COADJOINT ORBITS AND KÄHLER STRUCTURE:
EXAMPLES FROM COHERENT STATES

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ABSTRACT. Do co-adjoint orbits of Lie groups support a Kähler structure? We
study this question from a point of view derived from coherent states. We examine
three examples of Lie groups: the Weyl-Heisenberg group, SU(2) and SU(1, 1). In
cases, where the orbits admit a Kähler structure, we show that coherent states
give us a Kähler embedding of the orbit into projective Hilbert space. In contrast,
squeezed states, (which like coherent states, also saturate the uncertainty bound)
only give us a symplectic embedding. We also study geometric quantisation of
the co-adjoint orbits of the group SU(2, R) of real, special, upper triangular
matrices in two dimensions. We glean some general insights from these examples.
Our presentation is semi-expository and accessible to physicists.

Keywords: Coherent states, Squeezed States, Coadjoint orbits, Toda system

1. INTRODUCTION

Coadjoint orbits of Lie groups have a symplectic structure [6]. It is natural to
ask: do they have a Kähler structure? This question has been raised before [21]
in a different context. In a recent thesis [21], Villa studied the coadjoint orbits of
semisimple Lie groups. The motivation was to understand the geometry of the con-
vex hull of coadjoint orbits. In Theorem 2.27, [21] gives a necessary and sufficient
condition for the existence of a Kähler structure for semi-simple Lie groups. In this
semi-expository paper we study this question from the point of view of Perelomov coherent states in three examples. These are the Weyl-Heisenberg group, SU(2) and SU(1,1). In each of these cases we also discuss squeezed states using a definition of squeezing that is motivated by the Robertson-Schrödinger uncertainty relations [16, 19]. Our paper also includes a discussion of the group of special upper triangular matrices (non-zero entries only on the diagonal and above) and geometric quantization of its coadjoint orbits.

Coherent states were first studied by Schrödinger in an effort to construct quantum states with classical-like behaviour [18, 7, 20, 8, 9, 10]. He searched for states which saturate the inequality expressed by the Heisenberg uncertainty principle

\[ \Delta q \Delta p \geq \hbar / 2. \]

These states are called minimum uncertainty states. By minimising the quantum uncertainty, one finds quantum states that are the closest one can get to classical states. States of a classical system are described as points in phase space, which is a symplectic manifold. Coherent states can be thought of as localised around these points. They have equal uncertainty in position and momentum. These states are called Schrödinger coherent states (SCS) and are central to the physics of harmonic oscillators and optics.

However, requiring minimum uncertainty is an incomplete characterisation of coherent states. There exist other states which also saturate the Heisenberg bound. These states are called squeezed states [17] and they play an important role in physics. Squeezed states have a larger uncertainty in one variable (either position or momentum) and a smaller uncertainty in the conjugate variable. From the viewpoint of minimising uncertainty, squeezed states are just as good as coherent states. Not only do squeezed and coherent states saturate the Heisenberg uncertainty relation, but in fact, they both saturate a slightly stronger version, the Robertson-Schrödinger inequality[16, 19]. We will elaborate on this in Appendix 1.

Squeezed states are extremely important in quantum optics and quantum metrology. Squeezed states of light are used, for example, in the LIGO detector to give an accurate determination of position, while sacrificing accuracy in the conjugate variable, momentum. Although, both saturate the uncertainty bound, squeezed states are physically quite different from coherent states.

Coherent states have been studied mathematically from advanced group theoretic points of view[15], using co-adjoint orbits of Lie Groups. The symplectic structure on the co-adjoint orbit gives the orbit the character of a classical phase space. Coherent states then give us an embedding of the classical phase space into the ray space of quantum mechanics. The ray space of quantum mechanics admits a Kähler structure. Can one pull back this structure to the coadjoint orbit? Does the pullback agree with the Kähler structure (if it exists) on the coadjoint orbit? Can coherent states be used to define a Kähler structure on coadjoint orbits? These are the questions which motivate the present study. We will show in three pertinent
examples that coherent states give us a Kähler embedding of the classical phase space into the ray space of quantum mechanics, while squeezed states give us only a symplectic embedding. This property sets apart coherent states from the other minimum uncertainty states.

We show from first principles the fact that the Weyl-Heisenberg coherent states, SU(2) coherent states and SU(1,1) coherent states embed the respective coadjoint orbits into projective Hilbert space such that the pull back of the Fubini Study form is Kähler, thus illustrating a general result in [5, 12, 14]. The coadjoint orbits of interest are \( \mathbb{C} \), SU(2)/U(1) \( \equiv \mathbb{S}^2 \) and SU(1,1)/U(1) \( \equiv \mathbb{H} \) (upper half plane), respectively. Next we show that the pull back by squeezed states, however, yields only a symplectic structure on the coadjoint orbit.

Our exposition includes a detailed study of the geometric quantization of the coadjoint orbits of the special upper triangular matrices. A coadjoint orbit of SUT is SUT\(^+\) which is intimately connected to the 2-dimensional Toda system. In [1] Adler showed that a finite \( n \)-dimensional Toda system has a coadjoint orbit description of the group of lower triangular matrices of non-zero diagonal. In fact, one can restrict the action to that of lower triangular matrices of determinant 1 and positive diagonal elements. The orbit is homeomorphic to \( \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \), just described by \( a_i > 0, i = 1, ..., n-1 \) and \( b_i, i = 1, ..., n \) such that \( b_1 + b_2 + ... + b_n = c \), \( c \) a constant. In this paper we deal with the case \( n = 2 \). In [3], the authors geometrically quantized this system and studied the coherent states. Coherent and squeezed states pulled back from projective spaces have been considered in [4] in a somewhat more general context.

The paper is structured as follows: In section 2, (and in appendix 1) we introduce coherent and squeezed states and describe how Schrödinger coherent states emerge from geometric quantisation of co-adjoint orbits of the Weyl-Heisenberg group. In section 3, we define the ray space of quantum mechanics and describe the Kähler structure of the ray space. In section 4 we show that the Schrödinger coherent states give us a Kähler embedding of the complex plane into the ray space. and we show a similar embedding of the unit disc (SU(1,1) coherent states) and the sphere (SU(2) coherent states). We show squeezed states give only a symplectic embedding. In section 5, we study a coadjoint orbit of the group of special upper triangular matrices. The phase space is the upper half plane. Section 6 is a concluding discussion. An appendix (section 7) describes the origin of our definition of squeezed states and another appendix treats the Berezin quantisation of the upper half plane which is already known for the unit disc.

\(^1\)In fact, this orbit is also a group, and represents the AN part of the KAN or Iwasawa decomposition of SL(2, \( \mathbb{R} \)).
2. Squeezed and Coherent States

Schrödinger coherent states: Schrödinger coherent states are defined as solutions to the eigenvalue equation

\[(\hat{q} + i\hat{p}) |\alpha \rangle = \alpha |\alpha \rangle ,\]

where the eigenvalue \(\alpha\) is a complex number. These states saturate the uncertainty bound eq.(1). The eigenvalue equation is easily solved to give an expression for coherent states in terms of the oscillator eigenstates \(|n\rangle\).

\[|\alpha \rangle = \exp[-|\alpha|^2/2] \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle ,\]

\(|n\rangle\) are orthonormal eigenstates of the Hermitian operator \(\hat{H} = (\hat{q}^2 + \hat{p}^2)/2\).

Squeezed states: As mentioned above, coherent states are not the only states which saturate the uncertainty inequality. Squeezed states, defined by

\[\left(\hat{q}_\lambda + i\hat{p}_\lambda\right) |\alpha, \lambda \rangle = \alpha |\alpha, \lambda \rangle ,\]

also share this property. We describe the orgin of this defintion of squeezed states in appendix 1. Squeezed states are anisotropic in phase space being “squeezed" in one direction and expanded in the other. This squeezing operation in phase space preserves the symplectic structure but alters the complex structure of the plane.

Mathematically, Schrödinger coherent states emerge naturally from the Weyl-Heisenberg group \(W\), which is the exponential of the nilpotent W-H Lie algebra, generated by \(e_1, e_2, e_3\) with the only nonzero commutation relation being

\[[e_1, e_2] = e_3.\]

The mathematical theory of coherent states starts with an irreducible, unitary represent-ation of the W-H group in a Hilbert space \([15]\). Taking a fiducial vector \(|\psi_0\rangle\) satisfying

\[(\hat{q} + i\hat{p}) |\alpha \rangle = 0,\]

we have

\[|\alpha \rangle = D(\alpha) |\psi_0\rangle ,\]

where the complex number \(\alpha\) determines \(D(\alpha) = \exp[i\hat{q}\alpha_1 + i\hat{p}\alpha_2]\), and the group commutation relations ensure that \(2\) is satisfied \([15]\). The complex number \(\alpha\) (or equivalently the two real numbers \((\alpha_1, \alpha_2)\), where \(\alpha = \alpha_1 + i\alpha_2\), parametrises the points of a two dimensional co-adjoint orbit of \(W\). The co-adjoint orbit \(\Gamma\) has a natural symplectic structure from the Kirillov-Konstant- Souriau construction, \([23]\). \(\Gamma\) is in fact the classical phase space of the Harmonic oscillator, identified with the
complex plane. $\Gamma$ admits a Kähler structure, with the symplectic, Riemannian and complex structures coexisting compatibly.

