Meson effective mass in the isospin medium in hard-wall AdS/QCD model

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AdS/CFT first works

1. *The Large N limit of superconformal field theories and supergravity*  
   Juan Martin Maldacena (Harvard U.). Adv.Theor.Math.Phys. 2 (1998) 231-252

2. *Anti-de Sitter space and holography*  
   Edward Witten (Princeton, Inst. Advanced Study).  
   Published in Adv.Theor.Math.Phys. 2 (1998) 253-291

3. *Gauge theory correlators from noncritical string theory*  
   S.S. Gubser, Igor R. Klebanov, Alexander M. Polyakov (Princeton U.).  
   Published in Phys.Lett. B428 (1998) 105-114
I. INTRODUCTION

The AdS/CFT correspondence [1–3] discovered in recent decades is successfully applied for solving problems in different branches of physics. This correspondence establishes duality between the fields in the bulk of an anti-de Sitter (AdS) space with the field theory operators defined on the ultraviolet (UV) boundary of that space. For Quantum Chromodynamics this duality has special importance, since the ordinary perturbation theory does not work at a low energy limit and it needs non-perturbative methods for solving problems of strong interactions in this energy region. Application of AdS/CFT correspondence principle (or holography principle) to the QCD theory, which is named as holographic QCD, found to be a useful concept for the studies in QCD at low energies. In holographic QCD there are two approaches, which were named top-down and bottom-up ones. The top-down approach includes QCD models based on the string and D-brane theories.
Meanwhile, the bottom-up approach of the holographic QCD is based on direct application of the AdS/CFT principle to the theory of strongly interacting particles and so is frequently called AdS/QCD. Two main models of AdS/QCD, which are known as the hard-wall and soft-wall models, were constructed under the finiteness condition of the action for the model at the infrared (IR) boundary of AdS space. In the hard-wall model this condition is ensured by the cut off at this boundary and in the soft-wall one the condition is satisfied by introducing an exponential factor, which suppresses expressions under the integral over the extra dimension at infinite values of this dimension. The hard-wall model produces a linearly growing mass spectrum of mesons and the soft-wall one gives linearly growing spectrum for the squared mass. Both models are a powerful tool for the calculation of various coupling constants of the strongly interacting particles as well as the form factors for them. In the AdS/QCD models framework these coupling constants can be derived for both the ground and excited states of mesons and baryons, which enlarges the range of applicability of these models. There is light-front approach in AdS/QCD, within which these coupling constants and form factors could be derived as well. Besides calculations performed in vacuum, the AdS/QCD models are applied to research of physical effects and quantities in the dense nuclear medium.
In holographic QCD the two phases of nuclear matter - the confining and deconfining ones, in the dual gravity theory side are described by the different metrics. In the bottom-up approach, when the quark matter is absent the deconfining phase at the gravity hand is described by the Schwarzschild AdS black hole (SAdS BH) metric [E. Witten, Adv. Theor. Math. Phys. 2, 505 (1998)].

\[ ds^2 = \left( \frac{r^2}{b^2} + 1 - \frac{w_n M}{r^{n-2}} \right) dt^2 + \frac{dr^2}{\left( \frac{r^2}{b^2} + 1 - \frac{w_n M}{r^{n-2}} \right)} + r^2 d\Omega^2 \]

\[ w_n = \frac{16\pi G_N}{(n-1)\text{Vol}(S^{n-1})}. \]

The confined phase of the nuclear matter at the low temperature limit in the dual gravity theory is described by the thermal AdS space (tAdS) C. P. Herzog, Phys. Rev. Lett. 98, 091601 (2007)

\[ ds^2 = \frac{L^2}{z^2} \left( f(z) dt^2 + dx^2 + \frac{dz^2}{f(z)} \right) \]

\[ f(z) = 1 - \left( \frac{z}{z_h} \right)^4 \]
Dual Geometry for the Confining phase with quark field

What is the dual geometry describing the hadronic phase, in other words the confinement phase with quark matters?

