Uncertainties in the $\overline{\text{MS}}$ Bottom Quark Mass from Relativistic Sum Rules

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Abstract
A detailed compilation of uncertainties in the $\overline{\text{MS}}$ bottom quark mass $m_b(m_b)$ obtained from low-$n$ spectral sum rules at order $\alpha_s^2$ is given including charm mass effects and secondary $b\bar{b}$ production. The experimental continuum region above 11.1 GeV is treated conservatively. An inconsistency of the PDG averages for the electronic partial widths of $\Upsilon(4S)$ and $\Upsilon(5S)$ is pointed out. From our analysis we obtain $m_b(m_b) = 4.20 \pm 0.09$ GeV. The impact of future CLEO data is discussed.
Introduction

Present data from B factories on inclusive decays already require precise knowledge of the bottom quark mass parameter with a numerical precision of the order 50 MeV with a reliable estimate of the uncertainty. This will become even more acute in the future when more data becomes available. Using methods based on perturbative QCD, there have been several approaches in the past aiming at uncertainties of less than 100 MeV. The most frequently used method is based on large-$n$ ($n \gtrsim 4$) moments of the $b\bar{b}$ production cross section in $e^+e^-$ annihilation \[ P_n = \int \frac{ds}{s^{n+1}} R_{b\bar{b}}(s), \] where $R_{b\bar{b}} = \sigma(e^+e^- \rightarrow b\bar{b} + X)/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$ and the contributions from the virtual $Z$ are neglected. From fitting moments obtained from experimental data to the corresponding theoretical expressions the bottom mass can be determined in threshold schemes (for recent reviews see Refs. \[2\]).

Large-$n$ ("non-relativistic") moments have the advantage that the badly known experimental continuum $b\bar{b}$ cross section above the $\Upsilon(6S)$ is strongly suppressed in comparison to the rather well known resonance region. However, the order $\alpha_s^2$ corrections in the framework of the non-relativistic expansion are generally quite large and small uncertainties below 100 MeV can only be achieved with additional assumptions on higher order corrections. \[2\] This indicates that an improvement in the treatment of the non-relativistic bottom quark dynamics might be needed.

A different method uses low-$n$ ("relativistic") moments \[5, 6\] where $n \lesssim 4$. In the recent past, relativistic moments have been used less frequently because the badly known continuum region represents a major source of uncertainty that is not reducible without additional assumptions. Theoretically, the usual loop expansion in powers of $\alpha_s$ can be employed since for small $n$ the bottom dynamics is relativistic. Here, the $\overline{\text{MS}}$ mass is an appropriate mass definition to be used and extracted. In contrast to large-$n$ moments, the low-$n$ moments show a quite good perturbative behavior. A recent analysis by Kühn and Steinhauser \[7\] used moments at order $\alpha_s^2$. Adopting a theory-driven perspective, the experimental continuum data above the $\Upsilon(6S)$ was obtained from theoretical results for $R_{b\bar{b}}$, basically eliminating uncertainties from the continuum region. In Ref. \[8\] the important conclusion was drawn that, using the strategy of Ref. \[7\], a substantially more accurate measurement of the $\Upsilon(1S) - \Upsilon(6S)$ region at CLEO \[9\] could result in an uncertainty in $\overline{m}_b$ of only 30 MeV.

It is the main purpose of this paper to give a detailed compilation of all sources of uncertainties in the bottom $\overline{\text{MS}}$ mass $\overline{m}_b$ obtained from low-$n$ moments, including a more conservative treatment of the experimental continuum region. We

