Exact Foldy-Wouthuysen transformation of the Dirac-Pauli Hamiltonian by Kutzelnigg’s method

Dah-Wei Chiou¹ and Tsung-Wei Chen²

¹Department of Physics and Center for Condensed Matter Sciences, National Taiwan University, Taipei 10617, Taiwan
²Department of Physics, National Sun Yat-sen University, Kaohsiung 80424, Taiwan

We apply Kutzelnigg’s method for the Foldy-Wouthuysen (FW) transformation upon the Dirac-Pauli Hamiltonian. The exact FW transformations exist and agree with those obtained by Eriksen’s method for two special cases. In the weak-field limit of static and homogeneous electromagnetic fields, the long-held speculation is rigorously proven, by mathematical induction on the orders of $1/c$ in the power series, that the FW transformed Dirac-Pauli Hamiltonian is in full agreement with the classical counterpart, which is the sum of the orbital Hamiltonian for the Lorentz force equation and the spin Hamiltonian for the Thomas-Bargmann-Michel-Telegdi equation.

PACS numbers: 03.65.Pm, 11.10.Ef, 71.70.Ej

---

¹ This article deliberately contains the same introductory and review materials in ².

¹ dwchiou@gmail.com
² twchen@mail.nsysu.edu.tw
CONTENTS

I. Introduction ................................................. 3

II. Classical relativistic spinor .......................... 4

III. Dirac-Pauli spinor ....................................... 6

IV. Foldy-Wouthuysen transformation .................. 2

   A. Kutzelnigg’s method ................................ 9

   B. Special case I ........................................ 11

   C. Special case II ...................................... 12

   D. Weak-field limit .................................... 13

V. Dirac Hamiltonian ........................................ 14

   A. $X_n$ .................................................. 14

   B. $X$ and $X^\dagger$ .................................. 17

   C. $H_{FW}$ .............................................. 19

VI. Dirac-Pauli Hamiltonian ............................... 22

   A. $X'_n$ ............................................... 23

   B. $X'$ and $X''^\dagger$ ................................ 27

   C. $\mathcal{H}_{FW}$ .................................... 28

VII. Summary and discussion .............................. 30

Acknowledgments ............................................. 31

A. Useful formulae and lemmas ......................... 31

References .................................................. 32
I. INTRODUCTION

The relativistic quantum theory for a spin-1\(\frac{1}{2}\) particle is described by the Dirac equation \([1, 2]\), which, in the rigorous sense, is self-consistent only in the context of quantum field theory as particle-antiparticle pairs can be created and annihilated. The question that naturally arises is whether in the low-energy limit the particle and antiparticle can be treated separately without taking into account the field-theory interaction between them on the grounds that the probability of particle-antiparticle pair creation and annihilation is negligible. It turns out that such separation is possible and indeed gives an adequate description of the relativistic quantum dynamics whenever the relevant energy (the particle’s energy interacting with external, e.g. electromagnetic, fields) is much smaller than the Dirac energy gap \(2mc^2\) (\(m\) is the particle’s mass).

The Foldy-Wouthuysen (FW) transformation is the method devised to achieve the particle-antiparticle separation via a series of successive unitary transformations, each of which block-diagonalizes the Dirac Hamiltonian to a certain order of \(1/m\) \([3]\) (see \([4]\) for a review). In the same spirit of the standard FW method, many different approaches have been developed for various advantages \([5–14]\). Particularly, Kutzelnigg proposed a self-consistent equation that allows one to obtain the block-diagonalized Dirac Hamiltonian without explicitly evoking decomposition of even and odd Dirac matrices \([8]\).

Furthermore, to account phenomenologically for any presence of the anomalous magnetic moment, the Dirac equation, augmented with extra terms explicitly dependent on electromagnetic field strength, is extended to the Dirac-Pauli equation to describe the relativistic quantum dynamics of a spin-1\(\frac{1}{2}\) particle of which the gyromagnetic ratio is different from \(q/(mc)\) (\(q\) is the particle’s charge) \([15]\). The FW methods for the Dirac equation can be straightforwardly carried over to the Dirac-Pauli equation without much difficulty \([14]\).

On the other hand, the classical (non-quantum) dynamics for a relativistic point particle endowed with charge and intrinsic spin in electromagnetic fields is well understood. The orbital motion is governed by the Lorentz force equation and the precession of spin by the Thomas-Bargmann-Michel-Telegdi (T-BMT) equation \([16, 17]\) (see Chapter 11 of \([18]\) for a review). The orbital Hamiltonian for the Lorentz force equation plus the spin Hamiltonian for the T-BMT equation provides a low-energy description of the relativistic spinor dynamics. It is natural to conjecture that, in the weak-field limit, the Dirac or, more generically, Dirac-Pauli Hamiltonian, after block diagonalization, corresponds to the sum of the classical orbital and spin Hamiltonians.

The classical-quantum correspondence has been investigated from different aspects with various
degrees of rigor \([19, 22]\) and explicated in \([23]\). In the case of static and homogeneous electromagnetic fields, it has been shown that the FW transformed Dirac-Pauli Hamiltonian is in full agreement with the classical Hamiltonian up to the order of \(1/m^8\), if nonlinear terms of electromagnetic fields are neglected in the weak-field limit \([24]\). Recently, the work of \([24]\) was extended to the order of \(1/m^{14}\) by applying Kutzelnigg’s method \([8]\) with a further simplification scheme \([25]\).

Although the result of \([25]\) is very impressive, a rigorous proof for the full agreement to any arbitrary order is still missing. Thanks to the laborious task in \([25]\) up to the high order of \(1/m^{14}\), it is possible to conjecture the generic expression for terms of any given order in Kutzelnigg’s method and then provide a proof by mathematical induction on the orders of power series expansion. In this paper, we elaborate on Kutzelnigg’s method and present the rigorous proof. As a secondary result, we also show that the exact FW transformations by Kutzelnigg’s method exist and agree with those obtained by Eriksen’s method \([7]\) for two special cases of arbitrary magnetostatic field and arbitrary electrostatic field. Various conceptual issues of the FW transformation are also addressed and clarified.

This paper is organized as follows. After briefly reviewing the classical and Dirac-Pauli spinors in Sec. \(\text{II}\) and Sec. \(\text{III}\) respectively, we look into the FW transformation with the emphasis on Kutzelnigg’s method in Sec. \(\text{IV}\). We then present the proof for the exact classical-quantum correspondence in the weak-field limit for the Dirac Hamiltonian in Sec. \(\text{V}\) and then for the Dirac-Pauli Hamiltonian in Sec. \(\text{VI}\). Conclusions are summarized and discussed in Sec. \(\text{VII}\).

II. CLASSICAL RELATIVISTIC SPINOR

In this section, we briefly review the classical dynamics of a classical relativistic spinor, which is detailed in \([23]\).

For a relativistic point particle endowed with electric charge \(q\) and intrinsic spin \(s\) subject to external electromagnetic fields \(\mathbf{E}\) and \(\mathbf{B}\) (the corresponding 4-potential is denoted as \(A^\mu = (\phi, \mathbf{A})\) and the electromagnetic tensor by \(F_{\mu\nu}\), the orbital motion, which is governed by the Lorentz force equation, and the spin precession, which is governed by the T-BMT equation, are simultaneously described by the total Hamiltonian

\[
H(x, p, s; t) = H_{\text{orbit}}(x, p; t) + H_{\text{spin}}(s, x, p; t) + O(F_{\mu\nu}^2, \hbar^2)
\]

(2.1)

with the orbital Hamiltonian given by

\[
H_{\text{orbit}}(x, p; t) = \sqrt{m^2c^4 + c^2\pi^2} + q\phi(x, t)
\]

(2.2)
and the spin Hamiltonian given by

\[ H_{\text{spin}}(s, x, p; t) = -s \cdot \left[ \left( \gamma_m' + \frac{q}{mc \gamma_\pi} \frac{1}{\gamma_\pi(1 + \gamma_\pi)} \right) B(x) - \gamma_m' \frac{1}{\gamma_\pi(1 + \gamma_\pi)} \left( \frac{\pi}{mc} \cdot B(x) \right) \frac{\pi}{mc} \right. \]

\[ \left. - \left( \gamma_m' + \frac{q}{mc \gamma_\pi(1 + \gamma_\pi)} \left( \frac{\pi}{mc} \times E(x) \right) \right) \right], \]

(2.3)

where the kinematic momentum \( \pi \) is defined as

\[ \pi := p - \frac{q}{c} A(x, t), \]

(2.4)

the Lorentz factor associated with \( \pi \) is defined as

\[ \gamma_\pi := \sqrt{1 + \left( \frac{\pi}{mc} \right)^2}, \]

(2.5)

and \( \gamma_m' \) is the anomalous gyromagnetic ratio

\[ \gamma_m' := \gamma_m - \frac{q}{mc} \]

(2.6)

with \( \gamma_m \) being the total gyromagnetic ratio.

It should be remarked that the classical theory described by (2.1) respects Lorentz invariance only within a high degree of accuracy, unless the terms of \( O(F_{\mu\nu}^2, \hbar^2) \) are appropriately supplemented by a more fundamental quantum theory such as the Dirac-Pauli theory. In the weak-field limit, the nonlinear electromagnetic corrections of \( O(F_{\mu\nu}^2) \) can be neglected, and the particle's velocity is given by

\[ \mathbf{v} \equiv \frac{d\mathbf{x}}{dt} = \nabla_p H_{\text{orbit}} + \nabla_p H_{\text{spin}} \approx \frac{\pi}{m \gamma_\pi} \]

(2.7)

provided

\[ H_{\text{spin}} \ll mc^2, \]

(2.8)

which is true in the weak-field limit. Consequently, \( \pi \) remains to be the kinematic momentum associated with \( \mathbf{v} \), i.e.,

\[ \pi \approx m U \equiv \gamma m \mathbf{v}, \]

(2.9)

and \( \gamma_\pi \) is to be identified with the ordinary Lorentz boost factor, i.e.,

\[ \gamma_\pi \approx \gamma := \frac{1}{\sqrt{1 - \mathbf{v}^2/c^2}}. \]

(2.10)

Furthermore, the Dirac-Pauli theory also gives rise to the Darwin term of \( O(h^2) \), which has no classical (non-quantum) correspondence and does not show up in the case of homogeneous fields.
III. DIRAC-PAULI SPINOR

The relativistic quantum theory of a spin-1/2 particle subject to external electromagnetic fields is described by the Dirac equation

$$\tilde{\gamma}^\mu D_\mu |\psi\rangle + i \frac{mc}{\hbar} |\psi\rangle = 0,$$  \hspace{1cm} (3.1)

where the Dirac bispinor $|\psi\rangle = (\chi, \varphi)^T$ is composed of two 2-component Weyl spinors $\chi$ and $\varphi$, the covariant derivative $D_\mu$ is given by

$$D_\mu := \partial_\mu + i \frac{q}{\hbar c} A_\mu \equiv - i \frac{\hbar}{\hbar} \pi_\mu := - i \frac{\hbar}{\hbar} \left( p_\mu - \frac{q}{c} A_\mu \right) = - i \frac{\hbar}{\hbar} \left( E - q \phi - \left( \mathbf{p} - \frac{q}{c} \mathbf{A} \right) \right)$$  \hspace{1cm} (3.2)

with $p^\mu = (E/c, \mathbf{p})$ being the 4-vector of canonical energy and momentum and $\pi^\mu = (W/c, \pi)$ being the 4-vector of kinematic energy and momentum, and $\tilde{\gamma}^\mu$ are 4 $\times$ 4 matrices that satisfy

$$\tilde{\gamma}^\mu \tilde{\gamma}^\nu + \tilde{\gamma}^\nu \tilde{\gamma}^\mu = 2 g^{\mu\nu}. \hspace{1cm} (3.3)$$

The Dirac equation gives rise to the magnetic moment with $\gamma_m = q/(mc)$ (i.e., the g-factor is given by $g = 2$). To incorporate any anomalous magnetic moment $\mu' = \gamma'_m \hbar/2$, one can modify the Dirac equation to the Dirac-Pauli equation with augmentation of explicit dependence on field strength $\mathbf{A}$$\hspace{1cm} [14, 15]$:

$$\tilde{\gamma}^\mu D_\mu |\psi\rangle + i \frac{mc}{\hbar} |\psi\rangle + \frac{i \mu'}{2c} \tilde{\gamma}^\mu \tilde{\gamma}^\nu F_{\mu\nu} |\psi\rangle = 0.$$

(3.4)