B. -H. Lee, C. Park and S. -J. Sin, JHEP 0907, 087 (2009) [arXiv:0905.2800 [hep-th]].

It was shown on the background geometry of the dual gravity for the confinement phase containing the quark matter fields as well. This metric is named the thermal charged AdS space (tcAdS), and it can be obtained from the Reissner-Nordstrom black hole (RNAdS BH) metric by taking a zero of the mass of black hole and cutting of the fifth dimension at infrared (IR) boundary.
the thermal charged AdS (tcAdS) geometry, which is a non-black brane solution satisfying the Einstein and Maxwell equations simultaneously, is given by

\[ ds^2 = \frac{R^2}{z^2} \left( -f(z) dt^2 + \frac{1}{f(z)} dz^2 + dx^2 \right), \]

\[ f(z) = 1 + \sum_{\alpha=u,d} q_{\alpha} z^6, \]

where \( R \) is the AdS radius. The bulk gauge field satisfying the Maxwell equations becomes

\[ A_0^\alpha = 2\pi^2 \mu_\alpha - Q_\alpha z, \]

\( \mu_\alpha \) and \( Q_\alpha \) corresponds to the chemical potential and number density of u- and d-quark.

Note that \( q_\alpha \) in the tcAdS space is related to the number density \( Q_\alpha \)

\[ q_\alpha^2 = \frac{2\kappa^2}{3g^2R^2} Q_\alpha^2. \]
Isospin medium is the simplification of the dense nuclear medium, where the net baryon charge of the medium is taken zero while it’s isospin chemical potential remains to be non-zero. Such a simplified model in QCD was introduced in

[1] D. T. Son and M. A. Stephanov, Phys. Rev. Lett. 86, 592 (2001) [arXiv:0005225 [hep-ph]].

Such a model is useful to separate the effects taking place due to isospin from the ones occurring under the influences of other quantities of the dense medium [1, 34, 38, 39]. One of such effects is the splitting of the mass spectra of the meson states from the same isospin triplet in the medium due to their isospin interaction with the non-zero isospin of the medium. This effect was considered by a number of authors within the holographic QCD.

K. -I. Kim, Y. Kim and S. H. Lee, J. Korean Phys. Soc. 55, 1381 (2009). arXiv:0709.1772 [hep-ph].

B.-H. Lee, Sh. Mamedov, S. Nam and C. Park, JHEP 08 (2013) 045

B.-H. Lee, Sh. Mamedov and C. Park, Int. J. Mod. Phys. A 29, 1450170 (2014) [arXiv: hep-th/1402.6061].

H. Nishihara and M. Harada, Phys.Rev. D89, 076001 (2014) arXiv:1401.2928 [hep-ph]

H. Nishihara, M. Harada, Phys. Rev. D 90, 115027 (2014). arXiv:1407.7344 [hep-ph]
ISOSPIN MEDIUM IN HARD-WALL MODEL

\[ \mathcal{A}^{(L)} \text{ and } \mathcal{A}^{(R)} \]

\[ SU(2)_L \times SU(2)_R \]

\[ \mathcal{F}_{MN} = \partial_M \mathcal{A}_N - \partial_N \mathcal{A}_M - i [\mathcal{A}_M, \mathcal{A}_N], \]

\[ \mathcal{A}_M = \mathcal{A}^a_M T^a, \quad T^a = \frac{1}{2} \sigma^a. \]

\[ \mathcal{A}_z = 0. \]

\[ S = \int d^5 x \sqrt{-G} \left[ \frac{1}{2\kappa^2} (\mathcal{R} - 2\Lambda) - \frac{1}{4g^2} Tr \left( \mathcal{F}^{(L)}_{MN} \mathcal{F}^{(L)MN} + \mathcal{F}^{(R)}_{MN} \mathcal{F}^{(R)MN} \right) \right] \]

\[ \Lambda = -6/R^2 \quad 1/g^2 = N_c/(4\pi^2R) \text{ and } 1/(2\kappa^2) = N_c^2/(8\pi^2R^3). \]

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\[ A_M^{(L)} = L_M + l_M, \]
\[ A_M^{(R)} = R_M + r_M. \]

\[ V_0^3 = \frac{1}{2} \left( L_0^3 + R_0^3 \right), \]
\[ A_0^3 = \frac{1}{2} \left( L_0^3 - R_0^3 \right). \]

\[ L_0^a = R_0^a, \quad SU(2)_L \leftrightarrow SU(2)_R, \quad A_0^3 = 0 \]

\[ F_{MN}^3 = \partial_M V_N^3 - \partial_N V_M^3 \]

\( l_M, r_M \) remain to be non-abelian.

