Driven quadrature and spin squeezing in a cavity-coupled ensemble of two-level states

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The generated magnitude of quadrature squeezing in a cavity-coupled ensemble, which is continuously driven using a coherent off-axis field, is theoretically explored. Using a truncated set of equations-of-motion derived from a Dicke Hamiltonian, steady-state quadrature squeezing of the cavity field is numerically calculated to approach a limit of -3 dB, while frequency-modulated quadrature squeezing approaches a limit of -14 dB, in the absence of pure-dephasing, and as a function of the ensemble’s size and detuning. The impact of pure-dephasing on steady-state quadrature squeezing is shown to be mitigated by increased detuning of the driving field, while frequency-modulated squeezing is only shielded in a regime where the cumulative coupling and driving rates are in excess of the pure-dephasing rate. Spin-squeezed entanglement is also calculated to occur simultaneously with weakly-driven frequency-modulated quadrature squeezing.

I. INTRODUCTION

Quadrature squeezed light is an important experimental resource in quantum optics with a number of applications ranging from enhancing interferometry beyond the shot-noise limit [1, 2], to its use in generating entangled continuous-variable states for quantum information protocols [3, 4]. Due to its utility, there is justifiable motivation to not only generate larger squeezing magnitudes, necessary in particular for fault-tolerant continuous-variable quantum computing [5], but also in expanding its bandwidth [6], and in optimizing its experimental efficiency and integrability [7]. By making squeezed light sources more accessible and practically implementable, their benefits may be reaped in both routine spectroscopy and interferometry [8, 9], while further spurring the development of hybrid continuous/discrete variable quantum information protocols [10], and optical sensing schemes that go beyond the classical limits [11].

State-of-the-art sources of quadrature squeezed light are based on cavity-assisted χ(2) parametric down-conversion [12], while much effort is currently being invested in developing alternative on-chip integrated sources based on χ(3) four-wave mixing schemes [13, 14]. An alternative quadrature squeezing mechanism, which is technically simpler but less explored, is based on the resonant fluorescence of weakly-driven optical dipoles, first proposed by Walls and Zoller [15].

The maximum measurable degree of quadrature squeezing in free-space from such two-level systems is predicted to be in the order of -1.25 dB, without accounting for optical losses and realistic detection efficiencies. Experimental attempts so far have successfully substantiated this prediction; however, due to limited detection efficiencies and cumulative optical losses, the measured quadrature-squeezing has been far below the predicted value, ranging between the orders of -10 to -100 mDB for cavity-coupled atoms and quantum dots, respectively [16, 17].

Compared to the measured -15 dB from state-of-the-art parametric cavity systems [12], and considering the -15 to -17 dB desired for fault-tolerant continuous variable quantum computing [5], the motivation for pursuing resonance-fluorescence based quadrature squeezing lies rather in the possible technical advantages and accessible wavelengths. The appeal of their potentially small technical foot-print, and in providing squeezed light sources at wavelengths towards the higher energy end of the visible spectrum, makes exploring this approach worthwhile. The later point is particularly interesting, given the technical challenges of frequency-converting squeezed vacuum states [18], and the difficulty of engineering suitable non-linear systems for generating such states at wavelengths shorter than 600 nm.

Quadrature squeezing through resonance fluorescence is based on an established proportionality between the scattered field’s quadrature fluctuations and the dipole moment’s fluctuations, such that they may be considered interchangeable [19–21]. When considering an ensemble of non-interacting dipoles, this relationship may be transposed into a relationship between the collective angular momentum operator and the far-field quadrature, thereby highlighting a possible link between far-field quadrature squeezing and ensemble spin-squeezing [22]. In turn, given the direct relationship between spin-squeezing and multipartite entanglement [23, 24], any observable non-classical fluctuation of the far-field quadratures can be considered an unambiguous witness of multipartite entanglement, for certain experimental configurations.

Spin-squeezed states are an important class of metrological probes, usually employed in interferometric schemes that revolve around assessing a phase-shift of the collective spin-state, imparted by an external physical quantity. The smallest uncertainty when measuring such phase-shifts, and therefore the spin states ultimate sensitivity, is directly proportional to its degree of multipartite entanglement [25, 26]. Experimentally determining large-scale entanglement
is a principal goal in many areas of quantum information science. Therefore, exposing and delineating relationships between metrologically useful multipartite entangled states and directly measurable quantities, such as the quadrature fluctuations of coupled fields in this work, is fundamentally informative.

Here, an indirectly driven cavity-ensemble system is numerically explored to determine the conditions needed for generating quadrature-squeezing magnitudes beyond the free-space limit, and the consequential degree and type of spin squeezing. Analogous to the cavity-mediated detuned scheme studied in [27] and the Raman-based scheme in [28], the cavity-ensemble system is numerically solved for varying detuned configurations that address the side-bands, or dressed states, of the coupled system, with the various rates and detuning framed in relation to the two-level state’s longitudinal relaxation rate.

The numerical results highlight how the generation of both steady-state and frequency-modulated squeezing of the cavity output quadrature can exceed the free-space limit of single two-level systems while accounting for pure-dephasing. Furthermore, they highlight how the generation of frequency-modulated quadrature squeezing can simultaneously generate entangled spin-squeezing.

The investigation begins with the delineation of the studied Hamiltonian, the assumptions made, and considered solid-state ensemble systems in sec.II. This is followed by a discussion of the numerical results of steady-state quadrature squeezing from off-axis driven and cavity-coupled single, and ensembles, of two-level emitters in sec.III. Finally the numerical results for generated frequency-modulated quadrature squeezing and the simultaneous occurrence of entangled spin squeezing is presented and discussed in sec.IV.

