Quantum effects for the Dirac field in a Reissner–Nordström–AdS black hole background

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Abstract

The behavior of a charged massive Dirac field on a Reissner–Nordström–AdS black hole background is investigated. We first analyze the problem of the essential self-adjointness of the Dirac Hamiltonian, which is made difficult by the boundary-like behavior of spatial infinity, and we find that the Hamiltonian is essentially self-adjoint iff \( \mu L \geq \frac{1}{2} \); moreover, we determine the essential spectrum of the Hamiltonian. Then we focus on the analysis of the discharge problem for the case \( \mu L \geq \frac{1}{2} \). We follow the Ruffini–Damour–Deruelle approach and, as in the standard Reissner–Nordström black hole case, we find that the existence of level-crossing between the positive and negative energy solutions of the Dirac equation is at the root of the pair-creation process associated with the discharge of the black hole.

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1. Introduction

AdS background geometry has been considered a challenging field for quantum field theory in different frameworks, including supersymmetry and string theory. In this paper, we take into account the problem of the discharge of a Reissner–Nordström–AdS black hole by quantum effects, by extending the analysis carried out by \([1, 2]\) on the case of a charged black hole which is asymptotically AdS. This kind of analysis, which is addressed in the case of massive charged Dirac fields, also represents a completion on the quantum side of the analysis carried out in \([3]\), where classical superradiance effects in a Kerr–Newman–AdS black hole background are considered for the case of an uncharged scalar field, both in the case of reflecting boundary conditions and in the case of transparent boundary conditions. Herein, the behavior of a charged massive Dirac field on a Reissner–Nordström–AdS black hole background is investigated. The essential self-adjointness of the Dirac Hamiltonian is
studied. Then, an analysis of the discharge problem is carried out in analogy with the standard Reissner–Nordström black hole case.

2. Dirac Hamiltonian

In this section, we check if the one-particle Hamiltonian is well defined in the sense that no boundary conditions are required in order to obtain a self-adjoint operator. In other terms, we check if the Hamiltonian is essentially self-adjoint, that is, if a unique self-adjoint extension and a uniquely determined physics occur. A general study for the Dirac equation on the Kerr–Newman–AdS black hole background is in progress [4].

We first define the one-particle Hamiltonian for Dirac massive particles on the universal covering space of the Reissner–Nordström–AdS black hole geometry (in such a way that closed timelike curves do not occur; see, e.g., [5]), to which we refer for simplicity as RN–AdS in the following. We use natural units $\hbar = c = G = 1$ and unrationalized electric units. The metric of the RN–AdS manifold $(t \in \mathbb{R}; r \in (r_+, +\infty); \Omega \in S^2)$ is

$$ds^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega^2 \quad f(r) = 1 + \frac{r^2}{L^2} - \frac{2M}{r} + \frac{Q^2}{r^2}. \quad (1)$$

$r_+$ is the radius of the black hole event horizon (see also appendix A), $L$ is a length associated with the cosmological constant $\Lambda$ by $\Lambda = -3/L^2 \Leftrightarrow L = \sqrt{|\Lambda|}$; $M$ is the mass and $Q$ is the electric charge. The vector potential associated with the RN–AdS solution is $A_\mu = (-Q/r, 0, 0, 0)$. The spherical symmetry of the problem allows us to separate the variables and to study a reduced problem on a fixed eigenvalue sector of the angular momentum operator. For a complete deduction of the variable separation see, e.g., [6, 7]. We get the following reduced Hamiltonian:

$$H_{\text{red}} = \left[ \sqrt{\mathcal{J}} + eQ \right] - f \partial_\tau + k \sqrt{\mathcal{J}} \quad \left[ -f \partial_\tau + k \frac{\sqrt{\mathcal{J}}}{r} \right].$$

where $f(r)$ is the same as in (1), $k = \pm(j + 1/2) \in \mathbb{Z} - \{0\}$ is the angular momentum eigenvalue and $\mu$ is the mass of the Dirac particle. The Hilbert space in which $H_{\text{red}}$ is formally defined is the Hilbert space $L^2((r_+, +\infty), 1/f(r) dr)^2$ of the two-dimensional vector functions $\vec{g} \equiv (g_1, g_2)$ such that

$$\int_{r_+}^{+\infty} \frac{dr}{f(r)} (|g_1(r)|^2 + |g_2(r)|^2) < \infty.$$ 

As a domain for the minimal operator associated with $H_{\text{red}}$ we can choose the following subset of $L^2((r_+, +\infty), 1/f(r) dr)^2$: the set $C_0^\infty((r_+, +\infty))^2$ of the two-dimensional vector functions $\vec{g}$ whose components are smooth and of compact support [8]. It is useful to define a new tortoise-like variable $y$

$$\frac{dy}{dr} = -\frac{1}{f(r)} \quad (2)$$

and to choose an arbitrary integration constant in such a way that $y \in (0, +\infty)$. The reduced Hamiltonian becomes

$$H_{\text{red}} = D_0 + V(y), \quad (3)$$

where

$$D_0 = \begin{bmatrix} 0 & \partial_y \\ -\partial_y & 0 \end{bmatrix}$$
The given linear system displays a singularity of the first kind at \( x = 0 \) if for every \( \lambda \) equations ([8], theorem 5.6) states that the so-called limit circle case (LCC) occurs at \( x = 0 \) if for every \( \lambda \in \mathbb{C} \) all the solutions of \( (H_{\text{red}} - \lambda)g = 0 \) lie in \( L^2([0, b), dy)^2 \) in a right neighborhood \((0, b)\) of \( y = 0 \). If at least one solution not square integrable exists for every \( \lambda \in \mathbb{C} \), then no boundary condition is required and the so-called limit point case (LPC) is verified. The occurrence of LCC implies the necessity to introduce boundary conditions in order to obtain a self-adjoint operator. If at least one solution not square integrable exists for every \( \lambda \in \mathbb{C} \), then no boundary condition is required and the so-called limit point case (LPC) is verified. The same arguments can be applied for \( y = +\infty \).

The Hamiltonian operator is essentially self-adjoint if the LPC is verified both at \( y = 0 \) and at infinity (cf [8], theorem 5.7).

\[
H_{\text{red}}g = \lambda g
\]

are square integrable in a right neighborhood of \( y = 0 \) and in a left neighborhood of \( y = +\infty \).

The so-called Weyl alternative generalized to a system of first-order ordinary differential equations ([8], theorem 5.6) states that the so-called limit circle case (LCC) occurs at \( y = 0 \) if for every \( \lambda \in \mathbb{C} \) all the solutions of \( (H_{\text{red}} - \lambda)g = 0 \) lie in \( L^2([0, b), dy)^2 \) in a right neighborhood \((0, b)\) of \( y = 0 \). If at least one solution not square integrable exists for every \( \lambda \in \mathbb{C} \), then no boundary condition is required and the so-called limit point case (LPC) is verified.

