Abstract

Under mild conditions, it is possible to obtain, from almost purely measure-theoretic considerations and without any specific reference to stochastic processes, a change-of-measures result, resembling the usual Radon-Nikodým change of measures, associated with a variant of stochastic integration for a spectral representation of covariance stationary processes; the ideas are naturally embedded in the Hilbert space theory of $L^2$ spaces. The intended main contribution, including a complete proof of change of measures for spectral stochastic integrals, is the refined, self-contained developments of spectral stochastic integration toward change of measures.

Keywords: change of measures; orthogonal stochastic measures; spectral representation for covariance stationary processes; stochastic integration

MSC 2020: 60A10; 60H05; 60G10; 37M10

1 Introduction

For and only for our motivating account, a stochastic process is called a covariance stationary process if and only if the process is a random element of $\mathbb{C}^\mathbb{Z}$ with components i) having finite second moment, i.e. being $L^2$, ii) having a constant mean, and iii) having the property that the covariance of any pair of the components depends at most on the difference of their indexes. A well-known (classical) version of Herglotz representation theorem (e.g. Section 1, Chapter 6, Shiryaev [2]) asserts that, for every centered covariance stationary process, there is some complex measure on the Borel sigma-algebra $\mathcal{B}_{[-\pi,\pi]}$ of $\mathbb{R}$ relativized to $[-\pi,\pi]$, whose total variation measure is concentrated on $[-\pi,\pi]$, such that for every $n \in \mathbb{Z}$ the covariance function of the process at $n$ equals the integral of the function $\lambda \mapsto e^{i\lambda n}$ over $[-\pi,\pi]$ with respect to the complex measure.

The Herglotz spectral representation result suggests, in a mathematically natural way, if, for every $n \in \mathbb{Z}$, one can express the $n$th component, rather than the covariance function, of a centered covariance stationary process in terms of (modulo some underlying probability measure) an integral of the function $\lambda \mapsto e^{i\lambda n}$ over $[-\pi,\pi]$ in a suitable sense. To this end, special attention would be required as i) such an integral has to be “stochastic” and as ii) it is not clear that a naive pointwise definition could always circumvent the possibilities of encountering functions that are not of bounded variation. These well-known potential difficulties, together with the desire to obtain a spectral representation result for the components of a covariance stationary process parallel to the aforementioned Herglotz representation for...
the covariance function of the process, thus logically motivate the developments of a type of stochastic integration, which we refer to as *spectral stochastic integration*. The modifier “spectral” signifies the purpose-specific aspect of that kind of stochastic integration concerning us.

To the best of the author’s knowledge, the corresponding theory to spectral stochastic integration is *de facto* scattered in the related literature with relatively incomplete or cursory characterizations. One of the most complete treatments of spectral stochastic integration without loss of mathematical rigor known to the author would be Shiryaev [2], which leverages the fact that every $L^2$ space of $\mathbb{C}$-valued functions is a Hilbert space without bringing in too many context-unnecessary concepts and results, and furnishes an outline of the framework. The other would be Gikhman and Skorokhod [1]. Although a change-of-measures result for spectral stochastic integration is given therein (with only a brief proof), their definition for spectral stochastic integration depends on several deeper results in analysis, and hence might unintentionally obscure the simple nature of spectral stochastic integration. Moreover, in some directions (with other directions fixed) their definition is narrower than Shiryaev [2].

Based on what is outlined in Shiryaev [2], the present paper intends to complete a corner of the theory of spectral stochastic integration by redeveloping a systematic, unified, and non-redundant treatment to proving a natural change-of-measures result for spectral stochastic integrals that resembles the usual Radon-Nikodým version. Our proof for change of measures, intended as a complete one, is different than the proof idea sketched in the aforementioned Gikhman and Skorokhod [1]. Although a working knowledge in stochastic processes would be helpful in appreciating the theory of spectral stochastic integration, we motivate the building concepts of such integration and arrange the developments so that literally no working knowledge in stochastic processes is demanded. As a whole, these novel treatments of “known” ideas are also intended both as a compact, citable reference and as a contribution inclined to the pedagogical side.

