BRAIDING SURFACE LINKS WHICH ARE COVERINGS OF A TRIVIAL TORUS KNOT

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Abstract. We consider surface links in the 4-space which can be deformed to simple branched coverings of a trivial torus knot, which we call torus-covering-links. Torus-covering-links contain spun $T^2$-knots, turned spun $T^2$-knots, symmetry-spun tori and torus $T^2$-knots. In this paper we study the braid indices of torus-covering-links. In particular we show that the turned spun $T^2$-knot of the torus $(2, p)$-knot has the braid index four.

0. Introduction

Locally flatly embedded closed 2-manifolds in the 4-space $\mathbb{R}^4$ are called surface links. It is known that any oriented surface link can be deformed to the closure of a simple surface braid, that is, a simple branched covering of the 2-sphere ([10]).

As surface knots of genus one which can be made from classical knots, there are spun $T^2$-knots, turned spun $T^2$-knots, symmetry-spun tori and torus $T^2$-knots. Consider $\mathbb{R}^4$ as obtained by rotating $\mathbb{R}^3_+$ around the boundary $\mathbb{R}^2$. Then a spun $T^2$-knot is obtained by rotating a classical knot ([2]), a turned spun $T^2$-knot by turning it once while rotating ([2]), a symmetry-spun torus by turning a classical knot with periodicity rationally while rotating ([16]), and a torus $T^2$-knot is a surface knot on the boundary of a neighborhood of a solid torus in $\mathbb{R}^4$ ([6]). Symmetry-spun tori include spun $T^2$-knots, turned spun $T^2$-knots and torus $T^2$-knots. We call the link version of a symmetry-spun torus, a spun $T^2$-link, a turned spun $T^2$-link and a torus $T^2$-link respectively.

Now we consider surface links in the 4-space which can be deformed to simple branched coverings of a trivial torus knot, which we define as torus-covering-links ([14]). By definition, a torus-covering-link is described by a torus-covering-chart, which is a chart on the trivial torus knot. Torus-covering-links include symmetry $T^2$-links (and spun $T^2$-links, turned spun $T^2$-links, and torus $T^2$-links). A torus-covering-link has no 2-knot component. Each component of a torus-covering-link is of genus at least one.

In this paper we deform the torus-covering-link associated with a torus-covering-chart of degree $m$ to the closure of a simple surface braid. Then we obtain its (surface link) chart description, which is of degree $2m$, and gives an upper estimate of its braid index (Theorem 3.1). In particular, we show that the turned spun $T^2$-knot of the torus $(2, p)$-knot has the braid index four (Corollary 3.3).

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1. Definitions and Preliminaries

Definition 1.1. A locally flatly embedded closed 2-manifold in $\mathbb{R}^4$ is called a surface link. A surface link with one component is called a surface knot. A surface link whose each component is of genus zero (resp. one) is called a 2-link (resp. $T^2$-link). In particular a surface knot of genus zero (resp. one) is called a 2-knot (resp. $T^2$-knot).

An orientable surface link $F$ is trivial (or unknotted) if there is an embedded 3-manifold $M$ with $\partial M = F$ such that each component of $M$ is a handlebody.

An oriented surface link $F$ is called pseudo-ribbon if there is a surface link diagram of $F$ whose singularity set consists of double points and ribbon if $F$ is obtained from a trivial 2-link $F_0$ by 1-handle surgeries along a finite number of mutually disjoint 1-handles attaching to $F_0$. By definition, a ribbon surface link is pseudo-ribbon.

Two surface links are equivalent if there is an ambient isotopy or an orientation-preserving diffeomorphism of $S^4$ or $\mathbb{R}^4$ which deforms one to the other.

Definition 1.2. A compact and oriented 2-manifold $S$ embedded properly and locally flatly in $D^2_1 \times D^2_2$ is called a braided surface of degree $m$ if $S$ satisfies the following conditions:

(i) $\text{pr}_2|_S : S \longrightarrow D^2_2$ is a branched covering map of degree $m$,
(ii) $\partial S$ is a closed $m$-braid in $D^2_1 \times \partial D^2_2$, where $D^2_1, D^2_2$ are 2-disks, and $\text{pr}_2 : D^2_1 \times D^2_2 \rightarrow D^2_2$ is the projection to the second factor.