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1 Recently, the bottom mass has been determined in different threshold schemes from moments of inclusive semileptonic and radiative B meson decay spectra with an uncertainty of about 100 MeV, which is dominated by experimental errors. \[3\] (See also Ref. \[4\].)
believe that this compilation can contribute to a more differentiated view on the current uncertainties in $m_b(m_b)$ from low-$n$ moments and on the impact of new more precise data in the $\Upsilon$ resonance region from CLEO. In our analysis we also include the contributions from secondary $b\bar{b}$ production from gluon splitting and the effects of the non-zero charm quark mass, which have to our knowledge not been taken into account before. Both effects turn out to be small. Finally, we point out an inconsistency in the way the PDG has treated the original results for the electronic partial widths of $\Upsilon(4S)$ and $\Upsilon(5S)$ from CUSB \cite{10} and CLEO \cite{11}, which leads to a contribution to the experimental moments $P_n$ that is smaller than the contributions obtained from the original data in that energy region, both from CUSB and CLEO.

**Theoretical Moments**

For the QCD parameters used in this work we adopt the $\overline{\text{MS}}$ renormalization scheme and the convention that the bottom quark participates in the running ($n_f = 5$). The masses of the quarks in the first two generations are set to zero. In terms of $m_b(\mu)$ the moments in the OPE, including the known perturbative corrections to order $\alpha_s^2$ and the contribution from the dimension four gluon condensate, take the form

$$P_n = \frac{1}{(4m_b(\mu))^n} \left\{ f_n^0 + \left( \frac{\alpha_s(\mu)}{\pi} \right) \left( f_n^{10} + f_n^{11} \ln \left( \frac{m_b^2(\mu)}{\mu^2} \right) \right) \right. 
+ \left( \frac{\alpha_s(\mu)}{\pi} \right)^2 \left( f_n^{20}(r) + f_n^{21} \ln \left( \frac{m_b^2(\mu)}{\mu^2} \right) + f_n^{22} \ln^2 \left( \frac{m_b^2(\mu)}{\mu^2} \right) \right) 
+ \left. \left( \frac{\langle \alpha_s G^2 \rangle}{4m_b(\mu)} \right) \left[ g_n^0 + \left( \frac{\alpha_s(\mu)}{\pi} \right) \left( g_n^{10} + g_n^{11} \ln \left( \frac{m_b^2(\mu)}{\mu^2} \right) \right) \right] \right\} ,$$

where $r \equiv m_c/m_b$. In terms of the more specific choice of $m_b(m_b)$, the moments have the simpler form

$$P_n = \frac{1}{(4m_b(m_b))^n} \left\{ f_n^0 + \left( \frac{\alpha_s(\mu)}{\pi} \right) f_n^{10} \right. 
+ \left( \frac{\alpha_s(\mu)}{\pi} \right)^2 \left( f_n^{20}(r) - \frac{1}{4} \beta_0 f_n^{10} \ln \left( \frac{m_b^2(m_b)}{\mu^2} \right) \right) 
+ \left. \left( \frac{\langle \alpha_s G^2 \rangle}{4m_b(m_b)} \right) \left[ g_n^0 + \left( \frac{\alpha_s(\mu)}{\pi} \right) g_n^{10} \right] \right\} ,$$

where $\beta_0 = 11 - 2/3n_f$. The Born and order $\alpha_s$ terms of the moments are known since a long time \cite{5,6} and the order $\alpha_s^2$ contributions for primary $b\bar{b}$ production for massless light quarks have been determined in Ref. \cite{12}. We have cross-checked these contributions with the explicit expressions for the corresponding contributions to $R_{bb}$ given in Ref. \cite{13}. The order $\alpha_s^2$ contributions to the moments from secondary
\[ f_n^0 \]
\[ f_n^{10} \]
\[ f_n^{11} \]
\[ f_n^{20}(0) \]
\[ f_n^{21} \]
\[ f_n^{22} \]
\[ g_n^0 \]
\[ g_n^{10} \]
\[ g_n^{11} \]

Table 1: Coefficients of the theoretical expressions for the moments \( P_n \) to order \( \alpha_s^2 \) for massless light quarks.

