A set optimization approach to zero-sum matrix games with multi-dimensional payoffs

Andreas H. Hamel¹ · Andreas Löhne²

Received: 1 October 2017 / Published online: 28 May 2018
© Springer-Verlag GmbH Germany, part of Springer Nature 2018

Abstract A new solution concept for two-player zero-sum matrix games with multi-dimensional payoffs is introduced. It is based on extensions of the vector order in \( \mathbb{R}^d \) to order relations in the power set of \( \mathbb{R}^d \), so-called set relations, and strictly motivated by the interpretation of the payoff as multi-dimensional loss for one and gain for the other player. The new concept provides coherent worst case estimates for games with multi-dimensional payoffs. It is shown that—in contrast to games with one-dimensional payoffs—the corresponding strategies are different from equilibrium strategies for games with multi-dimensional payoffs. The two concepts are combined into new equilibrium notions for which existence theorems are given. Relationships of the new concepts to existing ones such as Shapley and vector equilibria, vector minimax and maximin solutions as well as Pareto optimal security strategies are clarified.

Keywords Zero-sum game · Multi-dimensional payoff · Multi-objective programming · Set relation · Set optimization · Incomplete preference

Mathematics Subject Classification Primary 91A05; Secondary 91A10 · 62C20 · 91A35

The work of the first author was supported by a generous start-up grant from Free University of Bozen.

Andreas H. Hamel
andreas.hamel@unibz.it

Andreas Löhne
andreas.loehne@uni-jena.de

¹ Faculty of Economics and Management, Free University Bozen-Bolzano, Bolzano, Italy
² Department of Mathematics, Friedrich Schiller University, Jena, Germany
1 Introduction

This paper is an attempt to provide novel answers to the question what should actually be played in games with multi-dimensional payoffs. In general, this situation involves a non-total order relation for the payoff vectors, i.e., there are non-comparable outcomes. Such games were first discussed by Blackwell (1956) and Shapley (1959). Similar setups with so-called incomplete preferences for players and/or decision makers were also discussed in economics, compare for example Bade (2005) and Park (2015).

The focus is on two-player, zero-sum matrix games with multi-dimensional payoffs where the utility function for each player just is, admitting mixed strategies, her/his expected vector-valued gain.

Already Shapley (1959, p. 58) [see also Aumann (1962, p. 447)] gave a motivation for studying such games: ‘The payoff of a game sometimes most naturally takes the form of a vector having numerical components that represent commodities (such as men, ships, money, etc.) whose relative values cannot be ascertained. The utility spaces of the players can therefore be given only a partial ordering (representable as the intersection of a finite number of total orderings), and the usual notions of solution must be generalized.’ However, the theory even for “simple” zero-sum matrix games with multi-dimensional payoffs is far from being complete in a way which parallels the elegant von Neumann approach in the scalar case. The earliest proposal by Blackwell (1956) generalized the notion of the value of a game to an “approachable set” apparently motivated by the fact that, in contrast to the classic von Neumann situation, insisting on a single payoff (vector) as value of the game does not make much sense for games with multi-dimensional payoffs.

Shapley (1959) introduced an equilibrium notion which relies on the order generated by the positive cone in the images space. His concept was put into the context of vector optimization as vector equilibrium points or vector saddle points, see, for example (Nieuwenhuis 1983, Definition 3.1), (Corley 1985, (5)), (Luc and Vargas 1992; Tanaka 1994). Compare the survey (Tanaka 2000) and the references therein as well as (Zhao 1991) where general games with multi-dimensional payoffs are considered and more references can be found.

However, it turned out that, despite this effort, vector optimization did not yet provide a satisfying equilibrium theory, not even for two-player, zero-sum matrix games with vector-valued payoffs/utilities.

What are the reasons for the lack of an appealing and applicable, multi-dimensional equilibrium theory? In Puerto and Perea (2018, p. 1634), the authors attribute the difficulties ‘to the lack of total orderings among players’ payoffs.’ Digging a little deeper one is confronted with a few conceptual problems which indeed appear due to the non-totalness of a general vector order.

One difficulty is the sheer number of “Shapley equilibrium values.” Shapley’s main result (Shapley 1959, Theorem) reduces a zero-sum matrix game with vector payoffs to infinitely many scalar non-zero-sum games. In most cases, this produces just too many candidates for a solution and ‘the impossible task of solving all possible scalarizations’ as remarked by Corley (1985, Section 4). In a more general context, Bade (2005, p. 328) observed the same: ‘Since Nash equilibrium sets of games with incomplete preferences can be large, it is of interest to consider refinements of equilibria . . .’
to De Marco and Morgan (2007) for a discussion of several approaches and references concerning refinements. The idea is to impose additional requirements to the equilibrium—often via scalarizations—'which adds to the original problem new endogenous parameters that are typical for the vector-valued form’ De Marco and Morgan (2007, p. 171). However, often it does not become clear which strategy the players should pick since it depends on the scalarization—which, in turn, has to be selected first, thus one is in the same situation as for Shapley equilibria.

In the context of vector optimization, similar scalarization results can be found, for example, in Zeleny (1975) (basically, already Shapley’s result), Cook (1976) (minimizing the weighted underachievement of goals) and Wierzbicki (1995) (via nonlinear scalarization functions). The notion of Nash equilibria in Zhao (1991, Definition 12) is based on properly efficient points, which can also be found via linear scalarizations.

Another, even more important difficulty is that Shapley equilibria are missing two main features of equilibrium strategies for scalar games. They are not interchangeable, and they do not provide worst case estimates.

The missing interchangeability was observed by Corley (1985, Example 3.2) (see Example 3.15 below), and contradicts Aumann’s belief that ‘the interchangeability property holds,’ cf. (Aumann 1962, p. 455) for Shapley equilibrium strategies. Therefore, a “best possible answer” to an equilibrium strategy is not necessarily a “best possible answer” to another one which leaves the question completely open what kind of strategy protects a player best against the opponent’s choices. In this paper, the meaning of “best” will be scrutinized, and it will be given a new meaning via a set relation approach.

The missing worst case insurance motivated (sightly deviating from each other) concepts of vector minimax and maximin strategies as discussed, for example, by Nieuwenhuis (1983), Corley (1985) and Tanaka (1988). The relationship to the previously introduced vector equilibrium/saddle points remained obscure as observed by Corley (1985, Section 4): ‘Second, it is not clear how minimax and maximin points are related to equilibrium points, except that a joint minimax and maximin point is obviously an equilibrium point.’ Moreover, the interpretation as a worst case insurance failed as remarked by Nieuwenhuis (1983, p. 473): ‘Whereas the notions of minimax point and maximin point do not seem to have an easy game-theoretic interpretation, the notion of saddle point has.’

An attempt to resolve this issue by “brutal force” was given by Ghose and Prasad (1989), called Pareto optimal security strategies (POSS). It relies on the idea of providing insurance against the component-wise worst cases, see (Ghose 1991; Fernández and Puerto 1996; Fernández et al. 1998) and also (Puerto and Perea 2018) for more details, and it leads to very conservative strategies which make little sense in some cases (see Sect. 4 below).

The third major difficulty is more mathematical in nature. It boils down to the fact that the infimum and the supremum with respect to vector orders (or even more general preferences) are of no use in most cases. Either they do not exist since the (vector) order is not a lattice order, or even if they do, they produce “ideal points” which are not attainable payoffs in general. The two difficulties described above of generalizing the von Neumann minimax theory to non-total orders can also be attributed to this feature.
Our approach via set relations opens a way out of this dilemma and is strictly motivated by the interpretation of the payoff as multi-dimensional loss for one and gain for the other player. We introduce a new solution concept for two-player, zero-sum matrix games where main tools are taken from the recently developed theory of set optimization (Hamel et al. 2015), but different “set orders” as, for instance, in (Maeda 2015) are used. Examples illustrate relationships to previous solution concepts and provide evidence that optimal strategies as introduced below indeed should be played if the players wish to achieve “best” protection against losses, and it is easily seen that they enjoy the interchangeability property. By combining the new concept with Shapley’s equilibrium notion, new equilibrium concepts are obtained for which existence theorems are given.

The set optimization approach proposed here re-introduces the infimum and the supremum in appropriate complete lattices of sets, and it is shown that in this way one obtains complete set optimization analogs to the scalar minimax and maximin problems. On the other hand, minimality/maximality and attainment of the infimum/supremum become two different concepts for set optimization problems, and therefore our new equilibrium concept bunches both features. The reader is referred to (Löhne 2011; Hamel et al. 2015) for more details and references concerning this approach. It should be noted that it is very distinct from the “vector approach” to set-valued optimization problems which was the basic concept in most contributions for the special issue (Chen and Jahn (eds.) 1998) on set-valued optimization.

The paper is organized as follows. In the next section, a few crucial concepts from set optimization are introduced along with some notation. In Sect. 3, the new solution concept for games with multi-dimensional payoffs is presented and existence theorems for corresponding equilibrium notions are provided. In Sect. 4, the new concept is compared with others from the literature.

An algorithm based on Löhne and Weißing (2017) is already available which admits to compute optimal strategies and the corresponding payoffs. It has been used to perform the computations for some of the examples of this paper.

2 Notation and a few basics on set optimization

For the readers convenience, basic concepts related to vector and set orders as well as the complete lattice approach to set optimization are summarized. The main reference is Hamel et al. (2015).

Let \( d \geq 1 \) be an integer and \( z \in \mathbb{R}^d \) an element of the \( d \)-dimensional vector space of column vectors with real components. We write \( z = (z_1, z_2, \ldots, z_d)^T \) where the upper \( T \) indicates the transpose of the row vector. In the following, only the component-wise partial order for elements of \( \mathbb{R}^d \) is used which is generated by the closed convex cone \( \mathbb{R}^d_+ = \{ z \in \mathbb{R}^d \mid z_1 \geq 0, \ldots, z_d \geq 0 \} \). This partial order is denoted by \( \leq_{\mathbb{R}^d_+} \) and defined through

\[
y \leq_{\mathbb{R}^d_+} z \iff z - y \in \mathbb{R}^d_+.
\]
Although \((\mathbb{R}^d, \leq_{\mathbb{R}^d})\) is a lattice, the infimum and the supremum of a set \(A \subseteq \mathbb{R}^d\) are not very useful: Even if they exist, they can be “far away” from \(A\): consider \(d = 2\) and \(A = \{z \in \mathbb{R}^2_+ \mid z_1 + z_2 \geq 2\}\) whose infimum with respect to \(\leq_{\mathbb{R}^2_+}\) is \(z = 0 \in \mathbb{R}^2\). Therefore, the predominant optimality notion in vector optimization and multi-criteria decision making is based on minimal (or maximal) points.

