THE RING OF ARITHMETICAL FUNCTIONS WITH UNITARY CONVOLUTION: THE \([n]-TRUNCATION\)

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Abstract. We study a certain truncation \(A_{[n]}\) of the ring of arithmetical functions with unitary convolution, consisting of functions vanishing on arguments \(> n\). The truncations \(A_{[n]}\) are artinian monomial quotients of a polynomial ring in finitely many indeterminates, and are isomorphic to the “artinified” Stanley-Reisner ring \(\mathbb{C}[\Delta([n])]\) of a simplicial complex \(\Delta([n])\).

1. Introduction

The set \(A\) of all complex sequences \((c_i)_{i=1}^{\infty}\) is in a natural way a complex vector space. There are several ways an algebra structure can be introduced on this vector space: the most intuitive way are through the so-called regular convolutions of Narkiewicz [14]. Among those, the Dirichlet convolution and the unitary convolution are the best known specimen. They are in a way the two extremes, in that they form the maximal and minimal elements in a certain natural partial order on the set of all regular convolutions.

The ring given by Dirichlet convolution is isomorphic to the formal power series ring on countably many indeterminates, hence a domain; in fact, it is a UFD, as proved by Cashwell and Everett [4]. As a contrast, the ring given by Dirichlet convolution is an epimorphic image of the formal power series ring on countably many indeterminates: it has zero-divisors and nilpotent element. In [17] we gave a conjectural characterization of the zero-divisors. We also established some divisorial properties: a given element can have factorizations (into irreducibles) of different length, but there is always a bound for those lengths.

Let us denote by \(A_{[n]}\) the subset of \(A\) consisting of those sequences \((c_i)_{i=1}^{\infty}\) for which \(c_i = 0\) when \(i > n\). Restricting any convolution product on \(A\), and modifying it so that the product of two elements in \(A_{[n]}\) stays in \(A_{[n]}\), we get a new algebra, which is an artinian monomial quotient of the original one. These quotients, for the case of the Dirichlet convolution, were studied in [16]. It turned out that in this case...

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the defining ideals are *strongly stable*, hence we can apply the Eliahou-Kervaire resolution \[3\] to get information about various homological invariants.

In this article, we study the algebra \(A_n\) equipped with the multiplication given by the unitary convolution. It turns out that the defining ideals, apart from being monomial ideals, are *almost square-free*, i.e. they are the sum of a square-free ideal and the ideal generated by the squares of the indeterminates. The ring is therefore the quotient of \(\mathbb{C}[x_1, \ldots, x_r, x_1^2, \ldots, x_r^2]\) by a square-free monomial ideal, hence it is the *artinified Stanley-Reisner ring* \(\mathbb{C}[\Delta]\), in the sense of Sköldberg \[15\], of a simplicial complex \(\Delta\). In short, the artinified Stanley-Reisner ring is like the indicator algebra \(\mathbb{C}\{\Delta}\) (see for instance \[1\] for a definition) but it is a quotient of \(\mathbb{C}[x_1, \ldots, x_r, x_1^2, \ldots, x_r^2]\) rather than a quotient of the exterior algebra, so it is commutative rather than skew-commutative.

The simplicial complex \(\Delta\) is easy to describe: the underlying vertex set consists of all prime powers \(p^a \leq n\), and a set \(\{q_1, \ldots, q_s\}\) of such prime powers is a simplex in \(\Delta\) if and only if the \(q_i\)'s are relatively prime, and their product is \(\leq n\). A figure is supposedly worth a thousand words, so the reader may want to look at Figure 3 on page 8, where the complex for \(n = 10\) is displayed. Understanding this complex, particularly for large \(n\), is the purpose of this article. In particular, we determine the dimension and the largest non-vanishing homology of \(\Delta\), and show that all homology groups are torsion-free, by exhibiting a (non-pure) shelling. We also give an asymptotic estimate of the socle dimension of \(A_n\).

In \[18\] a study was made of the truncation of \(A\) to some subset \(V \subset \mathbb{N}^+\) which is closed under taking unitary divisors. The results obtained for the corresponding algebras \(A_V\), in particular for finite \(V\), will be of use for us in our study of the special case \(V = [n] = \{1, \ldots, n\}\), so this article starts with a review of the pertinent definitions and results from \[18\].

1.1. **Acknowledgement.** I used the GAP-package *Simplicial Homology* \[5\] by Frank Heckenbach, Jean-Guillaume Dumas, Dave Saunders, and Volkmar Welker to calculate the homology of \(\Delta\). Having noted that the homology was torsion-free, I wrote a small GAP \[8\] programme to check if it was lex-shellable: it was, and it was then easy to prove that. I also benefitted from the programme Macaulay 2 \[10\] to calculate Poincaré-Betti series of \(A_n\).

2. **Preliminaries**

This article is a direct continuation of \[18\], from which we recall some definitions and basic results.
2.1. The ring of arithmetical functions with unitary convolution. Let $\mathbb{N}$ denote the non-negative integers and $\mathbb{N}^+$ the positive integers. Let $\mathbb{P}$ denote the set of primes, with $p_i \in \mathbb{P}$ the $i$'th prime. Let $\mathbb{PP}$ denote the set of prime powers. $\mathcal{A}$ denotes the set of arithmetical functions, i.e. functions $\mathbb{N}^+ \to \mathbb{C}$. It is a $\mathbb{C}$-vector space under point-wise addition and multiplication by scalars, and it has a natural topology given by the norm

$$|f| = \frac{1}{\min \text{supp}(f)},$$

where

$$\text{supp}(f) = \{ k \in \mathbb{N}^+ | f(k) \neq 0 \}.$$

$\mathcal{A}$ becomes an associative, commutative $\mathbb{C}$-algebra under unitary convolution

$$(f \oplus g)(n) = \sum_{d \mid |n} f(d)g(n/d) = \sum_{d \oplus m = n} f(d)g(m), \quad (1)$$

where the unitary multiplication for positive integers is defined by

$$d \oplus m = \begin{cases} \quad dm & \text{if } \gcd(d, m) = 1 \\ \quad 0 & \text{otherwise} \end{cases} \quad (2)$$

and where we write $d \mid |n$ (or sometimes $d \leq \oplus n$) when $d$ is a unitary divisor of $n$, i.e. when $n = d \oplus m$ for some $m$. For any $k \in \mathbb{N}^+$, $e_k$ denotes the characteristic function on $\{k\}$. Then $e_1$ is the multiplicative identity, the set of all $e_k$ is a Schauder basis for $\mathcal{A}$, and

$$e_a \oplus e_b = e_{a \oplus b} = \begin{cases} e_{ab} & \text{if } \gcd(a, b) = 1 \\ \quad 0 & \text{otherwise} \end{cases}$$

so $\{ e_k | k \in \mathbb{PP} \}$ generates a dense subalgebra of $\mathcal{A}$.

