Slowly Rotating Boson-Fermion Star

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Abstract: Relativistic prescription is used to study the slow rotation of stars composed by self-gravitating bosons and fermions (fermions may be considered as neutrons). Previous results demand that purely boson stars are unable to display slow rotation, if one uses relativistic prescription with classical scalar fields. In contrast to this, the present work shows that a combined boson-fermion star in its ground-state can rotate. Their structure and stability are analysed under slow rotation approximations.

I. INTRODUCTION

Stars composed by neutrons (fermions) has been first suggested by Oppenheimer and Volkoff [1]. After that, many theoretical studies have arisen on this theme and some years later many stellar objects has been identified with neutron stars. Inside neutron stars, fermions are attracted each other gravitationally but the configuration is prevented from collapsing via Pauli exclusion principle pressure.

Another possibility for stars configurations has been suggested [2] independently by Kaup, and Ruffini and Bonazzola. Boson stars are compact configurations made with bosons gravitationally bounded. Bosons are described via a scalar field, and large interest has been shown on this subject since properties of that kind of configurations can be obtained directly from the Lagrangian treatment, without the need to define an equation of state, as it is usually done to ordinary stars (see also Breit et al [3]). Boson stars are prevented from collapsing gravitationally due to a pressure that arises from the Heisenberg uncertainty principle. Boson stars formation by Jeans instability has been studied, e.g., by Grasso [17] and its stability has been addressed by Gleiser [7] and by Torres [9]. Some reviews [4] present the relativistic and astrophysical interest on this subject, properties and detection expectations.

Since in the primordial gas possibly bosons and fermions coexisted, one could expect stable configurations of self-gravitating bosons and fermions formed. A model boson fermion star has been first proposed by Henriques, Liddle and Moorhouse [5]. Some studies on stability [8] and on configurations with interactions between bosons and fermions [15] has been performed.

Boson star rotation has been first addressed by Kobayashi et al [11]. They used a perturbative approach and have shown that boson stars could not display slow rotation. But, as shown by Silveira and de Sousa [12], considering the scalar field quantum nature with axial symmetry it is possible to obtain stable configurations with \( l \neq 0 \), using Newtonian approach. After that, many works have contributed to the comprehension of the properties of spinning boson stars, as Ryan [13] which shows the large self-interaction case, and Yoshida and Eriguchi [14] studied static relativistic axisymmetric solutions.

But it seems there is no explicit proof that boson fermion stars can display rotation. Hence, in this article we present our results for relativistic slowly rotating stars composed with bosons and fermions. In section II we present the construction of a boson star in II.A and of a boson fermion star in section II.B. In section III we present the construction of a rotating fermion star in III.A, and an extension for studying rotating boson fermion stars in III.B. Results are shown in section IV.

II. NON-ROTATING BOSON-FERMION STARS

The study of composed boson-fermion stars has been first addressed by Henriques et al [5]. In this section we present a summary for boson stars model and its enhance for boson-fermion stars by introducing fermions with Chandrasekhar perfect fluid model. 

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A. Boson Stars

A non-rotating configuration is spherically symmetric and one can build the metric into the form:

\[ ds^2 = -B(r) d\tau^2 + A(r) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \]  

Our sign and conventions are the same as those used by Liddle and Madsen [4]. The Lagrangian density for the complex scalar field with mass \( m \), is:

\[ \mathcal{L} = \frac{R}{16\pi G} - \partial_\mu \Phi^* \partial^\mu \Phi - m^2 \Phi^* \Phi \]  

and, as usual for boson stars, we use the *ansatz*:

\[ \Phi(r, \tau) = \phi(r) e^{-i\omega \tau} \]  

The complex field \( \phi(r) \) may be written as a sum of real and imaginary parts. In absence of interaction between bosons and fermions, real and imaginary parts generate similar sets of equations. This way one can consider \( \phi(r) \) purely real; this is not possible if fermions and bosons interact [15] since \( \phi_1(r) \) and \( \phi_2(r) \) present different behaviour (considering \( \phi(r) = \phi_1(r) + i\phi_2(r) \) above).

