Wobbling motion coupled to gamma vibration at high spin

Makito Oi\textsuperscript{1,2,*} and Philip M. Walker\textsuperscript{1}

\textsuperscript{1} Department of Physics, University of Surrey, Guildford, Surrey, GU2 7XH, United Kingdom.
\textsuperscript{2} Department of Applied Physics, Fukui University, 3-9-1 Bunkyo, Fukui 910-8507, Japan.

(March 30, 2022)

We report a solution of the tilted-axis cranked HFB equation for \textsuperscript{164}Hf, which shows wobbling motion coupled to gamma vibration at high spin \((J \simeq 60\hbar)\). Possible anharmonicity and splitting of energy levels are also discussed as a consequence of the wobbling motion with large amplitude.

PACS number(s): 27.70.+q, 21.10.-k

In understanding the excited states of a quantum many-body system with finite degrees of freedom, the concept of dynamic collective modes has been playing a leading role. For example, in nuclear physics, shape vibrations such as \(\gamma\) and \(\beta\) vibrations \([1]\), octupole vibrations in super-deformed states \([2]\), and giant dipole resonances \([3]\) are well known. In a dilute atomic gas in a Bose-Einstein condensate, which is also a finite system, the dynamic motion displays the scissors mode, which is an out-of-phase vibration among states in the super and normal fluidity \([4]\).

As well as these vibrations, rotations have been also considered and studied in detail. As a result, rotational bands are observed in nuclei, as in atomic and molecular systems \([5,1]\). The relevant rotational forms are essentially uniform and one-dimensional, and they are sometimes called “static rotation” because the rotation axis is fixed.

Advances in experimental techniques these days allow us to obtain data for excited levels built on very high spin states. For understanding these levels, a concept of “dynamic” nuclear rotation is introduced and recently has attracted much interest. Indeed, the concept is helpful to comprehend several novel phenomena, such as nuclear wobbling motions in triaxial deformed nuclei \([4]\), dynamical coupling modes (wobbling modes) between high- and low-\(K\) bands \([5]\), and chiral vibrations through quantum tunnelling between right- and left-handed chiral rotating states \([6]\). These phenomena have recently been studied from both experimental and theoretical approaches.

Dynamic rotation is usually considered in terms of the classical mechanics as time-dependent and three-dimensional rotation, such as the precession in the rotation of the earth \([1]\), or motions of a tippe top in which the direction of the rotation axis varies time-dependently in the intrinsic frame \([10]\). The discussion of these rotations in nuclear physics was first presented by Bohr and Mottelson in the context of wobbling motion \([1]\). They considered the motion as an analogy of the small orientation fluctuations of a triaxially deformed classical rigid rotor from its static and one-dimensional rotation about the principal axis of the inertia tensor. They quantize the Hamiltonian of the rotor under the condition \(|I_1| \simeq |I| \gg 1\), by using the technique similar to the second quantization of the harmonic oscillator. \((I\) is the total angular momentum and \(I_1\) is the \(x\) component of the angular momentum vector.) It is then shown that the excited levels corresponding to the nuclear wobbling motion have a vibrational character on the rotational band, \(E(I, n_w) = \frac{I(I+1)}{2J} + \hbar \omega_w (n_w + 1/2)\), where \(\hbar \omega_w\) is calculated by the moments of inertia of the rotor and \(J_1\) is the moment of inertia around the \(x\)-axis. The oscillation quanta \(n_w\) take the values \(n_w = 0, 1, 2, \ldots\). In spite of their theoretical prediction, it had been difficult to identify the wobbling states in experiments until the experimental evidence recently reported in \textsuperscript{164}Lu with triaxial super-deformation \([1]\). With this discovery, there has been established a new research field in high-spin nuclear physics.

