Angular Momentum and Gravimagnetization of the $\mathcal{N} = 2$ SYM vacuum

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Abstract

In this note we discuss the gravimagnetization of the $\mathcal{N} = 2$ SYM vacuum in the $\Omega$-background. It is argued that the Seiberg-Witten prepotential is related to the vacuum density of the angular momentum in the Euclidean $\mathbb{R}^4$ space. The possible role of the dyonic instantons as the microscopic angular momentum carriers which could yield the spontaneous vacuum gravimagnetization is conjectured. We interpret the dyonic instanton as a kind of the Euclidean bounce in $\mathbb{R}^4$ similar to one responsible for the Schwinger pair creation. The induced angular momentum in $\mathbb{R}^4$ is also briefly considered in the dual Liouville formulation of $SU(2)$ theory via AGT relation.
1 Introduction

The common strategy while investigating the properties of a complicated system is to put it in the proper background fields and look at response. The background usually plays two-fold role, first it provides the regularization of the possible divergences and secondly it helps to extract the effective degrees of freedom via the response functions. For example, one can investigate the magnetization \( \langle M \rangle = \frac{d \log Z}{dB} \) in the external magnetic field \( B \) or some transport coefficients like conductivity in the external electric field. Such strategy has been applied to the analysis of the ground state of \( \mathcal{N} = 2 \) SYM theory in \cite{1}. Formally the Nekrasov partition function counts the instanton contributions to the holomorphic prepotential in the \( \Omega \)-deformed \( \mathcal{N} = 2 \) SYM theory \cite{1}. The background is provided by the graviphoton field which has managed to cure the singularity in the integration over the instanton moduli space.

In spite of being the effective tool to get the explicit answer the localization approach is a little bit formal and can not immediately provide the answers on more ”naive” questions. In particular it is interesting to know if instantons are organized in some effective degrees of freedom of higher dimensions. On the other hand there is the old suspicion that instanton itself can be considered as a kind of composite (see \cite{2} and references therein for the recent discussion).

In this note we focus on the second aspect of the background field analysis namely the response functions of the system. The simplest response function corresponds to the derivative of the Nekrasov partition function with respect to the graviphoton field. By the construction the parameters of the \( \Omega \)-background play the role of the chemical potentials for the rotations in the four-dimensional Euclidean space-time as well as the R-symmetry rotations. Hence we are in position to get the corresponding vev of the angular momentum in \( R^4 \) just differentiating the \( \log Z_{\text{Nek}} \) with respect to the graviphoton field that is from the gravimagnetization of the \( \mathcal{N} = 2 \) SYM vacuum.

Since we would like to analyze the properties of the instanton matter it is convenient to have in mind the \( D = 5 \) picture where the instantons are the particles charged with respect to the external graviphoton field. The \( D = 4 \) prepotential is interpreted as particular limit of generalized index in the \( D = 5 \) SYM theory on \( R^4 \times S^1 \) and the very Seiberg-Witten prepotential \cite{3} can be derived from the one-loop calculation with all particles in the loop including instantons are taken into account \cite{4}.

The simple semiclassical limit of the partition function provides us with a kind of surprise since it can be recognized immediately that the density of the angular momentum diverges at arbitrarily weak graviphoton field. This looks like an absolute
instability of the $\mathcal{N} = 2$ SYM matter with respect to the generation of the angular momentum in the external graviphoton field very much similar to absolute instability of the QED matter in the external electric field due to the Schwinger pair creation. The $D = 5$ one-loop derivation of the prepotential suggests that the analogy with the QED in the external field can be considered more carefully. The QED example provides us also with the toy model for the angular momentum generation. We shall compare QED and SYM phenomena in the external fields using the Euclidean interpretation of the pair production process via the bounce solution.

Having in mind the non-vanishing vev of the angular momentum it is natural to question on the microscopic states which are its carriers in the Euclidean space-time. The natural particle charged with respect to the graviphoton is D0 brane. When the graviphoton field is switched on the stabilization of the Euclidean bounce is due to the topological D0 charge dissolved on the D2 brane worldvolume. Moreover since we handle with the instanton matter the intrinsic object carrying the angular momentum is the dyonic instanton $[5, 6, 7, 8]$ that is the circular D2 brane with the electric and topological charges. This solution exists even without the graviphoton background and gets stabilized by the angular momentum of the fields proportional to the product of the instanton and electric charges. Hence generically the phenomena of the spontaneous gravimagnetization of the instanton matter could happen.

