Percolation Processes and Wireless Network Resilience

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Abstract—We study the problem of wireless network resilience to node failures from a percolation-based perspective. In practical wireless networks, it is often the case that nodes with larger degrees (i.e., more neighbors) are more likely to fail. We model this phenomenon as a degree-dependent site percolation process on random geometric graphs. In particular, we obtain analytical conditions for the existence of phase transitions within this model. Furthermore, in networks carrying traffic load, the failure of one node can result in redistribution of the load onto other nearby nodes. If these nodes fail due to excessive load, then this process can result in cascading failure. We analyze this cascading failures problem in large-scale wireless networks, and show that it is equivalent to a degree-dependent site percolation on random geometric graphs. We obtain analytical conditions for cascades in this model.

I. INTRODUCTION

In large-scale wireless networks, nodes are often vulnerable to attacks, natural hazards, and resource depletion. The ability of wireless networks to maintain their functionality in the face of these challenges is a central concern for network designers. One measure of the resilience of wireless networks to attacks and failures is the fraction of operational nodes in the largest connected component of the network. A network may be said to be resilient if the largest connected component spans almost the entire network even in the face of many node failures. From this perspective, the characterization of network resilience corresponds to the study of the qualitative and quantitative properties of the largest connected component. A powerful tool for this study stems from the theory of percolation [1]–[5]. Recently, percolation theory, especially continuum percolation, has been widely used to study the coverage, connectivity, and capacity of large-scale wireless networks [6]–[13].

A percolation process resides in a random graph structure, where nodes or links are randomly designated as either “occupied” or “unoccupied.” When the graph structure resides in continuous space, the resulting model is described by continuum percolation [1]–[3]. A major focus of continuum percolation theory is the random geometric graph induced by a Poisson point process with constant density \( \lambda \). A fundamental result for continuum percolation concerns a phase transition effect whereby the macroscopic behavior of the system is very different for densities below and above some critical value \( \lambda_c \). For \( \lambda < \lambda_c \) (subcritical), the connected component containing the origin (or any other fixed point) contains a finite number of points almost surely. For \( \lambda > \lambda_c \) (supercritical), the connected component containing the origin (or any other node) contains an infinite number of points with a positive probability [1]–[4].

In this paper, we study the problem of large-scale wireless network resilience to node failures from the percolation perspective. We first consider wireless networks with random, independent node failures. To see why this problem can be described by a percolation process on the network, note that in a network with random node failures, nodes are randomly occupied (operational) or unoccupied (failed), and the number of operational nodes that can successfully communicate with an extensive portion of the network is precisely the largest component of the corresponding percolation model. Hence, the phase transition phenomena of the percolation model directly translates to a description of the random failures model.

In practical wireless networks, it is often the case that nodes with large degrees (large number of neighbors) are more likely to fail. For instance, a wireless sensor node which must communicate with a large number of neighbors is more likely to deplete its energy reserve. A communication node directly connected to many other nodes in a military network is more likely to be attacked by an enemy seeking to break down the whole network. Such phenomenon can be described by a more general model where each node fail with a probability depending on its degree. In this paper, we study such degree-dependent node failure problems. Specifically, by modelling the problem as a degree-dependent site percolation process on random geometric graphs, we obtain analytical conditions on the existence of percolation in this model. Based on this general model, we further study a special case of the degree-dependent node failures problem—the so-called fixed-degree node failure problem.

In networks which carry load, distribute a resource or aggregate data, such as wireless sensor networks and electrical power networks, the failure of one node often results in the redistribution of the load from the failed node to other nearby nodes. If nodes fail when the load on them exceeds some maximum capacity or when the battery energy is depleted, then a cascading failure or avalanche may occur because the redistribution of the load causes other nodes to exceed their

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thresholds and fail, thereby leading to a further redistribution of the load. An example of such a cascading failure is the power outage in the western United States in August 1996, which resulted from the spread of a small initial power shutdown in El Paso, Texas. The power outage spread through six states as far as Oregon and California, leaving several million customers without electronic power [14], [15]. In this paper, we study such cascading node failures in large-scale wireless networks. We show that such problems can also be mapped to a percolation process on random geometric graphs. Using our degree-dependent site percolation model, we obtain analytical conditions on the occurrence of cascading failures.

This paper is organized as follows: In Section II, we outline some preliminary results for random geometric graphs and continuum percolation. In Section III, we first review independent node failures and fixed-degree node failures. We provide analytical conditions for the existence of percolation in these models. In Section IV, we show the equivalence between cascading failures in large-scale wireless networks and degree-dependent site percolation, and investigate the conditions under which a small exogenous event can trigger global cascading failures. In Section V, we present simulation results, and finally, we conclude this paper in Section VI.

II. RANDOM GEOMETRIC GRAPHS AND CONTINUUM PERCOLATION

A. Random Geometric Graphs

We use random geometric graphs to model wireless networks. Assume a communication link exists between two nodes if the distance between them is sufficiently small, so that the received power is large enough for successful decoding. A mathematical model for this is as follows. Let || · || be the Euclidean norm and \( f \) be some probability density function (p.d.f.) on \( \mathbb{R}^d \). Let \( X_1, X_2, \ldots, X_n \) be independent and identically distributed (i.i.d.) \( d \)-dimensional random variables with common density \( f \), where \( X_i \) denotes the random location of node \( i \) in \( \mathbb{R}^d \). The ensemble of graphs with undirected links connecting all those pairs \( \{x_i, x_j\} \) with \( ||x_i - x_j|| \leq r, r > 0 \), is called a random geometric graph [3], denoted by \( G(\mathcal{X}_n, r) \). The parameter \( r \) is called the characteristic radius.

