Persistent current and Drude weight in one-dimensional rings with substitution potentials

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Abstract

Persistent currents and Drude weights are investigated for the tight-binding approximation to one-dimensional rings threaded by a magnetic flux and with potential given by some almost-periodic substitution sequences with different degrees of randomness, and for various potential strengths. The Drude weight $D$ distinguishes correctly conductors and insulators, in accordance with the results shown by the currents. In the case of insulators the decay of $D(N)$ for large ring lengths $N$ provides an estimate for the localization length of the system. It is shown that the more random the sequence does not imply the smaller conducting properties. This discrepancy between the hierarchy of disorder of the sequences and the capacity of conduction of the system is explained by the gaps in the energy spectra.

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1 Introduction

The ability of quantum mechanical systems in conducting, and their transport properties in general, have been greatly considered in recent years in connection with anomalous transport and localization phenomena [1]. This consideration has been also intense in the mathematical-physics community where one would like to know the spectral and state properties of Schrödinger operators – even one-dimensional, as will always be the case here – with potentials lying between the hallmarks set by periodic potentials with absolutely continuous spectra and Bloch states at one extreme, and random potentials with pure point spectra and localized states at the other [2]. In between there appears a variety of behaviours which seem to be dictated in great measure by the randomness of the potential. To cite just two properties, it is accepted that the mean square displacement $d(t)$ and the quantum return probability $C(t)$ for an initial delta-function state are algebraic functions of time ruled by some exponents, $d(t) \sim t^{2\beta}$ and $C(t) \sim t^{-\alpha}$. These exponents by themselves reflect the more or less random the potential is: ballistic motion in periodic potentials is characterized by $\beta = \alpha = 1$, localization in random potentials by $\beta = \alpha = 0$, while anomalous diffusion in non-periodic potentials shows $0 < \alpha, \beta < 1$. On the mathematical side, a considerable amount of the efforts have sought to relate these dynamical exponents to the dimensions of the spectral measure [2]. However, up to the moment these relations have been very difficult to be applied in practical calculations: on the one hand, the information dimension $D_1$ provides only lower bounds for the exponent $\beta$ ruling the algebraic decay of the mean square displacement; on the other hand, in the case of the return probability $C(t)$ the associated correlation dimension $D_2$ is not in general attainable by means of analytical calculations, and its numerical computation is extremely difficult due mainly to the limiting process involved. In this scenario one-dimensional non-periodic potentials generated as substitution sequences have emerged as a convenient theoretical laboratory since the sequences are neither completely random nor periodic, even though they are formed from well defined rules. In addition, they can be classified by their autocorrelation measures, revealing a hierarchy with respect to disorder. These two aspects of the substitution sequences, a sort of correlated disorder, allow us to investigate transport properties, as well as mathematical ones, in a more controlled way. In this spirit it was shown recently the counter-intuitive result that a more random potential can give rise to better transport properties than a less random one [3]. From another point
of view, with the recent techniques of growing super-lattices one could think of the experimental and technological exploitation of systems characterized by these non-periodic substitution potentials [4].

It seems therefore desirable to have results for concrete physical situations, in order to test theoretical predictions concerning not only disordered systems and localization phenomena but also the almost-periodic substitution potentials themselves. Towards this end, we investigate here the tight-binding approximation to the problem of mesoscopic rings threaded by a time-independent magnetic flux, and calculate the persistent current and Drude weight in cases where the on-site potential is given by some substitution sequences with different degrees of randomness. The system is described by the Hamiltonian for a one-dimensional ring with spinless fermions

\[ H = -\sum_{k=1}^{N} \left[ \exp \left( \frac{2\pi i}{N} \phi \right) c_k^+ c_{k+1} + \exp \left( -\frac{2\pi i}{N} \phi \right) c_{k+1}^+ c_k \right] + \sum_{k=1}^{N} W_k c_k^+ c_k. \] (1)

In this expression \( N \) is the number of sites in a ring with lattice spacing equal to one, the magnetic flux \( \phi \) is measured in units of the quantum of flux \( \phi_0 = \hbar c/e \), \( W_k \) is the potential energy at site \( k \), and the hopping constant is set equal to one. This model is particularly convenient for our purposes for two reasons: first, it admits non-periodic substitution sequences in a simple way; second, the association of the wave vector in the one-dimensional lattice to the parameter \( \phi \) allows us to easily explore the transport properties as functions of the sensitivity of the energy bands to variations in the magnetic flux. In this way the Drude weight, shown by Kohn to be useful as a quantitative characterization of the insulating state [5], and also the persistent current, predicted by Büttiker, Imry and Landauer [6], can both be calculated as derivatives of the ground-state energy with respect to the flux.