Since the Nekrasov partition function is identified with the Liouville conformal block $[9]$ we could question the issue of the angular momentum in the Liouville theory as well. The parameters of the background enter the expression for the conformal weights and the central charge. That is we shall get the interesting interpretation for the derivatives of the conformal block with respect to the conformal weights of the operators involved as the angular momentum in the physical space. We shall also comment on the additional angular momentum due to the surface operators. As a byproduct remark we also relate the one-loop calculation in the external field with the auxiliary problem in the AdS space which provides us with the objects similar to conformal blocks in the Liouville theory.

The note is organized as follows. In the Section 2 we discuss the induced momentum in the $Ω$-deformed $\mathcal{N} = 2$ SYM theory and compare it with the Euclidean description of the pair production in QED in external field in Section 3. Section 4 concerns the possible role of the dyonic instantons in the spontaneous gravimagnetization. Section 5 is devoted to the discussion of the angular momentum at the Liouville side while in the last Section some discussion can be found.
2 Angular momentum from Nekrasov partition function

Let us remind the main features of the Nekrasov partition function. One considers the $\mathcal{N} = 2$ SYM theory and introduces the $\Omega$-background switching on the graviphoton RR field in $R^4$. The most simple way to describe $\Omega$-background is to start with the six-dimensional space-time and introduce $R^4$ bundle with the nontrivial $SO(4)$ connection over the two-dimensional torus. The corresponding metric reads as

$$ds^2 = Adzd\bar{z} + g_{ij}(dx^i + V^i dz + \bar{V}^i d\bar{z})(dx^j + V^j d\bar{z} + \bar{V}^j d\bar{z})$$

(2.1)

where $V^i = \Omega^i_j x^j$ and $A$ is the torus area. The components of the graviphoton field strength tensor $\Omega^i_j$, $\epsilon_{1,2}$ provide the regularization of the integration over the instanton moduli space. The alternative description of the $\Omega$-background is recently suggested in [11].

The localization of the integral over the instanton moduli space with respect to the natural torus action yields the exact partition function of the $\mathcal{N} = 2$ SYM in the $\Omega$ background [1]. It can be interpreted in terms of the weighted equivariant volumes of the instanton moduli spaces $\mathcal{M}_N$.

$$Z_Nek(a, \epsilon_1, \epsilon_2) = \sum_N q^N \int_{\mathcal{M}_N} "1"$$

(2.2)

It is important that the conventional Seiberg-Witten prepotential $F$ [3] can be derived at the semiclassical limit $\epsilon_{1,2} \to 0$

$$Z_{Nek} \propto \exp\left(\frac{F}{\epsilon_1 \epsilon_2} + ...\right)$$

(2.3)

Note that in the $\Omega$ background the effective volume of the system is

$$Vol_{eff} = \frac{1}{\epsilon_1 \epsilon_2}$$

(2.4)

hence to some extend in the semiclassical limit the prepotential can be treated as the "action density" of the vacuum state.

In what follows it will be useful for our purposes to consider the different realization of the Nekrasov partition function as the generalized index in the $D = 5$ SYM theory [1]

$$Z_{Nek} = Tr_H(-1)^{2(J_L + J_R)}(\exp((\epsilon_1 - \epsilon_2)J^3_L + (\epsilon_1 + \epsilon_2)J^3_R + (\epsilon_1 + \epsilon_2)J^3_I + \beta H))$$

(2.5)
where \( J_L, J_R \) are the generators of independent rotations in \( R^4 \) while \( J_I \) corresponds to the R-symmetry rotation. This representation clearly shows that the parameters of the deformation play the role of the generating parameters for the angular momenta and R-symmetry rotation.

This representation fits with the derivation of the Seiberg-Witten prepotential from the theory on \( R^4 \times S^1 \) [4] which involves explicit one-loop calculation where all massive BPS particles with the topological and electric charges are summed over. Since instantons are particles in five dimensions they are treated as degrees of freedom propagating in the loop. Upon taking the proper limit one-loop calculation in \( D = 5 \) yields precisely the instanton Seiberg-Witten prepotential [4]. The representation of \( Z_{Nek} \) as the generalized index fits with this one-loop calculation in the \( \Omega \) background. Somewhat similar representation of the generalized indexes has been also explored in [12].

Given the index representation it is evident that the derivatives of \( \log Z \) with respect to the equivariant parameters yield the vev of the physical angular momentum \( < J_{L,R} > \). The arguments providing the similar interpretation in the \( D = 4 \) framework are based on the particular form of the metric (2.1) in the \( \Omega \) background. Indeed the four-dimensional part of the metric is deformed by the terms proportional to the coordinates \( x_\nu \). The response of the Lagrangian on this deformation canonically yields the four-dimensional angular momentum hence the \( D = 4 \) and \( D = 5 \) pictures are consistent with each other.

