A Zero-stealthy Attack for Sampled-data Control Systems via Input Redundancy

Jihan Kim, Gyunghoon Park, Hyungbo Shim, Senior Member, IEEE, and Yongsoon Eun, Member, IEEE

Abstract—In this paper, we introduce a new vulnerability of cyber-physical systems to malicious attack. It arises when the physical plant, that is modeled as a continuous-time LTI system, is controlled by a digital controller. In the sampled-data framework, most anomaly detectors monitor the plant’s output only at discrete time instants, and thus, nothing abnormal can be detected as long as the sampled output behaves normal. This implies that if an actuator attack drives the plant’s state to pass through the kernel of the output matrix at each sensing time, then the attack compromises the system while remaining stealthy. We show that this type of attack always exists when the sampled-data system has an input redundancy, i.e., the number of inputs being larger than that of the outputs or the sampling rate of the actuators being higher than that of the sensors. Simulation results for the X-38 vehicle and for the other numerical examples illustrate this new attack strategy possibly brings disastrous consequences.

Index Terms—Networked control system, Cyber-physical system, Sampled-data system, Actuator attack, Multi-rate control, Cyber-physical attack.

I. INTRODUCTION

RECENT development of communication capabilities and computational resources has led to the integration of cyber-technologies and physical processes, which improves efficiency and flexibility of the system. These Cyber-Physical Systems (CPS) include not only simple or small devices, but also a variety of critical infrastructures that are closely related to public health and numerous financial costs. Examples include nuclear facilities, power grid (smart grid), supervisory control and data acquisition (SCADA) system, and networked transportation. For this reason, the security problem of CPS has received a lot of attention in recent years.

In particular, cyber-attacks on CPS may bring disastrous consequences, and their impacts are well illustrated by subsequent incidents, such as the Stuxnet attack on Iran’s nuclear plant [1], massive power blackouts in South America [2], Maroochy water breach in Australia [3], and cyber-attack on the Ukrainian power grid [4]. These instances highlight the need for measuring the vulnerabilities of CPS against malicious attacks and unexpected errors. There have been several researches that examine the vulnerabilities of CPS from the control-theoretic point of view. For instance, the weakness of electric power grids, possibly caused by false data injection attacks, was studied in [5]. An undetectable sensor attack to the unstable system was presented in [6]. More recently, the authors of [7] explored the question which resources should be utilized for the attack design, also focusing on various attack scenarios including denial of service (DoS) attack [8], replay attack [9], zero-dynamics attack [10], local zero-dynamics attack, and bias injection attack.

It is worth mentioning that most of the researches on security problems of CPS have been studied either in continuous-time or in discrete-time domain. From a practical standpoint, however, usual cyber-physical systems are composed of continuous-time physical plants and discrete-time digital controllers. It means that, for thorough understanding of cyber-security, interaction between the continuous-time and discrete-time components should come into the picture. In this regard, we are concerned with the security problem for sampled-data control system that consists of a multi-input multi-output (MIMO) continuous-time plant, samplers, and zero-order hold (ZOH) devices. Specifically, we allow that the sampling rate of the actuators be different from that of the sensors. These multi-rate sampling schemes have been widely studied in the literature for specific purposes. For example, a faster actuation than sensing has been adopted to improve control performance such as inter-sample behavior, disturbance rejection, and so on [13]–[17]. On the other hand, faster sensing has advantages on state feedback control design, acceleration control, and security problem [18]–[20].

In this paper, we show that the sampled-data systems are possibly vulnerable to a malicious adversary who utilizes an input redundancy of the systems. This redundancy becomes available to the attacker when (a) the sampling rate of the actuator is faster than that of the sensor, or (b) the number of inputs is larger than that of the outputs. Using the input redundancy, we present a new type of stealthy attack in the sampled-data framework. The underlying idea for the attack design is to express the sampled-data system as an extended lifted system with a stacked state variable, and to enforce the state to remain a (nontrivial) kernel of its output matrix (at sampling times). In doing so, the attack cannot be detected by any (discrete-time) anomaly detector that is built upon the sampled measurements of the output; at the same time, the inter-sample behavior of the physical plant is compromised. We will show that all of these can be done with the input redundancy. It should be pointed out that, unlike the well-known zero-dynamics attack [7], [10], [11], the proposed attack policy is applicable even when there is no unstable...
zero (either for continuous-time model or for its sampled-data counterpart).

The remainder of this paper is organized as follows. Section III presents the problem formulation. Section IV provides an attack design and studies when and how the adversary successfully spoils the sampled-data control systems. A few numerical examples and case studies can be found in Section V. Concluding remarks and further discussions are given in Section VI.

Notation: For two vectors $a$ and $b$, $\text{col}(a, b)$ stands for $[a^T \ b^T]^T$. The sets of natural, rational, and real numbers are denoted by $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$, respectively. The notation $\|x\|$ denotes the Euclidean norm for vector $x$. For a real number $r \in \mathbb{R}$, $[r]$ denotes the largest integer which is smaller than or equal to $r$. For a matrix $A$, $\ker A$ implies the null space of $A$ and $\text{im} A$ is the range space of $A$.

II. PROBLEM FORMULATION

We consider a compromised continuous-time physical system modeled as

$$
\dot{x}(t) = Ax(t) + B(u(t) + a(t)), \quad y(t) = Cx(t)
$$

where $x \in \mathbb{R}^n$ is the system state, $u, a \in \mathbb{R}^p$, and $y \in \mathbb{R}^q$ are the input, a malicious attack, and the output of the system, respectively, and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{q \times p}$, and $C \in \mathbb{R}^{q \times n}$.

Throughout the paper we suppose that the plant (1) is connected with a discrete-time controller through a communication network as seen in Fig. 1. Specifically, it is assumed that the discrete-time control is performed with the “sampler” for the output $y(t)$ with the sampling period $T_s$, and the “zero-order holder (ZOH)” for both the input $u(t)$ and the attack signal $a(t)$ with the sampling period $T_a$. Hence, $u(t)$ and $a(t)$ are piecewise constant functions such that $u(t) = u(iT_a)$ and $a(t) = a(iT_a)$ for $iT_a \leq t < (i + 1)T_a$. It is supposed that $u(iT_a) = \bar{u}[i]$ and $a(iT_a) = \bar{a}[i]$ where $\bar{u}[i]$ and $\bar{a}[i]$ are the output of a discrete-time controller and $\bar{a}[i]$ is a discrete-time attack signal injected through the vulnerable input communication network.

