RAMANUJAN'S CUBIC TRANSFORMATION AND
GENERALIZED MODULAR EQUATION

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Abstract. We study the quotient of hypergeometric functions
\[ \mu_a(r) = \frac{\pi}{2 \sin(\pi a)} \frac{F(a, 1-a; 1; 1-r^3)}{F(a, 1-a; 1; r^3)} \quad (r \in (0, 1)) \]
in the theory of Ramanujan’s generalized modular equation for \( a \in (0, 1/2) \),
find an infinite product formula for \( \mu_{1/3}(r) \) by use of the properties of \( \mu_a(r) \)
and Ramanujan’s cubic transformation. Besides, a new cubic transformation
formula of hypergeometric function is given, which complements the Ramanujan’s cubic transformation.

1. Introduction

For real numbers \( a, b \) and \( c \) with \( c \neq 0, -1, -2, \cdots \), the Gaussian hypergeometric
function is defined by
\[ F(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{x^n}{n!} \]
for \( x \in (-1, 1) \), where \( (a, n) \) denotes the shifted factorial function
\[ (a, n) = a(a+1)(a+2)(a+3)\cdots(a+n-1) \]
for \( n = 1, 2, \cdots \), and \( (a, 0) = 1 \) for \( a \neq 0 \). In particular, \( F(a, b; c; x) \) is called
zero-balanced if \( c = a + b \).

It is well known that \( F(a, b; c; x) \) has many important applications in various
fields of the mathematical and natural sciences, and many classes of special function
in mathematical physics are particular cases of this function. For these, and
properties of \( F(a, b; c; x) \) see [1, 2, 4-6, 9, 16, 22, 27, 31, 39]. Here we recall one of the
most important properties of \( F(a, b; c; x) \), the Ramanujan’s cubic transformation,
\[ F\left(\frac{1}{3}, \frac{2}{3}; 1; 1 - \left(\frac{1-r}{1+2r}\right)^3\right) = (1+2r)F\left(\frac{1}{3}, \frac{2}{3}; 1; r^3\right) \]
or
\[ F\left(\frac{1}{3}, \frac{2}{3}; 1; \left(\frac{1-r}{1+2r}\right)^3\right) = \frac{1+2r}{3}F\left(\frac{1}{3}, \frac{2}{3}; 1; 1-r^3\right), \]
which was raised by S. Ramanujan in his unpublished notebooks. In 1989, J. M.
and P. M. Borwein [19] provided a new proof of equation (1.2) or (1.3).

2000 Mathematics Subject Classification. Primary 33C05; Secondary 11F03.
Key words and phrases. Gaussian hypergeometric function, Ramanujan’s cubic transformation, generalized modular equation, infinite product, modular function.

This work was supported by the Natural Science Foundation of China (Grant Nos. 11071069, 11171307) and PhD Students Innovation Foundation of Hunan Province (CX2012B153).
The solution of (1.8) is given by
\[
(1.7)
\]
Then when \(a = 1\) and \(b = x \in (0, 1)\), the common limit \(F(1, x)\) of \(\{a_n\}\) and \(\{b_n\}\) is given by
\[
\frac{1}{F(1, x)} = \sum_{n=0}^{\infty} \frac{(1/3, n)(2/3, n)}{(n!)^2} (1 - x^3)^n = F\left(\frac{1}{3}, \frac{2}{3}; 1; 1 - x^3\right).
\]
For \(a \in (0, 1/2]\), \(r \in (0, 1)\), \(p > 0\), Ramanujan’s generalized modular equation (1.6) is defined by
\[
(1.6)
\]
Making use of the decreasing homeomorphism \(\mu_a : (0, 1) \to (0, \infty)\) defined by
\[
(1.7)
\]
we can rewrite (1.6) as
\[
(1.8)
\]
The solution of (1.8) is given by
\[
s = \varphi_K(a, r) = \mu_a^{-1}(\mu_a(r)/K), \quad K = 1/p.
\]
In the particular case \(a = 1/2\), Ramanujan’s generalized modular equation (1.6) reduces to the classical case, and the modular functions \(\mu_a(r)\) and \(\varphi_K(a, r)\) become \(\mu(r)\) and \(\varphi_K(r)\), respectively.

In this note, for convenience, we denote \(r^* = \sqrt{1 - r^3}\),
\[
(1.9)
\]
and
\[
(1.10)
\]
and
\[
(1.11)
\]
Then from (1.2), (1.3) and (1.9)-(1.11) we conclude that
\[
(1.12)
\]
\[
(1.13)
\]
\[
(1.14)
\]
\[ \varphi_3^*(r) = \frac{\sqrt[3]{3r(1 + r + r^2)}}{1 + 2r}, \quad \varphi_{1/3}^*(r) = \frac{1 - r^*}{1 + 2r^*}, \quad \varphi_3^*(r)^3 + \varphi_{1/3}^*(r^*)^3 = 1. \]

It follows from (1.9) and (1.10) that in order to study the modular functions \( \mu_a(r) \) and \( \varphi_K(a, r) \), we only need to consider the functions \( \mu_a^*(r) \) and \( \varphi_K^*(a, r) \).

As is known to all, Ramanujan’s cubic transformation and generalized modular equation have been developed for over a century. In 1900s, S. Ramanujan studied extensively \( F(a, b; c; x) \) and the modular equation (1.6), and gave a lot of statements concerning them in his unpublished notebooks [28-30], but no original proof have remained. Later, Ramanujan’s theories have been developed by many authors, such as J.M. and P.B. Borwein [17-20], K. Venkatachaliengar [34] and B. C. Berndt [11-15].

The greatest advances toward establishing Ramanujan’s theories have been made by J. M. and P. B. Borwein [20]. In searching for analogues of the classical arithmetic-geometric mean of Gauss, J. M. and P. B. Borwein discovered an elegant cubic analogue, namely, (1.4) and (1.5). Thus a cubic transformation formula (1.2) or (1.3) for \( F(1/3, 2/3; 1; x) \) was derived. In fact, equations (1.2) and (1.3) can be found on page 258 of Ramanujan’s second notebook [28], and they were rediscovered by the Borweins.

