Quantum evolution with a large number of negative decoherence rates

Katarzyna Siudzińska\(^1\) and Dariusz Chruściński\(^2\)

Institute of Physics, Faculty of Physics, Astronomy and Informatics, Nicolaus Copernicus University, ul. Grudziądzka 5/7, 87–100 Toruń, Poland

E-mail: kasias@umk.pl

Received 6 March 2020, revised 3 July 2020
Accepted for publication 21 July 2020
Published 18 August 2020

Abstract

We analyze the evolution of open quantum systems governed by time-local master equations beyond the Markovian semigroup. Non-Markovian effects are usually attributed to the negativity of some decoherence rates in the time-dependent generator. For the qubit dynamics, it is well known that there can be one permanently negative rate in the so-called eternally non-Markovian evolution. We show that for qudits one can have \((d - 1)^2\) out of the total \(d^2 - 1\) rates that are always negative, and the evolution is still physically legitimate—that is, represented by a completely positive, trace-preserving dynamical map.

Keywords: open quantum systems, quantum channels, generalized Pauli channels, non-Markovian evolution, decoherence rates

(Some figures may appear in colour only in the online journal)

1. Introduction

No existing quantum systems can be completely isolated from the influence of environment. Therefore, to describe their behavior, one is forced to use methods and techniques from the theory of open quantum systems [1–3]. Quantum processes that describe the evolution of open quantum systems are represented by dynamical maps \(\{\Lambda(t)\mid t \geq 0\}\). For any \(t \geq 0\), \(\Lambda(t) : L(\mathcal{H}) \to L(\mathcal{H})\) is completely positive and trace-preserving (CPTP), where \(L(\mathcal{H})\) denotes the space of linear operators acting on the Hilbert space \(\mathcal{H}\). Additionally, at \(t = 0\), \(\Lambda(0) = 1\) is the identity map. A dynamical map transforms an arbitrary initial state \(\rho\) into an evolved state \(\rho(t) = \Lambda(t)[\rho]\). In this paper, we consider only these quantum systems that have a finite number of levels, and we denote this number by \(\dim \mathcal{H} = d\).

\(^1\)Author to whom any correspondence should be addressed.
The Markovian master equation
\[ \dot{\Lambda}(t) = \mathcal{L}\Lambda(t) \]  
(1)
is derived by assuming weak coupling between the system and its environment, as well as a separation of characteristic time scales [1]. In the above formula, \( \mathcal{L} \) is the celebrated Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) generator [4, 5],
\[ \mathcal{L}[\rho] = -\frac{i}{\hbar}[H, \rho] + \sum_{k,l=1}^{d^2-1} K_{kl} \left( F_k \rho F_l^\dagger - \frac{1}{2} \{ F_k^\dagger F_k, \rho \} \right). \]  
(2)
\( H \) denotes the effective system Hamiltonian (including a possible Lamb shift), and \( F_k, l = 0, \ldots, d^2 - 1 \), with \( F_0 = \mathbb{I}/\sqrt{d} \) define an orthonormal basis in \( \mathcal{L}(\mathcal{H}) \), i.e. \( \text{Tr}(F_\alpha^\dagger F_\beta) = \delta_{\alpha\beta} \). Finally, \( K_{kl} \) is a positive-definite, \((d^2 - 1)\)-dimensional matrix. By diagonalizing the matrix \( K_{kl} \), one arrives at the so-called diagonal form of \( \mathcal{L} \), which reads
\[ \mathcal{L}[\rho] = -\frac{i}{\hbar}[H, \rho] + \sum_{k=1}^{d^2-1} \gamma_k \left( V_k \rho V_k^\dagger - \frac{1}{2} \{ V_k^\dagger V_k, \rho \} \right). \]  
(3)

Now, \( V_k \) are called the noise (Lindblad) operators, and the eigenvalues of \( K_{kl} \), denoted by \( \gamma_k \), are positive and referred to as decoherence rates. Going beyond the Markovian semigroup, one usually considers the master equation with a time-dependent generator \( \mathcal{L}(t) \). The generator \( \mathcal{L}(t) \) has the exact same form as \( \mathcal{L} \) in equation (3), but now \( H(t), V_k(t), \) and \( \gamma_k(t) \) are all time-dependent. The formal solution of this equation is the dynamical map
\[ \Lambda(t) = \mathcal{T} \exp \left( \int_0^t \mathcal{L}(\tau) d\tau \right); \quad t \geq 0, \]  
(4)
where \( \mathcal{T} \) is the time-ordering operator [1]. This dynamics is much more complicated, and the general conditions for \( \mathcal{L}(t) \) that guarantee complete positivity of \( \Lambda(t) \) for all \( t \geq 0 \) are not known. A sufficient condition is the positivity of all decoherence rates \( \gamma_k(t) \geq 0 \). This requirement is very strong and implies not only that \( \Lambda(t) \) is CPTP but also that all propagators
\[ V(t, s) = \mathcal{T} \exp \left( \int_s^t \mathcal{L}(\tau) d\tau \right); \quad t \geq s, \]  
(5)
are CPTP, as well. Such evolution is called CP-divisible, and it satisfies the composition law \( \Lambda(t) = V(t, s)\Lambda(s) \). CP-divisibility of \( \Lambda(t) \) is often considered to be a definition of Markovianity [6]. Recently, much attention has been given to various approaches to define and quantify non-Markovian effects in quantum evolution (see recent reviews [7–10]).

Now, if for all \( t \geq s \) the propagators \( V(t, s) \) define just positive, trace-preserving maps, then the corresponding evolution is P-divisible. Actually, one can introduce a whole hierarchy of divisibilities [11]. One calls \( \Lambda(t) \) \( k \)-divisible if and only if \( V(t, s) \) defines a \( k \)-positive map, meaning that \( \mathbb{1}_k \otimes V(t, s) \) is positive, where \( \mathbb{1}_k \) is the identity map in the space of \( k \times k \) complex matrices.