\[ S = \int d^5x \sqrt{-G} \left[ \frac{1}{2\kappa^2} (R - 2\Lambda) - \frac{1}{4g^2} F_{MN}^3 F^{3MN} \right] \]

\[ A_0^{(u),(d)} = \pm \frac{1}{\sqrt{2}} V_0^3 \]
In the isospin medium case the number densities $Q_\alpha$ are taken zero and consequently, then

$$f(z) = 1$$

the metric

$$ds^2 = \frac{R^2}{z^2} \left( -f(z) \, dt^2 + \frac{1}{f(z)} \, dz^2 + d\vec{x}^2 \right).$$

$$f(z) = 1 + z^6 \sum_{\alpha=u,d} q_{\alpha}^2.$$  

$$q_{\alpha} = \sqrt{2}\kappa Q_\alpha / (\sqrt{3}gR)$$

returns into the metric of ordinary AdS space:

$$ds^2 = \frac{R^2}{z^2} \left( -dt^2 + dz^2 + d\vec{x}^2 \right),$$

$$0 < z \leq z_{IR}$$

$$A^{(\alpha)}_0 = 2\pi^2 \mu_\alpha, \quad \alpha = u, d$$

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The boundary value of the $V_0^3$ maps to the isospin chemical potential of the medium $u$ and $d$ quarks. Here we shall deal with the case $L_0^a = R_0^a$, which means Lagrangian invariance under changing the left and right flavor groups $SU(2)_L \leftrightarrow SU(2)_R$. Obviously, $A_0^3 = 0$ for this case. This $L \leftrightarrow R$ invariance in the dual boundary theory means that the medium particles, i.e. the nucleons, are required to be in the parity-even states. Thus, we have

$$F_{MN}^3 = \partial_M V_N^3 - \partial_N V_M^3.$$ 

$$S = \int d^5 x \sqrt{-G} \left[ \frac{1}{2 \kappa^2} (R - 2 \Lambda) - \frac{1}{4 g^2} F_{MN}^3 F_{3MN} \right].$$

The holographic dual of the $A_0^{(u),(d)} = \pm \frac{1}{\sqrt{2}} V_0^3$ field will be $u$- and $d$-quarks of the boundary medium. Imposing the hard-wall cut-off on the bulk radial coordinate $z$ makes these quarks a confined ones in the boundary theory. It should be noted, the number densities of such defined quarks (isoquarks) are zero.
Here we are going to deal with the medium in the confining phase, where the fundamental excitations are nucleons but not quarks. So, the solution $V_0^3$ should be expressed in terms of the chemical potentials of nucleons. Taking into account the quark content of nucleons the isospin chemical potentials of nucleons may be defined as a sum of quark chemical potentials $\mu_P = 2\mu_u + \mu_d$ and $\mu_N = \mu_u + 2\mu_d$. For isospin matter with two flavors the number densities of nucleons are zero and $V_0^3 = \sqrt{2}\pi^2 (\mu_u - \mu_d)$ in deconfinement phase and

$$V_0^3 = \sqrt{2}\pi^2 (\mu_P - \mu_N).$$
\[ v^i_\mu = \frac{1}{\sqrt{2}} \left( l^i_\mu + r^i_\mu \right), \]

The total vector field \( \mathcal{V}_M = V_M + v_M = \frac{1}{\sqrt{2}} \left( L_M + R_M + l_M + r_M \right) \)

\[ \mathcal{F}_{MN}^\mathcal{V} = \partial_M \mathcal{V}_N - \partial_N \mathcal{V}_M - i [\mathcal{V}_M, \mathcal{V}_N] \]

\[ S_\mathcal{V} = -\frac{1}{4g^2} \int d^5 x \sqrt{-G} \left\{ \sum_{i=1}^{3} G^{\mu\nu} \left[ G^{zz} \partial_z v^i_\mu \partial_z v^i_\nu + G^{mn} \partial_\mu v^i_m \partial_\nu v^i_n \right] + G^{00} G^{mn} V_0^3 \left( v^1_m v^1_n + v^2_m v^2_n \right) \right. \]

\[ \left. + 2 \left( v^1_m \partial_0 v^2_n - v^2_m \partial_0 v^1_n \right) \right\}. \]

\[ \rho_0^m = v^3_m, \]

