Preparation of NOON State Induced by Macroscopic Quantum Tunneling in an Ising Chain

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In this brief report, we propose a possible way, theoretically and experimentally, to generate a NOON state of the two degenerate ferromagnetic ground states of the Transverse Ising Model. In our scheme we employ the macroscopic quantum tunneling (MQT) effect between the two degenerate ferromagnetic ground states to realize the NOON state. Our calculation about the MQT process is based on a higher-order degenerate perturbation method. After doing a transformation, the MQT process could also be treated as the hopping of individual virtual fermions in the spin chain, which will lead to an analytical description of tunneling process. The experimental feasibility for generating the NOON state is discussed in the setup of linear ion trap.

The NOON states are a kind of special quantum states with two orthogonal component states in maximal superposition, where one is all spin-up and the other is all spin-down, such as \(|\text{NOON}\rangle = (|↑↑\cdots↑↑⟩ ± |↓↓\cdots↓↓⟩)/\sqrt{2}\). The recent interests in production of NOON states mainly arise from the crucial role they play in quantum optical lithography \([1]\) and quantum metrology \([2,4]\) as well as quantum information processing \([3]\), including quantum gating \([6]\) and precision measurement of phases \([7–9]\). Many efforts have been devoted to preparation of NOON states theoretically and experimentally by using trapped ions \([10–12]\), cold atoms \([13,14]\), atomic ensembles \([15]\), superconducting quantum circuits \([16]\), cavity quantum electrodynamics \([17]\), and linear or nonlinear optical elements \([18,19]\).

In this Brief Report, we propose an idea to create the NOON states formed by two degenerate ferromagnetic ground states of a transverse Ising model by the mechanism of macroscopic quantum tunneling (MQT). In our case, the transverse Ising model is given by

\[
\hat{H} = \hat{H}_0 + \hat{H}_I, \tag{1}
\]

where \(\hat{H}_0 = -J \sum_i \sigma_i^x \sigma_{i+1}^x\) and \(\hat{H}_I = \hbar^2 \sum_i \sigma_i^z\). In general, the transverse Ising model includes two quantum limits at absolute zero temperature: the ferromagnetic (FM) limit \((J > \hbar^2)\) and the paramagnetic limit \((J < \hbar^2)\), separated by the strength of the transverse field \(J, 21\). In the present model, \(\hat{H}_I\) is the perturbation term to induce quantum tunneling, as discussed later. The global phase diagram of the transverse Ising model at zero temperature, as shown in Fig. 1, has been well known \([21,22]\), where quantum phase transition occurs from the ferromagnetic state with \(J > \hbar^2\) to the gapped spin-polarized state with \(J < \hbar^2\) at the quantum transition point \(J/\hbar^2 = 1\). As a result, for large \(J\) \((J > \hbar^2)\), the ground state owns a spontaneous non-zero magnetization in \(z\) direction due to its FM order. For small \(J\) \((J < \hbar^2)\), the ground state is the uncorrelated spin state, also called the spin-polarized state.

The key idea in our proposal is the quantum tunneling between degenerate ground states induced by a transverse perturbation field. In a macroscopic system, such a quantum tunneling is usually referred to MQT, which has appeared in a wide range of research fields, such as quantum oscillations between two degenerate wells of NH3, quantum coherence in one-dimensional charge density waves, the ferromagnetic single domain magnets and biased Josephson junctions \([23]\). In general, the MQT can be found in systems with two or more separated ‘classical’ states that are macroscopically distinct. For instance, the MQT in macroscopic systems at low temperature, such as the superconducting quantum interference device, might be feasible for observation \([24]\).

There have been some useful theoretical methods to deal with the MQT, such as semiclassical WKB (Wentzel, Kramers and Brillouin) method \([25]\) and Instanton method in path integration \([26]\). In the present work, we study the MQT in a transverse Ising chain by the perturbation method \([27,28]\), based on the fact that the transverse Ising model in a finite size owns the degeneracy on the ground state, which can be removed by the MQT process. This mechanism can result in production of NOON states in the transverse Ising model.

