Periodic points of algebraic functions related to a continued fraction of Ramanujan

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Abstract. A continued fraction \( v(\tau) \) of Ramanujan is evaluated at certain arguments in the field \( K = \mathbb{Q}(\sqrt{-d}) \), with \(-d \equiv 1 \pmod{8}\), in which the ideal \((2) = \mathfrak{p}_2 \mathfrak{p}_2'\) is a product of two prime ideals. These values of \( v(\tau) \) are shown to generate the inertia field of \( \mathfrak{p}_2 \) or \( \mathfrak{p}_2' \) in an extended ring class field over the field \( K \). The conjugates over \( \mathbb{Q} \) of these same values, together with \( 0, -1 \pm \sqrt{2} \), are shown to form the exact set of periodic points of a fixed algebraic function \( \hat{F}(x) \), independent of \( d \). These are analogues of similar results for the Rogers-Ramanujan continued fraction.

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1. Introduction

This paper is concerned with values of Ramanujan’s continued fraction

\[
v(\tau) = \frac{q^{1/2}}{1 + q} + \frac{q^2}{1 + q^3} + \frac{q^4}{1 + q^5} + \frac{q^6}{1 + q^7} + \ldots, \quad q = e^{2\pi i \tau},
\]

Received October 18, 2022.
2020 Mathematics Subject Classification. 14H05, 37F05, 11R37, 11D88, 11R29.
Key words and phrases. Periodic points, algebraic function, 2-adic field, extended ring class fields, Ramanujan continued fraction.
sometimes referred to as the Ramanujan-Göllnitz-Gordon continued fraction, which is also given by the infinite product

\[ v(\tau) = q^{1/2} \prod_{n=1}^{\infty} \left(1 - q^n\right)^{\left(\frac{\tau}{\pi}\right)}, \quad q = e^{2\pi i \tau}, \]

for \( \tau \) in the upper half-plane. Here, \( \left(\frac{\tau}{n}\right) \) is the Kronecker symbol. See [12], [9, p. 153], [5], [6]. The continued fraction \( v(\tau) \) is analogous to the Rogers-Ramanujan continued fraction

\[ r(\tau) = q^{1/5} \prod_{n=1}^{\infty} \left(1 - q^n\right)^{\left(\frac{\tau}{5}\right)}, \quad q = e^{2\pi i \tau}, \]

whose properties were considered in the papers [17], [18]. In [17] it was shown that certain values of \( r(\tau) \), for \( \tau \) in the imaginary quadratic field \( K = \mathbb{Q}(\sqrt{-d}) \) with \(-d \equiv \pm 1 \pmod{5}\), are periodic points of a fixed algebraic function, independent of \( d \), and generate certain class fields \( \Sigma \Omega \) over \( K \). Here \( \Sigma \) is the ray class field of conductor \( \mathfrak{p}_2 \) over \( K \), and \( \Omega \) is the ring class field of conductor \( f \) corresponding to the order \( R_{-d} \) of discriminant \(-d = \mathfrak{d}_K f^2 \) in \( K \) (\( \mathfrak{d}_K \) is the discriminant of \( K \)).

Here we will show that a similar situation holds for certain values of the continued fraction \( v(\tau) \). We consider discriminants of the form \(-d \equiv 1 \pmod{8}\) and arguments in the field \( K = \mathbb{Q}(\sqrt{-d}) \). Let \( R_K \) be the ring of integers in this field and let the prime ideal factorization of \( (2) \) in \( R_K \) be \( (2) = \mathfrak{p}_2 \mathfrak{p}_2' \). We define the algebraic integer \( w \) by

\[ w = \frac{a + \sqrt{-d}}{2}, \quad a^2 + d \equiv 0 \pmod{2^5}, \quad (N(w), f) = 1, \quad (1.1) \]

where \( \mathfrak{p}_2 = (2, w) \). Also, the positive (and odd) integer \( f \) is defined by \(-d = \mathfrak{d}_K f^2 \), where \( \mathfrak{d}_K \) is the discriminant of \( K / \mathbb{Q} \).

We will show that

\[ v(w/8) = \pm \frac{1 \pm \sqrt{1 + \pi^2}}{\pi}, \]

where \( \pi \) is a generator in \( \Omega_f \) of the ideal \( \mathfrak{p}_2 \) (or rather, its extension \( \mathfrak{p}_2 R_\Omega \) in \( \Omega_f \)). The algebraic integer \( \pi \) and its conjugate \( \xi \) in \( \Omega_f \) were studied in [14] and shown to satisfy

\[ \pi^4 + \xi^4 = 1, \quad (\pi) = \mathfrak{p}_2, \quad (\xi) = \mathfrak{p}_2', \quad \xi = \frac{\pi^2 + 1}{\pi^2 - 1}, \quad (1.2) \]

where \( \tau = \left(\frac{\Omega_f/K}{\mathfrak{p}_2} \right) \) is the Artin symbol (Frobenius automorphism) for the prime ideal \( \mathfrak{p}_2 \) and the ring class field \( \Omega_f \) over \( K \) whose conductor is \( f \). It follows from results of [14] that

\[ \pi = (-1)^c \mathfrak{p}(w), \]
where $c$ is an integer satisfying the congruence
\[ c \equiv 1 - \frac{a^2 + d}{32} \pmod{2} \]
and $p(\tau)$ is the modular function $p(\tau) = \frac{f_2(\tau/2)}{f(\tau/2)}$, defined in terms of the Weber-Schl"afl"i functions $f_2(\tau), f(\tau)$. (See [20], [8], [19].) The above formula for $v(w/8)$ follows from the identity
\[ \frac{2}{p(8\tau)} = \frac{1 - v^2(\tau)}{v(\tau)} = \frac{1}{v(\tau)} - v(\tau), \]
for $\tau$ in the upper half-plane, which we prove in Proposition 4.1. (Also see [7, Thm. 8.6, p. 475].)

As in [17], we consider a diophantine equation, namely
\[ E_2 : X^2 + Y^2 = \sigma^2(1 + X^2Y^2), \quad \sigma = -1 + \sqrt{2}. \]

An identity for the continued fraction $v(\tau)$ implies that
\[ (X, Y) = (v(w/8), v(-1/w)) \]
is a point on $E_2$. We prove the following theorem relating the coordinates of this point.

**Theorem A.** Let $w$ be given by (1.1) with $q_2 = (2, w)$ in $R_K$ and $-d = d_K f^2 \equiv 1 \pmod{8}$.

(a) The field $F_1 = \mathbb{Q}(v(w/8)) = \mathbb{Q}(v^2(w/8))$ equals the field $\Sigma_{q_2^3} \Omega_f$, where $\Sigma_{q_2^3}$ is the ray class field of conductor $f = q_2^3$ and $\Omega_f$ is the ring class field of conductor $f$ over the field $K$. The field $F_1$ is the inertia field for $q_2$ in the extended ring class field $L_{0,8} = \Sigma_8 \Omega_f$ over $K$, where $O = R_{-d}$ is the order of discriminant $-d$ in $K$.

(b) We have $F_2 = \mathbb{Q}(v(-1/w)) = \Sigma_{q_2} \Omega_f$, the inertia field of $q_2$ in $L_{0,8}/K$.

(c) If $\tau_2$ is the Frobenius automorphism $\tau_2 = \left( \frac{F_1/K}{q_2} \right)$, then
\[ v(-1/w) = \frac{v(w/8)^2 + (-1)^c \sigma}{\sigma v(w/8)^2 - (-1)^c}. \quad (1.3) \]

See Theorems 6.1, 7.3 and 7.5 and their corollaries. From part (c) of this theorem we deduce the following.

**Theorem B.**

(a) If $w$ and $c$ are as above, then the generator $(-1)^{1+c} v(w/8)$ of the field $\Sigma_{q_2^3} \Omega_f$ over $\mathbb{Q}$ is a periodic point of the multivalued algebraic function $\hat{F}(x)$ given by
\[ \hat{F}(x) = -\frac{x^2 - 1}{2} \pm \frac{1}{2} \sqrt{x^4 - 6x^2 + 1}; \]
and \( v^2(w/8) \) is a periodic point of the algebraic function \( \mathcal{T}(x) \) defined by

\[
\mathcal{T}(x) = \frac{1}{2}(x^2 - 4x + 1) \pm \frac{x - 1}{2}\sqrt{x^2 - 6x + 1}.
\]

(b) The minimal period of \((-1)^{1+c}v(w/8)\) (and of \(v^2(w/8)\)) is equal to the order of the automorphism \( \tau_2 \) in \( \text{Gal}(F_1/K) \).

(c) Together with the numbers 0, \(-1 \pm \sqrt{2}\), the values \((-1)^{1+c}v(w/8)\) and their conjugates over \( \mathbb{Q} \) are the only periodic points of the algebraic function \( \mathcal{F}(x) \) in \( \overline{\mathbb{Q}} \) or \( \mathbb{C} \). The only periodic points of \( \mathcal{T}(x) \) in \( \overline{\mathbb{Q}} \) or \( \mathbb{C} \) are 0, \((-1 \pm \sqrt{2})^2\), and the conjugates of the values \( v^2(w/8) \) over \( \mathbb{Q} \).

We understand by a periodic point of the multivalued algebraic function \( \mathcal{F}(x) \) the following. Let \( f(x, y) = x^2 y + x^2 + y^2 - y \) be the minimal polynomial of \( \mathcal{F}(x) \) over \( \mathbb{Q}(x) \). A periodic point of \( \mathcal{F}(x) \) is an algebraic number \( a \) for which there exist \( a_1, a_2, \ldots, a_{n-1} \in \overline{\mathbb{Q}} \) satisfying

\[
f(a, a_1) = f(a_1, a_2) = \cdots = f(a_{n-1}, a) = 0.
\]

A similar definition can be given over any ground field \( k \). See \[15\], \[16\]. Thus, if \( a \in \overline{\mathbb{Q}} \) is a periodic point of \( \mathcal{F}(x) \), so are its conjugates over \( \mathbb{Q} \), because \( f(x, y) \) has coefficients in \( \mathbb{Q} \). We show in Section 8 that \( v^2(w/8) \) is actually a periodic point in the usual sense of the single-valued 2-adic function

\[
T(x) = x^2 - 4x + 2 - (x - 1)(x - 3) \sum_{k=1}^{\infty} C_{k-1} \frac{2^k}{(x - 3)^{2k}},
\]

defined on a subset of the maximal unramified, algebraic extension \( K_2 \) of \( \mathbb{Q} \). (\( C_k \) is the \( k \)-th Catalan number.) This follows from the fact that

\[
v(w/8)^2 \tau_2 = T(v(w/8)^2),
\]

in the completion \( F_{1,q} \subset K_2 \) of \( F_1 = \Sigma_{\varphi^2_2} \Omega_f \) with respect to \( q \) prime divisor \( q \) of \( \varphi_2 \) in \( F_1 \). This implies that the minimal period of \( v^2(w/8) \) with respect to the function \( T(x) \) is \( n = \text{ord}(\tau_2) \).

From Theorems A and B we conclude the following.

**Theorem C.** Let \( K = \mathbb{Q}(\sqrt{-d}) \), with \( -d \equiv 1 \mod 8 \) and \( \varphi_2 \varphi_2' \in R_K \). Then every class field over \( K \) of the form \( \Sigma_{\varphi^2_2} \Omega_f \) or \( \Sigma_{\varphi'^2_2} \Omega_f \) (with \( f \) odd) is generated over \( \mathbb{Q} \) by an individual periodic point of the function \( \mathcal{F}(x) \) (or of \( \mathcal{T}(x) \)). Furthermore, all but three periodic points of \( \mathcal{F}(x) \) in \( \overline{\mathbb{Q}} \) generate a class field \( \Sigma_{\varphi^2_2} \Omega_f \) in this family over some imaginary quadratic field \( K = \mathbb{Q}(\sqrt{-d}) \), for which \( -d = d_k f^2 \equiv 1 \) (mod 8).

These are all analogues of the corresponding facts for the Rogers-Ramanujan continued fraction \( r(\tau) \) which were proved in \[17\] and \[18\].

An important corollary of the fact that the conjugates of the values \( v(w/8) \) in Theorem B are, together with the three fixed points, all the periodic points of the algebraic function \( \mathcal{F}(x) \), is the following class number formula. In this
formula, \( h(-d) \) denotes the class number of the order \( R_{-d} \) of discriminant \(-d\) in the quadratic field \( K = K_d \), and \( D_{n,2} \) is the finite set of negative discriminants \(-d \equiv 1 \pmod{8}\) for which the Frobenius automorphism \( \tau_2 \) in Theorem A has order \( n \) in \( \text{Gal}(F_1 / K_d) \), where \( F_1 = F_{1,d} \) also depends on \( d \):

\[
\sum_{d \in D_{n,2}} h(-d) = \frac{1}{2} \sum_{k | n} \mu(n/k) 2^k, \quad n > 1. \tag{1.4}
\]

(\( \mu(n) \) is the Möbius function.) See Theorem 9.2. This fact is the analogue for the prime \( p = 2 \) of Theorem 1.3 in [18] for the prime \( p = 5 \), or of Conjecture 1 of that paper for a prime \( p > 5 \).

The layout of the paper is as follows. Section 2 contains a number of \( q \)-identities (following Ramanujan) and theta function identities which we use to prove identities for various modular functions in Sections 3-5. Most of these identities are known; straightforward proofs – which use theta functions, but not the theory of modular forms or functions – are included here for the sake of completeness. In Sections 6 and 7 we prove Theorem A. The proofs of Theorem B and (1.4) are given in Sections 8 and 9.

Sections 2-5 and portions of Sections 6-9 also appear in the Ph.D. dissertation [1] of the first author.

2. Preliminaries.

As is customary, let us set

\[
(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=0}^{n-1} (1 - a q^k), \quad n \geq 1
\]

and

\[
(a;q)_\infty := \prod_{k=0}^{\infty} (1 - a q^k), \quad |q| \leq 1.
\]

Ramanujan’s general theta function \( f(a,b) \) is defined as

\[
f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \tag{2.1}
\]

Three special cases are defined, in Ramanujan’s notation, as

\[
\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \tag{2.2}
\]

\[
\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2}, \tag{2.3}
\]

\[
f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}. \tag{2.4}
\]
Jacobi’s triple product identity, in Ramanujan’s notation, takes the form
\[ f(a,b) = (-a;ab)_{\infty}(-b;ab)_{\infty}(ab;ab)_{\infty}. \]  
(2.5)

Using this, the above three functions can be written as
\[ \varphi(q) = (-q;q^2)_{\infty}(q^2;q^2)_{\infty}, \]  
(2.6)
\[ \psi(q) = (-q;q)_{\infty}(q^2;q^2)_{\infty} = \frac{(q^2;q^2)_{\infty}}{(q;q^2)_{\infty}}, \]  
(2.7)
\[ f(-q) = (q;q)_{\infty}. \]  
(2.8)

The equality that relates the right hand sides of both the equations for \( f(-q) \) in (2.4) and (2.8) is Euler’s pentagonal number theorem.

Another important function that plays a prominent role is given by
\[ \chi(q) := (-q;q^2)_{\infty}. \]  
(2.9)

All the above four functions satisfy a myriad of relations, most of which are listed and proved in Berndt’s books on Ramanujan’s notebooks, and we will refer to them as needed.

Last but not least, the Dedekind-eta function is defined as
\[ \eta(\tau) = q^{1/24} f(-q), \quad q = e^{2\pi i \tau}, \quad \text{Im} \tau > 0. \]  
(2.10)

Most of the identities that we use later on are listed here in order, for the sake of convenience.

\[ \varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2), \]  
(2.11)
\[ \varphi^4(q) - \varphi^4(-q) = 16q\psi^4(q^2), \]  
(2.12)
\[ \varphi(q)\psi(q^2) = \psi^2(q), \]  
(2.13)
\[ \varphi(-q) + \varphi(q^2) = 2 \frac{f^2(q^3,q^5)}{\psi(q)}, \]  
(2.14)
\[ \varphi(-q) - \varphi(q^2) = 2q \frac{f^2(q,q^7)}{\psi(q)}, \]  
(2.15)
\[ \varphi(q)\varphi(-q) = \varphi^2(-q^2), \]  
(2.16)
\[ \varphi(q) + \varphi(-q) = 2\varphi(q^4), \]  
(2.17)
\[ \varphi^2(q) - \varphi^2(-q) = 8q\psi^2(q^4). \]  
(2.18)

All of the above identities and their proofs can be found in [2, p. 40, Entry 25] and in [2, p. 51, Example (iv)].

