1. Introduction

Let \((M, g)\) be a Riemannian manifold of dimension \(d\). We are interested in the Schrödinger
\[
\begin{aligned}
&i\partial_t u - \Delta_g u = 0 \\
u(0) = u_0
\end{aligned}
\]
and wave equations on \(M\)
\[
\begin{aligned}
&\partial^2_t u - \Delta_g u = 0 \\
&(u(0), \partial_t u(0)) = (f, g),
\end{aligned}
\]
where \(\Delta_g\) is the Laplace-Beltrami operator. A key to study the perturbative theory and the nonlinear problems associated with these equations is to understand the size and the decay of the linear flows. One tool to quantify these decays is the so-called Strichartz estimates
\[
\|u\|_{L^p(0,T)L^r(M)} \leq C_T \left(\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}\right), \quad \text{(Waves)}
\]
\[
\|u\|_{L^p(0,T)L^r(M)} \leq C_T \|u_0\|_{L^2}, \quad \text{(Schrödinger)}
\]
where \((p, q)\) has to follow an admissibility condition given by the scaling of the equation, respectively
\[
\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - s, \quad \frac{1}{q} + \frac{d-1}{2r} \leq \frac{d-1}{4},
\]
\[
\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \quad (q, r, d) \neq (2, \infty, 2),
\]
for the Schrödinger and wave equations.

These estimates have a long story, beginning with the work of [Str77] for the \(p = q\) case in \(\mathbb{R}^n\), extended to all exponents by [GV85], [LS95], and [KT98]. For the wave equation in a manifold without boundary, the finite speed of propagation shows that it suffices to work in local coordinates to obtain local Strichartz estimates. This path was followed by [Kap89], [MSS93], [Smi98], and [Tat02]. The case of a manifold with boundary, where reflections have to be dealt with, is more difficult. Estimates outside one convex obstacle for the wave equation were
obtained by [SS95], following the parametrix construction of Melrose and Taylor, which gives an explicit representation of the solution near diffractive points, and for the Schrödinger equation later by [Iva10].

The first local estimates for the wave equation on a general domain were shown by [BLP08] for certain ranges of indices, then extended by [BSS09]. These estimates cannot be as good as in the flat case: [Iva12] showed indeed that a loss has to occur if some concavity is met, because of the formation of caustics. Recently, [ILP14] and [ILLP] obtained almost sharp local Strichartz estimates inside a convex domain.

One obstruction to the establishment of global estimates without loss is the presence of trapped geodesics. Under a non trapping assumption, such estimates were established for the wave equation by the works of [SS00], [Bur03] and [Met04]. For the Schrödinger flow in the boundaryless case, [BT07], [Bou11], [HTW06], [ST02] obtained the estimates in several non-trapping geometries.

When trapped geodesics are met, [Bur04] showed that a loss with respect to the flat case has to occur for the wave equation in the global $L^2$ integrability of the flow, and his counterpart, the smoothing estimate, for the Schrödinger equation, which write respectively in the flat case as

$$\| (\chi u, \chi \partial_t u) \|_{L^2(\mathbb{R}, \dot{H}^s \times \dot{H}^{s-1})} \lesssim \| u_0 \|_{\dot{H}^s} + \| u_1 \|_{\dot{H}^{s-1}} \quad \text{(Waves)},$$

$$\| \chi u \|_{L^2(\mathbb{R}, \dot{H}^{1/2})} \lesssim \| u_0 \|_{L^2} \quad \text{(Schrödinger)}. $$

Despite this loss in the smoothing estimate, [BGH10] showed Strichartz estimates without loss for the Schrödinger equation in an asymptotically euclidian manifold without boundary for which the trapped set is sufficiently small and exhibits an hyperbolic dynamic.

Following this breakthrough, we recently proved in [Laf17b, Laf17a] global Strichartz estimates without loss for Schrödinger and wave equations outside two strictly convex obstacles, exhibiting in the boundary case the first trapped situation where no loss occurs. The goal of this paper is to extend this result to the case of the exterior of $N \geq 3$ convex obstacles, which is in many aspects a counterpart with boundaries of the framework studied in [BGH10].

In this $N$-convex obstacles setting, there is infinitely many trapped rays. Therefore, there is a competition between the large number of parts of the flow that remain trapped between the obstacles and the decay of each such part. For a sufficient decay to hold, this competition has to occur in a favorable way. This is the so called Ikawa condition:

**Definition 1.1** (Ikawa condition, 1: strong hyperbolicity). There exists $\alpha > 0$ such that the following condition holds

$$(1.3) \quad \sum_{\gamma \in \mathcal{P}} \lambda_\gamma d_\gamma e^{\alpha d_\gamma} < \infty.$$ 

Here $\mathcal{P}$ denotes the set of all primitive periodic trajectories, $d_\gamma$ the length of the trajectory $\gamma$ and $\lambda_\gamma = \sqrt{\mu_\gamma \mu'_\gamma}$, where $\mu_\gamma$ and $\mu'_\gamma$ are the two eigenvalues of modulus smaller than one of the Poincaré map associated with $\gamma$. This condition was first introduced by [Ika82] when investigating the decay of the local energy of the wave equation. Notice that it is in particular automatically verified when the obstacles are sufficiently far from each other. It is the analog of the topologic pressure condition arising in [BGH10].
We will moreover suppose the second part of the Ikawa condition to be verified, namely, denoting by $\Theta_i$ the obstacles:

**Definition 1.2** (Ikawa condition, 2: no obstacle in shadow). For all $i, j, k$ pairwise distincts,

\[(1.4) \quad \text{Conv}(\Theta_i \cup \Theta_j) \cap \Theta_k = \emptyset.\]

At the difference of the first one, and excepting the degenerated situation where a periodic trajectory is tangent to an obstacle, this condition may be purely technical (it permits to construct solutions without been preoccupied by the shadows induced by the obstacles) and should be avoided with a more careful analysis.

We are now in position to state our result.

**Theorem 1.1.** Let $(\Theta_i)_{1 \leq i \leq N}$ be a finite family of smooth strictly convex subsets of $\mathbb{R}^3$, such that Ikawa’s conditions (1.3) and (1.4) hold, and $\Omega = \mathbb{R}^3 \setminus \bigcup_{1 \leq i \leq N} \Theta_i$.

Then, under the non-endpoint admissibility conditions:

\[
\begin{align*}
\frac{1}{q} + \frac{3}{r} &= \frac{3}{2} - s, \quad \frac{1}{q} + \frac{1}{r} \leq \frac{1}{2}, \quad q \neq \infty, \quad \text{(Waves)} \\
\frac{2}{q} + \frac{3}{r} &= \frac{3}{2}, \quad (q, r) \neq (2, 6), \quad \text{(Schrödinger)}
\end{align*}
\]

**global Strichartz estimates without loss hold for both Schrödinger and wave equations in $\Omega$:**

\[
\begin{align*}
\|u\|_{L^q(\mathbb{R}, L^r(\Omega))} &\lesssim \|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}}, \quad \text{(Waves)} \\
\|u\|_{L^q(\mathbb{R}, L^r(\Omega))} &\lesssim \|u_0\|_{L^2}, \quad \text{(Schrödinger)}
\end{align*}
\]

**Overview of the proof.** We generalise the approach introduced in [Laf17a, Laf17b].

As we dealt with the Schrödinger equation outside two convex obstacles in [Laf17b] and showed in [Laf17a] how to adapt the work to the wave equation, the main novelty of this note is how to handle the $N$-convex framework, and therefore we present a detailed proof of our main result in the more intricate case of the Schrödinger equation, and briefly explain how to adapt it to the wave equation with the material of [Laf17a] in the last section.

In the flat case, the smoothing estimate permits to stack Strichartz estimates in time $\sim h$ for data of frequency $\sim h^{-1}$ to show global estimates. As remarked in [BGH10], the logarithmic loss that appears in our setting in the smoothing estimate can be compensated if we show Strichartz estimates in time $h |\log h|$ instead of $h$ near the trapped set, provided a smoothing estimate without loss in the non trapping region is at hand. Therefore, our first section is devoted to prove such an estimate, using a commutator argument together with the escape function construction of Morawetz,Ralston and Strauss [MRS77]. We then show that we can reduce ourselves to data micro-locally supported near trapped trajectories, and that remain in a neighbourhood of it in logarithmic times. We extend to the $N$-convex framework the construction of an approximate solution for such data done in [Laf17a, Laf17b] following ideas of [Ika88, Ika82] and [Bur93], and finally, we show that under the strong hyperbolicity assumption (1.3), this construction gives a sufficient decay.
Notations.

- We denote by \( K \subset T^*\Omega \cup T^*\partial\Omega \) the trapped set, which is composed of infinitely many periodic trajectories,
- and by \( P \) the set of all primitive periodic trajectories, that is, followed only once,
- the operator \( \psi(-h^2\Delta) \) localizes at frequencies \( |\xi| \in [\alpha_0 h^{-1}, \beta_0 h^{-1}] \), we refer to [Iva10] for the definition of this operator,
- the set \( I \) is the set of all stories of reflexions, that is all finite sequences \((j_1, \cdots, j_k)\) with values in \([1, \cdots, N]\) such that \( j_i \neq j_{i+1} \),
- moreover, we will adopt all the notations introduced in [Laf17b]. Let us in particular recall that

\[ \Phi_t : T^*\Omega \cup T^*\partial\Omega \rightarrow T^*\Omega \cup T^*\partial\Omega \]

denotes the billiard flow on \( \Omega \): \( \Phi_t(x,\xi) \) is the point attained after a time \( t \) from the point \( x \) in the direction \( \frac{\xi}{|\xi|} \) at the speed \( |\xi| \), following the laws of geometrical optics,
- finally, let us recall that the spatial and directional components of \( \Phi_t \) are respectively denoted \( X_t \) and \( \Xi_t \).

