Self-dual Yang–Mills fields
in pseudoeuclidean spaces

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Abstract
The self-duality Yang–Mills equations in pseudoeuclidean spaces of dimensions $d \leq 8$ are investigated. New classes of solutions of the equations are found. Extended solutions to the $D=10$, $N=1$ supergravity and super Yang-Mills equations are constructed from these solutions.

1 Introduction
In 1983 Corrigan et al. [1] have proposed a generalization of the self-dual Yang–Mills equations in dimension $d > 4$:

$$f_{mnps}F^{ps} = \lambda F_{mn},$$

where the numerical tensor $f_{mnps}$ is completely antisymmetric and $\lambda = \text{const}$ is a non-zero eigenvalue. By the Bianchi identity $D[pF_{mn}]=0$, it follows that any solution of (1) is a solution of the Yang–Mills equations $[D^m, F_{mn}]=0$. Some of these solutions have found in [2].

The many-dimensional Yang–Mills equations appear in the low-energy effective theory of superstring and supermembrane [3,4]. In addition, there is a hope that Higgs fields and supersymmetry can be understood through dimensional reduction from $d > 4$ dimensions down to $d = 4$ [5].

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The paper is organized as follows. Section 2 contains well-known facts about Cayley-Dickson algebras and connected with them Lie algebras. In Sections 3 and 4 the self-duality Yang–Mills equations in pseudoeuclidean spaces of dimensions $d \leq 8$ are investigated. In Section 5 extended solutions to the $D = 10$, $N = 1$ supergravity and super Yang-Mills equations are constructed from these solutions.

2 Cayley-Dickson algebras

Let us recall that the algebra $A$ satisfying the identities
\[ x^2 y = x(xy), \quad yx^2 = (yx)x \]
is called alternative. It is obvious that any associative algebra is alternative. The most important example of nonassociative alternative algebra is Cayley-Dickson algebra. Let us recall its construction (see [6]).

Let $A$ be an algebra with an involution $x \to \bar{x}$ over a field $F$ of characteristic $\neq 2$. Given a nonzero $\alpha \in F$ we define a multiplication on the vector space $(A,\alpha) = A \oplus A$ by
\[ (x_1, y_1)(x_2, y_2) = (x_1x_2 - \alpha \bar{y}_2 y_1, y_2 x_1 + y_1 \bar{x}_2). \]
This makes $(A,\alpha)$ an algebra over $F$. It is clear that $A$ is isomorphically embedded into $(A,\alpha)$ and $\text{dim}(A,\alpha) = 2 \text{dim}A$. Let $e = (0,1)$. Then $e^2 = -\alpha$ end $(A,\alpha) = A \oplus Ae$. Given any $z = x + ye$ in $(A,\alpha)$ we suppose $\bar{z} = \bar{x} - ye$. Then the mapping $z \to \bar{z}$ is an involution in $(A,\alpha)$.

Starting with the base field $F$ the Cayley-Dickson construction leads to the following sequence of alternative algebras:

1) $F$, the base field.
2) $\mathbb{C}(\alpha) = (F,\alpha)$, a field if $x^2 + \alpha$ is the irreducible polynomial over $F$; otherwise, $\mathbb{C}(\alpha) \simeq F \oplus F$.
3) $\mathbb{H}(\alpha, \beta) = (\mathbb{C}(\alpha), \beta)$, a generalized quaternion algebra. This algebra is associative but not commutative.
4) $\mathbb{O}(\alpha, \beta, \gamma) = (\mathbb{H}(\alpha, \beta), \gamma)$, a Cayley-Dickson algebra. Since this algebra is not associative the Cayley-Dickson construction ends here.