To classify the degeneracy of the two ground states of \(\hat{H}\), we introduce two string operators \(W_X = \prod_i \sigma_i^x\) and \(W_Z = \prod_i \sigma_i^z\) \([27,31]\), which commute with \(\hat{H}_0\), i.e.,

\[
[\hat{H}_0, W_X] = [\hat{H}_0, W_Z] = 0 \quad \text{and are anti-commutated with each other}, \ i.e., \ \{W_Z, W_X\} = 0. \quad \text{Due to these properties, we may represent} \ W_X \text{ and } W_Z \text{ by two-component pseudo-spin operators } \tau^+ \text{ and } \tau^−, \text{ respectively, as } W_X \rightarrow \tau^+ \text{ and } W_Z \rightarrow \tau^−, \text{ for which we have } \tau^z\{|↑↑\cdots↑↑⟩⟩ = |↓↓\cdots↓↓⟩⟩\rangle, \tau^z\{|↑↑\cdots↑↑⟩⟩ = |↑↑\cdots↑↑⟩⟩\rangle, \text{where } |↑↑\cdots↑↑⟩⟩ \text{ and } |↓↓\cdots↓↓⟩⟩ \text{ are two degenerate ferromagnetic many-body ground states, as show in Fig. \[3]\).}
In order to illustrate the MQT process in the Ising chain, we transform the Ising spin chain into the system of free fermions by a perturbative method [28]. In the absence of $\hat{H}_I$, the fermions in this model have flat bands with the energy spectrum of $E(k) = 2J$, which implies that there is a mass gap $2J$ of the fermions in the transformed system, and the fermion cannot move in such a case. To change this situation, we may introduce the transverse perturbation $\hat{H}_I$, which drives the fermion to hop in the model.

In the perturbative approach, the perturbative operators $\sigma^x_i$ can be represented by hopping terms of the fermions, $\sigma^x_i \rightarrow (\psi^\dagger_i \psi_{i+1} + \psi^\dagger_{i+1} \psi_i + h.c.)$, where $\psi^\dagger_i$ ($\psi_i$) is the creation (annihilation) operator of a spinless fermion at the site $i$. In addition, we introduce a constraint for single occupation per site (i.e., hard-core constraint),

$$\langle \psi^\dagger_i \psi^\dagger_i | \Psi \rangle = |\Psi\rangle \text{ or } |\psi^\dagger_i \psi_i | \Psi \rangle = |\psi_i | \Psi\rangle,$$

where $|\Psi\rangle$ denotes a many-body quantum state.

Therefore, by the perturbative method, the Hamiltonian in Eq. (1) can be rewritten by spinless fermions as,

$$\hat{H} = 2J \sum_i \psi^\dagger_i \psi_i - \hbar^2 \sum_{(i,j)} (\psi^\dagger_i \psi_j + \psi^\dagger_j \psi_i + h.c.). \quad (2)$$

Under the Fourier transformation, we may assume a periodic boundary condition for the Ising chain, and obtain the dispersion of the fermion system in Eq. (2) as

$$E(k) = \sqrt{(2\hbar^2 \cos k + 2J)^2 + 4(\hbar^2 \sin k)^2}, \quad (3)$$

where $k$ is the wave vector, and for simplicity we set the separation of two nearest neighbor spins to be unity. So the energy gap of the fermions is obtained as $m_F = 2J \sqrt{1 - \frac{J^2}{4}}$, and thereby we may manipulate the dispersion of the fermions by tuning the external field $\hbar^2$. In the language of the free fermion system, during the tunneling process, a pair of virtual fermions are created, one of which moves around the chain until annihilation by meeting another fermion [27–29].

In the thermodynamic limit, the two ferromagnetic ground states of Eq. (1) are degenerate. However, there is very tiny energy splitting $\Delta E$ between the two ground states in a finite size chain. Since $\Delta E$ is much smaller than the energy gap $m_F$ in the fermionic excitation, we may safely ignore those excited states and only concentrate on the MQT between the two degenerate ground states in our perturbative calculation.

Now we employ a high-order degenerate perturbative approach to calculate the MQT effect in our system [27–29]. Firstly we introduce the transformation operator $\hat{U}_I(0, -\infty)$ defined in units of $\hbar = 1$,

$$\hat{U}_I(0, -\infty) = \hat{T} \exp(-i \int_{-\infty}^0 \hat{H}_I'(t')dt'), \quad (4)$$

where $\hat{H}_I'(t) = e^{i\hat{H}_0 t} \hat{H}_I e^{-i\hat{H}_0 t}$ is for Heisenberg picture, and $\hat{T}$ is the time-ordering operator. The transformation operator $\hat{U}_I(0, -\infty)$ in Eq. (4) can be further written as

$$\hat{U}_I(0, -\infty) |m\rangle = \sum_{j=0}^{\infty} \hat{U}_I^{(j)}(0, -\infty) |m\rangle, \quad (5)$$

where

$$\hat{U}_I^{(0)}(0, -\infty) |m\rangle = |m\rangle, \quad \hat{U}_I^{(j)\neq 0}(0, -\infty) |m\rangle = \frac{1}{E_0 - \hat{H}_0} \hat{H}_I^j |m\rangle \quad (6)$$

