MODULAR REPRESENTATIONS AND INDICATORS FOR BISMASH PRODUCTS

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Abstract. We introduce Brauer characters for representations of the bismash products of groups in characteristic \( p > 0, p \neq 2 \) and study their properties analogous to the classical case of finite groups. We then use our results to extend to bismash products a theorem of Thompson on lifting Frobenius-Schur indicators from characteristic \( p \) to characteristic 0.

1. Introduction

In this paper we study the representations of bismash products \( H_k = k^G \# kF \), coming from a factorizable group of the form \( Q = FG \) over an algebraically closed field \( k \) of characteristic \( p > 0, p \neq 2 \). Our general approach is to reduce the problem to a corresponding Hopf algebra in characteristic 0.

In the first part of the paper, we extend many of the classical facts about Brauer characters of groups in char \( p > 0 \) to the case of our bismash products; our Brauer characters are defined on a special subset of \( H \) of non-nilpotent elements, using the classical Brauer characters of certain stabilizer subgroups \( F_x \) of the group \( F \). In particular we relate the decomposition matrix of a character for the bismash product in char 0 with respect to our new Brauer characters, to the ordinary decomposition matrices for the group algebras of the \( F_x \) with respect to their Brauer characters. As a consequence we are able to extend a theorem of Brauer saying that the determinant of the Cartan matrix for the above decomposition is a power of \( p \) (Theorem 4.14).

These results about Brauer characters may be useful for other work on modular representations. We remark that the only other work on lifting from characteristic \( p \) to characteristic 0 of which we are aware is that of [EG], and they work only in the semisimple case.

In the second part, we first extend known facts on Witt kernels for \( G \)-invariant forms to the case of a Hopf algebra \( H \), as well as some facts about \( G \)-lattices. We then use these results and Brauer characters to extend a theorem of J. Thompson [Th] on Frobenius-Schur indicators for representations of finite groups to the case of bismash product Hopf algebras. In particular we show that if \( H_\mathbb{C} = \mathbb{C}^G \# \mathbb{C}F \) is a bismash product over \( \mathbb{C} \) and \( H_k = k^G \# kF \) is the corresponding bismash product over an algebraically closed field \( k \) of characteristic \( p > 0 \), and if \( H_\mathbb{C} \) is totally orthogonal (that is, all Frobenius-Schur indicators are +1), then the same is true for \( H_k \) (Corollary 6.6).

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This paper is organized as follows. Section 2 reviews known facts about bismash products and their representations, and Section 3 summarizes some basic facts about Brauer characters for representations of finite groups. In Section 4 we prove our main results about Brauer characters for the case of bismash products.

In Section 5 we extend the facts we will need on Witt kernels and lattices, and in Section 6 we combine all these results to prove our extension of Thompson’s theorem. Finally in Section 7 we give some applications and raise some questions.

Throughout the setting of C*-algebras, in which case $E = \mathbb{C}$, and in general by Takeuchi [Ta], constructed from what he called a matched pair of groups. These Hopf algebras can also be constructed from a factorizable group, and that is the approach we use here. Throughout, we assume that $F$ and $G$ are finite groups.

**Definition 2.1.** A group $Q$ is called *factorizable* into subgroups $F, G \subset Q$ if $FG = Q$ and $F \cap G = 1$; equivalently, every element $q \in Q$ may be written uniquely as a product $q = ax$ with $a \in F$ and $x \in G$.

A factorizable group gives rise to actions of each subgroup on the other. That is, we have

$$\triangleright : G \times F \to F \quad \text{and} \quad \triangleleft : G \times F \to G,$$

where for all $x \in G, a \in F$, the images $x \triangleright a \in F$ and $x \triangleleft a \in G$ are the (necessarily unique) elements of $F$ and $G$ such that $xa = (x \triangleright a)(x \triangleleft a)$.

Although these actions $\triangleright$ and $\triangleleft$ of $F$ and $G$ on each other are not group automorphisms, they induce actions of $F$ and $G$ as automorphisms of the dual algebras $E^G$ and $E^F$. Let $\{p_x \mid x \in G\}$ be the basis of $E^G$ dual to the basis $G$ of $EG$ and let $\{p_a \mid a \in F\}$ be the basis of $E^F$ dual to the basis $F$ of $EF$. Then the induced actions are given by

$$a \cdot p_x := p_{x \triangleleft a^{-1}} \quad \text{and} \quad x \cdot p_a := p_{x \triangleright a},$$

for all $a \in F, x \in G$. We let $F_x$ denote the stabilizer in $F$ of $x$ under the action $\triangleleft$.

The *bismash product* Hopf algebra $H_\mathcal{E} := E^G \# E^F$ associated to $Q = FG$ uses the actions above. As a vector space, $H_\mathcal{E} = E^G \otimes E^F$, with $E$-basis $\{p_x \# a \mid x \in G, a \in F\}$. The algebra structure is the usual smash product, given by

$$\quad (p_x \# a)(p_y \# b) = p_x(a \cdot p_y)\# ab = p_xp_{y\triangleleft a^{-1}} \# ab = \delta_{y, x \triangleleft a}p_x \# ab.$$

The coalgebra structure may be obtained by dualizing the algebra structure of $H_\mathcal{E}$, although we will only need here that $H_\mathcal{E}$ has counit $\epsilon(p_x \# a) = \delta_{x, 1}$. Finally the antipode of $H$ is given by $S(p_x \# a) = p_{(x \triangleleft a)^{-1}} \# (x \triangleright a)^{-1}$. One may check that $S^2 = id$. 

2. Extensions arising from factorizable groups and their representations

The Hopf algebras we consider here were first described by G. Kac [Ka] in the
For other facts about bismash products, including the alternate approach of matched pairs of groups, see [Ma2], [Ma3]. We will consider the explicit example of a factorization of the symmetric group in Section 7.

Observe that for any field $E$, a distinguished basis of $H_E$ over $E$ is the set

$$B := \{ p_y^#a \mid y \in G, a \in F \},$$

and that $B$ has the property that if $b,b' \in B$, then $bb' \in B \cup \{0\}$. In particular, if $w = p_y^#a$, then (2.3) implies that for all $k \geq 2$,

$$w^k = \begin{cases} p_y^#a^k & \text{if } a \in F_y \\ 0 & \text{if } a \notin F_y. \end{cases}$$

Thus if $a \in F_y$ and has order $m$, the minimum polynomial of $p_y^#a$ is

$$f(Z) = Z^{m+1} - Z,$$

and so the characteristic roots of $p_y^#a$ are $\{0\} \cup \{m^{th} \text{ roots of 1}\}$.

**Lemma 2.6.** (1) $B$ is closed under the antipode $S$.

(2) The set $B' := \{ p_y^#a \in B \mid a \in F_y \}$ is also closed under $S$.

(3) If $w = p_y^#a \in B'$, then $S(w) = p_{y^{-1}}^#ya^{-1}y^{-1}$.

**Proof.** (1) is clear from the formula for $S$ above. For (2), formula (2.5) shows that $B'$ is exactly the set of non-nilpotent elements of $B$, so it is also closed under $S$.

For (3), $w \in B'$ implies that $a \in F_y$, and thus $y \triangleleft a = y$. Then

$$ya = (y \triangleright a)(y \triangleleft a) = (y \triangleright a)y$$

and so $y \triangleright a = yay^{-1}$. Substituting in the formula for $S$, we see $S(w) = p_{y^{-1}}^#ya^{-1}y^{-1}$. \qed

We review the description of the simple modules over a bismash product.

**Proposition 2.7.** Let $H = E^G \#EF$ be a bismash product, as above, where now $E$ is algebraically closed. For the action $\triangleleft$ of $F$ on $G$, fix one element $x$ in each $F$-orbit $O$ of $G$, and let $F_x$ be its stabilizer in $F$, as above. Let $V = V_x$ be a simple left $F_x$-module and let $\hat{V}_x = EF \otimes_{EF_x} V_x$ denote the induced $EF$-module.

