ON HYPERCOMPLEX PSEUDO-HERMITIAN MANIFOLDS

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The class of the hypercomplex pseudo-Hermitian manifolds is considered. The flatness of the considered manifolds with the 3 parallel complex structures is proved. Conformal transformations of the metrics are introduced. The conformal invariance and the conformal equivalence of the basic types manifolds are studied. A known example is characterized in relation to the obtained results.

Introduction

This paper is a continuation of the same authors’s paper which is inspired by the seminal work of D. V. Alekseevsky and S. Marchiafava. We follow a parallel direction including skew-Hermitian metrics with respect to the almost hypercomplex structure.

In the first section we give some necessary facts concerning the almost hypercomplex pseudo-Hermitian manifolds introduced in.

In the second one we consider the special class of (integrable) hypercomplex pseudo-Hermitian manifolds, namely pseudo-hyper-Kähler manifolds. Here we expose the proof of the mentioned in statement that each pseudo-hyper-Kähler manifold is flat.

The third section is fundamental for this work. A study of the group of conformal transformations of the metric is initiated here. The conformal invariant classes and the conformal equivalent class to the class of the pseudo-hyper-Kähler manifolds are found.

Finally, we characterize a known example in terms of the conformal transformations.
1 Preliminaries

1.1 Hypercomplex pseudo-Hermitian structures in a real vector space

Let $V$ be a real $4n$-dimensional vector space. By \( \{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial u^i}, \frac{\partial}{\partial v^i} \}, i = 1, 2, \ldots, n \), is denoted a (local) basis on $V$. Each vector $x$ of $V$ is represented in the mentioned basis as follows

\[
x = x^i \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial y^i} + u^i \frac{\partial}{\partial u^i} + v^i \frac{\partial}{\partial v^i}.
\]  

A standard complex structure on $V$ is defined as in [5]

\[
J_1 \frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i}, \quad J_1 \frac{\partial}{\partial y^i} = -\frac{\partial}{\partial x^i}, \quad J_1 \frac{\partial}{\partial u^i} = -\frac{\partial}{\partial v^i}, \quad J_1 \frac{\partial}{\partial v^i} = \frac{\partial}{\partial u^i};
\]

\[
J_2 \frac{\partial}{\partial x^i} = \frac{\partial}{\partial u^i}, \quad J_2 \frac{\partial}{\partial y^i} = \frac{\partial}{\partial v^i}, \quad J_2 \frac{\partial}{\partial u^i} = -\frac{\partial}{\partial v^i}, \quad J_2 \frac{\partial}{\partial v^i} = -\frac{\partial}{\partial u^i};
\]

\[
J_3 \frac{\partial}{\partial x^i} = -\frac{\partial}{\partial u^i}, \quad J_3 \frac{\partial}{\partial y^i} = -\frac{\partial}{\partial v^i}, \quad J_3 \frac{\partial}{\partial u^i} = -\frac{\partial}{\partial v^i}, \quad J_3 \frac{\partial}{\partial v^i} = \frac{\partial}{\partial u^i}.
\]

The following properties about $J_\alpha$ are direct consequences of [2]

\[
J_1^2 = J_2^2 = J_3^2 = -\text{Id},
\]

\[
J_1 J_2 = -J_2 J_1 = J_3, \quad J_2 J_3 = -J_3 J_2 = J_1, \quad J_3 J_1 = -J_1 J_3 = J_2.
\]  

If $x \in V$, i.e. $x(x^i, y^i, u^i, v^i)$ then according to [2] and [3] we have

\[
J_1 x(-y^i, x^i, v^i, -u^i), \quad J_2 x(-u^i, -v^i, x^i, y^i), \quad J_3 x(v^i, -u^i, y^i, -x^i).
\]

**Definition 1.1** (1) A triple $H = (J_1, J_2, J_3)$ of anticommuting complex structures on $V$ with $J_3 = J_1 J_2$ is called a hypercomplex structure on $V$;

A bilinear form $f$ on $V$ is defined as ordinary, $f : V \times V \to \mathbb{R}$. We denote by $B(V)$ the set of all bilinear forms on $V$. Each $f$ is a tensor of type $(0, 2)$, and $B(V)$ is a vector space of dimension $16n^2$.

Let $J$ be a given complex structure on $V$. A bilinear form $f$ on $V$ is called Hermitian (respectively, skew-Hermitian) with respect to $J$ if the identity $f(Jx, Jy) = f(x, y)$ (respectively, $f(Jx, Jy) = -f(x, y)$) holds true.

