ADAPTIVE THRESHOLD ESTIMATION BY FDR

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Abstract

This paper addresses the following simple question about sparsity. For the estimation of an $n$-dimensional mean vector $\theta$ in the Gaussian sequence model, is it possible to find an adaptive optimal threshold estimator in a full range of sparsity levels where nonadaptive optimality can be achieved by threshold estimators? We provide an explicit affirmative answer as follows. Under the squared loss, adaptive minimaxity in strong and weak $\ell_p$ balls with $0 \leq p < 2$ is achieved by a class of smooth threshold estimators with the threshold level of the Benjamini-Hochberg FDR rule or its a certain approximation, provided that the minimax risk is between $n^{-\delta_n}$ and $\delta_n n$ for some $\delta_n \to 0$. For $p = 0$, this means adaptive minimaxity in $\ell_0$ balls when $1 \leq \|\theta\|_0 \ll n$. The class of smooth threshold estimators includes the soft and firm threshold estimators but not the hard threshold estimator. The adaptive minimaxity in such a wide range is a delicate problem since the same is not true for the FDR hard threshold estimator at certain threshold and nominal FDR levels. The above adaptive minimaxity of the FDR smooth-threshold estimator is established by proving a stronger notion of adaptive ratio optimality for the soft threshold estimator in the sense that the risk for the FDR threshold level is uniformly within an infinitesimal fraction of the risk for the optimal threshold level for each unknown vector, when the minimum risk of nonadaptive soft threshold estimator is between $n^{-\delta_n}$ and $\delta_n n$. It is an interesting consequence of this adaptive ratio optimality that the FDR smooth-threshold estimator outperforms the sample mean in the common mean model $\theta_i = \mu$ when $|\mu| < n^{-1/2}$.

1 Introduction

Let $X = (X_1, \ldots, X_n)$ be a vector of independent variables with marginal distributions $X_i \sim N(\theta_i, 1)$, $i = 1, \ldots, n$. Statistical inference about the mean vector $\theta$, known as the Gaussian sequence problem, has been considered as a canonical or motivating model in many important areas in statistics. Examples include empirical Bayes, admissibility, nonparametric regression, variable selection, multiple testing and so on. It also carries immense practical relevance in statistical applications since observed data are often understood, represented, or summarized approximately as a Gaussian vector.

An important Gaussian sequence problem, which illuminates our understanding of more complicated models such as high-dimensional linear regression and matrix estimation, is the estimation of a sparse vector $\theta$ under the squared-error loss. Donoho and Johnstone [7] made a fundamental contribution by proving that when the sparsity
of $\theta$ is expressed as the membership of $\theta$ in a properly standardized small $\ell_p$ ball with $0 \leq p < 2$, asymptotic minimaxity can be achieved by threshold estimators but not linear estimators. Linear estimators do not even achieve the optimal risk rate. However, since optimal linear estimation in $\ell_2$ balls can be adaptively achieved by the James-Stein estimator [27, 17], it is a natural question whether optimal threshold estimation can be adaptively achieved with a data driven threshold level. The universal threshold level $\sqrt{2 \log n}$ [6], equivalent to controlling the familywise error rate in multiple testing, is suboptimal since it results in an extra logarithmic risk factor for moderately small $\ell_p$ balls.

Adaptive threshold estimation has been considered by many, including SURE [8], the generalized $C_p$ [3], FDR [1] and the parametric EB posterior median (EBThresh) [21]. A general picture of the existing analyses of these estimators is that adaptive exact minimax threshold estimation is achieved in $\ell_p$ balls when the order of the minimax risk is between $(\log n)^{1-p/2+\gamma}$ and $n^{1-\kappa}$ with $\gamma > 4.5$ and any $\kappa > 0$ [1, 30], while adaptive rate minimaxity is achieved when the minimax risk is of no smaller order than $O(1)$ [3]. We refer to Johnstone’s book [20] for a comprehensive discussion of the topic.

The state of the matter is not quite satisfactory in view of some recent advances in empirical Bayes. In the framework of compound estimation and empirical Bayes [24, 25], our problem can be considered as restricted or threshold empirical Bayes [26, 33] since it aims to approximately achieve the performance benchmark

$$\min_{t(\cdot) \in \mathcal{D}} \int \int (t(x) - \theta)^2 \varphi(x - \theta) \, dx \, G_n(d\theta)$$

with a class of functions $\mathcal{D}$ restricted to threshold functions, where $\varphi(z)$ is the $N(0,1)$ density and $G_n(u) = n^{-1} \sum_{i=1}^{n} I\{\theta_i \leq u\}$, the empirical Bayes nominal prior, is the empirical distribution of the true deterministic unknown mean vector $\theta$. As we have mentioned, the James-Stein estimator, which can be viewed as linear or parametric empirical Bayes [9, 10, 23], is not rate optimal for the estimation of sparse $\theta$. However, the general empirical Bayes [25, 26], which aims to approximately achieve the benchmark (1.1) with general (unrestricted) $\mathcal{D}$, enjoys optimality properties for sparse as well as dense signals [32, 34, 5, 18]. In fact, the general maximum likelihood empirical Bayes (GMLEB) [18] is guaranteed to possess the adaptive exact minimax and a stronger adaptive ratio optimal properties when the order of the risk is between $(\log n)^{5+3/p}$ and $n$ for sparse $\theta$ and when $n^{-1} \sum_{i=1}^{n} |\theta_i|^p \ll n^p/(\log n)^{1+9p/2}$ for dense $\theta$. Thus, as far as these first order optimality properties are concerned, the advantage of threshold estimation is expected to lie in a class of very sparse signals with risk of logarithmic order, in view of the optimality of the GMLEB for both sparse and dense signals. From this point of view, adaptive optimality properties for widest range of sparse signals, especially for risks at and below logarithmic rate, is highly desirable as a theoretically nontrivial justification for the use of adaptive threshold estimators when the signal is believed to be sparse.

The above discussion leads to the following interesting question. When the signal
is sparse, is the required lower bound for the risk for adaptive threshold estimation of logarithmic order, of order 1, or even smaller? Consider the case where minimax risk in $\ell_p$ balls is of smaller order than $n$ since threshold estimators do not asymptotically attain minimax risk anyway in $\ell_p$ balls when the minimax risk is of order $n$. In this case, the question can be phrased as whether adaptive optimal threshold estimation can be achieved in $\ell_p$ balls when the minimax risk is above a logarithmic order, above 1 or smaller.

A main objective of this paper is to give an explicit affirmative answer to the above question. We prove that with the threshold level of the Benjamini-Hochberg rule [2] for controlling the false discovery rate (FDR) or a suitable approximation of the FDR rule, smooth threshold estimators between the soft and firm threshold estimators uniformly achieve the adaptive exact minimaxity in strong and weak $\ell_p$ balls when the minimax risk is between $n^{-\delta_n}$ and $\delta_n/n$ for any given $\delta_n \to 0$. For $p = 0$, this means adaptive minimaxity in $\ell_0$ balls when $1 \leq ||\theta||_0 \ll n$.

An interesting consequence of our result and that of [1] is a proven advantage of smooth thresholding against hard thresholding in adaptive estimation when the signal is relatively weak and highly sparse. The importance of continuity was advocated in [11] among others.

Another interesting question is whether it is possible for an adaptive threshold estimator, designed to solve the nonparametric Gaussian sequence problem, to also outperform a parametric estimator when the signal belongs to a parametric family. Let $X = n^{-1} \sum_{i=1}^n X_i$ be the sample mean. In the common mean model $\theta_i = \mu \forall i \leq n$, the $\ell_2$ risk of the sample mean $\bar{\theta}_i = \bar{X}$ is 1 for the estimation of $n$ elements of the vector $\theta$. Our results imply that in the common mean model, the FDR smooth threshold estimator outperforms $\bar{X}$ when $|\mu| < (1 - c)/\sqrt{n}$ for any fixed $c \in (0,1)$. In fact, the FDR smooth threshold estimator is comparable to the optimal soft thresholded $\bar{X}$ when $1/n^{1+\delta_n/2} \ll |\mu| \ll 1/n^{1/2}$. See Example 1 in Section 2.

These results are nontrivial in view of the following. Foster and George [12] proved the existence of a risk inflation factor of $2 \log n$ between an optimal threshold estimator and an oracle risk. Abramovich et al [1] proved that the FDR hard threshold estimator is adaptive rate minimax when the order of the minimax risk is between $(\log n)^{\theta - p/2}$ and $n^{1-\kappa}$ for all $\kappa > 0$ but the same estimator is not adaptive minimax to the constant factor when the nominal FDR level is higher than 1/2. Both results raised the possibility of a lower risk bound of logarithmic order for adaptive estimation. The question below order 1 is even less clear. Birgé and Massart [3] raised the possibility of a requirement of a risk of at least $O(1)$ ($O(\epsilon^2)$ in their paper) for rate adaptive estimation of $\theta$. Moreover, when the minimax risk is of order $\log n$, the nonadaptive asymptotic minimaxity of threshold estimators was only proven recently [36].

Simultaneous adaptive threshold estimation for moderately and highly sparse mean vectors is an analytically demanding problem. Sophisticated machinery and clever arguments were deployed in [1] to prove the theorem for minimax risk of order between $(\log n)^{1-p/2+\gamma}$ and $n^{1-\kappa}$ with $0 \leq p < 2$, $\gamma \geq 5$ and $\kappa > 0$. Their result was improved
upon recently to $\gamma > 4.5$ in [30]. We take a different analytical approach by first proving a certain ratio optimality of the FDR threshold level when the order of the minimum risk of soft threshold estimation is between $n^{-\delta_n}$ and $\delta_n n$ at the true unknown mean vector with $\delta_n \to 0$.

The ratio optimality asserts that the risk of the adaptive soft threshold estimator is uniformly within an infinitesimal fraction of the minimum risk of soft threshold estimators over all threshold levels. This directly guarantees the uniform optimality of the adaptive threshold level. It is a stronger notion of optimality than adaptive minimaxity since it guarantees the performance of the adaptive estimator at the true unknown $\theta$ instead of the performance in the worst case scenario in a class of the unknown $\theta$.

The paper is organized as follows. A class of FDR smooth threshold estimators is described in Section 2, along with statements of its adaptive ratio optimality and minimaxity properties. Some preliminary analytical results are presented in Section 3, including some properties of the Bayes risk of smooth threshold estimators and its approximation, comparison of the FDR rule and its population version, and applications of the Gaussian isoperimetric inequalities to smooth thresholding at random threshold levels. Section 4 provides an oracle inequality and optimality properties for a more general class of threshold rules than those in Section 2. Section 5 contains some discussion. Mathematical proofs are provided in the Appendix.

2 Main results

Let $P_\theta$ denote probability measures under which $X_i$ are independent statistics with

$$X_i \sim N(\theta_i, 1), \quad i = 1, \ldots, n,$$

where $\theta = (\theta_1, \ldots, \theta_n)$ is an unknown signal vector. In the vector notation, it is convenient to state (2.1) as $X = (X_1, \ldots, X_n) \sim N(\theta, I_n)$ with $I_n$ being the identity matrix in $\mathbb{R}^n$.

Our problem is to estimate $\theta$ under the mean squared error

$$E_\theta \|\hat{\theta} - \theta\|^2 = E_\theta \sum_{i=1}^n (\hat{\theta}_i - \theta_i)^2$$

for any estimator $\hat{\theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_n)$.

Throughout the paper, boldface letters denote vectors and matrices, for example, $X = (X_1, \ldots, X_n)$, $\varphi(x) = e^{-x^2/2}/\sqrt{2\pi}$, $\Phi(t) = \int_{-\infty}^t \varphi(x)dx$ and $\Phi^{-1}(t)$ denote the standard normal density, distribution and quantile functions, $\|v\| = \sqrt{\sum_i v_i^2}$, $\|v\|_\infty = \max_i |v_i|$ and $\|v\|_p = \{i : v_i \neq 0\}$ denote the $\ell_p$ norm for vectors $v$ with components $v_i$, $x \vee y = \max(x, y)$, $x \land y = \min(x, y)$, $x_+ = x \vee 0$, $a_n \ll b_n$ means $a_n = o(b_n)$ and $a_n \approx b_n$ means $\lim_{n \to \infty} a_n/b_n = 1$. Univariate functions are applied to vectors per component. Thus, $\hat{\theta} = t(X)$ means $\hat{\theta}_i = t(X_i), i = 1, \ldots, n$. 

4
2.1 Adaptive threshold estimation by FDR

Given a sequence of null hypotheses $H_1, \ldots, H_n$, the false discovery rate (FDR) of a multiple testing method is defined as

$$
\text{FDR} = \frac{E\#\{\text{falsely rejected hypotheses}\}}{1 \vee \#\{\text{rejected hypothesis}\}}.
$$

Benjamini and Hochberg \cite{2} advocated the use of FDR to measure Type-I errors in multiple testing and proposed the following rule to control the FDR. Suppose independent test statistics with known null distribution are observed for testing the $n$ null hypotheses. Let $p(1) \leq p(2) \leq \cdots \leq p(n)$ be the ordered $p$-values and $H(1), \ldots, H(n)$ the corresponding hypotheses. The Benjamini-Hochberg rule controls the FDR at the level $q_0/n$ by rejecting hypotheses $H(1), \ldots, H(\hat{k})$, where $\hat{k} = \max\{i: p(k) \leq qk/n\}$ and $n_0$ is the number of true hypotheses. Since $n_0$ is unknown, the quantity $q$ is treated as the nominal FDR level for the Benjamini-Hochberg rule.

Let $0 < \alpha_2 \leq \alpha_1 < 1$. Define candidate threshold levels

$$
\xi_{j,k} = -\Phi^{-1}\left(\frac{\alpha_jk}{2n}\right), \quad j = 1, 2, \quad k = 1, \ldots, n,
$$

and data-driven threshold levels

$$
\hat{\xi}_1 = \min\left\{\xi_{1,k}: N(\xi_{1,k}) \geq k\right\}, \quad \hat{\xi}_2 = \max\left\{\xi_{2,k}: N(\xi_{2,k+1}) < k + 1\right\},
$$

where $N(t)$ is the number of observations above threshold $t$,

$$
N(t) = \sum_{i=1}^{n} I\{|X_i| \geq t\}.
$$

Let $H_i : \theta_i = 0$ and considered two-sided test statistics $|X_i|$. Since $X_i \sim N(\theta_i, 0)$, the ordered $p$-values are $p(k) = 2\Phi(-|X(\hat{k})|)$, where $|X(1)| \geq \cdots \geq |X(n)|$ are the ordered values of $|X_i|$. Since $N(\xi_{1,k}) \geq k$ if and only if $p(k) \leq 2\Phi(-\xi_{1,k}) = \alpha_1 k/n$, $\hat{\xi}_1$ is the threshold level of the Benjamini-Hochberg rule for controlling the FDR at the nominal level $\alpha_1$.

Likewise, $N(\xi_{2,k}) \geq k$ if and only if $|X(\hat{k})| > \xi_{2,k}$. However, unlike $\hat{\xi}_1$, which corresponds to a step-up rule, $\hat{\xi}_2$ is the threshold level of a step-down rule \cite{10} matched by the Benjamini-Hochberg rule. Since $\xi_{1,k} \leq \xi_{2,k}$, we have $\hat{\xi}_1 \leq \hat{\xi}_2$ by (2.4).

Let $t_{\lambda}(x)$ denote a smooth threshold function indexed by its threshold level $\lambda$. We study optimality properties of the adaptive threshold estimator

$$
\hat{\theta} = t_{\hat{\lambda}}(X) = (t_{\hat{\lambda}}(X_1), \ldots, t_{\hat{\lambda}}(X_n))
$$

with a threshold level $\hat{\lambda}$ satisfying

$$
\sqrt{1 + \delta_{1,n}\hat{\xi}_1} \leq \hat{\lambda} \leq \sqrt{1 + \delta_{2,n}\hat{\xi}_2}, \quad 0 \leq \delta_{1,n} \leq \delta_{2,n} \to 0.
$$
The estimator (2.6) is closely related to the $\ell_0$ penalized estimator

$$\hat{\theta} = \arg \min_{\theta} \left\{ \| \theta - X \|_2^2 + \sum_{k=1}^{\| \theta \|_0} \xi_k^2 \right\}$$

(2.8)

with nonincreasing threshold levels $\xi_k$. A local minimum of (2.8), say $\hat{\theta}$ with $\| \hat{\theta} \|_0 = \hat{k}$, is a hard threshold estimator at threshold level $\xi_{\hat{k}}$ such that $X^2_{(\hat{k})} \geq \xi_{\hat{k}}^2$ and $\xi_{\hat{k}+1}^2 \geq X^2_{(k+1)}$. When $\xi_k \in [\xi_{1,k}, \xi_{2,k}]$, (2.4) implies $\hat{\xi}_1 \leq \hat{\xi}_k \leq \hat{\xi}_2$ for such local minima, so that (2.7) is satisfied with $\delta_{1,n} = 0$. In addition, (2.6) allows the use of the penalty level of (2.8) with data-dependent $\xi_k \in [\xi_{1,k}, \xi_{2,k}]$. The $\ell_0$ penalized estimator (2.8) was considered in [31, 13, 28, 15, 3, 1, 30] among many others.

### 2.2 Adaptive ratio optimality of FDR rule in soft thresholding

Let $s_\lambda(x) = \text{sgn}(x)(|x| - \lambda)_+$ be the soft threshold estimator. Given a sequence of vector classes $\Theta^*_n \subset \mathbb{R}^n$, a threshold level $\hat{\lambda}$ is adaptive ratio optimal for soft threshold estimation if

$$\sup_{\theta \in \Theta^*_n} \frac{E_\theta \| s_\lambda(X) - \theta \|_2^2}{\inf_{\lambda \geq 0} E_\theta \| s_\lambda(X) - \theta \|_2^2} \leq 1 + o(1).$$

(2.9)

In words, the risk of $s_\lambda$ is uniformly within an infinitesimal fraction of the risk of $s_\lambda$ with the optimal threshold level $\lambda$ for each unknown vector $\theta$ in the class $\Theta^*_n$. This means the optimality of $\hat{\lambda}$ for the true $\theta$ whenever $\theta \in \Theta^*_n$.

