GEOMETRIC INEQUALITIES ON BI-WARPED PRODUCT SUBMANIFOLDS OF LOCALLY CONFORMAL ALMOST COSYMPLECTIC MANIFOLDS

RAMANDEEP KAUR, GAUREE SHANKER, ALEXANDER PIGAZZINI, SAEID JAFARI, CENAP ÖZEL, AND ABDULQADER MUSTAFA

Abstract. In this paper we present not only some properties related to bi-warped product submanifolds of locally conformal almost cosymplectic manifolds, but also we show how the squared norm of the second fundamental form and the bi-warped product’s warping functions are related when the bi-warped product submanifold has a proper slant submanifold as a base or fiber.

AMS Subject Classification (2020): 53C15, 53C18, 53C25, 53D15.

Keywords: Bi-warped product, slant submanifolds, almost cosymplectic manifolds.

1. Introduction

The warped product is one of the most fruitful generalizations of the notion of the Cartesian product and it was introduced by Bishop and O’Neill in [4]. The interest in warped products has gradually grown more and more, as they allow to find many exact solutions to Einstein’s field equation. Examples include the Robertson-Walker models and the Schwarzschild solution, the latter laying the foundation for describing the final stages of gravitational collapse and objects known today as black holes (see [10]).

Over the years many types of warped-products have been studied among others, for example the Einstein multiply warped-product manifolds, Einstein Warped-twisted product manifolds, Einstein sequential warped product manifolds which are those that cover the widest variety of exact solutions to the field equation of Einstein; Recently, in [12], Pigazzini et al. studied a special case of Einstein sequential warped-product manifolds, which cover a wider variety of exact solutions to Einstein’s field equation, than Einstein warped-product manifolds with Ricci-flat fiber F, without complicating the calculations. In particular they studied a conformal semi-Riemannian Einstein metrics showing the existence of solutions with positive scalar curvature. The extrinsic and intrinsic Riemannian invariants have broad applications in many different fields of science and differential geometry and they are also of considerable importance in general relativity [11]. Among the extrinsic invariants, second fundamental form and the squared mean curvatures
are the most important ones, while among the main intrinsic invariants, sectional, Ricci and scalar curvatures are the well-known ones, as well as \( \delta \)-invariant. In [7], Chen initiated a significant inequality in terms of the intrinsic invariant \( \delta \)-invariant) and more recently (in [6]), using Codazzi equation, he establishes an inequality for the second fundamental form in terms of warping function. Similar inequality has been studied for warped product immersions in cosymplectic space forms, in [13], by Uddin and Alqahtani. Over the years, the inequalities of submanifolds in warped-geometry have become the subject of study by many authors (among many, see for example [1], [2], [3], [5], [9], [14], [15], [16]).

The purpose of the present paper is not only show some properties related to bi-warped product submanifolds of locally conformal almost cosymplectic manifolds, but also show how the squared norm of the second fundamental form and the bi-warped product’s warping functions are related when the bi-warped product submanifold has a proper slant submanifold as a base or fiber.

The paper is organized as follows: In the "Preliminaries" section, we will address the definition of almost contact manifold and the relation with Slant manifolds. Subsequently, in the section "Bi-Warped Product Submanifolds of a locally conformal almost cosymplectic manifold", we will report some notions, theorems and lemmas that will be fundamental for the main result of the paper. The last section, namely "Chen type inequality for Bi-Warped product immersions", which contains the main result of this paper, i.e., we show how the square norm of the second fundamental form and the warping functions of the bi-warped product (of locally conformal almost cosymplectic manifolds), are correlated, carrying out a double analysis: when the bi-warped product submanifold has a proper slant submanifold as a base or as a fiber.

2. Preliminaries

An almost contact manifold \( \tilde{L} \) of dimension \( 2m + 1 \) satisfies the following with contact structure \( (\Phi, \xi, \eta, g) \)

\[ \Phi^2 = -I + \eta \otimes \xi \]

\[ \eta(\xi) = 1 \]

\[ \Phi \xi = 0 \]

\[ \eta \circ \Phi = 0 \]

\[ g(\Phi X_4, \Phi Y_4) = g(X_4, Y_4) - \eta(X_4)\eta(Y_4), \]  

for any \( X_4, Y_4 \in \Gamma(\tilde{T}L) \), where the \( \Gamma(\tilde{T}L) \) denotes the Lie algebra of vector fields on \( \tilde{L} \). An almost contact manifold \( \tilde{L} \) reduce to be a locally conformal almost cosymplectic manifold if and only if [8]

\[ (\tilde{\nabla}_X \Phi)Y_4 = \beta (g(\Phi X_4, Y_4)\xi - \eta(Y_4)\Phi X_4), \quad \tilde{\nabla}_X \xi = \beta(X_4 - \eta(X_4)\xi), \]
for any \( X_4, Y_4 \in \Gamma(T\tilde{L}) \), where \( \tilde{\nabla} \) denotes the Levi-Civita connection of \( g \).

The Gauss and Weingarten formulas for a submanifold \( M \) in a Riemannian manifold \( \tilde{L} \) are given by

\[
\tilde{\nabla}_{X_4} Y_4 = \nabla_{X_4} Y_4 + \Sigma(X_4, Y_4),
\]

\[
\tilde{\nabla}_{X_4} N = -A_N X_4 + \nabla_{\perp X_4} N,
\]

for any vector field \( X_4, Y_4 \in \Gamma(TM) \) and \( N \in \Gamma(T^\perp L) \). The shape operator and second fundamental form are denoted by \( A \) and \( \Sigma \), respectively, and have following relation

\[
g(\Sigma(X_4, Y_4), N) = g(A_N X_4, Y_4).
\]

A submanifold \( L \) is said to be totally geodesic if \( \Sigma = 0 \) and totally umbilical if \( \Sigma(X_4, Y_4) = g(X_4, Y_4)H, \forall X_4, Y_4 \in \Gamma(TM) \), where \( H = \frac{1}{n} \sum_{i=1}^{n} \Sigma(e_i, e_i) \) is the mean curvature vector of \( L \). For any \( x \in L, \{e_1, \ldots, e_n, \ldots, e_{2m+1}\} \) is an orthonormal frame of the tangent space \( T_x\tilde{L} \) such that \( \{e_1, \ldots, e_n\} \) are tangent to \( L \) at \( x \). Then, we have

\[
\Sigma_{ij} = g(\Sigma(e_i, e_j), e_r)
\]

\[
||\Sigma||^2 = \sum_{i,j=1}^{n} g(\Sigma(e_i, e_i), e_j, e_j).
\]

for any \( 1 \leq i, j \leq n \) and \( 1 \leq r \leq 2m + 1 \). Then length of the gradient \( \tilde{\nabla}(\phi) \) for a differentiable function \( \phi \) manifold \( L \) is defined as

\[
||\tilde{\nabla}(\phi)||^2 = \sum_{i=1}^{n} (e_i(\phi))^2,
\]

for the orthonormal frame \( \{e_1, \ldots, e_n\} \) on \( \tilde{L} \). The tangential \( TX \) and normal components \( FX \) of \( \Phi X_4 \) for any vector field \( X_4 \in \Gamma(TM) \), are decomposed as

\[
\Phi X_4 = TX_4 + \phi X_4.
\]

(1) A submanifold \( L \) tangent to \( \xi \) is said to be slant if for any \( x \in L \) and any \( X \in T_xL \), linearly independent to \( \xi \), the angle between \( \Phi X \) and \( T_xL \) is a constant \( \nu \in [0, \pi/2] \), called the slant angle of \( L \) in \( \tilde{L} \).

(2) Invariant and anti-invariant submanifolds are \( \nu \)-slant submanifolds with slant angle \( \nu = 0 \) and \( \nu = \pi/2 \), respectively. Proper slant is a slant submanifold that is neither invariant nor anti-invariant.

For slant submanifolds, we have the following characterisation theorem.
Theorem 2.1. Let $M$ be a submanifold of an almost contact metric manifold $\tilde{M}$, such that $\xi \in \Gamma(TM)$. Then $L$ is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that

$$T^2 = \lambda(-I + \eta \otimes \xi).$$

Moreover, if $\nu$ is slant angle, then $\lambda = \cos^2 \nu$.

