EXAMPLES OF HYPERBOLIC HYPERSURFACES
OF LOW DEGREE
IN PROJECTIVE SPACES

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Abstract

We construct families of hyperbolic hypersurfaces of degree $2n$ in the projective space $\mathbb{P}^n(\mathbb{C})$ for $3 \leq n \leq 6$.

Keywords: Kobayashi conjecture, hyperbolicity, Brody Lemma, Nevanlinna Theory

1 Introduction and the main result

The Kobayashi conjecture states that a generic hypersurface $X_d \subset \mathbb{P}^n(\mathbb{C})$ of degree $d \geq 2n - 1$ is hyperbolic. It is proved by Demailly and El Goul [5] for $n = 3$ and a very generic surface of degree at least $21$. In [17], Păun improved the degree to $18$. In $\mathbb{P}^3(\mathbb{C})$, Rousseau [18] was able to show that a generic three-fold of degree at least $593$ contains no Zariski-dense entire curve, a result from which hyperbolicity follows, after removing divisorial components [7]. In $\mathbb{P}^n(\mathbb{C})$, for any $n$ and for $d \geq 2^{(n-1)^5}$, Diverio, Merker and Rousseau [6] established algebraic degeneracy of entire curves in $X_d$.

Also, Siu [21] proposed a positive answer for arbitrary $n$ and degree $d = d(n) \gg 1$ very large. Most recently, Demailly [4] has announced a strategy that is expected to attain Kobayashi’s conjecture for very generic hypersurfaces of degree $d \geq 2n$.

Concurrently, many authors tried to find examples of hyperbolic hypersurfaces of degree as low as possible. The first example of a compact Kobayashi hyperbolic manifold of dimension $2$ is a hypersurface in $\mathbb{P}^3(\mathbb{C})$ constructed by Brody and Green [2]. Also, the first examples in all higher dimensions $n - 1 \geq 3$ were discovered by Masuda and Noguchi [16], with degree large. So far, the best degree asymptotic is the square of the degree $d = 16(n - 1)^2$ and by Shiffman and Zaidenberg [19] with $d = 4(n - 1)^2$. In $\mathbb{P}^3(\mathbb{C})$ many examples of low degree were given (see the reference of [23]). The lowest degree found up to date is 6, given by Duval [9]. There are not so many examples of low degree hyperbolic hypersurfaces in $\mathbb{P}^4(\mathbb{C})$. We mention here an example of a hypersurface of degree $16$ constructed by Fujimoto [13]. Various examples in $\mathbb{P}^5(\mathbb{C})$ and $\mathbb{P}^6(\mathbb{C})$ only appear in the cases of arbitrary dimension mentioned above.

Before going to introduce the main result, we need some notations and conventions. A family of hyperplanes $\{H_i\}_{1 \leq i \leq q}$ with $q \geq n + 1$ in $\mathbb{P}^n(\mathbb{C})$ is said to be in general position if any $n + 1$ hyperplanes in this family have empty intersection. A hypersurface $S$ in $\mathbb{P}^n(\mathbb{C})$ is said to be in general position with respect to $\{H_i\}_{1 \leq i \leq q}$ if it avoids all intersection points of $n$ hyperplanes in $\{H_i\}_{1 \leq i \leq q}$, namely if:

$$S \cap \left( \cap_{i \in I} H_i \right) = \emptyset, \quad \forall I \subset \{1, \ldots, q\}, |I| = n.$$  

Now assume that $\{H_i\}_{1 \leq i \leq q}$ is a family of hyperplanes of $\mathbb{P}^n(\mathbb{C})$ ($n \geq 2$) in general position. Let $\{H_i\}_{i \in I}$ be a subfamily of $n + 2$ hyperplanes. Take a partition $I = J \cup K$ such that $|J|, |K| \geq 2$. Then there exists a unique hyperplane $H_{JK}$ containing $\cap_{j \in J} H_j$ and $\cap_{k \in K} H_k$. We call $H_{JK}$ a diagonal hyperplane of $\{H_i\}_{i \in I}$. The family $\{H_i\}_{1 \leq i \leq q}$ is said to be generic if, for all disjoint subsets $I, J, J_1, \ldots, J_k$ of $\{1, \ldots, q\}$ such that $|I|, |J_i| \geq 2$ and $|I| + |J_i| = n + 2, 1 \leq i \leq k$, for every subset $\{i_1, \ldots, i_l\}$ of $I$, the intersection between the $|J|$ hyperplanes $H_j, j \in J$, the $k$ diagonal hyperplanes $H_{I_1J_1}, \ldots, H_{I_kJ_k}$, and the $l$ hyperplanes $H_{i_1}, \ldots, H_{i_l}$ is a linear subspace of codimension $\min\{k + l, |I|\} + |J|$, with the convention that when $\min\{k + l, |I|\} + |J| > n$, this intersection is empty. Such a generic condition naturally appears in our constructions, and it has the virtue of being preserved when passing to
smaller-dimensional subspaces

Our aim in this article is to prove that, for $3 \leq n \leq 6$, a small deformation of a union of generic $2n$ hyperplanes in $\mathbb{P}^n(\mathbb{C})$ is hyperbolic.

**Main Theorem.** Let $n$ be an integer number in $\{3, 4, 5, 6\}$. Let $\{H_i\}_{1 \leq i \leq 2n}$ be a family of $2n$ generic hyperplanes in $\mathbb{P}^n(\mathbb{C})$, where $H_i = \{h_i = 0\}$. Then there exists a hypersurface $S = \{s = 0\}$ of degree $2n$ in general position with respect to $\{H_i\}_{1 \leq i \leq 2n}$ such that the hypersurface

$$\Sigma_\varepsilon = \{\varepsilon s + \Pi_{i=1}^{2n} h_i = 0\}$$

is hyperbolic for sufficiently small complex $\varepsilon \neq 0$.

Our proof is based on the technique of Duval [10] in the case $n = 3$. By the deformation method of Zaidenberg and Shiffman[20], the problem reduces to finding a hypersurface $S$ such that all complements of the form

$$\cap_{i \in I} H_i \setminus (\cup_{j \notin I} H_j \setminus S)$$

are hyperbolic. This situation is very close to Theorem 2.5. To create such $S$, we proceed by deformation in order to allow points of intersection of $S$ with more and more linear subspaces coming from the family $\{H_i\}_{1 \leq i \leq 2n}$.

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2 Notations and preparation

2.1 Brody Lemma

Let $X$ be a compact complex manifold equipped with a hermitian metric $\| \cdot \|$. By an entire curve in $X$ we mean a nonconstant holomorphic map $f : \mathbb{C} \to X$. A Brody curve in $X$ is an entire curve $f : \mathbb{C} \to X$ such that $\|f'\|$ is bounded. Brody curves arise as limits of sequences of holomorphic maps as follows (see [1]).

**Brody Lemma.** Let $f_n : \mathbb{D} \to X$ be a sequence of holomorphic maps from the unit disk to a compact complex manifold $X$. If $\|f'_n(0)\| \to \infty$ as $n \to \infty$, then there exist a point $a \in \mathbb{D}$, a sequence $(a_n)$ converging to $a$, and a decreasing sequence $(r_n)$ of positive real numbers converging to 0 such that the sequence of maps

$$z \to f_n(a_n + r_n z)$$

converges toward a Brody curve, after extracting a subsequence.

From an entire curve in $X$, the Brody Lemma also produces a Brody curve in $X$. A second consequence is a well-known characterization of Kobayashi hyperbolicity.

**Brody Criterion.** A compact complex manifold $X$ is Kobayashi hyperbolic if and only if it contains no entire curve (or no Brody curve).

We shall repeatedly use the Brody Lemma under the following form.

**Sequences of entire curves.** Let $X$ be a compact complex manifold and let $(f_n)$ be a sequence of entire curves in $X$. Then there exist a sequence of reparameterizations $r_n : \mathbb{C} \to \mathbb{C}$ and a subsequence of $(f_n \circ r_n)$ which converges towards an entire curve (or Brody curve).
2.2 Nevanlinna theory and some applications

We recall some facts from Nevanlinna theory in the projective space $\mathbb{P}^n(\mathbb{C})$. Let $E = \sum \mu_\nu a_\nu$ be a divisor on $\mathbb{C}$ and let $k \in \mathbb{N} \cup \{ \infty \}$. Summing the $k$-truncated degrees of the divisor on disks by

$$ n^{[k]}(t, E) := \sum_{|a_\nu|<t} \min \{ k, \mu_\nu \} \quad (t>0), $$

the truncated counting function at level $k$ of $E$ is defined by

$$ N^{[k]}(r, E) := \int_1^r \frac{n^{[k]}(t, E)}{t} \, dt \quad (r>1). $$

When $k = \infty$, we write $n(t, E), N(r, E)$ instead of $n^{[\infty]}(t, E), N^{[\infty]}(r, E)$. We denote the zero divisor of a nonzero meromorphic function $\varphi$ by $\langle \langle \varphi \rangle \rangle_0$. Let $f: \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be an entire curve having a reduced representation $f = [f_0 : \cdots : f_n]$ in the homogeneous coordinates $[z_0 : \cdots : z_n]$ of $\mathbb{P}^n(\mathbb{C})$. Let $D = \{ Q = 0 \}$ be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ defined by a homogeneous polynomial $Q \in \mathbb{C}[z_0, \ldots, z_n]$ of degree $d \geq 1$. If $f(\mathbb{C}) \not\subset D$, we define the truncated counting function of $f$ with respect to $D$ as

$$ N_f^{[k]}(r, D) := N^{[k]}(r, (Q \circ f)_0). $$

The proximity function of $f$ for the divisor $D$ is defined as

$$ m_f(r, D) := \int_0^{2\pi} \log \| f(re^{i\theta}) \|^d \| Q \|_{f(re^{i\theta})} \, d\theta, $$

where $\| Q \|$ is the maximum absolute value of the coefficients of $Q$ and

$$ \| f(z) \| = \max \{|f_0(z)|, \ldots, |f_n(z)|\}. $$

Since $|Q(f)| \leq \| Q \| \cdot \| f \|^d$, one has $m_f(r, D) \geq 0$. Finally, the Cartan order function of $f$ is defined by

$$ T_f(r) := \frac{1}{2\pi} \int_0^{2\pi} \log \| f(re^{i\theta}) \| \, d\theta. $$

It is known that $f$ is a Brody curve, then its order

$$ \rho_f := \limsup_{r \to +\infty} \frac{T_f(r)}{\log r} $$

is bounded from above by 2. Furthermore, Eremenko $[1]$ showed the following.

**Theorem 2.1.** If $f: \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ is a Brody curve omitting $n$ hyperplanes in general position, then it is of order 1.

Consequently, we have the following theorem.

**Theorem 2.2.** If $f: \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ is a Brody curve avoiding the first $n$ coordinate hyperplanes $\{z_i = 0\}_{i=0}^{n-1}$, then it has a reduced representation of the form

$$ [1 : e^{\lambda_1} z^{\mu_1} : \cdots : e^{\lambda_{n-1}} z^{\mu_{n-1}} : g], $$

where $g$ is an entire function and $\lambda_i, \mu_i$ are constants. If $f$ also avoids the remaining coordinate hyperplane $\{z_n = 0\}$, then $g$ is of the form $e^{\lambda_n} z^{\mu_n}$.

The core of Nevanlinna theory consists of two main theorems.

**First Main Theorem.** Let $f: \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve and let $D$ be a hypersurface of degree $d$ in $\mathbb{P}^n(\mathbb{C})$ such that $f(\mathbb{C}) \not\subset D$. Then for every $r > 1$, the following holds

$$ m_f(r, D) + N_f(r, D) = dT_f(r) + O(1), $$

hence

$$ N_f(r, D) \leq dT_f(r) + O(1). \quad (2.1) $$
A holomorphic curve \( f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \) is said to be linearly nondegenerate if its image is not contained in any hyperplane. For non-negatively valued functions \( \varphi(r), \psi(r) \), we write
\[
\varphi(r) \leq \psi(r) \|
\]
if this inequality holds outside a Borel subset \( E \) of \( (0, +\infty) \) of finite Lebesgue measure. Next is the Second Main Theorem of Cartan [3].

**Second Main Theorem.** Let \( f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \) and let \( \{H_i\}_{1 \leq i \leq q} \) be a family of hyperplanes in general position in \( \mathbb{P}^n(\mathbb{C}) \). Then the following estimate holds:
\[
(q - n - 1)T_f(r) \leq \sum_{i=1}^{q} N_f^{[n]}(r, H_i) + S_f(r),
\]
where \( S_f(r) \) is a small term compared with \( T_f(r) \)
\[
S_f(r) = o(T_f(r)) \|
\]

The next three theorems can be deduced from the Second Main Theorem.

**Theorem 2.3.** Let \( \{H_i\}_{1 \leq i \leq n+2} \) be a family of hyperplanes in general position in \( \mathbb{P}^n(\mathbb{C}) \) with \( n \geq 2 \). If \( f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \setminus \bigcup_{i=1}^{n+2} H_i \) is an entire curve, then its image lies in one of the diagonal hyperplanes of \( \{H_i\}_{1 \leq i \leq n+2} \).

The following strengthened theorem is due to Dufresnoy [8].

**Theorem 2.4.** If a holomorphic map \( f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \) has its image in the complement of \( n + p \) hyperplanes \( H_1, \ldots, H_{n+p} \) in general position, then this image is contained in a linear subspace of dimension \( \left\lfloor \frac{n}{p} \right\rfloor \).

As a consequence, we have the classical generalization of Picard’s Theorem (case \( n = 1 \)), due to Fujimoto [12] (see also [14]).

**Theorem 2.5.** The complement of a collection of \( 2n + 1 \) hyperplanes in general position in \( \mathbb{P}^n(\mathbb{C}) \) is hyperbolic.

