The interaction of gravitational waves with strongly magnetized plasmas

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We study the interaction of a gravitational wave (GW) with a plasma that is strongly magnetized. The GW is considered a small disturbance, and the plasma is modeled by the general relativistic analogue of the induction equation of ideal MHD and the single fluid equations. The equations are derived without neglecting any of the non-linear interaction terms, and the non-linear equations are integrated numerically. We find that for strong magnetic fields of the order of $10^{15}$ G the GW excites electromagnetic plasma waves very close to the magnetosonic mode. The magnetic and electric field oscillations have very high amplitude, and a large amount of energy is absorbed from the GW by the electromagnetic oscillations, of the order of $10^{44}$ erg/cm$^2$ in the case presented here. The absorbed energy is proportional to $B_0^2$, with $B_0$ the background magnetic field. The energization of the plasma takes place on fast time scales of the order of milliseconds. The amount of absorbed energy is comparable to the energies emitted in the most energetic astrophysical events, such as giant flares on magnetars and possibly even short Gamma ray bursts (GRB), for which the mechanism analyzed here also has the fast time-scales required.

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Gravitational waves (GW) can carry a large amount of energy near the sources where they are generated (e.g. [10]). They tend not to interact much with matter under normal conditions, it has been shown though in a number of articles (e.g. [8], [1], [2], [19], [15], [12], [9], [13]) that GWs excite various kinds of plasma waves, the more efficient, the stronger the background magnetic field is and the more tenuous the plasma is. Most of these studies are analytical and the equations describing the GW-plasma interaction were linearized, only Ref. [8] made an analytical study of a non-linear model, Refs. [1] and [7] took some second order effects into account, and Ref. [8] performed a numerical study. The GW-plasma interaction is a totally non-linear effect, and there is so far no conclusive answer to the question of how much energy can be absorbed by a plasma from a GW.

Here, we study the GW-plasma interaction in its full non-linearity, solving the non-linear system of equations numerically. Our main interest is in the amount of energy absorbed by the plasma from the GW and in the kind of plasma waves excited by the GW, and we focus on the case of very strong magnetic fields of the order of $10^{15}$ G.

The interaction of gravitational waves (GWs) with plasmas, if it is efficient in transferring energy from the GW to the plasma, might well be the basic mechanism behind the most energetic astrophysical events, such as giant flares on magnetars (i.e. highly energetic outbursts in the stars' magnetospheres; see e.g. [20]) or possibly even short Gamma ray bursts (GRB; see e.g. [16]; magnetars are strongly magnetized neutron stars [see e.g. [3]], which at the same time appear as Soft Gamma Ray repeaters [SGR]). The origin of short GRBs is far less established than that of the long GRBs. One model recently discussed is that short GRB might be giant flares on magnetars. In this model, the flares are caused by a catastrophic reconfiguration of the stellar magnetic field (e.g. [5], [14], [17]). The energy released in short GRBs is currently estimated to be at least $10^{48}$ erg [12], the uncertainty is mainly due to the problem of associating short GRBs with physical objects. The duration of short GRBs is less than 2 seconds. Giant flares have similar durations and release energies in the range $10^{44}$ to $10^{46}$ erg (e.g. [20]).

Our results show that a GW generated by a magnetar might well be the source of the energy released in a giant flare and may-be even in a short GRB, through the absorption of the GW energy by a plasma in the vicinity of the magnetar. In this scenario, the GW-plasma interaction is the primary mechanism that energizes the plasma before a giant flare or may-be a short GRB. Giant flares and possibly short GRBs would then be an indirect observation of GWs.

The GW is considered as a small amplitude perturbation of the otherwise flat spacetime, and we assume it to be + polarized and to propagate along the $z$-direction, so that the metric has the form $g_{ab} = \delta_{ab} - h(z, t)_{\perp} (1 + h(z, t))$, with $h(z, t) << 1$ the amplitude of the GW $h$. Our aim is to express the final equations in terms of the potentially observable quantities (electric field $E$, magnetic field $B$, 3-velocity $\vec{V}$ of the fluid, and rest-mass density $\rho$), which can either be defined in a Local Inertial Frame (LIF) or in an orthonormal frame (ONF) [3]. Here, we use the ONF, because it is a global frame, and it can be shown that the ONF in our case is locally equivalent to a LIF when applying the particular coordinate transformation given in [11]. The ONF has the advantage that 4-vectors and tens.
sors take the same form as in flat space-time, the effects of curvature appear only through the covariant derivatives, which are calculated with use of the Ricci rotation coefficients \[^{11}\]. Indices of quantities in the ONF carry \[^{4}\]s.

