A BSDE Approach to Stochastic Differential Games Involving Impulse Controls and HJBI Equation

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Abstract This paper focuses on zero-sum stochastic differential games in the framework of forward-backward stochastic differential equations on a finite time horizon with both players adopting impulse controls. By means of BSDE methods, in particular that of the notion from Peng’s stochastic backward semigroups, the authors prove a dynamic programming principle for both the upper and the lower value functions of the game. The upper and the lower value functions are then shown to be the unique viscosity solutions of the Hamilton-Jacobi-Bellman-Isaacs equations with a double-obstacle. As a consequence, the uniqueness implies that the upper and lower value functions coincide and the game admits a value.

Keywords Dynamic programming principle (DPP), forward-backward stochastic differential equations (FBSDEs), Hamilton-Jacobi-Bellman-Isaacs (HJBI), impulse control, stochastic differential games, value function, viscosity solution.

1 Introduction

Fleming and Souganidis[1] first investigated two-player zero-sum stochastic differential games as a pioneering work in a rigorous manner and proved that the lower and the upper value functions of such games fulfill the dynamic programming principle shown to be the unique viscosity solutions of the associated HJBI equations and coincide under the Isaacs condition. This work developed the former results on differential games by Isaacs[2], Elliott and Kalton[3], Friedman[4], Evans and Souganidis[5] from the purely deterministic into the stochastic framework and has made a huge progress in the field of stochastic differential games. There are many works which extend the Fleming and Souganidis approach into new contexts. For instance, Buckdahn, et al.[6] proved the existence of Nash equilibrium points for stochastic nonzero-sum differential...
games and characterize them. Meanwhile, the theory of backward stochastic differential equations (BSDE) has been used to study stochastic differential games. In this direction, the reader can see Hamadène and Lepeltier\textsuperscript{[7]} and Hamadène, et al.\textsuperscript{[8]}. In particular, Buckdahn and Li\textsuperscript{[9]} developed the findings obtained in [7, 8] and generalized the framework in [1] to BSDE. Besides, the celebrated Pontryagin’s maximum principle for stochastic differential games within the framework of BSDE can be found in Wang and Yu\textsuperscript{[10]} and Wang, et al.\textsuperscript{[11]}. The related topic in this fields see Chen and Wu\textsuperscript{[20, 21]}. For nonzero sum impulse control games, see [22].

Different from continuous control, impulse control is also an interesting topic in stochastic control theory. There are three approaches to exploit it. For functional analysis methods, see Cosso and Bensoussan and Lions\textsuperscript{[14]} for direct probabilistic methods, see Robin\textsuperscript{[15]}. For viscosity solution approaches concerning the study of impulse control, see Lenhart\textsuperscript{[16]}, Tang and Yong\textsuperscript{[17]}, Yong\textsuperscript{[18]} and Kharroubi, et al.\textsuperscript{[19]}, etc. Particularly, impulse control has been found as a useful tool for realistic models in mathematical finance, for instance, transaction costs and liquidity risk (cf. see [20, 21]). For nonzero sum impulse control games, see [22].

Cosso\textsuperscript{[23]} El Asri and Mazid\textsuperscript{[24]} studied a two-player zero-sum stochastic differential game, with both players adopting impulse controls on a finite time horizon of the following type:

\begin{equation}
X^x_s = x + \int_t^s b \left( s, X^x_s \right) ds + \int_t^s \sigma \left( s, X^x_s \right) dW_s + \sum_{l \geq 1} \xi_l \mathbf{1}_{[\tau_l, T]} (s) \prod_{l \geq 1} \mathbf{1}_{(\tau_l \neq \rho_l)},
\end{equation}

for all \( s \in [t, T] \), \( P \)-a.s., with \( X_{t-} = x \), on some filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where \( b : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n \), \( \sigma (\cdot, \cdot) : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d} \) are given deterministic functions, \( (W_s)_{s \geq 0} \) is an \( d \)-dimensional Brownian motion, \((x, t)\) are initial time and state. \( u = \sum_{m \geq 1} \xi_m \mathbf{1}_{[\tau_m, T]} \) and \( v = \sum_{l \geq 1} \eta_l \mathbf{1}_{[\rho_l, T]} \) are the impulse controls of player I and player II, respectively. The infinite product \( \prod_{l \geq 1} \mathbf{1}_{(\tau_l \neq \rho_l)} \) has the following meaning: Whenever the two players act together on the system at the same time, we take into account only the action of player II.

The gain functional for player I (resp., cost functional for player II) of the stochastic differential game is given by

\begin{equation}
J (t, x; u, v) = \mathbb{E} \left[ \int_t^T f \left( s, X^x_s \right) ds - \sum_{m \geq 1} c (\tau_m, \xi_m) \mathbf{1}_{(\tau_m \leq T)} (s) \prod_{l \geq 1} \mathbf{1}_{(\tau_l \neq \rho_l)} + \sum_{l \geq 1} \chi (\rho_l, \eta_l) \mathbf{1}_{(\rho_l \leq T)} + g \left( X^x_T \right) \right],
\end{equation}

where \( f : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R} \) and \( g : \mathbb{R}^n \rightarrow \mathbb{R} \) are two given deterministic functions, \( f \) denoting the running function and \( g \) the payoff. The function \( c \) is the cost function for player I and the gain function for player II, representing that when player I performs an action he/she has to pay a cost, resulting in a gain for player II. Analogously, \( \chi \) is the cost function for player II and the gain function for player I. Cosso\textsuperscript{[23]} El Asri and Mazid\textsuperscript{[24]} (under weak assumptions) have shown that the upper and lower value functions coincide and the game admits a value.
The theory of BSDE can be traced back to Bismut \[25\] who studied linear BSDE motivated by stochastic control problems. Pardoux and Peng \[26\] proved the well-posedness for nonlinear BSDE. Subsequently, Duffie and Epstein \[27\] introduced the notion of recursive utilities in continuous time, which is actually a type of BSDE where the generator \(f\) is independent of \(z\). Then, El Karoui, et al. \[28\] extended the recursive utility to the case where \(f\) contains \(z\). The term \(z\) can be interpreted as an ambiguity aversion term in the market (see Chen and Epstein \[29\]). Particularly, the celebrated Black-Scholes formula indeed provided an effective way of representing the option price (which is the solution to a kind of linear BSDE) through the solution of the Black-Scholes equation. Since then, BSDE have been extensively studied and used in the areas of applied probability and optimal stochastic controls, particularly in financial engineering (cf. see \[28\]).

In our present work, employing BSDE methods, in particular, the notion of stochastic backward semigroups (Peng \[30\]), allows us to prove the dynamic programming principle for the upper and lower value functions of the game, with both players adopting impulse controls on a finite time horizon, and to derive from it with the help of Peng’s method (similar to \[30, 31\]) the associated HJBI equations with a double-obstacle. To the best of our knowledge, this is the first work studying impulse control games via BSDE.

Consider

\[
Y_t^{x,u,v} = \Phi(X_T^{x,u,v}) + \int_s^T f(r, X_r^{x,u,v}, Y_r^{x,u,v}, Z_r^{x,u,v}) \, dr - \int_s^T Z_r^{x,u,v} \, dW_r - \sum_{m \geq 1} c(\tau_m, \xi_m) 1_{\{\tau_m \leq T\}} \prod_{l \geq 1} 1_{\{\tau_l \neq \rho_l\}} + \sum_{l \geq 1} \chi(\rho_l, \eta_l) 1_{\{\rho_l \leq T\}}, \tag{3}
\]

where \(X_t^{x,u,v}\) is defined in (1). The existence and uniqueness of BSDE (3) under certain conditions can be guaranteed in the next section. We are interested in studying two-player zero-sum stochastic differential game, with both players adopting impulse controls on a finite time horizon driven by FBSDEs (1)–(3). Compared with above literature, our paper has several new features. The novelty of the formulation and the contribution in this paper can be stated as follows:

1) First, in the framework of BSDE, the terminal condition will turn out to be a traditional \(\Phi(X_T^{x,u,v})\) plus gains functions with impulses controls. This new trait makes the backward semigroup valid and avoids the Itô’s formula with jumps.

2) Second, in Cosso \[23\], El Asri and Mazid \[24\], the cost functional is defined by (2) via linear expectation. Our paper considers a more general running cost functional, which implies that the cost functionals will be supported by a BSDE, which in fact defines a nonlinear expectation.

3) At last, as a response to one of Cosso’s closing comments (Cosso \[23\], 2013): “We could apply backward stochastic differential equations methods to provide a probabilistic representation, known as the nonlinear Feynman-Kac formula, for the value function of the game”, our paper aims to further this perfect theory in the framework of BSDE, with more rigorous proofs and new techniques introduced.

The rest of this paper is organized as follows: After some preliminaries and notations in...
Section 2, we devote Section 3 to study the regularity properties of the upper and lower value functions. Moreover, we prove the dynamic programming principle for the stochastic differential game with some corollaries and generalizations, which are useful in proving that the two value functions are viscosity solutions to the HJBI equation in Section 4. Furthermore, under certain assumptions, we establish the comparison theorem for the HJBI equation, from which one may deduce that the game admits a value. Finally, in Section 5, we conclude and schedule possible generalizations in future work. Some proofs can be found in Appendix.

2 Preliminaries and Notations

Throughout this paper, we denote by $\mathbb{R}^n$ the space of $n$-dimensional Euclidean space, by $\mathbb{R}^{n \times d}$ the space the matrices with order $n \times d$. The probability space is the classical Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$, and the Brownian motion $W$ will be the coordinate process on $\Omega$. Precisely: $\Omega$ is the set of continuous functions from $[0, T]$ to $\mathbb{R}^d$ starting from 0 ($\Omega = C_0([0, T]; \mathbb{R}^d)$), $\mathcal{F}$ is the Borel $\sigma$-algebra over $\Omega$, completed with respect to the Wiener measure $\mathbb{P}$ on this space, and $\mathcal{F}$ denotes the coordinate process: $W_s(\omega) = \omega_s$, $s \in [0, T]$, $\omega \in \Omega$. By $\mathcal{F} = \{F_s, 0 \leq s \leq T\}$, we denote the natural filtration generated by $\{W_s\}_{0 \leq s \leq T}$ and augmented by all $\mathcal{P}$-null sets, i.e., $\mathcal{F}_s = \sigma \{W_r, r \leq s\} \vee \mathcal{N}_P$, $s \in [t, T]$, where $\mathcal{N}_P$ is the set of all $\mathcal{P}$-null subsets and $T > 0$ a fixed real time horizon. For each $t > 0$, we denote by $\{\mathcal{F}_s^t, t \leq s \leq T\}$ the natural filtration of the Brownian motion $\{W_s - W_t, t \leq s \leq T\}$, augmented by $\mathcal{N}_P$. $T$ appearing in this paper as superscript denotes the transpose of a matrix. $U$ and $V$ are two convex cones of $\mathbb{R}^n$ with $U \subset V$. In what follows, $C$ represents a generic constant, which can be different from line to line.

Now, we give the following definition.

Definition 2.1 An impulse control $u = \sum_{m \geq 1} \xi_m \mathbf{1}_{[\tau_m, T]}$ for player I (resp. $v = \sum_{l \geq 1} \eta_l \mathbf{1}_{[\rho_l, T]}$) on $[t, T]$ is such that

1) $(\tau_m)_m$ (resp., $(\rho_l)_l$), the action time, is a nondecreasing sequence of $\mathcal{F}$-stopping time, valued in $[t, T] \cup \{+\infty\}$.

2) $(\xi_m)_m$ (resp., $(\eta_l)_l$), the actions, is a sequence of $U$-valued (resp., $V$-valued) random variable, where each $\xi_m$ (resp., $\eta_l$) is $\mathcal{F}_{\tau_m}$-measurable (resp., $\mathcal{F}_{\rho_l}$-measurable).

Remark 2.2 Let $D([0, T]; \mathbb{R}^m)$ be the space of all functions $\xi : [0, T] \to \mathbb{R}^m$ that are right limit with left continuous. Then, the pure jump part of $\xi$ is defined by $\xi^j(t) = \sum_{0 \leq s \leq t} \Delta \xi(s)$, and the continuous part is $\xi^c(t) = \xi(t) - \xi^j(t)$. By Lebesgue decomposition theorem that we have $\xi^c(t) = \xi^{ac}(t) + \xi^{sc}(t)$, $t \in [0, T]$, where $\xi^{ac}(t)$ is called the absolutely continuous part of $\xi$, and $\xi^{sc}$ the singularly continuous part of $\xi$. Thus, we obtain that

$$\xi(t) = \xi^{ac}(t) + \xi^{sc}(t) + \xi^j(t) \quad t \in [0, T], \text{ unique!}$$

If we assume that $\xi^{ac}(t) + \xi^{sc}(t) \equiv 0$, $t \in [0, T]$, then the singular control performs a special form of a pure jump process, so-called impulse control (see [32] for details).
We now introduce the following spaces of processes:

\[ S^2(0, T; \mathbb{R}) \triangleq \left\{ \mathbb{R}^n\text{-valued } \mathcal{F}_t\text{-adapted process } \phi(t); \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\phi_t|^2 \right] < \infty \right\}, \]

\[ \mathcal{M}^2(0, T; \mathbb{R}) \triangleq \left\{ \mathbb{R}^n\text{-valued } \mathcal{F}_t\text{-adapted process } \varphi(t); \mathbb{E} \left[ \int_0^T |\varphi_t|^2 \, dt \right] < \infty \right\}, \]

and denote \( \mathcal{N}^2[0, T] = S^2(0, T; \mathbb{R}^n) \times S^2(0, T; \mathbb{R}) \times \mathcal{M}^2(0, T; \mathbb{R}^n) \). Clearly, \( \mathcal{N}^2[0, T] \) forms a Banach space.

We assume that the following conditions hold.

(A1) The coefficients \( b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n \) and \( \sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n \) are continuous on \( [0, T] \times \mathbb{R}^n \), Lipschitz continuous in the state variable \( x \), uniformly with respect to time, and bounded on \( [0, T] \times \mathbb{R}^n \).

(A2) The coefficients \( f : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \) and \( \Phi : \mathbb{R}^n \to \mathbb{R} \) are continuous on \( [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \), Lipschitz continuous in the state variable \( (x, y, z) \), uniformly with respect to time, and bounded on \( [0, T] \times \mathbb{R}^n \).

To get a well-defined gain functional, we add the following assumption and introduce the concept of admissible impulse controls. Meanwhile, to ensure that multiple impulses occurring at the same time are suboptimal, we put (like in Cosso\(^{[23]}\)) the following:

(A3) Let cost functions \( c : [0, T] \times U \to \mathbb{R} \) and \( \chi : [0, T] \times V \to \mathbb{R} \) be measurable and \( 1/2 \) Hölder continuous in time, uniformly with respect to the other variable. Furthermore,

\[ \inf_{[0, T] \times U} c(t, \xi) > 0, \quad \inf_{[0, T] \times V} \chi(t, \eta) > 0, \quad (4) \]

and there exists a function \( h : [0, T] \to (0, +\infty) \) such that for all \( t \in [0, T] \),

\[ c(t, y_1 + z + y_2) \leq c(t, y_1) - \chi(t, z) + c(t, y_2) - h(t) \quad (5) \]

and

\[ \chi(t, z_1 + z_2) \leq \chi(t, z_1) + \chi(t, z_2) - h(t) \quad (6) \]

for \( y_1, z, y_2 \in U \) and \( z_1, z_2 \in V \). Moreover,

\[ c(t, y) \geq c(t, y) \quad \text{and} \quad \chi(t, y) \geq \chi(t, y) \quad (7) \]

for all \( t, \hat{t} \in [0, T] \) satisfying \( t \leq \hat{t}, \ y \in U \) and \( z \in V \).

**Definition 2.3** Let \( u = \sum_{m \geq 1} \xi_m 1_{[\tau_m, \tau]} \) be an impulse control on \([t, T]\), and let \( \sigma, \tau \) be two \([t, T]\)-valued \( \mathcal{F}\)-stopping times. Then we define the restriction \( u_{[\tau, \sigma]} \) of the impulse control \( u \) by

\[ u_{[\tau, \sigma]}(s) = \sum_{m \geq 1} \xi_{m, \tau}(u) + m 1_{\{\tau_{m, \tau}(u) + m \leq s \leq \sigma\}}(s), \quad \tau \leq s \leq \sigma, \quad (8) \]

where \( \mu_{\tau, \tau}(u) \) is called the number of impulses up to time \( \tau \), namely, \( \mu_{\tau, \tau}(u) := \sum_{m \geq 1} 1_{\{\tau_m \leq \tau\}} \).
Definition 2.4  An admissible impulse control $u$ for player I (resp., $v$ for player II) on $[t, T]$ is an impulse control for player I (resp., II) on $[t, T]$ with a finite average number of impulses, i.e., $\mathbb{E}[\mu_{t,T}(u)] < \infty$, resp., $\mathbb{E}[\mu_{t,T}(v)] < \infty$, in which $\mu_{t,T}(u)$ is given by (8). The set of all admissible impulse controls for player I (resp., II) on $[t, T]$ is denoted by $\mathcal{U}_{t,T}$ (resp., $\mathcal{V}_{t,T}$). We identify two impulse controls $u = \sum_{m \geq 1} \xi_m 1_{[r_m, t]}$ and $\tilde{u} = \sum_{m \geq 1} \tilde{\xi}_m 1_{[r_m, T]}$ in $\mathcal{U}_{t,T}$, and we write $u \equiv \tilde{u}$ on $[t, T]$ if $P([u = \tilde{u}, \text{ a.e. on } [t, T]]) = 1$. Similarly, we interpret $u \equiv \tilde{u}$ on $[t, T]$ in $\mathcal{V}_{t,T}$.