$\hat{V}_x$ becomes an $H$-module in the following way: for any $y \in G$, $a,b \in F$, and $v \in V_x$,

$$(p_y^#a)[b \otimes v] = \delta_{y^{-1}(ab),x} (ab \otimes v).$$

Then $\hat{V}_x$ is a simple $H$-module under this action, and every simple $H$-module arises in this way.

**Proof.** In the case of characteristic 0, this was first proved for the Drinfel’d double $D(G)$ of a finite group $G$ over $\mathbb{C}$ by [DPR] and [M]. The case of characteristic $p > 0$ was done by [?].

For bismash products, extending the results for $D(G)$, the characteristic 0 case was done in [KMM, Lemma 2.2 and Theorem 3.3]. The case of characteristic $p > 0$ follows by extending the arguments of [?] for $D(G)$; see also [MoW]. \qed
Remark 2.8. The arguments for Proposition 2.7 also show that if we begin with an indecomposable module $V_x$ of $F_x$, then $\hat{V}_x$ is an indecomposable module for $H_2$, and all indecomposable $H_2$-modules arise in this way. This fact is discussed in [W2] after Proposition 4.4; it could also be obtained from [?] using the methods of Theorem 2.2 and Corollary 2.3 in that paper.

Now fix an irreducible $H_2$-module $\hat{V} = \hat{V}_x = \mathbb{E}F \otimes_{\mathbb{E}F_x} V_x$ as in Proposition 2.7. To compute the values of the character for $\hat{V}$, we use a formula from [JM]; it is a simpler version of [N2, Proposition 5.5] and is similar to the formula in [KMM, p 898]:

Lemma 2.9. [JM, Lemma 4.5] Fix a set $T_x$ of representatives for the right cosets of $F_x$ in $F$. Let $\chi_x$ be the character of $V_x$. Then the character $\hat{\chi}_x$ of $\hat{V}_x$ may be computed as follows:

$$\hat{\chi}_x(p_y#a) = \sum_{t \in T_x \text{ and } t^{-1}at \in F_x} \delta_{y, at} \chi_x(t^{-1}at),$$

for any $y \in G$, $a \in F$.

Next we review some known facts about Frobenius-Schur indicators for representations of Hopf algebras. For a representation $V$ of $H$, recall that a bilinear form $\langle -, - \rangle : V \otimes_{\mathbb{E}} V \to \mathbb{E}$ is $H$-invariant if for all $h \in H$ and $v, w \in V$,

$$\sum \langle h_1 \cdot v, h_2 \cdot w \rangle = \varepsilon(h) 1_{\mathbb{E}}.$$

It follows that the antipode is the adjoint of the form; that is, for all $h, l \in H$, $v, w \in V$,

$$\langle S(h) \cdot v, l \cdot w \rangle = \langle h \cdot v, S(l) \cdot w \rangle = \langle h \cdot v, S(l) \cdot w \rangle,$$

using that $S^2 = id$.

Theorem 2.10. [GM] Let $H$ be a finite-dimensional Hopf algebra over $\mathbb{E}$ such that $S^2 = id$ and $\mathbb{E}$ splits $H$. Let $V$ be an irreducible representation of $H$. Then $V$ has a well-defined Frobenius-Schur indicator $\nu(V) \in \{0, 1, -1\}$. Moreover

1. $\nu(V) \neq 0 \iff V^* \cong V.$
2. $\nu(V) = +1$ (respectively $-1$) $\iff$ $V$ admits a non-degenerate $H$-invariant symmetric (resp, skew-symmetric) bilinear form.

If in addition $H$ is semisimple and cosemisimple, then in fact $\nu(V)$ can be computed by the formula $\nu(V) = \chi(A_1A_2)$, where $\chi$ is the character belonging to $V$ and $A$ is a normalized integral of $H$ [LM]. This formula does not work in general, but still Theorem 2.10 applies to any bismash product since as noted above, it is always true that $S^2 = id$. Sometimes the indicator is called the type of $V$.

We remark that, unlike the case for groups, $\nu(V) = +1$ does not imply that the character $\chi_V$ is real-valued, even when $\mathbb{E} = \mathbb{C}$. However it is still true that if $V^* \cong V$, then $\chi^* = \chi$, that is, $\chi \circ S = \chi$.

Finally we fix the following notation:

Definition 2.11. [CR, p 402]. A $p$-modular system $(\mathbb{K}, R, \mathfrak{k})$ consists of a discrete valuation ring $R$ with quotient field $\mathbb{K}$, maximal ideal $\mathfrak{p} = \pi R$ containing the rational prime $p$, and residue class field $\mathfrak{k} = R/\mathfrak{p}$ of characteristic $p$. 

We will mainly be interested in the following special case, as in [Th] with a slight change in notation.

**Example 2.12.** Let $H\mathbb{Q}$ be a Hopf algebra over $\mathbb{Q}$ and let $\mathbb{L}$ be an algebraic number field which is a splitting field for $H\mathbb{Q}$. Let $\mathcal{P}$ be a prime ideal of the ring of integers of $\mathbb{L}$ containing the rational prime $p \neq 2$, let $R$ be the completion of the ring of $\mathcal{P}$-integers of $\mathbb{L}$, $\mathbb{K}$ be the field of fractions of $R$, $\pi$ be a generator for the maximal ideal $\mathfrak{p}$ of $R$, and $\mathbb{k} = R/\pi R$.

Then $(\mathbb{K}, R, \mathbb{k})$ is a $p$-modular system.

### 3. Brauer characters for $G$

In this section we review the definition of Brauer characters for a finite group [CR], [Nv] and summarize some of the classical results.

We fix the following notation, for a given finite group $G$. Let $|G|$ denote the order of $G$, and let $|G|_p$ denote the largest power of $p$ in $|G|$; thus $|G| = |G|_p m$ where $p \nmid m$.

Since $\mathbb{K}$ splits $G$, it contains a primitive $m$th root of 1, say $\omega$, which in fact is in $R$. Under the natural map $f: R \rightarrow \mathbb{k}$, $\omega = f(\omega)$ is a primitive $m$th root of 1 in $\mathbb{k}$.

Let $G_{p'}$ denote the set of $p$-regular elements of $G$, that is elements of $G$ whose order is prime to $p$. Thus for each $x \in G_{p'}$, all of the eigenvalues of $x$ on any (left) $\mathbb{k}G$-module $W$ are $m$th roots of 1, and hence may be expressed as a power of $\bar{\omega}$. Denote the eigenvalues of $x$ by $\{\bar{\omega}^{i_1}, \ldots, \bar{\omega}^{i_t}\}$.

**Definition 3.1.** For each (left) $\mathbb{k}G$-module $W$, the $\mathbb{K}$-valued function $\phi: G_{p'} \rightarrow \mathbb{K}$ defined for each $x \in G_{p'}$ by

$$
\phi(x) = \omega^{i_1} + \cdots + \omega^{i_t} = \sum_{j=1}^{t} f^{-1}(\bar{\omega}^{i_j}).
$$

is called the Brauer character of $G$ afforded by $W$.

**Remark 3.2.** Note that $\phi$ is a class function on the conjugacy classes of $p$-regular elements of $G$. $\phi$ can be extended to $\phi^\#: \text{a class function on all of } G$, by defining $\phi^\#(x) = 0$ for any $x$ in the complement of $G_{p'}$. It follows that $\phi^\#$ is a $\mathbb{K}$-linear combination of the ordinary irreducible characters $\chi_i$ of $\mathbb{K}G$ [CR, p 423]. Thus $\phi$ is a $\mathbb{K}$-linear combination of the $\chi_i|_{G_{p'}}$.

