Isomorphism between two realizations of the Yangian \( \mathcal{Y}(\mathfrak{so}_3) \)

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Abstract
The isomorphism between Drinfeld’s new realization and the FRT realization is proved for the Yangian algebra \( \mathcal{Y}(\mathfrak{so}_3) \) using Gauss decomposition.

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1. Introduction

When the concept of quantum groups was introduced in 1985 two main classes of examples were Drinfeld–Jimbo quantum enveloping algebras of symmetrizable Kac–Moody Lie algebras \([4, 6, 11]\) and Yangian algebra \([6, 8]\), associated with complex simple Lie algebra. These quantum algebras correspond respectively to the trigonometric and rational solutions of the quantum Yang–Baxter equation. They have been used extensively in mathematics and theoretical physics, in particular in the study of solvable statistical models and quantum field theory. The theory of Yangian algebras has ample applications in statistical mechanics, representation theory and algebraic combinatorics, for example many interesting combinatorial properties of the general linear groups have been generalized to the Yangian algebra of type \( A \). For a beautiful comprehensive introduction, see Molev’s monograph \([12]\) (also \([13, 14]\)).

There are three important realizations of the Yangians. The first was given by Drinfeld using generators and relations similar to Serre relations in his fundamental paper \([6]\) in 1985. Drinfeld pointed out that Yangian algebra is the unique homogeneous quantization of the half-loop algebra \( \mathfrak{g}[[u]] \) associated with a simple Lie algebra \( \mathfrak{g} \). Further along this development, Drinfeld \([7]\) gave the second realization called Drinfeld realization for both quantum affine algebras and Yangians, which can be viewed as the quantum analogue of the loop realization of the affine Lie algebras. Using this new realization of Yangians, Drinfeld developed and classified finite-dimensional irreducible representations of Yangians. The third realization is...
the generalization of the Faddeev–Reshetikhin–Takhtajan realization [9], which has a longer history starting from the quantum inverse scattering method [16]. Under this realization, the comultiplication formulas have a particularly simple form for both quantum affine algebras and Yangians.

Drinfeld stated that the FRT realization and Drinfeld’s new realization of Yangian and quantum affine algebras are isomorphic. In 1994, Ding and Frenkel [5] proved the isomorphism between the FRT realization and Drinfeld’s new realization for quantum affine algebras of type $A$. In 1996, Iohara [10] introduced the central extension of the Yangian double using the RTT realization and gave the bosonic representations using its Drinfeld generators. Later, in 2005 Brundan and Kleshchev [3] proved the isomorphism between the two realizations of Yangians in type $A$. For BCD-type Yangians, Arnaudon et al [2] studied the FRT realization of the Yangian and classified finite-dimensional representations and gave the PBW theorem for the Yangian. In [1], Arnaudon et al proved that the FRT realization of BCD-type Yangians is indeed a homogeneous quantization of $g[u]$ corresponding to its canonical bialgebra structure. Still less is known for the relations between Drinfeld’s new realization and the FRT realization of BCD-type Yangians. Several known constructions or isomorphisms of Lie algebras are usually exclusively for Lie algebras. For example, the Yangians associated with the orthogonal and symplectic Lie algebras are not known to be subalgebras of the Yangian algebra associated with the general linear algebras. In fact this is why Olshanskii’s twisted Yangians are introduced by extending the imbedding [15].

In the lower rank cases, Arnaudon et al proved the isomorphism between the FRT realization of $Y(so_3)$ and $Y(sl_2)$ in [2] using the fusion procedure for $R$-matrix. In this paper, we will prove the isomorphism between the two realizations of $Y(so_3)$ using the Gauss decomposition, similar to Ding–Frenkel’s method [5]. It would be interesting to generalize this construction to higher rank cases.

The paper is organized as follows. In the second section, we give a brief introduction of the FRT or RTT realization of $Y(so_3)$ and $Y(sl_2)$ in [2] using the fusion procedure for $R$-matrix. In this paper, we will prove the isomorphism between the two realizations of $Y(so_3)$ using the Gauss decomposition, similar to Ding–Frenkel’s method [5]. It would be interesting to generalize this construction to higher rank cases.

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### 2. RTT realization of Yangian algebras corresponding to orthogonal Lie algebras

In [2], the RTT presentation of Yangian corresponding to orthogonal Lie algebras was studied. Here we briefly recall the RTT realization of the Yangian with respect to orthogonal Lie algebras, especially the case of $so_3$. Before introducing the RTT presentation, we list the following preliminaries and notations.

For $N = 2n$ or $2n+1$, we enumerate the rows and columns of $N \times N$ matrices by the indices $-n, \ldots, -1, 1, \ldots, n$ and $-n, \ldots, -1, 0, 1, \ldots, n$, respectively. Let $t: \text{End} \mathbb{C}^N \to \text{End} \mathbb{C}^N$ be the transposition given by $(e_{ij})^t = e_{-j,-i}$.