It is evident from the model that we assume non-interacting electrons and zero temperature. Although electron-electron interactions and average effects give important contributions to the transport properties — and to the amplitude of persistent currents, as we briefly recall below and in section 3 — these approximations describe quite successfully the qualitative features of the model [7]. They are justifiable in the present case since we attempt mainly to characterize how the randomness of the potential affects the motion of the particles; in so doing we would like therefore to avoid the contributions from other effects.
The aim of this paper is to report and discuss the results, obtained for Hamiltonian (1), of the persistent current and Drude weight as functions of the degree of randomness of the potential and for various potential strengths. Many related works have been published concerning different aspects of the problem. Persistent currents, Drude weight, and also optical conductivity, in mesoscopic rings have been investigated predominantly in Hubbard models, where the on-site interaction between particles with spin plays a major role. This was done in a series of papers ending up in criteria for determining whether a state is metallic, insulating or super-conducting [8]. Interaction between spinless particles together with on-site potential have been considered, mostly with disordered potentials [3, 11, 14, 12] but also with the Aubry-André potential [13]. Recently, particles with spin interacting over finite open chains modulated along the almost-periodic Fibonacci potential have been considered as well [14]. The general picture emerging from these works is not yet completely clear and points to a competition between the effects caused by interaction, potential, charge and spin degrees of freedom; it has been shown, moreover, that the results are sensitive to whether one averages over realizations of the potential [10, 11] or whether one considers individual samples [12]. Concerning solely the effects of the potential, however, the ring with non-periodic substitution potentials has been much less considered; Fibonacci potential was treated by Jin et al. [15]. The properties of systems with substitution potentials have been otherwise intensively studied in open chains, although mainly in the cases of Fibonacci, Thue-Morse, and Rudin-Shapiro sequences [16].

The Fibonacci potential is by far the best studied one of the cited substitution potentials. It is generally considered to be the most ordered one either in the sense of its autocorrelation measure — pure point as for periodic potentials — or in the sense of its dynamical exponents [3]. In this paper, however, Fibonacci potentials will be absent since we would like to compare the results obtained for the different potentials in half- and quarter-filled rings; recall that the lengths of the potential sequences in that case are given by the Fibonacci numbers.

In what follows, section 2 presents the definitions and some properties of the potentials given by the sequences Thue-Morse, Rudin-Shapiro, paper-folding and period-doubling. In section 3.1 persistent currents are calculated for the various potentials (with different amplitudes) as functions of the magnetic flux. A more quantitative characterization of the conducting property for the various systems is presented in section 3.2, where the Drude weight
is calculated and where some attempt is made to fit its behaviour as a function of the length of the ring. Also, in section 3.3 the cases pointed by the Drude weight to be insulators are analyzed in terms of the gaps in the energy spectra. Section 4 finishes the article with a discussion of the results obtained.

2 Non-periodic substitution potentials

In the next sections Hamiltonian (1) will be considered with on-site potential energies $W_k$ given by some sequences which in turn are constructed using well defined substitution rules. We will be interested mainly in almost-periodic sequences which are convenient in the context of disordered systems because we have not only non-periodicity but also non-perfect correlation [17].

The sequences we will use are constructed using an alphabet of two letters \{a, b\} and a specific substitution rule for each sequence:

- $a \rightarrow ab$  $b \rightarrow ba$  Thue–Morse (TM)
- $a \rightarrow ab$  $b \rightarrow aa$  period–doubling (PD).

Starting with one of the letters and applying successively the substitution rules we generate almost-periodic sequences as, for example, the sequence PD

\[abaaabababaabaa\ldots\]

The cases Rudin-Shapiro (RS) and paper-folding (PF) are worked out of an alphabet of four letters \{a′, b′, c′, d′\}, with the rules

- $a′ \rightarrow a′b′$  $b′ \rightarrow a′c′$  $c′ \rightarrow d′b′$  $d′ \rightarrow d′c′$  Rudin–Shapiro (RS)
- $a′ \rightarrow a′b′$  $b′ \rightarrow c′b′$  $c′ \rightarrow a′d′$  $d′ \rightarrow c′d′$  paper–folding (PF),

and the identifications $a′, b′ \rightarrow a$ and $c′, d′ \rightarrow b$ in both cases. The first elements of the sequence RS are

\[aaabaabaaababbb\ldots\]

