Gravity’s Rainbow

João Magueijo† and Lee Smolin‡

† Theoretical Physics, The Blackett Laboratory, Imperial College, Prince Consort Road, London SW7 2BZ, UK
‡ Perimeter Institute for Theoretical Physics, Waterloo, Canada N2J 2W9 and Department of Physics, University of Waterloo

Abstract. Non-linear special relativity (or doubly special relativity) is a simple framework for encoding properties of flat quantum space-time. In this paper we show how this formalism may be generalized to incorporate curvature (leading to what might be called “doubly general relativity”). We first propose a dual to non-linear realizations of relativity in momentum space, and show that for such a dual the space-time invariant is an energy-dependent metric. This leads to an energy-dependent connection and curvature, and a simple modification to Einstein’s equations. We then examine solutions to these equations. We find the counterpart to the cosmological metric, and show how cosmologies based upon our theory of gravity may solve the “horizon problem”. We discuss the Schwarzschild solution, examining the conditions for which the horizon is energy dependent. We finally find the weak field limit.
1. Introduction

It is generally believed that the geometry of spacetime is fundamentally described by a quantum theory, in which the smooth manifold and metric of general relativity is replaced by a quantum mechanical description. Whatever the nature of that description, it is believed that the Planck energy \( E_{Pl} = \sqrt{\hbar c^5/G} \) plays the role of a threshold separating the classical description from the quantum description. When probed at energies above \( E_{Pl} \) we expect that a radically new picture of spacetime will be needed.

Several candidates for that description are under study, including loop quantum gravity [1, 2], string theory [3, 4], Lorentzian dynamical triangulations [5], non-commutative geometry [6], condensed matter analogues [7], etc. A key issue that arises in such studies is the nature of the transition between the fundamental quantum description and the effective low energy description in terms of classical general relativity [16, 17, 18]. This is needed not only theoretically. It has recently become clear that astronomical and cosmological observations make it possible to probe the leading effects in \( l_{Pl} = 1/E_{Pl} \) around the classical limit (from now on we shall set \( \hbar = c = 1 \)). These include possible modifications of the energy-momentum relations

\[
E^2 - p^2 = m^2
\]  
\[
E^2 = p^2 + m^2 + \alpha l_{Pl} E^3 + \ldots \tag{1}
\]

For instance to leading order they could take the form,

\[
E^2 = p^2 + m^2 + \alpha l_{Pl} E^3 + \ldots \tag{2}
\]

where \( \alpha \) is a dimensionless constant of order unity. The effects of such terms may, on a variety of assumptions, be observed, or even—on some assumptions—ruled out in observations including tests of thresholds for ultra high energy cosmic rays [8, 9, 11, 12, 13, 14, 15] and Tev photons [10], a possible energy dependence of the speed of light observable in gamma ray bursts [11], as well as tests involving synchrotron radiation [19, 20, 22] and nuclear physics experiments [24]. Related effects may also be detectable in the near future in CMB observations [25].

Key to the understanding of such effects is the fate of global Lorentz symmetry in the classical limit of the quantum theory of gravity. While many physicists expect that Lorentz invariance remains unbroken, this may be utopic. For one thing, Lorentz invariance, unlike rotational invariance, involves an unbounded parameter. No matter how well it has been tested, up to some boost parameter \( \gamma \), there is always an infinite range to go. More seriously, Lorentz invariance cannot be a fundamental symmetry of the quantum theory of gravity, for it plays no fundamental role in classical general relativity. From the point of view of general relativity, global Lorentz invariance is only an accidental symmetry of a particular solution to the field equations. Thus, whether it is broken or modified, global Lorentz invariance cannot be an exact symmetry of the theory; rather, it is an approximate symmetry that emerges at low energies from the quantum theory of gravity.

There are then four main possibilities for how Lorentz invariance may be realized when effects to leading order in \( l_{Pl}E \) are included, where \( E \) is the energy of some quanta
observed by an inertial observer:

(i) Lorentz invariance and the relativity of inertial frames are maintained exactly, without modification, but the fundamental matter degrees of freedom may be non-pointlike. This is assumed in the construction of perturbative string theory (see, however \cite{26}) and other perturbative approaches to quantum gravity. Given the observational possibilities just mentioned, this will likely be the first assumption involved in the construction of string theory to be tested experimentally.

(ii) There is a preferred frame, observable in effects at leading order in $l_{Pl}E$, which break Lorentz invariance \cite{11,12,13} (for more extreme examples see \cite{33,34}). This generally leads to a phenomenology in which there are corrections to the energy-momentum relations of the form (11). Other relations, such as energy-momentum conservation, remain as before, so long as they are computed in the preferred frame. It has been suggested that this scenario is, on certain assumptions regarding dynamics, already ruled out by present observations \cite{19,22,23}.