2. Smoothing effect without loss outside the trapped set

Let us recall the smoothing effect with logarithmic loss obtained in [Bur04] in our framework of a family of strictly convex obstacle verifying Ikawa’s condition:

**Proposition 2.1.** For any \( \chi \in C_c^\infty(\mathbb{R}^3) \) and any \( u_0 \in L^2(\Omega) \) such that \( u_0 = \psi(-h^2\Delta)u_0 \), we have

\begin{equation}
\| \chi e^{-it\Delta} u_0 \|_{L^2(\mathbb{R},L^2(\Omega))} \lesssim (h |\log h|)^{\frac{1}{2}} \| u_0 \|_{L^2}.
\end{equation}

The aim of this first section is to prove a smoothing effect without loss outside the trapped set:

**Proposition 2.2** (Local smoothing without loss in the non trapping region). Let \( \phi \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R}^3) \) be supported in the complementary of the trapped set, \( K^c \). Then we have, for \( u_0 = \psi(-h^2\Delta)u_0 \)

\begin{equation}
\| Op_h(\phi)e^{-it\Delta} u_0 \|_{L^2(\mathbb{R},H^{1/2}(\Omega))} \lesssim \| u_0 \|_{L^2}.
\end{equation}

**Proof.** We will use the same strategy as in [BGH10] lemma 2.2, adapting the proof in the case of a domain with boundary. Notice that, for any operator \( A \),

\begin{equation}
(Au, u)(T) - (Au, u)(0) = \int_0^T \int_\Omega \langle [i\Delta, A] u, u \rangle + \int_0^T \int_{\partial\Omega} \langle Au, \partial_n u \rangle.
\end{equation}

Thus, if we find an operator \( A \) of order 0 such that \( [i\Delta, A] \) is elliptic and positive on the support of \( \phi \) and such that the border term

\[ B = \int_0^T \int_{\partial\Omega} \langle Au, \partial_n u \rangle d\sigma dt \]

is essentially positive, we shall obtain the desired estimate.
Notations. If $b \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ is a real symbol such that $b \geq 0$, and $b \geq \alpha$ on $U$, we use Garding inequality on symbols of the form
\[
    b - \alpha \left( \frac{a \bar{a}}{\sup |a|} \right) \geq 0,
\]
where $a \in C^\infty_c(\mathbb{R}^3 \times \mathbb{R}^3)$ is supported in $U$. Notice that we have $\text{Op}_h(a\bar{a}) = \text{Op}_h(a)\text{Op}_h(a)^* + O(h)$.

Moreover, we will denote, in this section and this section only, $\Phi$ for the operator associated to $\phi$.

The symbol of $A$ at the border as an operator acting on Schrödinger waves. We perform the semi-classical time change of variable to write:
\[
    B = h^{-1} \int_0^{hT} \int_{\partial\Omega} (A(e^{ith}u_0), \partial_n(e^{ith}u_0)) \, d\sigma \, dt
\]

We use the strategy of [MRS77] to derive the symbol of $A$ at the border as an operator acting on Schrödinger waves. Let us consider $A$ as an operator acting on $\partial\Omega \times \mathbb{R}$. Notice that, because $\partial\Omega \times \mathbb{R}$ is nowhere characteristic for the semi-classical Schrödinger flow, there exists an operator $Q$ of order zero such that for any semi-classical Schrödinger wave $v$

(2.4)
\[
    Av|_{\partial\Omega \times \mathbb{R}} = Q(\partial_n v).
\]

Let $q$ be the symbol of this operator. Let $(x_0, t_0) \in \partial\Omega \times \mathbb{R}$ and $(\eta, \tau) \in T_{(x_0, t_0)}(\partial\Omega \times \mathbb{R})$. We denote by $\psi_{\pm}$ the two distinct solutions of the Eikonal equations
\[
    |\nabla \psi(x)|^2 = -\tau,
\]
\[
    \psi_{\pm}(x) = x \cdot \eta \pm \sqrt{\tau - \eta^2 n},
\]
that are well defined in a neighborhood of $x_0$ as soon as $\tau - \eta^2 > 0$: indeed, extending $n$ in a small neighborhood of the border, one can always take
\[
    \psi_{\pm} = x \cdot \eta \pm \sqrt{\tau - \eta^2 n}.
\]

For $\lambda > 0$, consider, extending $\partial_n \psi_{\pm}$ in a neighborhood of $x_0$ in $\Omega$
\[
    v_\lambda = \frac{e^{i\lambda(\psi_{+} + i\tau)} - e^{i\lambda(\psi_{-} + i\tau)}}{(i\lambda)(\partial_n \psi_{+} - \partial_n \psi_{-})},
\]
which is solution of an approximate semi-classical Schrödinger equation
\[
    i\partial_t v_\lambda - \lambda^{-1} \Delta v_\lambda = O(\lambda^{-1}),
\]
\[
    v_\lambda = 0 \text{ on } \partial\Omega.
\]

verifying, in a neighborhood of $x_0$ in $\partial\Omega$
\[
    \partial_n v_\lambda = e^{i\lambda(x \cdot \eta + i\tau)}.
\]

But, the principal symbol of $Q$ can be computed as
\[
    q(x_0, t_0, \eta, \tau) = \lim_{\lambda \to \infty} e^{-i\lambda(x_0 \cdot \eta + t_0 \cdot \tau)}Q(e^{i\lambda(x \cdot \eta + i\tau)})(x_0, t_0)
\]
\[
    = \lim_{\lambda \to \infty} e^{-i\lambda(x_0 \cdot \eta + t_0 \cdot \tau)}Q(\partial_n v_\lambda)(x_0, t_0).
\]

By the Duhamel formula, the difference between $v_\lambda$ and the solution of the actual equation $w_\lambda$ is bounded in a neighborhood of $(x_0, t_0)$ by
\[
    |w_\lambda - v_\lambda| \lesssim \lambda^{-1},
\]
therefore, we can replace \( v_\lambda \) by \( w_\lambda \), which is an exact Schrödinger wave, in the limit and make use of (2.4) to get:

\[
q(x_0, t_0, \eta, \tau) = \lim_{\lambda \to \infty} e^{-i\lambda(x_0 \cdot \eta + t_0 \cdot \tau)} Q(\partial_\lambda w_\lambda)(x_0, t_0) \\
= \lim_{\lambda \to \infty} e^{-i\lambda(x_0 \cdot \eta + t_0 \cdot \tau)} A(w_\lambda)(x_0, t_0) = \lim_{\lambda \to \infty} e^{-i\lambda(x_0 \cdot \eta + t_0 \cdot \tau)} A(v_\lambda)(x_0, t_0)
\]

\[
\left( \frac{a(x, d\psi_+) - a(x, d\psi_-)}{2(\partial_\lambda \psi_+ - \partial_\lambda \psi_-)} \right)(x_0, t_0).
\]

And we conclude that, this computation been valid for \( \tau - \eta^2 > 0 \)

\[
q(x_0, t_0, \eta, \tau) = \left( \frac{a(x_0, \xi) - a(x_0, \xi')}{(\xi' - \xi) \cdot n(x_0)} \right),
\]

\[
\xi' = \eta \pm \sqrt{\tau - \eta^2} n(x_0).
\]

Notice that \( \xi' \) is a pair of reflected rays.

**The escape function.** Let \((y, \eta) \notin K\). The generalized broken ray starting from \((y, \eta)\) is composed of a finite number of segments, thus, the construction of [MRS77], Section 5, holds to construct a ray function starting from \((y, \eta)\), that is, a function \( p_0 \) satisfying

\[
\xi \cdot \nabla p_0(x, \xi) \geq 0, \quad \frac{p_0(x, \xi) - p_0(x, \xi')}{(\xi' - \xi) \cdot n(x)} \geq 0,
\]

and

\[
\eta \cdot \nabla p_0(y, \eta) > 0, \quad \frac{p_0(y, \eta) - p_0(y, \eta')}{(\eta - \eta') \cdot n(y)} > 0.
\]

Therefore, by compactness, we can construct a function \( a \) such that

\[
\xi \cdot \nabla a(x, \xi) \geq 0, \quad \frac{a(x, \xi) - a(x, \xi')}{(\xi' - \xi) \cdot n(x)} \geq 0
\]

\[
\xi \cdot \nabla a(x, \xi) > 0, \quad \frac{a(x, \xi) - a(x, \xi')}{(\xi' - \xi) \cdot n(x)} > 0, \text{ on } V \supset \supp \phi.
\]

Finally, notice that, because the construction of [MRS77] follows the rays and because the trapped set is invariant by the flow, we can construct \( a \) in such a way that

\[
a = 0 \text{ near } K.
\]

Remark that, as in [MRS77], Section 1, such an \( a \) can be approximated by a polynomial in order to justify the above integration by parts.

**A first estimate.** Let \( \delta > 0 \). Because of (2.7), (2.5), \( q \) is real-valued and positive on \( \{ \tau - \eta^2 \geq 0 \} \), therefore, there exists \( \epsilon > 0 \) small enough so that, on \( \{ \tau - \eta^2 \geq -\epsilon \} \) we have, with the notations of (2.5)

\[
\Re a(x_0, \xi) - a(x_0, \xi') \geq -\delta/2.
\]

and, for \(|a| \leq 2(d + 1) = 8

\[
\left| \Im \partial_\tau \frac{a(x_0, \xi) - a(x_0, \xi')}{(\xi' - \xi) \cdot n(x_0)} \right| \leq \delta/2
\]
Now, let $\chi$ be positive and supported in $\{\tau - \eta^2 \geq -2\varepsilon\}$ and such that $\chi = 1$ in $\{\tau - \eta^2 \geq -\varepsilon\}$. We decompose $a$ as the sum

$$a = \chi a + (1 - \chi)a.$$  

Note that $(1 - \chi)a$ is supported away from the characteristic set $\{\tau = \eta^2\}$ of the semi-classical Schrödinger flow. Therefore,

$$\|\text{Op}_b((1 - \chi)a)u\|_{H^\sigma(\mathbb{R} \times \Omega)} = O(h^\infty)\|u_0\|_{L^2},$$

and using a trace theorem

$$B = \int_0^T \int_{\partial\Omega} \langle R(\partial_n(e^{it\Delta}u_0)), \partial_n(e^{it\Delta}u_0) \rangle \, d\sigma dt + O(h^\infty)\|u_0\|_{L^2},$$

where $R = \text{Op}(\chi a)$. Notice that a pair of reflected rays share the same norm, therefore, by (2.5), the symbol of $R$ is

$$(\chi(x_0,t_0,\eta,\tau) = \chi(\eta,\tau) \left( \frac{a(x_0,\xi_+(\eta,\tau)) - a(x_0,\xi_-(\eta,\tau))}{(\xi_+(\eta,\tau) - \xi_-(\eta,\tau)) \cdot n(x_0)} \right),$$

$$(\xi_\pm = \eta \pm \sqrt{\tau - \eta^2}n(x_0).$$

Therefore, by (2.10), (2.8) and (2.9), we can use the Garding inequality for the real part, the Calderon-Vaillancourt theorem for the imaginary part in order to write

$$B \geq -\delta \int_0^T \int_{\partial\Omega} |\tilde{\phi}u|^2 \, d\sigma dt - c_{\text{Gard}}\|\chi_b u\|_{L^2([0,T],H^{-1/2}(\partial\Omega))} + O(h^\infty)\|u_0\|_{L^2}.$$  

where $\tilde{\phi} \in \mathcal{C}_c^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ is supported in $\mathcal{K}^c$ and $\tilde{\phi} = 1$ on the support of $\phi$, and $\chi_b \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ is such that $\chi_b = 1$ on $\partial\Omega$.