3. Ray Space as a Kähler Manifold

The states of a quantum system are described projectively as rays in a Hilbert space $\mathcal{H}$. Let us consider the space of normalised states $\mathcal{N} = \{ \psi \in \mathcal{H} : ||\psi||^2 = 1 \}$. We can regard the Ray space as elements of $\mathcal{N}$, modulo a overall phase

$$\mathcal{R} = \mathcal{N} / \sim,$$

where $|\psi \rangle \sim |\psi' \rangle$ if $|\psi \rangle = \exp i\gamma |\psi' \rangle$. An equivalent definition of $\mathcal{R}$ is to view it as the space of one dimensional projections on $\mathcal{H}$. These can be written as $\rho = |\psi \rangle \langle \psi|$. $\rho$ is Hermitean, $\rho^\dagger = \rho$, a projection operator $\rho^2 = \rho$ and normalised $\text{Tr}(\rho) = 1$. In finite dimensional quantum systems such as occur physically in spin systems, the ray space is $\mathbb{C}P^n$. In two of the systems we deal with in this paper, the classical phase space is non-compact and has infinite symplectic volume. As a result the Hilbert space $\mathcal{H}$ as well as the ray space $\mathcal{R}$ are infinite dimensional. We can describe the ray space as $\mathbb{C}P^n$ or $\mathbb{C}P^\infty$ depending on whether the dimension is finite or infinite.

The ray space $\mathcal{R}$ is naturally endowed with a metrical structure, (with distances $\delta$ determined by shortest geodesics of the Fubini-Study metric)

$$\| \langle \psi_1 | \psi_2 \rangle \| = \cos \delta / 2,$$

where $\delta$ ranges from 0 to $\pi$ and gives the distance between rays. $\delta$ is directly measurable in a laboratory as a transition probability.

The ray space also has two more structures inherited from $\mathcal{H}$: a symplectic structure and a complex structure. The symplectic structure on the ray space $\mathcal{R}$ is described by a closed, non-degenerate two form, the curvature of the universal $U(1)$ connection on the bundle: $\mathcal{N} \to \mathcal{R}$. The symplectic structure on $\mathcal{R}$ has been interpreted as a geometric phase [22], which also is amenable to experiments. The complex structure is less evident in laboratory terms but follows mathematically from the complex structure on $\mathcal{H}$. Physically, the complex structure is crucial to a discussion of time reversal symmetry, which acts by conjugation on the complex structure. The three structures - metrical, symplectic and complex- give us a Kähler structure on the ray space.

The equation (3) defining a coherent state gives us an embedding of the classical phase space into the ray space $\mathcal{R}$. Each complex number $\alpha \in \Gamma$, determines an unique nonzero element $|\alpha\rangle \in \mathcal{H}$, which projects down to the ray space $[|\alpha\rangle]$. (The square brackets refer to the equivalence class of $|\alpha\rangle$.) This can also be expressed as a projection operator $\rho = |\alpha\rangle \langle \alpha|$)

$$\phi : \Gamma \to \mathcal{R}$$

$$\phi(\alpha) = \rho = |\alpha\rangle \langle \alpha|.$$
It is known that in some examples, the coherent states provide us with a Kähler embedding of a coadjoint orbit into the projective Hilbert space $\mathbb{C}P^n$ or $\mathbb{C}P^\infty$, [5], [12], [14]. In this paper, we investigate the nature of this embedding for coherent and squeezed states in three examples.

It is well known that the classical phase space $\Gamma$ of the oscillator is a Kähler manifold. As we remarked above, the ray space $\mathcal{R}$ has a Kähler structure which gives mutually compatible symplectic, complex and metrical structures. Given the embedding (10) it is reasonable to ask if the pullback of the structures on $\mathcal{R}$ agrees with the corresponding structure on $\Gamma$. We first discuss this question for the Schrödinger coherent states, then the coherent states of $\text{SU}(2)$ and $\text{SU}(1, 1)$.

Consider two tangent vectors $\dot{\alpha}$ and $\alpha'$ on the complex plane $\Gamma$. The symplectic and metrical structures on the $\alpha$ plane are given respectively by

\begin{equation}
\omega(\alpha', \dot{\alpha}) = \text{Im} \left( \alpha'^* \dot{\alpha} \right)
\end{equation}

and

\begin{equation}
g(\alpha', \dot{\alpha}) = \text{Re} \left( \alpha'^* \dot{\alpha} \right).
\end{equation}

We can evaluate the push-forward of $\alpha'$ and $\dot{\alpha}$ to the ray space and use the definition (3) of $\langle \alpha \rangle$, the metric and symplectic structure on $\mathcal{R}$, to find the real and imaginary parts of

\begin{equation}
\langle \alpha' | P | \dot{\alpha} \rangle,
\end{equation}

(where $P$ is the projector $1 - |\alpha\rangle \langle \alpha|$ orthogonal to the rays of $\mathcal{H}$)

A calculation given in detail in section 4 with (3) shows that this works out to the symplectic (11) and metrical structure (12) on $\Gamma$. Thus Schrödinger coherent states have the property that the pull back of the Kähler structure on $\mathcal{R}$ to $\Gamma$ agrees with naturally occurring structures on $\Gamma$: $\phi$ gives us a Kähler embedding of $\Gamma$ into $\mathcal{R}$.

This property distinguishes coherent states from squeezed states. Squeezed states are got by composing the map (10) with the transformation $z \rightarrow \lambda(z + \bar{z})/2 + \lambda^{-1}(z - \bar{z})/2$. This transformation is not complex analytic for $\lambda \neq 1$ and does not respect the complex structure of $\Gamma$. However it does preserve the symplectic form. Squeezed states only give us a symplectic embedding but not a Kähler embedding.

4. Kähler embedding of the classical phase space into projective Hilbert space.

In this section we explicitly show in examples that the coherent states provide us with a Kähler embedding of a coadjoint orbit into the projective Hilbert space $\mathbb{C}P^\infty$, which illustrates a general result in [5, 12, 14]. In each case we also show that squeezed states only give a symplectic embedding.
4.1. Kähler embedding of the classical phase space: Schrödinger Coherent states. The coherent state on the complex plane is given by:

\[ |\alpha\rangle = \exp \left( -\frac{||\alpha||^2}{2} \right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \]

Let \( \alpha(s) \) be a curve in the complex plane. We have the following tangent vector on \( \mathbb{C} \):

\[ \alpha' = \frac{d\alpha}{ds}. \]

This defines a curve \( |\alpha(s)\rangle \in \mathcal{N} \). We can project this down to the ray space to obtain a curve \( [|\alpha(s)|] \in \mathcal{R} \).

The push forward of \( \alpha' \) to \( \mathcal{N} \) is \( \frac{d}{ds}|\alpha(s)\rangle \). Composing with \( \pi : \mathcal{H} - \{0\} \rightarrow \mathcal{R} \) we have

\[ \left( \frac{d}{ds}|\alpha(s)\rangle \right)^\perp = P \left( \frac{d}{ds}|\alpha(s)\rangle \right), \]

where \( P = \mathbb{1} - |\alpha\rangle\langle\alpha| \) is the projector perpendicular to \(|\alpha\rangle\).

\[ \frac{d}{ds}|\alpha(s)\rangle = \frac{d}{ds} \left( e^{-\frac{||\alpha||^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \right) \]

\[ = \left( \frac{d}{ds} \left( -\frac{||\alpha||^2}{2} \right) \right) |\alpha\rangle + e^{-\frac{||\alpha||^2}{2}} \alpha' \sum_{n=0}^{\infty} \frac{n\alpha^{n-1}}{\sqrt{n!}} |n\rangle. \]

On projection the first term, vanishes. Letting \( |b\rangle = \frac{d}{ds}|\alpha(s)\rangle \), we have

\[ (14) \quad P |b\rangle = P \left( \frac{d}{ds}|\alpha(s)\rangle \right) = P \left( e^{-\frac{||\alpha||^2}{2}} \alpha' \sum_{n=0}^{\infty} \frac{n\alpha^{n-1}}{\sqrt{n!}} |n\rangle \right). \]

Similarly, we set \( |b\rangle = \frac{d}{dt}|\alpha(t)\rangle \), to describe another tangent vector in the \( \alpha \) plane. This leads to a similar expression as \( (14) \) with \( \alpha' \) replaced by \( \dot{\alpha} \). We need to calculate:

\[ \langle a | P |b\rangle = \langle a | (\mathbb{1} - |\alpha\rangle\langle\alpha|) |b\rangle \]

\[ = \langle a|b\rangle - \langle a|\alpha\rangle \langle \alpha|b\rangle. \]
Carrying out these calculations we obtain:

\[
\langle a|b \rangle = \overline{\alpha} \hat{\alpha} e^{-||\alpha||^2} \sum_{m,n=0}^{\infty} \frac{mnm^{-1}n^{-1}a^m\alpha^n}{\sqrt{m!n!}} \langle m|n \rangle 
\]

\[
= \overline{\alpha} \hat{\alpha} e^{-||\alpha||^2} \sum_{n=0}^{\infty} \frac{n^2(\overline{\alpha}\alpha)^{n-1}}{n!} 
\]

\[
= \overline{\alpha} \hat{\alpha} e^{-||\alpha||^2} \left[ \overline{\alpha}\alpha ||\alpha||^2 + 2||\alpha||^2 \right] 
\]

\[
= \overline{\alpha} \hat{\alpha}(1 + \overline{\alpha}\alpha), 
\]

\[
\langle a|b \rangle = \hat{\alpha} e^{-||\alpha||^2} \sum_{m,n=0}^{\infty} \frac{\overline{\alpha}^m m^{-1}n^{-1}}{\sqrt{m!n!}} \langle m|n \rangle 
\]

\[
= \hat{\alpha} e^{-||\alpha||^2} \sum_{n=0}^{\infty} \frac{n\overline{\alpha}^n a^{n-1}}{n!} 
\]

\[
= \overline{\alpha} \hat{\alpha}. 
\]

Similarly, \( \langle a|\alpha \rangle = \overline{\alpha} \hat{\alpha}' \).

