Duality of integral étale motivic cohomology

Thomas H. Geisser

Abstract

We discuss duality pairings on integral étale motivic cohomology groups of regular and proper schemes over algebraically closed fields, local fields, finite fields, and arithmetic schemes.

1 Introduction

Let $X$ be a smooth and projective variety of dimension $d$ over a perfect field $k$. Using duality theorems for étale cohomology with finite coefficients, we show duality results on integral étale motivic cohomology groups.

If $k$ is algebraically closed, and $m$ is an integer prime to the characteristic of $k$, we construct for all $n, u$ satisfying $w = n + u - d > 0$ a Galois invariant pairing

$$H^{2d+1-i}_{et}(X, \mathbb{Z}(u))/m \times mH^i_{et}(X, \mathbb{Z}(n)) \to \mathbb{Z}/m(w)$$

which is non-degenerate on the left. For the right kernel $mH^i_{et}(X, \mathbb{Z}(n))^0$, we obtain a secondary perfect pairing

$$mH^i_{et}(X, \mathbb{Z}(n))^0 \times mH^{2d+2-i}_{et}(X, \mathbb{Z}(u))^0 \to \mathbb{Z}/m(w),$$

and we show that for $n + u = d + 1, i = 2n$ and $2d + 2 - i = 2u$, this pairing becomes perfect when restricted to the divisible subgroups. Thus the pairing can be thought of as a generalization of the $e_m$-pairing between the Picard and Albanese abelian variety of $X$.

If $k$ is finite, we construct for all $m$ pairings

$$H^{2d+2-i}_{et}(X, \mathbb{Z}(d - n))/m \times mH^i_{et}(X, \mathbb{Z}(n)) \to \mathbb{Z}/m$$

which are non-degenerate on the left. We conjecture that the pairing is perfect, and relate this conjecture to Tate’s conjecture on the surjectivity of the cycle map.
If $k$ is local and $m$ prime to the characteristic of $k$, we construct pairings

$$H_{\text{et}}^{2d+3-i}(X,\mathbb{Z}(d+1-n))/m \times mH_{\text{et}}^i(X,\mathbb{Z}(n)) \to \mathbb{Z}/m$$

which are non-degenerate on the left. In case $d = 0$, the perfectness of the pairing is equivalent to the statement of local class field theory, and for $d = n = 1$ it amounts to Lichtenbaum’s duality between the Picard group and the Brauer group for curves over a $p$-adic field [14]. In contrast, we show that the pairing can have a right kernel for curves, and we give an example showing that the pairing can have a right kernel even in the good reduction case. In particular, there is no duality in some appropriate sense of the term” expected by Lichtenbaum [16, §6].

Notation: We denote Bloch’s motivic complex by $\mathbb{Z}(n)$, a complex of étale sheaves [4, Lemma 3.1]. When we need to emphasize that $\mathbb{Z}(n)$ is considered as a complex of Zariski sheaves, we write $\mathbb{Z}(n)^{\text{Zar}}$.

For an abelian group $A$, we denote by $mA$ its $m$-torsion, by $A\{l\} = \text{colim}_r lA$ its subgroup of $l$-power torsion elements, by $A^*$ its Pontrjagin dual $\text{Hom}(A,\mathbb{Q}/\mathbb{Z})$, by $A^\wedge = \lim_m A/m$ its completion, by $A^{\wedge l} = \lim_r A/l^r$ its the $l$-adic completion, and by $T_lA = \lim_r lA$ its $l$-adic Tate module.

## 2 Algebraically closed fields

Over an algebraically closed base field, Zariski and étale hypercohomology of the motivic complex agree in weights at least the dimension, i.e. if $\epsilon : X_{\text{et}} \to X_{\text{Zar}}$ is the change of topology map, then the adjunction map $\mathbb{Z}(n)^{\text{Zar}} \to R\epsilon_*\mathbb{Z}(n)$ is a quasi-isomorphism of complexes of sheaves for the Zariski-topology for $n \geq d$. This was deduced in [16] from a theorem of Suslin away from the characteristic and from [8] at the characteristic. Since the Zariski hypercohomology admits a push-forward map for the proper map $f : X \to k$, we obtain a Gysin map for étale motivic cohomology as the composition

$$Rf_*R\epsilon_*\mathbb{Z}(w+d)\mid X \cong Rf_*\mathbb{Z}(w+d)^{\text{Zar}} \to \mathbb{Z}(w)^{\text{Zar}} \cong R\epsilon_*\mathbb{Z}(w)_k.$$

If $k$ is a perfect field, then applying Galois cohomology $\Gamma(\text{Gal}(k),-)$ to this Gysin map over the base extension to the algebraic closure, we obtain a ”trace” map

$$H_{\text{et}}^{2d+v}(X,\mathbb{Z}(w+d)) \to H_{\text{et}}^v(k,\mathbb{Z}(w)).$$
for $X$ proper over $k$ and any $w \geq 0$.

The cup product pairing on higher Chow groups induces a product on \(\acute{e}tale\) hypercohomology, hence for $w = n + u - d$ we obtain a pairing

\[
H^{2d+v-i}_{\acute{e}t}(X,\mathbb{Z}(u)) \times H^i_{\acute{e}t}(X,\mathbb{Z}(n)) \rightarrow H^{2d+v}_{\acute{e}t}(X,\mathbb{Z}(u + n)) \xrightarrow{\partial} H^v_{\acute{e}t}(k,\mathbb{Z}(w)). \tag{4}
\]

If $k$ is algebraically closed, $m$ invertible in $k$, and $w \geq 1$, then the coefficient sequence gives an isomorphism

\[
\mathbb{Q}/\mathbb{Z}(w) \cong H^0_{\acute{e}t}(k,\mathbb{Q}/\mathbb{Z}(w)) \cong \text{Tor}H^1_{\acute{e}t}(k,\mathbb{Z}(w)).
\]