In 1995, B. C. Berndt, S. Bhargava, and F. G. Garvan published a landmark paper [15] in which they studied the generalized modular equation (1.6) with \( p \) an integer. For several rational values of \( a \) such as \( a = 1/3, 1/4, 1/6 \) and prime \( p \) (e.g. \( p = 2, 3, 5, 7, \ldots \)), they were able to give proofs for numerous algebraic identities stated by S. Ramanujan in his unpublished notebooks. Meanwhile, a generalization of Ramanujan’s cubic transformation for \( F(1/3, 2/3; 1; x) \) was given.

After the publication of [15] many papers have been written on modular equations [3, 10, 26, 32]. A new connection between geometric function theory and Ramanujan’s theory was derived by M. Vuorinen in [35]. He found that the functions \( \varphi_K^*(r) \) and \( \mu(r) \), as the Hersch-Pfluger distortion function and the plane Grötzsch ring function, play an important role in the theory of quasiconformal maps. Then the functions \( \varphi_K^*(r) \) and \( \mu(r) \), and their generalizations \( \varphi_K(a, r) \) and \( \mu_a(r) \) have been the subject of intensive research. In particular, many remarkable inequalities for them can be found in the literature [3, 7, 21, 24-26, 36, 38]. Especially, G. D. Anderson, S.-L. Qin, M. K. Vamanamurthy and M. Vuorinen [3] established several analytic properties for \( \varphi_K(a, r) \), as applications, some estimates to the solution of generalized modular equation (1.6) were obtained. Recently, G.-D. Wang, X.-H. Zhang and Y.-M. Chu [36] found the relation between the modular function \( \varphi_K(a, r) \) and \( \mu_a(r) \), and proved that, for \( a(r) \) is a real function defined on \((0, 1)\), \( r \in (0, 1) \) and \( K \in (1, \infty) \), inequality

\[ \varphi_{1/K}(a, r) > r^K \exp \left\{ (1 - K) a(r) \right\} \]

holds if and only if \( a(r) \geq \mu_a(r) + \log r \). Equivalently, by (1.9) and (1.10),

\[ \varphi_{1/K}^*(a, r) > r^K \exp \left\{ \frac{2}{3}(1 - K) a^*(r) \right\} \]

if and only if \( a^*(r) \geq \mu_a^*(r) + (3/2) \log r \), where \( a^*(r) \) is also a real functions defined on \((0, 1)\).
The main purpose of this paper is to find an infinite-product representation for \( \mu^*(r) \) (or \( \mu_{1/3}(r) \)) which only contains \( r \), and to extend representation to the function \( \mu_{3n}^*(r) \). We shall prove the following Theorem 1.2.

**Theorem 1.2.** For \( a \in (0, 1/2] \) and \( r \in (0, 1) \), let \( r_0 = r^* = \sqrt{1 - r^3} \), \( r_1 = \phi_3^*(r^*) = \sqrt{9r^*(1 + r^* + r^{*2})/(1 + 2r^*)} \), \( \cdots \), and

\[
(1.17) \quad r_n = \phi_3^*(r_{n-1}) = \frac{\sqrt{9r_{n-1}(1 + r_{n-1} + r_{n-1}^2)}}{1 + 2r_{n-1}} = \phi_3^n(r^*).
\]

Then

\[
(1.18) \quad \prod_{n=0}^{\infty} \left[ (1 + 2r_n)(1 + r_n + r_n^2) \right]^{\frac{3^n}{r}} \leq \exp(\mu_n^*(r) + \frac{3}{2} \log r)
\]

\[
\leq \frac{1}{\sqrt{27}} \exp(R(a)/2) \prod_{n=0}^{\infty} \left[ (1 + 2r_n)(1 + r_n + r_n^2) \right]^{\frac{3^n}{r}}
\]

for \( a \in (0, 1/3] \), and the revered inequality

\[
(1.19) \quad \frac{1}{\sqrt{27}} \exp(R(a)/2) \prod_{n=0}^{\infty} \left[ (1 + 2r_n)(1 + r_n + r_n^2) \right]^{\frac{3^n}{r}} \leq \exp(\mu_n^*(r) + \frac{3}{2} \log r)
\]

\[
\leq \prod_{n=0}^{\infty} \left[ (1 + 2r_n)(1 + r_n + r_n^2) \right]^{\frac{3^n}{r}}
\]

holds for \( a \in [1/3, 1/2] \). Moreover, each equality in (1.18) and (1.19) is reached if and only if \( a = 1/3 \). In particular, for all \( r \in (0, 1) \),

\[
(1.20) \quad \exp(\mu^*(r) + \frac{3}{2} \log r) = \exp(\mu_{1/3}(r^{3/2}) + \frac{3}{2} \log r) = \prod_{n=0}^{\infty} \left[ (1 + 2r_n)(1 + r_n + r_n^2) \right]^{\frac{3^n}{r}}
\]

or

\[
(1.21) \quad \mu^*(r) + \frac{3}{2} \log r = \mu_{1/3}(r^{3/2}) + \frac{3}{2} \log r = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{3^n} \log \left[ (1 + 2r_n)(1 + r_n + r_n^2) \right].
\]

Theorem 1.2 and inequality (1.16) lead to the following corollary.

**Corollary 1.3.** For \( a \in (0, 1/2] \), \( r \in (0, 1) \) and \( K \in (1, \infty) \), let \( r_n \) defined as in Theorem 1.2, and \( \Sigma \equiv \sum_{n=0}^{\infty} 3^{-n} \log \left[ (1 + 2r_n)(1 + r_n + r_n^2) \right] \). Then

\[
\varphi_{1/K}^*(a, r) > r^K \exp \left\{ \frac{(1 - K)}{3} (R(a) - \log 27 + \Sigma) \right\}
\]

for \( a \in (0, 1/3] \) and \( r \in (0, 1) \), and

\[
\varphi_{1/K}^*(a, r) > r^K \exp \left\{ \frac{\Sigma(1 - K)}{3} \right\},
\]

for \( a \in [1/3, 1/2] \) and \( r \in (0, 1) \).