We assume an ideal conducting fluid, so that the electric field is given by the ideal Ohm’s law, \(0 = F^{\alpha \beta} \delta_{\beta \gamma} / c\), which in the ONF takes the usual form, \(0 = \dot{\gamma} \left( \vec{E} + \frac{1}{c} \vec{V} \times \vec{B} \right)\), with \(\dot{\gamma} = 1/\sqrt{1 - V^2 / c^2}, F^{\alpha \beta}\). Faraday’s field tensor, \(u^\alpha\) the 4-velocity, \(u^\alpha = \dot{\gamma}(c, V_x, V_y, V_z)\), and \(c\) the speed of light. The evolution of the magnetic field is determined by the Maxwell’s equation, \(F_{\alpha \beta; \gamma} + F_{\beta \gamma; \alpha} = 0\) \[^{11}\]. The electromagnetic (EM) energy momentum tensor is defined as \(T_{\alpha \beta} = \frac{\pi G}{c^4} (F^{\alpha \gamma} F_{\gamma \beta} - \frac{1}{4} \eta^{\alpha \beta} F^{\gamma \delta} F_{\gamma \delta})\), and for the fluid, we have the energy momentum tensor \(T_{\alpha \beta} = H u^\alpha u^\beta + \eta^{\alpha \beta} p c^2\), where \(H\) is the enthalpy, \(p\) the pressure, and \(\eta^{\alpha \beta} = \text{diag}(-1, 1, 1, 1)\) the flat metric of flat spacetime \[^{11}\]. We assume an ideal and adiabatic fluid, so that \(H = pc^2 + \frac{\rho}{\Gamma}\), with \(\Gamma\) the adiabatic index. The total energy momentum tensor \(T_{\alpha \beta} = T_{\alpha \beta} + T_{\alpha \beta}^{(EM)}\) yields the momentum equations \(\partial_{\alpha} T_{\alpha \beta} = 0\), \[^{11}\]. Continuity is expressed by \((\rho u^\beta)_{; \beta} = 0\). The evolution of the GW is determined by the linearized Einstein equation, where we take the back-reaction of the plasma onto the GW into account

\[
-\partial_\alpha h + c^2 \nabla^2 h = -\frac{1}{2} \frac{16\pi G}{c^4} (\delta T_{xx} - \delta T_{yy}),
\]

where \(\delta T_{xx}, \delta T_{yy}\) are the non-background, fluctuating parts of the components \(T_{xx}, T_{yy}\) of the total energy momentum tensor, and \(G\) is the gravitational constant \[^{3}\]. To close the system of equations, we assume an adiabatic and isentropic equation of state, \(p = K \rho^\Gamma\), with \(K\) a constant.

We focus on the excitation of MHD modes which propagate in the \(z\)-direction, parallel to the propagation direction of the GW and perpendicular to the background magnetic field \(\vec{B}_0 = B_0 \vec{e}_z\). We let consequently \(\vec{B} \parallel \vec{e}_x\), \(\vec{E} \parallel \vec{e}_y\), \(\vec{B} \parallel \vec{e}_z\), and all variables depend spatially only on \(z\) (note that \(B_z\) in the following is the total magnetic field, it includes \(B_0\)). In specifying the general equations to this particular geometry, (i) we express all 4-vector and tensor components through the potentially observable \(B_z, V_z\), and \(p\); (ii) we expand the covariant derivatives; (iii) we keep all non-linear terms, no approximations are thus made (except for the linearized Einstein equation). In this way, we are led to a system of non-linear, coupled, partial differential equations in a spatially 1-D geometry: With the electric field \(E_y\) from Ohm’s law, \(E_y = -\frac{1}{c} V_z B_x\), Faraday’s equation is fully expanded to

\[
\partial_t B_z = c \partial_z E_y + \frac{1}{2} B_z \frac{\partial h}{1-h} - \frac{1}{2} c E_y \frac{\partial h}{1-h}.
\]