In the following, we consider random geometric graphs \( G(\mathcal{X}_n, r) \) in \( \mathbb{R}^2 \), with \( X_1, X_2, \ldots, X_n \) distributed i.i.d. according to a uniform distribution in a square area \( A = [0, \sqrt{\pi}r^2] \). Let \( A = |A| \) be the area of \( A \). There exists a link between two nodes \( i \) and \( j \) if and only if \( i \) lies within a circle of radius \( r \) around \( x_j \). Ignoring border effect, the probability for the existence of such a link is:

\[
P_{\text{link}} = \frac{\pi r^2}{A} = \frac{\lambda \pi r^2}{n}. \tag{1}
\]

It follows that the degree of any given node has the distribution Binomial\((n-1, P_{\text{link}})\):

\[
p_k = \binom{n-1}{k} P_{\text{link}}^k (1 - P_{\text{link}})^{n-1-k}. \tag{2}
\]

The mean degree of each node is

\[
E[k] = (n-1)P_{\text{link}} = \frac{(n-1)\lambda \pi r^2}{n}. \tag{3}
\]

As \( n \) and \( A \) both become large with the ratio \( \frac{n}{A} = \lambda \) kept constant, each node has an approximately Poisson degree distribution with an expected degree \( \mu \)

\[
\mu = \lim_{n \to \infty} \frac{(n-1)\lambda \pi r^2}{n} = \lambda \pi r^2. \tag{4}
\]

In this case, as \( n \to \infty \) and \( A \to \infty \) with \( \frac{n}{A} = \lambda \) fixed, \( G(\mathcal{X}_n, r) \) converges in distribution to an (infinite) random geometric graph \( G(\mathcal{H}_\lambda, r) \) induced by a homogeneous Poisson point process with density \( \lambda > 0 \). Due to the scaling property of random geometric graphs [2], [3], in the following, we focus on \( G(\mathcal{H}_\lambda, 1) \).

Consider a graph \( G = (V, E) \), where \( V \) and \( E \) denote the set of nodes and links, respectively. Given \( u, v \in V \), we say \( u \) and \( v \) are adjacent if there exists a link between \( u \) and \( v \), i.e., \( (u, v) \in E \). In this case, we also say that \( u \) and \( v \) are neighbors.

B. Critical Density for Continuum Percolation

Let \( \mathcal{H}_{\lambda, 0} = \mathcal{H}_\lambda \cup \{0\} \), i.e., the union of the origin and the infinite homogeneous Poisson point process with density \( \lambda \). Note that in a random geometric graph induced by a homogeneous Poisson point process, the choice of the origin can be arbitrary. As discussed before, the phase transition takes place at the critical density. More formally, we have the following definition:

**Definition 1:** For \( G(\mathcal{H}_{\lambda, 0}, 1) \), the percolation probability \( p_\infty(\lambda) \) is the probability that the component containing the origin has an infinite number of nodes of the graph. The critical density \( \lambda_c \) is defined as

\[
\lambda_c = \inf\{\lambda > 0 : p_\infty(\lambda) > 0\}. \tag{5}
\]

It is known that if \( \lambda > \lambda_c \), then there exists a unique connected component containing \( \Theta(n) \) nodes\(^2\) in \( G(\mathcal{X}_n, 1) \) a.a.s.\(^3\) This largest connected component is called the giant component [2]. A fundamental result of continuum percolation states that \( 0 < \lambda_c < \infty \). Exact values of \( \lambda_c \) and \( p_\infty(\lambda) \) are not yet known. Simulation studies show that \( 1.43 < \lambda_c < 1.44 \) [16], while the rigorous bounds \( 0.696 < \lambda_c < 3.372 \) are given in [2], [17]. Very recently, through the use of the cluster coefficient in random geometric graphs, the lower bound has been improved to 0.7698 [18], [19].

\(^2\)We say \( f(n) = O(g(n)) \) if there exists \( n_0 > 0 \) and constant \( c_0 \) such that \( f(n) \leq c_0 g(n) \) \( \forall n \geq n_0 \). We say \( f(n) = \Omega(g(n)) \) if \( g(n) = O(f(n)) \). Finally, we say \( f(n) = \Theta(g(n)) \) if \( f(n) = \Omega(g(n)) \) and \( f(n) = O(g(n)) \).