Another purpose of this paper is to complement Theorem 1.1. From (1.4) and (1.5) we clearly see that the iteration is positively homogeneous so is the limit function \( F(a, b) \). Without loss of generality, we only consider two cases: (A) \( a = 1, b = x \in (0, 1) \); (B) \( a = x \in (0, 1), b = 1 \). Theorem 1.1 gives the limit function of case A, while the following Theorem 1.4 presents the answer of case B.
Theorem 1.4. Let \( a, b > 0 \),
\[
a_{n+1} := \frac{a_n + 2b_n}{3}, \quad a_0 := a,
\]
\[
b_{n+1} := \sqrt[3]{\frac{b_n(a_n^2 + a_nb_n + b_n^2)}{3}}, \quad b_0 := b.
\]
Then when \( a = x \in (0, 1) \) and \( b = 1 \), the common limit \( F(x, 1) \) of \( \{a_n\} \) and \( \{b_n\} \) is
\[
\frac{1}{F(x, 1)} = \sum_{n=0}^{\infty} \frac{(1/3, n)^2}{(n!)^2} (1 - x^3)^n = F\left(\frac{1}{3}; \frac{1}{3}; 1; 1 - x^3\right).
\]
In particular, for \( x \in (0, 1) \), then
\[
F\left(\frac{1}{3}; \frac{1}{3}; 1; 1 - x^3\right) = \sqrt[3]{\frac{3}{x^2 + x + 1}} F\left(\frac{1}{3}; \frac{1}{3}; 1; \frac{(1-x)^3}{9(x^2 + x + 1)}\right).
\]

The methods of the proofs of Theorems 1.2 and 1.4 primarily come from S.-L. Qiu and M. Vuorinen in [24], and J. M. and P. B. Borwein in [19], respectively.

2. Preliminary results

In this section, we study some monotonicity properties of the modular function \( \mu_a^*(r) \), which will be used in the proof of Theorem 1.2. But first, we recall some known results for the function \( F(a, b; c; x) \).

It is well known that the properties of the hypergeometric functions are closely related to those of the gamma function \( \Gamma(x) \), the psi function \( \Psi(x) \), and the beta function \( B(x, y) \). For positive numbers \( x \) and \( y \), these functions are defined by
\[
\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad \Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},
\]
respectively (cf. [39]). It is well known that the gamma function satisfies the difference equation
\[
\Gamma(x + 1) = x \Gamma(x)
\]
if \( x \) is nonpositive integer and has the so-called reflection property
\[
\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} = B(x, 1 - x)
\]
if \( x \) is not an integer. We shall also need the function
\[
(2.2) \quad R(a, b) = -2\gamma - \Psi(a) - \Psi(b), \quad R(a) = R(a, 1 - a), \quad R(1/3, 2/3) = \log 27,
\]
where \( \gamma \) is the Euler-Mascheroni constant defined by
\[
\gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right) = 0.577215 \cdots .
\]
By [25, Lemma 2.14(2)], \( R(a) \) is strictly decreasing in \( a \in (0, 1/2) \). Thus \( R(a) > \log 27 \) for \( a \in (0, 1/3) \), and \( R(a) < \log 27 \) for \( a \in (1/3, 1/2) \).

One important tool we shall need in our work is the following Ramanujan’s derivative formula [12, Corollary, p.86]
\[
(2.3) \quad \frac{d}{dx} \left[ \frac{F(a, 1-a; 1; 1-x)}{F(a, 1-a; 1; x)} \right] = -\frac{\sin(\pi a)}{\pi x(1-x)F(a, 1-a; 1; x)^2},
\]
for $a, x \in (0, 1)$. Then from (2.3) we immediately get the derivative of $\mu_n^*(r)$ with respect to $r$: for $a \in (0, 1/2)$ and $r \in (0, 1)$,

\[
\frac{d\mu_n^*(r)}{dr} = - \frac{1}{2} \cdot \frac{1}{r(1-r^3)F(a, 1-a; 1; r^3)^2}.
\]

Another important tool in our work is the following Ramanujan’s cubic transformation for zero-balanced hypergeometric function.

**Theorem 2.1** (see [37, Theorem 2.4]). Let $B(a, b)$ and $R(a, b)$ are defined as in (2.1) and (2.2), respectively. Then for $(a, b) \in \{(a, b) | a, b > 0, a + b \leq 1, ab - 2(a + b)/9 \leq 0\}$, inequality

\[
0 \leq (1 + 2r)F(a, b; a + b; r^3) - F(a, b; a + b; 9r(1 + r + r^2)/(1 + 2r)^3)
\]

\[
\leq \frac{2(R(a, b) - \log 27)}{B(a, b)}
\]

holds for all $r \in (0, 1)$. Also, for $(a, b) \in \{(a, b) | a, b > 0, a + b \geq 1, ab - 2(a + b)/9 \geq 0\}$,

\[
0 \leq F(a, b; a + b; 9r(1 + r + r^2)/(1 + 2r)^3) - (1 + 2r)F(a, b; a + b; r^3)
\]

\[
\leq \frac{2(\log 27 - R(a, b))}{B(a, b)}.
\]

Other important tool in the rest of this paper is the following Lemma 2.2.

**Lemma 2.2** (see [33, Lemma 2.1]). Suppose that the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ have the radius of convergence $r > 0$, $a_n \in \mathbb{R}$ and $b_n > 0$ for all $n \in \{0, 1, 2, \cdots \}$. Let $h(x) = f(x)/g(x)$, then the sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is (strictly) increasing (decreasing), then $h(x)$ is also (strictly) increasing (decreasing) on $(0, r)$.