For simplicity we are not considering scalar field self-interaction term, \( \lambda \Phi^4 \), and we are not considering boson-fermion interaction although it should give more realistic results [16]. This way, Klein-Gordon equation has no source term:

\[ \Box \Phi - m^2 \Phi = 0 \]  

Equation (2) gives the scalar-field (bosons) energy-momentum tensor:

\[ T^\mu_\nu = \partial_\nu \Phi^* \partial^\mu \Phi + \partial_\mu \Phi^* \partial^\nu \Phi - g^\mu_\nu (\partial^\lambda \Phi^* \partial_\lambda \Phi + m^2 \Phi^* \Phi) \]  

From the Nöther theorem the action derived from (2) gives rise to a conserved current:

\[ J^\mu = ig^{\mu\nu} (\Phi^* \partial^\nu \Phi - \Phi \partial^\nu \Phi^*) \]  

and the conserved charge:

\[ N_B = \int d^3r \sqrt{-g} J^0 \]  

which can be identified with the number of bosons.

Once the scalar field describes particles a quantum analysis is required. If we take \( N_B \) bosons in their lowest state, \( |N_B\rangle \), the Einstein equations \( G_{\mu\nu} = \langle N_B | : T_{\mu\nu} : |N_B \rangle \) give exactly the same set of equations. This result is obtained with the scalar field \( \Phi \) changed into the operator:

\[ \Phi(r, t) = \sum_k \frac{1}{\sqrt{2\pi k}} \left[ a_k \phi_k(r) e^{-i\omega_k t} + a_k^\dagger \phi^*_k(r) e^{i\omega_k t} \right] \]  

where \( a_k^\dagger \) creates quanta associated with the field \( \Phi \) and \( a_k \) destroys them, obeying the commutation relation \( [a_i, a_j^\dagger] = \delta_{ij} \). Considering the ground-state, one can introduce a semiclassical scalar field:

\[ \Phi_c = \sqrt{\frac{N_B + 1/2}{\omega_0}} \phi_0(r) e^{-i\omega_0 \tau} \]  

where \( \phi_0 \) and \( \omega_0 \) are the eigenfunction of the corresponding lowest eigenenergy, which gives a nodeless eigenfunction \( \phi_0(r) \). Hence, forthcoming equations and numerical results are related to the semiclassical field \( \Phi_c \).

The requirement of the functions to be orthonormal:

\[ \int_0^\infty 4\pi r^2 \sqrt{\frac{A}{B}} \phi_i^* \phi_j dr = \delta_{ij} \]  

allows one to determine the number of bosons, \( N_B \), by using the classical field (8):

\[ \int_0^\infty 4\pi r^2 \sqrt{\frac{A}{B}} |\Phi_c|^2 dr = \frac{N_B + 1/2}{\omega_0} \]
B. Introducing Fermions

Boson-fermion model star has been originally proposed by Henriques et al\[5\], and are equilibrium compact cold configurations made from both bosons and fermions.

As proposed by Chandrasekhar\[6\] for neutron star models, an equation of state can describe a perfect fluid of degenerate Fermi gas, and density of energy and pressure are given by:

\[ \rho = K (\sinh t - t) \]

\[ p = \frac{K}{3} \left( \sinh t - 8 \sinh \frac{t}{2} - 3t \right) \]

where \( K = \frac{m_n^4}{32\pi^2} \), and \( m_n \) is the fermion mass (which is called neutron mass for neutron stars). The parameter \( t \) is related to the maximum momentum value, \( q_0 = q_0(r) \), in the Fermi distribution in a distance \( r \) from the centre:

\[ t = \ln \left\{ \frac{q_0}{m_n} + \sqrt{1 + \left( \frac{q_0}{m_n} \right)^2} \right\} \]

(13)

The density of fermion particles in the distribution is:

\[ n(r) = \frac{q_0^3}{3\pi^2} = \frac{m_n^3}{3\pi^2} \sinh \frac{t}{4} \]

(14)

Considering a compact spherically symmetric distribution with \( 1 \) and \( 14 \), the number of fermions is:

\[ N_F = \int 4\pi r^2 n \sqrt{Adr} \]

(15)

Oppenheimer and Volkoff\[1\] have found equilibrium configurations by using Einstein equations with \( T_{11}^1 = T_{22}^2 = T_{33}^3 = p \) and \( T_{00}^0 = -\rho \), and with the hydrostatic equilibrium equation for the pressure:

\[ p' = -\frac{(\rho + p) B'}{2} \]

(16)

where \( ' = d/dr \), and \( B = B(r) \) is defined on \( 1 \).

