BLOW-UP CRITERION AND EXAMPLES OF GLOBAL SOLUTIONS OF FORCED NAVIER-STOKES EQUATIONS

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ABSTRACT. In this paper we first show a blow-up criterion for solutions to the Navier-Stokes equations with a time-independent force by using the profile decomposition method. Based on the orthogonal properties related to the profiles, we give some examples of global solutions to the Navier-Stokes equations with a time-independent force, whose initial data are large.

1. INTRODUCTION

We consider the incompressible Navier-Stokes equations with a time independent external force in $\mathbb{R}^3$,

\[
\begin{cases}
\partial_t u_f - \Delta u_f + u_f \cdot \nabla u_f = f - \nabla p, \\
\nabla \cdot u = 0, \\
u_f|_{t=0} = u_0
\end{cases}
\]

for $(t, x) \in (0, T) \times \mathbb{R}^3$, where $u_f$ is the velocity vector field, $f(x)$ is the given external force defined in $\mathbb{R}^3$ and $p(t, x)$ is the associated pressure function. In this paper, we study the blow-up criterion for $(NSf)$.

1.1. Blow-up problem in critical spaces. To put our results in perspective, we first recall the Navier-Stokes equations (without external force) blow-up problem in critical spaces. Consider the Navier-Stokes system:

\[
\begin{cases}
\partial_t u - \Delta u + u \cdot \nabla u = -\nabla \pi, \\
\nabla \cdot u = 0, \\
u|_{t=0} = u_0
\end{cases}
\]

where $u(t, \cdot) : \mathbb{R}^3 \to \mathbb{R}^3$ is the unknown velocity field.

The spaces $X$ appearing in the chain of continuous embeddings

\[
\hat{H}^\frac{1}{2} \hookrightarrow L^3 \hookrightarrow \dot{B}^{-1+\frac{3}{p}}_{p,q} \hookrightarrow \dot{B}^{-1+\frac{3}{p'}}_{p',q'} 
\]

are all critical with respect to the Navier-Stokes scaling in that $\|u_{0,\lambda}\|_X \equiv \|u\|_X$ for all $\lambda > 0$, where $u_{0,\lambda} := \lambda u(\lambda x)$ is the initial data which evolves as $u_\lambda := \lambda u(\lambda^2 t, \lambda x)$, as long as $u_0$ is the initial data for the solution $u(t, x)$. While the larger spaces $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}$ and $\dot{B}^{-1}_{\infty,\infty}$ are also critical spaces and global well-posedness is known for the first two for small enough initial data in those spaces thanks to [4, 21, 24] (but only for finite $p$ in the Besov case, see [3]), the ones in the chain above guarantee the existence and uniqueness of local-in-time solutions for any initial data. Specifically, there exist corresponding spaces

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Key words and phrases. Navier-Stokes equation, Besov class, long-time behavior, regularity.
Profile Decomposition

\[ X_T = X_T((0, T) \times \mathbb{R}^3) \] such that for any \( u_0 \in X \), there exists \( T > 0 \) and a unique strong solution \( u \in X_T \) to the corresponding Duhamel-type integral equation,

\[
\begin{align*}
u(t) &= e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u(s) \otimes u(s)) ds \\
&= e^{t\Delta}u_0 + B(u, u),
\end{align*}
\]

where

\[
(v \otimes w)_{j,k} := v_j w_k, \quad [\nabla \cdot (v \otimes w)] := \sum_{k=1}^3 \partial_k (v_j w_k) \quad \text{and} \quad \mathbb{P} v := v + \nabla (-\Delta)^{-1} (\nabla \cdot v),
\]

which results from applying the projection onto divergence-free vector fields operator \( \mathbb{P} \) on \((NS)\) and solving the resulting nonlinear heat equation. Moreover, \( X_T \) is such that any \( u \in X_T \) satisfying \((NS)\) belongs to \( C([0,T], X) \). Setting

\[
T^*_X(u_0) := \sup \{ T > 0 | \exists u := NS(u_0) \in X_T \text{ solving } (NS) \}
\]

the Navier-Stokes blow-up problem is:

**Question:**

Does \( \sup_{0<t<T^*_X(u_0)} \|u(t, \cdot)\|_X < \infty \) imply that \( T^*_X(u_0) = \infty \)?

In the important work [10] of Escauriaza-Seregin-Šverák, it was established that for \( X = L^3(\mathbb{R}^3) \), the answer is yes. This extended a result in the foundational work of Leray [22] regarding the blow-up of \( L^p(\mathbb{R}^3) \) norms at a singularity with \( p > 3 \), and of the “Ladyzhenskaya-Prodi-Serrin” type mixed norms \( L^1_t(L^p_x) \), \( \frac{2}{p} + \frac{2}{q} = 1, p > 3 \), establishing a difficult “end-point” case of those results. In [15], based on the work [19], I. Gallagher, G. S. Koch, F. Planchon gave an alternative proof this result in the setting of strong solutions using the method of “critical elements” of C. Kenig and F. Merle. In [15], I. Gallagher, G. S. Koch, F. Planchon extended the method in [10] to give a positive answer to the above question for \( X = \tilde{B}^{-1+\frac{3}{p}}_{p,q}(\mathbb{R}^3) \) for all \( 3 < p, q < \infty \) (see Definition [23]). Also in [1], D. Albritton proved a stronger blow-up criterion in \( \tilde{B}^{-\frac{3}{p}}_{p,q} \) for \( 3 < p, q < \infty \) and his proof is based on elementary splitting arguments and energy estimates.

We recall the main steps of the method of “critical elements”: assume the above question’s answer is no for some \( X \) and define

\[
\infty > A_c := \inf \{ \sup_{t \in [0,T^*_X(u_0))] \|NS(u_0)(t)\|_X | u_0 \in X \text{ with } T^*_X(u_0) < \infty \},
\]

where \( NS(u_0) \) is a solution to \((NS)\) belonging to \( C([0,T^*_X(u_0)], X) \) with initial data \( u_0 \in X \). And define the set of initial data generating “critical elements” (possibly empty) as follows:

\[\mathcal{D}_c := \{ u_0 \in X | T^*(u_0) < \infty, \sup_{t \in [0,T^*(u_0))] \|NS(u_0(t))\|_X = A_c \}.\]

The main steps are:

1. If \( A_c < \infty \), then \( \mathcal{D}_c \) is non empty.
2. If \( A_c < \infty \), then any \( u_0 \in \mathcal{D}_c \) satisfies \( NS(u_0)(t) \to 0 \) in \( S' \) as \( t \to T^*(u_0) \).
3. If \( A_c < \infty \), by backward uniqueness of the heat equation (see [14]), for any \( u_0 \in \mathcal{D}_c \), there exists a \( t_0 \in (0, T^*(u_0)) \) such that \( NS(u_0(t_0)) = 0 \), which contradicts to the fact that \( A_c < \infty \).
Profile Decomposition

In this paper, we consider the blow-up problem for the Navier-Stokes equation with a time-independent external force $f$, where $\Delta^{-1}f$ is small in $L^3$ and the initial data belongs to $L^3(\mathbb{R}^3)$.

According to Theorem 5.2, we know that there exists a universal constant $c > 0$ such that, if the given external force satisfies $\|\Delta^{-1}f\|_{L^3} < c$, then for any initial data $u_0 \in L^3$, there exists a unique maximal time $T^*(u_0, f) > 0$ and a unique solution to $(NSf)$ $u_f$ belonging to $C([0, T^*]; L^3((\mathbb{R}^3)))$ for any $T < T^*$ with initial data $u_0$. Again by Theorem 5.2, we have that if $T^*(u_0, f) = \infty$, then $u_f \in C([0, \infty), L^3(\mathbb{R}^3)) \cap L^\infty(\mathbb{R}_+, L^3(\mathbb{R}^3))$, and if $T^*(u_0) < \infty$, we have for any $p > 3$ and $2 < r < \frac{2p}{p-3}$,

$$\lim_{t \to T^*(u_0, f)} \|u_f - U_f\|_{L^r([0, \infty) \cap \dot{B}^s_{p, p} + \frac{2}{r})} = \infty,$$

where $U_f \in L^3$ is the unique small steady-state solution to $(NSf)$ (for existence and uniqueness of small steady-state solution, see [2]) and the function space $L^r([0, \infty) \cap \dot{B}^s_{p, p} + \frac{2}{r})$ is defined in Definition 2.2. However, the above criterion is on the corresponding perturbation solution instead of solution $u_f$.

In this paper, we give the following blow-up criterion for $(NSf)$: Let $\Delta^{-1}f$ be small in $L^3$, then

$$(BC) \quad \limsup_{0 < t < T^*(u_0, f)} \|u_f(t, \cdot)\|_{L^3} < \infty \Rightarrow T^*(u_0, f) = \infty.$$  

We use a profile decomposition for the solutions to $(NSf)$ to prove the above result. Precisely, the decomposition enables us to construct a connection between the forced and the unforced equation, which provides the blow-up information from the unforced solution to the forced solution. More precisely, we can decompose $u_f$ in a form consisting of the sum of profiles of solutions to $(NS)$, a solution to $(NSf)$ and a remainder. We show that the blow-up information of $u_f$ is determined by the blow-up information of the profiles of solutions to $(NS)$ by an argument using the scaling property of those solutions. Compared with the “critical element” roadmap, we avoid using backward uniqueness of the heat equation (which is only true for the unforced case). We also mention that the method used in [11] can not be applied to our forced case, because the proof of [11] relies on the following scaling property: if $u$ is solution to $(NS)$ with initial data $u_0$, then $\lambda u(\lambda^2 t, \lambda x)$ is also a solution to $(NS)$ with initial data $\lambda u_0(\lambda \cdot)$. However the above scaling property is not true for the Navier-Stokes equation with a time-independent force $f$ satisfying $\Delta^{-1}f \in L^3$. In fact, for any solution $u_f$ to $(NSf)$ with initial data $u_0$, $\lambda u_f(\lambda^2 t, \lambda x)$ is no longer a solution to $(NSf)$, unless $f$ is self-similar (which means $f(t, x) \equiv \lambda^3 f(\lambda^2 t, \lambda x)$), hence does not satisfy $\Delta^{-1}f \in L^3$. (And his proof still relies on the backwards uniqueness of heat equation.)

