LOCAL SOLUBILITY FOR A FAMILY OF QUADRICS OVER A SPLIT QUADRIC SURFACE

TIM BROWNING, JULIAN LYCZAK, AND ROMAN SARAPIN

Abstract. We study the density of everywhere locally soluble diagonal quadric surfaces, parameterised by rational points that lie on a split quadric surface.

1. Introduction

A great deal of recent activity has been directed at the question of local solubility for families of varieties. Let $X$ and $Y$ be smooth, proper and geometrically irreducible varieties over $\mathbb{Q}$, equipped with a dominant morphism $\pi: X \to Y$, with geometrically integral generic fibre. Suppose that $Y$ is Fano and equipped with an anticanonical height function $H: Y(\mathbb{Q}) \to \mathbb{R}$. This short note is concerned with the behaviour of the counting function

$$N_{\text{loc}}(\pi, B) = \# \{ y \in Y(\mathbb{Q}) \cap \pi(X(\mathbb{A}_{\mathbb{Q}})) : H(y) \leq B \},$$

as $B \to \infty$, where $\mathbb{A}_{\mathbb{Q}}$ denotes the ring of adèles. We shall consider an explicit example, with $Y \subset \mathbb{P}^3$ a split quadric surface, which calls into question our expectations about the behaviour of $N_{\text{loc}}(\pi, B)$.

The quantity $N_{\text{loc}}(\pi, B)$ measures the density of fibres of $\pi$ which are everywhere locally soluble, and it has received the most attention when $Y$ is projective space. For example, it follows from work of Poonen and Voloch [10, Thm. 3.6] that

$$N_{\text{loc}}(\pi, B) \sim c_{d,n} B,$$

as $B \to \infty$, for an explicit constant $c_{d,n} > 0$, when $\pi: X \to \mathbb{P}^N$ is the family of degree $d$ hypersurfaces in $\mathbb{P}^n$ over $\mathbb{Q}$, with $(d, n) \neq (2, 2)$ and $N = \binom{n+d}{d} - 1$. Moreover, when $Y = \mathbb{P}^n$, work of Loughran and Smeets [7] shows that

$$N_{\text{loc}}(\pi, B) \ll \frac{B}{(\log B)^{\Delta(\pi)}},$$

for a certain quantity $\Delta(\pi) \geq 0$, whose definition will be recalled in §2 and which can be extended to arbitrary fibrations $\pi: X \to Y$. For $Y = \mathbb{P}^n$, it is conjectured in [7] that the upper bound (1.2) should be sharp whenever the fibre $\pi^{-1}(D)$ above each codimension one point $D$ in $\mathbb{P}^n$ contains a component of multiplicity one.

It is natural to ask what happens when $Y$ is a Fano variety over $\mathbb{Q}$ that is not projective space. We shall assume that $Y(\mathbb{Q})$ is Zariski dense in $Y$ and that

Date: March 15, 2022.
2010 Mathematics Subject Classification. 14G05 (11N36, 14D10) .
Y conforms to the version of the Manin conjecture \[2\] without accumulating subvarieties, so that

\[
\# \{ y \in Y(\mathbb{Q}) : H(y) \leq B \} \sim c_Y B (\log B)^{ρ_Y - 1},
\]
as \( B \to \infty \), where \( c_Y > 0 \) is a constant and \( ρ_Y \) is the rank of the Picard group of \( Y \).

Assuming that \( Y \) is a Fano variety over \( \mathbb{Q} \) without accumulating subvarieties, a naive combination of the Manin conjecture with the upper bound (1.2) might lead us to suppose that

\[
N_{\text{loc}}(π, B) \sim \frac{c_π B (\log B)^{ρ_Y - 1}}{\log B^{Δ(π)}},
\]
as \( B \to \infty \), for a suitable constant \( c_π > 0 \). This is compatible with (1.1), since in this case \( Δ(π) = 0 \) and \( ρ_{P^N} = 1 \). It also compatible with an upper bound of Browning and Loughran [1, Thm. 1.10], which concerns the case where \( Y \subset \mathbb{P}^n \) is a smooth quadric over \( \mathbb{Q} \) of dimension at least 3, since then \( ρ_Y = 1 \).

