Three-photon polarization ququarts: polarization, entanglement and Schmidt decompositions

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Abstract
We consider polarization states of three photons, propagating collinearly and having equal given frequencies but with arbitrary distributed horizontal or vertical polarizations of photons. A general form of such states is a superposition of four basic three-photon polarization modes, to be referred to as the three-photon polarization ququarts (TPPQ). All such states can be considered as consisting of one- and two-photon parts, which can be entangled with each other. The degrees of entanglement and polarization, as well as the Schmidt decomposition and Stokes vectors of TPPQ are found and discussed.

Keywords: three-photon states, entanglement, polarization Schmidt modes, Schmidt decomposition, Stokes vectors

(Some figures may appear in colour only in the online journal)

1. Introduction

The main objects of the modern science of quantum information are bipartite and, in particular, biphoton states. Characteristic features of such states are their entanglement and polarization, which are widely investigated and used in practical purposes, such as, e.g. transmission of information in quantum nets. The simplest biphoton states are purely polarization states of two photons belonging both to the same spatial and spectral mode to be referred to as biphoton polarization qutrits (BPQ). Biphoto states can be produced in different ways, but the most often used method is based on the phenomenon of spontaneous parametric down-conversion (SPDC) in nonlinear birefringent crystals. In such processes some pump photons are converted in crystals into pairs of photons of smaller frequencies, and the pump is taken not too strong to avoid parametric amplification or simultaneous production of four and higher numbers of photons. On the other hand, in stronger fields, multiphoton quantum states arise rather naturally in the process of parametric amplification [1, 2], and the first reports on analysis of entanglement achievable in such ‘macroscopic quantum states’ was given in the works [3, 4]. Note, however, that definitions of the degree of entanglement in multiphoton states can be not as simple as in the case of biphotons, and further investigations may be needed. In this paper we consider the simplest multiphoton states—the states of three photons. A general interest in three-photon quantum states has existed for rather a long time and persists until now [5–20]. In principle, production of three-photon states can be realized in the usual SPDC scheme but with not-birefringent crystals having central symmetry. In this case the second-order susceptibility equals zero, \( \chi^{(2)} = 0 \), and the usual SPDC is forbidden, whereas the third-order susceptibility does not equal zero, \( \chi^{(3)} \neq 0 \), and the three-photon decay of pump photons is possible. It is true that usually \( \chi^{(3)} \) is very small, and to have efficient three-photon SPDC generation one has to use too strong pump fields. But in some semiconductor crystals (GaAs, Si, InSb) the third-order susceptibility can be rather high and comparable with typical second-order susceptibilities in birefringent crystals. Another possibility of making the third-order processes efficient is related to the use of fibers [19],
where a small value of the third-order susceptibility can be compensated for by a long distance at which the three-photon decay of pump photons can occur. Finally, a further approach is based on the use of two independent SPDC pairs, with one of four photons subsequently set apart [17].

In this paper we consider theoretically the most general form of three-photon pure polarization states with collinearly propagating photons having coinciding given frequencies, but with arbitrary distributed polarizations of photons. The main characteristics of such states are their degrees of polarization and entanglement, which will be found and investigated below in their dependence on parameters of three-photon states. Two related aspects are the Schmidt decompositions and features of Stokes vectors of three-photon states and of their Schmidt modes.

2. Three-photon polarization ququarts

The most general form of three-photon states defined above is given by a superposition of four basic three-photon states

$$\Psi = C_3|3_H\rangle + C_2|2_H, V\rangle + C_1|1_H, 2_V\rangle + C_4|3_V\rangle,$$  
(1)

where $H$ and $V$ indicate vertical and horizontal polarization of photons (in the plane, perpendicular to the propagation direction) and $C_{1,2,3,4}$ are arbitrary complex constants obeying the normalization condition $\Sigma |C_i|^2 = 1$. State vectors of the basic states in equation (1) are given by

$$|3_H\rangle = \frac{a^+_H a^+_V}{\sqrt{6}} |0\rangle, |2_H, V\rangle = \frac{a^+_H a^+_V}{\sqrt{2}} |0\rangle, $$  
(2)

$$|1_H, 2_V\rangle = \frac{a^+_H a^+_V}{\sqrt{2}} |0\rangle, |3_V\rangle = \frac{a^+_V}{\sqrt{6}} |0\rangle,$$

where $a^+_H$ and $a^+_V$ are the photon creation operators.

Equation (1) is a direct generalization of the expression for the state vector of BPQ

$$|\Psi\rangle_{\text{BPQ}} = C_3|3_H\rangle + C_2|2_H, V\rangle + C_1|1_H, 1_V\rangle + C_4|2_V\rangle,$$  
(3)

which have been widely investigated earlier [21-26] and comparison with which will be pursued throughout the paper.

As the number of basic three-photon polarization states equals four, all states of the form (1) can be referred to as ququarts. If superpositions of one-photon polarization states $|1_H\rangle$ and $|1_V\rangle$ form one-qubit states and BPQ are two-qubit states, TPPQ are three-qubit states. Dimensionality of the corresponding Hilbert space is $3^3 = 8$. In principle, the 8-dimensional Hilbert space has eight basic state vectors. Because of indistinguishability of photons, TPPQ occupy only a part of the 8-dimensional Hilbert space characterized by four basic state vectors. In the case of three-particle states of some other distinguishable one-qubit particles each of two three-photon basic states $|2_H, 1_V\rangle$ and $|1_H, 2_V\rangle$ would be replaced by three independent basic states to give altogether eight basic states.

In addition to state vectors of equations (1) and (2), TPPQ can be characterized by their polarization wave functions, depending on three discrete polarization variables $\sigma_1$, $\sigma_2$ and $\sigma_3$, each of which can take independently only one of two meanings, $\sigma_i = H$ or $\sigma_i = V$:

$$\Psi(\sigma_1, \sigma_2, \sigma_3) = \langle \sigma_1, \sigma_2, \sigma_3 | \Psi \rangle,$$  
(4)

and the same for basic state vectors (2). Owing to indistinguishability of photons none of the variables $\sigma_i$ can be associated with any specific photon, but certainly, the number of variables equals to the number of photons. Also, owing to indistinguishability of photons, all TPPQ wave functions are obliged to be symmetric with respect to variable superpositions.

The wave function $\Psi(\sigma_1, \sigma_2, \sigma_3)$ (4) determines the TPPQ density matrix

$$\rho(\sigma_1, \sigma_2, \sigma_i^\prime, \sigma_j^\prime) = \Psi(\sigma_1, \sigma_2, \sigma_3) \Psi^\dagger(\sigma_1^\prime, \sigma_2^\prime, \sigma_3^\prime).$$  
(5)

This full TPPQ density matrix can be reduced with respect to one of three variables to give a two-photon reduced density matrix

$$\rho_r(\sigma_1, \sigma_2, \sigma_i^\prime, \sigma_j^\prime) = \sum_{\sigma_3} \rho(\sigma_1, \sigma_2, \sigma_i^\prime, \sigma_j^\prime, \sigma_3).$$  
(6)

Reduction of the density matrix can be continued to give the twice reduced one-photon density matrix of TPPQ

$$\rho_{rr}(\sigma_1, \sigma_2) = \sum_{\sigma_3} \rho_r(\sigma_1, \sigma_2, \sigma_i^\prime, \sigma_j^\prime).$$  
(7)

Of course, equations (6) and (7) can be written in the form of true matrices, with lines and columns, elements of which in the natural bases of one and two horizontally and vertically polarized photons will be given just by $\rho_r(\sigma_1, \sigma_2)$ and $\rho_r(\sigma_1, \sigma_2, \sigma_i^\prime)$. Appearances of two types of reduced density matrices, $\rho_r$ and $\rho_{rr}$, differs TPPQ from BPQ (3), which have only three basic states $|2_H\rangle$, $|1_H, 1_V\rangle$, $|2_V\rangle$, two polarization variables $\sigma_1$ and $\sigma_2$, and only one one-photon reduced density matrix $\rho_r[25]$.  

3. Definitions of entanglement

Two equivalent definitions of entanglement to be mentioned are related to features of wave functions and reduced density matrices of states. In terms of wave functions, a pure multipartite state is disentangled if its wave function can be factorized for a product of single-party wave functions, and otherwise the state is entangled. For TPPQ the condition of disentanglement takes the form

$$\Psi(\sigma_1, \sigma_2, \sigma_3) = \psi(\sigma_1) \psi(\sigma_2) \psi(\sigma_3),$$  
(8)

and if such factorization does not occur, the TPPQ is entangled.

