Spin Hydrodynamics and Symmetric Energy-Momentum Tensors
– A current induced by the spin vorticity –

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We discuss a puzzle in relativistic spin hydrodynamics; in the previous formulation the spin source from the antisymmetric part of the canonical energy-momentum tensor (EMT) is crucial. The Belinfante improved EMT is pseudo-gauge transformed from the canonical EMT and is usually a physically sensible choice especially when gauge fields are coupled as in magnetohydrodynamics, but the Belinfante EMT has no antisymmetric part. We find that pseudo-transformed entropy currents are physically inequivalent in nonequilibrium situations. We also identify a current induced by the spin vorticity read from the Belinfante symmetric EMT.

Introduction: Polarization measurements of Λ and ¯Λ baryons in the relativistic heavy-ion collision have attracted lots of theoretical interest [1], which is driven by a recent confirmation of the global polarization of Λ and ¯Λ that signifies “the most vortical fluid” [2], as predicted in Ref. [3] and thermally quantified in Ref. [4]. The underlying physics is intuitive: non-central collisions provide hot and dense matter with the orbital angular momentum as large as ~10^6ℏ at the collision energy \( \sqrt{s_{NN}} = 200 \text{ GeV} \) with the impact parameter \( b \sim 10 \text{ fm} \) [5, 6]. Because of the spin-orbital coupling in relativistic systems, a finite angular momentum can be transported from the orbital angular momentum (OAM) to the spin angular momentum (SAM). We note that only the total angular momentum (TAM) is a conserved quantity associated with rotational symmetry in relativistic theories but the spin degrees of freedom is dissipative in a relativistic fluid. Thus, only the particle intrinsic spin affects the thermal abundance at the last stage, but the relativistic spin hydrodynamics is needed for thorough understanding of the spin evolution from the initial condition. There are actually some theoretical estimates based on parton cross section [3, 7], and we should clarify a missing bridge between the partonic and the thermal estimates. For some developments, see recent reviews [8, 9] and the references therein.

For the spin hydrodynamics pioneering works are found in Refs. [10, 11] where the Lagrangian effective field theory is adopted to approach the spin polarized medium. In Ref. [12] the spin hydrodynamic equations are derived from the kinetic equations for particles and antiparticles with spin 1/2. More recently, the spin hydrodynamic equations have been derived from the decomposition of the energy-momentum tensor (EMT) and the entropy current analysis in Ref. [13]. See Ref. [14] for another approach and discussions in gravitational physics and also Refs. [15–17] for recent discussions based on the Bjorken hydrodynamics. Another possible hint to build the spin hydrodynamics comes from the quantum kinetic theory for massive fermions with collisions, e.g., see Refs. [18–24].

In the present work we focus on a controversy in regard to pseudo-gauge ambiguity of the EMT for the spin physics. Similar problems have been well known also in the context of the proton spin decomposition; conventionally, the spin decomposition based on the canonical EMT is called the Jaffe-Manohar decomposition, while an improved EMT that is symmetric and gauge invariant gives the Ji decomposition (see a recent essay [25] and references therein). In gauge theories such as quantum chromodynamics (QCD) the symmetric EMT is manifestly gauge invariant, and is directly related to physical observables. Also in the future electron-ion collider (EIC) experiment the proton EMT will be measured [26, 27]. For this the symmetric EMT is empirically assumed.

The central puzzle in the context of the spin hydrodynamics lies in the fact that the spin degrees of freedom seem to appear from the antisymmetric component of the canonical EMT. As we mentioned above, however, the canonical EMT is not gauge invariant if gauge fields are involved as in magnetohydrodynamics. One might think of a way to enforce gauge invariance on the canonical EMT [28–30], but such a prescription requires non-local gauge potentials, which is not systematically implemented in hydrodynamics. Therefore, it would be theoretically preferable to formulate the spin hydrodynamics based on the symmetric EMT or the Belinfante improved form of the EMT after an appropriate pseudo-gauge transformation. The technical problem is, however, that one can no longer extract the spin part from the antisymmetric component that identically vanishes in the Belinfante form. It is a common consensus that the canonical and the Belinfante forms are both qualified as physical EMT’s, and yet only the canonical EMT works for the derivation of the spin hydrodynamics as employed in Ref. [13]. We note that some inequivalent
properties between different EMTs have been revealed in nonequilibrium environments [31–33], but this difference would not necessarily exclude a possibility to derive the spin hydrodynamics from the Belinfante EMT. Actually we will pursue this possibility and eventually reach a conclusion to support discussions by those preceding works in an illuminating way.