The algebras in 1) – 4) are called composition. Any of them has the non-degenerate quadratic form (norm) $n(x) = x\bar{x}$, such that $n(xy) = n(x)n(y)$. In particular, over the field $\mathbb{R}$ of real numbers, the above construction gives
3 split algebras (e.g., if $\alpha = \beta = \gamma = -1$) and 4 division algebras (if $\alpha = \beta = \gamma = 1$): the fields of real $\mathbb{R}$ and complex $\mathbb{C}$ numbers, the algebras of quaternions $\mathbb{H}$ and octonions $\mathbb{O}$, taken with the Euclidean norm $n(x)$. Note also that any simple nonassociative alternative algebra is isomorphic to Cayley-Dickson algebra $\mathbb{O}(\alpha, \beta, \gamma)$.

Let $A$ be Cayley-Dickson algebra and $x \in A$. Denote by $R_x$ and $L_x$ the operators of right and left multiplication in $A$

$$R_x : a \rightarrow ax, \quad L_x : a \rightarrow xa.$$  

It follows from (2) that

$$R_{ab} - R_a R_b = [R_a, L_b] = [L_a, R_b] = L_{ba} - L_a L_b. \quad (3)$$  

Consider the Lie algebra $\mathcal{L}(A)$ generated by all operators $R_x$ and $L_x$ in $A$. Choose in $\mathcal{L}(A)$ the subspaces $R(A)$, $S(A)$, and $D(A)$ generated by the operators $R_x$, $S_x = R_x + 2L_x$, and $2D_{x,y} = [S_x, S_y] + S_{[x,y]}$ respectively. Using (3), it is easy to prove that

$$3[R_x, R_y] = D_{x,y} + S_{[x,y]}, \quad (4)$$  

$$[R_x, S_y] = R_{[x,y]}, \quad (5)$$  

$$[R_x, D_{y,z}] = R_{[x,y,z]}, \quad (6)$$  

$$[S_x, S_y] = D_{x,y} - S_{[x,y]}, \quad (7)$$  

$$[S_x, D_{y,z}] = S_{[x,y,z]}, \quad (8)$$  

$$[D_{x,y}, D_{z,t}] = D_{[x,y,z],t} + D_{[y,z,t],x} + D_{[z,x,t],y}, \quad (9)$$

where $[x, y, z] = [x, [y, z]] - [y, [z, x]] - [z, [x, y]]$. It follows from (4)–(9) that the algebra $\mathcal{L}(A)$ is decomposed in the direct sum

$$\mathcal{L}(A) = R(A) \oplus S(A) \oplus D(A)$$

of the Lie subalgebras $D(A)$, $D(A) \oplus S(A)$ and the vector space $R(A)$.

In particular, if $A$ is a real division algebra, then $D(A)$ and $D(A) \oplus S(A)$ are isomorphic to the compact Lie algebras $g_2$ and $so(7)$ respectively. If $A$ is a real split algebra, then $D(A)$ and $D(A) \oplus S(A)$ are isomorphic to noncompact Lie algebras $g_2'$ and $so(4, 3)$. 
3 Self-dual solutions in $d = 8$

Let $A$ be a real linear space equipped with a nondegenerate symmetric metric $g$ of signature $(8, 0)$ or $(4, 4)$. Choose the basis $\{1, e_1, ..., e_7\}$ in $A$ such that

$$g = \text{diag}(1, -\alpha, -\beta, \alpha\beta, -\gamma, \alpha\gamma, \beta\gamma, -\alpha\beta\gamma),$$

(10)

where $\alpha, \beta, \gamma = \pm 1$. Define the multiplication

$$e_i e_j = -g_{ij} + c_{ijk} k e_k,$$

(11)

where the structural constants $c_{ijk} = g_{ks} c_{ij}^s$ are completely antisymmetric and different from 0 only if

$$c_{123} = c_{145} = c_{167} = c_{246} = c_{275} = c_{374} = c_{365} = 1.$$

(12)

The multiplication (11) transform $A$ into a linear algebra. It can easily be checked that the algebra $A \simeq O(\alpha, \beta, \gamma)$. In the basic $\{1, e_1, ..., e_7\}$ the operators

$$R e_i = e_i 0 + \frac{1}{2} c_{ij} k e_j, \quad L e_i = e_i 0 - \frac{1}{2} c_{ij} k e_j,$$