Moreover, by the same procedure as in [MRS77], we may suppose that for $|x| \geq R \gg 1$, $a$ is given by $a(x,\xi) = hx \cdot \xi$. Let $\chi_R \in \mathcal{C}_c^\infty$ be such that $\chi_R = 1$ on $\{|x| \leq 2R\}$ and $\chi_R = 0$ on $\{|x| \geq 3R\}$. We decompose

$$\int_{\Omega} \langle [i\Delta, A]u, u \rangle = \int_{\Omega} \langle [i\Delta, A]\chi_R u, \chi_R u \rangle$$

$$+ \int_{\Omega} \langle [i\Delta, A]\chi_R u, (1 - \chi_R)u \rangle + \int_{\Omega} \langle [i\Delta, A](1 - \chi_R)u, \chi_R u \rangle$$

$$+ \int_{\Omega} \langle [i\Delta, A](1 - \chi_R)u, (1 - \chi_R)u \rangle.$$  

Because the commutator is truly non-negative for functions supported in $\{|x| \geq 2R\}$, the last term is non-negative. Moreover, the integrand of both intermediate terms are supported in $\{2R \leq |x| \leq 3R\}$. Therefore, taking $R$ large enough, the long-range smoothing estimate, which is for example a consequence of the long-range resolvent estimate of Cardoso and Vodev [CV02] by the procedure of [BGT04], allows us to control them:

$$\left| \int_{\Omega} \int_{\mathbb{R}} \langle [i\Delta, A]\chi_R u, (1 - \chi_R)u \rangle + \langle [i\Delta, A](1 - \chi_R)u, \chi_R u \rangle \right|$$

$$\lesssim \|\hat{\chi}u\|_{L^2}H^{1/2} \lesssim \|u_0\|_{L^2},$$
where $\tilde{\chi} \in C^\infty$ is equal to one in $\{2R \leq |x| \leq 3R\}$ and supported in $\{|x| \geq R\}$. Finally, by the Garding inequality again, using (2.8):

$$\int_0^T \int_Q \langle [i\Delta, A]\chi RU, \chi RU \rangle \geq C\|\Phi u\|_{L^2H^{1/2}} - c_{\text{Gard}}\|\chi RU\|_{L^2L^2},$$

Thus, combining (2.3), (2.12), and (2.13), using the trace theorem and controlling the lower order terms with the estimate with logarithmic loss we get

$$\|\Phi u\|_{L^2H^{1/2}} \leq C(\|u_0\|_{L^2} + \delta\|\Phi u\|_{L^2H^{1/2}}) + C\delta O(h^\infty).$$

**Iteration and conclusion.** To conclude, we would like to take $\delta > 0$ small enough and iterate (2.14). In order to do so, we have to take care of the potential dependency in $\phi, \tilde{\phi}, \phi, \ldots, \phi, \ldots$ and $\delta$ of the constants appearing in this estimate. Let us first remark that we take all the $\phi$ in a given small neighborhood of the support of $\phi$ - this neighborhood is a subset of $V$ of (2.8). Thus, there exists $A \geq 1$ such that, for $|\alpha + \beta| \leq N$

$$\|\partial_x^{\alpha, \beta} \phi \|_{L^\infty} \leq A^k.$$

Therefore, the Garding constants $c_{\text{Gard}}$ in (2.12), (2.13) at the $k$-th iteration can be taken as $A^k$. Moreover, by (2.8), $\xi \cdot \nabla a$ is bounded below by a constant $C$ uniformly on the support of all the $\phi$, so we can choose the same constant $C$ in (2.13) at all iteration. Finally, the $O(h^\infty)$ term depends only of $\delta$.

Therefore, we can precise the constants in (2.14) at the $k$-th iteration:

$$\|\Phi u\|_{L^2H^{1/2}} \leq (C + A^k)\|u_0\|_{L^2} + C\delta \|\Phi u\|_{L^2H^{1/2}} + C\delta O(h^\infty),$$

where $C$ and $A$ have no dependencies in $k$ and $\delta$ and $C_5$ depends only of $\delta$. Thus we get

$$\|\Phi u\|_{L^2H^{1/2}} \leq \left[ C \frac{1 - (C\delta)^{k+1}}{1 - C\delta} + \frac{(C\delta A)^{k+1}}{1 - C\delta A} \right] \|u_0\|_{L^2} + \left[ C \frac{1 - (C\delta)^{k+1}}{1 - C\delta} + \frac{(C\delta A)^{k+1}}{1 - C\delta A} \right] \|u_0\|_{L^2} + C\delta O(h^\infty)$$

where $\chi_0$ is compactly supported. We fix $\delta$ small enough so that $C\delta A < 1$ and let $k$ go to infinity to obtain the result.

\[\square\]

**Remark 2.1.** Notice that the exact same proof holds for any arbitrary domain for which a smoothing estimate with logarithmic loss holds. Moreover, as remarked by [DV13], we can iterate such a proof and therefore it suffices to assume a smoothing estimate with polynomial loss. More precisely, we initiate the argument controlling the lower order terms by the smoothing estimate with polynomial loss, and then iterate the proof and control the lower order terms by the previous estimate at each step, until we reach $h^0$. Thus we obtain the more general:
2.1 Proposition is a consequence of 2.2

1.4 in the seek of readability.

flow shown in [127x200] supported to the points that remain near the trapped trajectories in logarithmic aim of this section is to show that we can reduce ourselves to data micro-locally where we denoted, here and in the sequel of this section Φ a compact set strictly containing the obstacles, such that

\[ \| \chi e^{-it\Delta_d} u_0 \|_{L^2(\mathbb{R}, H^{1/2}(\Omega))} \lesssim h^{-k} \| u_0 \|_{L^2}. \]

Then, a smoothing estimate without loss holds outside the trapped set \( K \): that is, for all \( \phi \in C^\infty_c(\mathbb{R}^3 \times \mathbb{R}^3) \) supported in \( K^c \), we have

\[ \| Op_\phi(\phi)e^{-it\Delta_d} u_0 \|_{L^2(\mathbb{R}, H^{1/2}(\Omega))} \lesssim \| u_0 \|_{L^2}. \]

3. Reduction to the logarithmic trapped set

Because of Proposition 2.1 and Proposition 2.2, the exact same proof as in [Laf17b], section 2, show that the following proposition implies our main result for the Schrödinger equation:

Proposition 3.1 (Strichartz estimates on a logarithmic interval near the trapped set). There exists \( \epsilon > 0 \) such that for all \( \phi \in C^\infty_c(\mathbb{R}^3 \times \mathbb{R}^3) \) supported in a small enough neighborhood of \( K \cap \{ |\xi| \in [\alpha_0, \beta_0] \} \), we have

\[ \| Op_\phi(\phi)e^{-it\Delta_d} \psi(-h^2\Delta) u_0 \|_{L^p(0, \epsilon h| \log h|) L^q(\Omega)} \leq C \| u_0 \|_{L^2}. \]

Notice that, by a classical \( TT^* \) argument, Proposition 3.1 is a consequence of the following pointwise dispersive estimate:

\[ \| A e^{ith\Delta_d} \psi(-h^2\Delta) A^* \|_{L^1 \to L^\infty} \lesssim (ht)^{-3/2}, \forall 0 \leq t \leq \epsilon | \log h |, \]

where we denoted, here and in the sequel of this section

\[ A := Op_\phi(\phi) \]

in the seek of readability.

Thus, the rest of the paper will be devoted to prove such an estimate. The aim of this section is to show that we can reduce ourselves to data micro-locally supported to the points that remain near the trapped trajectories in logarithmic times. In order to do so, we first need to generalizes some properties of the billiard flow shown in [Laf17b]:

3.1. Regularity of the billiard flow. We first need the following lemma, where we denoted by \( W_{\tan, \eta} \) an \( \eta \)-neighborhood of the tangent rays:

Lemma 3.1. There exists \( \eta > 0 \) such that any ray cannot cross \( W_{\tan, \eta} \) more than twice.

Proof. If it is not the case, for all \( n \geq 0 \), there exists \( (x_n, \xi_n) \in K \times \mathcal{S}^2 \), where \( K \) is a compact set strictly containing the obstacles, such that \( \Phi_t(x_n, \xi_n) \) cross \( W_{\tan, \eta} \) at least three times. Extracting from \( (x_n, \xi_n) \) a converging subsequence, by continuity of the flow, letting \( n \) going to infinity we obtain a ray that is tangent to \( \cup \Theta_i \) in at least points. Therefore, it suffices to show that such a ray cannot exists.

Remark that, because of the non-shadows condition (1.4), if \( (x, \xi) \in W_{\tan} \), if we consider the ray starting from \( (x, \xi) \) and the ray starting from \( (x, -\xi) \), one of the two do not cross any obstacle in positive times. But, if there is a ray tangent to the obstacles in at least three points, if we consider the second tangent point \( (x_0, \xi_0) \), both rays starting from \( (x_0, \xi_0) \) and \( (x_0, -\xi_0) \) have to cross an obstacle, therefore, this is not possible. \( \square \)
Together with lemma 3.2 of [Laf17b], which gives the (Hölder) regularity of the billiard flow near tangent points for a domain with no infinite order of contact points, we obtain, with the exact same proof as in this previous paper - the only assumption made been which given by lemma 3.1:

**Lemma 3.2.** Let $V$ be a bounded open set containing the convex hull of $\cup \Theta_1$. Then, there exists $\mu > 0$, $C > 0$ and $\tau > 0$ such that, for all $x, \bar{x} \in V$, all $\xi, \bar{\xi}$ such that $|\xi|, |\xi'| \in [\alpha_0, \beta_0]$, for all $t > 0$ there exists $t'$ verifying $|t' - t| \leq \tau$ such that

$$d(\Phi_{t'}(\bar{x}, \bar{\xi}), \Phi_t(x, \xi)) \leq C^{\tau} d((\bar{x}, \bar{\xi}), (x, \xi))^\mu.$$  

**Remark 3.1.** It is crucial, in the proof of this previous lemma, that a ray cannot cross $W_{\text{tan,}n}$ infinitely many times: indeed, regularity is lost at each tangent point. Therefore, in the case which does not enter the framework of Ikawa condition, 2: no obstacle in shadow of a trapped ray which is tangent to an obstacle, this proof does not hold, and we do not know if such a regularity of the flow is true. As this regularity is crucial in the sequel, we think that this “non shadow” condition may not be only technical, at least in the degenerated situation previously mentioned.

Finally, let us remark that

**Lemma 3.3.** Let $\delta > 0$ and $D_\delta$ be a $\delta$-neighborhood of $\mathcal{P}$. Then, for all compact $K$, $\Phi_t(\rho) \rightarrow \infty$ as $t \rightarrow \pm \infty$ uniformly with respect to $\rho \in K \cap D_\delta$.

