MODULARITY AND HEIGHTS OF CM CYCLES ON KUGA–SATO
VARIETIES

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Abstract. We prove a higher weight general Gross–Zagier formula for CM cycles on Kuga–Sato
varieties over modular curves of arbitrary levels. To formulate and prove this result, we prove
several results on the modularity of CM cycles, in the sense that the Hecke modules they generate
are semisimple modules whose irreducible components are associated to higher weight holomorphic
cuspidal automorphic representations of GL2, Q. These two types of results provide evidence toward
two conjectures of Beilinson–Bloch. The higher weight general Gross–Zagier formula is proved
using arithmetic relative trace formulas. The proof of the modularity of CM cycles is inspired by
arithmetic theta lifting.

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1. Introduction

1.1. Gross–Zagier formula and generalizations. Let $K$ be an imaginary quadratic field. On modular curves, there are explicit algebraic points representing elliptic curves with CM, i.e. complex multiplication, by $K$. They were studied by Heegner in the case of the modular curve $X_0(N)$ for some $N$, and thus called Heegner points in this case. In [9], Gross and Zagier established a remarkable formula that relates Néron–Tate heights of Heegner points and central derivatives of base change $L$-functions associated to holomorphic modular forms of weight 2 for the modular curve $X_0(N)$. Explicitly, for a holomorphic newform $f$ of weight 2, let $J_0(N)_f$ be the $f$-isotypical component ($f$-component for short) of the Jacobian of the modular curve $X_0(N)$. For a degree-0 divisor $P$ of Heegner points modified by the cusps, the Gross–Zagier formula says that under the Heegner condition on $K$ and $N$, the $f$-component $P_f \in J_0(N)_f$ has height

$$\langle P_f, P_f \rangle = c \cdot L'(f, K, 1)$$

for an explicit nonzero constant $c$. Since the first day of its establishment, the Gross–Zagier formula has been the best evidence toward the Birch–Swinnerton-Dyer conjecture.

The Gross–Zagier formula has been generalized in many aspects. In the content of modular curves, the two definite aspects are the level aspect and the weight aspect. We summarize some of the relevant results in the following table, and explain them in the next paragraph.

Table 1.

| Weight | Level | $X_0(N)$ with Heegner condition | Arbitrary modular curves |
|--------|-------|---------------------------------|--------------------------|
| 2      |       | Gross and Zagier (1986)         | Yuan, S. Zhang, and W. Zhang (2013) |
| $2k > 2$ |      | S. Zhang (1997)                |                          |

In the weight 2 case, Yuan, S. Zhang and W. Zhang [37] proved the Gross–Zagier formula at arbitrary levels. (In fact, they proved it even for all quaternionic Shimura curves.) Moreover, their formula is in terms of general CM points and automorphic forms, and thus deserves to be called the general Gross–Zagier formula. It was built on previous works of S. Zhang [41, 42]. In the higher weight $2k > 2$, S. Zhang [40] proved a Gross–Zagier type formula for Heegner cycles on Kuga–Sato varieties over $X_0(N)$ and modular forms of weight $2k$. The purpose of this paper is to fill in the blank in Table 1.

1.2. Results. The main result (Theorem 2.4.5) of this paper is a higher weight analog of the general Gross–Zagier formula on Kuga–Sato varieties over modular curves. It serves as evidence toward the conjecture of Beilinson and Bloch [1, 3] that is a higher dimensional generalization of the Birch–Swinnerton-Dyer conjecture.

The formulation of this main result is automorphic and aligns to the one of Yuan, S. Zhang and W. Zhang [37]. To formulate this result, we need two modularity results (Theorem 2.3.3 and Theorem 2.3.7) for the Hecke modules of CM cycles. In this introduction, for simplicity, we
only state one modularity result (Theorem 1.2.1) which is enough to formulate a weaker version (Theorem 1.2.2) of our main result. The formulation of this weaker version is closer to the original Gross–Zagier formula.

Let $X$ be the modular curve $X(N)$ over $\mathbb{Q}$ where $N \geq 3$ so that $X(N)$ is a fine moduli space. Let $k \geq 2$ be an integer. The Kuga–Sato variety $Y$ is the canonical desingularization of the $(2k - 2)$-tuple product of the universal generalized elliptic curve over $X$ (see [40, 2.2]). For an elliptic curve $A$ with CM by $K$ that represents a CM point on $X$, identify $A$ as the fiber of $Y$ over this CM point and define the CM cycle $Z(A)$ on $A^{2k-2}$ as follows. Fix an embedding of $K$ to $C$. Choose a positive integer $D$ such that $\sqrt{-D} \in K$ and is an endomorphism of $A$. Let $\Gamma$ be the graph of $\sqrt{-D}$. Let $\Gamma_0$ be the divisor $\Gamma - A \times \{0\} - D\{0\} \times A$ on $A \times A$. Then $\Gamma_0^{k-1}$ is a $(k-1)$-cycle on $A^{2k-2}$. Define the CM cycle on $A^{2k-2}$ to be

$$Z(A) = c \sum_{\sigma} \text{sgn}(\sigma) \sigma^* \left( \Gamma_0^{k-1} \right),$$

where $\sigma$ runs over the symmetric group of $2k-2$ elements which acts on $A^{2k-2}$ naturally, and $c$ is a positive constant such that the self intersection number of $Z(A)$ in $A^{2k-2}$ is $(-1)^{k-1}$. Then the Chow cycle $[Z(A)]$ of $Z(A)$ does not depend on the choice of $D$ (see [40, Proposition 2.4.1]). We understand $Z(A)$ as a cycle on $Y$ via $A^{2k-2} \subset Y$. Recall that CM points (and thus CM cycles) are defined over $K^{ab}$ (see [37, 3.1.2]). Define

$$[Z] = \int_{\text{Gal}(K^{ab}/K)} [Z(A)] \, d\tau,$$

where $\text{Gal}(K^{ab}/K)$ is endowed with the Haar measure of total volume 1. In fact, we will allow the twist of $[Z]$ by a finite order character of $\text{Gal}(K^{ab}/K)$ and use an equivalent adelic definition (Remark 2.2.2).

Let $S$ be a finite set of prime numbers containing all prime factors of $N$. Let $H^S$ be the unramified Hecke algebra outside $S$, which acts on the Chow groups of $Y$ via Hecke correspondences (see 2.1.3). Let $CM^S$ be the $H^S$-module generated by $[Z]$. Let $\overline{CM}$ be the quotient of $CM$ by the kernel of the Beilinson–Bloch height pairing on $CM$ defined in §2.2.4.

**Theorem 1.2.1** (Theorem 2.3.4). Assume that $S$ has cardinality at least three, and contains the prime 2 and all finite places of $\mathbb{Q}$ ramified in $K$. The $H^S$-module $\overline{CM}$ is semi-simple whose irreducible components are the $H^S$-modules associated to weight $2k$ holomorphic cuspidal automorphic representations of $\text{GL}_2, \mathbb{Q}$ with trivial central character.

This theorem is predicted by the conjecture of Beilinson and Bloch on the injectivity of the Abel–Jacobi map, see Remark 2.3.3.

Let $\pi$ be a weight $2k$ holomorphic cuspidal automorphic representation of $\text{GL}_2, \mathbb{Q}$ with trivial central character. Then the $\pi$-component of $[Z]$ is well-defined by Theorem 1.2.1 and denoted by $[Z]_{\pi}$.
Theorem 1.2.2. Let $\pi$ be generated by modular forms of weight $2k$ on $X[1]$. Assume $L(1/2, \pi_K) = 0$. There is a nonzero constant $c_\pi$ such that

$$\langle [Z]_\pi, [Z]_\pi \rangle = c_\pi \cdot L'(1/2, \pi_K).$$

This is a special case of Corollary 2.4.7 of our main result (Theorem 2.4.5). In S. Zhang [40] over $X_0(N)$, the assumption $L(1/2, \pi_K) = 0$ is implicit by the Heegner condition, which implies that the root number of $L(s, \pi_K)$ is $-1$.

We also remark that the definition of Kuga–Sato varieties, CM cycles, Theorem 1.2.1 and Theorem 1.2.2 apply to any fine modular curve $X$ (for Theorem 1.2.1 assume that $X$ has maximal level structures outside $S$). Indeed, we can dominate $X$ by some $X(N)$ with $S$ containing all prime factors of $N$. The results for $X(N)$ imply the ones for $X$. If $X$ is not a fine moduli space, one may follow [40, 4.1].

1.3. Legacy from previous works. Our proof of Theorem 1.2.1 is strongly influenced by the previous works in Table 1.

A key step in Gross and Zagier [9] for $X_0(N)$ is the following explicit modularity of generating series of heights. Let $T_m$ be the standard $m$-th Hecke operator in $\text{Pic} \left( X_0(N)^2 \right)$, as a correspondence from $X_0(N)$ to $X_0(N)$. If $m$ is coprime to $N$, the Néron-Tate height pairing

$$\langle P, T_m P \rangle$$

is the $m$-th coefficient of an explicit holomorphic modular form of weight 2.

In weight $2k > 2$, S. Zhang [40] followed Gross and Zagier’s approach, and proved the modularity of generating series of heights with $Z$ replacing $P$. A consequence of this result is an analog of Theorem 1.2.1 in the setting of loc. cit.. We shall call Theorem 1.2.1 the modularity of Hecke modules of Heegner/CM cycles. Analogous modularity for Heegner/CM points follows from the Abel–Jacobi theorem for divisors on curves for free.

We establish a modularity result of generating series of heights (4.35). However, our result is weak from two aspects. First, our result is only for some “regular” Hecke operators (see Assumption 4.4.1). This can be remedied, as S. Zhang computed some complement Hecke operators. Second, our relevant modular forms, more precisely, automorphic forms, live on an adelic group with many connected components. Our modularity result is only on one component (see Remark 4.4.8). The same issue appears in Yuan, S. Zhang and W. Zhang [37]. They overcame it using another modularity result, proved in a separate work of them [36]. However, we do not (need to) overcome this issue in our work.

Before we explain how to prove the modularity of Hecke modules from the modularity of generating series, we summarize some key words in the proofs of the above modularity results.

| Modularity of | Generating series | Hecke module |
|---------------|-------------------|--------------|
| Gross–Zagier (1986) | Explicit computation | Abel–Jacobi theorem |
| YZZ (2013) | Another modularity in YZZ (2009) | Abel–Jacobi theorem |
| S. Zhang (1997) | Explicit computation | Jacquet–Langalnds–Shimizu |
| Current work | Weak version | Numerical interpretation |

[1] Equivalently, $\pi$ has nonzero invariant vectors by the adelic principal congruence subgroup of $\text{GL}_2(\mathbb{Q})$ of level $N$. 
From a representation theoretic viewpoint, S. Zhang’s proof is essentially an arithmetic version of the Jacquet–Langalnds–Shimizu correspondence from $GL_2$ to $GL_2$ itself. By only having a weak version, we can not establish such a correspondence. However, we can interpret the modularity of Hecke modules of CM cycles in terms of the vanishing of some height pairings (Proposition 4.5.1). This numerical statement can be tackled by our weak version of the modularity of generating series of heights.

Finally, we remark that the approach of Gross and Zagier has now been further developed to the theory of arithmetic theta lifting by Kudla. It is worth mentioning that recently Li and Liu used arithmetic theta lifting to prove an analog of the Gross–Zagier formula for higher dimensional unitary Shimura varieties.

1.4. Relative trace formulas. To prove the higher weight analog of the general Gross–Zagier formula, instead of arithmetic theta lifting (as in Yuan, S. Zhang and W. Zhang), we use the arithmetic variants of Jacquet’s relative trace formulas. In this arithmetic relative trace formula approach, another modularity result (Theorem 2.3.7) is also vital. This modularity result is for the action of the full Hecke algebra on CM cycles.

The arithmetic relative trace formula approach was proposed by W. Zhang to attack the arithmetic Gan–Gross–Prasad conjecture, which is an exact generalization of the Gross–Zagier formula to unitary Shimura varieties. Subsequently, we used this method to prove the general Gross–Zagier formula over function fields. Moreover, Yun and Zhang used a geometric version of the arithmetic relative trace formula approach to prove an analog of the Gross–Zagier formula for higher order derivatives.

1.5. Generalizations. The analog of S. Zhang’s work for general Shimura curves has not been established yet. This is the main obstruction to the generalizations of our results to Shimura curves. A key ingredient to the analog of S. Zhang’s work has been established by Wen. We hope to return to Shimura curves in the near future.

1.6. Structure of the paper. In §2, we first define Kuga–Sato varieties, Hecke operators and CM cycles. Then we state our modularity results for the Hecke action on CM cycles in §2.3 and our higher weight analog of the general Gross–Zagier formula in §2.4.

In §3, we first review the relative trace formulas and related backgrounds. Then we establish an arithmetic relative trace identity in §3.4. Finally in §3.5, we prove the higher weight general Gross–Zagier formula, providing our modularity results.

In §4, we first recall some basics of theta series and Eisenstein series. Then we prove an arithmetic mixed Siegel–Weil formula in §4.3. In §4.5–4.6, we prove the modularity of CM cycles.

1.7. Some notations and conventions. A local field $F$ is assumed to be equipped with an absolute value $|\cdot|_F$. Let $|\cdot|_p = |\cdot|_\mathbb{Q}_p$ be that assigns $1/p$ to $p \in \mathbb{Q}_p$. On $\mathbb{R}$, we use the usual absolute value. On $\mathbb{C}$, the absolute value is the square of the usual one. Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$. The symbol $|a|$ for $a \in \mathbb{A}$ means the product of all local absolute values of $a$.

For a set of places $S$ of $\mathbb{Q}$ and a decomposable adelic object $X$ over $\mathbb{A}$, we use $X_S$ (resp. $X^S$) to denote the $S$-component (resp. component away from $S$) of $X$ if the decomposition of $X$ into the

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[2] The general Jacquet–Langalnds–Shimizu correspondence is from the unit group of a quaternion algebra to $GL_2$. In this setting, the quaternion algebra the matrix algebra and the unit group is also $GL_2$! For us, we will use a quaternionic notation in our proof in 4.5.
product of $X_S$ and $X^S$ is clear from the context. If $S = \{v\}$, we write $X_v$ (resp. $X^v$) for $X_S$ (resp. $X^S$). For example, $\mathbb{A}^\infty$ is the ring of finite adeles.

We use $\mathcal{S}_c(X)$ resp. $\mathcal{S}(X)$ to denote the space of compactly supported smooth functions resp. Schwartz functions on a reasonable space $X$. We will only use the following cases. If $X$ is a real manifold, we will only use $\mathcal{S}_c(X)$, except when $X = \mathbb{R}^n$, in which case the definition of $\mathcal{S}(X)$ is classic. If $X$ is locally profinite, $\mathcal{S}_c(X) = \mathcal{S}(X)$ and is the space of compactly supported locally constant functions.

For a group $G$ and a function $f$ on $G$, let

$$f^\vee(g) := f(g^{-1}).$$

If $G$ is a locally profinite group, the convolution on $\mathcal{S}_c(G) = \mathcal{S}(G)$ is

$$f * f_1(h) = \int_G f(g)f_1(g^{-1}h)dg.$$

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2. CM cycles on Kuga–Sato varieties

We first define Kuga–Sato varieties and Hecke operators on Chow cycles on Kuga–Sato varieties. Then define CM cycles as well as their height pairings. Finally we state our theorems on the modularity of CM cycles, and the higher weight general Gross–Zagier formula.

2.1. Kuga–Sato varieties and Hecke operators.

2.1.1. Kuga–Sato varieties. For a positive integer $N$, let $X(N)^\circ$ be the moduli stack of elliptic curves $A$ over $\mathbb{Q}$-schemes with full level $N$ structure $(\mathbb{Z}/N\mathbb{Z})^2 \simeq A[N]$. For simplicity, let $N \geq 3$ so that $X(N)^\circ$ is representable by a smooth curve over $\mathbb{Q}$, which is connected but not geometrically connected. It might be helpful to remind the reader the complex uniformization of the $\mathbb{C}$-points of $X(N)^\circ$ as a Riemann surface:

$$X(N)^\circ(\mathbb{C}) \simeq \mathrm{GL}_2(\mathbb{Q})_{>0}\backslash \mathbb{H} \times \mathrm{GL}_2(\mathbb{A}^\infty)^\times / U(N).$$

Here $\mathrm{GL}_2(\mathbb{Q})_{>0} \subset \mathrm{GL}_2(\mathbb{Q})$ consists of elements with positive determinants, $\mathbb{H} \subset \mathbb{C}$ is the upper half plane and is equipped with the action of $\mathrm{GL}_2(\mathbb{R})_{>0}$ by fractional linear transformation, and $U(N) = 1 + N\mathbb{M}_2(\mathbb{Z})$ is the corresponding principal congruence subgroup of $\mathrm{GL}_2(\mathbb{A}^\infty)$ of level $N$, where $\mathbb{Z} \subset \mathbb{A}^\infty$ is the ring of integral adeles.

The modular curve $X(N)$ be the smooth compactification of $X(N)^\circ$, which is the moduli space of generalized elliptic curves. Let $E \to X(N)$ be the universal generalized elliptic curve so that $E|_{X^\circ} \to X^\circ$ is the universal elliptic curve.

Let $k \geq 2$ be an integer. Let $Y(N)^\circ = (E|_{X(N)^\circ})^{2k-2}$, the $(2k-2)$-tuple product of $E|_{X(N)^\circ}$ over $X(N)^\circ$. The Kuga–Sato variety $Y(N)$ is the canonical desingularization of $E^{2k-2}$ [40 2.2]. Indeed, the desingularization procedure only happens at the cusps so that $Y(N)^\circ = Y(N)|_{X(N)^\circ}$.

Below, we will use some results in [40], where the author takes a geometrically connected component of $X(N)$ (while using the notation $X(N)$ in loc. cit.). The discussion there can be carried on to our setting accordingly.
2.1.2. Group and semi-group action. We will define Hecke operators on Chow cycles by mimicking the Hecke operators in the setting of automorphic forms, rather than classical modular forms. Thus we need an adelic group action. It is defined on some infinite level modular curves, and related Kuga–Sato varieties.

Let $G \subseteq \text{GL}_2(\mathbb{A}^\infty)$ be the subset of elements with integral coefficients. Let

$$X_\infty^0 = \varinjlim_N X(N)^0, \quad Y_\infty^0 = \varinjlim_N Y(N)^0,$$

where the transition morphisms are finite flat. We define a left $\text{GL}_2(\mathbb{A}^\infty)$-action on $X_\infty^0$ and a left $G$-action on $Y_\infty^0$ following Drinfeld [4 5.D].

For $g \in G$, $g$ defines an endomorphism of $(\mathbb{Q}/\mathbb{Z})^2$. Let $S$ be a $\mathbb{Q}$-scheme and $A$ an elliptic curve over $S$ together with an infinite level structure $\psi : (\mathbb{Q}/\mathbb{Z})^2 \to A(S)$, i.e. for every positive integer $n$, the restriction of $\phi$ to $\mathbb{Z}[\frac{1}{n}]$ is a (Drinfeld) $\Gamma(n)$-structure [17 3.1]. This gives an $S$-point of $X_\infty^0$. Let $H \subset A$ be the sum of $\psi(\alpha)$’s with $\alpha$ running over the kernel of the endomorphism $g$ on $(\mathbb{Q}/\mathbb{Z})^2$, which is a subgroup scheme of $A$ [17 Proposition 1.11.2]. Define $\psi_1$ so that the following diagram commutes:

$$
\begin{array}{ccc}
(Q/\mathbb{Z})^2 & \xrightarrow{\psi} & A(S) \\
\downarrow{g} \downarrow{\psi_1} & & \downarrow{\psi_1} \\
(Q/\mathbb{Z})^2 & \xrightarrow{\psi_1} & A/H(S)
\end{array}
$$

Here the right vertical arrow is the natural quotient morphism.

Let $E_\infty$ be the universal elliptic curve over $X_\infty^0$ with an infinite level structure. Applying the above construction, we get a quotient elliptic curve $E'_\infty$ over $X_\infty^0$, with an infinite level structure, and a quotient map $E_\infty \to E'_\infty$ over $X_\infty^0$. The elliptic curve $E'_\infty$ over $X_\infty^0$ is the pullback of $E$ via a morphism $X_\infty^0 \to X_\infty^0$, which we define as the action of $g$ on $X_\infty^0$. For $g \in G \cap \mathbb{Q}^\times$, the action is trivial. Thus we have an action of $\text{GL}_2(\mathbb{A}^\infty)$ on $X_\infty^0$ by letting $\mathbb{Q}^\times$ act trivially.

Remark 2.1.1. The complex uniformization (2.1) gives

$$X_\infty^0(\mathbb{C}) \simeq \text{GL}_2(\mathbb{Q})_{>0} \backslash \mathbb{H} \times \text{GL}_2(\mathbb{A}^\infty)^{\times}.$$

Then $g$ acts as the “right multiplication” by $g^{-1}$ (compare with [37 3.1.1]).

The composition of the above morphisms $E_\infty \to E'_\infty \to E_\infty$ defines a morphism $g : E_\infty \to E_\infty$ for $g \in G$. However, the action of $g \in G \cap \mathbb{Q}^\times$ is not trivial. Indeed, if $g \in \mathbb{Z}$, it acts trivially on $X_\infty^0$, and by multiplying by $g$ on each fiberal elliptic curves. Since $Y_\infty^0 = E_\infty^{2k-2}$, we have an action of $G$ on $Y_\infty^0$. Again, the action of $g \in G \cap \mathbb{Q}^\times$ is not trivial.

2.1.3. Hecke action on cohomological Chow cycles. Let $\text{Ch}^k(Y(N))$ be the spaces of codimension $k$ Chow cycles on $Y(N)_{\overline{\mathbb{Q}}}$ with $\mathbb{C}$-coefficients. We define a right action of $\text{GL}_2(\mathbb{A}^\infty)$ on

$$\varinjlim_N \text{Ch}^k(Y(N))$$

via pullback (this is the reason for calling $\varinjlim_N \text{Ch}^k(Y(N))$ the group of cohomological Chow cycles).

For given $N$ and $g \in \text{GL}_2(\mathbb{A}^\infty)$, the above construction defines a morphism $g_{N'}(N)^0 : Y(N')^\circ \to Y(N)^\circ$ for a large enough multiple $N'$ of $N$. Let $\Gamma g_{N',N}$ be the Zariski closure of the graph of $g_{N',N}$.
in $Y(N') \times Y(N)$. For $z \in \text{Ch}^k(Y(N))$, let $g^*z \in \text{Ch}^k(Y(N')) = \text{Ch}_{k-1}(Y(N'))$ be the pullback of $z$ by $\Gamma_{g, n', n}$. The image of $g^*z$ in $\lim_N \text{Ch}^k(Y(N))$ does not depend on the choice of $N'$. Define the natural right action of $g$ on $\lim_N \text{Ch}^k(Y(N))$ to be

$$R^\text{coh}_g(z) = |\det(g)|^{k-1} g^* z.$$  

Then $R^\text{coh}_g$ is the identity for $g \in \mathbb{G} \cap \mathbb{Q}^\times$. Let $\mathbb{Q}^\times$ act trivially on $\lim_N \text{Ch}^k(Y(N))$. Thus we have a right action of $\text{GL}_2(\mathbb{A}^\infty)$ on $\lim_N \text{Ch}^k(Y(N))$. However, in view of Remark 2.1.1 (as well as the conventions for automorphic forms), it is better to use the opposite left action

$$L^\text{coh}_g = R^\text{coh}_{g^{-1}}.$$  

The induced action of $f \in S(\text{GL}_2(\mathbb{A}^\infty))$ on $\lim_N \text{Ch}^k(Y(N))$ is

$$L^\text{coh}_f := \int_{g \in \text{GL}_2(\mathbb{A}^\infty)} f(g)L^\text{coh}_g dg.$$  

Remark 2.1.2. A more well-known definition of the Hecke action (in the unramified case) is as follows [40 p 122]. For $i \in \mathbb{Z}_{\geq 0}$, let $A_{p^i}$ be the characteristic function of $\text{GL}_2(\mathbb{Z}_p) \left[ \begin{array}{cc} p & 0 \\ 0 & 1 \end{array} \right] \text{GL}_2(\mathbb{Z}_p)$. For a positive integer $n$ coprime to $N$, let $A_n = \bigotimes_{p|n} A_{p^\omega p|n}$. Let $Y(N, n)^0$ be the noncuspidal locus of the Kuga–Sato variety of elliptic curves with the canonical full level $N$ structure, and a subgroup of order $n$. Let $Z^0$ be the quotient of $Y(N, n)^0$ by the $(2k - 2)$-tuple product of the subgroup of order $n$ of the universal elliptic curve (and remember the full level $N$ structure). Then we have natural morphisms

$$Y(N)^0 \leftarrow Y(N, n)^0 \rightarrow Z^0 \rightarrow Y(N)^0.$$  

Let $C$ be the pushforward of the fundamental cycle of $Y(N, n)^0$ to $Y(N) \times Y(N)$. Then up to the choice of the measure on $\text{GL}_2(\mathbb{A}^\infty)$, the Hecke action $R(A_n)$ coincides with the pullback by $n^{-(k-1)}C$.

Let $\xi$ be a cusp form of $\text{SL}_2(\mathbb{Z})$ of level $N$ and weight $2k$, and $\phi_\xi$ the associated cusp form of $\text{GL}_2, \mathbb{Q}$ as in [8 (3.4)]. The pullback of $\xi$ by $n^{-(k-1)}C$ [15 1.11] and the usual Hecke action of $A_n$ on $\phi_\xi$ are compatible [6 Lemma 3.7].

2.1.4. Hecke action on homological Chow cycles. We define a left action of $\text{GL}_2(\mathbb{A}^\infty)$ on

$$\lim_N \text{Ch}^k(Y(N))$$

via push-forward (this is the reason for calling $\lim_N \text{Ch}^k(Y(N))$ the group of homological Chow cycles).

Let $g \in \mathbb{G}$ act on $\lim_N \text{Ch}^k(Y(N))$ by

$$L^\text{hom}_g := |\det(g)|^{k-1} g_*.$$  

Then $L^\text{hom}_g$ is the identity for $g \in \mathbb{G} \cap \mathbb{Q}^\times$. Let $\mathbb{Q}^\times$ act trivially on $\lim_N \text{Ch}^k(Y(N))$. Thus we have a left action of $\text{GL}_2(\mathbb{A}^\infty)$ on $\lim_N \text{Ch}^k(Y(N))$. For $f \in S(\text{GL}_2(\mathbb{A}^\infty))$, we have the induced action

$$L^\text{hom}_f := \int_{g \in \text{GL}_2(\mathbb{A}^\infty)} f(g)L^\text{hom}_g dg.$$  

(2.3)
on \( \lim_N \text{Ch}^k(Y(N)) \).

2.1.5. **Relation.** Consider the natural inclusion

\[ \iota : \lim_N \text{Ch}^k(Y(N)) \to \lim_N \text{Ch}^k(Y(N)) \]

\[ z = \{ z_N \}_N \mapsto \{ \text{Vol}(U(N))z_N \}_N. \]

Then the restriction of \( L_g^{\text{hom}} \) via \( \iota \) is \( L_g^{\text{coh}} \).

2.2. **CM cycles and height pairings.**

2.2.1. **CM cycles.** Let \( K \) be an imaginary quadratic field with a fixed embedding to \( \mathbb{C} \). For an elliptic curve \( A \) over an integral domain \( R \) of characteristic 0 with CM by \( K \), define the CM cycle \( Z(A) \) of \( A \) as follows. Choose a positive integer \( D \) such that \( \sqrt{-D} \in K \) and is an endomorphism of \( A \). Let \( \Gamma \) be the graph of \( \sqrt{-D} \). Let \( \Gamma_0 \) be the divisor

\[ \Gamma - A \times \{ 0 \} - D\{ 0 \} \times A \]

of \( A \times A \). Then \( \Gamma_{0}^{\times k-1} \) is a \(( k-1)\)-cycle on \( A^{2k-2} \). Define the CM cycle on \( A^{2k-2} \) to be

\[ Z(A) = c\sum_{\sigma} \text{sgn}(\sigma)\sigma^{*}\left( \Gamma_{0}^{k-1} \right), \]

where \( \sigma \) runs over the symmetric group of \( 2k-2 \) elements which acts on \( A^{2k-2} \) naturally, and \( c \) is a positive constant such that the self intersection number of \( Z(A)_{\text{Frac}} \) in \( A^{2k-2}_{\text{Frac}} \) is \((-1)^{k-1} \). Then the Chow cycle of \( Z(A)_{\text{Frac}} \) does not depend on the choice of \( D \) (see [40, Proposition 2.4.1]).

Let \( \text{GL}(\mathbb{A}_{\infty}) \) act on \( X_{\infty}^{0} \) as in [2.1.2]. Fix an embedding \( K \hookrightarrow \mathbb{M}_{2,\mathbb{Q}} \) of \( \mathbb{Q} \)-algebras. So we have an embedding \( \mathbb{A}_{\infty}^{\times} \hookrightarrow \text{GL}(\mathbb{A}_{\infty}) \). Fix a point \( P_{0} \in X_{\infty}(\mathbb{Q})^{K^{\times}} \), which is a CM point. For \( g \in \text{GL}_{2}(\mathbb{A}_{\infty}) \), let \( P_{g} \) be the image of \( g^{-1} \cdot P_{0} \in X_{\infty} \) to \( X(N) \) by \( X_{\infty} \to X(N) \). Then \( P_{g} \) is a CM point. Let \( A = E_{\mathbb{Q}}^{1}\cdot P_{g} \) so that \( Y(N)_{\mathbb{Q}}^{1}\cdot P_{g} = A^{2k-2} \). Let the cycle \( Z_{g} \) on \( Y(N)_{\mathbb{Q}}^{1} \) be the pushforward of the CM cycle \( Z(A) \) on \( A^{2k-2} \) under \( A^{2k-2} \to Y(N)_{\mathbb{Q}}^{1} \). Then \( [Z_{g}] \) does not depend on the choice of \( D \), and is cohomologically trivial [40, 3.1].

Let \( \Omega \) be a character of \( K^{\times} \setminus \mathbb{A}_{K}^{\times,\infty} \) trivial on \( K^{\times} \). Define the \( \Omega \)-isotypic CM cycle to be

\[ Z_{\Omega} = \int_{K^{\times} \setminus \mathbb{A}_{K}^{\times,\infty}} Z_{\Omega}(t)dt, \]

where we endow \( K^{\times} \setminus \mathbb{A}_{K}^{\times,\infty} \) with the Haar measure such that

\[ \text{Vol}(K^{\times} \setminus \mathbb{A}_{K}^{\times,\infty}) = 1. \]

Let \( Z^{k}(Y(N)) \) be the space of codimension \( k \) cycles on \( Y(N)_{\mathbb{Q}}^{1} \) with \( \mathbb{C} \)-coefficients. As \( N \) varies, \( Z_{g}, Z_{\Omega} \) are compatible under pushforward. Thus we consider them as elements in \( \lim_{\leftarrow N} Z^{k}(Y(N)) \).

For a cycle \( Z \), let \([Z]\) be its Chow cycle, so that \([Z_{g}], [Z_{\Omega}] \) are elements in \( \lim_{\leftarrow N} \text{Ch}^{k}(Y(N)) \). Then

\[ L_{k-1}^{\text{hom}}[Z_{g}] = [Z_{gh}]. \]
2.2.2. Hecke translation. For \( f \in S(\text{GL}_2(\mathbb{A}^\infty)) \) right \( U(N) \)-invariant, define the translation of \( Z_\Omega \) by \( f \) to be

\[
Z_{\Omega,f} = \frac{1}{\text{Vol}(U(N))} \int_{\mathbb{A}^\infty/K} \int_{\text{GL}_2(\mathbb{A}^\infty)} f(g)Z_{tg}dg \Omega(t)dt.
\]

(2.6)

\[
= \int_{K \times \mathbb{A}^\infty/K} \sum_{g \in \text{GL}_2(\mathbb{A}^\infty)/U(N)} f(g)Z_{tg} \Omega(t)dt
\]

(2.7)

As \( N \) varies, \( Z_{\Omega,f} \) is compatible under pullback. Thus we consider \( Z_{\Omega,f} \) as an element in \( \lim_{\to N} Z^k(Y(N)) \).