For \( \tau \in \mathcal{H} \), the upper half plane, and \( q = e(\tau) = e^{2\pi i \tau} \), the theta constant with characteristic \( \left[ \frac{c}{c'} \right] \in \mathbb{R} \) is defined as
\[ \theta \left[ \frac{c}{c'} \right] (\tau) = \sum_{n \in \mathbb{Z}} e \left( \frac{1}{2} \left( n + \frac{c}{2} \right)^2 \tau + \frac{c'}{2} \left( n + \frac{c}{2} \right) \right). \]  
(2.19)
It satisfies the following basic properties for \( l, m, n \in \mathbb{Z} \) with \( N \) positive:

\[
\theta \left[ \frac{\varepsilon}{\varepsilon'} \right] (\tau) = e^{\pm \frac{e m}{2}} \theta \left[ \pm \varepsilon + 2l \right] (\tau), \tag{2.20}
\]

\[
\theta \left[ \frac{\varepsilon}{\varepsilon'} \right] (\tau) = \sum_{k=0}^{N-1} \theta \left[ \frac{e^{+2k}}{N \varepsilon'} \right] (N^2 \tau). \tag{2.21}
\]

We also have the transformation law, for \((a \ b \ c \ d) \in SL_2(\mathbb{Z})\):

\[
\theta \left[ \frac{\varepsilon}{\varepsilon'} \right] (a \tau + b) = e^{\pm \frac{e m}{2}} \theta \left[ \frac{\varepsilon' \pm \varepsilon}{b \varepsilon' + bd} \right] (\tau), \tag{2.22}
\]

where

\[
\kappa = e \left( -\frac{1}{4} (ace + c'e')bd - \frac{1}{8} (ace^2 + cde^2 + 2cde'e') \right),
\]

and \( \kappa_0 \) is an eighth root of unity depending only on the matrix \((a \ b \ c \ d)\).

In particular, we have:

\[
\theta \left[ \frac{\varepsilon}{\varepsilon'} \right] (1 + \tau) = e^{-\frac{e}{4}(1 + \frac{\varepsilon}{2})} \theta \left[ \varepsilon + \varepsilon' + 1 \right] (\tau), \tag{2.23}
\]

\[
\theta \left[ \frac{\varepsilon}{\varepsilon'} \right] (-\frac{1}{\tau}) = e(-\frac{1}{8}) \sqrt{\tau} e^{(\frac{\varepsilon'}{4})} \theta \left[ \frac{\varepsilon'}{-\varepsilon} \right] (\tau). \tag{2.24}
\]

We also have the product formula:

\[
\theta \left[ \frac{\varepsilon}{\varepsilon'} \right] (\tau) = e^{\varepsilon' e} \frac{q^{\frac{e}{2}}}{q} \prod_{n=1}^{\infty} (1 - q^n)(1 + e^{\left(\frac{\varepsilon'}{2}\right) q^{n-\frac{1+e}{2}}}(1 + e^{\left(-\frac{\varepsilon'}{2}\right) q^{n-\frac{1-e}{2}}}), \tag{2.25}
\]

which follows from Jacobi’s triple product identity.

More information about these theta constants and the above formulas, as well as their proofs, can all be found in [10, pp. 71-81]. Also see [9, pp. 143, 158-159].

### 3. Identities for \( u(\tau) \) and \( v(\tau) \)

Let us define the functions \( u(\tau) \) and \( v(\tau) \) as

\[
u(\tau) = q^{1/2} \prod_{n=1}^{\infty} (1 - q^n)^{\left(\frac{3}{n}\right)}.
\]

The functions \( u(\tau) \) and \( v(\tau) \) satisfy the following identities.
**Proposition 3.1.** (a) If \( x = u(\tau) \) and \( y = u(2\tau) \), we have
\[
x^4(y^4 + 1) = 2y^2.
\]
(b) If \( x = v(\tau) \) and \( y = v(2\tau) \), we have
\[
x^2y + x^2 + y^2 = y.
\]

**Remark.** The curve \( E : f(x, y) = 0 \) defined by
\[
f(x, y) = x^2y + x^2 + y^2 - y
\]
is an elliptic curve with \( j(E) = 1728 \), so \( E \) has complex multiplication by \( R = \mathbb{Z}[i] \).

**Proof.** (a) From (2.11), we have
\[
\varphi^2(-q) = 2\varphi^2(q^2) - \varphi^2(q),
\]
where
\[
\varphi(q) = (-q; q^2)\infty(q^2; q^2)\infty \quad \text{and} \quad \psi(q) = (q^2; q^2)\infty \quad (q; q^2)\infty
\]
are as defined in (2.6) and (2.7). Squaring both sides gives us
\[
\varphi^4(-q) = 4\varphi^4(q^2) - 4\varphi^2(q^2)\varphi^2(q) + \varphi^4(q).
\]
Using
\[
\varphi^4(q) - \varphi^4(-q) = 16q\psi^4(q^2),
\]
which is (2.12), we obtain
\[
\varphi^4(q^2) + 4q\psi^4(q^2) = \varphi^2(q^2)\varphi^2(q^2).
\]
Dividing both sides by \( \varphi^4(q^2) \) and using the relation \( \psi^2(q^2) = \varphi(q)\psi(q^2) \) from (2.13) we get
\[
1 + 4q\frac{\psi^4(q^2)}{\varphi^4(q^2)} = \frac{\varphi^2(q)}{\varphi^2(q^2)} = \frac{\psi^2(q^2)}{\varphi^2(q^2)} \cdot \frac{\varphi^4(q)}{\psi^4(q)}. \tag{3.1}
\]
Since
\[
u(\tau) = \sqrt{2q^{1/8}} \prod_{n=1}^{\infty} (1 + q^n)^{(1-q^n)} = \sqrt{2q^{1/8}} \frac{(-q^2; q^2)\infty}{(-q; q^2)\infty} = \sqrt{2q^{1/8}} \frac{\psi(q)}{\varphi(q)},
\]
the result follows by substituting the last equality for \( u(\tau) \) into (3.1).

(b) From [9, p. 153, (9.7)] we have the following relation between \( u = u(\tau) \) and \( v = v(\tau) \):
\[
u^4(v^2 + 1)^2 + 4v(v^2 - 1) = 0; \tag{3.2}
\]
which we rewrite as $u^4 = \frac{4v(1-v^2)}{(v^2 + 1)^2}$. (See the proof of Proposition 10.1 in the Appendix.) Substituting this expression for $u^4$ into the relation $u^4(\tau)[u^4(2\tau) + 1] = 2u^2(2\tau)$, after squaring, we obtain

$$\frac{16x^2(1-x^2)^2}{(x^2 + 1)^4} \cdot \left[ \frac{4y(1-y^2)}{(y^2 + 1)^2} + 1 \right]^2 = 4 \cdot \frac{4y(1-y^2)}{(y^2 + 1)^2},$$

where $x = v(\tau), y = v(2\tau)$. Clearing the denominators gives us

$$x^2(1-x^2)^2(y^2 - 2y - 1)^4 = y(1-y^2)(y^2 + 1)^2(x^2 + 1)^4.$$

Now moving everything to one side and factoring the polynomial using Maple, we finally arrive at

$$(x^2y + x^2 + y^2 - y)(x^2y^2 - x^2y + y + 1)(x^2y^2 + 2xy^2 + x^2 - 4xy + y^2 - 2x + 1)$$

$$\times (x^2y^2 - 2xy^2 + x^2 + 4xy + y^2 + 2x + 1) = 0.$$  

From the definitions of $x$ and $y$, it is clear that $x = O(q^{1/2})$ and $y = O(q)$ as $q$ tends to 0. Hence, the first factor above (and none of the others) vanishes for $q$ sufficiently small. By the identity theorem of complex analysis, the first factor vanishes for $|q| < 1$. This proves the result.

**Remark.** The identity in part (b) of Proposition 3.1 can be written as

$$v^2(\tau) = v(2\tau) \frac{1 - v(2\tau)}{1 + v(2\tau)}.$$ 

See [5, Thm. 2.2]. This is analogous to the identity for the Rogers-Ramanujan continued fraction $r(\tau)$:

$$r^5(\tau) = r(5\tau) \frac{r^4(5\tau) - 3r^3(5\tau) + 4r^2(5\tau) - 2r(5\tau) + 1}{r^4(5\tau) + 2r^3(5\tau) + 4r^2(5\tau) + 3r(5\tau) + 1}.$$ 

Also see [4, p. 167], [3, pp. 19-20].

**Proposition 3.2.** The functions $x = v^2(\tau)$ and $y = v^2(2\tau)$ satisfy the relation

$$g(x, y) = y^2 - (x^2 - 4x + 1)y + x^2 = 0.$$ 

**Proof.** For $x = v(\tau)$ and $y = v(2\tau)$, we have the relation

$$x^2 + y^2 = y(1 - x^2).$$ 

Squaring both sides and moving all the terms to the left side, we obtain

$$x^4 + y^4 + 4x^2y^2 - x^4y^2 - y^2 = 0.$$ 

Hence, $x = v^2(\tau)$ and $y = v^2(2\tau)$ satisfy the relation

$$g(x, y) = x^2 + y^2 + 4xy - x^2y - y = 0.$$ 

□
Let $A, \bar{A}$ denote the linear fractional mappings
\[ A(x) = \frac{\sigma x + 1}{x - \sigma}, \quad \bar{A}(x) = \frac{-x + \sigma}{\sigma x + 1}, \quad \sigma = -1 + \sqrt{2}. \quad (3.3) \]

**Proposition 3.3.** The following identity holds:
\[ v\left(-\frac{1}{\tau}\right) = \bar{A}(v(\tau/4)) = \frac{\bar{\sigma}v(\tau/4) + 1}{v(\tau/4) - \bar{\sigma}} = \frac{-v(\tau/4) + \sigma}{\sigma v(\tau/4) + 1}, \]
where $\bar{\sigma} = -1 - \sqrt{2}$.

**Proof.** This follows from the formula
\[ v(\tau) = e^{-2\pi i/8} \frac{\theta[3/4][8\tau]}{\theta[1/4][8\tau]}, \]
using the formulas (2.20), (2.21), (2.24). (Also see [10].) Namely, we have:
\[ v\left(-\frac{1}{\tau}\right) = e^{-2\pi i/8} \frac{\theta[3/4][\tau/8]}{\theta[1/4][\tau/8]} = \frac{\sum_{k=0}^{3} \theta \left[ \frac{1 + 2k}{4} \right] (2\tau)}{\sum_{k=0}^{3} \theta \left[ \frac{1 + 2k}{4} \right] (2\tau)}, \]
which after some simplification yields
\[ v\left(-\frac{1}{\tau}\right) = \left[ -1 + e^{3\pi i/8} \right] v(\tau/4) + \left[ e^{2\pi i/8} + e^{3\pi i/2} \right] \frac{v(\tau/4) + 1}{e^{2\pi i/8} + e^{3\pi i/2}} v(\tau/4) + \left[ 1 + e^{7\pi i/8} \right]. \]
This yields that
\[ v\left(-\frac{1}{\tau}\right) = \frac{\bar{\sigma}v(\tau/4) + 1}{v(\tau/4) - \bar{\sigma}} = \frac{-v(\tau/4) + \sigma}{\sigma v(\tau/4) + 1}. \]

\[ \square \]

The set of mappings
\[ \tilde{H} = \{x, A(x), \bar{A}(x), -1/x\} \]
forms a group under composition. We also have the formula
\[ (\sigma x + 1)^2(\sigma y + 1)^2 f(\bar{A}(x), \bar{A}(y)) = 2^{3/2} \sigma^2 f(y, x). \]

**Proposition 3.4.** The function $v(\tau)$ satisfies the following:
\[ v^2\left(-\frac{1}{8\tau}\right) = \frac{v^2(\tau) - \sigma^2}{\sigma^2 v^2(\tau) - 1}, \quad \sigma = -1 + \sqrt{2}. \quad (3.4) \]
Proof. Replacing $\tau$ by $8\tau$ in Proposition 3.3 and squaring gives us

$$v^2\left(\frac{-1}{8\tau}\right) = \frac{(-v(2\tau) + \sigma)^2}{(\sigma v(2\tau) + 1)^2} = \frac{(-y + \sigma)^2}{(\sigma y + 1)^2} = \frac{y^2 - 2\sigma y + \sigma^2}{\sigma^2 y^2 + 2\sigma y + 1},$$

where $y = v(2\tau)$. Then, replace $2\sigma$ by $1 - \sigma^2$ to obtain

$$v^2\left(\frac{-1}{8\tau}\right) = \frac{y^2 - y + \sigma^2 y + \sigma^2}{\sigma^2 y^2 + y - \sigma^2 y + 1} = \frac{\sigma^2(y + 1) - (y - y^2)}{(y + 1) - \sigma^2(y - y^2)}.$$

Now replace $(y - y^2)$ by $x^2(y + 1)$, using Proposition 3.1(b), to get the result:

$$v^2\left(\frac{-1}{8\tau}\right) = \frac{x^2(y + 1) - x^2(y + 1)}{(y + 1) - \sigma^2 x^2(y + 1)} = \frac{(\sigma^2 - x^2)(y + 1)}{(1 - \sigma^2 x^2)(y + 1)} = \frac{x^2 - \sigma^2}{\sigma^2 x^2 - 1},$$

where $x = v(\tau)$. 

For later use we denote the linear fractional map which occurs in (3.4) by $t(x)$:

$$t(x) = \frac{x - \sigma^2}{\sigma^2 x - 1}. \quad (3.5)$$

A straightforward calculation shows that

$$(\sigma^2 x - 1)^2(\sigma^2 y - 1)^2 g(t(x), t(y)) = 2^5\sigma^4 g(y, x). \quad (3.6)$$

4. The relation between $v(\tau)$ and $p(\tau)$.

In this section and the next we shall prove several identities between $v(\tau)$ and the functions $p(\tau)$ and $b(\tau)$ defined as follows. Let $f, f_1, f_2$ denote the Weber-Schlӓfli functions (see [8, p. 233], [19, p. 148]). Then the functions $p(\tau)$ and $b(\tau)$ are given by

$$p(\tau) = \frac{f_2(\tau/2)^2}{f(\tau/2)^2} = 2^q 1/16 \prod_{n=1}^{\infty} \left( \frac{1 + q^{n/2}}{1 + q^{n/2 - 1/4}} \right)^2, \quad (4.1)$$

$$b(\tau) = \frac{f_1(\tau/2)^2}{f(\tau/2)^2} = 2^q \prod_{n=1}^{\infty} \left( \frac{1 - q^{n/2 - 1/4}}{1 + q^{n/2 - 1/4}} \right)^2. \quad (4.2)$$
Note that \( b(\tau) \) occurs in [14, §10, (10.3)].

**Proposition 4.1.** We have the identity

\[
\frac{2}{p(8\tau)} = \frac{1 - v^2(\tau)}{v(\tau)} = \frac{1}{v(\tau)} - v(\tau).
\] (4.3)

**Proof.** (See [2, pp. 221–222].) The function \( v(\tau) \) satisfies

\[
v(\tau) = q^{1/2} \prod_{n \geq 1} (1 - q^n)^{\frac{8}{n}} = q^{1/2} \prod_{n \geq 1} \frac{(1 - q^{8n-1})(1 - q^{8n-7})}{(1 - q^{8n-3})(1 - q^{8n-5})}
\]

\[
= q^{1/2} \frac{(q; q^8)_\infty(q^7; q^8)_\infty}{(q^3; q^8)_\infty(q^2; q^8)_\infty}.
\]

This gives that

\[
\frac{1}{v(\tau)} - v(\tau) = q^{-1/2} \frac{(q^3; q^8)_\infty(q^5; q^8)_\infty - q^7 (q; q^8)_\infty(q^7; q^8)_\infty}{(q; q^8)_\infty(q^2; q^8)_\infty}
\]

\[
= q^{1/2} \frac{(q^3; q^8)_\infty(q^5; q^8)_\infty - q (q; q^8)_\infty(q^7; q^8)_\infty}{(q^3; q^8)_\infty(q^5; q^8)_\infty - q (q; q^8)_\infty(q^7; q^8)_\infty}
\]

\[
= q^{1/2} (q; q^2)_\infty.
\]

Multiplying the numerator and the denominator by \((q^8; q^8)_\infty\) and applying Jacobi's triple product identity in the form

\[
f(a, b) = (-a; ab)_\infty(-b; ab)_\infty(ab; ab)_\infty,
\]

with \((a, b) = (-q^3, -q^5)\) for the first term in the numerator and \((a, b) = (-q, -q^7)\) for the second, we obtain

\[
\frac{1}{v(\tau)} - v(\tau) = \frac{q^2 (-q^3, -q^5) - q f^2(-q, -q^7)}{q^{1/2} (q; q^2)_\infty(q^8; q^8)_\infty}
\]

Now replace \( q \) by \(-q\) in (2.14), (2.15) and apply this to the numerator to get

\[
\frac{1}{v(\tau)} - v(\tau) = \frac{\psi(-q)[\varphi(q) + \varphi(q^2)] - \psi(-q)[\varphi(q) - \varphi(q^2)]}{2 q^{1/2} (q; q^2)_\infty(q^8; q^8)_\infty}
\]

\[
= \frac{\psi(-q) \times \varphi(q^2)}{q^{1/2} (q; q^2)_\infty(q^8; q^8)_\infty}.
\]

