LIMITS AND COLIMITS OF PARTIAL GROUPS

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ABSTRACT. The purpose of this short paper is to analyse limits and colimits in the category $\text{Part}$ of partial groups, algebraic structures introduced by A. Chermak. The class of partial groups contains a subclass of objects corresponding to the class of transporter systems as defined by Oliver and Ventura. Special cases of such transporter systems are the centric linking systems associated to saturated fusion systems that were defined by Broto, Levi and Oliver. We will prove that $\text{Part}$ is both complete and cocomplete and, in addition, that the full subcategory of finite partial groups is both finitely complete and finitely cocomplete.

A. González has proven that $\text{Part}$ is equivalent to a full subcategory of the category $\text{sSet}$ of simplicial sets. We will show that such subcategory is not closed under formation of colimits in $\text{sSet}$, so that a different construction must be taken into consideration.

INTRODUCTION

The notion of a fusion system was introduced by L. Puig (with the name of Frobenius categories) for investigating modular representations. If $S$ is a finite $p$-group, a fusion system $\mathcal{F}$ over $S$ is a category whose objects are the subgroups of $S$ and morphisms are injective group homomorphisms, including at least all those arising by conjugation with elements of $S$. Of particular interest are those fusion systems satisfying the additional saturation property over the morphisms, modelled on the behavior of the conjugation homomorphisms between subgroups of a Sylow subgroup $S$ in a finite group $G$.

Fusion systems are currently studied also by group theorists, as a means to obtain a simplified proof of the classification of the finite simple group, and by algebraic topologists, in connection to the Martino-Priddy conjecture (see [1, 1.6, Theorem 1.17]).

C. Broto, R. Levi and B. Oliver have defined in [2] the notion of a centric linking system, a category which resembles a fusion system, but is richer and with a more group-like structure. Centric linking systems were subsequently generalized to transporter systems by B. Oliver and J. Ventura in [8].

Existence and uniqueness of a centric linking system associated to a saturated fusion system had been an open conjecture for many years and it was affirmatively proven by A. Chermak in [3]. Chermak was able to translate the categorical language of transporter systems into the algebraic language of group-like structures, namely partial groups. These are non-empty sets endowed with a product which may not be defined on all pairs of elements, but possibly only over few finite strings, and that satisfy some additional axioms which guarantee properties similar to those of groups, such as existence of a unit and of inverses.

Chermak showed that a subclass of partial groups corresponds, in a technical way, to the class of transporter systems as defined by Oliver and Ventura and solved the conjecture by focusing on such partial groups. As he also defines, in [4], a notion of morphisms of partial groups, the result is a category, which will be denoted by $\text{Part}$. It contains localities, i.e. the algebraic structures corresponding to transporter systems, as well as the entire category $\text{Grp}$ of groups with group homomorphisms. We wish to study the behavior of limits and colimits of partial groups. In particular, interesting early examples of how colimits in $\text{Part}$ may be exploited and used as a tool are the elementary expansions defined by Chermak in [5] and, also, the construction realized by...
We will show that, when dealing with limits and colimits, the category Part behaves partially as Set (the category of sets and functions) and partially as Grp. First of all, we will replace the category Set by that of pointed sets, denoted by Set*; this replacement does not affect the categorical properties we are investigating (partial groups and groups have a natural point fixed by all homomorphisms, namely the unit), but it makes some constructions more natural and, thus, easier.

Generalizing Chermak’s argument in [4, Example 1.2], in Section 1 we will construct free partial groups over pointed sets. The resulting assignment will be functorial; if \( F : \text{Set}^* \to \text{Part} \) is the functor assigning to each pointed set the corresponding free partial group, we will show that \( F \) is a left adjoint to the forgetful functor \( U : \text{Part} \to \text{Set}^* \), just as it happens for groups. This will be used in Section 3 to compute limits of partial groups. Indeed, having a forgetful functor which is a right adjoint provides us with a candidate, as underlying set, over which we only need to define a suitable partial group structure in order to obtain the desired limit.

Section 2 will be devoted to the analysis of colimits. The situation is a little more complicated; we will show that difficulties arise when looking for an underlying set through which we can factor all the maps defining the products of the involved partial groups. We will provide a counterexample which shows that the pointed set-wise colimit is not, in general, a suitable underlying set. Thus, we will simplify our analysis by studying separately coproducts and coequalizers, as every colimit can be constructed through these two special ones.

With regard to coproducts, the category Part behaves very differently from Grp: the coproducts have, as underlying set, the corresponding coproduct in Set*. Coequalizers, instead, are much more similar to those in Grp. Nonetheless, quotients in the category Grp allow us to build coequalizers in a fairly smooth way; not having a general quotient construction in Part will force us to search for a suitable underlying set over which we can build a partial group structure. In detail, if \( \mathcal{L} \xrightarrow{f} \mathcal{L}' \) is a coequalizer diagram in Part, we will show that there always exists a suitable quotient set of \( \mathcal{L}' \), i.e. an equivalence relation on \( \mathcal{L}' \), which can be endowed with a partial group structure having the property of being a coequalizer in Part. As a consequence, we will achieve a proof of the following theorem.

**Theorem A.** The category Part is complete and cocomplete. In addition, if FinPart is the full subcategory of Part with objects finite partial groups, then FinPart is finitely complete and finitely cocomplete.

Section 4 is dedicated to the correspondence between partial groups and simplicial sets. It was initially conjectured by C. Broto, and then formalized by A. González in [6], that every partial group can be regarded as a simplicial set. In particular, Part can be embedded in sSet (the category of simplicial sets) as a full subcategory.

Providing a detailed description of this correspondence is beyond the scope of this paper; however, since limits and colimits of simplicial sets are very well understood, we will show that this is not sufficient for computing colimits of partial groups. Indeed, it turns out that the colimit, computed in sSet, of a diagram in the subcategory Part needs not be a partial group, i.e. the subcategory Part is not closed under formation of colimits in sSet.

Throughout the entire work, we will use right-hand notation for functions and functors, i.e. writing \((x)f\), or simply \(xf\), whenever \(f\) is a function (functor) and \(xf\) is the object (or morphism) associated to \(x\) by \(f\).
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1. Partial groups and free partial groups

Roughly speaking, a partial group is a set $L$ equipped with a partial operation defined on a subset of the possible words on $L$, that is, it may be the case that it is not possible to multiply all elements as it happens in groups. More precisely we have the following definition.