At each step these sequences have length $N = 2^m$ at the $m$-th iteration. We then define the potential on site $k$ taking $W_k = 0$ if the $k$-th letter in the sequence is $a$, and $W_k = \lambda$ in case it is $b$; in such a way the strength $\lambda$ of a given potential can also be varied.
These non-periodic sequences have a classification with respect to their degree of randomness, which is based on their autocorrelation measure \[17\]. As is the case for periodic sequences, PF and PD have pure point autocorrelation measures; while RS has absolutely continuous autocorrelation measure, as independent random sequences do. The TM case lies in an intermediate place since it has singular continuous autocorrelation measure. One would expect that the differences in the degree of randomness of the potentials would produce different spectral types for the energy levels, and in this way explain the more or less difficult motion of the particles. In other words, one would expect that the more random the potential the smaller the transport properties. The point is that all the rigorously studied cases produce singular continuous spectra \[18\] (RS and PF being open problems \[19\]); yet their dynamical properties are very different and, as it seems, do not respect the hierarchy of disorder proposed by the autocorrelation measures \[3\]. In order to have a clearer picture of the situation we would like to map the role of the potential as a function of its degree of randomness in physical quantities such as persistent currents and conductivity via the Drude weight.

In what follows the \(N\) eigenvalues \(E\) and corresponding eigenfunctions \(\psi_E(k)\) are obtained for Hamiltonian \(H\) using exact diagonalization for ring lengths from \(N = 16\) up to \(N = 256\). A gauge transformation can eliminate the flux in the Hamiltonian by transforming the usual periodic boundary conditions into \(\psi_E(k + N) = e^{i2\pi\phi}\psi_E(k)\). In this way it was shown that the energy bands associated to the lattice vector \(q = -2\pi\phi/N\) are periodic functions of the flux \(\phi\) \[20\]. The effect of the potential on the transport properties are thus seen as the energy bands being more or less flattened out.

3 Results

3.1 Persistent currents

Since its discovery in 1983, persistent currents have become an important subject, both theoretically and experimentally, which is still open in what the amplitude of the currents is concerned. Particularly difficult to treat has been the role of the various contributions to it: electron-electron interactions, disorder of the potential, statistical effects. Much of these issues have been reviewed in Ref. \[21\] (see also, Ref. \[10\]). As predicted by Büttiker, Imry
and Landauer \[3\], in the presence of a non-vanishing magnetic flux $\phi$, even if it is time-independent, each energy band $E_n(\phi)$ carries a persistent current proportional to its derivative with respect to the flux. At zero temperature, the contributions of the $M$ occupied levels below the Fermi energy sum up to

$$I(\phi) = \sum_{n=1}^{M} \frac{\partial E_n}{\partial \phi}.$$  \hspace{1cm} (2)

Figure 1 shows typical curves for the currents for half- and quarter-filled rings with the various substitution potentials at strength $\lambda = 0.2$. One can notice that the order of increasing amplitudes is kept all along the flux axis. Whence in order to estimate a sequence of increasing currents with respect to the potential disorder, we plot in Fig. 2 the mean currents $\langle I^2 \rangle^{1/2}$, averaged over the flux period, for the interval of potential strength $0.2 \leq \lambda \leq 1.0$. In these figures the unit of current is the maximum amplitude obtained for null potential.

It is interesting to note that this hierarchy of amplitudes does not follow what would be expected if one is guided solely by a measure of the potential disorder. Based on the results given by the autocorrelation measure, one could predict the sequence (PD, PF)/TM/RS for the decreasing current amplitudes carried by the different potentials. However, we see that the results obtained for half-filling suggest otherwise PF/RS/TM/PD, while for quarter-filling one would set TM/RS/(PF,PD).