II. HAMILTONIAN & FLUCTUATIONS

A. System Hamiltonian and Dynamics

A two-level system, representing either an optical or magnetic dipole, is typically defined using pseudo-spin operators for a ground and excited state basis \{\ket{g}, \ket{e}\} separated by an energy \(\hbar \omega_0\), such that \(\sigma = \ket{g}\bra{e} + \ket{e}\bra{g}\), \(\sigma = (\sigma^\dagger)^\dagger\), and the commutation \([\sigma^\dagger, \sigma] = \sigma_z\). A Hermitian quadrature operator \(U_\phi\) for a dipole may be similarly defined to that of a single optical mode \(X_\phi\), considering a relative measurement phase or the optical field’s instantaneous phase \((\theta \text{ or } \phi)\):

\[
U_\phi = (e^{i\phi}\sigma + e^{-i\phi}\sigma^\dagger), \quad X_\phi = (e^{-i\theta}a + e^{i\theta}a^\dagger).
\] (1)

Assuming that an ensemble’s constituents only interact indirectly via the external fields, the energy and dynamics of a cavity-coupled ensemble may be described by a Dicke Hamiltonian:

\[
\mathcal{H}/\hbar = \omega_c a^\dagger a + \frac{1}{2} \sum_k N \omega_k \sigma_k z + \sum_k g_k X_0 U_{z,k},
\] (2)

where \(\omega_c\) and \(\omega_k\) are the cavity and dipole transition frequencies, and \(g_k\) is the coupling strength of the dipole and cavity mode. In the scenario discussed here, the dipole ensemble is considered to be directly driven by an off-axis field which bypasses the cavity input, as described in [27]. This is easily facilitated using a ring-based cavity configuration, but may also be implemented using a linear cavity via either off-axis excitation (requiring consideration of the relative dipole alignment to the external and cavity fields), or detuning of the cavity from the dipole’s and external field transition.

Such a system is schematically pictured in fig.1. This configuration ensures that the cavity remains a passive element which acts as a coherence ‘purifier’ of the ensemble’s transition, in the sense of which the coherent cavity-coupling rate outcompetes the incoherent decay rates of the ensemble. Furthermore, this configuration collects a large part of the ensembles emitted fluorescence, and avoids the bandwidth restrictions of driving the ensemble through the cavity itself, in addition to being experimentally convenient for distinguishing the driving field light from the cavity-transmitted light during detection.

The driving field is described using a dipole approximated semi-classical term as the resonantly driven component of the Hamiltonian:

\[
\mathcal{H}_\Omega/\hbar = \frac{\Omega}{2} (e^{i\omega t} + e^{-i\omega t}) \sum_k U_{0,k},
\] (3)

which is defined as a function of a driving Rabi frequency \(\Omega\) and a field frequency \(\omega\), and for which \(\Omega\) represents a product of a linearly polarized plane-wave electric field and a linear transition-moment of the two-level state.

The resulting system dynamics are calculated by numerically integrating the system’s Markov approximated master equation \(\dot{\rho}\), while accounting for a cavity decay rate \(\kappa\), the radiative longitudinal relaxation rate \(\gamma_1\), and pure dephasing rate \(\gamma_2^\prime\), as detailed in the appendix. A thermal starting configuration of the system is consistently used here for all numerically derived results, such that \(\{\langle \sigma_z(0) \rangle, \langle a(0) \rangle \} = \{0, 0\}\), and \(\langle a^\dagger a(0) \rangle = \bar{n}\).

When deriving the coupled equations of motion, the system is treated symmetrically as carried out in [29, 30], such that for all \(k\), single dipole expectation values and their correlation with the cavity field are considered equal \(\langle \sigma_k \rangle = \langle \sigma_1 \rangle, \langle a^\dagger \sigma_k \rangle = \langle a^\dagger \sigma_1 \rangle\), while all pairs of spins are designated as \(\langle \sigma_k^\dagger \sigma_j \rangle = \langle \sigma_1^\dagger \sigma_1 \rangle\) for all \(j \neq k\), adhering to the commutation relation \([\sigma_j^\dagger, \sigma_k] = \sigma_j \delta_{jk}\).

A rotating frame with the driving field frequency \(\omega_1\) is considered, along with the rotating wave approximation. However, this is not employed for the cavity coupling term, as the rapidly oscillating terms become non-negligible for increasing ensemble sizes, especially considering the possibility of the collective coupling strength approaching the transition frequency.
B. Quadrature Squeezing

The variance of an operator’s fluctuations is defined with respect to its expectation value such that:

\[
\Delta a = a - \langle a \rangle,
\langle \Delta a^2 \rangle = \langle a^2 \rangle - \langle a \rangle^2.
\]

For a single-mode field, its quadrature fluctuations is defined as:

\[
\langle \Delta X^2_{\theta} \rangle = 2\left(\langle \Delta a^\dagger \Delta a \rangle + \Re\{e^{-i2\theta} \langle \Delta a^2 \rangle \}\right) + 1.
\]

This expression consists of a coherent and incoherent contribution \(\langle \Delta a^2 \rangle\) and \(\langle \Delta a^\dagger \Delta a \rangle\), respectively, which are both effectively zero for a vacuum state. This sets the minimum uncertainty (the shot-noise level) to a value of 1, which is a consequence of the chosen quadrature definitions in eq. (1).

A similar picture may be attributed to that of a single two-level emitter such that:

\[
\langle \Delta U^2_{\phi} \rangle = 2\left(\langle \Delta \sigma^2 \rangle + \Re\{e^{-i2\phi} \langle \Delta \sigma^2 \rangle \}\right) - \langle \sigma_{z} \rangle.
\]

This resulting expression can be understood in analogy with eq. (5), consisting of a dipole transition’s coherent and incoherent contributions. However, instead of a constant uncertainty level like that of the optical field, the fluctuation are limited by the instantaneous population inversion \(\langle \sigma_{z} \rangle\), which is a function of the external driving fields and intrinsic decay rates.

To obtain a consistent prognosis, \(\langle \sigma_{z} \rangle\) is set to -1/2, rather then being discarded via employing normal-ordering during the derivation of eq. (6). This corresponds to the minimum value it takes when the coherent fluctuations \(\langle \Delta \sigma^2 \rangle\) is at maximum, and for which the minimum uncertainty product abides by \(\langle \Delta U^2_{\phi} \rangle \langle \Delta U^2_{\phi+\pi/2} \rangle \geq 1/2\).