Let $Y = \{ y_{i,j} | i, j \in \mathbb{N}^+ \}$ be a doubly infinite set of indeterminates, and let $[Y]$ be the free abelian monoid on $Y$. Let $\mathcal{M} \subset [Y]$ consist of those monomials in the $y_{i,j}$’s that are separated, i.e. can be written $y_{i_1,j_1} \cdots y_{i_r,j_r}$ with $i_1 < i_2 < \cdots < i_r$. Then $\mathcal{M}$ can be regarded as a monoid-with-zero, with the multiplication

$$m_1 \cdot m_2 = \begin{cases} m_1m_2 & \text{if } m_1m_2 \in \mathcal{M} \\ 0 & \text{if } m_1m_2 \notin \mathcal{M} \end{cases}$$

and

$$\Phi : \mathcal{M} \to \mathbb{N}^+$$

$$y_{i_1,j_1} \cdots y_{i_r,j_r} \mapsto p_{i_1}^{j_1} \cdots p_{i_r}^{j_r} \quad (3)$$

is a bijection which is a monoid-with-zero isomorphism, if $\mathbb{N}^+$ is regarded as a monoid-with-zero with unitary multiplication. From this
follows that
\[ \mathcal{A} \simeq \mathbb{C}[[\mathcal{M}]] \simeq \frac{\mathbb{C}[[Y]]}{J} \]  
(4)
where \( \mathbb{C}[[\mathcal{M}]] \) and \( \mathbb{C}[[Y]] \) are the generalized power series rings on \( \mathcal{M} \) and \( [Y] \), respectively (so \( \mathbb{C}[[Y]] \) is the power series ring on bi-infinitely many variables) and \( J \) is the smallest closed ideal of \( \mathbb{C}[[Y]] \) which contains all \( y_{i,j}y_{i,k} \).

2.2. General truncations. For any \( V \subseteq \mathbb{N}^+ \), \( \mathcal{A}_V \subseteq \mathcal{A} \) is the \( \mathbb{C} \)-sub vector space of functions supported on \( V \). With the modified multiplication
\[
(f \oplus g)(n) = \sum_{d \oplus_V m = n} f(d)g(m)
\]
(5)
it becomes a \( \mathbb{C} \)-algebra, but in general not a sub-algebra of \( \mathcal{A} \); it is a sub-algebra if and only if
\[
a, b \in V \implies a \oplus b \in V \cup \{0\}.
\]
If \( V \) contains all unitary divisors of its elements, then the restriction map \( \mathcal{A} \to \mathcal{A}_V \), which is always a vector space epimorphism, is an algebra epimorphism. In particular, if \( n \) is a positive integer, then the set \( [n] = \{1, 2, \ldots, n\} \subset \mathbb{N}^+ \) has this property. If we denote the kernel of the restriction map by \( \mathcal{S}_V \), then

**Theorem 2.1.** The set
\[
M_V = \{ e_k | k \not\in V, \text{ but } d \in V \text{ for all proper unitary divisors } d \text{ of } k \}
\]
(6)
form a minimal generating set of an ideal \( I_V \) whose closure is \( \mathcal{S}_V \).

2.3. Finite truncations. Now suppose that \( V \) has this property, and is finite. Put
\[
Y(V) = \{ y_{i,j} | p_j^i \in V \cap \mathbb{P}P \}
\]
(7)
From [18], we have that
\[
\mathcal{A}_V \simeq \frac{\mathbb{C}[Y(V)]}{A_V + B_V + C_V}
\]
(8)
where
\[
A_V = \langle y_{i,j}^2 | y_{i,j} \in Y(V) \rangle
\]
\[
B_V = \langle y_{i,j}y_{i,k} | y_{i,j}, y_{i,k} \in Y(V) \rangle
\]
(9)
\[
C_V = \langle y_{i_1,j_i} \cdots y_{i_r,j_r} | y_{i_1,j_i} \cdots y_{i_r,j_r} \in Y(V), i_1 < i_2 < \cdots < i_r, p_{i_1}^{j_i} \cdots p_{i_r}^{j_r} \not\in V \rangle
\]
We also have that
\[ \mathcal{A}_V \simeq \mathbb{C}[\Delta(V)] \]
(10)
where \( \Delta(V) \) is the simplicial complex on the vertex set \( V \cap \mathbb{P} \mathbb{P} \) given by
\[ \sigma = \{ p_{i_1}^{j_1}, \ldots, p_{i_r}^{j_r} \} \in \Delta(V) \iff p_{i_1}^{j_1} \cdots p_{i_r}^{j_r} \in V \]
(11)
and where \( \mathbb{C}[\Delta(V)] \) is the “Artinified Stanley-Reisner ring” \([15]\) on \( \Delta(V) \): it is the artinian commutative \( \mathbb{C} \)-algebra with a \( \mathbb{C} \)-basis
\[ \{ e_\sigma | \sigma \in \Delta(V) \}, \]
and multiplication
\[ e_\sigma e_\tau = \begin{cases} e_{\sigma \cup \tau} & \text{if } \sigma \cap \tau = \emptyset \\ 0 & \text{if } \sigma \cap \tau \neq \emptyset \end{cases} \]
(12)
Thus, as a graded \( \mathbb{C} \)-vector space \( \mathbb{C}[\Delta(V)] \) is isomorphic to \( \mathbb{C}\{\Delta(V)\} \), the indicator algebra on \( \Delta(V) \) \([1]\), and as a cyclic \( \mathbb{C}[Y(V)] \)-module it is isomorphic to the quotient of the ordinary Stanley-Reisner ring \( \mathbb{C}[\Delta(V)] \) by the ideal generated by the squares of all variables.

2.4. The \([n] \)-truncation. This article is devoted to the case \( V = [n] = \{1, 2, \ldots, n\} \), for which the ring \( \mathcal{A}_V \) and the simplicial complex \( \Delta(V) \) have some interesting properties.

3. SOME SPECIAL ARITHMETICAL FUNCTIONS

We’ll make use of the following special arithmetical functions: For a positive integer \( n \), \( \pi(n) \) is the number of primes \( \leq n \), and \( \pi'(n) \) the number of prime powers \( \leq n \). Let \( \omega(n) \) denotes the number of distinct prime factors of \( n \), and let (for \( k \leq 0 \)) \( \pi_k(n) \) be the number of positive integers \( \leq n \) with \( k \) distinct prime factors, i.e.
\[ \sum_{k=0}^{\infty} \pi_k(n) t^k = \sum_{j=1}^{n} t^{\omega(j)}. \]

By \( \ell(n) \) we mean the unique integer such that
\[ \prod_{i=1}^{\ell(n)} p_i \leq n < \prod_{i=1}^{\ell(n)+1} p_i, \]
(13)
We define \( v(1) = v(2) = -1 \), and for \( n \geq 3 \), \( v(n) \) as the unique integer such that
\[ \prod_{j=2}^{v(n)+1} p_j \leq n < \prod_{j=2}^{v(n)+2} p_j \]
(14)
It follows that \( v(n) = \ell(2n) - 2 \). The values of \( \ell(n) \) and \( v(n) \) for small \( n \) are tabulated below.
4. The structure of $Y([n])$

We now determine the structure of $Y(V)$ for $V = [n]$.