**Definition 1.1.** Let $L$ be a non-empty set and $W(L) = W$ be the free monoid on $L$; denote concatenation of words in $W$ by the symbol $\circ$. Consider a subset $D = D(L) \subseteq W$, a map $\Pi : D \rightarrow L$, which will be referred to as the multivariable product of $L$ (and sometimes just as the product of $L$), and an involutory bijection $i : L \rightarrow L$. Then the quadruple $(L, D, \Pi, i)$ (often denoted just by $L$ when the context leaves no ambiguity) is a partial group provided that the following hold:

1. $L \subseteq D$ and moreover $u \circ v \in D \implies u, v \in D$;
2. $\Pi|_L = id_L$;
3. if $u \circ v \circ w \in D$, then $u \circ (v \circ w) = (u \circ v) \circ w$;
4. by extending $i$ to $W(L)$ defining $(x_1, \ldots, x_n)i = ((x_n)i, \ldots, (x_1)i)$, if $w \in D$ then we have $(w)i \circ w \in D$ and $((w)i \circ w)\Pi = 1 = (\emptyset)\Pi$, where $\emptyset$ is the empty word.

For a survey of some elementary properties of partial groups we refer to [4, Section 1].

Clearly, $L$ is a group if and only if $D(L) = W(L)$ and, in such a case, $\Pi$ gives the group multiplication thanks to associativity.

A. Chermak also defines a notion of morphisms of partial groups.

**Definition 1.2.** Given partial groups $(L, D, \Pi, i)$ and $(L', D', \Pi', i')$, a morphism $\beta$ of partial groups from $L$ to $L'$ is a set-wise map $\beta : L \rightarrow L'$ such that, if $\beta^* : W(L) \rightarrow W(L')$ is the componentwise extension, then:

(a) $(D)\beta^* \subseteq D'$;
(b) we have a commutative diagram

$$
\begin{array}{ccc}
D & \xrightarrow{\Pi} & L \\
\downarrow^{\beta^*} & & \downarrow^{\beta} \\
D' & \xrightarrow{\Pi'} & L'.
\end{array}
$$

We therefore obtain a category $\text{Part}$ whose objects are partial groups and whose morphisms are morphisms of partial groups according to the above definition, clearly with the usual composition of maps.

It is trivial to check that $\beta$ is an isomorphism (in the categorical sense) if and only if $\beta$ is a bijective morphism of partial groups satisfying $(D)\beta^* = D'$.

As partial groups have a preferential object, i.e. the unit, we will be working with pointed sets instead of sets; this will simplify the construction of free partial groups.

Denoting by $\text{Set}^*$ the category of pointed sets, note that we clearly have a forgetful functor
$U : \text{Part} \to \text{Set}^*$. We now show that we also have a free-construction functor $F : \text{Set}^* \to \text{Part}$ such that $F \dashv U$. A similar construction can also be realized over the category $\text{Set}$ in place of $\text{Set}^*$, just as for groups. It is very similar to the one we are providing, so it will become clear what changes are need for working in the category $\text{Set}$.

Suppose we have a pointed set $(X,1)$ (the base-point will become the unit of the partial group). Consider the set $Y := \{1\} \sqcup X^* \sqcup \hat{X}^*$, where $X^* = X \setminus \{1\}$, $\sqcup$ denotes the disjoint union and we use $\hat{X}^*$ to distinguish the two copies of $X^*$ (the same notation will also be adopted for the respective elements).

Define an involutory bijection $i$ on $Y$ by (1)

\[
(1) \quad i = 1 \quad \text{and} \quad (x)i = \hat{x}.
\]

Set $D \subseteq W(L)$ to be the set of words obtained from alternating finite strings of the form $(\ldots, x, \hat{x}, x, \hat{x}, \ldots)$, for $x \in X^*$, by adding in any position any finite number of copies of the element 1. We say that such strings are built on the element $x$.

For $w \in D$, built on the element $x \in X^*$, define

\[
(w)\Pi = \begin{cases} 
  x & \text{if the number of } x \text{ is greater than that of } \hat{x}, \\
  1 & \text{if the number of } x \text{ equals that of } \hat{x}, \\
  \hat{x} & \text{if the number of } x \text{ is lower than that of } \hat{x}.
\end{cases}
\]

It is trivial to check that $(Y, D, \Pi, i)$ is a partial group, so we define it as $(X)F$. Moreover if $f : (X,1_X) \to (Y,1_Y)$ is a morphism of pointed sets and $M = XF$ and $N = YF$, then we associate to $f$ the map $fF : M \to N$ defined by setting $(1_X)fF = 1_Y$, $(x)fF = (x)f$ and $(\hat{x})fF = (\hat{x})f$. It is straightforward to check that $fF$ is a partial groups morphism and that the defined associations make $F$ a functor.

$XF$ is the free partial group over the pointed set $(X,1)$; indeed consider the inclusion map $j : (X,1) \to XF$ sending $1 \mapsto 1$ and $x \mapsto x$ and a set map $f : X \to L$ to a partial group $L$ sending 1 in the unit of $L$ (i.e. $f$ is a pointed-set map $(X,1) \to LU$, where $U$ is the forgetful functor). In the following diagram

\[
\begin{array}{ccc}
(X,1) & \xrightarrow{j} & XF \\
\downarrow{f} & & \downarrow{\hat{f}} \\
L & \xrightarrow{\hat{x}} & L
\end{array}
\]

there clearly exists a unique morphism $\hat{f}$ of partial groups which makes the diagram commute, defined by $\hat{f} : \hat{x} \mapsto ((x)f)^{-1}$, where the exponent $-1$ denotes the inverse in $L$. This is true because the domain $D$ of $XF$ is the smallest possible, so universal initial with respect to morphisms of partial groups.

An immediate consequence is that

\[
\text{Hom}_{\text{Part}}(XF, L) \cong \text{Hom}_{\text{Set}^*}((X,1), LU)
\]

affording the following lemma.