As proposed by Henriques et al\[5\], the basic properties of the composed boson-fermion star can be studied with the use of an energy momentum tensor given by the sum of bosonic (B) and fermionic (F) counterparts:

\[ T_{\mu\nu} = T_{\mu\nu}^B + T_{\mu\nu}^F \]

(17)

where:

\[ T_{\mu\nu}^B = \partial_\mu \Phi \partial_\nu \Phi^* + \partial_\mu \Phi^* \partial_\nu \Phi - g_{\mu\nu} \left( \partial_\lambda \Phi^* \partial^\lambda \Phi + m^2 \Phi^* \Phi + \frac{\lambda}{4} \Phi^4 \right) \]

(18)

\[ T_{\mu\nu}^F = (\rho + p) u_\mu u_\nu + g_{\mu\nu} p \]

(19)

where \( u^\mu = (u^0, u_r, u_\theta, u_\phi) = \frac{dx^\mu}{ds} \) is the fermion fluid four vector.

The set of differential equations can be obtained by using Einstein equations together with \( 3 \) and \( 10 \). After some calculations and by using variables redefinitions:

\[ x = mr \ , \ \sigma(x) = \sqrt{8\pi G\phi(r)} \ , \ w = \frac{\omega}{m} \]

(20)
\[ \bar{\rho}(t) = \frac{4\pi G}{m^2} \rho(t), \quad \bar{p}(t) = \frac{4\pi G}{m^2} p(t) \]

we obtain:

\[ A' = xA^2 \left[ 2\bar{\rho} + \left( \frac{w^2}{B} + 1 \right) \sigma^2 + \frac{\sigma'^2}{A} \right] - \frac{A}{x}(A-1) \] (21)

\[ B' = xAB \left[ 2\bar{p} + \left( \frac{w^2}{B} - 1 \right) \sigma^2 + \frac{\sigma'^2}{A} \right] + \frac{B}{x}(A-1) \] (22)

\[ \sigma'' = -\left[ \frac{2}{x} + \frac{1}{2} \left( \frac{B'}{B} - \frac{A'}{A} \right) \right] \sigma' - A \left[ \left( \frac{w^2}{B} - 1 \right) \sigma - \Lambda \sigma^3 \right] \] (23)

\[ t' = -2 \frac{B'}{B} \frac{\sinh t - 2 \sinh(t/2)}{\cosh t - 4 \cosh(t/2) + 3} \] (24)

Also, any function \( F(r) \) is replaced with \( F(x/m) \). Hence, \( F'(x) = \frac{dF(x/m)}{dx} = \frac{1}{m} \frac{dF(r)}{dr} \), where now \( ' = d/dx \).

Equations (21) - (22) are obtained directly from Einstein equations using metric (1), and (23) is the Klein-Gordon equation (4). Equation (24) arises from hydrostatic equilibrium equation (16), rewritten in the form:

\[ t' = -2 \frac{B'}{B} \bar{\rho} \frac{d\bar{\rho}}{dt} \frac{B'}{2B}(\bar{\rho} + \bar{p}) \] (25)

where:

\[ \bar{\rho} = \bar{K}(\sinh t - t), \quad \bar{p} = \frac{\bar{K}}{3} (\sinh t - 8 \sinh(t/2) + 3t) \] (26)

with:

\[ \bar{K} = \frac{m_\gamma^4}{8\pi^3 m^2 M_{\odot}^2} \] (27)

For convenience we will use \( \bar{K} = 1/4\pi \), which is the same value used by Oppenheimer and Volkoff for neutron stars. Note that (27) displays a constraint between boson mass \( m \) and fermion mass \( m_n \). Hence, if one choose \( m_n \) as the neutron mass and \( \bar{K} = 1/4\pi \), we are setting \( m = 5, 11 \times 10^{-17} \text{MeV} \).