We also point out that one can obtain a profile decomposition of solutions to the forced Navier-Stokes equation with an external force $f \in L^r(\mathbb{R}_+, \dot{B}^s_{p, p} + \frac{2}{r} - 2)$ (Definition 2.2) with $s_p + \frac{2}{r} > 0$ and initial data bounded in $\dot{B}^s_{p, p}$ for any $3 < p < \infty$ with a similar proof as in [16]. And by the same argument as the proof of Theorem 2.4 one can show the blow-up criterion as $(BC)$ by replacing $L^3$ by $\dot{B}^s_{p, p}$.

1.2. Global Solutions to the Forced Navier-Stokes Equations. The second topic of this paper is about the global solutions to the incompressible Navier-Stokes equation with a small external force.

As we mention before, $(NS)$ has a global solution if the initial data is small enough in the critical initial data space $L^3$, $\dot{B}^s_{p, \infty}$ or $BMO^{-1}$. According to embedding $L^3 \hookrightarrow \dot{B}^s_{p, \infty}(p < \infty) \hookrightarrow BMO^{-1} \hookrightarrow \dot{B}^{-1}_{\infty, \infty}$ and the fact that $\dot{B}^{-1}_{\infty, \infty}$ is the critical initial data space for $(NS)$,
we mention that all of these global wellposedness results require the initial data small enough in $\dot{B}^{-1}_{\infty,\infty}$.

Let us point out that none of the results mentioned are specific to \((NS)\), as they do not use the special structure of the nonlinear term in \((NS)\). In [S] J.-Y Chemin and I. Gallagher proved the global wellposedness result of \((NS)\) under a nonlinear smallness assumption on the initial, which may hold despite the fact that the data is large in $\dot{B}^{-1}_{\infty,\infty}$.

Our purpose is different. Once an initial $u_0 \in L^3$ generates a global solution to \((NS)\), we want to construct a global solution to \((NSf)\) with a scaled enough initial data $\lambda^{-1}u_0(\lambda \cdot)$. This is done by using the perturbation equation of \((NSf)\) and the orthogonal property of scales/cores. Hence we prove that for any initial data $u_0 \in L^3$ (could be large in $\dot{B}^{-1}_{\infty,\infty}$) generating a global solution to \((NS)\), after scaling, generates a global solution to \((NSf)\).

The rest of this article is structured as follows. In Section 3, we give the proofs Theorem 2.4 and Theorem 2.6. Section 4 is devoted to showing the profile decomposition of solutions to \((NSf)\). In Section 5, a perturbation result for \((NS)\) is stated in an appropriate functional setting which provides the key estimate of Section 4. Finally in the Appendix, we recall some well-posedness results for \((NSf)\) and the corresponding steady-state equation. Also we collect standard Besov space estimates used throughout the paper in it.

2. Notation and Statement of the Result

Let us first recall the definition of Besov spaces, in dimension $d \geq 1$.

**Definition 2.1.** Let $\phi$ be a function in $\mathcal{S}(\mathbb{R}^d)$ such that $\hat{\phi} = 1$ for $|\xi| \leq 1$ and $\hat{\phi} = 0$ for $|\xi| > 2$, and define $\phi_{j} := 2^{dj}(2^j x)$. Then the frequency localization operators are defined by

$$
S_j := \phi_j \ast \cdot, \quad \Delta_j := S_{j+1} - S_j.
$$

Let $f$ be in $\mathcal{S}'(\mathbb{R}^d)$. We say $f$ belongs to $\dot{B}^{s}_{p,q}$ if

1. the partial sum $\sum_{j=-m}^{m} \Delta_j f$ converges to $f$ as a tempered distribution if $s < \frac{d}{p}$ and after taking the quotient with polynomials if not, and

2. $\|f\|_{\dot{B}^{s}_{p,q}} := \|2^{js} \|\Delta_j f\|_{L^p_x} \|t^q_j < \infty$.

We refer to [9] for the introduction of the following type of space in the context of the Navier-Stokes equations.

**Definition 2.2.** Let $u(\cdot,t) \in \dot{B}^s_{p,q}$ for a.e. $t \in (t_1,t_2)$ and let $\Delta_j$ be a frequency localization with respect to the $x$ variable (see Definition 2.1). We shall say that $u$ belongs to $\mathcal{L}^p([t_1,t_2], \dot{B}^s_{p,q})$ if

$$
\|u\|_{\mathcal{L}^p([t_1,t_2], \dot{B}^s_{p,q})} := \|2^{js} \|\Delta_j u\|_{L^p([t_1,t_2], L^p_x)} \|t^q_j < \infty
$$

Note that for $1 \leq \rho_1 \leq q \leq \rho_2 \leq \infty$, we have

$$
L^p([t_1,t_2], \dot{B}^s_{p,q}) \hookrightarrow \mathcal{L}^p([t_1,t_2], \dot{B}^s_{p,q}) \hookrightarrow \mathcal{L}^{p_2}([t_1,t_2], \dot{B}^s_{p,q}) \hookrightarrow L^{p_2}([t_1,t_2], \dot{B}^s_{p,q}).
$$

Let us introduce the following notations (introduced in [15]): we define $s_p := -1 + \frac{2}{p}$ and

$$
\mathbb{L}^a_{p}(t_1,t_2) := \mathcal{L}^a([t_1,t_2] ; \dot{B}^{s_p + \frac{2}{p}}_{p,p}),
$$

$$
\mathbb{L}^a_{p}(T) := \mathbb{L}^a_{p}(0,T) \quad \text{and} \quad \mathbb{L}^a_{p}[T < T^*] := \cap_{T < T^*} \mathbb{L}^a_{p}(T).
$$

(2)

$$
\mathbb{L}^a_{p} := \mathbb{L}^{aa}_{p}, \quad \mathbb{L}^{ab}_{p}(T) := \mathbb{L}^a_{p}(0,T) \quad \text{and} \quad \mathbb{L}^{ab}_{p}[T < T^*] := \cap_{T < T^*} \mathbb{L}^{ab}_{p}(T).
$$
Remark 2.3. We point out that according to our notations, \( u \in L^p_{a,b}[T < T^*] \) merely means that \( u \in L^p_{a,b}(T) \) for any \( T < T^* \) and does not imply that \( u \in L^p_{a,b}(T^*) \)(the notation does not imply any uniform control as \( T \nearrow T^* \)).

Now let us state our main result.

**Theorem 2.4 (Blow-up Criterion).** Suppose that \( \|\Delta^{-1}f\|_{L^3} < c \), where \( c \) is the small universal constant in Theorem 5.2. Let \( u_0 \in L^3(\mathbb{R}^3) \) be a divergence free vector field and \( u_f = NF(u_0) \in C([0,T^*(u_0,f)],L^3(\mathbb{R}^3)) \), where \( T^*(u_0,f) \) is the maximal life span of \( u_f \), be the unique strong solution of \((NSf)\) with initial data \( u_0 \). If \( T^*(u_0,f) < \infty \), then

\[
\lim_{t \to T^*(u_0,f)} \|u(t)\|_{L^3(\mathbb{R}^3)} = \infty.
\]

**Remark 2.5.** Our profile decomposition method is not only valid for a time-independent force, but also can be extended to more general time-dependent external force. For example, our method is valid for solutions belonging to \( C([0,T^*],L^3(\mathbb{R}^3)) \) constructed in [6] with initial \( u_0 \in L^3 \), where the external force \( f \) can be written as \( f = \nabla \cdot V \) and \( \sup_{0 < t < \infty} t^{1-\frac{3}{p}} \|V\|_{L^\infty} \) is small enough for some \( 3 < p \leq 6 \). Actually our method only depends on the smallness of \( U_f \) and the continuity in time of solutions in space \( L^3 \), which are similar (\( U_f \) can be replaced by some small solution with small initial data in \( L^3 \) constructed in [6]) with the solutions in [6], whose associated force is time-dependent. After that we can obtain (BC) for any fixed small external force as above by a similar argument of the case that \( f \) is time independent.

Under the smallness assumption on the given external force, the following result is an example for the existence of global solution to \((NSf)\) whose initial data is large.

**Theorem 2.6 (Examples of Global Solutions).** Suppose that the external force \( f \) is given and \( \|\Delta^{-1}f\|_{L^3} < c_1 \), where \( c_1 \) is a universal small enough positive constant in Proposition 5.7. Let \( u_0 \in L^3 \) be a divergence free vector field and its corresponding solution \( u \) to \((NS)\) belongs to \( C([0,\infty),L^3(\mathbb{R}^3)) \). Then there exist \( \lambda > 0 \) depending on \( u_0 \) and \( f \) such that

\[
\|\Delta^{-1}f\|_{L^3} < c_1 \text{ and } \|\Delta^{-1}f\|_{L^3} < c_1 \text{ are similar.}
\]

\[
\|\Delta^{-1}f\|_{L^3} < c_1 \text{ implies that } \|\Delta^{-1}f\|_{L^3} < c_1 \text{ by Proposition 5.7.}
\]

3. **Proof of the two main results**

3.1. **The blow-up criterion.** Suppose that \( \|\Delta^{-1}f\|_{L^3} < c \) is a fixed external force.

Let us define

\[
A_c := \sup\{A > 0 \mid \sup_{t \in [0,T^*(u_0,f))} \|NSf(u_0)(t)\|_{L^3} \leq A \}
\]

\[
\implies T^*(u_0,f) = \infty, \forall u_0 \in L^3(\mathbb{R}^3) \}
\]

Note that \( A_c \) is well-defined by small-data results. If \( A_c \) is finite, then \( A_c \) can be rewritten as

\[
A_c = \inf\{\sup_{t \in [0,T^*(u_0,f))} \|NSf(u_0)(t)\|_{L^3}|u_0 \in L^3 \text{ with } T^*(u_0,f) < \infty \}
\]

In the case when \( A_c < \infty \), we introduce the (possibly empty) set of initial data generating a critical element as follows:

\[
\mathcal{D}_c := \{u_0 \in L^3(\mathbb{R}^3)|T^*(u_0,f) < \infty, \sup_{t \in [0,T^*(u_0,f))} \|NSf(u_0)(t)\|_{L^3} = A_c \}
\]

Before proving Theorem 2.4, we prove the above set is empty.
Profile Decomposition

**Proposition 3.1** (\(\mathcal{D}_c\) is empty). Suppose that \(A_c < \infty\), then \(\mathcal{D}_c = \emptyset\).

**Proof.** We prove the proposition by contradiction. Assume \(\mathcal{D}_c \neq \emptyset\), we take a \(u_{0,c} \in \mathcal{D}_c\) and denote \(u_c = NSf(u_{0,c})\). By the definition of \(\mathcal{D}_c\), we have \(T^*(u_{0,c}, f) < \infty\) and

\[
\sup_{t \in [0,T^*(u_{0,c}, f))} \|NSf(u_{0,c})(t)\|_{L^3} = A_c.
\]

