On Dense Subsemimodules and Prime Semimodules

Ahmed H. Alwan1*, Asaad M. A. Alhossaini2
1Department of Mathematics, College of Education for Pure Sciences, Thi-Qar University, Thi-Qar, Iraq
2Department of Mathematics, College of Education for Pure Sciences, Babylon University, Babylon, Iraq

Received: 25/8/2019 Accepted: 22/10/2019

Abstract

In this paper, we study the class of prime semimodules and the related concepts, such as the class of \( \pi \) semimodules, the class of Dedekind semidomains, the class of prime semimodules which is invariant subsemimodules of its injective hull, and the compressible semimodules. In order to make the work as complete as possible, we stated, and sometimes proved, some known results related to the above concepts.

Keywords: Semimodule, Semiring, Dense subsemimodule, Invertible ideal, Prime semimodule, Dedekind semidomain.

Introduction

Throughout this paper, \( R \) will denote a commutative semiring with identity, and \( M \) is an \( R \)-semimodule.

This paper consists of three sections. In Section one, we introduce some definitions and remarks which we will use in the paper. In Section two, we introduce the concept of density of semimodules. A non-zero \( R \)-subsemimodule of an \( R \)-semimodule is said to be dense in \( M \), if \( M = \sum \phi(N) \), where the sum is taken over all \( \phi \in \text{Hom}(N, M) \). We use the density concept to define the class of \( \pi \) semimodules, as \( M \) is said to be \( \pi \) semimodule if each non-zero subtractive subsemimodule of \( M \) is dense in \( M \).

In Section three, we define the concept of prime semimodules, analogous to that in modules [4], where \( M \) is said to be prime if \( \text{ann}(N) = \text{ann}(M) \), for each non-zero subtractive subsemimodule \( N \) of \( M \). Similar to that in modules [1], we will show that every \( \pi \) semimodule is a prime semimodule.

The aim of this paper is to discuss the converse of this statement in the case of semimodules having injective hull. Also we generalize some types of prime modules for semimodules, such as the compressible type.

*Email: ahha7810@gmail.com

1446
1. Preliminaries

In this section, we introduce some of definitions, remarks, and examples that might be needed in the main results.

**Definition 1.1.**[19] A nonempty set $R$ with two operations of addition and multiplication (denoted by $+$ and $\cdot$, respectively) is called a semiring, provided that:
1. $(R, +)$ is a commutative monoid (A monoid is a semigroup with identity) with identity element $0$;
2. $(R, \cdot)$ is a monoid with identity element $1 \neq 0$;
3. Multiplication distributes over addition, i.e. $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$; for all $a, b, c \in R$.
4. The element $0$ is the absorbing element of the multiplication, i.e. $r \cdot 0 = 0$ for all $r \in R$.

The semiring $R$ is said to be commutative if its multiplication is commutative.

**Definition 1.2.**[18] A non-empty subset $I$ of a semiring $R$ will be called an ideal of $R$ if $a, b \in I$ and $r \in R$ imply $a + b \in I$, $ra$, and $ar \in I$.

**Definition 1.3.**[6] A semiring is said to be a semidomain if $ab = 0$, $(a, b \in R)$ then either $a = 0$ or $b = 0$.

**Definition 1.4.**[10] A semiring $R$ is semisubtractive if, for all $x, y \in R$, then $x + h = y$ or $x = y + h$ for some $h \in R$.

**Definition 1.5.**[10] Let $R$ be a semiring, a left $R$-semimodule is a commutative monoid with additive identity $0$ for which we have a function $R \times M \rightarrow M$ defined by $(r, x) \mapsto rx$ (scalar multiplication), which satisfies the following condition, for all $x, y \in M$ and for all $r, s \in R$:
1. $(rs)x = r(sx)$
2. $r(x + y) = rx + ry$
3. $(r + s)x = rx + sy$
4. $0_R m = 0 = 0x$

If the condition $1x = x$ for all $x \in M$ holds, then the semimodule $M$ is said to be unitary.

**Definition 1.6.**[12] A non-empty subset $N$ of a left $R$-semimodule $M$ is called a subsemimodule of $M$ if $N$ is closed under addition and scalar multiplication, that is $N$ is a semimodule itself (denoted by $N \lhd M$).

**Definition 1.7.**[6] Let $M$ be an $R$-semimodule. A subtractive subsemimodule (or k-subsemimodule) $N$ is a subsemimodule of $M$ such that if $x, x + y \in N$, then $y \in N$. We define subtractive ideals (k-ideals) of a semiring $R$ in an analogous manner.

**Definition 1.8.** Let $S$ be a non-empty subset of an $R$-semimodule $M$.

If $S = \emptyset$, then we denote $RS$ by $S$, i.e., $RS = \{rs | r \in R \}$. If $S = \{s_1, s_2, \ldots, s_m \}$, then

$$RS = \{ \sum_{i=1}^{m} r_i s_i | r_i \in R, s_i \in S \}.$$

The expression $\sum_{i=1}^{m} r_i s_i$ is called a linear combination of the elements $s_1, s_2, \ldots, s_m$. If $S = \{s_1, s_2, \ldots, s_m \}$, then

$$RS = \{ \sum_{i=1}^{m} r_i s_i | r_i \in R, s_i \in S \}.$$

Especially, if $S = \{s\}$, then we denote $RS$ by $Rs$, i.e., $Rs = \{rs | r \in R \}$.

If $RS = M$, then $S$ is called a generating set for $M$. An $R$-semimodule having a finite generating set is called finitely generated, if $RS = M$ then $M$ is called cyclic. A non-empty subset $S$ of $M$ is called an free set if for each $\{s_1, s_2, \ldots, s_m \} \subseteq S$, the linear combination $\sum_{i=1}^{m} r_i s_i = 0$ implies $r_0 = 0$, $\forall i$, where $r_i \in R$. An $R$-semimodule $M$ is called a free semimodule if $M$ has a free generating subset $S$. In this case, $S$ is said to be a basis for $M$.

