SPECTRAL ASYMPTOTICS AT THRESHOLDS FOR A DIRAC-TYPE OPERATOR ON $\mathbb{Z}^2$

PABLO MIRANDA, DANIEL PARRA, AND GEORGI RAIKOV†

Abstract. In this article, we provide the spectral analysis of a Dirac-type operator on $\mathbb{Z}^2$ by describing the behavior of the spectral shift function associated with a sign–definite trace–class perturbation by a multiplication operator. We prove that it remains bounded outside a single threshold and obtain its main asymptotic term in the unbounded case. Interestingly, we show that the constant in the main asymptotic term encodes the interaction between a flat band and whole non–constant bands. The strategy used is the reduction of the spectral shift function to the eigenvalue counting function of some compact operator which can be studied as a toroidal pseudo–differential operator.

1. Introduction and main results

We start by giving a streamlined presentation of our operators acting on $\mathbb{Z}^2$. First we define a graph structure by setting the set of vertices $V = \mathbb{Z}^2$ and the set of oriented edges $A = \{(x, y) \in \mathbb{Z}^2 \times \mathbb{Z}^2 : y = x \pm \delta_i\}$, where $\{\delta_1, \delta_2\}$ form the canonical basis of $\mathbb{Z}^2$. We note an edge in $A$ as $e = (x, y)$ and its transpose by $\bar{e} = (y, x)$. Given a vertex $x$, we set $A_x = \{e \in A : e = (x, y)\}$.

Let us denote by $X = (V, A)$ this graph structure defined on $\mathbb{Z}^2$, and consider the vector spaces of $0$–cochains $C^0(X)$ and $1$–cochains $C^1(X)$ given by:

$$C^0(X) := \{f : V \to \mathbb{C}\}; \quad C^1(X) := \{f : A \to \mathbb{C} \mid f(e) = -f(\bar{e})\}.$$ 

We will denote by $C(X)$ the direct sum $C^0(X) \oplus C^1(X)$ and will refer to an $f \in C(X)$ as a cochain. This suggests the notation $C^c(X)$ for the space of cochains that are finitely supported.

For $f, g \in C(X)$ we take the inner product

$$(f, g) := \sum_{x \in V} f(x)\overline{g(x)} + \frac{1}{2} \sum_{e \in A} f(e)\overline{g(e)}.$$ \hspace{1cm} (1)

The Hilbert space $l^2(X)$ is defined as the closure of $C_c(X)$ in the norm induced by (1). It coincides with $\{f \in C(X) \mid \|f\| := (f, f)^{\frac{1}{2}} < \infty\}$.

The coboundary operator (or difference operator) $d : C^0(X) \to C^1(X)$ is defined by:

$$df(e) := f(y) - f(x), \text{ for } e = (x, y).$$

Its formal adjoint $d^* : C^1(X) \to C^0(X)$ is given by the finite sum

$$d^* f(x) = -\sum_{e \in A_x} f(e).$$

Note that since $A_x = \{(x, x + \delta_1), (x, x + \delta_2), (x, x - \delta_1), (x, x - \delta_2)\}$, the sum has only four terms.

We can now define the operator on $l^2(X)$

$$H_0 := \begin{pmatrix} m & d^* \\ d & -m \end{pmatrix},$$
where $m$ is just the multiplication operator by the constant $m \geq 0$. It is not difficult to see that

$$H_0^2 = \begin{pmatrix} \Delta_V + m^2 & 0 \\ 0 & \Delta_A + m^2 \end{pmatrix},$$

where $\Delta_V$ and $\Delta_A$ are the discrete Laplacians in edges and vertices, respectively. So it is natural to say that $H_0$ is a Dirac-type operator [Eck45; AT15; Par17].

The operator $H_0$ is bounded self-adjoint, and is analytically fibered over $\mathbb{T}^2$ (see Section 2.1 for details). The analysis of its band functions shows that the spectrum of $H_0$ is

$$\sigma(H_0) = \sigma_{ac}(H_0) = [-\sqrt{m^2 + 8}, -m] \bigcup [m, \sqrt{m^2 + 8}].$$

Moreover, there exist a discrete set $\mathcal{T}$ of thresholds in the spectrum of $H_0$, given by

$$\mathcal{T} = \{ \pm m, \pm \sqrt{m^2 + 4}, \pm \sqrt{m^2 + 8} \}.$$  

In this article we understand the set of thresholds as those point in the spectrum of $H_0$ where the Mourre estimates does not hold and hence we do not have a priori a Limiting Absorption Principle (LAP) [Par17].

We will see that there are three types of thresholds and one could expect qualitatively different properties of the spectrum near each type: for $m > 0$ the points $\{ \pm \sqrt{m^2 + 8}, m \}$ are thresholds of elliptic type, $\{ \pm \sqrt{m^2 + 4} \}$ are of hyperbolic type, and $-m$ is elliptic and an eigenvalue of infinite multiplicity. The case $m = 0$ is particular: there is no finite gap in the spectrum and the threshold at zero is not of elliptic type. It is called a Dirac point (see Section 2.2).

In this article we will study spectral properties of $H_0 \pm V$, where $V \geq 0$ is a potential that is defined both on vertices and edges, and satisfies $V(e) = V(R)$. Use the notation

$$H_\pm := H_0 \pm V.$$  

Since the Dirac delta functions on vertices and edges lie in $l^2(X)$, we have that the range of $V$ coincides with its set of eigenvalues. From this it is clear that the trace class norm of $V$ is

$$\|V\|_1 = \sum_{x \in V} |V(x)| + \frac{1}{2} \sum_{e \in A} |V(e)|.$$  

When $V$ is of trace class, there exists a unique $\xi(\lambda; H_\pm, H_0)$ in $L^1(\mathbb{R})$ that satisfies the trace formula

$$\text{Tr}(f(H_\pm) - f(H_0)) = \int_{\mathbb{R}} d\lambda \xi(\lambda; H_\pm, H_0) f'(\lambda),$$

for all $f \in C_c^\infty(\mathbb{R})$ (see the original work [Kre53], or the monograph [Yaf92, Ch. 8]). The function $\xi(\lambda; H_\pm, H_0)$ is called the Spectral Shift Function (SSF) for the pair $(H_\pm, H_0)$. This function can be defined in an abstract setting and is an important object in the analysis of linear operators. For instance, it is related to the scattering matrix $S(\lambda; H_\pm, H_0)$ by the Birman-Krein formula [BK62; Ya92]

$$\det S(\lambda; H_\pm, H_0) = e^{-2\pi i \xi(\lambda; H_\pm, H_0)}, \quad a.e. \lambda \in \sigma_{ac}(H_0).$$

It also gives the number of discrete eigenvalues of $H_\pm$ outside the essential spectrum (see Remark 1.6 for more details).

Our main goal in this article is to describe the SSF near the thresholds in the spectrum of $H_0$. These types of results have been obtained in the literature for different models. In particular for the Laplacian in $\mathbb{R}^d$ perturbed by a decaying electric potential. These results are related to Levinson’s theorem (see [Yaf10; Rob99] and references therein). More recently, results for 2D and 3D magnetic Schrödinger and Dirac operators were obtained [FR04; Tie11; BM18; BR20]. In the discrete case, some trace formulas have been obtained for periodic graphs in settings very close to ours [KS22; IK12].
One of the novelties of this work is related to the rich structure of the spectrum of $H_0$, which is due to the remarkable form of the band functions. In particular, we have the interaction of a flat band with a non-constant band function at the maximal point of the last one, as well as the existence of saddle points which implies the appearance of hyperbolic thresholds. To the best of our knowledge, this is the first article where these situations are studied.

1.1. Main results. Let $\mu \in \mathbb{Z}^2$, and define the edges $e_1 = ((0,0),(1,0))$, $e_2 = ((0,0),(0,1))$. Take the following real-valued functions on $\mathbb{Z}^2$:

$$v_1(\mu) := V(\mu) \quad ; \quad v_2(\mu) := V(\mu e_1) \quad \text{and} \quad v_3(\mu) := V(\mu e_2),$$

where for $e = (x,y)$ we have the natural action $\mu e = (\mu + x, \mu + y)$.

Whenever is convenient, we will keep the notation $x, y$ for elements of $\mathbb{Z}^2$ seen as vertices and $\mu, \nu$ for elements of $\mathbb{Z}^2$ seen as elements of the group acting on $X$.

To study the most basic properties of the SSF it will suffice to assume that

$$\sum_{\mu \in \mathbb{Z}^2, \langle \mu \rangle > N} |v_j(\mu)| = O(N^{-2\beta_j}), \quad N \to \infty,$$

with $0 < \beta_j \leq 1$ for $j = 1, 2, 3$, where $\langle \mu \rangle = (1 + |\mu|^2)^{1/2}$. For instance, it is easy to see that this condition ensures the existence of the SSF, as it implies that $V$ is summable. Moreover we have the following theorem:

**Theorem 1.1.** Let us suppose that the perturbation $V$ is positive and each $v_j$ satisfies (6) with $\beta_j > 0$. Then, on any compact set $K \subset \mathbb{R}\setminus\{-m\}$,

$$\sup_{\lambda \in K} |\xi(\lambda; H_\pm, H_0)| < \infty,$$

i.e. the SSF is bounded away from $-m$.

**Remark 1.2.** This theorem implies in particular that the SSF $\xi(\lambda; H_\pm, H_0)$ is bounded at the hyperbolic thresholds, and the study of the SSF at hyperbolic thresholds seems to have not been carried out before in any model. However, there is a growing interest in the understanding of the spectral properties of discrete Hamiltonians inside their continuous spectrum, where these kind of thresholds appear [IJ19; IJ21].

One consequence of Theorem 1.1 is that the only possible point of unbounded growth of the SSF is $-m$. In our second main theorem we describe the explicit asymptotic behavior of the SSF at that point, for potentials $V$ that shows a power-like decay. More precisely, for $\gamma > 0$ and for any multi-index $\alpha$ we assume

$$|D^\alpha v_j(\mu)| \leq C_\alpha \langle \mu \rangle^{-\gamma - \rho|\alpha|}, \quad j = 2, 3,$$

where $D_{\mu_j} v(\mu) := v(\mu + \delta_j) - v(\mu)$, and $D^\alpha := D_{\mu_1}^{\alpha_1} \ldots D_{\mu_d}^{\alpha_d}$. Moreover,

$$\lim_{|\mu| \to \infty} |\mu|^{-\gamma} v_l(\mu) = \Gamma_l; \quad l = 2, 3,$$

exist and at least one $\Gamma_j \in \mathbb{R}$ is not equal to 0.

We define the matrix

$$\Gamma := \begin{pmatrix} \Gamma_2 & 0 \\ 0 & \Gamma_3 \end{pmatrix},$$

and the constant

$$C := \pi \int_{\mathbb{T}^2} \text{Tr} \left( \left( A(\xi)^* \Gamma A(\xi) \right)^{2/\gamma} \right) d\xi,$$
where
\[ A := \begin{pmatrix} b(\xi)r(\xi)^{-1/2} & 0 \\ a(\xi)r(\xi)^{-1/2} & 0 \end{pmatrix}, \]
and \( a, b, r = a^2 + b^2 \) are given by the analytic fibration of \( H_0 \) (see Proposition 2.1). Notice that in condition (8) we do not ask to \( v_2 \) and \( v_3 \) to have the same decaying rate. For instance, if \( v_2 \) decays faster that \( v_3 \), and hence \( \Gamma_2 = 0 \), the constant \( C \) will not depend on \( v_2 \).

**Theorem 1.3.** Let \( V \geq 0 \) and suppose that \( v_1 \) satisfies (6) with \( \beta_1 > 0 \), and that \( v_2 \) and \( v_3 \) satisfy (7) and (8), with \( \gamma > 2 \). Then,
\[
\xi(\lambda; H_{-}, H_0) = \begin{cases} -C|\lambda + m|^{-2/\gamma}(1 + o(1)) & \text{if } \lambda \uparrow -m, \\ O(1) & \text{if } \lambda \downarrow -m, \end{cases}
\]
and
\[
\xi(\lambda; H_{+}, H_0) = \begin{cases} O(|\ln(|\lambda + m|)|) & \text{if } \lambda \uparrow -m, \\ C|\lambda + m|^{-2/\gamma}(1 + o(1)) & \text{if } \lambda \downarrow -m. \end{cases}
\]

**Remark 1.4.** The asymptotic order \(-2/\gamma\) of the SSF is clearly determined by the perturbation \( V \). This may be interpreted as the contribution of the constant band at \(-m\). The contribution of the non-constant band is instead encoded in the constant \( C \). This constant contains an explicit interaction between the perturbation and the whole non-constant band functions (see Proposition 2.1). As far as we know, this is the first time that such a behavior has been observed.

**Remark 1.5.** Theorem 1.3 is proved by a reduction of the SSF to an eigenvalue counting function, and by using a general theorem on the spectral asymptotics for integral operators with toroidal symbol. The statement and proof of this last theorem appears in section Section 6. The result is, to the best of our knowledge, new and may be of independent interest. The proof uses an appropriate Cwikel estimate (Lemma 6.3).

**Remark 1.6.** Since \( V \) is compact, \( \sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0) \). Thus, when \( m > 0 \) we have that \((-m, m)\) is a gap in the essential spectrum of \( H_{\pm} \). Then, for \( \lambda \in (-m, m) \) we can consider the function
\[ N^\pm(\lambda) = \text{Rank} \mathbf{1}_{\Omega}(\lambda, m)(H_{\pm}) \]
with \( \mathbf{1}_{\Omega} \) being the characteristic function over the Borel set \( \Omega \). Clearly, this function counts the number of discrete eigenvalues of \( H_{\pm} \) on the interval \((-m + \lambda, m)\). By Theorem 1.1 these functions are well defined, as there are no accumulation of eigenvalues at \( m \) from below. Moreover, for \( \lambda \in (-m, m) \)
\[ \xi(\lambda; H_{\pm}, H_0) = \pm N^\pm(\lambda) + O(1), \]
(see [Pus98]). Similarly, we can write analogous equations for the eigenvalues of \( H_{\pm} \) on the intervals \((-\infty, -\sqrt{m^2 + 8})\) and \((\sqrt{m^2 + 8}, \infty)\). In consequence, Theorem 1.1 and Theorem 1.3 gives the asymptotic distribution of the discrete eigenvalues of \( H_{\pm} \) (under conditions (6) and (8)). In order to reduce the length of the article and to maintain the symmetry of the results, we plan to consider \( N^\pm(\lambda) \) for \( \gamma \in [0, 2] \) elsewhere.

1.2. **Notations and structure of the article.** Throughout the article we will denote by \( C \) different constants that appear which are non-essential to our purposes. Whenever \( \pm \) appears it refers to two independent statements that are valid independently. Also for non-negative real functions \( f, g \) we note
\[ f \asymp g \]
if there exists two non-negative constants \( C_1, C_2 \) such that
We denote by
\[ C_1 f \leq g \leq C_2 f. \]

The rest of the paper is organized as follows. In Section 2 we recall the integral representation and describe the spectrum of \( H_0 \). Then, in Section 3, we obtain an explicit expression for the LAP that will be the base of our further investigations. In Section 4 we study the operators that appear in the SSF representation given in Section 3. In Section 5 we prove our results for hyperbolic thresholds. In Section 6 we obtain some abstract results concerning the eigenvalue asymptotics for a class of integral operators. Finally, in Section 7 we put together these results to obtain the proofs of our main theorems in the parabolic and Dirac point case.

