UNITARY PRINCIPLE AND REAL SOLUTION OF DIRAC EQUATION

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ABSTRACT. The exact solution of the Dirac equation with the exact boundary condition is further investigated in the present paper. We introduce the unitary principle that can be used to disclose all logic paradoxes hidden in natural science and mathematics. By the unitary principle, we show that the existence of solution of differential equation requires an important modification to the constant of angular momentum in the original Dirac equation. Applying it to the hydrogen-like atoms, we derive the real solution involving the zero energy and the quantum proton radius as well as the neutron ionization energy. However, the expression for the discrete energy levels in the real solution only depends on the radial quantum number. How to treat the difference between the real expression for the discrete energy levels of the Dirac-Coulomb equation and the experimental observation of the atomic spectrums of hydrogen is not resolved yet.

1. Introduction

Darwin[1] and Gordon[2] first obtained the standard solution of the Dirac equation with a Coulomb potential in 1928. From then on, many different treatments to the Dirac equation were given one after another. In some literatures[3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14], it constructs the second-order Dirac equations or the imitated first-order Dirac equation for the new upper and lower components by introducing the function transformation to the original Dirac equations. In recent years, we analyzed many different procedures to treat the Dirac equation and found that there were some substantial difficulties of mathematics hidden in those formal solutions[15, 16, 17, 18, 19, 20, 21]. For examples, the existence and uniqueness of solution for the bound-state case were destroyed there[22, 23, 24, 25, 26, 27]. Even if for the standard solution to the Dirac-Coulomb equation, one can also verify the full radial wave function of the $S$ state at the origin not meeting the given boundary condition yet. In fact, the divergence of the wave function is one of the crucial logical difficulties for the relativistic wave equations. It should seek another treatment to the Dirac equation in order to obtain the real exact solution that satisfies all mathematical axioms and physical background.

In 2006, we introduced the exact boundary condition with considering the radius of atomic nucleus instead of the traditional rough boundary condition without considering the radius of atomic nucleus and found another exact treatment[28] to the Dirac-Coulomb equation. A different exact solution, involving the different wave functions and the different expressions for the discrete energy levels, was hence obtained. However, by further analyzing the details of mathematical procedures of the various solutions under the traditional rough boundary
condition and our new solution under the exact boundary condition respectively, we found there was still another mathematical difficulty hidden in the original Dirac equation to distinguish the true from the false for its various formal exact solutions. It deals with why and how to construct the procedure to treat the constant of angular momentum. 25 years ago, in order to directly determine the ideas of right and wrong for all formal logic paradoxes hidden in theoretical physics, we summarized a general principle called the unitary principle. It provides great help to find the real solution to the Dirac equation. All of these works led to this paper. Here we try to come to hard mathematics logic to solve all mathematical difficulties hidden in various formal solutions of the Dirac equation, with the result that the exact quantum proton radius and neutron ionization energy were obtained.

2. Traditional solution under rough boundary condition

Here, it should point out a fact on quantum mechanics that once a wave equation with the condition for determining solution is given the rest would be pure mathematics only[29]. The preciseness of mathematics requires that one should not ignore any logical difficult hidden in a famous theory. If the bound state problems were put forward as the corresponding problem for determining the solution of equations of mathematical physics, one would always, first notice those fundamental differences of the formal solutions and the real solutions, thus the Dirac equation would have a different tendency to develop. For the Dirac equation describing the hydrogen-like atoms, the problem was finally translated into solving a system of radial differential equations for the upper and lower components of the original radial wave function

\[ R(r) = \frac{1}{r} \left( \frac{F(r)}{G(r)} \right) \]

obviously, the problem under the rough boundary condition is a pure mathematics problem

\[
\begin{align*}
\left( \frac{E - mc^2}{\hbar c} + \frac{Z\alpha}{r} \right) F + \left( \frac{\kappa}{r} + \frac{d}{dr} \right) G &= 0 \\
\left( \frac{E + mc^2}{\hbar c} + \frac{Z\alpha}{r} \right) G + \left( \frac{\kappa}{r} + \frac{d}{dr} \right) F &= 0 \\
R(r \to 0) &\neq \pm \infty, \quad R(r \to \infty) = 0
\end{align*}
\]

