Multi-agent Gradient Descent with A Protocol

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Abstract
This essay gives a short introduction to the multi-agent gradient descent method with a protocol. Compared with most existing literature on gradient-based methods, this essay explores a new way to do global optimization, i.e., multiple agents with certain communication protocol will be used in the descent process.

1 Introduction

Gradient-based optimization is to forecast global macro-information based on local micro-information. A common way to do this is to initialize a point and constantly iterate it using the information from derivatives. However, the derivative of a point is the most local property, which can only describe the infinitely small neighborhood of it. That is why we always choose step-size carefully and introduce stochastic process. That is also why such kind of methods always get trapped at local optimal points and move slowly because of its unconfidence.

Different from gradient-based methods, there is another kind of methods which do not care derivatives at all. The most famous one is the bisection method, which initializes two points at first and then use the information from the function values at these two points to iterate. These methods mainly use macro-information, such as the function values at different points. But a major drawback is that, it only works when given a fixed known interval at first. Besides, it is hard to apply such in higher dimensional spaces.

Based on the above two kinds of methods, inspired by multi-agent system theory, a new optimization method is explored in this article. We will use multiple agents to do the gradient descent process, where they communicate with each other using a protocol. In other words, we combine the micro-information (derivatives) and macro-information (communications) together to generate the descent directions and step-sizes.

The key point of this method is to design a protocol such that, the agents are able to communicate with each other effectively so that they cooperatively converge to the global optimal point finally. For simplicity, the optimizations in this paper are assumed to be smooth and have a unique global optimal point (not necessary to be convex).

2 Preliminaries

Gradient descent is a first-order iterative optimization method to minimize a continuous function. It firstly initializes a point, then constantly iterates it with taking steps proportional to the negative of the gradient (approximated gradient) at the current point until convergence. As one of the most

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common optimization methods, gradient descent method also has many shortcomings, including slow convergence, zigzags, and its failure in non-convex optimization. To better illustrate the new idea of this essay, the iterative scheme of traditional gradient descent method is listed as follows:

\[ d^{(k)} = -\nabla f(x^{(k)}) \]
\[ x^{(k+1)} = x^{(k)} + \beta d^{(k)} \]

where \( d \) is the descent direction and \( \beta \) is the step-size.

3 Main Result

Now we first give the following algorithm without formal proofs.

Algorithm 1

Step 1 Fix the step-size as \( \beta \). Set precision \( \xi \) and tolerance \( N \). Choose protocol factor as \( \lambda \).
Initialize \( m \) points (agents) as \( x_1^{(0)}, x_2^{(0)}, \ldots, x_m^{(0)} \). Initialize iterator \( k = 0 \).

Step 2 For \( i = 1, 2, \ldots, m \), the descent direction for the \( i \)-th agent is

\[ d_i^{(k)} = -[ (1 - \lambda)\nabla f(x_i^{(k)}) + \lambda \sum_{j \neq i} (x_j^{(k)} - x_i^{(k)}) (f(x_j^{(k)}) - f(x_i^{(k)})) ] \]

Step 3 For \( i = 1, 2, \ldots, m \), update the \( i \)-th agent:

\[ x_i^{(k+1)} = x_i^{(k)} + \beta d_i^{(k)} \]

Step 4 Update \( k = k + 1 \). If \( \max_{i \neq j} \| x_i^{(k)} - x_j^{(k)} \| < \xi \) or \( k > N \), STOP and output \( x^* = \arg \min_{i} f(x_i), \forall i = 1, 2, \ldots, m \). Otherwise, go to Step 2.

Remark 1 In Step 2, a simple protocol is used:

\[ \lambda \sum_{j \neq i} (x_j^{(k)} - x_i^{(k)}) (f(x_j^{(k)}) - f(x_i^{(k)})) \]

where \( \lambda \) is the protocol factor to control the weights of protocol’s influence and agent’s local derivative’s influence. Observe that, the norm of the above term will become larger as \( \| f(x_j^{(k)}) - f(x_i^{(k)}) \| \) is larger or \( \| x_j^{(k)} - x_i^{(k)} \| \) is larger, vice versa.

Remark 2 This algorithm gives us some intuition on non-convex cases. The reason is that, if the agents are initialized far away from each other, it will be very difficult for them to converge to local optimal points or saddle points, since such situations only happen when they get stuck at the same point at the same time. In other words, if one agent gets trapped at a local optimal point or saddle point, it is very likely for it to be influenced by other outside agents to get out of this point. This is why we use protocol here to allow agents to communicate with each other.

Remark 3 For simplicity, the step-size is fixed and the protocol factor is also fixed in this algorithm. Actually, this method can be easily combined with other traditional single-agent gradient-based methods. In other words, we only need to add agents and a protocol to the traditional optimization methods, then we will figure out the corresponding multi-agent version method.