Here, let us summarize our notation. The metric is $g_{\mu\nu} = \text{diag}(+,-,-,-)$ and the projection operator in our convention is $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$ with the four-vector fluid velocity $u^\mu$. We use $T^{\mu\nu}$ to represent the Belinfante EMT, while $\Theta^{\mu\nu}$ is the canonical one. Also, we define $J^{\mu\nu\alpha\beta}$ for the TAM in the Belinfante form and $J_{\text{can}}^{\mu\nu\alpha\beta}$ for the TAM in the canonical form. For an arbitrary tensor $A^{\mu\nu}$, we define its symmetric and antisymmetric parts as $A_{(s)}^{\mu\nu} = \frac{1}{2}(A^{\mu\nu} + A^{\nu\mu})$ and $A_{(a)}^{\mu\nu} = \frac{1}{2}(A^{\mu\nu} - A^{\nu\mu})$. We also use a symbol, $< ... >$, to mean the traceless part, i.e., $A^{<\mu\nu>} = \frac{1}{2}[(\Delta^{\alpha\mu} \Delta^{\nu\beta} + \Delta^{\nu\mu} \Delta^{\alpha\beta}) A_{\alpha\beta} - \frac{1}{3} A^{\alpha\beta}(\Delta^{\alpha\mu} \Delta^{\nu\beta})]$.

Canonical vs. Belinfante formulations: To make our point clear we shall make a brief review of the spin hydrodynamics from the canonical EMT as discussed in Ref. [13]. In the canonical form the TAM can be decomposed into

$$J_{\text{can}}^{\mu\nu} = x^\mu \Theta^{\nu\alpha} - x^\nu \Theta^{\alpha\mu} + \Sigma^{\alpha\mu\nu},$$

where $x^\mu \Theta^{\nu\alpha} - x^\nu \Theta^{\alpha\mu}$ and $\Sigma^{\alpha\mu\nu}$ represent the OAM and the SAM tensors, respectively. From the conservation laws of the TAM and the EMT we readily find,

$$\partial_\alpha \Sigma^{\alpha\mu\nu} = -2 \Theta_{(a)}^{\mu\nu}$$

with $\Theta_{(a)}^{\mu\nu}$ being the antisymmetric part of the canonical EMT, which is understood as spin nonconservation in relativistic systems.

Recalling that the spin in the quantum field theory is $S^{\beta\gamma} \sim S^{\gamma} = e^{ijk} S^{kj}$, we can decompose the spin tensor in terms of hydrodynamical variables as follows:

$$\Sigma^{\alpha\mu\nu} = u^\alpha S^{\mu\nu} + \Sigma_{(1)}^{\mu\nu}.$$  

We can understand Eq. (3) in analogy to decomposition of the charge current: $j^{\mu} = n u^\mu + j_{(1)}^{\mu}$ (where $u \cdot j_{(1)} = 0$), with the charge density $n$ and the dissipative current $j_{(1)}^{\mu}$ from the higher order in the gradient expansion. Correspondingly, we can identify $S^{\mu\nu}$ as the spin density and $\Sigma_{(1)}^{\mu\nu}$ as the dissipative higher order correction. We can neglect $\Sigma_{(1)}^{\mu\nu}$ since only $\partial_\mu \Sigma_{(1)}^{\alpha\mu\nu} \sim O(\partial^2)$ appears that is beyond the order focused in this work.

In deriving the hydrodynamic equations the entropy current and the second law of thermodynamics are essential. For this purpose we need to express the entropy current involving the spin tensors. The thermodynamic relation in equilibrium reads,

$$e + p = Ts + \mu n + \omega_{\mu\nu} S^{\mu\nu},$$

where $e$, $p$, $T$, $s$, and $\mu$ are the energy density, the pressure, the temperature, the entropy density, and the chemical potential, respectively. We also introduced the spin potential, $\omega_{\mu\nu}$, according to the prescription of Ref. [13]. For simplicity, we only consider one $U(1)$ conserved charge, e.g., the electric charge or the baryon charge. If necessary, we can easily extend our discussion to multiple conserved charges. For actual calculations differential forms of Eq. (4) are convenient; namely, $de = Tds + \mu dn + \omega_{\mu\nu} S^{\mu\nu} + dp =sdT + n d\mu + S^{\mu\nu} d\omega_{\mu\nu}$. In the present convention $e$ is a function of $S^{\mu\nu}$, while $p$ is a function of $\omega_{\mu\nu}$.