For hyperplanes that are not in general position, we have the following result (see [15], Theorem 3.10.15).

**Theorem 2.6.** Let \( \{H_i\}_{1 \leq i \leq q} \) be a family of \( q \geq 3 \) hyperplanes that are not in general position in \( \mathbb{P}^n(\mathbb{C}) \). If \( f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \setminus \bigcup_{i=1}^{q} H_i \) is an entire curve, then its image lies in some hyperplane.

### 3 Starting lemmas

Let us introduce some notations before going to other applications. Let \( \{H_i\}_{1 \leq i \leq q} \) be a family of generic hyperplanes of \( \mathbb{P}^n(\mathbb{C}) \), where \( H_i = \{h_i = 0\} \). For some integer \( 0 \leq k \leq n - 1 \) and some subset \( I_k = \{i_1, \ldots, i_{n-k}\} \) of the index set \( \{1, \ldots, q\} \) having cardinality \( n - k \), the linear subspace \( P_{k,I_k} = \cap_{i \in I_k} H_i \cong \mathbb{P}^k(\mathbb{C}) \) is called a subspace of dimension \( k \). For a holomorphic mapping \( f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \), we define
\[
n_f(t, P_{k,I_k}) := \sum_{|z| < t, f(z) \in P_{k,I_k}} \min_{i \in I_k} \text{ord}_z(h_i \circ f) \quad (t > 0),
\]
where we take the sum only for \( z \) in the preimage of \( P_{k,I_k} \), and
\[
N_f(r, P_{k,I_k}) := \int_{1}^{r} \frac{n_f(t, P_{k,I_k})}{t} \, dt \quad (r > 1). \quad (3.1)
\]
We denote by $P_{k,I_k}^*$ the complement $P_{k,I_k} \setminus \bigcup_{i \in I_k} H_i$ which will be called a star-subspace of dimension $k$. We can also define $n_f(t, P_{k,I_k}^*)$ and $N_f(r, P_{k,I_k}^*)$. Assume now $q = 2n + 1 + m$ with $m \geq 0$. Consider complements of the form

$$\mathbb{P}^n(\mathbb{C}) \setminus \bigcup_{i=1}^{2n+1+m} H_i \setminus A_{m,n},$$

(3.2)

where $A_{m,n}$ is a set of at most $m$ elements of the form $P_{k,I_k}^*$ ($0 \leq k \leq n - 2$). We note that if $m = 0$, these complements are hyperbolic by Theorem 2.5.

In $\mathbb{P}^2(\mathbb{C})$, a union of lines $\bigcup_{i=1}^q H_i$ is in general position if any three lines have empty intersection, and it is generic if in addition any three intersection points between three distinct pairs of lines are not collinear.

**Lemma 3.1.** In $\mathbb{P}^2(\mathbb{C})$, if $m \leq 3$, all complements of the form (3.2) are hyperbolic.

**Proof.** Without loss of generality, we can assume that $A_{m,2}$ is a set of $m$ distinct points belonging to $\bigcup_{i=1}^{5+m} H_{i_1} \cap H_{i_2}$.

When $m = 1$, an entire curve $f: \mathbb{C} \to \mathbb{P}^2(\mathbb{C}) \setminus \bigcup_{i=1}^6 H_i \setminus A_{1,2}$, if it exists, must avoid at least four lines.

By Theorem 2.3, its image lies in a diagonal line, which does not contain the intersection point of the two remaining lines by the generic condition. Hence, $f$ must be contained in the complement of four points in a line. By Picard’s theorem, $f$ is constant, which is contradiction.

When $m = 2$, $A_{2,2}$ is a set consisting of two points $A, B$, where $A = H_{i_1} \cap H_{i_2}, B = H_{i_3} \cap H_{i_4}$. We denote by $I$ the index set $\{i_1, i_2, i_3, i_4\}$, which has three elements if both $A$ and $B$ belong to a single line $H_i$ and which has four elements otherwise.

Let $f: \mathbb{C} \to \mathbb{P}^2(\mathbb{C}) \setminus \bigcup_{i=1}^7 H_i \setminus A_{2,2}$ be an entire curve. If $z \in f^{-1}(A)$, we have

$$\text{ord}_z(h_{i_1} \circ f) \geq 1,$$

$$\text{ord}_z(h_{i_2} \circ f) \geq 1.$$

This implies

$$\min \left\{ \text{ord}_z(h_{i_1} \circ f), 2 \right\} + \min \left\{ \text{ord}_z(h_{i_2} \circ f), 2 \right\} \leq 3 \min_{1 \leq j \leq 2} \text{ord}_z(h_{i_j} \circ f).$$

(3.3)

Hence, by summing this inequality

$$\sum_{|z| < t, f(z) = A} \min \left\{ \text{ord}_z(h_{i_1} \circ f), 2 \right\} + \sum_{|z| < t, f(z) = A} \min \left\{ \text{ord}_z(h_{i_2} \circ f), 2 \right\} \leq 3 \sum_{|z| < t, f(z) = A} \min_{1 \leq j \leq 2} \text{ord}_z(h_{i_j} \circ f).$$

(3.4)
By taking the sum of both sides of these inequalities and by integrating, we obtain

\[ \sum_{|z|<t,f(z)=B} \min \{ \text{ord}_z(h_{i_3} \circ f), 2 \} + \sum_{|z|<t,f(z)=B} \min \{ \text{ord}_z(h_{i_4} \circ f), 2 \} \leq 3 \sum_{3 \leq j \leq 4} \min \text{ord}_z(h_{ij} \circ f). \] (3.5)

By taking the sum of both sides of these inequalities and by integrating, we obtain

\[ \sum_{i \in I} N_j^{[2]}(r, H_i) \leq 3(N_f(r, A) + N_f(r, B)). \] (3.6)

Now, let \( \mathcal{L} = \{ \ell = 0 \} \) be the line passing through \( A \) and \( B \). Since \( \ell = \alpha_1 h_{i_1} + \alpha_2 h_{i_2} = \alpha_3 h_{i_3} + \alpha_4 h_{i_4} \) for some \( \alpha_1, \ldots, \alpha_4 \in \mathbb{C} \), the following inequalities hold

\[
\begin{align*}
\min_{1 \leq j \leq 2} \text{ord}_z(h_{ij} \circ f) &\leq \text{ord}_z(\ell \circ f) \quad (z \in f^{-1}(A)), \\
\min_{3 \leq j \leq 4} \text{ord}_z(h_{ij} \circ f) &\leq \text{ord}_z(\ell \circ f) \quad (z \in f^{-1}(B)).
\end{align*}
\] (3.7)

Since \( f^{-1}(A) \) and \( f^{-1}(B) \) are two disjoint subsets of \( f^{-1}(\mathcal{L}) \), by taking the sum of both sides of these inequalities on discs and by integrating, we obtain

\[ N_f(r, A) + N_f(r, B) \leq N_f(r, \mathcal{L}). \] (3.8)

If \( f \) would be linearly nondegenerate, then starting from Cartan Second Main Theorem, and using (3.6), (3.8), we would get

\[
4 T_f(r) \leq \sum_{i=1}^{7} N_j^{[2]}(r, H_i) + S_f(r) \\
\leq \sum_{i \in J} N_j^{[2]}(r, H_i) + S_f(r) \\
\leq 3(N_f(r, A) + N_f(r, B)) + S_f(r) \\
\leq 3 N_f(r, \mathcal{L}) + S_f(r) \\
\leq 3 T_f(r) + S_f(r),
\] (3.9)

which is absurd. Thus, any entire curve \( f : \mathbb{C} \to \mathbb{P}^2(\mathbb{C}) \setminus (\cup_{i=1}^{7} H_i \setminus A_{2.2}) \) must be contained in some line \( L \). Furthermore, the number of points of intersection between \( L \) and \( \cup_{i=1}^{7} H_i \setminus \{ A, B \} \) is at least 3 by the generic condition. By Picard’s Theorem, this contradicts the assumption that \( f \) is nonconstant.

When \( m = 3 \), \( A_{3.2} \) is a set consisting of three points \( A, B, C \), where \( A = H_{i_1} \cap H_{i_2}, B = H_{i_3} \cap H_{i_4}, C = H_{i_5} \cap H_{i_6} \). In this case, the index set \( J = \{ i_1, i_2, i_3, i_4, i_5, i_6 \} \) may contain 4, 5 or 6 elements.
Suppose to the contrary that there is an entire curve \( f : \mathbb{C} \rightarrow \mathbb{P}^2(\mathbb{C}) \setminus (\cup_{i=1}^8 H_i \setminus A_{3,3}) \). Similarly as above, cf. (3.4), (3.5), (3.6), we can show in all the four illustrated cases
\[
\sum_{i \in J} N^2_f(r, H_i) \leq 3 \left( N_f(r, A) + N_f(r, B) + N_f(r, C) \right).
\]

Next, let \( C = \{ c = 0 \} \) be the degenerate cubic consisting of the three lines \( AB = \{ \ell_{AB} = 0 \} \), \( BC = \{ \ell_{BC} = 0 \} \), and \( CA = \{ \ell_{CA} = 0 \} \). Similarly as in (3.7), we have
\[
2 \min_{1 \leq j \leq 2} \ord_z(h_{ij} \circ f) \leq \ord_z(\ell_{AB} \circ f) + \ord_z(\ell_{CA} \circ f)
\]
\[
= \ord_z(c \circ f) \quad (z \in f^{-1}(A)).
\]

We also have two other inequalities for \( h_{i3}, h_{i4}, z \in f^{-1}(B) \) and for \( h_{i5}, h_{i6}, z \in f^{-1}(C) \). Summing these inequalities and integrating, we get
\[
2 \left( N_f(r, A) + N_f(r, B) + N_f(r, C) \right) \leq N_f(r, C).
\]

If the curve \( f \) is linearly nondegenerate, then by proceeding as we did in (3.9), we also get a contradiction.
\[
5T_f(r) \leq \sum_{i=1}^8 N^2_f(r, H_i) + S_f(r)
\]
\[
\leq 3 \left( N_f(r, A) + N_f(r, B) + N_f(r, C) \right) + S_f(r)
\]
\[
\leq \frac{3}{2} N_f(r, C) + S_f(r)
\]
\[
\leq \frac{9}{2} T_f(r) + S_f(r).
\]

Thus the curve \( f \) must be contained in some line. By analyzing the position of this line with respect to \( \{ H_i \}_{1 \leq i \leq 8} \setminus \{ A, B, C \} \) and by using Picard’s theorem, we conclude as above.

In \( \mathbb{P}^3(\mathbb{C}) \), the generic condition for the family of planes \( \{ H_i \}_{1 \leq i \leq q} \) excludes the following cases.

(1) There are three disjoint subsets \( I, J, K \) of \( \{ 1, \ldots, q \} \) with \( |I| = 3, |J| = 2, |K| = 3 \) such that the diagonal (hyper)plane \( H_{IJ} \) contains the point \( \cap_{k \in K} H_k \).

(2) There are four disjoint subsets \( I, J, K_1, K_2 \) of \( \{ 1, \ldots, q \} \) with \( |I| = 3, |J| = 2, |K_1| = |K_2| = 2 \) such that the three points \( (\cap_{k_1 \in K_1} H_{k_1}) \cap H_{IJ}, (\cap_{k_2 \in K_2} H_{k_2}) \cap H_{IJ} \) and \( \cap_{i \in I} H_i \) are collinear.
Lemma 3.2. In \( \mathbb{P}^3(\mathbb{C}) \), if \( m \leq 2 \), all complements of the form (3.2) are hyperbolic.

Proof. Without loss of generality, we can assume that \( A_{m,3} \) is a set of \( m \) elements belonging to:

\[
\left( \bigcup_{1 \leq i_1 < i_2 < i_3 \leq 7 + m} (H_{i_1} \cap H_{i_2})^* \right) \bigcup \left( \bigcup_{1 \leq i_1 < i_2 < i_3 \leq 7 + m} H_{i_1} \cap H_{i_2} \cap H_{i_3} \right).
\]

Suppose to the contrary that there exists a Brody curve \( f: \mathbb{C} \rightarrow \mathbb{P}^3(\mathbb{C}) \setminus (\bigcup_{i = 1}^{7+m} H_i \setminus A_{m,3}) \).

When \( m = 1 \), the curve \( f \) must avoid at least five planes. By Theorem 2.4, its image is contained in some line \( L \). By the generic condition, the number of intersection points between \( L \) and \( \bigcup_{i = 1}^{8} H_i \setminus A_{1,3} \) is at least 3. By Picard’s theorem, \( f \) must be constant, which is a contradiction.

Next, we consider the case \( m = 2 \). If \( A_{2,3} = \{ l_1, l_2 \} \) where \( l_1, l_2 \) are lines, then the curve \( f \) avoids five planes, say \( \{ H_i \}_{1 \leq i \leq 5} \). By Theorems 2.2 and 2.3, its image lands in some line \( L \), which is contained in a diagonal plane \( \mathcal{P} \) of the family \( \{ H_i \}_{1 \leq i \leq 5} \). We may assume that the plane \( \mathcal{P} \) passes through the point \( H_1 \cap H_2 \cap H_3 \) and contains the line \( H_4 \cap H_5 \). If the line \( L \) does not pass through the point \( H_1 \cap H_2 \cap H_3 \), then it intersects \( \{ H_i \}_{1 \leq i \leq 3} \) in three distinct points, hence \( f \) is constant by Picard’s theorem. Thus \( L \) must pass through the point \( H_1 \cap H_2 \cap H_3 \). In the plane \( \mathcal{P} \), the curve \( f \) can pass through the points \( l_1 \cap \mathcal{P} \) and \( l_2 \cap \mathcal{P} \). But by the generic condition, cf. (2) above, the three points \( H_1 \cap H_2 \cap H_3, l_1 \cap \mathcal{P} \), \( l_2 \cap \mathcal{P} \) are not collinear. Hence, \( f(\mathbb{C}) \) is contained in a complement of at least three points in the line \( L \), which is impossible by Picard’s theorem.

