18) into Equation (17), we find the following:

\[
U(\eta) = -l \lambda - l \lambda \left[ \frac{A \cos(\sqrt{\lambda} \eta) - B \sin(\sqrt{\lambda} \eta)}{A \cos(\sqrt{\lambda} \eta) + B \sin(\sqrt{\lambda} \eta) + \frac{\mu}{2}} \right] + \frac{l \mu}{A \cos(\sqrt{\lambda} \eta) + B \sin(\sqrt{\lambda} \eta) + \frac{\mu}{2}} \pm l \sqrt{\lambda^2 \rho_1 - \mu^2} \left[ \frac{A \cos(\sqrt{\lambda} \eta) - B \sin(\sqrt{\lambda} \eta)}{A \cos(\sqrt{\lambda} \eta) + B \sin(\sqrt{\lambda} \eta) + \frac{\mu}{2}} \right]^{-1}
\]  

(20)

Equation (20) is one of the solutions of Equation (16). After integrating Equation (20), we attain a generalized trigonometric function solution of Equation (15) and if we substitute \( \eta = l x + p y - \frac{p (p^2 a + l^4 \lambda)}{2 l^2 \beta} \) in the obtained integrand equation, we obtain the desired solution of Equation (1).

For a specific solution, we choose \( A = 0 \) and \( \mu = 0 \) but \( B \neq 0 \) in Equation (20), as follows:

\[
U(\eta) = -l \lambda \sec^2(\sqrt{\lambda} \eta) \pm l \lambda \tan(\sqrt{\lambda} \eta) \sec(\sqrt{\lambda} \eta).
\]  

(21)
Integrating the above equation, we obtain the solution Equation (15), as follows:

\[
  u(x, y, t) = -l\sqrt{\lambda} \tan(\sqrt{\lambda}\eta) \pm l\sqrt{\lambda} \sec(\sqrt{\lambda}\eta)
\]  

(22)

where \( \eta = lx + py - \frac{p(\rho^2 - \alpha^2)}{2\beta\beta} \) and both \( l \) and \( \beta \) are nonzero.

Again, by taking \( B = 0 \) but \( A \neq 0 \) in Equation (20), as follows:

\[
  U(\eta) = -\frac{A l \lambda^2 \cos(\sqrt{\lambda}\eta)}{(\mu - R_1 + 2A\lambda \cos(\sqrt{\lambda}\eta))^2}
\]

(23)

where \( R_1 = \sqrt{\lambda^2\rho_1 - \mu^2} \). Integrating the above equation, we obtain another solution of Equation (15), as follows:

\[
  u(x, y, t) = -A l\sqrt{\lambda} \left[ \frac{2\eta\sqrt{\lambda}}{A} + \frac{2(-2\mu^3 + A^2\lambda^2 (R_1 + 3\mu))\tan^{-1}\left(\frac{(-A\lambda + \mu)\tan\left(\sqrt{\lambda}\eta/2\right)}{A(2\beta)}\right)}{A R_2^2} + \frac{R_2\lambda\mu(R_1 + \mu)\sin(\sqrt{\lambda}\eta)}{R_2^2(\mu + A\lambda \cos(\sqrt{\lambda}\eta))} \right]
\]

(24)

where \( R_2 = \sqrt{A^2\lambda^2 - \mu^2} \), \( \eta = lx + py - \frac{p(\rho^2 - \alpha^2)}{2\beta\beta} \) and both \( l \) and \( \beta \) are nonzero.

Similarly, inserting the values from Equation (19) into Equation (17), we find the following:

\[
  U(\eta) = -\frac{2l\lambda}{3} - l\lambda \left[ \frac{A \cos(\sqrt{\lambda}\eta) - B \sin(\sqrt{\lambda}\eta)}{A \cos(\sqrt{\lambda}\eta) + B \sin(\sqrt{\lambda}\eta) + \pi} \right]^2 + \frac{l\mu}{[A \cos(\sqrt{\lambda}\eta) + B \sin(\sqrt{\lambda}\eta) + \pi]^2}
\]

(25)

\[\pm l\lambda^2\rho_1 - \mu^2 \left[ \frac{A \cos(\sqrt{\lambda}\eta) - B \sin(\sqrt{\lambda}\eta)}{A \cos(\sqrt{\lambda}\eta) + B \sin(\sqrt{\lambda}\eta) + \pi} \right]^2 \]

Equation (25) is one of the solutions of Equation (16). By integrating Equation (25), we obtain another generalized trigonometric function solution of Equation (15) and if we put \( \eta = lx + py - \frac{p(\rho^2 - \alpha^2)}{2\beta\beta} \) into the attained integrand equation, we reach the desired solution of Equation (1).

