Bi-particle entanglement and its quaternion representation

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Keywords: singlet states, symmetry group representation, antisymmetry, quaternions

Abstract
Using a non-standard model of a collective set theory, this paper proposes a description of a singlet state of quantum particles in terms of quaternions. The basic relation of this theory is the division relation, which is pre-ordering; antisymmetry is rejected. It turns out that the rejection of antisymmetry opens a new perspective for description of singlet states. We obtain two pairs of quaternions describing such states; they are the only quaternions generating these states because they are generators of the same finite group. Moreover, quaternions that form a pair have the same angles of rotation, but the rotations are in opposite directions, and both quaternions designate the same straight line in $\mathbb{R}^3$. Finally, we can determine a $SU(2)$ group for the obtained composite state and prove that the investigated state is inseparable; therefore, it is entangled.

1. Introduction

In several works published within the last few years [1–6] different aspects of collective set theory, i.e. mereology, were investigated. This current paper presents an application of a modified model of mereology in the field of quantum physics. The applied model is based on a relation, e.g. a division relation, which is pre-ordering. Although antisymmetry is assumed to be axiomatic in various theories, the reason for this assumption is not obvious. In different models of classical set theory, antisymmetry is assumed in the axiom establishing conditions under which two sets can be considered equal, i.e. in an extensionality axiom [7]. Also in mereology, which can be seen as a theory on power sets, antisymmetry is assumed for the parthood relation [8]. There is no doubt that antisymmetry is necessary for establishing order on elements, but in some cases it can be too restrictive. The problem is that it excludes duality, i.e. antisymmetry excludes symmetry because it glues objects together that are symmetric, e.g. $A \subseteq B \land B \subseteq A \implies A = B$.

Furthermore, since each mathematical object is defined in terms of sets, assumptions, like antisymmetry are important because they affect fundamental relations. For example, in the relation of division on Integers, we can observe that: $1 \div -1, -1 \div 1$, and $1 \neq -1$; therefore, antisymmetry does not hold for Integers (and not only for them). According to antisymmetry, 1 and $-1$ would be glued, but they in fact are not. Hence, if we want to maintain duality (symmetry), we have to reject antisymmetry, and try to reinterpret different phenomena within such frameworks. We stress this aspect because we know how important symmetry is in physics. Also some physicists [9] have doubts regarding the assumption of order in theories applied in physics. To address these problems we are proposing an interpretation of a singlet state of quantum particles within a theory where antisymmetry is rejected. Thus section 2 presents an outline of a non-standard collective set theory, and section 3 proposes a new description of singlet states.

2. Some remarks on mereology and antisymmetry

Mathematics has two main set theories: ZFC set theory based on Cantor’s concept of a set [10, 11]; and mereology, based on Leśniewski’s concept [8]. In comparison to ZFC set theory, mereology opens a totally new perspective of conceiving sets. In mereology, first we consider a set as a whole (therefore, we have sums of
elements); only from the perspective of the whole do we then distinguish elements. For example; let us take a circle: \( C = \{ \theta : 0 \leq \theta < 2\pi \} \). Mereologically, it can be considered as a set composed of two halves: 
\( S_1 = \{ \theta : 0 \leq \theta < \pi \}, \ S_2 = \{ \pi \leq \theta < 2\pi \} \), therefore \( C_1 = \{ S_1, S_2 \} \) or as a set composed of four quarters: 
\( C_2 = \{ Q_1, Q_2, Q_3, Q_4 \} \), where: 
\( Q_1 = \{ \theta : 0 \leq \theta < \frac{\pi}{2} \}, \ Q_2 = \{ \frac{\pi}{2} \leq \theta < \pi \}, \ Q_3 = \{ \pi \leq \theta < \frac{3\pi}{2} \}, \ Q_4 = \{ \frac{3\pi}{2} \leq \theta < 2\pi \} \).
In the case of halves, this set has two elements: \( S_1, S_2 \); in the case of quarters, the given set has four elements: \( Q_1, Q_2, Q_3, Q_4 \). Since in mereology the concept of a class is synonymous with the concept of sum, we have \( C_1 = C_2 \), i.e. \( \{ S_1, S_2 \} = \{ Q_1, Q_2, Q_3, Q_4 \} \). But in ZFC set theory, since first we take elements, and then form an abstract whole, we have \( C_1 \neq C_2 \), i.e. \( \{ S_1, S_2 \} \neq \{ Q_1, Q_2, Q_3, Q_4 \} \), by the extensionality principle. In fact, \( Q_3 \subsetneq C_1 \). Thus, the novelty of mereology provides a new way of conceiving objects: there are wholes formed of parts—proper or improper; and there are wholes which can be divided into parts in different ways, as exemplified by the decay of a particle *meson* \( \pi \) [12].