**Definition 4.1.** For all positive integers $n, i$, let

$$
\lambda_i^{[n]} = \max \{ j \mid p_i^j \leq n \}
$$

$$
\lambda^{[n]} = (\lambda_1^{[n]}, \lambda_2^{[n]}, \ldots)
$$

We may regard $\lambda^{[n]}$ as a partition, since $\lambda_1^{[n]} \geq \lambda_2^{[n]} \geq \cdots$. We have that

$$
Y([n]) = \{ y_{i,j} \in Y \mid j \leq \lambda_i^{[n]} \}
$$

Note that $\max(\{ i \mid y_{i,1} > 0 \}) = \pi'(n)$.

**Example 4.2.** If $n = 30$, then

$$
\lambda^{[30]} = (4, 3, 2, 1, 1, 1, 1, 1, 1, 1),
$$

and the variables $Y(V_{[30]})$ can be visualised as in figure 1.

**Remark 4.3.** For a fixed $i$, as $n \to \infty$ we have that asymptotically

$$
\lambda_i^{[n]} \sim \frac{\log(n)}{\log(p_i)}.
$$

Thus, for both $n$ and $i$ large we get that $\lambda_i^{[n]} \sim \frac{\log(n)}{\log(i \log(i))}$. This relation is illustrated in the graph figure 2, which shows (the start of) $\lambda^{(10^{50})}$.

5. The presentation of $\mathcal{A}_{[n]}$

Having determined the indeterminates occurring in $Y((n))$, we’ll consider the defining ideal $A_{[n]} + B_{[n]} + C_{[n]}$ of $\mathcal{A}_{[n]} \simeq \frac{C[Y([n])]\mathcal{A}_{[n]} + B_{[n]} + C_{[n]}}{A_{[n]} + B_{[n]} + C_{[n]}}$. Recall that (3) defines a bijection $\Phi$ between $\mathcal{M}$ and $\mathbb{N}^+$. 

**Theorem 5.1.** Let $V = [n]$. Then
1. The minimal generators of $C_{[n]}$ correspond under $\Phi$ with those $e_k$ for which
   - $k > n$,
   - all proper unitary divisors of $k$ are $\leq n$,
   - $k = \prod_{i=1}^{r} p_i^{a_i}$ with $r \leq \pi'(n)$ and $0 \leq a_i \leq \lambda_i^{[n]}$ for $1 \leq i \leq r$.

2. $A_{[n]} + B_{[n]} + C_{[n]}$ is a strongly $N + 1$-multi-stable monomial ideal in $\mathbb{C}[Y([n])]$. By this, we mean the following: first, let $N$ denote the number of components $> 1$ in $\lambda^{[n]}$. Group the variables $y_{i,j}$ in column $i$ into one group, for $1 \leq i \leq N$, and the remaining variables into one last group. Order the variables in each group using $\Phi$. Then for any monomial $m \in A_{[n]} + B_{[n]} + C_{[n]}$, if $y, y'$ belong to the same group, $\Phi(y) < \Phi(y')$, and $y|m$, then $m' = \frac{y'}{y} m \in A_{[n]} + B_{[n]} + C_{[n]}$.

Proof. 1. Follows from Theorem 2.1 and (8) and (9).

2. If $m \in A_{[n]} + B_{[n]}$ then it contains two variables “from the same column”. Since $y, y'$ belong to the same group, $m'$ must also contains two variables “from the same column”; thus $m' \in A_{[n]} + B_{[n]}$.

If $m$ is separated, then so is $m'$, and $\Phi(m) < \Phi(m')$ so if in addition $m \in C_{[n]}$ then $m' \in C_{[n]}$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{\(\lambda^{(10^{50})}\)}
\end{figure}

The following lemma gives a reasonably efficient way of calculating $C_{[n]}$. 

Lemma 5.2. The minimal generators of \( C_{[n]} \) are contained in the set
\[
\mathcal{M} \cap \{ y_{i,j} \Phi^{-1}(k) \mid p_k \leq n, 1 \leq k \leq n \} \setminus \Phi^{-1}(\{1, 2, \ldots, n\}) \quad (17)
\]
Proof. Take \( m = m_0 \in \mathcal{M} \cap Y([n]), \Phi(m) > n \). Thus \( m \) is a separated monomial in variables \( y_{i,j} \) with \( \Phi(y_{i,j}) \leq n \). Hence there is some \( y_{i_1,j_1} \) with \( \Phi(y_{i,j}) \leq n \) which divides \( m \), say \( m = y_{i_1;j_1}m_1, m_1 \in \mathcal{M} \cap Y([n]) \).

If \( \Phi(m_1) \leq n \) we are done, otherwise \( m_1 \), being a separated monomials in variables \( y_{i,j} \) with \( \Phi(y_{i,j}) \leq n \), can be written \( m_1 = y_{i_1;j_2}m_2 \), etc.

Example 5.3. For \( n = 10 \), \( \lambda^{(10)} = (3, 2, 1, 1) \), and
\[
Y(V_{[10]}) = \{ y_{1,1}, y_{1,2}, y_{1,3}, y_{2,1}, y_{2,2}, y_{3,1}, y_{4,1} \}
\]
\[
A_{10} = (y_{1,1}y_{1,1}, y_{1,2}y_{1,2}, y_{1,3}y_{1,3}, y_{2,1}y_{2,1}, y_{2,2}y_{2,2}, y_{3,1}y_{3,1}, y_{4,1}y_{4,1})
\]
\[
B_{10} = (y_{1,1}^2, y_{1,2}^2, y_{1,3}^2, y_{2,1}^2)
\]
\[
C_{10} = (y_{1,1}y_{2,2}, y_{1,1}y_{2,2}, y_{1,2}y_{3,1}, y_{1,2}y_{4,1}, y_{1,3}y_{3,1}, y_{1,3}y_{4,1}, y_{1,3}y_{2,2}, y_{2,2}y_{3,1}, y_{2,2}y_{4,1}, y_{2,3}y_{3,1}, y_{2,3}y_{4,1}, y_{3,1}y_{2,2}, y_{3,1}y_{4,1})
\]
where the last four generators are superfluous.

In Corollary 3.3, we show that the maximal degree of a minimal generator of \( C_{[n]} \) is \( v(n) \).

6. Properties of \( \Delta([n]) \)

As an example of \( \Delta([n]) \), \( \Delta([10]) \) is shown in Figure 3.

**Figure 3. \( \Delta([10]) \)**

6.1. Simple properties. Let \( r = \pi'(n) \). Clearly \( \Delta([n]) \) has \( r \) vertices.

Lemma 6.1. \( \Delta([n]) \) consists of some isolated vertices, and of one large component containing everything else.