We study an explicit example that addresses a case where the variety \( Y \) has Picard number \( ρ_Y > 1 \) and which contradicts the expectation (1.3). Let \( Y \subset \mathbb{P}^3 \) be the split quadric surface

\[
y_0 y_1 = y_2 y_3.
\]
Clearly \( Y \) is smooth and \( ρ_Y = 2 \). Let \( X \subset \mathbb{P}^3 \times \mathbb{P}^3 \) be the variety defined by the pair of equations

\[
y_0 x_0^2 + y_1 x_1^2 + y_2 x_2^2 + y_3 x_3^2 = 0
\]
\[
y_0 y_1 = y_2 y_3.
\]

Then \( X \) is equipped with a dominant morphism \( π : X \to Y \) whose generic fibre is geometrically irreducible. We shall see in \( \S 2 \) that the total space \( X \) is singular and so we let \( \tilde{X} \to X \) be a desingularisation. Taking \( \tilde{π} \) to be the composition with \( π : X \to Y \), we shall prove the following bounds for the associated counting function \( N_{\text{loc}}(\tilde{π}, B) \) in \( \S 3 \).

**Theorem 1.1.** We have \( B \ll N_{\text{loc}}(\tilde{π}, B) \ll B \).

Since the Hasse principle holds for quadratic forms, we deduce from Theorem 1.1 that the same upper and lower bounds hold for the quantity \( N_{\text{glob}}(\tilde{π}, B) \), counting the number of \( y \in Y(\mathbb{Q}) \cap π(\tilde{X}(\mathbb{Q})) \) such that \( H(y) \leq B \).

The next result, proved in \( \S 2 \) shows that our example is not compatible with the naive expectation (1.3), since \( ρ_Y = 2 \).

**Theorem 1.2.** We have \( Δ(\tilde{π}) = 2 \).

Just as thin sets have been found to have a large influence on modern formulations of the Manin conjecture, as discussed by Peyre \[8\], so we expect that the
naive expectation (1.3) should hold after the removal of appropriate thin sets from $Y(\mathbb{Q})$. In fact, for our example (1.5), we conjecture that

$$\# \left\{ y \in Y(\mathbb{Q}) \cap \pi(\tilde{X}(A_Q)) : H(y) \leq B, \ -y_0y_2 \neq \square, \ -y_0y_3 \neq \square \right\} \sim c \frac{B}{\log B},$$

for an appropriate constant $c > 0$. It seems plausible that the character sum method developed by Friedlander and Iwaniec [4] will prove useful for tackling this counting problem, although the hyperboloid height conditions that arise from parameterising the points in $Y(\mathbb{Q})$ will undoubtedly complicate the argument.

In this note we have focused attention on the quadric surface bundle $X \to Y$, where $X$ is given by (1.5). One can also study the conic bundle $\eta : W \to Y$, where $W \subset \mathbb{P}^2 \times \mathbb{P}^3$ is given by

$$y_0x_0^2 + y_1x_1^2 + y_2x_2^2 = 0 \quad y_0y_1 = y_2y_3.$$ 

This variety is singular and so we consider the morphism $\tilde{\eta} : \tilde{W} \to Y$, where $\tilde{\eta}$ is the composition of a desingularisation $\tilde{W} \to W$ with $\eta$. The methods of this paper carry over to this case with little adjustment and can be used to prove that $B \ll N_{\text{loc}}(\tilde{\eta}, B) \ll B$. Moreover, one also has $\Delta(\tilde{\eta}) = 2$ in this case, providing a further example that counters the expectation (1.3), but one which we believe can be repaired via the removal of appropriate thin sets. (In fact, for $i \in \{0, 1\}$, the fibre of $\eta$ above the line $y_i = y_2 = 0$ has multiplicity two in $W$, whereas the corresponding fibre of $\tilde{\eta}$ admits an additional component of multiplicity one.)

Acknowledgement. While working on this paper the first author was supported by FWF grant P 32428-N35. This paper was written while the third author was an intern at IST.

2. Geometric Input

We begin by extending the definition of $\Delta(\pi)$ from [7] to general fibrations. Let $X$ and $Y$ be arbitrary proper and geometrically irreducible varieties over $\mathbb{Q}$, equipped with a dominant morphism $\pi : X \to Y$, with geometrically integral generic fibre. Assume that $Y$ is smooth and associate the residue field $\kappa(D)$ to any codimension one point $D \in Y^{(1)}$. Let $S_D$ be the set of geometrically irreducible components of $\pi^{-1}(D)$ and choose a finite group $\Gamma_D$ through which the action of $\text{Gal}(\kappa(D)/\kappa(D))$ on $S_D$ factors. Then $\Delta(\pi)$ is defined to be

$$\Delta(\pi) = \sum_{D \in Y^{(1)}} (1 - \delta_D(\pi)), \quad (2.1)$$

where

$$\delta_D(\pi) = \frac{\# \{ \sigma \in \Gamma_D : \sigma \text{ acts with a fixed point on } S_D \}}{\# \Gamma_D}.$$
Henceforth, we take $Y \subset \mathbb{P}^3$ to be the split quadric (1.4) and $X \subset \mathbb{P}^3 \times \mathbb{P}^3$ to be the variety (1.3). Consider the fibration $\pi: X \to Y$. There are precisely 4 codimension one points in $Y$ which produce reducible fibres. These are the four lines

$$D_{i,j} = \{y_i = y_j = 0\}, \quad \text{for } i \in \{0, 1\}, j \in \{2, 3\}. \quad (2.2)$$

