Remarkable and Reversible Prime Number Patterns

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May 5, 2014

Abstract
Prime number multiplet classifications and patterns are extended to negative integers. The extension from prime numbers to single prime powers is also studied. Prime number septets at equal distance are given. It is also shown that each class of generalized twin primes of the classification contains a positive fraction of all prime pairs.

MSC: 11N05, 11N32, 11N80
Keywords: Prime number triplets, quintets, regular multiplets.

1 Introduction
The basic classification [1] of prime number twins relies on the following sieve principle. Every third number from 1 to $\infty$ is divisible by 3. If the greatest common divisor $(D, 3) = 1$, then one of 3 odd numbers at distance $2D$ is divisible by 3. Now 3 can be replaced by any odd prime number. This generalized sieve principle then yields exceptional prime triplets (and more general prime number
multiplets) in Theor. 7 of Ref. [1] or Theor. 2.3 of Ref. [2] and Cor. 10 of [1] explaining the often noted empirical rule that longer (than 3) prime number sequences at equal distance $2D$ must have $3|D$ also (so $D$ is excluded from Theor. 7 and Theor. 2.3).

As in Refs. [1],[2],[3], we ignore as trivial the prime pairs $(2, p)$ of odd distance $p - 2$ with $p$ any odd prime. In the following prime number multiplets will consist of odd primes only. We generally follow the notations of Refs. [1],[2].

## 2 Extension to Negative Integers

In the context of prime number generating polynomials it is often useful to include negative values of the argument and the polynomial. This motivates extending the twin prime classification to negative integers. We find at once that the basic classification of twin primes in Theor. 2 of Ref. [1] or Theor. 2.2 of Ref. [2], which generalizes to all prime number multiplets, remains valid when negative integers are included. The running integer variable $a$ then is simply allowed to assume negative values as well.

The classes of odd prime number twins at distance $2D$ [2],

$$I : p_{f,i} = 2a \pm D, \quad D = 1, 3, 5, \ldots$$

$$II : p_{f,i} = 3(2a - 1) \pm D, \quad D = 2, 4, 8, 10, 14, \ldots$$

$$III : p_{f,i} = 2a + 1 \pm D, \quad D = 6, 12, 18, \ldots, \quad (1)$$

contain all infinitely many prime pairs $(p_i < p_f)$. Special prime pairs are $3, 3+2D$ with $D$ of class II and median $3+D \neq 3(2a-1)$. However, if $D$ is odd, then $3 + D = 2a$ is even, and $2a \pm D$ is a pair in class I. All others are special pairs of class II with $D$ even. The proof is essentially the same as for Theor. 2 of Ref. [1] and Theor. 2.2 of Ref. [2].

The second classification [2] of prime pairs in terms of arithmetic progressions of conductor 6 also extends to negative numbers.
When exceptional prime triplets are examined one realizes that they sometimes continue as triplets to the left, and they reverse by multiplying by \(-1\).

**Example 1.** Exceptional triplets often become quintets (at most, composed of a triplet to the left and another to the right), such as

\[ (-7, -5, 3, 11, 13); \quad \text{and} \quad (-13, -11, -3, 5, 7); \]
\[ (-13, -5, 3, 11, 19); \quad \text{and} \quad (-19, -11, -3, 5, 13); \]
\[ (-17, -7, 3, 13, 23); \quad \text{and} \quad (-23, -13, -3, 7, 17); \]
\[ (-37, -17, 3, 23, 43); \quad \text{and} \quad (-43, -23, -3, 17, 37). \]

Exceptional quintets sometimes turn into exceptional reversible nonets (bi-quintet or superquintet composed of a quintet to the left and another to the right), such as

\[ (-43, -31, -19, -7, 5, 17, 29, 41, 53) \]
\[ (-53, -41, -29, -17, -5, 7, 19, 31, 43) \] \hspace{1cm} (2)

at equal distance 12.