Remark 2.2.1. The normalization factor \( \frac{1}{\text{Vol}(U(N))} \) is to achieve the compatibility as \( N \) varies. The normalization factor will be more complicated than for general Shimura curves. See [37, (3.4.4)] or [27, 3.1.1] if we pretend \( k = 1 \).

Now we interpret \( Z_{\Omega,f} \) by Hecke action. Under (2.3), we have

\[
\tau([Z_{\Omega,f}]) = L_{f/\mathbb{F}}^{\text{hom}}[Z_\Omega].
\]

(2.8)

The following relations will be useful later: for \( g \in \text{GL}_2(\mathbb{A}^\infty) \),

\[
L_{g}^{\text{coh}}[Z_{\Omega,f}] = [Z_{\Omega,fg}],
\]

(2.9)

where \( f^g(h) = f(hg) \), and

\[
L_{f_1}^{\text{coh}}[Z_{\Omega,f}] = [Z_{\Omega,f*f_1/\mathbb{F}}].
\]

(2.10)

Remark 2.2.2. CM points (and thus CM cycles) are defined over \( K^{ab} \) (see [37, 3.1.2]). Regarding \( \Omega \) as a character of \( \text{Gal}(K^{ab}/K) \) by the class field theory, then we can redefine

\[
[Z_\Omega] = \int_{\text{Gal}(K^{ab}/K)} [Z_1]^{\tau} \Omega(\tau)d\tau
\]

(2.11)

(and \( [Z_{\Omega,f}] \) similarly), where \( \text{Gal}(K^{ab}/K) \) is endowed with the Haar measure of total volume 1.

2.2.3. Integral models. To define height pairings, we need integral models of Kuga–Sato varieties defined as follows. Let \( N \) be the product of two relatively prime integers which are \( \geq 3 \) until 2.2.5.

Let the morphism \( \pi : \mathcal{E} \to \mathcal{X} \) of regular, flat, and projective \( \mathbb{Z} \)-schemes be as in [10, Theorem 2.1.1], such that \( \mathcal{X}_Q \simeq X(N) \) and \( \mathcal{E} \) is a generalized elliptic curve over \( \mathcal{X} \) whose generic fiber is \( E \).

Let \( \mathcal{Y} \) be the canonical desingularization of \( \mathcal{E}^{2k-2} \) over \( \mathcal{X} \) [10, 2.2]. The desingularization procedure only happens at the cusps in the following sense. By the cusps of \( \mathcal{X} \), we mean the Zariski closure of \( X(N) \setminus X(N)^\circ \), the closed subscheme of the cusps on the generic fiber. Let \( \mathcal{X}^\circ \) be the noncuspidal locus of \( \mathcal{X} \) and \( \mathcal{Y}^\circ = \mathcal{Y}|_{\mathcal{X}^\circ} \). Then \( \mathcal{E}|_{\mathcal{X}^\circ} \to \mathcal{X}^\circ \) is smooth and \( \mathcal{Y}^\circ = (\mathcal{E}|_{\mathcal{X}^\circ})^{2k-2} \).

For a number field \( F \), we construct regular proper models \( \mathcal{X}' \) and \( \mathcal{Y}' \) of \( X_F \) and \( Y_F \) over \( \text{Spec} \mathcal{O}_F \) as follows. Let \( \mathcal{X}' \) be the minimal desingularization of \( \mathcal{X}_{\mathcal{O}_F} \). Let \( \mathcal{N} \) be a neighborhood of the cusps of \( \mathcal{X}' \) which is smooth over \( Z \) [17, Theorem 8.6.8]. Then

\[
\mathcal{X}'|_{\mathcal{N}} = \mathcal{N}_{\mathcal{O}_F}.
\]

(2.11)

Let \( \mathcal{E}' = \mathcal{E} \times_{\mathcal{X}} \mathcal{X}' \). As the non-smooth locus of \( \mathcal{E} \to \mathcal{X} \) is supported at the cusps, it is contained in \( \mathcal{E}|_{\mathcal{N}} \). So by (2.11), the non-smooth locus of \( \mathcal{E}' \to \mathcal{X}' \) is contained in \( (\mathcal{E}|_{\mathcal{N}})_{\mathcal{O}_F} \to \mathcal{N}_{\mathcal{O}_F} \). Then the explicit desingularization procedure in [10, 2.2] still applies to \( \mathcal{E}^{2k-2} \). The desingularization \( \mathcal{Y}' \)
satisfies $\mathcal{Y}_F' \simeq Y(N)_F$ and comes with a natural morphism $\mathcal{Y}' \to \mathcal{Y}(N)_{O_F}$. On the non-cuspidal locus $\mathcal{X}^{\circ} = \mathcal{X}'|_{\mathcal{X}^{\circ}}$, its preimage $\mathcal{Y}^{\circ}$ in $\mathcal{Y}$ satisfies

$$
(\mathcal{Y}^{\circ} \to \mathcal{X}') = (\mathcal{Y}^{\circ} \times_{\mathcal{X}^{\circ}} \mathcal{X}^{\circ} \to \mathcal{X}^{\circ}) = \left((\mathcal{E}|_{\mathcal{X}^{\circ}})^{2k-2} \times_{\mathcal{X}^{\circ}} \mathcal{X}^{\circ} \to \mathcal{X}^{\circ}\right).
$$

In particular, $\mathcal{Y}'|_{x'} = \mathcal{E}|_{x'}^{2k-2} \times_{x'} x'$ for $x' \in \mathcal{X}'$ over non cuspidal $x \in \mathcal{X}$.

In fact, we only care about $\mathcal{X}^{\circ}$ as our integral CM cycles below sit in this locus.

2.2.4. Height pairings and integral CM cycles. We define the Beilinson–Bloch height pairing $\mathbb{1}$ using arithmetic intersection pairing $\mathbb{7}$. We need to define integral CM cycles.

Consider the CM elliptic curve $E|_{P_g}$ corresponding to the CM point $P_g \in X(N)(\mathbb{Q})$ that is defined over $F$. Then by CM theory [30, Theorem 11.2] and reduction theory [30, Proposition 5.4, 5.5] of elliptic curves, $E|_{P_g}$ has reductions outside the cusps. Then the Zariski closure $\overline{P}_g$ of the CM point $P_g$ in $X_{O_F}$ is an $\mathcal{O}_F$-point away from the cusps, and defines a CM elliptic curve $A = \mathcal{E}|_{\overline{P}_g}$.

So we have the CM cycle $Z(A)$ on $\mathcal{Y}_{O_F}|_{\overline{P}_g} = A^{2k-2}$ (see §2.2.1). Let $Z_g$ be the pushforward of $Z(A)$ to $\mathcal{Y}_{O_F}$. Let $Z'_g$ be the flat pullback of $Z_g$ to $\mathcal{Y}'$ (defined in §2.2.3). As the desingularization procedure in §2.2.3 only happens at the cusps and $Z'_g$ is supported away from the cusps, $Z'_g$ is compatible if we vary $F$.

More precisely, let $F_1, F_2$ be two choices of $F$ and let $Z'_{g,1}, Z'_{g,2}$ be defined accordingly. If $F_2$ is an extension of $F_1$, then $Z'_{g,2}$ is the natural pullback of $Z'_{g,1}$.

Let $Z_{g_1}, Z_{g_2}$ be two CM cycles. By taking $F$ large enough, we have two CM cycles $Z'_{g_1}$ and $Z'_{g_2}$ both defined over $\mathcal{Y}'/\mathcal{O}_F$. Then the restrictions of $Z'_{g_1}$ and $Z'_{g_2}$ to the irreducible components of the special fibers of $\mathcal{Y}'$ are cohomologically trivial [30, 3.1].

So the Beilinson–Bloch height pairing $\mathbb{1}$ is unconditionally defined, and given by an arithmetic intersection pairing $\mathbb{7}$:

$$
\langle Z_{g_1}, Z_{g_2}\rangle_{\mathcal{Y}} = \frac{1}{[F : \mathbb{Q}]}(-1)^k(Z'_{g_1,} G_{g_1}) \cdot (Z'_{g_2}, G_{g_2}).
$$

Here $G_g$ is the “admissible” Green current of $Z_g$, in the sense of the first paragraph of [30, p 125].

Note that if we replace one of $Z'_{g_1}$ and $Z'_{g_2}$ by its horizontal part, we still get the same height pairing through the arithmetic intersection pairing. And $\langle Z_{g_1}, Z_{g_2}\rangle_{Y(N)}$ is independent of choice of $F$ by the projection formula and the compatibility discussed in the end of the last paragraph.

Moreover, by [1], the height pairing between CM cycles factors through their images in $\text{Ch}^k(Y(N))$.

2.2.5. Vary the level. Let us first discuss the general case, and then CM cycles. We want to define a height pairing

$$
\langle \cdot, \cdot \rangle : \varprojlim_N \text{Ch}^k(Y(N))_0 \times \varprojlim_N \text{Ch}^k(Y(N))_0 \to \mathbb{C}.
$$

Here $\text{Ch}^k(Y(N))_0$ denotes the subspace of $\text{Ch}^k(Y(N))$ of cohomologically trivial cycles. For $(x_N) \in \varprojlim_N \text{Ch}^k(Y(N))_0$ and $(y_N) \in \varprojlim_N \text{Ch}^k(Y(N))_0$, by the projection formula for height pairing [1], the height pairing $\langle x_N, y_N \rangle_{Y(N)}$, if well-defined (this is conjecturally the case) for $N$ large enough, does not depend on the choice of $N$. Let

$$
\langle (x_N), (y_N) \rangle = \langle x_N, y_N \rangle_{Y(N)}.
$$

This in particular gives the height pairing between $x_N, y_N$ for any $N$.

[3] This is a special case of a more general fact [22, Lemma 1.5].
Lemma 2.2.3. Let \( \eta \) be the Hecke character of \( \mathbb{A}^\times \) associated to \( K \) by the class field theory, and consider it as an automorphic function on \( GL_2(\mathbb{A}) \) by composition with \( \det \). Assume that \( \eta \) and \( f \) are right \( U(N) \)-invariant. Then \( H(f) = H(\eta f) \).

Proof. By definition, \( \langle Z_{g_1}, Z_{g_2} \rangle_{Y(N)} \neq 0 \) only if they are on the same geometrically connected components of \( Y(N) \), equivalently, \( \det(g_1) \in \det(g_2) F^\times \det(U(N)) \). Thus, in the definition (2.3) and (2.8) of \( Z_{\Omega,f} \), only \( g \in F^\times \det(U(N)) Nm(\mathbb{A}_E^\times) \) contributes to \( H(f) \). On these \( g \)'s, \( \eta(g) = 1 \) by definition. So \( H(f) = H(\eta f) \).

2.3. Modularity. We first formulate the modularity conjecture for CM cycles. Then we prove some cases of it.

2.3.1. The conjecture. Consider the following subspace of \( \lim_N Z^k(Y(N)) \):

\[
CM(\Omega) := \{ Z_{\Omega,f} : f \in \mathcal{S}(GL_2(\mathbb{A}^\infty)) \}.
\]

Let \( \overline{CM}(\Omega) \) be the quotient of \( CM(\Omega) \) by the kernel of the normalized height pairing (2.13) with \( CM(\Omega^{-1}) \). Define \( \overline{CM}(\Omega^{-1}) \) similarly. By (2.8), the left \( GL_2(\mathbb{A}^\infty) \)-action \( L^{coh} \) on \( \lim_N Ch^k(Y(N)) \) stabilizes \( \overline{CM}(\Omega) \) and \( \overline{CM}(\Omega^{-1}) \), and thus induces left \( GL_2(\mathbb{A}^\infty) \)-actions on them. From now on, we fix this left action. Under the left action, the central character of \( \overline{CM}(\Omega) \) is \( \Omega \).

Let \( \mathcal{A} \) be the set of cuspidal automorphic representations \( \pi \) of \( GL_2(\mathbb{Q}) \) such that

- \( \pi_\infty \) is the holomorphic discrete series of weight \( 2k \);
- \( \pi \) has central character \( \Omega^{-1}|_{\mathbb{A}^\times} \).

Under the second condition, \( \pi_K \otimes \Omega \) is self-dual. In particular, the \( L \)-function

\[
L(s, \pi_K \otimes \Omega) = L(s, \bar{\pi}_K \otimes \Omega^{-1})
\]

with root number

\[
\epsilon(1/2, \pi_K \otimes \Omega) = \epsilon(1/2, \bar{\pi}_K \otimes \Omega^{-1}) \in \{ \pm 1 \}.
\]

For \( \pi \in \mathcal{A} \), let \( \pi^\infty \) be the finite component. Let \( \bar{\pi}_\infty \) be the admissible dual of \( \pi_\infty \). Let

\[
\overline{CM}[\pi] = \text{Hom}_{GL_2(\mathbb{A}^\infty)}(\bar{\pi}_\infty, \overline{CM}(\Omega)),
\]

\[
\overline{CM}[\bar{\pi}] = \text{Hom}_{GL_2(\mathbb{A}^\infty)}(\pi_\infty, \overline{CM}(\Omega^{-1})).
\]
We have natural embeddings of $GL_2(\mathbb{A}^\infty)$-modules
\begin{align}
\bigoplus_{\pi \in \mathcal{A}} \pi^\infty \otimes \overline{CM}[\pi] & \hookrightarrow \overline{CM}(\Omega), \\
\bigoplus_{\pi \in \mathcal{A}} \pi^\infty \otimes \overline{CM}[\pi] & \hookrightarrow \overline{CM}(\Omega^{-1}).
\end{align}
(2.15)

We propose the following modularity conjecture, following S. Zhang [43].

**Conjecture 2.3.1.** The embeddings (2.15) are surjective.

**Remark 2.3.2.** The reader should think of $\overline{CM}$ as a subgroup of the “Mordell-Weil group” of the submotive of $\{Y(N)\}_N$ that is spanned by CM cycles. We remind the reader of the weight 2 case in \[37, p. 78\], there is a decomposition of abelian varieties up to isogeny similar to (2.15):
\[
\text{Jac}(X(N)) = \bigoplus \widetilde{\Pi}^U(N) \otimes_{\text{End}(\Pi)} A_{\Pi},
\]
where $\Pi$ runs over irreducible admissible representations of $GL_2(\mathbb{A}^\infty)$ over $\mathbb{Q}$ such that $\Pi_{\mathbb{C}}$ is a direct sum of the finite components of Galois Conjugate cuspidal automorphic representations of weight 2, $\widetilde{\Pi}^U(N)$ is the space of $U(N)$-invariants, and the $L$-function $L(s, A_{\Pi}) = L(s - 1/2, \Pi)$. Here we consider $\text{Jac}(X(N))(\mathbb{Q})$’s as Chow groups of $X(N)$’s so that it is equipped with a left Hecke action by the subalgebra of bi-$U(N)$-invariant functions in $S(\text{GL}_2(\mathbb{A}^S, \mathbb{A}^\infty))$ as above. This decomposition respects the Hecke-actions on both sides.

**Remark 2.3.3.** It is a conjecture (attributed to Beilinson and Bloch, see [14]) that for a smooth projective variety $Y$ over a number field $F$, the Abel–Jacobi map from the Chow group of cohomologically trivial codimension-$d$ cycles with $\mathbb{Q}_l$-coefficients
\[
\text{Ch}^d(Y)_{0, \mathbb{Q}_l} \to H^1 \left( \text{Gal}(\overline{F}/F), H^{2d-1}(Y_{\overline{F}}, \mathbb{Q}_l) \right)
\]
should be injective [1]. Conjecture 2.3.1 is implied by this conjecture as follows.

On $Y(N)$, there is the natural action of $\Delta_k := \{\pm 1\} \times S_{2k-2} = \{\pm 1\}^{2k-2} \times S_{2k-2}$ as in \[40, 2.3\] with the obvious compatibility as $N$ varies. Then by the functoriality and conjectural injectivity of the Abel–Jacobi map, we have a $\Delta_k \times GL_2(\mathbb{A}^\infty)$-equivariant embedding
\[
\varprojlim N \text{Ch}^k(Y(N))_{0, \mathbb{Q}_l} \to \varprojlim N H^1 \left( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), H^{2k-1}(Y(N)_{\overline{\mathbb{Q}}}, \mathbb{Q}_l) \right).
\]

Let $\varepsilon$ be the character of $\Delta_k$ that is the product map on $\{\pm 1\}^{2k-2}$, and is the sign character on $S_{2k-2}$. By \[40\] Lemma 2.4.3, the image of $CM(\Omega)$ in $\varprojlim_N \text{Ch}^k(Y(N))_{0, \mathbb{Q}_l}$ is contained in the $\varepsilon$-component. By \[40\] Theorem 2.3.1 (due to Scholl), the $\varepsilon$-component of
\[
\varprojlim N H^1 \left( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), H^{2k-1}(Y(N)_{\overline{\mathbb{Q}}}, \mathbb{Q}_l) \right) \otimes_{\mathbb{Q}_l} \mathbb{C}
\]
is a direct sum of elements in $\mathcal{A}$. Then so are $\text{Ch}^d(Y)_{0}$ (with $\mathbb{C}$-coefficients) and its subquotient $CM(\Omega)$. Then Conjecture 2.3.1 follows.
2.3.2. Unramified modularity. We establish a version of Conjecture 2.3.1 for modules of unramified Hecke algebras, which we now define.

For a finite set \( S \) of finite places of \( \mathbb{Q} \), let \( \mathcal{H}^S \subset S(\text{GL}_2(\mathbb{A}^S)) \) be the usual unramified Hecke algebra, i.e. the sub-algebra of bi-\( \text{GL}_2 \)-invariant functions. Let

\[
CM(\Omega, \mathcal{H}^S) = \{ Z_{\Omega,f} : f \in S(\text{GL}_2(\mathbb{A}_S)) \otimes \mathcal{H}^S \}.
\]

Let \( \overline{CM}(\Omega, \mathcal{H}^S) \) be the image of \( CM(\Omega, \mathcal{H}^S) \) in \( \overline{CM}(\Omega) \). Then \( \overline{CM}(\Omega, \mathcal{H}^S) \) is a \( \mathcal{H}^S \)-submodule of \( \overline{CM}(\Omega) \).

Let \( \mathcal{A}^{\text{GL}_2(\mathbb{A})} \subset \mathcal{A} \) be the subset of representations with nonzero \( \text{GL}_2(\mathbb{A}^S) \)-invariant vectors. For \( \pi \in \mathcal{A}^{\text{GL}_2(\mathbb{A})} \), let \( L_{\pi,S} \) (resp. \( L_{\pi,S}^\prime \)) be the character of \( \mathcal{H}^S \) on the 1-dimensional space of \( \text{GL}_2(\mathbb{A}^S) \)-invariant vectors in \( \pi_{S,\infty} \) (resp. \( \pi_{S,\infty}^\prime \)). Let

\[
\overline{CM}[\pi] = \text{Hom}_{\mathcal{H}^S}(L_{\pi,S}, \overline{CM}(\Omega, \mathcal{H}^S)),
\]

\[
\overline{CM}[\pi'] = \text{Hom}_{\mathcal{H}^S}(L_{\pi,S}', \overline{CM}(\Omega^{-1}, \mathcal{H}^S)).
\]

By the strong multiplicity one theorem, \( L_{\pi,S}' \)'s are pairwise non-isomorphic. So the following natural maps of \( \mathcal{H}^S \)-modules are embeddings

\[
\bigoplus_{\pi \in \mathcal{A}^{\text{GL}_2(\mathbb{A})}} L_{\pi,S} \otimes \overline{CM}[\pi] \hookrightarrow \overline{CM}(\Omega, \mathcal{H}^S),
\]

\[
\bigoplus_{\pi \in \mathcal{A}^{\text{GL}_2(\mathbb{A})}} L_{\pi,S} \otimes \overline{CM}[\pi'] \hookrightarrow \overline{CM}(\Omega^{-1}, \mathcal{H}^S).
\]

(2.16)

**Theorem 2.3.4.** Assume that \( S \) has cardinality at least three, and contains the prime 2 and all finite places of \( \mathbb{Q} \) ramified in \( K \). Then the embeddings (2.16) are surjective.

We will prove Theorem 2.3.4 in §4.5.

**Remark 2.3.5.** As one may expect, the assumption that \( S \) contains 2 comes from local theta lifting, see Lemma 1.1.2

2.3.3. Full modularity. Let us reformulate Conjecture 2.3.1 to make it easier to attack. We use Theorem 2.3.4 for the reformulation.

For \( S \subset S' \), we naturally have \( \overline{CM}(\Omega, \mathcal{H}^S) \subset \overline{CM}(\Omega, \mathcal{H}^{S'}) \). Choose any isomorphism \( L_{\pi,S} \simeq L_{\pi,S'} \) (of 1-dimensional spaces). Then \( f \in \overline{CM}[\pi] \), understood as a linear map from \( L_{\pi,S} \) to \( \overline{CM}(\Omega, \mathcal{H}^{S'}) \), is also \( \mathcal{H}^{S'} \)-equivariant. Thus we have an embedding \( \overline{CM}[\pi] \hookrightarrow \overline{CM}[\pi_{S'}] \) which gives an inclusion

(2.17)

\[
L_{\pi,S} \otimes \overline{CM}[\pi] \subset L_{\pi,S'} \otimes \overline{CM}[\pi_{S'}]
\]

in \( \overline{CM}(\Omega) \). Note that the inclusion is independent of the choice of the isomorphism \( L_{\pi,S} \simeq L_{\pi,S'} \). Let

\[
\overline{CM}(\pi) = \lim_{\rightarrow S} L_{\pi,S} \otimes \overline{CM}[\pi] \subset \overline{CM}(\Omega),
\]

which is stable by the \( \text{GL}_2(\mathbb{A}^\infty) \)-action. Define \( \overline{CM}(\pi) \subset \overline{CM}(\Omega^{-1}) \) similarly. As \( \overline{CM}(\Omega) = \lim_{\rightarrow S} \overline{CM}(\Omega, \mathcal{H}^S) \) by definition, we have

(2.18)

\[
\overline{CM}(\Omega) = \bigoplus_{\pi \in \mathcal{A}} \overline{CM}(\pi)
\]
by Theorem 2.3.4. So Conjecture 2.3.6 is implied by the truth of the following conjecture for all \( \pi \in \mathcal{A} \).

**Conjecture 2.3.6.** Under the embedding (2.15), we have
\[
\overline{CM}(\pi) = \overline{\pi}_\infty \otimes \overline{CM}[\pi]
\]
in \( \overline{CM}(\Omega) \). The corresponding equality for \( \overline{CM}(\bar{\pi}) \) also holds.

Moreover, if Conjecture 2.3.6 holds, by the strong multiplicity one theorem [26], \( L_{\mathbb{R}\mathbb{S}} \otimes \overline{CM}[\pi^\vee] \) sits in \( \overline{\pi}_\infty \otimes \overline{CM}[\pi] \). Thus Conjecture 2.3.6 is in fact equivalent to Conjecture 2.3.0 for all \( \pi \in \mathcal{A} \).

Our main result on Conjecture 2.3.6 is as follows.

**Theorem 2.3.7.** Assume that \( L(1/2, \pi_K \otimes \Omega) = 0 \). Then Conjecture 2.3.6 holds. In particular, if \( \epsilon(1/2, \pi_K \otimes \Omega) = -1 \), then Conjecture 2.3.0 holds.

The theorem is proved in §4.6.

2.3.4. Rank. To motivate the next result, recall that the theorem of Tunnell [31] and Saito [29] implies the following result.

**Proposition 2.3.8.** Assume Conjecture 2.3.6. Then \( \overline{CM}(\pi) = \{0\} \) if \( \epsilon(1/2, \pi_K \otimes \Omega_p)\Omega_p(-1) = -1 \) for some \( p < \infty \).

Without Conjecture 2.3.6 we prove the following unconditional result.

**Theorem 2.3.9.** Assume that \( \epsilon(1/2, \pi_K \otimes \Omega_p)\Omega_p(-1) = -1 \) for more than one \( p < \infty \). Then \( \overline{CM}(\pi) = \{0\} \) and \( \overline{CM}(\bar{\pi}) = \{0\} \). In particular, Conjecture 2.3.6 holds.

The theorem is proved in §4.6.

By Proposition 2.3.8 if \( \overline{CM}(\pi) \neq \{0\} \), we should expect \( \epsilon(1/2, \pi_K \otimes \Omega_p)\Omega_p(-1) = 1 \) for every \( p < \infty \). As \( \epsilon(1/2, \pi_K \otimes \Omega_{\infty})\Omega_{\infty}(-1) = -1 \), \( \epsilon(1/2, \pi_K \otimes \Omega) = -1 \) and thus \( L(1/2, \pi_K \otimes \Omega) = 0 \). Motivated by the analog in the weight 2 case (see Remark 2.3.2), we predict the following.

**Conjecture 2.3.10.** For \( \pi \in \mathcal{A} \), \( \dim \overline{CM} [\pi] \leq 1 \). The equality holds if and only if \( \epsilon(1/2, \pi_K \otimes \Omega_p)\Omega_p(-1) = 1 \) for every \( p < \infty \) and \( L'(1/2, \pi_K \otimes \Omega) \neq 0 \).

The “if” part is proved in Corollary 2.4.9. The “only if” part may be proved similarly assuming Conjecture 2.3.6 see Proposition 2.3.8. However, the claim \( \dim \overline{CM} [\pi] \leq 1 \) seems out of reach.

Assuming injectivity of the Abel–Jacobi map, the subspace of \( \dim \overline{CM} [\pi] \) coming from CM cycles on Kuga-Sato varieties over \( X_0(N) \)'s have dimension at most 1, by Nekovár’s theorem [25].

2.4. Height pairing and derivatives. Now we state the higher weight analog of the general Gross–Zagier formula of Yuan, S. Zhang, and W. Zhang. We need to fix some notations first.

2.4.1. Local periods. Recall that \( \eta \) is the Hecke character of \( \mathbb{Q}^\times \) associated to \( K \) by the class field theory.

Fix the following measures. For a finite place \( p \), endow \( \mathbb{Q}_p^\times \) with the Haar measure such that \( \text{Vol}(\mathbb{Z}_p^\times) = 1 \). This gives the measure on \( K_p^\times / \mathbb{Q}_p^\times \) for \( p \) split in \( K \). For \( p \) unramified in \( K \), endow \( K_p^\times / \mathbb{Q}_p^\times \) with the Haar measure such that \( \text{Vol}(K_p^\times / \mathbb{Q}_p^\times) = 1 \). For \( p \) ramified in \( K \), endow \( K_p^\times / \mathbb{Q}_p^\times \) with the Haar measure such that \( \text{Vol}(K_p^\times / \mathbb{Q}_p^\times) = 2|D_p|^{-1/2} \) where \( D_p \) is the local discriminant. See also [37] 1.6.2.
Let $\pi_p$ be an infinite-dimensional irreducible unitary admissible representation of $GL_2(\mathbb{Q}_p)$. For $\phi_p \in \pi_p$ and $\phi_{\tilde{p}} \in \pi_{\tilde{p}}$, let
\[
\alpha_{\pi_p}^z(\phi_p \otimes \phi_{\tilde{p}}) = \alpha_{\pi_p}^z(\phi_p, \phi_{\tilde{p}}) = \frac{L(1, \eta_p) \cdot L(1, \pi_p, \text{ad})}{\zeta_{\mathbb{Q}_p}(2) \cdot L(1/2, \pi_p, K_p \otimes \Omega_p)} \int_{K_p^\times / \mathbb{Q}_p^\times} \left( \pi_p(t) \phi_p, \phi_{\tilde{p}} \right) \Omega_p(t) dt.
\]
Then $\alpha_{\pi_p}^z(\phi_p \otimes \phi_{\tilde{p}}) = 1$ if $K_p, \Omega_p, \pi_p$ are unramified, $\phi_p, \phi_{\tilde{p}}$ are $GL_2(\mathbb{Z}_p)$-invariant, and $(\phi, \phi_{\tilde{p}}) = 1$. Let
\[
\alpha_{\pi_p}^z = \prod_{p < \infty} \alpha_{\pi_p}^z.
\]

2.4.2. **Yuan-Zhang-Zhang type formula I**. For $f \in S(GL_2(\mathbb{A}_\infty))$, let $z_{\Omega_f}$ be the image of $Z_{\Omega_f}$ in $CM(\Omega)$. For $\pi \in A$, let $z_{\Omega_f,\pi} \in CM_{\pi}(\Omega)$ be the $\pi$-component of $z_{\Omega_f}$, defined by applying (2.18).

We give an explicit description of $z_{\Omega_f,\pi}$ and a consequential lemma that will only be used in later sections. The reader may skip them and go to Lemma 2.4.2. Choose $S$ large enough such that the conditions in Theorem 2.3.4 hold and
\[
f = f_S \otimes f^S \in \mathcal{S}(GL_2(\mathbb{A}_S)) \otimes \mathcal{H}^S.
\]
Then $z_{\Omega_f,\pi}$ is the projection of $z_{\Omega_f}$ to $L_{\pi_S} \otimes \overline{CM}_{\pi^S}$ under Theorem 2.3.4. The projection is $\mathcal{H}^S$-equivariant. In particular, as $z_{\Omega_f} = L_{f_S}^{coh}(z_{\Omega_f})$ (see (2.10)), we have
\[
z_{\Omega_f,\pi} = L_{f_S}^{coh}(z_{\Omega_f,\pi}) = L_{\pi_S}^{coh}(f^S)z_{\Omega_f,\pi}.
\]

More explicitly, choose a finite subset $A_1$ of $A$ containing $\pi$ such that of $z_{\Omega_f}$ lies in the sum of the $L_{\pi_S}$-components of $CM_{\pi}(\Omega, \mathcal{H}^S)$ over all $\pi_1 \in A_1$. The existence of $A_1$ is assured by Theorem 2.3.4. Choose $f_1 \in \mathcal{H}^S$ which acts as identity on $\pi_1$ if $\pi_1 = \pi$, and as 0 otherwise. The existence of $f_1$ follows from the strong multiplicity one theorem and the density theorem of Jacobson and Chevalley for semisimple modules, see for example [37, p 73]. Then
\[
z_{\Omega_f,\pi} = L_{f_1}^{coh}(z_{\Omega_f}) = z_{\Omega_f} f_1^\vee.
\]
Here the second “=” is (2.10). A consequential lemma is as follows.

**Lemma 2.4.1.** If $\det(\text{supp} f) \subset \text{Nm}(\mathbb{A}_K^{\infty, x})$, then
\[
\langle z_{\Omega_f,\pi}, z_{\Omega^{-1}} \rangle = \langle z_{\Omega_f,\pi \otimes \eta}, z_{\Omega^{-1}} \rangle.
\]

**Proof.** Let $f_1$ be as above the lemma. Then $z_{\Omega_f,\pi \otimes \eta} = z_{\Omega_f} (f_1^\vee \eta)$. As $\det(\text{supp} f) \subset \text{Nm}(\mathbb{A}_K^{\infty, x})$, it is easy to see that $f \ast (f_1^\vee \eta) = (f \ast f_1^\vee) \eta$. Now the lemma follows from Lemma 2.2.3. \square

Note that (2.18) is defined independent of any conjecture. Now we impose Conjecture 2.3.6.

**Lemma 2.4.2.** Assume that Conjecture 2.3.6 holds for $\pi$. If the image of $f^\vee$ in $\pi^\infty \otimes \pi^\infty \subset \text{Hom}(\pi^\infty, \pi^\infty)$ via the usual (left) Hecke action is 0 (i.e., $f^\vee$ acts on $\pi^\infty$ as 0), then $z_{\Omega_f,\pi} = 0$.