**Lemma 1.1.** With the notation above, we have the adjunction $F \dashv U$.

**Proof.** The bijections are those defining the adjunction. Naturality is a straightforward computation. \qed

However note that neither $UF$ nor $FU$ are the identity functor; as a consequence even though $F$ preserves colimits (so we can push colimits in $\text{Set}^*$ to colimits in $\text{Part}$), we cannot use this fact to prove that $\text{Part}$ is cocomplete as there is a change of objects.
2. Colimits in Part

2.1. A Set Theory result. Before discussing about colimits in Part, we shall study in detail colimits in the category $\text{Set}^*$. Recall that given a relation $R$ on a set $X$ there exists a unique smallest equivalence relation on $X$ containing $R$, exactly the intersection of equivalence relations on $X$ containing $R$. We will need the following version of the well known factorization theorem of set-functions via equivalence relations.

**Proposition 2.1.** Let $f : X \to Y$ be a function between sets and $\sim'$ a binary relation on $X$, i.e. $\sim' \subseteq X \times X$. Let $\sim$ be the smallest equivalence relation on $X$ containing $\sim'$. Then $f$ factors through $X/\sim$, as in the commutative diagram below, if and only if $(u)f = (v)f$ for every $u \sim' v$. 

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\sim \downarrow & \searrow^{\exists ! f} & \\
X & \sim
\end{array}
$$

**Proof.** First of all note that trivially $(u)f = (u)f$ and that $(u)f = (v)f$ is a symmetric relation; hence closing $\sim'$ with respect to reflexivity and symmetry does not affect our statement. Thus from now on we consider $\sim'$ to be a reflexive and symmetric relation.

Since $\sim' \subseteq \sim$, the only if part is trivial. As for the if part, we build a graph in order to properly keep trace of the relations $\sim'$ and $\sim$. Let $G = (X, E)$ be the graph with set of vertices $X$ and, for any pair of vertices $u, v \in X$, $\{u, v\} \in E$ if and only if $u \sim' v$ (well defined by symmetry of $\sim'$).

The fundamental fact to note is that the equivalence classes of $X/\sim$ are exactly the connected components of $G$. We therefore want to prove that every connected component of $G$ has the property that $f$ is constant on its vertices, so consider a connected component $C$ of $G$.

If $X$ is finite, a simple induction on the diameter of $C$ proves the result; however this does not work when $X$ is infinite. To deal with this case we use Zorn’s lemma. Let $S_C$ be the set of (nonempty) connected subgraphs of $C$ satisfying the property that $f$ is constant on the vertices. Clearly $S_C \neq \emptyset$ as it contains the singletons made of one vertex; moreover we can order $S_C$ via inclusion of the sets of vertices and edges of the connected subgraphs, obtaining in such a way a partially ordered set. Given a chain in $S_C$, clearly the subgraph of $C$ afforded by considering the union of all the elements of the chain is still an element of $S_C$ and a maximal element of the chain. Hence we can apply Zorn’s lemma and conclude that $S_C$ admits maximal elements.

Now, as by hypothesis $(u)f = (v)f$ whenever $u \sim' v$ and as $C$ is a connected component of $G$ whose edges are exactly defined at $\sim'$, then any two maximal elements of $S_C$ are connected by a path in $G$ over which $f$ is constant on vertices. By maximality, the only possibility is that there is a unique maximal element, exactly $C$. Thus $f$ is constant on connected components of $G$. 

Turning our attention to colimits, let $\mathcal{C}$ be a small category and $F : \mathcal{C} \to \text{Set}^*$ be a functor. For every $X \in \text{Ob}(\mathcal{C})$ denote by $(X, x)$ its image under $F$; similarly for every $j \in \text{Mor}_C(X, Y)$ denote by $jF$ the image of the arrow $j$ via $F$. Then we claim that, paired with the obvious maps,

$$colim(F) = \left( \bigcup_{X \in \text{Ob}(\mathcal{C})} (X, x), \sim, [x]_\sim \right) \quad (\star)$$

where $\sim$ is the smallest equivalence relation on $\bigcup_{X \in \text{Ob}(\mathcal{C})} (X, x)$ containing all the pairs $(x, y)$ for all images $(X, x), (Y, y)$ and all the pairs $(z, (z)jF)$ for $z \in (X, x)$ and $j \in \text{Mor}_C(X, Y)$.

Indeed consider the following diagram:
the arrow $\psi$ is defined by $[z] \sim \psi \rightarrow t_X(z)$ for $z \in (X, x)$. Computations verifying that $\psi$ is a well defined morphism of pointed sets and that it is unique are now simple.

2.2. Factorization of the multivariable product. One might be tempted to prove cocompleteness of Part through colimits of Set* in the following way. If $\mathcal{C}$ is a small category and we have a functor $T : \mathcal{C} \rightarrow \text{Part}$, denote, for each morphism $x : a \rightarrow b$ in $\mathcal{C}$, the image under $T$ by $\lambda_x : aT \rightarrow bT$.

Now consider the forgetful functor $U : \text{Part} \rightarrow \text{Set}^*$ and take $L = \text{colim}((\mathcal{C})TU)$ the colimit in the category of pointed sets. Then

$$\text{colim}((\mathcal{C})TU) = \bigsqcup_{a \in \text{Ob}(\mathcal{C})}(aTU) \equiv$$

where $\equiv$ is the smallest equivalence relation containing all pairs $(1_{aT}, 1_{bT})$ and $(u, (u)\lambda_x)$ for $x : a \rightarrow b$ morphism in $\mathcal{C}$.

The domain $D \in W(\mathcal{L})$ is defined as the set of words $[w] = ([w_1], \ldots, [w_n])$ for which there exist some $a \in \text{Ob}(\mathcal{C})$ such that $w_i \in aT$ for all $i \in \{1, \ldots, n\}$ and $(w_1, \ldots, w_n) \in D(aT)$. Clearly $D$ satisfies condition (1) of Definition 1.

We would like to define a product $\Pi$ by lifting to products $\Pi_a$ of the partial groups $aT$ for suitable $a \in \text{Ob}(\mathcal{C})$. Then one might consider the following diagram

$$\bigsqcup_{a \in \text{Ob}(\mathcal{C})} D(aT) \xrightarrow{\bigsqcup \Pi_a} \bigsqcup_{a \in \text{Ob}(\mathcal{C})} aT \xrightarrow{\rho} \mathcal{L}$$

with the hope that, if $\rho$ is the projection over the quotient and $\rho^*$ is the componentwise application of $\rho$, then the composition $(\sqcup \Pi_a)\rho$ factors through $\rho^*$.