Motivated by S. Simić and M. Vuorinen [33], and Á. Baricz [8], we will employ Lemma 2.2 to present some Ramanujan’s cubic transformation inequalities for Gaussian hypergeometric functions, Kummer hypergeometric functions, generalized Bessel functions and for general power series (Theorems 2.3 and 2.4, Corollary 2.5). These results complement some results in [37], and also will be used in the proof of Theorem 2.7.

**Theorem 2.3.** Let $a, b, c \in \mathbb{R}$ such that $c$ is not a negative integer or zero and consider the function $Q : (0, 1) \to (0, \infty)$, defined by

\[
Q(x) = F(a, b; c; x)/F(1/3, 2/3; 1; x).
\]

Then the following assertions are true:

1. If $a + b \geq c$, $9ab/2 \geq \max\{1, c\}$, then $Q(x)$ is increasing, and consequently

\[
F(a, b; c; \frac{9r(1 + r + r^2)}{(1 + 2r)^3}) \geq (1 + 2r)F(a, b; c; r^3),
\]

\[
F(a, b; c; \left(\frac{1 - r}{1 + 2r}\right)^3) \leq \frac{1 + 2r}{3}F(a, b; c; 1 - r^3)
\]

hold for each $r \in (0, 1)$. 
(2) If \(a + b \leq c\), \(9ab/2 \leq \min\{1, c\}\), then \(Q(x)\) is decreasing, and consequently

\[
F(a, b; c; \frac{9r(1 + r + r^2)}{(1 + 2r)^3}) \leq (1 + 2r)F(a, b; c; r^3),
\]

\[
F(a, b; c; \left(\frac{1 - r}{1 + 2r}\right)^3) \geq \frac{1 + 2r}{3}F(a, b; c; 1 - r^3)
\]

hold for each \(r \in (0, 1)\).

Proof. Since \(Q(x)\) can be written as

\[
Q(x) = \frac{F(a, b; c; x)}{F(\frac{a}{3}; \frac{b}{3}; 1; x)} = \sum_{n=0}^{\infty} \frac{(a,n)/(b,n)}{(c,n)} \frac{x^n}{n!} \frac{1}{\sum_{n=0}^{\infty} (1/3n)(2/3n) \cdot \frac{x^n}{n!}},
\]

by Lemma 2.2, we know that the monotonicity of \(Q\) depends on the monotonicity of the sequence \(\{\alpha_n\}\), defined by

\[
\{\alpha_n\} = \frac{(a,n)/(b,n)}{(c,n)} \cdot \frac{(1,n)}{(1/3,n)(2/3,n)}.
\]

Note that

\[
\frac{\alpha_{n+1}}{\alpha_n} = \frac{(n + a)(n + b)(n + 1)}{(n + c)(n + 1/3)(n + 2/3)} \geq 1
\]

if and only if

\[
A_n = (a + b - c)n^2 + (a + b - c + ab - \frac{2}{9})n + ab - \frac{2}{9} \geq 0.
\]

Thus, if \(a + b \geq c\) and \(9ab/2 \geq \max\{1, c\}\), then \(A_n \geq 0\) for all \(n \in \{0, 1, \cdots\}\), that is, the sequence \(\{\alpha_n\}\) is increasing, and consequently by Lemma 2.2 the function \(Q\) is increasing. Now, putting \(x = x(r) = r^3\) and \(y = y(r) = 9r(1 + r + r^2)/(1 + 2r)^3\), then \(0 < x < y < 1\) and

\[
\frac{F(a, b; c; r^3)}{F(\frac{a}{3}; \frac{b}{3}; 1; r^3)} \leq \frac{F(a, b; c; \frac{9r(1 + r + r^2)}{(1 + 2r)^3})}{F(\frac{a}{3}; \frac{b}{3}; 1; \frac{9r(1 + r + r^2)}{(1 + 2r)^3})},
\]

that is,

\[
F(a, b; c; r^3) \leq F(a, b; c; \frac{9r(1 + r + r^2)}{(1 + 2r)^3}) \cdot \frac{F(\frac{1}{3}, \frac{2}{3}; 1; r^3)}{F(\frac{1}{3}, \frac{2}{3}; 1; \frac{9r(1 + r + r^2)}{(1 + 2r)^3})},
\]

which in view of (1.2) is equivalent to (2.7). Similarly, by choosing \(x = x(r) = [(1 - r)/(1 + 2r)]^3\) and \(y = y(r) = 1 - r^3\) we get the inequality

\[
\frac{F(a, b; c; \left(\frac{1-r}{1+2r}\right)^3)}{F(\frac{1}{3}, \frac{2}{3}; 1; \left(\frac{1-r}{1+2r}\right)^3)} \leq \frac{F(a, b; c; 1 - r^3)}{F(\frac{1}{3}, \frac{2}{3}; 1; 1 - r^3)},
\]

that is, by (1.3),

\[
F(a, b; c; \left(\frac{1-r}{1+2r}\right)^3) \leq \frac{1 + 2r}{3}F(a, b; c; 1 - r^3).
\]

This proves the part (1). The proof of part (2) is similar, and thus we omit further details. □
Remark. If we change $r$ to $(1-r)/(1+2r)$ in (2.7) and (2.9), then we have (2.8) and (2.10), respectively. Thus the Ramanujan’s cubic transformation inequalities (2.7) and (2.8) are equivalent as well as the inequalities (2.9) and (2.10).

Now, let us consider the sequence $\{\omega_n\}$, defined by

$$\omega_n = \frac{(nl)^2}{(1/3, n)(2/3, n)}.$$

Then making use of Lemma 2.2 together with the similar argument in Theorem 2.3 we will get a more general result of Theorem 2.3 as follows.

**Theorem 2.4.** Suppose that the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is convergent in all $x \in (0, 1)$, where $a_n \in \mathbb{R}$ for all $n \in \{0, 1, \ldots\}$, and assume that the sequence $\{a_n \cdot \omega_n\}$ is increasing. Then the function $x \to f(x)/F(1/3, 2/3; 1; x)$ is increasing on $(0, 1)$, and by use of the notation $\chi_f(x) = f(x^3)$ we have the Ramanujan’s cubic transformation inequalities for all $r \in (0, 1)$,

$$\chi_f \left( \frac{\sqrt[3]{9r(1 + r + r^2)}}{1 + 2r} \right) \geq (1 + 2r) \chi_f(r).$$

Moreover, if the sequence $\{a_n \cdot \omega_n\}$ is decreasing, then $x \to f(x)/F(1/3, 2/3; 1; x)$ is decreasing on $(0, 1)$, and consequently (2.11) is reversed.