2. Analysis of the unperturbed operator \( H_0 \)

2.1. Integral decomposition. In this section we construct an unitary operator \( \mathcal{W} : l^2(X) \to L^2(T^2, C^3) \) in order to write \( H_0 \) as an analytic fibered operator. We take the convention that \( T^2 = \mathbb{R}^2/[0,1]^2 \) and recall that \( T^2 \) is the dual of \( \mathbb{Z}^2 \) by \( \xi(\mu) = e^{2\pi i \mu} \), for \( \xi \in T^2 \) and \( \mu \in \mathbb{Z}^2 \). The construction of \( \mathcal{W} \) is a particularization of the general construction obtained in [Par17]. We define \( \mathcal{W} : C_c(X) \to L^2(T^2, C^3) \) by setting, for \( f \in C_c(X) \) and \( \xi \in T^2 \),
\[
(\mathcal{W} f)(\xi) = \left( \sum_{\mu \in \mathbb{Z}^2} e^{-2\pi i \mu f(\mu)}, \sum_{\mu \in \mathbb{Z}^2} e^{-2\pi i \mu f(\mu_1)}, \sum_{\mu \in \mathbb{Z}^2} e^{-2\pi i \mu f(\mu_2)} \right).
\]
We denote by Trig Pol\((T^2; C^3)\) the subspace of \( L^2(T^2, C^3) \) composed by functions \( \varphi \) that admit
\[
\varphi(\xi) = \sum_{\mu \in \mathbb{Z}^2} e^{2\pi i \mu} \varphi_{\mu} ; \text{ with } 0 \neq \varphi_{\mu} \in C^3 \text{ for only finitely many } \mu.
\]

This is a conveniently dense space in \( L^2(T^2, C^3) \) and coincides with \( \mathcal{W}(C_c(X)) \). To write the adjoint of \( \mathcal{W} \) define the index \( \eta((x, y)) = y - x \) and fix an orientation on the graph by setting \( A^+ = \{ e \in A : \eta(e) = \delta_i, i = 1,2 \} \). Set also the integer part of an edge by \( [(x, y)] := x \). Therefore, for \( \varphi \in \text{Trig Pol}(T^2; C^3) \)
\[
(\mathcal{W}^* \varphi)(x) = \int_{T^2} d\xi e^{2\pi i \xi \cdot x} \varphi_1(\xi),
\]
\[
(\mathcal{W}^* \varphi)(e) = \int_{T^2} d\xi e^{2\pi i \xi \cdot e} \varphi_{\eta(e)}(\xi) \quad \text{if } e \in A^+,
\]
\[
(\mathcal{W}^* \varphi)(e) = -\int_{T^2} d\xi e^{2\pi i \xi \cdot (|e| + \eta(e))} \varphi_{\eta(e)}(\xi) \quad \text{if } e \notin A^+.
\]
where \( i(e) = j + 1 \) if there exist \( \mu \in \mathbb{Z}^2 \) such that either \( \mu e_j = e \) or \( \mu \sigma_j = e \). Then \( \mathcal{W} \) extends to a unitary operator, still denoted by \( \mathcal{W} \), from \( l^2(X) \) to \( L^2(T^2, C^3) \).

Note that this definition of \( \mathcal{W} \) correspond to the following choice of Fourier transform in \( \mathbb{Z}^2 \):
\[
\mathcal{F}^* : l^2(\mathbb{Z}^2) \to L^2(T^2) ; \quad (\mathcal{F}^* f)(\xi) := \sum_{\mu \in \mathbb{Z}^2} e^{-2\pi i \mu f(\mu)}.
\]
We stress the fact that this choice can be considered as not standard, but it is rather natural in this context. We now reproduce, adapted to our setting, the result from [Par17] that will be the starting point of our investigation. To simplify the computations, we introduce for \( \xi = (\xi_1, \xi_2) \)
\[
a(\xi) := (-1 + e^{-2\pi i \xi_1}), \quad b(\xi) := (-1 + e^{-2\pi i \xi_2}).
\]
Proposition 2.1 ([Par17, Prop. 3.5]). The operator \( H_0 \) is, by conjugation by \( \mathcal{U} \), unitarily equivalent to a matrix-valued multiplication operator in \( L^2(\mathbb{T}^2, \mathbb{C}^4) \) given by the real-analytic function

\[
h_0(\xi) = \begin{pmatrix} m a(\xi) & b(\xi) \\ a(\xi) & -m & 0 \\ b(\xi) & 0 & -m \end{pmatrix}.
\]

2.2. Spectral theory for \( H_0 \). To compute the spectrum of \( \mathcal{U} H_0 \mathcal{U}^* \) we will obtain its band functions. Because for every \( \xi \in \mathbb{T}^2 \), \( h_0(\xi) \) has three eigenvalues, we will have three band functions. Note that, even if \( h_0(\xi) \) is real-analytic, in contrast with the case when the base is 1-dimensional, one can not a priori choose the band functions \( \{\lambda_j\} \) to be analytic. We will have however an explicit expression for them and be able to compute \( \sigma(H_0) = \bigcup_j \lambda_j(\mathbb{T}^2) \).

One can see that for \( \xi \in \mathbb{T}^2 \), the characteristic polynomial associated to \( h_0(\xi) \) is given by

\[
p_\xi(z) = (m - z)(m + z) + (m + z) \left| b(\xi) \right|^2 + \left| a(\xi) \right|^2.
\]

We notice the identities

\[
\left| a(\xi) \right|^2 = 2(1 - \cos(2\pi \xi_1)) = 4\sin^2(\pi \xi_1) \quad \text{and} \quad \left| b(\xi) \right|^2 = 2(1 - \cos(2\pi \xi_2)) = 4\sin^2(\pi \xi_2).
\]

There are three band functions:

\[
z_0(\xi) = -m, \quad z_\pm(\xi) = \pm \sqrt{m^2 + \left| a(\xi) \right|^2 + \left| b(\xi) \right|^2}.
\]

This gives us (2). For convenience we set \( r_m(\xi) := m^2 + \left| a(\xi) \right|^2 + \left| b(\xi) \right|^2 \geq 0 \) and assume the convention \( r_0 \equiv r \).

To compute the thresholds in \( T \) notice that

\[
\nabla z_\pm(\xi) = \pm \frac{2\pi}{r_m(\xi)} (\sin(2\pi \xi_1), \sin(2\pi \xi_2)).
\]

Then, both \( z_+ \) and \( z_- \) have the same critical points and they give (3).

First assume \( m > 0 \). From (13) we see that the thresholds \( \{\pm \sqrt{m^2 + 8}, m\} \) are of elliptic type. These thresholds are situated at the edges of the bands. The thresholds \( \{\pm \sqrt{m^2 + 4}\} \) are of hyperbolic type and correspond to embedded thresholds (see [IJ19] for this distinction for the Laplacian). Finally, there is \( -m \) that corresponds to the degenerated eigenvalue, but it coincides also with the maximum of \( z_- \).

When \( m = 0 \) there is no gap in the spectrum and the thresholds are \( \{0, \pm 2, \pm \sqrt{8}\} \). However, since \( r((0, 0)) = 0 \), \( z_\pm \) are not everywhere differentiable, which indicates that 0 is a different type of threshold. We call this a Dirac point.

One of our focus of attention in the sequel will be the spectral properties near the energy corresponding to a degenerated eigenvalue of \( H_0 \). From the Floquet-Bloch theory described above, this eigenvalue appears due to the existence of a flat band. However, one can also understand directly in \( L^2(X) \) its appearance: let suppose for simplicity that \( m = 0 \). Let \( x' \in \mathcal{V} \) be fixed and consider the closed path \( \Gamma_{x'} \subset \mathcal{A} \) given by \( \{(x', x' + \delta_1), (x' + \delta_1, x' + \delta_1 + \delta_2), (x' + \delta_1 + \delta_2, x' + \delta_2), (x' + \delta_2, x')\} \). We define \( f_{x'} \in L^2(X) \) by \( f_{x'}(e) = 1 \) if \( e \in \Gamma_{x'} \) and such that it vanishes elsewhere. Remember that by definition of \( C^1(X) \), we have that \( f_{x'}(\pi) = -1 \) for \( e \in \Gamma_{x'} \). Since \( f_{x'} \) vanishes in vertices we have \( (H_0 f_{x'})(e) = 0 \) for every \( e \). Furthermore,

\[
(H_0 f_{x'})(x') = -f_{x'}(x', x' + \delta_1) - f_{x'}(x', x' + \delta_2) - f_{x'}(x', x' - \delta_1) - f_{x'}(x', x' - \delta_2)
\]

\[
= -1 + 1 + 0 + 0 = 0.
\]

Similar computations hold for \( x' + \delta_1, x' + \delta_1 + \delta_2 \) and \( x' + \delta_2 \) while for the other vertices \( f_{x'} \) vanishes in every edge around them. If follows that \( H_0 f_{x'} \equiv 0 \). Repeating the procedure for different \( x' \in \mathcal{V} \) we get an infinitely dimensional kernel.
3. SSF representation

Let us start this section by giving some notation that we will use henceforth. Denote by $\mathcal{S}_\infty$ the class of compact operators. For $K \in \mathcal{S}_\infty$ self-adjoint and $r > 0$, we set

$$n_\pm(r; K) := \text{Rank} \mathbb{I}_{(r, \infty)}(\pm K).$$

Thus, the functions $n_\pm(\cdot; K)$ are respectively the counting functions of the positive and negative eigenvalues of the operator $K$. If $K$ is compact but not necessarily self-adjoint we will use the notation

$$n_+(r; K) := n_+(r^2; K^* K), \quad r > 0;$$

thus $n_+(\cdot; K)$ is the counting function of the singular values of $K$, which we denote by $\{s_j(K)\}$. Since $n_+(r^2; K^* K) = n_+(r^2; KK^*)$

$$n_+(r; K) = n_+(r; K^*), \quad r > 0.$$ 

Besides, for $r_1, r_2 > 0$, we have the Weyl inequalities

$$n_\pm(r_1 + r_2; K_1 + K_2) \leq n_\pm(r_1; K_1) + n_\pm(r_2; K_2),$$

where $K_j$, $j = 1, 2$, are self-adjoint compact operators (see e.g. [BS87, Theorem 9.2.9]), as well as the Ky Fan inequality

$$n_+(r_1 + r_2; K_1 + K_2) \leq n_+(r_1; K_1) + n_+(r_2; K_2),$$

for compact but not necessarily self-adjoint $K_j$, $j = 1, 2$, (see e.g. [BS87, Subsection 11.1.3]).

Denote by $\mathcal{S}_p$, $p \in [1, \infty)$, the Schatten-von Neumann class of compact operators, equipped with the norm

$$\|K\|_p^p = \sum s_j(K)^p.$$ 

Further, since for $K \in \mathcal{S}_p$, $\|K\|_p = (- \int_0^\infty \! \! d n_+(r; K) r^p)^{1/p}$, the Chebyshev-type estimate

$$n_+(r; K) \leq r^{-p}\|K\|_p^p$$

holds true for any $r > 0$ and $p \in [1, \infty)$.

Now, define for $z \in \mathbb{C}^+$ the operator $K(z) = V^{1/2}(H_0 - z)^{-1}V^{1/2}$. By Proposition 3.3 below, the norm limits

$$\lim_{\delta \downarrow 0} K(\lambda + i\delta) =: K(\lambda + i0)$$

exist for any $\lambda \in \mathbb{R} \setminus \mathcal{T}$. Moreover, recalling that $H_{\pm} = H_0 \pm V$, by [Pus98, Theorem 1.1] we have that the SSF admits the representative

$$\xi(\lambda; H_\pm, H_0) = \pm \frac{1}{\pi} \int_\mathbb{R} \! \! \frac{dt}{1 + t^2} n_\mp(1; \text{Re} K(\lambda + i0) + t \text{Im} K(\lambda + i0)).$$

All the results of this article about the SSF will concern its representative given by (19).

From this representation it is possible to obtain the following lemma, which is a small modification of [Pus98, Lemma 2.1] that we state for the ease of the reader.

**Lemma 3.1.** Let $T(\lambda)$ be a family of finite rank operators. Then, for any $\epsilon \in (0, 1)$

$$\xi(\lambda; H_\pm, H_0) \leq \pm n_\mp(1 \mp \epsilon; \text{Re} K(\lambda + i0)) + \frac{1}{\pi \epsilon} ||\text{Im} K(\lambda + i0) - T(\lambda)||_1 + \text{Rank} T(\lambda);$$

$$\xi(\lambda; H_\pm, H_0) \geq \pm n_\mp(1 \pm \epsilon; \text{Re} K(\lambda + i0)) - \frac{1}{\pi \epsilon} ||\text{Im} K(\lambda + i0) - T(\lambda)||_1 - \text{Rank} T(\lambda).$$
Motivated by Lemma 3.1, we now look for a more explicit expression of the operator \( \mathcal{V} K(\lambda + i0) \mathcal{W}^* \). Throughout this section we will consider a fixed \( \lambda \in \sigma(H_0) \setminus T \), write \( z = \lambda + i\delta \) and study the limit as \( \delta \downarrow 0 \). Furthermore, for simplicity we will assume \( \lambda > 0 \) but, by symmetry, the results also holds for the lower band.

We have that for \( z \in \mathbb{C} \setminus \sigma(H_0) \)

\[
(h_0 - z)^{-1}(\xi) = \frac{1}{p_c(z)} \begin{pmatrix} (m + z)^2 & (m + z)a(\xi) & (m + z)b(\xi) \\ (m + z)a(\xi) & z^2 - m^2 - |b(\xi)|^2 & a(\xi)b(\xi) \\ (m + z)b(\xi) & a(\xi)b(\xi) & z^2 - m^2 - |a(\xi)|^2 \end{pmatrix},
\]

the characteristic polynomial \( p_c(z) \) given in (12).

We are interested in studying, for \( z \) with \( \text{Im}(z) > 0 \), \( \mathcal{V} V^{1/2}(H_0 - z)^{-1} V^{1/2} \mathcal{W}^* \). We can readily see that

\[
\mathcal{V} V^{1/2}(H_0 - z)^{-1} V^{1/2} \mathcal{W}^* = \frac{\mathcal{V} V^*}{m + z} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \frac{1}{m + z} \begin{pmatrix} (m + z)^2 & (m + z)a(\cdot) & (m + z)b(\cdot) \\ (m + z)a(\cdot) & |a(\cdot)|^2 & a(\cdot)b(\cdot) \\ (m + z)b(\cdot) & a(\cdot)b(\cdot) & |b(\cdot)|^2 \end{pmatrix} \mathcal{V} V^{1/2} \mathcal{W}^*.
\]

Let \( g_j \) be defined by (5) but corresponding to the potential \( G := V^{1/2} \). Now, for each \( \xi' \in \mathbb{T}^2 \) and \( z \in \mathbb{C} \) define \( t_{z,m}(\xi') : L^2(\mathbb{T}^2; \mathbb{C}^3) \to \mathbb{C}^3 \) by

\[
\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \mapsto \begin{pmatrix} (m + z) \int_{\mathbb{T}^2} d\eta \, \tilde{g}_1(\xi' - \eta) f_1(\eta) \\ a(\xi') \int_{\mathbb{T}^2} d\eta \, \tilde{g}_2(\xi' - \eta) f_2(\eta) \\ b(\xi') \int_{\mathbb{T}^2} d\eta \, \tilde{g}_3(\xi' - \eta) f_3(\eta) \end{pmatrix},
\]

and \( t^1_{z,m}(\xi') : \mathbb{C}^3 \to L^2(\mathbb{T}^2; \mathbb{C}^3) \) by

\[
\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \mapsto \begin{pmatrix} \omega_1(m + z) \tilde{g}_1(\cdot - \xi') \\ \omega_2 \tilde{g}_2(\cdot - \xi') \\ \omega_3 \tilde{g}_3(\cdot - \xi') \end{pmatrix}.
\]

We also note \( t_{z,0}(\xi') =: t_z(\xi') \) and notice that \( t^1_{z,m}(\xi') = t_{z,m}(\xi')^* \). Then, on can see that

\[
\mathcal{V} V^{1/2}(H_0 - z)^{-1} V^{1/2} \mathcal{W}^* = \frac{\mathcal{V} V^*}{m + z} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \frac{1}{m + z} \int_{\mathbb{T}^2} d\xi' \frac{t^1_{z,m}(\xi') A t_{z,m}(\xi')}{r_m(\xi') - z^2},
\]

where

\[
A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.
\]

In (22) define \( T_{z,m}(\xi) = t^1_{z,m}(\xi') A t_{z,m}(\xi) \). As before, we also denote \( T_{z,0}(\xi) =: T_z(\xi) \).