In this paper, we are interested in general multi-rate sampled-data systems where $T_s$ and $T_a$ are not necessarily the same. In particular, the ratio between $T_s$ and $T_a$ is assumed to satisfy

$$
R := \frac{T_s}{T_a} \in \mathbb{Q}.
$$

In what follows, we often use the coprime fraction $R = \beta/\alpha$ with $\alpha, \beta \in \mathbb{N}$ (rather than (2)). It should be noticed that, while the actuation times are $t = iT_a$ with $i = 0, 1, \ldots$, there is no reason that the sensing time (when the output $y(t)$ is sampled) is synchronized with the actuation time in practice. So, let us suppose that the sensing times are $t = jT_s + \Delta$ with $j = 0, 1, \ldots$, where $0 \leq \Delta < T_s$ is called an offset in this paper. Note that, while actuation times and sensing times are asynchronous, distribution of their times exhibits a pattern that repeats in every $\alpha T_s = \beta T_a$ seconds (see Fig. 2) since $R = T_s/T_a = \beta/\alpha$.

For convenience, we define a normalized offset $\delta := \Delta/T_s$ (so that $0 \leq \delta < 1$), and a new index (which is a real number) as $j_\delta := j + \delta - \lfloor j + \delta \rfloor$, $j = 1, 2, \ldots$.

Then $j_\delta = (j - 1) + \delta$ if $\delta > 0$ and $j_\delta = j$ if $\delta = 0$. Using the index $j_\delta$, the sampled-data system in terms of the sensing times can be written in the discrete-time domain as

$$
x((j_\delta + 1)T_s) = e^{AT_s}x(j_\delta T_s)
$$

$$
+ \int_{j_\delta T_s}^{(j_\delta + 1)T_s} e^{A((j_\delta + 1)T_s - \tau)} B(u(\tau) + a(\tau))d\tau,
$$

$$
y(j_\delta T_s) = Cx(j_\delta T_s) = : \bar{y}[j]
$$

for $j = 1, 2, \ldots$, while $x(0T_s)$ is given by $x(0T_s) = e^{AT_s}x(0) + \int_0^{1T_s} e^{A((1T_s - \tau)} B(u(\tau) + a(\tau))d\tau$.

Without loss of generality, it is assumed that $t = 0$ be the time when the attack is initiated. Now, for comparison, let $x_o$ be the solution of (1) without any attack (i.e., $a(t) \equiv 0$) and let $y_o = Cx_o$. It is noted that $x_o(0) = x(0)$ since the attack starts at $t = 0$. Then, with the error variables

$$
\bar{x}(t) := x(t) - x_o(t), \quad \bar{y}(t) := C\bar{x}(t),
$$

we have the error dynamics (obtained from (3)) as

$$
\bar{x}((j_\delta + 1)T_s) = e^{AT_s}\bar{x}(j_\delta T_s)
$$

$$
+ \int_{j_\delta T_s}^{(j_\delta + 1)T_s} e^{A((j_\delta + 1)T_s - \tau)} Ba(\tau)d\tau,
$$

$$
\bar{y}(j_\delta T_s) = C\bar{x}(j_\delta T_s)
$$

with $\bar{x}(1T_s) = \int_0^{1T_s} e^{A((1T_s - \tau)} Ba(\tau)d\tau$.

The problem to be studied is to generate an attack signal $\bar{a}[i]$ having the following two important features simultaneously.

Definition 1: An attack sequence $\{\bar{a}[i]\}_{i=0}^{\infty}$ is said to have zero-stealthy property if $\bar{y}(j_\delta T_s) \equiv 0$ for all $j \geq 1$.

This property directly implies that $y(j_\delta T_s) \equiv y_o(j_\delta T_s)$ for all $j \geq 1$, and thus, the plant (3) under the attack seemingly operates normally as if it is attack-free. Thus, no anomaly detector that uses $\bar{u}[i]$ and $\bar{y}[j]$ can detect the attack.

Definition 2: For a given sequence of positive thresholds $\{H_k\}_{k=1}^{\infty}$, an attack sequence $\{\bar{a}[i]\}_{i=0}^{\infty}$ is said to have disruptive property with $\{H_k\}_{k=1}^{\infty}$, if $\|\bar{x}(t_k)\| \geq H_k$ for all $k \geq 1$ with a time sequence $\{t_k\}_{k=1}^{\infty}$ satisfying $(k - 1)(\beta T_a) < t_k \leq k(\beta T_a)$.

The disruptive property indicates that the size of the error state $\bar{x}(t)$ becomes larger than $H_k$ at least once within the $k$-th time interval of the length $\beta T_a$ (that is, the time interval for

1Refer to [12] for synchronous case. While [12] has more limitations such as $R$ being integer, the derivation of [12] is simpler than this paper.
β times of actuations, or α times of measurements). Strength of the attack can be considered as the values of the sequence \( \{ H_k \}_{k=1}^\infty \), whose selection is fully upon the adversary.

A conventional solution to this problem is, as widely studied in the literature, the so-called zero-dynamics attack \([10]\). However, this attack is effective only when the system is of non-minimum phase, and the strength of the attack is determined solely by the plant’s zero-dynamics, and so the attacker is not able to assign the speed of divergence. Moreover, it is not a completely stealthy attack in the sense that its initiation causes a transient that can be observed from the output. (Therefore, in practice, the initial condition of the zero-dynamics attacker is set to be small enough so that the transient can hide below the alarm level in the anomaly detector.) On the other hand, the proposed attack is ‘zero-stealthy’ implying that the attacked output is exactly the same as the normal one at every sampling times.

In this paper, we propose a zero-stealthy disruptive attack for the sampled-data system that is possibly more lethal than the conventional zero-dynamics attack. Our proposal is based on the assumption that the sampled-data system \([3]\) has a kind of input redundancy. This is the case when the zero-order holder works faster than the sampler (that is, \( R \) is larger than 1), or the number \( p \) of the input channel is larger than that of the output channel, \( q \). Then, as we shall see below, the adversary can generate a new type of stealthy attack that has disruptive behavior with arbitrarily large thresholds.