It is convenient to extract the pure angular momentum without the admixture of the R-rotation. To this aim one can consider the difference \( \partial_\epsilon_1 - \partial_\epsilon_2 \)

\[
< J_L > = (\partial_\epsilon_1 - \partial_\epsilon_2)\log Z \tag{2.6}
\]

which in the semiclassical limit behaves as

\[
< J > \propto \frac{(\epsilon_1 - \epsilon_2)\mathcal{F}}{\epsilon_1 \epsilon_2^2} + ... \tag{2.7}
\]

Hence the density of the angular momentum at weak field and \( \epsilon_1 = -\epsilon_2 = \epsilon \) reads as

\[
< J_L > \propto \epsilon^{-1} \mathcal{F} \tag{2.8}
\]

which indicates the instability of the SYM vacuum in the external graviphoton RR 1-form field. A little bit surprisingly the prepotential itself is proportional to the density of the angular momentum.

To complete this Section note that having in mind the finite effective volume in the \( \Omega \) background we can write for the "vacuum state" the relation

\[
< E >_{\text{vac}} \propto < J_L >_{\text{vac}} \tag{2.9}
\]
which implies that it effectively behaves as the system of rotators in the external field. In what follows we shall suggest the degrees of freedom which could serve as such rotators.

### 3 The toy example

Consider the simplest example of the QED in the external electric field $E$. The theory is unstable with respect to the charged pairs creation. The tunneling can be described via the first quantized picture in terms of the classical particle trajectories in the Euclidean space-time [13]. The trajectories are just the circles in the $x, t_E$ plane whose radii are fixed by the extremization of the simple effective action

$$S_{\text{eff}} = 2\pi Rm - E\pi R^2$$

The coordinate $x$ is selected by the direction of the external field. This semiclassical motion in the Euclidean space-time can be interpreted as the motion in the constant magnetic field. Similarly the creation of the generic dyons in the external field can be described in terms of the closed classical trajectories. The radius of the trajectory is fixed by the external field moreover the configuration is unstable. The negative mode in the spectrum of the fluctuations implies the bounce interpretation of such Euclidean solution.

The probability of the pair production in the unit time per unit volume is obtained from the action calculated at the bounce solution

$$w \propto \exp(-S_{\text{eff}}(R_{\text{cr}})) \propto \exp\left(-\frac{m^2}{eE}\right)$$

Since the classical trajectory is the circle with the fixed radius it implies the nontrivial angular momentum $< J >$ from the $R^4$ Euclidean viewpoint in the $x, t_E$ plane. The density of the loop trajectories in the $t_E, x$ plane corresponds to the angular momentum density from the $R^4$ viewpoint. Since we are interested in the derivative of the prepotential with respect to the external field let us examine the similar expression in QED

$$\frac{\partial S_{\text{eff}}}{\partial E} \propto < R^2 >$$

that is the area inside the semiclassical trajectory. In the semiclassical approximation $< R^2 > \propto R_{\text{cr}}^2 \propto E^{-2}$ and therefore diverges at weak field. This behavior reflects the instability of the system.

The above picture can be generalized for the multiple pair production problem when the multiple Euclidean bounces have to be taken into account. Contrary to
the single pair case the generic probability involves the account of the interaction between the bounces which makes the problem more complicated. The density of the bounces at the 2-dimensional plane selects the phase corresponding to ”gas” or the ”liquid” state of the bounce rotators system.

To make the link with \( \mathcal{N} = 2 \) SYM case let us represent the pair production in the brane terms [14]. The bounce corresponding to the generic \((p, q)\) dyon production is the tubular \((p, q)\) string stretched between two D3 branes in IIB setup. In the field theory limit the deformation of the cylinder due to the finite string tension is neglected. In what follows we shall consider very similar configuration of the tubular D2 branes between the parallel D4 branes in IIA setup.

The brane viewpoint allows to discuss the tunneling problem from the viewpoint within worldvolume theory of the created brane. Recall that the Schwinger -type process involves two steps - the tunneling described by Euclidean bounce and the Minkowski evolution upon the analytic continuation at the turning point. For instance, cutting the string cylinder at the turning point we get the disc worldsheet which represents the open string nucleated in the Minkowski space-time. The function describing the nucleated branes can be considered as the Hartle-Hawking wave function \( \Psi_{HH} \) in the worldvolume theory

\[
Z(a) \propto \Psi_{HH}(a) \tag{3.13}
\]

where \( a \) is some boundary characteristics of the emerging object and \( Z(a) \) corresponds to the probability rate of the brane creation. Typically the argument of the wave function can be size, metric or some boundary holonomy. In the case under consideration the argument of the Hartle-Hawking wave function \( \Psi_{HH} \) can be identified with the boundary length and therefore \( \Psi_{HH}(L) \) plays the role of the ”conformal block”. We shall comment on this interpretation later on. Somewhat related discussion can be found in [15].