In a non-standard mereology [3], the relation of division becomes the fundamental relation since it is pre-ordering; therefore, parts are correlated in a specific way. The division relation is useful because it so naturally corresponds to real phenomena. In fact, in science, we often speak about a division of cells, decomposition of waves into sums of *sine* and *cosine* functions, etc. The division relation is pre-ordering because it is reflexive, transitive, and non-antisymmetric. Thus the following postulates hold:

\[
\forall x \; x|x, \quad (\text{EPT1})
\]
\[
\forall x \forall y \forall z (x|y \land y|z \implies x|z), \quad (\text{EPT2})
\]
\[
\exists x \exists y (x|y \land y|x \land x \neq y). \quad (\text{EPT3})
\]

(EPT3) assures that there is an object in an investigated universe, e.g. a quantum particle, a wave, which has its specifically correlated corresponding part. We can assume the stronger condition, i.e. that each object has its specifically correlated part:

\[
\forall x \exists y (x|y \land y|x \land x \neq y). \quad (\text{EPT3A})
\]

However, this modification does not influence the obtained results [3, 4]. The only advantage is that the system becomes symmetric in respect to the division relation. Clay and Loeb have shown that a model of classical mereology with an order relation and additional two postulates forms a model isomorphic to a Boolean algebra without a null element [13, 14]. Also a mereology without antisymmetry could become a classical model of mereology if we define an order relation in terms of the division relation, thus yielding a classical algebraic structure.

The advantage of the denial of antisymmetry is that we have a full theory based on pre-ordering, and not an extensional classical theory with pre-ordered sets. Such a framework opens new possibilities for description of different strange phenomena. Let us now apply this specific correlation to entangled particles.

### 3. Singlet states of quantum particles

In quantum mechanics two particles are considered indiscernible if the probability of finding each of them in a given state is identical; as a result the change of place or the change of spin of one of two indiscernible particles does not influence the probability [15–17]. Hence, if \( \psi \) is the wave function describing the state of a particle, then the probability \( P[x_i, x_j](t) \) of finding this particle in a given space at a given time, can be also expressed as \( |e^{i\theta}|P[x_i, x_j](t) \), where \( |e^{i\theta}| = 1 \) and \( e^{i\theta} \) is a complex number. If \( x \) is a complex number, then \( z = a + bi \), \( a, b \in R \) and \( |z| = \sqrt{a^2 + b^2} \). Since we know, that the subgroup \( \{ z \in C : |z| = 1 \} \) of the multiplicative group of the field \( C \) is isomorphic to the special group of rotations on a plane—\( SO(2) \), we can devise examples of possible pairs of reals: \( \left( \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right), \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right), (1, 0), (0, -1), \) etc, corresponding to different complex numbers \( z_1, \ldots, z_4 \), etc. Therefore, the probability of finding a given particle in a stated described by the functions \( z_i\psi \) or \( z_j\psi \) for \( i \neq j \) is equal.

Let us take a singlet state of a particle, i.e. a system composed of two entangled particles. Once these are created, they remain intimately linked to each other, even if they are separated by huge distances [18]. At the moment when, for example, the spin of a particular particle is measured in a given direction, the spin of the other entangled particle (measured in the same direction) always assumes the value exactly correlated to the other [19].

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1. \( z \in A \iff z \in B \implies A = B \).
2. For quantum particles, the term 'indiscernible' particles is applied, but this concept is synonymous with the term 'indistinguishable' particles.
3. The probability of finding a given particle \( x \) in the interval \([x_i, x_j]\) at the time \( t \) is: \( P[x_i, x_j](t) = \int_{x_i}^{x_j} |\psi(x, t)|^2 dx \).
Within a non-standard mereology, we can interpret a singlet state as a state when one particle is split into a pair of entangled particles, i.e. one particle is annihilated and in its place, two new indiscernible particles appear\(^4\). This new development, with the entangled particles constituting an indecomposable whole, is a singlet state.

Since each quantum state is described by the wave function \(\alpha \cdot \psi\), where \(|\alpha| = 1, \alpha \in C\), then the same formula also describes a whole being in a singlet state. We know that \(\psi\) is strictly connected to the probability \(P\), and \(P\) remains unchanged; therefore, let us investigate the operator \(\alpha\); specifically, its internal structure. Since \(\alpha\) is composed of two real parts \(a\) and \(b\), and intuitively, to split means to divide, by (EPT3), for some \(a\) and \(b\), we have: \(a|b \wedge b|a \neq b\), hence, we obtain two operators: \(A = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\) and \(B = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\), where \(A = -B\).