**Definition 1.2** (1) A bilinear form $f$ on $V$ is called an Hermitian bilinear form with respect to $H = (J_\alpha)$ if it is Hermitian with respect to any complex structure $J_\alpha$, $\alpha = 1, 2, 3$, i.e.

\[
f(J_\alpha x, J_\alpha y) = f(x, y) \quad \forall \, x, y \in V.
\]  

We denote by $L_0 = B_H(V)$ the set of all Hermitian bilinear forms on $V$. The notion of pseudo-Hermitian bilinear forms is introduced by the following
Definition 1.3 (4) A bilinear form \( f \) on \( V \) is called a pseudo-Hermitian bilinear form with respect to \( H = (J_1, J_2, J_3) \), if it is Hermitian with respect to \( J_\alpha \) and skew-Hermitian with respect to \( J_\beta \) and \( J_\gamma \), i.e.

\[
f(J_\alpha x, J_\alpha y) = -f(J_\beta x, J_\beta y) = -f(J_\gamma x, J_\gamma y) = f(x, y) \quad \forall \ x, y \in V,
\]

where \((\alpha, \beta, \gamma)\) is a circular permutation of \((1, 2, 3)\).

We denote \( f \in L_\alpha \subset \mathcal{B}(V) \) \((\alpha = 0, 1, 2, 3)\) when \( f \) satisfies the conditions (5) and (6), respectively.

In 1 is introduced a pseudo-Euclidian metric \( g \) with signature \((2n, 2n)\) as follows

\[
g(x, y) := \sum_{i=1}^{n} (-x^i a^i - y^i b^i + u^i c^i + v^i d^i),
\]

where \( x(x^i, y^i, u^i, v^i), \ y(a^i, b^i, c^i, d^i) \in V, \ i = 1, 2, \ldots, n \). This metric satisfies the following properties

\[
g(J_1 x, J_1 y) = -g(J_2 x, J_2 y) = -g(J_3 x, J_3 y) = g(x, y).
\]

This means that the pseudo-Euclidean metric \( g \) belongs to \( L_1 \).

The form \( g_1 : g_1(x, y) = g(J_1 x, y) \) coincides with the Kähler form \( \Phi \) which is Hermitian with respect to \( J_\alpha \), i.e.

\[
\Phi(J_\alpha x, J_\alpha y) = \Phi(x, y), \quad \alpha = 1, 2, 3, \quad \Phi \in L_0.
\]

The attached to \( g \) associated bilinear forms \( g_2 : g_2(x, y) = g(J_2 x, y) \) and \( g_3 : g_3(x, y) = g(J_3 x, y) \) are symmetric forms with the properties

\[
-g_2(J_1 x, J_1 y) = -g_2(J_2 x, J_2 y) = g_2(J_3 x, J_3 y) = g_2(x, y),
-g_3(J_1 x, J_1 y) = g_3(J_2 x, J_2 y) = -g_3(J_3 x, J_3 y) = g_3(x, y),
\]

i.e. \( g_2 \in L_3, \ g_3 \in L_2 \).

It follows that the Kähler form \( \Phi \) is Hermitian regarding \( H \) and the metrics \( g, g_2, g_3 \) are pseudo-Hermitian of different types with signature \((2n, 2n)\).

Now we recall the following notion:

Definition 1.4 (4) The structure \((H, G) := (J_1, J_2, J_3, g, \Phi, g_2, g_3)\) is called a hypercomplex pseudo-Hermitian structure on \( V \).

1.2 Structural tensors on an almost \((H, G)\)-manifold

Let \((M, H)\) be an almost hypercomplex manifold\(\text{1}\). We suppose that \( g \) is a symmetric tensor field of type \((0, 2)\). If it induces a pseudo-Hermitian inner product in \( T_p M, \ p \in M \), then \( g \) is called a pseudo-Hermitian metric on \( M \).
The structure \((H, G) := (J_1, J_2, J_3, g, \Phi, g_2, g_3)\) is called an \textit{almost hypercomplex pseudo-Hermitian structure on} \(M\) or in short an \textit{almost \((H, G)\)-structure on} \(M\). The manifold \(M\) equipped with \(H\) and \(G\), i.e. \((M, H, G)\), is called an \textit{almost hypercomplex pseudo-Hermitian manifold}, or in short an \textit{almost \((H, G)\)-manifold}.\(^4\)

The 3 tensors of type \((0, 3)\) \(F_\alpha : F_\alpha(x, y, z) = g(\nabla_x J_\alpha y, z), \alpha = 1, 2, 3\), where \(\nabla\) is the Levi-Civita connection generated by \(g\), is called \textit{structural tensors of the almost \((H, G)\)-manifold}.\(^4\)