Two substantial cases remain:

(a) \( A_{2,3} = \{ A, l^* \} \), where \( A \) is a point and \( l \) is a line;

(b) the set \( A_{2,3} \) consists of two points.

We treat case (a). If both \( A \) and \( l \) are contained in some common plane \( H_i \), then \( f \) avoids five planes. By Theorems 2.2 and 2.3, its image must be contained in some diagonal plane, which does not contain the point \( A \) by the generic condition. Hence \( f \) must avoid seven planes in general position, which is absurd by Theorem 2.3. Thus, we can assume that \( A = H_1 \cap H_2 \cap H_3 \) and \( l^* = (H_4 \cap H_5) \setminus \bigcup_{i \neq 4, 5} H_i \). Hence \( f \) avoids four planes \( H_i \) (\( 6 \leq i \leq 9 \)).

First, we show that \( f \) is linearly nondegenerate. Suppose to the contrary that \( f(\mathbb{C}) \) is contained in some plane \( \mathcal{P} \). If \( A \notin \mathcal{P} \), then \( f \) also avoids \( H_1, H_2, H_3 \), which is impossible by Theorem 2.5. Hence the plane \( \mathcal{P} \) must pass through the point \( A \). If \( f(\mathbb{C}) \) is contained in some line \( L \subset \mathcal{P} \), then \( L \) must also pass through \( A \), for the same reason. Note that the number of intersection points between \( L \) and \( \{ H_i \}_{6 \leq i \leq 9} \) is at least 2, and it equals 2 only if either \( L \) passes through some point \( H_{i_1} \cap H_{i_2} \cap H_{i_3} \) (\( 6 \leq i_1 < i_2 < i_3 \leq 9 \)) or \( L \) intersects two lines \( H_{i_1} \cap H_{i_2}, H_{i_3} \cap H_{i_4} \) (\( \{ i_1, i_2, i_3, i_4 \} = \{ 6, 7, 8, 9 \} \)). If \( L \) has empty intersection with \( H_4 \cap H_5 \), then \( f \) avoids at least four points in the line \( L \), hence it is constant. If \( L \) intersects \( H_4 \cap H_5 \), then by considering the diagonal plane passing through \( A \) and containing \( H_4 \cap H_5 \), the two cases where \( |L \cap \{ H_i \}_{6 \leq i \leq 9}| = 2 \) are excluded by the generic condition. Thus \( f \) always avoids three distinct points in \( L \), hence it is constant.

Consequently, we can assume that \( f \) does not land in any line in the plane \( \mathcal{P} \). There are two possible positions of \( \mathcal{P} \):

(a1) it is a diagonal plane containing \( A \) and some line in \( \bigcup_{6 \leq i_1 < i_2 \leq 9} H_{i_1} \cap H_{i_2} \);
(a2) it does not contain any line in $\cup_{6 \leq i_1 < i_2 < i_3 \leq 9} H_{i_1} \cap H_{i_2}$.

In case (a1), assume that $\mathcal{P}$ contains the line $H_6 \cap H_7$. Among $\{ H_i \cap \mathcal{P} \}_{1 \leq i \leq 9}$, two lines $H_6 \cap \mathcal{P}$, $H_7 \cap \mathcal{P}$ coincide, and dropping the line $H_1 \cap \mathcal{P}$, by the generic condition, it remains seven lines $\{ H_i \cap \mathcal{P} \}_{i \neq 1,7}$ in general position in $\mathcal{P}$.

Letting $B$ be the intersection point of the line $l = H_4 \cap H_5$ with the plane $\mathcal{P}$, the curve $f$ lands in $\mathcal{P} \setminus (\cup_{1 \leq i \leq 9} H_i \cap \mathcal{P}) \setminus \{ A, B \})$. As in (3.9), $f(\mathbb{C})$ is contained in some line, which is a contradiction.

Next, consider case (a2).

If $\mathcal{P}$ contains some point in $\cup_{6 \leq i_1 < i_2 < i_3 \leq 9} H_{i_1} \cap H_{i_2} \cap H_{i_3}$, say $H_6 \cap H_7 \cap H_8$, then the curve $f$ avoids three lines $H_i \cap \mathcal{P}$ ($6 \leq i \leq 8$), which are not in general position. By Theorem 2.6 $f(\mathbb{C})$ must be contained in some line, which is a contradiction. Thereby, $\mathcal{P}$ does not contain any point in $\cup_{6 \leq i_1 < i_2 < i_3 \leq 9} H_{i_1} \cap H_{i_2} \cap H_{i_3}$. But then the curve $f$ avoids a collection of four lines $\{ H_i \cap \mathcal{P} \}_{6 \leq i \leq 9}$, which are in general position. By Theorem 2.3 its image must land in some diagonal line of this family, which is a contradiction.

Still in case (a), we can therefore assume that $f$ is linearly nondegenerate. Assume that the omitted planes $H_6$, $H_7$, $H_8$, $H_9$ are given in the homogeneous coordinates $[z_0 : z_1 : z_2 : z_3]$ by equations $\{ z_i = 0 \}$ ($0 \leq i \leq 3$). By Theorems 2.2 $f$ has a reduced representation of the form

$$[1 : \lambda_1 z + \mu_1 : \lambda_2 z + \mu_2 : \lambda_3 z + \mu_3],$$

where $\lambda_i, \mu_i$ are constants with $\lambda_i \neq 0$ ($1 \leq i \leq 3$ and $\lambda_i \neq \lambda_j$ ($i \neq j$)). Let $\mathcal{D}$ be the diagonal plane passing through the point $A = H_1 \cap H_2 \cap H_3$ and containing the line $l = H_4 \cap H_5$. By similar arguments as in Lemma 3.1 cf. (3.7), (3.8), we can show that

$$N_f(r, A) + N_f(r, l^*) \leq N_f(r, \mathcal{D}).$$

From the elementary inequality

$$\min \{ \text{ord}_z(h_4 \circ f), 3 \} + \min \{ \text{ord}_z(h_5 \circ f), 3 \} \leq 4 \min_{4 \leq i \leq 5} \text{ord}_z(h_i \circ f) \quad (z \in f^{-1}(l^*)),$$

by taking the sum on disks and then by integrating, we get

$$N_f^{[3]}(r, H_4) + N_f^{[3]}(r, H_5) \leq 4 N_f(r, l^*).$$

Next, we try to bound $N_f^{[3]}(r, H_i)$ ($1 \leq i \leq 3$) from above in terms of $N_f(r, A)$. Since $f$ is of the form (3.10), for any $z_1, z_2 \in f^{-1}(A)$, we have

$$f^{(k)}(z_1) = f^{(k)}(z_2) \quad (k \in \mathbb{N})$$
hence
\[ \text{ord}_2(h_i \circ f) = \text{ord}_2(h_i \circ f) \quad (1 \leq i \leq 3). \] (3.13)

Thus, it suffices to consider the two cases:

(a3) \( \text{ord}_2(h_i \circ f) \leq 2 \) for all \( 1 \leq i \leq 3 \) and for all \( z \in f^{-1}(A) \);

(a4) \( \text{ord}_2(h_i \circ f) \geq 3 \) for some \( i \) with \( 1 \leq i \leq 3 \) and for all \( z \in f^{-1}(A) \).

In case (a3), the elementary inequality
\[ \sum_{i=1}^{3} \min \{ \text{ord}_2(h_i \circ f), 3 \} \leq 5 \min_{1 \leq i \leq 3} \text{ord}_2(h_i \circ f) \quad (z \in f^{-1}(A)), \]
yields
\[ N_j^{[3]}(r, H_1) + N_j^{[3]}(r, H_2) + N_j^{[3]}(r, H_3) \leq 5 N_f(r, A). \] (3.14)

Since \( f \) is linearly nondegenerate, we can proceed similarly as in (3.9)
\[ 5 T_f(r) \leq \sum_{i=1}^{9} N_j^{[3]}(r, H_i) + S_f(r) \]
\[ \leq 5 N_f(r, A) + 4 N_f(r, l^*) + S_f(r) \]
\[ = 5 (N_f(r, A) + N_f(r, l^*)) - N_f(r, l^*) + S_f(r) \]
\[ \leq 5 N_f(r, D) - N_f(r, l^*) + S_f(r) \]
\[ \leq 5 T_f(r) - N_f(r, l^*) + S_f(r). \] (3.15)

This implies
\[ N_f(r, l^*) = S_f(r) \]
and hence, by (3.12), we have
\[ N_j^{[3]}(r, H_4) + N_j^{[3]}(r, H_5) = S_f(r). \]

Therefore, the first inequality of (3.15) can be rewritten as
\[ 5 T_f(r) \leq \sum_{i=1}^{3} N_j^{[3]}(r, H_i) + S_f(r). \]

By the First Main Theorem, the right-hand side of the above inequality is bounded from above by
\[ 3 T_f(r) + S_f(r). \] Thus we get
\[ 5 T_f(r) \leq 3 T_f(r) + S_f(r), \]
which is absurd.

Next, we consider case (a4). Assume that \( \text{ord}_2(h_1 \circ f) \geq 3 \) for all \( z \in f^{-1}(A) \). Since \( f \) is of the form (3.10), we claim that
\[ \text{ord}_2(h_i \circ f) \leq 2 \quad (z \in f^{-1}(A), \quad 2 \leq i \leq 3). \] (3.16)

Indeed, if \( \text{ord}_2(h_i \circ f) \geq 3 \) for some \( z \in f^{-1}(A) \) and for some \( 2 \leq i \leq 3 \), say \( i = 2 \), then
\( (e^{\lambda_1 z+\mu_1}, e^{\lambda_2 z+\mu_2}, e^{\lambda_3 z+\mu_3}) \) is a solution of a system of six linear equations of the form
\[
\begin{align*}
0 &= a_{10} + a_{11} u + a_{12} v + a_{13} w, \\
0 &= a_{11} \lambda_1 u + a_{12} \lambda_2 v + a_{13} \lambda_3 w, \\
0 &= a_{11} \lambda_1^2 u + a_{12} \lambda_2^2 v + a_{13} \lambda_3^2 w, \\
0 &= a_{20} + a_{21} u + a_{22} v + a_{23} w, \\
0 &= a_{21} \lambda_1 u + a_{22} \lambda_2 v + a_{23} \lambda_3 w, \\
0 &= a_{21} \lambda_1^2 u + a_{22} \lambda_2^2 v + a_{23} \lambda_3^2 w,
\end{align*}
\]
where \( u, v, w \) are unknowns, and where \( a_{ij} \) \((0 \leq i \leq 3)\) are the coefficients of \( h_i \) \((1 \leq i \leq 2)\) in the homogeneous coordinate \([z_0 : z_1 : z_2 : z_3]\). Since \( \lambda_i \) are nonzero distinct constants, this forces the two linear forms \( h_1, h_2 \) to be linearly dependent, which is a contradiction.

It follows from (3.16) that

\[
\min \{ \text{ord}_z(h_2 \circ f), 3 \} + \min \{ \text{ord}_z(h_3 \circ f), 3 \} \leq 3 \min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f) \quad (z \in f^{-1}(A)).
\]

By taking the sum on disks and by integrating, we get

\[
N_f^3(r, H_2) + N_f^3(r, H_3) \leq 3 N_f(r, A).
\] (3.17)

We may therefore proceed similarly as in (3.15)

\[
5 T_f(r) \leq \sum_{i=1}^{9} N_f^3(r, H_i) + S_f(r)
\]

\[
\leq N_f^3(r, H_1) + 3 N_f(r, A) + 4 N_f(r, l^*) + S_f(r)
\]

\[
\leq N_f(r, H_1) + 4 \left( N_f(r, A) + N_f(r, l^*) \right) - N_f(r, A) + S_f(r)
\]

\[
\leq T_f(r) + 4 N_f(r, D) - N_f(r, A) + S_f(r)
\]

\[
\leq 5 T_f(r) - N_f(r, A) + S_f(r).
\] (3.18)

This implies

\[
N_f(r, A) = S_f(r).
\]

By (3.17), we have

\[
N_f^3(r, H_2) + N_f^3(r, H_3) = S_f(r).
\]

Hence we can rewrite the first inequality of (3.18) and use First Main Theorem to get a contradiction

\[
5 T_f(r) \leq N_f^3(r, H_1) + N_f^3(r, H_4) + N_f^3(r, H_5) + S_f(r)
\]

\[
\leq 3 T_f(r) + S_f(r).
\]

Let us consider case (b). Assume now \( A_{2,3} = \{ A, B \} \), where \( A, B \) are two points contained in \( \cup_{1 \leq i < j < l \leq 9} H_{i_1} \cap H_{i_2} \cap H_{i_3} \). There are three possibilities for the positions of \( A \) and \( B \):

(b1) both \( A \) and \( B \) are contained in some line \( H_i \cap H_j \);

(b2) both \( A \) and \( B \) are contained in some plane \( H_i \) but they are not contained in any line \( H_i \cap H_j \);

(b3) there is no plane \( H_i \) containing both points \( A \) and \( B \).

In case (b1), the curve \( f \) avoids a family of five planes and, therefore, its image is contained in some diagonal plane of this family, which contains neither \( A \) nor \( B \) by the generic condition. Hence \( f \) avoids all planes \( H_i \), which is absurd by Theorem 2.5.

Next, we consider case (b2). Assume that \( A = H_1 \cap H_2 \cap H_3 \) and \( B = H_1 \cap H_4 \cap H_5 \), hence \( f \) avoids the 4 planes \( H_i \) \((6 \leq i \leq 9)\). Similarly as in case (a), the generic condition allows us to assume that \( f \) is linearly nondegenerate.

