Degenerate semigroups and stochastic flows of mappings in foliated manifolds

Paulo Henrique P. da Costa
Paulo R. Ruffino

Departamento de Matemática, Universidade Estadual de Campinas, 13.083-859- Campinas - SP, Brazil.

Abstract

Let \((M, \mathcal{F})\) be a compact Riemannian foliated manifold. We consider a family of compatible Feller semigroups in \(C(M^n)\) associated to laws of the \(n\)-point motion. Under some assumptions (Le Jan and Raimond, [16]) there exists a stochastic flow of measurable mappings in \(M\). We study the degeneracy of these semigroups such that the flow of mappings is foliated, i.e. each trajectory lays in a single leaf of the foliation a.s, hence creating a geometrical obstruction for coalescence of trajectories in different leaves. As an application, an averaging principle is proved for a first order perturbation transversal to the leaves. Estimates for the rate of convergence are calculated.

Key words: Feller semigroup, \(n\)-point motion, foliated space, stochastic flow of mappings, averaging principle.

MSC2010 subject classification: 60H10, 60H30, 57C12.

1 Introduction

The understanding of the geometry and dynamics in a foliated space plays a quite important role in the intersection of many areas. Among many others contemporary good references, we mention few of them which are closer to the approach we are interested in this paper, e.g. Tondeur [22], Candel and Conlon [6], Plante[20], Walczak [21]. On what regards stochastic systems, their interlace with foliations received a boost with the paper by L. Garnnet [10], where she has constructed harmonic measures introducing foliated Brownian motions on the leaves. Since them, many works intertwining stochastics, semigroups and foliations have appeared, see e.g. Kaimanovich [13], Candel [7], [8] and references therein.

In this article we are interested on constructing and studying properties of a certain class of stochastic flow adapted to a foliation in the sense that each trajectory lays in a single leaf of the the foliation. Classically, for stochastic (Stratonovich) differential equation in a differentiable manifold with smooth vectors fields, the existence of a flow of (local) diffeomorphisms has been well studied and has been applied in a vast range of topics in the literature. We mention mainly Kunita [14] [15] and [2] which contain in the references many other authors that contributed in the construction of these flows, each of them has considered different relevant aspects and properties. Extending this classical flow with regularity in the space and continuity of trajectories with respect to time \(t \geq 0\), many constructions have appeared, see e.g. a survey on some of these nonclassical stochastic flow in Tsirelson [23]. In particular, Chap.7], also [3], [4], [9] among others. In this article we are particularly interested
on studying degenerate dynamical properties of the stochastic flow of mappings as constructed by Le Jan and Raimond [16], where coalescence also can happen.

The Le Jan-Raimond stochastic flow, which here is going to called by the acronym LJR-flow, is constructed over a (locally) compact metric space \( M \), based on a family of Feller compatible semigroups of operators in \( M^n \) for integers \( n \geq 1 \). That is, for each \( n \geq 1 \), there exists a semigroup which determines the law of a Markovian process of the \( n \)-point motion. Given compatibility and diagonal preserving conditions on these semigroups, there exists a stochastic flow of measurable mappings in \( M \) which generate this family of semigroups. These concepts and results are going to be precisely stated and recalled below.

In this article we consider \( M \) a Riemannian compact manifold. Motivated by the concept of invariant submanifolds for flows, we look initially for conditions on the family of Feller semigroups which guarantee that a certain submanifold \( N \) is invariant by the corresponding LJR-flow. In this case, we call the semigroups \( N \)-degenerate, cf. Definition 2.1. With appropriate hypothesis, Theorem 3.2 guarantees that trajectories starting in \( N \) lays in \( N \) for all time \( t \geq 0 \) a.s.. Suppose now that the manifold \( M \) has a regular foliation \( \mathcal{F} \). Following the ideas ideas for a submanifold, an appropriate degeneracy on the semigroups with respect to the leaves of the foliation (Definition 2.2) will imply that each trajectory lays in a single leaf of the foliation, as stated in Theorem 3.2. This foliated behaviour of the flow, in particular, introduces a geometrical obstruction to coalescence of points in different leaves.

The class of examples on which this foliated phenomenon happens includes stochastic dynamical systems generated by classical Stratonovich differential equations whose vector fields belong to the tangent space of the leaves. Degenerate stochastic systems in differentiable manifolds also illustrates this context if the dimension of the Hörmander Lie algebra space has constant dimension, hence, in this case the foliation is determined a posteriori by the vector fields of the original SDE.

The article is organized in the following way. In the next paragraphs we recall the concepts and results on the construction of the LJR-flow, as in [16], which we are going to use in the next sections. In Section 2 we introduce the definitions of \( N \)-degenerate and foliated semigroups. In Section 3 we proof the main result on foliated flow in \( M \) which guarantees the equivalence between the foliated semigroups and LJR foliated flows. The reader will notice that for noncompact \( M \), the same result holds if one assume some further, rather natural conditions (see also [16]). Coalescent foliated flow is discussed at the end of this section. Finally in Section 4, as an application of this approach, assuming that the leaves are compact, we investigate the effective behaviour of a small transversal perturbation of order \( \epsilon \) in a foliated LJR-flow which destroys the foliated structure. An averaging principle is proved for a perturbation given by a vector field \( \epsilon K \), with \( K \) transversal to the leaves. We assume hypothesis on convergence of average functions along the perturbed process (Hypothesis H1 and H2) which are natural in most stochastic systems, see [11, 12, 18]. Essentially, for small \( \epsilon > 0 \), the transversal behaviour, with time rescaled by \( \frac{t}{\epsilon} \), is approximate by an ODE in this space whose vector field is given by the ergodic average of the transversal component of \( K \) on each leaf. We find estimates on the rate of convergence when \( \epsilon \) goes to 0.
1.1 Le Jan-Raimond stochastic flow of measurable mappings

We summarize below the main definitions and results on LJR-flows cf. [16] section 1. We apply this theory to our specific context here, where the state space $M$ is a Riemannian manifold. In the next sections $M$ will additionally be endowed with a regular foliation. We shall consider Markovian semigroups acting on $B_b(M^n)$, the space of bounded measurable functions in $M^n$ and Feller continuous semigroups in $C_0(M^n)$, the space of continuous functions in $M^n$ which goes to zero at infinity, see e.g. among others Revuz and Yor [21].

The results in the next sections are based on the fact that a consistent system of $n$-point indivisible points thrown into a fluid in $M$ e.g. among others Revuz and Yor [21].

We shall abbreviate and say that a family of Feller semigroups $\{P_t^{(n)}\}$ is compatible and diagonal preserving as defined above. For a fixed $k$, $k < n$, we have that

$$P_t^{(n)} f(x_1, \ldots, x_n) = P_t^{(k)} f(x_{i_1}, \ldots, x_{i_k}).$$

1. The family $\{P_t^{(n)}\}$ is called compatible if for all $f \in C_0(M^n)$ which can be written in terms of fewer variables, i.e. there exists an $\bar{f} \in C_0(M^k)$, $k < n$, with $f(x_1, \cdots, x_n) = \bar{f}(x_{i_1}, \cdots, x_{i_k})$, we have that $P_t^{(n)} f(x_1, \cdots, x_n) = P_t^{(k)} \bar{f}(x_{i_1}, \cdots, x_{i_k})$.

2. The family $\{P_t^{(n)}\}$ is diagonal preserving in the sense that for all $f \in C_0(M)$

$$P_t^{(2)} f \otimes^2 (x, x) = P_t f^2(x)$$

where here and along the article, $P_t$ stands for $P_t^{(1)}$. We shall abbreviate and say that a family of Feller semigroups $\{P_t^{(n)}\}$ is CDP if it is compatible and diagonal preserving as defined above. For a fixed $n \in \mathbb{N}$, the Markov process associated to $P_t^{(n)}$ starting at a certain initial condition $(x_1, \ldots, x_n)$ is called the $n$-point motion of this family of semigroups, it is defined on the set of càdlàg paths on $M^n$. See more on $n$-point motion also in Kunita [15].

Let $(F, \mathcal{E})$ be the space of measurable mappings on $M$ endowed with the $\sigma$-algebra generated by the application map $\varphi \mapsto \varphi(x)$ for every $x \in M$. More precisely, $\mathcal{E} = \sigma \{ \{ \varphi \in F : \varphi(x) \in A \} \}$, for all $A \in \text{Borel}(M)$, and $x \in M$.

**Definition 1.2.** Consider a convolution semigroup $\{Q_t : t \geq 0\}$ of regular probability measure on $(F, \mathcal{E})$, i.e $Q_{s+t} = Q_s \ast Q_t$ for all $0 \leq s, t$. This family is called a Feller convolution semigroup if for all $f \in C_0(M)$ we have that:

a) $\lim_{t \to 0} \sup_{x \in M} \int (f \circ \varphi(x) - f(x))^2 Q_t(d \varphi) = 0$;

d) For all $x \in M$ and $t \geq 0$

$$\lim_{t \to 0} \int (f \circ \varphi(y) - f \circ \varphi(x))^2 Q_t(d \varphi) = 0;$$

and

$$\lim_{t \to \infty} \int (f \circ \varphi(y))^2 Q_t(d \varphi) = 0.$$
The third and last equivalent definition in this context is the following. Let $(\Omega, F, P)$ be a probability space.