From now on, we will assume that each \( \nu_j \) satisfies (6) with \( 0 < \beta_j \leq 1 \). Further, we set

\[
\tilde{\beta} := \min\{\beta_2, \beta_3\}; \quad \beta := \min\{\beta_1, \tilde{\beta}\}.
\]
Lemma 3.2. For every $z \in \mathbb{C}$ and $\xi \in T^2$ we have that $T_{z,m}(\xi) \in \mathfrak{S}_1(L^2(T^2; \mathbb{C}^3))$. Furthermore, $T_{z,m}$ is locally Hölder with exponent $\beta$ in $\xi$ in the $\mathfrak{S}_1$-norm, and it depends analytically in $z$.

Proof. First, it is easy to see that

$$||t_{z,m}(\xi)||^2 = \int_{T^2} d\eta |m + z|^2 |\hat{g}_1(\xi - \eta)|^2 + |a(\xi)|^2 |\hat{g}_2(\xi - \eta)|^2 + |b(\xi)|^2 |\hat{g}_3(\xi - \eta)|^2$$

which in particular gives us that

$$||T_{z,m}(\xi)||_1 = ||t_{z,m}^\dagger(\xi)A_{z,m}(\xi)||_1 \leq C||t_{z,m}(\xi)||_2 < \infty .$$

Now, let us fix $z \in \mathbb{C}$ and $m \geq 0$. For $\xi$ and $\xi'$ in $T^2$ we have

$$||T_{z,m}(\xi') - T_{z,m}(\xi)||_1 \leq ||t_{z,m}^\dagger(\xi') - t_{z,m}^\dagger(\xi)||_2 ||A_{z,m}(\xi')||_2 + ||t_{z,m}^\dagger(\xi)A||_2 ||t_{z,m}(\xi') - t_{z,m}(\xi)||_2 .$$

Using

$$||t_{z,m}(\xi') - t_{z,m}(\xi)||_2 = \int_{T^2} d\eta |m + z|^2 |\hat{g}_1(\xi' - \eta) - \hat{g}_1(\xi - \eta)|^2 + |a(\xi')\hat{g}_2(\xi' - \eta) - a(\xi)\hat{g}_2(\xi - \eta)|^2 + |b(\xi')\hat{g}_3(\xi' - \eta) - b(\xi)\hat{g}_3(\xi - \eta)|^2 ,$$

together with the fact that condition (6) ensures us (see [DDR19, Theorem 3.5])

$$\int_{T^2} d\eta |\hat{g}_j(\eta + h) - \hat{g}_j(h)|^2 = O(|h|^{\beta_j}), \quad |h| \to 0, \quad 1 \leq j \leq 3 ,$$

we can conclude that, for $|\xi' - \xi| \to 0$,

$$||t_{z,m}(\xi') - t_{z,m}(\xi)||_2 \leq C |m + z|^2 |\xi' - \xi|^{\beta_1} + |a(\xi')|^2 |\xi' - \xi|^{\beta_2} + |b(\xi')|^2 |\xi' - \xi|^{\beta_3} + |a(\xi') - a(\xi)|^2 ||\hat{g}_2||^2 + |b(\xi') - b(\xi)|^2 ||\hat{g}_3||^2 ,$$

$$\leq C|\xi'| - \xi|^{\beta} .$$

Taking into account (26) we obtain the Hölder property. Finally, the analyticity in $z$ can be directly observed in (20) and (21).

For $z \in \mathbb{C} \setminus \sigma(H_0)$ we want to use the coarea formula

$$\int_{T^2} d\xi \frac{T_{z,m}(\xi)}{(r_m(\xi) - z^2)} = \int_{m^2}^{M_m^2} \frac{d\rho}{\rho - z^2} \int_{r_m^{-1}(\rho)} d\gamma \frac{T_{z,m}(\xi)}{\nabla r_m(\xi)} ,$$

where $d\gamma = d\gamma_\rho$ is the one dimensional Hausdorff measure over the level curve $r_m^{-1}(\rho)$, and $M_m := \sqrt{m^2 + 8}$. To this end it is enough to show that

$$\frac{||T_{z,m}(\cdot)||_1}{|r_m(\cdot) - z^2||\nabla r_m(\cdot)|}$$

is in $L^1(T^2)$,

which easily follows from the fact that $z \in \mathbb{C} \setminus \sigma(H_0)$, the boundedness of $||T_{z,m}(\cdot)||_1$ and

$$||\nabla r_m(\xi_1, \xi_2)||^2 = 16\pi^2 (\sin^2(2\pi \xi_1) + \sin^2(2\pi \xi_2)) .$$

In Fig. 1 we show the level curves of $r_m$. One can readily see a periodicity that will simplify some of our computations below.

The main result of this section is the following proposition which together with (22) gives an explicit description of $K(\lambda + i0)$.
**Proposition 3.3.** Let $\lambda \in \mathbb{R} \setminus \mathcal{T}$ and set $z = \lambda + i\delta$. Then

$$
\int_{m^2}^{M_m^2} \frac{d\rho}{\rho - z^2} \int_{r_m^{-1}(\rho)}^1 d\gamma \frac{T_{\lambda,m}(\xi)}{|\nabla r_m(\xi)|} \rightarrow \text{p.v.} \int_{m}^{M_m} \frac{2u}{(u^2 - \lambda^2)} \int_{r_m^{-1}(u^2)}^1 d\gamma \frac{T_{\lambda,m}(\xi)}{|\nabla r_m(\xi)|} \left(-i\pi \int_{r_m^{-1}(\lambda^2)} d\gamma \frac{T_{\lambda,m}(\xi)}{|\nabla r_m(\xi)|}\right),
$$

(33)
as $\delta \downarrow 0$ in the $S_1$-norm.

We present some preparatory lemmata before given the proof. After making the change $u^2 = \rho$ in (31), let us set

$$
B_m(u, z) := \frac{2u}{u + z} \int_{r_m^{-1}(u^2)} d\gamma \frac{T_{\lambda,m}(\xi)}{|\nabla r_m(\xi)|}.
$$

(34)

Note that this operator-valued function admits a limit when $\delta \downarrow 0$ and so we can write $B(u, \lambda)$. Moreover, $B(\cdot, \lambda)$ satisfies the following lemma:

**Lemma 3.4.** Outside $\mathcal{T}$, the function $u \mapsto B_m(u, \lambda)$ is locally Hölder with exponent $\beta$ (defined in (23)).

Proof. Let us consider an interval $I \in (m, \sqrt{m^2 + 4})$. For $u \in I$, and taking the function $\arcsin : [-1, 1] \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$ we define

$$
f_u(\xi_1) = \frac{1}{\pi} \arcsin \left( \sqrt{\frac{u^2 - m^2}{4} - \sin^2(\pi \xi_1)} \right),
$$
for \( |ξ| \leq \frac{1}{π} \arcsin \left( \sqrt{\frac{u^2-m^2}{4}} \right) =: ξ_1(u) \). Since \((ξ, ± f_ν(ξ))\) parametrize \( r_m^{-1}(u^2) \) we obtain:

\[
\int_{r_m^{-1}(u^2)} dγ \frac{T_{λ,m}(ξ)}{|∇r_r(ξ)|} = \int_{ξ_1(u)}^{ξ_1(v)} dξ_1 \left( \frac{|∇r_r(ξ,f_v(ξ))|}{|∇r_r(ξ,f_u(ξ))|} \times T_{λ,m}(ξ,f_u(ξ)) \right)
\]

For the second term, from (35), we have that

\[
0 = \frac{1}{8π} \int_{ξ_1(u)}^{ξ_1(v)} dξ_1 \left( \frac{T_{λ,m}(ξ,f_u(ξ))}{F_u(ξ)(1-F_u(ξ))} + \frac{T_{λ,m}(ξ,-f_u(ξ))}{F_u(ξ)(1-F_u(ξ))} \right),
\]

where

\[
F_u(ξ) = \frac{u^2 - m^2}{4} - \sin^2(πξ) .
\]

Now, let us consider \( v < u \) in \( I \) and bound \( ||B(u,λ) - B(v,λ)||\):

\[
\left\| \frac{2u}{u + λ} \int_{ξ_1(u)}^{ξ_1(v)} dξ_1 \left( \frac{T_{λ,m}(ξ,f_u(ξ))}{F_u(ξ)(1-F_u(ξ))} - \frac{2v}{v + λ} \int_{ξ_1(v)}^{ξ_1(u)} dξ_1 \left( \frac{T_{λ,m}(ξ,f_v(ξ))}{F_v(ξ)(1-F_v(ξ))} \right) \right) \right\|_1
\]

\[
\leq \frac{2u}{u + λ} \int_{ξ_1(u)}^{ξ_1(v)} dξ_1 \left( \frac{2u}{(u + λ)F_u(ξ)(1-F_u(ξ))} - \frac{2v}{(v + λ)F_v(ξ)(1-F_v(ξ))} \right) \left\| T_{λ,m}(ξ,f_v(ξ)) \right\|_1
\]

First we notice that

\[
u → \int_{ξ_1(u)}^{ξ_1(v)} dξ_1 \frac{2u}{(u + λ)F_u(ξ)(1-F_u(ξ))}
\]

is bounded on \( I \) since

(35) \[0 < \min_{u \in I} \{ \min_{|ξ| \leq ξ_1(u)} \{ F_u(ξ) \} \} \text{ and } \max_{u \in I} \{ \max_{|ξ| \leq ξ_1(u)} \{ F_u(ξ) \} \} < 1 .\]

Then, to treat the first term, if we take \( I \) suitably small, one can use Lemma 3.2 to obtain

\[
||T_{λ,m}(ξ,f_u(ξ)) - T_{λ,m}(ξ,f_v(ξ))||_1 \leq C(\max \{ F_u(ξ) \} )^β .
\]

One concludes by noticing that \( f_ν \) is Lipschitz in \( u \), for \( u \in I \), with a uniform constant in \( ξ_1 \). For the second term, from (35), we have that \( \frac{2u}{(u + λ)F_u(ξ)(1-F_u(ξ))} \in C^1(I) \). Using the same argument one can check that the norm in the third term is uniformly bounded and then conclude by noticing that \( u → ξ_1(u) \) has a bounded derivative on \( I \).

As a direct consequence, by applying Sokhotski-Plemelj formula we obtain the following.

**Corollary 3.5.** Let \( λ \) and \( z = λ + iδ \) be as before. Then

\[
\left\| \int_m^{M_m} \frac{dB_m(u,λ)}{u-λ} - \frac{p.u.}{u-λ} \int_m^{M_m} \frac{dB_m(u,λ)}{u-λ} - iπB_m(λ,λ) \right\|_1 = O(δ) \text{ as } δ ↓ 0 .
\]

The last lemma we need is the following.
We now treat each term of (36) separately. For the first term one only needs Lemma 3.4 and check that

\[ \epsilon > \text{for } \]

\[ \text{Lemma 3.6.} \]

for all \( u \). Hence

\[ \|B_m(u, z) - B_m(u, \lambda)\|_1 \leq C\delta \int_{r_m^{-1}(u^2)} \frac{d\gamma}{|\nabla r_m(\xi)|} \]

\[ \leq C\delta \int_{r_m^{-1}(u^2)} \frac{d\gamma}{|\nabla r_m(\xi)|} \]

(36)

We now treat each term of (36) separately. For the first term one only needs Lemma 3.4 and check that

\[ \left( \sup_{u \in [m, M_m]} \frac{2\epsilon}{u + z |u + \lambda|} \right) \left( \sup_{\xi \in \mathbb{T}^2} \|T_z(m)(\xi)\|_1 \right) < \infty, \]

uniformly on \( \delta \). For the second term, Lemma 3.2 gives us that

\[ \|B_m(u, z) - B_m(u, \lambda)\|_1 \leq C\delta \int_{r_m^{-1}(u^2)} \frac{d\gamma}{|\nabla r_m(\xi)|} =: C\delta R(u). \]

Note that \( R : [m; M_m] \setminus \mathcal{T} \to \mathbb{R} \) remains bounded near \( \{m, M_m\} \) (see (40) below). Furthermore, for \( \epsilon > 0 \), it satisfies \( R(\sqrt{m^2 + 4} \pm \epsilon) = O(|\ln(\epsilon^{-1})|) \). Hence \( R \in L^1([m; M_m]) \). The result follows by applying the dominated convergence theorem once one notice that

\[ \frac{\delta R(u)}{\sqrt{(u - \lambda)^2 + \delta^2}} \to 0 \text{ as } \delta \to 0, \]

for all \( u \notin \{\lambda, \sqrt{m^2 + 4}\}. \)

We have gathered all the result needed for proving Proposition 3.3.

\[ \Phi \left[ \text{Im } K(\lambda + i0) \right] = \frac{\pi}{m + \lambda} \int_{r_m^{-1}(\lambda^2)} \frac{d\gamma}{|\nabla r_m(\xi)|}, \]

As a consequence of Proposition 3.3, for \( \lambda \in \mathbb{R} \setminus \mathcal{T} \) we have
and

\[ \mathcal{W} \left[ \text{Re} K(\lambda + i0) \right] \mathcal{W}^* = \mathcal{W} V \mathcal{W}^* \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \]

(38)

\[ + \frac{1}{m + \lambda} \text{p.v.} \int_m^{M_m} du \frac{2u}{(u^2 - \lambda^2)} \int_{r_m^{-1}(u^2)} d\gamma \frac{T_{\lambda,m}(\xi)}{|\nabla r_m(\xi)|}. \]

4. Preliminary computations: parabolic thresholds and Dirac point case

Having in mind Lemma 3.1, in this section we study some asymptotic properties of the operators given by (37) and (38), for \( \lambda \) close to \( \pm m \). When \( m \neq 0 \) and \( |\lambda| < m \), the imaginary part obtained in (37) vanishes and the p.v. part of (38) is just the integral over the interval \([m, M_m]\). Hence, we start by considering in detail the case that \( \lambda \to \pm m \) with \( |\lambda| > m \). Several notations will depend on a \( \pm \) subscript. We do this to shorten notations and study both limits, \( \pm m \), at the same time. Notice that the case \( m = 0 \) needs also to be considered. In that case, the subscript \( \pm \) will indicate if we are approaching 0 by the positive or negative side.