For a particular solution, we choose \( A = 0 \) and \( \mu = 0 \) but \( B \neq 0 \) in Equation (25), as follows:

\[
  U(\eta) = -\frac{2l\lambda}{3} - l\lambda \tan^2(\sqrt{\lambda}\eta) \pm l\lambda \tan(\sqrt{\lambda}\eta) \sec(\sqrt{\lambda}\eta).
\]

(26)

By integrating Equation (26), we obtain a special solution of Equation (15), as follows:

\[
  u(x, y, t) = \frac{l\lambda}{3}\eta - l\sqrt{\lambda} \tan(\sqrt{\lambda}\eta) \pm l\sqrt{\lambda} \sec(\sqrt{\lambda}\eta).
\]

(27)

where \( \eta = lx + py - \frac{p(\rho^2 - \alpha^2)}{2\beta\beta} \) and both \( l \) and \( \beta \) are nonzero.

Again, for \( B = 0 \) and \( \mu = 0 \) but \( A \neq 0 \), we have the following:

\[
  U(\eta) = -\frac{2l\lambda}{3} - l\lambda \cot^2(\sqrt{\lambda}\eta) \pm l\lambda \cot(\sqrt{\lambda}\eta) \cos(\sqrt{\lambda}\eta).
\]

(28)

By integrating Equation (28), we obtain a particular solution of Equation (15), as the following:

\[
  u(x, y, t) = \frac{l\lambda}{3}\eta + l\sqrt{\lambda} \cot(\sqrt{\lambda}\eta) \pm l\sqrt{\lambda} \cos(\sqrt{\lambda}\eta).
\]

(29)
where \( \eta = lx + py - \frac{p(t^2a - l^4\lambda)}{2\beta} \) and both \( l \) and \( \beta \) are nonzero.

**Condition II.** For \( \lambda < 0 \).

By the same way, in Condition I, we have also two sets of values of those arbitrary constants, as follows:

**Set 1:**

\[
a_0 = -l\lambda, \ a_1 = 0, \ a_2 = -l, \ b_1 = l\mu, \ b_2 = \pm \frac{l\sqrt{\mu^2 + \lambda^2p^2}}{\sqrt{-\lambda}}, \ c = \frac{p(p^2a + l^4\lambda)}{2l^2\beta}
\]  

(30)

**Set 2:**

\[
a_0 = -\frac{2l\lambda}{3}, \ a_1 = 0, \ a_2 = -l, \ b_1 = l\mu, \ b_2 = \pm \frac{l\sqrt{\mu^2 + \lambda^2p^2}}{\sqrt{-\lambda}}, \ c = \frac{p(p^2a - l^4\lambda)}{2l^2\beta}
\]  

(31)

After inserting these values into Equation (17), we obtain the solutions of Equation (16). At first, inserting the values from Equation (30) into Equation (17), we find the following:

\[
U(\eta) = -l\lambda + l\lambda \left[ \frac{\text{Acosh}\left(\sqrt{-\lambda}\eta\right) + B\sinh\left(\sqrt{-\lambda}\eta\right)}{B\cosh\left(\sqrt{-\lambda}\eta\right) + A\sinh\left(\sqrt{-\lambda}\eta\right) + \beta} \right]^2 + \frac{l\mu}{B\cosh\left(\sqrt{-\lambda}\eta\right) + A\sinh\left(\sqrt{-\lambda}\eta\right) + \beta} \pm \frac{l\sqrt{\mu^2 + \lambda^2p^2}}{B\cosh\left(\sqrt{-\lambda}\eta\right) + A\sinh\left(\sqrt{-\lambda}\eta\right) + \beta} \left[ \frac{\text{Acosh}\left(\sqrt{-\lambda}\eta\right) + B\sinh\left(\sqrt{-\lambda}\eta\right)}{B\cosh\left(\sqrt{-\lambda}\eta\right) + A\sinh\left(\sqrt{-\lambda}\eta\right) + \beta} \right]^2
\]  

(32)

Equation (32) is hyperbolic type solution of Equation (16). By integrating Equation (32), we attain a generalized hyperbolic function solution of Equation (15) and if we substitute \( \eta = lx + py - \frac{p(t^2a + l^4\lambda)}{2\beta} \) in the obtained integrand equation, we obtain the desired solution of Equation (1).