## 2 Result

### 2.1 Preliminary Developments

To assign a suitable sense to that kind of stochastic integration serving the purpose of “spectrally” representing covariance stationary processes, i.e. to our spectral stochastic integration, it surprisingly suffices to employ a few natural requirements. Let $\Omega$ be a probability space with probability measure $P$; let $S \subset \Omega$ be nonempty; let $\mathcal{A}$ be an algebra of subsets of $S$. By an *orthogonal elementary stochastic measure*, which is the building block for our spectral stochastic integral, is meant a family $M \equiv \{ M(A) \}_{A \in \mathcal{A}}$ of complex random variables $\in L^2(P)$ on $\Omega$ such that i) $M(\emptyset) = 0$ a.s.-$P$, ii) $M(A_1 \cup A_2) = M(A_1) + M(A_2)$ a.s.-$P$ and $EM(A_1)\overline{M(A_2)} = 0$ for all disjoint $A_1, A_2 \in \mathcal{A}$, and iii) $A_1, A_2, \cdots \in \mathcal{A}$ being disjoint and $\cup_{n \in \mathbb{N}} A_n \in \mathcal{A}$ imply $E[|M(\cup_{n} A_n) - \sum_{j=1}^{n} M(A_j)|]^2 \to 0$. Since $(X, Y) \mapsto \int_{\Omega} X Y dP = EXY$ is an inner product for $L^2(P)$, the defining properties of an orthogonal elementary stochastic measure explain the terminology. Since every component of $M$ is by definition $L^2$, we can define the function $m : A \mapsto E[M(A)]^2$ on $\mathcal{A}$. To justify the existence of an orthogonal elementary stochastic measure,$^4$

---

$^4$Exploring the connections between spectral stochastic integration and other existing notions of stochastic integration is outside the scope of the present paper. Spectral stochastic integration might be likened to Paley-Wiener-Zygmund integration with a careful distinction, the latter of which is also concerned with assigning a suitable sense to integrating a “deterministic” function with respect to a stochastic object.
we give a stronger result with a handy construction without any reference to the theory of stochastic processes:

**Proposition 1.** For every probability space \((\Omega, \mathcal{F}, \mathbb{P})\) there is some orthogonal elementary stochastic measure on \(\Omega\).

**Proof.** Since \(\mathcal{F}\) is also an algebra, we consider the family \((\mathbf{1}_A)_{A \in \mathcal{F}}\) of the indicators of sets of \(\mathcal{F}\). If \(M(A) \equiv \mathbb{E}(A) := \mathbf{1}_A\) on \(\Omega\) for all \(A \in \mathcal{F}\), it is readily checked from the usual properties of indicator functions that \(M\) is an orthogonal elementary stochastic measure on \(\Omega\).

Indeed, as a side remark, the probability measure \(\mathbb{P}\) plays the role of \(m\) above.

Now the definition of an orthogonal elementary stochastic measure ensures the existence of the Carathéodory extension \(M\) for \(\mathcal{F}\) to \(\sigma(\mathcal{A})\). This observation enables us to employ the completeness of \(L^2(S, \sigma(\mathcal{A}), M)\) to define our spectral stochastic integrals. The function \(M\) will be referred to as the \textit{structural function} \(^2\) for \(\mathbb{M}\); and, later on, a triple of the form \((\mathbb{M}, M, \mathcal{A})\) declares \(\mathbb{M}\) to be an orthogonal elementary stochastic measure with \(M\) being its structural function defined on the sigma-algebra generated by a given algebra \(\mathcal{A}\) of sets.