We can observe, that the pair \((A, B)\) defines a quaternion \([20]\); therefore, in this case we obtain two quaternions \(-\frac{1}{\sqrt{2}}(1, -1, -1, 1)\) and \(\frac{1}{\sqrt{2}}(-1, 1, 1, -1)\), each describing entangled particles. When we apply the normalization\(^5\), we will have the following unit quaternions: \(Q = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\) and \(P = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\).

Let us examine the quaternion \(Q^8\). It turns out that:

(i) \(Q^2 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -Q^4\),

(ii) \(Q^3 = (-1, 0, 0, 0)\),

(iii) \(Q^4 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = Q\),

(iv) \(Q^5 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = Q^2\),

(v) \(Q^6 = (1, 0, 0, 0)\).

Therefore, \([Q, Q^2, Q^3, Q^4, Q^5, Q^6] \cong \mathbb{Z}_6\). Since \(|Q| = 1\) then \(Q^{-1} = \frac{Q}{||Q||} = Q = Q^8\). In other terms: \([Q, -Q^{-1}, -I, -Q, Q^{-1}, I] \cong \mathbb{Z}_6[21]\).

For each of these quaternions we can calculate angles and axes of rotations in \(R^3\):

\[
Q: \gamma = \frac{2\pi}{3}, \quad \vec{u} = \frac{1}{\sqrt{3}}(-1, -1, 1);
\]

\[
Q^2: \gamma = \frac{4\pi}{3}, \quad \vec{u} = \frac{1}{\sqrt{3}}(-1, -1, 1);
\]

\[
Q^3: \gamma = 2\pi, \quad \vec{u} = \vec{0};
\]

\[
Q^4: \gamma = \frac{4\pi}{3}, \quad \vec{u} = \frac{1}{\sqrt{3}}(1, 1, -1);
\]

\[
Q^5: \gamma = \frac{2\pi}{3}, \quad \vec{u} = \frac{1}{\sqrt{3}}(1, 1, -1);
\]

\[
Q^6: \gamma = 0, \quad \vec{u} = \vec{0}.
\]

Therefore, for \(Q\) and \(Q^8\) we have the same angles of rotation equal to \(\frac{2\pi}{3}\), and both vectors designate the same straight line in \(R^3\), being the axis of rotation; however, the rotations are in opposite directions. An analogous situation happens for \(-Q^8\) and \(-Q\). Hence, the pair: \(Q, Q^8\) could represent one singlet state, and the pair: \(-Q, -Q^8\) another singlet state\(^7\). Moreover, it is interesting to observe that, since each of these quaternions is a generator of the same cyclic group, then each of them contains complete information about the whole system, i.e. about a singlet state.

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\(^4\) By analogy, we can think of the famous Banach-Tarski theorem on the paradoxical division of a ball.

\(^5\) \(|\psi| = \sqrt{||\psi||}\).

\(^6\) The results obtain for both quaternions are equal.

\(^7\) In literature we also find two different descriptions of singlet states [17, 19, 22].
To verify whether the obtained results actually do describe a singlet state, we have to figure out a SU(2) group, and verify that the wave function describing our bi-particle system is indecomposable.

Let us begin with the symmetry group SU(2). Since \( Q = (s, v) \), where \( s \in \mathbb{R} \) and \( v \) is a vector in \( \mathbb{R}^3 \), we can express the vector \( v \) as a linear combination of the Pauli matrices [22] as follows\(^8\)

\[
v \sigma = v_1 \sigma_1 + v_2 \sigma_2 + v_3 \sigma_3 = \begin{pmatrix} v_1 - iv_2 \\ v_1 + iv_2 \end{pmatrix}.
\]

If we apply the normalization to \( v \), i.e. \( w = \frac{v}{|v|} \), we can form the following exponential function for \( \theta \in [0, 2\pi] \):

\[
e^{i\theta w} = \cos \theta I + i w \sigma \sin \theta = \begin{pmatrix} \cos \theta + iw_3 \sin \theta & (w_2 + iw_1) \sin \theta \\ -(w_2 + iw_1) \sin \theta & \cos \theta - iw_3 \sin \theta \end{pmatrix}
\]

(5)

It can be easily verified that:

\[
(e^{i\theta w})^{-1} = (e^{i\theta w})^*, \quad \det (e^{i\theta w}) = 1.
\]

(6)

hence \( e^{i\theta w} \) forms a SU(2) group [19, 22].