**Proof.** Choose $f_1 \in \mathcal{S}(GL_2(\mathbb{A}_\infty))$ such that $f_1 \ast f = f$. By (2.10), $z_{\Omega_f} = L_{f_1}^{coh}(z_{\Omega_f})$. As each direct summand of (2.18) is $GL_2(\mathbb{A}_\infty)$-stable (and thus Hecke stable), $z_{\Omega_f,\pi} = L_{f_1}^{coh}(z_{\Omega_f,\pi})$. The right hand side is 0 by Conjecture 2.3.6 and the condition on $f^\vee$. \square
For \( \phi \in \pi^\infty \) and \( \tilde{\phi} \in \pihat^\infty \), let \( f \in \mathcal{S}(\text{GL}_2(\mathbb{A}^\infty)) \) such that the image of \( f^\vee \) in \( \pi^\infty \otimes \pihat^\infty \subset \text{End}(\pihat^\infty) \) via the usual (left) Hecke action is \( \phi \otimes \tilde{\phi} \). Define
\[
(z_{\phi\tilde{\phi}}, z_{\Omega^{-1}}) = \frac{L'(1/2, \pi_K \otimes \Omega)}{\text{Vol}(\text{GL}_2(\mathbb{Z}))} \frac{(2k - 2)!}{2L(1, \eta)L(1, \pi, \text{ad})(k - 1)! \cdot (k - 1)!} \alpha_{\pi^\infty}^{\times}(\phi \otimes \tilde{\phi}).
\]
which does not depend on the choice of \( f \) under Conjecture \( \text{2.3.6} \) by Lemma \( \text{2.4.2} \) (but depends on the measure on \( \text{GL}_2(\mathbb{A}^\infty)) \). Similarly, we define \( z_{\phi\hat{\phi}} \in \text{CM}(\pihat) \).

The following formula is the counterpart of the projector version of the general Gross–Zagier formula in weight 2 \([37, \text{Theorem 3.15}]\).

**Theorem 2.4.3.** Assume that Conjecture \( \text{2.3.6} \) holds for \( \pi \). For \( \phi \in \pi^\infty \) and \( \tilde{\phi} \in \pihat^\infty \), we have
\[
(2.22)
\]
\[
(z_{\phi\tilde{\phi}}, z_{\Omega^{-1}}) = \frac{L'(1/2, \pi_K \otimes \Omega)}{2L(1, \eta)^2L(1, \pi, \text{ad})(k - 1)! \cdot (k - 1)!} \alpha_{\pi^\infty}^{\times}(\phi \otimes \tilde{\phi}).
\]

All global L-functions (or zeta functions) in this paper are the complete ones containing the archimedean parts. The theorem is proved in \( \text{§3.5} \).

**Remark 2.4.4.** We will explain the following assertions, which one may expect, in \( \text{§3.5} \) before the proof of the theorem.

1. The number \( \frac{(2k - 2)!}{(k - 1)! (k - 1)!} \) is part of a local factor \( \text{3.32} \) at \( \infty \).
2. If we pretend \( k = 1 \), the formula should (formally) be the same with \( \text{37, Theorem 3.15} \) (up to the choices of measures), see Remark \( \text{3.5.2} \).

2.4.3. **Yuan-Zhang-Zhang type formula II.** We define a height pairing between \( \text{CM}[^\pi] \) and \( \text{CM}[^\pihat] \), following S. Zhang \([43]\). For \( l \in \text{CM}[\pi] \) and \( \tilde{l} \in \text{CM}[\pihat] \), \( \phi \in \pi^\infty \) and \( \tilde{\phi} \in \pihat^\infty \) with \((\phi, \tilde{\phi}) \neq 0\), let
\[
(2.23)
\]
\[
\langle l, \tilde{l} \rangle = \frac{(l(\phi), \tilde{l}(\phi))^2}{(\phi, \phi)}.
\]

By Schur’s lemma, the pairing \( (2.23) \) does not depend on the choices of \( \phi, \tilde{\phi} \). By definition, the pairing \( (2.23) \) is non-degenerate.

Assume that Conjecture \( \text{2.3.6} \) holds for \( \pi \). For \( \phi \in \pi \), let
\[
l_{\phi} \in \text{CM}[\pi] = \text{Hom}_{\text{GL}_2(\mathbb{A}^\infty)}(\pihat^\infty, \text{CM}(\Omega))
\]
be defined by \( \tilde{\phi} \mapsto z_{\phi\tilde{\phi}} \). Since \( z_{\phi\tilde{\phi}} \)'s span \( \text{CM}(\pi) \), every element in \( \text{CM}[\pi] \) equals some \( l_{\phi} \). Similarly, every element in \( \text{CM}[\pihat] \) equals \( l_{\hat{\phi}} : \phi \mapsto z_{\phi\hat{\phi}} \) for some \( \hat{\phi} \in \pihat \).

The following theorem is a consequence of Theorem \( \text{2.4.3} \). It is in the form of the general Gross–Zagier formula \([37, \text{Theorem 1.2}] \). This is suggested by S. Zhang.

**Theorem 2.4.5.** Assume that Conjecture \( \text{2.3.6} \) holds for \( \pi \). For \( \phi \in \pi^\infty \) and \( \tilde{\phi} \in \pihat^\infty \), we have
\[
(2.24)
\]
\[
\langle l_{\phi}, l_{\tilde{\phi}} \rangle = \frac{L'(1/2, \pi_K \otimes \Omega)}{L(1, \eta)^2L(1, \pi, \text{ad})(2k - 1)! \cdot (k - 1)!} \alpha_{\pi^\infty}^{\times}(\phi, \tilde{\phi}).
\]

**Remark 2.4.6.** (1) Remark \( \text{2.4.4} \) still applies, replacing \([37, \text{Theorem 3.15}] \) in loc. cit. by \([37, \text{Theorem 1.2, 3.13}] \).

(2) Theorem \( \text{2.4.5} \) can be viewed as an evidence toward the conjecture of Beilinson and Bloch \([11, \text{Conjecture 5.4}] \), which is a higher dimensional generalization of the Birch–Swinnerton-Dyer conjecture. Compare with Remark \( \text{2.3.2} \).
Let us draw some unconditional corollaries from Theorem 2.4.5 by using Theorem 2.3.7 to remove the dependence on Conjecture 2.3.6 in Theorem 2.4.5.

**Corollary 2.4.7.** If \( L(1/2, \pi_K \otimes \Omega) = 0 \), then (2.24) holds.

**Corollary 2.4.8.** If \( L(1/2, \pi_K \otimes \Omega) = 0 \) and \( L'(1/2, \pi_K \otimes \Omega) = 0 \), then \( \overline{CM}(\pi) = \{0\} \).

Concerning Conjecture 2.3.10, we have the following.

**Corollary 2.4.9.** Assume that \( L'(1/2, \pi_K \otimes \Omega) \neq 0 \). If \( \epsilon(1/2, \pi_K \otimes \Omega) = -1 \) for all \( p < \infty \), then \( \dim \overline{CM}[\pi] = 1 \). Otherwise, \( \overline{CM}[\pi] = \{0\} \).

**Proof.** As \( L'(1/2, \pi_K \otimes \Omega) \neq 0 \), \( \epsilon(1/2, \pi_K \otimes \Omega) = -1 \). Thus \( L(1/2, \pi_K \otimes \Omega) = 0 \). By Theorem 2.3.7 Conjecture 2.3.6 holds for \( \pi \). Thus the “otherwise” part follows from Proposition 2.3.8. Now assume that \( \epsilon(1/2, \pi_K \otimes \Omega_p) \Omega_p(-1) = 1 \) for all \( p < \infty \). By the theorem of Tunnell [31] and Saito [29], \( \dim \text{Hom}_{A_{\infty} \times (\pi_{\infty} \otimes \Omega_{\infty}) \otimes \mathbb{C}} = 1 \). Let \( l \neq 0 \) be in this Hom space. Let \( \phi_0 \in \pi_{\infty} \) such that \( l(\phi_0) \neq 0 \). Then by Theorem 2.4.5 and the discussion above it, \( l(\phi) = l(\phi_0)l(\phi_0) \).

3. Relative trace formulas and height pairing

We use the arithmetic relative trace formula approach to prove Theorem 2.4.3. There will be a relative trace formula for each one of two (types of) groups. One group is \( \text{GL}_{2,K} \), the other is a quaternionic group, i.e., the unit group of a quaternion algebra over \( \mathbb{Q} \) containing \( K \). We will further specify these data later when necessary. Each relative trace formula is an identity between the orbital integral side and the automorphic distribution side of this group. The two relative trace formulas are the two vertical equations in the diagram (3.1) below. They are related by comparing their orbital integrals.

\[
\begin{array}{c}
\text{orbital integrals for a quaternionic group} & \text{comparison} & \text{orbital integrals for } \text{GL}_2 \\
\text{after summation} & & \text{after summation}
\end{array}
\]

Call the resulted bottom identity a relative trace identity.

The structure of this section is as follows. We introduce the trace formulas, including orbital integrals and automorphic distributions (plus local distributions) in the first three subsections. We do not establish the two vertical equations in the diagram (3.1), but will refer to literature when we need them. The arithmetic variants and the comparison are then done in §3.4. The conclusion of the comparison is an arithmetic relative trace identity (3.30). Finally in §3.5 we use the comparison to prove the higher weight general Gross–Zagier formula.

3.1. Orbital integrals. We discuss orbits and (derivatives of) orbital integrals on two groups separately, and then compare them. The main result is Proposition 3.1.8.

We start with a more general setup. Let \( F \) be a field, \( E \) a separable quadratic field extension of \( F \), \( z \mapsto \bar{z} \) the Galois conjugation, and \( \text{Nm} : E^\times \to F^\times \) the norm map. (We omit the case \( E = F \oplus F \), and refer to [27, 4.3].)
3.1.1. *Orbits I.* We first deal with the orbits for $GL_2$.

Let $G = GL_2(E)$. Let $\mathcal{V} \subset G$ be the subset of invertible Hermitian matrices over $F$ with respect to $E$. Let $E^\times \times F^\times$ act on $\mathcal{V}$ via

$$(a, z) \cdot s = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} s \begin{bmatrix} \bar{a} & 0 \\ 0 & 1 \end{bmatrix} z.$$ 

Define

$$\text{Inv} : \mathcal{V} \to \mathbb{P}^1(F) - \{1\} = (F^\times - \{1\}) \cup \{0, \infty\},$$

which maps $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ to

$$\frac{ad}{bd} = \frac{\text{det}(A)}{bd} + 1.$$ 

Call $\gamma \in \mathcal{V}$ regular for this action if $\text{Inv}(\gamma) \in F^\times - \{1\}$. Let $\mathcal{V}_{\text{reg}} \subset \mathcal{V}$ be the regular locus. The restriction of $\text{Inv}$ to the regular orbits $E^\times \\mathcal{V}_{\text{reg}} / F^\times$ are bijective to $F^\times - \{1\}$.

To relate functions on $G$ and $\mathcal{V}$, consider $g \in G$ acting on $\mathcal{V}$ by $g \cdot s := gs\bar{g}^t$, where $\bar{g}^t$ is the Galois conjugate of the transpose of $g$. Let $H_0 \subset G$ be the unitary group associated to $w = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathcal{V}$, i.e., $H_0$ is the stabilizer of $w$ for the above action. If $F$ is a local field (archimedean or non-archimedean) so that $\mathcal{V}$ is equipped with the corresponding topology, then $G \cdot w \subset \mathcal{V}$ is an open subspace. If $H_0$ is given a Haar measure, for $f \in S_c(G)$, let $\Phi_f \in S_c(G \cdot w)$, such that

$$(3.2) \quad \Phi_f(g \cdot w) = \int_{H_0} f(gh)dh.$$ 

The map $f \mapsto \Phi_f$ is surjective to $S_c(G \cdot w)$. If $F$ is a global field, the same definition applies to $f \in S_c(G(A_F))$.

Let $(1 - \text{Nm}(E^\times)) = \{1 - \text{Nm}(a) : a \in E^\times\}$. Then

$$\text{Inv}(G \cdot w) = (1 - \text{Nm}(E^\times)) \cup \{\infty\}.$$ 

We have the following easy lemma.

**Lemma 3.1.1.** (1) The map $\text{Inv}$ realizes $(G \cdot w) \cap \mathcal{V}_{\text{reg}}$ as a trivial (topological) $E^\times \times F^\times$-bundle over $(1 - \text{Nm}(E^\times)) - \{0\}$.

3.1.2. *Orbits II.* Then we discuss the orbits on a quaternionic group.

Let $B$ be a quaternion algebra over $F$ containing $E$. There exists $j \in B$ such that $B = E \oplus Ej$, $j^2 = \epsilon \in F^\times$ and $jz = \bar{z}j$ for $z \in E$. Then the reduced norm on $B$ can be expressed by $q(a + bj) = \text{Nm}(a) - \text{Nm}(b) \cdot \epsilon$. Note that different choices of $j$ give the same $\epsilon$ in $F^\times / \text{Nm}(E)^\times$. In fact, $F^\times / \text{Nm}(E)^\times$ classifies quaternion algebras over $F$ containing $E$ (see [27, 4.1]). And $\epsilon = 1$ if and only if $B = M_{2,F}$.

Let $E^\times \times E^\times$ act on $B^\times$ by

$$(h_1, h_2) \cdot \gamma = h_1^{-1}\gamma h_2.$$ 

[4][Here, for simplicity, we abuse the notion of “regular” for the more standard notion of “regular semi-simple” for reductive group actions.]
Define an invariant for this action: $\text{inv}(a + bj) = c\text{Nm}(b)/\text{Nm}(a)$. This induces a bijection

$$\text{inv} : E^x \backslash B^x / E^x \simeq (c\text{Nm}(E^x) - \{1\}) \cup \{0, \infty\}.$$ 

Call $\delta = a + bj$ regular for this action if $\text{inv} \neq 0, \infty$, equivalently $ab \neq 0$. Let $B^x_{\text{reg}} \subset B^x$ be the regular locus. We have the following easy lemma.

**Lemma 3.1.2.** For $g \in B^x_{\text{reg}}$, $\text{inv}(g) \in (1 - \text{Nm}(E^x))$ if and only if $q(g) \in \text{Nm}(E^x)$.

3.1.3. *Orbital integrals I.* Now we deal with the orbital integrals for $\text{GL}_2$.

Let $F$ be a local field (archimedean or non-archimedean). Let $\eta$ be the quadratic character of $F^\times$ associated to $E$ by the class field theory. Let $\Omega$ be a continuous character of $E^x$ and $\omega = \Omega|_{F^x}$.

For $\Phi \in \mathcal{S}(V)$, $\gamma \in \mathcal{V}_{\text{reg}}$, and $s \in \mathbb{C}$, define the orbital integral

$$O(s, \gamma, \Phi) = \int_{E^x} \int_{F^x} \Phi \left( z \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \gamma \begin{bmatrix} \bar{a} & 0 \\ 0 & 1 \end{bmatrix} \right) \eta \omega^{-1}(z) \Omega^{-1}(a)|a|_E^s dzda. \tag{3.3}$$

The integral (3.3) converges absolutely and defines a holomorphic function of $s$. For $x \in F^x - \{1\}$, let

$$\gamma(x) = \begin{bmatrix} x & 1 \\ 1 & 1 \end{bmatrix} \in \mathcal{V}_{\text{reg}}.$$ 

Then $\text{Inv}(\gamma(x)) = x$. Define

$$O(s, x, \Phi) = O(s, \gamma(x), \Phi), \quad x \in F^x - \{1\}.$$ 

Let $O'(0, x, \Phi)$ be the $s$-derivative at $s = 0$ (for the legitimacy of taking $s$-derivative, see [27, Lemma 4.2.2]). They are all smooth functions on $x$.

**Lemma 3.1.3.** (1) There is a neighborhood of $1$ in $F$ on which $O(s, x, \Phi)$ (thus $O(0, x, \Phi)$ and $O'(0, x, \Phi)$) vanishes for all $s$ simultaneously.

(2) If $\Phi$ is further compactly supported on $\mathcal{V}_{\text{reg}}$, there is a compact subset of $F^x - \{1\}$ on which $O(s, x, \Phi)$ (thus $O(0, x, \Phi)$ and $O'(0, x, \Phi)$) is supported for all $s$ simultaneously. In particular, $O(0, x, \Phi)$ and $O'(0, x, \Phi)$ extends to a Schwartz function on $F$ by extension by $0$.

(3) For $\phi \in \mathcal{S}_c((1 - \text{Nm}(E^x)))$ compactly supported, there is $\Phi \in \mathcal{S}_c((G \cdot w) \cap \mathcal{V}_{\text{reg}})$ such that $O(0, x, \Phi) = \phi(x)$, both understood as Schwartz functions on $F$ by extension by $0$.

*Proof.* As $\Phi$ is compactly supported on $V$ and $\text{Inv} : V \to \mathbb{P}^1 - \{1\}$ is continuous, $\text{Inv}$ maps the support of $\Phi$ to a compact set. (1) follows. (2) is similar. (3) is an immediate corollary of Lemma 3.1.1. \hfill \square

**Lemma 3.1.4.** (1) Let $A(x)$ be a smooth function defined on a neighborhood of $0 \in F$. There is $\Phi \in \mathcal{S}_c(G \cdot w)$ such that

- both $O(0, x, \Phi), O'(0, x, \Phi)$ extend to smooth functions on $F$ with compact supports, and
- $O(0, x, \Phi) = A(x)$ for $x \neq 0$ in a (smaller) neighborhood of $0$ in $(1 - \text{Nm}(E^x))$.

(1') Let $A(x)$ be a smooth function defined on a neighborhood of $0 \in F$. There is $\Phi \in \mathcal{S}_c(G \cdot w)$ such that

- both $O(0, x, \Phi), O'(0, x, \Phi)$ have compact supports on $F$ and are smooth outside $0$, and
- for $x \neq 0$ in a neighborhood of $0$ in $(1 - \text{Nm}(E^x))$, we have $O(0, x, \Phi) = \eta(x)A(x)$ and

$$O'(0, x, \Phi) = \eta(x) \left( -\frac{1}{2} A(x) \log |x|_E + C(x) \right).$$
where $C(x)$ is a smooth function.

(2) Let $A(x)$ be a smooth function defined on a neighborhood of $0$ in $E$. There is $\Phi \in \mathcal{S}_c(G \cdot w)$ such that

- both $O(0, x, \Phi), O'(0, x, \Phi)$ vanish in a neighborhood of $0$ and are smooth outside $0$, and
- for $x \in (1 - \text{Nm}(E^x))$ with large enough absolute value so that we can write $x$ as $-\text{Nm}(b)$ with $b^{-1}$ in a small enough neighborhood of $0$ in $E$, we have $O(0, x, \Phi) = \Omega(b)A(b^{-1})$.

**Proof.**

(1) Let $\phi \in \mathcal{S}_c(G \cdot w)$ be supported in a small enough neighborhood of $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and nonzero at $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. A direct computation shows that $O(0, x, \phi)$ extends to a smooth nonvanishing function on a neighborhood of $0 \in F$. Then the pullback of $A(x)/O(0, x, \phi)$, by the map Inv, to a neighborhood of $\text{Inv}^{-1}(\{0\})$ in $G \cdot w$ extends to a smooth function on $G \cdot w$. Let $\Phi$ be the product of $\phi$ with this smooth extension, and we proved the first part of the lemma. A direct computation shows that $O'(0, x, \phi)$ extends to a smooth function on a neighborhood of $0 \in F$. So does $O'(0, x, \Phi)$.

(1') Consider $\phi \in \mathcal{S}_c(G \cdot w)$ supported in a small enough neighborhood of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. The argument is similar to the one in the proof of (1).

(2) Consider $\phi \in \mathcal{S}_c(G \cdot w)$ supported in a small enough neighborhood of $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. The argument is similar to the one in the proof of (1). \qed

3.1.4. **Orbital integrals II.** Then we deal with the orbital integrals on the quaternionic group.

For $f \in \mathcal{S}_c(B^x)$ and $\delta \in B^x_{\text{reg}}$, define the orbital integral

$$O(\delta, f) := \int_{E^x/F^x} \int_{E^x} f(h_1^{-1}\delta h_2)\Omega(h_1)\Omega^{-1}(h_2)dh_2dh_1.$$ 

For $x \in \epsilon\text{Nm}(E^x) - \{1\}$ (where $\epsilon$ is as in \[3.1.2\]), choose $b \in E^x$ such that $x = \epsilon\text{Nm}(b)$. Let $\delta(x) = 1 + bj \in B^x \in B^x_{\text{reg}}$.

Then $\text{inv}(\delta(x)) = x$. Define

$$O(x, f) = O(\delta(x), f), \ x \in \epsilon\text{Nm}(E^x) - \{1\}.$$ 

Easy to check that $O(x, f)$ does not depend the choice of $b$.

We have the following characterization of $O(x, f)$ by [12, p. 322, Proposition] (which contains an error: it swapped the behaviors of $O(x, f)$ for $x$ near $0$ and near $\infty$).

**Proposition 3.1.5.** Let $\phi$ be a function on $\epsilon\text{Nm}(E^x) - \{1\}$. Then $\phi(x) = O(x, f)$ for some $f \in \mathcal{S}_c(B^x)$ if and only if the following conditions hold:

1. the function $\phi$ is smooth on $\epsilon\text{Nm}(E^x) - \{1\}$;
2. the function $\phi$ vanishes in a neighborhood of $1$;
3. there exists a smooth function $A_1$ defined on a neighborhood $U_1$ of $0$ in $F$, such that $\phi(x) = A_1(x)$ for $x \in U \cap (\epsilon\text{Nm}(E^x) - \{1\})$.
4. there exists a smooth function $A_2$ defined on a neighborhood $U_2$ of $0$ in $E$, such that $\phi(x) = \Omega(b)A_2(b^{-1})$ for $x = \epsilon\text{Nm}(b)$ with $b^{-1} \in U_2$.


3.1.5. **Transfer.** Now we compare the orbital integrals on the two groups.

**Definition 3.1.6.** For \( f \in \mathcal{S}(B^\times) \), \( f' \in \mathcal{S}(G) \) is a transfer of \( f \) if \( O(0, x, \Phi_{f'}) = O(x, f) \) for \( x \in \epsilon \text{Nm}(E^\times) - \{1\} \), and \( O(0, x, \Phi) = 0 \) for \( x \in F^\times - \{1\} - \epsilon \text{Nm}(E^\times) \).

We also have the following sufficient condition for a transfer to exist.

**Lemma 3.1.7.** (1) Let \( f \in \mathcal{S}(B^\times_{\text{reg}}) \) such that \( O(x, f) \) vanishes outside \( (1 - \text{Nm}(E^\times)) \), there is a transfer \( \Phi \in \mathcal{S}((G \cdot w) \cap V_{\text{reg}}) \).

(2) For \( \Phi \in \mathcal{S}((G \cdot w) \cap V_{\text{reg}}) \), there is \( e \in \mathcal{S}(B^\times_{\text{reg}}) \) such that \( O(x, e) = O'(0, x, \Phi) \).

**Proof.** (1) Similar to Lemma 3.1.3 (1) (2), \( O(x, f) \) has compact support contained in \( (1 - \text{Nm}(E^\times)) \) (see also Proposition 3.1.5 (2)). The lemma follows from Lemma 3.1.3 (3) and Proposition 3.1.5 (1).

(2) By Lemma 3.1.3 (2), \( O'(0, x, \Phi) \) has compact support contained in \( (1 - \text{Nm}(E^\times)) \). The rest of the proof is similar to (1) and omitted.

As \( |F^\times/\text{Nm}(E^\times)| = 2 \), by our discussion in the beginning of 3.1.2, there is a unique quaternion algebra \( B^\dagger \) over \( F \) and containing \( E \) that is non-isomorphic to \( B \). Let \( e^\dagger = e \in F^\times/\text{Nm}(E^\times) \) be associated to \( B^\dagger \) as in 3.1.2.

**Proposition 3.1.8.** Let \( f \in \mathcal{S}(B^\times) \) such that \( O(x, f) \) vanishes outside \( (1 - \text{Nm}(E^\times)) \). Let \( A_1(x) \) be as in Proposition 3.1.5 (3).

(1) There is a transfer \( f' \in \mathcal{S}(G) \) of \( f \).

(2) We can choose a transfer \( f' \) as in (1), \( f' \in \mathcal{S}(B^{1\times}) \) and a neighborhood \( U \) of \( 0 \) in \( F \) on which \( A_1 \) is defined, such that for \( x \in U \cap e^{1\dagger}\text{Nm}(E^\times) \), we have

\[
\frac{1}{2} A_1(x) \log |x|_E = -O'(0, x, \Phi_{f'}) + O(x, f') \tag{3.4}
\]

(3) Assume that \( A_1 \) extends to a smooth function on \( F \) such that

(a) \( \supp(A_1) \cap e^{1\dagger}\text{Nm}(E^\times) \) is bounded from above;

(b) \( 1 \notin \supp(A_1) \cap e^{1\dagger}\text{Nm}(E^\times) \).

If \( -1 \notin e^{1\dagger}\text{Nm}(E^\times) \), then (3.3) holds for all \( x \in e^{1\dagger}\text{Nm}(E^\times) \), i.e., we can remove \( U \).

**Proof.** (1) Apply Lemma 3.1.4 (1) resp. (1’) to \( A(x) := A_1(x) \) and get \( \Phi_1 \) resp. \( \Phi'_1 \). Let \( A_2(x) \) be as in Proposition 3.1.5 (4). Apply Lemma 3.1.3 (2) to \( A(x) := A_2(x) \) and get \( \Phi_2 \). Let \( \Phi = \Phi_1 - \eta(e^{1\dagger})\Phi'_1 + \Phi_2 \). By Lemma 3.1.4 (1) (2) and Proposition 3.1.5 we have

\[
\phi := O(x, f) - O(0, x, \Phi) \in \mathcal{S}_c((1 - \text{Nm}(E^\times)))
\]

Apply Lemma 3.1.3 (3) to \( \phi \) to get \( \Phi_3 \). Let \( f' \in \mathcal{S}(G) \) such that \( \Phi_{f'} = \Phi + \Phi_3 \) (recall the map \( f \mapsto \Phi_f \) is surjective to \( \mathcal{S}_c(G \cdot w) \)). Then \( f' \) is a transfer of \( f \).

(2) By Lemma 3.1.4 (1)(2), \( O'(0, x, \Phi_{f'}) + 2A_1(x) \log |x| \) is a smooth function for \( x \) in a neighborhood \( U \) of \( 0 \) in \( (1 - \text{Nm}(E^\times)) \). Choosing \( U \) small enough, by Proposition 3.1.5 for \( x \in U \cap e^{1\dagger}\text{Nm}(E^\times) \), we have \( O'(0, x, \Phi_{f'}) + 2A_1(x) \log |x| \) equals \( O(x, f') \) for some \( f' \in \mathcal{S}_c(B^{1\times}) \).

(3) Let us prove (3) by modifying \( f' \) in (2). In (2), we may assume that the support of \( O(x, f') \) is bounded from above. Moreover, it does not contain 1 by Proposition 3.1.5. If \( -1 \notin e^{1\dagger}\text{Nm}(E^\times) \), then \( (1 - \text{Nm}(E^\times)) \cap e^{1\dagger}\text{Nm}(E^\times) \) is bounded. Then by Lemma 3.1.3 (1) and conditions (a) and (b),
the support of $O'(0, x, \Phi_F) + 2A_1(x) \log |x| - O(x, f^\dagger)$, as a function on $\epsilon^\dagger \text{Nm}(E^\times)$ is a compact subset of $\epsilon^\dagger \text{Nm}(E^\times)$ not containing 1. By Proposition 3.4.1, there is $f^\dagger \in S_c(B_t, E^\times)$ such that

$$O'(0, x, \Phi_F) + 2A_1(x) \log |x| - O(x, f^\dagger) = O(x, f^\dagger)$$

on $\epsilon^\dagger \text{Nm}(E^\times)$. Replace $f^\dagger$ by $f^\dagger + f^\dagger$.

\[ \square \]

3.2. Automorphic distributions. Let $F$ be a global field, and let $E$ be a separable quadratic field extension. Let $\Omega$ be a character $E^\times \backslash \mathbb{A}_E^\times$, $\omega = \Omega|_{\mathbb{A}_E^\times}$, and $\omega_E = \omega \circ \text{Nm}$.

3.2.1. General linear side. Let $G = \text{GL}_2, E$. Let $A$ be the diagonal torus of $G$ and let $Z$ be the center. Let $H \subset G$ be the similitude unitary group associated to $w$, and let $\kappa$ be the similitude character.

For $f' \in S_c(G(\mathbb{A}_E))$, define a kernel function on $G(\mathbb{A}_E) \times G(\mathbb{A}_E)$:

$$K(x, y) = \int_{Z(E) \backslash Z(\mathbb{A}_E)} \left( \sum_{g \in G(E)} f'(x^{-1} g z) \right) \omega_E^{-1}(z) dz.$$  

For $a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \in A(\mathbb{A}_E)$, let $\Omega(a) = \Omega(a_1 \overline{a_2})$ and let $|a_E| = |a_1/a_2|_E$. (Compare with [27, 4.4.1], where there is a typo.) For $s \in \mathbb{C}$, formally define the distribution $O(s, \cdot)$ on $G(\mathbb{A}_E)$ by assigning to $f' \in S_c(G(\mathbb{A}_E))$ the integral

$$O(s, f') = \int_{Z(\mathbb{A}_E)A(E) \backslash A(\mathbb{A}_E)} \int_{Z(\mathbb{A}_E)H(F) \backslash H(\mathbb{A}_F)} K(a, h) \Omega(a) \eta \omega^{-1}(\kappa(h)) |a|^s dh \eta \omega^{-1}(\kappa(h)) |a|^s dh.$$

The invariance of the integrand under the inner (resp. outer) $Z(\mathbb{A}_E)$ follows from the definition of $K(x, y)$ and that the restriction of $\omega \circ \kappa$ (resp. $\Omega$) to $Z(\mathbb{A}_E) \simeq \mathbb{A}_E^\times$ is $\omega_E$.

We always assume the following assumption.

Assumption 3.2.1. Assume that $\Phi_F(g) = 0$ for $g \in \mathbb{A}_E^\times (\mathcal{V} - \mathcal{V}_{\text{reg}})\mathbb{A}_E^\times$.

Lemma 3.2.2. The integral $O(s, f')$ in (3.5) converges absolutely under Assumption 3.2.1. And we have a decomposition

$$O'(0, f') = \sum_{x \in F^\times - \{1\}} O'(0, x, \Phi_F).$$

Proof. Let us establish the equation in the lemma formally, and refer the convergence to [27, Lemma 4.4.2]. Unfolding $K(a, h)$, applying the definition of $\Phi_F'$ and then using $\kappa : H(\mathbb{A}_F)/H_0(\mathbb{A}_F) \simeq \mathbb{A}_F^\times$, we have

$$O(s, f') = \int_{Z(\mathbb{A}_E)A(E) \backslash A(\mathbb{A}_E)} \int_{H(F) \backslash H(\mathbb{A}_F)} \left( \sum_{g \in G(E)} f'(a^{-1} gh) \right) \Omega(a) \eta \omega^{-1}(\kappa(h)) |a|^s dh \eta \omega^{-1}(\kappa(h)) |a|^s dhda.$$

= \int_{Z(\mathbb{A}_E)A(E) \backslash A(\mathbb{A}_E)} \int_{H(F) \backslash H(\mathbb{A}_F)} \sum_{\gamma \in \mathcal{V}} \Phi_F'(a^{-1} z \cdot \gamma) \Omega(a) \eta \omega^{-1}(z) |a|^s dz da.$$

By Assumption 3.2.1, only regular $\gamma$ contributes to the inner sum. The inner sum is now

$$\sum_{\gamma \in \mathcal{E}^\times \mathcal{V}_{\text{reg}} / F^\times} \sum_{(a', z') \in \mathcal{E}^\times \times F^\times} \Phi_F'(a^{-1} z \cdot \gamma)$$
Identifying $Z(\mathbb{A}_E)A(E)\backslash A(\mathbb{A}_E) \simeq E^\times \backslash \mathbb{A}_E^\times$ via $\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \mapsto a_1/a_2$, and change the order of the summation, the equation in the lemma follows. \qed

Further assume that $f'$ is a pure tensor (so is $\Phi_{f'}$). Then we have a decomposition

\begin{equation}
O'(0, f') = \sum_{x \in F^\times} \sum_{v} O'(0, x, \Phi_{f',x})O(x, \Phi_{f'}),
\end{equation}

where the sum is over the set of places of $F$. The sum (3.6) converges absolutely.