The second definition, following from the first one, reads that a pure multipartite state is disentangled if all its reduced density matrices describe pure state, i.e., if entropies of all reduced states are equal zero. For TPPQ this means that the state is disentangled if

$$S_r \equiv S(\rho_r) = 0 \quad \text{and} \quad S_{rr} \equiv S(\rho_{rr}) = 0,$$  
(9)

and otherwise the state is entangled. As follows from this definition and as well known, entropies of reduced states can be considered as good entanglement quantifiers.
Note that sometimes analysis of entanglement in terms of wave functions and in terms of density matrices are referred to as related to particle- and mode-entanglement. But such differentiation is rather conventional because in fact these two approaches are absolutely equivalent to each other and they characterize the same feature of entanglement intrinsic to systems under consideration and independent of methods of their theoretical description.

Note also that very often entanglement is considered as a feature of systems consisting of two spatially separated parts, and reduction of density matrices is considered as the reduction over one of such remote subsystems. However, as follows from the derivation of the previous section, even if particles are not spatially separated, one can use the particle variables for reducing, e.g., a tree-photon state to its two-photon or one-photon parts. Such reduction reveals such fundamental features of systems under consideration as correlations (entanglement) between parts of systems even if these parts are not spatially separated. Measurement or practical use of such correlations is a different question partially addressed below in section 9 and in the works [26, 27].

For spatially separated subsystems the usual requirement to entanglement quantifiers is their invariance with respect to local unitary transformations in any of two subsystems. For systems of particles not separated in space this requirement does not have sense because all transformations are local and they affect all particles together. A more natural requirement for polarization states is the invariance of entanglement quantifiers with respect to all rotations of the Poincaré sphere, which include both rotations of the coordinate frame and all transformations between linear and circular polarizations (see section 8 below and discussions in the works [23, 26, 27]).

Specifically, as said above, in the case of TPPQ their states can be considered as consisting of two- and one-photon parts, characterized by the reduced density matrices \( \rho_r \) (6) and \( \rho_{rr} \) (7), and entanglement of the TPPQ states can be understood as entanglement between their two- and one-photon parts. Owing to indistinguishability of photons such ‘gedanken’ separation is unique for any given TPPQ and does not depend on which specific photons or polarization variables are taken belonging to one- or two-photon parts.

Note that in the cases of states with higher numbers of photons \( (N > 3) \), one can imagine more than one possibility to consider such states as consisting of pairs of states with smaller numbers of photons \( (N_1 < N \text{ plus } N - N_1 \text{ with different } N_1) \). For example, for four-photon states there are two possibilities: \( 4 = 2 + 2 \) and \( 4 = 3 + 1 \). Competition of such different ‘gedanken’ separations in terms of their degrees of entanglement is an interesting question to be analyzed elsewhere.

4. Eigenvalues of the reduced density matrices and the degrees of polarization and entanglement

The above-outlined procedure of finding the reduced density matrices via TPPQ wave functions is straightforward [28] though somewhat cumbersome. Instead, we can use the identity between the completely reduced one-photon density matrix of an arbitrary \( N \)-photon state and its polarization matrix, with the latter defined by correlators of creation and annihilation operators [29, 30] polarization matrix, with the latter defined by correlators of creation and annihilation operators [29, 30]

\[
\rho_{rr} = \frac{1}{N} \left( \frac{\langle a^*_i a^*_j \rangle}{\langle a^*_i a_j \rangle} \langle a^*_i a^*_j \rangle \right),
\]

where in the subscript to \( \rho \) the reduction symbol \( r \) is repeated \( N - 1 \) times and averaging in correlators is going on with the state vector of the \( N \)-photon state under consideration. In the case of TPPQ, \( N = 3 \) and the state vector to be used for averaging is given by equation (1), which yields immediately

\[
\rho_{rr} = \begin{pmatrix}
|C|^2 + \frac{2|C|C_2}{3} + \frac{|C|^2}{3} & \frac{GC_2^*}{\sqrt{3}} + \frac{2|C|^2C_2^*}{3} + \frac{2C_2^2}{3} \\
\frac{GC_2^*}{\sqrt{3}} & \frac{C_2^2}{3}
\end{pmatrix}
\]

(11)

Eigenvalues of this matrix are easily found to be given by

\[
\lambda_\pm = \frac{1}{2} \left( 1 \pm \sqrt{1 - C^2} \right)
\]

(12)

with the constant \( C_\pm = 2 \sqrt{\lambda_\pm} \) equal to

\[
C_\pm = 2 \sqrt{\lambda_\pm} = 2 \sqrt{(1 - \lambda_\pm)} = \sqrt{2 \left( \frac{GC_3}{\sqrt{3}} - \frac{C^2}{3} \right)}^2 + \left( \frac{GC_3 - C^2}{3} \right)^2 \]

(13)

If elements of the matrix \( \rho_{rr} \) (7) are written as \( \rho_{rr} (\sigma_1, \sigma'_1) \), then elements of the adjoint reduced matrix, \( \rho_r \), should be written as \( \rho_r (\sigma_2, \sigma_3, \sigma'_2, \sigma'_3) \). As a whole, in the matrix form, \( \rho_r \) is the \( 4 \times 4 \) two-photon matrix (received by reduction only over one photon variable, \( \sigma_1 \)). In principle, this matrix can be found either with the help of equations (1), (4) and (6) of section 2 or via the coherence matrix [30] of the fourth-order correlators of the creation and annihilation operators \( \langle a_i^* a_i^* a_j a_j \rangle \), where \( i, j = H \) or \( V \). Details of this last method will be described elsewhere. But here we do not dwell upon any derivations because, in fact, the two-photon reduced density matrix \( \rho_r \) has the same two nonzero eigenvalues \( \lambda_\pm \) (12) as its adjoint reduced one-photon density matrix \( \rho_{rr} \) plus two zero eigenvalues.

Equation (12) for the eigenvalues of the reduced density matrices is very similar to that occurring in the case of BPQ, namely

\[
\lambda_{\pm}^{(bph)} = \frac{1}{2} \left( 1 \pm \sqrt{1 - C^2} \right),
\]

where \( C \) is the concurrence [31] of BPQ (3) given by [25]

\[
C = 2 |G C_3 - C_2^2|.
\]

It’s true that \( C_\pm \) (13) is much more complicated than the BPQ concurrence. It is also true that the parameter \( C_\pm \) cannot be derived by the same Wootters method, that
has been used [25] for derivation of the BPQ concurrence $C$, because the latter is defined only for two-qubit states. But, on the other hand, as shown in figure 1, the parameter $C_\psi$ is perfectly compatible with other entanglement quantifiers such as entropy of the reduced density matrices

$$S_\psi = S_\lambda = -\lambda_+ \log \lambda_+ - \lambda_- \log \lambda_- = -\lambda_+ \log \lambda_+ + (1 - \lambda_-) \log (1 - \lambda_-)$$

(14)

and the Schmidt parameter

$$K = \frac{1}{\lambda_+^2 + \lambda_-^2} = \frac{2}{2 - C_\psi^2} = \frac{1}{(1 - \lambda_-)^2 + \lambda_-^2},$$

(15)

and this is one of reasons for treating $C_\psi$ as the generalized concurrence.

All three curves are monotonously growing functions of $\lambda_-$, and all three parameters $C_\psi$, $S_\psi = S_\lambda$ and $K$ can be considered as measuring the same degree of entanglement, though in different metrics. The minimal values of these parameters ($K = 1$ and $C_\psi = S_\lambda = S_\psi = 0$) are achieved at $\lambda_- = 0$ and they correspond to disentangled states. This specific case can be considered as providing the third definition of entanglement, additional to two definitions of section 3: the state is disentangled only if the minimal eigenvalue of its $2 \times 2$ reduced density matrix equals zero, $\lambda_- = 0$, and otherwise the state is entangled.