Note that, if for a fixed \( \lambda_0 \in \mathbb{C} \) all the solutions of \( (H_{\text{red}} - \lambda_0)g = 0 \) and of \( (H_{\text{red}} - \lambda_0)g = 0 \) lie in \( L^2([0, b), dy)^2 \) in a right neighborhood \((0, b)\) of \( y = 0 \), then this holds true for any \( \lambda \in \mathbb{C} \) ([8], theorem 5.3). The occurrence of LCC implies the necessity to introduce boundary conditions in order to obtain a self-adjoint operator. If at least one solution not square integrable exists for every \( \lambda \in \mathbb{C} \), then no boundary condition is required and the so-called limit point case (LPC) is verified. The same arguments can be applied for \( y = +\infty \).

The Hamiltonian operator is essentially self-adjoint if the LPC is verified both at \( y = 0 \) and at infinity (cf [8], theorem 5.7).

2.1. RN–AdS: behavior at \( r = \infty \)

In order to determine the essential self-adjointness properties of the Dirac operator in the RN–AdS black hole background, a study of the behavior of the first-order differential system (4) near \( r = \infty \) has to be done. The substitution \( x = 1/r \) allows us to map the interval \((r_\infty, \infty)\) into the interval \((0, 1/r_\infty)\). The reduced Hamiltonian is formally self-adjoint in \( L^2((0, 1/r_\infty), 1/(x^2 f(x)) dx)^2 \). Then the behavior as \( x \to 0 \) of the solutions has to be determined. In particular the eigenvalue equation can be re-written as follows:

\[
x \tilde{g} = \begin{bmatrix}
\frac{\sqrt{f}}{\sqrt{f}} - \mu L - \frac{e \mu L^2}{\sqrt{f}} + \lambda L x^2 \\
-\frac{1}{\sqrt{f}} + \frac{\sqrt{f}}{\sqrt{f}} + \lambda L x^2 \\
\end{bmatrix}^2 \tilde{g},
\]

where the prime indicates a derivative with respect to \( x \) and \( h(x) \) is defined by

\[
h(x) = 1 + L^2 x^2 - 2 M L^2 x^3 + Q^2 L^2 x^4 = L^2 x^2 f(x).
\]

The matrix on the left of the above system is regular at \( x = 0 \) and its limit as \( x \to 0 \) is given by the constant matrix

\[
A_0 = \begin{bmatrix}
0 & -\mu L \\
-\mu L & 0
\end{bmatrix}.
\]

The given linear system displays a singularity of the first kind at \( x = 0 \) [9], also called a weakly singular point [10]. Then the eigenvalues \( \epsilon_\pm = \pm \mu L \) of the matrix \( A_0 \) determine the asymptotic behavior of the solutions of the system near \( x = 0 \) (cf theorem V-1-2 p.110 of [9]), which, near \( x = 0 \) behave as follows: if \( \epsilon_+ - \epsilon_- = 2 \mu L \neq m \), with \( m \in \mathbb{N} - \{0\} \), then two linearly independent solutions are of the form

\[
\tilde{g}(x) = x^{\epsilon_+} \tilde{w}(x)
\]
where $\vec{w}(x) = \sum_{k=0}^{\infty} \vec{w}_k x^k$, $\vec{v}(x) = \sum_{k=0}^{\infty} \vec{v}_k x^k$ are (formal) series. If $\epsilon_+ - \epsilon_- = m$, with $m \in \mathbb{N} - \{0\}$, then two linearly independent solutions are of the form

$$\vec{g}_+(x) = x^{\epsilon_+} \vec{w}(x)$$

$$\vec{g}_-(x) = x^{\epsilon_-} \vec{v}(x) + c \vec{g}_+(x) \log(x).$$

One finds the essential self-adjointness condition

$$\mu L \geq \frac{1}{2}.$$  

It is worth pointing out that this condition coincides with the one occurring in the pure AdS case (cf [11] and see also appendix A). Note that, if the condition $\mu > 0$ on the mass is relaxed, then the Dirac Hamiltonian is self-adjoint for $|\mu L| \geq \frac{1}{2}$, as is easy to show.

### 2.2. RN–AdS: behavior at $r = r_+$

We introduce again the tortoise-like coordinate $y$. We find that

$$H_{\text{red}} = \begin{bmatrix} \sqrt{T} \mu + \frac{\epsilon_0}{r} & \epsilon_+ k \frac{\sqrt{T}}{r} + \partial_+ k \frac{\sqrt{T}}{r} \\ -\partial_+ k \frac{\sqrt{T}}{r} & -\sqrt{T} \mu + \frac{\epsilon_0}{r} \end{bmatrix}.$$  

In order to study the behavior on the horizon, i.e. for $y \to \infty$, one can simply apply the corollary to theorem 6.8 (p 99) of [8] and see that the LPC occurs on the horizon. This holds true both in the non-extremal case and in the extremal one. Thus, the Dirac operator is essentially self-adjoint on the RN–AdS black hole background if the bound on $\mu$ which is found in the pure AdS case is implemented. An alternative proof is found in appendix B.

### 2.3. Essential spectrum from near the horizon

One expects that, in the presence of an event horizon, i.e. of a so-called ergosurface, the mass gap vanishes and that the continuous spectrum includes the whole real line. We recall that qualitative spectral methods for the Dirac equation (see, e.g. [7, 8]) have been applied to Dirac fields on a black hole background in [12, 13]. In order to verify this property, we adopt the decomposition method [8]. We introduce the restriction of the Hamiltonian near the horizon, say to the interval $(r_+, r_0)$, where $r_0$ is arbitrary, and near infinity, say to the interval $(r_0, \infty)$. For our purposes, it is sufficient to consider the former restriction. In the tortoise-like coordinate $y$ one finds a potential $P$ such that

$$P = \begin{bmatrix} \sqrt{T} \mu + \frac{\epsilon_0}{r} & k \frac{\sqrt{T}}{r} \\ k \frac{\sqrt{T}}{r} & -\sqrt{T} \mu + \frac{\epsilon_0}{r} \end{bmatrix}$$  

and it holds

$$\lim_{y \to \infty} P(y) = P_0 = \begin{bmatrix} \Phi_+ & 0 \\ 0 & \Phi_- \end{bmatrix}.$$  

4 Theorem 6.8 of [8] states that, given a Dirac system $\tau \vec{u} = B^{-1} [J \vec{u} + P \vec{u}]$, with $x \in (a, b)$ and $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, if $B = k(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, with $k(x) \not\in L^1(c, b)$ for all $c \in (a, b)$, then $\tau$ is in the LPC at $b$. As a corollary, if $b = \infty$ and $k(x) = d > 0$, with $d = \text{const.}$, then $\tau$ is in the LPC at $b = \infty$. 