**Definition 1.3.** A family of $(F, E)$-valued random variables $\varphi = (\varphi_{s,t}, 0 \leq s \leq t)$ is called a measurable stochastic flow of mappings if $(x, \omega) \mapsto \varphi_{s,t}(x)$ is measurable, stationary, with independent increments, satisfies the cocycle property, i.e. for all $0 \leq s \leq u \leq t$ and $x \in M$, $P$-a.s. $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}(x)$ and for every $f \in C_0(M)$ we have that:

a) $\lim_{(u,v)\to(s,t)} \sup_{x \in M} E\left[(f \circ \varphi_{u,v}(x) - f \circ \varphi_{s,t}(x))^2\right] = 0$;

b) For all $x \in M$ and $t \geq 0$

$$\lim_{y \to x} \left[(f \circ \varphi_{0,t}(y) - f \circ \varphi_{0,t}(x))^2\right] = 0;$$

and

$$\lim_{y \to \infty} \left[(f \circ \varphi_{0,t}(y))^2\right] = 0.$$

A family of $(F, E)$-valued random variables $\varphi = (\varphi_{s,t}, 0 \leq s \leq t)$ is called a stochastic flow of mappings if there exists a measurable stochastic flow of mappings $\varphi' = (\varphi'_{s,t}, 0 \leq s \leq t)$ such that for all $x \in M$ and $0 \leq s \leq t$, we have that $\varphi_{s,t}(x) = \varphi'_{s,t}(x)$ $P$-a.s.

Given a stochastic flow of mappings $\varphi_{s,t}$, the law of $\varphi_{0,t}$ determines a semigroup of convolution $Q_t$; which in turn determines a family of Feller CDP semigroups given by, for $f \in C_0(M^n)$ and $x \in M^n$,

$$P^n_t f(x) = \int f \circ \varphi_{0,t}(x) Q_t(d\varphi),$$

see Propositions 1.2 and 1.3 in [16, p.1252-1254]. In fact the converse also hold, hence Definitions 1.1, 1.2 and 1.3 are associated one to each other, as stated in the next theorem.

Denote by $(\Omega, A)$ the space $(\Pi_{s \leq t} F, \bigotimes_{s \leq t} E)$ and by $\varphi_{s,t}$ the canonical stochastic process $\omega \mapsto \omega(s, t)$.

**Theorem 1.4** (Le Jan-Raimond). A family of Feller CDP semigroups $(P^n_t, n \in \mathbb{N})$ determines a unique Feller convolution semigroup of probability measures $(Q_t)_{t \geq 0}$ on $(F, E)$ which satisfies Equation (7). A Feller convolution semigroup $(Q_t)_{t \geq 0}$ in turn determines a unique shift invariant probability measure $P_Q$ on $(\Omega, A)$ such that the canonical stochastic process $(\varphi_{s,t} : s \leq t)$ is a stochastic flow of mappings with law $Q_{t-s}$.

A stochastic flow of measurable mappings on $M$, $(\varphi_{s,t} : s \leq t)$, is called a coalescing flow if for some point $(x, y) \in M^2$, $T_{x,y} = \inf\{t \geq 0 : \varphi_{0,t}(x) = \varphi_{0,t}(y)\}$ is finite with positive probability and for any $t \geq T_{x,y}$ we have that $\varphi_{0,t}(x) = \varphi_{0,t}(y)$. In other words, a stochastic flow is coalescing if there exists $x \neq y$ such that their trajectories stick together after a finite time with positive probability.
2 Definitions and Preliminary results

Consider a compact Riemannian manifold $M$. We shall use the extended definition of a submanifold of $M$, as in Candel and Conlon [6], in the following sense: we say that $N \subset M$ is a submanifold when $N$ is the image of a connected manifold $W$ by an injective immersion $i : W \to M$, that is $N = i(W)$. Whence $N$ is endowed with two topologies: the intrinsic topology $\tau_{\text{int}}$ of $W$ and the induced topology $\tau_{\text{ind}}$ as a subset of $M$. Obviously $\tau_{\text{ind}} \subseteq \tau_{\text{int}}$, and equality holds if $i$ is an embedding. Most of the time we are interested in the case of $N$ being a complete submanifold with respect to the induced Riemannian metric. In this case we have that

$$C_{0}(N, \tau_{\text{ind}}) = \{ f|_{N} : f \in C(M) \}$$

if and only if $N$ is compact with respect to $\tau_{\text{int}}$, i.e., $W$ is compact. Hence, in this case, we have also that $\tau_{\text{int}} = \tau_{\text{ind}}$ and $C_{0}(N, \tau_{\text{int}}) = C(N, \tau_{\text{int}})$. Note that if $N$ is dense in $M$ then $C_{0}(N, \tau_{\text{int}}) \cap C(N, \tau_{\text{ind}}) = \{0\}$.

The geometrical idea of invariant manifolds for certain dynamics, say e.g. stable submanifold, motivates the following definition.

**Definition 2.1.** Let $N$ be a submanifold of $M$, a Feller semigroup $P_{t}$ is $N$-degenerate if there exists a Markovian semigroup $P_{t}^{N}$ such that for all $f \in C(M)$ we have that:

$$P_{t}f(x) = P_{t}^{N}f|_{N}(x), \quad (2)$$

for all $x \in N$.

Since the set of test functions for the definition above reduces to a subset of $C(N, \tau_{\text{int}})$, the Markovian semigroup $P_{t}^{N}$ can be non unique in $C_{0}(N, \tau_{\text{int}})$. If $N$ is compact then $P_{t}^{N}$ is unique and Feller.

**Foliations.** A regular foliation $\mathcal{F}$ in $M$ is a partition of $M$ into equivalent classes of complete submanifolds of the same dimension; it corresponds to assign a regular integrable differentiable $d$-dimensional distribution in $M$. Each of these submanifolds are called the leaves of the foliation $\mathcal{F}$. More precisely: an $(n+d)$-dimensional smooth manifold $M$ is foliated when there exists an atlas on $M$ such that for any pair $(U, \psi)$ and $(V, \phi)$ of coordinate maps we have that:

1. $\psi(U) = U_{1} \times U_{2}$, where $U_{1} \subseteq \mathbb{R}^{n}$ and $U_{2} \subseteq \mathbb{R}^{d}$ are open sets;
2. If $U \cap V \neq \emptyset$ then the map $\phi \circ \psi^{-1} : \psi(U \cup V) \to \phi(U \cup V)$ has the following form $\phi \circ \psi^{-1}(x, y) = (h_{1}(x, y), h_{2}(y))$.

Given a point $x \in M$, the leaf passing through $x$ is denoted by $L_{x}$. For further properties and details see e.g. [5], [6], [22], [24].

**Definition 2.2.** Let $(M, \mathcal{F})$ be a foliated space. A Feller semigroup $P_{t}$ in $M$ is foliated or $\mathcal{F}$-foliated if $P_{t}$ is $\mathcal{L}$-degenerate for every $\mathcal{L} \in \mathcal{F}$, i.e. there exists a family of Markovian semigroups $(P_{t}^{\mathcal{L}})_{\mathcal{L} \in \mathcal{F}}$ such that for all $f \in C(M)$

$$P_{t}f(x) = P_{t}^{\mathcal{L}_{x}}f|_{\mathcal{L}_{x}}(x), \quad (3)$$

for all $x \in M$. 

5
As before, if the leaves are compact then for each leaf $\mathcal{L}$, the semigroup $P^\mathcal{L}_t$ which satisfy Equation (3) is Feller and unique. In this case, there exists also an intrinsic way to verify whether a certain Feller semigroup $P_t$ in $M$ is foliated (degenerate) or not. For this purpose, given $f \in C(M)$ and a submanifold $N \subset M$, consider the set

$$I_{f,N} := \{ g \in C(M) : g|_N = f|_N \}.$$ 

Hence, by definition, if $P_t$ is an $N$-degenerate semigroup then for any $g \in I_{f,N}$ we have $P_t f(x) = P_t g(x)$. Reciprocally:

**Proposition 2.3.** Consider a Feller semigroup $P^M_t$ on $C(M)$.

1. Assume that the submanifold $N$ is compact. The semigroup $P_t$ is $N$-degenerate if and only if $P_t f(x) = P_t g(x)$ for all $g \in I_{f,N}$.

2. Assume that the leaves of $\mathcal{F}$ are compact. The semigroup $P_t$ is foliated if and only if $P_t f(x) = P_t g(x)$ for all $g \in I_{f,N}$ and all $\mathcal{L} \in \mathcal{F}$.

**Proof.** We prove the converse of item (1). We only have to show that there exists a Feller semigroup acting on $C(N)$ which satisfies Equation (2). In fact, given $g \in C(N)$, take an extension $f \in C(M)$ and define $P^N_t g(x) = P_t f(x)$ for all $x \in N$. Hypothesis guarantees that $P^N_t$ is well defined. Item (2) follows trivially now.