If the evolution is commutative, i.e. \( \mathcal{L}(t)\mathcal{L}(\tau) = \mathcal{L}(\tau)\mathcal{L}(t) \) for any times \( t, \tau \geq 0 \), then we can drop the chronological operator \( \mathcal{T} \) from equation (4). In this case, a sufficient condition for \( \mathcal{L}(t) \) to generate a CP-divisible dynamical map states that \( \int_0^t \mathcal{L}(\tau) d\tau \) is a time-dependent GKSL generator. If the dynamical map \( \Lambda(t) \) is invertible, then \( V(t, s) = \Lambda(t)\Lambda^{-1}(s) \), and the
positivity of all decoherence rates \( \gamma_{\alpha}(t) \geq 0 \) is the necessary and sufficient condition for CP-divisibility of \( \Lambda(t) \) [12]. The case of non-invertible maps has recently been analyzed in [13] (see also recent discussions in [14, 15]).

The simplest example of the evolution with a time-local generator contains only one time-dependent rate \( \gamma(t) \). Consider the well-known qubit decoherence governed by

\[
\mathcal{L}(t)[\rho] = \frac{1}{2} \gamma(t) (\sigma_3 \rho \sigma_3 - \rho) \tag{6}
\]

In this case, the necessary and sufficient condition for legitimacy of the solution states that \( \Gamma(t) = \int_0^t \gamma(\tau) d\tau \geq 0 \). This condition is no longer necessary if there are more decoherence channels. Indeed, the generator

\[
\mathcal{L}(t)[\rho] = \frac{1}{2} \sum_{\alpha=1}^{3} \gamma_{\alpha}(t) (\sigma_{\alpha} \rho \sigma_{\alpha} - \rho) \tag{7}
\]

gives rise to a CP-divisible evolution if and only if \( \gamma_{\alpha}(t) \geq 0 \). Now, if \( \mathcal{L}(t) \) generates a legitimate dynamical map \( \Lambda(t) \), then \( \Lambda(t) \) is P-divisible if and only if \([16, 17]\)

\[
\gamma_1(t) + \gamma_2(t) \geq 0, \quad \gamma_2(t) + \gamma_3(t) \geq 0, \quad \gamma_3(t) + \gamma_1(t) \geq 0. \tag{8}
\]

An instructive example was provided in [18], where the authors considered two positive \( \gamma_1(t) = \gamma_2(t) = 1 \) and one always negative \( \gamma_3(t) = -\tanh t \) decoherence rates. Since \( \gamma_3(t) < 0 \), its integral \( \Gamma_3(t) = \int_0^t \gamma_3(\tau) d\tau < 0 \). Nevertheless, as shown in [18], the corresponding map \( \Lambda(t) \) is CPTP. Because one of the rates is permanently negative, the authors called this evolution \textit{eternally non-Markovian}. Clearly, the CP-divisibility of \( \Lambda(t) \) is broken but P-divisibility still holds. Interestingly, this evolution, which seems to be highly non-Markovian, still enjoys the monotonicity property

\[
\frac{d}{dt} \| \Lambda(t)(\rho_1 - \rho_2) \|_1 \leq 0 \tag{9}
\]

for any pair of qubit density operators \( \rho_1, \rho_2 \). Here, \( \| X \|_1 = \text{Tr} \sqrt{XX^\dagger} \) denotes the trace norm of \( X \). The above property is considered as an alternative concept of Markovianity [19]. Moreover, Megier \textit{et al} [20] have shown that the Pauli channel \( \Lambda(t) \) generated by \( \mathcal{L}(t) \) from equation (7) arises from a simple mixture of two Markovian semigroups,

\[
\Lambda(t) = \frac{1}{2} (e^{2i\xi_1} + e^{2i\xi_2}), \quad \mathcal{L}_\alpha[\rho] = \frac{1}{2} (\sigma_{\alpha} \rho \sigma_{\alpha} - \rho) \tag{10}
\]

The characteristic feature of this evolution is its asymptotic behavior

\[
\rho(0) = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \longrightarrow \rho(\infty) = \frac{1}{2} \begin{pmatrix} 1 & \rho_{01} \\ \rho_{10} & 1 \end{pmatrix}, \tag{11}
\]

which shows that the asymptotic state \( \rho(\infty) \) does depend upon the initial state \( \rho(0) \). This provides a clear sign of non-Markovian memory effects [21]. Moreover, the asymptotic state is not invariant under \( \Lambda(\infty) \) unless \( \rho_{01} = \rho_{10} = 0 \).

In this paper, we generalize the eternally non-Markovian evolution of a qubit to a qudit case. The qubit dynamics is replaced with the qudit dynamics described by the generalized Pauli channels [22]. As a special case, we analyze the convex combination of Markovian semigroups. We provide interesting examples where (i) all decoherence rates are temporarily negative and (ii) \( (d - 1)^2 \) (out of the total of \( d^2 - 1 \) rates) are always negative while the evolution is still
physically legitimate. It turns out that the set of CP-divisible dynamical maps defined by the convex combination of Markovian semigroups is only a small subset among all admissible maps. For dimension \( d = 3 \), we explicitly plot the range of parameters that leads to CP-divisible dynamical maps.

2. Pauli channels—qubit evolution

Consider the qubit evolution represented by the dynamical map

\[
\Lambda(t)[\rho] = \sum_{\alpha=0}^{3} p_\alpha(t) \sigma_\alpha \rho \sigma_\alpha,
\]

(12)

with \( \sigma_0 = \mathbb{1} \), which is generated by the time-local generator from equation (7). Its eigenvalue equations are

\[
\Lambda(t)[\sigma_\alpha] = \lambda_\alpha(t) \sigma_\alpha; \quad \alpha = 0, 1, 2, 3,
\]

(13)

where the time-dependent eigenvalues read [16, 17]

\[
\lambda_\alpha(t) = e^{\Gamma_\alpha(t) - \Gamma_0(t)}.
\]

(14)

These are related to the decoherence rates via \( \Gamma_\alpha(t) = \int_0^t \gamma_\alpha(\tau) d\tau \) with \( \gamma_0(t) = \sum_{k=1}^{3} \gamma_k(t) \). The associated probability 4-vector \( p_\alpha(t) \) is given by [16, 17]

\[
p_0(t) = \frac{1}{4}(1 - \lambda_1(t) + \lambda_2(t) + \lambda_3(t))
\]