Nuclear rotation is closely related to the shape of a nucleus or its symmetries \([4]\), so that exotic rotations can be seen for exotic shapes. Nuclei possessing axial symmetry follow the static one-dimensional rotation, such as in the ground-state rotational band, or \(\gamma\)-band. These rotations are low-lying collective modes which are caused by the spontaneous symmetry breaking mechanism for the rotational symmetry, that is, the SU(2) symmetry is broken into the subgroup U(1) through the nuclear deformation. As the U(1) symmetry is further broken into its discrete sub-symmetries, more variations can be expected for the nuclear rotation. For instance, dynamic and three- (or two-) dimensional rotations are predicted to emerge in nuclei with triaxial symmetry (a discrete symmetry for a \(\pi\) rotation around any axis, denoted as \(D_2\)).

One theoretical method used for the analysis of the wobbling motion is based on the particle-rotor model (PRM) \([2,12,13]\). Although the model allows quantum mechanical studies, the assumption of a “rotor” brings a macroscopic treatment into the analysis. The microscopic formulations for the dynamic rotations were derived in the framework of the random phase approximation (RPA) \([14]\), and the time-dependent variational method \([13,16]\). Based on these

\(*\text{m.oi@surrey.ac.uk}
and all subsequent works \cite{17,18,21}, we are now able to study the dynamic nuclear rotations from a microscopic point of view. The combination of the tilted-axis cranking model \cite{17,19,21} and the generator coordinate method (GCM) \cite{22,23} is one of the methods which are very powerful and useful in numerical studies.

Despite the fact that the experimental discovery of the wobbling mode was made in an odd-mass nucleus (\(^{163}\)Lu) and the theoretical investigation that the wobbling motion is favoured in odd-A mass nuclei \cite{23}, it is essentially the “rotor” (in the PRM) that wobbles. Hence, it is necessary to study the wobbling motion in even-even nuclei (i.e., nuclei consisting of the even numbers of protons and neutrons, where all the nucleons are coupled in the ground state through the BCS-type pairing interaction). According to Ref. \cite{11}, it is calculated that the yrast state (the lowest energy state for a given spin) in \(^{164}\)Hf becomes triaxial super-deformed (TSD) at \(I \geq 40\hbar\), having gamma deformation \(\gamma \approx -20^\circ\). (Our convention for the gamma deformation parameter, \(\gamma\), is defined as in p.7 of Ref. \cite{3}.) Thus, \(^{164}\)Hf is expected to show the wobbling mode as a result of its triaxially deformed states at high spin.

In this paper, we thus investigate the nuclear wobbling motion for \(^{164}\)Hf, by means of self-consistent calculations based on the tilted-axis cranking model. However, before going forward to the microscopic analysis, it is worth reviewing rotation of a classical rigid body because the analogy with classical mechanics is quite helpful to understand nuclear wobbling motions.

Rotations of a classical rigid body can be studied through the corresponding angular momentum vector \cite{9,12,23}. If the intrinsic coordinates are chosen so as to diagonalize the inertia tensor, cross sections between a sphere, \(M^2 = M_1^2 + M_2^2 + M_3^2\) (angular momentum conservation), and an ellipsoid, \(E = \frac{M_1^2}{2J_1^2} + \frac{M_2^2}{2J_2^2} + \frac{M_3^2}{2J_3^2}\) (energy conservation), give the trajectory of the vector on the sphere. (\(M_i\) denotes the \(i\)-th component of the “classical” angular momentum.) Fig.\(\ref{fig:4}(a)\) shows examples of such trajectories. When the rigid body has axial symmetry (say, along the \(z\)-axis), it is well known that the trajectories follow “precession”, which means that motions of the vector are circular around the \(z\)-axis and in the plane parallel to the \(x-y\) plane. However, in the case of a triaxial rigid body (\(J_1 \neq J_2 \neq J_3\)), the trajectories deviate from the precession in two ways: (i) distortion from the circle and (ii) deviation from two-dimensional to three-dimensional movement. These deviations imply that additional vibrations are induced by breaking the axial symmetry of the rigid body. These vibrational motions in rotations are called “wobbling motions” \cite{23}. It should be noted that these deviations from the precession are small when the angular momentum is polarized near the axes (\(x\)- or \(z\)-axes in Fig.\ref{fig:4}).

