A note on “Sur le noyau de l’opérateur de courbure d’une variété finslérienne, C. R. Acad. Sci. Paris, sér. A, t. 272 (1971), 807-810”*

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Abstract. In this note, adopting the pullback formalism of global Finsler geometry, we show by a counterexample that the kernel $\text{Ker}_R$ of the h-curvature $R$ of Cartan connection and the associated nullity distribution $\mathcal{N}_R$ do not coincide, contrary to Akbar-Zadeh’s result [1]. We also give sufficient conditions for $\text{Ker}_R$ and $\mathcal{N}_R$ to coincide.

Keywords: Cartan connection, h-curvature tensor, Nullity distribution, Kernel distribution.

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1. Introduction and notations

Nullity distribution in Finsler geometry has been investigated in [1] (adopting the pullback formalism) and [5] (adopting the Klein-Grifone formalism). In 1971, Akbar-Zadeh [1] proved that the kernel $\text{Ker}_R$ of the h-curvature operator $R$ of Cartan connection coincides with the nullity distribution $\mathcal{N}_R$ of that operator. This result was reappeared again in [2] and was used to prove that the nullity foliation is auto-parallel. Moreover, Bidabad and Refie-Rad [3] generalized this result to the case of k-nullity distribution following the same pattern of proof as Akbar-Zadeh’s.

In this note, we show by a counterexample that $\text{Ker}_R$ and $\mathcal{N}_R$ do not coincide, contrary to Akbar-Zadeh’s result. In addition, we find sufficient conditions for $\text{Ker}_R$ and $\mathcal{N}_R$ to coincide.

In what follows, we denote by $\pi : TM \to M$ the subbundle of nonzero vectors tangent to $M$, $\pi_* : T(TM) \to TM$ the linear tangent map of $\pi$ and $V_z(TM) = (\text{Ker} \pi_*)_z$ the vertical space at $z \in TM$. Let $\mathfrak{F}(TM)$ be the algebra of $C^\infty$ functions on $TM$ and $\mathfrak{X}(\pi(M))$ the $\mathfrak{F}(TM)$-module of differentiable sections of the pullback bundle $\pi^{-1}(TM)$. The elements of $\mathfrak{X}(\pi(M))$
will be called $\pi$-vector fields and denoted by barred letters $\overline{X}$. The fundamental $\pi$-vector field is the $\pi$-vector field $\overline{\eta}$ defined by $\overline{\eta}(z) = (z, z)$ for all $z \in TM$.

Let $D$ be a linear connection on the pullback bundle $\pi^{-1}(TM)$. Let $K$ be the map defined by $K : T(TM) \rightarrow \pi^{-1}(TM) : X \mapsto D_X\overline{\eta}$. The vector space $H_z(TM) := \{X \in T_z(TM) : K(X) = 0\}$ is the horizontal space to $M$ at $z$. The restriction of $\pi_*$ on $H_z(TM)$, denoted again $\pi_*$, defines an isomorphism between $H_z(TM)$ and $T_{\pi_zM}$. The connection $D$ is said to be regular if $T_z(TM) = V_z(TM) \oplus H_z(TM) \ \forall z \in TM$. In this case $K$ defines an isomorphism between $V_z(TM)$ and $T_{\pi_zM}$. If $M$ is endowed with a regular connection, then the preceding decomposition permits to write uniquely a vector $X \in T_z(TM)$ in the form $X = hX + vX$, where $hX \in H_z(TM)$ and $vX \in V_z(TM)$. The $((h)hv)$-torsion tensor of $D$, denoted by $T$, is defined by $T(X, Y) = T(vX, hY)$, for all $X, Y \in \mathfrak{X}(\pi(M))$, where $T(X, Y) = D_XhY - D_YhX - \pi_*[X, Y]$ is the (classical) torsion associated with $D$ and $\overline{X} = \pi_*X$ (the fibers of the pullback bundle are isomorphic to the fibers of the tangent bundle). The $h$-curvature tensor of $D$, denoted by $R$, is defined by $R(X, Y)Z = K(hX, hY)Z$, where $K(X, Y)Z = D_XD_YZ - D_YD_XZ - D_{[X,Y]}Z$ is the (classical) curvature associated with $D$. The contracted curvature $\hat{R}$ is defined by $\hat{R}(X, Y) = R(X, Y)\overline{\eta}$.