A braided surface $S$ is called a surface braid if $\partial S$ is the trivial closed braid. Moreover, $S$ is called simple if every singular index is two.

Two braided surfaces are equivalent if there is a fiber-preserving ambient isotopy of $D^2_1 \times D^2_2$ rel $D^2_1 \times \partial D^2_2$ which carries one to the other.

There is a theorem which corresponds to Alexander’s theorem for classical oriented links.

Theorem 1.3 (Kamada [10]). Any oriented surface link can be deformed by an ambient isotopy of $\mathbb{R}^4$ to the closure of a simple surface braid.

There is a chart which represents a simple surface braid.

Definition 1.4. Let $m$ be a positive integer, and $\Gamma$ be a graph on a 2-disk $D^2_2$. Then $\Gamma$ is called a surface link chart of degree $m$ if it satisfies the following conditions:

(i) $\Gamma \cap \partial D^2_2 = \emptyset$.
(ii) Every edge is oriented and labeled, and the label is in $\{1, \ldots, m - 1\}$.
(iii) Every vertex has degree 1, 4, or 6.
(iv) At each vertex of degree 6, there are six edges adhering to which, three consecutive arcs oriented inward and the other three outward, and those six edges are labeled $i$ and $i + 1$ alternately for some $i$. 

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(v) At each vertex of degree 4, the diagonal edges have the same label and are oriented coherently, and the labels $i$ and $j$ of the diagonals satisfy $|i - j| > 1$.

Vertices of degree 1 and 6 are called a black vertex and a white vertex. A black vertex (resp. white vertex) of a chart corresponds to a branch point (resp. triple point) of the simple surface braid associated with the chart.

An edge without end points is called a loop. An edge whose end points are black vertices is called a free edge, and a configuration consisting of a free edge and a finite number of concentric simple loops such that the loops are surrounding the free edge is called an oval nest.

An unknotted chart is a chart presented by a configuration consisting of free edges. A trivial oriented surface link is presented by an unknotted chart (10).

A ribbon chart is a chart presented by a configuration consisting of oval nests. A ribbon surface link is presented by a ribbon chart (10).

A chart with a boundary represents a simple braided surface. There is a notion of C-move equivalence (10) between two charts of the same degree. The following theorem is well-known:

Theorem 1.5 (10). Two charts of the same degree are C-move equivalent if and only if their associated simple braided surfaces are equivalent.

2. Torus-covering-links

We give the definition of torus-covering-links in $\mathbb{R}^4$ (cf. [14]).

**Definition 2.1.** First, embed $D^2 \times S^1 \times S^1$ into $\mathbb{R}^4$ naturally, and identify $D^2 \times S^1 \times S^1$ with $D^2 \times I_3 \times I_4 / \sim$, where $(x, 0, v) \sim (x, 1, v)$ and $(x, u, 0) \sim (x, u, 1)$ for $x \in D^2$, $u \in I_3 = [0, 1]$ and $v \in I_4 = [0, 1]$.

Let us consider a surface link $S$ embedded in $D^2 \times S^1 \times S^1$ such that $S \cap (D^2 \times I_3 \times I_4)$ is a simple braided surface. We call such a surface link a torus-covering-link.

A torus-covering-link $S$ can be described by a chart on the trivial torus knot, i.e. by a chart $\Gamma_T$ on $D^2_3 = I_3 \times I_4$ with $\Gamma_T \cap (I_3 \times \{0\}) = \Gamma_T \cap (I_3 \times \{1\})$ and $\Gamma_T \cap (\{0\} \times I_4) = \Gamma_T \cap (\{1\} \times I_4)$. Let us denote the classical braids described by $\Gamma_T \cap (I_3 \times \{0\})$ and $\Gamma_T \cap (\{0\} \times I_4)$ by $\Gamma_T^v$ and $\Gamma_T^h$ respectively. We will call $\Gamma_T$ a torus-covering-chart with boundary braids $\Gamma_T^v$ and $\Gamma_T^h$.

Let $b(\Gamma_T)$ be the number of black vertices in the torus-covering-chart $\Gamma_T$. Then let us consider the case $b(\Gamma_T) = 0$. In this case the torus-covering-link associated with $\Gamma_T$ is determined by the boundary braids $\Gamma_T^v$ and $\Gamma_T^h$. We
will call such a $\Gamma_T$ a torus-covering-chart \textit{without black vertices and with boundary braids} $\Gamma_T^v$ and $\Gamma_T^b$.