Indeed, this follows by comparing to Suslin’s calculation of the $K$-theory of an algebraically closed field. Restricting the pairing (4) for $v = 1$ to the $m$-torsion on the right, we obtain for

\[
n + u > d, \quad i + j = 2d
\]
a pairing

\[
H^j_{\acute{e}t}(X,\mathbb{Z}(u))/m \times_m H^{i+1}_{\acute{e}t}(X,\mathbb{Z}(n)) \rightarrow_m H^{2d+1}_{\acute{e}t}(X,\mathbb{Z}(u + n)) \rightarrow \mathbb{Z}/m(w).
\]

This is compatible with Poincare-duality for \(\acute{e}tale\) cohomology with finite coefficients

\[
\begin{array}{cccc}
H^j_{\acute{e}t}(X,\mathbb{Z}(u))/m \times_m H^{i+1}_{\acute{e}t}(X,\mathbb{Z}(n)) & \longrightarrow & mH^1_{\acute{e}t}(k,\mathbb{Z}(w)) & \cong \mathbb{Z}/m(w) \\
\downarrow & & \downarrow \partial & \cong \\
H^j_{\acute{e}t}(X,\mathbb{Z}/m(u)) \times H^{i}_{\acute{e}t}(X,\mathbb{Z}/m(n)) & \longrightarrow & H^0_{\acute{e}t}(k,\mathbb{Z}/m(w)) & \cong \mathbb{Z}/m(w) \\
\end{array}
\tag{5}
\]

because $\partial(x \cup y) = x \cup \partial(y)$ if $\partial(x) = 0$. Consequently, we obtain a map of short exact sequences of finite abelian groups

\[
\begin{array}{cccc}
H^j_{\acute{e}t}(X,\mathbb{Z}(u))/m & \longrightarrow & H^j_{\acute{e}t}(X,\mathbb{Z}/m(u)) & \longrightarrow & mH^{i+1}_{\acute{e}t}(X,\mathbb{Z}(u)) \\
\downarrow & & \downarrow & & \downarrow \\
mH^{i+1}_{\acute{e}t}(X,\mathbb{Z}(n))^2 & \longrightarrow & H^i_{\acute{e}t}(X,\mathbb{Z}/m(n))^2 & \longrightarrow & (H^i(X,\mathbb{Z}(n))/m)^2, \tag{6}
\end{array}
\]

where for any abelian group $A$, $A^2 = \text{Hom}(A,\mathbb{Q}/\mathbb{Z}(w))$ is the twist the usual Pontrjagin dual. The middle map is an isomorphism by Grothendieck’s
Poincaré duality for étale cohomology, so that the snake Lemma gives an exact sequence

\[ 0 \to H^i_{\text{et}}(X, \mathbb{Z}(u))/m \to (mH^{i+1}_{\text{et}}(X, \mathbb{Z}(n)))^\sharp \to \delta \to mH^{i+1}_{\text{et}}(X, \mathbb{Z}(u)) \to (H^i_{\text{et}}(X, \mathbb{Z}(n))/m)^\sharp \to 0. \quad (7) \]

In particular, the pairings (1) are non-degenerate on the left, and \( \delta \) induces the pairing (2). It is easy to see that the sequence (7) is compatible with varying \( m \).

**Remark 2.1** The construction of the pairing requires \( u + n \geq d + 1 \). For example, for \( X = \text{Spec} \, k \) and \( u = n = 0 \), the diagram (3) does not commute. The construction also does not work for \( m \) a power of the characteristic of the base field, because then \( \mathbb{Z}/m^w(\omega) = 0 \) for \( w > 0 \).

**The case** \( u + n = d + 1, i = 2n - 1, j = 2u - 1 \)

**Example.** (Rojtman’s theorem) If \( n = d, u = 1, i = 2d - 1, j = 1 \), then \( H^d_{\text{et}}(X, \mathbb{Z}(u))/m = k^\times/m = 0 \). Moreover, it was shown in [7] that \( H^{2d-1}_{\text{et}}(X, \mathbb{Z}(d)) \) modulo its divisible subgroup is isomorphic to the dual of \( \text{Tor}\,\text{NS}\,X \). Hence the fact that \( (A^2/m)^\sharp = mA \) for a finite group \( A \) imply that (7) gives a short exact sequence

\[ 0 \to mH^{2d}_{\text{et}}(X, \mathbb{Z}(d))^\sharp \to m\text{Pic} \to m\text{NS} \to 0. \]

From this we can deduce Rojtman’s theorem away from the characteristic. Indeed, since \( CH_0(X) \cong H^{2d}_{\text{et}}(X, \mathbb{Z}(d)) \) as explained in the beginning of this section, and the Albanese map \( CH_0(X) \to \text{Alb}_X(k) \) is surjective, it suffices to show that the order of the \( m \)-torsion of both sides agrees. But from the duality of the Picard and Albanese variety we know that \( |m\text{Alb}_X(k)| = |m\text{Pic}_X(k)| = |m\text{Pic}X|/|m\text{NS}X| \).

**Proposition 2.2** Let \( u + n = d + 1 \) and assume that \( H^{2u-1}_{\text{et}}(X, \mathbb{Z}_l) \) and \( H^{2n-1}_{\text{et}}(X, \mathbb{Z}_l) \) are torsion free. Then we have a perfect pairing

\[ \nu H^{2n}_{\text{et}}(X, \mathbb{Z}(n)) \times \nu H^{2u}_{\text{et}}(X, \mathbb{Z}(u)) \to \mu_\nu. \]

**Proof.** This follows because \( H^{2u-1}_{\text{et}}(X, \mathbb{Z}(u)) \) is the extension of an \( l \)-divisible group by a finite group contained in \( H^{2u-1}_{\text{et}}(X, \mathbb{Z}(u)) \), see [7]. It follows that under the hypothesis the outer terms in (7) vanish. \( \square \)
Note that the $l$-adic cohomology groups in question are torsion free for almost all $l$, and they are torsion free for all $l$ if $X$ is an abelian variety. To get an unconditional pairing, consider the subgroup of divisible elements

$$H_{\text{hom}}^{2n}(X, \mathbb{Z}(n)) = \ker H_{\text{et}}^{2n}(X, \mathbb{Z}(n)) \to \lim_m H_{\text{et}}^{2n}(X, \mathbb{Z}(n))/m$$
$$= \ker H_{\text{et}}^{2n}(X, \mathbb{Z}(n)) \to \prod_l H_{\text{et}}^{2n}(X, \mathbb{Z}_l(n)).$$

