Symmetric Spaces with Conformal Symmetry

A. Wehner*
Department of Physics, Utah State University, Logan, Utah 84322
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Abstract

We consider an involutive automorphism of the conformal algebra and the resulting symmetric space. We display a new action of the conformal group which gives rise to this space. The space has an intrinsic symplectic structure, a group-invariant metric and connection, and serves as the model space for a new conformal gauge theory.

*Electronic-mail: sllg7@cc.usu.edu
1 Introduction

Symmetric spaces are the most widely studied class of homogeneous spaces. They form a subclass of the reductive homogeneous spaces, which can essentially be characterized by the fact that they admit a unique torsion-free group-invariant connection. For symmetric spaces the curvature is covariantly constant with respect to this connection. Many essential results and extensive bibliographies on symmetric spaces can be found, for example, in the standard texts by Kobayashi and Nomizu \cite{KobayashiNomizu} and Helgason \cite{Helgason}.

The physicists' definition of a (maximally) symmetric space as a metric space which admits a maximum number of Killing vectors is considerably more restrictive. A thorough treatment of symmetric spaces in general relativity, where a pseudo-Riemannian metric and the metric-compatible connection are assumed, can be found in Weinberg \cite{Weinberg}. The most important symmetric space in gravitational theories is Minkowski space, which is based on the symmetric pair (Poincaré group, Lorentz group). Since the Poincaré group is not semi-simple, it does not admit an intrinsic group-invariant metric, which would project in the canonical fashion to the Minkowski metric $\eta_{ab} = \text{diag}(-1,1\ldots1)$.

The indisputable importance of conformal symmetry in physics has led us to describe a particular class of symmetric spaces which are built from the conformal group and admit a group-invariant metric as well as rich additional structures. There are compelling physical arguments that the conformal group, not the Poincaré group, should be the fundamental symmetry group of spacetime, not the least of which is the fact that it includes scale transformations (expressing our freedom to choose units arbitrarily) as a fundamental symmetry of nature. In addition, the conformal group is mathematically more natural than the Poincaré group as it is a simple Lie group, which admits a group-invariant indefinite Riemannian metric.

In contrast to the standard homogeneous space built from the conformal group (briefly described in Sec. 2), our construction takes the symmetric structure of the conformal Lie algebra into account (Sec. 3). The result is an even-dimensional symmetric space with local Lorentz and scaling symmetry. It has a conformally invariant connection and a natural symplectic structure. It can serve as the model geometry for a generalized space (in the sense of Cartan) called biconformal space, which was first discussed by Wheeler \cite{Wheeler}. We present a nonlinear action of the conformal group on this space, which treats translations and special conformal transformations symmetrically.
2 Conformal Group and Homogeneous Spaces

The conformal group \(C(n)\) is the group of transformations preserving angles between vectors when acting on \(n\)-dimensional \((n > 1)\) compactified Minkowski space \((\mathcal{M}^n, \eta_{\mu\nu})\) with metric \(\eta_{ab} = \text{diag}(-1, 1 \ldots 1)\), \(a, b, \ldots = 1 \ldots n\). Let \(\{x^\mu, \mu = 1 \ldots n\}\) be local coordinates for \(\mathcal{M}^n\). For \(n > 2\), the case we are considering from now on, the conformal group possesses the following well-known nonlinear action on \(\mathcal{M}^n\):

\[
M^\mu_\nu = -\Delta^\alpha_{\beta\nu}x^\beta\partial_\alpha \quad (1)
\]

\[
D = x^\mu\partial_\mu \quad (2)
\]

\[
P_\mu = \partial_\mu \quad (3)
\]

\[
K^\mu = (x^\mu x^\nu - \frac{1}{2}x^2\eta^{\mu\nu})\partial_\nu \quad (4)
\]

where \(\partial_\mu \equiv \frac{\partial}{\partial x^\mu}\) and we defined the antisymmetrization operator

\[
\Delta^\alpha\beta_{\mu\nu} \equiv \frac{1}{2}(\delta^\alpha_\mu\delta^\beta_\nu - \eta^{\alpha\beta}\eta_{\mu\nu}).
\]

