A NOTE ON CERTAIN REAL QUADRATIC FIELDS WITH CLASS NUMBER UP TO THREE

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Abstract. We obtain criteria for the class number of certain Richaud–Degert type real quadratic fields to be three. We also treat a couple of families of real quadratic fields of Richaud–Degert type that were not considered earlier, and obtain similar criteria for the class number of such fields to be two and three.

1. Introduction

The size of the class group of an algebraic number field is one of the fundamental problems in algebraic number theory. Gauss conjectured that there are exactly nine imaginary quadratic fields with class number one. This conjecture was proved independently by Baker [2] and Stark [25]. However, Heegner had already proved this conjecture in [13]. Unfortunately, his proof was regarded as incorrect or, at the best, incomplete. Stark found that the gap in the proof is very minor and he had completed the same in [26]. In fact, Gauss gave a list of imaginary quadratic fields with given very low class numbers, and he believed them to be complete. The list of imaginary quadratic fields with class number two was completely classified by Baker and Stark independently in [3] and [27], respectively, and jointly in [4]. The analogous list of imaginary quadratic fields with class number three was computed by Oesterlé in [23]. Finally, Watkins [28] classified all the imaginary quadratic fields with class numbers up to 100.

On the other hand, very little is known about the class number of real quadratic fields. In 1801, Gauss conjectured the following:

(G1) There exist infinitely many real quadratic fields of class number one.

Or more precisely:

(G2) There exist infinitely many real quadratic fields of the form \( \mathbb{Q}(\sqrt{p}) \), \( p \equiv 1 \pmod{4} \), of class number one.

This conjecture is yet to be resolved. It seems that one of the most essential difficulties of this problem comes from the deep connection of the class number with the fundamental unit.

In connection to (G2), Chowla and Friedlander [11] posted the following conjecture.

(CF) If \( D = m^2 + 1 \) is a prime with \( m > 26 \), then the class number of \( \mathbb{Q}(\sqrt{D}) \) is greater than one.

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This conjecture says that there are exactly seven real quadratic fields of the form \( \mathbb{Q}(\sqrt{m^2 + 1}) \) with class number one, and they correspond to \( m \in \{1, 2, 4, 6, 10, 14, 26\} \). In 1988, Mollin and Williams [22] proved this conjecture under the generalized Riemann hypothesis.

Chowla also posted a conjecture analogous to (CF) on a general family of real quadratic fields. More precisely, he conjectured the following.

(C) Let \( D \) be a square-free rational integer of the form \( D = 4m^2 + 1 \) for some positive integer \( m \). Then there exist exactly six real quadratic fields \( \mathbb{Q}(\sqrt{D}) \) of class number one, viz. \( D \in \{5, 17, 37, 101, 197, 677\} \).

Yokoi [29] studied this conjecture and he posted one more conjecture on another family of real quadratic fields. More precisely, he posted the following conjecture.

(Y) Let \( D \) be a square-free rational integer of the form \( D = m^2 + 4 \) for some positive integer \( m \). Then there exist exactly six real quadratic fields \( \mathbb{Q}(\sqrt{D}) \) of class number one, viz. \( D \in \{5, 13, 29, 53, 173, 293\} \).

Kim, Leu and Ono [14] proved that at least one of (C) and (Y) is true, and that there are at most seven real quadratic fields \( \mathbb{Q}(\sqrt{D}) \) of class number one for the other case. The conjectures (C) and (Y) were proved by Biró [5, 6]. Hoque and Saikia [15] proved that there do not exist any real quadratic fields of the form \( \mathbb{Q}(\sqrt{9(8n^2 + r) + 2}) \), where \( n \geq 1 \) and \( r = 5, 7 \) with class number one. In [16], two of the present authors proved that there are no real quadratic fields \( \mathbb{Q}(\sqrt{d}) \) of class number one when \( d = n^2p^2 + 1 \) with \( p \equiv \pm 1 \pmod{8} \) a prime and \( n \) an odd integer. Recently, Chakraborty and Hoque [10] proved that, if \( d \) is a square-free part of \( an^2 + 2 \), where \( a = 9, 196 \) and \( n \) is an odd integer, then the class number of \( \mathbb{Q}(\sqrt{d}) \) is greater than one.