The following relationships are a straightforward result of (2.11).

$$g(TX, TY) = \cos^2 \nu \left\{ g(X_4, Y_4) - \eta(X_4)\eta(Y_4) \right\},$$

$$g(FX, FY) = \sin^2 \nu \left\{ g(X_4, Y_4) - \eta(X_4)\eta(Y_4) \right\},$$

for any $X_4, Y_4 \in \Gamma(TM)$.

3. Bi-Warped Product Submanifolds of a Locally Conformal Almost Cosymplectic Manifold

Let $L = L_1 \times L_2 \times L_3$ be the Cartesian product of Riemannian manifolds $L_1$, $L_2$ and $L_3$. The canonical projections of $L$ onto $L_i$ are denoted and defined by $\pi_i : L \to L_i$ for each $i = 1, 2, 3$. If $\phi_2, \phi_3 : L_1 \to \mathbb{R}^+$ are positive real valued functions, then

$$g(X_4, Y_4) = g(\pi_{1*}X_4, \pi_{1*}Y_4) + (\phi_2 \circ \pi_1)^2 g(\pi_{2*}X_4, \pi_{2*}Y_4) + (\phi_3 \circ \pi_1)^2 g(\pi_{3*}X_4, \pi_{3*}Y_4),$$

defines a Riemannian metric $g$ on $L$. A product manifold $L$ endowed with metric $g$ is called a bi-warped product manifold. In this case, $\phi_2, \phi_3$ are non-constant functions, called warping functions on $L$. $L$ is a simply Riemannian product manifold $L = L_1 \times L_2 \times L_3$ if both $\phi_2, \phi_3$ are constant on $L$. Let $\tilde{L} = \tilde{L}_1 \times \tilde{L}_2 \times \tilde{L}_3$ be a bi-warped product submanifold of a Riemannian manifold $\tilde{L}$. Then, we have

$$\nabla_{X_4} Z_4 = \sum_{i=2}^{3} (X_4(\ln \phi_i))Z_4^i,$$

for any $X_4 \in \mathcal{D}_1$, the tangent space of $L_1$ and $Z_4 \in TN$, where $N = \phi_2 L_2 \times \phi_3 L_3$ and $Z_4^i$ is $L_i$-component of $Z_4$ and $\nabla$ is Levi-Civita connection on $L$.

Let $\phi_1, \phi_2 : L_\nu \to \mathbb{R}^+$ be non-constant functions. Then we consider the bi-warped product submanifolds of the form $L = L_\nu \times_\phi L_{T} \times_\phi L_{\perp}$ in a locally conformal almost cosymplectic manifold $\tilde{L}$, where $L_T, L_\perp$ and $L_\nu$ are invariant, anti-invariant and proper slant submanifolds of $\tilde{L}$, respectively. In this case, the tangent and normal space of $L$ are decomposed as according to the integrals manifolds $L_T$, $L_\perp$ and $L_\nu$ of $\mathcal{D}$, $\mathcal{D}^\perp$ and $\mathcal{D}^\nu$, respectively.
\( \mathbf{T} = \mathfrak{D} \oplus \mathfrak{D}^\perp \oplus \mathfrak{D}'' \oplus \langle \xi \rangle \) (3.2)

and

\( \mathbf{T}^\perp = \Phi \mathfrak{D}^{\perp} \oplus \phi \mathfrak{D}'' \oplus \mu, \quad \Phi \mathfrak{D}^{\perp} \oplus \phi \mathfrak{D}'' \perp \mu, \) (3.3)

where \( \mu \) is an \( \Phi \)-invariant normal subbundle of \( \mathbf{T}^\perp \). Now we obtain some classification theorems as follows.

**Theorem 3.1.** If \( \xi \) is tangent to either \( \xi \in \Gamma(\mathfrak{D}) \) or \( \xi \in \Gamma(\mathfrak{D}^{\perp}) \), then a bi-warped product submanifold of the type \( \mathbf{L} = \mathbf{L}_\nu \times_{\phi_1} \mathbf{L}_T \times_{\phi_2} \mathbf{L}_\perp \) in a locally conformal almost cosymplectic manifold \( \tilde{\mathbf{L}} \) is a single warped product.

*Proof.* If \( \xi \in \Gamma(\mathfrak{D}) \), then for any \( U_4 \in \Gamma(\mathfrak{D}) \), we get

\[ \nabla_{U_4} \xi = \beta U_4. \]

Using (2.4) and (3.1), we obtain

\[ U_4(\ln \phi_1) \xi = \beta U_4. \]

We get the following by multiplying \( \xi \) by the inner product and utilizing the fact that \( \xi \in \Gamma(\mathfrak{D}) \),

\[ U_4(\ln \phi_1) = 0. \]

As a result, \( \phi_1 \) is constant, and so \( \mathbf{L} \) is a warped product manifold. In the same way, we may simply achieve that

\[ U_4(\ln \phi_2) = 0, \]

which implies \( \phi_2 \) is constant and thus the proof. \( \square \)

**Theorem 3.2.** Let \( \xi \) is tangent to \( \mathbf{L}_\nu \) on a bi-warped product submanifold of the type \( \mathbf{L} = \mathbf{L}_\nu \times_{\phi_1} \mathbf{L}_T \times_{\phi_2} \mathbf{L}_\perp \) in a locally conformal almost cosymplectic manifold \( \tilde{\mathbf{L}} \). Then we have

\[ \xi(\ln \phi_i) = \beta, \quad \forall i = 1, 2. \] (3.4)

*Proof.* For any \( X_4 \in \Gamma(\mathfrak{D}) \), we have

\[ \nabla_{X_4} \xi = \beta X_4. \]

Utilizing (2.4) and (3.1), we obtain

\[ \xi(\ln \phi_1) X_4 = \beta X_4. \]

Taking the inner product with \( X_4 \), we derive

\[ \xi(\ln \phi_1) = \beta. \]

Similarly, it can be obtain

\[ \xi(\ln \phi_2) = \beta. \]

\( \square \)
Lemma 3.1. Let $\mathcal{L} = \mathcal{L}_\nu \times \phi_1 \mathcal{L}_T \times \phi_2 \mathcal{L}_\perp$ be a bi-warped product submanifold of locally conformal almost cosymplectic manifold $\tilde{\mathcal{L}}$. Then, we have

(i) $g(\Sigma(X_4, Y_4), F U_4) = \left( (U_4 \ln \phi_1) - \beta \eta(U_4) \right) g(X_4, \Phi Y_4) + T U_4(\ln \phi_1) g(X_4, Y_4)$,

(ii) $g(\Sigma(Z_4, W_4), F U_4) = \beta g(\Sigma(U_4, W_4), \Phi Z_4) + T U_4(\ln \phi_2) g(Z_4, W_4)$,

(iii) $g(\Sigma(U_4, V_4), \Phi Z_4) = g(\Sigma(U_4, Z_4), F V)$,

(iv) $g(\Sigma(X_4, V_4), F U_4) = 0$,

for any $X_4, Y_4 \in \Gamma(\mathcal{D})$, $U_4, V_4 \in \Gamma(\mathcal{D}^\nu \oplus \xi)$ and $Z_4, W_4 \in \Gamma(\mathcal{D}^\perp)$.