The structural tensors satisfy the following properties:

\[
\begin{align*}
F_1(x, y, z) &= F_2(x, J_3 y, z) + F_3(x, y, J_2 z), \\
F_2(x, y, z) &= F_3(x, J_1 y, z) + F_1(x, y, J_3 z), \\
F_3(x, y, z) &= F_1(x, J_2 y, z) - F_2(x, y, J_1 z), \\
F_1(x, y, z) &= -F_1(x, z, y) = -F_1(x, J_1 y, J_1 z), \\
F_2(x, y, z) &= F_2(x, z, y) = F_2(x, J_2 y, J_2 z), \\
F_3(x, y, z) &= F_3(x, z, y) = F_3(x, J_3 y, J_3 z).
\end{align*}
\]  \(^{(10)}\)

Let us recall the Nijenhuis tensors \(N_\alpha(X, Y) = \frac{1}{2} [[J_\alpha, J_\alpha]] (X, Y)\) for almost complex structures \(J_\alpha\) and \(X, Y \in \mathfrak{X}(M)\), where

\[
[[J_\alpha, J_\alpha]] (X, Y) = 2 \{ [J_\alpha X, J_\alpha Y] - J_\alpha [J_\alpha X, Y] - J_\alpha [X, J_\alpha Y] - [X, Y] \}.
\]

It is well known that the almost hypercomplex structure \(H = (J_\alpha)\) is a hypercomplex structure if \([[J_\alpha, J_\alpha]]\) vanishes for each \(\alpha = 1, 2, 3\). Moreover it is known that one almost hypercomplex structure \(H\) is hypercomplex if and only if two of the structures \(J_\alpha\) \((\alpha = 1, 2, 3)\) are integrable. This means that two of the tensors \(N_\alpha\) vanish.\(^1\)

We recall also the following definitions. Since \(g\) is Hermitian metric with respect to \(J_1\), according to\(^3\) the class \(\mathcal{W}_4\) is a subclass of the class of Hermitian manifolds. If \((H, G)\)-manifold belongs to \(\mathcal{W}_4\), with respect to \(J_4\), then the almost complex structure \(J_1\) is integrable and

\[
F_1(x, y, z) = \frac{1}{2^{2n-1}} \left[ g(x, y) \theta_1(z) - g(x, z) \theta_1(y) - g(x, J_1 y) \theta_1(J_1 z) + g(x, J_1 z) \theta_1(J_1 y) \right],
\]

where \(\theta_1(\cdot) = g^{ij} F_1(e_i, e_j, \cdot) = \delta \Phi(\cdot)\) for the basis \(\{e_i\}_{i=1}^{4n}\), and \(\delta\) – the coDerivative.

On other side the metric \(g\) is a skew-Hermitian with respect to \(J_2\) and \(J_3\), i.e. \(g(J_2 x, J_2 y) = g(J_3 x, J_3 y) = -g(x, y)\). A classification of all almost complex manifolds with skew-Hermitian metric (Norden metric or B-metric) is given in\(^2\). One of the basic classes of integrable almost complex manifolds
with skew-Hermitian metric is \( \mathcal{W}_1 \). It is known that if an almost \((H, G)\)-manifold belongs to \( \mathcal{W}_1(J_1, \alpha) \), \( \alpha = 2, 3 \), then \( J_\alpha \) is integrable and the following equality holds

\[
F_\alpha(x, y, z) = \frac{1}{4^n} \left[ g(x, y)\theta_\alpha(z) + g(x, z)\theta_\alpha(y) + g(x, J_\alpha y)\theta_\alpha(J_\alpha z) + g(x, J_\alpha z)\theta_\alpha(J_\alpha y) \right],
\]

where \( \theta_\alpha(z) = g^{ij}F_\alpha(e_i, e_j, z), \alpha = 2, 3, \) for an arbitrary basis \( \{e_i\}_{i=1}^{4n} \).

When (12) is satisfied for \((M, H, G)\), we say that \((M, H, G)\) \( \in \mathcal{W}(J_1) \). In the case, \((M, H, G)\) satisfies (13) for \( \alpha = 2 \) or \( \alpha = 3 \), we say \((M, H, G)\) \( \in \mathcal{W}(J_2) \) or \( \in \mathcal{W}(J_3) \). Let us denote the class \( \mathcal{W} := \bigcap_{\alpha=1}^{3} \mathcal{W}(J_\alpha) \).

The next theorem gives a sufficient condition an almost \((H, G)\)-manifold to be integrable.

**Theorem 1.1** \((4)\) Let \((M, H, G)\) belongs to the class \( \mathcal{W}(J_1) \cap \mathcal{W}(J_3) \). Then \((M, H, G)\) is of class \( \mathcal{W}(J_\gamma) \) for all cyclic permutations \((\alpha, \beta, \gamma)\) of \( (1, 2, 3) \).