**Theorem 2.** (i) There is at most one prime number quintet with distances \([2D, 2D, 2D, 2D] \) for given \(D = 1, 2, 4, 5, \ldots\) and \((3, D) = 1\) composed of a triplet starting at 3 going to the right and another to the left.

(ii) When the distances are \([2d_1, 2d_2] \) with \(3 | d_2 - d_1\) and \(3 \not| d_1\), there is at most one prime number quintet \(3 - 2d_1 - 2d_2, 3 - 2d_2, 3 + 2d_1, 3 + 2d_1 + 2d_2 \) or \(3 - 2d_1 - 2d_2, 3 - 2d_1, 3 + 2d_2, 3 + 2d_1 + 2d_2\) for given natural numbers \(d_1, d_2\).

The proof is essentially the same as for Theor. 7 of Ref. [1] or Theor. 2.3 of Ref. [2].

The first line of Example 1 exhibits (ii) for the distance pattern \([2, 8, 8, 2]\), while the others are at equal distances 8, 10, 20, respectively, and correspond to (i). Remarkably, there is no case where the \([2, 8]\) distance pattern moves left to become \([2, 8, 2, 8]\).

Exceptional prime number multiplets generalize similarly. Now, 3 can be replaced by any odd prime number \(p > 3\), and \(3 | D\) is necessary to exclude the bi-triplets (quintets) of Theor. 2.
**Corollary 3.** For any prime $p > 3$ there is at most one $(2p - 1) - tuple \ p - 2(p - 1)D, \ldots, p - 2D, p, p + 2D, \ldots, p + 2(p - 1)D$ at a given distance $2D, 3|D, p \not| D$.

This bi-$p$-tuple is composed of a $p$-tuple to the right and one to the left, both starting at $p$. The sequence just ahead of Theor. 2 is a case for $p = 5$, and $D = 6$.

The proof is essentially the same as for Cor. 10 of Ref. [1] and is applied to the left and right.

**Example 4.** At distance $2 \cdot 3 \cdot 5$ a reversible decuplet is

$-157, -127, -97, -67, -37, -7, 23, 53, 83, 113$.

The reversed 6-tuple $7, 37, 67, 97, 127, 157$ at equal distance 30 has maximum length, being different from the exceptional 7-tuple below. Going left can at most yield 5 more primes. Thus, the decuplet is actually one short of optimal. Is there an 11-tuple starting from 7 at some distance divisible by 6? The exceptional septet of G. Lemaire (1909) [5]

$7, 157, 307, 457, 607, 757, 907$,

is followed by other exceptional septets

$7, 2767, 5527, 8287, 11047, 13807, 16567$;
$7, 3457, 6907, 10357, 13807, 17257, 20707$;
$7, 9157, 18307, 27457, 36607, 45757, 54907$;
$7, 14197, 28397, 42597, 56797, 70997, 85197$;
$7, 21247, 42487, 63727, 84967, 106207, 127447$;
$7, 63607, 127207, 190807, 254407, 318007, 381607$;
$7, 76717, 153427, 230137, 306847, 383557, 460267$;
$7, 117427, 234847, 352267, 469687, 587107, 704527$;
$7, 134257, 268507, 402757, 537007, 671257, 805507$ at larger distances $2760 = 6 \cdot 4 \cdot 5 \cdot 23$, $3450 = 6 \cdot 5^2 \cdot 23$, $6 \cdot 5^2 \cdot 61$, $6 \cdot 5 \cdot 11 \cdot 43, 6 \cdot 20 \cdot 3 \cdot 59, 6 \cdot 10^2 \cdot 106, 6 \cdot 5 \cdot 2557, 6 \cdot$
10 \cdot 19 \cdot 103, 6 \cdot 5^3 \cdot 179, \text{ respectively. None continues to the left. Are there exceptional (reversible) quintets, septets and 11-, 13-, 17-...-tuples at infinitely many distances? The distances are all multiples of the mandatory 6; this follows from Theor. 2 above or Theor. 2.3 of Ref. [2]. It is no accident that distances of exceptional septets are divisible by 5. This is needed to avoid quintets. This observation generalizes as follows.}

**Corollary 5.** An exceptional p-tuple of primes starting with the prime p at equal distance D must have \( p' \mid D \) for all primes \( p' < p \).