We record some facts about Brauer characters of groups. See [CR, 17.5], [I, Chapter 15].

**Proposition 3.3.**

1. $\lambda$ takes values in $R$ and $\lambda(x) = Tr(x, V)$, all $x \in G_{p'}$.
2. Let $W_0 \supset W_1 \supset 0$ be $\mathbb{k}G$-modules, let $\phi$ be the Brauer character afforded by $W_0/W_1$, $\phi_1$ the Brauer character afforded by $W_1$ and $\phi_0$ the Brauer character of $W_0$. Then $\phi_0 = \phi + \phi_1$.
3. Let $V$ be a $\mathbb{K}G$-module with $\mathbb{K}$-character $\chi$. Then for each $RG$-lattice $M$ in $V$, the restriction $\chi|_{G_{p'}}$ is the Brauer character of the $\mathbb{k}G$-module $\overline{M} := M/\mathfrak{p}M$.

We fix the following notation, as in [CR]:

1. $Irr(G) = \{\chi_1, \ldots, \chi_n\}$ denotes the irreducible characters of $\mathbb{K}G$;
(2) \( \text{Irr}_k(G) = \{\psi_1, \ldots, \psi_d\} \) denotes the irreducible characters of \( kG \);
(3) \( \text{IBr}(G) = \{\phi_1, \ldots, \phi_d\} \) denotes the Brauer characters corresponding to \( \{\psi_1, \ldots, \psi_d\} \).

By [CR, 16.7 and 16.20], there exists a homomorphism of abelian groups

\[
d : G_0(\mathbb{K}G) \rightarrow G_0(kG),
\]

called the decomposition map, such that for any class \( [V] \) in \( G_0(\mathbb{K}G) \), \( d([V]) = [\mathcal{M}] \in G_0(\mathbb{K}G) \), where \( \mathcal{M} \) is any \( RG \)-lattice in \( V \) and \( \mathcal{M} := M/\mathcal{P}M \).

Using Proposition 3.3(3), it follows that for any \( \chi \), there are integers \( d_{ij} \) such that

\[
\chi|_{G_{p'}} = \sum_j d_{ij} \phi_j.
\]

The multiplicities \( d_{ij} = d(\chi|_{G_{p'}}, \phi_j) \) are called decomposition numbers, and the matrix \( D = [d_{ij}] \) is called the decomposition matrix. The matrix \( C = D^T D \) is called the Cartan matrix.

From [CR], 17.12 - 17.15, the set \( Bch(\mathbb{K}G) \) of virtual Brauer characters, that is \( \mathbb{Z} \)-linear combinations of Brauer characters of \( kG \)-modules, is a ring under addition and multiplication of functions, and \( Bch(\mathbb{K}G) \cong G_0(\mathbb{K}G) \). Using that \( G_0(\mathbb{K}G) \cong ch(\mathbb{K}G) \), the ring of virtual characters of \( \mathbb{K}G \), it follows that the decomposition map \( d \) induces a map

\[
d' : ch(\mathbb{K}G) \rightarrow Bch(\mathbb{K}G),
\]

where \( d' \) is the restriction map \( \psi \rightarrow \psi|_{G_{p'}} \).

Consequently Equation (3.5) implies that if \( \chi \) is the character for \( V \) and \( \chi|_{G_{p'}} = \sum_j \alpha_j \phi_j \), where \( \phi_j \) is the Brauer character of the simple \( kG \)-module \( W_j \), and \( d([V]) = [\mathcal{M}] \in G_0(kG) \), then

\[
[M] = \sum_j \alpha_j [W_j].
\]

We will need the following theorem in our more general situation:

**Theorem 3.6.** (Brauer) [CR, 18.25][I, Ex (15.3)] \( \det(C) \) is a power of \( p \).

We will also need the analog of the following:

**Theorem 3.7.** [CR, (17.9)] The irreducible Brauer characters \( \text{IBr}(G) \) form a \( \mathbb{K} \)-basis of the space of \( \mathbb{K} \)-valued class functions of \( G_{p'} \).

One consequence of this theorem is:

**Corollary 3.8.** Let \( E \) be a splitting field for \( G \) of char \( p > 0 \). Then the number of simple \( EG \)-modules is equal to the number of \( p \)-regular conjugacy classes of \( G \).

A crucial ingredient of the proof of the theorem is the following elementary lemma.

**Lemma 3.9.** Let \( \rho : G \rightarrow GL_n(k) \) be a matrix representation of \( G \) over \( k \). For any \( x \in G \), we may write \( x = us \), where \( u \) is a \( p \)-element of \( G \) and \( s \) is a \( p' \)-element. Then \( x \) and \( s \) have the same eigenvalues, counting multiplicities.

The lemma follows since \( su = us \), and all eigenvalues of \( u \) will equal 1.
4. Brauer characters for $H_k$

In this section we define Brauer characters for our bismash products and show that they have properties analogous to those for finite groups discussed in Section 3.

Assume that $\mathbb{L}, \mathbb{K}, \pi, R$ and $k$ are as Example 2.12, with $k = R/\pi R$.

Fix an irreducible $H_k$-module $V_L$ whose indicator is non-zero. Since $L$ is a splitting field for $H_Q$, so is $K$, and thus

$$V := V_L \otimes_L K$$

is an irreducible $H_K$-module. Moreover the bilinear form on $V_L$ extends to a bilinear form on $V$, and thus there is a non-singular $H_K$-invariant bilinear form $\langle \cdot, \cdot \rangle$ on $V$, with values in $K$, which is symmetric or skew-symmetric by 2.10.

Recall the basis $B$ of $H_K$ from Section 2.

**Definition 4.1.** Let $V = V_L \otimes_L K$ be as above. Then an $RB$-lattice in $V$ is a finitely generated $RB$-submodule $L$ of $V$ such that $KL = V$.

From now on we also assume that $\mathbb{L}$ denotes an algebraic number field which is a splitting field for $H_Q = \mathbb{Q}^G \# \mathbb{Q}F$. Then $k$ is a splitting field for $H_k$.

Let $\hat{W} = \hat{W}_x$ be a simple $H_k$ module, as in Proposition 2.7. That is, for a given $F$-orbit $O$ of $G$ and fixed $x \in O = O_x$, with $F_x$ the stabilizer of $x$ in $F$ and $W = W_x$ a simple $kF_x$-module, $\hat{W} = kF \otimes_{kF_x} W$. Recall $\hat{W}$ becomes an $H$-module via

$$(p_y \# a)[b \otimes w] = \delta_y \cdot (ab)_x[ab \otimes w],$$

for any $y \in G$, $a, b \in F$, and $w \in W$.

As in Lemma 2.9, fix a set $T_x$ of representatives of the right cosets of $F_x$ in $F$.

**Lemma 4.2.** Consider the action of $p_y \# a$ on $\hat{W} = \hat{W}_x$ as above.

1. If $(p_y \# a)\hat{W} \neq 0$, then there exists $t \in T_x$ and $w \in W$ such that

$$(p_y \# a)[t \otimes w] = \delta_y \cdot (at)_x[at \otimes w] \neq 0.$$

Thus $y = x \prec (at)^{-1} \in O_x$.

2. If $p_y \# a$ has non-zero eigenvalues on $\hat{W}_x$, then $a \in F_y$ and $x = y \prec t$, where $t$ is as in (1).

3. For $t$ as in (1) and (2), $t^{-1}at \in F_x$ and so $at \otimes w = t \otimes (t^{-1}at)w$.

**Proof.** (1) By the formula for the action of $p_y \# a$ on $\hat{W}$, there exists $b \in F$ and $w \in W$ such that $(p_y \# a)[b \otimes w] = \delta_y \cdot (ab)_x[ab \otimes w] \neq 0$. Thus $y = x \prec (ab)^{-1} \in O_x$. Now for some $t \in T_x$, $b \in tF_x$. It is easy to see that $t$ satisfies the same properties as $b$.