**Remark 1.9.** If a semiring $R$ is a ring then any $R$-semimodule is an $R$-module.

**Proof:** Let $M$ be a semimodule over a ring $R$. Then $M$ is a commutative monoid (commutative semigroup with identity) which satisfies all the conditions in Definition 1.5. To show that $M$ is an $R$-module, we need only to prove that for all $m \in M$ there exists $-m \in M$ such that $m + (-m) = -m + m = 0$. Now let $m \in M$, since $R$ is a ring, i.e. $R$ is a ring with identity $1$. Hence $-1 \in R$, and so $-1(m) \in M$. Thus $-m \in M$, $\forall m \in M$. Therefore $M$ is a group, and hence $M$ is an $R$-module.

**Remark 1.10.** The only subtractive ideals of the semiring $(\mathbb{N}, +, \cdot)$ are the cyclic ideals.

**Proof:** Let $I$ be a non-cyclic ideal of $\mathbb{N}$, and let $a$ be the smallest non-zero element of $I$ and $b$ is the first element of $I$ which is greater than $a$ and not multiple of $a$. Then $b = a + k$ for some $k \in \mathbb{N}$ and $k \notin I$.
I(by the choice of a and b), hence I is not subtractive. On the other hand, it is clear that any cyclic ideal of N is subtractive.

**Remark 1.11.** Let A be a subsemimodule of the \( \mathbb{N} \)-semimodule \( \mathbb{N} \), and let \( a_0 \) be the smallest non-zero element of A, then either \( A = \mathbb{N} a_0 = \{ n a_0 | n \in \mathbb{N} \} \) or \( A = \{ 0, a_0, a_0 + 1, a_0 + 2, \ldots \} \).

**Proof:** Assume that \( A \neq \mathbb{N} a_0 \), then \( A \subset \mathbb{N} a_0 \), if \( b_0 \) is the smallest element greater than \( a_0 \) such that \( b_0 \in A \), \( b_0 \notin \mathbb{N} a_0 \), then \( \mathbb{N} a_0 \cup Nb_0 \cup \mathbb{N} (a_0 + b_0) \subset A \). Similarly proceeding, we have \( A = \{ 0, a_0, a_0 + 1, a_0 + 2, \ldots \} \).

**Remark 1.12.** Let R be a commutative semiring with identity. A set \( S \subseteq R \) is said to be a multiplicatively closed set of \( R \) provided that “if \( a, b \in S \), then \( ab \in S \)”. The localization of \( R \) at \( S \) is defined in the following way:

First define the equivalent relation \( \sim \) on \( R \times S \) by \( (a, b) \sim (c, d) \) if \( sad = sbc \) for some \( s \in S \). Then put \( R_S \) as the set of all equivalence classes of \( R \times S \) and define the addition and multiplication on \( R_S \), respectively, by \( [a,b] + [c,d] = [ad + bc, bd] \) and \( [a,b] \cdot [c,d] = [ac, bd] \), where \([a,b] \) is also denoted by \( a/b \), by which we mean the equivalence class of \( (a, b) \). It is, then, easy to see that \( R_S \) with the above mentioned operations of addition and multiplication on \( R_S \) is a semiring [15].

**Definition 1.13.** In Remark 1.12, if \( S \) is the set of all not zero divisors of \( R \), then the total quotient semiring \( Q(R) \) of the semiring \( R \) is defined as the localization of \( R \) at \( S \). Note that \( Q(R) \) is also an \( R \)-semimodule. For more details, see previous articles [11, 13].

**Definition 1.14.** A subset \( I \) of the total quotient semiring \( Q(R) \) of \( R \) is called fractional ideal of a semiring \( R \), if the following hold:
1. \( I \) is an \( R \)-subsemimodule of \( Q(R) \), that is, if \( a, b \in I \) and \( r \in R \), then \( a + b \in I \) and \( ra \in I \).
2. There exists a non-zero divisor element \( d \in R \) such that \( dI \subseteq R \).

Let \( I, J \) be two fractional ideals of a semiring \( R \). Then
\[ I = \{ b_1 a_1, b_2 a_2, \ldots, b_n a_n : b_i \in I, a_i \in I \} \]

It is clear that any ideal \( I \) of \( R \) is a fractional ideal of a semiring \( R \).

**Definition 1.15.** Let \( I \) be a fractional ideal of a semiring \( R \), then \( I \) is called invertible if there exists a fractional ideal \( J \) of \( R \) such that \( JI = R \). Note that \( J \) is unique and we denote that by \( I^{-1} \). For more details, see for example earlier works [10, 11].

2. \( \pi \) Semimodules

Let \( \Omega \) be a family of \( R \)-semimodules. The \( R \)-semimodule \( M \), as an \( R \)-module [14, Ex.17(b), page 241]) is said to be generator for the family \( \Omega \) if for each \( N \in \Omega \),
\[ N = \sum_{\phi \in \text{Hom}(M,N)} \phi(M) \]

In some cases, for simplicity, we put \( H = \text{Hom}(M,N) \).

The following theorem gives a different form for generators.

**Theorem 2.1.** Let \( M \) be an \( R \)-semimodule and \( \Omega \) be a family of \( R \)-semimodules. Then the following statements are equivalent:
1. \( M \) is a generator for \( \Omega \).
2. For all \( R \)-semimodules \( N \) and \( K \) in \( \Omega \), and \( f \in \text{Hom}(N,K) \) with \( f \neq 0 \), \( \exists g \in \text{Hom}(M,N) \) such that \( fg \neq 0 \), see the diagram below.

\[
\begin{array}{ccc}
M & \xrightarrow{fg \neq 0} & 0 \\
\downarrow & & \\
N & \rightarrow & K \\
\forall f \neq 0
\end{array}
\]

**Proof:** (1) \( \Rightarrow \) (2). Since \( f \neq 0 \), there is \( a \in N \) with \( f(a) \neq 0 \). As \( M \) is a generator, there is a representation \( a = \sum_{i=1}^{n} g_i(m_i) \), \( g_i \in \text{Hom}(M,N) \), \( m_i \in M \). Hence we have
\[ 0 \neq f(a) = \sum_{i=1}^{n} fg_i(m_i) \]
and consequently there is a \( g_i \) with \( fg_i \neq 0 \).
(2) ⇒ (1). Suppose that \( \sum_{\phi \in H} \phi(M) \neq N \), \( H = \text{Hom}(M,N), N \in \Omega \), then let \( \nu : N \rightarrow N/\sum_{\phi \in H} \phi(M) \) be the natural epimorphism. Since \( \nu \neq 0 \), there is a \( g \in H \) with \( \nu g \neq 0 \). Consequently, we have \( g(M) \not\subseteq \sum_{\phi \in H} \phi(M) \), in contradiction to the definition of \( \sum_{\phi \in H} \phi(M) \). This completes the proof.