The Pauli-Dirac equation can be cast in the Hamiltonian formalism as

$$i \hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle$$  \hspace{1cm} (3.5)

with the Dirac Hamiltonian $\hat{H}$ and the Dirac-Pauli Hamiltonian $\hat{H}$ defined as

$$\hat{H} = mc^2 \tilde{\beta} + c \tilde{\alpha} \cdot \left( \mathbf{p} - \frac{q}{c} \mathbf{A} \right) + q \phi \equiv \begin{pmatrix} mc^2 + q \phi & c \sigma \cdot \pi \\ c \sigma \cdot \pi & -mc^2 + q \phi \end{pmatrix}, \hspace{1cm} (3.6a)$$

$$\hat{H} = \hat{H} + \mu' \left( -\tilde{\beta} \cdot \mathbf{B} + i \tilde{\alpha} \cdot \mathbf{E} \right) \equiv \begin{pmatrix} mc^2 + q \phi - \mu' \sigma \cdot \mathbf{B} & c \sigma \cdot \pi + i \mu' \sigma \cdot \mathbf{E} \\ c \sigma \cdot \pi - i \mu' \sigma \cdot \mathbf{E} & -mc^2 + q \phi + \mu' \sigma \cdot \mathbf{B} \end{pmatrix}, \hspace{1cm} (3.6b)$$

where the 4 $\times$ 4 matrices are given explicitly by

$$\tilde{\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \hspace{0.5cm} \tilde{\alpha} = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \hspace{0.5cm} \tilde{\sigma} = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}, \hspace{1cm} (3.7)$$

\hspace{1cm} 1 Throughout this paper, a tilde is attached to denote a 4 $\times$ 4 matrix.
and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ are the $2 \times 2$ Pauli matrices. Accordingly, the $\tilde{\gamma}$ matrices are given by
\[
\tilde{\gamma}^0 = \tilde{\beta}, \quad \tilde{\gamma}^i = \tilde{\beta} \tilde{\alpha}^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}.
\] (3.8)

IV. FOLDY-WOUTHUYSEN TRANSFORMATION

The Dirac or Dirac-Pauli Hamiltonians (3.6) (or, more generally, with other corrections) can be schematically put in the form
\[
\tilde{\mathcal{H}} = \tilde{\beta} m c^2 + \tilde{\mathcal{O}} + \tilde{\mathcal{E}},
\] (4.1)
where $\tilde{\mathcal{E}}$ is the “even” part that commutes with $\tilde{\beta}$, i.e., $\tilde{\beta} \tilde{\mathcal{E}} \tilde{\beta} = \tilde{\mathcal{E}}$, while $\tilde{\mathcal{O}}$ is the “odd” part that anticommutes with $\tilde{\beta}$, i.e., $\tilde{\beta} \tilde{\mathcal{O}} \tilde{\beta} = -\tilde{\mathcal{O}}$. Because of the presence of the odd part, the Hamiltonian in the Dirac bispinor representation is not block-diagonalized, and thus the particle and antiparticle components are entangled in each of the Weyl spinors $\chi$ and $\varphi$. The question that naturally arises is whether we can find a representation in which the particle and antiparticle are separated, or equivalently, the Hamiltonian is block-diagonalized. Foldy and Wouthuysen have shown that such a representation is possible \[3, 4\]. The Foldy-Wouthuysen (FW) transformation is a unitary and nonexplicitly time-dependent transformation on the Dirac bispinor
\[
|\psi\rangle \rightarrow |\psi_{\text{FW}}\rangle = \tilde{\mathcal{U}}|\psi\rangle,
\] (4.2a)
\[
\tilde{\mathcal{H}} \rightarrow \tilde{\mathcal{H}}_{\text{FW}} = \tilde{\mathcal{U}} \tilde{\mathcal{H}} \tilde{\mathcal{U}}^\dagger,
\] (4.2b)
which leaves (3.5) in the form
\[
i\hbar \frac{\partial}{\partial t} |\psi_{\text{FW}}\rangle = \tilde{\mathcal{H}}_{\text{FW}} |\psi_{\text{FW}}\rangle \] (4.3)
and block-diagonalizes the Hamiltonian, i.e., $[\tilde{\beta}, \tilde{\mathcal{H}}_{\text{FW}}] = 0$. As the FW transformation separates the particle and antiparticle components, the two diagonal blocks of $\tilde{\mathcal{H}}_{\text{FW}}$ are adequate to describe the relativistic quantum dynamics of the spin-1/2 particle and antiparticle respectively.\footnote{If $\tilde{U}$ is explicitly time-dependent, instead of (4.2b), the diagonalized Hamiltonian is given by $\tilde{\mathcal{H}}_{\text{FW}} = \tilde{U} \tilde{\mathcal{H}} \tilde{U}^\dagger + \text{i}\hbar \tilde{U} \frac{\partial}{\partial t} \tilde{U}^\dagger$, which is beyond the scope of the standard FW scenario. Throughout this paper, we consider only the case in static fields. For the nonstandard FW transformation involving non-static fields, see \[26\] for more details.}

However, it should be remarked that, rigorously, the Dirac equation is self-consistent only in the context of quantum field theory, in which the particle-antiparticle pairs can be created and annihilated. On this account, it might not be legitimate to block-diagonalize the Dirac Hamiltonian or its phenomenological extension such as the Dirac-Pauli Hamiltonian. In fact, some doubts have
been thrown on the mathematical rigour of the FW transformation \cite{27, 28} (but also see \cite{29} for discussion on its validity). If the unitary FW transformation does not exist after all, the power series used in any order-by-order methods does not converge and high-order terms might be misleading and disagree with those obtained by different methods.\footnote{For example, for the Dirac theory in the presence of both electric and magnetic fields, the term of order $F_{\mu \nu}^2$, in Kutzelnigg’s method is given by $-\frac{2}{3\alpha \gamma c^2} B^2$, while it is given by $\frac{2}{3\alpha \gamma c^2} (E^2 - B^2)$ in the standard FW method (see \cite{23}). (Nevertheless, these two methods agree with each other on the terms linear in $F_{\mu \nu}$.)} However, as will be shown in Sections \ref{IV.B} and \ref{IV.C} the exact FW transformations do exist at least for two special cases, suggesting that particle-antiparticle separation is consistent and does not lead to any disagreement in these special situations.\footnote{As we will see shortly, Kutzelnigg’s method yields exactly the same results of Eriksen’s method for these two cases.}

Furthermore, in the regime of weak fields such that the energy interacting with electromagnetic fields does not exceed the Dirac energy gap $2mc^2$, we expect that the probability of pair creation and annihilation is negligible, and accordingly the FW transformation remains sensible and the block-diagonalized Hamiltonian is adequate to describe the relativistic quantum dynamics of the spin-1/2 particle and antiparticle separately without taking into account the field-theory interaction with each other. Starting from Sec. \ref{IV.D}, this paper is mainly devoted to this topic.

It should be noted that even if the unitary FW transformation exists, it is far from unique, as one can easily perform further unitary transformations that preserve the block decomposition upon the block-diagonalized Hamiltonian. The non-uniqueness does not lead to any ambiguity, as different block-diagonalization transformations are unitarily equivalent to one another and thus yield the same physics. While the physics is the same, however, the pertinent operators $\sigma$, $x$, and $p$ may represent very different physical quantities in different representations. To figure out the operators’ physical interpretations, it is crucial to compare the resulting FW transformed Hamiltonian with the classical counterpart in a certain classical limit via the correspondence principle. The comparison will be carried out explicitly in the weak-field limit for Kutzelnigg’s method; it turns out that, in Kutzelnigg’s method (and in fact in most FW methods in the literature), $\sigma$, $x$, and $p$ simply represent the spin, position, and conjugate momentum of the particle (as decoupled from the antiparticle) in the resulting FW representation. In other words, the method is “minimalist” in the sense that it does not give rise to further transformations that obscure the operators’ interpretations other than block diagonalization.

There are various methods for the FW transformation with different advantages. In this paper, we adopt Kutzelnigg’s method \cite{8} improved with a further simplification scheme \cite{25}.\footnote{As we will see shortly, Kutzelnigg’s method yields exactly the same results of Eriksen’s method for these two cases.}
A. Kutzelnigg’s method

In Kutzelnigg’s method, the FW unitary transformation is assumed to take the form

\[
\tilde{U} = \begin{pmatrix} \mathcal{Y} & \mathcal{X} \mathcal{Y} \dagger \\ -\mathcal{Z} \mathcal{X} \dagger & \mathcal{Z} \end{pmatrix}, \quad \tilde{U}^\dagger = \begin{pmatrix} \mathcal{Y} & -\mathcal{X} \mathcal{Z} \dagger \\ \mathcal{X} \mathcal{Y} & \mathcal{Z} \end{pmatrix},
\]

(4.4)

where the $2 \times 2$ hermitian operators $\mathcal{Y}$ and $\mathcal{Z}$ are defined as

\[
\mathcal{Y} = \mathcal{Y} \dagger = \frac{1}{\sqrt{1 + \mathcal{X} \mathcal{X} \dagger}}, \quad \mathcal{Z} = \mathcal{Z} \dagger = \frac{1}{\sqrt{1 + \mathcal{X} \mathcal{X} \dagger}}
\]

(4.5)

for some operator $\mathcal{X}$ to be determined. It is easy to show that

\[
\tilde{U} \tilde{U}^\dagger = \begin{pmatrix} \mathcal{Y} (1 + \mathcal{X} \mathcal{X}) \mathcal{Y} & 0 \\ 0 & \mathcal{Z} (1 + \mathcal{X} \mathcal{X}) \mathcal{Z} \end{pmatrix} = 1.
\]

(4.6)

Generically, we assume the Hamiltonian operator $\tilde{\mathcal{H}}$ takes the form

\[
\tilde{\mathcal{H}} = \begin{pmatrix} H_+ & H_0 \\ H_0^\dagger & H_-
\end{pmatrix}, \quad \text{with } H_+^\dagger = H_+, \ H_-^\dagger = H_-,
\]

(4.7)

and the FW transformed Hamiltonian is then given by

\[
\tilde{\mathcal{H}}_{\text{FW}} = \begin{pmatrix} \mathcal{H}_{\text{FW}} & 0 \\ 0 & \mathcal{H}_{\text{FW}} \end{pmatrix} \tilde{U} \tilde{U}^\dagger = \begin{pmatrix} H_+ + H_0 \mathcal{X} + \mathcal{X} \mathcal{H}_0^\dagger + \mathcal{X} \mathcal{H}_- \mathcal{X} \dagger & \mathcal{Y} \mathcal{Y} (H_0 - H_+ \mathcal{X} \mathcal{H}_0^\dagger + \mathcal{X} \mathcal{H}_- \mathcal{X} \dagger) \mathcal{Z} \\ \mathcal{Z} (H_0^\dagger - \mathcal{X} \mathcal{H}_+ \mathcal{X} \dagger - \mathcal{X} \mathcal{H}_0 \mathcal{X} \dagger) \mathcal{Y} & \mathcal{Z} (H_0 - H_+ \mathcal{X} \mathcal{H}_0^\dagger + \mathcal{X} \mathcal{H}_- \mathcal{X} \dagger) \mathcal{Z} \end{pmatrix}.
\]

The requirement that the off-diagonal blocks of $\tilde{\mathcal{H}}_{\text{FW}}$ vanish demands $\mathcal{X}$ to satisfy

\[
H_0 - H_+ \mathcal{X} \mathcal{X} \dagger + \mathcal{X} \mathcal{H}_+ \mathcal{X} \dagger = 0,
\]

(4.9a)

\[
H_0 - H_- \mathcal{X} \mathcal{X} \dagger + \mathcal{X} \mathcal{H}_- \mathcal{X} \dagger = 0,
\]

(4.9b)

and meanwhile the diagonal blocks read as

\[
\tilde{\mathcal{H}}_{\text{FW}} = \mathcal{Y} (H_+ + H_0 \mathcal{X} + \mathcal{X} \mathcal{H}_0^\dagger + \mathcal{X} \mathcal{H}_- \mathcal{X} \dagger) \mathcal{Y},
\]

(4.10a)

\[
\tilde{\mathcal{H}}_{\text{FW}} = \mathcal{Z} (H_0 - H_+ \mathcal{X} \mathcal{H}_0^\dagger + \mathcal{X} \mathcal{H}_- \mathcal{X} \dagger) \mathcal{Z},
\]

(4.10b)

which are manifestly hermitian. Under the condition of (4.9), (4.10) can be further simplified as

\[
\mathcal{H}_{\text{FW}} = \mathcal{Y} (H_+ + H_0 \mathcal{X} + \mathcal{X} \mathcal{H}_+ \mathcal{X} \mathcal{H}_0 \mathcal{X}) \mathcal{Y} = \mathcal{Y} (1 + \mathcal{X} \mathcal{X} \mathcal{X} \mathcal{X}) (H_+ + H_0 \mathcal{X}) \mathcal{Y}
\]

(4.11a)

\[
\tilde{\mathcal{H}}_{\text{FW}} = \mathcal{Z} (H_- - H_0^\dagger \mathcal{X} \mathcal{X} \dagger + \mathcal{X} \mathcal{H}_- \mathcal{X} \dagger - \mathcal{X} \mathcal{H}_0^\dagger \mathcal{X} \dagger) \mathcal{Z} = \mathcal{Z} (1 + \mathcal{X} \mathcal{X} \mathcal{X} \mathcal{X}) (H_- - H_0^\dagger \mathcal{X} \dagger)
\]

(4.11b)
In the Dirac or Dirac-Pauli theory, the Hamiltonian (4.7) is explicitly given by (3.6). Consider the formal replacement:

\[ p, \pi, \sigma, q, \mu', i \to -p, -\pi, -\sigma, -q, -\mu', -i, \]  

which corresponds to

\[ H_+ \to -H_- \quad H_0 \to H_0^\dagger, \]  

and accordingly, by (4.9),

\[ \mathcal{X} \to \mathcal{X}^\dagger. \]  

Comparison between (4.11a) and (4.11b) by reference to (4.13) and (4.14) then implies

\[ \bar{\mathcal{H}}_{FW}(x, \pi, \sigma; q, \mu') = -\mathcal{H}_{FW}(x, -\pi, -\sigma; -q, -\mu'). \]  

That is, \( \bar{\mathcal{H}}_{FW} \) takes the form of \( \mathcal{H}_{FW} \) by formally replacing \( \pi, \sigma, q, \mu' \) with \( -\pi, -\sigma, -q, -\mu' \) (which accounts for the charge conjugation) in addition to an overall minus sign (which account for the negative frequency).\(^5\) (Also see [23] for comments on the CPT symmetries.)