2. If $p_y \# a$ has non-zero eigenvalues on $\hat{W}$, then $(p_y \# a)^2 \neq 0$, and so $a \in F_y$ by (2.5). Now using (1), $x = y \prec (at) = (y \prec a) \prec t = y \prec t$.

3. Since $y = x \prec t^{-1}$ and $a \in F_y$, it follows that $t^{-1}at \in F_x$. Thus we can write $at \otimes w = t \otimes (t^{-1}at)w$. \qed

We next prove an analog of Lemma 3.9, although in our case the two factors do not necessarily commute in $H_k$.

**Lemma 4.3.** Consider $\rho : H_k \rightarrow \text{End}_k(\hat{W}) \cong M_n(k)$. For $a \in F$ write $a = su$, with $s$ the $p$-regular part and $u$ the $p$-part of $a$. Then $\rho(p_y \# a)$ and $\rho(p_y \# s)$ have the same eigenvalues, counting multiplicities.
Lemma 2.6(3), \( S \) the formula in Lemma 2.9. That is, if \( W \) then \( \hat{W} \) do commute: suppose \( b \otimes w \) is such that \( (p_y \# a) \cdot [b \otimes w] \neq 0 \). By Lemma 4.2, \( y \triangleleft b = x \) and \( a \in F_y \). Thus \( s \in F_y \) since \( s \) is a power of \( a \). Then

\[
(p_y \# s)(1 \# u) \cdot [b \otimes w] = (p_y \# a) \cdot [b \otimes w] = ab \otimes v
\]

and

\[
(1 \# u)(p_y \# s) \cdot [b \otimes w] = \delta_{y \triangleleft sb,x}(1 \# u) \cdot [sb \otimes w] = (1 \# u) \cdot [sb \otimes w] = usb \otimes w = ab \otimes w.
\]

Since the eigenvalues of \( 1 \# u \) are all 1, the eigenvalues of \( p_y \# a \) are the same as those of \( p_y \# s \).

The lemma shows that to find the character of some \( p_y \# a \), it suffices to look at the character of \( p_y \# s \), where \( s \) is the \( p' \)-part of \( a \). Moreover, by Lemma 4.2, the character of \( p_y \# a \) will be non-zero only if \( a \in F_y \).

Thus, as a replacement for the \( p' \)-elements of the group in the classical case, we consider the subset of the basis \( \mathcal{B} \) defined in (2.4) of those elements which are non-nilpotent element and have group element in \( F_{p'} \). That is, we define

\[
(4.4) \quad \mathcal{B}_{p'} := \{ p_y \# a \in \mathcal{B} \mid a \in F_{p'} \} = \{ p_y \# a \in \mathcal{B} \mid a \in F_y \cap F_{p'} \},
\]

where \( F_{p'} \) is the set of \( p \)-regular elements in \( F \). By Lemma 2.6, \( \mathcal{B}_{p'} \) is also closed under the antipode \( S \), since if \( a \in F_{p'} \) and \( w = p_y \# a \) is non-nilpotent, then by Lemma 2.6(3), \( S(w) = p_{y^{-1}} \cdot sa^{-1}y^{-1} \). Since \( ya^{-1}y^{-1} \) has the same order as \( a \), \( S(w) \) is also in \( \mathcal{B}_{p'} \).

The above remarks motivate our definition of Brauer characters for \( H_k \), by using the formula in Lemma 2.9. That is, if \( W = W_x \) is a simple \( kF_x \)-module with character \( \psi \), then the character of the simple \( H_k \)-module \( W \) is given by

\[
(4.5) \quad \hat{\psi}(p_y \# a) = \sum_{t \in T_x \text{ and } t^{-1}\mathfrak{t} \in F_x} \delta_{y \triangleleft t,x} \psi(t^{-1}a).
\]

**Definition 4.6.** Let \( W = W_x \) be a simple \( kF_x \)-module with character \( \psi \), and let \( \phi \) be the classical Brauer character of \( W \) constructed from \( \psi \). Then the **Brauer character** of \( \hat{W} \) is the function

\[
\hat{\phi} : \mathcal{B}_{p'} \to \mathbb{K}
\]

defined on any \( p_y \# a \in \mathcal{B}_{p'} \) by

\[
\hat{\phi}(p_y \# a) = \sum_{t \in T_x \text{ and } t^{-1}\mathfrak{t} \in F_x} \delta_{y \triangleleft t,x} \phi(t^{-1}a).
\]

**Remark 4.7.** If \( \hat{\phi} \) is a Brauer character, then also \( \hat{\phi}^* = \hat{\phi} \circ S \) is a Brauer character: namely if \( \hat{\phi} \) is the Brauer character for \( \hat{\psi} \), then \( \hat{\phi}^* \) is the Brauer character of \( \hat{\psi}^* \), using the fact that \( \mathcal{B}_{p'} \) is stable under \( S \).
We fix the following notation, as for groups:

1. \( \text{Irr}(H_K) = \{ \hat{\chi}_1, \ldots, \hat{\chi}_n \} \) denotes the irreducible characters of \( H_K \);
2. \( \text{Irr}_K(H_k) = \{ \hat{\psi}_1, \ldots, \hat{\psi}_d \} \) denotes the irreducible characters of \( H_k \);
3. \( \text{IBr}(H_k) = \{ \hat{\phi}_1, \ldots, \hat{\phi}_d \} \) denotes the Brauer characters corresponding to \( \{ \hat{\psi}_1, \ldots, \hat{\psi}_d \} \).

As for groups, the elements of \( \text{IBr}(H_k) \) are called irreducible Brauer characters.

4. \( \text{Bch}(H_k) \) denotes the ring of virtual Brauer characters of \( H_k \), that is, the \( \mathbb{Z} \)-linear span of the irreducible Brauer characters.

**Lemma 4.8.** \( \hat{\phi}_j \) is a \( \mathbb{K} \)-linear combination of the \( \hat{\chi}_i|_{B_{\nu'}} \). Consequently if all \( \hat{\chi}_i \) are self-dual, then all \( \hat{\phi}_j \) are also self-dual, and so are all \( \hat{\psi}_j \).

**Proof.** By Remark 3.2 applied to \( F_x \), the Brauer character \( \hat{\phi}_j \) may be written as \( \hat{\phi}_j = \sum_i \alpha_i \hat{\chi}_i|_{B_{\nu'}} \), for \( \alpha_i \in \mathbb{K} \). Lifting this equation through induction up to \( F_{\nu'} \) (and so to \( B_{\nu'} \)) as in Lemma 2.9, we obtain the first statement in the lemma.

Now if all \( \hat{\chi}_i \) are self-dual, then the same property holds for the \( \hat{\phi}_j \) since they are linear combinations of the \( \hat{\chi}_i|_{B_{\nu'}} \). Fix one of the \( \hat{\psi}_j \) and its Brauer character \( \hat{\phi}_j \). Since \( \hat{\phi}_j^* = \hat{\phi}_j \circ S \) and \( \hat{\psi}_j^* = \hat{\psi}_j \circ S \), using the formula for \( S \) as well as (4.5) and the formula in 4.6, we see that \( \hat{\phi}_j^* = \hat{\phi}_j \) if and only if \( \hat{\psi}_j^* = \hat{\psi}_j \). \( \square \)

We may follow exactly the proof of Proposition 3.3, that is [CR, 17.5, (2) - (4)], to show the following:

**Proposition 4.9.** (1) \( \hat{\phi} \) takes values in \( R \) and \( \hat{\phi}(p_y \# a) = Tr(p_y \# a, \hat{W}) \), for \( a \in F_{\nu'} \).
(2) Given \( H_k \)-modules \( \hat{W}_0 \supset \hat{W}_1 \supset 0 \), let \( \hat{\phi} \) be the Brauer character afforded by \( \hat{W}_0 / \hat{W}_1 \), \( \hat{\phi}_1 \) the Brauer character afforded by \( \hat{W}_1 \), and \( \hat{\phi}_0 \) the Brauer character of \( \hat{W}_0 \). Then \( \hat{\phi}_0 = \hat{\phi} + \hat{\phi}_1 \).
(3) Let \( V \) be a \( \mathbb{K}B \)-module with \( \mathbb{K} \)-character \( \chi \). Then for each \( RB \)-lattice \( M \) in \( V \), the restriction \( \chi|_{RB_{\nu'}} \) is the Brauer character of the \( H_k \)-module \( \overline{M} := M/pM \).