Finally, we have still to define the admissible strategies for the game.

Definition 2.5  A nonanticipative strategy for player I on $[t, T]$ is an impulse control for player I (resp., II) on $[t, T]$ such that for any stopping time $\tau : \Omega \to [t, T]$ and any $v_1, v_2 \in \mathcal{V}_{t,T}$ with $v_1 \equiv v_2$ on $[[t, \tau]]$, it holds that $\alpha(v_1) \equiv \alpha(v_2)$ on $[[t, \tau]]$. Nonanticipative strategies for player II on $[t, T]$, denoted by $\beta : \mathcal{U}_{t,T} \to \mathcal{V}_{t,T}$, are defined similarly. The set of all nonanticipative strategies $\alpha$ (resp., $\beta$) for player I (resp., II) on $[t, T]$ is denoted by $\mathcal{A}_{t,T}$ (resp., $\mathcal{B}_{t,T}$). (Recall that $[[s, \tau]] = \{(r, \omega) \in [0, T] \times \Omega, s \leq r \leq \tau(\omega)\}$).

Assume that (A1)–(A3) are in force, for any $u(\cdot) \times v(\cdot) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$, it is easy to check that FBSDEs (1)–(3) admit a unique $\mathcal{F}_t$-adapted strong solution denoted by the triple

$$(X^{t,x,u,v}, Y^{t,x,u,v}, Z^{t,x,u,v}) \in \mathcal{N}^2[0, T]$$

(see Pardoux and Peng [30]).

As in Peng in [30], given any impulse controls $u(\cdot) \times v(\cdot) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$, we introduce the following cost functional:

$$J(t, x; u(\cdot), v(\cdot)) = Y^{t,x,u,v}_s |_{s=t}, \quad (t, x) \in [0, T] \times \mathbb{R}^n.$$  

(9)

Under Assumptions (A1)–(A3), the gain functional $J(t, x; u(\cdot), v(\cdot))$, defined by (9) is well defined for every $(t, x) \in [t, T] \times \mathbb{R}^n$, $u \in \mathcal{U}_{t,T}$, and $v(\cdot) \in \mathcal{V}_{t,T}$. We are interested in two value functions of the stochastic differential games of the following type:

$$V^- (t, x) = \inf_{\alpha \in \mathcal{A}_{t,T}} \sup_{v \in \mathcal{V}_{t,T}} J(t, x; u(\cdot), v(\cdot)), \quad (t, x) \in [0, T] \times \mathbb{R}^n$$

(10)

and

$$V^+ (t, x) = \sup_{\beta \in \mathcal{B}_{t,T}} \inf_{u \in \mathcal{U}_{t,T}} J(t, x; u(\cdot), v(\cdot)), \quad (t, x) \in [0, T] \times \mathbb{R}^n$$

(11)

for every $(t, x) \in [0, T] \times \mathbb{R}^n$. When $V^- = V^+$, we say that the game admits a value and $V := V^- = V^+$ is called the value function of the game. Since the value functions (10) and (11) are defined by the solution of controlled BSDE (3), $V^-(V^+)$ is well-defined. Moreover, they are both bounded $\mathcal{F}_t$-measurable random variables. Nonetheless, we shall prove that $V^- (V^+)$ are even deterministic.

Note that inf and sup in this paper should be interpreted via the essential infimum and the essential supremum with respect to indexed families of random variables (see Karatzas and Shreve [33]). For reader’s convenience, we recall the notion of $\text{ess inf}$ of processes. Given a family
of real-valued random variables $\eta_\alpha$, $\alpha \in I$, a random variable $\eta$ is said to be $\text{essinf}_{\alpha \in I} \eta_\alpha$, if (i) $\eta \leq \eta_\alpha$, $P$-a.s., for any $\alpha \in I$; (ii) if there is another random variable $\xi$ such that $\xi \leq \eta_\alpha$, $P$-a.s., for any $\alpha \in I$, then $\xi \leq \eta$, $P$-a.s. The random variable $\text{esssup}_{\alpha \in I} \eta_\alpha$ can be introduced now by the relation $\text{esssup}_{\alpha \in I} \eta_\alpha = -\text{essinf}_{\alpha \in I} (-\eta_\alpha)$. Finally, recall that $\text{essinf}_{\alpha \in I} \eta_\alpha = \inf_{n \geq 1} \eta_{\alpha_n}$ for some countable family $(\alpha_n) \subset I$; $\text{esssup}_{\alpha \in I} \eta_\alpha$ has the same property. We need the following estimations for BSDE, whose proof can be seen in Proposition 3.2 of Briand, et al.[34].

Lemma 2.6 Let $(y^i, z^i)$, $i = 1, 2$, be the solution to the following

$$y^i(t) = \xi^i + \int_t^T f^i(s, y^i(s), z^i(s)) \, ds - \int_t^T z^i(s) \, dW_s,$$

(12)

where $\xi^i \in L^2(\Omega, \mathcal{F}_T, P)$ with $E\left[|\xi^i|^\beta\right] < \infty$, $f^i(s, y^i(s), z^i)$ satisfies the condition (A2), and

$$E \left[ \left( \int_0^T |f^i(s, y^i(s), z^i(s))| \, ds \right)^\beta \right] < \infty.$$

Then, for some $\beta \geq 2$, there exists a positive constant $C_\beta$ such that

$$E \left[ \sup_{0 \leq t \leq T} |y^1(t) - y^2(t)|^\beta + \left( \int_0^T |z^1(s) - z^2(s)|^2 \, ds \right)^{\frac{\beta}{2}} \right] \leq C_\beta E \left[ |\xi^1 - \xi^2|^\beta + \left( \int_0^T |f^1(s, y^1(s), z^1(s)) - f^2(s, y^2(s), z^2(s))| \, ds \right)^\beta \right].$$

Particularly, whenever putting $\xi^2 = 0$, $f^2 = 0$, one has

$$E \left[ \sup_{0 \leq t \leq T} |y^1(t)|^\beta + \left( \int_0^T |z^1(s)|^2 \, ds \right)^{\frac{\beta}{2}} \right] \leq C_\beta E \left[ |\xi^1|^\beta + \left( \int_0^T |f^1(s, 0, 0)| \, ds \right)^\beta \right].$$

We recall the following well-known comparison theorem (see Barles, et al.[35], Proposition 2.6) for BSDE.

Lemma 2.7 (Comparison theorem) Let $(y^i, z^i)$, $i = 1, 2$, be the solution to the following

$$y^i(t) = \xi^i + \int_t^T f^i(s, y^i_s, z^i_s) \, ds - \int_t^T z^i_s \, dW_s,$$

(13)

where $E\left[|\xi^i|^2\right] < \infty$, $f^i(s, y^i, z^i)$ satisfies the condition (A2), $i = 1, 2$. Under Assumption (A2), BSDE (13) admits a unique adapted solution $(y^i, z^i)$, respectively, for $i = 1, 2$. Furthermore, if (i) $\xi^1 \geq \xi^2$, a.s.; (ii) $f^1(t, y, z) \geq f^2(t, y, z)$, a.e., for any $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$. Then we have $y^1(t) \geq y^2(t)$, a.s.

3 Dynamic Programming Principle

In this section, we present the DPP for our stochastic differential games in the framework of BSDE. The following lemma announces that the values functions are deterministic, which is important to investigate the other properties of value functions.

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Lemma 3.1  Let \((t, x) \in [0, T] \times \mathbb{R}^n\). Under the assumptions \((A1)–(A3)\), we have \(V^- (t, x) = \mathbb{E}[V^- (t, x)], \) \(P\)-a.s.

The proof is displayed in Appendix.

Remark 3.2  Since \(V^- (t, x)\) coincides with its deterministic version \(\mathbb{E}[V^- (t, x)]\), we can consider \(V^- : [0, T] \times \mathbb{R}^n \to \mathbb{R}\) as a deterministic function. An analogous statement holds for the value function \(V^+\).

We shall consider the value functions obtained by no impulse controls, which is useful to prove the Hölder continuity of value functions in the sequel.

Lemma 3.3  Assume that \((A1)–(A3)\) are in force, then the lower and upper value functions are given by

\[
V^- (t, x) = \inf_{\beta \in \mathcal{B}_{t,T}} \sup_{u \in \mathcal{U}_{t,T}} J(t, x; u, \beta (u)), \quad (t, x) \in [0, T] \times \mathbb{R}^n
\]

and

\[
V^+ (t, x) = \sup_{\alpha \in \mathcal{A}_{t,T}} \inf_{v \in \mathcal{V}_{t,T}} J(t, x; \alpha (v), v), \quad (t, x) \in [0, T] \times \mathbb{R}^n,
\]

where \(\mathcal{U}_{t,T}\) and \(\mathcal{V}_{t,T}\) contain all the impulse controls in \(\mathcal{U}_{t,T}\) and \(\mathcal{V}_{t,T}\), respectively, which have no impulses at time \(t\). Similarly, \(\mathcal{A}_{t,T}\) and \(\mathcal{B}_{t,T}\) are subsets of \(\mathcal{A}_{t,T}\) and \(\mathcal{B}_{t,T}\), respectively. In particular, they contain all the nonanticipative strategies with values in \(\mathcal{U}_{t,T}\) and \(\mathcal{V}_{t,T}\), respectively.

Proof  We borrow the idea from Cosso's work\(^{[23]}\). Fix \(\varepsilon > 0\). Let \(u \in \mathcal{U}_{t,T} \setminus \mathcal{U}_{t,T}\) and \(\beta \in \mathcal{B}_{t,T} \setminus \mathcal{B}_{t,T}\). Then, let \(v := \beta (u) \in \mathcal{V}_{t,T}\) and \(\overline{\beta} (\hat{u}) = \overline{v} \in \mathcal{V}_{t,T}\) for any \(\hat{u} \in \mathcal{U}_{t,T}\) for some \(\overline{\beta} \in \mathcal{B}_{t,T}\). Hence, we have to prove that there exist \(\overline{v} \in \mathcal{V}_{t,T}\) and \(\overline{v} \in \mathcal{V}_{t,T}\) such that

\[
|J (t, x; u, v) - J (t, x; \overline{v}, \overline{v})|^2 \leq \varepsilon.
\]

We may suppose \(v := \mathcal{V}_{t,T} \setminus \mathcal{V}_{t,T}\); in the other case can be proved similarly.

At the beginning, let \(u\) and \(v\) have only a single impulse at time \(t\). Therefore, there exist two \([t, T]\)-valued \(\mathcal{F}\)-stopping times \(\tau\) and \(\rho\), with \(P (\tau = t) > 0\) and \(P (\rho = t) > 0\), such that \(u = \xi 1_{[\tau, T]} + \hat{u}\) and \(v = \tau 1_{[\tau, T]} + \hat{v}\), where \(\hat{u} = \sum_{i\geq 1} \xi_i 1_{[\tau_n, T]} \in \mathcal{U}_{t,T}\), \(\hat{v} = \sum_{i\geq 1} \eta_i 1_{[\rho_n, T]} \in \mathcal{V}_{t,T}\), \(\xi\) is an \(\mathcal{F}_{\tau}\)-measurable \(U\)-valued random variable and \(\eta\) is an \(\mathcal{F}_{\rho}\)-measurable \(V\)-valued random variable. Let us introduce the following stopping times:

\[
\tau_n = \left( \tau + \frac{1}{n} \right) 1_{[\tau, T]} + \tau 1_{[\tau, T]} \quad \text{and} \quad \rho_n = \left( \rho + \frac{1}{n} \right) 1_{[\rho, T]} + \rho 1_{[\rho, T]}
\]

Clearly, \(\tau_n \to \tau\) and \(\rho_n \to \rho\), as \(n\) approaches to infinity, \(P\)-a.s. Now we define the admissible impulse controls as follows:

\[
u_n = \xi 1_{[\tau_n, T]} + \hat{u} \in \mathcal{U}_{t,T} \quad \text{and} \quad \nu_n = \eta 1_{[\rho_n, T]} + \hat{v} \in \mathcal{V}_{t,T}.
\]
By Proposition 3.2 in [34] and Lemma 3.1, we have the following estimate:

\[
|J (t, x; u, v) - J (t, x; u, v)|^2
= |Y^t,x;u,v - Y^t,x;u_n,v_n|^2
\leq CE \left[ \Phi \left( X^t,x;u,v \right) - \Phi \left( X^t,x;u_n,v_n \right) \\
+ \sum_{l \geq 1} \chi (\rho_l, \eta_l) 1_{\{\rho_l \leq T\}} - \sum_{m \geq 1} c (\tau_m, \xi_m) 1_{\{\tau_m \leq T\}} \prod_{l \geq 1} 1_{\{\tau_m \neq \rho_l\}} \\
- \sum_{l \geq 1} \chi (\rho_l, \eta_l) 1_{\{\rho_l \leq T\}} - \sum_{m \geq 1} c (\tau_m, \xi_m) 1_{\{\tau_m \leq T\}} \prod_{l \geq 1} 1_{\{\tau_m \neq \rho_l\}} \right]^2 \\
+ \left( \int_t^T \left| f \left( s, X^t_s,x;u,v, Y^t_s,x;u,v, Z^t_s,x;u,v \right) - f \left( s, X^t_s,x;u_0,v_0, Y^t_s,x;u_0,v_0, Z^t_s,x;u_0,v_0 \right) \right| ds \right)^2 \right].
\]

Note that, for every \( s \in [t, T] \) by Burkholder-Davis-Gundy (B-D-G for short, see [36]) inequality, \( X^t,x;u_n,v_n \to X^t,x;u,v \) as \( n \to \infty \), \( P \)-a.s. Therefore, from Grönwall’s inequality and the dominated convergence theorem, there is an integer \( N \geq 1 \) such that \( |J (t, x; u, v) - J (t, x; u_n, v_n)|^2 \leq \varepsilon \). As for multiple impulses at time \( t \), one can show the same result by using the assumption (A3), which actually leads to the case of the previous one with only a single impulse at time \( t \). We thus complete the proof.

Now we prove the two values functions are bounded.

**Proposition 3.4.** Assume that the assumptions (A1)–(A3) are in force, Then the lower and upper value functions are bounded.