Since \( f \) avoids four planes, it is of the form (3.10) in some affine coordinates on \( \mathbb{P}^3(\mathbb{C}) \). Since \( f \) has no singular point, we have

\[
\min_{i \in \{1,2,3\}} \text{ord}_z(h_i \circ f) = 1 \quad (z \in f^{-1}(A)),
\]

\[
\min_{i \in \{1,4,5\}} \text{ord}_z(h_i \circ f) = 1 \quad (z \in f^{-1}(B)).
\] (3.19)

Hence by using these two equalities together with (3.16),

\[
\sum_{i \in \{1,2,3\}} \min \{ \text{ord}_z(h_i \circ f), 3 \} \leq 6 \quad \min_{i \in \{1,2,3\}} \text{ord}_z(h_i \circ f), \quad (z \in f^{-1}(A)),
\]
\[
\sum_{i \in \{1,4,5\}} \min \{ \text{ord}_z(h_i \circ f), 3 \} \leq 6 = 6 \min_{i \in \{1,4,5\}} \text{ord}_z(h_i \circ f), \quad (z \in f^{-1}(A)).
\]

Thus, by taking the sum on disks of both sides of these inequalities and by integrating,

\[
5 \sum_{i=1}^{5} N_f^[[3]](r, H_i) \leq 6 (N_f(r, A) + N_f(r, B)).
\]

Next, using again that \( f \) is of the form \([3,10]\), one can find two planes \( P_1 = \{p_1 = 0\}, P_2 = \{p_2 = 0\} \) containing the line \( AB \) such that

\[
\text{ord}_z(p_1 \circ f) \geq 2 \quad (z \in f^{-1}(A)),
\]
\[
\text{ord}_z(p_2 \circ f) \geq 2 \quad (z \in f^{-1}(B)).
\]

Let \( Q = \{q = p_1p_2 = 0\} \) be the degenerate quadric \( P_1 \cup P_2 \). We have

\[
3 = 3 \min_{i \in \{1,2,3\}} \text{ord}_z(h_i \circ f) \leq \text{ord}_z(p_1 \circ f) + \text{ord}_z(p_2 \circ f) = \text{ord}_z(q \circ f) \quad (z \in f^{-1}(A)),
\]

\[
3 = 3 \min_{i \in \{1,4,5\}} \text{ord}_z(h_i \circ f) \leq \text{ord}_z(p_1 \circ f) + \text{ord}_z(p_2 \circ f) = \text{ord}_z(q \circ f) \quad (z \in f^{-1}(B)),
\]

which implies, by integrating, that

\[
3 \left( N_f(r, A) + N_f(r, B) \right) \leq N_f(r, T).
\]

We proceed similarly as above to get a contradiction

\[
5 T_f(r) \leq \sum_{i=1}^{9} N_f^[[3]](r, H_i) + S_f(r)
\]
\[
\leq 6 \left( N_f(r, A) + N_f(r, B) \right) + S_f(r)
\]
\[
\leq 2 N_f(r, T) + S_f(r)
\]
\[
\leq 4 T_f(r) + S_f(r).
\]

Now, we consider case \((b3)\). Assume that \( A = H_1 \cap H_2 \cap H_3, B = H_4 \cap H_5 \cap H_6 \), when \( f \) avoids the three planes \( H_7, H_8, H_9 \). If \( f(C) \) is contained in some plane \( P \), then it is not hard to see that \( P \) must pass through both \( A \) and \( B \). Furthermore, by using Theorem \([2,4]\) one can show that \( P \) does not pass through the point \( C = H_7 \cap H_8 \cap H_9 \). One can then always find 7 lines in general position in \( P \) among \( \{H_i \cap P\}_{1 \leq i \leq 9} \). Hence one can use similar arguments as in Lemma \([31]\) case \( m = 2 \), to get a contradiction. Thus, we can suppose that \( f \) is linearly nondegenerate.

Assume that the omitted planes \( H_7, H_8, H_9 \) are given in the homogeneous coordinates \([z_0 : z_1 : z_2 : z_3]\) by the equations \( \{z_0 = 0\}, \{z_1 = 0\}, \{z_2 = 0\} \). Since \( \{H_i\}_{1 \leq i \leq 9} \) is a family of planes in general position, the planes \( H_i (1 \leq i \leq 6) \) are given by

\[
h_i = \sum_{j=0}^{3} a_{ij} z_j = 0,
\]

with \( a_{i3} \neq 0 (1 \leq i \leq 6) \). Set \( l_{i1,2} = H_{i1} \cap H_{i2} (1 \leq i_1 < i_2 \leq 3), l_{j1,2} = H_{j1} \cap H_{j2} (4 \leq j_1 < j_2 \leq 6) \). For \( 1 \leq i < j \leq 3 \) or \( 4 \leq i < j \leq 6 \), let \( R_{i,j} = \{r_{i,j} = 0\} \) be the plane containing the lines \( AB, l_{i,j} \) and let \( T_{i,j} = \{t_{i,j} = a_{i3} h_i - a_{i3} h_j = 0\} \) be the plane passing through the point \( C = [0 : 0 : 0 : 1] \) and containing the line \( l_{i,j} \). We note that all \( r_{i,j}, t_{i,j} \) are linear combinations of \( h_i \) and \( h_j \) with nonzero coefficients.

Since \( f \) avoids three planes, by Theorem \([2,2]\) it has a reduced representation of the form

\[
[1 : e^{\lambda_1 z + \mu_1} : e^{\lambda_2 z + \mu_2} : g],
\]
where $\lambda_1$, $\lambda_2$, $\mu_1$, $\mu_2$ are constants with $\lambda_1 \neq \lambda_2$, $\lambda_1, \lambda_2 \neq 0$ and where $g$ is an entire function. Since $f$ has no singular point, we have

$$
\begin{align*}
\min_{1 \leq i \leq 3} \ord_z (h_i \circ f) &= 1 \quad (z \in f^{-1}(A)), \\
\min_{4 \leq j \leq 6} \ord_z (h_j \circ f) &= 1 \quad (z \in f^{-1}(B)).
\end{align*}
$$

(3.21)

Since $f$ is of the form (3.20), we claim that

$$
\begin{align*}
\min \{ \ord_z (h_{i_1} \circ f), \ord_z (h_{i_2} \circ f) \} &\leq 2 \quad (z \in f^{-1}(A) \quad 1 \leq i_1 < i_2 \leq 3), \\
\min \{ \ord_z (h_{j_1} \circ f), \ord_z (h_{j_2} \circ f) \} &\leq 2 \quad (z \in f^{-1}(B) \quad 4 \leq j_1 < j_2 \leq 6).
\end{align*}
$$

(3.22)

Indeed, if one of these inequalities does not hold, say $\min \{ \ord_z (h_{i_1} \circ f), \ord_z (h_{i_2} \circ f) \} \geq 3$ for some $z \in f^{-1}(A)$, then $z$ is a solution of the following system of equations

$$
\begin{align*}
0 &= (t_{1,2} \circ f)(z), \\
0 &= (t_{1,2} \circ f)'(z), \\
0 &= (t_{1,2} \circ f)''(z).
\end{align*}
$$

Equivalently, $(e^{\lambda_1 z + \mu_1}, e^{\lambda_2 z + \mu_2})$ is a solution of a system of three linear equations of the form

$$
\begin{align*}
0 &= (a_{23} a_{10} - a_{13} a_{20}) + (a_{23} a_{11} - a_{13} a_{21}) x + (a_{23} a_{12} - a_{13} a_{22}) y, \\
0 &= (a_{23} a_{11} - a_{13} a_{21}) \lambda_1 x + (a_{23} a_{12} - a_{13} a_{22}) \lambda_2 y, \\
0 &= (a_{23} a_{11} - a_{13} a_{21}) \lambda_1^2 x + (a_{23} a_{12} - a_{13} a_{22}) \lambda_2^2 y,
\end{align*}
$$

where $x, y$ are unknowns. Since $\lambda_1 \neq \lambda_2$, $\lambda_1, \lambda_2 \neq 0$, this implies that the two linear forms $h_1$, $h_2$ must be linearly dependent, which is a contradiction.

It follows from (3.21) and (3.22) that

$$
\begin{align*}
\sum_{i=1}^{3} \min \{ \ord_z (h_i \circ f), 3 \} &\leq 6 \quad (z \in f^{-1}(A)), \\
\sum_{j=4}^{6} \min \{ \ord_z (h_j \circ f), 3 \} &\leq 6 \quad (z \in f^{-1}(B)).
\end{align*}
$$

(3.23)

Now we prove the following equality

Claim 3.1.

$$
T_f(r) = N_f(r, A) + N_f(r, B) + S_f(r).
$$

(3.24)

Proof. Since $f$ is of the form (3.20) and since $t_{i,j}$ does not contain the term $x_3$, we have

$$
\begin{align*}
\ord_z (t_{i_1,i_2} \circ f) &= \ord_z (t_{i_1,j_2} \circ f) \quad (z \in f^{-1}(A) \quad 1 \leq i_1 < i_2 \leq 3), \\
\ord_z (t_{j_1,j_2} \circ f) &= \ord_z (t_{j_1,j_2} \circ f) \quad (z \in f^{-1}(B) \quad 4 \leq j_1 < j_2 \leq 6).
\end{align*}
$$

(3.25)

Thus, it suffices to consider the four cases depending on $f$ and $t_{i,j}$:

(b3.1) $\ord_z (t_{i_1,i_2} \circ f) = 1$ for all $1 \leq i_1 < i_2 \leq 3$, for all $z \in f^{-1}(A)$ and $\ord_z (t_{j_1,j_2} \circ f) = 1$ for all $4 \leq j_1 < j_2 \leq 6$, for all $z \in f^{-1}(B)$;

(b3.2) $\ord_z (t_{i_1,i_2} \circ f) \geq 2$ for some $1 \leq i_1 < i_2 \leq 3$, for all $z \in f^{-1}(A)$ and $\ord_z (t_{j_1,j_2} \circ f) = 1$ for all $4 \leq j_1 < j_2 \leq 6$, for all $z \in f^{-1}(B)$;

(b3.3) $\ord_z (t_{i_1,i_2} \circ f) = 1$ for all $1 \leq i_1 < i_2 \leq 3$, for all $z \in f^{-1}(A)$ and $\ord_z (t_{j_1,j_2} \circ f) \geq 2$ for some $4 \leq j_1 < j_2 \leq 6$, for all $z \in f^{-1}(B)$;
(b3.4) \( \text{ord}_z(t_{i_1,i_2} \circ f) \geq 2 \) for some \( 1 \leq i_1 < i_2 \leq 3 \), for all \( z \in f^{-1}(A) \) and \( \text{ord}_z(t_{j_1,j_2} \circ f) \geq 2 \) for some \( 4 \leq j_1 < j_2 \leq 6 \), for all \( z \in f^{-1}(B) \).

Consider case (b3.1). Since \( t_{i,j} \) is a linear combination of \( h_i \) and \( h_j \) with nonzero coefficients, we have

\[
\begin{align*}
\min \{ \text{ord}_z(h_{i_1} \circ f), \text{ord}_z(h_{i_2} \circ f) \} &= 1 \quad (z \in f^{-1}(A) \quad 1 \leq i_1 < i_2 \leq 3), \\
\min \{ \text{ord}_z(h_{j_1} \circ f), \text{ord}_z(h_{j_2} \circ f) \} &= 1 \quad (z \in f^{-1}(B) \quad 4 \leq j_1 < j_2 \leq 6).
\end{align*}
\]

Using these equalities together with (3.21), we get

\[
\begin{align*}
\sum_{i=1}^{3} \min \{ \text{ord}_z(h_i \circ f), 3 \} \leq 5 \min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f) \quad (z \in f^{-1}(A)), \\
\sum_{i=4}^{6} \min \{ \text{ord}_z(h_i \circ f), 3 \} \leq 5 \min_{4 \leq i \leq 6} \text{ord}_z(h_i \circ f) \quad (z \in f^{-1}(B)).
\end{align*}
\] (3.26)

By taking the sum on disks and by integrating these two inequalities, we obtain

\[
N_f^3(r, H_1) + N_f^3(r, H_2) + N_f^3(r, H_3) \leq 5 N_f(r, A),
\]

\[
N_f^3(r, H_1) + N_f^3(r, H_5) + N_f^3(r, H_6) \leq 5 N_f(r, B).
\]

Letting \( B \) be a plane passing through \( A \) and \( B \), we proceed similarly as before

\[
5 T_f(r) \leq \sum_{i=1}^{9} N_f^3(r, H_i) + S_f(r) \\
\leq 5 N_f(r, A) + 5 N_f(r, B) + S_f(r) \\
\leq 5 N_f(r, B) + S_f(r) \\
\leq 5 T_f(r) + S_f(r).
\] (3.27)

Here, \( S_f(r) = o(T_f(r)) \) is negligible, hence all inequalities are equalities modulo \( S_f(r) \). This gives (3.24), as wanted.

