ON THE COSSEERAT MODEL FOR THIN RODS MADE OF THERMOELASTIC MATERIALS WITH VOIDS

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Abstract. In this paper we employ a Cosserat model for rod-like bodies and study the governing equations of thin thermoelastic porous rods. We apply the counterpart of Korn’s inequality in the three–dimensional elasticity theory to prove existence and uniqueness results concerning the solutions to boundary value problems for thermoelastic porous rods, both in the dynamical theory and in the equilibrium case.

1. Introduction. In the field of mathematical elasticity the existence theory is based on Korn’s inequality, which has a crucial importance, see e.g. [10, 11]. In three-dimensional linear elasticity, this inequality asserts that there exists a constant $c > 0$ such that

$$
\int_D \left[ u_{i,j} + \varepsilon_{ij}(\mathbf{u}) \varepsilon_{ij}(\mathbf{u}) \right] dv \geq c \int_D \left[ u_{i,i} + u_{i,j} u_{i,j} \right] dv, \quad \forall \mathbf{u} = (u_1, u_2, u_3) \in H^1(D),
$$

(1)

where $D$ is the domain occupied by the body and $\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2} (u_{i,j} + u_{j,i})$ represents the strain tensor. The subject has been extensively treated in [11, 12], where the inequalities of Korn type for deformable surfaces have been established and applied to analyze the boundary–value problems of linear shell theory. Other studies leading to inequalities of Korn type for different solid mechanics models have been reported in [19, 20, 2, 3]. The inequalities of Korn type for various models of rod-like structures have been presented in several works like [22, 27, 5].

To describe the mechanical behavior of thermoelastic curved rods, we employ in this paper a Cosserat model for rod-like bodies [13], in which the thin body is modeled as a deformable curve with a triad of rigidly rotating orthonormal vectors attached to every material point. This approach (also called the theory of directed curves) has been established by Zhilin [29, 30], who presented also the appropriate constitutive equations. It is related to the so-called theory of Cosserat curves.
developed by Green and Naghdi [16]. Our approach is different from the model of Cosserat curves, since the later is based on the assumption that each material point of the curve is connected to a set of deformable directors [24].

After describing the kinematical model and the appropriate deformation measures in Section 2, we present an inequality which is the counterpart of the Korn inequality (1), written for the theory of directed curves [5]. In Section 3 we present the basic field equations for thermoelastic porous rods, where the porosity of the rods is described using the theory of elastic materials with voids elaborated by Nunziato and Cowin [21, 14, 4]. We show that the new Korn–type inequality is useful to prove the existence and uniqueness of weak solutions to the dynamical equations of porous rods, using the method of semigroup of operators. In Section 4 we consider the equilibrium theory and employ again the Korn–type inequality to investigate the equations governing the deformation of porous rods, written in a weak variational form. The important case of boundary–value problems with pure traction boundary conditions is treated in Section 5, where the necessary and sufficient condition for the existence of solutions is obtained.

These results show that the approach of directed rods is mathematically well-formulated, and it is an appropriate model for investigating the mechanical behavior of thermoelastic porous rods. Some applications of this direct approach to rods for solving practical mechanical problems of thin structures have been presented in [30, 4, 1].

2. Basic equations for thermoelastic porous rods. As a model for thin rods we consider deformable curves with a triad of rigidly rotating vectors (also called directors) attached to each point. Let us denote by \( C \) the reference (initial) configuration of the deformable curve, and let \( r(s) \) be the position vector of any point of \( C \) and \( d_i(s), i = 1, 2, 3, \) be the three directors such that \( d_i \cdot d_j = \delta_{ij} \) (\( \delta_{ij} \) is the Kronecker symbol). Here, \( s \in [0, l] \) stands for the material coordinate along the curve, which coincides with the arclength parameter of \( C \). The total length of the curve is denoted by \( l \), while the derivative with respect to \( s \) is written as a prime: \( (\ )' = \frac{d}{ds} \). In what follows, we employ the direct tensor notation [18, 17] and the usual summation convention over repeated indices. A superposed dot denotes the material time derivative.

In the reference configuration, the director \( d_3 \) is taken to be the unit tangent to the curve \( C \), while \( d_1 \) and \( d_2 \) belong to the normal plane. The directors \( d_1 \) and \( d_2 \) are usually chosen along the principal axes of inertia for any cross-section. We also consider the orthonormal triad \( \{ t_1, t_2, t_3 \} \) attached to any point of the curve \( C \), consisting in the unit tangent vector \( t_3 \), the principal normal \( t_1 \) and the binormal vector \( t_2 \). Thus, we have \( d_3 = t_3 = r'(s) \), and the angle between \( d_1(s) \) and \( t_1(s) \) is called the angle of natural twisting. If we introduce the Darboux vector \( \tau \) given by

\[
\tau = \frac{1}{R_c} t_2 + \frac{1}{R_t} t_3, \quad \frac{1}{R_c} = |r''(s)|, \quad \frac{1}{R_t} = \frac{|(r', r'', r''')|}{|r''|^2},
\]

where \( R_c \) and \( R_t \) are the radii of curvature and twisting, respectively. Then the Frenet formulas can be written as

\[
t'_i = \tau \times t_i, \quad i = 1, 2, 3.
\]
is denoted by \( C_t \) and it is characterized by the position vector \( R(s,t) \) and the directors \( D_i(s,t), i = 1, 2, 3 \). The triad \( D_i(s,t) \) remains orthonormal after deformation, but the director \( D_3 \) is not necessarily tangent to the curve \( C_t \). Thus, the motion of the directors describes the rotations of the cross-sections of the rod during deformation.