Let us remark that necessary and sufficient conditions \((M, H, G)\) to be in \( \mathcal{W} \) are

\[
\theta_\alpha \circ J_\alpha = -\frac{2n}{2n-4} \theta_1 \circ J_1, \quad \alpha = 2, 3.
\]

**2 Pseudo-hyper-Kähler manifolds**

**Definition 2.1** \((4)\) A pseudo-Hermitian manifold is called a pseudo-hyper-Kähler manifold, if \( \nabla J_\alpha = 0 \) \((\alpha = 1, 2, 3)\) with respect to the Levi-Civita connection generated by \( g \).

It is clear, then \( F_\alpha = 0 \) \((\alpha = 1, 2, 3)\) holds or the manifold is Kählerian with respect to \( J_\alpha \), i.e. \((M, H, G)\) \( \in \mathcal{K}(J_\alpha) \).

Immediately we obtain that if \((M, H, G)\) belongs to \( \mathcal{K}(J_\alpha) \cap \mathcal{W}(J_3) \) then \((M, H, G)\) \( \in \mathcal{K}(J_\gamma) \) for all cyclic permutations \((\alpha, \beta, \gamma)\) of \( (1, 2, 3) \).

Then the following sufficient condition for a \( \mathcal{K} \)-manifold is valid.

**Theorem 2.1** \((4)\) If \((M, H, G)\) \( \in \mathcal{K}(J_\alpha) \cap \mathcal{W}(J_\beta) \) then \( M \) is a pseudo-hyper-Kähler manifold \((\alpha \neq \beta \in \{1, 2, 3\})\).

Let \((M^{4n}, H, G)\) be a pseudo-hyper-Kähler manifold and \( \nabla \) be the Levi-Civita connection generated by \( g \). The curvature tensor seems as follows

\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, \quad \forall \ X, Y, Z, W \in \mathfrak{X}(M).
\]

and the corresponding tensor of type \((0, 4)\) is

\[
R(X, Y, Z, W) = g \left( R(X, Y)Z, W \right), \quad \forall \ X, Y, Z, W \in \mathfrak{X}(M).
\]
Lemma 2.2 The curvature tensor of a pseudo-hyper-Kähler manifold has the following properties:

\[ R(X, Y, Z, W) = R(X, Y, J_1 Z, J_1 W) = R(J_1 X, J_1 Y, Z, W) \]
\[ = -R(X, Y, J_2 Z, J_2 W) = -R(J_2 X, J_2 Y, Z, W) \]  
\[ = -R(X, Y, J_3 Z, J_3 W) = -R(J_3 X, J_3 Y, Z, W). \]  

(17)

\[ R(X, Y, Z, W) = R(X, J_1 Y, J_1 Z, W) \]
\[ = -R(X, J_2 Y, J_2 Z, W) = -R(X, J_3 Y, J_3 Z, W). \]  

(18)

Proof. The equality (17) is valid, because of (15), (16), the condition \( \nabla J_\alpha = 0 \) (\( \alpha = 1, 2, 3 \)), the equality (8) and the properties of the curvature (0, 4)-tensor.

To prove (18), we will show at first that the property

\[ R(J_2 X, Y, Z, W) = R(X, J_2 Y, Z, W), \]
\[ R(X, J_2 Y, Z, W) = R(Y, X, Z, W), \]
\[ S_X Y Z R(X, J_2 Y, J_2 Z, W) = 0, \]

where \( S_X Y Z \) denotes the cyclic sum regarding \( X, Y, Z \). In the last equality we replace \( Y \) by \( J_2 Y \) and \( W \) by \( J_2 W \).

Replacing \( Y \) by \( Z \), and inversely, we get

\[ -R(X, J_2 Y, J_2 Z, W) + R(Z, X, Y, W) = 0. \]  

(19)

As we have

\[ -R(J_2 Z, Y, J_2 X, W) = -R(Z, J_2 Y, J_2 X, W) = R(J_2 Y, Z, X, W), \]

with the help of (19) and (20) we obtain

\[ -R(X, J_2 Y, J_2 Z, W) + R(Z, X, Y, W) = 0. \]  

(21)

According to the first Bianchi identity and (17), we obtain

\[ -R(X, J_2 Z, J_2 Y, W) = R(J_2 Z, J_2 Y, X, W) + R(J_2 Y, X, J_2 Z, W) \]
\[ = -R(Z, Y, X, W) - R(X, J_2 Y, J_2 Z, W). \]

Then the equality (21) seem as follows

\[ -2R(X, J_2 Y, J_2 Z, W) + R(Z, X, Y, W) - R(X, Y, Z, W) + R(Y, Z, X, W) = 0 \]