Proof. For any $X_4, Y_4 \in \Gamma(\mathcal{D})$ and $U_4 \in \Gamma(\mathcal{D}^\nu \oplus \xi)$ we have

$$g(\Sigma(X_4, Y_4), F U_4) = g(\hat{\nabla}_{X_4} Y_4, F U_4) = g(\hat{\nabla}_{X_4} Y_4, \Phi U_4) - (\hat{\nabla}_{X_4} Y_4, T U_4) = g(\hat{\nabla}_{X_4} \Phi Y_4, U_4) - g(\hat{\nabla}_{X_4} \Phi Y_4, U_4) + g(Y_4, \hat{\nabla}_{X_4} T U_4).$$

Using Eqs (2.3) and (3.1), we get

$$g(\Sigma(X_4, Y_4), F U_4) = \left( (U_4 \ln \phi_1) - \beta \eta(U_4) \right) g(X_4, \Phi Y_4) + T U_4(\ln \phi_1) g(X_4, Y_4).$$

This is a first part. For the second, we have

$$g(\Sigma(Z_4, W_4), F U_4) = g(\hat{\nabla}_{Z_4} W_4, \Phi U_4) - g(\hat{\nabla}_{Z_4} W_4, F U_4),$$

for any $Z_4, W_4 \in \Gamma(\mathcal{D}^\perp)$. From the virtue (2.2), we obtain

$$g(\Sigma(Z_4, W_4), F U_4) = g(\hat{\nabla}_{Z_4} \Phi W_4, U_4) - g(\hat{\nabla}_{Z_4} \Phi W_4, U_4) - g(W_4, \hat{\nabla}_{Z_4} F U_4).$$

Making use of (2.3), (2.4) and (3.1), we get

$$g(\Sigma(Z_4, W_4), F U_4) = (g(\Sigma(U_4, W_4)), \Phi Z_4) + T U_4(\ln \phi_2) g(Z_4, W_4).$$

Again $U_4, V_4 \in \Gamma(\mathcal{D}^\nu \oplus \xi)$ & $Z_4 \in \Gamma(\mathcal{D}^\perp)$. We derive

$$g(\Sigma(U_4, V_4), \Phi Z_4) = g(\hat{\nabla}_{U_4} V_4, \Phi Z_4) = g((\hat{\nabla}_{U_4} \Phi) V_4, Z_4) - g(\hat{\nabla}_{U_4} \Phi V_4, Z_4).$$

The term $g((\hat{\nabla}_{U_4} \Phi) V_4, Z_4) = 0$ using (2.3). Then by the orthogonality of vector fields, we obtain

$$g(\Sigma(U_4, V_4), \Phi Z_4) = g(\hat{\nabla}_{U_4} Z_4, \Phi V_4) = g(\hat{\nabla}_{U_4} Z_4, F V) + g(\hat{\nabla}_{U_4} Z_4, F V),$$

Eqs (2.5) and (3.1) imply the following

$$g(\Sigma(U_4, V_4), \Phi Z_4) = g(\Sigma(U_4, Z_4), F V).$$

Third is completed. For the last part, we have

$$g(\Sigma(X_4, V_4), F U_4) = g(\hat{\nabla}_{V_4} X_4, \Phi U_4) - g(\hat{\nabla}_{V_4} X_4, T U_4),$$

$$= -g(\hat{\nabla}_{V_4} \Phi X_4, U_4) + g((\hat{\nabla}_{V_4} \Phi) X_4, U_4) - V_4(\ln \phi_1) g(X_4, T U_4).$$
for any $X_4 \in \Gamma(\mathcal{D})$ and $U_4, V_4 \in \Gamma(\mathcal{D}^\nu \oplus \xi)$. Using (3.5), (3.1), we get

$$g(\Sigma(X_4, V_4), FU_4) = 0.$$ 

So, the required results are obtained. □

If we interchange $X_4$ by $\Phi X_4$ and $Y_4$ by $\Phi Y_4$ in (i) of Lemma 3.1 then we find the following relations.

(3.5)

$$g(\Sigma(\Phi X_4, Y_4), FU_4) = \left\{(U_4 \ln \phi_1) - \beta \eta(U_4)\right\}g(X_4, Y_4) - TU_4(\ln \phi_1)g(X_4, \Phi Y_4).$$

(3.6)

$$g(\Sigma(X_4, \Phi Y_4), FU_4) = -\left\{(U_4 \ln \phi_1) - \beta \eta(U_4)\right\}g(X_4, Y_4) + TU_4(\ln \phi_1)g(X_4, \Phi Y_4).$$

and

(3.7)

$$g(\Sigma(\Phi X_4, \Phi Y_4), FU_4) = \left\{(U_4 \ln \phi_1) - \beta \eta(U_4)\right\}g(X_4, \Phi Y_4) + TU_4(\ln \phi_1)g(X_4, Y_4).$$

Similarly, we interchange $U_4$ by $TU_4$ in Lemma 3.1 and in (3.5)-(3.7), and using Theorem 2.1 and (3.4), we derive

(3.8)

$$g(\Sigma(X_4, Y_4), FTU) = TU_4(\ln \phi_1)g(X_4, \Phi Y_4) - \cos^2 \nu \left(U_4(\ln \phi_1) - \beta \eta(U_4)\right)g(X_4, Y_4)$$

(3.9)

$$g(\Sigma(\Phi X_4, Y_4), FTU) = TU_4(\ln \phi_1)g(X_4, Y_4) + \cos^2 \nu \left(U_4(\ln \phi_1) - \beta \eta(U_4)\right)g(X_4, \Phi Y_4)$$

(3.10)

$$g(\Sigma(X_4, \Phi Y_4), FTU) = -TU_4(\ln \phi_1)g(X_4, Y_4) - \cos^2 \nu \left(U_4(\ln \phi_1) - \beta \eta(U_4)\right)g(X_4, \Phi Y_4),$$

and

(3.11)

$$g(\Sigma(\Phi X_4, \Phi Y_4), FTU) = TU_4(\ln \phi_1)g(X_4, \Phi Y_4) - \cos^2 \nu \left(U_4(\ln \phi_1) - \beta \eta(U_4)\right)g(X_4, Y_4).$$

On the other hand, we interchange $U_4$ by $TU_4$ in Lemma 3.1(ii) and using Theorem 2.1 and (3.4), we obtain

(3.12)

$$g(\Sigma(Z_4, W_4), FTU) = g(\Sigma(TU_4, W_4), \Phi Z_4) - \cos^2 \nu (V_4(\ln \phi_2) - \eta(U_4))g(Z_4, W_4).$$
Let $L = L_\nu \times_{\phi_1} L_T \times_{\phi_2} L_\perp$ be a bi-warped product submanifold of a locally conformal almost cosymplectic manifold $\tilde{L}$. Then, we have
\[
g(\Sigma(X_4, V_4), \Phi Z_4) = g(\Sigma(X_4, Z_4), FV) = 0,
\]
for any $X_4 \in \Gamma(\mathcal{D})$, $V_4 \in \Gamma(\mathcal{D}^\perp)$ and $Z_4 \in \Gamma(\mathcal{D}^\perp)$.

Proof. For any $X_4 \in \Gamma(\mathcal{D})$, $V_4 \in \Gamma(\mathcal{D}^\perp)$ and $Z_4 \in \Gamma(\mathcal{D}^\perp)$, we have
\[
g(\Sigma(X_4, Z_4), FV) = g(\nabla_{Z_4}X_4, \Phi V_4) + g(\tilde{\nabla}_{Z_4}TV, X_4),
\]
\[
= g((\nabla Z_4\Phi)X_4, V_4) - g((\tilde{\nabla} Z_4\Phi)X_4, V_4) + TV(\ln \phi_2)g(X_4, Z_4).
\]

Using (23), (31) and the orthogonality of vector fields, we get $g(\Sigma(X_4, Z_4), FV) = 0$, which is second equality. On the other hand, we have
\[
g(\Sigma(X_4, Z_4), FV) = g(\nabla_{X_4}Z_4, \Phi V_4) + g(\nabla_{X_4}TV, Z_4),
\]
\[
= g((\nabla X_4\Phi)Z_4, V_4) - g((\tilde{\nabla} X_4\Phi)Z_4, V_4) + TV(\ln \phi_1)g(X_4, Z_4).
\]

Again use of (23), (25) and the orthogonality of vector fields, we get
\[
g(\Sigma(X_4, V_4), \Phi Z_4) = g(\Sigma(X_4, Z_4), FV).
\]

Hence the claim. $\square$

4. Chen type inequality for Bi-Warped product immersions

In this section, we show how the squared norm of the second fundamental form and the bi-warped product’s warping functions are related. We will conduct a double analysis. Specifically, we will show a Chen type inequality for Bi-Warped product immersions, in two different settings (Theorem 4.1 and Theorem 4.2). In this regard, let’s start by considering the following orthogonal frame field to show our primary thesis and to be able to provide this first relationship.