For the Dirac-Pauli theory, (4.9) and (4.11) read explicitly as

\[ 2mc^2 \mathcal{X} = -\mathcal{X} c \sigma \cdot \pi \mathcal{X} + c \sigma \cdot \pi + q[\phi, \mathcal{X}] - i\mu' \sigma \cdot E - i\mu' \mathcal{X} \sigma \cdot E \mathcal{X} + \mu'[\mathcal{X}, \sigma \cdot B] \]  

and

\[ \mathcal{H}_{FW} = mc^2 + \sqrt{1 + \mathcal{X}^\dagger \mathcal{X}} \left( q\phi + c \sigma \cdot \pi \mathcal{X} - \mu' \sigma \cdot \mathcal{X} + i\mu' \sigma \cdot E \mathcal{X} \right) \frac{1}{\sqrt{1 + \mathcal{X}^\dagger \mathcal{X}}}. \]  

Particularly, for the Dirac theory, (4.16) and (4.17) reduce to (by simply setting \( \mu' = 0 \))

\[ 2mc^2 X = -X c \sigma \cdot \pi X + c \sigma \cdot \pi + q[\phi, X], \]  

and

\[ H_{FW} = mc^2 + \sqrt{1 + X^\dagger X} \left( q\phi + c \sigma \cdot \pi X \right) \frac{1}{\sqrt{1 + X^\dagger X}}, \]  

where we have used the notations \( X \) and \( H_{FW} \) in place of \( \mathcal{X} \) and \( \mathcal{H}_{FW} \) when the Dirac-Pauli theory is reduced to the Dirac theory.

As caveated previously, the Hamiltonian \( \bar{\mathcal{H}} \) might not be block-diagonalizable at all and on this account there is no guarantee that the operator \( \mathcal{X} \) satisfying (4.16) or \( X \) satisfying (4.18) exists. However, as we will see, \( \mathcal{X} \) or \( X \) does exist in two special cases as well as in the case of homogeneous fields in the weak-field limit; accordingly \( \bar{\mathcal{H}} \) is block-diagonalizable in these situations.

\(^5\) Since \( \mathcal{H}_{FW} \) can be easily obtained by (4.15) once \( \mathcal{H}_{FW} \) is found, we focus only on the part of \( \mathcal{H}_{FW} \) in the rest of this paper. When \( \mathcal{H}_{FW} \) and \( \mathcal{H}_{FW} \) are combined to form \( \bar{\mathcal{H}}_{FW} \), the matrix \( \hat{\beta} \) will appear accordingly in the expression of \( \bar{\mathcal{H}}_{FW} \) as can be seen in Equations (3.14), (3.23), and (3.29) in [23].
B. Special case I

As the first special case, let us consider a Dirac spinor ($\mu' = 0$) with charge $q$ subject to a static magnetic field ($\partial_t B = 0$, $\partial_t A = 0$) but with no electric field ($E = 0$, $\phi = 0$). The condition (4.18) becomes a quadratic equation in $X$:

$$2mc^2X = -X c\sigma \cdot \pi X + c\sigma \cdot \pi,$$

which admits an exact solution

$$X = X^\dagger = \frac{c\sigma \cdot \pi}{mc^2 + \sqrt{m^2c^4 + c^2(\sigma \cdot \pi)^2}}. \tag{4.21}$$

Equation (4.19) with $\phi = 0$ then yields

$$H_{FW} = mc^2 + c\sigma \cdot \pi X = \sqrt{m^2c^4 + c^2(\sigma \cdot \pi)^2} = \sqrt{m^2c^4 + c^2p^2 - q\hbar c\sigma \cdot B} \tag{4.22}$$

by (A3). The resulting FW transformed Hamiltonian in (4.22) is exactly the same as that obtained by Eriksen’s method [7, 23].

The fact that the Dirac Hamiltonian in a static magnetic field can be block-diagonalized suggests that it is legitimate to ignore creation or annihilation of particle-antiparticle pairs. In fact, it has been shown that, in the context of QED, the charged particle-antiparticle pairs are not produced by any static magnetic field no matter how strong the field strength is, since the instanton actions for tunneling probability for pair production are infinite [30, 31].

If we turn off both electric and magnetic fields, (4.22) reduces to

$$H_{FW} = \sqrt{m^2c^4 + c^2p^2}, \tag{4.23}$$

which is the FW transformed Hamiltonian of a free particle.

Another interesting case is of a massless spinor. When it is subject only to a static magnetic field or it carries no charge ($q = 0$, such as a massless neutrino), (4.22) with $m = 0$ yields $X = X^\dagger = 1$, which follows from (4.4) that

$$\tilde{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{4.24}$$

The trivial FW transformation (4.24) is nothing but the unitary transformation that transforms the Dirac basis to the Weyl basis.

---

6. However, when the magnetic field changes in time, particle-antiparticle pairs can be produced [32], but this situation is beyond the scope of the standard FW scenario, in which $\tilde{U}$ is assumed to be nonexplicitly time-dependent.

7. In the Weyl basis, it is well known that the upper two components are decoupled from the lower two components for an uncharged massless spinor.
C. Special case II

As the second special case, let us consider a Dirac-Pauli spinor with zero charge \( q = 0 \) but nonzero magnetic moment \( \mu' \neq 0 \) subject to a static electric field \( \partial_t E = 0, \partial_t \phi = 0 \) but with no magnetic field \( B = 0, A = 0 \). The condition (4.16) now reads as

\[
2mc^2 \mathcal{X} = -\mathcal{X} \Omega \mathcal{X} + \Omega^\dagger, \tag{4.25}
\]

where we define the operators

\[
\Omega := c \sigma \cdot \mathbf{p} + i \mu' \sigma \cdot \mathbf{E}, \tag{4.26a}
\]

\[
\Omega^\dagger := c \sigma \cdot \mathbf{p} - i \mu' \sigma \cdot \mathbf{E}. \tag{4.26b}
\]

Multiplying \( \Omega \) on (4.25) from the left yields a quadratic equation in \( \mathcal{X} \):

\[
(\Omega \mathcal{X})^2 + 2mc^2 (\Omega \mathcal{X}) - \Omega \Omega^\dagger = 0. \tag{4.27}
\]

This admits an exact solution

\[
\Omega \mathcal{X} = -mc^2 + \sqrt{m^2c^4 + \Omega \Omega^\dagger}, \tag{4.28}
\]

which is manifestly hermitian, i.e.,

\[
\Omega \mathcal{X} = (\Omega \mathcal{X})^\dagger \equiv \mathcal{X}^\dagger \Omega. \tag{4.29}
\]

Meanwhile, multiplying \( \mathcal{X}^\dagger \) on (4.25) from the left and applying (4.29), we have

\[
(2mc^2 + \Omega \mathcal{X}) \mathcal{X}^\dagger \mathcal{X} = \mathcal{X}^\dagger \Omega^\dagger = \Omega \mathcal{X}, \tag{4.30}
\]

which follows

\[
\mathcal{X}^\dagger \mathcal{X} = \frac{\Omega \mathcal{X}}{2mc^2 + \Omega \mathcal{X}}. \tag{4.31}
\]

As \( \mathcal{X}^\dagger \mathcal{X} \) is a function of \( \Omega \mathcal{X} \), \( \mathcal{X}^\dagger \mathcal{X} \) commutes with \( \Omega \mathcal{X} \). As a result, (4.17) gives

\[
\mathcal{H}_{FW} = mc^2 + \sqrt{1 + \mathcal{X}^\dagger \mathcal{X} (\Omega \mathcal{X})} \frac{1}{\sqrt{1 + \mathcal{X}^\dagger \mathcal{X}}} \\
= mc^2 + \Omega \mathcal{X} = \sqrt{m^2c^4 + \Omega \Omega^\dagger} \\
= \left[ m^2c^4 + c^2 \mathbf{p}^2 - \mu' \hbar c \nabla \cdot \mathbf{E} + \mu' c (\mathbf{p} \times \mathbf{E} - \mathbf{E} \times \mathbf{p}) \cdot \sigma + \mu'' \mathbf{E}^2 \right]^{1/2}, \tag{4.32}
\]

---

\(^8\) A Dirac-Pauli spinor with \( q = 0 \) but \( \mu' \neq 0 \) can be used to describe spin-1/2 uncharged baryons such as protons. However, this description only gives an effective theory as Pauli’s prescription for inclusion of anomalous magnetic moment is only phenomenological.
where we have used (A1) and (A3) to compute $\Omega\Omega^\dagger$.

As for the first special case, the resulting FW transformed Hamiltonian in (4.32) is exactly the same as that obtained by Eriksen’s method [7, 23]. Unlike the first special case, however, the physical interpretation and relevance of the fact that the Hamiltonian can be exactly block-diagonalized is not well understood, as the second special case is rather artificial. Closer investigations into the mathematical structure of QED might for further insight are needed.

D. Weak-field limit

When the external electromagnetic field is weak enough, we expect that the FW transformed Hamiltonian exists and agrees with the classical Hamiltonian given by (2.1)–(2.3) except for some quantum corrections that have no classical correspondence. Mathematically, as $\hat{H}(\phi, A, E, B)$ denotes the Dirac or Dirac-Pauli Hamiltonian, the rigorous statement is to say that the $4 \times 4$ unitary matrix $\tilde{U}$ exists such that the linear-field limit defined as

$$\lim_{\lambda \to 0} \frac{\tilde{U} \hat{H}(\lambda\phi, \lambda A, \lambda E, \lambda B)\tilde{U}^\dagger}{\lambda}$$

(4.33)

is block-diagonal and in agreement with the classical counterpart, even though $\hat{H}$ itself might not be diagonalizable. Physically, this means the particle-antiparticle separation remains legitimate when the electromagnetic field is weak enough so that the energy interacting with electromagnetic fields does not exceed the Dirac energy gap.

As detailed in [23], the two special cases in Sections IV B and IV C in conjunction suggest that, in the weak-field limit, the FW transformed Dirac-Pauli Hamiltonian takes the form

$$\hat{H}_{FW}(x, p, \sigma) = \sqrt{c^2 \pi^2 + m^2 c^4 + q\phi}$$

$$- \frac{\hbar}{2} \sigma \cdot \left( \left( \gamma_m + \frac{q}{mc \gamma_\pi} \right) B - \gamma_m \frac{1}{\gamma_\pi (1 + \gamma_\pi)} \frac{\pi \cdot B}{m^2 c^2} \right)$$

$$- \left( \gamma_m \frac{1}{\gamma_\pi} + \frac{q}{mc \gamma_\pi (1 + \gamma_\pi)} \right) \frac{\pi \times E}{mc}_{\text{Weyl}}$$

$$+ \frac{\hbar^2}{4mc} \left( \frac{q}{2mc} - \gamma_m \right) \left( \nabla \cdot E \right)_{\text{Weyl}},$$

(4.34)

where $(\cdots)$ and $(\cdots)_{\text{Weyl}}$ denote specific symmetrization for operator orderings defined in [23]. $\hat{H}_{FW}$ in (4.34) is in full agreement with the classical counterpart given by (2.1)–(2.3) with $s = \hbar\sigma/2$ except for the operator orderings and the Darwin term involving $\hbar^2$, both of which have no classical correspondence.
The form of (4.34) is conjectured from the two special cases, which are complementary to each other, and still requires further confirmation for the cases in the presence of both $E$ and $B$. Its validity has been confirmed in [25] by Kutzelnigg’s method up to the order of $(\frac{\pi}{mc})^{14}$ for the case of static and homogeneous electromagnetic fields, whereby the Darwin term vanishes and there are no complications arising from operator orderings thanks to homogeneity, and the FW transformation remains explicitly time-independent and thus in conformity with the standard FW scenario thanks to staticity [23]. Based on the results obtained in [25], we are able to prove by mathematical induction that, in static and homogeneous electromagnetic fields, the FW transformed Hamiltonian in the weak-field limit is completely in agreement with the classical counterpart. We present the proof first for the Dirac Hamiltonian in Sec. V and then for the Dirac-Pauli Hamiltonian in Sec. VI.