We choose a sequence \((s_n)_{n \in \mathbb{N}} \subset [0, T^*(u_{0,c}, f))\) such that \(s_n \nearrow T^*(u_{0,c}, f)\). Let \(u_{0,n} := u_c(s_n)\) and \(u_n := NSf(u_{0,n})\). Since \(A_c < \infty\), we know that \((u_{0,n})_{n \in \mathbb{N}}\) is a bounded sequence in \(L^3(\mathbb{R}^3)\) and

\[
\sup_{t \in [0,T^*(u_{0,n}, f))} \|u_n(t)\|_{L^3} = A_c.
\]

By Theorem 4.3 with the same notation, for any \(t \leq \tau_n\), \(u_n\) has the following profile decomposition, for any \(J \geq J_0\) and \(n \geq n(J_0)\),

\[
u_n = U^1 + \sum_{j=2}^{J} \Lambda_{j,n} U^j + w_n^J + r_n^J,
\]

where \(\tau_n = \min_{j \in I} \{\lambda_{j,n}^2, J^j\}\). After reordering, we can write

\[
u_n = \sum_{j=1}^{J} \Lambda_{j,n} U^j + w_n^J + r_n^J
\]

with \(\Lambda_{j_0,n} \equiv Id\) for some \(1 \leq j_0 \leq J_0\) and for \(j \leq J\) and \(n\) large enough,

\[
\forall j \leq k \leq J_0, \quad \lambda_{j,n}^2 T_k^* \leq \lambda_{k,n}^2 T_k^*.
\]

First we claim that \(j_0 > 1\). In fact, by Theorem 4.3

\[
\lambda_{1,n} T_1^* \leq T^*(u_{0,n}, f) = T^*(u_{0,c}, f) - s_n \to 0, \quad \text{as} \quad n \to \infty,
\]

which implies that

\[
\lim_{n \to \infty} \lambda_{1,n} = 0.
\]

Hence \(j_0 > 1\), which implies that with the new ordering \(U^1 = NS(\phi_1)\), and \(T_1^* < \infty\).

Now we take \(s \in (0, T_1^*)\) and let \(t_n = \lambda_{1,n}^2 s\). According to Proposition 4.4 we have

\[
A_c^3 \geq \|u_n(t_n)\|_{L^3}^3 \geq \|U^1(s)\|_{L^3}^3 + \epsilon(n, s),
\]

where \(\lim_{n \to \infty} \epsilon(n, s) = 0\) for any fixed \(s\). By the blow-up criterion for the Navier-Stokes equation (see [16])

\[
\limsup_{t \to T_1^*} \|U^1(t)\|_{L^3(\mathbb{R}^3)} = \infty,
\]

then we choose a \(s_0 \in (0, T_1^*)\) such that

\[
\|U^1(s_0)\|_{L^3(\mathbb{R}^3)} > 2A_c.
\]

And we can take a corresponding \(n_0 := n(s_0)\) such that \(|\epsilon(n_0, s_0)| \leq A_c^3\). Then we get

\[
A_c^3 > 8A_c^3 - A_c^3 = 7A_c^3
\]

which contradicts the fact that \(A_c < \infty\). Then \(\mathcal{D}_c = \emptyset\). \(\square\)

Now we prove Theorem 2.3 by contradiction.
Profile Decomposition

Proof of Theorem 2.4. We suppose that $A_c < \infty$ which means (3) fails. Let us consider a sequence $u_{0,n}$ bounded in the space $L^3$ such that the life span of $NSf(u_{0,n})$ satisfies $T^*(u_{0,n}, f) < \infty$ for each $n \in \mathbb{N}$ and such that

$$A_n := \sup_{t \in [0, T^*(u_{0,n}, f))]} \|NSf(u_{0,n})\|_{L^3(\mathbb{R}^3)}$$

satisfies

$$A_c \leq A_n \quad \text{and} \quad A_n \rightarrow A_c, \quad n \rightarrow \infty.$$ 

Then by Theorem 4.3 and after reordering as above, we have for any $J \geq J_0$ and $n \geq n(J_0)$

$$u_n := NSf(u_{0,n}, f) = \sum_{j=1}^{J} \Lambda_{j,n} U^j + w_n^j + r_n^j, \forall t \in [0, \tau_n]$$

and for any $n \geq n_0(J_0)$, recalling that $T^*_j$ is the life span of $U^j$

$$\forall j \leq k \leq J_0, \quad \lambda_{j,n}^2 T^*_j \leq \lambda_{k,n}^2 T^*_k,$$

where $U^{j_0} = NSf(\phi_{j_0})$ ($j_0$ is such that $\Lambda_{j_0,n} \equiv 1$) and $U^j = NSf(\phi_j)$ for any $1 \leq j \leq J_0$ with $j \neq j_0$. Theorem 4.3 also ensures that there $J_0$ such that $T^*_0 < \infty$ (if not we would have $\tau_n \equiv \infty$ and hence $T^*(u_{0,n}, f) \equiv \infty$, contrary to our assumption). On the other hand, we recall that $U^{j_0} := NSf(\phi_{j_0})$ with $1 \leq j_0 \leq J_0$, where $\phi_{j_0}$ is a weak limit of $(u_{0,n})_{n \geq 1}$. Therefore by the above re-ordering, two different cases need to be considered:

- $j_0 = 1$: the lower-bound of the life span of $u_n$ is controlled by the life span of $U^{j_0} = U^1 = NSf(\phi_{j_0})$, which generates a critical element.
- $j_0 > 1$: the lower-bound of the life span of $u_n$ is controlled by the life span of $\Lambda_{1,n} NSf(\phi_1)$.

Case 1: $j_0 = 1$. In this case, by definition of $A_c$, we have $U^1 = NSf(\phi_1)$, $\Lambda_{1,n} \equiv Id$ and

$$\sup_{s \in [0, T^*_1)} \|NSf(\phi_1)\|_{L^3} \geq A_c.$$ 

For any $s \in (0, T^*_1)$, setting $t_n := \lambda_{1,n}^2 s$, by Proposition 4.3

$$A_n^3 \geq \sup_{t \in [0, T^*(u_{0,n}, f))]} \|NSf(u_{0,n})\|_{L^3}^3 \geq \|NSf(u_{0,n})(t_n)\|_{L^3}^3 \geq \|U^1(s)\|_{L^3}^3 + \varepsilon(n, s),$$

where for any fixed $s \in [0, T^*_1)$

$$\lim_{n \rightarrow \infty} \varepsilon(n, s) = 0.$$ 

According to (3) and the fact that $A_n \rightarrow A_c$ as $n \rightarrow \infty$, we infer that

$$\sup_{s \in [0, T^*_1)} \|NSf(\phi_1)\|_{L^3} = A_c,$$

which means $\phi_1 \in D_c$. This fact contradicts Proposition 5.1.

Case 2 $j_0 > 1$: In this case, $U^1 = NSf(\phi_1)$ and $U^1$ satisfies that

$$\lim_{t \rightarrow T^*_1} \|U^1(t)\|_{L^3} = \infty,$$

and $\Lambda_{1,n} \neq Id$. 

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On the other hand for any $s \in (0, T^*_s)$, setting $t_n := \lambda_{1,n}^2 s$,

$$A_n^3 \geq \sup_{t \in (0, T^*(u_{0,n}, f))} \| N S f(u_{0,n}) \|^2_{L^3} \geq \| N S f(u_{0,n})(t_n) \|^2_{H^{1/2}}$$

$$\geq \| U^3(s) \|^2_{L^3} + \varepsilon(n, s),$$

where

$$\lim_{n \to \infty} \varepsilon(n, s) = 0, \quad \forall s \in [0, T^*_s).$$

Thanks to (7), one can take $s_0$ such that

$$\| U^1(s_0) \|^2_{L^3} > 2A_c$$

and choose $n_0 := n(s_0)$ such that $\varepsilon(n_0, s_0) \leq A_{n_0}^3$ and $A_{n_0}^3 \leq 2A_c^3$, then we have

$$2A_c^3 \geq \| U^1(s_0) \|^2_{L^3} + \varepsilon(n_0, s_0)$$

$$> 7A_c^3,$$

which contradicts the fact that $A_c < \infty$. Then we prove that for any $u_0$, if $T^*(u_0, f) < \infty$

$$\lim_{t \to T^*(u_0, f)} \| N S f(u_0) \|^2_{L^3} = \infty.$$

Theorem 2.4 is proved.

\[ \square \]

3.2. The global solutions to $(NSf)$. In this part we focus on the existence of global solutions to $(NSf)$. In this paragraph we assume that the given external force $f$ satisfies $\| \Delta^{-1} f \|^2_{L^3} < c_1$, where $c_1$ is the small constant given in Proposition 5.1.

The proof of Theorem 2.6. Suppose that $f$ is the given external force and $\| \Delta^{-1} f \|^2_{L^3} < c_1$. According to Theorem 5.2 there exists a unique solution $U_f := N S f(0) \in C(\mathbb{R}_+, L^3)$ to $(NSf)$ with initial data 0. Let $u_0 \in L^3$ described in the theorem. Then its corresponding solution $u := N S (u_0)$ to $(NS)$ belonging to $L^\infty(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$, for any $3 < p < \infty$.

Denote $u_\lambda = \Lambda u$, where $\Lambda \Lambda(t, x) := \lambda^{-1} u(\lambda^{-2} t, \lambda^{-1} x)$. It is easy to check that $\Lambda u$ is a solution to $(NS)$ with the initial data $\lambda^{-1} u_0(\lambda \cdot)$ denoted by $u_{0,\lambda}$.

By Theorem 6.2, for any fixed $\lambda > 0$ there exist a unique $T^*(u_{0,\lambda}, f) > 0$ and a unique solution $u_{f,\lambda} := N S f(u_{0,\lambda})$ to $(NSf)$ such that $u_{f,\lambda} \in C([0, T^*(u_{0,\lambda}, f)), L^3(\mathbb{R}^3))$. We define $r_\lambda = u_{f,\lambda} - u_\lambda - U_f$, which is a solution to the following perturbation equation

$$\begin{cases} 
\partial_t r_\lambda - \Delta r_\lambda + \frac{1}{2} Q(r_\lambda, r_\lambda) + Q(r_\lambda, u_\lambda + U_f) = -Q(u_\lambda, U_f), \\
\nabla \cdot r_\lambda = 0, \\
\n| r_\lambda |_{t=0} = 0,
\end{cases}$$

where $Q(a, b) = \mathbb{P}(a \cdot \nabla b + b \cdot \nabla a)$. Hence to prove that there exists a $\lambda_0 > 0$ such that $u_f \in C(\mathbb{R}_+, L^3(\mathbb{R}^3))$ it is enough to prove there exists a $\lambda_0$ such that $r_{\lambda_0} \in L^p(\mathbb{R}^3)$ for some $3 < p < 5$.