**Lemma 4.1.** The following relation holds true

\[ \left| \left| \int_{r_m^{-1}(\lambda^2)} d\gamma \frac{T_{\lambda,m}(\xi)}{|\nabla r_m(\xi)|} \right| \right|_1 = O(\lambda(m + \lambda)), \quad \lambda \to \pm m, |\lambda| > m. \]

**Proof.** First notice that from (24) and (25), for all \( \lambda \) and \( \xi \)

(39)

\[ ||T_{\lambda,m}(\xi)||_1 \leq C \left( (m + \lambda)^2 + r(\xi) \right). \]

Now, recalling that

\[ r(\xi) = 2(1 - \cos(2\pi \xi)) + 2(1 - \cos(2\pi \xi_2)) = 4\sin^2(\pi \xi) + 4\sin^2(\pi \xi_2) \]

and comparing it with (32) one can easily deduce that for \( \xi \in [-1/4, 1/4] \times [-1/4, 1/4] \)

(40)

\[ \frac{1}{16\pi} |\nabla r(\xi)|^2 \leq r(\xi) \leq |\nabla r(\xi)|^2. \]

Furthermore, using

(41)

\[ \nabla r_m(\xi) = \nabla r(\xi) \quad \text{and} \quad r_m^{-1}(u^2) = r^{-1}(u^2 - m^2), \]

we see that for \( \lambda \) near \( \pm m \), the variable \( \xi \in r_m^{-1}(\lambda) = r^{-1}(\lambda - m^2) \) is small in modulus. Thus from (39) and (40)

\[ \left| \left| \int_{r_m^{-1}(\lambda^2)} d\gamma \frac{T_{\lambda,m}(\xi)}{|\nabla r(\xi)|} \right| \right|_1 \leq C \int_{r^{-1}(\lambda^2 - m^2)} d\gamma \frac{(m + \lambda)^2 + r(\xi)}{r(\xi)^{1/2}} \leq C \left( (m + \lambda)^2 + (\lambda^2 - m^2) \right) \]

\[ = C \lambda(m + \lambda). \]

From (38), we define

(42)

\[ S_0(\lambda) := \text{p.v.} \int_m^{M_m} du \frac{2u}{u^2 - \lambda^2} \int_{r_m^{-1}(u^2)} d\gamma \frac{T_{\lambda,m}(\xi)}{|\nabla r(\xi)|}. \]

Now, let us set

\[ B_{\lambda,m}(u) := \frac{2u}{(u \pm \lambda)} \int_{r_m^{-1}(u^2)} d\gamma \frac{T_{\lambda,m}(\xi)}{|\nabla r(\xi)|}. \]
Theorem 4.2.

One can readily check that for $S_0 - S_1$ as the sum of five terms, $R_j$, $j = 0, 1, 2, 3, 4$, the first of which is

$$R_0(\lambda) := \int_0^{\pm \lambda + m} \frac{d\rho}{\rho} \left( \frac{2\rho \pm 2\lambda}{\rho \pm 2\lambda} - \frac{-2\rho \pm 2\lambda}{\rho - 2\lambda} \right) \int_{r_m^{-1}((\pm\lambda + \rho)^2)} d\gamma \frac{|T_{\lambda,m}(\xi)|}{\|\nabla r(\xi)\|}.$$ 

Thus, we only need to consider the cases where $|\lambda| > m$. To prove this lemma we will write $S_0 - S_1$ as the sum of five terms, $R_j$, $j = 0, 1, 2, 3, 4$, the first of which is

$$R_0(\lambda) := \int_0^{\pm \lambda + m} \frac{d\rho}{\rho} \left( \frac{2\rho \pm 2\lambda}{\rho \pm 2\lambda} - \frac{-2\rho \pm 2\lambda}{\rho - 2\lambda} \right) \int_{r_m^{-1}((\pm\lambda + \rho)^2)} d\gamma \frac{|T_{\lambda,m}(\xi)|}{\|\nabla r(\xi)\|}.$$ 

Then, using (39) and (40) we can see that

$$||R_0(\lambda)||_1 \leq \int_0^{\pm \lambda + m} \frac{d\rho}{\rho} \left( \frac{2\rho \pm 2\lambda}{\rho \pm 2\lambda} - \frac{-2\rho \pm 2\lambda}{\rho - 2\lambda} \right) \int_{r_m^{-1}((\pm\lambda + \rho)^2-m^2)} d\gamma \frac{|T_{\lambda,m}(\xi)|}{\|\nabla r(\xi)\|} \leq \frac{C}{\pm 3\lambda + m} \int_0^{\pm \lambda + m} d\rho \int_{r_m^{-1}((\pm\lambda + \rho)^2-m^2)} d\gamma \left( \frac{(m + \lambda)^2}{\epsilon(\xi)^{\frac{4}{2}}} + \epsilon(\xi)^{\frac{4}{2}} \right) \leq \frac{C}{\pm 3\lambda + m} \int_0^{\pm \lambda + m} d\rho \left( (m + \lambda)^2 + (\pm\lambda + \rho)^2 - m^2 \right).$$

From the last expression we can deduce that

$$||R_0(\lambda)||_1 = \begin{cases} O((\lambda + m)^2) & \text{if } m \neq 0, \lambda \uparrow -m \\ O((\lambda - m)) & \text{if } m \neq 0, \lambda \downarrow m \\ O(\lambda^2) & \text{if } m = 0, \lambda \to 0. \end{cases}$$

In order to write the decomposition of $S_0(\lambda) - S_1(\lambda) - R_0(\lambda)$ set

$$\theta := \frac{(\pm\lambda + \rho)^2 - (\pm\lambda - \rho)^2}{4} = \pm \lambda \rho > 0,$$

and

$$D_\theta(\xi_1, \xi_2) = \begin{cases} \frac{1}{2\pi} \left( \arccos(-\theta + \cos(2\pi \xi_1)), \arccos(-\theta + \cos(2\pi \xi_2)) \right) & \text{if } \xi_1 \geq 0 \text{ and } \xi_2 \geq 0, \\ \frac{1}{2\pi} \left( \arccos(-\theta + \cos(2\pi \xi_1)), \arccos(-\theta + \cos(2\pi \xi_2)) \right) & \text{if } \xi_1 < 0 \text{ and } \xi_2 \geq 0, \\ \frac{1}{2\pi} \left( \arccos(-\theta + \cos(2\pi \xi_1)), \arccos(-\theta + \cos(2\pi \xi_2)) \right) & \text{if } \xi_1 \leq 0 \text{ and } \xi_2 < 0, \\ \frac{1}{2\pi} \left( \arccos(-\theta + \cos(2\pi \xi_1)), \arccos(-\theta + \cos(2\pi \xi_2)) \right) & \text{if } \xi_1 \geq 0 \text{ and } \xi_2 < 0. \end{cases}$$

One can readily check that for $\rho \not\equiv \lambda$, $D_\theta$ is injective on $r_m^{-1}((\lambda - \rho)^2)$ and satisfies $D_\theta(r_m^{-1}((\lambda - \rho)^2)) \subset r_m^{-1}((\lambda + \rho)^2)$. Let us denote by $D_\theta \gamma$ the pullback measure induced by $D_\theta$ on $r_m^{-1}((\lambda - \rho)^2)$ and set $\gamma_R := \gamma_m^{-1}((\lambda + \rho)^2) \setminus D_\theta(r_m^{-1}((\lambda - \rho)^2))$. In consequence, we can write $S_0 - S_1 - R_0 = \sum_{j=1}^4 R_j$ where
\[ R_1(\lambda) := \int_0^{2\lambda m} \frac{d\rho}{\rho} \frac{-2\rho \pm 2\lambda}{-\rho \pm 2\lambda} \left( \int_{r^{-1}_m((\pm \lambda - \rho)^2)} D_\theta^m d\gamma \frac{(T_{\lambda m}(D\theta(\xi)) - T_{-\lambda m}(\xi))}{|\nabla r(D\theta(\xi))|} \right) \]

\[ R_2(\lambda) := \int_0^{2\lambda m} \frac{d\rho}{\rho} \frac{-2\rho \pm 2\lambda}{-\rho \pm 2\lambda} \left( \int_{\gamma_R} T_{\lambda m}(\xi) \frac{d\gamma}{|\nabla r(\xi)|} \right) \]

\[ R_3(\lambda) := \int_0^{2\lambda m} \frac{d\rho}{\rho} \frac{-2\rho \pm 2\lambda}{-\rho \pm 2\lambda} \left( \int_{r^{-1}_m((\pm \lambda - \rho)^2)} (D_\theta^m d\gamma - d\gamma) \frac{(T_{\lambda m}(\xi))}{|\nabla r(D\theta(\xi))|} \right) \]

\[ R_4(\lambda) := \int_0^{2\lambda m} \frac{d\rho}{\rho} \frac{-2\rho \pm 2\lambda}{-\rho \pm 2\lambda} \left( \int_{r^{-1}_m((\pm \lambda - \rho)^2)} d\gamma T_{\lambda m}(\xi) \left( \frac{1}{|\nabla r(D\theta(\xi))|} - \frac{1}{|\nabla r(\xi)|} \right) \right). \]

To estimate \( R_1 \) compute

\[
|D_\theta(\xi) - \xi|^2 = \left| \int_{-\theta + \cos(2\pi \xi_1)}^{1} \frac{dt}{\sqrt{1 - t^2}} - \xi_1 \right|^2 + \left| \int_{-\theta + \cos(2\pi \xi_2)}^{\cos(2\pi \xi_1)} \frac{dt}{\sqrt{1 - t^2}} - \xi_1 \right|^2
\]

(45)

\[
\leq \int_{\cos(2\pi \xi_1) - \theta}^{\cos(2\pi \xi_1)} \frac{dt}{(1 - t)^{1/2}(1 + t)^{1/2}} \left[ \int_{\cos(2\pi \xi_1) - \theta}^{\cos(2\pi \xi_1)} \frac{dt}{(1 - t)^{1/2}(1 + t)^{1/2}} \right]^2
\]

\[
\leq \varepsilon \theta,
\]

since \( \xi_1, \xi_2 \) are small for \( \lambda \) near \( \pm m \). Particularizing (24), (26) and (29) we can also obtain

\[
||T_{\lambda m}(D\theta(\xi)) - T_{\lambda m}(\xi)||_1 \leq C \left( 2|m + \lambda|^2 + r(D\theta(\xi)) + r(\xi) \right)^{1/2} \left( |m + \lambda| |D_\theta(\xi) - \xi|^\beta_1 + r(D_\theta(\xi))^{1/2} |D_\theta(\xi) - \xi|^{\beta_1} + |D_\theta(\xi) - \xi| \right)
\]

and hence

\[
||R_1(\lambda)||_1 \leq C \int_0^{2\lambda m} \frac{d\rho}{\rho} \frac{(4\lambda m + 2\rho^2)^{1/2}}{r^{-1}((\pm \lambda - \rho)^2 - m^2)} \int_{r^{-1}_m((\pm \lambda - \rho)^2)} d\gamma \frac{|m + \lambda| |\theta^m + r(\xi)\frac{\theta^m}{\theta} + \theta^m|}{|\nabla r(\xi)|}.
\]

From the last expression we can estimate each term and deduce

(46)

\[
R_1(\lambda) = \begin{cases}
O(|\lambda + m|) & \text{if } m \neq 0, \lambda \uparrow -m \\
O(|\lambda - m|^{\frac{\beta_1}{2}}) & \text{if } m \neq 0, \lambda \downarrow m \\
O(\lambda^2) & \text{if } m = 0, \lambda \to 0
\end{cases}
\]

To estimate \( R_2 \) we need first an estimate on \( \gamma_R \). Since \( D_\theta \) does not change the quadrant of \( \xi \in r^{-1}_m((\lambda - \rho)^2) \), \( \gamma_R \) is composed of 8 parts. More precisely, if \( (\xi^+_1, 0) \in r^{-1}_m((\lambda - \rho)^2) \) and \( (\xi^-_1, 0) \in r^{-1}_m((\lambda + \rho)^2) \), one of these parts is the section of \( r^{-1}_m((\lambda + \rho)^2) \) going from \( D_\theta((\xi^-_1, 0)) \) to \( (\xi^+_1, 0) \). Hence, the convexity of the level curves gives us

\[
\int_{\gamma_R} d\gamma \leq C \left( |\arccos(1 - \theta)| + \left| \arccos \left( 1 + \frac{m^2 - (\lambda + \rho)^2}{2} \right) - \arccos \left( 1 + \frac{m^2 - (\lambda - \rho)^2}{2} \right) \right| \right)
\]

\[
\leq C\theta^{1/2}.
\]

Using this estimate we compute

\[
||R_2(\lambda)||_1 \leq \int_0^{2\lambda m} \frac{d\rho}{\rho} \frac{-2\rho \pm 2\lambda}{-\rho \pm 2\lambda} \left( \int_{\gamma_R} d\gamma \frac{||T_{\lambda m}(\xi)||_1}{|\nabla r(\xi)|} \right)
\]
\[ \leq C|\lambda|^{\frac{1}{2}} \int_{0}^{\frac{\pi}{2}} \frac{d\rho}{\rho^2} \left( \frac{2m\lambda + 2\lambda^2 + \rho^2 + 2\lambda\rho}{(\lambda^2 - m^2 + 2\lambda\rho + \rho^2)^{\frac{1}{2}}} \right) \leq C|\lambda|^{\frac{1}{2}} \int_{0}^{\frac{\pi}{2}} \frac{d\rho}{\rho^2} \left( \frac{2m\lambda + 2\lambda^2 + \rho^2 + 2\lambda\rho}{(\lambda^2 - m^2 + 2\lambda\rho + \rho^2)^{\frac{1}{2}}} \right) \leq C|\lambda|^{\frac{1}{2}} \left( 1 + \frac{\lambda + m}{\lambda - m} \right) \int_{0}^{\frac{\pi}{2}} d\rho (\rho \pm 2\lambda)^{\frac{1}{2}} + \left( \frac{\lambda^2 - m^2}{\rho} \right)^{\frac{1}{2}}. \]

From this one gets

\[ ||R_2(\lambda)||_1 = \begin{cases} O(|\lambda + m|) & \text{if } m \neq 0, \lambda \uparrow -m \\ O(1) & \text{if } m \neq 0, \lambda \downarrow m \\ O(\lambda^2) & \text{if } m = 0, \lambda \to 0 \end{cases} \tag{47} \]

Next, by (24) and (25) we get

\[ ||R_3(\lambda)||_1 \leq C \int_{0}^{\frac{\pi}{2}} \frac{d\rho}{\rho} \int_{r^{-1}} d\gamma \left( D_{\lambda}^0 d\gamma - d\gamma \right) \left( m + \lambda \right)^2 + r(\xi) \int_{r^{-1}} d\gamma \left( D_{\lambda}^0 d\gamma - d\gamma \right) \left( m + \lambda \right)^2 \tag{48} \]

Let \((\xi_1(t), \xi_2(t))\) be a parametrization of the curve \(r^{-1}((\lambda, -\rho)^2 - m^2)\). It is easy to see that

\[ |D_{\lambda}^0 d\gamma - d\gamma| = \left| \left( (\partial_1 D_{\theta,1})(\xi(t))^2 \xi_1'(t)^2 + (\partial_2 D_{\theta,2})(\xi(t))^2 \xi_2'(t)^2 \right)^{1/2} - (\xi_1(t)^2 + \xi_2(t)^2)^{1/2} \right| dt, \]

where \(D_{\theta}(\xi) = (D_{\theta,1}(\xi), D_{\theta,2}(\xi))\). Notice that

\[ (\partial_1 D_{\theta,1})(\xi) = \frac{\sin(2\pi \xi_1)}{(1 - \cos(2\pi \xi_1) - \theta)^{1/2}}; \quad (\partial_2 D_{\theta,2})(\xi) = \frac{\sin(2\pi \xi_2)}{(1 - \cos(2\pi \xi_2) - \theta)^{1/2}}. \]

Then, setting

\[ Q_0(\xi) := 1 - \frac{\sin^2(2\pi \xi_1)}{1 - \cos(2\pi \xi_1) - \theta^2} + 1 - \frac{\sin^2(2\pi \xi_2)}{1 - \cos(2\pi \xi_2) - \theta^2}. \]

it is easy to see that

\[ |D_{\lambda}^0 d\gamma - d\gamma| \leq Q_0(\xi) d\gamma. \tag{49} \]

Since \(\theta\) is positive and \(\xi\) is in a small neighborhood of \((0, 0)\), we can see that there exist positive constants \(C_1\) and \(C_2\) such that

\[ Q_0(\xi) \leq C\theta \left( \frac{1}{\sin^2(2\pi \xi_1) + C_1\theta} + \frac{1}{\sin^2(2\pi \xi_2) + C_2\theta} \right). \tag{50} \]

Now, write \(u^2 = (\pm \lambda - \rho)^2\) and parametrize the part of \(r^{-1}(u^2)\) in \([0, 1/2] \times [0, 1/2]\) by

\[ \xi_1(t) = \frac{1}{2\pi} \arccos \left( 1 - \frac{t}{2} \right); \quad \xi_2(t) = \frac{1}{2\pi} \arccos \left( \frac{t - u^2}{2} + 1 \right), \]

for \(t \in (0, u^2)\). Notice that

\[ \xi_1'(t) \simeq \frac{1}{\sqrt{t}}, \quad \xi_2'(t) \simeq \frac{1}{\sqrt{u^2 - t}}; \quad \text{for } t \in (0, u^2), \]
Thus quadrants.