This yields that

\[
\frac{1}{v(\tau)} - v(\tau) = q^{-1/2} \frac{(q^2; q^2)_\infty}{(-q; q^2)_\infty} \times \frac{(-q^2; q^2)_\infty(q^4; q^4)_\infty}{(q; q^2)_\infty(q^8; q^8)_\infty}.
\]
\[ q^{-1/2} \left( -q^2; q^4 \right)_\infty \left( q^2; q^4 \right)_\infty \]
\[ = q^{-1/2} \left( -q^2; q^4 \right)_\infty \left( q^4; q^4 \right)_\infty \]
\[ = q^{-1/2} \left( -q^2; q^4 \right)_\infty \left( q^8; q^8 \right)_\infty \]
\[ = q^{-1/2} \left( -q^2; q^4 \right)_\infty \left( q^2; q^4 \right)_\infty \]
\[ = q^{-1/2} \left( -q^2; q^4 \right)_\infty \left( q^8; q^8 \right)_\infty \]
\[ = q^{-1/2} \left( -q^2; q^4 \right)_\infty \left( q^2; q^4 \right)_\infty \]
\[ = q^{-1/2} \left( -q^2; q^4 \right)_\infty \left( q^8; q^8 \right)_\infty \]

Since
\[ p(8\tau) = q^{1/2} \prod_{n \geq 1} \left( 1 + q^{4n} \right)^2 = 2 q^{1/2} \frac{(-q^4; q^8)_\infty}{(-q^2; q^4)_\infty}, \]
we get the result by substituting into the last equality. \( \square \)

**Proposition 4.2.** The function \( p(\tau) \) satisfies the identity
\[ p^2(\tau)p^2(2\tau) + p^2(\tau) - 2p(2\tau) = 0. \]

**Proof.** We use the relation between \( x = v(\tau) \) and \( y = v(2\tau) \) from Proposition 3.1(b): \( x^2 = \frac{y(1-y)}{(1+y)} \). This gives
\[ \left( \frac{2x}{1 - x^2} \right)^2 = \frac{4x^2}{(1 - x^2)^2} = \frac{4 \cdot \frac{y(1-y)}{(1+y)}}{(1 - \frac{y(1-y)}{(1+y)})^2} \]
\[ = \frac{4y(1-y)(1+y)}{((1+y) - y(1-y))^2} \]
\[ = \frac{4y(1-y^2)}{(1 + y^2)^2} \]
\[ = \frac{4y(1-y^2)}{4y^2 + (1 - y^2)^2}. \]

Now divide both the numerator and the denominator by \( (1 - y^2)^2 \) to obtain
\[ \left( \frac{2x}{1 - x^2} \right)^2 = \frac{4y}{1-y^2} \cdot \frac{(2y)}{1-y^2} = \frac{2y}{1-y^2}. \] (4.4)

From Proposition 4.1, we know that
\[ p(8\tau) = \frac{2v(\tau)}{1 - v^2(\tau)} = \frac{2x}{1 - x^2}, \]
and
\[ p(16\tau) = \frac{2v(2\tau)}{1 - v^2(2\tau)} = \frac{2y}{1 - y^2}. \]
Thus, (4.4) becomes
\[ p^2(8\tau) = \frac{2p(16\tau)}{p^2(16\tau) + 1}. \]
Replacing \( \tau \) by \( \tau/8 \) and rearranging gives us the result. \( \square \)

**Proposition 4.3.**

a) The functions \( x = b(\tau) \) and \( y = b(2\tau) \) satisfy the relation
\[ x^2y^2 + 4y^2 - 16x = 0. \]

b) The following identity holds between \( x = b(\tau) \) and \( z = b(4\tau) \):
\[ (b(\tau) + 2)^4b^4(4\tau) = 2^8(b^3(\tau) + 4b(\tau)). \]

**Proof.**
a) On putting \( 4\tau \) for \( \tau \) in \( x \), we have
\[ b(4\tau) = 2 \prod_{n=1}^{\infty} \left(1 - \frac{q^{2n-1}}{1 + q^{2n-1}}\right)^2 = 2 \frac{(q; q^2)_{\infty}^2}{(-q; q^2)_{\infty}^2} = 2 \frac{\varphi(-q)}{\varphi(q)}. \]
From (2.11), we have
\[ \varphi^2(-q) + \varphi^2(q) = 2\varphi^2(q^2). \]
Multiplying both sides by \( \varphi^2(-q^2) = \varphi(q)\varphi(-q) \) from (2.16), we obtain
\[ \varphi^2(-q)\varphi^2(-q^2) + \varphi^2(q)\varphi^2(-q^2) = 2\varphi(q)\varphi(-q)\varphi^2(q^2). \]
Now dividing both sides by \( \varphi^2(q)\varphi^2(q^2) \) gives us
\[ \frac{\varphi^2(-q)}{\varphi^2(q)} \cdot \frac{\varphi^2(-q^2)}{\varphi^2(q^2)} + \frac{\varphi^2(-q^2)}{\varphi^2(q^2)} = 2 \frac{\varphi(-q)}{\varphi(q)}. \]
Hence, we see that \( x = b(4\tau) \) and \( y = b(8\tau) \) satisfy the relation
\[ x^2y^2 + 4y^2 - 16x = 0. \]
Now replace \( \tau \) by \( \tau/4 \).

b) From (2.17), upon taking fourth powers, we get
\[ \left[ \varphi(-q) + \varphi(q) \right]^4 = 16 \varphi^4(q^4). \]
Multiplying both sides by \( \varphi^4(-q^4)/[\varphi^4(q)\varphi^4(q^4)] \) gives us
\[ \frac{\left[ \varphi(-q) + \varphi(q) \right]^4}{\varphi^4(q)} \cdot \frac{\varphi^4(-q^4)}{\varphi^4(q^4)} = 16 \frac{\varphi(-q)\varphi(q)}{\varphi^4(q)}. \]
Then using (2.16) twice for the right side, we obtain
\[ \frac{\left[ \varphi(-q) + \varphi(q) \right]^4}{\varphi^4(q)} \cdot \frac{\varphi^4(-q^4)}{\varphi^4(q^4)} = 16 \frac{\varphi(-q)\varphi(q)}{\varphi^4(q)} \cdot \varphi^2(q^2). \]
Now use (2.11) for the last factor on the right side to get
\[ \frac{\left[ \varphi(-q) + \varphi(q) \right]^4}{\varphi^4(q)} \cdot \frac{\varphi^4(-q^4)}{\varphi^4(q^4)} = 8 \frac{\varphi(-q)}{\varphi^3(q)} \cdot [\varphi^3(-q) + \varphi^3(q)]. \]
This implies that
\[
\left[ \frac{\varphi(-q)}{\varphi(q)} + 1 \right]^4 \cdot \left[ \frac{\varphi(-q^4)}{\varphi(q^4)} \right]^4 = 8 \cdot \frac{\varphi(-q)}{\varphi(q)} \cdot \left[ \left( \frac{\varphi(-q)}{\varphi(q)} \right)^2 + 1 \right].
\]
The result follows on multiplying through by \(2^8\) and substituting
\[
b(4\tau) = 2 \frac{\varphi(-q)}{\varphi(q)} \quad \text{and} \quad b(16\tau) = 2 \frac{\varphi(-q^4)}{\varphi(q^4)}
\]
into the above equation, and then replacing \(\tau\) by \(\tau/4\). \(\square\)

5. The relation between \(v(\tau)\) and \(b(\tau)\).

We begin this section by proving the following identity.

**Proposition 5.1.**

\[
\frac{(v^2(\tau) + 1)^2}{v^4(\tau) - 6v^2(\tau) + 1} = \frac{4}{b^2(4\tau)}. \tag{5.1}
\]

**Proof.** We prove (5.1) using the identity relating the Weber-Schläfli functions from [20, p. 86, (12)] (see also [8, p. 234, (12.18)]):

\[
f_8^8(\tau) + f_8^8(\tau) = f_8^8(\tau).
\]

From the definitions (4.1) and (4.2) of \(p(\tau)\) and \(b(\tau)\), this identity translates to

\[
\frac{b^4(4\tau)}{16} = 1 - p^4(4\tau).
\]

Using the result of Proposition 4.1, we write this equation as

\[
\frac{b^4(4\tau)}{16} = 1 - \left( \frac{2v(\tau/2)}{1 - v^2(\tau/2)} \right)^4 = 1 - \frac{16v^4(\tau/2)}{(1 - v^2(\tau/2))^4}.
\]

Setting \(x = v(\tau/2)\) and \(y = v(\tau)\) and using the relation between \(x\) and \(y\) from Proposition 3.1(b) in the form \(x^2 = \frac{y(1-y)}{1+y}\) gives that

\[
\frac{b^4(4\tau)}{16} = 1 - \frac{16 x^4}{(1 - x^2)^4} = 1 - \frac{16 \left( \frac{y(1-y)}{1+y} \right)^2}{(1 - \frac{y(1-y)}{1+y})^4}
\]

\[
= 1 - \frac{16 y^4 (1 - y^2)^2}{(1 + y^2)^4} = \frac{(y^2 + 1)^4 - 16 y^2 (y^2 - 1)^2}{(y^2 + 1)^4}
\]

\[
= \frac{(y^2 - 1)^2 - 4 y^2}{(y^2 + 1)^4}
\]

\[
= \frac{(y^2 - 1)^2 - 4 y^2}{(y^2 + 1)^4}
\]
which is equivalent to (5.1). (The plus sign holds on taking the square-root because $b(i\infty) = 2, v^2(i\infty) = 0$.) □

Proposition 5.1 will now be used to prove the following formula for the function $j(\tau)$ in terms of $v(\tau)$.

**Proposition 5.2.** If $v = v(\tau)$ and $\tau$ lies in the upper half-plane, we have

$$j(\tau) = \frac{(v^{16} + 232v^{14} + 732v^{12} - 1192v^{10} + 710v^8 - 1192v^6 + 732v^4 + 232v^2 + 1)^3}{v^2(v^2 - 1)^2(v^2 + 1)^4(v^4 - 6v^2 + 1)^8}.$$  

**Proof.** Let

$$G(x) = \frac{(x^2 - 16x + 16)^3}{x(x - 16)}.$$  

Then from [14, p. 1967, (2.8)] the function

$$\alpha(\tau) = \zeta_8^{-1} \frac{\eta(\tau/4)^2}{\eta(\tau)^2}, \quad \zeta_8 = e^{2\pi i/8}, \quad (5.2)$$

satisfies the relation

$$j(\tau) = \frac{(\alpha^8 - 16\alpha^4 + 16)^3}{\alpha^4(\alpha^4 - 16)} = G(\alpha^4(\tau)). \quad (5.3)$$

Moreover, $\alpha(\tau)$ and $b(\tau)$ satisfy

$$16\alpha^4(\tau) + 16b^4(\tau) = \alpha^4(\tau)b^4(\tau),$$

so that

$$\alpha^4(\tau) = \frac{16b^4(\tau)}{b^4(\tau) - 16}. \quad (5.4)$$

Setting $b = b(\tau)$, we substitute for $\alpha = \alpha(\tau)$ in (5.3) and find that

$$j(\tau) = G \left( \frac{16b^4}{b^4 - 16} \right) = \left( \frac{b^8 + 224b^4 + 256}{b^4(b^4 - 16)^4} \right), \quad b = b(\tau).$$

Now replace $\tau$ by $4\tau$ and use (5.1) to replace $b^4(4\tau)$ by

$$b^4(4\tau) = \frac{16(v^4 - 6v^2 + 1)^2}{(v^2 + 1)^4},$$

giving

$$j(4\tau) = \frac{(v^{16} - 8v^{14} + 12v^{12} + 8v^{10} + 230v^8 + 8v^6 + 12v^4 - 8v^2 + 1)^3}{v^8(v^2 + 1)^4(v^2 - 1)^8(v^4 - 6v^2 + 1)^2}, \quad (5.5)$$

with $v = v(\tau)$. Replacing $v(\tau)$ by $\tilde{A}(v(-1/4\tau))$ from Proposition 3.3 gives that

$$j(4\tau) = j_2(x^2).$$
where \( x = v(-1/4\tau) \) and \( j_2(x) \) is the rational function
\[
j_2(x) = \frac{(x^8 + 232x^7 + 732x^6 - 1192x^5 + 710x^4 - 1192x^3 + 732x^2 + 232x + 1)^3}{x(x - 1)^2(x + 1)^4(x^2 - 6x + 1)^8}. \tag{5.6}
\]
Finally, replace \( \tau \) by \( \tau/4 \) to give that
\[
j(\tau) = j_2(v^2(-1/\tau)),
\]
which implies that \( j_2(v^2(\tau)) = j(-1/\tau) = j(\tau) \), completing the proof. \( \square \)

We highlight the relation
\[
j(\tau) = j_2(v^2(\tau)), \tag{5.7}
\]
which we will make use of in Section 7. Using the linear fractional map \( t(x) \) from (3.5) and the identity \( v^2(-1/8\tau) = t(v^2(\tau)) \) in (3.4) yields
\[
j\left(\frac{-1}{8\tau}\right) = j_2\left(v^2\left(\frac{-1}{8\tau}\right)\right) = j_2(t(v^2(\tau))).
\]
A calculation on Maple shows that
\[
j_{22}(x) = j_2(t(x)) = \frac{(x^8 - 8x^7 + 12x^6 + 8x^5 - 10x^4 + 8x^3 + 12x^2 - 8x + 1)^3}{x^8(x - 1)^4(x + 1)^2(x^2 - 6x + 1)}. \tag{5.8}
\]
Therefore,
\[
j\left(\frac{-1}{8\tau}\right) = j_{22}(v^2(\tau)).
\]
We take this opportunity to prove the following known identity (see [9, p. 154]) from the results we have established so far.

**Proposition 5.3.**
\[
v^{-2}(\tau) + v^2(\tau) - 6 = \frac{\eta^4(\tau)\eta^2(4\tau)}{\eta^2(2\tau)\eta^4(8\tau)}. \tag{5.9}
\]

**Proof.** We will show that (5.9) follows from (5.1). We first have that
\[
v^{-2}(\tau) + v^2(\tau) - 6 = \frac{v^4(\tau) - 6v^2(\tau) + 1}{v^2(\tau)} \]
\[
= \frac{8}{\left(\frac{(v^2(\tau)+1)^2}{v^4(\tau)+6v^2(\tau)+1}\right) - 1} \]
\[
= \frac{8}{\left(\frac{4}{\eta^2(4\tau)}\right) - 1} = \frac{8b^2(4\tau)}{4 - b^2(4\tau)},
\]
where \( b(4\tau) = v^2(\tau) \).
by (5.1). Using the expression $b(4\tau) = 2\varphi(-q)/\varphi(q)$ from the proof of Proposition 4.3a) and (2.18) gives

$$v^{-2}(\tau) + v^2(\tau) - 6 = \frac{8(4\varphi(-q))}{\varphi^2(q)} - \frac{8\varphi^2(-q)}{\varphi^2(q) - \varphi^2(-q)} = \frac{8\varphi^2(-q)}{8q\psi^2(q^2)}.$$  

Now putting $\varphi(-q) = (q; q^2)^2_\infty(q^2; q^2)_\infty = \frac{(q^4; q^8)^2_\infty}{(q^2; q^4)^2_\infty}$ and $\psi(q) = \frac{(q^2; q^4)^2_\infty}{(q^4; q^8)^2_\infty}$ yields

$$v^{-2}(\tau) + v^2(\tau) - 6 = \varphi^2(-q) \cdot \left(\frac{1}{q\psi^2(q^4)}\right)$$

$$= (q; q^2)^4_\infty(q^2; q^2)_\infty \cdot \frac{(q^4; q^8)^2_\infty}{(q^2; q^4)^2_\infty}$$

$$= \frac{q^{1/6}(q; q^4)^4_\infty \cdot q^{1/3}(q^4; q^4)^4_\infty}{q^{1/6}(q^2; q^2)_\infty \cdot q^{1/3}(q^2; q^2)_\infty}$$

$$= \frac{\eta^4(\tau)\eta^2(4\tau)}{\eta^2(2\tau)\eta^4(8\tau)},$$

using that $\eta(\tau) = q^{1/24}(q; q^4)_\infty$.  

6. The field generated by $u(w/8)$.

As in the Introduction, let $-d \equiv 1 \pmod{8}$ and set $-d = b_Kf^2$, where $b_K$ is the discriminant of the field $K = \mathbb{Q}(\sqrt{-d})$. Further, let $2 \equiv \varphi_2\varphi_2'$ in the ring of integers $R_K$ of $K$. We denote by $\Sigma_f$ the ray class field of conductor $\mathfrak{f}$ over $K$ and $\Omega_f$ the ring class field of conductor $f$ over $K$.