### 3.2 Drude weight

In order to give a more quantitative description of the observations above, we calculate in what follows the Drude weight for the various situations. As prescribed originally by Kohn \[4\], this quantity — useful in the characterization of conducting properties as we will see below — can be obtained as the second derivative of the ground-state energy level with respect to the lattice vector $q = -2\pi\phi/N$, which in our case reduces to

$$D = N \sum_{n=1}^{M} \frac{\partial^2 E_n}{\partial \phi^2} \bigg|_{\phi=\phi_{\text{min}}}.$$ \hspace{1cm} (3)

This expression is equivalent to that obtained using Kubo linear response formulae \[5\]. In the first works following Kohn, $D$ was calculated at $\phi = 0$ and it was realized that its signal depends on the number of occupied levels.
$M$ being even or odd. In this way diamagnetic response of various systems were discussed in the literature, as well as the less common paramagnetic one for $M = 4p$ (see the work of 1991 by Fye et al. in Ref. [8]). However, as was done by Bouzerar, Poilblanc and Montambaux [10] and clearly explained by Giamarchi and Shastry [11], $D$ describes a situation of equilibrium between the charge carriers and should be calculated at the value of $\phi$ where the ground-state energy has a minimum ($\phi_{min}$ in the expression above). Due to the characteristic lengths of our sequences, we adopt here the latter point of view and take $M = 4p$ for either half- or quarter-fillings, with $M = N/2$ and $M = N/4$ respectively. In this way $\phi_{min} = 1/2$ in all cases and we avoid different kinds of response, thus comparing all systems in the same situation.

The behaviour of $D$ as a function of $N$ gives a criterion for distinguishing between conductor and insulator [5]. We restrict the remarks to our one-dimensional case. In metallic conductors $D$ tends to a finite non-zero value as $N \to \infty$; whereas for insulators $D$ vanishes in that limit. This criterion has a more intuitive description if one realizes that $D$ is inversely proportional to the effective mass tensor, which in one-dimension is simply the second derivative written above. We present in Fig. 3 the results of $D(N)$ for the sequences considered in section 2. It is seen that the overall aspect of $D(N)$ and its rate of decaying in the various situations confirm the conduction ordering suggested by the persistent currents shown in Fig. 2, that is PF/RS/TM/PD for half-filling and TM/RS/(PF,PD) for quarter-filling.

It has been proposed in the context of Hubbard models [3, 4, 11] that, in general, insulators show an exponential decay, $D(N) \propto \exp(-N/\xi)$, governed by the localization length $\xi$. The behaviours here vary from well defined insulator (as, for example, in the case PD) to conductor (as quarter-filling TM shows itself), with transitions between these states in some cases. We attempted therefore a more general fitting

$$D(N) = A + BN^\gamma e^{-N/\xi}$$

(4)

where the parameters $A, B, \gamma$ and $\xi$ passed a $\chi^2$-test, and $D(N)$ is measured in units of the value $D_0$ attained for each filling in the largest ring with null potential. Some justifications for the fitting procedure are in order. Concerning the sizes of the system, we avoided filling the range between $N = 16$ and $N = 256$ with more points because this would mean to take incomplete sequences, since each iteration in their construction have a definite size (a power of 2). On the other hand, going to greater sizes does not alter
the results. Concerning the number of fitting parameters, formula (4) was chosen to take into account all possible behaviours, and also to check for meaningless fitting results. We think this strategy was successful since there was no situation in which the fitting curve could show dependence on all parameters. In fact, whenever it showed an exponential decay, the free-constant parameter $A$ was absent. The results are detailed in Table 1. We see that most of the cases showed no dependence on parameter $A$, except for PF half-filling for which $D = A + BN^\gamma$, with the exponent $\gamma$ varying from $-0.29$ to $-0.95$. In this situation, therefore, $D$ shows no dependence on $\xi$, presenting instead a polynomial decay. On the other hand, the case TM quarter-filling did not admit the fitting above.

The fitting curves $D(N)$ appear in Fig. 3 as solid lines, together with the numerical results obtained from eq. (3) (filled circles). Dashed lines are for the cases where the fitting expression given by eq. (4) does not work. We see excellent agreement in PD and PF cases, and in TM at half-filling; good agreement in RS case, and none in TM quarter-filling. The fitting procedure is interesting at this point because it provides information — via the localization length $\xi$, when it occurs — concerning the link between the insulating property and disorder under the action of the various substitution potentials. In figure 4, $\xi$ is plotted as a function of the potential strength $\lambda$. As expected the localization length diminishes with the increase of potential strength, thus meaning more localized wave functions. Obviously if $\xi > N$ for a given length $N$ the corresponding wave functions will cover the whole ring and the system can conduct. The finite value of $\xi$ suggests, however, that if $N$ is sufficiently large the system will show itself as an insulator. This is the case for TM and RS at half-filling, and also for RS at quarter-filling, for potential strengths $\lambda < 0.4$.