Proof. If the vertex \( p^a \) is not isolated, but connected to \( q^b \), we can assume that \( p > q \). Furthermore, since \( \{p^a, q^b\} \in \Delta([n]) \), it follows that \( p^aq^b \leq n \), so \( p^aq \leq n \), hence \( p^a \) is connected to \( q \). If \( q > 2 \) then \( p > q > 2 \) so \( \{p^a, 2\} \in \Delta([n]) \). Thus \( p^a \) is connected to 2. \( \Box \)
Lemma 6.2. \( \dim \Delta([n]) = \ell(n) - 1. \)

Proof. The maximal cardinality \( s \) of \((q_1, \ldots, q_s) \in \Delta([n])\) is the maximal \( s \) such that
\[
\prod_{i=1}^{s} q_i \leq n,
\]
with \( q_i \in \mathbb{P} \) pair-wise relatively prime. Clearly, the best we can do is to take the first \( s \) prime numbers, so \( s = \ell(n) \). Hence \( \dim \Delta([n]) = s - 1 = \ell(n) - 1. \)

Lemma 6.3. The \( f \)-vector of \( \Delta([n]) \) is
\[
(f_{-1}, f_0, f_1, \ldots, f_{\ell(n)-1}) = (\pi_0(n), \pi_1(n), \pi_2(n), \ldots, \pi_{\ell(n)}(n))
\]  
(18)

Proof. We have that \( f_i \) is the number of simplicies \( \sigma \in \Delta([n]) \) of dimension \( i \), i.e. of cardinality \( i + 1 \). Such a \( \sigma = \{p_1^{a_1}, \ldots, p_{i+1}^{a_{i+1}}\} \) correspond to \( k = p_1^{a_1} \cdots p_{i+1}^{a_{i+1}} \) with \( \omega(k) = i + 1 \). There are \( \pi_{i+1}(n) \) such simplices, so \( f_i = \pi_{i+1}(n) \).

Lemma 6.4. Let \( v(n) \) be defined by (14). Then

(i) The homological degree\(^1\) of \( \Delta([n]) \) is \( v(n) \).
(ii) The maximal homological degree of all \( \Delta([n])_U \), as \( U \) ranges among the non-empty subsets of \([n]\), is \( v(n) \).

Sketch of proof. When \( n = N = \prod_{j=2}^{s} p_j \); we have that
\[
\{p_1, \ldots, p_s\} \notin \Delta([n]), \quad \text{but } \forall i : \{p_1, \ldots, \widehat{p_i}, \ldots, p_s\} \in \Delta([n])
\]
so we get \((s - 1)\)-homology. As \( n \) increases and reaches
\[
2N = \prod_{j=1}^{s} p_j,
\]
this homology is killed off, but already when
\[
n = N \frac{p_{s+1}}{p_s} = p_{s+1} \prod_{j=2}^{s-1} p_j
\]
new \( s-1 \) homology appears, since all \((s-1)\)-subsets of \( \{p_1, \ldots, p_{s-1}, p_{s+1}\} \) belong to \( \Delta([n]) \), whereas the whole set doesn’t. Filling in this homology, i.e. increasing \( n \) to \( p_1 \cdots p_{s-1} \cdot p_{s+1} \), we have introduced new homology already at
\[
n = p_2 \cdots p_{s-1} \cdot p_{s+2}
\]

\(^1\)The maximal \( i \) such that the \( i \)’th reduced homology group (with coefficients in \( \mathbb{Z} \)) of \( \Delta([n]) \) is non-zero.
(use the fact that there is a prime number between \( q \) and \( 2q \) for all \( q \), and so on, until we reach
\[
n = \prod_{j=2}^{s+1} p_j,
\]
where \( s \)-homology occurs.

The asymptotic growth of \( \ell(n) \) as \( n \to \infty \) is very slow, as the following lemma shows. Thus \( \dim \Delta([n]) = \ell(n) \) and \( v(n) = \ell(2n) - 2 \), the homological degree, grows very slowly with \( n \).

**Lemma 6.5.** There are positive real constants \( A, B, C, D \) such that for all \( n \),
\[
A \frac{\log(n)}{W(A \log(n))} < \ell(n) < B \frac{\log(n)}{W(B \log(n))}
\]
where \( W(z) \) denotes the real-valued principal branch of the Lambert \( W \)-function, defined as the root of \( W(z) \exp(W(z)) = z \).

**Proof.** There are positive real constants\(^2\) \( A_1, A_2 \) such that
\[
A_1 x < \sum_{p \leq x} \log(p) < A_2 x,
\]
hence
\[
A_1 p_n < \sum_{i=1}^{n} \log(p_i) < \sum_{i=1}^{n+1} \log(p_i) < A_2 p_{n+1}.
\]

There are constants\(^3\) \( B_1, B_2 \) such that
\[
B_1 n \log(n) < p_n < p_{n+1} < B_2 (n + 1) \log(n + 1).
\]

Put \( m = \ell(n) \). Then
\[
\sum_{i=1}^{m} \log(p_i) \leq \log(n) < \sum_{i=1}^{m+1} \log(p_i),
\]
so
\[
A_1 B_1 m \log(m) < \log(n) < A_2 B_2 (m + 1) \log(m + 1).
\]

We claim that
\[
C m \log(m) = \log(n)
\]
has the solution
\[
m = \frac{\log(n)}{CW\left(\frac{\log(n)}{C}\right)}.
\]

From this claim, the assertion follows by monotonicity.

\(^2\)See [11]
\(^3\)See [11]
Putting \( \log(n) = z \), \( \log(m) = a(z) \), (24) becomes
\[
Ce^{a(z)}a(z) = z,
\]
which has a solution \( a(z) = W(z/C) \). Hence
\[
m = \exp(a(z)) = \exp(W(z/C)) = \frac{z/C}{W(z/C)} = \frac{\log n}{C W(\log n/C)}.
\]
\[\square\]

6.2. The \( h \)-vector. Using a result by Fröberg, we have that
\[
\mathbb{C}[\Delta([n])](t) = \mathbb{C}[\Delta(n)](\frac{t}{1-t}) = \frac{h_0 + h_1 t + h_2 t^2 + \cdots + h_{\ell(n)} t^{\ell(n)}}{(1-t)^{\ell(n)}}
\]
(21)
where \( h_0 + h_1 + \cdots + h_{\ell(n)} \neq 1 \). The vector \((h_0, \ldots, h_{\ell(n)})\) is called the \( h \)-vector of \( \Delta([n]) \). It can be expressed in terms of the \( f \)-vector as follows (see [14]).