\(^3\)An event is said to be asymptotic almost sure (abbreviated a.a.s.) if it occurs with a probability converging to 1 as \( n \to \infty \).
III. RANDOM NODE FAILURES

A. Independent Random Node Failures

As we mentioned in the introduction, the problem of network resilience to random node failures can be described by a percolation process on the graph modelling the network. Suppose the network is modelled by \( G(\mathcal{H}_\lambda,1) \) is subject to random node failures where each node fails, along with all associated links, with probability \( q \), independently of other nodes. When \( q \) stays below a certain threshold \( q_c \), there still exists a connected component of operational nodes that spans the entire network. When \( q > q_c \), the network disintegrates into smaller, disconnected operational parts. Since each node fails randomly and independently with probability \( q \), according to Thinning Theorem [2], [3], the remaining graph is still a random geometric graph with density \( (1-q)\lambda \). Thus the remaining graph is percolated if \( (1-q)\lambda > \lambda_c \), and not percolated if \( (1-q)\lambda < \lambda_c \). Therefore we have

\[
q_c = 1 - \frac{\mu_c}{\mu} = 1 - \frac{\lambda_c}{\lambda}, \tag{6}
\]

where \( \mu_c \) (\( \mu_c = \lambda_c/\pi \)) and \( \mu \) are the critical mean degree and the mean degree of \( G(\mathcal{H}_\lambda,1) \), respectively.

B. Degree-Dependent Node Failures

We have thus far considered wireless network resilience to independent random node failures. As we mentioned before, in practical wireless networks, it is often the case that nodes with large degrees (large number of neighbors) are more likely to fail. We therefore study network resilience in the face of degree-dependent node failures. Let the original random geometric graph be \( G(\mathcal{H}_\lambda,1) \) with density \( \lambda > \lambda_c \). Suppose each node with degree \( k \geq 1 \) in \( G(\mathcal{H}_\lambda,1) \) fails, along with all associated links, with probability \( q(k) \), \( 0 \leq q(k) \leq 1 \). A reasonable assumption on \( q(k) \) would be that it is non-decreasing in \( k \). Denote the remaining graph by \( G(\mathcal{H}_\lambda,1,q(\cdot)) \).

To study the percolation-based connectivity of \( G(\mathcal{H}_\lambda,1,q(\cdot)) \), we consider a degree-dependent site percolation problem for random geometric graphs. Similar problems have been studied in the context of Erdős-Rényi random graphs and random graphs with given degree distributions using generating function methods [20]–[23]. Due to clustering effects and geometric constraints, however, generating function methods are not applicable for random geometric graphs. The SINR-based models studied in [11], [12] involve dependent percolation but not degree-dependent percolation. In [24], a degree-dependent site percolation model was studied. There, the authors propose a topology control mechanism for sensor networks where each sensor stays active with \( \frac{\phi}{\lambda} \) fraction of the time for some constant \( \phi \) when \( k > \phi \). The authors obtain a sufficient condition for percolation within this model. Unlike [24], we do not have specified requirements on \( q(\cdot) \). Indeed, \( q(k) \) need not even be non-decreasing in \( k \). Moreover, in addition to a sufficient condition, we also obtain a necessary condition for percolation in our model. The main results are as follows.

Theorem 1: (i) For any \( \mu_1 > \mu_c \), there exists \( K_1 < \infty \) such that for any \( G(\mathcal{H}_\lambda,1) \) with \( \mu > \mu_1 \), if

\[
q(k) \leq 1 - \frac{\mu_1}{\mu}, \quad \text{for all } 1 \leq k \leq K_1, \tag{7}
\]

then \( G(\mathcal{H}_\lambda,1,q(\cdot)) \) is percolated, i.e., there is a giant component consisting of operational nodes in \( G(\mathcal{H}_\lambda,1) \) a.a.s.;

(ii) Given \( G(\mathcal{H}_\lambda,1) \) with \( \mu > \mu_c \), if \( q(k) \) is non-decreasing in \( k \) and

\[
e^{-q} + \sum_{k=1}^{\infty} \frac{q(k)}{k!} e^{-q} q(k-1)^k > 1 - \frac{1}{27}, \tag{8}
\]

then \( G(\mathcal{H}_\lambda,1,q(\cdot)) \) is not percolated, i.e., there is no giant component consisting of operational nodes in \( G(\mathcal{H}_\lambda,1) \) a.a.s.

An interesting implication of Theorem 1-(i) is that even if all nodes with degree larger than \( K_1 \) fail with probability 1, as the result of energy depletion or enemy attack, a giant component still exists in the remaining graph as long as (7) is satisfied.

To prove Theorem 1, we employ a mapping between the continuum model and a discrete percolation model. A similar technique was used in [10], [12]. The mapping is as follows. Consider a square lattice \( \mathcal{L} = d \cdot \mathbb{Z}^2 \), where \( d \) is the edge length. All the vertices of \( \mathcal{L} \) are located at \( (d \times i, d \times j) \) where \((i,j) \in \mathbb{Z}^2 \). For each horizontal edge \( a \), let the two end vertices be \((d \times x_a, d \times y_a)\) and \((d \times x_a + d, d \times y_a)\). Define event \( A_a \) as the set of outcomes for which both of the following occur:

(i) The rectangle \( R_a \) \([a_x d - \frac{d}{4}, a_x d + \frac{3d}{4}] \times [a_y d - \frac{d}{4}, a_y d + \frac{3d}{4}]\) is crossed\(^4\) from left to right by a connected component in \( G(\mathcal{H}_\lambda,1) \);

(ii) The left square \( S_a^- \) \([a_x d - \frac{d}{4}, a_x d + \frac{d}{4}] \times [a_y d - \frac{d}{4}, a_y d + \frac{d}{4}]\) and the right square \( S_a^+ \) \([a_x d + \frac{3d}{4}, a_x d + \frac{5d}{4}] \times [a_y d - \frac{d}{4}, a_y d + \frac{3d}{4}]\) are both crossed from top to bottom by connected components in \( G(\mathcal{H}_\lambda,1) \).