Property (2.9) is called weak adaptive ratio optimality when the following strong adaptive ratio optimality is also considered:

$$\sup_{\theta \in \Theta^*_n} \frac{E_\theta \| s_\lambda(X) - \theta \|_2^2}{\inf_{\lambda \geq 0} E_\theta \| s_\lambda(X) - \theta \|_2^2} \leq 1 + o(1).$$

(2.10)

The weak and strong adaptive ratio optimality properties are of a more appealing type than adaptive minimaxity since it applies directly to the true unknown, instead of the worst case scenario in individual classes of unknowns. The strong adaptive ratio optimality is even more appealing since it applies to both the given parameter vector $\theta$ and given data $X$. Define

$$L_{0,n} = (\log n)^{-3/2} \left( \delta_{1,n}(\log n)^{3/2} + (\log \log n) \sqrt{\log n} \right)^{1+\delta_{1,n}}.$$  

(2.11)

**Theorem 1** Let $X \sim N(\theta, I_n)$ under $P_\theta$ with unknown $\theta \in \mathbb{R}^n$ and $\hat{\theta} = s_\lambda(X)$ be the soft threshold estimator (2.6) with a threshold level $\hat{\lambda}$ satisfying (2.4). Suppose

$$\frac{L_{0,n}}{n^{\delta_{1,n}}} \ll \inf_{\lambda \geq 0} E_\theta \| s_\lambda(X) - \theta \|_2^2 \ll n.$$  

(2.12)
Then, \( \hat{\lambda} \) approximates the optimal fixed threshold level in the sense that
\[
\limsup_{n \to \infty} \frac{E_{\theta}\|s_{\hat{\lambda}}(X) - \theta\|^2}{\inf_{\lambda \geq 0} E_{\theta}\|s_{\lambda}(X) - \theta\|^2} \leq 1. \tag{2.13}
\]

Theorem 1 allows the FDR rule with \( \hat{\lambda} = \hat{\xi}_1 \) and \( \delta_{1,n} = \delta_{2,n} = 0 \). In this case, the lower risk bound for adaptive estimation is equivalent to
\[
\|\theta\|^2 \gg L_{0,n} = \frac{\log \log n}{\log n}.
\]

We prove in Lemma 2 in the next section that (2.12) holds if and only if
\[
\|\theta\|^2 \gg \frac{L_{0,n}}{n^{\delta_{1,n}}}, \quad \int_{|u| \leq \varepsilon} G_n(du) = \frac{\#\{i \leq n: |\theta_i| \leq \varepsilon\}}{n} \to 1 \tag{2.14}
\]
for all \( \varepsilon > 0 \) where \( G_n \) is the nominal empirical prior as in (1.1). In fact, we prove in Lemma 2 that for any constant \( c \in (0, 1) \)
\[
\sup \left\{ \frac{\inf_{\lambda \geq 0} E_{\theta}\|s_{\lambda}(X) - \theta\|^2}{\|\theta\|^2 - 1}: \frac{\|\theta\|^2}{2\log n} \leq 1 - c \right\} \to 0,
\]
so that the lower bounds in (2.12) and (2.13) are equivalent.

**Example 1** Consider the common mean model where \( \theta_i = \mu \) for all \( i \leq n \) and \( \|\theta\|^2 = n\mu^2 \). The risk of \( \bar{X} = n^{-1} \sum_{i=1}^{n} X_i \) is
\[
\text{risk}(\bar{X}) = E_{\theta} \sum_{i=1}^{n} (\bar{X} - \theta_i)^2 = nE_{\theta}(\bar{X} - \mu)^2 = 1.
\]

When \( |\mu| < (1 - c)/n^{1/2} \) for a fixed \( c \in (0, 1) \), the FDR soft threshold estimator \( s_{\lambda}(X) \) outperforms \( \bar{X} \) since the risk of the FDR soft threshold estimator is no greater than \( (1 + o(1))(1 - c)^2 < 1 = \text{risk}(\bar{X}) \). Moreover, for \( L_{0,n}/n^{\delta_{1,n}} \ll n\mu^2 \ll 1 \)
\[
E_{\theta}\|s_{\lambda}(X) - \theta\|^2 \approx n\mu^2 \approx \inf_{\lambda} E_{\theta}(s_{\lambda}(X) - \mu)^2.
\]

We will follow from our general theory in the next subsection that \( L_{0,n}/n^{\delta_{1,n}} \ll n\mu^2 \ll 1 \) also implies \( E_{\theta}\|t_{\lambda}(X) - \theta\|^2 \approx n\mu^2 \ll 1 = \text{risk}(\bar{X}) \) for the FDR firm threshold estimator and a class of FDR smooth threshold estimators between the soft and firm.

Theorem 1 provides the adaptive ratio optimality in classes
\[
\Theta_n^* = \left\{ \theta \in \mathbb{R}^n: M_n L_{0,n}/n^{\delta_{1,n}} \leq \inf_{\lambda \geq 0} E_{\theta}\|s_{\lambda}(X) - \theta\|^2 \leq \eta_n n \right\} \tag{2.15}
\]
for any constant sequences satisfying \( M_n \to \infty \) and \( \eta_n \to 0 \). This result is a consequence of an oracle inequality in Section 4 which uniformly bounds
\[
\text{regret}_\theta(\hat{\theta}) = \frac{1}{n} E_{\theta}\|s_{\hat{\lambda}}(X) - \theta\|^2 - \frac{1}{n} \inf_{\lambda \geq 0} E_{\theta}\|s_{\lambda}(X) - \theta\|^2 \tag{2.16}
\]
for the soft threshold estimator \( \hat{\theta} = s_\lambda(X) \) in (2.6) and (2.7).

It follows from the fundamental theorem of empirical Bayes in the compound decision theory that for any estimating function \( t(x) \),

\[
\frac{1}{n} E_\theta \| t(X) - \theta \|^2 = \int \int \left( t(x) - \theta \right)^2 \varphi(x - \theta) \, dx \, G_n(d\theta),
\]

(2.17)

where \( G_n(du) = n^{-1} \sum_{i=1}^{\infty} I\{ \theta_i \in du \} \) is the unknown nominal empirical prior [24, 33]. Let \( \lambda_{G_n} \) be the minimizer of (2.17) given \( G_n \) for the soft threshold estimator \( t(x) = s_\lambda(x) \). If \( G_n \) were known, \( s_{\lambda, G_n}(X) \) could be used to achieve \( \inf_{\lambda \geq 0} E_\theta \| s_\lambda(X) - \theta \|^2 \). Thus, as in [24, 25], (2.16) can be viewed as the regret of not knowing the nominal prior \( G_n \) when one is confined to using a soft threshold estimator.

We prove that the strong adaptive ratio optimality holds for (2.6) and (2.7) in a slightly smaller range of the minimum soft threshold risk.

**Theorem 2** Let \( X, \theta, P_\theta \) and \( \hat{\lambda} \) be the same as in Theorem 1. Suppose

\[
\log n \ll \inf_{\lambda \geq 0} E_\theta \| s_\lambda(X) - \theta \|^2 \ll n.
\]

(2.18)

Then, \( \hat{\lambda} \) approximates the optimal threshold level for the true \( \theta \) and almost all realizations of data \( X \) in the sense that

\[
\lim_{n \to \infty} \frac{E_\theta \| s_{\hat{\lambda}}(X) - \theta \|^2}{E_\theta \inf_{\lambda \geq 0} \| s_\lambda(X) - \theta \|^2} = 1.
\]

(2.19)

An immediate consequence of Theorem 2 is the strong adaptive ratio optimality in all classes in (2.15) with \( L_{0,n}/n^{\beta_1,n} \) replaced by \( \log n \).

### 2.3 Adaptive smooth threshold estimation

Consider smooth threshold estimators \( t_\lambda(x) \) satisfying the following conditions:

\[
\begin{cases}
  s_\lambda(x) \leq t_\lambda(x) \leq h_\lambda(x), & x \geq 0, \\
  h_\lambda(x) \leq t_\lambda(x) \leq s_\lambda(x), & x < 0,
\end{cases}
\]

(2.20)

\[
\begin{cases}
  0 \leq t_\lambda(y) - t_\lambda(x) \leq \kappa_0(y-x), & x < y, \\
  |t_\lambda(x) - t_\lambda'(x)| \leq \kappa_1|\lambda - \lambda'|,
\end{cases}
\]

(2.21)

with constants \( \kappa_0 \in [1, 2] \) and \( \kappa_1 < \infty \), where \( s_\lambda(x) = \text{sgn}(x)(|x| - \lambda)_+ \) and \( h_\lambda(x) = xI\{|x| > \lambda \} \) are the soft and hard threshold estimators. The following theorem asserts that given the unknown mean vector \( \theta \), the risk of the FDR smooth threshold estimator,

\[
\hat{\theta} = t_\lambda(X),
\]

(2.22)

with the threshold level \( \hat{\lambda} \) in (2.7), is within a small fraction of the minimum risk of nonadaptive soft threshold estimator when the risk is between \( n^{-\delta_n} \) and \( \delta_n n \) for any \( \delta_n \to 0^+ \).
Theorem 3  Let $X \sim N(\theta, I_n)$ under $P_{\theta}$ with unknown $\theta \in \mathbb{R}^n$ and $\hat{\theta} = t_\lambda(X)$ be the smooth threshold estimator (2.22) with a threshold level $\lambda$ satisfying (2.7) and threshold functions satisfying (2.20) and (2.21). Suppose that 
\[ L_{0,n} \ll \inf_{\lambda \geq 0} E_{\theta} \| s_\lambda(X) - \theta \|^2 \ll n. \] (2.23)
Then, the FDR smooth threshold estimation is no worse than the optimal nonadaptive soft threshold estimation in the sense that
\[ \limsup_{n \to \infty} \frac{E_{\theta} \| t_\lambda(X) - \theta \|^2}{\inf_{\lambda \geq 0} E_{\theta} \| s_\lambda(X) - \theta \|^2} \leq 1. \] (2.24)

Condition (2.20) confines the estimator $t_\lambda(x)$ to the interval between the soft and hard threshold estimators, while condition (2.21) implies the Lipschitz condition on $t_\lambda(x)$. Both conditions hold for the soft threshold estimator with $\kappa_0 = \kappa_1 = 1$ and the firm threshold estimator [14]
\[ f_\lambda(x) = \text{sgn}(x) \min \left\{ |x|, \kappa_0 (|x| - \lambda)_+ \right\} \] (2.25)
with $1 < \kappa_0 = \kappa_1 < 2$. In fact conditions (2.20) and (2.21) imply that the estimator $t_\lambda(x)$ must lie between the soft and firm threshold estimators. The firm threshold estimator can be written in the penalized form as
\[ f_\lambda(x) = \inf_{\mu} \left\{ (x - \mu)^2/2 + \rho_{\lambda}(\mu) \right\} \]
where $\rho_{\lambda}(\mu) = \lambda^2 \int_0^{|\mu|/\lambda} (1 - x/\gamma)_+ dx$ is the minimax concave penalty [35] with $\gamma = 1 + 1/\kappa_0$. The smoothness condition (2.21) with $c_0 < 2$ rules out the hard threshold estimator, which has discontinuities at $x = \pm \lambda$.

2.4 Adaptive minimaxity with FDR smooth thresholding

For vector classes $\Theta \subset \mathbb{R}^n$, the minimax risk is
\[ R(\Theta) = \inf_{\delta} \sup_{\theta \in \Theta} E_{\theta} \| \delta(X) - \theta \|^2, \] (2.26)
where the infimum is taken over all Borel mappings $\delta : \mathbb{R}^n \to \mathbb{R}^n$. An estimator $\hat{\theta}$ is asymptotically minimax with respect to a sequence of vector classes $\Theta_n \subset \mathbb{R}^n$ if
\[ \frac{\sup_{\theta \in \Theta_n} E_{\theta} \| \hat{\theta} - \theta \|^2}{R(\Theta_n)} = 1 + o(1). \] (2.27)

The estimator is adaptive minimax if (2.27) holds uniformly with a broad collection of sequences $\{\Theta_n \subset \mathbb{R}^n\}$ of parameter classes.
Define $\ell_p$ balls

$$\Theta_{p,C,n} = \left\{ \theta = (\theta_1, \ldots, \theta_n) : \frac{1}{n} \sum_{i=1}^{n} |\theta_i|^p \leq C \right\},$$ (2.28)

with the interpretation \(n^{-1} \# \{ i \leq n : \theta_i \neq 0 \} \leq C \) for the \(\ell_0\) ball. The quantity \(C\) is the length-normalized or standardized radius of the \(\ell_p\) ball. For \(p > 0\), (2.28) is called the strong \(\ell_p\) ball and denoted by \(\Theta_{p,C,n}^s\) when the following weak \(\ell_p\) ball is also considered as in [19, 1]:

$$\Theta_{p,C,n}^w = \left\{ \theta : |\theta_k|(k/n)^{1/p} \leq C \right\},$$ (2.29)

where \(|\theta_{(1)}| \geq \ldots \geq |\theta_{(n)}|\) are the ordered absolute values of the components of \(\theta\). We use \(\Theta_{p,C,n}^s,w\) to denote both strong and weak \(\ell_p\) balls when a statement applies to both types of balls.

**Theorem 4** Let \(X, \theta, P_\theta\) and \(\hat{\lambda}\) be the same as in Theorem 1. Let \(R(\Theta)\) be the minimax risk (2.27), \(M_n' \to \infty, \eta_n' \to 0, L_{0,n} \) be as in (2.11), and

$$\Omega_{n}^{s,w} = \left\{ (p,C) : 0 < p \leq 2 - c^{s,w}, M_n'(L_{0,n}/n^{\delta_1,n})^{p/2}n^{-1} \leq C \leq \eta_n' \right\},$$

$$\Omega_{0,n}^{s,w} = \left\{ (p,C) : p = 0, 1/n \leq C \leq \eta_n' \right\},$$

with \(c^{s,w} = c^s = 0\) for strong balls and any \(c^{s,w} = c^w \in (0, 1)\) for weak balls. Let \(t_\lambda(x)\) satisfy (2.27) and (2.21). Then, for both strong and weak \(\ell_p\) balls,

$$\lim_{n \to \infty} \sup_{(p,C) \in \Omega_{n}^{s,w} \cup \Omega_{0,n}^{s,w}} \frac{\sup_{\Theta_{p,C,n}^s} E_\theta \| t_\lambda(X) - \Theta \|^2}{R(\Theta)} = 1.$$ (2.30)

For \(0 < p \leq 2 - c^{s,w}\), Theorem 4 asserts the adaptive minimaxity of the FDR smooth threshold estimator in strong and weak \(\ell_p\) balls when

$$\left( L_{0,n}/n^{1+\delta_1,n} \right)^{p/2} \ll n^{Cp} \ll n$$

with \(\delta_1,n \to 0\) and logarithmic \(L_{0,n}\) satisfying \(L_{0,n} = (\log \log n)/\log n\) for \(\delta_1,n = 0\). For \(p = 0\), Theorem 4 asserts the adaptive minimaxity of the FDR smooth threshold estimator in \(\ell_0\) balls when \(1 \leq \|\theta\|_0 \ll n\).

Let \(p' = nI\{p > 0\} + I\{p = 0\}\) and \(\lambda_{p,C,n} = \sqrt{2 \log \left( \min(n, 1/Cp') \right)}\). The minimax risk for the strong and weak \(\ell_p\) balls can be expressed as

$$R(\Theta_{p,C,n}^s,w) = (1 + \epsilon_{(1)}^{p,C,n}) \sup_{\Theta_{p,C,n}^s,w} \left\{ E_\theta \|s_{\lambda_{p,C,n}}(X) - \Theta\|^2 : \Theta \in \Theta_{p,C,n}^{s,w} \right\}$$

$$= (1 + \epsilon_{(2)}^{p,C,n}) \sup_{\Theta_{p,C,n}^{s,w}} \left\{ \|\Theta\|^2 : \Theta \in \Theta_{p,C,n}^{s,w}, \|\Theta\|_\infty \leq \lambda_{p,C,n} \right\}$$

$$= (1 + \epsilon_{(3)}^{p,C,n}) \inf_{\lambda} \sup_{\Theta_{p,C,n}^{s,w}} \left\{ E_\theta \|h_{\lambda}(X) - \Theta\|^2 : \Theta \in \Theta_{p,C,n}^{s,w} \right\}$$ (2.31)
such that for the $c^{s,w}$ and $\eta_n'$ in Theorem 4
\[
\min_{n \to \infty} \max_{k=1,2,3} \sup \left\{ |\epsilon_{p,C,n}^{(k)}| : 0 \leq p \leq 2 - c^{s,w}, C^{p'} \leq \eta_n' \right\} = 0,
\]
where $h_\lambda(x) = x I\{|x| \geq \lambda\}$ is the hard threshold estimator \cite{7, 19, 36}. Since $\inf_\lambda \sup_{\theta \in \Theta} \lambda \geq \sup_{\theta \in \Theta} \inf_\lambda$ for any risk function, Theorem 4 is almost a direct consequence of Theorem 1 and the first part of (2.31).

Statement (2.31) asserts that the minimax risk is uniformly approximately attained by either the soft or hard threshold estimator and that the least favorable configuration in $\Theta_{p,C,n}^{s,w}$ is approximately attained when individual $|\theta_i|$ concentrate at the largest possible values below a nearly optimal threshold level. The results in (2.31) were proved in \cite{7} for strong balls with $C^{p'} \gg (\log n)^{p'/2}/n$, in \cite{19} for weak balls with $C^{p'} \gg (\log n)^{p'/2}/n$, and in \cite{36} for both strong and weak balls with $C^{p'} = O(1)(\log n)^{p'/2}/n$ and explicitly stated uniformity for the entire range of $(p, C)$.

The second part of (2.31) provides an approximate formula for the minimax risk in terms of $(p, C, n)$. For $C^{p'} \gg (\log n)^{p'/2}/n$, the formula can be more explicitly written as
\[
R(\Theta_{p,C,n}^{s,w} = (1 + \epsilon_{p,C,n})M_{p,C,n}^{s,w}nC^{p'}\lambda^{2-p}, (2.32)
\]
where $M_{p,C,n}^{s,w} = 1$ for strong balls and $M_{p,C,n}^{s,w} = 2/(2 - p)$ for weak balls.

As mentioned in the introduction, adaptive minimax estimation of normal means in $\ell_p$ balls with $0 \leq p < 2$ have been considered in \cite{8, 3, 21, 34, 1, 18, 30} and many others. The following results are most closely related to Theorem 4: adaptive minimaxity of the FDR hard threshold estimator for $(\log n)^{\gamma}/n \leq C^{p'} \leq n^{-\kappa}$ with $\kappa > 0$, $\gamma = 5$ in \cite{1}, and $\gamma = 4.5$ in \cite{30}; adaptive minimaxity of the GMLEB for $(\log n)^{4+p/2+3/p}/n \ll C^{p} \ll n^p/(\log n)^{4+p/2}$ for $p > 0$ in \cite{18}; adaptive rate minimaxity of the generalized $C_p$ for $1/n \leq O(1)C^{p'}$ in \cite{3}; adaptive rate minimaxity of the EBThresh for $(\log n)^2/n \leq O(1)C^{p'}$ and a modified EBThresh for $(\log n)^{p'/2}/n = O(1)C^{p'}$ in \cite{21}. The uniformity in $(p, C)$ of the results in \cite{21, 1, 18, 30} seems to follow from (2.31) and their proofs, possibly with some careful modification. It follows from (2.32) that the ranges of $C^{p'}$ here and those of the risk given in the introduction are equivalent for the respective cited results.

