Embeddings of Representations

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Abstract We derive “numerical” criteria for the existence of embeddings of representations of finite dimensional algebras.

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1 Introduction

By a classical result of M. Auslander [1, 2], a finite dimensional representation $M$ of a finitely generated algebra $A$ is determined up to isomorphism by the dimensions of homomorphism spaces to it, that is, two such representations $M$ and $N$ of $A$ are isomorphic if and only if $\dim \text{Hom}(U, M) = \dim \text{Hom}(U, N)$ for all (indecomposable) representations $U$ of $A$.

Presented by Claus Michael Ringel.

Dedicated to Klaus Bongartz on the occasion of his 65th birthday

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In light of this fact, one can ask for “numerical” criteria for representation-theoretic properties. One example is the characterization of degenerations [3] \( M \leq_{\text{deg}} N \) of representations of algebras of finite representation type by the condition \( \dim \text{Hom}(U, M) \leq \dim \text{Hom}(U, N) \) for all \( U \) [13].

The aim of the present paper is to prove numerical criteria for situations related to embeddings of representations. This question is motivated by a study of quiver Grassmannians for representations of Dynkin quivers, for which specific geometric properties can be expected (in contrast to arbitrary quiver Grassmannians, see [11]). The first step in this direction is a criterion for nonemptyness of a quiver Grassmannian, which will be proven in the following form:

**Theorem 1.1** A representation \( M \) of a Dynkin quiver \( Q \) with associated Euler form \( \langle \cdot, \cdot \rangle \) admits a subrepresentation of dimension vector \( e \) if and only if \( \dim \text{Hom}(U, M) \geq \langle \dim U, e \rangle \) for all (indecomposable) representations \( U \) of \( Q \).

Not directly related, but in the same spirit, we find a quite general sufficient criterion for irreducibility of a Dynkin quiver Grassmannian:

**Theorem 1.2** Given a dimension vector \( e \) and a representation \( M \) as before such that the following inequalities hold:

(i) \( \dim \text{Hom}(M, U) \leq \langle e, \dim U \rangle \) for all non-injective indecomposable \( U \),

(ii) \( \dim \text{Hom}(U, M) \leq \langle \dim U, \dim M - e \rangle \) for all non-projective indecomposables \( U \),

the quiver Grassmannian \( Gr_e(M) \) is irreducible of dimension 

\[ \dim Gr_e(M) = \langle e, \dim M - e \rangle. \]

Both results were predicted by extensive numerical experiments for a type \( A_3 \) quiver in the first named author’s master thesis [7].

The other main topic of this paper concerns the much finer problem to numerically characterize embeddings between two given representations. In this direction, we prove

**Theorem 1.3** Let \( A \) be an arbitrary finite dimensional algebra over an algebraically closed field \( k \), and let \( M \) and \( N \) be finite dimensional representations of \( A \). Then the following are equivalent:

(i) For all large enough \( r \geq 1 \), there exists an embedding \( N^r \to M^r \),

(ii) for all surjections \( U \to V \) of representations of \( A \), we have

\[ \dim \text{Hom}(U, N) - \dim \text{Hom}(V, N) \leq \dim \text{Hom}(U, M) - \dim \text{Hom}(V, M), \quad (1.1) \]

(iii) the estimate (1.1) holds for all quotients \( U = N^k \to N^k/S = V \), where \( S \) is a simple subrepresentation of \( N^k \) and \( k \leq \dim \text{Hom}(S, N) \).

Note that the numerical condition (1.1) is insensitive to multiplicities, so that one cannot expect to characterize existence of an actual embedding \( N \subset M \) in general. In fact, in Section 3, we will exhibit a (low-dimensional) example of representations \( N \) and \( M \) over the three-arrow Kronecker quiver such that \( N^2 \) can be embedded into \( M^2 \), but \( N \) cannot be embedded into \( M \). It is rather natural to ask for which algebras the condition (1.1) already characterizes embeddings \( N \subset M \); at least this holds for an equioriented type \( A \) quiver, see Section 3.