\textbf{Remark.} In the case $b(\Gamma_T) = 0$, the boundary braids $\Gamma_T^v$ and $\Gamma_T^b$ are commutative.

By definition, torus-covering-links contain symmetry-spun tori (and spun $T^2$-knots, turned spun $T^2$-knots and torus $T^2$-knots).

As we stated in Theorem 1.5 if there are two surface link charts of the same degree, their associated surface links are equivalent if and only if their charts are C-move equivalent. It follows that if two torus-covering-charts are C-move equivalent, their associated torus-covering-links are equivalent.

A torus-covering-link has no 2-knot component. In particular, if a torus-covering-chart has no black vertices, then each component of the associated torus-covering-link is of genus one.

Let $\Gamma_T$ be a torus-covering-chart of degree $m$ and with the trivial boundary braids. Let $F$ be the surface link associated with $\Gamma_T$ by assuming $\Gamma_T$ to be a surface link chart. Then the torus-covering-link associated with the torus-covering-chart $\Gamma_T$ is obtained from $F$ by applying $m$ trivial 1-handle surgeries.

\textbf{Example 2.2.} (Example 2.2 in [14])

(2.2.1) Let $\Gamma_T$ be a torus-covering-chart of degree 2 without black vertices and with boundary braids $\sigma_1^3$ and $e$ (the trivial braid). Then the torus-covering-knot associated with $\Gamma_T$ is the spun $T^2$-knot of a right-handed trefoil.

(2.2.2) Let $\Gamma_T$ be a torus-covering-chart of degree 2 without black vertices and with boundary braids $\sigma_1^3$ and $\sigma_1^3$ (or $\sigma_1^3$ and $\sigma_1^{-3}$). Then the torus-covering-knot associated with $\Gamma_T$ is the turned spun $T^2$-knot of a right-handed trefoil.

(2.2.3) Let $\Gamma_T$ be a torus-covering-chart without black vertices and with boudary braids $\beta^2$ and $\beta$. Then the torus-covering-knot associated with $\Gamma_T$ is a symmetry-spun torus.
3. Braiding torus-covering-links over the 2-sphere

For a (classical) braid $\beta$, let $\iota_k(\beta)$ be the braid obtained from $\beta$ by adding $k$ (resp. $l$) trivial strings before (resp. after) $\beta$, and

\[
\Pi_i = \sigma_{m+1}\sigma_{m+2}\cdots\sigma_{m+i}, 
\Pi_i^{\Lambda} = \sigma_{m-1}\sigma_{m-2}\cdots\sigma_{m-i},
\Delta_m = \Pi_{m-1}\Pi_{m-2}\cdots\Pi_1, 
\Delta'_m = \Pi_{m-1}\Pi_{m-2}\cdots\Pi_1,
\Theta_m = \sigma_m \cdot \Pi_{m-1} \cdot \Pi_{m-2} \cdots \sigma_m \cdot \Pi_{m} \cdot \sigma_m.
\]

**Remark.** Let $\Delta$ be Garside’s $\Delta$ for the braid group $B_m$. Then $\iota_m(\Delta) = \Delta_m$ (cf. [4]).

**Theorem 3.1.** Let $\Gamma_T$ be a torus-covering-chart of degree $m$ with boudary braids $a$ and $b$. Then the torus-covering-link associated with $\Gamma_T$ can be described by a surface link chart $\Gamma_S$ of degree $2m$ as in Fig. 3.1 where $H_b$ is a chart describing the simple braided surface as follows:

\[
i_0^m(b) \to (\Delta'_m)^{-1} \cdot \Delta_m^{-1} \cdot \Delta'_m \cdot \Delta_m \to i_0^m(b) \cdot (\Delta'_m)^{-1} \cdot \Delta_m^{-1} \cdot \Theta_m \to (\Delta'_m)^{-1} \cdot \Delta_m^{-1} \cdot i_0^m(b^*) \cdot \Theta_m \to (\Delta'_m)^{-1} \cdot \Delta_m^{-1} \cdot \Theta_m \cdot i_0^m(b^*) \to i_0^m(b^*),
\]

where $\to$ means an isotopy transformation and $\to$ a hyperbolic transformation along bands corresponding to the $m$ $\sigma_m$’s (Fig. 3.2), and $-(H_b)^*$ is the orientation-reversed mirror image of $H_b$, and $b^*$ is the braid obtained from the classical braid $b$ by taking its mirror image and reversing all the crossings (Fig. 3.3).