A point \(\bar{z} \in A \subseteq \mathbb{R}^d\) is called minimal with respect to \(\leq_{\mathbb{R}^d_+}\) if

\[
z \in A,\ z \leq_{\mathbb{R}^d_+} \bar{z} \implies z = \bar{z}.
\]

The set of minimal points of \(A\) is denoted by \(\text{Min} A\). Likewise, the set \(\text{Max} A\) of maximal points is introduced.

The lack of a reasonable infimum/supremum with respect to vector orders is a major motivation for introducing so-called set relations, see Kuroiwa et al. (1997) as well as Hamel et al. (2015) for a recent survey with many references.

Let \(A, B \subseteq \mathbb{R}^d\). By

\[
A \preceq_{\mathbb{R}^d_+} B :\iff B \subseteq A + \mathbb{R}^d_+ \quad \text{and} \quad A \preceq_{\mathbb{R}^d_+} B :\iff A \subseteq B - \mathbb{R}^d_+
\]

two “set relations” are defined which both are reflexive and transitive, but not anti-symmetric in general. Moreover, they are two different extensions of \(\leq_{\mathbb{R}^d_+} : y \leq_{\mathbb{R}^d_+} z \iff \{y\} \leq_{\mathbb{R}^d_+} \{z\} \iff \{y\} \leq_{\mathbb{R}^d_+} \{z\}\). Per se, these relations just shift the difficulty of defining optimality from \(\mathbb{R}^d\) to its power set \(\mathcal{P}(\mathbb{R}^d)\). Their value lies in the possibility to construct complete lattices of sets based on their symmetric parts.

Two sets \(A, B \subseteq \mathbb{R}^d\) are equivalent with respect to \(\preceq_{\mathbb{R}^d_+}\), written \(A \sim B\), if \(A \preceq_{\mathbb{R}^d_+} B \preceq_{\mathbb{R}^d_+} A\). It can easily be shown that \(A \sim B\) if and only if \(A + \mathbb{R}^d_+ = B + \mathbb{R}^d_+\). Therefore, the set of equivalence classes with respect to \(\sim\) can be identified with \(\mathcal{P}(\mathbb{R}^d, \mathbb{R}^d_+) := \{A \subseteq \mathbb{R}^d \mid A = A + \mathbb{R}^d_+\}\). Moreover, on \(\mathcal{P}(\mathbb{R}^d, \mathbb{R}^d_+)\) the relation \(\preceq_{\mathbb{R}^d_+}\) coincides with \(\supseteq\). Likewise, \(\mathcal{P}(\mathbb{R}^d, -\mathbb{R}^d_+) := \{A \subseteq \mathbb{R}^d \mid A = A - \mathbb{R}^d_+\}\) can be identified with the set of equivalence classes with respect to the symmetric part of \(\preceq_{\mathbb{R}^d_+}\), and on \(\mathcal{P}(\mathbb{R}^d, -\mathbb{R}^d_+)\) the relation \(\preceq_{\mathbb{R}^d_+}\) coincides with \(\subseteq\).

Moreover, both \((\mathcal{P}(\mathbb{R}^d, \mathbb{R}^d_+), \supseteq)\) and \((\mathcal{P}(\mathbb{R}^d, -\mathbb{R}^d_+), \subseteq)\) are complete lattices, i.e., every set \(A \subseteq \mathcal{P}(\mathbb{R}^d, \mathbb{R}^d_+)\) has an infimum and a supremum in \(\mathcal{P}(\mathbb{R}^d, \mathbb{R}^d_+)\) as well as every set \(B \subseteq \mathcal{P}(\mathbb{R}^d, -\mathbb{R}^d_+)\). Thus, infimum and supremum become again available without any restrictions, and this fact constitutes the major difference to more traditional approaches in multi-objective and even set-valued optimization, the latter only based on minimality/maximality with respect to the set relations defined above.

The applications in this paper involve convex (set-valued) functions and closed convex sets. Therefore, the following two sets are introduced:

\[
\mathcal{G}(\mathbb{R}^d, \mathbb{R}^d_+) := \{A \subseteq \mathbb{R}^d \mid A = \text{cl co}(A + \mathbb{R}^d_+)\}
\]

\[
\mathcal{G}(\mathbb{R}^d, -\mathbb{R}^d_+) := \{A \subseteq \mathbb{R}^d \mid A = \text{cl co}(A - \mathbb{R}^d_+)\}
\]
where the usual conventions for the Minkowski addition of sets are used with the extension \( \emptyset + A = A + \emptyset \) for all \( A \in \mathcal{P}(\mathbb{R}^d) \). The symbol \( \text{cl co} \) stands for the closure of the convex hull of the set \( D \subseteq \mathbb{R}^d \).

**Proposition 2.1** The pairs \( (\mathcal{G}(\mathbb{R}^d, \mathbb{R}_+^d), \supseteq) \) and \( (\mathcal{G}(\mathbb{R}^d, -\mathbb{R}_+^d), \subseteq) \) are complete lattices with the following formulas for infimum and supremum: For \( A \subseteq \mathcal{G}(\mathbb{R}^d, \mathbb{R}_+^d) \),

\[
\inf A = \text{cl co} \bigcup_{A \in \mathcal{A}} A \quad \text{and} \quad \sup A = \bigcap_{A \in \mathcal{A}} A.
\]

For \( B \subseteq \mathcal{G}(\mathbb{R}^d, -\mathbb{R}_+^d) \),

\[
\inf B = \bigcap_{B \in \mathcal{B}} B \quad \text{and} \quad \sup B = \text{cl co} \bigcup_{B \in \mathcal{B}} B.
\]

**Proof** See, for example, Hamel et al. (2015).

Note that the formulas for inf and sup in \( (\mathcal{G}(\mathbb{R}^d, \mathbb{R}_+^d), \supseteq) \) and \( (\mathcal{G}(\mathbb{R}^d, -\mathbb{R}_+^d), \subseteq) \) are exchanged due to the change of the ordering relation. The above result also shows that the infimum in \( (\mathcal{G}(\mathbb{R}^d, \mathbb{R}_+^d), \supseteq) \) as well as the supremum in \( (\mathcal{G}(\mathbb{R}^d, -\mathbb{R}_+^d), \subseteq) \) are unions and hence “close” to the defining sets whereas this is not true for the supremum in \( (\mathcal{G}(\mathbb{R}^d, \mathbb{R}_+^d), \supseteq) \) and the infimum in \( (\mathcal{G}(\mathbb{R}^d, -\mathbb{R}_+^d), \subseteq) \).

A set \( A \subseteq \mathbb{R}^d \) is said to enjoy the lower domination property if for each \( z \in A \) there is \( \bar{z} \in \text{Min} A \) with \( \bar{z} \leq R_d^+ z \). The upper domination property is defined parallel.

If \( A, B \subseteq \mathbb{R}^d \) satisfy the lower domination property, then

\[
A \preceq_{\mathbb{R}^d_+} B \quad \iff \quad \text{Min} B \subseteq \text{Min} A + \mathbb{R}_+^d,
\]

and if they satisfy the upper domination property, then

\[
A \preceq_{\mathbb{R}^d_+} B \quad \iff \quad \text{Max} A \subseteq \text{Max} B - \mathbb{R}_+^d.
\]

It is well-known that \( A \subseteq \mathbb{R}^d \) satisfies the lower as well as the upper domination property if it is compact, see Henig (1986).

The complete-lattice approach admits to provide coherent solution concepts for optimization problems with a vector- or set-valued objective function. The definition below is due to Heyde and Löhne (2011).

**Definition 2.2** Let \( X \) be a nonempty set, \( (L, \leq) \) a complete lattice and \( f : X \rightarrow L \) a function.

(a) A set \( M \subseteq X \) is called an infimizer for \( f \) if

\[
\inf_{x \in M} f(x) = \inf_{x \in X} f(x).
\]
(b) A point \( \bar{x} \in X \) is called a minimizer for \( f \) if
\[
x \in X, \quad f(x) \leq f(\bar{x}) \quad \Rightarrow \quad f(x) = f(\bar{x}).
\]
(c) A set \( M \subseteq X \) is called a solution of the problem
\[
\text{minimize} \quad f(x) \quad \text{over} \quad x \in X \quad (P)
\]
if \( M \) is an infimizer and each \( x \in M \) is a minimizer for \( f \). A solution \( M \) is called full, if \( M \) includes all minimizers for \( f \).

Supremizers, maximizers and solutions of complete lattice-valued maximization problems are defined in a parallel way.

3 A new solution concept for vector games

Let \( G = (g_{ij})_{m \times n} \) be an \( m \times n \) matrix whose entries
\[
g_{ij} = (g_{ij}^1, g_{ij}^2, \ldots, g_{ij}^d)^T \in \mathbb{R}^d
\]
are \( d \)-dimensional column vectors of real numbers. We interpret \( G \) as a loss matrix for the row-choosing player I. Independently, player I and the column-choosing player II select a row \( i \in \{1, \ldots, m\} \) and a column \( j \in \{1, \ldots, n\} \), respectively, which results in player I delivering \( g_{ij} \) to player II with the usual convention that negative delivery means that I receives the corresponding amount from II. An interpretation of the payoff vectors \( g_{ij} \) could be that the \( d \) components of \( g_{ij} \) denote units of \( d \) different assets, so I hands II a portfolio instead of an amount of one particular currency.

As usual in the theory of finite matrix games, we consider mixed strategies. Player I chooses row \( i \) with probability \( p_i \) and player II column \( j \) with probability \( q_j \). The two sets
\[
P = \left\{ p \in \mathbb{R}_+^m \mid \sum_{i=1}^{m} p_i = 1 \right\} \quad \text{and} \quad Q = \left\{ q \in \mathbb{R}_+^n \mid \sum_{j=1}^{n} q_j = 1 \right\}
\]
model the admissible (mixed) strategies of the two players. If player I chooses strategy \( p \in P \) and player II \( q \in Q \), then the expected (vector) loss of player I is
\[
v(p, q) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i g_{ij} q_j \in \mathbb{R}^d.
\]
The set
\[
V = \{ v(p, q) \mid p \in P, \ q \in Q \}
\]
is in fact a (possibly non-convex) subset of the convex hull of the entries of the matrix $G$.

Different from the scalar case, it is not a priori clear what kind of order should be used for comparing the (expected) payoffs. It would be reasonable to assume that each player has a preference for the payoffs and that these preferences are different from each other—as in Bade (2005), but in contrast to references like Nieuwenhuis (1983), Tanaka (1994) where a general partial order in $\mathbb{R}^d$ is used which is the same for both players. The point of view in this note is that it is only known to the players that player I prefers “less loss” and player II prefers “more gain.”

Therefore, it is assumed that both players’ decisions are consistent with the partial order $\leq_{\mathbb{R}_+^d}$ and that no other information about the preferences of the players is available. This is “Shapley’s assumption” (Shapley 1959, p. 59): ‘It is also assumed that the first player wants to increase the components of the vector, and the second player wants to decrease them. Finally, it is assumed that neither player has an a priori opinion concerning the relative importance to himself of the different components.’

