Abstract. In this paper, we prove Shelukhin’s conjecture on the translated points on any compact contact manifold \((M, \xi)\) which reads that for any choice of function \(H = H(t, x)\) and contact form \(\lambda\) the contactomorphism \(\psi^1_H\) carries a translated point in the sense of Sandon, whenever
\[
\|H\| \leq 2T(\lambda, M)
\]
This improves the result stated in [Oh21c, Theorem 1.19] by a factor of 2. Main analytical and geometrical tools are the ones employed in the paper arXiv:2205.12351 entitled Geometry and analysis of contact instantons and entanglement of Legendrian links I.

Additional analytical ingredients are the gluing transversality and some dimension counting argument based on the evaluation map transversality which we also establish in the present paper.

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1. Introduction

Let \((M, \xi)\) be a coorientable contact manifold. The notion of translated points was introduced by Sandon [San12] in the course of her applications of Legendrian spectral invariants to the problem of contact nonsqueezing initiated by Eliashberg-Kim-Polterovich [EKP06] and further studied by [Chi17], [Fra16].

We first recall the definition of translated points. For each given coorientation preserving contact diffeomorphism \(\psi\) of \((M, \xi)\), we call the function \(g\) appearing in \(\psi^*\lambda = e^g \lambda\) the conformal exponent for \(\psi\) and denote it by \(g = g_\psi\) [Oh21b].

**Definition 1.1** (Sandon [San12]). Let \((M, \xi)\) be a contact manifold equipped with a contact form \(\lambda\). A point \(x \in M\) is called a \(\lambda\)-translated point of a contactomorphism \(\psi\) if \(x\) satisfies

\[
\begin{cases}
g_\psi(x) = 0 \\
\psi(x) = \phi^\eta_{R, \lambda}(x) \text{ for some } \eta \in \mathbb{R}.
\end{cases}
\]

We denote the set of \(\lambda\)-translated points of \(\psi\) by \(\text{Fix}^\text{trn}_\lambda(\psi)\).

1.1. **Statement of Main results.** We recall the following standard definition \(T(M, R; \lambda)\) in contact geometry.

**Definition 1.2.** Let \(\lambda\) be a contact form of contact manifold \((M, \xi)\) and \(R \subset M\) a connected Legendrian submanifold. Denote by \(\mathcal{R}_{\text{Reeb}}(M, \lambda)\) (resp. \(\mathcal{R}_{\text{Reeb}}(M, R; \lambda)\)) the set of closed Reeb orbits (resp. the set of self Reeb chords of \(R\)).

(1) We define \(\text{Spec}(M, \lambda)\) to be the set

\[
\text{Spec}(M, \lambda) = \left\{ \int_\gamma \lambda \mid \lambda \in \mathcal{R}_{\text{Reeb}}(M, \lambda) \right\}
\]

and call the action spectrum of \((M, \lambda)\).

(2) We define the period gap to be the constant given by

\[
T(M, \lambda) := \inf \left\{ \int_\gamma \lambda \mid \lambda \in \mathcal{R}_{\text{Reeb}}(M, \lambda) \right\} > 0.
\]

We define \(\text{Spec}(M, R; \lambda)\) and the associated \(T(M, R; \lambda)\) similarly using the set \(\mathcal{R}_{\text{Reeb}}(M, R; \lambda)\) of Reeb chords of \(R\).

We set \(T(M, \lambda) = \infty\) (resp. \(T(M, R; \lambda) = \infty\)) if there is no closed Reeb orbit (resp. no \((R_0, R_1)\)-Reeb chord). Then we define

\[
T_\lambda(M; R) := \min\{T(M, \lambda), T(M, R; \lambda)\} \quad (1.1)
\]

and call it the (chord) period gap of \(R\) in \(M\).
The main goal of the present paper is to give the proof of the following conjecture of Shelukhin [She17]. This result is an improvement of [Oh21a, Theorem 1.19] by a factor of 2.

**Theorem 1.3** (Theorem 3.6; Conjecture 31, [She17]). Assume \((M, \lambda)\) is a compact contact manifold. Let \(T(M, \lambda) > 0\) be the minimal period of closed Reeb orbits of \(\lambda\). Then we have the following:

1. Provided \(|H| \leq 2T(M, \lambda)\), we have\[\# \text{Fix}_{\lambda}^{\text{tn}}(\psi) \neq \emptyset\]
2. Provided \(|H| < 2T(M, \lambda)\) and \(H\) is nondegenerate, we have\[\# \text{Fix}_{\lambda}^{\text{tn}}(\psi) \geq \dim H^*(M; \mathbb{Z}_2).\]

The main geometric operation entering in the study of Sandon-Shelukhin’s conjecture employed in [Oh21a] is the general operation what we recall the Legendrian-ization of contactomorphism \(\psi\). Such an operation was used by Chekanov [Che96], Bhupal [Bhu01] and Sandon [San12], and will be again the main geometric operation which converts the problem of translated points into that of Reeb chords. This being said, we prove the following.

**Theorem 1.4** (Theorems 1.16 & 3.6). Let \((M, \xi)\) be a contact manifold equipped with a tame contact form \(\lambda\). Let \(\psi \in \text{Cont}_0(M, \xi)\) and consider any Hamiltonian \(H = H(t, x)\) with \(H \mapsto \psi\). Assume \(R\) is any compact Legendrian submanifold of \((M, \xi)\). Then the following hold:

1. Provided \(|H| \leq 2T_\lambda(M, R)\), we have\[\# \text{Reeb}(\psi(R), R) \neq \emptyset.\]
2. Provided \(|H| < 2T_\lambda(M, R)\) and \(\psi = \psi_1^H\) is nondegenerate to \((M, R)\), then\[\# \text{Reeb}(\psi(R), R) \geq \dim H^*(R; \mathbb{Z}_2).\]

We specialize Theorem 1.3 to the case of the contact manifold \((M_Q, \mathcal{A})\) given by
\[M_Q = Q \times Q \times \mathbb{R}, \quad \mathcal{A} = -e^\eta \pi_1^* \lambda + \pi_2^* \lambda.\]
We have one-to-one correspondence between \(\text{Fix}_{\lambda}^{\text{tn}}(\psi)\) and the set \(\text{Reeb}(\Gamma_\psi, \Gamma_{id})\) of Reeb chords between \(\Gamma_\psi\) and \(\Gamma_{id}\).

**Remark 1.5.** The present paper should be regarded as a continuation of [Oh21a] in that all the main geometro-analytical framework, such as Gromov-Floer-Hofer style convergence result, is established in the paper which will not be repeated in the present paper and which we refer readers to. Frequently we will be very brief when we mention these analytical foundation and refer readers to [Oh21a] for relevant details with the precise locations thereof pointed out. We also use the same notations and terminologies therefrom.

However in addition to those used in [Oh21a], we need to establish one more crucial analytical ingredient, the generic evaluation transversality result and the accompanied intersection theory argument, to prove the complete version of Shelukhin’s conjecture above. Once the analytical foundations are set up, the main argument of the proof is largely geometric. Similar practice of dimension counting argument has been often used in the Gromov-Witten theory and its applications to symplectic topology. See [Oh93, RT95, HS95], to name a few.
A key observation to apply the above generic transversality result mentioned in this remark is the following.

**Corollary 1.6** (Key observation). Any Reeb chord \( \gamma \in \Reeb(M, \lambda; R) \) with action satisfying
\[
T_\lambda(M, R) < A(\gamma) < 2T_\lambda(M, R)
\]
is a primary chord, i.e., is somewhere injective.

Theorem 1.4 and 1.3 can be restated by introducing the following notion of Reeb-untangling energy of one subset from the Reeb trace of the other. We call the following union
\[
Z_S := \bigcup_{t \in \mathbb{R}} \phi_t^R(S)
\]
the Reeb trace of a subset \( S \subset M \).

**Definition 1.7** (Definition 1.28 [Oh21c]). Let \((M, \xi)\) be a contact manifold, and let \( S_0, S_1 \) of compact subsets \((M, \xi)\).

1. We define
\[
e_{\lambda}^{\text{trn}}(S_0, S_1) := \inf_{H} \{ \|H\| \mid \psi_H(S_0) \cap Z_{S_1} = \emptyset \}.
\]
We put \( e_{\lambda}^{\text{trn}}(S_0, S_1) = \infty \) if \( \psi_H(S_0) \cap Z_{S_1} \neq \emptyset \) for all \( H \). We call \( e_{\lambda}^{\text{trn}}(S_0, S_1) \) the \( \lambda \)-untangling energy of \( S_0 \) from \( S_1 \) or just of the pair \((S_0, S_1)\).

2. We put
\[
e^{\text{trn}}(S_0, S_1) = \sup_{\lambda \in \mathcal{C}(\xi)} e_{\lambda}^{\text{trn}}(S_0, S_1).
\]
We call \( e^{\text{trn}}(S_0, S_1) \) the Reeb-untangling energy of \( S_0 \) from \( S_1 \) on \((M, \xi)\).

We mention that the quantity \( e_{\lambda}^{\text{trn}}(S_0, S_1) \) is not symmetric, i.e.,
\[
e_{\lambda}^{\text{trn}}(S_0, S_1) \neq e_{\lambda}^{\text{trn}}(S_1, S_0)
\]
in general. Theorem 1.4 implies
\[
e_{\lambda}^{\text{trn}}(R, R) \geq T_\lambda(M, R) > 0
\]
for all compact Legendrian submanifolds \( R \).

1.2. **Legendrianization of contactomorphisms.**

**Definition 1.8** (\( \Gamma_\psi \)). Let \( \psi : Q \to Q \) be a contactomorphism with its conformal exponent function \( g_\psi \). We define the contact graph
\[
\Gamma_\psi := \{(x, y, \eta) \in Q \times Q \times \mathbb{R} \mid y = \psi(x), \eta = g_\psi(x)\}.
\]

The following corollary provides a link between the intersection set \( \psi(R) \cap Z_R \) and Sandon’s notion of translated point of a contactomorphism \( \psi \) on \( Q \).

**Corollary 1.9** (Corollary 4.8 [Oh21c]). Let \((Q, \xi)\) be a contact manifold equipped with a contact form \( \lambda \). Then

1. \( x \in Q \) is a translated point of a contactomorphism \( \psi \) if and only if \((x, x, 0) \in \Gamma_\psi \cap Z_{\Gamma_{id}} \).
2. \( \psi \) is nondegenerate if and only if \( \Gamma_\psi \cap Z_{\Gamma_{id}} \).

We have also have the following special property of the structure of chord spectrum of the pair \((M_Q, \mathcal{A})\).
Lemma 1.10 (Lemma 2.3 [Oh21c]). Let \((M, \mathcal{A})\) be as above. Then we have
\[ T(Q, \lambda) = T(M_Q, \mathcal{A}; \Gamma_{id}). \]

An immediate corollary of this lemma is the following.

Corollary 1.11. Any Reeb chord \(\gamma \in \Reeb(M, \lambda; \Gamma_{id})\) with action satisfying
\[ T(Q, \lambda) < A(\gamma) < 2T(Q, \lambda) \]
is a primary chord, i.e., is somewhere injective.

We also recall that \(\Gamma_\psi\) is contact isotopic to \(\Gamma_{id}\): In fact, we have
\[ \Gamma_\psi = \psi_1^1 \tilde{H}(\Gamma_{id}) \]
for any lifted Hamiltonian \(\tilde{H}\) of \(H\) generating the isotopy \(\psi^t\) with \(\psi^1 = \psi, \psi^0 = id\).

We derive the following inequality from this.

Proposition 1.12 (Proposition 2.4 [Oh21c]). Let \((Q, \xi)\) and \((M_Q, \mathcal{A})\) be as above. Then
\[ e^{trn}_\mathcal{A}(\Gamma_{id}, \Gamma_{id}) \leq e^{trn}_\lambda(Q, \xi). \]

Corollary 1.13. We have the following implication:
\[ e^{trn}_\mathcal{A}(\Gamma_{id}, \Gamma_{id}) \geq 2T(M_Q, \mathcal{A}; \Gamma_{id}) \implies e^{trn}_\lambda(Q, \xi) \geq 2T(Q, \lambda). \]

We mention that even if \(Q\) is compact, \(M_Q\) is not but it is tame in the sense of [Oh21c, Definition 1.8].

Proposition 1.14 (Proposition 11.7 [Oh21c]). The contact manifold \((M_Q, \mathcal{A})\) is tame in the sense of [Oh21c, Definition 1.8].

This enables us to apply the maximum principle which is a crucial ingredient in the study of the analysis of the moduli space of contact instantons.

1.3. Generic evaluation transversality. A crucial ingredient that enters in the proof of Theorem 1.3 is a general analytical ingredient, the evaluation map transversality, for the moduli space of contact instantons. Such evaluation transversality will be important for the application to contact topology, for example, in the construction of Fukaya-type category of contact instantons generated by Legendrian submanifolds in [Oh10], similarly as in the study of pseudoholomorphic curves in symplectic geometry. A rigorous proof of evaluation transversality is rather subtle even in the pseudoholomorphic curve theory as already mentioned in [Oh11]. A conceptually canonical proof of the evaluation map transversality given in [Oh15, Section 10.5], which in turn followed the scheme provided by Le and Ono [LO96] and Zhu and the author [OZ09] in their studies of one-jet evaluation transversality. This proof is based on a standard structure theorem of the distributions with point support. (See Theorem 5.4 below.) Naturality of the proof in [OZ09, Oh11] enables us to adapt it to the current context of contact instantons. However the proof of the evaluation transversality study for contact instantons is significantly more nontrivial in its details from the case of pseudoholomorphic curves thanks to the different nature of the equation which involves a system of partial differential equations of mixed degree.