### 2. Kernel and nullity distributions: Counterexample

Let $(M, F)$ be a Finsler manifold. Let $\nabla$ be the Cartan connection associated with $(M, F)$. It is well known that $\nabla$ is the unique metrical regular connection on $\pi^{-1}(TM)$ such that $g(T(X, Y), Z) = g(T(X, Z), Y)$ [2], [3]. Note that the bracket $[X, Y]$ is horizontal if and only if $\hat{R}(X, Y) = 0$, where $\hat{R}$ is the contracted curvature of the $h$-curvature tensor of $\nabla$.

**Lemma 2.1.** [2] Let $T$ and $K$ be the (classical) torsion and curvature tensors of $\nabla$ respectively. We have:

$$\mathcal{S}_{X,Y,Z}\{K(X,Y)Z - \nabla_ZT(X,Y) - T(X,[Y,Z])\} = 0,$$

where the symbol $\mathcal{S}_{X,Y,Z}$ denotes cyclic sum over $X, Y, Z \in \mathfrak{X}(TM)$.

Let us now define the concepts of nullity and kernel spaces associated with the curvature $K$ of $\nabla$, following Akbar-Zadeh’s definitions [1].

**Definition 2.2.** The subspace $N_K(z)$ of $H_z(TM)$ at a point $z \in TM$ is defined by

$$N_K(z) := \{X \in H_z(TM) : K(X, Y) = 0, \ \forall Y \in H_z(TM)\}.$$

The dimension of $N_K(z)$ is denoted by $\mu_K(z)$. The subspace $N_K(x) := \pi_*(N_K(z)) \subset T_xM, \ x = \pi z$, is linearly isomorphic to $N_K(z)$. This subspace is called the nullity space of the curvature operator $K$ at the point $x \in M$.

**Definition 2.3.** The kernel of $K$ at the point $x = \pi z$ is defined by

$$\ker_K(x) := \{\overline{X} \in \{z\} \times T_xM \simeq T_xM : K(Y, Z)\overline{X} = 0, \ \forall Y, Z \in H_z(TM)\}.$$
Theorem 2.4. The nullity space \( N_R(x) \) and the kernel space \( \text{Ker}_R(x) \) do not coincide.

Let \( M = \mathbb{R}^3 \), \( U = \{(x_1, x_2, x_3; y_1, y_2, y_3) \in \mathbb{R}^3 \times \mathbb{R}^3 : x_3 y_1 > 0, y_2^2 + y_3^2 \neq 0 \} \subseteq T M \). Let \( F \) be the Finsler function defined on \( U \) by

\[
F := \sqrt{x_3 y_1 \sqrt{y_2^2 + y_3^2}}.
\]

Using MAPLE program, we can perform the following computations. We write only the coefficients \( \Gamma^i_j \) of Barthel connection and the components \( R^{ij}_k \) of the h-curvature tensor \( R \).

The non-vanishing coefficients of Barthel connection \( \Gamma^i_j \) are:

\[
\Gamma_2^2 = \frac{y_3}{x_3}, \quad \Gamma_3^2 = \frac{y_2}{x_3}, \quad \Gamma_2^3 = -\frac{y_2}{x_3}, \quad \Gamma_3^3 = \frac{y_3}{x_3}.
\]

The independent non-vanishing components of the h-curvature \( R^{ij}_k \) of Cartan connection are:

\[
R^{1}_{23} = \frac{y_1 y_3}{2x_3(y_2^2 + y_3^2)}, \quad R^{1}_{32} = -\frac{y_1 y_2}{2x_3(y_2^2 + y_3^2)}, \quad R^{2}_{13} = -\frac{y_3}{2x_3 y_1}, \quad R^{3}_{12} = \frac{1}{2x_3}.
\]

Now, let \( X \in N_R \), then \( X \) can be written in the form \( X = X^1 h_1 + X^2 h_2 + X^3 h_3 \), where \( X^1, X^2, X^3 \) are the components of the vector \( X \) with respect to the basis \( \{h_1, h_2, h_3\} \) of the horizontal space; \( h_i := \frac{\partial}{\partial x^i} - \Gamma^m_i \frac{\partial}{\partial y^m} \), \( i, m = 1, \ldots, 3 \). The equation \( R(X, Y)Z = 0 \), \( \forall Y, Z \in H(TM) \), is written locally in the form \( X^i R^{ij}_k = 0 \). This is equivalent to the system of equations \( X^2 = 0, X^3 = 0 \) having the solution \( X^1 = t (t \in \mathbb{R}), X^2 = X^3 = 0 \). As \( \pi_*(h_i) = \frac{\partial}{\partial y^i} \), we have

\[
N_R(x) = \left\{ t \frac{\partial}{\partial x^i} \big| t \in \mathbb{R} \right\}.
\]