III. DESIGN OF ZERO-STEALTHY ATTACK WITH DISRUPTIVE PROPERTY

The first task for the attack design is to rewrite the sampled-data system \([4]\) in the actuation time frame with \( T_a \) as

\[
\tilde{x}(iT_a) = A_d \tilde{x}(0) + \sum_{m=0}^{i-1} A_d^{i-1-m} B_d \bar{a}[m]
\]

\[
= \sum_{m=0}^{i-1} A_d^{i-1-m} B_d \bar{a}[m]
\]

where the last equality follows from \( \tilde{x}(0) = 0 \), and

\[
A_d := e^{A T_s} \in \mathbb{R}^{n \times n}, \quad B_d := \left( \int_0^{T_a} e^{A T} dT \right) B \in \mathbb{R}^{n \times p}.
\]

For progression, we need generalized notations about \( A_d \) and \( B_d \), which are related to both \( T_a \) and \( T_s \) as follows:

\[
A_d^{(l,m)} := e^{A(lT_s - mT_a)}, \quad B_d^{(l,m)} := \left( \int_0^{(lT_s - mT_a)} e^{A T} dT \right) B.
\]

From the above definition, \( A_d \) and \( B_d \) also can be denoted as \( A_d^{(0,-1)} \) and \( B_d^{(0,-1)} \), respectively.

A. Clustering the Time Frame

To construct the attack sequence \( \bar{a}[i] \) efficiently, let us introduce the concept of ‘cluster.’ The \( k \)-th cluster is defined as the time period \( (k-1) \beta T_a < t < k \beta T_a \) (and sometimes we indicate the left-closed interval \( (k-1) \beta T_a \leq t < k \beta T_a \) by calling it the \( k \)-th input-cluster). It can be seen that each cluster contains exactly \( \alpha \) sensing times and \( \beta \) actuation times.

See Fig. 2 for the case of \( R = 4/7 \) with \( \Delta > 0 \). By exploiting these clusters, we will consider the error dynamics \([4]\) in terms of the clusters. For this, let us define stacked attack vector in the \( k \)-th input-cluster, and the stacked states and the stacked measurements in the \( k \)-th cluster, as follows: for \( k = 1, 2, \ldots \),

\[
\tilde{a}(k) := \begin{bmatrix}
\tilde{a}[(k-1)\beta] \\
\tilde{a}[(k-1)\beta + 1] \\
\vdots \\
\tilde{a}[k\beta - 1] \\
\end{bmatrix} \in \mathbb{R}^{\beta p},
\]

\[
\tilde{x}(k) := \begin{bmatrix}
\tilde{x}(((k-1)\alpha + 1)\delta T_s) \\
\tilde{x}(((k-1)\alpha + 2)\delta T_s) \\
\vdots \\
\tilde{x}(((k-1)\alpha + \alpha)\delta T_s) \\
\end{bmatrix} \in \mathbb{R}^{\alpha n},
\]

\[
\tilde{y}(k) := C \tilde{x}(k) \in \mathbb{R}^{\alpha q}
\]

where \( C := I_\alpha \otimes C \) (\( I_\alpha \) is the identity matrix of size \( \alpha \) and \( \otimes \) is the Kronecker product). It is noted that the vector \( \tilde{y}(k) \) is the collection of \( \alpha \) measurements within one cluster.

Now, let us focus on the terminal state of each cluster, which is denoted by \( \tilde{x}_c[k] := \tilde{x}(k \beta T_a) \in \mathbb{R}^n \). Then, from \([5]\), one can derive that

\[
\tilde{x}_c[k] = A_d^\beta \tilde{x}_c[k-1] + \begin{bmatrix} A_d^{\beta-1} B_d & A_d^{\beta-2} B_d & \cdots & B_d \end{bmatrix} \tilde{a}(k)
\]

\[
= : A_d^\beta \tilde{x}_c[k-1] + \Phi_c \tilde{a}(k)
\]

with \( \tilde{x}_c[0] = 0 \). Similarly, one can derive the following for \( \tilde{y}(k) \).

Lemma 1: It follows that

\[
\tilde{x}(k) = \tilde{A}_\alpha \tilde{x}_c[k-1] + \Pi \tilde{a}(k)
\]

\[
\tilde{y}(k) = C \tilde{A}_\alpha \tilde{x}_c[k-1] + C\Pi \tilde{a}(k)
\]

where

\[
\tilde{A}_\alpha := \begin{bmatrix}
e^{A_1 T_s} & e^{A_2 T_s} & \cdots & e^{A_{\beta} T_s} \\
\end{bmatrix},
\]

and the \((l,m)\)-th block of \( \Pi \in \mathbb{R}^{\alpha n \times \beta p} \) is defined as follows:

\[
\Pi(l,m) := \begin{cases}
A_d^{(l,m)} B_d, & m = 1, \ldots, |l R|, \\
B_d^{(l,m)}, & m = |l R| + 1, \\
0, & m = |l R| + 2, \ldots, \beta,
\end{cases}
\]

for \( l = 1, \ldots, \alpha \).
Proof: Consider the first cluster $k = 1$, in which the state $\tilde{x}(0)$ at the beginning of the cluster is zero and $\tilde{z}_2[0] = 0$. With the property $\{j_3R\}T_a \leq j_3RT_a = j_3T_a$, one can compute the state $\tilde{x}(j_3T_a)$, whose sensing time $j_3T_a$ belongs to this cluster, by the variation of constant formula as follows:

$$\tilde{x}(j_3T_a) = \int_0^{j_3T_a} e^{A(j_3T_a - \tau)}d\tau B\tilde{a}[0] + \int_0^{j_3T_a} e^{A(j_3T_a - \tau)}d\tau B\tilde{a}[1] + \cdots + \int_0^{(j_3R)T_a} e^{A(j_3T_a - \tau)}d\tau B\tilde{a}[[j_3R] - 1] + \int_{(j_3R)T_a}^{j_3T_a} e^{A(j_3T_a - \tau)}d\tau B\tilde{a}[[j_3R]]$$

$$= \sum_{m=1}^{j_3T_a} e^{A(j_3T_a - mT_a)} \int_0^{mT_a} e^{A(mT_a - \tau)}d\tau B\tilde{a}[m - 1] + \int_0^{j_3T_a - j_3R} e^{A\tau}d\tau B\tilde{a}[[j_3R]]$$

$$= \sum_{m=1}^{j_3T_a} A_d^{(j_3,m)} B_d\tilde{a}[m - 1] + B_d^{(j_3,j_3R)}\tilde{a}[[j_3R]]$$

When $[1_jR] = 0$ (which happens if $j = 1$ and $1_jT_s < T_a$), it should be interpreted that the summation term in the above equation is zero or null. The discussion so far verifies (7) and it should be interpreted that the summation term in the above equation becomes zero. Thus, (11) and (12) are verified for the first cluster.