Expansion of the \(z\)-component of the momentum equation yields

\[
\partial_t q_z + \partial_z \left[ \left( q_z - \frac{c}{4\pi} (-E_y B_z) \right) V_z \right] + \frac{c^2}{8\pi} \left( B_z^2 + E_y^2 \right) + c^2 \partial_z p = 0
\]

\[
+ \frac{c}{8\pi} B_x (cB_x + E_y V_z) \frac{\partial h}{(1+h)}
\]

\[
- \frac{c}{8\pi} E_y (cE_y + B_z V_z) \frac{\partial h}{(1-h)}
\]

\[
- q_z \left( \partial_t V_z + \partial_z h \right) \frac{h}{1-h^2} = 0, \quad (3)
\]

where we defined the new momentum variable \(q_z\) as \(q_z := H V_z \dot{\gamma}^2 + \frac{c}{4\pi} (-E_y B_z)\). The continuity equation takes the form

\[
\partial_t D + \partial_z (D V_z) - (D \partial_t h + D V_z \partial_z h) \frac{h}{1-h^2} = 0, \quad (4)
\]

with the new density variable \(D := \dot{\gamma} \rho\). The GW evolves according to

\[
\partial_t h = c^2 \partial_{zz} h + \frac{2G}{c^2} \left( E_y^2 - (B_z^2 - B_0^2) \right) + \frac{16\pi G}{c^2} (p - p_0) h,
\]

where the background magnetic field \(B_0\) and background pressure \(p_0 = K \rho_0^\Gamma\) have been subtracted.

To recover \(\dot{\gamma}, \rho, p, H, \) and \(V_z\) from the explicitly evolving variables \(q_z, D, B_y\) we solve the definition of \(q_z\) for \(V_z\) and insert it into a reformulated definition of \(\dot{\gamma}\), which yields a non-linear equation for \(\dot{\gamma}\), which we solve numerically. Once \(\dot{\gamma}\) is recovered, all the other primary variables follow in a straightforward way.

We solve the GW-plasma system of equations applying a pseudo-spectral method that is based on Chebyshev polynomials (see e.g. \[^{2}\]). Time stepping is done with the method of lines, using a fourth order Runge-Kutta method with adaptive step-size control. The one-dimensional grid along the \(z\)-direction consists of 256 grid-points and corresponds to a physical domain along the \(z\)-axis of length \(L = 5.410^7\) cm. The sampling time step \(\Delta t\) is set to \(\Delta t = T_{gw}/14\), with \(T_{gw} = 1/f_{gw}\) the GW frequency.

We assume a background magnetic field \(B_0 = 10^{15} G\), a background density \(\rho_0 = 10^{-14} g \ cm^{-3}\), and an adiabatic index \(\Gamma = 4/3\). The initial conditions are \(B_z(z, 0) = B_0, V_z(z, 0) = 0, \rho(z, 0) = \rho_0, \) and \(h(z, 0) = 0\). The GW has as boundary condition at the left end \(z_L\) of the box \(h(z_L, t) = h_0(t) \cos(k_{gw} z_L - \omega_{gw} t)\), so that a monochromatic plane wave is entering the box. The amplitude \(h_0\) rises within roughly 1 ms from 0 to \(10^{-4}\), at which value it stays constant for about 3 ms, where after it decays to 0 again. At the right edge \(z_R\) of the box, we apply non-reflecting boundary conditions to \(h(z_R, t)\), \(B_z, V_z, \) and \(\rho\) have free outflow boundary conditions at both edges of the box. The GW frequency is set to \(f_{gw} = 5 kHz\) and the GW dispersion relation is of the
density $\varepsilon$ with $L$ to tract the constant background magnetic field $B_0$ and whose amplitudes increase linearly towards the out-flowing edge of the box. Maximum amplitudes attained are $3 \times 10^{12}$ statvolt/cm ($9 \times 10^{16}$ V/m) for the electric field, $3 \times 10^{12}$ G for the oscillating magnetic field, roughly three orders of magnitude less than $B_0$, and finally the fluid velocity oscillations have a maximum amplitude $8 \times 10^7$ cm/sec. The fluid motions thus remain non-relativistic, so that our non-relativistic estimate of the kinetic energy is justified.

The waves excited in $E_y$, $B_x$, and $V_z$ have wave-number $k_z$, and frequency $\omega$ that cannot be distinguished from $k_{gw}$ and $\omega_{gw}$ within the numerical precision of the simulation, see Fig. 2. In particular, we do not find any harmonics to be excited. The relativistic Alfvén speed $u_A^2 = B_0^2/(4\pi \rho_0 + B_0^2/c^2)$ is very close to the speed of light $((c-u_A)/c = 2 \times 10^{-24})$ so that the excited plasma modes are indistinguishable from magneto-sonic modes.