Let player I choose a strategy $p \in P$. If for another strategy $p' \in P$ it holds

$$\forall q \in Q : v(p', q) \leq_{\mathbb{R}_+^d} v(p, q),$$

then clearly $p'$ is better than $p$ for player I. In this case, we write $p' \leq_I p$. In case of

$$\forall q \in Q : v(p', q) = v(p, q),$$

the strategies $p, p', \in P$ are considered to be equivalent and we write $p' =_I p$. It is natural to determine minimal elements with respect to the order $\leq_I$ and the equivalence relation $=_I$. We say $p \in P$ is $\leq_I$-minimal if

$$(p' \in P, \ p' \leq_I p) \Rightarrow p' =_I p.$$  

Likewise, we introduce $\leq_{II}$, $=_II$ and $\leq_{II}$-maximality for player II.

It turns out, however, that the orders $\leq_I$ and $\leq_{II}$ are not “rich enough”—there are just too many $\leq_I$-minimal and $\leq_{II}$-maximal elements since many cannot be compared with each other.

**Example 3.1** In the game given by

$$G = \begin{pmatrix} 0 & 4 \\ 0 & 4 \\ 3 & 1 \\ 1 & 3 \end{pmatrix}$$

$\odot$ Springer
any two strategies \( p, p' \in P \) are not comparable with each other with respect to the \( \leq_I \)-order, and hence they are all \( \leq_I \)-minimal. Indeed, choosing \( \bar{q} = (1, 0)^T \) we have
\[
v(p, \bar{q}) = \sum_{i=1}^{m} p_i g_{i1} = p_2 \cdot \left( \frac{3}{1} \right) = (1 - p_1) \cdot \left( \frac{3}{1} \right),
\]
i.e., the larger \( p_1 \in [0, 1] \) the smaller with respect to \( \leq_{\mathbb{R}^d_+} \) is \( v(p, \bar{q}) \). On the other hand, taking \( \hat{q} = (0, 1)^T \) we get
\[
v(p, \hat{q}) = \sum_{i=1}^{m} p_i g_{i2} = p_1 \cdot \left( \frac{4}{4} \right) + p_2 \cdot \left( \frac{1}{3} \right),
\]
i.e., the larger \( p_1 \in [0, 1] \) the larger with respect to \( \leq_{\mathbb{R}^d_+} \) is \( v(p, \hat{q}) \). This proves that any two strategies \( p, p' \in P \) are not comparable and hence all \( p \in P \) are \( \leq_I \)-minimal.

On the other hand, consider the strategy \( \bar{p} = (0, 1)^T \) for player I. Then
\[
v(\bar{p}, q) = \sum_{j=1}^{n} q_j g_{2j} = q_1 \cdot \left( \frac{3}{1} \right) + q_2 \cdot \left( \frac{1}{3} \right).
\]
For any two different elements \( q, q' \in Q \), we have \( v(\bar{p}, q) \not\leq_{\mathbb{R}^d_+} v(\bar{p}, q') \). Thus, elements in \( q, q' \in Q \) are never comparable with respect to \( \leq_{II} \), which shows that all elements \( q \in Q \) are \( \leq_{II} \)-maximal.

Even though, based on the order relations \( \leq_I \) and \( \leq_{II} \), any strategy \( p \in P \) is optimal for player I and any \( q \in Q \) is optimal for player II, a look at the payoff matrix may convince the reader that it might be better for player I to favor the second row and clearly better for II to have a bias towards the second column.

The above example provokes the question which order relation should be used instead. A minimal requirement certainly is that the order for player I maintains the \( \leq_I \)-order, but strategies should be comparable more often. If player I picks strategy \( p \in P \), then her/his expected payoff belongs to the bounded convex polyhedron
\[
v_I(p) = \{ v(p, q) \mid q \in Q \} = \text{co} \left\{ \sum_{i=1}^{m} p_i g_{i1}, \ldots, \sum_{i=1}^{m} p_i g_{in} \right\}.
\] (3.1)
The main idea is to compare the two sets \( v_I(p), v_I(p') \) for \( p, p' \in P \) by means of the set relations defined in the previous section. In fact, \( \leq_{\mathbb{R}^d_+} \) turns out to be the appropriate choice since the following statement is obvious.

**Proposition 3.2** Let \( p, p' \in P \). If \( p' \leq_I p \), then \( v_I(p') \leq_{\mathbb{R}^d_+} v_I(p) \).

Another aspect is uncovered if one asks which expected payoffs player II can generate if I plays \( \bar{p} \in P \). The worst case scenario for player I is a maximal point of
$v_I(\bar{p})$, which means $\bar{q} \in Q$ satisfying $v(\bar{p}, \bar{q}) \in \text{Max } v_I(\bar{p})$, i.e.,

$$\left( q \in Q, \quad v(\bar{p}, \bar{q}) \leq_{\mathbb{R}_+^d} v(p, q) \right) \implies v(\bar{p}, \bar{q}) = v(\bar{p}, q).$$

If player II knows the strategy $\bar{p}$ of player I, then (s)he certainly will pick such a maximal (= non-dominated) strategy. Thus, it might not be necessary to compare all values $\{v(\bar{p}, q) \mid q \in Q\}$, but only those which produce maximal elements.

The following result establishes a partial counterpart to Proposition 3.2.

**Proposition 3.3** If $p, p' \in P$ with $v_I(p') \leq_{\mathbb{R}_+^d} v_I(p)$ and $q \in Q$ with $v(p, q) \in \text{Max } v_I(p)$, then either $v(p', q) \leq_{\mathbb{R}_+^d} v(p, q)$ or $v(p', q), v(p, q)$ are not comparable with respect to $\leq_{\mathbb{R}_+^d}$.

**Proof** Assume $v_I(p') \subseteq v_I(p) - \mathbb{R}_+^d$ and $v(p, q) \leq_{\mathbb{R}_+^d} v(p', q)$ for some $v(p, q) \in \text{Max } v_I(p)$. Then, by the first part of the assumption, there is $\bar{q} \in Q$ such that $v(p', q) \leq_{\mathbb{R}_+^d} v(p, \bar{q})$. Hence

$$v(p, q) \leq_{\mathbb{R}_+^d} v(p', q) \leq_{\mathbb{R}_+^d} v(p, \bar{q}).$$

Since $v(p, q) \in \text{Max } v_I(p)$ this implies $v(p, q) = v(p', q) = v(p, \bar{q})$. $\square$

Propositions 3.2 and 3.3 may be interpreted as follows: The transition from the order $\leq_I$ in $P$ to the set order $\leq_{\mathbb{R}_+^d}$ for the comparison of the values $v_I(p)$ does not loose comparability, but may absorb some incomparable maximal elements. Thus, the relation $\leq_{\mathbb{R}_+^d}$ produces never more, but often fewer minimal elements: in Example 3.1, all strategies $p \in P$ are minimal with respect to the order $\leq_I$, but not all are for $\leq_{\mathbb{R}_+^d}$ (likewise for player II) as is explained in Example 3.7 below.

The above discussion almost forces the following set optimization approach to zero-sum matrix games with vector payoff. Consider the set-valued map defined by

$$V_I(p) := v_I(p) - \mathbb{R}_+^d.$$

Formally, $V_I$ can (and will) be understood as a function mapping $P$ into $\mathcal{G}(\mathbb{R}_+^d, -\mathbb{R}_+^d)$.

The set $V_I(p)$ includes all potential losses for the first player which are less than or equal to payoffs in $v_I(p)$ and thus also absorbs losses generated by “gifting” something to player II. This is just a version of the standard free disposal condition in economics, see Mas-Colell et al. (1995, p. 131). A strategy $p'$ clearly is preferable over $p$ for player I if $V_I(p') \subseteq V_I(p)$, i.e., one can reach the losses in $V_I(p')$ from those in $V_I(p)$ by gifting—which usually does not happen. This motivates the following definition.

**Definition 3.4** A strategy $\bar{p} \in P$ is said to be minimal for player I if there is no $p \in P$ with

$$V_I(p) \subseteq V_I(\bar{p}) \quad \text{and} \quad V_I(p) \neq V_I(\bar{p}).$$

The set of minimal strategies of player I is denoted by $\text{MIN}(I)$. 

$\copyright$ Springer
In the light of the previous discussion, a minimal strategy is a worst case insurance: The maximal loss, anticipated to be a maximal point in $v_I(p)$, should be as “small” as possible which is, in the sense of Definition 3.4, the case if the set $V_I(\bar{p})$ is as “small” as possible. Since $\subseteq$ is, in general, a non-total partial order, this means $V_I(\bar{p})$ is a minimal element of $\{V_I(p) \mid p \in P\}$ in $G(\mathbb{R}^d, -\mathbb{R}^d_+)$ with respect to $\subseteq$.

It is important to note that minimal strategies are independent of the choice of the second player—in contrast to all Nash-type equilibrium notions in the literature (unless the preference is complete).

**Remark 3.5** If $d = 1$, then $v_I(p) = \{v(p, q) \mid q \in Q\}$ and

$$V_I(p) = v_I(p) - \mathbb{R}_+ \leq \max_{q \in Q} v(p, q) - \mathbb{R}_+.$$ 

Thus, a minimal strategy $\bar{p} \in P$ satisfies

$$V_I(\bar{p}) = \min_{p \in P} \max_{q \in Q} v(p, q) - \mathbb{R}_+ = \max_{q \in Q} v(\bar{p}, q) - \mathbb{R}_+,$$

i.e., it is a minimax-strategy for player I.

A parallel discussion can be done for player II. With

$$v_{II}(q) = \{v(p, q) \mid p \in P\} = \text{co} \left\{ \sum_{j=1}^{n} q_j g_{ij}, \ldots, \sum_{j=1}^{n} q_j g_{mj} \right\}$$

it is clear that player II wants to “maximize” the set-valued function $q \mapsto v_{II}(q)$. The extension of the order $\leq_{II}$ on $Q$ is the set relation $\leq_{d}^{+}$ for comparing the values of $v_{II}$ (see the previous section), which are again convex bounded polyhedra. Of course, there are analogous statements to Propositions 3.2 and 3.3.

A set-valued map is defined by

$$V_{II}(q) := v_{II}(q) + \mathbb{R}^d_+$$

which formally is a function $V_{II}: P \rightarrow G(\mathbb{R}^d, \mathbb{R}^d_+)$.

**Definition 3.6** A strategy $\bar{q} \in Q$ is said to be maximal for player II if there is no $q \in Q$ with

$$V_{II}(q) \subseteq V_{II}(\bar{q}) \quad \text{and} \quad V_{II}(q) \neq V_{II}(\bar{q}).$$

The set of maximal strategies of player II is denoted by $\text{MAX}(II)$.