Contrasting with last Proposition, if $N$ is dense in $M$ then each set $I_{f,N}$ has a single element.

### 3 Foliated flow

At this point we return to the family $(P^{(n)}_t, n \in \mathbb{N})$ of Feller CDP semigroups as defined in Section 1.1. Initially note that if $\mathcal{F}$ is a foliation of $M$, then $\mathcal{L} = \mathcal{L}_1 \times \ldots \times \mathcal{L}_n$ with $\mathcal{L}_j \in \mathcal{F}$, $j = 1, \ldots, n$, are leaves of a foliation in $M^n$ which we are going to denote by $\mathcal{F}^n$. The main results in this section show that for a family of Feller CDP semigroups $(P^{(n)}_t, n \in \mathbb{N})$, if $P^{(i)}_t$ is $\mathcal{F}^i$-foliated for a certain $i \in \mathbb{N}$, then $P^{(n)}_t$ is $\mathcal{F}^n$-foliated for all $n \in \mathbb{N}$, moreover the associated LJR-flow $\varphi_{s,t}$ is foliated in the following sense:

**Definition 3.1.** A stochastic flow of mappings $\varphi_{s,t}$ in $M$ is called an $\mathcal{F}$-foliated stochastic flow if for all $0 \leq s, t$ we have that $\varphi_{s,t}(x) \in \mathcal{L}_x$ a.s..

The result below states that if the lowest level Feller semigroup $P^{(1)}_t$ is $\mathcal{F}$-foliated then the corresponding LJR-flow of mappings $\varphi_{s,t}$ is $\mathcal{F}$-foliated.

**Theorem 3.2** (Foliated flow). Let $(P^{(n)}_t : n \in \mathbb{N})$ be a family of Feller CDP semigroups in $M$.

1. **(Invariant submanifold)** Let $N$ be a submanifold of $M$. If $P^{(1)}_t$ is $N$-degenerate then the LJR stochastic flow of mappings preserves $N$, i.e. for $0 \leq s \leq t$ and $x \in N$ we have that $\varphi_{s,t}(x) \in N$, $\mathbb{P}$.a.s.

2. **(Foliated Flow)** If $P^{(1)}_t$ is $\mathcal{F}$-foliated then the corresponding LJR stochastic flow of mappings $\varphi_{s,t}$ is an $\mathcal{F}$-foliated flow.
Proof. For item (1), consider an increasing sequence of compact sets with respect to the intrinsic topology $\tau_{\text{int}}$ in $N$ such that $N = \cup_n K_n$. The idea of the proof is to control the probabilities with which the process exits the sets $K_n$. For fixed $0 \leq t$ and $x \in N$ consider the measurable set

$$B = \{\omega \in \Omega : \varphi_{0,t}(x,\omega) \in M \setminus N\}.$$  

We prove that $P(B) = 0$ writing $B = \cap_n D_n$ with

$$D_n = \{\omega \in \Omega : \varphi_{0,t}(x,\omega) \in M \setminus K_n\}$$

and proving that $P(D_n)$ goes to zero when $n$ tends to infinity. The semigroup $P_t^N$ has an associated transition probability measure $\mu_{P_t^N}(x,dy)$ with support in $(N, \tau_{\text{int}})$, see e.g. Revuz and Yor [21, Chap. III.2], which in general does not coincide with the support of $\mu(1)$

The key point here is to link these two probability measures using continuous functions in $C(M)$, as demanded in Definition 2.1.

Consider closed sets $F_{j,n} \subset M \setminus K_n$ which are increasing in $j$ and such that $M - K_n = \cup_j F_{j,n}$. For each pair $(j,n)$ take a continuous function $f_{j,n} \in C(M, [0, 1])$ such that $f_{j,n}(x) = 1$ for all $x \in F_{j,n}$ with support in $M \setminus K_n$. Therefore, for a fixed $x \in K_n$, by formula (1) and the fact that $P_t$ is $N$-degenerate we have that

$$E[f_{j,n} \circ \varphi_{0,t}(x)] = P_t^N f_{j,n}\big|_N(x)$$

$$= \int_N f_{j,n}\big|_N(y) \mu_{P_t^N}(x,dy).$$

Hence, using the fact that $f_{j,n}$ converges pointwise to the characteristic function $1_{M \setminus K_n}$ when $j$ goes to infinity we conclude that

$$P(D_n) = \lim_{j \to \infty} E[f_{j,n} \circ \varphi_{0,t}(x)]$$

$$= \int_N \lim_{j \to \infty} f_{j,n}\big|_N(y) \mu_{P_t^N}(x,dy)$$

$$= \mu_{P_t^N}(x, N \setminus K_n).$$

Last term goes to zero when $n$ goes to infinity since $\mu_{P_t^N}(x,dy)$ is a Radon measure.

Item (2) of the statement follows directly from item (1) applied in each leave of the foliation.

If the submanifold $N$ in the item (1) of Theorem 3.2 above is compact, a proof purely analytical functional on $C(M)$ shows nuances of the technique: there exists a countable sequence of closed sets $F_n$ such that $\cup_n F_n = M \setminus N$. For each $n \in \mathbb{N}$, consider a corresponding continuous function $f_n \in C(M, [0, 1])$ with support in $M \setminus N$ and such that $f_n(x) = 1$ for all $x \in F_n$. For fixed $0 \leq t$ and $x \in N$ we show that the measurable set

$$B = \{\omega \in \Omega : \varphi_{0,t}(x,\omega) \in M \setminus N\}$$

has probability zero. Now we write $B = \cup_n B_n$ where

$$B_n = \{\omega \in \Omega : \varphi_{0,t}(x,\omega) \in F_n\}.$$
We prove that $P(B_n) = 0$ for every $n \in \mathbb{N}$. In fact, by Chebyshev inequality, Theorem 3.2 and the definition of foliated semigroup we calculate:

$$P(B_n) \leq E(f_n \circ \varphi_{0,t}(x)) = P_t^{(1)} f_n(x) = P_t^N f_n \bigg|_N = 0.$$ 

Next result exploits the fact that, although in Theorem 3.2 for simplicity, we have assumed that the first element of the family, i.e. with $n = 1$, $P_t^{(1)}$ is foliated, the same result holds if one assumes, instead, that $P_t^{(k)}$ is $F^k$-foliated for a certain $k \geq 1$.

**Corollary 3.3.** Let $(P_t^{(n)} : n \in \mathbb{N})$ be a Feller CDP-family of semigroups in $C(M)$.

1. **(Invariant submanifold)** If for a positive integer $k \geq 1$ the semigroup $P_t^{(k)}$ is $N^k$-degenerate, then all members of the family $(P_t^{(n)} : n \in \mathbb{N})$ are $N^n$-degenerate.

2. **(Foliated Flow)** If for a positive integer $k \geq 1$ the semigroup $P_t^{(k)}$ is $F^k$-foliated, then all members of the family $(P_t^{(n)} : n \in \mathbb{N})$ are foliated in the corresponding $F^n$-foliation of $M^n$.

**Proof.** We prove Item (2). Let $\varphi_{s,t}$ be the LJR-flow associated to the CDP-family of Feller semigroups $(P_t^{(n)} : n \in \mathbb{N})$. Assuming that $P_t^{(k)}$ is $F^k$ foliated, fix a leaf $\mathcal{L} = \mathcal{L}_1 \times \ldots \times \mathcal{L}_k \in \mathcal{F}^k$, and a point $x = (x_1, \ldots, x_k) \in \mathcal{L}$. Let $K_n$ be a sequence of compact sets with respect to the intrinsic topology in $\mathcal{L}$ such that $\mathcal{L} = \bigcup_n K_n$. Controlling the probability with which the process $\varphi_{0,t}(x) = (\varphi_{0,t}(x_1), \ldots, \varphi_{0,t}(x_k))$ exits the sets $K_n$, following the same argument as in the proof of Theorem 3.2, we have that for all $t \geq 0$, $\varphi_{0,t}(x) \in \mathcal{L}$ a.s.. This implies that the leaves in $\mathcal{F}$ are invariant by the flow $\varphi_{s,t}$ a.s.

Now, for any $n \geq 1$, given a leaf (using the same notation) $\bar{\mathcal{L}} \in \mathcal{F}^n$, a point $x = (x_1, \ldots, x_n) \in \bar{\mathcal{L}}$ and $g \in B_b(\bar{\mathcal{L}})$, define the Markovian semigroup $P_t^{\bar{\mathcal{L}}}$ in the leaf $\bar{\mathcal{L}}$ by the formula

$$P_t^{\bar{\mathcal{L}}} g(x) = E \left[ g \left( \varphi_{0,t}(x_1), \ldots, \varphi_{0,t}(x_n) \right) \right].$$

For a function $f \in C(M^n)$, we have obviously that

$$P_t^{(n)} f = E \left[ f \big| \mathcal{L} \left( \varphi_{0,t}(x_1), \ldots, \varphi_{0,t}(x_n) \right) \right] = P_t^{\bar{\mathcal{L}}} f | \bar{\mathcal{L}}.$$ 

Hence $P_t^{(n)}$ satisfies Definition 2.2 of an $\mathcal{F}^n$-foliated semigroup. Item (1) follows directly by the same argument.