We first recall the off-shell setting of the study of linearized operator in Theorem A.12. For given Legendrian link \(\vec{R} = (R_1, \cdots, R_k)\), we consider the moduli space
\[ \mathcal{M}_k((\vec{\Sigma}, \partial \vec{\Sigma}), (M, \vec{R})) \]
of finite energy maps $w : \hat{\Sigma} \to M$ satisfying the equation (4.1) as before.

We will treat the two cases, evaluation at an interior marked point and one at a boundary marked point, separately. We denote by the subindex $(\ell, k)$ the number of interior and boundary marked points respectively. Consider the universal moduli space

$$M_{(1,0)}((\hat{\Sigma}, \partial \hat{\Sigma}, (M, R)) = \{(j, w) \in \mathcal{H}((\hat{\Sigma}, \partial \hat{\Sigma}), (M, R)) | w(\partial \hat{\Sigma}) \subset \vec{R}, z \in \text{Int } \hat{\Sigma}\}.$$  

The evaluation map $ev^+ : M_{(1,0)}((\hat{\Sigma}, \partial \hat{\Sigma}, (M, R)) \to M$ is defined by $ev^+((j, w), z) = w(z)$. We then have the fibration

$$\tilde{M}_{(1,0)}((\hat{\Sigma}, \partial \hat{\Sigma}, (M, \vec{R})) = \bigcup_{J \in \mathcal{J}} \tilde{M}_{(1,0)}((\hat{\Sigma}, \partial \hat{\Sigma}, (M, \vec{R}); J) \to \mathcal{J},$$

and

$$\tilde{M}^{\text{inj}}_{(1,0)}(\hat{\Sigma}, \partial \hat{\Sigma}, (M, \vec{R}))$$

to be the open subset of $\tilde{M}_{(1,0)}((\hat{\Sigma}, \partial \hat{\Sigma}, (M, \vec{R}))$ consisting of somewhere injective contact instantons. We have the universal $(0$-jet$)$ evaluation map

$$Ev^+ : \tilde{M}_{(1,0)}((\hat{\Sigma}, \partial \hat{\Sigma}, (M, \vec{R})) \to M.$$  

We also consider the boundary evaluation map

$$Ev_\partial : \tilde{M}_{(0,1)}((\hat{\Sigma}, \partial \hat{\Sigma}, (M, \vec{R})) \to \vec{R}.$$  

The basic generic transversality is the following.

**Theorem 1.15** (0-jet evaluation transversality). Both evaluation maps $Ev^+$ and $Ev_\partial$ are submersions on the open subset consisting of somewhere injective elements of $M((\hat{\Sigma}, \partial \hat{\Sigma}), (M, \vec{R}))$.

**Part 1. Proof of Shelukhin’s conjecture**

In this part, we give the proof of Theorem 1.3 utilizing the parameterized moduli space defined in [Oh21c, Part 2] and analytical foundation set up therein. In addition, we will use some dimension counting argument based on the generic evaluation transversality result established in Part 2. Once these analytical foundations are set up, the remaining proof is largely geometric whose practice should not be alien to the experts in pseudoholomorphic curve theory and its applications to symplectic topology.

The following is the main result of the present paper

**Theorem 1.16.** Let $(M, \lambda)$ be a tame contact manifold and let $R \subset (M, \lambda)$ be a compact Legendrian submanifold. Consider any smooth function $H = H(t, x)$ with $\|H\| \leq 2T_\lambda(M, R)$. Then we have

$$\Reeb(\psi(R), R) \neq \emptyset.$$  

(1.6)

The rest of this part will be occupied by the proof of this theorem leaving the proof of evaluation transversality to Part 2.
2. Setting up a pointed parameterized moduli space

In [Oh21c, Section 8.1], we take the family $H = H(s, t, x)$ given by

$$H^s(t, x) = sH(t, x)$$

(2.1)

and consider its contact Hamiltonian isotopies $\Psi_{s, t} := \psi_s^t H$. Obviously we have the $t$-developing Hamiltonian $\text{Dev}_\lambda(t \mapsto \Psi(s, t)) = H^s$. We then consider the elongated two parameter family

$$H_K(\tau, t, x) = \chi_K(\tau)H(t, x)$$

and write the $\tau$-developing Hamiltonian

$$G_K(\tau, t, x) = \text{Dev}_\lambda(\tau \mapsto \Psi^\tau_K t, x)$$

where $\Psi^\tau_K t, x = \Psi^{\chi_K(\tau)} t, x$. Then we consider the 2-parameter perturbed contact instanton equation given by

$$\begin{cases}
(du - X_{H_K(u)} dt + X_{G_K(u)} ds)\pi(0, 1) = 0, \\
d\left(e^{g_K(u)}(u^* \lambda + u^* H_K dt - u^* G_K d\tau) \circ j\right) = 0, \\
u(\tau, 0) \in R, u(\tau, 1) \in R. 
\end{cases}$$

(2.2)

where $g_K(u)$ is the function on $\Theta_{K_0+1}$ defined by

$$g_K(u)(\tau, t) := g_{\psi^K_{t}(\psi_{\lambda K}^{-1} u(\tau, t))}$$

(2.3)

for $0 \leq K \leq K_0$. We note that if $|\tau| \geq K + 1$, the equation becomes

$$\overline{\nabla} u = 0, \quad d(u^* \lambda \circ j) = 0.$$  

(2.4)

We define the parameterized moduli space

$$\mathcal{M}_{\text{para}}^{[0, K_0]}(M, R; J, H) = \bigcup_{K \in [0, K_0]} \{K\} \times \mathcal{M}_K(M, R; J, H)$$

whose elements are defined on the domain

$$\Theta_{K_0+1} := \Theta_- \cup \Theta_{K_0+1}^\times \cup \Theta_{K_0+1}^\times \subset \mathbb{C}$$

(2.5)

and equip it with the natural complex structure induced from $\mathbb{C}$. (See Equation (8.5) and (8.6) [Oh21c].) We can also decompose $\Theta_{K_0+1}$ into the union

$$\Theta_{K_0+1} := D^- \cup [-2K_0 - 1, 2K_0 + 1] \cup D^+$$

(2.6)

where we denote

$$D^\pm = D^\pm_{K_0} := \{z \in \mathbb{C} \mid |z| \leq 1, \pm \text{Im}(z) \leq 0\} \pm (2K_0 + 1)$$

(2.7)

respectively.

The following a priori $\pi$-energy identity is a key ingredient in relation to the lower bound of the Reeb-untangling energy. We recall the definition of oscillation

$$\text{osc}(H_t) = \max H_t - \min H_t.$$  

Proposition 2.1 (Theorem 1.26 [Oh21c]). Let $u$ be any finite energy solution of (2.2). Then we have

$$E_{(J_K, H)}^\pi(u) \leq \int_0^1 \text{osc}(H_t) dt =: \|H\|$$

(2.8)

and

$$E_{(J_K, H)}^\perp(u) \leq \|H\|$$

(2.9)
for all $K \geq 0$.

Let

$$E_{J_{K_{w}},H}(u) = E_{J_{K_{w}},H}^{+}(u) + E_{J_{K_{w}},H}^{-}(u)$$

be the total energy. With the $\pi$-energy bound (2.8) and the $\lambda$-energy bound (2.9), we are now ready to make a deformation-cobordism analysis of $\mathcal{M}_{\nu\nu+1}^{0\nu}(M, \lambda; R, H)$.

We consider the $n+1$ dimensional component of the parameterized moduli space

$$\mathcal{M}_{\nu\nu+1}^{\text{para}}(M, R; J, H) = \bigcup_{K \in [0, \nu]} \{K\} \times \mathcal{M}_{K}(M, R; J, H)$$

continued from $\mathcal{M}_{0}(M, R; J, H) \cong R$.

Now we consider a pointed parameterized moduli space

$$\mathcal{M}_{(0,1);[0,\nu]}^{\text{para}}(M, R; J, H) := \bigcup_{K \in [0, \nu]} \mathcal{M}_{1;K}(M, R; J, H).$$

More explicitly, we have

$$\mathcal{M}_{(0,1);K}(M, R; J, H) = \{(u, z) \mid u \in \mathcal{M}_{K}(M, R; J, H), z \in \partial \Theta_{K}\}.$$ 

This is an $n+1$ dimensional smooth manifold. (See Theorem 3.4 and 3.3 for the relevant transversality results.)

We have the natural evaluation map

$$\text{Ev}_{\theta} : \mathcal{M}_{(0,1);K}(M, R; J, H) \to R \times \mathbb{R}_{+} \times \partial \Theta_{K}; \quad (u, K, z) \mapsto (u(z), K, z).$$ (2.10)

We also consider the gauge transformation of $u$

$$\overline{\pi}_{K}(\tau, t) := \psi_{\chi_{K}(\tau)H}(\psi_{\chi_{K}(\tau)H}^{-1}(u(\tau, t)))$$ (2.11)

and make the choice of the family

**Choice 2.2** (Choice 9.6 [Oh21c]). We consider the following two parameter families of $J'$ and $\lambda$:

$$J'_{(s, t)} = ((\psi_{H'}^{-1}(\psi_{H'}^{-1})^{-1})^{s}J = (\psi_{H'}^{-1}(\psi_{H'}^{-1})^{-1})^{s}J, \quad (2.12)$$

$$\lambda'_{(s, t)} = ((\psi_{H'}^{-1}(\psi_{H'}^{-1})^{-1})^{s}\lambda = (\psi_{H'}^{-1}(\phi_{H'}^{-1})^{-1})^{s}\lambda. \quad (2.13)$$

A straightforward standard calculation also gives rise to the following.

**Lemma 2.3** (Lemma 6.7 [Oh21c]). For given $J$, consider $J'$ defined as above. We equip $(\Sigma, j)$ a Kähler metric $h$. Let $g_{H}(u)$ be the function defined in (2.8). Suppose $u$ satisfies (2.2) with respect to $J$. Then $\overline{\pi}$ satisfies

$$\begin{cases}
\overline{\pi}(\tau, 0) \in \psi_{H}(R_{0}), \overline{\pi}(\tau, 1) \in R_{1} \\
\overline{\pi}(\tau, 0) = 0, d(\overline{\pi} \circ j) = 0
\end{cases}$$ (2.14)

for $J$. The converse also holds. And $J' = J'(\tau, t)$ satisfies $J'(\tau, t) = J_{0}$ for $|\tau|$ sufficiently large.

Then $\overline{\pi}_{K}$ also satisfies the energy identity

$$E_{J_{K},H}(u) = E_{J'}^{+}(\overline{\pi}), \quad E_{J_{K},H}^{-}(u) := E_{J'}^{-}(\overline{\pi}).$$

(See [Oh21c] Proposition 7.7.)
**Proposition 2.4** (Proposition 9.7 [Oh21c]). Let \((K, u, u_\alpha)\) be a bubbling-off sequence with
\[
u_\alpha \in \mathcal{M}_{K,\alpha}(J, H)
\]
with
\[
u_\alpha \to (u, v, w)
\]
in the sense of Theorem 1.27 [Oh21c]. Then any bubble must have positive asymptotic action less than \(\|H\|\).

**Proposition 2.5.** Let \(u\) be any finite energy solution of (2.2). Then we have
\[
E(\nu_K) \leq \|H\|.
\]

3. **Proof of main theorem, Theorem 1.16**

We will focus on the general case without nondegeneracy hypothesis because the proof of nondegeneracy case is not much different from that of [Oh21c, Part 2].

We first state the following which is equivalent to [Oh21c, Corollary 9.4].

**Lemma 3.1** (Compare with Corollary 9.4 [Oh21c]). The evaluation map
\[
ev_0 : \mathcal{M}(0,1)_0(M, R; J_0, H) \to R
\]
is a diffeomorphism. In particular its \(Z_2\)-degree is nonzero.

Suppose to the contrary that there is a Hamiltonian \(H\) with
\[
\|H\| < 2T_\lambda(M, R)
\]
such that \(\psi(R) \cap Z_R = \emptyset\) which is equivalent to
\[
2\text{Reeb}(\psi(R), R) = \emptyset.
\]
We choose smooth embedded paths \(\Gamma : [0,1] \to R \times \mathbb{R}_+ \times (\mathbb{R} \times \{0, 1\})\) with
\[
\Gamma(s) = (\gamma(s), K(s), z(s))
\]
such that
\[
K(0) = 0, \quad K_0 \leq K(1) \leq 2K_0
\]
where \(K_0 > 0\) is the constant given in the following proposition.

**Proposition 3.2** (Proposition 8.9 [Oh21c]). Let \(H_t\) be the Hamiltonian such that
\[
\psi_1^J(R) \cap Z_R = \emptyset \quad \text{and} \quad H = sH_t \quad \text{and} \quad J \text{ as before. Suppose } \|H\| < T_\lambda(M, R).
\]
Then there exists \(K_0 > 0\) sufficiently large such that \(\mathcal{M}_K(M, R; J, H)\) is empty for all \(K \geq K_0\).