If \( A_a \) occurs, we say that rectangle \( R_a \) is a good rectangle, and edge \( a \) is a good edge. Let

\[
p_g \equiv \Pr\{A_a\}.
\]

Define \( A_a \) similarly for all vertical edges by rotating the rectangle by 90°. An example of a good rectangle and a good edge is illustrated in Figure 1.

According to Corollary 4.1 in [2], the probability \( p_g \) can be made arbitrarily close to 1 by choosing \( d \) large enough when the continuum model is in the supercritical phase. The events \( \{A_a\} \) are not independent in general. However, if two edges \( a \) and \( b \) are not adjacent, i.e., they do not share any common end node, then \( A_a \) and \( A_b \) are independent.

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\(^4\)Here, a rectangle \( R = [x_1, x_2] \times [y_1, y_2] \) being crossed from left to right by a connected component in \( G(\mathcal{H}_\lambda,1) \) means that there exists a sequence of nodes \( v_1, v_2, \ldots, v_m \in G(\mathcal{H}_\lambda,1) \) contained in \( R \), with \( ||x_i - x_{i+1}|| \leq 1, i = 1, \ldots, m - 1, \) and \( 0 < x(v_1) - x_1 < 1, 0 < x_2 - x(v_m) < 1, \) where \( x(v_1) \) and \( x(v_m) \) are the \( x \)-coordinates of nodes \( v_1 \) and \( v_m \), respectively. A rectangle being crossed from top to bottom is defined analogously.
in $G$ proof, we focus on the nodes in $G$ edges, and events
where $d$ are satisfied: the set of outcomes for which both of the following conditions
$G\lambda$ where $m$ in the random failures model are also active in percolated. When (7) is satisfied, almost all the active nodes
is located at the center of one square. Let the number of $G$ model in $1$ random geometric graph with density $G$,
It is clear by the Thinning Theorem that $G\lambda$, described above, for any given $\lambda\prime$ 5
Define another event $H\lambda$, 1) $\lambda\prime$ 1)
Before giving the proof, we need the following lemma,
Lemma 2: Given a square lattice $L\prime$, suppose that the origin
is located at the center of one square. Let the number of circuits5 surrounding the origin with length $2m$ be $\gamma(2m)$, where $m \geq 2$ is an integer, then we have
\[
\gamma(2m) \leq \frac{4}{27} (m - 1) 3^{2m}.
\]
Proof of Theorem 1-(i): We first consider a random failures model in $G(H_\lambda, 1)$ where each node fails independently with probability $1 - \frac{\mu_\lambda}{\mu}$. Let $G\prime(H_\lambda, 1)$ be the remaining graph. It is clear by the Thinning Theorem that $G\prime(H_\lambda, 1)$ is a random geometric graph with density $\lambda_1 = \frac{\mu_\lambda}{\mu} > \lambda_c$. Hence, $G\prime(H_\lambda, 1)$ is percolated.

Now we map $G\prime(H_\lambda, 1)$ to a square lattice $L = d_\epsilon(\lambda_1) \cdot \mathbb{Z}^2$, where $d_\epsilon(\lambda_1)$ is defined in (9). Define good rectangles and edges, and events $\{A_a\}$ as described above. Throughout the proof, we focus on the nodes in $G\prime(H_\lambda, 1)$, and their degrees in $G(H_\lambda, 1)$.

Define another event $B_a$ for each horizontal edge $a$ in $L$ as the set of outcomes for which both of the following conditions are satisfied:

5A circuit in a lattice $L\prime$ is a closed path with no repeated vertices in $L\prime$.\]
Denote by $N$ the number of nodes of $G'(\mathcal{H}_\lambda, 1)$ in $R_o$. Then $N$ has a Poisson distribution with mean
\[ E[N] = \frac{3}{4} d_\nu(\lambda_1)^2 \lambda_1. \]

By the Chebychev’s inequality, we have
\[
\Pr\{N < N_1\} = 1 - \Pr\{N \geq N_1\} = 1 - \Pr\{N \geq (1 + 1/3)E[N]\}
\geq 1 - \frac{\text{Var}(N)}{(1/3)^2 E[N]^2}
= 1 - \frac{12}{d_\nu(\lambda_1)^2 \lambda_1}.
\]

Since the degree of each node in $G(\mathcal{H}_\lambda, 1)$ also has a Poisson distribution with mean $\mu_K$, by the same argument, the probability that one node has degree larger than or equal to $K_1$ is upper bounded by $p_0 = \frac{1}{K_0^2 \mu}$ as follows:
\[
\Pr\{k \geq K_1\} = \Pr\{k \geq (1 + K_0^2)\mu\}
= \Pr\{k \geq (1 + K_0^2)E[k]\}
\leq \frac{\text{Var}(k)}{K_0^2 E[k]^2}
= \frac{1}{K_0^2 \mu}.
\]