Polarization features of TPPQ are determined by their Stokes vectors $\vec{S}$ and the degree of polarization $P = |\vec{S}|$. Components of the Stokes vectors $S_{1,2,3}$ are determined directly by the twice reduced density matrix (11) because, as said above, this matrix is identical to the polarization matrix [26]

$$\rho_{rr} = \rho_{pol} = \frac{1}{2} \begin{pmatrix} 1 + S_3 & S_1 - i S_2 \\ S_1 + i S_2 & 1 - S_3 \end{pmatrix}.$$ 

(16)

This equation gives immediately exact expressions for the TPPQ Stokes vectors in a general form

$$\vec{S} = \begin{pmatrix} 2 \text{Re} \left( \frac{C_1 C_2^* + C_3 C_4^*}{\sqrt{3}} + \frac{2 C_2 C_3^*}{3} \right) \\ -2 \text{Im} \left( \frac{C_1 C_2^* + C_3 C_4^*}{\sqrt{3}} + \frac{2 C_2 C_3^*}{3} \right) \\ |C_1|^2 + |C_2|^2 - \frac{|C_3|^2}{3} - |C_4|^2 \end{pmatrix}.$$ 

(17)

With a simple algebra, we find from these equations that the TPPQ degree of polarization $P$ is related to the generalized concurrence $C_\psi$ by the relation

$$P^2 + C_\psi^2 = 1.$$ 

(18)

This is the same relation which has been found earlier [25] for the degree of polarization and concurrence of BPQ, $P_{\text{BPQ}}^2 + C^2 = 1$. This relation is fundamentally important because it shows explicitly that the degrees of entanglement and polarization anticorrelate with each other so that maximally entangled states are not polarized and maximally polarized states are disentangled. Equation (18) shows that this general rule is equally valid for both TPPQ and BPQ. Note that in terms of eigenvalues of the matrix $\rho_{rr}$ (12) and in accordance with equation (18), the degree of polarization of TPPQ per one photon is given by

$$P = \lambda_+ - \lambda_- = \sqrt{1 - C_\psi^2}.$$ 

(19)

The same equality occurs in the case of BPQ though with the substitution of the generalized concurrence $C_\psi$ (13) by the true concurrence $C$ of BPQ [25].

5. Schmidt decomposition

Eigenfunctions of the reduced density matrices $\rho_{rr}$ (7), (11) and $\rho_\psi$ (6) are the one-photon and two-photon Schmidt modes $\psi_{\pm}$ and $\chi_\pm$, obeying equations

$$\rho_{rr} \psi_{\pm} = \lambda_{\pm} \psi_{\pm}, \quad \rho_{\psi} \chi_{\pm} = \lambda_{\pm} \chi_{\pm}.$$ 

(20)

The Schmidt modes are orthogonal and normalized

$$\langle \psi_\pm |\psi_\pm \rangle = 1, \quad \langle \psi_\pm |\chi_\pm \rangle = 0;$$ 

(21)

$$\langle \chi_\pm |\psi_\pm \rangle = 0, \quad \langle \chi_\pm |\chi_\pm \rangle = 1.$$ 

(22)

Equation (20) yields immediately the well-known Schmidt decompositions for the reduced density matrices

$$\rho_{rr} = \sum \lambda_{\pm} |\psi_\pm \rangle \langle \psi_\pm |, \quad \rho_\psi = \sum \lambda_{\pm} |\chi_\pm \rangle \langle \chi_\pm |.$$ 

(23)

The same Schmidt modes determine the Schmidt decomposition of the TPPQ wave function

$$\Psi(\sigma_1, \sigma_2, \sigma_3) = \sum |\lambda_{\pm} \psi_\pm (\sigma_1) \chi_\pm (\sigma_2, \sigma_3) \rangle.$$ 

(24)

In accordance with the original derivation by Schmidt [32], the decomposition of a bipartite wave function gives rise to the way of finding Schmidt modes alternative to that based on equations with density matrices equations (like
equations (20)). Directly from equation (24) we find that the functions \( \psi_x \) and \( \chi_x \) obey the following set of two equations:

\[
\begin{align*}
\sum_{\sigma_1} \psi_x^\sigma_1(\sigma_1) \psi(\sigma_1, \sigma_2, \sigma_3) &= \sqrt{\lambda_+} \chi_x(\sigma_2, \sigma_3), \\
\sum_{\sigma_2, \sigma_3} \chi_x^+ (\sigma_2, \sigma_3) \psi(\sigma_1, \sigma_2, \sigma_3) &= \sqrt{\lambda_-} \psi_x(\sigma_1).
\end{align*}
\]

(25) These equations can be considered as a transfer for the case of discrete variables of the integral equations derived by Schmidt for a function of two continuous variables \( \mathcal{F}(x, y) \), with the kernel of the integral equations given by the function \( \mathcal{F}(x, y) \) itself. In the case of TPPQ variables are discrete, which gives sums instead of integrals, and the variable substitutions are \( x = \sigma_1 \) and \( y = \{\sigma_2, \sigma_3\} \).

Equations of the type (25) and (26) were used in the works [27, 33] for finding Schmidt modes of BPQ.

From the Schmidt decomposition of the wave function (24) we find immediately the general relations between the Stokes vectors of TPPQ and Stokes vectors of Schmidt modes. The TPPQ Stokes vector (per one photon) can be defined as \( \vec{S} = \langle \psi|\mathcal{P}^{(0)}|\psi\rangle \), where \( \mathcal{P}^{(0)} \) is the vector of Pauli matrices acting on any rth polarization variable, \( \sigma_1, \sigma_2, \) or \( \sigma_3 \). By choosing \( i = 1 \) and \( i = 2 \) or \( i = 3 \), we find

\[
\vec{S} = \lambda_+ \vec{S}_{+}^{\psi} + \lambda_- \vec{S}_{-}^{\psi} = \lambda_+ \vec{S}_{+}^{\chi} + \lambda_- \vec{S}_{-}^{\chi},
\]

(27) where \( \vec{S}_{+}^{\psi} = (\psi_1|\mathcal{P}^{(1)}|\psi_1) \) and \( \vec{S}_{+}^{\chi} = (\chi_1|\mathcal{P}^{(2)}|\chi_1) \equiv (\chi_1|\mathcal{P}^{(3)}|\chi_1) \) are the Stokes vectors of one-photon and two-photon Schmidt modes \( \psi_x \) and \( \chi_x \). Presentation of the TPPQ Stokes vector \( \vec{S} \) in the form of sums of Schmidt-mode Stokes vectors \( \vec{S}_{+}^{\psi} \) or \( \vec{S}_{+}^{\chi} \) with the weighting coefficients \( \lambda_+ \) is analogous to similar results occurring in the case of BPQ [26]. Note that in spite of identity of the expressions for \( \vec{S} \) (27) in terms of Stokes vectors one- and two-photon Schmidt modes, in a general case of arbitrary TPPQ these Schmidt-mode Stokes vectors do not coincide with each other, \( \vec{S}_{+}^{\psi} \neq \vec{S}_{+}^{\chi} \). This statement will be illustrated by some specific examples in section 8 below.

It may be interesting to compare in more detail features of the Schmidt decompositions occurring in the cases of TPPQ and BPQ. In the case of qudits there are three forms of the Schmidt decompositions: decomposition of the reduced density matrix, of the wave function and of the state vector:

\[
\rho^{\text{biph}} = \sum_{\pm} \lambda_{\pm} |\psi_{\pm}\rangle \langle \psi_{\pm}|,
\]

(28)\[\psi^{\text{biph}}(\sigma_1, \sigma_2) = \sum_{\pm} \sqrt{\lambda_{\pm}} |\psi_{\pm}(\sigma_1)| \psi_{\pm}(\sigma_2),
\]

(29)\[|\psi^{\text{biph}}\rangle = \frac{1}{\sqrt{2}} \left( \sqrt{\lambda_+} a_1^+ |2_+\rangle + \sqrt{\lambda_-} a_1^- |2_-\rangle \right) |0\rangle
\]

(30)where \( a_1^+ \) and \( a_1^- \) are the photon creation operators for the orthogonal Schmidt modes \( \psi_{\pm} \) and \( \psi_{\pm} \).

A remarkable feature of BPQ is identity of the adjoint Schmidt modes, \( \chi_x \equiv \psi_x \), which is related to the obligatory symmetry of BPQ wave function with respect to the variable transposition. This means that two photons occupying adjoint Schmidt modes both belong to the same mode \( \psi_x \) or \( \psi_{\pm} \). This is the reason why the Schmidt decomposition for the wave function (29) can be transformed immediately to the Schmidt decomposition for the biphon decomposition of the state vector (30) indicates clearly a possibility of separation of the ‘+’ and ‘-’ Schmidt modes as well as the possibility of manipulations with Schmidt modes [27, 33] including revelation of operational features of BPQ related to their entanglement.

In the case of TPPQ, evidently, \( \chi_x \neq \psi_x \) because \( \chi_x \) is biphon and \( \psi_x \) one-phonon wave functions. For this reason, in a general case, there is no way to obtain the Schmidt decomposition of the TPPQ state vector similar to that of equation (30) occurring in the case of biphons. This means also that in a general case separation of the ‘+’ and ‘-’ Schmidt modes is problematic. The only exception occurs in the case when two-photon TPPQ Schmidt modes appear to be factorized for products of one-photon Schmidt modes

\[
\chi_x(\sigma_2, \sigma_3) = \psi_x(\sigma_1) \psi(\sigma_1, \sigma_2, \sigma_3),
\]

(31)\[|\psi\rangle = \frac{1}{\sqrt{6}} \left( \sqrt{\lambda_+} a_1^+ |3_+\rangle + \sqrt{\lambda_-} a_1^- |3_-\rangle \right) |0\rangle
\]

(33)This form of the decomposition is very similar to that of equation (30) and opens for TPPQ the same possibilities of manipulations, analysis and applications which are known for BPQ (see [26, 27, 33] and section 9 below). In the following section we find general conditions for the TPPQ parameters under which its Schmidt decompositions can be reduced to the forms (32) and (33).

