INVERSE RESONANCE SCATTERING FOR DIRAC OPERATORS ON THE HALF-LINE

EVLGENY KOROTYAEV AND DMITRII MOKEEV

Abstract. We consider massless Dirac operators on the half-line with compactly supported potentials. We solve the inverse problems in terms of Jost function and scattering matrix (including characterization). We study resonances as zeros of Jost function and prove that a potential is uniquely determined by its resonances. Moreover, we prove the following:
1) resonances are free parameters and a potential continuously depends on a resonance,
2) the forbidden domain for resonances is estimated,
3) asymptotics of resonance counting function is determined.

1. Introduction and main results

1.1. Introduction. We consider inverse problem for Dirac operators on the half-line with compactly supported potentials. Such operators have many physical and mathematical applications. In particular, they arise in study of stationary Dirac equations in $\mathbb{R}^3$ with spherically symmetric potentials (see e.g. [31]). Dirac operators are also known as $2 \times 2$ Zakharov-Shabat (or AKNS) system, which were used by Zakharov and Shabat [32] to study nonlinear Schrödinger equation (see also [1, 4, 6]). They used inverse scattering theory to describe NLS equations. In our paper, we consider the self-adjoint Dirac operator $H$ on $L^2(\mathbb{R}_+, \mathbb{C}^2)$ given by
\begin{equation}
Hy = -i\sigma_3 y' + i\sigma_3 Qy, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\end{equation}
with the boundary condition
\begin{equation}
e^{-i\alpha} y_1(0) - e^{i\alpha} y_2(0) = 0, \quad \alpha \in [0, \pi),
\end{equation}
where the parameter $\alpha$ is fixed throughout this paper. Note that if $\alpha = 0$, then (1.2) is the Dirichlet boundary condition, and if $\alpha = \pi/2$, then (1.2) is the Neumann boundary condition. The potential $Q$ has the following form
\begin{equation}
Q = \begin{pmatrix} 0 & q \\ \overline{q} & 0 \end{pmatrix}, \quad q \in \mathcal{P},
\end{equation}
where the class $\mathcal{P}$ is defined for some $\gamma > 0$ fixed throughout this paper by
\begin{equation}
\text{Definition.} \quad \mathcal{P} \text{ is the set of all functions } q \in L^2(\mathbb{R}_+) \text{ such that } \sup \text{supp } q = \gamma.
\end{equation}

It is well known that $\sigma(H) = \sigma_{ac}(H) = \mathbb{R}$ (see e.g. [27]). We introduce the $2 \times 2$ matrix-valued Jost solution $f(x, z) = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}(x, z)$ of the Dirac equation
\begin{equation}
f'(x, z) = Q(x)f(x, z) + iz\sigma_3 f(x, z), \quad (x, z) \in \mathbb{R}_+ \times \mathbb{C},
\end{equation}

Date: March 30, 2020.

Key words and phrases. Dirac operators, inverse problems, resonances, canonical systems, compactly supported potentials.
which satisfies the standard condition for compactly supported potentials:

\[ f(x, z) = e^{izx\sigma_3}, \quad \forall \ (x, z) \in [\gamma, +\infty) \times \mathbb{C}. \]

We define the Jost function \( \psi : \mathbb{C} \to \mathbb{C} \) by

\[ \psi(z) = e^{-i\alpha}f_{11}(0, z) - e^{i\alpha}f_{21}(0, z), \quad z \in \mathbb{C}. \]  

(1.5)

It is well-known that \( \psi \) is entire, \( \psi(z) \neq 0 \) for any \( z \in \mathbb{C}_+ \) and it has zeros in \( \mathbb{C}_- \), which are called resonances and they are also zeros of the Fredholm determinant and poles of the resolvent of the operator \( H \) (see e.g. [16]). We also define a scattering matrix \( S : \mathbb{R} \to \mathbb{C} \) by

\[ S(z) = \frac{\overline{\psi(z)}}{\psi(z)} = e^{-2i\arg \psi(z)}, \quad z \in \mathbb{R}. \]  

(1.6)

The function \( S \) admits a meromorphic continuation from \( \mathbb{R} \) onto \( \mathbb{C} \), since \( \psi \) is entire. Moreover, poles of \( S \) are zeros of \( \psi \) and then they are resonances. We sometimes write \( \psi(\cdot, q), S(\cdot, q), \ldots \) instead of \( \psi(\cdot), S(\cdot), \ldots \), when several potentials are being dealt with.

Our main goal is to solve inverse problems for the Dirac operator \( H \) with different spectral data: the scattering matrix, the Jost function, and the resonances. In general, an inverse problem is to determine the potential by some data, and it consists at least of the four parts:

(i) **Uniqueness.** Do data uniquely determine a potential?

(ii) **Reconstruction.** Give an algorithm to recover a potential by data.

(iii) **Characterization.** Give necessary and sufficient conditions that data correspond to a potential.

(iv) **Continuity.** Is the potential a continuous function of data and how can data be changed so that they remain data for some potential?

We solve the parts (i) – (iii) for the inverse resonance problem, when the data are resonances. In order to get these results, we solve the inverse problem for the operator \( H \) in terms of the Jost function and prove that it is uniquely determined and recovered by its zeros, i.e. by resonances of \( H \). We also solve the part (iv) for a finite number of resonances. Firstly, we show that a resonance of \( H \) is a free parameter, i.e. if we arbitrarily shift a zero of Jost function, then we obtain Jost function for some potential from \( \mathcal{P} \). Secondly, we prove that the potential continuously depends on one resonance, where all other resonances are fixed. Thirdly, we show that if we arbitrarily shift all zeros of Jost function along the real line or reflect them across the imaginary line, then we obtain Jost function for some potential from \( \mathcal{P} \). Note that we solve the inverse problem for the operator \( H \) in terms of the Jost function by using the relation between the scattering matrix and the Jost function and solution of the inverse scattering problem for compactly supported potential.

There are a lot papers about resonances in the different settings, see articles [8, 12, 19, 30, 33] and the book [15] and the references therein. The inverse resonance problem for Schrödinger operators with compactly supported potentials was solved in [21] for the case of the real line and in [19] for the case of the half line. In these papers, the uniqueness, reconstruction, and characterization problems were solved, see also Zworski [31], Brown-Knowles-Weikard [2] concerning the uniqueness. The stability problem for one-dimensional Schrödinger operators was considered in the papers [20, 28]. Moreover, there are other results about perturbations of the following model (unperturbed) potentials by compactly supported potentials: step potentials [3], periodic potentials [22], and linear potentials (corresponding to one-dimensional Stark operators) [23].
In the theory of resonances, one of the basic results is the asymptotics of the counting function of resonances, which is an analogue of the Weyl law for eigenvalues. For Schrödinger operators on the real line with compactly supported potentials, such result was first obtained by Zworski in [33]. The "local resonance" stability problems were considered in [20, 28] and results about the Carleson measures for resonances were obtained in [24].

In our paper, we discuss the inverse resonance problem for Dirac operators on the half-line. As far as we know, this problem has not been studied enough. Now, we discuss the known results on the resonances of one-dimensional Dirac operators. Global estimates of resonances for the massless Dirac operators on the real line were obtained in [23]. Resonance for Dirac operators was also studied in [16] for the massive Dirac operators on the half-line and in [15] for the massless Dirac operators on the real line. In these papers, the following results were obtained:

(i) asymptotics of counting function of the resonances;
(ii) estimates on the resonances and the forbidden domain;
(iii) the trace formula in terms of resonances for the massless case.

In [17], the radial Dirac operator was considered. There are a number of papers dealing with other related problems for the one-dimensional Dirac operators, for instance, the resonances for Dirac fields in black holes was described, see e.g., [14]. Note that Dirac operators can be rewritten as canonical systems, which are widely studied (see e.g. p. 389, eq. (5.12-15) in [10]).