Next corollary establishes sufficient conditions for the semigroups in the leaves to be Feller.

**Corollary 3.4** (Feller in the leaves). If the foliated flow of measurable mappings established by Theorem 3.2 is such that $\varphi_{s,t}|_{\mathcal{L}}$ satisfies Conditions (a) and (b) of Definition 1.3 for all $f \in C_0(\mathcal{L})$ then the semigroups in the leaves $P_t^{\mathcal{L}}$ for $\mathcal{L} \in \mathcal{F}^n$ are also Feller for all $n \geq 1$. (Analogous for the $N$-invariant flows).
Proof. By Equation 1 and Theorem 1.4 (alternatively [16, Prop. 1.3]) applied in each leaf \( \mathcal{L} \), it follows that the semigroups \( P^t_{\mathcal{L}} \) are Feller. □

Remark: Compactness of the leaves or continuity of \( \varphi_{0,t}|_{\mathcal{L}}(x) \) in \( x \) and \( t \) with respect to the intrinsic topology of \( \mathcal{L} \) imply Conditions (a) and (b) in the hypothesis of Corollary 3.4 above.

Example: Foliated semigroups in dense leaves. We consider the canonical example of a flat 2-torus \( T = \mathbb{R}^2/\mathbb{Z}^2 \) and a unitary vector in the plane \( v = (v_1, v_2) \in \mathbb{R}^2 \). We take a foliation in \( T \) such that at each point \( x = (a, b) \in T \), the leaf passing thorough \( x \) is given by the winding of a line passing thorough it in direction of \( v \):

\[
\mathcal{L}(x,y) = \{(a + \lambda v_1, b + \lambda v_2) \pmod{\mathbb{Z}^2}, \text{ for all } \lambda \in \mathbb{R}\}.
\]

Leaves are compact if \( v_2/v_1 \) is rational. We are going to explore the case of \( v_2/v_1 \) irrational, which implies that all the leaves are dense in \( T \). In this case, the leaf passing through \( x \) has intrinsic topology \( \tau_{\text{int}} \) given by the real line. Precisely, the bijective immersion \( i : \mathbb{R} \to (\mathcal{L}_x, \tau_{\text{int}}) \) defined by \( \lambda \mapsto (a + \lambda v_1, b + \lambda v_2) \pmod{\mathbb{Z}^2} \) is a homeomorphism.

The family of foliated semigroups \( P^{(n)}_t \) in \( T \) will be described via the Stratonovich differential equation:

\[
dx_t = v \, dB_t,
\]

where \( B_t \) is the standard Brownian motion and the flow \( \varphi_t(x) = (a + v_1 B_t, b + v_2 B_t) \pmod{\mathbb{Z}^2} \) preserves each leaf of this dense foliation in the sense that each trajectory lays in a single leaf. The family of semigroups \( P^{(n)}_t \) are given by, for \( f \in C(T^n) \),

\[
P^{(n)}_t f(x_1, \ldots, x_n) = E[f(\varphi_t(x_1), \ldots, \varphi_t(x_n))]
\]

with \( (x_1, \ldots, x_n) \in T^n \). In particular, the first semigroup

\[
P^{(1)}_t f(x) = \int_T f((x + vz) \pmod{\mathbb{Z}^2}) \, g_t(z) \, dz
\]

is foliated with semigroups in the leaves \( P^t_{\mathcal{L}} \) given as follows: given a function \( f \in C_0(\mathcal{L}_x, \tau_{\text{int}}) \),

\[
P^t_{\mathcal{L}} f(x) = \int_{\mathbb{R}} f(iz + i^{-1}(x)) \, g_t(z) \, dz.
\]

Here we have used that for \( z \in \mathbb{R} \),

\[
g_t(z) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{z^2}{2t}\right)
\]

is the one dimensional heat kernel associated to linear Brownian motion, with \( t > 0 \). In this particular example the Markovian semigroups in the leaves \( P^t_{\mathcal{L}} \) are Feller in \( C_0(\mathcal{L}) \), see Remark after Corollary 3.4. The main differences between \( P_t \) and \( P^t_{\mathcal{L}} \) which are relevant for our technique here are:

1. The domain of \( P_t \) restricts to \( \mathbb{Z}^2 \)-periodic function on \( \mathbb{R}^2 \) while the domain of \( P^t_{\mathcal{L}} \), with \( \mathcal{L} \in \mathcal{F} \) extends to the non-compact topology of \( \mathbb{R} \). Intersection of the domains is the unitary set of the null function.
2. The support of the transition probability measures $\mu_{P_t}(x, dy)$ associated to $P_t$ is the whole manifold $T$; while the support of the probability measures $\mu_{P_t^L}(x, dy)$ is the leaf $L_x$.

The second item above is precisely the phenomenon that the supports of the measures associated to the semigroups on the leaves restrict to the leaves themselves, even if the leaves are dense in $M$. This property has been exploited in the proof of Theorem 3.2.

We finish this section with a remark on the existence of intrinsic LJR flow on each leaf.

**Remark 3.5.** Let $(P_t^{(n)} : n \in \mathbb{N})$ be, as before, a CDP-family of Feller semigroups in $M$. Denote by $\varphi_{s,t}$ the corresponding LJR-flow in $M$. Introduce the foliated structure $\mathcal{F}$ in $M$ and assume that the family of semigroups is not only foliated but also that the semigroups $P_t^L$ in the leaves $L \in \mathcal{F}$ are also Feller (either by topological reasons, e.g. compact leaves, or by more general condition as in Corollary 3.4). Note that diagonal preserving condition (2) in Definition 1.1 is trivially satisfied when beforehand we have a flow associated to the semigroup. Hence, in each leaf $L$, it exists a CDP-family of Feller semigroups in $L$ which satisfies again the hypothesis of Theorem 1.4. It means that each leaf $L$ has intrinsically their own LJR-flow $\psi_{s,t}$ defined in the probability space $\Omega^L = (\Pi_{s \leq t} F)$, where $F$ is the space of measurable mappings on $L$ with the appropriate $\sigma$-algebra, as described in Section 1.1. An alternative and natural choice of $\psi_{s,t}$ is $\varphi_{s,t}|_L$, $\omega$-wise based on the same previous probability space $\Omega$. But in general the relation between $\psi_{s,t}$ and $\varphi_{s,t}$ weakens to the average: for $x \in L$ and $f \in C(M)$:

$$E^{\Omega^L} (f|_L \circ \psi_{s,t}(x)) = E (f \circ \varphi_{s,t}(x)).$$

### 3.1 Coalescing foliated flows

In this section, we consider coalescent foliated semigroups. As before, let $(P_t^{(n)})$ be a foliated family of Feller semigroups in a compact Riemannian manifold $M$ endowed with a regular foliation $\mathcal{F}$.

We denote by $X_t^{(n)} = (X_1^{(n)}, \ldots, X_n^{(n)})$, $n \geq 1$, with $X_0^{(n)} = (x_1, \ldots, x_n)$ the Markovian processes in $M^n$ associated to the laws of $P_t^{(n)}$ starting at the point $(x_1, \ldots, x_n)$. Consider the partial diagonals $\Delta_n = \{x \in M^n : \text{there exists a pair } i \neq j \text{ with } x_i = x_j\}$ and the entry times $T_{\Delta_n} = \inf\{t \geq 0, X_t^{(n)} \in \Delta_n\}$. With the same notation as in [10]:

**Theorem 3.6 (Le Jan–Raimond).** There exists a unique compatible family of Markovian semigroups $\{P_t^{(n)}|_L, n \geq 1\}$ on $M$ such that if $X^{(n)}|_L$ is the associated $n$-point motion and $T_{\Delta_n}^L = \inf\{t \geq 0, X^{(n)}|_L \in \Delta_n\}$, then:

(a) $(X_t^{(n)}|_L, t \leq T_{\Delta_n}^L)$ is equal in law to $(X_t^{(n)}, t \leq T_{\Delta_n})$;

(b) for $t \geq T_{\Delta_n}^L$, we have that $X_t^{(n)}|_L \in \Delta_n$.

With further condition it is possible to guarantee that the coalescent semigroup $P_t^{(n)}|_L$ is Feller. In fact, denoting by $P_{(x,y)}^2$ the transition probability associated to $P_t^{(2)}$ at the point $(x,y)$, introduce:
**Condition C:** For all $t > 0, \epsilon > 0$ and $x \in M$, assume that
\[
\lim_{y \to x} P^2_{(x,y)} \left[ \{ t < T_{\Delta_2} \} \cap \{ d(X^1_t, X^2_t) > \epsilon \} \right] = 0,
\]
and for some pair $x, y \in M$, $P^2_{(x,y)}[T_{\Delta_2} < \infty] > 0$.