Now, let us introduce a nonequilibrium entropy current $S_{\text{can}}^{\mu\nu}$ following a prescription of Ref. [34]. In the presence of the spin density and the spin potential we can postulate:

$$S_{\text{can}}^{\mu\nu} = \frac{u_{\nu}}{T} \Theta^{\mu\nu} + \frac{p}{T} u^{\mu} - \frac{\mu}{T} j^{\mu} + \frac{1}{T} \omega_{\rho\sigma} S^{\rho\sigma} u^{\mu} + O(\partial^2)$$

$$= s u^{\mu} + \frac{u_{\nu}}{T} \Theta^{\mu\nu} - \frac{\mu}{T} j^{\mu} + O(\partial^2),$$

where $\Theta_{(a)}^{\mu\nu}$ as well as $j_{(1)}^{\mu}$ denotes dissipative terms. This explicit form clearly shows that the entropy current has a definite relation to the equilibrium entropy up to the leading order, but the higher orders are not uniquely constrained. Therefore, Eq. (5) should be regarded as an Ansatz.

Using Eq. (4) and $u_{\nu} \partial_\mu \Theta^{\mu\nu} = 0$, we can prove $T \partial_\mu (s u^{\mu}) = \mu \partial_\mu j_{(1)}^{\mu} + \omega_{\rho\sigma} \partial_\mu (S^{\rho\sigma} u^{\mu}) + u_{\nu} \partial_\mu \Theta^{\mu\nu} = 0$. This significantly simplifies the divergence of the entropy current into

$$\partial_\mu S_{\text{can}}^{\mu\nu} = -j_{(1)}^{\mu} \partial_\mu \frac{\mu}{T} - \omega_{\rho\sigma} \partial_\mu (u^{\mu} S^{\rho\sigma}) + \Theta^{\mu\nu}_{(1)} \partial_\mu \frac{u_{\nu}}{T}.$$  

In the right-hand side we can use $\partial_\mu (u^{\mu} S^{\rho\sigma}) = -2 \Theta^{\rho\sigma}_{(a)} + O(\partial^2)$ which comes from Eqs. (2) and (3). At the first order, moreover, the tensor decomposition leads to $\Theta_{(1)}^{\mu\nu} = \Theta^{\mu\nu}_{(1s)} + \Theta^{\mu\nu}_{(1a)}$ with

$$\Theta^{\mu\nu}_{(1s)} = 2 h [u^\nu + \pi^{\mu\nu}], \quad \Theta^{\mu\nu}_{(1a)} = 2 q [u^\nu + \phi^{\mu\nu}].$$

As usual $\pi^{\mu\nu}$ is the viscous tensor and $\phi^{\mu\nu}$ is its antisymmetric counterpart. Likewise, $h^\mu$ is the heat flow and $q^\mu$ is its counterpart in the antisymmetric sector. In calculational steps $u_{\mu} \pi^{\mu\nu} = u_{\mu} \phi^{\mu\nu} = u \cdot q = u \cdot h = 0$ will be useful. As discussed in Ref. [13] we can collect terms involving $\pi^{\mu\nu}$, $\phi^{\mu\nu}$, $h^\mu$, and $q^\mu$ and identify their tensorial forms from the sufficient condition for the second law of thermodynamics, $\partial_\mu S_{\text{can}}^{\mu\nu} \geq 0$, as realized in a form of sum of squares.

Then, $\pi^{\mu\nu}$ and $h^\mu$ are found to have no spin corrections, while $q^\mu$ and $\phi^{\mu\nu}$ are found to have terms $\propto \omega^{\mu\nu}$ as

$$q^\mu = \lambda [T^{-1} \Delta^{\alpha\nu} \partial_\mu T + (u \cdot \partial) u^\mu - 4 \omega^{\mu\nu} u_{\nu}],$$

$$\phi^{\mu\nu} = -\gamma (\Theta^{\mu\nu} - 2 T^{-1} \Delta^{\alpha\beta} \omega_{\alpha\beta}).$$
where $\Omega^{\mu\nu} \equiv -\Delta^{\mu\nu} \Delta^{\rho\sigma} \partial_\rho (\beta u_\sigma)$ is usually referred to as the thermal vorticity [33], and $\lambda$ and $\gamma$ are nonnegative transport coefficients. We can reasonably understand the physical interpretation: The rotation carried by the fluid velocity and the thermal gradient together with the spin chemical potential plays a role of the source to produce/absorb the spin. Then, the spin hydrodynamics dictates the evolution of $\omega_{\mu\nu}$ or $S^{\mu\nu}$ and the local thermal equilibrium relation, $S^{\mu\nu} = \partial p/\partial \omega_{\mu\nu}$, imposes a connection between them.