The rotation tensor \( P \), the velocity vector \( v \) and the angular velocity vector \( \omega \) are defined by the relations

\[
P(s,t) = D_k(s,t) \otimes d_k(s), \quad v(s,t) = \dot{R}(s,t), \quad \dot{P}(s,t) = \omega(s,t) \times P(s,t).
\]

The last equation shows that \( \omega \) is the axial vector of the antisymmetric tensor \( P \cdot P^T \).

In the theory of directed curves, the deformation is described by means of the vectors of deformation \([29, 30]\): the vector of extension–shear \( \mathbf{E} \) and the vector of bending–twisting \( \Phi \) are defined by the relations

\[
\mathbf{E} = R' - P \cdot t_3, \quad P' = \Phi \times P \quad \text{(i.e. } \Phi = \text{axial}(P' \cdot P^T)),
\]

while the energetic vectors of deformation \( \mathbf{E}_* \) and \( \Phi_* \) are given by

\[
\mathbf{E}_* = P^T \cdot \mathbf{E}, \quad \Phi_* = P^T \cdot \Phi.
\]

The geometrical interpretations of these deformation vectors and their relation to the corresponding quantities in simplified beam theories have been presented in \([30]\), Section 5. We mention that the same measures of deformation have also been considered in other different approaches to rod theory, such as \([26, 25]\).

In the linear theory, the displacements and rotations associated to the deformation of rods are assumed to be infinitesimal. The displacement vector is \( u(s,t) = R(s,t) - r(s) \), and we can show the existence of the vector of small rotations \( \psi(s,t) \) such that

\[
P(s,t) = 1 + \psi(s,t) \times 1, \quad \omega(s,t) = \dot{\psi}(s,t), \quad \Phi(s,t) = \psi'(s,t),
\]

where \( 1 \) is the unit tensor of second order. The kinematical independent variables in the linear theory are \( u(s,t) \) and \( \psi(s,t) \). In this case, the vectors of deformation are denoted by \( \varepsilon \equiv \mathbf{E} = \mathbf{E}_* \) for extension–shear and by \( \kappa \equiv \Phi = \Phi_* \) for bending–twisting, and they are expressed by

\[
\varepsilon = u' + t_3 \times \psi, \quad \kappa = \psi'.
\]

Let us decompose the vectors \( u, \psi, \varepsilon \) and \( \kappa \) in the vector basis \{ \( t_1, t_2, t_3 \) \}, i.e. \( u = u_i \ t_i, \psi = \psi_i \ t_i, \varepsilon = \varepsilon_i \ t_i \) and \( \kappa = \kappa_i \ t_i \). Using the definition (4) and the Frenet formulas (3), we obtain the following expressions for the components of the deformation vectors

\[
\varepsilon_1 = u'_1 - \frac{u_2}{R_c} + \frac{u_3}{R_c} - \psi_2, \quad \varepsilon_2 = u'_2 + \frac{u_1}{R_c} + \psi_1, \quad \varepsilon_3 = u'_3 - \frac{u_1}{R_c},
\]

\[
\kappa_1 = \psi'_1 - \frac{\psi_2}{R_c}, \quad \kappa_2 = \psi'_2 + \frac{\psi_1}{R_c}, \quad \kappa_3 = \psi'_3 - \frac{\psi_1}{R_c}.
\]

The relations (8) will be useful to formulate and to prove the inequalities of Korn type for our model. Throughout this paper, we shall decompose any vector \( f \) in the same vector basis \{ \( t_1, t_2, t_3 \) \} (whenever it is necessary) and we denote its components by \( f_i = f \cdot t_i \).

Let us present the inequalities of Korn type corresponding to the deformation of directed curves. In order to formulate these inequalities in a rigorous manner, we shall make distinction between the ordered set of functions \( y = (u_i, \psi_i) \) representing
the set of components, and the vector fields of displacement and rotation given by \( \mathbf{u} = \mathbf{u}(y) = u_i \mathbf{t}_i \) and \( \psi = \psi(y) = \psi_i \mathbf{t}_i \). Let us denote by \( \| \cdot \|_1 \) the usual norm on the Sobolev space \( H^1[0,l] \) and by \( | \cdot |_0 \) the usual norm on the space \( L^2[0,l] \). The functional spaces \( H^1 \), \( L^2 \) are written with bold letters if their elements have several components. Thus, for example, for any \( \mathbf{y} = (u_i(s), \psi_i(s)) \in H^1[0,l] \) we have

\[
\| \mathbf{y} \|^2 = | \mathbf{y} |^2_0 + \int_0^l (u'_i u'_i + \psi'_i \psi'_i) \, ds = \int_0^l (u_i \psi_i + u'_i \psi_i + u'_i u'_i + \psi'_i \psi'_i) \, ds.
\]

The inequality of Korn type “without boundary conditions” is stated by the following result [5].

**Theorem 2.1.** Assume that the position vector \( \mathbf{r}(s) \) for the reference configuration \( \mathcal{C} \) is of class \( C^4[0,l] \), such that the radius of curvature \( R_c \) and the radius of twisting \( R_t \) given by (2) exist at any point. For every \( \mathbf{y} = (u_i(s), \psi_i(s)) \in H^1[0,l] \) we define the components of the deformation vectors \( \varepsilon_i(y) \) and \( \kappa_i(y) \) through the relations (8).