**Proof.** This is required to avoid all \( p' \)-tuples with one number divisible by \( p' \) by Cor. 10 of Ref. [1] or Cor. 3.

**Example 6.** Exceptional quintets at equal distance 6k are too numerous to be listed. They occur for \( k = 1, 2, 7, 8, 16, 21, 42, 71, 79, 99, \ldots \).

The first exceptional 11-plet is at equal distance 1536160080 = 210 \cdot 8 \cdot 13 \cdot 37 \cdot 1901:

\[
11, 1536160091, 3072320171, 4608480251, 6144640331, 7680800411, 9216960491, 10753120571, 12289280651, 13825440731, 15361600811. \tag{3}
\]

The next six (at distances 2 \cdot 3^4 \cdot 144379, 2100 \cdot 23 \cdot 519763, \ldots) are

\[
11, 4911773591, 9823547171, 14735320751, 19647094331, 2455867911, 29470641491, 34382415071, 39294188651, 44205962231, 49117735811; \]

\[
11, 25104552911, 50209105811, 75313658711, 100418211611, 125522764511, 150627317411, 175731870311, 200836423211, 225940076111, 251045529011; \]

\[
11, 75275138671, 150550279331, 225825418991, 301100558651, 376375698311, 451650837971, 526925977631, 60220117291, 677476256951, 752751396611; \]

\[
11, 83516678501, 167033356991, 250550035481, \ldots
\]
The first 13-tuple comes at the enormous distance 9918821194590 and is

\[13, 9918821194603, 19837642389193, 29756463583783, 39675284778373, 49594105972963, 59512927167553, 69431748362143, 79350569556733, 89269390751323, 99188211945913, 109107033140503, 119025854335093.\]

Other patterns such as \([6, 6, 6, 6] : 5, 11, 17, 23, 29\) are exceptional, but not \([6, 6, 6] : 61, 67, 73, 79\) which is rather common. The distance pattern \([4, 2, 4]\) is very common, too, probably repeats infinitely often, appears embedded in \([6, 4, 2, 4, 6] : 31, 37, 41, 43, 47, 53\) and doubled in \([4, 2, 4, 2, 4] : 7, 11, 13, 17, 19, 23\) that repeats as \(97, 101, 103, 107, 109, 113.\)

Another chaotic property of primes are their gaps. The largest gap in the interval \((1, 98)\) is \(8\). Gaps increase erratically and go way up to \(34\) in \((1300, 1400)\) and again in \((2110, 2200)\), not to be reached again until much later in \((8390, 8502)\). Up to about 10000, rare gap values are large single prime gaps, such as \(2 \cdot 13, 2 \cdot 17, \) less rare are \(2 \cdot 5, 2 \cdot 7, 2 \cdot 11\) and common are \(2^n \cdot 3, \quad n = 1, 2, 3, \ldots\) and \(2^n \cdot 3^n \cdot 5.\)

Finally, we list a few rules for forming new and remarkable patterns of prime number multiplets.

**Example 7.** From 641, 643, 647, 653, 659 at the distances \([2, 4, 6, 6]\) we drop 643 forming the quartet 641, 647, 653, 659 at
equal distance 6. From 601, 607, 613, 617, 619 we drop 617 to get
601, 607, 613, 619 at equal distance [6, 6, 6].

**Contraction:** Omitting an intermediate prime number leads
to a contraction of the multiplet. If the distance pattern is
[\ldots, n_1, n_0, n_2, \ldots] then omitting the intermediate prime yields
[\ldots, n_1, n_0 + n_2, \ldots].

**Insertion:** If there exists an intermediate prime number it
can be re-inserted to yield a multiplet with a longer distance pat-
tern: [\ldots, n_1, n_2, \ldots] \rightarrow [\ldots, n_1, n_0, n_2, \ldots] or [\ldots, n_0, n_1, n_2, \ldots]
or [\ldots, n_1, n_2, n_0, \ldots].