\[ \rho_\pm^m = \frac{1}{\sqrt{2}} \left( v^1_m \pm i v^2_m \right). \]

\[ \rho_0^m (t, z) = \int \frac{d\omega_0}{2\pi} e^{-i\omega_0 t} \rho_0^m (\omega_0, z), \]

\[ \rho_\pm^m (t, z) = \int \frac{d\omega_\pm}{2\pi} e^{-i\omega_\pm t} \rho_\pm^m (\omega_\pm, z). \]
\[
\partial_z \left( \sqrt{-G} G^{zz} G^{mn} \partial_z \rho_n^0 \right) - \omega_0^2 \sqrt{-G} G^{00} G^{mn} \rho_n^0 = 0,
\]
\[
\partial_z \left( \sqrt{-G} G^{zz} G^{mn} \partial_z \rho_n^\pm \right) - \left( \omega_\pm^2 + \left( V_0^3 \right)^2 \mp 2\omega_\pm V_0^3 \right) \sqrt{-G} G^{00} G^{mn} \rho_n^\pm = 0.
\]
\[
\partial^2 z \rho_n^0 - \frac{1}{z} \partial_z \rho_n^0 + \omega_0^2 \rho_n^0 = 0,
\]
\[
\partial^2 z \rho_n^\pm - \frac{1}{z} \partial_z \rho_n^\pm + \left( \omega_\pm \mp \sqrt{2} \pi^2 \left( \mu_P - \mu_N \right) \right)^2 \rho_n^\pm = 0,
\]
\[
L v_M^i + V_0^3 = -\frac{1}{4 g^2} \left( \mathcal{F}^i_{MN} \mathcal{F}^{iMN} \right).
\]
\[
S_{\rho_n^a + V_0^3} = -\frac{1}{4 g^2} \int d^5 x \sqrt{-G} \left( D_M^{(a)} \rho_n^a \right)^* \left( D^{(a)M} \rho_n^a \right).
\]
\[
D^{(a)}_\mu = \partial_\mu + ie^{(a)} V_0^3, \quad D^{(a)}_z = \partial_z \quad \text{with} \quad e^{(a)} = \pm 1, 0.
\]
\[ D_M^{(a)} \left( \sqrt{-G} G^{M'M} G^{mn} D_M^{(a)} \rho_n^a \right) = 0. \]

This is a five-dimensional D’Alembert equation in a curved space-time for a vector field having zero fifth component and zero five-dimensional mass \(M_5\). Boundary terms which arise on obtaining it lead to boundary conditions \(\partial_z \rho_n^a|_{z_{IR}} = 0\) and \(\rho_n^a(z)|_{\varepsilon} = 0\), which are same with free field ones.

\[ \frac{1}{\sqrt{-G}} \partial_z \left( \sqrt{-G} G^{zz'} G^{mn} \partial_z \rho^a_n \right) + \frac{1}{\sqrt{-G}} D_{\mu}^{(a)} \left( \sqrt{-G} G^{\mu\mu'} G^{mn} D_{\mu'}^{(a)} \rho^a_n \right) = 0, \]

where we divided it by \(\sqrt{-G}\). Now, let us write the D’Alembert equation in a four-dimensional curved space-time for the massive vector field in the constant external background gauge field:

\[ \frac{1}{\sqrt{-G}} D_{\mu}^{(a)} \left( \sqrt{-G} G^{\mu\mu'} G^{mn} D_{\mu'}^{(a)} \rho^a_n \right) - (m^a)^2 G^{mn} \rho^a_n = 0, \]

From this correspondence it is seen that the eigenvalues of the operator

\[ -\frac{1}{\sqrt{-G} G^{mn}} \partial_z \left( \sqrt{-G} G^{zz'} G^{mn} \partial_z \right) \]

correspond to the squared mass \((m^a)^2\).
\[
- \frac{1}{\sqrt{-G} G_{mn}} \partial_z \left( \sqrt{-G} G^{zz'} G^{mn} \partial_{z'} \rho_n^a \right) = (m^a)^2 \rho_n^a.
\]

\[
\partial_z \left( \frac{1}{z} \partial_z \rho_n^a \right) + \frac{1}{z} D^{(a)} \mu D^{(a)\mu} \rho_n^a = 0,
\]

\[
D^{(0)} \mu D^{(0)\mu} \rho_n^0 = \omega_0^2 \rho^0 = (m_0^*)^2 \rho_n^0,
\]