Next, we consider case (b3.2). Let us set

\[
\begin{align*}
E_{t,A} &= \{ z \in \mathbb{C} : |z| < t, f(z) = A \}, \\
E_{t,A,i} &= \{ z \in \mathbb{C} : |z| < t, f(z) = A, \text{ord}_z(h_i \circ f) = 1 \} \quad (1 \leq i \leq 3), \\
E_{\geq 2,t,A,i} &= \{ z \in \mathbb{C} : |z| < t, f(z) = A, \text{ord}_z(h_i \circ f) \geq 2 \} \quad (1 \leq i \leq 3), \\
E_{t,B} &= \{ z \in \mathbb{C} : |z| < t, f(z) = B \}, \\
E_{1,t,B,i} &= \{ z \in \mathbb{C} : |z| < t, f(z) = B, \text{ord}_z(h_i \circ f) = 1 \} \quad (4 \leq i \leq 6), \\
E_{\geq 2,t,B,i} &= \{ z \in \mathbb{C} : |z| < t, f(z) = B, \text{ord}_z(h_i \circ f) \geq 2 \} \quad (4 \leq i \leq 6).
\end{align*}
\]

Assume that \( \text{ord}_z(t_{1,2} \circ f) \geq 2 \) for all \( z \in f^{-1}(A) \). Since \( t_{1,2}, r_{1,2} \) are linear combinations of \( h_1 \) and \( h_2 \) with nonzero coefficients, we have

\[
E_{\geq 2,t,A,1} = E_{\geq 2,t,A,2},
\]

\[
\text{ord}_z(r_{1,2} \circ f) \geq 2 \quad (z \in E_{\geq 2,t,A,1}).
\] (3.28)

For the same reason

\[
E_{1,t,A,1} = E_{1,t,A,2},
\]

which yields

\[
\sum_{i=1}^{3} \min \{ \text{ord}_z(h_i \circ f), 3 \} \leq 5 \quad (z \in E_{1,t,A,1}).
\] (3.29)
Letting $R = \{ r = r_{1,2} r_{4,5} r_{5,6} r_{4,6} = 0 \}$ be the degenerate quartic $R_{1,2} \cup R_{4,5} \cup R_{5,6} \cup R_{4,6}$ whose four components pass through $A$ and $B$, we have

$$\text{ord}_z (r \circ f) \geq 4 \quad (z \in E\text{t},A \cup E\text{t},B). \quad (3.30)$$

Furthermore, it follows from (3.28) that

$$\text{ord}_z (r \circ f) \geq 5 \quad (z \in E\text{t},A \cup E\text{t},B).$$

Using this inequality together with (3.23) and (3.21), we get

$$3 \sum_{i=1}^3 \min \{ \text{ord}_z (h_i \circ f), 3 \} \leq 6 = 6 \min_{1 \leq i \leq 3} \text{ord}_z (h_i \circ f) \leq \frac{6}{5} \text{ord}_z (r \circ f) \quad (z \in E\text{t},A \cup E\text{t},B).$$

Combining (3.29), (3.21) and (3.30), we receive

$$3 \sum_{i=1}^3 \min \{ \text{ord}_z (h_i \circ f), 3 \} \leq 5 = 5 \min_{1 \leq i \leq 3} \text{ord}_z (h_i \circ f) \leq \frac{5}{4} \text{ord}_z (r \circ f) \quad (z \in E\text{t},A \cup E\text{t},B).$$

Since $\text{ord}_z (t_{j_1,j_2} \circ f) = 1$ for all $4 \leq j_1 < j_2 \leq 6$, for all $z \in f^{-1}(B)$, by similar arguments as in (3.26) and by using (3.30), we also have

$$6 \sum_{i=4}^6 \min \{ \text{ord}_z (h_i \circ f), 3 \} \leq 5 = 5 \min_{4 \leq i \leq 6} \text{ord}_z (h_i \circ f) \leq \frac{5}{4} \text{ord}_z (r \circ f) \quad (z \in E\text{t},B).$$

By taking the sum on disks and by integrating these three inequalities, we obtain

$$3 \int_1^r \frac{\sum_{z \in E\text{t},A \cup E\text{t},B} \min \{ \text{ord}_z (h_i \circ f), 3 \}}{t} \ dt \leq 6 \int_1^r \frac{\sum_{z \in E\text{t},A \cup E\text{t},B} \min_{1 \leq i \leq 3} \text{ord}_z (h_i \circ f)}{t} \ dt \leq \frac{6}{5} \int_1^r \frac{\sum_{z \in E\text{t},A \cup E\text{t},B} \text{ord}_z (r \circ f)}{t} \ dt, \quad (3.31)$$

$$3 \int_1^r \frac{\sum_{z \in E\text{t},A \cup E\text{t},B} \min \{ \text{ord}_z (h_i \circ f), 3 \}}{t} \ dt \leq 5 \int_1^r \frac{\sum_{z \in E\text{t},A \cup E\text{t},B} \min_{1 \leq i \leq 3} \text{ord}_z (h_i \circ f)}{t} \ dt \leq \frac{5}{4} \int_1^r \frac{\sum_{z \in E\text{t},A \cup E\text{t},B} \text{ord}_z (r \circ f)}{t} \ dt, \quad (3.32)$$

$$6 \int_1^r \frac{\sum_{z \in E\text{t},B} \min \{ \text{ord}_z (h_i \circ f), 3 \}}{t} \ dt \leq 5 N_f (r, B) \leq \frac{5}{4} \int_1^r \frac{\sum_{z \in E\text{t},B} \text{ord}_z (r \circ f)}{t} \ dt. \quad (3.33)$$
We then proceed similarly as before:

\[ 5T_f(r) \leq \sum_{i=1}^{9} N_f^{[3]}(r, H_i) + S_f(r) \]

\[ \leq \sum_{i=1}^{3} N_f^{[3]}(r, H_i) + \sum_{i=4}^{6} N_f^{[3]}(r, H_i) + S_f(r) \]

\[ = \sum_{i=1}^{3} \int_1^r \frac{\sum_{z \in E_{i,A}} \min \{ \text{ord}_z(h_i \circ f), 3 \} }{t} \, dt + \sum_{i=4}^{6} \int_1^r \frac{\sum_{z \in E_{i,B}} \min \{ \text{ord}_z(h_i \circ f), 3 \} }{t} \, dt + S_f(r) \]

\[ = \sum_{i=1}^{3} \left( \int_1^r \frac{\sum_{z \in E_{i,A}^{(2)}} \text{ord}_z(r \circ f) }{t} \, dt + \int_1^r \frac{\sum_{z \in E_{i,A}^{(1)}} \text{ord}_z(r \circ f) }{t} \, dt \right) + \sum_{i=4}^{6} \int_1^r \frac{\sum_{z \in E_{i,B}} \text{ord}_z(r \circ f) }{t} \, dt + S_f(r) \]

\[ \leq \frac{6}{5} \int_1^r \frac{\sum_{z \in E_{i,A}^{(2)}} \text{ord}_z(r \circ f) }{t} \, dt + \frac{5}{4} \int_1^r \frac{\sum_{z \in E_{i,A}^{(1)}} \text{ord}_z(r \circ f) }{t} \, dt + S_f(r) \]

\[ = \frac{5}{4} \left( \int_1^r \frac{\sum_{z \in E_{i,A}} \text{ord}_z(r \circ f) }{t} \, dt + \int_1^r \frac{\sum_{z \in E_{i,B}} \text{ord}_z(r \circ f) }{t} \, dt \right) + \left( \frac{6}{5} - \frac{5}{4} \right) \int_1^r \frac{\sum_{z \in E_{i,A}^{(2)}} \text{ord}_z(r \circ f) }{t} \, dt + S_f(r) \]

\[ \leq \frac{5}{4} N_f(r, R) - \frac{1}{20} \int_1^r \frac{\sum_{z \in E_{i,A}^{(2)}} \text{ord}_z(r \circ f) }{t} \, dt + S_f(r) \]

\[ \leq 5T_f(r) - \frac{1}{20} \int_1^r \frac{\sum_{z \in E_{i,A}^{(2)}} \text{ord}_z(r \circ f) }{t} \, dt + S_f(r). \quad (3.34) \]

This implies

\[ \int_1^r \frac{\sum_{z \in E_{i,A}^{(2)}} \text{ord}_z(r \circ f) }{t} \, dt = S_f(r) \quad (3.35) \]

and whence all inequalities in (3.34) become equalities modulo \( S_f(r) \), which gives

\[ \sum_{i=1}^{6} N_f^{[3]}(r, H_i) = 5T_f(r) + S_f(r), \quad (3.36) \]

\[ \sum_{i=1}^{3} N_f^{[3]}(r, H_i) = \frac{5}{4} \int_1^r \frac{\sum_{z \in E_{i,A}} \text{ord}_z(r \circ f) }{t} \, dt + S_f(r), \quad (3.37) \]

\[ \sum_{i=4}^{6} N_f^{[3]}(r, H_i) = \frac{5}{4} \int_1^r \frac{\sum_{z \in E_{i,B}} \text{ord}_z(r \circ f) }{t} \, dt + S_f(r). \quad (3.38) \]

It follows from (3.33) and (3.38) that

\[ \sum_{i=4}^{6} N_f^{[3]}(r, H_i) = 5 N_f(r, B) + S_f(r). \quad (3.39) \]

Owing to (3.36), the two inequalities (3.31) become

\[ \sum_{i=1}^{3} \int_1^r \frac{\sum_{z \in E_{i,A}^{(2)}} \min \{ \text{ord}_z(h_i \circ f), 3 \} }{t} \, dt = S_f(r) \]

Owing to (3.36), the two inequalities (3.31) become
\begin{equation*}
\int_1^r \frac{\sum_{z \in E_{t,A,1}^{\geq 2}} \min_{1 \leq i \leq 3} \ord_z(h_i \circ f)}{t} \ dt = S_f(r).
\end{equation*}

Hence
\begin{equation}
\sum_{i=1}^3 N_f^{(3)}(r, H_i) = \sum_{i=1}^3 \int_1^r \frac{\sum_{z \in E_{t,A,1}^1} \min \{\ord_z(h_i \circ f), 3\}}{t} \ dt + S_f(r),
\end{equation}

\begin{equation}
N_f(r, A) = \int_1^r \frac{\sum_{z \in E_{t,A,1}^1} \min_{1 \leq i \leq 3} \ord_z(h_i \circ f)}{t} \ dt + S_f(r).
\end{equation}

Combining (3.32), (3.40), (3.41), we get
\begin{equation}
\sum_{i=1}^3 N_f^{(3)}(r, H_i) = 5 N_f(r, A) + S_f(r).
\end{equation}

The equality (3.24) follows from (3.36), (3.39), (3.42).

Case (b3.3) can be treated by similar arguments as for case (b3.2).

Next, we consider case (b3.4). Assume that
\[
\ord_z(t_{1,2} \circ f) \geq 2 \quad (z \in f^{-1}(A)),
\]
\[
\ord_z(t_{4,5} \circ f) \geq 2 \quad (z \in f^{-1}(B)).
\]

By similar argument as in (3.28), we have\[ E_{t,A,1}^{\geq 2} = E_{t,A,2}^{\geq 2} = E_{t,B,4}^{\geq 2} = E_{t,A,1}^1, \]
which implies
\begin{equation}
\ord_z(r_{1,2} \circ f) \geq 2 \quad (z \in E_{t,A,1}^{\geq 2}),
\end{equation}
\begin{equation}
\ord_z(r_{4,5} \circ f) \geq 2 \quad (z \in E_{t,B,4}^{\geq 2}),
\end{equation}
\begin{equation}
\sum_{i=1}^3 \min \{\ord_z(h_i \circ f), 3\} \leq 5 \quad (z \in E_{t,A,1}^1),
\end{equation}
\begin{equation}
\sum_{i=4}^6 \min \{\ord_z(h_i \circ f), 3\} \leq 5 \quad (z \in E_{t,B,4}^1).
\end{equation}

Letting $S = \{ s = r_{12} r_{4,5} = 0 \}$ be the degenerate quadric $R_{1,2} \cup R_{4,5}$, we see that
\[
\ord_z(s \circ f) = \ord_z(r_{1,2} \circ f) + \ord_z(r_{4,5} \circ f) \geq 2 \quad (z \in E_{t,A} \cup E_{t,B}).
\]

Furthermore, by using (3.43) and (3.44), we have
\[
\ord_z(s \circ f) = \ord_z(r_{1,2} \circ f) + \ord_z(r_{4,5} \circ f) \geq 3 \quad (z \in E_{t,A,1}^{\geq 2} \cup E_{t,B,4}^{\geq 2}).
\]

Similarly as in the previous case, by using these inequalities together with (3.21) and (3.28), we receive
\begin{equation}
\sum_{i=1}^3 \min \{\ord_z(h_i \circ f), 3\} \leq 6 = 6 \min_{1 \leq i \leq 3} \ord_z(h_i \circ f) \leq 6 \frac{3}{3} \ord_z(s \circ f) \quad (z \in E_{t,A,1}^{\geq 2}),
\end{equation}
\begin{equation}
\sum_{i=1}^3 \min \{\ord_z(h_i \circ f), 3\} \leq 5 = 5 \min_{1 \leq i \leq 3} \ord_z(h_i \circ f) \leq 5 \frac{2}{2} \ord_z(s \circ f) \quad (z \in E_{t,A,1}^1),
\end{equation}
\begin{equation}
\sum_{i=4}^6 \min \{\ord_z(h_i \circ f), 3\} \leq 6 = 6 \min_{4 \leq i \leq 6} \ord_z(h_i \circ f) \leq 6 \frac{3}{3} \ord_z(s \circ f) \quad (z \in E_{t,B,4}^{\geq 2}),
\end{equation}
\begin{equation}
\sum_{i=4}^6 \min \{\ord_z(h_i \circ f), 3\} \leq 5 = 5 \min_{4 \leq i \leq 6} \ord_z(h_i \circ f) \leq 5 \frac{2}{2} \ord_z(s \circ f) \quad (z \in E_{t,B,4}^1).
\]
By taking the sum on disks and by integrating these four inequalities, we obtain

\[
\sum_{i=1}^{3} \int_1^r \frac{\sum_{z \in E_{i,1}^2} \min \{\text{ord}_z(h_i \circ f), 3\}}{t} \, dt \leq 6 \int_1^r \frac{\sum_{z \in E_{i,A,1}^2} \min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f)}{t} \, dt \\
\leq 2 \int_1^r \frac{\sum_{z \in E_{i,A,1}^2} \text{ord}_z(s \circ f)}{t} \, dt,
\]

\[
\sum_{i=1}^{3} \int_1^r \frac{\sum_{z \in E_{i,1}^1} \min \{\text{ord}_z(h_i \circ f), 3\}}{t} \, dt \leq 5 \int_1^r \frac{\sum_{z \in E_{i,A,1}^1} \min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f)}{t} \, dt \\
\leq \frac{5}{2} \int_1^r \frac{\sum_{z \in E_{i,A,1}^1} \text{ord}_z(s \circ f)}{t} \, dt,
\]

\[
\sum_{i=1}^{6} \int_1^r \frac{\sum_{z \in E_{i,B,1}^2} \min \{\text{ord}_z(h_i \circ f), 3\}}{t} \, dt \leq 6 \int_1^r \frac{\sum_{z \in E_{i,B,1}^2} \min_{4 \leq i \leq 6} \text{ord}_z(h_i \circ f)}{t} \, dt \\
\leq 2 \int_1^r \frac{\sum_{z \in E_{i,B,1}^2} \text{ord}_z(s \circ f)}{t} \, dt,
\]

\[
\sum_{i=1}^{6} \int_1^r \frac{\sum_{z \in E_{i,B,1}^1} \min \{\text{ord}_z(h_i \circ f), 3\}}{t} \, dt \leq 6 \int_1^r \frac{\sum_{z \in E_{i,B,1}^2} \min_{4 \leq i \leq 6} \text{ord}_z(h_i \circ f)}{t} \, dt \\
\leq \frac{5}{2} \int_1^r \frac{\sum_{z \in E_{i,B,1}^1} \text{ord}_z(s \circ f)}{t} \, dt + S_f(r).
\]