By the first Bianchi identity the equality is transformed in the following

\[ -2R(X, J_2 Y, J_2 Z, W) - 2R(X, Y, Z, W) = 0, \]
which is equivalent to
\[ R(X, J_2 Y, J_2 Z, W) = -R(X, Y, Z, W). \] (22)

As the tensor \( R \) has the same properties with respect to \( J_3 \), and to \( J_2 \), it follows that the next equality holds, too.
\[ R(X, J_3 Y, J_3 Z, W) = -R(X, Y, Z, W). \] (23)

Using (22) and (23) for \( J_1 = J_2 J_3 \) we get successively that
\[ R(X, J_1 Y, J_1 Z, W) = R(X, J_2 (J_3 Y), J_2 (J_3 Z), W) = -R(X, J_3 Y, J_3 Z, W), \]
which completes the proof of (18).

Now we will prove a theorem which gives us a geometric characteristic of the pseudo-hyper-Kähler manifolds.

**Theorem 2.3** Each pseudo-hyper-Kähler manifold is a flat pseudo-Riemannian manifold with signature \((2n, 2n)\).

**Proof.** Lemma 2.2 implies the properties
\[ -R(X, Y, Z, W) = R(X, J_1 Y, J_1 Z, J_1 W) = R(X, J_2 Y, Z, J_2 W) = R(X, J_3 Y, Z, J_3 W). \] (24)

As \( J_1 = J_2 J_3 \), we also have the following
\[ R(X, J_1 Y, J_1 Z, J_1 W) = R(X, J_2 (J_3 Y), Z, J_2 (J_3 W)) = -R(X, J_3 Y, Z, J_3 W) = R(X, Y, Z, W). \]

Comparing (24) with the last equality we receive
\[ -R(X, Y, Z, W) = R(X, J_1 Y, Z, J_1 W) = R(X, Y, Z, W), \]
or \( R \equiv 0 \).

### 3 Conformal transformations of the pseudo-Hermitian metric

The usual conformal transformation \( c : \bar{g} = e^{2u} g \), where \( u \) is a differential function on \( M^{4n} \), is known. Since \( g_\alpha(\cdot, \cdot) = g(J_\alpha \cdot, \cdot) \), the conformal transformation of \( g \) causes the same changes of the pseudo-Hermitian metrics \( g_2, g_3 \) and the Kähler form \( \Phi \equiv g_1 \). Then we say that it is given a conformal transformation \( c \) of \( G \) to \( \bar{G} \) determined by \( u \in \mathcal{F}(M) \). These conformal transformations form a group denoted by \( C \). The hypercomplex pseudo-Hermitian manifolds \((M, H, G)\) and \((M, H, \bar{G})\) we call \( C \)-equivalent manifolds or conformal-equivalent manifolds.
Let $\nabla$ and $\bar{\nabla}$ be the Levi-Civita connections determined by the metrics $g$ and $\bar{g}$, respectively. The known condition for a Levi-Civita connection implies the following relation

$$\bar{\nabla}_XY = \nabla_XY + du(X)Y + du(Y)X - g(X,Y)\text{grad}(u).$$

Using (25) and the definitions of structural tensors for $\nabla$ and $\bar{\nabla}$ we obtain

$$\bar{F}_1(X,Y,Z) = e^{2u}[F_1(X,Y,Z) - g(X,Y)du(J_1Z) + g(X,Z)du(J_1Y)$$

$$+ g(J_1X,Y)du(Z) - g(J_1X,Z)du(Y)],$$

$$\bar{F}_\alpha(X,Y,Z) = e^{2u}[F_\alpha(X,Y,Z) + g(X,Y)du(J_\alpha Z) + g(X,Z)du(J_\alpha Y)$$

$$- g(J_\alpha X,Y)du(Z) - g(J_\alpha X,Z)du(Y)],$$

for $\alpha = 2, 3$. The last two equalities imply the following relations for the corresponding structural 1-forms

$$\bar{\theta}_1 = \theta_1 - 2(2n - 1)du \circ J_1, \quad \bar{\theta}_\alpha = \theta_\alpha + 4n du \circ J_\alpha, \quad \alpha = 2, 3.$$

Let us denote the following (0,3)-tensors.

$$P_1(x,y,z) = F_1(x,y,z) - \frac{1}{2(2n - 1)} [g(x,y)\theta_1(z) - g(x,z)\theta_1(y)$$

$$- g(x,J_1y)\theta_1(J_1z) + g(x,J_1z)\theta_1(J_1y)],$$

$$P_\alpha(x,y,z) = F_\alpha(x,y,z) - \frac{1}{4n} [g(x,y)\theta_\alpha(z) + g(x,z)\theta_\alpha(y)$$

$$+ g(x,J_\alpha y)\theta_\alpha(J_\alpha z) + g(x,J_\alpha z)\theta_\alpha(J_\alpha y)], \quad \alpha = 2, 3.$$