**Lemma 6.6.** The \( h \)-vector of \( \Delta([n]) \) is
\[
(h_0, h_1, \ldots, h_{\ell(n)})
\]
(22)
where
\[
h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{\ell(n) - i}{k - i} f_{i-1}
\]
(23)
\[
= \sum_{i=0}^{k} (-1)^{k-i} \binom{\ell(n) - i}{k - i} c_{n,i}
\]
with the convention that \( f_{-1} = c_{n,0} = 1 \).
In particular,
\[
h_0 = 1 \]
\[
h_1 = -\ell(n) + \pi'(n) \]
\[
h_2 = \left( \frac{\ell(n)}{2} \right) - (\ell(n) - 1)\pi'(n) + c_{n,2}
\]
(24)

**Example 6.7.** Let \( \Delta = \Delta([10]) \). Then \( \Delta \) is 2-dimensional, looks like Figure 3, and has \( f \)-vector \((f_{-1}, f_0, f_2) = (1, 7, 2)\) and \( h \)-vector \((h_0, h_1, h_2) = (1, 5, -4)\). Furthermore
\[
\mathbb{C}[\Delta](t) = 1 + 7t + 2t^2,
\]
and
\[
\mathbb{C}[\Delta](t) = 1 + \frac{t}{1-t} + 2 \frac{t^2}{(1-t)^2} = \frac{1+5t-4t^2}{(1-t)^2}.
\]
6.2.1. The $h_2$ coefficient. Clearly, $h_1 > 0$ for $n > 2$. Furthermore, we have (see [20])

$$
\pi_k(x) \sim \frac{x}{\log x} \frac{(\log_2 x)^{k-1}}{(k-1)!} \quad x \to \infty
$$

Thus we can write (somewhat sloppily)

$$
h_2 = \binom{\ell(n)}{2} - (\ell(n) - 1)\pi(n) + c_{n,2}
\approx \left( \frac{C}{W(C \log(n))} \right)^2 - \left( \frac{C}{W(C \log(n))} \right) \frac{n}{\log n} + \frac{n}{\log n} \log_2 n
= -\frac{Cn}{W(C \log(n))} + \frac{C^2 (\log(n))^2}{W(C \log(n))^2} + \frac{n \log 2}{(\log(n))^2}
$$

We claim that $h_2 < 0$ for large $n$, i.e. that

$$
n > \frac{C (\log(n))^2}{W(C \log(n))} + \frac{n \log 2}{C} \frac{W(C \log(n))}{(\log(n))^2}
$$

for large $n$. Since $n \gg C^2 (\log(n))^2$, we need only show that

$$
\frac{W(C \log(n))}{(\log(n))^2} \to 0 \quad \text{as } n \to \infty
$$

However, since $\ell(n) \to \infty$ as $n \to \infty$, ([19]) shows that

$$
\frac{W(C \log(n))}{C \log n} \to 0 \quad \text{as } n \to \infty,
$$

hence ([20]) follows.

6.3. Related questions. We display below how $h_2$ varies with $n$. Note that $h_2$ has local maxima when $\ell(n) < \ell(n+1)$, i.e. when $n$ is of the form $n = -1 + \prod_{k=1}^{j} p_k$. Is the $h_2$ coefficient negative for all $n$?
One can also ask: what is the sign (and magnitude) of the \( h_k \) coefficient, for very large \( n \)?

### 6.4. Shellability

Recall that the maximal (with respect to inclusion) faces of a simplicial complex \( \Delta \) are called **facets**, and that a simplicial complex is **pure** if all its facets have the same dimension (which is then also the dimension of the simplicial complex itself). For a face \( \sigma \in \Delta \), we let \( \sigma = 2\sigma \), the set of all subsets of \( \sigma \), including \( \sigma \) itself, and the empty set.

Björner and Wachs \cite{2,3} defines \( \Delta \) to be **shellable** if its facets can be arranged in linear order \( F_1, F_2, \ldots, F_t \) in such a way that the subcomplex \( \bigcup_{i=1}^{k-1} F_i \cap \overline{F_k} \) is pure and of dimension \( \dim F_k - 1 \) for all \( k = 2, \ldots, t \). They proved the following \cite[Lemma 2.3]{2}:

**Lemma 6.8** (Björner-Wachs). An order \( F_1, F_2, \ldots, F_t \) of the facets of \( \Delta \) is a shelling if and only if for every \( i \) and \( k \) with \( 1 \leq i \leq k \) there is a \( j \) with \( 1 \leq j < k \) and an \( x \in F_k \) such that \( F_i \cap F_k \subseteq F_j \cap F_k = F_k \setminus \{x\} \).

We define the **lexicographic order** on finite subsets of \( \mathbb{PP} \) in the following way: if

\[
\sigma = a_1, \ldots, a_r \subset \mathbb{PP} \quad \text{with } 1 \leq a_1 < a_2 < \cdots < a_r,
\]

\[
\tau = \{b_1, \ldots, b_s\} \in \Delta([n]), \quad \text{with } 1 \leq b_1 < b_2 < \cdots < b_s,
\]
then \( \sigma \geq_{\text{lex}} \tau \) if either \( \tau = \emptyset \) or \( a_1 > b_1 \) or \( \sigma \setminus \{a_1\} \geq_{\text{lex}} \tau \setminus \{b_1\} \). Note that \( \geq_{\text{lex}} \) is a Boolean term-order in the sense of Maclagan [13], so
\[
\sigma \geq_{\text{lex}} \tau \implies \sigma \cup \{w\} \geq_{\text{lex}} \tau \cup \{w\} \geq_{\text{lex}} \tau.
\]

(27)

**Theorem 6.9.** \( \Delta([n]) \) is shellable; the lexicographic order on the facets is a shelling order.

**Proof.** Let \( F_1, \ldots, F_t \) be the facets in \( \Delta([n]) \) ordered lexicographically. Using the previous lemma, we’ll show that this is a shelling. So pick \( 1 \leq i < k \leq t \), and let
\[
\begin{align*}
F_i \cap F_k &= \{a_1, \ldots, a_r\}, \quad 1 \leq a_1 < a_2 < \cdots < a_r \leq n \\
F_i &= \{a_1, \ldots, a_r\} \cup \{b_1, \ldots, b_t\}, \quad 1 \leq b_1 < b_2 < \cdots < b_t \leq n \\
F_k &= \{a_1, \ldots, a_r\} \cup \{c_1, \ldots, c_s\}, \quad 1 \leq c_1 < c_2 < \cdots < c_s \leq n
\end{align*}
\]

We always have that
\[
\begin{align*}
r &= |F_i \cap F_k| < |F_k| = r + s,
\end{align*}
\]

since by definition, no facet is contain in another facet. If
\[
|F_k| - |F_i \cap F_k| = 1,
\]
then we are done, by taking \( j = i \).

So suppose that
\[
|F_k| - |F_i \cap F_k| \geq 2.
\]

This means that \( s \geq 2 \). Since \( i < k \), \( F_i >_{\text{lex}} F_k \). We distinguish two cases: some \( c_v \) is a prime power which is not a prime (case 1), or all \( c_v \)'s are prime (case 2).