3.2.2. Quaternion side. Let $B$ be a quaternion algebra over $F$. For $f \in S_c(B^\times(\mathbb{A}_F))$, define a kernel function on $B^\times(\mathbb{A}_F) \times B^\times(\mathbb{A}_F)$:

\begin{equation}
k(x, y) = \sum_{g \in B^\times} f(x^{-1}gy).
\end{equation}

Define a distribution $O(\cdot)$ on $B^\times(\mathbb{A}_F)$ by assigning to $f \in S_c(B^\times(\mathbb{A}_F))$ the integral

\[
O(f) = \int_{E^\times \backslash \mathbb{A}_E^\times} \int_{E^\times \backslash \mathbb{A}_E^\times} k(h_1, h_2)\Omega(h_1)\Omega^{-1}(h_2)dh_2dh_1.
\]

This integral converges absolutely under the following assumption.

**Assumption 3.2.3.** Assume that $f$ vanishes on $\mathbb{A}_E^\times(B^\times - B_{\text{reg}}^\times)\mathbb{A}_E^\times$.

We always assume Assumption 3.2.3. Then we have a decomposition

\[
O(f) = \sum_{x \in \text{Nm}(E^\times) - \{1\}} O(x, f),
\]

where $\epsilon$ is as in 3.1.2 and

\[
O(x, f) = \int_{\mathbb{A}_E^\times/\mathbb{A}_E^\times} \int_{\mathbb{A}_E^\times} f(h_1^{-1}\delta(x)h_2)\Omega(h_1)\Omega^{-1}(h_2)dh_2dh_1.
\]

3.2.3. The case of an incoherent transfer. Let $\mathbb{B}$ be a quaternion algebra over $\mathbb{A}_F$ such that for

an odd number of places $v$ of $F$, $\mathbb{B}_v$ is the division quaternion algebra over $F_v$ (such $\mathbb{B}$ is called

incoherent). Assume that $f \in S_c(\mathbb{B}^\times)$ is a pure tensor. Assume that $f' \in S_c(\mathbb{G}(\mathbb{A}_E))$ is a pure

tensor and a transfer of $f$ (at every place), and satisfies Assumption 3.2.1. Let $\Phi = \Phi_{f'}$.

We rearrange the decomposition of $O'(0, f')$ in (3.6) according to the decomposition

\[
F^\times - \{1\} = \prod \text{inv}(B_{\text{reg}}^\times),
\]

where the union is over all quaternion algebras over $F$ containing $E$ as an $F$-subalgebra. Let $B$ be such a quaternion algebra, and $x \in \text{inv}(B_{\text{reg}}^\times)$. If $B$ and $\mathbb{B}$ are not isomorphic at more than one
place, then by the transfer condition, for every place $u$, the product

\[
O(0, x, \Phi^u) := \prod_{v \neq u} O(0, x, \Phi_v)
\]

contains at least one local component with value 0 so that $O(0, x, \Phi^u) = 0$. Otherwise, $B$ is as follows. For a place $v$ of $F$, let $B(v)$ be the unique quaternion algebra over $F$ such that $\mathbb{B}^v \simeq B(v)(\mathbb{A}_F^v)$ (so that $\mathbb{B}_v \neq B(v)_v$), whose existence is assured by the Hasse principle. Moreover, if $B(v)$ contains $E$ as an $F$-subalgebra, then $v$ is nonsplit in $E$. (Indeed, otherwise, $E_v = F_v \oplus F_v$
and can not be contained in both \( \mathbb{B}_v \) and \( B(v) \). Let \( O(x, f^v) \) be the orbital integral defined by regarding \( f^v \) as a function on \( B^\times(\mathbb{A}_E^\times) \). Then by the transfer condition, for every place \( u \) of \( F \), \( O(0, x, \Phi^u) \neq 0 \) only if \( u = v \). In this case, \( O(0, x, \Phi^v) = O(x, f^v) \).

To sum up, we have the following lemma.

**Lemma 3.2.4.** Let \( \Xi_{\text{nsp}l} \) be the set of places of \( F \) nonsplit in \( E \). There is a decomposition

\[
O'(0, f') = \sum_{v \in \Xi_{\text{nsp}l}} \sum_{x \in \text{inv}(B(v)\mathbb{A}_E)} O'(0, x, \Phi^v) O(x, f^v).
\]

**Definition 3.2.5.** Let \( O'(0, f')_v \) be the summand corresponding to \( v \) in the equation in Lemma 3.2.4.

3.2.4. **Globalization.** Let \( B \) be a quaternion algebra over \( F \).

**Lemma 3.2.6.** Let \( v_1, \ldots, v_n \) be distinct places of \( F \) such that \( B_{v_i} \) is a division algebra, and for \( i = 1, \ldots, n \), let \( \pi_i \) be an irreducible representation of \( B^\times_{v_i} \). Assume that \( \text{Hom}_{E/v_i} (\pi_i \otimes \Omega_{v_i}, \mathbb{C}) \neq 0 \) (in particular, \( \pi_i \) has central character \( \omega_{v_i}^{-1} \)). Then there is an automorphic representation \( \pi \) of \( B^\times \) with central character \( \omega^{-1} \), and \( \phi \in \pi \) such that \( \pi_{v_i} \simeq \pi_i \) for \( i = 1, \ldots, n \), and

\[
\int_{E^\times/\mathbb{A}_E^\times} \phi(h)\Omega(h)dh \neq 0.
\]

**Proof.** The proof mimics the argument in [10]. We apply the relative trace formula associated to the automorphic distribution in (3.2.2). In particular, we choose a pure tensor \( f \) in (3.2.2) such that the twisted average of \( f_{v_i} \) by \( \omega_{v_i} \) along the center is a matrix coefficient of \( \pi_{v_i} \). Moreover, we choose another place \( v_0 \) such that \( \text{supp} f_{v_0} \subset B_{v_0,\text{reg}}^\times \). The rest of the proof is the same as in [10].

3.3. **Local distributions.** The automorphic distributions \( O(s, f') \) (and thus \( O(0, f'), O'(0, f') \)) and \( O(f) \) above can be decomposed into sums over automorphic representations. Each summand is further a product (when \( f, f' \) are pure tensors) of local distributions. For transfers \( f, f' \), the local distributions for \( O(0, f') \) and \( O(f) \) can be compared. We will not dive into the decomposition, but will refer to previous works. Here we only introduce the local distributions.

Let \( F \) be a local field and \( E/F \) a separable quadratic field extension. Let \( \Omega \) be a unitary character of \( E^\times, \omega = \Omega|_{F^\times}, \text{and } \omega_E = \omega \circ \text{Nm} \). Let \( \psi \) be a nontrivial additive character of \( F \), and \( \psi_E = \psi \circ \text{Tr} \) where \( \text{Tr}: E \to F \) is the trace map. (We omit the case \( E = F \oplus F \), and refer to [27 6.4].)

3.3.1. **Local distributions on \( \text{GL}_2, E \).** Let \( \sigma \) be an infinite dimensional irreducible unitary representation of \( G = \text{GL}_2, E \) with central character \( \omega_E^{-1} \). Let \( W(\sigma, \psi_E) \) be the \( \psi_E \)-Whittaker model of \( \sigma \). For \( W \in W(\sigma, \psi_E) \), define the local Rankin-Selberg period

\[
\lambda(s, W) = \int_{E^\times} W \left( \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} \right) |x|_E^s \Omega(x)dx
\]

and the local base change period

\[
\mathcal{P}(W) = \int_{E^\times} W \left( \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} \right) \eta\omega(x)dx.
\]

Then the integral (3.3.1) converges (see for example [13 p.52 Remark]). Assume that \( \sigma \) is tempered, then the local Rankin-Selberg integral (3.3.1) converges for \( \text{Re}(s) > -1/2 \). Moreover, when the data are “unramified”, \( \lambda(s, W) = L(1/2, \pi_E \otimes \Omega) \), see [27 (8.1)].
Fix some (unique up to scalar) $G$-invariant inner product (see the remark after Proposition 3.3.1 below). For $f \in S_c(G)$, define

$$I_\sigma(s, f) = \sum_W \lambda(s, \pi(f)W)P(W)$$

where the sum is over an orthonormal basis of $W(\sigma, \psi_E)$

3.3.2. Quaternion side. Let $B$ be a quaternion algebra over $F$ containing $E$. Let $\pi$ be an irreducible unitary representation of $B^\times$ with central character $\omega^{-1}$. For $u \in \pi$ and $v \in \tilde{\pi}$, define

$$\alpha_\pi(u, v) = \int_{E^\times/F^\times} (\pi(t)u, v)\Omega(t)dt.$$ 

By abuse of notation, for $f \in S_c(B^\times)$, let

(3.10) \hspace{1cm} \alpha_\pi(f) = \sum_u \alpha_\pi(\pi(f)u, \tilde{u})

where the sum is over an orthonormal basis $\{u\}$ of $\pi$, and $\{\tilde{u}\}$ is the dual basis of $\tilde{\pi}$.

**Proposition 3.3.1.** Assume that $F$ has characteristic 0, and $\alpha_\pi \neq 0$. Then for a transfer $f' \in S_c(G')$ of $f$, we have

$$I_{\pi_E}(0, f') = c_\pi \alpha_\pi(f),$$

where $c_\pi$ is an explicit nonzero constant.

**Proof.** The proposition follows from the globalization argument in [2, Proposition 5.7.1] and the proof of [13, Proposition 5] (see also [27, Proposition 6.3.1]). \hfill \Box

The constant $c_\pi$ in Proposition 3.3.1 depends on the choices of the $G$-invariant inner product on $W(\sigma, \psi_E)$ and the measures. In the non-archimedean case, it can be read from the equation in [27, Proposition 6.3.3]. It has the same shape in the archimedean case by the same proof. We will not need the constant $c_\pi$ explicitly, rather refer to [27] later.

3.3.3. A quaternionic test function at $\infty$. Let $D$ be the unique division quaternion algebra over $R$, and fix an embedding $C \hookrightarrow D$. In the notation of 3.1.2, we choose $\epsilon = -1$. (Then $D = C \oplus Cj$ is the usual way of expressing the Hamilton quaternion algebra.)

Let $\rho_{2k}$ be the irreducible $(2k-1)$-dimensional representation of $D^\times$ with trivial central character whose Jacquet-Langlands correspondence to $GL_2(\mathbb{R})$ is the holomorphic discrete series of weight $2k$. Fix an invariant inner product $(\cdot, \cdot)$ on $\rho_{2k}$, and let $e$ be the unique unit vector which is $C^\times$-invariant (see [8]). Let $f_\infty \in S_c(D^\times)$ be such that

(3.11) \hspace{1cm} \int_{D^\times} f_\infty(\rho_{2k}(g)e, e) dg = \int_{\mathbb{R}^\times} f_\infty(zg)dz = (\rho_{2k}(g)e, e).$

Then by the discussion in [27, 6.2], we have the first equality of the following

$$\alpha_{\rho_{2k}}(f_\infty) = \text{Vol}(C^\times/\mathbb{R}^\times) \int_{D^\times} f_\infty(g)(\rho_{2k}(g)e, e) dg = \frac{\text{Vol}(C^\times/\mathbb{R}^\times)}{d_{\rho_{2k}}}.$$
Here $d_{\rho_{2k}}$ is the formal degree of $\rho_{2k}$ and the second equality is by definition. It is a standard fact that $d_{\rho} \text{Vol}(D^\times / \mathbb{R}^\times) = \dim \rho$ for any irreducible representation $\rho$ of the compact group $D^\times / \mathbb{R}^\times$. So

$$\alpha_{\rho_{2k}}(f_\infty) = \frac{\text{Vol}(C^\times / \mathbb{R}^\times) \text{Vol}(D^\times / \mathbb{R}^\times)}{2k - 1}.$$  

Let $P_{k-1}(t)$ be the $(k - 1)$-th Legendre polynomial, i.e. the multiple of $\frac{d^{k-1}}{dt^{k-1}}(t^2 - 1)^{k-1}$ whose value at 1 is 1. Let $\Omega = 1$. By [5, Lemma 4.14] and a direct computation, we have

$$O(\delta, f_\infty) = P_{k-1} \left( \frac{1 + \text{inv}(\delta)}{1 - \text{inv}(\delta)} \right) \text{Vol}(C^\times / \mathbb{R}^\times)^2$$

for $\delta \in \mathbb{D}^\times_{\text{reg}}$. We remind the reader that there is a sign mistake in the statement of [5, Lemma 4.14], which is easy to spot by checking [5, Lemma 4.8] (the proof of [5, Lemma 4.14] is based on [5, Lemma 4.8]).

### 3.4. Local comparison I.

We compare the height distribution

$$H(f) = \langle Z_{\Omega, f}, z_{\Omega^{-1}} \rangle$$

on $\text{GL}_2(\mathbb{A}_\infty)$ with the automorphic distribution on $\text{GL}_2(\mathbb{A}_K)$. The conclusion is an arithmetic relative trace identity (3.30).

The proof of (3.30) consists of 10 steps, each is a subsubsection. We briefly sketch them as follows:

1. recall/set up notations;
2. decompose $H(f)$ into a sum of local heights $H(f)_v$’s over places of $\mathbb{Q}$;
3. express the local height $H(f)_p, p < \infty$, into a sum of intersection numbers using a description of the integral model;
4. for $p$ nonsplit in $K$, decompose the intersection numbers using formal uniformization of the integral model, and rewrite $H(f)_p$ in a form like a sum of orbital integrals (with the local orbital integrals at $p$ replaced by intersection numbers);
5. choose test functions and finish the comparison for $p < \infty$;
6. deal with the infinite place.

#### 3.4.1. Notations.

We will need the following notations for quaternion algebras. For a quaternion algebra $B$ over a field $F$, we always use $q$ to denote the reduced norm map. (If $B$ is the matrix algebra, then $q$ is the determinant, and we will consistently use $q$ to replace det from now on.) Let $E$ be a separable quadratic field extension of $F$. For an embedding of $E \hookrightarrow B$, define an invariant on $B^\times$ for the bi-$E^\times$-action

$$\lambda : B^\times \to \mathbb{Q},$$

$$\lambda(a + bj) = \frac{q(bj)}{q(a + bj)},$$

where the notation is the same as in §3.1.2. The relation with the invariant defined in §3.1.2 is

$$\lambda(\delta) = \frac{-\text{inv}(\delta)}{1 - \text{inv}(\delta)}.$$
We will use quaternion algebras over \(\mathbb{Q}\). For a finite place \(p\) of \(\mathbb{Q}\), let \(B(p)\) be the unique quaternion algebra over \(\mathbb{Q}\) such that \(B(p)_v\) is division only for \(v = p\) and \(v = \infty\). Let \(B(\infty)\) be the matrix algebra \(M_2(\mathbb{Q})\).

Also need Legendre functions. Let \(P_{k-1}(t)\) be the \((k-1)\)-th Legendre polynomial, i.e. the multiple of \(\frac{du}{du}((t^2-1)^{k-1})\) whose value at 1 is 1. Let \(Q_{k-1}\) be the Legendre function of the second kind, i.e.,

\[
Q_{k-1}(t) = \int_{u=0}^{\infty} (t + \sqrt{t^2 - 1} \cosh u)^{-k} du, \ t > 1.
\]

Then (see [9, p. 294, (5.7)])

\[
(3.14) \quad Q_{k-1}(t) = \frac{1}{2} P_{k-1}(t) \log \frac{t+1}{t-1} + \text{(polynomial in } t),
\]

and

\[
(3.15) \quad Q_{k-1}(t) = O(t^{-k}), \text{ for } t \to \infty.
\]

Finally, let \(N\) be a positive integer as in [2] and let \(U = U(N)\), the corresponding principal congruence subgroup of \(\text{GL}_2(\mathbb{A}^\infty)\). Let us rewrite \(H(f)\). Let

\[
c_K = \text{Vol}(K^x \backslash A_K^x / A^x) \text{Vol}(K^x \backslash A_K^x / \mathbb{R}^x).
\]

By definition and a simple change of variable \(h = t^{-1_1}t_2g\), we have

\[
H(f)_p = \frac{1}{c_K} \int_{K^x \backslash A_K^x / A^x} \int_{K^x \backslash A_K^x / \mathbb{R}^x} \sum_{h \in \text{GL}(A^\infty) / U} f(h)
\]

\[
(3.16) \quad \langle Z_{t_1b}, Z_{t_2} \rangle_{Y(N)} \Omega^{-1}(t_2) \Omega(t_1) dt_2 dt_1 = \frac{1}{c_K} \int_{K^x \backslash A_K^x / A^x} \int_{K^x \backslash A_K^x / \mathbb{R}^x} \sum_{g \in \text{GL}(A^\infty) / U} f(t_1^{-1}t_2g)
\]

\[
\langle Z_{t_2g}, Z_{t_2} \rangle_{Y(N)} \Omega^{-1}(t_2) \Omega(t_1) dt_2 dt_1.
\]

### 3.4.2. Local-global decomposition.

Let \(S\) be a finite set of finite places of \(\mathbb{Q}\) which contains all places of \(\mathbb{Q}\) ramified in \(K\) or ramified for \(\Omega\), and all prime factors of \(N\). Fix \(f_S = \otimes_{p \in S} f_p \in \mathcal{S}(\text{GL}_2(\mathbb{A}_{S}))\) where each \(f_p \in \mathcal{S}(\text{GL}_2(\mathbb{Q}_p))\) is right \(U_p\)-invariant. Let \(f = f_S \otimes f^S\) where \(f^S \in \mathcal{H}^S\). Such test functions will be enough for our proof of Theorem 2.4.4 in [3.5]

For a given \(f_S\), let us consider the defining fields and height pairings of the involved CM cycles as \(f^S \in \mathcal{H}^S\) varies. Let \(F\) be the finite abelian extension of \(K\) corresponding to

\[
\mathbb{A}_{K}^\infty \cap \bigcap_{g_S \in \text{supp} f_S} g_S U g_S^{-1}
\]

by the class field theory. Then by the CM theory, all CM cycles \(Z_{tgs}\)’s, \(t \in \mathbb{A}_{K}^\infty\) and \(g_S \in \text{supp} f_S\), are defined over \(F\). Moreover, for \(g^S \in \text{GL}_2(\mathbb{A}_{S}^\infty)\), \(Z_{tgs}\) is defined over a finite extension \(F'\) of \(F\) unramified at \(S\). So we can define height pairing \(\langle Z_{tgs}, Z_{t'} \rangle, t' \in \mathbb{A}_{K}^\infty\), as in [2.2.4] using the integral model \(Y'\) of \(Y(N)\) over \(\text{Spec} \mathcal{O}_F\) which is smooth outside \(S\).

Fix an \(f_S\) satisfying the following assumption (we will further specify such an \(f_S\) in [3.5]).

**Assumption 3.4.1.** For every \(p \in S\), we have \(\text{supp} f_p \subset \text{GL}_2(\mathbb{Q}_p)_{\text{reg}}\), the regular locus for the \(K_p^x \times K_p^x\)-action.
Then $Z_{\Omega, f_S}$ and $Z_{\Omega - 1}$ do not intersect. So we have the decomposition of the height pairing into local heights over places of $\mathbb{Q}$:
\begin{equation}
H(f) = \sum_v H(f)_v.
\end{equation}
Here the local height at $v$ comes from all places of $F$ over $v$.

3.4.3. **Integral CM cycles and desingularization.** Let $Y' \to X'$ be the integral model of $Y(N) \to X(N)$ in §2.2.3. Recall that a cycle on $Y'$ or $X'$ is called vertical if it is supported on the special fibers. It is called horizontal if it has no vertical component, equivalently, it is the Zariski closure of its generic fiber. For $t \in \mathbb{A}^1_{K^\infty}$, let $Z'_t$ be the integral CM cycle on $Y'$ as in §2.2.4 and we describe it as follows.

Let $p$ a prime number and $F_p = F \otimes_{\mathbb{Q}} \mathbb{Q}_p$, which is a product of finite extensions of $\mathbb{Q}_p$. Let $\mathcal{O}_{F_p}$ be the product of the integer rings of the completed maximal unramified extensions of the components of $F_p$. We work on $\mathcal{O}_{F_p}$ for later convenience. Let $P_{t, 0}$ (resp. $P'_{t, 0}$) be a reduction of the CM point $P_t$ in $\mathcal{X}_{\mathcal{O}_{F_p}}$ (resp., $\mathcal{X}'_{\mathcal{O}_{F_p}}$). Let $Y'_{P'_{t, 0}}$ be the fiber of $Y'$ over $P'_{t, 0} \in \mathcal{X}'_{\mathcal{O}_{F_p}}$. Then as we have seen in §2.2.3 the natural map $Y' \to Y$ gives the identification
\begin{equation}
Y'_{P'_{t, 0}} = \mathcal{E}|_{P'_{t, 0}}^{2k-2}.
\end{equation}
Let $Z_{t, 0} = Z(\mathcal{E}|_{P'_{t, 0}})$ be the CM cycle as in §2.2.1. Then
\begin{equation}
Z_t|_{Y'_{P'_{t, 0}}} = \overline{Z_t} + Z_{t, 0} \times D_t,
\end{equation}
where $\overline{Z_t}$ is the horizontal part of $Z'_t$ and $D_t$ is the base change to $\mathcal{O}_{F_p}$ of a divisor of $X'$ supported on the exceptional divisor of the desingularization $X' \to \mathcal{X}_{\mathcal{O}_p}$ at $p$. The special fiber of $\overline{Z_t}$ sits in $Y'_{P'_{t, 0}} = \mathcal{E}|_{P'_{t, 0}}^{2k-2}$, and is identified as $Z_{t, 0}$. We have a similar description for $Z_{t, g}$. Then we have
\begin{equation}
H(f)_p = i(f)_p + j(f)_p,
\end{equation}
where $i(f)_p$ comes from the intersections of horizontal cycles, $j(f)_p$ comes from the intersections between horizontal and vertical cycles.

As $X$ is smooth over $Z$ at ordinary loci (i.e., the loci of points representing ordinary elliptic curves over finite fields, see [17] or [22]), the desingularization of $\mathcal{X}_{\mathcal{O}_p}$ (to get $X'$) only happens at supersingular locus. So vertical parts of integral CM cycles are supported on supersingular loci.

3.4.4. **Split $p$.** Let $p$ be split in $K$. Under Assumption 3.4.11 the underlying CM points of $Z_{\Omega, f_S}$ and $Z_{\Omega - 1}$ do not have any common reduction (see [37, 8.4] or [27, 9.5.1]). So $i(f)_p = 0$. Since the CM points have ordinary reductions at such $p$ (see for example [9]) while the vertical parts of $Z'_g$ is supported over supersingular loci, $j(f)_p = 0$.

3.4.5. **Non-split $p$, $i$-part.** Let $p$ be non-split in $K$. By definition and (3.16), we have
\begin{equation}
i(f)_p = \frac{1}{c_K} \int_{K^\times \backslash K^\times / K^\times} \int_{K^\times \backslash K^\times / K^\times} \sum_{g \in \text{GL}_2(A^\infty) / U} f(t_1^{-1}t_2)(\overline{Z_{t_2 g}} : \overline{Z_{t_2}})_{p}\Omega^{-1}(t_2)\Omega(t_1)dt_2dt_1 \cdot \log p
\end{equation}
Here $\overline{Z_{t_2 g}}$ and $\overline{Z_{t_2}}$ are horizontal parts of $Z'_{t_2 g}$ and $Z'_{t_2}$ in $Y'_{\mathcal{O}_{F_p}}$ and $(\overline{Z_{t_2 g}} : \overline{Z_{t_2}})_p$ is $1/|F : \mathbb{Q}|$ times their intersection number.
To proceed, we need some preparations for computing \((Z_{t_{2g}} \cdot Z_{t_2})_p\). Note that the CM points have supersingular reductions at such \(p\) (see for example [9]). Let \(\mathcal{M}_{U_p}\) be the supersingular Lubin-Tate formal deformation space of level \(U_p\) [27, 9.3.2], which is defined over the integer rings of the completed maximal unramified extension of \(\mathbb{Q}_p\). Then the corresponding base change of the formal neighborhood of the supersingular locus in \(\mathcal{X}\) is given as follows:

\[
B(p)^{\times} \backslash \mathcal{M}_{U_p} \times B(p)^{\times}(\mathbb{A}^{p, \infty})/U^p.
\]

Let \(\mathcal{M}'_{U_p}\) be the minimal desingularization of the base change of \(\mathcal{M}_{U_p}\) to \(O_{F^{ur}}\).

Let \(t \in K_p^{\times}\) act on \(B(p)^{\times}\) by right multiplication by \(t^{-1}\), and act on \(GL_2(\mathbb{Q}_p)\) via left multiplication by \(t\). Then there is a natural map from the contracted product \(B(p)^{\times} \times K_p^{\times} GL_2(\mathbb{Q}_p)\) to \(\mathcal{M}_{U_p}\) parametrizing CM liftings [42, 5.5]. So, we have a map from \(B(p)^{\times} \times K_p^{\times} GL_2(\mathbb{Q}_p)\) to \(\mathcal{M}'_{U_p}\) by strict transform. Abusing notation, we use \((\delta, g) \in B(p)^{\times} \times K_p^{\times} GL_2(\mathbb{Q}_p)\) to denote the corresponding CM point in \(\mathcal{M}'_{U_p}\). The multiplicity function \(m_p(\delta, g)\) on \(B(p)^{\times} \times K_p^{\times} GL_2(\mathbb{Q}_p) - \{(1, 1)\}\) is defined as \(1/[F : \mathbb{Q}]\) times the intersection multiplicity between \((\delta, g)\) and \((1, 1)\) (see [27, Definition 9.3.6] or [37, 8.2.1]). It is a smooth function. Note that \(m_p\) in fact depends on \(U_p\) though we do not indicate this dependence in the notation.

We recall the following properties of the multiplicity function (see [27, 9.3.3]).

**Lemma 3.4.2.** (1) If \(m_p(\delta, g) \neq 0\), then \(q(\delta)q(g) \in q(U_p)\).

(2) We have \(m_p(\delta^{-1}, g^{-1}) = m_p(\delta, g)\).

As in [21, 2] first consider \(g \in G\). Let \(\psi_g : \mathcal{E}|_{P_{t_{2g}}} = E|_{P_{t_{2g}}} \to \mathcal{E}|_{P_{t_2}} = E|_{P_{t_2}}\) be the unique morphism that fits into the right downward arrow in (2.2) with \(A = \mathcal{E}|_{P_{t_{2g}}}\) there. (Here we use \(g\) rather than \(g^{-1}\) due to that \(P_{t_{2g}}\) is defined “via the left action by \(g^{-1}\),” see [22.2.1].) Let \(\psi_{g, 0}\) be its reduction. Consider

\[
\text{Hom}(\mathcal{E}|_{P_{t_{2,0}}}, \mathcal{E}|_{P_{t_{2g,0}}})_\mathbb{Q} \simeq \text{End}(\mathcal{E}|_{P_{t_{2,0}}})_\mathbb{Q} \simeq B(p)
\]

where the first isomorphism is \(\phi \mapsto \psi_{g, 0} \circ \phi\) and the second is well-known. For \(\delta \in B(p)^{\times}\), let \(\phi(\delta) \in \text{Hom}(\mathcal{E}|_{P_{t_{2,0}}}, \mathcal{E}|_{P_{t_{2g,0}}})_\mathbb{Q}\) be its preimage. By [40, Proposition 3.3.1] and [27, Lemma 9.3.9], if \(P_{t_{2g}} \neq P_{t_2}\), then

\[
(3.21) \quad (Z_{t_{2g}} \cdot Z_{t_2})_p = (-1)^k \sum_{\delta \in B(p)^{\times}} m_p(t_{2g, \delta}^{-1} t_{2, \delta}^{-1} g_{\delta}^{-1}) \left(1 + ((t_{2g}^{\infty} g^{\infty})^{-1} t_{2, \delta}^{-1} g_{\delta}^{-1})\right) \left(\phi(\delta)^{k-1} Z_{t_{2g,0}} \cdot Z_{t_2,0}\right) y_{P_{t_{2,0}}}.
\]

Here \((\phi(\delta)^{k-1} Z_{t_{2g,0}} \cdot Z_{t_2,0}) y_{P_{t_{2,0}}}\) is the intersection number on \(y_{P_{t_{2,0}}}\) (understood by the isomorphism (3.18)). By [40, 4.5.3] (for the CM cycle \(Z_{t_{2,0}}\), \(\sigma = \text{id}\) and \(a = (1)\) in the notations in loc. cit., and the proof is purely over \(\mathbb{F}_p\)), we have

\[
(3.22) \quad (\delta^{k-1} Z_{t_{2,0}} \cdot Z_{t_2,0}) y_{P_{t_{2,0}}} = (-q(\delta))^{k-1} P_{k-1}(1 - 2\lambda(\delta)).
\]

Then since \(\psi_{g,0}^{k-1} Z_{t_{2,0}} = |q(g)|^{1-k} Z_{t_{2g}}\) (the reduction of the corresponding equation on the generic fiber),

\[
(\phi(\delta)^{k-1} Z_{t_{2g,0}} \cdot Z_{t_2,0}) y_{P_{t_{2,0}}} = (-q(\delta) \cdot |q(g)|)^{k-1} P_{k-1}(1 - 2\lambda(\delta))
\]
By Lemma 3.4.2, the nonzero contribution in (3.21) is only from $\delta$ such that $q(\delta) \in q(t_2^\infty g_2^\infty, -1)q(U)$. Since $B(p)_\infty$ is division, $q(\delta) > 0$. So $q(\delta) = |q(g)|^{-1}$. Thus, we have

$$
  i(f)_p = -\frac{1}{c_K} \int_{K^{\times} \setminus A_K^{\times}/K^{\times}} \int_{K^{\times} \setminus A_K^{\times}/\mathbb{R}^{\times}} \sum_{g \in GL_2(\mathbb{A}^{\infty})/U} f(g) \sum_{\delta \in B(p)^{\times}} P_{k-1}(1-2\lambda(\delta))m_p(t_1^{-1}\delta t_2, g_2^{-1})1_{U_p}((t_1^{p^{\infty}}g^{p^{\infty}})^{-1}\delta t_2^{p^{\infty}})\Omega^{-1}(t_2)\Omega_p(t_1)dt_2dt_1 \cdot \log p.
$$

(3.23)

Note that $m_p(\cdot, g_2^{-1})$ only depends on $g_2U_p$ by Lemma 3.4.2(2). So the sum over $g$ is well-defined. Let $f_\infty$ be as in (3.11). Changing the order of summation in (3.23), we formally have

$$
  i(f)_p = -\frac{1}{c_K} \sum_{\delta \in K^{\times} \setminus B(p)^{\times}/K^{\times}} O(\delta, f^p \otimes f_\infty)
$$

$$
  \int_{K_p^{\times}/\mathbb{Q}_p^{\times}} \int_{K_p^{\times}} \sum_{g \in GL_2(\mathbb{Q}_p)/U_p} f(p)(g)m_p(t_1^{-1}\delta t_2, g_2^{-1})\Omega^{-1}(t_2)\Omega_p(t_1)dt_2dt_1 \cdot \log p.
$$

(3.24)

Here we use the orbital integrals of $f_\infty$ computed in (3.13). Moreover, the equality is verified by Assumption 3.4.1 and Fubini’s theorem, see [27, Lemma 9.3.19].