Choosing a generic \(\Gamma\), we can make the map (2.10) transverse to the path \(\Gamma\) so that
\[
N_\Gamma := \text{Ev}^{-1}(\Gamma)
\]
becomes a one dimensional manifold with its boundary consisting of
\[
\mathcal{M}_{K(0)}(M, R; J, H) \times \{z(0)\} \coprod \mathcal{M}_{K(1)}(M, R; J, H) \times \{z(1)\}.
\]
We fix a generic point \(x \in \psi(R)\) and choose \(\Gamma\) whose image is as small as we want near the given point \(x\).

There exists some \(K_0 > 0\) such that
\[
\mathcal{M}_K(M, R; J, H) = \emptyset
\]
for all $K \geq K_0$ by Proposition 3.2. Obviously this implies that the same holds for the pointed moduli space 
\[ \mathcal{M}_{(0,1):K}(M, R; J, H) = \emptyset \]
Choose any sequence $K_\alpha \to \infty$ as $\alpha \to \infty$ and $u_\alpha \in \mathcal{M}_{K_\alpha}(M, R; J, H)$. Select a sequence still denoted by $K_\alpha$ with $K_\alpha \to \infty$ and elements 
\[ u_\alpha \in \mathcal{M}_{K_\alpha}(M, R; J, H). \]
By the energy estimate given in (2.8) (and $E^v(u_\alpha) \leq \|H\| < \infty$), we have
\[ E^\pi(u_\alpha) \leq \|H\| \]
for all $\alpha \to \infty$. And we also have the uniform bound of the vertical energy from (2.9).

Under the assumption (3.2), the above boundary is a single point, i.e, $(u_0, z(0))$ where $u_0 \equiv \gamma(0)$ is the constant map. Therefore $N_\Gamma$ cannot be compact. The only source of non-compactness of $N_\Gamma$ lies in the bubbling off of either a spherical contact instanton of $J_{(K_1, z_1)}$ for some $z_1 \in \Theta_{K_1}$, or a disc-type open contact instanton with boundary on $R$.

The compactness theorem, [Oh21c, Theorem 8.8] implies that there exists a sequence $\{s_i, u_i\}$ with $s_i \to s_0$ and $0 < s_0 < 1$ such that $u_i \in \mathcal{M}_K(M, R; J, H)$ weakly converges to the cusp curve
\[ u_\infty = u_0 + \sum w_k + \sum v_\ell \]
where $u_0$ is an element in $\mathcal{M}_K(M, R; J, H)$ for $K = K(s_0)$, and $w_k$’s and $v_\ell$’s are spherical and a disc-type contact instantons respectively. Here we note that $s_0$ cannot be either 0 or 1, because the corresponding moduli spaces are Fredholm regular. This is because for $s = 1$, $\mathcal{M}_{K(1)}(M, R; J, H) = \emptyset$ by the choice of $\Gamma$ and for $s = 0$, $\mathcal{M}_{K(0)}(M, R; J, H)$ is regular.

We analyze the bubblings more closely. By definition of $N_\Gamma$, we have
\[ \Gamma(s) \in \text{Ev}_\partial(\mathcal{M}_{(0,1):K}(M, R; J, H)). \]
If the contact instantons attached to $N_\Gamma$ bubbles off, then the point $\Gamma(s_0)$ is contained in the set
\[ \text{Ev}_\partial \left( \partial \mathcal{M}_{(0,1):K} \right) \cap \Gamma = \emptyset. \]

The following proposition rules out that this will not occur for a generic choice of $(J, H, \Gamma)$.

**Proposition 3.3.** Assume that $H$ is a nondegenerate Hamiltonian. Then for a generic choice of $J$ and the embedded path $\Gamma$,
\[ \text{Ev}_\partial \left( \partial \mathcal{M}_{(0,1):[0,K]}(M, R; J, H) \right) \cap \Gamma = \emptyset. \]

**Proof.** This is essentially a dimension counting argument based on the following generic transversality results from [Oh22] and Theorem 3.5 below: We have only to note
\[ \dim \partial \mathcal{M}_{1:[0,K]}(M, R; J, H) = \dim R, \quad \dim \Gamma = 1, \quad \dim(R \times \mathbb{R}_+ \times \partial \Theta_K) = \dim R + 2 \]
and hence
\[ \dim \partial \mathcal{M}_{1:[0,K]}(M, R; J, H) + \dim \Gamma < \dim(R \times \mathbb{R}_+ \times \partial \Theta_K). \]
Theorem 3.4 (Generic mapping transversality \[\text{\cite{Oh22}}\]). Let \((T_\pm, \gamma_\pm)\) and \((T', \gamma')\) be nondegenerate Reeb chords for the pair \((\psi(R), R)\). Then the moduli spaces

\[
\mathcal{M}(\psi(R), R; \gamma_-, \gamma'), \quad \mathcal{M}(\psi(R), R; \gamma', \gamma_+)
\]

are transversal. Similar generic transversality holds for the moduli space of bubbles

\[
\mathcal{M}(M; \gamma), \quad \mathcal{M}(M, R; \gamma).
\]

Proof. This is proved in \[\text{\cite{Oh22} Theorem 4.4}\]. (See also Appendix B of the present paper.)

The following evaluation transversality is established in Part 2.

Theorem 3.5 (Generic evaluation transversality). Let

\[
\mathcal{M}^{\text{simp}}_{(0,1)}(M, R; J, H) \subset \mathcal{M}^{\text{para}}_{(0,1)}(M, R; J, H)
\]

be the set of somewhere injective (perturbed) contact instantons. Then for a generic choice of \(J\) and \(\Gamma\), the evaluation map

\[
\text{Ev}_\Theta : \mathcal{M}^{\text{para}}_{(0,1);[0,K]}(M, R; J, H) \rightarrow R \times \mathbb{R}_+ \times \partial \Theta_K; \quad (u, K, z) \mapsto (u(z), K, z).
\]

is transversal to \(\Gamma\).

Once these two generic transversality results are proved, the rest of the argument is by a dimension counting argument based on some intersection theory of the moduli spaces, the explanation of which is now in order.

Firstly, by the energy estimate Proposition 2.1, any asymptotic Reeb orbit of finite energy bubble contact instantons satisfy the bound

\[
\int \gamma^* \lambda \leq \|H\|.
\]

Secondly, we note that by the hypothesis

\[
\|H\| < 2T(M, \lambda),
\]

the spherical bubble cannot have multiple Reeb orbit as its asymptotic limit at the interior puncture: Otherwise, there will be another closed Reeb orbit \(\gamma_0\) such that

\[
2 \int \gamma_0^* \lambda \leq \|H\| < 2T_M(M) \leq 2T(M, \lambda)
\]

which would imply

\[
\int \gamma_0^* \lambda < T(M, \lambda; R)
\]

a contradiction by definition of \(T(M, \lambda; R)\). This proves that sphere bubble cannot occur. By the same reason, any disc bubble also has its asymptotic chord that must be somewhere injective. Otherwise it will again violate \(\|H\| < 2T(M, \lambda; R)\).

Then it in turn implies that the associated instanton is somewhere injective near the asymptotic chords and so we can apply the above mapping and evaluation transversality results.

Combining the above, we prove that for a generic choice of \(J\) no bubble whatsoever can develop in the moduli space

\[
\mathcal{M}^{\text{para}}_{(0,1);[0,K]}(M, R; J, H)
\]
as $K \to \infty$. This gives rise to a solution for the equation
\[
\overline{\partial}_u^\infty = 0, \quad u(\tau,0) \in \psi(R), \; u(\tau,1) \in R
\]
of $\mathbb{R} \times [0,1]$ with energy bound $E^\infty(u), \; E^\infty \leq \|H\|$. Therefore its asymptotic limit $\gamma_\pm(t) := u(\pm \infty, t)$ become Reeb chords which violates the standing hypothesis (3.2). This completes the proof of Theorem 1.16 for the case without the nondegenerate hypothesis.

For the nondegenerate case, the same proof as in [Oh21c, Section 10.2] with the similar bubbling analysis as above implies that under the hypothesis
\[
\|H\| < 2T \lambda(M,R).
\]
we have
\[
HI_\lambda^*(\psi(R), R) \cong HI_\lambda^*(R, R).
\]
Furthermore, we also have $HI_\lambda^*(R, R) \cong H^*(R)$. This then proves the following.

**Theorem 3.6.** Suppose $\psi$ is nondegenerate and let $H \mapsto \psi$ with $\|H\| < 2T \lambda(M,R)$. Then
\[
\#(\Reeb(\psi(R), R)) \geq \dim H_*(R; \mathbb{Z}_2).
\]

**Part 2. Generic evaluation transversality**

In this part we give an important general ingredient of the applications of contact instantons to contact topology, the evaluation map transversality, similarly as in the case of pseudoholomorphic curves. However the analytic framework of the evaluation transversality study is significantly different from the case of the latter due to the different nature of the contact instanton equation.

As already mentioned in [Oh11], a rigorous proof of this evaluation transversality is rather subtle. We imitate the conceptually canonical proof of the evaluation map transversality given in [Oh15, Section 10.5]. This proof is based on a standard structure theorem of the distributions with point support. (See Theorem 5.1.)

4. THE EVALUATION TRANSVERSALITY: STATEMENT

We first recall the off-shell setting of the study of linearized operator in Theorem A.12: Let $(M, \xi)$ be a contact manifold and consider contact triads $(M, \lambda, J)$ and let $\vec{R} = (R_1, R_2, \ldots, R_k)$ be a Legendrian link. We consider the associated contact instanton equation
\[
\begin{cases}
\overline{\partial}_u^\infty w = 0, & d(w^* \lambda \circ j) = 0 \\
w(\Sigma_i, \Sigma_{i+1}) \subset R_i, & i = 1, \ldots, k
\end{cases}
\]
for a map $w : (\Sigma, \partial \Sigma) \to (M, \vec{R})$ with the boundary condition given as above.

We consider the moduli space
\[
\mathcal{M}_k((\Sigma, \partial \Sigma), (M, \vec{R})), \quad \vec{R} = (R_1, \cdots, R_k)
\]
of finite energy maps $w : \Sigma \to M$ satisfying the equation (4.1) while it is more general than what we use in Part 1. (In the proof, the presence of $H$ does not play much role and so omitted for the clarity and simplicity of the exposition.)

We also consider the space given in A.13
\[
\mathcal{F}(\Sigma, \partial \Sigma), (M, \vec{R}); (\gamma, \overline{\gamma})
\]
consisting of smooth maps satisfying the boundary condition (A.2) and the asymptotic condition (A.3).

This being said, since the asymptotic conditions will be fixed, we simply write the space (A.1) as
\[ F((\dot{\Sigma}, \partial\dot{\Sigma}), (M, \bar{R})) \]
and the corresponding moduli space as
\[ \mathcal{M}((\dot{\Sigma}, \partial\dot{\Sigma}), (M, \bar{R})). \]

We again consider the covariant linearized operator
\[ D\Upsilon(w) : \Omega^0(w^*TM, (\partial w)^*T\bar{R}) \to \Omega^{(0,1)}(w^*\xi) \oplus \Omega^2(\Sigma) \]
of the section
\[ \Upsilon : w \mapsto (\partial\pi w, d(w^*\lambda \circ j)), \quad \Upsilon := (\Upsilon^1, \Upsilon^2) \]
as before. For the simplicity of notation, we also introduce the following notation.

**Notation 4.1** (Codomain of \( \Upsilon \)).
\[ CD_{(J,(j,w))} := \Omega^{(0,1)}_w(w^*\xi) \oplus \Omega^2(\Sigma) = H^{(0,1)}_w \oplus \Omega^2(\Sigma) \quad (4.2) \]
and
\[ CD = \bigcup_{(J,(j,w)) \in \mathcal{F}} \{w\} \times CD_{(J,(j,w))}. \]
(Here \( CD \) stands for ‘codomain’.)

We will treat the two cases, evaluation at an interior marked point and one at a boundary marked point, separately. We denote by the subindex \((\ell, k)\) the number of interior and boundary marked points respectively.

Consider the universal moduli space
\[ \mathcal{M}_{(1,0)}((\dot{\Sigma}, \partial\dot{\Sigma}), (M, \bar{R})) = \{((j, w), J, z) \mid w : \Sigma \to M, \Upsilon(J, (j, w)) = 0, \ w(\partial\dot{\Sigma}) \subset \bar{R}, \ z \in \text{Int} \dot{\Sigma}\}. \]
The evaluation map \( \text{ev}^+ : \mathcal{M}_{(1,0)}((\dot{\Sigma}, \partial\dot{\Sigma}), (M, \bar{R})) \to M \) is defined by
\[ \text{ev}^+((j, w), z) = w(z). \]
We then have the fibration
\[ \widetilde{\mathcal{M}}_{(1,0)}((\dot{\Sigma}, \partial\dot{\Sigma}), (M, \bar{R})) = \bigcup_{J \in \mathcal{J}_\lambda} \widetilde{\mathcal{M}}_{(1,0)}((\dot{\Sigma}, \partial\dot{\Sigma}), (M, \bar{R}); J) \to \mathcal{J}_\lambda \]
and
\[ \widetilde{\mathcal{M}}_{(1,0)}^{\text{inv}}((\dot{\Sigma}, \partial\dot{\Sigma}), (M, \bar{R})) \]
to be the open subset of \( \widetilde{\mathcal{M}}_{(1,0)}((\dot{\Sigma}, \partial\dot{\Sigma}), (M, \bar{R})) \) consisting of somewhere injective contact instantons. We have the universal (0-jet) evaluation map
\[ \text{Ev}^+ : \widetilde{\mathcal{M}}_{(1,0)}((\dot{\Sigma}, \partial\dot{\Sigma}), (M, \bar{R})) \to M. \]
The basic generic transversality is the following.

