CONCENTRATION OF MEASURE, CLASSIFICATION OF SUBMEASURES, AND DYNAMICS OF $L_0$

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Abstract. Exhibiting a new type of measure concentration, we prove uniform concentration bounds for measurable Lipschitz functions on product spaces, where Lipschitz is taken with respect to the metric induced by a weighted covering of the index set of the product. Our proof combines the Herbst argument with an entropic version of the weighted Loomis–Whitney inequality. We give a quantitative “geometric” classification of diffused submeasures into elliptic, parabolic, and hyperbolic. We prove that any non-elliptic submeasure (for example, any measure, or any pathological submeasure) has a property that we call covering concentration. Our results have strong consequences for the dynamics of the corresponding topological $L_0$-groups.

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1. Introduction

The present paper makes contributions to three areas: the probabilistic theme of concentration of measure in product spaces; the set theoretic and measure theoretic theme of submeasures; and the topological dynamical theme of extreme amenability.

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Concentration of measure in products. We introduce a generalization of the Hamming metric on product spaces and prove concentration of measure for it. (The book [Led01] is a rich source of information on concentration of measure.) Generalizations of the Hamming metric in the context of concentration of measure were considered by Talagrand [Tal95, Tal96]. Our approach appears to be orthogonal to Talagrand’s. We start with a sequence of sets \( C = (C_0, \ldots, C_{m-1}) \) covering a non-empty set \( N \) together with a sequence of positive real numbers, weights, \( w = (w_0, \ldots, w_{m-1}) \). The sequences \( C \) and \( w \) will be the parameters determining the metric. Given a family of sets \( \Omega_j, j \in N \), we define a metric \( d_{C,w} \) on \( \prod_{j \in N} \Omega_j \) as follows: for two points \( x = (x_0, \ldots, x_{m-1}) \) and \( y = (y_0, \ldots, y_{m-1}) \) in the product, let

\[
d_{C,w}(x, y) := \inf_I \sum_{i \in I} w_i,
\]

where \( I \) runs over all \( I \subseteq \{0, \ldots, m-1\} \) with

\[
\{j \in N | x_j \neq y_j\} \subseteq \bigcup_{i \in I} C_i.
\]

Note that if the sets \( C_i, i < m \), form a partition of \( N \) into one-element sets (so \( m = |N| \)) and \( w_i = 1/|N| \) for each \( i < m \), then \( d_{C,w} \) coincides with the normalized Hamming metric.

We prove a concentration of measure theorem in product spaces for the above metric \( d_{C,w} \). This result was inspired geometrically by the Loomis–Whitney theorem. Our interest in such a concentration of measure theorem comes from applications in topological dynamics in proving extreme amenability of certain Polish groups. To state the concentration of measure theorem, we extract a natural number \( k \) from the sequence \( C \); we call \( C \) a \( k \)-cover of \( N \) if each element of \( N \) belongs to at least \( k \) entries of the sequence \( C \). We consider now a family of standard Borel probability spaces indexed by the set \( N \): \( (\Omega_j, \mu_j)_{j \in N} \). Let \( P \) be the product measure on \( \prod_{j \in N} \Omega_j \). Assuming that \( C \) is a \( k \)-cover of \( N \), we prove in Theorem 3.6 that for each measurable function \( f: \prod_{j \in N} \Omega_j \to \mathbb{R} \) that is 1-Lipschitz with respect to \( d_{C,w} \) and for every \( r \in \mathbb{R}_{>0} \),

\[
\mathbb{P}(\{x | f(x) - \mathbb{E}_P(f) \geq r\}) \leq \exp\left(-\frac{kr^2}{4 \sum_{i<m} w_i^2}\right).
\]

The advancement consists of the presence of \( k \) in the exponent on the right-hand side of the above inequality. Our proof of concentration of measure extends the entropy method introduced by Marton [Mar96] and Ledoux [Led96], central ingredient of which is the so-called Herbst argument. Our main contribution here is Lemma 3.3, which relates entropy on product spaces with covering numbers of covers of the underlying index sets and which can be viewed as an entropic version of the geometric weighted Loomis–Whitney theorem. The latter geometric result is due to Finner [Fin92] and Bollobas–Thomason [BT95]. For a broader background on concentration of measure, the reader may consult [Led01].

Submeasures as pseudo-metrics. A real-valued function \( \phi \) on an algebra \( \mathcal{A} \) of subsets of a set \( X \) is a submeasure if it is subadditive, monotone with respect to the inclusion relation, and assigns the value 0 to the empty set. For some background on submeasures the reader may consult, for example, the papers [HC75, KR83, Sol99].
A submeasure can be viewed as a metric, or a pseudo-metric, on an algebra of sets that respects the structure of the algebra, namely, \( \phi \) induces a pseudo-metric on \( A \) by the formula

\[
d_\phi(A, B) := \phi((A \setminus B) \cup (B \setminus A)).
\]

(1)

Of course, \( d_\phi \) is a metric precisely when \( \phi \) is strictly positive on non-empty sets in \( A \). Seeing submeasures as pseudo-metrics yields connections between submeasures and nets of \( mm \)-spaces, on the one hand, and submeasures and Polish topological groups, on the other, which, in turn, connects the concentration of measure result above with extreme amenability of certain Polish groups. Before we explain these relationships, we describe our classification of submeasures, which will be important in our considerations.

**Classification of submeasures.** With each submeasure \( \phi \) defined on a subalgebra of subsets of \( X \), we associate a function \( h_\phi : \mathbb{R}^+ \to \mathbb{R}^+ \), whose value at \( \xi > 0 \) measures how thickly, relative to \( \xi \), the family of sets with submeasure not exceeding \( \xi \) covers the underlying set \( X \). More precisely, we consider the covering number of a family as introduced by Kelley [Kel59]: for a family \( B \) of subsets of \( Y \), the covering number of \( B \) is the supremum of the ratios

\[
\max \left\{ \frac{k}{|\{i < n \mid y \in B_i\}|} \geq k \text{ for each } y \in Y \right\}
\]

where \( (B_0, \ldots, B_{n-1}) \) varies over all sequence of elements of \( B \) with \( n \geq 1 \). Now, \( h_\phi(\xi) \) is defined to be equal to the covering number of the family \( A_{\phi, \xi} := \{A \in A \mid \phi(A) \leq \xi\} \) divided by \( \xi \). In Theorem 4.6, we show that the asymptotic behavior of \( h_\phi \) at 0 is rather restricted, for example, the quantity \( h_\phi(\xi) \) tends to a limit, possibly infinite, as \( \xi \) tends to 0. A key point in this proof is Lemma 4.9, which is analogous to certain convergence results on subadditive sequences, but appears not to be derivable from these results. We classify submeasures into hyperbolic, parabolic, and elliptic according to the asymptotic behavior of \( h_\phi \); using Landau’s big \( O \) notation, the submeasure \( \phi \) is hyperbolic if \( \frac{1}{h_\phi(\xi)} = O(\xi) \) as \( \xi \to 0 \), elliptic if \( h_\phi(\xi) = O(\xi) \) as \( \xi \to 0 \), and parabolic otherwise. In Theorem 4.6, we relate this classification to the two well-studied classes of submeasures: measures and pathological submeasures. In particular, using a result of Christensen [Chr78], we show that a submeasure is hyperbolic precisely when it is pathological. (Recall that a submeasure that is additive on pairs of disjoint sets is called a **measure**; a submeasure is called **pathological** if it does not have a non-zero measure below it.)

**Submeasures as functors from probability spaces to nets of \( mm \)-spaces.** An \( mm \)-space, or a metric measure space, is a standard Borel space equipped with a probability measure and a pseudo-metric that are compatible with each other. Assume we have a submeasure \( \phi \) defined on an algebra \( A \) of subsets of some set \( X \). The family of all partitions of the underlying set \( X \) into sets in \( A \) with the relation of refinement forms a directed partial order. Given a standard Borel probability space \((\Omega, \mu)\), we associate with each such partition \( B \) an \( mm \)-space by equipping the product space...
\( \Omega^B \) of all functions from \( B \) to \( \Omega \) with the product measure arising from \( \mu \) and a pseudo-metric \( \delta_{\phi,B} \) that naturally extends formula (1) by setting
\[
\delta_{\phi,B}(x,y) := \phi \left( \bigcup \{ A \in B \mid x(A) \neq y(A) \} \right).
\]
This procedure associates with \( \phi \) a net of \( mm \)-spaces indexed by finite partitions of \( X \) into elements of \( A \). A natural question arises whether the nets of \( mm \)-spaces obtained this way are Lévy, that is, whether they exhibit concentration of measure. Using our concentration of measure result, we prove in Theorem 5.6 that the nets of \( mm \)-spaces associated with hyperbolic and parabolic submeasures are Lévy. On the other hand, in Example 5.7, we exhibit an elliptic submeasure such that the net of \( mm \)-spaces associated with it is not Lévy, showing that Theorem 5.6 is essentially sharp.

Submeasures as functors from topological groups to topological groups. Given a topological group \( G \), we consider the topological group \( L_0(\phi,G) \) of all functions \( f \) from \( X \) to \( G \) that are constant on the elements of a finite partition \( B \subseteq A \) of \( X \), with \( B \) depending on \( f \). The group \( L_0(\phi,G) \) is equipped with pointwise multiplication. The topology on it is defined again by extending formula (1). Given \( \varepsilon > 0 \) and a neighborhood \( U \) of the neutral element in \( G \), a basic neighborhood of \( f \in L_0(\phi,G) \) consists of all \( g \in L_0(\phi,G) \) such that
\[
\phi \{ x \in X \mid g(x) \notin U f(x) \} < \varepsilon.
\]
A construction of this type was first carried out by Hartman–Mycielski [HM58], in the case of \( \phi \) being a measure, and by Herer–Christensen [HC75], in the case of a general submeasure. We ask when \( L_0(\phi,G) \) is extremely amenable, that is, for what \( \phi \) and \( G \), does each continuous actions of \( L_0(\phi,G) \) on a compact Hausdorff space have a fixed point? Results pertaining to this questions were obtained by Herer–Christensen [HC75], Glasner [Gla98], Pestov [Pes02], Farah–Solecki [FS08], Sabok [Sab12], and Pestov–Schneider [PS17]. For a broader background on extreme amenability the reader may consult [Pes06]. Our classification of submeasures plays a role here, too. In Theorem 7.3, we connect covering concentration of submeasures \( \phi \) and extreme amenability of groups \( L_0(\phi,G) \) for amenable \( G \). Using this theorem and our result on Lévy nets described above, we show in Corollary 7.4 that if \( \phi \) is hyperbolic or parabolic and \( G \) is amenable, then \( L_0(\phi,G) \) is extremely amenable, in fact, it is even whirlly amenable. This gives a common strengthening of the results from [HC75, Gla98, Pes02, PS17] and also of a large portion of the results from [FS08, Sab12]. In the other direction, by extending an argument from [PS17], we show in Proposition 7.5 that if \( \phi \) is parabolic or elliptic and \( G \) is not amenable, then \( L_0(\phi,G) \) is not extremely amenable, in fact, it is not even amenable.

2. Measure concentration and entropy

The purpose of this preliminary section is to provide the background material necessary for stating and proving the results of Section 3. This will include both a quick review of generalities concerning concentration of measure (Section 2.1) and a
discussion of a specific information-theoretic method for establishing concentration inequalities (Section 2.2).

2.1. A review of measure concentration. Let us briefly recall some of the general background concerning the phenomenon of measure concentration [Lév22, Mil67, MS86, GM83]. For more details, the reader is referred to [Led01, Mas07]. For a start, let us clarify some bits of notation: if \((X, d)\) is a pseudo-metric space, then, for any \(A \subseteq X\) and \(\varepsilon \in \mathbb{R}_{>0}\), we let

\[
B_d(A, \varepsilon) := \{x \in X \mid \exists a \in A: d(a, x) < \varepsilon\}.
\]

Let us note that, if \(X\) is a standard Borel space and \(d\) is a measurable pseudo-metric on \(X\), then the Measurable Projection Theorem of Castaing and Valadier [CV77, Theorem III.23] (see also [Cra02, Theorem 2.12]) entails the following: for any measurable \(A \subseteq X\) and \(\varepsilon \in \mathbb{R}_{>0}\), the set \(B_d(A, \varepsilon)\) is universally measurable in \(X\), that is, \(B_d(A, \varepsilon)\) is \(\mu\)-measurable for every probability measure \(\mu\) on \(X\).

**Definition 2.1.** Let \((X, d, \mu)\) be a metric measure space, that is, \(X\) is a standard Borel space, \(d\) is a measurable pseudo-metric on \(X\) and \(\mu\) is a probability measure on \(X\). The mapping \(\alpha_{(X, d, \mu)} : \mathbb{R}_{>0} \to [0, 1]\) defined by

\[
\alpha_{(X, d, \mu)}(\varepsilon) := 1 - \inf\{\mu(B_d(A, \varepsilon)) \mid A \subseteq X \text{ measurable}, \mu(A) \geq 1/2\}
\]

is called the concentration function of \((X, d, \mu)\). A net \((X_i, d_i, \mu_i)_{i \in I}\) of metric measure spaces is said to be a Lévy net if, for every family of measurable sets \(A_i \subseteq X_i\) \((i \in I)\),

\[
\liminf_{i \in I} \mu_i(A_i) > 0 \implies \forall \varepsilon \in \mathbb{R}_{>0}: \lim_{i \in I} \mu_i(B_d(A_i, \varepsilon)) = 1.
\]

Let us recollect some basic facts about concentration. Given two measurable spaces \(S\) and \(T\) as well as a measure \(\mu\) on \(S\), the push-forward measure of \(\mu\) along a measurable map \(f : S \to T\) will be denoted by \(f_\ast(\mu)\), that is, \(f_\ast(\mu)\) is the measure on \(T\) defined by \(f_\ast(\mu)(B) := \mu(f^{-1}(B))\) for every measurable subset \(B \subseteq T\).

**Remark 2.2.** The following hold.

1. For every metric measure space \((X, d, \mu)\), the map \(\alpha_{(X, d, \mu)} : \mathbb{R}_{>0} \to [0, 1]\) is antitone.
2. Let \((X_0, d_0, \mu_0)\) and \((X_1, d_1, \mu_1)\) be metric measure spaces. If there exists a measurable 1-Lipschitz map \(f : (X_0, d_0) \to (X_1, d_1)\) with \(f_\ast(\mu_0) = \mu_1\), then

\[
\alpha_{(X_1, d_1, \mu_1)} \leq \alpha_{(X_0, d_0, \mu_0)}
\]

(see [Pes06, Lemma 2.2.5]).
3. A net \((X_i, d_i, \mu_i)_{i \in I}\) of metric measure spaces is a Lévy net if and only if

\[
\lim_{i \in I} \alpha_{(X_i, d_i, \mu_i)}(\varepsilon) = 0
\]

for every \(\varepsilon \in \mathbb{R}_{>0}\) (see [Pes06, Remark 1.3.3]).
In this work, we deduce concrete estimates for concentration functions of a large family of metric measure spaces by bounding the measure-theoretic entropy of their 1-Lipschitz functions. Fundamental to this approach is the following elementary observation, where we let \( \mathbb{E}_\mu(f) := \int f \, d\mu \) for a probability space \((X, \mu)\) and a \(\mu\)-integrable function \(f: X \to \mathbb{R}\).

**Proposition 2.3** ([Led01], Proposition 1.7). Let \((X, d, \mu)\) be a metric measure space and consider any function \(\alpha: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}\). Suppose that, for every bounded measurable 1-Lipschitz function \(f: (X, d) \to \mathbb{R}\) and every \(r \in \mathbb{R}_{>0}\),
\[
\mu(\{x \in X \mid f(x) - \mathbb{E}_\mu(f) \geq r\}) \leq \alpha(r).
\]
Then \(\alpha(X, d, \mu)(r) \leq \alpha\left(\frac{r}{2}\right)\).

The concentration results to be proved in Section 3 will be shown to have interesting applications in topological dynamics (see Section 7). As this will require us to connect concentration of measure with the study of general topological groups, we conclude this section by briefly recalling and commenting on the concept of measure concentration in uniform spaces, as introduced by Pestov [Pes02, Definition 2.6].

**Definition 2.4.** Let \(X\) be a uniform space. For an entourage \(U\) in \(X\) and \(A \subseteq X\), let
\[
U[A] := \{y \in X \mid \exists x \in A: (x, y) \in U\}.
\]
A net \((\mu_i)_{i \in I}\) of Borel probability measures on \(X\) is said to concentrate in \(X\) (or called a Lévy net in \(X\)) if, for every family \((A_i)_{i \in I}\) of Borel subsets of \(X\) and any open entourage \(U\) of \(X\),
\[
\liminf_{i \in I} \mu_i(A_i) > 0 \implies \lim_{i \in I} \mu_i(U[A_i]) = 1.
\]

**Remark 2.5** ([GM83], 2.1; [Pes02], Lemma 2.7). Let \((X_i, d_i, \mu_i)_{i \in I}\) be a Lévy net of metric measure spaces, let \(Y\) be a uniform space, and let \(f_i: X_i \to Y\) for each \(i \in I\). If the family \((f_i)_{i \in I}\) is uniformly equicontinuous, that is, for every entourage \(U\) of \(Y\) there exists \(\varepsilon \in \mathbb{R}_{>0}\) such that
\[
\forall i \in I \forall x, y \in X_i: \quad d_i(x, y) \leq \varepsilon \implies (f_i(x), f_i(y)) \in U,
\]
then the net \(((f_i)_*, (\mu_i))_{i \in I}\) concentrates in \(X\).