These \(\frac{1}{2}(n+1)(n+2)\) vector fields are the solutions to the conformal Killing equation in \(\mathcal{M}^n\), i.e., the vector fields that preserve \(\eta_{\mu\nu}\) up to an overall function. They generate Lorentz transformations, scalings (dilatations), translations, and special conformal transformations, respectively. The latter are simply translations of the inverse coordinate \(z^\mu \equiv -x^\mu/x^2\) (hence the need for \(\mathcal{M}^n\) to be compact). The commutation relations of the vector fields (1)-(4) are

\[
[M^\alpha_\beta, M^\mu_\nu] = \Delta^\gamma_{\beta\nu}M^\mu_\gamma - \Delta^\alpha_{\beta\nu}M^\gamma_\nu \quad [D, P_\alpha] = -P_\alpha
\]

\[
[M^\alpha_\beta, P_\mu] = \Delta^\gamma_{\mu\beta}P_\gamma \quad [D, K^\alpha] = K^\alpha
\]

\[
[M^\alpha_\beta, K^\mu] = -\Delta^\alpha_{\gamma\beta}K^\gamma \quad [P_\alpha, K^\beta] = D\delta^\beta_\alpha - 2M^\beta_\alpha
\]

All other commutators vanish. These commutators are the same as those of the infinitesimal generators of the pseudo-orthogonal group \(O(n, 2)\) in an appropriate basis, i.e., \(C(n)\) and \(O(n, 2)\) are locally isomorphic. We note that our mixed index positioning, which originates from the \(O(n, 2)\) matrix notation, does not imply any use of the metric, but rather indicates the conformal weight of the operators as measured by the dilatation generator \(D\). Every upper index contributes +1 to the conformal weight of the operator, while every lower index contributes −1. Thus, \(K_\alpha\) and \(P^\alpha\) have conformal weight +1 and −1, respectively, while \(D\) and \(M^\alpha_\beta\) are weightless.
The isotropy subgroup of the infinitesimal action (1)-(4) at the origin is generated by those vector fields that vanish at the origin, namely \( \{ M_\mu^{\nu}, D, K^{\mu} \} \). It is the Poincaré group extended by scalings, which is commonly called the (inhomogeneous) similarity group, \( IS(n) \). Since the action is transitive on \( \mathcal{M}^n \), the space \( \mathcal{M}^n \) is isomorphic to the \( n \)-dimensional homogeneous space \( C(n)/IS(n) \). The space \( \mathcal{M}^n \) has long been known as conformal (or Möbius) space; it is topologically equivalent to a (pseudo-)sphere.

Curved generalizations of this space were first considered in 1923 by Elie Cartan \[5\]. In modern language, Cartan’s generalization of conformal space amounts to a principal bundle \( P \rightarrow M^n \) of dimension

\[
\dim P = \dim C(n) = \frac{1}{2}(n + 1)(n + 2),
\]

with fiber \( IS(n) \) and an \( n \)-dimensional curved base space \( M^n \) equipped with a conformal structure, i.e., an equivalence class of conformally related metrics. The bundle space \( P \) may be interpreted as the first differential prolongation (in the sense of Kobayashi \[6\]) of the conformal structure. The tangent space at any point of \( P \) is isomorphic to the conformal algebra, with the isomorphism given by the conformal connection, historically one of the first examples of what is today known as a Cartan connection. Later on, spaces with conformal connections were considered by many other authors \[7\].

Physicists have extensively used generalized conformal spaces in the 1970s as background geometries for conformal gauge theories \[8\]. Such theories were all shown to reduce to gauge theories formulated on Weyl geometries, in the sense that the connection forms (gauge fields) corresponding to the special conformal generators are algebraically removable in any field theory. In fact, it has been claimed on geometrical and physical grounds that the gauge transformations generated by special conformal transformations are redundant altogether, as their inclusion seems to amount either to gauging the same symmetry twice or to associating them with an unknown external symmetry of nature \[9\]. We do not wish to comment on this claim, but remark that even though general Weyl geometries incorporate local scale-invariance as an important physical principle, they do not give rise to general relativity in arbitrary dimensions without the use of additional structures such as compensating fields. Even more problematic is the fact that they predict unphysical size changes.
3 Symmetric Spaces