Applying Theorem 2.4 to the generalized Bessel function $u_v : (0, \infty) \to \mathbb{R}$ and the Kummer hypergeometric function $\Phi(p, q; \cdot) : (0, \infty) \to \mathbb{R}$, defined by

$$u_v(x) = \sum_{n=0}^{\infty} \frac{(-c)^n}{(n, n)} \frac{x^n}{n!}, \quad \Phi(p, q; x) = \sum_{n=0}^{\infty} \frac{(p, n)}{(q, n)} \frac{x^n}{n!},$$

where $v, b, c, p, q \in \mathbb{R}$, $\kappa = v + \frac{b+1}{2} \notin \{0, -1, \ldots\}$ and $q \notin \{0, -1, \ldots\}$. Then we have the following Corollary 2.5.

**Corollary 2.5.** Let $v, b, c, p, q \in \mathbb{R}$ such that $\kappa \geq \max\{-1, -9c/8, -2/9 - c/4\}$ and $q \geq \max\{0, 9p/2, p + 7/9\}$. Then $x \to u_v(x)/F(1/3, 2/3; 1; x)$ and $x \to \Phi(p, q; x)/F(1/3, 2/3; 1; x)$ are decreasing on $(0, 1)$ and consequently for all $r \in (0, 1)$ we get

$$\chi_{u_v} \left( \frac{\sqrt[3]{9r(1 + r + r^2)}}{1 + 2r} \right) \leq (1 + 2r) \chi_{u_v}(r), \chi_{\Phi} \left( \frac{\sqrt[3]{9r(1 + r + r^2)}}{1 + 2r} \right) \leq (1 + 2r) \chi_{\Phi}(r).$$

Next we prove the monotonicity properties and inequalities for the modular function $\mu^*_a(r)$ defined as in (1.9).

**Lemma 2.6.** (1) The function $\mu^*_a(r) + (3/2) \log r$ is strictly decreasing from $(0, 1)$ onto $(0, R(a)/2)$;

(2) If $a \in (0, 1/2]$, then the inequality

$$\frac{3\log 3}{2} < \mu^*_a(r) + \frac{1}{2} \log \left( \frac{1 - r^a}{1 + 2r^a} \right) < \frac{R(a)}{2}$$

holds for all $r \in (0, 1)$.

**Proof.** Part (1) directly follows from [26, Theorem 5.5(2)].
Then we have the following statements:

1. For \( a \in (0, 1/2] \), define the function \( g \) on \((0, 1)\) by
   \[
g(r) = 3\mu_a^\ast \left( \frac{\sqrt[3]{9r(1 + r + r^2)}}{1 + 2r} \right) - \mu_a^\ast(r).
   \]
   Then \( g \) is strictly decreasing from \((0, 1)\) onto \((0, R(a) - \log 27)\) if \( a \in (0, 1/3) \), is strictly increasing from \((0, 1)\) onto \((R(a) - \log 27, 0)\) if \( a \in (1/3, 1/2) \), and \( f(r) \equiv 0 \) if \( a = 1/3 \). Moreover, for \( a \in (0, 1/3) \) and all \( r \in (0, 1) \),
   \[
   (2.13) \quad \mu_a^\ast(r) < 3\mu_a^\ast \left( \frac{\sqrt[3]{9r(1 + r + r^2)}}{1 + 2r} \right) < \min\{\mu_a^\ast(r) + R(a) - \log 27, C_1\mu_a^\ast(r)\}.
   \]

2. For \( a \in (0, 1/2] \), then the function
   \[
f(r) = \mu_a^\ast \left( \frac{1 - r}{1 + 2r} \right) - 3\mu_a^\ast(r^*)
   \]
   is strictly decreasing from \((0, 1)\) onto \((\log 27 - R(a), 0)\) if \( a \in (0, 1/3) \), and is strictly increasing from \((0, 1)\) onto \((0, \log 27 - R(a))\) if \( a \in (1/3, 1/2) \). Moreover, for all \( a \in (0, 1/3) \) and \( r \in (0, 1) \),
   \[
   (2.15) \quad \max \left\{ \frac{3\pi^2}{4\sin^2(\pi a)} - (R(a) - \log 27)\mu_a^\ast(r), \frac{1}{C_1} \frac{3\pi^2}{4\sin^2(\pi a)} \right\} \leq
   \mu_a^\ast(r) \mu_a^\ast \left( \frac{1 - r}{1 + 2r} \right) \leq \frac{3\pi^2}{4\sin^2(\pi a)}.
   \]
   Also, for \( a \in [1/3, 1/2] \) and \( r \in (0, 1) \),
   \[
   (2.16) \quad \frac{3\pi^2}{4\sin^2(\pi a)} \leq \mu_a^\ast(r) \mu_a^\ast \left( \frac{1 - r}{1 + 2r} \right) \leq
   \min \left\{ \frac{3\pi^2}{4\sin^2(\pi a)} - (R(a) - \log 27)\mu_a^\ast(r), \frac{1}{C_1} \frac{3\pi^2}{4\sin^2(\pi a)} \right\}.
   \]
   Equality is reached in each inequality of (2.15) and (2.16) if and only if \( a = 1/3 \).