1.2. Main results. We introduce the class of all Jost functions.

**Definition.** $(\mathcal{J}, \rho_\mathcal{J})$ is a metric space, where $\mathcal{J}$ is the set of all entire functions $\psi$ such that
\[
\psi(z) = e^{-i\alpha} + \int_0^\gamma g(s)e^{2izs}ds, \quad z \in \mathbb{C}, \quad (1.7)
\]
for some $g \in \mathcal{P}$ and $\psi(z) \neq 0$ for any $z \in \mathbb{C}_-$, and the metric $\rho_\mathcal{J}$ is given by
\[
\rho_\mathcal{J}(\psi_1, \psi_2) = \|g_1 - g_2\|_{L^2(0,\gamma)}, \quad \psi_1, \psi_2 \in \mathcal{J}. \quad (1.8)
\]

**Remark.** This class is similar to the case of Schrödinger operators from [19], but there are the following differences from the case of Schrödinger operators:

(i) there are no zeros in $\mathbb{C}_+$;
(ii) zeros in $\mathbb{C}_-$ are not symmetric with respect to the imaginary line;
(iii) $\psi(z) - e^{-i\alpha}$ decreases more slowly as $z \rightarrow \pm \infty$.

Maybe the last point is the main problem.

Let $\mathbb{S}^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$ and let $g : \mathbb{R} \to \mathbb{S}^1$ be a continuous function such that $g(x) = C + o(1)$ as $x \to \pm \infty$ for some $C \in \mathbb{S}^1$. Then $g = e^{-2i\phi}$ for some continuous $\phi : \mathbb{R} \to \mathbb{R}$. We introduce a winding number $W(g) \in \mathbb{Z}$ by
\[
W(g) = \frac{1}{\pi} \left( \lim_{x \to +\infty} \phi(x) - \lim_{x \to -\infty} \phi(x) \right),
\]
i.e. $W(g)$ is a number of revolutions of $g(x)$ around 0 when $x$ runs through $\mathbb{R}$. We introduce a class of the scattering matrices by
Definition. \((S, \rho_S)\) is a metric space, where \(S\) is the set of all continuous functions \(S: \mathbb{R} \to \mathbb{S}\) such that \(W(S) = 0\) and there exist \(F \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})\) such that \(\inf \text{supp} F = -\gamma\) and

\[
S(z) = e^{2\alpha z} + \int_{-\gamma}^{+\infty} F(s)e^{2izs} \, ds, \quad z \in \mathbb{R};
\]

and the metric \(\rho_S\) is given by

\[
\rho_S(S_1, S_2) = \|F_1 - F_2\|_{L^2(-\gamma, +\infty)} + \|F_1 - F_2\|_{L^1(-\gamma, +\infty)}, \quad S_1, S_2 \in S.
\]

Remark. It follows from well-known properties of the Fourier transform that \(S\) given by (1.9) is continuous and \(S(x) = e^{2\alpha x} + o(1)\) as \(x \to \pm\infty\). Thus, the winding number \(W(S)\) is correctly defined.

Note that the metric spaces \((S, \rho_S)\) and \((J, \rho_J)\) are not complete. Moreover, we equip the class \( \mathcal{P} \) with the metric \(\rho_P\) given by \(\rho_P(q_1, q_2) = \|q_1 - q_2\|_{L^2(0, \gamma)}, q_1, q_2 \in \mathcal{P}\). Thus, \((\mathcal{P}, \rho_P)\) is a metric space, which is not complete. Now, we present our first result.

Theorem 1.1. i) The mapping \(q \mapsto S(\cdot, q)\) from \(\mathcal{P}\) to \(S\) is a homeomorphism;

ii) The mapping \(q \mapsto \psi(\cdot, q)\) from \(\mathcal{P}\) to \(J\) is a homeomorphism.

iii) The following identity holds true:

\[
S = \{ S(z) = \overline{\psi(z)}\psi^{-1}(z), \, z \in \mathbb{C} \mid \psi \in J \}.
\]

Remark. We describe an algorithm to recover a potential from the Jost function or scattering matrix in Section[2]

We recall well-known facts about entire functions. An entire function \(g(z)\) is said to be of exponential type if there exist constants \(\tau, C > 0\) such that \(|g(z)| \leq Ce^{\tau|z|}, z \in \mathbb{C}\). We introduce the Cartwright class of entire functions by

Definition. \(\mathcal{E}_{\text{Cart}}\) is a class of entire functions of exponential type \(g\) such that

\[
\int_{\mathbb{R}} \frac{\log(1 + |g(x)|) \, dx}{1 + x^2} < \infty, \quad \tau_+(g) = 0, \quad \tau_-(g) = 2\gamma,
\]

where \(\tau_\pm(g) = \limsup_{y \to +\infty} \frac{\log|g(\pm iy)|}{y}\).

Let \(g \in \mathcal{E}_{\text{Cart}}\) and let \(g(0) \neq 0\). We denote by \(z_n, n \geq 1\), the zeros of \(g\) counted with multiplicity and arranged that \(0 < |z_1| \leq |z_2| \leq \ldots\). Then \(g\) has the Hadamard factorization

\[
g(z) = g(0)e^{\gamma z} \lim_{r \to +\infty} \prod_{|z_n| \leq r} \left(1 - \frac{z}{z_n} \right), \quad z \in \mathbb{C},
\]

see e.g. p.130 in [26], where the product converges uniformly on compact subsets of \(\mathbb{C}\) and

\[
\sum_{n \geq 1} \frac{\text{Im} \, z_n}{|z_n|^2} < +\infty.
\]

For any entire function \(g\) and \((r, \delta) \in \mathbb{R}_+ \times [0, \frac{\pi}{2}]\), we introduce the following counting functions

\[
N_\pm(r, \delta, g) = \#\{ z \in \mathbb{C} \mid |g(z)| = 0, \, |z| \leq r, \, \pm \text{Re} \, z \geq 0, \, \delta < |\text{arg} \, z| < \pi - \delta \}.
\]

We need the Levinson’s result about zeros of functions from \(\mathcal{E}_{\text{Cart}}\), see e.g. p. 58 in [18].
**Theorem** (Levinson). Let \( g \in \mathcal{E}_{\text{Cart}} \). Then for each \( \delta > 0 \) we have
\[
N_{\pm}(r, 0, g) = \frac{\gamma}{\pi} r + o(r), \quad N_{\pm}(r, \delta, g) = o(r),
\]
as \( r \to +\infty \).

It follows from the Paley-Wiener theorem (see e.g. p.30 in [18]), that an entire function having form (1.7) belongs to the Cartwright class. Thus, we get the following corollary.

**Corollary 1.2.** Let \( q \in \mathcal{P} \). Then \( q \) is uniquely determined by its resonances, \( \psi(\cdot, q) \in \mathcal{E}_{\text{Cart}} \) and it satisfies (1.11-13).

In Corollary 1.2, we improve the result from [16], where a similar result was obtained for differentiable potentials. We describe the position of resonances and the forbidden domain.

**Theorem 1.3.** Let \( q \in \mathcal{P} \) and let \( z_n, n \geq 1 \), be its resonances. Let \( \varepsilon > 0 \). Then there exists a constant \( C = C(\varepsilon, q) \geq 0 \) such that the following inequality holds true for each \( n \geq 1 \):
\[
2\gamma \Im z_n \leq \ln \left( \varepsilon + \frac{C}{|z_n|} \right).
\]
In particular, for any \( A > 0 \), there are only finitely many resonances in the strip
\[
\{ z \in \mathbb{C} \mid 0 > \Im z > -A \}.
\]

**Remark.** If \( q' \in L^1(\mathbb{R}_+) \), then estimate (1.14) and the forbidden domain (1.15) can be improved (see Theorem 2.7 in [16]).

Now, we show how the scattering matrix can be constructed directly by resonances.

**Theorem 1.4.** Let \( q \in \mathcal{P} \). Then its scattering matrix \( S \) has the following form
\[
S(z) = e^{-2i\phi_{\text{sc}}(z)}, \quad z \in \mathbb{R},
\]
where \( \phi_{\text{sc}} \) is a real-valued function such that \( \phi_{\text{sc}} \in L^\infty(\mathbb{R}) \cap C^\infty(\mathbb{R}) \), \( \phi_{\text{sc}}(\cdot) + \alpha \in L^2(\mathbb{R}) \), and \( \phi_{\text{sc}}(z) \to -\alpha \) as \( z \to \pm \infty \).

Moreover, let \( z_n, n \geq 1 \), be zeros of \( \psi(\cdot, q) \). Then we have
\[
\phi_{\text{sc}}(z) = \phi_{\text{sc}}(0) + \int_0^z \phi'_{\text{sc}}(s)ds, \quad \phi'_{\text{sc}}(z) = \gamma + \sum_{n \geq 1} \frac{\Im z_n}{|z - z_n|^2}, \quad z \in \mathbb{R},
\]
\[
\phi_{\text{sc}}(0) = -\alpha - \lim_{z \to +\infty} \int_0^z \phi'_{\text{sc}}(s)ds = -\alpha + \lim_{z \to -\infty} \int_0^z \phi'_{\text{sc}}(s)ds,
\]
where the sum converges absolutely and uniformly on compact subsets of \( \mathbb{R} \).