We can construct now a foliated coalescing flow in $M$:

**Proposition 3.7.** Let $(P_{t}^{(n)}, n \geq 1)$ be a foliated family of Feller semigroups on $(M, \mathcal{F})$ which satisfies Condition C above. Then the coalescing semigroups $P_{t}^{(n),c}$ are CDP Feller foliated semigroups for all $n \geq 1$, hence associated to a coalescing foliated flow $\varphi_{s,t}$.

**Proof.** For every $n \geq 1$, by concatenating a Markov process which stops when it hits a partial diagonal set $\Delta_n$ with a process starting from this corresponding point, Le Jan and Raimond [16, Thm 4.1] have constructed explicitly a Markov process $X^{(n),c}$ with the properties established by Theorem [5.4]. Additional Condition C implies also that the family $P_{t}^{(n),c}$ is a CDP Feller semigroup, [16, Thm 4.1].

We only have to note that, by construction, for $n = 1$, the law of $X^{(1),c}$ and $X^{(1)}$ are equal, hence $P_{t}^{(1)} = P_{t}^{(1),c}$ for all $t \geq 0$. This implies that $P_{t}^{(1),c}$ is foliated. Hence, the result follows by Theorem [5.2] and Corollary [3.3].

Proposition [3.7] in particular implies that the pair of points $x, y \in M$ such that $P^2_{(x,y)}[T_{\Delta_2} < \infty] > 0$, in Condition C, must be in the same leaf. In fact, $\varphi_{s,t}(x) \in \mathcal{L}_x$ and $\varphi_{s,t}(y) \in \mathcal{L}_y$ and the intersection $\mathcal{L}_x \cap \mathcal{L}_y$ is non empty if and only if $x$ and $y$ are in the same leaf.

### 4 An averaging principle for foliated LJR flow

In this section we apply the technique of foliated semigroup and foliated flow to obtained an averaging principle for foliated LJR flows when the leaves of the foliation are compact. Initially we introduce appropriate foliated coordinates such that the leaves are going to be mapped in horizontal plaques of Euclidean space and we will have also a coordinate system for a direction which is transversal to the leaves. For convenience, in this coordinate system the leaves will be called horizontal and the transversal direction will be called the vertical direction.

Given a family of CDP foliated semigroups $(P_{t}^{(n)}, n \in \mathbb{N})$ in $(M, \mathcal{F})$ we are going to consider a small first order perturbation in the associated LJR-flow, corresponding to a family of CDP semigroups $(P_{t}^{(n),c}, n \in \mathbb{N})$, generically no longer foliated. Precisely, if $P_{t}^{c}$ denotes the first perturbed semigroup, then there exists a vector field $K$ in $M$ (generically transversal to the leaves) such that for a function $f \in C^\infty(M),$

\[
\lim_{t \searrow 0} \frac{P_t^c f - P_t f}{t} = \epsilon K f.
\]

We also localize our hypothesis such that $P_{t}^{c} f(x)$ is determined by $f$ restricted to a neighbourhood of the leaf $\mathcal{L}_x$ for small $t$. Equivalently, the support of the probability measure associated to the Feller semigroup $P_{t}^{c} f(x)$ stays in a neighbourhood of $\mathcal{L}_x$ for sufficiently small $t \geq 0$. 
Our main result in this section establishes an averaging principle for the dynamics induced by the family $P^{(n)}_t$, i.e., by its LJR flow in the transversal component. More precisely, as $\epsilon$ goes to zero, the average of the vertical component of the perturbed flow $\varphi^\epsilon_{0,t}$ approaches in a certain topology the solution of the ODE in the vertical space given by the average of the perturbing vector field $K$ in each leaf, where the average is taken according to the invariant measure in each leaf of the unperturbed system generated by the family $P^{(n)}_t$.

The approach here generalizes to semigroups the results for continuous diffusions [11], foliated Lévy processes [12] and for Hamiltonian/symplectic structures in X.-M.-Li [18].

**The foliated coordinate system.** Given an initial condition $x_0 \in M$, let $U \subset M$ be a bounded neighborhood of $x_0$ which is diffeomorphic to $L_{x_0} \times V$, with $V$ a connected open set in $\mathbb{R}^d$ containing the origin. By compactness of $L_{x_0}$, there exists a finite number of local foliated coordinate systems $\psi_i : U_i \rightarrow W_i \times V \subset \mathbb{R}^n \times \mathbb{R}^d$, where $W_i$ and $V$ are open sets, say with $1 \leq i \leq k$ and $x_0 \in U_1$ such that:

1) $U = \bigcup_{i=1}^k U_i$;

2) The leaf $L_{x_0} = \bigcup_{i=1}^k \psi_i^{-1}(W_i \times \{0\})$, i.e. each $U_i$ is diffeomorphic to the product of an open set in the leaf $L_{x_0}$ and the vertical component $V$;

3) If a pair of points $p \in U_i$ and $q \in U_j$ in $U$ belong to the same leaf then their transversal coordinates in $V$ are the same; i.e. $\pi(\psi_i(p)) = \pi(\psi_j(q))$ where $\pi$ is the projection on the transversal space $V$.

Note that for a fixed $y \in V$, the finite union $\bigcup_{i=1}^k \psi_i^{-1}(W_i \times \{y\})$ is the leaf $L_{\psi_i^{-1}(x,y)}$ for any $x \in W_i$. Natural examples of this scheme of coordinates systems appear if we consider compact foliation given by the inverse image of submersions: values in the image space provide local coordinates for the vertical space $V$.

Item (3) above also allows to simplify the notation in such a way that we can omit the coordinate system when dealing with the vertical directions, i.e. we shall write $\pi : M \rightarrow V$ to denote $\pi \circ \psi_i$, independently of the (finitely many) index $i$. In coordinates, we write

$$\pi(\cdot) = (\pi_1(\cdot), \ldots, \pi_d(\cdot)) \in V \subset \mathbb{R}^d.$$

**Hypotheses on the perturbed semigroup.** We are going to assume that following behaviour in the transversal dynamics. We shall denote by $\varphi_{s,t}$ and $\varphi^\epsilon_{s,t}$ the LJR flows associated to $P_t$ and $P^\epsilon_t$ respectively.

**H1)** Vertical regularity a.s. of the perturbed flow: For all $i = 1, \ldots, d$,

$$\frac{d}{dt}\pi_i(\varphi^\epsilon_{0,t}(x_0))\bigg|_{t=0} = \epsilon d\pi_i(K)(x_0).$$

**H2)** Transversal weak boundedness of the perturbation: Denote by $y_t = \varphi_{0,t}(y_0)$ and $y^\epsilon_t = \varphi^\epsilon_{0,t}(y_0)$ the trajectories of the perturbed and unperturbed systems respectively, both starting at $y_0$. Suppose there exists a common probability space where the random flows $\varphi_{s,t}$ and $\varphi^\epsilon_{s,t}$ are based such that, for a $p \in [1, \infty)$
and any \( g \in C(M) \), there exists a positive function \( h(\epsilon, t) \geq 0 \), defined for \( \epsilon, t \geq 0 \) which is continuous, \( h(0, t) = h(\epsilon, 0) = 0 \) and satisfies
\[
\left[ E \left( \sup_{0 \leq s < t} |g(y_s) - g(y'_s)|^p \right) \right]^\frac{1}{p} \leq h(\epsilon, \sqrt{\epsilon t}).
\] (5)

Note that Hypothesis (H1) states for each trajectory a property which always holds in the average, in fact: Denoting by \( A \) the infinitesimal generator of \( P_t \) and by \( A^\epsilon \) the infinitesimal generator of \( P^\epsilon_t \) just note that the projections into the vertical coordinates \( \pi_i, i = 1, \ldots, d \), are in the kernel of \( A \) and \( A^\epsilon \pi_i = \epsilon d\pi_i(K) \). This hypothesis is canonically satisfied by semigroups generated by foliated stochastic differential equations with an \( \epsilon \) perturbation of the drift in the direction \( K \). This can be easily verified by the fact that the kernel of the derivative \( d\pi_i \) includes the tangent spaces to the leaves, hence, the differentiability follows by Itô formula, cf. [11, p. 15], also [12].

Without lost of generality we can assume that function \( h(\epsilon, t) \) in Hypothesis (H2) is increasing in \( t \) for a fixed \( \epsilon \). In fact, given such a function \( h \) we have that \( \sup_{0 \leq s \leq t} h(\epsilon, s) \) also satisfies Hypothesis (H2). This hypothesis holds if the semigroups \( \hat{P}_t \) and \( P'_t \) are generated by perturbation of foliated stochastic (Lévy) differential equations, [11, Lemma 2.1], [12, Prop. 2.1] for \( p \geq 2 \), also completely integrable stochastic Hamiltonian system, Li [18]. See also Remark 1 after Lemma 4.5 for more generality. Another class of examples includes perturbing vector fields \( K \) which commute with the infinitesimal generator \( A \) of \( P_t \), which in this case makes \( P'_t = P_t \circ K_t \), where \( K_t \) is the flow of local diffeomorphisms associated to a vector field \( K \) in this class. In this case function
\[
h(\epsilon, t) = \sqrt{\epsilon} \ t \ \sup_{x \in U} |K(x)|
\]
satisfies the inequality (5).