This argument would at least be complicated to prove and we will actually show it wrong. Indeed suppose we have $u \in D(aT)$ and $v \in D(bT)$ such that $u\rho^* = v\rho^*$. Then $u$ and $v$ have equal length and for all the components we have $[u_i] = [v_i]$. Now we need $[(u)\Pi_a] = [(v)\Pi_b]$, however since there is no guarantee that the components $u_i$ and $v_i$ are identified through a unique morphism of partial groups $aT \rightarrow bT$, we cannot use property (b) of Definition 2 to prove our factorization.

We will provide a counterexample to the factorization above; to build a not too complicated counterexample we will need to split the formation of colimits via the following result about categories.

**Proposition 2.2.** Consider a locally small category $\mathcal{C}$; then every (co)limit is the (co)equalizer of a (co)product. In particular $\mathcal{C}$ is (finitely) (co)complete if and only if $\mathcal{C}$ admits all (finite) (co)products and all (co)equalizers.

**Proof.** See [7, Chapter V, section 2], where the author proves the statement for limits; the statement for colimits is obtained by duality. \qed
We start by showing that $Part$ has coproducts; however the situation with coequalizers is more complex. Indeed, as partial groups are a structure in between sets and groups, we shall see that coproducts in $Part$ behave very similarly to those in $Set$ (and actually are given by those in $Set^*$), whereas coequalizers have a behavior more similar to that of groups.

We will analyze coequalizers in order to provide a counterexample to the factorization above.

**Lemma 2.3.** The category $Part$ has all coproducts.

**Proof.** Consider a family $\{L_a\}_{a \in I}$ of partial groups indexed on $I$. In this case, as there are no morphisms, the coproduct taken in $Set^*$, that is $(\star)$, actually works as a coproduct in $Part$.

If $u \in D(L_a)$ and $v \in D(L_b)$ are such that $(u)\rho^* = (v)\rho^*$, then either all the components of $u$ and $v$ are the unit element or $L_a = L_b$ and $u = v$. Hence we can surely factor $(\bigcup \Pi)\rho$ through $\rho^*$ obtaining a product $\Pi : D(L) \to L$, with $L$ the pointed-setwise colimit. The inversion map $i$ on $L$ is given exactly by the inversion maps on all the $L_a$ and clearly factors to $L$.

It is a simple exercise to prove that $(L, D, \Pi, i)$ is a partial group. We clearly have inclusion morphisms $j_a : L_a \to L$ given by $(x)j_a = [x] = x$ and are all, trivially, morphisms of partial groups. Then $(L, j_a)$ is the coproduct; indeed consider the following diagram

\[
\begin{array}{ccc}
L_a & \xrightarrow{f_a} & L \\
\downarrow{\wr} & \searrow{j_a} & \nwarrow{\wr} \\
L & \xrightarrow{\exists!\psi} & M \\
\downarrow{\wr} & \searrow{j_b} & \nwarrow{\wr} \\
L_b & \xrightarrow{f_b} & L
\end{array}
\]

Clearly we can define $\psi$ by $([x])\psi = (x)f_a$ for $x \in L_a$ and $\psi$ is a well defined morphism of partial groups (note that we know $D(L_a)f_a \subseteq D(M)$, so that also $D(L) \subseteq D(M)$). Moreover, it is trivially unique such that all the triangles commute, so we have the universal property of the coproduct. \qed

**Remark.** Note, in particular, that coproducts of a finite number of finite partial groups are again finite; this will translate, after proving existence of coequalizers, in the fact that the full subcategory of $Part$ of finite partial groups is finitely cocomplete. This is probably among the major differences between the categories $Part$ and $Grp$.

We now begin our analysis of coequalizers. Consider two morphisms of partial groups $f, g : L \to L'$. The setwise coequalizer is given by

$$C = \text{coeq}_{Set}(f, g) = \frac{L'}{\sim}$$

paired with the trivial quotient map $q : L' \to (L' / \sim)$, where $\sim$ is the smallest equivalence relation on $L'$ containing the pairs $(xf, xg)$ for $x \in L$. This is a simplified realization of the same object constructed in $[\star]$; indeed according to it we would have to consider

$$C' = \frac{L \sqcup L'}{R}$$

with $R$ the smallest equivalence relation containing all pairs $(xf, xg)$ for $x \in L$. In particular $R$ identifies all $xf$ with $xg$; it is a simple exercise to prove that there is a bijection

$$\frac{L'}{\sim} \cong \frac{L \sqcup L'}{R}$$

which is the identity over representatives in $L'$.

Let’s try to define a partial group structure on $C = \frac{L'}{\sim}$. Let $q : L' \to C$ be the canonical projection, $D = D(C)$ be the set of words $[w]$ which are the pointwise projection of a word $w \in D(L')$ and
$q^* : D(\mathcal{L}') \rightarrow D(\mathcal{C})$ be the pointwise projection induced by $q$, so that $[w] = (w)q^*$; define $\pi : D \rightarrow \mathcal{C}$ by $[w]\pi = (w)\Pi q$ and the inversion $l$ on $\mathcal{C}$ by $[w]l = (w)iq^*$, where $\Pi$ is the product and $i$ the inversion of $\mathcal{L}'$.

That $l$ is well defined is an immediate consequence of the fact that $f$ and $g$ are partial group morphisms, so they send inverses to inverses. For $\pi$ to be well defined, observe that it must make the following diagram commutative

$$
\begin{array}{ccc}
D(\mathcal{L}') & \xrightarrow{q^*} & D(\mathcal{C}) \\
\downarrow \Pi & & \downarrow \pi \\
L' & \xrightarrow{q} & L' \\
\end{array}
$$

Moreover, commutativity of such diagram is a necessary condition since $q$ has to be a morphism of partial groups, i.e. $\Pi q$ must factor via the map $q^*$ to a map $\pi$. Thus we need to prove $uq^* = vq^* \Rightarrow u\Pi \sim v\Pi$, where $u, v \in D(\mathcal{L}')$.

The relation $\sim$ is generated by the weaker relation $\sim'$ made exactly of the pairs $(xf, xg)$ for $x \in \mathcal{L}$. If $P$ denotes the product in $\mathcal{L}$, being $fg = gg$ we get

$$(xf)\Pi q = (xP)fq = (xP)gq = (xg)\Pi q,$$

that is $(xf)\Pi \sim (xg)\Pi$. Combining with Proposition 2.1 we obtain that $\pi$ is well defined on words of length 1.