4
which is in diagonal form and whose eigenvalues coincide. We apply theorem 16.6 on p 249 of [8], which implies that, if \( \nu_-, \nu_+ \), with \( \nu_- \leq \nu_+ \), are the eigenvalues of the matrix \( P_0 \), then \( [\mathbb{R} - (\nu_-, \nu_+)] \subset \sigma_e(H_{\text{red}}(\gamma(r_0), \infty)) \) if

\[
\lim_{y \to \infty} \frac{1}{y} \int_{x_0}^{y} \text{d}t \| P(t) - P_0 \| = 0,
\]

where \( \| \cdot \| \) stays for any norm in the set of \( 2 \times 2 \) matrices (we choose the Euclidean norm). In our case one has to find the limit as \( y \to \infty \) for the following expression:

\[
\frac{1}{y} \int_{r(y)}^{r_1} \frac{1}{h(r)} \sqrt{r - r_+} \left( 2\mu^2 h(r) + 2(\Phi_2^2(r - r_+) + k^2 h(r)) \right) \nu^2,
\]

where we put \( h(r) = f(\mu L) \). In the non-extremal case \( r_+ > r_- \), the above integral is finite as \( r \to r_+ \) and then the limit is zero. In the extremal case \( r_+ = r_- \) the integral above displays a logarithmic divergence as \( r \to r_+ \); a trivial use of the l’Hospital’s rule allows us to find that the aforementioned limit is still zero. As a consequence, we can state that

\[
\sigma_e(H_{\text{red}}) = \mathbb{R}.
\]

\[3. \text{Pair creation and level-crossing}\]

In this section, we limit ourselves to the study of the case where essential self-adjointness is implemented. In particular, in the case of the Dirac field we impose \( \mu L \geq 1/2 \).

In the RN black hole background, the presence of an effect of particle creation can be related to the Klein paradox, i.e. to the possibility of finding regions where negative energy states overlap with positive energy states.

We follow the Ruffini–Damour–Deruelle approach [14–17], in which one identifies the effective potentials \( E_0^\pm(r) \) for the positive and negative energy states respectively; they represent the classical turning points for the ptc. motion and lead to the definition of the so-called effective ergosphere. These potentials enter the Hamilton–Jacobi (HJ) equation for a classical particle. They can be interpreted also at the quantum level, as in [17]. In particular, they indicate the regions of level-crossing between positive and negative energy states [15, 16]. See also [18]. We give some details below.

As far as the Dirac equation is concerned, it is known that the HJ equation corresponds to a WKB approximation to the Dirac equation at the lowest order [19, 20]. Compare also [21] and [22] for a semiclassical approach to the Dirac equation. We first recall the HJ equation and find \( E_0^\pm(r) \), then we show that it is possible to find a level-crossing even in the RN–AdS case. The HJ equation for a classical ptc. is

\[
g^{\mu\nu}(\partial_\mu S - e A_\mu)(\partial_\nu S - e A_\nu) + \mu^2 = 0.
\]

Variable separation leads to \( S = -\omega t + \Theta(\theta, \phi) + R(z) \), where \( dz = -dr/\Delta \), and

\[
\left( \frac{dR}{dz} \right)^2 = -Z,
\]

where

\[
Z \equiv \Delta (\mu^2 r^2 + K) - [\omega r^2 - e Q r]^2,
\]

and, in the RN case, \( \Delta = r^2 - 2Mr + Q^2 \) (\( K \) is the square of the angular momentum vector) [15]. The classical region of accessibility is defined by \( Z \leq 0 \). One has

\[
\Delta^2 \left( \frac{dR}{dr} \right)^2 = r^4 (E - E_0^+(r))(E - E_0^-(r)).
\]
where
\[ E_0^\pm(r) = \frac{eQ}{r} \pm \frac{1}{r^2} \sqrt{\Delta(\mu^2 r^2 + K)}, \quad (18) \]

From a classical point of view, \( E_0^\pm(r) \) represent effective potentials for the solutions having positive and negative energy, and particles states are defined for \( E > E_0^+(r) \). Classical bound states correspond to circular or elliptical orbits and require the presence of a couple of turning points. States such that \( E_0^-(r) < E < E_0^+(r) \) are forbidden (they correspond to particles having imaginary momentum). States with \( E < E_0^-(r) \) are also classically forbidden, because they correspond to particles with negative mass (and negative energy). The latter states are meaningful at the quantum level (anti-particles) \([15, 16]\).

Variable separation in the quantum case allows us to obtain an obvious improvement of the above classical formulae, amounting to replacing the classical value of \( K \) with the quantum eigenvalues of the corresponding quantum operator \( \hat{K} \). The analogy with the replacement of the classical centrifugal term \( \mu^2/r^2 \) with the quantum term \( \hbar^2/(2m^2) \) in the spherosymmetric case for the Schrödinger equation is self-evident. In subsection 3.2, the meaning of the aforementioned improved potentials will be discussed.

It can be shown that it is possible to find level-crossing in the RN–AdS case, where \( \Delta = r^2 - 2Mr + Q^2 + r^4/L^2 \). In particular, a level-crossing surely occurs when \( E_0^+(r) < E_0^-(r) \), see below. Note also that \( E_0^-(r) = eQ/r \), i.e., both the positive and negative energy states assume the same value on the horizon (this corresponds to the well-known spectral property that the contribution to the essential spectrum arising from near the horizon is \( \mathbb{R} \)).

### 3.1. Level-crossing

It is useful to define for \( eQ > 0 \)
\[ P(r) \equiv E_0^+(r) - \frac{eQ}{r_s} = -\frac{eQ}{rr_s}(r-r_s) + \frac{1}{r^2} \sqrt{\Delta(\mu^2 r^2 + K)}, \quad (19) \]
and
\[ N(r) \equiv E_0^-(r) - \frac{eQ}{r_s} = -\frac{eQ}{rr_s}(r-r_s) - \frac{1}{r^2} \sqrt{\Delta(\mu^2 r^2 + K)}. \quad (20) \]