Parallel to Remark 3.5, if $d = 1$ a maximal strategy is a maximizer of the function $q \mapsto \min_{p \in P} v(p, q)$, i.e., a maximin strategy. This shows that for $d = 1$ the above concepts boil down to the classic von Neumann approach. An additional (duality) argument is needed to show that the two problems are dual to each other and have the same value.
Example 3.7 Consider the game of Example 3.1 above. An easy calculation shows

$$\text{MIN}(I) = \left\{ p \in P \mid 0 \leq p_1 \leq \frac{1}{3} \right\} \quad \text{and} \quad \text{MAX}(II) = \left\{ q \in Q \mid 0 \leq q_1 \leq \frac{1}{2} \right\}.$$ 

The strategy $\hat{\rho} = \left( \frac{2}{3}, \frac{1}{3} \right)^T$ is not minimal for player I; the worst case expected payoff is $\text{Max} v_I(\hat{\rho}) = \left( 3, \frac{11}{3} \right)^T$. By playing the minimal strategy $\bar{\rho} = \left( \frac{1}{3}, \frac{2}{3} \right)^T$ with $V_I(\bar{\rho}) \subseteq V_I(\hat{\rho})$ player I can reduce her/his worst case expected payoff to $\text{Max} v_I(\bar{\rho}) = \left( 2, \frac{10}{3} \right)^T$. On the other hand, there does not exist another strategy $\tilde{\rho} \in P$ satisfying $\text{Max} v_I(\tilde{\rho}) \subseteq v_I(\bar{\rho}) - \mathbb{R}_d^+$, i.e., it is not possible for player I to guarantee a worst case expected payoff strictly better than $\left( 2, \frac{10}{3} \right)^T$ without generating other potential expected payoffs which are not comparable to $\left( 2, \frac{10}{3} \right)^T$ (and hence might be chosen by II).

Another link to set relations should be pointed out. It holds

$$V_I(p) = \bigcup_{q \in Q} \left[ v(p, q) - \mathbb{R}_d^+ \right] = \sup_{q \in Q} \left\{ v(p, q) - \mathbb{R}_d^+ \right\}$$

where the supremum is understood in $(\mathcal{G}(\mathbb{R}_d, -\mathbb{R}_d^+), \subseteq)$: the closure and the convex hull can be dropped from the supremum formula as $v(p, \cdot)$ is linear and $Q$ is a convex polyhedron. Since according to Definition 3.4, player I looks for minimizers of the function $V_I$, her/his problem can be understood as a version of worst case analysis: the worst loss (as described by the supremum) should be minimized.

Completely parallel to the scalar case, one can look for the infimum of $V_I$, i.e.

$$V_I = \inf_{p \in P} \sup_{q \in Q} \left\{ v(p, q) - \mathbb{R}_d^+ \right\} = \bigcap_{p \in P} \bigcup_{q \in Q} \left[ v(p, q) - \mathbb{R}_d^+ \right]. \quad (3.2)$$

This makes sense since $(\mathcal{G}(\mathbb{R}_d, -\mathbb{R}_d^+), \subseteq)$ is a complete lattice (see Proposition 2.1). The corresponding problem for player II is to look for the supremum of $V_{II}$ in $(\mathcal{G}(\mathbb{R}_d, \mathbb{R}_d^+), \supseteq)$, i.e., for

$$V_{II} = \sup_{q \in Q} \inf_{p \in P} \left\{ v(p, q) + \mathbb{R}_d^+ \right\} = \bigcap_{q \in Q} \bigcup_{p \in P} \left[ v(p, q) + \mathbb{R}_d^+ \right]. \quad (3.3)$$

These two problems are posed in two different image spaces and with respect to different order relations. Therefore, they do not produce a common equilibrium value and cannot be dual—in the sense of linear programming duality—at least not in the same way as the corresponding problems in the scalar case.

In general, the outer infimum in Eq. (3.2) and the outer supremum in Eq. (3.3) are not “attained” in a single strategy. Therefore, it cannot be expected that there is a single payoff (vector) which can be considered as the value of the game, and hence there is a multitude of optimal strategies for each player leading to different (non-comparable) payoffs. This is the reason why infimum and supremum in Eqs. (3.2) and (3.3) are replaced by minimality and maximality notions as introduced in...
Definitions 3.4 and 3.6. However, one can show that the sets of minimal and maximal strategies are non-empty and form solutions of the set optimization problem (3.2) and (3.3), respectively, in the sense of Heyde and Loehne (2011, Definition 2.7) (compare Definition 2.2 above). The following theorem provides the essence of the argument.

**Theorem 3.8** For each \((p, q) \in P \times Q\) there exists \((\bar{p}, \bar{q}) \in \text{MIN}(I) \times \text{MAX}(II)\) with \(V_I(\bar{p}) \subseteq V_I(p)\) and \(V_{II}(\bar{q}) \subseteq V_{II}(q)\).

**Proof** This follows from Proposition 5.15 in Heyde and Loehne (2011), which states that the domination property (i.e., for every \(x \in X\) there exists a minimal point \(w \in f[X] := \{f(x) \mid x \in X\}\) with \(w \leq f(x)\)) holds for a function \(f : X \to W\), where \(X\) is a compact topological space and \((W, \leq)\) a partially ordered set, whenever \(f\) is level-closed. The latter means that for all \(w \in W\) the level sets \(L_f(w) := \{x \in X \mid f(x) \leq w\}\) are closed. The proof for level-closedness is subject to the following lemma. \(\square\)

**Lemma 3.9** The two functions \(V_I : P \to (\mathcal{G}(\mathbb{R}^d, -\mathbb{R}^d_+), \subseteq)\) and \(V_{II} : Q \to (\mathcal{G}(\mathbb{R}^d, \mathbb{R}^d_+), \supseteq)\) are level-closed.

**Proof** The proof is given for \(V_I\) and runs in a similar way for \(V_{II}\).

Take \(A \in \mathcal{G}(\mathbb{R}^d, -\mathbb{R}^d_+)\) and \(\{p^\ell\}_{\ell=1,2,...} \subseteq P\) with \(\lim_{\ell \to \infty} p^\ell = \bar{p}\) in \(P\) such that \(V_I(p^\ell) \subseteq A\) for all \(\ell = 1, 2, \ldots\). Then,

\[
\forall p \in P : V_I(p) = \text{co} \left\{ \sum_{i=1}^{m} p_i g_{i1}, \ldots, \sum_{i=1}^{m} p_i g_{in} \right\} - \mathbb{R}^d_+. \tag{3.4}
\]

Thus, for all \(\ell = 1, 2, \ldots\),

\[
\forall j = 1, \ldots, n : \sum_{i=1}^{m} p^\ell_{ij} g_{ij} \in A.
\]

Since \(A\) is closed, the same holds for the limit:

\[
\forall j = 1, \ldots, n : \sum_{i=1}^{m} \bar{p}_{ij} g_{ij} \in A.
\]

Now, \(V_I(\bar{p}) \subseteq A\) follows from (3.4) and \(A \in \mathcal{G}(\mathbb{R}^d, -\mathbb{R}^d_+)\). \(\square\)

**Remark 3.10** In the sense of Definition 2.2, the set \(\text{MIN}(I)\) even is a full solution of problem (3.2), and the set \(\text{MAX}(II)\) is a full solution of problem (3.3).

Assume that the players pick \((\bar{p}, \bar{q}) \in P \times Q\) and they both realize it. What could be an incentive for player I to switch to another strategy in \(P\)?

First, there is a strategy \(\tilde{p} \in P\) satisfying \(\text{Max} v_I(\tilde{p}) - \mathbb{R}^d_+ \subset \text{Max} v_I(\bar{p}) - \mathbb{R}^d_+\) (strict inclusion). This means \(v_I(\tilde{p}) - \mathbb{R}^d_+ \subset v_I(\bar{p}) - \mathbb{R}^d_+\) since, due to the upper
domination property, \( v_I(p) - \mathbb{R}_+^d = \max v_I(p) - \mathbb{R}_+^d \) for all \( p \in P \). The switch to \( \tilde{p} \) would avoid some potential losses for player I and thus improve her/his worst case estimate (compare Example 3.7).

Secondly, such an incentive is \( v(\tilde{p}, \tilde{q}) \notin \min v_{II}(\tilde{q}) \), i.e., there is a strategy \( p \in P \) such that \( v(p, \tilde{q}) \in v_I(\tilde{q}) \) with \( v(p, \tilde{q}) \leq \mathbb{R}_+^d v(p, \tilde{q}) \) and \( v(p, \tilde{q}) \neq v(\tilde{p}, \tilde{q}) \). In this case, it makes sense to look for \( \hat{p} \in P \) such that \( v(\hat{p}, \tilde{q}) \in \min v_{II}(\tilde{q}) \). Such a \( \hat{p} \) always exists since the set \( v_{II}(q) \) satisfies the lower domination property for all \( q \in Q \).

The first case means that \( \tilde{p} \) is not minimal for player I. In the second case \( (\tilde{p}, \tilde{q}) \) is not a Shapley equilibrium strategy (compare the following definition) which may happen even if \( \tilde{p} \) is minimal (see Example 4.5). On the other hand, a strategy can produce a Shapley equilibrium, but not be minimal (see Example 4.4). The following two definitions are motivated by these considerations.

**Definition 3.11** A pair \((\tilde{p}, \tilde{q}) \in P \times Q\) is called a Shapley equilibrium if

\[
v(\tilde{p}, \tilde{q}) \in \max v_I(\tilde{p}) \cap \min v_{II}(\tilde{q}).
\]

It is called a strong Shapley equilibrium if

\[
V_I(\tilde{p}) \cap V_{II}(\tilde{q}) \subseteq \max v_I(\tilde{p}) \cap \min v_{II}(\tilde{q}).
\]

Since

\[
\max v_I(p) \cap \min v_{II}(q) \subseteq V_I(p) \cap V_{II}(q)
\]

is always true, the condition for a strong Shapley equilibrium in \((\tilde{p}, \tilde{q})\) actually means

\[
V_I(\tilde{p}) \cap V_{II}(\tilde{q}) = \max v_I(\tilde{p}) \cap \min v_{II}(\tilde{q}).
\]

While Shapley equilibria have been defined in Shapley (1959), strong Shapley equilibria seem to be a new concept. Clearly, a strong Shapley equilibrium also is a Shapley equilibrium. While the former produces a payoff which cannot be improved by either player with respect to the chosen strategies, the latter produces payoffs which cannot be improved with respect to the worst case estimate.

The reader is referred to Example 3.15 (v) which shows that a set Shapley equilibrium does not need to be strong. Clearly, the feature of being “strong” can be considered as a refinement of a Shapley equilibrium. However, no additional exogenous or ‘endogenous parameters’ (De Marco and Morgan 2007, p. 171) are introduced to the problem which is different from previous approaches.