**Proof.** We only consider the lower value function; the other case is analogous. Let \( \varepsilon > 0 \); then, by the definition of lower value function (10), we have, for any \( (t, x) \in [0, T] \times \mathbb{R}^n \), there exists some \( \beta_t (u_0) = \sum_{l \geq 1} \eta_l (u_0) 1_{\{\rho_l \leq T\}} \in V_{t,T} \), where \( u_0 \in U_{t,T} \) denotes the control with no impulses,

\[
V^- (t, x) = \inf_{\beta \in B_{t,T}} \sup_{u \in U_{t,T}} J (t, x; u, \beta (u))
= \inf_{\beta \in B_{t,T}} \sup_{u \in U_{t,T}} Y^t,x;u,\beta(u)
= \inf_{\beta \in B_{t,T}} \sup_{u \in U_{t,T}} \mathbb{E} \left[ \Phi \left( X^t,x;u,\beta(u) \right) + \int_s^T f \left( r, X^t,s,x;u,\beta(u), Y^t,s,x;u,\beta(u), Z^t,s,x;u,\beta(u) \right) dr \\
- \int_s^T Z^t,s,x;u,\beta(u) dW_r - \sum_{m \geq 1} c (\tau_m, \xi_m) 1_{\{\tau_m \leq T\}} \prod_{l \geq 1} 1_{\{\tau_m \neq \rho_l(u)\}} \\
+ \sum_{l \geq 1} \chi (\rho_l (u), \eta_l (u)) 1_{\{\rho_l (u) \leq T\}} \right] \\
\geq Y^t,x;u_0,\beta_t(u_0) - \varepsilon
= \mathbb{E} \left[ \Phi \left( X^t_T,x;u_0,\beta_t(u_0) \right) + \int_s^T f \left( r, X^t,r,x;u_0,\beta_t(u_0), Y^t,r,x;u_0,\beta_t(u_0), Z^t,r,x;u_0,\beta_t(u_0) \right) dr \right].
\]
for the solution \( u, v \) of SDE (1). Then, obviously, for the solution \( u, v \) of SDE (1), we define
\[
\begin{align*}
G^{t,x,u,v}_{s,t+\delta} [\eta + \Theta^{u,v}_{t+\delta}] := Y^{t,x,u,v}_{s},
\end{align*}
\] for \( s \in [t, t + \delta] \) where
\[
\Theta^{u,v}_{s,x} := \sum_{l \geq 1} \chi ( \rho_l, \eta_l ) 1_{\{ \rho_l \leq s \}} - \sum_{m \geq 1} c ( \tau_m, \xi_m ) 1_{\{ \tau_m \leq s \}} \prod_{l \geq 1} 1_{\{ \tau_m \neq \rho_l \}},
\] the couple \((Y^{t,x,u,v}_{s}, Z^{t,x,u,v}_{s})_{1 \leq s \leq t+\delta}\) is the solution of the following BSDE with the time horizon \( t + \eta \):
\[
\begin{align*}
Y^{t,x,u,v}_{s} &= \eta + \Theta^{u,v}_{t+\delta} + \int_{s}^{t+\delta} f ( r, X^{t,x,u,v}_{r}, Y^{t,x,u,v}_{r}, Z^{t,x,u,v}_{r} ) dr - \int_{s}^{t+\delta} Z^{t,x,u,v}_{r} dW_{r},
\end{align*}
\] for \( s \in [t, t + \delta] \) and \( X^{t,x,u,v} \) is the solution to SDE (1). Then, obviously, for the solution \((Y^{t,x,u,v}, Z^{t,x,u,v})\) to BSDE (3), we have
\[
G^{t,x,u,v}_{t,t+\delta} [\Phi (X^{t,x,u,v}_{t}) + \Theta^{u,v}_{t+\delta}] = G^{t,x,u,v}_{t,t+\delta} [Y^{t,x,u,v}_{t+\delta} + \Theta^{u,v}_{t+\delta}].
\] Indeed,
\[
J ( t, x; u, v ) = Y^{t,x,u,v}_{t} = G^{t,x,u,v}_{t,T} [\Phi (X^{t,x,u,v}_{T}) + \Theta^{u,v}_{T}]
\]
\[
= G^{t,x,u,v}_{t,t+\delta} [Y^{t,x,u,v}_{t+\delta} + \Theta^{u,v}_{t+\delta}]
\]
\[
= G^{t,x,u,v}_{t,t+\delta} [J ( t + \delta, X^{t,x,u,v}_{t+\delta}; u, v )].
\] Now we are ready to derive the the dynamic programming principle (DPP for short), by virtue of backward semigroups introduced above, in which the impulse control can be regarded as a
terminal condition. This principle is really important tool to character the viscosity solution property of corresponding H-J-B equation (see Section 4). The main technique is combining the comparison theorem of BSDE and the construction of impulse control.

**Theorem 3.5** Suppose that the assumptions (A1)–(A3) hold. Then, the value function \( V^- \) admits the following DPP: For any \( 0 \leq t < t + \delta \leq T, \ x \in \mathbb{R}^n \),

\[
V^-(t,x) = \inf_{\beta \in B_{t,t+\delta}} \sup_{u \in \mathcal{U}_{t,t+\delta}} G^{t,x;u,v}(u) \left[ V^- \left( t + \delta, X_{t+\delta}^{t,x;u,v}(u) \right) + \Theta_{t+\delta}^{u,v}(u) \right], \ P\text{-a.e.}
\]

An analogous statement holds for the value function \( V^+ \).

**Proof** We prove only the DPP only for \( V^- \); the other case is analogous.

Put

\[
V^-_\delta(t,x) = \inf_{\beta \in B_{t,t+\delta}} \sup_{u \in \mathcal{U}_{t,t+\delta}} G^{t,x;u,v}(u) \left[ V^- \left( t + \delta, X_{t+\delta}^{t,x;u,v}(u) \right) + \Theta_{t+\delta}^{u,v}(u) \right].
\]

We proceed the proof that \( V^-_\delta(t,x) \) coincides with \( V^- (t,x) \) into the following steps.

In the first step, we shall prove \( V^-_\delta(t,x) \geq V^- (t,x) \). To this end, we have

\[
V^-_\delta(t,x) = \inf_{\beta \in B_{t,t+\delta}} \sup_{u \in \mathcal{U}_{t,t+\delta}} I_\delta(t,x,\beta),
\]

where the notation \( I_\delta(t,x,\beta) = \sup_{u \in \mathcal{U}_{t,t+\delta}} I_\delta(t,x,u,\beta(u)) \) with

\[
I_\delta(t,x,u,\beta) = G^{t,x,u,v}(u) \left[ V^- \left( t + \delta, X_{t+\delta}^{t,x;u,v}(u) \right) + \Theta_{t+\delta}^{u,v}(u) \right], \ P\text{-a.s.}
\]

and for some sequences \( \{\beta_i\}_{i \geq 1} \subset B_{t,t+\delta} \) such that \( V^-_\delta(t,x) = \inf_{i \geq 1} I_\delta(t,x,\beta_i), \ P\text{-a.s.} \). Let \( \varepsilon > 0 \) and set \( T_i := \{I_\delta(t,x,\beta_i) - \varepsilon \leq V^-_\delta(t,x)\} \in \mathcal{F}_t, \ i \geq 1 \). Construct

\[
T_i := T_i \setminus \cup_{k=1}^{i-1} T_k.
\]

Certainly, \( \{T_i\}_{i \geq 1} \) forms an \( (\Omega, \mathcal{F}) \)-partition, moreover, \( \beta^{i,1} := \sum_{i \geq 1} 1_{T_i} \beta_i \in B_{t,t+\delta} \). According the existence and uniqueness of FBSDEs (3), it follows that \( I_\delta(t,x,u,\beta^{i,1}(u)) = \sum_{i \geq 1} 1_{T_i} I_\delta(t,x,u,\beta_i(u)), \ P\text{-a.s., for each } u \in \mathcal{U}_{t,t+\delta} \).

Next, for \( \forall u \in \mathcal{U}_{t,t+\delta} \),

\[
V^-_\delta(t,x) \geq \sum_{i \geq 1} 1_{T_i} I_\delta(t,x,\beta_i) - \varepsilon
\]

\[
= \sum_{i \geq 1} 1_{T_i} I_\delta(t,x,u,\beta^{i,1}(u)) - \varepsilon
\]

\[
= G^{t,x,u,v}(u) \left[ V^- \left( t + \delta, X_{t+\delta}^{t,x;u,v}(u) \right) + \Theta_{t+\delta}^{u,v}(u) \right] - \varepsilon, \ P\text{-a.s.} \quad (17)
\]

We now focus on the time interval \([t + \delta, T]\). From the definition of \( V^-_\delta(t,x) \), we also deduce that, with help of previous idea and Lemma 3.3, for any \( y \in \mathbb{R}^n \), there exists \( \beta_y^+ \in \overline{B}_{t+\delta,T} \) for each \( u^2 \in \mathcal{U}_{t+\delta,T} \) such that

\[
V^- (t + \delta, y) \geq \sup_{u^2 \in \mathcal{U}_{t+\delta,T}} J \left( t + \delta, y, u^2, \beta_y^+(u^2) \right) - \varepsilon, \ P\text{-a.s.} \quad (18)
\]
Now consider a decomposition of \( \mathbb{R}^n \), namely, \( \sum_{i \geq 1} \mathcal{O}_i = \mathbb{R}^n \) such that \( \text{diam}(\mathcal{O}_i) \leq \varepsilon \), for each \( i \geq 1 \). Take any \( y_i \in \mathcal{O}_i \) fixed, \( i \geq 1 \) and define \( X^{t,x,u_1,\beta^{-1}(u^1)}_{t+\delta} = \sum_{i \geq 1} y_i 1 \{ X^{t,x,u_1,\beta^{-1}(u^1)}_{t+\delta} \in \mathcal{O}_i \} \).

Clearly, we always have \( \left| X_{t+\delta}^{t,x,u_1,\beta^{-1}(u^1)} - X_{t+\delta}^{t,x,u_1,\beta^{-1}(u^1)} \right| \leq \varepsilon \), almost on \( \Omega \), for each \( u^1 \in \mathcal{U}_{t,t+\delta} \). For every \( y_i \in \mathcal{O}_i \), one can seek \( \beta_{y_i}^\varepsilon \in \mathcal{B}_{t+\delta,T} \) such that (18) holds true.

We introduce the strategy \( \beta_{u^1}^{2,\varepsilon} \in \mathcal{B}_{t+\delta,T} \) as follows:

\[
\beta_{u^1}^{2,\varepsilon} := \sum_{i \geq 1} 1 \{ X^{t,x,u_1,\beta^{-1}(u^1)}_{t+\delta} \in \mathcal{O}_i \} \beta_{y_i}^\varepsilon \in \mathcal{B}_{t+\delta,T}.
\] (19)

Set \( \beta_{y}^\varepsilon (u^2) = \sum_{i \geq 1} \eta^\varepsilon_i (u^2) 1_{[\rho_i^\varepsilon (u^2), T]} \) for \( u^2 \in \mathcal{U}_{t+\delta,T} \); then

\[
\beta_{u^1}^{2,\varepsilon} (u^2) := \sum_{i \geq 1} \eta^\varepsilon_i (u^2) 1_{[\rho_i^\varepsilon (u^2), T]},
\] (20)

where

\[
\eta^\varepsilon_i (u^2) := \sum_{i \geq 1} 1 \{ X^{t,x,u_1,\beta^{-1}(u^1)}_{t+\delta} \in \mathcal{O}_i \} \eta^\varepsilon_i (u^2)
\] (21)

It is possible to define a new strategy \( \beta^\varepsilon (u) \) from \( \beta^{-1}(u^1) \in \mathcal{B}_{t,t+\delta} \) and \( \beta_{u^1}^{2,\varepsilon} (u^2) \in \mathcal{B}_{t+\delta,T} \) where \( u^1 = u_{[t,t+\delta]} \), \( u^2 = u_{(t+\delta,T)} \) (see Definition 2.3), in the following way.

Let

\[
\beta^{-1}(u^1) = \sum_{I \geq 1} \eta^\varepsilon_i (u^1) 1_{[\rho^\varepsilon_i (u^1), T]}, \quad \beta_{u^1}^{2,\varepsilon} (u^2) = \sum_{i \geq 1} \eta^\varepsilon_i (u^2) 1_{[\rho^\varepsilon_i (u^2), T]},
\] (22)

then \( \beta^\varepsilon (u) = \sum_{i \geq 1} \eta^\varepsilon_i (u) 1_{[\rho^\varepsilon_i (u), T]} \), where

\[
\eta^\varepsilon_i (u) = \eta^\varepsilon_i (u^1) 1_{I \leq \mu_{t,t+\delta}(\beta^\varepsilon(1)))} + \eta^\varepsilon_i (u^2) 1_{I > \mu_{t,t+\delta}(\beta^\varepsilon(1)))}
\] (23)

and

\[
\rho^\varepsilon_i (u) = \rho_i^\varepsilon (u^1) 1_{I \leq \mu_{t,t+\delta}(\beta^\varepsilon(1)))} + \rho_i^\varepsilon (u^2) 1_{I > \mu_{t,t+\delta}(\beta^\varepsilon(1)))},
\] (24)

where \( \mu_{t,t+\delta} \) is defined in (8).

Next we shall show that \( \beta^\varepsilon (u) \) is nonanticipating: Indeed, let \( \kappa : \Omega \to [t, T] \) be an \( \mathcal{F} \)-stopping time and \( u, u' \in \mathcal{U}_{t,T} \) be such that \( u \equiv u' \) on \( [t, \kappa] \). Decomposing \( u, u' \) into \( u_1, u_1' \in \mathcal{B}_{t,t+\delta}, u_2, u_2' \in \mathcal{B}_{t+\delta,T} \) such that \( u = u_1 \oplus u_2, u' = u_1' \oplus u_2' \) where \( u_1 \oplus u_2 \) (the same for \( u_1' \oplus u_2' \)) is defined as follows:

Let

\[
u_1 = \sum_{i \geq 1} \eta^\varepsilon_i 1_{[\rho_i^\varepsilon, T]}, \quad \nu_2 = \sum_{i \geq 1} \eta^\varepsilon_i 1_{[\rho_i^\varepsilon, T]},
\] (25)

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Thus, $\beta_f(u) = \beta_{f,1} (u^1) \oplus \beta^2_{u^2}(u^2)$ for $u^1 = u_{[t,t+\delta]}$, $u^2 = u_{[t,T]}$. We immediately have $\beta_{f,1} (u_1) = \beta_{f,1} (u'_1)$ since $u_1 = u'_1$ on $[t, \kappa \land t + \delta]$. On the other hand, $u_2 = u'_2$ on $(t + \delta, \kappa \lor t + \delta]$ and on $\{ \kappa > t + \delta \}$, we have $X^{t,x,u_1,\beta_{f,1}(u_1)}_{t+\delta} = X^{t,x,u'_1,\beta_{f,1}(u'_1)}_{t+\delta}$. This yields our desired result.

Fix $u \in U_{t,T}$ arbitrarily and decompose into $u^1 = u_{[t,t+\delta]}$, $u^2 = u_{[t+\delta,T]}$. Then, from (17) and Proposition 2.6 in [35], we obtain

$$V_\delta^-(t,x) \geq G^{t,x,u^1,\beta_{f,1}(u^1)}_{t,t+\delta} \left[ V^-(t + \delta, X^{t,x,u^1,\beta_{f,1}(u^1)}_{t+\delta}) + \Theta^{u^1,\beta_{f,1}(u^1)}_{t+\delta} \right] - \varepsilon$$

$$\geq G^{t,x,u^1,\beta_{f,1}(u^1)}_{t,t+\delta} \left[ V^-(t + \delta, X^{t,x,u^1,\beta_{f,1}(u^1)}_{t+\delta}) + \Theta^{u^1,\beta_{f,1}(u^1)}_{t+\delta} \right] - C\varepsilon$$

$$= G^{t,x,u^1,\beta_{f,1}(u^1)}_{t,t+\delta} \left[ \sum_{i \geq 1} \mathbf{1}_{\{X^{t,x,u^1,\beta_{f,1}(u^1)}_{t+\delta} \in \Theta_i\}} V^-(t + \delta, y_i) + \Theta^{u^1,\beta_{f,1}(u^1)}_{t+\delta} \right] - C\varepsilon, \text{ P-a.s.} \quad (28)$$

From (28) and Proposition 2.6 in [35], it follows

$$V_\delta^-(t,x) \geq G^{t,x,u^1,\beta_{f,1}(u^1)}_{t,t+\delta} \left[ \sum_{i \geq 1} \mathbf{1}_{\{X^{t,x,u^1,\beta_{f,1}(u^1)}_{t+\delta} \in \Theta_i\}} J \left( t + \delta, y_i, u^2, \beta_{y_i} (u^2) \right) + \Theta^{u^1,\beta_{f,1}(u^1)}_{t+\delta} \right] - C\varepsilon$$

$$= G^{t,x,u^1,\beta_{f,1}(u^1)}_{t,t+\delta} \left[ J \left( t + \delta, X^{t,x,u^1,\beta_{f,1}(u^1)}_{t+\delta}, u^2, \beta_{y_i} (u^2) \right) + \Theta^{u^1,\beta_{f,1}(u^1)}_{t+\delta} \right] - C\varepsilon$$

$$\geq G^{t,x,u^1,\beta_{f,1}(u^1)}_{t,t+\delta} \left[ J \left( t + \delta, X^{t,x,u^1,\beta_{f,1}(u^1)}_{t+\delta}, u^2, \beta_{y_i} (u^2) \right) + \Theta^{u^1,\beta_{f,1}(u^1)}_{t+\delta} \right] - C\varepsilon$$

$$= G^{t,x,u^1,\beta_{f,1}(u^1)}_{t,t+\delta} \left[ J \left( t, X^{t,x,u^1,\beta_{f,1}(u^1)}_{t+\delta}, u^2, \beta_{y_i} (u^2) \right) + \Theta^{u^1,\beta_{f,1}(u^1)}_{t+\delta} \right] - C\varepsilon$$

$$= V^{t,x,u^1,\beta_{f,1}(u^1)}_{t+\delta} - C\varepsilon, \text{ P-a.s., for every } u \in U_{t,T}.$$
Let $\beta \in \mathcal{B}_{t,T}$ be arbitrarily chosen and $u_2 \in \overline{\mathcal{U}}_{t+\delta, T}$. Define the restriction of $\beta$ to $\mathcal{U}_{t, t+\delta}$ as

$$
\beta^1 (u_1) := (u_1 \oplus u_2)_{[t, t+\delta]}, \ u_1 \in \mathcal{U}_{t, t+\delta}.
$$