By [7, Cor. 2.2], Tor$H_{\text{et}}^{2n}(X, \mathbb{Z}(n))$ is a direct summand of $H_{\text{et}}^{2n}(X, \mathbb{Z}(n))$. This property is shared with its subgroup of divisible elements $H_{\text{hom}}^{2n}(X, \mathbb{Z}(n))$, and from this one easily concludes that $H_{\text{hom}}^{2n}(X, \mathbb{Z}(n))$ is in fact the maximal divisible subgroup of $H_{\text{et}}^{2n}(X, \mathbb{Z}(n))$.

**Proposition 2.3** We have a perfect pairing

$$mH_{\text{hom}}^{2n}(X, \mathbb{Z}(n)) \times mH_{\text{hom}}^{2n}(X, \mathbb{Z}(u)) \to \mu_m.$$

**Proof.** We can assume that $m = l^r$ is a prime power. Let

$$Q = (H_{\text{et}}^{2n}(X, \mathbb{Z}(n))/H_{\text{hom}}^{2n}(X, \mathbb{Z}(n)) \otimes \mathbb{Z}(l)) \subseteq \lim_r H_{\text{et}}^{2n}(X, \mathbb{Z}(n))/l^r \subseteq H_{\text{et}}^{2n}(X, \mathbb{Z}_l(n))$$

be the canonical inclusion. Both inclusions are isomorphisms on torsion subgroups: The former because the cokernel of the completion map $A \to \lim_r A/l^r$ is uniquely divisible, and the latter because the Tate-module $T_l H_{\text{et}}^{2n+1}(X, \mathbb{Z}(n))$ is torsion free.

Since $H_{\text{hom}}^{2n}(X, \mathbb{Z}(n))$ is divisible, we obtain a short exact sequence

$$0 \to (mQ)^\sharp \to (mH_{\text{et}}^{2n}(X, \mathbb{Z}(n)))^\sharp \to (mH_{\text{hom}}^{2n}(X, \mathbb{Z}(n)))^\sharp \to 0.$$

The group $H_{\text{et}}^{2n-1}(X, \mathbb{Z}(u))$ is an extension of a finite group by a divisible group [7, Thm. 1.1], and the $l$-primary part of this finite group is $H_{\text{et}}^{2n-1}(X, \mathbb{Z}_l(u))\{l\}$, which is dual to $H_{\text{et}}^{2n}(X, \mathbb{Z}_l(n))\{l\}$, [7 Prop. 1.2 (1)], the $l$-primary part of $Q$ by the above. We conclude that in the sequence (7), the image of $H_{\text{et}}^{2n-1}(X, \mathbb{Z}(u))/m$ in $(mH_{\text{et}}^{2n}(X, \mathbb{Z}(n)))^\star$ is exactly $(mQ)^\star$.

**Example.** We have $H_{\text{hom}}^2(X, \mathbb{Z}(1)) \cong \text{Pic}^0(X)$, and $H_{\text{hom}}^{2d}(X, \mathbb{Z}(d)) \cong CH_0(X)^d$, and obtain another proof of Rojtman’s theorem.
A class is algebraically equivalent to zero if it lies in the image of some map
\[ H^{2n}_{et}(T \times X, \mathbb{Z}(n)) \xrightarrow{t_1^*-t_0^*} H^{2n}_{et}(X, \mathbb{Z}(n)) \]
for a smooth connected scheme \( T \) (which we can assume to be a smooth curve) and closed points \( t_0, t_1 \in T \). The subgroup of classes algebraically equivalent to zero is written \( H^{2n}_{alg}(X, \mathbb{Z}(n)) \). It is a subgroup of \( H^{2n}_{hom}(X, \mathbb{Z}(n)) \), hence we can restrict the pairing above.

**Definition 2.4** A homomorphism from \( H^{2n}_{alg}(X, \mathbb{Z}(n)) \) to the \( k \)-rational points of an abelian variety \( A \) is regular, if for every pointed smooth connected variety \( t_0 \in T \) and element \( \Gamma \in H^{2n}_{alg}(T \times X, \mathbb{Z}(n)) \), the composition with
\[
T(k) \to H^{2n}_{alg}(X, \mathbb{Z}(n)), \quad t \mapsto t^*\Gamma - t_0^*\Gamma
\]
is the map induced on closed points by a morphism of varieties \( T \to A \).

In [19], [20], Murre studied the situation for Chow groups, and he proved that a universal homomorphism to an abelian variety exists for dimension 0, and codimensions 1 and 2.

**Theorem 2.5** There is a universal object \( \rho_n : H^{2n}_{alg}(X, \mathbb{Z}(n)) \to A_n \) for regular homomorphisms from \( H^{2n}_{alg}(X, \mathbb{Z}(n)) \) to abelian varieties.

**Proof.** This follows by the argument of Serre-H. Saito [21] because the dimension of surjective maps to abelian varieties is bounded, see also [12].

