Hawking-Moss bounces and vacuum decay rates

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The conventional interpretation of the Hawking-Moss (HM) solution implies a transition rate between vacua that depends only on the values of the potential in the initial vacuum and at the top of a potential barrier, leading to the implausible conclusion that transitions to distant vacua can be as likely as those to a nearby one. I analyze this issue using a nongravitational example with analogous properties. I show that such HM bounces do not give reliable rate calculations, but are instead related to the probability of finding a quasistable configuration at a local potential maximum.

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A quantum field theory may have, in addition to the “true vacuum” of minimum energy, one or more metastable “false vacua” of higher energy. The latter can decay by nucleating, through either quantum tunneling or thermal fluctuation, bubbles of true vacuum that then expand and coalesce. In a series of papers [1, 2, 3], Coleman and collaborators developed a formalism, based on “bounce” solutions of the Euclideanized field equations, for calculating the rate of such bubble nucleation. When this formalism is extended to take gravity into account, one finds not only the Coleman-De Luccia (CDL) bounces [3] that seem analogous to the flat-space bounces, but also the spatially homogeneous Hawking-Moss (HM) bounce [4] that appears to represent a fluctuation from the false vacuum to the top of the potential barrier separating the true and false vacua.

A striking feature of the HM bounce is that, by the conventional interpretation, it implies a transition rate that depends only on the values of the potential at the top of the barrier and at the false vacuum, but not at intermediate points. For a theory with only a single false vacuum, this may perhaps be seen as just a curiosity, although it is certainly a bit troubling. However, it becomes particularly salient when viewed in the context of a possible string theory landscape. If the transition rate depends only on the initial and final values of the potential, then transitions to distant parts of the landscape could be as likely as those to nearby points. Although the probability of any given transition might be exponentially small, the exponentially large number of potential final states could make the lifetime of any one vacuum, even one with a small cosmological constant, quite short. In this letter, I will re-examine the reasoning that leads to this conclusion, and show where it fails.

To begin, let us recall the main features of the bounce formalism. Consider the theory of a single scalar field $\phi$ with a potential such as that shown in Fig. 1. In the absence of gravity, and at zero temperature, the bounce is a solution of the four-dimensional Euclidean field equations with a region of approximate true vacuum surrounded by an infinite false vacuum region. For the case of an $O(4)$-symmetric bounce that depends on a single radial variable $s$, we have $\phi(\infty) = \phi_{tv}$, while $\phi(0)$ lies on the true vacuum side of the potential barrier. Note that $\phi(0) \neq \phi_{tv}$. The rate per unit volume at which true vacuum bubbles nucleate within a false vacuum region is of the form $\Gamma = A e^{-B}$, where $B = S_E(\phi_{bounce}) - S_E(\phi_{tv})$ is the difference between the Euclidean action of the bounce solution and that of the homogeneous configuration with $\phi = \phi_{tv}$ everywhere. (We will not need the detailed expression for $A$ in this discussion.)

Gravity can be incorporated into this formalism by including the Euclidean action for gravity in $S_E$, and solving for both the scalar field and the metric. If $V(\phi)$ is everywhere non-negative, then the bounce solution necessarily has the topology of a four-sphere. In the CDL bounce the scalar field at one pole has a value that is on the true vacuum side of the barrier, while at the antipodal point it is on the false vacuum side. In Fig. 1 $\phi_1$ and $\phi_2$ indicate these values for a “typical” CDL bounce.

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In limits where one would expect gravitational effects on bubble nucleation to be small, $|\phi - \phi_{\text{fv}}|$ is exponentially small and the CDL result for $B$ approaches that of the flat space calculation.