(15)

and

\[
p_k(t) = \frac{1}{4}(1 + 2\lambda_k(t) - \lambda_1(t) - \lambda_2(t) - \lambda_3(t)); \quad k = 1, 2, 3.
\]

(16)

Let us analyze the following convex combination of legitimate qubit dynamics,

\[
\Lambda(t) = x_1 e^{w_1(t) \mathcal{L}_1} + x_2 e^{w_2(t) \mathcal{L}_2} + x_3 e^{w_3(t) \mathcal{L}_3},
\]

(17)

where \( w_\alpha(t) \geq 0 \) for all \( t \geq 0 \), and \((x_1, x_2, x_3)\) is a probability vector. This model has been studies in [20] in the special case where \( w_1(t) = w_2(t) = w_3(t) = 2t \). One finds

\[
\begin{pmatrix}
\gamma_1 \\
\gamma_2 \\
\gamma_3
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
-1 & 1 & 1 \\
1 & -1 & 1 \\
1 & 1 & -1
\end{pmatrix} \begin{pmatrix}
\mu_1 \\
\mu_2 \\
\mu_3
\end{pmatrix}
\]

(18)

with

\[
\begin{align*}
\mu_1(t) &= \frac{x_2 \hat{w}_2(t) e^{-w_2(t)} + x_3 \hat{w}_3(t) e^{-w_3(t)}}{x_1 + x_2 e^{-w_2(t)} + x_3 e^{-w_3(t)}}, \\
\mu_2(t) &= \frac{x_1 \hat{w}_1(t) e^{-w_1(t)} + x_3 \hat{w}_3(t) e^{-w_3(t)}}{x_2 + x_1 e^{-w_1(t)} + x_3 e^{-w_3(t)}}, \\
\mu_3(t) &= \frac{x_1 \hat{w}_1(t) e^{-w_1(t)} + x_2 \hat{w}_2(t) e^{-w_2(t)}}{x_3 + x_1 e^{-w_1(t)} + x_2 e^{-w_2(t)}}.
\end{align*}
\]

(19)
The dynamical map given by equation (17) is P-divisible if and only if conditions (8) hold, which is equivalent to

$$\mu_k(t) \geq 0; \quad k = 1, 2, 3. \quad (20)$$

It is, therefore, clear that if all $\dot{w}_k(t) \geq 0$, then $\Lambda(t)$ is P-divisible. However, if $\dot{w}_k(t) \not\geq 0$, then it can happen that temporarily all rates $\gamma_k(t)$ are strictly negative. Note that for $w_1(t) = w_2(t) = w_3(t) \equiv w(t)$, $\Lambda(t)$ is P-divisible if and only if $\dot{w}(t) \geq 0$ for all $t \geq 0$.

Interestingly, if $\dot{w}(t) \geq 0$, then $\gamma_1(t) = \gamma_2(t) \geq 0$ and $\gamma_3(t) \leq 0$. However, if $\dot{w}(t) \leq 0$, then $\gamma_1(t) = \gamma_3(t) \leq 0$ and $\gamma_3(t) \geq 0$. One easily checks that for $w(t) = 2t$ and $x_1 = x_2 = 1/2$, $x_3 = 0$, the dynamical map from equation (17) recovers the eternally non-Markovian evolution [18]

$$\Lambda(t) = \frac{1}{2} (e^{w(t) \mathcal{L}_1} + e^{w(t) \mathcal{L}_2}). \quad (21)$$

Now, the map from equation (17) corresponds to the Pauli channel with

$$p_0(t) = \frac{1}{2} \left(1 + x_1 e^{-w_1(t)} + x_2 e^{-w_2(t)} + x_3 e^{-w_3(t)}\right) \quad (22)$$

and

$$p_k(t) = \frac{x_k}{2} \left(1 - e^{-w_k(t)}\right), \quad k = 1, 2, 3. \quad (23)$$

It turns out that $p_k(t)$ satisfy a time-local rate equation of their own.

**Proposition 1.** A probability 4-vector $p_\alpha(t)$ is a solution of the time-local rate equation

$$\frac{d}{dt} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \mathcal{L}(t) \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} \quad (24)$$

with the time-dependent generator

$$\mathcal{L} = \frac{1}{2} \begin{pmatrix} -W & 2\dot{w}_1 - W & 2\dot{w}_2 - W & 2\dot{w}_3 - W \\ \dot{w}_1 x_1 & -(2 - x_1)\dot{w}_1 & \dot{w}_1 x_1 & \dot{w}_1 x_1 \\ \dot{w}_2 x_2 & \dot{w}_2 x_2 & -(2 - x_2)\dot{w}_2 & \dot{w}_2 x_2 \\ \dot{w}_3 x_3 & \dot{w}_3 x_3 & \dot{w}_3 x_3 & -(2 - x_3)\dot{w}_3 \end{pmatrix} \quad (25)$$

where $W(t) := \dot{w}_1(t)x_1 + \dot{w}_2(t)x_2 + \dot{w}_3(t)x_3$.