It is evident that, for any \( r > r_s \), one has \( N(r) < 0 \) (for \( eQ < 0 \) an obvious change in the definitions above is required). This means that \( P(r) < 0 \) is the condition which has to be satisfied in order to find a level-crossing \( P(r) \geq 0 \) for all \( r > r_s \), means \( E_0^+ > E_0^- \) in the same region, i.e., no level-crossing. One has to study the roots of the equation \( P(r) = 0 \) for \( r \geq r_s \) and then find out the sign of \( P(r) \) between two consecutive roots. Of course \( r = r_s \) is a root of \( P(r) \). It is necessary to find another root for \( r > r_s \) in order to find level-crossing. It is useful to write
\[ \Delta = (r-r_+)(r-r_-)G(r), \quad (21) \]
where \( G(r) \equiv (1/L^2)[r^2 + (r_e + r_+)r + L^2 + r_s^2 + r^2 + r_s r_-] \), \( r_s \) and \( r_- \) are the event horizon radius and the Cauchy horizon radius, respectively. This re-writing takes into account that a term proportional to \( r^3 \) is missing in \( \Delta \), and it amounts to the following re-parametrization of \( f(r) \) in terms of \( L^2, r_s, r_- \):
\[ f(r) = \frac{1}{L^2 r^2}(r-r_+)(r-r_-)[r^2 + (r_e + r_-)r + L^2 + r_s^2 + r^2 + r_s r_-], \quad (22) \]
with
\[
M = \frac{(r_+ + r_-)(\eta - r_+r_-)}{2L^2},
\]
\[
Q^2 = \frac{\eta r_-}{L^2},
\]
where
\[
\eta = L^2 + r_+^2 + r_-^2 - (r_+ + r_-)(\eta - r_+r_-).
\]
One has to implement \( P(r) < 0 \), which amounts to
\[
\frac{1}{r^2L^2}(r - r_+)(r - r_-)(r^2 + (r_+ + r_-)r + \eta)(\mu^2r_+^2 + k^2) < \left(\frac{eQ}{r_+}\right)^2 (r - r_+)^2;
\]
then, for the extremal case one obtains
\[
\frac{1}{r^2L^2}(r^2 + 2r_+r + 3r_+^2 + L^2)(\mu^2r_+^2 + k^2) < \left(\frac{eQ}{r_+}\right)^2,
\]
i.e.
\[
\mu^2r_+^4 + 2r_+\mu^2r_+^3 + \left(3r_+^2 + L^2\right)\mu^2 + k^2 - \left(\frac{eQ}{r_+}\right)^2 r_+^2 + 2k^2r_+r_+ + k^2(3r_+^2 + L^2) < 0.
\]
From a naive inspection, one sees that the following necessary condition has to be implemented:
\[
(3r_+^2 + L^2)\mu^2 + k^2 - \left(\frac{eQ}{r_+}\right)^2 L^2 < 0.
\]
A more accurate study of the first derivative of \( E_0^+(r) \) shows that
\[
\frac{dE_0^+}{dr}(r = r_+) < 0
\]
is a sufficient condition for the existence of a level-crossing, which amounts to
\[
\left(\frac{eQ}{r_+}\right)^2 > \mu^2 + \frac{6\mu^2r_+^2}{L^2} + \frac{6k^2}{L^2} + \frac{k^2}{r_+^2}.
\]
When this sufficient condition is implemented, the standard condition
\[
\left(\frac{eQ}{r_+}\right)^2 > \mu^2,
\]
for the standard Reissner–Nordström case is also implemented.

In the general case, it is difficult to find out analytically the overlap region between the positive and negative energy states. It is still possible to find out level-crossing by means of graphical tools. We show two examples below. They concern the case of an extremal RN–AdS black hole and of a non-extremal one, respectively. In figure 1 the extremal case is displayed in the case that the particle and the black hole have charge with the same sign; for the same parameters, in figure 2 the opposite sign case is shown. Analogously, in figures 3 and 4 the non-extremal case is displayed.

3.2. The meaning of \( E_0^+ (r) \) in the case of the Dirac Hamiltonian

Let us consider the potential term in the Dirac Hamiltonian
\[
V(r) = \begin{bmatrix} p_{11}(r) & p_{12}(r) \\ p_{21}(r) & p_{22}(r) \end{bmatrix}
\]
One can formally calculate the eigenvalues of the above matrix, which are found by solving
\[(p_{11}(r) - \lambda)(p_{22}(r) - \lambda) - p_{12}(r)p_{21}(r) = 0; \quad (32)\]
then, defining \[S(r) \equiv \sqrt{(p_{11}(r) + p_{22}(r))^2 - 4p_{11}(r)p_{22}(r) + 4p_{12}(r)p_{21}(r)}\]
one finds
\[\lambda^\pm(r) = \frac{1}{2}(p_{11}(r) + p_{22}(r) \pm S(r)). \quad (33)\]
One has
\[\lambda^\pm(r) = E_0^\pm(r). \quad (34)\]
Figure 3. Level-crossing in the case of a non-extremal RN–AdS black hole, with $L = 10, M = 10000, Q = 100, m = 10^{-3}, e = 1, k = 1$. The particle and the black hole have charges with the same sign. Black hole horizon occurs for $r \sim 125.56$. The straight line represents the value $eQ/r$. The upper potential is $E^+_0(r)$, the lower one is $E^-_0(r)$. Level-crossing occurs where $E^+_0(r) < eQ/r$. The figure on the right displays the potentials near $r = r_*$. 

Figure 4. Level-crossing in the case of a non-extremal RN–AdS black hole, with $L = 10, M = 10000, Q = 100, m = 10^{-3}, e = -1, k = 1$. The particle and the black hole have charges with opposite sign. Black hole horizon occurs for $r \sim 125.56$. The straight line represents the value $eQ/r$. The upper potential is $E^+_0(r)$, the lower one is $E^-_0(r)$. Level-crossing occurs where $E^+_0(r) < eQ/r$. The figure on the right displays the potentials near $r = r_*$. 

i.e., the potentials coincide with the eigenvalues of the matrix potential term in the Dirac Hamiltonian. Moreover, one has

$$E^\pm_0(r) = \frac{eQ}{r} \pm \frac{1}{r^2} \sqrt{\Delta (\mu^2 r^2 + k^2)}. \quad (35)$$

Note again that the square of the classical angular momentum term is replaced by the square of the eigenvalues $k = \pm (j + 1/2)$ for the quantum angular momentum. Then, in the case of a Dirac particle, the angular momentum contribution $k$ cannot vanish.

From the point of view of qualitative spectral analysis, the eigenvalues–potentials $\lambda^\pm(r)$ play a relevant role in the following sense. Let us consider for simplicity the reduced Dirac Hamiltonian in the case of flat spacetime; referring, e.g. to theorems 16.5 and 16.6 of [8], which concern the essential spectrum of Dirac systems, one finds that: in theorem 16.5 a
fundamental role is played by the eigenvalues $\nu_- \leq \nu_+$ of the matrix $P_0 \equiv \lim_{r \to \infty} P(r)$. It is evident that

$$\nu_{\pm} = \lim_{r \to \infty} \lambda_{\pm}(r).$$

In theorem 16.6 the norm $||P(t) - P_0||$ can be replaced with the norm of the difference $(\lambda^+(t) - \lambda^-(t)) - U(t)P_0U^\dagger(t)$, where the unitary matrix $U(t)$ diagonalizes $P(t)$ and where $\lim_{r \to \infty} U(t)P_0U^\dagger(t) = \begin{pmatrix} \nu_+ & 0 \\ 0 & \nu_- \end{pmatrix}$ (this is quite obvious if one chooses to define the norm of any matrix $P$ as a norm in a suitable Hilbert space $\mathcal{H}$: $||P|| \equiv \sup_{v \in \mathcal{H}} \frac{||Pv||}{||v||}$, because in this case $||P|| = ||U P U^\dagger||$ for any unitary matrix $U$).