Set $\kappa = N/2 - 1$, and consider the $R$-matrix

$$R(u) = 1 - \frac{P}{u} + \frac{Q}{u - \kappa},$$

where $P = \sum_{i,j=-n}^{n} e_{ij} \otimes e_{ji}$, and $Q = P^t = \sum_{i,j=-n}^{n} e_{ij} \otimes e_{-i,-j}$. It is easy to check that $R(u)$ is a rational solution of Yang–Baxter equation:

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v).$$

We define an algebra $Y_{so_3}$ by using the $R$-matrix $R(u)$ as follows.
Definition 2.1. The algebra \( Y_\mathfrak{g} (\mathfrak{so}_3) \) is a unital associative algebra generated by \( t_{ij}^{(r)} \), \( r \in \mathbb{Z}_+ \), \( i, j \in \{-n, -n + 1, \ldots, n - 1, n\} \), the defining relations can be written as the following RTT form by the matrix of generators \( T(u) \):

\[
R(u - v)T_i(u)T_j(v) = T_j(v)T_i(u)R(u - v),
\]

\[
T(u)T^\dagger (u + \kappa) = T^\dagger (u + \kappa)T(u) = 1,
\]

where \( T(u) = \sum_{i,j=-n}^n t_{ij}(u) \otimes e_{ij}, t_{ij}(u) = \sum_{r=0}^\infty t_{ij}^{(r)} u^{-r}, t_{ij}^{(0)} = \delta_{ij}. \)

In particular, for the \( \mathfrak{so}_3 \) case the algebra \( Y_\mathfrak{g} (\mathfrak{so}_3) \) is generated by \( t_{ij}^{(r)}, i, j \in \{-1, 0, 1\} \) and subject to the RTT relations.

Proposition 2.2. For the case \( Y(\mathfrak{so}_3) \), the RTT generating relation (2.2) can be written equivalently in terms of generating series as following:

\[
[t_{ij}(u), t_{kl}(v)] = \frac{1}{u - v} (t_{ij}(u)t_{kl}(v) - t_{kl}(v)t_{ij}(u))
\]

\[
- \frac{1}{u - v} \left( \delta_{k,i} \sum_{p=1}^{\infty} t_{pj}(u)t_{pk}(v) - \delta_{l,j} \sum_{p=1}^{\infty} t_{kl}(v)t_{lp}(u) \right).
\]

The RTT defining relation (2.2) can be rewritten equivalently as follows:

\[
T^{-1}(v)R(u - v)T_1(u) = T_1(u)R(u - v)T^{-1}(v).
\]

Denote by \( t_{ij}^{(r)}(u) \) the \( ij \)th element of \( T^{-1}(u) \).

Proposition 2.3. The defining relation (2.2) is equivalent to

\[
[t_{pq}(u), t_{rs}^{(r)}(v)] = \frac{1}{u - v - \frac{1}{2}} (t_{rs}^{(r)}(v)t_{pq}(u) - t_{pq}(u)t_{rs}^{(r)}(v))
\]

\[
+ \frac{1}{u - v} \left( \delta_{qr} \sum_{i=1}^{\infty} t_{ip}(u)t_{iq}^{(r)}(v) - \delta_{ps} \sum_{i=1}^{\infty} t_{ir}^{(r)}(v)t_{ip}(u) \right).
\]

3. Gauss decomposition of \( Y_\mathfrak{g} (\mathfrak{so}_3) \)

In this section, we study the Gauss decomposition of \( T(u) \) and the commutation relations between the ‘Gauss generators’.

Theorem 3.1. In \( Y_\mathfrak{g} (\mathfrak{so}_3) \) the matrix \( T(u) \) has the following unique decomposition:

\[
T(u) = \begin{pmatrix}
1 & 0 & 0 & k_{-1}(u) & 1 & e_{-1,0}(u) & e_{-1,1}(u) \\
0 & 1 & 0 & k_0(u) & 0 & 1 & e_{0,1}(u) \\
0 & 0 & 1 & k_1(u) & 0 & 0 & 0 \\
\end{pmatrix}.
\]

where the entries are defined by the matrix decomposition and for \( i < j \)

\[
e_{ij}(u) = \sum_{r=1}^{\infty} e_{ij}^{(r)} u^{-r} \in Y_\mathfrak{g} (\mathfrak{so}_3)[[u^{-1}]].
\]

\[
f_{ij}(u) = \sum_{r=1}^{\infty} f_{ij}^{(r)} u^{-r} \in Y_\mathfrak{g} (\mathfrak{so}_3)[[u^{-1}]].
\]

\[
k_i(u) = 1 + \sum_{r=1}^{\infty} k_i^{(r)} u^{-r} \in Y_\mathfrak{g} (\mathfrak{so}_3)[[u^{-1}]].
\]

The elements \( e_{ij}(u), f_{ij}(u), k_i(u) \) are called the Gauss generators of \( Y_\mathfrak{g} (\mathfrak{so}_3) \).
Proof. The decomposition is obtained formally by the matrix factorization. The matrix entries in the decomposition can be computed as in the usual matrix algebra, which also provides the definition for each of the Gauss generators iteratively. Since $k_{-1}(u)$ is invertible, it is easy to prove that the Gauss decomposition is unique.

The commutation relations between the Gauss generators are directly computed from the matrix equation.