Interestingly, if all $\dot{w}_k(t) \geq 0$ and $\dot{w}_k(t) \geq \frac{1}{2}W(t)$ for $k = 1, 2, 3$, then $\Sigma_{\alpha \beta} \geq 0$ for $\alpha \neq \beta$, and hence $p_\alpha(t)$ satisfy the classical Pauli rate equation. Now, for $w_1(t) = w_2(t) = w_3(t) = w(t)$, equation (24) reduces to

$$\frac{d}{dt} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & 1 & 1 & 1 \\ x_1 & -1 & 0 & 0 \\ x_2 & 0 & -1 & 0 \\ x_3 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}. \quad (26)$$
Indeed, starting from equation (25), one finds that

\[
\mathcal{L} = \frac{\dot{w}}{2} \begin{pmatrix}
-1 & 1 & 1 & 1 \\
\gamma_1 & -1 & 1 & x_1 \\
\gamma_2 & x_2 & 1 & x_2 \\
\gamma_3 & x_3 & x_3 & 1 \\
\end{pmatrix} = \frac{\dot{w}}{2} \begin{pmatrix}
-1 & 1 & 1 & 1 \\
\gamma_1 & -1 & 1 & x_1 \\
\gamma_2 & x_2 & 1 & x_2 \\
\gamma_3 & x_3 & x_3 & 1 \\
\end{pmatrix}
\]

\[
+ \frac{\dot{w}}{2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & x_1 - 1 & x_1 & x_1 \\
0 & x_2 & x_2 - 1 & x_2 \\
0 & x_3 & x_3 & x_3 - 1 \\
\end{pmatrix},
\]

(27)

where

\[
\begin{pmatrix}
x_1 - 1 & x_1 & x_1 \\
x_2 & x_2 - 1 & x_2 \\
x_3 & x_3 & x_3 - 1 \\
\end{pmatrix}
\begin{pmatrix}
p_1 \\
p_2 \\
p_3 \\
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
\end{pmatrix}
\]

(28)

due to equation (23).

3. Generalized Pauli channels—qudit evolution

Consider the evolution of a qudit in the \(d\)-dimensional Hilbert space \(\mathcal{H}\) that admits the maximal number \(N(d) = d + 1\) of mutually unbiased bases (MUBs) \(\{\psi_0^{(\alpha)}, \ldots, \psi_{d-1}^{(\alpha)}\}\) [23, 24]. Recall that two bases are mutually unbiased if

\[
\langle \psi_k^{(\alpha)} | \psi_l^{(\beta)} \rangle = \delta_{kl}, \quad \left| \langle \psi_k^{(\alpha)} | \psi_l^{(\beta)} \rangle \right|^2 = \frac{1}{d}, \quad \alpha \neq \beta.
\]

(29)

Now, assume that the generator of the evolution has the form

\[
\mathcal{L}(t) = \sum_{\alpha=1}^{d+1} \gamma_\alpha(t) \mathcal{L}_\alpha, \quad \mathcal{L}_\alpha = \Phi_\alpha - 1,
\]

(30)

where

\[
\Phi_\alpha[X] = \sum_{k=0}^{d-1} P_k^{(\alpha)} X P_k^{(\alpha)},
\]

(31)

and \(P_k^{(\alpha)} := | \psi_k^{(\alpha)} \rangle \langle \psi_k^{(\alpha)} |\) denote the rank-1 projectors onto the MUB vectors. The solution of the time-local master equation \(\Lambda(t) = \mathcal{L}(t) \Lambda(t)\) with generator (30) is the generalized Pauli channel [22, 25]

\[
\Lambda(t) = \frac{dp_0(t) - 1}{d - 1} + \frac{d}{d - 1} \sum_{\alpha=1}^{d+1} p_\alpha(t) \Phi_\alpha,
\]

(32)

where \(p_\alpha(t)\) denotes the probability \((d + 2)\)-vector with the components

\[
p_\alpha(t) = \frac{1}{d^2} \left[ 1 + (d - 1) \sum_{\alpha=1}^{d+1} \lambda_\alpha(t) \right],
\]

(33)
\[ p_\alpha(t) = \frac{d - 1}{d^2} \left[ 1 + d \lambda_\alpha(t) - \sum_{\beta=1}^{d+1} \lambda_\beta(t) \right], \quad \alpha = 1, \ldots, d + 1. \] (34)

The eigenvalues of \( \Lambda(t) \) to the eigenvectors

\[ U_a^k = \sum_{i=0}^{d-1} \omega^k P_i^{(0)}, \quad k = 1, \ldots, d - 1, \] (35)

with \( \omega = e^{2\pi i / d} \) are given by

\[ \lambda_\alpha(t) = \exp \left[ \Gamma_\alpha(t) - \Gamma_0(t) \right], \] (36)

where \( \Gamma_0(t) = \sum_{k=1}^{d+1} \Gamma_k(t) \). For \( d = 2 \), \( \Lambda(t) \) reproduces the Pauli channel in equation (12).

The qubit map from equation (17) can be straightforwardly generalized to

\[ \Lambda(t) = \sum_{\alpha=1}^{d+1} x_\alpha e^{w_\alpha(t) L_\alpha} \] (37)

with \( w_\alpha(t) \geq 0 \) for all \( t \geq 0 \) and \( w_\alpha(0) = 0 \). The associated decoherence rates are given by

\[ \gamma_\alpha(t) = \frac{1}{d} \sum_{\beta=1}^{d+1} \mu_\beta(t) - \mu_\alpha(t), \] (38)

where

\[ \mu_\alpha(t) = \frac{\sum_{\beta=1}^{d+1} x_\beta \dot{w}_\beta(t) e^{-w_\beta(t)} - x_\alpha \dot{w}_\alpha(t) e^{-w_\alpha(t)}}{x_\alpha (1 - e^{-w_\alpha(t)}) + \sum_{\beta=1}^{d+1} x_\beta e^{-w_\beta(t)}}. \] (39)

**Example 1.** If \( d = 3 \), then formula (38) can be represented in the matrix form,

\[
\begin{pmatrix}
\gamma_1 \\
\gamma_2 \\
\gamma_3 \\
\gamma_4
\end{pmatrix} = \frac{1}{3}
\begin{pmatrix}
-2 & 1 & 1 & 1 \\
1 & -2 & 1 & 1 \\
1 & 1 & -2 & 1 \\
1 & 1 & 1 & -2
\end{pmatrix}
\begin{pmatrix}
\mu_1 \\
\mu_2 \\
\mu_3 \\
\mu_4
\end{pmatrix}.
\] (40)

This provides a generalization of equation (18) to a higher-dimensional system.

Let us recall that if \( L(t) \) is a legitimate time-local generator, then the corresponding dynamical map \( \Lambda(t) \) is P-divisible if and only if

\[ \text{Tr}(P L(t) Q) \geq 0 \] (41)

for any pair of rank-1 mutually orthogonal projectors \( P \) and \( Q \) [26]. The orthogonality is checked with respect to the Hilbert-Schmidt inner product \( \langle P | Q \rangle := \text{Tr}(P^\dagger Q) \). For a recent discussion, see [27].