Problems arise when considering words of length greater than 1; if we can write $u = xf^* = (x_1 f, \ldots, x_n f)$ and $v = xg^* = (x_1 g, \ldots, x_n g)$ for $x \in W(\mathcal{L})$, then we clearly get

$$u\Pi = (x)f^*\Pi = xPf \sim xPg = (x)g^*\Pi = v\Pi.$$ 

However if $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are words in $D(\mathcal{L}')$ such that $uq^* = vq^*$, we only know that $u_i \sim v_i$ for each $1 \leq i \leq n$. So even if we may be able to represent $uq^*$ through an element of the form $\hat{u} = (x_i h_i)$, where $h_i = f$ for some $i$ and $h_j = g$ for $j \neq i$, there is no guarantee that $u\Pi \sim \hat{u}\Pi$.

To better understand the obstruction to our factorization, suppose $n = 2$, so $u = (u_1, u_2)$ and $v = (v_1, v_2)$, with $u_1 = xf, v_1 = xg, u_2 = yf$ and $v_2 = yg$. Clearly the pairs $(xf, yf), (xf, yg), (xg, yg), (xg, yf)$ are all representatives of the same word in $D(\mathcal{L}'/\sim)$; we have already observed that $(xf, yf)\Pi \sim (xg, yg)\Pi$, however there is no evident reason, for example, for $(xf, yf)\Pi \sim (xf, yg)\Pi$. We are now ready for our counterexample.

**Counterexample 2.4.** Consider the pointed set $(A, 1)$ with $A = \{1, a, b\}$ and let $\mathcal{L}$ be the free partial group on $(A, 1)$; recall that $\mathcal{L} = \{1, a, \hat{a}, b, \hat{b}\}$. Consider the Klein group $M = \mathbb{Z} / 2\mathbb{Z} \times \mathbb{Z} / 2\mathbb{Z}$ ($(M, 1)$ as a pointed-set) with generators $(x, 1) = x$ and $(1, y) = y$ and define the pointed-set functions

$$
\begin{array}{l}
\tilde{f} : (A, 1) \longrightarrow M \\
a \longmapsto x \\
b \longmapsto y
\end{array}
\quad
\begin{array}{l}
\tilde{g} : (A, 1) \longrightarrow M \\
a \longmapsto xy \\
b \longmapsto x
\end{array}
$$

Then we get induced morphisms of partial groups $f, g : \mathcal{L} \longrightarrow M$ making the diagram below commutative

$$
\begin{array}{ccc}
(A, 1) & \xrightarrow{\tilde{f}} & M \\
\downarrow \tilde{g} & & \downarrow \\
(A, 1) & \xrightarrow{f} & M \\
\end{array}
$$
We can therefore consider the coequalizer of morphisms \( f \) and \( g \)
\[
\begin{array}{c}
\mathcal{L} \\
\downarrow f \\
M \\
\downarrow g \\
\end{array}
\]

with \((M/ \sim, q) = \text{coeq}_{\text{Set}}(f, g)\). If \( \Pi \) is the multivariable product on \( M \), we want to show that \( \Pi q \) does not factorize via \( q^* \) to a product of \( M/ \sim \).

First of all note that

\[
x = af \sim ag = xy;
\]

\[
x = bg \sim bf = y;
\]

Since \( af = \hat{a}f, bf = \hat{b}f \) and similarly for \( g \), then \( \sim \) identifies exactly the non-identity elements of \( M \), leaving us with \( \frac{M}{\sim} \cong \mathbb{Z}/2\mathbb{Z} \), which has a unique structure of partial group coinciding with that of the group.

As words \((af, bf)\) and \((af, bg)\) have equal image under \( q^* \), they represent the same word on \( M/ \sim \). However

\[
(af, bf)\Pi = (x, y)\Pi = xy,
\]

\[
(af, bg)\Pi, = (x, x)\Pi = x^2 = 1,
\]

but \( 1 \not\sim xy \) in \( M/ \sim \).

2.3. The category \( \text{Part} \) is cocomplete. We can think at the previous counterexample by analogy with the situation in the category of groups and group homomorphism.

When building coequalizers in groups, we need to factor out more than just the relations given by the morphisms in the diagram (namely \( f \) and \( g \)), i.e. more than the relation \( \sim \): we need to factor out the smallest normal subgroup containing those relations. Similarly, when considering a coequalizer in \( \text{Part} \), we need to factor out through a relation in such a way that the multivariable product also factors through the quotient; however, with partial groups we do not have a substructure controlling this factorization process, so we will need to identify the proper equivalence relation in a different way.

Let’s consider the situation where we have an equivalence relation \( \sim \) on a partial group \( \mathcal{L}' \); for \( q : \mathcal{L}' \rightarrow \frac{\mathcal{L}'}{\sim} \) and \( q^* : D(\mathcal{L}') \rightarrow D\left(\frac{\mathcal{L}'}{\sim}\right) \) the induced componentwise application of \( q \), the fibers of \( q^* \) induce an equivalence relation \( \equiv \) on \( D(\mathcal{L}') \). Moreover, clearly \( q^* \) is surjective being \( q \) surjective.

For \( u = (u_i)_i \) and \( v = (v_i)_i \), the relation \( \equiv \) is given by

\[
u \equiv v \iff u, v \in (u)(q^*)^{-1} \iff u_i \sim v_i \forall i,
\]

in particular \( u \) and \( v \) have equal length. Hence we get

\[
\begin{array}{c}
D(\mathcal{L}') \xrightarrow{q^*} D\left(\frac{\mathcal{L}'}{\sim}\right) = D(\frac{\mathcal{L}'}{\equiv}) \\
\downarrow \Pi \downarrow \downarrow \\
\mathcal{L}' \xrightarrow{q} \frac{\mathcal{L}'}{\sim}
\end{array}
\]

The map \( \Pi q \) is surjective as well and we can consider the equivalence relation \( \approx \) on \( D(\mathcal{L}') \) afforded by fibers of \( \Pi q \), so that \( \frac{\mathcal{L}'}{\sim} \cong \frac{D(\mathcal{L}')}{\approx} \). The square diagram above then becomes
so that $\Pi q$ factors through $q^*$ if and only if $\equiv \subseteq \approx$.