For the case $k > 1$, by taking into account the initial condition $\tilde{x}_e[k-1]$ and by noting that the matrix $\Phi_k^*$ is obtained exactly the same way as for $k = 1$, equation (11) is easily verified.

Note that, if all $T_k^*$ are chosen as a constant for all $k \geq 1$, then both $A_k^*$ and $\Phi_k^*$ are constant matrices. Now, with Lemma 1 and Lemma 2, the problem of our interest is reformulated in a cluster-wise sense; i.e., our interest becomes designing an attack sequence $\hat{a}(k)$ that satisfies $||\tilde{x}_e[k]|| \geq H_k$ (disruptive property), and at the same time, $\tilde{y}(k) \equiv 0$ for each $k$-th cluster (zero-stealthy property) for all $k \geq 1$.

B. Conditions for Attack Design

With equations (7), (8), and (11) at hand, conditions for attack design can be established. First of all, by (8), stealthiness of the attack is obtained if the attack sequence $\hat{a}(k)$ for the $k$-th cluster belongs to the kernel of $\Phi_k^*$, and so, we require the kernel to be non-trivial. Second, for the disruptive property of the state $\tilde{x}_e[k]$ in (11), we ask the kernel of $\Phi_k^*$ not to include the kernel of $\Phi_k^*$ of the $k$-th cluster. Because, if $\ker \Phi_k^* \ni \ker \Phi_k^*$, then any stealthy attack has no affect on $\tilde{x}_e[k]$. Finally, as the attack is initiated, the state $\tilde{x}(t)$ becomes non-zero, and therefore, even if the attack $\hat{a}(k-1)$ is designed to be stealthy from the measurement vector $\tilde{y}(k-1)$ for the $(k-1)$-th cluster, it may become detectable through non-zero $\tilde{x}_e[k-1] = \tilde{x}((k-1)T_a)$ in the $k$-th cluster. See (7) and (8). In order to counteract it, we require the range space of $\tilde{C}_A^*$ to belong to the range space of $\Phi_k^*$ so that some component of the attack sequence $\hat{a}(k)$ is designed to cancel the effect of $\tilde{x}_e[k-1]$ on $\tilde{y}(k)$. These discussions yield the following formal assumption.

Assumption 1: The following conditions hold:

(a) ker $\Phi_k^* \neq \{0\}$,
(b) ker $\Phi_k^* \not\subset \ker \Phi_k^*$, for all $k \geq 1$, with disruption times $\{T_{k}^*\}_{k=1}^{\infty}$,
(c) im $\tilde{C}_A^* \subset \ker \Phi_k^*$.

A few sufficient conditions for Assumption 1 can be derived. For example, since $\Phi_k^* \in R^{a_k \times b_k}$ so that $a_k < b_k$ implies ker $\Phi_k^* \neq \{0\}$, the item (a) is satisfied either when the number $p$ of inputs is large, or when the actuator works faster than the sensor (i.e., $R = T_s/T_a = \beta/\alpha$ is large enough). Hence, a sufficient condition for the item (a) is obviously $qT_s < pT_a$, which is simpler to check than item (a). On the other hand, it is noted that the condition (c) holds if the matrix $\tilde{C}_I$ has full
row rank or if the matrix $\Pi$ has full row rank. Finally, for the condition (b), we have the following.

**Proposition 1:** If the condition (a) of Assumption 1 holds and $B_d$ has full column rank (i.e., rank $B_d = p$), then there exists a sequence $\{t^*_k\}_{k=1}^\infty$ with which the condition (b) holds.

**Proof:** By the condition (a), pick any non-zero $z = \text{col}(z_1, \ldots, z_\beta) \in \ker \mathcal{C}$ where $z_i \in \mathbb{R}^p$. Define the index $i^* := \min\{i : z_i \neq 0, i = 1, \ldots, \beta\}$, and pick the disruption time $t^*_k \in (0, 1]$ such that $i^* = \beta t^*_k$. Then, it follows from (12) that $\Phi^*_i z = B_d z_{i^*} \neq 0$ since $B_d$ has full column rank. This implies that $\ker \mathcal{C} \not\subset \ker \Phi^*_i$, i.e., the item (b).

**Remark 1:** As a special case, let us consider the case when $R$ is a positive integer (i.e., $R = N \geq 1$ so that $\alpha = 1$ and $\beta = N$), and $\delta = 0$. This is the case that has been studied in [12]. In this case, we have $\mathcal{C} = C$, $A_d = e^{AT_d} = A_d^N$, and $\Pi = \left(e^{AT_d - 2N} \right) B_d$, $e^{AT_d - 2N} \right) B_d$, $e^{AT_d - (N-1)T_d} \right) B_d$, $B_d = [A_d^{N-1} B_d, A_d^{N-2} B_d, \ldots, B_d]$, and the conditions (a) and (c) of Assumption 1 are read as

(a) $\{0\} \neq \ker [A_d^{N-1} B_d, \ldots, B_d]$,
(b) $\ker [A_d^{N-1} B_d, \ldots, B_d] \not\subset \ker \Pi$.