Then, there exists a constant \( c_1 = c_1(\mathbf{r}) > 0 \) such that

\[
\int_{\mathcal{C}} [u_i \psi_i + \varepsilon_i(y) + \varepsilon_i(y) \varepsilon_i(y) + \kappa_i(y) \kappa_i(y)] \, ds \geq c_1 \int_{\mathcal{C}} (u_i \psi_i + u'_i \psi_i + u'_i u'_i + \psi'_i \psi'_i) \, ds,
\]

for any \( \mathbf{y} = (u_i, \psi_i) \in H^1[0,l] \).

The proof follows the same steps as in the case of deformable surfaces [11, 12]. The inequality of Korn type (9) is valid for arbitrary functions \( \mathbf{y} = (u_i(s), \psi_i(s)) \in H^1[0,l] \), which are not restricted by any conditions in the end points \( s = 0, l \) and, hence, it is also called Korn’s inequality “without boundary conditions”. But as we know, in many mechanical problems the functions \( u_i(s) \) and \( \psi_i(s) \) have to satisfy some boundary conditions for \( s = 0, l \). For this reason, we shall establish in the remaining of this section a Korn inequality “with boundary conditions”.

Let us denote by \( \Gamma = \{0,l\} \) the boundary of the domain \( (0,l) \subset \mathbb{R} \). We denote by \( \Gamma_u \) and \( \Gamma_\psi \) the subsets of \( \Gamma \) where the displacements \( u_i \) and the rotations \( \psi_i \) are prescribed, respectively. Assuming zero boundary conditions for simplicity, we have the following restrictions imposed to the functions \( u_i \) and \( \psi_i \)

\[
u_i(s) = 0 \quad \text{for } s \in \Gamma_u, \quad \psi_i(s) = 0 \quad \text{for } s \in \Gamma_\psi.
\]

We introduce the subspace \( V \) of all the displacement and rotation fields which satisfy the boundary conditions (10), i.e.

\[
V = \{ \mathbf{y} = (u_i, \psi_i) \in H^1[0,l] \mid u_i = 0 \text{ on } \Gamma_u, \psi_i = 0 \text{ on } \Gamma_\psi \}.
\]

in the sense of traces. We notice that \( V \) is a closed subspace of \( H^1[0,l] \), and hence \( (V, \| \cdot \|_1) \) is a Banach space. The following result, concerning infinitesimal rigid body displacements and rotations of directed curves, will be useful in the sequel.

**Lemma 2.2.** Assume that the hypotheses of Theorem 2.1 are satisfied. Let \( \mathbf{y} = (u_i, \psi_i) \in H^1[0,l] \) be such that

\[
\varepsilon_i(y) = 0, \quad \kappa_i(y) = 0 \quad \text{on } [0,l].
\]

Then the displacement vector \( \mathbf{u}(y) = u_i \mathbf{t}_i \) and the rotation vector \( \psi(y) = \psi_i \mathbf{t}_i \) represent a rigid body displacement of the rod, i.e. there exist two constant vectors.
\( \mathbf{a} \) and \( \mathbf{b} \) such that

\[
\mathbf{u}(\mathbf{y}) = \mathbf{a} + \mathbf{b} \times \mathbf{r}, \quad \psi(\mathbf{y}) = \mathbf{b}.
\]  

Moreover, if \( \Gamma_u \) and \( \Gamma_\psi \) are nonempty sets and we have \( \mathbf{y} = (u_i, \psi_i) \in \mathbf{V} \), then the relations (12) imply that \( u_i = 0 \), \( \psi_i = 0 \) on \([0, l]\).

**Proof.** In view of (7)_2 and (12)_1 it follows that \( \psi'(\mathbf{y}) = 0 \), so that there exists a constant vector \( \mathbf{b} \) with \( \psi(\mathbf{y}) = \mathbf{b} \). Then, from (7)_1 and (12)_1 we deduce that \( (\mathbf{u}(\mathbf{y}) + \mathbf{r} \times \mathbf{b})' = 0 \), so that there exists a constant vector \( \mathbf{a} \) which satisfies the relation (13)_1. If the sets \( \Gamma_u \) and \( \Gamma_\psi \) are nonempty, then from (11) and (13) we deduce \( \mathbf{a} = 0 \), \( \mathbf{b} = 0 \) and hence \( u_i = 0 \), \( \psi_i = 0 \). The proof is complete.

On the basis of Theorem 2.1 and Lemma 2.2 we can prove the Korn inequality “with boundary conditions” [5].

**Theorem 2.3.** Assume that the hypotheses of Theorem 2.1 are satisfied and that \( \Gamma_u \) and \( \Gamma_\psi \) are nonempty sets. Let \( \mathbf{V} \) be the space defined by (11). Then, there exists a constant \( c_2 = c_2(\mathbf{r}, \Gamma_u, \Gamma_\psi) > 0 \) such that for any \( \mathbf{y} = (u_i, \psi_i) \in \mathbf{V} \) we have

\[
\int_\mathcal{C} [\varepsilon_i(\mathbf{y}) \varepsilon_i(\mathbf{y}) + \kappa_i(\mathbf{y}) \kappa_i(\mathbf{y})]d\mathbf{s} \geq c_2 \int_\mathcal{C} (\mathbf{u}_i \mathbf{u}_i + \psi_i \psi_i + u'_i u'_i + \psi'_i \psi'_i)d\mathbf{s}.
\]  

In the next sections we present the usefulness of the Korn-type inequality to prove existence results for the solution to the equations of thin thermoelastic porous rods.

