The direct scattering problem for perturbed Kadomtsev-Petviashvili multi line solitons

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Abstract

Regular Kadomtsev-Petviashvili II (KPII) line solitons have been investigated and classified successfully by the Grassmannians. The inverse scattering method provides a promising and powerful approach to study the stability properties of \( \text{Gr}(N,M)_{>0} \) KP solitons. In this paper, we complete rigorous analysis for the direct scattering problem of perturbed \( \text{Gr}(N,M)_{>0} \) KP solitons.

1. Introduction

The KPII equation

\[
-4u_{x_3} + u_{x_1x_1x_1} + 6uu_{x_1} + 3u_{x_2x_2} = 0,
\]

introduced by Kadomtsev and Petviashvili \cite{15}, is a two-spatial dimensional integrable generalization of the Korteweg-de Vries (KdV) equation. It is an asymptotic model, where \( u = u(x) = u(x_1, x_2, x_3) \) represents the wave amplitude at the point \((x_1, x_2)\) for a fixed time \(x_3\), for dispersive systems in the weakly nonlinear, long wave regime, when the wavelengths in the transverse direction are much larger than in the direction of propagation (see \cite{4} for a formal derivation and \cite{20} for a rigorous one in the water wave context).

For \( u(x) \) decaying at infinity, the Cauchy problem or the well-posedness problem has been studied intensively by many scientists via the inverse scattering method or PDE techniques since late 1980’s (see \cite{16} or \cite{28} for references). On the other hand, interesting features of the water wave can be reproduced by the \( \text{Gr}(N,M)_{>0} \) KP solitons which are regular KP solutions in the entire \( x_1x_2 \) - plane and with peaks localized and nondecaying along certain line segments and rays (see \cite{17}, \cite{18} for a history). They are constructed as follows \cite{19}, given \( \kappa_1 < \cdots < \kappa_M \) and \( A = (a_{ij}) \in \text{Gr}(N,M)_{>0} \) (full rank \( N \times M \) matrices with non negative minors),

\[
(1.2) \quad u_0(x) = 2 \partial_{x_1}^2 \ln \tau(x),
\]
where the \( \tau \)-function is the Wronskian determinant

\[
\tau(x) = \begin{vmatrix}
    a_{11} & a_{12} & \cdots & a_{1M} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{N1} & a_{N2} & \cdots & a_{NM}
\end{vmatrix}
\begin{vmatrix}
    E_1 & \cdots & \kappa_1^{N-1}E_1 \\
    E_2 & \cdots & \kappa_2^{N-1}E_2 \\
    \vdots & \ddots & \vdots \\
    E_M & \cdots & \kappa_M^{N-1}E_M
\end{vmatrix}
\]

\[
= \sum_{1 \leq j_1 < \cdots < j_N \leq M} \Delta_{j_1,\ldots,j_N}(A) E_{j_1,\ldots,j_N}(x),
\]

where \( E_j(x) = \exp \theta_j = \exp(\kappa_j x_1 + \kappa_2^2 x_2 + \kappa_3^3 x_3) \), the coefficients \( \Delta_{j_1,\ldots,j_N}(A) \), called the Plücker coordinates, is the \( N \times N \) minor of the matrix \( A \) whose columns are labelled by the index set \( J = \{ j_1 < \cdots < j_N \} \), and the exponential term \( E_{j_1,\ldots,j_N}(x) \) is given by \( W(1, \ldots, E_{j_N}) \), i.e.,

\[
E_j(x) = E_{j_1,\ldots,j_N}(x) = \prod_{l<j} (\kappa_{j_l} - \kappa_{j_l}) \exp \left( \sum_{n=1}^{N} \theta_{j_n}(x) \right).
\]

Note that the simplest \( \text{Gr}(1,2)_{>0} \) KP solitons, corresponding to \( \kappa_1, \kappa_2 \), and \( A = (1, a) \), \( a > 0 \), are

\[
u_0(x) = (\kappa_1 - \kappa_2)^2 \frac{2}{2 \text{sech}^2(x) - \theta_2(x) - \theta_1(x) - \ln a}.
\]

They are the 1-line solitons discussed in literature. There has been important progress in combinatoric properties, classification theory, and wave resonant theory of KPII line solitons [17, 18] since 2000’s. Different perspectives about \( \text{Gr}(N,M)_{>0} \) KP solitons, the connection to real finite gap KP solutions, are investigated in [1, 2, 3].

Our interest is the stability problem of \( \text{Gr}(N,M)_{>0} \) KP solitons. The \( H^s \)-global well posedness of the KPII equation with initial data \( u_c(x_1, x_2) \) where \( u_c(x_1 - cx_3, x_2) \) is a KP solution has been solved by Molinet-Saut-Tzvetkov [21]. Their result shows that the deviation of the KPII solution from the initial data could evolve exponentially. Taking \( \kappa_1 = -\kappa_2 \), \( A = (1, 1) \), Mizumachi establishes \( L^2 \)-orbital stability and \( L^2 \)-instability theories by showing that the amplitude of the line soliton converges to that of the line soliton at initial time whereas jumps of the local phase shift of the crest propagate in a finite speed toward \( x_2 = \pm \infty \) [22, 23].

An alternative approach to study the stability problem of \( \text{Gr}(N,M)_{>0} \) KP solitons is the inverse scattering theory (IST) based on the Lax pair

\[
\begin{cases}
    (-\partial_{x_2}^2 + \partial_{x_1}^2 + u)\Phi(x, \lambda) = 0, \\
    (-\partial_{x_3} + \partial_{x_1}^3 + \frac{3}{4} u \partial_{x_1} + \frac{3}{4} u_{x_1} + \frac{3}{4} \partial_{x_1}^{-1} u_{x_2} - \lambda^3)\Phi(x, \lambda) = 0
\end{cases}
\]

of the KPII equation, where \( u(x) = \nu_0(x) + v_0(x) \) is a perturbation of the \( \text{Gr}(N,M)_{>0} \) KP soliton \( \nu_0(x) \) and \( \Phi(x, \lambda) \) is called the Jost solution. Pioneering research on the IST
with data nondecaying along a single line for KPII were derived by [4], [25]. For data being a perturbed Gr($N, M \geq 0$) KP soliton, Boiti-Pempenelli-Pogrebkov-Prinari introduce various methods, the Darboux transform [24], [5], [6], the twisting transformations [7], [8], and the $\tau$-function formulation [8], [13] to set foundations for the IST for KPII. Most significant achievements include deriving explicit formula of the Green function [9], [12], boundedness of $G_d$ (the discrete summand in the Green function) [9], [12], and deriving the $D^{\flat}$-symmetry of the eigenfunctions [13, (4.38)].

In the previous works [27], [28], via a KdV theory approach and a Sato theory approach, the direct scattering problem for a perturbed Gr($1, 2 \geq 0$) KP soliton, $u(x) = u_0(x) + v_0(x)$, is rigorously completed by establishing a $\lambda$-uniform estimate of the Green function $G$ of the heat operator $-\partial_{x_2}^2 + \partial_{x_1}^2 + 2\lambda \partial_{x_1} + u_0(x),\ (1.7) \quad |G(x, x', \lambda)| \leq C(1 + \frac{1}{\sqrt{|x_2 - x'_2|}}), \quad C \text{ a constant}$

the forward scattering transformation [28 Definition 3.2]

(1.8) \quad \mathcal{S}(u(x)) = \{0, \kappa_1, \kappa_2, s_d, s_c(\lambda)\},

and a Cauchy integral equation (or a $\overline{\partial}$ equation) with controllable singularities,

(1.9) \quad m(x, \lambda) = 1 + \frac{m_{\text{res}}(x)}{\lambda} + CTm,\ (1.10) m \in W, (see [27], [28] for definitions of $m_{\text{res}}, \mathcal{S}, T, W$). We remark that, with the help of (1.7), a Picard iteration can be utilized to solve the eigenfunction of the heat equation $(-\partial_{x_2}^2 + \partial_{x_1}^2 + 2\lambda \partial_{x_1} + u(x))m = 0$. Besides, the discrete scattering data $0, \kappa_j$ are determined by the Sato eigenfunction (see (2.2)), the Gr($1, 2 \geq 0$) KP soliton; and the continuous scattering data $s_c(\lambda)$ is a nonlinear Fourier transform of the initial perturbation $v_0(x)$. Finally, (1.9) and (1.10), a dynamic reformulation of the forward scattering transformation (1.8), serve as an analogue of the Gelfand-Levitan-Marchenko equation for the KdV equation in solving the inverse problem.

The goal of this paper is to demonstrate there is a strong analogy between rigorous analysis of the direct problem for perturbed Gr($1, 2 \geq 0$) and perturbed Gr($N, M \geq 0$) KP solitons. The key observation is that, based on a deep Sato theory, the kernel, defining the Green functions for a Gr($N, M \geq 0$) KP soliton, is found to share similar singular structures as that for a Gr($1, 2 \geq 0$) KP soliton. Hence one can adapt techniques in [28] to derive corresponding (1.7) and (1.9) for perturbed Gr($N, M \geq 0$) KP solitons. In defining corresponding forward scattering transformation $\mathcal{S}$ and eigenfunction space $W$, difficulties occur in search for a notion generalizing the norming constant $s_d$ in (1.8). This obstruction...
has been removed by the $D^b$-symmetry introduced in [13] (4.38)]. Heavily building on the $D^b$-symmetry, and taking it in directions for being linearisable and compatible with a Cauchy integrable equation possessing controllable singularities, we generalize norming constants and define corresponding $S$, $W$.

The contents of the paper are as follows. In Section 2, we justify the unique solvability of an associated heat equation with proper boundary value (see (2.1)). Precisely, a simplified argument of Boiti et al [13] to derive an explicit form of the Green function will be provided first. Then we will prove a $\lambda$-uniform estimate for the Green function, Proposition 2.1, which yields the existence of the eigenfunction $m(x, \lambda)$.