**Remark 3.12** Shapley equilibria can be found by solving scalar non-zero-sum games. More precisely, a pair \((\tilde{p}, \tilde{q}) \in P \times Q\) is a Shapley equilibrium if, and only if, there are \( \alpha, \beta \in \mathbb{R}_+^d \) such that \( \tilde{p} \) is optimal for player I for the scalar game with the payoff matrix containing the entries \( \alpha^T g_{ij} \), and \( \tilde{q} \) is optimal for player II for the game with the matrix \( \beta^T g_{ij} \). This was established in Shapley (1959, Theorem). Note that Shapley denoted such an equilibrium as a Strong Equilibrium Point (SEP).
As already remarked, minimal/maximal strategies provide worst case payoff estimates for each player independent of the choice of the other. Therefore, minimal/maximal strategies are (trivially) interchangeable. On the other hand, Shapley equilibrium strategies involve both players and finding them is a recursive procedure; interchangeability is violated as already shown by Corley (1985, Example 3.2).

**Definition 3.13** A pair \((\bar{p}, \bar{q}) \in P \times Q\) is called a set relation equilibrium if \(\bar{p}\) is minimal and \(\bar{q}\) is maximal.

A set relation equilibrium \((\bar{p}, \bar{q})\) is called a set Shapley equilibrium if it also is a Shapley equilibrium.

A set relation equilibrium \((\bar{p}, \bar{q})\) is called a strong set Shapley equilibrium if it also is a strong Shapley equilibrium.

Definition 3.13 can be understood as an equilibrium version of Definition 2.2. The two features “being minimal/maximal” and “attaining an equilibrium value” are no longer equivalent as they are in the scalar case. The following theorem ensures the existence of (strong) set Shapley equilibria.

**Theorem 3.14** For every zero-sum matrix game with vector payoffs there exists a strong set Shapley equilibrium.

**Proof** Consider the scalar zero-sum matrix game given by the matrix

\[
\left( \sum_{k=1}^{d} g_{ij}^k \right)_{m \times n} = (e^T g_{ij})_{m \times n},
\]

where \(e = (1, \ldots, 1)^T \in \mathbb{R}^d\) and let \((\hat{p}, \hat{q})\) be an equilibrium point for this game, the existence of which follows from linear programming duality. The expected payoff \(v\) for the scalar game is related to the payoff \(v\) of the game with vector payoffs by

\[
v(p, q) := \sum_{i,j} p_i e^T g_{ij} q_j = e^T \sum_{i,j} p_i g_{ij} q_j = e^T v(p, q).
\]

Since \((\hat{p}, \hat{q})\) is an equilibrium for the scalar game, we have

\[
t := v(\hat{p}, \hat{q}) = \max_{q \in Q} v(\hat{p}, q) = \min_{p \in P} v(p, \hat{q}).
\]

This can be written as

\[
t = \max_{y \in V_I(\hat{p})} e^T y = \min_{y \in V_{II}(\hat{q})} e^T y.
\]

Because of \(\max_{y \in -\mathbb{R}^d_+} e^T y = \min_{y \in \mathbb{R}^d_+} e^T y = 0\), this is equivalent to

\[
t = \max_{y \in V_I(\hat{p})} e^T y = \min_{y \in V_{II}(\hat{q})} e^T y.
\]

 Springer
We conclude

\[ V_I(\hat{p}) \subseteq H_- := \left\{ y \in \mathbb{R}^d \mid e^T y \leq t \right\} \quad \text{and} \quad V_{II}(\hat{q}) \subseteq H_+ := \left\{ y \in \mathbb{R}^d \mid e^T y \geq t \right\}. \]

By Theorem 3.8 there exists \((\hat{p}, \hat{q}) \in \text{MIN}(I) \times \text{MAX}(II)\) with \(V_I(\hat{p}) \subseteq V_I(\hat{p})\) and \(V_{II}(\hat{q}) \subseteq V_{II}(\hat{q})\). Thus

\[ V_I(\hat{p}) \subseteq H_- \quad \text{and} \quad V_{II}(\hat{q}) \subseteq H_+. \]

One has \(v(\hat{p}, \hat{q}) \in V_I(\hat{p}) \cap V_{II}(\hat{q}) \subseteq H_- \cap H_+ =: H\). Hence

\[ t = e^T v(\hat{p}, \hat{q}) = \max_{y \in V_I(\hat{p})} e^T y = \min_{y \in V_{II}(\hat{q})} e^T y. \]

The well-known characterization of a vector minimum (and a vector maximum) by a weighted sum scalarization, see, e.g., Zeleny (1974) and Ehrgott (2005), yields

\[ v(\hat{p}, \hat{q}) \in \text{Max} v_I(\hat{p}) \cap \text{Min} v_{II}(\hat{q}). \]

Thus, \((\hat{p}, \hat{q})\) is an equilibrium. \(\square\)

The new concepts are illustrated by means of the following example which is a version of Corley (1985, Example 3.2) adapted to our setting.

**Example 3.15** The following facts can be verified for the game given by

\[ G = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{array} \right). \]

(i) The set of minimal strategies is \(\text{MIN}(I) = \{p \in P \mid 0 < p_1 \leq 1\}\) and the set of maximal strategies is \(\text{MAX}(II) = \{q \in Q \mid \frac{1}{2} \leq q_1 \leq 1\}\).

(ii) The pairs \((\bar{p}, \bar{q}) = ((1, 0)^T, (1, 0)^T)\) and \((\hat{p}, \hat{q}) = \left(\left(\frac{1}{4}, \frac{3}{4}\right)^T, \left(\frac{3}{4}, \frac{1}{4}\right)^T\right)\) are Shapley equilibria. Moreover, \((\bar{p}, \bar{q})\) is a strong set Shapley equilibrium, see Fig. 1 (left).

(iii) The pair \((\bar{p}, \bar{q})\) is minimal/maximal, but not a Shapley equilibrium, hence not a set Shapley equilibrium. Therefore, there is an incentive for player II to change to \(\hat{q}\) with \(v(\bar{p}, \hat{q}) = (1, 0)^T\) instead of \(v(\bar{p}, \bar{q}) = (\frac{3}{4}, 0)^T\), see Fig. 1 (central).

(iv) The pair \((\bar{p}, \hat{q})\) with \(\bar{p} = (0, 1)^T\) and \(v(\bar{p}, \hat{q}) = (0, 1)^T\) is a Shapley equilibrium, but \(\bar{p}\) is not minimal. There is an incentive for player I to switch. Since player II can generate any payoff in \(v_I(\bar{p})\) by an appropriate choice of \(q \in Q\), a switch to \(p \in P\) with \(p_1 \in (0, \frac{1}{2})\) would reduce the degree of freedom for player II considerably, and if \(p_1 \in \left[\frac{1}{2}, 1\right]\) is chosen by player I, the choice of player II is even forced (if player II always plays "rational," i.e., maximal points in \(v_I(p)\)) to generate \(v(p, q) = (1 - p_1, p_1)^T\) as this is the only maximal point in \(v_I(p)\) for \(p_1 \geq \frac{1}{2}\).
A set optimization approach to zero-sum...

Fig. 1 The sets $V_I(p)$ and $V_{II}(q)$ for certain strategy pairs $(p, q)$. Left: strong set Shapley equilibrium from Example 3.15 (ii); central: the minimal/maximal strategy pair of Example 3.15 (iii) is not a Shapley equilibrium; right: the set Shapley equilibrium from Example 3.15 (v) is not a strong set Shapley equilibrium.

(v) The pair $(p, q)$ with $p = (1/8, 7/8)^T$ and $q = (5/8, 3/8)^T$ is minimal/maximal, a Shapley equilibrium, but not a strong one (see central part of Fig. 1). Moreover, $p$ is minimal, $\hat{q}$ is maximal, so $(p, \hat{q})$ also is a set Shapley equilibrium, but not a strong set Shapley equilibrium. Figure 1 (right) shows $V_I(p)$ and $V_{II}(\hat{q})$ whose intersection has a non-empty interior and thus includes non-minimal and non-maximal points of $v_I(p)$ and $v_{II}(\hat{q})$, respectively.

Again, a strong set Shapley equilibrium can be considered as a refinement of a set Shapley equilibrium.

The question arises if the players have an incentive to switch if they find themselves in a (set) Shapley equilibrium which is not strong. In particular, the players might be tempted to aim for a payoff in $(V_I(\bar{p}) \cap V_{II}(\bar{q})) \{v(\bar{p}, \bar{q})\}$ in case $(\bar{p}, \bar{q})$ is a (set) Shapley equilibrium, but not a strong one. Although it is not possible to achieve a strictly better payoff, a player might be attracted by an alternative payoff which is not comparable to $v(\bar{p}, \bar{q})$ for reasons which are not part of the mathematical model so far. The limit is set by the following: If $y \in (V_I(\bar{p}) \cap V_{II}(\bar{q})) \{v(\bar{p}, \bar{q})\}$, then $y$ is not comparable to $v(\bar{p}, \bar{q})$. Indeed, if $v(\bar{p}, \bar{q}) \leq_{R_+^d} y$, then there would be $\bar{y} \in \max v_I(\bar{p})$ with $v(\bar{p}, \bar{q}) \leq_{R_+^d} y \leq_{R_+^d} \bar{y}$, so $v(\bar{p}, \bar{q}) \not\in \max v_I(\bar{p})$ which contradicts the assumption. A similar argument works for $y \leq_{R_+^d} v(\bar{p}, \bar{q})$.

The potential transition from such a payoff $v(\bar{p}, \bar{q})$ to another one in $V_I(\bar{p}) \cap V_{II}(\bar{q})$ might occur due to the ‘taste’ of a player in the sense of Ok et al. (2012) if one interprets the expected gain as her/his utility function.

4 Relationships to other solution concepts

In this section, the new solution concepts are compared to previously introduced concepts from the literature. Examples illustrate the differences.
4.1 Minimax and maximin strategies

First, we will compare minimal and maximal strategies to so-called vector minimax and maximin strategies. Both concepts are in some sense a transfer of worst case estimates from the one-dimensional to the multi-dimensional payoff case.