**Theorem 4.2** (0-jet evaluation transversality). The evaluation map
\[ \text{Ev}^+ : \widetilde{\mathcal{M}}_{(1,0)}((\dot{\Sigma}, \partial\dot{\Sigma}), (M, \bar{R})) \to M \]
is a submersion. The same holds for the boundary evaluation map
\[ \text{Ev}_\partial : \widetilde{\mathcal{M}}_{(0,1)}((\dot{\Sigma}, \partial\dot{\Sigma}),(M, \bar{R})) \to \bar{R}. \]
5. THE INTERIOR EVALUATION TRANSVERSALITY: PROOF

We closely follow the scheme exercised for the proof of evaluation transversality given in [Oh15, Section 10.5] which in turn follows the scheme of the generic 1-jet transversality results proved in [OZ09], [Oh11] for the case of pseudoholomorphic curves in symplectic geometry.

An important ingredient in their proofs is the following the structure theorem of the distributions with point support from [GS68, Section 4.5], [Rud73, Theorem 6.25] whose proof we refer readers thereto.

**Theorem 5.1** (Distribution with point support). Suppose \( \psi \) is a distribution on open subset \( \Omega \subset \mathbb{R}^n \) with \( \text{supp} \psi \subset \{ p \} \) and of finite order \( N < \infty \). Then \( \psi \) has the form
\[
\psi = \sum_{|\alpha| \leq N} D^\alpha \delta_p
\]
where \( \delta_p \) is the Dirac-delta function at \( p \) and \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is the multi-indices.

We start with the case of interior marked point and consider the map
\[
\aleph_0 : J_\lambda \times M_{\Sigma} \times \tilde{F}_{(1,0)}(\Sigma, M) \to CD \times M
\]
\[
(J, (j, w), z_0) \mapsto (\Upsilon(J, (j, w)), w(z_0)).
\]
(5.1)

Here the subindex 0 in \( \aleph_0 \) stands for the ‘0-jet’.

We denote by \( \pi_i \) the projection from \( J_\lambda \times \tilde{F}_{(1,0)}(\Sigma, M) \) to the \( i \)-th factor with \( i = 1, 2 \). Then we introduce
\[
\tilde{M}_{(1,0)}(\Sigma, M; \{ p \}; J) = \aleph_0^{-1}(o_{CD} \times \{ p \}); J)
\]
\[
\tilde{M}_{(1,0)}(\Sigma, M; \{ p \}; J) = \aleph_0^{-1}(o_{CD} \times \{ p \}); J).
\]

The following is a fundamental proposition for the proof of Theorem 4.2 as in the standard strategy exercised in the similar transversality result for the study of pseudoholomorphic curves in [Oh15, Section 10.5] which in turn follows the scheme used in [OZ09] for the 1-jet transversality proof for the case of pseudoholomorphic curves. We apply the same scheme with the replacement of pseudoholomorphic curves by contact instantons first for the 0-jet case in this part and for the 1-jet case in the next part. Because the nature of equation is different, especially because the contact instanton equation involves the second derivatives, the proof involves additional complication beyond that of [OZ09].

**Proposition 5.2.** The map \( \aleph_0 \) is transverse to the submanifold
\[
o_{CD} \times \{ p \} \subset CD \times M.
\]

**Proof.** Its linearization \( D\aleph_0(J, (j, w), z) \) is given by the map
\[
(L, (b, Y), v) \mapsto (D_{J, (j, w)}(L, (b, Y)), Y(w(z)) + dw(z)(v))
\]
for
\[
L \in T_J J_\lambda, \ b \in T_j M_{\Sigma}, \ v \in T_{z_0} \Sigma, \ Y \in T_w F(\Sigma, M; \beta).
\]

This defines a linear map
\[
T_J J_\lambda \times T_j M_{\Sigma} \times T_w F(\Sigma, M; \beta) \times T_{z_0} \Sigma \to CD_{(J, (j, w))} \times T_{w(z)} M
\]
on \( W^{1,p} \). But for the map \( \aleph_0 \) to be differentiable, we need to choose the completion \( W^{k,p}(\Sigma, M) \) of \( F(\Sigma, M) \) with \( k \geq 2 \).
We take the Sobolev completion in the $W^{k,p}$-norm for at least $k \geq 2$. We take $k = 2$. We would like to prove that this linear map is a submersion at every element $(J, j, w, z_0) \in \tilde{M}_1(\tilde{\Sigma}, M)$ i.e., at the pair $(w, z_0)$ satisfying

$$\Upsilon(J, (j, w)) = 0, \quad w(z_0) = p.$$ 

For this purpose, we need to study solvability of the system of equations

$$D_{J, (j, w)} \Upsilon(L, (b, Y), v) = (\gamma, \omega), \quad Y(w(z_0)) + dw(v) = X_0$$

for any given $(\gamma, \omega) \in CD_w$ and $X_0$, i.e.,

$$\gamma \in \Omega^{p(0,1)}_J(w^*TM), \quad \omega \in \Omega^2(\tilde{\Sigma}), \quad X_0 \in T_{w(z_0)}M.$$ 

For the current study of evaluation transversality, the domain complex structure $j$ does not play much role in our study. Especially it does not play any role throughout our calculations except that it appears as a parameter. Therefore we will fix $j$ throughout the proof. Then it will be enough to consider the case $b = 0$. Then the above equation is reduced to

$$D_{J, w} \Upsilon(L, Y) = (\gamma, \omega), \quad Y(w(z_0)) + dw(v) = X_0.$$ 

Firstly, we study (5.4) for $Y \in W^{2,p}$. We regard

$$CD_{J, (j, w)} \times T_{w(z_0)}M$$

as a Banach space with the norm $\| \cdot \|_1 + \| \cdot \|_p + | \cdot |$, where $| \cdot |$ is any norm induced by an inner product on $T_xM$.

We will show that the image of the map (7.2) restricted to the elements of the form

$$(L, (0, Y), v)$$

is onto as a map

$$T_{J, J} \cup \Omega^{0,1}_p(w^*TM) \to CD_{J, w}^{1,p} \times T_{w(z_0)}M$$

where $(w, j, z_0, J)$ lies in $\Upsilon^{-1}_0(o_{H''} \times \xi)$, and we set

$$CD_{J, w}^{1,p} := \Omega^{p(0,1)}_1(w^*\xi) \times \Omega^2_p(\tilde{\Sigma}).$$

For the clarification of notations, we denote the natural pairing

$$\mathcal{B} \times \mathcal{B}^* \to \mathbb{R}$$

by $\langle \cdot, \cdot \rangle$ for any Banach space $\mathcal{B}$ and the inner product on $T_xM$ by $(\cdot, \cdot)_x$.

We now prove the following which will then finish the proof by the ellipticity of the linearization map. The remaining part of proof will be occupied by the proof of this statement.

**Proposition 5.3.** The subspace

$$\text{Image } K_0 \subset CD_0 \oplus T_{w(z_0)}R$$

is dense.

**Proof.** For the proof, we will use the Hahn-Banach lemma in an essential way. Let $((\eta, f), X_p) \in CD^* \times T_pM$ satisfy

$$\left\langle D_w \Upsilon(J, (j, w)) Y + \left( \frac{1}{2} L \cdot d^*w \circ j, 0 \right), (\eta, f) \right\rangle + \langle Y, \delta_{w(z_0)} X_p \rangle = 0$$

(5.6)
for all $Y \in \Omega^d_{2,p}(w^*TM)$ and $L$ where $\delta_{z_0}$ is the Dirac-delta function supported at $z_0$. By the Hahn-Banach lemma, it will be enough to prove

$$ (\eta, f) = 0 = X_p. \quad (5.7) $$

In the derivation of (5.6), we have used the formula

$$ \Upsilon(J, (j, w)) = (\partial^\pi_J w, d(w^* \circ \lambda \circ j)) $$

to compute the linearization of $N_0$. In particular, we have

$$ D_1N_0(L) = \left( \frac{1}{2} L \cdot d^\pi w \circ j, 0 \right) $$

in the direction of $J$ where the second factor of the value of $\Upsilon$ does not depend on $J$. Obviously, we have

$$ D_2N_0(Y) = D_\Upsilon_{J,j}(Y). $$

Under this assumption, we would like to show (5.7). Without loss of any generality, we may assume that $Y$ is smooth since $C^\infty(w^*TM) \hookrightarrow \Omega^0_{2,p}(w^*TM)$ is dense.

Taking $L = 0$ in (7.6), we obtain

$$ \langle D_w \Upsilon(J, (j, w))Y, (\eta, f) \rangle + \langle Y, \delta_{z_0}X_p \rangle = 0 \quad \text{for all } Y \text{ of } C^\infty. \quad (5.8) $$

Therefore by definition of the distribution derivatives, $\eta$ satisfies

$$ (D_w \Upsilon(J, (j, w)))^\dagger(\eta, f) - \delta_{z_0}X_p = 0 $$

as a distribution, i.e.,

$$ (D_w \Upsilon(J, (j, w)))^\dagger(\eta, f) = \delta_{z_0}X_p $$

where $(D_w \Upsilon(J, (j, w)))^\dagger$ is the formal adjoint of $D_w \Upsilon(J, (j, w))$ whose symbol is of the same type as $D_w \Upsilon(J, j)$ and so is an elliptic first order differential operator. (See A.1.7 for the linearization formula and recall that $(\partial^\pi_J)^\dagger = -\partial^\pi_j$ modulo zero order operators.) By the elliptic regularity, $(\eta, f)$ is a classical solution on $\Sigma \setminus \{z_0\}$. On the other hand, by setting $Y = 0$ in (7.6), we get

$$ \langle L \cdot dw \circ j, (\eta, f) \rangle = 0 \quad (5.9) $$

for all $L \in T_JJ\lambda$. From this identity, the argument used in the transversality proven in the previous section shows that $\eta = 0$ in a small neighborhood of any somewhere injective point in $\Sigma \setminus \{z_0\}$. Such a somewhere injective point exists by the hypothesis of $w$ being somewhere injective and the fact that the set of somewhere injective points is open and dense in the domain under the given hypothesis. Then by the unique continuation theorem, we conclude that $\eta = 0$ on $\Sigma \setminus \{z_0\}$ and so the support of $\eta$ as a distribution on $\Sigma$ is contained at the one-point subset $\{z_0\}$ of $\Sigma$.

The following lemma will conclude the proof of Proposition 5.3. We postpone the proof of the lemma till the next section.

**Lemma 5.4.** $(\eta, f)$ is a distributional solution of $(D_w \Upsilon(J, (j, w)))^\dagger(\eta, f) = 0$ on $\Sigma$ and so continuous. In particular, we have $(\eta, f) = 0$ in $(CD)^*$. Once we know $(\eta, f) = 0$, the equation (7.6) is reduced to the finite dimensional equation

$$ (Y(z_0), X_p)_{z_0} = 0 \quad (5.10) $$

It remains to show that $X_p = 0$. For this, we have only to show that the image of the evaluation map

$$ Y \mapsto Y(z_0) $$

is surjective onto $T_pM$, which is now obvious.
Now it remains to prove Lemma 5.2.

6. Proof of Lemma 5.2

Our primary goal is to prove
\[ \langle D_w Y(J, (j, w)) Y, (\eta, f) \rangle = 0 \] (6.1)
for all smooth \( Y \in \Omega^0(w^*TM) \), i.e., \( \eta \) is a distributional solution of
\[ (D_w Y(J, (j, w)))^\dagger(\eta, f) = 0 \]
on the whole \( \Sigma \), not just on \( \Sigma \setminus \{z_0\} \). This will imply that \((\eta, f)\) is a solution smooth everywhere by the elliptic regularity.

We start with (5.8)
\[ \langle D_w Y(J, (j, w)) Y, (\eta, f) \rangle + \langle Y, \delta_{z_0} X_p \rangle = 0 \] for all \( Y \in \mathcal{C}^\infty \).

We first simplify the expression of the pairing \( \langle D_w Y(J, (j, w)) Y, (\eta, f) \rangle \) knowing that \( \text{supp}(\eta, f) \subset \{z_0\} \).

Let \( z \) be a complex coordinate centered at a fixed marked point \( z_0 \) and \((x_1, y_1, x_2, y_2, \cdots, x_n, y_n, \eta)\) be a Dabroux coordinates so that \( \lambda = d\eta - \sum_{i=1}^n y_i dx_i \) on a neighborhood of \( p \in M \).

Remark 6.1. In the proof of [Oh15, Section 10.5], we chose a complex coordinates \( (w_1, \cdots, w_n) \) identifying a neighborhood of \( p \) with an open subset of \( \mathbb{C}^n \).