For \( Q = \left( \frac{1}{2}, -\frac{i}{2}, -\frac{1}{2}, \frac{i}{2} \right) \), \( w = \left( -\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right) \), and

\[
e^{i\theta w} = \begin{pmatrix} \cos \theta, -\frac{\sqrt{3}}{3} \sin \theta, -\frac{\sqrt{3}}{3} \sin \theta, -\frac{\sqrt{3}}{3} \sin \theta \end{pmatrix}.
\]

(7)

For \( Q^* = \left( \frac{1}{2}, \frac{i}{2}, \frac{i}{2}, -\frac{1}{2} \right) \), \( w^* = \left( \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3} \right) \), and

\[
e^{i\theta w^*} = \begin{pmatrix} \cos \theta, -\frac{\sqrt{3}}{3} \sin \theta, \frac{\sqrt{3}}{3} \sin \theta, \frac{\sqrt{3}}{3} \sin \theta \end{pmatrix}.
\]

(8)

To create a SU(2) group for a bi-particle system, it suffices to take a tensor product of representations of SU(2) for \( Q \) and \( Q^* \) [22]. Hence,

\[
e^{i\theta w} \otimes e^{i\theta w^*}.
\]

(9)

Moreover, we can verify, that quaternions \( Q \) and \( Q^* \) represent particles with \( \frac{1}{2} \) spin. For the operator representing spinors, i.e. \( S_j = \frac{\sigma_j}{2} \), we have \( wS = \left( \begin{array}{cc} \frac{w_3}{2} & \frac{w_1 - iw_2}{2} \\ \frac{w_1 + iw_2}{2} & -\frac{w_3}{2} \end{array} \right) \), and

\[
\det (wS - \lambda I) = 0 \iff \lambda^2 - \frac{1}{4} = 0 \iff \lambda = \pm \frac{1}{2}.
\]

This means that we can pick out a specific direction \( w = \frac{v}{|v|} \), towards which a particle is split in two components of eigenvalues equal to \( + \frac{1}{2} \) and \( - \frac{1}{2} \).

Since any quaternion could be the limit of an infinite number of other quaternions expressed in a polar form [23], then the exponential function \( e^{i\theta w} \) could represent a quaternion wave function. Therefore, to prove that \( Q \) represents a singlet state, i.e. an entangled state, it is enough to show that \( Q \) is indecomposable.

Let us assume that \( Q \) is decomposable; hence, \( Q \) represented in a 4x4 matrix form\(^9\) would be equal to a tensor product of two 2x2 matrices:

\[
Q = \frac{1}{2} \left( \begin{array}{cccc} 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{array} \right) = \left( \begin{array}{cc} a_0 & a_1 \\ a_2 & a_3 \end{array} \right) \otimes \left( \begin{array}{cc} b_0 & b_1 \\ b_2 & b_3 \end{array} \right)
\]

When we solve the above system of equations, we obtain a contradiction:

\[
b_2 = -b_3, \quad b_2 = b_1, \quad b_2 \neq 0, \quad b_3 = 0.
\]

Moreover, one can verify that \( Q \) changes a decomposable state into an indecomposable state; therefore, \( Q \) is entangled in the same way as the universal quantum gate–\( M_{XX} \) [24]:

\(^8\)The Pauli matrices form a basis of a Hilbert space applied in quantum mechanics.

\(^9\) \( Q = q_0 + iq_1 + jq_2 + kq_3 = \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) \), where \( \alpha = q_0 - iq_3, \beta = -q_2 - iq_1; \alpha = \left( \begin{array}{cc} q_0 & q_1 \\ -q_3 & q_3 \end{array} \right) \).
\[
Q\left(\frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)\right) = Q\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)
\]

In conclusion, there exist specific pairs of quaternions: \(Q\), \(Q^*\) and \(-Q\), \(-Q^*\), which describe singlet states. Moreover, they are the only quaternions generating such states because they are generators of the same finite group. Finally, once quaternions for singlet states were created, we are able to generalize the method, and create pairs of quaternions for any, finite number of entangled particles. Such research might be useful for experimental physics, quantum cryptography and industry.

Acknowledgments

I would like to thank professor Pawel Horodecki for his valuable remarks which helped in improving this paper.

This work was performed thanks to the financial support of the Polish Ministry of Arts and Higher Education, no. 493/S/17.

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