This type of small fluctuation corresponds to the nuclear wobbling motion in Ref. \cite{1} if the total angular momentum \(|M|\) is large enough \(||M|| \simeq |M_1| >> 1\). Dashed lines in Fig.\ref{fig:4}(a) correspond to special trajectories called “separatrices”, which divide the area into four topologically different domains. The existence of such domains is evidence of non-linearity in the system. In fact, in Fig.\ref{fig:4}(a), there are two separatrices separated at points \(P\) (\(M_1, M_2, M_3\) = \((0, \pm 1, 0)\), and they do not cross each other, as shown in Fig.\ref{fig:4}(b). This subset figure shows the time-dependence of the angular momentum vector for the separatrix in the domain \(z > 0\), which is obtained by solving the Euler equations \((\frac{d\omega}{dt} = \frac{\partial}{\partial J_i} \omega^i \omega^k, )\) with proper initial conditions. (Indices \(i, j, k\) should be taken in a cyclic manner.) Most of the time, the vector stays at \(P\) and the motion between the two points is comparatively fast once the vector is apart from these points. The other separatrix has similar motion except the sign of \(M_3\).

Now, let us begin the microscopic analysis by means of the tilted-axis cranked HFB method. Our Hamiltonian consists of two terms: a spherical part (the spherical Nilsson Hamiltonian) and a residual part (pairing-plus-Q-Q force). The model space and force parameters are chosen in the same manner as in our previous work \cite{21}. The HFB equation is solved by means of the method of steepest descent, under constraints on the angular momentum and the particle number. Particularly, the angular momentum constraints are expressed as, \(\langle J_1 \rangle = J \cos \varphi, \langle J_2 \rangle = 0, \langle J_3 \rangle = J \sin \varphi\), where \(\varphi\) denotes the tilt angle. For details about the method, see Ref. \cite{17}. An advantage of the present method is that the physical quantities like the deformation parameters are calculated self-consistently. It is thus possible to describe the deformations as functions of the tilt angle, for instance, \(\gamma = \gamma(\varphi)\).

First of all, let us see briefly the results obtained by the principal-axis (one-dimensional) cranking calculations (\(\varphi = 0^\circ\)). In the present calculations, there are no super-deformed solutions appearing in the yrast band throughout the range \(0 \leq J \leq 80\hbar\). This can be explained by that fact that our model space is not large enough to reproduce super-deformation. Nevertheless, at \(J \approx 60\hbar\), the yrast states take on gamma deformation (\(\gamma \approx -20^\circ, \beta \approx 0.13\)). For the purpose of studying the wobbling motion theoretically, emergence of triaxility is enough. The gamma deformed states appear after of the gap energy vanishes at \(J \geq 50\hbar\) (20h) for protons (neutrons). In the present calculations, major contributions to the gamma deformation are alignments of \(i_{13/2}\) and \(h_{11/2}\) protons in addition to neutron alignments at lower spins (\(J \approx 20\hbar\); \(i_{13/2}, f_{7/2}\), and \(h_{9/2}\).

Next, let us see the results in the tilted-axis (two-dimensional) cranking calculations. The tilted-axis cranked HFB states are created from the principal-axis cranked HFB states (\(\varphi = 0^\circ\)) at \(J = 60\hbar\) as the initial states. The range of the gamma deformation is self-consistently determined and given in the range \(180^\circ \leq \gamma \leq 300^\circ\). Due to the symmetry in the definition of \(\gamma\), the range may be converted into \(-60^\circ \leq \gamma \leq 60^\circ\). The conversion is equivalent to renaming...
the axes in a cyclic manner, for instance, “x-axis → y-axis”. As a result, the wobbling motion in this study is in the x-y plane. In other words, it can be said that the tilt angle in the present work corresponds to the azimuthal angle \( \phi \) while in the previous works \[ \text{[1][2][3]} \] it corresponds to the polar angle \( \theta \), if the z-axis is chosen for the direction of elongation of the nucleus.