It is more interesting to find necessary and sufficient conditions that a real quadratic field has a given fixed class number \( g \). Yokoi [29] proved using an algebraic method that, for a positive integer \( m \), the class number of \( \mathbb{Q}(\sqrt{3m^2 + 1}) \) is one if and only if \( m^2 - t(t + 1) \) is a prime for every \( 1 \leq t \leq m - 1 \). Lu [20] obtained this result using the theory of continued fractions. Kobayashi [18] obtained stronger conditions for this as well as some other families of real quadratic fields to be of class number one. In [7], Byeon and Kim established certain necessary and sufficient conditions for the class number of real quadratic fields of Richaud–Degert type to be one. They obtained [8] these conditions by comparing the special zeta values attached to a real quadratic field determined by two different methods of computation. Analogously, they also obtained some necessary and sufficient conditions for the class number of real quadratic fields of Richaud–Degert type to be two. Mollin [21] also obtained some analogous conditions for the class number to be two using the theory of continued fractions and algebraic arguments.

In this paper, we consider all real quadratic fields of narrow Richaud–Degert type with two exceptions. More precisely, we consider the real quadratic fields \( k = \mathbb{Q}(\sqrt{d}) \), where \( d = n^2 + r \) and \( |r| \in \{1, 4\} \) with the exceptions when \( n^2 - 1 \equiv 3 \pmod{4} \) and \( n^2 - 4 \equiv 5 \pmod{8} \). We also consider wide Richaud–Degert type real quadratic fields \( \mathbb{Q}(\sqrt{d}) \), where \( d = n^2 + r \) and \( r \notin \{1, 4\} \) with \( d \equiv 1 \pmod{8} \). We obtain some criteria for the class number of these fields to be three. We also obtain similar criteria for \( \mathbb{Q}(\sqrt{n^2 + r}) \), \( r \in \{1, 4\} \) to have class number two, which were not covered by Byeon and Kim [8]. We largely follow the method of Byeon and Kim [7, 8].
2. Values of Dedekind zeta function

In this section, we discuss two different ways of computing special values of zeta functions attached to a real quadratic field that are due to Siegel [24] and Lang [19]. Let $k$ be a real quadratic field, and $\zeta_k(s)$ be the Dedekind zeta function of $k$. By specializing Siegel’s formula [24] for $\zeta_k(1-2n)$ for general $k$, Zagier [30] described this formula by direct analytic methods when $k$ is a real quadratic field. For $n = 1$, it takes the following form (see Section 3 in [30]).

**Theorem 2.1.** (Zagier [30]) Let $k$ be a real quadratic field with discriminant $D$. Then

$$\zeta_k(-1) = \frac{1}{60} \sum_{|t| < \sqrt{D}} \sigma\left(\frac{D - t^2}{4}\right),$$

where $\sigma(n)$ denotes the sum of divisors of $n$.

Another method of computing special values of $\zeta_k(s)$ is due to Lang whenever $k$ is a real quadratic field. Let $k = \mathbb{Q}(\sqrt{D})$ be a real quadratic field with discriminant $D$, and let $\mathfrak{A}$ be an ideal class in $k$. Let $\mathfrak{a}$ be an integral ideal in $\mathfrak{A}^{-1}$ with an integral basis $\{r_1, r_2\}$. We define

$$\delta(\mathfrak{a}) = r_1 r_2' - r_1' r_2,$$

where $r_1'$ and $r_2'$ are the conjugates of $r_1$ and $r_2$, respectively.

Let $\varepsilon$ be the fundamental unit of $k$. Then $\{\varepsilon r_1, \varepsilon r_2\}$ is also an integral basis of $\mathfrak{a}$, and thus we can find a matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with integer entries satisfying

$$\varepsilon \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = M \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}.$$

We can now recall the following result of Lang [19], which is one of the main ingredients to prove our results.