### 2.2. The entropy method and the Herbst argument

The idea of applying information-theoretic arguments to derive concentration inequalities has its origin in the pioneering work of Marton [Mar96] and Ledoux [Led96]. The presentation here will focus on the results necessary for the purposes of Section 3. For a comprehensive introduction to this method, the reader is referred to [Mas07, Section 1.2.3]. We start off with a definition.

**Definition 2.6** ([Mas07], Definition 2.11; or [Led01], page 91). Let \((\Omega, \mu)\) be a probability space and let \(f: \Omega \to \mathbb{R}_{\geq 0}\) be \(\mu\)-integrable. The entropy of \(f\) with respect to \(\mu\) is defined as
\[
\text{Ent}_\mu(f) := \int f(x) \ln f(x) \, d\mu(x) - \left(\int f(x) \, d\mu(x)\right) \ln \left(\int f(x) \, d\mu(x)\right).
\]
We recall the following dual characterization of entropy, where \( \mathbb{R} := \mathbb{R} \cup \{ -\infty, \infty \} \).

**Proposition 2.7** ([Mas07], Proposition 2.12; or [Led01], page 98). Let \((\Omega, \mu)\) be a probability space and let \( f : \Omega \to \mathbb{R}_{\geq 0} \) be \( \mu \)-integrable. Then

\[
\operatorname{Ent}_\mu(f) = \sup \left\{ \int g f \, d\mu \left| g : \Omega \to \mathbb{R} \text{ measurable}, \int \exp \circ g \, d\mu \leq 1 \right\}.
\]

We note a slight variation of Proposition 2.7.

**Corollary 2.8.** Let \((\Omega, \mu)\) be a probability space and let \( f : \Omega \to \mathbb{R}_{\geq 0} \) be \( \mu \)-integrable. Then

\[
\operatorname{Ent}_\mu(f) = \sup \left\{ \int g f \, d\mu \left| g : \Omega \to \mathbb{R} \text{ measurable}, \int \exp \circ g \, d\mu \leq 1 \right\}.
\]

**Proof.** Clearly, if \( \int f \, d\mu = 0 \), then \( \operatorname{Ent}_\mu(f) = 0 \) and \( f(x) = 0 \) for \( \mu \)-almost every \( x \in \Omega \), so that the desired equality holds trivially. Therefore, we may and will assume that \( \alpha := \int f \, d\mu > 0 \). Moreover, thanks to Proposition 2.7, it suffices to verify that

\[
\operatorname{Ent}_\mu(f) \leq \sup \left\{ \int g f \, d\mu \left| g : \Omega \to \mathbb{R} \text{ measurable}, \int \exp \circ g \, d\mu \leq 1 \right\}.
\]

(2)

For this, let \( \varepsilon \in \mathbb{R}_{>0} \). Put \( \beta := \mu(B) \) for the measurable set \( B := \{ x \in \Omega \mid f(x) = 0 \} \). Choose any \( \delta \in \mathbb{R}_{>0} \) with \( \alpha \delta \leq \varepsilon \) and then \( n \in \mathbb{N} \) such that \( \exp(-n) \leq 1 - \exp(-\delta) \). Consider the measurable function \( g : \Omega \to \mathbb{R} \) defined by

\[
g(x) := \begin{cases} 
\ln f(x) - \ln \alpha - \delta & \text{if } x \in \Omega \setminus B, \\
-\alpha & \text{otherwise}
\end{cases}
\]

for all \( x \in \Omega \). We observe that

\[
\int \exp \circ g \, d\mu = \exp(-\delta) \alpha^{-1} \int_{\Omega \setminus B} f \, d\mu + \exp(-n) \beta \leq \exp(-\delta) + \exp(-n) \leq 1
\]

and

\[
\int f g \, d\mu = \int f(x)(\ln f(x) - \ln \alpha - \delta) \, d\mu(x)
\]

\[
= \int f(x) \ln f(x) \, d\mu(x) - \alpha \ln \alpha - \alpha \delta = \operatorname{Ent}_\mu(f) - \alpha \delta \geq \operatorname{Ent}_\mu(f) - \varepsilon.
\]

This proves (2) and hence completes the argument. \( \square \)

When estimating entropy in Section 3, we will moreover make use of the following.

**Lemma 2.9** ([Led01], Corollary 5.8). Let \((\Omega, \mu)\) be a probability space and \( f : \Omega \to \mathbb{R} \) be \( \mu \)-integrable. Then

\[
\operatorname{Ent}_\mu(\exp \circ f) \leq \int \int_{f(x) \geq f(y)} (f(x) - f(y))^2 \exp(f(x)) \, d\mu(y) \, d\mu(x).
\]
Proof. Applying Jensen’s inequality and Fubini’s theorem, we see that

\[
\begin{align*}
\text{Ent}_\mu(\exp \circ f) &= \int f(x) \exp(f(x)) \, d\mu(x) - \mathbb{E}_\mu(\exp \circ f) \ln \mathbb{E}_\mu(\exp \circ f) \\
&\leq \int f(x) \exp(f(x)) \, d\mu(x) - \left( \int \exp(f(x)) \, d\mu(x) \right) \left( \int f(x) \, d\mu(x) \right) \\
&= \frac{1}{2} \int \int (f(x) - f(y))(\exp(f(x)) - \exp(f(y))) \, d\mu(y) \, d\mu(x) \\
&= \int_{f(x) \geq f(y)} (f(x) - f(y))(\exp(f(x)) - \exp(f(y))) \, d(\mu \otimes \mu)(x, y).
\end{align*}
\]

Furthermore, a straightforward application of the mean value theorem shows that, if \(a, b \in \mathbb{R}\) and \(a \geq b\), then \(\exp(a) - \exp(b) \leq \exp(a)(a - b)\), thus

\[(a - b)(\exp(a) - \exp(b)) \leq (a - b)^2 \exp(a)\].

Combining this inequality with Fubini’s theorem, we conclude that

\[
\text{Ent}_\mu(\exp \circ f) \leq \int_{f(x) \geq f(y)} (f(x) - f(y))^2 \exp(f(x)) \, d(\mu \otimes \mu)(x, y) = \int \int_{f(x) \geq f(y)} (f(x) - f(y))^2 \exp(f(x)) \, d\mu(y) \, d\mu(x).
\]

\[
\square
\]

Our interest in entropy is due to the following fact, known as the Herbst argument.

**Proposition 2.10** (Herbst argument, [Mas07], Proposition 2.14). Let \((\Omega, \mu)\) be a probability space, let \(f: \Omega \to \mathbb{R}\) be \(\mu\)-integrable, and let \(D \in \mathbb{R}_{>0}\). Suppose that, for each \(\lambda \in \mathbb{R}_{>0}\),

\[
\text{Ent}_\mu(\exp(\lambda f)) \leq \frac{1}{2} \lambda^2 D \int \exp(\lambda f) \, d\mu.
\]

Then, for each \(\lambda \in \mathbb{R}_{>0}\),

\[
\int \exp(\lambda(f(x) - \mathbb{E}_\mu(f))) \, d\mu(x) \leq \exp\left(\frac{1}{2} \lambda^2 D\right).
\]

The Herbst argument provides a technique for proving concentration of measure, via combining it with Proposition 2.3 and the following well-known fact.

**Proposition 2.11.** Let \((\Omega, \mu)\) be a probability space, let \(f: \Omega \to \mathbb{R}\) be \(\mu\)-integrable, and let \(D \in \mathbb{R}_{>0}\). Suppose that for each \(\lambda \in \mathbb{R}_{>0}\)

\[
\int \exp(\lambda(f(x) - \mathbb{E}_\mu(f))) \, d\mu(x) \leq \exp\left(\frac{1}{2} \lambda^2 D\right).
\]

Then, for each \(r \in \mathbb{R}_{>0}\),

\[
\mu(\{x \in \Omega \mid f(x) - \mathbb{E}_\mu(f) \geq r\}) \leq \exp\left(-\frac{r^2}{2D}\right).
\]
Proof. Let \( r \in \mathbb{R}_{>0} \). By Markov’s inequality, our hypothesis implies that
\[
\mu(\{x \in \Omega \mid f(x) - \mathbb{E}_\mu(f) \geq r\}) = \mu(\{x \in \Omega \mid \lambda(f(x) - \mathbb{E}_\mu(f)) \geq \lambda r\}) \\
\leq \exp(-\lambda r) \int \exp(\lambda(f(x) - \mathbb{E}_\mu(f))) \, d\mu(x) \leq \exp\left(\frac{1}{2} \lambda^2 D - \lambda r\right)
\]
for every \( \lambda \in \mathbb{R}_{>0} \). Choosing \( \lambda := \frac{r}{D} \), we conclude that
\[
\mu(\{x \in \Omega \mid f(x) - \mathbb{E}_\mu(f) \geq r\}) \leq \exp\left(\frac{1}{2} \left(\frac{r}{D}\right)^2 D - \left(\frac{r}{D}\right)r\right) = \exp\left(-\frac{r^2}{2D}\right). \quad \Box
\]

3. Covering concentration

In this section, we prove concentration of measure for a new class of metric measure spaces, namely for products of probability spaces equipped with a pseudo-metric naturally arising from any weighted covering of the underlying index set (Theorem 3.6 and Corollary 3.7). In addition to the tools outlined in Section 2.2, the main technical ingredient is given by Lemma 3.3 below. Our concentration inequalities will be formulated in terms of Kelley’s covering number [Kel59], the definition of which we recall next. Given a set \( X \), let us denote by \( \mathcal{P}(X) \) the power set of \( X \).

Definition 3.1. Let \( X \) be a set, \( m \in \mathbb{N}_{\geq 1} \). Consider any \( \mathcal{C} = (C_i)_{i<m} \in \mathcal{P}(X)^m \). Then
\[
t_X(\mathcal{C}) := \sup\{k \in \mathbb{N} \mid \forall x \in X : |\{i < m \mid x \in C_i\}| \geq k\}
\]
is called the covering number of \( \mathcal{C} \) with respect to \( X \). Moreover, \( \mathcal{C} \) is called a \( k \)-cover of \( X \) if \( t_X(\mathcal{C}) \geq k \). A cover of \( X \) is defined to be a 1-cover of \( X \). Finally, the sequence \( \mathcal{C} \) is said to be uniform (over \( X \)) if \( |\{i < m \mid x \in C_i\}| = t_X(\mathcal{C}) \) for every \( x \in X \).

Evidently, a finite sequence of subsets of a set \( X \) constitutes a cover of \( X \) in the sense of Definition 3.1 if and only if its union coincides with \( X \). Let us point out the following simple observation about uniform refinements of covers. For a set \( X \) and some \( \mathcal{B} \subseteq \mathcal{P}(X) \), we will denote by \( \langle \mathcal{B} \rangle_X \) the partition of \( X \) generated by \( \mathcal{B} \), that is,
\[
\langle \mathcal{B} \rangle_X := \left\{ \bigcap\{B \in \mathcal{B} \mid x \in B\} \cap \bigcap\{X \setminus B \mid x \notin B \in \mathcal{B}\} \mid x \in X \right\}.
\]

Lemma 3.2. Let \( X \) be a set and let \( \mathcal{A} \) be a Boolean subalgebra of \( \mathcal{P}(X) \). Let \( m \in \mathbb{N}_{\geq 1} \) and let \( \mathcal{C} = (C_i)_{i<m} \in \mathcal{A}^m \) be a cover of \( X \). Then there exists a uniform \( t_X(\mathcal{C}) \)-cover \( \mathcal{C}^* = (C^*_i)_{i<m} \in \mathcal{A}^m \) of \( X \) such that \( C^*_i \subseteq C_i \) for each \( i < m \).

Proof. Let \( k := t_X(\mathcal{C}) \) and let us denote by \( \mathcal{P}_k(m) \) the set of all \( k \)-element subsets of \( \{0, \ldots, m-1\} \). Evidently, \( \mathcal{B} := \langle \{C_i \mid i < m\} \rangle_X \) is a finite subset of \( \mathcal{A} \). Moreover,
\[
\forall B \in \mathcal{B} \forall x, y \in B \forall i < m: \quad x \in C_i \iff y \in C_i.
\]
The latter entails the existence of a map \( \pi: \mathcal{B} \to \mathcal{P}_k(m) \) such that
\[
\forall B \in \mathcal{B} \forall i \in \pi(B): \quad B \subseteq C_i.
\]
For each \( i < m \), let \( C^*_i := \bigcup\{B \in \mathcal{B} \mid i \in \pi(B)\} \). Clearly, \( \mathcal{C}^* := (C^*_i)_{i<m} \in \mathcal{A}^m \) and \( C^*_i \subseteq C_i \) whenever \( i < m \). Furthermore, \( \mathcal{C}^* \) is a uniform \( k \)-cover of \( X \): if \( x \in X \), then there is \( B \in \mathcal{B} \) with \( x \in B \), which implies that \( \{i < m \mid x \in C^*_i\} = \pi(B) \) and thus
\[
|\{i < m \mid x \in C^*_i\}| = |\pi(B)| = k. \quad \Box
\]
Let us now proceed to the afore-mentioned Lemma 3.3, which may be considered an entropic version of the weighted Loomis-Whitney inequality (for the latter, the reader is referred to [Fin92, Corollary 2.2] or [BT95, page 419]). It generalizes a result due to Ledoux [Led01, Proposition 5.6].

To clarify some notation, let $N$ be a finite set and let $(\Omega_j)_{j \in N}$ be a family of measurable spaces. If $x \in \prod_{j \in S} \Omega_j$ and $y \in \prod_{j \in T} \Omega_j$ for disjoint subsets $S, T \subseteq N$, then we will write $(x, y)$ for the unique element of $\prod_{j \in S \cup T} \Omega_j$ that projects to $x$ and $y$. Furthermore, if $f: \prod_{j \in N} \Omega_j \to \mathbb{R}$ is a measurable function, then, for any subset $S \subseteq N$ and $z \in \prod_{j \in N \setminus S} \Omega_j$, the map

$$f_z: \prod_{j \in S} \Omega_j \to \mathbb{R}, \quad x \mapsto f(x, z)$$

is measurable, too. (Note that $S$ can be recovered from $z$, so there is no ambiguity about the domain of $f_z$.) Now, for each $j \in N$, let $\mu_j$ be a probability measure on $\Omega_j$. Set $\mu := (\mu_j)_{j \in N}$. Given a subset $B \subseteq N$, we consider the probability measure

$$P^\mu_B := \bigotimes_{j \in B} \mu_j$$

on the measurable space $\prod_{j \in B} \Omega_j$. We set

$$\mathbb{P}^\mu := P^\mu_N.$$  

With this notation, Fubini’s theorem states that, for every $\mathbb{P}^\mu$-integrable function $f: \prod_{j \in N} \Omega_j \to \mathbb{R}$ and every $B \subseteq N$, the map $f_z$ is $P^\mu_B$-integrable for $P^\mu_{N \setminus B}$-almost every $z \in \prod_{j \in N \setminus B} \Omega_j$, and

$$\int f \, d\mathbb{P}^\mu = \int \int f_z \, dP^\mu_B \, dP^\mu_{N \setminus B}(z).$$

By a standard Borel probability space, we mean a pair $(\Omega, \mu)$ consisting of a standard Borel space $\Omega$ and a probability measure $\mu$ on $\Omega$.

**Lemma 3.3.** Let $N$ be a finite non-empty set. Let $k, m \in \mathbb{N}_{\geq 1}$ and suppose that $C = (C_i)_{i \leq m} \subset \mathcal{P}(N)^m$ is a uniform $k$-cover of $N$. Consider any family of standard Borel probability spaces $(\Omega_j, \mu_j)_{j \in N}$ and let $\mu := (\mu_j)_{j \in N}$. Then, for every bounded measurable function $f: \prod_{j \in N} \Omega_j \to \mathbb{R}_{\geq 0}$,

$$\operatorname{Ent}_{\mathbb{P}^\mu}(f) \leq \frac{1}{k} \sum_{1 \leq i \leq m} \int \operatorname{Ent}_{P^\mu_{C_i}}(f_z) \, dP^\mu_{N \setminus C_i}(z).$$

**Proof.** Without loss of generality, we may assume that $N = \{0, \ldots, n - 1\}$ for some $n \in \mathbb{N}_{\geq 1}$. We abbreviate $X = \prod_{j \in N} \Omega_j$, $\mathbb{P} := \mathbb{P}^\mu$ and $P_B := P^\mu_B$ for any $B \subseteq N$. We use Corollary 2.8. To this end, let $g: X \to \mathbb{R}$ be measurable such that $\int \exp \circ g \, d\mathbb{P} \leq 1$. Since $\exp \circ g$ takes only positive values, $\int \exp(g(y, x)) \, dP_{\{0, \ldots, j\}}(y) > 0$ for all $j \in N$ and $x \in \prod_{i=1}^{j-1} \Omega_i$. Furthermore, invoking Fubini’s theorem, we find some measurable subset $S \subseteq X$ with $\mathbb{P}(S) = 1$ such that $\int \exp(g(y, x|_{\{j+1, \ldots, n-1\}})) \, dP_{\{0, \ldots, j\}}(y) < \ldots
\(\infty\) for all \(j \in \mathbb{N}\) and \(x \in S\). For each \(j \in \mathbb{N}\), consider the measurable map \(g^j: X \rightarrow \mathbb{R}\) given by

\[
g^j(x) := \ln \left( \frac{\int \exp(g(y, x_{\{j,...,n-1\}})) \, dP_{\{0,...,j-1\}}(y)}{\int \exp(g(y, x_{\{j+1,...,n-1\}})) \, dP_{\{0,...,j\}}(y)} \right)
\]

for all \(x \in S\) and \(g(x) := 0\) for all \(x \in X \setminus S\). Note that, by Fubini’s theorem, for each \(j \in \mathbb{N}\) and \(P_{N \setminus \{j\}}\)-almost every \(z \in \prod_{j' \in N \setminus \{j\}} \Omega_{j'}\),

\[
\int \exp \circ g^j \, dm_j = \int \frac{\int \exp(g(y, x_{\{j+1,...,n-1\}})) \, dP_{\{0,...,j\}}(y)}{\int \exp(g(y, x_{\{j+1,...,n-1\}})) \, dP_{\{0,...,j\}}(y)} \, dm_j(x) = 1. \tag{3}
\]