(iii) Lorentz invariance is spontaneously broken by a vector field taking on a vacuum expectation value, leading to a (possibly locally varying) preferred frame. This option has been explored in \cite{27,28}.

(iv) The relativity of inertial frames is maintained, but the transformation laws now act on momentum space non-linearly, picking up new terms in $l_{Pl}E$ \cite{16,17,18,15}. As a consequence the invariant norm on momentum space is no longer bilinear, leading to corrections to the usual energy-momentum relations (11). This proposal is variously known as non-linear, deformed, or doubly special relativity (DSR) \footnote{It has been argued (e.g. \cite{21}) that the non-linearities in DSR can be removed by making a redefinition of energy and momenta. This has been refuted by two developments: the example of 2+1 gravity \cite{29} and the realization that DSR’s phase space has an invariant curvature \cite{31}. Redefinitions of energy and momentum are also unphysical for certain position space formulations \cite{32}.}.

As a consequence of the modification proposed in \cite{18,15}, the classical description of spacetime, may become an invariant, in the sense that all inertial observers agree on whether a particle has more or less than this energy. This resolves an otherwise troubling issue: how a threshold between a quantum and a classical description can depend on the motion of an observer.

This paper is a contribution to the last proposal. We examine the question of how the modification of special relativity proposed in \cite{18,15} can be extended to general relativity. Our inquiry is meant to hold in the sense of the leading corrections to a limit in which the classical spacetime description is recovered for low energy quanta from a purely quantum spacetime geometry. Thus we are concerned with the effects of the propagation of quanta with energies smaller than $E_{Pl}$ but whose wavelengths are much shorter than the local radius of curvature.
Our main result is that we do find that there is a sensible modification of the principles and equations of general relativity that makes sense in this regime. This is characterized by the feature that the geometry of spacetime becomes energy dependent. Thus, quanta of different energies see different classical geometries. These classical geometries share the same inertial frames, and so the equivalence principle can be maintained, in a modified form proposed below. But measurements of distance and time now pick up a new dependence on the energy of the quanta used in the measurements.

This conclusion is in part motivated by [32]. Deformed special relativity was initially proposed in momentum space. With loss of linearity the dual position space no longer mimicks momentum space. One possible reconstruction of position space [32] leads to spacetime positions subject to energy dependent transformation laws. Concomitantly, the metric become energy dependent.

One consequence of this new picture is that the velocity of light and other massless quanta naturally becomes energy dependent. Thus, when we turn to the study of cosmological models we find naturally the variable speed of light cosmologies (VSL), previously proposed in [27, 33] (see [34] for a review of the field). It has been speculated that there may be a connection between modified or doubly special relativity and variable speed of light cosmologies [35, 36, 37]. What we show here is that this connection does indeed follow from a natural extension to general relativity.

2. Deformed special relativity and the rainbow metric

Deformed or doubly special relativity is a class of theories that implement a modified set of principles of special relativity. These are

(i) The relativity of inertial frames.
(ii) In the limit \( E/E_{Pl} \to 0 \) the speed of a photon or massless quanta goes to a universal constant, \( c \), which is the same for all inertial observers.
(iii) \( E_{Pl} \) in the above condition is also a universal constant, and is the same for all inertial observers.

As a result the invariant of energy and momentum is modified to

\[
E^2 f^2(E/E_{Pl}) - p \cdot pg^2(E/E_{Pl}) = m^2 \tag{3}
\]

This can be realized by the action of a non-linear map from momentum space to itself, denoted, \( U : \mathcal{P} \to \mathcal{P} \) given by

\[
U \cdot (E, p_i) = (U_0, U_i) = \left( f\left( \frac{E}{E_{Pl}} \right) E, g\left( \frac{E}{E_{Pl}} \right) p_i \right) \tag{4}
\]

which implies that momentum space has a non-linear norm, given by

\[
|p|^2 = \eta^{ab} U_a(p) U_b(p) \tag{5}
\]

This norm is preserved by a non-linear realization of the Lorentz group, given by

\[
\tilde{L}_b^a = U^{-1} \cdot L_b^a \cdot U \tag{6}
\]
where $L^b_a$ are the usual generators. Some examples of theories of this type are described in [13, 15]. The presence of a singularity in $U$ marks the emergence of an invariant energy scale, as required by principle (iii) above [15].