It is clear that the above conditions hold if $q < Np$ and if either $C[A_d^{N-1} B_d, \ldots, B_d]$ or $[A_d^{N-1} B_d, \ldots, B_d]$ has full row rank. On the other hand, in [12], the disruption time $t^*_k$ is determined in the assumption as one of $\{1/N, 2/N, \ldots, 1\}$. It is noted from (12) that, for each candidate of $t^*_k = j/N$, $j = 1, \ldots, N$, the matrix $\Phi^*_i$ becomes as $\Phi^*_i |_{i=j/N} = [A_d^{N-1} B_d, A_d^{N-2} B_d, \ldots, B_d, 0, \ldots, 0]$. To facilitate selection of $t^*_k$ among the candidates, the condition of (12) reads as

$$\ker [A_d^{N-1} B_d, \ldots, B_d] \not\subset \ker \Pi$$

When this condition holds, one can pick suitable $t^*_k$ among the candidates for the condition (b) of Assumption 1. Another sufficient condition for (b) in this special case is: (b') $\ker C \cap \text{im} \Pi \neq \{0\}$. This is because (b') means that there exists a vector $v$ such that $\Pi v \neq 0$ and $\Pi v \in \ker C$. This implies that the vector $v$ belongs to $\ker \mathcal{C}$ while it does not belong to $\ker \Pi$, which guarantees (b') with $t^*_k = 1$. In Section IV-A, we demonstrate this case with $N = 1$.

**C. Off-line Construction of Attack Signal**

In this subsection, based on Assumption 1, we design an attack sequence $\tilde{a}^{(k)}$, or equivalently $\tilde{a}_{(k)}$, that solves the reformulated problem; i.e., make $\|\tilde{x}_{a_{(k)}}\| \geq H_k$ and $\tilde{y}_{(k)} \equiv 0$ for $k = 1, 2, \ldots$. In particular, we propose the sequence $\tilde{a}_{(k)}$ in the following form:

$$\tilde{a}_{(k)} = \frac{\eta_{(k)}}{\kappa_{(k)}} + \zeta_{(k)} \in \mathbb{R}^{\beta p}$$

where $\kappa_{(k)}$ is a positive constant and $\eta_{(k)}, \zeta_{(k)} \in \mathbb{R}^{\beta p}$. The idea is to pick $\eta_{(k)}$ such that $\Pi \eta_{(k)}$ is stealthy (i.e., belongs to $\ker \mathcal{C}$ but disruptive (i.e., $\Phi^*_i \eta_{(k)} \neq 0$) while $\kappa_{(k)}$ decides the intensity of disruption, and to pick $\zeta_{(k)}$ to counteract the effect of non-zero $\tilde{x}_{c[k-1]}$ on $\tilde{y}_{(k)}$ (i.e., $\tilde{x}_{c[k-1]} + \Pi \zeta_{(k)} \in \ker \mathcal{C}$). See Fig. 3.

![Fig. 3: Graphical interpretation of attack components](image)

The attack signal is designed sequentially, i.e., in the order of $a_{(1)}$, $a_{(2)}$, and so on. As the first step, let $\zeta_{(1)} = 0$ (since there is no attack before the time $t = 0$, and pick $\eta_{(1)} \in \ker \Pi$ such that $\Phi^*_i \eta_{(1)} \neq 0$ (whose existence is guaranteed by Assumption 1(a)). Then, stealthiness follows since

$$\tilde{y}_{(1)} = \mathcal{C} \Pi \eta_{(1)} = \kappa_1 \mathcal{C} \Pi \eta_{(1)} = 0.$$  (13)

For the disruptive property, pick $\kappa_1 > 0$ such that

$$\kappa_1 \Phi^*_i \eta_{(1)} \geq H_1.$$  (14)

By this, a stealthy and disruptive attack signal $a_0^{(1)}, \ldots, a_{\beta - 1}^{(1)}$ is obtained for the first cluster $(0, \beta T_a]$.

In order to design $a_2^{(2)}$ for the second cluster, consider

$$\tilde{y}_{(2)} = \mathcal{C} \tilde{A}_2 \tilde{x}_{e[1]} + \mathcal{C} \Pi \eta_{(2)}$$

where $\tilde{x}_{e[1]}$ is computed by (6) and $\tilde{a}_{(2)} = \kappa_2 \eta_{(2)} + \zeta_{(2)}$. Similarly as before, pick $\eta_{(2)}$ such that

$$\mathcal{C} \Pi \eta_{(2)} = 0 \quad \text{and} \quad \Phi^*_2 \eta_{(2)} \neq 0.$$  (15)

and pick $\zeta_{(2)}$ such that

$$\mathcal{C} \Pi \zeta_{(2)} = -\mathcal{C} \tilde{A}_2 \tilde{x}_{e[1]}.$$  (16)

Since

$$\tilde{x}_{e[2]} = \tilde{A}_2 \tilde{x}_{e[1]} + \Phi^*_2 \tilde{a}_{(2)} = \tilde{A}_2 \tilde{x}_{e[1]} + \Phi^*_2 \zeta_{(2)} + \Phi^*_2 \kappa_2 \eta_{(2)},$$  (17)

we take $\kappa_2$ such that

$$\kappa_2 \Phi^*_2 \eta_{(2)} \geq H_2 + \|\tilde{A}_2 \tilde{x}_{e[1]} + \Phi^*_2 \zeta_{(2)}\|.$$  (18)

This ensures stealthiness and disruptive property of the attack in the second cluster $(\beta T_a, 2\beta T_a]$.

We now generalize the procedure.

**Procedure of Attack Signal Generation:**

**Step k** ($k = 1, 2, \ldots$): Take $\zeta_{(k)}$ so that the following equation holds:

$$\mathcal{C} \Pi \zeta_{(k)} = -\mathcal{C} \tilde{A}_k \tilde{x}_{[k-1]}.$$  (19)
(for \( k = 1 \), \( \tilde{x}_c[0] = 0 \) so that \( \zeta_{(1)} = 0 \)). Pick \( \eta_{(k)} \) such that
\[
\eta_{(k)} \in \ker \mathcal{C} \Pi \quad \text{and} \quad \eta_{(k)} \notin \ker \Phi_k^* \]
and select a positive \( \kappa_k \) such that
\[
\kappa_k \geq \frac{H_k + ||\hat{A}_k^* \hat{x}_c[k-1] + \Phi_k^* \zeta_{(k)}||}{||\Phi_k^* \eta_{(k)}||}. \quad (20)
\]
With these terms, construct \( \tilde{a}(k) = \kappa_k \eta_{(k)} + \zeta_{(k)} \) and update \( \hat{x}_c[k] = A_d^k \hat{x}_c[k-1] + \Phi_a \tilde{a}(k) \) by (6).