Given any non-empty subset \(B \subseteq N\), define the measurable function

\[
h^B := \sum_{j \in B} g^j: X \rightarrow \mathbb{R}.
\]

Note that \(h^B\) does not depend on the \(j\)-th coordinates with \(j < \min B\). We claim that, for every non-empty \(B \subseteq N\) and \(P_{N \setminus B}\)-almost every \(z \in \prod_{j \in N \setminus B} \Omega_j\),

\[
\int \exp \circ h^B_z \, dP_B = 1. \tag{4}
\]

The proof of (4) proceeds by induction. For a start, let \(B \subseteq N\) with \(|B| = 1\), that is, \(B = \{j\}\) for some \(j \in \mathbb{N}\). Then, for \(P_{N \setminus B}\)-almost every \(z \in \prod_{\ell \in N \setminus B} \Omega_{\ell}\),

\[
\int \exp \circ h^B_z \, dP_B = \int \exp \circ g^j \, dm_j \tag{3} = 1.
\]

For the inductive step, let \(B \subseteq N\) with \(|B| > 1\) and suppose that (4) holds for every non-empty proper subset of \(B\). Denote by \(j\) the smallest element of \(B\) and let \(B' := B \setminus \{j\}\). Then there exists a measurable subset \(T \subseteq \prod_{\ell \in N \setminus B'} \Omega_{\ell}\) with \(P_{N \setminus B'}(T) = 1\) such that, for every \(z \in T\),

\[
\int \exp \circ h^B_z \, dP_{B'} = 1. \tag{5}
\]

Due to the Measurable Projection Theorem of Castaing and Valadier [CV77, Theorem III.23] (see also [Cra02, Theorem 2.12]), the set \(T' := \{z \in N \setminus B \mid z \in T\}\) is a \(P_{N \setminus B'}\)-measurable subset of \(\prod_{\ell \in N \setminus B} \Omega_{\ell}\). For each \(z \in T'\), there exists some \(\omega \in \Omega_j = \prod_{\ell \in \{j\}} \Omega_{\ell}\) with \((\omega, z) \in T\), so that Fubini’s theorem yields that

\[
\int \exp \circ h^B_z \, dP_B = \int (\exp \circ g^j_{(y,z)}) (\exp \circ h^B_{(y,z)}) \, d(\mu_j \otimes P_{B'})
\]

\[
= \int \int (\exp \circ g^j_{(y,z)}) (\exp \circ h^B_{(y,z)}) \, d\mu_j \, dP_{B'}(y)
\]

\[
= \int \int (\exp \circ g^j_{(y,z)}) \, d\mu_j \exp (h^B_{(\omega,z)}(y)) \, dP_{B'}(y)
\]

\[
= \int \exp (h^B_{(\omega,z)}(y)) \, dP_{B'}(y) \tag{3} = 1,
\]
where the third equality follows from \( h_{\mathcal{B}'} \) not depending on the \( j \)-th coordinate. Since \( P_{N \setminus \mathcal{B}'}(T') \geq P_{N \setminus \mathcal{B}'}(T) = 1 \), this completes our induction and therefore proves (4).

Thanks to Proposition 2.7, our assertion (4) implies that, for every non-empty \( B \subseteq N \) and \( P_{N \setminus \mathcal{B}} \)-almost every \( z \in \prod_{j \in N \setminus \mathcal{B}} \Omega_j \),

\[
\int h_z^B f_z \, dP_B \leq \text{Ent}_{P_B}(f_z) .
\]

Furthermore, for each \( x \in S \),

\[
\sum_{j \in N} g^j(x) = \sum_{j \in N} \ln \left( \int \exp(g(y, x_{[j, \ldots, n-1]})) \, dP_{\{0, \ldots, j-1\}}(y) \right) - \sum_{j \in N} \ln \left( \int \exp(g(y, x_{[j+1, \ldots, n-1]})) \, dP_{\{0, \ldots, j\}}(y) \right) = g(x) - \ln \left( \int \exp \circ g \, d\mathbb{P} \right) \geq g(x) - \ln(1) = g(x).
\]

Since \( \mathcal{C} \) is a uniform \( k \)-cover of \( N \), this entails that

\[
\sum_{i<m} h^{C_i}(x) = \sum_{i<m} \sum_{j \in C_i} g^j(x) = k \sum_{j \in N} g^j(x) \geq kg(x)
\]

for every \( x \in S \), that is, \( g \leq \frac{1}{k} \sum_{i<m} h^{C_i} \) \( \mathbb{P} \)-almost everywhere. Combining this with Fubini’s theorem and (6), we conclude that

\[
\int g \, d\mathbb{P} \leq \frac{1}{k} \sum_{i<m} \int h^{C_i} \, d\mathbb{P} = \frac{1}{k} \sum_{i<m} \int (h^{C_i} f_z) \, dP_{C_i} \, dP_{N \setminus C_i}(z) = \frac{1}{k} \sum_{i<m} \int \text{Ent}_{P_{C_i}}(f_z) \, dP_{N \setminus C_i}(z).
\]

By Proposition 2.7, the conclusion follows. \( \square \)

**Corollary 3.4.** Let \( N \) be a finite non-empty set. Let \( k, m \in \mathbb{N}_{\geq 1} \) and suppose that \( \mathcal{C} = (C_i)_{i<m} \in \mathcal{P}(N)^m \) is a uniform \( k \)-cover of \( N \). Consider any family of standard Borel probability spaces \((\Omega_j, \mu_j)_{j \in N}\) and let \( \mu := (\mu_j)_{j \in N} \). Then, for every bounded measurable function \( f : \prod_{j \in N} \Omega_j \rightarrow \mathbb{R} \),

\[
\text{Ent}_\mu(\exp \circ f) \leq \frac{1}{k} \sum_{i<m} \iint (f_{x}(x) - f_z(y))^2 \exp(f_z(x)) \, dP_{C_i}(y) \, dP_{N \setminus C_i}(z).
\]

**Proof.** This is an immediate consequence of Lemma 3.3 and Lemma 2.9. \( \square \)

Next up, we introduce a pseudo-metric on the product of a family of sets naturally associated with any weighted covering of the underlying index set.

**Definition 3.5.** Let \( N \) be a finite non-empty set. Let \( m \in \mathbb{N}_{\geq 1} \) and suppose that \( \mathcal{C} = (C_i)_{i<m} \in \mathcal{P}(N)^m \) is a cover of \( N \). Moreover, let \( w = (w_i)_{i<m} \) be a sequence of non-negative reals. For a family of sets \((\Omega_j)_{j \in N}\), we define the pseudo-metric

\[
d_{\mathcal{C}, w} : \prod_{j \in N} \Omega_j \times \prod_{j \in N} \Omega_j \rightarrow \mathbb{R}_{\geq 0}
\]
by setting
\[ d_{C,w}(x, y) := \inf \left\{ \sum_{i \in I} w_i \left| I \subseteq m, \{ j \in N \mid x_j \neq y_j \} \leq \bigcup_{i \in I} C_i \right\} \]
for all \( x, y \in \prod_{j \in N} \Omega_j \).

Now everything is prepared to state and prove our first main result.

**Theorem 3.6.** Let \( N \) be a finite non-empty set. Let \( k, m \in \mathbb{N}_{\geq 1} \) and suppose that \( C = (C_i)_{i < m} \subseteq \mathcal{P}(N)^m \) is a \( k \)-cover of \( N \). Let \( w = (w_i)_{i < m} \) be a sequence of non-negative reals. Consider any family of standard Borel probability spaces \((\Omega_j, \mu_j)_{j \in N}\) and set \( \mu := (\mu_j)_{j \in N} \). Let \( f : \prod_{j \in N} \Omega_j \to \mathbb{R} \) be measurable and 1-Lipschitz with respect to \( d_{C,w} \). Then, for every \( r \in \mathbb{R}_{>0} \),
\[
\mathbb{P}^\mu \left( \left\{ x \in \prod_{j \in N} \Omega_j \mid f(x) - \mathbb{E}_{\mathbb{P}^\mu}(f) \geq r \right\} \right) \leq \exp \left( -\frac{kr^2}{4\|w\|_2^2} \right).
\]

**Proof.** Of course, the desired statement holds trivially if \( w = 0 \). Therefore, we may and will assume that \( w \neq 0 \). Due to Lemma 3.2, there exists a uniform \( k \)-cover \( C^* = (C^*_i)_{i < m} \subseteq \mathcal{P}(N)^m \) of \( N \) such that \( C^*_i \subseteq C_i \) for each \( i < m \). Since \( f \) is 1-Lipschitz with respect to \( d_{C,w} \),
\[
|f_z(x) - f_z(y)| \leq d_{C,w}((x,z),(y,z)) \leq w_i
\]
whenever \( i < m, x, y \in \prod_{j \in C_i} \Omega_j \) and \( z \in \prod_{j \in N \setminus C_i} \Omega_j \). As the pseudo-metric \( d_{C,w} \) is bounded, \( f \) being 1-Lipschitz with respect to \( d_{C,w} \) moreover implies that \( f \) is bounded. By Corollary 3.4 and Fubini’s theorem, it follows that, for every \( \lambda \in \mathbb{R}_{>0} \),
\[
\text{Ent}_{\mathbb{P}^\mu}(\exp(\lambda f)) \leq \frac{1}{k} \sum_{i < m} \iint_{f_z(x) \geq f_z(y)} (\lambda w_i)^2 \exp(\lambda f_z(x)) \, dP^\mu_{C_i}(y) \, dP^\mu_{C_i}(x) \, dP^\mu_{N \setminus C_i}(z)
\]
\[
\leq \frac{1}{k} \sum_{i < m} (\lambda w_i)^2 \iint \exp(\lambda f_z(x)) \, dP^\mu_{C_i}(x) \, dP^\mu_{N \setminus C_i}(z)
\]
\[
= \frac{1}{k} \sum_{i < m} (\lambda w_i)^2 \iint \exp(\lambda f) \, d\mathbb{P}^\mu = \frac{\lambda^2 \|w\|_2^2}{k} \iint \exp(\lambda f) \, d\mathbb{P}^\mu.
\]

Using Proposition 2.10 and Proposition 2.11 with \( D := \frac{2\|w\|_2^2}{k} \) gives the conclusion. \( \square \)

**Corollary 3.7.** Let \( N \) be a finite non-empty set. Let \( k, m \in \mathbb{N}_{\geq 1} \) and suppose that \( C = (C_i)_{i < m} \subseteq \mathcal{P}(N)^m \) is a \( k \)-cover of \( N \). Let \( w = (w_i)_{i < m} \) be a sequence of non-negative reals. Consider any family of standard Borel probability spaces \((\Omega_j, \mu_j)_{j \in N}\). Let \( X := \prod_{j \in N} \Omega_j \) and \( \mathbb{P} := \bigotimes_{j \in N} \mu_j \). Then, for every \( r \in \mathbb{R}_{>0} \),
\[
\alpha(X,d_{C,w},\mathbb{P})(r) \leq \exp \left( -\frac{kr^2}{8\|w\|_2^2} \right).
\]

**Proof.** This is an immediate consequence of Theorem 3.6 and Proposition 2.3. \( \square \)
4. A CLASSIFICATION OF SUBMEASURES

Our objective in this section is to give a quantitative classification of diffused submeasures in terms of the asymptotics of weighted covering ratios (as detailed in Definition 4.5 and Theorem 4.6). We start with recalling the notion of submeasure and various standard definitions concerning this concept.

**Definition 4.1.** Let $X$ be a set and denote by $\mathcal{P}(X)$ the power set of $X$. Consider a Boolean subalgebra $\mathcal{A}$ of $\mathcal{P}(X)$. A function $\phi: \mathcal{A} \to \mathbb{R}$ is called a submeasure if

- $\phi(\emptyset) = 0$,
- $\phi$ is monotone, that is, $\phi(A) \leq \phi(B)$ for all $A, B \in \mathcal{A}$ with $A \subseteq B$, and
- $\phi$ is subadditive, that is, $\phi(A \cup B) \leq \phi(A) + \phi(B)$ for all $A, B \in \mathcal{A}$.

Let $\phi: \mathcal{A} \to \mathbb{R}$ be a submeasure. Then $\phi$ is called a measure if $\phi(A \cup B) = \phi(A) + \mu(B)$ for any two disjoint $A, B \in \mathcal{A}$. The submeasure $\phi$ is called pathological if there does not exist a non-zero measure $\mu: \mathcal{A} \to \mathbb{R}$ with $\mu \leq \phi$. Furthermore, $\phi$ is said to be diffused if, for every $\varepsilon > 0$, there exists a finite subset $B \subseteq \mathcal{A}$ such that $X = \bigcup B$ and $\phi(B) \leq \varepsilon$ for each $B \in B$.

Our classification of diffused submeasures will be formulated in terms of the asymptotic behavior of a certain function associated with any such submeasure. The definition of the latter relies on the notion of covering number (Definition 3.1).

**Definition 4.2.** Let $X$ be a set. Let $\mathcal{A}$ be a Boolean subalgebra of $\mathcal{P}(X)$ and let $\phi: \mathcal{A} \to \mathbb{R}$ be a diffused submeasure. For $\xi \in \mathbb{R}_{>0}$, let

$$A_{\phi, \xi} := \{ A \in \mathcal{A} \mid \phi(A) \leq \xi \}.$$

Define $h_{\phi}: \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ by

$$h_{\phi}(\xi) := \sup \left\{ \frac{t_X(C)}{m_\xi} \mid m \in \mathbb{N}_{\geq 1}, C \in (A_{\phi, \xi})^m \right\}.$$

Evidently, for any diffused submeasure $\phi$, the function $h_{\phi}$ is well defined, that is, $h_{\phi}$ indeed only takes values in $\mathbb{R}_{>0}$. In the definition of $h_{\phi}$, the ratio $\frac{t_X(C)}{m_\xi}$ measures how thickly $C$ covers $X$, and this ratio is divided by a normalizing factor $\xi$ to compensate for the fact that the sets in $A_{\phi, \xi}$ become smaller as $\xi$ approaches 0. As a result of Lemma 3.2, we have the following reformulation in terms of uniform covers.

**Corollary 4.3.** Let $X$ be a set. Consider a Boolean subalgebra $\mathcal{A}$ of $\mathcal{P}(X)$ and let $\phi: \mathcal{A} \to \mathbb{R}$ be a diffused submeasure. Then, for every $\xi \in \mathbb{R}_{>0}$,

$$h_{\phi}(\xi) = \sup \left\{ \frac{t_X(C)}{m_\xi} \mid m \in \mathbb{N}_{\geq 1}, C \in (A_{\phi, \xi})^m \text{ uniform over } X \right\}.$$

Furthermore, an application of the Hahn–Banach extension theorem yields the subsequent description, where $\frac{1}{0} := \infty$. Given two sets $A \subseteq X$, let $\chi_A: X \to \{0, 1\}$ denote the corresponding indicator function defined by $\chi_A(x) := 1$ for all $x \in A$ and $\chi_A(x) := 0$ for all $x \in X \setminus A$. 
**Proposition 4.4.** Let $X$ be a set. Let $\mathcal{A}$ be a Boolean subalgebra of $\mathcal{P}(X)$ and let $\phi: \mathcal{A} \to \mathbb{R}$ be a diffused submeasure. For every $\xi \in \mathbb{R}_{>0},$

$$h_\phi(\xi) = \min \left\{ \frac{1}{p(X)} \left\| \mu : \mathcal{A} \to \mathbb{R} \text{ measure with } \mathcal{A}_{\phi,\xi} \subseteq \mathcal{A}_{\mu,\xi} \right\| \right\}.$$  

**Proof.** Let $\xi \in \mathbb{R}_{>0}$ be fixed.

$(\leq)$ Consider any measure $\mu : \mathcal{A} \to \mathbb{R}$ with $\mathcal{A}_{\phi,\xi} \subseteq \mathcal{A}_{\mu,\xi}$. If $\mathcal{C} = (C_i)_{i \leq m} \in (\mathcal{A}_{\phi,\xi})^m$ for some $m \in \mathbb{N}_{\geq 1}$, then

$$t_X(\mathcal{C}) \mu(X) \leq \sum_{i < m} \mu(C_i) \leq m \xi$$

and thus $\frac{t_X(\mathcal{C})}{m \xi} \leq \frac{1}{p(X)}$. Therefore, $h_\phi(\xi) \leq \frac{1}{p(X)}$ as desired.