3 Analysis of the FDR smooth threshold estimator

As mentioned in the introduction, the compound estimation of normal means is closely related to the Bayes estimation of a single normal mean. Let $G$ be a prior distribution. In the Bayes problem, we estimate a univariate random parameter $\theta$ based on a univariate observation $X$ such that
\[
X|\theta \sim N(\theta, 1), \quad \theta \sim G.
\]
The Bayes risk of the soft threshold estimator $s_\lambda(X)$, with fixed $\lambda$, is

$$R_G(\lambda) = \int R(u, \lambda)G(du),$$

where $R(\mu, \lambda)$ is the conditional risk given $\theta = \mu$,

$$R(\mu, \lambda) = E_\mu(s_\lambda(N(\mu,1)) - \mu)^2 = \int \left(s_\lambda(x + \mu) - \mu\right)^2 \varphi(x)dx. \quad (3.2)$$

The nominal empirical Bayes prior, which naturally matches the unknown mean vector $\theta = (\theta_1, \ldots, \theta_n)$, is defined as

$$G_n(t) = \frac{1}{n} \sum_{i=1}^{n} I\{\theta_i \leq t\}. \quad (3.3)$$

With the above notation, (2.17) with $t(x) = s_\lambda(x)$ can be written as

$$R_{G_n}(\lambda) = \frac{1}{n} E_\theta \|s_\lambda(X) - \theta\|^2. \quad (3.4)$$

We denote the Bayes optimal soft threshold risk and level for prior $G$ by

$$\eta_G = \min_\lambda R_G(\lambda), \quad \lambda_G = \arg \min_\lambda R_G(\lambda). \quad (3.5)$$

It follows immediately from (3.4) that $\lambda_{G_n}$ is the optimal deterministic soft threshold level when $\theta$ is the true mean vector and

$$\eta_{G_n} = R_{G_n}(\lambda_{G_n}) = \frac{1}{n} \inf_{\lambda \geq 0} E_\theta \|s_\lambda(X) - \theta\|^2. \quad (3.6)$$

For smooth threshold functions satisfying (2.20) and (2.21), we define

$$R_{G}^{(sm)}(\lambda) = \int \int \left(t_\lambda(x + u) - u\right)^2 \varphi(x) dx G(du). \quad (3.7)$$

Our analysis requires a concentration inequality to bound from the above the difference $\|t_\lambda(X) - \theta\|^2/n - R_{G_n}^{(sm)}(\hat{\lambda})$ and exponential inequalities to bound from the above $R_{G_n}^{(sm)}(\hat{\lambda}) - R_{G_n}(\lambda_{G_n})$. Define

$$R_{1,n}(\theta, \lambda) = E_\theta \|t_\lambda(X) - \theta\|/\sqrt{n}. \quad (3.8)$$

The concentration inequality actually bounds the difference between random variables $\|t_\lambda(X) - \theta\|/\sqrt{n}$ and $R_{1,n}(\theta, \lambda)$ based on the fact that for each deterministic $\lambda$, the Lipschitz norm of $\|t_\lambda(X) - \theta\|$ is no greater than $\kappa_0$ as a function of the error $X - \theta$. Since $\hat{\lambda}$ is bounded by functions of the FDR rules $\{\hat{\xi}_1, \hat{\xi}_2\}$ in (2.4) and the FDR rules
are defined through the counting process $N(t)$ in [2,5], exponential inequalities for $N(t)$ are used to bound the difference between $R_{G_n}^{(sm)}(\hat{\lambda})$ and $R_{G_n}(\lambda_{G_n})$.

We provide below preliminary analysis of the Bayes risk function $R_G(\lambda)$ for the soft threshold estimator, that of $R_{G_n}^{(sm)}(\hat{\lambda})$ for the smooth threshold estimator, that of the FDR rules $\{\hat{\xi}_1, \hat{\xi}_2\}$, and the concentration inequality. Throughout the analysis, we denote by $M^*$ a positive numerical constant which may take different values in different appearances.

### 3.1 Risk properties of soft thresholding at fixed level

With a numerical constant $B_0 \geq 4$, let

$$\rho_G(\lambda) = \int (u^2 \lor \lambda^2) G(du), \quad r_G(\lambda) \equiv \rho_G(\lambda) + B_0 \Phi(-\lambda).$$

We carry out an analysis of the Bayes risk $R_{G_n}(\lambda)$ by studying the relationship between $R_{G_n}(\lambda)$ and the more explicit $r_G(\lambda)$. Parallel to (3.5), we define

$$\eta^*_G = \min_{\lambda} r_G(\lambda), \quad \lambda^*_G = \arg \min_{\lambda} r_G(\lambda).$$

**Lemma 1** Let $R_G(\lambda)$, $R(\mu, \lambda)$, $\eta_G$, $\lambda_G$, $r_G(\lambda)$, $\rho_G(\lambda)$, $\eta^*_G$ and $\lambda^*_G$ be as in (3.1), (3.2), (3.5), (3.9) and (3.10) respectively.

(i) The soft threshold risk $R(0, \lambda)$ at $\mu = 0$ is decreasing in $\lambda$

$$\frac{4\Phi(-\lambda)}{\lambda^2 + 5} \leq R(0, \lambda) \leq \frac{4\Phi(-\lambda)}{\lambda^2 + 2}.$$  

The Bayes risk $R_G(\lambda)$ is bounded by

$$R_G(\lambda) \leq \rho_G(\sqrt{\lambda^2 + 1}) + R(0, \lambda) \leq \rho_G(\lambda) + r_G(\lambda)/(\lambda^2 \lor 1).$$  

Consequently, the minimum Bayes risk is bounded by

$$\eta_G \leq \left(1 + \frac{1}{(\lambda^*_G)^2 \lor 1}\right) \eta^*_G.$$  

(ii) There exists a constant $M_0^*$ depending on $B_0$ only such that

$$\max\{\rho_G(1), \eta^*_G\} \leq \left(1 + \frac{\sqrt{8\log(\lambda_G \lor e) + M_0^*}}{\lambda_G \lor e}\right) \eta_G.$$  

Lemma 1 provides an approximation of the optimal risk $\eta_G \approx \eta^*_G$ for large $\lambda_G \lor \lambda^*_G$.

We now consider small Bayes soft threshold risk.
Lemma 2  Let $R_G(\lambda)$, $G_n$, $\eta_G$, $\lambda_G$, $r_G(\lambda)$, $\rho_G(\lambda)$, $\eta^*_G$ and $\lambda^*_G$ be as in (3.1), (3.3), (3.5), (3.9) and (3.10) respectively.

(i) The quantity $\lambda^*_G$ is an increasing function of $B_0$. There exists a numerical constant $M^*$ such that

$$
\sqrt{2 \log(1/\eta_n)} \leq \left(1 + \frac{\log \log(1/\eta_n) - \log(8/7)}{4 \log(1/\eta_n)}\right) \lambda^*_G,
$$

whenever $\min(\eta_G, \eta^*_G) \leq \eta_n \leq 1/M^*$. Moreover, under the same condition

$$
\max \left\{ \eta_G, \rho_G(\lambda^*_G) \right\} \leq \left(1 + \frac{1}{\log(1/\eta_n)}\right) \eta^*_G
$$

$$
\leq \left(1 + \frac{2 \sqrt{\log \log(1/\eta_n)} + M^*_0}{\sqrt{\log(1/\eta_n)}}\right) \eta_G
$$

with a constant $M^*_0$ depending on $B_0$ only.

(ii) If $\sqrt{2 \log(1/\eta_n)} \geq B_0/\sqrt{2\pi}$ and $\rho_G(1) \leq \eta_n$, then

$$
\max \left( \eta_G - \eta_n, \eta^*_G \right) \leq \eta_n \left(1 + 2 \log(1/\eta_n)\right).
$$

(iii) Let $z_n > 0$ satisfy $z_n^{-2}\Phi(-z_n) = 1/(4n)$. Then, $z_n^2 > \log n$ for $n \geq 2$, $z_n^2 > 2 \log(4n/\sqrt{2\pi}) - 3 \log \left(2 \log(4n/\sqrt{2\pi})\right)$ for $n \geq 7$, and

$$
\lambda^*_G = \infty \quad \text{and} \quad \eta^*_G = \|\theta\|^2/n
$$

for $\eta^*_G \leq z_n^2/n$. Moreover, there exists a numerical positive integer $n^*$ such that

$$
\eta^*_G \leq z_n^2/n
$$

holds whenever $\sqrt{m\eta_G} \leq \sqrt{2 \log n - 2 \sqrt{2 \log \log n}}$ and $n \geq n^*$.

Lemma 2 (i) provides the approximation of the optimal nonadaptive soft threshold risk, $\eta_G \approx \eta^*_G$, when $\min(\eta_G, \eta^*_G)$ is small, compared with the less explicit condition of having large $\lambda_G \wedge \lambda^*_G$ in Lemma 1 (ii). Lemma 2 (ii) and Lemma 1 (ii) imply the equivalence of the condition $\min(\eta_G, \eta^*_G) \to 0$ and the even more explicit $\rho_G(1) \to 0$, which implies the equivalence of the upper risk bound conditions in (2.12) and (2.14) in Proposition 1 (i) below. Lemma 2 (iii) gives explicit expression of $\eta^*_G$ when the optimal risk $\eta_{G_n}$ is smaller than a critical risk level near $2(\log n)/n$, which implies the equivalence of the lower risk bound conditions in (2.12) and (2.14) as described in Proposition 1 (ii). Define $G(t) = \int_{|u|>t} G(du)$. For $0 < \epsilon \leq 1$, we have

$$
e^2\bar{G}(\epsilon) \leq \rho_G(1) \leq e^2 + \bar{G}(\epsilon).
$$
Proposition 1 Let $R_G(\lambda), \eta_G = \inf_\lambda R_G(\lambda), r_G(\lambda), \rho_G(\lambda)$ and $\eta_G^* = \inf_\lambda r_G(\lambda)$ be as in (3.1), (3.5), (3.9) and (3.10) respectively. Let $G_n$ be the nominal empirical prior in (3.5) for the unknown vector $\theta$.

(i) For fixed $B_0$ the following conditions are equivalent to each other: (a) $\eta_G \to 0$; (b) $\eta_G^* \to 0$; (c) $\rho_G(1) \to 0$; and (d) $\hat{G}(\epsilon) \to 0$ for all $\epsilon > 0$.

(ii) There exists a numerical positive integer $n^*$ such that

$$\eta_{G_n}^* = \|\theta\|^2/n = (1 + o(1))\eta_{G_n}$$

whenever $\min(\eta_{G_n}, \eta_{G_n}^*) \leq (\sqrt{2\log n} - 3\sqrt{2\log \log n})^2/n$ and $n \geq n^*$.

We omit the proof of Proposition 1 since it is a direct consequence of Lemmas 1 and 2 as discussed above its statement.

### 3.2 Risk properties of smooth thresholding at fixed level

Let $t_\lambda(x)$ be threshold functions between the soft and firm threshold estimators:

$$
\begin{cases}
  s_\lambda(x) \leq t_\lambda(x) \leq f_\lambda(x), & x \geq 0 \\
  f_\lambda(x) \leq t_\lambda(x) \leq s_\lambda(x), & x < 0
\end{cases}
$$

(3.19)

where $s_\lambda(x) = \text{sgn}(x)(|x| - \lambda)_+$ and $f_\lambda(x)$ is as in (2.25) with $\kappa_0 \in [1, 2]$.

Lemma 3 Suppose (3.19) holds with $\kappa_0 \in [1, 2]$. Let $C_0 = \kappa_0/(2 - \kappa_0)$.

(i) For all $\mu$ and $\lambda \geq 0$,

$$|t_\lambda(x) - \mu| \leq \max\left(|\mu|, C_0(|x - \mu| - \lambda)_+\right).$$

(ii) Suppose $EX = \mu$. Then, for all $\lambda \geq 0$,

$$E\left(t_\lambda(X) - \mu\right)^2 \leq \lambda^2 + 2\text{Var}(X).$$

(iii) Let $1 \leq q \leq 2$. Suppose $EX = \mu$. Then, for all $\lambda \geq 0$,

$$E\left|t_\lambda(X) - \mu\right|^q \leq \min\left(|\mu|^q + C_0^qE(|X - \mu| - \lambda)_+^q, (\lambda^2 + 2\text{Var}(X))^{q/2}\right).$$

(iv) Let $X \sim N(\theta, I)$ under $P_\theta$ and $R(\mu, \lambda), G_n, r_G(\lambda), \rho_G(\lambda)$ be as in (3.2), (3.3) and (3.4) respectively with $B_0 \geq 4 \sqrt{2C_0^2}$. Then,

$$n^{-1}E_{\theta}\|t_\lambda(X) - \theta\|_2^2 \leq \rho_{G_n}(\sqrt{\lambda^2 + 2} + C_0^2R(0, \lambda) \leq \rho_{G_n}(\lambda) + 2r_{G_n}(\lambda)/(\lambda^2 \lor 1).$$
It follows from Lemma 3 (iv) that for the optimal \( \lambda^*_G \) and \( \eta^*_G \) in (3.10),

\[
    n^{-1} E_{\theta} \| t_{\lambda^*_G}(X) - \theta \|_2^2 \leq \left(1 + 2/(\lambda^*_G \vee 1)^2\right) \eta^*_G.
\]

Thus, as in Lemma 2 condition \( \min(\eta_G, \eta^*_G) \leq \eta_n \leq 1/M^* \) implies

\[
    E_{\theta} \| t_{\lambda^*_G}(X) - \theta \|_2^2 
    \leq \left(1 + 2\sqrt{\log \log(1/\eta_n)} + M^*_0 \right) \inf_{\lambda} E_{\lambda} \| s_\lambda(X) - \theta \|_2^2.
\]

This asserts that at a proper threshold level, the risk of smooth thresholding satisfying (3.19) can not be significantly larger than that of the optimal soft thresholding. The reverse is not true in view of the following example.

**Example 2** Let \#\{i : \theta_i = 0\} = n - 1 and \#\{i : \theta_i = \mu\} = 1 with \( \mu = 4\sqrt{2 \log n} \). Lemma 2 yields \( n\eta^*_G \geq n\rho_n(\lambda^*_G) \geq (\lambda^*_G)^2 \wedge \mu^2 \), so that

\[
    n\eta_G \geq \inf_{\lambda} E_{\theta} \| s_\lambda(X) - \theta \|_2^2 \geq (1 + o(1))2 \log n.
\]

On the other hand, for the firm threshold estimation (2.25) with \( \kappa_0 = 3/2 \) and \( \lambda = \sqrt{2 \log n} \), we have \( C_0 = \kappa_0/(2 - \kappa_0) = 3 \) and

\[
    E_{\theta} \| f_\lambda(X) - \theta \|_2^2 \leq 9nR(0, \lambda) + E\left(f_\lambda(N(\mu, 1)) - \mu\right)^2 = O(1).
\]

### 3.3 Analysis of the FDR threshold level

We discuss the relationship between the FDR threshold levels (2.4) and their population version.

A population version of the FDR can be defined as

\[
    \text{FDR}_{\text{pop}} = \frac{E\#\{\text{falsely rejected hypotheses}\}}{E\#\{\text{rejected hypothesis}\}}.
\]

Let \( G_n \) be the nominal empirical prior in (3.3). Define

\[
    S_G(t) = \int P\{|N(u, 1)| > t\}G(du)
\]

for any probability distribution \( G \). If \( \theta \) has \( n_0 \) zero components and \( H_i : \theta_i = 0 \) is tested by thresholding \( |X_i| \) at level \( t \), the population FDR is

\[
    \text{FDR}_{\text{pop}}(t) = \frac{n_0 P\{|N(0, 1)| > t\}}{\sum_{i=1}^{n} P\{|N(\theta_i, 1)| > t\}} = \frac{n_0 2\Phi(-t)}{nS_G(t)}.
\]
We call \( 2\Phi(0) / S_{G_n}(t) \) the nominal FDR function as its sample version.

Since this paper is concerned with estimation, the \( \ell_0 \) sparsity of \( \theta \) is covered but not assumed. Actually, we allow \( n_0 = \#\{i \leq n : \theta_i = 0\} = 0 \). Still, the nominal FDR function \( 2\Phi(0) / S_{G_n}(t) \) plays a crucial role in studying the FDR threshold level (2.4).

Given two nominal FDR levels \( \alpha' \) and \( \alpha'' \), the population version of the threshold levels (2.4) is

\[
\xi_{1,*} = \inf \left\{ t : \frac{2\Phi(0)}{S_{G_n}(t)} \leq \alpha' \right\}, \
\xi_{2,*} = \sup \left\{ t : \frac{2\Phi(0)}{S_{G_n}(t)} \geq \alpha'' \right\}
\]  

(3.21)

with the \( G_n \) in (3.3). We consider fixed \( 0 < \alpha'' < \alpha' \leq \alpha_1 < \alpha'_1 < 1 \). Let

\[
\tilde{G}(t) = \int_{|u|>t} G(du).
\]

As we have mentioned in the introduction, we will present an oracle inequality in Section 4 for a more general class of threshold rules. This class involves certain functions \( g_{1,n} \) satisfying

\[
0 \leq g_{1,n}(x) \leq x, \quad R(0, g_{1,n}(x)) \leq 4\Phi(-x), \quad \forall x > 0.
\]

(3.23)

Lemma 4 Let \( S_G(t) \), \( \tilde{G}(t) \), \( \rho_G(t) \), \( G_n \), \( \xi_{1,*} \), \( \xi_{2,*} \) be as in (3.20), (3.22), (3.4), (3.3) and (3.21) respectively.

(i) Suppose \( S_G(t) \neq 2\Phi(0) \) for some \( t > 0 \), i.e. \( G(t) = 0 \) for some \( t > 0 \). Then, the nominal population FDR level, \( 2\Phi(0) / S_G(t) \), is strictly decreasing in \( t \) from 1 at \( t = 0 \) to 0 as \( t \to \infty \), and that for all \( t > 0 \),

\[
\tilde{G}(t)/2 \leq S_G(t) \leq 2\Phi(0) - \rho_G(1).
\]

(3.24)

Moreover, \( \lambda_{G_n} > \xi_{2,*} \) when \( B_0 \geq 8/\alpha'' \).