Above each of these the fibre of $\pi$ is split by a quadratic extension, so that $\delta_{D_{i,j}}(\pi) = \frac{1}{2}$. It follows that $\Delta(\pi) = 4 \cdot \frac{1}{2} = 2$ in (2.1).

However, the Jacobian of $X$ is

$$J = \begin{pmatrix} x_0^2 & x_1^2 & x_2^2 & x_3^2 & 2y_0x_0 & 2y_1x_1 & 2y_2x_2 & 2y_3x_3 \\ y_1 & y_0 & -y_3 & -y_2 & 0 & 0 & 0 & 0 \end{pmatrix}.$$ 

Thus $X$ is singular, with singular locus supported on the union of subschemes

$$\Sigma_{i,j} = V(y_i, y_j, x_{i'}, x_{j'}, y_jx_i^2 + y_ix_j^2), \quad \text{for } \{i, i'\} = \{0, 1\} \text{ and } \{j, j'\} = \{2, 3\},$$

where we note that a choice of $i$ and $j$ uniquely determines $i'$ and $j'$. In fact, away from the intersection of two such subschemes, the singular locus $\Sigma$ is precisely the union of these four reduced subschemes. Let $\tilde{X} \to X$ be a desingularisation of $X$ and let $\tilde{\pi}$ be the composition with $\pi: X \to Y$. The main aim of this section is to prove that

$$\Delta(\tilde{\pi}) = 2,$$ 

as claimed in Theorem 1.2.

We begin by establishing the following geometric lemma that will be useful in our treatment.

**Lemma 2.1.** Let $C$ and $C'$ be two integral curves smooth and projective over a field $k$. Let us write $\kappa$ and $\kappa'$ for their function fields. Consider a proper morphism $\rho: W \to C \times C'$.

If $W \times C_{\kappa'}$ is smooth over $\kappa$ and $W \times C'_{\kappa'}$ is smooth over $\kappa'$, then the image of the singular locus of $W/k$ under $\rho$ has dimension 0.

**Proof.** Consider the composite morphism

$$W \to C \times_{k'} C' \to C,$$

which has a smooth generic fibre by assumption. By spreading out [9, Thm. 3.2.1], we see that $W_U \to U$ is smooth for a dense open subset $U \subset C$. We conclude that $W_U \to U \to \text{Spec } k$ is smooth. Similarly, we have a $V \subset C'$ such that $W_V \to V$ is smooth. We conclude that the image of the smooth locus of $W \to \text{Spec } k$ under $\rho$ contains both $U \times C'$ and $C \times V$. $\square$

It suffices to prove (2.3) for a single desingularisation.

**Definition 2.2.** Let $X' \to X$ be the blowup in the union of the closed subschemes $\Sigma_{i,j}$. Let $\tilde{X} \to X'$ be a fixed strong desingularisation.
We have seen that the image of the singular locus $\Sigma$ under $X \to Y$ is one-dimensional and supported on the union of lines $D_{i,j} \subset Y$ in (2.2). Let $\eta_{i,j}$ be the generic point of the line $D_{i,j}$. We will prove that the image of the singular locus of $X'$ under $\pi' \colon X' \to Y$ has lower dimension.

**Lemma 2.3.** The image of the singular locus $\Sigma'$ of $X'$ under $X' \to Y$ is (at most) zero-dimensional. The fibre $X''_{i,j}$ has four geometrically irreducible components, of which two make up the strict transform of $X''_{i,j}$ and two come from the exceptional divisor of the blowup $X' \to X$. These components are conjugated in pairs.

**Proof.** We identify $Y$ with $\mathbb{P}^1 \times \mathbb{P}^1$, where the projective lines have coordinates $u = \frac{y_i}{y_j} = \frac{y_{i'}}{y_{j'}}$ and $t = \frac{y_i}{y_{j'}} = \frac{y_{i'}}{y_j}$, respectively. By Lemma 2.1 and symmetry we only need to show that $X' \times_Y \mathbb{P}^1_{Q(u)}$ is smooth over $\mathbb{P}^1_{Q(u)}$. Again, by symmetry, we can restrict to the affine opens where $t \neq \infty$.