These theories are typically formulated in momentum space, and to discuss how general relativity might be set up, one must first discuss how to identify a dual space representing positions [32]. Since the momentum transformation laws are no longer linear, the definition of a dual space is non-trivial. A number of different answers have been proposed to the question of what is the modified spacetime geometry consistent with deformed or doubly special relativity. Among the possible answers are non-commutative geometry, for example, $\kappa$ deformed Minkowski spacetime [38, 39].

We take the view here that this is the wrong question to ask and that, instead, there is no single classical spacetime geometry when effects of order $l_P E$ are taken into account. Instead, we propose that classical spacetime is to leading order in $l_P$ represented by a one parameter family of metrics, parameterized by the ratio $E/E_P$. That is, just as the properties of a material may depend on the energy of a phonon in it, we take the view that the geometry of spacetime may depend on the energy of a particle moving in it. Thus, spacetime geometry has an effective description; in the language of the renormalization group, geometry “runs.” Hence there is no single spacetime dual to momentum space; the dual to momentum space is the energy dependent family of metrics.

We stress that the argument $E$ in the metric $g_{ab}(E)$ is not the energy of the spacetime. Instead it is the scale at which the geometry of spacetime is probed. That is, if a freely falling inertial observer uses the motion of a particle, or system of particles, to measure the geometry of the spacetime, $E$ is the total energy of that particle or system of particles, as measured by that observer. The construction should be such that the metric is co-variant with regards to the dual of the non-linear representation of the Lorentz group [6] encoding deformed special relativity. That such covariance may be achieved is proved by the second example below.

Another way of describing these properties is by saying that in the absence of gravity spacetime has an energy-dependent geometry in the sense that particles of energy $E$ move in a geometry given by an energy dependent set of orthonormal frame fields

$$e_0 = f^{-1}(E/E_P)\tilde{e}_0, \quad e_i = g^{-1}(E/E_P)\tilde{e}_i$$

where the tilde quantities refer to the energy independent frame fields that specify the geometry probed by low energy quanta. The metric given by

$$g(E) = \eta^{ab}e_a \otimes e_b$$

is flat for all $E$. It can be considered to be a one parameter family of flat metrics, parameterized by $E$. The metrics share the same set of inertial frames, but due to the scalings, generally they do not share all their geodesics; instead geodesics are generally energy dependent. This is equivalent to saying that the energy momentum relations are modified, so they are no longer quadratic. We refer to such a one parameter energy-
dependent family of metrics as a single "rainbow metric", and the metric above is the flat rainbow metric.

2.1. Two well-known motivations: string theory and loop quantum gravity

The statement made in the previous Section is a postulate, and as such it is not derived from anything. But, even if it is not derived from anything, there are examples in which such scale dependent geometries are present in quantum theories of gravity. One well studied example comes from conformal field theory and string theory. The target space metric $G_{AB}$ in the standard string action

$$I = \int \sqrt{\hbar} G_{AB} \partial_i \Phi^A \partial_j \Phi^B$$

is in fact energy dependent, and has been treated as such since early works of Friedan and Polyakov. That is, if one regards the theory as fundamentally two dimensional, the target space metric $G^{AB}$ is just a matrix of coupling constants, and as any coupling constants in quantum field theory, these run, and become dependent on the ratio of an energy scale $E$ to the cut off scale $M$.

This may seem nonsensical if one regards the geometry of spacetime as primary. However, in the currently well studied theories of quantum gravity, including loop quantum gravity and string theory, the classical geometry of spacetime is not primary. Rather, it emerges as a low energy coarse grained description of a very different quantum geometry. In loop quantum gravity this is completely explicit. The classical metric is as much an emergent quantity as the thermodynamic variables in ordinary statistical physics. As such, the spacetime metric must depend on the ratio of a cutoff scale, $M$, to the scale probed, $E$; hence $g_{ab}(E/M)$. The usual classical metric is only defined as the limit

$$\lim_{E \to 0} g_{ab}(E/M) = g_{ab}^{\text{classical}}$$

The energy dependence of $g_{ab}$ is then not just consistent with the present viewpoint in quantum gravity, it is required by it.

2.2. Another motivation

The rainbow metric is closely related to the method developed in [32] for constructing position space in deformed special relativity. In this approach one requires that free field theories in flat space-time have plane wave solutions (examples of such field theories are presented in [32] and elsewhere), even though the 4-momentum they carry satisfies deformed dispersion relations [33]. For this to be possible the contraction between position and momentum (providing the phase for such waves) must remain linear (so that the waves remain “plane”). That is:

$$dx^a p_a = dx^0 p_0 + dx^i p_i$$

If momentum transforms non-linearly (from a non-linear action derived from the generators Eq. [6]) then this requires that the $dx^a$ tranformation be energy dependent,
Gravity's Rainbow

as explained in [32]. It is not difficult to show, that for a $U$ of the form $[4]$ the space-time dual has invariant:

$$ds^2 = -\frac{(dx^0)^2}{f^2} + \frac{(dx^i)^2}{g^2}$$

Thus, the dual space $dx^a$ is endowed with an energy dependent quadratic invariant, that is, an energy-dependent metric.