V. DIRAC HAMILTONIAN

For the Dirac theory, we first solve the operator $X$ by the power series expansion and then obtain the FW transformed Hamiltonian $H_{FW}$. As we assume the applied electromagnetic fields to be static and homogeneous, we have $[\pi_i, E_j] = [\pi_i, B_j] = 0$. Moreover, because we focus on the weak-field limit, we neglect all the terms nonlinear in $F_{\mu\nu}$.

A. $X_n$

The operator $X$ used in Kutzelnigg’s method satisfies the condition (4.18) for the Dirac theory. Consider the power series of $X$ in powers of $c^{-1}$:

$$X = \sum_{j=1}^{\infty} \frac{X_j}{c^j}.$$  \hspace{1cm} (5.1)

For the orders of $1/c$ and $1/c^2$, (4.18) yields

$$2mX_1 = \sigma \cdot \pi,$$ \hspace{1cm} (5.2a)

$$2mX_2 = 0.$$ \hspace{1cm} (5.2b)

According to (4.18), the higher-order terms in the power series of $X$ can be determined by the following recursion relations (for $j \geq 1$):

$$2mX_{2j} = - \sum_{k_1+k_2=2j-1} X_{k_1} \sigma \cdot \pi X_{k_2} + q[\phi, X_{2j-2}],$$ \hspace{1cm} (5.3a)

$$2mX_{2j+1} = - \sum_{k_1+k_2=2j} X_{k_1} \sigma \cdot \pi X_{k_2} + q[\phi, X_{2j-1}].$$ \hspace{1cm} (5.3b)
Explicitly, the leading terms \( X_j \) read as

\[
X_1 = \frac{\sigma \cdot \pi}{2m},
\]

\[
X_3 = -\frac{1}{8} \frac{(\sigma \cdot \pi)^3}{m^3} - \frac{1}{4 m^2} i q h \sigma \cdot E,
\]

\[
X_5 = \frac{1}{16} \frac{(\sigma \cdot \pi)^5}{m^3} + \frac{3}{16 m^4} \pi^2 (\sigma \cdot E) + \frac{1}{8 m^4} (\sigma \cdot \pi)(E \cdot \pi),
\]

\[
X_7 = -\frac{5}{128} \frac{(\sigma \cdot \pi)^7}{m^7} - \frac{5}{32 m^6} \pi^4 (\sigma \cdot E) - \frac{3}{16 m^6} \pi^2 (\sigma \cdot \pi)(E \cdot \pi),
\]

\[
X_9 = \frac{7}{256} \frac{(\sigma \cdot \pi)^9}{m^9} + \frac{35}{256 m^8} \pi^6 (\sigma \cdot E) + \frac{29}{128 m^8} \pi^4 (\sigma \cdot \pi)(E \cdot \pi),
\]

\[
X_{11} = -\frac{21}{1024} \frac{(\sigma \cdot \pi)^{11}}{m^{11}} - \frac{63}{1024 m^{10} \pi^8 (\sigma \cdot E)} + \frac{65}{256 m^{10} \pi^6 (\sigma \cdot \pi)(E \cdot \pi)},
\]

\[
X_{13} = \frac{33}{2048} \frac{(\sigma \cdot \pi)^{13}}{m^{13}} + \frac{231}{2048 m^{12} \pi^{10} (\sigma \cdot E)} + \frac{281}{1024 m^{12} \pi^8 (\sigma \cdot \pi)(E \cdot \pi)},
\]

and \( X_{2j} = 0 \) for all \( j \). (These were laboriously calculated in [25].)

Based on the result of (5.4), we can conjecture the following theorem and provide its proof by mathematical induction.

**Theorem 1.** In the weak-field limit, we neglect nonlinear terms in \( E \) and \( B \). If the electromagnetic field is homogeneous (thus, \( [\pi_i, E_j] = [\pi_i, B_j] = 0 \)), the generic expression for \( X_{n \geq 0} \) is given by

\[
X_{2j} = 0,
\]

\[
X_{2j+1} = a_j \frac{(-1)^j}{(2m)^{2j+1}} (\sigma \cdot \pi)^{2j+1} + b_j \frac{i q h (-1)^j}{(2m)^{2j}} \pi^{2j-2} (\sigma \cdot E) + c_j \frac{i q h (-1)^j}{(2m)^{2j-2}} \pi^{2j-4} (\sigma \cdot \pi)(E \cdot \pi),
\]

where the coefficients are defined as

\[
a_{j \geq 0} = \frac{(2j)!}{j!(j+1)!},
\]

\[
b_{j \geq 1} = \frac{(2j-1)!}{j!(j-1)!} \equiv (2j-1)a_{j-1}, \quad b_{j=0} = 0,
\]

\[
c_{j \geq 0} = 2 \sum_{j_1 + j_2 = j} b_{j_1} b_{j_2}, \quad \text{(particularly, } c_{j=0,1} = 0). \]

**Proof (by induction).** It is trivial to prove (5.5a) by applying (5.3a) on (5.2b) inductively. To prove (5.5b), we first note that it is valid for \( j = 1 \) by (5.4b). Suppose (5.5b) is true for all \( X_{2k+1} \) with \( k < j \). Since \( X_{2k} = 0 \), the recursive relation (5.3b) reads as

\[
2m X_{2j+1} = - \sum_{j_1 + j_2 = j-1} X_{2j_1+1} (\sigma \cdot \pi) X_{2j_2+1} + q[\phi, X_{2j-1}],
\]
which, by applying the inductive hypothesis for \( k < j \), yields

\[
2mX_{2j+1} = - \sum_{j_1+j_2=j-1} X_{2j_1+1}(\sigma \cdot \pi)X_{2j_2+1} + q \left[ \phi, a_{j-1} \frac{(-1)^{j-1}}{(2m)^{2j-1}}(\sigma \cdot \pi)^{2j-1} \right] \quad (5.8a)
\]

\[
= - \sum_{j_1+j_2=j-1} a_{j_1}a_{j_2} \frac{(-1)^{j_1+j_2}}{(2m)^{2(j_1+j_2)+2}}(\sigma \cdot \pi)^{2(j_1+j_2)+3}
- 2iqh \sum_{j_1+j_2=j-1} a_{j_1}b_{j_2} \frac{(-1)^{j_1+j_2}}{(2m)^{2(j_1+j_2)+1}}(\sigma \cdot \pi)^{2j_1+2}(\sigma \cdot E)
- 2iqh \sum_{j_1+j_2=j-1} a_{j_1}c_{j_2} \frac{(-1)^{j_1+j_2}}{(2m)^{2(j_1+j_2)+1}}(\sigma \cdot \pi)^{2j_2-2}(\sigma \cdot \pi)(\mathbf{E} \cdot \pi)
+ q a_{j-1} \frac{(-1)^{j-1}}{(2m)^{2j-1}}[\phi, (\sigma \cdot \pi)^{2j-1}] \quad (5.8b),
\]

where in (5.8a) we have neglected nonlinear terms in \( \mathbf{E} \) and in (5.8b) adopted \([\pi_i, E_j] = 0\). Next, applying (A3) and (A6b) and dropping out the second term in (A3) whenever it is accompanied by \( \mathbf{E} \), we then have

\[
2mX_{2j+1} = - \sum_{j_1+j_2=j-1} a_{j_1}a_{j_2} \frac{(-1)^{j_1+j_2}}{(2m)^{2(j_1+j_2)+2}}(\sigma \cdot \pi)^{2(j_1+j_2)+3}
- 2iqh \sum_{j_1+j_2=j-1} a_{j_1}b_{j_2} \frac{(-1)^{j_1+j_2}}{(2m)^{2(j_1+j_2)+1}}(\sigma \cdot \pi)^{2(j_1+j_2)}(\sigma \cdot \mathbf{E})
- 2iqh \sum_{j_1+j_2=j-1} a_{j_1}c_{j_2} \frac{(-1)^{j_1+j_2}}{(2m)^{2(j_1+j_2)+1}}(\sigma \cdot \pi)^{2j_2-2}(\sigma \cdot \pi)(\mathbf{E} \cdot \pi)
- iqh a_{j-1} \frac{(-1)^{j-1}}{(2m)^{2j-1}}(\sigma \cdot \mathbf{E})
- 2iqh a_{j-1}(j-1) \frac{(-1)^{j-1}}{(2m)^{2j-1}}(\sigma \cdot \pi)^{2j-4}(\sigma \cdot \pi)(\mathbf{E} \cdot \pi). \quad (5.9)
\]

Consequently, we have

\[
X_{2j+1} = \left( \sum_{j_1+j_2=j-1} a_{j_1}a_{j_2} \right) \frac{(-1)^{j}}{(2m)^{2j+1}}(\sigma \cdot \pi)^{2j+1}
+ iqh \left( 2 \sum_{j_1+j_2=j-1} a_{j_1}b_{j_2} + a_{j-1} \right) \frac{(-1)^{j}}{(2m)^{2j}}(\sigma \cdot \pi)^{2j-2}(\sigma \cdot \mathbf{E})
+ iqh \left( 2 \sum_{j_1+j_2=j-1} a_{j_1}c_{j_2} + 2(j-1)a_{j-1} \right) \frac{(-1)^{j}}{(2m)^{2j}}(\sigma \cdot \pi)^{2j-4}(\sigma \cdot \pi)(\mathbf{E} \cdot \pi), \quad (5.10)
\]

which can be shown to take the form of (5.5b) by the combinatorial identities (their proofs will be
provided shortly):

\[
\begin{align*}
\text{for } j \geq 1 : & \quad \sum_{j_1 + j_2 = j - 1} a_{j_1} a_{j_2} = a_j, \\
& 2 \sum_{j_1 + j_2 = j - 1} a_{j_1} b_{j_2} = b_j - a_{j-1} \equiv 2(j-1)a_{j-1}, \\
& 2 \sum_{j_1 + j_2 = j - 1} a_{j_1} c_{j_2} = 4 \sum_{j_1 + j_2 + j_3 = j-1} a_{j_1} b_{j_2} b_{j_3} \\
& \quad = c_j - b_j + a_j \equiv c_j - 2(j-1)a_{j-1}. 
\end{align*}
\] (5.11)

We therefore have proved the theorem by mathematical induction.

\[\square\]

B. \(X\) and \(X^\dagger\)

We have the Taylor series with the radius of convergence \(|x| < 1\):

\[
\begin{align*}
\sum_{j=0}^{\infty} a_j \frac{(-1)^j}{2^{2j+1}} x^{2j+1} &= \frac{x}{1 + \sqrt{1 + x^2}} \equiv x^{-1} \left( \sqrt{1 + x^2} - 1 \right), \\
\sum_{j=1}^{\infty} b_j \frac{(-1)^j}{2^{2j-1}} x^{2j-2} &= \frac{1}{2} \left( \frac{1}{1 + \sqrt{1 + x^2}} - \frac{1}{\sqrt{1 + x^2}} \right), \\
\sum_{j=2}^{\infty} c_j \frac{(-1)^j}{2^{2j-4}} x^{2j-4} &= \frac{1}{8} \left( \frac{1}{1 + \sqrt{1 + x^2}} - \frac{1}{\sqrt{1 + x^2}} \right)^2,
\end{align*}
\] (5.12)

where (5.12a) and (5.12b) are obtained by the binomial series: \((1 + x)^{\pm 1/2} = \sum_{n=0}^{\infty} \binom{\pm 1/2}{n} x^n\). Meanwhile, with \(c_j\) defined by (5.6c), taking squares on both sides of (5.12b) immediately yields (5.12c).

The combinatorial identities (5.11) can be proven by the above Taylor series. Taking squares on both side of (5.12a) gives

\[
\begin{align*}
\sum_{j_1, j_2=0}^{\infty} a_{j_1} a_{j_2} \frac{(-1)^{j_1 + j_2}}{2^{2(j_1 + j_2) + 2}} x^{2(j_1 + j_2) + 1} &= \sum_{j=0}^{\infty} \sum_{j_1 + j_2 = j} a_{j_1} a_{j_2} \frac{(-1)^j}{2^{2j+2}} x^{2j+1} \\
&= \sum_{j=1}^{\infty} \sum_{j_1 + j_2 = j - 1} a_{j_1} a_{j_2} \frac{(-1)^{j-1}}{2^{2j+2}} x^{2j+2} - \frac{2}{x} \sum_{j=0}^{\infty} \sum_{j_1 + j_2 = j-1} a_{j_1} a_{j_2} \frac{(-1)^j}{2^{2j+1}} x^{2j+1} \\
&= \left( \frac{x}{1 + \sqrt{1 + x^2}} \right)^2, 
\end{align*}
\] (5.14)

9 Conversely, we have

\[
\begin{align*}
\sqrt{1 + x^2} &= 1 + \sum_{j=0}^{\infty} a_j \frac{(-1)^j x^{2(j+1)}}{2^{2j+1}}, \\
\frac{1}{\sqrt{1 + x^2}} &= \sum_{j=0}^{\infty} (a_j + b_{j+1}) \frac{(-1)^j x^{2j}}{2^{2j+1}} = \sum_{j=0}^{\infty} (j+1) a_j \frac{(-1)^j x^{2j}}{2^{2j}}. 
\end{align*}
\] (5.13)
which leads to

\[
\sum_{j=0}^{\infty} \sum_{j_1+j_2=j-1} a_j a_{j_2} \frac{(-1)^j}{2^{2j+1}} x^{2j+1} = -\frac{x}{2} \left( \left(\frac{x}{1+\sqrt{1+x^2}}\right)^2 - 1 \right) = \frac{x}{1+\sqrt{1+x^2}}. \tag{5.15}
\]

By (5.12a) again, we obtain (5.11a). The identity (5.11b) can be proved similarly, and (5.11c) follows immediately from (5.11b) with the definition (5.6c). Additionally, exploiting (5.12) in a similar way enables us to prove one more combinatorial identity:

for \( j \geq 0 \):

\[
b_{j+1} + c_{j+1} = 4b_j + 4c_j + a_j, \tag{5.16}
\]

which will be useful later.