In this section we take $\tau = w/8$, where

$$w = \frac{a + \sqrt{-d}}{2}, \text{ with } a^2 + d \equiv 0 \pmod{2^5}, \ (N(w), f) = 1. \quad (6.1)$$

For this value of $w$,

$$b^4(8\tau) = b^4(w)$$

is the fourth power of the number

$$\beta = i^{-a}b(w) \quad (6.2)$$

from [14, (10.3), Thms. 10.6, 10.7]. We also need the number $\pi$ from [14, (10.2),(10.9)], which is given by

$$\pi = \frac{i^6 \varphi_2(w/2)^2}{\mathfrak{f}(w/2)^2} = i^6p(w),$$
\[ \bar{c} \equiv a \left( 2 - \frac{a^2 + d}{16} \right) \pmod{4}. \]

(We have replaced \( v \) in the formulas of [14] by \( a \) and \( a \) by \( \bar{c} \).) But here the integer \( a^2 + d \) is divisible by 32, by (6.1), so \( \bar{c} \) is even. Replacing \( \bar{c} \) by the integer \( c = \frac{\bar{c}}{2} \), satisfying
\[ c \equiv 1 - \frac{a^2 + d}{32} \pmod{2} \]
yields
\[ \pi = (-1)^c p(w), \quad w = \frac{a + \sqrt{-d}}{2}. \] (6.3)

It follows from the results of [14] that \( \xi = \beta/2 \) and \( \pi \) lie in the ring class field \( \Omega_f \) of the quadratic field \( K = \mathbb{Q}(\sqrt{-d}) \) (where \( -d = b_K f^2 \) and \( b_K \) is the discriminant of \( K/\mathbb{Q} \)) and \( \xi^4 + \pi^4 = 1 \). Furthermore, \( \mathbb{Q}(\pi) = \mathbb{Q}(\pi^4) = \Omega_f \). We also note that \((\xi) = \wp'_2 \) and \((\pi) = \wp_2 \) in \( \Omega_f \), so that \((\xi \pi) = (2)\).

From (4.3) and (6.3) we have that
\[ (-1)^c \frac{2}{\pi} = \frac{1}{v(w/8)} - v(w/8) = \frac{1 - v^2(w/8)}{v(w/8)}. \] (6.4)

In particular, \( v(w/8) \) satisfies a quadratic equation over \( \Omega_f \) and the map \( \rho : v(w/8) \to \frac{-1}{v(w/8)} \) leaves the right side of (6.4) invariant. On squaring (6.4), we see that \( X = v^2(w/8) \) satisfies the equation
\[ X^2 - \left( 2 + \frac{4}{\pi^2} \right) X + 1 = 0, \] (6.5)
and therefore
\[ v^2(w/8) = \frac{\pi^2 + 2 \pm 2\sqrt{\pi^2 + 1}}{\pi^2} = \left( \frac{1 \pm \sqrt{1 + \pi^2}}{\pi} \right)^2. \]

Hence
\[ v(w/8) = \pm \frac{1 \pm \sqrt{1 + \pi^2}}{\pi}. \] (6.6)

It follows from these expressions that
\[ \Omega_f(v(w/8)) = \Omega_f(v^2(w/8)) = \Omega_f(\sqrt{1 + \pi^2}). \]

We now prove the following.

**Theorem 6.1.** If
\[ w = \frac{a + \sqrt{-d}}{2}, \quad \text{with } a^2 + d \equiv 0 \pmod{2^5}, \]
and \( \wp_2 = (2, w) \) in \( R_K \), then the field \( \mathbb{Q}(v(w/8)) = \mathbb{Q}(\sqrt{1 + \pi^2}) \) coincides with the class field \( \Sigma_{\wp_2'^3} \Omega_f \) over \( K = \mathbb{Q}(\sqrt{-d}) \). The units \( v(w/8) \) and \( v^2(w/8) \) have degree \( 4h(-d) \) over \( \mathbb{Q} \).
Proof. Let $\Lambda = \mathbb{Q}(\sqrt{1 + \pi^2})$. It is clear that $\Lambda$ contains the ring class field $\Omega_f$, since $\mathbb{Q}(\pi^2) = \Omega_f$. We use the fact that $1 + \pi^2 \equiv \wp'_{\Omega_f}$ from [16, Lemma 5]. From this fact it is clear that $1 + \pi^2$ is not a square in $\Omega_f$, since $\wp'_{\Omega_f}$ is unramified in $\Omega_f/K$. Hence, $[\Lambda : \Omega_f] = 2$. Further, the prime divisors $\mathfrak{q}$ of $\wp'_{\Omega_f}$ in $\Omega_f$ are certainly ramified in $\Lambda$. Equation (6.5) implies that $x = \wp^2(w/8)$ satisfies $(x - 1)^2/(4x) = 1/\pi^2$, and therefore $\mathbb{Q}(\wp^2(w/8)) = \mathbb{Q}(\sqrt{1 + \pi^2})$. This implies that $[\mathbb{Q}(\wp^2(w/8)) : \mathbb{Q}] = 4h(-d)$, since

$$[\Lambda : \mathbb{Q}] = [\Lambda : \Omega_f][\Omega_f : K][K : \mathbb{Q}] = 4h(-d).$$

Since $\wp^2(\tau)$ is a modular function for $\Gamma_1(8)$ ([9, p.154]), it follows from Schertz [19, Thm. 5.1.2] that $\wp^2(w/8) \in \Sigma_{\wp_f}$, the ray class field of conductor $8f$ over $K$. More precisely, $\wp^2(w/8) \in L_{\wp^8}$, where $L_{\wp^8} = \Sigma_8\Omega_f$ is an extended ring class field corresponding to the order $O = R_d$. See [8, p. 315]. Thus, $\Lambda \subseteq L_{\wp^8}$ is an abelian extension of $K$, whose conductor $f$ divides $8f$ in $K$. The discriminant of the polynomial $X^2 - (1 + \pi^2)$ is of course $4(1 + \pi^2) \equiv \wp'_{\Omega_f}^2\wp_1^3$. Since the ramification index of each $\mathfrak{q} | \wp'_{\Omega_f}$ is $e_q = 2$ in $\Lambda/f$, Dedekind's discriminant theorem says that at least $\wp'_{\Omega_f}$ divides the discriminant $b = b_{\Lambda/O_f}$, and since the power of $q$ in $b$ is odd and at most 3 ($\Omega_f/K$ is unramified over 2), it follows that $\wp'_{\Omega_f}$ exactly divides $b$. We claim now that $\wp'_{\Omega_f}$ is unramified in $\Lambda$.

From above $x = \wp^2(w/8)$ satisfies $(x - 1)^2 - 4 \pi^2 x = 0$. Thus $x_1 = x - 1$ satisfies $h(x_1) = 0$, with

$$h(X) = X^2 - \frac{4}{\pi^2}(X + 1), \quad \text{disc}(h(X)) = \frac{16}{\pi^4} + \frac{4}{\pi^2},$$

where the ideal $\left(\frac{16}{\pi^4}\right) = \left(\frac{2}{\pi}\right)^4 = (\xi)^4 = \wp'_{\Omega_f}^4$ is not divisible by $\wp'_{\Omega_f}$. This shows that $\text{disc}(h(X))$ is not divisible by $\wp'_{\Omega_f}$ and therefore that $\wp'_{\Omega_f}$ is unramified in $\mathbb{Q}(\wp^2(w/8))$. Thus $b = \wp'_{\Omega_f}^3$.

Now $[\Sigma_8 : \Sigma_1] = \frac{1}{2}\phi_K(\wp'_{\Omega_f}^3\wp_1^3) = 8$, where $\phi_K$ is the Euler function for the quadratic field $K$, and $\mathbb{Q}(\zeta_8) \subseteq \Sigma_8$. Since the prime divisors of 2 do not ramify in $\Omega_f$, we have that $\Omega_f \cap \Sigma_8 = \Sigma_1$ and therefore

$$[L_{\wp^8} : \Omega_f] = [\Sigma_8\Omega_f : \Omega_f] = [\Sigma_8 : \Sigma_1] = 8,$$

from which we obtain

$$\text{Gal}(\Sigma_8\Omega_f/\Omega_f) \cong \text{Gal}(\Sigma_8/\Sigma_1).$$

By this isomorphism the intermediate fields $L\Omega_f$ of $\Sigma_8\Omega_f/\Omega_f$ are in $1 - 1$ correspondence with the intermediate fields $L$ of $\Sigma_8/\Sigma_1$.

The ray class field $\Sigma_{\wp'_{\Omega_f}\wp_1^3}$ has degree 4 over the Hilbert class field $\Sigma_1$, and two of its quadratic subfields are $\Sigma_{\wp'_{\Omega_f}}$ and $\Sigma_{\wp'_{\Omega_f}\wp_1^3} = \Sigma_4 = \Sigma_1(i)$. It follows that $\text{Gal}(\Sigma_{\wp'_{\Omega_f}\wp_1^3}/\Sigma_1) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, and the third quadratic subfield has conductor equal to $f' = \wp'_{\Omega_f}\wp_1^3$ over $K$. The other quadratic intermediate fields of
\( \Sigma_8/\Sigma_1 \) are \( \Sigma_1(\sqrt{2}) \) and \( \Sigma_1(\sqrt{-2}) \), both of which have conductor \( (8) = \wp_2^3 \wp_2' \wp_2'' \) over \( K \), the field \( \Sigma_{\wp_2^3} \), and a field whose conductor over \( K \) is \( \wp_2^2 \wp_2' \wp_2'' \). Hence, \( L = \Sigma_{\wp_2''} \) is the only quadratic intermediate field whose conductor is not divisible by \( \wp_2 \). This proves that \( \mathbb{Q}(\upsilon^2(2/8)) = \Sigma_{\wp_2^2} \Omega_f \) and (6.6) shows that \( \mathbb{Q}(\upsilon^2(8/8)) = \Sigma_{\wp_2^2} \Omega_f \).

**Corollary 6.2.** The field \( \mathbb{Q}(\upsilon(w/8)) = \Sigma_{\wp_2^3} \Omega_f \) is the inertia field for the prime ideal \( \wp_2 \) in the extension \( L_{\wp_8}/K = \Sigma_{\wp_8} \Omega_f/K \).

**Proof.** The above proof implies that \( \text{Gal}(\Sigma_{\wp_8} \Omega_f / \Omega_f) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \), since there are 7 quadratic intermediate fields. Any subfield containing \( \Omega_f \) which properly contains \( \Sigma_{\wp_2^3} \) must also contain another quadratic subfield, in which \( \wp_2 \) must ramify.

**Corollary 6.3.** If \(-d \equiv 1 \pmod{8}\) and \( w \) is given by (6.1), then the quantity

\[
A = \frac{\eta^2(w/8)\eta(w/2)}{\eta(w/4)\eta^2(w)}
\]

generates the class field \( \Sigma_{\wp_2^3} \Omega_f \) for \( K = \mathbb{Q}(\sqrt{-d}) \) over \( \mathbb{Q} \).

**Proof.** We appeal to equation (5.9). Setting \( \eta = \upsilon(w/8) \), first use the equation preceding (6.6) to see that

\[
A^2 = \eta^{-2} + \eta^2 - 6 = \frac{\pi^2 + 2 + 2\sqrt{1 + \pi^2}}{\pi^2} + \frac{\pi^2 + 2 + 2\sqrt{1 + \pi^2}}{\pi^2} - 6
\]

\[
= 4 \frac{1 - \pi^2}{\pi^2}.
\]

This gives that \( A = \pm \frac{2}{\pi} \sqrt{1 - \pi^2} \). Since \( \sqrt{1 - \pi^2} \sqrt{1 + \pi^2} = \sqrt{1 - \pi^4} = \pm \xi^2 \in \Omega_f \) and \( \mathbb{Q}(A^2) = \Omega_f \), we get that \( \mathbb{Q}(A) = \mathbb{Q}(1 + \pi^2) = \Sigma_{\wp_2^3} \Omega_f \), by the result of Theorem 6.1.

The fact that \( \upsilon^2(2/8) \in L_{\wp_8} \) in the above proof is derived in [8, p. 317] using Shimura’s Reciprocity Law. We can give a more elementary proof of this fact by showing that \( \sqrt{1 + \pi^2} \in L_{\wp_8} \), as follows. We focus on the elliptic curve

\[
E_1(\alpha) : Y^2 + XY + \frac{1}{\alpha^4} Y = X^3 + \frac{1}{\alpha^4} X^2,
\]

which is the Tate normal form for a point of order 4, with

\[
\alpha^4 = \alpha(w)^4 = \left( \frac{\eta(w/4)}{\eta(w)} \right)^8,
\]

as in (5.2). From [14, (2.10), Prop. 3.2, p. 1970], the curve \( E_1 = E_1(\alpha) \) has complex multiplication by the order \( \mathcal{O} = R_{-d} \) of discriminant \( -d \) in \( K \). Now,
with \( \beta = i^{a}b(w) \) as in (6.2),

\[
\frac{1}{\alpha^4} = \frac{\beta^4 - 16}{16\beta^4} = \frac{1}{16} - \frac{1}{\beta^4},
\]

and Lynch [13] has given explicit expressions for the points of order 8 on \( E_1 \) in terms of \( \beta \). Lynch [13, Prop. 3.3.1, p. 38] defines the following expressions:

\[
\begin{align*}
b_1 &= \frac{\beta\sqrt{2} + (\beta^2 + 4)^{1/2} + (\beta^2 - 4)^{1/2}}{2\beta\sqrt{2}}, \\
b_2 &= \frac{\beta\sqrt{2} + (\beta^2 + 4)^{1/2} - (\beta^2 - 4)^{1/2}}{2\beta\sqrt{2}}, \\
b_3 &= \frac{\beta\sqrt{2} - (\beta^2 + 4)^{1/2} + (\beta^2 - 4)^{1/2}}{2\beta\sqrt{2}}, \\
b_4 &= \frac{\beta\sqrt{2} - (\beta^2 + 4)^{1/2} - (\beta^2 - 4)^{1/2}}{2\beta\sqrt{2}}.
\end{align*}
\]

With these expressions, Lynch shows [13, Thm. 3.3.1, p. 41] that the points \((X, Y) = P_1 = (b_1b_3b_4, -b_1b_3b_4) \) and \( P_2 = (b_2b_3b_4, -b_2b_3b_4) \) are points of order 8 on \( E_1(\alpha) \). By [11, Satz 2] or [14, Prop. 6.4] the corresponding Weber functions satisfy

\[
\frac{g_2g_3}{\Delta}(X(P_i) + \frac{4b + 1}{12}) \in \Sigma_8\Omega_f, \quad b = \frac{1}{\alpha^4}.
\]

(See [14, (6.1)]. The expression inside the parentheses arises from putting the curve \( E_1(\alpha) \) in standard Weierstrass form.) As in [14, p. 1976], \( b, g_2, g_3, \Delta \in \Omega_f \), so that \( X(P_i) = b_ib_3b_4 \in L_{\mathcal{O}_8} \) for \( i = 1, 2 \). This implies that

\[
(b_1 + b_2)b_3b_4 = \left(\frac{\sqrt{2} \beta + (\beta^2 + 4)^{1/2}}{\sqrt{2}\beta}\right) \left(\frac{\beta^2 + 4 - \sqrt{2}\beta(\beta^2 + 4)^{1/2}}{4\beta^2}\right)
= \frac{4 - \beta^2}{4\sqrt{2}\beta^3}(\beta^2 + 4)^{1/2}
\]

lies in \( L_{\mathcal{O}_8} \). But we know that \( 4 - \beta^2 \neq 0 \). In addition, \( \sqrt{2} \in \mathbb{Q}(\xi_8) \subset \Sigma_8 \) and \( \beta \in \Omega_f \), so that \( (\beta^2 + 4)^{1/2} = 2\sqrt{\xi^2 + 1} \in L_{\mathcal{O}_8} \), with \( \xi = \beta / 2 \). Now \( \pi \) and \( \xi \) are conjugate over \( \mathbb{Q} \), hence \( \pm \sqrt{1 + \pi^2} \) is conjugate to \( \sqrt{1 + \xi^2} \) over \( \mathbb{Q} \). Since \( \Sigma_8\Omega_f \) is normal over \( \mathbb{Q} \), this implies that \( \sqrt{1 + \pi^2} \in L_{\mathcal{O}_8} \), which proves the assertion.

**Proposition 6.4.** Assume \( c \) in (6.3) is odd. The map \( A(x) = \frac{\alpha x + 1}{x - \sigma} \) (see (3.3)) fixes the set of conjugates of \( v(w/8) \). If \( f_d(x) \) is the minimal polynomial of \( v(w/8) \) over
Q, then

\[(x - \sigma)^{4h(-d)} f_d(A(x)) = 2^{3h(-d)} \sigma^{2h(-d)} f_d(x)\].

**Proof.** Note that (6.4) implies that the minimal polynomial of \(v(w/8)\) is

\[f_d(x) = 2^{-h(-d)}(x^2 - 1)^{2h(-d)}b_d \left( (-1)^h \frac{2x}{1 - x^2} \right), \quad (6.7)\]

where \(b_d(x)\) is the minimal polynomial of \(\pi\). Note that the degree of \(b_d(x)\) is \(2h(-d)\) and the constant term of \(b_d(x)\) is \(N_{\Omega_j/\Omega}(\pi) = N_{\Omega_j/\Omega}(q_2^{h-d}) = 2^{h-d}\)

from [14]. Thus, \(\deg(f_d(x)) = 4h(-d)\), which implies by Theorem 6.1 that \(f_d(x)\) is irreducible.