**Proof.** It suffices to prove that the length of all trajectories in $K \cap D_\delta$ are uniformly bounded. If it is not the case, there exists $\rho_n \in D_\delta \cap K$ such that

$$\text{length}\{\Phi_t(\rho_n)\}_{t \geq 0} \cap K \rightarrow +\infty$$

as $n$ goes to infinity. Up to extract a subsequence, $\rho_n \rightarrow \rho^* \in D_\delta$. Necessarily, length $\{\Phi_t(\rho^*)\}_{t \geq 0} \cap K = \infty$, thus $\rho^* \in \mathcal{P}$, this is not possible. \hfill $\square$

**Lemma 3.4.** $K$ is closed.

**Proof.** Let $\rho_n \in K$, $\rho_n \rightarrow \rho$. There exists $A > 0$ such that for any $t$, $d(\pi_x \Phi_t(\rho_n), 0_{\mathbb{R}^3}) \leq A$. $\pi_x \Phi_t(\cdot)$ be continuous for any fixed $t$, it suffices to pass to the limit $n \rightarrow \infty$ in the previous inequality to obtain $\rho \in K$. \hfill $\square$

### 3.2. Reduction of the problem.

We now show that we can reduce ourselves to points that remain near trapped trajectories in logarithmic times $T_0 \leq t \leq \epsilon |\log h|$ in order to prove the pointwise dispersive estimate (3.2) in times $[T_0, \epsilon |\log h|]$. In contrast to [Laf17b], where we used a translation argument in the spirit of [Iva10], we are here inspired by [BGH10].

Let $\delta > 0$. By lemma 3.4, the projection on $\mathbb{R}^3 \times \mathcal{S}_2^2$ of the trapped set is compact, thus there exists a finite number of phase-space segments $(S_k)_{1 \leq k \leq N_\delta} \subset S_k = s_k \times \mathbb{R} \subseteq T^* \Omega$, $s_k$ be a segment of $\mathbb{R}^3$, such that $K$ is contained in a $\delta$-neighborhood of $\cup S_k$. The small quantity $\delta > 0$ may be reduced a finite number of time in the sequel.

We will now define a microlocal partition of unity $(\Pi_k)$. Let $p_k \in C_0^\infty(T^* \Omega)$, $0 \leq p_k \leq 1$ be a family of functions such that $p_k$ is supported in a neighborhood $W_k$ of $S_k$ and

$$\sum_{1 \leq k \leq N_\delta} p_k = 1 \text{ in a neighborhood of } K.$$

Let us define

$$\Pi_k = \text{Op}_h(p_k), \; \forall 1 \leq k \leq N_\delta.$$
Now, let \( \chi_0 \in C^\infty(\mathbb{R}^3) \), \( 0 \leq \chi_0 \leq 1 \) such that \( \chi_0 \) is supported sufficiently far from \( \text{Con} \cup \Theta \), and equal one far from the origin. Notice that any broken bicharacteristic entering the support of \( \chi_0 \) from its complement remains in it for all times. We take
\[
\Pi_0 = \chi_0
\]
and let
\[
\Pi_{-1} = \text{Op}_h \left( 1 - \chi_0 - \sum_{1 \leq k \leq N_s} p_k \right).
\]
\( \Pi_{-1} \) is defined in such a way that his symbol verifies
\[
d(\text{Supp}_{\Pi_{-1}}, K) \geq d_1 > 0,
\]
therefore, by lemma 3.3, there exists \( T_0 > 0 \) such that
\[
\pi_x \Phi_t(\text{Supp}_{\Pi_{-1}}) \subset \text{Supp}\chi_0, \forall |t| \geq T_0.
\]
Now, let \( \tau > 0 \). It will be fixed in the sequel. In the spirit of [BGH10], we decompose \( T = (L - 1)\tau + s_0 \), where \( L \in \mathbb{N} \) and \( s_0 \in [0, \tau) \). We have
\[
e^{i\tau h \Delta} = e^{its_0} \left( e^{i\tau h \Delta} \right)^{L-1}, e^{i\tau h \Delta} = e^{i\tau h \Delta} \sum_{-1 \leq k \leq N_s} \Pi_k,
\]
and thus
\[
e^{i\tau h \Delta} = \sum_{k=(k_1, \ldots, k_L)} e^{its_0}\Pi_{k_1} e^{i\tau h \Delta}\Pi_{k_{L-1}} \cdots \Pi_{k_L} e^{i\tau h \Delta},
\]
where the sum is taken over all multi-indice \( k \in [-1, N_s]^L \). Let us remark that, because the wavefront set of the semi-classical Schrödinger flow is invariant by the generalized bicharacteristic flow, denoting
\[
\sigma_k = Ae^{its_0}\Pi_{k_1} e^{i\tau h \Delta}\Pi_{k_{L-1}} \cdots \Pi_{k_L} e^{i\tau h \Delta} \psi(\tau^2 h^2)A^*,
\]
it holds that
\[
(3.4) \quad \rho \in \text{WF}_h(\sigma_k) \implies \begin{cases} 
\pi_x \rho \in \text{Supp}\phi, \\
\Phi_{j\tau}(\rho) \in \text{Supp}\phi_{k_j} \quad \forall 1 \leq j \leq L, \\
\pi_x \Phi_{j\tau}(\rho) \in \text{Supp}\phi.
\end{cases}
\]
Thus we have

**Lemma 3.5.** Let \( k \in [-1, N_s]^L \). If there exists \( 1 \leq j \leq L \) such that \( k_j = 0 \) or \( k_j = -1 \), then \( \sigma_k = O(h^\infty) \) as an \( L^1 \to L^\infty \) operator.

**Proof.** As remarked in [BGH10], by virtue of Sobolev embeddings it suffices to show that \( \sigma_k = O(h^\infty) \) as an \( L^2 \to L^2 \) operator, thus has null operator wavefront set. Let us suppose first that there exists \( j \) such that \( k_j = 0 \). We choose \( j \) to be the the first such indice. Suppose that \( \rho \in \text{WF}_h(\sigma_k) \). There exists \( t_0 \in [(j-1)\tau, j\tau] \) such that the spatial projection of \( \Phi_{j\tau}(\rho) \) enters the support of \( \chi_0 \) from its complementary, thus it does not leave it. Therefore \( \pi_x \Phi_{j\tau}(\rho) \in \text{Supp}\chi_0 \), this is not possible. Thus \( \text{WF}_h(\sigma_k) = \emptyset \).

Now, suppose that there exists \( j \in [1, L - \frac{2}{\tau}] \) such that \( k_j = -1 \). Let \( \rho \in \text{WF}_h(\sigma_k) \). \( \Phi_{j\tau}(\rho) \in \text{Supp}\Pi_{-1} \), hence
\[
\pi_x \Phi_{j\tau+t}(\rho) \in \text{Supp}\chi_0, \forall t \geq T_0,
\]
and we are reduced thus to the previous case. In the same way, we exclude \( j \in \left[ \frac{2}{\tau}, L \right] \) using the property for all \( t \leq -T_0 \). \( \square \)
But, as the $k$-sum contains at most $(N_\delta + 2)|\log h|$ — that is, a negative power of $h$ — terms, we have
\[
\sum_k O(h^\infty) = O(h^\infty),
\]
and therefore we deduce from the previous lemma that, as an $L^1 \to L^\infty$ operator
\[
Ae^{t \Delta h} \psi(-h^2 \Delta) A^* = \sum_{k,k_j \geq 1} \sigma_k + O(h^\infty).
\]

Now, we will choose $\tau > 0$ small enough given by the following lemma:

**Lemma 3.6.** For all $\delta > 0$, there exists $\tau > 0$ small enough so that, for every trajectory $\gamma \in \mathcal{P}$, we have
\[
d(\rho, \gamma) < \delta, \quad d(\Phi_T(\rho), \gamma) < \delta \implies \forall t \in [0, \tau], d(\Phi_T(\rho), \gamma) < 3\delta.
\]

**Proof.** Let $\tilde{\rho}$ realizing the distance from $\rho$ to $\gamma$. We denote
\[
t_0 = \inf \{ t \geq 0, \text{ s.t. } \pi_x \Phi_t(\rho) \in \Theta \}, \quad \tilde{t}_0 = \inf \{ t \geq 0, \text{ s.t. } \pi_x \Phi_t(\tilde{\rho}) \in \Theta \}.
\]
We assume that, for example, $\tilde{t}_0 > t_0$. Notice that, by the proof of lemma 3.2 from [Laf17b], we have
\[
\forall t \in [0, \tau] \setminus (t_0, \tilde{t}_0), \quad d(\Phi_T(\rho), \Phi_T(\tilde{\rho})) \leq C^* \delta.
\]
Moreover, for $t \in [t_0, \tilde{t}_0]$, 
\[
d(\Phi_T(\rho), \Phi_T(\tilde{\rho})) \leq d(\Phi_T(\rho), \Phi_{t_0}(\rho)) + d(\Phi_{t_0}(\rho), \Phi_{t_0}(\tilde{\rho})) + d(\Phi_{t_0}(\rho), \Phi_{t_0}(\tilde{\rho})),
\]
but, as $\{t \in [t_0, \tilde{t}_0] \}$ and $\{\Phi_t(\tilde{\rho})\}_{t \in [t_0, \tilde{t}_0]}$ are straight lines
\[
d(\Phi_T(\rho), \Phi_{t_0}(\rho)) \leq |t - t_0| \pi_x \rho | \leq \tau \beta_0,
\]
and similarly for $\tilde{\rho}$. Therefore
\[
d(\Phi_T(\rho), \Phi_T(\tilde{\rho})) \leq 2\tau \beta_0 + C^* \delta.
\]
We take $\tau > 0$ small enough so that $2\tau \beta_0 \leq \delta$ and $C^* \leq 2$ and we get the result. \hfill \square

The segment $S_{k_j}$ joins the obstacles $\Theta_{a_j}$ and $\Theta_{b_j}$. Choosing $\delta > 0$ small enough, by (3.4), $\sigma_k$ is not $O(h^\infty)$ only if, for all $j$
\[
(a_j = a_{j+1} \text{ and } b_j = b_{j+1}) \text{ or } (a_{j+1} = b_j).
\]
that is, only if $\gamma_k = S_{k_1} \circ S_{k_2} \circ \cdots \circ S_{k_k}$ is a trajectory. Let, if it is the case, $J_k$ be the corresponding story of reflexions. We extract from $J_k$ the primitive story $I_k$, that is, $J_k = I_k + r$, $I_k$ been primitive.