Since blowing up commutes with pulling back along flat morphisms we find that $X' \times_Y \mathbb{A}^1_{Q(u)}$ and the blowup of $X \times_Y \mathbb{A}^1_{Q(u)}$ in the pullback of $\Sigma$ are naturally isomorphic over $\mathbb{A}^1_{Q(u)}$. The three subschemes $\Sigma_{i',j}, \Sigma_{i,j'}$ and $\Sigma_{i',j'}$ pull back to the empty scheme under $X \times_Y \mathbb{A}^1_{Q(u)}$ to $X$ and it will therefore suffice to compute the blowup of $X \times_Y \mathbb{A}^1_{Q(u)}$ in the pullback of $\Sigma_{i,j}$.

We see that $X \times_Y \mathbb{A}^1_{Q(u)} \subset \mathbb{P}^3 \times_Q \mathbb{A}^1_{Q(u)}$ is given by $utx_i^2 + tx_j^2 + tx_{i'}^2 + u x_{j'}^2 = 0$. The pullback of $\Sigma_{i,j}$ equals $V(t, x_{i'}, x_{j'}, ux_i^2 + x_j^2)$, which is contained in the affine open given by $x_i \neq 0$. By abuse of notation we will use the same notation for the variables on this affine open.

We can compute the blowup $Z$ of $\{ut + x_{i'}^2 + tx_j^2 + ux_{j'}^2 = 0\} \subset \mathbb{A}^3 \times_Q \mathbb{A}^1_{Q(u)}$ in the subscheme $V(t, x_{i'}, x_{j'}, u + x_j^2)$ explicitly. We find

$$Z = \text{Proj } \mathbb{Q}(u)[t][x_{i'}, x_j, x_{j'}][A, B, C, D]/I$$

where the grading comes from $A$, $B$, $C$ and $D$, and $I$ is the ideal generated by the $2 \times 2$-minors of

$$\begin{pmatrix} A & B & C & D \\ t & x_{i'} & x_{j'} & u + x_j^2 \end{pmatrix},$$

together with the polynomials

$$ut + x_{i'}^2 + tx_j^2 + ux_{j'}^2, \quad \left(x_{i'}^2 + ux_{j'}^2\right)B + tx_{i'}D, \quad ux_j C + x_{i'} B + t D, \quad AD + B^2 + u C^2.$$  

One can now directly check that $Z \to \text{Spec } \mathbb{Q}(u)$ is indeed smooth. Thus the first statement follows from Lemma 2.1.

For the second part we note that the fibre of $Z \to \mathbb{A}^1_{Q(u)}$ over $t = 0$ is an affine open of $X''_{i,j}$. It consists of the strict transform of $X''_{i,j}$, which we already know to consist of two conjugate components. Any other component comes from the exceptional divisor, which we can compute from the equations above to be

$$\text{Proj } \mathbb{Q}(u)[x_j][A, B, C, D]/(u + x_j^2, AD + B^2 + u C^2).$$

Hence, $X''_{i,j}$ has four geometrical components which are conjugated in pairs. \qed
It follows from this result $\delta_{D_{i,j}}(\pi') = \frac{1}{2}$. Moreover, the singular locus of $X'$ does not meet the fibres $X'_{n_{i,j}}$ over the generic points $\eta_{i,j}$ of the $D_{i,j}$. By using a strong desingularisation, which was proven to exist by Hironaka [5], we can conclude that $\hat{X}_{n_{i,j}} \xrightarrow{\sim} X'_{n_{i,j}}$ over $Y$. It therefore follows that $\delta_{D_{i,j}}(\pi) = \frac{1}{2}$, which thereby completes the proof of (2.3), via (2.1).