Now, we proceed similarly as above

\[
5T_f(r) \leq \sum_{i=1}^{9} N_f^3(r, H_i) + S_f(r)
\]

\[
= \sum_{i=1}^{3} \int_1^r \frac{\sum_{z \in E_{i,A}^2} \min \{\text{ord}_z(h_i \circ f), 3\}}{t} \, dt + \sum_{i=4}^{6} \int_1^r \frac{\sum_{z \in E_{i,B}^2} \min \{\text{ord}_z(h_i \circ f), 3\}}{t} \, dt + \sum_{i=1}^{3} \left( \int_1^r \frac{\sum_{z \in E_{i,A,1}^2} \min \{\text{ord}_z(h_i \circ f), 3\}}{t} \, dt + \int_1^r \frac{\sum_{z \in E_{i,A,1}^1} \min \{\text{ord}_z(h_i \circ f), 3\}}{t} \, dt \right) + S_f(r)
\]

\[
\leq \frac{5}{2} \left( \int_1^r \frac{\sum_{z \in E_{i,A}^2} \text{ord}_z(s \circ f)}{t} \, dt + \int_1^r \frac{\sum_{z \in E_{i,B}^2} \text{ord}_z(s \circ f)}{t} \, dt \right) + S_f(r)
\]

\[
= \frac{5}{2} N_f(r,S) - \frac{1}{2} \left( \int_1^r \frac{\sum_{z \in E_{i,A,1}^2} \text{ord}_z(s \circ f)}{t} \, dt + \int_1^r \frac{\sum_{z \in E_{i,B,1}^2} \text{ord}_z(s \circ f)}{t} \, dt \right) + S_f(r)
\]

\[
\leq 5T_f(r) - \frac{1}{2} \left( \int_1^r \frac{\sum_{z \in E_{i,A,1}^2} \text{ord}_z(s \circ f)}{t} \, dt + \int_1^r \frac{\sum_{z \in E_{i,B,1}^2} \text{ord}_z(s \circ f)}{t} \, dt \right) + S_f(r).
\]
This implies
\[
\int_1^r \sum_{z \in E_{i,A,B}^{2,2}} \frac{\text{ord}_z(s \circ f)}{t} \, dt = S_f(r),
\]
\[
\int_1^r \sum_{z \in E_{i,A,B}^{2,2}} \frac{\text{ord}_z(s \circ f)}{t} \, dt = S_f(r),
\]
\[
\sum_{i=1}^6 N_j^{[3]}(r, H_i) = 5 T_f(r) + S_f(r),
\]
\[
\sum_{i=1}^6 N_j^{[3]}(r, H_i) = 5 \int_1^r \sum_{z \in E_{i,A}^3} \text{ord}_z(r \circ f) \, dt + S_f(r),
\]
\[
\sum_{i=1}^6 N_j^{[3]}(r, H_i) = 5 \int_1^r \sum_{z \in E_{i,B}^3} \text{ord}_z(r \circ f) \, dt + S_f(r).
\]
By proceeding similarly as in (3.42), we receive
\[
\sum_{i=1}^3 N_j^{[3]}(r, H_i) = 5 N_f(r, A) + S_f(r),
\]
\[
\sum_{i=1}^6 N_j^{[3]}(r, H_i) = 5 N_f(r, B) + S_f(r).
\]
Hence, the equality (3.24) also holds in this case. Claim 3.1 is thus proved.

Next, since \( f \) is of the form (3.20), one can find a plane \( K = \{ k = 0 \} \) passing through \( A \) and \( C \) such that
\[
\text{ord}_z(k \circ f) \geq 2 \quad (z \in f^{-1}(A)).
\]
Let \( B_i = \{ b_i = 0 \} \) be the plane containing the two lines \( AB, H_i \cap K \) \((1 \leq i \leq 3)\). Since \( b_i \) is a linear combination of \( h_i \) and \( k \) with nonzero coefficients, we have
\[
\text{ord}_z(b_i \circ f) \geq 2 \quad (z \in E_{i,A}^{2,2}),
\]
which yields
\[
\sum_{i=1}^3 \text{ord}_z(b_i \circ f) \geq 4 \quad (z \in \bigcup_{i=1}^3 E_{i,A}^{2,2}). \tag{3.45}
\]
Let \( C = \{ c = b_1 b_2 b_3 = 0 \} \) be the degenerate cubic \( \bigcup_{1 \leq i \leq 3} B_i \). It follows from (3.24) and (3.45) that
\[
\min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f) = 1 \leq \frac{1}{4} \sum_{i=1}^3 \text{ord}_z(b_i \circ f) = \frac{1}{4} \text{ord}_z(c \circ f) \quad (z \in \bigcup_{i=1}^3 E_{i,A}^{2,2}),
\]
\[
\min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f) = 1 \leq \frac{1}{3} \sum_{i=1}^3 \text{ord}_z(b_i \circ f) = \frac{1}{3} \text{ord}_z(c \circ f) \quad (z \in E_{i,A} \setminus \bigcup_{i=1}^3 E_{i,A}^{2,2}),
\]
\[
\min_{4 \leq i \leq 6} \text{ord}_z(h_i \circ f) = 1 \leq \frac{1}{3} \sum_{i=1}^3 \text{ord}_z(b_i \circ f) = \frac{1}{3} \text{ord}_z(c \circ f) \quad (z \in E_{i,B}).
\]
By taking the sum on disks and by integrating these inequalities,
\[
\int_1^r \sum_{z \in E_{i,A}^{2,2}} \min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f) \, dt \leq \frac{1}{4} \int_1^r \sum_{z \in E_{i,B}^{2,2}} \text{ord}_z(c \circ f) \, dt,
\]
\[
\int_1^r \sum_{z \in E_{i,A} \setminus \bigcup_{i=1}^3 E_{i,A}^{2,2}} \min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f) \, dt \leq \frac{1}{3} \int_1^r \sum_{z \in E_{i,A} \setminus \bigcup_{i=1}^3 E_{i,A}^{2,2}} \text{ord}_z(c \circ f) \, dt,
\]
\[
N_f(r, B) = \int_1^r \sum_{z \in E_{i,B}} \min_{4 \leq i \leq 6} \text{ord}_z(h_i \circ f) \, dt \leq \frac{1}{3} \int_1^r \sum_{z \in E_{i,B}} \text{ord}_z(c \circ f) \, dt.
\]
By using these inequalities together with (3.24), we receive

\[ 5T_f(r) = 5N_f(r, A) + 5N_f(r, B) + S_f(r) \]

\[ = 5 \left( \int_1^r \frac{\sum_{z \in U_{j=1}^3 E_{i,A}^{<2}} \min_{1 \leq i \leq 3} \ord_z(h_i \circ f)}{t} \, dt + \int_1^r \frac{\sum_{z \in E_{i,A} \cup U_{j=1}^3 E_{i,A}^{<2}} \min_{1 \leq i \leq 3} \ord_z(h_i \circ f)}{t} \, dt \right) \]

\[ + 5 \left( \int_1^r \frac{\sum_{z \in E_{i,B}} \min_{1 \leq i \leq 6} \ord_z(h_i \circ f)}{t} \, dt + S_f(r) \right) \]

\[ \leq \frac{5}{3} \left( \int_1^r \frac{\sum_{z \in E_{i,A}} \ord_z(c \circ f)}{t} \, dt + \int_1^r \frac{\sum_{z \in E_{i,B}} \ord_z(c \circ f)}{t} \, dt \right) \]

\[ + \left( \frac{5}{4} - \frac{5}{3} \right) \int_1^r \frac{\sum_{z \in U_{j=1}^3 E_{i,A}^{<2}} \ord_z(c \circ f)}{t} \, dt + S_f(r) \]

\[ \leq \frac{5}{3} N_f(r, C) - \frac{5}{12} \int_1^r \frac{\sum_{z \in U_{j=1}^3 E_{i,A}^{<2}} \ord_z(c \circ f)}{t} \, dt + S_f(r) \]

\[ \leq 5T_f(r) - \frac{5}{12} \int_1^r \frac{\sum_{z \in U_{j=1}^3 E_{i,A}^{<2}} \ord_z(c \circ f)}{t} \, dt + S_f(r). \]

This implies

\[ \int_1^r \frac{\sum_{z \in U_{j=1}^3 E_{i,A}^{<2}} \ord_z(c \circ f)}{t} \, dt = S_f(r). \]

By using (3.23) and (3.45), we get

\[ \sum_{i=1}^3 \min \{ \ord_z(h_i \circ f), 3 \} \leq 6 \leq \frac{3}{2} \ord_c(c \circ f) \quad (z \in U_{j=1}^3 E_{i,A}^{<2}), \]

which yields

\[ \int_1^r \frac{\sum_{z \in U_{j=1}^3 E_{i,A}^{<2}} \min_{1 \leq i \leq 3} \ord_z(h_i \circ f)}{t} \, dt \leq \frac{3}{2} \int_1^r \frac{\sum_{z \in U_{j=1}^3 E_{i,A}^{<2}} \ord_z(c \circ f)}{t} \, dt = S_f(r). \]

Moreover, we also have

\[ \sum_{i=1}^3 \min \{ \ord_z(h_i \circ f), 3 \} = 3 = \min_{1 \leq i \leq 3} \ord_z(h_i \circ f) \quad (z \in E_{i,A} \cup U_{j=1}^3 E_{i,A}^{<2}), \]

which implies, by integrating, that

\[ \sum_{i=1}^3 \int_1^r \frac{\sum_{z \in E_{i,A} \cup U_{j=1}^3 E_{i,A}^{<2}} \min \{ \ord_z(h_i \circ f), 3 \}}{t} \, dt \leq \int_1^r \frac{\sum_{z \in E_{i,A} \cup U_{j=1}^3 E_{i,A}^{<2}} \min_{1 \leq i \leq 3} \ord_z(h_i \circ f)}{t} \, dt. \]

By combining (3.47) and (3.48), we get

\[ \sum_{i=1}^3 N_f^{[3]}(r, H_i) = \sum_{i=1}^3 \int_1^r \frac{\sum_{z \in E_{i,A}} \ord_z(h_i \circ f)}{t} \, dt \]

\[ = \sum_{i=1}^3 \left( \int_1^r \frac{\sum_{z \in E_{i,A} \cup U_{j=1}^3 E_{i,A}^{<2}} \min \{ \ord_z(h_i \circ f), 3 \}}{t} \, dt \right) \]

\[ + \int_1^r \frac{\sum_{z \in U_{j=1}^3 E_{i,A}^{<2}} \min \{ \ord_z(h_i \circ f), 3 \}}{t} \, dt + S_f(r) \]

\[ \leq 3 \int_1^r \frac{\sum_{z \in E_{i,A} \cup U_{j=1}^3 E_{i,A}^{<2}} \min_{1 \leq i \leq 3} \ord_z(h_i \circ f)}{t} \, dt + S_f(r) \]

\[ \leq 3 N_f(r, A) + S_f(r). \]
By symmetry, we also have
\[
\sum_{i=4}^{6} N_f^3(r, H_i) \leq 3 N_f(r, B) + S_f(r).
\]
Hence we can rewrite (3.27) to get a contradiction:
\[
5 T_f(r) \leq \sum_{i=1}^{9} N_f^3(r, H_i) + S_f(r) \\
\leq 3 N_f(r, A) + 3 N_f(r, B) + S_f(r) \\
\leq 3 T_f(r) + S_f(r).
\]

\[
\square
\]

In \( \mathbb{P}^4(\mathbb{C}) \), by the generic condition for the family of hyperplanes \( \{H_i\}_{1 \leq i \leq q} \), when \( q \geq 10 \), we see that, for all three disjoint subsets \( I, J, K \) of \( \{1, \ldots, q\} \) with \( |I| \geq 2, |J| \geq 2, |I| + |J| = 6, |K| = 4 \), the diagonal hyperplane \( H_{I,J} \) does not contain the point \( \cap_{k \in K} H_k \).