Now, for a specific solution, we choose \( A = 0 \) but \( B \neq 0 \) in Equation (32), as follows:

\[
U(\eta) = -\frac{B l \lambda^2 (B\lambda + \mu \cos h(\sqrt{-\lambda}\eta) - R_3\sinh(\sqrt{-\lambda}\eta))}{(\mu + B\lambda \cos h(\sqrt{-\lambda}\eta))^2}
\]  

(33)

where \( R_3 = \sqrt{\lambda^2p^2 + \mu^2} \). By integrating Equation (33), we obtain a particular solution of Equation (15), as the following:

\[
u(x,y,t) = \frac{l\sqrt{-\lambda}(R_3 + B\lambda\sinh(\sqrt{-\lambda}\eta))}{(\mu + B\lambda \cos h(\sqrt{-\lambda}\eta))}
\]  

(34)

where \( \eta = lx + py - \frac{p(t^2a + l^4\lambda)}{2\beta} \) and both \( l \) and \( \beta \) are nonzero.

Again, inserting the values from Equation (31) into Equation (17), we find the following:

\[
U(\eta) = -\frac{2l\lambda}{3} + l\lambda \left[ \frac{\text{Acosh}\left(\sqrt{-\lambda}\eta\right) + B\sinh\left(\sqrt{-\lambda}\eta\right)}{B\cosh\left(\sqrt{-\lambda}\eta\right) + A\sinh\left(\sqrt{-\lambda}\eta\right) + \beta} \right]^2 + \frac{l\mu}{B\cosh\left(\sqrt{-\lambda}\eta\right) + A\sinh\left(\sqrt{-\lambda}\eta\right) + \beta} \pm \frac{l\sqrt{\mu^2 + \lambda^2p^2}}{B\cosh\left(\sqrt{-\lambda}\eta\right) + A\sinh\left(\sqrt{-\lambda}\eta\right) + \beta} \left[ \frac{\text{Acosh}\left(\sqrt{-\lambda}\eta\right) + B\sinh\left(\sqrt{-\lambda}\eta\right)}{B\cosh\left(\sqrt{-\lambda}\eta\right) + A\sinh\left(\sqrt{-\lambda}\eta\right) + \beta} \right]^2
\]  

(35)
Equation (35) is another hyperbolic-type solution of Equation (16). After integrating Equation (35), we have a generalized hyperbolic function solution of Equation (15) and if we substitute \( \eta = lx + py - \frac{p(\alpha \eta + l \lambda)}{2l^2 \beta} \) in the obtained integrand equation, we reach the desired solution of Equation (1).

Now, we take \( A = 0, \mu = 0 \) but \( B \neq 0 \) in Equation (35), as follows:

\[
U(\eta) = \frac{\lambda l \sec^2 \left( \sqrt{-\lambda} \eta \right) \left( -B + B \sinh \left( \sqrt{-\lambda} \eta \right) \right)}{B}
\]  

(36)

where \( \lambda < 0 \). By integrating Equation (36), we obtain a particular solution of Equation (15), as the following:

\[
u(x, y, t) = \frac{\sqrt{-\lambda} l \sec \left( \sqrt{-\lambda} \eta \right) \left( \sqrt{\rho^2 + B \sinh \left( \sqrt{-\lambda} \eta \right)} \right)}{B}
\]

(37)

where \( \eta = lx + py - \frac{p(\alpha \eta + l \lambda)}{2l^2 \beta} \) and both \( l \) and \( \beta \) are nonzero.

**Condition III.** For \( \lambda = 0 \). By the same process in the previous two cases, we have only one set of values of the given arbitrary constants, as follows:

Set 1:

\[
a_0 = 0, \ a_1 = 0, \ a_2 = -l, \ b_1 = l \mu, \ b_2 = \pm \sqrt{A^2 - 2 \mu B}, \ c = \frac{l^3 \alpha}{2l^2 \beta}
\]

(38)

Putting these values from into Equation (17), we find the solution of Equation (16), as follows:

\[
U(\eta) = -l \left( \frac{\mu \eta + A}{2 \eta^2 + A \eta + B} \right)^2 + \frac{l \mu}{2 \eta^2 + A \eta + B} \pm l \sqrt{A^2 - 2 \mu B} \left( \frac{\mu \eta + A}{2 \eta^2 + A \eta + B} \right)^2
\]

(39)

Equation (39) is the rational-type solution of Equation (16). After integrating Equation (39), we attain a generalized rational function solution of Equation (15) and if we substitute \( \eta = lx + py - \frac{p(\alpha \eta + l \lambda)}{2l^2 \beta} \) in the obtained integrand equation, we obtain the desired solution of Equation (1).

Setting \( B = 0, \mu = 0 \) but \( A \neq 0 \) in Equation (39), as follows:

\[
U(\eta) = -\frac{2l}{\eta^2}
\]

(40)

By integrating Equation (40), we obtain a particular solution of Equation (15), as the following:

\[
u(x, y, t) = \frac{2l}{\eta}
\]

(41)

where \( \eta = lx + py - \frac{p(\alpha \eta + l \lambda)}{2l^2 \beta} \) and both \( l \) and \( \beta \) are nonzero.

