Einstein-Born-Infeld on Taub-NUT Spacetime in $2k+2$ Dimensions

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Abstract

We wish to construct solutions of Taub-NUT spacetime in Einstein-Born-Infeld gravity in even dimensions. Since Born-Infeld theory is a nonlinear electrodynamics theory, it leads to nonlinear differential equations. However a proper analytical solution was not obtained, we try to solve it numerically (by the Runge-Kutta method) with initial conditions coinciding with those of our previous work in Einstein-Maxwell gravity. We solve equations for 4, 6 and 8 dimensions and do data fitting by the least-squares method. For $N = l = b = 1$, the metric turns to the NUT solution only in 8 dimensions, but in 4 and 6 dimensions the spacetime does not have any Nut solution.

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I. INTRODUCTION

In the recent years, a great deal of attention has been focuse d on the BornInfeld theory. This theory emerges from string theory with a parameter $b$ with dimension of length. The covariant action of this field is

$$I = \frac{1}{4\pi b^2} \int d^4x \sqrt{-g} \left\{ 1 - \sqrt{1 - \frac{1}{2} b^2 F_{\mu\nu} F^{\mu\nu} - \frac{1}{16} b^4 (\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma})^2} \right\}$$

(1)

where $F_{\mu\nu}$ is derived from a vector potential $A_{\mu}$ as $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$. Since this theory is a nonlinear version of electromagnetism, the self energy of the electron in this field is finite. This fact was a first motivation for Born and Infeld who formulated this theory in 1934 [1]. In 1935, Hoffman [2] joined general relativity to Born-Infeld electrodynamics to obtain a spherically symmetric solution representing the gravitational field of a charged object.

The properties of Taub-Nut spacetime are described in Refs. [3, 4, 5]. The spherically symmetric solutions and rotating solutions in (Einstein/Gause-Bonnet/Lovelock)-Born-Infeld gravity with or without cosmological constant and dilaton field have been considered by many authors [6, 7, 8, 9, 10, 11]. In this paper, we are interested in NUT solutions of Einstein-Born-Infeld (EBI) gravity. The thermodynamic properties of this spacetime have been investigated in Einstein-Maxwell and Gause-Bonnet gravity [5, 12].

The outline of my paper is as follows. A brief review of the field equations of EBI gravity and field equations of taub-NUT spacetime in EBI gravity are described in Section II. Also in this section we obtain non linear equations to be solved numerically in Section III. At last, the paper ends with some concluding remarks.

II. FIELD EQUATIONS

The gravitational action for Einstein gravity in $n + 1$ dimension in the presence of BI field with a cosmological constant $\Lambda = -n(n - 1)/2l^2$ is

$$I_G = -\frac{1}{16\pi} \int d^{n+1}x \sqrt{-g} [R - 2\Lambda + L(F)] + \frac{1}{8\pi} \int_{\partial M} d^n x \sqrt{\gamma} K(\gamma).$$

(2)

The first term is the Einstein-Hilbert action and second term is the Gibbons-Hawking boundary term which is chosen such that the variational principle is well-defined. The manifold $\mathcal{M}$ has the metric $g_{\mu\nu}$. $K$ is trace of the extrinsic curvature $K^{\mu\nu}$ of any boundaries $\partial \mathcal{M}$ of
the manifold $\mathcal{M}$ with the induced metric $\gamma_{ij}$. In Eq. (2) $L(F)$ is the Lagrangian of BI field given as

$$L(F) = \frac{4}{b^2} \left( 1 - \sqrt{1 + \frac{b^2 F^2}{2}} \right).$$

In these equations $F^2 = F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu}$ is the electromagnetic tensor field and $b$ is the BI parameter. As one can see in the limit $b \to 0$, $L(F)$ reduces to the standard Maxwell Lagrangian $-F^2$. As $b \to \infty$, $L(F) \to 0$.

The Field equations are obtained by varying the action (2) with respect to the gauge field $A_\mu$ and the gravitational field $g_{\mu\nu}$ as

$$\partial_\mu \left( \frac{\sqrt{-g} F^{\mu\nu}}{\sqrt{1 + \frac{b^2 F^2}{2}}} \right) = 0,$$

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{1}{2\pi} \left( g_{\mu\nu} L(F) + \frac{4F_{\mu\lambda}F_{\nu}^\lambda}{\sqrt{1 + \frac{b^2 F^2}{2}}} \right),$$

where $G_{\mu\nu}$ is the Einstein tensor.