By (5.5), we obtain the Taylor series of the \( X \) operator:

\[
X = \sum_{k=1}^{\infty} \frac{X_k}{k^2} = \sum_{j=0}^{\infty} X^{2j+1} = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2mc)^{2j+1}} (\sigma \cdot \pi)^{2j+1} + \frac{iqh}{c} \sum_{j=1}^{\infty} b_j \frac{(-1)^j}{(2mc)^{2j}} \pi^{2j-2} (\sigma \cdot \mathbf{E}) + \frac{iqh}{c} \sum_{j=2}^{\infty} c_j \frac{(-1)^j}{(2mc)^{2j}} \pi^{2j-4} (\sigma \cdot \pi)(\mathbf{E} \cdot \pi). \tag{5.17}
\]

By (5.12), the Taylor series of the operator \( X \) given in (5.17) converges to a closed form provided that

\[
|\langle \sigma \cdot \pi \rangle| = \left| \pi^2 - \frac{qh}{c} \sigma \cdot \mathbf{B} \right| < m^2 c^2. \tag{5.19}
\]

We will discuss the condition for convergence in the end of Sec. VICT
Adopting \([\pi_i, E_j] = 0\) again and neglecting nonlinear terms in \(E\), (5.17) and (5.18) then give

\[
X^\dagger X = \sum_{j_1, j_2=0}^{\infty} a_{j_1} a_{j_2} \frac{(-1)^{j_1+j_2}}{(2mc)^{2(j_1+j_2)+2}} (\sigma \cdot \pi)^{2(j_1+j_2)+2}
\]

\[
+ \frac{iq\hbar}{c} \sum_{j_1=0}^{\infty} a_{j_1} b_{j_2} \frac{(-1)^{j_1+j_2}}{(2mc)^{2(j_1+j_2)+1}} \pi^{2(j_1+j_2)-2} [\sigma \cdot \pi, \sigma \cdot E]
\]

\[
= \sum_{j=0}^{\infty} \sum_{j_1+j_2=j} a_{j_1} a_{j_2} \frac{(-1)^{j}}{(2mc)^{2j+2}} (\sigma \cdot \pi)^{2j+2}
\]

\[
+ 2 \frac{q\hbar}{c} \sum_{j=1}^{\infty} \sum_{j_1+j_2=j} a_{j_1} b_{j_2} \frac{(-1)^{j}}{(2mc)^{2j+1}} \pi^{2j-2} (E \times \pi) \cdot \sigma
\]

\[
= \sum_{j=0}^{\infty} a_{j+1} \frac{(-1)^{j}}{(2mc)^{2j+2}} (\sigma \cdot \pi)^{2j+2}
\]

\[
+ \frac{q\hbar}{c} \sum_{j=1}^{\infty} (b_{j+1} - a_j) \frac{(-1)^{j}}{(2mc)^{2j+1}} \pi^{2j-2} (E \times \pi) \cdot \sigma,
\]

(5.20)

where \([A1], [A3]\), and (5.11) have been used.

C. \(H_{FW}\)

Before we calculate \(H_{FW}\), let us investigate the operators \([q\phi, (X^\dagger X)]\) and \([c \sigma \cdot \pi X, (X^\dagger X)^n]\) beforehand.

First, by (5.20) and (A6a), we have

\[
[q\phi, X^\dagger X] = \sum_{j=0}^{\infty} a_{j+1} \frac{(-1)^{j}}{(2mc)^{2j+2}} [q\phi, (\sigma \cdot \pi)^{2j+2}]
\]

\[
= iq\hbar \sum_{j=0}^{\infty} 2(j+1)a_{j+1} \frac{(-1)^{j}}{(2mc)^{2j+2}} \pi^{2j}(E \cdot \pi).
\]

(5.21)

Note that \([X^\dagger X, \pi^{2j}(E \cdot \pi)] = 0\) if we neglect nonlinear terms in \(F_{\mu\nu}\) and adopt \([\pi_i, E_j] = 0\). Consequently, by induction, we have

\[
[q\phi, (X^\dagger X)^n] = n[q\phi, X^\dagger X](X^\dagger X)^{n-1},
\]

(5.22)

for \(n \geq 1\). Expanding \((1 + x)^{1/2} = \sum_{n=0}^{\infty} \frac{1}{n!}x^n \equiv \sum_{n=0}^{\infty} c_n x^n\), we can then compute

\[
\sqrt{1 + X^\dagger X} (q\phi) \equiv \sum_{n=0}^{\infty} c_n (X^\dagger X)^n(q\phi)
\]

\[
= \sum_{n=0}^{\infty} d_n(q\phi)(X^\dagger X)^n - \sum_{n=1}^{\infty} n c_n[q\phi, X^\dagger X](X^\dagger X)^{n-1}
\]

\[
= (q\phi)\sqrt{1 + X^\dagger X} - [q\phi, X^\dagger X] \frac{1}{2\sqrt{1 + X^\dagger X}}.
\]

(5.23)
where we have used $\frac{d}{dx}(1 + x)^{1/2} = \frac{1}{2}(1 + x)^{-1/2} = \sum_{n=1}^{\infty} n e_n x^{n-1}.$

Second, from (5.17), we get

$$c(\sigma \cdot \pi) X = c \sum_{j=0}^{\infty} a_j \frac{(-1)^j}{(2mc)^{2j+1}} (\sigma \cdot \pi)^{2j+2} + qh \sum_{j=1}^{\infty} b_j \frac{(-1)^j}{(2mc)^{2j}} \pi^{2j-2}(E \times \pi) \cdot \sigma$$

$$+ iqh \sum_{j=1}^{\infty} (b_j + c_j) \frac{(-1)^j}{(2mc)^{2j}} \pi^{2j-2}(E \cdot \pi),$$

(5.24)

where (A1) and (A3) have been used and the superfluous term involving $c_j = 0$ is added for bookkeeping convenience. Note that, up to the linear terms in $F_{\mu\nu}$, the $\sigma \cdot B$ piece of (A3) can be dropped out for the factors $(\sigma \cdot \pi)^{2j+2}$ in both (5.20) and (5.24) when we compute $[c \sigma \cdot \pi X, X^\dagger X]$. Consequently we have

$$\left[ c \sigma \cdot \pi X, X^\dagger X \right] = 0.$$  

(5.25)

We are now ready to calculate $H_{FW}$. With (5.23) and (5.25), (4.19) leads to

$$H_{FW} = mc^2 + \sqrt{1 + X^\dagger X} \left( q\phi + c \sigma \cdot \pi X \right) \frac{1}{\sqrt{1 + X^\dagger X}}$$

$$= mc^2 + q\phi - [q\phi, X^\dagger X] \frac{1}{2(1 + X^\dagger X)} + c \sigma \cdot \pi X.$$  

(5.26a)

(5.26b)

Substituting (5.21) and (5.24) into (5.26) gives

$$H_{FW} = mc^2 + q\phi + c \sum_{j=0}^{\infty} a_j \frac{(-1)^j}{(2mc)^{2j+1}} (\sigma \cdot \pi)^{2j+2} + qh \sum_{j=1}^{\infty} b_j \frac{(-1)^j}{(2mc)^{2j}} \pi^{2j-2}(E \times \pi) \cdot \sigma$$

$$+ iqh \sum_{j=1}^{\infty} (b_j + c_j) \frac{(-1)^j}{(2mc)^{2j}} \pi^{2j-2}(E \cdot \pi)$$

$$- iqh \left( \sum_{j=0}^{\infty} (j + 1)a_{j+1} \frac{(-1)^j}{(2mc)^{2j+2}} \pi^{2j}(E \cdot \pi) \right) \frac{1}{1 + X^\dagger X}.$$  

(5.27)

Because $H_{FW}$ is hermitian, the last two terms in (5.27), which give the anti-hermitian part, are expected to cancel each other exactly. This can be seen explicitly by checking vanishing of the
following composition of operators:

\[
\left( \sum_{j=1}^{\infty} (b_j + c_j) \frac{(-1)^j}{(2mc)^{2j}} \pi^{2j-2} \right) \left( 1 + X^\dagger X \right) + \sum_{j=0}^{\infty} (j+1) a_{j+1} \frac{(-1)^j}{(2mc)^{2j+2}} \pi^{2j} \\
- \sum_{j=1}^{\infty} ja_j \frac{(-1)^j}{(2mc)^{2j}} \pi^{2j-2} \\
= \sum_{j=1}^{\infty} (b_j + c_j) \frac{(-1)^j}{(2mc)^{2j}} \pi^{2j-2} + \sum_{j_1,j_2=1}^{\infty} (a_{j_1} b_{j_2} + a_{j_1} c_{j_2}) \frac{(-1)^{j_1+j_2+1}}{(2mc)^{2(j_1+j_2)}} \pi^{2(j_1+j_2)-2} \\
- \sum_{j=1}^{\infty} ja_j \frac{(-1)^j}{(2mc)^{2j}} \pi^{2j-2} \\
= \sum_{j=1}^{\infty} (b_j + c_j) \frac{(-1)^j}{(2mc)^{2j}} \pi^{2j-2} + \sum_{j=2}^{\infty} \sum_{j_1+j_2=j \atop j_1,j_2 \neq 0} (a_{j_1} b_{j_2} + a_{j_1} c_{j_2}) \frac{(-1)^{j_1+1}}{(2mc)^{2j}} \pi^{2j-2} \\
- \sum_{j=1}^{\infty} ja_j \frac{(-1)^j}{(2mc)^{2j}} \pi^{2j-2} \\
= a_1 - b_1 - c_1 \frac{1}{(2mc)^2} + \sum_{j=2}^{\infty} \left( b_j + c_j - ja_j - \sum_{j_1+j_2=j \atop j_1,j_2 \neq 0} (a_{j_1} b_{j_2} + a_{j_1} c_{j_2}) \right) \frac{(-1)^j}{(2mc)^{2j}} \pi^{2j-2}, \quad (5.28)
\]

where in the second line we have dropped out the $\sigma \cdot B$ piece of (A3) for the factors $(\sigma \cdot \pi)^{2j+2}$ in (5.20). For each coefficient factor of the summand, we have

\[
b_j + c_j - ja_j - \sum_{j_1+j_2=j \atop j_1,j_2 \neq 0} (a_{j_1} b_{j_2} + a_{j_1} c_{j_2}) \equiv 2b_j + 2c_j - ja_j - \sum_{j_1+j_2=j} (a_{j_1} b_{j_2} + a_{j_1} c_{j_2}) \\
= 2b_j + 2c_j - \frac{1}{2} (b_{j+1} + c_{j+1} - a_j) \quad (5.29)
\]

by (5.11), and it vanishes identically by (5.16). Also note that $a_1 - b_1 - c_1 = 0$. We thus show that (5.28) vanishes, thereby affirming hermiticity of $H_{FW}$. 