Substituting these values in the equation above we obtain:

\[
\langle a|P|b \rangle = \overline{\alpha} \hat{\alpha}, 
\]

which exactly agrees with the Hermitian metric on \( \mathbb{C} \).

4.1.1. **Symplectic embedding of the classical phase space of the harmonic oscillator by squeezed states:** To briefly recapitulate the main formulae, recall that \( a = \frac{(q+i\varphi)}{\sqrt{2}} \) and \( a^+ = \frac{(q-i\varphi)}{\sqrt{2}} \). Coherent states are defined by the formula \( a|\alpha \rangle = \alpha|\alpha \rangle \). The fiducial (or ‘vacuum’) state \(|0\rangle\) is defined by \( a|0\rangle = 0 \), and the displacement operator by \( D(\alpha) = \exp(\alpha a^+ - \overline{\alpha} a) \). The displacement operator \( D(\alpha) \) satisfies \( D^{-1}(\alpha) a D(\alpha) = a + \alpha \), so that \(|\alpha\rangle = D(\alpha)|0\rangle\).

Consider a deformed operator \( \tilde{a} = \frac{1}{\sqrt{2}} \left( \lambda\hat{q} + \frac{i}{\lambda}\hat{p} \right) = \frac{1}{2} (\lambda + \frac{1}{\lambda}) a + \frac{1}{2} (\lambda - \frac{1}{\lambda}) a^+ \), where \( \lambda \) (real, positive) is the squeezing parameter. It follows that \( \tilde{a} = \cosh(\varphi)a + \sinh(\varphi)a^+ \). We now define the squeezed vacuum by \( \tilde{a}|0;\lambda\rangle = 0 \). By Robertson-Schrödinger, (see appendix 1), this is a minimum uncertainty state. Define translates of the squeezed vacuum as \( |\alpha;\lambda\rangle = D(\alpha)|0;\lambda\rangle \). Then \( D^{-1}(\alpha) \tilde{a} D(\alpha) = \tilde{a} + \alpha \cosh(\varphi) + \overline{\alpha} \sinh(\varphi) \). It follows that \( \tilde{a} D(\alpha)|0;\lambda\rangle = \alpha D(\alpha)|0;\lambda\rangle \) where \( \tilde{a} = \alpha \cosh(\varphi) + \overline{\alpha} \sinh(\varphi) = e^{\varphi}a_1 + i e^{\varphi} a_2 \) where \( \alpha = a_1 + i a_2 \). We write \( D(\alpha)|0;\lambda\rangle \) as \( |\tilde{\alpha};\lambda\rangle \). Below we drop the tilde and just write \(|\alpha;\lambda\rangle\) for the squeezed states.

Fix \( \lambda \) and consider the embedding of the complex plane into the ray space: \( \alpha \mapsto |\alpha;\lambda\rangle \mapsto [|\alpha;\lambda\rangle] \). The translation group acts on the complex plane which is a homogenous space. To calculate the pull back of the Fubini-Study form by
the above embedding, it is enough to work at the origin. Consider \( \alpha = \alpha(s) \), a
curve in the complex plane such that \( \alpha(0) = 0 \) and \( \frac{d\alpha(s)}{ds}|_{s=0} = \alpha' \). |\( \alpha(s); \lambda \rangle = D(\alpha(s))|0; \lambda \rangle \). Let |\( \psi' \rangle = \frac{d\alpha(s);\lambda}{d\alpha(s)} = \frac{dD(\alpha(s))|0;\lambda \rangle}{d\alpha(s)}|0; \lambda \rangle \). From \( \frac{dD(\alpha(s))}{d\alpha(s)}|_{s=0} = \frac{d\bar{a}}{ds}a - \frac{d\bar{a}}{ds}a \), we find that |\( \psi' \rangle = (\alpha'a - \bar{a}'\bar{a})|0; \lambda \rangle \). Likewise for a curve parametrised by \( t \),
\[ |\psi' \rangle = (\alpha' e\alpha + i\alpha'e^{-\alpha}) \bar{a}^\dagger|0; \lambda \rangle \] and |\( \bar{\psi} \rangle = (\bar{\alpha}_1 e\alpha + i\bar{\alpha}_2 e^{-\alpha}) \bar{a}^\dagger|0; \lambda \rangle \).

To evaluate the pullback of the Kähler form, we need to find \( \langle \psi | \bar{P} | \psi' \rangle \) where

\[ P = 1 - |0; \lambda \rangle \langle 0; \lambda | \]

\[ \langle \psi | \bar{P} | \psi' \rangle = 0; \lambda | \bar{a}a + |0; \lambda \rangle \langle 0; \lambda ; | \psi' \rangle \]. Using \( [\bar{a}, \bar{a}^\dagger] = 1 \) we have (since \( \langle 0; \lambda ; | \psi' \rangle = 0 \) \( \langle \phi | \psi' \rangle = (\alpha' \alpha_1 e^{2\alpha} + \alpha'_2 \alpha_2 e^{-2\alpha}) + i(\alpha_1 \alpha'_2 - \alpha_2 \alpha'_1) \).
The pullback of the Riemannian metric (the real part of the above expression) agrees with the Riemannian metric on the complex plane only for \( v = 0 \). However, the pullback of the symplectic form (the imaginary part of the above expression) does agree with the symplectic form on the complex plane for all \( v \). Thus the pull back by squeezed states gives a symplectic structure but not a Kähler structure.

4.2. Symplectic and Kähler embedding of the sphere in Projective Hilbert space. Let the generators of the Lie algebra of SU(2) be \( L_x, L_y, L_z \), all Hermitian. Then the non-vanishing commutators are \( [L_x, L_y] = iL_z \), \( [L_y, L_z] = iL_x \)
and \( [L_z, L_x] = iL_y \). Let \( L_+ = (L_x + iL_y)/\sqrt{2} \) and \( L_- = (L_x - iL_y)/\sqrt{2} \). Then
\[ [L_+; L_-] = L_+, \ [L_+; L_-] = -L_- \] and \( [L_+; L_-] = L_0 \). With the parameter \( \lambda = e\alpha \), let us define “squeezed operators” \( \tilde{L}_\pm = e^\alpha L_\pm = e^\alpha i L_x \pm i e^{-\alpha} L_y \).

In analogy with the earlier case, we define the squeezed vacuum state as the kernel of \( \tilde{L}_- \)

|\( \tilde{L}_- |0; \lambda \rangle = 0 \).

Let \( \alpha \in \mathbb{C} \) be the stereographic coordinate of a point on the sphere. Then \( D(\alpha) = \exp(\bar{a}L_- - \alpha L_+) \) is the displacement operator \([15]\). A general squeezed state can be got by displacing the squeezed vacuum using \( D(\alpha) \). The squeezed operators \( \tilde{L} \) are related to the \( L \)s as follows:

\[ \tilde{L}_+ = \cosh(v)L_+ + \sinh(v)L_- \]
\[ \tilde{L}_- = \sinh(v)L_+ + \cosh(v)L_- \]

Or in inverse form,

\[ L_+ = \cosh(v)\tilde{L}_+ - \sinh(v)\tilde{L}_- \]
\[ L_- = -\sinh(v)\tilde{L}_+ + \cosh(v)\tilde{L}_- \]

Let us now evaluate the pullback of the Kähler form on the projective Hilbert space. Because the coadjoint orbit is a homogeneous space, it is sufficient to do the calculation at a single point, which we conveniently choose to be the squeezed vacuum. Let \( \alpha = \alpha(s) \) be a curve in the complex plane such that \( \alpha(0) = 0 \). Then |\( \alpha(s); \lambda \rangle = D(\alpha)|0; \lambda \rangle \), where as before |\( 0; \lambda \rangle \) is the squeezed vacuum and
Consider a curve $\alpha$ where $\langle \alpha \rangle$. Now, we have the projection operator, $P$. The corresponding tangent vector in $N$ is the displaced squeezed state. Then one sees that $|\psi\rangle = \frac{d}{ds}|\alpha; \lambda\rangle = (\frac{d\alpha}{ds}L_- - \frac{d\alpha}{ds}L_+)|0; \lambda\rangle = -\left(\alpha^* \sinh(v) + \alpha \cosh(v)\right)\tilde{L}_+|0; \lambda\rangle$.