**Question 2.6** Is there a duality between the abelian varieties \( A_n \) and \( A_u \) of Theorem [2.5] such that the diagram below arising from the \( \epsilon_m \)-pairing is commutative?

\[
\begin{array}{ccc}
\rho_n & \downarrow & \rho_u \\
\mu_m \\
mA_n \times mA_u & \longrightarrow & \mu_m.
\end{array}
\]
3 Finite fields

Over a finite field, the pairing (4) for $v = 2$ and $w = 0$ becomes for

$$u + n = d, \quad i + j = 2d + 1$$

the pairing

$$H^j_{\text{et}}(X, \mathbb{Z}(u)) \times H^{i+1}_{\text{et}}(X, \mathbb{Z}(n)) \to H^{2d+2}_{\text{et}}(X, \mathbb{Z}(d)) \xrightarrow{tr} H^2_{\text{et}}(k, \mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}.$$ 

This is compatible with Poincaré duality

$$H^i_{\text{et}}(X, \mathbb{Z}/m(n)) \times H^{i+1}_{\text{et}}(X, \mathbb{Z}/m(u)) \to \mathbb{Q}/\mathbb{Z}$$

for all integers $m$ (the pairing for $m$ a power of $p$ is discussed in [17] using the isomorphism $\mathbb{Z}/m(n) \cong \nu_r(n)$ from [8]), i.e. the diagram (5) commutes. We obtain as in the previous case an exact sequence

$$0 \to H^j_{\text{et}}(X, \mathbb{Z}(u))/m \to (mH^{i+1}_{\text{et}}(X, \mathbb{Z}(n)))^* \xrightarrow{\delta} mH^{j+1}_{\text{et}}(X, \mathbb{Z}(u)) \to (H^i_{\text{et}}(X, \mathbb{Z}(n))/m)^* \to 0, \quad (8)$$

hence the resulting pairing

$$H^j_{\text{et}}(X, \mathbb{Z}(u))/m \times mH^{i+1}_{\text{et}}(X, \mathbb{Z}(n)) \to \mathbb{Z}/m \quad (9)$$

is non-degenerate on the left. It is non-degenerate if and only if $\delta$ vanishes, and we show that this is equivalent to Tate’s conjecture:

**Theorem 3.1** The map $\delta$ vanishes for $i \neq 2n, 2n+1$. It vanishes for $i = 2n$ if and only if $H^{2n+1}_{\text{et}}(X, \mathbb{Z}(u))$ is finite. It vanishes for $i = 2n+1$ if and only if $H^{2n+1}_{\text{et}}(X, \mathbb{Z}(u))$ is finite.

Note that $i = 2n \Leftrightarrow j = 2u+1$ and $i = 2n+1 \Leftrightarrow j = 2u$.

**Proof.** We want to show that the map $\delta_m$ in the following diagram is the zero map:

$$
\begin{array}{ccc}
\text{(Tor}H^{i+1}_{\text{et}}(X, \mathbb{Z}(n)))^* & \xrightarrow{\delta_m} & TH^{j+1}_{\text{et}}(X, \mathbb{Z}(u)) \\
\downarrow v & & \downarrow \\
(mH^{i+1}_{\text{et}}(X, \mathbb{Z}(n)))^* & \xrightarrow{\delta_m} & mH^{j+1}_{\text{et}}(X, \mathbb{Z}(u)).
\end{array}
$$
Since $v$ is surjective, the vanishing of $\delta_\infty$ is equivalent to the vanishing of $\delta_m$ for all $m$. The first statement of the Proposition follows because $H^j_{\text{et}}(X, \mathbb{Q}/\mathbb{Z}(u))$ is finite for $j \neq 2u, 2u + 1$ for weight reasons, and this group surjects onto $\text{Tor} H^{j+1}_{\text{et}}(X, \mathbb{Z}(u))$, so that $TH^{j+1}_{\text{et}}(X, \mathbb{Z}(u)) = 0$ for $j \neq 2u, 2u + 1$.

If $i = 2n$, then finiteness of $H^{2n+1}_{\text{et}}(X, \mathbb{Z}(n))$ implies that the source of $\delta_\infty$ is finite, hence cannot map non-trivially to a Tate-module. Conversely, if $\delta_m = 0$ for all $m$, then the duality between the two cotorsion groups $H^{2u+1}_{\text{et}}(X, \mathbb{Z}(u))/m$ and $mH^{2n+1}_{\text{et}}(X, \mathbb{Z}(n))$ implies that $H^{2n+1}_{\text{et}}(X, \mathbb{Z}(n))$ is finite. Reversing the roles of $i$ and $j$ we obtain the result for $i = 2n + 1$.

The connection to Tate’s conjecture is given by the following (well-known) Proposition.

**Proposition 3.2** The finiteness of $H^{2n+1}_{\text{et}}(X, \mathbb{Z}(n))$ is equivalent to Tate’s conjecture on the surjectivity of the cycle map in degree $n$ for $X$.

**Proof.** Consider the coefficient sequence

$$0 \to H^{2n}_{\text{et}}(X, \mathbb{Z}(n))^\wedge \to H^{2n}_{\text{et}}(X, \mathbb{Z}_l(n)) \to T_lH^{2n+1}_{\text{et}}(X, \mathbb{Z}(n)) \to 0.$$ 

The middle group surjects onto $H^{2n}(\bar{X}, \mathbb{Z}_l(n))^G$ with finite kernel. On the other hand, in the composition

$$CH^n(X) \otimes \mathbb{Z}_l \to H^{2n}_{\text{et}}(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_l \to H^{2n}_{\text{et}}(X, \mathbb{Z}(n))^\wedge,$$

the left map is an isomorphism up to torsion, and the right map is surjective (as the target is a subgroup of $H^{2n}_{\text{et}}(X, \mathbb{Z}_l(n))$, hence a finitely generated $\mathbb{Z}_l$-module). We conclude that the cycle map $CH^n(X) \otimes \mathbb{Z}_l \to H^{2n}_{\text{et}}(X, \mathbb{Z}_l(n))^G$ is rationally surjective if and only if $T_lH^{2n+1}_{\text{et}}(X, \mathbb{Z}(n)) = 0$ if and only if (the group of cofinite type) $H^{2n+1}_{\text{et}}(X, \mathbb{Z}(n))\{l\}$ is finite. Finally, by Gabber’s theorem [3], the cotorsion of $H^{2n+1}_{\text{et}}(X, \mathbb{Z}(n))\{l\}$ vanishes for almost all $l$.