In Section 3, with the help of the $D^b$-symmetry, we shall characterize analytic and algebraic constraints of the eigenfunction $m(x, \lambda)$. To formulate a Cauchy integral equation with controllable singularities, a special renormalization, $\widetilde{m}(x, \lambda)$, is introduced to resolve the multiplicity of poles property of $m(x, \lambda)$ caused by the Sato eigenfunction in (2.1). Then we extract the scattering data $\widetilde{s}_c$, $\widetilde{D}$, define the forward scattering transform $S$, and prove that the scattering data can linearize the Kadomtsev-Petviashvili equation (1.1).

In Section 4, we define the eigenfunction space $W$ based on properties of $\widetilde{m}$. Then, for $x$ fixed, after providing important $L^\infty$ estimates on $CT\widetilde{m}$ (Theorem 6), we will derive a Cauchy integral equation with controllable singularities for $\widetilde{m}(x, \lambda)$ (Theorem 7). Together with the algebraic constraints in $W$, a closeness property is addressed at the end of the paper.

Throughout this paper, we set $x_3 = 0$ and $x = (x_1, x_2)$ unless special mention and denote $C$ various uniform constants which are independent of $x$, $\lambda$. For convenience, we provide a table of notations here.

### Notation and Terminology

| Notation | Explanation | Examples in text |
|----------|-------------|------------------|
| $\tau(x), \kappa_j, A, \theta_j(x)$, $J, \Delta_j(A), E_j(x)$; $\text{Gr}(N, M)\geq 0$ KPII solitons | definition of the tau function; regular multi-line soliton | $\text{(1.2)-(1.4)}$ |
| $u(x) = u_0(x) + v_0(x)$ | initial data with $u_0$ a $\text{Gr}(N, M)\geq 0$ KPII soliton and $v_0$ a perturbation | Theorem 6 |
| $L; L$ | the heat operator associated to $u_0$ | $\text{(2.3)}$ |
| $G(x, x', \lambda)$, $G(x, x', \lambda)$; $G_c(x, x', \lambda), G_d(x, x', \lambda)$, $G_c(x, x', \lambda), G_d(x, x', \lambda)$; $\theta(s), \delta(s)$ | Green functions; continuous Green functions; discrete Green functions | $\text{Definition 2.1}$ |
| | Heaviside function, Dirac function | $\text{Lemma 2.3}$ |
\begin{align*}
\varphi(x, \lambda), \chi(x, \lambda); & \quad \text{Sato eigenfunction, Sato normalized eigenfunction; values of Sato eigenfunction at } \kappa_j, \\
\bar{\varphi}_j(x), \bar{\chi}_j(x) & \quad \text{values of Sato normalized eigenfunction at } \kappa_j
\end{align*}

\begin{align*}
\psi(x, \lambda), \xi(x, \lambda); & \quad \text{Sato adjoint eigenfunction, Sato normalized adjoint eigenfunction; the residue of Sato adjoint eigenfunction at } \kappa_j, \\
\bar{\psi}_j(x), \bar{\xi}_j(x) & \quad \text{residue of normalized adjoint eigenfunction at } \kappa_j
\end{align*}

\begin{align*}
\Phi(x, \lambda), \eta(x, \lambda); & \quad \text{Jost solution, eigenfunction; renormalized eigenfunction, } \bar{\Phi}(x, \lambda) = e^{ax} \eta(x, \lambda); \\
\bar{\Phi}(x, \lambda), \bar{\eta}(x, \lambda); & \quad \text{satisfy (3.14) and (3.15); } \bar{\Phi}(x, \lambda) = (\Phi(\lambda), \ldots, \Phi(\lambda))
\end{align*}

\begin{align*}
D, D' & \quad \text{symmetries associated to Sato eigenfunction, and to Sato adjoint eigenfunction; symmetries associated to } \Phi; \text{symmetries associated to } \bar{\Phi}
\end{align*}

\begin{align*}
G \ast f & \quad \text{convolution operator; } G(x, \lambda) \ast f(y, \lambda) = \int G(x - y, \lambda) f(y, \lambda) \, dy
\end{align*}

\begin{align*}
m_{x, \alpha \beta}(x), \bar{m}_{x, \alpha \beta}(x, \lambda); & \quad \text{coefficients of the Laurent series of } m \text{ at } \lambda = 0; \\
\bar{m}_{x, \alpha}(x, \lambda); & \quad \text{residue and remainder of } m \text{ at } \lambda = z_n
\end{align*}

\begin{align*}
D_z, D_z^+; & \quad \text{disks and punctured disks at } z; \text{characteristic functions of } D_z, \text{and of a non single point set } \Lambda; \quad \bar{a} = a/N; \\
E_{\Delta}(\lambda), \bar{E}_{\Delta}(\lambda); & \quad \text{leading continuous of } G \text{ at } \lambda = \kappa_j
\end{align*}

\begin{align*}
\bar{\Theta}_j(x, \gamma_j); & \quad \text{factors of leading discontinuities of } m \text{ at } \lambda = \kappa_j
\end{align*}

\begin{align*}
s_c(\lambda), \bar{s}_c(\lambda); & \quad \text{continuous scattering data; poles of } m; \\
T, S & \quad \text{continuous scattering operator, forward scattering transform}
\end{align*}

\begin{align*}
\mathcal{C} = \mathcal{C}_\lambda & \quad \text{Cauchy integral operator}
\end{align*}

\begin{align*}
W & \quad \text{the eigenfunction space}
\end{align*}

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## 2. Direct Problem : the eigenfunction

The direct scattering problem starts with finding the Jost solution solving an associated heat equation with proper boundary value (see (2.11)). In this section, we shall simplify the argument of Boiti et al \[13\] to derive an explicit form of the Green function first. Then we provide a $\lambda$-uniform estimate for the Green function which yields the existence of the Jost solution.

Let $u(x) = u_0(x) + v_0(x)$, $u_0(x) = 2\partial_{x_1}^2 \ln \tau(x)$ be a Gr($N, M$)$_{\geq 0}$ KP soliton defined by the data $\kappa_1 < \cdots < \kappa_M$, $A = (a_{ij}) \in \text{Gr}(N, M)_{\geq 0}$ through (1.2), (1.3). Consider the
boundary value problem

\[
(2.1) \quad \begin{cases} 
(\partial^2_{x_2} + \partial^2_{x_1} + u(x))\Phi(x, \lambda) = 0, \\
\lim_{|x|\to \infty} e^{-\lambda x_1 + \lambda^2 x_2} (\Phi(x, \lambda) - \varphi(x, \lambda)) = 0,
\end{cases}
\]

for any fixed \( \lambda \neq 0, \lambda_R \notin \{\kappa_1, \ldots, \kappa_M\} \), where

\[
\varphi(x, \lambda) = e^{\lambda x_1 + \lambda^2 x_2} \sum_{1 \leq j_1 < \cdots < j_N \leq M} \Delta_{j_1, \ldots, j_N}(A) \frac{(1 - \frac{\kappa_{j_1}}{\lambda}) \cdots (1 - \frac{\kappa_{j_N}}{\lambda}) E_{j_1, \ldots, j_N}(x)}{\tau(x)}
\]

(2.2) \equiv e^{\lambda x_1 + \lambda^2 x_2} \chi(x, \lambda)

is the Sato eigenfunction and \( \chi(x, \lambda) \) is the Sato normalized eigenfunction \( \| \), \( \| \) Theorem 6.3.8., (6.3.13) \( , \| \) Proposition 2.2, (2.21) \( ) \) satisfying

\[
(2.3) \quad L\varphi(x, \lambda) = (\partial^2_{x_2} + \partial^2_{x_1} + u_0(x)) \varphi(x, \lambda) = 0,
\]
\[
L\chi(x, \lambda) = (\partial^2_{x_2} + \partial^2_{x_1} + 2\lambda \partial_{x_1} + u_0(x)) \chi(x, \lambda) = 0.
\]

Renormalizing \( \Phi(x, \lambda) = e^{\lambda x_1 + \lambda^2 x_2} m(x, \lambda) \), the boundary value problem (2.1) turns into

\[
(2.4) \quad \begin{cases} 
Lm(x, \lambda) = -v_0(x)m(x, \lambda), \\
\lim_{|x|\to \infty} (m(x, \lambda) - \chi(x, \lambda)) = 0,
\end{cases}
\]

for any fixed \( \lambda \neq 0, \lambda_R \notin \{\kappa_1, \ldots, \kappa_M\} \).

Definition 2.1. Define the Green functions, associated to a Gr(\( N, M \)\) \( \geq 0 \) KP soliton \( u_0(x) \), by \( G(x, x', \lambda) \) and \( G(x, x', \lambda) \) satisfying

\[
(2.5) \quad \begin{align*}
\mathcal{L}G(x, x', \lambda) &= \delta(x - x'), \\
L(G(x, x', \lambda)) &= \delta(x - x'), \\
G(x, x', \lambda) &= e^{\lambda(x_1 - x'_1) + \lambda^2(x_2 - x'_2)} G(x, x', \lambda).
\end{align*}
\]

Here \( \delta(x) \) is the dirac function at \( x = 0 \).