(2.1)

On the other hand, let \( Z \in \text{Ker}_R \). The equation \( R(X, Y)Z = 0 \), \( \forall X, Y \in H(TM) \), is written locally in the form \( Z^i R^{ij}_k = 0 \). This is equivalent to the system:

\[
y_3 Z^2 - y_2 Z^3 = 0, \quad y_3 Z^1 + y_1 Z^3 = 0, \quad y_2 Z^1 + y_1 Z^2 = 0.
\]

This system has the solution \( Z^1 = t, Z^2 = -\frac{y_2}{y_1} t \) and \( Z^3 = -\frac{y_3}{y_1} t \), \( t \in \mathbb{R} \). Thus,

\[
\text{Ker}_R(x) = \left\{ t \left( \frac{y_2}{y_1} \frac{\partial}{\partial x^2} - \frac{y_3}{y_1} \frac{\partial}{\partial x^3} \right) \big| t \in \mathbb{R} \right\}.
\]

(2.2)

Comparing (2.1) and (2.2), we note that there is no value of \( t \) for which \( N_R(x) = \text{Ker}_R(x) \). Consequently, \( N_R(x) \) and \( \text{Ker}_R(x) \) do not coincide.

According to Akabr-Zadeh’s proof, if \( X \in N_R \), then, by Lemma 2.1, we have \( R(Y, Z)X = T(X, [Y, Z]) \). But there is no guarantee for the vanishing of the right-hand side. Even the equation \( g(R(Y, Z)X, \pi_* W) = g(T(X, [Y, Z]), \pi_* W), W \in H(TM) \), is true only for \( X \in N_R \) and, consequently, we cannot use the symmetry or skew-symmetry properties in \( X \) and \( W \) to conclude that \( g(R(Y, Z)X, W) = 0 \). This can be assured, again, by the previous example: if we take \( X = h_1 \in N_R(z) \) and \( Y = h_2, Z = h_3 \), then the bracket \( [Y, Z] = -\frac{y_3}{y_1} \frac{\partial}{\partial y_2} + \frac{y_2}{y_1} \frac{\partial}{\partial y_3} \) is vertical and \( T(h_1, [h_2, h_3]) = -\frac{1}{2x_3 y_1} (y_3 \partial_2 - y_2 \partial_3) \neq 0 \), where \( \partial_i \) is the basis of the fibers of the pullback bundle.

As has been shown above, \( N_R(x) \) and \( \text{Ker}_R(x) \) do not coincide in general. Nevertheless, we have
Theorem 2.5. Let \((M, F)\) be a Finsler manifold and \(R\) the \(h\)-curvature of Cartan connection. If
\[
\mathcal{S}_{X,Y,Z} R(X,Y)Z = 0,
\] (2.3)
then the two distributions \(\mathcal{N}_R\) and \(\text{Ker}_R\) coincide.

Proof. If \(X \in \mathcal{N}_R\), then, from (2.3), we have \(R(Y,Z)X = 0\) and consequently \(X \in \text{Ker}_R\). On the other hand, it follows also from (2.3) that \(g(R(X,Y)Z,W) = R(X,Y,Z,W) = R(Z,W,X,Y)\). This proves that if \(X \in \text{Ker}_R\), then \(X \in \mathcal{N}_R\).

The following corollary shows that there are nontrivial cases in which (2.3) is verified and consequently the two distributions coincide.

Corollary 2.6. Let \((M, F)\) be a Finsler manifold and \(g\) the associated Finsler metric. If one of the following conditions holds:
(a) \(\hat{R} = 0\) (the integrability condition for the horizontal distribution),
(b) \(\hat{R}(X,Y) = \lambda F(\ell(X)Y - \ell(Y)X)\), where \(\lambda(x,y)\) is a homogenous function of degree 0 in \(y\) and \(\ell(X) := F^{-1}g(X,\eta)\) (the isotropy condition),
then the two distributions \(\mathcal{N}_R\) and \(\text{Ker}_R\) coincide.

Proof. (a) We have \(\mathcal{S}_{X,Y,Z}\{R(X,Y)Z - T(X,\hat{R}(Y,Z))\} = 0\). Then, if \(\hat{R} = 0\), (2.3) holds.
(b) If \(\hat{R}(X,Y) = \lambda F(\ell(X)Y - \ell(Y)X)\), then, by [4], (2.3) is satisfied.

Remark 2.7. It should be noted that the identity (2.3) is a sufficient condition for the validity of the identity (2.1) of [1].

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