$(\geq)$ Consider the seminorm $p : \ell^\infty(X) \to \mathbb{R}_{\geq 0}$ defined by

$$p(f) := \inf \left\{ \xi \sum_{i < m} r_i \left| m \in \mathbb{N}, (r_i)_{i < m} \in (\mathbb{R}_{\geq 0})^m, (B_i)_{i < m} \in (\mathcal{A}_{\phi,\xi})^m, \right. \right.$$  

$$\left| f \right| \leq \sum_{i < m} r_i \chi_{B_i} \right\}$$

for every $f \in \ell^\infty(X)$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, it follows that

$$p(\chi_X) = \inf \left\{ \frac{\xi m}{m k} \left| m, k \in \mathbb{N}_{\geq 1}, (B_i)_{i < m} \in (\mathcal{A}_{\phi,\xi})^m, \chi_X \leq \frac{1}{k} \sum_{i < m} \chi_{B_i} \right. \right\} = h_\phi(\xi)^{-1}.$$  

Concerning the linear functional $I : \mathbb{R}\chi_X \to \mathbb{R}, \ r\chi_X \mapsto rh_\phi(\xi)^{-1}$, we note that

$$|I(r\chi_X)| = |rh_\phi(\xi)^{-1}| = |r|h_\phi(\xi)^{-1} = |r|p(\chi_X) = p(r\chi_X)$$

for all $r \in \mathbb{R}$. Therefore, the Hahn–Banach extension theorem asserts the existence of a linear functional $J : \ell^\infty(X) \to \mathbb{R}$ such that $J(\chi_X) = h_\phi(\xi)^{-1}$ and $|J(f)| \leq p(f)$ for every $f \in \ell^\infty(X)$. Let us define

$$\mu : \mathcal{A} \to \mathbb{R}, \ A \mapsto J(\chi_A)$$

and observe that $\mu(\emptyset) = 0$ and $\mu(X) = h_\phi(\xi)^{-1}$, and moreover $\mu(A \cup B) = \mu(A) + \mu(B)$ for any two disjoint $A, B \in \mathcal{A}$. Straightforward calculations now show that

$$\mu^+ : \mathcal{A} \to \mathbb{R}_{\geq 0}, \ A \mapsto \sup \{ \mu(B) \mid A \supseteq B \in \mathcal{A} \}$$

constitutes a measure (we refer to [RR83, Theorem 2.2.1(4)] for the details). Furthermore, since $p(\chi_B) \leq p(\chi_A)$ for any $B \subseteq A \subseteq X$, it follows that

$$\mu^+(A) = \sup \{ J(\chi_B) \mid A \supseteq B \in \mathcal{A} \} \leq \sup \{ p(\chi_B) \mid A \supseteq B \in \mathcal{A} \} \leq p(\chi_A)$$

for every $A \in \mathcal{A}$. Therefore, if $A \in \mathcal{A}_{\phi,\xi}$, then $\mu^+(A) \leq p(\chi_A) \leq \xi$, hence $A \in \mathcal{A}_{\mu^+,\xi}$. Finally, let us observe that $\mu^+(X) \leq p(\chi_X) = h_\phi(\xi)^{-1} = \mu(X) \leq \mu^+(X)$, which means that $\mu^+(X) = h_\phi(\xi)^{-1}$. This completes the proof. \[\square\]

The asymptotic behavior of $h_\phi(\xi)$ as $\xi \to 0$ will be fundamental to our considerations. We introduce the following terminology using Landau’s big $O$ notation. Let us recall that, for two functions $f, g : \mathbb{R}_{>0} \to \mathbb{R}_{>0},$

$$f(x) = O(g(x)) \text{ as } x \to 0 \iff \limsup_{x \to 0} \frac{f(x)}{g(x)} < \infty.$$  

**Definition 4.5.** A diffused submeasure $\phi$ is called
— elliptic if \( h_\phi(\xi) = O(\xi) \) as \( \xi \to 0 \),
— hyperbolic if \( \frac{1}{h_\phi(\xi)} = O(\xi) \) as \( \xi \to 0 \),
— parabolic if \( \phi \) is neither elliptic, nor hyperbolic.

Evidently, the three notions defined above are mutually exclusive. We note that a diffused submeasure \( \phi \) is elliptic if and only if
\[
\sup_{\xi \in \mathbb{R}^+} \frac{h_\phi(\xi)}{\xi} < \infty.
\]

Clearly, the latter implies the former. Conversely, \( \frac{h_\phi(\xi)}{\xi} \leq \frac{1}{\xi^2} \) for all \( \xi \in \mathbb{R}^+ \), so that
\[
\limsup_{\xi \to 0} \frac{h_\phi(\xi)}{\xi} < \infty \implies \sup_{\xi \in \mathbb{R}^+} \frac{h_\phi(\xi)}{\xi} < \infty.
\]

The subsequent theorem is the main result of this section. It gives initial justification to the importance of the function introduced in Definition 4.5.

**Theorem 4.6.** Let \( \phi \) be a diffused submeasure.

(A) The following conditions are equivalent.

— \( \phi \) is hyperbolic;
— \( \phi \) is pathological;
— \( h_\phi \) is unbounded;
— \( \lim_{\xi \to 0} \xi h_\phi(\xi) = 1 \).

(B) If \( \phi \) is parabolic, then \( \lim_{\xi \to 0} h_\phi(\xi) \) exists and is finite.

(C) If \( \phi \) is elliptic, then \( \lim_{\xi \to 0} h_\phi(\xi) = 0 \).

(D) If \( \phi \) is a measure, then \( \lim_{\xi \to 0} h_\phi(\xi) = \frac{1}{\phi(X)} = \infty \).

It follows immediately from (A), (C) and (D) in Theorem 4.6 that every non-zero diffused measure constitutes a parabolic submeasure. Of course, a zero measure is hyperbolic. The converses to (B) and (C) do not hold. A family of elliptic submeasures, the existence of which clearly witnesses that the implication in (B) cannot be reversed, is constructed in Example 5.7. For an instance of a parabolic submeasure \( \phi \) with \( \lim_{\xi \to 0} h_\phi(\xi) = 0 \), illustrating the failure of the converse to (C), see Example 4.10.

We remark here that (A) in Theorem 4.6 is essentially a reformulation of the following characterization of pathological submeasures due to Christensen [Chr78].

**Theorem 4.7 ([Chr78], Theorem 5).** Let \( X \) be a set and \( \mathcal{A} \) be a Boolean subalgebra of \( \mathcal{P}(X) \). If \( \phi: \mathcal{A} \to \mathbb{R} \) is a pathological submeasure, then for every \( \xi \in \mathbb{R}^+ \) there exist \( m \in \mathbb{N} \geq 1 \), \( C_0, \ldots, C_{m-1} \in \mathcal{A}_{\phi,\xi} \) and \( a_0, \ldots, a_{m-1} \in \mathbb{R}^+ \geq 0 \) such that \( \sum_{i < m} a_i = 1 \) and \( \sum_{i < m} a_i \chi_{C_i} \geq 1 - \xi \).

Since \( \mathbb{Q} \) is dense in \( \mathbb{R} \), Christensen’s Theorem 4.7 immediately entails the following corollary, which constitutes the essential ingredient in the proof of (A) in Theorem 4.6.

**Corollary 4.8.** Let \( X \) be a set and let \( \mathcal{A} \) be a Boolean subalgebra of \( \mathcal{P}(X) \). If \( \phi: \mathcal{A} \to \mathbb{R} \) is a pathological submeasure, then for every \( \xi \in \mathbb{R}^+ \) there exist \( m \in \mathbb{N} \geq 1 \) and \( \mathcal{C} \in (\mathcal{A}_{\phi,\xi})^m \) such that \( \frac{1_{\mathcal{C}}(\xi)}{m} \geq 1 - \xi \).
The proof of (B) in Theorem 4.6 relies on the following general convergence result.

**Lemma 4.9.** Let \( f : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0} \). If \( \sup_{\xi \in \mathbb{R}_{>0}} \frac{f(\xi)}{\xi} < \infty \) and, for all \( \xi, \zeta \in \mathbb{R}_{>0} \),
\[
 f(\xi + \zeta) \geq f(\xi) + f(\zeta) - f(\xi)f(\zeta),
\]
then \( \lim_{\zeta \to 0} \frac{f(\xi)}{\zeta} \) exists and is finite.

**Proof.** Let \( M := \sup \left\{ \frac{f(\xi)}{\xi} \mid \xi \in \mathbb{R}_{>0} \right\} \). For a start, we prove that
\[
\forall \xi \in \mathbb{R}_{>0} \forall k \in \mathbb{N}_{\geq 1}: \quad \frac{f(k\xi)}{k\xi} \geq \frac{f(\xi)}{\xi} - M^2 k\xi. \tag{7}
\]
Let \( \xi \in \mathbb{R}_{>0} \). We prove the inequality by induction over \( k \in \mathbb{N}_{\geq 1} \). Clearly, if \( k = 1 \), then the desired statement holds trivially. Furthermore, if \( \frac{f(k\xi)}{k\xi} \geq \frac{f(\xi)}{\xi} - M^2 k\xi \) for some \( k \in \mathbb{N}_{\geq 1} \), then
\[
 f((k + 1)\xi) \geq f(k\xi) + f(\xi) - f(k\xi)f(\xi) \geq f(k\xi) + f(\xi) - M^2 k^2 k
\[
 \geq k f(\xi) - M^2 k^2 \xi^2 + f(\xi) - M^2 \xi^2 k = (k + 1) f(\xi) - M^2 \xi^2 (k^2 + k)
\[
 \geq (k + 1) f(\xi) - M^2 \xi^2 (k + 1)^2,
\]
that is, \( \frac{f((k+1)\xi)}{(k+1)\xi} \geq \frac{f(\xi)}{\xi} - M^2 \xi (k+1) \). This completes our induction and therefore proves (7).

Let \( L := \limsup_{\zeta \to 0} \frac{f(\xi)}{\zeta} \). Clearly, \( L \leq M < \infty \). We prove that \( \frac{f(\xi)}{\xi} \to L \) as \( \xi \to 0 \). Of course, if \( \frac{f(\xi)}{\xi} \geq L \), then \( \xi \) holds trivially if \( L = 0 \). So, assume that \( L > 0 \). Fix \( \varepsilon \in (0, L) \). It will suffice to show that
\[
\xi \in \left(0, \frac{\varepsilon}{2M^2 + 1}\right) \implies \frac{f(\xi)}{\xi} > (1 - \varepsilon)(L - \varepsilon). \tag{8}
\]
By definition of \( L \), there exists \( \xi \in (0, \xi) \) such that \( \frac{f(\xi)}{\xi} > L - \frac{\varepsilon}{2} \) and \( \frac{\xi}{\zeta} + 1 > 1 - \varepsilon \). Let \( k := \lfloor \xi/\zeta \rfloor \), so that \( \xi = k\zeta + r \) for some \( r \in [0, \zeta) \). Note that \( \frac{k\zeta}{\xi} \geq \frac{k}{k+1} > 1 - \varepsilon \).

It follows that
\[
\frac{f(\xi)}{\xi} \geq \frac{1}{k\zeta + r} (f(k\zeta) + f(r) - f(k\zeta)f(r)) \geq \frac{k\zeta}{k(\zeta + r)} \left( \frac{f(k\zeta)}{k\zeta} - \frac{f(k\zeta)}{k\zeta}f(r) \right)
\[
\overset{(7)}{\geq} \frac{k\zeta}{k\zeta + r} \left( \frac{f(\xi)}{\xi} - M^2 k\zeta \right) \geq \frac{k\zeta}{k\zeta + r} \left( \frac{f(\xi)}{\xi} - M^2 \xi \right) > (1 - \varepsilon)(L - \varepsilon).
\]
This proves (8) and thus completes our proof. \( \square \)

**Proof of Theorem 4.6.** Let \( X \) be a set and let \( \mathcal{A} \) be a Boolean subalgebra of \( \mathcal{P}(X) \). Consider a diffused submeasure \( \phi : \mathcal{A} \to \mathbb{R} \).

(A) For a start, let us note that \( h_\phi(\xi) \leq \frac{1}{\xi} \) for every \( \xi \in \mathbb{R}_{>0} \). Now, if \( \phi \) is pathological, then Corollary 4.8 yields that
\[
h_\phi(\xi) \geq \frac{1-\varepsilon}{\xi}
\]
for all \( \xi \in \mathbb{R}_{>0} \), which therefore entails that \( \xi h_\phi(\xi) \to 1 \) as \( \xi \to 0 \). The latter condition clearly implies that \( \phi \) is hyperbolic. Furthermore, if \( \phi \) is hyperbolic, then \( h_\phi \) must be unbounded. It only remains to argue that, if \( \phi \) is unbounded, then \( \phi \)
will be pathological. To this end, let us assume that \( \phi \) is non-pathological, that is, there exists a measure \( \mu : A \to \mathbb{R} \) with \( 0 \neq \mu \leq \phi \). Then Proposition 4.4 entails that \( h_\phi(\xi) \leq \frac{1}{\mu(X)} \) for all \( \xi \in \mathbb{R}_{>0} \). In particular, \( h_\phi \) is bounded. This proves (A).

(B) Suppose that \( \phi \) is parabolic. Since \( \phi \) is not hyperbolic, \( h_\phi \) is bounded by (A). Consider the function

\[
f : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}, \quad \xi \mapsto \xi h_\phi(\xi).
\]

We prove that, for all \( \xi, \zeta \in \mathbb{R}_{>0} \),

\[
f(\xi + \zeta) \geq f(\xi) + f(\zeta) - f(\xi) \cdot f(\zeta).
\]

(9)

For this purpose, fix \( \varepsilon \in \mathbb{R}_{>0} \). Due to Lemma 3.2, there exist \( k_\xi, k_\zeta, m_\xi, m_\zeta \in \mathbb{N}_{\geq 1} \), some uniform \( k_\xi \)-cover \( C_\xi = (C_{\xi,i})_{i \leq m_\xi} \in (A_\phi)_m \) of \( X \), as well as some uniform \( k_\zeta \)-cover \( C_\zeta = (C_{\zeta,i})_{i \leq m_\zeta} \in (A_\phi)_m \) of \( X \) such that

\[
(1 - \varepsilon)f(\xi) \leq \frac{k_\xi}{m_\xi} \leq f(\xi), \quad (1 - \varepsilon)f(\zeta) \leq \frac{k_\zeta}{m_\zeta} \leq f(\zeta).
\]

(10)

Put \( m := m_\xi \cdot m_\zeta \). Let us define a sequence \( B := (B_\xi)_{\xi \leq m} \in \mathbb{A}^m \) by setting, for each pair \( (i, j) \in \{0, \ldots, m_\xi - 1\} \times \{0, \ldots, m_\zeta - 1\} \),

\[
B_{m_\xi + j} := C_{\xi,i} \cup C_{\zeta,j}.
\]

As \( \phi \) is a submeasure, \( B \) belongs to \( (A_\phi)_m \). Since \( C_\xi \) is a uniform \( k_\xi \)-cover of \( X \) and \( C_\zeta \) is a uniform \( k_\zeta \)-cover of \( X \), it follows that, for each \( x \in X \),

\[
|\{ \ell \leq m \mid x \in B_\ell \}| = |\{(i, j) \mid i \leq m_\xi, j \leq m_\zeta, x \in C_{\xi,i} \cup C_{\zeta,j}\}|
\]

\[
= |\{i \leq m_\xi \mid x \in C_{\xi,i}\}| \cdot m_\zeta + m_\xi \cdot |\{j \leq m_\zeta \mid x \in C_{\zeta,j}\}|
\]

\[
- |\{i \leq m_\xi \mid x \in C_{\xi,i}\}| \cdot |\{j \leq m_\zeta \mid x \in C_{\zeta,j}\}|
\]

\[
= k_\xi \cdot m_\zeta + m_\xi \cdot k_\zeta - k_\xi \cdot k_\zeta.
\]

Thus, appealing to (10), we conclude that

\[
f(\xi + \zeta) \geq \frac{k_\xi \cdot m_\zeta + m_\xi \cdot k_\zeta - k_\xi \cdot k_\zeta}{m_\xi \cdot m_\zeta} = \frac{k_\xi}{m_\xi} + \frac{k_\zeta}{m_\zeta} \cdot \frac{k_\xi}{m_\xi} \cdot \frac{k_\zeta}{m_\zeta} \geq (1 - \varepsilon)(f(\xi) + f(\zeta)) - f(\xi) \cdot f(\zeta).
\]

This proves (9). Since the function \( h_\phi \) is bounded, assertion (9) and Lemma 4.9 together imply the desired conclusion.

(C) is obvious. 

(D) Of course, if \( \phi = 0 \), then \( \phi \) is pathological, thus hyperbolic by (A), and therefore

\[
\lim_{\xi \to 0} h_\phi(\xi) = \lim_{\xi \to 0} \frac{1}{\xi} \cdot \frac{1}{\xi} = \infty.
\]

Suppose now that \( \phi \) is a non-zero measure. In particular, \( \phi \) is non-pathological. This implies, by (B) and (C), the existence of the limit \( a := \lim_{\xi \to 0} h_\phi(\xi) \in \mathbb{R} \). We will prove that \( a = \frac{1}{\phi(X)} \). By Proposition 4.4, we have \( h_\phi(\xi) \leq \frac{1}{\phi(X)} \) for every \( \xi \in \mathbb{R}_{>0} \). Hence, \( a \leq \frac{1}{\phi(X)} \). To prove the reverse inequality, we will show that

\[
\forall \theta \in \mathbb{R}_{>1} \forall n \in \mathbb{N}_{\geq 1} : \quad h_\phi \left( \frac{\theta \phi(X)}{n} \right) \geq \frac{1}{\theta \phi(X)}.
\]

(11)
To this end, let $\theta \in \mathbb{R}_{>1}$ and $n \in \mathbb{N}_{\geq 1}$. Since $\phi$ is diffused, $X$ admits a partition $B \subseteq A$ such that $\phi(B) \leq (\theta - 1)\frac{\phi(X)}{n}$ for every $B \in B$. Note that, if $B' \subseteq B$ and $\phi(\bigcup B') < \frac{\phi(X)}{n}$, then

$$\phi\left(B \cup \bigcup B'\right) \leq \phi(B) + \phi\left(\bigcup B'\right) < (\theta - 1)\frac{\phi(X)}{n} + \frac{\phi(X)}{n} = \theta \frac{\phi(X)}{n}$$

for any $B \in B \setminus B'$. Using this observation, one can select a sequence of pairwise disjoint subsets $B_0, \ldots, B_{n-1} \subseteq B$ such that $B_i \subseteq B$ and $\phi(\bigcup B_i) < \frac{\theta \phi(X)}{n}$ for each $i < n$. Consider the sequence $C := (C_i)_{i<n} \subseteq A$ given by $C_i := \bigcup B_i$ for each $i < n$. As $\phi(C_i) < \frac{\theta \phi(X)}{n}$ for all $i < n$,

$$h_\phi\left(\frac{\theta \phi(X)}{n}\right) \geq \frac{t_{\phi}(C)}{n(\theta \phi(X))/n} = \frac{1}{\theta \phi(X)}.$$  

This proves (11). From (11), we now infer that

$$a = \lim_{n \to \infty} h_\phi\left(\frac{\theta \phi(X)}{n}\right) \geq \frac{1}{\theta \phi(X)}$$

for every $\theta \in \mathbb{R}_{>1}$. Thus, $a \geq \frac{1}{\theta \phi(X)}$ as desired. 