We consider the standard metric
\[ h = \frac{\sqrt{-1}}{2} dz d\bar{z} \]
on a neighborhood \( U \subset \tilde{\Sigma} \) of \( z_0 \).

The following lemma will be crucial in our proof.

Lemma 6.2. Let \( \eta \) be as above. For any smooth section \( Y \) of \( w^*(TM) \) and \( \eta \) of \( (\Omega^{0,1}_{1,p}(w^*\xi))^* \)
\[ \langle D\bar{\partial}_j^\pi(Y), \eta \rangle = \langle \bar{\partial} Y^\pi, \eta \rangle \]
where \( \bar{\partial} \) is the standard Cauchy-Riemann operators on \( \mathbb{R}^{2n} \cong \mathbb{C}^n \) in the above coordinate.

Proof. We have already shown that \( \eta \) is a distribution with \( \text{supp}(\eta, f) \subset \{z_0\} \). By the structure theorem on the distribution supported at a point \( z_0 \) Theorem 5.1, we have
\[ \eta = P \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right) (\delta_{z_0}) \]
where \( z = s + it \) is the given complex coordinates at \( z_0 \) and \( P \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right) \) is a differential operator associated by the polynomial \( P \) of two variables with coefficients in
\[ \Lambda^{(0,1)}_{(j_{z_0}, t_p)}(w^*\xi). \]

Furthermore since \( \eta \in (W^{1,p})^* \cong W^{-1,q} \), the degree of \( P \) must be zero and so we obtain
\[ \eta = \beta_{z_0} \cdot \delta_{z_0} \] (6.3)
for some constant vector \( \beta_{z_0} \in \Lambda^{(0,1)}(\xi_p) \).
We then have the expression
\[ D\partial J = \partial Y + E \cdot \partial Y + F \cdot Y \]
near \( z_0 \) in coordinates where \( E \) and \( F \) are zero-order matrix operators satisfying
\[ E(z_0) = 0 = F(z_0). \]
(See [OZ09, p.331] and [Sik94] for such a derivation.) Therefore by (6.3), we derive
\[ \langle E \cdot \partial Y + F \cdot Y, \eta, f \rangle = \langle \partial Y, \eta \rangle \]
and we obtain
\[ \langle D\partial J, \eta \rangle = \langle \partial Y, \eta \rangle \]
since \( \text{supp} \eta \subset \{ z_0 \} \). This finishes the proof. \( \square \)

By this lemma, (6.2) becomes
\[ \langle \partial Y, \eta \rangle + \langle -\Delta(\lambda Y) dA + df \circ j, \eta \rangle + \langle Y, \delta z_0 X_p \rangle = 0 \]
for all \( Y \). We next rewrite the middle summand by integration by parts.

**Lemma 6.3.** We have
\[ \langle -\Delta(\lambda Y) dA + df \circ j, \eta \rangle = -\int \lambda Y \Delta f dA + \int df \circ j \wedge (Y \lambda) d\lambda \]
Recalling \( Y = Y^\pi + \lambda R \lambda \) and noting that \( \lambda \) and \( Y^\pi \) are independent and arbitrary, we rearrange the summand of (6.4) into
\[ 0 = -\int \lambda Y \Delta f dA + \langle \lambda \lambda R \lambda, \delta z_0 X_p \rangle + \int (\partial Y^\pi, \eta) + \int df \circ j \wedge (Y \lambda) d\lambda \]
Therefore we have derived
\[ 0 = -\int \lambda Y \Delta f dA + \langle \lambda \lambda R \lambda, \delta z_0 X_p \rangle \]
\[ 0 = \int (\partial Y^\pi, \eta) + \int df \circ j \wedge (Y \lambda) d\lambda \]
We decompose \( Y \) as
\[ Y(z) = (Y(z) - \chi(z)Y(z_0)) + \chi(z)Y(z_0) \]
on \( U \) where \( \chi \) is a cut-off function with \( \chi \equiv 1 \) in a small neighborhood \( V \subset U \) of \( z_0 \) and satisfies \( \text{supp} \chi \subset U \). It induces the corresponding decomposition of \( Y^\pi \) and \( \lambda Y \).

We first examine the equation (6.6). Then the first summand \( \tilde{Y}^\pi \) defined by
\[ \tilde{Y}^\pi(z) := Y^\pi(z) - \chi(z)Y^\pi(z_0) \]
is a smooth section on \( \Sigma \), and satisfies
\[ \tilde{Y}^\pi(z_0) = 0, \quad \overline{\partial} \tilde{Y}^\pi = \overline{\partial} Y^\pi \quad \text{on } V \]
PROOF OF SHELUKHIN’S CONJECTURE

since \( \chi(z)Y(\pi(z_0)) = Y(\pi(z_0)) \) on \( V \). Therefore applying (6.6) to \( \widetilde{Y} \) instead of \( Y \) and recalling supp(\( \eta, f \)) \( \subset \{ p \} \), we obtain
\[
(\partial \widetilde{Y}, \eta) + (\tilde{Y}, \delta_{\pi_0} X_p) = 0.
\]
Again using the support property supp \( \eta \subset \{ z_0 \} \) and (6.2), we derive
\[
(\widetilde{Y}, \delta_{\pi_0} X_p) = (\tilde{Y}(\pi(z_0)), X_p) = 0
\]
and so \( (\partial \tilde{Y}, \eta) = 0 \). But we also have
\[
(\partial \pi \tilde{Y}, \eta) = (\partial \pi Y, \eta)
\]
for all \( Y \).

Applying similar reasoning to (6.5), we have derived
\[
\int \lambda(Y) \Delta f dA = 0
\]
for all \( \lambda(Y) \). Combining the two, we have proved \( (\eta, f) \) is a weak solution of
\[
(\partial \pi Y, \eta) = 0 \quad (6.8)
\]
on whose \( \Sigma \). Therefore a we have finished the proof of (6.1) by Lemma 6.2. By the elliptic regularity, \( (\eta, f) \) is a smooth solution. In particular it is continuous. Since we have already shown \( (\eta, f) = 0 \) on \( \Sigma \setminus \{ z_0 \} \), continuity of \( \eta \) proves \( (\eta, f) = 0 \) on the whole \( \Sigma \). This finishes the proof.

This in turn finishes the proof of Proposition 5.3.

This then finishes the proof of Proposition 5.3.

7. The case of the boundary evaluation map

In this section, we explain how we can augment the arguments used in the proof of generic evaluation transversality to handle the case of boundary evaluation maps. We will also write \( R \) for \( \tilde{R} \) in the present section.

We now consider the map
\[
M_0^0 : J_\lambda \times M_\Sigma \times \bar{F}(0, 1)(\tilde{\Sigma}, R) \to CD \times R,
\]
\[
(J, (j, w), z_0) \mapsto (T(J, (j, w)), w(z_0)).
\]
Then for a given point \( p \in R \),
\[
M(0, 1)(\Sigma, M; \{ p \}) = N_0^{-1}(o_{CD} \times \{ p \})
\]
\[
M(1, 0)(\Sigma, M; \{ p \}; J) = M(1, 0)(\Sigma, M; \{ p \}) \cap \pi^{-1}(J).
\]
We now establish the following boundary analog to Proposition 5.2

Proposition 7.1. The map \( N_0 \) is transverse to the submanifold
\( o_{CD} \times \{ p \} \subset CD \times R \).

Proof. Its linearization \( DN_0(J, (j, w), z) \) is given by the map
\[
(L, (b, Y), v) \mapsto \left( \frac{D_L}{(j, w)} T(L, (b, Y)), Y(w(z)) + dw(z)(v) \right)
\]
for \( L \in T_J J_\lambda, b \in T_j M_\Sigma, v \in T_{\Sigma^0} \Sigma, Y \in T_w F((\Sigma, \partial \Sigma), (M, \tilde{R})). \)
But this time, \((L, (b, Y), v)\) satisfies the boundary condition
\[
Y(\partial \hat{\Sigma}) \subset T\hat{R}, \quad v \in T\partial \hat{\Sigma}.
\] (7.3)
This defines a linear map
\[
T_J \mathcal{J}_\lambda \times T_J \mathcal{M}_{\hat{\Sigma}} \times T_w F((\hat{\Sigma}, \partial \hat{\Sigma}), (M, R)) \times T_z \hat{\Sigma} \to CD(J, (j, w)) \times T_w(z_0) R.
\]

We take the Sobolev completion in the \(W^{k,p}\)-norm for with \(k = 2\). We would like to prove that this linear map is a submersion. For this purpose, we again need to study solvability of the system of equations
\[
D_J, (j, w) Y(L, (b, Y), v) = (\gamma, \omega), \quad Y(w(z_0)) + dw(v) = X_0
\] (7.4)
for any given \((\gamma, \omega) \in CD_\omega\) and \(X_0\), i.e.,
\[
\gamma \in \Omega^{0,1}(w^*TM), \quad \omega \in \Omega^2(\hat{\Sigma}), \quad X_0 \in T_w(z_0) R.
\]
Again we put \(b = 0 = v\) obtain the equation
\[
D_J, w Y(L, Y) = (\gamma, \omega), \quad Y(w(z_0)) = X_0
\] (7.5)
for \(Y \in T_w F((\hat{\Sigma}, \partial \hat{\Sigma}), (M, R))\).

We will show that the image of the map (7.2) restricted to the element of the form
\[
(L, (0, Y), v)
\]
is onto as a map
\[
T_J \mathcal{J}_\lambda \times \Omega_{2,p}^0(w^*TM) \to CD_{J, w}^{1,p} \times T_w(z_0) R
\]
where \((w, j, z_0, J)\) lies in \(\mathcal{Y}^{-1}(\alpha_H' \times \xi)\).

We now prove the following boundary analog to Proposition 5.3 which will then finish the proof.

**Proposition 7.2.** The subspace
\[
\text{Image } \mathcal{N}_{0}^J \subset CD \oplus T_w(z_0) R
\]
is dense.

**Proof.** Let \(((\eta, f), X_p) \in (CD)^* \times T_p \hat{R}\) satisfy
\[
\left< D_w Y(J, (j, w) Y, (\eta, f)) + \left( \frac{1}{2} L \cdot d^w w \circ j, 0 \right), (\eta, f) \right> + \langle Y, \delta_{z_0} X_p \rangle = 0
\] (7.6)
for all \(Y \in \Omega_{2,p}^0(w^*TM, (\partial w)^* TR)\) and \(L\) where \(\delta_{z_0}\) is the Dirac-delta function supported at \(z_0\). We again would like to show
\[
(\eta, f) = 0 = X_p.
\] (7.7)
Taking \(L = 0\) in (7.6), we obtain
\[
(\eta, f) = 0 = X_p
\] (7.8)
for all \(Y \in C^\infty\) satisfying the boundary condition
\[
Y(\partial \hat{\Sigma}) \subset TR.
\] (7.9)
Therefore by definition of the distribution derivatives, \(\eta\) satisfies
\[
(D_w Y(J, (j, w)))^\dagger(\eta, f) - \delta_{z_0} X_p = 0
\]
as a distribution, i.e.,
\[
(D_w Y(J, (j, w)))^\dagger(\eta, f) = \delta_{z_0} X_p
\]
where \((D_w \Upsilon(J, (j, w)))^\dagger\) is the formal \(L^2\)-adjoint of \(D_w \Upsilon(J, (j, w))\). The following lemma provides a description of the formal adjoint.

**Lemma 7.3.** The \(L^2\)-adjoint \((D_w \Upsilon(J, (j, w)))^\dagger\) is a linear elliptic operator whose domain is given by the pairs \((\eta, f)\) such that

\[
\eta \in \Omega_{-1,q}^{(1,0)}, \quad f \in L^q
\]

satisfying the elliptic boundary condition.

By the similar reasoning by considering the variations \(L\) with \(Y = 0\), we again arrive at the following which will finish the proof by the same reason as for the interior case.

**Lemma 7.4.** \(\eta\) is a distributional solution of \((D_w \Upsilon(J, (j, w)))^\dagger(\eta, f) = 0\) on \(\Sigma\) and so continuous. In particular, we have \((\eta, f) = 0\) in \((CD)^\ast\).

Now it remains to prove Lemma 7.4. (In fact, we have only to establish just near \(\{z_0\}\) since the support of \((\eta, f)\) is concentrated at a point \(z_0\).) Our primary goal is to prove \(\eta\) is a distributional solution of \((D_w \Upsilon(J, (j, w)))^\dagger(\eta, f) = 0\) on an open set including \(z_0\) (and so on the whole \(\Sigma\)) by the same reason. The rest of the section will be occupied by the proof of this goal.

We start with (7.8)

\[
\langle D_w \Upsilon(J, (j, w)) Y, (\eta, f) \rangle + \langle Y, \delta_{z_0} X_p \rangle = 0
\]

for all \(Y \in C^\infty\) satisfying (A.2), \(Y(\partial \tilde{\Sigma}) \subset T\tilde{R}\). Again knowing that \(\text{supp}(\eta, f) \subset \{z_0\}\), we can simplify the expression of the pairing to

\[
\langle \partial^\pi Y^\pi, \eta \rangle + \langle -\Delta(\lambda(Y)) dA + d((Y^\pi | d\lambda) \circ j, f) + \langle Y, \delta_{z_0} X_p \rangle = 0 \quad (7.10)
\]

for all \(Y\) satisfying \(Y(\partial \tilde{\Sigma}) \subset T\tilde{R}\).