It follows from theorem 3.1 that

$$T(u) = \begin{pmatrix} k_{-1}(u) & k_{-1}(u)e_{-1,0}(u) & k_{-1}(u)e_{-1,1}(u) \\ f_{0,-1}(u)k_{-1}(u) & k_0(u) + f_{0,-1}(u)k_{-1}(u)e_{-1,0}(u) & * \\ f_{1,-1}(u)k_{-1}(u) & * & * \end{pmatrix},$$

(3.2)

since $k_i(u)$ are invertible, we also obtain that

$$T^{-1}(u) = \begin{pmatrix} * & k_0^{-1}(u) + e_{01}(u)k_1^{-1}(u)f_{10}(u) & * \\ * & -k_1^{-1}(u)f_{10}(u) & * \\ * & k_1^{-1}(u) & * \end{pmatrix}.$$

(3.3)

Moreover, from the defining relation (2.1), we have $T^t\left(u + \frac{1}{2}\right) = T^{-1}(u)$ and

$$T^t\left(u + \frac{1}{2}\right) = \begin{pmatrix} * & * & * \\ * & \Delta & k_{-1}\left(u + \frac{1}{2}\right)e_{-1,0}\left(u + \frac{1}{2}\right) \\ * & k_{-1}\left(u + \frac{1}{2}\right) & * \end{pmatrix},$$

(3.4)

where $\Delta = k_0\left(u + \frac{1}{2}\right) + f_{0,-1}\left(u + \frac{1}{2}\right)k_{-1}\left(u + \frac{1}{2}\right)e_{-1,0}\left(u + \frac{1}{2}\right)$.

Comparing the matrices (3.3) and (3.4), we obtain the following equations in $Y_0(su_3)$:

$$k_1^{-1}(u) = k_{-1}\left(u + \frac{1}{2}\right)$$

(3.5)

$$-e_{01}(u)k_1^{-1}(u) = k_{-1}\left(u + \frac{1}{2}\right)e_{-1,0}\left(u + \frac{1}{2}\right)$$

(3.6)

$$-k_1^{-1}(u)f_{10}(u) = f_{0,-1}\left(u + \frac{1}{2}\right)k_{-1}\left(u + \frac{1}{2}\right)$$

(3.7)

$$k_0^{-1}(u) + e_{01}(u)k_1^{-1}(u)f_{10}(u) = k_0\left(u + \frac{1}{2}\right) + f_{0,-1}\left(u + \frac{1}{2}\right)k_{-1}\left(u + \frac{1}{2}\right)e_{-1,0}\left(u + \frac{1}{2}\right).$$

(3.8)

Let us return back to the defining relations (2.4) to obtain the commutation relations among $k_{-1}(u), k_0(u)$ and $e_{-1,0}(u), f_{0,-1}(u)$.

Proposition 3.2. In $Y_0(su_3)$, we have

$$[k_{-1}(u), k_{-1}(v)] = 0, [k_{-1}(u), k_0(v)] = 0,$$

(3.9)

$$[k_{-1}(u), e_{-1,0}(v)] = \frac{k_{-1}(u)(e_{-1,0}(v) - e_{-1,0}(u))}{u - v},$$

(3.10)

$$[k_{-1}(u), f_{0,-1}(v)] = \frac{(f_{0,-1}(u) - f_{0,-1}(v))k_{-1}(u)}{u - v}.$$

(3.11)

Proof. We just show (3.10) as the other relations are proved in the same way. It follows from equation (2.4) that

$$[t_{-1,-1}(u), t_{-1,0}(v)] = \frac{1}{u - v}(t_{-1,-1}(u)t_{-1,0}(v) - t_{-1,-1}(v)t_{-1,0}(u)),$$

i.e. $[k_{-1}(u), k_{-1}(v)e_{-1,0}(v)] = \frac{1}{u - v}(k_{-1}(u)k_{-1}(v)e_{-1,0}(v) - k_{-1}(v)k_{-1}(u)e_{-1,0}(u))$, then formula (3.10) follows immediately. We obtain equation (3.11) similarly. □

In the following proposition, we will give the relations between $e_{-1,0}$ and $f_{0,-1}$. 

4
Proposition 3.3.

\[ [e_{-1,0}(u), f_{0_{-1}}(v)] = \frac{k_{-1}^{-1}(u)k_0(u) - k_{-1}^{-1}(v)k_0(v)}{u - v}. \] (3.12)

Proof. Since \( t_{-1,0}(u) = k_{-1}(u)e_{-1,0}(u) \) and \( t_{-1,1}(u) = k_{-1}(v) \), relation (2.4) implies that
\[
k_{-1}(u)e_{-1,0}(u)k_{-1}(v) - k_{-1}(v)k_{-1}(u)e_{-1,0}(u)
= \frac{1}{u - v} (k_{-1}(u)e_{-1,0}(u)k_{-1}(v) - k_{-1}(v)e_{-1,0}(u)k_{-1}(u)).
\]

Thus, we have
\[
- f_{0_{-1}}(v)k_{-1}(v)k_{-1}(u)e_{-1,0}(u) = - \frac{u - v - 1}{u - v} f_{0_{-1}}(v)k_{-1}(u) - \frac{1}{u - v} f_{0_{-1}}(v)e_{-1,0}(u)k_{-1}(u).
\]