Recall that an involutive automorphism (of order 2) of a Lie group \(G\) is a Lie group automorphism \(\sigma : G \to G\) such that \(\sigma^2 = 1, \sigma \neq 1\). If \(H\) is a Lie subgroup of \(G\) with involutive automorphism \(\sigma\) such that \(H\) is fixed by \(\sigma\), then the coset space \(G/H\) is called a symmetric space. Let \(g\) and \(h\) be the Lie algebras of \(G\) and \(H\) and the Lie algebra automorphism \(\sigma_g : g \to g\) the one induced by \(\sigma\). We have the decomposition \(g = h \oplus m\) (direct sum), where \(h \equiv \{X \in g : \sigma(X) = X\}\) forms a Lie subalgebra and \(m \equiv \{X \in g : \sigma(X) = -X\}\) is the complementary space. As a consequence, \([h, m] \subset m\) and \([m, m] \subset h\). The pair \((g, h)\) is are known as a symmetric pair. The vector space spanned by \(m\) can be identified in a natural way with both the homogeneous space \(g/h\) and the tangent space at the point \(H\) of \(G/H\).

Homogeneous conformal spaces have a Lie algebra decomposition such that \(h = \{M^\alpha_\mu, D, K^\mu\}\) and \(m = \{P^\mu\}\). Because \([P^\alpha, K^\beta] = D\delta^\beta_\alpha - 2M^\beta_\alpha\), \([h, m] \not\subset m\), so these spaces are not symmetric. The reason is that homogeneous conformal spaces, while retaining the largest possible continuous symmetry on the fibers, do not take the discrete symmetries of the conformal algebra into account. The conformal algebra admits the following two involutive automorphisms:

\[
\begin{align*}
\sigma_1 : & \quad P^\alpha \to -P^\alpha \\
& \quad K^\alpha \to -K^\alpha \\
& \quad D \to D \\
& \quad M^\alpha_\beta \to M^\alpha_\beta.
\end{align*}
\]

\[
\begin{align*}
\sigma_2 : & \quad P^\alpha \to -\eta^{\alpha\beta}K^\beta \\
& \quad K^\alpha \to -\eta^{\alpha\beta}P^\beta \\
& \quad D \to -D \\
& \quad M^\alpha_\beta \to M^\alpha_\beta.
\end{align*}
\]

Both transformations leave the Lorentz sector of the conformal algebra invariant, but only the first incorporates in addition scale-invariance.\(^1\) Thus, we shall be interested in symmetric spaces based on \(\sigma_1\). In this case, the Lie subgroup \(H\) is the (non-compact) similarity, homothety, or conformal orthogonal group \(S(n)\), i.e., the direct product of the Lorentz group and the scaling group, \(S(n) = O(n-1,1) \otimes \mathbb{R}^+\). The resulting symmetric space,

\[
B(n) := C(n)/S(n),
\]

\(^1\) A third involutive automorphism is given by the composition of \(\sigma_1\) and \(\sigma_2\). Since the conformal group is non-compact and semisimple, its Lie algebra admits in addition a Cartan involution, i.e. an involutive automorphism whose fixed set is the Lie algebra of the maximal compact subgroup of \(C(n)\). Symmetric spaces defined by Cartan involutions are known to be homeomorphic to Euclidean spaces.\(^2\)
is of dimension 2n. The conformal group is simple, so \( S(n) \) cannot contain a non-trivial normal subgroup of \( C(n) \). Thus, \( C(n) \) acts not only transitively, but also effectively on \( B(n) \). The tangent space \( T_e(B(n)) \) at the identity is isomorphic to the vector space \( \mathfrak{m} \) spanned by \( P_\alpha \) and \( K_\alpha \). The space \( B(n) \) has an intrinsic metric which follows from the Killing form of the conformal algebra. The Killing form of the conformal group in the chosen basis is