Let $L = L_\nu \times_{\phi_1} L_T \times_{\phi_2} L_\perp$ be an $n$-dimensional bi-warped product submanifolds of a $(2n + 1)$-dimensional locally conformal almost cosymplectic manifold $\tilde{L}$ such that $\xi$ is tangent to the base manifold $L_\nu$. If the dimensions $\dim(L_T) = n_1$, $\dim(L_\perp) = n_2$ and $\dim(L_\nu) = n_3$, then the orthogonal frames of the corresponding tangent spaces $\mathcal{D}, \mathcal{D}^\perp$ and $\mathcal{D}^\perp$, respectively, are given by $\{e_1, \ldots, e_p, e_{p+1} = \Phi e_1, \ldots, e_{n_1} = e_2 = \Phi e_p\}$, $\{e_{n_1+1} = \tilde{e}_1, \ldots, e_{n_1+n_2} = \tilde{e}_n\}$ and $\{e_{n_1+n_2+1} = e_1^*, \ldots, e_{n_1+n_2+q} = e_q^*, e_{n_1+n_2+q+1} = e_{q+1}^* = \sec \nu T e_1^*, \ldots, e_{n_1+n_2+2q} = e_{2q}^* = \sec \nu T e_q^*, e_{n_3}^* = \tilde{e}_{2q+1} = \tilde{\xi}\}$. Then the orthonormal frame fields of the normal subbundles of $\Phi \mathcal{D}^\perp, F\mathcal{D}^\perp$ and $\mu$, respectively, are $\{e_{n_1+1} = \tilde{e}_1 = \Phi e_1, \ldots, e_{n_1+n_2} = \tilde{e}_n = \Phi \tilde{e}_n\}$,
\[
\begin{align*}
\{e_{m+n_2+1} = \tilde{\tilde{e}}_{n_2+1} = \csc \nu \Phi e_1^*, \ldots, e_{m+n_2+q} = \tilde{\tilde{e}}_{n_2+q} = \csc \nu \Phi e_q^*, e_{m+n_2+q+1} = \tilde{\tilde{e}}_{n_2+q+1} = \csc \nu \Phi \sec \nu T e_1^*, \ldots, e_{m+n_2+n_3} = \tilde{\tilde{e}}_{n_2+n_3} = \csc \nu \sec \nu T e_q^*\} \quad \text{and} \quad \{e_{m+n_2+n_3} = \tilde{\tilde{e}}_{n_2+n_3}, \ldots, e_{2n+1} = \tilde{\tilde{e}}_{2(n_2+n_3+1)-1}\}.
\end{align*}
\]
Theorem 4.1. Let $L = L_\nu \times \phi_1 L_T \times \phi_2 L_\perp$ be a $\mathcal{D}^\perp \subset \mathcal{D}^\nu$ mixed totally geodesic bi-warped product submanifold of a locally conformal almost cosymplectic manifold $\tilde{L}$ such that $\xi$ is tangent to $L_\nu$. Then

(i) The squared norm of the second fundamental form $\Sigma$ of $L$ satisfies

$$||\Sigma||^2 \geq n_1 \left(1 + \cot^2 \nu \right) \left(1 + \cos^2 \nu \right) \left(||\nabla (\ln \phi_1)||^2 - \beta^2\right)$$

$$+ \ n_2 \left(\csc^2 \nu - 1 \right) \left(||\nabla (\ln \phi_2)||^2 - \beta^2\right) \quad (4.1)$$

where $n_1 = \dim(L_T), n_2 = \dim(L_\perp)$ and $\nabla (\ln \phi_1)$ is the gradient of $\ln \phi_1$ along $L_T$ and $\nabla (\ln \phi_2)$ is the gradient of $\ln \phi_2$ along $L_\perp$.

(ii) If the equality sign in (i) holds identically, then $L_\nu$ is totally geodesic submanifold of $\tilde{L}, L_T$ and $L_\perp$ are totally umbilical submanifolds of $\tilde{L}$. In addition, $L$ is a completely $\mathcal{D}^\nu$-geodesic submanifold of $\tilde{L}$.

Proof. From (2.7) and (2.8), we get

$$||\Sigma||^2 = \sum_{i,j=1}^{m} g(\Sigma(e_i, e_j), \Sigma(e_i, e_j)) = \sum_{r=m+1}^{2n+1} \sum_{i,j=1}^{m} g(\Sigma(e_i, e_j), e_r)^2.$$ 

The above relation can take the form:

$$||\Sigma||^2 = \sum_{r=1}^{n_2} \left(\sum_{i,j=1}^{m} g(\Sigma(e_i, e_j), \tilde{e}_r)^2\right)$$

$$+ \sum_{r=n_2+1}^{n_2+n_3-1} \left(\sum_{i,j=1}^{m} g(\Sigma(e_i, e_j), \tilde{e}_r)^2\right)$$

$$+ \sum_{r=n_2+n_3}^{2(n-n_2-n_3+1)-n_1} \left(\sum_{i,j=1}^{m} g(\Sigma(e_i, e_j), \tilde{e}_r)^2\right). \quad (4.2)$$

We derive $\mathcal{D}, \mathcal{D}^\perp$ and $\mathcal{D}^\nu$, by using the constructed frame fields of $\mathcal{D}, \mathcal{D}^\perp$ and $\mathcal{D}^\nu$, leaving the third $\mu$-components positive terms because we could not find any
connection for bi-warped products in terms of the \( \mu \)-components.

\[
(4.3)
||\Sigma||^2 \geq \sum_{r=1}^{n_2} \sum_{i,j=1}^{n_1} g(\Sigma(e_i, e_j), \Phi \bar{e}_r)^2 + \sum_{r=1}^{n_2} \sum_{i,j=1}^{n_1} g(\Sigma(\bar{e}_i, \bar{e}_j), \Phi \bar{e}_r)^2
\]

\[
+ \sum_{r=1}^{n_2} \sum_{i,j=1}^{n_3} g(\Sigma(e_i^*, e_j^*), \Phi \bar{e}_r)^2 + 2 \sum_{r=1}^{n_1} \sum_{i,j=1}^{n_1} g(\Sigma(e_i, \bar{e}_j), \Phi \bar{e}_r)^2
\]

\[
+ 2 \sum_{r=1}^{n_2} \sum_{i,j=1}^{n_3} g(h(e_i, e_j^*), \Phi \bar{e}_r)^2 + 2 \sum_{r=1}^{n_2} \sum_{i,j=1}^{n_1} g(\Sigma(e_i, e_j^*), \Phi \bar{e}_r)^2
\]

\[
+ \sum_{r=1}^{n_2} \sum_{i,j=1}^{n_3} g(\Sigma(e_i, \bar{e}_j^*), \Phi \bar{e}_r)^2 + \sum_{r=1}^{n_1} \sum_{i,j=1}^{n_1} g(\Sigma(\bar{e}_i, \bar{e}_j^*), \Phi \bar{e}_r)^2
\]

\[
+ 2 \sum_{r=1}^{n_2} \sum_{i,j=1}^{n_3} g(e_i^*, e_j^*), \Phi \bar{e}_r)^2 + 2 \sum_{r=1}^{n_1} \sum_{i,j=1}^{n_1} g(\Sigma(\bar{e}_i, \bar{e}_j^*), \Phi \bar{e}_r)^2.
\]

Using Lemma 3.1(iii) with the mixed completely geodesic condition \( \mathcal{D}^1 - \mathcal{D}^\nu \), the third term in the right-hand side is identically zero. Similarly, the fifth and tenth terms vanish when Lemma 3.2 is used, whereas the sixth and twelfth terms are zero when the \( \mathcal{D}^1 - \mathcal{D}^\nu \) mixed totally condition is used. Using Lemma 3.1, the eleventh term is also zero (iv). However, we were unable to identify a relationship for the first, second, fourth, and ninth terms in the right-hand side of the above inequality for these bi-warped products, so we will leave them positive. After that, just the seventh and eighth terms must be assessed, and the statement above can be expressed as

\[
||\Sigma||^2 \geq \sum_{r=1}^{q} \sum_{i,j=1}^{n_1} g(\Sigma(e_i, e_j), \csc \nu Fe_i^*)^2 + \sum_{r=1}^{q} \sum_{i,j=1}^{n_1} g(\Sigma(e_i, e_j), \csc \nu \sec \nu \mathcal{F} e_i^*)^2
\]

\[
+ \sum_{r=1}^{q} \sum_{i,j=1}^{n_2} g(\Sigma(\bar{e}_i, \bar{e}_j), \csc \nu Fe_r^*)^2 + \sum_{r=1}^{q} \sum_{i,j=1}^{n_2} g(\Sigma(\bar{e}_i, \bar{e}_j), \csc \nu \sec \nu \mathcal{F} e_r^*)^2.
\]
Using Lemma 3.1(i) and the relations (3.10)-(3.11) in first two terms and using Lemma 3.1(ii) and (3.12) in last two terms, we get