**Example.** If $X$ is a surface, we obtain $H^6_{\text{et}}(X, \mathbb{Z}(1)) = 0$, and pairings

$$k^\times \times H^5_{\text{et}}(X, \mathbb{Z}(1)) \to \mathbb{Q}/\mathbb{Z},$$

$$m \text{Pic}(X) \times H^4_{\text{et}}(X, \mathbb{Z}(1))/m \to \mathbb{Z}/m,$$

$$\text{Pic}(X)/m \times mH^4_{\text{et}}(X, \mathbb{Z}(1)) \to \mathbb{Z}/m,$$

$$\text{Br}(X)/m \times m\text{Br}(X) \to \mathbb{Z}/m.$$
The pairings are all perfect, except for the last one, which is perfect if and only if the Brauer group is finite. In [25, Thm. 5.1], Tate defines a skew-symmetric pairing on the Brauer group whose kernel consists exactly of the divisible elements. It is easy to see from the construction that Tate’s pairing is obtained by composing the above pairing with the canonical map \( \text{Br}(X) \to \text{Br}(X)/m \). In the limit, this become the composition

\[
\text{Br}(X) \to \text{Br}(X)^\wedge \to \text{Br}(X)^*.
\]

The first map has kernel exactly the divisible elements and the second map is injective.

**Remark 3.3** A small modification of étale motivic cohomology yields Weil-étale motivic cohomology groups \( H^i_W(X,\mathbb{Z}(n)) \) which are expected to be finitely generated for all \( i,n \) and smooth and projective \( X \) [5]. Assuming finite generation, they satisfy dualities

\[
H^i_W(X,\mathbb{Z}(n))/\text{Tor} \times H^i_W(X,\mathbb{Z}(u))/\text{Tor} \to \mathbb{Z}
\]

\[
\text{Tor}H^i_W(X,\mathbb{Z}(n)) \times \text{Tor}H^{i+1}_W(X,\mathbb{Z}(u)) \to \mathbb{Q}/\mathbb{Z}.
\]

Under the finite generation conjecture, we have \( \text{Tor}H^i_{et}(X,\mathbb{Z}(n)) \cong \text{Tor}H^i_W(X,\mathbb{Z}(n)) \) for \( i \neq 2n+2 \), and \( \text{Tor}H^{2n+2}_{et}(X,\mathbb{Z}(n)) \cong H^{2n+2}_W(X,\mathbb{Z}(n)) \oplus (\mathbb{Q}/\mathbb{Z})^r \), where \( r \) is the rank of \( H^{2n}_{et}(X,\mathbb{Z}(n)) \), as one sees from the long exact sequence [5, Thm. 7.1].

**Arithmetic schemes**

The same discussion as for finite fields should also apply to arithmetic schemes, i.e. schemes which are regular, and proper over the spectrum \( B \) of the ring of integers of a number field or a smooth and proper curve over a finite field. See [4] for properties of Bloch’s higher Chow groups on smooth schemes over a Dedekind ring. In order to get the correct 2-torsion in the presence of real embeddings, one has to consider cohomology with compact support. It is defined as the étale cohomology with compact support on \( B \) of \( Rf_!\mathbb{Z}(n) \), see [11, §3]. There is an exact sequence

\[
\cdots \to H^i_c(X,\mathbb{Z}(n)) \to H^i_{et}(X,\mathbb{Z}(n)) \to \bigoplus_{v \in S_\infty} H^i_f(\mathbb{R}, R\Gamma_{et}(X_C,\mathbb{Z}(n))) \to \cdots ,
\]

(10)
where the last term is Tate-modified cohomology, a finite 2-group. To use the same argument as above, two ingredients are missing, see also the discussion in [2, §6]:

1) The cup-product

\[ H^i_{et}(X, \mathbb{Z}(u)) \times H^i_{c}(X, \mathbb{Z}(n)) \rightarrow H^{2i+1}_{c}(X, \mathbb{Z}(d)) \rightarrow H^{4}_{c}(B, \mathbb{Z}(1)) \cong \mathbb{Q}/\mathbb{Z} \]

is conjectured to exist, but this is currently unknown for higher Chow groups [14]. The problem is that if two cycles are located in the same special fiber, they do not intersect in the correct codimension, so that one cycle has to be moved to lie horizontal or in another special fiber. Spitzweck [24] has announced a construction of this pairing in case that \( X \) is smooth over \( B \).

2) Duality with finite coefficients, i.e., a perfect pairing of finite groups,

\[ H^i_{et}(X, \mathbb{Z}/m(n)) \times H^j_{c}(X, \mathbb{Z}/m(u)) \rightarrow H^3_{c}(B, \mathbb{Z}/m(1)) \cong \mathbb{Z}/m \]  

is not known to exist. If the fibers at all places dividing \( m \) are normal crossing schemes, then Sato proved a duality as above for \( \mathbb{Z}/m(n) \) replaced by his \( p \)-adic Tate-twists \( \mathcal{T}_m(n) \) [23, Thm. 1.2.1]. It is expected that \( \mathcal{T}_m(n) \) and \( \mathbb{Z}/m(n) \) are quasi-isomorphic, but this is only known if \( \mathbb{Z}/m(n) \) is acyclic in degrees larger than \( n \) (because \( \mathcal{T}_m(n) \) has this property by construction) [27]. This would follow, for example, from a Gersten resolution for \( \mathbb{Z}/m(n) \), but this is only known for smooth schemes [4]. In particular, we obtain such a pairing localized away from all \( p \) where \( X \) has bad reduction at a place above \( p \).

**Conjecture 3.1 (Lichtenbaum)** The groups \( H^i_{et}(X, \mathbb{Z}(n)) \) are finitely generated for \( i \leq 2n \), finite for \( i = 2n + 1 \), and of cofinite type for \( i \geq 2n + 2 \).

If follows from the long-exact sequence (10) that then the same statement holds for cohomology with compact support. On the other hand, the statement of the conjecture is wrong if one removes points from the base \( B \).