**Proof.** For part (1), if \( a = 1/3 \), then \( f(r) = 0 \) by (1.13). And let \( x = \sqrt[3]{9r(1 + r + r^2)}/(1 + 2r) \), then \( x^* = (1 - r)/(1 + 2r) \) and
   \[
   (2.17) \quad \frac{dx}{dr} = \frac{x^*}{3x^2}(1 + 2x^*)^2.
   \]
It follows from Lemma 2.6(1) that

\[
\lim_{r \to 0^+} g(r) = \lim_{r \to 0^+} \left( 3\mu_a^*(x) + \frac{9}{2} \log x - \left[ \mu_a^*(r) + \frac{3}{2} \log r - \frac{3}{2} \log (1 + 2r) \right] \right)
\]

\[
= \frac{3R(a)}{2} - \frac{R(a)}{2} - \log 27
\]

\[= R(a) - \log 27.\]

Clearly \( \mu_a^*(1^-) = 0 \) so that

\[
\lim_{r \to 1^-} g(r) = 0.
\]

Next, by differentiation and (2.4), we get

\[
g'(r) = -\frac{9}{2} \frac{x(1 - x^3)F(a, 1 - a; 1; x^3)^2}{r(1 - r^3)F(a, 1 - a; 1; x^3)^2} \frac{3x^2(1 + 2r)}{3} + \frac{3}{2} \frac{1}{r} + \frac{1}{r^3}
\]

\[
= \frac{3}{2} \frac{1}{r} \frac{1}{(1 - r^3)}F(a, 1 - a; 1; x^3)^2 \frac{F(a, 1 - a; 1; 1 - r^3)}{F(a, 1 - a; 1; 1 - r^3)}
\]

\[
\times \left[ F(a, 1 - a; 1; x^3)^2 - (1 + 2r)^2 F(a, 1 - a; 1; r^3)^2 \right].
\]

Therefore, the monotonicity and range of \( g \) follows from (2.18)-(2.20) and Theorem 2.3. The first inequality and the first upper bound in (2.13), and the first lower bound and the second inequality in (2.14) are clear. For other inequalities in (2.13) and (2.14), by (2.5) and (2.6) we have, when \( a \in (0, 1/3) \) \( (a \in [1/3, 1/2] \) resp.),

\[
3\mu_a^* \left( \frac{\sqrt[3]{4r(1 + r + r^2)}}{1 + 2r} \right) / \mu_a^*(r)
\]

\[
= 3 \frac{F(a, 1 - a; 1; r^3)}{F(a, 1 - a; 1; 1 - r^3)} \frac{F(a, 1 - a; 1; \left( \frac{1 - r}{1 + 2r} \right)^3)}{F(a, 1 - a; 1; \left( \frac{1 - r}{1 + 2r} \right)^3)}
\]

\[
\leq \left[ 1 + \frac{\sin(\pi a)(R(a) - \log 27)}{\pi} \right] \left[ 1 + \frac{\sin(\pi a)(R(a) - \log 27)}{\pi} \right]
\]

\[
\leq \left[ 1 + \frac{\sin(\pi a)(R(a) - \log 27)}{\pi} \right] .
\]

Equality holds in each of above inequalities if and only if \( a = 1/3 \). On the other hand, since \( x > r \), it follows from the monotonicity of \( \mu_a^*(r) \) with respect to \( r \) on \( (0, 1) \) that \( \mu_a^*(x) < \mu_a^*(r) \). Hence, the remaining bounds in (2.13) and (2.14) follow.

For part (2), let \( t = (1 - r)/(1 + 2r) \). Then \( t^* = \sqrt[3]{4r(1 + r + r^2)} / (1 + 2r) \) and \( f(r) = -g(t) \). Hence the assertion about \( f \) follows from part (1).
It follows from (1.12), (1.15) and (2.21) that
\[\mu_a^*(r)\mu_a^*(1 - r) = \frac{3\pi^2}{4\sin^2(\pi a)} \frac{\mu_a^*(t)}{3\mu_a^*(\sqrt{9r_t(1+r^2)/1+2r})} \geq \frac{1}{C_1} \left[ \frac{3\pi^2}{4\sin^2(\pi a)} \right] \]
for all \(a \in (0, 1/3]\) and \(r \in (0, 1),\) and inequality
\[\mu_a^*(r)\mu_a^*(1 - r) = \frac{3\pi^2}{4\sin^2(\pi a)} \frac{\mu_a^*(t)}{3\mu_a^*(\sqrt{9r_t(1+r^2)/1+2r})} \leq \frac{1}{C} \left[ \frac{3\pi^2}{4\sin^2(\pi a)} \right] \]
holds for \(a \in [1/3, 1/2]\) and \(r \in (0, 1),\) with equality of (2.22) or (2.23) if and only if \(a = 1/3.\)

On the other hand, for \(a \in (0, 1/3]\) \((a \in [1/3, 1/2]\) resp.), then from (1.12) and Theorem 2.1 we have
\[\mu_a^*(r)\mu_a^*(1 - r) = \frac{\pi}{2\sin(\pi a)} \frac{F(a, 1 - a; 1; \frac{9r_t(1+r^2)}{(1+2r)^3})}{F(a, 1 - a; 1; \frac{1-r}{1+2r})} \leq (\geq \text{resp.}) \frac{\pi}{2\sin(\pi a)} \frac{(1+2r)F(a, 1 - a; 1; r^2)}{(1+2r)^3} F(a, 1 - a; 1; 1-r^2) \mu_a^*(r) \]
\[= 3\mu_a^*(r)\mu_a^*(r^*) = \frac{3\pi^2}{4\sin^2(\pi a)}. \]
The second equality in (2.24) holds if and only if \(a = 1/3.\) Thus inequalities (2.15) and (2.16) follows from (1.12) and (2.22)-(2.24) together with the monotonicity of \(f. \]

**Remark.** Theorem 2.7 extends the formulas (1.13) and (1.14) to the function \(\mu_a^*(r)\) for \(a \in (0, 1/2]\).

### 3. Proofs of Theorems 1.2 and 1.4

In this section, we prove our main results stated in Section 1.