6. Conditions under which the TPPQ state vector (1) can be reduced to the form (33)

Let us consider the TPPQ state vector of a general form (1) with arbitrary coefficients \( C_{1,2,3,4} \neq 0 \). The question is whether it is possible to transform it to the form of equation (33) and under which conditions? Let us make a basis transformations corresponding to arbitrary rotations in the Poincaré sphere

\[
a^{\pm} = a_1^+ \cos \theta - \sin \theta a_1^-, \\
a^{\phi} = e^{i\phi} (a_1^+ \sin \theta + \cos \theta a_1^-),
\]

(34)where \( \theta \) and \( \phi \) are arbitrary real transformation parameters, whereas \( a_1^+ \) and \( a_1^- \) are photon creation operators in two orthogonal modes in the transformed basis. With \( a_1^+ \) and \( a_1^- \) of equations (34) substituted into equation (1), the latter takes a similar form but with modified coefficients.
\[ |\Psi\rangle = \left( \frac{C_1}{\sqrt{6}} a^+_{13} + \frac{C_2}{\sqrt{2}} a^+_{12}a^+_{12} + \frac{C_4}{\sqrt{6}} a^+_{13} \right) |0\rangle . \] (35)

If we want the transformed expression to have the form (33), we have to require
\[ \tilde{C}_1 = 0 \text{ and } \tilde{C}_3 = 0 . \] (36)

Explicitly \( \tilde{C}_2 \) and \( \tilde{C}_3 \) are given by
\[ \tilde{C}_2 = -\sqrt{\frac{3}{2}} C_1 \cos^3 \theta \sin \theta + \frac{e^{-i\phi}}{\sqrt{2}} C_2 (\cos^3 \theta - 2 \cos \theta \sin^2 \theta) + \frac{e^{-2i\phi}}{\sqrt{2}} C_3 (\sin^3 \theta + 2 \sin \theta \cos^2 \theta) + \frac{3}{2} e^{-3i\phi} C_4 \sin^2 \theta \cos \theta \] and
\[ \tilde{C}_3 = \sqrt{\frac{3}{2}} C_1 \sin^3 \theta \cos \theta + \frac{e^{-i\phi}}{\sqrt{2}} C_2 (\sin^3 \theta - 2 \sin \theta \cos^2 \theta) + \frac{e^{-2i\phi}}{\sqrt{2}} C_3 (\cos^3 \theta - 2 \cos \theta \sin^2 \theta) + \frac{3}{2} e^{-3i\phi} C_4 \cos^2 \theta \sin \theta . \] (37)

With real \( \theta \) and \( \phi \), equations (36) can be satisfied only if \( \tilde{C}_2 \) and \( \tilde{C}_3 \) are real. This condition puts limitations for phases \( \varphi_{1,2,3,4} \) of the constants \( C_{1,2,3,4} \). For example these conditions for phases can be taken in the form
\[ \varphi_1 = 0, \varphi_2 = \varphi, \varphi_3 = 2\varphi, \varphi_4 = 3\varphi . \] (39)

With these phases all phase factors in equations (37) and (38) disappear and all constants \( C_{1,2,3,4} \) are replaced by their absolute values. Then equations (37) and (38) take the form of two cubic equations for \( \tan \theta \). By assuming \( \theta \neq 0 \), we get two quadratic equations for \( \tan \theta \) having the form
\[ A \tan^2 \theta + B \tan \theta + D = 0, \]
\[ D \tan^2 \theta - B \tan \theta + A = 0, \] (40)

where
\[ A = |C_1|^2 + |C_2|^2, \]
\[ B = \sqrt{3} (|C_1||C_2| - |C_3||C_4|), \]
\[ D = -2 (|C_2|^2|C_4|^2 + \sqrt{3} (|C_1||C_2||C_4| + |C_1||C_3||C_4|)). \] (41)

Two equations (40) are compatible only if \( A = -D \), which yields
\[ |C_1|^2 + |C_3|^2 = \sqrt{3} (|C_2||C_4| + |C_3||C_4|) . \] (42)

Equations (39) and (42) determine the complete set of conditions under which the TPPQ state vector can be reduced to the form of the Schmidt-decomposition (33). Under the same conditions the solutions of equations (40) are given by
\[ \tan \theta = \frac{-B \pm \sqrt{B^2 + 4A^2}}{2A} . \] (43)

These solutions together with equations (34), (39), (41) and (42) can be used for finding explicitly the Schmidt-mode creation operators \( a^+_{14} \) and \( a^+_{13} \), as well as the Schmidt modes themselves \( |\varphi_n\rangle = a^+_{14} |0\rangle \) and \( |\varphi_c\rangle = a^+_{13} |0\rangle \).

**Figure 2.** Degrees of entanglement \( C_s \) and polarization \( P \) (per one photon) as functions of the parameter \( \theta \) of equation (46).

Note that together with the normalization condition, the constraints (39) and (42) leave three free parameters: the phase \( \varphi = \varphi_2 \) and two of four absolute values of the constants \( C_{1,2,3,4} \), e.g. \( |C_1| \) and \( |C_3| \). Thus, the described procedure defines a three-parametric manifold of states of TPPQ which can be reduced to the Schmidt-decomposition of the form (33).

Below we consider two relatively simple examples of TPPQ:
\[ |\Psi_{14}\rangle = C_2|3_H\rangle + C_4|3_v\rangle \] with \( C_2 = C_3 = 0 \) \( (44) \)
and
\[ |\Psi_{23}\rangle = C_2|2_H\rangle + C_3|2_v\rangle \] with \( C_1 = C_2 = 0 \). \( (45) \)

As easily checked, the first of these states, \( |\Psi_{14}\rangle \), obeys the conditions (39) and (42), whereas the second state, \( |\Psi_{23}\rangle \), does not obey.

**7. The state \( |\Psi_{14}\rangle \)**

Let us write the state vector \( |\Psi_{14}\rangle \) in the form
\[ |\Psi\rangle = \cos \theta |3_H\rangle + e^{i\phi} \sin \theta |3_v\rangle , \] \( (46) \)
where the phase of \( C_1 \) is taken equal zero and the variation ranges of \( \theta \) and \( \varphi \) are \( \pi > \theta > 0 \) and \( \pi/2 > \varphi > -\pi/2 \). In this case the general equations (12), (13) and (18) yield
\[ \lambda_+ = \max \{ |C_1|^2, |C_4|^2 \} , \lambda_- = \min \{ |C_1|^2, |C_4|^2 \} . \] (47)

\[ C_s = 2|C_1||C_4| |\sin 2\theta| \] \( (48) \)

\[ P = \lambda_+ - \lambda_- = |C_1|^2 - |C_4|^2 = |\cos 2\theta| . \] (49)

The functions \( C_s(\theta) \) and \( P_s(\theta) \) are shown in figure 2.

Maximally entangled unpolarized state \( (C_s = 1, P = 0) \) occurs when \( |C_1| = |C_4| \) or \( C_1 = 0 \). In contrast, the TPPQ (46) is disentangled and maximally polarized \( (C_s = 0, P = 1) \) if \( |C_1| = 0 \) or \( |C_4| = 0 \), i.e. if \( \theta = 0 \) or \( \theta = \pi/2 \).

As follows from the general expression (17) for the Stokes vector (per one photon) of TPPQ, in the case \( C_2 = C_3 = 0 \) the vector \( \vec{S} \) has only one non-zero component: \( S_1 = S_2 = 0 \) and \( S_3 = |C_1|^2 - |C_4|^2 = \cos 2\theta \). In the Poincaré sphere the vector \( \vec{S} \) is directed along the \((V-H)\) axis and is given by 1/3 of the algebraic sum of all one-photon Stokes vectors of all photons presented in the state (17) with weighting factors \( |C_1|^2 = \cos^2 \theta \) and \( |C_4|^2 = \sin^2 \theta \), correspondingly, for horizontally and vertically polarized photons.
One-photon Schmidt modes of the state $|\psi_{1}\rangle$ are $|\psi_{+}\rangle = a_{1}^{+}|0\rangle$ and $|\psi_{-}\rangle = a_{1}^{-}|0\rangle$ if $|\cos \theta| > |\sin \theta|$ and the same but with $\pm \rightarrow \mp$ in the opposite case. Two-photon Schmidt modes of the state $|\psi_{2}\rangle$ are factorized for the product of one-photon Schmidt modes $\psi_{2}(\sigma_{2}, \sigma_{3}) = \psi_{2}(\sigma_{2})\psi_{4}(\sigma_{3})$. The Stokes vectors of the Schmidt modes are directed along the line $V-H$ in the Poincaré sphere.