| \( n \) | 1 | 2 | 3 | 4 |
|-------|---|---|---|---|
| \( f_n^0 \) | 0.2667 | 0.1143 | 0.0677 | 0.0462 |
| \( f_n^{10} \) | 0.6387 | 0.2774 | 0.1298 | 0.0508 |
| \( f_n^{11} \) | 0.5333 | 0.4571 | 0.4063 | 0.3694 |
| \( f_n^{20}(0) \) | 0.9446 | 0.8113 | 0.5172 | 0.3052 |
| \( f_n^{21} \) | 0.8606 | 1.2700 | 1.1450 | 0.8682 |
| \( f_n^{22} \) | 0.0222 | 0.4762 | 0.8296 | 1.1240 |
| \( g_n^0 \) | -4.011 | -6.684 | -9.722 | -13.088 |
| \( g_n^{10} \) | -4.876 | 1.386 | 16.964 | 44.081 |
| \( g_n^{11} \) | -24.063 | -53.473 | -97.224 | -157.055 |

Table 2: Corrections due to the non-zero charm quark mass to the order \( \alpha_s^2 \) coefficient \( f_n^{20} \) for \( r = m_c/m_b \).

| \( r \) | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|-------|-----|-----|-----|-----|-----|
| \( f_n^{20}(r) - f_n^{20}(0) \) | -0.0021 | -0.0078 | -0.0164 | -0.0266 | -0.0382 |
| \( f_n^{21}(r) - f_n^{21}(0) \) | -0.0028 | -0.0091 | -0.0187 | -0.0302 | -0.0430 |
| \( f_n^{22}(r) - f_n^{22}(0) \) | -0.0024 | -0.0101 | -0.0204 | -0.0330 | -0.0466 |
| \( f_n^{30}(r) - f_n^{30}(0) \) | -0.0030 | -0.0109 | -0.0219 | -0.0348 | -0.0491 |

\( b \bar{b} \) production, where the \( b \bar{b} \) pair is produced through gluon radiation off light quarks, has been computed from the corresponding results for the \( R \)-ratio given in Refs. [14, 15]. These contributions only affect the coefficient \( f_n^{20} \). The coefficients of the gluon condensate have been taken from Ref. [16, 17]. For convenience, the numerical results for the coefficients in Eqs. (2) and (3) for \( n = 1, 2, 3, 4 \) and \( m_c = 0 \) (\( r = 0 \)) are collected in Tab. 1. The numbers for the \( f_n \)'s agree with Ref. [8] up to a convention dependent factor of 1/4, except for the results for \( f_n^{20}(0) \), which are slightly larger accounting for the contributions from secondary \( b \bar{b} \) production. The effects of the non-zero charm quark mass are generated either through virtual gluon self-energy effects or through real primary or secondary associated charm production. The corresponding contributions to \( R_{b \bar{b}} \) for arbitrary mass constellations have been given in Refs. [13, 14]. Numerical values of the charm quark mass corrections to the coefficient \( f_n^{20} \) are displayed in Tab. 2 for values of \( r \) between 0.1 and 0.5. We note that the numbers given in Tab. 2 also include the non-zero charm mass effects in the bottom quark pole-MS mass relation [14].
Table 3: Individual contributions to the experimental moments including uncertainties. The contribution from a resonance \( k \) has been determined in the narrow width approximation, \((P_n)_k = \frac{9\pi}{\Gamma_k} [\alpha(10 \text{ GeV}) M_n^2 + 1]\), where for the electromagnetic coupling \([\alpha(10 \text{ GeV})]\) has been adopted.