The aforementioned results can be applied also to the black hole case. See, e.g. [12] and also [13].

Some physical considerations are in order. In the presence of an external field (like, e.g. an external electrostatic field and a static non-flat spacetime metric) the properties of the second-quantized theory cannot be related in a straightforward way to the spectral decomposition of the one-particle Hamiltonian $H$ into the positive and negative energy states. This, of course, happens in the free case in flat spacetime, but, in the presence of external fields, an intrinsic relevance of the classical external potentials in determining the structure of the quantum field theory associated with the given one-particle Hamiltonian has to be stressed. In the given case, the potentials $\lambda^+$ and $\lambda^-$ not only delimit the allowed regions of the classical motion, but also give physical insights about the characteristics of the quantum field theory one can build up by starting from the given $H$. On the one hand, their role is analogous to the one of the potential in non-relativistic quantum mechanics for a single particle: the potential defines the allowed regions of the classical motion, gives rise to quantum resonances and to quantum tunneling phenomena and its behavior at infinity fixes also the limits of the essential spectrum (see, e.g. theorems 16 and 17, pp 1448–9 of [23] for simple mathematical results which can be applied to one-dimensional quantum mechanical systems and in particular to the radial Schrödinger equation). On the other hand, there is a further requirement which has to be satisfied if a well-defined second quantization of the Dirac field has to be obtained. It is the requirement that positive and negative continuous energy states must be separated from each other. It may happen that a phenomenon of level-crossing occurs, i.e. would-be negative energy states can overlap with positive energy ones. This lack of a clear separation between the positive and the negative energy solutions of the Dirac equation can be physically interpreted as the occurrence of an instability phenomenon which leads to pair creation [15]. In particular, defining as in [15] a complete set of positive and negative ‘in’ modes $p^\text{in}_i$ and $n^\text{in}_i$ and a complete set of ‘out’ modes $p^\text{out}_i$ and $n^\text{out}_i$, with the channel for the transition $n^\text{in}_i \rightarrow p^\text{out}_i$ a non-vanishing transition amplitude

$$T_{ik} = \langle p^\text{out}_i, n^\text{in}_k \rangle$$

(37)

can be associated. The mean number of pairs which are created is [15]

$$\langle N \rangle = \sum_{\text{channels}i,k} |T_{ik}|^2.$$  

(38)

This should be compared with the transmission coefficient in the Klein paradox situation, where the transmission coefficient is

$$|T|^2 = \frac{\text{transmitted flux}}{\text{incident flux}}.$$  

(39)

The absence of overlapping between the ranges of $\lambda^+$ and $\lambda^-$ avoids the occurrence of level-crossing. When an overlap occurs, the possibility that a positron state can emerge in a suitable
range as a positive energy scattering state is at the root of the Klein paradox and also of the pair-creation process (cf also [7]). Semi-quantitative estimates for this phenomenon can be given, e.g. in the WKB approximation. In the case analyzed herein, numerical estimates show that the presence of a non-vanishing $k$ makes it more difficult to find a level-crossing (which, in the scalar particle case occurs more easily for $l = 0$); nevertheless it is still possible to find an overlap which is not limited to $r = r_+$:

$$\lambda^+(r_+, \infty) \cap \lambda^-(r_+, \infty) \supset \lambda^+(r_+).$$  \hspace{1cm} (40)

Some further consideration has to be addressed to the problem of the choice of the quantum state playing the role of vacuum. In the extremal black hole case, the so-called Boulware vacuum, which corresponds to the positive and negative frequencies arising from the Hamiltonian, is allowed and then the previous analysis is enough. In the non-extremal case, one has to take into account the fact that the regular quantum state corresponds to the so-called Hartle–Hawking state, which is the thermal state at the Hawking temperature, whereas the Boulware vacuum is singular on the horizon. Suitable analyticity requirements for the fields on the extended manifold lead to the thermal state [1]. Notwithstanding, the thermal state can be obtained by ‘heating up’ the Boulware one at least in the Schwarzschild case, where the construction can be implemented rigorously for the scalar field [24]. For the Dirac field see [25]. In the presence of an electric charge, as in the present case, there is a further difficulty which is associated with the presence of level-crossing (it has manifestly the characteristics of the Klein paradox in the Reissner–Nordström case) which signals the presence of a further source of particle creation which is different from the one associated with the Hawking effect.

### 3.3. A picture for a discharge

According to the classical potentials described above, the negative energy state, for definiteness, the positron with suitable energy can tunnel beyond the barrier and reach the external region, i.e. in the classically allowed region where an observer can measure it and also determine that, by charge conservation, the charge of the black hole is diminished. This transition, in a WKB approximation, is exponentially suppressed. Near infinity, the positron meets a classical turning point where it is reflected back toward the barrier. The possibility of a further passage of the barrier is exponentially suppressed, and then the whole process of emission-re-absorption of the positron by the black hole is suppressed with respect to the simple emission process. As a consequence, the phenomenon of pair creation and emission of positrons by a positively charged black hole is allowed also in the RN–AdS black hole manifold. See also figure 5.

The emitted positrons are mostly confined in the classically allowed region, and they classically behave as bound states with a minimal and a maximal distance from the black hole. They do not correspond to stable quantum mechanical bound states, indeed there is no discrete spectrum for the Hamiltonian, and, as a consequence, they cannot be identified with isolated eigenvalues of the Hamiltonian. They could at most correspond to resonances (cf also [15]). This picture holds as far as back-reaction effects by the emitted positrons are negligible. In a WKB approximation, one finds [15]

$$|T|^2 = \exp \left( -2 \int_{\text{barrier}} d\zeta \sqrt{Z} \right).$$  \hspace{1cm} (41)

It is worth mentioning that in the case of non-extremal RN black holes [26], and more generally in the case of non-extremal stationary axisymmetric black holes [27] it has been shown that there is no solution of the Dirac equation which is both normalizable and time periodic. The interpretation of the aforementioned result is that Dirac particles either disappear
Figure 5. Pair creation in the case of a non-extremal RN–AdS black hole, with $L = 10$, $M = 10000$, $Q = 100$, $m = 10^{-3}$, $e = -1$, $k = 1$. The particle and the black hole have charges with opposite sign. Black hole horizon occurs for $r \sim 125.56$. The straight line represents the value $e Q/r_c$. The upper potential is $E^+_0(r)$, the lower one is $E^-_0(r)$.