Similarly, one may follow the first part of the proof of Theorem 3.7 [CR, 17.9], replacing Lemma 3.9 with Lemma 4.3, to show

**Theorem 4.10.** The irreducible Brauer characters \( \text{IBr}(H_k) \) are \( \mathbb{K} \)-linearly independent.

We may also extend the decomposition map \( d \) in Section 3 to obtain a homomorphism of abelian groups

\[
(4.11) \quad \hat{d} : G_0(H_K) \to G_0(H_k),
\]

called the decomposition map, such that for any class \( [\hat{V}] \) in \( G_0(H_K) \), \( \hat{d}([\hat{V}]) = [\overline{M}] \in G_0(H_k) \), where \( M \) is any \( RB \)-lattice in \( \hat{V} \) and \( \overline{M} := M/pM \).

Again using the facts about groups, the set \( \text{Bch}(H_k) \) of virtual Brauer characters, that is \( \mathbb{Z} \)-linear combinations of Brauer characters of \( kG \)-modules, is a ring under addition and multiplication of functions, and \( \text{Bch}(H_k) \cong G_0(H_k) \). Using that
\( G_0(\mathbb{K}G) \cong \text{ch}(\mathbb{K}G) \), the ring of virtual characters of \( H_{\mathbb{K}} \), it follows that the decomposition map \( \hat{d} \) induces a map

\[
\hat{d}^\prime : \text{ch}(H_{\mathbb{K}}) \to B\text{ch}(H_{\mathbb{K}}),
\]

where \( \hat{d}^\prime \) is the restriction map \( \hat{\psi} \to \hat{\psi}|_{B_{\rho'}} \).

Consequently, Equation (3.5) implies that if \( \hat{\chi} \) is the character for \( \hat{V} \) and \( \hat{\chi}|_{B_{\rho'}} = \sum_j \alpha_j \hat{\phi}_j \), where \( \hat{\phi}_j \) is the Brauer character of the simple \( H_{\mathbb{K}} \)-module \( \hat{W}_j \), and \( \hat{d}(\hat{V}) = [M] \in G_0(H_{\mathbb{K}}) \), then

\[
(4.11) [M] = \sum_j \alpha_j [W_j].
\]

From now on we wish to distinguish the characters (over \( \mathbb{k} \) or \( \mathbb{K} \)) which arise from stabilizers of elements in different \( F \)-orbits of \( G \). Assume that there are exactly \( s \) distinct orbits of \( F \) on \( G \) and that we fix \( x_q \in O_q \), the \( q \)th orbit. Thus for a fixed \( x = x_q \in G \) with stabilizer \( F_x = F_{x_q} \), we will write \( \chi_{i,x} \) for an irreducible character of \( kF_x \), and \( \hat{\chi}_{i,x} \) for its induction up to \( \mathbb{K}F_x \), which becomes an irreducible character of \( H_{\mathbb{K}} \).

Similarly, \( \psi_{j,x} \) denotes an irreducible character of \( kF_x \), and \( \hat{\psi}_{j,x} \) its induction up to \( \mathbb{K}F_x \), which becomes an irreducible character of \( H_{\mathbb{K}} \). Also \( \phi_{j,x} \) denotes the Brauer character corresponding to \( \psi_{j,x} \), and \( \hat{\phi}_{j,x} \) the Brauer character corresponding to \( \hat{\psi}_{j,x} \).

**Lemma 4.12.** Let \( \phi_x \) be a virtual Brauer character of \( kF_x \) and assume that \( \phi_x = \sum_j z_{j,x} \phi_{j,x} \), where as above the \( \phi_{j,x} \) are the Brauer characters of \( kF_x \).

Then \( \hat{\phi}_x = \sum_j z_{j,x} \hat{\phi}_{j,x} \).

The lemma follows from Definition 4.6 of a Brauer character \( \hat{\phi} \) for \( H_{\mathbb{K}} \) in terms of a Brauer character \( \phi \) for \( kF_x \). Moreover, Lemma 2.9 becomes

\[
\hat{\chi}_{i,x}(p_y \# a) = \sum_{t \in T_x \text{ and } t^{-1} at \in F_x} \delta_{y \triangleleft t,x} \chi_{i,x}(t^{-1} at).
\]

Applying Equation (3.5) to \( F_x \), there are integers \( d_{ij,x} \) such that

\[
\chi_{i,x}|(F_x)_{\rho'} = \sum_j d_{ij,x} \phi_{j,x},
\]

where the \( \phi_{j,x} \) are in \( IBr(kF_x) \).

Lifting the \( \chi_{i,x} \) to \( \hat{\chi}_{i,x} \) on \( B_{\rho'} \), we see that

\[
\hat{\chi}_{i,x}|_{B_{\rho'}} = \sum_j d_{ij,x} \hat{\phi}_{j,x}.
\]

That is, the decomposition numbers for the \( \hat{\chi}_{i,x}|_{B_{\rho'}} \) with respect to the \( \hat{\phi}_{j,x} \) are the same as the decomposition numbers for the \( \chi_{i,x}|(F_x)_{\rho'} \) with respect to the \( \phi_{j,x} \) for the group \( F_x \). Thus the decomposition matrix \( \hat{D}_x = [d_{ij,x}] \) for the \( \hat{\chi}_{i,x}|_{B_{\rho'}} \) with respect to the \( \hat{\phi}_{j,x} \) is the same as the decomposition matrix \( D_x \) for the \( \chi_{i,x}|(F_x)_{\rho'} \) with respect to the \( \phi_{j,x} \).
The above discussion proves

**Proposition 4.13.** As above, assume that there are exactly \( s \) distinct orbits \( \mathcal{O} \) of \( F \) on \( G \) and choose \( x_q \in \mathcal{O}_q \), for \( q = 1, \ldots, s \). Then

1. \( \hat{D}_{x_q} = D_{x_q} \)

2. The decomposition matrix for the \( \hat{\chi}_{i|B\rho} \) with respect to \( \hat{\phi}_j \) is the block matrix

\[
\hat{D} = \begin{bmatrix}
\hat{D}_{x_1} & 0 & \cdots & 0 \\
0 & \hat{D}_{x_2} & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & \hat{D}_{x_s}
\end{bmatrix}
\]

where \( \hat{D}_{x_q} \) is the decomposition matrix of \( \hat{\chi}_{i,x_q|B\rho} \) with respect to \( \hat{\phi}_{j,x_q} \).

As for groups, \( \hat{C} = \hat{D}^t \hat{D} \) is called the Cartan matrix. We are now able to extend the theorem of Brauer we need (3.6).