3.4.6. Non-split $p$, $j$-part. Let $p \in S$ which is non-split in $K$. Let $\mathcal{P}$ be the union of the exceptional divisors of $X'$ over $p$. Let $\mathcal{V}$ be the exceptional divisor of $\mathcal{M}'_{U_p}$, which inherits a $B(p)^{\times}$-action from the $B(p)^{\times}$-action on $\mathcal{M}'_{U_p}$. Then

$$
  \mathcal{P} \simeq B(p)^{\times} \times \mathcal{V} \times GL_2(\mathbb{A}^{p, \infty})/U^p.
$$

(3.25)

For a vertical divisor $C$ of $\mathcal{M}'_{U_p}$ supported on $\mathcal{V}$ and $g \in GL_2(\mathbb{A}^{p, \infty})$, let $[C, g]$ be the corresponding divisor of $\mathcal{P}$ via (3.25). Define a function $l_C$ on $B(p)_p^{\times} \times K_p^{\times}$ by letting $l_C(\delta, g)$ be the intersection number of $C$ and $(\delta, g)$ in $\mathcal{M}'_{U_p}$.

There is a natural map $\mathcal{M}'_{U_p} \rightarrow \mathcal{Z}$ given by the degree of the quasi-isogeny in the definition of the deformation space $\mathcal{M}'_{U_p}$ [27, 9.3.2]. Thus we have a map $\mathcal{M}'_{U_p} \rightarrow \mathcal{Z}$. It induces a map $\mathcal{V} \rightarrow \mathcal{Z}$.

Lemma 3.4.3 ([27, Lemma 9.3.19]). The function $l_C$ satisfies the following properties:

1. for $h \in B(p)_p^{\times} \times K_p^{\times}$ GL$_2(\mathbb{Q}_p)$, $l_C(h) \neq 0$ only if the image of the support of $C$ in $\mathbb{Z}$ by the map $\mathcal{V} \rightarrow \mathcal{Z}$ contains the image of $h$ by $B(p)_p^{\times} \times K_p^{\times}$ GL$_2(\mathbb{Q}_p) \rightarrow \mathbb{Z}$;
2. for $b \in B(p)_p^{\times}$ and $h \in B(p)_p^{\times} \times K_p^{\times}$ GL$_2(\mathbb{Q}_p)$, $l_{C}(bh) = l_{C}(h)$;
3. $l_C$ is smooth.

Let $D_t$ be as in (3.19). Write $D_1$ as $[C, 1]$ for a divisor $C$ of $\mathcal{M}'_{U_p}$ supported on $\mathcal{V}$. Then $D_t = [t_2^\infty C, t_2^t]$. By Lemma 3.4.3(2), a similar process of unfolding as in (3.23) and refolding as in (3.24), we have

$$
  j(f)_p = -\frac{1}{c_K} \sum_{\delta \in K^{\times} \setminus B(p)^{\times}/K^{\times}} O(\delta, f^p f_\infty)
$$

$$
  \int_{K_p^{\times}/\mathbb{Q}_p^{\times}} \int_{K_p^{\times}} \sum_{g \in GL_2(\mathbb{Q}_p)/U_p} f(p)l_{C}(t_1^{-1}\delta t_2, g^{-1})\Omega^{-1}_p(t_2)\Omega_p(t_1)dt_2dt_1 \cdot \log p.
$$

(3.26)
3.4.7. Test functions. For $p \in S$, besides Assumption 3.4.1, further assume that $q(\text{supp} f_p) \subset \text{Nm}(K_p^\times), p \in S$. Then by Lemma 3.1.2 we can use Lemma 3.1.7 (1). Let $f'_p$ be a transfer of $f_p$ given by Lemma 3.1.7 (1).

For $v = \infty$, $q(\text{supp} f_\infty) \subset \text{Nm}(K_\infty^\times)$ holds. So we can use Proposition 3.1.8 (for now Proposition 3.1.8 (1)) and we will specify $f'_\infty$ in 3.4.10.

Assume that $f^S$ has an unramified transfer $f'^S \in S_c \left( \text{GL}_2 \left( \mathbb{A}_K^{S,\infty} \right) \right)$ given by the fundamental lemma [27, Proposition 7.3.1] (see also [12, Section 4] [13, Proposition 3]). Explicitly, let $\mathcal{H}^S_E$ be unramified-outside-$S$ Hecke algebra of $\text{GL}_2,E$ (defined in the same way as $\mathcal{H}^S$). So we have the base change homomorphism $bc : \mathcal{H}^S_E \to \mathcal{H}^S$ (see [18, Section 1]). Then $f^S$ is a multiple of $bc(f'^S)$, depending on the measures. Moreover, as $q(\text{supp} bc(f'^S) \subset \text{Nm}(\mathbb{A}_K^{S,\infty,\times}), q(\text{supp} f) \subset \text{Nm}(\mathbb{A}_K^{\infty,\times})$.

Let

$$f' = \bigotimes_{p \in S} f'_p \bigotimes f'_\infty \bigotimes f'^S.$$

We will further specify the consequences of this choice of test function in the later paragraphs.

3.4.8. Matching for $p \notin S$. The full arithmetic fundamental lemma (proved in [27, Proposition 10.2.3]) equates

$$-2 \int_{K_p^\times/Q_p} \int_{K_p^\times} \sum_{g \in \text{GL}_2(Q_p)/U_p} f_p(g)m_p(t_1^{-1}\delta t_2,g^{-1})\Omega_p^{-1}(t_2)\Omega_p(t_1)dt_2dt_1 \cdot \log p$$

with the derivative $O'(0, \Phi_{f_p})$ of the local orbital integral for $f'_p$. Combining it with (3.24), we have the following lemma. (Note that the $j(f)_p = 0$ here.)

Lemma 3.4.4. For a finite place $p \notin S$ and nonsplit, we have

$$2H(f)_p = \frac{1}{c_K} O'(0, f'_p),$$

where the right hand side is defined in Definition 3.2.5.

3.4.9. Coherence for $p \in S$. For $p \in S$, we do not need anything like the arithmetic fundamental lemma, as the local terms are “coherent” (a term borrowed from [37], whose meaning is indicated in the proof of the lemma below).

Lemma 3.4.5. For $p \in S$ and nonsplit, there exists $f_p \in S_c(B(p)_p)$ such that

$$2H(f)_p - \frac{1}{c_K} O'(0, f'_p) = O(f_{S-\{p\}} \otimes f_p) \otimes f^S \otimes f^\infty).$$

Proof. Choose $e \in S_c(B(p)_p)$ by Lemma 3.1.7 (2) so that $O(x,e) = O'(0, x, \Phi_{f'_p})$. Let

$$f_p(\delta) = -\frac{1}{c_K} \left( 2 \sum_{g \in \text{GL}_2(Q_p)/U_p} f_p(g) \left( (m_p + l c)(\delta, g^{-1}) + e(\delta) \right) \right)$$

which is well-defined on $B(p)_p$ by Assumption 3.4.1. It is automatically in $S_c(B(p)_p)$ (so is “coherent”) by Lemma 3.4.2 and Lemma 3.4.3. Then the lemma follows from 3.2.4 and 3.2.6. \qed
3.4.10. **Local height at \( \infty \).** The comparison at \( \infty \) is similar to the comparison at \( p \) nonsplit in \( K \) and not in \( S \). Namely, we first write \( H(f)_{\infty} \) in a form like a sum of orbital integrals (with the local orbital integral at \( \infty \) replaced by the Legendre function). Besides, we need a truncation process.

Recall the complex uniformization (2.1.1) of the noncuspidal locus \( X(N)^{\circ} \) of \( X(N) \):

\[
X(N)^{\circ}(\mathbb{C}) \simeq B(\infty)_{>0} \backslash \mathbb{H} \times B(\infty)(A^{\infty})^{\times} / U(N)
\]

Here \( B(\infty) \) is the matrix algebra \( M_{2,\mathbb{Q}} \), \( B(\infty)_{>0} \subset B(\infty) \) consists of elements with positive norms (i.e., positive determinants), and \( \mathbb{H} \subset \mathbb{C} \) is the upper half plane. For \( (z_1, g_1), (z_2, g_2) \in X(N)^{\circ}(\mathbb{C}) \), let

\[
G_{k}((z_1, g_1), (z_2, g_2)) = - \sum_{\delta \in B(\infty)_{>0}} Q_{k-1}(d(z_1, \delta z_2)) 1_{U}(g_1^{-1} \delta g_2),
\]

where \( d \) is the hyperbolic distance. The sum is absolutely convergent by (3.15). If \( (z_1, g_1), (z_2, g_2) \) are distinct CM points, the local height pairing between the CM cycles over them defined as in (2.1.1) is \( G_{k}((z_1, g_1), (z_2, z g_2)) \), see [40] Proposition 3.4.1.

To use (3.27) to compute \( H(f)_{\infty} \), we need more details about the terms on the right-hand side of (3.27).

First, the hyperbolic distance has the following formula. Recall that \( B(\infty) \) is the matrix algebra \( M_{2,\mathbb{Q}} \), and we have fixed an embedding \( K \hookrightarrow M_{2,\mathbb{Q}} \) of \( \mathbb{Q} \)-algebras to define CM points and cycles in 2.2.1. Let \( z_0 \in \mathbb{H} \) be the unique fixed point under the action of \( K_{\infty}^{\times} \subset B(\infty)_{\infty,>0} \). Let \( \delta \in B(\infty)_{\infty,>0} \cap B(\infty)_{\infty,\text{reg}}^{\times} \). The hyperbolic distance between \( z_0 \in \mathbb{H} \) and \( \delta z_0 \), is

\[
d(z_0, \delta z_0) = 1 - 2\lambda(\delta),
\]

where \( \lambda \) is defined with respect to the embedding \( K_{\infty}^{\times} \subset B(\infty)_{\infty} \) see [37] 8.1.1.

Second, we truncate \( Q_{k-1} \), since \( Q_{k-1} \) does not vanish near infinity (see (3.15)). Note that

\[
\lambda \left( B(\infty)_{\infty,>0} \cap B(\infty)_{\infty,\text{reg}}^{\times} \right) = (-\infty, 0)
\]

so that \( 1 - 2\lambda \) takes values in \((1, \infty)\). Let \( \phi \) be a smooth function on \([1, \infty)\) with compact support such that \( \phi([1, 2]) = \{1\} \). Let \( Q_{k-1}^{\phi} = Q_{k-1} \phi \), and

\[
G_{k}^{\phi}((z_1, g_1), (z_2, g_2)) = - \sum_{\delta \in B(\infty)_{>0}} Q_{k-1}^{\phi}(d(z_1, \delta z_2)) 1_{U}(g_1^{-1} \delta g_2).
\]

For \( Z = \sum_{i=1}^{n} a_n z_n \) be a divisor on \( X(N)^{\circ}(\mathbb{C}) \), let

\[
D_{Z}(x) = \sum_{i=1}^{n} a_n \left( G_{k}(x, z_n) - G_{k}^{\phi}(x, z_n) \right).
\]

Then \( D_{Z} \) is a smooth bounded function on \( X(N)^{\circ}(\mathbb{C}) \) by (3.15).

Now we can use (3.27) to compute \( H(f)_{\infty} \). Under the complex uniformization (see Remark 2.1.1)

\[
X_{\infty}^{\circ}(\mathbb{C}) \simeq \text{GL}_{2}(\mathbb{Q})_{>0} \backslash \mathbb{H} \times \text{GL}_{2}(A^{\infty})^{\times},
\]

we may take \( P_{0} \in X_{\infty}^{\circ}(\mathbb{Q})^{K^{\times}} \) to be \( (z_0, 1) \). Then under the complex uniformization (2.1) of \( X(N)^{\circ}(\mathbb{C}) \), we have \( P_{g} = (z_0, g) \). By a similar process of unfolding as in (3.23) (now we use the
Lemma 3.4.6. Let \( m \)orphic function on \( X \) and where \( X \) function on \( t \)ations as in these lemmas, we have

\[
(3.28) \quad H(f)_{\infty} = \frac{1}{c_K} \int_{K^{\times}\backslash A_K^{\times}/A^{\times}} \int_{K^{\times}\backslash A_K^{\times}/A^{\times}} \sum_{g \in GL_2(A^{\infty})/U} f(g) G_k(P_{19}, P_{22}) \Omega^{-1}(t_2) \Omega(t_1) \, dt_2 \, dt_1
\]

Finally, we choose \( f'_{\infty} \) and compare \( Q_{k-1}^\phi (d(z_0, \delta(x)z_0)) \) to \( O'(0, x, \Phi_{f'_{\infty}}) \). Since \( 1 - 2\lambda = \frac{1 + \text{inv}}{1 - \text{inv}} \) (see (3.11), \( \lambda(\delta) \in (-\infty, 0) \) corresponds to \( \text{inv}(\delta) \in (0, 1) \). So condition (a) in Proposition 3.1.8 (3) holds. By the truncation above, condition (b) in Proposition 3.1.8 (3) holds. Choose \( f'_{\infty} \) by Proposition 3.1.8 (3), and let us look at its consequence. Recall that \( f_{\infty} \) satisfies (3.13). By (3.14) and Proposition 3.1.8 (3),

\[
(3.29) \quad 2 \text{Vol}(\mathcal{C}^{\times}/\mathbb{R}^{\times})^2 Q_{k-1}^\phi (d(z_0, \delta(x)z_0)) = -O'(0, x, \Phi_{f'_{\infty}}) + O(x, f_{\infty})
\]

and where \( f_{\infty} \in \mathcal{S}_c(B(\infty)_\infty) \) is provided by Proposition 3.1.8 (3).

By (3.28) and (3.29), we have the following lemma.

**Lemma 3.4.6.** Let \( f \) and \( f' \) be as in (3.4.7). Then there exists \( f_{\infty} \in \mathcal{S}_c(B(\infty)_\infty) \) and an auto-
morphic function on \( X_{\infty}'(\mathbb{C}) \) such that

\[
2H(f)_{\infty} = \frac{1}{c_K} O'(0, f'_{\infty}) + O(f \otimes f_{\infty}) + \int_{K^{\times}\backslash A_K^{\times}/A^{\times}} \rho(f) D(P) \Omega(t) \, dt.
\]

Here, \( D \) is obtained from the third line of (3.28), i.e.,

\[
D(P) = \frac{2}{\text{Vol}(K^{\times}\backslash A_K^{\times}/A^{\times}) \text{Vol}(U)} D_{\Omega}(P),
\]

which is a priori a smooth bounded function on \( X(N)^m(\mathbb{C}) \), and understood as an automorphic function on \( X_{\infty}'(\mathbb{C}) \).

3.4.11. **Conclusion.** By Lemma 3.2.4 Lemma 3.4.4 Lemma 3.4.5 and Lemma 3.4.6 with the no-
tations as in these lemmas, we have

\[
(3.30) \quad 2H(f) = \frac{1}{c_K} O'(0, f') + \sum_{p \in S_{\text{nspl}}} O(f_{S-\{p\}} \otimes f(p) \otimes f^S \otimes f_{\infty})
\]

Here \( S_{\text{nspl}} \subseteq S \) is the subset of places nonsplit in \( K \).

3.5. **Proof of Theorem 2.4.3.** Before the proof of Theorem 2.4.3 let us first restate it with Tamagawa measures as in [27, 37]. This is also convenient for the proof as we will refer to [27] for some details about local-global decomposition of the automorphic distribution \( \frac{1}{c_K} O'(0, f') \), and it is important to keep the same measures as loc. cit..
Let the Hamilton quaternion algebra $\mathbb{D}$ and the representation $\rho_{2k}$ of $\mathbb{D}^\times / \mathbb{R}^\times$ be as in §3.3.3. Choose measures so that

\begin{equation}
\text{Vol}(\text{GL}_2(\mathbb{Q}_p)) = \zeta_{\mathbb{Q}_p}(2)^{-1},
\end{equation}

and

$\text{Vol}(\mathbb{C}^\times / \mathbb{R}^\times) = 2, \text{Vol}(\mathbb{D}^\times / \mathbb{R}^\times) = 4\pi^2.$

Here we use $\varpi$ to denote the usual $\pi$, $3.1415926...$.

Let $\alpha_{\rho_{2k}}^\sharp$ be the normalization of $\alpha_{\rho_{2k}}$ by the local $L$-factors as in §2.4.1 (with $\mathbb{Q}_p$ replaced by $\mathbb{R}$ and so on). For a local admissible representation $\rho$ of the unit group of a quaternion algebra, at a finite or infinite place, make $\alpha_{\rho}^\sharp$ into a distribution on the local Schwartz functions as in (3.10) (with $\alpha$ replaced by $\alpha^\sharp$). Then by (3.12) and [24, p 169] on local $L$-factors, we have

\begin{equation}
\zeta_{\mathbb{R}}(2)\alpha_{\rho_{2k}}^{\sharp}(f_{\infty}) = 4\frac{(2k-2)!}{(k-1)!} \cdot (k-1)!.
\end{equation}

Now by (3.31) and (3.32), Theorem 2.4.3 is equivalent to the following.

**Theorem 3.5.1.** Assume that Conjecture 2.3.4 holds for $\pi$. For $\phi \in \pi^\infty$ and $\tilde{\phi} \in \pi^\infty$, we have

\begin{equation}
\langle z_{\phi \otimes \tilde{\phi}}, z_{\Omega-1} \rangle = \frac{1}{2} \frac{\zeta_{\mathbb{R}}(2)L'(1/2, \pi_K \otimes \Omega)}{L(1, \pi, \text{ad})} \alpha_{\rho_{2k}}^{\sharp}(f_{\infty})\alpha_{\pi^\infty}^{\sharp}(\phi \otimes \tilde{\phi}).
\end{equation}

**Remark 3.5.2.** If we pretend $k = 1$, (3.33) is the same as the formula in [37, Theorem 3.15], despite we have the extra $\frac{1}{4}$ in (3.33) and $f_{\infty}$ in local factor at $\infty$ here. The reason is as follows.

First, here we take representations $\pi^\infty, \pi^\infty$ of weight $2k$ dual to each other tautologically while we use Hecke action to define $\pi^\infty \otimes \pi^\infty \to \text{CM}(\Omega)$ by sending $\phi \otimes \phi$ to $z_{\phi \otimes \tilde{\phi}}$. However, in [37], the authors use a geometric realization of the representations $\pi^\infty, \pi^\infty$ of weight $2$ so that a similar map to $\pi^\infty \otimes \pi^\infty \to \text{lim sup} \text{Jac}(X(N))$ is tautological. But they have to define a pairing between the representations. They choose a normalization factor $1/\text{Vol}(X(N))$ (see [37, 3.1.3] for the definition).

Unwinding the definitions, we see that the difference between the definition of CM points or cycles is by $\frac{1}{2\text{Vol}(X(N))} = 2\varpi^2$. Second, our local factor at $\infty$, if we let $k = 1$, is $\text{Vol}(\mathbb{D}^\times / \mathbb{R}^\times) = 4\pi^2$ times the one in [37]. So the extra $\frac{1}{4}$ is cancelled by $4\pi^2/2\pi^2 = 2$.

**Proof of Theorem 2.5.1.** Let

$P_{\pi^\infty} := \text{Hom}_{\mathcal{A}K^\times \times \mathcal{A}K^\times}((\pi^\infty \otimes \Omega^\infty) \otimes (\pi^\infty \otimes \Omega^{-1,\infty}), \mathbb{C}),$

and for each $\pi_v$, define $P_{\pi_v}$ similarly. Then

$P_{\pi^\infty} = \otimes_{p < \infty} P_{\pi_p}.$

The theorem of Tunnell [31] and Saito [29] implies that $\dim P_{\pi_v} \leq 1$ so that $\dim P_{\pi^\infty} = 1$.

Since Conjecture 2.3.4 holds for $\pi$, both sides of (3.33) are in $P_{\pi^\infty}$. We may assume that $P_{\pi^\infty}$ is nonzero, otherwise, (2.22) becomes trivial. So $\dim P_{\pi_p} = 1$ for all $p$ so that $\dim P_{\pi^\infty} = 1$. Thus we only need to prove (2.22) for one element $\Psi \in \pi^\infty \otimes \pi^\infty$ that is not annihilated by $\alpha_{\pi^\infty}^{\sharp}$. (In fact, it is known $\alpha_{\pi^\infty}^{\sharp} \neq 0$ in this case, see for example [27, 6.2]. Moreover, $\alpha_{\pi^\infty}^{\sharp} \neq 0$, as its evaluation at a vector is an infinite product with almost all factors $1$.) We will choose this element $\Psi \in \pi^\infty \otimes \pi^\infty$ to be the image of $f^\vee$ in $\pi^\infty \otimes \pi^\infty \subset \text{End}(\pi^\infty)$ for some $f \in S_c(\text{GL}_2(\mathbb{A}_\infty)).$ (Then $\Psi = \sum_{\phi} \pi(f)\phi \otimes \tilde{\phi}$).
where the sum is over a basis \( \{ \phi \} \) of \( \pi^\infty \), and \( \{ \bar{\phi} \} \) is the dual basis of \( \bar{\pi}^\infty \). Then the value \( \alpha_{\pi^\infty}(\Psi) \) is \( \prod_{p<\infty} \alpha_{\pi_p}(f_p) \) if \( f \) is a pure tensor.

**Lemma 3.5.3.** For every finite place \( p \), there exists \( f_p \) such that \( \text{supp} f_p \subset \text{GL}_2(\mathbb{Q}_p)_{\text{reg}} \), \( q(\text{supp} f_p) \subset \text{Nm}(K_p^\times) \), and \( \alpha_{\pi_p}(f_p) \neq 0 \).

**Proof.** For \( p \) nonsplit in \( K \), by [27, 6.2], there exists a smooth function \( \omega \) on \( \text{GL}_2(\mathbb{Q}_p) \) nonzero at 1, such that

\[
\alpha_{\pi_p}(f_p) = \int_{\text{GL}_2(\mathbb{Q}_p)} f_p(g)\omega(g)dg.
\]

Let \( V \) be an open neighborhood of 1 on which \( \omega \) is constant. Let \( V^\circ \subset V \cap \text{GL}_2(\mathbb{Q}_p) \cap q^{-1}(\text{Nm}(K_p^\times)) \) be open compact and nonempty. Let \( f_p = 1_{V^\circ} \).

For \( p \) split in \( K \), it always holds that \( q(\text{supp} f_p) \subset \text{Nm}(K_p^\times) \). Then we apply [45, Theorem A.2].

Let \( N \) be large enough such that the subspace of \( U(N) \)-invariants \( \pi^{U(N)} \neq \{0\} \). For \( p \in S \), let \( f_p \) be as in Lemma 3.5.3. Choose \( f^S \in \mathcal{H}_E^S \) and \( f \in \mathcal{H}^S \) as in (3.47). For \( f \) being a pure tensor, we need to establish

\[
(\zeta_{\Omega, f^S, \pi^\infty, z_{\Omega^{-1}}} = \frac{1}{2} \frac{\zeta(2)}{\zeta(1/2, \pi_K \otimes \Omega)} \prod_{\rho_k} \alpha_{\pi_p}(f_p) \prod_{p<\infty} \alpha_{\pi_p}(f_p).
\]

Then let \( f_p = 1_{\text{GL}_2(z_p)} \) for \( p \notin S \) so that \( \alpha_{\pi_p}(f_p) = 1 \) and thus \( \prod_{p<\infty} \alpha_{\pi_p}(f_p) \neq 0 \). Theorem 3.5.1 follows.

Recall that we use the base change homomorphism bc : \( \mathcal{H}_E^S \to \mathcal{H}^S \) to choose \( f^S \) to be a multiple of \( bc(f^S) \), (see 3.47). We regard (3.30) as an equation between linear forms on \( f^S \in \mathcal{H}_E^S \). By Theorem 2.3.4 and choosing \( S \) large enough, we can perform the spectral decomposition on (3.30), i.e., decomposed into a sum of characters \( \mathcal{H}_E^S \) (see [13, 27]). Let us deal with the case that \( \pi_E \) is cuspidal (equivalently, \( \pi \not\equiv \pi \otimes \eta \) and the other case is similar, see [27, Proof of Theorem 3.1.9].

For a cuspidal automorphic representation \( \sigma \) of \( \mathcal{H}_E^S \), we have its \( \mathcal{H}^S \)-character \( L_{\sigma}^S \). Below, we write \( L_{\sigma} \) for \( L_{\sigma}^S \) and \( L_{\pi} \) for \( L_{\pi}^S \) to ease the notations. Then \( L_{\pi} \) and \( L_{\pi \otimes \eta} \) are the only two characters of \( \mathcal{H}^S \) whose restrictions to \( \mathcal{H}_E^S \) are \( L_{\pi_E} \). Let us consider the \( L_{\pi_E} \)-components of both sides of (3.30).

For the left hand side of (3.30), by (2.19), its \( L_{\pi_E} \)-component is 2(\( \zeta_{\Omega, f^S, \pi^\infty, z_{\Omega^{-1}}} \)) (see [34.47], this also equals the \( L_{\pi \otimes \eta} \)-component. So its the \( L_{\pi_E} \)-component is

\[
(3.35) 4(\zeta_{\Omega, f^S, \pi^\infty, z_{\Omega^{-1}}}).
\]

(Note that here we do not need Conjecture 2.3.6 for \( \pi \otimes \eta \).

For the right hand side, we claim that for each of the second, third and fourth term, the \( L_{\pi_E} \)-component, i.e., the sum of the \( L_{\pi} \)-component and \( L_{\pi \otimes \eta} \)-component vanishes. Let us consider the \( L_{\pi} \)-component first. For \( p \in S_{\text{nsp}} \), the \( L_{\pi} \)-component of \( O(f_{S_{\text{nsp}}} \otimes f(\rho) \otimes f^S \otimes f(\infty)) = 0 \). Indeed, \( B(p) \) is the unique division quaternion algebra over \( \mathbb{Q}_p \) and let \( \pi_p \) be the Jacquet-Langlands correspondence of \( \pi_p \) to \( B(p)^\times \). By definition, the \( L_{\pi} \)-component of \( O(f_{S_{\text{nsp}}} \otimes f(\rho) \otimes f^S \otimes f(\infty)) = 0 \) only depends on the image of \( f(\rho) \) in \( \text{Hom}(\pi_p^\vee, \pi_p^\vee) \), and then defines an element in \( P_{\pi^\vee} \). As \( P_{\pi^\vee} \neq \{0\} \), by the theorem of Tunnell [31] and Saito [29], \( P_{\pi^\vee} = \{0\} \). So the \( L_{\pi} \)-component of the second term on the right hand side of (3.30) is 0. Similarly, since \( \text{Hom}_{K^\times}(\pi_\infty \otimes \Omega_\infty, \mathbb{C}) = \{0\} \) (see [8] and recall
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\( \Omega_{\infty} \) is trivial) the \( L_{\pi} \)-component of the third and fourth term on the right hand side of (3.30) is 0. (Surely, the vanishing of the fourth term also follows by considering weights.) Now as \( P_{\pi} = P_{\pi} \otimes \eta_{\pi} \) by definition, the \( L_{\pi \otimes \eta} \)-components vanish too. So the \( L_{\pi E} \)-component of the right hand side of (3.30) only comes from the first term, the automorphic distribution \( \frac{1}{c_K} \Omega' (0, f') \).

We want to relate the \( L_{\pi E} \)-component of \( \frac{1}{c_K} \Omega' (0, f') \) to the right hand side of (3.5.1). We only give a sketch and refer to [27, Proof of Theorem 3.1.9] for more details. Take \( f \) to be a pure tensor. By the local-global decomposition of the \( L_{\pi E} \)-component of \( \frac{1}{c_K} \Omega' (0, f') \) (see [27, Section 8] or [13]), the \( L_{\pi E} \)-component of the right hand side of (3.30) is the product of \( \frac{\zeta(2) L'(1/2, \pi_K \otimes \Omega)}{L(1, \pi, \text{ad})} \) and local distributions \( I_{\pi E, v} (0, f_v) \), \( v \in S \cup \{ \infty \} \) on \( \text{GL}_2, E \) divided the local \( L \)-factors of \( \frac{\zeta(2) L'(1/2, \pi_K \otimes \Omega)}{L(1, \pi, \text{ad})} \) at \( v \) (compare with our normalization of \( \alpha^z \) in 2.4.1). Then by Proposition 3.3.1 (and the discussion below it), the \( L_{\pi E} \)-component of the right hand side of (3.30) finally becomes

(3.36) \[ \frac{1}{c_K} \frac{\zeta(2) L'(1/2, \pi_K \otimes \Omega)}{L(1, \pi, \text{ad})} \prod_v \alpha_{\pi_v}^z (f_v), \]

where the product is over the set of places of \( \mathbb{Q} \). Since \( \text{Vol}(K^\times \mathbb{A}_K^\times / \mathbb{A}^\times) = 2L(1, \eta) \) (see [37, 1.6.3]), we have

\[ c_K = (2L(1, \eta))^2 \text{Vol}(\mathbb{Q}^\times \mathbb{A}_K^\times / \mathbb{R}^\times) = 2L(1, \eta)^2. \]

Now the equality between the \( L_{\pi E} \)-components of both sides of (3.30) gives the equality between (3.35) and (3.36). This is (3.34).

4. Theta lifting and modularity

In this section, we prove the modularity of CM cycles using an arithmetic mixed Siegel–Weil formula.

The classical Siegel–Weil formula relates theta series and Eisenstein series, and its arithmetic variant is central in Kudla’s program. In fact, an arithmetic Siegel-Weil formula already implicitly appeared in the work of Gross and Zagier [9, p 233, (9.3)], where the authors use a linear combination of products of theta series and Eisenstein series, which we call a mixed theta-Eisenstein series.

In §4.1-§4.4, we first recall some basics of theta series and Eisenstein series, and form mixed theta-Eisenstein series of \( \text{GL}_2, \mathbb{Q} \) of weight \( 2k \). Then we study Whittaker functions of mixed theta-Eisenstein series, locally and globally. Finally, we compare the height pairing \( H(f) \) with the value of the Whittaker function at \( g = 1 \). The conclusion is an arithmetic mixed Siegel–Weil formula (4.35). We refer to our another work [28] for the systematic treatment of such a formula for unitary groups.

In §4.5-§4.6, we prove the modularity of CM cycles based on the arithmetic mixed Siegel–Weil formula. After that, in §4.7 we prove some technical local results used in the proof of the modularity.

4.1. Theta series and Eisenstein series.

4.1.1. Weil representation. Let \( F \) be a local field with an absolute value \( | \cdot | \). Let \( \psi \) be a nontrivial additive character of \( F \) and endow \( F \) with the self-dual Haar measure. Let \( (V, q) \) be a nondegenerate quadratic space over \( F \) of even dimension and endow \( V \) with the self-dual Haar measure. Let \( \nu \)
be the similitude character of similitude orthogonal group $GO(V)$. The Weil representation of $GL_2(F) \times GO(V)$ on $S(V \times F^\times)$ is defined as follows [37 2.1.3]:

- $r(1, h)\Phi(x, u) = \Phi(h^{-1}x, \nu(h)u)$ for $h \in GO(V)$;
- $r\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, 1\right)\Phi(x, u) = \chi(V, u^q)(a)|a|^{\dim(V)/2}\Phi(ax, u)$ for $a \in F^\times$, where $\chi(V, u^q)$ is the quadratic character associated to the quadratic space $(V, u)$.\[[5]\]
- $r\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, 1\right)\Phi(x, u) = \psi(buq(x))\Phi(x, u)$ for $b \in F$;
- $r\left(\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, 1\right)\Phi(x, u) = |a|^{-\dim(V)/4}\Phi(x, a^{-1}u)$ for $a \in F^\times$;
- $r\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1\right)\Phi(x, u) = \gamma(V, u^q)(\mathcal{F}(V, u^q)\Phi)(x, u)$ for $a \in F^\times$, where $\gamma(V, u^q)$ (resp. $\mathcal{F}(V, u^q)$) is the Weil index (resp. Fourier transform on the $x$-variable) for the quadratic space $(V, u)$.

**Remark 4.1.1.** (1) The action of $GO(V)$ extends to all functions on $V \times F^\times$.

(2) The action of $GL_2(F) \times GO(V)$ extends to all functions $\Phi$ such that for $u_0 \in F^\times$, $\Phi(x, u_0) \in S(V)$.

Let $O(V)$ act on $S(V)$ in the usual way: $h \in O(V)$ sends $\phi(g) \in S(V)$ to $\phi(h^{-1}g)$.