### 3. Existence results for the dynamical equations.

To describe the thermal effects in thin rods, we denote by \( \theta(s, t) > 0 \) the absolute temperature field in the points of the deformed curve \( \mathcal{C}_i \). For the reference configuration \( \mathcal{C} \), the temperature is assumed to have the constant value \( \theta_0 \). For the porosity effects we employ the Nunziato–Cowin theory of materials with voids [21, 14]. This type of materials can also be regarded as media with microstructure, in which each micromedium has only dilatational degree of freedom, in addition to the usual translational and rotational degrees of freedom [8, 9]. The so-called *volume fraction field* \( \nu(s, t) \) \((0 < \nu \leq 1)\) is an additional independent kinematical variable, which is defined by the multiplicative decomposition

\[
\rho(s, t) = \nu(s, t) \gamma(s, t),
\]

where \( \rho \) is the mass density of the porous rod and \( \gamma \) is the mass density of the matrix thermoelastic material. In the reference configuration \( \mathcal{C} \) the above relation read \( \rho_0(s) = \nu_0(s) \gamma_0(s) \). The volume fraction field \( \nu \) describes the continuous distribution of pores along the middle curve of the rod.

In the linear theory, we shall assume that the variations of temperature and the variations of volume fraction field are very small. If we denote by

\[
\varphi(s, t) = \nu(s, t) - \nu_0(s), \quad T(s, t) = \theta(s, t) - \theta_0,
\]

then the porosity function \( \varphi \) and the thermal field \( T \) are infinitesimal.

In the direct approach for rods the usual balance laws of continuum mechanics are postulated, but they are formulated directly for the one-dimensional continua, i.e. for deformable curves. Thus, we employ the balances of mass, momentum, angular momentum, energy and entropy. In addition, we adopt the principle of equilibrated force [21, 14] which is needed to govern the porosity variables. In what follows we record the linearized basic field equations, which have been presented in
The local forms of the balances of momentum and angular momentum are, respectively
\[ N' + \rho_0 F = \rho_0 \left( \ddot{u} + \Theta_1 \cdot \ddot{\psi} \right), \quad M' + t_1 \times N + \rho_0 L = \rho_0 \left( \ddot{u} \cdot \Theta_1 + \Theta_2 \cdot \ddot{\psi} \right), \] (15)
where \( N \) is the force vector, \( M \) is the moment vector, while \( F \) and \( L \) are the external body force and moment per unit mass. In relations (15), \( \Theta_1 \) and \( \Theta_2 \) are second order tensors which are called inertia tensors (per unit mass).

Remark 1. The expressions of the inertia tensors \( \Theta_\alpha(s) \) in terms of the three-dimensional mass density and the geometry of the rod are given in [29, 4]. We notice that \( \Theta_1 \) is antisymmetric, \( \Theta_2 \) is symmetric and the tensor \( \Theta_2 - \Theta_1^T \cdot \Theta_1 \) is positive definite [6].

The equation of equilibrated force is written in the form
\[ h' - g + \rho_0 p = \rho_0 \kappa \ddot{\varphi}, \] (16)
where \( h(s,t) \) is the equilibrated stress, \( g(s,t) \) represents the internal equilibrated body force and \( p(s,t) \) is the assigned equilibrated body force per unit mass. The factor \( \kappa = \kappa(s) > 0 \) is an inertia coefficient associated to the porosity variable. This principle of equilibrated force has been introduced for the first time in [21, 15], and it can be seen as the special case of a balance equation which arises in the microstructural theories of elastic materials, when only the dilatation of the micromedium is considered [14, 8].

The energy balance equation can be written as
\[ q' + \rho_0 S = \rho_0 \theta_0 \dot{\eta}, \] (17)
where \( \eta(s,t) \) designates the specific entropy function, \( S(s,t) \) is the external rate of heat supply per unit mass and \( q(s,t) \) is the heat flux along the rod. The heat flux \( q \) is expressed by the constitutive equation
\[ q = KT', \] (18)
where the scalar \( K \) is the thermal conductivity of the rod. By virtue of the entropy inequality, the relation \( K \geq 0 \) holds.