\[
D^{(\pm)} \mu D^{(\pm)\mu} \rho_n^\pm = \left( \omega_\pm \mp \sqrt{2} \pi^2 (\mu_P - \mu_N) \right)^2 \rho_n^\pm = (m_\pm^*)^2 \rho_n^\pm,
\]

\[
\rho_n^a (t, z) = \sum_{s=0}^{\infty} r_n^a (s) (t) R_s^a (z), \quad a = 0, \pm.
\]

\[
R_s^a (z) = z [c_1 J_1 (m^s_a z) + c_2 Y_1 (m^s_a z)].
\]
\[ m_0^s = \omega_0^s, \]
\[ m_\pm^s = \omega_\pm^s \mp \sqrt{2}\pi^2 (\mu_P - \mu_N) \]

The Dirichlet boundary condition on solution \( c_2 \) relates the normalisation constant \( c_2 \) with \( c_1 \) going to zero at UV boundary \( z = \varepsilon \):

\[ c_2^a = -c_1^a J_1 (m_a^s \varepsilon) / Y_1 (m_a^s \varepsilon) \to 0. \]

\[ (m_a^s)^2 \rho_n^a(s) = \partial_\mu \partial^\mu \rho_n^a(s). \]

\[ \partial_\mu \partial^\mu \rho_n^i = \omega_i^2 \rho_n^i. \]

\[ \left( \partial_\mu + ie^{(a)} V_\mu^3 \right) \left( \partial^\mu + ie^{(a)} V^{3\mu} \right) \rho_n^a = D^{(a)}_\mu D^{(a)}^\mu \rho_n^a = (m_a^*)^2 \rho_n^a; \]

\[ (m_a^*)^2 = (m_a)^2. \]
\[ m^*_{\pm} = \omega_{\pm} \mp V_0^3, \]

\[ m^*_0 = \omega_0. \]

Applying the Neumann boundary condition at IR boundary \( \partial_z \rho^a(z) \mid_{z_{IR}} = 0 \) on solution (29) yields the following formula for the mass spectra \( m^s_a \)

\[ m^s_0 \simeq \left( s - \frac{1}{4} \right) \frac{\pi}{z^0_{IR}}, \]

\[ m^s_{\pm} \simeq \left( s - \frac{1}{4} \right) \frac{\pi}{z^*_{IR}} \quad s = 1, 2, 3... \]

\[ (s - \frac{1}{4}) \pi / z^0_{IR} = (s - \frac{1}{4}) \pi / z^0_{IR} \]

\[ \omega^s_0 \mp \sqrt{2} \pi^2 (\mu_P - \mu_N) = \omega^s_0 \]
\[ S_{\rho^0 V_0^3} = \int d^4x \left[ \int_0^{z_{IR}} dz \sqrt{-G} \, \mathcal{L}^{(0)} + \int_0^{z_{IR}^+} dz \sqrt{-G} \, \mathcal{L}^{(+)} + \int_0^{z_{IR}^-} dz \sqrt{-G} \, \mathcal{L}^{(-)} \right], \]

\[ \mathcal{L}^{(0)} = G^{zz} G^{mn} \left( \partial_z \left( \rho_m^0 \right)^* \right) \left( \partial_z \rho_n^0 \right) + \omega_0^2 G^{00} G^{mn} \left( \rho_m^0 \right)^* \rho_n^0, \]

\[ \mathcal{L}^{(\pm)} = G^{zz} G^{mn} \left( \partial_z \left( \rho_m^{\pm} \right)^* \right) \left( \partial_z \rho_n^{\pm} \right) + \left( \omega_\pm^2 + \left( V_0^3 \right)^2 \mp 2 \omega_\pm V_0^3 \right) G^{00} G^{mn} \left( \rho_m^{\pm} \right)^* \rho_n^{\pm}. \]

\[ \left( \rho^a \right)^* \partial_z \rho^a \bigg|_{z_{UV}}^{z_{IR}} \]

\[ \partial_z \rho^0(z) \bigg|_{z_{IR}^0} = 0, \quad \partial_z \rho^+(z) \bigg|_{z_{IR}^+} = 0, \quad \partial_z \rho^-(z) \bigg|_{z_{IR}^-} = 0. \]
\[ S_{\pm} = \pm V_0^3 \int d^4 x \sqrt{-G} G^{00} G^{mn} \left( (\rho^\pm_m)^* \rho^\pm_n \right)_{z=z_{IR}}. \]