**Omission of prime at start or end** leads to shorter multi-
plet and distance pattern.

### 3 Admission of Prime Powers

When single prime powers are admitted as equivalent to prime
numbers, this amounts to inserting extra numbers from which mul-
tiplets can start to the left and right, with dramatic consequenc-
es. Now, there are far fewer exceptional multiplets because they may
repeat starting at a prime power and, within positive numbers,
they may go left as well.

**Example 8.** The triplet 3, 7, 11 at distances [4, 4] now repeats
as 19, 23, 3^3 and goes on to 31, forming a quartet at equal distance
4. The exceptional triplet 3, 5, 13 at distances [2, 8] now repeats as
a quintet 17, 19, 3^3, 29, 37 with distance pattern [2, 8, 2, 8], which
is new. The exceptional triplet 3, 5, 7 repeats as a longer quintet
5, 7, 3^2, 11, 13 followed, remarkably, by another one 23, 5^2, 3^3, 29, 31
and the triplet 79, 3^4, 83. In the former cases, we have triplets to
the left and right starting at the prime powers 3^2, 3^3, respectively.
Both have distance patterns [2, 2, 2, 2] and are almost exceptional,
as this pattern never repeats.

**Corollary 9.** *The quintet −13, −5, 3, 11, 19 at equal distance 8
in Example 1 stays exceptional. The quintets at distances [2, 2, 2, 2]
in Example 5 do not repeat further, etc.*
That’s why there are only the following quartets 73, 3^4, 89, 97; 6553, 3^8 = 6561, 6569, 6577 at equal distance 8.

**Proof.** For the quintet to repeat requires that the two prime powers 3^m, 5^n be within 4 times the distance 8, or 3^m ≈ 5^n within ≤ 2^5,

\[
\frac{3^m}{5^n} = 1 + \mathcal{O}\left(\frac{2^5}{5^n}\right), \quad m, n \to \infty,
\]

where the constant in \( \mathcal{O} \) is ≤ 1. This does not happen for exponents \( m, n \leq 9 \). In fact, after \( 5^3 - 3^2 = 2 \), these prime powers have their close encounter \( 3^3 - 5^2 = 2, \frac{3^{3m}}{5^{2m}} = (1.08)^m \to \infty, \quad m \to \infty \),

(6)
to diverge exponentially from each other at the rate 1.08. ⋄

**Lemma 10.** Two prime powers \( p_2^{m_2} = p_1^{m_1} + 2D, \quad D \geq 1, \quad p_2 > 5 \) have at most one close encounter and never meet again.

**Proof.** Because

\[
\frac{p_2^{m_2}}{p_1^{m_1}} > 1, \quad \left(\frac{p_1^{m_1}}{p_2^{m_2}}\right)^n = (1 + \varepsilon)^n \to \infty, \quad \varepsilon > 0,
\]

they diverge exponentially from each other. ⋄

**Example 11.** The numbers 9 and 7 meet as \( 3^2 - 7 = 2 \) and then diverge at the rate \( \frac{3^{2m}}{7^{m}} = (1.2857 \ldots)^m \), similarly \( 3^4 - 79 = 2 \) at \( \frac{3^{4m}}{79^{m}} = (1.025 \ldots)^m, \quad 5^3 - 11^2 = 4 \) at \( \frac{5^{3m}}{112^{m}} = (1.033 \ldots)^m \), etc.

**Corollary 12.** Let \( p_2^{m_2}, p_1^{m_1} = p_2^{m_2} + 2D \) be the only adjacent prime power members of a quintet with equal distance pattern \([2D, 2D, 2D, 2D]\). If \( p_2 - p_1 > 2D \) then this quintet is exceptional.

**Proof.** The sequence of the form \( p_1^{m_1 + 1}, p_2^{m_2 + 1} = p_1^{m_1 + 1} + 2D, \ldots \) is ruled out because this implies \( (p_2 - p_1)p_2^{m_2} = 2D(p_1 + 1) < 2Dp_2 \), whereas \( (p_2 - p_1)p_2^{m_2} > 2Dp_2^{m_2} \), q.e.a.