Considering an ensemble of non-interacting dipoles, the field-mediated intra-ensemble fluctuations is linearly summed, rather then summed in quadrature, as all the individual fluctuations are correlated via their identical coupling to the same cavity:

\[
\Sigma_{inc} = N\langle \Delta \sigma_{1}^2 \rangle + \langle N-1 \rangle \langle \Delta \sigma_{1} \Delta \sigma_{2} \rangle,
\Sigma_{coh} = N\langle \Delta \sigma_{1}^2 \rangle + \langle N-1 \rangle \langle \Delta \sigma_{1} \Delta \sigma_{2} \rangle,
\langle \Delta U^2_{\phi} \rangle' = 2\left(\Sigma_{inc} + \Re\{e^{-i2\phi}\Sigma_{coh}\}\right) + \frac{N}{2}.
\]

The degree of quadrature squeezing is conventionally characterized using homodyne detection, where the signal of interest is mixed with a local oscillator at a given phase \(\phi\) and a frequency \(\omega_{LO}\), generating sidebands at the frequencies \(\omega_{LO} \pm \nu\). Upon detection with a suitable bandwidth detector, these are converted into a low-frequency photocurrent, whose spectral density \(S_{\theta}(\nu)\) directly measures the \(\nu\)-dependent noise variance.

Given the weak-sense stationary nature of the rate equations, \(S_{\theta}(\nu)\) is by definition the Fourier transform of the field quadratures auto-correlation function \(\langle g^{(1)} \rangle\), via the Wiener-Khintchine theorem. However, in the case where an analytical expression is not sought, it is numerically convenient to directly estimate the spectral density of the integrated rate equations using a periodogram-based computation (e.g. Welch’s method [31]).

The periodogram of the normally ordered variance, \(\langle \Delta X^2_{\theta} \rangle\) (which excludes the constant term), is scaled as a product of the collection and detection efficiencies \(\varepsilon\) and \(\varphi\) to obtain an estimate of \(S_{\theta}(\nu)\):

\[
S_{\theta}(\nu) = \varepsilon \varphi \left(1 + \hat{P}\left(\Delta X^2_{\theta}\right)(t)\right)
\]

For constant steady-state quadrature variances, a power spectrum is not meaningful as there are no frequency
components other than what is introduced. The power spectrum is therefore only calculated for oscillating solutions, which directly conveys the distribution and magnitude of the quadrature fluctuations frequency components.

C. Spin Squeezing

The fluctuations of an ensemble of two-level states are conventionally assessed via a collective angular momentum operator \( \langle J \rangle = \{ \langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle \} \), also referred to as the collective spin. While there are a few definitions of spin-squeezing depending on the experimental settings and the observables of interest [32, 33], the variance and basic uncertainty relationships of the collective spin components is defined in terms of the three orthogonal coordinates of the collective state Bloch sphere:

\[
\langle J_\ell \rangle = \sum_k \frac{\langle \sigma_{\ell k} \rangle}{2}
\]

where \( \ell = \{ x, y, z \} \) designates the given Pauli matrix, which are defined in term of the pseudo-spin operators. Their variance, and the basic uncertainty relation, are further defined with respect to the previously described symmetric treatment of the ensemble:

\[
\langle \Delta J_\ell^2 \rangle = \frac{N}{4} \left( \langle \sigma_{11} \rangle + (N-1)\langle \sigma_{12} \rangle \right) - \langle J_\ell \rangle^2, \\
\langle \Delta J_\ell^2 \rangle < \frac{1}{4} \sqrt{\langle J_k \rangle^2 + \langle J_l \rangle^2},
\]

where \( \{ j, k, l \} \) represent the three orthogonal spin-coordinates.

For this uncertainty relationship, it is possible for squeezing to occur simultaneously in the two orthogonal directions of the Bloch sphere’s equatorial plane (see fig.1), generating planar spin-squeezed states as opposed to standard squeezed state along only one of the orientations \( J_{x,y} \), and Dicke spin-squeezed states which represent un-polarized ensembles where only the spin coordinate in the axial plane \( J_z \) is squeezed [26, 34]. Planar spin-squeezed states are particularly interesting as they enable the simultaneous measurement of imparted phase and amplitude changes beyond the classical limit, unlike their standard counterpart [35].

It has been established that spin-squeezing directly implies multipartite entanglement, which is deemed metrologically useful [34] when below a size-dependent threshold [23, 25]:

\[
\xi_j^2 = \frac{\langle \Delta J_j^2 \rangle}{\langle J_j \rangle^2 + \langle J_l \rangle^2} < \frac{1}{N},
\]

Comparing eq.(10) and eq.(9), it is evident that multipartite-entanglement and spin-squeezing are not necessarily correspondent [33], however any degree of spin-squeezing immediately implies some magnitude of multipartite entanglement [36].

D. Solid-state ensemble densities & coupling strengths

Considering solid-sate ensembles, the symmetric description of a two-level ensemble employed here implies a uniformity which counters the typical in-homogeneity associated with such systems. As the particular spectral information is not sought here, the specifics of the inhomogeneous distribution is not needed for the following analysis; the individual coupling strengths are considered identical, while the direct effect of density-dependent spectral and pure-dephasing inhomogeneity can be crudely accounted for to first order by setting \( \gamma_2^1 > \gamma_1 \).

Generally, the cavity coupling strength is proportional to an effective mode volume \( V_{ef} \) and scaled by the relative alignment of the transition-moment and the resonant field \( \zeta \), such that a collective coupling strength \( \mathcal{G} \) is defined proportionality for a given ensemble density \( N_d \) and longitudinal decay rate \( \gamma_1 \). For optical dipoles, this may be expressed via:

\[
\mathcal{G} = \sqrt{N} g_1 \propto \zeta \left[ (N_d V_{ef}) \left( \frac{3\pi c^2 \gamma_1}{2 \omega V_{ef} n^3} \right) \right]^{1/2},
\]

where \( c \) is the speed of light and \( n \) is the refractive index of the cavity medium. Based on this, the relationship between \( g_1 \) and \( V_{ef} \) may be considered constant for any given \( N_d \). Instead, the allowed values of \( g_1 \) and \( N \) may be delineated for any given \( N_d \) with respect to the considered cavity system, and the type of ensemble used.