where \( c \) is the velocity of light in the vacuum, \( m \) the rest mass of electron, \( \alpha = e^2/(4\pi\varepsilon_0\hbar c) \) is the fine-structure constant, \( h = h/2\pi \) with the plank constant and \( \kappa = \pm 1, \pm 2, \pm 3, \cdots \) is the artificial constant of motion. The conventional viewpoints have been that the above problem for determining solution only has the following solution[30, 31, 32, 33]

\[
E = \sqrt{1 + \frac{m^2c^4}{(n_r + \sqrt{\kappa^2 - Z^2\alpha^2})^2}}
\]

\[
F = \exp \left( -\frac{\sqrt{m^2c^4 - E^2}}{\hbar c} r \sum_{\nu=0}^{n} b_{\nu}r^{\sqrt{\kappa^2 - Z^2\alpha^2} + \nu} \right)
\]

\[
G = \exp \left( \frac{\sqrt{m^2c^4 - E^2}}{\hbar c} r \sum_{\nu=0}^{n} d_{\nu}r^{\sqrt{\kappa^2 - Z^2\alpha^2} + \nu} \right)
\]
where \( n_\nu = 0, 1, 2, \cdots \) and the coefficient \( b_\nu \) and \( d_\nu \) satisfy the following system of recurrence relations

\[
\begin{align*}
(s + \nu - \kappa)b_\nu + Z\alpha d_\nu - ab_{\nu-1} - \frac{mc^2 - E}{\hbar c}d_{\nu-1} &= 0 \\
Z\alpha b_\nu - (s + \nu + \kappa)d_\nu - \frac{m_0 c^2 + E}{\hbar c}b_{\nu-1} + ad_{\nu-1} &= 0
\end{align*}
\]

where \( s = \sqrt{\kappa^2 - Z^2\alpha^2} \) and \( a = \sqrt{m^2c^4 - E^2}/\hbar c \).

(2.3) with (2.4) is often called the standard solution of Dirac-Coulomb equation, because the formula of the energy levels in (2.3) agrees with the experimental observation to the atomic spectrum of hydrogen, this standard solution has been regarded as succeeding in describing the fin-structure of the hydrogen-like atoms. However, as \( \kappa = \pm 1 \), \( R(r \to 0) = \pm \infty \), the wave function for the \( S \) state at the origin does not meet the boundary condition, implying that the wave function in (2.3) should be a formal key to the problem for determining the solution (2.2). It usually explains the above mathematical paradox as that the Dirac wave function for the hydrogen ground state has a mild divergence at the origin. We consider that the “mild divergence” is not a law of nature! On the other hand, there exist other contradictions hidden in the standard solution of the Dirac-Coulomb equation such as the domain difficulty of the constant of motion. From 1988, we have tried to find an exacter treatment to the Dirac equation. It was found that the rough boundary condition in (2.2), which was first given in the Schrödinger equation, was not applicable to the Dirac equation.

3. A DIFFERENT SOLUTION UNDER EXACT BOUNDARY CONDITION

Usually, a theory does not accept any deduction concealing some mathematics errors, even though it agrees with the experimental observation. If the Dirac equation were the only exact equation for describing the hydrogen-like atoms, a correct method to derive the Dirac formula for the energy levels would exist. We should seek the correct treatment but not various specious explanations. In the different formal theories, in fact, the cited Dirac formulas for the energy levels are not the real deductions of the corresponding logic. However, the topic is often transferred to that the formula for the energy levels agrees with the energy spectrum of the Dirac equation. Such actions have become a huge resistance force so it is difficult to expose and criticize the errors of the formal logic in theoretical physics, even if it is completely wrong. Experiment is a main norm for verifying a physical deduction, but what if the experiment data did not deny some deductions derived from the incorrect theories?

Malenfant made the contributions[34, 35] to overcome the divergence of relativistic wave functions at the origin. For the divergence of standard wave function of the Dirac equation for the \( S \) state at the origin, it usually considers the fact that the atomic nucleus of the hydrogen-like atoms has finite radius. As one of the reliable treatments, it is generally assumed that equation (2.2) is correct only for \( r > \delta \) (radius of proton). Inside \( \delta \), the potential has to be modified from a Coulomb \( 1/r \) potential to one corresponding to a spread-out charge distribution. So, to do a completely correct calculation, one solves the Dirac equation separately outside of and inside of \( \delta \), with two different potentials, and then matches, at \( r = \delta \), the outside solution (i.e., the standard Dirac-Coulomb one) to the inside solution. The inside solution is finite at \( r = 0 \); its effect is to modify the energy-level
formal by a small correction that takes into account the finite radius of the proton. This idea just supports introducing the exact boundary condition to the wave equation. For the hydrogen-like atoms, by the exact boundary condition, it should change the standard problem for determining the solution (2.2) into the exacter problem for determining the solution to the Dirac-Coulomb equation outside \( \delta \)