(ii) For \( j = 1, 2 \), \( 2\Phi(-\xi_{j,*}) \leq \rho_{G_n}(1) \alpha_j / (1 - \alpha_j) \). If (3.23) holds, then

\[
\alpha_j S_G(g_{1,n}(t)) \leq (5 + t^2)2\Phi(0), \quad \forall t \leq \xi_{j,*}
\]

(3.25)

(iii) Let \( A_1 \geq 1 \) and \( \theta_{*,n} = \max\{t : A_1^{1/2} \xi_{1,1} t + t^2/2 \leq \beta_0 \log n\} \) with a certain \( \beta_0 \in (0,1/2) \) and the \( \xi_{1,1} \) in (2.3). Suppose

\[
\eta_{G_n}^* \leq \theta_{*,n}^2 / n, \quad n \geq 2, \quad 1 + \sqrt{\beta_0-1/2} \leq 1 / \alpha_1.
\]

(3.26)

Then, \( \xi_{1,*} > A_1^{1/2} \xi_{1,1} \) with the population FDR threshold level (3.21). Moreover, for all \( t \) satisfying \( 2\theta_{*,n} \leq t \leq A_1^{1/2} \xi_{1,1} \),

\[
\int_t^\infty S_{G_n}(x) dx \leq t^{-1} \sqrt{2\Phi(0)} \left( 2 + 4n^{-1/2} + n^{\beta_0-1/2} \right).
\]

(3.27)
Lemma 4 (i) provides the monotonicity of the population FDR as a function of the threshold level $t$ and lower and upper bounds for the population rejection probability $S_G(t)$. Lemma 4 (ii) provides a lower bound for the population FDR threshold level $\xi_{j*,}$ and an upper bound for the population rejection probability at level $g_{1,n}(t)$. Lemma 4 (iii) provides a condition under which the population FDR threshold level is greater than the highest possible sample FDR threshold level $\xi_{1,1} = -\Phi^{-1}(\alpha_1/n)$ at the nominal FDR level $\alpha_1$. We note that the third condition in (3.26) holds for all $n$ and $\beta_0 \leq 1/2$ when $\alpha'_1 \leq 2/3$.

Since $0 < \alpha'_2 < \alpha_2 \leq \alpha_1 < \alpha'_1 < 1$ and smaller false discovery error requires higher threshold level, we expect $\hat{\xi}_1 \leq \xi_{1,*} \leq \xi_{2,*} \leq \xi_2$ with large probability. This is verified in the following lemma.

**Lemma 5** Let $X \sim N(\theta, I_n)$ under $P_\theta$. Let $N(t)$ be as defined in (2.20). Let $\xi_{j,k}$, $\hat{\xi}_j$ and $\xi_{j,*}$ be as defined in (2.3), (2.4) and (3.21). Then

(i) For all $\xi_k > 0$ (e.g. $\xi_k = \xi_{1,k}$ or $\xi_k = \xi_{2,k}$),
\[
P_\theta \{ \text{sgn}(k - E_\theta N(\xi_k))(N(\xi_k) - k) \geq 0 \} \leq \exp(-\nu_k k), \tag{3.28}
\]

where $\nu_k = E_\theta N(\xi_k)/k - 1 - \log(E_\theta N(\xi_k)/k) > 0$.

(ii) Let $\nu_{1,*} = \alpha_1/\alpha'_1 - 1 - \log(\alpha_1/\alpha'_1)$. For all $\xi_{1,k} \leq \xi_{1,*}$,
\[
P_\theta \{ \hat{\xi}_1 \leq \xi_{1,k} \} \leq P_\theta \{ N(\xi_{1,k}) \geq k \} \leq \exp(-\nu_{1,*} k). \tag{3.29}
\]

Let $\nu_{2,*} = \alpha_2/\alpha'_2 - 1 - \log(\alpha_2/\alpha'_2)$. For all $\xi_{2,k} \geq \xi_{2,*}$,
\[
P_\theta \{ \hat{\xi}_2 \geq \xi_{2,k} \} \leq P_\theta \{ N(\xi_{2,k}) \leq k \} \leq \exp(-\nu_{2,*} k). \tag{3.30}
\]

### 3.4 Gaussian isoperimetric inequality

Here we provide large deviation bounds, based on the Gaussian isoperimetric inequality [4, 29, 22], for the difference between the loss and risk functions of the smooth threshold estimators satisfying (2.20) and (2.21) at an arbitrary random threshold level.

For real-valued functions $f$ on $\mathbb{R}^n$, the Lipschitz norm is defined as
\[
\|f\|_{\text{Lip}} = \sup_{u \neq v} \frac{|f(u) - f(v)|}{\|u - v\|}.
\]

The Gaussian isoperimetric inequality asserts that for $Z \sim N(0, I_n)$ and functions $f: \mathbb{R}^n \to \mathbb{R}$ with $\|f\|_{\text{Lip}} \leq 1$,
\[
P \{ |f(Z) - Ef(Z)| > t \} \leq 2e^{-t^2/2}. \tag{3.31}
\]

By (2.21), the smooth threshold estimator $t_\lambda(x)$ has Lipschitz norm $\kappa_0$ as a function of $x$, so that the $\ell_2$ norm of the loss $\|t_\lambda(X) - \theta\|$ also has Lipschitz norm $\kappa_0$ as a real
valued function of $Z = X - \theta$. This leads to the following lemma. Let $L(X, \theta, \lambda) = \|t_\lambda(X) - \theta\|/\sqrt{n}$ and define

$$R_{1,n}(\theta, \lambda) = E_\theta L(X, \theta, \lambda),$$

$$H_\theta(a, b) = \left| L(X, \theta, Y) - R_{1,n}(\theta, Y) \right|^2 I\{a \leq Y < b\},$$

with a nonnegative random variable $Y$, and

$$H_\theta^*(\lambda) = \max_{y \geq \lambda} \left| L(X, \theta, y) - \|\theta\|/\sqrt{n} \right|^2. \quad (3.33)$$

**Lemma 6** Let $X \sim N(\theta, I_n)$ under $P_\theta$ and $\{S_G(t), G_n\}$ be as in (3.20) and (3.3). Then, for all $a = c_0 < \cdots < c_m = b$ and $\pi_j \geq P_\theta\{c_{j-1} \leq Y < c_j\}$,

$$\sqrt{E_\theta H_\theta(a, b)} \leq 2\kappa_0 \left\{ \frac{2}{n} \sum_{j=1}^m \pi_j \log \left( \frac{2e}{\pi_j} \right) \right\}^{1/2} + 2\kappa_1 \left\{ \sum_{j=1}^m \pi_j (c_j - c_{j-1})^2 S_{G_n}(c_j) \right\}^{1/2}. \quad (3.34)$$

In particular, for all integers $m \geq 1$,

$$\sqrt{E_\theta H_\theta(a, b)} \leq 2\kappa_0 \left\{ \frac{2}{n} \log(2em) \right\}^{1/2} + 2\kappa_1 \left\{ \frac{(b-a)^2}{m^2} S_{G_n}(a) \right\}^{1/2}. \quad (3.35)$$

Moreover, for all $\lambda > 0$,

$$\sqrt{E_\theta H_\theta^*(\lambda)} \leq \kappa_1 \int_\lambda^\infty S_{G_n}^{1/2}(t) dt \quad (3.36)$$

and with the $R_{G_n}^{(sm)}(\lambda)$ in (3.7)

$$R_{1,n}^2(\theta, \lambda) \leq E_\theta L^2(X, \theta, \lambda) = R_{G_n}^{(sm)}(\lambda) \leq R_{1,n}^2(\theta, \lambda) + \frac{4\kappa_0^2}{n}. \quad (3.37)$$

### 4 An oracle inequality

We provide an oracle inequality for a more general class of threshold levels. Let $g_{1,n}(x)$ be a sequence of functions satisfying the following conditions:

$$0 \leq g_{1,n}(x) \leq x, \ 0 \leq (d/dx)g_{1,n}(x) \leq M_0, \ \forall x > 0,$$

$$R(0, g_{1,n}(x)) \leq \min \left\{ 4\Phi(-x), \frac{M_0 \Phi(-x)}{(x^{c_{1,n}} + 2)(1 \vee \log x)^{c_{2,n}}} \right\}$$

$$0 < c_{1,n} \leq 2, \ |c_{2,n}| \leq M_0, \ c_{2,n} \leq 0 \ \text{for} \ c_{1,n} = 2,$$
where $M_0$ is a numerical constant. Recall that $R(0, x) = E(|N(0, 1)| - x)^2$. Since $R(0, x) \leq 4\Phi(-x)/(x^2 + 2)$ for all $x > 0$ by (3.11) of Lemma 1(i), (4.1) holds for $g_{1,n}(x) = x$ with $M_0 = 4$, $c_{1,n} = 2$ and $c_{2,n} = 0$.

Threshold levels of the following form will be considered:

$$\sqrt{1 + \delta_{1,n}g_{1,n}(\hat{\xi}_1)} \leq \lambda \leq \sqrt{1 + \delta_{2,n}\hat{\xi}_2}$$

(4.2)

with $0 \leq \delta_{1,n} \leq \delta_{2,n}$. Compared with (2.7), the lower bound in (4.2) is smaller with $g_{1,n}(\hat{\xi}_1) \leq \hat{\xi}_1$ and the upper bound in (4.2) is larger since $\delta_{2,n} \to 0$ is no longer assumed. We note that $\delta_{2,n} \to 0$ is necessary for attaining the optimal constant factor in our analysis but not for rate optimality.

We prove in the Appendix that for large $x$, condition (4.1) implies

$$\Phi\left(-\sqrt{1 + \delta_{1,n}g_{1,n}(x)}\right) \leq x^{-1}\left(\frac{M^*x^3\Phi(-x)}{(x^2, n)(\log x)^{c_{2,n}}}\right)^{1+\delta_{1,n}}$$

(4.3)

with a numerical constant $M^*$ depending on $M_0$ only. Define

$$L_{2,n} = (\log n)^{3/2}\left(\frac{\delta_{1,n}(\log n)^{(5-c_{1,n})/2}}{(\log \log n)^{c_{2,n}}} + \frac{(\log n)^{(3-c_{1,n})/2}}{(\log \log n)^{c_{2,n}-1}}\right)^{1+\delta_{1,n}}.$$

(4.4)

**Theorem 5** Let $\alpha'_2 < \alpha_2 \leq \alpha_1 < \alpha'_1 < 1$, $\beta_0 \leq 1/2$ and $A$ be fixed positive constants. Let $X \sim N(\theta, I_n)$ under $P_\theta$ and $\lambda$ be a threshold level satisfying (4.2) with $\hat{\xi}_1$ and $\hat{\xi}_2$ in (2.4), constants $0 \leq \delta_{1,n} \leq \delta_{2,n} \wedge A$ and functions $g_{1,n}$ satisfying (4.1). Let $t_\lambda(x)$ be threshold functions satisfying (2.20) and (2.21) with $G_\lambda = \kappa_0/(2 - \kappa_0)$, $G_n$ be as (3.3), $r_\lambda(\lambda)$ as in (3.9) with $B_0 = (8/\alpha'_2) \vee (2C_0^2)$, and $\eta_{G_n}$ as in (3.10). Assume $\max_{1 \leq n < n_*}(1 + \delta_{2,n}) \leq A$ with $n_* = \min\{n : 1 + n^{\beta_0-1/2}/2 \leq 1/\alpha'_1, n \geq 2\}$. Then,

$$\sqrt{E_\theta||t_\lambda(X) - \theta||^2}/n \leq (1 + \delta_{2,n})\eta_{G_n}^* + M^*\sqrt{(1 + \delta_{2,n})\tau_{1,n}\eta_{G_n}^* + \tau_{2,n}^*},$$

(4.5)

where $M^*$ is a constant depending on $\{\alpha'_1, \alpha'_2, \beta_0, A, M_0\}$ only with the $M_0$ in (4.1), $\tau_{2,n}^* = L_{2,n}/n^{1+\delta_{1,n}}$ and with $L_{1,n}^* = e \vee \log(1/\eta_{G_n}^*)$

$$\tau_{1,n}^* = \max\left(\frac{(\log e \vee \log n)}{1 \vee \log n}, \frac{(\log L_{1,n}^*)^{-c_{2,n}}}{(L_{1,n}^*)^{c_{1,n}/2}}, \frac{1}{L_{1,n}^*}\right).$$

(4.6)

We note that $n_* = 2$ when $\alpha'_1 \leq 2/3$.

Let $\eta_{G_n} = \inf_{\alpha'_0 > 0} E_\theta||s_\lambda(X) - \theta||^2/n$ defined through (3.4), (3.5) and (3.3) and $\eta_{G_n}^* = \inf_\lambda r_{G_n}(\lambda)$ defined through (3.9) and (3.10). It follows from (3.12) of Lemma 2 that $\eta_{G_n}$ and $\eta_{G_n}^*$ are within a small fraction of each other when $\eta_{G_n} \wedge \eta_{G_n}^*$ is small, so that Theorem 5 with $t_\lambda(x) = s_\lambda(x)$ implies adaptive ratio optimality and minimaxity of the FDR soft threshold estimator when $\eta_{G_n} \wedge \delta_{2,n} \to 0$ and $\tau_{2,n}^* \ll \eta_n$. The following corollaries provide a more general and more explicit version of Theorems 1, 3 and 4.
Corollary 1 Let $X \sim N(\theta, I_n)$ under $P_\theta$ and $\hat{\lambda}$ be a threshold level satisfying (4.2) with $0 \leq \delta_{1,n} \leq \delta_{2,n} \to 0$ and functions $g_{1,n}$ satisfying (4.1). Let $\eta_n \geq \eta_{G_n} = R_{G_n} (\lambda_{G_n}) = \min_{\lambda \geq 0} E_\theta \|s_\lambda(X) - \theta\|^2 / n$. Let

$$
\tau_{1,n} = \max \left( \delta_{2,n}, \frac{\log(e \lor \log n)}{1 \lor \log n}, \frac{(\log L_{1,n})^{-c_{2,n}}}{L_{1,n}^{c_{1,n}/2}}, \frac{\sqrt{\log L_{1,n}}}{\sqrt{L_{1,n}}} \right)
$$

with $L_{1,n} = e \lor \log(1/\eta_n)$ and $\tau_{2,n}^* = L_{2,n}/n^{1+\delta_{1,n}}$ with $L_{2,n}$ in (4.4). Let $t_\lambda(x)$ be functions satisfying (2.20) and (2.21) with $C_0 = \kappa_0/(2 - \kappa_0)$. Then,

$$
\sqrt{E_\theta \|t_\lambda(X) - \theta\|^2 / n} \leq \sqrt{\eta_{G_n} + M^*} \sqrt{\tau_{1,n} \eta_{G_n} + \tau_{2,n}^*}, \quad (4.7)
$$

where $M^*$ is a constant depending on $\{\alpha'_1, \alpha'_2, \beta_0, A, M_0\}$ only. Consequently, the adaptive ratio optimality (2.9) for

$$
\Theta_n^* = \left\{ \theta : M_n L_{2,n}/n^{1+\delta_{1,n}} \leq R_{G_n} (\lambda_{G_n}) \leq \eta_n \right\}
$$

as long as $M_n \to \infty$ and $\eta_n \to 0$.

Corollary 2 Let $X$, $\theta$, $P_\theta$, $t_\lambda(x)$ and $\hat{\lambda}$ be as in Corollary 1. Then, the conclusion of Theorem 4 holds for $\hat{\theta} = t_\lambda(X)$ when $L_{0,n}$ is replaced by the $L_{2,n}$ in (4.4).

5 Discussion

Although the focus of this paper is adaptive optimality sharp to the constant, the oracle inequality in Theorem 5 also implies the following rate optimality properties as corollaries.

Corollary 3 Let $X$, $\theta$, $P_\theta$, $t_\lambda(x)$ and $\hat{\lambda}$ be as in Theorem 4. Then,

$$
E_\theta \|t_\lambda(X) - \theta\|^2 \leq M^* (1 + \delta_{2,n}) \min_{\lambda \geq 0} E_\theta \|s_\lambda(X) - \theta\|^2 \quad (5.1)
$$

for all vectors $\theta$ satisfying $\|\theta\|^2 \geq L_{2,n}/n^{1+\delta_{1,n}}$, where $M^*$ is a constant depending on $\{\alpha'_1, \alpha'_2, \beta_0, A, M_0\}$ only and $L_{2,n}$ is as in (4.4).

Corollary 4 Let $X$, $\theta$, $P_\theta$, $t_\lambda(x)$ and $\hat{\lambda}$ be as in Theorem 4. $R(\Theta)$ be the minimax risk (2.20), and

$$
\Omega_{n}^{s,w} = \left\{ (p, C) : 0 < p \leq 2 - c_{s,w}^p \geq (L_{2,n}/n^{\delta_{1,n}})^{p/2} n^{-1} \right\},
$$

$$
\Omega_{0,n} = \left\{ (p, C) : p = 0, C \geq 1/n \right\},
$$
with $c^{s,w} = c^s = 0$ for strong balls and any $c^{s,w} = c^w \in (0, 1)$ for weak balls. Then, the adaptive rate minimaxity holds in the following sense:

$$\sup_{(p,C) \in \Omega_{s,w}^n \cup \Omega_{0,n}} \sup_{\theta \in \Theta_{s,w}^{p,C,n}} \frac{E_\theta \| \hat{X}^\lambda(X) - \theta \|^2}{R(\Theta_{s,w}^{p,C,n})} \leq M^* (1 + \delta_{2,n}),$$

where $M^*$ and $L^2_{n}$ are as in Corollary 3.

For $0 \leq p \leq 2$, the minimax rate in $\ell_p$ balls can be expressed as

$$R(\Theta_{s,w}^{p,C,n}) \asymp \begin{cases} M^*_{s,w} \min \left( n^p \lambda_{p,C,n}^{2-p}, n, (nC^p)^{2/p} \right), & 0 < p \leq 2 \\
C\lambda_{0,C,n}^2, & p = 0 < C \\
0, & p = 0 = C, \end{cases}$$ (5.2)

where $\lambda_{p,C,n} = 1 \vee \sqrt{2 \log(n \wedge (1/C^p')}$ with $p' = p$ for $p > 0$ and $p' = 1$ for $p = 0$.

Here $a \asymp b$ means $a/b = O(1)$ and this $O(1)$ is uniform in (5.2). It follows from (2.32) that for small $C^p'$, the constant factor in (5.2) is accurate in the sense of its uniform validity when $\asymp$ is replaced by $\approx$, provided that $p = 0$ or $nC^p \lambda_{p,C,n}^{2-p} \ll (nC^p)^{2/p}$. When $nC^p \lambda_{p,C,n}^{2-p} \asymp (nC^p)^{2/p}$, or equivalently $nC^p \asymp \lambda_{p,C,n}^p$, the constant factor in (5.2) is no longer accurate for $p > 0$ by (2.31). Moreover, for $p > 0$, $nC^p \lambda_{p,C,n}^{2-p}$ is of greater order than the minimax rate when $(nC^p)^{2/p} \ll nC^p \lambda_{p,C,n}^{2-p}$.