We use (6.7) to prove the proposition, as follows. Setting \(h = h(-d)\) and assuming \(c\) is odd, we have that

\[(x - \sigma)^{4h} f_d(A(x)) = 2^{-h} (x - \sigma)^{4h} (A(x)^2 - 1)^{2h} b_d \left( \frac{2A(x)}{A(x)^2 - 1} \right) \]

\[= 2^{-h} (x - \sigma)^{4h} \left( \frac{-2\sigma(x^2 - 2x - 1)}{(x - \sigma)^2} \right)^{2h} b_d \left( \frac{x^2 + 2x - 1}{x^2 - 2x - 1} \right) \]

\[= 2^h \sigma^{2h} (x^2 - 2x - 1)^{2h} b_d \left( \frac{P(x) + 1}{P(x) - 1} \right), \]

where

\[P(x) = \frac{2x}{x^2 - 1} \quad \text{and} \quad \frac{P(x) + 1}{P(x) - 1} = -\frac{x^2 + 2x - 1}{x^2 - 2x - 1} = R(x).\]

We also know from [14] that the map \(x \to \frac{x+1}{x-1}\) permutes the roots of \(b_d(x)\) and

\[(x - 1)^{2h} b_d \left( \frac{x + 1}{x - 1} \right) = 2^h b_d(x).\]

This gives that \(b_d \left( \frac{P(x)^{h+1}}{P(x)^{h-1}} \right) = (P(x) - 1)^{-2h} 2^h b_d(P(x))\) and therefore that

\[(x - \sigma)^{4h} f_d(A(x)) = 2^h \sigma^{2h} (x^2 - 2x - 1)^{2h} (P(x) - 1)^{-2h} 2^h b_d(P(x)) \]

\[= 2^h \sigma^{2h} (x^2 - 2x - 1)^{2h} \left( \frac{x^2 - 1}{x^2 - 2x - 1} \right)^{2h} b_d(P(x)) \]

\[= 2^{3h} \sigma^{2h} 2^{-h} (x^2 - 1)^{2h} b_d(P(x)) \]

\[= 2^{3h} \sigma^{2h} f_d(x).\]

\[\square\]
We also check that
\[ x^{4h} f_d \left( \frac{-1}{x} \right) = 2^{-h} x^{4h} \left( \frac{1}{x^2} - 1 \right)^{2h} b_d(P(-1/x)) \]
\[ = 2^{-h} (x^2 - 1)^{2h} b_d(P(x)) = f_d(x). \]

We conclude the following. Recall the definition of \( \bar{A}(x) \) from (3.3).

**Proposition 6.5.** If \( c \) is odd, the mappings in the group
\( \bar{H}_1 = \{ x, A(x), \bar{A}(x), -1/x \} \)
permute the roots of \( f_d(x) \).

Now let \( c \) be even, \( \delta = 1 + \sqrt{2} \), and \( B(x) = \frac{\delta x + 1}{x - \delta} = \frac{x + \sigma}{\sigma x - 1} = -\bar{A}(-x) \).

Then we have
\[ (x - \delta)^h f_d(B(x)) = 2^{-h}(x - \delta)^{4h}(B^2(x) - 1)^{2h} b_d(\frac{2B(x)}{1 - B^2(x)}) \]
\[ = 2^{-h}(x - \delta)^{4h} \left( \frac{2\delta(x^2 + 2x - 1)}{(x - \delta)^2} \right)^{2h} b_d\left( \frac{x^2 - 2x - 1}{x^2 + 2x - 1} \right) \]
\[ = 2^h \delta^{2h} (x^2 + 2x - 1)^{2h} b_d\left( \frac{2x}{1 - x^2} + \frac{1}{x} \right) \]
\[ = 2^h \delta^{2h} (x^2 + 2x - 1)^{2h} \cdot 2^h \left( \frac{2x}{1 - x^2} - 1 \right)^{-2h} b_d\left( \frac{2x}{1 - x^2} \right) \]
\[ = 2^h \delta^{2h} (x^2 + 2x - 1)^{2h} \cdot \left( \frac{1 - x^2}{x^2 + 2x - 1} \right)^{2h} b_d\left( \frac{2x}{1 - x^2} \right) \]
\[ = 2^h \delta^{2h} \cdot (x^2 - 1)^{2h} b_d\left( \frac{2x}{1 - x^2} \right) \]
\[ = 2^h \delta^{2h} f_d(x) \]
\[ = 2^3 \delta^{2h} f_d(x). \]

Setting \( \bar{B}(x) = B(-1/x) = \frac{-\sigma x + 1}{x + \sigma} = -\bar{A}(-x) \), we have the following.

**Proposition 6.6.** If \( c \) is even, the mappings in the group
\( \bar{H}_0 = \{ x, B(x), \bar{B}(x), -1/x \} \)
permute the roots of \( f_d(x) \).

7. The diophantine equation.

From (3.4) we know that \((X, Y) = (\nu(w/8), \nu(-1/w))\) is a solution of the diophantine equation
\[ e_2 : X^2 + Y^2 = \sigma^2(1 + X^2Y^2), \quad \sigma = -1 + \sqrt{2}. \]
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This seems to be an analogue of the equation $C_5$ in [17]. Set

$$F_2(X, Y) = X^2 + Y^2 - \sigma^2(1 + X^2Y^2).$$

Then

$$(\sigma Y + 1)^2 F_2(X, \tilde{A}(Y)) = 4\sqrt{2}\sigma^2(X^2Y + X^2 + Y^2 - Y) = 4\sqrt{2}\sigma^2 f(X, Y).$$

Since

$$\tilde{A}(x) = \frac{-x + \sigma}{\sigma x + 1} = \frac{-\delta x + 1}{x + \delta}, \quad \delta = \frac{1}{\sigma} = 1 + \sqrt{2},$$

the linear fractional map $\tilde{A}(x)$ is the analogue of the map $T(x)$ in [17, p. 1199]. Considering Thm. 5.1 in [17, p. 1205] suggests the following conjecture.

**Conjecture 7.1.** Assume $c$ is odd. If $\tau_2 = \left(\Sigma_{\phi_2}^\phi f/K\right)$, then

$$-v(-1/w) = \tilde{A}(v(w/8)^{\tau_2}) = -\frac{v(w/8)^{\tau_2} + \sigma}{\sigma v(w/8)^{\tau_2} + 1},$$

where $w$ is given by (6.1).

To prove this conjecture, we first appeal to Proposition 4.2, which implies that

$$p(2\tau) = \frac{1 \pm \sqrt{1 - p^4(\tau)}}{p^2(\tau)}.$$

Setting $\tau = w$, (6.3) gives that

$$p(2w) = \frac{1 \pm \sqrt{1 - \pi^4}}{\pi^2} = \frac{1 \pm \xi^2}{\pi^2}.$$

Note that

$$\frac{1 + \xi^2}{\pi^2} - \frac{1 - \xi^2}{\pi^2} = \frac{1 - \xi^4}{\pi^4} = 1$$

and

$$\frac{1 - \xi^2}{\pi^2} = -\pi^{\tau_2}$$

from [16, p. 333]. Thus, $\frac{1 + \xi^2}{\pi^2} = -\pi^{\tau_2}.$

**Theorem 7.2.** If $w$ is given by (6.1) we have

$$p(2w) = \frac{1 + \xi^2}{\pi^2} = \frac{-1}{\pi^{\tau_2}}.$$

**Proof.** We use an argument from [14, Section 10]. With the number $\beta = i^{-a} b(w)$ from (6.2) we have [14, eq. (8.0), p. 1980]

$$j(w) = \frac{(\beta^8 + 224\beta^4 + 256)^3}{\beta^4(\beta^4 - 16)^4}.$$

(See the proof of Proposition 5.2.) Furthermore, the roots of the equation

$$0 = (X - 16)^3 - j(w)X = (X - 16)^3 - \frac{(\beta^8 + 224\beta^4 + 256)^3}{\beta^4(\beta^4 - 16)^4} X$$

are $X = \frac{1 + \xi^2}{\pi^2}$.
are, on the one hand, given by the values
\[ X = f_2^{24}(w), \quad -f_1^{24}(w), \quad -f_2^{24}(w); \]
(see [8, p. 233, Th. 12.17]) and on the other, are equal to the expressions
\[ X = \frac{(\beta^2 - 4)^4}{\beta^2(\beta^2 + 4)^2}, \quad \frac{(\beta^2 + 4)^4}{\beta^2(\beta^2 - 4)^2}, \quad -\frac{2^{12}\beta^4}{(\beta^4 - 16)^2}. \]
See [14, p. 2000]. From [14, p. 2000] we also have (since our value \( w \) satisfies the conditions for \( w \) in [14, Prop. 3.1])
\[ f_2^{24}(w) = -\frac{(\beta^2 + 4)^4}{\beta^2(\beta^2 + 4)^2}, \quad (7.1) \]
since \( f_2^{24}(w) \) must be a unit (from the results of [21]). There are two cases to consider.

**Case 1.** First assume that
\[ f_2^{24}(w) = -\frac{(\beta^2 - 4)^4}{\beta^2(\beta^2 + 4)^2}, \quad (7.2) \]
\[ f_1^{24}(w) = \frac{2^{12}\beta^4}{(\beta^4 - 16)^2}. \]
In this case, (7.1) and (7.2) give the following formula:
\[ p^{12}(2w) = \frac{f_2(w)^{24}}{f(w)^{24}} = \frac{(\beta^2 + 4)^6}{(\beta - 2)^6(\beta + 2)^6}. \]
Now we use the following ideal factorizations in the ring class field \( \Omega_f \):
\[ (\beta^2 + 4) = \wp_1^2\wp_2^2, \quad (\beta - 2) = \wp_1^2\wp_2^2, \quad (\beta + 2) = \wp_2^3\wp_2^2. \quad (7.3) \]
See [16, Lemma 4]. These factorizations imply that
\[ p^{12}(2w) \cong \left( \frac{\wp_1^2\wp_2^2}{\wp_1^2\wp_2^2}\right)^6 = \frac{1}{\wp_1^2} \in \Omega_f, \]
which implies that
\[ p(2w) \cong \frac{1}{\wp_1}. \quad (7.4) \]
By the remarks preceding the statement of the theorem, this shows that \( p(2w) \)
is not an algebraic integer, giving that \( p(2w) = \frac{1 + 2\xi^2}{\pi^2} = -\pi^{-\xi_2}. \)

**Case 2.** The alternative to (7.2) is
\[ f_2^{24}(w) = -\frac{2^{12}\beta^4}{(\beta^4 - 16)^2}, \quad (7.5) \]
\[ f_1^{24}(w) = \frac{(\beta^2 - 4)^4}{\beta^2(\beta^2 + 4)^2}. \]
In this case we have

\[ p^{12}(2w) = \frac{f_2(w)^{24}}{f(w)^{24}} = \left( \frac{\beta^2 + 4}{2^2 \beta} \right)^6 \cong \left( \frac{\varphi_2^3 \varphi_2^2}{\varphi_2^2 \varphi_2^2 \varphi_2^2} \right)^6 = \frac{1}{\varphi_2^{12}}, \]

giving that \( p(2w) \cong \frac{1}{\varphi_2^2} \). However, this is impossible, since the above remarks show that the only prime divisors occurring in the factorization of \( p(2w) \) are prime divisors of \( \varphi_2 \). This shows that Case 2 is impossible, and Case 1 proves the formula of the theorem. □

Now we set

\[ \eta = \nu(w/8), \quad \lambda = -\nu(-1/w), \quad \nu = \nu(w/4). \] (7.6)

We first show \( \lambda \) is a root of the minimal polynomial \( f_d(x) \) of \( \nu(w/8) \) (c odd). We have from Proposition 3.3 that

\[ \frac{2\lambda}{\lambda^2 - 1} = -\frac{2\bar{A}(\nu)}{\bar{A}^2(\nu) - 1} = \frac{\nu^2 + 2\nu - 1}{\nu^2 - 2\nu - 1}. \]

Proposition 4.1 and Theorem 7.2 give further that

\[ \frac{2\lambda}{\lambda^2 - 1} = \frac{\nu - \frac{1}{\nu} + 2}{\nu - \frac{1}{\nu} - 2} = \frac{-2}{\nu(2w) - 2} + \frac{2}{\nu(2w)^2 + \nu^2 - 1} = \frac{\pi^{\tau_2} + 1}{\pi^{\tau_2} - 1}. \] (7.7)

Since \( \pi^{\tau_2} + 1 \) is a root of \( b_d(x) \), we have from (6.7) that

\[ f_d(\lambda) = 2^{-h(-d)}(\lambda^2 - 1)^{2h(-d)} b_d \left( \frac{2\lambda}{\lambda^2 - 1} \right) = 0. \]

Hence, \( \lambda = -\nu(-1/w) \) is a conjugate of \( \nu(w/8) \).

**Theorem 7.3.** If \( c \) is odd, we have the formula

\[ \lambda = -\nu(-1/w) = \bar{A}(\nu(w/8)^{\tau_2}) = -\frac{\nu(w/8)^{\tau_2} + \sigma}{\sigma \nu(w/8)^{\tau_2} + 1}, \quad \sigma = -1 + \sqrt{2}, \]

where \( w \) is given by (6.1).

**Proof.** We will prove that \( \bar{A}(\lambda) = \nu(w/8)^{\tau_2} = \eta^{\tau_2} \) by showing that

\[ \bar{A}(\lambda) - \eta^{\tau_2} \equiv 0 \pmod{\varphi_2}. \]

We have \( \eta^{\tau_2} + \lambda^2 = \sigma^2(1 + \eta^{\tau_2} \lambda^2) \), which implies that

\[ \bar{A}(\lambda) - \eta^{\tau_2} = \frac{-\lambda + \sigma}{\sigma \lambda + 1} - \frac{-\lambda^2 + \sigma^2}{\lambda^2 - \sigma^2 \lambda^2} = \frac{-\lambda + \sigma}{\sigma \lambda + 1} + \frac{\sigma^2 - \lambda^2}{\sigma^2 \lambda^2 - 1} \]

\[ = \frac{(\sigma + 1)\lambda^2 + (\sigma^2 + 1)\lambda + \sigma^2 - \sigma}{\sigma^2 \lambda^2 - 1} \]

\[ = -\frac{\sigma^2 \lambda^2 - 1}{\sigma^2 \lambda^2 - 1} \]
\[
\frac{810}{SUSHMANTHJ\text{.}AKKARAPAKAMANDPATRICKMORTON} = -\sqrt{2}\lambda^2 + (4 - 2\sqrt{2})\lambda + 4 - 3\sqrt{2} \\
= -\sqrt{2}(\lambda - \sigma)^2 \\
\sigma^2(\lambda - \sigma)(\lambda + \sigma).
\]

We multiply the last expression by

\[
A(\lambda) - \frac{1}{\eta^2} = \frac{(-4 + 3\sqrt{2})(\lambda - \sigma)^2}{\lambda^2 - \sigma^2} = \frac{\sqrt{2}\sigma^2(\lambda - \sigma)^2}{\lambda^2 - \sigma^2},
\]

which is obtained from the last calculation by fixing \(\lambda\) and mapping \(\sqrt{2}\) to \(-\sqrt{2}\).