Considering the simple case of a near-concentric optical cavity, where the mode volume is estimated as a product of a zeroth-order Laguerre-Gaussian beam-waist and the cavity length, the resulting ensemble sizes and expected single emitter coupling strength is plotted in fig.2 for a range of concentrations. A choice of appropriate coupling rates and ensemble sizes can thereby be based on experimentally determined \( \gamma_1 \).

For single two-level systems, experimentally achieved coupling strength have typically been four to six orders of magnitude lower than the transition frequency [37], while the ratio \( g_1 / \{ \gamma_1, \kappa \} \) can span between 0.1 to 100 for highly optimized systems, but are usually two to three orders of magnitude lower than \( \gamma_1 \).

For solid-state optical defects such as tin vacancy centers in diamond, the average decay rates in the order of 200 MHz have been measured from ensembles with densities estimated in the order of 1 ppm [38, 39]. Alternatively, for rare-earth ion systems such as europium-doped yttrium silicate or prasodymium-doped yttrium aluminum garnet, decay rates down to 100 Hz below 10 K, and up to 50 MHz at room temperature have been measured [40, 41].

Further accounting for the crystal symmetry e.g. the tetrahedral symmetry of diamond and how the ensembles dipole orientations will be distributed over four distinct orientations, the dipole alignment factor \( \zeta \) ranges
between 0.5-0.75, such that a realistic collective coupling strength can be considered to range from $G \propto 10^{-8}\omega_k$ for diamond-based ensembles up to $\propto 10^{-3}\omega_k$ for rare-earth ion ensembles, considering the rough scaling in fig. (2).

Conceptually, the proportion between $N$ and $g_1$ can be modified by varying the concentration, but the issues associated with larger concentrations are non-trivial. As well as increased inhomogeneous broadening, a hard limit on the feasible density exists, beyond which the defect loses its integrity and desired transition properties.

However, as demonstrated using dense rare-earth ion ensembles, inhomogeneous broadening can be circumvented via spectral-hole burning techniques e.g. [42]. Ideally, a system where the ground- and excited-state hyperfine transition frequencies exceed the inhomogeneous broadening frequency, such as in Ho$^{3+}$ [43] is desired, in order to avoid issues related to the modification of the cavity’s free spectral range and decay rate by the hole-burning procedure [44, 45].

Another possibility could involve preparing highly concentrated colloidal quantum dot aggregates. Such systems characteristically possess much faster decoherence and longitudinal decay rates, but provide the advantage of facilitating the creation of comparably homogeneous concentrations exceeding 100 ppm, which span larger volumes, with the appealing potential for wavelength tune-ability by adjusting their size. Considering a recent example demonstrating discrete single photon emission from colloidal perovskite-based quantum dots [46], the longitudinal decay rates are measured to be an order of magnitude faster then those for diamond defects, which projects possible rates in the order of $G \propto 10^{-4}\omega_k$, considering the scaling in fig. (2).

### III. CONTINUOUSLY DRIVEN SQUEEZING

#### A. Single Dipole

For the case of a single emitter without a cavity ($N = 1, g = 0$), a direct analytical solution may be obtained for the steady-state quadrature fluctuations, in a rotating frame with the driving field frequency $\omega_1$. The steady-state expression in terms of a scaled Rabi frequency $z = (\Omega/|\Gamma_1|)^2$ is derived as:

$$\langle \Delta U^2 \rangle_s = \frac{z\alpha}{(2z\alpha + 1)} - \frac{z(1 + \text{e}^{-2\phi})}{4(2z\alpha + 1)^2} + \frac{1}{2},$$

where the subscript $s$ denotes the steady-state, $\Gamma_1 = \Gamma + i\Delta_0, \alpha = \Gamma/2\gamma_1, \Gamma = \gamma_1 + \gamma_2 + \bar{n}\gamma_1$ is the total temperature-dependent dephasing rate, and $\Delta_0 = (\omega_0 - \omega_1)$ is the detuning with respect to the driving field.

Neglecting heat ($\bar{n} = 0$), eq.(12) is plotted in fig.3(a), and shows that the minimum squeezed variance is obtained for $z = 1/6$, in the order of $-1.25$ dB. Introducing dephasing drastically reduces the difference between the two orthogonal quadratures, such that no squeezing may be generated when $\gamma_2 > \gamma_1$.

When coupling a single dipole to a cavity and coherently driving its transition resonantly ($\Delta_c = (\omega_c - \omega_1) = 0 = \Delta_0$) using an off-axis field, a simultaneous increase in the fluctuations of both orthogonal quadratures is generated as $\Omega$ is increased, which is plotted in fig.3(b).

Based on a cursory analysis of the steady-state form of the coupled rate equations (see appendix), this may be considered a consequence of the enhanced exchange rate of quanta between the cavity field and the dipole $\langle a^\dagger a \rangle_s$, which leads to the simultaneous reduction of the coherent fluctuations $\langle \Delta \sigma^2 \rangle_s$, and the increase of incoherent fluctuations $\langle \Delta \sigma^2 \Delta \sigma^\dagger \rangle_s$:

$$\langle \Delta \sigma^2 \rangle_s \propto -\frac{g_1^2}{\Gamma_1^2} \langle a \sigma_1 \rangle^2,$$

$$\langle \Delta \sigma^2 \Delta \sigma^\dagger \rangle_s \propto \frac{g_1}{\gamma_1} \langle a^\dagger a \rangle_s.$$  

(15)

Evidently, these counteractive mechanisms may be mitigated by detuning the dipole from the driving frequency, in particular towards $\Delta_0 > g_1$ which reduces the coherence-reducing correlation $\langle a \sigma_1 \rangle$ quadratically compared to the detuning-insensitive (to first-order) exchange of quanta $\langle a^\dagger a \rangle_s$.