\[ S_{\rho+V_0^3} = \int d^4 x \int_0^{z_{IR}} dz \sqrt{-G} \sum_a \mathcal{L}^{(a)} + S_+ + S_- , \]

\[ \left( \partial_z \rho^0 (z) \right) |_{z_{IR}} = 0, \quad \left( \partial_z \rho^+ (z) + V_0^3 \rho^+ (z) \right) |_{z_{IR}} = 0, \quad \left( \partial_z \rho^- (z) - V_0^3 \rho^- (z) \right) |_{z_{IR}} = 0. \]

\[ m^s_0 = \omega^s_0 \simeq \left( s - \frac{1}{4} \right) \frac{\pi}{z_{IR}} , \]

\[ m^s_\pm \pm V_0^3 = \omega^s_0 \pm V_0^3 = \omega^s_0 \simeq \left( s - \frac{1}{4} \right) \frac{\pi}{z_{IR}} \quad s = 1, 2, 3... \]

\[ m_{\rho^\pm} \simeq \frac{3}{4} \pi \frac{1}{z_{IR}^0} + \sqrt{2} \pi^2 (\mu_P - \mu_N) \simeq \frac{2.4}{z_{IR}^{-1}} + \sqrt{2} \pi^2 (\mu_P - \mu_N) \]

\[ m_{\rho^0} \approx \frac{2.4}{z_{IR}^0} \]
IV. A MASS AND IR BOUNDARY SPLITTING FOR THE $a_1$ AND $\pi$ MESONS

$$a_{\mu}^{i} = \frac{1}{\sqrt{2}} \left( l_{\mu}^{i} - r_{\mu}^{i} \right)$$

a complex scalar field $\Phi$, which performs the chiral symmetry breaking

$$SU(2)_L \times SU(2)_R \rightarrow SU(2)_V$$

$$S_\phi = -\int d^5x \sqrt{-\text{g}} \quad r \left[ |D\Phi|^2 + m_5^2 |\Phi|^2 \right].$$

Here $D_M \Phi = \partial_M \Phi - i L_M \Phi + i \Phi R_M$ and $m_5^2 = -3$.

$$\Phi = \phi \exp \left[ i \sqrt{2} \pi^a T^a \right]$$

$$\phi(z) = \frac{1}{2} m_q z + \frac{1}{2} \sigma z^3.$$  

$$S_a = -\frac{1}{4g^2} \int d^5x \sqrt{-G} \quad Tr \left( F^a_{MN} F^a MN \right)$$
the $a_1$ mesons

\[
a_{1m}^0 = \bar{a}_m^3,
\]

\[
a_{1m}^\pm = \frac{1}{\sqrt{2}} \left( \bar{a}_m^1 \pm i\bar{a}_m^2 \right).
\]

\[
a_{1m}^0(t, z) = \int \frac{d\bar{\omega}_0}{2\pi} e^{-i\bar{\omega}_0 t} a_{1m}^0(\bar{\omega}_0, z),
\]

\[
a_{1m}^\pm(t, z) = \int \frac{d\bar{\omega}_\pm}{2\pi} e^{-i\bar{\omega}_\pm t} a_{1m}^\pm(\bar{\omega}_\pm, z),
\]

\[
\partial_z \left( \sqrt{-G} G^{zz} G^{mn} \partial_z a_{1n}^0 \right) - \left( G^{00} \bar{\omega}_0^2 + 4g^2 \phi^2 \right) \sqrt{-G} G^{mn} a_{1n}^0 = 0,
\]

\[
\partial_z \left( \sqrt{-G} G^{zz} G^{mn} \partial_z a_{1n}^\pm \right) - \left[ \left( \bar{\omega}_\pm + V_0^3 \right)^2 G^{00} + 4g^2 \phi^2 \right] \sqrt{-G} G^{mn} a_{1n}^\pm = 0.
\]
\[
\frac{1}{\sqrt{-G}} D^{(a)}_{\mu} \left( \sqrt{-G} G^{\mu\mu'} G^{mn} D^{(a)}_{\mu'} a^a_{1n} \right) + 4g^2 \phi_0^2 G^{mn} a^a_{1n} - (\bar{m}^a)^2 G^{mn} a^a_{1n} = 0.
\]

\[
(\bar{m}^a)^2 G^{mn} a^a_{1n} = - \frac{1}{\sqrt{-G}} \partial_z \left( \sqrt{-G} G^{zz'} G^{mn} \partial_{z'} a^a_{1n} \right).
\]