\[
\left( \frac{E - mc^2}{\hbar c} + \frac{Z\alpha}{r} \right) F + \left( \frac{\kappa}{r} + \frac{d}{dr} \right) G = 0
\]

\[
\left( \frac{E + mc^2}{\hbar c} + \frac{Z\alpha}{r} \right) G + \left( \frac{\kappa}{r} + \frac{d}{dr} \right) F = 0
\]

\( R (r \to \delta) \neq \pm \infty, \quad R (r \to \infty) = 0 \)

where \( \delta > 0 \) is defined as the quantum radius of the atomic nucleus, it is an intrinsic parameter of the original Dirac equation. Using the variable substitution \( r = \xi + \delta \), the exact boundary condition becomes

\( R (\xi \to 0) \neq \pm \infty, \quad R (\xi \to \infty) = 0 \). Just like finding the standard solution to the problem for determining the solution (2.2), it easily obtained a different formal solution to the Dirac-Coulomb equation as follows

\[
E = \frac{mc^2}{\sqrt{1 + \frac{Z^2\alpha^2}{n^2}}}
\]

\[
F (r) = \exp \left( -\frac{\sqrt{m^2c^4 - E^2}}{\hbar c} (r - \delta) \right) \sum_{\nu=1}^{n_r} b_\nu (r - \delta)^\nu
\]

\[
G (r) = \exp \left( -\frac{\sqrt{m^2c^4 - E^2}}{\hbar c} (r - \delta) \right) \sum_{\nu=1}^{n_r} d_\nu (r - \delta)^\nu
\]

where \( n_r = 0, 1, 2, \cdots, r \geq \delta \) and the coefficient \( b_\nu \) and \( d_\nu \) are determined by the following new coupled recurrence relations

\[
\frac{E - mc^2}{\hbar c} b_{\nu-1} + \left( \frac{E - mc^2}{\hbar c} \delta + Z\alpha \right) b_\nu - ad_{\nu-1} + (\kappa + \nu - a\delta) d_\nu + (\nu + 1) \delta d_{\nu+1} = 0
\]

\[
a b_{\nu-1} + (\kappa - \nu + a\delta) b_\nu - (\nu + 1) \delta b_{\nu+1} + \frac{E + mc^2}{\hbar c} d_{\nu-1} + \left( \frac{E + mc^2}{\hbar c} \delta + Z\alpha \right) d_\nu = 0
\]

This new form of solution of the radial Dirac-Coulomb equation is different from the standard solution (2.3). First, the full wave function \( R (r) \) determined by (3.2) has not any divergence for all states, because it only holds for \( r \geq \delta > 0 \). It has relation to the quantum proton radius \( \delta \). Therefore, \( \delta \) is an intrinsic constant of the radial Dirac equation. Second, there is one big difference between the new expression for the discrete energy levels (the first formula in (3.2)) and the Dirac formula (the first formula in (2.3)). The new expression for the discrete energy levels cannot describe the fine structure of hydrogen-like atoms. However, in mathematics, the above new solution is a necessary deduction of the original Dirac-Coulomb equation. In the standard solution of the Dirac equation, in order to obtain the Dirac formula for the energy levels, some mathematics rules had to be destroyed. Further studies show that the solution (3.2) is just the real form of mathematical deduction of the Dirac-Coulomb equation with the exact boundary condition. Of course, there, we show that it needs to determine the \( \delta \) value and make an important modification to the constant of motion by the existence of solution of differential equation.
4. Unitary principle and real solution under exact boundary condition

On the same Dirac wave equation describing the hydrogen-like atoms, the exact solutions obtained differ in different line of reasoning because of using the rough and exact boundary condition respectively. Two different forms of solution (2.3) and (3.2) form a logic paradox. One with a very good agreement with the experimental observation conceals the mathematical contradictions; another without any mathematical contradiction remains a sizable gap between the logic deduction and the experimental observation. It has been difficult to treat such a logic paradox in theoretical physics.