Let $E_k$ be the event that the $k$-th node of $G'(\mathcal{H}_\lambda, 1)$ in $R_o$ has degree greater than or equal to $K_1$ in $G(\mathcal{H}_\lambda, 1)$. Then $E_k$ is a decreasing event\(^6\) for all $k$. By the FKG inequality [2]–[4], we have
\[
\Pr\{\bigcap_{k=1}^N E_k\} \geq \prod_{k=1}^N \Pr(E_k).
\]

Then, the probability that no node of $G'(\mathcal{H}_\lambda, 1)$ in $R_o$ has degree larger than or equal to $K_1$ in $G(\mathcal{H}, r)$ given that there are strictly fewer than $N_1$ nodes of $G'(\mathcal{H}_\lambda, 1)$ in $R_o$ is lower bounded by
\[
\prod_{k=1}^N \Pr(E_k) \geq \prod_{k=1}^{N_1} \Pr(E_k)
\geq (1 - p_0)^{N_1}
= \left(1 - \frac{1}{K_0^2 \mu}\right)^{d_\nu(\lambda_1)^2 \lambda_1}
\]

Therefore,
\[
p_e \geq \left(1 - \frac{12}{d_\nu(\lambda_1)^2 \lambda_1}\right) \left(1 - \frac{1}{K_0^2 \mu}\right)^{d_\nu(\lambda_1)^2 \lambda_1} = 1 - \frac{12}{d_\nu(\lambda_1)^2 \lambda_1} - \epsilon, \quad (16)
\]

By (9) and (16), we have
\[
p_o \geq p_o + p_e - 1 > 1 - q_0. \quad (17)
\]

Now consider the dual lattice $L'$ of $L$. The construction of $L'$ is as follows: let each vertex of $L'$ be located at the center of a square of $L$. Let each edge of $L'$ be open if and only if it crosses an open edge of $L$, and closed otherwise. It is clear that each edge in $L'$ is open also with probability $p_o$. Let
\[
q = 1 - p_o,
\]
and choose $2m$ edges in $L'$. Because the states (i.e., open or closed) of any set of non-adjacent edges are independent, we can choose $m$ edges among these $2m$ edges such that their states are independent. As a result,
\[
\Pr\{\text{All the } 2m \text{ edges are closed}\} \leq q^m.
\]

Now a key observation is that if the origin belongs to an infinite open edge cluster in $L$, then there cannot exist a closed circuit surrounding the origin in $L'$, and vice versa. This is demonstrated in Figure 3. Hence
\[
\Pr\{E_L\} > 0 \iff \Pr\{E_{L'}\} < 1.
\]

Furthermore, we have
\[
\Pr\{E_{L'}\} = \sum_{m=2}^{\infty} \Pr\{\exists \mathcal{O}_c(2m)\} \leq \sum_{m=2}^{\infty} \gamma(2m)q^m,
\]
where $\mathcal{O}_c(2m)$ is a closed circuit having length $2m$ surrounding the origin, and $\gamma(2m)$ is the number of such circuits.

By Lemma 2, we have
\[
\sum_{m=2}^{\infty} \gamma(2m)q^m \leq \sum_{m=2}^{\infty} \frac{4}{27} (m - 1)(9q)^m
= \frac{4}{27} \left[ \sum_{m=2}^{\infty} m(9q)^m - \sum_{m=2}^{\infty} (9q)^m \right]
= \frac{4}{27} \left[ 2(9q)^2 - (9q)^3 \right]
= \frac{12q^2}{(1 - 9q)^2}, \quad (18)
\]
we have $\frac{2\sqrt{2}q}{b - 9q} < 1$, and hence $\frac{12q^2}{(1-9q)^2} < 1$. Thus the origin belongs to an infinite open edge cluster in $\mathcal{L}$ with a positive probability. Because the existence of an infinite open edge cluster in $\mathcal{L}$ implies the existence of an infinite connected component in $G(\mathcal{H}_\lambda, 1, q(\cdot))$, $G(\mathcal{H}_\lambda, 1, q(\cdot))$ is percolated. □

The first part of Theorem 1 provides a sufficient condition for $G(\mathcal{H}_\lambda, 1, q(\cdot))$ to be percolated. The second part of Theorem 1 provides a sufficient condition for $G(\mathcal{H}_\lambda, 1, q(\cdot))$ to be not percolated. Thus, it provides a necessary condition for $G(\mathcal{H}_\lambda, 1, q(\cdot))$ to be percolated. To show this, we use another mapping between the continuum model and a discrete model.

Proof of Theorem 1-(ii): Map $G(\mathcal{H}_\lambda, 1)$ to a square lattice $\mathcal{L}$ with edge length $d = \frac{\sqrt{3}}{2}$. Let the square centered at node $a$ with edge length $d$ be $S_a$. Denote by $N(S_a)$ and $N'(S_a)$ the number of nodes of $G(\mathcal{H}_\lambda, 1)$ and $G(\mathcal{H}_\lambda, 1, q(\cdot))$ contained in $S_a$, respectively. We say $S_a$ is open if and only if it satisfies either one of the following conditions:

(i) $N'(S_a) \geq 1$;
(ii) There is a link of $G(\mathcal{H}_\lambda, 1, q(\cdot))$ crossing $S_a$ which directly connects two nodes of $G(\mathcal{H}_\lambda, 1, q(\cdot))$ outside $S_a$.