**Theorem 2.2.** (Lang [19]) By keeping the above notation, we have

$$\zeta_k(-1, \mathfrak{A}) = \frac{\text{sgn} \delta(\mathfrak{a})}{360 N(\mathfrak{a}) c^3} \left\{ (a + d)^3 - 6(a + d) N(\varepsilon) - 240c^3 (\text{sgn} c) S^3(a, c) ight.$$

$$\left. + 180ac^3 (\text{sgn} c) S^2(a, c) - 240c^3 (\text{sgn} c) S^3(d, c) + 180dc^3 (\text{sgn} c) S^2(d, c) \right\},$$

where $N(\mathfrak{a})$ is the norm of $\mathfrak{a}$ and $S^i(\cdot, \cdot)$ denotes the generalized Dedekind sum as defined in [1].

We need to determine the values of $a$, $b$, $c$ and $d$ and the generalized Dedekind sum in order to apply Theorem 2.2. The following result of Kim [17] helps us to determine the values of $a$, $b$, $c$ and $d$. 

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LEMMA 2.1. (Kim [17]) The entries of $M$ are given by
\[
a = \text{Tr}\left(\frac{r_1 r'_2 \epsilon}{\delta(a)}\right), \quad b = \text{Tr}\left(\frac{r_1 r'_2 \epsilon}{\delta(a)}\right), \quad c = \text{Tr}\left(\frac{r_2 r'_1 \epsilon}{\delta(a)}\right) \quad \text{and} \quad d = \text{Tr}\left(\frac{r_1 r'_2 \epsilon}{\delta(a)}\right).
\]
Moreover, $\det(M) = N(\epsilon)$ and $bc \neq 0$.

Kim [17] obtained the following expressions for the generalized Dedekind sum by using the reciprocity law. These expressions are also needed to compute the values of zeta functions for ideal classes of the respective real quadratic fields.

LEMMA 2.2. (Kim [17]) For any positive integer $m$, we have
(i) $S^3(\pm 1, m) = \pm(-m^4 + 5m^2 - 4)/(120m^3),$
(ii) $S^3(\pm 1, m) = (m^4 + 10m^2 - 6)/(180m^3).$

LEMMA 2.3. (Kim [17]) For any positive even integer $m$, we have
(i) $S^3(m \pm 1, 2m) = \pm S^1(m + 1, 2m) = \mp(m^4 - 50m^2 + 4)/(960m^3),$
(ii) $S^2(m - 1, 2m) = S^2(m + 1, 2m) = (m^4 + 100m^2 - 6)/(1440m^3),$
(iii) $S^3(m + 1, 4m) = (-m^4 - 180m^3 + 310m^2 - 4)/(7680m^3),$
(iv) $S^3(m - 1, 4m) = (m^4 - 180m^3 - 310m^2 + 4)/(7680m^3),$
(v) $S^2(m - 1, 4m) = S^2(m + 1, 4m) = (m^4 + 820m^2 - 6)/(11520m^3).$

3. Real quadratic fields with class number three

In this section, we compute the value $\zeta_k(-1, \mathfrak{A})$ for some ideal class $\mathfrak{A}$ in $k$, and then compare these values to $\zeta_k(-1)$ to derive our results. Throughout this section, $k$ is a real quadratic field of Richaud–Degert (RD) type; more precisely, $k = \mathbb{Q}(\sqrt{d})$ with radicand $d = n^2 + r$ satisfying $r|4n$ and $-n < r \leq n$. Degert [12, Satz 1] shows that the fundamental unit $\epsilon$ of $k$ and its norm $N(\epsilon)$ are

$$
\epsilon = \begin{cases} 
\frac{n + \sqrt{n^2 + r}}{2}, & N(\epsilon) = -\text{sgn} \, r, \text{ if } |r| = 1, \\
\frac{2n^2 + r + \sqrt{n^2 + r}}{|r|}, & N(\epsilon) = -\text{sgn} \, r, \text{ if } |r| = 4, \\
\frac{2n^2 + r + \sqrt{n^2 + r}}{|r|}, & N(\epsilon) = 1, \quad \text{if } |r| \neq 1, 4.
\end{cases}
$$

(3.1)