In order to reach a consensus on treating those recognized theories concealing the mathematical contradictions, 20 years ago, we generalized an important principle that is called the unitary principle. Describing the law of nature can choose different metrologies, there are definite transforms among different metrologies, but the law of nature does not change per se because of choosing different metrologies. As the different mathematical forms in the different metrologies for describing the same law of nature are transformed into the same metrologies, it must be the same as the form in the present metrologies, $1=1$, the transformation is unitary. This principle can be used to disclose all logic paradoxes hidden in science theory as well as mathematics and find the correct treatments to the corresponding problems. By the unitary principle, we ever used rough boundary condition to solve the second-order Dirac equations for the original components $F(r)$ and $G(r)$ of the wave function; it still derives the formal formula of energy eigenvalue for the hydrogen-like atoms. However, the formal wave function of the second-order differential equations is not equivalent to the formal wave function of the original first-order differential equations. In this logic connection, the first order Dirac equation is chosen as one metrologies, and the second order Dirac equation for the original radial components of wave function is chosen as another metrologies. In fact, we have proven that this equivalent second-order Dirac equation, all other second-order Dirac equations and all kinds of imitated first-order Dirac equations has not any real solution under the rough boundary condition. However, in the corresponding formal solution, for obtaining the Dirac formula for the energy levels, the boundary condition for the original Dirac equation and the uniqueness theorem of solution for the equation and so on were used to be destroyed. This case still appeared in the initial value problem of the Maxwell equation, seeing a paper in Chinese, in which as one of real solutions of the Maxwell equation the real mode of plane transverse electromagnetic was given [36].

According to the unitary principle, if the standard solution (2.3) were correct, the new exact solution (3.2) would deduce $\delta = 0$. However, by further investigating the existence of solution to the system of recurrence relations (3.3), we find that the system of recurrence relations (3.3) does not hold for any artificial constant of motion $\kappa = \pm 1, \pm 2, \pm 3, \cdots$. This leads to the important modification to the radial Dirac-Coulomb equation, but the corresponding wave function and the expression for the discrete energy levels are still (3.2). It deduces the quantum proton radius $\delta$ and the neutron ionization energy.

We focus on the existence of solution to the systems of recurrence relations (2.4) and (3.3). Recently, in writing other papers, we proved that in history constructing the constant of angular momentum for the radial Dirac equation is unreliable. This can be viewed from a hidden paradox in recurrence relations (2.4) for the standard solution (2.3). As $n_r = 0,$
\( \delta \) solution to the Dirac-Coulomb equation outside \( (4.2) \) by using the undetermined parameter Dirac theory. The existence of solution needs the modification to the constant of angular momentum by using the undetermined parameter \( C \) to replace the artificial constant \( \kappa \). \( C \) is another intrinsic parameters of the Dirac equation. Thus, the problem for determining the solution to the Dirac-Coulomb equation outside \( \delta \) takes the following correct form

\[
\begin{align*}
\left( \frac{E - mc^2}{\hbar c} + \frac{Z\alpha}{r} \right) F + \left( \frac{C}{r} + \frac{d}{dr} \right) G &= 0 \\
\left( \frac{E + mc^2}{\hbar c} + \frac{Z\alpha}{r} \right) G + \left( \frac{C}{r} + \frac{d}{dr} \right) F &= 0
\end{align*}
\]

(4.2)

where \( C \) is determined by the existence of solution of the differential equation with the exact boundary condition itself. The above problem for determining the solution has the solution as the same as expressions (3.2), which is rewritten as follows

\[
E = \frac{mc^2}{\sqrt{1 + (Z\alpha/n_r)^2}}
\]