Thus $|\psi\rangle = \left(\alpha_1' e^v + i\alpha_2' e^{-v}\right)\tilde{L}_+|0; \lambda\rangle$, where $\alpha = \alpha_1 + i\alpha_2$. Similarly, $|\psi\rangle = \left(\alpha_1 e^v - i\alpha_2 e^{-v}\right)\tilde{L}_-|0; \lambda\rangle$. We calculate $\langle \psi | P | \psi \rangle$ where $P = 1 - |0; \lambda\rangle\langle 0; \lambda|$. Since $\langle 0; \lambda | \psi\rangle = 0$ and $\langle \dot{\psi} | \psi \rangle = \left(\alpha_1' e^v + i\alpha_2' e^{-v}\right)\left(\alpha_1 e^v - i\alpha_2 e^{-v}\right)\langle 0; \lambda | \tilde{L}_- \tilde{L}_+ | 0; \lambda\rangle$ and $[\tilde{L}_-, \tilde{L}_+] = -L_z$, we find that the pull back Hermitian metric is (apart from an overall constant $-\langle 0; \lambda | L_z | 0; \lambda\rangle$),

$$\langle \psi | P | \psi \rangle = (\alpha_1' \dot{\alpha}_1 e^{-2v} + \alpha_2' \dot{\alpha}_2 e^{2v}) + i(\alpha_2' \dot{\alpha}_1 - \alpha_1' \dot{\alpha}_2).$$

The real part corresponds to the Riemannian metric on the complex plane and the imaginary part corresponds to the symplectic form. For $v = 0$ it is a Kähler embedding, but if $v \neq 0$, we get only a symplectic embedding since the Riemannian metric is not compatible with the symplectic form and the complex structure.

4.3. Kähler embedding of the unit disc: $SU(1, 1)$ case. This following calculation repeats for the upper half plane the calculation we did for Schrödinger coherent states. The argument is very similar. It is actually more convenient to work on the unit disc. The expression for coherent states are taken from section 2 of [2]. There is a parameter $k$ which is continuous and $k > \frac{1}{2}$.

The coherent states are given as:

$$|\alpha, k\rangle = (1 - \overline{\alpha} \alpha)^k \sum_{n=0}^{\infty} \left[ \frac{\Gamma(n + 2k)}{n! \Gamma(2k)} \right]^{\frac{1}{2}} \alpha^n |n, k\rangle,$$

where $\langle m, k | n, k\rangle = \delta_{mn}$. $k$ is fixed from this point on.

Consider a curve $\alpha(s)$ in the unit disc. The tangent vector is given by:

$$\alpha'(s) = \frac{d}{ds} \alpha(s).$$

The corresponding tangent vector in $N$ is the pushforward,

$$\frac{d}{ds} |\alpha(s), k\rangle = u |\alpha(s), k\rangle + (1 - \overline{\alpha} \alpha)^k \sum_{n=0}^{\infty} \left[ \frac{\Gamma(n + 2k)}{n! \Gamma(2k)} \right]^{\frac{1}{2}} n \alpha^{n-1} |n, k\rangle \alpha'.$$

Now, we have the projection operator, $P = 1 - |\alpha\rangle \langle \alpha|$. Denote:

$$|b\rangle = P \frac{d}{ds} |\alpha(s), k\rangle$$

$$= (1 - \overline{\alpha} \alpha)^k \sum_{n=0}^{\infty} \left[ \frac{\Gamma(n + 2k)}{n! \Gamma(2k)} \right]^{\frac{1}{2}} n \alpha^{n-1} |n, k\rangle \alpha'.$$
\[ |a\rangle = P \frac{d}{dt} |\alpha(t), k\rangle \]
\[ = (1 - \overline{\alpha}\alpha)^k \sum_{n=0}^{\infty} \left[ \frac{\Gamma(n + 2k)}{n!\Gamma(2k)} \right]^{\frac{1}{2}} n\alpha^{n-1} |n, k\rangle \hat{\alpha}. \]

We need to compute
\[ \langle a|P|b \rangle = \langle a|b \rangle - \langle a|\alpha \rangle \langle \alpha|b \rangle. \]

The terms are calculated using binomial expansion.
\[ \frac{1}{(1 - x)^{2k}} = \sum_{n=0}^{\infty} \binom{2k + n - 1}{n} x^n = \sum_{n=0}^{\infty} \frac{\Gamma(n + 2k)}{n!\Gamma(2k)} x^n. \]

Now,
\[ \langle a|b \rangle = (1 - \overline{\alpha}\alpha)^{2k} \overline{\alpha}\hat{\alpha} \sum_{n=0}^{\infty} \frac{\Gamma(n + 2k)}{n!\Gamma(2k)} n^2 (\overline{\alpha}\alpha)^{n-1} \]
\[ = \frac{2k}{(1 - \overline{\alpha}\alpha)^2} \overline{\alpha}\hat{\alpha}(1 + 2k\overline{\alpha}\alpha), \]
\[ \langle a|\alpha \rangle = (1 - \overline{\alpha}\alpha)^{2k} \overline{\alpha}\hat{\alpha} \sum_{n=0}^{\infty} \frac{\Gamma(n + 2k)}{n!\Gamma(2k)} n(\overline{\alpha}\alpha)^{n-1} \]
\[ = \frac{2k}{1 - \overline{\alpha}\alpha} \overline{\alpha}\hat{\alpha}, \]
\[ \langle \alpha|b \rangle = \frac{2k}{1 - \overline{\alpha}\alpha} \overline{\alpha}\hat{\alpha}. \]

Putting it all together, we calculate:
\[ \langle a|b \rangle - \langle a|\alpha \rangle \langle \alpha|b \rangle = \overline{\alpha}\hat{\alpha} \left[ \frac{2k}{(1 - \overline{\alpha}\alpha)^2} (1 + 2k\overline{\alpha}\alpha) - \frac{(2k)^2\overline{\alpha}\alpha}{(1 - \overline{\alpha}\alpha)^2} \right] \]
\[ = \frac{2k\overline{\alpha}\hat{\alpha}}{(1 - \overline{\alpha}\alpha)^2}, \]
which is the Kähler metric on the unit disc apart from a constant depending on \( k \).

**Note:** The calculations above for SU(1, 1) and those for SU(2) given above can be adapted to each other after supplying the required signs. We do not repeat the calculations for the squeezed states for SU(1, 1).
5. An Example: Upper Triangular Matrices of unit determinant

In this section we study the geometric quantisation of the coadjoint orbits of the group of $2 \times 2$ upper triangular matrices with a different point of view from [3]. In the process we give elementary proofs of some known results in [1].

Let $\text{SUT}(2, \mathbb{R})$ denote the Lie group of $2 \times 2$ upper triangular matrices over the reals with determinant 1.

\[ \text{SUT}(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix} : a \in \mathbb{R}\backslash\{0\}, b \in \mathbb{R} \right\}. \]

$\text{SUT}(2, \mathbb{R})$ is not a connected group. It has two disjoint components, matrices of positive diagonal and negative diagonal elements respectively. Let $\mathfrak{su}t$ denote the Lie algebra and $\mathfrak{su}t^*$ denote its dual. Then

\[ \mathfrak{su}t = \left\{ \begin{pmatrix} u & v \\ 0 & -u \end{pmatrix} : u, v \in \mathbb{R} \right\}, \]

\[ \mathfrak{su}t^* = \left\{ \begin{pmatrix} u & 0 \\ v & -u \end{pmatrix} : u, v \in \mathbb{R} \right\}. \]

The group $\text{SUT}(2, \mathbb{R})$ acts on its Lie algebra $\mathfrak{su}t$ by the adjoint action. This induces an action on the dual space, $\mathfrak{su}t^*$, called the co-adjoint action. Take the following basis for $\mathfrak{su}t$, $\mathcal{B} = \{E_1, E_2\}$

\[ E_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \]

\[ E_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

The corresponding dual basis of $\mathfrak{su}t^*$ is given by $\mathcal{B}^* = \{E_1^*, E_2^*\}$

\[ E_1^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \]

\[ E_2^* = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

The factor of $\frac{1}{2}$ in $E_2^*$ comes from the the normalization $\text{Tr}(E_2E_2^*) = 1$.

Let $g = \begin{bmatrix} g_1 & g_2 \\ 0 & \frac{1}{g_1} \end{bmatrix}$ be an element in $\text{SUT}(2, \mathbb{R})$ and $X = \begin{bmatrix} u_0 & 0 \\ v_0 & -u_0 \end{bmatrix}$ be an element of $\mathfrak{su}t^*$.

**Lemma 5.1.** The co-adjoint action of $g \in \text{SUT}(2, \mathbb{R})$ on $\mathfrak{su}t^*$, denoted by $\text{Ad}^*_g$, is given by the following:

\[ \text{Ad}^*_g : \mathfrak{su}t^* \to \mathfrak{su}t^* \]
Lemma 5.3. Every element of $V$ is given by $\text{Ad}_g^*[u_0 \ 0 \ v_0 \ -u_0] = \left[ \begin{array}{cc} u_0 + g_2g_1^{-1}v_0 & 0 \\ v_0g_1^{-1} & -(u_0 + g_2g_1^{-1}v_0) \end{array} \right]$. 