Using the equation (3.11), we obtain
\[
f_{0_{-1}}(v)k_{-1}(u) = \frac{u - v}{u - v - 1} k_{-1}(u) f_{0_{-1}}(v) - \frac{1}{u - v} f_{0_{-1}}(v)e_{-1,0}(u)k_{-1}(u).
\]

Then from the equation (3.13), we have
\[
- f_{0_{-1}}(v)k_{-1}(v)k_{-1}(u)e_{-1,0}(u) = - k_{-1}(u) f_{0_{-1}}(v)e_{-1,0}(u)k_{-1}(v)
+ \frac{1}{u - v} f_{0_{-1}}(u)k_{-1}(u)e_{-1,0}(u)k_{-1}(v) - \frac{1}{u - v} f_{0_{-1}}(v)e_{-1,0}(u)k_{-1}(u).
\]

Similarly, using \( t_{0_{-1}}(u) = k_{-1}(u)e_{-1,0}(u) \) and \( t_{-1,0}(u) = f_{0_{-1}}(u)k_{-1}(u) \) we obtain the following from equation (2.4):
\[
k_{-1}(u)e_{-1,0}(u)f_{0_{-1}}(v)k_{-1}(v) - f_{0_{-1}}(v)k_{-1}(v)k_{-1}(u)e_{-1,0}(u)
= \frac{1}{u - v} (k_0(u)k_{-1}(v) - k_0(v)k_{-1}(u)) + \frac{1}{u - v} f_{0_{-1}}(u)k_{-1}(u)e_{-1,0}(u)k_{-1}(v)
- \frac{1}{u - v} f_{0_{-1}}(v)e_{-1,0}(u)k_{-1}(u).
\]

Plugging equation (3.14) into (3.15), we have
\[
k_{-1}(u)e_{-1,0}(u)f_{0_{-1}}(v)k_{-1}(v) - k_{-1}(u)f_{0_{-1}}(v)e_{-1,0}(u)k_{-1}(v)
= \frac{1}{u - v} (k_0(u)k_{-1}(v) - k_0(v)k_{-1}(u)).
\]

Since the \( k_i(u) \) are invertible, we obtain equation (3.12). \( \square \)

The following relations between \( e_{-1,0}(u) \) and \( e_{01}(u) \) \( (f_{0_{-1}}(u) \) and \( f_{10}(u)) \) will be useful.

Proposition 3.4. In the algebra \( Y_R(sso_3) \)
\[
e_{01}(u) = - e_{-1,0} \left( u - \frac{1}{2} \right) \] (3.17)
\[
f_{10}(u) = - f_{0_{-1}} \left( u - \frac{1}{2} \right). \] (3.18)
Proof. From equation (3.10), we easily obtain
\[ k_{-1}(u + \frac{1}{2})e_{-1,0}(u + \frac{1}{2}) = e_{-1,0}(u - \frac{1}{2})k_{-1}(u + \frac{1}{2}). \] (3.19)
Taking into account (3.5) and (3.6), we have \( e_{01}(u) = -e_{-1,0}(u - \frac{1}{2}) \). Equation (3.18) is proved similarly.

Now we can give the relations between \( k_0(u) \) and \( k_{-1}(u) \).

**Proposition 3.5.** In \( Y_{\mathfrak{p}(\mathfrak{so}_3)} \), we have
\[ k_0(u) = k_{-1}(u)k_{-1}^{-1}(u + \frac{1}{2}). \] (3.20)

**Proof.** If follows from equation (3.8) that
\[ k_0(u + \frac{1}{2}) - k_0^{-1}(u) = e_{01}(u)k_{-1}(u)f_{01}(u) - f_{0,-1}(u + \frac{1}{2})k_{-1}(u + \frac{1}{2})e_{-1,0}(u + \frac{1}{2}). \]
Moreover, using equations (3.7), (3.17) and (3.19), we can obtain
\[ k_0(u + \frac{1}{2}) - k_0^{-1}(u) = [e_{-1,0}(u - \frac{1}{2}), f_{0,-1}(u + \frac{1}{2})]k_{-1}(u + \frac{1}{2}). \]
Proposition 3.3 implies that
\[ [e_{-1,0}(u - \frac{1}{2}), f_{0,-1}(u + \frac{1}{2})] = -\left(k_{-1}^{-1}(u - \frac{1}{2})k_0(u - \frac{1}{2}) - k_{-1}^{-1}(u + \frac{1}{2})k_0(u + \frac{1}{2})\right), \]
thus,
\[ k_0^{-1}(u) = k_{-1}^{-1}(u - \frac{1}{2})k_0(u - \frac{1}{2})k_{-1}(u + \frac{1}{2}), \]
which is equivalent to
\[ k_0(u - \frac{1}{2})k_0(u) = k_{-1}(u - \frac{1}{2})k_{-1}(u)k_{-1}(u)k_{-1}^{-1}(u + \frac{1}{2}). \]
Since \( k_i(u) \) are formal series, we obtain
\[ k_0(u) = k_{-1}(u)k_{-1}^{-1}(u + \frac{1}{2}). \]