**Proposition 3.4** Assume Conjecture [5, 1] and the existence of the pairings with finite coefficients [11]. Then we have perfect pairings

\[ H^i_{et}(X, \mathbb{Z}(u)) \wedge \operatorname{Tor} H^{i+1}_{c}(X, \mathbb{Z}(n)) \rightarrow \mathbb{Q}/\mathbb{Z}, \]

\[ H^i_{c}(X, \mathbb{Z}(u)) \wedge \operatorname{Tor} H^{i+1}_{et}(X, \mathbb{Z}(n)) \rightarrow \mathbb{Q}/\mathbb{Z}. \]
Proof. (see also [2, Prop. 3.4]) We show the first statement, the proof of the
other statement is identical. If \( j \leq 2u \), then \( T \mathcal{H}^{j+1}_e(X, Z(u)) = 0 \) (as the Tate
module of a finitely generated group vanishes) and \( H^i_c(X, Z(u)) \otimes \mathbb{Q}/\mathbb{Z} = 0 \)
(as the cohomology group is torsion) and we obtain
\[
H^j_e(X, Z(u))^\wedge \cong \lim H^j_1(X, Z(m(u))) \cong H^i_c(X, \mathbb{Q}/\mathbb{Z}(n))^* \cong (\text{Tor} \mathcal{H}^{i+1}_e(X, Z(n)))^*.
\]
If \( j > 2u \), then \( H^j_e(X, Z(u)) \wedge \) is finite, \( (H^i_c(X, Z(n)) \otimes \mathbb{Q}/\mathbb{Z})^* \) is torsion free, and
\[
H^j_e(X, Z(u))^\wedge \cong \text{Tor} \lim H^j_1(X, Z(m(u))) \cong \text{Tor}(H^i_c(X, \mathbb{Q}/\mathbb{Z}(n))^*) \cong (\text{Tor} \mathcal{H}^{i+1}_e(X, Z(n)))^*.
\]
□

4 Local fields

Let \( k \) be a local field of characteristic \( p \geq 0 \), i.e. a complete discrete valuation
field with finite residue field. If we take \( w = 1 \) and \( j = 3 \) in (1), then setting
\[
n + u = d + 1, \quad i + j = 2d + 2,
\]
we obtain a pairing
\[
H^j_e(X, Z(u)) \times H^{i+1}_e(X, Z(n)) \to H^{2d+3}_e(X, Z(d+1)) \xrightarrow{\text{tr}} H^3_e(k, Z(1)) \cong \text{Br} k \cong \mathbb{Q}/\mathbb{Z}.
\]
Combining this with the duality over local fields for \( p \nmid m \) (see [18, I Cor. 2.3] combined with Poincaré duality over algebraically closed fields),
\[
H^j_e(X, Z/m(u)) \times H^i_e(X, Z/m(n)) \to \mathbb{Q}/\mathbb{Z},
\]
we again obtain an exact sequence
\[
0 \to H^j_e(X, Z(u))/m \to (mH^{i+1}_e(X, Z(n)))^* \\
\quad \quad \delta \xrightarrow{m} H^{i+1}_e(X, Z(u)) \to (H^i_e(X, Z(n))/m)^* \to 0, \quad (12)
\]
and pairings which are non-degenerate on the left
\[
H^j_e(X, Z(u))/m \times mH^{i+1}_e(X, Z(n)) \to \mathbb{Q}/\mathbb{Z}.
\]
If $\delta$ is the zero-map, then this induces in the limit a duality

$$H^j_{et}(X, \mathbb{Z}(u))^\wedge \times \text{Tor}H^{i+1}_{et}(X, \mathbb{Z}(n)) \to \mathbb{Q}/\mathbb{Z},$$

i.e. the duality “in some appropriate sense of the term” expected by Lichtenbaum [16, §6].

Example. The vanishing of $\delta$ for $X$ the spectrum of a local field is equivalent to class field theory. Indeed, for $u = 1, j = 1$ it states that the injection

$$H^1_{et}(k, \mathbb{Z}(1))^\wedge \cong (k^\times)^\wedge \to H^2_{et}(k, \mathbb{Z})^* \cong H^1_{et}(k, \mathbb{Q}/\mathbb{Z})^* \cong \text{Gal}(k)^{ab}$$

is an isomorphism. For $u = 0, j = 0$ it states that the injection

$$H^0_{et}(k, \mathbb{Z})^\wedge \cong \hat{\mathbb{Z}} \to \text{Br}(k)^* \cong H^3_{et}(k, \mathbb{Z}(1))^*$$

is an isomorphism.

Example. If $X$ is a curve over a $p$-adic field and $n = 1$, then the statement for $i = 1, 2$ is the duality between $\text{Pic}(X)$ and $\text{Br}(X)$ proven by Lichtenbaum [15]. For $i = 0, 3$ it follows from

$$H^5_{et}(X, \mathbb{Z}(1)) \cong H^2(k, H^2_{et}(\bar{X}, \mathbb{Q}/\mathbb{Z}(1)) \cong H^2(k, \mathbb{Q}/\mathbb{Z}) = 0.$$ 

Proposition 4.1 Assume that either $i \notin \{n, \ldots, n + d + 1\}$, or that $X$ has good reduction and $i \neq 2n - 1, 2n, 2n + 1$. Then $\delta = 0$.

Proof. Again the vanishing of $\delta_m$ for all $m$ is equivalent to the vanishing of $\delta_\infty$ in the diagram

$$\begin{array}{ccc}
(T \text{Tor}H^{i+1}_{et}(X, \mathbb{Z}(n)))^* & \xrightarrow{\delta_\infty} & TH^{i+1}_{et}(X, \mathbb{Z}(u)) \\
\downarrow & & \downarrow \\
(mH^{i+1}_{et}(X, \mathbb{Z}(n)))^* & \xrightarrow{\delta_m} & mH^{i+1}_{et}(X, \mathbb{Z}(u))
\end{array}$$

because the left vertical map is surjective. By [11], $H^i_{et}(X, \mathbb{Q}/\mathbb{Z}[1/p])$ is finite for $i \notin \{n, \ldots, n + d + 1\}$ for general $X$, and $i \neq 2n - 1, 2n, 2n + 1$ for $X$ with good reduction. This implies that $\text{Tor}H^{i+1}_{et}(X, \mathbb{Z}(n))$ is finite for $i < n$ and $i < 2n - 1$, respectively, hence its dual cannot map non-trivially to the torsion free Tate-module. On the other hand, the Tate module $TH^{i+1}_{et}(X, \mathbb{Z}(u))$ vanishes for $j < u \iff i > n + d + 1$ and $j < 2u - 1 \iff i > 2n + 1$, respectively. □
We believe that an improvement is possible:

**Conjecture 4.1** If $i \not\in \{n + 1, \ldots, n + d\}$ or if $X$ has good reduction and $i \neq 2n$, then $\delta = 0$.