In Figure 4, we illustrate all possibilities of open squares in $\mathcal{L}$. If $S_a$ is open only because $S_a$ satisfies condition (ii), we say it is type-2 open; otherwise, we say it is type-1 open.

Since $d = \frac{\sqrt{3}}{2}$, all the nodes of $G(\mathcal{H}_\lambda, 1)$ in a square are directly connected to each other. Hence if there are $k$ nodes in one square, every node has degree greater than or equal to $k - 1$. Therefore, the probability that $S_a$ is type-1 open is

$$p_1 = \Pr\{N'(S_a) \geq 1\} = \sum_{k=1}^{\infty} \Pr\{N(S_a) = k, \text{at least one is operational}\} \leq \sum_{k=1}^{\infty} \Pr\{N(S_a) = k\} \cdot (1 - q(k - 1)^k) = \sum_{k=1}^{\infty} \frac{(\mu^k)}{k!} e^{-\mu} (1 - q(k - 1)^k) = 1 - e^{-\mu} - \sum_{k=1}^{\infty} \frac{(\mu^k)}{k!} e^{-\mu} q(k - 1)^k < \frac{1}{27},$$

where the first inequality holds because $q(k)$ is non-decreasing and there are $k$ nodes of $G(\mathcal{H}_\lambda, 1)$ in $S_a$, with each of them having degree greater than or equal to $k - 1$. The last inequality is due to (8).

Note that if there is an infinite component in $G(\mathcal{H}_\lambda, 1, q(\cdot))$, there must exist a path passing through an infinite number of open nodes in $\mathcal{L}$, as illustrated in Figure 5. This is because, along the infinite component in $G(\mathcal{H}_\lambda, 1, q(\cdot))$, each square of $\mathcal{L}$ contains at least one node of $G(\mathcal{H}_\lambda, 1, q(\cdot))$ or is crossed by a link of $G(\mathcal{H}_\lambda, 1, q(\cdot))$ that directly connects two nodes of $G(\mathcal{H}_\lambda, 1, q(\cdot))$ outside $S_a$.

Now choose a path in $\mathcal{L}$ starting from the origin having length $3m$. From Figure 4, we can see that a link from any given node in $G(\mathcal{H}_\lambda, 1, q(\cdot))$ can go through at most three open squares in addition to the open square containing the given node. As a result, along the path, among every three consecutive open squares, there exists at least one type-1 open square. Thus, we have

$$\Pr\{\text{All the 3m edges are open}\} \leq p_1^{m + 1}.$$
Now
\[ \Pr\{o_p(3m)\} \leq (3m)p_1^{m+1}, \]
where \( o_p(3m) \) is an open path in \( L \) starting from the origin with length \( 3m \), and \( (3m) \) is the number of such paths. For a path in \( L \) from the origin, the first edge has four choices for its direction, and all other edges have at most three choices for their directions. Therefore, we have
\[ \xi(3m) \leq 4 \cdot 3^{3m-1}, \]
and
\[ \Pr\{o_p(3m)\} \leq 4 \cdot 3^{3m-1}p_1^{m+1} = \frac{4}{3}p_1(3^3p_1)^m. \]
When \( p_1 < \frac{1}{3^{27}} \), (21) converges to 0 as \( m \to \infty \). This implies that there is no infinite path in \( L \) a.a.s. and therefore there is no infinite component in \( G(H_\lambda, 1, q(\cdot)) \) either.

### C. Fixed-Degree Node Failures

From an attacker’s point of view, an efficient and effective way to break down a network is to destroy nodes with a large number of neighbors. One such strategy is to set a threshold \( \phi \) and destroy all nodes having degree strictly greater than \( \phi \). This is a special case of the degree-dependent node failures problem—the so-called fixed-degree node failure problem. Given \( G(H_\lambda, 1) \) and an integer \( \phi \), all nodes with degree strictly greater than \( \phi \) and their associated links fail, and all other nodes remain operational. That is
\[ q(k) = \begin{cases} 0 & \text{if } k \leq \phi \\ 1 & \text{if } k \geq \phi + 1 \end{cases} \]

Let the remaining graph be denoted by \( G(H_\lambda, 1, \phi) \). By directly applying Theorem 1-(i), we know that there exists \( K_1 < \infty \), which depends on \( \mu \) and is given by (13), such that when \( \phi \geq K_1 \), \( G(H_\lambda, 1, \phi) \) is percolated.

We can also apply Theorem 1-(ii) to obtain a lower bound on the critical value of \( \phi \). By substituting (22) into (8), we see that if \( \phi' \) satisfies
\[ e^{-\frac{\phi'}{27}} + \sum_{k=\phi'+2}^{\infty} e^{-\frac{k}{27}} > 1 - \frac{1}{27}, \]
then for any \( \phi \leq \phi' \), \( G(H_\lambda, 1, \phi) \) is not percolated.