Following the approach of Boiti et al \( \| \), the key ingredient to derive the Green function is an orthogonality relation between Sato eigenfunctions and Sato adjoint eigenfunctions, defined by

\[
\psi(x, \lambda)
\]

(2.6) \equiv \sum_{1 \leq j_1 < \cdots < j_N \leq M} \Delta_{j_1, \ldots, j_N}(A) \frac{E_{j_1, \ldots, j_N}(x)}{1 - \frac{\kappa_{j_1}}{\lambda} \cdots 1 - \frac{\kappa_{j_N}}{\lambda}}
\]
\[
\equiv e^{-(\lambda x_1 + \lambda^2 x_2)} \xi(x, \lambda).
\]
Here $\xi(x, \lambda)$ is called the **Sato normalized adjoint eigenfunction** \cite[(2.12)]{13}, \cite[(6.3.13)]{14} Theorem 6.3.8., (6.3.13). Note

\begin{align}
\mathcal{L}^\dagger \psi(x, \lambda) &\equiv (\partial_{x_2} + \partial_{x_1}^2 + u_0(x)) \psi(x, \lambda) = 0,

L^\dagger \xi(x, \lambda) &\equiv (\partial_{x_2} + \partial_{x_1}^2 - 2\lambda \partial_{x_1} + u_0(x)) \xi(x, \lambda) = 0.
\end{align}

Moreover, for $\forall x \in \mathbb{R}^2$ fixed, $\chi(x, \cdot)$ is a rational function normalized at $\infty$ and with a pole at 0 of multiplicity $N$; and $\xi(x, \cdot)$ is a rational function normalized at $\infty$ with a zero at 0 of multiplicity $N$, and simple poles at $\kappa_1, \cdots, \kappa_M$. Therefore, values of $\varphi$ and residues of $\psi$ at $\kappa_j$ are significant in deriving the orthogonality relation. Let

\begin{align}
\varphi_j(x) &= \varphi(x, \kappa_j) = e^{\kappa_j x_1 + \kappa_j^2 x_2} \chi_j(x),

\psi_j(x) &= \text{res}_{\lambda=\kappa_j} \psi(x, \lambda) = e^{-(\kappa_j x_1 + \kappa_j^2 x_2)} \xi_j(x),
\end{align}

and

\begin{align}
\varphi(x, \kappa) &= (\varphi_1(x), \cdots, \varphi_M(x)),

\psi(x, \kappa) &= (\psi_1(x), \cdots, \psi_M(x)).
\end{align}

**Definition 2.2.** For a $\text{Gr}(N,M)_{\geq 0}$ KP soliton $u_0(x)$, write

\begin{equation}
A = \begin{pmatrix} I_N, & d \end{pmatrix} \pi,
\end{equation}

where $\pi$ is an $M \times M$ permutation matrix and $d$ is an $N \times (M - N)$ matrix. Define

\begin{align}
\mathcal{D} &= \text{diag}(\kappa_1^N, \cdots, \kappa_M^N) A^T,

\mathcal{D}' &= \begin{pmatrix} -d^T, & I_{M-N} \end{pmatrix} \pi \text{diag}(\kappa_1^{-N}, \cdots, \kappa_M^{-N}).
\end{align}

In \cite{8}, Boiti et al prove the following $\mathcal{D}$, $\mathcal{D}'$ symmetries about the Sato eigenfunction and the Sato adjoint eigenfunction for $\text{Gr}(N,M)_{\geq 0}$ KP solitons and an orthogonality relation between $\mathcal{D}$ and $\mathcal{D}'$. They yield an explicit form of the Green function and will induce full symmetries of $m(x, \lambda)$ (see \cite[3.17]{6}). In the following lemma we provide a direct and simplified proof via an approach suggested by Yuji Kodama.

**Lemma 2.1.** \cite{8} Define $\mathcal{D}$ and $\mathcal{D}'$ by Definition 2.2. Then

\begin{equation}
\mathcal{D}' \mathcal{D} = 0, \quad \varphi(x, \kappa) \mathcal{D} = 0, \quad \mathcal{D}' \psi(x, \kappa)^T = 0.
\end{equation}

**Proof.** From Definition 2.2

\begin{align*}
\mathcal{D}' \mathcal{D} &= \begin{pmatrix} -d^T, & I_{M-N} \end{pmatrix} \pi \text{diag}(\kappa_1^{-N}, \cdots, \kappa_M^{-N}) \text{diag}(\kappa_1^N, \cdots, \kappa_M^N) \pi^T \begin{pmatrix} I_N \\ d^T \end{pmatrix} \\
&= \begin{pmatrix} -d^T, & I_{M-N} \end{pmatrix} \begin{pmatrix} I_N \\ d^T \end{pmatrix} = 0.
\end{align*}
To prove \( \varphi \mathcal{D} = 0 \), writing
\[
A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1M} \\
\vdots & \vdots & \ddots & \vdots \\
a_{N1} & a_{N2} & \cdots & a_{NM}
\end{pmatrix},
\]
and using (1.4), for \( 1 \leq n \leq N \), we obtain

*the n-th column of \( \varphi(x, \kappa) \mathcal{D} \)*

\[
\begin{align*}
&= \sum_{m=1}^{M} \varphi_m \kappa_m^N \phi_{nm} \\
&= \frac{1}{\tau(x)} \sum_{m=1}^{M} a_{nm} \sum_{j_1, \ldots, j_N} \Pi_{k=1}^{N} \left( \kappa_m - \kappa_{j_k} \right) \Pi_{\beta<\alpha} \left( \kappa_{j_\alpha} - \kappa_{j_\beta} \right) \Delta_{j_1, \ldots, j_N}^{N} (A) e^{\theta_{j_1} + \cdots + \theta_{j_N} + \theta_m} \\
&= (-1)^{N-1} \sum_{k=1}^{N+1} \sum_{j_1, \ldots, j_{N+1}} \Pi_{\beta<\alpha} \left( \kappa_{j_\alpha} - \kappa_{j_\beta} \right) \Delta_{j_1, \ldots, j_{N+1}}^{N} (A) (-1)^k a_{n j_k} e^{\theta_{j_1} + \cdots + \theta_{j_{N+1}}} \\
&= (-1)^{N-1} \sum_{j_1, \ldots, j_{N+1}} E_{j_1, \ldots, j_{N+1}} \left[ \sum_{k=1}^{N+1} \Delta_{j_1, \ldots, j_{N+1}}^{N} (A) (-1)^k a_{n j_k} \right] = 0.
\end{align*}
\]

Here \((i_1, \ldots, i_k, \ldots, i_N) \equiv (i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_N)\) and for the last equality, we have used the following Plücker relation
\[
0 = \begin{vmatrix}
a_{1j_1} & \cdots & a_{nj_1} & a_{nj_1} \\
a_{1j_2} & \cdots & a_{nj_2} & a_{nj_2} \\
\vdots & \ddots & \vdots & \vdots \\
a_{1j_{N+1}} & \cdots & a_{nj_{N+1}} & a_{nj_{N+1}}
\end{vmatrix} = (-1)^{N-1} \sum_{k=1}^{N+1} \Delta_{j_1, \ldots, j_{N+1}}^{N} (A) (-1)^k a_{n j_k}.
\]

Hence the \( \mathcal{D} \)-symmetry of the Sato eigenfunction \( \varphi \mathcal{D} = 0 \) is verified.

For the \( \mathcal{D}' \)-symmetry, write
\[
B \equiv \left( -d^T, I_{M-N} \right) \pi = \begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1M} \\
\vdots & \vdots & \ddots & \vdots \\
b_{(M-N),1} & b_{(M-N),2} & \cdots & b_{(M-N),M}
\end{pmatrix},
\]
then
\[
\Delta_J (B) = \sigma (J, I) \Delta_J (A),
\]
\[
I = \{ i_1 < i_2 < \cdots < i_{M-N} \}, \quad J = \{ j_1 < j_2 < \cdots < j_N \},
\]
\[
I \cup J = \{ 1, \ldots, M \}, \quad \sigma (J, I) = (-1)^{\frac{N(N+1)}{2} + j_1 + \cdots + j_N}
\]

(2.13)
Here I associated to a Grassmannian which are orthogonal projectors, i.e.,

\[ (2.15) \]

The lemma follows by applying Lemme 2.1 and the projection

\[ P = \mathcal{D}(\mathcal{D}^T \mathcal{D})^{-1} \mathcal{D}^T, \]

\[ P' = (\mathcal{D}')^T (\mathcal{D}' \mathcal{D}'^T)^{-1} \mathcal{D}', \]

which are orthogonal projectors, i.e.,

\[ P^2 = P, \quad (P')^2 = P', \quad PP' = 0 = P'P, \]

\[ P + P' = I_{M \times M}. \]
Lemma 2.3. \[13\] Let \( \theta \) be the Heaviside function and \( \lambda = \lambda_R + i \lambda_I, \lambda_R, \lambda_I \in \mathbb{R} \). Then the Green function, associated to a \( Gr(N, M) \geq 0 \) KP soliton \( u_0(x) \), has the explicit form

\[
\mathcal{G}(x, x', \lambda) = \mathcal{G}_c(x, x', \lambda) + \mathcal{G}_d(x, x', \lambda),
\]

\[
\mathcal{G}_c = -\frac{\text{sgn}(x_2 - x_2')}{2\pi} \int R \theta((s^2 - \lambda^2_1)(x_2 - x_2')) \varphi(x, \lambda_R + is) \psi(x', \lambda_R + is) ds,
\]

\[
\mathcal{G}_d = -\theta(x_2' - x_2) \sum_{j=1}^M \theta(\lambda_R - \kappa_j) \varphi_j(x) \psi_j(x').
\]

Proof. After deriving the orthogonality relation, Lemma \[2.2\], one can apply Fourier inversion formula, meromorphic properties of normalized eigenfunctions \( \chi(x, \lambda) \) and \( \xi(x, \lambda) \), and the residue theorem to derive the explicit formula of the Green function \( \mathcal{G}(x, x', \lambda) \). For a direct proof, we refer to \[28, Lemma 2.2\].

\[\square\]

Definition 2.3. For \( z \in \mathcal{I} = \{0, \kappa_1, \cdots, \kappa_M\} \), define \( a = \frac{1}{2} \min_{j \neq j'} \{ |\kappa_j|, |\kappa_j - \kappa_{j'}| \} \),

\[
D_z = \{ \lambda \in \mathbb{C} : |\lambda - z| < a \}, \quad D_x^z = \{ \lambda \in \mathbb{C} : 0 < |\lambda - z| < a \};
\]

\[
D_{z, x} = \{ \lambda \in \mathbb{C} : |\lambda - z| < ra \}, \quad D_{z, x}^x = \{ \lambda \in \mathbb{C} : 0 < |\lambda - z| < ra \},
\]

characteristic functions \( \mathcal{E}_x(z, \lambda) \equiv 1 \) on \( D_z \), \( \mathcal{E}_x(z, \lambda) \equiv 0 \) on \( D_x^z \), \( \mathcal{E}_A(z, \lambda) \equiv 1 \) on \( A \), \( \mathcal{E}_A(z, \lambda) \equiv 0 \) elsewhere.