Below, we describe an example of a diffused submeasure that shows that the converse to the implication in (C) of Theorem 4.6 fails to hold. It is a parabolic submeasure that is far from being a measure.

**Example 4.10.** There exists a diffused submeasure $\phi$ such that

(i) $\phi$ is parabolic, and  
(ii) $\lim_{\xi \to 0} h_\phi(\xi) = 0$.

The submeasure $\phi$ will be defined on the Boolean algebra $A$ of all clopen subsets of the topological product space $X := \coprod_{n=0}^{\infty} K_n$ for an appropriate choice of positive integers $(K_n)_{n \in \mathbb{N}}$. To guarantee that $\phi$ is not elliptic, as implied by point (i), we need to make sure that

$$\limsup_{\xi \to 0} \frac{h_\phi(\xi)}{\xi} = \infty,$$

which will follow if we find a sequence $(B_n)_{n \in \mathbb{N}}$ of partitions of $X$ into clopen sets and a sequence $(\xi_n)_{n \in \mathbb{N}}$ of positive real numbers such that

$$\phi(A) \leq \xi_n \text{ for all } n \in \mathbb{N} \text{ and } A \in B_n, \text{ and}$$

$$\lim_{n \to \infty} \frac{|B_n|^2}{\xi_n} = 0. \quad (12)$$

Note that the above condition implies that $\lim_{n \to \infty} \xi_n = 0$ and, in turn, that $\phi$ will be diffused. To furthermore guarantee point (ii) and, in turn, prove the remaining part of point (i), by Proposition 4.4 and the convergence established in Theorem 4.6, it will suffice to find a sequence $(\mu_n)_{n \geq 1}$ of measures on $A$ such that, for the sequence $(\xi_n)_{n \in \mathbb{N}}$ as above,

for all $A \in A$ and $n \geq 1$, if $\phi(A) \leq \xi_n$, then $\mu_n(A) \leq \xi_n$, and

$$\lim_{n \to \infty} \mu_n(X) = \infty. \quad (13)$$
We take a sequence \((M_n)_{n \geq 1}\) of natural numbers such that, for each \(n \geq 1\),
\[
n|M_n|, 1 \leq \frac{M_n}{n} \leq \frac{\sqrt{M_{n+1}}}{n+1}, \quad \text{and} \lim_{n \to \infty} \frac{M_n}{n} = \infty.
\] (14)
So, for example, letting \(M_n := n^3\) for each \(n \geq 1\) will work. We set
\[
K_0 := 1 \quad \text{and} \quad K_n := \frac{M_n}{n} \quad \text{for each} \quad n \geq 1
\]
in the above definition of \(X\). We also set
\[
\xi_0 := 1 \quad \text{and} \quad \xi_n := \frac{n}{\sqrt{M_n}} \quad \text{for each} \quad n \geq 1.
\]

For \(n \in \mathbb{N}\) and \(i < K_n\), let
\[
[i, n] := \{x \in X \mid x_n = i\}.
\]
Furthermore, consider the set of finite sequences
\[
S := \{(i_k, n_k)_{k=1}^p \mid p \in \mathbb{N}, n_1, \ldots, n_p \in \mathbb{N}, i_1 < K_{n_1}, \ldots, i_p < K_{n_p}\}.
\]
We define \(\phi : \mathcal{A} \to \mathbb{R}\) by setting
\[
\phi(A) := \inf \left\{ \left(\sum_{k=1}^p \xi_{n_k} \right) \mid (i_k, n_k)_{k=1}^p \in S, A \subseteq \bigcup_{k=1}^p [i_k, n_k] \right\}
\] (15)
for every \(A \in \mathcal{A}\). Clearly, \(\phi : \mathcal{A} \to \mathbb{R}\) is a submeasure. We have the following claim that asserts that the infimum in (15) is attained.

**Claim.** Let \(A \in \mathcal{A}\). There exists \((i_k, n_k)_{k=1}^p \in S\) such that
\[
A \subseteq \bigcup_{k=1}^p [i_k, n_k] \quad \text{and} \quad \phi(A) = \sum_{k=1}^p \xi_{n_k}.
\]

**Proof of Claim.** Since \(A\) is clopen, there exists a natural number \(N\) such that, for \(x, y \in X\), if \(x_n = y_n\) for all \(n \leq N\), then \(x \in A\) if and only if \(y \in A\). Fix such an \(N\) for the remainder of the proof of the claim.

If \(\phi(A) \geq 1\), it suffices to take \(p = 1\) and \(n_1 = i_1 = 0\). So, let us assume \(\phi(A) < 1\). It will suffice to show that, for every sequence \((i_k, n_k)_{k=1}^p \in S\), if
\[
A \subseteq \bigcup_{k=1}^p [i_k, n_k] \quad \text{and} \quad \sum_{k=1}^p \xi_{n_k} < 1,
\] (16)
then
\[
A \subseteq \bigcup \{[i_k, n_k] \mid k \in \{1, \ldots, p\}, n_k \leq N\}.
\]

Assume, towards a contradiction, that there is a sequence \((i_k, n_k)_{k=1}^p \in S\) for which the above implication fails. By the choice of \(N\), we can find \(s \in \prod_{n=0}^N K_n\) such that
\[
B \subseteq A \quad \text{and} \quad B \cap \bigcup \{[i_k, n_k] \mid k \in \{1, \ldots, p\}, n_k \leq N\} = \emptyset,
\] (17)
where \(B := \{x \in \prod_{n=0}^N K_n \mid x|_N = s\}\). Note that there is \(n > N\) such that \(\forall i < K_n \exists k \in \{1, \ldots, p\} : i = i_k \text{ and } n = n_k\). (18)
Otherwise, we can produce \(y \in \prod_{n=0}^N K_n\) such that
\[
y|_N = s \quad \text{and} \quad y \not\in \bigcup \{[i_k, n_k] \mid k \in \{1, \ldots, p\}, n_k > N\},
\]
which, by (17), implies that \( y \in A \) and \( y \notin \bigcup_{k=1}^{p}[i_k, n_k] \), leading to a contradiction with (16). So, fix \( n > N \) such that (18) holds. Then, by (14), we have
\[
\sum_{k=1}^{p} \xi_{n_k} \geq K_n \xi_n = \frac{M_n}{n \sqrt{M_n}} = \frac{\sqrt{M_n}}{n} \geq 1,
\]
contradicting (16). The claim follows.

We claim that \( \phi \) satisfies conditions (12) and (13), and therefore (i) and (ii). To see (12), for each \( n \in \mathbb{N} \), consider the partition
\[
\mathcal{B}_n := \{ [i, n] \mid i < K_n \} \in \Pi(A),
\]
and note that \( \phi([i, n]) \leq \xi_n \) for all \( i \in K_n \), and moreover
\[
\lim_{n \to \infty} |\mathcal{B}_n| \xi_n^2 = \lim_{n \to \infty} \frac{M_n}{n} \left( \frac{1}{\sqrt{M_n}} \right)^2 = \lim_{n \to \infty} \frac{1}{n} = 0.
\]
To see (13), for each \( n \geq 1 \), we consider the product measure
\[
\mu_n := \bigotimes_{j=1}^{\infty} \nu_{n,j},
\]
where for \( j \neq n \), \( \nu_{n,j} \) is the measure on \( K_j \) assigning weight \( \frac{1}{K_j} = \frac{1}{M_n} \) to each singleton \( \{i\} \) for \( i < K_j \), while \( \nu_{n,n} \) is the measure on \( K_n \) assigning weight \( \frac{1}{\sqrt{M_n}} \) to each singleton \( \{i\} \) for \( i < K_n \). So, for \( j \neq n \), \( \nu_{n,j} \) is a probability measure, while the total mass of \( \nu_{n,n} \) is equal to
\[
K_n \frac{1}{\sqrt{M_n}} = \frac{M_n}{\sqrt{M_n}} = \frac{\sqrt{M_n}}{n}.
\]
It follows that
\[
\mu_n(X) = \frac{\sqrt{M_n}}{n},
\]
so \( \lim_{n \to \infty} \mu_n(X) = \infty \).

It only remains to see that, for each \( n \geq 1 \) and each \( A \in \mathcal{A} \), if \( \phi(A) \leq \xi_n \), then \( \mu_n(A) \leq \xi_n \); we will actually show that
\[
\phi(A) \leq \xi_n \implies \mu_n(A) \leq \phi(A).
\]
To this end, fix \( n \geq 1 \), which will remain fixed for the remainder of the example. First, we point out that since \( \mu_n \) is a measure, it follows from (19) that, for all \( j \geq 1 \) and \( i < K_j \),
\[
\mu_n([i, j]) = \frac{\sqrt{M_n}}{K_j} = \frac{\sqrt{M_n}}{\sqrt{M_n}} \frac{1}{n}.
\]
Now, let us call a sequence \( (i_k, n_k)_{k=1}^{p} \in S \) tight if
\[
\phi\left( \bigcup_{k=1}^{p} [i_k, n_k] \right) = \sum_{k=1}^{p} \xi_{n_k}.
\]
We claim that, for every tight sequence \( (i_k, n_k)_{k=1}^{p} \in S \),
\[
\phi\left( \bigcup_{k=1}^{p} [i_k, n_k] \right) \leq \xi_n \implies \mu_n\left( \bigcup_{k=1}^{p} [i_k, n_k] \right) \leq \phi\left( \bigcup_{k=1}^{p} [i_k, n_k] \right).
\]
We prove (22) by induction on \( p \), with the usual convention for \( p = 0 \): the sequence is empty, it is tight, and the implication (22) holds since \( \bigcup_{k=1}^{p} [i_k, n_k] = \emptyset \). So, fix
\[ p \geq 0 \text{ and assume that (22) holds for } p; \text{ we prove it for } p + 1. \text{ Let } (i_k, n_k)_{k=1}^{p+1} \in S \text{ be a tight sequence. Set}\]
\[ C := \bigcup_{k=1}^{p+1} [i_k, n_k] \text{ and } B := \bigcup_{k=1}^{p} [i_k, n_k]. \]
Note that \( (i_k, n_k)_{k=1}^{p+1} \) is tight since otherwise, \( p > 0 \) and \( \phi(B) < \sum_{k=1}^{p} \xi_n, \) so
\[ \phi(C) \leq \phi(B) + \phi([i_{p+1}, n_{p+1}]) \leq \phi(B) + \xi_{n_{p+1}} < \sum_{k=1}^{p+1} \xi_n, \]
a contradiction. Thus, by inductive assumption, it follows that
\[ \phi(C) = \sum_{k=1}^{p} \xi_n + \xi_{n_{p+1}} = \phi(B) + \xi_{n_{p+1}} \geq \mu_n(B) + \xi_{n_{p+1}} = \mu_n(B) + \frac{1}{\sqrt{M_{n_{p+1}}}}. \] (23)

Note that since \( \xi_n \geq \phi(C) = \sum_{k=1}^{p+1} \xi_n, \) we have \( n_{p+1} \geq n. \) Using this inequality and (14), we see that
\[ \frac{1}{\sqrt{M_{n_{p+1}}}} \geq \frac{1}{M_{n_{p+1}}} \cdot \frac{n}{n_{p+1}}. \]
Thus, continuing with (23) and using (21), we get
\[ \phi(C) \geq \mu_n(B) + \frac{\sum \xi_n}{M_{n_{p+1}}} \cdot \frac{n_{p+1}}{n} = \mu_n(B) + \mu_n([i_{p+1}, n_{p+1}]) \geq \mu_n(C). \]

The inductive argument for (22) is completed.

Now, we prove (20). Fix any \( A \in \mathcal{A} \) with \( \phi(A) \leq \xi_n. \) By our Claim above, there exists a sequence \( (i_k, n_k)_{k=1}^{p+1} \) such that \( A \subseteq \bigcup_{k=1}^{p} [i_k, n_k] \) and \( \phi(A) = \sum_{k=1}^{p} \xi_n. \)
It is clear that this sequence is tight. Therefore, by (22), we have
\[ \phi(A) = \phi\left(\bigcup_{k=1}^{p} [i_k, n_k]\right) \geq \mu_n\left(\bigcup_{k=1}^{p} [i_k, n_k]\right) \geq \mu_n(A), \]
as required.

5. Lévy nets from submeasures

In this section, we combine the quantitative classification from Section 4 with the results of Section 3 to exhibit new examples of Lévy nets: we prove that any non-elliptic submeasure gives rise to a Lévy net (Theorem 5.6). For this purpose, let us introduce the following family of pseudo-metrics, the definition of which may be compared with Definition 3.5

**Definition 5.1.** Let \( X \) be a set and let \( \mathcal{A} \) be a Boolean subalgebra of \( \mathcal{P}(X). \) Define \( \Pi(\mathcal{A}) \) to be the set of all finite partitions of \( X \) into elements of \( \mathcal{A}. \) Let \( \phi: \mathcal{A} \rightarrow \mathbb{R} \) be a submeasure. For \( B \in \Pi(\mathcal{A}) \) and a set \( \Omega, \) we define a pseudo-metric
\[ \delta_{\phi,B}: \Omega^B \times \Omega^B \rightarrow \mathbb{R}_{\geq 0} \]
by setting
\[ \delta_{\phi,B}(x, y) := \phi\left(\bigcup\{B \in \mathcal{B} \mid x_B \neq y_B\}\right). \]
Given a standard Borel probability space \((\Omega, \mu)\), we let 
\[ X(\Omega, \mu, \mathcal{B}, \phi) := (\Omega^\mathcal{B}, \delta_{\phi^\mathcal{B}}, \mu^\mathcal{B}) \]

Let \(\lambda\) denote the Lebesgue measure on the standard Borel space \([0, 1] \subseteq \mathbb{R}\).

**Remark 5.2.** Let \(X\) be a set and let \(\mathcal{A}\) be a Boolean subalgebra of \(\mathcal{P}(X)\). Consider a submeasure \(\phi: \mathcal{A} \to \mathbb{R}\) and let \(\mathcal{B} \in \Pi(\mathcal{A})\). If \((\Omega_0, \mu_0)\) and \((\Omega_1, \mu_1)\) are two standard Borel probability spaces and \(\pi: \Omega_0 \to \Omega_1\) is a measurable map with \(\pi_* (\mu_0) = \mu_1\), then

\[ \hat{\pi}: (\Omega_0^\mathcal{B}, \delta_{\phi^\mathcal{B}}) \to (\Omega_1^\mathcal{B}, \delta_{\phi^\mathcal{B}}), \quad x \mapsto \pi \circ x \]

is a 1-Lipschitz map and \(\hat{\pi}_*(\mu_0^\mathcal{B}) = \mu_1^\mathcal{B}\), thus Remark 2.2(2) asserts that

\[ \alpha_{X(\Omega_1, \mu_1, \mathcal{B}, \phi)} \leq \alpha_{X(\Omega_0, \mu_0, \mathcal{B}, \phi)} \]

In particular, since for every standard Borel probability space \((\Omega, \mu)\) there exists a measurable map \(\psi: I \to \Omega\) with \(\psi_* (\lambda) = \mu\) (for instance, see [Shi16, Lemma 4.2]), this entails that

\[ \alpha_{X(\Omega, \mu, \mathcal{B}, \phi)} \leq \alpha_{X(\Omega, \lambda, \mathcal{B}, \phi)} \]

**Definition 5.3.** Let \(X\) be a set and let \(\mathcal{A}\) be a Boolean subalgebra of \(\mathcal{P}(X)\). For any two \(\mathcal{B}, \mathcal{C} \in \Pi(\mathcal{A})\),

\[ \mathcal{B} \preceq \mathcal{C} \iff \forall \mathcal{C} \in \mathcal{C} \exists \mathcal{B} \in \mathcal{B}: \mathcal{C} \subseteq \mathcal{B} \]

We say that a submeasure \(\phi: \mathcal{A} \to \mathbb{R}\) has **covering concentration** if, for every \(\varepsilon \in \mathbb{R}_{>0}\), there exists \(\mathcal{B} \in \Pi(\mathcal{A})\) such that

\[ \sup \{ \alpha_{X(\Omega, \mu, \mathcal{B}, \phi)}(\varepsilon) \mid \mathcal{C} \in \Pi(\mathcal{A}), \mathcal{B} \preceq \mathcal{C} \} \leq \varepsilon \]

**Remark 5.4.** Let \(X\) be a set and let \(\mathcal{A}\) be a Boolean subalgebra of \(\mathcal{P}(X)\). By Remark 2.2(1), a submeasure \(\phi: \mathcal{A} \to \mathbb{R}\) has covering concentration if and only if there exists a sequence \((\mathcal{C}_\ell)_{\ell \in \mathbb{N}} \in \Pi(\mathcal{A})^\mathbb{N}\) such that, for every \(\varepsilon \in \mathbb{R}_{>0}\),

\[ \sup \{ \alpha_{X(\Omega, \lambda, \mathcal{B}, \phi)}(\varepsilon) \mid \mathcal{B} \in \Pi(\mathcal{A}), \mathcal{C}_\ell \preceq \mathcal{B} \} \to 0 \text{ as } \ell \to \infty \]

For clarification, let us point out the following.