The Hartle-Hawking interpretation of the Nekrasov partition function in the semiclassical approximation is possible as well. To this aim the prepotential \( F(a) \) has to be interpreted as the action of some Hamiltonian system. This is true indeed for the Hamiltonian system with the phase space \((a, a_D)\) [17, 16] due to the set of relations

\[
a_D = \frac{\partial F(a)}{\partial a}, \quad H(a, a_D) = \frac{\partial F(a)}{\partial \tau} \tag{3.14}
\]

where \( a, a_D \) are the conventional variables defining the central charge in \( \mathcal{N} = 2 \) SYM theory [3]. Hence prepotential defines the semiclassical Whitham wave function in the particular polarization and admits the interpretation as the HH function of the non-perturbatively created object. Indeed being the integral over the \( A \) cycle the
coordinate variable $a$ could be treated as the proper boundary data argument of the HH wave function.

4 On Dyonic instantons and spontaneous vacuum gravimagnetization

In the previous Section we have explained the analogy between the effective actions in QED in the constant background and $\mathcal{N} = 2$ SYM in $\Omega$ background. The Euclidean bounce is responsible for the pair production and just these loops of the electrons or generic dyons in the $R^4$ Euclidean space-time provide the microscopic description of the angular momentum which is captured by the derivatives with respect to the external electric or magnetic field. The induced angular momentum in the $R^4$ Euclidean space-time is due to the Euclidean dyonic loops.

The natural question concerns the possible microscopic description of the angular momentum in the $\Omega$ background in terms of some bounces in the Euclidean space similar to QED. Since in the $D = 5$ SYM one-loop derivation of the Seiberg-Witten prepotential the sum in the loop goes over the states with instanton and electric charges $[4]$ the dyonic instantons are the relevant objects to think about. Similar states in $\Omega$-background can be thought of as the blown instantons in the graviphoton field. The D0 branes get expanded into the tubular D2 branes with the density of the D0 branes on their worldvolumes. In the T-dual picture the dyonic instanton gets represented by the D1-helix.

The D0 branes are charged with respect to the graviphoton field and prevent D2 branes from collapsing to a point. Hence in the $\Omega$ background we have a kind of bounce configurations in Euclidean space-time. Note that somewhat related picture has been developed in $[19]$ where the Nekrasov partition function was derived in terms of the blowing of instantons in the proper background into the $D2-\bar{D}2$ state.

Let us emphasize that we necessarily should clarify the origin of the induced angular momentum in the absence of the external field since the Seiberg-Witten prepotential which we have related to the density of the angular momentum can be obtained in the $\epsilon_{1,2} = 0$ case as well. Hopefully contrary to the conventional QED case the phenomena of the spontaneous gravimagnetization can happen that is generation of the angular momentum in $R^4$ without the graviphoton field. In what follows we shall briefly discuss the solution responsible for this phenomena. Instead of the external field the angular momentum stabilizes the extended brane solution in the Euclidean space-time.

The dyonic instantons can be considered as the instantons (D0 branes) with
attached fundamental strings \[5, 6, 7\]. Due to the Myers effect the D0+fundamental string state gets blown into the tubular D2 branes with the electric field and the instanton charge. That is in the \( D = 4 \) space-time the point-like instanton becomes the circular dyonic loop which carries the topological charge due to D0 density and electric field due to the fundamental string.

More precisely the D0 branes are localized at the D4 branes stretched between two NS5 branes in the standard Hanany-Witten type brane geometry and their world-line are extended along the coordinate \( x_6 \) transverse to the NS5 branes. The fundamental string connects two parallel D4 branes with the geometry \( R^4 \times I \) where \( D = 5 \) gauge theory lives on. The dyonic instantons are solutions to the conventional \( D = 5 \) equation of motion

\[
F_{\mu\nu} = *F_{\mu\nu}, \quad D_\mu \phi = E_\mu, \quad D_0 \phi = 0 \tag{4.15}
\]

where \( \phi \) is the scalar field and \( E_\mu \) is electric field in \( R^4 \). The shape of the loop can be arbitrary \[6\] however the extremization of the angular momentum yields the circular loop. The dyonic instanton is the BPS state and keeps 1/2 of the initial SUSY. The BPS formula yields for its mass

\[
M = \frac{4\pi^2}{g^2} |I| + |vQ_e| \tag{4.16}
\]

where \( |I| \) is the instanton charge and \( v \) is the vev of the scalar. The BPS-ness is provided by the combined effect of the scalar, electric field and the "running" of the instantons in the loop. Note that the symmetries supported by the dyonic instantons, that is diagonal \( SU(2) \) from left rotations and R-symmetry, coincide with the symmetries kept by the \( \Omega \) background.