**Proposition 2.** If the dynamical map \( \Lambda(t) \) generated by time-local generator (30) is P-divisible, then

\[ \mu_\alpha(t) = \sum_{\beta=1}^{d+1} \gamma_\beta(t) - \gamma_\alpha(t) \geq 0 \] (42)
for all $\alpha = 1, \ldots, d + 1$.

Condition (42) immediately follows if one applies equation (41) to $P = P_k^{(\alpha)}$ and $Q = P_l^{(\alpha)}$ for $k \neq l$. It should be stressed that condition (42) for P-divisibility is necessary but not sufficient. It becomes also sufficient in the qubit case, where $d = 2$.

The dynamical map in equation (37) corresponds to the generalized Pauli channel with

$$p_0(t) = \frac{1}{d} \left( 1 + [d - 1] \sum_{\alpha=1}^{d+1} x_\alpha e^{-\omega_\alpha(t)} \right)$$

(43)

and

$$p_k(t) = \frac{d-1}{d} x_k \left( 1 - e^{-\omega_k(t)} \right), \quad k = 1, \ldots, d + 1.$$  

(44)

Interestingly, if all $\dot{\omega}_k(t) \geq 0$, then $p_\alpha(t)$ satisfy the classical Pauli rate equation

$$\dot{p}_\alpha(t) = \frac{1}{d} \sum_{\beta=1}^{d+1} L_{\alpha\beta}(t) p_\beta(t)$$

(45)

with

$$L_{00}(t) = - W(t),$$

$$L_{0k}(t) = - W(t) + d \dot{\omega}_k(t),$$

$$L_{0l}(t) = L_{kl}(t) = (d - 1) x_k \dot{\omega}_l(t),$$

$$L_{kk}(t) = -(d(1 - x_k) + x_k) \dot{\omega}_k(t),$$

(46)

where $W(t) := (d - 1) \sum_{\alpha=1}^{d+1} \dot{\omega}_\alpha(t) x_\alpha$. For example, if $d = 3$ and all $\omega_k(t) = \omega(t)$, then

$$L = \frac{\dot{\omega}}{3} \begin{pmatrix}
-1 & 1 & 1 & 1 & 1 \\
2 x_1 & -1 & 0 & 0 & 0 \\
2 x_2 & 0 & -1 & 0 & 0 \\
2 x_3 & 0 & 0 & -1 & 0 \\
2 x_4 & 0 & 0 & 0 & -1
\end{pmatrix}. $$

(47)

**Remark 1.** Actually, one can temporally have $\gamma_\alpha(t) \leq 0$ for all $\alpha = 1, \ldots, d + 1$. Indeed, if $w_\alpha(t) = \omega(t)$ and $x_\alpha = \frac{1}{d+1}$, then

$$\gamma_\alpha(t) = \frac{\dot{\omega}(t)}{d + e^{\omega(t)}}.$$  

(48)

Now, assume that, for some $t_*$, one has $\dot{\omega}(t_*) < 0$ even though $\omega(t_*) > 0$. Then, the evolution is non-Markovian with all $d + 1$ decoherence rates that are negative for $t = t_*$, while the dynamical map $\Lambda(t)$ is still legitimate (CPTP). It should be stressed that the rates do not stay negative forever because $\omega(t) \geq 0$ for any $t \geq 0$. 


4. A case study–convex combination of Markovian semigroups

As a special case, let us analyze the dynamical map being a convex combination of Markovian dynamical semigroups

$$\Lambda(t) = \sum_{\alpha=1}^{d+1} x_\alpha e^{\alpha t} L_\alpha = e^{-rt} \mathbb{1} + (1 - e^{-rt}) \sum_{\alpha=1}^{d+1} x_\alpha \Phi_\alpha$$

with $r > 0$. It corresponds to the choice

$$p_0(t) = \frac{1}{d}(1 + [d - 1]e^{-rt}),$$

$$p_\alpha(t) = \frac{d}{d} - 1 \frac{1}{d}(1 - e^{-rt}) x_\alpha, \quad \alpha = 1, \ldots, d + 1.$$  

(50)

The spectrum of $\Lambda(t)$ is $\lambda_0(t) = 1$, as well as

$$\lambda_\alpha(t) = e^{-rt} + (1 - e^{-rt}) x_\alpha, \quad \alpha = 1, \ldots, d + 1,$$

with multiplicity $d - 1$. The resulting map is no longer a semigroup. Instead, it solves the master equation

$$\dot{\Lambda}(t) = L(t) \Lambda(t),$$

where the time-local generator has the decoherence rates

$$\gamma_\alpha(t) = \frac{(1 - x_\alpha)r}{1 + (e^{rt} - 1)x_\alpha} + \sum_{\beta=1}^{d+1} \frac{1 - x_\beta}{1 + (e^{rt} - 1)x_\beta}$$

(52)

that no longer have to be positive. There holds a simple relation,

$$\begin{bmatrix}
\gamma_1 \\
\gamma_2 \\
\vdots \\
\gamma_{d+1}
\end{bmatrix} = \frac{r}{d} \begin{bmatrix}
-(d - 1) & 1 & \cdots & 1 \\
1 & -(d - 1) & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & -(d - 1)
\end{bmatrix} \begin{bmatrix}
\mu_1 \\
\mu_2 \\
\vdots \\
\mu_{d+1}
\end{bmatrix},$$

where

$$\mu_\alpha(t) = \frac{1 - x_\alpha}{1 + (e^{rt} - 1)x_\alpha}.$$  

(53)

Now, the resulting dynamical map is CP-divisible if and only if all $\gamma_\alpha(t) \geq 0$ for any $t \geq 0$.