**Lemma 3.3.** In \( \mathbb{P}^4(\mathbb{C}) \), all complements of the form (5.2) are hyperbolic if \( m = 1 \).

**Proof.** We can assume that \( A_{1,4} \) is a set consisting of one element in
\[
(\bigcup_{1 \leq i_1 < i_2 \leq 10} (H_{i_1} \cap H_{i_2})^*) \cup \left( \bigcup_{1 \leq i_1 < i_2 < i_3 < i_4 \leq 10} H_{i_1} \cap H_{i_2} \cap H_{i_3} \cap H_{i_4} \right)^*.
\]
Suppose to the contrary that there is an entire curve \( f : \mathbb{C} \to \mathbb{P}^4(\mathbb{C}) \setminus \left( \bigcup_{i=1}^{10} H_i \setminus A_{1,4} \right) \). If \( A_{1,4} \) is not a set of a point, then \( f \) avoids at least seven hyperplanes. By Theorem 2.4, its image is contained in a line \( L \) and we can continue to analyze the position of \( L \) with respect to \( \bigcup_{i=1}^{10} H_i \setminus A_{1,4} \) to get a contradiction. Consider the remaining case where \( A_{1,4} \) consists of a point, say \( \cap_{i=1}^{4} H_i \). By Theorem 2.5 the curve \( f \) lands in some diagonal hyperplane of the family \( \{H_i\}_{5 \leq i \leq 10} \), which does not contain the point \( \cap_{i=1}^{4} H_i \) by the generic condition. Hence, \( f \) must avoid all \( H_i \) (\( 1 \leq i \leq 10 \)), which is impossible by Theorem 2.5. \( \square \)

**3.1 Stability of intersections**

We will also invoke the following known complex analysis fact.

**Stability of intersections.** Let \( X \) be a complex manifold and let \( H \subset X \) be an analytic hypersurface. Suppose that a sequence \( (f_n) \) of entire curves in \( X \) converges toward an entire curve \( f \). If \( f(\mathbb{C}) \) is not contained in \( H \), then we have
\[
f(\mathbb{C}) \cap H \subset \lim f_n(\mathbb{C}) \cap H.
\]

**4 Proof of the Main Theorem**

We keep the notation of the previous section. Let \( S \) be a hypersurface of degree \( 2n \), which is in general position with respect to the family \( \{H_i\}_{1 \leq i \leq 2n} \). We would like to determine what conditions \( S \) should satisfy for \( \Sigma_t \) to be hyperbolic. Suppose that \( \Sigma_t \) is not hyperbolic for a sequence \( (\epsilon_k) \) converging to 0. Then we can find entire curves \( f_{\epsilon_k} : \mathbb{C} \to \Sigma_{\epsilon_k} \). By the Brody lemma, after reparameterization and extraction, we may assume that the sequence \( (f_{\epsilon_k}) \) converges to an entire curve \( f : \mathbb{C} \to \bigcup_{i=1}^{2n} H_i \).

The curve \( f(\mathbb{C}) \) lands inside some hyperplane \( H_i \). Moreover, it cannot land inside any subspace of dimension 1 (a line). Indeed, if \( f(\mathbb{C}) \subset \cap_{i \in I} H_i \) for some subset \( I \) of the index set \( Q = \{1, \ldots, 2n\} \) having cardinality \( n - 1 \), then for all \( j \in Q \setminus I \), by stability of intersections, one has
\[
f(\mathbb{C}) \cap H_j \subset \lim f_{\epsilon_k}(\mathbb{C}) \cap H_j \subset \lim \Sigma_{\epsilon_k} \cap H_j \subset S \cap H_j.
\]
Thus \( f(\mathbb{C}) \) and \( H_j \) have empty intersection by the general position. Hence the curve \( f(\mathbb{C}) \) lands in
\[
\cap_{i \in I} H_i \setminus \left( \bigcup_{j \in Q \setminus I} H_j \right).
\]
This is a contradiction, because the complement of \( n + 1 \) \((n \geq 3)\) points in a line is hyperbolic by Picard’s theorem.

Now, let \( I \) be the largest subset of \( \mathbb{Q} \) such that the curve \( f(\mathbb{C}) \) lands in \( \cap_{i \in I} H_i \). We have \(|I| \leq n-2\).

By stability of intersections, \( f(\mathbb{C}) \cap H_j \) is contained in \( S \) for all \( \ell \in \mathbb{Q} \setminus I \). Therefore the curve \( f(\mathbb{C}) \) lands in

\[
\cap_{i \in I} H_i \setminus \left( \cup_{j \in \mathbb{Q} \setminus I} H_j \setminus S \right).
\]

(4.1)

So, the problem reduces to finding a hypersurface \( S \) of degree \( 2n \) such that all complements of the form \( \cap_{I, J} \) are hyperbolic, where \( I \) is a subset of \( \mathbb{Q} \) of cardinality at most \( n-2 \). For example when \( n = 3 \), we need to find a sextic curve \( S \) such that all complements of the form \( H_i \setminus \left( \cup_{j \neq i} H_j \setminus S \right) \) are hyperbolic. In this case, we have the complement of five lines in the hyperplane \( H_i \) on which all points of intersection with \( S \) are deleted.

We will construct such \( S \) by deformation, step by step. For \( 2 \leq l \leq n-1 \), let \( \Delta_l \) be a finite collection of subspaces of dimension \( n-l \), in the sense of section 3. Let \( D_l \in \Delta_l \) be another subspace of dimension \( n-l \), defined as \( D_l = \cap_{i \in I_d} H_i \). For a hypersurface \( S = \{s = 0\} \) in general position with respect to the family \( \{H_i\}_{1 \leq i \leq 2n} \) and \( \epsilon \neq 0 \), we set

\[
S_{\epsilon} = \{\epsilon s + \Pi_{i \notin I_d} h_i^{n_i} = 0\},
\]

where \( n_i \geq 1 \) are chosen (freely) so that \( \sum_{i \notin I_d} n_i = 2n \). It is not hard to see that the hypersurface \( S_{\epsilon} \) is also in general position with respect to the family \( \{H_i\}_{1 \leq i \leq 2n} \). We denote by \( \overline{\Delta}_l \) the family of all subspaces of dimension \( n-l \) \((2 \leq l \leq n)\) with the convention \( \overline{\Delta}_n = \emptyset \).

**Lemma 4.1.** Assume that all complements of the form

\[
\cap_{i \in I} H_i \setminus \left( \cup_{j \in J} H_j \setminus (\{(\Delta_l \cup \overline{\Delta}_l) \cap S) \cup A_{m,n-|I|}) \right)
\]

(4.2)

are hyperbolic where \( I, J \) are two disjoint subsets of \{\(1, \ldots, 2n\)\} such that \(|I| \leq n-2, |J|+2|I| \geq 2n+1 \) and \( m \leq |J|+2|I| - (2n+1) \). Here, \( A_{m,n-|I|} \) is a set of at most \( m \) star-subspaces coming from the family of hyperplanes \( \{\cap_{i \in I} H_i \cap H_j\}_{j \in J} \) in \( \cap_{i \in I} H_i \cong \mathbb{P}^{n-|I|}(\mathbb{C}) \). Then all complements of the form

\[
\cap_{i \in I} H_i \setminus \left( \cup_{j \in J} H_j \setminus (\{(\Delta_l \cup D_l \cup \overline{\Delta}_l) \cap S_l) \cup A_{m,n-|I|}) \right)
\]

(4.3)

are also hyperbolic for sufficiently small \( \epsilon \neq 0 \).

**Proof.** By the definition of \( S_{\epsilon} \), we see that \( S_{\epsilon} \cap \{\cap_{m \in M} H_m\} = S \cap \{\cap_{m \in M} H_m\} \) when \( M \cap (\mathbb{Q} \setminus I_d) \neq \emptyset \), hence

\[
(\Delta_l \cup D_l \cup \overline{\Delta}_l) \cap S_{\epsilon} = (\Delta_l \cup \overline{\Delta}_l) \cap S) \cup (D_l \cap S_{\epsilon}).
\]

When \(|I| \geq l\), using this, we observe that two complements \(4.2, 4.3\) coincide.
Assume therefore $|I| \leq l - 1$. Suppose by contradiction that there exists a sequence of entire curves $(f_{\epsilon_k}(\mathbb{C}))_k$, $\epsilon_k \to 0$ contained in the complement

$$
\cap_{i \in I} H_i \setminus \left( \cup_{j \in J} H_j \setminus \left( \left( \Delta I \cup D_I \cup \Sigma_{l+1} \cap S_{\epsilon_k} \right) \cup A_{m,n-|I|} \right) \right).
$$

By the Brody Lemma, we may assume that $(f_{\epsilon_k})$ converges to an entire curve $f(\mathbb{C}) \subset \cap_{i \in I} H_i$. Our aim is to prove that the curve $f(\mathbb{C})$ lands in some complement of the form $(4.2)$. Let $\cap_{k \in K} H_k$ be the smallest subspace containing $f(\mathbb{C})$. It is clear that $K \supset I$. Take an index $j$ in $J \setminus K$. By stability of intersections, one has

$$
f(\mathbb{C}) \cap H_j \subset \lim f_{\epsilon_k}(\mathbb{C}) \cap H_j 
\subset \left( \left( \Delta I \cup \Sigma_{l+1} \cap S \right) \cup A_{m,n-|I|} \cup \lim(D_I \cap S_{\epsilon_k}) \right).
$$

(4.4)

If the index $j$ does not belong to $I_D$, then $H_j \cap D_I \cap S_{\epsilon_k} \subset \Sigma_{l+1} \cap S$. It follows from (4.4) that

$$
f(\mathbb{C}) \cap H_j \subset \left( \left( \Delta I \cup \Sigma_{l+1} \cap S \right) \cup A_{m,n-|I|} \right).
$$

(4.5)

If the index $j$ belongs to $I_D$, noting that $\lim(D_I \cap S_{\epsilon_k})$ is contained in $D_I \cap (\cup_{i \notin I_D} H_i)$, again from (4.4), one has

$$
f(\mathbb{C}) \cap H_j \subset \left( \left( \Delta I \cup \Sigma_{l+1} \cap S \right) \cup A_{m,n-|I|} \cup \left( D_I \cap (\cup_{i \notin I_D} H_i) \right) \right).
$$

(4.6)

Assume first that $K = I$. We claim that (4.5) also holds when the index $j \in J \setminus I$ belonging to $I_D$. Indeed, for the supplementary part in (4.4), we have

$$
f(\mathbb{C}) \cap H_j \cap \left( D_I \cup \cup_{i \notin I_D} H_i \right) \subset \cup_{i \notin I_D} f(\mathbb{C}) \cap H_j \cap H_i,
$$

so that (4.5) applies here to all $i \notin I_D$. Hence, the curve $f(\mathbb{C})$ lands inside

$$
\cap_{i \in I} H_i \setminus \left( \cup_{j \in J} H_j \setminus \left( \left( \Delta I \cup \Sigma_{l+1} \cap S \right) \cup A_{m,n-|I|} \right) \right),
$$

contradicting the hypothesis.

Assume now that $I$ is a proper subset of $K$. Let us set

$$
A_{m,n-|I|,K} = \{ X \cap (\cap_{k \in K} H_k) | X \in A_{m,n-|I|} \}.
$$

This set consists of star-subspaces of $\cap_{k \in K} H_k \cong [\mathbb{P}^{n-|K|}]$. Let $B_{m,K}$ be the subset of $A_{m,n-|I|,K}$ containing all star-subspaces of dimension $n - |K| - 1$ (i.e., of codimension 1 in $\cap_{k \in K} H_k$), and let $C_{m,K}$ be the remaining part. A star-subspace in $B_{m,K}$ is of the form $(\cap_{k \in K} H_k \cap H_j)^*$ for some index $j \in J \setminus K$. Then let $R$ denote the set of such indices $j$, so that

$$
|R| = |B_{m,K}|.
$$

We consider two cases separately, depending on the dimension of the subspace $Y = \cap_{k \in K} H_k \cap D_I$.

**Case 1.** $Y$ is a subspace of dimension $n - |K| - 1$. In this case, $Y$ is of the form $(\cap_{k \in K} H_k) \cap H_y$ for some index $y$ in $I_D$. It follows from (4.4), (4.5), (4.6) that the curve $f(\mathbb{C})$ lands inside the set

$$
\cap_{k \in K} H_k \setminus \left( \cup_{j \in (J \setminus K) \setminus (R \cup \{y\})} H_j \setminus \left( \left( \Delta I \cup \Sigma_{l+1} \cap S \right) \cup C_{m,K} \right) \right).
$$

Now we need to show that this set is of the form (4.2). First, we verify the corresponding required inequality between cardinalities

$$
|\mathbb{J} K \setminus (R \cup \{y\})| \geq |\mathbb{J} \setminus K| - |B_{m,K}| - 1
\geq |\mathbb{J} - |J \cap K| - (|\mathbb{I}| + 2|I| - 2n - 1 - |C_{m,K}|) - 1
= 2(n - |K|) + |C_{m,K}| + 2|K \setminus I| - |J \cap K|
\geq 2(n - |K|) + 1 + |C_{m,K}|.
$$
where the last inequality holds because \( I \) and \( J \) are two disjoint sets and \( I \) is a proper subset of \( K \). Secondly, we verify that the set \( K \) is of cardinality at most \( n - 2 \). Indeed, if \( |K| = n - 1 \), then since \( S \) is in general position with respect to \( \{ H_{i} \}_{1 \leq i \leq 2n} \), we see that

\[
\cap_{k \in I} H_{k} \setminus \left( \cup_{j \in (J \setminus K) \cup R_{j}} H_{j} \right) = \cap_{k \in I} H_{k} \setminus \left( \left( \Delta_{I} \cup \Delta_{I+1} \right) \cap S \cup C_{m,K} \right) = \cap_{k \in I} H_{k} \setminus \left( \cup_{j \in (J \setminus K) \cup R} H_{j} \right) \setminus C_{m,K}.
\]

Owing to the inequality \(|J \setminus K| \geq 3 + |C_{m,K}|, the curve \( f \) lands in a complement of at least three points in a line. By Picard’s theorem, \( f \) is constant, which is a contradiction.