**Lemma 5.5.** *Every submeasure having covering concentration is diffused.*

**Proof.** Let \(X\) be a set and let \(\mathcal{A}\) be a Boolean subalgebra of \(\mathcal{P}(X)\). Suppose that \(\phi: \mathcal{A} \to \mathbb{R}\) is a submeasure with covering concentration. Let \(\varepsilon \in \mathbb{R}_{>0}\). By assumption, there exists \(\mathcal{B} \in \Pi(\mathcal{A})\) with \(\alpha_{X(\Omega, \lambda, \mathcal{B}, \phi)}(\varepsilon) < \frac{1}{2}\). We claim that \(\phi(B) < \varepsilon\) for each \(B \in \mathcal{B}\). To see this, let \(\mathcal{B} \in \mathcal{B}\). Note that \(\lambda^\mathcal{B}(T) = \frac{1}{2}\) for the measurable subset

\[ T := \{ x \in \mathbb{B} \mid x_B \leq \frac{1}{2} \} \subseteq \mathbb{B} \]

Now, if \(\phi(B) \geq \varepsilon\), then \(B_{\phi^\mathcal{B}}(T, \varepsilon) = T\), which implies that \(\lambda^\mathcal{B}(B_{\phi^\mathcal{B}}(T, \varepsilon)) = \frac{1}{2}\), so \(\alpha_{X(\Omega, \lambda, \mathcal{B}, \phi)}(\varepsilon) = \frac{1}{2}\), contradicting our choice of \(\mathcal{B}\). Hence, \(\phi(B) < \varepsilon\) as desired. \(\square\)

By force of Corollary 3.7, we arrive at our third main result.

**Theorem 5.6.** *Every hyperbolic or parabolic submeasure has covering concentration.*
Proof. Consider any non-elliptic diffused submeasure $\phi: \mathcal{A} \to \mathbb{R}$ and set $X := \bigcup \mathcal{A}$. Let $\varepsilon \in \mathbb{R}_{>0}$. Fix any $r \in \mathbb{R}_{\geq 0}$ with $\exp\left(-\frac{r\varepsilon^2}{16}\right) \leq \varepsilon$. By our assumption, there exists some $\xi \in \mathbb{R}_{>0}$ such that
\[
\frac{h_\phi(\xi)}{\xi} \geq r.
\] (24)

Now, we find $m \in \mathbb{N}_{>0}$ and a sequence $\mathcal{C} = (C_i)_{i<m} \in (\mathcal{A}_{\phi,\xi})^m$ such that
\[
\frac{t_X(C)}{m\varepsilon^2} \geq \frac{h_\phi(\xi)}{2\xi}.
\] (25)

Denote by $\langle C \rangle$ the partition of $X$ generated by $\mathcal{C}$, that is, $\langle C \rangle := \{\{C_i \mid i < m\}\}$, and observe that $\langle C \rangle$ belongs to $\Pi((\mathcal{A})$. If $\mathcal{B} \in \Pi((\mathcal{A})$ and $\langle C \rangle \preceq \mathcal{B}$, then the sequence $\mathcal{C}_\mathcal{B} := (C_{\mathcal{B},i})_{i<m}$, given by $C_{\mathcal{B},i} := \{B \in \mathcal{B} \mid B \subseteq C_i\}$ for all $i < m$, constitutes a $t_X(C)$-cover of $\mathcal{B}$. Furthermore, note that, by subadditivity of the submeasure $\phi$, we have $\delta_{\phi,\mathcal{B}} \leq d_{\mathcal{C}_\mathcal{B},(\phi(C_i))_{i<m}}$ on $\mathbb{P}^\mathcal{B}$. Consequently, combined with (25) and (24), Corollary 3.7 asserts that
\[
\sup\{\alpha_{X(1,\lambda,\mathcal{B},\phi)}(\varepsilon) \mid \mathcal{B} \in \Pi((\mathcal{A}), \langle C \rangle \preceq \mathcal{B}\} \leq \exp\left(-\frac{t_X(C)\varepsilon^2}{8\sum_{i<m} \phi(C_i)\xi}\right) \leq \exp\left(-\frac{h_\phi(\xi)\varepsilon^2}{16\xi}\right) \leq \exp\left(-\frac{r\varepsilon^2}{16}\right) \leq \varepsilon. \quad \Box
\]

We conclude this section by exhibiting a family of elliptic submeasures without covering concentration: in fact, we construct a diffused submeasure $\phi$ that does not have concentration and is such that $h_\phi(\xi)/\xi$ converges to 0, as $\xi \to 0$, as slowly as we wish. The example involves an application of the Berry–Esseen theorem [Ber41, Ess42] (see also [Fel71, Chapter XVI.5]). A precise statement is given below.

Example 5.7. Fix any function $\theta: \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that $\lim_{\xi \to 0} \theta(\xi) = 0$. There exists a diffused submeasure $\phi$ such that

(i) $\phi$ does not have covering concentration, and

(ii) $\limsup_{\xi \to 0} \frac{h_\phi(\xi)}{\theta(\xi)} = \infty$.

A consequence of the Berry–Esseen theorem. As a result of the Berry–Esseen theorem, there exists an increasing function $C: [1/2, 1) \to \mathbb{R}_{>0}$ with the following property: for all $a, b \in \mathbb{R}_{>0}$ with $b \leq a$ and $a + b = 1$, for every $d \in \mathbb{R}_{>0}$, and for every finite sequence $X_1, \ldots, X_n$ of independent random variables such that
\[
\forall i \in \{1, \ldots, n\}: \quad \mathbb{P}[X_i = 0] = a, \quad \mathbb{P}[X_i = 1] = b,
\]
we have
\[
\mathbb{P}\left[\frac{1}{\sqrt{n}} \sum_{i=1}^n (X_i - b) < d\right] < \frac{1}{2} + C(a)\left(d + \frac{1}{\sqrt{n}}\right). \tag{26}
\]
It follows from (26) that, if $a \in \left[\frac{1}{2}, \frac{3}{4}\right]$ and $\delta \in \mathbb{R}_{>0}$, then
\[
\mathbb{P}[\{i \in \{1, \ldots, n\} \mid X_i = 1\}] < \frac{1}{2} + \delta\sqrt{n} - \frac{1}{2} < K\left(\delta + (a - \frac{1}{2})\sqrt{n} + \frac{1}{\sqrt{n}}\right), \tag{27}
\]
where $K := \max\{C(\frac{3}{4}), 1\}$. Indeed, assuming that $a \leq \frac{3}{4}$ and substituting
\[
d := \delta + (a - \frac{1}{2})\sqrt{n}
\]
A quick calculation shows that the condition
\[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - b) < \delta + (a - \frac{1}{2}) \sqrt{n} \]
is equivalent to the condition
\[ \sum_{i=1}^{n} X_i < \frac{n}{2} + \delta \sqrt{n}, \]
which, in turn, is equivalent to the condition
\[ |\{i \in \{1, \ldots, n\} \mid X_i = 1\}| < \frac{n}{2} + \delta \sqrt{n}. \]

Putting the above equivalences together with (28), we arrive at (27).

**Defining a submeasure.** For any sequence of positive integers \( M = (M_i)_{i \in \mathbb{N}_1} \) and any sequence of positive reals \( w = (w_i)_{i \in \mathbb{N}} \), we define the submeasure
\[ \phi_{M,w}: \mathcal{P} \left( \prod_{i \in \mathbb{N}_1} M_i \right) \rightarrow \mathbb{R} \]
by setting
\[ \phi_{M,w}(A) := \inf \left\{ \sum_{s \in S} w_s \mid S \subseteq \bigcup_{i \in \mathbb{N}_1} \prod_{j=1}^{i-1} M_j, \ A \subseteq \bigcup_{s \in S} [s]_M \right\}, \]
where \([s]_M := \left\{ x \in \prod_{i \in \mathbb{N}_2} M_i \mid x|_{\{1, \ldots, i-1\}} = s \right\} \) for any \( s \in \prod_{j=1}^{i-1} M_j \) with \( i \in \mathbb{N}_1 \).

**Choosing the parameters.** To determine the submeasure \( \phi_{M,w} \) we only need to specify the two sequences \( M \) and \( w \). We pick \( M \) and \( w \) in agreement with the following three conditions:
\[ \lim_{i \to \infty} w_i = 0; \]
\[ \lim_{i \to \infty} w_i^2 M_1 \cdots M_i \theta(w_i) = 0; \]
and there exists a sequence \((\varepsilon_k)_{k \in \mathbb{N}}\) of positive reals such that
\[ \varepsilon_0 < \frac{1}{4} \quad \text{and} \quad \varepsilon_{k-1} = K \left( \frac{1}{w_k \sqrt{M_k}} + \sqrt{M_k \varepsilon_k + \frac{1}{\sqrt{w_k}}} \right) \quad \text{for all} \ k \in \mathbb{N}_1. \] (31)

Note that the last equation in (31) determines \((\varepsilon_k)_{k \in \mathbb{N}}\) from \( \varepsilon_0 \). So, given \( \varepsilon_0 \), we can define the whole sequence \((\varepsilon_k)_{k \in \mathbb{N}}\); the only issue in question is whether \( \varepsilon_k > 0 \) for all \( k \in \mathbb{N}_1 \).

The sequences \( M \) and \( w \) are constructed as follows. The constant \( K \geq 1 \) was defined above. Let \( w_0 := 1 \). Since \( \lim_{\xi \to 0} \theta(\xi) = 0 \), for each \( i \in \mathbb{N}_1 \), we find a positive real \( w_i \) so that
\[ w_i \leq 2^{-i} \quad \text{and} \quad 2^{2i+5} M_1 \cdots M_{i-1} K^i \sqrt{\theta(w_i)} < 1, \] (32)
with the usual convention that the product \( M_1 \cdots M_{i-1} \) equals 1 if \( i = 1 \). Then, using (32) and the fact that \( 1 \leq \sqrt{\frac{m+1}{\sqrt{m}}} \leq 2 \) for all \( m \in \mathbb{N}_1 \), we find a positive
integer $M_i$ so that
\[
2^i \sqrt{M_1 \cdots M_{i-1}} \sqrt{\theta(w_i)} \leq \frac{1}{w_i \sqrt{M_i}} \leq 2^{i+1} \sqrt{M_1 \cdots M_{i-1}} \sqrt{\theta(w_i)}.
\] (33)

Let us check that the chosen sequences $w$ and $M$ meet the three conditions imposed above. Evidently, (29) is satisfied due to the first assertion of (32). Also, the first inequality in (33) gives (30). The second one, together with (32), guarantees that, for each $k \in \mathbb{N}$, the series
\[
\varepsilon_k := \sum_{i=k+1}^{\infty} \left( \frac{1}{w_i} + 1 \right) \frac{\sqrt{M_{k+1} \cdots M_{i-1}}}{\sqrt{M_i}} K^{i-k}
\]
covers, and that $\varepsilon_0 < \frac{1}{4}$, again with the usual convention that the product $M_{k+1} \cdots M_{i-1}$ is equal to 1 if $i = k + 1$. It is clear that $\varepsilon_k > 0$ for each $k \in \mathbb{N}$. It is also easy to check that the sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ satisfies the equation in (31).

Let $M$ and $w$ be sequences as above. Consider the Boolean algebra $\mathcal{A}$ of all clopen subsets of topological product space $Z := \prod_{k \in \mathbb{N}_{\geq 1}} M_k$, and note that the submeasure
\[
\phi := \phi_{M,w} \lceil \mathcal{A} : \mathcal{A} \to \mathbb{R}
\]
is diffused due to (29). Additionally, for each $k \in \mathbb{N}_{\geq 1}$, let
\[
\delta_k := \frac{1}{w_k \sqrt{M_k}}
\] (34)
and consider the partition
\[
\mathcal{B}_k := \{ [s]_M \mid s \in M_1 \times \cdots \times M_k \} \subseteq \Pi(\mathcal{A}).
\]

**Lack of covering concentration.** Denote by $\mu$ the normalized counting measure on $2 = \{0,1\}$. We will prove that, for each $k \in \mathbb{N}_{\geq 1}$,
\[
\alpha_{\mathcal{X}(2,\mu,\mathcal{B}_k,\phi)}(1) \geq \frac{1}{4}.
\] (35)
By Remark 5.2 and $\{ \mathcal{B}_k \mid k \in \mathbb{N}_{\geq 1} \}$ being cofinal in $(\Pi(\mathcal{A}), \preceq)$, this will imply that $\phi : \mathcal{A} \to \mathbb{R}$ does not have covering concentration. Inequality (35) will be witnessed by the sets $\mathcal{A}'$ and $\mathcal{B}'$ defined below. The idea for the definitions of these two sets comes from [FS08, Theorem 4.2].

To prove (35), let $k \in \mathbb{N}_{\geq 1}$. Define
\[
T := M_1 \times \cdots \times M_k, \quad T_{\preceq} := \bigcup_{i=0}^{k} M_1 \times \cdots \times M_i.
\]
Furthermore, let $\mathcal{B} := \mathcal{B}_k$ and consider the bijection $f : T \to \mathcal{B}, s \mapsto [s]_M$.

First, for each $y \in 2^T$, we define an extension $\bar{y} \in 2^T_{\preceq}$ recursively as follows: let $\bar{y}(t) := y(t)$ for all $t \in T$; and if $i \in \{0, \ldots, k - 1\}$ and $\bar{y}(s)$ is defined for all $s \in T_{\preceq}$ with $|s| \geq i + 1$, then let us define
\[
\bar{y}(t) := \begin{cases} 0 & \text{if } |\{ j < M_{i+1} \mid \bar{y}(t_j) = 1\}| \leq \frac{M_{i+1}}{2}, \\ 1 & \text{otherwise}, \end{cases}
\]
for each $t \in T_{\preceq}$ with $|t| = i$. Define
\[
\mathcal{A} := \{ y \in 2^T \mid \bar{y}(\emptyset) = 0 \}.
\]
We point out that
\[ |A| \geq 2^{|T| - 1}. \quad (36) \]
To see the above inequality, consider the bijection
\[ 2^T \rightarrow 2^T, \quad y \mapsto 1 - y \]
where, for each \( s \in T \), \((1 - y)(s) := 1 - y(s)\). Now (36) is an immediate consequence (with \( t = \emptyset \)) of the implication
\[ \bar{y}(t) = 1 \implies \overline{1 - y(t)} = 0, \]
which holds for all \( t \in T \) and is proved by induction on \( k - |t| \).

Second, for each \( y \in 2^T \), define another extension \( \hat{y} \in 2^{T \leq} \) recursively as follows:

For each \( t \in T \), we let \( \hat{y}(t) = y(t) \); and if \( i \in \{0, \ldots, k - 1\} \) and \( \hat{y}(s) \) is defined for all \( s \in T \) with \( |s| \geq i + 1 \), then we define
\[ \hat{y}(t) := \begin{cases} 0 & \text{if } |\{j < M_{i+1} \mid \hat{y}(tj) = 1\}| < \frac{M_{i+1}}{2} + \delta_{i+1} \sqrt{M_{i+1}}, \\ 1 & \text{otherwise} \end{cases} \]
for each \( t \in T \) with \( |t| = i \). Define
\[ B := \{ y \in 2^T \mid \hat{y}(\emptyset) = 0 \}. \]

Since \( \delta_i \sqrt{M_i} = \frac{1}{w_c} \geq 1 \) for all \( i \in \mathbb{N}_{\geq 1} \), we have \( A \subseteq B \). We will prove that
\[ \{ y \in 2^B \mid \exists x \in A' : \delta_{\phi, B}(x, y) < 1 \} \subseteq B', \quad (37) \]
where
\[ A' := \{ x \in 2^B \mid x \circ f \in A \} \quad \text{and} \quad B' := \{ x \in 2^B \mid x \circ f \in B \}. \]

We also aim to prove that
\[ |B| \leq \frac{3}{4} 2^{|T|}. \quad (38) \]

Formulas (37) and (38) together with (36) will show (35).

We start with showing (37). To this end, let us define a binary relation \( \sim \subseteq 2^T \times 2^T \) as follows. For \( x, y \in 2^T \), we write \( x \sim y \) precisely if there exists a subset \( S \subseteq T \setminus \{\emptyset\} \) such that
\[ \forall i < k \quad \forall s \in T : \quad (|s| = i \implies |S \cap \{sj \mid j < M_{i+1}\}| < \delta_{i+1} \sqrt{M_{i+1}}), \quad (39) \]
\[ \forall t \in T : \quad (x(t) \neq y(t) \implies (\exists i \in \{0, \ldots, k\} : t \mid_{\{1, \ldots, i\}} \in S)). \]

The relation \( \sim \) is symmetric and reflexive. We prove that
\[ \{ y \in 2^T \mid \exists x \in A : x \sim y \} \subseteq B. \quad (40) \]

Inclusion (40) is proved by induction. We show the inductive step. Given the sequence \( (M_i)_{i=1}^k \) with \( k > 1 \), we consider the sequence \( (M_i)_{i=2}^k \). Define
\[ T_0 := M_2 \times \cdots \times M_k, \quad T_0 := \bigcup_{i=1}^k M_2 \times \cdots \times M_i. \]
Let $A^0$, $B^0$, and $\sim_0$ be defined in the manner analogous to $A$ and $B$ but for the sequence $(M_i)_{i=2}^k$. By induction, we assume that inclusion (40) holds for this sequence, that is,
\[ \{ y \in 2^{T_0} \mid \exists x \in A^0: x \sim_0 y \} \subseteq B^0. \] (41)

For $x \in 2^T$ and $j < M_1$, let $x_j \in 2^{T^0}$ be defined by
\[ x_j(s) := x(js). \]

The following three implications hold for all $x, y \in 2^T$:
\[ x \in A \implies \left( \lvert \{ j < M_1 \mid x_j \in A^0 \} \rvert \geq \frac{M_1}{2} \right), \] (42)
\[ \left( \lvert \{ j < M_1 \mid y_j \in B^0 \} \rvert > \frac{M_1}{2} - \delta_1 \sqrt{M_1} \right) \implies y \in B; \] (43)
\[ x \sim y \implies \left( \lvert \{ j < M_1 \mid x_j \sim_0 y_j \} \rvert > M_1 - \delta_1 \sqrt{M_1} \right). \] (44)

Implications (42) and (43) follow directly from the definitions. To get (44), observe that if $S \subseteq T_\leq \setminus \{ \emptyset \}$ witnesses that $x \sim y$, then, for $j < M_1$, if the one-element sequence whose only entry is $j$ is not in $S$, then the set
\[ \{ s \in T_0^0 \mid js \in S \} \]

witnesses that $x_j \sim_0 y_j$; thus, (44) follows since $S$ fulfills (39) (for $i = 0$). Now we show (40), that is, we aim to prove $y \in B$ assuming that $x \in A$ and $x \sim y$. By (42) and (44) we get
\[ \lvert \{ j < M_1 \mid x_j \in A^0 \text{ and } x_j \sim_0 y_j \} \rvert > \frac{M_1}{2} - \delta_1 \sqrt{M_1}. \]

Applying our inductive assumption (41) to this inequality, we get
\[ \lvert \{ j < M_1 \mid y_j \in B^0 \} \rvert > \frac{M_1}{2} - \delta_1 \sqrt{M_1}, \]
which yields $y \in B$ by (43), as required. Therefore, (40) holds.