### 4. Figures and Discussion

In this section, the physical illustration of the obtained wave solutions of the CDJMK equation is investigated. The solutions are trigonometric, hyperbolic, and rational solutions, which are sketched in 3D and 2D shapes in Figures 1–6. The 3D figures indicate the singular periodic solitary, kink soliton, and anti-kink soliton wave solutions. In Figure 1, solution (24) indicates the singular periodic solitary wave solution for the values \( \alpha = 2, \ l = 1, \ p = 1, \ \lambda = 4, \ \mu = 5, \ \beta = 1, \ A = 1, \ \theta = 0.5, \) and \( \rho_1 = 2 \). The corresponding 2D shape for the solution (24) is depicted in Figure 2 for different values of the fractional parameter \( \theta \). In Figure 3, the solution given by (34) implies the kink soliton wave solution for the values...
\( \alpha = 2, l = 1, p = 1, \lambda = -4, \mu = 4, \beta = 1, B = -1, \theta = 0.5, \) \( \) and \( \rho_2 = 1. \) The corresponding 2D shape for the solution (34) is depicted in Figure 4 for different values of the fractional parameter \( \theta. \) In Figure 5, the solution (37) implies an anti-kink shape soliton wave solution when \( \alpha = 2, l = 1, p = 1, \lambda = -4, \mu = 0, \beta = 1, B = -1, \theta = 0.5, \) \( \) and \( \rho_2 = 1. \) The corresponding 2D shape for the solution (34) is depicted in Figure 6 for different values of the fractional parameter \( \theta. \) It is clear that in all 2D figures, for small fractional time derivative, wave progressions are changed significantly. Additionally, we notice that the amplitude of the solution given in Figures 1 and 2 is very high in the medium through which the wave is propagated, thus this singular periodic wave is a shock wave.

![Figure 1. The 3D shape of the solution (24) indicates the singular periodic solitary wave solution for \( \alpha = 2, \beta = 1, l = p = 1, \lambda = 4, \mu = 5, A = 1, \rho_1 = 2, \) and \( \theta = 0.5. \)](image)

![Figure 2. The 2D shape of the solution (24) for \( \alpha = 2, \beta = 1, l = p = 1, \lambda = 4, \mu = 5, A = 1, \) and \( \rho_1 = 2. \)](image)
Figure 1. The 3D shape of the solution (24) indicates the singular periodic solitary wave solution for $\alpha = 2$, $\beta = 1$, $\rho_1 = 2$, and $\theta = 0.5$.

Figure 2. The 2D shape of the solution (24) for $\alpha = 2$, $\beta = 1$, $\rho_1 = 2$.

Figure 3. The 3D shape of the solution (34) implies the kink shape soliton for $\alpha = 2$, $\beta = 1$, $\rho_2 = 1$, and $\theta = 0.5$.

Figure 4. The 2D shape of the solution (34) for $\alpha = 2$, $\beta = 1$, $\rho_2 = 1$.

Figure 5. The 3D shape of the solution (37) implies the anti-kink shape soliton for $\alpha = 2$, $\beta = 1$, $\rho_2 = 1$, and $\theta = 0.5$.

Figure 6. The 2D shape of the solution (37) for $\alpha = 2$, $\beta = 1$, $\rho_2 = 1$.

5. Conclusion

In this paper, the two-variable $(\varphi', \varphi/\rho)$-expansion method has been applied as a powerful tool to obtain a variety of exact solutions for the CDJMK equation. The obtained solutions are trigonometric, hyperbolic, and rational function solutions. These solutions have different wave structures, such as singular periodic solitary, kink soliton, and anti-kink soliton.
In this paper, the two-variable \((G'/G, 1/G)\)-expansion method has been applied as a powerful tool to obtain a variety of exact solutions for the CDJMK equation. The obtained solutions are trigonometric, hyperbolic, and rational function solutions. These solutions have different wave structures, such as singular periodic solitary, kink soliton, and anti-kink soliton wave solutions, which have many applications in the modern science and engineering sectors. It is worth mentioning that we attained nine new exact traveling wave solutions by using this eminent method, which are very effective for further investigation into the CDJMK equation. The attained solutions in this article may have momentous effects on future understandings of the natural world, and it is proven that the two-variable \((G'/G, 1/G)\)-expansion method is an adequate, useful, compatible, and important mathematical instrument for obtaining solutions to other NLEEs. The process of the \((G'/G, 1/G)\)-expansion method is easier, more expansive, and more effective than...
any other well-known methods. In future works, the two-variable $(G'/G, 1/G)$-expansion method can be implemented into different fractional nonlinear systems with variable coefficients; thus, the new exact solutions and their dynamical behaviors can be acquired.

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