(13)

where $e_{ij}$ are generators of the Lie algebra $L(A)$ satisfying the switching relations

$$[e_{mn}, e_{ps}] = g_{mp} e_{ns} - g_{ms} e_{np} - g_{np} e_{ms} + g_{ns} e_{mp}.$$

(14)

Now, let $G$ be a matrix Lie group constructed by the Lie algebra $D(A) \oplus S(A)$. In the space $A$ equipped with the metric (10) we define the completely antisymmetric $G$-invariant tensor $f_{mnps}$ (cp. [7]):

$$f_{ijk0} = c_{ijk},$$

$$f_{ijkl} = g_{il} g_{jk} - g_{ik} g_{jl} + c_{ijm} c_{kl}^m,$$

where $i, j, k, l \neq 0$. Representing the nonzero components of $f_{mnps}$ in the explicit form

$$f_{0123} = f_{0145} = f_{0167} = f_{0246} = f_{0275} = f_{0374} = f_{0365} = 1,$$

$$f_{4567} = f_{2367} = f_{2345} = f_{1357} = f_{1364} = f_{1265} = f_{1274} = 1,$$

we see that the tensor $f_{mnps}$ satisfies the identity

$$f_{mnij} f_{ps}^{ij} = 6(g_{mp} g_{ns} - g_{ms} g_{np}) + 4 f_{mnps}.$$

(15)
Define the projectors $\tilde{f}_{mnps}$ of $\mathcal{L}(A)$ onto the subalgebra $D(A) \oplus S(A)$ and its generators $\tilde{f}_{mn}$ by

$$\tilde{f}_{mnps} = \frac{3}{8}(g_{mp}g_{ns} - g_{ms}g_{np}) - \frac{1}{8}f_{mnps},$$
$$\tilde{f}_{mn} = \tilde{f}_{mn}^{ij}e_{ij}.$$  

It follows from (15) that

$$f_{mni}f_{ps}^{ij} = -2 \tilde{f}_{mnps}, \quad (16)$$
$$f_{mni}f^{ij} = -2 \tilde{f}_{mn}.$$  

Using the identities (7)–(9) and (13), we get the switching relations

$$[\tilde{f}_{mn}, \tilde{f}_{ps}] = \frac{3}{4}(\tilde{f}_{m[p}g_{s]n} - \tilde{f}_{n[p}g_{s]m}) - \frac{1}{8}(f_{mn}^{k[p}\tilde{f}_{s]k} - f_{ps}^{k[m}\tilde{f}_{n]k}). \quad (18)$$

Now we can find solutions of (1). We choose the ansatz (cp. [2])

$$A_m(x) = \frac{4}{3} \frac{\tilde{f}_{mi}x^i}{\rho^2 + r^2}, \quad (19)$$

where $r^2 = x_kx^k$ and $\rho \in \mathbb{R}$. Using the switching relations (18), we get

$$F_{mn}(x) = \frac{4}{9} \frac{(6\rho^2 + 3r^2)\tilde{f}_{mn} + 8\tilde{f}_{mn}^s\tilde{f}_{s}x^ix^j}{(\rho^2 + r^2)^2}. \quad (20)$$

It follows from (16)–(17) that the tensor $F_{mn}$ is self-dual. If the metric (10) is Euclidean, then we have the well-known solution of equations (1) (see. [2]). If the metric (10) is pseudoeuclidean, then we have a new solution.