We claim that
\[ \forall x, y \in 2^B: \ d_{B,\phi}(x, y) < 1 \implies (x \circ f) \sim (y \circ f). \] (45)

To see this, let $x, y \in 2^B$ with $d_{B,\phi}(x, y) < 1$. Then there exists $S \subseteq T_\leq$ such that
\[ \forall t \in T: \ x([t]_M) \neq y([t]_M) \implies (\exists i \in \{ 0, \ldots, k \}: \ t|\{i_1,\ldots,i\} \in S) \]
and
\[ \sum_{s \in S} w_{|s|} < 1. \]

In particular, $\emptyset \notin S$ since $w_0 = 1$, and if $i \in \{ 0, \ldots, k - 1 \}$, then, for each $s \in T_\leq$ with $|s| = i$, we have
\[ w_{i+1} \left| S \cap \{ sj \mid j < M_{i+1} \} \right| = \sum_{sj \in S} w_{|sj|} < 1 = \delta_{i+1} w_{i+1} \sqrt{M_{i+1}}, \]
which implies that
\[ \left| S \cap \{ sj \mid j < M_{i+1} \} \right| < \delta_{i+1} \sqrt{M_{i+1}}. \]
Thus, \( S \) witnesses that \((x \circ f) \sim (y \circ f)\). This proves (45). Clearly, from (45) together with (40), the inclusion (37) follows immediately.

Now we prove (38). To this end, choose any family of independent random variables \((X_t)_{t \in T}\) defined on a common domain \(\Omega\) such that, for each \(t \in T\), we have
\[
P[X_t = 0] = \frac{1}{2} = P[X_t = 1].
\]
We define a family of random variables \((Y_s)_{s \in T_{\leq}}\) on the same domain \(\Omega\) recursively as follows. For each \(t \in T\), let \(Y_t := X_t\). Furthermore, if \(i \in \{0, \ldots, k - 1\}\) and \(Y_s\) is defined for all \(s \in T_{\leq}\) with \(|s| \geq i + 1\), then, for each \(t \in T_{\leq}\) with \(|t| = i\), we define
\[
Y_t(\omega) := \begin{cases} 0 & \text{if } \{|j \in M_{i+1} \mid Y_{ij}(\omega) = 1\} < \frac{M_{i+1}}{2} + \delta_{i+1} \sqrt{M_{i+1}}, \\ 1 & \text{otherwise.} \end{cases}
\]
for all \(\omega \in \Omega\). Define also, for \(t \in T_{\leq}\), the set
\[
B_t := \{ y \in 2^T \mid \hat{y}(t) = 0 \}.
\]
We leave it to the reader to verify by induction on \(k - |t|\) that, for each \(t \in T_{\leq}\),
\[
\frac{|B_t|}{2^{|t|}} = P[Y_t = 0].
\]
Since \(B_\emptyset = B\), the equation above gives
\[
\frac{|B_\emptyset|}{2^{|\emptyset|}} = P[Y_\emptyset = 0].
\]
Therefore, to prove (38), it remains to show that \(P[Y_\emptyset = 0] \leq \frac{3}{4}\). In fact, we will prove that
\[
P[Y_\emptyset = 0] - \frac{1}{2} \leq \varepsilon_0,
\]
which will suffice by (31). To this end, let us note that, for every \(i \in \{0, \ldots, k\}\), there are real numbers \(0 < b_i \leq a_i\) with \(a_i + b_i = 1\) and such that, for all \(t \in T_{\leq}\),
\[
|t| = i \implies (P[Y_t = 0] = a_i \text{ and } P[Y_t = 1] = b_i).
\]
Evidently, \(a_k = b_k = 1/2\). Furthermore, for each \(i \in \{0, \ldots, k\}\), \((Y_t \mid t \in T_{\leq}, |t| = i)\) is a family of independent random variables. Observe now that, by (31), the sequence \((\varepsilon_i)_{i \in \mathbb{N}}\) is decreasing from \(\varepsilon_0 < \frac{1}{4}\), so that in particular
\[
\forall i \in \{0, \ldots, k\}: \quad \varepsilon_i < \frac{1}{4}.
\]
Using (27), (31), (34) and (47), we see by induction on \(k - i\) that
\[
\forall i \in \{0, \ldots, k\}: \quad a_i - \frac{1}{2} < \varepsilon_i.
\]
Now, (48) and (47) together imply that
\[
\forall i \in \{0, \ldots, k\}: \quad a_i < \frac{3}{4},
\]
which gives (46) for \(i = 0\), as required.

The submeasure is barely elliptic. For every \(i \in \mathbb{N}_{\geq 1}\), considering the partition of \(Z\) into the sets \([s]_M \in \mathcal{A}\) with \(s \in M_1 \times \cdots \times M_i\), we conclude that
\[
\frac{h_a(w_a)}{w_i} \geq \frac{1}{w_i^M M_1 \cdots M_i},
\]
From (30) and (29), it follows that
\[
\limsup_{\xi \to 0} \frac{h_\phi(\xi)}{\theta(\xi)} \geq \limsup_{i \to \infty} \frac{1}{w_i M_1 \cdots M_i \theta(w_i)} = \infty,
\]
as required.

6. Dynamical background

The purpose of this section is to provide some background material necessary for the topological applications of our concentration results, which are given in the subsequent Section 7. These applications will concern topological dynamics, that is, the structure of topological groups reflected by their flows. To be more precise, if \( G \) is a topological group, then a \( G \)-flow is any non-empty compact Hausdorff space \( X \) together with a continuous action of \( G \) on \( X \). The study of such objects is intimately linked with properties of certain function spaces naturally associated with the acting group. Some aspects of this correspondence, in particular concerning amenability, extreme amenability, and the connection with measure concentration, will be summarized below. For more details, we refer to [Pes06, Pac13].

Now let \( G \) be a topological group. Denote by \( U(G) \) the neighborhood filter of the neutral element in \( G \) and endow \( G \) with its right uniformity defined by the basic entourages \( \{(x, y) \in G \times G \mid xy^{-1} \in U\} \), where \( U \in U(G) \). In particular, a function \( f: G \to \mathbb{R} \) is called right-uniformly continuous if for every \( \varepsilon \in \mathbb{R}_{>0} \) there exists \( U \in U(G) \) such that
\[
\forall x, y \in G: \quad xy^{-1} \in U \implies |f(x) - f(y)| \leq \varepsilon.
\]
The set \( \text{RUCB}(G) \) of all right-uniformly continuous, bounded real-valued functions on \( G \), equipped with the pointwise operations and the supremum norm, constitutes a commutative unital real Banach algebra. A subset \( H \subseteq \text{RUCB}(G) \) is called UEB (short for uniformly equicontinuous, bounded) if \( H \) is \( \| \cdot \|_{\infty}\)-bounded and right-uniformly equicontinuous, that is, for every \( \varepsilon \in \mathbb{R}_{>0} \) there is \( U \in U(G) \) such that
\[
\forall f \in H \forall x, y \in G: \quad xy^{-1} \in U \implies |f(x) - f(y)| \leq \varepsilon.
\]
The set \( \text{RUEB}(G) \) of all UEB subsets of \( \text{RUCB}(G) \) forms a convex vector bornology on \( \text{RUCB}(G) \). The UEB topology on the dual Banach space \( \text{RUCB}(G)^* \) is defined as the topology of uniform convergence on the members of \( \text{RUEB}(G) \). This is a locally convex linear topology on the vector space \( \text{RUCB}(G)^* \) containing the weak*-topology, that is, the initial topology generated by the maps \( \text{RUCB}(G)^* \to \mathbb{R}, \mu \mapsto \mu(f) \) where \( f \in \text{RUCB}(G) \). More detailed information on the UEB topology is to be found in [Pac13]. Furthermore, let us recall that the set
\[
\text{M}(G) := \{ \mu \in \text{RUCB}(G)^* \mid \mu \text{ positive}, \mu(1) = 1 \}
\]
of all means on \( \text{RUCB}(G) \) constitutes a compact Hausdorff space with respect to the weak*-topology. The set \( \text{S}(G) \) of all (necessarily positive, linear) unital ring homomorphisms from \( \text{RUCB}(G) \) to \( \mathbb{R} \) is a closed subspace of \( \text{M}(G) \), called the Samuel
compactification of $G$. For $g \in G$, let $\lambda_g: G \to G, x \mapsto gx$ and $\rho_g: G \to G, x \mapsto xg$. Note that $G$ admits an affine continuous action on $M(G)$ given by

$$(g\mu)(f) := \mu(f \circ \lambda_g),$$

where $g \in G$, $\mu \in M(G)$, $f \in \text{RUCB}(G)$, and that $S(G)$ constitutes a $G$-invariant subspace of $M(G)$. Let us recall that $G$ is amenable (resp., extremely amenable) if $M(G)$ (resp., $S(G)$) admits a $G$-fixed point. It is well known that $G$ is amenable (resp., extremely amenable) if and only if every continuous action of $G$ on a non-void compact Hausdorff space admits a $G$-invariant regular Borel probability measure (resp., a $G$-fixed point). For a comprehensive account on (extreme) amenability of topological groups, the reader is referred to [Pes06]. Below we recollect two specific results in that direction (Theorem 6.1 and Theorem 6.4), relevant for Section 7.

First, regarding amenability of topological groups, we recall the following result from [ST18], which will be used in the proof of Theorem 7.3. Given a measurable space $\Omega$, let us denote by $\text{Prob}(\Omega)$ the set of all probability measures on $\Omega$ and by $\text{Prob}_{\text{fin}}(\Omega)$ the convex envelope of the set of Dirac measures in $\text{Prob}(\Omega)$.

**Theorem 6.1** ([ST18], Theorem 3.2). A topological group $G$ is amenable if and only if, for every $\varepsilon \in \mathbb{R}_{>0}$, every $H \in \text{RUEB}(G)$ and every finite subset $E \subseteq G$, there exists $\mu \in \text{Prob}_{\text{fin}}(G)$ such that, for $g \in E$ and $f \in H$,

$$\left| \int f \, d\mu - \int f \circ \lambda_g \, d\mu \right| \leq \varepsilon.$$

The result above suggests the following definition.

**Definition 6.2.** Let $G$ be a topological group. A net $(\mu_i)_{i \in I}$ of Borel probability measures on $G$ is said to $\text{UEB}$-converge to invariance (over $G$) if, for all $g \in G$ and $H \in \text{RUEB}(G)$,

$$\sup_{f \in H} \left| \int f \, d\mu_i - \int f \circ \lambda_g \, d\mu_i \right| \to 0, \text{ as } i \to I.$$ 

Second, let us recall that concentration of measure (Section 2.1) provides a very prominent method for proving extreme amenability of topological groups. This approach goes back to the seminal work of Gromov and Milman [GM83] and has since been used in establishing extreme amenability for many concrete examples of Polish groups (see [Pes06, Chapter 4] for an overview). Below we mention a refined version of this method, as developed in [Pes10, PS17]. As usual, we define the support of a Borel probability measure $\mu$ on a topological space $X$ to be

$$\text{spt} \mu := \{x \in X \mid \forall U \subseteq X \text{ open: } x \in U \implies \mu(U) > 0\},$$

which is easily seen to constitute a closed subset of $X$. The following notion first appeared in [Pes10], but originates in [GTW05, GW05].

**Definition 6.3.** A topological group $G$ is called whirly amenable if

— $G$ is amenable, and
any $G$-invariant regular Borel probability measure on a $G$-flow has support contained in the set of $G$-fixed points.

Of course, whirly amenability implies extreme amenability. Note that the converse does not hold: the Polish group $\text{Aut}(\mathbb{Q},<)$, carrying the topology of pointwise convergence, is extremely amenable [Pes98], but not whirly amenable [GTW05, Remark 1.3].

In order to establish whirly (hence extreme) amenability of topological groups of measurable maps the next section, we will combine the results of Section 5 with the strategy provided by the following theorem, which generalizes earlier results by Pestov [Pes10, Theorem 5.7] and Glasner–Tsirelson–Weiss [GTW05, Theorem 1.1].

**Theorem 6.4** ([PS17], Theorem 3.9). Let $G$ be a topological group. If there exists a net $(\mu_i)_{i \in I}$ of Borel probability measures on $G$ such that

- $(\mu_i)_{i \in I}$ concentrates in $G$ (with respect to the right uniformity),
- $(\mu_i)_{i \in I}$ UEB-converges to invariance over $G$,

then $G$ is whirly amenable.

For a quantitative generalization of Theorem 6.4 in the context of Gromov’s observable diameters [Gro99, Chapter 3.4], the reader is referred to [Sch19, Theorem 1.2].

7. Topological groups of measurable maps

This final section is devoted to applications of our results in topological dynamics. More precisely, we will establish whirly amenability of topological groups of measurable maps over parabolic or hyperbolic submeasures, with coefficients in any amenable topological group. Such groups, introduced for the Lebesgue measure by Hartman–Mycielski [HM58] and later studied for pathological submeasures by Herer–Christensen [HC75], have more recently attracted growing attention [Gla98, Pes02, FS08, Pes10, Sab12, Sol14, KLM15, PS17, KM19].

Let $X$ be a set and let $A$ be a Boolean subalgebra of $\mathcal{P}(X)$. Let $\phi: A \to \mathbb{R}$ be a submeasure and let $G$ be a topological group. Consider the topological group

$$L_0(\phi, G) := \{ f \in G^X \mid \exists B \in \Pi(A) \forall B \in \mathcal{B} : f \text{ is constant on } B \}$$

equipped with the pointwise multiplication, that is, the subgroup structure inherited from $G^X$, and the topology of convergence in $\phi$. To be precise about the topology, let

$$N_{\phi}(f, U, \varepsilon) := \{ h \in L_0(\phi, G) \mid \phi(\{ x \in X \mid h(x) \notin Uf(x) \}) < \varepsilon \}$$

for any $f \in L_0(\phi, G)$, $U \in \mathcal{U}(G)$ and $\varepsilon \in \mathbb{R}_{>0}$. Then a subset $M \subseteq L_0(\phi, G)$ is open if and only if

$$\forall f \in M \exists U \in \mathcal{U}(G) \exists \varepsilon \in \mathbb{R}_{>0} : N_{\phi}(f, U, \varepsilon) \subseteq M.$$
where $U \in \mathcal{U}(G)$ and $\varepsilon \in \mathbb{R}_{>0}$. Given $\mathcal{B} \in \Pi(A)$, let us denote by $\pi_{\mathcal{B}} : X \to \mathcal{B}$ the associated projection. For every $\mathcal{B} \in \Pi(A)$, the map $\gamma_{\mathcal{B}} : G^\mathcal{B} \to L_0(\phi, G)$, $f \mapsto f \circ \pi_{\mathcal{B}}$ is a continuous homomorphism. Furthermore, if $I$ is a set, $i \in I$ and $a \in G^{\mathcal{B}\setminus\{i\}}$, then we define $\eta_{i,a} : G \to G^I$ by

$$\eta_{i,a}(g)(j) := \begin{cases} g & \text{if } j = i, \\ a(j) & \text{otherwise} \end{cases}$$

for all $g \in G$ and $j \in I$. Moreover, for a subset $H \subseteq \text{RUEB}(L_0(\phi, G))$, let

$$[H] := \{ f \circ \gamma_{\mathcal{B}} \circ \eta_{i,a} \mid f \in H, \mathcal{B} \in \Pi(A), \mathcal{B} \in \mathcal{B}, a \in G^{\mathcal{B}\setminus\{\mathcal{B}\}} \}.$$ 

The following two lemmata are straightforward adaptations of the corresponding results in [PS17]. We include the proofs for the sake of convenience.