Figs.2(a) and (b) show, respectively, changes of gamma deformation, \( \gamma(\varphi) \), and energy, \( E(\varphi) \). From (a), coupling of the tilting and gamma degrees of freedom is clearly seen. It should be noted that the curvatures of \( \gamma(\varphi) \) are flat around \( \varphi \approx 0^\circ \) and \( \pm 90^\circ \). This result implies that the assumption in the PRM studies (fixing the gamma values) \[ \text{[1][4]} \] is reasonable as long as the wobbling amplitude is small. For large wobbling amplitude, it seems that the \( \gamma \) vibration is cooperatively induced by the fluctuation of the orientation of the rotation axis, that is, the wobbling motion. This situation is schematically depicted in Fig.2(c). \( \text{(The } \beta \text{ values are almost constant, } 0.130 \lesssim \beta \lesssim 0.135, \text{ for the entire range of } \varphi. \) The nuclear shape becomes prolate when the tilt angle reaches \( \varphi = \pm 45^\circ \), which correspond to unstable extrema in Fig.2(b). At these points, the \( \varphi \) range is divided into several domains from a dynamical point of view. Let us call the domains for \(-45^\circ \leq \varphi < 45^\circ \) domain-I, in which the wobbling motion is around the y-axis, while the other for \(45^\circ < \varphi \leq 135^\circ \) domain-II, in which the wobbling is around the x-axis. Domain-I and domain-II are related by a discrete symmetry operation called \( C_{4z} \) \( \text{a symmetry for } \frac{\pi}{2} \text{-rotation about the } z \text{-axis), due to the dynamical coupling between } \varphi \text{ and } \gamma. \) The wobbling motions are topologically classified through these domains. Therefore, it can be said that the rotations at \( \varphi = \pm 45^\circ, \pm 135^\circ \) correspond to “separatrices” in classical mechanics. As seen in Fig.2(b), the energy curve looks like a harmonic potential near \( \varphi \approx 0^\circ \) and \( \approx 90^\circ \). From the present calculation, it is obtained as \( V(\varphi) \approx a \varphi^2 \) where \( a = 1.64 \) (MeV). As \( \varphi \) approaches the value for the separatrix, the curve deviates from the harmonic shape. Similar features are seen in the wobbling motion of a classical rigid body with triaxiality, that is, the deviation from planar motion becomes larger as the trajectory reaches a separatrix.

Let us discuss the possible manifestation of the excited structures of the wobbling motion. In Ref. [1], the wobbling excitations are given as a harmonic structure like \( \hbar \omega_w (n_w + 1/2) \). Although the harmonic levels are expected to be seen for the wobbling motion with small wobbling amplitude, i.e., small \( n_w \), anharmonicity can be expected for wobbling motions with large amplitude, i.e., large \( n_w \), according to the shape of the energy curve. To see the excited structure schematically, we solve the one-dimensional Schrödinger equation for a wobbling phonon, i.e., the quantized wobbling motion. It is assumed that the tilt angle can be treated as a dynamical variable, and that \( w \) excitations are given as a harmonic structure like \( \hbar \omega_w \). The wobbling motions are topologically classified through these domains. Therefore, it is obtained as \( V(\varphi) \approx a \varphi^2 \) where \( a = 1.64 \) (MeV). As \( \varphi \) approaches the value for the separatrix, the curve deviates from the harmonic shape. Similar features are seen in the wobbling motion of a classical rigid body with triaxiality, that is, the deviation from planar motion becomes larger as the trajectory reaches a separatrix.