This example further elucidates the meaning of $E$ in $g_{ab}(E)$. If a given observer sees a particle (or plane wave, or wave-packet) with energy $E$, then he concludes that this particle feels the metric $g_{ab}(E)$. If this particle’s energy is $E' \neq E$ for a different observer, then the latter will assign to the particle a different metric, $g_{ab}(E')$. It may also happen that the first observer sees a different particle in the same place but with a different energy - and accordingly assign to it a different metric. So not only different observers may see a given particle being affected by different metrics, but the same observer may assign different metrics to different particles moving in the same region at the same time. This is required by covariance, once we allow for a non-linear representation of the Lorentz group in momentum space.

This argument, valid in flat space-time, carries over locally to curved space-times using the equivalence principle as detailed in the next Section.

3. The deformed equivalence principle

Having defined the position space dual to deformed relativity in momentum space, we are now ready to consider general relativity. We start by stating the deformed equivalence and correspondence principles:

- **Modified equivalence principle**
  Consider a region of spacetime in which the radius of curvature $R$ is much larger than $E_{Pl}^{-1}$. Then freely falling observers, making measurements of particles and fields with energies $E$ which satisfy $1/R << E << E_{Pl}$ observe the laws of physics to be, to first order in $1/R$, the same as in modified special relativity. Hence freely falling observers to first order in $1/R$ can describe themselves as being inertial observers in rainbow flat spacetime describe by $[8]$. In particular, they use a family of energy dependent orthonormal frames given locally by $[8]$. We insist on the restriction $1/R << E$ because otherwise we may have to take into account terms in $R(\partial p/p)$ coming from the fact that the wavelength of a quanta is not much smaller than the radius of curvature. The upper limit $E << E_{Pl}$ comes from the expectation that the geometry of quantum spacetime does not have a smooth, classical description for energies of Planck scales and higher.

- **Correspondence principle**
  In the limit $E/E_{Pl} \rightarrow 0$ ordinary classical general relativity is recovered.
The modified equivalence principle implies that spacetime is described by a one parameter family of metrics given in terms of a one parameter family of orthonormal frame fields

\[ g(E) = \eta^{ab} e_a(E) \otimes e_b(E) \]  

(13)

where the energy dependence of the frame fields is given by

\[ e_0(E) = \frac{1}{f(E/E_{Pl})} \tilde{e}_0, \quad e_i(E) = \frac{1}{g(E/E_{Pl})} \tilde{e}_i \]  

(14)

The correspondence principle then requires that

\[ \lim_{E/E_{Pl} \to 0} f(E/E_{Pl}) \to 1, \]  

and likewise for \( g(E/E_{Pl}) \).

This then leads to a one parameter family of connections \( \nabla(E) \), and curvature tensors \( R(E)_{\mu\nu}^\lambda \) defined by the usual formulas. One defines also a one parameter family of energy-momentum tensors \( T_{\mu\nu}(E) \) and the Einstein equations are replaced by a one parameter family of equations

\[ G_{\mu\nu}(E) = 8\pi G(E) T_{\mu\nu}(E) + g_{\mu\nu} \Lambda(E) \]  

(16)

where \( G(E) \) is an energy dependent Newton’s constant, defined so that \( G(0) \) is the physical Newton’s constant. The energy dependence of \( G(E) \) reflects the expectation that the effective gravitational coupling will depend on the energy scale and will satisfy a renormalization group equation. Similarly we expect an energy dependent cosmological constant \( \Lambda(E) \).

As in the usual theory, these equations must satisfy a number of consistency conditions: Bianchi’s identities. This leads to local energy conservation and the geodesic equation.

4. Modified FRW solutions

We illustrate the basic principles of this new theory with the simplest cosmological solutions. We begin with flat FRW solutions, whose family of metrics are given according to (13) and the usual symmetry requirements, by,

\[ ds^2(E) = -\frac{dt^2}{f^2(E)} + \frac{a^2(t)}{g^2(E)} \gamma_{ij}dx^i dx^j \]  

(17)

where \( \gamma_{ij} \) represents the spatially homogeneous and isotropic metric of a sphere (positive curvature \( K = 1 \)), pseudo-sphere (with negative curvature \( K = -1 \)), or euclidean space (\( K = 0 \), so that \( \gamma_{ij} = \delta_{ij} \)).