### III. SLOWLY ROTATING BOSON-FERMION STAR

Slowly rotating relativistic stars has been studied for long in applications with neutron stars [10]. Meanwhile, until Kobayashi et al [11], rotation configurations for boson stars has never been studied, and until Silveira et al [12], stable boson stars rotating configurations has never been found. Indeed, Kobayashi et al show that there is no stable configuration with nonvanishing angular momentum for pure boson star studied perturbatively. Meanwhile, considering the quantum background for an axisymmetric scalar field one obtains stable configurations in the Newtonian framework.

In this section we study relativistic equations for slowly rotating boson-fermion stars, in a similar framework presented by Kobayashi et al.

#### A. Rotating Fermion Stars

Using Hartle’s prescription [10] for slowly rotating neutron stars, one introduces the metric:

\[ ds^2 = -H^2d\tau^2 + Q^2dr^2 + r^2K^2 \left[ d\theta^2 + \sin^2 \theta (d\varphi - Ld\tau)^2 \right] \] (28)
where $H$, $Q$, $K$ and $L$ are functions of $r$ and $\theta$. When considering slow rotation we have $R\Omega \ll c$, where $R$ is the average radius and $\Omega$ is the angular velocity as seen by an observer at infinity. In this section we expand the metric (28) only up to order $\Omega^2$. This way, energy and pressure can be approximated using:

$$E = \rho + O(\Omega^2)$$

(29)

$$P = p + O(\Omega^2)$$

When rotation is considered this metric introduces a dragging of inertial frames. Thence, $L = d\phi/d\tau$ is the angular velocity acquired by one observer free falling the infinity to the point $(r, \theta)$. The metric (28) must be invariant under rotation reversion, i.e., $\varphi \to -\varphi$ and $\Omega \to -\Omega$. Thus, functions $H$, $Q$ and $K$ are even in powers of $\Omega$; meanwhile $L$ is odd. Since for slow rotation we study terms only up to order $\Omega^2$, one can consider effects up to order $\Omega$ in $L(r, \theta)$, represented by $C(r, \theta)$:

$$L(r, \theta) = C(r, \theta) + O(\Omega^3)$$

(30)

In cases when there is no rotation, $\Omega = 0$ and the energy-momentum tensor for the perfect fluid is:

$$T^\mu_\nu = (\rho + p) u^\mu u_\nu + p \delta^\mu_\nu$$

(31)

But, when $\Omega \neq 0$:

$$T^\mu_\nu = (\mathcal{E} + \mathcal{P}) u^\mu u_\nu + P \delta^\mu_\nu$$

(32)

In the case analysed by Hartle [10], the first order term in (30), $C(r, \theta)$, can be solved analytically, using the equation:

$$R^0_3 = 8\pi G T^0_3$$

(33)

we are considering $u^\mu = (u^\tau, u^r, u^\theta, u^\phi) = (u^1, 0, 0, u^3)$ due to symmetries involved. Since $d\varphi = \Omega d\tau$, one can use $u^3 = \Omega u^0$. With the normalization condition $u^\mu u_\mu = -1$, one obtains $(u^0)^2 [g_{00} + 2\Omega g_{03} + \Omega^2 g_{33}] = -1$. Thus, the energy-momentum tensor component $T^0_3$ is:

$$T^0_3 = (\mathcal{E} + \mathcal{P}) u_3 u^0 = (\mathcal{E} + \mathcal{P}) (u^0)^2 g_{03} + u^0 u_3 g_{33} = (\mathcal{E} + \mathcal{P})(u^0)^2 (g_{03} + \Omega g_{33})$$

(34)

Expanding this term up to order $\Omega^2$ yields:

$$T^0_3 = (\rho + p)e^{-\nu}(\Omega - C)r^2 \sin^2 \theta + O(\Omega^3)$$

(35)

Therefore, the field equation (33) becomes:

$$\frac{1}{r^4} \partial_r \left[ r^4 e^{-(\nu + \lambda)/2} \partial_r \overline{C} \right] + \frac{e^{(\lambda - \nu)/2}}{r^4 \sin^3 \theta} \partial_\theta \left( \sin^3 \theta \partial_\theta \overline{C} \right) = 16\pi G (\rho + p)e^{(\lambda - \nu)/2}\overline{C}$$