We describe some automorphisms of the class \( \mathcal{J} \). Firstly, we show that the resonances are free parameters and prove that the Jost function continuously depends on a resonance.

**Theorem 1.5.** Let \( q^o \in \mathcal{P} \) and let \( z^o_n \in \mathbb{C}_- \), \( n \geq 1 \), be its resonances. Let \( N = \# \{ z^o_j \mid |z^o_j| < r \} \) for some \( r > 1 \). Let \( z_j \in \mathbb{C}_- \), \( |z_j| < r \), \( j = 1, \ldots, N \). Then there exist a unique \( q \in \mathcal{P} \) such that
\[
\psi(z, q) = \psi(z, q^o) \prod_{j=1}^N \frac{z - z_j}{z - z^o_j}, \quad z \in \mathbb{C}.
\]
In particular, any point on $\mathbb{C}_-$ can be a resonance with any multiplicity for some $q \in \mathcal{P}$. Moreover, if each $z_j \to z_j^o$, $j = 1, \ldots, N$, then we have
$$
\rho_p(q^o, q) \to 0, \quad \rho_{\mathcal{J}}(\psi(\cdot, q^o), \psi(\cdot, q)) \to 0, \quad \rho_S(S(\cdot, q^o), S(\cdot, q)) \to 0.
$$

**Remark.** For Schrödinger operators, similar results are obtained in [19].

Secondly, we show that $\mathcal{J}$ and $\mathcal{S}$ are invariant with respect to shifts along the real line and with respect to the reflection across the imaginary line.

**Theorem 1.6.** Let $q \in \mathcal{P}$. Then the following identities hold true:
$$
\begin{align*}
\psi(z + k, q) &= \psi(z, e_k q), \quad \overline{\psi(-z, q)} = e^{2i\alpha} \psi(z, e^{4i\alpha} q), \quad z \in \mathbb{C}, \\
S(z + k, q) &= S(z, e_k q), \quad \overline{S(-z, q)} = e^{-4i\alpha} S(z, e^{4i\alpha} q), \quad z \in \mathbb{R},
\end{align*}
$$
(1.19)

where $e_k(x) = e^{2i\alpha x}$ for any $k \in \mathbb{R}$.

**Remark.** 1) Due to (1.19), we can arbitrarily shift all resonances along the real line.

2) If $f$ is a meromorphic or entire function, then the function $h$ such that $h(z) = f(-z)$ is also meromorphic or entire and its zeros are zeros of $f$ reflected across the imaginary line. Thus, using (1.19), we can reflect all resonances across the imaginary line.

3) For Schrödinger operators, such theorem does not hold. It follows from the fact that the resonances of Schrödinger operators are symmetric with respect to the reflection across the imaginary line.

4) If we solve the inverse problem for $\psi \in \mathcal{J}$, then, by (1.19), we can solve the inverse problem for $\psi(\cdot + k)$ for any $k \in \mathbb{R}$ and for $\psi_1$ such that $\psi_1(z) = \psi(-z, q), z \in \mathbb{C}$. 

5) Note that any combination of shifts and reflections is just one shift and at most one reflection.

## 2. Preliminary

### 2.1. Notations

In these section, we recall results about inverse scattering problem for the Dirac operator on the half-line. We introduce the following Banach spaces
$$
\begin{align*}
\mathcal{B} &= L^2(\mathbb{R}_+) \cap L^1(\mathbb{R}_+), \quad \| \cdot \|_{\mathcal{B}_+} = \| \cdot \|_{L^2(\mathbb{R}_+)} + \| \cdot \|_{L^1(\mathbb{R}_+)}, \\
\mathcal{B} &= L^2(\mathbb{R}) \cap L^1(\mathbb{R}), \quad \| \cdot \|_{\mathcal{B}} = \| \cdot \|_{L^2(\mathbb{R})} + \| \cdot \|_{L^1(\mathbb{R})}.
\end{align*}
$$

We also define the following unital Banach algebras with pointwise multiplication
$$
\begin{align*}
\mathcal{A}_+ &= \{ c + \mathcal{F}g(x), x \in \mathbb{C}_+ \mid c \in \mathbb{C}, g \in \mathcal{B}_+ \}, \quad \| c + \mathcal{F}g \|_{\mathcal{A}_+} = |c| + \|g\|_{\mathcal{B}_+}, \\
\mathcal{A} &= \{ c + \mathcal{F}g(x), x \in \mathbb{R} \mid c \in \mathbb{C}, g \in \mathcal{B} \}, \quad \| c + \mathcal{F}g \|_{\mathcal{A}} = |c| + \|g\|_{\mathcal{B}}.
\end{align*}
$$

where $\mathcal{F}g(x) = \int_{\mathbb{R}} g(s)e^{2i\pi x s}ds$ is the Fourier transform of $g$. It is well-known that $\mathcal{A}_+$ and $\mathcal{A}$ are unital Banach algebras (see e.g. Chapter 17 in [9]). At last, by $\mathcal{M}_2(\mathbb{C})$ we denote the Banach space of $2 \times 2$ matrices with complex entries.

### 2.2. Jost functions

We consider Dirac operator $H y = -i\sigma_3 y^f + i\sigma_3 Q y$, where potential $Q$ has form (1.3) and $q \in \mathcal{B}_+$. If $q \in \mathcal{B}_+$, then we introduce the Jost solution $f(x, z)$ of Dirac equation (1.4) satisfying the following asymptotic condition:
$$

f(x, z) = e^{i\pi z} (1 + o(1)), \quad \text{as } x \to +\infty.
$$
(2.1)

It is well known that for every $z \in \mathbb{R}$ there exists a unique Jost solution and it has an integral representation in terms of the transformation operator. We recall these known results, see p.39 in [6] and Proposition 3.5 in [7].
Lemma 2.1. Let $q \in \mathcal{B}_+$. Then there exists a function $\Gamma : \mathbb{R}_+^2 \to \mathcal{M}_2(\mathbb{C})$ such that

$$\Gamma(x, s) = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix}(x, s), \quad (x, s) \in \mathbb{R}_+^2,$$

and

$$f(x, z) = e^{ixz\sigma_3} + \int_0^{+\infty} \Gamma(x, s)e^{i(2s+x)z\sigma_3}ds, \quad (x, z) \in \mathbb{R}_+ \times \mathbb{R}. \quad (2.2)$$

Moreover, the following holds:

i) For each $n,m = 1,2$, the mapping $x \to \Gamma_{nm}(x, \cdot)$ from $\mathbb{R}_+$ into $\mathcal{B}_+$ is continuous and for any $x \in \mathbb{R}_+$, the following estimate holds true:

$$\|\Gamma_{nm}(x, \cdot)\|_{\mathcal{B}_+} \leq e^{q_1(x)}(1 + q_2(x))^{-1}, \quad (2.3)$$

where

$$q_1(x) = \int_x^{+\infty} |q(s)|ds, \quad q_2(x) = \left(\int_x^{+\infty} |q(s)|^2ds\right)^{1/2};$$

ii) For each $n,m = 1,2$ and each fixed $x \in \mathbb{R}_+$, the mapping $q \to \Gamma_{nm}(x, \cdot, q)$ from $\mathcal{B}_+$ into $\mathcal{B}_+$ is continuous;

iii) For almost all $x \in \mathbb{R}_+$, we have

$$q(x) = -\Gamma_{12}(x, 0). \quad (2.4)$$

Recall that the Jost function $\psi(z)$ is defined by (1.5). We introduce the following class of all Jost functions for potentials from $\mathcal{B}_+$.

**Definition.** ($\mathcal{J}_+, \rho_{\mathcal{J}_+}$) is a metric space, where $\mathcal{J}_+$ is the set of all functions $\psi : \mathbb{T}_+ \to \mathbb{C}$ such that

$$\psi(z) = e^{-i\alpha} + \int_0^{+\infty} g(s)e^{2izs}ds, \quad z \in \mathbb{T}_+, \quad (2.5)$$

for some $g \in \mathcal{B}_+$ and $\psi(z) \neq 0$ for any $z \in \mathbb{T}_+$, and the metric $\rho_{\mathcal{J}_+}$ is given by

$$\rho_{\mathcal{J}_+}(\psi_1, \psi_2) = \|g_1 - g_2\|_{\mathcal{B}_+}, \quad \psi_1, \psi_2 \in \mathcal{J}_+. \quad (2.6)$$

**Remark.** It follows from (2.5-6) that $\mathcal{J}_+ \subset \mathcal{A}_+$ isometrically. Moreover, each $\psi \in \mathcal{J}_+$ is invertible in $\mathcal{A}_+$ (see e.g. Lemma 2.9 in [13]).