**Case 2.** \( Y \) is a subspace of dimension at most \( n - |K| - 2 \). In this case, the curve \( f(\mathbb{C}) \) lands inside

\[
\cap_{k \in R} H_{k} \setminus \left( \cup_{j \in (J \setminus K) \setminus R} H_{j} \right) \setminus \left( \left( \Delta_{I} \cup \Delta_{I+1} \right) \cap S \cup C_{m,K \cup Y^{*}} \right).
\]

This set is of the form \((4.2)\) since

\[
|\{ j \in (J \setminus K) \setminus R \}| \geq 2(n - |K|) + 1 + |C_{m,K \cup Y^{*}}|
\]

which also implies \( |K| \leq n - 2 \) by similar argument as in **Case 1**.

The lemma is thus proved. \( \square \)

**End of proof of the Main Theorem.** We now come back to the proof of the Main Theorem. Keep the notation as in Lemma 4.1. We claim that \( \{ \cap_{i \in I} H_{i} \cap H_{j} \}_{j \in J} \) is also a family of generic hyperplanes in the projective space \( \cap_{i \in I} H_{i} \cong \mathbb{P}^{n-I}(\mathbb{C}) \). Indeed, let \( I, J, J_{1}, \ldots, J_{k} \) be disjoint subsets of \( J \) such that \( |I|, |J_{i}| \geq 2 \), \( |I| + |J_{i}| = (n - |I|) + 2 \), \( 1 \leq i \leq k \) and let \( \{ i_{1}, \ldots, i_{r} \} \) be a subset of \( I \). Let us set \( I = I \cup J \); then the intersection between the \( |J| \) hyperplanes \( H_{j}, j \in J \), the \( k \) diagonal hyperplanes \( H_{1}J_{1}, \ldots, H_{k}J_{k} \), and the \( |I| + l \) hyperplanes \( H_{i} \) (\( i \in I \)), \( H_{i_{1}} \ldots, H_{i_{r}} \) is a linear subspace of codimension \( \min\{k + |I| + l, |I| \} + |J| \), with the convention that when \( \min\{k + |I| + l, |I| \} + |J| > n \), this intersection is empty. Since

\[
\min\{k + |I| + l, |I| \} + |J| = \min\{k + l, |I| \} + |I| + |J|
\]

we deduce that in the projective space \( \cap_{i \in I} H_{i} \), the intersection between the \( |J| \) hyperplanes \( H_{j}, j \in J \), the \( k \) diagonal hyperplanes \( H_{I}J_{1}, \ldots, H_{I}J_{k} \), and the \( l \) hyperplanes \( H_{i_{1}} \ldots, H_{i_{r}} \) is a linear subspace of codimension \( \min\{k + l, |I| \} + |J| \), with the convention that when \( \min\{k + l, |I| \} + |J| > n - |I| \), this intersection is empty.

**Starting point of the process by deformation:** We start with the hyperbolicity of all complements of the forms

\[
\cap_{i \in I} H_{i} \setminus \left( \cup_{j \in J} H_{j} \setminus A_{m,n-|I|} \right),
\]

where \( I, J, A_{m,n-|I|} \) are as in Lemma 4.1. More precisely,

- when \( n = 3 \), we start with the hyperbolicity of all complements \( H_{i} \setminus \left( \cup_{j \neq i} H_{j} \right) \), which follows from Theorem 2.5 in \( \mathbb{P}^{3}(\mathbb{C}) \);
- when \( n = 4 \), we start with the hyperbolicity of all complements

\[
H_{i} \setminus \left( \cup_{j \neq i} H_{j} \right),
\]

\[
\cap_{i \in I} H_{i} \setminus \left( \cup_{j \in J} H_{j} \setminus A_{1,2} \right) \quad (|I| = 2, 5 + |A_{1,2}| \leq |J| \leq 6),
\]

which follows from Theorem 2.5 in \( \mathbb{P}^{3}(\mathbb{C}) \) and Lemma 3.1 for \( m = 1 \);
- when \( n = 5 \), we start with the hyperbolicity of all complements

\[
H_{i} \setminus \left( \cup_{j \neq i} H_{j} \right),
\]

\[
\cap_{i \in I} H_{i} \setminus \left( \cup_{j \in J} H_{j} \setminus A_{1,3} \right) \quad (|I| = 2, 7 + |A_{1,3}| \leq |J| \leq 8),
\]

\[
\cap_{i \in I} H_{i} \setminus \left( \cup_{j \in J} H_{j} \setminus A_{2,2} \right) \quad (|I| = 3, 5 + |A_{2,2}| \leq |J| \leq 7),
\]

which follows from Theorem 2.5 in \( \mathbb{P}^{4}(\mathbb{C}) \), Lemma 3.2 for \( m = 1 \), and Lemma 3.1 for \( m = 2 \);
• when \( n = 6 \), we start with the hyperbolicity of all complements

\[
H_i \setminus \left( \bigcup_{j \neq i} H_j \right),
\]

\[
\cap_{i \in I} H_i \setminus \left( \bigcup_{j \in J} H_j \setminus A_{1,4} \right) \quad (|I| = 2, 9 + |A_{1,4}| \leq |J| \leq 10),
\]

\[
\cap_{i \in I} H_i \setminus \left( \bigcup_{j \in J} H_j \setminus A_{2,3} \right) \quad (|I| = 3, 7 + |A_{2,3}| \leq |J| \leq 9),
\]

\[
\cap_{i \in I} H_i \setminus \left( \bigcup_{j \in J} H_j \setminus A_{3,2} \right) \quad (|I| = 4, 5 + |A_{3,2}| \leq |J| \leq 8),
\]

which follows from Theorem 2.5 in \( \mathbb{P}^5(\mathbb{C}) \), Lemma 3.3 for \( m = 1 \), Lemma 3.2 for \( m = 2 \), and Lemma 5.1 for \( m = 3 \).

**Details of the process by deformation:** In the first step, we apply inductively Lemma 4.1 for \( l = n - 1 \) and get at the end a hypersurface \( S_1 \) such that all complements of the forms

\[
\cap_{i \in I} H_i \setminus \left( \bigcup_{j \in J} H_j \setminus (S_1 \cup A_{m,n-|I|}) \right) \quad (|I| = n - 2),
\]

\[
\cap_{i \in I} H_i \setminus \left( \bigcup_{j \in J} H_j \setminus ((\bigcap_{n-1} S_1) \cup A_{m,n-|I|}) \right) \quad (|I| \leq n - 3)
\]

are hyperbolic. Considering this as the starting point of the second step, we apply inductively Lemma 4.1 for \( l = n - 2 \). Continuing this process, we get at the end of the \((n - 2)\)th step a hypersurface \( S = S_{n-2} \) satisfying the required properties.

\[ \square \]

## 5 Some discussion

Actually, our method works for a family of at least \( 2n \) generic hyperplanes in \( \mathbb{P}^n(\mathbb{C}) \). We hope that the Main Theorem is true for all \( n \geq 3 \). As we saw above, the problem reduces to proving the following conjecture.

**Conjecture.** All complements of the form (3.2) are hyperbolic.

We already know it to be true for \( n = 2 \), since Lemma 3.1 holds generally, without restriction on \( m \).

**Lemma 5.1.** In \( \mathbb{P}^2(\mathbb{C}) \), all complements of the form (3.2) are hyperbolic.

**Proof.** Assume now \( m \geq 4 \) and \( A_{m,2} = \{ A_1, \ldots, A_m \} \), where \( A_i = H_{i_1} \cap H_{i_2} \) (\( 1 \leq i \leq m \)). We denote by \( I \) the index set \( \{ i_j : 1 \leq i \leq m, 1 \leq j \leq 2 \} \). Suppose to the contrary that there exists an entire curve \( f : \mathbb{C} \to \mathbb{P}^2(\mathbb{C}) \setminus (\bigcup_{i=1}^{5+m} H_i \setminus A_{m,2}) \). By the generic condition, we can assume that \( f \) is linearly nondegenerate. By similar arguments as in Lemma 3.1 (cf. (3.6)), we have

\[
\sum_{i \in I} N_f^{[2]}(r, H_i) \leq 3 \sum_{i=1}^{m} N_f(r, A_i).
\]

Let \( C_m = \{ c_m = 0 \} \) be an algebraic curve in \( \mathbb{P}^2(\mathbb{C}) \) of degree \( d \) passing through all points in \( A_{m,2} \) with multiplicity at least \( k \) which does not contain the curve \( f(\mathbb{C}) \). Starting from the inequality

\[
\min_{1 \leq i \leq 2} \text{ord}_z(h_{i_j} \circ f) \leq \frac{1}{k} \text{ord}_z(c_m \circ f) \quad (z \in f^{-1}(A_i))
\]

and proceeding as in (3.8), we get

\[
\sum_{i=1}^{m} N_f(r, A_i) \leq \frac{1}{k} N_f(r, C_m).
\]
We may then proceed similarly as in (3.9)

\[(m + 2)T_f(r) \leq \sum_{i=1}^{5+m} N_f^{[2]}(r, H_i) + S_f(r)\]

\[\leq 3 \sum_{i=1}^{m} N_f(r, A_i) + S_f(r)\]

\[\leq \frac{3}{k} N_f(r, C_m) + S_f(r)\]

\[\leq \frac{3d}{k} T_f(r) + S_f(r).\]  

(5.1)

When \(m \geq 5\), the following claim yields a concluding contradiction.

**Claim 5.1.** If \(m \geq 5\), we can find some curve \(C_m\) which does not contain \(f(C)\) such that

\[k > \frac{3d}{m + 2}.\]  

(5.2)

Indeed, the degree of freedom for the choice of a curve of degree \(d\) is

\[\frac{(d + 1)(d + 2)}{2} - 1.\]

We want \(C_m\) to pass through all points in \(A_{m,2}\) with multiplicity at least \(k\). The number of equations (with the coefficients of \(C_m\) as unknowns) for this is not greater than

\[m \frac{k(k + 1)}{2}.\]

Thus, for the existence of \(C_m\), it is necessary that

\[\frac{(d + 1)(d + 2)}{2} - 1 > m \frac{k(k + 1)}{2}.\]  

(5.3)

We try to find two natural numbers \(k, d\) satisfying (5.2) and (5.3). This can be done by choosing \(d = (m + 2)M\) and \(k = 3M + 1\) for large enough \(M\). Using the remaining freedom in the choice of \(C_m\), we can choose it not containing \(f(C)\), which proves the claim.

Next, we consider the remaining case where \(m = 4\).

If there exists a collinear subset \(\{A_{i_1}, A_{i_2}, A_{i_3}\}\) of \(A_{4,2}\), then by the generic condition, it must be contained in some line \(H_i\). Let \(A_{i_4}\) be the remaining point of the set \(A_{4,2}\) and let \(C_4\) be the degenerate quintic consisting of the three lines \(A_{i_j}A_{i_4}\) (1 ≤ \(i\) ≤ 3) and of the line \(H_i\) with multiplicity 2. Since \(C_4\) passes through all points in \(A_{4,2}\) with multiplicity at least 3, the inequality (5.2) is satisfied. By using (5.1), we get a contradiction.

Now we assume that any subset of \(A_{4,2}\) containing three points is not collinear. Let \(E_{i_j} = \{e_{i_j} = 0\}\) (1 ≤ \(i\) ≤ 4, 1 ≤ \(j\) ≤ 2) be the eight conics passing through all points of \(A_{4,2}\), tangent to the line \(H_{i_j}\) at the point \(A_{i_j}\) (1 ≤ \(i\) ≤ 4, 1 ≤ \(j\) ≤ 2). Let \(E = \{e = 0\}\) be the degenerate curve of degree 16 consisting of all these \(E_{i_j}\). We claim that \(f\) does not land in \(E\). Otherwise, it lands in some conic \(E_{i_j}\). Since the number of intersection points between \(E_{i_j}\) and \(\bigcup_{i=1}^{4} H_i \setminus A_{4,2}\) is > 3 and since any complement of three distinct points in an irreducible curve is hyperbolic, \(f\) must be constant, which is a contradiction.
Letting $z$ be a point in $f^{-1}(A_i)$, we have
\[
\text{ord}_z(e_{i_j} \circ f) \geq 1 \quad (1 \leq i \leq 4, 1 \leq j \leq 2).
\]
By the construction of $E_i$, if $\text{ord}_z(h_{i_j} \circ f) \geq 2$ for some $1 \leq j \leq 2$, then we also have $\text{ord}_z(e_{i_j} \circ f) \geq 2$. Furthermore, if $\text{ord}_z(h_{i_j} \circ f) \geq 2$ for all $1 \leq j \leq 2$, then $\text{ord}_z(e_{i_j} \circ f) \geq 2$ for all $1 \leq i \leq 4, 1 \leq i \leq 2$.

Thus, the following inequality holds:
\[
\min \left\{ \text{ord}_z(h_{i_1}) \circ f, 2 \right\} + \min \left\{ \text{ord}_z(h_{i_2}) \circ f, 2 \right\} \leq \frac{1}{3} \sum_{i=1}^{4} \sum_{j=1}^{2} \text{ord}_z(e_{i_j} \circ f)
= \frac{1}{3} \text{ord}_z(e \circ f) \quad (z \in f^{-1}(A_i)).
\]
This implies
\[
\sum_{i \in I} N_{f}^{[2]}(r, H_i) \leq \frac{1}{3} N_f(r, \mathcal{E}).
\]
We proceed similarly as before to derive a contradiction
\[
6 T_f(r) \leq \sum_{i=1}^{9} N_{f}^{[2]}(r, H_i) + S_f(r)
\leq \frac{1}{3} N_f(r, \mathcal{E}) + S_f(r)
\leq \frac{16}{3} T_f(r) + S_f(r).
\]
Lemma \ref{5.1} is thus proved.

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\square
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