Let us present the remaining constitutive equations. In the linear theory, the free energy (Helmholtz) function \( \Psi \) depends on the variables \( \{ \varepsilon, \kappa, \varphi, \varphi', T \} \) and has the expression
\[ \rho_0 \Psi(\varepsilon, \kappa, \varphi, \varphi', T) = \rho_0 U(\varepsilon, \kappa, \varphi, \varphi') - \mathcal{H}(\varepsilon, \kappa, \varphi, \varphi') T - \frac{1}{2} G T^2, \] (19)
where the deformation energy \( U \) is a quadratic form of its arguments, while \( \mathcal{H} \) is a linear function of its arguments. The fields \( N, M, h, g \) and \( \eta \) are then given by the constitutive relations
\[ N = \frac{\partial (\rho_0 \Psi)}{\partial \varepsilon}, \quad M = \frac{\partial (\rho_0 \Psi)}{\partial \kappa}, \quad g = \frac{\partial (\rho_0 \Psi)}{\partial \varphi}, \quad h = \frac{\partial (\rho_0 \Psi)}{\partial \varphi'}, \quad \eta = - \frac{\partial \Psi}{\partial T}. \] (20)
All the constitutive coefficients are bounded measurable functions of \( s \) which belong to \( H^1[0, l] \). The scalar \( G \) is a constitutive coefficient which is assumed to satisfy the restriction
\[ G > 0, \] (21)
while the deformation energy \( U \) is considered to be positive definite, i.e. there exists a constant \( c_3 > 0 \) such that
\[ U(\varepsilon, \kappa, \varphi, \varphi') \geq c_3 (\varepsilon \cdot \varepsilon + \kappa \cdot \kappa + \varphi^2 + (\varphi')^2). \] (22)
To resume, the basic field equations for the deformation of thermoelastic porous rods consists in the balance equations (15)–(17), the constitutive equations (18)–(20), and the geometrical relations (7). To these equations we must adjoin boundary conditions and initial conditions.

In what follows we show the existence of weak solutions to this problem, both in the dynamical and in the statical cases. The inequality of Korn type for rods presented in Section 2 will be useful in both situations. Although more general boundary conditions could be considered, we restrict our attention to the following zero boundary conditions

$$u = 0, \quad \psi = 0, \quad \varphi = 0, \quad T = 0 \quad \text{on } \Gamma. \quad (23)$$

The initial conditions (at $t = 0$) are taken in the form

$$u(s, 0) = u_0(s), \quad u'(s, 0) = v_0(s), \quad \psi(s, 0) = \psi_0(s), \quad \varphi(s, 0) = \omega_0(s), \quad T(s, 0) = T_0(s), \quad \text{for } s \in [0, l], \quad (24)$$

where the functions in the right-hand sides are prescribed.

In order to show the existence of weak solutions $\{u, \psi, \varphi, T\}$ to the boundary–initial–value problem formulated above, we first write our boundary–initial–value problem in the form of an abstract Cauchy problem in an appropriate functional framework, and then we apply the theory of semigroup of linear operators. Consider the following Banach space $\mathcal{W}, \| \cdot \|$ endowed with the usual (product) norm

$$\mathcal{W} = \{ Y = (u_i, v_i, \psi_i, \omega_i, \varphi, \lambda, T) \mid u_i, \psi_i, \omega_i, \varphi, \lambda, T \in L^2[0, l] \},$$

$$\| Y \|^2 = \sum_{i=1}^{3} (\| u_i \|_1^2 + \| \psi_i \|_1^2 + |v_i|^2 + |\omega_i|^2) + \| \varphi \|^2 + \| \lambda \|^2 + |T|^2.$$  

We introduce on the space $\mathcal{W}$ the following scalar product: for every $Y = (u_i, v_i, \psi_i, \omega_i, \varphi, \lambda, T)$ and $Z = (\hat{u}_i, \hat{v}_i, \hat{\psi}_i, \hat{\omega}_i, \hat{\varphi}, \hat{\lambda}, \hat{T})$ in $\mathcal{W}$, let

$$\langle Y, Z \rangle_{\mathcal{W}} = \frac{1}{2} \int_{C} \rho_0 (v \cdot \dot{v} + v \cdot \Theta_1 \cdot \hat{\omega} + \dot{v} \cdot \Theta_1 \cdot \omega + \omega \cdot \Theta_2 \cdot \hat{\omega} + \kappa \lambda \dot{\lambda}) ds$$

$$+ \frac{1}{2} \int_{C} [N(z) \cdot \varepsilon(y) + M(z) \cdot \kappa(y) + g(z) \varphi + h(z) \varphi' + \rho_0 \eta(y) \hat{T}] ds,$$  

(26)

where we have denoted by $y = (u_i, \psi_i, \varphi, T), \ z = (\hat{u}_i, \hat{\psi}_i, \hat{\varphi}, \hat{T})$ for brevity. By virtue of the inequality of Korn type established in Theorem 2.3, we can prove that the scalar product (26) induces a norm $\| \cdot \|_{\mathcal{W}}$ on the space $\mathcal{W}$, which is equivalent to the norm $\| \cdot \|$ given by (25). Consequently, $(\mathcal{W}, \langle \cdot, \cdot \rangle_{\mathcal{W}})$ is a Hilbert space.

To write the balance equations (15), (16) and the energy equation (17) into an operator form, we introduce the second order tensor $Q$ which is the inverse of the symmetric and positive definite tensor $\Theta_2 - \Theta_1^T \cdot \Theta_1$, i.e.

$$Q = (\Theta_2 - \Theta_1^T \cdot \Theta_1)^{-1}.$$  

Suggested by the balance equations (15)–(17), we define the operator $P : D(P) \subseteq \mathcal{W} \rightarrow \mathcal{W}$ by: for every $Y = (u_i, v_i, \psi_i, \omega_i, \varphi, \lambda, T) \in \mathcal{W}$ we have
Lemma 3.1. The operator $\mathcal{P}$ is defined by (27) satisfies:

i) the domain $D(\mathcal{P})$ is dense in $\mathcal{W}$;
ii) for every $\mathbf{Y} \in D(\mathcal{P})$ the following inequality holds:

$$\langle \mathcal{P} \mathbf{Y}, \mathbf{Y} \rangle_\mathcal{W} \leq 0;$$

(30)

iii) the range condition $\text{Range}(I - \mathcal{P}) = \mathcal{W}$ holds true, where $I$ designates the identity operator.