(36)

where $\overline{C} = \Omega - C$. Defining the quantity:

$$j(r) = e^{-(\nu + \lambda)/2}$$

(37)

one obtains:

$$\frac{dj}{dr} = -\frac{1}{2} (\nu' + \lambda') e^{-(\nu + \lambda)/2}$$

(38)

From Einstein equations:

$$\lambda' = 8\pi Gr e^\lambda \rho - \frac{1}{r}(e^\lambda - 1)$$
\[ \nu' = 8\pi Ge^{\lambda}p + \frac{1}{r}(e^{\lambda} - 1) \]

one is able to rewrite (38) as:

\[ \frac{4}{r} \frac{dj}{dr} = -16\pi G(\rho + p)e^{(\lambda - \nu)/2} \]  

(40)

Thus (39) becomes, up to first order in \( \Omega \):

\[ \frac{1}{r^4} \partial_r \left[ r^4 j(r) \partial_r C \right] + 4 \left( \frac{\lambda - \nu}{r^2} \right) \partial_\theta \partial_\phi \left( \sin^2 \theta \partial_\theta C \right) = 0 \]  

(41)

Expanding \( \overline{C} \) in vector spherical harmonics:

\[ C(r, \theta) = \sum_{l=1}^{\infty} \overline{C}_l(r) \left[ -\frac{1}{\sin \theta} \partial_\theta P_l(\cos \theta) \right] \]  

(42)

and thus:

\[ \frac{1}{r^4} \partial_r \left[ r^4 j(r) \partial_r \overline{C}_l \right] + \overline{C}_l \left[ \frac{4}{r} \partial_r j - e^{-\left(\lambda - \nu\right)/2} \frac{l(l+1) - 2}{r^2} \right] = 0 \]  

(43)

When \( r \to \infty \), \( \lambda \) and \( \nu \) vanish, and \( j \to 1 \). Hence, when \( r \to \infty \) in (43), \( \overline{C}_l \) behaves like:

\[ \overline{C}_l(r) \to k_1 r^{-l-2} + k_2 r^{l-1} \quad (r \to \infty) \]  

(44)

where \( k_1 \) and \( k_2 \) are constants. Since fields decrease very fast the solution when \( r \to \infty \) is of the Kerr-Newman form:

\[ C_l \propto \frac{1}{r^3} \]

in such way that \( \overline{C}_l = \Omega - C_l \approx \Omega \). This is reproduced by (14) by choosing \( l = 1 \), and appropriate values for \( k_1 \) and \( k_2 \). Taking again correct values for the constants in such wise the solution to be regular, one can observe that terms with \( l > 1 \) decrease even faster and \( l = 1 \) mode can be considered as the dominant one. Hence, \( \overline{C}(r, \theta) \approx \overline{C}_1(r) \), and \( \overline{C} \) becomes function of \( r \) alone. Now, \( \overline{C}_1 \) is:

\[ \frac{1}{r^4} \partial_r \left[ r^4 j(r) \partial_r \overline{C}_1 \right] + 4 \left( \frac{\lambda - \nu}{r^2} \right) \partial_\theta \partial_\phi \left( \sin^2 \theta \partial_\theta \overline{C}_1 \right) = 0 \]  

(45)

Since the solution must be regular close to the origin, \( \overline{C}_1(0) = \text{const.} \), and considering the exterior solution, one obtains:

\[ \overline{C}_1(r) = \Omega - \frac{2J}{r^3} \]  

(46)

where \( J \) is the total angular momentum of the fermion star. The quantity \( J/\Omega \) defines its moment of inertia.

**B. Introducing Bosons**

After reviewing Hartle’s perturbative approach for rotating fermion stars (which are usually referred as neutron stars) it is convenient to introduce bosons to obtain the composite boson-fermion star model, which is the original part of this article.