Unlike the classical wobbling motion of a triaxial rigid body, tunnelling effects should be taken into account in the nuclear wobbling motion. In other words, each wobbling mode classified by domains can jump from one domain to another over the separatrices. As a result of the coupling between \( \varphi \) and \( \gamma \) degrees of freedom, \( E(\varphi) \) has discrete \( C_{4z} \) symmetry, implying that a wobbling phonon moves in the four potential wells. It is well known that the double well potentials cause the splitting in energy levels due to the parity symmetry in the corresponding Hamiltonian \[ \text{[20]}. \] The states are then expressed by symmetric and antisymmetric linear combinations of the localized states in each potential. In the present case, there would be four degenerate states localized in each potential, if the barrier height were infinitely high. Let us denote these localized states as \( |L_1 \rangle, |L_2 \rangle, |R_2 \rangle, \) and \( |R_1 \rangle \), for the further left potential to the further right. For the parity operation \( \hat{\pi} \), “\( R \)” and “\( L \)” are exchanged. For instance, \( \hat{\pi} |L_1 \rangle = |R_1 \rangle \). In the case of finite barrier height, these four localized states are superposed through the tunnelling to produce two symmetric and two antisymmetric linear combinations. These states might form a quartet, and be written as,

\[
|S1 \rangle = \frac{1}{2} (|L_1 \rangle + |L_2 \rangle + |R_2 \rangle + |R_1 \rangle)
\]

\[
|S2 \rangle = \frac{1}{2} (|L_1 \rangle - |L_2 \rangle - |R_2 \rangle + |R_1 \rangle)
\]

\[
|A1 \rangle = \frac{1}{2} (|L_1 \rangle + |L_2 \rangle - |R_2 \rangle - |R_1 \rangle)
\]

3
|A2⟩ = \frac{1}{2} (|L1⟩ − |L2⟩ + |R2⟩ − |R1⟩) \quad (4)

“S” and “A” denote symmetric and antisymmetric states. If each localized state is in the ground state, the nodal numbers for the superposed states are 0,1,2, and 3 for “S1”, “A1”, “S2” and “A2”, respectively. The order of the corresponding energy levels is thus expected to be \( E_{S1} < E_{A1} < E_{S2} < E_{A2} \).

In summary, we have investigated the possible wobbling motion in \(^{164}\text{Hf}\) by means of the tilted-axis cranked HFB states. The calculations show that the nucleus has sizeable gamma deformation at high spin (\( J \approx 60\hbar \)), and wobbling motion coupled to the gamma vibration can be expected. A possible anharmonicity in the wobbling excitations is predicted for higher wobbling excitations, based on the calculated energy curve with respect to the tilt angle. In addition, a brief discussion is presented to show that a “wobbling quartet” can be expected for the nuclear wobbling motion coupled to the gamma vibration at high spin, as a consequence of the tunnelling effect.

M.O. thanks to Drs. I. Hamamoto, N. Tajima, T. Døssing, and P. Ring for fruitful discussions and hospitalities during his stay in Lund University, Fukui University, Niels Bohr Institute, and Technische Universität München. Discussions with Drs. R. Bengtsson, G.B. Hagemann, A. Hayashi, A. Ansari, and R. Hilton are also appreciated. He would like to acknowledge financial support from the Japan Society for the Promotion of Sciences (JSPS), and technical assistance from Center for Nuclear Sciences (CNS), University of Tokyo, for parts of numerical calculations in this work.