We are then led to search for an equivalence relation $\sim$ on $\mathcal{L}'$ such that the induced relations $\equiv$ and $\approx$ satisfy $\equiv \subseteq \approx$; we would then consider such a smallest equivalence relation as a candidate for our colimit. Let’s also describe $\approx$ in terms of $\sim$;

\[ u \sim v \iff (u)\Pi q = (v)\Pi q \iff u \Pi \sim v \Pi. \]

Combining (1) and (2), we need to find $\sim$ such that, for $u = (u_i), v = (v_i) \in D(\mathcal{L}')$:

\[ u_i \sim v_i \forall i \iff u \Pi \sim v \Pi. \tag{\*} \]

Let now $\sim_0$ be the equivalence relation on $\mathcal{L}'$ generated by pairs of the form $(xf, xg)$ for some $x \in \mathcal{L}$; clearly the set

\[ E := \{ \mathcal{R} \subseteq \mathcal{L}' \times \mathcal{L}' | \mathcal{R} \text{ is an eq. rel. containing } \sim_0 \text{ and satisfying } [\mathcal{R}] \} \]

is not empty, since it contains $\mathcal{L}' \times \mathcal{L}'$. Since inclusion of $\sim_0$ and property $[\mathcal{R}]$ are clearly preserved by intersections, there exists the smallest equivalence relation we are searching for, namely

\[ \mathcal{R} = \bigcap_{\mathcal{R} \in E} \mathcal{R}. \]

**Lemma 2.5.** The category $\text{Part}$ has all coequalizers.

**Proof.** Consider morphisms between partial groups $f, g : \mathcal{L} \to \mathcal{L}'$; build the setwise coequalizer $\mathcal{L}'_{/\mathcal{R}}$ and the relation $\mathcal{R}$ (which depends on $\sim$) on $\mathcal{L}'$ as above. Consider the projection $t : \mathcal{L}' \to \mathcal{L}'_{/\mathcal{R}}$ and the diagram

\[ \mathcal{L} \xrightarrow{f} \mathcal{L}' \xrightarrow{t} \mathcal{L}'_{/\mathcal{R}} \]

Then we prove that:

1. $\mathcal{L}'$ induces a structure of partial group on $\mathcal{L}'_{/\mathcal{R}}$;
2. $t$ is a morphism of partial groups and $ft = gt$;
3. \[ \left( \frac{\mathcal{L}'_{/\mathcal{R}}}{t}, t \right) = \text{coeq}_{\text{Part}}(f, g). \]

Define $D := D(\mathcal{L}'_{/\mathcal{R}})$ as the set of images of elements of $D(\mathcal{L}')$ via the map $t^* : W(\mathcal{L}') \to W(\mathcal{L}'_{/\mathcal{R}})$ induced by $t$ in the usual way. By construction $R$ is the smallest equivalence relation on $\mathcal{L}'$ such that $\Pi t$ factors as in the following commutative diagram:

\[ \begin{tikzcd}
D(\mathcal{L}') \ar{r}{t^*} & D\left( \frac{\mathcal{L}'_{/\mathcal{R}}}{t} \right) \\
\mathcal{L}' \ar[-, very near start, shift left=0.75]{u}{\Pi} & \frac{\mathcal{L}'_{/\mathcal{R}}}{t} \ar[d, two heads, shift left=1.5] \ar[lu, shift right=0.5, shift left=0.5, near start, shift left=0.5, near start, shift left=0.5, near start, shift left=0.5] \ar[u, two heads, shift right=0.5, shift left=0.5, near start, shift left=0.5, near start, shift left=0.5, near start, shift left=0.5] \end{tikzcd} \]

We take $\pi$ as the product map.

If $i$ is the inversion map on $\mathcal{L}'$, define an inversion map $j$ by \([x]_\mathcal{R}j := [x]_i|_\mathcal{R}$. Since $\mathcal{R}$ is by definition the smallest equivalence relation containing $\sim$ and such that $[\mathcal{R}]$ holds, if $x\mathcal{R}y$, then we can reduce either to $x = zf$ and $y = zg$ for some $z \in \mathcal{L}$ or $x = (u_k)_k \Pi$ and $y = (v_k)_k \Pi$ such that $u_k \sim v_k$ for every $k$. In the first case we get $(x)i\mathcal{R}(y)j$ since $f$ and $g$ are morphisms of partial groups; in the other case it is a consequence of the fact that $((u_k)_k \Pi)i = ((u_k)_k j)\Pi$, that inverting $(u_k)$ and $(v_k)$ consists in considering inverses
in opposite order and that \((u_k)i \sim (v_k)i\). This gives us that \(j\) is well defined.

Properties (1) to (4) of Definition 1.1 are easily verified, so that \((\mathcal{E}, D, \pi, j)\) is a partial group.

(2) That \(t\) satisfies conditions (a) and (b) of Definition 1.2 is a trivial consequence of the definitions of \(D\) and \(t\). As \(\sim \subseteq R\), we also have \(ft = gt\).

(3) If \(\tau : \mathcal{L}' \rightarrow M\) is a partial group morphism such that \(f\tau = g\tau\), define \(\psi\) as in the diagram below by:

\[
\psi : [x]_R \mapsto (x)\tau
\]

\[
\begin{array}{ccc}
\mathcal{L} & \xrightarrow{f} & \mathcal{L}' \\
\downarrow{g} & & \downarrow{t} \\
\mathcal{M} & \xleftarrow{\psi} & \mathcal{E}'
\end{array}
\]

If \(\psi\) is well defined, then it is clearly unique such that the diagram above commutes.

Now let \(R_\tau\) be the equivalence relation induced by the fibers of \(\tau\) on \(\mathcal{L}'\); it induces equivalence relations \(\equiv_\tau\) and \(\approx_\tau\) on \(D(\mathcal{L}')\) via fibers of, respectively, \(\tau^*\) and \(\Pi\tau\). As \(\tau\) is a morphism of partial groups, it satisfies condition (b) of Definition 1.2, so \(R_\tau\) fulfills \(\cdot\).

Then \(R \subseteq R_\tau\), which means exactly that \(\psi\) is well defined.

We obtain

**Theorem 2.6.** The category \(\text{Part}\) is cocomplete.

**Proof.** Combine Theorems 2.3, 2.5 and Proposition 2.2. \(\square\)

3. Limits in \(\text{Part}\)

We now deal with limits; the situation is definitely simpler. For example, both the categories \(\text{Set}\) and \(\text{Grp}\) have the same object (up to the forgetful functor \(U : \text{Grp} \rightarrow \text{Set}\)) as a product.