A sequence of the form \( p_1^{m_1 + 1}, p_3 = p_1^{m_1 + 1} + 2D, p_2^{m_2 + 1} = p_1^{m_1 + 1} + 4D, \ldots \) is ruled out because this implies \( (p_2 - p_1)p_2^{m_2} = 2D(p_1 + 2) < 2Dp_2 \), whereas \( (p_2 - p_1)p_2^{m_2} > 2Dp_2^{m_2} \), q.e.a.
A sequence of the form $p_{m_1+1}^1, p_3, p_4, p_2^{m_2+1} = p_{m_1+1}^1 + 6D, \ldots$ is ruled out because this implies $(p_2 - p_1)p_2^{m_2} = 2D(p_1 + 3) < 2Dp_2$, whereas $(p_2 - p_1)p_2^{m_2} > 2Dp_2^{m_2}$, q.e.a.

A sequence of the form $p_{m_1+1}^1, p_3, p_4, p_5, p_2^{m_2+1} = p_{m_1+1}^1 + 8D, \ldots$ is ruled out because this implies $(p_2 - p_1)p_2^{m_2} = 2D(p_1 + 4) \leq 2Dp_2$, whereas $(p_2 - p_1)p_2^{m_2} > 2Dp_2^{m_2}$, q.e.a.

Since $m_2 \geq 1, m_1 > m_2$ we know that $m_1 \geq 2$. So $p_{m_1+1} = p_1(p_2^{m_2} + 2D) \geq 3p_2^{m_2} + 6D$. If the exponent of $p_2$ stays $m_2$ then the next quintet must start like the first: $p_2^{m_2}, p_2^{m_2} + 2D = p_{m_1+1}, \ldots$, and then another power of $p_1$ cannot occur. So the next quintet has to have $p_{m_1+1}^1$ and $p_2^{m_2+1}$. Higher powers are ruled out because they diverge, as shown in Lemma 10 and Example 11. This proves that no other quintet at equal distance $2D$ is possible. $\diamond$

Prime multiplets at such meeting points of prime powers are the quintets in Example 8 and Corollary 9, or 41, 43, 47, $7^2, 53$ at distances $2, 4, 2, 4$ that is a part-repeater of the longer sequence 3, 5, 7, 11, 13, 17, 19, 23, just as 11$^2, 5^3, 127, 131$ with reversed distance pattern $[4, 2, 4]$ and 163, 167, 13$^2, 173$ are repeaters of 7, 11, 13, 17. The distance pattern $[4, 2, 4]$ is very common and probably repeats infinitely often, it appears also embedded in $[6, 4, 2, 4, 6]: 31, 37, 41, 43, 47, 53$ or $[4, 2, 4, 2, 4]: 7, 11, 13, 17, 19$ that repeats as 97, 101, 103, 107, 109, 113.

Another consequence is that some almost-optimal prime number generating polynomials become optimal, a case in point being

$$Q_{13}(x) = x^2 - 3x + 43 = E_{41}(x - 2), \quad Q_{13}(42) = 41^2.$$ (8)

A related case is $Q_{14}(j) = \text{prime}, j = 0, \ldots, 11$ forming a 12-plet, where

$$Q_{14}(x) = x^2 - 3x + 13, \quad Q_{14}(12) = 11^2,$$ (9)

now forming a 13-tuple, i.e. becoming optimal. In addition, $Q_{14}(-x) = x^2 + 3x + 13 = \text{prime}$ for $x = 0, \ldots, 8$; $Q_{14}(-9) = 11^2$ forming a 23-plet over positive and negative arguments.
4 Number of Primes in Class I,II,III

With $a, D$ running in each class I,II,III of the classification, it should to be easier to settle the question whether or not there are infinitely many generalized twins in each class than any twin prime conjecture with fixed distance $2D$.