All these features of the three-photon state (46) are practically identical to those of BPQ of a special form, $C_{1}|2_{H}| + C_{2}|2_{V}|$ [25, 26]. This direct analogy between the three-photon and biphoton states does not occur in other configurations of three-photon states, and one example of such configurations considered in the following section is the state $|\psi_{23}\rangle$ (45).

8. The state $|\psi_{23}\rangle$

In the case $C_{1} = C_{4} = 0$ the state vector of TPPQ $|\psi_{23}\rangle$ can be written in the form similar to that of equation (46)

$$|\psi_{23}\rangle = \cos \theta |2_{H}, 1_{V}\rangle + e^{i\phi} \sin \theta |1_{H}, 2_{V}\rangle,$$

(50)

where $|\theta| \leq \pi/2$, which corresponds to the constant $C_{2}$ taken real and positive, i.e. having a zero phase.

8.1. Polarization, entanglement and Stokes vector of the state $|\psi_{23}\rangle$

As follows from general expressions of equations (12), (13) and (18), the eigenvalues of the reduced density matrix, generalized concurrence and degree of polarization of the state $|\psi_{23}\rangle$ (50) are given by

$$\lambda_{2} = \frac{1}{2} \left( 1 \pm \frac{1}{3} \sqrt{4 - 3 \cos^{2} 2\theta} \right),$$

(51)

$$C_{S} = \frac{1}{3} \sqrt{8 - 12 |C_{2}|^{2} |C_{4}|^{2}} = \frac{\sqrt{5 + 3 \cos^{2} 2\theta}}{3},$$

(52)

$$P = \frac{1}{3} \sqrt{4 + 12 |C_{2}|^{2} |C_{4}|^{2}} = \frac{\sqrt{4 - 3 \cos^{2} 2\theta}}{3}.$$  

(53)

The variation ranges of these parameters are

$$\frac{\sqrt{5}}{3} \leq C_{S} \leq \frac{2\sqrt{2}}{3} \quad \text{and} \quad \frac{2}{3} \geq P \geq \frac{1}{3}.$$  

(54)

Within these ranges, entanglement is maximal and degree of polarization is minimal at $\theta = 0$ or $\theta = \pi/2$, i.e. at $C_{3} = 0$ or $C_{4} = 0$. And, in contrast, entanglement is minimal and degree of polarization is maximal at $\theta = \pi/4$ when $|C_{3}| = |C_{4}| = 1/\sqrt{2}$. In other words, the single states $|2_{H}, 1_{V}\rangle$ and $|1_{H}, 2_{V}\rangle$ are more entangled and less polarized than their superpositions. This behavior is somewhat unexpected and contrasts with that of the states (46) considered in the previous subsection. For the states (50) the dependencies of the generalized concurrence and the degree of polarization on the parameter $\theta$ are shown in figure 3.

The polarization Stokes vector of the state $|\psi_{23}\rangle$ is determined by the general expression (17) with $C_{1} = C_{4} = 0$

$$\vec{S} = \frac{1}{3} \begin{pmatrix} 2 \sin(2\theta) \cos \phi \\ 2 \sin(2\theta) \sin \phi \\ \cos(2\theta) \end{pmatrix} \equiv P \begin{pmatrix} \sin(2\alpha) \cos \phi \\ \sin(2\alpha) \sin \phi \end{pmatrix}.$$  

(55)

Here $P = ||\vec{S}||$ is the degree of polarization (53) and $\alpha$ is an angle related to but somewhat different from $\theta$ and determined by the equation

$$\cos 2\alpha = \frac{\cos 2\theta}{\sqrt{4 - 3 \cos^{2} 2\theta}}.$$  

(56)

The dependence of $\cos 2\alpha$ on $\cos 2\theta$ is shown in figure 4.

As follows from the second expression in equation (55) for the TPPQ Stokes vector $\vec{S}$, $2\alpha$ is the angle between its direction and the $(V-H)$ axis in the Poincaré sphere shown in figure 5. In this picture, the solid blue lines in the horizontal and vertical planes determine the ending positions of the Stokes vector $\vec{S}$ in two cases: $\phi = 0$, $0 \leq \theta \leq \pi/2$ and $\theta = \pi/4$, $0 \leq \phi \leq \pi/2$, whereas the dashed blue line in the vertical plane corresponds to the general case, $\phi \neq 0$ and $\theta \neq \pi/4$. In the case $\theta = 0$ equation (55) yields: $S_{1} = S_{3} = 0$ and $S_{2} = 1/3$. This means that in this case the TPPQ Stokes vector $\vec{S}$ is directed along the $(V-H)$ axis, and its length equals $1/3$, which has a very simple explanation. At $\theta = 0$ the state (50) turns into the single basic state $|2_{H}, 1_{V}\rangle$. In this state the lengths of collinear Stokes vectors of individual photons are equal 1 for horizontally and −1 for vertically polarized photons. The sum of these three individual Stokes vectors equals $1 + 1 - 1 = 1$. This is the length of the total Stokes vector of the state $|2_{H}, 1_{V}\rangle$ as a whole. The Stokes vector per one photon is obtained from the total Stokes vector per one photon as functions of the parameter $\theta$ of equation (50).
vector by means of division by the number of photons, which gives 1/3.

In the other special case, \( \theta = \pi/4 \) and \( \varphi = 0 \), the total Stokes vector \((55)\) of the state \((50)\) is directed along the axis \((-45^\circ, 45^\circ)\) in the Poincaré sphere, and its length equals 2/3. This specific result does not have such a simple explanation as that occurring in the case \( \theta = 0 \).

8.2. Factorizing creation operators

As is known \([23, 24, 26]\), the state vectors of BPQ can be presented in the form \(|\Psi_\text{biph}\rangle = NA^i B^j|0\rangle\), where \(N\) is the normalizing factor and \(A^i\) and \(B^j\) are the single-photon creation operators, factorizing the BPQ state vector. A simple way of finding \(A^i\) and \(B^j\) and analysis of their features are given in \([26]\). In particular, of course, the operator factorization of the expression for the BPQ state vector does not indicate disentanglement of BPQ. As shown in \([26, 27]\), the degree of entanglement of BPQ is determined by commutation features of the operators \(A^i\) and \(B^j\). The one-photon states \(|A^i\rangle\) and \(|B^j\rangle\) generated by these operators are characterized by their Stokes vectors \(\vec{S}_A\) and \(\vec{S}_B\). As known \([23, 24, 26]\), the biphon Stokes vector \(\vec{S}_\text{biph}\) is always located in the plane \([\vec{S}_A, \vec{S}_B]\) and is directed along the bisector of the angle between \(\vec{S}_A\) and \(\vec{S}_B\), i.e. \(\vec{S}_\text{biph} \parallel (\vec{S}_A + \vec{S}_B)\). In the case of TPPQ, too, the operators \(A^i\) and \(B^j\) can be found in a way similar to that described in \([26]\) for BPQ. Here we drop this derivation which will be described elsewhere. Note, however, that in analogy with BPQ, it might be natural to think that the TPPQ Stokes vector \(\vec{S}\) is parallel to the sum of three one-photon Stokes vectors \(\vec{S}_A + \vec{S}_B + \vec{S}_D\). But in a general case this assumption appears to be wrong. This is clearly seen in the example of the state \((50)\) we consider here. For this state the factorizing operators are evident: \(A^i = a_{0i}^i, B^j = a_{0j}^j\), and \(D^i = \frac{1}{\sqrt{2}} (\sin \theta a_{0i}^i + e^{i\varphi} \sin \theta a_{0j}^j)\). The Stokes vectors \(\vec{S}_A\) and \(\vec{S}_B\) are directed along the \((V-H)\) axis of the Poincaré sphere, they have equal absolute values \((=1)\) but are oppositely directed. They cancel each other in the sum of three Stokes vectors \(\vec{S}_A + \vec{S}_B + \vec{S}_D = \vec{S}_D\). In the case \(\varphi = 0\) both \(\vec{S}_D\) and TPPQ Stokes vector \(\vec{S}\) are located in the horizontal plane of the Poincaré sphere, but they are not parallel to each other, as shown in figure 6. The angles between these vectors and the \((V-H)\) axis are equal to \(2\theta\) and \(2\alpha\) correspondingly, for \(\vec{S}_D\) and \(\vec{S}\).

Thus, this example shows clearly that in a general case the TPPQ Stokes vector is not parallel to the vectorial sum of one-photon Stokes vectors \(\vec{S}_{A,B,D}\) and the assumed simple analogy with biphons does not work for three-photon states. An alternative interpretation and other results arise in the approach based on the Schmidt decompositions, Schmidt modes and their Stokes vectors (see sections 6 and 7).