**Lemma 4.1.2.** Assume that $F$ is non-archimedean of residue characteristic not 2. Assume that $V$ has a self-dual lattice $L$. Let $O(L) \subset O(V)$ be the stabilizer of $L$. Let $\phi \in S(V)$ be $O(L)$-invariant.

Then for every $a \in F^\times$, there exists $f \in S(GL_2(F))$, right $GL_2(O_F)$-invariant, such that

$$
\phi \otimes 1_{O_F^\times} = \int_{GL_2(F)} f(g)r(g, 1)(1_L \otimes 1_{O_F^\times}) dg.
$$

**Proof.** The lemma is a simple corollary of [11 Theorem 10.2].\[\square\]

**4.1.2. Theta series and Eisenstein series.** We follow [37 4.1, 6.1]. Fix the standard additive character $\psi$ of $Q \setminus A$ (see [37 1.6.1]) for defining the global Weil representations.

Let $V_1$ be a nondegenerate quadratic space over $Q$ of even dimension. For $\Phi_1 = \Phi_1^\infty \otimes \Phi_{1, \infty}$, where $\Phi_1^\infty \in S(V_1(A^\infty) \times A^{\infty, x})$ and $\Phi_{1, \infty}$ is a smooth function on $V_1(\mathbb{R}) \times \mathbb{R}^\times$ such that $\Phi_{1, \infty}(x, u) \in S(V_1(\mathbb{R}))$ for every $u$, the theta series

$$
\theta(g, u, \Phi_1) = \sum_{x \in V_1(Q)} r(g, 1)\Phi_1(x, u)
$$

is absolutely convergent for every $g \in GL_2(A), u \in A^\times$. Note that $\theta(g, u, \Phi_1)$ in only left $SL_2(Q)$-invariant. (Later, we will specify a concrete $\Phi_{1, \infty}$.)

For $g \in GL_2(A), h \in GO(V_A)$, define

$$
\theta(g, h, \Phi_1) = \sum_{(x, u) \in V_1(Q) \times Q^\times} r(g, h)\Phi_1(x, u),
$$

The sum (4.2) only involves finitely many $u$ and converges absolutely. Then (4.2) defines a smooth $GL_2(Q) \times GO(V)$-invariant function on $GL_2(A) \times GO(V_A)$.
Let $V_2$ be a nondegenerate quadratic space over $\mathbb{A}$ of even dimension. For $\Phi_2 = \Phi_2^{\infty} \otimes \Phi_{2,\infty}$ where $\Phi_2^{\infty} \in \mathcal{S}(V_2^{\infty} \times \mathbb{A}^{\infty,x})$ and $\Phi_{2,\infty}$ is a smooth function on $V_{2,\infty} \times \mathbb{R}^\times$ such that $\Phi_{2,\infty}(x,u) \in \mathcal{S}(V_{2,\infty})$ for every $u$, the Eisenstein series

\begin{equation}
E(s,g,u,\Phi_2) = \sum_{\gamma \in P(\mathbb{Q}) \setminus \text{SL}_2(\mathbb{Q})} \delta(\gamma g)^s \eta(\gamma g,1) \Phi_2(0,u)
\end{equation}

is absolutely convergent for $\text{Re}s$ large enough and it has a meromorphic continuation to the whole complex plane, and is holomorphic at $s = 0$. Here $P$ is the standard Borel subgroup of $\text{SL}_2$ and $\delta$ the standard modular character $[37, 1.6.6]$. Note that $E(s,g,u,\Phi_2)$ in only $\text{SL}_2(\mathbb{Q})$-invariant.

Define a function on $V_1(\mathbb{A}) \times V_2 \times \mathbb{A}^\times$ by

$$
\Phi_1 \otimes \Phi_2((x,y),u) = \Phi_1(x,u) \Phi_2(y,u).
$$

Define the mixed theta-Eisenstein series $[37, 2.3, 6.1]$

\begin{equation}
I(s,g,\Phi_1 \otimes \Phi_2) = \sum_{u \in \mathbb{Q}^\times} \theta(g,u,\Phi_1) E(s,g,u,\Phi_2),
\end{equation}

which is a smooth $\text{GL}_2(\mathbb{Q})$-invariant function on $\text{GL}_2(\mathbb{A})$.

4.1.3. Local Whittaker function. We will use the local Whittaker functions $W_a(s,g,u,\phi)$ of Eisenstein series $[37]$ p. 186], indeed only its value at $g = 1$. We denote its value at $g = 1$ to be $W_a(s,u,\phi)$, which is given as follows.

Let $v$ be a place of $\mathbb{Q}$. For $a \in \mathbb{Q}_v$, $u \in \mathbb{Q}_v^\times$ and $\phi \in \mathcal{S}(V_{2,v} \times \mathbb{Q}_v^\times)$, define a (not necessarily convergent) Whittaker integral

\begin{equation}
W_a(s,u,\phi) = \int_{F_v} \delta \left( \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} b \\ 0 \end{bmatrix} \right) s \left( \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \right) \phi(0,u) \psi_v(-ab) db.
\end{equation}

4.1.4. Gaussian. We need special Schwartz functions at the infinite place. Fix the additive character of $\mathbb{R}$ to be the archimedean component $\psi_\infty$ of $\psi$, that is $\psi_\infty(x) = e^{2\pi ix}$.

By the Iwasawa decomposition, any element in $\text{GL}_2(\mathbb{R})$ can be uniquely written as

$$
\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \left[ \begin{array}{cc} y & x \\ 0 & 1 \end{array} \right] \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix},
$$

where $c \in \mathbb{R}_{>0}$ and $y \in \mathbb{R}^\times$.

For non-negative integer $l$, let

$$
p_l(t) = \sum_{j=0}^{l} \binom{l}{j} \frac{(-t)^j}{j!}.
$$

For $V = \mathbb{C}$ and $q(z) = z\bar{z}$, define the standard Gaussian of weight $2l+1$

$$
\Psi(z,u) = p_l(4\pi uq(z)) e^{-2\pi uq(z)} 1_{\mathbb{R}_{>0}}(u).
$$

By $[33]$ p 350, A2], $r(g,1)\Psi(0,u)$ is of weight $2l+1$. Then a direct computation shows that

\begin{equation}
r(g,1)\Psi(z,u) = \text{sgn}(y) |y|^{1/2} p_l(4\pi uq(z)y) e^{2\pi uq(z)(x+yi)} e^{(2l+1)i\theta} 1_{\mathbb{R}_{>0}}(uy), \ z \neq 0,
\end{equation}

\begin{equation}
r(g,1)\Psi(0,u) = \text{sgn}(y) |y|^{1/2} e^{(2l+1)i\theta} 1_{\mathbb{R}_{>0}}(uy),
\end{equation}

where $g$ has Iwasawa decomposition as in $[41,6]$. 


4.2. Local Whittaker functions. We specialize the above definitions using quaternion algebras. To study the Whittaker function of our mixed theta-Eisenstein series $I(s, g, \Phi)$ on $\text{GL}_2(\mathbb{A})$ defined in (4.4), we first study (a function which will be) its (non-archimedean) local components.

4.2.1. Quaternion algebra as a quadratic space. Let $B$ be a quaternion algebra over a field $F$ with reduced norm $q$. Let $(b_1, b_2) \in B^\times \times B^\times$ act on $B$ by

$$x \mapsto b_1xb_2^{-1}.$$ 

This gives an embedding of $B^\times \times B^\times$, modulo the diagonal embedding of $F^\times$, into the similitude orthogonal group $GO(B)$ of the quadratic space $(B, q)$. We use $[b_1, b_2]$ to denote the image of $(b_1, b_2)$ in $GO(B)$. Then $GO(B) = B^\times \times B^\times/F^\times \times \{1, \iota\}$ with $\iota([b_1, b_2]) = [b_2, b_1]$. And $SO(B) = \{[b_1, b_2] : q(b_1) = q(b_2)\}$, $O(B) = SO(B) \times \{1, \iota\}$.

Let $\mathbb{B}$ be a quaternion algebra over $\mathbb{A}_F$. Fix an embedding $\mathbb{A}_F \hookrightarrow \mathbb{B}$ and a decomposition $\mathbb{B} = \mathbb{A}_K \oplus \mathbb{A}_K \iota$ as in 3.1.2. Then clearly, it is an orthogonal decomposition.

4.2.2. Nonsplit case. Let $p$ be a finite place of $\mathbb{Q}$. Let $\text{ord}_p$ be the standard discrete valuation on $\mathbb{Q}_p$.

Assume that $p$ is nonsplit in $K$. Let $K_1^p \subset K_p$ be the subset of norm 1 elements, with the measure as in [37, 1.6.2]. Let $\eta_p$ be the quadratic character of $\mathbb{Q}_p$ associated to $K_p$ by the class field theory. For $a \in \mathbb{Q}_p$, $u \in \mathbb{Q}_p^\times$ and $\phi \in \mathcal{S}(K_pj_p \times \mathbb{Q}_p^\times)$, let

$$W_a^\phi(s, u, \phi) = \gamma_{(K_pj_p, uq)}^{-1} W_a(s, u, \phi)$$

where $\gamma_{(K_pj_p, uq)}$ is the Weil index for the quadratic space $(K_pj_p, uq)$, and $W_a$ is the Whittaker integral (1.5).

Remark 4.2.1. We remind the reader that in [37, 6.1.1], the normalization of the Whittaker integral for $a = 0$ differs from the one for $a \neq 0$. Our normalization here is uniform for $a = 0$ and $a \neq 0$. This uniformity will be useful in 4.2.3.

By [37] Proposition 6.10 (1)], we have

$$W_a^0(s, u, \phi) = (1 - p^{-s}) \sum_{n=0}^{\infty} p^{-ns+n} \int_{D_u(a)} \phi(t, u)du_t, \tag{4.9}$$

where $du_t$ is the self-dual measure on the quadratic space $(K_pj_p, uq)$ and

$$D_u(a) = \{t \in K_pj_p : uq(t) \in a + p^n\mathbb{Z}_p\}.$$

Lemma 4.2.2. If $a \neq 0$ or $\phi(0, u) = 0$, then (4.9) converges absolutely and defines a holomorphic function on $s$.

Proof. If $a \neq 0$, then (4.9) is a finite sum. For the case $\phi(0) = 0$, see Lemma 4.2.3 (2). \qed

Lemma 4.2.3. Fix $u \in \mathbb{Q}_p^\times$. Let $W(a) = \frac{d}{ds}|_{s=0} W_a^\phi(s, u, \phi)$, as a function on $\mathbb{Q}_p^\times$.

1. Regard $W(a)$ as a function on $\mathbb{Q}_p$ with singularity at 0. Then it has compact support.
2. If $\phi(0, u) = 0$, then $W(a)$ extends to a Schwartz function function on $\mathbb{Q}_p$.
3. Let $u \in \mathbb{Z}_p^\times$. If $\phi(0, u) = 1$, then $\frac{d}{\text{vol}(K_pj_p)} W(a) - \frac{\text{ord}_p(a)}{2} \cdot 1_{\mathbb{Z}_p} \cdot \log p$ extends to a Schwartz function on $\mathbb{Q}_p$. (One may replace $\mathbb{Z}_p$ by any open compact neighborhood of 0.)
Proof. We use (4.10).

(1) For $a$ large enough, $\phi(\cdot,u)$ is 0 on $D_a(a)$.

(2) Without loss of generality, assume that $ug(\text{supp}\phi(\cdot,u)) \subset p^A\mathbb Z_\mathbb F^\times$ where $A$ is an integer. If $\text{ord}_p(a) \geq A + 1$, then the nonzero summands in (4.9) are $n = 0,...,A$ (if $A < 0$, then every summands in (4.9) is 0). And for these $n$’s, $D_n(a)$ does not depend on $a$ for $a$. So $W_n(s,1,u)$ does not depend on $a$. Thus the extension follows.

(3) follows from a direct computation. For example, see [37, p. 195] and [35, p. 598]. Or one may bypass the computation as follows: by [37, p. 195] and [35, p. 598], (3) holds if $\phi(\cdot,u) = 1_{O_{K_j,p}}$. The general case follows by applying (2).

For $\phi_1 \in \mathcal S(K_p \times \mathbb Q_p^\times), \phi_2 \in \mathcal S(K_{p,j} \times \mathbb Q_p^\times)$, and $(y_1,x_2) \in \mathbb B_p = K_p \oplus K_{p,j}$, let
\[
\phi_1 \otimes \phi_2((y_1,x_2),u) = \phi_1(y_1,u)\phi_2(x_2,u).
\]

Let $B(p)_p$ be the quaternion algebra over $\mathbb Q_p$ non-isomorphism to $\mathbb B_p$. Fix an embedding $K_p \hookrightarrow B(p)_p$ and a decomposition $B(p)_p = K_p \oplus K_{p,j}(p)$ as in 3.1.2.

First, let $p$ be nonsplit in $K$. For $(y_1,y_2) \in B(p)_p - K_p = K_p \oplus (K_{p,j}(p) - \{0\})$, and $u \in \mathbb Q_p^\times$, define
\[
(4.10) \quad k_{\phi_1 \otimes \phi_2}((y_1,y_2),u) = \frac{L(1,\eta_p)}{\text{Vol}(K_{p,j}^1)} \cdot \phi_1(y_1,u) \cdot d|_{s=0}W^\circ_{uq(y_2),p}(s,u,\phi_2).
\]

For a general $\phi \in \mathcal S(\mathbb B_p \times \mathbb Q_p^\times)$, we define $k_\phi$ by linear extension. Then $k_\phi$ is a smooth function on $(B(p) - K_p) \times \mathbb Q_p^\times$.

Lemma 4.2.3 (2) implies the following lemma.

**Lemma 4.2.4.** Assume that $\phi$ vanishes on $K_p \times \mathbb Q_p^\times$. Then $k_\phi$ extends to a Schwartz function on $B(p)_p \times \mathbb Q_p^\times$.

4.2.3. Split case. Assume that $p$ is split in $K$. For $u \in \mathbb Q_p^\times$ and $\phi \in \mathcal S(K_{p,j} \times \mathbb Q_p^\times)$, let
\[
W^\circ_{0,p}(s,u,\phi) = \gamma^{-1}_{(K_{p,j},uq)} \frac{\zeta_{\mathbb Q_p}(s+1)}{\zeta_{\mathbb Q_p}(s)}W_0(s,u,\phi).
\]

**Remark 4.2.5.** Here our normalization of the Whittaker integral is the same with the one in [37, 6.1.1], and differs from the one for in the nonsplit case, see Remark 4.2.4.

Note that $\frac{\zeta_{\mathbb Q_p}(s+1)}{\zeta_{\mathbb Q_p}(s)}$ has a zero at $s = 0$, and $W^\circ_{0,p}(s,u,\phi)$ has analytic continuation to $s = 0$. In fact, later we will only consider $\phi$ such that $\phi(0,u) = 0$. In this case, we have an analog of Lemma 4.2.2 so that $W_0(s,u,\phi)$ is automatically holomorphic at $s = 0$.

For $y \in K_p$ and $u \in \mathbb Q_p^\times$, define
\[
(4.11) \quad c_{\phi_1 \otimes \phi_2}(y,u) = \phi_1(y,u) \cdot d|_{s=0}W^\circ_{0,p}(s,u,\phi_2).
\]

For a general $\phi \in \mathcal S(\mathbb B_p \times \mathbb Q_p^\times)$, we define $c_\phi$ by linear extension.

We have the following analog of Lemma 4.2.4 which is deduced from an analog of Lemma 4.2.3 (2).
Lemma 4.2.6. Assume that $\phi$ vanishes on $K_p \times Q_p^\times$. Then $c_\phi$ extends to a Schwartz function on $K_p \times Q_p^\times$.

4.2.4. A local analytic kernel $k_p(y,x)$. Let $p$ be a finite place of $\mathbb{Q}$ nonsplit in $K$. Let $U_p$ be an open compact subgroup of $B_p^\times$. Let $t \in K_p^\times$ act on $B(p)_p^\times$ by right multiplication by $t^{-1}$, and act on $B_p^\times$ via left multiplication by $t$. Define a function $k_p$ on $B(p)_p^\times \times K_p^\times B_p^\times / U_p - \{(1,1)\}$ by

$$k_p(y,x) = k_{x-1U_p \times q(x)q(U_p)}(y, q(y^{-1})).$$

(If $y \in K_p^\times$ and $x \notin K_p^\times$, we use Lemma 4.2.4.) The invariance by $K_p^\times$ can be checked directly using (4.5) and (4.10). Note that $k_p$ in fact depends on $U_p$ though we do not indicate this dependence in the notation. Later, we will fix a $U_p$, and compare $k_p$ with the multiplicity function $m_p$ defined in 3.4.5, see Corollary 4.4.6 and Corollary 4.5.8.

Now we study properties of $k_p$. First, we consider its support.

Lemma 4.2.7. If $k_p(y,x) \neq 0$, then $q(y)q(x) \in q(U_p)$.

Proof. Express $k_{\phi_p}$ using (4.9) and (4.10), consider the $Q_p^\times$-components of $B_p^\times \times Q_p^\times$ and $B(p)_p^\times \times Q_p^\times$. The lemma follows directly. \hfill \Box

Second, we consider $k_p(y,x)$ with $x \in K_p^\times$. By the $K_p^\times$-action, we only need to consider the case $x = 1$.

Lemma 4.2.8. There is an open compact subgroup $U'$ of $B(p)_p^\times$ such that:

1. $U' \cap K_p^\times = U_p \cap K_p^\times$ as a subgroup of $K_p^\times$;
2. the function $k_p(y,1) - \text{ord}_{p}(\lambda(y)) \cdot 1_{U'_p} \cdot \log p$ on $B(p)_p^\times - K_p^\times$ extends to a smooth function on $B(p)_p^\times$. Here $\lambda$ is the invariant defined in 3.4.4.

Proof. Let $U_0 \subset U_p$ be an open compact subgroup of the form

$$(4.12) \quad U_0 = (1 + U) \times U j \subset B_p = K_p \oplus K_p j_p$$

where $U$ is an open compact subgroup of $K_p^\times$ (small enough so that $U_0$ is indeed a group). Let \{t_1, ..., t_n\} be a set of representatives of $(K_p^\times U_0 \cap U_p)/U_0$. Let

$$k_i(y) = k_{t_iU_0 \times q(U_p)}(y, q(y^{-1}))$$

and express $k_i(y)$ using (4.9) and (4.10). By Lemma 4.2.3 (3), $k_i(y)$ satisfies the conditions in the lemma. Let $k(y) = \sum_{i=1}^{n} k_i(y)$. By the $K_p^\times$-invariance of $\lambda$, $k(y)$ also satisfies the conditions in the lemma. Note that

$$1_{U_p \times q(U_p)} - \sum_{i=1}^{n} 1_{t_iU_0 \times q(U_p)}$$

vanishes on $K_p \times Q_p^\times$. By Lemma 4.2.4, $k_p(y,1) - k(y)$ extends to a Schwartz function on $B(p)_p \times Q_p^\times$. Then the lemma follows. \hfill \Box

By Lemma 4.2.4 and Lemma 4.2.8, we have the following corollary.

Corollary 4.2.9. The function $k_p$ on $B(p)_p^\times \times K_p^\times B_p^\times / U_p - \{(1,1)\}$ is smooth.

Finally, we consider $k_p(y,x)$ with $x$ far away from $K_p^\times$, in the sense that $\lambda(x)$ is far away from 0.
Lemma 4.2.10. Assume that $\mathbb{B}_p$ is the matrix algebra. Let $V$ be an open compact neighborhood of 0 in $\mathbb{Q}_p$. For $U_p$ small enough and $V$ large enough, $k_p(y, x) = 0$ for all $y \in B(p)_p^\times$ and $x \in \mathbb{B}_p^\times$ with $\lambda(x) \not\in V$.

Proof. Fix a compact subset $C$ of $B(p)_p^\times$ such that

$$C \times \mathbb{B}_p^\times \to B(p)_p^\times \times K_p^\times \times \mathbb{B}_p^\times$$

is surjective (here we use the fact that $B(p)_p$ is the division algebra, which follows from the assumption that $\mathbb{B}_p$ is the matrix algebra). Since $\lambda$ is $K_p^\times$-invariant, we may and we do assume that $y \in C$.

By Lemma 4.2.7, $k_p(y, x) = 0$ unless $q(x^{-1}) \in q(C)q(U_p)$. Now assume $q(x^{-1}) \in q(C)q(U_p)$. Write $x = (x_1, x_2)$ under $\mathbb{B}_p = K_p \oplus K_pJ_p$. Then $\lambda(x) = q(x_2)/q(x) \not\in V$ with $V$ large enough is equivalent to that $\text{ord}_p(\lambda(x_2))$ is small enough. Since $q(x)$ is bounded, $\text{ord}_p(q(x_2)) = \text{ord}_p(q(x_1))$.

Choose $U_p$ to be as in (4.12). For $u_1 \in U, u_2 \in U_j$,

$$x(u_1 + u_2) = (x_1u_1 + x_2u_2) + (x_2u_1 + x_1u_2)$$

Choose $U$ small enough such that $\text{ord}_p(q(x_2u_1 + x_1u_2)) = \text{ord}_p(q(x_2))$ for every pair of $u_1 \in 1 + U, u_2 \in U_j$. Thus for the integrand $\phi_2(t, 1)$ in (4.9) to be nonzero, we necessarily have $\text{ord}_p(q(t)) = \text{ord}_p(q(x_2))$. Now we check the condition $q(t) \in q(y_2) + p^n\mathbb{Z}_p$ defining the integration domain $D_n(q(y_2))$ of (4.9), where $n \geq 0$. Let $V$ be large enough such that $\text{ord}_p(q(t)) = \text{ord}_p(q(x_2)) < 0$. Since $B(p)_p$ is the division algebra, $q(y_2) \not\in \text{Nm}(K_p^\times)$. Thus $D_n(q(y_2))$ is empty, and $k_p(y, x) = 0$. □

4.3. Global Whittaker functions. Now we study the Whittaker functions of the mixed theta-Eisenstein series, in particular, their holomorphic projections.

4.3.1. Schwartz functions. We further require $\mathbb{B}_\infty$ to be division (so the Hamilton quaternion algebra). Then in the decomposition $\mathbb{B} = \mathbb{A}_K \bigoplus \mathbb{A}_{K_j}, j^2_\infty > 0$. Recall that we fixed an integer $k > 1$. Let

$$(4.13) \quad \Phi_\infty = \Phi_{\infty, 1} \otimes \Phi_{\infty, 2}$$

where $\Phi_{\infty, 1}$ is the standard Gaussian on $K_\infty$ of weight 1 and $\Phi_{\infty, 2}$ is the standard Gaussian on $K_{\infty, j}$ of weight $2k - 1$ defined in [4.1.2]. For $\Phi_\infty \in \mathcal{S}(\mathbb{B}_\infty \times \mathbb{A}_\infty^\times)$, let $\Phi = \Phi_\infty \otimes \Phi_{\infty}$.

4.3.2. Holomorphic projection. For an automorphic form $f$ on $GL_2(\mathbb{A})$, let $\text{Pr} f$ be the orthogonal projection of $f$ to the space of holomorphic cusps forms on $GL_2(\mathbb{A})$ of weight $2k$. I.e. for every holomorphic cusp form $\varphi$ on $GL_2(\mathbb{A})$ of weight $2k$, the Petersson inner product of $f$ and $\varphi$ equals the one of $\text{Pr} f$ and $\varphi$. Recall that we have fixed the standard additive character $\psi$ of $\mathbb{A}$ (see [37 1.6.1]). Denote the $\psi$-Whittaker function of $f$ by $f_\psi$. Then $\text{Pr} f_\psi$ can be explicitly computed as follows.

For a positive integer $m$, the standard holomorphic Whittaker function on $GL_2(\mathbb{R})$ of weight $m$ is

$$W^{(m)}(h) = |y|^{m/2} e^{2\pi i (x + y)} e^{mi\theta} 1_{\mathbb{R}_{>0}}(y)$$

where $h$ has Iwasawa decomposition as in [4.1.6].
Proposition 4.3.1. Assume
\[ f\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} g\right) = O_2\left(||a|| ||b||\right)^{k-\epsilon} \]
for some \(\epsilon > 0\). Then
\[ f_\psi(g) = W(2k)(g_\infty) \int_{Z\setminus GL_2(\mathbb{R})} f_\psi(gh) W(2k)(h) dh, \]
where \(Z\) is the center of \(GL_2(\mathbb{R})\) and \(N\) is the upper triangular unipotent subgroup of \(GL_2(\mathbb{R})\).

Proof. This is the adelic version of [9, p. 288, (5.1)]. One may also prove it in a way similar to [37, Proposition 6.12] which is in the weight 2 case. \(\square\)

4.3.3. Theta series. Before the holomorphic projection of the Whittaker function of the mixed theta-Eisenstein series, we study the one of a theta series, which is simpler and will also be used.

Assume that \(B = B_h\) where \(B\) is a quaternion algebra over \(\mathbb{Q}\). Then we have the theta series
\[ \theta\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} g, h, \Phi\right) \]
for \(a, b \in \mathbb{R}^\times\), we have
\[ \theta\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} g, h, \Phi\right) = O_{g,h}(||a|| ||b||) \]
where \(||a|| = \max\{||a||, ||a^{-1}||\}\).

Proof. Only need to check the constant term of \(\theta(g, h, \Phi)\), which is the finite sum
\[ \sum_{u \in \mathbb{Q}^\times} r(g,h)\Phi(0,u). \]
The growth is checked directly by definition. \(\square\)

Then the \(\psi\)-Whittaker function \(Pr\theta(g, h, \Phi)_\psi\) of the holomorphic projection of \(\theta(g, h, \Phi)\) can be computed using Proposition 4.3.1 and (4.7). Then we have
\[ (4.14) \ Pr\theta(g, h, \Phi)_\psi = \frac{(4\pi)^{k-1}(k-1)!}{(2k-2)!} \sum_{y \in \mathbb{B}^\times} r(g_\infty, h_\infty)\Phi_\infty(y, q(y)^{-1})W_1^{(2k)}(g_\infty) \cdot P_{k-1}(1-2\lambda(y)) \]
if \(h_\infty\) fixes \(\Phi_\infty\). Here \(W_1^{(2k)}\) is the standard holomorphic Whittaker function (see §3.1.4), \(P_{k-1}(t)\) is the \((k-1)\)-th Legendre polynomial (see §3.4.1) and \(\lambda\) is the invariant defined in §3.4.3.

Remark 4.3.3. The computation is essentially the first integral in the second displayed formula on [9, p. 293], except now we are the adelic setting.

4.3.4. Mixed theta-Eisenstein series. Assume that \(B\) is an incoherent quaternion algebra over \(\mathbb{A}_F\) (see §3.2.3). In other words, for an odd number of places \(v\) of \(\mathbb{Q}\) (including \(\infty\)), \(B_v\) is the division quaternion algebra over \(\mathbb{Q}_v\). For a place \(v\) of \(\mathbb{Q}\) nonsplit in \(K\), let \(B(v)\) be the unique quaternion algebra over \(\mathbb{Q}\) such that \(B(v)(\mathbb{A}_F^v) \simeq B_v\), and fix an embedding \(K \hookrightarrow B(v)\) so that \(\mathbb{A}_K^v \subset B(v)(\mathbb{A}_F^v)\) is identified with the \(\mathbb{A}_K^v \subset B_v\). Let \(B(p)\) be the quaternion algebra over \(\mathbb{Q}_p\) non-isomorphism to \(\mathbb{B}_p\). Fix a decomposition \(B(v) = K \bigoplus K_j(v)\) as in §3.1.2.

Recall the mixed theta-Eisenstein series \(I(s, g, \Phi)\) on \(GL_2(\mathbb{A})\) defined in (4.4). Let \(I'(0, g, \Phi)\) be the derivative at \(s = 0\).
Lemma 4.3.4. For \( a, b \in \mathbb{R}^x \), we have
\[
I'(0, \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} g, \Phi) = O_g\left(\|a\|(\log \|a\|)\|b\|(\log \|b\|)\right)
\]
where \( \|a\| = \max\{|a|, |a^{-1}|\} \).

Proof. The lemma follows from the same reasoning as in the proof of [37, Proposition 6.7] (with \( W \) where \( Q \) and \( K \)).

We want to give an expression of \( \Pr I'(0, g, \Phi)_\psi \) that is similar to (4.14). To lighten notations, we were only consider \( \Pr I'(0, 1, \Phi)_\psi \), which will be enough for purpose. We further assume the following assumption.

Assumption 4.3.5. There is a finite places \( p_0 \) nonsplit in \( K \) such that \( \Phi = \Phi^{p_0} \otimes \Phi_{p_0} \) with \( \Phi_{p_0} \) vanishes on \( K_{p_0} \times \mathbb{Q}_{p_0} \).

One effect of Assumption 4.3.5 is the following lemma.

Lemma 4.3.6. Let \( \phi = \phi_\infty \otimes \phi_\infty \) where \( \phi_\infty \in S(\mathbb{V}_2^\infty \times \Lambda^\infty_x) \) is a pure tensor and \( \phi_\infty \) is a smooth function on \( \mathbb{V}_2^\infty \times \mathbb{R}^x \) such that \( \phi_\infty(x, u) \in S(\mathbb{V}_2^\infty) \) for every \( u \). Let \( E'_0(0, 1, u) \) be the value at \( g = 1 \) of the constant term of \( \frac{d}{ds}|_{s=0} E(s, g, u, \phi) \) (see 4.3). Assume that \( \phi = \phi^{p_0} \otimes \phi_{p_0} \) such that \( \phi_{p_0}(0, u) = 0 \). Then
\[
E'_0(0, 1, u) = \frac{L(1, \eta_{p_0})}{\text{Vol}(K_{p_0}^\times)} \cdot \frac{d}{ds}|_{s=0} W_{0, p_0}^\infty(s, u, \phi_{p_0}) \cdot \phi_{p_0}^\infty(0, u).
\]

Proof. The lemma follows from the same reasoning as in the proof of [37, Proposition 6.7] (with different normalization, see Remark 4.2.1).

For a finite place \( p \) nonsplit in \( K \), let
\[
K_{\Phi}^{(p)} = \sum_{y \in B(p)^x} (\Phi^{p, \infty} \cdot k_{\Phi_p}) (y, q(y)^{-1}) \cdot P_{k-1}(1 - 2\lambda(y)) \cdot W_1^{(2k)}(1)
\]
where \( W_1^{(2k)} \), \( P_{k-1} \), and \( \lambda \) are the ones in (4.14). This is a finite sum. If \( p = p_0 \), \( k_{\Phi_p}(y, q(y)^{-1}) \) is well-defined for \( y \in K^x \) by Assumption 4.3.5 and Lemma 4.2.1. If \( p \neq p_0 \), the summand in (4.15) corresponding to \( y \in K^x \) is understood as 0 by Assumption 4.3.5. For the infinite place, let
\[
K_{\Phi}^{(\infty)} = \sum_{y \in B(\infty)^{>0-K^x}} \Phi^{\infty}(y, q(y)^{-1}) \cdot Q_{k-1}(1 - 2\lambda(y)) \cdot W_1^{(2k)}(1),
\]
where \( Q_{k-1} \) is the Legendre function of the second kind (see 3.4.1).

The \( \psi \)-Whittaker function \( \Pr I'(0, g, \Phi)_\psi \) of the (cuspidal) holomorphic projection \( \Pr I'(0, g, \Phi) \) of \( I'(0, g, \Phi) \) is computed using Proposition 4.3.1. By (4.8), [34, Lemma 2.3 (2) (3)] and the proof
of (which is the discussion above) \[\text{p. 294, (5.8)},\] we have

\[
(4.17) \quad \Pr I'(0, 1, \Phi)_\psi = -2 \sum_v \int_{A_K^\times \setminus A_K^\times} K_r^{(v)} \cdot r(1, [t, t]) \Phi dt
\]

where the sum is over all nonsplit places of \(Q\) and the volume of \(A_K^\times \setminus A_K^\times\) is chosen to be 1. Note that \(r(1, [t, t])\Phi\) also satisfies the condition in Assumption 4.3.5.