**Theorem 4.14.** \( \text{Det}(\hat{C}) \) is a power of \( p \).

**Proof.** First, \( \hat{C} \) is also a block matrix, with blocks \( \hat{C}_{x_q} = (\hat{D}_{x_q})^t \hat{D}_{x_q} \). By Brauer’s theorem applied to each group \( F_{x_q} \), we know that \( \text{Det}(\hat{C}_{x_q}) \) is a power of \( p \). Thus \( \text{Det}(\hat{C}) \) is a power of \( p \). \( \square \)

### 5. Invariant Forms: Witt kernels and Lattices

A first step in the direction of extending Thompson’s theorem concerns the Witt kernel of a module with a bilinear form as in Theorem 2.10. We will show that, for an arbitrary field \( \mathbb{E} \), the notion of Witt kernel of an \( \mathbb{E}G \)-module extends to \( H_\mathbb{E} \)-modules. One can then follow the argument in [Th].

Let \( V \) be a finitely-generated \( H_\mathbb{E} \)-module which is equipped with a non-degenerate \( H_\mathbb{E} \)-invariant bilinear form \( \langle -, - \rangle : V \otimes_\mathbb{E} V \to \mathbb{E} \), which is either symmetric or skew symmetric. For example, if \( \mathbb{E} \) is algebraically closed, then any irreducible self-dual \( H_\mathbb{E} \)-module has such a form by Theorem 2.10. For any submodule \( U \) of \( V \),

\[
U^\perp = \{ v \in V \mid \langle v, U \rangle = 0 \}.
\]

Since the form is \( H \)-invariant and \( S \) is the adjoint of the form, for all \( u \in U^\perp \),

\[
\langle h \cdot v, U \rangle = \langle v, S(h) \cdot U \rangle = \langle v, U \rangle = 0.
\]

Thus \( U^\perp \) is also a submodule of \( V \). Note also that \( U^{\perp \perp} = U \) since \( V \) is finite-dimensional over \( \mathbb{E} \). Let

\[
\mathcal{M} = \mathcal{M}_V = \{ V_0 \mid V_0 \text{ is an } H_\mathbb{E} \text{-submodule of } V \text{ and } \langle V_0, V_0 \rangle = 0 \},
\]

that is, \( V_0 \subsetneq V_0^\perp \).

Obviously, \( \{ 0 \} \in \mathcal{M} \), and \( \mathcal{M} \) is partially ordered by inclusion. If \( V_0 \in \mathcal{M} \), then \( V_0^\perp / V_0 \) inherits a non-degenerate form given by

\[
\langle v_0 + V_0, v_0' + V_0 \rangle_{V_0^\perp / V_0} := \langle v_0, v_0' \rangle, \quad v_0, v_0' \in V_0.
\]
If \( V_1 \) is a maximal element of \( \mathcal{M} \), it is not difficult to see that \( \frac{V_1^\perp}{V_1} \) is a completely reducible \( H_E \)-module, and the restriction of \(( , )_{V_1^\perp/V_1} \) to any \( H_E \)-submodule of \( \frac{V_1^\perp}{V_1} \) is non-degenerate.

**Definition 5.1.** Let \( V_1 \) be a maximal element of \( \mathcal{M} \). Then the **Witt kernel** of \( V \) is 
\[ V' := \frac{V_1^\perp}{V_1}. \]

It is not clear from this definition that the Witt kernel is independent of the choice of the maximal element of \( \mathcal{M} \). However, we have

**Lemma 5.2.** If \( V_1, V_2 \) are maximal elements of \( \mathcal{M} \), then there is an \( H_\mathbb{k} \)-isomorphism 
\[ \Phi : \frac{V_1^\perp}{V_1} \to \frac{V_2^\perp}{V_2}, \]

such that 
\[ (v_1, v'_1)v_{V_1^\perp/V_1} = (\Phi(v_1), \Phi(v'_1))v_{V_2^\perp/V_2}, \text{ for all } v_1, v'_1 \in V_1^\perp/V_1. \]

The proof follows exactly the proof of [Th, Lemma 2.1] for group algebras.

We next extend the facts shown in [Th] about \( \mathbb{G} \)-lattices to the case of lattices for bismash products. Our proofs follow [Th] very closely.

Assume that \( \mathbb{L}, \mathbb{K}, \pi, \mathbb{R} \) and \( \mathbb{k} \) are as at the end of Section 2, with \( \mathbb{k} = \mathbb{R}/\pi\mathbb{R}. \)

Fix an irreducible \( H_\mathbb{L} \)-module \( V_\mathbb{L} \) whose indicator is non-zero. Since \( \mathbb{L} \) is a splitting field for \( H_\mathbb{Q} \), so is \( \mathbb{K} \), and thus
\[ V := V_\mathbb{L} \otimes_\mathbb{L} \mathbb{K} \]
is an irreducible \( H_\mathbb{K} \)-module. Moreover the bilinear form on \( V_\mathbb{L} \) extends to a bilinear form on \( V \), and thus there is a non-singular \( H_\mathbb{K} \)-invariant bilinear form \( \langle , \rangle \) on \( V \), with values in \( \mathbb{K} \), which is symmetric or skew-symmetric by 2.10.

Recall the basis \( \mathcal{B} \) of \( H_\mathbb{K} \) from Section 2.

**Definition 5.3.** Let \( V = V_\mathbb{L} \otimes_\mathbb{L} \mathbb{K} \) as above. Then an **\( \mathbb{R}\mathbb{B} \)-lattice** in \( V \) is a finitely generated \( \mathbb{R}\mathbb{B} \)-submodule \( L \) of \( V \) such that \( \mathbb{K}L = V. \)

Let \( \mathcal{L} = \mathcal{L}_V \) be the family of \( \mathbb{R}\mathbb{B} \)-sublattices of \( V \). If \( L \in \mathcal{L} \), then \( L^* \) denotes the **dual lattice** defined by 
\[ L^* = \{ l \in V | \langle L, l \rangle \subseteq R \}. \]

Since \( R \) is Noetherian, \( L^* \) is also an \( \mathbb{R}\mathbb{B} \)-lattice by [CR, 4.24]. In particular \( L^* \) is also finitely-generated. We also let 
\[ \mathcal{L}_I = \mathcal{L}_{V,I} = \{ \mathbb{R}\mathbb{B} \text{-lattices } L \in \mathcal{L}_V | \langle L, L \rangle \subseteq R \} \]
denote the set of integral lattices. If \( L \) is any element of \( \mathcal{L} \), there is an integer \( n \) such that \( \pi^n L \in \mathcal{L}_I \). Obviously, \( \mathcal{L}_I \) is partially ordered by inclusion and if \( L_1, L_2 \in \mathcal{L}_I \) with \( L_1 \subseteq L_2 \), then \( L_2 \subseteq L_1^* \). Thus any chain of sublattices starting with \( L_1 \) is contained in \( L_1^* \), which is a Noetherian \( \mathbb{R}\mathbb{B} \)-module, and so the chain must stop. Thus every element of \( \mathcal{L}_I \) is contained in a maximal element of \( \mathcal{L}_I \).

In the following discussion, \( L \) denotes a **fixed** maximal element of \( \mathcal{L}_I \).

**Lemma 5.4.** (1) \( \pi L^* \subseteq L. \)

(2) Let \( M = L^*/L. \) There is a non-singular \( \mathbb{R}\mathbb{B} \)-invariant form \( \langle , \rangle_M \) on \( M \), with values in \( \mathbb{k} \), defined as follows: if \( m_1, m_2 \in M, m_i = x_i + L \) then \( \langle m_1, m_2 \rangle_M := \text{image in } \mathbb{k} \text{ of } \pi \langle x_1, x_2 \rangle. \)
Proof. (1) Let $h$ be the smallest integer $\geq 0$ such that $\pi^hL^* \subseteq L$. If $h \leq 1$, then (1) holds. So suppose $h \geq 2$.