In fact if $r_{\lambda_0} \in L^p(\mathbb{R}^3)$, we notice that $r_{\lambda_0}$ has the following integral form

$$r_{\lambda_0}(t) = B(r_{\lambda_0}, r_{\lambda_0}) + 2B(r_{\lambda_0}, u_{\lambda_0} + U_f) + 2B(u_{\lambda_0}, U_f),$$

where $B$ is defined in (1). By the smooth effect of heat kernel and the product law of Besov space introduced in Proposition 5.3, we have that

$$\| B(r_{\lambda_0}, r_{\lambda_0}) \|_{L^p_{\frac{3}{2}, \infty}(\mathbb{R}^3)} + 2\| B(r_{\lambda_0}, u_{\lambda_0}) \|_{L^p_{\frac{3}{2}, \infty}(\mathbb{R}^3)} \lesssim \| r_{\lambda_0} \|_{L^p_{\frac{3}{2}, \infty}(\mathbb{R}^3)} (\| r_{\lambda_0} \|_{L^p_{\frac{3}{2}, \infty}(\mathbb{R}^3)} + \| u_{\lambda_0} \|_{L^p_{\frac{3}{2}, \infty}(\mathbb{R}^3)}),$$
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and there exists some $2 < q < 3$ such that

$$\|B(r_{\lambda_0}, U_f)\|_{L_p^\infty(\infty)} + \|B(u_{\lambda_0}, U_f)\|_{L_p^\infty(\infty)} \lesssim (\|r_{\lambda_0}\|_{L_p^\infty(\infty)} + \|u_{\lambda_0}\|_{L_p^\infty(\infty)}) \|U_f\|_{L^3}.$$  

According to the fact that for any $1 \leq \tilde{p} < 3$, $\mathcal{L}^\infty(\mathbb{R}^+, \dot{B}_p^{\tilde{p}}(\mathbb{R}^3)) \rightarrow L^\infty(\mathbb{R}^+, L^3(\mathbb{R}^3))$ and the smooth effect of heat kernel, we prove $r \in C(\mathbb{R}^+, L^3(\mathbb{R}^3))$.

Now we turn to prove that there exists a $\lambda_0 > 0$ such that $r_{\lambda_0} \in \mathbb{L}_p^{3, \infty}(\infty)$ with $3 < p < 5$. According to Proposition 4.7, we have

$$\lim_{\lambda \to 0} \|Q(u_{\lambda}, U_f)\|_{L^p(\mathbb{R}^+, \dot{B}_p^{s, p + \frac{3}{2} - 2})} \leq \lim_{\lambda \to 0} \|u_{\lambda} \otimes U_f\|_{L^p(\mathbb{R}^+, \dot{B}_p^{s, p + \frac{3}{2} - 1})} = 0.$$ 

Hence there exists a $\lambda_0 > 0$ such that

$$\|Q(u_{\lambda_0}, U_f)\|_{L^p(\mathbb{R}^+, \dot{B}_p^{s, p + \frac{3}{2} - 2})} \leq \varepsilon_0 \exp\left(-C\|u_{\lambda_0}\|_{L^p(\mathbb{R}^+, \dot{B}_p^{s, p + \frac{3}{2}})}\right)$$

provided that $\|u_{\lambda}\|_{L^p(\mathbb{R}^+, \dot{B}_p^{s, p + \frac{3}{2}})} = \|u\|_{L^p(\mathbb{R}^+, \dot{B}_p^{s, p + \frac{3}{2}})}$ is independent of $\lambda$. Here $\varepsilon_0$ and $C$ are constant in Proposition 4.1.

Applying Proposition 5.1 we prove $r_{\lambda_0} \in \mathbb{L}_p^{3, \infty}(\infty)$ with $3 < p < 5$. Then we prove the theorem.

4. Profile decomposition

In [16] a profile decomposition of solutions to the Navier-Stokes equations associated with data in $\dot{B}_p^{s, p}$ is proved for $d < p < 2d + 3$, thus extending the result of [19]. In this section we use the idea of [16] to give a decomposition of solutions to the Navier-Stokes equations with a small external force and associated with initial data in $L^3$.

4.1. Profile decomposition of bounded sequence in $L^3$. Before stating the main result of this section, let us recall the following definition.

**Definition 4.1.** We say that two sequences $(\lambda_{j,n}, x_{j,n})_{n \in \mathbb{N}} \in ((0, \infty) \times \mathbb{R}^3)^\mathbb{N}$ for $j \in \{1, 2\}$ are orthogonal, and we write $(\lambda_{1,n}, x_{1,n})_{n \in \mathbb{N}} \perp (\lambda_{2,n}, x_{2,n})_{n \in \mathbb{N}}$, if

$$\lim_{n \to +\infty} \frac{\lambda_{1,n}}{\lambda_{2,n}} + \frac{\lambda_{2,n}}{\lambda_{1,n}} + \frac{|x_{1,n} - x_{2,n}|}{\lambda_{1,n}} = +\infty.$$  

Similarly we say that a set of $(\lambda_{j,n}, x_{j,n})_{n \in \mathbb{N}}$, for $j \in \mathbb{N}$, $j \geq 1$, is orthogonal if for all $j \neq j'$, $(\lambda_{j,n}, x_{j,n})_{n \in \mathbb{N}} \perp (\lambda_{j',n}, x_{j',n})_{n \in \mathbb{N}}$.

Next let us define, for any set of sequences $(\lambda_{j,n}, x_{j,n})_{n \in \mathbb{N}}$ (for $j \geq 1$), the scaling operator

$$\Lambda_{j,n} U_j(t, x) := \frac{1}{\lambda_{j,n}} U_j(t \lambda_{j,n}^2, \frac{x - x_{j,n}}{\lambda_{j,n}}).$$  

It is proved in [20] that any bounded (time-independent) sequence in $\dot{B}_p^{s, p}(\mathbb{R}^3)$ may be decomposed into a sum of rescaled functions $\Lambda_{j,n}\phi_j$, where the set of sequences $(\lambda_{j,n}, x_{j,n})_{n \in \mathbb{N}}$ is orthogonal, up to a small remainder term in $\dot{B}_q^{s, q}$, for any $q > p$. Since in this paper we only consider the initial data in $L^3$, we only state the profile decomposition result of bounded sequences in $L^3$ in [20]. The precise statement is in the spirit of the pioneering work [17].

**Theorem 4.2.** Let $(\varphi_{n})_{n \geq 1}$ be a bounded sequence of functions in $L^3(\mathbb{R}^3)$ and let $\phi_1$ be any weak limit point of $(\varphi_{n})_{n \in \mathbb{N}}$. Then, after possibly replacing $(\varphi_{n})_{n \in \mathbb{N}}$ by a subsequence which we relabel $(\varphi_{n})_{n \geq 1}$, there exists a subsequence of profiles $(\phi_{j})_{j \geq 2}$ in $L^3(\mathbb{R}^3)$, and a set of
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sequences \((\lambda_{j,n}, x_{j,n})_{n \in \mathbb{N}}\) for \(j \in \mathbb{N}\) with \((\lambda_{1,n}, x_{1,n}) \equiv (1,0)\) which are orthogonal in the sense of Definition 4.1 such that, for all \(n, J \in \mathbb{N}\), if we define \(\psi_n^J\) by

\[\varphi_n = \sum_{j=1}^{J} \Lambda_{j,n} \phi_j + \psi_n^J,\]

the following properties hold:

- the function \(\psi_n^J\) is a remainder in the sense that for any \(p > 3\),

\[
\lim_{J \to \infty} \left( \limsup_{n \to \infty} \|\psi_n^J\|_{\ell^p_{\Omega_p}} \right) = 0;
\]

- There is a norm \(\| \cdot \|_{L^3}\) which is equivalent to \(\| \cdot \|_{L^3}\) such for each \(n \in \mathbb{N}\),

\[
\sum_{j=1}^{\infty} \|\phi_j\|_{L^3(\mathbb{R}^3)} \leq \liminf_{n \to \infty} \|\varphi_n\|_{L^3(\mathbb{R}^3)}
\]

and, for any integer \(J\),

\[
\|\psi_n^J\|_{L^3} \leq \|\varphi_n\|_{L^3} + o(1)
\]

as \(n\) goes to infinity.

We mention that, in particular, for any \(j \geq 2\), either \(\lim_{n \to \infty} |x_{j,n}| = \infty\) or \(\lim_{n \to \infty} \lambda_{j,n} \in \{0, \infty\}\) due to the orthogonality of scales/cores with \((\lambda_{1,n}, x_{1,n}) \equiv (1,0)\), and also that

\[
\sum_{j=1}^{\infty} \|\phi_j\|_{L^3} \leq \liminf_{n \to \infty} \|\varphi_n\|_{L^3}.
\]

4.2. Profile decomposition of solutions to \((NSf)\).

**Theorem 4.3.** Suppose that \(\|\Delta^{-1} f\|_{L^3} < c\), where \(c\) is the small universal constant in Theorem 6.2.