| contribution          | \( P_1 \times 10^3 \text{ GeV}^2 \) | \( P_2 \times 10^5 \text{ GeV}^4 \) | \( P_3 \times 10^7 \text{ GeV}^6 \) | \( P_4 \times 10^9 \text{ GeV}^8 \) |
|-----------------------|--------------------------------------|--------------------------------------|--------------------------------------|--------------------------------------|
| \( \Upsilon(1S) \)    | 0.766(29)                            | 0.856(32)                            | 0.956(36)                            | 1.068(40)                            |
| \( \Upsilon(2S) \)    | 0.254(16)                            | 0.252(16)                            | 0.251(15)                            | 0.250(15)                            |
| \( \Upsilon(3S) \)    | 0.211(29)                            | 0.196(27)                            | 0.183(26)                            | 0.171(24)                            |
| \[\Upsilon(4S) - \Upsilon(5S)\]_{PDG} | 0.222(40)                            | 0.192(34)                            | 0.167(29)                            | 0.145(25)                            |
| \[\Upsilon(4S) - \Upsilon(5S)\]_{CUSB} | 0.257(42)                            | 0.223(36)                            | 0.194(31)                            | 0.169(27)                            |
| \[\Upsilon(4S) - \Upsilon(5S)\]_{CLEO} | 0.244(95)                            | 0.213(82)                            | 0.186(72)                            | 0.162(62)                            |
| \[\Upsilon(4S) - \Upsilon(5S)\]_{our} | 0.251(95)                            | 0.218(82)                            | 0.190(72)                            | 0.165(62)                            |
| \( \Upsilon(6S) \)    | 0.048(11)                            | 0.039(9)                             | 0.032(7)                             | 0.027(6)                             |
| 11.1 GeV – 12.0 GeV   | 0.418(57)                            | 0.314(44)                            | 0.236(34)                            | 0.178(27)                            |
| 12.0 GeV – \( M_Z \) | 2.467(26)                            | 0.886(21)                            | 0.414(13)                            | 0.217(8)                             |
| \( M_Z \rightarrow \infty \) | 0.047(1)                             | 0.000(0)                             | 0.000(0)                             | 0.000(0)                             |

Table 3: Individual contributions to the experimental moments including uncertainties. The contribution from a resonance \( k \) has been determined in the narrow width approximation, \((P_n)_k = \frac{9\pi}{\Gamma_k} [\alpha(10 \text{ GeV}) M_n^2 + 1]\), where for the electromagnetic coupling \([\alpha(10 \text{ GeV})]\) has been adopted.

**Experimental Moments**

For the contributions to the experimental moments from the \( \Upsilon(1S) \), \( \Upsilon(2S) \), \( \Upsilon(3S) \) and \( \Upsilon(6S) \) we use the averages for masses and \( e^+e^- \) widths given by the PDG \[18\]. In Table 3 a collection of all contributions to the moments including uncertainties is given. The averages for the \( \Upsilon(1S) \), \( \Upsilon(2S) \) and \( \Upsilon(3S) \) are dominated by data from ARGUS \[19\] and the averages for the \( \Upsilon(6S) \) are from results from CUSB \[10\] and CLEO \[11\].

In the 4S–5S region between 10.5 and 10.95 GeV there have been measurements from CUSB \[10\] and CLEO \[11\]. We find it remarkable that both experiments observed an additional resonance-like enhancement between the \( \Upsilon(4S) \) and \( \Upsilon(5S) \) at about 10.7 GeV. Whereas CUSB fitted for \( \Upsilon(4S) \) and \( \Upsilon(5S) \) resonances within in a coupled channel model, the CLEO experiment was fitting an additional resonance, called “B∗”, at \( m_{B^*} = 10.684 \pm 0.013 \text{ GeV} \) with an \( e^+e^- \) width of \( \Gamma_{B^*}^{e^+e^-} = 0.20 \pm 0.11 \text{ keV} \). As a consequence the \( e^+e^- \) widths for \( \Upsilon(4S) \) and \( \Upsilon(5S) \) obtained from CLEO were systematically smaller than those from CUSB. In the PDG compilation, however, the existence of the “B∗” contribution in the CLEO analysis was ignored when the averages for \( \Upsilon(4S) \) and \( \Upsilon(5S) \) have been determined. As a consequence, the contribution to the moments from the region between 10.5 and 10.95 GeV using