Birge Zimmermann-Huisgen [3] introduced the definition of self-generator for \( R \)-modules. In this paper, we recall this definition for \( R \)-semimodules. Let \( N \) be the natural epimorphism. Since there is a with \( . Consequently, we have \( \sum \) in contradiction to the definition of \( \sum \). This completes the proof.

Birge Zimmermann-Huisgen [3] introduced the definition of self-generator for \( R \)-modules. In this paper, we recall this definition for \( R \)-semimodules. If \( M \) generates each of its subsemimodules. In other words, an \( R \)-semimodule \( M \) is called self-generator, if for any subsemimodule \( N \) of \( M \),

\[
N = \sum_{\phi \in \text{Hom}(M,N)} \phi(M).
\]

In this section, we study the semimodules which can be generated by each of their non-zero subsemimodules. This is a "dual problem" of self-generator concept. Now, for any two \( R \)-semimodules \( M_1, M_2 \), let \( \pi(M_1, M_2) = \sum_{\phi} \phi(M_1) \) where the sum is taken over all \( \phi \in \text{Hom}(M_1, M_2) \). If \( N \) is a subsemimodule of \( M \), then we may put \( \pi(N) \) instead of \( \pi(N, M) \). Note that if \( M_2 = R \) then \( \pi(M_1, R) \) is just the trace of \( M_1 \). For more details see a previously published study [1, page 7].

Now we introduce the following definition.

**Definition 2.2.** A non-zero subsemimodule \( N \) of an \( R \)-semimodule \( M \) is said to be dense in \( M \), if \( N \) generates each of its subsemimodules. In other words, an \( R \)-semimodule \( M \) is called dense in \( M \), if for any subsemimodule \( N \) of \( M \),

\[
M = \sum_{\phi \in \text{Hom}(N,M)} \phi(N).
\]

In the following lemma, we give other forms of dense subsemimodules, with the proof as in Theorem 2.1.

**Lemma 2.3.** Let \( N \) be a non-zero subsemimodule of an \( R \)-semimodule \( M \). Then the following statements are equivalent:

1. \( N \) is dense in \( M \).
2. For any \( R \)-semimodule \( K \) and, and with \( f \neq 0 \), \( \exists g \in \text{Hom}(N,M) \), such that \( fg \neq 0 \).

**Proposition 2.4.** Let \( N \) be a non-zero subsemimodule of an \( R \)-semimodule \( M \). If \( N \) is dense in \( M \), then \( \text{ann}(N) = \text{ann}(M) \).

**Proof:** We have \( \text{ann}(M) \subseteq \text{ann}(N) \), thus it is enough to show that \( \text{ann}(N) \subseteq \text{ann}(M) \). Let \( r \in \text{ann}(N) \). Since \( N \) is dense in \( M \), then by definition 2.2, \( \forall m \in M \), \( \exists \phi_1, \phi_2, \ldots, \phi_n \in \text{Hom}(N, M) \), and \( \exists x_1, x_2, \ldots, x_n \in N \) such that \( m = \sum_{i=1}^{n} \phi_i(x_i) \). Then \( rm = \sum_{i=1}^{n} \phi_i(rx_i) \), but \( r \in \text{ann}(N) \), hence \( rx_i = 0 \), and \( rm = 0 \). Therefore, \( r \in \text{ann}(M) \) and \( \text{ann}(N) = \text{ann}(M) \).

**Remark 2.5.**

1. \( \text{Hom}_2(\mathbb{Q}, \mathbb{Z}) = 0 \).
2. \( \text{Hom}_2(\mathbb{Q}, \mathbb{N}) = 0 \).

**Proof:** For(1), assume that \( 0 \neq f(1) = n \) and \( m \) is any integer with \( g, c, d(m, n) = 1 \), then \( n = f(1) = f(m/m) = mf(1/m) \), \( f(1/m) = n/m \notin \mathbb{Z} \) (which is not possible). Hence \( f(1) \) must equal zero. Therefore \( \text{Hom}_2(\mathbb{Q}, \mathbb{Z}) = 0 \). Using the same way we prove(2).

The following example shows that the condition in Proposition 2.4 is not sufficient.

**Example 2.6.** Let \( M = \mathbb{Z} \oplus \mathbb{Q} \) be considered as a \( \mathbb{Z} \)-semimodule, where \( \mathbb{Z} \) and \( \mathbb{Q} \) are the groups of integers and rationals, respectively. Let \( N = 0 + \mathbb{Q} \) be a non-zero subsemimodule of \( M \). It is clear that, \( \text{ann}(N) = \text{ann}(M) = (0) \). If \( (n, 0) \in M \), \( n \neq 0 \). From Remark 2.5, we have \( \text{Hom}(\mathbb{Q}, \mathbb{Z}) = 0 \), then \( (n, 0) \notin \pi(N) \). Thus \( N \) is not dense in \( M \).

Note that, in Example 2.6, if we put \( M = \mathbb{N} \oplus \mathbb{Q} \) considered as a \( \mathbb{N} \)-semimodule, where \( \mathbb{N} \) is a semigroup of natural numbers, we will get \( N = 0 + \mathbb{Q} \) is not dense in \( M = \mathbb{N} \oplus \mathbb{Q} \).

The following lemma shows that the condition of Proposition 2.4 is sufficient to make a subsemimodule dense if a subsemimodule is cyclic.

**Lemma 2.7.** Let \( Ra \) be a non-zero cyclic subsemimodule of an \( R \)-semimodule \( M \), then the following statements are equivalent:

1. \( M = \pi(Ra) \)
2. \( \text{ann}(M) = \text{ann}(Ra) \)
3. \( \forall m \in M, \exists \) is a homomorphism \( \phi_m : Ra \to M \) such that \( \phi_m(a) = m \).