### 4.1 Averaging functions on the leaves

By compactness, the leaf \( L_p \) passing through a point \( p \in M \) contains the support of an invariant measure \( \mu_p \) for the unperturbed semigroup \( P_t \). We assume that \( \mu_p \) is ergodic. Consider a continuous function \( g : M \to R \). We shall work with the \( \mu_p \)-average of \( g \), \( Q^p : V \subset R^d \to R \) defined for each leaf, i.e. if \( v \) is the vertical coordinate of \( p \), \( \pi(p) = v \in V \), then:
\[
Q^p(v) = \int_{L_p} g(x) \, d\mu_p(x).
\]

We assume the following hypothesis on the invariant measures on the leaves:

**H3. Regularity of \( Q^p \):** For any Lipschitz continuous function \( g \) on \( M \), its corresponding average function \( Q^p \) on the transversal space \( V \) is Lipschitz.

Hypothesis (H3) means that the invariant measures \( \mu_p \) for the unperturbed foliated system has some weakly continuity with respect to the vertical component of \( p \); say, for instance, locally there is no sort of bifurcation of the horizontal foliated dynamics performed by \( (P_t^{(n)}) \) when one varies the vertical parameter, as in [11].
We use the derivative of each component of \( \pi(\cdot) = (\pi_1(\cdot), \ldots, \pi_d(\cdot)) \in V \subset \mathbb{R}^d \) to get the averages \( Q^{d\pi_i(K)}(x) \) of the real functions \( g = d\pi_i(K) \), \( i = 1, \ldots, d \) on each leaf \( L_x \). The proposition below gives an ergodic estimation of the error which occurs when one considers the average \( Q^g \) instead of the original function \( g \) in a time integration.

**Proposition 4.1.** For \( i = 1, 2, \ldots, d, \ t \geq 0 \) and \( \epsilon > 0 \) let

\[
\delta_i(\epsilon, t) = \int_0^t d\pi_i(K)(y_{\epsilon r}) - Q^{d\pi_i(K)}(\pi(y_{\epsilon r})) \, dr.
\]

We have the following estimates for the difference \( \delta_i(\epsilon, t) \)

\[
\left( \mathbb{E}|\delta_i(\epsilon, t)|^p \right)^{\frac{1}{p}} \leq \sqrt{t} H(\epsilon, t)
\]

where \( H(\epsilon, t) \) is continuous in \( \epsilon, t \geq 0 \) and \( H(0, 0) = 0 \).

**Remark:** Precisely, in terms of function \( h(\epsilon, t) \) in Hypothesis (H2) we have that

\[
H(\epsilon, t) = \min \left\{ h(\epsilon, t)\sqrt{t}, C_1 \epsilon^{\frac{1}{4}}, C_2 \sqrt{\epsilon t}^{\frac{1}{2}}, C_3 \sqrt{\epsilon t} \right\}
\]

for some positive constants \( C_1, C_2 \) and \( C_3 \).

**Proof.** The proof consists of changing variables to get an integration in the interval \([0,t/\epsilon]\) such that considering a convenient partition of this interval we estimate by comparing in each subinterval the average of the flow of the original system (on the corresponding leaf) with the average of the perturbed flow (transversal to the leaves) using Hypothesis (H2). For sufficiently small \( \epsilon \), we take the following assignment of increments of our partition:

\[
\Delta t = \frac{t}{\sqrt{\epsilon}}.
\]

We consider the partition \( t_n = n\Delta t \), for \( 0 \leq n \leq N \) such that

\[
0 = t_0 < t_1 < \cdots < t_N \leq \frac{t}{\epsilon},
\]

with \( N = N(\epsilon) = \lceil \epsilon^{-1/2} \rceil \) where here \( \lceil x \rceil \) denotes the integer part of \( x \).

To simplify the notation, denote by \( g(x) \) the function \( d\pi_i(K)(x) \). Hence, the first integrand can be written as the sum:

\[
\epsilon \int_0^t g(y_{\epsilon r})dr = \epsilon \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} g(y_{\epsilon r})dr + \epsilon \int_t^\infty g(y_{\epsilon r})dr.
\]

Let \( \varphi_{s,t} \) denote the LJR-flow presented in the previous section for the unperturbed foliated semigroups \( (P_t^{(n)}) \), i.e. such that each trajectory stays in a single leaf of the foliation. By triangular inequality, we divide our calculation into four parts:

\[
|\delta(\epsilon, t)| \leq |A_1| + |A_2| + |A_3| + |A_4|,
\]

(6)
where

\begin{align*}
A_1 &= \epsilon \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left[ g(y_r^s) - g(\varphi_{t_n,r}(y_{t_n}^s)) \right] dr,
A_2 &= \epsilon \sum_{n=0}^{N-1} \left[ \int_{t_n}^{t_{n+1}} g(\varphi_{t_n,r}(y_{t_n}^s)) dr - \Delta t Q^g(\pi(y_{t_n}^s)) \right],
A_3 &= \sum_{n=0}^{N-1} \epsilon \Delta t Q^g(\pi(y_{t_n}^s)) - \int_0^t Q^g(\pi(y_r^s)) dr,
A_4 &= \epsilon \int_{t_N}^t g(y_r^s) dr.
\end{align*}

We proceed by showing that each of the processes $A_1, A_2, A_3$ and $A_4$ above tends to zero on compact intervals.

**Lemma 4.2.** Process $A_1$ converges to zero on compact intervals when $\epsilon$ goes to zero. More precisely, we have the following estimates on the rate of convergence:

\[
\left( \mathbb{E} |A_1|^p \right)^{\frac{1}{p}} \leq h(\epsilon, t) t,
\]

where $h(\epsilon, t)$ is given by Hypothesis (H2).

**Proof.** If $\frac{1}{p} + \frac{1}{q} = 1$, by Hölder and triangular inequalities we have that

\[
\left( \mathbb{E} |A_1|^p \right)^{\frac{1}{p}} \leq \epsilon \sum_{n=0}^{N-1} \left( \mathbb{E} \left[ \int_{t_n}^{t_{n+1}} |g(y_r^s) - g(\varphi_{t_n,r}(y_{t_n}^s))| dr \right]^p \right)^{\frac{1}{p}}
\]

\[
\leq \epsilon \sum_{n=0}^{N-1} \left( \mathbb{E} \left[ \left( \int_{t_n}^{t_{n+1}} dr \right)^{\frac{1}{2}} \left( \int_{t_n}^{t_{n+1}} |g(y_r^s) - g(\varphi_{t_n,r}(y_{t_n}^s))|^p dr \right)^{\frac{1}{p}} \right] \right)^{\frac{1}{p}}
\]

\[
\leq \epsilon (\Delta t)^{\frac{1}{2}} \sum_{n=0}^{N-1} \left( \mathbb{E} \left[ \Delta t \sup_{t_n \leq r \leq t_{n+1}} |g(y_r^s) - g(\varphi_{t_n,r}(y_{t_n}^s))|^p \right] \right)^{\frac{1}{p}}
\]

\[
\leq \epsilon (\Delta t)^{\frac{1}{2}} \sum_{n=0}^{N-1} \left( \mathbb{E} \left[ \sup_{t_n \leq r \leq t_{n+1}} |g(y_r^s) - g(\varphi_{t_n,r}(y_{t_n}^s))|^p \right] \right)^{\frac{1}{p}}
\]

Hypothesis (H2) together with the fact that the law of the flow $\varphi_{s,t}$ depends only on the difference $t - s$, imply that for each $0 \leq n \leq N - 1$ above, the function $g$ evaluated along trajectories of the perturbed system compared with $g$ evaluated along the unperturbed trajectories, both starting at $y_{t_n}^s$ satisfies:

\[
\left[ \mathbb{E} \sup_{t_n \leq r \leq t_{n+1}} |g(y_r^s) - g(\varphi_{t_n,r}(y_{t_n}^s))|^p \right]^{\frac{1}{p}} \leq h(\epsilon, \sqrt{\epsilon \Delta t}).
\]

15
Hence
\[
\left[ \mathbb{E}|A_1|^p \right]^{\frac{1}{p}} \leq \epsilon \Delta t \ N h(\epsilon, \sqrt{\epsilon} \Delta t) \\
= \ h(\epsilon, t) \ t.
\]

\[ \square \]

**Lemma 4.3.** Process \(A_2\) in equation (6) goes to zero with the following rate of convergence:
\[
\left[ \mathbb{E}|A_2|^p \right]^{\frac{1}{p}} \leq C_1 \sqrt{t} \ \epsilon^{\frac{1}{4}}.
\]
for a positive constant \(C_1\).