Remark 4 If we use back-tracking to determine the step-sizes for each agent in each iteration, the functions value at any agent will always non-increase in this process. Observing that, this algorithm gets stuck (all agents stop moving) only when \( d_i^{(k)} = 0 \), \( \forall i \). Notice that, 

\[ d_i^{(k)} = -[ (1 - \lambda)\nabla f(x_i^{(k)}) + \lambda \sum_{j \neq i} (x_j^{(k)} - x_i^{(k)}) (f(x_j^{(k)}) - f(x_i^{(k)})) ] \]

Since we assume function \( f \) has unique global optimal point, it is nearly impossible for \( d_i^{(k)} = 0 \), \( \forall i \) unless all agents attain local optimal points at the same time (\( \nabla f(x_i^{(k)}) = 0 \)) and they have the same
function values \((f(x_i^k) = f(x_j^k), \forall i, j)\), or the vector-sum equals zero at all agents at the same time. Thus, it is not easy for the algorithm to get stuck. With back-tracking strategy, the above algorithm will easily converge to the global optimal point or nice sub-optimal points (influenced by the initialization). This is a sketch proof of the algorithm’s convergence in non-convex cases.

**Remark 5** Observe that, the protocol in terms tends to move the current agent further away from the other agents with higher function values and move the current agent closer to the other agents with lower function values. This will enhance the robustness of the algorithm in non-convex cases and sometimes speed up the convergence speed of the algorithm.

### 4 Simulations

In this section, we give some MATLAB numerical simulations to reveal the effectiveness of the proposed algorithm. Since the multi-agent optimization without protocol will be reduced to a traditional gradient descent methods, we will compare the algorithm’s convergence speeds of two different cases, i.e., with a protocol \((\lambda \neq 0)\) and without protocol \((\lambda = 0)\).

**Experiment 1 (convex case)**
Consider the unconstrained minimization problem \(\min f(x)\) where objective function \(f\) is defined as follows:

\[
   f(x, y) = (x - 2)^2 + (y + 4)^2.
\]

Using Algorithm 1, we use three agents in this case and initialize them as:

\[
   x_1^{(0)} = [-10, 20]^T, \quad x_2^{(0)} = [10, 10]^T \quad \text{and} \quad x_3^{(0)} = [30, -20]^T.
\]

The step-size is fixed as \(\beta = 0.1\). The protocol factor is fixed as \(\lambda = 0.005\) in 'protocol' case, and \(\lambda = 0\) in 'no protocol' case. The convergence process of the two cases is illustrated in the following figure, where the x-axis denotes the iteration \(k\) and y-axis denotes the attained optimal value \(f(x^*)\).

**Experiment 2 (non-convex case)**
In this experiment, we use the famous Rosenbrock function to test our algorithm. The Rosenbrock function has a narrow curved valley which contains the minimum. The bottom of the valley is very flat. Because of the curved flat valley the traditional gradient descent method is zig-zagging slowly with small step sizes towards the minimum.

Consider the unconstrained minimization problem \(\min f(x)\) where objective function \(f\) is defined as follows:

\[
   f(x, y) = [(x - 1)^2 + 100(y - x^2)^2] \times 10^{-6}.
\]

Using Algorithm 1, we use three agents in this case and randomly initialize them as:

\[
   x_1^{(0)} = [-10, 20]^T, \quad x_2^{(0)} = [10, 10]^T \quad \text{and} \quad x_3^{(0)} = [30, -20]^T.
\]
The step-size is fixed as $\beta = 0.1$. The protocol factor is fixed as $\lambda = 0.005$ in 'protocol' case, and $\lambda = 0$ in 'no protocol' case. The convergence process of the two cases is illustrated in the above figure, where the x-axis denotes the iteration $k$ and y-axis denotes the attained optimal value $f(x^*)$.

5 Conclusion

This essay proposed a new method to do gradient descent optimization. Using multiple agents and certain protocol, the new algorithm can also be combined with most existing traditional single-agent gradient descent method. The algorithm in this essay is much simplified to better show the core idea of this method, and further work may be done on the improvements of this rough algorithm, such as step-sizes and protocol designs. The numerical simulations illustrate the effectiveness of the algorithm and reveal that, using multi-agent with protocol, many old illnesses in traditional gradient descent methods such as zigzags and saddle point problems, will be avoided. Because of the limited time, formal convergence proof of this algorithm is skipped in this essay (in both convex and non-convex cases), and more efforts may focus on this later.

References

[1] Ferber, Jacques & Weiss, Gerhard. (1999). Multi-agent systems: an introduction to distributed artificial intelligence. Addison-Wesley Reading.

[2] Boyd, S., & Vandenberghe, L. (2004). Convex optimization. Cambridge University Press.

PS: The MATLAB codes can be found at [http://i.cs.hku.hk/~h3532972/MAS/m.html](http://i.cs.hku.hk/~h3532972/MAS/m.html)