**Remark 2:** It is noted that the construction of attack sequence can be done off-line, or *a priori* before the attack begins, because the procedure does not need any real-time information. Moreover, if the normalized disruption time \( t^*_k \in (0, 1] \) is chosen as a fixed constant for all \( k \geq 1 \), then the matrices \( \Phi^*_k \) are the same for all \( k \geq 1 \). Then, the vector \( \eta_{(k)} \) can also be chosen as a constant \( \eta \).

We close this section by summarizing the discussions so far.

**Theorem 1:** Suppose that the adversary has the information of \( T_s, T_a, \) and \( \Delta \) as well as the system information of \( A, B, \) and \( C \). If \( R = T_s/T_a \in \mathbb{Q} \) and Assumption 1 holds with normalized disruption times \( \{t^*_k\}_{k=1}^\infty \), then an attack sequence \( \{\tilde{a}[k]\}_{k=0}^\infty \) constructed via the proposed procedure has the zero-stealthy property and the disruptive property for any given \( \{H_k\}_{k=1}^\infty \).

IV. EXAMPLES

A. Numerical Example: \( R = 1 \) with \( \delta = 0 \)

In this subsection, we study a simple example in order to illustrate the attack generation procedure for the case \( R = 1 \) without offset, as discussed in Remark 1. For this, let us consider the error dynamics (4) with
\[
A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -5 & -3 \\ 0 & 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = [1 \ 0 \ 1].
\]
With a zero-order holder and sampler whose sampling periods are \( T_s = T_a = 1 \) sec (and thus \( R = 1 \)), its sampled-data system is given by (5) with
\[
A_d = \begin{bmatrix} 0.368 & 0 & 0 \\ 0 & -0.121 & -0.257 \\ 0 & 0.171 & 0.306 \end{bmatrix}, \quad B_d = \begin{bmatrix} 0.632 & 0 \\ 0 & 0.086 \\ 0 & 0.231 \end{bmatrix}.
\]
It is easily seen that Assumption 1 holds for the above system (see Remark 1). In particular, \( q < R \rho \) so that (a) holds and the matrix \( \mathcal{C} \Pi = CB \) has full row rank so that (c) holds. Also, the matrix \( B \) has full column rank, and so, by (the proof of) Proposition 1, the condition (b) holds with \( t^*_k = t/\beta = 1/1 \).

Now, for given \( H_k = k \), an attack sequence \( \tilde{a}(k) = \kappa_{(k)} \eta_{(k)} + \zeta_{(k)} \) is constructed as follows:

**Step 1:** Set \( \zeta_{(1)} = \text{col}(0, 0) \) and \( \eta_{(1)} = \text{col}(-0.343, 0.939) \in \ker \mathcal{C} \Pi \) such that \( \Phi^*_1 \eta_{(1)} = B_d \eta_{(1)} \neq 0 \). Then, select \( \kappa_1 = 3.15 \) to satisfy (14) (i.e., \( \kappa_1 ||B_d \eta_{(1)}|| \geq H_1 \)). Set \( \hat{x}_c[1] = B_d (\kappa_1 \eta_{(1)} + \zeta_{(1)}) \).

**Step 2:** Choose \( \zeta_{(2)} \) such that (10) holds (i.e., \( CB \zeta_{(2)} = -CA_d \hat{x}_c[1] \)). For convenience, let \( \eta_{(2)} = \tilde{\eta}_{(1)} \) as discussed in Remark 2. Then, select \( \kappa_2 \) for (18) and set \( \hat{x}_c[2] = A_d \hat{x}_c[1] + B_d (\kappa_2 \eta_{(2)} + \zeta_{(2)}) \).

Similarly, the remaining steps proceed with \( \eta_{(k)} = \eta_{(1)} \).

The designed attack sequence is injected into the control input at \( t = 0 \) sec. Fig. 4 shows the continuous-time state \( \hat{x}(t) \) from its initial condition \( \hat{x}(0) = \text{col}(0, 0, 0) \). Note that \( \hat{x}(t) \) is the error between the attack-free state \( x_a(t) \) and the state \( x(t) \) under attack. In this figure, it is observed that the error \( \hat{x}(t) \) moves far from the origin while it repeatedly encounters \( \ker C \). The sampled error output \( \tilde{y}(jT_s) \) remains zero as seen in Fig. 5 (zero-stealthy property). On the other hand, from Fig. 6 one can see that the disruptive property is satisfied; that is, \( ||\hat{x}(t_k)|| \geq H_k \) for \( (k-1)T_n < t_k \leq kT_n \) (here, \( t_k = kT_n \)).
B. X-38 Vehicle Example: \( R = 4 \) with \( \delta = 0 \)

As another example, we consider the X-38 vehicle model which is a prototype flight test vehicles for crew return vehicle [13]. In [13], the X-38 is operated by a multi-rate digital controller whose holder operates four times faster than the sampler (i.e., \( R = T_a/T_s = 4 \)) with \( T_a = 0.04 \) sec and \( T_s = 0.16 \) sec. The X-38 model has 3 inputs, 9 outputs, and 11 states \((A \in \mathbb{R}^{11 \times 11}, B \in \mathbb{R}^{11 \times 3}, \text{ and } C \in \mathbb{R}^{9 \times 11})\). More detailed information on the X-38 plant is provided in [13], [23].

From the information of X-38 model in [13] (that is omitted in this paper), one can verify that Assumption 1 holds by the following reasons:

- \( R_p = 12 \) and \( q = 9 \), and so, the condition (a) holds (i.e., \( C \Pi \in \mathbb{R}^{9 \times 12} \) so that \( \ker C \Pi \neq \{0\} \)),
- the matrix \( B_d \) has full column rank, and there exists a non-zero vector \( z \) such that \( C \Pi z = 0 \) where the first 3 components are a non-zero vector in \( \mathbb{R}^3 \). Then, by the proof of Proposition 1, \( i^* = 1 \). Therefore, the condition (b) holds with \( t_{k}^{*} = t_{k}^{*}/\beta = 1/4 \),
- the matrix \( C \Pi \) has full row rank so that \( \text{im } C \Pi = \mathbb{R}^9 \) and the condition (c) holds.