As the antihermitian part vanishes, (5.27) leads to

\[ H_{\text{FW}} = mc^2 + q\phi + c \sum_{j=0}^{\infty} a_j \frac{(-1)^j}{(2mc)^{2j+1}} \left( \sigma \cdot \pi \right)^{2j+2} + \frac{q\hbar}{mc} \sum_{j=1}^{\infty} b_j \frac{(-1)^j}{(2mc)^{2j}} \pi^{2j-2} (E \times \pi) \cdot \sigma \]

\[ = mc^2 + q\phi + mc^2 \left( \sqrt{1 + \left( \frac{\sigma \cdot \pi}{mc} \right)^2} - 1 \right) \]

\[ + \frac{q\hbar}{2(mc)^2} \left( \frac{1}{1 + \left( \frac{\pi}{mc} \right)^2} - \frac{1}{\sqrt{1 + \left( \frac{\pi}{mc} \right)^2}} \right) \sigma \cdot (E \times \pi), \quad (5.30) \]

where the Taylor series (5.12a) and (5.12b) are used. Note that, up to the linear order in \( B \), we have

\[ \sqrt{1 + \left( \frac{\sigma \cdot \pi}{mc} \right)^2} = \sqrt{1 + \left( \frac{\pi}{mc} \right)^2} - \frac{q\hbar}{mc^2} \sigma \cdot B \]

\[ = \sqrt{1 + \left( \frac{\pi}{mc} \right)^2} \left( 1 - \frac{q\hbar}{2mc^2} \frac{\sigma \cdot B}{1 + \left( \frac{\pi}{mc} \right)^2} + \cdots \right). \quad (5.31) \]

Taking this back into (5.30), we obtain

\[ H_{\text{FW}} = q\phi + \sqrt{m^2c^4 + \pi^2} - \frac{q\hbar}{2mc} \frac{1}{\gamma_\pi} \sigma \cdot B \]

\[ + \frac{q\hbar}{2mc} \left( \frac{1}{\gamma_\pi} - \frac{1}{1 + \gamma_\pi} \right) \sigma \cdot \left( \frac{\pi}{mc} \times E \right), \quad (5.32) \]

where the Lorentz factor associated with the kinematic momentum \( \pi \) is defined as

\[ \gamma_\pi := \sqrt{1 + \left( \frac{\pi}{mc} \right)^2} \equiv \sum_{n=0}^{\infty} \left( \frac{1/2}{n} \right) \left( \frac{\pi}{mc} \right)^{2n} \quad (5.33) \]

in accordance with the classical counterpart (2.5). The FW transform of the Dirac Hamiltonian given in (5.32) fully agrees with the classical counterpart (2.1)–(2.3) with \( s = \frac{\hbar}{2} \sigma \) and \( \gamma'_m = 0 \) (or \( \gamma_m = \frac{q}{mc} \)).

**VI. DIRAC-PAULI HAMILTONIAN**

As we have proved the exact correspondence between the Dirac Hamiltonian and the classical counterpart in the weak-field limit, we now extend the result to the Dirac-Pauli theory. Again, we first solve the operator \( X \) by the power series expansion and then obtain the FW transformed Hamiltonian \( H_{\text{FW}} \). We again assume \( [\pi_i, E_j] = [\pi_i, B_j] = 0 \) for homogeneous fields and neglect all the terms nonlinear in \( F_{\mu\nu} \) in the weak-field limit.
A. \( X'_n \)

For the Dirac-Pauli theory, the operator \( X \) used in Kutzelnigg’s method satisfies the condition (4.16), which reads as

\[
2mc^2X = -Xc\sigma \cdot \pi X + c\sigma \cdot \pi + q[\phi, X] \\
- \frac{i\mu''}{c}\sigma \cdot \mathbf{E} - \frac{i\mu''}{c}X\sigma \cdot \mathbf{E}X + \frac{\mu''}{c}\{X, \sigma \cdot \mathbf{B}\},
\]

(6.1)

where we define

\[
\mu'' := c\mu',
\]

(6.2)

as it is more convenient to factor out the dimensionality of \( c^{-1} \) in \( \mu' \) for the power series method in powers of \( c^{-1} \).

Consider the power series of \( X \) in powers of \( c^{-1} \):

\[
X := X + X' = \sum_{j=1}^{\infty} \frac{X_j}{c^j} = \sum_{j=1}^{\infty} \frac{X_j}{c^j} + \sum_{j=1}^{\infty} \frac{X'_j}{c^j},
\]

(6.3)

where \( X \) and \( X_j \) have been detailed in Sec. V. For the orders of \( 1/c \), \( 1/c^2 \) and \( 1/c^3 \), we have

\[
2mX_1 = \sigma \cdot \pi, \quad \Rightarrow \quad X_1 = (5.4a), \quad X'_1 = 0,
\]

(6.4a)

\[
2mX_2 = 0, \quad \Rightarrow \quad X_2 = 0, \quad X'_2 = 0
\]

(6.4b)

\[
2mX_3 = -X_1\sigma \cdot \pi X_1 + q[\phi, X_1] - i\mu''\sigma \cdot \mathbf{E}, \quad \Rightarrow \quad X_3 = (5.4b), \quad X'_3 = -\frac{i\mu''}{2m} \sigma \cdot \mathbf{E}.
\]

(6.4c)

The higher-order terms in the power series of \( X \) can be determined by the following recursion relations \((j \geq 2)\):

\[
2mX_{2j} = - \sum_{k_1+k_2=2j-1} X_{k_1}\sigma \cdot \pi X_{k_2} + q[\phi, X_{2j-2}] \\
- i\mu'' \sum_{k_1+k_2=2j-3} X_{k_1}\sigma \cdot \mathbf{E}X_{k_2} + \mu''\{X_{2j-3}, \sigma \cdot \mathbf{B}\}
\]

(6.5a)

\[
2mX_{2j+1} = - \sum_{k_1+k_2=2j} X_{k_1}\sigma \cdot \pi X_{k_2} + q[\phi, X_{2j-1}] \\
- i\mu'' \sum_{k_1+k_2=2j-2} X_{k_1}\sigma \cdot \mathbf{E}X_{k_2} + \mu''\{X_{2j-2}, \sigma \cdot \mathbf{B}\},
\]

(6.5b)
which together with (5.3) lead to the recursion relation for \( X'_n \) \((j \geq 2)\):

\[
2mX'_{2j} = - \sum_{k_1+k_2=2j-1} (X_{k_1} \sigma \cdot \pi X'_{k_2} + X'_{k_1} \sigma \cdot \pi X_{k_2} + X'_{k_1} \sigma \cdot \pi X'_{k_2}) \\
- i\mu'' \sum_{k_1+k_2=2j-3} (X_{k_1} \sigma \cdot E X_{k_2} + X'_{k_1} \sigma \cdot E X'_{k_2} + X'_{k_1} \sigma \cdot E X_{k_2} + X'_{k_1} \sigma \cdot E X'_{k_2}) \\
+ q \left[ \phi, X'_{2j-2} \right] + \mu'' \left\{ X_{2j-3} + X'_{2j-3}, \sigma \cdot B \right\},
\]

(6.6a)

\[
2mX'_{2j+1} = - \sum_{k_1+k_2=2j} (X_{k_1} \sigma \cdot \pi X'_{k_2} + X'_{k_1} \sigma \cdot \pi X_{k_2} + X'_{k_1} \sigma \cdot \pi X'_{k_2}) \\
- i\mu'' \sum_{k_1+k_2=2j-2} (X_{k_1} \sigma \cdot E X_{k_2} + X'_{k_1} \sigma \cdot E X'_{k_2} + X'_{k_1} \sigma \cdot E X_{k_2} + X'_{k_1} \sigma \cdot E X'_{k_2}) \\
+ q \left[ \phi, X'_{2j-1} \right] + \mu'' \left\{ X_{2j-2} + X'_{2j-2}, \sigma \cdot B \right\}.
\]

(6.6b)

Neglecting nonlinear terms in \( E \) and \( B \), the leading terms \( X'_j \) read as

\[
X'_1 = 0, \quad X'_2 = 0,
\]

(6.7a)

\[
X'_3 = - \frac{i\mu''}{2m} \sigma \cdot E, \quad X'_4 = \frac{\mu''}{2m^2} B \cdot \pi,
\]

(6.7b)

\[
X'_5 = \frac{3i\mu''}{8m^3} \pi^2 (\sigma \cdot E) - \frac{i\mu''}{4m^3} (\sigma \cdot \pi)(E \cdot \pi), \quad X'_6 = - \frac{3\mu''}{8m^4} \pi^2 (B \cdot \pi),
\]

(6.7c)

\[
X'_7 = - \frac{5i\mu''}{16m^5} \pi^4 (\sigma \cdot E) + \frac{i\mu''}{4m^5} \pi^2 (\sigma \cdot \pi)(E \cdot \pi), \quad X'_8 = \frac{5\mu''}{16m^6} \pi^4 (B \cdot \pi),
\]

(6.7d)

\[
X'_9 = \frac{35i\mu''}{128m^7} \pi^6 (\sigma \cdot E) - \frac{15i\mu''}{64m^7} \pi^4 (\sigma \cdot \pi)(E \cdot \pi), \quad X'_{10} = - \frac{35\mu''}{128m^8} \pi^6 (B \cdot \pi),
\]

(6.7e)

\[
X'_{11} = - \frac{63i\mu''}{256m^9} \pi^8 (\sigma \cdot E) - \frac{7i\mu''}{32m^9} \pi^6 (\sigma \cdot \pi)(E \cdot \pi), \quad X'_{12} = \frac{63\mu''}{256m^{10}} \pi^8 (B \cdot \pi),
\]

(6.7f)

(These where laboriously calculated in [25].)

Based on the result of (6.7), we can conjecture the following theorem and provide its proof by mathematical induction.

**Theorem 2.** In the weak-field limit, we neglect nonlinear terms in \( E \) and \( B \). If the electromagnetic field is homogeneous (thus, \([\pi_i, E_j] = [\pi_i, B_j] = 0\)), the generic expression for \( X'_{n \geq 2} \) is given by

\[
X'_{2j} = 2b_{j-1} \frac{\mu'' (-1)^{j}}{(2m)^{2j-2}} \pi^{2j-4} (B \cdot \pi),
\]

(6.8a)

\[
X'_{2j+1} = b_{j} \frac{i\mu'' (-1)^{j}}{(2m)^{2j-1}} \pi^{2j-2} (\sigma \cdot E) \\
+ d_{j} \frac{i\mu'' (-1)^{j+1}}{(2m)^{2j}} \pi^{2j-4} (\sigma \cdot \pi)(E \cdot \pi),
\]

(6.8b)

where the coefficients \( b_j \) are given by (5.6b) and \( d_j \) are defined as

\[
d_{j \geq 2} = \sum_{j_1+j_2+j_3=j-2} 2(j_1+1)a_{j_1}a_{j_2}a_{j_3}, \quad d_{j=0} = d_{j=1} = 0.
\]

(6.9)
Proof (by induction). Note that (6.8) is valid for \( j = 1 \) and \( j = 2 \) by (6.7). Suppose (6.8) is true for all \( X_{2k} \) and \( X_{2k+1} \) with \( k < j \), we will prove \( X'_{2j} \) and \( X'_{2j+1} \) to be true for \( j \geq 2 \) by induction.

First, we prove (6.8a) for \( j \geq 2 \). With the inductive hypothesis and (5.5), the recursive relation (6.6a) yields

\[
2mX'_{2j} = - \sum_{j_1 + j_2 = j-1} (X_{2j_1+1}(\sigma \cdot \pi)X'_{2j_2} + X'_{2j_2}(\sigma \cdot \pi)X_{2j_1+1}) + \mu'' \{ X_{2j-3}, \sigma \cdot B \}, \tag{6.10}
\]

where we have neglected nonlinear terms in \( E \) and \( B \). Applying the inductive hypothesis for \( k < j \) and (5.5b), we have

\[
2mX'_{2j} = -\mu'' \sum_{j_1 + j_2 = j-1} 2a_{j_1}b_{j_2-1} \frac{(-1)^{j_1+j_2}}{(2m)^2(j_1+j_2)-1} \left( (\sigma \cdot \pi)^{2j_1+2} \pi^{2j_2-4} (B \cdot \pi) + \pi^{2j_2-4} (B \cdot \pi)(\sigma \cdot \pi)^{2j_1+2} \right) + \mu'' a_{j-2} \left( \frac{-1}{2m} \right)^{j-2} \left( (\sigma \cdot \pi)^{2j-3}(\sigma \cdot B) + (\sigma \cdot B)(\sigma \cdot \pi)^{2j-3} \right) \\
= -2\mu'' \sum_{j_1 + j_2 = j-2} 2a_{j_1}b_{j_2} \frac{(-1)^{j_1+j_2}}{(2m)^2(j_1+j_2)+1} \pi^{2j_1+j_2} (B \cdot \pi) + \mu'' a_{j-2} \left( \frac{-1}{2m} \right)^{j-2} \sigma^{2j-4} \left( (\sigma \cdot \pi)(\sigma \cdot B) + (\sigma \cdot B)(\sigma \cdot \pi) \right) (\sigma \cdot \pi)^{2j-4} \\
= 2\mu'' \left( 2 \sum_{j_1 + j_2 = j-2} a_{j_1}b_{j_2} + a_{j-2} \right) \frac{(-1)^j}{(2m)^2j-3} \pi^{2j-4} (B \cdot \pi), \tag{6.11}
\]

where we have used (A3) to throw away nonlinear terms in \( B \) and used (A1) with \( [\pi_i, B_j] = 0 \) to get

\[
(\sigma \cdot \pi)(\sigma \cdot B) + (\sigma \cdot B)(\sigma \cdot \pi) = \pi \cdot B + B \cdot \pi + i(\pi \times B + B \times \pi) \cdot \sigma = 2(B \cdot \pi). \tag{6.12}
\]

By the combinatorial identity (5.11b), it follows from (6.11) that \( X'_{2j} \) for \( j \geq 2 \) takes the form of (6.8a).