Given any \(\mathcal{A}\)-simple function \(f : S \to \mathbb{C}\) of the form \(f = \sum_{j=1}^n a_j \mathbf{1}_{A_j}\) where \(a_1, \ldots, a_n \in \mathbb{C}\) are distinct and where \(A_1, \ldots, A_n \in \mathcal{A}\), we define

\[
\int_S f \, d\mathbb{M} := \sum_{j=1}^n a_j M(A_j),
\]

and refer to \(\int_S f \, d\mathbb{M}\) as the spectral stochastic integral of \(f\) with respect to \(\mathbb{M}\). The convention of omitting the domain of integration applies here, and we remark that \(\int f \, d\mathbb{M} \in L^2(\mathbb{P})\) for all \(\mathcal{A}\)-simple \(f: S \to \mathbb{C}\) by the very definition of \(\mathbb{M}\). The finite additivity and the orthogonality of \(\mathbb{M}\) imply that \(\mathbb{E}[M(A \cap A')^2] = \mathbb{E}[M(A)\overline{M}(A')]\) for all \(A, A' \in \mathcal{A}\), which follows from a consideration over the partition \(A = (A \cap A') \cup (A \setminus A')\) for \(A\) and the same partition for \(A'\); with more notation it then holds that

\[
\mathbb{E}\left( \int f \, d\mathbb{M} \right) = \int f \overline{d\mathbb{M}}
\]

for all \(\mathcal{A}\)-simple \(f, g: S \to \mathbb{C}\). Thus the space of all \(\mathcal{A}\)-simple functions \(f : S \to \mathbb{C}\) in \(L^2(M)\) is inner-product homomorphic to the space of their spectral stochastic integrals with respect to \(\mathbb{M}\).

To extend the definition of our spectral stochastic integration for arbitrary elements of \(L^2(M)\), we claim that \(\mathcal{A}\)-simple functions \(S \to \mathbb{C}\) are \(L^2\)-dense in \(L^2(M)\). This is not immediate as the “measurability” of the approximating sequence of simple functions is now restricted to the algebra \(\mathcal{A}\); the restriction is reasonable as we have thus far defined our spectral stochastic integration for and only for \(\mathcal{A}\)-simple functions. Fortunately, the do-ability is not so covert; we begin by showing that for every \(B \in \sigma(\mathcal{A})\) and every \(\varepsilon > 0\) there is some \(A \in \mathcal{A}\) such that \(M(A \Delta B) < \varepsilon\), which is a proposition usually left as an exercise in textbooks. To see this, let \(\mathcal{G}\) be the collection of all such \(B \in \sigma(\mathcal{A})\). It is immediate that \(\mathcal{A} \subset \mathcal{G}\) and that \(\emptyset \in \mathcal{G}\). If \(B \in \mathcal{G}\), then the identity \(A \Delta B = A^c \Delta B^c\) implies that \(B^c \in \mathcal{G}\). If \(B_1, B_2 \in \mathcal{G}\), then, since \((A_1 \cup A_2) \Delta (B_1 \cup B_2) \subset (A_1 \Delta B_1) \cup (A_2 \Delta B_2)\) for all \(A_1, A_2 \subset S\), choosing respectively the suitable approximating \(A_1, A_2 \in \mathcal{A}\) for \(B_1, B_2\) ensures that

\[\text{This terminology would be in a sense self-evident once we recall that the spectral measure present in the Herglotz representation for the covariance function of a covariance stationary process happens to play the role of the structural function for an orthogonal stochastic measure employed to represent the components of the covariance stationary process.}\]
$B_1 \cup B_2 \in \mathcal{G}$. If $B_1, B_2, \cdots \in \mathcal{G}$, then, as $M$ is a finite measure by the definition of $m$, there is some $N \in \mathbb{N}$ such that $M(\cup_{n \geq N+1} B_n)$ is as small as desired; since $\cup_{n=1}^N B_n \in \mathcal{G}$, it follows from the inclusion $\cup_{n \in \mathbb{N}} B_n \setminus \cup_{n=1}^N A_n \subset (\cup_{n=1}^N B_n \setminus \cup_{n=1}^N A_n) \cup (\cup_{n \geq N+1} B_n)$ that $\cup_{n \in \mathbb{N}} B_n \in \mathcal{G}$. But then $\mathcal{G}$ is a sigma-algebra, and hence $\sigma(\mathcal{A}) \subset \mathcal{G}$. From the identity $\int (1_A - 1_B)^2 \, dM = M(A \Delta B)$, it holds that the space of all $\mathcal{A}$-simple functions $S \to \mathbb{C}$ is $L^2$-dense in that of all simple measurable functions $S \to \mathbb{C}$. On the other hand, if $f \in L^2(M)$, and if $| \cdot |_{L^2(M)}$ denotes the $L^2$-norm of the space $L^2(M)$, then the $L^2$-denseness of simple measurable functions in $L^2(M)$ and the (formal) triangle inequality