**Proposition 3.6.** In \( Y_{\mathfrak{p}(\mathfrak{so}_3)} \), we have
\[ [k_{-1}^{-1}(u)k_0(u), e_{-1,0}(v)] = -\left(\frac{1}{2} \frac{1}{u - v}k_{-1}^{-1}(u)k_0(u), e_{-1,0}(u) - e_{-1,0}(v)\right) \] (3.21)
\[ [k_{-1}^{-1}(u)k_0(u), f_{0,-1}(v)] = -\left(\frac{1}{2} \frac{1}{u - v}k_{-1}^{-1}(u)k_0(u), f_{0,-1}(u) - f_{0,-1}(v)\right). \] (3.22)

**Proof.** Since two formulas are treated similarly, we only prove the formula (3.21). From proposition 3.5, we obtain \( k_{-1}^{-1}(u)k_0(u) = k_{-1}^{-1}(u + \frac{1}{2}). \) Therefore,
\[ [k_{-1}^{-1}(u)k_0(u), e_{-1,0}(v)] = [k_{-1}^{-1}(u + \frac{1}{2}), e_{-1,0}(v)]. \]
Furthermore, from equation (3.10), we obtain
\[ [k_{-1}^{-1}(u + \frac{1}{2}), e_{-1,0}(v)] = \frac{1}{u - v + \frac{1}{2}}(e_{-1,0}(u + \frac{1}{2}) - e_{-1,0}(v))k_{-1}^{-1}(u + \frac{1}{2}). \]
Now the proposition will follow if the following is true:
\[
\frac{1}{u - v + \frac{1}{2}} \left(e_{-1,0}(u + \frac{1}{2}) - e_{-1,0}(v)\right)k_{-1}^{-1}\left(u + \frac{1}{2}\right) = \frac{1}{2} \frac{1}{u - v} \left[k_{-1}^{-1}\left(u + \frac{1}{2}\right), e_{-1,0}(u) - e_{-1,0}(v)\right].
\]
In fact, using equation (3.10) one has
\[ k^{-1}_{-1} \left( u + \frac{1}{2} \right) e_{-1,0}(u) = 2e_{-1,0} \left( u + \frac{1}{2} \right) k^{-1}_{-1} \left( u + \frac{1}{2} \right) - e_{-1,0}(u) k^{-1}_{-1} \left( u + \frac{1}{2} \right). \]

\[ k^{-1}_{-1} \left( u + \frac{1}{2} \right) e_{-1,0}(v) = \frac{1}{u-v+\frac{1}{2}} e_{-1,0} \left( u + \frac{1}{2} \right) k^{-1}_{-1} \left( u + \frac{1}{2} \right) + \frac{u-v-\frac{1}{2}}{u-v+\frac{1}{2}} e_{-1,0}(v) k^{-1}_{-1} \left( u + \frac{1}{2} \right). \]

Hence, we can obtain the difference of the above two equations
\[ k^{-1}_{-1} \left( u + \frac{1}{2} \right) (e_{-1,0}(u) - e_{-1,0}(v)) = \frac{2(u-v)}{u-v+\frac{1}{2}} e_{-1,0} \left( u + \frac{1}{2} \right) k^{-1}_{-1} \left( u + \frac{1}{2} \right) - e_{-1,0}(u) k^{-1}_{-1} \left( u + \frac{1}{2} \right). \]

Thus, we obtain
\[ k^{-1}_{-1} \left( u + \frac{1}{2} \right) \cdot e_{-1,0}(u) - e_{-1,0}(v) = \frac{2(u-v)}{u-v+\frac{1}{2}} \left( e_{-1,0} \left( u + \frac{1}{2} \right) - e_{-1,0}(v) \right) k^{-1}_{-1} \left( u + \frac{1}{2} \right), \]

which is just what we need. \( \square \)

**Proposition 3.7.** In \( Y_\delta(so_3) \), we have
\[ [e_{-1,0}(u), e_{-1,0}(v)] = \frac{1}{2} \frac{(e_{-1,0}(u) - e_{-1,0}(v))^2}{u-v}. \] (3.23)

\[ [f_{-1,-1}(u), f_{0,-1}(v)] = -\frac{1}{2} \frac{(f_{-1,-1}(u) - f_{0,-1}(v))^2}{u-v}. \] (3.24)

Before proving proposition 3.7, we first derive the following useful result.