Next, we prove (6.8b) for \( j \geq 2 \). With the inductive hypothesis and (5.5) again, the recursive relation (6.6b) yields

\[
2mX'_{2j+1} = - \sum_{j_1 + j_2 = j-1} (X_{2j_1+1}(\sigma \cdot \pi)X'_{2j_2+1} + X'_{2j_2+1}(\sigma \cdot \pi)X_{2j_1+1}) \\
- i\mu'' \sum_{j_1 + j_2 = j-2} X_{2j+1}(\sigma \cdot E)X_{2j+1}, \tag{6.13}
\]
where we have neglected nonlinear terms in \( E \) and \( B \). Applying the inductive hypothesis for \( k < j \) and (5.3b), we have

\[
2mX'_{2j+1} = -i\mu'' \sum_{j_1+j_2=j-1} a_{j_1} b_{j_2} \frac{(-1)^{j_1+j_2}}{(2m)^{2(j_1+j_2)}} \pi^{2j_1+j_2} (\sigma \cdot E)
\]

\[
- i\mu'' \sum_{j_1+j_2=j-1} a_{j_1} d_{j_2} \frac{(-1)^{j_1+j_2+1}}{(2m)^{2(j_1+j_2)}} \pi^{2j_1+j_2-2}(\sigma \cdot \pi)(\sigma \cdot E)
\]

\[
- i\mu'' \sum_{j_1+j_2=j-1} a_{j_1} b_{j_2} \frac{(-1)^{j_1+j_2+2}}{(2m)^{2(j_1+j_2)+2}} \pi^{2j_1+j_2-2}(\sigma \cdot \pi)(\sigma \cdot E)
\]

\[
- i\mu'' \sum_{j_1+j_2=j-2} a_{j_1} a_{j_2} \frac{(-1)^{j_1+j_2+2}}{(2m)^{2(j_1+j_2)+2}} \pi^{2j_1+j_2+2}(\sigma \cdot \pi)(\sigma \cdot E)
\]

(6.14)

By using (A3) to throw away nonlinear terms in \( B \) and using (A1) with \([\pi_i, B_j] = 0\) to get

\[
(\sigma \cdot \pi)(\sigma \cdot E)(\sigma \cdot \pi) = (\pi \cdot E + i(\pi \times E) \cdot \sigma)(\sigma \cdot \pi)
\]

\[
= (\pi \cdot E)(\sigma \cdot \pi) + i(\pi \times E) \cdot \pi - ((\pi \times E) \times \pi) \cdot \sigma
\]

\[
= (\pi \cdot E)(\sigma \cdot \pi) + ((\pi \cdot E)\pi - \pi^2 E) \cdot \sigma
\]

\[
= 2(\sigma \cdot \pi)(E \cdot \pi) - \pi^2(\sigma \cdot E),
\]

(6.15)

(6.15) then leads to

\[
2mX'_{2j+1} = -i\mu'' \sum_{j_1+j_2=j-1} a_{j_1} b_{j_2} \frac{(-1)^{j_1+j_2}}{(2m)^{2(j_1+j_2)}} \pi^{2j_1+j_2} (\sigma \cdot E)
\]

\[
- i\mu'' \sum_{j_1+j_2=j-1} a_{j_1} d_{j_2} \frac{(-1)^{j_1+j_2+1}}{(2m)^{2(j_1+j_2)}} \pi^{2j_1+j_2-2}(\sigma \cdot \pi)(\sigma \cdot E)
\]

\[
- i\mu'' \sum_{j_1+j_2=j-1} a_{j_1} b_{j_2} \frac{(-1)^{j_1+j_2+2}}{(2m)^{2(j_1+j_2)+2}} \pi^{2j_1+j_2-2}(\sigma \cdot \pi)(\sigma \cdot E)
\]

\[
- 2i\mu'' \sum_{j_1+j_2=j-2} a_{j_1} a_{j_2} \frac{(-1)^{j_1+j_2+2}}{(2m)^{2(j_1+j_2)+2}} \pi^{2j_1+j_2+2}(\sigma \cdot \pi)(E \cdot \pi)
\]

\[
+ i\mu'' \sum_{j_1+j_2=j-2} a_{j_1} a_{j_2} \frac{(-1)^{j_1+j_2}}{(2m)^{2(j_1+j_2)+2}} \pi^{2j_1+j_2+2}(\sigma \cdot E),
\]

(6.16)
and consequently
\[ X'_{2j+1} = i\mu'' \left( \sum_{j_1 + j_2 = j - 1} 2a_{j_1}b_{j_2} + \sum_{j_1 + j_2 = j - 1} a_{j_1}a_{j_2} \right) \left( \frac{-1)^j}{(2m)2j-2} \pi^{2j-2}(\sigma \cdot E) \right) \] (6.17)
\[ + 2i\mu'' \left( \sum_{j_1 + j_2 = j - 1} a_{j_1}a_{j_2} \right) \left( \frac{(-1)^{j+1}}{(2m)2j-2} \pi^{2(j_1+j_2)-2}(\sigma \cdot \pi)(\sigma \cdot E). \right) \]

The combinatorial identities (5.11a) and (5.11b) immediately imply that the summations inside the first pair of parentheses in (6.17) are equal to \( b_j \). Furthermore, by (5.11a) and the new combinatorial identity (its proof will be provided shortly)

\[ \sum_{j_1 + j_2 = j - 1} a_{j_1}d_{j_2} + \sum_{j_1 + j_2 = j - 1} a_{j_1}a_{j_2} = (\sigma \cdot \pi)(\sigma \cdot E). \] (6.18)

the summations inside the second pair of parentheses in (6.17) are equal to \( d_j \). Consequently, it follows from (6.17) that \( X'_{2j+1} \) for \( j \geq 2 \) takes the form of (6.8b).

We have proved both (6.8a) and (6.8b) by mathematical induction.

**B. \( X' \) and \( X'^{\dagger} \)**

We have the Taylor series with the radius of convergence \( |x| < 1 \):
\[
\sum_{j=2}^{\infty} d_j \frac{(-1)^j}{2^{2j-1}} x^{2j-4} = \frac{1}{\sqrt{1+x^2}} \left( \frac{1}{1+\sqrt{1+x^2}} \right)^2,
\] (6.19)
which, with \( d_j \) defined by (6.9), can be proven by taking squares on both sides of (5.12a) and then multiplying both sides by (5.13b). Similarly, exploiting (5.12) and (6.19) also enables us to prove the combinatorial identities (6.18) and

\[ \sum_{j=1}^{\infty} b_{j+1} + a_j = d_{j+1}. \] (6.20)

By (6.9), we obtain the Taylor series of the \( X' \) operator:
\[
X' = \sum_{j=1}^{\infty} X'_{2j} = \sum_{j=1}^{\infty} \frac{X'_{2j}}{c^{2j}} + \sum_{j=1}^{\infty} \frac{X'_{2j+1}}{c^{2j+1}}
= -2\mu'' \sum_{j=1}^{\infty} b_j \frac{(-1)^j}{(2mc)^{2j}} \pi^{2j-2}(B \cdot \pi) + i\mu'' \sum_{j=1}^{\infty} b_j \frac{(-1)^j}{(2mc)^{2j-1}} \pi^{2j-2}(\sigma \cdot E)
- i\mu'' \sum_{j=2}^{\infty} d_j \frac{(-1)^j}{(2mc)^{2j-1}} \pi^{2j-4}(\sigma \cdot \pi)(E \cdot \pi).
\] (6.21)
Adopting \([\pi_i, E_j] = [\pi_i, B_j] = 0\), we have
\[
X'' = -2\mu'' \sum_{j=1}^{\infty} b_j \frac{(-1)^j}{(2mc)^{2j-2}} \pi^{2j-2} (B \cdot \pi) - i\mu'' \sum_{j=1}^{\infty} b_j \frac{(-1)^j}{(2mc)^{2j-1}} \pi^{2j-2} (\sigma \cdot E)
+ i\mu'' \sum_{j=2}^{\infty} d_j \frac{(-1)^j}{(2mc)^{2j-1}} \pi^{2j-4} (\sigma \cdot \pi)(E \cdot \pi).
\] (6.22)

By (5.12b) and (6.19), the Taylor series of the operator \(X\) given in (6.24) converges to a closed form provided that
\[
|\pi^2| < m^2 c^2.
\] (6.23)

We will discuss the condition for convergence in the end of Sec. 6 C.

C. \(\mathcal{H}_{FW}\)

We have (4.17) with
\[
\mathcal{X} = X + X'.
\] (6.24)

Because \(X'\) is of the order \(O(F_{\mu\nu})\) as shown in (6.24), up to \(O(F_{\mu\nu})\), (4.17) leads to
\[
\mathcal{H}_{FW} = mc^2 + \sqrt{1 + X'X} (q\phi + c\sigma \cdot \pi X) \frac{1}{\sqrt{1 + X'X}}
+ (c\sigma \cdot \pi X' - \mu' \sigma \cdot B + i\mu' \sigma \cdot E X)
= : H_{FW} + H'_{FW},
\] (6.25)

where the first half part is identified as \(H_{FW}\) by (5.26a), and the second half is called \(H'_{FW}\).

By (5.17) and (6.21), we have
\[
H'_{FW} = c\sigma \cdot \pi X' - \mu' \sigma \cdot B + i\mu' \sigma \cdot E X
= -2\mu' \sum_{j=1}^{\infty} b_j \frac{(-1)^j}{(2mc)^{2j}} \pi^{2j-2} (\sigma \cdot \pi)(B \cdot \pi)
+ i\mu' \sum_{j=1}^{\infty} b_j \frac{(-1)^j}{(2mc)^{2j-1}} \pi^{2j-2} (E \cdot \pi) + \mu' \sum_{j=1}^{\infty} b_j \frac{(-1)^j}{(2mc)^{2j-1}} \pi^{2j-2} (E \times \pi) \cdot \sigma
- i\mu' \sum_{j=2}^{\infty} d_j \frac{(-1)^j}{(2mc)^{2j-1}} \pi^{2j-2} (E \cdot \pi)
- \mu' \sigma \cdot B
- i\mu' \sum_{j=0}^{\infty} a_j \frac{(-1)^j}{(2mc)^{2j+1}} \pi^{2j} (E \cdot \pi) - \mu' \sum_{j=0}^{\infty} a_j \frac{(-1)^j}{(2mc)^{2j+1}} \pi^{2j} (E \times \pi) \cdot \sigma,
\] (6.26)
where we have used (A1) and (A3) and neglected nonlinear terms in $F_{\mu\nu}$. Equation (6.26) leads to

$$H'_{FW} = -2\mu' \sum_{j=1}^{\infty} b_j \frac{(-1)^j}{(2mc)^{2j}} \pi^{2j-2} (\sigma \cdot \pi)(B \cdot \pi)$$

$$+ \mu' \left( 1 + \sum_{j=0}^{\infty} \frac{(-1)^j}{(2mc)^{2j}} \pi^{2j-2} - \sum_{j=0}^{\infty} a_j \frac{(-1)^j}{(2mc)^{2j+1}} \pi^{2j} \right) (E \times \pi) \cdot \sigma$$

$$- \mu' \sigma \cdot B$$

$$- i\mu' \sum_{j=0}^{\infty} \left( b_{j+1} - d_j + a_j \right) \frac{(-1)^j}{(2mc)^{2j+1}} \pi^{2j} (E \cdot \pi).$$

(6.27)

By (6.20), we find that the antihermitian part in (6.27) vanishes identically. Furthermore, by (5.12a) and (5.12b), we have

$$H'_{FW} = -\mu' \left( 1 + \frac{1}{\sqrt{1 + \frac{(\pi \cdot \pi)^2}{mc^2}}} - \frac{1}{\sqrt{1 + \left( \frac{\pi \cdot \pi}{mc} \right)^2}} \right) \frac{(\sigma \cdot \pi)(B \cdot \pi)}{(mc)^2}$$

$$- \mu' \left( \frac{1}{\sqrt{1 + \left( \frac{\pi \cdot \pi}{mc} \right)^2}} \right) \frac{(E \times \pi) \cdot \sigma}{mc} - \mu' \sigma \cdot B$$

$$= \mu' \left( \frac{1}{\gamma_\pi} - \frac{1}{1 + \gamma_\pi} \right) \sigma \cdot \frac{\pi \cdot \pi}{mc} B + \mu' \frac{1}{\gamma_\pi} \sigma \cdot \left( \frac{\pi \cdot \pi}{mc} \times E \right) - \mu' \sigma \cdot B,$$

(6.28)

where $\gamma_\pi$ is defined in (5.33).