We now introduce the trapped set of an open subset in time $T$:

**Definition 3.1.** Let $D$ be an open subset of $(T^* \Omega \cup T^* \partial \Omega) \cap \{ |\xi| \in [\alpha_0, \beta_0] \}$ and $T > 0$. We define the trapped set of $D$ in time $T$, denoted $\mathcal{T}_T(D)$, in the following way
\[
\rho \in \mathcal{T}_T(D) \iff \forall t \in [0, T], \Phi_T(\rho) \in D.
\]

Let us denote by $D_{k, \delta}$ a $\delta$-neighborhood of $\gamma_k \cap \{ |\xi| \in [\alpha_0, \beta_0] \}$. For $I$ a primitive story of reflexions, let $q_{I,T} \in C_0^\infty$ be such that
\[
q_{I,T} = 0 \text{ outside } \mathcal{T}_T(D_{I,\delta}), \quad q_{I,T} = 1 \text{ in } \mathcal{T}_T(D_{I,3\delta}),
\]
and denote
\[
Q_I^T := \text{Op}_h(q_{I,T}).
\]
We have, by (3.4) and the choice of \( \tau > 0 \) permitted by lemma 3.6
\[
\sigma_k = \sigma_k Q^T_{I_k} + O(h^\infty).
\]
Now, remark that for \( I \) a primitive story of reflexions
\[
A e^{i T h \Delta} \psi(-h^2 \Delta) A^* Q^T_I = \sum_{k, I_k = I} \sigma_k Q^T_{I_k} + O(h^\infty),
\]
and therefore we recover
\[
\sum_I A e^{i T h \Delta} \psi(-h^2 \Delta) A^* Q^T_I = A e^{i T h \Delta} \psi(-h^2 \Delta) A^* + O(h^\infty).
\]
Let us finally remark that for \( T \leq \epsilon |\log h| \), we have \( h \leq e^{-\frac{T}{4}} \), thus the \( O(h^\infty) \) term verifies the dispersive estimate. Therefore, we have proven that:

**Lemma 3.7.** If the following dispersive estimate holds true
\[
\| \sum_I A e^{i T h \Delta} \psi(-h^2 \Delta) A^* Q^T_I \|_{L^1 \to L^\infty} \lesssim (hT)^{-\frac{3}{2}}, \quad \forall T_0 \leq T \leq \epsilon |\log h|,
\]
then the dispersive estimate (3.2) is true in times \([T_0, \epsilon |\log h|]\).

### 3.3. Times \( 0 \leq t \leq T_0 \) and conclusion of the section.

Finally, notice that the construction of \( Q^T_I \) does not depend of \( \phi \). We choose \( \phi \) supported in a small enough neighborhood of \( K \) so that, in times \( 0 \leq t \leq T_0 \) and for \( |\xi| \in [\alpha_0, \beta_0] \), the bicharacteristic flow \( \Phi_t(\rho) \) starting from \( \rho \) has only hyperbolic points of intersection with the boundary. But, for such points, we can use the parametrix construction of Ikawa [Ika82, Ika88], adapted to this problem in [Laf17b] and explained in the next section in the \( N \)-convex framework to show that the dispersive estimate holds true in times \( 0 \leq t \leq T_0 \), with a constant depending on \( T_0 \): indeed, the flow can be written as a finite (depending on \( T_0 \)) sum of reflected waves, each of them verifying the dispersive estimate.

Thus, by lemma 3.7, we are reduced to show the following dispersive estimate in order to obtain our main result, namely, we have

**Lemma 3.8.** If the following dispersive estimate holds true
\[
(3.6) \quad \| \sum_I A e^{i T h \Delta} \psi(-h^2 \Delta) A^* Q^T_I \|_{L^1 \to L^\infty} \lesssim (hT)^{-\frac{3}{2}}, \quad \forall T_0 \leq T \leq \epsilon |\log h|,
\]
then Strichartz estimates of Theorem 1.1 hold true for the Schrödinger equation.

where the symbols of \( Q^T_I \) were defined by (3.5). The sequel of the paper is devoted to doing so.

Let us remark that, with the same proof as in [Laf17b], we have, as a consequence of lemma 3.2,
\[
d(TT(\hat{D})^c, T_T(D)) \geq \frac{1}{4} e^{-cT} d(\hat{D}^c, D), \quad \forall D \subset \hat{D}
\]
and therefore \( q^T_I \) can, and will be constructed in such a way that, for \( 0 \leq T \leq \epsilon |\log h| \)
\[
|\partial_\alpha q^T_I | \lesssim h^{-2|\alpha|_\infty}.
\]
4. Construction of an approximate solution

4.1. The microlocal cut off. We will use the reflected-phase construction of [Ika88, Ika82] and [Bur93]. It is summed up in [Laf17b], let us recall that $\varphi_J$ is the reflected phase obtained from $\varphi$ after the story of reflexions $J$.

According to [Bur93] (remark 3.17’) there exists $M > 0$ such that if $J \in I$, $J = rI + l$ verifies $|J| \geq M$, and $\varphi$ verifies $(P)$, $\varphi_J$ can be defined in $U^\infty_{rI,l}$. We choose $\delta > 0$ small enough so that, according to the construction of the previous section

$$D_{I,4\delta} \subset \bigcup_{|l| \leq |l|-1} U^\infty_{l,l},$$

moreover, we will take $T_0 \geq 2\beta_0 M$.

Let us recall that we are reduced to show the following dispersive estimate:

$$\| \sum_{I \ \text{primitive}} A e^{iTh\Delta} \psi(-h^2\Delta)A^*Q_J^T \|_{L^1 \rightarrow L^\infty} \lesssim (hT)^{-\frac{3}{2}}, \forall T_0 \leq T \leq \epsilon |\log h|.$$ 

For all primitive story $I$, let us define

$$\delta_J^y(x) = \frac{1}{(2\pi h)^3} \int e^{-i(x-y) \cdot \xi / h} p_{1,T}(x,\xi) d\xi,$$

where $p_{1,T}$ is the symbol associated with $P_T^T := \psi(-h^2\Delta)A^*Q_T^T$. Then we have, for $u_0 \in L^2$

$$\psi(-h^2\Delta)A^*Q_T^T u_0(x) = \int \delta_J^y(x) u_0(y) dy.$$

Then, by linearity of the flow

$$A e^{ith\Delta} \psi(-h^2\Delta)A^*Q_T^T u_0 = \int A e^{ith\Delta} \delta_J^y u_0(y) dy,$$

and it therefore suffices to show that

$$\sum_{I \ \text{primitive}} |A e^{ith\Delta} \delta_J^y(x)| \lesssim (hT)^{-3/2}, \forall T_0 \leq T \leq \epsilon |\log h|.$$ 

Finally, notice that as the operator $A$ is bounded in $L^\infty \rightarrow L^\infty$ in the same way as in [Laf17b], it suffices only to show that

$$\sum_{I \ \text{primitive}} |\chi e^{ith\Delta} \delta_J^y(x)| \lesssim (hT)^{-3/2}, \forall T_0 \leq T \leq \epsilon |\log h|,$$

where $\chi \in C^\infty_c(\mathbb{R}^3)$ is supported in a neighborhood of the spatial projection of the support of $\varphi$ and equal to one on it.

In order to do so, we will construct a parametrix, that is, an approximate solution, in time $0 \leq t \leq \epsilon |\log h|$ for the semi-classical Schrödinger equation with data $\delta_J^y$. The first step will be to construct an approximate solution of the semi-classical Schrödinger equation with data

$$e^{-i(x-y) \cdot \xi / h} p_{1,T}(x,\xi)$$

where $\xi \in \mathbb{R}^n$ is fixed and considered as a parameter. Now that we are localized around a trajectory, the construction is exactly the same as in [Laf17b]. Let us sum it up briefly. In the sequel of this section, $p_{1,T}$ will be denoted $p$ in the seek of conciseness.
4.2. Approximate solution. We look for the solution in positives times of the equation

\[
\begin{cases}
  (i\partial_t w - h\Delta w) = 0 \text{ in } \Omega \\
  w(t = 0)(x) = e^{-i(x-y)\cdot \xi/h} p(x, \xi) \\
  w|_{\partial \Omega} = 0
\end{cases}
\]

as the Neumann serie

\[
w = \sum_{J \in I} (-1)^{|J|} w^J
\]

where

\[
\begin{cases}
  (i\partial_t w^0 - h\Delta w^0) = 0 \text{ in } \mathbb{R}^n \\
  w^0(t = 0)(x) = e^{-i(x-y)\cdot \xi/h} p(x, \xi)
\end{cases}
\]

and, for \( J \neq \emptyset, J = (j_1, \cdots, j_n), J' = (j_1, \cdots, j_{n-1}) \)

\[
\begin{cases}
  (i\partial_t w^J - h\Delta w^J) = 0 \text{ in } \mathbb{R}^n \setminus \Theta_{j_n} \\
  w^J(t = 0) = 0 \\
  w^J|_{\partial \Theta_{j_n}} = w^{J'}|_{\partial \Theta_{j_n}}.
\end{cases}
\] (4.2)

We will look for the \( w^J \)'s as power series in \( h \). In the sake of conciseness, these series will be considered at a formal level in this section, and we will introduce their expression as a finite sum plus a reminder later, in the last section.

We look for \( w^\emptyset \) as

\[
w^\emptyset = \sum_{k \geq 0} h^k w^\emptyset_k e^{-i((x-y)\cdot \xi - t\xi^2)/h},
\]

\[
w^\emptyset_0(t = 0) = q(x, \xi), \ w^\emptyset_k(t = 0) = 0.
\]

Solving the transport equations gives immediately

\[
w^\emptyset_0 = p(x - 2t\xi, \xi),
\]

\[
w^\emptyset_k = -i \int_0^t \Delta w^\emptyset_{k-1}(x - 2(s-t)\xi, s) ds \quad k \geq 1.
\]

Now, starting from the phase \( \varphi(x) = \frac{(\xi - p(x))\cdot \xi}{|\xi|} \), we define the reflected phases as before and we look for \( w^J \) as:

\[
w^J = \sum_{k \geq 0} h^k w^J_k e^{-i(\varphi_J(x,\xi)|\xi - t\xi^2)/h},
\]

\[
w^J_k|_{t \leq 0} = 0, \ w^J_k|_{\partial \Theta_{j_n}} = w^{J'}_k|_{\partial \Theta_{j_n}}.
\]

For \( x \in U_J(\varphi) \), we have

\[
\begin{cases}
  (\partial_t + 2|\xi| \nabla \varphi_J \cdot \nabla + |\xi| \Delta \varphi_J) w^J_0 = 0 \\
  w^J_0|_{\partial \Theta_{j_n}} = w^{J'}_0|_{\partial \Theta_{j_n}} \\
  w^J_0|_{t \leq 0} = 0
\end{cases}
\]

and

\[
\begin{cases}
  (\partial_t + 2|\xi| \nabla \varphi_J \cdot \nabla + |\xi| \Delta \varphi_J) w^J_k = -i\Delta w^J_{k-1} \\
  w^J_k|_{\partial \Theta_{j_n}} = w^{J'}_k|_{\partial \Theta_{j_n}} \\
  w^J_k|_{t \leq 0} = 0.
\end{cases}
\]
Solving the transport equations along the rays by the procedure explained in [Laf17b], we get the exact same following expressions of $w_k^l$ for $x \in U_J(\varphi)$:

**Proposition 4.1.** We denote by $\hat{X}_{-2l}(x, |\xi| \nabla \varphi_J)$ the backward spatial component of the flow starting from $(x, |\xi| \nabla \varphi_J)$, defined in the same way as $X_{-2l}(x, |\xi| \nabla \varphi_J)$, at the difference that we ignore the first obstacle encountered if it’s not $\Theta_{J_n}$, and we ignore the obstacles after $|J|$ reflections. Moreover, for $J = (j_1, \ldots, j_n) \in \mathcal{I}$, we denote by

$$J(x,t,\xi) = \begin{cases} (j_1, \cdots, j_k) & \text{if } \hat{X}_{-2l}(x, |\xi| \nabla \varphi_J) \text{ has been reflected } n-k \text{ times,} \\ \emptyset & \text{if } \hat{X}_{-2l}(x, |\xi| \nabla \varphi_J) \text{ has been reflected } n \text{ times}. \end{cases}$$

Then, the $w_k^l$'s are given by, for $t \geq 0$ and $x \in U_J(\varphi)$

$$w_0^l(x,t) = \Lambda \varphi_J(x,\xi)p(\hat{X}_{-2l}(x, |\xi| \nabla \varphi_J), \xi)$$

where

$$\Lambda \varphi_J(x,\xi) = \left( \frac{G \varphi_J(x)}{G \varphi_J(X^{-1}(x, |\xi| \nabla \varphi_J))} \right)^{1/2} \times \cdots \times \left( \frac{G \varphi(X^{-|J|-1}(x, |\xi| \nabla \varphi_J))}{G \varphi(X^{-|J|}(x, |\xi| \nabla \varphi_J))} \right)^{1/2},$$

and, for $k \geq 1$, and $x \in U_J(\varphi)$

$$w_k^l(x,t) = -i \int_0^t g_{\varphi_J}(x,t-s,\xi) \Delta w_{k-1}^{l(x,\xi,t-s)}(\hat{X}_{-2l(t-s)}(x, |\xi| \nabla \varphi_J), s) ds$$

where

$$g_{\varphi_J}(x,\xi,t) = \left( \frac{G \varphi_J(x)}{G \varphi_J(X^{-1}(x, |\xi| \nabla \varphi_J))} \right)^{1/2} \times \cdots \times \left( \frac{G \varphi_j(x,t,\xi)}{G \varphi_j(X^{-|(x,t,\xi)|}(x, |\xi| \nabla \varphi_J))} \right)^{1/2}.$$  

And, by the same proof again as in [Laf17b] it implies in particular the following three results. The first of them is about the support of the solutions:

**Lemma 4.1.** For $x \in U_J(\varphi)$

$$w_0^l(x,t) \neq 0 \implies (\hat{X}_{-2l}(x, |\xi| \nabla \varphi_J), \xi) \in \text{Suppp}.$$  

And moreover

$$\text{Supp}w_k^l \subset \{J(x,\xi,t) = \emptyset\}.$$  

It implies that we can extend it by zero outside the domains of definition of the phases:

**Proposition 4.2.** For $x \notin U_J(\varphi)$ and $0 \leq t \leq T$ we have $w_k^l(x,t) = 0$.

And that the have $|J| = t$:

**Lemma 4.2.** There exists $c_1, c_2 > 0$ such that for every $J \in \mathcal{I}$, the support of $w_k^l$ is included in $\{c_1 |J| \leq t\}$ and which of $\chi w_k^l$ is included in $\{c_1 |J| \leq t \leq c_2(|J| + 1)\}$.

Now, let us recall that $q = q_{I,T}$ where $I$ is a given primitive trajectory. We have:

**Lemma 4.3.** If $J$ is not of the form $rI + l$, then $w_k^l = 0$ for $0 \leq t \leq \epsilon |\log h|$.
Proof. If \( w^J_\xi(x,\xi) \neq 0 \), it follows from lemma 4.1 that there exists a broken ray joining \( (x,\xi|\nabla \varphi_I) \) and a point of the support of \( p_{I,T} \) in time \( t \) following the complete story of reflexions \( J \). By definition of the trapped set and because \( \text{Supp} \subset T_f(D_{I,4\delta}) \), this broken ray remains in a neighborhood of the trajectory \( \gamma \) corresponding to \( I \), thus \( J \) can only be of the form \( rI + l \).

Finally, let us notice that

**Lemma 4.4.** In times \( 0 \leq t \leq T \), for \( J = rI + l \), \( \xi w^J_\xi \) is supported in \( U_{I,T}^\infty \).

**Proof.** From (4.3), the support of \( w^J_\xi \) consists of the support of \( q(.,\xi) \), transported along the billiard flow with initial direction \( \xi \) along the story of reflexion \( J \) and then ignoring the obstacles. Because of the non-shadow condition (1.4), the part ignoring the obstacles is cut off by \( \chi \), thus we obtain the result. \( \square \)

### 4.3. The \( \xi \) derivatives

The following results about the directional derivatives of the phase and the solution has been proven in [Laf17b], where the proof does not involve the particular two obstacles geometry. The first one involves the critical points of the phase and its non-degeneracy:

**Lemma 4.5.** Let \( J \in I \) and \( S_J(x,t,\xi) := \varphi_J(x,\xi)|\xi| - t\xi^2 \). For all \( t > 0 \) and there exists at most one \( s_J(x,t) \) such that \( D^2_S J(x,t,s_J(x,t)) = 0 \). Moreover, for all \( t_0 > 0 \), there exists \( c(t_0) > 0 \) such that, for all \( t \geq t_0 \) and all \( J \in I \)

(4.5) \[ w^J(x,t,\xi) \neq 0 \implies |\det D^2_S J(x,t,\xi)| \geq c(t_0) > 0. \]

The last two permits to control the directional derivatives of the solutions:

**Proposition 4.3.** For all multi-indices \( \alpha, \beta \) there exists a constant \( D_{\alpha,\beta} > 0 \) such that the following estimate holds on \( U_{I,T}^\infty \):

\[ |D^2_{\xi} D^\beta_x \nabla \varphi_J| \leq D_{\alpha,\beta}^{|J|}. \]

**Corollary 4.1.** We following bounds hold on \( U_{I,T}^\infty \)

\[ |D^2_{\xi} w^J_\xi| \lesssim C_{\alpha,|J|} h^{-(2k+|\alpha|)\epsilon}. \]

### 4.4. Decay of the reflected solutions

The principal result which permits us to estimate the decay of the reflected solutions is the convergence of the product of the Gaussian curvatures \( \Lambda \varphi_J \) obtained by [Ika88, Ika82] and [7]. It writes, in this setting

**Proposition 4.4.** Let \( 0 < \lambda_1 < 1 \) be the square-root of the product of the two eigenvalues lesser than one of the Poincaré map associated with the periodic trajectory \( I \). Then, there exists \( 0 < \alpha < 1 \) and a \( C^\infty \) function \( a_{I,I} \) defined in \( U_{I,I}^\infty \), such that, for all \( J = rI + l \), we have

\[ \sup_{U_{I,I}^\infty} |\Lambda \varphi_J - \lambda_I a_{I,I}| \leq C_m \lambda_I^{|J|}. \]

In the same way as in [Laf17b], it implies in particular:

**Proposition 4.5.** If \( J = rI + l \), where \( I \) is a primitive trajectory and \( l \leq |I| \), then the following bounds hold on \( U_{I,I}^\infty \):

\[ |w^J_\xi| \leq C_k^{\lambda_I^{|J|}} h^{-(2k+m)\epsilon}. \]

Moreover, on the whole space, \( |w^J_\xi| \leq C_k h^{-(2k+m)\epsilon}. \)
5. Proof of the Main Result

Let $K \geq 0$. By the previous section, the function

$$(x, t) \to \frac{1}{(2\pi h)^3} \sum_{j=r+l} \int \sum_{k=0}^{K} h^k w^j_{k}(x, t, \xi)e^{-i(\varphi_{j}(x, \xi))(|\xi|-t\xi^2)/h} d\xi$$

satisfies the approximate equation

$$\partial_t u - ih \Delta u = -ih^K \frac{1}{(2\pi h)^3} \sum_{j=r+l} \int \Delta w^j_{K-1}(x, t, \xi)e^{-i(\varphi_{j}(x, \xi))(|\xi|-t\xi^2)/h} d\xi$$

with data $\delta^y_{I,T}$. Because $e^{-i(t-s)h}\Delta$ is an $H^m$-isometry and by the Duhamel formula, the difference from the actual solution $e^{-i\theta h}\delta^y$ is bounded in $H^m$ norm by

$$C \times |t| \times h^{K-3} \times \sup_{t, \xi} \sum_{j=r+l} \| \Delta w^j_{K-1}(\cdot, t, \xi)e^{-i(\varphi_{j}(\cdot, \xi))(|\xi|-t\xi^2)/h} \|_{H^m}.$$ 

Therefore,

$$\sum_{I \text{ primitive}} e^{-i\theta h}\delta^y_I(x) = S_K(x, t) + R_K(x, t)$$

with

$$S_K(x, t) = \frac{1}{(2\pi h)^3} \sum_{j=r+l} \int \sum_{k=0}^{K} h^k w^j_{k}(x, t, \xi)e^{-i(\varphi_{j}(x, \xi))(|\xi|-t\xi^2)/h} d\xi$$

and, for $0 \leq t \leq \epsilon |\log h|$,

$$\| R_K(\cdot, t) \|_{H^m} \lesssim |\log h|h^{K-3} \times \sup_{t, \xi} \sum_{j=r+l} \| \Delta w^j_{K-1}(\cdot, t, \xi)e^{-i(\varphi_{j}(\cdot, \xi))(|\xi|-t\xi^2)/h} \|_{H^m},$$

where $w^j_k$ is understood to be constructed from $p_{I,T}$ when $J = rI + l$.