Let \((u_{0,n})_{n \in \mathbb{N}}\) be a bounded sequence of divergence-free vector fields in \(L^3(\mathbb{R}^3)\), and \(\phi_1\) be any weak limit point of \(\{u_{0,n}\}\). Then, after possibly relabeling the sequence due to the extraction of a subsequence following an application of Theorem 6.2 with \(\varphi_n := u_{0,n}\), defining \(u_n := NSf(u_{0,n})\), \(U^1 := NSf(\phi_1) \in C([0,T_1], L^3)\) and \(U^j := NSf(\phi_j) \in C([0,T_j], L^3)\) for any \(j \geq 2\) (where \(T_j\) is any real number smaller than \(T_j^*\)), where \(T_j^*\) is the life span of \(U^j\) for \(j \geq 1\), and \(T^j = \infty\) if \(T_j^* = \infty\), the following properties hold:

- there is a finite (possibly empty) subset \(I\) of \(\mathbb{N}\) such that

\[
\forall j \in I, \quad T^j < \infty \quad \text{and} \quad \forall j' \in \mathbb{N} \setminus I, \quad U^{j'} \in C(\mathbb{R}^+, L^3(\mathbb{R}^3)).
\]

Moreover setting \(\tau_n := \min_{j \in I} \lambda_{j,n}^2 T^j\) if \(I\) is nonempty and \(\tau_n = \infty\) otherwise, we have

\[
\sup_n \|u_n\|_{L^\infty_t L^3_x(\mathbb{R}^3)} < \infty;
\]

- there exists some large \(J_0 \in \mathbb{N}\) such that for each \(J > J_0\), there exists \(N(J) \in \mathbb{N}\) such that for all \(n > N(J)\), all \(t \leq \tau_n\) and all \(x \in \mathbb{R}^3\), setting \(w_n^J := e^{\Delta J} (\psi_n^J)\) and defining \(r_n^J\) by

\[
u_n(t, x) = U^1 + \sum_{j=2}^{J} \Lambda_{j,n} U^j + w_n^J + r_n^J,
\]
then $w_n^j$ and $r_n^j$ are small remainders in the sense that, for any $3 < p < 5$,
\begin{equation}
\lim_{J \to \infty} \left( \lim_{n \to \infty} \|w_n^J\|_{L_p^1(\infty)} \right) = \lim_{J \to \infty} \left( \lim_{n \to \infty} \|r_n^J\|_{L_p^\infty(\tau_n)} \right) = 0.
\end{equation}

We recall the following important orthogonality result without proof. Its proof is the same as the proof of Claim 3.3 of [16], as it just depends on orthogonality property on scales/core. To state the result, note first that an application of Theorem 4.3 yields a non-empty blow-up as the proof of Claim 3.3 of [16], as it just depends on orthogonality property on scales/core.

**Proposition 4.4.** Let $(u_{0,n})_{n \geq 1}$ be a bounded sequence in $L^3$ and for which the set $I$ of blow-up profile indices resulting from an an application of Theorem 4.3 is non-empty. After re-ordering the profiles in the profile decomposition of $u_n$ := $NSf(u_{0,n})$ such that (12) holds for some $J_0$, setting $t_n := \lambda_{j,n}^2 s$ for $s \in [0, T_j^*)$ one has (after possibly passing to a subsequence in $n$):
\begin{equation}
\|u_n(t)\|_{L^3}^3 = \|(\Lambda_{1,n}U^1)(t_n)\|_{L^3}^3 + \|u(t_n) - (\Lambda_{1,n}U^1)(t_n)\|_{L^3}^3 + \varepsilon(n, s),
\end{equation}
where $\varepsilon(n, s) \to 0$ as $n \to \infty$ for each fixed $s \in [0, T_j^*)$.

**Proof of Theorem 4.3.** Let $(u_{0,n})_{n \geq 1}$ be a bounded sequence in $L^3$. We first use Theorem 4.2 to decompose the above sequence.

Then with the notation of Theorem 4.3
\begin{equation}
u_{0,n} = \sum_{j=1}^J \Lambda_{j,n} \phi_j + \psi_n^J.
\end{equation}

We define
\begin{align*}
u_n := NSf(u_{0,n}), \quad U^1 := NSf(\phi_1) \in C([0, T_1^*), L^3(\mathbb{R}^3)), \\
u_j := NS(\phi_j) \in C([0, T_j^*), L^3(\mathbb{R}^3)) \quad \text{and} \quad w_n^j := e^{t\Delta}(\psi_n^j).
\end{align*}

By (10) and standard linear heat estimates we have
\begin{equation}
\lim_{J \to \infty} \left( \lim_{n \to \infty} \|w_n^J\|_{L_p^1(\infty)} \right) = 0.
\end{equation}

According to (11), we have for any $p > 3$
\begin{equation}
\left\|\|\phi_j\|_{B_{p,p}^\infty}\right\|_{L_p^1} \lesssim \left\|\|\phi_j\|_{L^3}\right\|_{L_p^1} \lesssim \liminf_{n \to \infty} \|u_{0,n}\|_{L^3},
\end{equation}
which implies that, for any $j \geq 2$,
\begin{equation}
U_j \in L_p^1(\infty) \quad \text{and} \quad \|U_j\|_{L_p^1(\infty)} \lesssim \|\phi_j\|_{B_{p,p}^\infty}.
\end{equation}

Hence, $I$ will be a subset of $\{1, \ldots, J_0\}$ which proves the first part of the first statement in Theorem 4.3.
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From now on, we restrict \( p \in (3, 5) \). By the local Cauchy theory we can solve \((NSf)\) with initial data \( u_{0,n} \) for each integer \( n \), and produce a unique solution \( u_n \in C([0, T_n^\ast), L^3(\mathbb{R}^3)) \), where \( T_n^\ast \) is the life span of \( u_n \). Now we define, for any \( J \geq 1 \),

\[
  r_n^J := u_n - \sum_{j=1}^{J} \Lambda_{j,n} U^J - w_n^J,
\]

where we recall that \( \Lambda_1 U^1 = U^1 \). We mention that the life span of \( \Lambda_{j,n} U^J \) is \( \lambda_{j,n}^2 T_n^* \).

Therefore, the function \( r_n^J(t, x) \) is defined a priori for \( t \in [0, t_n] \), where

\[
  t_n := \min(T_n^*, \tau_n)
\]

with the notation of Theorem 4.3. Our main goal is to prove that \( r_n^J \) is actually defined on \([0, \tau_n]\) (at least if \( J \) and \( n \) are large enough), which will be a consequence of perturbation theory for the Navier-Stokes equations, see Proposition 5.1. In the process, we shall obtain the uniform limiting property, namely,

\[
  \lim_{J \to \infty} \left( \limsup_{n \to \infty} \|r_n^J\|_{L^p_t L^\infty_x}(\tau_n) \right) = 0.
\]

Let us write the equation satisfied by \( r_n^J \). We adapt the same method as \cite{15} and \cite{16}. It turns out to be easier to write that equation after a re-scaling in space-time. For convenience, let use re-order the functions \( \Lambda_{j,n} U^J \), for \( 1 \leq j \leq J_0 \), in such a way that, for some \( n_0 = n_0(J_0) \) sufficiently large, we have as in \cite{10},

\[
  \forall n \geq n_0, \quad j \leq j' \leq J_0 \Rightarrow \lambda_{j,n} T_j^* \leq \lambda_{j',n} T_{j'}^*.
\]

And we define \( 1 \leq j_0 \leq J_0 \) as the integer such that \( \Lambda_{j_0,n} = \text{Id} \). And \( \Lambda_{j_0,n} U^{j_0} = NS f(\phi_{j_0}) = U_f + V^{j_0} \) (see Theorem 6.2). We note that \( \lambda_{j,n}^2 T_j^* \) is the life span of \( \Lambda_{j,n} U^J \).

The inverse of our dilation/translation operator \( \Lambda_{j,n} \) is

\[
  \Lambda^{-1}_{j,n} f(s, y) := \lambda_{j,n} f(\lambda_{j,n}^2 s, \lambda_{j,n} y + x_{j,n}).
\]

Then we define, for any integer \( J \),

\[
  j \leq J, \quad U^{j,1}_n := \Lambda_{1,n}^{-1} \Lambda_{j,n} U^J, \quad R^{J,1}_n := \Lambda_{1,n}^{-1} r_n^J, \quad V^{j_0,1}_n := \Lambda_{1,n}^{-1} V^{j_0}
\]

\[
  U^J_f := \Lambda_{1,n}^{-1} U_f, \quad W^{J,1}_n := \Lambda_{1,n}^{-1} w_n^J \quad \text{and} \quad U^J_n := \Lambda_{1,n}^{-1} u_n.
\]

Clearly we have

\[
  R^{J,1}_n = U^1_n - \left( \sum_{j=1}^{J} U^{j,1}_n + W^{J,1}_n \right)
\]

and \( R^{J,1}_n \) is a divergence free vector field, solving the following system:

\[
  \left\{ \begin{array}{l}
    \partial_t R^{J,1}_n - \Delta R^{J,1}_n + \mathbb{P}(R^{J,1}_n \cdot \nabla R^{J,1}_n) + Q(R^{J,1}_n, U^J_f + G^{J,1}_n) = F^{J,1}_n, \\
    R^{J,1}_n|_{t=0} = 0,
  \end{array} \right.
\]

where we recall that \( \mathbb{P} := \text{Id} - \nabla \Delta^{-1}(\nabla \cdot) \) is the projection onto divergence free vector fields, and where\[
  Q(a, b) := \mathbb{P}((a \cdot \nabla)b + (b \cdot \nabla)a)
\]

for two vector fields \( a, b \). Finally we have defined

\[
  G^{J,1}_n := \sum_{j \neq j_0} U^{j,1}_n + W^{J,1}_n + V^{j_0,1}_n,
\]

and...
Lemma 4.5. Fix $T_1 < T^*_1$. The sequence $(G_n^{J,1})_{n \geq 1}$ is bounded in $L^p([0,T_1], B_{p,p}^{s_p+\frac{2}{p}})$, uniformly in $J$, which means that

$$\lim_{J \to \infty} \limsup_{n \to \infty} \|G_n^{J,1}\|_{L^p([0,T_1], B_{p,p}^{s_p+\frac{2}{p}})} = 0.$$ 

The proof of the above lemma is the same as the proof of Lemma 2.5 in [16], as it just depends on orthogonality property on scales/core.

Lemma 4.6. Fix $T_1 < T^*_1$. The source term $F_n^{J,1}$ goes to zero for each $J \in \mathbb{N}$, as $n$ goes to infinity, in the space $F := L^p([0,T_1], B_{p,p}^{s_p+\frac{2}{p}} + L^2([0,T_1], B_{p,p}^{s_p+\frac{2}{p}-2})$. In precisely,

$$\lim_{J \to \infty} \limsup_{n \to \infty} \|F_n^{J,1}\|_F = 0.$$ 

Assuming these lemmas to be true, the end of the proof of the theorem is a direct consequence of Proposition 5.1.

Now let us prove Lemma 4.6.