In addition to the exact adaptive minimaxity literature discussed earlier, adaptive rate minimaxity in $\ell_p$ balls was proved in [3] for generalized $C_p$ when the minimax $\ell_2$ risk is of no smaller order than $O(1)$, and in [21] for EBThresh when the risk is of no smaller order than $(\log n)^{2+(2-p)/2}$ and for a modified EBThresh when the risk is of no smaller order than $\log n$, among many important contributions to the problem. It follows from [3, 34] that a hybrid between the Fourier general empirical Bayes estimator and universal soft threshold estimators is also adaptive rate minimax in $\ell_p$ balls when the minimax $\ell_2$ risk is of no smaller order than $O(1)$.

The results in [21, 1] are valid for the $\ell_q$ loss with $q \geq p$. It is unclear at the moment of this writing if our analysis can be extended to hard threshold estimators and the $\ell_q$ loss for $q > 2$. The continuity of the soft threshold estimator is a significant element in our analysis.

**Appendix**

**Proof of Lemma 1** (i) The risk of soft thresholding $N(0,1)$,

$$R(0, \lambda) = E(|N(0,1)| - \lambda)_+^2 = 2 \int_\lambda^\infty (x - \lambda)^2 \varphi(x) dx,$$
is clearly decreasing in $\lambda \in [0, \infty)$. To prove (3.11), we define

$$J_k(\lambda) = \int_0^\infty u^k \exp(-u - u^2/(2\lambda^2))du.$$  

With $u = \lambda(x - \lambda)$, we find that $\varphi(x) = \varphi(\lambda) \exp(-u - u^2/(2\lambda^2))$ and

$$\lambda^3R(0, \lambda)/2 = \lambda^3 \int_0^\infty (x - \lambda)^2 \varphi(x) dx = \varphi(\lambda)J_2(\lambda),$$

$$\lambda \Phi(-\lambda) = \lambda \int_\lambda^\infty \varphi(x) dx = \varphi(\lambda)J_0(\lambda).$$  

(A.1)

Integrating by parts yields

$$(k + 1)J_k(\lambda) = \int_0^\infty \exp(-x - x^2/(2\lambda^2))dx^{k+1}$$

$$= \int_0^\infty (x^{k+1} + x^{k+2}/\lambda^2) \exp(-x - x^2/(2\lambda^2))dx$$

$$= J_{k+1}(\lambda) + J_{k+2}(\lambda)/\lambda^2.$$  

(A.2)

It follows that $J_0(\lambda) = J_1(\lambda) + J_2(\lambda)/\lambda^2 \geq J_2(\lambda)/2 + J_2(\lambda)/\lambda^2$, so that

$$J_2(\lambda)/J_0(\lambda) \leq 1/(1/2 + 1/\lambda^2).$$

In addition, (A.2) also implies that $J_3(\lambda) \leq 3J_2(\lambda)$, so that

$$\frac{J_2(\lambda)}{J_0(\lambda)} = \frac{J_2(\lambda)}{J_2(\lambda)/2 + J_3(\lambda)/(2\lambda^2) + J_2(\lambda)/\lambda^2} \geq \frac{1}{1/2 + 3/(2\lambda^2) + 1/\lambda^2}.$$  

We complete the proof of (3.11) by simple algebra after applying the above two displayed inequalities to (A.1).

It follows from (3.2) that for $\mu \neq 0$,

$$R(\mu, \lambda) = E\left(\mu^2I_{|z_+|\leq\lambda} + (Z - \lambda)^2I_{Z_+\geq\lambda} + (Z + \lambda)^2I_{Z_-\leq-\lambda}\right)$$  

(A.3)

with $Z \sim N(0, 1)$. This implies $(\partial/\partial \mu)R(\mu, \lambda) = 2\mu P\{|N(\mu, 1)| \leq \lambda\}$. Since $R(\mu, \lambda)$ is even in $\mu$ and $\lim_{\mu\to\infty} R(\mu, \lambda) = \lambda^2 + 1$, we find

$$R(\mu, \lambda) \leq R(0, \lambda) + \mu^2 \land (\lambda^2 + 1), \quad \mu \land \lambda \geq 0.$$  

Integration of this inequality with $dG$ gives the first inequality in (3.12). Since $\rho_G(b) \leq (b/a)^2 \rho_G(a)$ for $0 \leq a \leq b$ and $B_0 \geq 4$ in (3.9), for $\lambda \geq 1$ the second inequality in (3.12) follows from the first and (3.11) via

$$\rho_G(\sqrt{\lambda^2 + 1} + R(0, \lambda) \leq (1 + \lambda^{-2})\rho_G(\lambda) + \frac{4\Phi(-\lambda)}{\lambda^2} \leq \rho_G(\lambda) + \frac{r_G(\lambda)}{\lambda^2}.$$  

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Let \( g_0(\lambda) = 4\Phi(-\lambda) + \lambda^2 - 1 - R(0, \lambda) \). For \( 0 \leq \lambda \leq 1 \), we have \( 1 - \lambda^2 \geq (\lambda^2 + 1) \wedge u^2 - 2(\lambda^2 \wedge u^2) \), so that

\[
\rho_G(\lambda) + r_G(\lambda) - \left\{ \rho_G(\sqrt{\lambda^2 + 1}) + R(0, \lambda) \right\} = \int \left\{ 2(\lambda^2 \wedge u^2) + 4\Phi(-\lambda) - (\lambda^2 + 1) \wedge u^2 - R(0, \lambda) \right\} G(du) \geq g_0(\lambda).
\]

Since \( g_0(0) = 0 \), \( g_0'(0) = -4\varphi(0) + 2E|N(0, 1)| = 0 \), and \( g_0''(\lambda) = 4\lambda\varphi(\lambda) + 2 - 2\Phi(\lambda) \geq 0 \), we have \( g_0(\lambda) \geq 0 \). Thus, the second inequality in (3.12) also holds for \( 0 \leq \lambda \leq 1 \).

(ii) Since \( \Phi(-\lambda - \tau\lambda) \leq e^{-\lambda^2(\tau + \tau^2/\lambda)}\Phi(-\lambda) \), (3.11) implies

\[
R(0, \lambda) \geq \frac{4\Phi(-\lambda)}{\lambda^2 + 5} \geq B_0\Phi(-\lambda - \tau_1\lambda), \tag{A.4}
\]

where \( \tau_1 \) is the solution of \( \tau_1 + \tau_1^2/2 = \lambda^{-2}\log\{(\lambda^2 + 5)B_0/4\} \). For \( \lambda > 1 \), this implies \( \tau_1 \leq \lambda^{-2}\{2\log\lambda + \log(5B_0/4)\} \). We also need a lower bound for the difference \( R_G(\lambda) - R(0, \lambda) \). Again, since \( (\partial/\partial \mu)R(\mu, \lambda) = 2\mu P\{|N(\mu, 1)| \leq \lambda\} \) by (A.3), for \( 0 \leq M \leq \lambda \) we have

\[
\int_{\mu \wedge (\lambda - M)}^{|\mu| \wedge (\lambda - M)} 2uP\left\{ |\lambda - 2M \leq N(\lambda - M, 1) \leq \lambda \right\} du \\
\leq \int_{\mu \wedge (\lambda - M)}^{|\mu| \wedge (\lambda - M)} 2uP\left\{ |N(\lambda - M, 1) \leq \lambda \right\} du \\
\leq \int_{0}^{|\mu| \wedge (\lambda - M)} 2uP\left\{ |N(u, 1) \leq \lambda \right\} du \\
\leq R(\mu, \lambda) - R(0, \lambda).
\]

Let \( \Phi(-M) = 1/(2 \vee \lambda) \). Since \( \Phi(-M) \leq e^{-M^2/2}/2 \), \( M < \sqrt{2\log((\lambda/2) \vee 1)} \). Integrating over \( G(du) \), we find that

\[
R_G(\lambda) - R(0, \lambda) \geq \left( 1 - \frac{2}{\lambda \vee 2} \right) \rho_G(\lambda - \sqrt{2\log((\lambda/2) \vee 1)}).
\]

Let \( \tau_2 = \sqrt{2\log((\lambda/2) \vee 1)/(\lambda \vee e)} \). Since \( \rho_G(a) \leq \rho_G(b) \leq (b/a)^2\rho_G(a) \) for \( 0 < a \leq b \), for \( \lambda \geq e \) we have

\[
R_G(\lambda) - R(0, \lambda) \geq \left( 1 - \frac{2}{\lambda \vee 2} \right)^2 (1 - \tau_2^2) (1 + \tau_1^2)^{-1} \rho_G(\lambda + \lambda\tau_1) \\
\geq \left( 1 + 2\tau_2 + M_0^*/(\lambda) \right) \rho_G(\lambda + \lambda\tau_1) \tag{A.5}
\]
with an $M_0^*\lambda$ depending on $B_0$ only. We are allowed to incorporate higher order terms in $M_0^*/\lambda$ for $\lambda \geq e$ in (A.5) since $\tau_1 \leq \lambda^{-2}\log((\lambda^2 + 5)B_0/4)$ and $\tau_2 \leq \lambda^{-1}\sqrt{2}\log\lambda$. Combining (A.4) and (A.5), we find that for $\lambda \geq e$,

$$ \rho_G(1) \leq \rho_G(\lambda + \lambda\tau_1) + B_0\Phi(-\lambda - \lambda\tau_1) \leq \left(1 + \lambda^{-1}\sqrt{8}\log\lambda + M_0^*/\lambda\right)R_G(\lambda). $$

For $\lambda_G \geq e$, this implies (3.14) with $\lambda = \lambda_G$ due to $\eta_G^* \leq \rho_G(\lambda + \lambda\tau_1) + B_0\Phi(\lambda + \lambda\tau_1)$.

For $0 \leq \lambda_G \leq e$, we have $\eta_G = R_G(\lambda_G) \geq R(0,e) > 0$, $\eta_G \leq r_G(0) = B_0/2$ and $\rho_G(1) \leq 1$, so that (3.14) also holds.

PROOF OF LEMMA 2. (i) We first proof the monotonicity of $\lambda_G^*$ in $B_0$. Let $\tilde\lambda_G^* = \arg\min_\lambda\{\rho_G + \tilde B_0\Phi(-\lambda)\}$ with $\tilde B_0 \leq B_0$. For all $\lambda \leq \tilde\lambda_G^*$,

$$ \rho_G(\tilde\lambda_G^*) - \rho_G(\lambda) \leq \tilde B_0\left\{\Phi(-\lambda) - \Phi(-\tilde\lambda_G^*)\right\} \leq B_0\left\{\Phi(-\lambda) - \Phi(-\tilde\lambda_G^*)\right\}, $$

so that $\lambda_G^* \geq \tilde\lambda_G^*$. We assume without loss of generality that $B_0 = 4$ in the proof of (3.15) and (3.16) since they only involve lower bounds for $\lambda_G^*$.

Let $o(1)$ denote uniformly small quantity when $\eta_n$ is sufficiently small. Suppose $\eta_G^* \leq \eta_n$. By (3.9), $4\Phi(-\lambda_G^*) \leq \eta_n$. Since $(d/dt)\{((\sqrt{\pi}/2 + t)\Phi(-t) - \varphi(t)\} = -\sqrt{\pi}/2\varphi(t) + \Phi(-t) \leq 0$, we have $\Phi(-t) \geq \varphi(t)/(\sqrt{\pi}/2 + t)$. Thus, with $t = \lambda_G^*$, we find $\varphi(\lambda_G^*) \leq (\sqrt{\pi}/2 + \lambda_G^*)\eta_n/4$, or equivalently

$$ (\lambda_G^*)^2 + \log\left(\left(\sqrt{\pi}/2 + \lambda_G^*\right)^2\right) \geq 2\log\left(1/\eta_n\right) + \log(8/\pi). $$

Since $(1 + x)^{1/2} = 1/(1 + x/2 + O(x^2))$ for small $x$, this implies

$$ \lambda_G^* \geq \left(2\log\left(1/\eta_n\right) + \log(8/\pi) - \log(2\log(1/\eta_n)) + o(1)\right)^{1/2} = \sqrt{2\log(1/\eta_n)}\left(1 + \frac{\log\log(1/\eta_n) - \log(4/\pi) + o(1)}{4\log(1/\eta_n)}\right)^{-1}. $$

This and (3.13) implies $R(0, \lambda_G) \leq \eta_n \leq (1 + o(1))\eta_n$. It follows from (3.11) that $R(0, t) \geq 4\Phi(-t)/(t^2 + 5) \geq 4\varphi(t)/\{(t^2 + 5)(t + \sqrt{\pi}/2)\}$, so that

$$ \lambda_G \geq \left(2\log\left(\frac{4 + o(1)}{\eta_n\sqrt{2\pi}}\right) - 2\log\left\{\left(\lambda_G^2 + 5\right)\left(\lambda_G + \sqrt{\pi}/2\right)\right\}\right)^{1/2} = \left(2\log(1/\eta_n) + \log(8/\pi) + o(1) - \log \lambda_G^6\right)^{1/2} \geq \sqrt{2\log(1/\eta_n)}\left(1 + \frac{3\log\log(1/\eta_n) + \log \pi + o(1)}{4\log(1/\eta_n)}\right)^{-1}. $$

This and (3.14) implies $\eta_G \leq (1 + o(1))\eta_n$. This completes the proof of (3.15).
Consequently, the first inequality in (3.16) follows from (3.13), (3.15) and the fact that $\rho_G(\lambda) \leq r_G(\lambda)$, while the second inequality in (3.16) follows from (3.14) and (3.15).

(ii) Let $\lambda_n = \sqrt{2\log(1/\eta_n)}$. Since $\lambda_n \geq 4\sqrt{2\pi} \geq 1$, $\rho_G(\lambda_n) \leq \lambda_n^2\eta_n$ and $\rho_G(\sqrt{\lambda_n^2 + 1}) \leq (1 + \lambda_n^{-2})\rho_G(\lambda_n) \leq \rho_G(\lambda_n) + \rho_G(1)$. We also have

$$B_0\Phi(-\lambda_n) \leq (B_0/\lambda_n)\varphi(\lambda_n) = \eta_nB_0/(\lambda_n\sqrt{2\pi}) \leq \eta_n.$$ 

Thus, by (3.12), $R_G(\lambda_n) - \rho_G(1) \leq \rho_G(\lambda_n) + B_0\Phi(-\lambda_n) \leq (1 + \lambda_n^2)\eta_n$.

(iii) Let $t_n = 2\log(4n/\sqrt{2\pi})$ and $y_n = t_n - 3\log t_n$. For $n \geq 10$, $t_n$ is increasing in $n$ with $3\log t_n > 5.13$ and $y_n$ is increasing in $t_n$ with $y_n\{\exp(2/(3y_n)) - 1\} \leq (2/3)\exp(2/(3y_n)) \leq 3.5 \leq 3\log t_n$. By algebra,

$$y_n + 3\log y_n + 2/y_n < t_n = y_n + 3\log t_n, \quad \forall n \geq 10.$$

By the Jensen inequality, $\int_0^\infty e^{-u-tu^2/2}du > e^{-t}$, so that

$$\frac{1}{4n} = z_n^{-2}\Phi(-z_n) = z_n^{-3}\varphi(z_n)\int_0^\infty e^{-u-(u/z_n)^2/2}du > z_n^{-3}\varphi(z_n)e^{-1/z_n^2}.$$ 

This gives $z_n^2 + 3\log z_n^2 + 2/z_n^2 > t_n$, so that $z_n^2 > y_n$ for $n \geq 10$. We also numerically verify $z_n^2 > y_n$ for $7 \leq n \leq 9$.

For $n \geq 1500$, $(\partial/\partial n)(y_n - \log n) = 2/n - (3/t_n)(2/n) - 1/n = (1 - 6/t_n)/n > 0.0004 > 0$ and $y_n - \log n \geq 0.01$, so that $z_n^2 \geq y_n > \log n$. We also verify $z_n^2 > \log n$ numerically for $2 \leq n < 1500$.

Suppose $\eta_{G_n}^* \leq z_n^2/n$. Since $z_n^{-2}\Phi(-z_n) = 1/(4n)$, we have

$$B_0\Phi(-\lambda_n^*) \leq \eta_{G_n}^* \leq 4\Phi\left(-\sqrt{n\eta_{G_n}^*}\right).$$ 

This inequality and the constraint $B_0 \geq 4$ yield

$$\lambda_n^* \geq \sqrt{n\eta_{G_n}^*} > \sqrt{n\rho_G(\lambda_n^*)} = \sqrt{\sum_{i=1}^n (\theta_i)^2 \wedge (\lambda_n^*)^2}.$$ 

Consequently, $\rho_{G_n}(\lambda_n^*) = \|\theta\|^2/n = r_{G_n}(\infty) = \eta_{G_n}^*$ and $\lambda_{G_n}^* = \infty$.

Let $\eta_n = (\sqrt{2\log n} - 2\sqrt{2\log \log n})^2/n$ with sufficiently large $n \geq n^*$. Suppose $\eta_{G_n} \leq \eta_n$. We need to prove $\eta_{G_n}^* \leq z_n^2/n$. Since $\lambda_n^*$ is increasing in $B_0$, it suffices to consider $B_0 = 4$. Since $\log(1/\eta_n) \geq \log n - \log(2\log n)$ it follows from (3.16) and the condition $\eta_{G_n} \leq \eta_n$ that

$$n\eta_{G_n}^* \leq \left(1 + \frac{2\sqrt{\log \log n + M^*}}{\log n}\right)n\eta_n$$

$$= \left(1 + \frac{2\sqrt{\log \log n + M^*}}{\log n}\right)\left(1 - \frac{2\sqrt{\log \log n}}{\log n}\right)^2 2\log n$$

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Since $z_n^2 \geq 2 \log \left( \frac{4n}{\sqrt{2\pi}} \right) - 3 \log \left( 2 \log \left( \frac{4n}{\sqrt{2\pi}} \right) \right) = 2 \log n - 3 \log \log n + O(1)$, we have $nn_G^* \leq z_n^2$. 

PROOF OF LEMMA 3. Assume $\mu \geq 0$ by symmetry. 