In the HM solution the metric is the standard round metric on a four-sphere of radius $H_{\text{top}}^{-1} \equiv [8\pi V(\phi_{\text{top}})/3M_{\text{Pl}}^2]^{-1/2}$, with $\phi = \phi_{\text{top}}$ everywhere on the four-sphere. A calculation of its Euclidean action leads to a tunneling exponent

$$B_{\text{HM}} = S_E(\phi_{\text{top}}) - S_E(\phi_{\text{fv}}) = -\frac{3M_{\text{Pl}}^4}{8V(\phi_{\text{top}})} + \frac{3M_{\text{Pl}}^4}{8V(\phi_{\text{fv}})}.$$  \hspace{1cm} (1)

In the case where $|V(\phi_{\text{top}}) - V(\phi_{\text{fv}})|/V(\phi_{\text{fv}}) \ll 1$, this can be approximated by

$$B_{\text{HM}} \approx \frac{(4\pi/3)H_{\text{fv}}^{-3}[V(\phi_{\text{top}}) - V(\phi_{\text{fv}})]}{T_{\text{fv}}}, \hspace{1cm} (2)$$

where $H_{\text{fv}}^{-1} \equiv [8\pi V(\phi_{\text{fv}})/3M_{\text{Pl}}^2]^{-1/2}$ and $T_{\text{fv}} = H_{\text{fv}}/2\pi$ is the de Sitter temperature of the false vacuum. The form of the latter expression suggests an interpretation of the HM bounce as corresponding to a thermal fluctuation of a horizon-sized region up to the top of the barrier, following which the field can roll (either homogeneously or spinodally), down to either the false or the true vacuum [8]. [There is no contribution from gravitational energy in Eq. (2) because it is obtained by taking the limit in which the difference in $V$, and hence in the geometry, between the false vacuum and the top of the barrier is small.]

The difficulty, alluded to above, is that for a potential with many local maxima there are HM bounces corresponding to each of these maxima. Thus, for the potential shown in Fig. 2 there are HM solutions associated with the local maxima of $V$ at $\phi_A$ and $\phi_B$, in addition to the one associated with $\phi_{\text{top}}$. If any of these maxima are degenerate with the one at $\phi_{\text{top}}$, then the standard interpretation would imply that the transitions from $\phi_{\text{fv}}$ to these more distant points would occur at the same rate as transitions to $\phi_{\text{top}}$.

Since this conclusion seems intuitively untenable, one is led to ask what might be wrong with the reasoning involved. The first thought might be that the HM bounce is somehow pathological, and that one should ignore it and use only CDL bounces. The difficulty with this is that it is always possible to continuously deform the potential in such a way that the endpoints of the CDL bounce move up toward, and eventually meet at, $\phi_{\text{top}}$ [4, 7]. The CDL bounces with endpoints near $\phi_{\text{top}}$ will have counterparts that are concentrated near the maxima at $\phi_A$ and $\phi_B$, and these will lead to conclusions similar to, and as discomforting as, those following from the HM bounces.

At this point one might decide that gravity, especially in its Euclidean form, is too subtle and that one should not trust the CDL formalism; further support for this idea might come from noting that this formalism was obtained by analogy with the flat space case rather than through a rigorous derivation. However, one is faced with the fact that the CDL bounces seem to give reasonable results in the limit where gravitational effects are small.

Furthermore, there is a non-gravitational example that also suffers from the same difficulties. Consider the scalar field theory described above on a compact space that is a three-sphere of radius $L$, with a nonzero temperature $T$. In the finite temperature extension of the bounce formalism [8, 9], one looks for Euclidean solutions with periodicity $1/T$ in the imaginary time. For $T \gg L^{-1}$, the dominant bounces are constant in imaginary time. This effectively reduces the problem to a three-dimensional one that will have homogeneous solutions completely analogous to the four-dimensional HM bounces and, for suitable potentials, CDL-type solutions that are confined to the region near the maxima of the potential [10]. Thus, it is clear that the difficulties we have found cannot be attributed to Euclidean gravity.

These remarks lead us to ask the following questions:

1) Does the bounce formalism indeed go wrong for some or all of the HM, and nearby CDL, bounces and, if so, why?

2) If a HM bounce doesn’t calculate a transition rate, then what does it calculate?