Here, by an abuse of notation, we are still calling \( |\gamma| |\) so we can deduce

\[
|\gamma(t)| = 1 - \left( 1 - \frac{t}{2} \right)^2, \quad \text{for } t \text{ near } 0
\]

Further,

\[
\sin^2(2\pi \xi_1(t)) = 1 - \left( \frac{t - u^2}{2} \right)^2, \quad \text{for } t \text{ near } u^2.
\]

We can see from (49) to (52) that

\[
\int_{r^{-1}(\pm \lambda - \rho)^2}} |d\gamma - D_\theta^\delta d\gamma| \leq \int_{r^{-1}(\pm \lambda - \rho)^2}} Q_\theta(\xi)d\gamma
\]

\[
\leq C \theta \left( \int_0^{u^2/2} \frac{dt}{(t + A\theta)^{1/2}} + \int_{u^2/2}^{u^2} \frac{dt}{(u^2 - t + B\theta)(u^2 - t)^{1/2}} \right)
\]

\[
\leq C\theta^2.
\]

Here, by an abuse of notation, we are still calling \( r^{-1}(\pm \lambda - \rho)^2 \) the part of this curve that is in \([0, 1/2] \times [0, 1/2]\). The same estimates can be obtained for the parts of this curve in the other quadrants.

Using this last estimate on (48) we obtain

\[
||R_3(\lambda)||_1 \leq C(|\lambda + m|) \int_0^{\frac{2\lambda + m}{2}} \frac{d\rho}{\bar{\rho}^2((\pm \lambda + \rho)^2 - m^2)^{1/2}}
\]

\[
\leq C(|\lambda + m|) \int_0^{\frac{2\lambda + m}{2}} \frac{d\rho}{\bar{\rho}^2((\lambda + m + \rho)(\lambda + m - \rho))^{1/2}}
\]

\[
\leq C(|\lambda + m|) \frac{1}{2}(\lambda + m + (\lambda - m)^2).
\]

Thus

\[
||R_3(\lambda)||_1 = \begin{cases} O(|\lambda + m|) & \text{if } m \neq 0, \lambda \uparrow -m \\ O(1) & \text{if } m \neq 0, \lambda \downarrow m \\ O(\lambda^2) & \text{if } m = 0, \lambda \rightarrow 0 \end{cases}
\]

For \( R_4 \) we start by noticing

\[
|\nabla r(D_\theta(\xi))|^2 = |\nabla r(\xi)|^2 - 16\pi^2 \theta(-2 \cos(2\pi \xi_1) - 2 \cos(2\pi \xi_2) + \theta^2),
\]

so we can deduce

\[
\left| \frac{1}{|\nabla r(\xi)|} - \frac{1}{|\nabla r(D_\theta(\xi))|} \right| \leq \frac{C \theta}{|\nabla r(\xi)|^2|\nabla r(D_\theta(\xi))|}.
\]

Also, from Lemma 3.4 and (39) and (40) we obtain,

\[
||R_4(\lambda)||_1 \leq C \int_0^{\frac{2\lambda + m}{2}} \frac{d\rho}{\bar{\rho}^2((\lambda + m)^2 + (\pm \lambda - \rho)^2 - m^2)((\pm \lambda - \rho)^2 - m^2)^{1/2}}
\]

\[
\leq C \frac{1}{(\pm \lambda - m)^{1/2}(\pm \lambda + 3m)^{1/2}(\pm \lambda - m)^{1/2}(\pm \lambda + m)^{1/2}}
\]
which together with (44), (46), (47) and (53) finishes the proof. □

Let us start by defining Lemma 4.3.

\[
\begin{cases}
\int_{m^2, M^2_{\lambda}} d\rho \frac{1}{\rho - \lambda^2} \int_{r_m^1(\rho)} d\gamma \left( \frac{T_{\lambda,m}(\xi)}{|\nabla r(\xi)|} \right) & 0 \leq \lambda < m \\
\int_{m^2, M^2_{\lambda}} d\rho \frac{1}{\rho - \lambda^2} \int_{r_m^1(\rho)} d\gamma \left( \frac{T_{\lambda,m}(\xi)}{|\nabla r(\xi)|} \right) & \lambda > m
\end{cases}
\]

Define \( \hat{T}_{\lambda,m}(\xi) = T_{\lambda,m}(\xi) - T_{\lambda,m}(\xi_0) \). Taking into account (24), (26) and (30) we obtain for \( \xi \) small enough

\[
\|\hat{T}_{\lambda,m}(\xi)\|_1 \leq C(|m + \lambda| |\xi|^\beta + r(\xi)^2)(|\lambda + m| + r(\xi)^{1/2})
\]

Set

\[
\begin{cases}
\int_{m^2, M^2_{\lambda}} d\rho \frac{1}{\rho - \lambda^2} \int_{r_m^1(\rho)} d\gamma \left( \frac{\hat{T}_{\lambda,m}(\xi)}{|\nabla r(\xi)|} \right) & 0 \leq \lambda < m \\
\int_{m^2, M^2_{\lambda}} d\rho \frac{1}{\rho - \lambda^2} \int_{r_m^1(\rho)} d\gamma \left( \frac{\hat{T}_{\lambda,m}(\xi)}{|\nabla r(\xi)|} \right) & \lambda > m
\end{cases}
\]

Notice that, since \( T_{\lambda,m}(\xi_0) \) is of finite rank, for all \( r > 0 \) we have

\[
n_\pm(r; S_1(\lambda)) = n_\pm(r; S_2(\lambda)) + O(1).
\]

Further, set

\[
\begin{cases}
\int_{m^2, M^2_{\lambda}} d\rho \frac{1}{\rho - \lambda^2} \int_{r_m^1(\rho)} d\gamma \left( \frac{\hat{T}_{\lambda,m}(\xi)}{|\nabla r(\xi)|} \right) & 0 \leq \lambda < m \\
\int_{m^2, M^2_{\lambda}} d\rho \frac{1}{\rho - \lambda^2} \int_{r_m^1(\rho)} d\gamma \left( \frac{\hat{T}_{\lambda,m}(\xi)}{|\nabla r(\xi)|} \right) & \lambda > m
\end{cases}
\]

Lemma 4.3. For \( \lambda \) near \( \pm m \) and \( \xi \) small enough we have

\[
\|\hat{T}_{\lambda,m}(\xi) - \hat{T}_{\lambda,m}(\xi)\|_1 \leq C(|\lambda \mp m| r(\xi)^{\frac{\beta}{2}})((\pm m + m) + |\lambda + m| + r(\xi)^{1/2})
\]

Proof. Let us start by defining

\[
s_{\lambda,m}(\xi) := t_{\lambda,m}(\xi) - t_{\lambda,m}(\xi_0) - t_{\pm,m}(\xi) + t_{\pm,m}(\xi_0).
\]

One can readily see that \( s_{\lambda,m}(\xi) : L^2(T^2; \mathbb{C}^3) \rightarrow \mathbb{C}^3 \) is given by

\[
\begin{pmatrix}
f_1 \\
f_2 \\
f_3
\end{pmatrix} \mapsto \frac{(\lambda \mp m) \int_{T^2} d\eta (\hat{g}_1(\xi - \eta) - \hat{g}_1(-\eta)) f_1(\eta)}{0}.
\]

from which together with (28) we obtain

\[
\|s_{\lambda,m}(\xi)\|_2 \leq |\lambda \mp m| |\xi|^\beta.
\]
Noticing
\[ \hat{T}_{\lambda,m}(\xi) - \hat{T}_{\pm m,m}(\xi) \]
\begin{align}
&= (s^+_{\lambda,m}(\xi)^* - t_{\pm m,m}(\xi_0)^*) A t_{\lambda,m}(\xi) + t_{\lambda,m}(\xi_0)^* A (s^+_{\lambda,m}(\xi) + s^-_{\lambda,m}(\xi)) \\
&+ t_{\pm m,m}(\xi)^* A (s^+_{\lambda,m}(\xi) + t_{\lambda,m}(\xi_0)) + (s^+_{\lambda,m}(\xi)^* - t_{\lambda,m}(\xi_0)^*) A t_{\pm m,m}(\xi_0) \\
&= s^+_{\lambda,m}(\xi)^* A t_{\lambda,m}(\xi) + t_{\lambda,m}(\xi_0)^* A s^+_{\lambda,m}(\xi) + t_{\pm m,m}(\xi)^* A s^+_{\lambda,m}(\xi) + s^+_{\lambda,m}(\xi) A t_{\pm m,m}(\xi_0),
\end{align}

one concludes taking into account (24) and (57). \(\square\)

From this lemma one can easily see that
\[ ||S_\lambda(\lambda) - S_\lambda(-m)|| = O((\lambda + m)), \quad \lambda \to \pm m. \]

Using (55) we can deduce that there exists \(\rho_0\) such that for all \(m^2 < \rho \leq \rho_0\)
\[ \int_{r_m(\rho)} d\gamma \frac{||\hat{T}_{\pm m,m}(\xi)||}{|\nabla r(\xi)|} \leq C \left( |\pm m + m|^{2(\rho - m^2)^{-1/2}} + |\pm m + m|^{(\rho - m^2)^{1/2}} + (\rho - m^2) \right), \]
thus
\[ S_\lambda(\pm m) := \int_{m^2}^{M_m^2} \frac{d\rho}{\rho - m^2} \int_{r_m(\rho)} d\gamma \frac{\hat{T}_{\pm m,m}(\xi)}{|\nabla r(\xi)|} \]
is well defined in the trace class.

**Lemma 4.4.** For \(m > 0\)
\[ ||S_\lambda(\lambda) - S_\lambda(-m)|| = O((\lambda + m) \ln((\lambda + m))) \quad \lambda \to -m. \]
\[ ||S_\lambda(\lambda) - S_\lambda(0)|| = O(\ln(\lambda)), \quad \lambda \to 0. \]
\[ ||S_\lambda(\lambda) - S_\lambda(m)|| = O(1), \quad \lambda \to m. \]

**Proof.** We only give the proof of (61) as the others use similar ideas. Let us fix \(\rho_0\) such that (60) holds. Without loss of generality suppose \((\frac{3\lambda + m}{2})^2 < \rho_0\). We start by comparing
\[ \left\| \int_{\rho_0}^{M_m^2} d\rho \left( \frac{1}{\rho - \lambda^2} \int_{r_m(\rho)} d\gamma \frac{\hat{T}_{\lambda,m}(\xi)}{|\nabla r(\xi)|} - \frac{1}{\rho - m^2} \int_{r_m(\rho)} d\gamma \frac{\hat{T}_{-m,m}(\xi)}{|\nabla r(\xi)|} \right) \right\|_1. \]

By Lemma 4.3 we see that it is bounded by
\[ C|\lambda + m| \int_{\rho_0}^{M_m^2} d\rho \left( \frac{\lambda - m}{\rho - \lambda^2}(\rho - m^2) \int_{r_m(\rho)} d\gamma \frac{||\hat{T}_{\lambda,m}(\xi)||}{|\nabla r(\xi)|} + \frac{1}{\rho - m^2} \int_{r_m(\rho)} d\gamma \right), \]
where the last inequality follows by recalling the proof of Lemma 3.4. Next, for \(\lambda < -m < 0\), the remainder term is given by
\[ \int_{m^2}^{\lambda - m} d\rho \left( \frac{1}{\rho - \lambda^2} - \frac{1}{\rho - m^2} \right) \int_{r_m(\rho)} d\gamma \frac{\hat{T}_{\lambda,m}(\xi)}{|\nabla r(\xi)|} \]
\[ - \int_{(\frac{\lambda + m}{2})^2}^{\frac{3\lambda - m}{2}} d\rho \frac{1}{\rho - m^2} \int_{r_m(\rho)} d\gamma \frac{\hat{T}_{\lambda,m}(\xi)}{|\nabla r(\xi)|} \]
\[ + \int_{\rho_0}^{\rho_1} d\rho \frac{1}{\rho - m^2} \int_{r_m(\rho)} d\gamma \frac{\hat{T}_{\lambda,m}(\xi) - \hat{T}_{-m,m}(\xi)}{|\nabla r(\xi)|}. \]
Notice that for $-m < \lambda < 0$ the second term vanishes and the first integral is over $[m^2, \rho_0]$. First we see from (55) that,

\[
\left\| \int_{r_m^{-1}(\rho)} \frac{\dot{T}_{\lambda,m}(\xi)}{|\nabla r(\xi)|} \right\| \leq C \int_{r_m^{-1}(\rho)} \frac{\lambda + \rho}{r(\xi)} \frac{d\gamma}{r(\xi)^{\frac{3}{2}}} + \lambda + \rho r(\xi)^{\frac{1}{2}} + r(\xi) \leq C \left( |\lambda + \rho| + |\rho - m^2|^{\frac{3}{2}} + |\rho - m^2|^{\frac{1}{2}} + |\rho - m^2| \right).
\]

To estimate the first term of (64) for $\lambda \uparrow -m$, we use this last inequality and that for $\lambda \in [m^2, (\frac{m^2 - m}{2})^2] \cup [(\frac{m^2 + m}{2})^2, \rho_0]$ we have that $|\lambda^2 - \rho| \geq C|\lambda + m|$. So

\[
\int_{m^2}^{(\frac{m^2 - m}{2})^2} \frac{d\rho}{|\rho - \lambda^2| |\rho - m^2|} \left( |\lambda + m| + |\rho - m^2|^{\frac{3}{2}} + |\lambda + m| + |\rho - m^2|^{\frac{1}{2}} + |\rho - m^2| \right) \leq C \int_{m^2}^{(\frac{m^2 - m}{2})^2} d\rho \left( (\lambda + m)^2 + |\rho - m^2|^{\frac{3}{2}} + |\rho - m^2|^{\frac{1}{2}} + 1 \right) \leq C|\lambda + m|,
\]

and

\[
\int_{(\frac{m^2 + m}{2})^2}^{\rho_0} \frac{|\lambda^2 - m^2|}{|\rho - \lambda^2| |\rho - m^2|} \left( |\lambda + m| + |\rho - m^2|^{\frac{3}{2}} + |\lambda + m| + |\rho - m^2|^{\frac{1}{2}} + |\rho - m^2| \right) \leq C \left( |\lambda + m|^2 + |\lambda + m| + \int_{(\frac{m^2 + m}{2})^2}^{\rho_0} \frac{|\lambda^2 - m^2|}{|\rho - \lambda^2|} \right) \leq C(|\lambda + m| \ln(|\lambda + m|)) .
\]

Finally, the previous computation also gives us for $-m < \lambda < 0$ (and hence $0 < \lambda^2 < m^2 < \rho$)

\[
\int_{m^2}^{\rho_0} \frac{|\lambda^2 - m^2|}{|\rho - \lambda^2| |\rho - m^2|} \left( |\lambda + m|^2 + |\rho - m^2|^{\frac{3}{2}} + |\lambda + m| + |\rho - m^2|^{\frac{1}{2}} + |\rho - m^2| \right) \leq C(|\lambda + m| \ln(|\lambda + m|)) .
\]

5. Proof of the main results for hyperbolic thresholds

We turn now our attention to the hyperbolic thresholds. For ease of notation we will set $m = 0$ and consider the positive hyperbolic threshold $\tau = 2$.