8.3. One-photon Schmidt modes and Stokes vectors of the state \(|\Psi_{23}\rangle\)

The twice reduced density matrix of the state \(|\Psi_{23}\rangle\) can be obtained directly from its general expression (11) with \(C_1\) and \(C_2\) taken equal zero, and in terms of \(\theta\) and \(\varphi\) parameters it takes the form

\[
\rho_{\psi} = \frac{1}{6} \left( \begin{array}{cc} 3 + \cos 2\theta & 2e^{-i\varphi} \sin 2\theta \\ 2e^{i\varphi} \sin 2\theta & 3 - \cos 2\theta \end{array} \right). \tag{57}
\]

Eigenfunctions of this matrix are one-photon Schmidt modes, and they are easily found to be given by

\[
|\psi_i\rangle = \left( \begin{array}{c} \cos \alpha \\ e^{i\varphi} \sin \alpha \end{array} \right), \quad |\psi_e\rangle = \left( \begin{array}{c} -\sin \alpha \\ e^{i\varphi} \cos \alpha \end{array} \right). \tag{58}
\]

where \(\alpha\) is determined by equation (56). Though these equations are written in the matrix form, they can be rewritten...
equivalently in the functional form to be applicable directly for being used in the Schmidt decomposition of the TPPQ wave function (24):

\[
\psi_+(\sigma_l) = \cos \alpha \langle \sigma_l | \ell \rangle + e^{i\varphi} \sin \alpha \langle \sigma_l | v \rangle, \tag{59}
\]

\[
\psi_-(\sigma_l) = -\sin \alpha \langle \sigma_l | \ell \rangle + e^{i\varphi} \cos \alpha \langle \sigma_l | v \rangle
\]

with \(\langle \sigma_l | \ell \rangle = \delta_{\sigma_l, \ell} \) and \(\langle \sigma_l | v \rangle = \delta_{\sigma_l, v} \).

The Stokes vectors of the one-photon Schmidt modes are easily found from the definition (58) of their wave functions by means of constructing the corresponding density matrices of these states and identifying them with the polarization matrices. The results are given by

\[
S^x_{\ell} = \pm \cos 2\alpha, \quad S^y_{\ell} = \pm \sin 2\alpha \cos \varphi, \quad S^z_{\ell} = \pm \sin 2\alpha \sin \varphi. \tag{60}
\]

Comparison with the second expression of equation (55) shows that \(S^x_W = \text{const} \neq \tilde{S} \), i.e., the Stokes vectors of one-photon Schmidt modes are collinear to each other and to the polarization Stokes vector of the TPPQ as a whole. Besides, the vectors \(S^x_W\) and \(S^y_W\) are counter-directed. In contrast to \(\tilde{S}\), the Stokes vectors of one-photon Schmidt modes have a unit length, \(|S^x_W| = 1\), which indicates that one-photon states \(|\psi_{\sigma_l}\rangle\) are pure and maximally polarized. The angles between these vectors and the \((V-H)\) axis in the Poincaré sphere are equal to \(2\alpha\) and \(\pi + 2\alpha\) (correspondingly, for \(S^x_W\) and \(S^y_W\)). In accordance with the general equation (27), the sum of the Schmidt-mode Stokes vectors \(S^x_W\) and \(S^y_W\) with the weighting coefficients \(\lambda_+\) and \(\lambda_-\) gives the TPPQ Stokes vector \(\tilde{S}\):

\[
\lambda_+ S^x_W + \lambda_- S^y_W = (\lambda_+ - \lambda_-) \frac{\tilde{S}}{P} = \tilde{S}. \tag{61}
\]

As the projection of the Stokes vector \(\tilde{S}, \tilde{S}_W\) on the \((V-H)\) axis equals \(S^x_W = \cos^2 \alpha\), this last value determines the degree of linear polarization along the horizontal axis of the one-photon ‘+’ Schmidt mode of the TPPQ \(|\psi_{23}\rangle\). Moreover, if \(\varphi = 0\), all Stokes vectors are located in the horizontal plane of the Poincaré sphere, and the ‘+’ one-photon Schmidt mode is linearly polarized along the direction in this plane having the angle \(\alpha\) with respect to the horizontal axis. This clarifies the physical meaning of the angle \(\alpha\) determined by equation (56) and shown in figure 4.

8.4. Two-photon Schmidt modes of the state \(|\psi_{23}\rangle\)

As said earlier, two-photon Schmidt modes can be found as eigenfunction of the reduced matrix \(\rho_r\) (6), which is not written down here explicitly. Instead, we use equation (25), which permits to find the two-photon Schmidt modes immediately if the one-photon Schmidt modes are known. For the state \(|\psi_{23}\rangle\) the results are given by

\[
\chi_+ = \frac{1}{\sqrt{3\lambda_+}}(e^{-i\varphi} \sin \alpha \cos \theta \psi_{23} + \cos (\alpha - \theta) \sqrt{2} \psi_{0_r, 0_v} + e^{i\varphi} \cos \alpha \sin \theta \psi_{0_v, 0_r}) \tag{62}
\]

and

\[
\chi_- = \frac{1}{\sqrt{3\lambda_-}}(e^{-i\varphi} \sin \alpha \cos \theta \psi_{23} - \sin (\alpha - \theta) \sqrt{2} \psi_{0_r, 0_v} - e^{i\varphi} \cos \alpha \sin \theta \psi_{0_v, 0_r}). \tag{63}
\]

In these equations \(\psi_{0_r, 0_v}, \psi_{0_v, 0_r}\), are the wave functions of the BPQ basic states. They can be written either in the matrix form as direct products (or sums of products) of two two-line columns [25] or as functions of two polarization variables. In the last option \(\chi_+ = \langle \sigma_2, \sigma_3 | 2_H \psi_{0_r, 0_v} | \sigma_2, \sigma_3 | 2_H \psi_{0_v, 0_r} \rangle\). The two-photon Schmidt modes of equations (62) and (63) obey the normalization and orthogonality conditions \(\langle \chi_+ | \chi_+ \rangle = \langle \chi_- | \chi_- \rangle = 1\) and \(\langle \chi_+ | \chi_- \rangle = 0\). Interesting enough, in spite of orthogonality of the functions \(\chi_+\) and \(\chi_-\), as shown below, the Stokes vectors of these two-photon states can be noncollinear. This contrasts with orthogonal one-photon states (such as \(\psi_+\) and \(\psi_-\)), the Stokes vectors of which are collinear and counter-directed.

As \(\chi_+\) and \(\chi_-\) are the wave functions of pure states of BPQs, analysis of their polarization features does not require any new derivations and can be based on known results [23, 26, 27]. Indeed, for any BPQ of the form (3) the polarization Stokes vector is given by

\[
\widetilde{S} = \begin{pmatrix} \sqrt{2} \text{Re} (C_1 C_2^* + C_2 C_1^*) \\ -\sqrt{2} \text{Im} (C_1 C_2^* + C_2 C_1^*) \end{pmatrix}. \tag{64}
\]

By extracting the BPQ parameters \(C_1, C_2, 3\) from the definitions of \(\chi_+\) (62) and \(\chi_-\) (63) and substituting them into equation (64), we find expressions for the Stokes vectors (per one photon) of the two-photon Schmidt modes of the state \(|\psi_{23}\rangle\):

\[
\tilde{S}_x = \frac{1}{3\lambda_+} \begin{pmatrix} 2 \cos \varphi \cos (\alpha - \theta) \sin (\alpha + \theta) \\ 2 \sin \varphi \cos (\alpha - \theta) \sin (\alpha + \theta) \end{pmatrix} \tag{65}
\]

and

\[
\tilde{S}_x = \frac{1}{3\lambda_-} \begin{pmatrix} -2 \cos \varphi \sin (\alpha - \theta) \cos (\alpha + \theta) \\ -2 \sin \varphi \sin (\alpha - \theta) \cos (\alpha + \theta) \end{pmatrix}. \tag{66}
\]

These general expressions verify the above mentioned conclusion that as a rule the Stokes vectors of the two-photon Schmidt modes are not collinear to each other and are not collinear with the TPPQ Stokes vector (55). But nevertheless, the Stokes vectors \(\tilde{S}_W\) obey the general relation (27): their sum with weighting coefficients \(\lambda_+\) gives the TPPQ Stokes vector \(\tilde{S}\). This situation is shown in a qualitative picture of figure 7 for the state \(|\psi_{23}\rangle\) with \(\varphi = 0\).