The nonanticipativity property of $\beta$ indicates that $\beta^1$ is independent of the special choice of $u_2 \in \overline{\mathcal{U}}_{t+\delta, T}$. From the definition of $V^\delta_-$, we have

$$
V^\delta_- (t,x) \leq \sup_{u_1 \in \mathcal{U}_{t, t+\delta}} G^1_{t, t+\delta} (u_1) \left[ V^\delta \left( t + \delta, X_{t+\delta}^1, \beta (u_1) \right) \right], \text{ P-a.s.}
$$

Consider $\mathcal{I}_\delta (t,x, \beta^1) = \sup_{u_1 \in \mathcal{U}_{t, t+\delta}} \mathcal{I}_\delta (t,x, u_1^1, \beta^1 (u_1^1))$; then there exists a sequence $\{ u_1^1 \}_{i \geq 1} \subset \mathcal{U}_{t, t+\delta}$ such that $\mathcal{I}_\delta (t,x, \beta^1) = \sup_{i \geq 1} \mathcal{I}_\delta (t,x, u_1^i, \beta^1 (u_1^i))$, P-a.s. With the same technique as before, for any $\varepsilon > 0$, set $\overline{\mathcal{A}}_i := \{ \mathcal{I}_\delta (t,x, \beta^1) \leq \mathcal{I}_\delta (t,x,u_1^i, \beta^1 (u_1^i)) + \varepsilon \} \in \mathcal{F}_t$, $i \geq 1$. Construct $A_i := \overline{\mathcal{A}}_i \setminus (\cup_{k=1}^{i-1} \mathcal{A}_k)$. Certainly, $\{ A_i \}_{i \geq 1}$ forms an $(\Omega, \mathcal{F})$-partition, moreover, $u_1^i := \sum_{i \geq 1} 1_{A_i} u_1^i \in \mathcal{U}_{t, t+\delta}$. From the existence and uniqueness of FBSDEs (3), we deduce that

$$
\mathcal{I}_\delta (t,x, u_1^i, \beta^1 (u_1^i)) = \sum_{i \geq 1} \mathcal{I}_\delta (t,x,u_1^i, \beta^1 (u_1^i)), \text{ P-a.s.}
$$

Then,

$$
\mathcal{V}^\delta_- (t,x) \leq \mathcal{V}^\delta (t,x, \beta^1) \leq \sum_{i \geq 1} 1_{A_i} \mathcal{I}_\delta (t,x,u_1^i, \beta^1 (u_1^i)) + \varepsilon
$$

Noting that $\beta^1 (\cdot) := \beta (\cdot \oplus u_2) \in B_{t, t+\delta}$ does not depend on $u_2 \in \overline{\mathcal{U}}_{t+\delta, T}$, we can construct $\beta^2 (u_2) := (u_1^i \oplus u_2)_{[t, t+\delta]}$; for each $u_2 \in \overline{\mathcal{U}}_{t+\delta, T}$ such that $\beta^2 : \overline{\mathcal{U}}_{t+\delta, T} \rightarrow \overline{\mathcal{V}}_{t+\delta, T}$ belongs to $\mathcal{B}_{t, T}$. Therefore, from the definition of $V^\delta (t+\delta, y)$ and Lemma 3.3, we have, for any $y \in \mathbb{R}^n$,

$$
V^\delta \left( t+\delta, X_{t+\delta}^1, \beta^1 (u_1^i) \right) \leq \sup_{u_2 \in \overline{\mathcal{U}}_{t+\delta, T}} J \left( t+\delta, X_{t+\delta}^1, \beta^1 (u_1^i); u_2, \beta^2 (u_2) \right).
$$

There exists a sequence $\{ u_2^i \}_{i \geq 1} \subset \mathcal{U}_{t+\delta, T}$ such that

$$
\sup_{u_2 \in \overline{\mathcal{U}}_{t+\delta, T}} J \left( t+\delta, X_{t+\delta}^1, \beta^1 (u_1^i); u_2, \beta^2 (u_2) \right)
$$

$$
= \sup_{i \geq 1} J \left( t+\delta, X_{t+\delta}^1, \beta^1 (u_1^i); u_2^i, \beta^2 (u_2^i) \right).
$$

Then with the same technique as before, for any $\varepsilon > 0$, set

$$
\Pi_i := \left\{ \sup_{u_2 \in \overline{\mathcal{U}}_{t+\delta, T}} J \left( t+\delta, X_{t+\delta}^1, \beta^1 (u_1^i); u_2, \beta^2 (u_2) \right) \right\} \in \mathcal{F}_{t+\delta}, \ i \geq 1.
$$

Construct $\Pi_i := \Pi_i \setminus (\cup_{k=1}^{i-1} \Pi_k)$. Certainly, $\{ \Pi_i \}_{i \geq 1}$ also forms an $(\Omega, \mathcal{F})$-partition, moreover, $u_2^i := \sum_{i \geq 1} 1_{\Pi_i} u_2 \in \overline{\mathcal{U}}_{t+\delta, T}$. Then, $\beta^2 (u_2^i) = \sum_{i \geq 1} 1_{\Pi_i}$. We construct a new strategy.
\[ \beta(u^1_1 \oplus u^2_2) = \beta^1(u^1_1) \oplus \beta^2(u^2_2). \]

From the existence and uniqueness of FBSDEs (3), we have
\[
J \left( t + \delta, X_{t+\delta}^{t,x,u^1_1,\beta^1(u^1_1)}; u^2_2, \beta^2(u^2_2) \right) = Y_{t+\delta}^{t,\delta} \left( X_{t+\delta}^{t,x,u^1_1,\beta^1(u^1_1)}; u^2_2, \beta^2(u^2_2) \right)
\]
\[
= \sum_{j \geq 1} \mathbf{1}_n Y_{t+\delta}^{t,\delta} \left( X_{t+\delta}^{t,x,u^1_1,\beta^1(u^1_1)}; u^2_2, \beta^2(u^2_2) \right)
\]
\[
= \sum_{j \geq 1} \mathbf{1}_n J \left( t + \delta, X_{t+\delta}^{t,x,u^1_1,\beta^1(u^1_1)}; u^2_2, \beta^2(u^2_2) \right).
\] (29)

Therefore,
\[
V^-(t + \delta, X_{t+\delta}^{t,x,u^1_1,\beta^1(u^1_1)}) \leq \sup_{u^2 \in U_{t+\delta,T}} J \left( t + \delta, X_{t+\delta}^{t,x,u^1_1,\beta^1(u^1_1)}; u^2_2, \beta^2(u^2_2) \right)
\]
\[
\leq \sum_{j \geq 1} \mathbf{1}_n Y_{t+\delta}^{t,\delta} \left( X_{t+\delta}^{t,x,u^1_1,\beta^1(u^1_1)}; u^2_2, \beta^2(u^2_2) \right) + \varepsilon
\]
\[
= Y_{t+\delta}^{t,x,u^1_1,\beta(u^1_1)} + \varepsilon
\]
\[
= Y_{t+\delta}^{t,x,u^1_1,\beta(u^1_1)} + \varepsilon.
\] (30)

where \( u^x = u^1_1 \oplus u^2_2 \in U_{t,T}. \) Repeating the method before, from (29) and (30) and Proposition 2.6 in [35], we have
\[
V^-_\delta(t, x) \leq C_{t,x} \left( \gamma_{t,x}^{u^x,\beta(u^x)} + \Theta_{t,x}^{u^x,\beta(u^x)} \right) + C\varepsilon
\]
\[
= C_{t,x} \left( \gamma_{t,x}^{u^x,\beta(u^x)} + \Theta_{t,x}^{u^x,\beta(u^x)} \right) + C\varepsilon
\]
\[
= Y_{t,x}^{t,x,u^x,\beta(u^x)} + C\varepsilon
\]
\[
\leq \sup_{u \in U_{t,T}} Y_{t,x}^{t,x,u,\beta(u)} + C\varepsilon, \text{ P-a.s.},
\]
which holds for all \( \beta \in \mathcal{B}_{t,T}. \)
\[
V^-_\delta(t, x) \leq \inf_{\beta \in \mathcal{B}_{t,T}} \sup_{u \in U_{t,T}} Y_{t,x}^{t,x,u,\beta(u)} + C\varepsilon = V^-(t, x) + C\varepsilon.
\]

Now letting \( \varepsilon \to 0, \) we get the desired result, \( V^-(t, x) \leq V^-(t, x). \) The proof is completed.

Next, we will show that the continuity of value functions with respect to \( x \) and \( t. \) Due to the influence of Brownian motion, the value function will be Lipschitz continuity in \( x, \) but \( \frac{1}{2} \)-Hölder continuous in \( t. \)

**Proposition 3.6** Assume that the assumptions (A1)–(A3) are in force. Then the lower value function \( V^- (t, x) \) is \( \frac{1}{2} \)-Hölder continuous in \( t. \) There exists a constant \( C > 0 \) such that, for every \( (t, x) \in [0, T) \times \mathbb{R}^n \)
\[
|V^-(t, x) - V^-(t, x')| \leq C |x - x'|, \quad (31)
\]
\[
|V^-(t, x) - V^-(t', x)| \leq C |t - t'|^{\frac{1}{2}}. \quad (32)
\]
Proof The first property of the lower value function $V^-(t, x)$ which we present is an immediate consequence of Proposition 3.2 in [34].

Let $(t, x) \in [0, T] \times \mathbb{R}^n$ and $\delta > 0$ be arbitrarily given such that $0 < \delta < T - t$. Then for every $\varepsilon > 0$, according to Lemma 3.3, there exist $u_\varepsilon \in \mathcal{U}_{t,T}$ and $\beta_\varepsilon \in \mathcal{B}_{t,T}$ (This ensures no impulse on initial state) such that

$$V^-(t, x) - V^-(t + \delta, x) \leq G_{t,T}^{u_\varepsilon, \beta_\varepsilon}(\phi \left( X^T_T, \xi, \nu \right)) - G_{t,T}^{u_\varepsilon, \beta_\varepsilon}(\phi \left( X^T_T, \xi, \nu \right)) + \varepsilon, \quad (33)$$

where $\hat{u}_\varepsilon \in \mathcal{U}_{t,T}$ and $\hat{\beta}_\varepsilon \in \mathcal{B}_{t,T}$ will be determined soon. Indeed, from (14) and (15), there exist $u_\varepsilon \in \mathcal{U}_{t,T}$ and $\beta_\varepsilon \in \mathcal{B}_{t,T}$ such that

$$G_{t,T}^{u_\varepsilon, \beta_\varepsilon}(\phi \left( X^T_T, \xi, \nu \right)) \geq V^-(t, x) - \frac{\varepsilon}{2} \quad (34)$$

and

$$G_{t,T}^{u_\varepsilon, \beta_\varepsilon}(\phi \left( X^T_T, \xi, \nu \right)) \leq V^-(t + \delta, x) + \frac{\varepsilon}{2}. \quad (35)$$

From (34) and (35), one can obtain (33) easily.

We postulate that $u_\varepsilon = \sum_{m \geq 1} \xi_m^\varepsilon \mathbf{1}_{[\tau_m, \tau_m]} \in \mathcal{U}_{t,T}$; now define $\hat{u}_\varepsilon$ as $\hat{u}_\varepsilon = \sum_{\tau_m \leq t+\delta} \xi_m^\varepsilon \mathbf{1}_{[\tau_m, \tau_m]} + \sum_{\tau_m > t+\delta} \xi_m^\varepsilon \mathbf{1}_{[\tau_m, T]}$. Observe that $\hat{u}_\varepsilon$ is an impulse control constructing from $u_\varepsilon$ via gathering all the impulses in the interval $[t, t+\delta]$. To eliminate the impulses on time $t+\delta$ of player II, we define $v_\varepsilon = \beta_\varepsilon(u) = \beta_\varepsilon(\hat{u}_\varepsilon) \in \mathcal{Y}_{t+\delta,T}$ for any $u \in \mathcal{U}_{t,T}$. After this work, (33) can be written as

$$V^-(t, x) - V^-(t + \delta, x) \leq G_{t,T}^{u_\varepsilon, v_\varepsilon}(\phi \left( X^T_T, \xi, \nu \right)) - G_{t,T}^{u_\varepsilon, v_\varepsilon}(\phi \left( X^T_T, \xi, \nu \right)) + \varepsilon. \quad (36)$$

We deal with

$$G_{t,T}^{u_\varepsilon, v_\varepsilon}(\phi \left( X^T_T, \xi, \nu \right)) - G_{t,T}^{u_\varepsilon, v_\varepsilon}(\phi \left( X^T_T, \xi, \nu \right)) = \phi (T, X^T_T, \xi, \nu) - \phi (X^T_T, \xi, \nu) \int_s^T Z_r^{x, u_\varepsilon, v_\varepsilon} dW_r - \int_s^T Z_r^{x, u_\varepsilon, v_\varepsilon} dW_r \quad (37)$$

but from the conditions (5) and (7), we have

$$c \left( t + \delta, \sum_{\tau_m \leq t+\delta} \xi_m^\varepsilon \right) \leq \sum_{\tau_m \leq t+\delta} c \left( \tau_m, \xi_m^\varepsilon \right) \mathbf{1}_{t+\delta}.$$
Hence, (37) yields
\[
G_{t,T}^{r,x,u,v_\varepsilon} \left( \phi \left( X_T^{r,x,u,v_\varepsilon} \right) \right) - G_{t+\delta,T}^{r,x,\tilde{u},v_\varepsilon} \left( \phi \left( X_T^{r,x,\tilde{u},v_\varepsilon} \right) \right)
\leq \phi \left( T, X_T^{r,x,u,v_\varepsilon} \right) - \phi \left( X_T^{r,x,\tilde{u},v_\varepsilon} \right) dr + \int_s^T Z^r_s \left( X_s^{r,x,\tilde{u},v_\varepsilon}, X_s^{r,x,u,v_\varepsilon} \right) dW_r - \int_s^T Z^r_s \left( X_s^{r,x,u,v_\varepsilon}, X_s^{r,x,\tilde{u},v_\varepsilon} \right) dW_r
- \int_s^T f \left( r, X_r^{r,x,u,v_\varepsilon}, Y_r^{r,x,u,v_\varepsilon}, Z^r_r \left( X_r^{r,x,u,v_\varepsilon}, X_r^{r,x,\tilde{u},v_\varepsilon}, Z^r_r \left( X_r^{r,x,\tilde{u},v_\varepsilon}, X_r^{r,x,u,v_\varepsilon} \right) \right), Z^r_r \left( X_r^{r,x,u,v_\varepsilon}, X_r^{r,x,\tilde{u},v_\varepsilon} \right) \right) dr. 
\]  
(38)

Taking the expectation on both sides of (38) and noting Lemma 3.1, we have
\[
E \left[ \phi \left( X_T^{r,x,u,v_\varepsilon} \right) - \phi \left( X_T^{r,x,\tilde{u},v_\varepsilon} \right) dr + \int_s^T Z^r_s \left( X_s^{r,x,\tilde{u},v_\varepsilon}, X_s^{r,x,u,v_\varepsilon} \right) dW_r - \int_s^T Z^r_s \left( X_s^{r,x,u,v_\varepsilon}, X_s^{r,x,\tilde{u},v_\varepsilon} \right) dW_r
- \int_s^T f \left( r, X_r^{r,x,u,v_\varepsilon}, Y_r^{r,x,u,v_\varepsilon}, Z^r_r \left( X_r^{r,x,u,v_\varepsilon}, X_r^{r,x,\tilde{u},v_\varepsilon}, Z^r_r \left( X_r^{r,x,\tilde{u},v_\varepsilon}, X_r^{r,x,u,v_\varepsilon} \right) \right), Z^r_r \left( X_r^{r,x,u,v_\varepsilon}, X_r^{r,x,\tilde{u},v_\varepsilon} \right) \right) dr \right]. 
\]  
(39)

Set \( \widehat{\Xi}_r = \Xi_r^{r,x,u,v_\varepsilon} - \Xi_r^{r,x,\tilde{u},v_\varepsilon} \), \( r \in [t + \delta, T] \) for \( \Xi = X, Y, Z \). By B-D-G inequality and classical method, we have
\[
E \left[ \sup_{t+\delta \leq r \leq T} \left| \widehat{X}_r \right|^2 + \sup_{t+\delta \leq r \leq T} \left| \widehat{Y}_r \right|^2 + \int_s^T \left| \widehat{Z}_r \right|^2 dr \right] \leq C \left| t - t' \right|^{\frac{7}{2}}.
\]

Therefore,
\[
V^- \left( t, x \right) - V^- \left( t + \delta, x \right) \leq G_{t,T}^{r,x,u,v_\varepsilon} \left( \phi \left( X_T^{r,x,u,v_\varepsilon} \right) \right) - G_{t+\delta,T}^{r,x,\tilde{u},v_\varepsilon} \left( \phi \left( X_T^{r,x,\tilde{u},v_\varepsilon} \right) \right) \leq C \left| t - t' \right|^{\frac{7}{2}} + \varepsilon.
\]

Letting \( \varepsilon \to 0 \), we get the desired result. We thus complete the proof.