3.4. Discussion

There is a problem in the definition of a sensible notion of a particle in the RN–AdS background. This kind of difficulty also occurs in the case of the pure AdS manifold. The above discussion about the presence of level-crossing in the RN–AdS case relies on the fact that the classical potentials $E^\pm_0(r)$ still maintain their heuristic meaning in the RN–AdS case as in the standard asymptotically flat RN case. The difference is that one cannot define asymptotic scattering in the black hole or escape to infinity [26]. One might wonder if the present analysis of the discharge mechanism implies a different behavior in the RN–AdS black hole case with respect to the results of [26]. We limit ourselves to point out herein that, in order to consider the problem of normalizable and time-periodic solutions for the RN–AdS case at hand, a further study is necessary: one has to determine the point spectrum of the Hamiltonian (see, e.g. Winklmeier and Yamada [13] for black holes of the Kerr–Newman family; see also [28] for the extremal Kerr case). We shall deal with this analysis in [4] for the more general case of a non-extremal Kerr–Newman–AdS black hole, and show that also in this case there is no time-periodic and normalizable solution of the Dirac equation. This means that a full agreement with the results of [26] occurs in the present RN–AdS case. As to the above qualitative description of the behavior of the positrons in the classically allowed region, as a consequence we can point out that they cannot correspond to time-periodic and normalizable solutions of the Dirac equation in the non-extremal case. Still, they are allowed to correspond to lasting (non-time-periodic) solutions in the classically allowed region (a classical analogy could be found with bounded open orbits in a potential corresponding to central potentials different from the Newton (Coulomb) potential or from the spatial oscillator potential). Moreover, quantum effects can allow the disappearance of the positrons in the black hole, as well as further pair creation.
states for the Hamiltonian as \( r \to \infty \) in the RN–AdS case as well as in the AdS one. Note that, in the latter case, one has

\[
E^\pm_0(r) = \pm \frac{1}{r^2} \sqrt{\left( \frac{r^4}{L^2} + r^2 \right) (\mu^2 r^2 + k^2)}.
\]

(42)

No level-crossing occurs, as expected. The behavior as \( r \to \infty \) of \( E^\pm_0(r) \) is the same as for the RN–AdS background, at the leading order.

3.5. A note on the case \( \mu L < \frac{1}{2} \)

In the case where essential self-adjointness is missing, one can in the scalar field case impose reflective boundary conditions (Dirichlet) and also Neumann boundary conditions as in the pure AdS case. In the case of the Dirac field, one can, e.g. impose the MIT bag boundary conditions which play the same role for the Dirac field as the Dirichlet boundary conditions for the scalar field; if \( \Psi \) is the four spinor and \( \vec{n} \) is the (outer) normal to the boundary \( \partial \Omega \) of the domain \( \Omega \), the MIT boundary conditions amount to

\[
\left. i \vec{\gamma} \cdot \vec{n} \Psi \right|_{\partial \Omega} = \left. \Psi \right|_{\partial \Omega}.
\]

(43)

It is easy to show that, in the spherosymmetrical case, these boundary conditions amount to

\[
g_1|_{\Omega} = g_2|_{\Omega}.
\]

(44)

By using the same coordinates as in section 2.1, one finds the condition

\[
g_1(x = 0) = g_2(x = 0).
\]

(45)

One can also introduce other local boundary conditions, see, e.g. [11] for the case of pure AdS. We do not take into account this topic any longer herein.

4. Conclusions

We have studied the self-adjointness properties of the Dirac Hamiltonian in the RN–AdS black hole background. We have found that the horizon does not require any boundary condition, whereas \( r = +\infty \) may behave as a boundary also in the Dirac case. Nevertheless, if the bound \( |\mu L| \geq 1/2 \) is implemented, then essential self-adjointness is ensured. In the latter case, we have taken into account the spectral properties of the Hamiltonian, and we have found that the essential spectrum is the whole real line. This means that the presence of a horizon also in this case is sufficient for ensuring that no discrete spectrum (which is by definition made of isolated eigenvalues of finite multiplicity) occurs. Moreover, a study of the potentials \( \lambda \pm \) which arise in a semi-classical approximation enabled us to verify that a level-crossing between states with energy \( E > \lambda^+ \), which correspond to classically allowed states, and states with energy \( E < \lambda^- \), which correspond to the classical counterpart of the Dirac sea states, can occur. This overlapping, as well known from the previous literature on this topic, indicates the possibility of finding out a phenomenon of pair-creation by means of tunneling from the Dirac sea to the classically allowed states. The WKB approximation can be implemented in order to describe the tunneling. As a consequence, a way to describe the discharge of the black hole by means of a quantum effect which has a different origin with respect to the Hawking effect (and, in fact, it occurs also in the case of extremal black holes) has been shown to be available also in the case of Dirac fields on RN–AdS black hole backgrounds.
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Appendix A. \( r_r \) for an RN–AdS black hole

The radius \( r_r \) of an RN–AdS black hole is determined by the higher real root of

\[
1 + \frac{r^2}{L^2} - 2\frac{M}{r} + \frac{Q^2}{r^2} = 0,
\]

(A.1)

or

\[
x^4 + x^2 - 2\mu x + q^2 = 0,
\]

(A.2)

where

\[
x = \frac{r}{L}, \quad \mu = \frac{M}{L}, \quad q = \frac{Q}{L}.
\]

(A.3)

The standard way to solve this equation is to introduce a free parameter \( y \) and rewrite the equation in the form

\[
(x^2 - y)^2 + 2y x^2 - y^2 + x^2 - 2\mu x + q^2 = 0,
\]

(A.4)

which is true for any \( y \) and then

\[
(x^2 - y)^2 = -(2y + 1) \left[ x^2 - \frac{2\mu}{2y+1} x + \frac{q^2 - y^2}{2y+1} \right].
\]

(A.5)

The strategy is to choose \( y \) so that the rhs of the equation is a square too. This happens if

\[
\left( \frac{\mu}{2y+1} \right)^2 = \frac{q^2 - y^2}{2y+1},
\]

(A.6)

Then

\[
y^3 + \frac{y^2}{2} - q^2 y + \frac{\mu^2 - q^2}{2} = 0,
\]

(A.7)

or, using \( z = y + 1/6 \)

\[
3 \left( q^2 + \frac{1}{12} \right) - \left( q^2 + \frac{\mu^2}{2} + \frac{1}{2^233} \right) = 0.
\]

(A.8)

This has the form

\[
z^3 = az + b,
\]

(A.9)

which can be solved by the Cardano formula

\[
z = \left( \frac{b}{2} + \sqrt{\frac{b^2}{4} - \frac{a^3}{27}} \right)^{\frac{1}{3}} + \left( \frac{b}{2} - \sqrt{\frac{b^2}{4} - \frac{a^3}{27}} \right)^{\frac{1}{3}}.
\]

(A.10)

We get

\[
y(\mu, q) = -\frac{1}{6} + \left( \frac{q^2}{6} - \frac{\mu^2}{4} - \frac{1}{6^3} + \sqrt{\sigma(\mu, q)} \right)^{\frac{1}{3}} + \left( \frac{q^2}{6} - \frac{\mu^2}{4} - \frac{1}{6^3} - \sqrt{\sigma(\mu, q)} \right)^{\frac{1}{3}},
\]