The equations (58) can be solved at UV \((z \to 0)\) and at IR \((z \to z_{IR})\) limits. At UV limit the equations (58) are usual Bessel equations

\[
a^a_{1s}(z) = c_1 z J_1(\bar{m}^s a z) + c_2 z Y_1(\bar{m}^s a z).
\]

Dirichlet boundary condition on the Kaluza-Klein state \(a^a_{1s}(z)\) at UV boundary only \(J_1\) part of it remains:

\[
a^a_{1s}(z) = c_1 z J_1(\bar{m}^s a z).
\]
\[ a_{1s}^a = c z J_1 (\tilde{m}_{1s}^s z). \]

\[ \tilde{m}_0^s = \sqrt{(\tilde{\omega}_0^s)^2 - g^2 \left[ m_q + \sigma (z_{IR})^2 \right]^2} \approx \tilde{\omega}_0^s - \frac{g^2 \left[ m_q + \sigma (z_{IR})^2 \right]^2}{2 \tilde{\omega}_0^s}, \]

\[ \tilde{m}_\pm^s = \sqrt{(\tilde{\omega}_\pm^s \mp V_0^3)^2 - g^2 \left[ m_q + \sigma (z_{IR})^2 \right]^2} \approx |\tilde{\omega}_\pm^s \mp V_0^3| - \frac{g^2 \left[ m_q + \sigma (z_{IR})^2 \right]^2}{2 \tilde{\omega}_\pm^s \mp V_0^3}. \]

The mass splitting formula has form similar to one for the UV solution

\[ \tilde{m}_\pm^s \approx |\tilde{m}_0^s \mp V_0^3 + \frac{g^2 \left( m_q + \sigma (z_{IR})^2 \right)^2}{2 \tilde{\omega}_0^s} | - \frac{g^2 \left( m_q + \sigma (z_{IR})^2 \right)^2}{2 \tilde{\omega}_\pm^s \mp V_0^3} |. \]
\( \pi \) mesons in isospin medium

\[
\pi_m^0 = \pi_m^3, \quad \chi_m^0 = \chi_m^3, \\
\pi_m^\pm = \frac{1}{\sqrt{2}} (\pi_m^1 \pm i\pi_m^2), \quad \chi_m^\pm = \frac{1}{\sqrt{2}} (\chi_m^1 \pm i\chi_m^2).
\]

\[
\pi_m^0(t, z) = \int \frac{d\tilde{\omega}_0}{2\pi} e^{-i\tilde{\omega}_0 t} \pi_m^0(\tilde{\omega}_0, z), \quad \chi_m^0(t, z) = \int \frac{d\tilde{\omega}_0}{2\chi} e^{-i\tilde{\omega}_0 t} \chi_m^0(\tilde{\omega}_0, z),
\]

\[
\pi_m^\pm(t, z) = \int \frac{d\tilde{\omega}_\pm}{2\pi} e^{-i\tilde{\omega}_\pm t} \pi_m^\pm(\tilde{\omega}_\pm, z), \quad \chi_m^\pm(t, z) = \int \frac{d\tilde{\omega}_\pm}{2\pi} e^{-i\tilde{\omega}_\pm t} \chi_m^\pm(\tilde{\omega}_\pm, z),
\]

\[
\partial_z \left[ \frac{1}{\phi^2 \sqrt{-GG^00}} \partial_z \left( \phi^2 \sqrt{-GG^zz} \partial_z \pi^0 \right) \right] - \left( \tilde{\omega}_0^2 + 4g^2\phi^2G_{00} \right) \partial_z \pi^0 = 0,
\]

\[
\partial_z \left[ \frac{1}{\phi^2 \sqrt{-GG^00}} \partial_z \left( \phi^2 \sqrt{-GG^zz} \partial_z \pi^\pm \right) \right] - \left[ \left( \tilde{\omega}_\pm \mp V_0^3 \right)^2 + 4g^2\phi^2G_{00} \right] \partial_z \pi^\pm = 0.
\]
\[ G_{MM} = \phi^{-4/3}(z)G_{MM}, \quad G^{MM} = \phi^{4/3}(z)G^{MM} \]