Using integral representation (2.2) and the fact that $\psi \in \mathcal{J}_+$ is invertible, we obtain the following corollary.

**Corollary 2.2.** Let $q \in \mathcal{B}_+$. Then $\psi \in \mathcal{J}_+$ and the following integral representation holds:

$$\psi(z) = e^{-i\alpha} + \int_0^{+\infty} g(s)e^{2izs}ds, \quad z \in \mathbb{T}_+, \quad (2.7)$$

where

$$g \in \mathcal{B}_+, \quad g(s) = e^{-i\alpha}\Gamma_{11}(0,s) - e^{i\alpha}\Gamma_{21}(0,s), \quad s \in \mathbb{R}_+. \quad (2.8)$$

Moreover, there exists a unique $h \in \mathcal{B}_+$ such that

$$\psi^{-1}(z) = e^{i\alpha} + \int_0^{+\infty} h(s)e^{2izs}ds, \quad z \in \mathbb{T}_+. \quad (2.9)$$
2.3. Direct scattering. Recall that the scattering matrix \( S(z) \) was defined by \( (1.6) \). Now, we introduce the class of all scattering matrices for potentials from \( \mathcal{B}_+ \).

**Definition.** \((S_+, \rho_{S_+})\) is a metric space, where \( S_+ \) is the set of all continuous functions \( S : \mathbb{R} \to \mathbb{S}^1 \) such that \( W(S) = 0 \) and there exists \( F \in \mathcal{B} \) such that

\[
S(z) = e^{2ia} + \int_{-\infty}^{+\infty} F(s)e^{2isz}ds, \quad z \in \mathbb{R};
\]

and the metric \( \rho_{S_+} \) is given by

\[
\rho_{S_+}(S_1, S_2) = \| F_1 - F_2 \|_B, \quad S_1, S_2 \in S.
\]

**Remark.** Note that \( S \subset S_+ \subset \mathcal{A} \) isometrically and each \( S \in S_+ \) is invertible in \( \mathcal{A} \).

We need the following lemma.

**Lemma 2.3.** The mapping \( \psi \mapsto S = \overline{\psi}\psi^{-1} \) from \( \mathcal{J}_+ \) to \( S_+ \) is continuous.

**Proof.** Firstly, we show that \( S \in S_+ \). Let \( \psi \in \mathcal{J}_+ \) and let \( S(z) = \psi(z)\psi^{-1}(z) \), \( z \in \mathbb{R} \). Using the definition of \( S \), we have \( |S(z)| = \frac{|\psi(z)|}{|\psi(z)|} = 1 \), \( z \in \mathbb{R} \). Let \( \psi = e^{ia} + Fg \) for some \( g \in \mathcal{B}_+ \). Changing variables in the Fourier transform, we get \( \psi = e^{ia} + Fh \), where \( r(x) = g(-x) \), \( x \in \mathbb{R} \). Since \( \psi \) is invertible in \( \mathcal{B}_+ \subset \mathcal{A}_+ \), there exists \( h \in \mathcal{B}_+ \) such that \( \psi^{-1} = e^{ia} + Fh \). Substituting these representations in \( S \), we get

\[
S = e^{2ia} + FF, \quad F(s) = e^{-ia} (h(s) + r(s)) + (r \ast h)(s), \quad s \in \mathbb{R}.
\]

Due to \( r, h \in \mathcal{B} \), we have \( F \in \mathcal{B} \). Now, we show that \( W(S) = 0 \). It follows from the definition of \( S \) that \( W(S) = -2W(\psi) \). We introduce the conformal mapping \( w \mapsto z(w) = \frac{i - w}{1 + w} \) from \( \{|z| \leq 1\} \) into \( \mathbb{C}_+ \) and the function \( \phi(w) = \psi(z(w)), \quad w \in \{|z| \leq 1\} \). Thus, \( \phi \) is analytic on \( \{|z| < 1\} \), continuous up to the boundary \( \mathbb{S}^1 \), and does not vanish in \( \{|z| \leq 1\} \). Moreover, \( W(\psi) \) equals the winding number of \( \phi(w) \) around zero when \( w \) runs throughout \( \mathbb{S}^1 \). Using the Cauchy argument principle, it is easy to see that the winding number of \( \phi(w) \) around zero when \( w \) runs throughout \( |z| = r \) equals zero for each \( r < 1 \). Considering \( r \to 1 \), we have \( W(\psi) = 0 \).

Secondly, we show that the mapping \( \psi \mapsto S \) is continuous. Since \( \psi, \psi^{-1} \in \mathcal{A}_+ \subset \mathcal{A} \), and the mapping \( x \mapsto x^{-1} \) is continuous on the subspace of invertible elements, it follows that \( \psi \mapsto \psi^{-1} \) is a continuous mapping from \( \mathcal{A} \) to \( \mathcal{A} \). Moreover, the multiplication and the complex conjugate are also continuous mappings from \( \mathcal{A} \) to \( \mathcal{A} \). Then we have that \( \psi \mapsto S = \overline{\psi}\psi^{-1} \) is a continuous mapping from \( \mathcal{A} \) to \( \mathcal{A} \). Due to \( S_+ \subset \mathcal{A} \) and \( \mathcal{J}_+ \subset \mathcal{A} \) isometrically, we have that the mapping \( \psi \mapsto S \) is a continuous mapping from \( \mathcal{J}_+ \) to \( S_+ \). \( \square \)

2.4. Inverse scattering. We need the following theorem about inverse scattering problem (see Theorem 3.1 in [13]).

**Theorem 2.4.** The mapping \( q \mapsto S(\cdot, q) \) from \( \mathcal{B}_+ \) to \( S_+ \) is a homeomorphism.

It follows from this theorem that a potential is uniquely determined by a scattering matrix. Moreover, one can recover a potential from a scattering matrix using the Gelfand-Levitan-Marchenko (GLM) equation. For any scattering matrix \( S \in S_+ \), we introduce the matrix-valued function

\[
\Omega(x) = \begin{pmatrix} 0 & F(-x) \\ F(-x) & 0 \end{pmatrix}, \quad x \in \mathbb{R},
\]

(2.12)
where $F$ is given by (2.10). The following result was also obtained in [13].

**Lemma 2.5.**  
(i) Let $\Gamma(x, s) = \Gamma(x, s, q)$ and $\Omega(s) = \Omega(s, q)$ for some $q \in \mathcal{B}_+$. Then $\Gamma$ and $\Omega$ satisfy the GLM equation

$$\Gamma(x, s) + \Omega(x + s) + \int_0^{+\infty} \Gamma(x, t)\Omega(x + t + s)dt = 0$$  
(2.13)

for almost all $x, s \in \mathbb{R}_+$.  
(ii) Let $\Omega$ be given by (2.12) for some $S \in \mathcal{S}_+$. Then equation (2.13) has a unique solution $\Gamma(x, \cdot) \in \mathcal{B}_+ \otimes \mathcal{M}_2(\mathbb{C})$ and this solution depends continuously on $x \in \mathbb{R}_+$. Moreover, the mapping $s \mapsto \Gamma_{12}(\cdot, s)$ from $\mathbb{R}_+$ into $\mathcal{B}_+$ is continuous.

**Remark.** One can recover a potential $q$ from the scattering matrix $S \in \mathcal{S}_+$ as follows:

(i) Construct $\Omega$ by $S$ as in (2.12);  
(ii) Construct $\Gamma(x, s)$ as a solution of (2.13);  
(iii) Recover a potential by using $q(x) = -\Gamma_{12}(x, 0)$, $x \in \mathbb{R}_+$.

2.5. **Compactly supported potentials.** Now, we show that a potential is compactly supported if and only if the associated kernel $\Gamma$ is compactly supported.

**Lemma 2.6.** Let $q \in \mathcal{B}_+$ and let $\delta > 0$. Then $\sup \supp q \leq \delta$ if and only if $\Gamma(x, s) = 0$ for almost all $x, s \in \mathbb{R}_+$ such that $x + s > \delta$.  