First, taking into account (38), we will need to study the operator

\[
R(\lambda) := \text{p.v.} \int_0^{M_0} \frac{2u}{u^2 - \lambda^2} \int_{r^{-1}(u^2)} d\gamma \frac{T_\lambda(\xi)}{|\nabla r(\xi)|}.
\]

Since $V$ is trace class, for all $r > 0$

\[
n_{+}(r; \text{Re}(K(\lambda + i0))) = n_{+}(r; \frac{1}{\lambda} R(\lambda)) + O(1).
\]

When integrating on $\mathbb{T}^2$, due to the periodicity, we will focus on the rectangle $[0, 1/2] \times [0, 1/2]$. Let us take the critical points $\xi_a = (1/2, 0)$ and $\xi_b = (0, 1/2)$ in $\mathbb{T}^2$. Notice that the level curve defined by the threshold $\tau = 2$, namely $r^{-1}(4)$, pass through these points.

Define the triangles $\Delta_a := \{ \xi \in [0, 1/2] \times [0, 1/2] : \xi_a \leq \xi \}$ and $\Delta_b := \{ \xi \in [0, 1/2] \times [0, 1/2] : \xi_b \leq \xi \}$ and set

\[
\tilde{T}_\lambda(\xi) := 1_{\Delta_a}(\xi)(T_\lambda(\xi) - T_\lambda(\xi_a)) + 1_{\Delta_b}(\xi)(T_\lambda(\xi) - T_\lambda(\xi_b)).
\]

For simplicity, all the following computations will be performed on the triangle $\Delta_a$. We will also assume that $\lambda \geq 2$. 

\[
\text{(65)}
\]
Denote by $\gamma_\lambda$ the part of the curve $r^{-1}(\lambda^2)$ that lies in $\triangle_a$. It admits the following parametrization:

\[(66)\quad \xi_1(t) = \frac{1}{2\pi} \arccos \left(1 - \frac{t}{2}\right); \quad \xi_2(t) = \frac{1}{2\pi} \arccos \left(1 + \frac{t - \lambda^2}{2}\right),\]

for $t \in [\frac{\lambda^2}{2}, 4]$. We can easily compute

\[(67)\quad \xi'_1(t) = \frac{1}{2\pi t^{1/2}(4 - t)^{1/2}}; \quad \xi'_2(t) = -\frac{1}{2\pi (\lambda^2 - t)^{1/2}(4 + t - \lambda^2)^{1/2}},\]

so

\[(68)\quad d\gamma \leq C((4 - t)^{-1/2})dt .\]

Most computations will make use of this explicit parametrization, but other parts of $\mathbb{T}^2$ can be dealt with accordingly, as can be the case $\lambda \leq 2$.

**Lemma 5.1.** The following asymptotic relation holds true

\[
\left\| \int_{\gamma_\lambda} \frac{\partial \mathcal{L}(\xi)}{\partial r(\xi)} \right\|_1 = O(1), \quad \lambda \to \tau. \tag{69}
\]

**Proof.** For $\xi \in \triangle_a$, from (26) and (30) we can see that

\[
(68)\quad ||\mathcal{T}(\xi)||_1 \leq C|\xi - \xi_a|^\beta .
\]

Further, we have that $|\nabla r(\xi)|^2 = r(\xi - \xi_a) \times |\xi - \xi_a|^2$ so

\[(69)\quad \left\| \frac{\partial \mathcal{L}(\xi)}{\partial r(\xi)} \right\|_1 \leq C|\xi - \xi_a|^\beta - 1 ,
\]

when $|\lambda - \tau|$ is small enough. Therefore

\[
\left\| \int_{\gamma_\lambda} d\gamma \frac{\partial \mathcal{L}(\xi)}{\partial r(\xi)} \right\|_1 \leq C \int_{\gamma_\lambda} d\gamma |\xi - \xi_a|^\beta - 1 .
\]

A direct inspection gives us that for $t \in [\frac{\lambda^2}{2}, 4]$

\[(70)\quad |\xi(t) - \xi_a| \leq C(4 - t)^{\frac{1}{2}} .
\]

From this we obtain

\[
\int_{\gamma_\lambda} d\gamma |\xi - \xi_a|^\beta - 1 \leq C \int_{\frac{\lambda^2}{2}}^{4} dt \frac{(4 - t)^{\frac{3}{2} - 1}}{(4 - t)^{\frac{3}{2}}} \leq C \int_{\frac{\lambda^2}{2}}^{4} dt(4 - t)^{\frac{3}{2} - 1} = O(1) \text{ as } \lambda \to \tau . \quad \square
\]

Set

\[
(71)\quad \mathcal{R}_\lambda(\lambda) := \text{p.v.} \int_0^{M_\lambda} \frac{2u}{u^2 - \lambda^2} \int_{t^{-1}(u^2)} d\gamma \frac{T_\lambda(\xi)}{|\nabla r(\xi)|}.
\]

Denote by $\gamma_{\lambda,b}$ the part of the curve $r^{-1}(\lambda^2)$ that lies in $\triangle_b$, then

\[
\int_{t^{-1}(\lambda^2)} d\gamma \frac{T_\lambda(\xi) - T_\lambda(\xi)}{|\nabla r(\gamma)|} = T_\lambda(\xi_a) \int_{\gamma_\lambda} d\gamma \frac{1}{|\nabla r(\gamma)|} + T_\lambda(\xi_b) \int_{\gamma_{\lambda,b}} d\gamma \frac{1}{|\nabla r(\gamma)|}.
\]
is an operator with rank at most twelve. Thus, we can use Lemma 3.1 together with (37) and (65) and Lemma 5.1 to obtain that for any $\epsilon \in (0, 1)$

$$\pm n_\mp \left(1 \pm \epsilon; \frac{1}{\lambda} R_1(\lambda)\right) + O(1) \leq \xi(\lambda; H_\pm, H_0)$$

(72)

$$\leq \pm n_\mp \left(1 \mp \epsilon; \frac{1}{\lambda} R_1(\lambda)\right) + O(1).$$

as $\lambda \to \tau$.

Lemma 5.2. There exists $u_0 > 0$ such that for $u, u'$ in $(2 - u_0, 2)$ or in $[2, 2 + u_0)$

$$\frac{1}{|u - u'|^{1/2}} \ln |u - u'|^{-1}, \quad |u - u'| \to 0.$$

Proof. Assume that $\sqrt{8} > u' > u \geq \tau = 2$. As before we consider the triangle $\Delta_a$. Recall that we denote by $\gamma_a$ the part of the curve $r^{-1}(u^2)$ that lies in $\Delta_a$.

Set $\zeta := (u^2 - u'^2)/2$ and define the function

$$D_\zeta(\xi_1, \xi_2) = \left(\frac{1}{2\pi} \arccos(\zeta + \cos(2\pi \xi_1)), \xi_2\right).$$

Notice that $D_\zeta(\gamma'_a) \subset \gamma_u$ and

$$\frac{d}{d\gamma} \left| \frac{T_\lambda(\xi)}{\nabla r(\xi)} \right| (\gamma_u) = \int_{\gamma_u} D_\zeta d\gamma \frac{T_\lambda(D_\zeta)}{|\nabla r(D_\zeta)|} + \int_{\gamma_u \setminus D_\zeta} d\gamma \frac{T_\lambda(\xi)}{|\nabla r(\xi)|},$$

so we need to estimate

(73)

$$\int_{\gamma_u} (D_\zeta d\gamma - d\gamma) \frac{\hat{T}_\lambda(D_\zeta)}{|\nabla r(D_\zeta)|} + \int_{\gamma_u \setminus D_\zeta} d\gamma \frac{T_\lambda(D_\zeta) - \hat{T}_\lambda(\xi)}{|\nabla r(D_\zeta)|} + \int_{\gamma_u \setminus D_\zeta} d\gamma \frac{T_\lambda(\xi)}{|\nabla r(\xi)|} - \frac{1}{|\nabla r(\xi)|} = (I) + (II) + (III),$$

and

$$\int_{\gamma_u \setminus D_\zeta} d\gamma \frac{T_\lambda(\xi)}{|\nabla r(\xi)|} := (IV).$$

We start by estimating (IV). It is easy to see that $\gamma_u \setminus D_\zeta$ can be parametrized by (66) with $t \in [4 + u^2 - u'^2, 4]$. Then, by (67), (69) and (70)

(74)

$$\left| \int_{\gamma_u \setminus D_\zeta} d\gamma \frac{T_\lambda(\xi)}{|\nabla r(\xi)|} \right| \leq C \int_{4 + u^2 - u'^2} dt (4 - t)^{\beta/2 - 1} \leq C |u' - u|^\beta/2.$$

To estimate $(I) + (II) + (III)$ a partition of $\Delta_a$ is needed, namely $\Delta_{a,1} := \{\xi \in \Delta_a : \xi_1 \geq 3/8, \xi_2 \leq 1/8\}$ and $\Delta_{a,2} := \Delta_a \setminus \Delta_{a,1}$. Accordingly we write $\gamma_{a,j}$ for the part of the curve $r^{-1}(u^2)$ that lies in $\Delta_{a,j}$, $j = 1, 2$. The same for $(I_1), (II_1), (III_1)$.

Let us start by studying $(I_1), (II_1), (III_1)$. Consider first $(I)$ on $\Delta_{a,1}$. We have that $|d\gamma - D_{\xi} d\gamma| \leq |\partial_1 D_{\xi,1}(\xi)^2 - 1| d\gamma$, and

(75)

$$|\partial_1 D_{\xi,1}(\xi)^2 - 1| = \frac{|\zeta(2 \cos 2\pi \xi_1 + \zeta)|}{|\sin^2 2\pi \xi_1 - \zeta(2 \cos 2\pi \xi_1 + \zeta)|}.$$

On $\Delta_{a,1}$ we have $-\zeta(2 \cos 2\pi \xi_1 + \zeta) > \tilde{C} \xi_1$, for a positive $\tilde{C}$ (when $\zeta$ is sufficiently small). Putting all this together with 69

(76)

$$\| (I_1) \|_1 \leq C \int_{\gamma_{u',1}} |d\gamma - D_{\xi} d\gamma| |\xi - \xi_a|^{\beta - 1} \leq C \xi \int_{\gamma_{u',1}} d\gamma \frac{|\xi - \xi_a|^{\beta - 1}}{|\sin^2 2\pi \xi_1 + \zeta|}.$$
and since
\[ \sin^2 2\pi \xi(t) = \frac{t(4-t)}{4}, \]
by (67) and (70)
\[ \| (I_1) \|_1 \leq C \xi \int_{u^2/2}^4 \frac{dt}{4-t + \xi} \leq C \xi^{3/2} \int_0^\infty ds \frac{s^{3/2-1}}{s + C}. \]

Now, for \((I_1)_1\), proceeding as in (45) we can show that
\[ \| \hat{T}_\lambda(D_\xi \xi) - \hat{T}_\lambda(\xi) \|_1 \leq C|\xi - D_\xi \xi|^\beta \leq C|\xi|^\beta. \]
Further
\[ |\nabla r(D_\xi \xi)|^2 = |\nabla r(\xi)|^2 - 16\pi \xi(\xi + 2 \cos 2\pi \xi), \]
and using the same parametrization as before
\[ |\nabla r(D_\xi \xi(t))|^2 = 4\pi^2 t(4-t) + 4\pi^2 (4-u^2 + t)(u^2 - t) - 16\pi^2 \xi(\xi + 2 - t)) \]
for some \(\hat{B}, \hat{C} > 0\). Then,
\[ \| (II_1) \|_1 \leq \left| \int_{\gamma_{u',1}} d\gamma \frac{\hat{T}_\lambda(D_\xi \xi) - \hat{T}_\lambda(\xi)}{|\nabla r(D_\xi \xi)|} \right|_1 \leq C|\xi|^{3/2} \int_{\gamma_{u',1}} d\gamma \frac{1}{|\nabla r(D_\xi \xi)|} \leq C|\xi|^{3/2} \int_{\gamma_{u',1}} d\gamma \frac{(4-t)^{-1/2}}{4-t + \hat{C} \xi} \leq C|\xi|^{3/2} \ln(\xi^{-1}). \]
Finally consider \((III_1)_1\). Since
\[ \left| \int_{\gamma_{u',2}} d\gamma \frac{\xi - \xi_u |^{\beta-1}}{|\nabla r(D_\gamma)|^2} \right| \leq \left| \int_{\gamma_{u',2}} d\gamma \frac{(4-t)^{3/2-1}}{(4-t + \hat{C} \xi)^{1/2}} \right| \leq C|\xi|^{3/2}. \]

Similar arguments can be used to deal with \((I_2), (II_2), (III_2)\). In fact, for these cases the computations are simpler because if \(u_0 \) is small enough all the functions appearing on the integrals of (73) are H"older continuous (see Lemma 3.2). Moreover, the denominators are non-vanishing, so the result follows immediately when we restrict to \(\Delta_{a,2}\).