Now we are concerned on a special case of DPP, that is \( s = t \), according to the condition (A3), the multiple impulses can be neglected. It will be useful in proving that the two value functions are viscosity solutions to the associated HJBI equation and deriving the so called lower and upper obstacles. Whilst, it announces that our games problems can interpreted via optimal stopping times.

**Lemma 3.7** Assume the assumptions (A1)–(A3) are in force. Given any \((t, x) \in [0, T] \times \mathbb{R}^n \), we have
\[
V^- \left( t, x \right) = \inf_{\rho \in T_{t+\infty}^{t+\infty}} \sup_{\eta \in F_{t+\infty}^{t+\infty}} \left[ -c \left( t, \xi \right) \mathbf{1}_{\tau = t} \mathbf{1}_{\rho = +\infty} + \chi \left( t, \eta \right) \mathbf{1}_{\rho = t} + V^- \left( t, X_t^{r,x,\xi \mathbf{1}_{\tau = t}, \eta \mathbf{1}_{\rho = t}} \right) \right],
\]  
(40)
where $T_{t,+\infty}$ is the set of $\mathcal{F}$-stopping times with values in $\{t,+\infty\}$, $\tau \in T_{t,+\infty}$, $\xi \in \mathcal{F}_\tau$, $u = \xi_{[\tau,T]}$ and $\rho \in T_{t,+\infty}$, $\eta \in \mathcal{F}_\rho$, $\beta(u) = \eta_{[\rho,T]}$. An analogous statement holds for the upper value function $V^+(t,x)$.

**Proof** In Theorem 3.5, consider $V^-$ with $\delta = 0$:

$$V^-(t,x) = \inf_{\beta \in \mathcal{B}_t,T \ u \in \mathcal{U}_{t,T}} G_{t,t}^{t,u,\beta(u)} \left[ V^-(t,X_t^{t,x,u,\beta(u)}) + \sum_{l \geq 1} \chi(\rho, \eta) \textbf{1}_{\{\rho = t\}} - \sum_{m \geq 1} c(\tau_m, \xi_m) \prod_{l \geq 1} \textbf{1}_{\{\tau_m = t\}} \right].$$

Given any $u \in \mathcal{U}_{t,T}$, consider the strategy $\beta(u) = \eta_{[\rho,T]}$. Let $u = \sum_{m \geq 1} \xi_m 1_{[\rho_m,T]}$, then construct a new control $\overline{\eta} = 1_{[\tau,T]}$, where

$$\tau = \inf \left( 1 - \prod_{m \geq 1} 1_{\{\tau_m > t\}} \right) + \infty \prod_{m \geq 1} 1_{\{\tau_m > t\}} = m 1_{\{\tau_m = t\}}.$$

Apparently, $\tau \in T_{t,+\infty}$ and $\xi \in \mathcal{F}_\tau$. Meanwhile, we deduce that $X_t^{t,x,u,\beta(u)} = X_t^{t,x,\xi_{[\tau,T]},\eta_{[\rho,T]}}$, $P$-a.s. By means of (A3), it follows that

$$G_{t,t}^{t,x,u,\beta(u)} \left[ \sum_{m \geq 1} c(t, \xi_m) 1_{\{\tau_m = t\}} 1_{\{\rho = +\infty\}} + \chi(t, \eta) 1_{\{\rho = t\}} \right] \leq G_{t,t}^{t,x,u,\beta(u)} \left[ \sum_{m \geq 1} c(t, \xi_m) 1_{\{\tau_m = t\}} 1_{\{\rho = +\infty\}} + \chi(t, \eta) 1_{\{\rho = t\}} \right].$$

As a result, we have

$$V^-(t,x) \leq \inf_{\rho \in T_{t,+\infty}} \sup_{\eta \in \mathcal{F}_\rho \tau \in T_{t,+\infty}, \xi \in \mathcal{F}_\tau} \mathbb{E} \left[ - c(t, \xi) 1_{\{\tau = t\}} 1_{\{\rho = +\infty\}} + \chi(t, \eta) 1_{\{\rho = t\}} + V^- \left( t, X_t^{t,x,\xi_{[\tau,T]},\eta_{[\rho,T]}} \right) \right].$$

The reverse inequality can be proved in the analogous way. We end the proof.

In order to prove the the two value functions satisfy, in the viscosity sense, the terminal condition to the HJBI equation. We need a useful technical lemma.

**Lemma 3.8** Let the assumptions (A1)--(A3) hold. Given any $(t,x) \in [0,T] \times \mathbb{R}^n$, we have

$$V^-(t,x) = \inf_{\rho \in T_{t,+\infty}} \sup_{\eta \in \mathcal{F}_\rho \tau \in T_{t,+\infty}, \xi \in \mathcal{F}_\tau} G_{t,t}^{t,x,u,\beta(u)} \left[ - c(t, \xi) 1_{\{\tau = t\}} 1_{\{\rho = +\infty\}} + \chi(t, \eta) 1_{\{\rho = t\}} + V^- \left( t, X_t^{t,x,\xi_{[\tau,T]},\eta_{[\rho,T]}} \right) \right] + \int_t^T f(s, X_s^{t,x,u_0,v_0}, Y_s^{t,x,u_0,v_0}, Z_s^{t,x,u_0,v_0}) \, ds + \phi \left( X_T^{t,x,u_0,v_0} \right) 1_{\{T = +\infty, \rho = +\infty\}},$$

where $u_0$, $v_0$ are the controls with no impulses. An analogous statement holds for the upper value function $V^+$. 

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Proof For any $\varepsilon > 0$, from the definition of inf, there exist $\rho_{t,T}^{\varepsilon,1} \in \mathcal{T}_{t,T}$, $\eta_{t,T}^{\varepsilon,1} \in \mathcal{F}_{t,T}$ such that

the right side of (41)

\[
G_{t,t}^{t,x,u,\beta(u)} \left[ \left( -c(t,\xi)1_{\{\tau=t\}}1_{\{\rho_{t,T}^{\varepsilon,1}=+\infty\}} \right. \right. \\
\left. \left. + \chi(t,\eta_{t,T}^{\varepsilon,1})1_{\{\rho_{t,T}^{\varepsilon,1}=t\}} + V^{-} \left( t, X_{t}^{t,x,t_{1}[\tau,T],\eta_{t,T}^{\varepsilon,1}[\rho_{t,T}^{\varepsilon,1},T]} \right) \right. \right. \\
\left. \left. \left( 1 - 1_{\{\tau=+\infty,\rho_{t,T}^{\varepsilon,1}=+\infty\}} \right) \right. \right. \\
\left. \left. + \left( \int_{t}^{T} f \left( s, X_{s}^{t,x,u_{0},v_{0}}, Y_{s}^{t,x,u_{0},v_{0}}, Z_{s}^{t,x,u_{0},v_{0}} \right) ds + \phi \left( X_{T}^{t,x,u_{0},v_{0}} \right) \right) 1_{\{\tau=+\infty,\rho_{t,T}^{\varepsilon,1}=+\infty\}} \right] \right] \\
- \varepsilon.
\]

(42)

To deal with $V^{-} \left( t, X_{t}^{t,x,t_{1}[\tau,T],\eta_{t,T}^{\varepsilon,1}[\rho_{t,T}^{\varepsilon,1},T]} \right)$, let $\tilde{u} \in \mathcal{U}_{t,T}$. From Theorem 3.5, there exists a strategy $\beta_{t,T}^{\varepsilon,2} \in \mathcal{B}_{t,T}$ such that

\[
V^{-} \left( t, X_{t}^{t,x,t_{1}[\tau,T],\eta_{t,T}^{\varepsilon,1}[\rho_{t,T}^{\varepsilon,1},T]} \right) \geq \mathbb{E} \left[ J \left( t, X_{t}^{t,x,t_{1}[\tau,T],\eta_{t,T}^{\varepsilon,1}[\rho_{t,T}^{\varepsilon,1},T]}, \tilde{u}, \beta_{t,T}^{\varepsilon,2} (\tilde{u}) \right) \right] - \varepsilon.
\]

Define

\[
u_{t} = \left( \xi_{1_{\tau=t}} + v_{t}1_{\{\tau=+\infty\}} \right) t_{t} + \tilde{u}_{t} \left( 1 - 1_{\{\tau=+\infty,\rho_{t,T}^{\varepsilon,1}=+\infty\}} \right) + u_{0}1_{\{\tau=+\infty,\rho_{t,T}^{\varepsilon,1}=+\infty\}},
\]

\[
\beta_{t} = \left( \eta_{t,T}^{\varepsilon,1}1_{\{\rho_{t,T}^{\varepsilon,1}=t\}} + u_{0}1_{\{\tau=+\infty\}} \right) t_{t} + \beta_{t,T}^{\varepsilon,2} \left( 1 - 1_{\{\tau=+\infty,\rho_{t,T}^{\varepsilon,1}=+\infty\}} \right) + v_{t}1_{\{\tau=+\infty,\rho_{t,T}^{\varepsilon,1}=+\infty\}}.
\]

It is easy to check that $u_{t} \in \mathcal{U}_{t,T}$ and $\beta_{t} \in \mathcal{B}_{t,T}$. Hence, from (41), (42) and Theorem 3.5, we have

\[
\inf_{\rho \in \mathcal{U}_{t,T}} \sup_{\eta \in \mathcal{F}_{t,T}} \mathbb{E} \left[ G_{t,t}^{t,x,u,\beta(u)} \left[ \left( -c(t,\xi)1_{\{\tau=t\}}1_{\{\rho_{t,T}^{\varepsilon,1}=+\infty\}} \right. \right. \\
\left. \left. + \chi(t,\eta_{t,T}^{\varepsilon,1})1_{\{\rho_{t,T}^{\varepsilon,1}=t\}} + V^{-} \left( t, X_{t}^{t,x,t_{1}[\tau,T],\eta_{t,T}^{\varepsilon,1}[\rho_{t,T}^{\varepsilon,1},T]} \right) \right. \right. \\
\left. \left. \left( 1 - 1_{\{\tau=+\infty,\rho_{t,T}^{\varepsilon,1}=+\infty\}} \right) \right. \right. \\
\left. \left. + \left( \int_{t}^{T} f \left( s, X_{s}^{t,x,u_{0},v_{0}}, Y_{s}^{t,x,u_{0},v_{0}}, Z_{s}^{t,x,u_{0},v_{0}} \right) ds + \phi \left( X_{T}^{t,x,u_{0},v_{0}} \right) \right) 1_{\{\tau=+\infty,\rho_{t,T}^{\varepsilon,1}=+\infty\}} \right] \right] \right.
\]

\[
\geq \mathbb{E} \left[ J \left( t, X_{t}^{t,x,t_{1}[\tau,T],\eta_{t,T}^{\varepsilon,1}[\rho_{t,T}^{\varepsilon,1},T]}, \tilde{u}, \beta_{t,T}^{\varepsilon,2} (\tilde{u}) \right) \right] - 2\varepsilon.
\]
Let \( \varepsilon \to 0 \), we get
\[
V^-(t, x) \leq \inf_{\rho \in \mathcal{T}, \eta \in \mathcal{F}_\rho \tau \in \mathcal{T}, \xi \in \mathcal{F}_\tau} \sup_{u \in \mathcal{U}} G_{t,t}^{x;u,\beta(u)} \left[ \left( -c(t, \xi) 1_{\{\tau=t\}} 1_{\{\rho=+\infty\}} + \chi(t, \eta) 1_{\{\rho=t\}} + V^-(t, X^t_{t,x;u,\xi}, Y^t_{t,x;u,\xi}, Z^t_{t,x;u,\xi}) \right) - \left( 1 - 1_{\{\tau=+\infty, \rho=+\infty\}} \right) + \left( \int_t^T f(s, X^t_{s,x;u,\xi}, Y^t_{s,x;u,\xi}, Z^t_{s,x;u,\xi}) ds + \Phi (X^t_{T,x;u,\xi}) \right) 1_{\{\tau=+\infty, \rho=+\infty\}} \right].
\]

The reverse part can be obtained in the same way. We complete the proof.

\[\square\]

4 \textbf{HJBI Equation: Viscosity Approach}

In the stochastic optimal control theory, the value function is a solution to the corresponding Hamilton-Jacobi-Bellman equation (H-J-B in short) whenever it has sufficient regularity (Fleming and Soner\cite{Fleming92}, Krylov\cite{Krylov92}). In other word, it requires that the HJB equation admit classical solutions, meaning that the solutions be smooth enough (to the order of derivatives involved in the equation). Unfortunately, this is not necessarily the case even for some very simple situations. In the stochastic environment where the diffusion is possibly degenerate, the HJB equation may in general have no classical solutions either. To overcome this difficulty, Crandall and Lions introduced the so-called viscosity solutions in the early 1980s (see also \cite{Crandall83}). This new notion is a kind of non-smooth solutions (the value function is continuous, valued) super-/sub-differentials while maintaining the uniqueness of solutions under very mild conditions. These make the theory a powerful tool in tackling optimal control problems.

In this section, we consider the following HJBI equation associated to our stochastic differential games, in which lead to be the same expression for the two value functions since the two players can not operate at the same time in the systems, is described by
\[
\begin{cases}
\max \left\{ V - \mathcal{H}^\chi_{\sup} V, \min \left( -\frac{\partial}{\partial t} V(t, x) - H(t, x, V, DV, D^2V) , V - \mathcal{H}^\chi_{\inf} V \right) \right\} = 0, \\
V(T, x) = \Phi(x), \quad (t, x) \in [0, T) \times \mathbb{R}^n,
\end{cases}
\tag{43}
\]
where associated with the Hamiltonians:
\[
H(t, x, y, p, Q) = \langle b(t, x), p \rangle + \frac{1}{2} \text{tr} \left( \sigma \sigma^T (t, x) Q \right) + f(t, x, y, p^T \sigma(t, x))
\tag{44}
\]
and the nonlocal operators \( \mathcal{H}^\chi_{\sup} V \) and \( \mathcal{H}^\chi_{\inf} V \) are defined by
\[
\mathcal{H}^\chi_{\sup} V(t, x) = \sup_{y \in \mathcal{U}} \left[ V(t, x + y) - c(t, y) \right],
\]
\[
\mathcal{H}^\chi_{\inf} V(t, x) = \inf_{z \in \mathcal{V}} \left[ V(t, x + z) + \chi(t, z) \right],
\]
for any \( (t, x) \in [0, T) \times \mathbb{R}^n, \ y \in \mathbb{R}, \ p \in \mathbb{R}^n, \ Q \in \mathbb{S}^n \) where \( \mathbb{S}^n \) denotes the set of \( n \times n \) symmetric matrices. The coefficients \( b, \sigma, f, \Phi, \chi \) and \( c \) are supposed to satisfy (A1)-(A3).
We next prove that the lower value function $V(t, x)$ introduced by (43) is the viscosity solution of (43). We extend Cosso's work for stochastic differential games involving impulse controls into Peng’s BSDE’s framework. The difficulties related with this extension come from the fact that now, contrarily to the framework of stochastic control theory studied by Peng, we have to do with stochastic differential games in which strategies are played versus controls. In order to overcome these difficulties in the proof that $V^-$ is a viscosity supersolution, we have, in particular, to enrich Peng’s BSDE method. On the other hand, the proof that $V^+$ is a viscosity subsolution is not covered by Peng’s BSDE method and requires a quite new approach. The uniqueness of the viscosity solution will be shown in the next section for the class of bounded continuous functions. We first recall the definition of a viscosity solution of (43). The interested reader is referred to Crandall, et al. [39].

**Definition 4.1** Let $u(t, x) \in C([0, T] \times \mathbb{R}^n)$ and $(t, x) \in [0, T] \times \mathbb{R}^n$. For every $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^n)$,

1) for each local maximum point $(t_0, x_0)$ of $u - \varphi$ in the interior of $[0, T] \times \mathbb{R}^n$, we have

$$\max \left\{ V - H_{\inf} \varphi, \min \left[ -\frac{\partial}{\partial t} \varphi - H(t, x, x, D\varphi, D^2\varphi), V - H_{\sup} \varphi \right] \right\} \leq 0,$$

and for each $x \in \mathbb{R}^n$, we have

$$\max \left\{ V(T, x) - H_{\inf} \varphi(T, x), \min \left[ V(T, x) - \Phi(x), V(T, x) - H_{\inf} \varphi(T, x) \right] \right\} \leq 0,$$

i.e., $u$ is a subsolution to HJBI equation (43).