Proof. Let us denote by $\langle A, B \rangle = \text{Tr}(AB)$, $A \in \text{sut}^*$, $B \in \text{sut}$. By definition, $\langle \text{Ad}_g^*X, Y \rangle = \langle X, \text{Ad}_gY \rangle$. Where $\text{Ad}_g^{-1}Y = g^{-1}Yg$.

It is easy to see that if $Y = E_1 \text{Ad}_g^{-1}Y = g_1^{-2}E_1$.

Similarly if $Y = E_2$, $\text{Ad}_g^{-1}Y = 2g_2g_1^{-1}E_1 + E_2$.

Then if $Y = E_1$, $\langle \text{Ad}_g^*(2u_0E_2^* + v_0E_1^*), Y \rangle = v_0g_1^{-2}$.

If $Y = E_2$, then $\langle \text{Ad}_g^*(2u_0E_2^* + v_0E_1^*), Y \rangle = 2(u_0 + v_0g_1^{-1})$.

Let $\text{Ad}_g^*[u_0 \ 0 \ v_0 \ -u_0] = \left[ \begin{array}{cc} a & 0 \\ c & -a \end{array} \right]$. Then clearly, $\text{Tr}(a \ 0 \ 0 \ 1) = c = v_0g_1^{-2}$.

Similarly, $\text{Tr}(a \ 0 \ 1 \ -1) = 2a = 2(u_0 + v_0g_1^{-1})$ and thus $a = u_0 + v_0g_1^{-1}$. \hfill \Box

Lemma 5.2. The orbit of the co-adjoint action on $X \in g^*$, denoted by $O_X$ is given as follows.

$$O_X = \{ \text{Ad}_g^*X : g \in \text{SUT}(2, \mathbb{R}) \} = \left\{ \left[ \begin{array}{cc} u_0 - \lambda v_0 & 0 \\ \mu v_0 & -u_0 + \lambda v_0 \end{array} \right] : \lambda, \mu \in \mathbb{R}, \mu > 0 \right\}.$$ 

$O_X$ is a 2-plane when $v_0 \neq 0$ (homeomorphic to the upper half plane). It is a point when $v_0 = 0$.

Proof. Take $\mu = g_1^{-2}$ and $\lambda = -g_2g_1^{-1}$.

From now on, we take $v_0 \neq 0$.

Fix a point $A = \left[ \begin{array}{cc} a_1 & 0 \\ a_2 & -a_1 \end{array} \right]$ in $O_X$. $O_X$ is the upper half plane and any point on this plane is given as $\left[ \begin{array}{cc} s \ 0 \\ t & -s \end{array} \right]$. This is diffeomorphic to $\mathbb{R}^2$ with global charts given as $\Psi_{v_0} : \left[ \begin{array}{cc} s \ 0 \\ t & -s \end{array} \right] \mapsto (\lambda, \mu)$, where $\lambda = \frac{u_0 - s}{v_0}$ and $\mu = \frac{t}{v_0}$.

Let $v_0 > 0$. The upper half plane has a symplectic (Kähler) form given by $d\lambda \wedge \frac{d\mu}{\mu^2}$.

The pullback form on $O_X$ is given by $\Psi_{v_0}^*(d\lambda \wedge \frac{d\mu}{\mu^2}) = \frac{dt}{v_0^2} \wedge ds$. We will use this later.

The tangent space at $A$ is denoted by $T_A O_X$. With the chart above, elements of the tangent space $T_A O_X$ are given as $\mu_1 \frac{\partial}{\partial t}|_A + \mu_2 \frac{\partial}{\partial s}|_A$.

Lemma 5.3. Every element of $\xi \in T_A O_X$ has a representation given as $\xi = \text{ad}_V^* A$ (where $V \in \text{sut}$ and $\text{ad}_V^*$ is the differential of $\text{Ad}^*$ at $V$).
Proof.

\[ \text{Ad}^* : \text{SUT}(2, \mathbb{R}) \to \text{End}(\mathfrak{su}^*). \]

\[ d(\text{Ad}^*) = \text{ad}^* : \mathfrak{su} \to \text{End}(\mathfrak{su}^*). \]

Denote, \( \text{ad}^*(V) = \text{ad}^*_V. \) Now take \( \mu \in \mathfrak{su}^* \) and \( Y \in \text{SUT}(2, \mathbb{R}). \)

\[ \langle \text{ad}^*_V \mu, Y \rangle = \langle \mu, -\text{ad}_V(Y) \rangle = -\langle \mu, [V, Y] \rangle. \]

Consider the following curve:

\[ \gamma : \mathbb{R} \to \mathcal{O}_X \]

\[ t \mapsto \text{Ad}_{e^{tV}}(A). \]

\[ \gamma(0) = \text{Ad}_{e^{0V}}(A) = \text{Ad}^*_V(A) = A \]

\[ \gamma'(0) = \text{ad}^*_V A. \]

Since \( \gamma \) is a smooth curve in \( \mathcal{O}_X \) with \( \gamma(0) = A, \gamma'(0) \in T_A \mathcal{O}_X \), we deduce that \( \text{ad}^*_V A \in T_A \mathcal{O}_X \). Conversely, given \( Y \in T_A \mathcal{O}_X \), there exists \( V^* \in \mathfrak{g} \) s.t \( \text{ad}^*_{V^*}(A) = Y \).

We make the identification more precise. Let \( V = \begin{bmatrix} v_1 & v_2 \\ 0 & -v_1 \end{bmatrix} \).

\[ \langle \text{ad}^*_V(A), E_i \rangle = -\langle A, [V, E_i] \rangle = -\text{tr}(A, [V, E_i]). \]

\[ \text{ad}^*_V(A) = \sum_{i=1}^{2} \langle \text{ad}^*_V(A), E_i \rangle E^*_i. \]

The identification of the two definition of tangent spaces is given as \( E_1^* \mapsto -\frac{\partial}{\partial t} \) and \( E_2^* \mapsto -\frac{\partial}{\partial s} \). Hence

\[ \sum_{i=1}^{2} \langle \text{ad}^*_V(A), E_i \rangle E^*_i \mapsto 2a_2 v_2 \frac{\partial}{\partial s} \bigg|_A + 2a_2 v_1 \frac{\partial}{\partial t} \bigg|_A. \]

We have the following well known symplectic form.

**Lemma 5.4.** Let \( \xi_1, \xi_2 \in T_A \mathcal{O}_X \). Then from lemma 5.3 we know we can write,

\[ \xi_1 = \text{ad}^*_{V_1} A, \]

\[ \xi_2 = \text{ad}^*_{V_2} A. \]
where $V_1, V_2 \in \mathfrak{g}$. The Kirillov-Kostant-Souriau symplectic form, $\omega \in \Lambda^2(\mathcal{O}_X)$ is defined as follows:

$$
\omega_A(\xi_1, \xi_2) := \omega_A(\text{ad}^*_V A, \text{ad}^*_V A) \\
= \langle A, [V_1, V_2] \rangle \\
= A([V_1, V_2]).
$$

Then we have the following well known theorem (see [1], [3] for instance) for which we give a simple proof.

**Theorem 5.1.** \(\mathcal{O}_X\) is a symplectic manifold] For $v_0 \neq 0$, $\mathcal{O}_X$ has a natural structure of a symplectic manifold given by the Kirillov-Kostant-Souriau symplectic form, $\omega$.

**Proof.** Since $\omega$ is a symplectic form on $\mathcal{O}_X$, $\omega_A \in \Lambda^2(T_A \mathcal{O}_X)$.
Now using the definition of tangent space, since the manifold is two dimensional,

$$
T_A(\mathcal{O}_X) = \{(A, \mu_1, \mu_2) : \mu_1, \mu_2 \in \mathbb{R}\} \\
= \{\text{ad}^*_V (A) : V \in \mathfrak{su}\}.
$$

Take two tangent vectors $\mu, \xi$.

$$
\mu = \mu_1 \left. \frac{\partial}{\partial s} \right|_A + \mu_2 \left. \frac{\partial}{\partial t} \right|_A, \\
\xi = \xi_1 \left. \frac{\partial}{\partial s} \right|_A + \xi_2 \left. \frac{\partial}{\partial t} \right|_A.
$$

We calculate $X_\mu, X_\xi$ such that

$$
\mu = \text{ad}^*_A (\mu_1 \left. \frac{\partial}{\partial s} \right|_A - \mu_2 \left. \frac{\partial}{\partial t} \right|_A), \\
\xi = \text{ad}^*_A (\xi_1 \left. \frac{\partial}{\partial s} \right|_A - \xi_2 \left. \frac{\partial}{\partial t} \right|_A).
$$

Using the identification given above,

$$
X_\mu = \begin{bmatrix} \frac{\mu_1}{2 \mu_2} & \frac{\mu_1}{2 \mu_2} \\
\frac{\mu_2}{2 \mu_1} & \frac{-\mu_1}{2 \mu_2} \end{bmatrix}, \\
X_\xi = \begin{bmatrix} \frac{\xi_2}{2 \xi_1} & \frac{-\xi_2}{2 \xi_1} \\
\frac{\xi_1}{2 \xi_2} & \frac{-\xi_1}{2 \xi_2} \end{bmatrix}.
$$
Now we calculate the symplectic form:

\[ \omega_A(\mu, \xi) = \omega_A \left( \text{ad}_{X_{\mu}}^*(A), \text{ad}_{X_{\xi}}^*(A) \right) = \langle [X_{\mu}, X_{\xi}] \rangle = -\frac{1}{2a_2}(\xi_1 \mu_2 - \mu_1 \xi_2). \]

Hence,

\[ \omega_A = \frac{1}{a_2} ds|_A \wedge dt|_A, \]
\[ \omega = \frac{1}{t} ds \wedge dt. \]

Now, let \( P = \begin{bmatrix} s & 0 \\ t & -s \end{bmatrix} \in O_X \). Define the functions \( J_1, J_2 \) on \( O_X \) as follows:

\[ J_i = \text{tr}(PE_i). \]

Then \( J_1 = t \) and \( J_2 = 2s \).