Definition 2.4. For any Schwartz function \( f(x) \), define the convolution operator

\[
G \ast f(x, \lambda) \equiv \iint G(x, x', \lambda) f(x') dx', \quad dx' = dx_1 dx_2.'
\]

So the boundary value problem \[2.4\] turns into

\[
m(x, \lambda) = \chi(x, \lambda) - G \ast v_0 m(x, \lambda)
\]

formally. Since the initial potential \( u(x) = u_0(x) + v_0(x) \) is independent of \( \lambda \). To utilize a Picard iteration to solve the above integral equation for \( m(x, \lambda) \), a \( \lambda \)-uniform estimate for the Green function \( G \) is necessary. We provide a \( \lambda \)-uniform estimate in the following proposition. Note that, throughout the paper, we use \( C \) to denote different uniform constants.

Proposition 2.1. The Green function \( G \), associated to a totally positive \( Gr(N, M) \geq 0 \) KP soliton \( u_0(x) \), satisfies, for \( \forall x_2 - x_2' \neq 0, \lambda \neq \kappa_1, \cdots, \kappa_M \),

\[
|G(x, x', \lambda)| \leq C(1 + \frac{1}{|x_2 - x_2'|}),
\]
and for any Schwartz function \( f \),

\[
\lim_{|x| \to \infty} G(x, x', \lambda) * f(x') \to 0.
\]

**Proof.** From (2.2), (2.6), and Lemma 2.3

\[
G_c(x, x', \lambda)
\]

\[
= -\frac{1}{2\pi} \int_{\mathbb{R}} ds \text{ sgn}(x_2 - x_2') \theta((s^2 - \lambda_1^2)(x_2 - x_2')) \chi(x, \lambda_R + is)
\]

\[
\times \xi(x', \lambda_R + is) e^{[\lambda - (\lambda_R + is)][(x_1' - x_1) + [\lambda^2 - (\lambda_R + is)^2](x_2' - x_2)]}
\]

\[
= -\left[ \frac{e^{i(\lambda R(x_1' - x_1) + 2\lambda R(x_2' - x_2))}}{2\pi} \right] \int_{\mathbb{R}} ds \text{ sgn}(x_2 - x_2') \theta((s^2 - \lambda_1^2)(x_2 - x_2'))
\]

\[
\times e^{(s^2 - \lambda_1^2)(x_2' - x_2) - is[(x_1' - x_1) + 2\lambda R(x_2' - x_2)]}
\]

\[
\times \sum_{J} \prod_{m \in J}(\lambda_R + is - \kappa_m) \Delta_J(A) E_J(x) \sum_{I} \prod_{n \in I}(\lambda_R + is - \kappa_n) \tau(x'),
\]

where \( J = \{j_1 < \cdots < j_N\}, I = \{i_1 < \cdots < i_M\}, \) \( 1 \leq j_m, i_n \leq M, \) and \( \Delta_J(A), \Delta_I(A) \) are the Plücker coordinates (see (1.3) and (1.1)). Since

\[
\Delta_J(A), \Delta_I(A), \left| \frac{E_J(x)}{\tau(x)} \right|, \left| \frac{E_I(x')}{\tau(x')} \right| < C,
\]

replacing 2 by general \( M \geq 2 \), (2.22) is in a complete analogy with [27] (2.21), (2.30), (2.31) and one can follow [27, Proposition 2.1] to decompose (2.22) into a combination of discontinuous, oscillatory, exponentially decaying factors and derive

\[
|G_c(x, x', \lambda)| \leq C(1 + \frac{1}{\sqrt{|x_2 - x_2'|}}),
\]

\[
\lim_{|x| \to \infty} G_c(x, x', \lambda) * f(x') \to 0.
\]

Via the Sato theory and ingenious ideas, Boiti, Pempinelli, and Pogrebkov derive a decomposition formula for \( G_d \) in [9] (3.15)-(3.17) or [12] (3.20)-(3.22). Then applying the totally positive condition \( A \in \text{Gr}(N, M)_{>0} \) to that decomposition formula and similar argument as that in [27, Proposition 2.1], one can justify

\[
|G_d(x, x', \lambda)| < C,
\]

\[
\lim_{|x| \to \infty} G_d(x, x', \lambda) * f(x') \to 0.
\]

\( \Box \)

Thanks to Proposition 2.1, especially, the technical reasons in the derivation of analytical properties of \( G_d \), we restrict our focus to totally positive \( \text{Gr}(N, M)_{>0} \) KP solitons from now on.
**Theorem 1.** Suppose $u(x) = u_0(x) + v_0(x)$ with $u_0$ a totally positive $Gr(N,M)_{>0}$ KP soliton, $\partial_x^k v_0 \in L^1 \cap L^\infty$, $|k| \leq 2$, $|v_0|_{L^1 \cap L^\infty} \ll 1$, and $v_0(x)$ a real-valued function. Then, for fixed $\lambda \in \mathbb{C} \setminus \{0, \kappa_1, \cdots, \kappa_M\}$, there is a unique solution $m(x,\lambda)$ to the boundary value problem (2.1).

**Proof.** Applying Proposition 2.1 and the assumption $\partial_x^k v_0 \in L^1 \cap L^\infty$, $0 \leq |k| \leq 2$, $|v_0|_{L^1 \cap L^\infty} \ll 1$, for $\lambda \neq 0$, $\lambda R \notin \{\kappa_1, \cdots, \kappa_M\}$, one can prove the unique solvability of an $L^\infty$ solution to (2.19). Moreover, from (2.5) and Lemma 2.3 the unique solvability of (2.1) is equivalent to that of (2.19).

### 3. Direct Problem : scattering data

The goal of the inverse scattering theory is to prove that the eigenfunction $m(x,\lambda)$ can be characterized by special analytical and algebraic constraints (scattering data) and these constraints will evolve with time linearly. In this section, we shall construct the forward scattering transformation which maps the potential $u(x)$ to these scattering data and prove that it can linearise the Kadomtsev-Petviashvili equation (1.1).

To start, we present analytical and algebraic constraints of the Green function.

**Lemma 3.1.** The Green function $G$, associated to a totally positive $Gr(N,M)_{>0}$ KP soliton $u_0(x)$, satisfies

\begin{equation}
(i) \quad G(x, x', \lambda) = G(x, x', \frac{\lambda}{\lambda_0});
\end{equation}

\begin{equation}
(ii) \quad \text{nearly the pole of the Sato normalized eigenfunction } \chi, \text{ namely, } \lambda \in D_0^x, \quad G(x, x', \lambda) = \sum_{m=0}^{N-1} G_0^{(m)}(x, x') \lambda^m + \omega_0(x, x', \lambda),
\end{equation}

\begin{align*}
&\left|\lambda^m G_0^{(m)}\right|_{L^\infty(D_0)} \leq \frac{G_0^{(m)}(x, x')}{1 + |x - x'|^m} \left|\lambda^{n-k} (1 + |x - x'|^{n-k})\right|_{L^\infty(D_0)} \\
&\leq C(1 + \frac{1}{\sqrt{|x_2 - x_1|}}), \quad 0 \leq m \leq N - 1, \quad 0 \leq k \leq n \leq N;
\end{align*}

\begin{equation}
(iv) \quad \text{nearly the poles of the Sato normalized adjoint eigenfunction } \xi, \text{ i.e., } \lambda \in D_{\kappa_j}^x, \quad G(x, x', \lambda) = \mathcal{G}_j(x, x') + \frac{1}{\pi} \chi_j(x) \xi_j(x') \cot^{-1} \frac{\lambda R - \kappa_j}{|\lambda|} + \omega_j(x, x', \lambda),
\end{equation}

\begin{align*}
&\cot^{-1} \frac{\lambda R - \kappa_j}{|\lambda|} = \left\{ \begin{array}{ll}
\alpha, & 0 < \alpha \leq \pi, \quad \lambda \in D_{\kappa_j}^x; \\
2\pi - \alpha, & \pi \leq \alpha < 2\pi, \quad \lambda \in D_{\kappa_j}^x,
\end{array} \right.
\end{align*}

\begin{align*}
&\left|\mathcal{G}_j\right|_{L^\infty(D_{\kappa_j})}, \quad \left|\omega_j\right|_{L^\infty(D_{\kappa_j})}, \quad \left|\frac{\omega_j(x, x', \lambda)}{(\lambda - \kappa_j)(1 + |x' - x|)}\right|_{L^\infty(D_{\kappa_j})}
\end{align*}
Moreover, then the Green function \( G \) satisfies the symmetry (under the mod \( M \)-condition) \([13]\)

\[
G(x, x', \kappa_j^+) = G(x, x', \kappa_{j-1}^+) + \phi_j(x)\psi_j(x'), \; \kappa_j^+ = \kappa_j + 0^+.
\]

**Proof.** We omit the details for the proof of (3.1)-(3.3) and refer [27, Lemma 2.3, Theorem 1] for a similar detailed proof.