$$|f - \psi|_{L^2(M)} \leq |f - \varphi|_{L^2(M)} + |\varphi - \psi|_{L^2(M)}$$

(for $\psi$ being $\mathcal{A}$-simple and for $\varphi$ being simple measurable) together imply the claim.

If we return to complete the definition of our spectral stochastic integration, consider an arbitrary $f \in L^2(M)$. Since there are some $\mathcal{A}$-simple functions $f_1, f_2, \cdots : S \to \mathbb{C}$ such that $|f_n - f|_{L^2(M)} \to 0$, the sequence $(f_n)_{n \in \mathbb{N}}$ is $L^2$-Cauchy. But, if $| \cdot |_{L^2(P)}$ denote the $L^2$-norm of the space $L^2(P)$, then the fact that

$$\left| \int f_n \, dM - \int f_m \, dM \right|_{L^2(P)} = |f_n - f_m|_{L^2(M)}$$

for all $n, m \in \mathbb{N}$ and the completeness of $L^2(P)$ jointly imply that the sequence $(\int f_n \, dM)_{n \in \mathbb{N}}$ converges in the space $L^2(P)$ in the corresponding $L^2$ sense. The spectral stochastic integral of $f$, denoted $\int f \, dM$, is then defined as the $L^2$-limit of the sequence $(\int f_n \, dM)$. Indeed, intuitively, the spectral stochastic integral of $f$ is invariant in the choice of $(f_n)$ as the principal ingredient of the above construction is still a simple measurable $L^2$-approximating sequence $(\varphi_n)$ for $f$. We have completed the definition, based on Shiryaev [2], of spectral stochastic integration with respect to an orthogonal elementary stochastic measure for elements of the space of $\mathbb{C}$-valued functions that are square-integrable with respect to the structural function for the orthogonal elementary stochastic measure.

### 2.2 Change of Measures

In this particular paragraph, we use the same notation as in the previous subsection. If $g \in L^2(M)$, then $1_B g \in L^2(M)$ for all $B \in \sigma(\mathcal{A})$, and our definition of a spectral stochastic integral implies that

$$\int 1_B g \, dM \in L^2(P)$$

for all $B \in \sigma(\mathcal{A})$.

Now we prove the change-of-measures result for our spectral stochastic integration:

**Theorem 1.** Let there be given a probability space with probability measure $\mathbb{P}$; let $\mathcal{A}$ be an algebra of subsets of a given subset of the probability space; let $(M_1, M_2, \mathcal{A})$ be an orthogonal elementary stochastic measure on the probability space; let $g \in L^2(M_1)$. If $M_2(B) := \int 1_B g \, dM_1$ for all $B \in \mathcal{A}$, and if $m_2 : B \mapsto E[M_2(B)]^2$ on $\mathcal{A}$, then i) the family $M_2 := (M_2(B))_{B \in \sigma(\mathcal{A})}$ is an orthogonal elementary stochastic measure on the given probability space with the Carathéodory extension $M_2$ of $m_2$ to $\sigma(\mathcal{A})$ being its structural function; and ii)

$$\int f \, dM_2 = \int fg \, dM_1 \quad a.s.-\mathbb{P}$$

for all $f \in L^2(M_2)$, where the $\mathbb{P}$-null set of the points at which the equality possibly fails may depend on the choice of $f$. 

---

4
Proof. Indeed, as readily seen from our definition of spectral stochastic integration, a spectral stochastic integration operator acting on the vector space $L^2$ with respect to the structural function enjoys linearity and preserves inner product.