**Case 1:** There is some \( c_v \) which is not a prime, thus \( c_v = p^\delta \) with \( \delta > 1 \), \( p \) a prime. Put
\[
G = (F_k \cup \{p\}) \setminus \{c_v\}.
\]

Then \( G >_{\text{lex}} F_k \), and since \( p < c_v, \prod_{u \in G} u \leq n \), so \( G \in \Delta([n]) \). Clearly,
\[
G \cap F_k = F_k \setminus \{c_v\}.
\]

Now \( G \) need not be a facet, but it is contained in one, say \( G \subseteq F_j \), and by (27) it follows that \( F_j >_{\text{lex}} F_k \), whence \( j < k \). Since
\[
F_j \cap F_k \supseteq G \cap F_k = F_k \setminus \{c_v\},
\]

and since, as noted, \( F_k \) can not be contained in another facet, we must have that
\[
F_j \cap F_k = F_k \setminus \{c_v\},
\]
as desired.

**Case 2:** Since \( F_i >_{\text{lex}} F_k \) then \( b_1 < c_1 < c_2 < \cdots < c_s \). If all \( c_1, \ldots, c_s \) are prime, then \( \gcd(b_1, c_v) = 1 \) for \( 1 \leq v \leq s \). Hence
\[
G = (F_k \cup \{b_1\}) \setminus \{c_1\} \in \Delta([n]).
\]
As before, it suffices to note that $G >_{\text{lex}} F_k$ and that $G \cap F_k = F_k \setminus \{c_1\}$ to be able to conclude that there is some $j < k$ such that $G \subseteq F_j,$

$$F_j \cap F_k = G \cap F_k = F_k \setminus \{c_1\}.$$ 

It follows that all the homology groups of $\Delta([n])$ are torsion-free.

7. Socle degree, the Gorenstein property, and symmetric Hilbert function for $A_{[n]}$

7.1. The socle of $A_{[n]}$. The following easy Lemma was proved in [18].

Lemma 7.1. Socle($A_{[n]}$) is spanned as a $C$-vector space by the set

$$\{ e_k | 1 < k \leq n, \ e_k \oplus f = 0 \text{ for all } f \text{ with } f(0) \neq 0 \} =$$

$$\{ e_k | 1 < k \leq n, \ kp > n \text{ for all } p \in \mathbb{P} \text{ such that } \gcd(k, p) = 1 \} \quad (28)$$

Furthermore, the $e_k$'s which span the socle correspond precisely to the facets (maximal faces) $\sigma \in \Delta([n]), \sigma = \{p_{i_1}^{a_1}, \ldots, p_{i_r}^{a_r}\}, k = p_{i_1}^{a_1} \cdots p_{i_r}^{a_r}$.

Theorem 7.2. Let $\dim_C \text{Socle}(A_{[n]})$ denote the vector space dimension of the socle of $A_{[n]}$. Then

$$\lim_{n \to \infty} \frac{\dim_C \text{Socle}(A_{[n]})}{n} = \frac{1}{2} + \sum_{i=1}^{\infty} \frac{1}{p_i} - \frac{1}{p_{i+1}} \prod_{j=1}^{i} p_j \approx 0.60771435951661818$$

Proof. Let $\mathbb{N}^{++} = \{ k \in \mathbb{N} | k > 1 \}$. For all $n, k \in \mathbb{N}^{+},$ put

$$I_{n,0} = \mathbb{N}^{++} \cap \left[ \frac{n}{2}, n \right]$$

$$I_{n,k} = \mathbb{N}^{++} \cap \left[ \frac{n}{p_{k+1}}, \frac{n}{p_k} \right]$$

By (28), the integers $v \in I_{n,k}$ correspond to $e_v \in \text{Socle}(A_{[n]})$ whenever $pv > n$ for all prime numbers relatively prime to $v$. When $2p_k > n,$ $I_{n,k} = \emptyset.$ For the remaining $k$'s, we have that $I_{n,k}$ contains approximately

$$\frac{n}{p_k} - \frac{n}{p_{k+1}}$$

integers. Of those integers, only those that are divisible by $p_1, \ldots, p_k$ correspond to $e_k \in \text{Socle}(A_{[n]})$. Thus, the contribution to the socle from $I_{n,k}$ is approximately

$$\frac{n}{p_k} - \frac{1}{p_{k+1}} \prod_{j=1}^{i} p_j.$$ 

Clearly, all integers in $I_{n,0}$ are in the socle, which gives a contribution of approximately $\frac{n}{2}$. Furthermore, for the interval $I_{n,k}$ to contain any
integers, it must have length $\geq 1$, i.e. $\frac{n}{p_k} - \frac{n}{p_{k+1}} \geq 1$. In particular, we must have that $\frac{n}{p_i} > 1$, that is $p_i < n$. Hence, we need only consider $\pi(n)$ such intervals. For each interval $I_{n,k}$ that does contain integers, the error

$$-1 < \left( \sum_{x \in I_{n,k} \cap \mathbb{N}} 1 \right) - \frac{n}{p_k} - \frac{n}{p_{k+1}} < 1.$$  

Thus, we get that

$$\dim_{\mathbb{C}} \text{Socle}(A_{[n]}) \approx \frac{n}{2} + \frac{n}{\prod_{j=1}^{k} p_j},$$

with an error $< \pi(n) \approx n/\log(n)$, from which (29) follows.

In [18] we defined the multiplicative syzygies of $A_V$ as the kernel $K_2(V)$ of the $\mathbb{C}$-linear map

$$A_+^\times \otimes A_+^\times \rightarrow A_+^\times$$

$$f \otimes g \mapsto fg$$

We call elements in $K_2(V)$ of the form $e_a \otimes e_b$ monomial multiplicative syzygies.

**Lemma 7.3.** The monomial multiplicative syzygies of $A_{[n]}^\times$ correspond to the lattice points

$$M([n]) = \{ (i, j) \mid 1 < i, j \leq n, ij > n \} \cup \{ (i, j) \mid 1 < i, j \leq n, \gcd(i, j) > 1 \}$$

(S2)

Socle elements correspond to an integer on the $x$-axis such that the column supported on it is contained in $M([n])$.

Almost all syzygies are monomial, in the sense that

$$\lim_{n \to \infty} \frac{|M([n])|}{\dim_{\mathbb{C}} K_2([n])} = 1$$

where $K_2([n])$ is defined as in (31).

**Proof.** The first two assertions are obvious. By elementary linear algebra we have that

$$\dim_{\mathbb{C}} K_2([n]) = (n - 1)^2 - (n - 1).$$

On the other hand,

$$|M([n])| \geq (n - 1)^2 - \int_{2}^{n} \frac{nt}{t} \geq (n - 1)^2 - n \log n,$$

so (33) follows.

Below we have plotted the monomial syzygies of $A_{[30]}$. One can see that 12 is in the socle.
7.2. The Gorenstein property for $A_{[n]}$. A graded Artinian algebra is Gorenstein if and only if the socle is 1-dimensional. A direct computation shows that $A_{[2]} \simeq \mathbb{C}[t]/(t^2)$ is Gorenstein. We note that for $n > 2$, $e_{n-1}$ and $e_n$ must both belong to the socle, which is then at least 2-dimensional, so then $A_{[n]}$ is not Gorenstein.