The only non-vanishing components of the associated connection are:

\[ \Gamma^0_{ij} = \left( \frac{f}{g} \right)^2 a \dot{\gamma}_{ij} \]  

(18)

\[ \Gamma^i_{0j} = \delta^i_j \frac{\dot{a}}{a} \]  

(19)

\[ \Gamma^i_{jk} = \tilde{\Gamma}^i_{jk} \]  

(20)
where $\tilde{\Gamma}^i_{jk}$ are the standard spatial connection coefficients. This leads to Riemann tensor components:

\[
R^0_{i0j} = \left(\frac{f}{g}\right)^2 a\ddot{a}\gamma_{ij}
\]

\[
R^i_{00j} = \delta^i_j \frac{\ddot{a}}{a}
\]

\[
R^i_{jkm} = \tilde{R}^i_{jkm} + \left(\frac{f}{g}\right)^2 \ddot{a}^2 (\delta^i_k \gamma_{jm} - \delta^i_m \gamma_{jk})
\]

with all other components zero or trivially derived from these ones using the symmetries of the Riemann tensor. The non-trivial Ricci tensor components are:

\[
R_{00} = -3\frac{\ddot{a}}{a}
\]

\[
R_{ij} = \gamma_{ij} \left(\left(\frac{f}{g}\right)^2 (\ddot{a}a + 2\dot{a}^2) + 2K\right)
\]

The energy momentum tensor has a perfect fluid form,

\[
T_{\mu\nu} = \rho u_\mu u_\nu + p(g_{\mu\nu} + u_\mu u_\nu)
\]

but now $u_\mu$ depends on $E$ through $u_\mu = (f^{-1}(E), 0, 0, 0)$, since it is a unit vector, $u_\mu u_\nu g^{\mu\nu} = -1$, and $g$ depends on $E$. The energy density and pressure, $\rho$ and $p$, do not depend on $E$, but they do depend on the temperature, and through it on $a$. However the equation of state, $p = w\rho$, is modified due to the deformed energy momentum relations, as discussed extensively in [35, 36].

The Einstein’s equations become,

\[
\left(\frac{\ddot{a}}{a}\right)^2 = \frac{8\pi}{3} G(E) \frac{\rho}{f^2} - \frac{K}{a^2} \left(\frac{g}{f}\right)^2 + \frac{\Lambda(E)}{3}
\]

\[
\frac{\ddot{a}}{a} = -\frac{4\pi}{3} G(E) \rho + 3p f^2 + \frac{\Lambda(E)}{3}
\]

These may be combined to produce a conservation equation:

\[
\dot{\rho} + 3\frac{\ddot{a}}{a} (\rho + p) = 0
\]

It is simple to investigate the consequences in some simple cases. For example, assume that $G$ and $\Lambda$ have in fact the same energy dependence,

\[
G(E) = h^2(E) G, \quad \Lambda(E) = h^2(E) \Lambda.
\]

The effect of the energy dependence can be mocked up by introducing an energy dependent time coordinate

\[
\tau(E) = \frac{h(E)}{f(E)} t
\]

Then the effect is that a high energy quanta of energy $E$, high enough to make the functions $h$ and $f$ differ from unity, sees itself to be in a universe which is older or
younger than a low energy quanta. In this way the horizon problem may be solved, or worsened, depending on the forms of the functions.

Indeed this theory of gravity opens up the doors for a new type of solution to the horizon problem. Recall that the comoving horizon is given by

\[ r_h = \frac{cH^{-1}}{a} \]  

and that the horizon problem is that this quantity increases in time. Two possible solutions, rendering \( r_h \) a decreasing function of time, are accelerated expansion (\( \ddot{a} > 0 \)) and a decreasing speed of light (\( \dot{c} < 0 \)). Our theory may realize either of these solutions \[36, 35\]. As shown in \[36\], radiation subject to deformed dispersion relations may satisfy \( \rho + 3p < 0 \), leading to inflation. Also, since the speed of light may be obtained by setting \( ds^2 = 0 \) in \[12\] we find

\[ c = \frac{dx}{dt} = \frac{g}{f} \]  

(note that we are setting the low energy value of \( c_0 = 1 \); also \( dx^0 = c_0 dt \)). If \( f \neq g \), we may then realize the VSL scenario \[35\].