Let $L_1 = L + \pi^{h-1}L^*$. Then $L_1 \in \mathcal{L}$. Moreover, if $u_1, u_2 \in L_1$, say $u_i = l_i + \pi^{h-1}l_i^*$, $l_i \in L, l_i^* \in L^*$, then

$$\langle u_1, u_2 \rangle = \langle l_1, l_2 \rangle + \pi^{h-1}(\langle l_1^*, l_2^* \rangle + \langle l_2^*, l_1^* \rangle) + \pi^{h-2}\langle \pi^h l_i^*, l_2^* \rangle \in R$$

by definition of $L^*$ and of $h$. Thus, $L_1 \in \mathcal{L}_I$. Since $L \subseteq L_1$, this violates the maximality of $L$. So (1) holds.

(2) If $l_1^*, l_2^* \in L^*$, then $\pi l_1^* \in L$, so $\langle \pi l_1^*, l_2^* \rangle \in R$. Since $\langle L, L^* \rangle$ and $\langle L^*, L \rangle$ are contained in $R$, and since $\pi$ is a generator for the maximal ideal of $R$, it follows that $\langle \cdot, \cdot \rangle_M$ is well defined. To see that this form is non singular, suppose $l^* \in L^*$ and $\langle l^*, L^* \rangle = 0$. Then $l^* \in L^{**} = L$, so $l^* + L = 0$ in $M$. This proves (2).

Lemma 5.5. \{l \in L \mid \langle l, L \rangle \subseteq \pi R\} = \pi L^*.

Proof. By Lemma 5.4(1), $\pi L^* \subseteq L$. By definition of $L^*$, $\pi L^* \subseteq \{l \in L \mid \langle l, L \rangle \subseteq \pi R\}$. Thus it suffices to show that if $l \in L$ and $\langle l, L \rangle \subseteq \pi R$, then $l \in \pi L^*$. This is clear, since $\langle 1/\pi l, L \rangle \subseteq R$, so that by the definition of $L^*$, we have $1/\pi l \in L^*$.

6. Indicators and Brauer characters

In this section we combine our work in the previous sections to prove the analog of a theorem of Thompson.

Theorem 6.1. Thompson [Th] Let $k$ be an algebraically closed field of odd characteristic, let $G$ be a finite group, and let $W$ be an irreducible $kG$-module. If $W$ has non-zero Frobenius-Schur indicator, then $W$ is a composition factor (of odd multiplicity) in the reduction mod $p$ of an irreducible $kG$-module with the same indicator as $W$.

By reduction mod $p$, we mean to use the $p$-modular system $(k, R, k)$ as described in Example 2.12, and then the induced decomposition map as in (4.11).

We first prove the analog of [Th, Lemma 3.3]. Recall the notation in Section 4:

$V_L$ is a fixed irreducible $H_L$-module which is self-dual and thus $V = V_L \otimes_L k$ is an irreducible self-dual $H_k$-module, with character $\chi$. $V$ has a non-degenerate $H_k$-invariant bilinear form $\langle \cdot, \cdot \rangle$ with values in $k$, which is symmetric or skew-symmetric by Theorem 2.10.

As in Section 5, $\mathcal{L}$ is the family of $RB$-sublattices of $V$ and $\mathcal{L}_I$ is the subset of integral lattices. Let $L$ denote a fixed maximal element of $\mathcal{L}_I$ with dual lattice $L^*$. Consider the following $H_k$-modules: let $X = L^*/\pi L^*, Y = L/\pi L^*, Z = L^*/L$. Note that $Y$ is a submodule of $X$. Then there is an exact sequence of $H_k$-modules

$$0 \to Y \to X \to Z \to 0.$$
Proposition 6.2. Let $V$, $L$, $X$, $Y$, and $Z$ be as above. Suppose that $P$ is an irreducible $H_k$-module with Brauer character $\hat{\phi}$, such that

1. $\hat{\phi}^* = \hat{\phi}$;
2. $d(\hat{\chi}|_{B_P}, \hat{\phi})$ is odd.

Let $Y', Z'$ be the Witt kernels of $Y$, $Z$ respectively. Then the multiplicity of $P$ in $Y' \oplus Z'$ is odd. Consequently $P$ has the same type as $V$.

Proof. We let $M = L^*$; then $X = L^*/\pi L^* = M/pM$. By Proposition 4.9, the restriction $\chi|_{RB_P}$ is the Brauer character of the $H_k$-module $M := M/pM$.

By hypothesis, the multiplicity of $\hat{\phi}$ in $\hat{\chi}|_{B_P}$ is odd, and thus using the decomposition map, the multiplicity of $P$ in $X = M$ is odd. Thus the multiplicity of $P$ in $Y \oplus Z$ is odd. Since $\hat{\phi} \circ S = \hat{\phi}$, it follows from the definition of Brauer characters that also $\hat{\psi} \circ S = \hat{\psi}$ on $P$, and so $P \cong P^*$ as $H_k$-modules.

As in Section 5, let $Y_1$ be an $H_k$-submodule of $Y$ which is maximal subject to $\langle Y_1, Y_1 \rangle_Y = 0$. Then the multiplicity of $P$ in $Y_1$ equals the multiplicity of $P$ in $Y_1^\perp$ (since $Y_1^* \cong Y_1^\perp$ and $P^* = P$), and so the parity of the multiplicity of $P$ in $Y$ equals the parity of the multiplicity of $P$ in the Witt kernel $Y' = Y_1^\perp/Y_1$.

The same argument applies to $Z$, and thus the multiplicity of $P$ in $Y' \oplus Z'$ is odd. For the second part, by Section 5 we know that $Y''$ is completely reducible, and thus if $P$ appears in $Y''$, the non-degenerate bilinear form on $Y''$ restricts to a non-degenerate form on $P$. By uniqueness, this form must agree with the given form on $P$, and thus $P$ and $Y''$ have the same type.

Similarly, if $P$ appears in $Z'$, then $P$ and $V$ have the same type. But since $P$ appears an odd number of times in $Y' \oplus Z'$, it must appear in either $Y'$ or $Z'$.

Theorem 6.3. Let $\hat{P}$ be a self-dual simple $H_k$-module, and let $\hat{\phi}$ be its Brauer character. Then there is an irreducible $\mathbb{K}$-character $\hat{\chi}$ of $H_{\mathbb{K}}$ such that

1. $\hat{\chi}^* = \hat{\chi}$, and
2. $d(\hat{\chi}|_{B_{\mathbb{K}}}, \hat{\phi})$ is odd.

Moreover if $\hat{\chi}$ is any irreducible $\mathbb{K}$-character of $H_{\mathbb{K}}$ satisfying (1) and (2), then $\nu(\hat{\chi}) = \nu(\hat{P})$.

To prove the theorem, it will suffice to show that $\hat{\chi}$ exists, since the equality $\nu(\hat{\chi}) = \nu(\hat{P})$ follows from Proposition 6.2.

We follow the outline of Thompson’s argument, although we must look at the $RF_x$-blocks separately. We know that for some $x = x_q$, $\hat{P}$ is induced from a simple $kF_x$-module $P$. Let $B_x$ be the block of $RF_x$ containing the Brauer character $\phi$ of $P$, let $\{\chi_1, \ldots, \chi_m\}$ be all of the irreducible $\mathbb{K}$-characters in $B_x$, and let $\{\phi_1, \ldots, \phi_n\}$ be all of the irreducible Brauer characters in $B_x$.

Let $D_x$ be the decomposition matrix of the $\chi_i$ with respect to the $\phi_j$, and $C_x = D_x^tD_x$ the Cartan matrix. From Brauer’s theorem 3.6, $Det(C_x)$ is a power of $p$ and so is odd since $p$ is odd.