**Proof:** From Proposition 2.4, (1) gives (2). Suppose that (2) holds and \( m \in M \). We define \( \phi_m : Ra \to M \) as follows: \( \phi_m(ra) = rm \), in particular \( \phi_m(a) = m \). The assumption implies that \( \phi_m \) is well-defined. Finally, suppose that (3) holds, then it is clear that \( \pi(Ra) \subseteq M \), let \( m \in M \) by (3), then for all \( \phi \in \text{Hom}(Ra, M) \), we have \( m \in \sum \phi \), \( \phi(Ra) \), thus \( M \subseteq \pi(Ra) \). Thus \( M = \pi(Ra) \).

After defining the concept of a dense subsemimodule, as previously described in the modules [1, page 11], we are ready now to give the concept of a \( \pi \) semimodule, which is a dual, in some sense, to the concept of self-generator semimodule, given in modules.

**Definition 2.8.** An \( R \)-semimodule \( M \) is said to be a \( \pi \) semimodule if for each non-zero subtractive subsemimodule \( N \) of \( M \), \( \pi(N) = M \), i.e., each non-zero subtractive subsemimodule of \( M \) is dense in \( M \).

Note that \( M \) is a \( \pi \) semimodule if it is generated by each of its nonzero subtractive subsemimodule, while \( M \) is a self-generator if it generates each of its subtractive subsemimodules.

**Example 2.9.** Here we introduce some examples to explain \( \pi \) semimodules:

1. Any simple semimodule is a \( \pi \) semimodule.
2. Let \( \mathbb{N} \) be the semiring of natural numbers, and let \( \mathbb{A} \) be a any non-zero ideal in \( \mathbb{N} \). Define a \( \mathbb{N} \)-homomorphism \( f : \mathbb{N} \to \mathbb{N} \) by putting \( f(an) = n \), \( \forall an \in \mathbb{A} \). In particular, \( f(a) = 1 \). Hence \( \mathbb{A} \) is dense in \( \mathbb{N} \). Thus \( \mathbb{N} \) is a \( \pi \) semimodule.
3. Let \( \mathbb{Q}^+ \) be the \( \mathbb{N} \)-semimodule of non-negative rational numbers, and let \( K \) be any non-zero subsemimodule of \( \mathbb{Q}^+ \). Then \( 3a/b \in K \) with \( a, b \neq 0 \). Let \( m/n \in \mathbb{Q}^+ \). Define a map \( f : K \to \mathbb{Q}^+ \) by putting \( f(x) = (bm/an)x \), \( \forall x \in K \). It is clear that \( f \) is an \( \mathbb{N} \)-homomorphism and \( f(a/b) = m/n \). Thus \( K \) is dense in \( \mathbb{Q}^+ \), and \( \mathbb{Q}^+ \) is a \( \pi \) \( \mathbb{N} \)-semimodule.
4. Let \( p \) be a prime number, and let \( \mathbb{N}_{p^n} \) be the set of rationals of the form \( m/n \), with \( m \) and \( n \) are in \( \mathbb{N} \) and \( n \) is not divisible by \( p \). Then \( \mathbb{N}_{p^n} \) is a subsemigroup of \( \mathbb{Q}^+ \). As a \( \mathbb{Z} \)-module, \( \mathbb{Z}_{p^n} \) [17]. We put \( \mathbb{N}_{p^n} = \mathbb{Q}^+/\mathbb{N}_{p^n} \).

Then \( \mathbb{N}_{p^n} \) is a \( \mathbb{N} \)-semimodule. It is known that each proper non-zero subsemigroup of \( \mathbb{N}_{p^n} \) is cyclic of the form \( \mathbb{N}_{p^n} \). Note that, since each element of \( f(\mathbb{N}_{p^n}) \) where \( f \in \text{Hom}(\mathbb{N}_{p^n}, \mathbb{N}_{p^n}) \) is of order less than or equal to \( p^n \), then \( \mathbb{N}_{p^n} \) is not dense in \( \mathbb{N}_{p^n} \). Thus \( \mathbb{N}_{p^n} \) is not a \( \pi \) semimodule.

A subsemimodule \( N \) of an \( R \)-semimodule \( M \) is called **invariant** subsemimodule if \( f(N) \subseteq N \), \( \forall f \in \text{Hom}(M, M) \), and \( N \) is called a **stable** subsemimodule if \( f(N) \subseteq N \), \( f \in \text{Hom}(M, M) \) [2].

**Remark 2.10.** Let \( N \) be a non-zero subsemimodule of an \( R \)-semimodule \( M \), then
1. \( N \subseteq \pi(N) \subseteq M \).
2. \( N \) is a stable subsemimodule of \( M \) iff \( \pi(N) = N \).
3. \( \pi(N) \) is a stable subsemimodule of \( M \).

**Proof:** (1) and (2) are clear. (3) Let \( f : \pi(N) \to M \). We want to show that \( f(m) \in \pi(N) \), \( \forall m \in \pi(N) \). Since \( m \in \pi(N) \), then \( m = \sum_{i=1}^{n} \phi_i(x_i) \), where \( \phi_i \in \text{Hom}(M, M) \), and \( x_i \in N \), \( \forall i \leq i \leq n \). Thus

\[
\text{f(m) = } \sum_{i=1}^{n} \phi_i(x_i)
\]

Since \( f \phi_i \in \text{Hom}(M, M) \), then \( f(m) \in \pi(N) \), so \( f(\pi(N)) \subseteq \pi(N) \), \( \forall f \in \text{Hom}(\pi(N), M) \). Then \( \pi(N) \) is a stable subsemimodule of \( M \).

The following proposition relates the concept of a \( \pi \) semimodule and the concept of stability.

**Proposition 2.11.** Let \( M \) be an \( R \)-semimodule, then \( M \) is a \( \pi \) semimodule iff \( M \) has no non-trivial stable subsemimodules.