[1] A.Bohr and B.Mottelson, *Nuclear Structure Vol.II* (Benjamin, Massachusetts, 1975).
[2] T.Nakatsukasa, S.Mizutori, and K.Matsuyanagi, Prog. Theor. Phys. **89**, 847 (1993).
[3] P.Ring and P.Schuck, *Nuclear Many-body Problem* (Springer Verlarg, Berlin, 1980).
[4] D.Guéry-Odelin and S.Stringari, Phys. Rev. Lett. **83**, 4452 (1999); O.M. Marragó, et al., Phys. Rev. Lett. **84**, 2056 (2000).
[5] J. Rainwater, Phys. Rev. **79**, 432 (1950).
[6] S. Ørdegård, et al., Phys. Rev. Lett. **86**, 5866 (2001).
[7] M.Oi, A.Ansari, T.Horibata, and N.Onishi, Phys. Lett. **B480**, 53 (2000).
[8] K. Starosta, et al., Phys. Rev. Lett. **86**, 971 (2001).
[9] H.Goldstein, *Classical Mechanics*, p.212, (Addison-Wesley, 1980).
[10] A.R. Del Campo, Am. J. Phys. **23**, 544 (1955); R.Cohen, Am. J. Phys. 45, 12 (1977).
[11] R.Bengtsson, www.matfys.lth.se/∼ragnar/TSD.html
[12] I. Hamamoto and B. Mottelson, Phys. Lett. **127 B**, 281 (1983).
[13] I. Hamamoto, Phys. Lett. **B193**, 399 (1987).
[14] D.Janssen and I.N.Mikhailov, Nucl. Phys. **A318**, 390 (1977).
[15] E.R.Marhaelk, Nucl. Phys. **A331**, 429 (1979).
[16] A.K.Kerman and N.Onishi, Nucl. Phys. **A361**, 179 (1981).
[17] T.Horibata, M. Oi, and N.Onishi, Phys. Lett. **B355**, 433 (1995).
[18] F.Dönau, Jing-ye Zhang, L.L. Riedinger, Phys. Lett. **B450**, 313 (1999).
[19] S. Frauendorf, Nucl. Phys. **A677**, 115 (2000); Nucl. Phys. **A557**, 259c, (1993).
[20] R. Bengtsson, Nucl. Phys. **A557**, 277c (1993).
[21] H.Madokoro, J.Meng, M.Matsuzaki, and S.Yamaji, Phys. Rev. C62,061301(R) (2000).
[22] H.Flocard and D.Vautherin, Phys. Lett. 55 B (1975).
[23] N. Onishi, *Butsurigaku Saizensen* Vol.4, p.130, (Kyoritsu Shuppan, Tokyo, 1984).
[24] M. Oi, P.M. Walker, and A.Ansari, Phys. Lett. B. 505, 75 (2001).
[25] P. Möller, J. R. Nix, W. D. Myers, and W. J. Swiatecki, Atomic Data Nucl. Data Tables **59**, 185 (1995).
[26] J.J.Sakurai, *Modern Quantum Mechanics* (revised edition), p.256, (Addison-Wesley, 1994).
FIG. 1. (a): Trajectories of the angular momentum vector of a classical rigid body (with a quadrupole shape; \( \beta, \gamma = (0.4, -20^\circ) \)). Separatrices are denoted by chains. (b): Time dependence of angular momentum components on the separatrix in the region \( z > 0 \) (written as “Separatrix A (z > 0)” in the figure).
FIG. 2. (a) Gamma deformation with respect to tilt angle, $\gamma(\varphi)$; (b) Energy curve with respect to tilt angle, $E(\varphi)$; and (c) a schematic picture of wobbling motion coupled with gamma vibration. At $\varphi = \pm 45^\circ$ the nucleus has the axial symmetry, and the state corresponds to the separatrix in classical mechanics.
FIG. 3. Excited levels of the wobbling phonon. Left: the solid curve is the energy (corresponding to Domain-I) calculated by the tilted-axis cranked HFB method for $J = 60\hbar$, while the dashed line is the harmonic approximation of the above line. The range for the tilt angle corresponds to $|\varphi| \leq 45^\circ$. Horizontal lines denote the quantized levels for the wobbling motion. Right: A simple line indicates $\hbar \omega_w (n_w + 1/2)$, while the asterisks connected by lines show the quantized wobbling energies calculated in this study.