According to (12) and (13) it is clear that

$$(M, H, G) \in \mathcal{W}(J_\alpha) \iff P_\alpha = 0 \quad (\alpha = 1, 2, 3).$$

The equalities (26)–(28) imply the following two interconnections

$$\bar{P}_\alpha = e^{2u}P_\alpha, \quad \alpha = 1, 2, 3;$$

$$\bar{\theta}_\alpha \circ J_\alpha = \frac{2n}{2n - 1} \bar{\theta}_1 \circ J_1 = \theta_\alpha \circ J_\alpha + \frac{2n}{2n - 1} \theta_1 \circ J_1, \quad \alpha = 2, 3.$$

From (31) we receive that each of $\mathcal{W}(J_\alpha) (\alpha = 1, 2, 3)$ is invariant with respect to the conformal transformations of $C$, i.e. they are $C$-invariant classes. Having in mind also (32), we state the validity of the following

**Theorem 3.1** The class $\mathcal{W}$ of hypercomplex pseudo-Hermitian manifolds is $C$-invariant.

Now we will determine the class of the (locally) $C$-equivalent $K$-manifolds. Let us denote the following subclass $\mathcal{W}^0 := \{W | d(\bar{\theta}_1 \circ J_1) = 0\}$. 

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Theorem 3.2 A hypercomplex pseudo-Hermitian manifold belongs to $\mathcal{W}^0$ if and only if it is $C$-equivalent to a pseudo-hyper-Kähler manifold.

Proof. Let $(M, H, G)$ be a pseudo-hyper-Kähler manifold, i.e. $(M, H, G) \in \mathcal{K}$. Then $F_\alpha = \theta_\alpha = 0$ ($\alpha = 1, 2, 3$). Hence (28) has the form
\[
\bar{\theta}_1 = -2(2n - 1)du \circ J_1, \quad \bar{\theta}_\alpha = 4ndu \circ J_\alpha, \quad \alpha = 2, 3.
\] (33)

From (26), (27) and (33) and having in mind (12) and (13) we obtain that $(M, H, G)$ is a $\mathcal{W}$-manifold. According to (33) the 1-forms $\bar{\theta}_\alpha \circ J_\alpha$ ($\alpha = 1, 2, 3$) are closed. Because of (14) the condition $d(\bar{\theta}_1 \circ J_1) = 0$ is sufficient.

Conversely, let $(M, H, G)$ be a $\mathcal{W}$-manifold with closed $\bar{\theta}_1 \circ J_1$. Because of (14) the 1-forms $\bar{\theta}_\alpha \circ J_\alpha$ ($\alpha = 2, 3$) are closed, too. We determine the function $u$ as a solution of the differential equation $du = -\frac{1}{2(2n-1)}\bar{\theta}_1 \circ J_1$. Then by an immediate verification we state that the transformation $c^{-1} : g = e^{-2u}\bar{g}$ converts $(M, H, G)$ into $(M, H, G) \in \mathcal{K}$. This completes the proof.

Let us remark the following inclusions
\[
\mathcal{K} \subset \mathcal{W}^0 \subset \mathcal{W} \subset \mathcal{W}(J_\alpha), \quad \alpha = 1, 2, 3.
\]

Let $R, \rho, \tau$ and $\bar{R}, \bar{\rho}, \bar{\tau}$ be the curvature tensors, the Ricci tensors, the scalar curvatures corresponding to $\nabla$ and $\bar{\nabla}$, respectively. The following tensor is curvature-like, i.e. it has the same properties as $R$.
\[
\psi_1(S)(X, Y, Z, U) = g(Y, Z)S(X, U) - g(X, Z)S(Y, U)
+ g(X, U)S(Y, Z) - g(Y, U)S(X, Z),
\]
where $S$ is a symmetric tensor.

Having in mind (25) and (15), we obtain

**Proposition 3.3** The following relations hold for the $C$-equivalent $(H, G)$-manifolds
\[
\bar{R} = e^{2u}\{R - \psi_1(S)\},
\bar{\rho} = \rho - \text{tr}Sg - 2(2n - 1)S,
\bar{\tau} = e^{-2u}\{\tau - 2(4n - 1)\text{tr}S\},
\] (34)

where
\[
S(Y, Z) = S(Z, Y) = (\nabla_Y du) Z + du(Y)du(Z) - \frac{1}{2}du(\text{grad}(du))g(Y, Z).
\] (35)

If $(M, H, G)$ is a $C$-equivalent $\mathcal{W}$-manifold to a $\mathcal{K}$-manifold, i.e. $(M, H, G) \in \mathcal{W}^0$, then Proposition 3.3 implies

**Corollary 3.4** A $\mathcal{W}^0$-manifold has the following curvature characteristic
\[
R = \frac{1}{2(2n - 1)}\left\{\psi_1(\rho) - \frac{\tau}{4n - 1} - \pi_1\right\},
\]
where $\pi_1(X, Y, Z, U) = \frac{1}{2}\psi_1(g) = g(Y, Z)g(X, U) - g(X, Z)g(Y, U)$. 