#### 3.1 Proof of Theorem 1.2
Consider the function
\[f(r) = \mu_a^*(r) + \frac{1}{2} \log \left( \frac{1 - r^*}{1 + 2r^*} \right) = \mu_a^*(r) + \frac{3}{2} \log r - \frac{1}{2} \log(1+2r^*)(1+r^*+r^{*2}) \]
for \(a \in (0, 1/2]\) and \(r \in (0, 1).\) Let \(r_0 = r^*, \ r_1 = \varphi_3^*(r^*) = \sqrt{9r^*(1+r^*+r^{*2})/(1+2r^*)}, \ r_2 = \varphi_3^*(r_1) = \varphi_5^*(r^*),\) then \(r^* = \varphi_1^* / (1 + 2r_1^*), \ r_1^* = (1 - r^*)/(1 + 2r_1^*) = \varphi_1^*/(1 + 2r_1^*),\) and \(r = \varphi_3^*(r_1^*)\) so that
\[f(r) = \mu_a^* \left( \sqrt{9r_1^*(1 + r_1^* + r_1^{*2}) / 1 + 2r_1^*} \right) + \frac{1}{2} \log \left( \frac{1 - \varphi_1^*(r_1^*)}{1 + 2\varphi_1^*(r_1^*)} \right) \]
\[= \mu_a^* \left( \sqrt{9r_1^*(1 + r_1^* + r_1^{*2}) / 1 + 2r_1^*} \right) + \frac{1}{2} \log r_1^*. \]
Let $g(x) = 3\mu^*_a \left( \sqrt[3]{x(1+x+x^2)/(1+2x)} \right) - \mu^*_a(x)$ for $x \in (0, 1)$ and $a \in (0, 1/2)$. Then (3.2) can be written as

$$f(r) - \frac{1}{2} \log r^*_1 + \frac{1}{6} \log \left( \frac{1-r_1}{1+2r_1} \right) = \frac{1}{3} [g(r^*_1) + f(r^*_1)],$$

that is

$$f(r) - \frac{1}{6} \log ((1+2r_1)(1+r_1+r^*_1)) = \frac{1}{3} [g(r^*_1) + f(r^*_1)].$$

Similarly, putting $r_2 = \varphi^*_a(r_1) = \varphi^*_a(r^*_1)$, we get

$$f(r^*_1) - \frac{1}{6} \log ((1+2r_2)(1+r_2+r^*_2)) = \frac{1}{3} [g(r^*_2) + f(r^*_2)],$$

and hence, by (3.3),

$$f(r) - \frac{1}{6} \log ((1+2r_1)(1+r_1+r^*_1)) - \frac{1}{18} \log ((1+2r_2)(1+r_2+r^*_2)) = \frac{1}{3} g(r^*_1) + \frac{1}{9} g(r^*_2) + \frac{1}{9} f(r^*_2).$$

Generally, assuming

$$f(r) - \frac{1}{2} \sum_{k=1}^{n-1} \frac{1}{3^k} \log ((1+2r_k)(1+r_k+r^*_k)) = \frac{1}{3} \sum_{k=1}^{n-1} g(r^*_k) + \frac{1}{3^{n-1}} f(r^*_n)$$

for $n \in \mathbb{N}$ and $n \geq 2$, we let $r_n = \varphi^*_a(r_{n-1}) = \varphi^*_a(r^*_n)$ in (3.3), and from (3.6) it follows that

$$f(r) - \frac{1}{2} \sum_{k=1}^{n} \frac{1}{3^k} \log ((1+2r_k)(1+r_k+r^*_k)) = \frac{1}{3} \sum_{k=1}^{n} g(r^*_k) + \frac{1}{3^n} f(r^*_n).$$

Hence, by induction, (3.7) holds for all $n \in \mathbb{N}$, $a \in (0, 1/2]$, and $r \in (0, 1)$.

Next, we divide the proof into two cases.

**Case 1** $a \in (0, 1/3]$. Then from (3.7), Lemma 2.6(2) and Theorem 2.7(1) we have

$$\frac{1}{3^n} \left( -\frac{3 \log 3}{2} \right) \leq f(r) - \frac{1}{2} \sum_{k=1}^{n} \frac{1}{3^k} \log ((1+2r_k)(1+r_k+r^*_k)) \leq \sum_{k=1}^{n} \frac{1}{3^k} (R(a) - \log 27) + \frac{1}{2} R(a) - \frac{1}{2} \sum_{k=1}^{n} \frac{1}{3^k} \log (1+2r_k)(1+r_k+r^*_k)$$

$$= \frac{1}{2} (R(a) - \log 27) + \frac{\log 27}{2 \cdot 3^n}.$$

Letting $n \to \infty$, we get

$$\frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{3^k} \log ((1+2r_k)(1+r_k+r^*_k)) \leq f(r) \leq \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{3^k} \log ((1+2r_k)(1+r_k+r^*_k)) + \frac{1}{2} (R(a) - \log 27).$$

The double inequality (1.18) follows from (3.1) and (3.8).
**Case 2** \(a \in [1/3, 1/2]\). It follows from (3.7), Lemma 2.6 and Theorem 2.7(1) that

\[
\frac{1}{2}(R(a) - \log 27) - \frac{R(a)}{2 \cdot 3^n} = \sum_{k=1}^{n} \frac{1}{3^k} (R(a) - \log 27) + \frac{1}{3^n} \left( -\frac{3 \log 3}{2} \right) \\
\leq f(r) - \frac{1}{2} \sum_{k=1}^{n} \frac{1}{3^k} \log[(1 + 2r_k)(1 + r_k + r_k^2)] \\
\leq \frac{R(a)}{2 \cdot 3^n}.
\]

Letting \(n \to \infty\), we get

\[
\frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{3^k} \log[(1 + 2r_k)(1 + r_k + r_k^2)] + \frac{1}{2}(R(a) - \log 27) \leq f(r) \\
\leq \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{3^k} \log[(1 + 2r_k)(1 + r_k + r_k^2)].
\]

The double inequality (1.19) follows from (3.1) and (3.9), and the remaining results are clear. □

The following corollary follows easily from Theorem 1.2.