**Proof.** By Minkowsky inequality we have that
\[
\left[ \mathbb{E}|A_2|^p \right]^{\frac{1}{p}} \leq \epsilon \left[ \mathbb{E} \left| \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} g(\varphi_{t_n,r}(y_{t_n}^\epsilon)) \ dr - \Delta t Q^g(\pi(y_{t_n}^\epsilon)) \right|^p \right]^{\frac{1}{p}}
= \epsilon \Delta t \sum_{n=0}^{N-1} \left[ \mathbb{E} \left| \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} g(\varphi_{t_n,r}(y_{t_n}^\epsilon)) \ dr - Q^g(\pi(y_{t_n}^\epsilon)) \right|^p \right]^{\frac{1}{p}}.
\]
For all \(n = 0, \ldots, N - 1\), the ergodic theorem implies that the two terms inside the modulus converges to each other when \(\Delta t\) goes to infinity. Moreover, as in [18, Lemma 3.2] by Markovian property and central limit theorem, the rate of convergence has order \(1/\sqrt{\Delta t}\) when \(\Delta t\) goes to infinity. Hence, for small \(\epsilon\) we have
\[
\left[ \mathbb{E}|A_2|^p \right]^{\frac{1}{p}} \leq C_1 \epsilon \ N(\Delta t) \frac{1}{\sqrt{\Delta t}}
= \ C_1 \epsilon \left( \epsilon^{-\frac{1}{2}} \right) \sqrt{t \epsilon^{-\frac{1}{2}}}
\leq \ C_1 \sqrt{t} \ \epsilon^{\frac{1}{4}}.
\]

\[ \square \]

**Lemma 4.4.** \(A_3\) converges to zero when \(t\) or \(\epsilon\) go to 0. We have the following rate of convergence:
\[
\left( \mathbb{E}|A_3|^p \right)^{\frac{1}{p}} \leq C_2 \sqrt{\epsilon} \ t^2,
\]
for a positive constant \(C_2\).

**Proof.** Consider the partition \(t_n\) of the interval \([0, t]\), whose mesh goes to zero. Then, the sum in the expression of \(A_3\) is the Riemman sum of the integral which appears in second term. Hence, the convergence to zero corresponds to the existence of the Riemann integral, which is guaranteed by continuity of \(\pi(y_r^\epsilon)\) (Hypothesis H1).
We calculate now an estimate for the rate of convergence to zero. Let \( C \) be the Lipschitz constant of \( Q^g \). Then

\[
|A_3| \leq \epsilon \sum_{n=0}^{N-1} \Delta t \sup_{\ell t_n < s \leq \ell t_{n+1}} |Q^g(\pi(y_{\ell t_n}^\epsilon)) - Q^g(\pi(y_{\ell t_{n+1}}^\epsilon))| \\
\leq \epsilon (\Delta t) C \sum_{n=0}^{N-1} \sup_{\ell t_n < s \leq \ell t_{n+1}} |\pi(y_{\ell t_n}^\epsilon) - \pi(y_{\ell t_{n+1}}^\epsilon)|.
\]

(7)

By Hypothesis (1) we have the following inequality which is independent of \( \epsilon \):

\[
|\pi(y_{\ell t_n}^\epsilon) - \pi(y_{\ell t_{n+1}}^\epsilon)| \leq \sup_{x \in U} K(x) |u - v|
\]

for all \( u, v \geq 0 \). Hence, continuing the estimates for \( |A_3| \), Inequality (7) above implies that

\[
|A_3| \leq C_2 (\epsilon \Delta t)^2 N \\
= C_2 \left( \epsilon \frac{t}{\sqrt{\epsilon}} \right)^2 \epsilon^{-\frac{1}{2}} \\
= C_2 \sqrt{\epsilon} t^2,
\]

for a positive constant \( C_2 \).

\[\square\]

**Lemma 4.5.** Process \( A_4 \) converges to zero with

\[
\left( \mathbb{E} |A_4|^p \right)^{\frac{1}{p}} \leq C_3 t \sqrt{\epsilon}.
\]

**Proof.** Denote

\[ C_3 = \sup_{x \in U} |g(x)|. \]

The result follows straightforward since

\[
\epsilon \left| \int_{\ell t_N}^{\ell t} g(y_{\ell t}^\epsilon) d\ell \right| \leq C_3 \epsilon \Delta t = C_3 t \sqrt{\epsilon}.
\]

Now, going back to the proof of Proposition 4.1. Note that each of the four estimates of Lemmas 4.2–4.5 allows a factorization which has a common factor \( \sqrt{t} \) times a continuous function which goes to zero when \( (t, \epsilon) \to 0 \). Explicitly, take

\[
H(\epsilon, t) = \min \left\{ h(\epsilon, t) \sqrt{t}, C_1 \epsilon^4, C_2 \sqrt{\epsilon t^2}, C_3 \sqrt{\epsilon t} \right\}.
\]

Proposition 4.1 now follows by inequality (6).

\[\square\]

**Remark:** The technique we have used to prove Proposition 4.1 can be extended in fact to a larger class of functions \( h \) in inequality (5). Let \( f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) be a continuous function with \( f(0) = 0 \),

\[
\lim_{\epsilon \to 0} f(\epsilon)^{-1} = +\infty \quad \text{and} \quad \lim_{\epsilon \to 0} \epsilon f(\epsilon)^{-1} = 0.
\]
Then inequality (5) of Hypothesis (H2) can be restated for $h(\epsilon, f(\epsilon)t)$. In this case, in the proof of Proposition 4.1 one has to consider the partition $\Delta t = \frac{1}{N}$, $N = [f(\epsilon)^{-1}]$ and the results follow by the same arguments. As state before, for stochastic Hamiltonian systems, using the Liouville coordinate systems on invariant torus, one can use $f(\epsilon) = \sqrt{\epsilon}$, see [13]. For general stochastic equations on foliated manifolds $f(\epsilon) = |\ln \epsilon|^{-\frac{1}{2}}$ satisfies this extended Hypothesis (H2) hence Proposition 4.1 also holds in this case, see [11, Lemma 3.1].

4.2 An averaging principle

**Theorem 4.6.** Assume that the unperturbed foliated semigroups on $M$ satisfies hypotheses (H1), (H2) and (H3) above. Let $v(t)$ be the solution of the deterministic ODE in the transversal component $V \subset \mathbb{R}^n$,

$$\frac{dv}{dt} = (Q^{d\pi_1(K)}, \ldots, Q^{d\pi_d(K)}(v(t)))$$

with initial condition $v(0) = \pi(x_0) = 0$. Let $T_0$ be the time that $v(t)$ reaches the boundary of $V$. Then, for all $0 < t < T_0$ we have that

$$[E \left( \left| \pi_{y_{t}}(y_{t}) - v(t) \right|^p \right)]^{\frac{1}{p}} \leq G(\epsilon, t)$$

where $G(\epsilon, t) \geq 0$ is continuous for nonnegative $\epsilon$ and $t$, it is decreasing in $t$ for a fixed $\epsilon$ and $G(\epsilon, 0) = G(0, t) = 0$.

**Remark:** Precisely, in terms of function $h(\epsilon, t)$ in Hypothesis (H2) we have that the estimates above are given by

$$G(\epsilon, t) = \sqrt{t}e^{Ct} \min \left\{ 1, \epsilon^{\frac{1}{2}}, C_1 \epsilon^{\frac{1}{2}}, C_2 \epsilon^{\frac{1}{2}}, C_3 \epsilon^{\frac{1}{2}} \right\}.$$ 

for some positive constants $C, C_1, C_2$ and $C_3$.

**Proof.** Most of the calculations have been done in Proposition 4.1. We only have to note that for each $i = 1, 2, \ldots, d$, by Hypothesis (H1), Jensen’s inequality and Proposition 4.1 we have

$$\left| \pi_{i} \left( y_{t} \right) - v_{i}(t) \right| \leq \int_{0}^{t \wedge T^\epsilon} \left| Q^{d\pi_{i}(K)}(\pi_{i}(s)) - Q^{d\pi_{i}(K)}(\pi_{i}(s)) \right| ds + |\delta_{i}(\epsilon, t)|$$

$$\leq C_i \int_{0}^{t} \left| \pi_{i}(y_{s}) - v_{i}(s) \right| ds + |\delta_{i}(\epsilon, t)|,$$

where each $C_i$ is the Lipschitz constant of $Q^{d\pi_{i}(K)}$. Summing up the $i$’s and using Gronwall’s lemma we have, for a constant $C$:

$$\left| \pi_{i} \left( y_{t} \right) - v_{i}(t) \right| \leq e^{Ct} \sum_{i=1}^{n} |\delta_{i}(\epsilon, t)|.$$ 

And the result follows by Proposition 4.1. \qed
4.3 Example:

We present a simple example to illustrate the framework where the averaging principle for perturbed foliated semigroups holds. Consider \( M = \mathbb{R}^3 - \{(0, 0, z), z \in \mathbb{R}\} \) with the 1-dimension horizontal circle foliation of \( M \) where the leaf passing through a point \( p = (x, y, z) \) is given by the circle

\[
L_p = \left\{ (\sqrt{x^2 + y^2} \cos \theta, \sqrt{x^2 + y^2} \sin \theta, z), \ \theta \in [0, 2\pi] \right\}.
\]

For a point \( p_0 = (x_0, y_0, z_0) \), say with \( x_0 \geq 0 \) consider the local foliated coordinates in the neighbourhood \( U = \mathbb{R}^3 \setminus \{(x, 0, z); x \leq 0; z \in \mathbb{R}\} \) given by cylindrical coordinates. Hence, the coordinate system is defined by \( \psi \) say, suppose that in cylindrical coordinates \( K \)

The sum in the angles are taken module \( 2\pi \).