To prove i), we first observe that $M_2(\varnothing) = 0$. The linearity of a spectral stochastic integration operator implies $M_2(A_1 \cup A_2) = \int 1_{A_1}g \, dM_1 + \int 1_{A_2}g \, dM_1 = M_2(A_1) + M_2(A_2)$ for all disjoint $A_1, A_2 \in \mathcal{A}$. The orthogonality of $M_2$ follows from the trivial equality $1_{A_1 \cap A_2}(g)^2 = 0$ for all disjoint $A_1, A_2 \in \mathcal{A}$, and from the inner-product-preserving property of a spectral stochastic integration operator. To see the countable additivity in the $L^2(\mathbb{P})$ sense only takes the following (formal) relations

$$M_2(\bigcup_{n=1}^N A_n) = \sum_{j=1}^N M_2(A_j) = \int 1_{\bigcup_{n=1}^N A_n}g \, dM_1 = \int \sum_{j=1}^N 1_{A_j}g \, dM_1;$$

$$\left| \int 1_{\bigcup_{n=1}^N A_n \setminus \bigcup_{j=1}^N A_j}g \, dM_1 \right|_{L^2(\mathbb{P})}^2 = \int 1_{\bigcup_{n=1}^N A_n \setminus \bigcup_{j=1}^N A_j}\left|g\right|^2 \, dM_1 \rightarrow 0,$$

where we have acknowledged the linearity and the norm-preserving properties of a spectral stochastic integration operator and the monotone convergence theorem. By the argument present in the beginning of the present subsection, it follows that $M_2$ is an orthogonal elementary stochastic measure on the given probability space. Now the definition of $m_2$ and a Carathéodory extension applied to $m_2$ together complete the proof of i).

To finish the whole proof, we bring in the fact that our spectral stochastic integration operators enjoy, from the linearity and the norm-preserving property, a “continuity” property in the sense that a sequence of suitable integrands converging in the suitable $L^2$ sense to an $L^2$ integrand implies the convergence of the corresponding sequence of spectral stochastic integrals in the suitable $L^2$ sense to the spectral stochastic integral of the limiting integrand. Let $f \in L^2(M_2)$. Then, as was shown, there are some $\mathcal{A}$-simple $f_1, f_2, \ldots$ such that $f_n \to f$ in $L^2$; so $\int f_n \, dM_2 \to \int f \, dM_2$ in $L^2(\mathbb{P})$. Since

$$M_2(B) = \left| \frac{M_2(B)}{\chi_{L^2(\mathbb{P})}} \right|_{L^2(\mathbb{P})}^2 = \left| \int 1_B g \, dM_1 \right|_{L^2(\mathbb{P})}^2 = \int 1_B \left|g\right|^2 \, dM_1$$

for all $B \in \sigma(\mathcal{A})$, we have

$$\left| \int f_n \, dM_2 - \int f \, dM_2 \right|_{L^2(\mathbb{P})}^2 = \left| \int (f_n - f) \, dM_2 \right|_{L^2(\mathbb{P})}^2 = \left| f_n - f \right|_{L^2(M_2)}^2 = \int |f_n - f|^2 \, dM_2 = \int |f_n - f|^2 |g|^2 \, dM_1;$$

so $(f_n - f)g \to 0$ in $L^2(M_1)$, and the continuity property of our spectral stochastic integration implies
that
\[ \left| \int (f_n - f) g \, d\mathbb{M}_1 \right|_{L^2(P)} \to 0. \]

But
\[ \int f_n \, d\mathbb{M}_2 = \int f_n g \, d\mathbb{M}_1 \]

for all \( n \in \mathbb{N} \) by the definition of \( \mathbb{M}_2 \) and the linearity of our spectral stochastic integration operators; the essential uniqueness of \( L^2 \)-limit then completes the proof.

\[ \square \]

References

[1] Gikhman, I.I. and Skorokhod, A.V. (1980). *The Theory of Stochastic Processes I*, translated by S. Kotz. Springer.

[2] Shiryaev, A.N. (1996). *Probability*, second edition, translated by R.P. Boas. Springer.