and $|m\rangle = \left( \begin{array}{c} \uparrow \uparrow \cdots \uparrow \uparrow \\ \downarrow \downarrow \cdots \downarrow \downarrow \end{array} \right)$. The element of the transformation matrix from the state $|m\rangle$ to $|n\rangle$ becomes
the site \(|n|\hat{U}_1(0,-\infty)|m\rangle\), corresponding to the energy,

\[ E = \langle n|\hat{H}\hat{U}_1(0,-\infty)|m\rangle = E_0 + \delta E, \quad (7) \]

with \(E_0\) the eigenvalue of the Hamiltonian \(\hat{H}_0\) for the eigenstate \(|m\rangle\).

When the MQT occurs between the two degenerate many-body ground states \(|\uparrow\uparrow \cdots \uparrow\uparrow\rangle\) and \(|\downarrow\downarrow \cdots \downarrow\downarrow\rangle\), we may describe the process in spinless fermion language as that, a pair of spinless fermions are created, and then one of them travels around the chain and leads to a string operator. As a result, the dominant term in Eq. (5) is labeled by \(j = N - 1\), where \(N\) is the site number of the Ising chain. Considering the tunneling process, we obtain the perturbative energy as

\[ \delta E = \langle n|\hat{H}_1\hat{U}_1(0,-\infty)|m\rangle \]

\[ = \langle n|\hat{H}_1\sum_{j=0}^{\infty}\hat{U}_j^{(j)}(0,-\infty)|m\rangle \]

\[ = \langle n|\hat{H}_1\hat{U}_1^{(N-1)}(0,-\infty)|m\rangle, \]

where the operator \(\hat{H}_1\hat{U}_1^{(N-1)}(0,-\infty)\) is proportional to a string operator \(W_{X,2}\).

To calculate the MQT between the two ground states, we consider a virtual fermion propagates around the chain, leading to the quantum state \(|\uparrow\uparrow \cdots \uparrow\downarrow\rangle\) flipped as

\[ \begin{pmatrix} \uparrow\uparrow \cdots \downarrow \downarrow \\ \uparrow\uparrow \cdots \uparrow \uparrow \end{pmatrix} = \sigma^x \begin{pmatrix} \uparrow\uparrow \cdots \uparrow \downarrow \\ \uparrow\uparrow \cdots \uparrow \uparrow \end{pmatrix}. \quad (9) \]

Using Eq. (8), we may obtain the energy shift \(\delta E\) as

\[ \delta E = U_1^{(N)} \]

\[ = \langle \uparrow\uparrow \cdots \uparrow\uparrow |\hat{H}_1(\frac{1}{E_0 - \hat{H}_0})^{N-1}|\downarrow\downarrow \cdots \downarrow\downarrow \rangle. \quad (10) \]

Due to the translation invariance of Eq. (1), we choose the site \(i\) as the starting point of the MQT process and we have

\[ \frac{1}{E_0 - \hat{H}_0} \hat{H}_1|\downarrow\downarrow \cdots \downarrow\downarrow \rangle \]

\[ \rightarrow N(\frac{h^x}{E_0 - \hat{H}_0})\sigma^x_i|\downarrow\downarrow \cdots \downarrow\downarrow \rangle \]

\[ = N(\frac{h^x}{E_0 - \hat{H}_0})|\Psi_i\rangle, \quad (11) \]

where \(|\Psi_i\rangle\) is the excited state of the two fermions at the sites \(i\) and \(i-1\) with the energy \(E_0 + 4J\). Using \(\hat{H}_0|\Psi_i\rangle = (E_0 + 4J)|\Psi_i\rangle\), we have

\[ \frac{1}{E_0 - \hat{H}_0} \hat{H}_1|\downarrow\downarrow \cdots \downarrow\downarrow \rangle = N(\frac{h^x}{-4J})|\Psi_i\rangle. \]