From above discussions it is clear that Eq. (2) is crucial for constructing hydrodynamics with spin degrees of freedom, and it seems to be indispensable to keep $\Theta^{\mu\nu}_{(a)}$. The EMT, however, has pseudo-gauge invariance, and one can always choose a symmetrized or Belinfante improved EMT form without losing physics contents.

**Spin strikes back:** The confusion lies in the absence of the antisymmetric part of the Belinfante EMT which implies no spin degrees of freedom at all. We obtain the symmetric Belinfante EMT by the following pseudo-gauge transformation:

$$T^{\mu\nu} = \Theta^{\mu\nu} + \frac{1}{2} \partial_\lambda \left( u^\lambda S^{\mu\nu} - u^\nu S^{\mu\lambda} + u^\mu S^{\nu\lambda} \right),$$

$$K^{\lambda\mu\nu} = \frac{1}{2} \left( \Lambda^{\lambda\mu\nu} - \Lambda^{\mu\lambda\nu} + \Lambda^{\nu\lambda\mu} \right).$$

With this choice we can get rid of the spin source and it is easy to confirm that $T^{\mu\nu}$ is symmetric; $T^{\mu\nu} = T^{\nu\mu}$. Here, $K^{\lambda\mu\nu}$ is antisymmetric with respect to $\lambda$ and $\mu$, so that $\partial_\mu T^{\nu\mu} = 0$ still holds as long as $\partial_\lambda \Theta^{\mu\nu} = 0$. In other words we have an identity,

$$\partial_\mu \partial_\lambda \left( u^\lambda S^{\mu\nu} + u^\nu S^{\mu\lambda} + u^\mu S^{\nu\lambda} \right) = 0,$$

from Eqs. (3) and (11). This equation corresponds to the “quantum spin vorticity principle” in the quantum spin vorticity theory [35].

The Belinfante improved TAM, which is a counterpart of Eq. (1), reads,

$$J^{\alpha\mu\nu} = x^{\nu} T^{\alpha\nu} - x^{\nu} T^{\alpha\mu},$$

where $J^{\alpha\mu\nu} \equiv J^{\alpha\mu\nu} + \partial_\rho (x^\rho K^{\alpha\mu\nu} - x^\nu K^{\rho\alpha\mu})$. Equation (13) looks like an OAM relation [see the first part in Eq. (1)] and it is often said that the spin is identically vanishing in the Belinfante form. Precisely speaking, since the energy-momentum conservation, $\partial_\mu T^{\mu\nu} = 0$, leads to the TAM conservation, $\partial_\alpha J^{\alpha\mu\nu} = 0$, in the Belinfante form, one cannot find a counterpart of Eq. (2). Our point is that we do not have to go through the EMT to write down such a tensor decomposition.

Before addressing the entropy analysis, we shall discuss a possibility to introduce terms with $S^{\mu\nu}$ in the symmetric EMT form; the tensor indices we can use are not only $\Theta^{\mu\nu}$, $u^\mu$, $\partial^\mu$, but also $S^{\mu\nu}$ in general. The guiding principle is provided from a transformation between $T^{\mu\nu}$ and $\Theta^{\mu\nu}$. We can utilize Eq. (10) together with $\Sigma^{\mu\alpha\beta} = u^\mu S^{\alpha\beta} + O(\partial)$, to find,

$$T^{\mu\nu} = \Theta^{\mu\nu} + \frac{1}{2} \partial_\lambda \left( u^\lambda S^{\mu\nu} - u^\nu S^{\mu\lambda} + u^\mu S^{\nu\lambda} \right) + O(\partial^2)$$

$$= \Theta^{\mu\nu}_{(a)} + \frac{1}{2} \left[ \partial_\lambda \left( u^\mu S^{\nu\lambda} + u^\nu S^{\mu\lambda} \right) \right] + O(\partial^2).$$

If we need to construct the hydrodynamics using the symmetric EMT as demanded in the case with gauge fields, we must employ the above form of symmetric spin corrections. One might think that $\partial_\mu T^{\mu\nu} = 0$ may look different from $\partial_\mu \Theta^{\mu\nu} = 0$, but they are equivalent thanks to Eq. (12); therefore, Eq. (12) constitutes an evolution equation. The hydrodynamic expansion leads to

$$T^{\mu\nu} = (e + p) u^\mu u^\nu - p g^{\mu\nu} + \frac{1}{2} \partial_\lambda \left( u^\mu S^{\nu\lambda} + u^\nu S^{\mu\lambda} \right) + O(\partial^2),$$

where

$$T^{\mu\nu}_{(1)} = 2 h^{(\mu} u^{\nu)} + \pi^{\mu\nu} + \frac{1}{2} \partial_\lambda \left( u^\mu S^{\nu\lambda} + u^\nu S^{\mu\lambda} \right).$$

We should emphasize that $T^{\mu\nu}_{(1)}$ is still symmetric with respect to $\mu$ and $\nu$ even with spin involving terms.