A strategy $\tilde{p} \in P$ is called a minimax strategy if there is $\tilde{q} \in Q$ such that

$$v(\tilde{p}, \tilde{q}) \in \text{Min} \bigcup_{p \in P} \text{wMax} \, v_I(p).$$

Vice versa, a strategy $\tilde{q} \in Q$ is called a maximin strategy if there is $\tilde{p} \in P$ such that

$$v(\tilde{p}, \tilde{q}) \in \text{Max} \bigcup_{q \in Q} \text{wMin} \, v_{II}(q).$$

Here $\text{wMin}$ and $\text{wMax}$ refer to maximal and minimal points with respect to the cone $\{0\} \cup \text{int} \, \mathbb{R}^d_+$, which usually are called weakly maximal and weakly minimal, respectively. Minimax and maximin strategies can be understood as “vector criterion solutions” of the set-valued optimization problems

$$\text{minimize } p \mapsto \text{wMax}_w v_I(p) \quad \text{and} \quad \text{maximize } q \mapsto \text{wMin}_w v_{II}(q)$$

over $p \in P$ and $q \in Q$, respectively. This means, one looks for minimal and maximal points of the union of all function values, compare (Hamel et al. 2015, p. 80, (III)) and the references therein for a discussion of this concept. The above definition can be found in Tanaka (1994), for example. Earlier definitions differ insofar as sometimes minimax strategies include “outer” weakly minimal and weakly maximal points as in Nieuwenhuis (1983) or drop the “weak” concept altogether as in Corley (1985). The latter case is denoted as strong minimax and maximin strategies.

The sets of minimax and maximin payoffs do not coincide in general, and they are also different from the vector saddle point payoffs, i.e., payoffs of Shapley equilibria. This has already been observed by Corley (1985) and Nieuwenhuis (1983) (see Sect. 1). Tanaka (1994) provides sufficient conditions for the existence of two—in general different—minimax and maximin payoffs.

The following examples show that minimax/maximin strategies on the one hand and minimal/maximal strategies on the other hand are different concepts in general.

**Example 4.1** Consider again the game of Example 3.1 above.

The set of minimax strategies is $\{(0, 1)^T, (\frac{1}{3}, \frac{2}{3})^T\} \subset \text{MIN}(I)$, the set of maximin strategies is $\{q \in Q \mid 0 \leq q_1 \leq \frac{1}{2}\} = \text{MAX}(II)$ (this is also the set of all strong maximin strategies).

The set of strong minimax strategies is $P \not\subseteq \text{MIN}(I)$ since $(2, 2)^T \not\in v_I(p)$ for all $p \in P$, and this point belongs to Max $v_I(p)$ for all $p \in P$ with $0 \leq p_1 < \frac{1}{3}$.

**Example 4.2** Consider again the game from Example 3.15.
The set of all minimax strategies is \( \{(1, 0)^T\} \) which is included in the set of all minimal solutions. The set \( \{p \in P \mid 0 \leq p_1 < \frac{1}{2}\} \) is strongly minimax, and this set is neither included, nor does it include the set of minimal solutions. This also shows the possible “jump” behavior of the change from minimax to strongly minimax.

The set of all maximin as well as the set of strongly maximin strategies is \( Q \), so \( \text{MAX}(II) \) is a strict subset of the set of maximin strategies.

By the way of conclusion, the minimax and maximin concepts from vector optimization do not provide a coherent worst case analysis, i.e., they do not yield bounds for worst possible payoffs. Moreover, they can differ significantly from minimal and maximal solutions. On the other hand, minimal and maximal strategies in the sense of Definitions 3.4, and 3.6 have a clear game-theoretic interpretation since they provide worst case bounds for the players.

4.2 Shapley equilibria

As already mentioned in Remark 3.12, Shapley equilibria can be found by solving scalar non-zero-sum games: For \( \alpha, \beta \in \mathbb{R}^d_+ \), we define

\[
s^\alpha_I := \inf_{p \in P} \sup_{q \in Q} \sum_{i, j} p_i \alpha^T g_{ij} q_j \quad \text{and} \quad s^\beta_{II} := \sup_{q \in Q} \inf_{p \in P} \sum_{i, j} p_i \beta^T g_{ij} q_j.
\]

A strategy \( \bar{p} \in P \) is called Shapley solution for player I if there is \( \alpha \in \text{int} \mathbb{R}^d_+ \) such that

\[
s^\alpha_I = \sup_{q \in Q} \sum_{i, j} \bar{p}_i \alpha^T g_{ij} q_j,
\]

and \( \bar{q} \in Q \) is called Shapley solution for player II if there is \( \beta \in \text{int} \mathbb{R}^d_+ \) such that

\[
s^\beta_{II} = \inf_{p \in P} \sum_{i, j} p_i \alpha^T g_{ij} \bar{q}_j.
\]

A pair \((\bar{p}, \bar{q})\) \( \in P \times Q \) of Shapley solutions forms a Shapley equilibrium (compare Remark 3.12).

For \( A \subseteq \mathbb{R}^d \), we consider the following scalarization functionals which are both versions of the support function of \( A \):

\[
\sigma^+_A(\alpha) := \sup_{x \in A} \alpha^T x \quad \text{and} \quad \sigma^-_A(\beta) := \inf_{x \in A} \beta^T x.
\]

For a collection \( \{A_i\}_{i \in I} \) of sets in \( \mathbb{R}^d \) we have [compare, e.g., Hamel et al. (2015, Lemma 4.14)]

\[
\sigma^+_\bigcup_{i \in I} A_i = \sup_{i \in I} \sigma^+_A \quad \text{and} \quad \sigma^-_\bigcup_{i \in I} A_i = \inf_{i \in I} \sigma^- A_i.
\]
It follows that
\[ \sigma^+_{V_1(p)}(\alpha) = \sup_{q \in Q} \sum_{i,j} p_i \alpha^T g_{ij} q_j \quad \text{and} \quad \sigma^-_{V_{11}(q)}(\beta) = \inf_{p \in P} \sum_{i,j} p_i \beta^T g_{ij} q_j. \]

Hence
\[ s^\alpha_I = \inf_{p \in P} \sigma^+_{V_1(p)}(\alpha) \quad \text{and} \quad s^\beta_{11} = \sup_{q \in Q} \sigma^-_{V_{11}(q)}(\beta). \]

On the one hand, this shows that Shapley equilibria are related to a scalarization of \( V_1(p) \) and \( V_{11}(q) \), respectively. On the other hand our optimality concepts can also be characterized in terms of \( \sigma^+, \sigma^- \).

**Proposition 4.3** A strategy \( \tilde{p} \in P \) is minimal for player I if, and only if,
\[ (p \in P, \ V_1(p) \neq V_1(\tilde{p})) \Rightarrow \left( \exists \alpha \in \mathbb{R}^d_+ : \sigma^+_{V_1(p)}(\alpha) > \sigma^+_{V_1(\tilde{p})}(\alpha) \right). \]

Likewise, a strategy \( \tilde{q} \in Q \) is maximal for player II if and only if
\[ (q \in Q, \ V_{11}(q) \neq V_{11}(\tilde{q})) \Rightarrow \left( \exists \beta \in \mathbb{R}^d_+ : \sigma^-_{V_{11}(q)}(\beta) < \sigma^-_{V_{11}(\tilde{q})}(\beta) \right). \]

**Proof** This follows from the equivalence
\[ A \preceq \mathbb{R}^d_+ B \iff \sigma^+_A \leq \sigma^+_B, \]
for \( \sigma^+_A \) defined on \( \mathbb{R}^d_+ \). Likewise, we have
\[ A \preceq \mathbb{R}^d_+ B \iff \sigma^-_A \leq \sigma^-_B \]
for \( \sigma^-_A \) defined on \( \mathbb{R}^d_+ \).

\[ \square \]

The next two examples show that our optimality notions are essentially different from Shapley’s optimality concept.

**Example 4.4** If \( \tilde{p} \in P \) is a Shapley solution for player I, then it is not necessarily minimal. Indeed, if
\[ G = \begin{pmatrix} (0) & (0) & (0) \\ (0) & (0) & (0) \\ (1, -1) & (-1, 1) \end{pmatrix}, \]
then $\bar{p} = (0, 1)^T$ is a Shapley solution for player I (with respect to $\alpha = (\frac{1}{2}, \frac{1}{2})$), but it is not minimal in the sense of Definition 3.4, since $V_I(1, 0) \subsetneq V_I(0, 1)$. In fact,

$$V_I(1, 0) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} - \mathbb{R}^2_+ \quad \text{and} \quad V_I(0, 1) = \text{co} \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\} - \mathbb{R}^2_+.$$  

**Example 4.5** If $\bar{p} \in P$ is minimal for player I in the sense of Definition 3.4, it is not necessarily a Shapley solution. Indeed, if

$$G = \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 3 \\ -3 \end{pmatrix} \\ \begin{pmatrix} -3 \\ 3 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{pmatrix},$$

then the strategy $\bar{p} = (0, 0, 1)^T$ is minimal, but not Shapley: Consider the sets

$$V_I(1, 0, 0) = \text{co} \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -3 \end{pmatrix} \right\} - \mathbb{R}^2_+, \quad V_I(0, 1, 0) = \text{co} \left\{ \begin{pmatrix} -3 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} - \mathbb{R}^2_+,$$

as well as

$$V_I(0, 0, 1) = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} - \mathbb{R}^2_+$$

and take into account the above scalarization results. It is easy to see that $\bar{p}$ is not a Shapley solution. Assume that $\bar{p}$ is not minimal. Then there is some $p \in P$ with $V_I(p) \subseteq V_I(\bar{p})$ and $V_I(p) \neq V_I(\bar{p})$. Then we have

$$\forall q \in Q : \quad p_1 \begin{pmatrix} 3 \\ -3 \end{pmatrix} q_2 + p_2 \begin{pmatrix} -3 \\ 3 \end{pmatrix} q_1 + p_3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} - \mathbb{R}^2_+.$$  

Since $q_1 + q_2 = 1$ we obtain

$$(-3 p_1 - 3 p_2)q_1 \leq 1 - p_3 - 3 p_1,$$

$$(3 p_1 + 3 p_2)q_1 \leq 1 - p_3 + 3 p_1$$

for all $q_1 \in [0, 1]$. This system is satisfied if and only if the first inequality holds for $q_1 = 0$ and the second one for $q_1 = 1$. Thus, it is equivalent to

$$p_3 + 3 p_1 \leq 1,$$

$$p_3 + 3 p_2 \leq 1.$$  

It follows that $p = (0, 0, 1)^T$, which shows that $\bar{p}$ is minimal.
4.3 Maeda’s bi-matrix games with set payoffs

Maeda (2015) introduced different concepts of “equilibrium points” for bi-matrix games with set payoffs. If one specializes these concepts to the setting of this note, one may see that Maeda’s maximal Nash equilibrium is closely related to, but still different from the optimality concepts introduced and motivated in the previous section. We shortly recall some definitions and results from Maeda (2015) using our notation and our setting of assumptions.