**Lemma 3.8.** In \( Y_\delta(so_3) \) one has
\[ 3e_{-1,1} \left( u + \frac{1}{2} \right) - e_{-1,1}(u) + 3e_{-1,0} \left( u + \frac{1}{2} \right) e_{-1,0}(u) - 2e^{-1}_{-1,0}(u) = 0. \]

**Proof.** First we recall the usual generators from equation (2.4)
\[ [t_{-1,-1}(u), t_{-1,-1}(u + \frac{1}{2})] = -2t_{-1,-1}(u) t_{-1,-1}(u + \frac{1}{2}) + 2t_{-1,-1}(u + \frac{1}{2}) t_{-1,-1}(u) - t_{-1,1} \left( u + \frac{1}{2} \right) t_{-1,0}(u) - t_{-1,-1}(u + \frac{1}{2}) t_{-1,1}(u), \]

which is actually equivalent to the following equation:
\[ 3t_{-1,-1} \left( u + \frac{1}{2} \right) t_{-1,0}(u) - t_{-1,-1}(u + \frac{1}{2}) t_{-1,1}(u) + t_{-1,0}(u + \frac{1}{2}) t_{-1,1}(u) = 0. \]

Rewriting the above equation in terms of Gauss generators we obtain
\[ 3k_{-1}(u) k_{-1} \left( u + \frac{1}{2} \right) e_{-1,1} \left( u + \frac{1}{2} \right) - k_{-1} \left( u + \frac{1}{2} \right) k_{-1}(u) e_{-1,1}(u) + k_{-1} \left( u + \frac{1}{2} \right) e_{-1,0} \left( u + \frac{1}{2} \right) k_{-1}(u) e_{-1,0}(u) = 0. \] (3.25)

Then it follows from equation (3.10) that
\[ e_{-1,0} \left( u + \frac{1}{2} \right) k_{-1}(u) = 3k_{-1}(u) e_{-1,0} \left( u + \frac{1}{2} \right) - 2k_{-1}(u) e_{-1,0}(u). \]

Thus, equation (3.25) is equivalent to
\[ k_{-1}(u) k_{-1} \left( u + \frac{1}{2} \right) \left( 3e_{-1,1} \left( u + \frac{1}{2} \right) - e_{-1,1}(u) + 3e_{-1,0} \left( u + \frac{1}{2} \right) e_{-1,0}(u) - 2e^{-1}_{-1,0}(u) \right) = 0. \]

This implies the result as \( k_{-1}(u) \) are invertible. \( \square \)
Lemma 3.9. In \( Y_R(\mathfrak{so}_3) \), we have
\[
e_{-1,1}(u) = [e_{-1,0}^{(1)}, e_{-1,0}(u)] - e_{-1,0}^2(u).
\]

**Proof.** It follows from equation (2.4) that
\[
[t_{-1,0}(u), t_{01}(v)] = \frac{1}{u - v} (t_{00}(u)t_{-1,1}(v) - t_{00}(v)t_{-1,1}(u)).
\]
Taking \( v \to \infty \), we obtain
\[
[t_{-1,0}(u), t_{01}^{(1)}] = t_{-1,1}(u).
\]
Using the Gauss decomposition of \( T(v) \), we know that
\[
t_{01}(v) = k_0(v)e_{01}(v) + f_{0,-1}(v)k_{-1}(v)e_{-1,0}(v).
\]
So \( t_{01}^{(1)} = e_{01}^{(1)} \). Moreover, proposition 3.4 implies that \( e_{01}^{(1)} = -e_{-1,0}^{(1)} \). Subsequently,
\[
[k_{-1}(u)e_{-1,0}(u), e_{-1,0}^{(1)}] = -k_{-1}(u)e_{-1,1}(u).
\]
From equation (3.10) one then obtains that
\[
[k_{-1}(u), e_{-1,0}^{(1)}] = k_{-1}(u)e_{-1,0}(u).
\]
Since
\[
k_{-1}(u)e_{-1,0}(u), e_{-1,0}^{(1)} = k_{-1}(u)[e_{-1,0}(u), e_{-1,0}^{(1)}] + [k_{-1}(u), e_{-1,0}^{(1)}]e_{-1,0}(u),
\]
we have
\[
k_{-1}(u)[e_{-1,0}(u), e_{-1,0}^{(1)}] + k_{-1}(u)e_{-1,0}^2(u) = -k_{-1}(u)e_{-1,1}(u).
\]
Dividing \( k_{-1}(u) \) we prove the lemma. \( \Box \)

Lemma 3.10. In \( Y_R(\mathfrak{so}_3) \) we have
\[
e_{-1,0}(u) = e_{-1,0}^2(u) - e_{-1,0}(u + \frac{i}{2}) e_{-1,0}(u) - e_{-1,1}(u + \frac{i}{2}).
\]

**Proof.** From equation (2.6), we compute that
\[
[t_{-1,0}(u), t_{01}(v)] = \frac{1}{u - v} (t_{01}(v)t_{-1,0}(u) - t_{-1,0}(u)t_{01}'(v))
\]
\[
\quad + \frac{1}{u - v} (t_{-1,1}(u)t_{-1,1}'(v) + t_{-1,0}(u)t_{01}'(v) + t_{-1,1}(u)t_{11}'(v))
\]
which is equivalent to
\[
(u - v) \left( \frac{u - v + \frac{i}{2}}{u - v} \right) [t_{-1,0}(u), t_{01}'(v)] = t_{-1,1}(u)t_{-1,1}'(v) + t_{-1,0}(u)t_{01}'(v) + t_{-1,1}(u)t_{11}'(v).
\]
Taking \( u \to \infty \) and invoking Gauss decomposition of \( T(u) \) and \( T'(u) \), we obtain that
\[
- [e_{-1,0}^{(1)}, e_{01}(v)k_1^{-1}(v)] = e_{-1,0}(v)e_{01}(v)k_1^{-1}(v) - e_{-1,1}(v)k_1^{-1}(v). \quad (3.26)
\]
Note that from equation (3.10), we have
\[
[e_{-1,0}^{(1)}, k_1^{-1}(v)] = e_{01}(v)k_1^{-1}(v),
\]
then we derive
\[
- [e_{-1,0}^{(1)}, e_{01}(v)k_1^{-1}(v)] = -[e_{-1,0}^{(1)}, e_{01}(v)]k_1^{-1}(v) - e_{01}^2(v)k_1^{-1}(v). \quad (3.27)
\]
Combining equations (3.26) and (3.27), we obtain
\[-[e_{-1,0}(u), e_0(v)] - e_0^2(v) = e_{-1,0}(v)e_0(v) - e_{-1,1}(v).\]