We give examples for $\delta$ to be non-zero, thus giving counterexamples to duality of étale motivic cohomology over local fields. Since $H^1_{et}(X, \mathbb{Z}) = 0$, we get from the limit of (12) a short exact sequence

$$0 \rightarrow H^{2d+1}_{et}(X, \mathbb{Z}(d+1))^\wedge \rightarrow (\text{Tor}H^2_{et}(X, \mathbb{Z}))^* \rightarrow TH^{2d+2}_{et}(X, \mathbb{Z}(d+1)) \rightarrow 0.$$  

For $X$ a curve, $H^3_{et}(X, \mathbb{Z}(2)) \cong H^3_{\text{M}}(X, \mathbb{Z}(2)) \cong SK_1(X)$, and it follows from S. Saito’s result [22, Thm. 2.6] that the right hand side has rank equal to $\text{rank} H^1_{et}(Y, \mathbb{Z})$, where $Y$ the special fiber of a smooth and proper model. For arbitrary dimension, Yoshida proved that its rank is the dimension of the maximal split torus of the Neron model of $\text{Alb}_X$ [26]. Hence $\delta$ can be non-zero for a curve (with bad reduction) in weights $n = 2, u = 0$.

We now give an example, obtained with the help of S. Saito and K. Sato, showing that $\delta$ can be non-zero even for schemes with good reduction. For $n = d, i = 2d$, the limit of (12) gives a sequence

$$0 \rightarrow \text{Pic}(X)^\wedge \rightarrow (H^{2d+1}_{et}(X, \mathbb{Z}(d))\{l\})^* \rightarrow T_l\text{Br}(X) \rightarrow (H^{2d}_{et}(X, \mathbb{Z}(d)) \otimes \mathbb{Q}_l/\mathbb{Z}_l)^* \rightarrow 0.$$  

**Proposition 4.2** Assume that $X$ admits a smooth and proper model $\mathcal{X}$. Then we have a commutative diagram with exact rows:

$$
\begin{array}{cccc}
0 & \rightarrow & T_l\text{Br} \mathcal{X} & \rightarrow & T_l\text{Br} X & \rightarrow & (\text{CH}_0(X) \otimes \mathbb{Q}_l/\mathbb{Z}_l)^* \\
& & \downarrow & & \downarrow & & \\
& & 0 & \rightarrow & (H^{2d}_{et}(X, \mathbb{Z}(d)) \otimes \mathbb{Q}_l/\mathbb{Z}_l)^* & \rightarrow & (\text{CH}_0(X) \otimes \mathbb{Q}_l/\mathbb{Z}_l)^* \\
\end{array}
$$

Here the upper row is induced by the Brauer-Manin pairing and the lower row by the change of topology map.

**Proof.** By Colliot-Thélène and Saito [1 Cor. 2.4], the kernel of the Brauer-Manin pairing is $T_l\text{Br} \mathcal{X}$. The lower row is exact because the cokernel of $\text{CH}_0(X) \rightarrow H^{2d}_{et}(X, \mathbb{Z}(d))$ is torsion, hence it vanishes after tensoring with $\mathbb{Q}/\mathbb{Z}$. It remains to show that the diagram is commutative.
Lemma 4.3 The following diagram is commutative, where the upper pairing is the Brauer-Manin pairing and the lower pairing the cup-product pairing:

\[
\begin{array}{ccc}
CH_0(X)/m & \times_m Br(X) & \longrightarrow \ Z/m \\
\downarrow & \downarrow & \downarrow \\
H_{et}^2d(X, \mathbb{Z}(d))/m \times_m Br(X) & \longrightarrow & \ Z/m.
\end{array}
\]

Proof. By definition of the lower pairing, it suffices to show this after adding the following commutative diagram on the bottom

\[
\begin{array}{ccc}
H_{et}^2d(X, \mathbb{Z}(d))/m \times_m Br(X) & \longrightarrow & \ Z/m \\
\downarrow & \downarrow & \downarrow \\
H_{et}^2d(X, \mathbb{Z}/m(d)) \times H_{et}^2d(X, \mathbb{Z}/m(1)) & \longrightarrow & \ Z/m.
\end{array}
\]

because the middle vertical map is surjective. The Brauer-Manin pairing is defined point by point. But if \( i : \text{Spec} \ k \to X \) is a closed point, then the commutativity follows from the projection formula, i.e. the commutativity of the following diagram

\[
\begin{array}{ccc}
H^0_{et}(k, \mathbb{Z}/m(0)) \times H^2_{et}(k, \mathbb{Z}/m(1)) & \longrightarrow & \mathbb{Z}/m \cong H^2_{et}(k, \mathbb{Z}/m(1)) \\\n\downarrow i_* & & \downarrow i_* \sim \\
H^2_{et}(X, \mathbb{Z}/m(d)) \times H^2_{et}(X, \mathbb{Z}/m(1)) & \longrightarrow & \mathbb{Z}/m \cong H^{2d+2}_{et}(X, \mathbb{Z}/m(d + 1)).
\end{array}
\]