In order to emphasize that the origin of these difficulties does not lie in the gravitational setting, as well as to avoid some of the conceptual issues associated with quantum field theory in curved spacetime, I will address these questions using the finite volume, finite temperature example (although I will adopt the HM and CDL terminology to refer to the corresponding bounces).

To begin, let us recall the reasoning that led to the bounce formalism. At zero temperature and infinite volume, with gravity ignored, bubble nucleation is a quantum tunneling process in which one tunnels through the potential energy barrier that separates a spatially homogeneous configuration with $\phi(x) = \phi_{\text{fv}}$ from a configuration that has a bubble of approximate true vacuum surrounded by false vacuum exterior. The one-dimensional WKB treatment of quantum tunneling is
readily extended to systems with many degrees of freedom [11,12]. For each possible path through the barrier one can define a WKB tunneling exponent $B$. The tunneling rate is obtained from the path that minimizes $B$. By arguments familiar (apart from some changes of sign) from classical mechanics, one can show [1] that this minimization problem is equivalent to finding a stationary point of the Euclidean action. This stationary point is the bounce solution.

A more precise formulation is obtained by a path integral argument [2]. Before describing this, it will be best to specify more clearly what is meant by the false vacuum. This is unambiguous classically, where it is the homogeneous state with $\phi(x) = \phi_{\text{FV}}$. In the quantum field theory, however, things are not so simple. It is not sufficient to require that the expectation value of the field $\langle \phi(x) \rangle = \phi_{\text{FV}}$, because this condition is also satisfied by states corresponding to mixed regions of approximate true and approximate false vacua. In addition, we want to require that the wave functional be concentrated on the true and approximate false vacua. This can be made more precise by considering a deformed potential that agrees with $V(\phi)$ in the potential well containing $\phi_{\text{FV}}$ but that increases monotonically outside this interval. The theory defined by this deformed potential has a stable ground state. This is unambiguous classically, where it is the homogeneous three-dimensional configuration with exponentially small overlap with $|\phi_{\text{FV}}\rangle$. However, the integration should only be over periodic configurations with nontrivial overlap with the $|\phi_{\text{FV}}\rangle$.

Now consider the path integral

$$I(T) = \int [d\phi] e^{-S_{\text{E}}[\phi]} = \langle \phi_{\text{FV}} | e^{-H_{\text{E}} T} | \phi_{\text{FV}} \rangle,$$  \hspace{1cm} (3)

where the integration is over configurations satisfying $\phi(x, \pm T/2) = \phi_{\text{FV}}$, while $|\phi_{\text{FV}}\rangle$ denotes the state whose wave functional is a delta functional at $\phi(\phi(x) = \phi_{\text{FV}}$. The standard procedure would now be to expand the matrix element on the right-hand side in terms of a complete set of energy eigenstates and to note that in the limit $T \to \infty$ the sum would be dominated by the term corresponding to the lowest energy state. This would be the true vacuum, whereas we are interested in the false vacuum. Note, however, that the overlap of the true vacuum with $|\phi_{\text{FV}}\rangle$ is exponentially small. Indeed, there is a similar suppression for any state that is orthogonal to (or that has exponentially small inner product with) the $|\nu_{\text{FV}}\rangle$ defined above. Hence, for a large range of $T$ the sum is dominated not by the absolutely lowest lying state, but rather by the lowest of the $|\nu_{\text{FV}}\rangle$ (i.e., by the false vacuum $|\text{FV}\rangle$), giving

$$I(T) \approx |\langle \text{FV} | \phi_{\text{FV}} \rangle|^2 e^{-E_{\text{FV}} T}.$$ \hspace{1cm} (4)

After evaluating $I(T)$ by expanding about its stationary points (the classical false vacuum, the bounce, and the multibounce solutions), one can read off $E_{\text{FV}}$ from its large $T$ behavior. Because $|\text{FV}\rangle$ (like all of the $|\nu_{\text{FV}}\rangle$) is not quite an eigenstate of the full Hamiltonian, $E_{\text{FV}}$ is complex. Its imaginary part corresponds to a decay rate and yields the desired nucleation rate $\Gamma$.