**Proposition 3.** The decoherence rates from equation (52) satisfy conditions (42).

**Proof.** Indeed, one has

$$\gamma_\alpha(t) = \frac{r}{d} \sum_{\beta=1}^{d+1} \mu_\beta(t) - r \mu_\alpha(t),$$

and hence

$$\sum_{\beta=1}^{d+1} \gamma_\beta(t) - \gamma_\alpha(t) = r \mu_\alpha(t) = \frac{r(1 - x_\alpha)}{1 + (e^{rt} - 1)x_\alpha} \geq 0,$$

which ends the proof. □
These conditions are necessary for P-divisibility. However, they are not sufficient. A system of sufficient conditions has been found, and they depend on the number of negative rates.

**Proposition 4** ([28]). Suppose that, at a given time \( t \), there are \( k \leq \frac{d+1}{2} \) negative rates

\[
\gamma_\alpha(t) < 0, \quad \alpha \in \Gamma \subset \{1, 2, \ldots, d+1\}, \quad |\Gamma| = k.
\]

If for all \( \beta \notin \Gamma \),

\[
\gamma_\beta(t) \geq \frac{d + 2(k-1)}{d - 2(k-1)} \max_{\alpha \in \Gamma} |\gamma_\alpha(t)|,
\]

then \( \Lambda(t) \) is P-divisible.

**Corollary 1.** Assume that \( k = 1 \), which means there is only one negative rate \( \gamma_\alpha(t) \) at any given time \( t \). If

\[
\gamma_\beta(t) + \gamma_\alpha(t) \geq 0
\]

for all \( \beta \neq \alpha \), then \( \Lambda(t) \) is P-divisible. Note that this condition provides a generalization of equation (8) to the qudit case.

**Example 2.** For the maximally mixed probability vector \( x_\alpha = \frac{1}{d+1} \), the time-local generator \( \mathcal{L}(t) \) is determined by

\[
\gamma_\alpha(t) = \frac{r}{d + e^r} \geq 0,
\]

which always correspond to a CP-divisible dynamical map.

**Example 3.** If \( x_\alpha = 1/d \) for \( \alpha = 1, \ldots, d \) and \( x_{d+1} = 0 \), the dynamical map is generated by \( \mathcal{L}(t) \) with

\[
\gamma_\alpha(t) = \frac{r}{d}, \quad \alpha = 1, \ldots, d,
\]

\[
\gamma_{d+1}(t) = -\frac{r}{d} \left( \frac{d-1}{e^r - 1 + d} \right) \leq 0,
\]

where one decoherence rate is eternally negative. Observe that condition (42) is satisfied but (55) is not. A simple numerical test shows that condition (41) is not satisfied, hence the dynamical map is P-divisible. The characteristic feature of this evolution is the behavior of its asymptotic state \( \rho(\infty) \). If \( U_{k+1}^d \) are the only diagonal operators among \( U_\alpha^d \), then

\[
\rho_i(0) \to \rho_i(\infty) = \frac{1}{d}, \quad \rho_{ij}(t) \to \rho_{ij}(\infty) = \frac{1}{d} \rho_{ij}(0); \quad i \neq j.
\]

For \( d = 3 \), we have

\[
\rho(0) = \begin{pmatrix}
\rho_{00} & \rho_{01} & \rho_{02} \\
\rho_{10} & \rho_{11} & \rho_{12} \\
\rho_{20} & \rho_{21} & \rho_{22}
\end{pmatrix} \quad \rightarrow \quad \rho(\infty) = \frac{1}{3} \begin{pmatrix}
1 & \rho_{01} & \rho_{02} \\
\rho_{10} & 1 & \rho_{12} \\
\rho_{20} & \rho_{21} & 1
\end{pmatrix}.
\]

Similarly to the qubit case, this dynamics possesses a clear sign of non-Markovian memory effects, as the asymptotic state \( \rho(\infty) \) depends on the initial state \( \rho(0) \) [21]. Moreover, the asymptotic state is not invariant under \( \Lambda(\infty) \) unless \( \rho_{kl} = 0 \) for \( k \neq l \).
**Example 4.** For the choice \( x_1 = x_2 = 1/2 \) and \( x_\alpha = 0 \) when \( \alpha = 3, \ldots, d + 1 \), the associated \( \mathcal{L}(t) \) has

\[
\gamma_1(t) = \frac{r d - 1 + e^{-\alpha t}}{d + e^{-\alpha t}} \geq 0,
\]

\[
\gamma_\alpha(t) = -\frac{r}{d} \tanh \frac{rt}{2} \leq 0, \quad \alpha = 3, \ldots, d + 1.
\]

Hence, there exists a legitimate time-local generator with \( d - 1 \) identical, eternally negative \( \gamma_\alpha(t) \).

Actually, there is an entire family of channels describing the eternally non-Markovian evolution with \( 1 \leq k \leq d - 1 \) negative decoherence rates.

**Example 5.** Let us take \( x_\alpha = 0 \) for \( \alpha = 1, \ldots, k \) and \( x_\alpha = \frac{1}{d!+1} \) whenever \( \alpha = k + 1, \ldots, d + 1 \). The corresponding decoherence rates are given by

\[
\gamma_\alpha(t) = \frac{r d + k(e^{\alpha t} - 1)}{d - k + e^{\alpha t}} \geq 0, \quad \alpha = k + 1, \ldots, d + 1,
\]

\[
\gamma_\alpha(t) = -\frac{r (d - k)(e^{\alpha t} - 1)}{d - k + e^{\alpha t}} \leq 0, \quad \alpha = 1, \ldots, k.
\]

Hence, the resulting evolution has \( k \) eternally negative rates, where \( 1 \leq k \leq d - 1 \).

Recalling that each \( \gamma_\alpha(t) \) has multiplicity \( d - 1 \), example 3 shows that there is a legitimate (CPTP) dynamical map with \((d - 1)^2\) rates that are negative for all \( t > 0 \). It would be interesting to show that this is the maximal number of eternally negative rates compatible with completely positive evolution.

None of the above examples is generated by \( \mathcal{L}(t) \) with \( \gamma_\alpha(t) \) for which sufficient P-divisibility conditions (55) hold. Let us provide examples of P-divisible dynamics in \( d = 3 \).