2) for each local minimum point $(t_0, x_0)$ of $u - \varphi$ in the interior of $[0, T] \times \mathbb{R}^n$, we have

$$\max \left\{ V - H_{\inf} \varphi, \min \left[ -\frac{\partial}{\partial t} \varphi - H(t, x, x, D\varphi, D^2\varphi), V - H_{\sup} \varphi \right] \right\} \geq 0,$$

and for each $x \in \mathbb{R}^n$, we have

$$\max \left\{ V(T, x) - H_{\inf} \varphi(T, x), \min \left[ V(T, x) - \Phi(x), V(T, x) - H_{\inf} \varphi(T, x) \right] \right\} \geq 0,$$

i.e., $u$ is a supersolution to HJBI equation (43).

3) $u(t, x) \in C([0, T] \times \mathbb{R}^n)$ is said to be a viscosity solution of (43) if it is both a viscosity sub and supersolution.

We have the other definition which will be useful to verify the viscosity solutions.

**Definition 4.2** Let $u(t, x) \in C([0, T] \times \mathbb{R}^n)$ and $(t, x) \in [0, T] \times \mathbb{R}^n$. We denote by $\mathcal{P}^{2,+}u(t, x)$, the “parabolic superjet” of $u$ at $(t, x)$ the set of triples $(p, q, X) \in R \times \mathbb{R}^n \times S^n$ which are such that

$$u(s, y) \leq u(t, x) + p(s - t) + q\cdot(x - y) + \frac{1}{2} \langle X \rangle(X, y - x) + o(|s - t| + |y - x|^2).$$
Similarly, we denote by $\mathcal{P}^{2,-}u(t,x)$, the “parabolic subjet” of $u$ at $(t,x)$ the set of triples $(p,q,X) \in \mathbb{R} \times \mathbb{R}^n \times S^n$ which are such that
\[
u(s,y) \geq u(t,x) + p(s-t) + \langle q, x-y \rangle + \frac{1}{2} \langle X(y-x), y-x \rangle + o \left( |s-t| + |y-x|^2 \right).
\]

**Definition 4.3**
(i) It can be said $V(t,x) \in C([0,T] \times \mathbb{R}^n)$ is a viscosity subsolution of (43) if at any point $(t,x) \in [0,T] \times \mathbb{R}^n$, for any $(p,q,X) \in \mathcal{P}^{2,+}V(t,x),$ 
\[
\max \left\{ V - \mathcal{H}_{\inf}^{c}V, \min \left[ -p - \Phi(t,x,V(t,x),q,X), V - \mathcal{H}_{\sup}^{c}V \right] \right\} \leq 0,
\]
and for each $x \in \mathbb{R}^n$, it holds
\[
\max \left\{ V(T,x) - \mathcal{H}_{\inf}^{c}V(T,x), \min \left[ V(T,x) - \Phi(x), V(T,x) - \mathcal{H}_{\inf}^{c}V(T,x) \right] \right\} \leq 0.
\]
(ii) It can be said $V(t,x) \in C([0,T] \times \mathbb{R}^n)$ is a viscosity supersolution of (43) if at any point $(t,x) \in [0,T] \times \mathbb{R}^n$, for any $(p,q,X) \in \mathcal{P}^{2,+}V(t,x),$ 
\[
\max \left\{ V - \mathcal{H}_{\inf}^{c}V, \min \left[ -p - \Phi(t,x,V(t,x),q,X), V - \mathcal{H}_{\sup}^{c}V \right] \right\} \geq 0,
\]
and for each $x \in \mathbb{R}^n$, we have
\[
\max \left\{ V(T,x) - \mathcal{H}_{\inf}^{c}V(T,x), \min \left[ V(T,x) - \Phi(x), V(T,x) - \mathcal{H}_{\inf}^{c}V(T,x) \right] \right\} \geq 0.
\]
(iii) It can be said $u(t,x) \in C([0,T] \times \mathbb{R}^n)$ is a viscosity solution of (43) if it is both a viscosity sub and super solution.

**Remark 4.4** Definitions 4.1 and 4.3 are equivalent to each other. For more details, see Fleming and Soner [1], Lemma 4.1 (page 211).

We now introduce the lower and upper obstacles with the help of the following lemmas.

**Lemma 4.5** Assume (A1)–(A3) are in force. Given any $(t,x) \in (0,T] \times \mathbb{R}^n$, the lower and upper value functions satisfy the following equation:
\[
\max \left\{ \min \left[ 0, V(t,x) - \mathcal{H}_{\sup}^{c}V(t,x) \right], V(t,x) - \mathcal{H}_{\inf}^{c}V(t,x) \right\} = 0.
\]

**Proof** From Lemma 3.1, (40) can be expressed as
\[
V^{-}(t,x) = \inf_{\rho \in T_{t,x}, \tau \in T_{t,x}, \xi \in \mathcal{F}_{\tau}} \sup_{\eta \in \mathcal{F}_{\eta}} E \left[ -c(t, \xi) 1_{\{\tau=\tau\}} 1_{\{\rho=+\infty\}} + \chi(t, \eta) 1_{\{\rho=\tau\}} + V^{-} \left( t, \chi(t,x,1_{\{\tau\}}, \eta, \xi, \tau, x \right) \right].
\]
The remainder of the proof is the same as Lemma 5.3 from [23]. We omit it.

**Remark 4.6** We have $V^{-}(t,x) \leq \mathcal{H}_{\inf}^{c}V(t,x)$ on $(0,T] \times \mathbb{R}^n$ from Lemma 4.5. Besides, whenever $V(t,x) \leq \mathcal{H}_{\inf}^{c}V(t,x)$, then $\mathcal{H}_{\sup}^{c}V(t,x) \leq V^{-}(t,x)$ and $\mathcal{H}_{\sup}^{c}V(t,x) \leq \mathcal{H}_{\inf}^{c}V(t,x).$ So we may regard $\mathcal{H}_{\sup}^{c}V(t,x)$ as a lower obstacle and $\mathcal{H}_{\inf}^{c}V(t,x)$ as an upper obstacle. Both of them are implicit forms, since they depend on $V^{-}$. The same remark applies to $V^{+}$ likewise.
We shall prove that the two value functions satisfy, in the viscosity sense, the terminal condition.

**Lemma 4.7** Assume Assumptions (A1)–(A3) are in force. The lower value function $V^-(T, x)$ is a viscosity solution of (43).

**Proof** We shall prove

$$
\max \{V(T, x) - \mathcal{H}^c_{in}V(T, x), \min [V(T, x) - \Phi(x), V(T, x) - \mathcal{H}^c_{sup}V(T, x)]\} \geq 0.
$$

From Lemma 3.8, we have

$$
V^-(t, x) = \inf_{\rho \in \mathcal{T}_i \cup \mathcal{H}_{\rho}} \sup_{\tau \in \mathcal{T}_i \cup \mathcal{H}_{\rho}, \xi \in \mathcal{F}_{\tau}} E \left[ \left( -c(t, \xi) \mathbf{1}_{\{\tau = t\}} \mathbf{1}_{\{\rho = +\infty\}} + \chi(t, \eta) \mathbf{1}_{\{\rho = t\}} + V^-(t, X^t_{\xi}[\tau, t], Y^t_{\eta}[\tau, t]) \right) (1 - \mathbf{1}_{\{\tau = +\infty, \rho = +\infty\}}) + \left( \int_t^T f(s, X^s_{\xi}[\rho, \rho], Y^s_{\eta}[\rho, \rho], Z^s_{\eta}[\rho, \rho]) \, ds \right) + \Phi(X^T_{\xi}[\rho, \rho]) \right] \mathbf{1}_{\{\tau = +\infty, \rho = +\infty\}}.
$$

Thanks to (A1)–(A2), it follows that

$$
E \left[ \int_t^T f(s, X^s_{\xi}[\rho, \rho], Y^s_{\eta}[\rho, \rho], Z^s_{\eta}[\rho, \rho]) \, ds \right] \\
\leq (T - t)^{\frac{1}{2}} E \left[ \int_t^T |f(s, X^s_{\xi}[\rho, \rho], Y^s_{\eta}[\rho, \rho], Z^s_{\eta}[\rho, \rho])|^2 \, ds \right]^{\frac{1}{2}} \\
\leq C (T - t)^{\frac{1}{2}}
$$

and

$$
E \left[ \left| X^T_{\xi}[\rho, \rho] - X(t) \right| \right] \leq C E \left[ \left| X^T_{\xi}[\rho, \rho] - x \right|^2 \right]^{\frac{1}{2}} \\
\leq C (T - t)^{\frac{1}{2}}, \text{ uniformly in } u_0, v_0.
$$

Therefore,

$$
V^-(t, x) \geq -\inf_{\rho \in \mathcal{T}_i \cup \mathcal{H}_{\rho}} \sup_{\tau \in \mathcal{T}_i \cup \mathcal{H}_{\rho}, \xi \in \mathcal{F}_{\tau}} E \left[ \left( -c(t, \xi) \mathbf{1}_{\{\tau = t\}} \mathbf{1}_{\{\rho = +\infty\}} + \chi(t, \eta) \mathbf{1}_{\{\rho = t\}} + V^-(t, X^t_{\xi}[\tau, t], Y^t_{\eta}[\tau, t]) \right) (1 - \mathbf{1}_{\{\tau = +\infty, \rho = +\infty\}}) + \Phi(x) \mathbf{1}_{\{\tau = +\infty, \rho = +\infty\}} \right] - C (T - t)^{\frac{1}{2}}.
$$

Repeating the method in Lemma 4.5, we have

$$
\max \{\min [V^-(t, x) - \Phi(x), V^-(t, x) - \mathcal{H}^c_{sup}V^-(t, x)], V^-(t, x) - \mathcal{H}^c_{in}V^-(t, x)]\} \\
\geq -C (T - t)^{\frac{1}{2}}.
$$

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According to (A3), namely the 1/2-Hölder continuity in time for $c$, $\chi$ and $V^-$, we deduce that
\[
\max \{ V(T,x) - \mathcal{H}_{\text{inf}}^c V(T,x), \min [V(T,x) - \Phi(x), V(T,x) - \mathcal{H}_{\text{inf}}^c V(T,x)] \} + C(T-t)^{1/2} \\
\geq \max \{ \min [V^- (t,x) - \Phi(x), V^- (t,x) - \mathcal{H}_{\text{sup}}^c V^- (t,x)] , V^- (t,x) - \mathcal{H}_{\text{inf}}^c V^- (t,x) \} \\
\geq -C_1(T-t)^{1/2}
\]
for some $C_1 > 0$. Then, letting $t = T$ in (53) ends the proof.

We first prove that the lower value function $V^-(t,x)$ is a viscosity solution of (43).

**Theorem 4.8** Assume the assumptions (A1)–(A3) are in force, the lower value function $V^-(t,x)$ is a viscosity solution of (43).

**Proof** We first show that the lower value function $V^-$ is a viscosity solution to (43); the other case is analogous.

In Lemma 4.7, we have proved that $V^-$ satisfies, in the viscosity sense, the terminal condition, namely (50) and (50). Therefore, we have only to address (49). From Proposition 3.6, $V^-$ is continuous on $[0,T) \times \mathbb{R}^n$. Thus we begin by proving that $V^-$ is a viscosity supersolution. By virtue of Lemma 4.5, we have to show that, given $(\tilde{t},\tilde{x}) \in [0,T) \times \mathbb{R}^n$ such that $\mathcal{H}_{\text{sup}}^c V(\tilde{t},\tilde{x}) \leq V^- (\tilde{t},\tilde{x})$ and $V^- (\tilde{t},\tilde{x}) \leq \mathcal{H}_{\text{inf}}^c V(\tilde{t},\tilde{x})$, then for every $\varphi \in C^{1,2}([0,T) \times \mathbb{R}^n)$, such that $(\tilde{t},\tilde{x})$ is a local minimum of $V^- - \varphi$, we have
\[
\frac{\partial}{\partial \tilde{t}} \varphi (\tilde{t},\tilde{x}) + H (t,x,\varphi (\tilde{t},\tilde{x}),D\varphi (\tilde{t},\tilde{x}),D^2 \varphi (\tilde{t},\tilde{x})) \leq 0,
\]
where $H$ is defined in (44).

Without loss of generality, postulate $V^- (\tilde{t},\tilde{x}) = \varphi (\tilde{t},\tilde{x})$. Let
\[
\lambda + V^- (\tilde{t},\tilde{x}) = \mathcal{H}_{\text{inf}}^c V(\tilde{t},\tilde{x}) = \inf_{y \in \mathcal{Y}} \left[ V^- (\tilde{t},\tilde{x} + y) + \chi (\tilde{t},y) \right].
\]
We proceed as in [23] to derive the following result: For every random variable $\eta$, $\mathcal{F}_\tau$-measurable and values in $V$, there exists $C > 0$,
\[
\mathbb{E}^{\mathbb{F}_\tau} [ V \left( s, X^\tau_{\tau} \right) ] \leq \mathbb{E}^{\mathbb{F}_\tau} [ V \left( s, X^\tau_{\tau} + \eta \right) + \chi (s,\eta) ] + C \left| s - \tilde{t} \right|^{1/2} - \lambda
\]
with $X^\tau_{\tau} = X^\tau_{\tau} u_0, v_0$ for all $s \in [\tilde{t},T]$, P-a.s., where $u_0$ and $v_0$ denote the controls with no impulses.

Next recall
\[
V^- (\tilde{t},\tilde{x}) = \inf_{\beta \in \mathcal{U}_{\tau,\tilde{t}}^e} \sup_{u \in \mathcal{U}_{\tau,\tilde{t}}} G^\tau_{\tau + \delta} \left[ V^- \left( \tilde{t} + \delta, X^\tau_{\tau + \delta} u, \beta (u) \right) + \Theta^u_{\tau + \delta} \right],
\]
where
\[
\Theta^u_{\tau + \delta} = \sum_{l \geq 1} \chi (\rho_l,\eta_l) 1_{\{ \rho_l \leq \tau + \delta \}} - \sum_{m \geq 1} e (\tau_m, \xi_m) 1_{\{ \tau_m \leq \tau + \delta \}} \prod_{l \geq 1} 1_{\{ \tau_m \neq \rho_l \}}.
\]
From the definition of $V^-(\bar{t}, \bar{\tau})$, we have
\[
V^-(\bar{t}, \bar{\tau}) = \inf_{\beta \in \mathcal{B}_{\bar{t}, \bar{\tau}}} \sup_{\gamma \in \mathcal{B}_{\bar{t}, \bar{\tau}}} G^\gamma_{t, t+\delta} \left[ V^-(\bar{t} + \delta, X^\gamma_{t+\delta}) + \Theta^\gamma_{t+\delta} \right]
\]
\[
\geq G^\delta_{t, t+\delta} \left[ V^-(\bar{t} + \delta, X^\delta_{t+\delta}) + \sum_{l \geq 1} \chi(\rho_l, \eta_l) 1_{\{\rho_l \leq \tau + \delta\}} \right] - \varepsilon
\]
for some $\beta^\varepsilon \in \mathcal{B}_{\bar{t}, \bar{\tau}}$, with $\beta^\varepsilon(\epsilon) = \sum_{l \geq 1} \eta_l 1_{[\rho_l, T]} \in \mathcal{V}_{\bar{t}, \bar{\tau}}$. Note that
\[
\sum_{l \geq 1} \chi(\rho_l, \eta_l) 1_{\{\rho_l \leq \tau + \delta\}} = \sum_{l \geq 1} \chi(\rho_l, \eta_l).
\]
By (A1)–(A2), we have the following estimate
\[
\mathbb{E}^\mathcal{F}_{t+\delta} \left[ V^-(\bar{t} + \delta, X^\delta_{t+\delta}) - V^-(\bar{t} + \delta, X^\delta_{t+\delta} + \sum_{l \geq 1} \chi(\rho_l, \eta_l) 1_{\{\rho_l \leq \tau + \delta\}}) \right] \leq C\delta^2 \mathbb{E}^\mathcal{F}_{t+\delta} 1_{\{\mu_{\mathcal{F}_{t+\delta}} \geq 1\}}.
\]
As a consequence, using (A3) and (54), we deduce
\[
\mathbb{E}^\mathcal{F}_{t+\delta} \left[ V^-(\bar{t} + \delta, X^\delta_{t+\delta} + \sum_{l \geq 1} \chi(\rho_l, \eta_l)) \right] \geq \mathbb{E}^\mathcal{F}_{t+\delta} \left[ V^-(\bar{t} + \delta, X^\delta_{t+\delta}) + (\lambda - C\delta^2) 1_{\{\mu_{\mathcal{F}_{t+\delta}} \geq 1\}} \right].
\]
Therefore, applying comparison theorem (Proposition 2.6 in [35]), we find
\[
G^\delta_{t, t+\delta} \left[ V^-(\bar{t} + \delta, X^\delta_{t+\delta}) + \sum_{l \geq 1} \chi(\rho_l, \eta_l) 1_{\{\rho_l \leq \tau + \delta\}} \right] \geq G^\delta_{t, t+\delta} \left[ V^-(\bar{t} + \delta, X^\delta_{t+\delta}) + (\lambda - C\delta^2) 1_{\{\mu_{\mathcal{F}_{t+\delta}} \geq 1\}} \right].
\]
Thus,
\[
V^-(\bar{t}, \bar{\tau}) \geq G^\delta_{t, t+\delta} \left[ V^-(\bar{t} + \delta, X^\delta_{t+\delta}) + (\lambda - C\delta^2) 1_{\{\mu_{\mathcal{F}_{t+\delta}} \geq 1\}} \right] - \varepsilon.
\]
From the boundedness of $f$ we deduce
\[
f(s, X^\delta_{t+\delta}, y, z) \geq f(s, X^\delta_{t+\delta}, y, z) - f(s, X^\delta_{t+\delta}, y, z) = f(s, X^\delta_{t+\delta}, y, z) + f(s, X^\delta_{t+\delta}, y, z) - C, \text{ for } (y, z) \in \mathbb{R} \times \mathbb{R}.
\]
Applying comparison theorem (Proposition 2.6 in [35]) again, we have $V^- (\bar{t}, \bar{\pi}) \geq \mathcal{Y}_{\bar{t}}$, where $\mathcal{Y}_{\bar{t}}$ is the solution to the following BSDE:

$$
\begin{align*}
\mathcal{Y}_{\bar{t}} &= V^- (\bar{t} + \delta, X_{\bar{t} + \delta}^\pi) + \left( \lambda - C \delta + \delta \right) s, x, y, z \mathbb{1}_{\{\mathcal{F}_{\bar{t} + \delta} \geq 1\}} \\
&\quad + \int_{\bar{t}}^{\bar{t} + \delta} f \left( s, X_s^\pi, \mathcal{Y}_s, Z_s \right) ds - \int_{\bar{t}}^{\bar{t} + \delta} Z_s dW_s.
\end{align*}
$$