Following the perturbative approach used by Hartle, we expand the metric up to terms in \( \Omega^2 \), and the Kerr metric with no charge can be written as:

\[ ds^2 = -B(r)dr^2 + A(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 - 2C(r)\epsilon^2 \sin^2 \theta dr d\phi + O(\Omega^2) \]  

(47)
where \( C(r) = C_1(r) \) in the previous section. The total energy-momentum tensor is again:

\[
T_{\mu\nu} = T_{\mu\nu}^B + T_{\mu\nu}^F \tag{48}
\]
as in equation (17). Up to order \( \Omega^2 \), \( \dot{\varphi} = \Omega \), and the nonvanishing components of \( T_{\mu\nu} \) are:

\[
T^0_0 = -\rho - \left( \frac{\omega^2}{B} + m^2 \right) \phi^2 - \frac{\phi'^2}{A} \tag{49}
\]

\[
T^1_1 = p + \left( \frac{\omega^2}{B} - m^2 \right) \phi^2 + \frac{\phi'^2}{A} \tag{50}
\]

\[
T^2_2 = T^3_3 = p + \left( \frac{\omega^2}{B} - m^2 \right) \phi^2 - \frac{\phi'^2}{A} \tag{51}
\]

\[
T^0_3 = \frac{(\rho + p)}{B} r^2 \sin^2 \theta [\Omega - C] \tag{52}
\]

\[
T^3_0 = (\rho + p)\Omega \tag{53}
\]

Using redefinitions (20) and after some straightforward calculations, one obtains up to order \( \Omega^2 \), the set of differential equations:

\[
A' = x A^2 \left[ 2\bar{\rho} + \left( \frac{w^2}{B} + 1 \right) \sigma^2 + \frac{\sigma'^2}{A} \right] - \frac{A}{x} (A - 1) \tag{54}
\]

\[
B' = x A B \left[ 2\bar{\rho} + \left( \frac{w^2}{B} - 1 \right) \sigma^2 + \frac{\sigma'^2}{A} \right] + \frac{B}{x} (A - 1) \tag{55}
\]

\[
\sigma'' = - \left[ \frac{2}{x} + \frac{1}{2} \left( \frac{B' - A'}{B} \right) \right] \sigma' - A \left( \frac{w^2}{B} - 1 \right) \sigma \tag{56}
\]

\[
t' = -2 \frac{B'}{B} \sinh t - 2 \sinh(t/2) \tag{57}
\]

\[
C'' = - \left[ \frac{4}{x} - \frac{1}{2} \left( \frac{B' + A'}{B} \right) \right] C' - 4A(\bar{\rho} + \bar{p}) (\Omega - C(r)) \tag{58}
\]

where now \( t' = d/dx \).

Equations (54) to (57) are the same as those obtained for boson-fermion stars. Equation (58) gives the function \( C(x) \), corresponding to rotation of the boson-fermion star. Note that \( C(x) \) depends not only on \( x \), but the parameter \( t \) and metric fields also contributes to the rotation. Thus, step-by-step results that numerically arise from (54) - (57) can modify the rotational term \( C(x) \), via changes on the initial values, \( \sigma_0 \) and \( t_0 \), which represents bosons and fermions contributions respectively. On the other hand, \( C(x) \) does not appear on equations (54) - (57). Hence, changes on the initial value of \( C(x) \) does not produce changes on \( \sigma(x) \), \( t(x) \), \( A(x) \) or \( B(x) \), up to order \( \Omega^2 \).
IV. RESULTS

After obtaining perturbatively, up to order $\Omega^2$, the equations that govern the rotation of a boson-fermion star, the results from section 3.2 are obtained numerically. Before that, equations (56) and (58) ought be rewritten to form a first order set of differential coupled equations. Hence, equation (56) splits into:

\[ s = \sigma' \] (59)

\[ s' = \left( xA\sigma^2 - \frac{A+1}{x} \right) s - A\sigma \left( \frac{w^2}{B} - 1 \right) \] (60)

and taking:

\[ \overline{C}(x) = \Omega - C(x) \] (61)

equation (58) becomes:

\[ \overline{C}' = -U \] (62)

\[ U' = \left[ xA \left( \bar{\rho} + \bar{p} + \frac{w^2}{B} \sigma^2 + \frac{s^2}{A} \right) - \frac{4}{x} \right] U - 4A(\bar{\rho} + \bar{p})\overline{C} \] (63)