To prove (3.4), one can follow the argument in [13, §3] which relies on the condition \(|x - x'|\) is bounded. Precisely, from (2.2), (2.6), (2.8), Lemma 2.3, we decompose

\[
g_M(x, x', \lambda) = g_M(x, x', \lambda) + f(x, x', \lambda),
\]

\[
g_M(x, x', \lambda) = -\frac{\theta(x_2 - x'_2)}{2\pi} \int_{\mathbb{R}} e^{(\lambda_R + i\kappa)(x_1 - x'_1) + (\lambda_R + i\kappa)^2(x_2 - x'_2)} \left( 1 + \sum_{j=1}^M \frac{\chi_j(x)\xi_j(x')}{i\sigma + \lambda_R - \kappa_j} \right) ds
\]

\[
- 2\pi \sum_{j=1}^M \theta(\lambda_R - \kappa_j)\phi_j(x)\psi_j(x'),
\]

\[
f(x, x', \lambda) = \frac{1}{2\pi} \int_{-|\lambda|}^{|\lambda|} e^{(\lambda_R + i\kappa)(x_1 - x'_1) + (\lambda_R + i\kappa)^2(x_2 - x'_2)} \left( 1 + \sum_{j=1}^M \frac{\chi_j(x)\xi_j(x')}{i\sigma + \lambda_R - \kappa_j} \right) ds
\]

\[
- \sum_{j=1}^M \theta(\lambda_R - \kappa_j)\phi_j(x)\psi_j(x').
\]

If \(|x - x'|\) is bounded, applying the residue theorem,

\[
g_M(x, x', \lambda)
\]

\[
= -\frac{\theta(x_2 - x'_2)}{2\pi} \int_{\mathbb{R}} e^{(\lambda_R + i\kappa)(x_1 - x'_1) + (\lambda_R + i\kappa)^2(x_2 - x'_2)} ds
\]

\[
+ \sum_{j=1}^M \chi_j(x)\xi_j(x') \int_{\mathbb{R}} \frac{1}{i\sigma + \lambda_R - \kappa_j} ds - 2\pi \sum_{j=1}^M \theta(\lambda_R - \kappa_j)\phi_j(x)\psi_j(x')]
\]

\[
= -\frac{\theta(x_2 - x'_2)}{2\pi} \int_{\mathbb{R}} e^{(\lambda_R + i\kappa)(x_1 - x'_1) + (\lambda_R + i\kappa)^2(x_2 - x'_2)} ds
\]

\[
+ \sum_{j=1}^M \chi_j(x)\xi_j(x') \int_{\mathbb{R}} \frac{1}{i\sigma + \lambda_R - \kappa_j} ds
\]

\[
\equiv g_M(x, x').
\]
Using a similar argument, Lemma 2.2, and
\[
\int_{-|\lambda_I|}^{|\lambda_I|} \frac{1}{s - i(\lambda_R - \kappa_j)} ds = 2\pi i[\theta(\lambda_R - \kappa_j) - 1] + 2i \cot^{-1} \frac{\lambda_R - \kappa_j}{|\lambda_I|}, \quad \lambda \in D_{\kappa_j},
\]
one obtains
\[
(3.7) \quad f(x, x', \lambda) = \frac{1}{2\pi} \int_{-|\lambda_I|}^{|\lambda_I|} \frac{e^{(\lambda_R+is)(x_1-x'_1)+(\lambda_R+is)^2(x_2-x'_2)}}{is + \lambda_R - \kappa_j} ds
\]
\[
+ \frac{1}{2\pi} \sum_{j=1}^{M} \lambda_j(x) \xi_j(x') \int_{-|\lambda_I|}^{|\lambda_I|} \frac{e^{(\lambda_R+is)(x_1-x'_1)+(\lambda_R+is)^2(x_2-x'_2)} - e^{\kappa_j(x_1-x'_1)+\kappa_j^2(x_2-x'_2)}}{is + \lambda_R - \kappa_j} ds
\]
\[
+ \frac{1}{2\pi} \sum_{j=1}^{M} \varphi_j(x) \psi_j(x') \{2\pi[\theta(\lambda_R - \kappa_j) - 1] + 2 \cot^{-1} \frac{\lambda_R - \kappa_j}{|\lambda_I|}\}
\]
\[
- \sum_{j=1}^{M} \theta(\lambda_R - \kappa_j) \varphi_j(x) \psi_j(x').
\]
Combining (3.5) - (3.7), and applying Lemma 2.2 if \(|x - x'|\) is bounded, we obtain
\[
(3.8) \quad \mathcal{G}(x, x', \lambda) = g_M(x, x') + \frac{1}{2\pi} \int_{-|\lambda_I|}^{|\lambda_I|} \frac{e^{(\lambda_R+is)(x_1-x'_1)+(\lambda_R+is)^2(x_2-x'_2)}}{is + \lambda_R - \kappa_j} ds
\]
\[
+ \frac{1}{2\pi} \sum_{j=1}^{M} \lambda_j(x) \xi_j(x') \int_{-|\lambda_I|}^{|\lambda_I|} \frac{e^{(\lambda_R+is)(x_1-x'_1)+(\lambda_R+is)^2(x_2-x'_2)} - e^{\kappa_j(x_1-x'_1)+\kappa_j^2(x_2-x'_2)}}{is + \lambda_R - \kappa_j} ds
\]
\[
+ \frac{1}{2\pi} \sum_{j=1}^{M} \varphi_j(x) \psi_j(x') \cot^{-1} \frac{\lambda_R - \kappa_j}{|\lambda_I|}.
\]
Consequently, using \(|x - x'| < \infty\),
\[
(3.9) \quad \mathcal{G}(x, x', \lambda) = g_M(x, x') + \sum_{j=1}^{M} \varphi_j(x) \psi_j(x') \theta(\kappa_j - \lambda), \quad \forall \lambda \in \mathbb{R}.
\]
So (3.4) is verified.

The following lemma will be used to show that \(m(x, \lambda)\) satisfies a \(\overline{\partial}\) problem.

**Lemma 3.2.** [13, 25] For \(\lambda_I \neq 0\),
\[
\partial_{\lambda} \mathcal{G}(x, x', \lambda) = -\frac{\text{sgn}(\lambda_I)}{2\pi i} e^{(\overline{\lambda} - \lambda)(x_1-x'_1)+(\overline{\lambda}^2-\lambda^2)(x_2-x'_2)} \chi(x, \overline{\lambda}) \xi(x', \overline{\lambda}).
\]
Proof. Via the same argument as that in the proof in [28, Lemma 2.4], one obtains

\begin{equation}
\partial_\lambda G_c(x, x', \lambda) = -\frac{\text{sgn}(\lambda)}{2\pi i} e^{(x-x')^T(\lambda x'-\lambda x'')} \chi(x, \lambda) \xi(x', \lambda) + \frac{1}{2} \theta(x_2' - x_2) \sum_{j=1}^{M} e^{-\lambda(x_2-x_2')^T-\lambda^2(x_2-x_2')} \varphi_j(x) \psi_j(x') \delta(\lambda R - \kappa_j),
\end{equation}

\begin{equation}
\partial_\lambda G_d(x, x', \lambda) = -\frac{1}{2} \theta(x_2' - x_2) \sum_{j=1}^{M} e^{-\lambda(x_2-x_2')^T-\lambda^2(x_2-x_2')} \varphi_j(x) \psi_j(x') \delta(\lambda R - \kappa_j).
\end{equation}

Hence follows the lemma.

Based on the characterization of the Green function $G$, we can provide analytic constraints, algebraic constraints, and the $\tilde{\mathcal{D}}$ data of $m$ in Theorem \[ and \[.

**Theorem 2.** Let $u(x) = u_0(x) + v_0(x)$, $u_0(x)$ a totally positive $Gr(N, M) > 0$ KP soliton, $v_0(x)$ a real-valued function with $\partial_x^2 v_0 \in L^1 \cap L^\infty$ for $|k| \leq 2$ and $|v_0|_{L^1 \cap L^\infty} \ll 1$. Then the eigenfunction $m(x, \lambda)$ derived from Theorem \[ satisfies

\begin{equation}
(i) \ m(x, \lambda) = \overline{m(x, \lambda)};
\end{equation}

\begin{equation}
(ii) \ |1 - E_0 m|_{L^\infty} \leq C;
\end{equation}

\begin{equation}
(iii) \ near \ the \ pole, \ namely, \ \lambda \in D_0^\times,
\end{equation}

\begin{equation}
m(x, \lambda) = \sum_{n=1}^{N} \frac{m_{\text{res}, n}(x)}{\lambda^n} + m_{0, r}(x, \lambda), \ m_{\text{res}, n}(x) \in \mathbb{R},
\end{equation}

\begin{equation}
\left| \frac{m_{\text{res}, n}}{1 + |x|^{N-n}} \right|_{L^\infty(D_0)}, \left| \frac{m_{0, r}}{1 + |x|^N} \right|_{L^\infty(D_0)} \leq C |v_0|_{L^1 \cap L^\infty}, \ 1 \leq n \leq N;
\end{equation}

\begin{equation}
(iv) \ near \ the \ discontinuities \ \kappa_j, \ namely, \ \lambda \in D_{\kappa_j}^\times,
\end{equation}

\begin{equation}
m(x, \kappa_j + 0^+ e^{i\alpha}) \equiv m(x, \lambda_0) = \frac{\Theta_j(x)}{1 - \gamma_j \cot^{-1} \frac{\lambda_0 R - \kappa_j}{\lambda_0},}
\end{equation}

\begin{equation}
|\partial_x m(x, \lambda)|_{L^\infty(D_{\kappa_j})}, \left| \frac{\partial_x m(x, \lambda)}{1 + |x|} \right|_{L^\infty(D_{\kappa_j})} \leq C |v_0|_{L^1 \cap L^\infty}, \ 1 \leq j \leq M,
\end{equation}

\begin{equation}
\Theta_j(x) = [1 + \Theta_j(x, x') + v_0(x')]^{-1} \chi_j(x'), \ \gamma_j = -\frac{1}{\pi} \int \xi_j(x) v_0(x) \Theta_j(x) dx.
\end{equation}

Moreover, the Jost function $\Phi$ satisfies the algebraic constraints \[.

\begin{equation}
\Phi(x, \kappa) D^\phi = 0,
\end{equation}

where

\begin{equation}
\Phi(x, \kappa) = (\Phi_1(x), \ldots, \Phi_M(x)), \ \Phi_j(x) = e^{\kappa_j x_1 + \kappa_j^2 x_2} m(x, \kappa_j^+), \ k_j^+ = k_j + 0^+,
\end{equation}
\[ D_{ji} = D_{ji} + \sum_{l=j}^{M} c_{jl} D_{li} \frac{1}{1 - c_j}, \quad 1 \leq j \leq M, \quad 1 \leq i \leq N \]

\[ c_{jl} = -\int v_0(x) \Psi_j(x) \varphi_l(x) dx, \quad c_j = c_{jj}, \quad \Psi_j(x) = res_{\lambda \to \kappa_j^+} \Psi(x, \lambda), \]

\[ L^\dagger \Psi(x, \lambda) \equiv (\partial_{x_2} + \partial_{x_1}^2 + u_0(x)) \Psi(x, \lambda) = -v_0(x) \Psi(x, \lambda) \]

and \( D \) defined by Definition 2.2.