Condition (23) can be simplified as
\[ \sum_{k=\phi'+1}^{\phi'+1} e^{-\frac{k}{27}} < 1 + e^{-\frac{\phi'}{27}}, \]
which can be further expressed as
\[ \sum_{k=0}^{\phi'+1} e^{-\frac{k}{27}} < 1 + e^{-\frac{\phi'}{27}} + 1. \]

For any given \( \mu \), we can use (25) to find the critical value of \( \phi \). Figure 6 plots the maximal \( \phi' \) satisfying (25).

Moreover, when \( \phi \leq 2 \), \( G(H_\lambda, 1, \phi) \) is not percolated. To see why this is true, suppose after removing all the nodes with degree strictly greater than two, there is still an infinite component in \( G(H_\lambda, 1, 2) \). Then according to the argument for the proof of Theorem 1, for large enough \( d \), \( \Pr\{A_d\} \to 1 \). However, as all nodes with degree strictly greater than 2 have been removed, \( \{A_d\} \) cannot occur since no matter how large \( d \) is, the nodes at the intersections of the left-to-right crossing and top-to-bottom crossings must have at least three neighbors. This contradiction ensures that if \( G(H_\lambda, 1, \phi) \) is percolated, we must have \( \phi > 2 \).

To summarize, we have proved the following result:

**Corollary 3:** Given \( G(H_\lambda, 1) \) with \( \mu > \mu_c (\lambda > \lambda_c) \), let
\[ \alpha(\mu) = (1 + \delta_1)\mu, \]
where \( \delta_1 = \frac{1}{\sqrt{\mu}} \left[ 1 - \left( 1 - e^{-\frac{\phi'}{27}} \right)^{\frac{1}{\arctan(\alpha(\mu)/\lambda)}} \right]^{-\frac{1}{2}} \) with \( d_c(\lambda) \) defined by (9), and
\[ \beta(\mu) = \max \left\{ \phi' : \sum_{k=0}^{\phi'+1} e^{-\frac{k}{27}} < 1 + e^{-\frac{\phi'}{27}} + 1 \right\}. \]

(i) If \( \phi \geq \alpha(\mu) \), then \( G(H_\lambda, 1, \phi) \) is percolated, i.e., there exists a giant component consisting of operational nodes in \( G(H_\lambda, 1, \phi) \) a.a.s.

(ii) If \( \phi \leq \max\{2, \beta(\mu)\} \), then \( G(H_\lambda, 1, \phi) \) is not percolated, i.e., there is no giant component consisting of operational nodes in \( G(H_\lambda, 1, \phi) \) a.a.s.

### IV. Cascading Node Failures

Cascades have been studied in social networks to model phenomena such as epidemic spreading, belief propagation, etc. Although they are generated by different mechanisms, cascades in social and economic systems are similar to cascading failures in physical infrastructure networks [14], [15] in that initial failures increase the likelihood of subsequent failures, leading to eventual dramatic global outages. Usually, such cascading failures are extremely difficult to predict, even when
the properties of individual components are well understood. In [20], [25], the author investigate such cascading failures in social networks by modeling the problem as a binary decision percolation process on random networks where the links between distinct pairs of nodes are independent.

In contrast to previous work, we study cascading failures in large-scale wireless networks. In particular, we consider the following model. Given \( G(\mathcal{H}_0, 1) \) with density \( \lambda > \lambda_0 \), the network is seeded by initial failures corresponding to some nonzero density \( \lambda_0 \) of nodes. That is, each node in \( G(\mathcal{H}_0, 1) \) is an initial failure independently with probability \( \frac{\lambda_0}{\lambda} \). Assume that \( \lambda_0 \ll 1 \), so that the initial seeds consist of single isolated nodes. These initial failures are exogenous events (shocks) that are very small relative to the whole network.

We assumed that due to redistribution of load, each node \( i \) fails if a given fraction \( \psi_1 \) of its neighbors has failed, where the quantities \( \psi_i \) are i.i.d. random variables drawn from a distribution \( f(\psi) \).

Now observe that the initial failure can grow only when a neighbor, say \( i \), of the initial seed has a threshold satisfying \( \psi_i < \frac{1}{k} \), where \( k \) is the degree of \( i \). We call such a node vulnerable. The probability of a node being vulnerable is

\[
\rho_k = F_\psi\left( \frac{1}{k} \right) = \int_0^{\frac{1}{k}} f(\psi) d\psi, \tag{28}
\]

where \( F_\psi(\cdot) \) is the distribution function of \( \psi_i \). When the initial failures are connected to a component of vulnerable nodes, all nodes in this component will fail. Clearly, the more vulnerable nodes exist in the network, the more likely it is that an cascading failure may occur. Moreover, the extent of its growth, and hence the resilience of the network, depends not only on the number of vulnerable nodes, but on how connected they are to one another. In the context of this model, a cascade of failed nodes will form if vulnerable nodes form a giant component in the network, i.e., the largest connected vulnerable cluster must occupy a non-zero fraction of the entire infinite network.

On the other hand, if node \( i \) has a threshold satisfying \( \psi_i > \frac{k-1}{k} \), where \( k \) is the degree of \( i \), then node \( i \) will not fail as long as at least one neighbor is operational. We call such a node reliable. The probability of a node being reliable is

\[
\sigma_k = 1 - F_\psi\left( \frac{k-1}{k} \right) = \int_{\frac{k-1}{k}}^{1} f(\psi) d\psi. \tag{29}
\]

When reliable nodes in the network form a giant component, no cascades can take place. To see this, note that when two reliable nodes are adjacent and neither is an initial failure seed, then no matter what else happens in the network, they remain functional. When the reliable nodes form a giant component initially, no cascade of failures result.