We shall take $\delta$ sufficiently small. Indeed, there exists $\delta > 0$ such that for $\zeta \in (0, \delta)$, we have $\lambda - C \delta + \delta \geq 0$. Immediately, by Proposition 2.6 in [35], it follows

$$
V^- (\bar{t}, \bar{\pi}) \geq G_{\bar{t}, \bar{t} + \delta}^{\pi, \zeta} \left[ V^- \left( \bar{t} + \zeta, X_{\bar{t} + \zeta}^\pi \right) \right] - \varepsilon.
$$

(56)

To abbreviate notations we set, for some arbitrarily chosen but fixed $\varphi \in C^{1,2} (\{0, T\} \times \mathbb{R}^n)$,

$$
F (s, x, y, z) = \frac{\partial}{\partial s} \varphi (s, x) + \frac{1}{2} \text{Tr} \left( \sigma \sigma^T (s, x) D^2 \varphi \right) + \langle D \varphi, b (s, x) \rangle + f (s, x, y + \varphi (s, x), z + D \varphi (s, x) \cdot \sigma (x), x),
$$

for $(s, x, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d$.

Let us consider the following BSDE:

$$
\begin{align*}
- dY_{s}^{1} &= F \left( s, X_{s}^{\pi}, Y_{s}^{1}, Z_{s}^{1} \right) ds - Z_{s}^{1} dW_s, \\
Y_{\bar{t} + \zeta}^{2} &= 0.
\end{align*}
$$

(57)

It is not hard to check that $F \left( s, X_{s}^{\pi}, y, z \right)$ satisfies (A1) and (A2). Thus, BSDE (57) admits a unique adapted strong solution. We can characterize the solution process $Y_{s}^{1}$ as follows:

$$
Y_{s}^{1} = G_{s, \bar{t} + \zeta}^{\pi} \left[ \varphi \left( \bar{t} + \zeta, X_{\bar{t} + \zeta}^{\pi} \right) \right] - \varphi \left( s, X_{s}^{\pi} \right).
$$

(58)

Indeed, $G_{s, \bar{t} + \zeta}^{\pi, \zeta} \left[ \varphi \left( \bar{t} + \zeta, X_{\bar{t} + \zeta}^{\pi} \right) \right]$ is defined by the solution of the following BSDE:

$$
\begin{align*}
- dY_{s}^{1} &= f \left( s, X_{s}^{\pi}, Y_{s}, Z_{s} \right) ds - Z_{s} dW_s, \\
Y_{\bar{t} + \zeta}^{2} &= \varphi \left( \bar{t} + \zeta, X_{\bar{t} + \zeta}^{\pi} \right).
\end{align*}
$$

(59)

Therefore, one just need to prove $Y_{s} - \varphi \left( s, X_{s}^{\pi} \right) = Y_{s}^{1}$. Applying Itô’s formula to $\varphi \left( s, X_{s}^{\pi} \right), we obtain $d \left[ Y_{s} - \varphi \left( s, X_{s}^{\pi} \right) \right] = dY_{s}^{1}$, and at the terminal time $Y_{\bar{t} + \zeta} - \varphi \left( \bar{t} + \zeta, X_{\bar{t} + \zeta}^{\pi} \right) = Y_{\bar{t} + \zeta}^{2}$, as a result, they are equal in the interval $[\bar{t}, \bar{t} + \zeta]$.

Now let us introduce a more simpler BSDE than (57), i.e., $X_{s}^{\pi}$ of the equation (57) is taken place by $x$:

$$
\begin{align*}
- dY_{s}^{2} &= F \left( s, x, Y_{s}^{2}, Z_{s}^{2} \right) ds - Z_{s}^{2} dW_s, \\
Y_{\bar{t} + \zeta}^{2} &= 0.
\end{align*}
$$

(60)
Notice that \( F \) is a deterministic function of \((s, x, y, z)\) therefore, \( (Y_s^2, Z_s^2) = (Y_0(s), 0) \) where \( Y_0(s) \) is the solution of the ODE:

\[
\begin{align*}
-\dot{Y}_0(s) &= F(s, \bar{x}, Y_0(s), 0) \, ds, \quad s \in [\bar{t}, \bar{t} + \zeta], \\
Y_0(\bar{t} + \zeta) &= 0.
\end{align*}
\]

The following result indicates that the difference of the solutions of (57) and (60) can be neglected whenever \( \zeta \) is sufficiently small enough. From the classical estimate on SDE, we have

\[
E \left[ \sup_{s \in [\bar{t}, \bar{t} + \zeta]} |X_t^\pi|^p \right] \leq C (1 + |\pi|^p).
\]

Moreover, applying B-D-G inequality, we get

\[
E \left[ \sup_{s \in [\bar{t}, \bar{t} + \zeta]} |X_t^\pi - \bar{x}|^2 \right] \leq C \zeta. \quad \text{Hence, when } \zeta \to 0, \text{ the following random variable } \kappa^\zeta := \sup_{s \in [\bar{t}, \bar{t} + \zeta]} |X_t^\pi - \bar{x}| \text{ converges monotone to } 0.
\]

On the one hand, employing Proposition 3.2 in [34] to BSDEs (57) and (60), we have

\[
E \left[ \int_{\bar{t}}^{\bar{t} + \zeta} |Y_s^1 - Y_s^2|^2 + |Z_s^1|^2 \, ds \right] \leq C E \left[ \int_{\bar{t}}^{\bar{t} + \zeta} \varpi \left( |X_t^\pi - x| \right)^2 \, ds \right] \leq C \zeta E \varpi (\kappa^\zeta)^2.
\]

On the other hand, from Lemma 3.1, we have

\[
|Y_\bar{t}^1 - Y_\bar{t}^2| = |E Y_\bar{t}^1| = E \left[ \int_{\bar{t}}^{\bar{t} + \zeta} \left( F(s, X_s^\pi, Y_s^1, Z_s^1) - F(s, \bar{x}, Y_s^2) \right) \, ds \right] \leq C E \left[ \int_{\bar{t}}^{\bar{t} + \zeta} \varpi \left( |X_t^\pi - x| \right) + |Y_s^1 - Y_s^2| + |Z_s^1| \, ds \right] \\
\leq C \zeta E \varpi (\kappa^\zeta)^2 + C \zeta^{1/2} \left\{ E \left[ \int_{\bar{t}}^{\bar{t} + \zeta} \left( |Y_s^1 - Y_s^2|^2 + |Z_s^1|^2 \right) \, ds \right] \right\}^{1/2} \\
\leq C \zeta E \left[ \varpi (\kappa^\zeta)^2 + \varpi (\kappa^\zeta) \right],
\]

with \( \varpi (\varepsilon) \to 0 \) as \( \varepsilon \to 0 \). Note that, for each \( \zeta > 0 \), \( \varpi (\kappa^\zeta) \) is square integrable, we set

\[
\varpi_0(\zeta) = E \left[ \varpi (\kappa^\zeta)^2 + \varpi (\kappa^\zeta) \right].
\]

Hence,

\[
|Y_\bar{t}^1 - Y_\bar{t}^2| \leq C \zeta \varpi_0(\zeta). \quad (63)
\]

From the monotonicity of \( G [\cdot] \),

\[
\varphi(\bar{t}, \bar{x}) = V^- (\bar{t}, \bar{x}) \geq G_{\bar{t}, \bar{t} + \zeta}^{\pi} \left[ V^- (\bar{t} + \zeta, X_{\bar{t} + \zeta}^\pi) \right] - \varepsilon \\
\geq G_{\bar{t}, \bar{t} + \zeta}^{\pi} \left[ \varphi (\bar{t} + \zeta, X_{\bar{t} + \zeta}^\pi) \right] - \varepsilon.
\]

From (58) and letting \( \varepsilon \to 0 \),

\[
0 \geq G_{\bar{t}, \bar{t} + \zeta}^{\pi} \left[ \varphi (\bar{t} + \zeta, X_{\bar{t} + \zeta}^\pi) \right] - \varphi (\bar{t}, \bar{x}) = Y_\bar{t}^1.
\]
By (63) we further have \( Y_0(\overline{t}) = Y_T^2 \leq C\zeta\omega_0(\zeta) \). Therefore, it follows easily that \( F(\overline{t}, \pi, 0, 0) \leq 0 \) and from the definition of \( F \) we see that \( V^- \) is a viscosity supersolution of (43). The proof is similar for the viscosity sub-solution.

Next, we shall prove that the HJBI equation (43) has a unique viscosity solution. Consequently, the lower and upper value functions coincide, since they are both viscosity solutions to (43). Thus, the stochastic differential game admits a value.

Before introducing the comparison principle, we need the following two technical lemmas, mainly taken from [23].

**Lemma 4.9** Assume that (A3) is in force. Let \( U, V : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R} \) be a viscosity supersolution and a viscosity subsolution to the HJBI equation (43), respectively. Let \( \hat{t}, \hat{x} \in [0, T] \times \mathbb{R}^n \) such that

\[
V(\hat{t}, \hat{x}) \leq \mathcal{H}_{\sup}^c V(\hat{t}, \hat{x}) , \quad U(\hat{t}, \hat{x}) \leq \mathcal{H}_{\inf}^c U(\hat{t}, \hat{x}) ,
\]

or

\[
U(\hat{t}, \hat{x}) \geq \mathcal{H}_{\inf}^c U(\hat{t}, \hat{x}) .
\]

Then for every \( \varepsilon > 0 \), there exists \( \hat{x} \in \mathbb{R}^n \) such that

\[
V(\hat{t}, \hat{x}) - U(\hat{t}, \hat{x}) \leq V(\hat{t}, \hat{x}) - U(\hat{t}, \hat{x}) + \varepsilon
\]

and

\[
V(\hat{t}, \hat{x}) > \mathcal{H}_{\sup}^c V(\hat{t}, \hat{x}) , \quad U(\hat{t}, \hat{x}) < \mathcal{H}_{\inf}^c U(\hat{t}, \hat{x}) .
\]

**Lemma 4.10** Assume that (A3) is in force. Let \( U, V : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R} \) be a viscosity supersolution and a viscosity subsolution to the HJBI equation (43), respectively. Let \( \hat{t}, \hat{x} \in [0, T] \times \mathbb{R}^n \) such that

\[
V(\hat{t}, \hat{x}) > \mathcal{H}_{\sup}^c V(\hat{t}, \hat{x}) , \quad U(\hat{t}, \hat{x}) < \mathcal{H}_{\inf}^c U(\hat{t}, \hat{x}) ,
\]

then there exists \( \varepsilon > 0 \) for which

\[
V(t, x) > \mathcal{H}_{\sup}^c V(t, x) , \quad U(t, x) < \mathcal{H}_{\inf}^c U(t, x) ,
\]

where \( (t, x) \in [(\hat{t} - \delta) \vee 0, (\hat{t} + \delta) \wedge T] \times \overline{B}_\delta(\hat{x}) \).

In order to get the uniqueness, we add the following assumption:

(A4) Assume that \( f \) is strictly monotone in \( y \), that is, \( f(t, x, y_1, z) < f(t, x, y_2, z) \), for \( \forall y_1, y_2 \in \mathbb{R} \) with \( y_1 < y_2 \), \( \forall (t, x, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^d \).

**Theorem 4.11** Let \( U, V : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R} \) be a viscosity supersolution and a viscosity subsolution to the HJBI equation (43), respectively. Assume that (A1)–(A4) are in force and that \( U, V \) are uniformly continuous on \( [0, T] \times \mathbb{R}^n \). Then, we have \( U \geq V \) on \( [0, T] \times \mathbb{R}^n \).

**Proof** We prove our result by contradiction. Suppose that

\[
\sup_{[0, T] \times \mathbb{R}^n} (V - U) > 0.
\]
Fix $\theta > C_F > 0$ where $C_F$ denotes the Lipschitz constant of $f$.

Define
\[
\overline{U}(t, x) = e^{\theta t} U(t, x), \quad \overline{V}(t, x) = e^{\theta t} V(t, x), \quad (t, x) \in [0, T] \times \mathbb{R}^n.
\]

It is fairly easy to check that $\overline{U}(t, x)$ ($\overline{V}(t, x)$) is a viscosity supersolution (subsolution) to the following HJBI equation:
\[
\begin{cases}
\max \left\{ W - \mathcal{H}^x_{\sup} W, \min \left\{ \theta W - \frac{\partial W}{\partial t} - \mathcal{L} W - f, W - \mathcal{H}^c_{\inf} W \right\} \right\} = 0, \\
W(T, x) = \Phi(x), \quad (t, x) \in [0, T] \times \mathbb{R}^n,
\end{cases}
\]

where
\[
\mathcal{L} W(t, x) = \langle b(t, x), D W(t, x) \rangle + \frac{1}{2} \text{tr} \left[ \sigma \sigma^T (t, x) D^2 W(t, x) \right],
\]
\[
f(t, x, W, D W \cdot \sigma(t, x)) = e^{\theta t} f(t, x, e^{-\theta t} W, e^{-\theta t} D W \cdot \sigma(t, x)),
\]
\[
\Phi(x) = e^{\theta T} \phi(x),
\]
\[
\mathcal{H}^x_{\sup} W(t, x) = \sup_{x \in V} \left[ W(t, x + z) + e^{\theta t} \chi(t, z) \right],
\]
\[
\mathcal{H}^c_{\inf} W(t, x) = \inf_{y \in \mathcal{U}} \left[ W(t, x + y) - e^{\theta t} c(t, z) \right].
\]

Assume that there exists $x_0 \in \mathbb{R}^n$ such that $(\overline{U} - \overline{V})(T, x_0) < 0$. Then, from Lemma 4.9, there exists $\tilde{x} \in \mathbb{R}^n$ such that $(\overline{U} - \overline{V})(T, \tilde{x}) < 0$. On the other hand, from the subsolution property of $\overline{V}$, we know $\overline{V}(T, \tilde{x}) \leq \Phi(\tilde{x})$. Similarly, utilizing the supersolution property of $\overline{U}$, we have $\overline{U}(T, \tilde{x}) \geq \Phi(\tilde{x})$. Therefore, $\overline{U}(T, \tilde{x}) \geq \overline{V}(T, \tilde{x})$ which leads a contradiction to $(\overline{U} - \overline{V})(T, \tilde{x}) < 0$.