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the PDG averages is, although compatible within errors, systematically lower than the contributions one obtains using the numbers given in the original CLEO and CUSB publications (see Tab. 3). For our analysis we decided to ignore the PDG averages and to take the average of the original CLEO (including the “B”) and the CUSB contributions to the moments for this energy region. We adopted the larger CLEO error assuming that the respective uncertainties are correlated. We believe that this conservative treatment is justified as long as the situation is not clarified. We note that the PDG treatment of the 4S–5S region does not affect the results for the hadronic vacuum polarization effects of $\alpha(M_Z)$ and $(g - 2)_\mu$, because the corresponding differences are much smaller than the total uncertainties. For the $\Upsilon(6S)$ we used the PDG averages since it is unlikely that the different treatment of the enhancement observed between $\Upsilon(4S)$ and $\Upsilon(5S)$ has affected the fits above the $\Upsilon(5S)$.

There is no direct experimental data for $\sigma(e^+e^- \to \gamma^* \to b\bar{b} + X)$ in the region above 11.1 GeV. However, there are measurements of the total hadronic cross section taken by a number of experiments up to energies close to $M_Z$ that are compatible with the Standard Model predictions. Furthermore, from measurements of $R_b$ at the Z pole by LEP and in the region between 133 and 207 GeV by LEP$^2$, it is known that perturbative QCD agrees with the data for $b\bar{b}$ production to about 1% at $M_Z$ and to about 10% in the LEP2 region [21]. It is therefore not unreasonable to estimate the experimental contribution to the moments from above the $\Upsilon(6S)$ from perturbation theory itself. Although order $\alpha_s^3$ corrections to $R_b$ in the high energy expansion are known, we use the perturbative $b\bar{b}$ cross section to order $\alpha_s^2$ to estimate the continuum contributions because the theoretical moments are likewise only available to order $\alpha_s^2$. A much more subtle question is how to estimate the “experimental” uncertainties in this region. The approach of Ref. [7] assumes that the experimental data for the $b\bar{b}$ cross section including errors lie within the theoretical predictions and uses the small theoretical errors. This approach is quite similar to using finite energy sum rules [21] where an upper “duality” cutoff $s_{\text{max}} \gtrsim 11.1$ GeV is used in the integral in Eq. (1).

In Tab. 3 we have displayed the continuum contributions to the moments to order $\alpha_s^2$. The charm quark mass has been set to zero. We have subdivided the continuum contribution into three parts coming from $11.1 - 12.0$ GeV (region 1), $12$ GeV $- M_Z$ (region 2) and $M_Z - \infty$ (region 3) in order to visualize the impact of the various energy regions. The theoretical errors shown in Tab. 3 come from varying the strong coupling in the range $\alpha_s(M_Z) = 0.118 \pm 0.003$ and the $\overline{\text{MS}}$ bottom mass in the conservative PDG range $m_b(m_b) = 4.2 \pm 0.2$. The renormalization scale $\mu$ has been varied between 2.5 and 10 GeV. For the running of $\alpha_s$ and $m_b$ four-loop renormalization group equations have been used. Our theoretical errors are larger than in Ref. [4] where also order $\alpha_s^3$ contributions have been included [22].

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3 At the Z pole $R_b$ is defined as the ratio of the total b quark partial width of the Z to its total hadronic partial width, $\Gamma_{b\bar{b}}/\Gamma_{\text{had}}$, and for LEP2 energies it is defined as the ratio of the total $b\bar{b}$ cross section to the total hadronic one.
Our central values have been obtained from the average of the respective extremal values. Region 1 has been displayed separately because data for the $b\bar{b}$ cross section could potentially be collected there by CLEO [4] in the near future.

Since the continuum region above 11.1 GeV is unsuppressed and constitutes a sizeable contribution to the experimental moments [5], using the small theory errors leads to a considerable model-dependence of the bottom quark mass. For our final error estimate of $m_b(m_b)$ we adopt a 10% correlated (relative) error for the continuum regions 2 and 3 and ignore the respective theory errors given in Tab. 3. This choice is, in principle, as arbitrary as using the theoretical errors (or no errors at all), but should reduce the model-dependence to an acceptable level. For region 1 we use the theory error since here the variation of the $\overline{\text{MS}}$ mass in the conservative PDG bounds has the largest impact and leads to a variation of more than 10%.