**Proof:** Assume that \( M \) is a \( \pi \) semimodule, and \( N \) is a proper non-zero stable subsemimodule of \( M \). By Remark 2.10, \( \pi(N) = N \). Since \( M \) is a \( \pi \) semimodule, hence \( M = \pi(N) = N \), which is a contradiction. Conversely, since \( \pi(N) \) is a stable non-zero subsemimodule of \( M \), see Remark 2.10, thus by assumption, \( M = \pi(N) \). Therefore \( M \) is a \( \pi \) semimodule.

Now, we study when an ideal is dense in semiring.

**Remark 2.12.** A non-zero ideal \( I \) of a semiring \( R \) is dense in \( R \) iff \( \text{trace}(I) = R \).

Golan [9, page 39] proved that an ideal \( I \) of a ring \( R \) is a direct summand iff \( I = Re \) for some idempotent element \( e \) of \( R \). Here, we use another proof for a semirings.

**Lemma 2.13.** An ideal \( I \) of \( R \) is a direct summand iff \( I = Re \) for some idempotent element \( e \) of \( R \).

**Proof:** \( (\Rightarrow) \) Assume that \( I \) is a direct summand of \( R \), that is \( R = I \oplus J \), then \( 1 = e + \bar{e} \) for some \( e \in I \).
and $\epsilon \in J$. For each $x \in I, x = xe + x\epsilon$. Since $I$ is subtractive $x \in I \land xe + x\epsilon \in I$, imply $x\epsilon \in I$, hence $x\epsilon \in I \cap J = \{0\}$. Then, $x = xe \ (\forall x \in I)$, that is $I = Re$. Now if we put $x = e \ in$ the expression $x = xe$, we get $e = e\epsilon = e^2$, and $e$ is idempotent.

$(\Leftarrow)$ Assume that $e$ is an idempotent element of $R$ and $I = Re$. If $e$ is a non-zero divisor, then $a: R \to Re$, defined by $a \mapsto re$, is an isomorphism, so $Re$ is a direct summand of $R$. If $e$ is a zero divisor, and $ee = 0$ (for some $\epsilon \in R$). Claim that $R = Re + R\epsilon$ for some $\epsilon$ such that $ee = 0$. We need to consider that $R$ is semisubtractive. In this case, either $e + \epsilon = 1$ or $e = 1 + \epsilon$ for some $\epsilon \in R$. If $e + \epsilon = 1$, then $R = Re + R\epsilon$, and since $(e + \epsilon)e = e^2 + e\epsilon = e \to e + e\epsilon = e \to ee = 0$, then $R = Re \oplus R\epsilon$. In the case that $e = 1 + \epsilon$, we also get $ee = 0$ and $Re \cap R\epsilon = 0$. On the other hand, $re = r + r\epsilon, \forall r \in R$. Now $re \in Re + R\epsilon \land r\epsilon \in Re + R\epsilon$, by subtractivity, $r \in Re + R\epsilon \to R = Re + R\epsilon \to Re \oplus R\epsilon$. Therefore, $I = Re$ is a direct summand of $R$.

As in the modules, we give the following lemma without proof, since it is already included in the modules [9, page 61].

**Lemma 2.14.** A left $R$-semimodule is isomorphic to a direct summand of a free left $R$-semimodule iff it is projective.

**Theorem 2.15.** Let $I$ be a non-zero subtractive ideal of $R$, then $I$ is dense in $R$ iff $I$ is a faithful finitely generated projective ideal.

**Proof:** $(\Rightarrow)$ Suppose that $I$ is dense in $R$, by Remark 2.12, $1 = \sum \varphi_i(x_i), x_i \in I$, for finite $i$. Thus, $\forall x \in I, x = \sum \varphi_i(x_i) = \sum x\varphi_i(x)$. Hence $I$ is a direct summand of a free left $R$-semimodule. As in the rings theory [8], we have $I$ is dense in $R$.

$(\Leftarrow)$ Suppose that $I$ is a faithful finitely generated projective ideal. Since $I$ is faithful, then $\text{ann}(I) = \text{ann}(R) = 0$, thus $I$ is faithful.

**Corollary 2.16.** If $I$ is a subtractive dense ideal of a semiring $R$, then $I$ is an invertible in $R$.

**Proposition 2.17.** If $I$ is an invertible ideal of a semiring $R$, then $I$ is dense in $R$.

**Proof: Since $I$ is a subtractive dense ideal of $R$, then by Theorem 2.15, we have $I$ is a faithfully generated projective ideal.**

As in the rings theory [8], we have $I$ is invertible.

**Theorem 2.18.** Let $R$ be a semiring, then $R$ is a $\pi R$-semimodule iff $R$ is a Dedekind semidomain.

**Proof: (\Rightarrow) Assume that $R$ is a $\pi$-semimodule, then $Ra$ is dense in $R, \forall a \in R$, and by Theorem 2.15, $Ra$ is faithful and $\text{ann}(Ra) = 0$. Hence, $R$ is a semidomain. Moreover, every non-zero subtractive ideal $I$ of $R$ is dense, thus by Corollary 2.16, $I$ is invertible. Then $R$ is a Dedekind semidomain.**

$(\Leftarrow)$ The converse follows immediately from Proposition 2.17. Thus $R$ is a $\pi R$-semimodule.

**Remark 2.19.** Let $R$ be a semiring and $a \in R$. Then the principal ideal $(a)$ is invertible iff $a$ is not zero divisor.
Then, $\exists x \in J$ such that $ax = 1 = xa$. But $x(ab) = x0 = 0, \rightarrow 0 = (xa)b = 1b = b$ and hence $a$ is not a zero divisor.

($\Rightarrow$) Assume that $a$ is not zero divisor element of $R$, and let $(a)$ be a principal ideal of $R$. Since $a$ is not zero divisor, then $I = (s/a)$ is a fractional ideal of $R$. Now, $R \subseteq I = (a)(s/a)$. Let $y \in I$, then

\[
y = \left( r_1 a \right) \left( \frac{y}{a} \right) + \left( r_2 a \right) \left( \frac{y}{a} \right) + \cdots + \left( r_n a \right) \left( \frac{y}{a} \right).
\]

Hence, $I$ is an ideal of $R$, and $I = R$. Therefore, $I$ is invertible ideal of $R$.