Proof of Lemma 4.6. We first notice that

$$F_n^{J,1} = -\frac{1}{2} Q(W_n^{J,1}, W_n^{J,1}) - \frac{1}{2} \sum_{j \neq j', j \neq j_0}^J Q(U_n^{j,1}, U_n^{j',1}) - \sum_{j = 1, j \neq j_0}^J Q(U_n^{j_0,1}, W_n^{J,1})$$

and

$$A_n^{J} := -\frac{1}{2} Q(W_n^{J,1}, W_n^{J,1}) - \frac{1}{2} \sum_{j \neq j', j \neq j_0}^J Q(U_n^{j,1}, U_n^{j',1}) - \sum_{j = 1, j \neq j_0}^J Q(U_n^{j_0,1}, W_n^{J,1})$$

is the same as the $G_n^{J,0}$ of Lemma 2.7 in [16]. As a consequence of Lemma 2.7 in [16], we obtain

$$\lim_{J \to \infty} \limsup_{n \to \infty} \|A_n^{J}\|_F = 0.$$ 

Hence to finish the proof of Lemma 4.6 we need to show

$$\lim_{J \to \infty} \limsup_{n \to \infty} \|B_n^{J}\|_F = 0,$$

where

$$B_n^{J} := -\sum_{j = 1, j \neq j_0}^J Q(U_n^{j_0,1}, U_n^{J,1}) - Q(U_n^{j_0,1}, W_n^{J,1}).$$
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By product laws and scaling invariance, we first have
\[
\| Q(U_{n}^{j_{0}}, W_{n}^{j_{1}}) \|_{L^{p}([0,T_{1}], B_{p,p}^{s_{p}+\frac{a}{2}})} \lesssim \| W_{n}^{j_{1}} \|_{L^{p}([0,T_{1}], B_{p,p}^{s_{p}+\frac{a}{2}})} \| U_{n}^{j_{0}} \|_{L^{\infty}([0,T_{0}], B_{p,p}^{s_{p}})}
\]
implies that
\[
\lim_{n \to \infty} \lim_{J \to \infty} \| Q(U_{n}^{j_{0},1}, W_{n}^{j_{1},1}) \|_{L^{p}([0,T_{1}], B_{p,p}^{s_{p}+\frac{a}{2}})} = 0.
\]
Now we are left with proving that
\[
\lim_{J \to \infty} \lim_{n \to \infty} \| \sum_{j=1,j \neq j_{0}}^{J} Q(U_{n}^{j_{0},1}, U_{n}^{j_{1},1}) \| = 0.
\]
We can write \( \sum_{j=1,j \neq j_{0}}^{J} Q(U_{n}^{j_{0},1}, U_{n}^{j_{1},1}) \) as the following way:
\[
\sum_{j=1,j \neq j_{0}}^{J} Q(U_{n}^{j_{0},1}, U_{n}^{j_{1},1}) = \sum_{j \neq j_{0}} Q(V_{n}^{j_{0},1}, U_{n}^{j_{1},1}) + \sum_{j \neq j_{0}} Q(\Lambda_{1,n}^{-1} U_{f}, U_{n}^{j_{1},0}).
\]
Since for any \( j \neq j_{0} \), \( U_{n}^{j_{0},1} \in L^{1}(T_{0}), V_{n}^{j_{0},1} \in L^{p}_{p}(T_{0}) \) and \( 3 < p < 5 \), by (21) in Proposition 4.7, we have for all \( j' \), \( j \neq j_{0} \),
\[
\lim_{n \to \infty} \| Q(U_{n}^{j_{0},1}, V_{n}^{j_{1},1}) \|_{L^{p}(0, T_{1}, B_{p,p}^{s_{p}+\frac{a}{2}})} = 0.
\]
And according to \( U_{f} \in L^{3}(\mathbb{R}^{3}) \), we have
\[
\lim_{n \to \infty} \| Q(U_{n}^{j_{0},1}, \Lambda_{1,n}^{-1} U_{f}) \|_{L^{p}(0, T_{1}, B_{p,p}^{s_{p}+\frac{a}{2}})} = 0
\]
by Proposition 4.7. By the above two relations, we have
\[
\lim_{J \to \infty} \lim_{n \to \infty} \| \sum_{j=1,j \neq j_{0}}^{J} Q(U_{n}^{j_{0},1}, U_{n}^{j_{1},1}) \|_{L^{p}(0, T_{1}, B_{p,p}^{s_{p}+\frac{a}{2}})} = 0.
\]
Lemma 4.6 is proved. \( \square \)

4.3. Orthogonality Property. In this paragraph, we show the orthogonality properties used in the proof of Lemma 4.6. The first statement of Proposition 4.7 is just a particular case of orthogonality property given in 14 (see the proof Lemma 3.3 in 14). By the same idea in 15, we give an orthogonality property in the case that one of the element in the product is time-independent.

Proposition 4.7. We assume that \( (\lambda_{1,n}, x_{1,n})_{n \in \mathbb{N}} \) and \( (\lambda_{2,n}, x_{2,n})_{n \in \mathbb{N}} \) are orthogonal. Let \( T \in \mathbb{R}_{+} \cup \{+\infty\} \). Then the following properties hold:

1. Let \( p > 3 \) and \( 1 - \frac{2}{p} < \frac{a}{2} < 1 \). Suppose that \( v, w \in L^{2a}([0, T], B_{p,p}^{s_{p}+\frac{a}{2}}) \). Then we have
\[
(21) \lim_{n \to \infty} \| (\Lambda_{1,n} v)(\Lambda_{2,n} w) \|_{L^{a}(0, T_{n}, B_{p,p}^{s_{p}+\frac{a}{2}})} = 0,
\]
where \( T_{n} := \min\{\lambda_{1,n}^{2} T, \lambda_{2,n}^{2} T\} \).

2. Let \( p > 3 \) and \( 2 < r < \frac{2p}{p-3} \). Suppose that \( U \in L^{3}(\mathbb{R}^{3}) \) and \( v \in L^{r}([0, T], B_{p,p}^{s_{p}+\frac{a}{2}}) \).
\[
(22) \lim_{n \to \infty} \| (\Lambda_{1,n} U)(\Lambda_{2,n} v) \|_{L^{r}(0, T_{n}', B_{p,p}^{s_{p}+\frac{a}{2}})} = 0,
\]
where \( T_{n}' := \lambda_{2,n} T \).
Proof. As we mentioned above, (21) is a particular case of orthogonality property given in [15], we only need to prove the second statement of the proposition.

For any given \( \varepsilon > 0 \) one can find two compactly supported (in space and time) functions \( v_\varepsilon \) and \( U_\varepsilon \) such that
\[
\| v - v_\varepsilon \|_{L^2([0,T], \dot{B}^{s_p+\frac{2}{p}}_{p,p})} + \| U - U_\varepsilon \|_{L^3} \leq \varepsilon.
\]

Product rules (along with the scale invariance of the scaling operators) gives that
\[
\| (\Lambda_{1,n}v)(\Lambda_{2,n}(U - U_\varepsilon)) \|_{L^6([0,T_n], \dot{B}^{s_p+\frac{2}{p}-1}_{p,p})} + \| (\Lambda_{1,n}(v - v_\varepsilon))(\Lambda_{2,n}(U)) \|_{L^6([0,T_n], \dot{B}^{s_p+\frac{2}{p}-1}_{p,p})} + \| (\Lambda_{1,n}(v - v_\varepsilon))(\Lambda_{2,n}(U - U_\varepsilon)) \|_{L^6([0,T_n], \dot{B}^{s_p+\frac{2}{p}-1}_{p,p})} \lesssim \varepsilon.
\]

Then it is enough to prove that for fixed \( \varepsilon > 0 \)
\[
\lim_{n \to \infty} \| (\Lambda_{1,n}U_\varepsilon)(\Lambda_{2,n}(v_\varepsilon)) \|_{L^6([0,T_n], \dot{B}^{s_p+\frac{2}{p}-1}_{p,p})} = 0.
\]
Again by Proposition 6.3 we have for some \( 3 < q < \frac{3p}{p-3} \) and small enough \( \delta > 0 \),
\[
\| (\Lambda_{1,n}U_\varepsilon)(\Lambda_{2,n}(v_\varepsilon)) \|_{L^6([0,T_n], \dot{B}^{s_p+\frac{2}{p}-\delta}_{p,p})} \lesssim \|\Lambda_{1,n}U_\varepsilon\|_{\dot{B}^{s_p+\delta}_{p,q}} \|\Lambda_{2,n}v_\varepsilon\|_{L^2([0,T_n], \dot{B}^{s_p+\frac{2}{p}}_{p,p})}.
\]
According to the fact that
\[
\|\Lambda_{1,n}U_\varepsilon\|_{\dot{B}^{s_p+\delta}_{p,q}} \lesssim \lambda_{1,n}^{-\delta} \|U\|_{L^3},
\]
and
\[
\|\Lambda_{2,n}v_\varepsilon\|_{L^2([0,T_n], \dot{B}^{s_p+\frac{2}{p}}_{p,p})} \lesssim \lambda_{2,n}^\delta \|v_\varepsilon\|_{L^2([0,T_n], \dot{B}^{s_p+\frac{2}{p}}_{p,p})},
\]
we have
\[
\| (\Lambda_{1,n}U_\varepsilon)(\Lambda_{2,n}(v_\varepsilon)) \|_{L^6([0,T_n], \dot{B}^{s_p+\frac{2}{p}-\delta}_{p,p})} \lesssim \left(\frac{\lambda_{2,n}}{\lambda_{1,n}}\right)^\delta \to 0, \quad n \to \infty,
\]
if \( \frac{\lambda_{2,n}}{\lambda_{1,n}} \to 0 \). Hence we prove (22).

\[\Box\]

5. Estimates on perturbation equations

Now we consider the following perturbation equation,
\[
\begin{aligned}
\partial_t w - \Delta w + \frac{1}{2}Q(w,w) + Q(w,g) + Q(w,U) &= f, \\
w|_{t=0} &= w_0,
\end{aligned}
\]
Let us state the following perturbation result.