The resulting enhancement is demonstrated in fig.4 which compares both resonant and detuned configurations. In the resonant case, the quadrature fluctuations can be seen to be reduced when the cavity is on resonance with the dressed-states generated by the coherent driving field. In the detuned case, squeezing in the order of $-2.5$ dB may be achieved when the dressed state (generalized Rabi) frequency matches that of the
FIG. 3. (a) Quadrature fluctuations of a free-space single emitter as a function of the scaled Rabi frequency \( z\) for various dephasing rates \( \gamma_2/\gamma_1 = \{0, 0.1, 1\} \), and (b) for various cavity-coupling rates \( g_1/\gamma_1 = \{0, 0.4, 0.6, 1\} \), where \( \kappa/\gamma_1 = 10 \). Blue and red traces represent orthogonal quadrature variances.

Compellingly, the quadrature variance of the cavity output is also modified, showing non-negligible squeezing in the anti-detuned case. Further analysis of the steady-state expressions points towards a mechanism based on the detuning-dependent relationship between the intra-cavity coherence and the dipole coherence:

\[
\langle \Delta a^2 \rangle_s \propto g_1 N \left( \frac{1}{T_c} \right) \langle a \sigma_1 z \rangle_s - \frac{g_1^2 N^2}{T_c} (\Delta \sigma^2)_s
\]

\[
= \propto \frac{g_1 N}{T_c} \langle a \sigma_{1z} \rangle_s + \frac{g_1^2 N^2}{T_c^2} (\sigma_{1z})^2_s. \quad (16)
\]

This highlights how the correlation between the intra-cavity field and the population inversion \( \langle a \sigma_z \rangle \) is strongly enhanced or suppressed when the relative detunings are of opposite signs, such that the real and imaginary components of the denominator increase and decrease, respectively.

Physically, this illustrates how a coherent side-band-driven process of the coupled system generates coupled photons without incoherently populating the cavity (via \( \langle a^\dagger a \rangle \) and thereby \( \langle \Delta a^\dagger \Delta a \rangle \)). A rough proportionality may thus be defined between the ensemble and cavity quadrature fluctuations such that:

\[
\langle \Delta U^2_0 \rangle_s \propto \left( \frac{g_1^2 N}{T_c \Gamma_t} \right)^2 \langle \Delta U^2_{\pi/2} \rangle_s', \quad \text{for } \kappa < \Delta_c. \quad (17)
\]

This indicates how, for low driving rates such that
FIG. 5. Plot of the cavity field steady-state quadrature variance as a function of coupling strengths $g_1$ for a single emitter for various cavity field detuning. The parameters $\Omega/\kappa = 1$, $\gamma_1/\kappa = 0.1$, and $\Delta_0/\kappa = 25$, as those used for the detuned case in fig.4(b).

$\langle \Delta U_{\pi/2}^2 \rangle_s$ is squeezed (cf. eq.(14)) and at detunings beyond the cavity decay rate, matching the product of oppositely signed cavity and emitter detuning to the square of the coupling strength (i.e. the product of the denominator in eq.(17) is maximized for $\Im\{\Gamma_c\}<0$, $\Im\{\Gamma_t\}>0$ to exceed the numerator), any squeezing generated in the ensemble can be proportionally transferred to the cavity field.

This relation can be understood in terms of how the direct exchange of quanta via the coherent coupling term can be regulated by compensating for the difference between the cavity and two-level relaxation rates through the relative detuning. This resulting squeezing is thereby measurable in the cavity output, and enhanced by appropriately set relative detunings to offset larger coupling rates.

Going further, the generated virtual dressed state via the detuned-driving, in addition to the direct transition coupled to the cavity, can be understood to constitute a three-level scheme. This can thereby facilitate lasing beyond both a given coupling or driving rate, which manifests as an exponential increase in both the cavity occupation $\langle a^\dagger a \rangle$ and quadrature variances. This is demonstrated in fig.5 as a function of $g_1$ for varying values of $\Delta_c$.

The onset of lasing occurs prominently in a far-detuned regime ($\kappa < 2\Delta_c$) when the coupling exceeds the product of the ensemble and cavity decay rates $g_1^2 N \geq \frac{1}{2}\Gamma_c \Gamma_t$. By varying the cavity detuning for a given coupling strength, the system can be tuned to reside just before the threshold where squeezing is optimized. Thus for lower $g_1$ coupling values and larger $\Delta_c$ detuning values, squeezing in the cavity-output quadrature may approach the free-space limit of a single two-level state, at the expense of a reduced cavity-field amplitude.

FIG. 6. Steady-state map of the minimum quadrature variance as a function of cavity detuning $\Delta_c$ and the Rabi frequency $\Omega$, for a detuned ensemble of emitters, using the same parameters as for Fig.4, except for $g_1$ which is adjusted such that $\mathcal{G}/\kappa = 1$.

B. Dipole Ensemble

Increasing the ensemble size leads to an enhancement of all cavity-field related correlations by a factor $N$, which augments the proportionality highlighted in eq.(17). However, this is counteracted by a reducing lasing threshold, beyond which the quadrature variance of both the cavity field and the ensemble far-field increase by an order of magnitude. This driving- and coupling-dependent threshold can however be pushed to higher values at the expense of the cavity field amplitude.

Replacing the single emitter with an ensemble, the resulting detuning-dependence is plotted in fig.6, for which the collective coupling strength is set to equal the value of $g_1$ used in fig.4, such that $\mathcal{G}/\kappa = 1$ ($N = 10^6$, $g_1 = 10^{-3}$).

Given the assumption of a non-interacting ensemble, the reduction of $g_1$ implies that $\langle \Delta U_{\pi/2}^2 \rangle_s$ will resemble that of the single free-space emitter. In this case, provided a weakly-driven regime where $\Omega < 2\Delta_0$, the intra-ensemble correlations are negligible, with the exception of when the cavity is tuned into anti-resonance with the ensemble’s driven dressed state ($\Delta_c \approx -\sqrt{\Omega^2 + \Delta_0^2}$). The effect on the cavity output quadrature remains near identical,
as expected considering the proportionality described in eqs.(16,17).

The influence of ensemble size is explored in fig.7, highlighting how the cavity field may be squeezed towards a limit of -3 dB in the absence of pure-dephasing. In particular, it shows how this is reached by increasing the ensemble size under weak driving. In terms of the proportionality in eq.(17), this results in a decreased ensemble quadrature variance $\langle \Delta U^2 \rangle_s$ by virtue of the increased coherent intra-ensemble fluctuations $\langle \Delta \sigma_1 \Delta \sigma_2 \rangle$ (eq.7).