(4.3)

\[
\begin{align*}
F(r) &= \exp \left( -\sqrt{\frac{m^2c^4 - E^2}{\hbar c}} (r - \delta) \right) \sum_{\nu=1}^{n_r} f_{\nu} (r - \delta)^{\nu} \\
G(r) &= \exp \left( -\sqrt{\frac{m^2c^4 - E^2}{\hbar c}} (r - \delta) \right) \sum_{\nu=1}^{n_r} g_{\nu} (r - \delta)^{\nu}
\end{align*}
\]

where \( n_r = 0, 1, 2, \ldots, r \geq \delta \). The expression for the discrete energy levels and the wave function is as same as (3.2), and the energy formula has no relation to the undermined parameter \( C \). Replacing \( \kappa \) by \( C \), \( b_{\nu} \) by \( f_{\nu} \), and \( d_{\nu} \) by \( g_{\nu} \) in the recurrence relations in (3.3), it obtains the system of recurrence relations for determining the coefficients of the series \( f_{\nu} \) and \( g_{\nu} \)

\[
\begin{align*}
&\frac{E - mc^2}{\hbar c} f_{\nu-1} + \left( \frac{E - mc^2}{\hbar c} \delta + Z\alpha \right) f_{\nu} - ag_{\nu-1} + (C + \nu - a\delta) g_{\nu} + (\nu + 1) \delta g_{\nu+1} = 0 \\
&af_{\nu-1} + (C - \nu + a\delta) f_{\nu} - (\nu + 1) \delta f_{\nu+1} + \frac{E + mc^2}{\hbar c} g_{\nu-1} + \left( \frac{E + mc^2}{\hbar c} \delta + Z\alpha \right) g_{\nu} = 0
\end{align*}
\]

(4.4)
The intrinsic parameter $\delta$ and $C_0$, $C_1$, $C_2$, $\cdots$, $C_n$, corresponding to $n_r = 0$, 1, 2, $\cdots$ will be uniquely determined by this system of recurrence relations, but only unique value of minimum quantum proton radius $\delta$ is permitted. The expression for the discrete energy levels and wave function (4.3) with the recurrence relations (4.4) constitutes all real solutions of the radial Dirac equation for hydrogen-like atoms.

5. Quantum proton radius and neutron ionization energy

Seeing about the nonlinear homogeneous[37] system of recurrence relations (4.4), in which each of quantities $E$, $C$, $f_\nu$, $g_\nu$ may depend on $n$. A sequence of numbers $f_1$, $f_2$, $\cdots$, $f_n$ and $g_1$, $g_2$, $\cdots$, $g_n$ satisfying the system of recurrence relations (4.4) is uniquely determined once one of the values of $f_0$ and $g_0$, the so-called initial value, and the intrinsic constants $\delta$ and $C_n$, are prescribed. According to the unitary principle, the values of $\delta$ and $C_n$ will play a decisive role to affirm that the standard solution under the rough boundary condition is not the real solution of the Dirac-Coulomb equation. We now discuss the special solution of the system of recurrence relations (4.4) for zero energy state. In the real solution (4.3), as $n_r = 0$, $b_1 = b_2 = \cdots = 0$ and $d_1 = d_2 = \cdots = 0$. In order to obtain the untrivial solution, order $b_0 \neq 0$ and $d_0 \neq 0$. Inserting $n_r = 0$ into the recurrence relations (4.4) yields

\[
\begin{align*}
\left(\frac{mc^2 - E_0}{\hbar c} \delta - Z\alpha\right) b_0 - (C_0 - a_0 \delta) d_0 &= 0 \\
(C_0 + a_0 \delta) b_0 + \left(\frac{mc^2 + E_0}{\hbar c} \delta + Z\alpha\right) d_0 &= 0 \\
\frac{mc^2 - E_0}{\hbar c} b_0 + a_0 d_0 &= 0 \\
a_0 b_0 + \frac{mc^2 + E_0}{\hbar c} d_0 &= 0
\end{align*}
\]

where $a_0 = \sqrt{m_0^2 c^4 - E_0^2}/\hbar c$. The last two relations are equivalent to each other. By the first and second relations, one easily obtains $\delta = \hbar c \left(\frac{C_0^2 - Z^2 \alpha^2}{C_0^2 + Z^2 \alpha^2} \right) (2Z\alpha E_0)^{-1}$. Substituting the first relation into the third or fourth relation yields

\[
E_0 = \frac{C_0^2 - Z^2 \alpha^2}{C_0^2 + Z^2 \alpha^2} mc^2
\]

(5.2)

$$\delta = \frac{(C_0^2 + Z^2 \alpha^2) \hbar}{2Z\alpha mc}$$

Substituting the second formula into the third formula in (5.1) gives

\[
C_0 = \frac{Z\alpha \sqrt{mc^2 - E_0}}{mc^2 + E_0}
\]