Indeed if the \(G_i\) are groups, for \(i \in I\) and \(I\) a set of indices, then

\[
\text{lim}_{\text{Set}}(G_iU) = \prod_{i \in I} (G_iU) = \left(\prod_{i \in I} G_i\right)U = (\text{lim}_{\text{Grp}}G_i)U.
\]

We will also see that they share equalizers as well, therefore limits of partial groups will have no space to move.

Indeed, this rigidity (if we think at groups again) is due to the fact that there are a free construction functor \(F : \text{Set} \rightarrow \text{Grp}\) and a forgetful functor \(U : \text{Grp} \rightarrow \text{Set}\) such that \(F \dashv U\). As \(U\) is a right adjoint, it preserves limits; hence a limit of a diagram in \(\text{Grp}\) is pushed to a limit in \(\text{Set}\) over the diagram obtained by considering the involved groups just as sets. Thus the only possible candidate for a limit in \(\text{Grp}\) is the setwise limit endowed with a suitable group structure. Similarly, with partial groups we have seen that we have a free construction functor \(F : \text{Set}^* \rightarrow \text{Part}\) and a forgetful functor \(U : \text{Part} \rightarrow \text{Set}^*\), again with \(F \dashv U\). Hence, again, a limit in \(\text{Part}\) must have as underlying set the pointed setwise limit.

We now proceed with the proof of existence of limits in \(\text{Part}\).

**Theorem 3.1.** The category \(\text{Part}\) is complete.

**Proof.** Again we prove that \(\text{Part}\) has all products and equalizers.
Step 1. If \( \{ \mathcal{L}_a \}_{a \in I} \) is a family of partial groups, consider the setwise cartesian product \( \mathcal{L} = \prod \mathcal{L}_a \); the projections \( p_a : \mathcal{L} \to \mathcal{L}_a \) extend in the usual way to \( p^*_a : W(\mathcal{L}) \to W(\mathcal{L}_a) \). Define \( D(\mathcal{L}) = D \) as

\[
D := \{ w = ((w_a^1), \ldots, (w_a^n)) \in W(\mathcal{L}) \mid \forall a \in I, wp_a^* = (w_a^1, \ldots, w_a^n) \in D(\mathcal{L}_a) \}.
\]

and a product \( \Pi \) on \( \mathcal{L} \) by \( (w) \Pi := ((wp_a^*) \Pi) \). The inversion map \( i \) on \( \mathcal{L} \) is trivially defined componentwise; it is a straightforward exercise to check that \( (\mathcal{L}, D, \Pi, i) \) is a partial group. Considering the diagram below

\[
\begin{array}{ccc}
\mathcal{L}_a & \xrightarrow{t_a} & \mathcal{L} & \xleftarrow{t_b} & \mathcal{L}_b \\
p_a \downarrow & & \downarrow & & \downarrow \\
& \mathcal{L} & \xleftarrow{i} & \mathcal{M} & \xrightarrow{t} \\
p_b \uparrow & & \uparrow & & \uparrow \\
& \mathcal{L}_b & & \mathcal{L}_a & \\
\end{array}
\]

\( t \) is defined by \( t : m \mapsto (mt_a) \). Notice that, for \( w = (m_1, \ldots, m_n) \in D(\mathcal{M}) \), we have \( wt_a^* = (m_1t_a, \ldots, m_nt_a) \in D(\mathcal{L}_a) \) for all \( a \in I \), so \( wt \in D \). Clearly \( t \) is a morphism of partial groups, unique satisfying the universal property of products.

Step 2. Consider morphisms of partial groups \( f, g : \mathcal{L} \to \mathcal{L}' \); define

\[
\mathcal{E} := \{ x \in \mathcal{L} \mid xf = xg \}.
\]

We have

\[
\mathcal{E} \xrightarrow{j} \mathcal{L} \xrightarrow{f} \mathcal{L}' \xrightarrow{g} \mathcal{L};
\]

it is a simple exercise to prove that \( \mathcal{E} \) is a partial group, \( j \) a partial groups morphism and that the pair \( (\mathcal{E}, j) = eq(f, g) \).

As a consequence of Proposition 2.2, we obtain that \( \text{Part} \) is complete. In addition, it is now a trivial observation the fact that the full subcategory of \( \text{Part} \) whose objects are the finite partial groups is finitely complete.

Combining Theorems 2.6 and 3.1 and the remark after Lemma 2.3, we obtain precisely Theorem A.

4. Partial groups as simplicial sets

The symbol \( sSet \) will denote the category of simplicial sets, with the usual morphisms given by natural transformations between functors. The idea of viewing partial groups as simplicial sets was first conjectured by C. Broto and formalized in joint work with A. González. We will provide a quick introduction to simplicial sets and describe their relation with partial groups; details of the construction can be found in [6]. As it turns out that \( \text{Part} \) is equivalent to a subcategory of \( sSet \), we will show that such subcategory is not closed under formation of colimits in \( sSet \); moreover controlling colimits in the simplicial sets setting might be more challenging than the computations made in section 2.

Presheaves and simplicial sets. Let \( \mathcal{C} \) and \( \mathcal{A} \) be locally small categories; an \( \mathcal{A} \)-valued presheaf is a functor

\[
X : \mathcal{C}^{\text{op}} \longrightarrow \mathcal{A}.
\]

We will be only interested in \( \text{Set} \)-valued presheaves, so from now on we will consider \( \mathcal{A} = \text{Set} \). Let now \( \mathcal{C} = \Delta \) be the category of finite partially ordered sets with monotone, non decreasing...
functions as morphisms. It is possible to show that $\Delta$ is equivalent to the category whose objects are the finite ordered sequences $[n] = \{0 < 1 < \cdots < n\}$ together with non-decreasing functions; this is known as the skeletal subcategory, we will denote it again by $\Delta$ and from now on we will consider this.

**Definition 4.1.** A simplicial set is a presheaf 

$$X : \Delta^{\text{op}} \longrightarrow \text{Set};$$

hence we may think at a simplicial set as a sequence of sets $X_n := [n]X$ with morphisms induced by the monotone functions in $\Delta$, taken with opposite direction.