A bi-matrix game with set payoffs is defined by an \( m \times n \) matrix the entries of which are pairs \((A_{ij}, B_{ij})\) of sets \(A_{ij}, B_{ij} \subseteq \mathbb{R}^d\). Mixed strategies are considered which results for player I in an expected payoff

\[
 v_I(p, q) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i A_{ij} q_j \subseteq \mathbb{R}^d
\]

and for player II in

\[
 v_{II}(p, q) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i B_{ij} q_j \subseteq \mathbb{R}^d.
\]

Both players maximize their expected payoffs. By considering the singleton sets

\[
 A_{ij} = \{-g_{ij}\} \quad B_{ij} = \{g_{ij}\}
\]

we obtain a zero-sum game with vector payoff, as considered in this article.

Maeda (2015) argues that each player can choose a set relation. In particular, the set relation \(\preceq_{\mathbb{R}^d_+}\) (“L-type”) and \(\succeq_{\mathbb{R}^d_+}\) (“U-type”) as introduced in Sect. 3 are suggested as candidates for preferences of the players. Moreover, the “LU-type” relation, which is defined by the requirement that both \(\preceq_{\mathbb{R}^d_+}\) and \(\succeq_{\mathbb{R}^d_+}\) are satisfied, is suggested. The definitions of various types of equilibrium points as well as corresponding existence results are given in Maeda (2015, p. 320) under the assumption that ‘both players […] are LU type and this is a common knowledge for the players.’

In Maeda (2015, Definition 4.1), a Nash equilibrium strategy is introduced. In our setting this is a pair \((p^*, q^*) \in P \times Q\) such that

\[
 \forall p \in P : p^* \preceq_I p \quad \text{and} \quad \forall q \in Q : q \preceq_{II} q^*.
\]

The existence of a Nash equilibrium strategy was shown under a very strong assumption, which is not fulfilled, for instance, in Example 3.1. We have shown that any two strategies for I are not comparable (likewise for II) and hence there is no Nash equilibrium strategy. This issue has also been addressed in Maeda (2015, Example 4.2), and for this reason other concepts have been introduced. A maximal Nash equilibrium strategy [compare (Maeda 2015, Definition 4.2)] is (in our setting) a pair \((p^*, q^*)\) such that \(v_I(p^*)\) is minimal and \(v_{II}(q^*)\) is maximal with respect to a set relation. But
as mentioned above, this set relation is supposed to be of LU-type, whereas in our concept player I has to use \( \preceq_{\mathbb{R}^d_+} \) (U-type) and player II has to use \( \preceq_{\mathbb{R}^d_+} \) (L-type). This choice is forced by the interpretation of the payoff as loss for player I and gain for player II, and it is also of practical relevance as the following example shows.

**Example 4.6** Consider the game
\[
G = \begin{pmatrix}
2 & 2 & 0 & 0 \\
-1 & -1 & 0 & 0 \\
-1 & 2 & 0 & 0 \\
2 & -1 & 0 & 0
\end{pmatrix}.
\]

For \( p \in P \) and \( q \in Q \) we have
\[
v_I(p) = \text{co} \left\{ p_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix}, p_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\},
\]
\[
v_{II}(q) = \text{co} \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, q_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} + q_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\}.
\]

One can easily verify that for any two different \( p, p' \in P \), the sets \( v_I(p), v_I(p') \) are not comparable with respect to \( \preceq_{\mathbb{R}^d_+} \). But using the relation \( \preceq_{\mathbb{R}^d_+} \), for \( p^* = (0, 1)^T \) we have
\[
\forall p \in P : \ v_I(p^*) \preceq_{\mathbb{R}^d_+} v_I(p) \quad \text{and} \quad \forall p \in P \setminus \{p^*\} : \ v_I(p^*) \preceq_{\mathbb{R}^d_+} v_I(p).
\]

On the other hand, for any two different \( q, q' \in Q \), the sets \( v_{II}(q), v_{II}(q') \) are not comparable with respect to \( \preceq_{\mathbb{R}^d_+} \). For the relation \( \preceq \), setting \( Q^* = \{ q \in Q \mid \frac{1}{3} \leq q_1 \leq \frac{2}{3} \} \) we have
\[
\forall q, q' \in Q^* : \ v_{II}(q) \preceq_{\mathbb{R}^d_+} v_{II}(q') \quad \text{and} \quad v_{II}(q') \preceq_{\mathbb{R}^d_+} v_{II}(q).
\]

Moreover we have
\[
\forall q \in Q, \forall q^* \in Q^* : \ v_{II}(q) \preceq_{\mathbb{R}^d_+} v_{II}(q^*).
\]

We conclude the following: On the one hand, \( \{p^*\} = \text{MIN}(I) \) whereas all \( p \in P \) are optimal for player I with respect to the LU-type order. On the other hand, \( Q^* = \text{MAX}(II) \) whereas all \( q \in Q \) are optimal with respect to the LU-type order. This illustrates that the LU-order may produce “too many” optimal strategies due to the fact that “too few” comparisons are possible. In this respect, the LU-order behaves very similar to \( \preceq_I \) (see Sect. 3).

In this game, it is quite obvious that a loss averse player I should prefer to choose the second row. Similarly, a strategy in \( Q^* \) avoids any long-term losses for the second player.
Example 4.6 also shows that existence results for optimality notions of games with vector and set payoffs are not sufficient for their justification. It is rather important to ensure that not too many optimal strategies exist and that the concepts come along with a clear motivation and interpretation.

4.4 Pareto optimal security strategies (POSS)

Assume that player I picks strategy \( p \in P \). The set

\[
W_I(p) := \bigcap_{q \in Q} \left[ v(p, q) + \mathbb{R}_+^d \right]
\]

includes all expected losses which player I can suffer by choosing strategy \( p \) independently of player II’s choice. The addition of the convex cone \( \mathbb{R}_+^d \) reflects the fact that player I always can “gift something” to II. It makes sense to make this set as “big” as possible, i.e., include as many potential losses as possible since then the chance that there are “small ones” among them is bigger. Thus, it makes sense to look for

\[
\bigcup_{p \in P} \bigcap_{q \in Q} \left[ v(p, q) + \mathbb{R}_+^d \right].
\]

Up to a closure and a convex hull, this expression coincides with

\[
\inf_{p \in P} \sup_{q \in Q} \left[ v(p, q) + \mathbb{R}_+^d \right]
\]

where inf and sup are understood in \((G(\mathbb{R}^d, \mathbb{R}_+^d), \supseteq)\). The expression \( \bigcap_{q \in Q} \left[ v(p, q) + \mathbb{R}_+^d \right] \) is not changed if \( Q \) is replaced by the set if its vertices, hence we get

\[
\bigcup_{p \in P} \bigcap_{q \in Q} \left[ v(p, q) + \mathbb{R}_+^d \right] = \bigcup_{p \in P} \bigcap_{j \in \{1, \ldots, n\}} \left[ \sum_{i=1}^m p_i g_{ij} + \mathbb{R}_+^d \right].
\]

We also conclude that the set \( \{(z, p) \in \mathbb{R}^d \times P \mid z \in W_I(p)\} \) is closed and (polyhedral) convex. Moreover, \( P \) is compact. Now it is easy to see that the closed convex hull in the definition of the infimum can be omitted here, i.e., we have

\[
W_I := \inf_{p \in P} \sup_{q \in Q} \left[ v(p, q) + \mathbb{R}_+^d \right] = \bigcup_{p \in P} \bigcap_{q \in Q} \left[ v(p, q) + \mathbb{R}_+^d \right].
\]

If, as usual, player I tries to minimize her/his maximal expected loss (s)he is led to the following \( G(\mathbb{R}^d, \mathbb{R}_+^d) \)-valued problem: Find

\[
\mathcal{W}_I = \inf_{p \in P} \sup_{j \in \{1, \ldots, n\}} \left[ \sum_{i=1}^m p_i g_{ij} + \mathbb{R}_+^d \right] = \bigcup_{p \in P} \bigcap_{j \in \{1, \ldots, n\}} \left[ \sum_{i=1}^m p_i g_{ij} + \mathbb{R}_+^d \right].
\]
Definition 4.7 A strategy $\bar{p} \in P$ is called a Pareto optimal security strategy (POSS) for player I if there is no $p \in P$ satisfying

$$W_I(p) \supseteq W_I(\bar{p}) \quad \text{and} \quad W_I(p) \neq W_I(\bar{p}).$$

The set of POSS for player I is denoted by POSS(I).

Player II proceeds in a similar way. The set

$$W_{II}(q) := \bigcap_{p \in P} \left[ v(p, q) - \mathbb{R}^d_+ \right] \in \mathcal{G} \left( \mathbb{R}^d, -\mathbb{R}^d_+ \right)$$

includes all potential gains for her/him including those obtained by “giving up something for free,” and this set should be “as big as possible.” So, player II is faced with the problem to find

$$W_{II} = \sup_{q \in Q} \inf_{i \in \{1, \ldots, m\}} \left[ \sum_{j=1}^{n} q_j g_{ij} - \mathbb{R}^d_+ \right] = \bigcup_{q \in Q} \bigcap_{i \in \{1, \ldots, m\}} \left[ \sum_{j=1}^{n} q_j g_{ij} - \mathbb{R}^d_+ \right].$$

Definition 4.8 A strategy $\bar{q} \in Q$ is called a Pareto optimal security strategy (POSS) for player II if there is no $q \in Q$ satisfying

$$W_{II}(q) \supseteq W_{II}(\bar{q}) \quad \text{and} \quad W_{II}(q) \neq W_{II}(\bar{q}).$$

The set of POSS for player II is denoted by POSS(II).

The previous two definitions are versions of Definition 4.1 in Ghose and Prasad (1989) adopted to our setting. The following results are well-known, see Fernandez and Puerto (1996, Theorem 3.1).

Proposition 4.9 Define the two sets

$$S_I = \left\{ (p, y) \in \mathbb{R}^m \times \mathbb{R}^d \mid y \geq \sum_{i=1}^{m} p_i g_{ij}, \; j = 1, \ldots, n, \; p \geq 0, \; e^T p = 1 \right\}$$

$$S_{II} = \left\{ (q, y) \in \mathbb{R}^n \times \mathbb{R}^d \mid y \leq \sum_{j=1}^{n} g_{ij} q_j, \; i = 1, \ldots, m, \; q \geq 0, \; e^T q = 1 \right\}.$$

Then,

$$\mathcal{W}_I = \{ y \mid (p, y) \in S_I \} + \mathbb{R}^d_+ \quad \text{and} \quad \mathcal{W}_{II} = \{ y \mid (q, y) \in S_{II} \} - \mathbb{R}^d_+.$$

The result means that Pareto optimal security strategies as well as the sets $\mathcal{W}_I, \mathcal{W}_{II}$ can be obtained by solving two linear multi-criteria optimization problems (MLOP). This is important for computational approaches. Moreover, the POSS approach is related to the concepts introduced in Sect. 3 as follows.
Proposition 4.10  It holds
\[\forall p \in P, \forall q \in Q: W_{II}(q) \subseteq V_I(p),\]
\[\forall p \in P, \forall q \in Q: V_{II}(p) \supseteq W_I(q).\]

Moreover, \(V_{II} \supseteq W_I\) and \(W_{II} \subseteq V_I\).