**Remark 4.3.7.** Formally, the above result is the same as \[\text{[37, Proposition 6.5, Proposition 6.15]},\] which is in the weight 2 case.

4.4. **Local comparison II.** We compare the height pairing \(H(f) = \langle Z_{\Omega, f}, z_{\Omega^{-1}} \rangle\) with \(\Pr I'(0, 1, \Phi)_\psi\), the derivative of the mixed Eisenstein series (more precisely, the value at \(q = 1\) of its \(\psi\)-Whittaker function) computed in (4.17). The main conclusion is an arithmetic mixed Siegel–Weil formula (4.35).

The proof of (4.35) consists of 9 steps, each is a subsubsection. We briefly sketch them as follows:

1. recall/set up notations;
2. review local heights \(H(f)_p, p < \infty\), especially when \(p\) is nonsplit in \(K\);
3. for \(p\) is nonsplit in \(K\), rewrite local height \(H(f)_p\) using a Schwartz function \(\Phi\) on \(B^\infty\);
4\&5\&6\&7 finish the comparison for \(p\) nonsplit in \(K\) (we will give instructions there in the beginning);
8\&9 finish the comparison for \(p\) split in \(K\) and at \(\infty\).

4.4.1. **Notations.** Let \(B\) be the quaternion algebra \(B = D \times M_2(A^\infty)\) over \(A\) where \(D\) is the unique division quaternion algebra over \(A\). We consistently use \(B^\infty, \times\) to replace \(GL_2(A^\infty)\) used before (see §3.4). Thus we do not confuse this \(GL_2(A^\infty)\) with the \(GL_2\) used in the Weil representation in §1.1.

With the given \(B\), let \(B(v)\) be the quaternion algebra over \(Q\) as in §4.3.4 i.e. for a finite place \(p\) of \(Q\), let \(B(p)\) be the unique quaternion algebra over \(Q\) such that \(B(p)_v\) is division only for \(v = p\) and \(v = \infty\), and \(B(\infty)\) is the matrix algebra. Then \(B(v)'\)s are the same as in §3.4.1.

For a quaternion algebra \(B\) over a local field, let \(h \in B^\times\) act on \(S(B \times F^\times)\) by \(r(1, [h, 1])\), where \(r\) is the Weil representation and \([h, 1]\) is defined in §1.2.1. Then the induced action of \(f \in S(B^\times)\) on \(\phi \in S(B \times F^\times)\) is given by

\[
(4.18) \quad r(f)\phi(x, u) := \int_{h \in B^\times} \phi(h^{-1}x, q(h)u) f(h) dh.
\]

We choose measures so that

\[
\text{Vol}(K^\times A_K^\infty / A_K^\infty) = \text{Vol}(K^\times A_K^\infty) = 1.
\]

4.4.2. **Review height decomposition.** Let \(S\) be a finite set of finite places of \(Q\) which contains 2 and all finite places of \(Q\) ramified in \(K\) or ramified for \(\Omega\). Recall that \(S_{\text{nspl}} \subset S\) is the subset of places nonsplit in \(K\). Let \(N\) be the product of two relatively prime integers which are \(\ge\) such that the prime factors of \(N\) are contained in \(S\). Let \(U = U(N)\), the corresponding principal congruence subgroup of \(GL_2(A^\infty)\).

We consider right \(U\)-invariant Schwartz functions of the form \(f = f_S \otimes f^S\). (Note that we do not require the bi-\(U^S\)-invariance of \(f^S\) as in §3.4) Assume that \(f_S = \otimes_{p \in S} f_p\) is a pure tensor for simplicity. Also assume the following assumption until 4.5 where we prove Theorem 2.3.4 and Theorem 2.3.7

**Assumption 4.4.1.** There exists \(p_0 \in S\) nonsplit in \(K\) such that \(f_{p_0}\) vanishes on \(K_K^\times\).
Though Assumption 4.4.1 is weaker than Assumption 3.4.1 (and we relaxed the invariance on \(f^S\)), the decompositions of \(H(f)\) in 3.4 still hold after suitable modifications, which are given below. By Assumption 4.4.1 \(Z_{U,f_S}\) and \(Z_{U,-}\) do not intersect. So we have the decomposition of \(H(f)\) into local heights as in (3.17) and (3.20). We continue to use the notations in (3.17) and (3.20).

We treat the case that \(p\) is split in \(K\) in 4.4.8.

Now let \(p\) be a finite place of \(\mathbb{Q}\) nonsplit in \(K\). Let \(m_p\) be the multiplicity function on \(B(p)^{\times} \times K_p^\times \mathbb{B}_p^\times - \{(1, 1)\}\) defined in §3.4.5. For \(t_1, t_2 \in \mathbb{A}_K^{\times}\), let

\[
\begin{aligned}
h(f, t_1, t_2) & = \sum_{\delta \in B(p)^{\times}} P_{k-1}(1 - 2\lambda(\delta)) \\
& \quad \sum_{x \in \mathbb{B}_K^{\times}\times /U} f(x_p)m_p(t_1^{-1}\delta t_2, x_p^{-1})f(x_p^1 t_1^{-1}\delta t_2^1)^1.
\end{aligned}
\]

(4.19)

Here \(P_{k-1}(t)\) is the \((k - 1)\)-th Legendre polynomial (see §3.4.1) and \(\lambda\) is the invariant defined in §3.4.1.

**Remark 4.4.2.** (1) Note that \(m_p(\cdot, x_p^{-1})\) only depends on \(x_p U_p\) by Lemma 3.4.2 (2). So the sum over \(x\) is well-defined.

(2) If \(p = p_0\) and \(x_p \in K_p^{\times}\), \(f(x_p) = 0\) by Assumption 4.4.1. Then \(f(x_p)m_p(t_1^{-1}\delta t_2, x_p^{-1})\) is understood as 0. If \(p \neq p_0\) and \(\delta \in K^{\times}\), \(f(x_p^1 t_1^{-1}\delta t_2^1)^1 = 0\) by Assumption 4.4.1. Then the summand in (4.19) corresponding to \(\delta \in K^{\times}\) is understood as 0.

Then (3.23) is rewritten as follows:

\[
\begin{aligned}
i(f) & = -\int_{K^{\times}\setminus \mathbb{A}_K^{\times} / \mathbb{A}_K^{\times}} \int_{K^{\times}\setminus \mathbb{A}_K^{\times}} h(f, t_1, t_2) \log p \cdot \Omega^{-1}(t_2) \Omega(t_1) dt_2 dt_1.
\end{aligned}
\]

(4.20)

4.4.3. **Rewrite the local height following [37].** Consider the following basic Schwartz function on \(\mathbb{B}_\infty\):

\[
\Phi_0 = 1_{U_S \times q(U_S)} \bigotimes 1_{M_2(\mathbb{Z}^S) \times \mathbb{Z}^S} \bigotimes \Phi_\infty,
\]

where \(\Phi_\infty\) is the Gaussian on \(\mathbb{B}_\infty\) as in 3.13. Let

\[
\Phi = r(f) \Phi_0.
\]

Then \(\Phi\) is right invariant by \(U\), and

\[
\Phi^\infty(x, q(x)^{-1}) = f(x).
\]

(4.22)

Moreover, if \(u_S q(x_S) \notin q(U_S)\), then

\[
\Phi_S(x_S, u_S) = 0.
\]

(4.23)

In particular, Assumption 4.3.5 holds for \(\Phi\) by Assumption 4.4.1.

For \(\phi \in \mathcal{S}(\mathbb{B}_p \times \mathbb{Q}_p^\times)\) right invariant by \(U_p\), define a function \(m_\phi\) on \((B(p)^{\times} - K_p^{\times}) \times \mathbb{Q}_p^{\times}\) as in [37, Notation 8.3]:

\[
m_\phi(y, u) = \sum_{x \in \mathbb{B}_p^\times / U_p} m_p(y, x^{-1}) \phi(x, u q(y)/q(x)).
\]

(4.24)
The sum in (4.24) only involves finitely many nonzero terms. Moreover, if $\phi(x) = 0$ for $x \in K_p \times \mathbb{Q}_p^\times$, then $m_\phi(y, u)$ extends to $B(p)^\times \times \mathbb{Q}_p^\times$ by the same formula.

By Lemma 4.4.5 (4.19), we may assume that $\Phi$ is a linear combination of functions in the form of the right hand side of (4.1) in Lemma 4.1.1. By □, then the lemma follows [37, Lemma 6.6, Proposition 8.8].

4.4.5. Matching for $p \notin S$. Assume that $p \notin S$ and nonsplit in $K$.

**Lemma 4.4.3.** For $y \in B(p)^\times$, we have

$$k_{r(1, t_1, t_2)} \phi_p(y, q(y)^{-1}) = m_{r(1, t_1, t_2)} \phi_p(y, q(y)^{-1}) \cdot \log p.$$  

**Proof.** By the descriptions of $GO(B_p)$ and $O(B_p)$ in [4.1.1] and a direct computation, we see that $\Phi_p$ is a linear combination of functions in the form of the right hand side of [4.1] in Lemma 4.1.1. By Lemma 4.4.4, we may assume that $\Phi_p = r(g, 1) \Phi_{0, p}$ for some $g$ in the standard Borel subgroup of $GL_2(\mathbb{Q}_p)$. By definition, for $a \in F^\times$, we have

$$k_\phi(ay, a^{-1}q(y)^{-1}) = k_{r(a, 1)}(y, q(y)^{-1}).$$

Then the lemma follows [37, Lemma 6.6, Proposition 8.8].

Then by (4.15) and (4.25), we have

$$W_{1}^{(2k)}(1) h(f, t_1, t_2) \log p = K_{r(1, t_1, t_2)} \phi.$$

Since $j(f) = 0$, by (4.20), we have

$$W_{1}^{(2k)}(1) H(f) \log p = -\int_{K_1 \times \mathbb{A}_K^\times} \int_{\mathbb{A}_K^\times} K_{r(1, t_1, t_2)} \phi \Omega^{-1}(t_2) \Omega(t_1) dt_2 dt_1.$$  

4.4.6. Coherence for $p \in S_{nspl}$: difference between $i$-part and the analytic kernel. Let $p \in S_{nspl}$.

**Lemma 4.4.4 ([27, Lemma 9.3.16]).** There is an open compact subgroup $U'$ of $B(p)^\times$ such that:

1. $U' \cap K_p^\times = U \cap K_p^\times$ as a subgroup of $K_p^\times$;

2. $m_p(y, 1) - \frac{\text{ord}_p(\lambda(y))}{2} \cdot 1_{U'}$ extends to a smooth function on $B(p)^\times$. Here $\lambda$ is the invariant defined in (3.4.7).

By Lemma 3.4.3 (1), Lemma 4.2.7 (2), Lemma 4.2.8 and Lemma 4.4.4 we have the following corollary. (This corollary could be regarded as an analog of the “arithmetic smooth matching” [27, Proposition 10.4.1].)

**Corollary 4.4.5.** The function $k_p(y, 1) - m_p(y, 1) \cdot \log p$ on $y \in B(p)^\times - K_p^\times$ extends to a smooth function on $B(p)^\times$. 
Combined with Corollary 4.4.2.3, we have the following corollary.

**Corollary 4.4.6.** The function $k_p(y, x) - m_p(y, x) \cdot \log p$ extends to a smooth function on $B(p)^{\times} \times K_p^{\times}$.

Let $d_p$ be such a smooth extension. Then $d_p$ is supported on the union of the supports of $k_p$ and $m_p$. For $\phi \in \mathcal{S}(\mathbb{B}_p \times \mathbb{Q}_p^{\times})$, define a smooth function $d_\phi$ on $B(p)^{\times} \times \mathbb{Q}_p^{\times}$ by

$$d_\phi(y, u) = \sum_{x \in \mathbb{B}_p^{\times} / U_p} d_p(y, x^{-1}) \phi(x, uq(y)/q(x)).$$

The sum in (4.27) only involves finitely many nonzero terms. As we can recover (part of) $k_\phi$ from $k_p$ by the following tautological formula

$$k_\phi(y, q(y)^{-1}) = \sum_{x \in \mathbb{B}_p^{\times} / U_p} k_p(y, x^{-1}) \phi(x, 1/q(x)),$$

we have

$$d_\phi(y, q(y)^{-1}) = k_\phi(y, q(y)^{-1}) - m_\phi(y, q(y)^{-1}).$$

Moreover, by Lemma 3.4.2 (1), Lemma 4.2.7, Corollary 4.4.6, and that $\Phi_p \in \mathcal{S}(\mathbb{B}_p^{\times} \times \mathbb{Q}_p^{\times})$ (see (4.22) and (4.23)), we have

$$d_{r(1, [t_1, t_2])}\Phi_p \in \mathcal{S}(B(p)^{\times} \times \mathbb{Q}_p^{\times}).$$

(6) Lemma 6.6, Lemma 8.5], we have $k_{r(1, [t_1, t_2])}\Phi_p = r(1, [t_1, t_2])k_\Phi$ and $m_{r(1, [t_1, t_2])}\Phi_p = r(1, [t_1, t_2])m_\Phi$ (see Remark 4.1.1). Thus $d_{r(1, [t_1, t_2])}\Phi_p = r(1, [t_1, t_2])d_\Phi$. Then by (4.14), (4.15), (4.20), (4.23), we have

$$W_1^{2k}(1)j(f)_p$$

$$= -\int_{K^{\times} \backslash \mathbb{A}_K^{\times} / \mathbb{A}_{\infty}^{\times}} \int_{K^{\times} \backslash \mathbb{A}_K^{\times}} K_{r(1, [t_1, t_2])}\Phi \Omega^{-1}(t_2)\Omega(t_1)dt_1dt_2$$

$$+ \int_{K^{\times} \backslash \mathbb{A}_K^{\times} / \mathbb{A}_{\infty}^{\times}} \int_{K^{\times} \backslash \mathbb{A}_K^{\times}} \text{Pr} \theta(1, [t_1, t_2], d_\Phi \otimes \Phi^p)\psi^{-1}(t_2)\Omega(t_1)dt_1dt_2.$$
Lemma 4.4.7. There exists an open compact subgroup $U'_p$ of $B(p)_p^\times$ such that $l_C$ is left $U'_p$-invariant. In particular, $l_\phi$ is left $U'_p$-invariant for all $\phi \in \mathcal{S}(\mathbb{R}_p \times \mathbb{Q}_p^\times)$.

Proof. Let $B(p)_p^\times$ act on the desingularized deformation space $\mathcal{M}'_{U_p}$ by the functoriality of the minimal desingularization. Let $U'_p$ be the stabilizer of $C$. Then the lemma follows from Lemma 3.4.3 (2). □

4.4.8. Split $p$. Let $p$ be split in $K$. Then $i(f)_p = 0$, $j(f)_p = 0$. The proof is the same as the proof in §3.4.4.

4.4.9. Local height at $\infty$. By (3.27) and a similar (and easier, since we do not need holomorphic projection,) process as in [37, 8.1] (which contains a sign mistake due to the sign mistake in the definition of the Green function in [37, 7.1.3]), we have

\[(4.31) \quad W_1^{(2k)}(1)H(f)_\infty = -\int_{K_\times \backslash \mathbb{A}_K^\times / \mathbb{A}_\infty^\times} \int_{\mathbb{A}_\infty^\times} K^{(\infty)}_{r(1,[t_1,t_2])}\phi \Omega^{-1}(t_2)\Omega(t_1)dt_2dt_1.\]

See (4.16) for the definition of $K^{(\infty)}_{\phi}$.

4.4.10. Conclusion. Define the following holomorphic cusp forms on $\text{GL}_2(\mathbb{A})$ of weight $2k$ and central character $\Omega|_{A^\times}$:

\[(4.32) \quad I(g, \Phi) = \int_{K_\times \backslash \mathbb{A}_K^\times} \text{Pr} I'(0, g, r(1, [t, 1])\Phi)\Omega(t)dt;\]

\[(4.33) \quad \theta_d,p(g, \Phi) = \int_{K_\times \backslash \mathbb{A}_K^\times / \mathbb{A}_\infty^\times} \int_{\mathbb{A}_\infty^\times} \text{Pr} \theta(g, [t_1, t_2], d\phi_p \otimes \Phi^p)\Omega^{-1}(t_2)\Omega(t_1)dt_2dt_1;\]

\[(4.34) \quad \theta_l,p(g, \Phi) := \int_{K_\times \backslash \mathbb{A}_K^\times / \mathbb{A}_\infty^\times} \int_{\mathbb{Q} \times \mathbb{A}_\infty^\times} \text{Pr} \theta(g, [t_1, t_2], l\phi_p \otimes \Phi^p)\Omega^{-1}(t_2)\Omega(t_1)dzdt_1.\]

Assume Assumption 4.4.1. Combining (4.17), (4.26), (4.28), (4.30) and (4.31), we have

\[(4.35) \quad 2W_1^{(2k)}(1)H(f) = I(1, \Phi)_{\psi} + \sum_{p \in \mathcal{S}_{\text{aspl}}} (\theta_{d,p}(1, \Phi)_{\psi} - \theta_{l,p}(1, \Phi)_{\psi}).\]

Here each term on the right hand is the value of the corresponding $\psi$-Whittaker function at $g = 1$.

Remark 4.4.8. In the current set-up, modularity of generating series of heights means a generalization of (4.35) where on the right hand side, the variable $1$ is replaced by a general $g \in \text{GL}_2(\mathbb{A})$, and the left hand side is modified accordingly. Such a generalization is not hard to prove for $g \in \text{GL}_2(\mathbb{A}_S)\text{GL}_2(\mathbb{A}^S)$, see [37, 1.5.10, 7.4.3] for the weight $2$ case. Moreover, in the weight $2$ case, such a generalization is proved in [37, 1.5.10, 7.4.3] for general $g \in \text{GL}_2(\mathbb{A})$ by using another modularity result, proved in a separate work of them [36]. In higher weights, S. Zhang [40] established such a generalization for some $\Phi$, even without Assumption 4.4.1. From his result, we deduce an analog of (4.35) for some $\Phi$, see (4.40).
4.5. **Proof of Theorem 2.3.4.** The proof of Theorem 2.3.4 is done in §4.5.5. Our strategy to prove the modularity is to show that $H(f)$ vanishes if $f$ acts as 0 on some representations, precisely Proposition 4.5.1 below. This is done by showing the vanishing of the right hand side of (4.35). The proof of this proposition is the focus of this subsection.

We continue to use the notations in §4.3. Recall that $B$ is the quaternion algebra $B = D \times M_2(A_{\infty})$ over $A$ where $D$ is the unique division quaternion algebra over $\mathbb{R}$ and $M_2$ denotes the matrix algebra over $\mathbb{Z}$. So $B_{\infty, \times} \simeq \text{GL}_2(A_{\infty})$, but we use the former consistently (unless we need to specify the isomorphism). Thus we do not confuse this $\text{GL}_2(A_{\infty})$ with the $\text{GL}_2$ used in the Weil representation in §4.3.

Also recall that $A$ is our set of cuspidal automorphic representations of $\text{GL}_2(Q)$ defined in §2.3 of weight $2k$. In view our convention on the use of the quaternion algebra $B$, we consistently consider $A$ as a set of admissible representations of $B_{\times}$ by the Jacquet-Langlands correspondence to $B_{\times}$, if $\pi \in A$, then it is simply viewed as $1_{B_{\infty}} \otimes \pi_{\infty}$ as a representation of $B_{\times}$ (under $B_{\infty, \times} \simeq \text{GL}_2(A_{\infty})$).

For an open compact subgroup $V \subset B_{\infty, \times}$, let $A^V \subset A$ be the finite subset of representations with nonzero $V$-invariant vectors.

Let $S$ be as in Theorem 2.3.4. Let $N$ be a product of two relatively prime integers which are $\geq 3$ and assume that the prime factors of $N$ are contained in $S$. Let $U = U(N)$, the corresponding principal congruence subgroup of $B_{\infty, \times} = \text{GL}_2(A_{\infty})$. Then $H^S$ is the Hecke algebra of bi-$U^S$-invariant Schwartz functions on $(B_{\infty, \times})^\times$.

Recall that $S_{\text{nspl}} \subset S$ is the subset of places nonsplit in $K$.

**Proposition 4.5.1.** There is an open compact subgroup $V^S_{\text{nspl}} \subset U_{S_{\text{nspl}}}$ such that if $f^S \in H^S$ acts as 0 on all representations in $A^{S_{\text{nspl}}} U^S_{\text{nspl}}$, then for every pure tensor $f_S = \otimes_{p \in S} f_p$ right invariant by $U_S$, we have $H(f_S \otimes f^S) = 0$.

We will prove Proposition 4.5.1 in §4.5.4. We will first prove it under Assumption 4.4.1 using the comparison (4.35) in §4.5.3 and then remove the assumption. Before the proof, we need some preparations on the analytic kernels (i.e. the right hand side) in (4.35). Below we always let $f = f_S \otimes f^S$. Assume that $f_S = \otimes_{p \in S} f_p$ and $f^S \in H^S$. The right invariance of $f_S$ will be imposed in §4.5.3.

### 4.5.1. Integral representation of L-function.

We study the first term $I(g, \Phi)$ on the right hand side of the comparison (4.35), using the fact that it is a kernel function for integral representations of the twisted base change L-functions. The main results are Lemma 4.5.3-4.5.5.

Let $\Phi_0^{\infty}$ be a Schwartz function on $B_{\infty, \times} A_{\infty, \times}^{\times}$ invariant by $r(1, [1, h])$ for $h \in U$. (Recall that $r$ is the Weil representation and $[1, h]$ is defined in (4.21).) Let

$$\Phi_0 = \bigotimes_{p \in S} \Phi_{0, p} \bigotimes \Phi_0^{\infty} \bigotimes \Phi_{\infty},$$

where $\Phi_\infty$ is the Gaussian as in (4.13) (generalizing (4.21)). Let

$$\Phi = r(f) \Phi_0.$$

Here $r(f)$ is defined in (4.18). Then $\Phi$ is still invariant by $r(1, [1, h])$ for $h \in U$.

Let $I(g, \Phi)$ be the holomorphic cusp forms on $\text{GL}_2(A)$ of weight $2k$ and central character $\Omega|_{A^\times}$ as in (4.32) (with the current $\Phi$ as the input). Let $\sigma$ be a cuspidal automorphic representation of $\text{GL}_{2, \mathbb{Q}}$ and $\phi \in \sigma$. Consider the Petersson pairing (not taking complex conjugate of the second...
term) $\langle I(g, \Phi), \phi \rangle$ of $I(g, \Phi)$ and $\phi$. We assume that the central character of $\sigma$ is $\Omega^{-1}|_{A_\infty}$ and $\sigma_\infty$ is holomorphic discrete of weight $2k$. Otherwise, $\langle I(g, \Phi), \phi \rangle = 0$. Let $\pi$ be the Jacquet-langlands correspondence of $\sigma$ to $\mathbb{B}$.

Let $W$ be the $\psi^{-1}$-Whittaker function of $\phi$. Without loss of generality, assume that $W$ is a pure tensor. Also assume that $\Phi$ is a pure tensor. For a place $v$ of $\mathbb{Q}$, define a local integral

$$P_v(s, \Omega_v, \Phi_v, W_v) = \int_{K_\infty^v/Q_v^\infty} \Omega_v(t)dt \int_{N_v/GL_2(Q_v)} \delta(g)^s W_v(g)r((g, 1))\Phi_v(t^{-1}, q(t))dg.$$ 

Here $N_v$ is the upper triangular unipotent subgroup and $\delta$ the standard modular character $[37, 1.6.6]$. Let

$$P_v^\circ(s, \Omega_v, \Phi_v, W_v) = \frac{L((s+1)/2, \pi_{\kappa} \otimes \Omega_v)}{L(s+1, \eta_v)}P_v(s, \Omega_v, \Phi_v, W_v),$$

where $\eta$ is the Hecke character of $\mathbb{Q}_\infty$ associated to $K$ by the class field theory. Then (up to normalizing the measures)

$$P_v^\circ(0, \Omega_v, \Phi_v, W_v) = \alpha_{\kappa_v}^2(\Theta_v(\Phi_v \otimes W_v)),$$

where $\alpha_{\kappa_v}^2$ is defined in [24.41] (and $\alpha_{\kappa_\infty}^2$ is defined similarly), and $\Theta_v(\Phi_v \otimes W_v) \in \pi_v \otimes \pi_v$ is the normalized local Shimizu lifting $[37, 2.2.2]$. Let $\Theta = \prod_v \Theta_v$ where the product is over all places of $\mathbb{Q}$.

By $[37, 2.3]$, (up to normalization of the measures) we have

$$\langle I(\cdot, \Phi), \phi \rangle = \frac{d}{ds}|_{s=0} \left( \prod_v P_v(s, \Omega_v, \Phi_v, W_v) \right),$$

where the product is over all places of $\mathbb{Q}$.

If $L(1/2, \pi_K \otimes \Omega) = 0$, then

$$\langle I(\cdot, \Phi), \phi \rangle = \frac{L'(1/2, \pi_K \otimes \Omega)}{L(1, \eta)} \alpha_{\kappa}^2(\Theta(\Phi \otimes W)).$$

Assume that $L(1/2, \pi_K \otimes \Omega) \neq 0$. Then $\epsilon(1/2, \pi_K \otimes \Omega) = 1$. Since $\epsilon(1/2, \pi_{\kappa_\infty} \otimes \Omega_\infty) \Omega_\infty(-1) = -1$ (indeed, $\Omega_\infty$ is trivial), there exists $p \in S$ (necessary nonsplit in $K$) such that $\epsilon(1/2, \pi_{\kappa_p} \otimes \Omega_p)\Omega_p(-1) = -1$. Let $\pi'_{p}$ be the Jacquet-langlands correspondence of $\sigma_p$ to $B(p)^\times$. By the theorem of Tunnell $[31]$ and Saito $[29]$, $\text{Hom}_{K_p}(\pi_p \otimes \Omega_p, \mathbb{C}) = 0$ and $\text{Hom}_{K_p}(\pi'_p \otimes \Omega_p, \mathbb{C}) \neq 0$. In particular, $\alpha^2_{\pi_p}(\Theta_p(\Phi_p \otimes W_p)) = 0$ and $\pi'_p \neq \{0\}$. From $\alpha_{\pi_p}(\Theta_p(\Phi_p \otimes W_p)) = 0$, we have

$$\langle I(\cdot, \Phi), \phi \rangle = \frac{L'(1/2, \pi_K \otimes \Omega)}{L(1, \eta)} (P_p^p)'(0, \Omega_p, \Phi_p, W_p)\alpha_{\pi_p}^2(\Theta(\Phi^p \otimes W_p)).$$

We need to control $(P_p^p)'(0, \Omega_p, \Phi_p, W_p)$ using $\pi'_p$, as follows.

**Lemma 4.5.2.** Assume that $p$ is nonsplit in $K$ such that $\epsilon(1/2, \pi_{\kappa_p} \otimes \Omega_p)\Omega_p(-1) = -1$. There is an open compact subgroup $U'_p$ of $B(p)^\times$ such that $(P_p^p)'(0, \Omega_p, \Phi_p, W_p) = 0$ unless $\pi'_p$ has a nonzero $U'_p$-invariant vector. Here $U'_p$ is uniform for all such $\pi_p$ and all $f_p \in S(\text{GL}_2(\mathbb{Q}_p))$. (Note that $\Phi_p = r(f_p)\Phi_{0,p}$, see (4.36).)

We will prove Lemma 4.5.2 in §4.7.
Lemma 4.5.3. There is an open compact subgroup $V_{S_{\text{nspl}}} \subset U_{S_{\text{nspl}}}$ such that if $f^S \in \mathcal{H}^S$ acts as 0 on all representations in $\mathcal{A}^{V_{S_{\text{nspl}}} U_{S_{\text{nspl}}}}$, then $I(g, \Phi) = 0$.

Proof. We show that there exists $V_{S_{\text{nspl}}}$ such that for every $f^S$ as in the lemma and every cusp form $\phi$ of $\text{GL}_2 \otimes \text{GL}_2$, $\langle I(g, \Phi), \phi \rangle = 0$. Then the lemma follows. We may and we do assume that $\phi \in \sigma$ as above. Let $\pi$ and $W$ be as above.

Since $\Theta$ is $\mathbb{B}^\times \times \mathbb{B}^\times$-equivariant [22, 2.2.2] (we only use the second $\mathbb{B}^\times$ here) and $\Phi$ is right $U$-invariant, $\Theta(\Phi \otimes W) \in \pi \mathbb{B} \tilde{\pi}$ is $U$-invariant, where $U$ acts on $\tilde{\pi}$. By (4.37) and (4.38), a necessary condition for $\langle I(\cdot, \Phi), \phi \rangle$ to be nonzero is:

1. if $L(1/2, \pi_K \otimes \Omega) = 0$, then $\pi \in \mathcal{A}^U$;
2. if $L(1/2, \pi_K \otimes \Omega) \neq 0$, then there exists $p \in S_{\text{nspl}}$ such that the Jacquet-Langlands correspondence of $\sigma$ to $B(p)^\times$ has a nonzero $U_p' U_p^\times$-invariant vector, where $U_p'$ is as in Lemma 4.5.2. There are only finitely many such $\sigma$ (independent of $\pi$). Let $V_p$ an open compact subgroup of $U_p$ such that the Jacquet-Langlands correspondence of $\sigma_p$, for every such $\sigma$, to $\mathbb{B}_p^\times$ has a nonzero $V_p$-invariant vector. Thus $\pi \in \mathcal{A}^{V_{S_{\text{nspl}}} U_{S_{\text{nspl}}}}$.

Let $f^S$ act as 0 on $\pi$, we only need to show that $\langle I(\cdot, \Phi), \phi \rangle = 0$. Note that by the $\mathbb{B}^\times \times \mathbb{B}^\times$-equivariance of $\Theta$ (we only use the first $\mathbb{B}^\times$ here), we have

$$\Theta^{S, \infty}(\Phi^{S, \infty} \otimes W^{S, \infty}) = \pi(f^S) \Theta^{S, \infty}(\Phi_0^{S, \infty} \otimes W^{S, \infty}).$$

Then $\langle I(\cdot, \Phi), \phi \rangle = 0$ by (4.37) and (4.38). □

Lemma 4.5.4. Assume that $\Phi_0^{S, \infty}$ is invariant by the standard maximal compact subgroups of $\text{GL}_2(\mathbb{A}^{S, \infty})$ and $(\mathbb{B}^{S, \infty})^\times$, where $g \in \text{GL}_2(\mathbb{A}^{S, \infty})$ acts by $r(g, 1)$ and $h \in (\mathbb{B}^{S, \infty})^\times$ acts by $r(1, [h, 1])$. We have $I(g, \Phi) = r_{GL_2}(f^S) I(g, \Phi_0)$. Here $r_{GL_2}(f^S)$ is the usual Hecke action of $f^S$, regarded as a Schwartz function on $\text{GL}_2(\mathbb{A}^{S, \infty})$ via $(\mathbb{B}^{S, \infty})^\times = \text{GL}_2(\mathbb{A}^{S, \infty})$.

Proof. It is enough to show that for every cusp form $\phi$ of $\text{GL}_2 \otimes \text{GL}_2$, we have

$$\langle I(g, \Phi), \phi \rangle = \langle I(g, \Phi_0), r_{GL_2}(f^S) \phi \rangle$$

Again, we assume that $\phi \in \sigma$ as above, and let $\pi$ and $W$ be as above. By (4.37) and (4.38), the lemma follows from the following equations for $p \notin S$:

$$\Theta_p(\Phi_p \otimes W_p) = \pi_p(f_p) \Theta_p(\Phi_{0,p} \otimes W_p) = \Theta_p(\Phi_{0,p} \otimes \sigma_p(f_p) W_p),$$

where the first equation is the $\mathbb{B}_p^\times \times \mathbb{B}_p^\times$-equivariance of $\Theta_p$, and the second equation is the unramified Shimizu lifting. □

Define a “special value” version of $I(g, \Phi)$ by

$$J(g, \Phi) = \int_{K^\times \backslash \mathbb{A}_{K, K}^\times} \Pr I(0, g, r(1, [t, 1]) \Phi) \Omega(t) dt.$$

Lemma 4.5.5. In Lemma 4.5.3 and Lemma 4.5.4 replacing $I(g, \Phi)$ by $J(g, \Phi)$, the same results hold.