In view of Lemma 3.1, we can apply the Lumer–Phillips theorem (see e.g. [23], p. 14) and obtain that the operator $\mathcal{P}$ is the infinitesimal generator of a semigroup of contractions $\{R(t) ; t \geq 0\}$ in the Hilbert space $\mathcal{W}$. Using the general results of the semigroup of operators theory (see [28], Section 8.1) we deduce the existence and uniqueness of weak solutions as stated in the next theorem.

Theorem 3.2. Assume that the hypotheses of Theorem 2.1 are satisfied, and let $t_0$ be an arbitrary moment of time such that the external body loads and heat supply satisfy the condition $\Phi(t) \in C^1([0, t_0], L^2([0, l]))$. If the initial data are such that $\mathbf{Y}_0 \in D(\mathcal{P})$, then there exists a unique solution $\mathbf{Y}(t) \in C^1([0, t_0], \mathcal{W}) \cap C^0([0, t_0], D(\mathcal{P}))$ to the problem (28). The solution $\mathbf{Y}(t)$ is expressed by

$$\mathbf{Y}(t) = R(t) \mathbf{Y}_0 + \int_0^t R(t - \tau) \Phi(\tau) \, d\tau, \quad t \in [0, t_0].$$

(31)
Remark 2. The continuous dependence of the weak solution $Y(t)$ on the initial data $Y_0$ and the external body loads and heat supply $\Phi(t)$ is stated by the estimate

$$\|Y(t)\|_W \leq \|Y_0\|_W + \int_0^t \|\Phi(\tau)\|_W d\tau, \quad t \in [0,t_0].$$

which is obtained from (31), since $\{R(t) : t \geq 0\}$ is a semigroup of contractions.

4. Equilibrium equations for rods. For the static theory, the functions defined previously do not depend on time. In this case, we observe that the energy equation (17) decouples from the balance equations (15), (16), and it can be solved separately. Thus, the thermo-conductivity problem can be treated separately. For this reason, we consider in what follows the isothermal theory for porous rods. Our boundary-value problem reduces to find the unknown fields $\{u, \psi, \varphi\}$ which satisfy the equilibrium equations

$$N' + \rho_0 F = 0, \quad M' + t_3 \times N + \rho_0 L = 0, \quad h' - g + \rho_0 p = 0,$$

(33)

together with the constitutive equations

$$N = \frac{\partial (\rho_0 U)}{\partial \varepsilon}, \quad M = \frac{\partial (\rho_0 U)}{\partial \kappa}, \quad g = \frac{\partial (\rho_0 U)}{\partial \varphi}, \quad h = \frac{\partial (\rho_0 U)}{\partial (\varphi')}$$

(34)

and the boundary conditions

$$u = 0 \quad \text{for} \quad s \in \Gamma_u, \quad N = 0 \quad \text{for} \quad s \in \Gamma \setminus \Gamma_u,$$

$$\psi = 0 \quad \text{for} \quad s \in \Gamma_\psi, \quad M = 0 \quad \text{for} \quad s \in \Gamma \setminus \Gamma_\psi,$$

$$\varphi = 0 \quad \text{for} \quad s \in \Gamma_\varphi, \quad h = 0 \quad \text{for} \quad s \in \Gamma \setminus \Gamma_\varphi.$$

(35)

Let us assume that the set of functions $z = (\hat{u}_i, \hat{\psi}_i, \hat{\varphi}) \in C^2[0, l]$ satisfies the balance equations (33), the constitutive equations (34) and the geometrical relations (7). Then, for every $y = (u_i, \psi_i, \varphi) \in C^1[0, l]$ the following equality holds

$$\int_C [N(z) \cdot \varepsilon(y) + M(z) \cdot \kappa(y) + g(z) \varphi + h(z) \varphi'] ds$$

$$= \int_C \rho_0 [F \cdot u(y) + L \cdot \psi(y) + p \varphi] ds + (N(z) \cdot u(y) + M(z) \cdot \psi(y) + h(z) \varphi)'|_0^l,$$

(36)

where we employ the notation $\int_0^f = f(l) - f(0)$, for any $f$. To prove the relation (36), we insert the geometrical relations (7) for $\varepsilon(y)$ and $\kappa(y)$ in the left-hand side, then we use integration by parts and finally we employ the equilibrium equations (33) written for $z$. The identity (36) can be seen as a principle of virtual work for porous rods.

Suggested by the equality (36), we define next the weak solution of our boundary-value problem. We denote by $V$ the subspace of $H^1[0, l]$ given by

$$V = \{y = (u_i, \psi_i, \varphi) \in H^1[0, l] \mid u_i = 0 \text{ on } \Gamma_u, \psi_i = 0 \text{ on } \Gamma_\psi, \varphi = 0 \text{ on } \Gamma_\varphi\},$$

(37)

in the sense of traces. The body loads $F, L$ and $p$ are assumed to be functions of class $L^2[0, l]$. In these conditions, we say that the set of functions $z = (\hat{u}_i, \hat{\psi}_i, \hat{\varphi})$ is a weak solution of the boundary-value problem for porous rods if $z \in V$ and the following relation holds for every $y = (u_i, \psi_i, \varphi) \in V$:

$$\int_C [N_i(z) \varepsilon_i(y) + M_i(z) \kappa_i(y) + g(z) \varphi + h(z) \varphi'] ds = \int_C \rho_0 (F_i u_i + L_i \psi_i + p \varphi) ds.$$

(38)
Concerning the existence of such solutions we give the next result.