Proposition 5.1. Let \( T \in \mathbb{R}_+ \cup \{+\infty\} \) and \( 3 < p < 5 \). Suppose that \( U \in L^3(\mathbb{R}^3) \), \( g \in L^p([0,T], \dot{B}^{s_p+\frac{2}{p}}_{p,p}(\mathbb{R}^3)) \) and \( f \in \mathcal{F}([0,T]) := L^p([0,T], \dot{B}^{s_p+\frac{2}{p}-2}_{p,p}) + \mathcal{L}^2([0,T], \dot{B}^{s_p+\frac{2}{p}-2}_{p,p}). \)

Suppose that for any \( h \in L^p([0,T], \dot{B}^{s_p+\frac{2}{p}}_{p,p}(\mathbb{R}^3)) \),
\[
\|Q(h,U)\|_{L^p([0,T], \dot{B}^{s_p+\frac{2}{p}-2}_{p,p})} \lesssim c_1 \|h\|_{L^p([0,T], \dot{B}^{s_p+\frac{2}{p}}_{p,p})},
\]
where \( c_1 > 0 \) is a universal small constant. Then there exists a constant \( C \) independent of \( T \) and \( \varepsilon_0 \) such that the following is true. If
\[
\|w_0\|_{\dot{B}^{s_p}_{p,p}} + \|f\|_{\mathcal{F}([0,T])} \leq \varepsilon_0 \exp\left(-C\|g\|_{L^p([0,T], \dot{B}^{s_p+\frac{2}{p}}_{p,p})}\right),
\]
then the solution \( w \) of the above perturbation problem satisfies
\[
\|w\|_{L^\infty([0,T], \dot{B}^{s_p+\frac{2}{p}}_{p,p})} \lesssim \|w_0\|_{\dot{B}^{s_p}_{p,p}} + \|f\|_{\mathcal{F}([0,T])} + \|g\|_{L^p([0,T], \dot{B}^{s_p+\frac{2}{p}}_{p,p})}.
\]
Profile Decomposition

then \( w \in L^p_p(T) \) and

\[
\|w\|_{L^p_p(T)} \leq C(\|w_0\|_{B^{s_p}_{p,p}} + \|f\|_{\mathcal{F}(0,T)}) \exp(C\|g\|_{L^p([0,T],B^{s^p+\frac{2}{p}}_{p,p})}).
\]

The proof of the proposition follows the estimates of [13] (see in particular Proposition 4.1 and Theorem 3.1 of [14]). The main difference is the absence of an exterior force and a small time-independent drift term in [13], but those terms are added with no difficulty to the estimates.

**Proof.** By Proposition 4.1 of [14], for any \( \alpha, \beta \in [0, T] \), we have the following estimates

\[
\|w\|_{L^p_p([\alpha,\beta])} \leq K \|w(\alpha)\|_{B^{s_p}_{p,p}} + K \|f\|_{\mathcal{F}([\alpha,\beta])} + K \|w\|^2_{L^p_p([\alpha,\beta],B^{s^p+2/p}_{p,p})}
\]

\[
+ K(c_1 + \|g\|_{L^p_p([\alpha,\beta],B^{s^p+2/p}_{p,p})}) \|w\|_{L^p_p([\alpha,\beta],B^{s^p+2/p}_{p,p})}.
\]

(24)

We recall that \( c_1 \) is a small enough number such that

\[
Kc_1 < \frac{1}{4}.
\]

And we claim that there exist \( N \) real numbers \( (T_i)_{1 \leq i \leq N} \) such that \( T_1 = 0 \) and \( T_N = T \), satisfying \( [0, T] = \bigcup_{i=1}^{N-1} [T_i, T_{i+1}] \) and

\[
\|g\|_{L^p_p([T_i,T_{i+1}],B^{s^p+\frac{2}{p}}_{p,p})} \leq \frac{1}{4K}, \quad \forall i \in \{1, \ldots, N-1\}
\]

Suppose that

(25)

\[
\|w_0\|_{B^{s_p}_{p,p}} + \|f\|_{\mathcal{F}(0,T)} \leq \frac{1}{8KN(4K)^N}.
\]

By time continuity we can define a maximal time \( \hat{T} \in \mathbb{R}_+ \cup \{\infty\} \) such that

\[
\|w\|_{L^p_p([0,\hat{T}],B^{s^p+\frac{2}{p}}_{p,p})} \leq \frac{1}{4K}.
\]

If \( \hat{T} \geq T \) then the proposition is proved. Indeed, by (23), we have,

\[
\|w\|_{L^p_p([T_i,T_{i+1}],B^{s^p+\frac{2}{p}}_{p,p})} \leq K \|w(T_i)\|_{B^{s_p}_{p,p}} + K \|f\|_{\mathcal{F}([T_i,T_{i+1}])} + \frac{3}{4} \|w\|_{L^p_p([T_i,T_{i+1}],B^{s^p+\frac{2}{p}}_{p,p})},
\]

which deduces that

\[
\|w\|_{L^p_p([T_i,T_{i+1}],B^{s^p+\frac{2}{p}}_{p,p})} \leq 4K(\|w(T_i)\|_{B^{s_p}_{p,p}} + \|f\|_{\mathcal{F}([T_i,T_{i+1}])}).
\]

Hence according to (23),

\[
\|w\|_{L^p_p([T_i,T_{i+1}],B^{s_p}_{p,p})} \leq K \|w(T_i)\|_{B^{s_p}_{p,p}} + K \|f\|_{\mathcal{F}([T_i,T_{i+1}])} + \frac{3}{4} \|w\|_{L^p_p([T_i,T_{i+1}],B^{s^p+\frac{2}{p}}_{p,p})}
\]

\[
\leq 4K(\|w(T_i)\|_{B^{s_p}_{p,p}} + \|f\|_{\mathcal{F}([T_i,T_{i+1}])}).
\]

Therefore,

\[
\|w(T_i)\|_{B^{s_p}_{p,p}} \leq (4K)^{i-1}(\|w(0)\|_{B^{s_p}_{p,p}} + \|f\|_{\mathcal{F}(0,T)}),
\]

which implies that

\[
\|w\|_{L^p_p([T_i,T_{i+1}],B^{s^p+\frac{2}{p}}_{p,p})} \leq (4K)^i(\|w(0)\|_{B^{s_p}_{p,p}} + \|f\|_{\mathcal{F}(0,T)}).
\]

Hence,

\[
\|w\|_{L^p_p([0,T],B^{s^p+\frac{2}{p}}_{p,p})} \leq N(4K)^N(\|w(0)\|_{B^{s_p}_{p,p}} + \|f\|_{\mathcal{F}(0,T)}).
\]
Profile Decomposition

Take $N \sim \|g\|_{L^p([0,T],B^{sp+\frac{2}{p}}_p)}$, we have

$$\|w\|_{L^p([0,T],B^{sp+\frac{2}{p}}_p)} \lesssim (\|w(0)\|_{B^{sp}_p} + \|f\|_{\mathcal{F}([0,T])})\exp(C\|g\|_{L^p([0,T],B^{sp+\frac{2}{p}}_p)}).$$

And by [23], we have

$$\|w\|_{L^\infty([0,T],B^{sp}_p)} \lesssim (\|w_0\|_{B^{sp}_p} + \|f\|_{\mathcal{F}([0,T])})\exp(C\|g\|_{L^p([0,T],B^{sp+\frac{2}{p}}_p)}).$$

Thus the proposition is proved in the case $\tilde{T} < T$.

Now we turn to the proof in the case of $\tilde{T} < T$. We define an integer $K \in \{1, \ldots, N-1\}$ such that

$$T_k \leq \tilde{T} < T_{k+1}.$$

Then for any $i \leq k-1$, we have

$$\|w\|_{L^p([T_i,T_{i+1}],B^{sp+\frac{2}{p}}_p)} \leq (4K)^i(\|w(0)\|_{B^{sp}_p} + \|f\|_{\mathcal{F}([0,T])}),$$

and

$$\|w(T_i)\|_{B^{sp}_p} \leq (4K)^{i-1}(\|w(0)\|_{B^{sp}_p} + \|f\|_{\mathcal{F}([0,T])}).$$

The same arguments as above also apply on the interval $[T_k, T]$ and yield,

$$\|w\|_{L^p([T_k,T],B^{sp+\frac{2}{p}}_p)} \leq (4K)^N\|w_0\|_{B^{sp}_p} + CK^2\|f\|_{\mathcal{F}([0,T])},$$

and

$$\|w\|_{L^\infty([T_k,T],B^{sp}_p)} \leq (4K)^N\|w_0\|_{B^{sp}_p} + CK^2\|f\|_{\mathcal{F}([0,T])}.$$ 

Therefore we have

$$\|w\|_{L^p([0,T],B^{sp+\frac{2}{p}}_p)} \leq N(4K)^N(\|w_0\|_{B^{sp}_p} + \|f\|_{\mathcal{F}([0,T])}) < \frac{1}{4K},$$

which contradicts to the maximality of $\tilde{T}$.

\[\square\]

6. Appendix

6.1. Some results on the steady-state Navier-Stokes equations. In this part, we recall some existence results on the steady state Navier-Stokes equations, and the Navier-Stokes equations equipped with the same time-independent external force. The steady state Navier-Stokes system is defined as follows,

$$(SNS) \begin{cases} -\Delta U + U \cdot \nabla = f - \nabla \Pi, \\ \nabla \cdot U = 0, \end{cases}$$

where $f(x)$ is the external force defined on $\mathbb{R}^3$. Since we only care about the case of $U \in L^3$, we state the following result for $\Delta^{-1}f \in L^3$ without proof, which is a consequence of Theorem 2.2 in [2].

Proposition 6.1. There exists an absolute constant $\delta > 0$ with the following property. If $f \in S'$ satisfies $\Delta^{-1}f \in L^3(\mathbb{R}^3)$ and

$$\|\Delta^{-1}f\|_{L^3(\mathbb{R}^3)} < \delta,$$

then there exists a unique solution to $(SNS)$ such that

$$\|U\|_{L^3} \leq 2\|\Delta^{-1}f\|_{L^3} < 2\delta.$$
Theorem 6.2. Suppose that \( f \) is a time-independent external force such that \( \| \Delta^{-1} f \|_{L^3} < c \), where \( c < \delta \) is a universal small constant. Let \( U_f \in L^3(\mathbb{R}^3) \) be the unique solution to (SNS) with \( \| U_f \|_{L^3} < 2 \| \Delta^{-1} f \|_{L^3} \) (the existence of \( U_f \) is provided by Proposition 2.7 in [26]). Then we have

1. For any initial data \( u_0 \in L^3(\mathbb{R}^3) \), there exists a unique maximal time \( T^*(u_0, f) > 0 \) and a unique solution \( u_f \) to (NSf) with initial data \( u_0 \) such that for any \( T < T^*(u_0, f) \),

\[
u_f \in C([0, T], L^3(\mathbb{R}^3)).\]

Moreover there exists a constant \( \delta_2(f) \) such that if \( \| u_0 - U \|_{L^3} < \delta_2, \) then \( u_0 \in C_0(\mathbb{R}_+, L^3(\mathbb{R}^3)) \). The solution \( u_f \) satisfies that for \( 3 < p < 5 \)

\[
\lim_{T \to T^*(u_0, f)} \| u_f - U_f \|_{L^p(T)} = \infty.
\]

(2) Let \( u_f \in C(\mathbb{R}_+, L^3(\mathbb{R}^3)) \) with initial data \( u_0 \in L^3(\mathbb{R}^3) \). Then \( u_f \in L^\infty(\mathbb{R}_+, L^3(\mathbb{R}^3)) \)

\[
\| u_f - U_f \|_{L^p(\mathbb{R}^3)} = 0.
\]

6.2. Product laws and heat estimates. We first recall the following standard product laws in Besov space, which use the theory of para-products (for details, see [7, 14]).