**Proof.** Let $\sup \supp q \leq \delta$. Then it follows from (2.3) that $\|\Gamma_{nm}(x, \cdot)\|_{\mathcal{B}_+} = 0$ for each $x > \delta$, $n, m = 1, 2$. Thus, $\Gamma(x, s) = 0$ for each $x > \delta$ and for almost all $s \in \mathbb{R}_+$. Substituting this identity in (2.13), we get $\Omega(x + s) = 0$ for each $x > \delta$ and for almost all $s \in \mathbb{R}_+$, i.e. $\Omega(x) = 0$ for almost all $x > \delta$. Now substituting this identity in (2.13), we get $\Gamma(x, s) = 0$ for almost all $x, s \in \mathbb{R}_+$ such that $x + s > \delta$.

Let $\Gamma(x, s) = 0$ for almost all $x, s \in \mathbb{R}_+$ such that $x + s > \delta$. By Lemma 2.5 the mapping $s \mapsto \Gamma_{12}(\cdot, s)$ is continuous. Combining these facts, we get $\Gamma_{12}(x, 0) = 0$ for almost all $x > \delta$, which yields, by Lemma 2.4 that $q(x) = 0$ for almost all $x > \delta$. 

The support of a potential is also related to the support of the Fourier transform of the scattering matrix and the Jost function.

**Lemma 2.7.** Let $q \in \mathcal{B}_+$ and let $g$ and $F$ be given by (2.7) and (2.10). Then we have

$$\sup \supp q = -\inf \supp F = \sup \supp g.$$  
(2.14)

**Proof.** Firstly, we show that $\sup \supp q \leq -\inf \supp F$. If $\inf \supp F = -\infty$, then the inequality is evident. Let $\inf \supp F = -\delta < 0$ for some $\delta < +\infty$. Due to (2.12), we have $\Omega(x) = 0$ for any $x > \delta$. Substituting this identity in (2.13), we get $\Gamma(x, s) = 0$ for almost all $x, s \in \mathbb{R}_+$ such that $x + t > \delta$. Thus, by Lemma 2.3, $\sup \supp q \leq \delta$.

Secondly, we show that $-\inf \supp F \leq \sup \supp g$. Let $\sup \supp g = \delta$. It follows from (2.11) that

$$F(s) = e^{-i\alpha}(h(s) + r(s)) + (r \ast h)(s), \quad s \in \mathbb{R},$$  
(2.15)

where $r(s) = \overline{g(-s)}$, $s \in \mathbb{R}$ and $h \in \mathcal{B}_+$. Using $\inf \supp r = -\delta$, $\inf \supp h \geq 0$, and well-known property $\supp(r \ast h) \subset \supp r + \supp h$, we get $\inf \supp r \ast h \geq -\delta$. Substituting these inequalities in (2.15), we have $-\inf \supp F \leq \delta$.

Thirdly, we show that $\sup \supp g \leq \sup \supp q$. Let $\sup \supp g = \delta$. Then, by Lemma 2.6 $\Gamma(0, s) = 0$ for almost all $s > \delta$.

Combining these three inequalities, we obtain (2.14). 

\qed
3. Proof of the main theorems

Proof of Theorem 1.1.  

i) Since $\mathcal{P} \subset \mathcal{B}_+$ and the metrics $\rho_\mathcal{P}$ and $\| \cdot \|_{\mathcal{B}_+}$ are equivalent on $\mathcal{P}$, it follows from Theorem 2.2 that the mapping $q \mapsto S(\cdot, q)$ from $\mathcal{P}$ to $\mathcal{S}_+$ is continuous. Moreover, by Lemma 2.7, this mapping is bijection between $\mathcal{P}$ and $\mathcal{S}$. Due to the fact that $\mathcal{S} \subset \mathcal{S}_+$ isometrically, we have that the inverse mapping is also continuous and then it is a homeomorphism between $\mathcal{P}$ and $\mathcal{S}$.

ii) We show that the mapping $q \mapsto \psi(\cdot, q)$ is a bijection between $\mathcal{P}$ and $\mathcal{J}$. Let $q \in \mathcal{P} \subset \mathcal{B}_+$. Then, by Lemma 2.2, we have $\psi \in \mathcal{J}_+$. Due to sup supp $q = \gamma$, it follows from Lemma 2.7 that $\psi \in \mathcal{J}$.

Let $\psi \in \mathcal{J} \subset \mathcal{J}_+$. It follows from Lemma 2.3 that $S = \overline{\psi}\psi^{-1} \in \mathcal{S}_+$. Moreover, using Lemma 2.7 and $\psi \in \mathcal{J}$, we get $S \in \mathcal{S}$. Thus, by i), there exists a unique $q \in \mathcal{P}$ such that $S(\cdot) = S(\cdot, q)$. It is easy to see that $\psi(\cdot) = \psi(\cdot, q)$, since they are entire functions and their zeros coincide.

Now, we show that the mapping $q \mapsto \psi(\cdot, q)$ from $\mathcal{P}$ into $\mathcal{J}$ and its inverse are continuous. Let $q_1, q_2 \in \mathcal{P}$. Then using (1.8), (2.8), and the fact that the norms $\| \cdot \|_{L^2(0, \gamma)}$ and $\| \cdot \|_{\mathcal{B}_+}$ are equivalent on $\mathcal{P}$, we get

$$
\rho_\mathcal{J}(\psi(\cdot, q_1), \psi(\cdot, q_2)) = \|e^{-i\alpha}(\Gamma^{(1)}_{11}(0, \cdot) - \Gamma^{(2)}_{11}(0, \cdot)) - e^{-i\alpha}(\Gamma^{(1)}_{21}(0, \cdot) - \Gamma^{(2)}_{21}(0, \cdot))\|_{L^2(0, \gamma)} \\
\leq \|\Gamma^{(1)}_{11}(0, \cdot) - \Gamma^{(2)}_{11}(0, \cdot)\|_{\mathcal{B}_+} + \|\Gamma^{(1)}_{21}(0, \cdot) - \Gamma^{(2)}_{21}(0, \cdot)\|_{\mathcal{B}_+},
$$

(3.1)

where $\Gamma^{(j)}_{nm}(0, \cdot) = \Gamma_{nm}(0, \cdot, q_j)$ for any $n, m, j = 1, 2$. Due to Lemma 2.1 the mappings $q \mapsto \Gamma_{11}(0, \cdot, q)$ and $q \mapsto \Gamma_{12}(0, \cdot, q)$ from $\mathcal{B}_+$ into $\mathcal{B}_+$ are continuous. The embedding $\mathcal{P} \subset \mathcal{B}_+$ is also continuous. Thus, it follows from (3.1) that $\rho_\mathcal{J}(\psi(\cdot, q_1), \psi(\cdot, q_2)) \to 0$ as $\rho_\mathcal{P}(q_1, q_2) \to 0$.

Let $\rho_\mathcal{J}(\psi(\cdot, q_1), \psi(\cdot, q_2)) \to 0$, where $q_1, q_2 \in \mathcal{P}$. Then it follows from Lemma 2.3 that $\rho_\mathcal{S}(S(\cdot, q_1), S(\cdot, q_2)) \to 0$. Since the mapping $q \mapsto S(\cdot, q)$ from $\mathcal{P}$ to $\mathcal{S}$ is a homeomorphism, we get $\rho_\mathcal{P}(q_1, q_2) \to 0$.

iii) By i) and ii), the mappings $\psi(\cdot, q) \mapsto q$ from $\mathcal{J}$ into $\mathcal{P}$ and $q \mapsto S(\cdot, q)$ from $\mathcal{P}$ into $\mathcal{S}$ are homeomorphisms. Combining these mappings, we get that the mapping $\psi \mapsto S = \overline{\psi}\psi^{-1}$ from $\mathcal{J}$ into $\mathcal{S}$ is a homeomorphism. Moreover, since $\psi \in \mathcal{J}$ is entire, it follows that $\mathcal{S}$ admits a meromorphic continuation from $\mathbb{R}$ onto $\mathbb{C}$.

Proof of Corollary 1.2. It follows from Theorem 1.1 that $q \in \mathcal{P}$ is uniquely determined by its Jost function $\mathcal{J}$. Each function from $\mathcal{J}$ is entire and then it is uniquely determined by its zeros. Since the resonances are zeros of the Jost function, we get that $q \in \mathcal{P}$ is uniquely determined by its resonances. By the Paley-Wiener theorem (see e.g. p.30 in [18]), the Jost function belongs to $\mathcal{E}_{\text{cart}}$ and then it satisfies (1.11,13).