Since \( \omega = ds \wedge \frac{dt}{t} \), we get

**Lemma 5.5.** The Hamiltonian vectors fields corresponding to \( J_i \)'s are denoted by \( X_{J_i} \), satisfying, \( \omega(X_{J_i}, -) = dJ_i(-) \) are

\[ X_{J_1} = t \frac{\partial}{\partial s}, \]
\[ X_{J_2} = -2t \frac{\partial}{\partial t}. \]

\( O_X \) is a symplectic manifold and we can perform geometric prequantization on it. Denote the Hilbert space of square integrable sections as \( H \). Let \( \psi \in H \).

Let a choice of a symplectic potential be \( \theta = -|\log(t)|ds \).

**Proposition 5.1.** The operators corresponding to the \( J_i \)'s on the Hilbert space after geometric prequantization are denoted by \( \hat{J}_i \) and satisfy, \( \hat{J}_i \psi = -i\hbar \nabla_{X_{J_i}} + J_i \psi \), definition as in [23].

\[ \hat{J}_1 \psi = -i\hbar t \frac{\partial}{\partial s} \psi + t \log|t|\psi + t \psi, \]
\[ \hat{J}_2 \psi = 2i\hbar t \frac{\partial}{\partial t} \psi + 2s \psi. \]

If we exponentiate these we get the operators corresponding to the group. Take \( a = \log(t) \), then

\[ \frac{\partial}{\partial a} = t \frac{\partial}{\partial t}. \]
Note that $s$ and $\frac{\partial}{\partial a}$ commute. Hence we can use the property of exponentials to obtain:

**Proposition 5.2.**

$$e^{\tau J_1} \psi(a, s) = e^{\tau (\log|t| + 1)} \psi(a, s - i\hbar \tau),$$

$$e^{\tau J_2} \psi(a, s) = e^{\tau \psi(a + 2i\hbar \tau, s))}.$$

We have the following inclusion of $O_X$ in $\mathbb{R}^2$:

$$\begin{pmatrix} u & 0 \\ v & -u \end{pmatrix} \mapsto (u, v).$$

Then the corresponding action of $\text{SU}(2, \mathbb{R})$ on $\mathbb{R}^2$ is $(u_0, v_0) \mapsto (u_0 - \lambda v_0, \mu v_0)$.

**Lemma 5.6.** The stabilizer of the action, $H = \pm I$.

**Proof.**

$$\begin{pmatrix} u_0 & 0 \\ v_0 & -u_0 \end{pmatrix} = \begin{pmatrix} g_1 & g_2 \\ 0 & \frac{1}{g_1} \end{pmatrix} \cdot \begin{pmatrix} u_0 & 0 \\ v_0 & -u_0 \end{pmatrix}$$

$$= \begin{pmatrix} u_0 - \frac{\sqrt{t}}{g_1}v_0 & 0 \\ \frac{1}{g_1}v_0 & -u_0 + \frac{\sqrt{t}}{g_1}v_0 \end{pmatrix}.$$  

Hence, $\frac{\sqrt{t}}{g_1} = 0$ and $\frac{1}{g_1} = 1 \implies g_2 = 0, g_1 = \pm 1$. So $H = \pm I$.  

We work with $\text{SU}(2, \mathbb{R})$ matrices with positive diagonals from now on, so that $H$ is trivial. Note that this is one of the connected components. Since $H$ is trivial we have, $\text{SU}^+ \cong O_X$.

We then have direct proof of the following result in the next proposition. The result can be found for instance in [1].

**Proposition 5.3.** $\text{SU}^+$ is a Kähler manifold.

**Proof.** Let

$$\Phi_{v_0} : O_X \rightarrow \text{SU}^+$$

$$\begin{pmatrix} s & 0 \\ t & -s \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{\frac{a^2}{t}} & \frac{u_0-s}{\sqrt{v_0t}} \\ 0 & \sqrt{\frac{t}{v_0}} \end{pmatrix}. $$

$$\Phi_{v_0}^{-1} : \text{SU}^+ \rightarrow O_X$$

$$\begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix} \mapsto \begin{pmatrix} u_0 - \frac{b v_0}{a^2} & 0 \\ \frac{v_0}{a^2} & -u_0 + \frac{b v_0}{a} \end{pmatrix}.$$  

where $a = \sqrt{\frac{a^2}{t}}$ and $b = \frac{u_0-s}{\sqrt{v_0t}}$.

Recall there is a map $\Psi_{v_0} : \begin{pmatrix} s & 0 \\ t & -s \end{pmatrix} \mapsto (\lambda, \mu)$, where $\lambda = \frac{u_0-s}{v_0}$ and $\mu = \frac{t}{v_0}$. 

Let $\chi_{v_0} = \Psi_{v_0} \circ \Phi_{v_0}^{-1}$ be a homeomorphism of $SUT^+$ to the upper half plane.

It is given by $\chi_{v_0} : SUT^+ \mapsto H$, as follows:

$$\left( \begin{array}{cc} a & b \\ 0 & \frac{1}{a} \end{array} \right) \mapsto \left( \begin{array}{cc} u_0 - \frac{bv_0}{a} \\ \frac{v_0}{a^2} \end{array} \right),$$

$$\mapsto \left( \begin{array}{c} b \\ \frac{1}{a} \end{array} \right) = (\lambda, \mu) \in H.$$

Here $s = u_0 - \frac{bv_0}{a}$ and $t = \frac{v_0}{a^2}$. Then calculating the pull-back of the usual Kahler form $\Omega = d\lambda \wedge \frac{du}{\mu^2}$ on the upper half plane, we have,

$$\chi_{v_0}^* (\Omega) = \Phi_{v_0}^{-1} \circ \Psi_{v_0}^* \left( d\lambda \wedge \frac{d\mu}{\mu^2} \right) \quad \mapsto \Phi_{v_0}^{-1} \left( \frac{dt}{t^2} \wedge ds \right)$$

$$= 2da \wedge db = 2db \wedge \frac{\tilde{a} d\tilde{a}}{a^2},$$

where $\tilde{a} = \frac{1}{a}$. Here we have used the fact that $a = \sqrt{\frac{v_0}{t}}$ and $b = \frac{v_0 - s}{\sqrt{v_0 t}}$.

Let the map $F_1$ be as follows.

$$F_1 : (b, \tilde{a}) \in H \mapsto \left( \begin{array}{cc} a & b \\ 0 & \frac{1}{a} \end{array} \right) \in SUT^+, \quad \text{where} \quad \tilde{a} = \frac{1}{a}, \quad b + i\tilde{a} \in H.$$

$$F_1^* \circ \chi_{v_0}^* \left( d\lambda \wedge \frac{du}{\mu^2} \right) = 2db \wedge \frac{d\tilde{a}}{a}$$

which is a Kahler form on $H$. Then it is easy to see that $\chi_{v_0}^* (\Omega)$ is Kahler via the map $F_1$.

\[ \square \]

6. Conclusion

In this paper we have considered some examples of Lie groups and studied coherent states as embeddings of the classical phase space into the quantum ray space. We find that the states saturating the Heisenberg uncertainty fall into two classes: coherent states which yield Kahler embeddings and squeezed states which only give us symplectic embeddings. Our motivation was to understand from these examples, which coadjoint orbits naturally admit a Kahler structure and whether this agrees with the pullback of the structure on projective Hilbert space. Let us consider this aspect in more detail.

Do co-adjoint orbits of Lie groups naturally admit Kahler structures? Let us see how our examples illuminate this question. In the case of compact semisimple groups, there is a positive definite Cartan-Killing metric on the Lie algebra. This metric permits us to identify $\mathfrak{g}$ with its dual and we get a Riemannian metric on $\mathfrak{g}^*$. The generic co-adjoint orbit is a submanifold of $\mathfrak{g}^*$ and we get an induced
Riemannian metric on $\mathcal{O}_X$. This leads to a metric and then a Kähler structure on $\mathcal{O}_X$. This is in fact what happens with the SU(2) example.

Let us now consider noncompact semisimple groups. While it is true that the Lie algebra has a non-degenerate Cartan-Killing metric, this metric has indefinite signature. In general the orbits do not necessarily have a Riemannian metric: the induced metric may be Lorentzian or even degenerate. It is still possible however, that the orbits inherit a Riemannian metric. This is just what occurs in the case of SU(1, 1)/U(1) that we considered. While the signature of the Cartan-Killing form is $\{-, -, +\}$, the $+$ direction is in the stabiliser U(1) of the group action on the orbit. As a result there is a (negative) definite metric on the co-adjoint orbit. Reversing the sign of the metric gives us a Riemannian metric. This example as well as the previous one fall within the framework of [21].