For \(\lambda > 2\) we write principal value part of the operator \(\mathcal{R}_1(\lambda)\) (defined in (71)) as follows
\[ \text{p.v.} \int_0^{M_0} \frac{2u}{(u^2 - \lambda^2)} \int_{r^{-1}(u^2)} d^4 \frac{\hat{T}_\lambda(\xi)}{|\nabla r(\xi)|} \]
\[ = \int_{[0, (\lambda + 2)/2] \cup [(3\lambda - 2)/2, M_0]} \frac{2u}{(u^2 - \lambda^2)} \int_{r^{-1}(u^2)} d^4 \frac{\hat{T}_\lambda(\xi)}{|\nabla r(\xi)|} + \int_0^{(\lambda - 2)/2} \frac{d\rho}{\rho} \left( \int_{r^{-1}(\rho(\lambda + \rho)^2)} d^4 \frac{\hat{T}_\lambda(\xi)}{|\nabla r(\xi)|} - \frac{2(\lambda - \rho)}{(2\lambda + \rho)} \int_{r^{-1}(\lambda - \rho)^2} d^4 \frac{\hat{T}_\lambda(\xi)}{|\nabla r(\xi)|} \right) \]
\[ =: \tilde{\mathcal{R}}_1(\lambda) + \mathcal{R}_{1, \text{pv}}(\lambda). \]

Thus from Lemma 5.2 we can readily see that
\[ \| \mathcal{R}_{1, \text{pv}}(\lambda) \|_1 = o(2 - \lambda), \quad \lambda \to 2. \]
Set
\[ \mathcal{R}_2(\lambda) := \int_{[0,(\lambda+2)/2] \cup [(3\lambda-2)/2, M_0]} \frac{2u}{(u^2 - \lambda^2)} \int_{r^{-1}(\lambda^2)} \frac{\hat{T}_\lambda(\xi)}{\left| \nabla r(\xi) \right|} \, du. \]

From Lemma 5.2 we can see that
\[ ||\hat{R}_1(\lambda) - \mathcal{R}_2(\lambda)||_1 = O(1). \]

Finally, the function
\[ \lambda \mapsto \int_{[0,(\lambda+2)/2] \cup [(3\lambda-2)/2, M_0]} \frac{2u}{(u^2 - \lambda^2)} = \ln \left( \frac{(3\lambda + 2)(M_0^2 - \lambda^2)}{\lambda^2(5\lambda - 2)} \right) \]

is bounded for \( \lambda \to \tau = 2 \), so \( \mathcal{R}_2(\tau) \) is well defined and compact. Moreover, we notice that for \( \xi \in \mathcal{U} \)
\[ ||\hat{T}_\lambda(\xi) - \hat{T}_2(\xi)||_1 = \left| \left| \left( (t_\lambda(\xi))^{-1} - (t_2(\xi))^{-1} \right) A t_\lambda(\xi) \right| \right|_{1,1} \]
\[ + \left| \left| \left( (t_2(\xi))^{-1} - (t_2(\xi))^{-1} \right) A (t_\lambda(\xi) - t_2(\xi)) \right| \right|_{1,1} \]
\[ + \left| \left| \left( t_\lambda(\xi) - t_2(\xi) \right) A (t_\lambda(\xi) - t_2(\xi)) \right| \right|_{1,1} \]
\[ \leq C|\lambda - 2| ||\xi - \xi_2||^2. \]

An analogous computations holds when \( \xi \in \mathcal{U}_0 \). From (83), together with Lemma 5.2 we obtain that
\[ \mathcal{R}_1(\lambda) \to \mathcal{R}_1(\tau), \quad \lambda \to \tau = 2, \]
in the trace class norm.

In consequence, putting together with (72), (78), (80), (81) and (84) we conclude that if \( V \) satisfies (6) then
\[ \xi(\lambda; H_\pm, H_0) = O(1), \quad \lambda \to 2. \]

6. Eigenvalue Asymptotics for Integral Operators with Toroidal Kernel

Denote by \( S^*_\gamma(\mathbb{Z}^d) \) the class of symbols given by the functions \( v : \mathbb{Z}^d \to \mathbb{C} \) that satisfies for any multi-index \( \alpha \)
\[ |D^\alpha v(\mu)| \leq C_\alpha (\mu)^{-\gamma - \rho|\alpha|}, \]
where \( D_\mu v(\mu) := v(\mu + \delta_j) - v(\mu) \), and \( D^\alpha := D_{\alpha_1} \cdots D_{\alpha_d} \).

Condition 1: For \( \gamma > d, \rho > 0 \) assume \( \{v_k\}_{k=1}^N \in S^*_\gamma(\mathbb{Z}^d) \). We suppose also that
\[ v_k(\mu) = v_0(\mu)(T_k + o(1)), \quad |\mu| \to \infty, \]
for a function \( v_0 : \mathbb{Z}^d \to \mathbb{C} \) which, viewed as a multiplication operator, satisfies
\[ n_\pm(\lambda; v_0) = \lambda^{-d/\gamma}(c_\pm + o(1)), \quad \lambda \downarrow 0, \]
with \( c_\pm > 0 \).

Let \( \{B_k\}_{k=1}^N \) be a family of functions in \( L^2(\mathbb{T}^d) \). Define
\[ \Psi := \sum_{k=1}^N B_k(\xi)\mathcal{F}v_k(\xi)\mathcal{F}(\xi), \]
This is a compact operator on \( L^2(\mathbb{T}^d) \) and has integral kernel given by
\[ \sum_{k=1}^N B_k(\xi)\hat{v}_k(\xi - \eta)\mathcal{F}(\eta). \]
Theorem 6.1. Assume Condition 1 and Ψ as above. Then
\[ n_{\pm}(\lambda; \Psi) = C_{B\pm} \lambda^{-d/\gamma}(1 + o(1)), \quad \lambda \downarrow 0, \]
where
\[ C_{B\pm} = c_{\pm} \int_{\mathbb{T}^d} \, \mathrm{d}f \left( \sum_{k=1}^{N} \Gamma_k |B_k(\xi)|^2 \right)^{d/\gamma}. \]

Remark 6.2. Using the ideas of [BS70], Theorem 6.1 can be easily extended to cover more general kernels. In particular, in (88) one could consider matrix-valued \( w \) or introduce another hermitian matrix-valued function depending on \( \xi \) and \( \eta \). Here we limit ourselves to setting we were aiming to apply it. Although Theorem 6.1 bears resemblance with [BS70, Theorem 1.], we would like to stress out that it does not follows directly from it or it’s related results (see for instance [BS77b; BKS91]).

Set \( \square := [0,1]^d \). In order to prove the following lemma we need to use the spaces of compact operators \( S_{p,w} \) defined for \( 0 < p < \infty \) by
\[ S_{p,w} : \{ K \in S_{\infty} : s_{n}(K) = O(n^{-1/p}) \} \]
with the quasi-norm
\[ ||K||_{p,w} := \sup_{n} \{ n^{1/p} s_{n}(K) \} = \left( \sup_{s > 0} \{ s^{p} n_{s}(s; K) \} \right)^{1/p}. \]
These spaces satisfy the *weakened triangle inequality* \( ||K_1 + K_2||_{p,w} \leq 2^{1/p}(||K_1||_{p,w} + ||K_2||_{p,w}) \) and the *weakened Hölder inequality* \( \]
\[ ||K_1 K_2||_{r,w} \leq c(p, q)||K_1||_{p,w}||K_2||_{q,w}, \]
for \( r^{-1} = p^{-1} + q^{-1} \) and \( c(p, q) = \frac{1}{p/r} \left( \frac{q/r}{1,q} \right) \) (see [Chapter 11][BS87]).

In order to prove Theorem 6.1 we need the following Cwikel--type estimate.

Lemma 6.3. Assume that \( |v(\mu)| \leq C(\mu)^{-\gamma} \). Then, if \( p, q \in L^2(\square) \), there exists a positive constant \( C(v, d) \) such that
\[ ||p \mathcal{F} v; \mathcal{F}^* q||_{d/\gamma,w} \leq C(v, d)||p||_{L^2(\square)}||q||_{L^2(\square)} \]

Proof. By (90)
\[ ||p \mathcal{F} v; \mathcal{F}^* q||_{d/\gamma,w} \leq ||p \mathcal{F} v||^{1/2}_{2d/\gamma,w}||v||^{1/2}_{2d/\gamma,w} ||\mathcal{F}^* q||_{2d/\gamma,w}. \]
Then it is enough to prove \( ||p \mathcal{F} v||^{1/2}_{2d/\gamma,w} \leq C||p||_{L^2(\square)}. \) First notice that by the min-max principle the singular values satisfies
\[ s_{n}(p \mathcal{F} v) = \lambda_{n}(p \mathcal{F} v; \mathcal{F}^* \mathcal{P}) \leq C \lambda_{n}(p \mathcal{F} w; \mathcal{F}^* \mathcal{P}), \]
where \( w(\mu) = |\mu|^{-\gamma} \), and \( \{\lambda_{n}(K)\} \) denotes the non increasing sequence of eigenvalues of \( 0 \leq K \in S_{\infty} \).

Set \( g(x) := |x|^{-\gamma} \). Since \( \gamma > d \), we have that \( g \in L^1(\mathbb{R}^d) \) and \( \hat{g} \in C_{0}(\mathbb{R}^d) \). Here we use the standard notation \( \hat{g} \) for
\[ \hat{g}(y) := \int_{\mathbb{R}^d} \, \mathrm{d}x e^{-2\pi iy \cdot x} g(x). \]

Further, \( \hat{g} \) is smooth in \( \mathbb{R}^d \setminus \{0\} \) and decay at infinity faster than \( |y|^{-n} \) for any \( n \in \mathbb{Z}^+ \) (same for its derivatives). Then, we can use the Poisson summation formula (see for instance [Gra08]) to obtain
\[ \sum_{\mu \in \mathbb{Z}^d} \omega(\mu) e^{-2\pi i \mu \cdot x} = \sum_{\mu \in \mathbb{Z}^d \setminus \{0\}} \hat{\omega}(\mu) e^{-2\pi i \mu \cdot x} + \omega(0) \]
\begin{equation}
(94) \quad \sum_{\mu \in \mathbb{Z}^d} \hat{g}(-x + \mu) - \hat{g}(0) + \omega(0).
\end{equation}

Set \( G(x) = g(x) \sum_{\mu \in (-2, 2)^d} e^{2\pi i \mu \cdot x} \). Then, it is clear that \( \hat{G}(x) = \sum_{\mu \in \mathbb{Z}^d \setminus (-2, 2)^d} \hat{g}(-x + \mu) \) is smooth in \( \mathbb{R}^d \setminus (-2, 2)^d \). Further, \( \phi(x) := \sum_{\mu \in \mathbb{Z}^d \setminus (-2, 2)^d} \hat{g}(x - \mu) - \hat{g}(0) + \omega(0) \) is smooth in \( (-2, 2)^d \). To see this take \( x \in (-2, 2)^d \). Then \( x - \mu \neq 0 \) for all \( \mu \in \mathbb{Z}^d \setminus (-2, 2)^d \) and so, each function \( \hat{g}(\cdot - \mu) \) is smooth in \( (-2, 2)^d \) when \( \mu \in \mathbb{Z}^d \setminus (-2, 2)^d \). Moreover, since each partial derivative of \( \hat{g} \) decays fast, using the Lebesgue dominated convergence theorem permit us to derivate term by term the series of \( \phi \).

Further, from (94) we make the decomposition
\begin{equation}
(95) \quad p \mathcal{F}_\omega \mathcal{F}^* p = p \mathcal{F} G \mathcal{F}^* p + p \Phi p,
\end{equation}
where \( \Phi \) is the operator in \( L^2(\mathbb{R}) \) with integral kernel \( \phi(\xi - \eta) \). Notice that the non-zero eigenvalues of \( p \mathcal{F} G \mathcal{F}^* p \) are the same as those of the operator \( 1 \mathbb{I}[p \mathcal{F} G \mathcal{F}^* p] \) in \( L^2(\mathbb{R}^d) \). Then, by (92) and (95)
\[
||p \mathcal{F}||^{1/2} \leq C (||p \mathcal{F}||^{1/2} + ||p \Phi||^{1/2})
\]

The following definition is taken from [BKS91, Subsection 5.6] and we recall it for convenience of the reader. Let \( \phi \in L^1_{\text{loc}}(\mathbb{R}^d) \) and for \( \mu \in \mathbb{Z}^d \) define
\[
a_{\phi}(\mu) = (\mu)^2 (\mathcal{F} \phi(x) dx)^{1/2}.
\]
For \( \alpha > 0 \) we introduce the spaces \( l_{\alpha, w}(\mathbb{Z}^d, L^2(\mathbb{R})) \) as the set of functions \( \phi \in L^2_{\text{loc}}(\mathbb{R}^d) \) that satisfies
\[
\#\{\mu \in \mathbb{Z}^d : |a_{\phi}(\mu)| > t\} = O(t^{-\alpha}).
\]

We want to show that \( G^{1/2} \in l_{2d/\gamma, w}(\mathbb{Z}^d, L^2(\mathbb{R})) \). For this it is enough to notice that the sequence \((\mu)^{-d} dx)^{1/2} \leq C (\mu)^{-d/\gamma} \) and
\begin{equation}
(96) \quad \#\{\mu : (\mu)^{-d/\gamma} > t\} = O(t^{-2d/\gamma}).
\end{equation}

Then, since \( 2d/\gamma < 2 \), by [BKS91, Subsection 5.7]
\begin{equation}
(97) \quad ||\mathbb{1}[p \mathcal{F} G^{1/2}]||_{2d/\gamma, w} \leq C ||G^{1/2}||_{2d/\gamma, w} ||\mathbb{1}[p]||_{L^{2d/\gamma}(\mathbb{R}^d)}.
\end{equation}

On the other side, since \( \phi \) is smooth in \( B_0(\sqrt{d}) \), it is in any Besov Space \( B^0_{p, \infty}((-1, 1)^d) \) (notice that \((-1, 1)^d = \mathbb{R}^d \)). By [BS77, Theorem 6, page 274] or [BS77a, Theorem 6.1]
\begin{equation}
(98) \quad ||p \Phi p||_{d/\gamma, w} \leq C ||p||_{L^2(\mathbb{R}^d)}^2.
\end{equation}
Since \( 2d/\gamma < 2 \), we can conclude from (89), (91), (95), (97) and (98) that
\[
||p \mathcal{F} q||_{d/\gamma, w} \leq C (||p||_{L^2(\mathbb{R}^d)} + ||p||_{L^{2d/\gamma}(\mathbb{R}^d)}) (||q||_{L^2(\mathbb{R}^d)} + ||q||_{L^{2d/\gamma}(\mathbb{R}^d)})
\]
\[
\leq C ||p||_{L^2(\mathbb{R}^d)} ||q||_{L^2(\mathbb{R}^d)}
\]

\( \Box \)

Lemma 6.3 enable us to adapt the proof of [BS70, Theorem 1] to our case.
Lemma 6.4. Let $X, Y \subset \mathbb{R}^d$ such that they do not have interior points in common. Then, if $v \in S^0_0(\mathbb{R}^d)$

$$n_+(s; 1_X v F^* 1_Y) = o(s^{-d/\gamma}), \quad s \downarrow 0.$$  

Proof. By [RT09, Theorem 4.3.6] the kernel of the operator $F v F^*$ is smooth in $\mathbb{R}^d$ when $x \neq y$. Suppose first that $\text{dist}(X,Y) > 0$ and consider smooth functions $\nu_X, \nu_Y$ such that $\nu_X = 1$ in a neighborhood $O_X$ of $X$, $\nu_Y = 0$ in a neighborhood $O_Y$ of $Y$ and $\text{dist}(O_X, O_Y) > 0$. Similarly for $\nu_Y$. Then $1_X F v F^* 1_Y = 1_X \nu_X F v F^* \nu_Y 1_Y$ and the kernel of $\nu_X F v F^* \nu_Y$ is smooth in $\mathbb{R}^d$.

Then, invoking again [BS87, Theorem 6, page 274], for any $r > 1$

$$n_+(s; \nu_X F v F^* \nu_Y) = o(s^{-r}), \quad s \downarrow 0.$$  

Suppose now that $\text{dist}(X,Y) = 0$, fix $\epsilon > 0$ and set $X_\epsilon := \{ x \in X : \text{dist}(x,Y) > \epsilon \}$. Write $1_X F v F^* 1_Y = 1_{X_\epsilon} F v F^* 1_Y + 1_{X \setminus X_\epsilon} F v F^* 1_Y$.