This yields the formula

\[
(A(\lambda) - \frac{1}{\eta^2}) = -2(\lambda - \sigma)(\lambda - \sigma) \\
\lambda^2 - 2\lambda - 1 = -2\frac{\lambda^2 + 2\lambda - 1}{\lambda^2 - 2\lambda - 1}.
\]

Now

\[
\frac{\lambda^2 + 2\lambda - 1}{\lambda^2 - 2\lambda - 1} = \frac{1 + \frac{2\lambda}{\lambda^2 - 1}}{1 - \frac{2\lambda}{\lambda^2 - 1}},
\]

where

\[
\frac{2\lambda}{\lambda^2 - 1} = \frac{\pi^2 + 1}{\pi^2 - 1}
\]

from (7.7). It follows from (7.9) that

\[
\frac{\lambda^2 + 2\lambda - 1}{\lambda^2 - 2\lambda - 1} = \frac{1 + \frac{\pi^2 + 1}{\pi^2 - 1}}{1 - \frac{\pi^2 + 1}{\pi^2 - 1}} = -\pi^2.
\]

Thus, (7.8) becomes

\[
(\tilde{A}(\lambda) - \eta^2) \left( A(\lambda) - \frac{1}{\eta^2} \right) = 2\pi^2
\]

and therefore \((\pi^2) = (\pi) = \mathcal{O}_2\) yields that

\[
(\tilde{A}(\lambda) - \eta^2) \left( A(\lambda) - \frac{1}{\eta^2} \right) \equiv 0 \pmod{\mathcal{O}_2^2}.
\]

It follows that

\[
\tilde{A}(\lambda) \equiv \eta^2 \quad \text{or} \quad A(\lambda) \equiv \frac{1}{\eta^2} \pmod{q},
\]

for each prime divisor \(q\) of \(\mathcal{O}_2\) in \(F_1 = \mathbb{Q}(\eta)\). But \(A(\lambda) = -1/\tilde{A}(\lambda)\) and \(\eta\) are units, so the second congruence in (7.10) implies the first. This proves that

\[
\tilde{A}(\lambda) \equiv \eta^2 \pmod{\mathcal{O}_2}(7.11)
\]

in \(F_1\). Note that \(\tilde{A}(\lambda)\) and \(\lambda = -v(-1/w)\) are roots of \(f_\lambda(x)\) (Proposition 6.5), so \(F_2 = \mathbb{Q}(\lambda)\) is isomorphic to \(F_1 = \mathbb{Q}(\eta) = \mathbb{Q}(v(w/8))\). However, by (3.4),

\[
\lambda^2 = v^2(-1/w) = \frac{-v(w/8)^2 + \sigma^2}{1 - \sigma^2 v(w/8)^2}.
\]
does not lie in \( F_1 \), since \( \sqrt{2} \not\in F_1 \) (otherwise \( \wp_2 \) would be ramified in \( F_1 \); note that \( v(w/8) \) is not a fourth root of unity, so the determinant of the linear fractional transformation in \( \sigma^2 \) is nonzero). It follows that from Theorem 6.1 that

\[
F_2 = \mathbb{Q}(\lambda) = \Sigma_{\wp_2}^{\Omega_f}.
\]

The same argument now shows that \( \tilde{A}(\lambda) = \frac{\lambda + \sigma}{\sigma^2 + 1} \not\in F_2 \), so \( \tilde{A}(\lambda) \in F_1 \). Therefore, \( \psi : \eta \to \tilde{A}(\lambda) \) is an automorphism of \( F_1 \), and since \( \wp_2 \) is not ramified in \( F_1 \) but \( \wp_2' \) is, it follows that \( \psi \) fixes \( \wp_2 \), implying that it fixes the field \( K \).

Recalling the rational function \( j_2(x) \) from (5.6), a computation on Maple shows that

\[
j_2\left(\left(\frac{1 - v}{1 + v}\right)^2\right) = j_2(v^2) = j_2(v^2(w/4)) = j(w/4),
\]

by (5.7). Now Proposition 3.3 and the fact that \( \tilde{A}(x) \) has order 2 imply that \( v(w/4) = \tilde{A}(v(-1/w)) \) and

\[
\frac{1 - v(w/4)}{1 + v(w/4)} = \frac{1 - \tilde{A}(v(-1/w))}{1 + \tilde{A}(v(-1/w))} = \frac{v(-1/w) + \sigma}{-\sigma v(-1/w) + 1} = \tilde{A}(-v(-1/w)) = \tilde{A}(\lambda).
\]

This implies that

\[
j_2(\tilde{A}(\lambda)^2) = j_2\left(\left(\frac{1 - v}{1 + v}\right)^2\right) = j(w/4).
\]

On the other hand, equation (5.7) gives

\[
j(w/8)^\psi = j_2(\eta^{2\psi}) = j_2(\tilde{A}(\lambda)^2) = j(w/4) = j(w/8)^{\tau_2}.
\]

Hence \( \psi|_{\Omega_f} = \tau_2|_{\Omega_f} \). It follows that \( \psi = \tau_2 \) or \( \psi = \rho \tau_2 \), where \( \rho : \eta \to -1/\eta \) is the nontrivial automorphism of \( F_1/\Omega_f \). If \( \psi = \rho \tau_2 \), then by (7.11)

\[
\eta^\psi = \tilde{A}(\lambda) \equiv \eta^2 (\mod \wp_2)
\]

and \( \eta^{\tau_2} \equiv \eta^2 (\mod \wp_2) \) imply that

\[
\eta^2 \equiv \eta^{\rho \tau_2} = -1 \eta^{\tau_2} \equiv 1 (\mod \wp_2).
\]

It follows from this congruence that \( \eta^4 + 1 \equiv (\eta + 1)^4 \equiv 0 (\mod \wp_2) \). This implies in turn that \( z = \eta - \eta^{-1} \equiv 0 (\mod \wp_2) \). But this contradicts (4.3) (with \( \tau = w/8 \)) and (6.3), according to which \( z = 2/\pi \) is relatively prime to \( \wp_2 \). Hence, \( \psi = \tau_2 \) must be the Artin symbol for \( \wp_2 \) in \( F_1/K \). This completes the proof. \( \square \)
Corollary 7.4. Assume $c$ is odd. If $\tau_2 = \left(\frac{\Sigma \wp_2 \Omega_f / K}{\wp_2}\right)$, then

$$v(w/8) \tau_2 = \frac{1 - v(w/4)}{1 + v(w/4)}$$

and

$$f(v(w/8), v(w/8) \tau_2) = 0.$$ 

Proof. The first formula is immediate from $\eta^\psi = \eta^\tau_2 = \tilde{A}(\lambda)$ and (7.12). The second follows from Proposition 3.1 and

$$f(v(w/8), v(w/4)) = 0 = f\left(v(w/8), \frac{1 - v(w/4)}{1 + v(w/4)}\right),$$

since

$$f\left(x, \frac{1 - y}{1 + y}\right) = \frac{2f(x, y)}{(1 + y)^2}.$$ 

□

Theorem 7.5. If $c$ is even, then

$$v(w/8) \tau_2 = \frac{v(w/4) - 1}{v(w/4) + 1}$$

and

$$v(-1/w) = B(v(w/8) \tau_2) = \frac{v(w/8) \tau_2 + \sigma}{\sigma v(w/8) \tau_2 - 1}.$$ 

Proof. From Proposition 3.3, we have that

$$v(-1/w) = \tilde{A}(v(w/4)) = -B(-v(w/4)),$$

where

$$B(x) = \frac{x + \sigma}{\sigma x - 1} = -\frac{-(-x) + \sigma}{\sigma(-x) + 1} = -\tilde{A}(-x).$$

Hence, according to (7.12), we obtain

$$v(w/8) \tau_2 = \frac{v(w/4) - 1}{v(w/4) + 1} = B(v(-1/w)) \iff v(-1/w) = B(v(w/8) \tau_2),$$

showing that both the statements in the theorem are equivalent. We now show that Proposition 6.6 implies that $v(w/8)$ and $v(-1/w)$ are conjugate algebraic integers.

In similar fashion to (7.6), we set

$$\eta = v(w/8), \quad \tilde{\lambda} = v(-1/w) = -\lambda, \quad \nu = v(w/4).$$

Then, according to (7.7), we get

$$\frac{2\tilde{\lambda}}{1 - \tilde{\lambda}^2} = -\frac{2 - (1 - \nu)}{2 + (1 - \nu)} = -\frac{2 - \frac{2}{\wp(2w)}}{2 + \frac{2}{\wp(2w)}} = -\frac{1 + \pi^\tau_2}{1 - \pi^\tau_2} = \frac{\pi^\tau_2 + 1}{\pi^\tau_2 - 1},$$
Since $\frac{\pi^{\tau_2 + 1}}{\pi^{\tau_2 - 1}}$ is a root of $b_d(x)$, we have that

$$f_d(\lambda) = 2^{-h}(\lambda^2 - 1)^{2h} b_d \left( \frac{2\lambda}{1 - \lambda^2} \right) = 0,$$

showing that $\lambda = v(-1/w)$ is a conjugate of $\eta = v(w/8)$.

Now,

$$B(\lambda) - \eta^2 = \frac{\lambda + \sigma}{\sigma \lambda - 1} - \frac{\sigma^2 - \lambda^2}{1 - \sigma^2 \lambda^2} = \frac{\lambda - \sigma}{\sigma \lambda + 1} - \frac{\sigma^2 - \lambda^2}{1 - \sigma^2 \lambda^2}$$

$$= \frac{(\lambda - \sigma)(\sigma \lambda - 1) + (\sigma^2 - \lambda^2)}{(\sigma \lambda + 1)(\sigma \lambda - 1)}$$

$$= \frac{-\sqrt{2} \sigma(\lambda^2 + 2\lambda - 1)}{\sigma^2(\lambda - \sigma)(\lambda + \sigma)}$$

$$= \frac{-\sqrt{2} \sigma(\lambda - \sigma)(\lambda - \sigma)}{\sigma^2(\lambda - \sigma)(\lambda + \sigma)}$$

$$= \sqrt{2} \sigma(\lambda + \sigma)(\lambda - \sigma).$$

In the above calculation, mapping $\sqrt{2}$ to $-\sqrt{2}$, while fixing $\lambda$, gives us

$$B(\lambda) - \frac{1}{\eta^2} = -\frac{\sqrt{2} \sigma(\lambda + \sigma)}{(\lambda - \sigma)}.$$

Multiplying the above two expressions gives us

$$(B(\lambda) - \eta^2) \left( B(\lambda) - \frac{1}{\eta^2} \right) = \frac{2(\lambda + \sigma)(\lambda + \sigma)}{(\lambda - \sigma)(\lambda - \sigma)} = \frac{2\lambda^2 - 2\lambda - 1}{\lambda^2 + 2\lambda - 1}$$

$$= \frac{2 + \left( \frac{\lambda^2}{1 - \lambda^2} \right)}{1 - \left( \frac{\lambda^2}{1 - \lambda^2} \right)} = \frac{2 + \left( \frac{\pi^{\tau_2} + 1}{\pi^{\tau_2} - 1} \right)}{1 - \left( \frac{\pi^{\tau_2} + 1}{\pi^{\tau_2} - 1} \right)} = -2\pi^{\tau_2}.$$

Now a similar argument to the end of the proof of Theorem 7.3 applies here and shows that the automorphism $\psi$ on $F_1$ taking $\eta$ to $\lambda$ is $\eta^\psi = B(\lambda)$. As before, $\psi$ coincides with $\tau_2$, giving that $\lambda = v(-1/w) = B(\eta^{\tau_2}) = B(v(w/8)^{\tau_2})$. Also see the argument below.

**Corollary 7.6.** If $c$ is even, the point $(x, y) = (-\eta, -\eta^{\tau_2})$ lies on the curve $f(x, y) = 0$:

$$f(-v(w/8), -v(w/8)^{\tau_2}) = 0, \quad \tau_2 = \frac{\sum_{g_2 \not= 0} \Omega_f/K}{\varphi_2}.$$
Proof. We have

\[0 = f(\nu(w/8), \nu(w/4)) = f\left(\nu(w/8), -\frac{\nu(w/4) - 1}{\nu(w/4) + 1}\right)\]

\[= f(\nu(w/8), -\nu(w/8)^{\tau_2}) = f(-\nu(w/8), -\nu(w/8)^{\tau_2}).\]

\[\square\]

Combining the arguments in the proofs of Theorems 7.3 and 7.5 for \(c\) odd and \(c\) even yields the following corollary.

Corollary 7.7. The field \(F_2 = \mathbb{Q}(\nu(-1/w)) = \sum_{\wp_2} \Omega_f\) is the inertia field for the prime ideal \(\wp_2\) in the extension \(L_{\wp_2}/K\).

We also give an alternate argument to show \(\psi = \tau_2\) in the proofs of Theorems 7.3 and 7.5. We first note that the modular function \(j(\tau)\) can be expressed in terms of \(z = \nu(\tau) - \frac{1}{\nu(\tau)}\), namely

\[j(\tau) = J(z) = \frac{(z^8 + 240z^6 + 2144z^4 + 3840z^2 + 256)^3}{z^2(z^2 + 4)(z - 2)^8(z + 2)^8},\]

using Proposition 5.2. Now set \(z = \eta - \frac{1}{\eta} = \pm \frac{2}{\pi}\), so that \((z, \wp_2) = 1\). This allows us to reduce the above formula modulo \(\wp_2\), giving that

\[j(w/8) \equiv \frac{z^2}{z^2 + 4} \equiv z^2 \pmod{\wp_2}.\]

This shows that \(j(w/8)^{\tau_2}\) is conjugate to \(z^2\) modulo each prime divisor \(\wp\) of \(\wp_2\) in \(\Omega_f\), for each automorphism \(\tau \in \text{Gal}(\Omega_f/K)\); and this implies that the class equation \(H_{-d}(X)\) and the minimal polynomial \(\mu_d(X)\) of \(z\) over \(K\) are congruent:

\[H_{-d}(X) \equiv \mu_d(X) \pmod{\wp_2}.\]

A theorem of Deuring says that the discriminant of \(H_{-d}(X)\) is odd (since \(\left(\frac{-d}{2}\right) = +1\)), so the discriminant of \(\mu_d(X)\) is not divisible by \(\wp_2\). This implies that the discriminant of the minimal polynomial \(\mu_d(X) = X^{h(-d)}\mu_d\left(X - \frac{1}{X}\right)\) of \(\eta\) over \(K\) is relatively prime to \(\wp_2\), as well. This is because

\[\mu_d(X) = \prod_{i=1}^{h(-d)} (X - (\eta_i - \frac{1}{\eta_i}))\]

is a product over the conjugates \(z_i = \eta_i - \frac{1}{\eta_i}\) of \(z\), so that

\[X^{h(-d)}\mu_d\left(X - \frac{1}{X}\right) = \prod_{i=1}^{h(-d)} (X^2 - (\eta_i - \frac{1}{\eta_i})X - 1),\]

\[= \prod_{i=1}^{h(-d)} (X^2 - z_iX - 1), \quad z_i = \eta_i - \frac{1}{\eta_i}.\]
Hence,
\[
\text{disc}(\tilde{\mu}_d(X)) = \prod_{i=1}^{h(-d)} (z_i^2 + 4) \prod_{i<j} \text{Res}(X^2 - z_iX - 1, X^2 - z_jX - 1)^2
\]
\[
= \prod_{i=1}^{h(-d)} (z_i^2 + 4) \prod_{i<j} (z_i - z_j)^4
\]
\[
= \prod_{i=1}^{h(-d)} (z_i^2 + 4) \left(\text{disc}(\mu_d(X))\right)^2.
\]

Now the $z_i$ are conjugate over $K$, so each $z_i$ is relatively prime to $\wp_2$, which implies that $(z_i^2 + 4, \wp_2) = 1$ for each $i$. This proves the claim that $(\text{disc}(\tilde{\mu}_d(X)), \wp_2) = 1$. This proves

**Theorem 7.8.** Let $R_{\wp_2}$ denote the ring of elements of $K$ which are integral for $\wp_2$. Then the powers of $\eta = \nu(w/8)$ form a basis over $R_{\wp_2}$ for the ring $\overline{R}$ of elements of $F_1 = \mathbb{Q}(\eta)$ which are integral for $\wp_2$.

Given this theorem, the congruence
\[
\eta^\psi \equiv \eta^2 \pmod{\wp_2}
\]
implies that
\[
\alpha^\psi \equiv \alpha^2 \pmod{\wp_2},
\]
for all $\alpha \in F_1$ which are integral for $\wp_2$. Since $F_1/K$ is abelian and $\wp_2$ is unramified in this extension, this implies by definition of the Artin symbol that $\psi = \tau_2$.

**8. Values of $\nu(\tau)$ as periodic points.**

We now define the following algebraic functions. The roots of $f(x, y) = y^2 + (x^2 - 1)y + x^2$ (see Proposition 3.1) as a function of $y$ are
\[
\hat{\nu}(x) = -\frac{y^2 - 1}{2} \pm \frac{1}{2} \sqrt{y^4 - 6y^2 + 1}.
\]
Also, the roots of $g(x, y) = y^2 - (x^2 - 4x + 1)y + x^2$ (see Proposition 3.2) are given by
\[
\hat{\nu}(x) = \frac{1}{2}(x^2 - 4x + 1) \pm \frac{1}{2} \sqrt{(x^2 - 2x + 1)(x^2 - 6x + 1)}
\]
\[
= \frac{1}{2}(x^2 - 4x + 1) \pm \frac{x - 1}{2} \sqrt{x^2 - 6x + 1}.
\]

We prove the following.

**Theorem 8.1.** If $w \in R_K$ is the algebraic integer defined by
\[
w = \frac{a + \sqrt{-d}}{2}, \quad \text{with } a^2 + d \equiv 0 \pmod{2^5}
\]
and the integer \( c \) satisfies
\[
\begin{align*}
c &\equiv 1 - \frac{a^2 + d}{32} \pmod{2},
\end{align*}
\]
then the generator \((-1)^{1+c} \nu(w/8)\) of the field \( \Sigma_{\wp_1^c} \Omega_f \) over \( \mathbb{Q} \) is a periodic point of the algebraic function \( \hat{F}(x) \) defined by \( (8.1) \) and \( \hat{v}^2(w/8) \) is a periodic point of the function \( \hat{T}(x) \) defined by \( (8.2) \). If \( c \) is even, then \( \nu(w/8) \) is a pre-periodic point of \( \hat{F}(x) \).