**Proof. Step 1 (Proof of (3.12)-(3.14)):** The reality condition (3.12) follows from (2.19), (3.1) and \( v_0(x) \) is real-valued. Applying (2.19) and Proposition 2.1,

\[ |(1 - E_0)m(x, \lambda)| = |1 + G * v_0)^{-1}(1 - E_0)\chi| \leq C. \]

So (3.13) is justified. Besides, for \( \lambda \in D_0 \), (3.2) and the resolvent identity imply, there exist operators \( P_0^{(k)}(x), P_0^{(N)}(x, \lambda), 0 \leq k \leq N - 1 \),

\[ P_0^{(k)} f = \int R \Phi_0^{(k)}(x, x') f(x') dx', \]

\[ \left| \frac{\Phi_0^{(k)}}{1 + |x - x'| k} \right|, \quad \left| \frac{\Phi_0^{(N)}}{1 + |x - x'| N} \right| \leq C \left( 1 + \frac{1}{\sqrt{|x_2 - x_2'|}} \right), \]

such that

\[ (1 + G * v_0)^{-1} \]

\[ = (1 + \left[ \sum_{k=0}^{N-1} \lambda^k G_0^{(k)} + \omega_0 \right] * v_0)^{-1} \]

\[ = \sum_{k=0}^{N-1} (1 + G_0^{(k)} * v_0)^{-1} \left[ \Delta G_0^{(0)} * v_0(1 + G_0^{(0)} * v_0)^{-1} \right]^k \]

\[ + \left[ \Delta G_0^{(0)} * v_0(1 + G_0^{(0)} * v_0)^{-1} \right]^N (1 + G * v_0)^{-1} \]

\[ = \sum_{k=0}^{N-1} P_0^{(k)}(x) \lambda^k + P_0^{(N)}(x, \lambda). \]

From (2.21), one can set \( \chi(x, \lambda) = \sum_{m=0}^{N} \frac{\chi_{res,m}(x)}{\lambda^m} \) for \( \chi_{res,m}(x) \in L^\infty \). Along with (3.18), (3.19), and Proposition 2.1,

\[ m(x, \lambda) = \left( \sum_{k=0}^{N-1} P_0^{(k)}(x) \lambda^k + P_0^{(N)}(x, \lambda) \right) \sum_{m=0}^{N} \frac{\chi_{res,m}(x)}{\lambda^m}, \]
which yields

\[ m_{res,n}(x) = \sum_{l=0}^{N-n} P_{0}^{(l)} \chi_{res,n+l}(x), \quad 1 \leq n \leq N, \]

\[ m(x, \lambda) = \sum_{n=1}^{N} \frac{m_{res,n}(x)}{\lambda^n} + m_{0,r}(x, \lambda). \]

(3.20)

So (3.14) follows from (3.18) and (3.20).

**Step 2 (Proof of (3.15) - (3.17))**

Proving (3.15) can be derived similarly as that in the proof of [28, Theorem 1, Step 3].

The \( D^N \)-symmetry (3.16) is crucial for defining the discrete data (norming constants) for perturbed \( \text{Gr}(N,M) \geq 0 \) solitons when \( M > 2 \). For convenience, we sketch the proof and refer details to [13, §4].

First of all, Proposition 2.1 implies there exists a unique \( K(x,x',\lambda) \) satisfying the following integral equations,

\[ K(x,x',\lambda) = G(x,x',\lambda) - G(x,y,\lambda) * v_0(y) K(y,x',\lambda) \]

\[ \equiv G - G * v_0 K, \]

(3.21)

\[ K(x,x',\lambda) = G(x,x',\lambda) - K(x,y,\lambda) * v_0(y) G(y,x',\lambda) \]

\[ \equiv G - K * v_0 G. \]

The function \( K(x,x',\lambda) \) is called the total Green function since

\[ \overrightarrow{L_{v_0}} K = K \overrightarrow{L_{v_0}} = \delta(x - x'), \]

(3.22)

\[ L_{v_0} = L + v_0, \quad L = -\partial_{x_2} + \partial_{x_1}^2 + u_0(x). \]

Here \( \overrightarrow{L} \) denotes the operator \( L \) applying to the \( x \) variable of \( K \) and \( \overrightarrow{L} \) denotes the operator applying to the \( x' \) variable of \( K \). Moreover, the eigenfunction \( \Phi(x,\lambda) \) and adjoint eigenfunction \( \Psi(x,\lambda) \) defined by (2.1) and (3.17) satisfy

\[ \Phi(x,\lambda) = K(x,x',\lambda) *_{x'} \overrightarrow{L} \varphi(x',\lambda) \equiv K * \overrightarrow{L} \varphi, \]

(3.23)

\[ \Psi(x',\lambda) = \psi(x,\lambda) *_{x} \overrightarrow{L} K(x,x',\lambda) \equiv \psi * \overrightarrow{L} K. \]

Taking \( \lambda \) limits of (3.21) and (3.23) yield

\[ K_j = G_j - G_j * v_0 K_j, \]

(3.24)

\[ \Phi_j = K_j * \overrightarrow{L} \varphi_j, \quad \Psi_j = \psi_j * \overrightarrow{L} K_j, \]

where we use the convention \( G_j = G(x,x',\kappa_j^+), \) \( K_j = K(x,x',\kappa_j^+), \) and \( \Phi_j = \Phi(x,\kappa_j^+) \).

Using (3.21)-(3.24), (3.4), and

\[ c_{jl} = - \int v_0(x) \Psi_j(x) \varphi_l(x) dx = \int \psi_j(x) \overrightarrow{L} \varphi_l(x) dx = \Psi_j * \overrightarrow{L} \varphi_l, \]
one can derive

\[ K_{j-1} = K_j + \frac{\Phi_j(x)\Psi_j(x')}{1 - c_j}, \]

which implies

\[ \sum_{j=1}^{M} \frac{\Phi_j(x)\Psi_j(x')}{1 - c_j} = 0. \]

and

\[ K_l = K_i + \sum_{j=l+1}^{i+M} \frac{\Phi_j(x)\Psi_j(x')}{1 - c_j}. \]

Here \( c_j = c_{jj} \) and the mod \( M \)-condition is adopted.

Applying \( \mathcal{L} \varphi_i \) to (3.27) from the right and using (3.24), we obtain

\[ K_l \leftrightarrow \mathcal{L} \varphi_i = \Phi_i + \sum_{j=l+1}^{i+M} \frac{\Phi_j(x)c_{ji}}{1 - c_j}. \]

Summing (3.28) up with the matrix \( D_{im} \) and using the symmetry (2.12), we derive

\[ \sum_{i=1}^{M} \Phi_i D_{im} + \sum_{i=1}^{M} \sum_{j=l+1}^{i+M} \frac{\Phi_j(x)c_{ji}D_{im}}{1 - c_j} = 0. \]

Taking \( l = M \) in (3.29) and using (3.26), we obtain (3.16).

We emphasize that the remarkable \( D^\sharp \)-symmetry (3.16), introduced by Boiti, Pempenny, and Pogrebkov [13], depends not only the data \( \kappa_1 < \cdots < \kappa_M, A = (a_{ij}) \in \text{Gr}(N,M)_{\geq 0} \), but also on the perturbation \( v_0(x) \). In case \( (N,M) = (1,2) \), it defines a different but equivalent symmetry from the KdV symmetry [27, Theorem 2, Theorem 4, (3.5), (3.7), (3.40)], [28, (3.4)].

**Theorem 3.** Let \( u(x) = u_0(x) + v_0(x) \), \( u_0(x) \) a totally positive \( \text{Gr}(N,M)_{\geq 0} \) KP soliton, \( v_0(x) \) a real-valued function with \( \partial^k_x v_0 \in L^1 \cap L^\infty \) for \( |k| \leq 2 \) and \( |v_0|_{L^1 \cap L^\infty} \ll 1 \). Then the solution \( m(x,\lambda) \) obtained from Theorem 1 satisfies

\[ \partial^\lambda m(x,\lambda) = s_c(\lambda)c^{(\lambda-\lambda)x_1 + (\lambda^2-\lambda^2)x_2}m(x,\lambda), \lambda_I \neq 0, \]

with

\[ s_c(\lambda) = \frac{\text{sgn} (\lambda_I)}{2\pi i} \int e^{-(\lambda-\lambda)x_1 + (\lambda^2-\lambda^2)x_2} \xi(x,\lambda)v_0(x)m(x,\lambda)dx, \]

\[ s_c(\lambda) = \overline{s(\lambda)}. \]
Moreover,
\[(3.32)\quad |(1 - E_{\cup_{1 \leq j \leq M} D_{x_j}}) s_c|_{L^2(\{|\lambda| > \lambda_0\} \cap L^\infty)} \leq C \sum_{|k| \leq 2} |\partial^k_x v_0|_{L^1 \cap L^\infty},\]
and
\[(3.33)\quad s_c(\lambda) = \begin{cases} \frac{\text{sgn}(\lambda I)}{\lambda^{\kappa_j}} - \frac{\gamma_I}{\lambda^{\kappa_j} - \lambda_{\lambda I}} + \text{sgn}(\lambda_I) h_j(\lambda), & \lambda \in D_{\kappa_j}^\infty, \\ \text{sgn}(\lambda) h_0(\lambda), & \lambda \in D_0^\infty, \end{cases}
\]
for \(1 \leq j \leq M,\) with
\[(3.34)\quad |\gamma_j|_{L^\infty} \leq |v_0|_{L^1}, \quad \sum_{0 \leq |l| \leq 1} (|\partial^l_{\lambda R} \partial^l_{\lambda I} h_j|_{L^\infty} + |\partial^l_{\lambda R} \partial^l_{\lambda I} h_0|_{L^\infty}) \leq C (1 + |x|^2) |v_0|_{L^1 \cap L^\infty},
\]
h_j(\lambda) = -\overline{h_j(\lambda)}, \quad h_0(\lambda) = -\overline{h_0(\lambda)}.