(5.3)

this formula is combined with (5.2) to read

\[
E_0 = 0, \quad C_0 = Z\alpha, \quad \delta = \frac{Z\alpha \hbar}{mc},
\]

(5.4)

Thus, as it can be seen, the constant of the angular momentum in Dirac equations should be $C = C_n$ with $n = 0$, 1, 2, $\cdots$ but not the fabricated constant $\kappa = \pm 1, \pm 2, \pm 3, \cdots$, and the quantum proton radius is not zero, $\delta \neq 0$. According to the unitary principle, the rough boundary condition is unsuitable for the Dirac equation describing hydrogen-like atoms and
the so-called standard solution (2.3) with (2.4) should be one of the formal solutions of the Dirac-Coulomb equations. Otherwise, it would read $C = \kappa$ and $\delta = 0$.

The special energy eigenvalue $E_0 = 0$ corresponds to the unique energy state of neutron and neutron-like $\Delta E = E_\infty - E_0$. As $Z = 1$, it deduces the neutron ionization energy and the quantum proton radius as follows

$$\Delta E = mc^2, \quad \delta = \frac{e^2}{4\pi\varepsilon_0 mc^2}$$

where $E_\infty$ is the energy of hydrogen-like atoms corresponding to $n_r = \infty$ in the formula (4.3). The neutron ionization energy implies that the neutron is so hard that it would be broken up only by a photon with the energy $mc^2$, and combining an electron and a proton into a neutron would emit a photon with the energy $mc^2$. The term $\delta = e^2 \left(4\pi\varepsilon_0 mc^2\right)^{-1}$ (=2.8117940285 fm) has been regarded as the classical electron radius. However, the above analysis shows that it should be the quantum proton radius. This is about tripliation of the recent values that are respectively reported to be 0.88015 fm [38], 0.895±0.018 fm [39] and 0.89014 fm, which the value is actually calculated by Lamb shift [40]. Admittedly, the accurate calculation (5.5) is the necessary deduction of the Dirac equation. The difference between the calculation and the result of indirect measurement strongly suggests the direct measurement to the proton radius.

6. Summary and expectation

As one of the important criteria for testing logic of nature science and mathematics, the unitary principle is introduced in this paper. We show that the existence of solution for the Dirac-Coulomb equation demands the correction for undetermined parameter $C_n$. Replacing the rough boundary condition by the exact boundary condition and replacing the artificial constant $\kappa = \pm1, \pm2, \cdots$ corresponding to angular momentum by strictly deriving the intrinsic undermined constant $C_n$ are the mainly ameliorations to the Dirac theory for the hydrogen-like atoms. Being different from the traditional standard solution, the real exact solution of the Dirac equation is obtained by solving the Dirac equation with the exact (real) boundary condition, without any mathematical paradox. It predicts the quantum proton radius and the neutron ionization energy. The traditional standard solution is obtained by solving the Dirac equation with the rough boundary condition and the fictive constant of motion. It conceals some paradoxes that destroy the fundamental mathematical rules but have been covered up by those specious exploitations or mathematical distortion technique. Just like that the divergence of the Dirac wave function of the $S$ state at the origin was called the “mild divergence”. As one of the unreal solutions to the Dirac-Coulomb equation, the Dirac formula for the energy levels agrees pretty accurately with the experimentally observed hydrogen spectral. It seems to be a brainteaser! However, as the real solution of the Dirac Coulomb equation, the new formula for the energy levels does not give the fine structure splitting. In this case, out of question, it must choose the real solution and then make further correction. Otherwise, the physic would continually use much more wrong logic or false science to coin the anticipant deduction. We conclude, therefore, that the Dirac equation succeeds in describing the character of neutron combined by the proton and the electron. We know that the exact boundary condition will influence some conclusions of quantum field theory and quantum electrodynamics. Why cannot the
real solution of the Dirac equation with the exact boundary condition describe the fine structure of hydrogen atom? Does the Dirac equation contain another stricter deduction for energy eigenvalue having not been found? By the unitary principle, if the real solution presents no other evidence that the real wave function predicts results that agree with any other experimental data, then it would inspire new possibilities to look for the exacter amelioration to the Dirac equation.

ACKNOWLEDGMENTS

I wish to thank Jiewen-Liang, Caitlin Jackson and Emily Hsieh for English help.

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