**Theorem 4.1.** Assume that the hypotheses of Theorem 2.1 are satisfied, and the sets $\Gamma_u$ and $\Gamma_\psi$ are nonempty. Then, there exists a unique weak solution $z$ of the boundary–value problem for porous rods. The weak solution $z$ is characterized as the minimizer on the space $V$ of the following functional

$$J(y) = \int_C \rho_0 [\mathcal{U}(\varepsilon(y), \kappa(y), \varphi, \varphi') - \mathcal{F}_1 u_i - \mathcal{L}_1 \psi_i - p \varphi'] ds, \quad \forall y = (u_i, \psi_i, \varphi) \in V.$$  

(39)

**Proof.** The proof resides in the use of the Korn inequality and the application of the Lax–Milgram lemma. Let us introduce the bilinear form $B(\cdot, \cdot)$ on $V \times V$ and the linear functional $F(\cdot)$ on $V$ defined by

$$B(y, z) = \int_C [N_i(z) \varepsilon_i(y) + M_i(z) \kappa_i(y) + g(z) \varphi + h(z) \varphi'] ds,$$  

$$F(y) = \int_C \rho_0 (\mathcal{F}_1 u_i + \mathcal{L}_1 \psi_i + p \varphi) ds, \quad \forall y, z \in V, \quad y = (u_i, \psi_i, \varphi).$$  

(40)

Then, the relation (38) can be written as

$$B(y, z) = F(y), \quad \forall y \in V.$$  

(41)

We remark that $B$ and $F$ are continuous and that $B$ is symmetric. In order to show that $B$ is also $V$–elliptic, we observe that

$$B(y, y) = 2 \int_C \rho_0 \varepsilon_i(y) \kappa_i(y) ds, \quad \forall y = (u_i, \psi_i, \varphi) \in V.$$  

(42)

Using the relation (22) and the Korn inequality (14), then from (42) we deduce that there exists a constant $c_4 > 0$ such that

$$B(y, y) \geq c_4 \|y\|^2_1, \quad \forall y \in V.$$  

(43)

Applying the Lax–Milgram lemma [7] for the problem (41) we obtain the existence and uniqueness of the solution $z$, and its characterization as the minimizer of the functional $J(y)$ on $V$.  

**Remark 3.** The hypothesis that $\Gamma_u$ and $\Gamma_\psi$ are nonempty sets excludes the possibility of a “pure traction” problem, i.e. the case when the forces $N$ and the moments $M$ are prescribed on both ends of the rod. Since this case is of great importance in engineering, it will be analyzed in details in the next section.

5. **Pure traction problems.** For the treatment of boundary–value problems with pure traction boundary conditions we need to establish first a Korn–type inequality “over the quotient space” $H^1[0, l] / R$. Here, $R$ is the subspace of $H^1[0, l]$ given by

$$R = \{ y = (u_i, \psi_i) \in H^1[0, l] \mid \varepsilon_i(y) = 0, \kappa_i(y) = 0 \}.$$  

We can see that $R$ represents the subspace of rigid–body displacements and rotations which can also be characterized by the relations $u_i t_i = a + b \times r, \psi_i t_i = b$, where $a$ and $b$ are two constant vectors, according to Lemma 2.2.

We introduce the quotient space $V = H^1[0, l] / R$, and we denote by $\dot{y}$ the equivalence class of any element $y \in H^1[0, l]$, i.e.

$$\dot{y} = \{ w \in H^1[0, l] \mid (y - w) \in R \}.$$
The space $\hat{V}$ is a Banach space, equipped with the quotient norm $\| \cdot \|_{\hat{V}}$ defined by

$$\| \hat{y} \|_{\hat{V}} = \inf_{w \in R} \| y + w \|_1, \quad \forall \hat{y} \in \hat{V}.$$  \hspace{1cm} (44)

The inequality of Korn–type “over the quotient space $\hat{V}$” is given by the next result.

**Theorem 5.1.** Assume that the hypotheses of Theorem 2.1 are satisfied. Then, there exists a constant $\hat{c} > 0$ such that the following inequality holds

$$\left\{ \int_C [\varepsilon_i(y)\varepsilon_i(y) + \kappa_i(y)\kappa_i(y)]ds \right\}^{1/2} \geq \hat{c} \| y \|_{H^1}, \quad \forall y \in H^1[0, l].$$  \hspace{1cm} (45)

**Proof.** To prove this inequality, we follow the same lines as in Theorem 4.3-5 of [12]. First, we apply the Hahn–Banach theorem and deduce the existence of 6 continuous linear forms $\ell_k$ defined on the space $H^1[0, l]$, $1 \leq k \leq 6$, with the following property: an element $w \in R$ is equal to 0 if and only if $\ell_k(w) = 0$, $1 \leq k \leq 6$.