Let us describe the dynamical quantum numbers of solution in some details. To this aim it is useful to discuss the worldvolume Lagrangian of the tubular D2 brane of constant radius \( R \) in flat Euclidean space-time

\[
L = -\sqrt{R^2(1 - E^2) + B^2} \tag{4.17}
\]

where \( E, B \) are the worldvolume electric and magnetic fields. The corresponding canonical momentum reads as

\[
\Pi = \frac{\partial L}{\partial E} = \frac{R^2 E}{\sqrt{R^2(1 - E^2) + B^2}} \tag{4.18}
\]

and Hamiltonian density is

\[
\mathcal{H} = R^{-1} \sqrt{(\Pi^2 + R^2)(B^2 + R^2)} \tag{4.19}
\]
There are evident integrals

\[ Q_F = \frac{1}{2\pi} \oint d\phi \Pi, \quad Q_0 = \frac{1}{2\pi} \oint d\phi B \]  

(4.20)

corresponding to the F1 and D0 conserved charges per unit length carried by the D2 tube.

The tension of the tube is

\[ T = Q_F = \frac{1}{2\pi} \oint d\phi \mathcal{H} \]  

(4.21)

which equals at the solution to the equation of motion to

\[ T = |Q_F| + |Q_0| \]  

(4.22)

This formula implies that the D2 brane tension does not contribute hence the cross section of the tube behaves as the ”tensionless” object which explains the arbitrary form of the tube cross section. We have crossed electric and magnetic fields on the tube that is the Poynting vector does not vanish and yields the angular component

\[ M_\phi = \Pi B \]  

(4.23)

Hence we get the non-vanishing angular momentum of dyonic instanton per unit length

\[ J = Q_F Q_0 \]  

(4.24)

directed along the axis of the cylinder.

Remind that the dyonic instanton is extended along \( x_6 \) hence looks as closed loop in \( R^4 \). The solution carries the angular momentum which has been identified in [6]. We can calculate the angular momentum exactly on the cross-section on the supertube [6] which reads as

\[ L = \oint ds \left( x_3 \frac{\partial x_4}{\partial s} - x_4 \frac{\partial x_3}{\partial s} \right) \]  

(4.25)

It is this angular momentum which provides the stabilization of the radius of the tubular D2 brane stretched between two D4 branes.

What physics the dyonic instanton represents in the four-dimensional space-time? The solution is defined in \( R^4 \times I \) space-time that is it presumably corresponds to some tunneling process in the Minkowski space. There are a few arguments supporting such interpretation. First note that the dyonic instanton can be generalized to the
multiple circular tubular D2 branes \[8\]. The solution has zero net R-R D2 brane charge however there is non-vanishing dipole four-form field strengths

\[ G_4 = dC_3 - dB_2 \wedge C_1 \]  

(4.26)

and 3-form dipole moment is proportional to the angular momentum. Hence there could be the process of the droppping out the angular momentum and correspondingly the $G_4$ field via the shell repulsion \[8\].

In the Euclidean space-time one could search for the negative mode which upon the analytic continuation would provide instability in the Minkowski space. Such negative mode responsible for the expanding of the radius of the solution has been found for the large radii in \[8\]. Similar negative mode in the supertube worldvolume theory for the large radius has been also found in the Goedel metrics \[23\]. Hence at least at large radius of the dyonic instanton fixed by its angular momentum there is negative mode in $R^4$ which supports the bounce interpretation. Remark that usually the instability reflects the attempt of the system to screen the external field like in Schwinger mechanism. In the dyonic instanton case the background also involves the angular momentum density hence the emerging objects should have the angular momentum to screen the background one as well.

Assuming that we are in the region where the negative mode exists the fate of the solution upon the analytic continuation can be questioned. The most subtle point concerns the fate of the topological charge. Indeed since the solution is just blown up instanton it carries the topological charge. What happens with it upon the analytic continuation from $R^4$ to $R^{3,1}$? Naively we get the dressed monopole-antimonopole pair hence the initial instanton charge has to be somehow "divided" between them. Since from Minkowski viewpoint the instanton corresponds to the tunneling between two states with the different Chern numbers we can conjecture that two "dressed monopoles" with the fractional instanton numbers are involved. Such states are familiar in the theory with the compact dimensions. Certainly this issue deserves for further investigation.