It is well known that the $C$-invariant tensor of each pseudo-Riemannian manifold is the so-called Weil tensor $W$. From (34) we receive immediately

$$W = e^{2u}W, \quad W = R - \frac{1}{2(2n-1)} \left( \psi_1(\rho) - \frac{\tau}{4n-1} \pi_1 \right). \quad (36)$$

Let us remark that the vanishing of the Weil tensor is a necessary and sufficient condition a pseudo-Riemannian manifold to be conformal equivalent to a flat manifold with dimension greater than 3.

This is confirmed by the combining of Theorem 2.3, Theorem 3.2 and Corollary 3.4, i.e. $(M, H, G) \in \mathcal{W}^0$ iff $W = 0$ on $(M, H, G)$.

Since each conformal transformation determines uniquely a symmetric tensor $S$ by (35) then it takes an interest in the consideration $S$ as a bilinear form on $T_pM$ belonging to each of the components $L_\alpha, \ (\alpha = 0, 1, 2, 3)$.

Let $S \in L_0$. In view of (33) $\text{tr} S = 0$ holds and according to (33) we receive $\bar{\tau} = e^{-2u}\tau$ and an invariant tensor $W_0 = R - \frac{1}{2(2n-1)} \psi_1(\rho)$. When $W_0$ vanishes on $(M, H, G)$ then the curvature tensor has the form $R = \frac{1}{4n-1} \psi_1(\rho)$.

In the cases when $S \in L_\alpha \ (\alpha = 1, 2, 3)$ we consider $(M, H, G)$ as an $\mathcal{W}^0$-manifold. Then according to Theorem 2.3 and Theorem 3.2 we have $R = 0$ on the $C$-equivalent $K$-manifold of $(M, H, G)$.

Now let $S \in L_1$. By reason of $g \in L_1$ we have a cause for the consideration of the possibility $S = \lambda g$. Hence $\lambda = \frac{\text{tr} S}{4n} = \frac{\bar{\tau}}{4n(4n-1)}$. Then having in mind (34) $R = \frac{\bar{\tau}}{4n(4n-1)} \pi_1$ holds true. From here it is clear that if $S \in L_1$ then $(M, H, G)$ is an Einstein manifold.

Let us consider the case when $S \in L_2$. Then according to (33) $\text{tr} S$ vanishes, and from (34) $\tau$ vanishes, too. Because of $g_3 \in L_2$ we consider $S = \lambda g_3$, whence $\lambda = -\frac{\text{tr} (2Sg_3)}{4n}$. Then (33) implies $R = \frac{\text{tr} (2Sg_3)}{4n} \pi_3$, where $\pi_3$ is the following tensor $\pi_3$ with respect to the complex structure $J = J_3$

$$\pi_3(X, Y, Z, U) = -\pi_1(X, Y, JZ, U) - \pi_1(X, Y, JZ, U).$$

It is known that $\pi_3$ is a Kähler curvature-like tensor, i.e. it satisfies the property $\pi_3(X, Y, JZ, U) = -\pi_3(X, Y, Z, U)$. Therefore in this case $R$ is Kählerian with respect to $J_3$ and the tensor $R^{*J_3} : R^{*J_3}(X, Y, Z, U) = R(X, Y, Z, J_3U)$ is curvature-like. Then we obtain immediately

$$R = \frac{\tau(R^{*J_3})}{8n(2n-1)} \pi_3, \quad \rho = -\frac{\tau(R^{*J_3})}{4n} g_3.$$

Hence if $S \in L_2$ then $(M, H, G)$ is a $\ast$-Einstein manifold with respect to $J_3$.

By an analogous way, in the case when $S \in L_3$ we receive that $(M, H, G)$ is a $\ast$-Einstein manifold with respect to $J_2$. 

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4  A 4-dimensional pseudo-Riemannian spherical manifold with 
\((H,G)\)-structure

In \cite{4} is considered a hypersurface \(S^4_2\) in \(\mathbb{R}^5_2\) by the equation
\[
-(z^1)^2 - (z^2)^2 + (z^3)^2 + (z^4)^2 + (z^5)^2 = 1, \tag{37}
\]
where \(Z (z^1, z^2, z^3, z^4, z^5)\) is the positional vector of \(p \in S^4_2\).