Finally using equation (3.17), we obtain the following equation:
\[e_{-1,0}(u) - e_{-1,0}(v - \frac{1}{2}) - e_{-1,0}^2(v - \frac{1}{2}) = -e_{-1,0}(v)e_{-1,0}(v - \frac{1}{2}) - e_{-1,1}(v).\]

The lemma is obtained if we take \(u = v - \frac{1}{2}\). □

Next we can prove an interesting relation between \(e_{-1,1}(u)\) and \(e_{-1,0}(u)\).

**Lemma 3.11.** In \(Y_\delta(s\delta_3)\), we have
\[e_{-1,1}(u) = -\frac{1}{2}e_{-1,0}(u).\]

**Proof.** Using previous lemmas, we have
\[e_{-1,1}(u) = -e_{-1,0}(u + \frac{1}{2})e_{-1,0}(u) - e_{-1,1}(u + \frac{1}{2}).\]
Moreover, we also have
\[-e_{-1,0}(u + \frac{1}{2})e_{-1,0}(u) - e_{-1,1}(u + \frac{1}{2}) = -\frac{1}{2}(e_{-1,1}(u) + 2e_{-1,0}(u)).\]
The lemma then follows immediately. □

Similarly, we can obtain the relation between \(f_{-1,1}(u)\) and \(f_{1,0}(u)\).

**Lemma 3.12.** In \(Y_\delta(s\delta_3)\), we have
\[f_{-1,1}(u) = -\frac{1}{2}f_{1,0}(u).\]

Now we are ready to prove proposition 3.7.

**Proof.** It is enough to check equation (3.23), as equation (3.24) can be treated similarly.

Using the generating relations (2.4), we obtain
\[[t_{-1,0}(u), t_{-1,0}(v)] = \frac{1}{u - v}(t_{-1,0}(u)t_{-1,0}(v) - t_{-1,0}(v)t_{-1,0}(u))
+ \frac{1}{u - v - \frac{1}{2}}(t_{-1,1}(v)t_{-1,1}(u) + t_{-1,0}(v)t_{-1,0}(u) + t_{-1,0}(v)t_{-1,0}(u)),\]
\[[t_{-1,1}(u), t_{-1,1}(v)] = \frac{1}{u - v}(t_{-1,1}(u)t_{-1,1}(v) - t_{-1,1}(v)t_{-1,1}(u))
+ \frac{1}{u - v - \frac{1}{2}}(t_{-1,1}(v)t_{-1,1}(u) + t_{-1,0}(v)t_{-1,0}(u) + t_{-1,0}(v)t_{-1,0}(u)).\]

Comparing these two equations, we obtain that
\[[t_{-1,0}(u), t_{-1,0}(v)] = t_{-1,1}(u)t_{-1,1}(v) + \frac{1}{u - v - \frac{1}{2}}t_{-1,1}(v)t_{-1,1}(u)
- \frac{u - v}{u - v - \frac{1}{2}}t_{-1,1}(v)t_{-1,1}(u).\]

Using the defining relations (2.4) again, we have
\[[t_{-1,1}(v), t_{-1,0}(u)] = \frac{1}{v - u}(t_{-1,1}(v)t_{-1,0}(u) - t_{-1,1}(u)t_{-1,0}(v)).\]
In terms of Gauss generators they are
\[k_{-1}(u)e_{-1,0}(u)k_{-1}(v) = k_{-1}(u)k_{-1}(v)e_{-1,0}(u) + \frac{1}{u - v}k_{-1}(u)k_{-1}(v)(e_{-1,0}(u) - e_{-1,0}(v)).\]
Since \( t_{-1,0}(u)t_{-1,0}(v) = k_{-1}(u)e_{-1,0}(u)k_{-1}(v)e_{-1,0}(v) \), we can write

\[
t_{-1,0}(u)t_{-1,0}(v) = k_{-1}(u)k_{-1}(v)e_{-1,0}(u)e_{-1,0}(v) + \frac{1}{u-v}k_{-1}(u)k_{-1}(v)(e_{-1,0}(u)e_{-1,0}(v) - e_{-1,0}^2(v)),
\]

and

\[
t_{-1,0}(v)t_{-1,0}(u) = k_{-1}(v)k_{-1}(u)e_{-1,0}(v)e_{-1,0}(u) + \frac{1}{v-u}k_{-1}(v)k_{-1}(u)(e_{-1,0}(v)e_{-1,0}(u) - e_{-1,0}^2(u)).
\]