**Theorem 13.** Each class I,II,III contains a positive fraction of all prime pairs. Altogether they encompass all prime pairs, except for special prime pairs from class II, $(3, 3 + 2D)$ with median $3 + D \neq 3(2a - 1)$.

**Proof.** We construct the prime pair Dirichlet series using Golomb’s arithmetic formula [4] for class III

$$2\Lambda(2a + 1 - 6D)\Lambda(2a + 1 + 6D) = \sum_{(2a+1)^2 - 6^2D^2} \mu(d) \log^2 d, \quad (10)$$

in conjunction with the product theorem for Dirichlet series as in Ref. [3]. Let’s pick a prime number $p'$ from class III and hold it fixed. Write it as $p' = 2a + 1 - 6D$ and let $a, D$ run so that $p = 2a + 1 + 6D$ is a matching prime twin in class III for all appropriate $a, D$. Now we apply the methods of Ref. [3] to construct the Dirichlet series for these pairs of twins. The constraint Dirichlet series for this case is

$$q(s) = \frac{1}{p'^s} \sum_{a,D} \frac{\delta_{p', 2a + 1 - 6D}}{(2a + 1 + 6D)^s}$$

$$= \frac{1}{p'^s} \sum_{D=1}^{\infty} \frac{1}{(12D + p')^s}$$

$$= \frac{1}{4p'^s} \sum_{\chi_{12}} \chi_{12}(p') L(s, \chi_{12}) - \frac{1}{p'^{2s}}, \quad \sigma > 1, \quad (11)$$

involving an arithmetic progression of conductor 12. The twin prime series is given by

$$2\log p' \sum_{p'w} \Lambda(12D + p') \sum_{D=1}^{\infty} \frac{\Lambda(12D + p')}{(12D + p')^w}$$

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\[-\log \frac{p'}{2p'w} \sum_{\chi_{p'}}\chi_{12}(p') \left( \frac{L'(w, \chi_{12}) + \chi_{12}(p') \log \frac{p'}{w}}{p'w} \right) = \lim_{T \to \infty} \frac{1}{2\pi T} \int_{-T}^{T} Z(w - \sigma - it)q(\sigma + it)dt, \sigma > 3. \quad \text{(12)}\]

Here \(Z(s)\) is the twin prime sieve function \([3]\) based on Golomb’s formula \([3]\)

\[Z(s) = \zeta(s) \frac{d^2}{ds^2} \frac{1}{\zeta(s)} = \sum_{n=2}^{\infty} \frac{1}{n^s} \sum_{d \mid n} \mu(d) \log^2 d \quad \text{(13)}\]

for \(\sigma > 1\). Although the product formula \([12]\) holds for \(\sigma > 3\) only, the \(\text{lhs}\) analytically continues to the left-hand complex \(w\)-plane. The series on the \(\text{lhs}\) of Eq. \((12)\) contains the primes (and powers) of the arithmetic progression \(12D + p'\). With \(p' = 2a + 1 - 6D\) and matching prime \(2a + 1 + 6D\) the constraint \((2a + 1 - 6D, 2a + 1 + 6D) = 1\) is fulfilled. Here \(\chi_{12}\) are the characters (mod 12). We now invoke the prime number theorem for arithmetic progressions due to Siegel and Walfisz which states that, for any conductor \(q > 1\), the primes are evenly distributed among the congruence classes coprime to \(q\). The series \((12)\) has a simple pole at \(w = 1\) with a positive residue. The nonnegative coefficients on the \(\text{lhs}\) allow applying a Tauberian theorem implying that the density of these twin members in class III is a positive fraction of all primes.

Letting \(p'\) run covers all prime pairs of class III.