Proof of Theorem 1.3. It follows from Theorem 1.1 that $\psi(\cdot, q) \in \mathcal{J}$ and then there exists $g \in \mathcal{P}$ such that $\psi(\cdot, q) = e^{-i\alpha} + \mathcal{F}g$. It is well-known that the set of smooth compactly supported functions $C^\infty_0(0, \gamma)$ is dense in $L^2(0, \gamma)$. Thus, for any $\varepsilon > 0$, there exist $g_1 \in C^\infty_0(0, \gamma)$ such that $g = g_1 + g_2$ and $\|g_2\|_{L^2(0, \gamma)} < \varepsilon|\gamma|^{-1/2}$. Let $\psi(z) = 0$ for some $z \in \mathbb{C}_-$. Then we get $\mathcal{F}g(z) = -e^{-i\alpha}$. Estimating the left-hand side of this identity, we get

$$
|\mathcal{F}g_1(z)| + |\mathcal{F}g_2(z)| \geq 1.
$$

(3.2)
Due to $g_1 \in C_0^\infty(0, \gamma)$, we get
\[
|\mathcal{F}g_1(z)| \leq \left| \int_0^\gamma g_1(s)e^{2izs}ds \right| = \left| \frac{-1}{2iz} \int_0^\gamma g'_1(s)e^{2izs}ds \right|
\leq \frac{1}{2|z|} \int_0^\gamma |g'_1(s)|e^{-2\gamma \text{Im}zs}ds \leq \frac{e^{-2\gamma \text{Im}z}}{2|z|} \left\|g'_1\right\|_{L^1(0,\gamma)} = Ce^{-2\gamma \text{Im}z} \frac{1}{|z|}.
\] (3.3)

Due to $\|g_2\|_{L^1(0,\gamma)} \leq \sqrt{\gamma}\|g_2\|_{L^2(0,\gamma)} = \varepsilon$, we have
\[
|\mathcal{F}g_2(z)| \leq \int_0^\gamma |g_2(s)|e^{-2\gamma \text{Im}zs}ds \leq e^{-2\gamma \text{Im}z}\|g_2\|_{L^1(0,\gamma)} = \varepsilon e^{-2\gamma \text{Im}z}.
\] (3.4)

Substituting (3.3–4) in (3.2), we get
\[
e^{-2\gamma \text{Im}z} \left( \varepsilon + \frac{C}{|z|} \right) \geq 1,
\]
which yields (1.14). Now we consider (1.14) for $|z_n| \to \infty$. For $\varepsilon > 0$ and $C \geq 0$ fixed, we get
\[
2\gamma \text{Im} z_n \leq \ln(\varepsilon) + O(|z|^{-1}).
\]
Thus, there are finitely many resonances such that $\text{Im} z_n > \ln(\varepsilon)$. Since it holds for any $\varepsilon > 0$, we complete the proof of the lemma.

**Proof of Theorem 1.4.** Let $q \in \mathcal{P}$. Then, by Theorem 1.4, $S(\cdot) = S(\cdot, q) \in \mathcal{S}$, which yields that $|S(z)| = 1$, $z \in \mathbb{R}$, and $W(S) = 0$. Then there exist a real-valued function $\phi_{sc} \in L^\infty(\mathbb{R})$ such that (1.16) holds. By Theorem 1.4, $S$ is a meromorphic function without poles on the real line. Thus, we get $\phi_{sc} \in C^\infty(\mathbb{R})$.

By (1.2), $S(\cdot) - e^{2i\alpha}$ is a Fourier transform of function from $L^1(\mathbb{R})$. Then the application of the Riemann-Lebesgue lemma (see e.g. Theorem IX.7 in [29]) yields $S(z) - e^{2i\alpha} \to 0$ as $z \to \pm \infty$ and then $\phi_{sc}(z) \to -\alpha + 2\pi n_+ z$ as $z \to \pm \infty$ for some $n_+, n_- \in \mathbb{Z}$. Since $W(S) = 0$, it follows that $n_+ = n_-$. Thus, we can choose $\phi_{sc}$ such that $\phi_{sc}(z) \to -\alpha$ as $z \to \pm \infty$.

It follows from (1.9) and the Plancherel theorem (see e.g. Theorem IX.6 in [29]) that $S(\cdot) - e^{2i\alpha} \in L^2(\mathbb{R})$. Thus, there exist $g \in L^2(\mathbb{R})$ such that
\[
1 - e^{-2i(\phi_{sc}(z)+\alpha)} = g(z), \quad z \in \mathbb{R}.
\] (3.5)

Above, we show that $\phi_{sc}(z) \to -\alpha$ as $z \to \pm \infty$. Then using the Taylor series for the exponential function in (3.3), we get
\[
\phi_{sc}(z) + \alpha = O(g(z)) \quad \text{as} \quad z \to \pm \infty.
\]

Since $g \in L^2(\mathbb{R})$ and $\phi_{sc}(\cdot) + \alpha \in L^\infty(\mathbb{R})$, it follows that $\phi_{sc}(\cdot) + \alpha \in L^2(\mathbb{R})$.

Recall that $\psi(z) \neq 0$ for any $z \in \mathbb{R}$. Thus, we have
\[
\int_0^z \frac{\psi'(s)}{\psi(s)}ds = \ln \psi(z) - \ln \psi(0), \quad z \in \mathbb{R},
\]
which yields
\[
\arg \psi(z) = \arg \psi(0) + \int_0^z \text{Im} \frac{\psi'(s)}{\psi(s)}ds, \quad z \in \mathbb{R}.
\] (3.6)
It follows from (1.6) that \( \phi_{sc}(z) = \arg \psi(z) \), \( z \in \mathbb{R} \). Due to (3.6), we get \( \phi'_{sc}(z) = \text{Im} \frac{\psi'(z)}{\psi(z)} \), \( z \in \mathbb{R} \). It follows from Corollary 1.2 that \( \psi \in \mathcal{E}_{\text{Cart}} \). Using the Hadamard factorization (1.11) for \( \psi \), we obtain
\[
\text{Im} \frac{\psi'(z)}{\psi(z)} = \gamma + \sum_{n=1}^{\infty} \frac{\text{Im} z_n}{|z - z_n|^2}, \quad z \in \mathbb{C} \setminus \{z_n\}_{n=1}^{\infty},
\]
where the series converges absolutely and uniformly on compact subsets of \( \mathbb{C} \setminus \{z_n\}_{n=1}^{\infty} \), since \( \text{Im} z_n < 0 \) for each \( n \geq 1 \) and (1.12) holds. Considering (3.6) as \( z \to \pm \infty \) and using \( \phi_{sc}(z) \to -\alpha \) as \( z \to \pm \infty \), we obtain
\[
\phi_{sc}(0) = -\alpha - \lim_{z \to +\infty} \int_{0}^{z} \phi'_{sc}(s) ds = -\alpha + \lim_{z \to -\infty} \int_{z}^{0} \phi'_{sc}(s) ds.
\]

\( \square \)