The following two corollaries are immediate from Remark 2.19 and Proposition 2.17.

**Corollary 2.20.** Every principal ideal in a semiring $R$ generated by a non-zero divisor is dense in $R$.

**Corollary 2.21.** Let $R$ be a semiring, then the following statements are equivalent:

1. $R$ is a semidomain.
2. Each non-zero principal ideal of $R$ is an invertible ideal of $R$.
3. Each non-zero principal ideal of $R$ is dense in $R$.

### 3. Prime Semimodules Having Injective Hull

In Proposition 2.4, we saw that for every dense subsemimodule $N$ of $M$, $\text{ann}(N) = \text{ann}(M)$, thus in a $R$-semimodule $M$, for every non-zero subtractive subsemimodule $N$ of $M$, $\text{ann}(N) = \text{ann}(M)$. And in Lemma 2.7, we observed that a cyclic subsemimodule $Ra$ is dense in $M$ iff $\text{ann}(Ra) = \text{ann}(M)$.

These observations lead us to study prime semimodules. Analogous to the concept of prime modules [4], we define a prime semimodules as follows:

**Definition 3.1.** An $R$-semimodule $M$ is said to be prime if $\text{ann}(N) = \text{ann}(M)$, for every non-zero subtractive subsemimodule $N$ of $M$.

We observed that the class of prime semimodules contains the class of $\pi$ semimodules. But the converse is false. Note that the $\mathbb{Z}$-semimodule $M = \mathbb{Z} \oplus \mathbb{Q}$ is easily seen to be a prime semimodule. Anyway, any direct summand of semimodule is subtractive, [11, page 184], hence $\mathbb{Q}$ is a subtractive subsemimodule of $M$ which is not dense in $M$ (see Example 2.6). Thus, $M$ is not a prime semimodule. One can ask when a prime semimodule can possibly be a $\pi$ semimodule. We will show later that, in the class of quasi-injective semimodule, the two concepts of $\pi$ semimodule and prime semimodule are equivalent.

It is well known that, for every $R$-module $M$, $M$ can be embedded in an injective $R$-module. $\tilde{M}$ is called an injective hull of $M$, if $\tilde{M}$ is an essential extension of $M$, i.e $M \cap N \neq 0$ for every non-zero submodule $N$ of $\tilde{M}$ [17].

It is well known, however, that injective hulls always exist if $R$ is a ring. But, Golan[10] proved that injective hulls of non-zero $R$-semimodules need not exist for every semiring $R$[10, prop.17.21, page 198]. If $R$ is a semiring then any cancellative $R$-semimodule can be embedded in an injective $R$-module $\tilde{M}$, [10, Ex.17.35, page 202], Wang [19] proved that every semimodule over an additively-idempotent semiring has an injective hull. For more details on an injective hull of semimodules over semiring, see for example information described previously [13].

**Lemma 3.2.** Let $R$ be a semisubtractive semiring, and let $M$ and $N$ be cancellative $R$-semimodules. If $x \in M$ and $y \in N$ with $\text{ann}(Rx) = \text{ann}(Ry)$, then $f: Rx \rightarrow Ry$ defined by $rx \mapsto ry$ is well-defined $R$-homomorphism.

**Proof:** Assume $rx = fx$, then either $r = f + s$, for some $s \in R$. Hence $(f + s)x = fx, \rightarrow fx + sx = fx, \rightarrow sx = 0, \rightarrow s \in \text{ann}(Rx), \rightarrow s \in \text{ann}(Ry), \rightarrow sy = 0, \rightarrow ry = (f + s)y = fy + sy = fy$. Or $r + s = f$, by similar process $rx = fx, \rightarrow ry = fy$, and then $f$ is well-defined. On the other hand, it is clear that $f$ is $R$-homomorphism.

Note that it is considered in this work that all semiring $R$ is a semisubtractive and all $R$-semimodules are cancellative. The following proposition gives another characterization of prime semimodules, which is analogous for modules [4].

**Proposition 3.3.** Let $M$ be a non-zero $R$-semimodule having an injective hull $\tilde{M}$, then the following statements are equivalent:

1. $M$ is a prime semimodule.
2. $M$ is contained in every non-zero invariant subsemimodule of $\tilde{M}$.

**Proof:** (1)$\Rightarrow$(2) Let $N$ be a non-zero invariant subsemimodule of $\tilde{M}$. We want to prove that $M \subseteq N$. Since $\tilde{M}$ is an essential extension of $M$, then $M \cap N \neq 0$. Thus $0 \neq x \in M \cap N$. Since $M$ is prime, then
∀0 ≠ y ∈ M, ann(Rx) = ann(Ry). We define f: Rx → Ry as follows: f(rx) = ry, ∀r ∈ R. By Lemma 3.2 we have that f is a well-defined R-homomorphism. Since M is injective R-semimodule then f can be extended to F: M → M, as in the following diagram.

\[
\begin{array}{ccc}
Rx & f & Ry \\
i_1 & & i_3 \\
& N & \downarrow F \\
& i_2 & M
\end{array}
\]

where \(i_1, i_2\) and \(i_3\) are the inclusion R-homomorphisms. Since N is an invariant subsemimodule of M, then \(F(Rx) \subseteq N\), but \(f(Rx) = Ry\), then \(y \in N\), hence \(M \subseteq N\).

(2) \(\Rightarrow\) (1). Let N be a non-zero subsemimodule of M. Since \(\text{ann}(M) \subseteq \text{ann}(N)\), we want to show that \(\text{ann}(N) \subseteq \text{ann}(M)\). Assume that \(\exists r \in R\) such that \(r \in \text{ann}(N)\), and \(\exists x \in M\) with \(rx \neq 0\). Since \(0 \neq \exists \neq y \in N\). Now, \(\pi(Ry, \overline{M}) = \sum_{\phi} \phi(Ry)\), \(\phi \in \text{Hom}(Ry, \overline{M})\). Since \(\pi \subseteq M \subseteq \overline{M}\), so \(\pi(Ry, \overline{M})\) is a non-zero subsemimodule of \(\overline{M}\), and it is easy to check that \(\pi(Ry, \overline{M})\) is an invariant nonzero submodule of \(\overline{M}\). Thus by assumption \(M \subseteq \pi(Ry, \overline{M})\). Then \(\exists r_1, r_2, \ldots, r_n \in R\), and \(\exists \phi_1, \phi_2, \ldots, \phi_n \in \text{Hom}(Ry, \overline{M})\) such that, \(x = \sum_{i=1}^{n} \phi_i(r_iy)\). Thus, \(rx = \sum_{i=1}^{n} r\phi_i(r_iy) = \sum_{i=1}^{n} \phi_i(r_iy) = 0\), which is a contradiction. Then \(\text{ann}(N) \subseteq \text{ann}(M)\), and hence \(M\) is a prime semimodule.