The Euclidean section of the $(2k + 2)$ dimensional Taub-NUT spacetime can be written as

$$ds^2 = F(r)(d\tau + NA)^2 + F^{-1}(r)dr^2 + (r^2 - N^2)d\Xi_B$$

where $\tau$ is the coordinate of the fibers $S^1$ and $A$ is the Kähler form of the base space $B$, $N$ is the NUT charge and $F(r)$ is a function of $r$. The metric $d\Xi_B$ is a $2k$-dimensional base space Einstein-Kähler manifold $B$.

Here, we consider only the cases where all the factor spaces of $B$ have positive curvature. Thus, the base space $B$ may be the product of 2-sphere $S^2$ and/or $\mathbb{C}P^k$ spaces for all values of $k$. For completeness, we give the 1-forms and the metrics of these factor spaces. The 1-forms and the metrics of $S^2$ are

$$A_i = 2 \cos \theta_i d\phi_i,$$

$$d\Omega_i^2 = d\theta_i^2 + \sin^2 \theta_i d\phi_i^2,$$

and those of $\mathbb{C}P^k$ are

$$A_k = 2(k + 1) \sin^2 \xi_k (d\psi_k + \frac{1}{2k}A_{k-1}),$$

$$d\Sigma_k^2 = 2(k + 1) \left\{ d\xi_k^2 + \sin^2 \xi_k \cos^2 \xi_k (d\psi_k + \frac{1}{2k}A_{k-1})^2 + \frac{1}{2k} \sin^2 \xi_k d\Sigma_{k-1}^2 \right\}.$$
where $A_{k-1}$ is the Kähler potential of $\mathbb{C}P^{k-1}$. In Eqs. (8) $\xi_k$ and $\psi_k$ are the extra coordinates corresponding to $\mathbb{C}P^k$ with respect to $\mathbb{C}P^{k-1}$. The metric $\mathbb{C}P^k$ is normalized so that the Ricci tensor is equal to the metric, $R_{\mu\nu} = g_{\mu\nu}$. The 1-form and the metric of $\mathbb{C}P^1$ is

$$A_1 = 4\sin^2 \xi_1 d\psi_1$$

$$d\Sigma_1^2 = 4\left(d\xi_1^2 + \sin^2 \xi_1 \cos^2 \xi_1 d\psi_1^2\right)$$

and those of $\mathbb{C}P^k$ can be constructed through the use of Eqs. (8). One can see that $\mathbb{C}P^1 = S^2$ by changing parameter $\theta = 2\xi$.

The gauge potential has the form

$$A = g(r)(d\tau + N A),$$

where $g(r)$ is a function of $r$. The electromagnetic field equation (11) for the metric (6) with the vector potential (11) in $(2k + 2)$ dimensions is

$$[1 + 4kb^2 h(r)^2 N^2] (r^2 - N^2) h''(r) + 2krb^2 (r^2 - N^2) h'(r)^3 + 4kb^2 h(r)(r^2 - N^2)(3r^2 - 2N^2) h'(r)^2 + 8k \left[2r^2 + (2k + 1)N^2\right] b^2 h(r)^3 + 2r \left\{4k[(k - 1)N^2 + 3r^2] b^2 h(r)^2 + (k + 2)\right\} h'(r) + 2(2k + 1) h(r) = 0$$

(12)

where $h(r) = g(r)(r^2 - N^2)$ and prime denotes a derivative with respect to $r$. To find the function $F(r)$ in the metric (6), one may use any components of Eq. (5). The simplest equation is the $tt$ component of these equations which can be written as

$$krb^2 p(r) F'(r) + kb^2 \frac{[(2k - 1)r^2 + N^2]}{(r^2 - N^2)} F(r) + 2(r^2 - N^2)^2$$

$$- \left\{2 + k(2k + 1) \frac{b^2}{l^2} (r^2 - N^2) + kb^2\right\} p(r) + 8kN^2 b^2 g(r)^2 = 0$$

(13)

where $p(r)$ is a function of $r$ as

$$p(r) = \sqrt{[g'(r)^2b^2 + 1](r^2 - N^2)^2 + 4kb^2 N^2 g(r)^2}. \quad (14)$$

Eqs. (12) and (13) are nonlinear. These equations must be solved to find $F(r)$. An analytic solution of these equations has not been found and at last we try to solve them numerically.

III. NUMERICAL SOLUTIONS

The solutions of Eqs. (12 and 13) describe NUT solutions, if...
(I) $F(r = N) = 0$,
(II) $g(r = N) = 0$,
(III) $g'(r = N) = S(k)$,
(IV) $F(r)$ should have no positive roots at $r > N$.