By Lemma 6.3

(99) \[ s^{d/\gamma} n_+(s; 1_{X \setminus X_\epsilon} F v F^* 1_Y) \leq C \| 1_{X \setminus X_\epsilon} \|_{L^2(\mathbb{R}^d)}^d = O(\epsilon), \]

uniformly for $s > 0$. Further, by the first part of this proof, since $\gamma > d$,

(100) \[ n_+(s; 1_{X \setminus X_\epsilon} F v F^* 1_Y) = o(s^{-d/\gamma}), \quad s \downarrow 0. \]

Putting (16) together with (99) and (100) we conclude the proof. \qed

Lemma 6.5. Let \{\square_j\}_{j=1}^l be a partition of $\mathbb{R}^d$ into finite cubes of equal size. Let $\Psi_{\text{diag}} : L^2(\square) \to L^2(\square)$ be the integral operator with kernel

$$\sum_{j=1}^l 1_{\square_j}(x) \left( \sum_{k=1}^N |B_{k,j}|^2 \hat{\epsilon}_k(x-y) \right) 1_{\square_j}(y),$$

where $B_{k,j} \in \mathbb{C}$. Then, if $v_k$ satisfy the conditions of Theorem 6.1 we have

$$n_\pm(\lambda; \Psi_{\text{diag}}) = C_{\text{diag}} \lambda^{-d/\gamma}(1 + o(1)), \quad \lambda \downarrow 0,$$

with $C_{\text{diag}} = c_{\pm} \sum_{j=1}^l |\square_j| \left( \sum_{k=1}^N |B_{k,j}|^2 \Gamma_k \right)^{d/\gamma}$ and $|\square_j|$ is the Lebesgue measure of $\square_j$.

Proof. To simplify the notation assume $N = 1$. Evidently $\Psi_{\text{diag}} = \bigoplus_j |B_{1,j}|^2 \Psi_{j,\text{diag}}$, where the integral kernel of $\Psi_{j,\text{diag}}$ is $1_{\square_j}(x) \hat{\epsilon}_1(x-y) 1_{\square_j}(y)$. Therefore

$$n_\pm(\lambda; \Psi_{\text{diag}}) = \sum_{j=1}^l n_\pm(\lambda; |B_{1,j}|^2 \Psi_{j,\text{diag}}).$$

Now, since the kernel $\hat{\epsilon}_1(x-y) = \sum_{j=1}^l 1_{\square_j}(x) \hat{\epsilon}_1(x-y) 1_{\square_j}(y) + \sum_{j \neq i} 1_{\square_j}(x) \hat{\epsilon}_1(x-y) 1_{\square_i}(y)$, we have that

$$F^* v_1 F = \bigoplus_j \Psi_{j,\text{diag}} + R,$$

where $\sum_{j \neq i} 1_{\square_j}(x-y) 1_{\square_i}(y)$ is the kernel of $R$.

By (15) and Lemma 6.4, for any $\delta \in (0,1)$$n_\pm(\lambda; F^* v_1 F) \leq n_\pm(\lambda(1-\delta); \bigoplus_j \Psi_{j,\text{diag}}) + o(\lambda^{-d/\gamma})$$and$$n_\pm(\lambda; F^* v_1 F) \geq n_\pm(\lambda(1+\delta); \bigoplus_j \Psi_{j,\text{diag}}) + o(\lambda^{-d/\gamma}).$
We can take the square $\square_0 = [0, 1/m]^d$, for some $m \in \mathbb{Z}_0$, and assume that each $\square_j$ is just a translation of $\square_0$. Notice that in this case $l = m^d$ and $|\square_j| = 1/m^d$. Moreover, each operator $\Psi_{j, \text{diag}}$ is unitary equivalent with each other, so the two previous inequalities imply that for any $1 \leq j \leq m^d$ and $\delta \in (0, 1)$

$$n(\lambda(1 + \delta); v_1) + o(\lambda^{-d/\gamma}) \leq m^dn_{\pm}(\lambda; \Psi_{j, \text{diag}}) \leq n(\lambda(1 - \delta); v_1) + o(\lambda^{-d/\gamma}).$$

By (86) for all $\epsilon > 0$ there exists $M$ such that $|\mu| > M$ implies

$$|v_1(\mu) - \Gamma_1v_0(\mu)| < \epsilon|v_0(\mu)|.$$  

Then, for any $\delta \in (0, 1)$

$$n_+ (\lambda(1 + \delta); v_0^+(\mu) (\Gamma_1|B_{1,j}|^2 - \epsilon|B_{1,j}|^2)) + o(\lambda^{-d/\gamma}) \leq |\square_j|^{-1}n_{\pm} (\lambda; |B_{1,j}|^2\Psi_{j, \text{diag}}) \leq n_+ (\lambda(1 - \delta); v_0^-(\mu) (\Gamma_1|B_{1,j}|^2 + \epsilon|B_{1,j}|^2)) + o(\lambda^{-d/\gamma}).$$

Here $v_0^\pm$ are the positive and negative parts of $v_0$.

Now, multiplying by $\lambda^{-d/\gamma}$ and taking into account (87), the limit when $\lambda$ goes to zero is

$$c_\pm ((|\Gamma_k| - \epsilon)|B_{1,j}|^2)^{d/\gamma} \leq |\square_j|^{-1} \lim_{\lambda \downarrow 0} \lambda^{d/\gamma}n_{\pm} (\lambda; |B_{1,j}|^2\Psi_{j, \text{diag}}) \leq c_\pm ((|\Gamma_k| + \epsilon)|B_{1,j}|^2)^{d/\gamma}.$$  

Taking the limit when $\epsilon \downarrow 0$ we finish the proof.  
\hfill \Box

**Proof of Theorem 6.1**

*Proof.* Let $\epsilon > 0$ and for any $1 \leq k \leq N$ take a step function in $\mathbf{I}$ of the form

$$B_{k, \epsilon}(x) = \sum_{j=1}^l B_{k, \epsilon, j} 1_{\square_j}(x),$$

$$\|B - B_{k, \epsilon}\|_{L^2, \mathbf{I}} < \epsilon,$$  

and where $\{\square_{k, j}\}$ are cubes chosen as in the previous lemma. Set $\Psi_\epsilon$ as the operator with integral kernel $\sum_{k=1}^NB_{k, \epsilon}(x)\hat{\epsilon}_k(x - y)\overline{B_{k, \epsilon}}(y)$. Using Lemma 6.3 we obtain

$$\|\Psi_\epsilon - \Psi_0\|_{L^2, \mathbf{I}} \leq C\epsilon.$$  

Now, let $\Psi_{\epsilon, \text{diag}}$ be the operator with integral kernel

$$\sum_{j=1}^l \sum_{k=1}^M 1_{\square_j}(x) \left(|B_{k, \epsilon, j}|^2\hat{\epsilon}_k(x - y)\right) 1_{\square_j}(y),$$

then the difference $\Psi_\epsilon - \Psi_{\epsilon, \text{diag}}$ is the operator with kernel

$$\sum_{i \neq j}^l \sum_{k=1}^N 1_{\square_{i, j}}(x) \left(B_{k, \epsilon, i}\hat{\epsilon}_k(x - y)\overline{B_{k, \epsilon, j}}\right) 1_{\square_{i, j}}(y).$$

Thus, applying Lemma 6.4

$$\lim_{s \downarrow 0} s^{d/\gamma} n_\pm (s; \Psi_\epsilon - \Psi_{\epsilon, \text{diag}}) = 0.$$  

Using the last inequality together with (16) gives that for any $\delta > 0$

$$n_\pm ((1 + \delta)\lambda; \Psi_{\epsilon, \text{diag}}) + o(\lambda^{d/\gamma}) \leq n_\pm (\lambda; \Psi) \leq n_\pm ((1 - \delta)\lambda; \Psi_{\epsilon, \text{diag}}) + o(\lambda^{d/\gamma}), \ \lambda \downarrow 0.$$
Finally, use Lemma 6.5

\[
\lim_{\lambda \downarrow 0} \lambda^{d/\gamma} n_{\pm}(\lambda; \Psi) = \lim_{\epsilon \downarrow 0} \epsilon \sum_{j=1}^l |\Theta_{\epsilon,j}| \left( \sum_{k=1}^N |B_{k,\epsilon,j}|^2 \Gamma_k \right)^{d/\gamma}
\]

\[
= c_{\pm} \int \xi \left( \sum_{k=1}^N |\Gamma_k| B_k(\xi) |^2 \right)^{d/\gamma}
\]

\[\square\]

7. Proofs of the main results for parabolic thresholds and Dirac point cases

7.1. Proof for the bounded part of the SSF. The boundedness of the SSF near the hyperbolic thresholds was proved in Section 5. Further, for the elliptic thresholds we will only make explicit computations for \( m \), the case of \( \pm M_m \) being analogous. Set

\[
B := \mathcal{V} V^* + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},
\]

and \( Q(\lambda) := B + S_3(\lambda) \). Then, starting from Lemma 3.1, using (37), (38), (56) and (59) and Lemma 4.2, together with the Weyl inequalities (15) and the Chebyshev-type estimate (17) we obtain

\[
\pm n_{\mp} \left( 1 \pm \epsilon; \frac{Q(\lambda)}{\lambda + m} \right) + O(1) \leq \xi(\lambda; H_{\pm}, H_0)
\]

\[
\leq \pm n_{\mp} \left( 1 \pm \epsilon; \frac{Q(\lambda)}{\lambda + m} \right) + O(1),
\]

for \( |\lambda| \to m \).

The following formula is valid for \( \lambda \leq 0 \) when \( m > 0 \)

\[
Q(\lambda) = \mathcal{V} G^* \mathcal{V}^* \left( \begin{array}{ccc} 0 & 0 & 0 \\ -|b|^2 + \lambda^2 - m^2 & \frac{\pi b}{\bar{b}} & 0 \\ 0 & 0 & -|b|^2 + \lambda^2 - m^2 \end{array} \right) \mathcal{V} G^* \mathcal{V}^*
\]

\[
= \mathcal{V} G^* \mathcal{V}^* \left( \begin{array}{ccc} 0 & 0 & 0 \\ \bar{b} a & -|a|^2 + \lambda^2 - m^2 & 0 \\ 0 & 0 & \lambda^2 - m^2 \end{array} \right) + \left( \begin{array}{ccc} 0 & 0 & 0 \\ -|b|^2 & \frac{\pi b}{\bar{b}} & 0 \\ 0 & \bar{b} a & -|a|^2 \end{array} \right) \mathcal{V} G^* \mathcal{V}^*.
\]

Moreover, it is easy to see that

\[
\left( \begin{array}{cc} -|b|^2 & \frac{\pi b}{\bar{b}} \\ \bar{b} a & -|a|^2 \end{array} \right) = - \left( \begin{array}{cc} b & \pi \bar{b} \\ -a & \bar{b} \end{array} \right) \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \bar{b} a & -|a|^2 \end{array} \right) \leq 0.
\]

Therefore, for \( 0 > \lambda > -m \), the operator \( Q(\lambda) \) is nonpositive. Thus, (103) immediately implies the second assertion in (10) when \( m > 0 \).

Next, let \( k \) be \(-m, 0\) or \( m \), and let us set

\[
l_k(\lambda) = \left\{ \begin{array}{ll} \ln(|\lambda + m|) & \text{if } k = -m \\ 1 & \text{if } k = 0, m \end{array} \right.
\]
Thus, from Lemma 4.4 and (15), (17), (103) and (104)

\[(106) \quad \pm n_\pm \left((1 \pm \epsilon); \frac{Q(k)}{\lambda + m}\right) + O(l_k(\lambda)) \leq \xi(\lambda; H_\pm, H_0)\]

\[(107) \quad \leq \pm n_\pm \left((1 \mp \epsilon); \frac{Q(k)}{\lambda + m}\right) + O(l_k(\lambda))\]

for \(\lambda \to \pm m\). Now, if \(m > 0\), by a change of variable we obtain that

\[(108) \quad S_3(-m) = S_3(0) = \int_0^{M_0} \frac{d\eta}{\rho} \int_{r^{-1}(\rho)} d\gamma \frac{T_0(\gamma)}{\sqrt{r(\gamma)}} = \int_{T^2} d\xi \frac{T_0(\xi)}{r(\xi)},\]

and so

\[(109) \quad Q(-m) = Q(0) = \mathcal{V}\mathcal{G}\mathcal{V}^* \begin{pmatrix} 0 & 0 & 0 \\ -|b|^2 & \pi b & 0 \\ 0 & -a & -|a|^2 \end{pmatrix} \mathcal{V}\mathcal{G}\mathcal{V}^*.

We already know that this operator is non-positive, then the second statement in (10) for \(m = 0\) and the first statement in (11) follow from (107).

**7.2. Proof for the unbounded part of the SSF.** Now we turn to the other two statements of Theorem 1.3. Take into account (106) to (109). By an abuse of notation, we write \(Q(0) = \mathcal{V}\mathcal{G}\mathcal{V}^* \begin{pmatrix} 0 & 0 & 0 \\ -|b|^2 & \pi b & 0 \\ 0 & -a & -|a|^2 \end{pmatrix} \mathcal{V}\mathcal{G}\mathcal{V}^*\). Then, if we define

\[q = \mathcal{V}\mathcal{G}\mathcal{V}^* \begin{pmatrix} 0 & 0 & 0 \\ -|b|^2 & \pi b & 0 \\ 0 & -a & -|a|^2 \end{pmatrix} \mathcal{V}\mathcal{G}\mathcal{V}^* \begin{pmatrix} b \\ \pi \eta \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\]

we have that \(Q(0) = -qq^*\), and for all \(s > 0\)

\[n_\mp(s; Q(0)) = n_\pm(s; q^*q)\.

In particular we have

\[(110) \quad \pm n_\pm \left((1 \pm \epsilon); \frac{q^*q}{\lambda + m}\right) + O(|\ln(|\lambda + m|)|) \leq \xi(\lambda; H_\pm, H_0)\]

\[(111) \quad \leq \pm n_\pm \left((1 \mp \epsilon); \frac{q^*q}{\lambda + m}\right) + O(|\ln(|\lambda + m|)|),\]

for \(\mp \lambda \downarrow \mp m\).

Now, set \(V_{eff}: \mathbb{Z}^2 \to M_n(\mathbb{C}^2)\) to be

\[V_{eff}(\mu) := \begin{pmatrix} v_2(\mu) & 0 \\ 0 & v_3(\mu) \end{pmatrix}.

The positive eigenvalues of the operator \(q^*q = A^*\mathcal{P}V_{eff}\mathcal{P}^*A\) coincide with the ones of the integral operator in \(L^2(\mathbb{T}^2; \mathbb{C})\) with kernel

\[\frac{b(\xi)}{\sqrt{r(\xi)}} \overline{v}(\xi - \eta) \frac{b(\eta)}{\sqrt{r(\eta)}} + \frac{a(\xi)}{\sqrt{r(\xi)}} \overline{v}(\xi - \eta) \frac{a(\eta)}{\sqrt{r(\eta)}}.

This kernel is of the form (88), then it defines an operators \(\Psi\) as in Section 6. To finish the proof it is enough to use Theorem 6.1 in the case \(d = 2\). To this end, it only remains to show that \(v_2\) and \(v_3\) satisfy Condition 1. First, (86) correspond to condition (8) by considering \(v_0(\mu) = (\mu)^{-\gamma}\).
Second, \( \left< \mu \right>^{1-\gamma} \) satisfies (87) as is shown by (96). Finally, that \( v_2, v_3 \) are in \( S_\gamma(Z^2) \) is just condition (7).

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Departamento de Matemática y Ciencia de la Computación, Universidad de Santiago de Chile, Las
Sophoras 173. Santiago, Chile.
Email address: pablo.miranda.r@usach.cl

Departamento de Matemática y Ciencia de la Computación, Universidad de Santiago de Chile, Las
Sophoras 173. Santiago, Chile.
Email address: daniel.parra.v@usach.cl

Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Av. Vicuña Mackenna 4860,
Santiago, Chile.