After the two fermions created, one of them moves along the Ising chain with following steps,

\[ \frac{h^x}{E_0 - \hat{H}_0} \sum_i \sigma^x_i |\downarrow\downarrow \cdots \downarrow\downarrow \rangle \]

\[ = \left( \frac{h^x}{E_0 - \hat{H}_0} \sum_i \sigma^x_i \right) N(\frac{h^x}{4J})|\Psi_i\rangle \]

\[ = N(\frac{h^x}{4J})\left( \frac{h^x}{E_0 - \hat{H}_0} \sum_i \sigma^x_i \right)|\Psi_i\rangle \]

\[ = N(\frac{h^x}{4J})\left( \frac{h^x}{E_0 - \hat{H}_0} \sigma^x_{i+1} \right)|\Psi_i\rangle, \]

where \(|\Psi'_i\rangle\) is the excited state with two fermions at sites \(i + 1\) and \(i - 1\). By this way, the fermion moves around the chain step by step. When coming back to the starting point the fermion annihilates with the partner fermion, which implies that the system changes with the ground state flipped from \(|\downarrow\downarrow \cdots \downarrow\downarrow \rangle\) to \(|\uparrow\uparrow \cdots \uparrow\uparrow \rangle\). So finally we get the energy splitting

\[ \Delta E = 2\delta E = 2U_1^{(N)} \]

\[ = 2\langle \uparrow\uparrow \cdots \uparrow\uparrow |\hat{H}_1(\frac{1}{E_0 - \hat{H}_0})^{N-1}|\downarrow\downarrow \cdots \downarrow\downarrow \rangle \]

\[ = 2 \times N(\frac{h^x}{-4J})^{N-1} = 8NJ(\frac{h^x}{-4J})^N. \quad (12) \]

For the chain with open boundary condition, the tunneling process is much different from our discussion above under the periodic boundary condition. In this case, the dominant tunneling of a single virtual fermion is from one side to the other with the tunneling splitting

\[ \Delta E = 2\delta E = 2(\frac{h^x}{2J})^N. \quad (13) \]

By this high-order perturbative approach we have studied the tunneling splitting between the two degenerate ground states. Fig. 3 shows the MQT calculated numerically by the exact diagonalization technique, which indicates that our theoretical results agree well with the numerical results. After adding the perturbation field \(h^x\) to the Ising chain, we are able to generate two fermions and even drive one of them hopping around the chain by considering high-order perturbation terms. In this way we could manipulate the tunneling splitting of the two degenerate ground states. In addition, in the sense of ‘perturbative treatment’, our approach can only be applied to the cases under a small external field. Nevertheless, the tunneling induced by the transverse disturbance field \(h^x\) in our model makes it possible to realize a string NOON state of the many-body state components \(|\uparrow\uparrow \cdots \uparrow\uparrow \rangle\) and \(|\downarrow\downarrow \cdots \downarrow\downarrow \rangle\).

Now we discuss how to generate a NOON state in a linear trapped-ion system. We may employ \(S_{1/2} \langle m_j = \)}
the dynamical evolution caused by the small transverse perturbation field \( h_x \) can be expressed as \( N - 1 \)-th order perturbation of \( \hat{U}_f(0,-\infty) \), which is proportional to the action of \( e^{i\Delta W x} \). As we had mentioned before, the purpose we introduce \( h_x \) is to induced the tunneling process between the two degenerate ferromagnetic limit ground states \( |\uparrow\uparrow\cdots\uparrow\rangle \) and \( |\downarrow\downarrow\cdots\downarrow\rangle \). It implies that the transverse field \( h_x \) is just a little disturbance compared to the big Ising coupling \( J \) along \( z \)-direction. Based upon the consideration above, we may globally flip the whole ions in the chain in an adiabatic way to generate the perturbative string operator \( W_x = \prod \sigma_i^x \), which can be complemented by applying a Gaussian beam with large width experimentally. This key step takes time of \( \tau = \frac{h}{\Delta E} = \frac{(2J)^2}{3h^2\Delta^2 h} \) \[33\], where \( \Delta E \) can be calculated using \( \text{Eq.}(7) \) by high order degenerate perturbation method. If we take an entanglement involving 14 ions as an example\[33\], the time we need for the Gaussian beam is on the order of \( \tau \approx 10^{-12} \)s. The ferromagnetic domain wall in the chain implies that two fermions are created and every fermion has a \( 2J \) excitation energy which is much greater than \( \Delta E \). In this way, we have achieved a NOON state \( |\text{NOON}\rangle = \alpha |\uparrow\uparrow\cdots\uparrow\rangle + \beta e^{i\phi} |\downarrow\downarrow\cdots\downarrow\rangle \) in the trapped ions under this transverse Ising model, the three parameters: \( \alpha, \beta \) and \( \phi \) could be determined by experimental measurements.

In summary, we have proposed a scheme to generate the NOON state of the two degenerate ferromagnetic ground states in the transverse Ising model through MQT process. This processes could be effectively manipulated by the perturbative field which is experimentally feasible in ion-trap systems. We think that this Ising string NOON state may have potential application in quantum computation and even in topological quantum computation if we could introduce topological protection to this Ising string.

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