It is easy to see that $n^2 + 1 \not\equiv 3 \pmod{4}$, $n^2 - 1 \not\equiv 1 \pmod{4}$, $n^2 + 4 \not\equiv 2 \pmod{4}$, $n^2 - 4 \not\equiv 1 \pmod{8}$. Thus to cover all real quadratic fields of narrow RD type, it is enough to consider the following cases:
(i) $n^2 + 1 \equiv 1, 2 \pmod{4},$
(ii) $n^2 - 1 \equiv 3 \pmod{4},$
(iii) $n^2 + 4 \equiv 1 \pmod{4},$
(iv) $n^2 - 4 \equiv 5 \pmod{8}.$

We consider the real quadratic field $k = \mathbb{Q}(\sqrt{d})$ of RD type with $d \equiv 1 \pmod{8}$. Then 2 splits in $k$, that is,

$$
(2) = \left(2, \frac{1 + \sqrt{d}}{2}\right)\left(2, \frac{1 - \sqrt{d}}{2}\right).
$$

Note that $n$ is even if $|r| \neq 1, 4$. In fact, when $n$ is odd, one has that $1 \equiv d = n^2 + r \equiv r + 1 \pmod{8}$, which is contrary to the assumption $r \mid 4n$. The case $|r| = 4$ cannot occur since $n^2 \pm 4 \neq 1 \pmod{8}$. We extract the following result from Theorem 2.3 of [7].

**Theorem 3.1.** (Byeon and Kim [7]) Let $d = n^2 + r$, and let $k = \mathbb{Q}(\sqrt{d})$ be a real quadratic field of RD type. Let $\mathfrak{P}$ denote the ideal class of principal ideals of $k$. If $d \equiv 1 \pmod{8}$, then

$$\zeta_k(-1, \mathfrak{P}) = \begin{cases} \frac{n^3 + 14n}{360}, & \text{if } |r| = 1, \\ \frac{2n^3(r^2 + 1) + 3n(3r^3 + 50r^2 + 3r)}{720r^2}, & \text{if } |r| \neq 1, 4. \end{cases}$$

The following result can be extracted from [8, Theorem 2.5]. However, for the sake of completeness, we provide a detailed proof.

**Theorem 3.2.** (Byeon and Kim [8]) Let $d = n^2 + r$, and let $k = \mathbb{Q}(\sqrt{d})$ be a real quadratic field of RD type. Let $\mathfrak{A}$ be the ideal class containing $(2, (1 + \sqrt{d})/2)$ or $(2, (1 - \sqrt{d})/2)$. If $d \equiv 1 \pmod{8}$, then

$$\zeta_k(-1, \mathfrak{A}) = \begin{cases} \frac{n^3 + 104n}{1440}, & \text{if } |r| = 1, \\ \frac{2n^3(r^2 + 1) + 3n(3r^3 + 410r^2 + 3r)}{2880r^2}, & \text{if } |r| \neq 1, 4. \end{cases}$$

**Proof.** Let us assume that $a := (2, (1 + \sqrt{d})/2) \in \mathfrak{A}^{-1}$. Then $\{r_1 = (1 + \sqrt{d})/2, r_2 = 2\}$ is an integral basis for $a$ and thus $\delta(a) = 2\sqrt{d}$. We will give computations in detail for $|r| = 1$, and similar argument goes through for other cases. By Lemma 2.1, we get

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} n + 1 & (d - 1)/4 \\ 4 & n - 1 \end{pmatrix}.$$ 

Since $n^2 + 1 \equiv 1 \pmod{8}$, so that $4 | n$, and thus $n \pm 1 \equiv \pm 1 \pmod{4}$. Hence by Lemma 2.2, we obtain

\[
\begin{align*}
240c^3(\text{sgn } c)S^3(a, c) &= 240c^3S^3(n + 1, 4) = 240 \times 4^3S^3(1, 4) = -360, \\
240c^3(\text{sgn } c)S^3(d, c) &= 240c^3S^3(n - 1, 4) = 240 \times 4^3S^3(-1, 4) = 360, \\
180ac^3(\text{sgn } c)S^2(a, c) &= 180ac^3S^2(n + 1, 4) = 180 \times 4^3aS^2(1, 4) = 410(n + 1), \\
180dc^3(\text{sgn } c)S^2(d, c) &= 180dc^3S^2(n - 1, 4) = 180 \times 4^3dS^2(-1, 4) = 410(n - 1).
\end{align*}
\]

By Theorem 2.2, we get

$$\zeta_k(-1, \mathfrak{A}) = \frac{n^3 + 104n}{1440}. \qed$$

Let $h(d)$ denote the class number of $\mathbb{Q}(\sqrt{d})$. 