Lifting this set-up to $H_{\mathbb{K}}$, $\hat{B}_x$ is the block of $RB$ containing the Brauer character $\hat{\phi}$ of $\hat{P}$, $\{\hat{\chi}_1, \ldots, \hat{\chi}_m\}$ are all the irreducible $\mathbb{K}$-characters in $\hat{B}_x$, and $\{\hat{\phi}_1, \ldots, \hat{\phi}_n\}$ are all of the irreducible Brauer characters in $\hat{B}_x$. 
Choose notation so that \( n = 2n_1 + n_2 \), where \( \{ \hat{\phi}_1, \hat{\phi}_2 \}, \{ \hat{\phi}_3, \hat{\phi}_4 \}, \ldots, \{ \hat{\phi}_{2n_1-1}, \hat{\phi}_{2n_1} \} \), are pairs of non self-dual characters, that is, \( (\hat{\phi}_{2i-1})^* = \hat{\phi}_{2i} \), and \( \hat{\phi}_{2n_1+1}, \ldots, \hat{\phi}_n \) are self-dual. By hypothesis, \( n_2 \neq 0 \) since \( \hat{\phi} \) is one of the \( \hat{\phi}_i \).

Write \( C_x \) in block form as \( C_x = \begin{bmatrix} C_0 & C_2 \\ C_2^t & C_1 \end{bmatrix} \), where \( C_0 \) is \( 2n_1 \times 2n_1 \) and \( C_1 \) is \( n_2 \times n_2 \).

The Theorem will now follow from the next two lemmas:

**Lemma 6.4.** \( \text{Det}(C_1) \) is odd.

**Proof.** For \( i = 1, 2, \ldots, n \), let \( P_i \) be the projective indecomposable \( kF \)-module whose socle has Brauer character \( \phi_i \), and let \( \Phi_i \) be the Brauer character of \( P_i \). Then \( c_{ij} = (\Phi_i, \Phi_j) \).

Let \( \sigma = (1, 2)(3, 4) \cdots (2n_1 - 1, 2n_1) \in S_n^* \); also let \( \sigma \) denote the corresponding permutation matrix. Let \( \tilde{S}_n \) be the set of all permutations in \( S_n \) which do not fix \( \{ 2n_1 + 1, \ldots, n \} \). Since \( \tilde{S}_n \) is the complement in \( S_n \) of the centralizer of \( \sigma \), it follows that \( \sigma^{-1} \tilde{S}_n \sigma = \tilde{S}_n \), and \( \sigma \) has no fixed points on \( \tilde{S}_n \).

Looking at the matrix \( C_x \), it follows that \( \sigma^{-1}C_x \sigma = C_x \) since \( (\hat{\phi}_i)^* = \hat{\phi}_{i+1} \) for \( i = 1, 3, \ldots, 2n_1 - 1 \) and \( (\hat{\phi}_i)^* = \hat{\phi}_i \) for \( i = 2n_1 + 1, \ldots, n \). Then

\[
\text{Det}(C_x) = \text{Det}(C_0)\text{Det}(C_1) + \sum_{\tau \in \tilde{S}_n} \text{sgn}(\tau)c_{1\tau(1)}c_{2\tau(2)} \cdots c_{n\tau(n)}.
\]

Moreover \( c_{ij} = c_{\sigma(i)\sigma(j)} \), again since \( \sigma^{-1}C_x \sigma = C_x \). Choose \( \tau \in \tilde{S}_n \) and set \( \tau' = \sigma \tau \sigma \). Then \( \tau' \neq \tau \), and it follows that

\[
\prod_{i=1}^{n} c_{\tau(i)} = \prod_{i=1}^{n} c_{\tau'(i)}.
\]

Thus \( \text{Det}(C_x) \equiv \text{Det}(C_0)\text{Det}(C_1) \pmod{2} \). This proves the Lemma. \( \square \)

**Lemma 6.5.** For each \( j = 2n_1 + 1, \ldots, n \), there exists \( i \in \{ 1, 2, \ldots, m \} \) such that \( \hat{\chi}_i^* = \hat{\phi}_i \) and the decomposition number \( d_{ij} = d(\hat{\chi}_i|_{B_j}, \hat{\phi}_j) \) is odd.

**Proof.** Let \( m = 2m_1 + m_2 \), where the notation is chosen so that \( \{ \hat{\chi}_1, \hat{\chi}_2 \}, \{ \hat{\chi}_3, \hat{\chi}_4 \}, \ldots, \{ \hat{\chi}_{2m_1-1}, \hat{\chi}_{2m_1} \} \), are pairs of non self-dual characters, that is, \( (\hat{\chi}_{2i-1})^* = \hat{\chi}_{2i} \), and \( \hat{\chi}_{2m_1+1}, \ldots, \hat{\chi}_m \) are self-dual.

Suppose \( d_{ij} \equiv 0 \pmod{2} \), for all \( i = 2m_1+1, \ldots, m \). Then for each \( k \in \{ 1, 2, \ldots, n \} \), we have

\[
c_{jk} = \sum_{i=1}^{m} d_{ij}d_{ik} \equiv \sum_{i=1}^{2m_1} d_{ij}d_{ik}.
\]

On the other hand, \( \phi_j^* = \phi_j \) and if \( k \in \{ 2n_1 + 1, \ldots, n \} \) then \( \phi_k^* = \phi_k \) and so

\[
d_{ij} = d_{i+1,k}, \quad d_{ik} = d_{i+1,k}, \quad i = 1, 3, \ldots, 2m_1 - 1,
\]

hence \( c_{jk} \equiv 0 \pmod{2} \), for all such \( k \). This means that some row of \( C_1 \) consists of even entries. This violates the previous lemma. \( \square \)
As in [GM, Theorem 4.4], we have the following consequence:

**Corollary 6.6.** Consider the bismash products as above.

1. If all irreducible $H_C$-modules have indicator +1, the same is true for all irreducible $H_k$-modules.
2. If all irreducible $H_C$-modules have indicator 0 or 1, the same is true for all irreducible $H_k$-modules.
3. If all irreducible $H_C$-modules are self dual, the same is true for all irreducible $H_k$-modules.

**Proof.** (3) This follows by Lemma 4.8.

Now consider (2). By Theorem 6.3 there are no irreducible $kG$-modules $V$ with $\nu(V) = -1$. So (2) follows immediately.

Now (1) follows by (2) and (3).

□

7. Applications to the symmetric group

In this section we apply the results of Section 6 to bismash products constructed from some specific groups.

Let $S_n$ be the symmetric group of degree $n$, consider $S_{n-1} \subset S_n$ by letting any $\sigma \in S_{n-1}$ fix $n$, and let $C_n = \langle z \rangle$, the cyclic subgroup of $S_n$ generated by the $n$-cycle $z = (1, 2, \ldots, n)$. Then $S_n = S_{n-1}C_n = C_nS_{n-1}$ shows that $Q = S_n$ is factorizable. Thus we may construct the bismash product $H_n, E := E C_n \# E S_{n-1}$. It was shown in [JM] that if $E$ is algebraically closed of characteristic 0, then $H_n$ is totally orthogonal; that is, every irreducible module has indicator +1.

**Corollary 7.1.** Let $k$ be algebraically closed of characteristic $p > 0$ and let $H_{n,k} := kC_n \# kS_{n-1}$. Then $H_{n,k}$ is totally orthogonal.

**Proof.** Apply Corollary 6.6 to the characteristic 0 result of [JM] mentioned above. □

**Remark 7.2.** In [GM] it is proved that $D(G)$ is totally orthogonal for any finite real reflection group $G$ over any algebraically closed field. Corollary 6.6 shows that this result in characteristic $p > 0$ follows from the case of characteristic 0, which is somewhat easier to prove. When $G = S_n$, the characteristic 0 case was shown in [KMM].

We close with a question.

**Question 7.3.** It would be interesting to know if our results could be extended to bicrossed products. However to extend our proof one would need a theory of Brauer characters for twisted group algebras (that is, for projective representations) which includes a version of Brauer’s theorem on the Cartan matrix.

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