3. Analytic input

In this section we prove Theorem 1.1. We note that the map $\hat{X} \rightarrow X$ is an isomorphism outside the locus of $y \in \mathbb{P}^3$ such that $y_0 \ldots y_3 = 0$. Let $|y| = \max_{0 \leq i \leq 3} |y_i|$ if $y \in \mathbb{P}^3(\mathbb{Q})$ is represented by a vector $y \in \mathbb{Z}_{\text{prim}}^4$. The anticanonical height function on $Y(\mathbb{Q})$ is then $H(y) = |y|^2$. Since there are $O(\sqrt{B})$ choices of $y \in \mathbb{P}^3(\mathbb{Q})$ of height at most $B$ for which $y_0 \ldots y_3 = 0$, it follows that

$$N_{\text{loc}}(\pi, B) = N_{\text{loc}}(\pi, B) + O(\sqrt{B}),$$

where $\pi: X \rightarrow Y$. For a given vector $y$, let $X_y \subset \mathbb{P}^3$ be the diagonal quadric

$$y_0x_0^2 + y_1x_1^2 + y_2x_2^2 + y_3x_3^2 = 0.$$

Then

$$N_{\text{loc}}(\pi, B) = \frac{1}{2} \# \left\{ y \in \mathbb{Z}_{\text{prim}}^4 : |y| \leq \sqrt{B}, \; X_y(A_\mathbb{Q}) \neq \emptyset \right\}.$$

In order to prove Theorem 1.1 it will suffice to prove that

$$B \ll N_{\text{loc}}(\pi, B) \ll B. \quad (3.1)$$

To see the lower bound, we take $y = (u, v, -u, -v)$, for coprime $u, v \in \mathbb{Z}$ such that $|u|, |v| \leq \sqrt{B}$. For such $y$ we obviously have $(1: 1: 1: 1) \in X_y(A_\mathbb{Q})$. It easily follows that $N_{\text{loc}}(\pi, B) \gg B$.

Remark 3.1. The lower bound for $N_{\text{loc}}(\pi, B)$ arises from points on the anti-diagonal line $D \subset Y$ defined by the equations $y_0 + y_2 = y_1 + y_3 = 0$. Consider the fibration $\pi_D: X_D \rightarrow D$. The reducible fibres occur precisely above the closed points $y_0 = 0$ and $y_1 = 0$ in $D \cong \mathbb{P}^1$. However, the fibres above these points are split and so it follows from (2.1) that $\Delta(\pi_D) = 0$. On the other hand $\rho_D = 1$ and so our lower bound is consistent with (1.3), which would predict that $N_{\text{loc}}(\pi_D, B) \sim cB$ for an appropriate constant $c > 0$.

It remains to prove the upper bound in (3.1). Let us first dispatch the contribution to $N_{\text{loc}}(\pi, B)$ from $y$ for which $y_0y_1y_2y_3 = 0$. But then at least two components of $y$ must vanish and the fibre over any such point has an obvious rational point. There are $\frac{4}{\zeta(2)}B + O(\sqrt{B})$ vectors $(a, b) \in \mathbb{Z}_{\text{prim}}^2$ with $|a|, |b| \leq B$.

We denote by $N^c(B)$ the contribution from $y$ for which $y_0y_1y_2y_3 \neq 0$. Then our work so far shows that

$$N_{\text{loc}}(\pi, B) = N^c(B) + \frac{8}{\zeta(2)}B + O(\sqrt{B}). \quad (3.2)$$
For each \( y \in \mathbb{Z}_{\neq 0}^4 \) with \( y_0y_1 = y_2y_3 \), there exist \( t = (t_0, t_1, t_2, t_3) \in \mathbb{Z}_{\neq 0}^4 \) such that
\[
y_0 = t_0t_2, \quad y_1 = t_1t_3, \quad y_2 = t_0t_3, \quad y_3 = t_1t_2.
\]
In this way it follows that
\[
N^o(B) \ll \# \left\{ t \in \mathbb{Z}_{\neq 0}^4 : |t_0t_2|, |t_0t_3|, |t_1t_2|, |t_1t_3| \leq \sqrt{B} \right\}.
\] (3.3)

We shall begin by studying a related counting function in which only square-free coefficients appear, before later deducing our final estimate for \( N^o(B) \).

### 3.1. Square-free coefficients.

For given \( U_0, \ldots, U_3 > 0 \), let
\[
\mathcal{U} = \{ u \in \mathbb{Z}^4 : U_i/2 < |u_i| \leq U_i \text{ for } 0 \leq i \leq 3 \}.
\]
In this section we shall be concerned with estimating
\[
N^*(U_0, \ldots, U_3) = \# \left\{ u \in \mathcal{U} : \mu^2(u_0) \cdots \mu^2(u_3) = 1 \right\},
\] (3.4)
where \( \mu \) is the Möbius function. We shall prove the upper bound
\[
N^*(U_0, \ldots, U_3) \ll \frac{U_0U_1U_2U_3}{(1 + \log U_{\text{min}})^2},
\] (3.5)
if \( U_{\text{min}} = \min\{U_0, \ldots, U_3\} \geq 1 \).