\[ ||\Sigma||^2 \geq n_1 \csc^2 \nu(1 + \sec^2 \nu) \sum_{r=1}^{q} (T e_r^* (\ln \phi_1))^2 + \nu \csc^2 \nu \sum_{r=1}^{q} (e_r^* (\ln \phi_1) - \eta(e_r^*))^2 \]

\[ + n_2 \csc^2 \nu \sum_{r=1}^{q} (T e_r^* (\ln \phi_2))^2 + \nu \csc^2 \nu \sum_{r=1}^{q} (e_r^* (\ln \phi_2) - \eta(e_r^*))^2 \]

\[ = n_1 \csc^2 \nu(1 + \sec^2 \nu) \sum_{r=1}^{2q+1} (T e_r^* (\ln \phi_1))^2 - n_1 \csc^2 \nu \sum_{r=1}^{2q+1} g(e_r^*, T \nabla (\ln \phi_1))^2 \]

\[ - n_1 \csc^2 \nu(1 + \sec^2 \nu)(T e_{2q+1}^* (\ln \phi_1))^2 + \nu \csc^2 \nu \sum_{r=1}^{2q} (e_r^* (\ln \phi_1))^2 \]

\[ + n_2 \csc^2 \nu \sum_{r=1}^{2q+1} (T e_r^* (\ln \phi_2))^2 - n_2 \csc^2 \nu \sum_{r=1}^{2q+1} g(e_r^*, T \nabla (\ln \phi_2))^2 \]

\[ - n_2 \csc^2 \nu \sum_{r=1}^{2q} (e_r^* (\ln \phi_2))^2. \]

The third and seventh terms in the right-hand side of the last relation vanish identically because \( e_{2q+1}^* = \xi \) and \( T \xi = 0 \). As a result, using (2.9), the aforementioned inequality has the form

\[ ||\Sigma||^2 \geq n_1 \csc^2 \nu(1 + \sec^2 \nu)||T \nabla (\ln \phi_1)||^2 - n_1 \csc^2 \nu(1 + \sec^2 \nu) \sec^2 \nu \sum_{r=1}^{q} g(T e_r^*, T \nabla (\ln \phi_1))^2 \]

\[ + \nu \csc^2 \nu \sum_{r=1}^{q} (e_r^* (\ln \phi_1))^2 + n_2 \csc^2 \nu||T \nabla (\ln \phi_2)||^2 + \]

\[ - n_2 \csc^2 \nu \sec^2 \nu \sum_{r=1}^{q} g(T e_r^*, T \nabla (\ln \phi_2))^2 + \nu \csc^2 \nu \sum_{r=1}^{q} (e_r^* (\ln \phi_2))^2. \]

Using (2.9), (2.12) and the fact that \( \xi(\ln f_i) = \beta, i = 1, 2 \) (Theorem 3.2), we get

\[ ||\Sigma||^2 \geq \nu \csc^2 \nu(1 + \sec^2 \nu)(||\nabla (\ln \phi_1)||^2 - 1) + \nu \csc^2 \nu(||\nabla (\ln \phi_2)||^2 - 1), \]

which is inequality (i).

Now, we discuss the equality case, from the leaving third \( \mu \)-components terms in the right-hand side of (4.2), we have

(4.4) \[ \Sigma(T L, T L) \bot \mu. \]
We may deduce from the first, second, and fourth terms in (4.3) that
\[
\Sigma(\mathcal{D}, \mathcal{D}) \perp \Phi \mathcal{D}^\perp, \\
\Sigma(\mathcal{D}^\perp, \mathcal{D}^\perp) \perp \Phi \mathcal{D}^\perp, \\
\Sigma(\mathcal{D}, \mathcal{D}^\perp) \perp \Phi \mathcal{D}^\perp.
\]
(4.5)
The leaving ninth term of (4.3), we get
\[
\Sigma(\mathcal{D}, \mathcal{D}) \perp F \mathcal{D}^\nu.
\]
(4.6)
Then (4.4) and (4.5), give the following
\[
\Sigma(\mathcal{D}, \mathcal{D}) \in \mathcal{V}, \\
\Sigma(\mathcal{D}^\perp, \mathcal{D}^\perp) \in \mathcal{V}, \\
\Sigma(\mathcal{D}, \mathcal{D}^\perp) \in \mathcal{V}.
\]
(4.7)
As \(L\) is \(\mathcal{D}^\perp - \mathcal{D}^\nu\) mixed totally geodesic, we conclude
\[
\Sigma(\mathcal{D}^\perp, \mathcal{D}^\nu) = 0.
\]
(4.8)
From the deleting third term of (4.3), we arrive
\[
\Sigma(\mathcal{D}^\nu, \mathcal{D}^\nu) \perp \Phi \mathcal{D}^\perp.
\]
(4.9)
From (4.4), (4.6) and (4.9), we get
\[
\Sigma(\mathcal{D}^\nu, \mathcal{D}^\nu) = 0.
\]
(4.10)
Similarly, from the deleting fifth, tenth and eleventh terms of (4.3), we find, respectively,
\[
\Sigma(\mathcal{D}, \mathcal{D}^\nu) \perp \Phi \mathcal{D}^\perp, \\
\Sigma(\mathcal{D}, \mathcal{D}^\perp) \perp \mathcal{V}, \\
\Sigma(\mathcal{D}, \mathcal{D}^\nu) \perp \mathcal{V}.
\]
(4.11)
Thus, (4.4), (4.5) and (4.11), we obtain
\[
\Sigma(\mathcal{D}, \mathcal{D}^\nu) = 0, \\
\Sigma(\mathcal{D}, \mathcal{D}^\perp) = 0.
\]
(4.12)
Since \(L_T\) is a totally geodesic submanifold of \(L\). Combining with (4.8), (4.10) and (4.12), we reached that \(L_\nu\) is totally geodesic in \(\tilde{L}\).

On the other hand, since \(L_T\) and \(L_\perp\) are totally umbilical, using (4.7), we conclude that \(L_T\) and \(L_\perp\) are totally umbilical submanifolds of \(L\), which yields to (ii).

Moreover, all conditions including (4.5)-(4.12) imply that \(L\) is a \(\mathcal{D}^\nu\)-totally geodesic submanifold of \(L\).

\(\square\)

**Lemma 4.1.** Let \(L = L_T \times_\phi L_\perp \times_\phi L_\nu\) be a bi-warped product submanifold of locally conformal almost cosymplectic manifold \(\tilde{L}\) such that \(\xi \in (\mathcal{D})\). Then, we have

(i) \(g(\Sigma(X_4, Y_4), \Phi V_4) = 0\);

(ii) \(g(\Sigma(X_4, V_4), \Phi U_4) = -\Phi X_4(\ln \phi_1)g(U_4, V_4)\).

for any \(X_4, Y_4 \in \mathcal{D}\) and \(U_4, V_4 \in \mathcal{D}^\perp\).
Proof. The first component is straightforward to prove using Gauss-Weingarten formulae and the structure equation of a locally conformal almost cosymplectic manifold with (3.1). Now, we get (ii) for any \(X_4 \in \mathcal{D}\) and \(U_4, V_4 \in \mathcal{D}^\perp\),

\[
g(\Sigma(X_4, V_4), \Phi U_4) = g(\nabla_{V_4} X_4, \Phi U_4) = g((\nabla_{V_4} \Phi) X_4, U_4) - g(\nabla_{V_4} \Phi L X_4, U_4).
\]

Now, using (2.3) and (3.1) and the perpendicularity of the vector fields, we get

\[
g(\Sigma(X_4, V_4), \Phi U_4) = -\Phi X_4(\ln \phi_1) g(U_4, LV).
\]

Hence the claim. \(\square\)

**Lemma 4.2.** Let \(L = L_T \times_\phi L_\perp \times_\phi L_\nu\) be a bi-warped product submanifold of locally conformal almost cosymplectic manifold \(\tilde{L}\) such that \(\xi \in (\mathcal{D})\). Then, we have

(i) \(g(\Sigma(X_4, Y_4), FZ) = 0;\)

(ii) \(g(\Sigma(X_4, Z_4), FW) = -\Phi X_4(\ln \phi_2) g(Z_4, W_4) + \left\{X_4(\ln \phi_2) - \beta \eta(X_4)\right\} g(TZ_4, W_4).\)

for any \(X_4, Y_4 \in \mathcal{D}\) and \(Z_4, W_4 \in \mathcal{D}^\nu\).