Proof. The proofs are similar to the proofs of Lemma 4.5.3 and Lemma 4.5.4. It is in fact easier since we only need to use $\langle J(\cdot, \Phi), \phi \rangle = \frac{L(1/2, \pi_K \otimes \Omega)}{L(1, \pi)} \alpha_q^* (\Theta(\Phi \otimes W))$ instead of (4.37) and (4.38). □
4.5.2. Invariance of \(d_\phi\). We study the second term \(\theta_d, p(g, \Phi)_\psi\) on the right hand side of the comparison (1.35). The main result is Corollary 4.5.8 about its local test function \(d_\phi\) (see (1.27)). The same result for the local test function \(l_\phi\) of the third term \(\theta_l, p(g, \Phi)_\psi\) was already done in Lemma 4.4.7.

Let \(\lambda\) be the invariant defined in §3.4.1. We abuse notation and use \(\lambda\) to denote this invariant on both \(\mathbb{B}_p\) and \(B(p)_p\).

**Lemma 4.5.6.** Let \(V\) be an open compact neighborhood of \(0\) in \(\mathbb{Q}_p\). For \(V\) large enough, there exists an open compact subgroup \(U'_1\) of \(B(p)^\times_p\) such that for all \(x \in \mathbb{B}_p^\times\) with \(\lambda(x) \notin V\), \(m_p(y, x)\) is left \(U'_p\)-invariant as a function on \(y\).

**Proof.** We need some notations defined in §3.4.5. Let \(\mathcal{M}_{U_p}\) be the base change to \(\mathcal{O}_{F_p}\) of the supersingular formal deformation space of level \(U_p\). (This notation is slightly different from 3.4.3 for convenience.) Let \(\mathcal{M}'_{U_p}\) be the minimal desingularization of \(\mathcal{M}_{U_p}\). For \((y, x) \in B(p)^\times \times K_p^\times \mathbb{B}_p\), let \((y, x)'\) denote its image in \(\mathcal{M}_{U_p}\) and let \((y, x)''\) denote its image in \(\mathcal{M}'_{U_p}\). Then \(m_p(y, x)\) is the intersection number of \((y, x)''\) and \((1, 1)''\) on \(\mathcal{M}'_{U_p}\), and we let \(\mu_p(y, x)\) be the intersection number of \((y, x)''\) and \((1, 1)''\) on \(\mathcal{M}'_{U_p}\).

We have the following explicit formula of \(\mu_p(y, x)\) [21, Theorem 1.3]:

\[
\mu_p(y, x) = c \int_{g \in U_p x} |(1 - \lambda(y)) - (1 - \lambda(g))|_p dg,
\]

where \(c\) is a nonzero constant independent of \(y, g\). Recall \(1 - \lambda(y) = \frac{1}{1 - \text{inv}(y)}\) (see §3.4.1) where \(\text{inv}(y) \in \epsilon \text{Nm}(K_p^\times) \cup \{0, \infty\}\) with \(\epsilon \notin \text{Nm}(K_p^\times)\) (see §3.1.2). Since \(|1 - \text{inv}(y)|_p \geq 1, |1 - \lambda(y)|_p \leq 1\), thus \(\mu_p(y, x)\) does not depend on \(y\) for \(\lambda(x)\) large enough.

By the projection formula, \(m_p(y, x) - \mu_p(y, x)\) is the intersection number of \((y, x)''\) in \(\mathcal{M}'_{U_p}\) with the vertical part \(C\) of the preimage of \((1, 1)''\) in \(\mathcal{M}_{U_p}\) by the morphism \(\mathcal{M}'_{U_p} \to \mathcal{M}_{U_p}\). Let \(U'_p\) be the stabilizer of \(C\). Then the lemma follows from Lemma 3.4.3 (2). \(\square\)

**Remark 4.5.7.** The condition that \(\mathbb{B}_p\) is a matrix algebra is essential in Lemma 4.5.6. If \(\mathbb{B}_p\) is a division algebra, an explicit formula of the corresponding \(\mu_p(y, x)\) is not known yet.

**Corollary 4.5.8.** For \(U_p\) small enough, there exists an open compact subgroup \(U'_p\) of \(B(p)^\times_p\) such that \(d_p = k_p - m_p\) is left \(U'_p\)-invariant. In particular, \(d_\phi\) is left \(U'_p\)-invariant for all \(\phi \in \mathcal{S}(\mathbb{B}_p \times \mathbb{Q}_p^\times)\).

**Proof.** Let \(V\) be an open compact neighborhood of \(0\) in \(\mathbb{Q}_p\) such that both Lemma 4.2.10 and Lemma 4.5.6 hold. We only need to show that for \(x \in \lambda^{-1}(V) \subset \mathbb{B}_p^\times\), \(d_p(y, x)\) is left \(U'_p\)-invariant for some \(U'_p\). By Lemma 3.4.2 (1), Lemma 4.2.7 and Corollary 4.4.6 we only need to show that

\[
\{(y, x) \in B(p)^\times \times K_p^\times \mathbb{B}_p^\times : \lambda(x) \in V, q(y)g(x) \in q(U_p)\}
\]

is compact. Up to the \(K_p^\times\)-action, we may assume that \(y\) is in a fixed compact subset of \(B(p)^\times_p\) (here we use that \(B(p)_p\) is a division algebra), so that \(q(x)\) is in a fixed compact subset \(U_1 \subset \mathbb{Z}_p^\times\). Write \(x = a + bj\) as in §3.1.2. Then \(b\) is in a fixed compact subset \(U_2 \subset K^\times\) by \(\lambda(x) \in V\). By \(q(x) \in U_1, a\) is in a fixed compact subset \(U_3 \subset K\). Let \(U_3 = U_{3,1} \cup U_{3,2}\) where \(U_{3,1}\) is a compact subset of \(K_p^\times\) and \(U_{3,2}\) is a small compact neighborhood of \(0\). If \(a \in U_{3,1}\), \(a + bj \in U_{3,1}(1 + U_{3,1}U_{3,2})\), which is compact. If \(a \in U_{3,1}\), then by \(q(x) \in U_1, b\) is in a compact subset \(U_{2,1} \subset K_p^\times\). Then \(a + bj \in U_{2,1}(U_{2,1}U_{3} + j)\), which is compact. \(\square\)
4.5.3. Proposition 4.5.1 under Assumption 4.4.1. Still, let $\Phi = r(f)\Phi_0$ be as in (4.36). Assume that $f_S = \otimes_p S f_p$ is right invariant by $U_S$. For $p \in S_{\text{nspl}}$, let the holomorphic cusp forms $\theta_{d,p}(g, \Phi)$ and $\theta_{l,p}(g, \Phi)$ of $GL_2, \mathbb{Q}$ on $GL_2(\mathbb{A})$ of weight $2k$ and central character $\Omega_{\mathbb{A}}^\times$ be as in (4.33) and (4.34).

We have the following analog of Lemma 4.5.3.

Lemma 4.5.9. There is an open compact subgroup $V_{\text{nspl}} \subset U_{\text{nspl}}$ such that if $f_S \in \mathcal{H}^S$ acts as 0 on all representations in $A^{V_{\text{nspl}}} U_{\text{nspl}}^{\Phi}$, then for every $f_S$ satisfying Assumption 4.4.1, $\theta_{d,p}(g, \Phi) = \theta_{l,p}(g, \Phi) = 0$ for all $p \in S_{\text{nspl}}$.

Proof. Let $U_p' \subset B(p)^\times$ be such that both Lemma 4.5.1 and Corollary 4.5.8 hold. There are only finitely many automorphic representations $\pi$ of $B(p)^\times$ with nonzero $U_p' U_p^\times$-invariant vectors such that the central character of $\pi$ is $\Omega^{-1}|_{\mathbb{A}}^\times$ and the Jacquet-Langlands correspondence of $\pi_\infty$ to $GL_2(\mathbb{R})$ is holomorphic discrete of weight 2k. Choose $V_p \subset U_p$ such that for every such $\pi$, the Jacquet-Langlands correspondence of $\pi_p$ to $\mathbb{B}_p^\times$ has nonzero $V_p$-invariant vectors. Let $V_{\text{nspl}} = \prod_{p \in S_{\text{nspl}}} V_p$.

We show that for every $f = f_S \otimes f_S$ as in the lemma and every cusp form $\phi$ of $GL_2, \mathbb{Q}$, the Petersson pairings $\langle \theta_{d,p}(g, \Phi), \phi \rangle$ and $\langle \theta_{l,p}(g, \Phi), \phi \rangle$ are 0, where $p \in S_{\text{nspl}}$. Let $\phi$ be in a cuspidal automorphic representation $\sigma$. We assume that the central character of $\sigma$ is $\Omega^{-1}|_{\mathbb{A}}^\times$ and $\sigma_\infty$ is holomorphic discrete of weight 2k. (Otherwise, $\langle \theta_{d,p}(g, \Phi), \phi \rangle = \langle \theta_{l,p}(g, \Phi), \phi \rangle = 0$ already holds.) Consider $\langle \text{Pr} \theta(\cdot, h, d_{\Phi_p} \otimes \Phi_p^0), \phi \rangle$ as a function on $h \in (B(p)^\times \times B(p)^\times)(\mathbb{A})$. Let us show that

$$\langle \text{Pr} \theta(\cdot, h, d_{\Phi_p} \otimes \Phi_p^0), \phi \rangle = 0.$$

Then $\langle \theta_{d,p}(g, \Phi), \phi \rangle = 0$ by definition.

Note that $\langle \text{Pr} \theta(\cdot, h, d_{\Phi_p} \otimes \Phi_p^0), \phi \rangle$ lies in the global Shimizu lifting $\pi' \otimes \pi'$ of $\sigma$ [37, 2.2]. Here $\pi'$ is the Jacquet-Langlands correspondence of $\sigma$ to $B(p)^\times$. By the $(B(p)^\times \times B(p)^\times)(\mathbb{A})$-equivariance of the formation of the theta series, $\langle \text{Pr} \theta(\cdot, h, d_{\Phi_p} \otimes \Phi_p^0), \phi \rangle = 0$. The lemma is proved.

Corollary 4.5.10. Proposition 4.3.1 holds under Assumption 4.4.1 on $f_S$.

4.5.4. Proof of Proposition 4.5.1. For a positive integer $N_0$ with prime factors in $S$, let $U_0(N_0)$ be the open compact subgroup of $\mathbb{B}^{\times, \times} = GL_2(\mathbb{A}^\infty)$ corresponding to the usual congruence subgroup $\Gamma_0(N_0)$ of $SL_2(\mathbb{Z})$.

Theorem 4.5.11. Assume that $N_0$ is the product of two relatively prime integers which are $\geq 3$ whose prime factors are all split in $K$. Then Proposition 4.5.1 holds for $f_S = U_0(N_0)^S$.

Proof. The theorem follows from a result of S. Zhang [40, Theorem 0.2.1], explained as follows.
4.5.5. Proof of Theorem 2.3.4.

Proof. Indeed, span \((\sigma, K)\) and \(S\). Fourier coefficients and Hecke eigenvalues. On \(r\), appropriately, see [9, p. 271]). Regard \(A\) as \([4.40]\) and Lemma 4.5.5, we have \(a, b\) are constants independent of \(a\), \(b\) are multiples of \(L(1, \eta^\infty)\) and \(L'(1, \eta^\infty)\) respectively, see [9, p. 271]). Regard \(A\) as a Schwartz function on \(GL_2(\mathbb{A}^S_\infty)\) via \((\mathbb{A}^S_\infty)^{\times} = GL_2(\mathbb{A}^S_\infty)\). Let \(\rho_{GL_2}\) denote the usual Hecke action by Schwartz functions. Then

\[
I(1, \Phi_N) \psi_n = (\rho_{GL_2}(A_n) I(1, \Phi_N)) \psi,
\]

and

\[
J(1, \Phi_N) \psi_n = (\rho_{GL_2}(A_n) J(1, \Phi_N)) \psi.
\]

Indeed, span \(I(g, \Phi_N)\) (resp. \(J(g, \Phi_N)\)) as a linear combination of Hecke eigenforms (for all Hecke operators \(A_n\)’s). Then the corresponding equation follows from the classical relation between Fourier coefficients and Hecke eigenvalues. On \(S_{2k}(N_0)\) (as \(\rho_{GL_2}(A_n) = \rho_{GL_2}(A_n')\)). Then by Lemma 4.5.4 and Lemma 4.5.5 we have

\[
(4.40) \quad H(1_{U_0(N)_S} \otimes A_n) = a \cdot I(1, \Phi) + b \cdot J(1, \Phi),
\]

where \(\Phi = r(1_{U_0(N)_S} \otimes A_n) \Phi_0\), see [4.36]. As \(A_n\)’s span \(\mathcal{H}^S\) modulo the center of \(GL_2\), the theorem is implied by Lemma 4.5.3 and Lemma 4.5.5.

\[\Box\]

Lemma 4.5.12. Functions satisfying Assumption 4.4.1 and left translations of \(1_{U_0(N)_S}\) by \(\mathbb{A}_{K,S}^\times\) span \(S(\mathbb{B}_S^\times)\).

Proof. Let \(W \subset S(\mathbb{B}_S^\times)\) be the span. Let \(p \in S\). Then \(\{f_p \in S(\mathbb{B}_p^\times - K_p^\times)\}\) and left translations of \(1_{U_0(N)_p}\) by \(K_p^\times\) span \(S(\mathbb{B}_p^\times)\). Thus \(W\) contains \(S(\mathbb{B}_p^\times) \otimes \{1_{U_0(N)_S} - (p)\}\) and its left translations by \(\mathbb{A}_{K,S}^\times\). Thus the lemma is reduced to the same statement for \(S - \{p\}\). So the lemma is proved by induction.

\[\Box\]

Note that the truth of \(H(f_S \otimes f^S) = 0\) and the right invariance of \(f_S\) do not change if we left translate \(f_S\) by \(\mathbb{A}_{K,S}^\times\). Thus Proposition 4.5.1 follows from Corollary 4.5.10 and Theorem 4.5.11.

4.5.5. Proof of Theorem 2.3.4. We shall prove Theorem 2.3.4 for \(\overline{CM(\pi)}\), for which our previous set-up is more handy to use.

For \(f_S \in GL_2(\mathbb{A}_S)\), define a subspace of \(CM(\Omega^{-1}, \mathcal{H}^S)\) by

\[
CM(\Omega^{-1}, f_S \mathcal{H}^S) = \{Z_{\Omega^{-1}, f_S \mathcal{H}^S} : f_S \in \mathcal{H}^S\}.
\]

Let \(\overline{CM(\Omega^{-1}, f_S \mathcal{H}^S)}\) be the image of \(CM(\Omega^{-1}, f_S \mathcal{H}^S)\) in \(\overline{CM(\Omega^{-1})}\). Recall that by the strong multiplicity one theorem [26], \(L_\pi\)'s are pairwise non-isomorphic. Claim: there is an open compact subgroup \(V_S \subset GL_2(\mathbb{A}_S)\) such that \(\overline{CM(\Omega^{-1}, f_S \mathcal{H}^S)}\), as an \(\mathcal{H}^S\)-module, is a direct sum of \(L_\pi\)'s of multiplicity at most one, where \(\pi \in A^{V_S U} S\). Then Theorem 2.3.4 follows.
For the claim, we want to show that the $\mathcal{H}^S$ action on $\overline{CM}(\Omega^{-1}, f_S \mathcal{H}^S)$ factors through its action on a direct sum of $L_{ns}$’s. By (2.10) and (2.14), it is enough to show that if $f^S$ acts on $\pi$ as 0 for all $\pi \in \mathcal{A}^{V_S U^S}$, then $H(f_0 * (f^S_2 \otimes f^S)) = 0$ for all $f_0 \in S(\mathbb{R}^{\infty, \times})$. This is Proposition 4.5.1.

4.6. Proofs of Theorem 2.3.7 and Theorem 2.3.9 Similar to (4.5), our strategy to prove the modularity is to show that $H(f)$ vanishes under suitable conditions, precisely Proposition 4.6.1 below. This is again done by showing the vanishing of the right hand side of (4.35).

**Proposition 4.6.1.** Let $f_S \in S(GL_2(\mathbb{A}_S))$ be right $U_S$-invariant such that Assumption 4.4.1 holds. Let $V_{nspl} \subset U_{nspl}$ be as in Lemma 4.5.3 and Lemma 4.5.9. Assume that $f^S \in \mathcal{H}^S$ acts as 0 on all representations in $\mathcal{A}^{V_{nspl} U^{nspl}}$ other than $\pi$. Then $H(f_S \otimes f^S) = 0$ under condition (1) or (2):

1. $L(1/2, \pi_K \otimes \Omega) = 0$ and $f_S$ acts as 0 on $\pi$;
2. $\epsilon(1/2, \pi_K \otimes \Omega, \Omega_v(-1)) = -1$ for more than one $v < \infty$.

We use the proofs of Lemma 4.5.3 and Lemma 4.5.9. The new ingredient is to apply the Waldspurger formula under the condition $L(1/2, \pi_K \otimes \Omega) = 0$.

**Proof.** Let $\Phi = r(f_S \otimes f^S) \Phi_0$ be as in (4.38). By (4.35), we only need to show that for every cusp form $\phi$ of $GL_2(\mathbb{Q})$, the Petersson pairings $(I(g, \Phi), g, (\theta, p, (g, \Phi), \phi)) = 0$ unless $\pi^{S, \infty} = \pi^{S, \infty}$ as representations of $B(p)^\times(\mathbb{A}^{S, \infty}) \simeq (\mathbb{B}^{S, \infty})^\times$. Assume this is the case. Then $\pi$ is the Jacquet-Langlands correspondence of $\sigma$ to $\mathbb{B}^\times$. In particular, $L(1/2, \pi_K \otimes \Omega) = L(1/2, \pi_K \otimes \Omega)$ and $\pi^p = \pi^p$ as representations of $B(p)^\times(\mathbb{A}^{S, \infty}) \simeq (\mathbb{B}^{S, \infty})^\times$. So under condition (1), as $L(1/2, \pi_K \otimes \Omega) = 0$, by the Waldspurger formula [37, 1.4.2] (or more directly, the fourth equation on [37, p 44]), the claim follows. Now assume condition (2), $\epsilon(1/2, \pi_K \otimes \Omega_v, \Omega_v(-1)) = -1$ for more than one $v < \infty$. (Such $v$ is necessary in $S_{nspl}$.) In particular, there is $v \in S_{nspl} - \{p\}$ such that $\epsilon(1/2, \pi_K \otimes \Omega_v, \Omega_v(-1)) = -1$. By the theorem of Tunnell [31] and Saito [29], $\text{Hom}_{K_v}(\pi_v \otimes \Omega_v, \mathbb{C}) = 0$. Recall that as a function on $h \in (B(p)^\times \times B(p)^\times)(\mathbb{A})$, $(\text{Pr} \theta(h, d_{\Phi_p} \otimes \Phi_p^p), \phi)$ lies in the global Shimizu lifting $\pi' \boxtimes \pi'$ of $\sigma$ [37, 2.2]. So by the definition (4.35) of $\theta_{d,p}(g, \Phi)$, $\langle \theta_{d,p}(g, \Phi), \phi \rangle = 0$. Similarly, $\langle \theta_{l,p}(g, \Phi), \phi \rangle = 0$. Thus the claim follows.

Now, we prove that $\langle I(g, \Phi), \phi \rangle = 0$. By the reasoning in the proof of Lemma 4.5.3 and the assumption on $f^S$, the Petersson pairing $(I(g, \Phi), \phi)$ is 0 unless the Jacquet-langlands correspondence of $\sigma$ to $\mathbb{B}$ is $\pi$. Assume (1) so that we are in the case of (4.37). Then $I(g, \Phi) = 0$ by the assumption on $f_S$. Assume (2). There is $v \in S_{nspl} - \{p\}$ such that $\epsilon(1/2, \pi_K \otimes \Omega_v, \Omega_v(-1)) = -1$. Then both (4.37) and (4.38) are 0 by the theorem of Tunnell [31] and Saito [29].

**Remark 4.6.2.** In condition (1), the part for $f_S$ is only used in proving that $\langle I(g, \Phi), \phi \rangle = 0$.

We shall prove Theorem 2.3.7 and Theorem 2.3.9 for $\overline{CM}(\pi)$, for which our previous set-up is more handy to use.

Let $T$ be a finite set of finite places of $\mathbb{Q}$ as in Theorem 2.3.4. (We save the notation $S$ for later use.) For $f \in S(GL_2(\mathbb{A}_T)) \otimes \mathcal{H}^T$, let $z_{\Omega^{-1}}f$ be the image of $Z_{\Omega^{-1}}f$ in $\overline{CM}(\Omega^{-1})$. For $\pi \in \mathcal{A}^{U^T}$, let
$z_{\Omega^{-1},f,\pi}$ be the $L_{\pi}T$-component of $z_{\Omega^{-1},f}$ via Theorem 2.3.4. Then the conclusion in Theorem 2.3.7 is then that the $H^T$ action on $\overline{CM}(\pi)$ factors through its action on $\pi_T$. By (2.10), Theorem 2.3.7 is equivalent to (1) of the following theorem. Similarly, Theorem 2.3.9 is equivalent to (2) of the following theorem.

**Theorem 4.6.3.** (1) Assume that $L(1/2, \pi_K \otimes \Omega) = 0$. If $f_T \in \text{GL}_2(k_T)$ acts as 0 on $\pi_T$, then $z_{\Omega^{-1},f,\pi_T,\pi}$ is defined as in translation by the generating series of Hecke operators (which is an automorphic form!), and the $cZ$ between automorphic forms of $\text{GL}_2$ we do not require the degeneracy condition [37, Assumption 5.3]. However, the analog of (4.28) in local and purely analytic, we prove it via a global method and use the work of Yuan, S. Zhang and W. Zhang [37] essentially. We expect that there should be a local proof of Lemma 4.5.2.

**Proposition 4.6.1.** From the weight 2 case.

**Proof.** By definition (see the beginning 2.4.2), $z_{\Omega^{-1},f,\pi_T,\pi}$ is given as follows. Let $A_1$ be a finite subset of $A$ containing $\pi$ such that for all $f_T \in H^T$, act as identity on $\pi_1$ if $\pi_1 = \pi$, and as 0 otherwise. Then $z_{\Omega^{-1},f,\pi_T,\pi} = z_{\Omega^{-1},(f,\pi),\pi_T}$. We want choose suitable $A_1$ and $f_T$ so that we can apply Proposition 4.6.1 to prove the theorem. Let $p_0 \not\in T$ be a finite place of $Q$ inert in $K$ such that $f_{p_0} = 1_{\text{GL}_2(Q)}$. Let $S = T \cup \{p_0\}$ and let $A_1$ be $A_{\text{Sap}}^V U_{\text{Sap}}^S$ in Proposition 4.6.1. Let $f^S_T \in H^S$ act as identity on $\pi_1$ if $\pi_1 = \pi$, and as 0 otherwise (the same requirement as $f^T$, replacing $T$ by $S$). Let $f_{1,p_0} \in H_{p_0}$, the unramified Hecke algebra at $p_0$, vanish on $K_{p_0}^\infty$, and act as identity on $\pi$ so that Assumption 4.4.1 holds. (The existence of such $f_{1,p_0}$ is easy to prove.) Let $f^T_T = f_{1,p_0} \otimes f^S_T$. By (2.11), the theorem follows from Proposition 4.6.1.

**4.7. Aid from the weight 2 case.** We prove Lemma 4.5.2 in this subsection. Though it is purely local and purely analytic, we prove it via a global method and use the work of Yuan, S. Zhang and W. Zhang [37] essentially. We expect that there should be a local proof of Lemma 4.5.2.

First, we recall [37 (1.5.6)] (see also [37 7.4.3], the second equation on page 227). Let $B$ be as in 4.4.1. Let $\Phi^{(2)} = \Phi^{(2)}_p \otimes \Phi^{(2)}_\infty \in S(B^\times)$ such that

1. $\Phi^{(2)}_p = \Phi_p$ where $\Phi_p$ is the one in Lemma 4.5.2.
2. $\Phi^{(2)}_\infty$ is the standard Gaussian used in [37 4.1.1] (same with our definition 4.11) if we let $k = 1$ in (4.11);
3. $\Phi^{(2),p,\infty}$ satisfies the degeneracy conditions in [37 5.2] such that $\alpha^{s}_{g(2),p}(\Theta^p(\Phi^{(2),p} \otimes W^{(2),p})) \neq 0$.

The existence of such $W^{(2)}_p$ and $\Phi^{(2),p,\infty}$ is established in [37 Proposition 5.8]. Note that at $p$, we do not require the degeneracy condition [37 Assumption 5.3]. However, the analog of (4.28) in this setting still holds by the same proof. Then [37 (1.5.6)] still holds, which is an equation between automorphic forms of $\text{GL}_2(Q)$ of weight 2:

\begin{equation}
(4.41) \quad cZ(g, \Omega, \Phi^{(2)}) = I(g, \Phi^{(2)}) + \sum_{v \in S_{\text{Sap}}} (\theta_{d,v}(g, \Phi^{(2)}) - \theta_{l,v}(g, \Phi^{(2)})).
\end{equation}

Here $c$ is a nonzero constant, $Z(g, \Omega, \Phi^{(2)})$ is the height pairing between a CM divisor and its translation by the generating series of Hecke operators (which is an automorphic form!), and the terms on the right hand side are defined as in 4.4.10 with holomorphic projection of weight 2.
Remark 4.7.1. (1) For $\theta_{d,v}(g, \Phi^{(2)})_\psi$ and $\theta_{l,v}(g, \Phi^{(2)})_\psi$, the holomorphic projection in (4.410) is redundant.

(2) To be precise, in the definition of $\theta_{l,v}(g, \Phi^{(2)})_\psi$, the $l_{\Phi_v}$ is the one in [37 YZZ]. It is formed in the same way as the one in (4.19), with a different vertical divisor of the desingularized deformation space $M'_{U_v}$. So it has the same properties with the one in (4.29). (Note that we do not need to know the vertical divisor explicitly.)

Let the notations be as in Lemma [14.5.2]. Then

- $B(p)$ is the quaternion algebra over $\mathbb{Q}$ only division at $v = p$ and $v = \infty$;
- $\pi_p^{(2)}$ is an irreducible admissible representation of $B(p)^\ast$ such that $\text{Hom}_{K_p}(\pi'_p \otimes \Omega_p, \mathbb{C}) \neq 0$;
- $\pi_p$ is the Jacquet-Langlands correspondence of $\pi'_p$ to $B_v^\ast$ with $\text{Hom}_{K_p}(\pi_p \otimes \Omega_p, \mathbb{C}) = 0$;
- $W_p$ is in the $\psi^{-1}$-Whittaker model of the Jacquet-Langlands correspondence of $\pi'_p$ to $GL_2(\mathbb{Q}_p)$.

By Lemma [3.2.6] there exists an automorphic representation $\pi^{(2)}$ of $B(p)^\ast$ such that

- $\pi_p^{(2)} = \pi'_p$, and $\pi^{(2)}_\infty$ is the trivial representation;
- $\text{Hom}_{K_p}(\pi^{(2)}_v \otimes \Omega_v, \mathbb{C}) \neq 0$ for every place $v$ of $\mathbb{Q}$.

Let $\sigma^{(2)}$ be the Jacquet-Langlands correspondence of $\pi^{(2)}$ to $GL_2(\mathbb{Q})$. Then $W$ is in the $\psi^{-1}$-Whittaker model of $\sigma^{(2)}_p$. Let $\phi^{(2)} \in \sigma^{(2)}$, and $\Phi^{(2)} \in S(B^\ast)$ such that the $\psi^{-1}$-Whittaker function $W^{(2)}$ of $\phi^{(2)}$ satisfies $W^{(2)} = W_p^{(2)} \otimes W^{(2)}$ with $W_p^{(2)} = W_p$. Consider the Petersson inner product of each term of (4.41) with $\phi^{(2)}$.

- Let $v \in S_{nspl}$. Recall that as a function on $h \in (B(v)^\ast \times B(v)^\ast)(A)$,

$$\langle \text{Pr} \theta(\cdot, h, d_{\Phi^{(2)}_v} \otimes \Phi^{(2), v}), \phi^{(2)} \rangle$$

lies in the global Shimizu lifting $\pi(v) \not\equiv \pi(v)$ of $\sigma^{(2)}$ [37 2.2]. Here $\pi(v)$ is the Jacquet–Langlands correspondence of $\sigma^{(2)}$ to $B(v)^\ast$. If $v \neq p$, then $\pi(v)_p = \pi_p$ as representations of $B(v)^\ast_p \simeq B_v^\ast$. Since $\text{Hom}_{K_p}(\pi_p \otimes \Omega_p, \mathbb{C}) = 0$, by the definition (4.33) of $\theta_{d,v}(g, \Phi^{(2)})$,

$$\langle \theta_{d,v}(g, \Phi^{(2)}), \phi^{(2)} \rangle = 0.$$  Similarly, $\langle \theta_{l,v}(g, \Phi^{(2)}), \phi^{(2)} \rangle = 0$.

- By [37 Theorem 3.22] and $\text{Hom}_{K_p}(\pi_p \otimes \Omega_p, \mathbb{C}) = 0$, $(Z(g, \Omega, \Phi^{(2)}), \phi^{(2)}) = 0$.

Thus (4.41) implies

$$\langle I(\cdot, \Phi^{(2)}), \phi^{(2)} \rangle + \langle \theta_{d,p}(\cdot, \Phi^{(2)}), \phi^{(2)} \rangle - \langle \theta_{l,p}(\cdot, \Phi^{(2)}), \phi^{(2)} \rangle = 0.$$  Then by (4.38) and the condition (3) on $\Phi$, Lemma [14.5.2] is implied by the following lemma

Lemma 4.7.2. There is an open compact subgroup $U'_p$ of $B(p)^\ast$ such that the Petersson pairings

$$\langle \theta(\cdot, h, d_{\Phi^{(2)}_p} \otimes \Phi^{(2), p}), \phi^{(2)} \rangle$$  and $\langle \theta(\cdot, h, l_{\Phi^{(2)}_p} \otimes \Phi^{(2), p}), \phi^{(2)} \rangle$ are 0 for all $h \in (B(p)^\ast \times B(p)^\ast)(A)$ unless $\pi_p^{(2)}$ has a nonzero $U'_p$-invariant vector.

Proof. The proof is similar to (part of) the proof of Lemma [14.5.9]. As functions on $h \in (B(p)^\ast \times B(p)^\ast)(A)$, $\langle \theta(\cdot, h, d_{\Phi^{(2)}_p} \otimes \Phi^{(2), p}), \phi^{(2)} \rangle$ and $\langle \theta(\cdot, h, l_{\Phi^{(2)}_p} \otimes \Phi^{(2), p}), \phi^{(2)} \rangle$ lie in the global Shimizu lifting [37 2.2] of $\sigma^{(2)}$, which is $\pi^{(2)} \not\equiv \pi^{(2)}$ [37 2.2]. Let $U'_p$ be as in Lemma [14.4.7] and Lemma [14.5.6]. By the $(B(p)^\ast \times B(p)^\ast)(A)$-equivariance of the formation of the theta series, $\langle \theta(\cdot, h, d_{\Phi^{(2)}_p} \otimes \Phi^{(2), p}), \phi^{(2)} \rangle$
and $\langle \theta(\cdot, h, t_{\Phi_p} \otimes \Phi^{(2)}(p)), \phi^{(2)} \rangle$ are invariant by $U'_p$, where $U'_p$ acts on the $\pi^{(2)}$ -component of $\pi^{(2)} \boxtimes \tilde{\pi}^{(2)}$. Then the lemma follows. $\square$

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