With (5.32) and (6.28), we have

$$\mathcal{H}_{FW}(x, p, \sigma) = H_{FW} + H'_{FW}$$

$$= \sqrt{m^2c^4 + c^2p^2} + q\phi(x)$$

$$- \sigma \cdot \left[ \left( \mu' + \frac{q}{2mc} \frac{1}{\gamma_\pi} \right) B - \mu' \frac{1}{\gamma_\pi} \left( \frac{\pi \cdot \pi}{mc} \cdot B \right) \frac{\pi}{mc} \right]$$

$$- \left( \mu' \frac{1}{\gamma_\pi} + \frac{q}{2mc} \frac{1}{\gamma_\pi(1 + \gamma_\pi)} \right) \left( \frac{\pi \cdot \pi}{mc} \times E \right),$$

(6.29)

which is exactly the same as (4.34) except that the Darwin term vanishes and the operator orderings are superfluous. This proves that, in the weak-field limit, the FW transform of the Dirac-Pauli Hamiltonian is in complete agreement with the classical counterpart (2.1)–(2.3) with $s = \frac{\hbar}{2}\sigma$ and $\mu' = \frac{\gamma_\mu}{2}$. Note that, by applying the Taylor series (5.12) and (6.19), the functions of the operator $\Omega = \sigma \cdot \pi/(mc)$ or $\Omega = \pi/(mc)$ are understood via the Taylor series as

$$f(1 + \Omega^2) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} \Omega^{2n},$$

(6.30)
which produces convergent results provided that the spectrum of $\Omega$ satisfies $|\Omega^2| < 1$. This requires that the conditions of (5.19) and (6.23) have to be satisfied. In comparison with the classical theory in the weak-field regime, in which $\pi$ remains to be the kinematic momentum associated with $v$ as indicated by (2.9) and (2.10), the conditions (5.19) and (6.23) correspond to $|v| < c/\sqrt{2}$ (which is well beyond the low-speed limit). Once the operators $X$ and $X'$ converge to closed forms for $|v| < c/\sqrt{2}$, their closed forms are in fact upheld even beyond the conditions of (5.19) and (6.23). This is because, instead of the Taylor series (5.12) and (6.19), the pertinent function $1/\sqrt{1 + \Omega^2}$ can be alternatively understood in terms of the integral

$$\frac{1}{\sqrt{1 + \Omega^2}} = \lim_{N \to \infty} \int_{-N}^{N} d\eta e^{-\pi\eta^2(1+\Omega^2)},$$

where the exponential operator is defined by means of its Taylor expansion. The form of (6.31) gives convergent results for all $\Omega$. Therefore, even though the Taylor series (5.12) and (6.19) break down when (5.19) and (6.23) do not hold, the resulting $H_{FW}$ in (6.29) nevertheless remains valid as long as the applied electromagnetic field is weak enough so that nonlinear terms in $F_{\mu\nu}$ can be neglected.

### VII. SUMMARY AND DISCUSSION

In Kutzelnigg’s method improved with a further simplification scheme, the FW transform of the Dirac-Pauli Hamiltonian is given by (4.17) with $X$ satisfying (4.16), which reduces to (4.19) with $X$ satisfying (4.18) for the Dirac Hamiltonian. For the two special cases studied in Sec. IV B and Sec. IV C, the exact FW transformed Hamiltonians exist and agree with those obtained by Eriksen’s method [7]. Existence of the exact FW transformation in the first special case is accordant with the fact that charged particle-antiparticle pairs are not produced by any static magnetic field no matter how strong the field strength is [30, 31]. On the other hand, the physical relevance of the exact FW transformation in the second case is unclear and requires further research.

The conditions for the operators $X$ and $X' \equiv X + X'$ give rise to the recursion relations (5.3), (6.5), and (6.6) for their power series. When the applied electromagnetic field is static and homogeneous, in the weak-field limit in which nonlinear terms in $F_{\mu\nu}$ are neglected, we have Theorem 1 and Theorem 2 which are proven by mathematical induction via the recursion relations and various combinatorial identities. Consequently, the resulting FW transformed Dirac-Pauli Hamiltonian in

\[\text{\footnotesize 10} \] Here, we have adopted the idea propounded in [7].
the weak-field limit is given by (6.29), which is in full agreement with the classical counterpart (2.1)–(2.3) with \( s = \frac{\hbar}{2} \sigma \) and \( \mu' = \frac{\hbar}{2} \gamma m' \).

If the applied electromagnetic field is inhomogeneous, it is suggested in [23] that the FW transform in the weak-field limit takes the form of (4.34), which is an extension of (6.29) with corrections of the Darwin term and operator orderings. A rigorous proof of (4.34) in the style of this paper is however much more difficult, as it is very cumbersome to keep track of operator orderings in an order-by-order scenario. Instead, applying the alternative block-diagonalization method via the expansion in powers of the Planck constant \( \hbar \) [10–13] might provide a better route to investigate the quantum corrections arising from zitterbewegung (which is responsible for the Darwin term) and operator orderings. Furthermore, as we have remarked that it might not be legitimate to block-diagonalize the Dirac or Dirac-Pauli Hamiltonian in strong fields except for special cases, the method of expansion in \( \hbar \) [14] may help to elucidate the breakdown of particle-antiparticle separation in strong fields (also see [29]).

ACKNOWLEDGMENTS

D.W.C. would like to thank Sang Pyo Kim for valuable discussions. D.W.C. was supported in part by the Ministry of Science and Technology of Taiwan under the Grants No. 101-2112-M-002-027-MY3 and No. 101-2112-M-003-002-MY3, and T.W.C. under the Grant No. 101-2112-M-110-013-MY3.

Appendix A: Useful formulae and lemmas

The Pauli matrices satisfy the identity

\[
(\sigma \cdot a)(\sigma \cdot b) = a \cdot b + i(a \times b) \cdot \sigma
\]

for arbitrary vectors \( a \) and \( b \). Meanwhile, we have

\[
(\nabla \times a + a \times \nabla)\psi = (\nabla \times a)\psi.
\]

By (A1) and (A2), we have

\[
(\sigma \cdot \pi)^2 = \pi^2 - \frac{q\hbar}{c} \sigma \cdot B.
\]

Consider the commutator between \( \phi \) and \( \sigma \cdot \pi \). We have

\[
[\phi, \sigma \cdot \pi] = i\hbar(\sigma \cdot \nabla)\phi = i\hbar(\sigma \cdot E),
\]
and consequently

\[
\begin{align*}
\left[ \phi, (\sigma \cdot \pi)^2 \right] &= \sigma \cdot \pi [\phi, \sigma \cdot \pi] + [\phi, \sigma \cdot \pi] \sigma \cdot \pi \\
&= i\hbar [\left( \sigma \cdot \pi \right) \cdot (\sigma \cdot E) + (\sigma \cdot E) \cdot (\sigma \cdot \pi)] \\
&= i\hbar \left[ \pi \cdot E + E \cdot \pi + i \left( \frac{\hbar}{i} \nabla - \frac{q}{c} A \right) \times E + E \times \left( \frac{\hbar}{i} \nabla - \frac{q}{c} A \right) \right] \cdot \sigma \\
&= i\hbar (\pi \cdot E + E \cdot \pi) = 2i\hbar (E \cdot \pi),
\end{align*}
\]

(A5)

where we have applied the identities (A1) and (A2) and assumed \( \mathbf{E} \) is homogeneous.

As we consider only the terms linear in \( \mathbf{E} \) and \( \mathbf{B} \), we neglect the second term in (A3) whenever it is multiplied by the terms containing \( \mathbf{E} \) or \( \mathbf{B} \). Consequently, by induction, we have

\[
\begin{align*}
\left[ \phi, (\sigma \cdot \pi)^{2n} \right] &= (2n)i\hbar \pi^{2(n-1)}(E \cdot \pi), \quad (A6a) \\
\left[ \phi, (\sigma \cdot \pi)^{2n+1} \right] &= i\hbar \pi^{2n}(\sigma \cdot E) + (2n)i\hbar \pi^{2n-2}(\sigma \cdot \pi)(E \cdot \pi). \quad (A6b)
\end{align*}
\]

[1] P. A. M. Dirac, “The Quantum Theory of the Electron,” Proc. R. Soc. London 117, 610 (1928).
[2] P. A. M. Dirac, Principles of Quantum Mechanics, 4th ed. (Clarendon, Oxford, 1982).
[3] L. L. Foldy and S. A. Wouthuysen, “On the Dirac theory of spin 1/2 particle and its nonrelativistic limit,” Phys. Rev. 78, 29 (1950).
[4] P. Strange, Relativistic Quantum Mechanics, 1st ed. (Cambridge University Press, Cambridge, United Kingdom, 2008).
[5] P. O. Löwdin, “A Note on the Quantum-Mechanical Perturbation Theory,” J. Chem. Phys. 19, 1396 (1951).
[6] J. M. Luttinger and W. Kohn, “Motion of Electrons and Holes in Perturbed Periodic Fields,” Phys. Rev. 97, 896 (1955).
[7] E. Eriksen, “Foldy-Wouthuysen Transformation. Exact Solution with Generalization to the Two-Particle Problem,” Phys. Rev. 111, 1011 (1958).
[8] W. Kutzelnigg, “Perturbation theory of relativistic corrections,” Z. Phys. D 15, 27 (1990).
[9] R. Winkler, Spin-Orbit Coupling Effects in Two-Dimensional Electron and Hole Systems, 1st ed. (Springer, New York, 2003).
[10] A. J. Silenko, “Foldy-Wouthuysen transformation for relativistic particles in external fields,” J. Math. Phys. 44, 2952 (2003) [math-ph/0404067].
[11] K. Y. Bliokh, “Topological spin transport of relativistic electron,” Europhys. Lett. 72, 7 (2005) [quant-ph/0501183].
[12] P. Gosselin, A. Berard and H. Mohrbach, “Semiclassical diagonalization of quantum Hamiltonian and equations of motion with Berry phase corrections,” Eur. Phys. J. B 58, 137 (2007) [hep-th/0603192].
[13] P. Gosselin, A. Berard and H. Mohrbach, “Semiclassical Dynamics of Dirac particles interacting with a Static Gravitational Field,” Phys. Lett. A 368, 356 (2007) [hep-th/0604012].
[14] A. J. Silenko, “Foldy-Wouthuysen Transformation and Semiclassical Limit for Relativistic Particles in Strong External Fields,” Phys. Rev. A 77, 012116 (2008) [arXiv:0710.4218 [math-ph]].
[15] W. Pauli, “Relativistic Field Theories of Elementary Particles,” Rev. Mod. Phys. 13, 203 (1941).
[16] L. H. Thomas, “The kinematics of an electron with an axis,” Phil. Mag. Ser. 7 3, 1 (1927).
[17] V. Bargmann, L. Michel and V. L. Telegdi, “Precession of the polarization of particles moving in a homogeneous electromagnetic field,” Phys. Rev. Lett. 2, 435 (1959).
[18] J. D. Jackson, Classical Electrodynamics, 3rd ed. (John Wiley & Sons, New York, 1999).
[19] S. I. Rubinow and J. B. Keller, “Asymptotic Solution of the Dirac Equation,” Phys. Rev. 131, 2789 (1963).
[20] K. Rafaelli and R. Schiller, “Classical Motions of Spin-1/2 Particles,” Phys. Rev. 135, B279 (1964).
[21] J. Fröhlich and U. M. Studer, “Gauge invariance and current algebra in nonrelativistic many body theory,” Rev. Mod. Phys. 65, 733 (1993).
[22] A. J. Silenko, “Dirac equation in the Foldy-Wouthuysen representation describing the interaction of spin 1/2 relativistic particles with an external electromagnetic field,” Theor. Math. Phys. 105, 1224 (1995) [Teor. Mat. Fiz. 105, 46 (1995)].
[23] T.-W. Chen and D.-W. Chiou, “Correspondence between classical and Dirac-Pauli spinors in view of the Foldy-Wouthuysen transformation,” Phys. Rev. A 89, 032111 (2014) [arXiv:1310.8513 [quant-ph]].
[24] T.-W. Chen and D.-W. Chiou, “Foldy-Wouthuysen transformation for a Dirac-Pauli dyon and the Thomas-Bargmann-Michel-Telegdi equation,” Phys. Rev. A 82, 012115 (2010) [arXiv:1005.4128 [quant-ph]].
[25] T.-W. Chen and D.-W. Chiou, “High-order Foldy-Wouthuysen transformations of the Dirac and Dirac-Pauli Hamiltonians in the weak-field limit,” Phys. Rev. A 90, 012112 (2014) [arXiv:1311.3432 [quant-ph]].
[26] A. J. Silenko, “Energy expectation values of a particle in nonstationary fields,” Phys. Rev. A 91, 012111 (2015) [arXiv:1410.0169 [math-ph]].
[27] B. Thaller, The Dirac Equation, (Springer, Berlin, 1992).
[28] Ch. Brouder, M. Alouani, K. H. Bennemann, “Multiple-scattering theory of x-ray magnetic circular dichroism: Implementation and results for the iron K edge,” Phys. Rev. B 54, 7334 (1996).
[29] A. J. Silenko, “General method of the relativistic Foldy-Wouthuysen transformation and proof of validity of the Foldy-Wouthuysen Hamiltonian,” Phys. Rev. A 91, 022103 (2015) [arXiv:1501.02052 [math-ph]].
[30] S. P. Kim and D. N. Page, “Schwinger pair production via instantons in a strong electric field,” Phys. Rev. D 65, 105002 (2002). [hep-th/0005078].
[31] S. P. Kim, “QED Effective Action in Magnetic Field Backgrounds and Electromagnetic Duality,” Phys. Rev. D 84, 065004 (2011). [arXiv:1109.1249 [hep-th]].
[32] S. P. Kim, “Landau Levels of Scalar QED in Time-Dependent Magnetic Fields,” Annals Phys. 344, 1 (2014) [arXiv:1305.2577 [hep-th]].