**Definition 3.2.** Let us call $H_b$ the 1-handle chart of $\Gamma_T$, and its corresponding braided surface the 1-handle braided surface of $\Gamma_T$.

**Remark.** The surface link chart $\Gamma_S$ is of degree $2m$ and well-defined, for the edges representing $i_0^m(a)$ have labels at most $m - 1$ and the edges representing $i_0^m(b^*)$ have labels at least $m + 1$. Note the 1-handle chart $H_b$ has $2m$ black vertices.

**proof.** (Step 1) Let us consider a trivial torus knot $T^2$ in $\mathbb{R}^4$ as the result of 1-handle surgeries of $S_1 \cup S_2$ along $h_1 \cup h_2$, where $S_1$ (resp. $S_2$)
Figure 3.2. The 1-handle braided surface of $\Gamma_T$

Figure 3.3. The classical braid $\overline{b}^*$

is a 2-sphere in $\mathbb{R}^4$ with a positive (resp. negative) orientation such that $S_1$ contains $S_2$, and $h_1$ (resp. $h_2$) is a 1-handle attaching to the two spheres trivially in a neighborhood of the north (resp. south) pole (Fig. 3.4).

(Step 2) Deform the two 1-handles and the inner sphere $h_1 \cup h_2 \cup S_2$ by an ambient isotopy of $\mathbb{R}^4$ to make $S_2$ have a positive orientation as in Fig. 3.5 where the 1-handle $h_1$ is as in Fig. 3.6 which has a double point curve with a branch point for each end, and the other 1-handle $h_2$ is the orientation-reversed mirror image of $h_1$.

(Step 3) Slide the 1-handle $h_2$ to the neighborhood of the north pole to make both $h_1$ and $h_2$ be in the neighborhood of the north pole, and cut off the two southern hemispheres to obtain the surface braid and the surface link chart of degree 2 as in Fig. 3.7.
Figure 3.8.

Figure 3.9.

(Step 4) Now, consider the trivial torus knot $T^2$ as the torus-covering-link associated with $\Gamma_T$ (of degree $m$) by drawing $\Gamma_T$ on $T^2$ (Fig. 3.8). Let us denote the $m$ 1-handles corresponding to $h_1$ (resp. $h_2$) by $H_1$ (resp. $H_2$). Then $H_1$ can be deformed to the 1-handle braided surface as in Fig. 3.2, and $H_2$ to the orientation-reversed mirror image of $H_1$, and the surface braid will be as in Fig. 3.9. Hence we obtain the surface link chart $\Gamma_S$ of degree $2m$ as in Fig. 3.1.

The braid index of an oriented surface link $F$ is the minimum degree of simple closed surface braids in $\mathbb{R}^4$ which are equivalent to $F$. Kamada showed in [8] that surface links whose braid index is at most three are indeed ribbon, and Shima showed in [15] that the turned spun $T^2$-knot of a non-trivial classical knot is not ribbon. Hence we obtain the following corollary:

**Corollary 3.3.** Let $S$ be the torus-covering-link associated with a torus-covering-chart of degree $m$. Then the braid index of $S$ is equal or less than $2m$. In particular, the braid index of the turned spun $T^2$-knot of the torus $(2, p)$-knot is four.

**Remark.** Hasegawa in [5] (10, Part3 “Chart description of twist-spun surface-links”) showed that for the turned spun $T^2$-link of a closed $m$-braid, its braid index is at most $3m$.

**Example 3.4.** Let us consider the torus-covering-chart $\Gamma_T$ of Example 2.2.2 (Fig. 3.10).
Figure 3.10. The torus-covering-chart $\Gamma_T$

Figure 3.11. The 1-handle chart $H_b$, where $b = \sigma_1^{-3}$

Figure 3.12. The surface link chart $\Gamma_S$ obtained from $\Gamma_T$ (degree 4)

The torus-covering-chart $\Gamma_T$ describes the turned spun $T^2$-knot of the right-handed trefoil. Its 1-handle chart is as in Fig. 3.11 and the surface link chart $\Gamma_S$ obtained from $\Gamma_T$ is as in Fig. 3.12.
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