**Proof.** For any \(Y_4 \in \mathcal{D}\) and \(Z_4 \in \mathcal{D}^\nu\),

\[
g(\Sigma(X_4, Y_4), FZ) = g(\nabla_{X_4} Y_4, FZ) = g(\nabla_{X_4} Y_4, \Phi Z_4) - g(\nabla_{X_4} Y_4, PZ).
\]

Since \(g(Y_4, TZ_4) = g(Y_4, \Phi Z_4) = -g(\Phi Y_4, Z_4) = 0\), from (4.13) and (2.3), we obtain

\[
g(\Sigma(X_4, Y_4), FZ) = \beta g(\Phi Y_4, \nabla_{X_4} Z_4) + g(\nabla_{X_4} Y_4, PZ) = \beta g(\Phi Y_4, \nabla_{X_4} Z_4) + g(Y_4, \nabla_{X_4} PZ).
\]

Hence using (3.1), we get the desired results. Again for any \(X_4, Y_4 \in \mathcal{D}\) and \(Z_4, W_4 \in \mathcal{D}^\nu\),

\[
g(\Sigma(X_4, Z_4), FW) = g(\nabla_{Z_4} X_4, \Phi W_4) - g(\nabla_{Z_4} X_4, TW) = g(\Phi \nabla_{Z_4} X_4, W_4) - g(\nabla_{Z_4} X_4, TW).
\]

By using the co-variant derivative property of \(\Phi\) and (3.1), we get

\[
g(\Sigma(X_4, Z_4), FW) = g((\nabla_{Z_4} \Phi) X_4, X_4, W_4) - g(\nabla_{Z_4} \Phi X_4, W_4) - X_4(\ln \phi_2) g(Z_4, TW).
\]

Applying (2.3) and (3.1), we obtained the required results. \(\square\)

Interchanging the following relations yields the following results \(X\) by \(\Phi X_4, Z_4\) by \(TZ_4\) and \(W_4\) by \(TW_4\) in Lemma (4.1),
(4.14) \[ g(\Sigma(\Phi X, Z), FW) = \left\{ X_3(\ln \phi_2) - \beta \eta(X_4) \right\} g(Z_4, W_4) + \Phi X_4(\ln \phi_2) g(TZ_4, W_4) \]

(4.15) \[ g(\Sigma(X_4, TZ), FW) = -\Phi X_4(\ln \phi_2) g(TZ_4, W_4) - \left\{ X_4(\ln \phi_2) - \beta \eta(X_4) \right\} \cos^2 \nu g(Z_4, W_4) \]

(4.16) \[ g(\Sigma(X_4, Z), FTW) = -\Phi X_4(\ln \phi_2) g(Z_4, TW_4) - \left\{ X_4(\ln \phi_2) - \beta \eta(X_4) \right\} \cos^2 \nu g(Z_4, W_4) \]

(4.17) \[ g(\Sigma(\Phi X, TZ), FW) = \left\{ X_4(\ln \phi_2) - \beta \eta(X_4) \right\} g(TZ_4, W_4) - \Phi X_4(\ln \phi_2) \cos^2 \nu g(Z_4, W_4) \]

(4.18) \[ g(\Sigma(\Phi X, Z), FTW) = \left\{ X_4(\ln \phi_2) - \beta \eta(X_4) \right\} g(Z_4, TW_4) + \Phi X_4(\ln \phi_2) \cos^2 \nu g(Z_4, W_4) \]

(4.19) \[ g(\Sigma(X, TZ), FTW) = -\Phi X_4(\ln \phi_2) \cos^2 \nu g(TZ, W_4) \]

(4.20) \[ g(\Sigma(\Phi X, TZ), FTW) = \left\{ X_4(\ln \phi_2) - \beta \eta(X_4) \right\} \cos^2 \nu g(Z_4, W_4). \]

**Lemma 4.3.** Let \( L = L_T \times \phi_1 L_\perp \times \phi_2 L_\nu \) be a bi-warped product submanifold of a locally conformal almost cosymplectic manifold \( L \), then, we have

(i) \( g(\Sigma(X, Z), \Phi V) = 0 \),

(ii) \( g(\Sigma(X, V), FZ) = 0 \).

for any \( X \in \mathcal{D}, V_4 \in \mathcal{D}_\perp \) and \( Z_4 \in \mathcal{D}_\nu \). Moreover, \( L_T \) is an anti-invariant submanifold, \( L_\perp \) is an anti-invariant submanifold and \( L_\nu \) is a proper slant submanifold of \( L \).

The following frame fields for a \( m \)-dimensional bi-warped product submanifold are now constructed. Let \( L = L_T \times \phi_1 L_\perp \times \phi_2 L_\nu \) be of a \((2n+1)\)-dimension locally conformal almost cosymplectic manifold \( L \) such that \( \xi \) is tangent to \( L_T \).

If the dimensions \( \dim(L_T) = 2t_1 + 1, \dim(L_\perp) = t_2, \text{ and } \dim(L_\nu) = 2t_3 \), then the orthonormal frames of the corresponding tangent spaces are \( \mathcal{D}, \mathcal{D}_\perp \) and \( \mathcal{D}_\nu \), respectively. Given by \( \{ e_1, \ldots, e_{t_1+1} = \Phi e_1, \ldots, e_{2t_1+1} = \xi \}, \{ e_{2t_1+2} = \tilde{e}_1, \ldots, e_{2t_1+t_2+1} = \tilde{e}_t \} \) and \( \{ e_{2t_1+t_2+2} = \epsilon_1, \ldots, e_{2t_1+t_2+t_3} = \epsilon_t, e_{t_2+t_3} = e_{t_2+t_3} = e_{t_3} = \sec \nu T \epsilon_{t_3}^{*}, \ldots, e_m = \sec \nu T e_I^{*}, \ldots, e_m = \sec \nu T e_{t_3}^{*}, \ldots, e_{m+t_2} = \tilde{e}_t \} \), then the orthonormal frame fields of the normal subbundles of \( \mathcal{D}_\perp, \mathcal{D}_\nu \) and \( \mu \), respectively, are \( \{ e_{m+1} = \tilde{e}_1 = \Phi \tilde{e}_1, \ldots, e_{m+t_2} = \tilde{e}_t = \Phi \tilde{e}_t \}, \{ e_{m+t_2+1} = \tilde{e}_{t+1} = \csc \nu \Phi e_I^{*}, \ldots, e_{m+t_2+t_3} = \tilde{e}_{t_2+t_3} = \tilde{e}_{t_3} = \csc \nu \Phi e_{t_3}^{*} \}. \)
Theorem 4.2. Let $L = L_T \times \phi_1 L_\perp \times \phi_2 L_\nu$ be a $\mathcal{D}^\perp - \mathcal{D}^\nu$ mixed totally geodesic bi-warped product submanifold of a locally conformal almost cosymplectic manifold $\tilde{L}$ such that $\xi$ is tangent to $L_T$. Then

(i) The squared norm of the second fundamental form $\Sigma$ of $L$ satisfies

\[
||\Sigma||^2 \geq 2t_2(||\tilde{\nabla}(\ln \phi_1)||^2 - \beta^2) + 4t_3 \csc^2 \nu \left( \sin^2 \nu + 2 \cos^2 \nu \right) \left( ||\tilde{\nabla}(\ln \phi_2)||^2 - \beta^2 \right)
\]

(4.21)

where $t_2 = \dim(L_\perp)$, $2t_3 = \dim(L_\nu)$ and $\tilde{\nabla}(\ln \phi_1)$ is the gradient of $\ln \phi_1$ along $L_\perp$ and $\tilde{\nabla}(\ln \phi_2)$ is the gradient of $\ln \phi_2$ along $L_\nu$.

(ii) If the equality sign in (i) holds identically, then $L_T$ is a totally geodesic submanifold of $\tilde{L}$, $L_\perp$ and $L_\nu$ are totally umbilical submanifolds of $\tilde{L}$ with $-\tilde{\nabla}(\ln \phi f_1)$ and $-\tilde{\nabla}(\ln \phi_2)$ as mean curvature vectors, respectively.