But more interestingly, there is a third solution. It is possible to solve the horizon problem with decelerated expansion and constant speed of light simply because the metric is energy dependent. Consider the various factors in \[32\]. The Hubble parameter, \( H = \dot{a}/a \), is energy independent because a factor of \( g \) appears in both numerator and denominator. The speed of light \( c(E) = g/f \), and \( a \) is replaced by \( a/g \) (see \[17\]). Hence the comoving horizon is

\[ r_h = \frac{c(E)H^{-1}}{a/g} \]  

If \( g(E) \rightarrow 0 \) fast enough at early times the comoving region containing the whole observable universe nowadays may therefore be much smaller than expected, solving the horizon problem. The conversion factor between comoving and proper distances in an expanding universe is energy dependent, and this is enough to bring the whole observed universe together at high energies.

An interesting case is the dispersion relation proposed in \[18\]:

\[ f = g = \frac{1}{1 + \lambda E} \]  

It does not produce a varying \( c \), and yet it may solve the horizon problem.

We shall return to this cosmological scenario in a future publication. Our theory of gravity justifies adhoc assumptions used in \[35, 37\]. Thus we may convert our theory into a scenario for structure formation based on thermal fluctuations \[37\].

5. The modified Schwarzschild solution

We begin with the general spherically symmetric metric. Given that the metric is energy dependent, when we specialize the coordinates to spherically symmetric form,
we may write the metric either in energy dependent coordinates, or energy independent coordinates. The energy independent coordinates will be denoted $\tilde{r}, \tilde{t}, \theta, \phi$. The time and radial coordinates can also absorb energy dependence, leading to energy dependent coordinates, which will be denoted $r(E), t(E)$. The angular coordinates $\theta, \phi$ are always energy independent. The energy independent coordinates will coincide with the energy dependent coordinates in the limit $E/E_{Pl} \to 0$.

In terms of energy independent coordinates the most general form for a spherically symmetric metric is

$$ds^2 = -\tilde{F}(\tilde{r}) d\tilde{t}^2 + \tilde{H}(\tilde{r}) \left( \frac{\tilde{r}^2}{g^2(E)} \right) dr^2 + \tilde{r}^2 \left( \frac{1}{g^2(E)} \right) d\Omega^2$$ \hspace{1cm} (36)

$\tilde{r}$ is hence the area coordinate, defined so it is proportional to the square root of a physical area measured at that radius, as measured by observers in the limit $E/E_{Pl} \to 0$. Now, because the metric is energy dependent, an observer using quanta of energy $E$ will measure a sphere at constant $\tilde{r}$ to have a different area than an observer using probes of energy $E = 0$. We see that the area coordinate appropriate to measurements made by quanta of energy $E$ is

$$r(E) = \frac{\tilde{r}}{g(E)}$$ \hspace{1cm} (37)

We can then define new energy dependent functions,

$$F(r(E), E) = \tilde{F}(\tilde{r}), \quad H(r(E), E) = \tilde{H}(\tilde{r})$$ \hspace{1cm} (38)

It is also natural to introduce an energy dependent time coordinate

$$t(E) = \frac{\tilde{t}}{f(E)}$$ \hspace{1cm} (39)

The metric then takes the form,

$$ds^2 = -F(r(E), E) dt(E)^2 + H(r(E), E) dr(E)^2 + r(E)^2 d\Omega^2$$ \hspace{1cm} (40)

This is then a one parameter family of metrics, each in the standard form. By Birkhoff’s theorem, we must have, for each $E$

$$F(r(E), E) = H^{-1}(r(E), E) = \left( 1 - \frac{C(E)}{r(E)} \right)$$ \hspace{1cm} (41)

where the constant of integration $C(E)$ is now energy dependent.

However, we determine the energy dependence of the constant of integration, because we recall that by (38) all the energy dependence in the energy-independent coordinates must be in the functions $f$ and $g$ in the original form (36). Hence we have,

$$F(r(E), E) = \left( 1 - \frac{C(E)g(E)}{\tilde{r}} \right) = \tilde{F}(\tilde{r})$$ \hspace{1cm} (42)

Thus, using the fact that at $E = 0$ we must have $C = 2G(0)M$,

$$C(E) = \frac{2G(0)M}{g(E)}$$ \hspace{1cm} (43)
where $M$ denotes the energy independent mass. Thus, in energy independent
coordinates the modified Schwarzschild metric must take the form,
\[
ds^2 = -\left(1 - \frac{2G(0)M}{\tilde{r}}\right)dt^2 + \frac{1}{g^2(E) \left(1 - \frac{2G(0)M}{\tilde{r}}\right)}d\tilde{r}^2 + \frac{\tilde{r}^2}{g^2(E)}d\Omega^2 \tag{44}
\]
In particular we see that the position of the horizon, in fixed, energy independent
coordinates, is fixed at the usual place. However, the area of the horizon is then energy
dependent. The implications of this result for black hole thermodynamics, and photon
dynamics around the horizon, are currently being investigated.