## 4 Solutions in $d < 8$

Now we’ll find solutions of the self-duality equations in dimension $d < 8$. If $B_\alpha$ is a subalgebra of the real Cayley-Dickson algebra $A$, then the subgroup $H_\alpha$ of automorphisms of $A$ leaving fixed the element of $B_\alpha$ is called the Galois group of $A$ over $B_\alpha$. Description of these groups is known [8]. In particular, if $A$ is the real division algebra and $B_1 \simeq \mathbb{R}, B_2 \simeq \mathbb{C}, B_3 \simeq \mathbb{H}$, then

$$G \simeq Spin(7), \quad H_1 \simeq G_2, \quad H_2 \simeq SU(3), \quad H_3 \simeq SU(2).$$

5
If $A$ is the real split algebra, then for the same choice of $B_i$,

$G \simeq Spin(4,3), \quad H_1 \simeq G'_2, \quad H_2 \simeq SU(2,1), \quad H_3 \simeq SU(1,1).$

Obviously, the orthogonal complement $B_{\alpha}^\perp$ of $B_\alpha$ in $A$ is $H_\alpha$-invariant subspace of dimension $d_\alpha = 8 - \dim H_\alpha$. Now it is easy to construct a completely antisymmetric $H_\alpha$-invariant $d_\alpha$-tensor $f^\alpha_{mnp}$.

It is a projection of the $d$-tensor $f_{mnp} \in \Lambda^4(A)$ onto the subspace $\Lambda^4(B_\alpha)$.

We can choose nonzero components of $f_{mnp}$ in the form

\[
\begin{align*}
    f^1_{4567} &= f^1_{2367} = f^1_{1357} = 1, \\
    f^2_{1364} &= f^2_{2345} = 1, \\
    f^3_{2345} &= 1.
\end{align*}
\]

Now we can easily get analogues of the identities (15)–(18). In particular, the switching relations (18) takes the form

\[
[j_{mn}^\alpha, j_{ps}^\alpha] = 3 - \alpha \frac{1}{4} (f^n_{m[ps]} - f^n_{p[s]m}) - \frac{1}{8 - 2\alpha} (f^k_{mn} f^k_{ps} - f^k_{pn} f^k_{ms}).
\]

Note that if we choose the ansatz $A_m(x)$ in the form

\[
A_m(x) = k \frac{\tilde{f}_{mi} x^i}{\rho^2 + r^2},
\]

then the corresponding field strength $F_{mn}$ is not self-dual for $\alpha = 2$. On the contrary, if $\alpha = 1$ or $\alpha = 3$, then choice of $A_m(x)$ in the form (21) reduce to a self-dual field strength. For example, if $\alpha = 1$, then $k = 3/2$ and

\[
F_{mn}(x) = -\frac{3}{2} \frac{(2\rho^2 + r^2) f^n_{mn} + 3 f^n_{mm} f^s_{sj} x^i x^j}{(\rho^2 + r^2)^2}.
\]  

(21)

For Euclidean metric this solution is known (see. [2]). For pseudoeuclidean metric we have a new solution. For $\alpha = 3$ we have the well-known instanton solution [9] or its noncompact analogue.

Note that in $d = 4$ there exist another solution of the Yang-Mills equations. It depends on coordinates of the Minkowski space. Indeed, we choose the ansatz $A_m(x)$ in the form

\[
A_m(x) = \frac{2\epsilon_{mn} x^n}{\lambda^2 + x_k x^k},
\]

(22)
where $e_{mn}$ are generators of the Lie algebra $so(p, q)$ satisfying the relations $so(p, q)$. Then the field strength

$$F_{mn}(x) = \frac{-4\lambda^2 e_{mn}}{(\lambda^2 + x_k x^k)^2},$$

and

$$\partial^m F_{mn} + [A^m, F_{mn}] = \frac{8\lambda^2 e_{mn} x^m}{(\lambda^2 + x_k x^k)^3} (4 - \delta_i^i).$$

Hence the anzatz (22) satisfies the Yang–Mills equations if $p + q = 4$. If $|p - q| = 4$ or $0$, then the algebra $so(p, q)$ is isomorphic to the direct sum $so(3) \oplus so(3)$ or $so(2, 1) \oplus so(2, 1)$ of proper subalgebras. Therefore projecting $A_m(x)$ on these subalgebras, we again get the instanton solution [9] or its noncompact analogue. If $|p - q| = 2$, then the algebra $so(p, q)$ is simple. In this case the solution (23) of the Yang–Mills equations is not self-dual.