**Proof.** Setting \( \eta = (-1)^{1+c} \nu(w/8) \) and \( F_1 = \mathbb{Q}(\eta) = \mathbb{Q}(\eta^2) \), we have from the corollaries to Theorems 7.3 and 7.5 that \( f(\eta, \eta^{2^z}) = 0 \), where \( \tau_2 = \left( \frac{F_1/K}{\wp_1^c} \right) \) is an automorphism in \( \text{Gal}(F_1/K) \). If the order of \( \tau_2 \) is \( n \), then applying powers of \( \tau_2 \) gives that
\[
\begin{align*}
f(\eta, \eta^{2^z}) &= f(\eta^{2^z}, \eta^{2^z^2}) = \cdots = f(\eta^{2^z^n}, \eta) = 0,
\end{align*}
\]
which implies that \( \eta \) is a periodic point of \( \hat{F}(x) \) of period \( n \). If \( c \) is even, then from Corollary 7.6 and the fact that \( f(x, y) = f(-x, y) \) we also have that
\[
f(\nu(w/8), -\nu(w/8)^{2^z}) = 0;
\]
thus, \( \nu(w/8) \) is a pre-periodic point of \( \hat{F}(x) \), since \( -\nu(w/8)^{2^z} \) is periodic.

It is straightforward to check that
\[
\hat{F}(x)^2 = \frac{1}{2}(x^4 - 4x^2 + 1) \pm \frac{1}{2}(x^2 - 1)v(\sqrt{x^4 - 6x^2 + 1}) = \hat{T}(x^2)
\]
and that the minimal polynomial of \( \hat{F}(x)^2 \) over \( \mathbb{Q}(x) \) is \( g(x^2, y) \). In particular, \( f(x, y) = 0 \) implies that \( g(x^2, y^2) = 0 \), since
\[
g(x^2, y^2) = (-x^2y + x^2 + y^2 + y)(x^2y + x^2 + y^2 - y) = f(x, -y)f(x, y).
\]
Hence, \( (8.3) \) implies that
\[
g(\eta^2, \eta^{2^z}) = g(\eta^{2^z^2}, \eta^{2^z^2}) = \cdots = g(\eta^{2^z^n}, \eta^2) = 0,
\]
which shows that \( \eta^2 = \nu(w/8)^2 \) is a periodic point of \( \hat{F}(x) \).

**Remarks.**

1. Note that if \( c \) is even, meaning that \( 2^5 \mid a^2 + d \), then \( 2^6 \mid (a + 16)^2 + d \), so that \( w + 8 = \frac{a+16+\sqrt{-d}}{2} = w' \) satisfies \( (6.1) \) with \( c \) odd. Then the infinite product formula for \( \nu(\tau) \) shows that \( \nu(w/8) = \nu(w'/8 - 1) = -\nu(w'/8) \), and \( -\nu(w/8) = \nu(w'/8) \) in Corollary 7.6.

2. Given that \( f(\nu(\tau), \nu(2\tau)) = 0 \), it is tempting to try to show that \( \nu(w/8) \) is a periodic point by considering the chain of equations
\[
f(\nu(w/8), \nu(w/4)) = f(\nu(w/4), \nu(w/2)) = \cdots = f(\nu(2^{n-1}w/8), \nu(2^n w/8)) = 0,
\]
and find an integer \( n \) for which \( 2^{n-1}w = M(w/8) = \frac{aw+8b}{cw+8d} \), for some unimodular matrix \( M \) for which \( \nu(M(w/8)) = \nu(w/8) \). However, this requires
that \( M \in \Gamma_1(8) \), so that \( a \equiv 1 \pmod{8} \) and \( 8 \mid c \). This condition leads to the equation

\[
2^{n-3}cw^2 + (2^nd - a)w - 8b = 0.
\]

Moreover, \( w \) is an algebraic integer, so the fact that \( 8 \mid c \) shows that \( 2^n \) must divide the other coefficients of this quadratic. Hence, \( 2^n \mid a \), which is impossible for \( n \geq 1 \). Thus, this approach does not yield an orbit leading back to \( \nu(w/8) \).

As in the papers [15]-[18], the minimal polynomials of periodic points of \( \hat{F}(x) \) can be computed using iterated resultants involving its minimal polynomial \( f(x, y) \). We set

\[
R_1(x, x_1) = f(x, x_1) = x^2 + x^2 + x_1^2 - x_1
\]

and define, inductively,

\[
R^{(n)}(x, x_n) = \text{Res}_{x_{n-1}}(R^{(n-1)}(x, x_{n-1}), f(x_{n-1}, x_n)) \; n \geq 2.
\]

Then the roots of the polynomial

\[
R_n(x) = R^{(n)}(x, x), \; n \geq 1,
\]

are the periodic points of \( \hat{F}(x) \) whose minimal periods divide \( n \). See [15, p. 727]. For example, we compute that

\[
R_1(x) = x(x^2 + 2x - 1),
\]
\[
R_2(x) = x(x^2 + 2x - 1)(x^4 - x^3 + x + 1),
\]
\[
R_3(x) = x(x^2 + 2x - 1)(x^{12} - 5x^{11} + 2x^{10} + 10x^9 + 5x^8 + 23x^7
\]
\[\quad - 8x^6 - 23x^5 + 5x^4 - 10x^3 + 2x^2 + 5x + 1),
\]
\[
R_4(x) = x(x^2 + 2x - 1)(x^4 - x^3 + x + 1)(x^8 - x^7 + x^6 - 5x^5 + 5x^3 + x^2 + x + 1)
\]
\[\times (x^{16} + 5x^{15} - 18x^{14} - 75x^{13} + 137x^{12} + 105x^{11} + 38x^{10} + 185x^9
\]
\[\quad - 300x^8 - 185x^7 + 38x^6 - 105x^5 + 137x^4 + 75x^3 - 18x^2 - 5x + 1).
\]

We now set \( x = z + 3 \) in the function \( \hat{T}(x) \), so that the square-root in \( \hat{T}(x) \) has the 2-adic expansion

\[
\sqrt{x^2 - 6x + 1} = \sqrt{z^2 - 8} = z\sqrt{1 - \frac{8}{z^2}} = z\sum_{k=0}^{\infty} (-1)^k \binom{1/2}{k} \frac{8^k}{z^{2k}}.
\]

We will show that this series is 2-adically convergent for (roughly) half of the primitive periodic points of the algebraic function \( \hat{T}(x) \) of a given period \( n \) in the field \( K_2(\sqrt{2}) \), where \( K_2 \) is the maximal unramified, algebraic extension of the 2-adic field \( \mathbb{Q}_2 \).

If we set

\[
T(x) = \frac{1}{2}(x^2 - 4x + 1) + \frac{x - 1}{2}\sqrt{x^2 - 6x + 1},
\]
then using the above series in \( T(x) \) and splitting off the \( k = 0 \) term, we find

\[
T(x) = x^2 - 4x + 2 + (x - 1)(x - 3) \sum_{k=1}^{\infty} (-1)^k 2^{2k-1} \binom{1/2}{k} \frac{2^k}{(x - 3)^{2k}},
\]

for \( x - 3 \in \mathcal{O}^* \), where \( \mathcal{O} \) is the ring of integers in \( K_2(\sqrt{2}) \). Since

\[
(-1)^{k-1} 2^{2k-1} \binom{1/2}{k} = C_{k-1} \in \mathbb{Z}
\]

is the Catalan sequence, it follows that

\[
T(x) \equiv x^2 \pmod{2}, \quad x - 3 \in \mathcal{O}^*.
\]

Hence, \( T(x) \) is a lift of the Frobenius automorphism for points \( x \) in the set

\[
\overline{D} = \{ x \in K_2(\sqrt{2}) : |x - 3|_2 = 1 \}.
\]

Furthermore,

\[
T(x) - 3 = (x - 3)^2 + 2(x - 3) - 4 - (x - 1)(x - 3) \sum_{k=1}^{\infty} C_{k-1} \frac{2^k}{(x - 3)^{2k}}.
\]

It follows that

\[
|T(x) - 3|_2 = |x - 3|^2 = 1, \quad (8.6)
\]

and \( T \) maps \( \overline{D} \) to itself.

We next prove

**Proposition 8.2.** We have the congruences

\[
R^{(n)}(x, x_n) \equiv (x^{2^n} + x_n)(x_n + 1)^{2^{n-1}} \pmod{2};
\]

\[
R_n(x) \equiv (x^{2^n} + x)(x + 1)^{2^{n-1}} \pmod{2}.
\]

**Proof.** We have \( f(x, y) = x^2y + x^2 + y^2 - y \). So, for \( n = 1 \), we get

\[
R^{(1)}(x, x_1) = f(x, x_1) = x^2x_1 + x^2 + x_1^2 - x_1 \equiv x^2x_1 + x^2 + x_1^2 + x_1 \pmod{2} \equiv (x^2 + x_1)(x_1 + 1) \pmod{2}.
\]

Hence,

\[
R_1(x) \equiv (x^2 + x)(x + 1) \pmod{2}.
\]

Now for the induction step, assume the result is true for \( n - 1 \). Then,

\[
R^{(n)}(x, x_n) = \text{Res}_{x_{n-1}}(R^{(n-1)}(x, x_{n-1}), f(x_{n-1}, x_n))
\]

\[
\equiv \text{Res}_{x_{n-1}}((x^{2^{n-1}} + x_{n-1})(x_{n-1} + 1)^{2^{n-1}-1}, (x_{n-1}^2 + x_n)(x_n + 1)) \pmod{2}.
\]
By definition, the resultant of two polynomials \( f = \sum_{i=0}^{n} a_i x^i \) and \( g = \sum_{i=0}^{m} b_i x^i \), having roots \( \alpha_1, \alpha_2, \ldots, \alpha_n \) and \( \beta_1, \beta_2, \ldots, \beta_m \), respectively, is given by

\[
\text{Res}(f, g) = a_n^m \prod_{i=1}^{n} g(\alpha_i),
\]

and

\[
\text{Res}(g, f) = (-1)^{mn} \text{Res}(f, g).
\]

The roots of \( (x^2 - 1 + x_n)(x_n + 1) \), as a polynomial in \( x_{n-1} \), are \( \pm \sqrt{-x_n} \). Hence,

\[
\begin{align*}
\text{Res}_{x_{n-1}}(x^2 - 1 + x_n)(x_{n-1} + 1)^{2^{n-1}-1}, (x^2 - 1 + x_n)(x_{n-1} + 1)) & = (-1)^{2^{n-1}-2}(x_n + 1)^{2^{n-1}}(x^2 - 1 + \sqrt{-x_n})(\sqrt{-x_n} + 1)^{2^{n-1}-1} \\
& \times (x^2 - 1 + \sqrt{-x_n})(-\sqrt{-x_n} + 1)^{2^{n-1}-1} \\
& = (-1)^{2^n(x_n + 1)}^{2^{n-1}}(x^2 + x_n)(x_n + 1)^{2^{n-1}-1} \\
& = (x^2 + x_n)(x_n + 1)^{2^{n-1}}.
\end{align*}
\]

Hence, we obtain

\[
R^{(n)}(x, x_n) \equiv (x^2 + x_n)(x_n + 1)^{2^{n-1}} \pmod{2}, \quad R_n(x) \equiv (x^2 + x)(x + 1)^{2^{n-1}} \pmod{2},
\]

completing the induction. \( \square \)

**Corollary 8.3.** The degree of \( R_n(x) \) is \( \deg(R_n(x)) = 2^{n+1} - 1 \).

**Proof.** This follows from the proposition, if the leading coefficient of \( R_n(x) \) is not divisible by 2. In fact, this follows from the relation

\[
R^{(n)}(x, x_n) = A_n(x_n) x^{2^n} + S_n(x, x_n),
\]

where for \( n \geq 3, \)

\[
A_n(x_n) = (x_n + 1)(x_n^2 + 1)(x_n^2 - 2x_n - 1)^2(x_n^2 + 2x_n - 1)^{2^{n-1}-4}
\]

and for \( n \geq 1, \)

\[
\deg(A_n(x_n)) = 2^n - 1, \quad \deg(S_n(x, x_n)) \leq 2^n - 2, \quad \deg_x(S_n(x, x_n)) = 2^n.
\]

We refer the reader to the lemma in [15, pp. 727-728] for a similar proof. \( \square \)

The roots of the factor \( x^{2^n} + x = x(x + 1) \frac{x^{2^{n-1}+1}}{x+1} = x(x + 1) h_n(x) \) other than \( x = 0, 1 \) have degree greater than 1, and therefore satisfy \( x - 3 \not\equiv 0 \pmod{2} \). It follows from Hensel’s Lemma that \( 2^{n-1} \) of the roots of \( R_n(x) \) over \( \mathbb{Q}_2 \) have the property that \( x - 3 \in \mathcal{O}^\times \), and for these roots the series for \( T(x) \) converges in \( \mathbb{K}_2 \).
Now the argument at the end of the proof of Theorem 7.3 shows that \( \eta \not\equiv 1 \pmod{\wp^2} \), so that the image of \( \eta \) in the completion \( F_{1,q} \subset K_2 \) of \( F_1 = \Sigma_{\wp^2} \Omega_f \) with respect to a prime divisor \( \wp \) of \( \wp^2 \) in \( F_1 \) satisfies \( \eta^2 - 3 \in O^\times \). Hence, the series for \( T(\eta^2) \) converges. We claim now that \( \eta^{2\tau_2} = T(\eta^2) \). But \( g(\eta^2, \eta^{2\tau_2}) = 0 \) implies that \( \eta^{2\tau_2} \) is one of the values of \( T(\eta^2) \). The value different from \( T(\eta^2) \) in \( K_2 \) is

\[
T_1(\eta^2) = \eta^4 - 4\eta^2 + 1 - T(\eta^2)
\]

\[
\equiv \eta^4 - 4\eta^2 + 1 - \eta^4 \pmod{\wp}
\]

\[
\equiv 1 \pmod{\wp}.
\]

But we also know \( \eta^{2\tau_2} - 3 = (\eta^2 - 3)^{\tau_2} \in O^\times \), so that \( \eta^{2\tau_2} \neq T_1(\eta^2) \). This yields the following.

**Theorem 8.4.** If \( w \) satisfies (6.1), then the value \( \eta = \nu(w/8) \) and the automorphism \( \tau_2 = \left( \frac{F_1/K}{\wp^2} \right) \) satisfy

\[
\eta^{2\tau_2} = T(\eta^2),
\]

in the completion \( F_{1,q} \subset K_2 \) of \( F_1 = \Sigma_{\wp^2} \Omega_f \) with respect to a prime divisor \( \wp \) of \( \wp^2 \) in \( F_1 \), where

\[
T(x) = x^2 - 4x + 2 - (x - 1)(x - 3) \sum_{k=1}^{\infty} C_{k-1} \frac{2^k}{(x - 3)^{2k}}
\]

converges for \( x \in \overline{D} = \{ x \in K_2(\sqrt{2}) : |x - 3|_2 = 1 \} \).

Since \( \tau_2 \) fixes the prime divisors of \( \wp^2 \), it extends naturally to an automorphism of \( F_{1,q} \), and can be applied to the individual terms of the series representing \( T(x) \). Thus, we see inductively that

\[
\eta^{2\tau_1} = T(\eta^{2\tau_1}) = T(T^{i-1}(\eta^2)) = T^i(\eta^2)
\]

is the \( i \)-th iterate of \( T(x) \) applied to \( \eta^2 \). From this and the fact that \( Q(\eta^2) = F_1 \) we see that the order of \( \tau_2 \) in \( \text{Gal}(F_1/K) \) is the minimal period of the periodic point \( \eta^2 \), and that \( \eta^2 \) is a periodic point in the ordinary sense of the 2-adic function \( T(x) \). This also shows that the minimal period of \( \eta \) with respect to \( \hat{F}(x) \) is \( n = \text{ord}(\tau_2) \), since if \( \eta \) had smaller minimal period \( m \), then by the proof of Theorem 8.1, \( \eta^2 \) would have period \( m < n \) with respect to the function \( T(x) \). This completes the proof of the assertions of Theorem B of the Introduction regarding minimal periods.

9. The periodic points of \( \hat{F}(x) \) and a class number formula.

In this section we show that the only periodic points of \( \hat{F}(x) \) are the values given in Theorem 8.1. In fact, we will prove the following.
Theorem 9.1. The only periodic points of the function $F(x)$ in $\overline{Q}$ are the fixed points $0, \sigma, \bar{\sigma}$ and the conjugates over $Q$ of the values $\nu(w/8)$ in Theorem 8.1 (for odd $c$).