**Proof of Theorem 1.5.** For simplicity, we consider the case when \( N = 1 \). We introduce
\[
B(z) = \frac{z - z_1}{z - z_1}, \quad z \in \mathbb{C} \setminus \{z_1\}.
\]
Firstly, we show that \( \psi_1 = B\psi \in \mathcal{J} \), where \( \psi(\cdot) = \psi(\cdot, q) \). By Corollary 1.2, \( \psi \) is entire. Due to \( \psi(z_0) = 0 \), it follows that \( \psi_1 \) is entire and \( \psi_1(z) \neq 0 \) for any \( z \in \mathbb{C} \). We show that (1.7) holds for \( \psi_1 \). We consider \( h(z) = \psi_1(z) - e^{-iz}, \quad z \in \mathbb{C} \), which is an entire function, since \( \psi_1 \) is entire. Using the definition of \( \tau_{\pm} \), we have
\[
\tau_{\pm}(h) = \tau_{\pm}(\psi_1) = \tau_{\pm}(B) + \tau_{\pm}(\psi) = \tau_{\pm}(\psi),
\]
where \( \tau_{\pm}(B) = 0 \), since \( B(z) \to 1 \) as \( |z| \to \infty \). Due to \( \psi \in \mathcal{E}_{\text{Cart}} \), we get
\[
\tau_{+}(h) = \tau_{+}(\psi) = 0, \quad \tau_{-}(h) = \tau_{-}(\psi) = 2\gamma.
\]
Now, we show that \( h \in L^2(\mathbb{R}) \). We have
\[
h(z) = e^{-iz}(B(z) - 1) + B(z)(\psi(z) - e^{-iz}), \quad z \in \mathbb{C}.
\]
By direct calculation, we get
\[
\|1 - B(\cdot)\|_{L^\infty(\mathbb{R})} = \frac{|z_0 - z_1|}{|\text{Im} z_0|}, \quad \|1 - B(\cdot)\|_{L^2(\mathbb{R})} = |z_0 - z_1| \frac{\pi}{|\text{Im} z_0|^{1/2}}.
\]
Moreover, it follows from (1.7) and the Plancherel theorem (see e.g. Theorem IX.6 in [29]) that \( \psi(\cdot) - e^{-i\pi} \in L^2(\mathbb{R}) \). Using these facts, we get from (3.8) that \( h \in L^2(\mathbb{R}) \). Recall that \( h \) is entire and (3.7) holds. Thus, it follows from the Paley-Wiener theorem (see e.g. p.30 in [18]) that \( h \) has the following form
\[
h(z) = \int_{0}^{\gamma} r(s)e^{2isz} ds, \quad z \in \mathbb{C},
\]
for some \( r \in \mathcal{P} \), which yields that \( \psi_1 \in \mathcal{J} \). Thus, it follows from Theorem 1.1 that there exists a unique \( q_0 \in \mathcal{P} \) such that \( \psi_1(\cdot) = \psi(\cdot, q_0) \).
Secondly, we show that \( \rho_\mathcal{J}(\psi_1, \psi) \to 0 \) as \( z_1 \to z_0 \). Let \( \psi = e^{-i\alpha} + \mathcal{F}g \) and \( \psi_1 = e^{-i\alpha} + \mathcal{F}r \) for some \( g, r \in \mathcal{P} \). Then using (1.8) and the Plancherel theorem, we get
\[
\rho_\mathcal{J}(\psi_1, \psi) = \| r - g \|_{L^2(\alpha, \gamma)} = \| \psi_1 - \psi \|_{L^2(\mathbb{R})} = \| B \psi - \psi \|_{L^2(\mathbb{R})} \\
= \|(\psi - e^{-i\alpha})(1 - B) + e^{-i\alpha}(1 - B)\|_{L^2(\mathbb{R})} \\
\leq \| 1 - B \|_{L^2(\mathbb{R})} + \| 1 - B \|_{L^\infty(\mathbb{R})} \| \psi - e^{-i\alpha} \|_{L^2(\mathbb{R})}.
\]
Substituting (3.9) in (3.10), we get
\[
\rho_\mathcal{J}(\psi_1, \psi) \leq |z_0 - z_1| \left[ \frac{\pi}{\text{Im} z_0} \right]^{1/2} \left( 1 + \frac{\| \psi - e^{-i\alpha} \|_{L^2(\mathbb{R})}}{\| \psi \|_{L^2(\mathbb{R})}} \right),
\]
which yields \( \rho_\mathcal{J}(\psi_1, \psi) \to 0 \) as \( z_1 \to z_0 \). By Theorem 1.1 the mappings \( q \mapsto \psi(\cdot, q) \) and \( q \mapsto S(\cdot, q) \) are homeomorphisms and then we have \( \rho_\mathcal{P}(q_o, q) \to 0 \) and \( \rho_\mathcal{S}(S(\cdot, q_o), S(\cdot, q)) \to 0 \) as \( z_1 \to z_0 \).

**Proof of Theorem 1.6**

i) Let \( \psi(\cdot) = \psi(\cdot, q) \) for some \( q \in \mathcal{P} \) and let \( k \in \mathbb{R} \). We show that \( \psi(\cdot + k) \in \mathcal{J} \). Since \( \psi(z) \neq 0 \), \( z \in \mathbb{C}_+ \), it follows that \( \psi(z + k) \neq 0 \) for each \( z \in \mathbb{C}_+ \). Using (1.7), we have
\[
\psi(z + k) = e^{-i\alpha} + \int_0^\gamma g(s)e^{2iks}e^{2i\alpha z}ds, \quad z \in \mathbb{C},
\]
where \( g(s)e^{2iks} \in \mathcal{P} \). Thus, \( \psi(\cdot + k) \in \mathcal{J} \) and it follows from Theorem 1.1 that there exists a unique \( q_k \in \mathcal{P} \) such that \( \psi(\cdot + k) = \psi(\cdot, q_k) \). Moreover, Theorem 1.1 gives that \( S(\cdot + k) = S(\cdot, q_k) \in \mathcal{S} \). Now, we recover \( q_k \). Using (1.9), we get
\[
S(z + k) = e^{2i\alpha} + \int_{-\gamma}^{-\gamma} F(s)e^{2i(z+k)s}ds = e^{2i\alpha} + \int_{-\gamma}^{+\gamma} F_k(s)e^{2izs}ds, \quad z \in \mathbb{C}.
\]
We introduce
\[
\Omega(x) = \begin{pmatrix} 0 & F(-x) \\ F(-x) & 0 \end{pmatrix}, \quad \Omega_k(x) = \begin{pmatrix} 0 & F_k(-x) \\ F_k(-x) & 0 \end{pmatrix}, \quad x \in \mathbb{R},
\]
where
\[
\Omega_k(x) = \Omega(x)e^{2iks\sigma_3} = e^{-2iks\sigma_3}\Omega(x), \quad x \in \mathbb{R}.
\]
It follows from (2.4) and Lemma 2.5 that there exists a unique solution \( \Gamma(x, s) \) of the GLM equation
\[
\Gamma(x, s) + \Omega(x + s) + \int_{-\gamma}^{+\gamma} \Gamma(x, t)\Omega(x + t + s)dt = 0
\]
such that \( \Gamma_{12}(x, 0) = -q(x) \) for almost all \( x \in \mathbb{R}_+ \). We introduce
\[
\Gamma_k(x, s) = e^{iks\sigma_3}\Gamma(x, s)e^{-i(2s + x)k\sigma_3}, \quad x, s \in \mathbb{R}_+.
\]
Substituting (3.13) in (3.12) and using (3.11), we get for almost all \( x, s \in \mathbb{R}_+ \)
\[
\Gamma_k(x, s) + \Omega_k(x + s) + \int_{-\gamma}^{+\gamma} \Gamma_k(x, t)\Omega_k(x + t + s)dt = 0.
\]
Thus, it follows from Lemma 2.5 that \( \Gamma_k(x, s) = \Gamma(x, s, q_k) \) and then, due to (2.4) and (3.13), we get for almost all \( x \in \mathbb{R}_+ \)
\[
q_k(x) = -(\Gamma_k)_{12}(x, 0) = -e^{2iks}\Gamma_{12}(x, 0) = e^{2iks}q(x).
\]
ii) Let \( \psi(\cdot) = \psi(\cdot, q) \) for some \( q \in \mathcal{P} \) and let \( \psi_1(z) = e^{-2i\alpha \psi(-\bar{z})}, z \in \mathbb{C} \). We show that \( \psi_1 \in \mathcal{J} \). Since \( \psi(z) \neq 0, z \in \mathbb{C}_+ \), it follows that \( \psi_1(z) \neq 0 \) for each \( z \in \mathbb{C}_+ \). Using (1.7), we have

\[
\psi_1(z) = e^{-i\alpha} + \int_{0}^{\gamma} e^{-2i\alpha \overline{g(s)} e^{2i\alpha z}} ds, \quad z \in \mathbb{C},
\]

where \( e^{-2i\alpha \overline{g(s)}} \in \mathcal{P} \). Thus, \( \psi_1 \in \mathcal{J} \) and it follows from Theorem 1.1 that there exists a unique \( q_o \in \mathcal{P} \) such that \( \psi_1(\cdot) = \psi(\cdot, q_o) \). Moreover, Theorem 1.1 gives that \( S_1 = \psi_1(\cdot) \psi_1^{-1}(\cdot) = S(\cdot, q_o) \in \mathcal{S} \). Now, we recover \( q_o \). Using the definition of \( \psi_1 \), we get

\[
S_1(z) = e^{2i\alpha} \frac{\psi(-\bar{z})}{e^{-2i\alpha} \psi(-\bar{z})} = e^{4i\alpha} S(-\bar{z}), \quad z \in \mathbb{R},
\] (3.14)

where \( S(\cdot) = S(\cdot, q) \in \mathcal{S} \). Substituting (1.9) in (3.14), we obtain

\[
S_1(z) = e^{2i\alpha} + \int_{-\infty}^{+\infty} e^{4i\alpha \overline{F(s)} e^{2i\alpha z}} ds = e^{2i\alpha} + \int_{-\infty}^{+\infty} F_o(s) e^{2i\alpha z} ds, \quad z \in \mathbb{R}.
\]