From Proposition 2.4, we have that every \(\pi\) semimodule is a prime semimodule. Thus we have the following corollary.

**Corollary 3.4.** Let M be a semimodule having an injective hull \(\overline{M}\). If M is a \(\pi\) semimodule then M is contained in every non-zero invariant submodule of \(\overline{M}\).

**Proposition 3.5.** Let M be a non-zero semimodule having an injective hull \(\overline{M}\). If M is invariant subsemimodule of \(\overline{M}\) then the following statements are equivalent:

1. M is a prime semimodule.
2. M has no non-trivial invariant subsemimodules.

**Proof:** (1) \(\Rightarrow\) (2). Let N be a non-zero invariant subsemimodule of M. Because M is an invariant subsemimodule of \(\overline{M}\), so it can easily seen that N is also invariant subsemimodule of \(\overline{M}\). Thus, by Proposition 3.3 we have \(M \subseteq N\), and hence \(M = N\).

(2) \(\Rightarrow\) (1). Let K be a non-zero invariant subsemimodule of \(\overline{M}\). By Proposition 3.3, it is enough to show that \(M \subseteq K\). Since \(\overline{M}\) is an essential extension of M, hence \(M \cap K \neq (0)\). Now we claim that \(M \cap K\) is an invariant subsemimodule of M. If this is proved, then by assumption M has no non-trivial invariant subsemimodules and thus \(M \cap K = M\), which implies that \(M \subseteq K\).

To prove the claim, consider f any homomorphism in \(\text{Hom}(M, M)\). Since \(f(M \cap K) \subseteq f(M) \cap f(K)\), and since \(f(M) \subseteq M\), so it is enough to show that \(f(K) \subseteq K\). Because \(\overline{M}\) is an injective semimodule, then f can be extended to \(F \in \text{Hom}(\overline{M}, \overline{M})\), but K is an invariant subsemimodule of \(\overline{M}\). Thus \(f(K) = F(K) \subseteq K\), hence \(M \cap K\) is an invariant subsemimodule of M.

Now, as in the modules [7, page 22], we say that an R-semimodule M is said to be quasi-injective if each homomorphism from any subsemimodule N into M can be extended to a homomorphism of M to M. Note that any simple semimodule, and any injective semimodule, is quasi-injective. However, a quasi-injective semimodule needs not to be injective. For example, for each prime number p, \(\mathbb{N}_{p^n}\) is considered as a \(\mathbb{N}\)-semimodule which is quasi-injective. In verity, the only non-zero subsemimodules of \(\mathbb{N}_{p^n}\) are \(\mathbb{N}_{p^k}\), \(1 \leq k \leq n\). Then, for each \(f \in \text{Hom}(\mathbb{N}_{p^k}, \mathbb{N}_{p^n})\), and all \(x \in \mathbb{N}_{p^k}\), the order of \(f(x)\) is less than or equal to \(p^k\), hence \(f(\mathbb{N}_{p^k}) \subseteq \mathbb{N}_{p^k}\). It is clear that \(f\) can be extended to a homomorphism in \(\text{Hom}(\mathbb{N}_{p^n}, \mathbb{N}_{p^n})\). Whereas, \(\mathbb{N}_{p^n}\) is not injective.

The following theorem gives the relation between invariant and quasi-injective Semimodules.

**Theorem 3.6.** Let M be a semimodule having an injective hull \(\overline{M}\). If M is an invariant subsemimodule
of $\hat{M}$ then $M$ is a quasi-injective.

**Proof:** Assume that $M$ is a non-zero invariant subsemimodule having an injective hull, and $\alpha \in \text{Hom}(\hat{M}, \hat{M})$. Since $\hat{M}$ is injective, it is enough to consider that $\alpha \in \text{Hom}(M, \hat{M})$. Let $X \hookrightarrow M$ and $\beta: X \rightarrow M$ be a homomorphism. Since $\hat{M}$ is injective, $\beta$ can be extended to $\alpha: M \rightarrow \hat{M}$. By assumption, $\alpha(M) \subseteq M$, and hence $\alpha: M \rightarrow M$ extends $\beta$. Therefore $M$ is Quasi-injective. See the diagram below.

![Diagram](image)

**Remark 3.7.** We showed in Example 2.6 that $M = \mathbb{Z} \oplus \mathbb{Q}$ is considered as a $\mathbb{Z}$-semimodule which is a prime semimodule, and we proved that $M$ is not a $\pi$ semimodule. We show now that $M$ is not a quasi-injective semimodule.

**Proof:** Let $N = (m/n)$ be a cyclic subsemimodule of $\mathbb{Q}$ generated by the non-zero element $m/n$, where $\text{g.c.d}(m, n) = 1$. We define $f: (m/n) \rightarrow \mathbb{Z}$ as follows: $f(r \cdot m/n) = rm$, $\forall r \in \mathbb{Z}$.

It is clear that $f$ is a well-defined $\mathbb{Z}$-semimodule. Consider the diagram.

![Diagram](image)

where $i_1$ is the inclusion into the first factor and $i_2$ is the inclusion in the second factor. Suppose that $f$ can be extended to $F \in \text{Hom}(\mathbb{Z} \oplus \mathbb{Q}, \mathbb{Z} \oplus \mathbb{Q})$. Let $\rho: \mathbb{Z} \oplus \mathbb{Q} \rightarrow \mathbb{Z}$ be the natural projection, and let $f_1 = \rho \circ f|_{\mathbb{Q}}$. It is easily seen that $f_1$ is a non-zero element in $\text{Hom}(\mathbb{Q}, \mathbb{Z})$. But $\text{Hom}(\mathbb{Q}, \mathbb{Z}) = (0)$, which is a contradiction. This completes the proof.