### 6. The Newtonian limit

To check the Newtonian and linear limits we expand around the metric of modified or
doubly special relativity to find,
\[
g_{\mu\nu} = \eta(E)_{\mu\nu} + h(E)_{\mu\nu} \tag{45}
\]
where the modified Minkowski metric is $\eta(E)_{\mu\nu} = \text{diag}(-f^{-2}, g^{-2}, g^{-2}, g^{-2})$. We define
the usual trace reversed coordinate
\[
h_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \tag{46}
\]
where $h = \eta^{\mu\nu}h_{\mu\nu}$ is easily seen to be energy independent. Fixing the standard gauge
conditions $\eta^{\mu\nu}\partial_{\mu}h_{\nu\lambda} = 0$ we find that the linearized Einstein equations reduce to
\[
\eta^{\mu\nu}(E)\partial_{\mu}\partial_{\nu}h_{00} = -16\pi G(E)T_{00} \tag{47}
\]
Let us write,
\[
h_{00} = \frac{\Phi}{f^2(E)} \tag{48}
\]
where $\Phi$ is the usual Newtonian gravitational potential. We note that this form is
required by the correspondence principle.

Using again the modified perfect fluid form of the energy momentum tensor we find
that, for static fields,
\[
\delta^{ij}\partial_i\partial_j\Phi = -16\pi G(E)g(E)^2\rho \tag{49}
\]
The energy dependence $g(E)$ corresponds to the fact that the euclidean coordinates
in which the spatial metric takes the form $g_{ij} = \text{diag}(1, 1, 1)$ are energy dependent. Alternatively, the coordinate differentials dual to $p_i = g(E)\partial_i$ are $e^i(E) = dx^i/g(E)$. If we transform to energy independent coordinates
\[
\tilde{x}^i = x^i g(E) \tag{50}
\]
Then we find in these coordinates,
\[
\delta^{ij}\partial_i\partial_j\Phi = -16\pi G(E)\rho \tag{51}
\]
However, the Newtonian limit corresponds to the limit $c \to \infty$. In this same limit $E_{pl} = \sqrt{\hbar c^5/G} \to \infty$. Hence, by the correspondence principle $G(E/E_{pl}) \to G(0) = G$. Hence we do find, in energy independent coordinates, the correct Newtonian limit,

$$\delta^{ij}\partial_i\partial_j \Phi = -16\pi G \rho$$

proving the weak field consistency of the theory.

7. Conclusions

We have proposed a way of incorporating spacetime curvature – and thus gravity – into non-linear realizations of relativity (also known as DSR). The gravity theory proposed in this paper is closely related to one view of how to dualize doubly special relativity. For a number of historical reasons (the theory is motivated by cosmic ray kinematics) non-linear relativity was first defined in momentum space. With loss of linearity duals no longer mimic each other, so that recovering position space in these theories is highly non-trivial.

One possible strategy is to impose linearity to the contraction between position and momentum space. This is equivalent to requiring that field theories still have plane wave solutions \cite{32}. This requirement is strong enough to fully fix the transformation laws of position space, which must still be linear, but acquire an energy dependence. Consequently a flat quadratic metric may still be defined, but the energy dependence of the new Lorentz transformations propagates into the metric. It is this “rainbow” metric that we gauged in this paper, using the usual techniques of differential geometry.

We explored some solutions to this theory of gravity, namely the cosmological, black hole, and weak field solutions. We found that cosmological distances, in an expanding universe, become energy dependent. Thus the physical distance associated with a given comoving distance depends on the energy scale at which it is measured. Seen in another way, the age of the universe is energy dependent. We used this fact to show how the horizon problem may be solved without inflation or a varying speed of light. We also considered the counterpart of the Schwarzschild solution, and found that the area of the horizon is energy dependent. This result may have important implications for black hole thermodynamics.

Other theories of gravity would follow from different realizations of position space. If the Lorentz group is non-linearly realized in position space \cite{32}, gravity follows from gauging a symmetry which is non-linearly realized. If position space is non-commutative, then yet another theory of gravity would emerge (however see \cite{40} for a possible obstruction). These alternative paths should be pursued and compared with the theory proposed in this paper.

Acknowledgements

We would like to thank G. Amelino-Camelia, J-L. Lehners, F. Markopoulou and J. Kowalski-Gilkman for useful discussions. JM thanks the Perimeter Institute for
hospitability and LS the Jesse Phillips Foundation for generous support which made this work possible.