Now the boundary analog to Lemma 6.3 involves the boundary contribution which is again by integration by parts combined with \(\lambda(Y) \equiv 0\) on \(\partial \tilde{\Sigma}\) by the Legendrian boundary condition of \(Y\).

**Lemma 7.5.** We have

\[
\langle -\Delta(\lambda(Y)) dA + d((Y^\pi | d\lambda) \circ j, f) \rangle
=\quad -\int \lambda(Y) \Delta f dA + \int df \circ j \land (Y^\pi | d\lambda) + \int_{\partial \tilde{\Sigma}} f \frac{\partial}{\partial \nu}(\lambda(Y)) d\theta - f Y^\pi | d\lambda.
\]

Now we derive the boundary analogs to (7.5) and (7.6) respectively:

\[
0 = -\int \lambda(Y) \Delta f dA + \int_{\partial \tilde{\Sigma}} f \frac{\partial}{\partial \nu}(\lambda(Y)) d\theta + \langle \lambda(Y) R_\lambda, \delta_{z_0} X_p \rangle \quad (7.11)
\]

\[
0 = \int \langle \partial^\pi Y^\pi, \eta \rangle + \int df \circ j \land (Y^\pi | d\lambda) - \int f Y^\pi | d\lambda + \langle Y^\pi, \delta_{z_0} X_p \rangle. \quad (7.12)
\]
Again by replacing $Y$ by $\tilde{Y}$ as before, we have now derived the following boundary analogs to (6.5) and (6.6)

$$\langle \nabla_{\tilde{Y}} \pi, \eta \rangle + \int df \circ j \wedge (Y^\pi]d\lambda) - \int f Y^\pi]d\lambda = 0 \quad (7.13)$$

and

$$\int \lambda(Y) \Delta fdA + \int_{\partial \Sigma} f \frac{\partial}{\partial \nu}(\lambda(Y)) d\theta = 0 \quad (7.14)$$

respectively for all $Y$ satisfying $Y(\partial \tilde{\Sigma}) \subset \tilde{R}$. Knowing that $Y^\pi$ and $\lambda(Y)$ are independent functions, we have derived the equation

$$\Delta f = 0, \quad f|_{\partial \tilde{\Sigma}} = 0.$$

By the elliptic regularity, $f$ is continuous and hence $f \equiv 0$. (Recall $\text{supp} f \subset \{p\}$.

We start with the case of closed orbits.

**Definition A.2.** Let $(\gamma,T)$ be an iso-speed closed Reeb orbit in the sense as above. When $|T|$ is minimal among such that $\gamma(1) = \gamma(0)$ and $\int \gamma^* \lambda \neq 0$, we call the pair $(\gamma,T)$ a *simple* iso-speed closed Reeb orbit.

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**Appendix A. Fredholm theory of (relative) contact instantons**

In this appendix, we review the mapping transversality result established in [Oh22] which will also provide the framework for the evaluation transversality considered in the present paper. Let $(M,\xi)$ be a contact manifold and consider contact triads $(M,\lambda,J)$ and let $\tilde{R} = (R_1,R_2,\ldots,R_k)$ be a Legendrian link. We consider the associated contact instanton equation (4.1)

$$\begin{cases}
\n\n\n\n\end{cases}$$

for a map $w : (\Sigma, \partial \Sigma) \to (M, \tilde{R})$ with the boundary condition given as above.

We start with recalling the framework for the study of generic nondegeneracy results for the Reeb orbits and chords given in [Oh22].

**A.1. Set-up for the study of generic nondegeneracy of Reeb orbits and chords.** We first introduce the following definition

**Definition A.1.** Let $T \geq 0$ and consider a curve $\gamma : [0,1] \to M$ be a smooth curve. We say $(\gamma,T)$ an *iso-speed Reeb trajectory* if the pair satisfies

$$\dot{\gamma}(t) = TR_\lambda(\gamma(t)), \quad \int \gamma^* \lambda = T$$

for all $t \in [0,1]$. If $\gamma(1) = \gamma(0)$, we call $(\gamma,T)$ an iso-speed closed Reeb orbit and $T$ the *action* of $\gamma$.

We start with the case of closed orbits.
We consider the relative version thereof.

**Definition A.3** (Iso-speed Reeb chords \cite{Oh21c}). Let \((R_0, R_1)\) be a pair of Legendrian submanifolds of \((M, \xi)\) and \(T \geq 0\). For given contact form \(\lambda\), we say \((\gamma, T)\) with \(\gamma : [0, 1] \to M\) is a Reeb chord from \(R_0\) to \(R_1\) if
\[
\dot{\gamma}(t) = T \dot{R}_\lambda(\gamma(t)), \quad \gamma(0) \in R_0, \gamma(1) \in R_1.
\]
We call any such \((\gamma, T)\) an iso-speed Reeb chord and nonnegative if \(T \geq 0\).

Let \((\gamma, T)\) be a closed Reeb orbit of action \(T\). By definition, we can write
\[
\gamma(T) = \phi^T_{R_\lambda}(\gamma(0))
\]
for the Reeb flow \(\phi^T_{R_\lambda}\) of the Reeb vector field \(R_\lambda\). In particular, \(p = \gamma(0)\) is a fixed point of the diffeomorphism \(\phi^T\). Since \(L_{R_\lambda} = 0\), \(\phi^T\) is a contact diffeomorphism and so induces an isomorphism
\[
\Psi_\gamma := \frac{d\phi^T}{dp}|_{\xi_p} : \xi_p \to \xi_p
\]
which is the linearization restricted to \(\xi_p\) of the Poincaré return map.

**Definition A.4.** Let \(T > 0\). We say a \(T\)-closed Reeb orbit \((T, \lambda)\) is nondegenerate if \(\Psi_\gamma : \xi_p \to \xi_p\) with \(p = \gamma(0)\) has not eigenvalue 1.

The following generic nondegeneracy result is proved in \cite{Oh22} for Reeb chords which extends the above generic nondegeneracy results to the case of open strings of Reeb chord and to the Bott-Morse situation of constant chords.

**Theorem A.5.** Let \((M, \xi)\) be a contact manifold. Let \((R_0, R_1)\) be a pair of Legendrian submanifolds allowing the case \(R_0 = R_1\).

1. For a given pair \((R_0, R_1)\), there exists a residual subset
\[
C^\text{reg}(\xi; R_0, R_1) \subset C(M, \xi)
\]
such that for any \(\lambda \in C^\text{reg}(\xi; R_0, R_1)\) all Reeb chords from \(R_0\) to \(R_1\) are nondegenerate for \(T > 0\) and Bott-Morse nondegenerate when \(T = 0\).

2. For a given contact form \(\lambda\), there exists a residual subset of pairs \((R_0, R_1)\) of Legendrian submanifolds such that all Reeb chords from \(R_0\) to \(R_1\) are nondegenerate for \(T > 0\) and Bott-Morse nondegenerate when \(T = 0\).

**A.2. Off-shell description of moduli spaces.** Let \((\Sigma, j)\) be a bordered compact Riemann surface, and let \(\Sigma\) be the punctured Riemann surface with \(\{z_1, \ldots, z_k\} \subset \partial \Sigma\), we consider the moduli space
\[
\mathcal{M}_k((\Sigma, \partial \Sigma), (M, \bar{R})), \quad \bar{R} = (R_1, \ldots, R_k)
\]
of finite energy maps \(w : \Sigma \to M\) satisfying the equation \([4.1]\).

We will be mainly interested in the two cases:

1. A generic nondegenerate case of \(R_1, \ldots, R_k\) which in particular are mutually disjoint,

2. The case where \(R_1, \ldots, R_k = R\).

The second case is transversal in the Bott-Morse sense both for the Reeb chords and for the moduli space of contact instantons, which is rather straightforward and easier to handle, and so omitted.

For the first case, all the asymptotic Reeb chords are nonconstant and have nonzero action \(T \neq 0\). In particular, the relevant punctures \(z_0\) are not removable.
Therefore we have the decomposition of the finite energy moduli space
\[ \mathcal{M}_k((\hat{\Sigma}, \partial\hat{\Sigma}), (M, \hat{R})) = \bigcup_{\gamma \in \prod_{i=0}^{k-1} \mathfrak{reb}(R_i, R_{i+1})} \mathcal{M}(\gamma), \quad \gamma = (\gamma_1, \ldots, \gamma_k). \]

Depending on the choice of strip-like coordinates we divide the punctures \( \{z_1, \ldots, z_k\} \subset \partial\Sigma \) into two subclasses \( p_1, \ldots, p_{s^+}, q_1, \ldots, q_{s^-} \in \partial\Sigma \) as the positive and negative boundary punctures. We write \( k = s^+ + s^- \).

Let \( \gamma_i^+ \) for \( i = 1, \ldots, s^+ \) and \( \gamma_j^- \) for \( j = 1, \ldots, s^- \) be two given collections of Reeb chords at positive and negative punctures respectively. We denote by \( \underline{\gamma} \) and \( \overline{\gamma} \) the corresponding collections
\[
\underline{\gamma} = \{\gamma_1^+, \ldots, \gamma_{s^+}\}, \quad \overline{\gamma} = \{\gamma_1^-, \ldots, \gamma_{s^-}\}.
\]

For each \( p_i \) (resp. \( q_j \)), we associate the strip-like coordinates \( (\tau, t) \in [0, \infty) \times S^1 \) (resp. \( (\tau, t) \in (-\infty, 0] \times S^1 \)) on the punctured disc \( D_{e^{-2\pi K_0}}(p_i) \setminus \{p_i\} \) (resp. on \( D_{e^{-2\pi R_0}}(q_i) \setminus \{q_i\} \)) for some sufficiently large \( K_0 > 0 \).

**Definition A.6.** We define
\[
\mathcal{F}(\hat{\Sigma}, \partial\hat{\Sigma}), (M, \hat{R}); J, \underline{\gamma}, \overline{\gamma}) \quad \text{(A.1)}
\]
to the set of smooth maps satisfying the boundary condition
\[
w(z) \in R_i \quad \text{for} \quad z \in \overline{z_{i-1} z_0} \subset \partial\hat{\Sigma} \quad \text{(A.2)}
\]
and the asymptotic condition
\[
\lim_{\tau \to \infty} w((\tau, t), i) = \gamma_i^+(T_i(t + t_i)), \quad \lim_{\tau \to -\infty} w((\tau, t), j) = \gamma_j^-(T_j(t - t_j)) \quad \text{(A.3)}
\]
for some \( t_i, t_j \in S^1 \), where
\[
T_i = \int_{S^1} (\gamma_i^+)^* \lambda, \quad T_j = \int_{S^1} (\gamma_j^-)^* \lambda.
\]
Here \( t_i, t_j \) depends on the given analytic coordinate and the parameterizations of the Reeb chords.

We will fix \( j \) and its associated Kähler metric \( h \). We regard the assignment
\[
\Upsilon : w \mapsto \left( \overline{\nabla} w, d(w^* \lambda \circ j) \right), \quad \Upsilon := (\Upsilon_1, \Upsilon_2)
\]
as a section of the (infinite dimensional) vector bundle: We first formally linearize and define a linear map
\[
D\Upsilon(w) : \Omega^0(w^*TM, (\partial w)^*T\hat{R}) \to \Omega^{(0,1)}(w^*\xi) \oplus \Omega^2(\Sigma)
\]
where we have the tangent space
\[
T_w\mathcal{F} = \Omega^0(w^*TM, (\partial w)^*T\hat{R}).
\]
For the optimal expression of the linearization map and its relevant calculations, we use the contact triad connection \( \nabla \) of \((M, \lambda, J)\) and the contact Hermitian connection \( \nabla^\pi \) for \((\xi, J)\) introduced in [OW14, OW18a].
Let $k \geq 2$ and $p > 2$. We denote by
\[ W^{k,p} := W^{k,p}((\tilde{\Sigma}, \partial \tilde{\Sigma}), (M, \tilde{R}); \gamma, \bar{\gamma}) \] (A.4)
the completion of the space $\mathcal{A}_1$. It has the structure of a Banach manifold modelled by the Banach space given by the following

**Definition A.7.** We define
\[ W^{k,p}(w^*TM, (\partial w)^*T\tilde{R}; \gamma, \bar{\gamma}) \] to be the set of vector fields $Y = Y^\pi + \lambda(Y)R_{\lambda}$ along $w$ that satisfy
\[
\begin{cases}
Y^\pi \in W^{k,p}((\tilde{\Sigma}, \partial \tilde{\Sigma}), \ldots), \\
\lambda(Y) \in W^{k,p}((\tilde{\Sigma}, \partial \tilde{\Sigma}), \ldots), \\
Y^\pi(\partial \tilde{\Sigma}) \subset T\tilde{R}, \quad \lambda(Y)(\partial \tilde{\Sigma}) = 0.
\end{cases}
\] (A.5)

Here we use the splitting $TM = \xi \oplus \text{span}_{\mathbb{R}}\{R_{\lambda}\}$ where $\text{span}_{\mathbb{R}}\{R_{\lambda}\} := \mathcal{L}$ is a trivial line bundle and so
\[ \Gamma(w^*\mathcal{L}) \cong C^\infty(\tilde{\Sigma}, \partial \tilde{\Sigma}). \]

The above Banach space is decomposed into the direct sum
\[ W^{k,p}((\tilde{\Sigma}, \partial \tilde{\Sigma}), \ldots) \bigoplus W^{k,p}((\tilde{\Sigma}, \partial \tilde{\Sigma}), \ldots) \otimes R_{\lambda} : \] (A.6)
by writing $Y = (Y^\pi, gR_{\lambda})$ with a real-valued function $g = \lambda(Y(w))$ on $\tilde{\Sigma}$. Here we measure the various norms in terms of the triad metric of the triad $(M, \lambda, J)$.