The upper bound (3.5) is trivial if \( 1 \leq U_{\text{min}} < 2 \) and so we may proceed under the assumption that \( U_{\text{min}} \geq 2 \). Our primary tool will be the following multidimensional form of the large sieve inequality, as worked out by Kowalski [6, Thm. 4.1].

**Lemma 3.1.** Let
\[
X = \{ (n_1, \ldots, n_r) \in \mathbb{Z}^r : -N_j \leq n_j \leq N_j \text{ for } 1 \leq j \leq r \}
\]
and let \( \Omega_p \subset \mathbb{F}_p^r \) for all \( p \). Then
\[
\# \{ x \in X : x \mod p \notin \Omega_p \forall p \leq L \} \ll \frac{\prod_{i=1}^r (N_i + L^2)}{F(L)},
\]
where
\[
F(L) = \sum_{n \leq L} \mu^2(n) \prod_{p | n} \frac{|\Omega_p|}{p^r - |\Omega_p|}.
\]
Define \( \Omega_p = \bigcup_{i=0}^3 \Omega_{p,i} \), where
\[
\begin{align*}
\Omega_{p,0} &= \{(r_0, r_1, r_2, r_3) \in \mathbb{F}_p^4 : r_0 = 0, \ r_1r_2r_3 \neq 0, \ -r_2r_3 \notin \mathbb{F}_p^{\times 2}\}, \\
\Omega_{p,1} &= \{(r_0, r_1, r_2, r_3) \in \mathbb{F}_p^4 : r_1 = 0, \ r_0r_2r_3 \neq 0, \ -r_2r_3 \notin \mathbb{F}_p^{\times 2}\}, \\
\Omega_{p,2} &= \{(r_0, r_1, r_2, r_3) \in \mathbb{F}_p^4 : r_2 = 0, \ r_0r_1r_3 \neq 0, \ -r_0r_1 \notin \mathbb{F}_p^{\times 2}\}, \\
\Omega_{p,3} &= \{(r_0, r_1, r_2, r_3) \in \mathbb{F}_p^4 : r_3 = 0, \ r_0r_1r_2 \neq 0, \ -r_0r_1 \notin \mathbb{F}_p^{\times 2}\}.
\end{align*}
\]
Let \( \mathbf{u} \) be a vector counted by \( N^*(U_0, \ldots, U_3) \). The following result shows that \( \mathbf{u} \mod p \not\in \Omega_p \) for any prime \( p \).

**Lemma 3.2.** Let \( p \) be a prime and let \( \mathbf{u} \in \mathbb{Z}^4 \) be such that \( \mu^2(u_0) \cdots \mu^2(u_3) = 1 \). If \( \mathbf{u} \mod p \in \Omega_p \) then \( X_{u_0u_2,u_1u_3,u_0u_3,u_1u_2}(\mathbb{Q}_p) = \emptyset \).

*Proof.* This is standard but we include a full proof for the sake of completeness. Assume without loss of generality that \( \mathbf{u} \mod p \in \Omega_{p,0} \). We suppose for a contradiction that there exists a solution \((x_0 : x_1 : x_2 : x_3)\) over \( \mathbb{Q}_p \) to the equation

\[
    u_0 u_2 x_0^2 + u_1 u_3 x_1^2 + u_0 u_3 x_2^2 + u_1 u_2 x_3^2 = 0.
\]

We can assume that \((x_0, x_1, x_2, x_3) \in \mathbb{Z}_p^4\) is primitive. Since \( p \mid u_0 \), it follows that \( u_1 u_3 x_1^2 + u_1 u_2 x_3^2 \equiv 0 \mod p \). But \( p \nmid u_1 \) and so

\[
    u_3 x_1^2 + u_2 x_3^2 \equiv 0 \mod p.
\]

But \(-u_2 u_3\) is a non-square modulo \( p \) and so \( x_1 \equiv x_3 \equiv 0 \mod p \). Write \( x_1 = px_1' \), \( x_3 = px_3' \) and \( u_0 = pu_0' \), where \( p \nmid u_0' \) since \( u_0 \) is square-free. Then it follows that

\[
    u_1' u_2 x_0^2 + pu_1 u_3 x_1' + u_0' u_3 x_2' + pu_1 u_2 x_3' = 0,
\]

whence \( u_2 x_0^2 + u_3 x_2' \equiv 0 \mod p \). But this implies that \( x_0 \equiv x_2 \equiv 0 \mod p \), which contradicts the primitivity of \((x_0, x_1, x_2, x_3)\). \( \square \)