5 Angular momentum in the Liouville theory

5.1 The logarithmic operators

According to the AGT correspondence \[9\] the Nekrasov partition function coincides with the conformal block in the Liouville theory. Hence we could inspect the derivatives with respect to the equivariant parameters at the Liouville side. Of course we
would like to gain some new insights from this formal procedure. The main goal is to identify the group of rotation of $R^4$ in Euclidean space or Lorentz group in the Minkowski version in the Liouville theory. The naive guess could be that the algebra generated by screening operators should play the key role. We shall not provide the complete answer but a few findings are quite inspiring.

The parameters $\epsilon_i$ enter conformal weights of the operators involved into conformal block
\begin{equation}
\alpha_i = Q/2 + a_i
\end{equation}
and the central charge
\begin{equation}
c = 1 + 6Q^2, \quad Q = b + b^{-1}, \quad b^2 = \frac{\epsilon_1}{\epsilon_2}
\end{equation}
The derivative amounts to the substitution of the one of the vertex operators by the operator
\begin{equation}
\tilde{V}_\alpha = \phi \exp \alpha \phi
\end{equation}
which is typical logarithmic operator in the CFT. Hence formally the insertion of the U(1) angular momentum in $R^4$ corresponds to the insertion of the Liouville field $\phi$ at the marked points.

Similar expression can be derived in the semiclassical approximation where the Liouville correlator $\Phi(z_i)$ can be expressed in terms of the classical Liouville action calculated at the solutions to the equation of motion with the prescribed behavior at the insertion points $z_i$
\begin{equation}
\Phi(z_i) \propto \exp(\frac{1}{b^2} S_{cl}(z_i))
\end{equation}
Hence the variation with respect to the equivariant parameters gets reduced to the variation of the classical action with respect to the moduli of the solutions. It is convenient to use the symplectic form on the moduli space of the solutions to the classical equation of motion in the Liouville theory [21]. It involves besides the terms term corresponding to the insertion positions $z_i$ the relevant terms at each point
\begin{equation}
\omega \propto \sum_i d\phi(z_i) \wedge d\alpha_i
\end{equation}
Therefore we can use this canonically conjugated pair to express the derivative with respect to the weights in terms of the values of the Liouville field
\begin{equation}
\frac{\partial S}{\partial \alpha_i} \propto \phi(z_i)
\end{equation}
in agreement with the formal arguments.
5.2 Adding the surface operator

Let us make some comments concerning the effect of the surface operators on the vacuum gravimagnetization. Remind that the surface operators correspond to the D2 branes filling the two-dimensional sub-manifold of $R^4$ which can be considered from the four-dimensional viewpoint as the worldsheet of the magnetic string. When both $\epsilon_{1,2}$ parameters are switched on to some extent one can interpret this background as the condensate of the magnetic strings. On the gauge theory side the insertion of the surface operator oriented in some plane in $R^4$ is described in the semiclassical weak field limit as

$$Z_{Nek,sur} = \exp\left( \frac{\mathcal{F}}{\epsilon_1\epsilon_2} + \frac{\mathcal{W}}{\epsilon_1} + \ldots \right)$$ (5.33)

hence the wave function of the D2 brane reads as

$$\Psi_{sur} \propto \exp\left( \frac{\mathcal{W}}{\epsilon_1} \right)$$ (5.34)

The derivative of the partition function with the insertion of the surface operator yields in the semiclassical limit the additional contribution into the angular momentum in $R^4$

$$\delta < J > \propto W(a, z)$$ (5.35)

where coordinate $z$ corresponds to $x_6$ coordinate of the D2 brane. Equivalently the $W$ has the interpretation of the twisted superpotential in the worldvolume theory on D2 brane or the Yang-Yang function in the integrable systems [25].

Let us turn to the Liouville side where the surface operator corresponds to the insertion of the $V_{1,2}$ primary operator [22, 23]. The contribution of the surface operator into gravimagnetization corresponds to the insertion of the logarithmic primary field

$$\tilde{V}_{1,2} = \phi V_{1,2}$$ (5.36)

into the Liouville correlator. It was found in [20] that logarithmic primaries in the semiclassical limit obey the following relations

$$D_m \tilde{D}_m \tilde{V}_{1,m} = A_m V(1,-m)$$ (5.37)

where $A_m$ are numerical constants and

$$D_m = \partial^m + d_m$$ (5.38)

where $d_m$ involves the energy stress-tensor in the Liouville theory and its derivatives.
The operators $\tilde{V}_{12}$ and $V_{1,-2}$ form the logarithmic pair of the operators with the same conformal dimension. The natural question concerns the interpretation of the object created by the operator $\tilde{V}_{1,2}$ that is dressing of the surface operator. Some analogy comes from the minimal string theory. In that case there are two types of the branes in the Liouville theory - ZZ and FZZT branes [31]. ZZ branes correspond to the D-instantons localized in the Liouville zero mode direction and get condensed. On the other hand the FZZT branes are extended in the Liouville coordinate and correspond to the surface operators. It was observed [24] that the instanton ZZ brane can be interpreted as the superposition of FZZT brane and anti-brane system.

$$|n, m >_{\text{ZZ}} = |n, m >_{\text{FZZT}} - |n, -m >_{\text{FZZT}}$$ (5.39)

We can speculate that the realization of ZZ brane in terms of the pair of FZZT branes is analogous to the blowing up of the instanton.