**Example 6.** For \( d = 3 \), the dynamical map \( \Lambda(t) \) is P-divisible and has exactly \( k = 1 \) negative decoherence rate if

\[
x_1 = \frac{3}{20}, \quad x_2 = \frac{1}{4}, \quad x_3 = \frac{1}{3}, \quad x_4 = \frac{4}{15}.
\]

The corresponding decoherence rates are equal to

\[
\gamma_1(t) = \frac{1122 e^{-3t} + 594 e^{-6t} - 53 e^{-3t} - 43}{1122 e^{-3t} + 1541 e^{-6t} + 764 e^{-3t} + 161 + 12 e^{3t}} \geq 0,
\]

\[
\gamma_2(t) = \frac{1122 e^{-3t} + 1122 e^{-6t} + 403 e^{-3t} + 53}{1122 e^{-3t} + 1541 e^{-6t} + 764 e^{-3t} + 161 + 12 e^{3t}} \geq 0,
\]

\[
\gamma_3(t) = \frac{1122 e^{-3t} + 1683 e^{-6t} + 706 e^{-3t} + 89}{1122 e^{-3t} + 1541 e^{-6t} + 764 e^{-3t} + 161 + 12 e^{3t}} \geq 0,
\]

\[
\gamma_4(t) = \frac{1122 e^{-3t} + 1224 e^{-6t} + 472 e^{-3t} + 62}{1122 e^{-3t} + 1541 e^{-6t} + 764 e^{-3t} + 161 + 12 e^{3t}} \geq 0,
\]

and they satisfy condition (55), which reduces to

\[
\gamma_1(t) + \gamma_\alpha(t) \geq 0, \quad \alpha = 2, 3, 4.
\]
Example 7. For $d = 3$, the dynamical map $\Lambda(t)$ is P-divisible and has exactly $k = 2$ negative decoherence rates if

$$x_1 = \frac{7}{40}, \quad x_2 = \frac{7}{40}, \quad x_3 = \frac{1}{3}, \quad x_4 = \frac{19}{60}. \quad (66)$$

For such a choice of $x_\alpha$, the decoherence rates are equal to

$$\gamma_1(t) = \gamma_2(t) = \frac{2706e^{-6t} + 1148e^{-3t} - 74}{2706e^{-6t} + 3181e^{-3t} + 1180 + 133e^t} \neq 0,$$

$$\gamma_3(t) = \frac{2706e^{-6t} + 3485e^{-3t} + 1009}{2706e^{-6t} + 3181e^{-3t} + 1180 + 133e^t} \geq 0,$$

$$\gamma_4(t) = \frac{2706e^{-6t} + 3188e^{-3t} + 946}{2706e^{-6t} + 3181e^{-3t} + 1180 + 133e^t} \geq 0. \quad (67)$$

In this case, condition (55) simplifies to

$$\gamma_3(t) + 5\gamma_1(t) \geq 0, \quad \gamma_4(t) + 5\gamma_1(t) \geq 0. \quad (68)$$

Straightforward calculations show that the above inequalities always hold.

5. Divisibility regions

Even though $\Lambda(t)$ given by equation (49) is a convex combination of CP-divisible dynamical maps, the resulting channel is CP-divisible only for certain values of $x_\alpha$. For $d = 2$, every convex combination of Markovian semigroups is P-divisible. This property is no longer true for $d \geq 3$. However, $\Lambda(t)$ always satisfies the necessary P-divisibility conditions from equation (55).

The CP-divisibility region can be characterized analytically. The necessary and sufficient conditions for CP-divisibility are $\gamma_\alpha(t) \geq 0$, which can be rewritten as

$$\left(\prod_{\nu=1}^{d+1} x_\nu\right) \left[\sum_{j=1}^{d+1} \frac{1}{x_\nu} - \frac{d}{x_\alpha} - 1\right] \geq 0 \quad (69)$$

after multiplying equation (52) by $e^{-\bar{d}t}\prod_{\nu=1}^{d+1} \left[1 + (e^t - 1) x_\nu\right]$ and taking the limit $t \to \infty$. The border of the CP-divisibility region corresponds to the equality. The hyperplanes $x_\alpha = x_\beta$ give $d + 1$ symmetry planes, which cross at $x_\alpha = 1/d$. If $d = 3$, one has

$$x_4 = \left\{ \begin{aligned}
-\frac{x_1 x_2 x_3}{x_1 x_2 x_3}, & \quad x_1 x_2 x_3 \\
0 \leq x_2 \leq x_3 \leq 1, & \quad \left(\frac{x_1 x_2 x_3}{x_1 x_2 x_3}\right) \leq x_1 \leq 1, \\
-\frac{x_1 x_2 - x_1 x_3 + 2 x_1 x_2 x_3}{x_1 x_2 x_3}, & \quad 0 \leq x_2, x_3 \leq 1, \quad 0 \leq x_1 \leq \frac{x_2 x_3}{x_2 + x_3 - x_2 x_3}, \\
& \quad \frac{x_1 x_2 x_3}{x_1 x_2 x_3}, \\
0 \leq x_3 \leq x_2 \leq 1, & \quad \left(\frac{x_1 x_2 x_3}{x_1 x_2 x_3}\right) \leq x_1 \leq 1, \\
-\frac{x_1 x_2 x_3}{x_1 x_2 x_3}, & \quad 0 \leq x_1, x_2, x_3 \leq 1.
\end{aligned} \right. \quad (70)$$
Figure 1. The range of parameters $x_\alpha$ in $d = 3$ for which $\gamma_\alpha(t) \geq 0$ at all times $t \geq 0$.