(i) Let $x_{\mu,\lambda}$ be the solution of $f_\lambda(x_{\mu,\lambda}) = 2\mu$. Since $f_\lambda(x) \leq \kappa_0(x - \lambda)$ for $x > 0$, we have $(2 - \kappa_0)\mu \leq \kappa_0(x_{\mu,\lambda} - \mu - \lambda)$. For $t_\lambda(x) > 2\mu$,

$$t_\lambda(x) - \mu \leq \kappa_0(x - \lambda) - \mu = \kappa_0(x - \mu - \lambda) + (\kappa_0 - 1)\mu \leq \kappa_0(x - \mu - \lambda) + \frac{(\kappa_0 - 1)}{2 - \kappa_0}\kappa_0(x_{\mu,\lambda} - \mu - \lambda).$$

Since $f_\lambda(x) \geq t_\lambda(x) > 2\mu$, we have $x > x_{\mu,\lambda}$, so that

$$|t_\lambda(x) - \mu| \leq C_0(x - \mu - \lambda) \leq C_0(|x - \mu| - \lambda).$$

For $x > -\lambda$ and $t_\lambda(x) \leq 2\mu$, $|t_\lambda(x) - \mu| \leq \mu$. For $x \leq -\lambda$,

$$|t_\lambda(x) - \mu| \leq \mu - f_\lambda(x) \leq \mu + \kappa_0(|x| - \lambda) \leq \kappa_0(|x - \mu| - \lambda).$$

Thus, $|t_\lambda(x) - \mu|$ is bounded by either $|\mu|$ or $C_0(|x - \mu| - \lambda)$. 

(ii) It suffices to consider the location model where the distribution of $X - \mu$ is fixed. Since $(\partial/\partial \mu)E(s_\lambda(X) - \mu)^2 = 2\mu P\{|X| \leq \lambda\}$, $E(s_\lambda(X) - \mu)^2 \leq \lim_{\mu \to \infty} E(s_\lambda(X) - \mu)^2 = E(X - \mu - \lambda)^2 = \lambda^2 + \text{Var}(X)$. Since the value of $t_\lambda(x)$ is between $s_\lambda(x)$ and $x$, it follows that

$$E\left(t_\lambda(X) - \mu\right)^2 \leq E\left(s_\lambda(X) - \mu\right)^2 + E(X - \mu)^2 \leq \lambda^2 + 2\text{Var}(X).$$

(iii) The $\ell_q$ error bound follows from (i) and (ii). 

(iv) Since $B_0 \geq 2C^2_0$, [3,11] gives $C^2_0 E(|X_i - \theta_i| - \lambda)^4 \leq C^2_0 R(0, \lambda) \leq 4C^2_0 \Phi(-\lambda)/\lambda^2 \leq 2B_0 \Phi(-\lambda)/\lambda^2$. Thus, (iii) with $q = 2$ and (2.17) yield

$$n^{-1}E_\theta ||t_\lambda(X) - \theta||_2^2 \leq \int \min\left(u^2 + C^2_0 R(0, \lambda), \lambda^2 + 2\right)G_n(du) \leq (1 + 2/\lambda^2)\rho_{G_n}(\lambda) + C^2_0 R(0, \lambda) \leq \rho_{G_n}(\lambda) + 2r_{G_n}(\lambda)/\lambda^2.$$ 

For $0 \leq \lambda \leq 1$, $(\lambda^2 + 2) \wedge u^2 - 3(\lambda^2 \wedge u^2) \leq 2(1 - \lambda^2) \leq (B_0/2)(1 - \lambda^2)$ gives

$$\rho_{G_n}(\lambda) + 2r_{G_n}(\lambda) - \rho_{G_n}(\sqrt{\lambda^2 + 2}) - C^2_0 R(0, -\lambda)$$

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\[ \geq \int \left\{ 3({\lambda}^2 \wedge u^2) + 2B_0\Phi(-\lambda) - ({\lambda}^2 + 2) \wedge u^2 - (B_0/2)R(0, \lambda) \right\} G_n(du) \]
\[ \geq 2B_0\Phi(-\lambda) + (B_0/2)(\lambda^2 - 1) - (B_0/2)R(0, \lambda), \]

which is nonnegative as in the proof of (3.12). \(\square\)

**Proof of Lemma 4.** (i) Let \(h_G(t) = \int e^{-u^2/2}\cosh(tu)G(du)\) where \(\cosh(t) = (e^t + e^{-t})/2\). Since \(P\{|N(u, 1)| > t\} = 1 - \Phi(t - u) + \Phi(-t - u)\),

\[ \frac{S_G(t)}{2\Phi(-t)} = \frac{1}{2\Phi(-t)} \int \int_{x > t} (\varphi(x + u) + \varphi(x - u)) dx G(du) \]
\[ = \frac{1}{2\Phi(-t)} \int \int_{x > t} (e^{-u^2/2}(e^{-xu} + e^{xu})) \varphi(x) dx G(du) \]
\[ = \frac{1}{\Phi(-t)} \int_{t}^{\infty} h_G(x) \varphi(x) dx. \]

Since \(G\) does not put the entire mass at 0, \(h_G(t)\) is strictly increasing in \(t\) for \(t \geq 0\), and the monotonicity of \(S_G(t)/\Phi(-t)\) follows from

\[ \frac{\partial}{\partial t} \log \left( \frac{S_G(t)}{2\Phi(-t)} \right) = -\frac{h_G(t)\varphi(t)}{ \int_{t}^{\infty} h_G(x) \varphi(x) dx } + \frac{\varphi(t)}{\Phi(-t)} \]
\[ = \frac{\varphi(t)}{\Phi(-t)} \int_{t}^{\infty} h_G(x) \varphi(x) dx \left( \int_{t}^{\infty} h_G(x) \varphi(x) dx - h_G(t)\Phi(-t) \right) > 0. \]

Since \(P\{|N(\mu, 1)| > t\}\) is even in \(\mu\) and

\[ \left( \frac{\partial}{\partial \mu} \right)^2 P\{|N(\mu, 1)| > t\} = -\Phi''(t - \mu) + \Phi''(-t - \mu) \leq 2 \max_{t} t \varphi(t) = 2\varphi(1) < 2, \]

we have \(S_G(t) \leq \int \left( P\{|N(0, 1)| > t\} + 1 \wedge u^2 \right) G(du) = 2\Phi(-t) + \rho_G(1)\). In addition,

\[ \tilde{G}(t)/2 \leq \int_{|u| \geq t} P\{|N(|u|, 1)| > t\} G(du) \leq S_G(t). \]

These imply the inequalities in (3.24).

To prove \(\lambda_{G_n}^* > \xi_{2,*}\), we observe from (3.9) that

\[ \frac{d\tilde{G}_n(\lambda)}{d\lambda} \bigg|_{\lambda = \lambda_{G_n}^*} = 2\lambda_{G_n}^* \tilde{G}_n(\lambda_{G_n}^*) - B_0\varphi(\lambda_{G_n}^*) = 0, \] (A.6)
so that by (3.24) and the condition $B_0 \geq 8/\alpha_2'$

$$\frac{S_{G_n}(\lambda_{G_n}^*)}{2\Phi(-\lambda_{G_n}^*)} \geq \frac{\dot{G}_n(\lambda_{G_n}^*)/2}{2\Phi(-\lambda_{G_n}^*)} = \frac{B_0\varphi(\lambda_{G_n}^*)}{8\lambda_{G_n}^* \Phi(-\lambda_{G_n}^*)} > \frac{1}{\alpha_2'}.$$  

The monotonicity of $S_{G_n}(t)/\Phi(-t)$ guarantees $\lambda_{G_n}^* > \xi_{2,*}$ by (3.21).

(ii) Since $S_{G_n}(\xi_{1,*})/(2\Phi(-\xi_{1,*})) = 1/\alpha_j'$, $1/\alpha_j' - 1 \leq \rho_{G_n}(1)/(2\Phi(-\xi_{j,*}))$ by (3.24) and simply algebra. For $t \leq \xi_{j,*}$, we have $g_{1,n}(t) \leq t \leq \xi_{j,*}$ and $R(0, g_{1,n}(t)) \leq 4\Phi(-t)$ by (3.23). Thus, by the monotonicity of $S_{G_n}(t)/\Phi(-t)$, (3.21) and (3.11), (3.25) follows from

$$\frac{\alpha_j' S_{G_n}(g_{1,n}(t))}{2\Phi(-t)} \leq \frac{\Phi(-g_{1,n}(t))}{\Phi(-t)} \leq \frac{(5 + g_{1,n}^2(t)) R(0, g_{1,n}(t))}{4\Phi(-t)} \leq 5 + t^2.$$  

(iii) Since $\theta_{s,n}^2 \leq 2\beta_0 \log n \leq \log n$, $\theta_{s,n} \leq \tilde{z}_n$ in Lemma 2 (iii), so that $\|\theta\|^2 = n\theta_{s,n}^2 \leq \theta_{s,n}^2$. It follows that $\|\theta\|_{\infty} \leq \theta_{s,n}$ and $\int |u| G_n(du) \leq \sqrt{\|\theta\|^2/n} \leq \theta_{s,n}/\sqrt{n}$ by Cauchy-Schwarz. By the convexity of $\Phi(x)$ in $x < 0$,

$$S_{G_n}(t) = \int \{\Phi(-t - |u|) + \Phi(-t + |u|)\} G_n(du)$$

$$\leq \Phi(-t) + \int \left\{\left(1 - \frac{|u|}{\theta_{s,n}}\right)\Phi(-t) + \frac{|u|}{\theta_{s,n}}\Phi(-t + \theta_{s,n})\right\} G_n(du) \quad (A.7)$$

$$\leq 2\Phi(-t) + n^{-1/2} \{\Phi(-t + \theta_{s,n}) - \Phi(-t)\}$$

for $t \geq \theta_{s,n}$. For $\theta_{s,n} \leq t \leq A_1^{1/2} \xi_{1,1}$, $(t\theta_{s,n} + \theta_{s,n}^2/2) \leq \beta_0 \log n$, so that

$$\Phi(-t + \theta_{s,n}) - \Phi(-t) \leq \Phi(-t) e^{t\theta_{s,n} + \theta_{s,n}^2/2} \leq \Phi(-t) n^{\beta_0}.$$  

Consequently, $S_{G_n}(t)/\{2\Phi(-t)\} \leq 1 + n^{\beta_0 - 1/2} / 2 \leq 1/\alpha_1'$ at $t = A_1^{1/2} \xi_{1,1}$. This gives $\xi_{1,*} \geq A_1^{1/2} \xi_{1,1}$ by (3.21) and the monotonicity of $S_{G_n}(t)/\Phi(-t)$.

The proof of (3.27) utilizes the following fact. For $x > t > 0$, $\Phi(-x) = \int_t^\infty \varphi(u + x - t) du \leq e^{-t(x-t)-(x-t)^2/2\Phi(-t)}$, so that

$$\int_t^\infty \Phi^{1/2}(-x) dx \leq \int_t^\infty \Phi^{1/2}(-x) e^{-(x-t)^2 - (x-t)^2/4} dx$$

$$\leq \Phi^{1/2}(-t) \min \left(2/t, \sqrt{n}\right).$$  

Since $t - \theta_{s,n} \geq t/2$ for $2\theta_{s,n} \leq t \leq A_1^{1/2} \xi_{1,1}$, (A.7) implies

$$\int_t^\infty S_{G_n}^{1/2}(x) dx \leq \int_t^\infty \sqrt{2\Phi(-x) + n^{-1/2}\Phi(-x + \theta_{s,n})} dx$$

$$\leq \sqrt{2\Phi(-t)2/t + \sqrt{n^{-1/2}\Phi(-t + \theta_{s,n})2/(t - \theta_{s,n})}}.$$  

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\[ \leq \sqrt{2\Phi(-t)(2/t)} \left( 1 + 2\sqrt{n^{-1/2} + n^{\beta_0 - 1/2}} \right). \]

This completes the proof of the lemma. \(\square\)

**Proof of Lemma 5.** (i) Let \(t_k = \log(k/E_\theta N(\xi_k))\). For \(k > E_\theta N(\xi_k)\), we have \(t_k > 0\) and

\[
P_\theta \{ N(\xi_k) \geq k \} \leq e^{-t_k} e^t \exp(t_k N(\xi_k))
= e^{-t_k} \prod_{i=1}^n E_\theta \exp(t_k |X_i| \geq \xi_k)
= e^{-t_k} \prod_{i=1}^n \left( 1 + (e^{t_k} - 1) P_\theta \{ |X_i| \geq \xi_k \} \right)
\leq \exp \{-t_k k + (e^{t_k} - 1) E_\theta N(\xi_k)\}
= \exp(-\nu_k k),
\]

where \(\nu_k = E_\theta N(\xi_k)/k - 1 - \log(E_\theta N(\xi_k)/k) > 0\). Similarly, for \(k < E_\theta N(\xi_k)\) and \(t_k < 0\),

\[
P_\theta \{ N(\xi_k) \leq k \} \leq e^{-t_k} e^t \exp(t_k N(\xi_k))
\leq \exp \{-t_k k + (e^{t_k} - 1) E_\theta N(\xi_k)\} = \exp(-\nu_k k).
\]

Thus, (3.28) holds in both cases.

(ii) Due to the monotonicity of \(S_{G_n}(t)/\Phi(-t)\), for \(\xi_{1,k} \leq \xi_{1,*}\)

\[
\frac{S_{G_n}(\xi_{1,k})}{k/n} - \frac{\alpha_1 S_{G_n}(\xi_{1,k})}{2\Phi(-\xi_{1,k})} \leq \frac{\alpha_1 S_{G_n}(\xi_{1,*})}{2\Phi(-\xi_{1,*})} = \frac{\alpha_1}{\alpha'_1}.
\]

Since \(\alpha_1 < \alpha'_1 < 1\), we have \(k/n > S_{G_n}(\xi_{1,k})\), so that by (3.28)

\[
P_\theta \{ N(\xi_{1,k}) - k \geq 0 \} \leq \exp(-\nu_{1,k} k).
\]

Since \(x - 1 - \log(x)\) is a decreasing function for \(0 < x < 1\), and \(E_\theta N(\lambda)/n = S_{G_n}(\lambda)\) by the definition these quantities in (2.5) and (3.20),

\[
\nu_{1,k} = \frac{S_{G_n}(\xi_{1,k})}{k/n} - 1 - \log \left( \frac{S_{G_n}(\xi_{1,k})}{k/n} \right)
\geq \frac{\alpha_1}{\alpha'_1} - 1 - \log \left( \frac{\alpha_1}{\alpha'_1} \right) = \nu_{1,*}.
\]

The above inequalities imply (3.29) in view of the definition of \(\hat{\xi}_1\) in (2.4). The proof of (3.30) is nearly identical and omitted. \(\square\)
Proof of Lemma 6. Let \( N(t) \) be as in (2.5) and define
\[
\Delta(x; a, b) = \max_{a \leq \lambda \leq b} \| t_\lambda(x) - t_b(x) \|_\sqrt{n}.
\]

The following inequalities follow directly from related definitions and (2.21):
\[
\begin{align*}
\sup_{a \leq \lambda \leq b} \left| L(X, \theta, \lambda) - L(X, \theta, b) \right| & \leq \Delta(X; a, b), \\
\sup_{a \leq \lambda \leq b} \left| R_{1,n}(\theta, \lambda) - R_{1,n}(\theta, b) \right| & \leq E_\theta \Delta(X; a, b), \\
E_\theta \Delta^2(x; a, b) & \leq \kappa_1^2 (b - a)^2 E_\theta N(a)/n = \kappa_1^2 (b - a)^2 S_{G_n}(a).
\end{align*}
\]

The last inequality in (A.8) follows from \(|t_\lambda(x) - t_b(x)| \leq \kappa_1 (b - a) I\{|x| > a\} \) for \( a \leq \lambda \leq b \).

Let \( \Delta_j(x) = \Delta(x; c_{j-1}, c_j) \) and \( B_j = \{c_{j-1} \leq Y < c_j\} \). It follows from the triangle inequality and the first two inequalities of (A.8) that
\[
\sqrt{H_\theta(a, b)} \leq \max_{1 \leq j \leq m} \left| L(X, \theta, Y) - R_{1,n}(\theta, Y) \right| I_{B_j}
\]
\[
\leq \max_{1 \leq j \leq m} \left| L(X, \theta, c_j) - R_{1,n}(\theta, c_j) \right| I_{B_j}
\]
\[
+ \max_{1 \leq j \leq m} \left\{ \Delta_j(X) + E_\theta \Delta_j(X) \right\} I_{B_j}
\]
\[
\leq \max_{1 \leq j \leq m} \left| L(X, \theta, c_j) - R_{1,n}(\theta, c_j) \right| I_{B_j}
\]
\[
+ \max_{1 \leq j \leq m} \left\{ \Delta_j(X) - E_\theta \Delta_j(X) \right\} + I_{B_j} + 2 \max_{1 \leq j \leq m} I_{B_j} E_\theta \Delta_j(X).
\]

Since \( \|t_\lambda(X) - \theta\| \) also has Lipschitz norm \( \kappa_0 \), \( L(x, \theta, \lambda) \) has Lipschitz norm \( \kappa_0/\sqrt{n} \).

Thus, by the Gaussian isoperimetric inequality,
\[
E_\theta \left\{ \max_{1 \leq j \leq m} \left| L(X, \theta, c_j) - R_{1,n}(\theta, c_j) \right| I_{B_j} \right\}^2
\]
\[
\leq \int_0^\infty \sum_{j=1}^m P_\theta \left\{ \left| L(X, \theta, c_j) - R_{1,n}(\theta, c_j) \right| I_{B_j} > x \right\} dx^2
\]
\[
\leq \sum_{j=1}^m \int_0^\infty \min (\pi_j, 2e^{-nx^2/(2\kappa_0^2)}) dx^2
\]
\[
= \frac{2\kappa_0^2}{n} \sum_{j=1}^m \left\{ \pi_j \log(2/\pi_j) + \pi_j \right\}.
\]

Similarly, due to \(|\Delta_j(u) - \Delta_j(v)| \leq \kappa_0 \|u - v\| / \sqrt{n} \),
\[
E_\theta \left\{ \max_{1 \leq j \leq m} \left( \Delta_j(X) - E_\theta \Delta_j(X) \right) I_{B_j} \right\}^2 \leq \frac{2\kappa_0^2}{n} \sum_{j=1}^m \pi_j \log(2e/\pi_j).
\]

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Inserting the above two inequalities and the third inequality of (A.8) to (A.9) after an application of the Minkowski inequality, we find

\[ \sqrt{E_{\theta}H_{\theta}(a, b)} \leq \left\{ E_{\theta}\left( \max_{1 \leq j \leq m} \left| L(X, \theta, c_j) - R_{1,n}(\theta, c_j) \right| I_{B_j} \right) \right\}^{1/2} \]

\[ + \left\{ E_{\theta}\left( \max_{1 \leq j \leq m} \left( \Delta_j(X) - E_{\theta}\Delta_j(X) \right) + I_{B_j} \right) \right\}^{1/2} \]

\[ + 2 \left\{ E_{\theta}\left( \max_{1 \leq j \leq m} I_{B_j} E_{\theta}\Delta_j(X) \right) \right\}^{1/2} \]

\[ \leq 2\kappa_0 \left\{ \frac{2}{n} \sum_{j=1}^{m} \pi_j \log(2e/\pi_j) \right\}^{1/2} + 2\kappa_1 \left\{ \sum_{j=1}^{m} \pi_j (c_j - c_{j-1})^2 S_Gn(c_{j-1}) \right\}^{1/2}. \]

This is (3.31). With \( c_j = a + (j/m)(b - a) \) and \( \pi_j = P\{ c_{j-1} \leq Y < c_j \} \),

\[ \sum_{j=1}^{m} \pi_j \log(2e/\pi_j) = \sum_{j=1}^{m} \int_{0}^{\infty} \min \{ \pi_j, 2e^{-t} \} dt \]

\[ \leq \int_{0}^{\infty} \min \{ 1, 2me^{-t} \} dt = \log(2em) \]

and \((c_j - c_{j-1})^2 S_Gn(c_{j-1}) \leq m^{-2}(b - a)^2 S_Gn(a)\), so that (3.31) implies (3.35).