An interesting interpretation also emerges from the Hamiltonian viewpoint. The surface operators provide the degrees of freedom for the particular Hamiltonian system. One of the equivariant parameters in the $D = 4$ SYM theory can be identified with the Planck constant in this Hamiltonian system [25]. The same interpretation can be derived in the worldvolume theory on the surface operator [26]. Hence we have to differentiate the surface operator wave function with respect to the Planck constant. From the quantum mechanical viewpoint the corresponding angular momentum in $R^4$ plays the role of a kind of the R-symmetry generator. Probably such picture is related to the Parisi-Sourlas type approach to the classical dynamics where the effective R-symmetry can be naturally defined in terms of auxiliary $\mathcal{N} = 2$ SUSY.

### 5.3 On the "Liouville type" representation of the effective action

To complete this Section let us discuss as byproduct the "Liouville type" representation of the conventional $D = 4$ effective action in the background of constant electromagnetic field. To start with let us remind the first quantized representation for the effective actions for the charged particle of mass $m$

$$Z = \sum_{\text{path}} \exp(-mL(C) + i\Phi(C))W(C)$$ (5.40)

where $\Phi(C)$ is the so called spin factor expressed in terms of the trajectories in $CP^3$ and $W(C)$ is the Wilson loop along the contour $C$. The potential need for the Liouville mode to appear is the restoration of the reparametrization invariance
in the summation over the contours. One could expect that the integral over the reparametrization of the contour appears in some form which is familiar in the stringy calculations of the Wilson loops.

However more convenient form of the effective action involves the integral over the Schwinger parameter. In the external self-dual field the one-loop effective action reads as

$$S_{\text{one-loop}} = \int ds \frac{e^{ism^2}}{s (\sinh esE)^2}$$

(5.41)

where we assume that $E = \pm H$. Let us emphasize that the effective action in QED is not holomorphic object hence we would like to represent (5.41) as a kind of "correlator" in the Liouville-type theory that is integrated product of the chiral conformal blocks over the intermediate conformal weights. Therefore the question is whether the integral over the Schwinger parameter can be treated as the integral over the intermediate conformal dimensions in the Liouville theory.

First note that the Schwinger parameter can be treated as the radial coordinate in the AdS space [28, 29]. On the other hand the radial coordinate in AdS geometry can be considered as the zero mode of the Liouville field [27] that is Schwinger parameter is related with the Liouville zero mode. The correspondence can be made more precise. To the aim let us use the observation from [29] that the integrand can be represented in terms of the wave function of the $SL(2, R)$ 2-dimensional YM theory on the disc which yields the solutions to the $AdS_2$ gravity and depends on the boundary holonomy as the argument. The boundary $SL(2, R)$ holonomy can be expressed in terms of the geodesic length

$$tr_{1/2} P \oint_C A = 2 \cosh \frac{l(C)}{2}$$

(5.42)

which in the case under consideration is proportional to $eEs$. Hence the Schwinger parameter is proportional to the geodesic boundary length.

Given these arguments let us remind that the boundary lengths $l_i$ are the natural coordinates on the Teichmueller phase space. It is useful to consider the coherent state representation on the Teichmueller phase space in the Kahler polarization (see [30] for the review). The Liouville conformal block has an interesting interpretation as the wave function in the quantum Teichmueller theory in the coherent representation which is eigenfunction of the length operator [30]

$$\Psi^T_{\text{Tei}}(q) = \Psi^{\text{Liouv}}_L(q)$$

(5.43)

Here $L$ is the length variable in the Teichmueller theory while simultaneously it
parameterizes the weights in the Liouville theory via identification

\[ \alpha = \frac{Q}{2} + \frac{iL}{4\pi b} \]  \hspace{1cm} (5.44)

Hence we see the clear analogy between the Liouville representation of the \( \mathcal{N} = 2 \) SYM in the \( \Omega \)-background and the QED in the constant external field. The integration over the dimension of the intermediate state in the conformal block corresponds to the integration over the Schwinger parameter in QED. Let us emphasize once again that the holomorphic prepotential in \( \mathcal{N} = 2 \) theory corresponds to the conformal block in Liouville theory while the QED effective action plays the role of the correlator. Moreover we have mentioned that \( s \) has the interpretation of the zero mode of the Liouville field therefore to some extent the quantized values of \( s_k \) in the imaginary part calculation correspond to the positions of Liouville ZZ branes \[31\] which are localized at this direction.