Proof. From the definition of $\Sigma$, we get

\[
||\Sigma||^2 = \sum_{i,j=1}^m g(\Sigma(e_i, e_j), \Sigma(e_i, e_j)) = \sum_{r=m+1}^{2n+1} \sum_{i,j=1}^m g(\Sigma(e_i, e_j), \tilde{e}_r)^2.
\]

So the above expression can be rewritten in the form

\[
||\Sigma||^2 = \sum_{r=1}^{t_2} \left( \sum_{i,j=1}^m g(\Sigma(e_i, e_j), \tilde{e}_r)^2 \right) + \sum_{r=t_2+1}^{t_2+2t_3} \left( \sum_{i,j=1}^m g(\Sigma(e_i, e_j), \tilde{e}_r)^2 \right) + \sum_{r=t_2+2t_3+1}^{2n+1-t_2-2t_3-m} \left( \sum_{i,j=1}^m g(\Sigma(e_i, e_j), \tilde{e}_r)^2 \right).
\]

(4.22)

Leaving the third $\mu$-components positive terms as we could not find any relation for bi-warped products in terms of the $\mu$-components, then, by using the constructed
Then, we assume the forth and eleventh terms which are evaluated. Thus, we have left these positive terms but we will consider them for equality case. We do not have any relations for bi-

terms are also identically zero by using Lemma (4.2) and Lemma (4.3)(ii) respectively. Similarly, seventh and tenth terms are also identically zero by using Lemma (4.1)(i) and Lemma (4.3)(ii), respectively. In addition, there is no relation for $g(\Sigma(\bar{e}_i, \bar{e}_j), \Phi\bar{e}_r)$, $i, j, r = 1, ..., t_2$, when the vectors are from the same space. We do not have any relations for bi-warped products of the following terms:

$g(\Sigma(e^*_i, e^*_j), \Phi\bar{e}_r)$, $i, j = 1, ..., 2t_3$ and $r = 1, ..., t_2$;

$g(\Sigma, \Phi\bar{e}_r)$, $i, j = 1, ..., 2t_3$ and $r = 1, ..., t_2$;

$g(\Sigma(e^*_i, e^*_j), \bar{e}_r)$, $i, j = 1, ..., t_2$ and $r = 1, ..., 2t_3$;

$g(\Sigma(e^*_i, e^*_j), \bar{e}_r)$, $i, j, r = 1, ..., 2t_3$;

$g(\Sigma(e^*_i, e^*_j), \bar{e}_r)$, $i = 1, ..., t_2$ and $j, r = 1, ..., 2t_3$;

Thus, we have left these positive terms but we will consider them for equality case. Then, we assume the forth and eleventh terms which are evaluated.
\[ ||\Sigma||^2 \geq 2 \sum_{j,r=1}^{t_2} \sum_{i=1}^{2t_1+1} g(\Sigma(e_i, e_j), \Phi\bar{e}_r)^2 + 2 \sum_{j,r=1}^{t_2} \sum_{i=1}^{2t_1+1} g(\Sigma(e_i, e^*_j), \bar{e}_r)^2 \\
+ 2 \sum_{j,r=1}^{2t_1} \sum_{i=1}^{t_1} g(\Sigma(e^*_i, e^*_j), \Phi\bar{e}_r)^2 + 2 \sum_{j,r=1}^{t_2} g(\Sigma(e_{2t_1+1}, e^*_j), \Phi\bar{e}_r)^2 \\
+ 2 \sum_{j,r=1}^{2t_1} \sum_{i=1}^{t_1} g(\Sigma(e_i, e^*_j), \bar{e}_r)^2 + 2 \sum_{j,r=1}^{t_2} g(\Sigma(e_{2t_1+1}, e^*_j), \Phi\bar{e}_r)^2. \]

Since \(e_{2t_1+1} = \xi\) and for a locally conformal almost cosymplectic manifold \(\Sigma(\xi, X^4) = 0\), for any \(X \in TM\), then the second and forth terms in the right hand side of the above expression vanish identically. Thus, by use of frame fields of \(D, D^\nu, \Phi D^\perp\) and \(\phi D^\nu\), we find

\[ ||\Sigma||^2 \geq 2 \sum_{j,r=1}^{t_2} \sum_{i=1}^{t_1} g(\Sigma(e_i, e_j), \Phi\bar{e}_r)^2 + 2 \sum_{j,r=1}^{t_2} \sum_{i=1}^{t_1} g(\Sigma(\Phi e_i, e_j), \Phi\bar{e}_r)^2 \\
+ 2 \csc^2 \nu \sum_{j,r=1}^{t_1} \sum_{i=1}^{t_1} g(\Sigma(e_i, e^*_j), F e^*_r)^2 + 2 \csc^2 \nu \sum_{j,r=1}^{t_1} \sum_{i=1}^{t_1} g(\Sigma(\Phi e_i, e^*_j), \phi e^*_r)^2 \\
+ 2 \sec^2 \nu \csc^2 \nu \sum_{j,r=1}^{t_3} \sum_{i=1}^{t_1} g(\Sigma(e_i, T e^*_j), \phi e^*_r)^2 + 2 \sec^2 \nu \csc^2 \nu \sum_{j,r=1}^{t_3} \sum_{i=1}^{t_1} g(\Sigma(\Phi e_i, T e^*_j), \phi e^*_r)^2 \\
+ 2 \sec^2 \nu \csc^2 \nu \sum_{j,r=1}^{t_3} \sum_{i=1}^{t_1} g(\Sigma(e_i, e^*_j), F T e^*_r)^2 + 2 \sec^2 \nu \csc^2 \nu \sum_{j,r=1}^{t_3} \sum_{i=1}^{t_1} g(\Sigma(\Phi e_i, e^*_j), F T e^*_r)^2 \\
+ 2 \sec^4 \nu \csc^2 \nu \sum_{j,r=1}^{t_3} \sum_{i=1}^{t_1} g(\Sigma(e_i, T e^*_j), F T e^*_r)^2. \]
Then from Lemma 4\((ii)\), Lemma 4\((ii)\) and the relations \((\ref{4.14})-(\ref{4.20})\), the above expression takes the form

\[
||\Sigma||^2 \geq 2t_2 \sum_{i=1}^{t_1} (e_i(\ln \phi_1))^2 + 2t_2 \sum_{i=1}^{t_1} (\Phi e_i(\ln \phi_1))^2
\]

\[
+ 4t_3 \csc^2 \nu \sum_{i=1}^{t_1} (e_i(\ln \phi_2))^2 + 4t_3 \cot^2 \nu \sum_{i=1}^{t_1} (e_i(\ln \phi_2))^2
\]

\[
+ 4t_3 \csc^2 \nu \sum_{i=1}^{t_1} (\Phi e_i(\ln \phi_2))^2 + 4t_3 \cot^2 \nu \sum_{i=1}^{t_1} (\Phi e_i(\ln \phi_2))^2
\]

\[
= 2t_2 \sum_{i=1}^{2t_1+1} (e_i(\ln \phi_1))^2 - 2t_2 (e_{2t_1+1}(\ln \phi_1))^2
\]

\[
+ 4t_3(1 + 2 \cot^2 \nu) \sum_{i=1}^{2t_1+1} (e_i(\ln \phi_2))^2 - 4t_3(1 + 2 \cot^2 \nu)(e_{2t_1+1}(\ln \phi_2))^2
\]

Since \(2t_1 + 1 = \xi\), by \(\xi(\ln \phi_1) = \beta, \xi(\ln \phi_2) = \beta\) and using \((\ref{2.9})\), we obtain

\[
||\Sigma||^2 \geq 2t_2(||\tilde{\nabla}(\ln \phi_1)||^2 - \beta^2) + 4t_3(1 + 2 \cot^2 \nu)(||\tilde{\nabla}(\ln \phi_2)||^2 - 4t_3(1 + 2 \cot^2 \nu)\beta^2,
\]

which is the inequality \((i)\).

For the equality case, from the leaving third term in the right hand side of equation \((\ref{4.22})\), we get

\[
(4.24) \quad \Sigma(X_4, Y_4) \perp \mu
\]

for any \(X_4, Y_4 \in TM\).