Now postulate that there exists $(\tilde{t}, \tilde{x}) \in [0, T] \times \mathbb{R}^n$ such that $(\overline{U} - \overline{V})(\tilde{t}, \tilde{x}) < 0$. Then, from Lemma 4.10, there exist $(\hat{t}, \hat{x}) \in [0, T] \times \mathbb{R}^n$ and $\delta > 0$ such that
\[
\sup_{t \in I \times \mathcal{B}_\delta(\hat{x})} (\overline{V} - \overline{U})(t, x) > 0
\]

and
\[
\overline{V}(t, x) > \mathcal{H}^c_{\sup} V(t, x), \quad \overline{U}(t, x) < \mathcal{H}^x_{\inf} U(t, x), \quad (t, x) \in I \times \mathcal{B}_\delta(\hat{x})
\]

with $I := [(\hat{t} - \delta) \vee 0, (\hat{t} + \delta) \wedge T]$.

We define
\[
\hat{V}(t, x) = \overline{V}(t, x) - \frac{16 |x - \tilde{x}|^4}{15 \delta^4} M 1_{\{ |x - \tilde{x}| \geq \delta \}} + \frac{M}{15},
\]
where $M := \sup_{t \in I \times \mathcal{B}_\delta(\hat{x})} (\overline{V} - \overline{U})(t, x)$. It is easy to check that
\[
(\hat{V} - \overline{V})(t, x) = (\overline{V} - \overline{U})(t, x) - \frac{16 |x - \tilde{x}|^4}{15 \delta^4} M 1_{\{ |x - \tilde{x}| \geq \delta \}} - \frac{M}{15} \leq M.
\]

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So without loss of generality, we may assume that
\[
\left( \tilde{V} - \overline{U} \right)(t, x) \leq 0, \quad (t, x) \in I \times \partial B_{\delta}(\tilde{x}).
\]
Note that \(\mathcal{P}^{2,+} \overline{V} (t, x) = \mathcal{P}^{2,+} \tilde{V} (t, x)\) for all \((t, x) \in [0, T] \times \mathbb{R}^n\), \(\tilde{V}\) can be replaced with \(\overline{V}\).

Now choose \((t', x') \in I \times B_{\delta}(\tilde{x})\) such that
\[
\sup_{I \times B_{\delta}(\tilde{x})} (\overline{V} - \overline{U})(t, x) = (\overline{V} - \overline{U})(t', x') > 0. \tag{66}
\]

Define the following text function:
\[
\phi_n(t, x, y) = \overline{V}(t, x) - \overline{U}(t, y) - \psi_n(t, x, y), \quad n \in \mathbb{N}
\]
with
\[
\psi_n(t, x, y) = \frac{n}{2} |x - y|^2 + |x - x'|^2 + |t - t'|^2
\]
for every \((t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n\). Clearly, given any \(n \geq 1\), there exists \((t_n, x_n, y_n) \in I \times B_{\delta}(\tilde{x}) \times B_{\delta}(\tilde{x})\) attaining the maximum of \(\phi_n\) on \(I \times B_{\delta}(\tilde{x}) \times B_{\delta}(\tilde{x})\). Up to a subsequence, \((t_n, x_n, y_n) \in I \times B_{\delta}(\tilde{x}) \times B_{\delta}(\tilde{x}) \rightarrow (t_0, x_0, y_0) \in I \times B_{\delta}(\tilde{x}) \times B_{\delta}(\tilde{x})\) as \(n \rightarrow \infty\). Nonetheless, for every \(n \geq 1\), we have
\[
(\overline{V} - \overline{U})(t', x') = \phi_n(t', x', x') \geq \phi_n(t_n, x_n, y_n).
\]

It yields that
\[
(\overline{V} - \overline{U})(t', x') \leq \sup_{n \rightarrow \infty} \liminf_{n \rightarrow \infty} \phi_n(t_n, x_n, y_n)
\]
\[
\leq \overline{V}(t_0, x_0) - \overline{U}(t_0, y_0) - \inf_{n \rightarrow \infty} |x_n - y_n|^2
\]
\[
- |x_0 - x'|^2 - |t_0 - t'|^2, \tag{67}
\]
from which, up to a subsequence, \(\inf_{n \rightarrow \infty} n |x_n - y_n|^2 < \infty\). Then it follows that \(x_0 = y_0\). From (67), we derive that
\[
\begin{cases}
1) \ (t_n, x_n, y_n) \rightarrow (t', x', x') , \\
2) \ n |x_n - y_n|^2 \rightarrow 0 , \\
3) \ \overline{V}(t_n, x_n) - \overline{U}(t_n, y_n) \rightarrow \overline{V}(t', x') - \overline{U}(t', x'),
\end{cases} \tag{68}
\]
as \(n \rightarrow \infty\). By virtue of Ishii’s lemma (Theorem 8.3 in [39]), up to a subsequence, we may find sequence \(p_n^1, q_n^1, Q_n^1 \in \mathcal{P}^{2,+} \overline{V}(t_n, x_n)\) and \(p_n^2, q_n^2, Q_n^2 \in \mathcal{P}^{2,-} \overline{U}(t_n, y_n)\) such that
\[
p_n^1 - p_n^2 = 2(t_n - t') , \\
q_n^1 = D_x \psi_n(t_n, x_n, y_n) = n(x_n - y_n) , \\
q_n^2 = -D_y \psi_n(t_n, x_n, y_n) = n(x_n - y_n) .
\]

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and
\[
\begin{pmatrix} Q_n^1 & O \\ O & -Q_n^2 \end{pmatrix} \leq A_n + \frac{1}{2n} A_n^2,
\]
where
\[
A_n = D_{xy} \psi_n (t_n, x_n, y_n) = n \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.
\]
Then,
\[
\begin{pmatrix} Q_n^1 & O \\ O & -Q_n^2 \end{pmatrix} \leq 2n \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.
\] (69)

From \(\overline{U} (t, x) (\overline{V} (t, x))\) is a viscosity supersolution (subsolution) to the following HJBI equation (64), we have
\[
\theta \overline{U} (t_n, y_n) - p_n^2 - \mathcal{L} \overline{U} (t_n, y_n) - \nabla \nabla (t_n, y_n, e^{-\theta t_n} \overline{U}, e^{-\theta t_n} \mathcal{D} \overline{U} \cdot \sigma (t_n, y_n)) \geq 0, \quad (70)
\]
\[
\theta \overline{V} (t_n, x_n) - p_n^1 - \mathcal{L} \overline{V} (t_n, x_n) - \nabla \nabla (t_n, x_n, e^{-\theta t_n} \overline{V}, e^{-\theta t_n} \mathcal{D} \overline{V} \cdot \sigma (t_n, x_n)) \leq 0,
\] (71)

where \(\mathcal{L}\) is defined in (65). From (70) and (71), we immediately get
\[
\begin{align*}
\theta \overline{U} (t_n, x_n) - \theta \overline{U} (t_n, x_n) + p_n^2 - p_n^1 + \mathcal{L} \overline{U} (t_n, x_n) - \mathcal{L} \overline{V} (t_n, y_n) \\
+ \nabla (t_n, y_n, e^{-\theta t_n} \overline{U} (t_n, y_n), e^{-\theta t_n} \mathcal{D} \overline{V} (t_n, y_n) \cdot \sigma (t_n, y_n)) \\
- \nabla (t_n, x_n, e^{-\theta t_n} \overline{V} (t_n, x_n), e^{-\theta t_n} \mathcal{D} \overline{U} (t_n, x_n) \cdot \sigma (t_n, x_n))
\end{align*}
\] \leq 0. \quad (72)

Clearly,
\[
p_n^1 - p_n^2 \to 0, \quad \text{as} \quad n \to \infty
\] (73)

and
\[
b (t_n, y_n), n (x_n - y_n)) - b (t_n, x_n), n (x_n - y_n)) \to 0,
\] (74)

since 1)-2) in (68).

For simplicity, set \(\sigma_1 = \sigma (t_n, y_n), \sigma_2 = \sigma (t_n, x_n)\). We deal with
\[
\begin{align*}
\frac{1}{2} \text{Tr} \left( \sigma \sigma^T (t_n, y_n) Q_n^1 \right) - \frac{1}{2} \text{Tr} \left( \sigma \sigma^T (t_n, x_n) Q_n^2 \right) \\
\leq \frac{1}{2} \text{Tr} \begin{pmatrix} \sigma_1^T & \sigma_1^T \\ \sigma_2^T & \sigma_2^T \end{pmatrix} \begin{pmatrix} Q_n^1 & 0 \\ 0 & -Q_n^2 \end{pmatrix} \\
\leq n \text{Tr} \begin{pmatrix} \sigma_1^T & \sigma_1^T \\ \sigma_2^T & \sigma_2^T \end{pmatrix} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \\
\leq n \text{Tr} \left[ \sigma_1^T - \sigma_2^T - \sigma_2^T + \sigma_2^T \right] \\
= n \text{Tr} \left[ (\sigma_1 - \sigma_2) (\sigma_1 - \sigma_2)^T \right] \\
\leq n C |\sigma_1 - \sigma_2|^2 \\
\leq n C |x_n - y_n|^2 \to 0, \quad \text{as} \quad n \to \infty,
\end{align*}
\] (75)
where we have used the assumption that Lipschitz condition on $\sigma$, (69) and (2) in (68).

According to (3) in (68), the left-hand side of the inequality (72) goes to $\theta \left[ V(t', x') - U(t', x') \right]$, as $n \to \infty$, moreover, by (A4), we have

$$
\theta \left[ V(t', x') - U(t', x') \right] \leq \mathcal{F} \left( t', x', e^{-\theta t'} U(t', x'), 0 \right) - \mathcal{F} \left( t', x', e^{-\theta t'} V(t', x'), 0 \right)
$$

which leads to a contradiction to (66). Our proof is thus completed. 

**Remark 4.12** To get a uniqueness result for viscosity solution of (43), we adapt some techniques from [23]. We have to mention that there is another approach developed by Barles, et al. [35]. The value function can be considered in given class of continuous functions satisfying

$$
\lim_{|x| \to \infty} |u(t, x)| \exp \left\{ -A \log (|x|)^2 \right\} = 0,
$$

uniformly for $t \in [0, T]$, for some $A > 0$. The space of continuous functions endowed with a growth condition is slightly weaker than the assumption of polynomial growth but more restrictive than that of exponential growth. This growth condition was first introduced by Barles, et al. [35] to prove the uniqueness of the viscosity solution of an integro-partial differential equation associated with a decoupled FBSDEs with jumps. It has been shown in [35] that this kind of growth condition is optimal for the uniqueness and can, in general, not be weakened. These techniques have been applied in [40] for the uniqueness for viscosity solutions of recursive control of the obstacle constraint problem and Hamilton-Jacobi-Bellman-Isaacs equations related to stochastic differential games, respectively. However, as you may have observed, in our HJBI equation, there appears two obstacles, which are implicit obstacles, in the sense that they depend on $V^-$. It is worth to pointing out that the smooth supersolution built in [35], namely

$$
\chi(t, x) = \exp \left[ \left( \hat{C} (T - t) + A \right) \psi(x) \right],
$$

whilst

$$
\psi(x) = \left[ \log \left( |x|^2 + 1 \right) + 1 \right]^2,
$$

where $\hat{C}$ and $A$ are positive constants. One can show

$$
\min_{v \in U} \left\{ \mathcal{L}(t, x, v) \chi(t, x) + C |\chi| + C |\nabla \chi \sigma(t, x, v)| \right\} \leq 0,
$$

where $C$ is the Lipschitz constant of $f$. Following the idea in [35], whenever considering the difference of $u_1 - u_2$ where $u_1$ ($u_2$) is a subsolution (supersolution) of (43). It is hard to check the obstacles of viscosity solution (45)–(48). This is the reason we borrow the idea from Fleming and Souganidis [1] and Cosso [23] to handle the uniqueness.

**Remark 4.13** As observed in our paper, we put somewhat strong assumptions on coefficients, namely, boundedness. On the one hand, it simplifies our proof of existence. Recently, El Asri and Mazid [24] also investigated the solution to the zero-sum stochastic differential games, but under rather weak assumptions on the cost functions ($c$ and $\chi$ are not decreasing in time).
In the future, we shall adopt the idea developed by El Asri and Mazid\cite{24} to exploit the recursive utilities.

5 Concluding Remarks

In this paper, we study on zero-sum stochastic differential games in the framework of backward stochastic differential equations on a finite time horizon with both players adopting impulce controls. By means of stochastic backward semigroups and comparison theorem of BSDE, we prove a dynamic programming principle for both the upper and the lower value functions of the game. The upper and the lower value functions are then shown to be the unique viscosity solutions of the Hamilton-Jacobi-Bellman-Isaacs equations with a double-obstacle. As a result, the uniqueness implies that the upper and lower value functions coincide and the game admits a value. In future work, we plan to relax our assumptions and to try to find a smooth solution for the HJBI (43) in order to obtain uniqueness as Remark 4.12. Besides, as in Zhang\cite{41}, we will consider problems in which one player adopts impulse controls and the other adopts continuous controls, finite/infinite horizons, etc. These possible extensions promise to be interesting research directions. We shall respond these challenging topics in our future work.

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Appendix

The Proof of Lemma 3.1

Proof We adopt the idea from [9]. Let \( H \) denote the Cameron–Martin space of all absolutely continuous elements \( h \in \Omega \) whose derivative \( \dot{h} \) belongs to \( L^2([0,T],\mathbb{R}^d) \). For any \( h \in H \), we define the mapping \( \tau_h : \omega \mapsto \omega + h, \omega \in \Omega \). Clearly, \( \tau_h : \Omega \rightarrow \Omega \) is a bijection, and its law is given by

\[
P \circ (\tau_h)^{-1} = \exp \left\{ \int_0^T \dot{h}_s dW_s - \frac{1}{2} \int_0^T |\dot{h}_s|^2 ds \right\} P.
\]

Let \((t,x) \in [0,T] \times \mathbb{R}^n\) be arbitrarily fixed, and put \( H_t = \{ h \in H | h(\cdot) = h(\cdot \wedge t) \} \). Let \( u \in U_t,T \), \( v \in V_t,T \), and \( h \in H_t \), we first show that \( J(t,x,u,v)(\tau_h) = J(t,x,u(\tau_h),v(\tau_h)) \), \( P \)-a.s. Indeed, substitute the transformed control processes \( u(\tau_h) \) and \( v(\tau_h) \) for \( u \) and \( v \) into FBSDEs (1)–(3) and take the Girsanov transformation to (1)–(3), finally compare the obtained equation with the previous ones. Then from the uniqueness of the solution of (1)–(3), we conclude with

\[
X^{t,x;u,v}_s(\tau_h) = X^{t,x;u(\tau_h),v(\tau_h)}_s,
\]

\[
Y^{t,x;u,v}_s(\tau_h) = Y^{t,x;u(\tau_h),v(\tau_h)}_s,
\]

\[
Z^{t,x;u,v}_s(\tau_h) = Z^{t,x;u(\tau_h),v(\tau_h)}_s.
\]
for any $s \in [t, T]$, $P$-a.s., which indicates that $J(t, x; u, v)(\tau_h) = J(t, x; u(\tau_h), v(\tau_h))$, $P$-a.s.

For $\beta \in B_{t,T}$, $h \in H_t$, let $\beta^h(u) := \beta(u(\tau_h)) (\tau_h)$, $u \in \mathcal{U}_{t,T}$. Then $\beta^h \in B_{t,T}$, which makes $\mathcal{U}_{t,T}$ into $\mathcal{V}_{t,T}$. Moreover, it is easy to check that this mapping is nonanticipating and verify

$$\left\{ \sup_{u \in \mathcal{U}_{t,T}} J(t, x; u, \beta(u)) \right\}(\tau_h) = \sup_{u \in \mathcal{U}_{t,T}} \{ J(t, x; u, \beta(u)) (\tau_h) \}, P\text{-a.s.}$$

Now let $h \in H_t$,

$$V^-(t, x)(\tau_h) = \inf_{\beta \in B_{t,T}} \sup_{u \in \mathcal{U}_{t,T}} \left\{ J(t, x; u, \beta(u)) (\tau_h) \right\}$$

$$= \inf_{\beta \in B_{t,T}} \sup_{u \in \mathcal{U}_{t,T}} \left\{ J(t, x; u(\tau_h), \beta^h(u(\tau_h))) \right\}$$

$$= \inf_{\beta \in B_{t,T}} \sup_{u \in \mathcal{U}_{t,T}} \left\{ J(t, x; u, \beta^h(u)) \right\}$$

$$= \inf_{\beta \in B_{t,T}} \sup_{u \in \mathcal{U}_{t,T}} \left\{ J(t, x; u, \beta(u)) \right\}$$

$$= V^-(t, x), P\text{-a.s.},$$

which holds even for all $h \in H$. Recall the definition of the filtration, the $\mathcal{F}_t$-measurable random variable $V^-(t, x)(\omega)$, $\omega \in \Omega$, depends only on the restriction of $\omega$ to the time interval $[0, t]$. We complete our proof with help of Lemma 3.4 in [9].