We introduce

\[
\Omega(x) = \begin{pmatrix} 0 & F(-x) \\ F(-x) & 0 \end{pmatrix}, \quad \Omega_o(x) = \begin{pmatrix} 0 & F_o(-x) \\ F_o(-x) & 0 \end{pmatrix}, \quad x \in \mathbb{R},
\]

where

\[
\Omega_o = U \Omega(x) U^{-1}, \quad U = e^{2i\alpha \sigma_3} \sigma_1, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\] (3.15)

As above, there exist a unique solution \( \Gamma(x, s) \) of the GLM equation

\[
\Gamma(x, s) + \Omega(x + s) + \int_{0}^{\infty} \Gamma(x, t) \Omega(x + t + s) dt = 0 \quad (3.16)
\]
such that \( \Gamma_{12}(x, 0) = -q(x) \) for almost all \( x \in \mathbb{R}_+ \). We introduce

\[
\Gamma_o(x, s) = U \Gamma(x, s) U^{-1}, \quad x, s \in \mathbb{R}_+.
\] (3.17)

Substituting (3.17) in (3.16) and using (3.15), we get for almost all \( x, s \in \mathbb{R}_+ \)

\[
\Gamma_o(x, s) + \Omega_o(x + s) + \int_{0}^{\infty} \Gamma_o(x, t) \Omega_o(x + t + s) dt = 0.
\]

Thus, it follows from Lemma 2.5 that \( \Gamma_o(x, s) = \Gamma(x, s, q_o) \) and then, due to (2.4) and (3.13), we get for almost all \( x \in \mathbb{R}_+ \)

\[
q_o(x) = -(\Gamma_o)_{12}(x, 0) = -e^{4i\alpha \Gamma_{12}(x, 0)} = e^{4i\alpha q(x)}.
\]

□

Acknowledgments. E. K. is supported by the RSF grant No. 18-11-00032. D. M. is supported by the RFBR grant No. 19-01-00094.
References

[1] M. J. Ablowitz, B. Prinari and A. D. Trubatch, *Discrete and continuous nonlinear Schrödinger systems*. London Mathematical Society Lecture Note Series, 302. Cambridge University Press, Cambridge, 2004.

[2] B.M. Brown, I. Knowles and R. Weikard, *On the inverse resonance problem*. J. London Math. Soc. (2) 68 (2003), no. 2, 383–401.

[3] T. Christiansen, *Resonances for steplike potentials: forward and inverse results*. Trans. Amer. Math. Soc. 358 (2006), no. 5, 2071–2089.

[4] K. R. Dodd, J. C. Eilbeck, J. D. Gibbon and H. C. Morris, *Solitons and nonlinear wave equations*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1982.

[5] S. Dyatlov and M. Zworski, *Mathematical theory of scattering resonances*. Graduate Studies in Mathematics, 200. American Mathematical Society, Providence, RI, 2019.

[6] L. D. Faddeev and L. A. Takhtajan, *Hamiltonian methods in the theory of solitons*. Translated from the 1986 Russian original by Alexey G. Reyman. Reprint of the 1987 English edition. Classics in Mathematics. Springer, Berlin, 2007.

[7] C. Frayer, R. O. Hryniv, Ya. V. Mykytyuk and P. A. Perry, *Inverse scattering for Schrödinger operators with Miura potentials. I. Unique Riccati representatives and ZS-AKNS systems*. Inverse Problems 25 (2009), no. 11, 115007, 25 pp.

[8] R. Froese, *Asymptotic distribution of resonances in one dimension*. J. Differential Equations 137 (1997), no. 2, 251–272.

[9] I. Gelfand, D. Raikov and G. Shilov, *Commutative normed rings*. Translated from the Russian, with a supplementary chapter. Chelsea Publishing Co., New York, 1964.

[10] I. C. Gohberg and M. G. Krein, *Theory and applications of Volterra operators in Hilbert space*. Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, Vol. 24 American Mathematical Society, Providence, R.I. 1970

[11] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*. Reprint of the 1952 edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1988.

[12] M. Hitrik, *Bounds on scattering poles in one dimension*. Comm. Math. Phys. 208 (1999), no. 2, 381–411.

[13] R. O. Hryniv and S. S. Manko, *Inverse scattering on the half-line for ZS-AKNS systems with integrable potentials*. Integral Equations Operator Theory 84 (2016), no. 3, 323–355.

[14] A. Iantchenko, *Quasi-normal modes for Dirac fields in the Kerr–Newman–de Sitter black holes*. Anal. Appl. (Singap.) 16 (2018), no. 4, 449–524.

[15] A. Iantchenko and E. Korotyaev, *Resonances for 1D massless Dirac operators*. J. Differential Equations 256 (2014), no. 8, 3038–3066.

[16] A. Iantchenko and E. Korotyaev, *Resonances for Dirac operators on the half-line*. J. Math. Anal. Appl. 420 (2014), no. 1, 279–313.

[17] A. Iantchenko and E. Korotyaev, *Resonances for the radial Dirac operators*. Asymptot. Anal. 93 (2015), no. 4, 327–369.

[18] P. Koosis, *The logarithmic integral. I*. Corrected reprint of the 1988 original. Cambridge Studies in Advanced Mathematics, 12. Cambridge University Press, Cambridge, 1998.

[19] E. Korotyaev, *Inverse resonance scattering on the half line*. Asymptot. Anal. 37 (2004), no. 3-4, 215–226.

[20] E. Korotyaev, *Stability for inverse resonance problem*. Int. Math. Res. Not. (2004), no. 73, 3927–3936.

[21] E. Korotyaev, *Inverse resonance scattering on the real line*. Inverse Problems 21 (2005), no. 1, 325–341.

[22] E. Korotyaev, *Resonance theory for perturbed Hill operator*. Asympt. Anal. 74 (2011), no. 3-4, 199–227.

[23] E. Korotyaev, *Global estimates of resonances for 1D Dirac operators*. Lett. Math. Phys. 104 (2014), no. 1, 43–53.

[24] E. Korotyaev, *Estimates of 1D resonances in terms of potentials*. J. Anal. Math. 130 (2016), 151–166.

[25] E. Korotyaev, *Resonances for 1d Stark operators*. J. Spectr. Theory 7 (2017), no. 3, 699–732.
[26] B. Ya. Levin, Lectures on entire functions. In collaboration with and with a preface by Yu. Lyubarskii, M. Sodin and V. Tkachenko. Translated from the Russian manuscript by Tkachenko. Translations of Mathematical Monographs, 150. American Mathematical Society, Providence, RI, 1996.

[27] B. M. Levitan and I. S. Sargsjan, Sturm-Liouville and Dirac operators. Translated from the Russian. Mathematics and its Applications (Soviet Series), 59. Kluwer Academic Publishers Group, Dordrecht, 1991.

[28] M. Marletta, R. Shterenberg and R. Weikard, On the inverse resonance problem for Schrödinger operators. Comm. Math. Phys. 295 (2010), no. 2, 465–484.

[29] M. Reed and B. Simon, Methods of modern mathematical physics. I. Functional analysis. Second edition. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1980.

[30] B. Simon, Resonances in one dimension and Fredholm determinants. J. Funct. Anal. 178 (2000), no. 2, 396–420.

[31] B. Thaller, The Dirac equation. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1992.

[32] V. E. Zakharov and A. B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. Soviet Physics JETP 34 (1972), no. 1, 62–69.; translated from Zh. Eksp. Teor. Fiz. 61 (1971), no. 1, 118–134 (in Russian).

[33] M. Zworski, Distribution of poles for scattering on the real line. J. Funct. Anal. 73 (1987), no. 2, 277–296.

[34] M. Zworski, A remark on isopolar potentials. SIAM, J. Math. Analysis, 82 (2002), no. 6, 1823–1826.