The finite temperature tunneling rate is obtained not from the imaginary part of the energy of the false vacuum, but rather from the imaginary part of the free energy of the false vacuum $\langle \text{FV} | E_{\text{FV}} | \text{FV} \rangle$. This can be gotten by replacing the path integral in Eq. (3) by a similar integral in which the integration is over configurations that are periodic with period $1/T$ in imaginary time. This gives the partition function $Z = \text{Tr} e^{-H/T}$, and thus the free energy [17]. An unrestricted integral over all periodic configurations would correspond to a trace over all states, and would give the full partition function for the theory. Since what we actually want is the partition function corresponding to the metastable false vacuum, the trace must be restricted to the subspace spanned by the $|\nu_{\text{FV}}\rangle$. Hence, the integration should only be over periodic configurations with nontrivial overlap with the $|\nu_{\text{FV}}\rangle$.

In the infinite volume case, this restriction is automatically imposed by the requirement that the three-dimensional configurations have finite energy, relative to that of the false vacuum, which implies that they must approach $\phi_{\text{FV}}$ at spatial infinity. When the volume is finite, on the other hand, any smooth configuration will have finite energy. Hence, a naive application of the bounce formalism could well admit configurations that do not contribute to $Z_{\text{FV}}$, and that are therefore not relevant for a calculation of the lifetime of the false vacuum.

We now see how the bounce formalism can go astray for a HM bounce. In order that the false vacuum be metastable (i.e., that it persist for a substantial length of time), homogeneous configurations lying outside the potential well surrounding $\phi_{\text{FV}}$ must be far out in the exponential tails of the wave functionals both of $|\text{FV}\rangle$ and of the excited false vacuum states $|\nu_{\text{FV}}\rangle$. Hence, a homogeneous three-dimensional configuration with $\phi$ at the top of a distant local maxima is necessarily among those that must be excluded, and so the corresponding HM bounce cannot lead to a reliable calculation of $\Gamma$.

The situation is more subtle for the HM bounces corresponding to the maxima that bound the false vacuum potential well. These, too, correspond to configurations with exponentially small overlap with $|\text{FV}\rangle$. However, they can have nontrivial overlap with those excited false vacuum $|\nu_{\text{FV}}\rangle$ states with energies comparable to the top of the barrier. At finite temperature these excited states will be populated, although that population will be thermally suppressed. In fact, for states with energy equal to the top of the barrier, this suppression factor is (up to possible pre-exponential factors) precisely $e^{-B_{\text{HM}}}$. Thus, one can view this HM bounce as representing an evaporation process rather than a tunneling one [13]. Alternatively, if the potential is sufficiently flat near $\phi_{\text{top}}$, stochastic evolution can bring the field to the top, again with a rate given by the HM bounce [14,15,16].
The situation with CDL bounces with endpoints near a potential maximum is similar. If the potential maximum does not bound the false vacuum potential well, then the CDL bounce is not relevant for the decay of the false vacuum. If, instead, the potential maximum is adjacent to the false vacuum well, the bounce, even if it has negligible overlap with the $|\text{FV}|$, will overlap with thermally excited false vacuum states and will contribute to $\Gamma$; it can be understood as representing thermally assisted tunneling.

Although a distant HM bounce may not be a reliable guide to the transition rate out of the false vacuum, this does not mean that it has no physical meaning; elegant answers in physics seldom lack a question. Indeed, the discussion above makes its significance clear. The potential maximum does not bound the false vacuum potential well. If, instead, the potential maximum is similar. If the potential maximum displays the same difficulties as the potential with an unconstrained maximum, since the gravitational HM bounce refers to a system with a finite horizon volume and a nonzero de Sitter temperature.

The arguments presented in this note have addressed a particularly striking anomaly associated with the HM solution. However, there remain a number of other unusual features associated with tunneling in curved spacetime including, among others, the fact that the CDL bounce is not sensitive to the shape of the potential in a finite region including the initial vacuum. It is clear that a fuller account of curved spacetime tunneling is needed. A partial step in that direction will be presented elsewhere [18].

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