It is possible to prove that simplicial sets (and more generally simplicial objects) are characterized as sequences of sets $X_0, X_1, \ldots$ paired with two classes of maps (face operators and degeneracy operators) with certain properties. We will not see this in detail, however an exhaustive treaty of the topic may be found in [10, Chapter 8 and Proposition 8.1.3]; we will only need the maps described in the following lines.

In the category $\Delta$, for $n \geq r, s$, we have the following important morphisms.

1. $r$-front-face maps, i.e. maps $f_{r,n} : [r] \longrightarrow [n]$ defined by $(i)f_{r,n} = i$.
2. $s$-back-face maps, i.e. maps $b_{s,n} : [s] \longrightarrow [n]$ defined by $(i)b_{s,n} = n - s + i$.

Let $F_{r,n} = (f_{r,n})X$ and $B_{s,n} = (b_{s,n})X$, for $r$ and $s$ fixed. The collection of maps $F_r = \{F_{r,n} \mid n \geq r\}$ is called the $r$-front-face operator; similarly the collection $B_s = \{B_{s,n} \mid n \geq s\}$ is called the $s$-back-face operator.

By a reiterated application of front and back face operators, one defines the enumerating operator $E$ as the collection of maps $\{E_n\}$ for $n \geq 1$, where 

$$E_n : X_n \xrightarrow{F_{n-1} \times B_1} X_{n-1} \times X_1 \xrightarrow{F_{n-2} \times B_1 \times \text{id}} \ldots \xrightarrow{F_1 \times B_1 \times \text{id}} X_1 \times \cdots \times X_1.$$ 

**Definition 4.2.** Let $X$ be a simplicial set and $E$ be the enumerating operator:

- $X$ is a $N$-simplicial set if it satisfies the property

  $$E \text{ is such that } E_n \text{ is injective for every } n \geq 1. \tag{N}$$

- $X$ is reduced if $X_0 = \{v\}$ is a singleton, i.e. $X$ has a single vertex.

González, in [6, Chapter 4], proceeds with the definition of an inversion in a reduced N-simplex, then proves the following theorem ([6 Theorem 4.8]).

**Theorem 4.1** (A. González). Let $s\text{Part}$ be the full subcategory of the category $s\text{Set}$ of simplicial sets whose objects are the reduced, $N$-simplicial sets with inversion.

Then the category Part of partial groups is equivalent to $s\text{Part}$. 

**Sketch of proof.** We provide here the realization of the equivalence, without proving the details of it.

If $L$ is a partial group, then we associate to it the simplicial set $M$ given by the sequence $\{M_i\}$, where each $M_i$ is the set of words in $D(L)$ of length $i$. In particular $M_0 = \{\emptyset\}$ and $M_1 = L$. The face operators are defined via the product of $L$, the degeneracy operators via insertion of the unit of $L$.

Vice versa, given a reduced N-simplicial set $M$ with inversion, one can prove that the set $M_1$ has a partial group structure. \qed
Limits and colimits of simplicial sets. As simplicial sets are presheaves, we will discuss about limits and colimits of presheaves.

Indeed it is a simple exercise of category theory to show that the limit (or colimit) of a diagram over presheaves is again a presheaf and is built levelwise. That is, if it is given a diagram in the category $[C^{op}, Set]$ of presheaves, then the limit (or colimit) in $[C^{op}, Set]$ is the presheaf obtained by assigning to each object in $C$ the limit (or colimit) taken in $Set$ over the images of the various presheaves.

With regard to simplicial sets, these are objects of the category $[\Delta^{op}, Set]$ by assigning to each object in $\Delta$ the limit (or colimit) taken in $Set$ over the images of the various presheaves.

Indeed it is a simple exercise of category theory to show that the limit (or colimit) of a diagram over presheaves is again a presheaf and is built levelwise. That is, if it is given a diagram in the category $[C^{op}, Set]$ of presheaves, then the limit (or colimit) in $[C^{op}, Set]$ is the presheaf obtained by assigning to each object in $C$ the limit (or colimit) taken in $Set$ over the images of the various presheaves.

With regard to simplicial sets, these are objects of the category $[\Delta^{op}, Set]$; if $D$ is a diagram with the simplicial sets $D_i$ images via $D$, the limit (colimit) of $D$ is provided by assigning to each object $[n]$ the limit (colimit) of the sets $[n]D_i$, where morphisms are naturally induced by $D$.

We now show that the category $sPart$ is not closed under formation of colimits in $sSet$. In detail, we will show that it is not closed under coproducts.

Indeed consider two partial groups $X, Y$ seen as simplicial sets according to Theorem 4.1 write $D(X)$ for the domain of the product of $X$, and similarly for $D(Y)$. In particular, $X$ and $Y$ are both reduced and the equivalence in Theorem 4.1 is realized so that $X_0 = \{0_X\}$, $Y_0 = \{0_Y\}$, $X_1 = X$ and $Y_1 = Y$. Then the coproduct of $X_0$ and $Y_0$ is $X_0 \sqcup Y_0 = \{0_X, 0_Y\}$ and the coproduct of $X_1$ and $Y_1$ is $X_1 \sqcup Y_1$. In particular, $X \sqcup Y$ is not reduced; moreover in the image $X_1 \sqcup Y_1$ of the object $[1]$ the units $1_X$ and $1_Y$, respectively of the partial groups $X$ and $Y$, are not identified.

However, as $sPart$ and $Part$ are equivalent, Theorem 2.6 yields to existence of colimits in $sPart$; thus, it simply means they are not formed as colimits in the entire category $sSet$.

Nonetheless, apart from the problems arising with the units of the partial groups, there is another reason why proving existence of colimits in the simplicial sets setting may be complicated; indeed, a colimit in $sPart$ must be a $N$-simplicial set, so the enumerating operator must consist of injective maps. However, as colimits of sets are formed by quotients over equivalence relations, proving injectivity might be hard without good control of the relations involved.

The situation is similar when the colimit is not a coproduct, that is coequalizing is involved. As shown in section 2, the equivalence relation providing the setwise coequalizer may be too small to also provide a coequalizer in $Part$. However we have seen that, when viewed as simplicial sets, partial groups $X$ and $Y$ are identified by the sets $X_1$ and $Y_1$; as taking the setwise coequalizer of $X_1$ and $Y_1$ is not generally enough, we notice again that the colimit taken in $sSet$ may not be a partial group.

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