In order to apply the large sieve we need to calculate the size of \( \Omega_{p,i} \). But for \( p > 2 \) we clearly have

\[
    |\Omega_{p,i}| = (p - 1)^2 \cdot \frac{p - 1}{2} = \frac{(p - 1)^3}{2},
\]

for \( 0 \leq i \leq 3 \). Hence it follows that

\[
    |\Omega_p| = 2p^3 \left( 1 - \frac{1}{p} \right)^3. \tag{3.6}
\]

Lemma 3.2 implies that

\[
    N^*(U_0, \ldots, U_3) \leq \# \{ \mathbf{u} \in \mathcal{W} : \mathbf{u} \mod p \not\in \Omega_p \ \forall p \leq L \},
\]

for any \( L \geq 1 \). On appealing to Lemma 3.1 we conclude that

\[
    N^*(U_0, \ldots, U_3) \ll \frac{\prod_{i=0}^3 (U_i + L^2)}{F(L)}, \tag{3.7}
\]

where

\[
    F(L) = \sum_{n \leq L} \mu^2(n) \prod_{p \mid n} \frac{|\Omega_p|}{p^4 - |\Omega_p|}.
\]

The following result gives a lower bound for this quantity.

**Lemma 3.3.** We have \( F(L) \gg (\log L)^2 \), for any \( L \geq 2 \).
Proof. It follows from (3.6) that
\[
\frac{\Omega_p}{p^4 - |\Omega_p|} = \frac{2p^3(1 - \frac{1}{p})^3}{p^4(1 - \frac{2}{p}(1 - \frac{1}{p})^3)} \geq \frac{2}{p} \left(1 - \frac{1}{p}\right)^3.
\]
Hence
\[
F(L) \geq \sum_{n \leq L} \frac{\mu^2(n)\tau(n)\phi^*(n)^3}{n},
\]
where \(\tau(n)\) is the divisor function and
\[
\phi^*(n) = \prod_{p|n} \left(1 - \frac{1}{p}\right).
\]
We now apply a standard result on the asymptotic evaluation of multiplicative arithmetic functions supported on square-free integers, as supplied by Friedlander and Iwaniec [3, Thm. A.5]. Let \(g(n) = \mu^2(n)\tau(n)\phi^*(n)^3/n\). Then
\[
\sum_{p \leq x} g(p) \log p = 2 \sum_{p \leq x} \frac{\log p}{p} + O(1) = 2 \log x + O(1),
\]
by Mertens’ theorem. Moreover, the conditions (A.16) and (A.17) of [3, Thm. A.5] are easily verified. Hence, on taking \(k = 2\) in this result, it follows that
\[
\sum_{n \leq L} g(n) \gg (\log L)^2,
\]
which thereby establishes the lemma. \(\Box\)

Let \(U_{\min} = \min\{U_0, \ldots, U_3\}\) and recall that we are assuming that \(U_{\min} \geq 2\). Inserting Lemma 3.3 into (3.7) and taking \(L = \sqrt{U_{\min}}\), we finally arrive at the upper bound (3.5).

3.2. Deduction of the general case. We now turn to the estimation of \(N^0(B)\), starting from (3.3). Pick \(T_0, \ldots, T_3\) such that
\[
T_0, \ldots, T_3 \geq 1 \quad \text{and} \quad T_0T_2, T_0T_3, T_1T_2, T_1T_3 \leq 4\sqrt{B}.
\]
We shall examine the contribution \(N(T_0, \ldots, T_3)\), say, from those \(t \in \mathbb{Z}_{\neq 0}^4\) for which \(T_i/2 < |t_i| \leq T_i\), for \(0 \leq i \leq 3\). Then
\[
N^0(B) \ll \sum_{T_0, \ldots, T_3} N(T_0, \ldots, T_3),
\]
where \(T_0, \ldots, T_3\) run over powers of 2 subject to the inequalities in (3.8).
Each non-zero integer $t_i$ admits a factorisation $u_im_i^2$ for square-free $u_i \in \mathbb{Z}$ and $m_i \in \mathbb{Z}_{>0}$. Moreover,

$$X_{t_0t_2,t_1t_3,t_0t_1t_2}(A_Q) \neq \emptyset \iff X_{u_0u_2,u_1u_3,u_0u_3,u_1u_2}(A_Q) \neq \emptyset.$$ 