Let \((u^1, u^2, u^3, u^4)\) be local coordinates of \(p \in S^4_2\). The hypersurface \(S^4_2\) is defined by the scalar parametric equations:
\[
\begin{align*}
z^1 &= \sinh u^1 \cos u^2, \\
z^2 &= \sinh u^1 \sin u^2, \\
z^3 &= \cosh u^1 \cos u^3 \cos u^4, \\
z^4 &= \cosh u^1 \cos u^3 \sin u^4, \\
z^5 &= \cosh u^1 \sin u^3.
\end{align*} \tag{38}
\]

Further we consider the manifold on \(\hat{S}^4_2 = S^4_2 \setminus \{(0,0,0,0,\pm 1)\}\), i.e. we omit two points for which \(\{u^1 \neq 0\} \cap \{u^3 \neq (2k + 1)\pi/2, k \in \mathbb{Z}\}\). The tangent space \(T_p\hat{S}^4_2\) of \(\hat{S}^4_2\) at \(p \in \hat{S}^4_2\) is determined by the vectors \(z_i = \frac{\partial}{\partial u^i}(i = 1, 2, 3, 4)\). The vectors \(z_i\) are linearly independent on \(\hat{S}^4_2\), defined by \((38)\), and \(T_p\hat{S}^4_2\) has a basis \((z_1, z_2, z_3, z_4)\) in every point \(p \in \hat{S}^4_2\).

The restriction of \((\cdot, \cdot)\) from \(\mathbb{R}^5_2\) to \(\hat{S}^4_2\) is a pseudo-Riemannian metric \(g\) on \(\hat{S}^4_2\) with signature \((2,2)\). The non-zero components \(g_{ij} = (z_i, z_j)\) are
\[
g_{11} = -1, \quad g_{22} = -\sinh^2 u^1, \quad g_{33} = \cosh^2 u^1, \quad g_{44} = \cosh^2 u^3 \cosh^2 u^3. \tag{39}
\]

The hypersurface \(S^4_2\) is equipped with an almost hypercomplex structure \(H = (J_\alpha), (\alpha = 1, 2, 3)\), where the non-zero components of the matrix of \(J_\alpha\) with respect to the local basis \(\{\frac{\partial}{\partial u^i}\}_{i=1}^4\) are
\[
\begin{align*}
(J_1)^1_2 &= -\frac{1}{\cosh u^1} = -\sinh u^1, \\
(J_1)^3_4 &= -\frac{1}{\cosh u^3} = \cos u^3, \\
(J_2)^1_3 &= -\frac{1}{\cosh u^1} = -\sinh u^1, \\
(J_2)^2_4 &= -\frac{1}{\cosh u^3} = -\cosh u^1 \cos u^3, \\
(J_3)^1_4 &= -\frac{1}{\cosh u^1} = \cosh u^1 \cos u^3, \quad (J_3)^2_3 = -\frac{1}{\cosh u^3} = \tanh u^1.
\end{align*} \tag{40}
\]

**Theorem 4.1** \cite{4} The spherical pseudo-Riemannian 4-dimensional manifold, defined by \((38)\), admits a hypercomplex pseudo-Hermitian structure on \(\hat{S}^4_2\), determined by \((39)\) and \((40)\), with respect to which it is of the class \(\mathcal{W}(J_1)\) but it does not belong to \(\mathcal{W}\) and it has a constant sectional curvature \(k = 1\).

Let us consider a conformal transformation determined by the function \(u\) which is a solution of the equation \(du = -\frac{1}{2(2n-1)}(\theta_1 \circ J_1)\), where the nonzero component of \(\theta_1\) with respect to the local basis \(\{\frac{\partial}{\partial u^i}\}_{i=1}^4\) is \(\theta_1 \left(\frac{\partial}{\partial u^i}\right) = 2\sinh^2 u^1 \cosh u^1\).

Since \(\hat{S}^4_2\) has a constant sectional curvature then the Weil tensor is vanishes, i.e. \(\hat{S}^4_2\) is \(C\)-equivalent to a flat \(\mathcal{K}(J_1)\)-manifold. If we admit that it is in
\(K\), then according to Theorem 3.2 we obtain that the manifold \((\tilde{S}_2^4, H, G) \in W\) which is a contradiction. Therefore the considered manifold is \(C\)-equivalent to a flat \(K(J_1)\)-manifold, but it is not a pseudo-hyper-Kähler manifold. By direct verification we state that the tensor \(S\) of this conformal transformation belongs to \(L_1\). Therefore \((\tilde{S}_2^4, H, G)\) is an Einstein manifold.

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