**Proof.** The theorem can be proved by the same approach as that in [28, Theorem 2, (3.10)-(3.12)]. \(\square\)

One can adapt the approach as that in [28, Theorem 3, Theorem 4] to derive the following Cauchy integral equation for \(m(x, \lambda),\)
\[(3.35)\quad m(x, \lambda) = 1 + \sum_{n=1}^{N} \frac{m_{\text{res}, n}(x)}{\lambda^n} + CTm, \quad \forall \lambda \neq 0.
\]
Nevertheless, if we use the above Cauchy integral equation to solve the inverse problem, then the multiple pole property at 0 could form an obstruction. Since it yields a singular integral with a kernel blowing up of order \(N + 1\) at 0 and causes troubles when \(N \geq 2.\) To remedy the situation, we introduce

**Definition 3.1.** A regularization eigenfunction is defined by
\[(3.36)\quad \tilde{m}(x, \lambda) = \frac{(\lambda - z_1)^{N-1}}{\prod_{2 \leq n \leq N} (\lambda - z_n)} m(x, \lambda),
\]
z_1 = 0, \(z_n = \frac{n-1}{N} a,\) \(2 \leq n \leq N;\)
\(\tilde{a} = a/N, \quad \tilde{D}_z = D_{z,\tilde{a}}, \quad \tilde{D}_z^\times = \tilde{D}_z/\{z\},\)
where \(a\) and \(D_{z,r}\) are defined by Definition 2.5.

**Theorem 4.** If \(u(x) = u_0(x) + v_0(x),\) \(u_0(x)\) is a totally positive \(\text{Gr}(N, M)_{>0}\) KP soliton, \(v_0(x)\) a real-valued function with \(\partial^k_x v_0 \in L^1 \cap L^\infty,\) \(|k| \leq 2,\) \(|v_0|_{L^1 \cap L^\infty} \ll 1,\) then for fixed \(\lambda \in \mathbb{C}\backslash\{z_n, \kappa_j\},\) \(1 \leq n \leq N, 1 \leq j \leq M,\) there is a unique solution \(\tilde{m}(x, \lambda)\) to the spectral equation
\[(3.37)\quad \left\{ \begin{array}{l}
L \tilde{m}(x, \lambda) = -v_0(x)\tilde{m}(x, \lambda), \\
\lim_{|x| \to \infty} (\tilde{m}(x, \lambda) - \frac{(\lambda - z_1)^{N-1}}{\prod_{2 \leq n \leq N} (\lambda - z_n)} \chi(x, \lambda)) = 0,
\end{array} \right\}
\]
and

\[(3.38)\quad (i) \quad \tilde{m}(x, \lambda) = \overline{m(x, \lambda)};\]

\[(3.39)\quad (ii) \quad |(1 - E_{1 \leq n \leq N} \tilde{m}(x, \lambda)| \leq C;\]

\[(3.40)\quad (iii)\text{ near the poles } z_n, \text{ namely, } \lambda \in \tilde{D}^0_{z_n}, 1 \leq n \leq N,\]

\[
\tilde{m}(x, \lambda) = \frac{\tilde{m}_{z_n, \text{res}}(x)}{\lambda - z_n} + \tilde{m}_{z_n, \text{rel}}(x, \lambda), \quad \tilde{m}_{z_n, \text{res}}(x) \in \mathbb{R},
\]

\[
|\tilde{m}_{z_n, \text{res}}(x)|_{L^\infty}, \quad |(\lambda - z_n)\tilde{m}_{z_n, \text{rel}}(x, \lambda)|_{L^\infty(\tilde{D}_{z_n})}, \quad \left|\frac{\tilde{m}_{z_n, \text{rel}}(x, \lambda)}{1 + |x|}\right|_{L^\infty(\tilde{D}_{z_n})} \leq C|v_0|_{L^1 \cap L^\infty};
\]

\[(3.41)\quad (iv)\text{ near the discontinuities, namely, } \lambda \in \tilde{D}^0_{\kappa_j}, 1 \leq j \leq M,\]

\[
\tilde{m}(x, \kappa_j + 0^+ e^{i\alpha}) \equiv \tilde{m}(x, \lambda_0) = \frac{\Pi_{2 \leq n \leq N}(\kappa_j - z_n)}{1 - \gamma_j \cot^{-1} \frac{\lambda_j - \kappa_j}{|\lambda_j - 1|}},
\]

\[
|\tilde{m}|_{L^\infty(\tilde{D}_{\kappa_j})}, \quad \left|\frac{\partial}{\partial x} \tilde{m}(x, \lambda)\right|_{L^\infty(\tilde{D}_{\kappa_j})} \leq C|v_0|_{L^1 \cap L^\infty}, \quad \Theta_j, \quad \gamma_j \text{ defined by } (3.15).
\]

Besides, \((\tilde{\Phi}_1(x), \ldots, \tilde{\Phi}_M(x))\mathcal{D}^3 = 0 \text{ with}

\[
\tilde{\Phi}_j(x) = e^{\kappa_j x_1 + \kappa_j^2 x_2} \tilde{m}(x, \kappa_j^+),
\]

\[
\mathcal{D}^3 = \text{diag}\left(\frac{\Pi_{2 \leq n \leq N}(\kappa_1 - z_n)}{(\kappa_1 - z_1)^{N-1}}, \ldots, \frac{\Pi_{2 \leq n \leq N}(\kappa_M - z_n)}{(\kappa_M - z_1)^{N-1}}\right)\mathcal{D}^3.
\]

So there exists uniquely \(b \in GL(N \times N)\) such that

\[
\mathcal{D}^3 \times b = \tilde{D} = \begin{pmatrix}
\kappa_1^N & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \kappa_N^N \\
\tilde{D}_{N+1,1} & \cdots & \tilde{D}_{N+1,N} \\
\vdots & \ddots & \vdots \\
\tilde{D}_{M,1} & \cdots & \tilde{D}_{M,N}
\end{pmatrix},
\]

\[
(\tilde{\Phi}_1(x), \ldots, \tilde{\Phi}_M(x))\tilde{D} = 0.
\]

Moreover,

\[
\partial_x \tilde{m}(x, y, \lambda) = \tilde{s}_c(\lambda)e^{(\lambda - \lambda)x_1 + (\lambda^2 - \lambda^2)x_2} \tilde{m}(x, \lambda), \quad \lambda I \neq 0,
\]

\[
\tilde{s}_c(\lambda) = \frac{(\lambda - z_1)^{N-1}\Pi_{2 \leq n \leq N}(\lambda - z_n)}{(\lambda - z_1)^{N-1}\Pi_{2 \leq n \leq N}(\lambda - z_n)} s_c(\lambda), \quad \tilde{s}_c(\lambda) = \overline{s(\lambda)}
\]
Theorem 2.

Proof. We only give the proof for (3.40). Other properties can be derived directly from the spectral transformation with $W$ and the spectral transformation

Combining (3.47) and (3.48), we justify (3.40).

Using the argument as in the proof of (3.2), for $\lambda \in \tilde{D}_z \subset D_0$, the Green function satisfies

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Combining (3.47) and (3.48), we justify (3.40).

Based on the characterization of the eigenfunction $\tilde{m}$, we define the eigenfunction space $W$ and the spectral transformation $T$ in Definition 4.2 and 3.2.

Definition 3.2. Given $u(x) = u_0(x) + v_0(x)$, $u_0(x)$ a totally positive $Gr(N, M) > 0$ KP soliton, $v_0(x)$ a real-valued function with $\partial_x^k v_0 \in L^1 \cap L^\infty$, $|k| \leq 2$, $|v_0|_{L^1 \cap L^\infty} \ll 1$, define $S$ as the forward scattering transform by

where $1 \leq n \leq N$, $1 \leq j \leq M$,

$z_n$, location of the simple poles of $\tilde{m}$,

$\kappa_j$, location of discontinuities of $\tilde{m}$,

$\bar{D}$, the norming constants (the $\bar{D}$-symmetry of $\tilde{\Phi}$),
defined by (3.43), are the discrete scattering data; and \( \tilde{s}_c(\lambda) \) defined by (3.45), is the continuous scattering data. Denote \( T \) as the continuous scattering operator by

\[
T(\phi)(x, \lambda) = \tilde{s}_c(\lambda)e^{(\lambda - \lambda)x_1 + (\lambda^2 - \lambda^2)x_2}\phi(x, x),
\]

So for general \( Gr(N, M) > 0 \) KP solitons, the discrete scattering data \( \tilde{D} \) defined in Definition 3.2 is determined by \( \kappa_j, z_n, A \), as well as the initial perturbation \( v_0 \). This is different from the situation in [28] (or [27]), where for \( Gr(1, 2) > 0 \) KP solitons, the discrete scattering data is replaced by \( D \), determined by \( \kappa_j, z_n, A \), and independent of \( v_0 \).

To conclude this section, we justify that for a KPII solution \( u(x) = u(x_1, x_2, x_3) \), the scattering data flow \( S(u)(x_3) \) is linear.

**Theorem 5.** For \( \forall x_3 \geq 0 \), if \( \tilde{\Phi}(x, \lambda) = \tilde{m}(x, \lambda)e^{\lambda x_1 + \lambda^2 x_2} \) satisfies the Lax pair (1.6) with the continuous and discrete scattering data

\[
\partial_\lambda \tilde{m}(x, \lambda) = \tilde{s}_c(\lambda, x_3)e^{(\lambda - \lambda)x_1 + (\lambda^2 - \lambda^2)x_2}\tilde{m}(x, x),
\]

(\( \tilde{\Phi}(x, \kappa_1^+), \cdots, \tilde{\Phi}(x, \kappa_M^+) \))\( \tilde{D} = 0 \).