\[ ds^2 = \frac{R^2}{z^2 \phi^{4/3}} \left( -dt^2 + dz^2 + d\bar{x}^2 \right). \]

\[ \phi^2 \sqrt{-G} G^{zz} = \sqrt{-G} G^{zz}, \quad \phi^2 \sqrt{-G} G^{00} = \sqrt{-G} G^{00} \]

\[ -\omega_0^2 \sqrt{-G} G^{00} \pi^0 + \partial_z \left( \sqrt{-G} G^{zz} \partial_z \pi^0 \right) = 0, \]

\[ -\sqrt{-G} G^{00} \left( \omega_{\pm} \mp V_0^3 \right)^2 \pi^\pm + \partial_z \left( \sqrt{-G} G^{zz} \partial_z \pi^\pm \right) = 0. \]

The equation (81) is the five dimensional Klein-Gordon equation for the free \( \pi^a \) field in the background geometry (79). Consequently, the equation of motion for the \( \pi^a \) field interacting with the \( \phi \) field equivalent to the one for the free \( \pi^a \) field in the modified geometry (79).
Using (81) we can derive a formula determining the effective mass of the $\pi$ fields. In order to make correspondence similarly to previous sections let us write the Klein-Gordon equation in a four-dimensional curved space-time with the new metric $G$ for the massive scalar field $\pi^a$ interacting with the constant external gauge field $V_0^3$ and with the field of condensate denoted by $\phi_0$:

$$\frac{1}{\sqrt{-G}} D_{\mu}^{(a)} \left( \sqrt{-G} G^{\mu \mu'} D_{\mu'}^{(a)} \pi^a \right) + G^{00} (m^a)^2 \pi^a = 0,$$

where the $m^a$ is the mass of the $\pi^a$ field in the background of both condensate and gauge fields. If to consider the equations (82) as the UV limit of the (81) ones, then the eigenvalues of the operator $-\frac{1}{\sqrt{-G}} \partial_z \left( \sqrt{-G} G^{zz'} \partial_{z'} \right)$ in (81) will correspond to the squared mass $(m^a)^2$ of the 4d vector field $\pi^a$ in (82). So, we may admit the equality of the eigenvalues of this operator to the effective mass of the $\pi$ field defined at this boundary:

$$-\frac{1}{\sqrt{-G}} \partial_z \left( \sqrt{-G} G^{zz'} \partial_{z'} \pi^a \right) = (m^a)^2 \pi^a.$$
at $z \rightarrow \tilde{z}_{IR}$ limit.

$$\pi_s^a (z) = c_1 z J_1 (\tilde{m}_a^s z) + \frac{c'_1}{z} J_1 (\tilde{M}_a^s z).$$

$$\tilde{M}_0^s = \sqrt{(\tilde{\omega}_0^s)^2 - g^2 \left[m_q^2 + \sigma^2 (\tilde{z}_{IR}^0)^4\right]} \approx \tilde{\omega}_0^s - \frac{g^2 \left[m_q^2 + \sigma^2 (\tilde{z}_{IR}^0)^4\right]}{2 \omega_0^s},$$

$$\tilde{M}_\pm^s = \sqrt{(\tilde{\omega}_\pm^s \mp V_0^3)^2 - g^2 \left[m_q^2 + \sigma^2 \left(\tilde{z}_{IR}^\pm\right)^4\right]} \approx \tilde{\omega}_\pm^s \mp V_0^3 - \frac{g^2 \left[m_q^2 + \sigma^2 \left(\tilde{z}_{IR}^\pm\right)^4\right]}{2 (\tilde{\omega}_\pm^s \mp V_0^3)}.$$

$$\tilde{M}_\pm^s \approx \tilde{M}_0^s + V_0^3 - \frac{1}{2} g^2 \left[\frac{m_q^2 + \sigma^2 \left(\tilde{z}_{IR}^\pm\right)^4}{\tilde{\omega}_0^s \mp V_0^3} - \frac{m_q^2 + \sigma^2 \left(\tilde{z}_{IR}^0\right)^4}{\tilde{\omega}_0^s}\right].$$
Thank you for attention!