Finally, we note that there are examples with non-vanishing \( T_i Br \mathcal{X} \): Let \( X/\mathbb{Q}_p \) be the self product of an elliptic curve \( E/\mathbb{Q}_p \) without complex multiplication and good reduction \( E_s \). Then the graph of the Frobenius of \( E_s \) in \( \text{Pic}(Y) \), \( Y \) the special fiber of the proper smooth model \( \mathcal{X}/\mathbb{Z}_p \), does not lift to \( \text{Pic}(\mathcal{X}) \), because \( \text{End}(E) \) has rank 1 \([13\ p.\ 331]\). Hence we conclude by the proper base change theorem and the following diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \text{Pic}(\mathcal{X})^{\wedge l} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \text{Pic}(Y)^{\wedge l}
\end{array} \quad \begin{array}{ccc}
\longrightarrow & \longrightarrow & T_i Br \mathcal{X} \\
\downarrow & \downarrow & \downarrow \\
\longrightarrow & \longrightarrow & T_i Br Y
\end{array} \quad \begin{array}{c}
\longrightarrow \end{array} \quad \begin{array}{c}
0
\end{array}
\]
Remark 4.4 We believe that there should be a better behaved duality theory for Weil-étale cohomology groups, see [10] in the case of curves. B. Morin expects that there are locally compact groups $H^i_W(X, \mathbb{Z}(n))$ and $H^j_W(X, \mathbb{R}/\mathbb{Z}(u))$, together with a trace map $H^{2d+2}_W(X, \mathbb{R}/\mathbb{Z}(d+1)) \to H^2_W(K, \mathbb{R}/\mathbb{Z}(1)) \cong \mathbb{R}/\mathbb{Z}$, such that there is a perfect Pontrjagin pairing of locally compact groups

$$H^i_W(X, \mathbb{Z}(n)) \times H^{2d+2-i}_W(X, \mathbb{R}/\mathbb{Z}(d+1-n)) \to H^{2d+2}_W(X, \mathbb{R}/\mathbb{Z}(d+1)) \to \mathbb{R}/\mathbb{Z}.$$ 

References

[1] J.L. Colliot-Thélène, S. Saito, Zéro-cycles sur les variétés $p$-adiques et groupe de Brauer. Internat. Math. Res. Notices 1996, no. 4, 151–160.

[2] M. Flach, B. Morin, Weil-étale cohomology and zeta-values of proper regular arithmetic schemes. http://arxiv.org/abs/1605.01277

[3] O. Gabber, Sur la torsion dans la cohomologie $l$-adique d’une variété. C. R. Acad. Sci. Paris Sér. I Math. 297 (1983), no. 3, 179–182.

[4] T. Geisser, Motivic cohomology over Dedekind rings. Math. Z. 248 (2004), no. 4, 773–794.

[5] T. Geisser, Weil-étale cohomology, Math. Ann. 330 (2004), 665–692.

[6] T. Geisser, Duality via cycle complexes, Ann. of Math. (2) 172 (2010), no. 2, 1095–1126.

[7] T. Geisser, On the structure of étale motivic cohomology. J. Pure Appl. Algebra 221 (2017), no. 7, 1614–1628.

[8] T. Geisser, M. Levine, The K-theory of fields in characteristic $p$. Invent. Math. 139 (2000), no. 3, 459–493.

[9] B. Kahn, Some finiteness results for étale cohomology. J. Number Theory 99 (2003), no. 1, 57–73.

[10] D.A. Karpuk, Weil-étale cohomology of curves over $p$-adic fields. J. Algebra 416 (2014), 122–138.
[11] K. Kato, A Hasse principle for two dimensional global fields. Journal für die reine und angewandte Mathematik 366, 142–180.

[12] T. Kohrita, Thesis, Nagoya University

[13] A. Langer, S. Saito, Torsion zero-cycles on the self-product of a modular elliptic curve. Duke Math. J. 85 (1996), no. 2, 315–357.

[14] M. Levine, The $K$-theory and motivic cohomology of schemes. http://www.math.uiuc.edu/K-theory/336/

[15] S. Lichtenbaum, Duality theorems for curves over $p$-adic fields. Invent. Math. 7 (1969) 120–136.

[16] S. Lichtenbaum, Values of zeta-functions at nonnegative integers. Number theory, Noordwijkerhout 1983 (Noordwijkerhout, 1983), 127–138, Lecture Notes in Math., 1068, Springer, Berlin, 1984.

[17] J. Milne, Values of zeta functions of varieties over finite fields. Amer. J. Math. 108 (1986), no. 2, 297–360.

[18] J. Milne, Arithmetic duality theorems. Second edition. BookSurge, LLC, Charleston, SC, 2006. viii+339 pp. ISBN: 1-4196-4274-X

[19] J. Murre, Applications of algebraic K-theory to the theory of algebraic cycles. Algebraic geometry, Sitges (Barcelona), 1983, 216–261, Lecture Notes in Math., 1124, Springer, Berlin, 1985.

[20] J. Murre, Algebraic cycles and algebraic aspects of cohomology and K-theory. Algebraic cycles and Hodge theory (Torino, 1993), 93–152, Lecture Notes in Math., 1594, Springer, Berlin, 1994.

[21] H. Saito, Abelian varieties attached to cycles of intermediate dimension. Nagoya Math. J. 75 (1979), 95–119.

[22] S. Saito, Class field theory for curves over local fields. J. Number Theory 21 (1985), no. 1, 44–80.

[23] K. Sato, $p$-adic étale Tate twists and arithmetic duality. Ann. Sci. École Norm. Sup. (4) 40 (2007), no. 4, 519–588.
[24] M. Spitzweck, A commutative $P^1$-spectrum representing motivic cohomology over Dedekind domains, arXiv:1207.4078.

[25] J. Tate, On the conjectures of Birch and Swinnerton-Dyer and a geometric analog, Sem. Bourbaki 1965/66, no. 306.

[26] T. Yoshida, Finiteness theorems in the class field theory of varieties over local fields. J. Number Theory 101 (2003), no. 1, 138–150.

[27] C. Zhong, Comparison of dualizing complexes. J. Reine Angew. Math. 695 (2014), 1–39.

Department of Mathematics, Rikkyo University, Nishi-Ikebukuro, Toshimaku, Tokyo, Japan

E-mail: geisser@rikkyo.ac.jp