For $d = r = 3$, the CP-divisibility region is presented in figure 1 (cf. similar plots for $d = 2$ in [20, 29]). Clearly, it is not a convex set. Moreover, it is symmetric with respect to any permutation of vertices. The figure on the left was plotted after fixing $x_4 = 1 - x_1 - x_2 - x_3$, whereas the one on the right shows the CP-divisibility region on a simplex in the simplex coordinates

$$x'_1 = \frac{-x_1 + x_2}{\sqrt{2}},$$
$$x'_2 = \frac{-x_1 - x_2 + 2x_3}{\sqrt{6}},$$
$$x'_3 = \frac{-x_1 - x_2 - x_3 + 3x_4}{2\sqrt{3}},$$
$$x'_4 = \frac{1}{2}.$$  \hspace{1cm} (71) \hspace{1cm} (72) \hspace{1cm} (73) \hspace{1cm} (74)

The vertices $X_k = (x'_1, x'_2, x'_3)$ of the simplex are located at

$$X_1 = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}}, -\frac{1}{2\sqrt{3}}\right), \quad X_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}}, -\frac{1}{2\sqrt{3}}\right),$$
$$X_3 = \left(0, \frac{2}{\sqrt{6}}, -\frac{1}{2\sqrt{3}}\right), \quad X_4 = \left(0, 0, \frac{\sqrt{3}}{2}\right).$$  \hspace{1cm} (75)

and its center is at the point $X_0 = (0, 0, 0)$. We see that the Markovian evolution constitutes a relatively small portion of the set of admissible quantum evolutions. Moreover, the region of CP-divisible dynamical maps has the same symmetry planes as the simplex.

6. Conclusions

We provided a generalization of eternally non-Markovian evolution of a qubit to the $d$-level systems represented by the generalized Pauli channels. In particular, the convex combination
of Markovian semigroups is analyzed in more details. Quantum evolution of any $d$-level system is characterized by $(d^2 - 1)$ time-dependent decoherence rates. Interestingly, we show that $(d - 1)^2$ rates can be negative for all $t > 0$, and the evolution is still physically legitimate. It turns out that, for the convex combination of Markovian semigroups, a set of CP-divisible dynamical maps is only a small subset of the set of CPTP dynamical maps. For dimension $d = 3$, we explicitly provide the plot range of parameters that lead to CP-divisible dynamical maps. It would be interesting to know whether one can have more eternally negative rates when considering a more general evolution. A natural candidate might be

$$\Lambda(t)[\rho] = \sum_{\alpha=1}^{d+1} \sum_{k=0}^{d-1} p_{\alpha,k}(t) U^{k}_{\alpha} \rho U^{k\dagger}_{\alpha}$$

(76)

with $U^{k}_{\alpha}$ defined in equation (35). The generalized Pauli channels considered in this paper correspond to the special case where $p_{\alpha,k} = p_\alpha$. For $d = 2$, this again reproduces the standard Pauli channel. However, for $d > 2$, the analysis of $\Lambda(t)$ from equation (76) becomes much more complicated due to the fact that the corresponding eigenvalues $\lambda_{\alpha,k}(t)$ are no longer real. We postpone this for future research.

Acknowledgments

KS and DC were supported by the Polish National Science Centre projects No. 2018/31/N/ST2/00250 and 2018/30/A/ST2/00837, respectively.

ORCID iDs

Katarzyna Siudzińska ⓒ https://orcid.org/0000-0002-1816-7242
Dariusz Chruściński ⓒ https://orcid.org/0000-0002-6582-6730

References

[1] Breuer H-P and Petruccione F 2003 The Theory of Open Quantum Systems (Oxford: Oxford University Press)
[2] Weiss U 2012 Quantum Dissipative Systems (Singapore: World Scientific)
[3] Rivas A and Huelga S F 2011 Open Quantum Systems. An Introduction (Heidelberg: Springer)
[4] Gorini V, Kossakowski A and Sudarshan E 1976 J. Math. Phys. 17 821
[5] Lindblad G 1976 Comm. Math. Phys. 48 119
[6] Rivas A, Huelga S F and Plenio M B 2010 Phys. Rev. Lett. 105 050403
[7] Rivas A, Huelga S F and Plenio M B 2014 Rep. Prog. Phys. 77 094001
[8] Breuer H-P, Laine E-M, Piilo J and Vacchini B 2016 Rev. Mod. Phys. 88 021002
[9] de Vega I and Alonso D 2017 Rev. Mod. Phys. 89 015001
[10] Li L, Hall M J W and Wiseman H M 2018 Phys. Rep. 759 1–51
[11] Chruściński D and Maniscalco S 2014 Phys. Rev. Lett. 112 120404
[12] Chruściński D and Kossakowski A 2012 J. Phys. B: At. Mol. Opt. Phys. 45 154002
[13] Chruściński D, Rivas A and Størmer E 2018 Phys. Rev. Lett. 121 080407
[14] Bylicka B, Johansson M and Acín A 2017 Phys. Rev. Lett. 118 120501
[15] Santis D D, Johansson M, Bylicka B, Bernardes N K and Acín A 2019 Phys. Rev. A 99 012303
[16] Chruściński D and Wudarski F A 2013 Phys. Lett. A 377 1425
[17] Chruściński D and Wudarski F A 2015 Phys. Rev. A 91 012104
[18] Hall M J W, Cresser J D, Li L and Andersson E 2014 Phys. Rev. A 89 042120
[19] Breuer H-P, Laine E-M and Piilo J 2009 Phys. Rev. Lett. 103 210401
[20] Megier N, Chruściński D, Piilo J and Strunz W T 2017 Sci. Rep. 7 6379
[21] Chruściński D, Kossakowski A and Pascazio S 2010 Phys. Rev. A 81 032101
[22] Nathanson M and Ruskai M B 2007 J. Phys. A: Math. Theor. 40 8171
[23] Wootters W K and Fields B D 1989 Ann. Phys. 191 363
[24] Bandyopadhyay S, Boykin P, Roychowdhury V and Vatan F 2002 Algorithmica 34 512
[25] Chruściński D and Siudzińska K 2016 Phys. Rev. A 94 022118
[26] Kossakowski A 1972 Bull. Acad. Polon. Sci. Sér. Math. Astr. Phys 20 1021–5
[27] Caiaffa M, Smirne A and Bassi A 2017 Phys. Rev. A 95 062101
[28] Siudzińska K and Chruściński D 2018 J. Math. Phys. 59 033508
[29] Jagadish V, Srikanth R and Petruccione F 2020 Phys. Rev. A 101 062304