Then, we employ the Korn–type inequality “without boundary conditions” given by Theorem 2.1 to show that there exists a constant $\hat{c} > 0$ such that we have

$$\left\{ \int_C [\varepsilon_i(y)\varepsilon_i(y) + \kappa_i(y)\kappa_i(y)]ds \right\}^{1/2} + \sum_{k=1}^{6} |\ell_k(y)| \geq \hat{c} \| y \|_1, \quad \forall y \in H^1[0, l].$$  \hspace{1cm} (46)

Further, for any $y \in H^1[0, l]$, we define the element $w = w(y) \in R$ by the relations $\ell_k(y + w) = 0$, $1 \leq k \leq 6$. If we write the inequality (46) for the element $y + w \in H^1[0, l]$ and take into account that $\varepsilon_i(y + w) = \varepsilon_i(y)$, $\kappa_i(y + w) = \kappa_i(y)$, then we obtain

$$\left\{ \int_C [\varepsilon_i(y)\varepsilon_i(y) + \kappa_i(y)\kappa_i(y)]ds \right\}^{1/2} \geq \hat{c} \| y + w(y) \|_1, \quad \forall y \in H^1[0, l].$$  \hspace{1cm} (47)

Finally, in view of the definition (44) for the quotient norm, we have

$$\| y + w(y) \|_1 \geq \| \hat{y} \|_{\hat{V}}, \quad \forall y \in H^1[0, l].$$  \hspace{1cm} (48)

From (47) and (48) we get the desired inequality (45). \hfill $\square$

The inequality of Korn–type established in Theorem 5.1 is useful to prove existence results in the case of pure traction problems. In what follows, we consider the boundary–value problem (33)–(35) for the equilibrium of porous rods in the case of pure traction boundary conditions, i.e. when $\Gamma_u = \Gamma_\psi = \emptyset$.

In this situation, the variational problem can be formulated as follows: find the weak solution $z = (\bar{u}_i, \bar{\psi}_i, \bar{\varphi}) \in H^1[0, l]$ such that the equation (38) holds for every $y = (u_i, \psi_i, \varphi) \in H^1[0, l]$.

Let us consider the subspace $\bar{R}$ of $H^1[0, l]$ given by

$$\bar{R} = \left\{ y = (u_i, \psi_i, \varphi) \in H^1[0, l] \mid \varepsilon_i(y) = 0, \kappa_i(y) = 0, \varphi = 0 \right\},$$  \hspace{1cm} (49)

which contains the infinitesimal rigid body displacements and rotations for porous rods. We observe that the left–hand side of the variational equation (38) vanishes for every $y \in \bar{R}$. Then, from (38) and (49) we obtain the following necessary condition for the existence of weak solutions

$$\int_C \rho_0 (\mathcal{F}_i u_i + \mathcal{L}_i \psi_i) ds = 0, \quad \forall (u_i, \psi_i) \in R.$$  \hspace{1cm} (50)
Let us denote by $\hat{V}$ the quotient space $\hat{V} = H^1[0, l] / \hat{R}$ and by $\hat{y} \in \hat{V}$ the equivalence class of any element $y = (u, \psi, \varphi) \in H^1[0, l]$. In the next theorem we show that the condition (50) is also sufficient for the existence of solutions.

**Theorem 5.2.** Assume that the hypotheses of Theorem 2.1 are fulfilled and that the external body loads $F_i$ and $L_i$ satisfy the condition (50). Then, there exists a weak solution $z = (\tilde{u}_i, \tilde{\psi}_i, \tilde{\varphi})$ of the pure traction boundary–value problem, which is unique up to an additive (rigid–body) field $w = (u, \psi, \varphi) \in \hat{R}$.

Moreover, the equivalence class of the weak solution $\hat{z} \in \hat{V}$ is also the unique solution of the minimization problem

$$J(\hat{z}) = \inf_{\hat{y} \in \hat{V}} J(\hat{y}) ,$$  

where $J$ is the functional defined on $\hat{V}$ by

$$J(\hat{y}) = \int_C \rho_0 \left[ U(e(\hat{y}), \kappa(\hat{y}), \varphi, \varphi') - F_i \tilde{u}_i - L_i \tilde{\psi}_i - p \varphi \right] ds , \quad \forall \hat{y} = (\tilde{u}_i, \tilde{\psi}_i, \tilde{\varphi}) \in \hat{V} .$$  

(52)

**Proof.** We observe that our problem can be written naturally as a variational equation on the quotient space $\hat{V}$ as follows: find the solution $\hat{z} \in \hat{V}$ which satisfies the equation

$$\int_C \left[ N_i(\hat{z}) e_i(\hat{y}) + M_i(\hat{z}) \kappa_i(\hat{y}) + g(\hat{z}) \varphi + h(\hat{z}) \varphi' \right] ds = \int_C \rho_0 \left[ F_i \tilde{u}_i + L_i \tilde{\psi}_i + p \varphi \right] ds ,$$  

(53)

for every $\hat{y} = (\tilde{u}_i, \tilde{\psi}_i, \tilde{\varphi}) \in \hat{V}$. By virtue of the inequality of Korn–type “over the quotient space” (45) established in Theorem 5.1, the variational problem (53) can be treated in the same manner as in the proof of Theorem 4.1 to show the existence of solutions and obtain the desired results (51), (52).

Concerning the boundary–value problem for rods we mention that one can use the same mathematical techniques as in the case of shells, and prove regularity results for the weak solutions, see e.g. Theorems 4.4-4 and 4.4-5 from [12].

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