**Lemma 7.1** (cf. [PS17], Lemma 4.3). If $\phi$ is a submeasure and $G$ a topological group, then, for each $H \in \text{RUEB}(L_0(\phi, G))$,

$$[H] \in \text{RUEB}(G).$$

**Proof.** Let $X$ be a set and let $\mathcal{A}$ be a Boolean subalgebra of $\mathcal{P}(X)$. Consider any submeasure $\phi : \mathcal{A} \to \mathbb{R}$. Let $H \in \text{RUEB}(L_0(\phi, G))$. Evidently, $[H]$ is $\| \cdot \|_{\infty}$-bounded as $H$ is. In order to prove that $[H]$ is right-uniformly equicontinuous, let $\varepsilon \in \mathbb{R}_{>0}$. Since $H \in \text{RUEB}(L_0(\phi, G))$, there exists $U \in \mathcal{U}(L_0(\phi, G))$ such that $|f(x) - f(y)| \leq \varepsilon$ for all $f \in H$ and $x, y \in L_0(\phi, G)$ with $xy^{-1} \in U$. Fix any $V \in \mathcal{U}(G)$ and $\varepsilon' \in \mathbb{R}_{>0}$ so that $N_{\phi}(V, \varepsilon') \subseteq U$. We claim that $|f'(x) - f'(y)| \leq \varepsilon$ for all $f' \in [H]$ and $x, y \in G$ with $xy^{-1} \in V$. To see this, let $f \in H, \mathcal{B} \in \Pi(A), \mathcal{B} \in \mathcal{B}$ and $a \in G^{\mathcal{B}\setminus\{\mathcal{B}\}}$. Then, for any $x, y \in G$ with $xy^{-1} \in V$, it follows that

$$\gamma_{\mathcal{B}}(\eta_{\mathcal{B},a}(x))\gamma_{\mathcal{B}}(\eta_{\mathcal{B},a}(y))^{-1} = \gamma_{\mathcal{B}}(\eta_{\mathcal{B},a}(x)\eta_{\mathcal{B},a}(y)^{-1}) = \gamma_{\mathcal{B}}(\eta_{\mathcal{B},a}(xy^{-1})) \in N_{\phi}(V, \varepsilon')$$

and therefore $|f(\gamma_{\mathcal{B}}(\eta_{\mathcal{B},a}(x))) - f(\gamma_{\mathcal{B}}(\eta_{\mathcal{B},a}(y))))| \leq \varepsilon$. Hence, $[H] \in \text{RUEB}(G)$. \hfill $\Box$

**Lemma 7.2** (cf. [PS17], Lemma 4.4). Let $X$ be a non-empty set and let $\mathcal{A}$ be a Boolean subalgebra of $\mathcal{P}(X)$. Consider a submeasure $\phi : \mathcal{A} \to \mathbb{R}$ and a topological group $G$. If $(\mathcal{B}_i, \mu_i)_{i \in I}$ is a net in $\Pi(A) \times \text{Prob}(G)$ such that

- $\forall \mathcal{B} \in \Pi(A) \exists i_0 \in I \forall i \in I : \ i_0 \leq i \implies \mathcal{B} \preceq \mathcal{B}_i$, 
- $\forall g \in G \forall H \in \text{RUEB}(G) : \sup_{f \in H} |\int \phi d\mu_i - \int \phi \circ \lambda_h d\mu_i| \to 0$, as $i \to I$,

then the net $((\gamma_{\mathcal{B}_i})_*(\mu_i^{\otimes \mathcal{B}_i}))_{i \in I}$ UEB-converges to invariance over $L_0(\phi, G)$.

**Proof.** For each $i \in I$, let us consider the corresponding push-forward Borel probability measure $\nu_i := (h_{\mathcal{Q}_i})_*(\mu_i^{\otimes \mathcal{Q}_i})$ on $L_0(\phi, G)$. We will show that $(\nu_i)_{i \in I}$ UEB-converges to invariance over $L_0(\phi, G)$. To this end, let $H \in \text{RUEB}(L_0(\phi, G))$, $g \in L_0(\phi, G)$ and $\varepsilon \in \mathbb{R}_{>0}$. Put $\mathcal{B} := \{ g^{-1}(h) \mid h \in g(X) \}$ and $E := g(X) \cup \{ e \}$. 

**Proof.** (cont.)
According to Lemma 7.1 and our assumptions, there exists \( i_0 \in I \) such that, for every \( i \in I \) with \( i \geq i_0 \), we have \( B \preceq B_i \) and

\[
\forall s \in E: \quad \sup_{f \in [H]} \left| \int f \, d\mu_i - \int f \circ \lambda_s \, d\mu_i \right| \leq \frac{\varepsilon}{|B_i|}. \tag{49}
\]

We claim that

\[
\forall i \in I, i \geq i_0: \quad \sup_{f \in H} \left| \int f \, d\nu_i - \int f \circ \lambda \, d\nu_i \right| \leq \varepsilon. \tag{50}
\]

To prove this, let \( i \in I \) with \( i \geq i_0 \). Since \( B \preceq B_i \), we find \( g' \in E^{B_i} \) with \( g = \gamma_{B_i}(g') \). Let \( n_0 := |B_i| \) and pick an enumeration \( B_i = \{ B_{ij} \mid j < n_i \} \). For each \( j < n_i \), let us define \( a_j \in E^{B_i} \) by

\[
a_j(B) := \begin{cases} 
    g'_{\ell} & \text{if } B = B_{\ell i} \text{ for } \ell \in \{0, \ldots, j\}, \\
    e & \text{otherwise}
\end{cases}
\]

for each \( B \in B_i \), and let \( b_{ij} := a_j|_{B_i \setminus \{ B_{ij} \}} \in E^{B_i \setminus \{ B_{ij} \}} \). Furthermore, let us define \( a_{-1} := e \in E^{B_i} \). For all \( j < n_i \) and \( z \in G^{B_i \setminus \{ B_{ij} \}} \), note that \( \lambda_{a_j} \circ \eta_{B_{ij}, z} = \eta_{B_{ij}, b_{ij} \circ \lambda_{g_j}} \) and \( \lambda_{a_{j-1}} \circ \eta_{B_{ij}, z} = \eta_{B_{ij}, b_{ij}} \). Combining these observations with (49) and Fubini’s theorem, we conclude that

\[
\left| \int f \circ \gamma_{B_{ij}(a_j)} \, d\nu_i - \int f \circ \gamma_{B_{ij}(a_{j-1})} \, d\nu_i \right| \\
= \left| \int \left( f \circ \gamma_{B_{ij}(a_j)} \circ \gamma_{B_i} \right) - \left( f \circ \gamma_{B_{ij}(a_{j-1})} \circ \gamma_{B_i} \right) \, d\mu_i \right| \\
= \left| \int \left( f \circ \gamma_{B_i} \circ \lambda_{a_{j-1}} - (f \circ \gamma_{B_i} \circ \lambda_{a_j}) \right) \, d\mu_i \right| \\
= \left| \int \left( \int f \circ \gamma_{B_i} \circ \lambda_{a_{j-1}} \circ \eta_{B_{ij}, z} \, d\mu_i \\
- \int f \circ \gamma_{B_i} \circ \lambda_{a_j} \circ \eta_{B_{ij}, z} \, d\mu_i \right) \, d\mu_i \right| \\
= \left| \int \left( \int f \circ \gamma_{B_i} \circ \eta_{B_{ij}, b_{ij} \circ \lambda_{g_j}} \, d\mu_i - \int f \circ \gamma_{B_i} \circ \eta_{B_{ij}, b_{ij} \circ \lambda_{g_j}} \, d\mu_i \right) \, d\mu_i \right| \\
\leq \left| \int \left( \int f \circ \gamma_{B_i} \circ \eta_{B_{ij}, b_{ij}} \, d\mu_i - \int f \circ \gamma_{B_i} \circ \eta_{B_{ij}, b_{ij}} \, d\mu_i \right) \, d\mu_i \right| \left| \int \frac{d\mu_i}{n_i} \right| \left| B_{ij} \right| (z) \leq \frac{\varepsilon}{n_i}
\]

for all \( j \in \{0, \ldots, n_i - 1\} \) and \( f \in H \). For every \( f \in H \), it follows that

\[
\left| \int f \, d\nu_i - \int f \circ \lambda \, d\nu_i \right| \leq \sum_{j=0}^{n_i-1} \left| \int f \circ \gamma_{B_{ij}(a_j)} \, d\nu_i - \int f \circ \gamma_{B_{ij}(a_{j-1})} \, d\nu_i \right| \leq \varepsilon,
\]

which proves (50) and hence completes the argument. \( \square \)
We arrive at our fourth and final main result.

**Theorem 7.3.** Let \( \phi \) be a submeasure and let \( G \) be a topological group. If \( \phi \) has covering concentration and \( G \) is amenable, then \( L_0(\phi, G) \) is whirlly amenable.

**Proof.** Let \( X \) be a set and let \( \mathcal{A} \) be a Boolean subalgebra of \( \mathcal{P}(X) \). Consider any submeasure \( \phi : \mathcal{A} \to \mathbb{R} \). Since the desired conclusion is trivial if \( X = \emptyset \), we may and will assume that \( X \) is non-void. According to Theorem 6.1, we find a net \( (\mathcal{B}_j, \mu_j)_{j \in J} \) in \( \Pi(\mathcal{A}) \times \text{Prob}_{\text{fin}}(G) \) such that

1. \( \forall \mathcal{B} \in \Pi(\mathcal{A}) \exists j_0 \in J \forall j \in J : j_0 \leq j \implies \mathcal{B} \preceq \mathcal{B}_j, \)
2. \( \forall g \in G \exists H \in \text{RUEB}(G) : \sup_{f \in H} |\int \int f(d\mu_j - f \circ \lambda_g d\mu_j)| = 0, \) as \( j \to J. \)

Suppose that \( \phi \) has covering concentration. By Remark 5.4, we find \( (C_\ell)_{\ell \in \mathbb{N}} \in \Pi(\mathcal{A})^\mathbb{N} \) such that, for every \( \varepsilon \in \mathbb{R}_{>0} \),

\[
\sup\{\alpha_N(\mathcal{A}, \mathcal{B}, \phi)(\varepsilon) \mid \mathcal{B} \in \Pi(\mathcal{A}), C_\ell \preceq \mathcal{B}\} \to 0, \quad \text{as } \ell \to \infty. \tag{51}
\]

Consider the directed set \( (I, \leq_J) \) where \( I := \{(\ell, j) \in \mathbb{N} \times J \mid C_\ell \preceq \mathcal{B}_j\} \) and

\[
(\ell_0, j_0) \leq_J (\ell_1, j_1) \iff \ell_0 \leq \ell_1, j_0 \leq_J j_1.
\]

For every \( (\ell, j) \in I \), define \( B_{(\ell, j)} := \mathcal{B}_j \) and \( \mu_{(\ell, j)} := \mu_j. \) For each \( i \in I \), let us consider

\[
\nu_i := (\gamma_{\mathcal{B}_i})_* \left( \mu_i^{\mathcal{B}_i} \right) \in \text{Prob}(L_0(\phi, G)).
\]

By Lemma 7.2, the net \( (\nu_i)_{i \in I} \) UEB-converges to invariance over \( L_0(\phi, G). \)

Thanks to Theorem 6.4, it remains to show that \( (\nu_i)_{i \in I} \) concentrates in \( L_0(\phi, G). \) For each \( i \in I \), we find a finite subset \( S_i \subseteq G \) and a probability measure \( \sigma_i \) on the discrete measurable space \( S_i \) such that \( \mu_i \) equals the push-forward measure of \( \sigma_i \) along the map \( S_i \to G, g \mapsto g. \) According to (51), Remark 5.2 and Remark 2.2(3), the net \( (\mathcal{X}(S_i, \sigma_i, B_i, \phi))_{i \in I} \) constitutes a Lévy net. Thus, by Remark 2.5, it suffices to verify that the family \( (\gamma_{\mathcal{B}_i})_{i \in I} \) is uniformly equicontinuous. For this purpose, let \( U \in \mathcal{U}(G) \) and \( \varepsilon \in \mathbb{R}_{>0}. \) For all \( i \in I \) and \( g, h \in G^{B_i}, \) we have

\[
\phi(\{x \in X \mid \gamma_{B_i}(g)(x)\gamma_{B_i}(h)(x)^{-1} \notin U\}) \leq \phi(\{x \in X \mid \gamma_{B_i}(g)(x) \neq \gamma_{B_i}(h)(x)\}) = \phi\left( \bigcup \{ B \in B_i \mid g_B \neq h_B \} \right) = \delta_{\phi, B_i}(g, h),
\]

and therefore

\[
\delta_{\phi, B_i}(g, h) < \varepsilon \implies \gamma_{B_i}(g)\gamma_{B_i}(h)^{-1} \in N_\phi(U, \varepsilon).
\]

Hence, due to Remark 2.5, the net \( (\nu_i)_{i \in I} \) concentrates in \( L_0(\phi, G), \) so that \( L_0(\phi, G) \) is whirlly amenable by Theorem 6.4. \( \square \)

**Corollary 7.4.** Let \( \phi \) be a parabolic or hyperbolic submeasure. If \( G \) is an amenable topological group, then \( L_0(\phi, G) \) is whirlly amenable.

**Proof.** This is an immediate consequence of Theorem 5.6 and Theorem 7.3. \( \square \)

We conclude with a partial converse of Corollary 7.4.
Proposition 7.5. Let $G$ be a topological group. If $\phi$ is an elliptic or parabolic submeasure and $L_0(\phi,G)$ is amenable, then $G$ is amenable.

Proof. We generalize an argument from [PS17, Theorem 1.1 (2)$\Rightarrow$(1)]. As $\phi: A \to \mathbb{R}$ is not pathological, we find a non-zero measure $\mu: A \to \mathbb{R}$ such that $\mu \leq \phi$. Of course, the set $L_0(\phi,\mathbb{R})$ forms a unital linear subspace of $\ell^\infty(X)$, where $X := \bigcup A$. Thanks to $\mu$ being a non-zero measure, the map $I_\mu: L_0(\phi,\mathbb{R}) \to \mathbb{R}$, given by

$$I_\mu(f) := \frac{1}{\mu(X)} \sum_{r \in \mathbb{R}} r \mu(f^{-1}(r)),$$

is a positive unital linear functional. Define $\Phi: \text{RUCB}(G) \to \text{RUCB}(L_0(\phi,G))$ by

$$\Phi(f) := I_\mu(f \circ h),$$

where $f \in \text{RUCB}(G)$ and $h \in L_0(\phi,G)$. We check that $\Phi$ is well defined. Note that, if $h \in L_0(\phi,G)$, then $f \circ h \in L_0(\phi,G)$ for any function $f: G \to \mathbb{R}$. Hence, the term $\Phi(f)(h)$ is well defined for all $h \in L_0(\phi,G)$ and $f \in \mathbb{R}^G$. Let $f \in \text{RUCB}(G)$. Since

$$\sup_{h \in L_0(\phi,G)} |\Phi(f)(h)| = \|f\|_\infty,$$

it follows that $\Phi(f) \in \ell^\infty(L_0(\phi,G))$. In order to show that $\Phi(f) \in \text{RUCB}(L_0(\phi,G))$, let $\varepsilon \in \mathbb{R}_{>0}$. As $f \in \text{RUCB}(G)$, there exists $U \in \mathcal{U}(G)$ such that $\|f - (f \circ \lambda_g)\|_\infty \leq \frac{\varepsilon}{2}$ for all $g \in U$. Consider

$$\varepsilon' := \frac{\varepsilon \mu(X)}{2\|f\|_\infty + 1}.$$

Then $V := N_\phi(U,\varepsilon')$ constitutes a neighborhood of the neutral element in $L_0(\phi,G)$. Now, let $h_0,h_1 \in L_0(\phi,G)$ with $h_0h_1^{-1} \in V$. Then $\phi(X \setminus A) \leq \varepsilon'$ for

$$A := \{ x \in X \mid h_0(x)h_1(x)^{-1} \in U \}.$$

Pick any $B \in \Pi(A)$ so that both $h_0$ and $h_1$ are constant on each member of $B$. Then

$$\Phi(f)(h_0) - \Phi(f)(h_1) = \frac{1}{\mu(X)} \sum_{B \in B} (f(h_0(B)) - f(h_1(B)))\mu(B)$$

$$= \frac{1}{\mu(X)} \sum_{B \in B} (f(h_0(B \cap A)) - f(h_1(B \cap A)))\mu(B \cap A)$$

$$+ \frac{1}{\mu(X)} \sum_{B \in B} (f(h_0(B \setminus A)) - f(h_1(B \setminus A)))\mu(B \setminus A),$$

which, as $\mu \leq \phi$, readily implies that

$$|\Phi(f)(h_0) - \Phi(f)(h_1)| \leq \frac{\varepsilon}{2} + 2\|f\|_\infty \varepsilon' \leq \varepsilon.$$

This shows that $\Phi(f) \in \text{RUCB}(L_0(\phi,G))$. Therefore, $\Phi$ is a well-defined mapping. Since $I_\mu$ is positive, unital, and linear, so is $\Phi$. If $f \in \text{RUCB}(G)$ and $g \in G$, then

$$\Phi(f \circ \lambda_g)(h) = I_\mu(f \circ \lambda_g \circ h) = I_\mu(f \circ (\gamma_X(g)h))$$

$$= \Phi(f)(\gamma_X(g)h) = \left(\Phi(f) \circ \lambda_{\gamma_X(g)}\right)(h)$$

for all $h \in L_0(\phi,G)$, that is, $\Phi(f \circ \lambda_g) = \Phi(f) \circ \lambda_{\gamma_X(g)}$. Assuming that $L_0(\phi,G)$ is amenable and considering a left-invariant mean $m: \text{RUCB}(L_0(\phi,G)) \to \mathbb{R}$, we deduce from the properties of $\Phi$ that $m \circ \Phi: \text{RUCB}(G) \to \mathbb{R}$ is a left-invariant mean, whence $G$ is amenable. \[\square\]
The subsequent corollary generalizes the main result of [PS17] from non-zero diffused measures to arbitrary parabolic submeasures.

**Corollary 7.6.** Let $\phi$ be a parabolic submeasure and let $G$ be a topological group. Then the following are equivalent.

- $G$ is amenable.
- $L_0(\phi, G)$ is amenable.
- $L_0(\phi, G)$ is whirlly amenable.

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