References

[1] C. Rovelli, Living Rev. Rel. 1 (1998) 1.
[2] S. Carlip, Rept.Prog.Phys. 64 (2001) 885.
[3] J. Polchinski, hep-th/9611050
[4] S. Forste, hep-th/0110055
[5] J. Ambjorn and R. Loll, Nucl. Phys. B536 (1998) 407 hep-th/9805108; J. Ambjorn, J. Jurkiewicz and R. Loll, Phys. Rev. Lett. 85 (2000) 924 [hep-th/0002050]; Nucl. Phys. B610 (2001) 347 hep-th/0105267; R. Loll, Nucl. Phys. B (Proc. Suppl.) 94 (2001) 96 hep-th/0011194; J. Ambjorn, J. Jurkiewicz and R. Loll, Phys. Rev. D64 (2001) 044011 hep-th/0011276.
[6] Alain Connes, Noncommutative geometry, Academic Press, Inc., San Diego, CA, 1994.
[7] R. B. Laughlin, gr-qc/0302028; Shou-Cheng Zhang, hep-th/0210162; Shou-Cheng Zhang, Jiangping Hu, cond-mat/0110572; Science 294 (2001) 823; G.E. Volovik, gr-qc/0005091; Physics Reports, 351, 195-348 (2001).
[8] P. Biermann and G. Sigl, Lect.Notes Phys. 576 (2001) 1-26.
[9] M. Takeda et al, Astrophy.s I. 522 (1999) 225; Phys.Rev.Lett. 81 (1998) 1163-1166.
[10] D. Finkbeiner, M. Davis and D. Schlegel, Astro. Ph. 544, 81 (2000).
[11] G. Amelino-Camelia et al, Int.J.Mod.Phys.A12:607-624,1997; G. Amelino-Camelia et al Nature 393:763-765,1998.
[12] J. Ellis et al, Astrophys.J.535, 139-151, 2000.
[13] J. Ellis, N.E. Mavromatos and D. Nanopoulos, Phys. Rev. D63, 124025, 2001; ibidem astro-ph/0108295.
[14] G. Amelino-Camelia and T. Piran, Phys.Rev. D64 (2001) 036005.
[15] J. Magueijo and L. Smolin, Phys. Rev. D67, 044017, 2003.
[16] G. Amelino-Camelia, Int. J. Mod. Phys. D11, 35, 2002; gr-qc/0012051; G. Amelino-Camelia, Phys. Lett. B510, 255-263, 2001.
[17] J. Kowalski-Glikman, Phys. Lett. A286, 391-394, 2001 hep-th/0102098; N.R. Bruno, G. Amelino-Camelia, J. Kowalski-Glikman, Phys. Lett. B522, 133-138, 2001.
[18] J. Magueijo and L. Smolin, Phys.Rev.Lett. 88 (190403) 2002.
[19] T. Jacobson, S. Liberati and D. Mattingly, astro-ph/0212190.
[20] G. Amelino-Camelia, gr-qc/0212087.
[21] D. V. Ahluwalia-Khalilova, gr-qc/0212128; gr-qc/0207004.
[22] T. Jacobson, S. Liberati and D. Mattingly, gr-qc/0303001.
[23] R. Myers and M. Pospelov, hep-ph/0301124.
[24] D. Sudarsky, L. Urrutia, H. Vucetich, Phys. Rev. Lett. 89 (2002) 231301.
[25] L. Mersini, M. Bastero-Gil, P. Kanti, Phys.Rev. D64 (2001) 043508; hep-ph/0101210.
[26] J. Magueijo and L. Smolin, hep-th/0401087.
[27] J. Moffat, Int. J. Mod. Phys. D 2, 351 (1993).
[28] J. W. Moffat, hep-th/0211167.
[29] L. Freidel, J. Kowalski-Glikman, L. Smolin, hep-th/0307085.
[30] J. Kowalski-Glikman, Phys.Lett. B547:291-296,2002.
[31] J. Kowalski-Glikman and S. Nowak, Class.Quant.Grav. 20 (2003) 4799-4816.
[32] D. Kimberley, J. Magueijo and J. Medeiros, gr-qc/0303067.
[33] A. Albrecht and J. Magueijo, Phys. Rev. D 59, 043516 (1999).
[34] J. Magueijo, Rept. Prog. Phys. 66, 2025 (2003); astro-ph/0305457.
[35] S. Alexander and J. Magueijo, hep-th/0104093.
[36] S. Alexander, R. Brandenberger and J. Magueijo, hep-th/0108190.
[37] J. Magueijo, L. Pogosian, Phys.Rev. D67, 043518 (2003).
[38] M. Daszkiewick, K. Imilkowska and J. Kowalski-Gilkman, hep-th/0304027.
[39] S. Mignemi, gr-qc/0304029.
[40] E. Hawkins, math.QA/0211203.