7.3. Symmetric Hilbert function. A Gorenstein Artinian algebra has a symmetric Hilbert function, hence $A_{[2]}(t) = 1 + t$ is symmetric. Can $A_{[n]}(t)$ be symmetric for other values of $n$? If $A_{[n]}(t)$ symmetric, then $c_{n,\ell(n)} = c_{n,0} = 1$, which can only occur when

$$\prod_{i=1}^{r} p_i \leq n < p_{r+1} \prod_{i=1}^{r-1} p_i$$

(34)

for some $r$: in this case, $\ell(n) = r$ and the only integer in the interval $[1, n]$ which is the product of $r$ primes is $\prod_{i=1}^{r} p_i$. Checking these intervals for $1 \leq r \leq 10$, we get the matches displayed below.

| $r$ | $n$ | $A_{[n]}(t)$ |
|-----|-----|-------------|
| 1   | 2   | $1 + t$     |
| 2   | 6   | $1 + 4t + t^2$ |
| 2   | 7   | $1 + 5t + t^2$ |
| 2   | 8   | $1 + 6t + t^2$ |
| 2   | 9   | $1 + 7t + t^2$ |
| 3   | 40  | $1 + 19t + 19t^2 + t^3$ |
Furthermore, we have [11, §§ 22.11] that the average order and the normal order of $\omega(n)$ is $\log \log n$. For (34) we have that

$$\log \log n < \log \log (p_{r+1} \prod_{i=1}^{r-1} p_i)$$

$$< \log r \log (p_{r+1})$$

$$= \log r + \log \log p_{r+1}$$

$$< \log r + \log ((r + 1) \log 2)$$

$$= \log r + \log (r + 1) + \log \log 2 \ll r/2$$

whenever $r$ is sufficiently large. If $A_{[n]}(t)$ were symmetric, it should be centred around $r/2$. Hence, for sufficiently large $r$, $A_{[n]}(t)$ is not symmetric. There can therefore be only a finite number of $n$ such that $A_{[n]}(t)$ is symmetric. We conjecture that the examples tabulated above are in fact all such examples.

8. Basic homological properties

8.1. $A_{[n]}$ as a cyclic $C[Y([n])]-$module. $A_{[n]}$ is an Artinian ring and a zero-dimensional module over $C[Y([n])]$, with embedding dimension $r = \pi'(n)$, and homological dimension $r$. Recall [11] that $r = \pi'(n) \approx \frac{n}{\log(n)}$. Furthermore, for the last Betti number we have that

$$\beta_r(C[Y([n])], A_{[n]}) = \dim_C \mathrm{Socle}(A_{[n]})$$

$$\approx \frac{n}{2} + \sum_{k \geq 1} \frac{n}{p_k} - \frac{n}{p_{k+1}} \prod_{j=1}^{k} p_j$$

$$\approx 0.60771435951661818n,$$

by [19, Theorem 12.4] and Theorem 7.2. In fact, the bijection

$$\text{Tor}^C_{r}(C[Y([n])], A_{[n]}, C) \simeq \mathrm{Socle}(A_{[n]})$$

is degree-preserving, so

$$\beta_{r,j}(C[Y([n])], A_{[n]}) = \dim_C \mathrm{Socle}(A_{[n]})_j.$$

For the first betti number we have

$$\beta_1(C[Y([n])], A_{[n]}) = \mu(A_{[n]} + B_{[n]} + C_{[n]}) = \mu(A_{[n]}) + \mu(B_{[n]}) + \mu(C_{[n]}),$$

(35)

the minimal number of generators of the defining ideal. Clearly

$$\mu(A_{[n]}) = r, \quad \mu(B_{[n]}) = \sum_{i=1}^{r} \binom{\lambda_{[n]}^i}{2}$$

(36)
8.2. \( A_{[n]} \) as a cyclic \( \mathbb{C}[Y([n])] \)-module. We can also consider \( A_{[n]} \) as a zero dimensional module over \( \mathbb{C}[Y([n])] \), with embedding dimension \( r \), and infinite homological dimension. We have that
\[
\beta_1(\mathbb{C}[Y([n])], A_{[n]}) = \mu(B_{[n]} + C_{[n]}) = \mu(B_{[n]}) + \mu(C_{[n]}).
\]

**Lemma 8.1.**
\[
P_{\mathbb{C}[Y([n])]}(t, u) = \text{the square-free part of } P_{\mathbb{C}[\Delta([n])]}(t, u) = t^{-1} \sum_{U \subseteq V} u_U t^{|U|} q_U
\]
where
\[
q_U = \sum_{i=0}^{|U|} t^{-i} \tilde{H}^i(\Delta([n])_U, \mathbb{C})
\]
\[
u_U = \nu_U(u_1, \ldots, u_r) = \prod_{j \in U} u_j,
\]
\[
u_U = \nu_U(u_1, \ldots, u_r) = \prod_{j \in U} \frac{u_j}{1-tu_j}
\]

**Proof.** It follows from the work of Gasharov, Peeva, and Welker \[9\] that
\[
\beta_{i,a}(\mathbb{C}[Y([n])], C[\Delta([n])]) = \dim_{\mathbb{C}} \tilde{H}^{i-2}(1, y^a)_L
\]
where \((1, y^a)_L\) is the order complex of the interval \((1, y^a)\) in the sublattice \(L\) of \(Y^*\) generated by the minimal generators of \(A_{[n]} + B_{[n]} + C_{[n]}\).

Similarly,
\[
\beta_{i,a}(\mathbb{C}[Y([n])], C[\Delta([n])]) = \dim_{\mathbb{C}} \tilde{H}^{i-2}(1, y^a)_{L'}
\]
where \((1, y^a)_{L'}\) is the order complex of the interval \((1, y^a)\) in the sublattice \(L'\) of \(Y^*\) generated by the minimal generators of \(A_{[n]} + B_{[n]} + C_{[n]}\).

Since the square-free monomials form a sublattice \(S\) of \(Y^*\), it follows that \(L' = L \cap S\), hence
\[
\beta_{i,a}(\mathbb{C}[Y([n])], C[\Delta([n])]) = \begin{cases} 0 & \text{a not square-free} \\ \beta_{i,a}(\mathbb{C}[Y([n])], C[\Delta([n])]) & \text{a square-free} \end{cases}
\]

The remaining results are immediate from the formulae in the appendix. \[\square\]

**Theorem 8.2.** The Castelnuovo-Mumford regularity of the \( \mathbb{C}[Y([n])] \)-module \( A_{[n]} \) (or, equivalently, the Castelnuovo-Mumford regularity of the \( \mathbb{C}[Y([n])] \)-module \( \mathbb{C}[\Delta([n])] \)) is \( 1 + v(n) \), where \( v(n) \) is as defined in \((14)\).
Proof. It follows from (37) and (14) that the Castelnuovo-Mumford regularity is equal to
\[ 1 + \max \left\{ i \left| \exists U : \tilde{H}^i(\Delta([n])_U, \mathbb{C}) \neq 0 \right. \right\} = 1 + v(n). \]

\[ \square \]