**The reminder.** We first deal with the reminder term $R_K$. Let us denote

$$W^j_{K-1}(x, t) = \Delta w^j_{K-1}(\cdot, t, \xi)e^{-i(\varphi_{j}(\cdot, \xi))(|\xi|-t\xi^2)/h}$$

Notice that, by construction of the $w^j_k$s, $w^j_k$ is supported in a set of diameter $(C + \beta_0 t)$. Therefore, using Proposition 4.5 to control the derivatives coming from $w^j_{K-1}$ and the estimate

$$|\nabla \varphi_{j}|_m \leq C_m |\nabla \varphi|_m$$

from [Ika88] to control the derivatives coming from the phase we get:

$$\| \partial^m W^j_{K-1} \|_{L^2} \lesssim C_K(1 + \beta_0 t)^\frac{1}{2} \| \partial^m W^j_{K-1} \|_{L^\infty} \lesssim C_K(1 + t)^\frac{1}{2}h^{-m} \times h^{-(2K+m+2)\epsilon}$$

and thus, by (5.2) and the Sobolev embedding $H^2 \hookrightarrow L^\infty$, for $0 \leq t \leq \epsilon |\log h|$,

$$\| R_K \|_{L^\infty} \lesssim |\log h|^\frac{2}{3}h^{K(1-2\epsilon\alpha)-5+4\epsilon \alpha} \{ J \in \mathcal{I}, \text{ s.t } w^j_{K-1} \neq 0 \}.$$

Note that $w^j_{K-1}(t) \neq 0$ implies by lemma 4.2 that $|J| \leq c_1 t$, and $\{ J \in \mathcal{I}, \text{ s.t } w^j_{K-1} \neq 0 \}$ is bounded by the number of elements in

$$\alpha_{[c_1 t]}$$

where

$$\alpha_k = \{ \text{sequences } s \text{ in } [1, N] \text{ of length } \leq k \text{ s.t } s_{i+1} \neq s_i \}$$

But
Lemma 5.1. The number of elements in \( \alpha_k \) admits the bound
\[
|\alpha_k| \leq C_N N^k.
\]

Proof. Let us denote
\[
\beta_k = \{ \text{sequences } s \text{ in } [1, N] \text{ of length } k \text{ s.t } s_{i+1} \neq s_i \}.
\]
We have
\[
|\beta_1| = N
\]
and
\[
|\beta_{k+1}| = (N - 1)|\beta_k|.
\]
Therefore
\[
|\beta_k| = N(N - 1)^{k-1}, \quad |\alpha_k| = \sum_{i=1}^{k} \beta_i + 1 = N \frac{(N - 1)^k - 1}{N - 2} + 1,
\]
and the bound holds. \( \square \)

Thus
\[
(5.4) \quad |\{ J \in I, \text{s.t } w^J_{K-1} \neq 0 \}| \lesssim N^t
\]
and therefore, according to (5.3), for \( 0 \leq t \leq \epsilon \log h \)
\[
\|R_K\|_{L^\infty} \lesssim C_K |\log h|^3 h^{K(1-2\epsilon) - 5 - 4\epsilon} h^{-\epsilon \log N}
\]
\[
\lesssim C_K h^{K(1-2\epsilon) - 6 - 4\epsilon - \epsilon \log N}.
\]
We take \( \epsilon > 0 \) small enough so that \( 2\epsilon \leq \frac{1}{2} \) and \( \epsilon \log N \leq 1 \) in order to get
\[
\|R_K\|_{L^\infty} \leq C_K h^{5/8}.
\]
Let us fix \( K = 15 \). Then, \( \|R_K\|_{L^\infty} \leq C_K h^{-3/4} \). Therefore, as \( t \leq \epsilon \log h \) implies \( h \leq e^{-\frac{t}{4}} \), we get
\[
(5.5) \quad \|R_K\|_{L^\infty} \leq C_K h^{-3/4} e^{-\frac{t}{4}}
\]
for \( 0 \leq t \leq \epsilon \log h \).

Times \( t \geq t_0 > 0 \). Let us now deal with the approximate solution \( S_K \), \( K \) been fixed and \( x \) in Supp\( \chi \). Let \( t_0 > 0 \) to be chosen later. For \( t \geq t_0 \), by lemma 4.5 we can perform a stationary phase on each term of the \( J \) sum, up to order \( h \). We obtain, for \( t \geq t_0 \)
\[
(5.6) \quad S_K(x, t) = \frac{1}{(2\pi h)^{3/2}} \sum_{J \in I} e^{-i(\phi_J(x, s_j(t, x)))} |s_j(t, x)| - ts_j(t, x)^2|/h \left( w^J_0(t, x, s_j(t, x)) + h\tilde{w}^J_1(t, x) \right)
\]
\[
+ \frac{1}{h^{3/2}} \sum_{J \in I} R^J_{\text{st.ph.}}(x, t) + \frac{1}{(2\pi h)^3} \sum_{J \in I} \int_{V} h^k w^J_k(x, t, \xi) e^{-i(\tilde{\phi}_J(x, \xi))} |\xi| - t\xi^2|/h \, d\xi
\]
where \( s_j(t, x) \) is an eventual unique critical point of the phase (if it does not exist, the corresponding term is \( O(h^{\infty}) \) and by (5.4) it does not contribute). The term \( \tilde{w}^J_1 \) is a linear combination of
\[
D^2_{\xi} w^J_0(t, x, s_j(t, x)), w^J_1(t, x, s_j(t, x)),
\]
and $R^j_{\text{st.ph.}}$ is the reminder involved in the stationary phase, who verifies (see for example to [Zwo12], Theorem 3.15)

\begin{equation}
|R^j_{\text{st.ph.}}(x,t)| \leq h^2 \sum_{|\alpha| \leq 7} \sup |D^\alpha \xi w^j_k(x,\cdot,t)|.
\end{equation}

We recall that by lemma 4.4, for $0 \leq t \leq \epsilon \log h$, $\chi w^j_k$ is supported in $U^\infty_{I,l}$. Therefore, for $0 \leq t \leq \epsilon \log h$ and all $0 \leq k \leq K - 1$, we have, if $x \in \text{Supp} \chi$, using the estimate of Proposition 4.5, because $w^j_k(x,\xi,\cdot)$ is supported in $\{c_1|J| \leq t \leq c_2(|J| + 1)\}$ by lemma 4.2,

$$
\sum_{J \in I} |w^j_k| \leq C_k h^{-2kce} \sum_{I \text{ primitive}, |s| \leq |I| - 1} \lambda^{|J|}_I.
$$

Thus

$$
\sum_{J \in I} |w^j_k| \leq C_k h^{-2kce} \sum_{I \text{ primitive}} \sum_{r \geq 0 \atop 0 \leq s \leq |I| - 1} \lambda^{|J|}_I \rho_k(I)^r \lambda^s_I,
$$

where we denoted

$$
\rho_k(I) = \inf \{r \geq 1 \text{ s.t. } \exists s, w^j_{k+s} \neq 0\},
$$

and we get

\begin{equation}
\sum_{J \in I} |w^j_k| \leq C_k h^{-2kce} \sum_{I \text{ primitive} \atop \rho_k(I) \neq \infty} \frac{1}{1 - \lambda^{|J|}_I} \rho_k(I)|I|.
\end{equation}

Moreover, as

\begin{equation}
\rho_k(I) \lesssim \frac{t}{|I|}
\end{equation}

and, because as remarked in [Bur93], if $\gamma$ is the trajectory associated to $I$

\begin{equation}
\frac{d_{\gamma}}{\text{diam} C} \leq \text{card} \gamma = |I| \leq \frac{d_{\gamma}}{d_{\text{min}}}
\end{equation}

where $C$ is the convex hull of $\cup \Theta_i$. Therefore, combining (5.8) with (5.9) and (5.10)

\begin{equation}
\sum_{J \in I} |w^j_k| \lesssim C_k h^{-2kce} \sum_{\gamma \text{ primitive}} d_{\gamma,\lambda^{|J|}_I} \frac{d_{\gamma}}{\lambda^{|J|}_I \rho_k(I)}.
\end{equation}

But, by Ikawa condition (1.3), there exists $\alpha > 0$ such that

$$
\sum_{\gamma \text{ primitive}} d_{\gamma,\lambda^{|J|}_I} e^{\alpha d_{\gamma}} < \infty.
$$

Let us denote

$$
C_{\gamma} = \lambda^{|J|}_I e^{\alpha d_{\gamma}}.
$$

Notice that, because $d_{\gamma}$ is bounded from below by $d_{\text{min}}$ uniformly with respect to $\gamma$, we have a fortiori

$$
\sum C_{\gamma} < \infty.
$$

Therefore, all $C_{\gamma}$ but a finite number are lesser than one. Reducing $\alpha$ if necessary and taking it small enough, we can thus assume that

$$
0 \leq C_{\gamma} \leq 1, \forall \gamma.
$$
Hence, for \( t \geq \frac{d_{\min}}{D_k} \) we have
\[
C_{\gamma} D_{\gamma} \leq C_{\gamma},
\]
thus, by (5.12), for \( t \geq \frac{d_{\min}}{D_k} \)
\[
\sum_{J \in \mathcal{I}} |w_{k}^{J}| \lesssim C_k h^{-2kc} \sum_{\gamma \text{ primitive}} d_{\gamma} (C_{\gamma} e^{-\alpha D_{\gamma}}) D_{\gamma} \frac{t}{\gamma} \leq C_k h^{-2kc} e^{-\alpha D_{\gamma} t} \sum_{\gamma \text{ primitive}} d_{\gamma} C_{\gamma},
\]
and hence, because of (1.3),
\[
(5.12) \quad \sum_{J \in \mathcal{I}} |w_{k}^{J}| \leq C_k h^{-2kc} e^{-\mu_k t} \quad \text{for} \quad \frac{d_{\min}}{D_k} \leq t \leq \epsilon |\log h|.
\]
for some \( \mu_k > 0 \). Now, remark that for \( t \leq \frac{d_{\min}}{D_k} \), by (5.11) we have
\[
\sum_{J \in \mathcal{I}} |w_{k}^{J}| \lesssim C_k h^{-2kc} \sum_{\gamma \text{ primitive}} d_{\gamma} \lambda_{\gamma}
\]
but because \( d_{\gamma} \) are bounded below, (1.3) implies a fortiori
\[
\sum_{\gamma \text{ primitive}} d_{\gamma} \lambda_{\gamma} < \infty
\]
and thus
\[
(5.13) \quad \sum_{J \in \mathcal{I}} |w_{k}^{J}| \lesssim C_k h^{-2kc} \quad \text{for} \quad t_0 \leq t \leq \frac{d_{\min}}{D_k}
\]
Combining (5.12) and (5.13) we get
\[
\sum_{J \in \mathcal{I}} |w_{k}^{J}| \leq C_k h^{-2kc} e^{-\mu_k t} \quad \text{for} \quad t_0 \leq t \leq \epsilon |\log h|
\]
Let us take \( \epsilon > 0 \) small enough so that \( 2Kcc \leq \frac{1}{2} \). We get, for \( t_0 \leq t \leq \epsilon |\log h| \)
\[
(5.14) \quad \sum_{J \in \mathcal{I}} |w_{k}^{J}| \leq C_k h^{-\frac{1}{2}} e^{-\mu t}, \quad 1 \leq k \leq K - 1,
\]
(5.15) \[ \sum_{J \in \mathcal{I}} |w_{0}^{J}| \leq e^{-\mu t}. \]
with
\[
\mu = \min_{0 \leq k \leq K - 1} \mu_k > 0.
\]
Moreover, using (5.7) together with (5.4), lemma 4.2 and Corollary 4.1 we obtain, for \( t \leq \epsilon |\log h| \)
\[
\sum_{J \in \mathcal{I}} |R_{\text{st},\text{ph.}}(x, t)| \leq h^{3} \sum_{J \in \mathcal{I}} \sup_{|a| \leq 7} |D_{\xi}^{\alpha} w_{k}^{J}(x, \cdot, t)| \leq h^{2} \sum_{J \in \mathcal{I}} C_{\gamma} \leq h^{2} \frac{N}{(2K + 7)c} \leq h^{2} \frac{N}{(2K + 7)c} h^{-\eta c}
\]
where $\eta > 0$ depends only of $\alpha_0, \beta_0$, and the geometry of the obstacles. Therefore, choosing $\epsilon > 0$ small enough
\begin{equation}
\sum_{J \in I} |R_{st.\text{ph.}}^{\,J}(x, t)| \lesssim h \leq e^{-t/\epsilon}.
\end{equation}
for $t \leq \epsilon |\log h|$. In the same way we get, taking $\epsilon > 0$ small enough and $t \leq \epsilon |\log h|
\begin{equation}
\sum_{J \in I} |D_{\xi}^2 w_{\xi}^{\,J}| \leq N^t C^\pi \lesssim h^{-1/4}
\end{equation}
and therefore
\begin{equation}
\sum_{J \in I} |D_{\xi}^2 w_{\xi}^{\,J}| \leq h^{-\frac{1}{2}} e^{-t/4\epsilon}.
\end{equation}
So, combining (5.14), (5.15), (5.16) and (5.17) with (5.6), we obtain, for some $\nu > 0$
\begin{equation}
|\chi_S K(x, t)| \lesssim e^{-\nu t} h^{3/2}
\end{equation}
for $t_0 \leq t \leq T$.