In addition from the vanishing first and seventh terms of \((\ref{4.23})\), we find

\[
(4.25) \quad \Sigma(D, D) \perp \Phi D, \quad \Sigma(D, D) \perp \phi D^\nu.
\]

Then from \((4.24)\) and \((4.25)\), we get

\[
(4.26) \quad \Sigma(D, D) = \{0\}.
\]

Similarly, from the leaving second and eighth terms of \((\ref{4.23})\), we find that

\[
(4.27) \quad \Sigma(D^\perp, D^\perp) \perp \Phi D^\perp, \quad \Sigma(D^\perp, D^\perp) \perp \phi D^\nu.
\]

Thus, from \((4.24)\) and \((4.27)\), we have

\[
(4.28) \quad \Sigma(D^\perp, D^\perp) = \{0\}.
\]

Now, from the leaving third and ninth terms of \((\ref{4.23})\), we find that

\[
(4.29) \quad \Sigma(D^\nu, D^\nu) \perp \Phi D^\perp, \quad \Sigma(D^\nu, D^\nu) \perp \phi D^\nu.
\]

Thus, from \((4.24)\) and \((4.28)\), we have

\[
(4.30) \quad \Sigma(D^\nu, D^\nu) = \{0\}.
\]
Since $M_T$ is totally geodesic submanifold of $L$, by using (4.26), (4.28) and (4.30), we conclude that $M_T$ is totally geodesic in $\tilde{L}$, which is the first part of the equality case.

On the other hand, the vanishing tenth term of (4.23) with (4.24), given by

\[(4.31) \quad \Sigma(D, D^\perp) \subset \Phi D^\perp\]

Similarly, from the vanishing fifth term in (4.23) with (4.24), we get

\[(4.32) \quad \Sigma(D, D^\nu) \subset \Phi D^\nu\]

In addition, from the leaving sixth and twelfth terms in (4.23), we get

\[(4.33) \quad \Sigma(D^\perp, D^\nu) \perp \Phi D^\perp, \Sigma(D^\perp, D^\nu) \perp \phi D^\nu.\]

Thus, from (4.24) and (4.33), we have

\[(4.34) \quad \Sigma(D^\perp, D^\nu) = \{0\}.\]

On the other hand, from (4.2), we know that, for any $1 \leq i \neq j \leq 2$, and any vector field $Z_{4i}$ in $D_i$ and $Z_{4j}$ in $D_j$, we have

\[\nabla_{Z_{4i}} Z_{4j} = 0,\]

which implies

\[g(\nabla_{Z_{4i}} W_{4i}, Z_{4j}) = 0.\]

Using this fact, if $h^\perp$ denotes the second fundamental form of $L_\perp$ in $L$, then we have

\[g(\Sigma^\perp(U_4, V_4), X_4) = g(\nabla_{U_4} V_4, X_4) = g(\tilde{\nabla}_{U_4} V_4, X_4) = -g(\tilde{\nabla}_{U_4} X_4, V).\]

for any $U, V \in D^\perp$ and $X_4 \in D$. Using (2.4) and (3.1), we find

\[g(\Sigma^\perp(U, V), X_4) = -X_4(\ln \phi_1)g(U, V).\]

or equivalently,

\[(4.35) \quad \Sigma^\perp(U, V) = -\tilde{\nabla}(\ln \phi_1)g(U, V).\]

Similarly, if $\Sigma^\nu$ is the second fundamental form of $L_\nu$ in $L$, then we can obtain

\[(4.36) \quad \Sigma^\nu(Z, W) = -\tilde{\nabla}(\ln \phi_2)g(Z, W),\]

for any $Z, W \in D^\nu$. Since $L_\perp$ and $L_\nu$ are totally umbilical in $L$, using this fact with (4.31), (4.32), (4.35) and (4.36), we conclude that $L_\perp$ and $L_\nu$ are totally umbilical submanifolds of $L$, which proves statement (ii) and we are done. \[\square\]
ACKNOWLEDGMENTS

The second author is thankful to CSIR for providing financial assistance in terms of JRF scholarship vide letter no. (09/1051(0026)/2018-EMR-1).

Statements and Declarations

The authors declare that they have no financial or non-financial interests, directly or indirectly related to the work presented for publication.

Data availability

Not applicable.

Author Contribution statement

All authors contributed equally to the paper.

REFERENCES

[1] Ali, A.; Mofarreh, F. Geometric inequalities of bi-warped product submanifolds of nearly Kenmotsu manifolds and their applications. *Mathematics* 2020, 8, 1805.

[2] Ali, A.; Othman, W.A.M.; Ozel, C.; Hajjari, T. A geometric inequality for warped product pseudo-slant submanifolds of nearly Sasakian manifolds. *C. R. Acad. Bulg. Sci.* 2017, 70, 175–182.

[3] Ali, A.; Othman, W.A.M.; Ozel, C. Some inequalities for warped product pseudo-slant submanifolds of nearly Kenmotsu manifolds. *J. Inequal. Appl.* 2015, 2015, 291.

[4] Bishop, R. L., O’Neill, B., *Manifolds of negative curvature*, Trans. Amer. Math. Soc, 145, 1–49.

[5] Chen, B.Y.; Dillen, F. Optimal inequalities for multiply warped product submanifolds. *Int. Electron. J. Geom.* 2008, 1, 1–11.

[6] Chen, B.Y., *Another general inequality for CR-warped products in complex space forms*, Hokkaido Math. J., 32 (2003), 415-444.

[7] Chen, B.Y., *Some pinching and classification theorems for minimal submanifolds*, Archiv der Math, 60 (1993), 568-578.

[8] K. Kenmotsu, A class of almost contact Riemannian manifolds, *Tohoku Math. J.*, 24 (1972), 93-103.

[9] Mustafa, A.; Uddin, S.; Wong, R.B. Generalized inequalities on warped product submanifolds in nearly trans-Sasakian manifolds. *J. Inequal. Appl.* 2014, 2014, 346.

[10] O’Neill, B., *Semi-Riemannian geometry with applications to relativity*, Academic Press, New York (1983).

[11] Osserman, R., *Curvature in the eighties*. Amer. Math. Monthly 97 (1990), 731-756

[12] Pigazzini, A., Özel, C., Jafari, S., Pincak, R., DeBenedictis,A., *A family of special case of sequential warped product manifolds with semi-Riemannian Einstein metrics*, https://doi.org/10.48550/arXiv.2203.04572, (2022).

[13] Uddin, S., Alqahtani, L. S., *Chen type inequality for warped product immersions in cosymplectic space forms*, J. Nonlinear Sci. Appl. 9 (2016), 2914–2921.

[14] Uddin, S.; Al-Solamy, F.R.; Shahid, M.H.; Saloom, A. B.-Y. Chen’s inequality for bi-warped products and its applications in Kenmotsu manifolds. *Mediterr. J. Math.* 2018, 15, 193.
[15] Uddin, S.; Khan, K.A. An inequality for contact CR-warped product submanifolds of nearly cosymplectic manifolds. *J. Inequal. Appl.* 2012, 2012, 304.

[16] Uddin, S.; Mustafa, A.; Wong, R.B.; Ozel, C. A geometric inequality for warped product semi-slant submanifolds of nearly cosymplectic manifolds. *Rev. Un. Mat. Argentina.* 2014, 55, 55–69.

Ramandeep Kaur - Department of Mathematics and Statistics, Central University of Punjab, Bathinda, Punjab-151 401, India; E-mail Address: ramanaulakh1966@gmail.com

Gauree Shanker - Department of Mathematics and Statistics, Central University of Punjab, Bathinda, Punjab-151 401, India; E-mail Address: gauree.shanker@cup.edu.in

Alexander Pigazzini∗ - Mathematical and Physical Science Foundation, 4200 Slagelse, Denmark; E-mail Address: pigazzini@topositus.com

Saeid Jafari - Mathematical and Physical Science Foundation, 4200 Slagelse, Denmark and College of Vestsjaelland South, Herrestaedte 11, 4200 Slagelse, Denmark; E-mail Address: saeidjafari@topositus.com

Cenap Özel - Department of Mathematics, Faculty of Science, King Abdulaziz University, 21589 Jeddah, Saudi Arabia; E-mail Address: cozel@kau.edu.sa

Abdulqader Mustafa - Department of Mathematics, Faculty of Arts and Science, Palestine Technical University, Kadoorei, Tulkarm, Palestine; E-mail Address: abdulqader.mustafa@ptuk.edu.ps