8.3. The Stanley-Reisner ring $\mathbb{C}[\Delta([n])]$. The Stanley-Reisner $\mathbb{C}[\Delta([n])]$ is a ring of dimension $\dim \Delta([n]) + 1 = \ell(n)$ and a $\mathbb{C}[Y([n])]$-module of homological dimension $r - 1$. From the formulaes in the appendix we get that
\[
\mu(B_{[n]}) + \mu(C_{[n]}) = \beta_1(\mathbb{C}[Y([n])], A_{[n]}) = \beta_1(\mathbb{C}[Y([n])], \mathbb{C}[\Delta([n])]) = \sum_{U \subset V} \dim \tilde{H}^{\lvert U \rvert - 2}(\Delta([n])_U, \mathbb{C}) = \sum_{i=0}^{r-2} \sum_{\lvert U \rvert = i+2} \dim \tilde{H}^i(\Delta([n])_U, \mathbb{C}) \tag{38}
\]

8.4. The Koszul property. Since $A_{[n]}$ and $B_{[n]}$ are quadratic, and we know for which $i$ the reduced simplicial homology $\tilde{H}^i(\Delta([n])_U, \mathbb{C})$ can be non-zero, we get

**Corollary 8.3.** The maximal degree of a minimal generator of $C_{[n]}$, for $n > 2$, is $\max \{2, v(n)\}$. In particular, $C_{[n]}$ is quadratic for $n < 15$. Furthermore,
\[
\mu(C_{[n]}) = \sum_{i=0}^{r-2} \sum_{\lvert U \rvert = i+2} \dim \tilde{H}^i(\Delta([n])_U, \mathbb{C}) - \sum_{i=1}^{r} \left( \lambda_{[n]}^i \right) \tag{39}
\]

**Proposition 8.4.** $A_{[n]}$ is Koszul if and only if $n < 15$.

**Proof.** $A_{[n]}$ is a monomial algebra, and is quadratic iff $n < 15$. Thus by a result of Fröberg \[7\], $A_{[n]}$ is Koszul iff $n < 15$. \[ \square \]

**Appendix A. Homological formulae for Stanley-Reisner rings and indicator algebras**

In this appendix, we assume that $\Delta$ is a simplicial complex on the finite set $W = \{1, \ldots, r\}$. For $U \subset W$, $\Delta_U = \{ \sigma \in \Delta | \sigma \subset U \}$; it is a simplicial complex on $U$. We denote by $\tilde{H}^i(\Delta_U; \mathbb{C})$ the reduced simplicial homology. Note that when $\emptyset \in \Delta$, $\tilde{H}^{-1}(\Delta_{\emptyset}; \mathbb{C}) \simeq \mathbb{C}$.

We write $S = \mathbb{C}[x_1, \ldots, x_r]$, $\bar{S} = \mathbb{C}[x_1, \ldots, x_r] / (x_1^2, \ldots, x_r^2)$, and $E$ for the exterior algebra on the vector space of linear forms in $S$. Then $\mathbb{C}[\Delta]$ is a cyclic $S$-module, $\mathbb{C}[\bar{\Delta}]$ a cyclic $\bar{S}$-module, and $\mathbb{C}\{\Delta\}$ a cyclic $E$-module. $C$ is a module over all these rings.
Theorem A.1 (Hochster, [12]). Let $\beta_i$ denote the $i$'th Betti number of $\mathbb{C}[\Delta]$ (in a minimal free resolution of $\mathbb{C}[\Delta]$ as an $S$-module), and let $\beta_{i,\alpha}$ denote the corresponding multi-graded Betti number. Then

$$\beta_{i,\alpha} = \begin{cases} 0 & \text{if } \alpha \text{ is not square-free,} \\ \dim \mathbb{C} \tilde{H}^{[\alpha]-i-1}(\Delta_U; \mathbb{C}) & \text{if } \alpha \text{ is square-free with } \text{supp}(\alpha) = U \end{cases}$$

Thus the Poincaré-Betti series is

$$P_{C[\Delta]}(t, u) = \sum_{U \subset W} \prod_{j \in U} u_j \sum_{i=-\infty}^{\infty} t^i \dim \mathbb{C} \tilde{H}^{[U]-i-1}(\Delta_U; \mathbb{C})$$

Note that $\tilde{H}^j(\Delta_U; \mathbb{C}) = 0$ for $j < -1, j \geq n - 1$, so the above sum is finite.

Theorem A.2 (Aramova-Herzog-Hibi [1]). Let $\beta_i$ denote the $i$'th Betti number of $\mathbb{C}\{\Delta\}$ (in a minimal free resolution of $\mathbb{C}\{\Delta\}$ as an $E$-module), and let $\beta_{i,\alpha}$ denote the corresponding multi-graded Betti number. Then

$$\beta_{i,\alpha} = \dim \mathbb{C} \tilde{H}^{[\alpha]-i-1}(\Delta_U; \mathbb{C}), \quad \text{where } U = \text{supp}(\alpha)$$

Corollary A.3. In the above situation,

$$\beta_i = \sum_{U \subset W} \sum_{\ell = -1}^{\lfloor |U| - 1 \rfloor} \left( \ell + i \right) \dim \mathbb{C} \tilde{H}^{[U]-i-1}(\Delta_U; \mathbb{C})$$

Lemma A.4 (Sköldberg [13]).

$$P_{E}(t, u_1, \ldots, u_r) = P_{S}[\Delta](t, \frac{u_i}{1 - tu_1}, \ldots, \frac{u_r}{1 - tu_r})$$

Corollary A.5.

$$P_{E}(t, u) = \sum_{U \subset W} \prod_{j \in U} \frac{u_j}{1 - tu_j} \sum_{i=-\infty}^{\infty} t^i \dim \mathbb{C} \tilde{H}^{[U]-i-1}(\Delta_U; \mathbb{C})$$

Definition A.6. For $U \subset W$, we introduce the notation

$$u_U = u_U(u_1, \ldots, u_r) = \prod_{j \in U} u_j,$$

$$v_U = v_U(u_1, \ldots, u_r) = \prod_{j \in U} \frac{u_j}{1 - tu_j},$$

$$p_U = \sum_{i=-1}^{\infty} t^i \tilde{H}^i(\Delta_U, \mathbb{C})$$
Corollary A.7.

\begin{align*}
P_S^{C[\Delta]}(t, u) &= t^{-1} \sum_{U \subset W} u_U t^{|U|} \mathcal{P}_U \\
E^{C[\Delta]}(t, u) &= t^{-1} \sum_{U \subset W} v_U t^{|U|} \mathcal{P}_U 
\end{align*}

Lemma A.8 (Sköldberg, [15]).

\begin{align*}
P_S^{C[\Delta]}(t, u) &= P_E^{C[\Delta]}(t, u) \\
P_C^{C[\Delta]}(t, u) &= P_{C'[\Delta]}(t, u) 
\end{align*}

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