We proceed similarly for class II, pick a fixed prime \(p' \in \{\text{II}\}\), write it as \(3(2a - 1) - D\) with \(a, D\) running so that \(3(2a - 1) + D = p\) with \(3 \mid D, 2 \nmid D\) is a matching twin prime \(\in \{\text{II}\}\). The constraint series is given by

\[q(s) = \frac{1}{p'^s} \sum_{a=1}^{\infty} \frac{1}{6(2a - 1) - p'^s}, \sigma > 1. \quad \text{(14)}\]

The twin prime Dirichlet series for class II becomes

\[2 \log \frac{p'}{p'w} \sum_{a \geq \lceil(p' + 6)/12\rceil} \frac{\Lambda(6(2a - 1) - p')}{6(2a - 1) - p'^w} \]
$$= - \log \frac{p'}{p''w} \left[ \sum_{\chi_6} \chi_6(-p') \frac{L'}{L}(w, \chi_6) - \sum_{\chi_{12}} \chi_{12}(-p') \frac{L'}{L}(w, \chi_{12}) \right]$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} Z(w - \sigma - it)q(\sigma + it)dt, \; \sigma > 3, \quad (15)$$

where $\chi_6, \chi_{12}$ are the characters (mod 6) and (mod 12), respectively, and $[x]$ is the largest integer below $x$. Again, the series is over the primes (and their powers) of the relevant arithmetic progression. The application of the prime number theorem in conjunction with the Tauberian theorem leads to the same conclusion.

For class I, $p' = 2a - D, 2 \not| D, p = 2a + D$, Golomb’s identity is given by

$$2\Lambda(2a - D)\Lambda(2a + D) = \sum_{d|4a^2 - D^2} \mu(d) \log^2 d. \quad (16)$$

The constraint Dirichlet series is defined as

$$q(s) = \frac{1}{p'^s} \sum_{a > [p'/4]} \frac{1}{(4a - p')^s}$$

$$= \frac{1}{2p'^s} \sum_{\chi_4} \chi_4(-p')[L(s, \chi_4) - \chi_4(1)], \; \sigma > 1. \quad (17)$$

The twin prime Dirichlet series becomes

$$2\log \frac{p'}{p''w} \sum_{a > [p'/4]} \frac{\Lambda(4a - p')}{(4a - p')^w}$$

$$= - \log \frac{p'}{p''w} \sum_{\chi_4} \chi_4(-p') \frac{L'}{L}(w, \chi_4)$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} Z(w - \sigma - it)q(\sigma + it)dt, \; \sigma > 3, \quad (18)$$

where $\chi_4$ are the characters (mod 4). Again, it has a simple pole at $w = 1$, the Tauberian theorem applies and gives the corresponding result. \diamond

Next, let us apply this method to the classes of the second generalized twin prime classification, Theor. 2.3 of Ref. [2].
Corollary 14. The subset of twin primes \( p', p \equiv 1 \pmod{6} \) for \( a \equiv 0 \pmod{3} \) in class III is a positive fraction of all prime pairs.

Proof. We have

\[
a = 3\alpha, \quad p' = 6\alpha + 1 - 6D, \quad p = 6\alpha + 1 + 6D,
\]

where \( p' \) is held fixed again, while \( \alpha \) and \( D \) run. The constraint Dirichlet series is

\[
q(s) = \frac{1}{p'^s} \sum_{D=1}^{\infty} \frac{1}{(12D + p')^s}
\]

\[
= \frac{1}{4p'^s} \sum_{\chi_12} \chi_12(p')L(s, \chi_12) - \frac{1}{p'^{2s}}, \quad \sigma > 1.
\]

The associated twin prime series is

\[
\frac{\log p'}{p'^w} \sum_{\chi_12} \frac{\Lambda(12D + p')}{(12D + p')^w}
\]

\[
= -\frac{\log p'}{2p'^w} \sum_{\chi_12} \chi_12(p') \left( \frac{L'(w, \chi_12)}{L(w, \chi_12)} + \frac{\chi_12(p') \log p'}{p'^w} \right)
\]

\[
= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} Z(w - \sigma - it)q(\sigma + it)dt, \quad \sigma > 3,
\]

where \( \chi_12 \) are the characters \( \pmod{12} \). Again, it has a simple pole at \( w = 1 \), the Tauberian theorem applies and gives the corresponding result.

The method applies similarly to other classes of the second generalized twin prime classification in Theor. 2.3 of Ref. [2].

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