8.5. Geometrical representation

A rather interesting and picturesque geometrical way for characterizing the degree of entanglement is related to the use of the barycentric or trilinear coordinates of points in triangles [34]. This method is applicable to special classes of TPPQ in which one of constants \(C_1, C_2, C_3, \) or \(C_4\) equals zero and
three remaining constants are real and positive. As shown in the inset of figure 8, for any point \( O \) inside a triangle \( ABC \) the sum of areas of smaller triangles \( AOB, BOC, \) and \( AOC \) does not depend of the position of the point \( O \) and equals the area of the triangle \( ABC \),

\[
S_{AOB} + S_{BOC} + S_{AOC} = S_{ABC}.
\]

Owing to this condition one can identify relative areas of smaller triangles with squared values of three nonzero TPPQ constants, e.g. as

\[
C_1^2 = \frac{S_{AOC}}{S_{ABC}}, \quad C_2^2 = \frac{S_{BOC}}{S_{ABC}}, \quad C_3^2 = \frac{S_{AOB}}{S_{ABC}}
\]

in the case \( C_4 = 0 \). If the triangle \( ABC \) is taken equilateral, the TPPQ constants \( C_i \) can be expressed in terms of distances \( h_{1,2,3} \) from the point \( O \) to the triangle sides \( AB, BC \) and \( AC \):

\[
C_i^2 = \frac{h_i}{\sum h_i}.
\]

Characterization of the point-\( O \) position by its distances from the triangle sides corresponds to the definition of trilinear or barycentric coordinates. For any given values of \( h_i \) we find constants \( C_i \) and, with the help of equations (12)–(14), a value of the reduced-state entropy \( S_r = S_rT \). Then we color from red to blue regions corresponding to higher or smaller values of \( S_r \) corresponding to higher or lower levels of entanglement. In addition to the picture of figure 8 and in a similar way we can construct three other pictures corresponding to cases \( C_1 = 0, C_2 = 0, \) or \( C_3 = 0 \). Combined together, all these four pictures form a tetrahedron, one side of which is just the triangle shown in figure 8.

9. Measurements

If parameters of TPPQ \( C_{1,2,3,4} \) are not known, an important problem is the possibility to measure them. In principle, this can be done in a way similar to that known for BPQ \[25\] and based on the use of a series of coincidence measurements. However, there is a rather important difference related to the number of photons in TPPQ and in BPQ. As TPPQ are states of three photons, the usual pair-coincidence measurements are insufficient and they have to be replaced by triple-coincidence measurements, when one registers only signals coming simultaneously to a computer from three single-photon channels. A scheme of such measurements is shown in figure 9. The scheme includes two nonselective beam splitters, a series of mirrors, three detectors and three polarizers. Each photon coming to a beam splitter has equal 50\% probabilities of transmission or reflection. As for triplets of photons coming to a beam splitter, some of them pass or are reflected unsplit, and some others are split for one photon and pair of photons moving in different directions. By counting all possible combinations, we find numbers of single photons \( N(s) \) in all three channels for a given number \( N \) of triplets coming to the first beamsplitter:

\[
N(s) = N/4\quad \text{after the first beamsplitter (in the upper channel in figure 9 and not shown in other channels indicates the numbers of single-photon states after beamsplitters).}
\]

\[
h_i \quad \text{of all triplets will be completely split between three channels and will give contributions to the triple-coincidence signals. Polarizers (P_1, P_2, P_3) can be installed differently to permit transmission of differently polarized photons. In the first series of measurements it is sufficient to use only the following four combinations of polarizer installations:}
\]

Figure 7. Horizontal plane of the Poincaré, the TPPQ Stokes vector (red) and Stokes vectors of the two-photon Schmidt modes for the states (50) with \( \phi = 0 \).

Figure 8. Entropy of the reduced states characterizing the degree of entanglement of TPPQ with \( C_4 = 0 \) and real and positive constants \( C_{1,2,3} \).

Figure 9. A scheme of triple coincidence measurements. \( BS_1, BS_2, \) beamsplitters; \( M, \) mirrors; \( P, \) polarizers; \( N \) is the number of photon triplets coming to the first beamsplitter during any given time \( T \); \( N(s) \) shown in the upper and not shown in other channels indicates the numbers of single-photon states after beamsplitters.
\[
\begin{pmatrix}
P_1 \\
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} \\
\end{pmatrix} \rightarrow 
\begin{pmatrix}
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} \\
\end{pmatrix}.
\] (67)

Note that there is a difference between the states \(|3\eta_i\rangle\) or \(|3\nu\rangle\) and \(|2\eta_i, 1\nu\rangle\) or \(|2\nu, 1\eta\rangle\). E.g. the first of these states can be registered with the installation of polarizers \(HHH\), and it does not matter which of three photons of completely split triplets goes to the upper or lower channels. Hence, all \(\frac{1}{16}\) of all 3η triplets will contribute to the coincidence counting. As for the state \(|2\eta_i, 1\nu\rangle\), it can be registered with the installation of polarizers \(HHV\), and only one third of all completely split triplets will give contributions to the coincidence counting. Specifically, in this case the contributing splittings of the triplets \(|2\eta_i, 1\nu\rangle\) are only those in which the vertically polarized photons go to the lower channel. The total probability of such events is \(\frac{1}{16}\), rather than \(\frac{1}{3}\). The same is true for the states \(|1\eta, 2\nu\rangle\). Finally, for TPPQ of a general form (1) the arising probabilities for triplets \(|3\eta_i\rangle\), \(|2\eta_i, 1\nu\rangle\), \(|2\nu, 1\eta\rangle\), and \(|3\nu\rangle\) are \(|C_1|^2\), \(|C_2|^2\), \(|C_3|^2\), \(|C_4|^2\).

With all these remarks taken into account, the expected numbers of coincidence clicks in the scheme of figure 9 are given by:

\[
N_{HHH} = \frac{3}{16} \eta_1 \eta_2 \eta_3 |N| |C_1|^2,
\]

\[
N_{HHV} = \frac{1}{16} \eta_1 \eta_2 \eta_3 |N| |C_2|^2,
\]

\[
N_{VHV} = \frac{1}{16} \eta_1 \eta_2 \eta_3 |N| |C_3|^2,
\]

\[
N_{VVV} = \frac{3}{16} \eta_1 \eta_2 \eta_3 |N| |C_4|^2,
\]

where \(\eta_1, \eta_2, \eta_3\) are efficiencies of detectors. As the constants \(C_{1,2,3,4}\) obey the normalization condition \(|C_1|^2 + |C_2|^2 + |C_3|^2 + |C_4|^2 = 1\), we can construct the sum of the results of measurement, which does not depend on the constants \(C_{1,2,3,4}\):

\[
\Sigma = N_{HHH} + 3N_{HHV} + 3N_{VHV} + N_{VVV} = \frac{3 \eta_1 \eta_2 \eta_3 N}{16},
\] (68)

which gives

\[
|C_1|^2 = \frac{N_{HHH}}{\Sigma},
\]

\[
|C_2|^2 = \frac{N_{HHV}}{\Sigma},
\]

\[
|C_3|^2 = \frac{N_{VHV}}{\Sigma},
\]

\[
|C_4|^2 = \frac{N_{VVV}}{\Sigma}.
\] (70)

Both the numerators and denominator of fractions in equations (70) are determined only by the results of measurements and they do not depend of either efficiencies of detectors \(\eta_{1,2,3}\) or the total number of triplets in the beam \(N\).

The next step is measuring phases \(\psi_{1,2,3}\) of the constants \(C_{1,2,3,4}\). Actually, as one of this phases can be taken equal zero, there are only three phases to be measured, e.g. \(\phi_2, \phi_3,\) and \(\phi_4\), with \(\phi_1 = 0\) if \(|C_1| \neq 0\). These phases can be found from a series of equations arising when the same measurements as described above are repeated with the polarizers turned for angles 45° and 135° (correspondingly, instead of the horizontal and vertical directions). The arising equation are identical to (68)–(70) with substitutions

\[
N_{HHH} \rightarrow N_{3\eta_1},
\]

\[
N_{HHV} \rightarrow N_{2\eta_2,1\eta_3},
\]

\[
N_{VHV} \rightarrow N_{2\eta_1,1\eta_3},
\]

\[
N_{VVV} \rightarrow N_{3\eta_3},
\]

and \(C_i \rightarrow C_i^{45}\), where \(i = 1, 2, 3, 4\) and \(C_i^{45}\) are the same TPPQ parameters as in equation (1) but in a basis turned for 45°:

\[
C_i^{45} = \frac{1}{2\sqrt{2}} \{C_i + C_i + \sqrt{3} (C_2 + C_3)\},
\]

\[
C_i^{45} = \frac{1}{2\sqrt{2}} \{\sqrt{3} (-C_i + C_i) C_i + (-C_2 + C_3)\},
\]

\[
C_i^{45} = \frac{1}{2\sqrt{2}} \{\sqrt{3} (C_i + C_i) C_i - (C_2 + C_3)\},
\]

\[
C_i^{45} = \frac{1}{2\sqrt{2}} \{(-C_i + C_i) + \sqrt{3} (C_2 - C_3)\}.
\] (71)