Then we get to the left-hand side of equation (3.28)

\[
[t_{-1,0}(u), t_{-1,0}(v)] = k_{-1}(u)k_{-1}(v)[e_{-1,0}(u), e_{-1,0}(v)] - \frac{1}{u-v}k_{-1}(u)k_{-1}(v)(e_{-1,0}(u) - e_{-1,0}(v))^2.
\]

Now we consider the right-hand side of equation (3.28). Using the Gauss decomposition and lemma 3.11 we have

\[
t_{-1,1}(v)t_{-1,1}(u) = -\frac{1}{2}k_{-1}(v)e_{-1,0}^2(v)k_{-1}(u).
\]

Using equation (3.10), we finally obtain

\[
e_{-1,0}^2(v)k_{-1}(u) = \frac{(u-v-1)^2}{(u-v)^2}k_{-1}(u) - \frac{1}{u-v}k_{-1}(u)e_{-1,0}(u)e_{-1,0}(v) + \frac{u-v-1}{(u-v)^2}k_{-1}(u)e_{-1,0}(u) - \frac{1}{(u-v)^2}k_{-1}(u)e_{-1,0}^2(u).
\]

Then using Gauss decomposition of \( T(u) \) and lemma (3.11) again, we obtain that the right-hand side of equation (3.28) equals

\[
-\frac{1}{2} \frac{u-v}{u-v}k_{-1}(u)k_{-1}(v)(e_{-1,0}(u) - e_{-1,0}(v))^2.
\]

Taking into consideration the LHS we prove the proposition. \( \square \)

4. The isomorphism between RTT realization and Drinfeld’s realization

We recall the Drinfeld realization of the Yangian associated with \( so_3 \), denoted by \( Y(so_3) \) in the following.

**Definition 4.1.** \( Y(so_3) \) is the associative algebra generated by infinite generators \( h_k, x_i^\pm \), \((l = 0, 1, 2, \ldots)\), subject to the following defining relations:

\[
[h_k, h_l] = 0, [x_k^+, x_l^-] = h_{k+l}, [h_0, x_l^+] = \pm x_l^+,
\]

\[
[h_k, x_l^+] - [h_k, x_{l+1}^+] = \pm \frac{1}{2} [h_k, x_l^+],
\]

\[
[x_k^+, x_{l+1}^-] - [x_{k+1}^+, x_l^-] = \pm \frac{1}{2} [x_k^+, x_l^-].
\]

Let \( X^\pm(u) = \sum_{k=0}^\infty x_k^\pm u^{-k-1} \), \( H(u) = 1 + \sum_{k=0}^\infty h_k u^{-k-1} \) be the generating series, then we have the following proposition.
Proposition 4.2. The defining relations of \( Y(\mathfrak{so}_3) \) are equivalent to the following form in terms of generating series:

\[
[H(u), H(v)] = 0, \tag{4.1}
\]
\[
[X^+(u), X^-(v)] = -\frac{H(u) - H(v)}{u - v}, \tag{4.2}
\]
\[
[H(u), X^+(v)] = -\frac{1}{2} \frac{\{H(u), (X^+(u) - X^+(v))\}}{u - v}, \tag{4.3}
\]
\[
[H(u), X^-(v)] = \frac{1}{2} \frac{\{H(u), (X^-(u) - X^-(v))\}}{u - v}, \tag{4.4}
\]
\[
[X^+(u), X^+(v)] = -\frac{1}{2} \frac{(X^+(u) - X^+(v))^2}{u - v}, \tag{4.5}
\]
\[
[X^-(u), X^-(v)] = \frac{1}{2} \frac{(X^-(u) - X^-(v))^2}{u - v}. \tag{4.6}
\]

Remark 4.3. The proof of proposition 4.2 can be derived by comparing the coefficients of the two sides of equations.

Now we can give the main result in the following theorem.

Theorem 4.4. The map \( \Phi : Y(\mathfrak{so}_3) \to Y_R(\mathfrak{so}_3) \) given by

\[
X^-(u) \mapsto e_{-1,0}(u), X^+(u) \mapsto f_{0,-1}(u), H(u) \mapsto k_{-1}^{-1}(u)k_0(u) \tag{4.7}
\]

is an isomorphism.

Proof. It is easy to verify that \( \Phi \) is a homomorphism from propositions 3.2, 3.3, 3.6, and 3.7. Furthermore, the surjectivity is obtained by using the Gauss decomposition of \( T(u) \) and equation (3.5), propositions 3.4, 3.5, 3.11 and lemma 3.12. Since the Gauss generators \( k_{-1}(u), e_{-1,0}(u), f_{0,-1}(u) \) can generate the algebra \( Y_R(\mathfrak{so}_3) \), the inverse map of \( \Phi \) can be given by

\[
e_{-1,0}(u) \mapsto X^-(u), f_{0,-1}(u) \mapsto X^+(u), k_{-1}(u) \mapsto H^{-1}\left(u - \frac{1}{2}\right). \tag{4.8}
\]

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