Hence it follows that

$$N(T_0, \ldots, T_3) \leq \sum_{m_0 \leq \sqrt{T_0}} \cdots \sum_{m_3 \leq \sqrt{T_3}} N^*(\frac{T_0}{m_0^2}, \ldots, \frac{T_3}{m_3^2}),$$

in the notation of (3.4). On appealing to (3.5), we deduce that

$$N(T_0, \ldots, T_3) \ll \sum_{m_0 \leq \sqrt{T_0}} \cdots \sum_{m_3 \leq \sqrt{T_3}} \sum_{j \in \{0, \ldots, 3\}} \frac{1}{m^2(1 + \log(T_j/m^2))^2} \ll \frac{T_0T_1T_2T_3}{(1 + \log T_{\min})^2},$$

where $T_{\min} = \min\{T_0, \ldots, T_3\}$. Returning to (3.9), we finally conclude that

$$N^0(B) \ll \sum_{T_0, \ldots, T_3} \frac{T_0T_1T_2T_3}{(1 + \log T_{\min})^2},$$

where $T_0, \ldots, T_3$ run over powers of 2 subject to the inequalities in (3.8).

By symmetry, we can suppose without loss of generality that $T_0 \leq T_1$ and $T_2 \leq T_3$. Summing first over $T_1 \leq 4\sqrt{B}/T_3$, we see that

$$\sum_{T_0, \ldots, T_3} \frac{T_0T_1T_2T_3}{(1 + \log T_{\min})^2} \ll \sqrt{B} \sum_{T_0, T_2, T_3} \frac{T_0T_2}{(1 + \log \min\{T_0, T_2\})^2}.$$ 

Suppose first that $T_0 \leq T_2$. Then we sum over $T_2 \leq T_3$ and then $T_3 \leq 4\sqrt{B}/T_0$ to get

$$\sum_{T_0, \ldots, T_3} \frac{T_0T_1T_2T_3}{(1 + \log T_{\min})^2} \ll \sqrt{B} \sum_{T_0, T_3} \frac{T_0T_3}{(1 + \log T_0)^2} \ll B \sum_{T_0} \frac{1}{(1 + \log T_0)^2},$$

where the sum is over $T_0 = 2^j$ with $1 \leq 2^j \leq 2B^{1/4}$. This sum is convergent, whence this case contributes $O(B)$. Alternatively, if $T_0 > T_2$, then we sum over
$T_0 \leq 4\sqrt{B}/T_3$ and then $T_3 \geq T_2$ to get
\[
\sum_{T_0,\ldots,T_3} \frac{T_0 T_1 T_2 T_3}{(1 + \log T_{\text{min}})^2} \ll \sqrt{B} \sum_{T_0,T_2,T_3} \frac{T_0 T_2}{(1 + \log T_2)^2}
\ll B \sum_{T_2,T_3} \frac{T_2}{T_3(1 + \log T_2)^2}
\ll B \sum_{T_2} \frac{1}{(1 + \log T_2)^2}.
\]
This also makes the satisfactory contribution $O(B)$.
Hence we have shown that $N^<(B) = O(B)$. Once inserted into (3.2), this therefore completes the proof of the upper bound in (3.1).

\section*{References}
[1] T.D. Browning and D. Loughran, Sieving rational points on varieties. \textit{Trans. Amer. Math. Soc.} \textbf{371} (2019), 5757–5785.
[2] J. Franke, Y.I. Manin and Y. Tschinkel, Rational points of bounded height on Fano varieties. \textit{Invent. Math.} \textbf{95} (1989), 421–435.
[3] J.B. Friedlander and H. Iwaniec, \textit{Opera de cribro}. American Math. Soc., 2010.
[4] J.B. Friedlander and H. Iwaniec, Ternary quadratic forms with rational zeros. \textit{J. Théor. Nombres Bordeaux} \textbf{22} (2010), 97–113.
[5] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero. I & II. \textit{Annals Math.} \textbf{79} (1964), 109–203 & 205–326.
[6] E. Kowalski, \textit{The large sieve and its applications}. Cambridge University Press, 2009.
[7] D. Loughran and A. Smeets, Fibrations with few rational points. \textit{Geom. Funct. Anal.} \textbf{26} (2016), 1449–1482.
[8] E. Peyre, Liberté et accumulation. \textit{Documenta Math.} \textbf{22} (2017), 1615–1659.
[9] B. Poonen, \textit{Rational Points on Varieties}. Volume 186 of \textit{Graduate Studies in Mathematics}. American Mathematical Society, Providence, RI, 2017.
[10] B. Poonen and J.F. Voloch, Random Diophantine equations. \textit{Arithmetic of higher-dimensional algebraic varieties} (Palo Alto, CA, 2002), 175–184, \textit{Progr. Math.} \textbf{226}, Birkhäuser, 2004.