The numerical results show that, for strong magnetic fields, a large amount of energy is absorbed by the plasma from the GW on a short time-scale, which is of the order of 10 GW periods, i.e. in the millisecond range. The energy absorbed by the plasma is thus not proportional to the duration of the GW-plasma interaction if this duration is longer than typically a few GW periods.

In a parametric study, we found that the energy absorbed by the plasma is proportional to $B_0^2$ (see Fig. 4) and to $\rho_0^2$. Varying $\rho_0$ in the range $10^{-20}$ g/cm$^3 \leq \rho_0 \leq 10^3$ g/cm$^3$, it turned out that the absorbed energy is independent of the value of the matter density $\rho_0$, see Fig. 4. The kinetic energy is actually negligible compared to the electromagnetic energy (this is in accordance with the fact that the background matter rest-energy density is much smaller than the background magnetic energy density in the entire range of values $\rho_0$ investigated). The absorbed energy density is furthermore proportional to $\omega_{gw}^2$, as we verified by varying the frequency in the kHz range (from 1 to 10 kHz). It also seems that the time for the plasma needed to reach the maximum level of energy absorption is related to the time needed for the GW to cross the box, $L/c$, which equals 0.002 sec for the case considered here.

The maximum amplitudes of the excited oscillations...
FIG. 3: The energy density $\epsilon_{pl}$ absorbed by the plasma (at maximum absorption) as a function of the background magnetic field $B_0$ (diamonds), together with a reference line of logarithmic slope 2 (dotted).

FIG. 4: The energy density $\epsilon_{pl}$ absorbed by the plasma (at maximum absorption) as a function of the background matter density $\rho_0$ (diamonds, connected with a dotted line).

are proportional to the box length $L$, as explained above. This implies that the amount of energy density absorbed is proportional to the squared box-length, $L^2$, with the physical meaning of $L$ to be the length along the propagation direction of the GW where the GW meets a constant magnetic field. We additionally verified this scaling behavior with numerical simulations in which the box-length $L$ was varied. The absorbed energy is thus proportional to $L^3$ and to the effective area $A_{eff}$ through which the GW is incident on the plasma, where

$$A_{eff} := \frac{V}{L},$$

with $V$ the volume in which the interaction takes place. We can summarize our numerical findings for the total energy $E_{pl}$ absorbed by the plasma as follows, noting that in the case presented here the absorbed energy through an effective area of 1 cm$^2$ is $E_{pl} = 3.4 \times 10^7 \text{erg}$, with $L = 5.4 \times 10^7$ cm,

$$E_{pl} = 3.4 \times 10^7 \left(\frac{L}{1 \text{ cm}}\right)^3 \left(\frac{A_{eff}}{1 \text{ cm}^2}\right) \left(\frac{B}{10^{15} \text{G}}\right)^2 \times \left(\frac{h_0}{10^{-4}}\right)^2 \left(\frac{f_{gw}}{5 \text{kHz}}\right)^2 \text{erg},$$

and $E_{pl}$ is independent of $\rho_0$.

**Astrophysical Application.** On the basis of our results, we can suggest a new model for the primary mechanism behind giant flares on magnetars and possibly even behind short GRBs. This mechanism deposits a large amount of energy in a plasma on time-scales of milliseconds, prior to the actual outburst of the giant flare or possibly the short GRB. In our model, we assume that a magnetar generates a GW, which travels away from the magnetar and interacts with a plasma in the vicinity of the magnetar, whereby the plasma efficiently absorbs energy from the GW. Once the plasma is energized, secondary mechanisms will be triggered that convert the energy ultimately to Gamma rays, the nature of these mechanisms is though not in the scope of this article.

The typical magnetic field $B_\star$ of a magnetar is currently estimated to be of the order of $10^{15}$ to $10^{16}$ G at the surface (e.g. 14, 17, 20). GWs are expected to be emitted by magnetars, as by usual neutron stars, when they undergo some deformation, e.g. as the result of the rearrangement of the strong stellar magnetic field. In such a case, the GW amplitude at 5 stellar radii typically is $10^{-4}$ (e.g. 10). At the same distance, the background spacetime can be considered flat. The magnetic field is poloidal within the light cylinder, so that, near the equatorial plane, a GW generated by the magnetar and traveling radially outwards propagates perpendicular to the magnetic field, as in the presented simulations.