The first condition comes from the fact that the metric (6) has a singularity at $r = N$ or in other word all extra dimensions should collapse to zero at a fixed point set of $\partial / \partial \tau$. The second one come from the regularity of vector potential at $r = N$, the third one from our previous work [5] for Einstein-Maxwell gravity, and the fourth condition means that $r = N$ should be the outer horizon. $S(k)$ is the value of $g'(N)$ in any dimension.

By using these conditions, we solve Eqs. (12 and 13) numerically by the following steps

1. Solve Eq. (12) with the Runge-Kutta method with two initial conditions (conditions (II) and (III) for NUT solutions).

2. By data fitting in the least squares method, an appropriate function for $g(r)$ is obtained. The function that we use for fitting is $g(r) = \sum_n a_n r^n \times f(r)$ where $f(r)$ is the corresponding function is obtained analytically in the previous work on Einstein-Maxwell (EM) gravity in this spacetime [5].

3. By substituting the data of $g(r)$ and $g'(r)$ into Eq. (14), $p(r)$ is calculated. Then data fitting is done as the previous step.

4. Insert $g(r)$ and $p(r)$ in Eq. (13). By solving this differential equation numerically, with initial condition, conditions and data fitting, $F(r)$ is obtained.

We solve these equations for $N = l = b = 1$. The values of $S_k$ are

$$S_k = \begin{cases} 
0.5 & \text{for } k = 1 \text{ (4 Dim.)} \\
-1 & \text{for } k = 2 \text{ (6 Dim.)} \\
1.25 & \text{for } k = 3 \text{ (8 Dim.)}
\end{cases}$$

In figures 1-5, the function $F(r)$ is plotted for 4, 6 and 8 dimensions. as we see in these figures, in 4D, there are three roots in the region $r \geq N$ at $r = 1, 1.12$ and 2.265. In 6D, there are two roots in the region $r \geq N$ at $r = 1$, and 2.318. Since the fourth condition is not satisfied, therefore, the metric is no NUT solution in these dimensions. But in 8D, as we see in Fig. 5, there are no roots greater than $N$. Therefore the metric has a Nut solution.
IV. CONCLUSION

The Born-Infeld theory on Taub-NUT spacetimes were investigated. Two field equations were driven after varying the gravitational action. We fined two nonlinear differential equations for Taub-NUT spacetime in $2k + 2$ dimensions. A Numerical solution for these equations was suggested. By having conditions for NUT solutions and for $N = l = b = 1$, we saw that there is no NUT solution for 4 and 6 dimensions. In 8 dimensions, there exist NUT solutions. As one can see, These solutions are checked only for $N = b = 1$. In the future, if these equations are solved for all values of ‘$N$’ and ‘$b$’, we may obtain regions of
FIG. 3: $F(r)$ in 6 dimensions ($0 < r < 1.15$)

FIG. 4: $F(r)$ in 6 dimensions ($0 < r < 2.3$)

FIG. 5: $F(r)$ in 8 dimensions ($0 < r < 1.15$)
NUT solutions in any dimensions.

[1] M. Born and L. Infeld, Proc. Roy. Soc. Lond. A 144, 425 (1934).
[2] B. Hoffmann, Phys. Rev. 47, 877 (1935).
[3] S. W. Hawking, C. J. Hunter and D. N. Page, Phys. Rev. D 59, 044033 (1999).
[4] R. B. Mann, Phys. Rev. D 60, 104047 (1999).
[5] M. H. Dehghani and A. Khodam-Mohammadi; Phys. Rev. D 73, 124039 (2006).
[6] M. Demianski, Found. Phys. 16, (1986) 187; D. Wiltshire, Phys. Rev. D 38, 2445 (1988).
[7] R. G. Cai, D. W. Pang and A. Wang, Phys. Rev. D 70, 124034 (2004).
[8] T. K. Dey, Phys. Lett. B 595, 484 (2004).
[9] M. H. Dehghani, G. H. Bordbar and M. Shamirzaei, Phys. Rev. D 74, 064023 (2006).
[10] M. H. Dehghani, S. H. Hendi, A. Sheykhi, H. Rastegar Sedehi, JCAP 0702, 020 (2007).
[11] M. H. Dehghani, A. Sheykhi, S. H. Hendi, Phys. Lett. B 659, 476 (2008).
[12] A. Khodam-Mohammadi and M. Monshizadeh; Phys. Rev. D 79, 044002 (2009).