**Conclusion.** Combining the above estimate (5.18) with the control of the reminder term (5.5) and taking $t = T$ gives (4.1) and therefore the dispersive estimate (3.6). By the work of reduction of the third section and summed up in lemma 3.8, Theorem 1.1 is therefore demonstrated for the Schrödinger equation.

### 6. The wave equation

In the case of the wave equation, the counterpart of the smoothing estimate without loss outside the trapped set, namely the following $L^2$- decay of the local energy
\begin{equation}
\|(Au, A\partial_t u)\|_{L^2(\mathbb{R}, H^\gamma \times H^{\gamma-1})} \lesssim \|u_0\|_{\dot{H}^\gamma} + \|u_1\|_{\dot{H}^\gamma},
\end{equation}
where $A$ has micro-support disjoint from $K$, is obtained using the same commutator argument, writing in the case of the wave equation as
\begin{equation}
0 = \int \int_{\mathbb{R} \times \Omega} (u, [\Box, P]u) + \int \int_{\mathbb{R} \times \partial \Omega} \langle Pu, \partial_n u\rangle,
\end{equation}
where $P$ is any pseudo-differential operator. Notice that the symbol of $P$ at the border, as an operator acting on waves, has been derivated in $\{\tau^2 - \eta^2 > 0\}$ by [MRS77]. Our method apply in the exact same way as for the Schrödinger equation.

Once (6.1) is obtained, it follows as in [Laf17a] that we can reduce ourselves to prove the Strichartz estimates near the trapped set in logarithmic times, namely
\begin{equation}
\|\text{Op}_h(\phi)u\|_{L^q(\mathbb{R}^n |\log h|, L^r(\Omega))} \lesssim \|u_0\|_{\dot{H}^r} + \|u_1\|_{\dot{H}^{r-1}},
\end{equation}
where $u_{0,1} = \psi(-h^2 \Delta) u_{0,1}$ and $\phi$ is supported in a small neighborhood of $K$. In order to reduce ourselves at points of the phase-space that remain near a periodic trajectory in logarithmic times, the exact same cuting as in the third section holds, at the difference that the flow is followed at constant speed one.

Then, the construction of an approximate solution is the same as in [Laf17a], with the adaptations of the $N$-convex framework presented in the fourth section. In particular, the results of non-degeneracy of the phase and stationary points of [Laf17a] hold, as their proof does not rely on the particular two-convex geometry. Thus, we can perform the same stationary phase argument as in [Laf17a], the difference with the Schrödinger equation been that the phase is now stationary on plain lines due to the constant speed of propagation, and we obtain the good scale.
in \( h \). Now, the only difference with the conclusion section of [Laf17a] is that we cannot deal with
\[
\sum_{J \in I}
\]
as in the two convex case. But we can do it in the exact same way as presented in the fifth section, using the strong hyperbolic setting assumption (1.3), in order to deduce the sufficient time decay. Thus the appropriate dispersive estimate for the waves is obtained and the theorem follows.

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References

[BGH10] Nicolas Burq, Colin Guillarmou, and Andrew Hassell, Strichartz estimates without loss on manifolds with hyperbolic trapped geodesics, Geom. Funct. Anal. 20 (2010), no. 3, 627–656. MR 2720226 (2012f:58068)

[BGT04] N. Burq, P. Gérard, and N. Tzvetkov, On nonlinear Schrödinger equations in exterior domains, Ann. Inst. H. Poincaré Anal. Non Linéaire 21 (2004), no. 3, 295–318. MR 2068304 (2005g:35264)

[BLP08] Nicolas Burq, Gilles Lebeau, and Fabrice Planchon, Global existence for energy critical waves in 3-D domains, Ann. Inst. H. Poincaré Anal. Non Linéaire 21 (2004), no. 3, 831–845. MR 2393429

[Bou11] Jean-Marc Bouclet, Strichartz estimates on asymptotically hyperbolic manifolds, Anal. PDE 4 (2011), no. 1, 1–84. MR 2783305

[BSS09] Matthew D. Blair, Hart F. Smith, and Christopher D. Sogge, Strichartz estimates for the wave equation on manifolds with boundary, Ann. Inst. H. Poincaré Anal. Non Linéaire 26 (2009), no. 5, 1817–1829. MR 2566711

[BT07] Jean-Marc Bouclet and Nikolay Tzvetkov, Strichartz estimates for long range perturbations, Amer. J. Math. 129 (2007), no. 6, 1565–1609. MR 2369889

[Bur93] Nicolas Burq, Contrôle de l’équation des plaques en présence d’obstacles strictement convexes, Mém. Soc. Math. France (N.S.) (1993), no. 55, 126. MR 1254820

[Bur03] N. Burq, Global Strichartz estimates for nontrapping geometries: about an article by H. F. Smith and C. D. Sogge: “Global Strichartz estimates for nontrapping perturbations of the Laplacian” [Comm. Partial Differential Equation 25 (2000), no. 11-12: 2171–2183; MR1789924 (2001j:35180)], Comm. Partial Differential Equations 28 (2003), no. 9-10, 1675–1683. MR 2001179

[Bur04] , Smoothing effect for Schrödinger boundary value problems, Duke Math. J. 123 (2004), no. 2, 403–427. MR 2066943 (2006e:35026)

[CV02] F. Cardoso and G. Vodev, Uniform estimates of the resolvent of the Laplace-Beltrami operator on infinite volume Riemannian manifolds. II, Ann. Henri Poincaré 3 (2002), no. 4, 673–691. MR 1933365

[DV13] Kiril Datchev and András Vasy, Propagation through trapped sets and semiclassical resonant states, Microlocal methods in asymptotic physics and global analysis, Trends Math., Birkhäuser/Springer, Basel, 2013, pp. 7–10. MR 3307787

[GV85] J. Ginibre and G. Velo, Scattering theory in the energy space for a class of nonlinear Schrödinger equations, J. Math. Pures Appl. (9) 64 (1985), no. 4, 363–401. MR 839728

[HTW06] Andrew Hassell, Terence Tao, and Jared Wunsch, Sharp Strichartz estimates on nontrapping asymptotically conic manifolds, Amer. J. Math. 128 (2006), no. 4, 963–1024. MR 2251591

[Iba82] Mitsuru Iwata, Decay of solutions of the wave equation in the exterior of two convex obstacles, Osaka J. Math. 19 (1982), no. 3, 459–509. MR 676233

[Iba88] , Decay of solutions of the wave equation in the exterior of several convex bodies, Ann. Inst. Fourier (Grenoble) 38 (1988), no. 2, 113–146. MR 949013

[ILLP] Oana Ivanovici, Richard Lascar, Gilles Lebeau, and Fabrice Planchon, Dispersion for the wave equation inside strictly convex domains II: the general case, Preprint.
Oana Ivanovici, Gilles Lebeau, and Fabrice Planchon, *Dispersion for the wave equation inside strictly convex domains I: the Friedlander model case*, Ann. of Math. (2) 180 (2014), no. 1, 323–380. MR 3194817

Oana Ivanovici, *On the Schrödinger equation outside strictly convex obstacles*, Anal. PDE 3 (2010), no. 3, 261–293. MR 2672795 (2011j:58037)

Oana Ivanovici, *Counterexamples to the Strichartz inequalities for the wave equation in general domains with boundary*, J. Eur. Math. Soc. (JEMS) 14 (2012), no. 5, 1357–1388. MR 2966654

L. V. Kapitanskiǐ, *Some generalizations of the Strichartz-Brenner inequality*, Algebra i Analiz 1 (1989), no. 3, 127–159. MR 1015129

Markus Keel and Terence Tao, *Endpoint Strichartz estimates*, Amer. J. Math. 120 (1998), no. 5, 955–980. MR 1646048

D. Lafontaine, *About the wave equation outside two strictly convex obstacles*, Preprint, https://arxiv.org/abs/1711.09734 (2017).

D. Lafontaine, *Strichartz estimates without loss outside two strictly convex obstacles*, Preprint, https://arxiv.org/abs/1709.03836 (2017).

Hans Lindblad and Christopher D. Sogge, *On existence and scattering with minimal regularity for semilinear wave equations*, J. Funct. Anal. 130 (1995), no. 2, 357–426. MR 1335386

Jason L. Metcalfe, *Global Strichartz estimates for solutions to the wave equation exterior to a convex obstacle*, Trans. Amer. Math. Soc. 356 (2004), no. 12, 4839–4855. MR 2084401

Cathleen S. Morawetz, James V. Ralston, and Walter A. Strauss, *Decay of solutions of the wave equation outside nontrapping obstacles*, Comm. Pure Appl. Math. 30 (1977), no. 4, 447–508. MR 0509770

Hart F. Smith, *A parametrix construction for wave equations with C^{1,1} coefficients*, Ann. Inst. Fourier (Grenoble) 48 (1998), no. 3, 797–835. MR 1644105

Hart F. Smith and Christopher D. Sogge, *On the critical semilinear wave equation outside convex obstacles*, J. Amer. Math. Soc. 8 (1995), no. 4, 879–916. MR 1308407

Hart F. Smith and Christopher D. Sogge, *Local smoothing of Fourier integral operators and Carleson-Sjölin estimates*, J. Amer. Math. Soc. 6 (1993), no. 1, 65–130. MR 1168960

Gigliola Staffilani and Daniel Tataru, *Strichartz estimates for a Schrödinger operator with nonsmooth coefficients*, Comm. Partial Differential Equations 27 (2002), no. 7-8, 1337–1372. MR 1924470

Gigliola Staffilani and Daniel Tataru, *Strichartz estimates for a Schrödinger operator with nonsmooth coefficients*, III. J. Amer. Math. Soc. 15 (2002), no. 2, 419–442. MR 1887639

Maciej Zworski, *Semiclassical analysis*, Graduate Studies in Mathematics, vol. 138, American Mathematical Society, Providence, RI, 2012. MR 2952218