In order to get a rough estimate of the energy absorbed by a plasma, we consider the plasma in the conical volume of radial size $L$ between the inner and outer radii $r_{in}$ and $r_{out} = r_{in} + L$, respectively (with $r$ the distance from the magnetar), which is limited to within the poloidal...
opening angle $\alpha$ and the toroidal range $\varphi$, as illustrated by the sketch in Fig. [3]. We assume the magnetic field and the GW amplitude to be constant over the range $L$, with values $B_0 = B_\alpha(r_\alpha/r_{in})^3$ and $h_0 = h(t)/r_{in}$, respectively, in order to be able to apply Eq. (7) (note that the magnitude of the magnetic field along the radial direction is proportional to $1/r^3$). The plasma volume is given as $V(r_{in}, L) = \frac{2}{\varphi} \sin(\alpha/2) \left( r_{out}^3 - r_{in}^3 \right)$, with corresponding effective area $A_{eff} = V(r_{in}, L)/L$.

For a numerical estimate, we set $B_\alpha = 10^{16}$ G, $h_\alpha = 10^{-4}$, $r_\alpha = 10$ km, $\alpha = 10^8$, and, to take possible anisotropic GW emission into account, $\varphi = 180^\circ$. Fig. [8] shows the absorbed energy as a function of $r_{in}$ for four different values of $L$. The energy decreases fast with distance, it roughly is proportional to $1/r_{in}^3$ so that at $r_{in} = 5 r_\alpha$, the energy has fallen to $E_{pl} = 10^{14}$ erg for e.g. $L = 500$ km. Extrapolating our results to regions closer to the star, we find at $r_{in} = 2 r_\alpha$ and again for $L = 500$ km a plasma energy $E_{pl} = 10^{15}$ erg, which is of the order of the energy released in giant flares on magnetars (e.g. [14], [7], [20]). Even closer to the star, at $r_{in} = r_\alpha$, the energy is of the order of $E_{pl} = 10^{17}$ erg, well in excess of the energy in giant flares, and approaching the energy observed in short GRBs according to latest estimates (e.g. [16], [14]). For different values of $B_\alpha$ and $h_\alpha$, the energy values given here scale according to Eq. 7 in a straightforward way.

Again for $L = 500$ km, the volume is $2 \times 10^7$ km$^3$ for $r_{in} = r_\alpha$ and increases to $4 \times 10^7$ km$^3$ for $r_{in} = 10 r_\alpha$, which corresponds to effective areas of $A_{eff} = 5 \times 10^4$ km$^2$ and $A_{eff} = 8 \times 10^4$ km$^2$, respectively. The involved volumes and areas are thus relatively small.

Our numerical simulations were done for the case of a flat background spacetime, which does not hold in the range $r_{in} \leq 2 r_\alpha$ anymore, to which we have extrapolated the energy estimates. The non-flatness of spacetime close to the star, together with the necessarily arbitrary choice of a value for $L$, imply that our energy estimates must be interpreted with some care, they can though be taken indicative of the fact that the GW-plasma interaction is efficient and is an important mechanism near the star. Also to mention is the uncertainty concerning the actual magnetic topology in the flaring magnetosphere, since any deviation from the dipole is likely to intensify the GW-plasma interaction through the enlargening of the possible interaction regions. It thus remains to be seen in how far the given numbers will be modified when a more realistic decay of the background magnetic field is used in the simulations and when the curvature of the background spacetime is included, which though only can be done when extending the equations to the case of three spatial dimensions. In favor of the model of giant flares on magnetars (and, less certain, but possibly also of short GRBs) driven by GWs is that the mechanism proposed has a fast enough time-scale, of the order of milli-seconds.

In summary, our results show that strongly magnetized plasmas are efficient absorbers of GWs, largely irrespective of the plasma density, and with an absorption time-scale of the order of milli-seconds. This implies that GWs may be the energy source for secondary, highly energetic phenomena. It also implies that GWs may eventually be strongly damped, if appropriate conditions are met. In particular, we can conclude that the GW-plasma interaction is an efficient and important mechanism in magnetar atmospheres close to the star.

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