Proof  Everything is immediate from the definitions. \(\square\)

The next result shows that, as a rule, optimal strategies are not worse than POSS with respect to their payoffs, compare also Examples 4.12, 4.13 and 4.14 below. Therefore, one may guess that optimal strategies even lead to better worst case estimates for the expected payoff, and this is indeed the case for many examples.

Theorem 4.11  It holds
\[\forall p \in \text{MIN}(I): V_I(p) \cap (W_I + \mathbb{R}^d_+ \{0\}) = \emptyset,\]
\[\forall q \in \text{MAX}(II): V_{II}(q) \cap (W_{II} - \mathbb{R}^d_+ \{0\}) = \emptyset.\]

Proof  Fix \(\bar{p} \in P\). According to (3.1), one has
\[V_I(\bar{p}) = \text{co}\left\{\sum_{i=1}^{m} \bar{p}_i g_{i1}, \ldots, \sum_{i=1}^{m} \bar{p}_i g_{in}\right\} - \mathbb{R}^d_+\].

Assume there is
\[z \in V_I(\bar{p}) \cap (W_I + \mathbb{R}^d_+ \{0\}).\]

Then there exist \(p \in P\) and \(c \in \mathbb{R}^d_+ \{0\}\) such that both is satisfied \(z \in V_I(\bar{p})\) and
\[z - c \in \bigcap_{j \in \{1, \ldots, n\}} \left[\sum_{i=1}^{m} p_i g_{ij} + \mathbb{R}^d_+\right].\]

This implies
\[V_I(p) = \text{co}\left\{\sum_{i=1}^{m} p_i g_{i1}, \ldots, \sum_{i=1}^{m} p_i g_{in}\right\} - \mathbb{R}^d_+ \subseteq [z - c] - \mathbb{R}^d_+ \subseteq [z] - \mathbb{R}^d_+ \subseteq V_I(\bar{p}).\]

Thus \(\bar{p}\) is not minimal for player I, which proves the first claim. The second statement can be shown analogously. \(\square\)

By the way of conclusion, POSS can also be obtained by a set optimization approach, but are different from minimal/maximal strategies in general. The following example shows this and, moreover, that sometimes POSS are not the best choices.
Example 4.12 Consider the game
\[
G = \begin{pmatrix}
(0 & -1) \\
(-1 & 0) \\
(0 & 0)
\end{pmatrix}.
\]
The only minimal strategy is \(\bar{p} = (1, 0)^T\) whereas all strategies in \(Q\) are maximal for player II. On the other hand, \(\text{POSS}(I) = P\) and \(\text{POSS}(II) = \text{MAX}(II) = Q\). Clearly, any POSS for player I which is not minimal does not make much sense: Why should player I refuse to gain something? Finally,
\[
V_I = \text{co}\{(0, -1)^T, (-1, 0)^T\} - \mathbb{R}^2_+ = \mathcal{W}_{II} \quad \text{and} \quad V_{II} = \mathbb{R}^2_+ = \mathcal{W}_I.
\]

Example 4.13 For the game
\[
G = \begin{pmatrix}
(-2 & 1) \\
(1 & -2) \\
(0 & 0)
\end{pmatrix},
\]
it holds \(\text{MIN}(I) = P\), and \(\bar{p} = (0, 1)^T\) is the only POSS for player I. On the other hand, \(\text{MAX}(II) = Q\) and \(\text{POSS}(II) = \{q \in Q \mid \frac{1}{3} \leq q_1 \leq \frac{2}{3}\}\). Although the POSS for both players are strict subsets of the sets of minimal/maximal strategies, we have
\[
V_I = \text{co}\{(-1, 0)^T, (0, -1)^T\} - \mathbb{R}^2_+ = \mathcal{W}_{II} \quad \text{and} \quad V_{II} = \mathcal{W}_I = \mathbb{R}^2_+.
\]
In this case, the POSS are overly conservative in the sense that they avoid losses at all costs and therefore rule out potential payoffs with one negative component which are not comparable to \((0, 0)^T\). Observe that \((\bar{p}, q)\) is a Shapley equilibrium precisely when \(q\) is not a POSS, and \((\hat{p}, q)\) is a strong set Shapley equilibrium for all \(q \in Q\) for \(\hat{p} = (1, 0)^T\).

The previous two examples show that the concept of POSS is different from the concept of minimal/maximal strategies and, moreover, that POSS do not make much sense in some cases. The following example shows that one can have strict inclusions for the relationships in Proposition 4.10.

Example 4.14 Let us again consider the game of Examples 3.1 and 3.7. We have \(\text{MIN}(I) = \text{POSS}(I)\) and \(\text{MAX}(II) = \text{POSS}(II)\). However, while \(V_I = \mathcal{W}_{II} = \text{co}\{(1, 3)^T, (2, 2)^T\} - \mathbb{R}^2_+\) we have strict inclusion in \(V_{II} \supset \mathcal{W}_I\) since
\[
V_{II} = (2, 3)^T + \mathbb{R}^2_+, \quad \mathcal{W}_I = \text{co}\{(3, 3)^T, (2, 10/3)^T\} + \mathbb{R}^2_+.
\]
5 Conclusions and perspectives

A new solution concept for zero-sum matrix games with multi-dimensional payoffs has been introduced, which, for the first time, provides coherent and computable worst case estimates for such games. This concept corresponds to minimax- and maximin-strategies in the one-dimensional case; it yields interchangeable strategy pairs which should be played if the players are “loss averse” and do not know anything about their preferences, but the fact that they prefer “less loss” and “more gain.” The new concept uses set relations, but is based on the complete-lattice approach to set optimization. Combining the minimal/maximal solutions with (strengthened) versions of Nash-type equilibria for multi-dimensional payoff games introduced by Shapley, new equilibrium concepts are obtained for which existence is shown.

Extensions are now possible to situations in which both players have preferences expressed by two potentially different convex cones $C_I, C_{II} \subset \mathbb{R}^d$. In fact, this seems to be just a mathematical exercise since the corresponding concepts, in particular set relations generated by arbitrary cones, are available, see Hamel et al. (2015). Moreover, even the general situation as considered in Bade (2005) is well within reach since every preorder can be extended to set relations (not just vector preorders).

References

Aumann RJ (1962) Utility theory without the completeness axiom. Econometrica 30(3):445–462
Bade S (2005) Nash equilibrium in games with incomplete preferences. Econ Theor 26(2):309–332
Blackwell D (1956) An analog of the minimax theorem for vector payoffs. Pac J Math 6(1):1–8
Chen GY, Jahn J (1998) Special issue “Set-valued optimization”. Math Methods Oper Res 48(2)
Cook WD (1976) Zero-sum games with multiple goals. Nav Res Logist Q 23(4):615–621
Corley HW (1985) Games with vector payoffs. J Optim Theory Appl 47(4):491–498
De Marco G, Morgan J (2007) A refinement concept for equilibria in multicriteria games via stable scalarizations. Int Game Theory Rev 9(02):169–181
Ehrgott M (2005) Multicriteria optimization, vol 491, 2nd edn. Lecture notes in economics and mathematical systems. Springer, Berlin
Fernández FR, Puerto J (1996) Vector linear programming in zero-sum multicriteria matrix games. J Optim Theory Appl 89(1):115–127
Fernández FR, Monroy L, Puerto J (1998) Multicriteria goal games. J Optim Theory Appl 99(2):403–421
Ghose D (1991) A necessary and sufficient condition for pareto-optimal security strategies in multicriteria matrix games. J Optim Theory Appl 68(3):463–481
Ghose D, Prasad UR (1989) Solution concepts in two-person multicriteria games. J Optim Theory Appl 63(2):167–189
Hamel AH, Heyde F, Löhne A, Rudloff B, Schrage C (eds) (2015) Set optimization—a rather short introduction. In: Set optimization and applications—the state of the art. From set relations to set-valued risk measures. Springer, Berlin, pp 65–141
Henig MI (1986) The domination property in multicriteria optimization. J Math Anal Appl 114(1):7–16
Heyde F, Löhne A (2011) Solution concepts in vector optimization: a fresh look at an old story. Optimization 60(10–12):1421–1440
Kuroiwa D, Tanaka T, Ha TXD (1997) On cone convexity of set-valued maps. Nonlinear Anal Theory Methods Appl 30(3):1487–1496
Löhne A (2011) Vector optimization with infimum and supremum. Springer, Berlin
Löhne A, Weißing B (2017) The vector linear program solver Bensolve—notes on theoretical background. Eur J Oper Res 260(3):807–813
Luc DT, Vargas C (1992) A saddlepoint theorem for set-valued maps. Nonlinear Anal Theory Methods Appl 18(1):1–7
Maeda T (2015) On characterization of Nash equilibrium strategy in bi-matrix games with set payoffs. In: Hamel AH, Heyde F, Löhne A, Rudloff B, Schrage C (eds) Set optimization and applications—the state of the art. From set relations to set-valued risk measures. Springer, Berlin, pp 313–331
Mas-Colell A, Whinston MD, Green JR (1995) Microeconomic theory. Oxford University Press, New York
Nieuwenhuis JW (1983) Some minimax theorems in vector-valued functions. J Optim Theory Appl 40(3):463–475
Ok EA, Ortoleva P, Riella G (2012) Incomplete preferences under uncertainty: indecisiveness in beliefs versus tastes. Econometrica 80(4):1791–1808
Park J (2015) Potential games with incomplete preferences. J Math Econ 61:58–66
Puerto J, Perea F (2018) On minimax and pareto optimal security payoffs in multicriteria games. J Math Anal Appl 457(2):1634–1648
Shapley LS (1959) Equilibrium points in games with vector payoffs. Nav Res Logist Q 6:57–61
Tanaka T (1988) Some minimax problems of vector-valued functions. J Optim Theory Appl 59(3):505–524
Tanaka T (1994) Generalized quasiconvexities, cone saddle points, and minimax theorem for vector-valued functions. J Optim Theory Appl 81(2):355–377
Tanaka T (2000) Vector-valued minimax theorems in multicriteria games. In: Yong S, Milan Z (eds) New frontiers of decision making for the information technology era. World Scientific, Singapore, pp 75–99
Wierzbicki AP (1995) Multiple criteria games-theory and applications. J Syst Eng Electron 6(2):65–81
Zeleny M (1974) Linear multiobjective programming, vol 95. Lecture notes in economics and mathematical systems. Springer, Berlin
Zeleny M (1975) Games with multiple payoffs. Int J Game Theory 4(4):179–191
Zhao J (1991) The equilibria of a multiple objective game. Int J Game Theory 20(2):171–182