### Uncertainties in $m_b(m_b)$

For the determination of $m_b(m_b)$ and the uncertainties we have used 4 methods:

1. The bottom mass $m_b(m_b)$ is determined from single moment fits ($n = 1, 2, 3$) using Eq. (3).

2. The bottom mass $m_b(\mu)$ is determined from single moment fits ($n = 1, 2, 3$) using Eq. (2) and $m_b(m_b)$ is computed subsequently using renormalization group equations.

3. The bottom mass $m_b(m_b)$ is determined from fits to ratios $P_n/P_{n+1}$ ($n = 1, 2$) using Eq. (3).

4. The bottom mass $m_b(\mu)$ is determined from fits to ratios $P_n/P_{n+1}$ ($n = 1, 2$) using Eq. (2) and $m_b(m_b)$ is computed subsequently using renormalization group equations.

For the analysis we employed only moments for $n = 1, 2, 3$ to avoid the large higher order contributions $\sim (\alpha_s \sqrt{n})^k$ that are characteristic for the large-$n$ moments and need to be summed. For method 3 and 4 we did not expand the perturbative series in the theoretical ratios $P_n/P_{n+1}$. We checked that expanding the theoretical ratios has only very small effects on the results. We employed four-loop renormalization group equations and used $\alpha_s(M_Z) = 0.118 \pm 0.003$, $m_c = 1.3 \pm 0.2$ GeV, $\langle G^2 \rangle = (0.024 \pm 0.024)$ GeV$^4$ as theoretical input. The renormalization scale $\mu$ was varied between 2.5 and 10 GeV.

In Tab. 4 the results of our analysis for $m_b(m_b)$ are displayed in detail. The table shows the respective central values (in units MeV), which were obtained using

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4 We note that the relative contribution to the low-$n$ $c\bar{c}$ spectral moments coming from continuum energies above 4.5 GeV is considerably smaller than for the continuum region above 11.1 GeV in $b\bar{b}$ spectral sum rules, see e.g. Ref. [5].
Table 4: Central values and uncertainties for $\bar{m}_b(m_b)$ in units of MeV based on the methods described in the text.

|       | Method 1 (2) | Method 3 (4) |
|-------|--------------|--------------|
|       | Central      |              |
|       | $\Upsilon(1S)$ | 14 (13)    | 11 (11)    |
|       | $\Upsilon(2S)$ | 12 (12)    | 11 (11)    |
|       | $\Upsilon(3S)$ | 11 (11)    | 9 (9)      |
|       | $4S-5S$      | 10 (10)    | 8 (8)      |
|       | $\Upsilon(6S)$ | 10 (10)    | 7 (7)      |
|       | Combined     | 5 (5)      | 3 (3)      |
|       |              | 50 (50)    | 38 (38)    |
|       |              | 15 (15)    | 15 (15)    |
| region 1th | 27th (26th) | 17th (17th) | 11th (11th) | 7th (7th) | 2th (2th) |
| region 2th | 12th (12th) | 8th (8th)  | 4th (4th)  | 4th (4th) | 4th (4th) |
| region 210% | 115 (114)  | 33 (33)    | 13 (13)    | 49 (49)   | 29 (29)   |
| region 3th | 1th (1th)   | 0th (0th)  | 0th (0th)  | 1th (1th) | 0th (0th) |
| region 310% | 2 (2)       | 0 (0)      | 0 (0)      | 2 (2)     | 0 (0)     |
| $\delta^{(\Xi \to \eta' G^2)}$ | 0 (0)       | 0 (0)      | 0 (0)      | 0 (1)     | 0 (1)     |
| $\delta m_c$ | 0 (0)       | 0 (0)      | 0 (0)      | 0 (0)     | 0 (0)     |
| $\delta \alpha_s(M_Z)$ | 17 (18)     | 10 (11)    | 6 (6)      | 3 (3)     | 2 (2)     |
| $\delta \mu$ | 23 (5)      | 16 (14)    | 11 (27)    | 15 (27)   | 3 (50)    |
| Combined   | 184 (166)   | 77 (75)    | 41 (57)    | 76 (88)   | 37 (85)   |
| Total      | 251 (233)   | 127 (125)  | 79 (95)    | 110 (121) | 51 (99)   |