**Corollary 3.1.** Let \(r \in (0, 1)\), then

\[
\mu^*(r) \leq \mu_a^*(r) \leq \mu^*(r) + \frac{1}{2}(R(a) - \log 27)
\]

if \(a \in (0, 1/3]\), and

\[
\mu^*(r) + \frac{1}{2}(R(a) - \log 27) \leq \mu_a^*(r) \leq \mu^*(r)
\]

if \(a \in [1/3, 1/2]\), each equality in (3.10) or (3.11) is reached if and only if \(a = 1/3\).

**3.2 Proof of Theorem 1.4.** Since the limit function \(F(a, b)\) satisfies

\[
F(a_0, b_0) = F(a_1, b_1) = \cdots,
\]

we get

\[
F(a_0, b_0) = F(a_1, b_1) = F\left(\frac{a_0 + 2b_0}{3}, \sqrt[3]{\frac{b_0(a_0^2 + a_0b_n + b_n^2)}{3}}\right)
\]

or

\[
F(x, 1) = F\left(\frac{x + 2}{3}, \sqrt[3]{\frac{x^2 + x + 1}{3}}\right)
\]

\[
= \sqrt[3]{\frac{x^2 + x + 1}{3}} F\left(\frac{x + 2}{\sqrt[3]{9(x^2 + x + 1)}}, 1\right).
\]

Let

\[
H(x) = \frac{x^{1/2}(1 - x)^{1/3}}{F((1 - x)^{1/3}, 1)},
\]

\[
t(x) = 1 - \frac{(\hat{x} + 2)^3}{9(\hat{x}^2 + \hat{x} + 1)} = \frac{(1 - \hat{x})^3}{9(\hat{x}^2 + \hat{x} + 1)}, \quad \hat{x} = (1 - x)^{1/3}.
\]
Then from equations (3.12) and (3.13) we have

\[
\frac{d\hat{x}}{dx} = -\frac{1}{3\hat{x}^2}, \quad \frac{dt(x)}{dx} = \frac{(1 - \hat{x})^2(\hat{x} + 2)^2}{27\hat{x}^2(\hat{x} + 1)^2},
\]

and

\[
\frac{H(x)}{H(t(x))} = \frac{x^{1/2}(1 - x)^{1/3}}{(1 - \hat{x})^{1/2} + 2(\hat{x} + 1)^{1/2}}, \quad \frac{\sqrt{3}}{\hat{x}^2 + \hat{x} + 1}.
\]

Thus the key point of the proof is to show that

\[
G(x) = x^{1/2}(1 - x)^{1/3} F\left(\frac{1}{3}, \frac{1}{3}; 1; x\right)
\]

also satisfies the function equation (3.15). From this we deduce that \(G(x) = H(x)\). In fact, if we let \(J(x) = G(x)/H(x)\), then \(J(x) = J(t(x))\). Note that

\[
t(x) = \frac{(1 - \hat{x})^3}{9(\hat{x}^2 + \hat{x} + 1)} < \frac{1}{9} x
\]

for \(x \in (0, 1)\), thus \(J(x) = J(0^+) = 1\) for \(x \in (0, 1)\).

The hypergeometric differential equation satisfied by \(G\) is

\[
\frac{G''(x)}{G(x)} = -\frac{9x^2 + 10x - 9}{36x^2(1 - x)^2} = a(x),
\]

since \(F\left(\frac{1}{3}; \frac{1}{3}; 1; x\right)\) satisfies the hypergeometric differential equation

\[
x(1 - x)y'' + \left(1 - \frac{5}{3}x\right)y' - \frac{1}{9}y = 0.
\]

Now it is a calculation that

\[
G^*(x) = \sqrt{\frac{3}{t'(x)}} G(t(x))
\]

also satisfies (3.17) exactly when

\[
a(x) = \left(t'(x)\right)^2 a(t(x)) - \frac{1}{2} \frac{t''(x)}{t'(x)} + \frac{3}{4} \left(\frac{t''(x)}{t'(x)}\right)^2.
\]

A tedious calculation gives

\[
t''(x) = \frac{2(1 - \hat{x})(\hat{x} + 2)(-\hat{x}^4 + 2\hat{x}^3 + 6\hat{x}^2 + 4\hat{x} + 2)}{81\hat{x}^3(\hat{x}^2 + \hat{x} + 1)^3},
\]

\[
t'''(x) = \frac{2(5\hat{x}^8 + 20\hat{x}^7 - 34\hat{x}^6 - 136\hat{x}^5 - 22\hat{x}^4 + 68\hat{x}^3 + 104\hat{x}^2 + 56\hat{x} + 20)}{243\hat{x}^5(\hat{x}^2 + \hat{x} + 1)^4},
\]

\[
a(t(x)) = -\frac{81(\hat{x}^2 + \hat{x} + 1)^2(2\hat{x}^6 + 4\hat{x}^5 + 76\hat{x}^4 + 152\hat{x}^3 + 248\hat{x}^2 + 176\hat{x} + 72)}{4(1 - \hat{x})^6(\hat{x}^3 + 6\hat{x}^2 + 12\hat{x} + 8)}.
\]
Putting the three equations above into the right hand of (3.19), by simplification using Maple 13, we get
\[ (t'(x))^2 a(t(x)) - \frac{1}{2} \frac{t''(x)}{t'(x)} + \frac{3}{4} \left( \frac{t''(x)}{t'(x)} \right)^2 = -\frac{9x^6 - 8x^3 + 8}{36x^2(1-x)^2(x^2 + x + 1)^2} = a(x). \]

Hence both \( G^*(x) \) and \( G(x) \) satisfy (3.17). Furthermore, since the roots of the indicial equation of (3.17) are \( (1/2, 1/2) \), there is a fundamental logarithmic solution. Since both \( G^* \) and \( G \) are asymptotic to \( \sqrt{x} \) at 0, they are in fact equal. Thus (3.18) shows that \( G \) satisfies (3.15). This implies \( F(x, 1) = 1/F(1/3, 1/3; 1; 1 - x^3) \), and equation (1.22) follows from (3.12). □

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