6 Discussion

In this Letter we have made the first step towards the investigation of the \( \mathcal{N} = 2 \) SYM vacuum response functions in the graviphoton background focusing on the simplest magnetization-like quantity. It is evident that more complicated response functions, say the graviconductivity, are interesting as well.

It was argued that the vacuum state is absolutely unstable with respect to the generation of the angular momentum in the Euclidean \( \mathbb{R}^4 \) space-time at arbitrarily weak graviphoton field. Moreover at the semiclassical approximation the Seiberg-Witten prepotential itself is proportional to the angular momentum density. Our primary goal was to get some new information concerning SYM vacuum state hence we have tempted to recognize the microscopic degrees of freedom in \( \mathbb{R}^4 \) responsible for the induced angular momentum. Since the instantons themselves can not be the proper degree of freedom we have looked at the dyonic instantons which are carriers of the angular momentum.

The analogy with the non-perturbative Schwinger monopole/dyons pair production turns out to be fruitful. The process is described by the bounces in the Euclidean \( \mathbb{R}^4 \) space-time which are the circular Wilson loop in the electric case and the circular t’Hooft loop in the magnetic case. One can naturally assign to these loops the angular momentum in \( \mathbb{R}^4 \). The dyonic instantons fit with this picture being the monopole loop in \( \mathbb{R}^4 \) with the additional electric and topological charges.

The key point concerns the stabilization mechanism of the Euclidean bounce solution. We have argued that in the graviphoton background the bounce stabilization
is supported by the D0 charge density of the tubular D2 brane. However since the Seiberg-Witten prepotential can be derived without any graviphoton background another source of the stabilization has to be presented. The dyonic instanton indeed provides such mechanism via the nontrivial angular momentum. Hence we arrive at the interesting picture of the Euclidean $R^4$ vacuum populated by the dyonic loops carrying the topological charge density. These loops emerge as the result of the spontaneous gravimagnetization of the instanton medium.

It would be interesting to analyze the possibility of the similar picture of Euclidean vacuum populated by the dyonic instantons in the theories with less amount of SUSY. The mechanism of stabilization of the closed curves from shrinking via the "Poynting angular momentum" seems to work in much more generic setting. In particular the QCD-like theory could support the flavored dyonic instantons stabilized in the similar manner. The role of the dyonic instantons in the monopole condensation upon the perturbation of the $\mathcal{N} = 2$ has to be investigated.

The subtle point concerns the interpretation of medium with the constant density of dyonic loops from the Minkowski viewpoint. In the conventional Schwinger mechanism the account of the multiple bounces corresponds to the sub-leading factors in the production rate however we do not know precisely how the medium of the dyonic instantons could describe the vacuum instability. In general when considering the multiple bounces the effect of their interaction has to be taken into account. Moreover the analytic continuation from the Euclidean to the Minkowski space appears to be the nontrivial problem. Indeed when considering the single bounce the selection of the continuation surface corresponding to the turning points is a simple issue. However in the case of the multiple interacting bounces the choice of the proper continuation surface is complicated problem.

Having in mind the Euclidean bounce picture the prepotential itself describes the semiclassical Hartle-Hawking wave function of the creating object. This is consistent with the interpretation of the prepotential as the logarithm of the Whitham wave function in the Hamiltonian picture. It would be interesting to realize the proper meaning of the attractor equations in our approach.

Using the AGT relation we commented on the angular momentum issue in terms of the Liouville conformal blocks and argued that the logarithmic operators in the Liouville theory are relevant. The logarithmic Liouville theory generically involves the infinite number of degenerate operators and the role of the Zamolodchikov higher equations of motion has to be clarified.

It is worth noting that the dyonic instanton medium in $R^4$ has evident similarities with the Quantum Hall system hence the methods and concepts developed there could be of some use in the SYM context. We can not exclude that the system of
strongly interacting dyonic loops could form a kind of QHE-like vacuum droplet. In this case the subtle transition from the Euclidean to the Minkowski picture would be more simple. Note that the analogy between the instability of the Goedel metrics and the motion in the magnetic field has been mentioned in [32, 33]. In particular the issue of CTC has been mapped into the problem of the closeness of the trajectories in the external field.

We also can speculate that the discussion above has something to do with the Regge calculus. Indeed we have argued that dyonic instantons are natural objects populating Yang-Mills $R^4$ Euclidean vacuum whose stabilization is supported by the angular momentum. This issue has some resemblance with the representation of the reggeons in the brane terms [34]. We hope to discuss these issues elsewhere.

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