the experimental and theoretical central values given before and $\mu = 5$ GeV. The central values obtained with the four methods are within 15 MeV around 4.20 GeV. All uncertainties (in units of MeV) are presented separately. The experimental errors correspond to the uncertainties given in Tab.3, where for the 4S-5S region our conservative CUSB-CLEO average has been used. The uncertainty from the 4S-5S region constitutes the largest experimental error from the resonance region. The errors from the continuum regions indicated with a subscript "th" are obtained from the corresponding theory errors shown Tab.3. For the continuum regions 2 and 3 also the errors coming from a 10% deviation from the theory prediction are displayed (having no subscript). We note that the latter errors scale roughly linearly, i.e. assuming a 5% (20%) deviation the error decreases (increases) by a factor of two, etc.. This illustrates how strongly the bottom quark mass depends on assumptions for the experimentally unknown $\bar{b}b$ continuum cross section above the $\Upsilon(6S)$. The theoretical errors have been obtained by varying each of the theoretical parameters in the ranges given above while the respective other parameters were fixed to their central values. For method 3 and 4, which are based on fitting ratios $P_n/P_{n+1}$, the
same theoretical input parameters have been used for the moments in the numerator and those in the denominator. If $\alpha_s(M_Z)$ is chosen independently for the moments in the numerator and denominator, the errors for method 3 (4) are 38 (26) MeV for $n = 1$ and 39 (13) MeV for $n = 2$. If $\mu$ is chosen independently for the moments in the numerator and denominator, the errors for method 3 are 52 MeV for $n = 1$ and 63 MeV for $n = 2$. We found that, except for the variations of $\mu$, all resulting errors scale linearly with changes of the input parameters. The errors coming from variations of $\mu$ have been chosen to be the larger ones of the two deviations obtained in the ranges 2.5 GeV $< \mu < 5$ GeV and 5 GeV $< \mu < 10$ GeV. Note that the overall shift in the central value of $m_b(m_b)$ coming from the gluon condensate contribution is between $-0.1$ and $-1$ MeV. The shift caused by the non-zero charm quark mass is between $-1$ and $-3$ MeV, which is an order of magnitude smaller than for large-$n$ moments, where the charm mass effects are enhanced by a factor $1/\alpha_s^2 \sim n \[23\]$. Using the PDG average for the 4S-5S region instead of the CUSB-CLEO average (see Tab. 3), $m_b(m_b)$ is shifted by 10 to 15 MeV for method 1 and 2 and by 2 to 6 MeV for method 3 and 4. We consider the numbers given in Tab. 4 as the main result of this work.