Now for each given $w \in W^{k,p}((\tilde{\Sigma}, \partial \tilde{\Sigma}), (M, \tilde{R}); J; \gamma, \bar{\gamma})$, we consider the Banach space
\[ \Omega^{(0,1)}_{k-1,p}(w^*\xi) := W^{k-1,p}(\Lambda^{(0,1)}(w^*\xi)) \]
the $W^{k-1,p}$-completion of $\Omega^{(0,1)}(w^*\xi) = \Gamma(\Lambda^{(0,1)}(w^*\xi))$ and form the bundle
\[ \bigcup_{w \in W^{k,p}} \Omega^{(0,1)}_{k-1,p}(w^*\xi) \] (A.7)
over $W^{k,p}$.

**Definition A.8.** We associate the Banach space
\[ H^{(0,1)}_{k-1,p}(M, \lambda)|_w := \Omega^{(0,1)}_{k-1,p}(w^*\xi) \oplus \Omega^2_{k-2,p}(\tilde{\Sigma}) \] (A.8)
to each $w \in W^{k,p}$ and form the bundle
\[ H^{(0,1)}_{k-1,p}(M, \lambda) := \bigcup_{w \in W^{k,p}} H^{(0,1)}_{k-1,p}(M, \lambda)|_w \]
over $W^{k,p}$.

Then we can regard the assignment
\[ \Upsilon_1 : w \mapsto \mathcal{F}^w \] as a smooth section of the bundle $H^{(0,1)}_{k-1,p}(M, \lambda) \to W^{k,p}$. Furthermore the assignment
\[ \Upsilon_2 : w \mapsto d(w^*\lambda \circ j) \]
defines a smooth section of the trivial bundle
\[ \Omega^2_{k-2,p}(\Sigma) \times \mathcal{W}^{k,p} \to \mathcal{W}^{k,p} \]
for the Banach manifold
\[ \mathcal{W}^{k,p} := \mathcal{W}^{k,p}((\dot{\Sigma}, \partial \dot{\Sigma}), (M, \vec{R}); J; \gamma, \bar{\gamma}). \]

We summarize the above discussion to the following lemma.

**Lemma A.9.** Consider the vector bundle
\[ \mathcal{H}^{(0,1)}_{k-1,p}((\dot{\Sigma}, \partial \dot{\Sigma}), (M, \vec{R}); J; \gamma, \bar{\gamma}) \]

The map \( \Upsilon \) continuously extends to a continuous section
\[ \Upsilon : \mathcal{W}^{k,p} \to \mathcal{H}^{(0,1)}_{k-1,p}(\xi; \vec{R}). \]

With these preparations, the following is a consequence of the exponential estimates established in [OW18a] for the case of vanishing charge.

**Proposition A.10** (Theorem 1.12 [OW18a]). Assume \( \lambda \) is nondegenerate and \( Q(p_i) = 0 \). Let \( w : \dot{\Sigma} \to M \) be a contact instanton and let \( w^* \lambda = a_1 d\tau + a_2 dt \). Suppose
\[
\begin{align*}
\lim_{\tau \to -\infty} a_{1,i} &= 0, & \lim_{\tau \to -\infty} a_{2,i} &= T(p_i) \\
\lim_{\tau \to \infty} a_{1,j} &= 0, & \lim_{\tau \to \infty} a_{2,j} &= T(p_j)
\end{align*}
\] (A.9)

at each puncture \( p_i \) and \( q_j \). Then \( w \in \mathcal{W}^{k,p}(\dot{\Sigma}, M; J; \gamma, \bar{\gamma}) \).

Now we are ready to define the moduli space of contact instantons with prescribed asymptotic condition.

**Definition A.11.** Consider the zero set of the section \( \Upsilon \)
\[
\tilde{M}((\dot{\Sigma}, \partial \dot{\Sigma}), (M, \vec{R}); J; \gamma, \bar{\gamma}) = \Upsilon^{-1}(0) \] (A.10)
in the Banach manifold \( \mathcal{W}^{k,p}((\dot{\Sigma}, \partial \dot{\Sigma}), (M, \vec{R}); J; \gamma, \bar{\gamma}) \), and
\[
\tilde{M}((\dot{\Sigma}, \partial \dot{\Sigma}), (M, \vec{R}); J; \gamma, \bar{\gamma}) = \tilde{M}((\dot{\Sigma}, \partial \dot{\Sigma}), (M, \vec{R}); J; \gamma, \bar{\gamma})/\sim \] (A.11)
to be the set of isomorphism classes of contact instantons \( w \).

This definition does not depend on the choice of \( k, p \) or \( \delta \) as long as \( k \geq 2, p > 2 \) and \( \delta > 0 \) is sufficiently small. One can also vary \( \lambda \) and \( J \) and define the universal moduli space. (See also [Oh21a] for the case of closed strings.)

**A.3. Linearized operator and its ellipticity.** Let \( (\dot{\Sigma}, \xi) \) be a punctured Riemann surface, the set of whose punctures may be empty, i.e., \( \dot{\Sigma} = \Sigma \) is either a closed or a punctured Riemann surface. In this subsection and the next, we lay out the precise relevant off-shell framework of functional analysis, and establish the Fredholm property of the linearization map.

Then we have the following explicit formula thereof.
Theorem A.12 (Theorem 10.1 [Oh21a]; See also Theorem 1.15 [OS22]). In terms of the decomposition $d\pi = d\pi w + w^*\lambda R_\lambda$ and $Y = Y^\pi + \lambda(Y)R_\lambda$, we have

$$
DT_1(w)(Y) = \bar{\nabla} \pi \circ D(Y) + B^{(0,1)} w \circ D(Y) + T_{dw}^{\pi(0,1)}(Y^\pi)
$$

$$
+ \frac{1}{2}\lambda(Y)(\mathcal{L}_R, J)(\bar{\nabla} \pi w)
$$

(A.12)

$$
DT_2(w)(Y) = -\Delta(\lambda(Y)) dA + d((Y^\pi|d\lambda) \circ j)
$$

(A.13)

where $B^{(0,1)}$ and $T_{dw}^{\pi(0,1)}$ are the $(0,1)$-components of $B$ and $T_{dw}^{\pi}$, where $B, T_{dw}^{\pi} : \Omega^0(w^*TM) \to \Omega^1(w^*\xi)$ are zero-order differential operators given by

$$
B(Y) = -\frac{1}{2}w^*\lambda \otimes ((\mathcal{L}_R, J)\pi Y)
$$

(A.14)

and

$$
T_{dw}^{\pi}(Y) = \pi T(Y, dw)
$$

(A.15)

respectively.

From the above expression of the covariant linearization of of the section $\Upsilon = (\Upsilon_1, \Upsilon_2)$, the linearization continuously extends to a bounded linear map

$$
D\Upsilon_{(\lambda, T)}(w) : TW^{k,p} \to \mathcal{H}_{k-1,p}(M, \lambda)
$$

(A.16)

where we recall

$$
TW^{k,p} = \Omega^0_{k,p}(w^*TM, (\partial w)^*T R) \circ \Upsilon(w, T)
$$

$$
\mathcal{H}_{k-1,p}(M, \lambda) = \Omega^{(0,1)}_{k-1,p}(w^*\xi) \oplus \Omega^{(0,2)}_{k-2,p}(\Sigma)
$$

for any choice of $k \geq 2$, $p > 2$. Using the decomposition

$$
\Omega^0_{k,p}(w^*TM, (\partial w)^*T R) \cong \Omega^{(0,1)}_{k,p}(w^*\xi, (\partial w)^*T R) \oplus \Omega^{(0,2)}_{k,p}(\Sigma, \partial\Sigma) : R_\lambda,
$$

$D\Upsilon(w)$ can be written into the matrix form

$$
\begin{pmatrix}
\bar{\nabla} \pi + T_{dw}^{\pi(0,1)} + B^{(0,1)} & \frac{1}{2}\lambda(Y)(\mathcal{L}_R, J)d\pi w \\
\frac{1}{2}\lambda(Y)(\mathcal{L}_R, J)d\pi w & -\Delta(\lambda(Y)) dA
\end{pmatrix}
$$

(A.17)

It follows that the map $D\Upsilon(w)$ is a partial differential operator whose principle symbol map is given by $\sigma(D\Upsilon) = \sigma(D\Upsilon_1) \oplus \sigma(D\Upsilon_2)$ where

$$
\sigma(D\Upsilon_1(w))(\eta) = J\pi^* \eta
$$

$$
\sigma(D\Upsilon_2(w))(\eta) = \langle \lambda, \eta \rangle^2 = \langle \eta(R_\lambda) \rangle^2
$$

(A.18)

where $\eta$ is a cotangent vector in $T^*M \setminus \{0\}$ and has decomposition

$$
\eta = \eta^\pi + \eta(R_\lambda(\pi(\eta)) \lambda(\pi(\eta)).
$$

(See [LMS5] for the discussion of general elliptic operators of mixed degree on noncompact manifolds with cylindrical ends.)

In particular we note that the restriction $D\Upsilon_1(w)|_{\Omega^p(w^*\xi)}$ has the same principle symbol as that of

$$
\bar{\nabla} \pi : \Omega^0(w^*\xi, (\partial w)^*\xi) \to \Omega^{(0,1)}(w^*\xi)
$$

which is the first order elliptic operator of Cauchy-Riemann type, and that $D\Upsilon_2(w)$ has the symbol of the Hodge Laplacian acting on zero forms

$$
\star \Delta : \Omega^0(\Sigma, \partial\Sigma) \to \Omega^2(\Sigma).
$$
A.4. Fredholm theory on punctured bordered Riemann surfaces. By the (local) ellipticity shown in the previous subsection, it remains to examine the Fredholm property of the linearized operator $DY(w)$. For this purpose, we need to examine the asymptotic behavior of the operator near punctures in strip-like coordinates.

We first decompose the section $Y \in w^*TM$ into

$$Y = Y^\pi + \lambda(Y)R_\lambda.$$  

Then we have

$$DY_1^i(w)(Y^\pi) = \overrightarrow{D}^\pi \ Y^\pi + B^{(0,1)}(Y^\pi) + T_{dw}^{(0,1)}(Y^\pi),$$  

$$DY_2^i(w)(Y^\pi) = d((Y^\pi) d\lambda) \circ j,$$  

$$DY_1^j(w)(\lambda(Y)R_\lambda) = \frac{1}{2} \lambda(Y)\mathcal{L}_{R_\lambda}J\partial^\pi w,$$  

$$DY_2^j(w)(\lambda(Y)R_\lambda) = -\Delta(\lambda(Y)) dA.$$  

Noting that $Y^\pi$ and $\lambda(Y)$ are independent of each other, we write

$$Y = Y^\pi + fR_\lambda, \quad f := \lambda(Y)$$

where $f: \hat{\Sigma} \to \mathbb{R}$ is an arbitrary function satisfying the boundary condition

$$Y^\pi(\partial\Sigma) \subset T\hat{\mathbf{R}}, \quad f(\partial\Sigma) = 0$$

by the Legendrian boundary condition satisfied by $Y$. The following is obvious from the expression of the $DY_i^j(w)$.

**Lemma A.13** (Lemma 3.17 [Oh89]). Suppose that $w$ is a solution to (4.1). The operators $DY_i^j(w)$ have the following continuous extensions:

$$DY_1^i(w)(Y^\pi) : \Omega^0_{k,p} w^*\xi,(\partial w)^*T\hat{\mathbf{R})} \to \Omega^{(0,1)}_{k-1,p}(w^*\xi)$$

$$DY_2^j(w)(Y^\pi) : \Omega^0_{k,p} w^*\xi,(\partial w)^*T\hat{\mathbf{R})} \to \Omega^2_{k-1,p}(\Sigma) \to \Omega^2_{k-2,p}(\Sigma)$$

$$DY_1^i(w)((\cdot)R_\lambda) : \Omega^0_{k,p}(\Sigma,\partial\Sigma) \to \Omega^2_{k-2,p}(\Sigma)$$

$$DY_2^j(w)((\cdot)R_\lambda) : \Omega^0_{k,p}(\Sigma,\partial\Sigma) \to \Omega^2_{k-2,p}(\Sigma).$$

We regard the domains of $DY_i^j$ for $i = 1, 2$ as $C^\infty(\Sigma,\partial\Sigma)$ using the isomorphism

$$C^\infty(\Sigma,\partial\Sigma) \cong \Omega^0(\Sigma,\partial\Sigma) \cong R_\lambda.$$  

We now establish the following Fredholm property of the linearized operator.