The boundary conditions are $A_0 = 1$, $s_0 = 0$, $U_0 = U(0) = 0$, considering $B_0$, $\sigma_0$, $t_0$ and $\overline{C}(0) = \overline{C}_0$ as arbitrary initial values to be ascribed. And, at the infinity conditions are $A(\infty) \to 1$, $B(\infty) \to 1$, $\sigma(\infty) \to 0$ and $s(\infty) \to 0$. As seem before, star properties as mass, radius and total number of particles does not differ from boson-fermion stars with no rotation up to $\Omega^2$. For example, a configuration with $\sigma_0 = 0.2$, $t_0 = 4.0$ we have obtained $M \simeq 0.41M_{\text{Pl}}^2/m$, $(mN_B + m_nN_F) \simeq 0.46M_{\text{Pl}}^2/m$ and $mR \simeq 0.94M_{\text{Pl}}^2/m$. Thus, if one finds an equilibrium configuration, changes on rotation parameter $C$ does not modify those results. Meanwhile, if one fix the initial value $C_0$, changes on values of $\sigma_0$ and $t_0$ provide modifications on the $\overline{C}$ values evolution, i.e., the referential frames dragging is modified.

Evolution of the scalar fields, its derivative and metric coefficients are shown in figure 1, and rotation parameters are shown in figure 2. Those figures are obtained with $\sigma_0 = 0.20$, $t_0 = 4.0$ and $\overline{C}_0 = 1.0$, for which $B_0 = 0.1422$ and $w = 0.79108$. In this case, $E_B < 0$, showing that configuration is probably stable. We can also observe that $\overline{C}$ stabilizes for $x \to \infty$. But this is expected and from equation (46) one can see that this maximum value is $\Omega$, up to the approximation order we assume. One can obtain this by using the boundary conditions at infinity, where asymptotic flatness is expected for (28), and $\lim_{r \to \infty} A(r) = 1 = \lim_{r \to \infty} B(r)$ and $\lim_{r \to \infty} C(r) = 0$. The last one is equivalent to:

\[ \lim_{x \to \infty} C(x) = 0 \] (64)

Since $C(x) = \Omega - \overline{C}(x)$, equation (64) is possible if and only if:

\[ \lim_{x \to \infty} \overline{C}(x) = \Omega \] (65)

In the case shown in the figures 1 and 2, angular momentum as seem by an observer far from the object is $\Omega = -0.1651$. Some other results are shown in the table I below for a configuration with $t_0 = 4.0$ and $\sigma_0 = 0.2$.

| $C_0$  | 3.0  | 2.0  | 1.0  | -1.0 | -2.0 |
|-------|------|------|------|------|------|
| $\Omega$ | -0.4954 | -0.3302 | -0.1651 | 0.1651 | 0.3302 |

TABLE I. Values obtained for $\Omega$ given an initial value of the rotational parameter $C_0$, with $\sigma_0 = 0.2$ e $t_0 = 4.0$.  

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All configurations obtained present $E_B < 0$, and the same values for mass and radius. Note that negative values just provide an inversion of rotation, as expected.

![Graph](image_url)

**FIG. 1.** Scalar field, its derivative and metric coefficients for a composite boson-fermion star, with $\sigma_0 = 0.20$, $t_0 = 4.0$ and $C_0 = 1.0$. Values obtained: $B_0 = 0.1422$, $w = 0.79108$ and $E_B \simeq -0.05$.

![Graph](image_url)

**FIG. 2.** Rotation parameter evolution for the case shown in figure 1. In this case $\lim_{x \to \infty} C(x) = \Omega = -0.1651$.

**V. CONCLUSIONS**

Given a star composed of both bosons and fermions, without interaction, we have show that it is possible to obtain stable configurations with slow rotation, by using the perturbative relativistic method that usually describes neutron stars. We have developed sample calculations considering fields up to the second order in angular velocity. With that, angular velocity, mass, number of particles and binding energy can be measured by an observer at the infinity.

But, if it is known that the perturbative prescription allows no rotation for boson stars, why have we obtained rotation with the same prescription when bosons are mixed to fermions inside the star? The answer is in equation...
Up to the order considered here, rotation is not directly influenced by the scalar field; instead, energy density and pressure of the fermi gas are explicit in the equation that describes the rotation of the configuration.

Since we have performed \( l = 0 \) case, which is the predominant one, these results may contribute to studies on deformations of the structure, e.g. \( l = 2 \), and so on, and studies on the emission of gravitational waves from those kind of objects. This study also opens some questions about spinning boson fermion stars with the use of axisymmetric scalar field.

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