### 3.3 Spectral gaps

The characteristics seen in the Drude weights for insulators can be explained in terms of the widths of the gaps in the energy spectra. In fact, systems with singular continuous spectra, as is commonly the case under almost-periodic potentials, show spectra which are Cantor sets; therefore gaps appear with different widths at different energy values. The question of where (large) gaps will appear has not yet a definite answer, with some clues having been provided by gap-labelling procedures [22]. For the cases investigated here, gaps open in accordance with the results shown by the calculations related to
the Drude weight. This is presented in Fig. 5 where the difference between
the last filled energy level and the next one, $\Delta = E_{M+1} - E_M$, is computed at
$\phi = \phi_{\text{min}}$ for the largest ring considered ($N = 256$) and plotted as a function
of the potential strength $\lambda$. We see two distinct types of gap. The cases
which are clearly insulators after the criterion put forth by the exponential
decay of the Drude weight $D$ have gaps increasing with $\lambda$ and much larger
than the gaps shown by those cases which are conductors for the values of
$N$ considered. Also, for the systems which present greater gaps in Fig. 5,
we have checked the linear relation $\Delta(\lambda) \sim \xi^{-1}(\lambda)$, not shown here (see the
article of 1991 by Fye et al. in ref. [8]). There is very good agreement
for TM and PD at half-filling and also for PD and PF at quarter-filling.
The exception is RS which appears as an insulator but for which the results
concerning the gaps are commonly intractable.

This relation between the Drude weight and the gaps in the spectra can
be pursued further. On the one hand, Fye et al. [8] obtained results pointing
to the independence of the Drude weight of the boundary conditions being
periodic or open. On the other hand, gap-labelling procedures have shown
that for open chains the integrated density of states ($IDS$) for the TM po-
tential has an open gap at $IDS = 1/2$ and a closed gap at $IDS = 1/4$; the
PD case presents open gaps at both $IDS = 1/2, 1/4$; and for the cases RS
and PF there are no conclusive results [22]. Accepting that these results for
open chains are valid for sufficiently large rings, and noting that the $IDS$ is
directly translated into our filling factor, we see that the results shown in fig-
ures 1 – 4 are consistent with gap-labelling predictions. In particular, these
predictions would explain why the current is maximal for TM quarter-filling,
since there would be a closed gap at $IDS = 1/4$. On the other way, the
vanishing currents for the PF case at quarter-filling could suggest a closed
gap at $IDS = 1/4$.

Although this relation clearly needs more investigation, the observations
above may provide a more intuitive understanding of the relation between
the transport properties of the system and its spectral type, as recalled in
the Introduction.

4 Discussion and conclusion

The results of the previous sections show that the behaviour of $D(N)$ dis-
tinguishes correctly the conductors and insulators as suggested by the rela-
tive amplitudes of the persistent currents. Indeed, the exponential decay of $D(N)$ in the case of insulators, and the resulting localization length $\xi$, are reinforced by the structure of gaps in the energy spectra. The other cases, where that decay is slower than exponential, clearly need more investigation to be modeled by the behaviour of $D(N)$. Chaves and Satija [8] have recently conjectured the existence of a new phase state between insulator and conductor for deterministic aperiodic systems, with a polynomial decay of $D$ as a function of the system size $N$. Our results, seen in the figures and in the fitting output, points to the same direction although suggesting a richer behaviour, perhaps with mixtures and transitions between the possibilities above.

Some care must be exercised in comparing the results presented in this paper and other results available in the literature dealing with the relations between transport properties or diffusion exponents and the degree of randomness of the potential [1, 2, 3]. The latter ones mostly deal with a single particle moving in an open chain, with an initial delta-function state on a given position. In that case all the energy eigenstates contribute to the quantum state of the system, contrary to the situation seen here where only the lowest levels, up to the filling, are present. Whether is applicable here the result of Fye et al. [8] that the Drude weight does not depend on the boundary conditions is to be investigated more systematically.

We have thus investigated in this work how different degrees of randomness in the potential affect the persistent currents and the conducting properties — via the Drude weight — of half- and quarter-filled rings threaded by a magnetic flux. This was done in the tight-binding approximation with the potential being given by some almost-periodic substitution sequences (Thue-Morse, Rudin-Shapiro, paper-folding and period-doubling). The results show that it is by no means obvious what potential allows better conducting properties, if one is guided solely by a measure of its degree of randomness as, for example, the autocorrelation measure of the sequence itself. More specifically, that measure predicts the following hierarchy of disorder (PD, PF)/TM/RS, while the persistent-current amplitudes obey PF/RS/TM/PD for half-filling and TM/RS/(PF,PD) for quarter-filling. A more quantitative estimate, given by the decay of the Drude weight as a function of the ring length, shows indeed that for half-filling TM and PD systems are insulators and PF ones can conduct, whereas for quarter-filling PD and PF systems are insulators and TM ones are conductors (RS systems have not so simple a classification). That discrepancy is, moreover,
confirmed and explained by the gaps opened in the energy spectra as functions of the potential strength for the various sequences.