**Proposition A.14.** Suppose that $w$ is a solution to (4.1). Consider the completion of $DY(w)$, which we still denote by $DY(w)$, as a bounded linear map from $\Omega^0_{k,p}(w^*TM,(\partial w)^*T\hat{\mathbf{R}})$ to $\Omega^{(0,1)}(w^*\xi) \oplus \Omega^2(\Sigma)$ for $k \geq 2$ and $p \geq 2$. Then

1. The off-diagonal terms of $DY(w)$ are relatively compact operators against the diagonal operator.

2. The operator $DY(w)$ is homotopic to the operator

$$\begin{pmatrix}
\overrightarrow{D}^\pi + T_{dw}^{(0,1)} + B^{(0,1)} \\
0 \\
-\Delta(\lambda(\cdot)) dA
\end{pmatrix}$$

via the homotopy

$$s \in [0,1] \mapsto \begin{pmatrix}
\overrightarrow{D}^\pi + T_{dw}^{(0,1)} + B^{(0,1)} s d(\cdot) d\lambda \circ j \\
\frac{\pi}{2} \lambda(\cdot)(\mathcal{L}_{R_\lambda}J\pi dw)^{(1,0)} \\
-\Delta(\lambda(\cdot)) dA
\end{pmatrix} =: L_s$$  

(A.24)
which is a continuous family of Fredholm operators.

(3) And the principal symbol
\[ \sigma(z, \eta) : w^*TM|_z \to w^*\xi|_z \oplus \Lambda^2(T_z\tilde{\Sigma}), \quad 0 \neq \eta \in T^*_z\Sigma \]
of (A.23) is given by the matrix
\[
\begin{pmatrix}
\frac{n+i\gamma}{2}Id & 0 \\
0 & |\eta|^2
\end{pmatrix}.
\]

Proof. Statement (1) is a consequence of compactness of Sobolev embeddings
\[ \Omega^2_{k-1,p}(\Sigma) \hookrightarrow \Omega^2_{k-2,p}(\Sigma), \quad \Omega^2_{k,p}(\Sigma) \hookrightarrow \Omega^2_{k-2,p}(\Sigma). \]
When \( \partial\Sigma = \emptyset \), the same kind of statement is proved in [Oh21a]. Essentially the same proof applies by incorporating the boundary condition. \(\square\)

With these preparations, the following is a corollary of exponential estimates established in [OW18b, Part II], [Oh21b] for bordered contact instantons with Legendrian boundary condition.

**Proposition A.15** (Section 7 [Oh21b]). Let
gamma = \{\gamma^+_1, \ldots, \gamma^+_s\}, \quad \gamma' = \{\gamma^-_1, \ldots, \gamma^-_{s'}\}
bef the given \( \lambda \)-Reeb chords of \( \bar{R} = (R_1, \ldots, R_{s+s'}) \) which are nondegenerate. If \( w \) has finite energy, then we have
\[ w \in W^{k,p}(\partial\Sigma, (M, \bar{R}); J; \gamma, \gamma'). \]

Now we are ready to wrap-up the discussion of the Fredholm property of the linearization map
\[ D\Upsilon_{(\lambda,T)}(w) : \Omega^0_{k,p}(w^*TM, (\partial w)^*T\bar{R}; J; \gamma, \gamma') \to \Omega^{(0,1)}_{k-1,p}(w^*\xi) \oplus \Omega^2_{k-2,p}(\tilde{\Sigma}) \]
by proving Statement (1) of Proposition A.14.

The following proposition can be derived from the arguments used by Lockhart and McOwen [LM85].

**Proposition A.16.** Assume that \( \gamma, \gamma' \) are nondegenerate. Then the operator (A.17) is Fredholm.

Then by the continuous invariance of the Fredholm index, we obtain
\[ \text{Index} D\Upsilon_{(\lambda,T)}(w) = \text{Index} \left( \mathcal{F}^\tau + T^\tau_{dw} + B^{(0,1)} \right) + \text{Index}(-\Delta). \] (A.25)
The computation of index is given in [Oh21a] for the closed string case and in [?] for the current open string case.

**Appendix B. Generic transversality of (relative) contact instantons**

In this section, we summarize two main generic transversality results for the contact instanton moduli spaces needed for the study of the moduli problem of contact instanton maps. The transversality results for the perturbation of contact forms and for the perturbation of boundary Legendrian submanifolds are studied in [Oh22].

In this section, we study the mapping trasversality under the perturbation of \( J \). Now we involve the set \( \mathcal{F}(M, \lambda) \) of \( \lambda \)-adaptable almost complex structures which we recall here.
Definition B.1 (Contact triad \[ \text{OW14} \]). Let \((M, \xi)\) be a contact manifold, and \(\lambda\) be a contact form of \(\xi\). An endomorphism \(J : \mathcal{T}M \to \mathcal{T}M\) is called a \(\lambda\)-adapted CR-almost complex structure if it satisfies

1. \(J(\xi) \subset \xi,\ J\mathcal{R}_\lambda = 0\) and \(J^2_\xi = -\text{id}\)|\(\xi\),
2. \(g_\xi := d\lambda(\cdot, J|_\xi \cdot)|_\xi\) defines a Hermitian vector bundle \((\xi, J|_\xi, g_\xi)\).

We call the triple \((M, \lambda, J)\) a contact triad.

We study the linearization of the map \(\Upsilon_{\text{univ}}\) which is the map \(\Upsilon\) augmented by the argument \(J \in J^\ell(M, \lambda)\). More precisely, we define the universal section \(\Upsilon_{\text{univ}} : M(\dot{\Sigma}) \times F \times J^\ell(M, \lambda) \to \mathcal{H}^{(0,1)}_{\pi}(0,1)(M, \lambda)\) given by

\[
\Upsilon_{\text{univ}}(j, w, J) = (\partial_\pi w, d(w^* \lambda \circ j)) =: \Upsilon_J(w, j) \quad (\text{B.1})
\]

We state the following standard statement that often occurs in this kind of generic transversality statement via the Sard-Smale theorem.

Theorem B.3. Let \(0 < \ell < k - \frac{2}{p}\). Consider the moduli space \(\mathcal{M}(M, \lambda, \bar{\gamma}, \gamma)\). Then
(1) $\mathcal{M}(\mathcal{M}, \lambda, \vec{R}; \gamma, \gamma)$ is an infinite dimensional $C^\ell$ Banach manifold.

(2) The projection

$$\Pi_2|_{(\Upsilon_{\text{univ}})^{-1}(0)} : (\Upsilon_{\text{univ}})^{-1}(0) \to \mathcal{J}(\mathcal{M}, \lambda)$$

is a Fredholm map and its index is the same as that of $D\Upsilon(w)$ for a (and so any) $w \in \mathcal{M}(\mathcal{M}, \lambda, \vec{R}; J; \gamma, \gamma)$.

An immediate corollary of Sard-Smale theorem is that for a generic choice of $J$

$$\Pi_2^{-1}(J) \cap (\Upsilon_{\text{univ}})^{-1}(0) = \mathcal{M}(\gamma, \gamma)$$

is a smooth manifold: One essential ingredient for the generic transversality under the perturbation of $J \in \mathcal{J}(\mathcal{M}, \lambda)$ is the usage of the following unique continuation result.

**Proposition B.4** (Unique continuation lemma; Proposition 12.3 [Oh21a]). Any non-constant contact Cauchy-Riemann map does not have an accumulation point in the zero set of $dw$.

A generic transversality under the perturbation of boundaries has been also established in [Oh22].

**Theorem B.5** (Theorem 4.4 [Oh22]). Let $(\mathcal{M}, \xi)$ be a contact manifold, and let $\lambda$ a contact form be given. We consider the same equation considered in Theorem B.3. Fix $J$ and $k$ and consider $\vec{R}$ that intersect transversally pairwise and no triple intersections.

Then there exists a residual subset of $\vec{R} = (R_0, \ldots, R_k)$ of Legendrian submanifolds such that the moduli space $\mathcal{M}(\mathcal{M}, \lambda, \vec{R}; \gamma, \gamma)$ is transversal.

**References**

[Bhu01] M. Bhupal, A partial order on the group of contactomorphisms of $\mathbb{R}^{2n+1}$ via generating functions, Turkish J. Math. 25 (2001), 125–235.

[Che96] Yu. V. Chekanov, Critical points of quasifunctions, and generating families of Legendrian manifolds, (Russian) Funktsional. Anal. i Prilozhen. 30 (1996)), no. 2, 56–69, translation in Funct. Anal. Appl. 30 (1996), no. 2, 118–128.

[Chi17] Sheng-Fu Chiu, Nonsqueezing property of contact balls, Duke Math. J. 166 (2017), no. 4, 605–655.

[EKP06] Y. Eliashberg, S. S. Kim, and L. Polterovich, Geometry of contact transformations and domains: orderability versus squeezing, Geom. Topol. 10 (2006), 1635–1747.

[Fra16] Maia Fraser, Contact non-squeezing at large scale in $\mathbb{R}^{2n} \times S^1$, Internat. J. Math. 27 (2016), no. 13, 1650107, 25 pp.

[GS68] I.M. Gelfand and G.E. Shilov, Generalized Functions, vol.2, Academic Press, New York, 1968.

[HS95] H. Hofer and D. A. Salamon, Floer homology and Novikov rings, Progr. Math., vol. 133, p. 483–524, Birkhäuser, Basel, 1995.

[LM85] R. Lockhard and R. McOwen, Elliptic differential operators on noncompact manifolds, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 12 (1985), no. 3, 409–447.

[LO96] H. V. Le and K. Ono, Perturbation of pseudo-holomorphic curves, addendum to “notes on symplectic 4-manifolds with $b^+_2 = 1$, II, Internat. J. Math. 7 (1996), no. 6, 771–774.

[MS04] Dusa McDuff and Dietmar Salamon, $J$-holomorphic curves and symplectic topology, American Mathematical Society Colloquium Publications, vol. 52, American Mathematical Society, Providence, RI, 2004. MR 2045629 (2004m:53154)

[Oha] Y.-G. Oh, Geometric analysis of perturbed contact instantons with legendrian boundary conditions, preprint, arXiv:2205.12351.

[Ohb] Y.-G. Oh, Geometry and analysis of contact instantons and entanglement of Legendrian links II, in preparation.
[Oh93] _____, Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks I, Comm. Pure Appl. Math. 46 (1993), no. 7, 949–993.
[Oh11] _____, Higher jet evaluation transversality of j-holomorphic curves, J. Korean Math. Soc. 48 (2011), no. 2, 341–365.
[Oh15] _____, Symplectic Topology and Floer Homology, vol. 1., New Mathematical Monographs, 28., Cambridge University Press, Cambridge., 2015.
[Oh21a] _____, Analysis of contact Cauchy-Riemann maps III: energy, bubbling and Fredholm theory, preprint, arXiv:2103.15376, 2021.
[Oh21b] _____, Contact Hamiltonian dynamics and perturbed contact instantons with Legendrian boundary condition, preprint, arXiv:2103.15390(v2), 2021.
[Oh21c] _____, Geometry and analysis of contact instantons and entanglement of Legendrian links I, preprint, arXiv:2111.02597, 2021.
[Oh22] _____, Gluing theories of contact instantons and of pseudoholomorphic curves in SFT, preprint, arXiv:2205.00370, 2022.
[OS22] Y.-G. Oh and Y. Savelyev, Pseudoholomorphic curves on the LCS-fication of contact manifolds, Advances in Geometry (2022), to appear.
[OW14] Y.-G. Oh and R. Wang, Canonical connection on contact manifolds, Real and Complex Submanifolds, Springer Proceedings in Mathematics & Statistics, vol. 106, 2014, (arXiv:1212.4817 in its full version), pp. 43–63.
[OW18a] _____, Analysis of contact Cauchy-Riemann maps I: A priori $C^k$ estimates and asymptotic convergence, Osaka J. Math. 55 (2018), no. 4, 647–679.
[OW18b] _____, Analysis of contact Cauchy-Riemann maps II: Canonical neighborhoods and exponential convergence for the Morse-Bott case, Nagoya Math. J. 231 (2018), 128–223.
[OZ09] Y.-G. Oh and K. Zhu, Embedding property of $J$-holomorphic curves in Calabi-Yau manifolds for generic $j$, Asian J. Math. 13 (2009), no. 3, 323–340.
[RT95] Yongbin Ruan and Gang Tian, A mathematical theory of quantum cohomology, J. Differential Geom. 42 (1995), no. 2, 259–367.
[Rud73] W. Rudin, Functional analysis, McGraw-Hill Book Co., New York, 1973.
[San12] S. Sandon, On iterated translated points for contactomorphisms of $\mathbb{R}^{2n+1}$ and $\mathbb{R}^{2n} \times S^1$, Internat. J. Math. 23 (2012), no. 2, 1250042, 14 pp.
[She17] E. Shelukhin, The Hofer norm of a contactomorphism, J. Symplectic Geom. 15 (2017), no. 4, 1173–1208.
[Sik94] J. C. Sikorav, Some properties of holomorphic curves in almost complex manifolds, Chapter V of Holomorphic Curves in Symplectic Geometry, ed., Audin, M. and Lafontaine, J., Birkhäuser, Basel.

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