With the above, the cascading failure problem is translated into a degree-dependent site percolation problem as studied before. Thus, when the network is percolated by vulnerable nodes, random initial shocks trigger global cascades, and when the network is percolated by reliable nodes, early failed nodes are isolated from each other and are unable to generate the

**Corollary 4:** Assume that initial failures are isolated. Then, (i) For any \( \mu_1 > \mu_c \), there exists \( K_1 < \infty \), such that for any \( G(\mathcal{H}_1, 1) \) with \( \mu > \mu_1 \), if

\[
\rho_{K_1} = F_\psi\left( \frac{1}{K_1} \right) \geq \frac{\mu_1}{\mu}, \tag{30}
\]

then there will be a cascading failure in \( G(\mathcal{H}_1, 1) \) with probability 1.

(ii) For any \( \mu_1 > \mu_c \), there exists \( K'_1 < \infty \), such that for any \( G(\mathcal{H}, 1) \) with \( \mu > \mu_1 \), if

\[
F_\psi\left( \frac{K'_1 - 1}{K'_1} \right) = 1 - \sigma_{K'_1} \leq 1 - \frac{\mu_1}{\mu}, \tag{31}
\]

then there will be no cascading failure in \( G(\mathcal{H}_1, 1) \) with probability 1.

**Proof:** Let \( \rho_k \) be defined as (28). We view this problem as a degree-dependent node failure problem, where each node with degree \( k \) fails with a probability \( \rho_k \). Then by applying Theorem 1-(i) directly, we have (30). The second part can be proved in the same manner. \( \Box \)

As in Theorem 1, Corollary 4 only provides sufficient conditions for the occurrences and non-occurrences of cascading...
failures, respectively. It is possible that the network admits (does not admit) cascading failures even when condition (30) ((31)) is not satisfied.

V. SIMULATION STUDIES

We illustrate degree-dependent node failures with two examples in Figure 8 and Figure 9. In Figure 8, $q(k) = \max\{0, 1 - \frac{\mu - 1}{\nu} - 1\}$. This function satisfies condition (7) and the remaining network still has a connected component spanning almost the whole network. In Figure 9, $q(k) = 0, k \leq 4$, and $q(k) = 1, k > 4$. This function satisfies condition (8) and the remaining network is broken into isolated parts.

In Figure 10 and Figure 11, we illustrate fixed-degree node failures in $G(\mathcal{H}_\lambda, 1)$ with examples where $\mu = 6$ and $\mu = 11$, respectively. Figure 10-(a) shows $G(\mathcal{H}_\lambda, 1, 8)$, the resulting graph after all nodes with degree strictly larger than 8 are removed. The remaining network still has a connected component spanning almost the entire network. Figure 10-(b) shows $G(\mathcal{H}_\lambda, 1, 7)$, the resulting graph after all nodes with degree strictly larger than 7 are removed, and the remaining network is broken into isolated parts. Similarly, Figure 11-(a) shows $G(\mathcal{H}_\lambda, 1, 10)$, and Figure 11-(b) shows $G(\mathcal{H}_\lambda, 1, 9)$.

VI. CONCLUSIONS

In this paper, we studied network resilience problems from a percolation-based perspective. To analyze realistic situations where nodes with larger degrees (i.e., more neighbors) are more likely to fail, we introduced the degree-dependent failures problem. We model this phenomenon as a degree-dependent site percolation process on random geometric graphs. Using coupling methods and renormalization arguments, we obtained analytical conditions for the occurrence of phase transitions within this model. As a special case of the general degree-dependent failures problem, we investigated the so-called fixed-degree failures problem. Furthermore, because in networks carrying traffic load, such as wireless sensor networks and electrical power networks, the failure of one node can result in redistribution of the load onto other nearby nodes, if these nodes fail due to excessive load, then this process can result in cascading failure. We analyzed this cascading failures problem in large-scale wireless networks, and showed that it is equivalent to a degree-dependent site percolation on random geometric graphs. We obtained analytical conditions for the occurrence and non-occurrence of cascading failures, respectively. Extensive simulation results confirming our theoretical predictions were also presented.

A. Proof for Lemma 2

Proof for Lemma 2: In Figure 12, an example of a circuit that surrounds the origin is illustrated. First note that the length of such a circuit must be even. This is because there is a one-to-one correspondence between each pair of edges above and below the line $y = 0$, and similarly for each pair of edges at the left and right of the line $x = 0$. Furthermore, the rightmost edge can be chosen only from the lines $l_i : x = i - \frac{1}{2}, i = 1, \ldots, m - 1$. Hence the number of possibilities for this edge is at most $m - 1$. Because this edge is the rightmost edge, each of the two edges adjacent to it has two choices for its direction. For all the other edges, each one has at most three choices for its direction. Therefore the number of total choices for all the other edges is at most


\[ \exists 2^{m-3} \]. Consequently, the number of circuits that surround the origin and have length \(2m\) must be less or equal to \((m-1)^2 2^{3m-3}\), and hence we have (11). □

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