The squared absolute values of these constants can be expressed via the measured numbers of triple-coincidence counts in the turned basis. These equalities will contain unknown phases in the form of superpositions of \(\cos \phi_i\) and \(\cos(\phi_i - \phi_j)\) with \(i, j = 2, 3, 4\), which can be solved numerically. To be specific, let us show only one (first) of these equations

\[
N_{3\eta_1} = |C_i^{45}|^2 = \frac{1}{8} \left\{ |C_i|^2 + 3|C_2|^2 + 3|C_3|^2 + |C_4|^2 + 2|C_i||C_2| \cos \phi_2 + 2\sqrt{3}|C_i||C_3| \cos \phi_3 + 2\sqrt{3}|C_i||C_4| \cos \phi_4 + 6|C_i||C_3| \cos (\phi_2 - \phi_3) \right\}.
\] (72)

All other equations are similar and they differ from this one by coefficients and signs in front of terms in equation (72). Only three of four equations of the type (turned basis) are independent from each other, and they may be sufficient for finding three unknown phases, \(\phi_2, \phi_3, \phi_4\). Though, some uncertainty in determining phases can remain because given values of cosines define two possible values of their arguments. For choosing one of these two solutions, additional, measurements can be needed, with polarizers turned for some angle \(\alpha \neq 0, 45°\). For BPQ similar sets of measurements providing determination of phases were discussed in [35].

A much simpler scheme of measurements can be used for finding parameters of TPPQ which can be reduced to the form (33). Both this form and a scheme of its measurement (figure 10) indicate a deep analogy between the TPPQ of the type (33) and BPQ [25, 26].

At first, the expression for the TPPQ state vector (33) can be further simplified with the help of properly installed \(\lambda/4\)- and \(\lambda/2\)-plates on a way of a three-photon beam to transform the Schmidt mode representation (33) this gives

\[
|\Psi\rangle - |\bar{\Psi}\rangle = \sqrt{\lambda} |3\eta_i\rangle + e^{i\phi} \sqrt{\lambda} |3\nu\rangle.
\] (73)

An experimental criterion that the \(\lambda/4\)- and \(\lambda/2\)-plates are installed correctly is the zero coincidence signal between two channels immediately after PBS (any wrong installations do
not provide the described transformation and give rise to a non-zero coincidence signal. After the transformation (73), as $\lambda_+ + \lambda_- = 1$, the state $|\bar{\Psi}\rangle$ is characterized by two parameters only, e.g. $\lambda_+$ and the phase $\phi$. In accordance with what was proposed for BPQ [26], for measuring $\lambda_+$ and $\phi$, one can send the beam to the polarizing beamsplitter and then to the triple-coincidence scheme in one of the channels after PBS. The triple-coincidence scheme can be simplified compared to that shown in figure 9 because now polarizers are not needed. The number of coincidence counts in the transmission channel of PBS installed in a standard way equals

$$N_{00} = \frac{3}{16} \eta \eta_2 \eta_3 N \lambda_+,$$  

(74)

where $N$ is the number of photon triplets coming to PBS per a given time $\Delta t$. If in the second series of measurements we turn PBS for 90°, the transmission channel will becomes open only for vertically polarized photons, and the number of coincidence counts equals

$$N_{00} = \frac{3}{16} \eta \eta_2 \eta_3 N \lambda_+.$$  

(75)

The sum of these two results is

$$\Sigma = N_+ + N_- = \frac{3}{16} \eta \eta_2 \eta_3 N.$$  

(76)

This sum does not depend of $\lambda_+$ and can be used for normalization to give expressions for $\lambda_+$ and $\lambda_-$ in terms of experimentally measurable numbers of coincidence counts

$$\lambda_+ = \frac{N_+}{\Sigma} = \frac{N_+}{N_+ + N_-}, \quad \lambda_- = \frac{N_-}{\Sigma} = \frac{N_-}{N_+ + N_-}.$$  

(77)

For measuring the phase $\phi$ PBS has to be turned for 45°. Then the transmission channel is open only for 45°-polarized photons, and the state describing such photons is

$$|\Psi\rangle_{trans} = \frac{1}{\sqrt{2}} \left( \sqrt{\lambda_+} + e^{3i\phi} \sqrt{\lambda_-} \right) |3_{45}\rangle.$$  

(78)

The corresponding number of the triple-coincidence counts is

$$N_{45^\circ} = \frac{3}{8 \times 16} \eta \eta_2 \eta_3 (1 + C_\phi \cos 3\phi),$$  

(79)

where $C_\phi = 2 \sqrt{\lambda_+ \lambda_-}$ is the generalized concurrence. The equation expressing the phase $\phi$ in terms of the experimentally measurable numbers of counts is given by

$$\frac{1}{8} \left( 1 + C_\phi \cos 3\phi \right) = \frac{N_{45^\circ}}{\Sigma} = \frac{N_{45^\circ}}{N_0 + N_{90^\circ}}.$$  

(80)

10. Conclusions

The states of TPPQ considered in this work are states of three photons with either coinciding or different polarizations but with identical given frequencies and propagation directions. Such states can be considered as consisting of two-photon and one-photon parts, and these parts can be entangled with each other or not. The two-photon and one-photon reduced density matrices of TPPQ $\rho_r$ and $\rho_{rr}$, their eigenvalues $\lambda_\pm$ (12), of the reduced density matrices $S_r = S_{rr}$ (14), the generalized concurrence $C_\phi$ (13), the Schmidt parameter $K$ (15) and the TPPQ Stokes vector $\vec{\tilde{S}}$ are found in a general form. The parameters $S_r = S_{rr}$, $C_\phi$ and $K$ are shown to determine the degree of entanglement of TPPQ and the degree of its polarization $P = |\vec{\tilde{S}}| = \sqrt{1 - C_\phi^2}$ (18) and (19). All these parameters, and their features are very similar to those occurring in the case of BPQ [25, 26], except the expression for $C_\phi$ in terms of the TPPQ constants $C_\phi$ (13) is much more complicated than the analogous expression for concurrence of BPQ [25]. The Schmidt decomposition for the TPPQ wave functions (24) and reduced density matrices (23) are also found in a general form. But in contrast to BPQ, the wave functions of the adjacent one-photon and two-photon Schmidt modes are not (and cannot be) identical to each other. The most important difference between TPPQ and BPQ concerns the Schmidt decomposition of the corresponding state vectors. Such decomposition always occurs for BPQ (30), but it occurs in the case of TPPQ (33) only under a series of special conditions consisting in the requirement of factorization of both two-photon Schmidt modes $\chi_+$ and $\chi_-$ for the products of the corresponding one-photon Schmidt modes (31). In terms of original parameters of TPPQ $C_{\phi}$, the conditions are given by equations (39) and (42). Actually these condition separate TPPQs for two big groups with different features. A three-parametric class of TPPQ obeying the conditions (39) and (42) are similar to BPQ with the only difference: instead of pairs of photons in the ‘+’ and ‘−’ Schmidt modes one has now triplets of photons. All known manipulations with BPQ remain possible for TPPQ obeying the conditions (39) and (42). Some schemes of such possible experiments are described and discussed. In particular, they include the transformation of the TPPQ state vector to the simplest form (73) characterized by two constants only, $\lambda_+$ and $\phi$. The methods of their measurements are described. Moreover, the phase $\phi$ can be easily modified and, in this way, used for encoding and transmission of information, this possibility illustrates operational features of TPPQ related to their entanglement.

As for TPPQ not obeying the conditions (39) and (42), their features are different. Such states can have some degree of entanglement and polarization. Moreover, polarization features of such states are rather interesting. This is illustrated above in section 8 on the example of the state $|\Psi_{32}\rangle$ (45). One specific example concerns two-photon Schmidt modes of this
state $\chi_+ (62)$ $\chi_- (63)$, which are orthogonal to each other but are characterized by the non-collinear Stokes vectors $\tilde{S}_+^\chi$ and $\tilde{S}_-^\chi$ (equations (65) and (66) and figure 7). But in the case of states not obeying the conditions (39) and (42), the Schmidt decomposition of their wave functions cannot be used to derive any kind of Schmidt decomposition for the state vectors. As the state vectors of such states cannot be presented in the form (33), separation of their $'+ '$ and $' − '$ Schmidt modes is impossible, and for such states all discussed suggestions of experiments are inapplicable. Probably more refined schemes of experiments have to be found for revealing their entanglement and peculiar polarization features, to which we hope to return elsewhere.

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