In contrast the Weyl-Heisenberg group is not semi-simple. In fact, the Cartan-Killing form vanishes identically. We might then suppose that the metric on the complex plane can be derived from the complex structure on the W-H group. A complex structure can be combined with the symplectic structure to give us a metric.

The case of SUT is also similar. G is upper triangular and non-compact. The Cartan-Killing form is degenerate (though not identically zero). We can regard G as a subgroup of SL(2, $\mathbb{R}$) where the Cartan-Killing metric is nondegenerate, but of Lorentzian signature. This metric does not pull back to a Riemannian metric on $\mathcal{O}_X$. We are led to ask what metrics naturally exist on $\mathcal{O}_X$? Since $\mathcal{O}_X$ is a homogeneous space $G/H$, we would expect that the metric must also be homogeneous and therefore have constant curvature. In two dimensions, constant curvature metrics can only be the plane, the sphere and the upper half plane. The second is ruled out since it is compact, unlike the coadjoint orbit of SUT. The others have isometry groups which are respectively $E(2)$ and $SO(2, 1)$. The Lie algebras of these groups would generate symmetries of $\mathcal{O}_X$. In our case the symmetry group of $\mathcal{O}_X$ is two dimensional and nonabelian. The Lie algebra of $E(2)$ does not admit such a subalgebra. We conclude that the natural Riemannian metric on $\mathcal{O}_X$ is the constant negative curvature metric. We can use such a metric to determine a complex structure on $\mathcal{O}_X$, thereby turning it into a Kähler manifold. We have seen that in cases where there is a natural Kähler structure on the co-adjoint orbit, the technique of pulling back using coherent states agrees with the natural structure up to an overall constant. It may be possible to use this technique more generally to induce Kähler structures on co-adjoint orbits.

7. Appendix

7.1. Appendix1: Minimum Uncertainty States: This appendix summarises the argument for the Robertson-Schrödinger uncertainty relations [16, 19] which were used in the text to motivate our definition of squeezed states in a broader context than SCS. Let $\hat{A}, \hat{B}$ be Hermitian operators on a Hilbert space $\mathcal{H}$. For any
fixed, normalized \( |\psi\rangle \in \mathcal{H} \), define \( a = \langle \psi | \hat{A} | \psi \rangle \) and \( b = \langle \psi | \hat{B} | \psi \rangle \) and \( \hat{A} = \hat{A} - aI \), \( \hat{B} = \hat{B} - bI \). Let \( |\sigma\rangle = \left( \hat{A} + \gamma \hat{B} \right) |\psi\rangle \) where \( \gamma = \gamma_1 + i\gamma_2 \) is a complex number. The function \( F(\gamma) = \langle \sigma | \sigma \rangle \geq 0 \) of \( \gamma \) is non-negative and vanishes only when \( |\sigma\rangle \) does. Then \( F(\gamma) = |\gamma|^2 \beta + \gamma_1 \langle \psi | \left\{ \hat{A}, \hat{B} \right\} |\psi\rangle + i\gamma_2 \langle \psi | \left[ \hat{A}, \hat{B} \right] |\psi\rangle + \alpha \), where \( \beta = \langle \psi | \hat{B}^2 |\psi\rangle \), \( \alpha = \langle \psi | \hat{A}^2 |\psi\rangle \). Defining \( C_+ = \langle \psi | \left\{ \hat{A}, \hat{B} \right\} |\psi\rangle \) and \( C_- = i \langle \psi | \left[ \hat{A}, \hat{B} \right] |\psi\rangle \), we have \( F(\gamma) = \beta |\gamma|^2 + \gamma_1 C_+ + \gamma_2 C_- + \alpha \geq 0 \).

To derive the uncertainty relations, we consider \( F(\gamma) \) as a quadratic in \( \gamma \) on the complex \( \gamma \) plane and look at special cases.

1. Heisenberg uncertainty relations: One sets \( \gamma_1 = 0 \). Then \( \beta \gamma_2^2 + \gamma_2 C_- + \alpha \geq 0 \) for all \( \gamma_2 \) (real). This implies \( C_-^2 - 4\beta \alpha \leq 0 \) or \( \beta \alpha \geq \frac{C_-^2}{4} \). For \( \hat{A} = \hat{x} \) and \( \hat{B} = \hat{p} \) we have \( \left\{ \hat{A}, \hat{B} \right\} = i\hbar \) and thus \( C_- = -\hbar \). Then \( \beta \alpha \geq \frac{\hbar^2}{4} \). It is easy to see that \( \alpha = (\Delta A)^2 \) and \( \beta = (\Delta B)^2 \) which leads to

\[
\Delta A \Delta B \geq \frac{\hbar}{2}.
\]

2. Set \( \gamma_2 = 0 \). Then \( \beta \gamma_1^2 + \gamma_1 C_+ + \alpha \geq 0 \) for all \( \gamma_1 \) real implies \( \beta \alpha \geq C_+^2 \) or

\[
\Delta A \Delta B \geq \frac{1}{2} \left| \left\{ \hat{A}, \hat{B} \right\} \right|^2.
\]

3. Robertson-Schrödinger uncertainty relations: The strongest version of the uncertainty relations comes from using complex \( \gamma \) and minimising \( F(\gamma) \) over the complex plane. Let \( \gamma = \rho e^{i\theta} \). Then \( F(\rho, \theta) = \beta \rho^2 + \rho \cos(\theta) C_+ + \rho \sin(\theta) C_- + \alpha \geq 0 \). Setting \( \frac{\partial F}{\partial \rho} = 0 \) and \( \frac{\partial F}{\partial \theta} = 0 \) we have (since \( \rho > 0 \)) \( \tan(\theta) = \frac{C_+}{C_-} \). In other words \( \cos(\theta) = \frac{C_+ \epsilon}{C_-} \) and \( \sin(\theta) = \frac{C_- \epsilon}{C_-} \) where \( \epsilon = \pm 1 \) and \( C_- = C_+^2 + C_-^2 \). For the minimum, we compute the second derivatives. We get \( \frac{\partial^2 F}{\partial \rho \partial \rho} = 2\beta \), \( \frac{\partial^2 F}{\partial \rho \partial \theta} = -\rho C \epsilon \) and \( \frac{\partial^2 F}{\partial \theta \partial \theta} = 0 \). So for a minimum, \( \epsilon = -1 \). The value of \( F(\rho, \theta) = \frac{-C_-^2}{4} + \alpha \geq 0 \) at the minimum. This gives \( \alpha \beta \geq \frac{C_-^2}{4} \) where \( C_- = C_+^2 + C_-^2 \) or

\[
\Delta A \Delta B \geq \frac{1}{2} \sqrt{\left| \left\{ \hat{A}, \hat{B} \right\} \right|^2 + \left| \left[ \hat{A}, \hat{B} \right] \right|^2}.
\]

The Robertson-Schrödinger inequality implies the Heisenberg inequality and the inequality in (2) as well. The inequality is saturated when \( |\sigma\rangle = (\hat{A} + \gamma_0 \hat{B}) |\psi\rangle = 0 \). Or with some rearrangement (setting \( i\lambda^{-2} = \gamma_0 \))

\[
\left( \lambda \hat{A} + i \hat{B} / \lambda \right) |\psi\rangle = (\lambda a + ib/\lambda) |\psi\rangle,
\]

the condition we use to define minimum uncertainty states. We need only consider \( \lambda \) real and positive.
7.2. Appendix 2: Coherent states and Berezin Quantization of the phase space. The phase space is identified with the upper half plane \( \mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \). In this section we show the Berezin Quantization of the upper half plane adapting the one for the unit disc as described in Perelomov [15], chapter 16.

Recall there is a biholomorphism from \( \mathbb{H} \) to the unit disc \( \mathbb{D} \) given by: \( \epsilon : \mathbb{H} \to \mathbb{D} \)

\[
\epsilon(w) = \frac{w - i}{w + i}.
\]

Let \( \chi \) be the inverse of \( \epsilon \). The Kähler form of \( \mathbb{D} \) is \( d\mu(z, \bar{z}) = \frac{1}{2\pi i} \frac{dz \wedge d\bar{z}}{(1 - |z|^2)^2} \). Using \( z = \epsilon(w) \) we get

\[
d\mu(z, \bar{z}) = \frac{1}{2\pi i} \frac{d\bar{w} \wedge dw}{4(\text{Im}(w))^2} = d\mu(w, \bar{w}).
\]

Also, \( (1 - |z|^2)^{1/\hbar} = \left( \frac{4\text{Im}(w)}{|w|^2 + 2\text{Im}(w) + 1} \right)^{1/\hbar} \).

Let \( f \in C^\infty(\mathbb{H}) \). Then \( f = \epsilon^* (\phi) \) where \( \phi \in C^\infty(\mathbb{D}) \) and \( \phi = \chi^*(f) \).