Beyond the threshold, a phase transition occurs pertaining to an increase in the incoherent intra-ensemble fluctuations $\langle \Delta \sigma_1 \Delta \sigma_2 \rangle$, up to the point where the scaled Rabi frequency $z$ matches the detuning frequency. Beyond this value, the strength of the driving field exceeds the rate of the enhanced collective process, and the proportionality outlined in eq.(17) is invalidated by higher-order correlations.

Interestingly, the transition from a conventional lasing character to a more superradiant one is reflected in the relative change of the ensemble quadrature fluctuations - as the ensemble size is increased, the quadrature fluctuations of the cavity increases while the ensemble fluctuations decrease. This is a result of the concurrent increase in both coherent and incoherent intra-ensemble fluctuations, which increases the number of cavity-photons, as the ensemble size and collective coupling rate increases.

Introducing a finite pure-dephasing rate, the degree of squeezing in the cavity-output quadrature is only weakly perturbed, as demonstrated in fig.8(a). In particular, the impact of pure-dephasing is observed to be mitigated by varying the cavity detuning, such that by keeping the detuning rate larger then the pure-dephasing rate, irrespective of the cavity and coupling-rate, a degree of squeezing can be maintained.

Provided that the pure-dephasing rate $\gamma^2$ does not exceed the ensemble’s detuning $\Delta_0$, (such that the proportionality defined in eq.(17) is optimized), the presence of pure-dephasing is therefore not completely detrimental to the generated squeezing of $\langle \Delta X^2 \rangle_s$. However, this mode of control is useful only so far as the intra-cavity field retains a experimentally detectable number of photons, shown in fig.8(b), which inadvertently decreases as a function of $\langle a^+a \rangle_s \propto 1/\Delta_0$.

### IV. FREQUENCY-MODULATED QUADRATURE AND SPIN SQUEEZING

Aside from the well-known phenomenon of coherent collapse-and-revival, there are other periodic dynamics, as shown in fig.9, which may uniquely generate a degree of quadrature and spin-squeezing. Despite starting from a thermally mixed state, it is possible to generate frequency-modulated quadrature fluctuations, where the periodic enhancement can significantly exceed the optimized steady-state squeezing discussed in the previous section.

Such periodic modulation transposes itself to the
FIG. 9. Time-dependent dynamics plotted for two different detuning regimes, which demonstrates how quadrature-squeezing may be generated as a steady-state or as a periodic modulation depending on the choice of relative detuning and driving field strength. Plots simulated for $N = 10^8$, $g_1/\kappa = 5 \times 10^{-3}$ and $\gamma_1/\kappa = 0.1$, and $\gamma_{1*} = 0$.

Despite a thermally mixed starting point, persistent modulation of the intra-cavity field quadratures and the collective angular momenta is generated, for which both the resulting rate and minimum-squeezed magnitude become a function of the driving field frequency and the detuning. In particular, the modulation is comprised of the cavity-detuning frequency enveloped by the much slower coherent driving rate.

As shown in fig.10, the lower the driving Rabi frequency $\Omega$ and the larger its difference with the cavity detuning $\Delta_c$, the more pronounced the modulation, albeit occurring at slower rates. This reaching an asymptotic limit in the order of -14 dB, while the simultaneous modulated spin-squeezing transitions from a Dicke-like state where only $\xi_z$ is squeezed and $\xi_{x,y}$ are highly uncertain, to a more planar-like state with squeezing of both $\xi_y$ and $\xi_z$.

This difference in scaling between the phase and population-related spin-squeezing can be understood in relation to how the correlations $\langle \sigma_1^x \sigma_2^z \rangle$ and $\langle \sigma_1(x,y) \sigma_2(x,y) \rangle$ scale with $\Omega$ and the ensemble size $N$. For larger ensembles, lower values of $\Omega$ are required to limit the noise contribution of $\langle \sigma_1^x \sigma_2^z \rangle$ to $\langle \Delta J_z^2 \rangle$ (eq.[10]). Conversely, the phase fluctuations along the...
The impact of introducing a finite $\gamma_2^*$, and the resulting spectral density of $\langle \Delta X_2^* \rangle$ are plotted in fig.11. Two regimes are delineated around the point where $\gamma_2^*$ equals the sum of the scaled off-axis driving rate and cavity-coupling rate. Unlike the steady-state case, the pure-dephasing rate can not be mitigated by increasing the cavity detuning. Instead the sum of the individual coupling rate and driving rate need to outcompete $\gamma_2^*$, to ensure that both quadrature and entangled spin-squeezing may be generated.

The spectral density plots in fig.11 highlights the well-known property of continuous-frequency modulated signals (e.g. sine-wave modulation), which distribute the time dependent amplitude over multiple frequency components, and for which the cumulative integrated power is commensurate with the minimum time-dependent squeezing. Experimentally, data acquisition can be locked at the instances of minimum squeezed variance, which can greatly exceeds that of the optimized steady-state value. However, for the small fixed bandwidths within the modulation rate generated here, the spectral density shows how squeezing for single-frequency components will not exceed the free-space limit for the detuning and driving rates explored here.

V. CONCLUSION

Using a Markov-approximated master equation derived for a Dicke-type Hamiltonian, describing a cavity-coupled ensemble driven by an off-axis field, the truncated equations of motion (via third-order cumulant expansion) were numerically integrated to explore the generation of quadrature and entangled spin-squeezing. A minimum steady-state and frequency-modulated quadrature squeezing was calculated to occur in the limit of -3 dB and -14 dB, respectively, while entangled spin-squeezing occurs at a similar order of magnitude alongside weakly-driven frequency-modulated squeezing.

A direct proportionality between the cavity-field quadrature and the ensembles dipole quadrature was described to scale as a function of the ratio between the collective coupling strength and the relative cavity- and ensemble-loss rates, which is modifiable via the relative detuning of the cavity and the external driving field. Consequently, the degradation of the cavity-field squeezing by the ensemble’s pure-dephasing rate was observed to be mitigated by increasing the relative detuning, at the expense of decreasing the intra-cavity amplitude.