Now, following the proposed attack generation procedure, we construct an attack sequence \( \tilde{a}(k) = \kappa_k \zeta(k) + \eta(k) \) with disruptive property \( H_k = 0.5k \). In particular, we have chosen

\[
\eta_k = \eta = \begin{pmatrix} -0.132 \\ 0.497 \\ 0.324 \\ 0.108 \end{pmatrix} \in \ker C \Pi \quad \text{and} \quad \Phi_k \eta \neq 0,
\]

and \( \zeta(k)_i \)'s and \( \kappa_k \)'s are selected to satisfy \([19]\) and \([20]\), respectively.

To see the effect of the attack, the attack sequence is injected into the input channel of the plant at \( t = 0 \) sec. Fig. 7 shows the injected attack and Fig. 8 illustrates the state error \( \tilde{x}(t) \). In spite of the disruptive property seen in Fig. 8, the measured output at sampling times look normal (Fig. 9(c)). In fact, the continuous-time output \( y(t) \) is not calm as seen in Fig. 9(b) while the attack-free continuout-time output \( y_a(t) \) is also depicted for comparison in Fig. 9(a).

C. Numerical Example: \( R = 0.4 \) with \( \delta = 0.75 \)

In this subsection, we show that the proposed attack is effective under Assumption 1 even if the sampling period of the sensor is shorter than that of the actuator. Let us consider the case where \( T_a = 1 \) sec and \( T_s = 0.4 \) sec, so
that $R = 0.4/1 = 2/5 = \beta/\alpha$ (i.e., there are 5 sensings and 2 actuations for each cluster). Moreover, let us assume an offset with $\delta = 0.3/0.4 = 0.75$ (i.e., the sensor starts 0.3 sec later than the actuator). The considered plant is described by a minimal realization of

$$G(s) = \begin{bmatrix} 1 & 2 \\ s + 1 & (s + 2)(s + 3) & 4 \\ s + 3 & (s + 4)(s + 5) \end{bmatrix}.$$  

From the minimal realization $A \in \mathbb{R}^{5 \times 5}$, $B \in \mathbb{R}^{5 \times 3}$, and $C \in \mathbb{R}^{1 \times 5}$, one can verify Assumption 1 as follows:

- The plant has 3 inputs, 1 output, and $R = 0.4$. Hence, $q = 1 < Rp = 1.2$, and so, the condition (a) holds (i.e., $C\Pi \in \mathbb{R}^{5 \times 3}$ so that $\ker C\Pi \neq \{0\}$).
- The matrix $B_d$ has full column rank, and there exists a non-zero vector $z$ such that $C\Pi z = 0$ where the first 3 components are a non-zero vector in $\mathbb{R}^3$. Then, by the proof of Proposition 1, $i^* = 1$. Therefore, the condition (b) holds with $t_k^* = t/\beta = 1/2$.
- The matrix $C\Pi$ has full row rank so that $\im C\Pi = \mathbb{R}^5$ and the condition (c) holds.

An attack sequence $\bar{a}_i[i]$ is constructed as proposed with $H_k = 10k$. In particular, we have chosen $\eta_k = \eta = \text{col}(-0.188, -0.163, 0.746, 0.138, -0.467, 0.379) \in \ker C\Pi$ for all $k \geq 1$, which satisfies $\Phi_k \eta \neq 0$. The quantities $\kappa_k$ and $\xi_k$ are selected appropriately by the attack generation procedure in Section III.

The simulation results illustrate the constructed attack signal in Fig. 10 the behavior of $\bar{x}(t)$ in Fig. 11 and the output signal $\tilde{y}(t)$ in Fig. 12 respectively. It is seen that, even if the error variable $\bar{x}(t)$ and the continuous-time output $\tilde{y}(t)$ diverge, the output measurements (represented as red circles in Fig. 12) remain zero, so that both stealthiness and disruptive property are achieved.

Out of curiosity, we have simulated the case where the actual $R = 0.4004$ but is estimated as $R = 0.4$, so that the attack signal is designed based on $R = 0.4$. As seen in Fig. 10, the measured output does not remain zero forever, but if the estimate is sufficiently close to the true value, it is expected that the detection of the attack is delayed until a fatal damage is incurred in the plant.

Finally, in order to detect such an attack, one may deploy a mechanism of intermittent output sampling in addition to periodic sampling. Clearly, Fig. 12 and 13 show that additional output sample will yield non-zero values that would call attention of the operator.

V. CONCLUDING REMARK AND FUTURE WORKS

It has been recently studied that the interconnection between continuous- and discrete-time components may make some CPS more vulnerable to cyber-physical attacks (for instance, the zero-dynamics attack targeting the sampling zeros [20], [21]). We have clarified in this paper that another type of zero-stealthy attack is also possible: if there exists enough input redundancy for the system in the multi-rate or multi-input sense. By taking a closer look at the state trajectory in both continuous- and discrete-time domains, we showed that how the additional input resources and full system knowledge enable the adversary to compromise the inter-sample behavior of the sampled-data system, while being perfectly undetected at each sampling time.
Future works include consideration of input saturation and investigation of the case when \( R \), the ratio of \( T_s \) and \( T_a \), is real number. By analyzing the proposed construction of attack signal, we expect to figure out quantitative relationship between \( H_k \) and the saturation level. When the ratio between sampling period and actuation period is a real number, by approximating it as a rational number sufficiently closely, we expect to delay the detection of attack as much as we want, under strictly positive error threshold of anomaly detector.

Finally, it is also necessary to develop a method to detect the proposed attack. At this moment, we just think that intermittent random sampling of the output in addition to periodic sampling, removing unnecessary input channels, or concealing the system knowledge may be helpful.

**REFERENCES**

[1] J. P. Conti, “The day the samba stopped,” *Engineering and Technology*, vol. 5, no. 4, pp. 46–47, 2010.

[2] J. P. Conti, “The day the samba stopped,” *Engineering and Technology*, vol. 5, no. 4, pp. 46–47, 2010.

[3] J. Slay and M. Miller, “Lessons learned from the Maroochy water breach,” in *Critical Infrastructure Protection*, Springer, pp. 73–82, 2007.

[4] R. M. Lee, M. J. Assante, and T. Conway, “Analysis of the cyber attack on the Ukrainian power grid,” *Joint work between SANS ICS and the Electricity Information Sharing and Analysis Center*, 2016.