We conclude that $M = \mathbb{Z} \oplus \mathbb{Q}$ is not an invariant subsemimodule of its injective hull $\hat{M} = \mathbb{Q} \oplus \mathbb{Q}$. Thus we arrive at the following main theorem.

**Theorem 3.8.** Let $M$ be any prime semimodule having an injective hull $\hat{M}$. If $M$ is an invariant subsemimodule of $\hat{M}$, then $M$ is a $\pi$ semimodule.

**Proof:** We use the characterization of $\pi$ semimodules given in Proposition 2.11. So let $N$ be a non-zero stable subsemimodule of $M$, then we have to show that $M$ is a contained in $N$. From the definition of stability, it is easy to see that $N$ is invariant subsemimodule of $M$. By assumption, $M$ is an invariant and prime semimodule and, using Proposition 3.5, $M$ has no non-trivial invariant subsemimodule.

Therefore, $M \subseteq N$. This completes the proof.

The following corollary is immediate from Corollary 3.4 and Theorem 3.8.

**Corollary 3.9.** Let $M$ be a semimodule having an injective hull $\hat{M}$, if $M$ is an invariant subsemimodule of $\hat{M}$, then $M$ is a $\pi$ semimodule.

Next, similar to the case in the modules [20], we can say that an $R$-semimodule $M$ is called compressible if every non-zero subsemimodule of $M$ contains an isomorphic copy of $M$. As a trivial example:

- Every simple $R$-semimodule is compressible.
- $\mathbb{N}$ as a $\mathbb{N}$-semimodule is compressible.
• \( \mathbb{Q}^+ \) as a \( \mathbb{N} \)-semimodule is not compressible since \( \mathbb{Q}^+ = \text{Hom}(\mathbb{Q}^+, \mathbb{N}) = (0) \).

The following shows that the class of prime semimodules contains the class of compressible semimodules.

**Theorem 3.10.** Every compressible \( R \)-semimodule is a prime \( R \)-semimodule.

**Proof:** Let \( M \) be a compressible \( R \)-semimodule, and let \( 0 \neq N \hookrightarrow M \). Now, we show that \( \text{ann}(N) = \text{ann}(M) \).

Since \( \text{ann}(M) \subseteq \text{ann}(N) \). So it is enough to prove that \( \text{ann}(N) = \text{ann}(M) \). Since \( M \) is compressible, then \( \exists \) a monomorphism \( \alpha: M \to N \). Hence, \( \forall r \in \text{ann}(N), r\alpha(M) = (0) \), thus \( \alpha(rM) = (0) \), which implies that \( rM = (0) \), and \( r \in \text{ann}(M) \), thus \( \text{ann}(N) \subseteq \text{ann}(M) \). This completes the proof.

**References**

1. Al-Alwan, F. H. 1993. Dedekind Modules and The Problem of Embeddability. Ph.D. Thesis, University of Baghdad, Baghdad, Iraq.
2. Alhossaini, A. M. A. and Abdul Al-Ameer, H. A. 2017. Fully Stable Semimodules. AL-Bahir Quarterly Adjudicated Journal for Natural and Engineering Research and Studies. 5(9 and 10): 13-20.
3. Birge Zimmermann-Huisgen. 1975. Endomorphism rings of self generators. *Pacific J. Math.*, 61: 587-602.
4. Beachy, J. A. 1976. Some Aspects of noncommutative Localization. Lecture Notes in Mathematics, Vol. 545, Springer-Verlag, New York.
5. Deore, R. P. and Patil, K. B. 2005. On the Dual Basis of Projective Semimodules and its Applications. *Sarajevo Journal of Mathematics*. 1(14): 161-169.
6. Ebrahimi Atani, R. and Ebrahimi Atani, S. 2010. On subsemimodules of semimodules. Buletinul Academiei De Stiinte. A Republicii Moldova. Matematica. 2(63): 20-30.
7. Faith, C. 1967. *Lectures on injective modules and quotient rings*. Springer-Verlag, Berlin, Heidelberg, New York.
8. Gordon, M. L. and Smith, P. F. 1990. *Multiplication Modules and Ideals*. Departmental report, University of Glasgow.
9. Golan, J. S. and Tom Head. 1991. *Modules and the Structure of Rings*: A Primer. Monographs and textbooks in pure and applied mathematics.
10. Golan, J. S. 1999. *Semirings and Their Applications*. Kluwer Academic Publishers, Dordrecht.
11. Ghalandarzadeh, S., Nasehpour, P. and Razavi, R. 2017. Invertible Ideals and Gaussian Semirings. *Math. Ac*, arXiv:1404.1901v4.
12. Hebisch, U. and Weinert, H. J. 1998. *Semirings-Algebraic Theory and Applications in Computer Science*. World Scientific. Singapore.
13. Il’in, S. N. 2016. On injective envelopes of semimodules over semirings. *Journal of Algebra and Its Applications*, 15(6): 1-13.
14. Kasch, F. 1982. *Modules and Rings*. Academic Press. London.
15. Kim, C. B. 1985. A Note on the Localization in Semirings. *Journal of Scientific Institute at Kookmin Univ.*, 3: 13-19.
16. Larsen, M. D. and McCarthy, P. J. 1971. *Multiplicative Theory of Ideals*. Academic Press, New York and London.
17. Sharpe, D. W. and Vamos, P. 1972. *Injective modules*. Cambridge University Press, London.
18. Tsiba, J. R. and Sow, D. 2010. On Generators and Projective Semimodules. *International Journal of Algebra*. 4(24): 1153-1167.
19. Wang, H. 1994. Injective Hulls of Semimodules over Additively-Idempotent Semirings. *Semigroup Forum*. Springer-Verlag New York Inc., 48: 377-379.
20. Zelmanowitz, J. 1977. Dense ring of linear transformations. *Ring Theory II. Proceedings of the Second Oklahoma Conference*, Marcel Dekker, New York.