Let \( f, g \in C^\infty(\mathbb{H}) \). Let \( \psi = \chi^*(g) \). Then \( (f, g)_\mathbb{H} = (\phi, \psi)_\mathbb{D} \) where

\[
(\phi, \psi)_\mathbb{D} = \left( \frac{1}{\hbar} - 1 \right) \int_\mathbb{D} \bar{\phi}(z) \psi(z) (1 - |z|^2)^{1/2} \frac{1}{\pi} d\mu(z, \bar{z})
\]

\[
= \left( \frac{1}{\hbar} - 1 \right) \int_\mathbb{H} \bar{f}(w) g(w) \left( \frac{4\text{Im}(w)}{|w|^2 + 2\text{Im}(w) + 1} \right)^{1/\hbar} d\mu(w, \bar{w}).
\]

Let \( \mathcal{F}_h \) be the space of all smooth square integrable functions with respect to the above inner product, i.e. \( \|f\|_\mathbb{H} < \infty \).

We proceed as in Perelomov, [15], chapter 16 where the Berezin quantization on the Lobachevsky plane is explained.

Let \( \psi_I(z) = (l!)^{-1/2} \left[ \left( \frac{1}{\hbar} \right) \cdots \left( \frac{1}{\hbar} - 1 + l \right) \right]^{1/2} \times z^I \) be the orthonormal basis for \( \chi^*(\mathcal{F}_h) \) (from 16.3.3, [15]).

Let \( f_l(w) = (l!)^{-1/2} \left[ \left( \frac{1}{\hbar} \right) \cdots \left( \frac{1}{\hbar} - 1 + l \right) \right]^{1/2} \times \left( \frac{w-l}{w+l} \right)^I \) be a basis for \( \mathcal{F}_h \).

\( (f_l, f_m)_\mathbb{H} = (\psi_I, \psi_m) = \delta_{lm} \).

Let \( \tau_p = \sum_l \bar{f}_I(p) f_l \) be the coherent state parametrized by \( p \), i.e. \( \tau_p(w) = \sum_l \bar{f}_I(p) f_l(w) \), \( w \in \mathbb{H} \).

One can show that for any function \( f \in \mathcal{F}_h \), \( (\tau_p, f)_\mathbb{H} = f(p) \).

Let \( \hat{P} \) be a bounded linear operator acting on \( \mathcal{F}_h \).

Then it is easy to check that there exists an operator \( \hat{A} \) acting on the square integrable smooth functions on \( \mathbb{D} \) such that:

\[
\hat{P}(f) = \hat{P}(\epsilon^*(\phi)) = \epsilon^* \left( \hat{A}(\phi) \right) = \epsilon^* \hat{A}\left( \chi^*(f) \right),
\]

where \( \hat{A} = \chi^* \hat{P} \epsilon^* \).

The symbol of \( \hat{P} \) is defined to be \( \mathcal{P}(p, \bar{q}) = (\tau_p, \hat{P} \tau_q)_\mathbb{H} \).
An easy calculation shows that $\chi^*(\tau_p) = \psi_\zeta$ is a coherent state on $\mathbb{D}$ parametrized by $\zeta = \epsilon(p)$. In fact all coherent states on $\mathbb{D}$ are of this form. Let the symbol of $\hat{A}$ be denoted by $A(\zeta, \bar{\eta})$ where $\eta = \epsilon(q)$. One can show that the symbol $\mathcal{P}(p, \bar{q}) = A(\zeta, \bar{\eta})$, i.e. the two symbols are in fact the same.

**The star product:** Let $\hat{P}_1, \hat{P}_2$ be two bounded linear operators acting on $\mathcal{F}_\hbar$ and $\hat{A}_1, \hat{A}_2$ be two bounded linear operators acting on $\chi^*(\mathcal{F}_\hbar)$. Then it is easy to show that the star product $(\mathcal{P}_1 * \mathcal{P}_2)(p, \bar{p}) = (A_1 * A_2)(\zeta, \bar{\zeta})$ is the symbol for the composition operator $\hat{P}_1 \circ \hat{P}_2$.

It has been proved in [15], chapter 16, that the correspondence principle holds for $A_1 * A_2$. Thus,

$$\lim_{\hbar \to 0} (\mathcal{P}_1 * \mathcal{P}_2)(p, \bar{p}) = \mathcal{P}_1(p, \bar{p}) \mathcal{P}_2(p, \bar{p}).$$

Also,

$$\lim_{\hbar \to 0} \frac{1}{\hbar} ((\mathcal{P}_1 * \mathcal{P}_2) - (\mathcal{P}_2 * \mathcal{P}_1)) = \{A_1, A_2\}_D = \{\mathcal{P}_1, \mathcal{P}_2\}_H,$$

where the Poisson brackets are defined as follows:

$$\{A_1, A_2\}_D = (1 - |z|^2)^2 \left( \frac{\partial A_1}{\partial \bar{z}} \frac{\partial A_2}{\partial z} - \frac{\partial A_2}{\partial \bar{z}} \frac{\partial A_1}{\partial z} \right)$$

$$= 4(\text{Im}(w))^2 \left( \frac{\partial \mathcal{P}_1}{\partial \bar{w}} \frac{\partial \mathcal{P}_2}{\partial w} - \frac{\partial \mathcal{P}_2}{\partial \bar{w}} \frac{\partial \mathcal{P}_1}{\partial w} \right)$$

$$= \{\mathcal{P}_1, \mathcal{P}_2\}_H.$$  \hspace{1cm} (22)

The last equality follows from the fact that $(1 - |z|^2)^2 = \frac{16(\text{Im}(w))^2}{(|w|^2 + 2\text{Im}(w) + 1)^2}$ and $|\frac{\partial w}{\partial \bar{z}}|^2 = \frac{1}{4} (|w|^2 + 2\text{Im}(w) + 1)^2$ where recall $z = \frac{w - \bar{w}}{w + \bar{w}}$.

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**References**

[1] M. Adler: On a Trace Functional for Formal Pseudo-Differential Operators and the Symplectic Structure of the Korteweg-Devries Type Equations, *Inventiones mathematicae* 50, 219 (1978).
[2] C. Brif, A. Vourdas and A. Mann: Analytic representations based on SU(1,1) coherent states and their applications *Journal of Physics A:Mathematical and General* 29, 5873 (1996).
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[3] R. Dey and S. Ganguli: Geometric Quantization of finite Toda systems and coherent states 
J. Geom. Symm. Phys. 44, 21 (2017).
[4] R. Dey, K. Ghosh: Pull back coherent states and squeezed states and quantization, Symmetry, Integrability and Geometry: Methods and Applications (SIGMA), 18, (2022), 028, 14 pages – arxiv: 2108.08082
[5] A. J. Di Scala, H. Ishi and A. Loi: Kähler immersions of homogeneous Kähler manifolds into complex space forms, Asian J. Math. 16 (3), 479 (2012); arXiv:1009.4045
[6] S. Dwivedi, J. Herman, L.C. Jeffrey and T. van den Hurk: The Symplectic Structure on Coadjoint Orbits, Hamiltonian Group Actions and Equivariant Cohomology, 27-29, Springer International Publishing 2019.
[7] V. A. Fock: Verallgemeinerung und Lasung der Diracschen statistischen Gleichung, Z. Phys.49, 339 (1928).
[8] R. J. Glauber: Coherent and Incoherent States of the Radiation Field, PhysRev. 131 (6), 2766 (1963).
[9] J. R. Klauder and B. Skagerstam: Coherent States, World Scientific, Singapore 1985.
[10] J. R. Klauder and E. C. G. Sudarshan: Fundamentals of Quantum Optics, W. A. Benjamin, New York 1968.
[11] J. G. Kuriyan, N. Mukunda and E. C. G. Sudarshan: Master Analytic Representation: Reduction of O(2,) in an O(1,1) Basis, J. Math. Phys. 9, 2100 (1968).
[12] A. Loi and R. Mossa: Some Remarks On Homogeneous Kähler Manifolds, Geometriae dedicata 179(1), 377 (2015). arXiv:1502.00011
[13] N. Mukunda, E. C. G. Sudarshan, J. K. Sharma and C. L. Mehta: Representations and properties of Bose oscillator operators. I. Energy position and momentum eigenstates, J. Math. Phys.21, 2386 (1980).
[14] A. Odzijewicz: Coherent states and geometric quantization, Commun. Math. Phys. 150 (2), 385 (1992).
[15] A. Perelomov: Generalized Coherent States and Their Applications, Springer-Verlag, Berlin 1986.
[16] H. P. Robertson: The Uncertainty Principle Phys. Rev. 34(1).163 (1929).
[17] R. Schnabel: Squeezed states of light and their applications in laser interferometers, Phys. Rep. 684,1 (2017).
[18] E. Schrödinger: Der stetige Abergang von der Mikro- zur Makromechanik, Die Naturwissenschaften 14, 664 (1926).
[19] E. Schrödinger: Zum Heisenbergschen Unschärfeprinzip Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalisch-mathematische Klasse 14, 296 (1930).
[20] E. C. G. Sudarshan: Equivalence of Semiclassical and Quantum Mechanical Descriptions of Statistical Light Beams Phys. Rev. Lett. 10 (7), 277 (1963).
[21] P. B. Villa: Kählerian structures of coadjoint orbits of semisimple Lie groups and their orbihedra, Dissertation, Ruhr-Universität Bochum, 2015.
[22] F. Wilczek and A. Shapere: Geometric Phases in Physics, World Scientific, Singapore 1989.
[23] N. M. J. Woodhouse: Geometric Quantization, Oxford Mathematical Monographs, Claredon Press , Oxford 2007.