(A.11)
with
\[
\sigma(\mu, q) = \frac{\mu^4}{24} - \frac{q^6}{3^3} - \frac{\mu^2 q^2}{12} + \frac{q^4}{54} + \frac{\mu^2 - q^2}{2^4 3^3}.
\] (A.12)
If \( q \) is small (take it very small) enough then \( \sigma(\mu, q) \) is positive and \( y \) is real and from (A.6) we see that then \( 2y + 1 \) is negative and (A.5) takes the form
\[
x^2 - y = \pm \left( x \sqrt{|2y + 1|} + \frac{\mu}{\sqrt{|2y + 1|}} \right).
\] (A.13)
The discriminant is
\[
\Delta = -(2y + 1) + 4y \pm 4 \sqrt{y^2 - q^2},
\] (A.14)
and is positive only if we take the plus sign. Then the real roots satisfy
\[
x^2 - x \sqrt{|2y + 1|} - y - \frac{\mu}{\sqrt{|2y + 1|}} = 0.
\] (A.15)
Note that
\[
y - \frac{\mu}{\sqrt{|2y + 1|}} = |y| - \sqrt{y^2 - q^2} > 0
\] (A.16)
so that both the roots are positive and
\[
r_{\pm} = \frac{L}{2} \left[ \sqrt{|2y + 1|} \pm \sqrt{|2y + 1| + 4 \left( y + \sqrt{y^2 - \frac{Q^2}{L^2}} \right)} \right],
\] (A.17)
with \( y = y(M/L, Q/L) \) given by (A.11).

Appendix B. \( M = 0 = Q \) case: pure AdS

It is useful to introduce AIS (Avis–Isham–Storey [29]) coordinates
\[
ds^2 = L^2 \frac{1}{\cos^2(\rho)} [-dt^2 + d\rho^2 + \sin^2(\rho) \, d\Omega^2],
\] (B.1)
where \( 0 \leq \rho < \pi / 2 \) and \( t \in \mathbb{R} \) (universal covering space, called also CAdS [29]). One finds
\[
H_{\text{red}} = \begin{bmatrix}
\frac{1}{\cos \rho} \mu & -\frac{1}{L} \partial_\rho + k \frac{1}{L} \frac{1}{\sin \rho} \\
\frac{1}{L} \partial_\rho + k \frac{1}{L} \frac{1}{\sin \rho} & -\frac{1}{\cos \rho} \mu
\end{bmatrix}.
\]
It is easy to see that, in the limit as \( \rho \to 0^+ \) the LPC is verified, that is, one does not need to impose a boundary condition at \( \rho = 0 \) (\( r = 0 \) in the original coordinates). In order to study the limit as \( \rho \to (\pi/2)^- \) it is useful to introduce a new coordinate \( x \equiv (\pi/2) - \rho \). Then
\[
H_{\text{red}} = \begin{bmatrix}
\frac{1}{\sin x} \mu & \frac{1}{L} \partial_x + k \frac{1}{L} \frac{1}{\cos x} \\
-\frac{1}{L} \partial_x + k \frac{1}{L} \frac{1}{\cos x} & -\frac{1}{\sin x} \mu
\end{bmatrix}.
\]
One gets the following system of first-order equations for the eigenvalue equation:
\[
x \vec{g}' = \begin{bmatrix}
k \frac{1}{\cos x} - \frac{\mu}{\sin x} \mu L + \lambda Lx & -\frac{k}{\sin x} \mu L - \lambda Lx \\
-k \frac{1}{\cos x} & -k \frac{1}{\cos x}
\end{bmatrix} \vec{g} \equiv A(x) \vec{g};
\]
the prime indicates the derivative w.r.t. \( x \). The matrix \( A(x) \) is regular as \( x \to 0 \) and one has
\[
\lim_{x \to 0} A(x) = A_0 = \begin{bmatrix}
0 & -\mu L \\
-\mu L & 0
\end{bmatrix}.
\]
The same considerations as in subsection 2.1 apply. Restoring the physical constants, one finds the following condition for essential self-adjointness (see also [11]):
\[
\mu c^2 \geq \frac{1}{2} (\hbar c) \frac{1}{L} = \mu_{\text{bound}}.
\] (B.2)
Appendix C. LPC at the horizon

For the non-extremal case, we define
\[ z = r - r_+, \quad (C.1) \]
then the eigenvalue equation for \( H_{\text{red}} \) can be re-written as follows:
\[
z \ddot{g}' = \begin{bmatrix}
-\frac{k \sqrt{\varepsilon}}{(r_+ + z) \sqrt{f(z)}} & \frac{\sqrt{\varepsilon}}{\sqrt{f(z)}} \mu - \left( \frac{\epsilon Q}{r_+ z} - \lambda \right) \frac{1}{h(z)} \\
\frac{\sqrt{\varepsilon}}{\sqrt{f(z)}} \mu + \left( \frac{\epsilon Q}{r_+ z} - \lambda \right) \frac{1}{h(z)} & -\frac{k \sqrt{\varepsilon}}{(r_+ + z) \sqrt{f(z)}} \end{bmatrix} \dot{g};
\]
the prime stays for the derivative with respect to \( z \) and \( h(z) \) is defined by
\[ h(z) = \frac{f(z)}{z}. \quad (C.2) \]
The matrix on the left of the above system is regular at \( z = 0 \) and its limit as \( z \to 0 \) is given by the constant matrix
\[ A_0 = \begin{bmatrix}
0 & -\frac{1}{h(0)} (\Phi_+ - \lambda) \\
\frac{1}{h(0)} (\Phi_+ - \lambda) & 0
\end{bmatrix}, \]
where \( \Phi_+ \) is the electrostatic potential at the horizon. The eigenvalues of the above matrix are
\[ \epsilon_{\pm} = \pm i \frac{1}{h(0)} (\Phi_+ - \lambda). \quad (C.3) \]
As in the previous subsection, one can find the asymptotic behavior of the solution as \( z \to 0 \). The leading order is given by \( z^{\epsilon_{\pm}} \). It is easy to see that no solution is integrable near the horizon. Let us consider
\[ \lambda = \lambda_R + i \lambda_I; \quad (C.4) \]
then, one finds
\[
\int_0^\delta \frac{dz}{z h(z)} |z^{\epsilon_{\pm}}|^2 = \int_0^\delta \frac{dz}{z h(z)} z^{2 |\epsilon_{\pm}|};
\]
for any choice of \( \lambda \in \mathbb{C} \), there is a solution which is divergent near \( z = 0 \). As a consequence, the LPC holds at the horizon.