It follows from the definition of \( H_{\theta}^*(a, b) \) and (A.8) that for \( \epsilon > 0 \),

\[ \sqrt{E_{\theta}H_{\theta}^*(\lambda)} - \sqrt{E_{\theta}H_{\theta}^*(\lambda + \epsilon)} \leq \sqrt{E_{\theta}\Delta^2(X; \lambda, \lambda + \epsilon)} \leq \epsilon \kappa_1 S_G^{1/2}(\lambda), \]

in view of the third inequality of (A.8). This implies \((\partial/\partial \lambda)\sqrt{E_{\theta}H_{\theta}^*(\lambda)} \leq \kappa_1 S_G^{1/2}(\lambda)\).

Since \( H_{\theta}^*(\lambda) \to 0 \) almost surely as \( \lambda \to \infty \), the monotone convergence theorem gives (3.36).

Finally, the Gaussian isoperimetric inequality gives

\[ \text{Var}[L(X, \theta, \lambda)] = \int_{0}^{\infty} P_{\theta} \left\{ \left| L(X, \theta, \lambda) - R_{1,n}(\theta, \lambda) \right| > t \right\} dt^2 \]

\[ \leq \frac{1}{n} \int_{0}^{\infty} P_{\theta} \left\{ \left\| t_\lambda(X) - \theta \right\| - E_{\theta}\| t_\lambda(X) - \theta \| \right\} > t \right\} dt^2 \]

\[ \leq \frac{2}{n} \int_{0}^{\infty} e^{-t^2/(2\kappa_0^2)} dt^2 = \frac{4\kappa_0^2}{n}. \]

This and \( E_{\theta}L^2(X, \theta, \lambda) = R_{1,n}^2(\theta, \lambda) + \text{Var}[L(X, \theta, \lambda)] \) give (3.37). \( \square \)

**Proof of (3.3).** Since \( R(0, x) \) is decreasing in \( t \), (4.1) implies \( g_{1,n}(x) \to \infty \) as \( x \to \infty \). For large \( x \), \( R(0, g_{1,n}(x)) = (4 + o(1))g_{1,n}(x)^{-3} \varphi(g_{1,n}(x)) \leq 4\Phi(-x) = (4 + o(1))x^{-1} \varphi(x) \) by (4.1), so that

\[ -g_{1,n}^2(x)/2 - 3 \log g_{1,n}(x) + o(1) \leq -x^2/2 - \log x. \]
This implies \( g_{1,n}(x) \geq (1 + o(1))x \). It follows that \( g_{1,n}(x) = (1 + o(1))x \) due to \( g_{1,n}(x) \leq x \). Thus, for \( A_1 = 1 + \delta_{1,n} \) and large \( x \)

\[
\Phi\left(-\sqrt{1 + \delta_{1,n}}g_{1,n}(x)\right)
\approx \left(\sqrt{A_1}2\pi x\right)^{-1}\left(\sqrt{2\pi}\varphi(g_{1,n}(x))\right)^{1+\delta_{1,n}}
\approx \frac{(2\pi)^{\delta_{1,n}/2}}{A_1^{1/2}x} \left(\left(x^3/4\right)R(0, g_{1,n}(x))\right)^{1+\delta_{1,n}}
\leq \frac{(1 + o(1))(2\pi)^{\delta_{1,n}/2}}{x} \left(\frac{x^3M_0\Phi(-x)}{4(x^{c_1,n} + 2)(\log_+ x)^{c_2,n}}\right)^{1+\delta_{1,n}}.
\]

This completes the proof of (4.3). \( \square \)

**Proof of Theorem 5.** Let \( A_j = 1 + \delta_{j,n} \), \( j = 1, 2 \). We denote by \( M^* \) a constant depending on \( \{\alpha'_1, \alpha'_2, \beta_0, A, M_0\} \) only which may take different values from one appearance to the next. We note that \( A_1 \leq A \) for all \( n \) and \( A_2 \leq A \) for \( n < n_\ast \).

Recall that in (3.10), \( \lambda_{G_n}^* = \arg \min_{\lambda} r_{G_n}(\lambda) \) and \( \eta_{G_n}^* = r_{G_n}(\lambda_{G_n}^*) \). Our plan is to prove that

\[
E_{\theta}\left(||\hat{\lambda}(X) - \theta||/\sqrt{n} - R_{1,n}(\theta, \hat{\lambda})\right) \leq M^*(A_2\tau_{1,n}^*\eta_{G_n}^* + \tau_{2,n}^*) \quad (A.10)
\]

and that with \( R_{G_n}^{(sm)}(\lambda) = E_{\theta}||t_{\lambda}(X) - \theta||^2/n \) as in (3.7),

\[
E_{\theta}R_{G_n}^{(sm)}(\hat{\lambda}) \leq A_2\eta_{G_n}^* + M^*(A_2\tau_{1,n}^*\eta_{G_n}^* + \tau_{2,n}^*). \quad (A.11)
\]

We first observe that (4.5) follows from (A.10) and (A.11); To wit,

\[
\sqrt{E_{\theta}||\hat{\lambda}(X) - \theta||^2/n}
\leq \sqrt{E_{\theta}R_{1,n}^2(\theta, \hat{\lambda})} + \sqrt{E_{\theta}\left(||\hat{\lambda}(X) - \theta||/\sqrt{n} - R_{1,n}(\theta, \hat{\lambda})\right)^2}
\leq \sqrt{A_2\eta_{G_n}^* + M^*(A_2\tau_{1,n}^*\eta_{G_n}^* + \tau_{2,n}^*)} + \sqrt{M^*(A_2\tau_{1,n}^*\eta_{G_n}^* + \tau_{2,n}^*)}
\leq \sqrt{A_2\eta_{G_n}^* + 2M^*(A_2\tau_{1,n}^*\eta_{G_n}^* + \tau_{2,n}^*)},
\]

due to the Cauchy-Schwarz inequality \( R_{1,n}^2(\theta, \lambda) \leq R_{G_n}^{(sm)}(\lambda) \).

Let \( \theta_{s,n} = \max\{t : A_1^{1/2}G_1t + t^2/2 \leq \beta_0(1 + \log n)\} \). Define

\[
\begin{cases}
\text{Case 1: } n < n_\ast \text{ or } \eta_{G_n}^* > \theta_{s,n}^2/n, \\
\text{Case 2: } n \geq n_\ast \text{ and } \eta_{G_n}^* \leq \theta_{s,n}^2/n.
\end{cases} \quad (A.12)
\]

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It follows from (2.3) and the definition of \( \theta_{s,n} \) that
\[
\frac{1}{\theta_{s,n}^2} = \frac{(A_{1}^{1/2} \xi_{1,1} + \sqrt{A_{1} \xi_{1,1}^2 + 2 \beta_0 \log n})^2}{(2\beta_0)^2(1 \vee \log n)^2} \leq \frac{M^* A_1}{1 \vee \log n}.
\]
Thus, due to \( \log(e \vee \log n)/(1 \vee \log n) \leq \tau_{s,n}^* \), by (4.6) and the boundedness of \( \max_{n < n^*} 1/\tau_{s,n}^* = \max_{n < n^*} n^{A_1}/L_{2,n} \) by (4.3), we have in Case 1
\[
\frac{1}{n} \leq \begin{cases} 
\frac{\eta^*_{G_n}/\theta_{s,n}^2}{M^* \tau_{1,n}^* \eta^*_{G_n}/\log(e \vee \log n)}, & n \geq n^* \\
M^* \tau_{2,n}^*/\{A_2 n \log(e \vee \log n)\}, & n < n^*.
\end{cases}
\]  
(A.13)

The conditions in Case 2 allows application of Lemma 4 (iii).
Let \( \nu_{1,s} = \alpha_1/\alpha_1' - 1 - \log(\alpha_1/\alpha_1') \) as in (3.29) and define
\[
k_0 = \begin{cases} 
1, & \text{Case 1}, \\
\left(\left(1 + (\delta_{1,n} \log n + 2 \log \log n)/\nu_{1,s}\right) \vee 1\right) \wedge n, & \text{Case 2}.
\end{cases}
\]  
(A.14)

Let \( \xi_{1,s} \) be as in (3.21). Define \( \xi_{1,n+1} = \xi_{1,s} \) for \( \xi_{1,s} < \xi_{1,n} \);
\[
k_{1,s} = \min \{ k : \xi_{1,k} \leq (\xi_{1,s} \wedge \xi_{1,k_0}) \}, \quad \lambda_{1,s} = \sqrt{1 + \delta_{1,n} g_{1,n}(\xi_{1,k_1})} = A_{1}^{1/2} g_{1,n}(\xi_{1,k_1}).
\]  
(A.15)

Note that \( k_{1,s} \geq k_0 \) and that \( \{k_{1,s}, \xi_{1,k_1}\} \) is the unique solution of
\[
\begin{cases} 
\xi_{1,k_1}^* \leq \xi_{1,s} < \xi_{1,k_1}^* - 1, & \xi_{1,n} \leq \xi_{1,s} \leq \xi_{1,k_0} \\
\xi_{1,k_1}^* = \xi_{1,k_0}, & \xi_{1,s} \geq \xi_{1,k_0} \\
\xi_{1,k_1}^* = \xi_{1,s}, & \xi_{1,s} < \xi_{1,n}.
\end{cases}
\]

We split the excess risk in 4 main terms:
\[
\zeta_{1,n} \equiv E_{\theta} \left\| t_{1,s}(X) - \theta \right\| / \sqrt{n} - R_{1,n}(\theta, \hat{\lambda}) \right\| \right|^2 I\{\hat{\lambda} \leq \lambda_{1,s}\},
\]  
(A.16)
\[
\zeta_{2,n} \equiv E_{\theta} \left\| t_{2,s}(X) - \theta \right\| / \sqrt{n} - R_{1,n}(\theta, \hat{\lambda}) \right\| \right|^2 I\{\hat{\lambda} > \lambda_{1,s}\},
\]  
(A.17)
\[
\zeta_{3,n} \equiv E_{\theta} \rho_{G_n}(\hat{\lambda}^2 + 2)^{1/2} - \rho_{G_n}(\lambda_{G_n}^s),
\]  
(A.18)
\[
\zeta_{4,n} \equiv C_0^2 E_{\theta} R(0, \hat{\lambda}).
\]  
(A.19)

We prove in four steps that
\[
\zeta_{1,n} \leq M^* \left( \tau_{1,n}^* \eta_{G_n}^* + \tau_{2,n}^* \right),
\]  
(A.20)
\[
\zeta_{2,n} \leq M^* \left( A_{2} \tau_{1,n}^* \eta_{G_n}^* + \tau_{2,n}^* \right),
\]  
(A.21)
\[
\zeta_{3,n} \leq \delta_{2,n} \eta_{G_n}^* + A_{2} M^* \tau_{1,n}^* \eta_{G_n}^*;
\]  
(A.22)
\[
\zeta_{4,n} \leq M^* \left( \tau_{1,n}^* \eta_{G_n}^* + \tau_{2,n}^* \right).
\]  
(A.23)
Inequalities (A.20) and (A.21) directly imply (A.10). By (3.9) and (3.10), \( \eta^{*}_{G,n} = \rho_{G,n}(\lambda^{*}_{G,n}) + B_{0}\Phi(-\lambda^{*}_{G,n}) \). By Lemma 3 (iv),

\[
R_{G}^{(sm)}(\hat{\lambda}) \leq \rho_{G}((\hat{\lambda}^{2} + 2)^{1/2}) + C_{0}^{2}R(0, \hat{\lambda}).
\]

Thus, (A.22) and (A.23) imply (A.11). It remains to prove (A.20), (A.21), (A.22) and (A.23). This is done in the following four steps respectively.

**Step 1.** In this step we prove (A.20). Since \( g'_1(n)(x) > 0 \), \( \xi^{*}_{1,n} < \xi_{1,n} \) implies \( \hat{\lambda} \geq A_{1}^{1/2}g_{1,n}(\xi_{1,n}) > A_{1}^{1/2}g_{1,n}(\xi^{*}_{1,n}) = \lambda^{*}_{1,n} \). Thus, this step only concerns the case of \( \xi^{*}_{1,n} \geq \xi_{1,n} \), where \( \xi_{1,n} \leq \xi_{1,k_{1},*} \leq \xi^{*}_{1,n} \).

It follows from (4.2) and (3.29) that for all \( k_{1} < n \),

\[
P\left\{ g_{1,n}(\xi_{1,k+1}) \leq A_{1}^{1/2}\lambda \leq g_{1,n}(\xi_{1,k}) \right\} \leq P\left\{ \hat{\xi}_{1} \leq \xi_{1,k} \right\} \leq e^{-\nu_{1},k_{1}}.
\]

Since \( 2\Phi(-\xi_{j,k}) = \alpha_{j}k/n \), we have

\[
\frac{\alpha_{j}/(2n)} = \frac{\Phi(-\xi_{j,k+1}) - \Phi(-\xi_{j,k})}{\Phi(-\xi_{j,k})} \geq \frac{\varphi(\xi_{j,k})}{\varphi(\xi_{j,k+1})} \geq \frac{\xi_{j,k} \Phi(-\xi_{j,k})(\xi_{j,k} - \xi_{j,k+1})}{\xi_{j,k}(\xi_{j,k} - \xi_{j,k+1})} \alpha_{j}k/(2n).
\]

Since \( 0 \leq (d/dx)g_{1,n}(x) \leq M_{0} \) by (1.1), this gives

\[
0 \leq g_{1,n}(\xi_{1,k}) - g_{1,n}(\xi_{1,k+1}) \leq M_{0}(\xi_{1,k} - \xi_{j,k+1}) \leq M_{0}/(k\xi_{1,k}).
\]

An application of (3.34) with \( \pi_{k} = e^{-\nu_{1},k_{1}} \) and \( c_{n+1-k} = A_{1}^{1/2}g_{1,n}(\xi_{1,k}) \) yields

\[
E_{\hat{\theta}} \left[ \frac{1}{\sqrt{n}} \| \hat{\theta} - \theta \| - R_{1,n}(\theta, \hat{\lambda}) \right]^{2}I\{ \hat{\lambda} \leq \lambda_{1,*} \} \leq 8 \sum_{k_{1},* < k < n} \left\{ \pi_{k} \log(2e/\pi_{k}) \right. \frac{n}{(2\kappa_{0}^{2})} + \pi_{k} \left( \frac{A_{1}^{1/2}M_{0}}{k\xi_{1,k}/\kappa_{1}} \right)^{2} \left. S_{G_{n}}(g_{1,n}(\xi_{1,k+1})) \right\}.
\]

For \( k_{1,*} \leq k < n \), we have \( \xi_{1,k+1} \leq \xi^{*}_{1,*} \) by (A.15), so that by (A.25)

\[
\frac{S_{G_{n}}(g_{1,n}(\xi_{1,k+1}))}{\alpha_{1}(k+1)/n} = \frac{S_{G_{n}}(g_{1,n}(\xi_{1,k+1}))}{2\Phi(-\xi_{1,k+1})} \leq \frac{5 + \xi_{1,k+1}^{2}}{\alpha_{1}^{2}} \leq \frac{5 + \xi_{1,k}^{2}}{\alpha_{1}^{2}}.
\]

It follows from this inequality, (A.16) and (A.25) that

\[
\zeta_{1,n} \leq \sum_{k_{1,*} \leq k} e^{-\nu_{1},k_{1}} \left\{ \frac{k\nu_{1,*} + \log(2e)}{n/(16\kappa_{0}^{2})} + \frac{A_{1}M_{0}^{2}(5 + \xi_{1,k}^{2})(k+1)}{k^{2}\xi_{1,k}^{2}n\alpha_{1}^{2}/(8\alpha_{1}^{2}\kappa_{1}^{2})} \right\} \leq \frac{M_{*}k_{1,*}\nu_{1,*}}{ne^{nu_{1,k_{1}}}}.
\]
We note that $\frac{1}{\xi_{1,k}} \leq \frac{1}{\xi_{1,n}} = 1/|\Phi^{-1}(\alpha_1/2)|$ is bounded.

In Case 1, (A.20) follows from (A.26) and (A.13) due to $x/e^x \leq 1/e$:

$$\zeta_{1,n} \leq \frac{M^*}{n} \leq M^* \left( \tau_{1,n}^* \eta_{G_n}^* + \tau_{2,n}^*/n \right).$$

In Case 2, $\nu_{1,k_1,n} \geq \nu_{1,k_0} \geq \delta_{1,n} \log n + 2 \log \log n$ by (A.14) and (A.15), so that (A.26) implies

$$\zeta_{1,n} \leq \frac{M^* k_0}{n^{1+\delta_{1,n} \log n} / \log n} \leq M^* \tau_{2,n}^*,$$

by the definition of the $\tau_{2,n}^*$. Note that by (4.4)

$$n^{1+\delta_{1,n}} \tau_{2,n}^* = L_2, n \geq (\log \log n)^{1-c_{2,n}} / (\log n)^{c_{1,n}/2} \geq \frac{\log \log n}{M^* \log n},$$

due to the constraints $c_{2,n} \leq 0$ for $c_{1,n} = 2$ and $0 < c_{1,n} \leq 2$.