Frequency-modulated quadrature squeezing was also shown to concur with entangled spin-squeezing in a weakly-driven regime, where the driving Rabi frequency was orders of magnitude lower than the collective...
coupling strength. Unlike the steady-state regime, frequency-modulated squeezing is more susceptible to the presence of a finite pure-dephasing rate, which is instead only mitigated by larger cavity coupling rates.

Albeit using a rudimentary Dicke Hamiltonian and Markov-approximated rate equations, this work highlights the possibility of continuously generating quadrature squeezed light from a cavity via applying an off-axis drive to a coupled ensemble, which exceeds the free-space limit of a single two-level emitter. Furthermore, the pure-dephasing rate of the ensemble constituents, and thereby by extension the ensembles inhomogeneous broadening, may be mitigated with an appropriately detuned and driven configuration, although optimized squeezing is obtained at the expense of the cavity’s output field amplitude and bandwidth. Notably, the generation of entangled spin-squeezed states is found to be inherent in a weakly-driven regime, but bandwidth-limited depending on the driving and pure-dephasing rates.

The motivating interest of this work has been in exploring the limits in optimizing quadrature squeezing from an ensemble of emitters, using a cavity and off-axis near-resonant driving. While more sophisticated theoretical approaches need to be considered, these results provide an informative basis for the experimental exploration of both practical aspects of near-resonance fluorescence based squeezing using solid-state ensembles. This is considered with the particular anticipation of providing solid-state systems with shorter-wavelength and smaller technical footprints, which can compliment established parametric oscillator sources.

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Appendix: System Dynamics

The quadrature fluctuations and steady-state expectation values are obtained by integrating the systems Markov-approximated master equation. Accounting for the cavity decay rate $\kappa$ and the radiative damping in the presence of a heat bath $\bar{n}$ with relaxation and pure dephasing rates $\gamma_1$ and $\gamma_2^\ast$, respectively, the master equation and the associated Lindblad operator terms take the form:

$$\dot{\rho} = -\frac{i}{\hbar} [\mathcal{H} + \mathcal{H}_\Omega, \rho] + \mathcal{L}_\kappa + \mathcal{L}_{\gamma_1} + \mathcal{L}_{\gamma_2^\ast}, \quad (A.1)$$

$$\mathcal{L}_\kappa = \kappa \left( \bar{n} + 1 \right) \left( 2a^\dagger a \rho - \{ a^\dagger a, \rho \} \right) + \bar{n} \left( 2a^\dagger a \rho - \{ aa^\dagger, \rho \} \right),$$

$$\mathcal{L}_{\gamma_1} = \sum_{k} \frac{\gamma_{1,k}}{2} \left[ \left( \bar{n} + 1 \right) \left( 2\sigma_k \rho \sigma_k^\dagger - \{ \sigma_k^\dagger \sigma_k, \rho \} \right) + \bar{n} \left( 2\sigma_k \rho \sigma_k^\dagger - \{ \sigma_k \sigma_k^\dagger, \rho \} \right) \right],$$

$$\mathcal{L}_{\gamma_2^\ast} = \sum_{k} \frac{\gamma_{2,k}^\ast}{2} \left( \sigma_k \rho \sigma_k^\dagger - \{ \sigma_k \sigma_k^\dagger, \rho \} \right),$$

$$\bar{n} = \left( e^{\hbar \omega_0 (k \alpha r)} - 1 \right)^{-1}.$$

This system is analytically solvable in the case of free-space emitters ($N \geq 1$, $g = 0$), but when $g \neq 0$, the process of extracting the equations of motion results in an infinite set of successively increasing correlation orders.

A common strategy for truncation usually involves assuming some form of weakly driven or perturbed systems where $\langle \sigma_z \rangle$ is set to -1 and is assumed to negligibly change. For ensemble systems this is usually accompanied with the Holstein-Primakoff approximation, subsequently enabling the simplification of higher-order correlations by mapping the spin operators onto bosonic operators. These approximations enable the derivation of a closed set of coupled equations, which have been experimentally validated in weakly driven systems, e.g [16].

However, these approximations are not appropriate when accounting for non-negligible amplitudes of near-resonant driving fields. Furthermore for an ensemble, the number of equations quickly increases to an un-workable amount dependent on the ensemble size.

As carried out in [29, 30], and described in section II.A, the ensemble is described symmetrically to decouple the $N$-dependence of the number of coupled equations of motion. Following this, an alternative strategy is employed to further simplify and reduce the number of coupled equations, based on expanding the correlations in terms of their cumulant expectation values [29, 48]. This avoids any direct restriction of the coupling/driving frequency strengths, but does assume that the third order cumulants are negligible, such that:

$$\langle abc \rangle \approx \langle ab \rangle \langle c \rangle + \langle ac \rangle \langle b \rangle + \langle bc \rangle \langle a \rangle - 2\langle a \rangle \langle b \rangle \langle c \rangle. \quad (A.2)$$