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**Real quadratic fields with class number up to three**
THEOREM 3.3. Let \( d = n^2 + 1 \equiv 1 \pmod{8} \) be a square-free integer. If \( h(d) = 3 \) then

\[
\sum_{|t|<\sqrt{d}} \sigma\left(\frac{d - t^2}{4}\right) = \frac{n^3 + 44n}{4}.
\]

The converse holds if \( h(d) \) is odd with one exception, viz. \( d = 17 \).

Proof. Let us assume that the class group of \( k = \mathbb{Q}(\sqrt{d}) \) is \( \mathcal{C}(k) = \{\mathfrak{P}, \mathfrak{A}, \mathfrak{B}\} \) with principal ideals class \( \mathfrak{P} \).

Then, by Theorem 3.1, we have

\[
\zeta_k(-1, \mathfrak{P}) = \frac{n^3 + 14n}{360}.
\]

If \( 2, (1 \pm \sqrt{d})/2 \in \mathfrak{A}^{-1} = \mathfrak{B} \), then, by Theorem 3.2, we see that \( \zeta_k(-1, \mathfrak{P}) = \zeta_k(-1, \mathfrak{A}) \) if and only if \( d = 17 \). Thus \( (2, (1 - \sqrt{d})/2) \) and \( (2, (1 + \sqrt{d})/2) \) are non-principal ideals except for \( d = 17 \).

Let \( (2, (1 - \sqrt{d})/2) \in \mathfrak{A} \) and \( (2, (1 + \sqrt{d})/2) \in \mathfrak{B} \). Then, by Theorem 3.2, we obtain

\[
\zeta_k(-1, \mathfrak{A}) = \zeta_k(-1, \mathfrak{B}) = \frac{n^3 + 104n}{1440}.
\]

As \( \mathcal{C}(k) = \{\mathfrak{P}, \mathfrak{A}, \mathfrak{B}\} \), we obtain

\[
\zeta_k(-1) = \zeta_k(-1, \mathfrak{P}) + \zeta_k(-1, \mathfrak{A}) + \zeta_k(-1, \mathfrak{B}) = \frac{n^3 + 44n}{240}.
\]

We now apply Theorem 2.1 to get

\[
\sum_{|t|<\sqrt{d}} \sigma\left(\frac{d - t^2}{4}\right) = \frac{n^3 + 44n}{4}.
\]

The converse part implies

\[
\zeta_k(-1) = \frac{n^3 + 44n}{240}.
\]

Then by [7, Theorem 2.4] and [8, Theorem 2.7], we obtain \( h(d) \geq 3 \). If \( h(d) > 3 \), then there exist at least five ideal classes in \( k \) since \( h(d) \) is odd. If \( \mathcal{C} \) and \( \mathcal{D} \) are another two ideal classes in \( k \), then

\[
\zeta_k(-1) \geq \zeta_k(-1, \mathfrak{P}) + \zeta_k(-1, \mathfrak{A}) + \zeta_k(-1, \mathfrak{B}) + \zeta_k(-1, \mathcal{C}) + \zeta_k(-1, \mathcal{D}),
\]

where the equality holds if \( h(d) = 5 \). Without loss of generality, let us assume that \( (2, (1 - \sqrt{d})/2) \in \mathfrak{A} \) and \( (2, (1 + \sqrt{d})/2) \in \mathfrak{B} \). Then, by Theorem 3.2, we obtain

\[
\zeta_k(-1, \mathfrak{A}) = \zeta_k(-1, \mathfrak{B}) = \frac{n^3 + 104n}{1440}.
\]

Since for any ideal class \( \mathfrak{D} \), \( \zeta_k(-1, \mathfrak{D}) > 0 \), thus by (3.3) we obtain

\[
\zeta_k(-1) > \zeta_k(-1, \mathfrak{P}) + \zeta_k(-1, \mathfrak{A}) + \zeta_k(-1, \mathfrak{B}) = \frac{n^3 + 44n}{240},
\]

which contradicts (3.2). This completes the proof.