**Step 2.** In this step, we prove (A.21). We first consider Case 1 as specified in (A.12) with $k_0 = 1$ as in (A.14). An application of the concentration inequality (3.35) yields that for all positive integer $m$,

$$\zeta_{2,n} = E \left[ \frac{1}{\sqrt{n}} \| \hat{\theta} - \theta \| - R_{1,n}(\theta, \hat{\lambda}) \right]^2 \leq M^* \left( \frac{\log m + S_{G_n}(\lambda_{1,n}) A_2 \xi_{2,1}^2}{m^2} \right),$$

By (2.3), $\xi_{1,k_{1,n}}^2 \leq \xi_{1,1}^2 \leq \xi_{2,1}^2 \leq 2 \log(2n/\alpha_2)$. By the upper bound for $S_{G_n}(g_{1,n}(t))$ in (3.25),

$$S_{G_n}(\lambda_{1,n}) \leq (5 + \xi_{1,k_{1,n}}^2) 2\Phi(-\xi_{1,k_{1,n}})/\alpha_1' \leq M^*(\log n) 2\Phi(-\xi_{1,k_{1,n}}).$$

Since $\xi_{1,k_{1,n}} \leq \xi_{1,1} \leq \xi_{1,k_{1,n}} \leq \xi_{1,k_0}$ for $k_{1,n} > k_0 = 1$ and $2\Phi(-\xi_{1,k}) = \alpha k/n$, the upper bound for $2\Phi(-\xi_{1,k})$ in Lemma 4(ii) yields

$$2\Phi(-\xi_{1,k_{1,n}}) \leq \frac{\alpha_1}{n} + 4\Phi(-\xi_{1,1}) \leq \frac{\alpha_1}{n} + \frac{2\alpha_1' \rho_{G_n}(1)}{1 - \alpha_1' \rho_{G_n}(1)}.$$

Moreover, (3.14) and (3.16) imply $\rho_{G_n}(1) \leq M^* \eta_{G_n}^*$. Thus, with $m = \lceil (\log n)^2 \rceil$ and an application of (A.13), the upper bound for $\zeta_{2,n}$ becomes

$$\zeta_{2,n} \leq M^* \left( \frac{\log m + S_{G_n}(\lambda_{1,n}) A_2 \xi_{2,1}^2}{m^2} \right),$$

$$\leq M^* \left\{ \frac{\log m + A_2 (\log n)^2}{(\log n)^4} \left( \frac{1}{n} + \rho_{G_n}(1) \right) \right\},$$

$$\leq M^* \left\{ \frac{A_2 + \log \log n}{n} + \frac{A_2 \eta_{G_n}^*}{(\log n)^2} \right\},$$

$$\leq M^* \left( A_2 \tau_{1,n}^* \eta_{G_n}^* + \tau_{2,n}^* \right).$$

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Now we consider Case 2 as specified in (A.12). By Lemma 4 (iii), \( \xi_{1,*} \geq A_1^{1/2} \xi_{1,1} \geq \xi_{1,k_0} \), so that \( \lambda_{1,*} = A_1^{1/2} g_{1,n}(\xi_{1,k_0}) \) by (A.15). Since \( 2\Phi(-\xi_{1,k_0}) = \alpha_1 k_0/n \) and \( \log k_0 = o(1) \log n \) by (A.14), \( \xi_{1,k_0} \approx \sqrt{2} \log n \). By (4.4), \( (1 + o(1)) x = g_{1,n}(x) \leq x \) for large \( x \) as in the proof of (4.3), so that \( \lambda_{1,*} \approx A_1 \sqrt{2} \log n \). Thus, by (4.3) and the definition of \( L_{2,n} \) in (4.4),

\[
\Phi(-\lambda_{1,*}) = \Phi\left(-\sqrt{1 + \delta_{1,n} g_{1,n}(\xi_{1,k_0})}\right)
\leq \frac{M^*}{\sqrt{1 + \delta_{1,n} \xi_{1,k_0}}} \left( \frac{\xi_{1,k_0}^3}{(\xi_{1,k_0})^2} \Phi(-\xi_{1,k_0}) \right)^{1+\delta_{1,n}}
\leq \frac{M^*}{\sqrt{1 + \delta_{1,n} \xi_{1,1}}} \left( \frac{\xi_{1,1}^3 \alpha_1 k_0/n}{(\xi_{1,1})^2} \right)^{1+\delta_{1,n}}
\leq \frac{M^*}{\sqrt{\log n}} \left( \frac{(\log n)^{(3-c_{1,n})/2} k_0}{n (\log n)^{c_{2,n}}} \right)^{1+\delta_{1,n}}
\leq M^* \left( \frac{\delta_{1,n} (\log n)^{(5-c_{1,n})/2}}{n (\log n)^{c_{2,n}}} + \frac{(\log n)^{(3-c_{1,n})/2}}{n (\log n)^{c_{2,n}-1}} \right)^{1+\delta_{1,n}}
\leq \frac{M^* L_{2,n}}{n^{1+\delta_{1,n}}} \lambda_{1,*}^2.
\]

Moreover, it follows from the definition of \( H_\theta^*(\lambda) \) in (3.33) that

\[
\left\| t_\lambda(X) - \theta \right\| / \sqrt{n} - R_{1,n}(\theta, \lambda) \left\{ \lambda > \lambda_{1,*} \right\}
\leq \sqrt{H_\theta^*(\lambda_{1,*})} + E_\theta \sqrt{H_\theta(\lambda_{1,*})}.
\]

Consequently, (3.36) of Lemma 6 and (3.27) of Lemma 4 (iii) yield

\[
\frac{\zeta_{2,n}}{\kappa_{1}} \leq 4 \left( \int_{\lambda_{1,*}}^{\infty} S_{G_n}(t) dt \right)^2 \leq \frac{(4C')^2}{\lambda_{1,*}^2} \Phi(-\lambda_{1,*}) \leq \frac{M^* L_{2,n}}{n^{1+\delta_{1,n}}} = M^* \tau_{2,n}^*.
\]

Thus, (A.21) holds in both cases.

**Step 3.** In this step we prove (A.22). Recall that \( \rho_G(\lambda) = \int (u^2 \wedge \lambda^2) G(du) \) in (3.33) and \( \tilde{\lambda} \leq \sqrt{1 + \delta_{2,n}\xi_2} \) in (4.2). We bound \( \zeta_{3,n} \) in (A.18) by

\[
\zeta_{3,n} \leq G_n(\lambda_{G_n}^*) E_\theta \left( 2 + (1 + \delta_{2,n}) \xi_2^2 - (\lambda_{G_n}^*)^2 \right) + \zeta_{3,n}.
\]

By (A.24), \( \xi_{2,k}^2 - \xi_{2,k+1}^2 = (\xi_{2,k} + \xi_{2,k+1})/k \xi_{2,k} \leq 2/k \). Since \( B_0 = 8/\alpha'_2, \lambda_{G_n}^* \geq \xi_{2,*} \) by Lemma 4 (i). Let \( k_{2,*} = \inf \{ k \geq 1 : \xi_{2,k} \geq \lambda_{G_n}^* \} \). By (3.30),

\[
E_\theta \left( \xi_2^2 - (\lambda_{G_n}^*)^2 \right) \leq E_\theta \left( \xi_2 - \xi_{2,k_{2,*}}^2 \right) + \zeta_{3,n}.
\]
Let \( \hat{\xi} \). Thus, by the analysis in Step 1 after (A.26),
\[
\xi
\]
we proved / formally bounded. Thus, 1 with the two cases specified in (A.12) and the definition of
\( k \). It follows from (A.6) that
\[
\tau(i) \quad \text{and the definition of} \quad \theta
\]
\( R \)
Since \( 1/\lambda \) is decreasing in \( \lambda \). Thus, 1/\( \lambda \) is uniformly bounded. Thus, \( 1/(\lambda \) is decreasing in \( \lambda \) by \( \lambda \) of Lemma 2 (i) and the definition of \( \tau \), and (A.22) follows.

**STEP 4.** In this step we prove (A.23). By Lemma 1 (i), \( R(0, \lambda) \) is decreasing in \( \lambda \).
Let \( \hat{\lambda} = A_{1/2} g_{1/n}(\hat{\xi}) \). Since \( \hat{\lambda} \geq \hat{\lambda} \), it suffices to prove
\[
E_\theta R \left( 0, \hat{\lambda} \right) \leq M^* (\tau_{1,n}^* \eta_{G_n}^* + \tau_{2,n}^*).
\]  (A.29)

By condition \( (i) \) on \( g_{1/n} \) and the definition of \( \xi_{1,k} \) in (2.3), \( R(0, g_{1/n}(\xi_{1,k})) \leq 4\Phi(-\xi_{1,k}) = 2\alpha_1 k/n \). By (A.15) and (3.29) of Lemma 5
\[
E_\theta R \left( 0, \hat{\lambda} \right) I\{\hat{\lambda} \leq \lambda_{1,*}\} \leq \sum_{k_1 \leq k \leq n} R(0, g_{1/n}(\xi_{1,k})) P_\theta \{\hat{\xi} = \xi_{1,k}\}
\]
\[
\leq \sum_{k_1 \leq k \leq n} 2\alpha_1 (k/n) e^{-\nu_{1,*} k}
\]
Thus, by the analysis in Step 1 after (A.26),
\[
E_\theta R \left( 0, \hat{\lambda} \right) I\{\hat{\lambda} \leq \lambda_{1,*}\} \leq M^* (\tau_{1,n}^* \eta_{G_n}^* + \tau_{2,n}^*).
\]  (A.30)

Since \( R(0, \lambda) \) is decreasing in \( \lambda \) and \( \lambda_{1,*} = \sqrt{1 + \delta_{1,n} g_{1/n}(\xi_{1,k_{1,*}})} \),
\[
E_\theta R \left( 0, \hat{\lambda} \right) I\{\hat{\lambda} > \lambda_{1,*}\} \leq R(0, \lambda_{1,*})
\]  (A.31)
\[
\leq \begin{cases} 
R(0, g_{1/n}(\xi_{1,*} \wedge \xi_{1,1})), & \text{Case 1} \\
R(0, \sqrt{1 + \delta_{1,n} g_{1/n}(\xi_{1,k_0})}), & \text{Case 2}
\end{cases}
\]
with the two cases specified in (A.12) and the definition of \( k_{1,*} \) in (A.15). Note that we proved \( \xi_{1,*} \geq \xi_{1,1} \geq \xi_{1,k_0} \) in Case 2 in Step 2.
We first consider Case 1. If \( \eta_{G_n} \geq 1/M^* \), then \( \tau_{1}^{*} \) is not small and \( R(0, g_{1,n}(\xi_{1,*})) \leq 1 \leq M^*\eta_{G_n}^* \leq (M^*)^2 \tau_{1,n}^{*} \eta_{G_n}^* \). Otherwise, \( \eta_{G_n} \leq 1/M^* \), so that the upper bound for \( 2\Phi(-\xi_{j,*}) \) in Lemma 4 (ii), (3.15) and (3.16) imply

\[
2\Phi(-\xi_{1,*}) \leq \frac{\alpha' \rho_{G_n}(1)}{1 - \alpha'1} \leq M^*\eta_{G_n}^*.
\]

Thus, as in the proof of (3.15),

\[
\sqrt{2\log(1/\eta_{G_n}^*)} \leq \left(1 + \frac{\log \log(1/\eta_{G_n}^*) + M^*}{4\log(1/\eta_{G_n}^*)}\right) \xi_{1,*} \leq 2\xi_{1,*}.
\]

The second bound for \( R(0, g_{1,n}(x)) \) in (4.1) gives

\[
R(0, g_{1,n}(\xi_{1,*})) \leq \frac{M_0 \Phi(-\xi_{1,*})}{(\xi_{1,*} + 2)(\log \xi_{1,*})^{c_{2,n}}} \leq \frac{\log(1/\eta_{G_n}^*)^{c_{1,n}/2} \log(1/\eta_{G_n}^*)}{M^*\tau_{1,n}^{*} \eta_{G_n}^*},
\]

in view of the definition of \( \tau_{1,n}^{*} \) in (4.6). Moreover, the first bound for \( R(0, g_{1,n}(x)) \) in (4.1) and (A.13) give

\[
R(0, g_{1,n}(\xi_{1,*})) \leq 4\Phi(-\xi_{1,1}) = \frac{2\alpha_1}{n} \leq M^*(\tau_{1,n}^{*} \eta_{G_n}^* + \tau_{2,n}^{*}).
\]

It follows that \( R(0, g_{1,n}(\xi_{1,*} \wedge \xi_{1,1})) \leq M^*(\tau_{1,n}^{*} \eta_{G_n}^* + \tau_{2,n}^{*}), \) and (A.23) follows from (A.31).

In Case 2, (3.11) of Lemma 1 (i) and (A.27) yield

\[
R(0, g_{1,n}(\xi_{1,1})) \leq \frac{4\Phi(-\lambda_{1,*})}{\lambda_{1,*}^2 + 2} \leq M^*\tau_{2,n}^{*}.
\]

Thus, (A.29) holds in both cases in view of (A.30) and (A.31). The proof of Theorem 5 is completed since we have already proved the oracle inequality based on (A.20), (A.21), (A.22) and (A.23).

**Proof of Theorems 1 and 3 and Corollary 1.** It follows from (3.16) that \( \eta_{G_n}^{*} \leq (1 + M^*\tau_{1,n})\eta_{G_n} \), so that (4.7) follows from Theorem 5. Since \( M_n L_{2,n} / n^{1+\delta_{1,n}} \leq \eta_n \to 0 \), we have \( n \to \infty \) and \( \tau_{1,n} \to 0 \). The adaptive ratio optimality (2.9) then follows from (4.7) with the special \( \tau_{ \lambda_n}(x) = s_{ \lambda_n}(x) \) since the risk range guarantees \( \tau_{2,n}^{*} \ll \eta_{G_n} \) uniformly in the specified class. Theorems 1 and 3 are consequences of Corollary 1 since (4.1) holds and \( L_{2,n} = L_{0,n} \) for \( g_{1,n}(x) = x, c_{1,n} = 2 \) and \( c_{2,n} = 0 \).
Proof of Theorem 4 and Corollary 2 Let $\Omega_{n}^{s,w}$ and $\Omega_{0,n}$ be as in Theorem 4 with $L_{0,n}$ replaced by $L_{2,n}$. It follows from the second part of (2.31), which implies (2.32), that for certain $M_{n} \to \infty$ and $\eta_{n} \to 0,$

$$M_{n}L_{2,n}/n^{\delta_{1,n}} \leq \mathcal{R}(\Theta_{p,C,n}^{s,w}) \leq n\eta_{n}$$

uniformly for all $(p, C) \in \Omega_{n}^{s,w} \cup \Omega_{0,n}$. Thus, Corollary 1 and the first part of (2.31) imply that uniformly for all $(p, C) \in \Omega_{n}^{s,w} \cup \Omega_{0,n},$

$$\sup \left\{ E_{\theta}\|t_{\lambda}(X) - \theta\|^{2} : \theta \in \Theta_{p,c,n}^{s,w} \right\} \leq (1 + o(1))\sup \left\{ n\eta_{G} : \theta \in \Theta_{p,c,n}^{s,w} \right\} \leq (1 + o(1))\mathcal{R}(\Theta_{p,c,n}^{s,w}).$$

This completes the proof of Corollary 2. Theorem 4 is a consequence of Corollary 2 since (4.1) holds for $g_{1,n}(x) = x$, $c_{1,n} = 2$ and $c_{2,n} = 0$ and $L_{2,n} = L_{0,n}$ for those $c_{1,n}$ and $c_{2,n}$. □

The proof of Theorem 2 requires the following lemma.

Lemma 7 For any real numbers $\lambda > b \geq 0$, $\mu$ and $\varepsilon$,

$$(s_{b}(\varepsilon + \mu) - \mu)^{2} - (s_{\lambda}(\varepsilon + \mu) - \mu)^{2} \leq (|\varepsilon| + b)^{2}I\{|\varepsilon| > b\}.$$

Proof. Let $\mu > 0$ without loss of generality due to symmetry. We have

$$\begin{align*}
(\varepsilon - b)^{2} & \quad \text{if } \varepsilon > b, \\
(\varepsilon + b)^{2} & \quad \text{if } \varepsilon + \mu \leq -b, \\
0 & \quad \text{if } |\varepsilon + \mu| \leq b, \\
(\varepsilon - b)^{2} & \quad \text{if } b < \varepsilon + \mu \leq \lambda \text{ and } \varepsilon \leq b, \\
(\varepsilon - b)^{2} - (\varepsilon - \lambda)^{2} & \quad \text{if } \varepsilon + \mu > \lambda \text{ and } \varepsilon \leq b.
\end{align*}$$

The upper bound is no greater than $(|\varepsilon| + b)^{2}I\{|\varepsilon| > b\}$. □

Proof of Theorem 2 Let $a = 0 < b = \sqrt{2}\log n$ and

$$Y_{1} = \arg\min_{\lambda \geq 0} \|s_{\lambda}(X) - \theta\|^{2}, \quad Y_{2} = \min(Y_{1}, b).$$

The risk difference between $s_{Y_{1}}$ and $s_{Y_{2}}$ is controlled by applying Lemma 7

$$\frac{1}{n}E_{\theta}\|s_{Y_{1}}(X) - \theta\|^{2} - \frac{1}{n}E_{\theta}\|s_{Y_{2}}(X) - \theta\|^{2}$$

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\[
\sum_{i=1}^{n} E_\theta \left\{ (|X_i - \theta_i| + b)^2 I\{|X_i - \theta_i| > b\} \right\} 
\leq 2 \int_{b}^{\infty} (t + b)^2 \varphi(t) dt 
\leq M^* \left( \log \frac{n}{n} \right). \tag{A.32}
\]

Since \(0 \leq Y_2 \leq b = \sqrt{2 \log n}\), an application of the concentration inequality (3.35) with \(m = n\) gives that
\[
\sqrt{\frac{E_\theta \| s_{Y_2}(X) - \theta \|}{\sqrt{n} - R_{1,n}(\theta, Y_2)}} \leq 2 \left\{ \frac{2}{n} \left( \log(2en) + 1 \right) \right\}^{1/2} + 2 \left\{ \frac{2 \log n}{n^2} \right\}^{1/2} 
\leq M^* \sqrt{\log(n)/n}. 
\]

This and (A.32) yield
\[
\sqrt{E_\theta R_{2,n}(\theta, Y_2)} \leq M^* \sqrt{\log(n)/n} + \sqrt{E_\theta \| s_{Y_1}(X) - \theta \|^2/n}. \tag{A.33}
\]

It follows from the optimality of \(\lambda_{G_n} = \arg \min_{\lambda} R_{G_n}(\lambda)\) and (3.37) that
\[
R_{G_n}(\lambda_{G_n}) \leq E_\theta R_{G_n}(Y_2) \leq E_\theta R_{1,n}^2(\theta, Y_2) + 4/n.
\]

Consequently,
\[
\sqrt{R_{G_n}(\lambda_{G_n})} \leq \sqrt{4/n + M^* \sqrt{\log(n)/n} + \sqrt{E_\theta \| s_{Y_1}(X) - \theta \|^2/n}} 
= \sqrt{4/n + M^* \sqrt{\log(n)/n} + \sqrt{E_\theta \inf_{\lambda} \| s_{\lambda}(X) - \theta \|^2/n}}.
\]

This and (2.13) yield Theorem 2. \(\square\)

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