The resulting coupled rate equations (not including the conjugate set of equations) are defined below with the complex loss rates denoted as $\Gamma_c = \kappa + i \Delta_c$, $\Gamma_1 = (\gamma_1/2 + \gamma_2^\ast + \bar{n} \gamma_1) + i \Delta_0$:
\[ \langle a \rangle = -\Gamma_c \langle a \rangle + g_1 N \left( \langle a \sigma_1 \rangle - \langle \sigma_1 \rangle \right), \]  
\[ \langle a \sigma_1 \rangle = -\Gamma_1 \langle a \sigma_1 \rangle + g_1 \left( \langle a a \sigma_1 \rangle + \langle a^\dagger a \sigma_1 \rangle \right) + \frac{i\Omega}{2} \langle \sigma_1 \rangle, \]  
\[ \langle a^2 \rangle = -2\Gamma_c \langle a^2 \rangle + 2g_1 N \left( \langle a \sigma_1 \rangle - \langle \sigma_1 \rangle \right), \]  
\[ \langle a^\dagger a \rangle = g_1 N \left( \langle a \sigma_1 \rangle + \langle a^\dagger \sigma_1 \rangle - \langle a \sigma_1 \rangle - \langle a^\dagger \sigma_1 \rangle \right) - 2\kappa \langle a^\dagger a \rangle + 2\kappa n, \]  
\[ \langle a^\dagger_1 \sigma_1 \rangle = -g_1 \left( \langle a \sigma_1 \rangle + \langle a^\dagger \sigma_1 \rangle + \langle a \sigma_1 \rangle + \langle a^\dagger \sigma_1 \rangle \right) - \gamma_1 \langle \sigma_1 \rangle - n \gamma_1 \langle \sigma_1 \rangle - \frac{i\Omega}{2} \left( \langle \sigma_1 \rangle - \langle \sigma_1 \rangle \right), \]  
\[ \langle a^\dagger_1 \sigma_2 \rangle = -2\Gamma_1 \langle a^\dagger_1 \sigma_2 \rangle + 2g_1 \left( \langle a \sigma_1 \sigma_2 \rangle + \langle a \sigma_1 \sigma_2 \rangle \right) + i\Omega \langle a^\dagger_1 \sigma_2 \rangle, \]  
\[ \langle a^\dagger_2 \sigma_2 \rangle = -2\Gamma_1 \langle a^\dagger_2 \sigma_2 \rangle + g_1 \left( \langle a \sigma_1 \sigma_2 \rangle + \langle a \sigma_1 \sigma_2 \rangle + \langle a \sigma_1 \sigma_2 \rangle \right) - \frac{i\Omega}{2} \left( \langle \sigma_1 \sigma_2 \rangle - \langle \sigma_1 \sigma_2 \rangle \right), \]  
\[ \langle a^\dagger_1 \sigma_2 \rangle = -4\gamma_1 \left( \langle a \sigma_1 \sigma_2 \rangle + \langle a \sigma_1 \sigma_2 \rangle + \langle a \sigma_1 \sigma_2 \rangle \right) - 4g_1 \left( \langle a \sigma_1 \sigma_2 \rangle + \langle a \sigma_1 \sigma_2 \rangle + \langle a \sigma_1 \sigma_2 \rangle \right) - 2\Omega \left( \langle \sigma_1 \sigma_2 \rangle - \langle \sigma_1 \sigma_2 \rangle \right), \]  
\[ \langle a^\dagger_2 \sigma_2 \rangle = -2\Gamma_1 \langle a^\dagger_2 \sigma_2 \rangle - 2\gamma_1 \left( \langle a \sigma_1 \sigma_2 \rangle + \langle a \sigma_1 \sigma_2 \rangle + \langle a \sigma_1 \sigma_2 \rangle \right) + g_1 \left( \langle a \sigma_1 \sigma_2 \rangle + \langle a \sigma_1 \sigma_2 \rangle - 2 \left( \langle a \sigma_1 \sigma_2 \rangle + \langle a \sigma_1 \sigma_2 \rangle + \langle a \sigma_1 \sigma_2 \rangle \right) \right) \]  
\[ - \frac{i\Omega}{2} \left( 2 \left( \langle \sigma_1 \sigma_2 \rangle - \langle \sigma_1 \sigma_2 \rangle \right) - \langle \sigma_1 \sigma_2 \rangle \right), \]  
\[ \langle a^\dagger_1 \sigma_1 \rangle = -\left( \Gamma_c + \Gamma_2 \right) a \sigma_1 + g_1 \left( \langle a \sigma_1 \rangle + \langle a^\dagger a \sigma_1 \rangle - \langle \sigma_1 \rangle \right) + g_1 \left( \langle a a \sigma_1 \rangle + \langle a^\dagger a \sigma_1 \rangle \right) + \frac{i\Omega}{2} \left( a \sigma_1 \right), \]  
\[ \langle a^\dagger_2 \sigma_1 \rangle = -\left( \Gamma_c + \Gamma_2 \right) a \sigma_1 + g_1 \left( \langle a \sigma_1 \rangle + \langle a^\dagger a \sigma_1 \rangle + \langle a^\dagger a \sigma_1 \rangle \right) + g_1 \left( \langle a a \sigma_1 \rangle - \langle \sigma_1 \rangle \right) + \frac{i\Omega}{2} \left( a \sigma_1 \right), \]  
\[ \langle a^\dagger_2 \sigma_2 \rangle = -\Gamma_c \langle a \sigma_1 \sigma_2 \rangle - 2\gamma_1 \left( \langle a \sigma_1 \sigma_2 \rangle + \langle a \sigma_1 \sigma_2 \rangle \right) - g_1 \left( \langle a \sigma_1 \sigma_2 \rangle + \langle a \sigma_1 \sigma_2 \rangle + \langle a \sigma_1 \sigma_2 \rangle \right) \]  
\[ + g_1 \left( \langle a \sigma_1 \sigma_2 \rangle - \langle \sigma_1 \rangle \right) - i\Omega \left( \langle a \sigma_1 \rangle - \langle a \sigma_1 \rangle \right). \]  

The employed approximations convert the resulting autonomous differential system of equations from an infinite linear set to a finite non-linear one, which does not always converge to an asymptotically stable solution given the presence of a continuous coherent drive.

While the global stability of the resulting non-linear system can not be analytically assessed using conventional stability theory, a rudimentary analysis of the linearized Jacobian indicates that the system's equilibrium solutions are all at least stable for the physically allowed values $\langle \sigma_1 \rangle \in [-1, 1]$ and $\langle a \rangle \in [0, +\infty)$, and will at least converge to a stable periodic solution.

On the other hand, the resulting system becomes severely numerically stiff, especially when detuning is introduced and the single cooperativity $C = g_1^2/\kappa\gamma_1 > 1$, resulting in the numerical instability for parameter configurations that can be difficult to predict. In light of this, some of the steady-state solutions are double-checked by numerically integrating the rate equations. This is carried out to map the derivatives beyond the point where they remain within the numerical solvers tolerance, for a duration corresponding to an order of magnitude longer than the reciprocal of the slowest defined rate.

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