We can prove the following result using a similar argument as in Theorem 3.3, and by using Lemma 2.3.
THEOREM 3.4. Let \( n \) be a positive integer and \( d = n^2 + 1 \equiv 2 \pmod{4} \) be square-free. Let \( p \) be a prime divisor of \( n \). If \( h(d) = 3 \), then
\[
\sum_{|t| < \sqrt{d}} \frac{\sigma\left(d - t^2\right)}{t^2 \equiv d \pmod{4}} = \frac{2n^3 + 13n}{3} + \frac{8n^3 + 2n(p^4 + 10p^2)}{3p^2}.
\]
The converse holds if \( h(d) \) is odd with one exception, viz. \( d = 2 \).

We now consider a family of real quadratic fields of wide RD type, and deduce similar criteria for class number three. The proof of the following result goes along similar lines to the proof of Theorem 3.3.

THEOREM 3.5. Let \( d = n^2 + r \equiv 1 \pmod{8} \) be a square-free integer with \(|r| \neq 1, 4\). If \( h(d) = 3 \), then
\[
\sum_{|t| < \sqrt{d}} \frac{\sigma\left(d - t^2\right)}{t^2 \equiv d \pmod{4}} = \frac{2n^3(r^2 + 1) + n(3r^3 + 170r^2 + 3r)}{8r^2}.
\]
The converse holds if \( h(d) \) is odd with one exception, viz. \( d = 33 \).

4. Real quadratic fields with class numbers two and three

In this section, we obtain class number two and three criteria for the real quadratic fields \( \mathbb{Q}(\sqrt{n^2 + r}) \) when \( r \in \{1, 4\} \) and \( n^2 + r \equiv 5 \pmod{8} \). These two families were not considered in [8] to obtain the class number two criteria.

Let \( d = n^2 + 4 \equiv 5 \pmod{8} \), and let \( p \) be an odd prime divisor of \( n \). Then \( p \) splits in \( k = \mathbb{Q}(\sqrt{d}) \), that is, \((p) = pp'\), where \( p = (p, (p + 2 + \sqrt{d})/2) \) and \( p' = (p, (p + 2 - \sqrt{d})/2) \). Similarly if \( d = n^2 + 1 \equiv 5 \pmod{8} \), and \( q \) is an odd prime divisor of \( n \), then \( q \) also splits in \( k = \mathbb{Q}(\sqrt{d}) \), that is, \((q) = qq'\), where \( q = (q, (1 + \sqrt{d})/2) \) and \( q' = (q, (1 - \sqrt{d})/2) \).

We can prove the following result using a similar argument to the proof of Theorem 3.2.

THEOREM 4.1. Let \( d = n^2 + r \equiv 5 \pmod{8} \) be square-free with \( r = 1, 4 \), and let \( k = \mathbb{Q}(\sqrt{d}) \). Let \( p \) be an odd prime divisor of \( n \). If \( \mathfrak{A} \) is the ideal class containing one of \( p \), \( p' \), \( q \) and \( q' \) (as defined above), then
\[
\zeta_k(-1, \mathfrak{A}) = \begin{cases} 
(n^3 + n(p^4 + 10p^2))/(360p^2), & \text{if } r = 4, \\
(n^3 + n(4q^4 + 10q^2))/(360q^2), & \text{if } r = 1.
\end{cases}
\]

Let \( \mathfrak{B} \) be the ideal class of principal ideals in \( k \). Then
\[
\zeta_k(-1, \mathfrak{B}) = \begin{cases} 
(n^3 + 11n)/360, & \text{if } r = 4, \\
(n^3 + 14n)/360, & \text{if } r = 1.
\end{cases}
\]

Thus \( h(d) > 1 \) if \( \zeta_k(-1, \mathfrak{A}) \neq \zeta_k(-1, \mathfrak{A}) \). On the other hand, \( \zeta_k(-1, \mathfrak{A}) = \zeta_k(-1, \mathfrak{A}) \) implies that
\[
n = \begin{cases} p, & \text{if } r = 4, \\
2q, & \text{if } r = 1.
\end{cases}
\]
Remark 1. Let $d$ be as in Theorem 4.1. If $h(d) = 1$, then $d$ must be of the form either $p^2 + 4$ or $4p^2 + 1$.