Proposition 6.3. (1) Let \( p > 3, q > 3 \) and \( r > 2 \). Moreover assume that \( s_q + s_p + \frac{2}{r} > 0 \).

We have, for any \( |\xi| < 1 \) such that \( 1 - \frac{2}{r} + \varepsilon > 0 \),

\[
\| vw \|_{L^r(\mathbb{R}^3)} \lesssim C(\varepsilon) \| v \|_{L^q(\mathbb{R}^3)} \| w \|_{L^2(\mathbb{R}^3)}.
\]

(2) Let \( p > 3 \) and \( 2 < r < \frac{2p}{p-3} \). Then for any \( \varepsilon \in \mathbb{R} \) such that \( 1 - \frac{2}{r} - |\varepsilon| > 0 \), we have

\[
\| vw \|_{L^r(\mathbb{R}^3)} \lesssim C(\varepsilon) \| v \|_{L^q(\mathbb{R}^3)} \| w \|_{L^2(\mathbb{R}^3)}.
\]

(3) Let \( p_1, p_2 \in (3, \infty), 2 < r < \frac{2p}{p-3} \) and \( T \in \mathbb{R}_+ \cup \{ \infty \}. \) Suppose that \( v \in L^{r, \infty}(T) \) and \( w \in L^{p_2, \infty}(T) \). Then we have

\[
\| vw \|_{L^r(\mathbb{R}^3)} \lesssim \| v \|_{L^{p_1}(\mathbb{R}^3)} \| w \|_{L^{p_2}(\mathbb{R}^3)}.
\]

where \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \).

(4) Let \( p > 3 \). Suppose that \( w \in L^\infty([0, T], L^3) \) and \( v \in L^{r_0}(\mathbb{R}^3; \dot{B}^{s_p + \frac{2}{r}}_{p, p}) \) for some \( T \in \mathbb{R}_+ \cup \{ \infty \} \) with \( r_0 = \frac{2p}{p-3} \). Then we have

\[
\| vw \|_{L^{r_0}(\mathbb{R}^3; \dot{B}^{s_p + \frac{2}{r}}_{p, p})} \lesssim C(p) \| v \|_{L^{r_0}(\mathbb{R}^3; L^3)} \| w \|_{L^{r_0}(\mathbb{R}^3; \dot{B}^{s_p + \frac{2}{r}}_{p, p})},
\]

where \( \frac{1}{p} = \frac{1}{r_0} + \frac{1}{p} \) and \( C(p) \to \infty \) as \( p \to \infty \).

Since the first three results in the proposition are standard and well-known, which can be found in [7, 14], we only give the proof of the last of the last of the proposition.

Proof. For simplicity, we treat \( w \) and \( v \) as functions. We have

\[
\Delta_j uv = \Delta_j T_u v + \Delta_j T_v w + \Delta_j R(u, v).
\]

We first take \( q_1 \) such that \( \frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_1} = \frac{1}{2} + \frac{1}{18} \) implying that \( q_1 = \frac{54}{2p-3} > 3 \).
About $\Delta_j T_w v$, we have
\[ \| \Delta_j T_w v \|_{L^0(\ell^p)} \lesssim \| (S_j w) (\Delta_j v) \|_{L^0(\ell^p)} \lesssim \| S_j w \|_{L^\infty(\ell^n)} \| v \|_{L^0(\ell^p)}. \]
And we notice that
\[ \| S_j w \|_{L^\infty(\ell^n)} \lesssim \sum_{j' \leq j} \| \Delta j w \|_{L^\infty(\ell^n)} \lesssim \sum_{j' \leq j} 2^{-j s_{q_1}} c_{j,q_1} \| w \|_{L^\infty([0,T],B_{q_1,q_1}^{s_{q_1}})}, \]
and
\[ \| v \|_{L^0(\ell^p)} \lesssim 2^{-j (s_p + \frac{2}{q_0})} c_{j,p} \| v \|_{L^0([0,T],B_{q_1,q_1}^{s_{q_1}+\frac{2}{q_0}})}. \]
Since $s_{q_1} < 0$, we have
\[ \| 2^{j (s_p + \frac{2}{q_0} - 1)} \| \Delta_j T_w v \|_{L^0(\ell^p)} \| \| v \|_{L^0([0,T],B_{q_1,q_1}^{s_{q_1}+\frac{2}{q_0}})} \]
This combined with $L^3 \hookrightarrow \dot{B}_{3,3}^{0} \hookrightarrow \dot{B}_{q_1,q_1}^{s_{q_1}}$, implies that
\[ \| 2^{j (s_p + \frac{2}{q_0} - 1)} \| \Delta_j T_w v \|_{L^0(\ell^p)} \| \| v \|_{L^0([0,T],B_{q_1,q_1}^{s_{q_1}+\frac{2}{q_0}})} \].
Now we choose $q := \frac{12p}{4q-1}$ and $p_1 := 4p$. It is easy to check such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q} = \frac{3}{4} + \frac{1}{4p}$ . We notice that
\[ \| \Delta_j T_v w \|_{L^0(\ell^p)} \lesssim \| S_j v \|_{L^0(\ell^p)} \| \Delta_j w \|_{L^\infty(\ell^p)}, \]
and
\[ \| S_j v \|_{L^0(\ell^p)} \lesssim \sum_{j' \leq j} \| \Delta j v \|_{L^0(\ell^p)} \lesssim 2^{-j (s_{p_1} + \frac{2}{q_0})} c_{j,p_1} \| v \|_{L^0([0,T],B_{p_1,p_1}^{s_{p_1}+\frac{2}{q_0}})} \]
\[ \lesssim 2^{-j (s_{p_1} + \frac{2}{q_0})} c_{j,p_1} \| v \|_{L^0([0,T],B_{p_1,p_1}^{s_{p_1}+\frac{2}{q_0}})}, \]
and
\[ \| \Delta_j w \|_{L^\infty(\ell^p)} \lesssim 2^{-j s_q} c_{j,q} \| w \|_{L^\infty([0,T],B_{q,q}^{s_q})}. \]
Since $s_{p_1} + \frac{2}{q_0} = -1 + \frac{3}{4p} + 1 - \frac{1}{p} = -\frac{1}{4p} < 0$, we have
\[ \| 2^{j (s_{p_1} + \frac{2}{q_0} - 1)} \| \Delta_j T_w v \|_{L^0(\ell^p)} \| \| v \|_{L^0([0,T],B_{q,q}^{s_q})} \]
Again by Lemma $L^3 \hookrightarrow \dot{B}_{3,3}^{0} \hookrightarrow \dot{B}_{q,q}^{s_q}$, we have
\[ \| 2^{j (s_{p_1} + \frac{2}{q_0} - 1)} \| \Delta_j T_w v \|_{L^0(\ell^p)} \| \| v \|_{L^0([0,T],L^3)} \].
Now we turn to the remainder $\Delta_j R(w, v)$. We denote that $\frac{1}{p} := \frac{1}{p_1} + \frac{1}{q} = \frac{3}{4} + \frac{11}{12p}$. Since
\[ \| \Delta_j R(w, v) \|_{L^0(\ell^p)} \lesssim \sum_{k \geq j - 1} \| \Delta k w \|_{L^\infty(\ell^p)} \| \Delta k v \|_{L^0(\ell^p)} \]
\[ \lesssim \sum_{k \geq j} 2^{-k (s_q + s_p + \frac{2}{q_0})} c_{k,q} c_{k,p} \| w \|_{L^\infty([0,T],B_{q,q}^{s_q})} \| v \|_{L^0([0,T],B_{q,q}^{s_q})}, \]
and
\[ s_p + s_q + \frac{2}{q_0} = \frac{7}{4p} > 0 \]
we have that, by applying $L^3 \hookrightarrow \dot{B}^0_{3,3} \hookrightarrow \dot{B}^s_{q,q}$,
\[
\|2^{(\frac{s}{p}+\frac{2}{q}-1)}\|\Delta_j R(w, v)\|_{L^p(\mathbb{R}^3)}\|_{\mathbb{R}^3} \lesssim p\|w\|_{L^\infty([0,T], B^{s}_{p,p})} \lesssim p\|w\|_{L^\infty([0,T], L^3)}\|v\|_{L^\infty([0,T], B^{s+rac{2}{p}}_{p,p})},
\]
which is $R(w, v) \in L^\infty([0,T], \dot{B}^{s+rac{2}{p}}_{p,p})$. And we have $R(w, v) \in L^\infty([0,T], \dot{B}^{s+p+rac{2}{q}}_{p,q})$, as $\tilde{p} < \tilde{q}$. Combining with (24) and (25) we get
\[
\|wv\|_{L^\infty([0,T], \dot{B}^{s+p+rac{2}{q}}_{p,q})} \leq C(p)\|w\|_{L^\infty([0,T], L^3)}\|v\|_{L^\infty([0,T], \dot{B}^{s+rac{2}{p}}_{p,p})},
\]
where $C(p) \to \infty$ as $p \to \infty$. The proposition is proved.

Now let us recall the following standard heat estimate. For any $p \in [1, \infty]$, there exists some $c_0, c > 0$ such that for any $f \in \mathcal{S}'$ and $j \in \mathbb{Z}$,
\[
\|\Delta_j (e^{t\Delta} f)\|_{L^p} \leq c_0 e^{-ct^{2j}}\|\Delta_j f\|_{L^p}.
\]
Hence for $0 < t \leq \infty$, recalling
\[
B(u, v) := \int_0^t e^{(t-s)\Delta} \mathbb{P} \cdot (u(s) \otimes v(s)) ds,
\]
Young’s inequality for convolutions implies that for any $\tilde{r} \in [r, \infty]$
\[
(30)\quad \|B(u, v)\|_{L^r([0,T], \dot{B}^{s+2\frac{1}{r}+\frac{1}{p}}_{p,p})} \lesssim \|u \otimes v\|_{L^r([0,T], \dot{B}^{s+1}_{p,p})},
\]
and we recall that $B$ is a bounded operator from $L^\infty([0,T], L^{3,\infty}) \times L^\infty([0,T], L^{3,\infty})$ to $L^\infty([0,T], L^{3,\infty})$ for any $T \in \mathbb{R}_+ \cup \{+\infty\}$ (see [2])

Acknowledgement. The author is grateful to Isabelle Gallagher for sharing many insights about the Navier-Stokes equation and entertaining discussions for overcoming the difficulties during this research.

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