## 5 Extended supersymmetric solutions

Let now show that the above instanton solutions can be extended to solutions of the $D = 10, N = 1$ supergravity and super Yang-Mills equations. Consider the purely bosonic sector of the theory

$$S = \frac{1}{2k^2} \int d^{10}x \sqrt{-\text{det}} \left( R + 4(\nabla)^2 - \frac{1}{3}\text{H}^2 - \frac{\alpha'}{30}\text{Tr}(F^2) \right).$$

Rather than directly solve the equations of motion for this action, it is much more convenient to look for bosonic backgrounds which are annihilated by some of $N = 1$ supersymmetry transformations. This requires that in ten dimensions there exist at least one Majorana-Weyl spinor $\epsilon$ such that the super symmetry variations

$$\delta \chi = F_{MN} \Gamma^{MN} \epsilon,$$

$$\delta \lambda = \left( \Gamma^M \partial_M \phi - \frac{1}{6} \text{H}_{MNP} \Gamma^{MNP} \right) \epsilon,$$

$$\delta \psi_M = \left( \partial_M + \frac{1}{4} (\omega_M^{AB} - H_M^{AB}) \Gamma_{AB} \right) \epsilon$$

of the fermionic fields vanish for such solutions. We will construct a simple ansatz for the bosonic fields which does just this (cp. [3]).
First, we choose $\epsilon$ to be $Spin(4,3)$-singlet of Majorana-Weyl spinor of $SO(5,5)$. Denote world indices of the eight-dimensional subspace indices by $\mu, \nu = 1, \ldots, 8$ and the corresponding tangent space indices by $m, n = 1, \ldots, 8$. We assume that no fields depend on the coordinates with indices $M, N = 0, 9$. It follows that 
\[
\tilde{f}_{mnp4} \Gamma^{ps} \epsilon = \tilde{f}_{mn} \epsilon = 0.
\]

Using the expression (20) for tensor field strength $F_{mn}$, we see that the supersymmetry variation $\delta \chi$ vanishes.

Further, we choose the antisymmetric tensor field strength $H_{mnp}$ and metric tensor $g$ in the form
\[
H_{mnp} = \frac{1}{7} f_{mnp}\partial^s \phi, \quad g_{\mu\nu} = e^{(6/7)\phi} g_{0\mu\nu},
\]
where $g_0$ is the pseudoeuclidean metric (10). Using the identities
\[
f_{mnp4} \Gamma^{mnp} = 42 \Gamma_s
\]
and the explicit form of spin connectedness
\[
\omega_{\mu mn} = \frac{3}{7} (\delta_{\mu m} \partial_n \phi - \delta_{\mu n} \partial_m \phi),
\]
we prove that the variations $\delta \lambda$ and $\delta \psi_M$ also vanish for any $\phi(x)$.

It follows from the Bianchi identity
\[
\partial_{[m} H_{nps]} = -\alpha' \text{Tr}_8 F_{[mn} F_{ps]} \]
that the tensor field strength
\[
H_{mnp} = -\alpha' \frac{3\rho^2 + r^2}{9(\rho^2 + r^2)^3} f_{mnp} x^s. \tag{26}
\]

If we compare (26) with (25), we find
\[
e^{(6/7)\phi} = e^{(6/7)\phi_0} + \alpha' \frac{2\rho^2 + r^2}{3(\rho^2 + r^2)^2}, \tag{27}
\]
where $\phi_0$ is the value of the dilaton $\phi$ on $\infty$. Similarly, if we choose $G'_2$-singlet of Majorana-Weyl spinor of $SO(5,5)$ and use the expression (21) for the tensor field strength $F_{mn}$, we get an analog of the solution (27).
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