[5] Y. Liu, P. Ning, and M. K. Reiter, “False data injection attacks against state estimation in electric power grids,” *ACM Transactions on Information and System Security*, vol. 14, no. 1, pp. 13:1–13:33, 2011.

[6] Y. Mo and B. Sinopoli, “False data injection attacks in control systems,” in *Proceedings of 1st Workshop on Secure Control Systems*, Stockholm, Sweden, Apr. 2010.

[7] A. Teixeira, I. Shames, H. Sandberg, and K. H. Johansson, “A secure control framework for resource-limited adversaries,” *Automatica*, vol. 51, pp. 135–148, 2015.

[8] S. Amin, A. A. Cardenas, and S. S. Sastry, “Safe and secure networked control systems under denial-of-service attacks,” in *Hybrid Systems: Computation and Control*, Springer, pp. 31–45, 2009.

[9] Y. Mo and B. Sinopoli, “Secure control against replay attacks,” in *Proceedings of 47th Annual Allerton Conference on Communication, Control, and Computing*, pp. 911–918, 2009.

[10] A. Teixeira, I. Shames, H. Sandberg, and K. H. Johansson, “Revealing stealthy attacks in control systems,” in *Proceedings of 50th Annual Allerton Conference on Communication, Control, and Computing*, pp. 1806–1813, 2012.

[11] G. Park, H. Shim, C. Lee, Y. Eun, and K. H. Johansson, “When adversary encounters uncertain cyber-physical systems: Robust zero-dynamics attack with disclosure resources,” in *Proceedings of IEEE Conference on Decision and Control*, Las Vegas, 2016.

[12] J. Kim, G. Park, H. Shim, and Y. Eun, “Zero-stealthy attack for sampled-data control systems: the case of faster actuation than sensing,” in *Proceedings of IEEE Conference on Decision and Control*, Las Vegas, 2016.

[13] L. S. Shieh and W. M. Wang, “Design of lifted dual-rate digital controllers for X-38 Vehicle,” *Journal of Guidance, Control, and Dynamics*, vol. 23, no. 4, pp. 629–639, 2000.

[14] H. Fujimoto and Y. Hori, “High-performance servo systems based on multirate sampling control,” *Control Engineering Practice*, vol. 10, no. 7, pp. 773–781, 2002.

[15] H. Fujimoto, F. Kawakami, S. Kondo, “Multirate repetitive control and applications,” in *Proceedings of American Control Conference*, pp. 2875–2880, 2003.

[16] K. Lavanya and B. Umamaheswari, “Design of digital multi-rate controller using frequency domain analysis,” *Journal of Circuits, Systems, and Computers*, vol. 17, no. 4, pp. 675–684, 2008.

[17] M. D. I. Sen and S. A. Quesada, “Model matching via multirate sampling with fast sampled input guaranteeing the stability of the plant zeros: extensions to adaptive control,” *IET Control Theory and Applications*, vol. 1, no. 1, pp. 210–225, 2007.

[18] T. Hagiwara and M. Araki, “Design of a stable state feedback controller based on the multirate sampling of the plant output,” *IEEE Transactions on Automatic Control*, vol. 33, no. 9, pp. 812–819, 2002.

[19] M. Mizuochi, T. Tsuji, and K. Ohnishi, “Multirate sampling method for acceleration control system,” *IEEE Transactions on Industrial Electronics*, vol. 53, no. 3, pp. 1462–1471, 2007.

[20] M. Naghnaeian, N. Hizallah, and P. G. Voulgaris, “Dual rate control for security in cyber-physical systems,” in *Proceedings of IEEE Conference on Decision and Control*, pp. 1415–1420, 2012.

[21] J. Back, J. Kim, C. Lee, G. Park, and H. Shim, “Enhancement of security against zero dynamics attack via generalized hold,” in *Proceedings of IEEE Conference on Decision and Control*, pp. 1350–1355, 2017.

[22] J. I. Yuz and G. C. Goodwin, *Sampled-Data Models for Linear and Nonlinear Systems*, Springer, 2014.

Jihan Kim received his B.S. degree in the School of Electronic Engineering from Sogang University in 2014. Since 2014, he has been working toward his Ph.D. degree at Seoul National University. His research interests include security of cyber-physical systems, and sampled-data system.

Gyunghoon Park received his B.S. degree in the School of Electrical and Computer Engineering from Sungkyunkwan University in 2011, and M.S. degree from the School of Electrical Engineering and Computer Science, Seoul National University in 2013, respectively. Since 2013, he has been working toward his Ph.D. degree at Seoul National University. His research interests include theory and application of disturbance observer, security of cyber-physical systems, and sampled-data system.

Hyungbo Shim received the B.S., M.S., and Ph.D. degrees from Seoul National University, Korea, and held the post-doc position at University of California, Santa Barbara till 2001. He joined Hanyang University, Seoul, Korea, in 2002. Since 2003, he has been with Seoul National University, Korea. He served as associate editor for Automatica, IEEE Trans. on Automatic Control, Int. Journal of Robust and Nonlinear Control, and European Journal of Control, and as editor for Int. Journal of Control, Automation, and Systems. He was the Program Chair of ICCAS 2014 and Vice-program Chair of IFAC World Congress 2008. His research interest includes stability analysis of nonlinear systems, observer design, disturbance observer technique, secure control systems, and synchronization.

Yongsoo Eun (M’03) received the B.A. degree in mathematics, and the B.S. and M.S.E. degrees in control and instrumentation engineering from Seoul National University, Seoul, Korea, in 1992, 1994, and 1997, respectively, and the Ph.D. degree in electrical engineering and computer science from the University of Michigan, Ann Arbor, MI, USA, in 2003. From 2003 to 2012, he was a Research Scientist with the Xerox Innovation Group, Webster, NY, USA, where he worked on a number of subsystem technologies in the xerographic marking process and image registration method in inkjet marking technology. Since 2012, he is an Associate Professor with the Department of Information and Communication Engineering, Daegu Gyeongbuk Institute of Science and Technology (DGIST), Daegu, Korea. His research interests include control systems with nonlinear sensors and actuators, networked control systems, cyber-physical systems, and resilient control systems.