The extremal case is more difficult because an irregular singularity appears for \( r = r_+ \). The change of variable
\[ r = r_+ + 1/x, \quad (C.6) \]
with \( x \in (0, \infty) \) allows us to find a system in the form \( \ddot{g}' = A \ddot{g} \), with
\[
A = \begin{bmatrix}
k \frac{\sqrt{\varepsilon}}{r_+} & \frac{1}{h(0)} \left( -\mu \sqrt{\tau} + \frac{\epsilon Q}{r_+} - \lambda \right) \\
-\frac{1}{h(0)} \left( \mu \sqrt{\tau} + \frac{\epsilon Q}{r_+} - \lambda \right) & -k \frac{\sqrt{\varepsilon}}{r_+}
\end{bmatrix};
\]
the prime stays for the derivative with respect to \( x \) and
\[ B = \frac{1}{L^2} \left( r^2 + 2r_r + 3r_r^2 + L^2 \right). \quad (C.7) \]
One has
\[
\lim_{x \to \infty} B = B_0 = \begin{bmatrix}
0 & -\frac{1}{h(0)} \left( \frac{\epsilon Q}{r_+} - \lambda \right) \\
\frac{1}{h(0)} \left( \frac{\epsilon Q}{r_+} - \lambda \right) & 0
\end{bmatrix},
\]
whose eigenvalues are
\[ \eta_{\pm} = \pm i \frac{1}{B(0)} \left( eQ \frac{r_+}{r_+^2} - \lambda \right). \] (C.8)

There exists a nonsingular matrix \( T \) such that \( A_{\text{diag}} \equiv T A T^{-1} \) tends to a diagonal matrix \( A_{\text{diag}}^0 \) with entries \( \eta_{\pm} \) for \( x \to \infty \). \( A_{\text{diag}} \) admits a power series expansion in some neighborhood of \( x = \infty \)
\[ A_{\text{diag}} = \sum_{k=0}^{\infty} A_{k}^{\text{diag}} \frac{1}{x^k}. \] (C.9)

Applying the general theory [30] to the present case, a formal solution matrix \( \Psi \) can be found in the form
\[ \Psi = P x^R \exp(Q x), \] (C.10)
where \( P \) is a formal power series
\[ P = \sum_{k=0}^{\infty} P_k \frac{1}{x^k}, \quad \det(P_0) \neq 0, \] (C.11)
\( Q \) coincides with \( A_{\text{diag}}^0 \) and \( R \) is a diagonal constant matrix whose entries coincide with the diagonal entries of \( A_{\text{diag}}^1 \). The diagonal entries of \( A_{\text{diag}}^1 \) are given by
\[ \xi_{\pm} = \pm i \left( L^2 eQ + \lambda - \frac{eQ}{r_+} \right) \left( 2r_+ L^2 - \frac{4r_+^3 L^2}{6r_+^2 + L^2} \right) - \frac{1}{6r_+^2 + L^2}. \] (C.12)

By using (C.4) one finds that the condition of \( L^2 \)-integrability of the leading order solution at \( x = \infty \) cannot be satisfied by both the solutions, because
\[ \int_{c}^{\infty} dx \frac{\pm r_+ L^2 \frac{\xi_{\pm}^2}{x^2 + r_+^2}}{B(x)} \exp \left( \pm \frac{2}{B(0)} \lambda x \right) \] (C.13)
is divergent for one of them for any choice of \( \lambda \in \mathbb{C} \). Then the LPC case occurs.

References

[1] Gibbons G 1975 Comm. Math. Phys. 44 245
[2] Khlopov I B 1999 Phys. Rep. 329 37
[3] Winstanley E 2001 Phys. Rev. D 64 104010
[4] Belgiorno F and Cacciatori S L 2008 Preprint 0803.2496
[5] Hawking S W and Page D N 1983 Comm. Math. Phys. 87 577
[6] Brill D R and Wheeler J A 1965 Rev. Mod. Phys. 29 465
  Soffel M, Müller B and Greiner W 1977 J. Phys. A: Math. Gen. 10 551
  Soffel M, Müller B and Greiner W 1982 Phys. Rep. 85 51
  Ansdrushkevich I E and Shishkin G V 1987 Theor. Math. Phys. 70 204
[7] Thaller B 1992 The Dirac Equation (Berlin: Springer)
[8] Weidmann J 1987 Spectral Theory of Ordinary Differential Operators (Lecture Notes in Mathematics vol 1258) (Berlin: Springer)
[9] Hsieh Po-Fang and Sibuya Y 1999 Basic Theory of Ordinary Differential Equations (New York: Springer)
[10] Walter W 1998 Ordinary Differential Equations (Graduated Text in Mathematics vol 182) (Berlin: Springer)
[11] Bachelot A 2007 Preprint 0706.1151v1
[12] Belgiorno F and Martellini M 1999 Phys. Lett. B 453 17
[13] Winklmeier M and Yamada O 2006 J. Math. Phys. 47 102503
  Batic D and Schmid H 2006 Prog. Theor. Phys. 116 517
  Batic D and Schmid H 2007 Preprint gr-qc/0703023
[14] Christodoulou D and Ruffini R 1971 Phys. Rev. D 12 3552
[15] Damour T 1977 Klein paradox and vacuum polarization Proc. 1st Marcel Grossmann Meeting on General Relativity (Trieste, 1975) ed R Ruffini (Amsterdam: North Holland) p 459
[16] Deruelle N 1977 Classical and quantum states in black hole physics Proc. 1st Marcel Grossmann Meeting on General Relativity (Trieste, 1975) ed R Ruffini (Amsterdam: North Holland) p 483
[17] Deruelle N and Ruffini R 1974 Phys. Lett. B 52 437
[18] Deruelle N and Ruffini R 1975 Phys. Lett. B 57 248

Damour T, Deruelle N and Ruffini R 1976 Lett. Nuovo Cimento 15 257

Damour T 1975 Lett. Nuovo Cimento 12 315
[19] Pauli W 1932 Helv. Phys. Acta 5 179
[20] Rubinow S I and Keller J B 1963 Phys. Rev. 131 2789
[21] Maslov V P 1972 Théorie des perturbations et méthodes asymptotiques Études Mathématiques (Paris: Dunod)
Maslov V P 1981 Semi-classical approximation in quantum mechanics Mathematical Physics and Applied Mathematics (Dordrecht: Reidel) pp 7
[22] Bolte J and Keppeler S 1999 Ann. Phys. 274 125

Keppeler S 2003 Ann. Phys. 304 40
[23] Dunford N and Schwartz J T 1963 Linear operators: part II spectral theory, self adjoint operators in Hilbert space Pure and Applied Mathematics (A Series of Texts and Monographs) (New York: Interscience)
[24] Kay B S and Wald R M 1991 Phys. Rep. 207 49
[25] Melnyk F 2004 Commun. Math. Phys. 244 483

Melnyk F 2004 J. Phys. A: Math. Gen. 37 9225
[26] Finster F, Smoller J and Yau S T 2000 J. Math. Phys. 41 2173
[27] Finster F, Smoller J and Yau S T 2000 Commun. Pure Appl. Math. 53 902
[28] Schmid H 2004 Math. Nachr. 274–275 117
[29] Avis S J, Isham C J and Storey D 1978 Phys. Rev. D 18 3565
[30] Coddington E A and Levinson N 1955 Theory of Ordinary Differential Equations (New York: McGraw-Hill)