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Figure Captions

Figure 1. Persistent currents for the various substitution potentials at strength $\lambda = 0.2$ and ring length $N = 256$: (a) half-filling; (b) quarter-filling.

Figure 2. Mean current amplitudes as a function of the potential strength: (a) half-filling; (b) quarter-filling.

Figure 3. Drude weight as a function of the number of sites in the ring (filled circles): (a) half-filling; (b) quarter-filling. Fitting curves are shown as solid lines.

Figure 4. Localization length $\xi$ as a function of the strength of potential $\lambda$. Results for RS are shown in the inset for clarity.

Figure 5. Gaps in the energy spectra (symbols) at (a) half-filling and (b) quarter-filling for flux value $\phi = \phi_{\text{min}}$

Table Caption

Table 1. Fitting parameters as in eq. (4) for the various substitution potentials. The indices $h$ and $q$ refer to half-filling and quarter-filling, respectively. The localization length $\xi$ is given in units of the (nearest integer) number of sites. The symbol — indicates no dependence of the curve on the corresponding parameter, whereas the symbol $- -$ points to the non-applicability of the fitting expression.
Table 1

| Potential | $A_h$ | $B_h$ | $\gamma_h$ | $\xi_h$ | $A_q$ | $B_q$ | $\gamma_q$ | $\xi_q$ |
|-----------|-------|-------|------------|--------|-------|-------|------------|--------|
| TM 0.2    | 0.96  | 0.01  | 1935       | -      | -     | -     | -          | -      |
| TM 0.4    | 0.65  | 0.17  | 150        | -      | -     | -     | -          | -      |
| TM 0.6    | 0.51  | 0.31  | 53         | -      | -     | -     | -          | -      |
| TM 0.8    | 0.42  | 0.47  | 25         | -      | -     | -     | -          | -      |
| TM 1.0    | 0.34  | 0.66  | 14         | -      | -     | -     | -          | -      |
| RS 0.2    | 0.96  | 0.01  | 1308       | -      | 0.98 | 0.00  | 1205       |
| RS 0.4    | 0.89  | 0.03  | 427        | -      | 1.02 | -0.03 | 773        |
| RS 0.6    | 0.51  | 0.23  | 125        | -      | 1.02 | -0.04 | 316        |
| RS 0.8    | 0.64  | 0.15  | 105        | -      | 0.73 | -0.06 | 125        |
| RS 1.0    | 0.09  | 1.03  | 20         | -      | 0.88 | -0.04 | 139        |
| PD 0.2    | 0.51  | 0.37  | 36         | -      | 0.51 | 0.32  | 55         |
| PD 0.4    | 0.74  | 0.36  | 17         | -      | 0.19 | 0.82  | 20         |
| PD 0.6    | 1.07  | 0.13  | 11         | -      | 0.07 | 1.31  | 12         |
| PD 0.8    | 4.88  | -0.21 | 9          | -      | 0.03 | 1.68  | 9          |
| PD 1.0    | 6.47  | -0.23 | 7          | -      | 0.01 | 1.97  | 7          |
| PF 0.2    | 0.98  | 0.16  | -0.95      | -      | 0.43 | 0.45  | 31         |
| PF 0.4    | 0.94  | 0.11  | -0.43      | -      | 0.29 | 0.78  | 13         |
| PF 0.6    | 0.88  | 0.16  | -0.30      | -      | 0.33 | 0.86  | 8          |
| PF 0.8    | 0.81  | 0.22  | -0.29      | -      | 0.48 | 0.81  | 6          |
| PF 1.0    | 0.76  | 0.29  | -0.35      | -      | 0.62 | 0.81  | 5          |
Fig. 1a

half-filling

I

\( \phi \)

• TM

-- RS

-- PD

• PF
Fig. 1b

quarter-filling
half-filling

Fig. 2a
Fig. 2b

quarter-filling

\( \langle I^2 \rangle^{1/2} \)

\( \lambda \)
Fig. 3b
Fig. 4a
Fig. 4b
quarter-filling

- TM
- RS
- PF
- PD

Fig. 5b