Proof. Let $\tilde{g}(x, y) = x^2y^2 + 2y + x^2$. Note that $\tilde{g}(x, y) = g(y, x)$ for the polynomial $g(x, y)$ in [16, Thm. 2, p. 327]. By the results of that paper the numbers $\pi, \xi$ and their conjugates over $Q$ (as $-d$ ranges over all discriminants $\equiv 1$ modulo 8) are, together with 0 and $-1$, the only periodic points of the algebraic function $f(z)$ defined by $\tilde{g}(z, f(z)) = 0$. The assertion of the theorem will follow from the identity

$$(x^2 - 1)^2(y^2 - 1)^2\tilde{g}\left(\frac{2x}{x^2 - 1}, \frac{2y}{y^2 - 1}\right) = 4f(x, y)(x^2y^2 - x^2y + y + 1). \tag{9.1}$$

Here, as in Proposition 3.1, $f(x, y) = x^2y + x^2 + y^2 - y$. Let $\eta$ be a periodic point of $\tilde{F}(x)$ in $\overline{Q}$ which is distinct from its fixed points $0, \sigma, \bar{\sigma}$. Then there are $\eta_1 = \eta, \eta_2, ..., \eta_n$ in $\overline{Q}$ for which

$$f(\eta_1, \eta_2) = f(\eta_2, \eta_3) = \cdots = f(\eta_n, \eta_1) = 0. \tag{9.2}$$

Setting $\lambda_i = \frac{2\eta_i}{\eta_i^2 - 1}$, equations (9.1) and (9.2) give that

$$g(\lambda_1, \lambda_2) = g(\lambda_2, \lambda_3) = \cdots = g(\lambda_n, \lambda_1) = 0. \tag{9.3}$$

Note that $\eta_i \neq \pm 1$ since $\pm 1$ are preperiodic (and not periodic) for $f(x, y)$, since

$$f(\pm 1, y) = y^2 + 1, \quad f(\pm i, y) = y^2 - 2y - 1, \quad f(1 \pm \sqrt{2}, y) = (y + 1 \pm \sqrt{2})^2.$$

Equation (9.3) implies that $\lambda_1$ is a periodic point of the function $f(z)$ defined above. Also, $\lambda_i \neq 0, -1$ since $\eta_i \not\in \{0, \sigma, \bar{\sigma}\}$. By the results of [16, Thm. 2], this shows that $\lambda_1$ must be a conjugate of the number $\pi$ for some discriminant $-d$ and is therefore a root of the polynomial $b_d(x)$. (See Proposition 6.4.) Since $\lambda_1 = 2\eta/(\eta^2 - 1)$, this shows that $\eta$ is a root of the minimal polynomial $f_d(x)$ of $\nu(w/8)$, for $c$ odd, by (6.7). This completes the proof.

Remark. We can use equation (9.1) to give an alternate proof of the Corollary to Theorem 7.3, as follows. We would like to show that $f(\eta, \eta^{\tau_2}) = 0$, where $\eta = \nu(w/8)$ and $\tau_2 = \left(\frac{F_1/K}{\varphi_2}\right)$, with $F_1 = \sum_{\varphi_2}^{\varphi_2'}\Omega_f$. Since $\tau_2|\Omega_f = \left(\frac{\Omega_f/K}{\varphi_2}\right)$, we know that $\tilde{g}(\pi, \pi^{\tau_2}) = 0$, by [16, pp. 332-333]. Using $\pi = \frac{2\eta}{\eta^2 - 1}$ from (6.4), equation (9.1) implies that $f(\eta, \eta^{\tau_2})k(\eta, \eta^{\tau_2}) = 0$, where $k(x, y) = x^2y^2 - x^2y + y + 1$. But $k(\eta, \eta^{\tau_2}) \equiv k(\eta, \eta^{\tau_2}) \mod \varphi_2$ in $F_1$. An easy computation shows that $k(x, x^2) \equiv (x + 1)^6 \mod 2$, so $k(\eta, \eta^{\tau_2}) \equiv (\eta + 1)^6 \mod \varphi_2$. If $\eta \equiv 1$ modulo some prime divisor $p$ of $\varphi_2$ in $F_1$, then the relation $\eta^{\tau_2} - \frac{2}{p} - 1 = 0$ would give that $\frac{2}{p} \equiv 0 \mod \varphi_2$, which is impossible since $\frac{2}{p} \not\equiv \varphi_2'$. Hence, $k(\eta, \eta^{\tau_2}) \equiv 0 \mod \varphi_2$, which implies $k(\eta, \eta^{\tau_2}) \neq 0$ and therefore $f(\eta, \eta^{\tau_2}) = 0$, as claimed.
Theorem 9.1 has the following consequence. As in the last remark, let \( F_1 = \Sigma \varpi \Omega_f \) be the field generated by \( \psi(w/8) \) in Theorem 6.1. Then \([F_1 : \mathbb{Q}] = 4h(-d)\) and \( F_1 \) is the inertia field for \( \varpi \Omega_2 \) in the field \( \Sigma \varpi \Omega_f \), an extended ring class field over \( K_d = \mathbb{Q}(\sqrt{-d}) \). As in Section 7, let \( \tau_2 = \left( \frac{F_1/K_d}{\varpi \Omega_2} \right) \) be the Artin symbol for \( \varpi \Omega_2 \) in the extension \( F_1/K_d \). Now define the set of discriminants

\[ \mathfrak{D}_{n,2} = \{ -d < 0 \mid -d \equiv 1 \pmod{8} \text{ and } \text{ord}(\tau_2) = n \text{ in } \text{Gal}(F_1/K_d) \}. \quad (9.4) \]

**Theorem 9.2.** If \( n \geq 2 \), we have the following relation between class numbers of discriminants in the set \( \mathfrak{D}_{n,2} \):

\[ \sum_{-d \in \mathfrak{D}_{n,2}} h(-d) = \frac{1}{2} \sum_{k \mid n} \mu(n/k)2^k. \quad (9.5) \]

**Proof.** This proof mirrors the arguments in [18, pp.792-793, 806]. First, define

\[ P_n(x) = \prod_{k \mid n} R_k(x)^{\mu(n/k)}. \quad (9.6) \]

We show that \( P_n(x) \in \mathbb{Z}[x] \). From Proposition 8.2 it is clear that \( R_n(x) \), for \( n > 1 \), is divisible \( \pmod{2} \) by the \( N \) irreducible (monic) polynomials \( h_i(x) \) of degree \( n \) over \( \mathbb{F}_2 \), where

\[ N = \frac{1}{n} \sum_{k \mid n} \mu(n/k)2^k, \]

and that these polynomials are simple factors of \( R_n(x) \pmod{2} \). It follows from Hensel’s Lemma that \( R_n(x) \) is divisible by distinct irreducible polynomials \( h_i(x) \) of degree \( n \) over \( \mathbb{Z}_2 \), the ring of integers in \( \mathbb{Q}_2 \), for \( 1 \leq i \leq N \), with \( h_i(x) \equiv h_i(x) \pmod{2} \). In addition, all the roots of \( h_i(x) \) are periodic of minimal period \( n \) and lie in the unramified extension \( K_2 \). Furthermore, \( n \) is the smallest index for which \( h_i(x) \mid R_n(x) \pmod{2} \).

Now consider the identity

\[ (\sigma x + 1)^2(\sigma y + 1)^2f(\bar{A}(x), \bar{A}(y)) = 2^2\sigma^2f(y, x), \quad (9.7) \]

where \( \bar{A}(x) = \frac{-x + \sigma}{\sigma x + 1} \), as in (3.3). If the periodic point \( a \) of \( \hat{F}(x) \), with minimal period \( n > 1 \), is a root of one of the polynomials \( h_i(x) \), then \( a \) is a unit in \( K_2 \), and for some \( a_1, \ldots, a_{n-1} \) we have

\[ f(a, a_1) = f(a_1, a_2) = \cdots = f(a_{n-1}, a) = 0. \quad (9.8) \]

Furthermore, \( a \not\equiv 1 \pmod{\sqrt{2}} \), since otherwise its reduction \( a \equiv \bar{a} \equiv 1 \pmod{2} \) would have degree 1 over \( \mathbb{F}_2 \) (using that \( K_2 \) is unramified over \( \mathbb{Q}_2 \)). Hence, \( a + 1 + \sqrt{2} \) is a unit in \( K_2(\sqrt{2}) \), which gives that \( \sigma a + 1 \) is a unit, as well. All of the \( a_i \) satisfy \( a_i \not\equiv 1 \pmod{\sqrt{2}} \), since the congruence \( f(1, y) \equiv (y + 1)^2 \pmod{2} \) has
only $y \equiv 1$ as a solution. Hence, if some $a_i \equiv 1 \pmod{\sqrt{2}}$, then $a_j \equiv 1$ for $j > i$, which would imply that $a \equiv 1 \pmod{\sqrt{2}}$, as well. The elements $b_i = \tilde{A}(a_i)$ are distinct and lie in $K_2(\sqrt{2})$ and satisfy
\[ b_i - 1 \equiv \frac{-a_i + \sigma - \sigma a_i - 1}{\sigma a_i + 1} \equiv \frac{-2}{\sigma a_i + 1} \equiv 0 \pmod{\sqrt{2}}. \]
The identity (9.7) yields that
\[ f(b, b_{n-1}) = f(b_{n-1}, b_{n-2}) = \cdots = f(b_1, b) = 0 \quad (9.9) \]
in $K_2(\sqrt{2})$. Hence, $b_i \equiv 1 \pmod{\sqrt{2}}$, and the orbit $\{b, b_{n-1}, \ldots, b_1\}$ is distinct from all the orbits in (9.8).

Now the map $\tilde{A}(x)$ has order 2, so it is clear that $b = \tilde{A}(a)$ has minimal period $n$ in (9.9), since otherwise $a = \tilde{A}(b)$ would have period smaller than $n$. It follows that there are at least $2N$ periodic orbits of minimal period $n > 1$. Noting that
\[ R_n(x) = f(x, x) = x(x^2 + 2x - 1), \]
these distinct orbits and factors account for at least
\[ 3 + \sum_{d|n, d > 1} \left( 2 \sum_{k|d} \mu(d/k)2^k \right) = -1 + 2 \sum_{d|n} \left( \sum_{k|d} \mu(d/k)2^k \right) = 2 \cdot 2^n - 1 \]
roots, and therefore all the roots, of $R_n(x)$. This shows that the roots of $R_n(x)$ are distinct and the expressions $P_n(x)$ are polynomials. Furthermore, over $K_2(\sqrt{2})$ we have the factorization
\[ P_n(x) = \pm \prod_{1 \leq i \leq N} h_i(x)\tilde{h}_i(x), \quad n > 1, \quad (9.10) \]
where $\tilde{h}_i(x) = c_i(\sigma x + 1)^n h_i(\tilde{A}(x))$, and the constant $c_i$ is chosen to make $\tilde{h}_i(x)$ monic.

By the results of Section 8, for each discriminant $-d \in \mathcal{D}_{n,2}$ we have that $f_d(x) \mid P_n(x)$. Furthermore, every root of $P_n(x)$ is a root of some $f_d(x)$, by Theorem 9.1, where $\text{ord}(\tau_2) = n$ in order for the roots of $f_d(x)$ to have minimal period $n$. It follows that
\[ P_n(x) = \tilde{c}_n \prod_{-d \in \mathcal{D}_{n,2}} f_d(x), \]
for some constant $\tilde{c}_n$, and taking degrees on both sides and using (9.10) gives the formula
\[ 2 \sum_{k|n} \mu(n/k)2^k = \sum_{-d \in \mathcal{D}_{n,2}} 4h(-d). \]
The formula of the theorem follows.

The result of Theorem 9.2 is the analogue of [18, Thm.1.3] for the prime 2 in place of 5. The factor $1/2$ in front is to be interpreted as $2/\phi(8)$, replacing the factor $2/\phi(5)$ in the result of [18]. Also, see Conjecture 1 in the Introduction of that paper.
Theorem 9.1 will now be used to prove the corresponding fact for the algebraic function $\tilde{T}(x)$ in Theorem 8.1.

**Theorem 9.3.** The periodic points of the function $\tilde{T}(x)$ of (8.2) in $\overline{\mathbb{Q}}$ (or $\mathbb{C}$) are exactly the squares of the periodic points of the function $\tilde{F}(x)$, i.e., the fixed points $0, \sigma^2, \bar{\sigma}^2$ and the conjugates over $\mathbb{Q}$ of the values $v^2(w/8)$, where $w$ is given by (6.1).

**Proof.** As in the proof of Theorem 8.1, the polynomials $g(x, y) = y^2 - (x^2 - 4x + 1)y + x^2$ and $f(x, y) = y^2 + (x^2 - 1)y + x^2$ defining $\tilde{T}$ and $\tilde{F}$, respectively, satisfy the identity

$$g(x^2, y^2) = f(x, -y)f(x, y).$$

Let $\eta^2$ be a periodic point of $g(x, y)$ of period $n$. Then there exist $\eta_1^2, \eta_2^2, ..., \eta_{n-1}^2 \in \overline{\mathbb{Q}}$ such that

$$g(\eta_1^2, \eta_2^2) = g(\eta_2^2, \eta_3^2) = \cdots = g(\eta_{n-1}^2, \eta^2) = 0.$$ 

This means that, for every $i = 0, 1, ..., n - 1$, either

$$f(\eta_i, \eta_{i+1}) = 0 \text{ or } f(\eta_i, -\eta_{i+1}) = 0, \text{ where } \eta_0 = \eta = \eta_n.$$

Now if $f(\eta_i, \eta_{i+1}) = 0$ for all $i$, then $\eta$ is a periodic point of $\tilde{F}(x)$.

Otherwise, there exists an $i$ such that $f(\eta_i, \eta_{i+1}) \neq 0$, but $f(\eta_i, -\eta_{i+1}) = 0$. In this case, if $i < n-1$, replace $\eta_{i+1}$ by $-\eta_{i+1}$ in the next equation of the sequence, yielding $f(-\eta_{i+1}, \eta_{i+2}) = 0$. And if this happens for $i = n - 1$, then simply replace $\eta$ by $-\eta$. This works because $f(-x, y) = f(x, y)$. In other words, in the chain of equations for $f$, whenever the second argument has a negative sign, choose the next first argument with the same negative sign. And in case the last equation has second argument $\eta$ with a negative sign, then choose the first argument of the first equation as $-\eta$ also. Hence, there is a chain of equations $f(\eta_1, \eta_{i+1}) = 0$ beginning and ending with $\pm \eta$. Hence, $\pm \eta$ is a periodic point of $\tilde{F}(x)$ in either case, which implies that $\eta^2$ is the square of a periodic point of $\tilde{F}(x)$. This completes the proof. \[\square\]

With this theorem, we have completely proved all the statements in Theorem B of the Introduction.

### 10. Appendix

Here we give a proof of the relation between $u(\tau)$ and $v(\tau)$ that was used in the proof of Proposition 3.1b).

**Proposition 10.1.** The following relation holds between $u(\tau)$ and $v(\tau)$:

$$u^4(v^2 + 1)^2 + 4v(v^2 - 1) = 0.$$
Proof. We have derived in the proof of Proposition 4.1 that
\[ \frac{1}{\nu(\tau)} = q^{-1/2} \frac{(-q^2; q^4)_{\infty}^2}{(-q^4; q^4)_{\infty}^2}. \]
Proceeding in a similar way, we obtain
\[ \frac{1}{\nu(\tau)} + \nu(\tau) = \frac{\psi(-q) \cdot \varphi(q)}{q^{1/2} (q; q^2)_{\infty} (q^8; q^8)_{\infty}} \]
\[ = q^{-1/2} \frac{(q^2; q^4)_{\infty}}{(-q^2; q^4)_{\infty}} \cdot \frac{(-q^2; q^4)^2_{\infty}}{(q^2; q^4)^2_{\infty}} \]
\[ = q^{-1/2} \frac{(-q^2; q^4)^2_{\infty}}{(q; q^2)_{\infty} (q^8; q^8)_{\infty}} \cdot \frac{(q^2; q^4)^2_{\infty}}{(q^2; q^4)_{\infty}} \]
\[ = q^{-1/2} \frac{(-q^2; q^4)^2_{\infty}}{(q; q^2)_{\infty}} \cdot \frac{(q^2; q^4)^2_{\infty}}{(q^2; q^4)_{\infty}} \]
\[ = q^{-1/2} \frac{(-q^2; q^4)^2_{\infty} (q^2; q^4)_{\infty}}{(q; q^2)_{\infty} (q^8; q^8)_{\infty}}. \]
(See [2, pp. 221-222].) Putting the above two expressions to use in \[ \frac{4u(1-u^2)}{(1+u^2)^2} = \]
\[ \frac{4(\frac{1}{\nu} - u)}{(\frac{1}{\nu} + u)}, \]
we find that
\[ \frac{4u(1-u^2)}{(1+u^2)^2} = 4q^{1/2} \frac{(-q^2; q^4)^2_{\infty}}{(-q^4; q^4)_{\infty}} \cdot \frac{(-q^2; q^4)^2_{\infty}}{(q^2; q^4)^2_{\infty}} \]
\[ = 4q^{1/2} \frac{(-q^2; q^4)^2_{\infty} (q^2; q^4)_{\infty}}{(-q^2; q^4)^4_{\infty}} \]
\[ = 4q^{1/2} \frac{(-q^2; q^4)^4_{\infty}}{(-q^2; q^4)^4_{\infty}} \]
\[ = 4q^{1/2} \frac{(-q^2; q^4)^4_{\infty}}{(-q^2; q^4)^4_{\infty}} \]
\[ = u^4(\tau), \]
completing the proof. \(\square\)

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PERIODIC POINTS OF ALGEBRAIC FUNCTIONS

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This paper is available via http://nyjm.albany.edu/j/2024/30-36.html.