This remark does not provide any information about the conjectures (C) and (Y). One can prove the following result using a similar argument to the proof of Theorem 3.3.

Theorem 4.2. Let $k$ and $p$ be as in Theorem 4.1. If $h(d) = 3$ then

$$\sum_{|t| < \sqrt{d}} \sigma \left( \frac{d - t^2}{4} \right) = \begin{cases} \frac{n^3 + 11n}{6} + \frac{n^3 + n(p^4 + 10p^2)}{3p^2}, & \text{if } r = 4 \text{ and } n \neq p, \\ \frac{n^3 + 14n}{6} + \frac{n^3 + n(4p^4 + 10p^2)}{3p^2}, & \text{if } r = 1 \text{ and } n \neq 2p. \end{cases}$$

The converse holds if $h(d)$ is odd.

Along the same line, we obtain the following criteria for class number two.

Theorem 4.3. Let $k$ and $p$ be as in Theorem 4.1. Then $h(d) = 2$ if and only if

$$\sum_{|t| < \sqrt{d}} \sigma \left( \frac{d - t^2}{4} \right) = \begin{cases} \frac{n^3 + 11n}{6} + \frac{n^3 + n(p^4 + 10p^2)}{6p^2}, & \text{if } r = 4 \text{ and } n \neq p, \\ \frac{n^3 + 14n}{6} + \frac{n^3 + n(4p^4 + 10p^2)}{6p^2}, & \text{if } r = 1 \text{ and } n \neq 2p. \end{cases}$$

Note that Byeon and Lee [9] proved that, if $d = n^2 + 1$ is an even square-free integer with $d > 362$, then $h(d) \geq 3$. In particular, they proved that $d = 10, 26, 122$ and $362$ are the only values of $d$ for which $h(d) = 2$.

5. Computations and concluding remarks

In this section, we give some numerical examples which verify our results in Sections 3 and 4. We use SAGE version 8.4 (2018-10-17) for all the computations in this paper. We have computed $h(d)$ and verified Theorem 3.3 for $d \leq 10^{10}$ when $d$ is composite, and $d \leq 10^{13}$ when $d$ is prime. We have obtained only one $d$, viz. $d = 257$, with $h(d) = 3$ under the assumptions of this theorem.

In the case of Theorem 3.5, we have computed $h(d)$ for $n \leq 10^4$ and $|r| \leq 4 \times 10^4$. Out of these, we have obtained only two fields with $h(d) = 3$. These values are listed in Table 1, and Theorem 3.5 is verified for all these values.

We have computed $h(d)$ for $d \leq 10^8$ satisfying the conditions in Theorem 4.2. We have listed in Table 2 only those values which correspond to $h(d) = 3$. We have verified the equation in Theorem 4.2 by computation for the values listed in Table 2. There are only five

| $n$ | $r$ | $d$ | $h(d)$ |
|-----|-----|-----|--------|
| 18  | -3  | 321 | 3      |
| 22  | -11 | 473 | 3      |
real quadratic fields of the form $\mathbb{Q}(\sqrt{n^2 + r})$ satisfying $n^2 + r \equiv 5 \pmod{8}$ with $r = 1, 4$ and $n^2 + r \leq 10^8$. Out of these fields, one field is of the form $\mathbb{Q}(\sqrt{n^2 + 1})$ and three fields are of the other form.

Similarly, we have computed $h(d)$ for $d \leq 10^{10}$ satisfying the conditions in Theorem 4.3. There are only three real quadratic fields of the form $\mathbb{Q}(\sqrt{n^2 + r})$ satisfying $n^2 + r \equiv 5 \pmod{8}$ with $r = 1, 4$, $h(d) = 2$ and $n^2 + r \leq 10^{10}$. Out of these fields, no field is of the form $\mathbb{Q}(\sqrt{n^2 + 1})$ and two fields are of the other form. We have listed these values in Table 3, and verified the equation in Theorem 4.2 for them.

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