In order to obtain combined errors from the uncertainties of the resonance data we treated one half of each error as correlated (being added linearly) and the other half uncorrelated (being added quadratically) because all data came from $e^+e^-$ machines with common systematic uncertainties and, roughly, systematic and statistical uncertainties were found to have comparable sizes [11]. In Tab. 4 the resulting combined resonance errors are shown in the line below the numbers for the $\Upsilon(6S)$. The theoretical uncertainties including the errors adopted for the three continuum regions have been combined linearly since they do not have any statistical meaning and, in particular, the division into three continuum regions is completely arbitrary. Note that for regions 2 and 3 the uncertainties from the 10% variation of the theory prediction have been adopted and not the smaller theoretical errors. The resulting combined error is displayed in the line below the numbers for $\delta \mu$. To obtain our total error (last line in Tab. 4) we added the resonance, the continuum and the theory errors linearly. As expected we find that the uncertainties from the continuum have the largest impact for the moments with $n = 1$ and 2 and that they are partially canceled when ratios are used for the fitting. However, the theoretical errors are larger for fits with ratios than with single moments, if the theoretical parameters are chosen independently for numerator and denominator. In general, the total error decreases for larger $n$. This trend does, however, not continue for higher values $n > 3$ particularly for methods 2 and 4. Compared to the results of Ref. [7] our total errors are much larger, particularly for fits involving $P_1$ and $P_2$. This is mainly because we adopted more conservative errors for the continuum region in the experimental moments, and we combined uncertainties linearly, when they cannot be treated statistically. Since we believe that $P_3$ can be computed reliably using Eqs. (2) and (3), we adopt the average of the total errors in the third column of Tab. 4 as our final
estimate for the uncertainty and obtain (rounded to units of 10 MeV)

\[ m_b(m_b) = 4.20 \pm 0.09 \text{ GeV} \, . \] (4)

We do not take into account the small 50 MeV error from \( P_2/P_3 \) for method 3 because the small error from variations of \( \mu \) only persists if the same \( \mu \) is chosen for both moments. For an independent choice the error is considerably larger (see text above). If a 20% relative uncertainty is assumed for the continuum regions 1, 2 and 3, the final error increases by 20 MeV. Our result in Eq. (4) is compatible with the result from Ref. [4]. Our result is also compatible with results from fitting large-\( n \) moments at NNLO in the non-relativistic expansion [4] and with recent results, using different methods, obtained in Refs. [21, 24, 25].

Based on the numbers given in Tab. 3 and 4 it is straightforward to discuss the impact of improved measurements of the resonance parameters at CLEO. Assuming improved measurements for the electronic widths of \( \Upsilon(1S), \Upsilon(2S) \) and \( \Upsilon(3S) \) at the level of 2\%, the combined resonance errors shown in Tab. 3 would be reduced by a factor 2/3, which would reduce the total error (obtained from the third column) by about 10 MeV. An improved measurement of the 4S-5S region and the \( \Upsilon(6S) \) with the same precision would result approximately in a reduction of the total error by an additional 10 MeV. A further reduction of the error below about 70 MeV, however, will be difficult to achieve without real experimental data for \( \sigma(e^+e^- \to b\bar{b}) \) in the continuum region above the \( \Upsilon(6S) \) with a precision of better than ten percent.

**Conclusion**

We have given a detailed analysis of the uncertainties in the \( \overline{\text{MS}} \) bottom quark mass \( m_b(m_b) \) using low-\( n \) spectral moments of the cross section \( \sigma(e^+e^- \to \gamma^* \to b\bar{b}) \). For the experimental moments we employed experimental data for the \( \Upsilon(1S) - \Upsilon(6S) \) resonances and the order \( \alpha_s^2 \) QCD predictions for the continuum region above 11.1 GeV. For the 4S-5S region between 10.5 and 10.95 GeV we found that the PDG averages for the electronic partial widths of \( \Upsilon(4S) \) and \( \Upsilon(5S) \) based on data from CUSB, CLEO and ARGUS contain an inconsistency, stemming from the fact CLEO also assumed the existence of an additional resonance at about 10.7 GeV which was ignored in the averaging procedure. For our analysis we therefore used the original CUSB and CLEO results. For the continuum region above the \( \Upsilon(6S) \) we assumed a 10\% error in our final error estimate. For the theoretical moments we used perturbative results at order \( \alpha_s^2 \) including also the contributions from secondary \( b\bar{b} \) production and finite charm mass effects. As our final result we get \( m_b(m_b) = 4.20 \pm 0.09 \text{ GeV} \) and we conclude that more precise data for the electronic partial widths of the \( \Upsilon \) resonances at CLEO could reduce the error by about 20 MeV.
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