\[
K_{conf} = 2n \begin{pmatrix}
\frac{1}{2} \Delta_{\mu\nu}^{\beta\alpha} & 0 & 0 & 0 \\
0 & 0 & \delta_\beta^\alpha & 0 \\
0 & \delta_\alpha^\beta & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Since the conformal group is semi-simple and non-compact, its Killing form is non-degenerate and indefinite and hence gives rise to a conformally invariant indefinite Riemannian metric on \( C(n) \). The vector spaces \( \mathfrak{h} \) and \( \mathfrak{m} \) are mutually orthogonal with respect to \( K_{conf} \). We restrict this quadratic form to the space \( B(n) \) and normalize it:

\[
K_{\alpha\beta} = \frac{1}{2} \begin{pmatrix}
0 & \delta_\beta^\alpha \\
\delta_\alpha^\beta & 0
\end{pmatrix}
\]

This restricted metric is still non-degenerate. Given a vector \( v = v^\alpha P_\alpha + v_\alpha K_\alpha \), we have \( K(v, v) = v^\alpha v_\alpha \) which shows that \( K \) is indefinite on \( B(n) \).

We now give an action of \( C(n) \) on the space \( B(n) \). Let \( \{x^\mu, y_\mu\} \) be local coordinates for \( B(n) \), and let \( (\eta_{\mu\nu}, -\eta^{\mu\nu}) \), which is the diagonalized form of the restricted Killing metric \( K_{con} \), be given on \( B(n) \). Then an infinitesimal action is given by the vector fields

\[
M_\mu^\nu = -\Delta_{\beta\mu}^{\mu\alpha} (x^\beta \partial_\alpha - y_\alpha \partial^\beta) \\
D = x^\alpha \partial_\alpha - y_\alpha \partial^\alpha \\
P_\mu = \partial_\mu + (y_\mu y_\nu - \frac{1}{2} y^2 \eta_{\mu\nu}) \partial^\nu \\
K_\mu = \partial^\mu + (x^\mu x^\nu - \frac{1}{2} x^2 \eta^{\mu\nu}) \partial_\nu
\]

with \( \partial_\mu \) as before and \( \partial^\mu = \frac{\partial}{\partial y_\mu} \). These vector fields generate the following
transformation:

\[
\begin{align*}
\tilde{x}^\mu &= \Lambda^\mu_\nu x^\nu \\
\tilde{y}_\mu &= y_\nu \Lambda^\nu_\mu \\
\tilde{x}^\mu &= \lambda x^\mu \\
\tilde{y}_\mu &= \lambda^{-1} y_\mu \\
\tilde{x}^\mu &= x^\mu + a^\mu \\
\tilde{y}_\mu &= y_\mu + b_\mu \\
\end{align*}
\]

\[\Lambda^\mu_\nu \in O(n - 1, 1)\]

\[\lambda \in \mathbb{R}^+\]

\[a^\mu \in \mathbb{R}^n\]

\[b_\mu \in \mathbb{R}^n\]

We see that this construction treats \(P_\mu\) and \(K_\mu\) on an equal footing: each generates translations in one sector and special conformal transformations in the other. We retain the name translations for the transformations generated by \(P_\mu\) and suggest the name co-translations for the ones corresponding to \(K_\mu\). This action does not mix the \(x\)- and \(y\)-sectors; angles are preserved in each of them separately. If one sets either \(x^\mu\) or \(y_\mu\) to zero, the action reduces to the standard action on compactified Minkowski space.

The isotropy subgroup at the origin \((x^\mu, y_\mu) = (0, 0)\) is generated by the vector fields \(M_\nu^\mu\) and \(D\); it is the similarity group \(S(n)\), so that indeed \(B(n) = C(n)/W(n)\). We will refer to \(B(n)\) as homogeneous biconformal space. In this geometry, symmetry of the fibers is exchanged for increased coordinate freedom for the base manifold. The tangent space at the origin of \(B(n)\) is spanned by \(P_\mu\) and \(K_\mu\). Notice that the quantity \(x^\mu y_\mu\) is a fiber invariant: \(D(x^\mu y_\mu) = 0 = M_\beta^\mu(x^\mu y_\mu)\).