Adding a first order perturbation.

We investigate the effective behaviour of a perturbed foliated semigroups. Consider \( \epsilon K \) where \( \epsilon \in (0, \pi) \) is angular and \( v = (r, z) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \) is such that \( \psi^{-1} : (u, v) \mapsto (r \cos u, r \sin u, z) \in M \). In this coordinate system, the transversal projections \( \pi_1 \) and \( \pi_2 \) correspond to the radial \( r \)-component and the \( z \)-coordinate, respectively.

Consider the semigroup \( P_t \) acting in \( C_0(M) \) given by the following: denoting a point \( \varphi(x) = (\theta, r, z) \) by its coordinates and writing the entries of \( f \in C_0(M) \) in these coordinates,

\[
P_t f(x) = \frac{1}{2} \left\{ f(\theta + t, r, z)(1 + e^{-2t}) + f(\theta + \pi + t, r, z)(1 - e^{-2t}) \right\}.
\]

The sum in the angles are taken module \( 2\pi \). This semigroup corresponds to a Lévy flow in each circular leaf diffeomorphic to \( S^1 \) of the foliation which has simultaneously two commutative behaviour: pure rotation and Poisson jumps to the antipodal. Its infinitesimal generator in cylindrical coordinates is given by \( A f(\theta, r, z) = \frac{\partial}{\partial \theta} f(\theta) + f(\theta + \pi, r, z) - f(\theta, r, z) \). See e.g. Applebaum [1] or Liao [19]. Hence it is obviously a foliated semigroup in the foliated space \( M \). The unique invariant probability measure is the normalized Lebesgue measure in each circle, hence Hypothesis (H3) is satisfied.

Adding a first order perturbation. We investigate the effective behaviour of a small transversal perturbation in the semigroup, such that the original infinitesimal generator is perturbed by \( \epsilon K \). Functions \( Q^{d\pi_1(K)} \) and \( Q^{d\pi_2(K)} \) are simply the integral along each circle of the radial and vertical components of \( K \), respectively. Theorem 4.6 says that in the average, the transversal behaviour of \( P_t^\epsilon \) is approximate by \( v(t) \) where \( v(t) \) is solution starting at zero of the EDO \( v'(t) = (Q^{d\pi_1(K)}(v_1), Q^{d\pi_2(K)}(v_2)) \), i.e.

\[
P_t^\epsilon |\pi_1(\cdot) - v_1(t)| \leq G(\epsilon, t).
\]

A class of examples appears if we consider a vector field \( K \) which commutes with \( A \), say, suppose that in cylindrical coordinates \( K \) is given by:

\[
K(\theta, r, z) = (0, \lambda_0, k_3(z))
\]

where \( k_3 : \mathbb{R} \to \mathbb{R} \) is a smooth function with bounded derivative. The perturbed semigroup in this case is given by

\[
P_t^\epsilon f(x) = \frac{f(\eta_1)(1 + e^{-2t}) + f(\eta_2')(1 - e^{-2t})}{2}.
\]

where

\[
\eta(t) = \left( \theta + t, r + c\lambda_0 t, \xi^\epsilon(z, t) \right);
\]
\[ \eta'_t = \left( \theta + \pi + t, \ r + \epsilon \lambda_0 t, \ \xi^\epsilon(z, t) \right); \]

and \( \xi^\epsilon(z, t) \) is the solution of the ODE in the real line

\[ \xi^\epsilon(z, t) = z + \int_0^t k_3(\xi^\epsilon(z, s)) \, ds. \]

Vector field \( K \) commutes with the infinitesimal generator \( A \), hence hypothesis (H1) and (H2) with exponent \( p = 1 \) are satisfied.

In this case we have the radial \( d\pi_1(K) = \lambda_0 \) and the vertical \( d\pi_2(k)(\theta, r, z) = k_3(z) \), hence their average with respect to Lebesgue measure on the leaves are \( Q^{d\pi_1(K)} = \lambda_0 \) and \( Q^{d\pi_2(k)} = k_3(\pi_2(x)) \). Hence the transversal components as stated in the main Theorem 4.6 are given by \( v(t) = (r_0 + \epsilon t \lambda_0, \xi^\epsilon(z, t)) \) for all \( t \geq 0 \), if \( \lambda_0 \geq 0 \), and \( 0 \leq t < \frac{\epsilon |\lambda_0|}{\epsilon \lambda_0} \) if \( \lambda_0 < 0 \). One checks easily that

\[ P^*_x \| \pi_i(\cdot) - v_i(t) \| = 0 \leq G(\epsilon, t). \]

\[ \square \]

**Acknowledgments:** This article has been written while the authors are visiting Humboldt University, Berlin. They would like to express their gratitude to Prof. Peter Imkeller and his research group for the nice and friendly hospitality. Author P.H.C. has been supported by CNPq 149688/2010-5 and 236640/2012-7 and P.R.R. has been partially supported by FAPESP 11/50151-0 and 12/03992-1.

**References**

[1] D. Applebaum – *Lévy processes and stochastic calculus*, Cambridge University press, 2004.

[2] L. Arnold – *Random dynamical systems*, Springer-Verlag, 1998.

[3] P. Baxendale – T. E. Harris’s contributions to recurrent Markov processes and stochastic flows. *Ann. Probab.* 39 (2011), no. 2, 417-428.

[4] J. Bertoin and J. F. Le Gall – Stochastic flows associated to coalescent processes. *Probab. Theory Related Fields*, 126 (2003), 261-288.

[5] C. Camacho and A. Lins-Neto – *Geometric theory of foliations*, Birkhäuser Boston, 1985.

[6] A. Candel and L. Conlon – *Foliations I and II*. Graduate Studies in Mathematics, American Mathematical Society, 1999.

[7] A. Candel – The harmonic measures of Lucy Garnett. *Adv. Math.* 176 (2003), no. 2, 187?247.

[8] P. Catuogno, D. Ledesma and P. Ruffino – Harmonic measures in embedded foliated manifolds. Submitted. (2012) ArXiv 1208.0629.
[9] J. Chen and K. N. Xiang – Natural flow not in Le Jan-Raimond framework. Stochastics and Dynamics, 12 (2012), no. 2, 1150014.

[10] L. Garnett, Foliation, the ergodic theorem and Brownian motion. Journal of Functional Analysis 51, (1983) pp. 285-311.

[11] I. I. Gonzales-Gargate and P. R. Ruffino – An averaging principle for diffusions in foliated spaces. Submitted (2012) ArXiv 1212.1587.

[12] M. Högele and P. R. Ruffino – Averaging along Lévy diffusions in foliated spaces. Preprint, Mathematics Department, Potsdam University 2 (2013) 10. Submitted.

[13] V. A. Kaimanovich, Brownian motion on foliations: Entropy, invariant measures, mixing Functional Analysis and Its Applications, Vol. 22, N° 4, (1988) pp. 326-328

[14] H. Kunita – Stochastic differential equations and stochastic flows of diffeomorphisms. In École d’Été de Probabilités de Saint-Flour XII-1982, pp. 143–303. Lecture Notes in Math. 1097, Springer-Verlag, Berlin, 1984.

[15] H. Kunita – Stochastic flows and stochastic differential equations. Cambridge University Press, 1988.

[16] Y. Le Jan and O. Raimond – Flows, coalescence and noise. Ann. Probab., 32 (2004) no. 2, 1247-1315.

[17] Y. Le Jan and O. Raimond – Flows associated to Tanaka’s SDE. ALEA Lat. Am. J. Probab. Math. Stat. 1 (2006), 21-34.

[18] X. M. Li – An averaging principle for a completely integrable stochastic Hamiltonian systems. Nonlinearity, 21 (2008) 803-822.

[19] M. Liao – Lévy processes in Lie groups. Cambridge University Press, 2004.

[20] J. F. Plante, Foliations with measure preserving holonomy. Annals of Mathematics, 102 (1975), 327-361.

[21] D. Revuz and M. Yor – Continuous martingales and Brownian motion. Springer-Verlag, Berlin 1999.

[22] P. Tondeur. Foliations on Riemannian manifolds. Universitext, Springer Verlag, Berlin-Heidelberg-New York, 1988.

[23] B. Tsirelson. Nonclassical stochastic flows and continuous products. Probab. Surv. 1 (2004), 173–298.

[24] P. Walczak – Dynamics of foliations, groups and pseudogroups. Birkhäuser Verlag 2004.