Then the evolution equations of the scattering data are

\[
\tilde{s}_c(\lambda, x_3) = e^{(\lambda^3 - \lambda^3)x_3}\tilde{s}_c(\lambda, 0),
\]

(3.52)

\[
\tilde{D}_{mn}(x_3) = e^{(\kappa_n^3 - \kappa_m^3)x_3}\tilde{D}_{mn}(0).
\]

**Proof.** We skip the proof of (3.51) since it can be proved by the same argument as that in [28 Lemma 4.1]. On the other hand, from the \( \tilde{D} \) symmetry,

(3.53)

\[
-k_1^N \tilde{\Phi}_1 = \sum_{n=N+1}^{M} \tilde{D}_{n1}\tilde{\Phi}_n,
\]

\[
\vdots
\]

\[
-k_N^N \tilde{\Phi}_N = \sum_{n=N+1}^{M} \tilde{D}_{nN}\tilde{\Phi}_n.
\]

Denote \( M_\lambda = -\partial_{x_3} + \partial_{x_1}^3 + \frac{3}{2}u\partial_{x_1} + \frac{3}{4}u_{x_1} + \frac{3}{4}\partial_{x_2}^{-1}u_{x_2} + \tau(\lambda) \) and \( \tau(\lambda) = -\lambda^3 \). So (1.6) implies

\[
0 = -M_{\kappa_j}(k_j^N \tilde{\Phi}_j) = \sum_{n=N+1}^{M} M_{\kappa_j}(\tilde{D}_{nj}\tilde{\Phi}_n)
\]

\[
= \sum_{n=N+1}^{M} \left( \tilde{\Phi}_n[-\partial_{x_3} + \tau(\kappa_j)]\tilde{D}_{nj} + \tilde{D}_{nj}[M_{\kappa_j} - \tau(\kappa_j)]\tilde{\Phi}_n \right) \]
\[
\sum_{n=N+1}^{M} (\tilde{\Phi}_n [-\partial_3 + \tau(\kappa_j)] \tilde{D}_{nj} + \tilde{D}_{nj} [M_{\kappa_n} - \tau(\kappa_n)] \tilde{\Phi}_n) = \sum_{n=N+1}^{M} \tilde{\Phi}_n [-\partial_3 - \tau(\kappa_n) + \tau(\kappa_j)] \tilde{D}_{nj}.
\]

Thus (3.52) is justified. \[\Box\]

4. The Cauchy integral equation

We will derive a Cauchy integral equation for the eigenfunction \(\tilde{m}(x, \lambda)\). We will first justify a crucial estimate for \(C_T \tilde{m}\), an \(L^\infty\) estimate for each \(x\) fixed. It is sufficient for deriving a Cauchy integral equation but is insufficient for solving the inverse problem (cf [26]). Finally, we show a closeness property implied by the Cauchy integral equation and the eigenfunction space characterization (Definition 4.2).

Definition 4.1. Let \(C\) be the Cauchy integral operator defined by

\[
C(\phi)(x, \lambda) = C_\lambda(\phi) = -\frac{1}{2\pi i} \int \int \frac{\phi(x, \zeta)}{\zeta - \lambda} d\zeta \wedge d\zeta.
\]

We shall apply Liouville’s theorem to derive a Cauchy equation. To this aim, boundedness of \(C_T \tilde{m}\) is important which is provided in the following theorem.

Theorem 6. Suppose \(u(x) = u_0(x) + v_0(x)\), \(u_0(x)\) a totally positive \(Gr(N, M)_{>0}\) KP soliton, \(v_0(x)\) a real-valued function with \(\partial_x^2 v_0 \in L^1 \cap L^\infty\), \(|k| \leq 2\), \(|v_0|_{L^1 \cap L^\infty} \ll 1\). Then the eigenfunction \(\tilde{m}(x, \lambda)\) obtained in Theorem 4 satisfies

\[
|C_T \tilde{m}|_{L^\infty} \leq C(1 + |x|) \sum_{|k| \leq 2} |\partial_x^k v_0|_{L^1 \cap L^\infty},
\]

\[
C_T \tilde{m}(x, \lambda) \to 0, \quad \text{as} \quad |\lambda| \to \infty, \quad \lambda_I \neq 0.
\]

Proof. One can adapt the argument as that in the proof of [28, Theorem 3] to prove the theorem. We emphasize that, determined by the heat operator \(-\partial_{x_2}^2 + \partial_{x_1}^2 + 2\lambda \partial_{x_1} + u_0(x)\), the continuous scattering operator \(T\) has poles at \(\kappa_j\), \(1 \leq j \leq M\), and is bounded in the weighted space \(L^2(|\lambda_I|d\lambda \wedge d\lambda) \cap L^\infty\). Consequently, in the proof

1. Estimates near \(\kappa_j\), a refined estimate of the classical Cauchy integral formula, depends crucially on boundedness of \(\tilde{m}(x, \lambda)\) near \(\kappa_j\) (see (3.15)). These estimates are not valid if \(\tilde{m}(x, \lambda)\) blows up at \(\lambda = \kappa_j\) which is the case for [6, Eq.(4.11)] or [25, Eq.(41)] due to a different boundary condition chosen in (2.1).

2. Introducing by the Sato normalized eigenfunction \(\chi(x, \lambda)\) in (2.2) and the renormalization Definition 3.1, \(\tilde{m} \in W\) has a simple pole at \(z_n\), \(1 \leq n \leq N\) (see (3.40)). Since \(T\) is regular at \(z_n\) (see (3.46)), similar analysis as above holds here.
(3) Thanks to (3.46), there is a missing direction in the \(\lambda\)-plane, i.e. the real axis, for the continuous scattering operator \(T\) to decay no matter how smooth the initial data \(v_0(x)\) is. Therefore, boundedness of \(m(x, \lambda)\) near \(\infty\) is vital to derive uniform estimates near \(\infty\). In particular, estimates near \(\infty\) can not hold if \(m(x, \lambda)\) blows up there, which is the case for [13, Eq. (4.9), (4.10), (4.11)] due to a different boundary condition chosen in (2.1).

\[\square\]

**Definition 4.2.** The eigenfunction space \(W \equiv W_x\) is the set of functions

\[(i)\quad \phi(x, \lambda) = \overline{\overline{\phi(x, \lambda)}};\]

\[(ii)\quad (1 - E_{\cup 1 \leq n \leq N} D_{z_n}) \phi(x, \lambda) \in L^\infty;\]

\[(iii)\quad \phi(x, \lambda) = \frac{\phi_{z_n, \text{res}}(x)}{\lambda - z_n} + \phi_{z_n, r}(x, \lambda), \quad \lambda \in D_{z_n}^\times;\]

\[\phi_{z_n, \text{res}}(x), \quad (\lambda - z_n) \phi_{z_n, r}(x, \lambda), \quad \frac{\phi_{z_n, r}(x, \lambda)}{1 + |x|} \in L^\infty(D_{z_n});\]

\[(iv)\quad (e^{x_1 + \kappa_1^2 x_2} \phi(x, \kappa_1^+), \ldots, e^{x_M + \kappa_M^2 x_2} \phi(x, \kappa_M^+)) \mathcal{D} = 0,\]

\(\mathcal{D}\) is defined by (3.43), \(\kappa_j^+ = \kappa_j + 0^+, \quad \frac{\partial \phi(x, \lambda)}{1 + |x|} \in L^\infty(D_{\kappa_j}).\)

**Theorem 7.** If \(u(x) = u_0(x) + v_0(x)\), \(u_0\) is a totally positive \(Gr(N, M)_> 0\) KP soliton, \(v_0(x)\) is a real valued function, and

\[(1 + |x|)^2 \partial_x^k v_0 \in L^1 \cap L^\infty, \quad |k| \leq 4, \quad |v_0|_{L^1 \cap L^\infty} \ll 1,\]

then the eigenfunction \(\tilde{m}\) derived from Theorem 4 satisfies

\[(4.1)\quad \tilde{m}(x, \lambda) \in W\]

and the Cauchy integral equation

\[(4.2)\quad \tilde{m}(x, \lambda) = 1 + \sum_{n=1}^N \frac{\tilde{m}_{z_n, \text{res}}(x)}{\lambda - z_n} + CT\tilde{m}, \quad \forall \lambda \neq 0.\]

**Proof.** One can adapt the argument as that in the proof of [28, Theorem 4] to prove the theorem.

\[\square\]

In the above theorem, the characterization \(\tilde{m} \in W\) is an important companion condition to the Cauchy equation (4.2). Observe that, evaluating (4.2) at \(\kappa_j^+\), we obtain \(M\) equations

\[\tilde{m}(x, \kappa_j^+) = 1 + \sum_{n=1}^N \frac{\tilde{m}_{z_n, \text{res}}(x)}{\kappa_j - z_n} + C_{\kappa_j^+} T\tilde{m},\]
\[
\tilde{m}(x, \kappa_M^+) = 1 + \sum_{n=1}^{N} \tilde{m}_{z_n, \text{res}}(x) \kappa_n - z_n + C_{\kappa_M^+} T \tilde{m},
\]
for \(M + N\) variables \(\{\tilde{m}(x, \kappa_j^+), \tilde{m}_{z_n, \text{res}}(x)\}, 1 \leq j \leq M, 1 \leq n \leq N\). The \(\tilde{D}\) -symmetry in \(W\), namely,

\[
(\tilde{\Phi}(x, \kappa_1^+), \ldots, \tilde{\Phi}(x, \kappa_M^+)) \times \tilde{D} = 0
\]

where \(\tilde{D}\) is an \(M \times N\) matrix, provides us extra \(N\) constraints. So \(\tilde{m}_{z_n, \text{res}}(x)\) are determined by discrete scattering data \(\kappa_j, z_n\), and \(C_{\kappa_j^+} T \tilde{m}\). Along with Theorem 5 this closeness property implies that (4.1) and (4.2) can serve as a good starting point for the inverse problem.

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