As every homogeneous space, \(B(n)\) may be considered the base space of a principal bundle \(C(n) \to B(n)\) with fiber \(W(n)\). A connection on this bundle follows from the canonical form on \(C(n)\). Switching to orthonormal indices \(a, b... = 1...n\), we write this form as

\[
\omega = M^a_b \omega^b_a + P_a \omega^a + K^a \omega_a + D \omega^0,
\]

where the index positions on the basis 1-forms \(\{\omega^b_a, \omega^a, \omega_a, \omega^0\}\) again indicate the conformal weight. The form \(\omega\) satisfies the Maurer-Cartan structure
equations of the conformal group,
\begin{align*}
\mathbf{d}\omega^a_b &= \omega^c_b\omega^a_c - 2\Delta^a_d\omega^d\omega^c \\
\mathbf{d}\omega^a &= \omega^a_b\omega^b_a + \omega^0\omega^a \\
\mathbf{d}\omega_a &= \omega^0\omega_a + \omega_a\omega^0 \\
\mathbf{d}\omega^0 &= -\omega^a\omega_a,
\end{align*}
where we leave off the wedges between adjacent forms. They are integrable by virtue of the Jacobi identity. Clearly, the structure equations are invariant under the involutive automorphisms (5).

It is well known that for a symmetric space the $\mathfrak{h}$-component of the canonical form $\omega$ defines a unique connection in the bundle $G \to G/H$ called the canonical connection, with the following properties: (a) It is invariant under both $G$ and the involutive automorphism defining the symmetric space, (b) it has vanishing torsion, (c) it has covariantly constant curvature, and (d) it is compatible with the $G$-invariant indefinite Riemannian metric (6). Thus the form $\tilde{\omega} = M^a_b\omega^b_a + D\omega^0$ defines the canonical connection on $C(n) \to B(n)$.

The cotangent space to $B(n)$ is spanned by the $2n$ differential forms \{\omega^a, \omega_a\}. Both $\omega^a$ and $\omega_a$ can be consistently set to zero in the structure equations (7)-(10), which yields two classes of $n$-dimensional subspaces of $B(n)$. We observe that the space $B(n)$ has a natural symplectic structure: Since the connection forms $\omega^a$ and $\omega_a$ are independent, the 2-form $\Omega := \omega^a\omega_a$ is non-degenerate and, by virtue of equation (10), closed.

In addition to the symplectic structure $\Omega$ and the conformally invariant metric $K_{\alpha\beta}$, there exists the canonical almost complex structure $J$ on homogeneous biconformal space. It is integrable and compatible with the symplectic structure in the sense that $\Omega(Ju, Jv) = \Omega(u, v)$, which implies that the 2n-dimensional real space under consideration is equivalent to an $n$-dimensional complex space with Kähler metric $g(u, v) = \Omega(Ju, v)$. Neither Minkowski spaces, flat Weyl geometries, nor homogeneous conformal spaces give rise to these structures in an intrinsic fashion.

Finally, we mention that a homogeneous biconformal space can serve as the underlying model geometry for a curved generalization called biconformal space, which was introduced in [4]. We have previously shown [10] that such generalized geometries permit linear scale-invariant Lagrangians in any dimension and reproduce general relativity on certain subspaces. They do not give rise to unphysical size changes.
In summary, we compare some of the properties of the homogeneous spaces mentioned in this article:

| Homogeneous Space   | Minkowski | Weyl | Conformal | Biconformal |
|---------------------|-----------|------|-----------|-------------|
| Group $G$           | Poincaré  | $IS(n)$ | Conformal | Conformal   |
| Subgroup $H$        | Lorentz   | $S(n)$ | $IS(n)$   | $S(n)$      |
| $\dim(G/H)$         | $n$       | $n$   | $n$       | $2n$        |
| Killing form non-degenerate? | no   | no   | yes       | yes         |
| Symmetric space?    | yes       | yes   | no        | yes         |
| Local scale-invariance? | no  | yes   | yes       | yes         |
| Symplectic structure? | no  | no    | no        | yes         |
| Kähler structure?   | no        | no    | no        | yes         |
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