The heat flow, $h^{\mu}$, is defined from the symmetric index structure involving $u^{\mu}$. Therefore, once $T^{\mu\nu}_{(1)}$ is given, one can identify $h^{\mu}$ from the tensor decomposition of $T^{\mu\nu}_{(1)}$. In the presence of spin correction terms, the tensor decomposition leads to the heat flow coupled to the spin. We can readily see this from the following decomposition:

$$2 h^{(\mu} u^{\nu)} + \pi^{\mu\nu} + \frac{1}{2} \partial_\lambda \left( u^\mu S^{\nu\lambda} + u^\nu S^{\mu\lambda} \right)$$

$$= \delta e u^\mu u^\nu + 2 \left( h^{(\mu} u^{\nu)} + \delta h^{(\mu} u^{\nu)} \right) + \pi^{\mu\nu} + \delta \pi^{\mu\nu}.$$  

Here, we have the energy density correction, $\delta e$, the heat flow correction, $\delta h^{\mu}$, and the viscous tensor correction, $\delta \pi^{\mu\nu}$, given respectively by

$$\delta e = u_\mu \partial_\sigma S^{\sigma\mu},$$

$$\delta h^{\mu} = \frac{1}{2} \left( \Delta^{\mu\rho} \partial_\mu S^{\rho\lambda} + u_\mu S^{\phi\lambda} \partial_\lambda u^{\phi} \right),$$

$$\delta \pi^{\mu\nu} = \partial_\lambda \left( u^{(\mu} S^\sigma_{\nu)} \right) + \partial \Pi \Delta^{\mu\nu},$$

$$\delta \Pi = \frac{1}{3} \partial_\lambda \left( u^{\sigma} S^{\rho\lambda} \Delta_{\rho\sigma} \right),$$

where $\delta \Pi$ is the bulk viscous correction. We note that the above correction of $\delta h^{\mu}$ is consistent with the momentum density induced by the spin vorticity that has been discussed in the quantum spin vorticity theory [35]. We will later discuss the physical meaning in more details.

One may wonder how $\delta e$ and $\partial \Pi$ can be retrieved in the Belinfante formalism at all, since they are extracted from the antisymmetric EMT as in Eq. (7), which is identically vanishing in the Belinfante form. As we exercised for the canonical EMT, let us consider the entropy current. The Belinfante counterpart of the thermodynamic...
with which the divergence of the entropy current takes the following form:

\[ \partial_\mu S^\mu = \left( \frac{n}{e + p} \tilde{h}^\mu - j_{(1)}^\mu \right) \Delta_{\mu \nu} \partial_\nu \frac{u^\mu}{T} + \frac{1}{T} \pi_{\mu \nu} \partial_\nu u_\mu + \Delta \]

with

\[ \Delta \equiv \frac{1}{2} \left[ \partial_\lambda (u^\mu S^{\nu \lambda} + u^\nu S^{\mu \lambda}) \right] \partial_\mu \frac{u_\nu}{T} - \frac{\omega_{\mu \sigma}}{T} \partial_\lambda (u^\lambda S^{\nu \sigma}) . \]

Here, we emphasize that Eqs. (20, 21) are not equivalent to Eq. (6) even with Eq. (12). For more clarification we will transform Eq. (21) using Eq. (12). We can add Eq. (12) to find,

\[ \Delta = \frac{1}{2} \partial_\mu \left[ \partial_\lambda (u^\mu S^{\nu \lambda} + u^\nu S^{\mu \lambda} + u^\nu S^{\lambda \mu}) \frac{u_\nu}{T} \right] - \frac{1}{2} \left[ \partial_\lambda (u^\lambda S^{\mu \nu}) \right] \partial_\mu \frac{u_\nu}{T} - \frac{\omega_{\mu \nu}}{T} \partial_\lambda (u^\lambda S^{\nu \sigma}) . \]

Therefore, the difference between Eqs. (20, 21) and (6) turns out to be a total derivative. We recall that Eq. (19) is an Ansatz and we could have defined an entropy, \( S' \), to absorb the total derivative and then we arrive at

\[ \Delta' = - \partial_\lambda (u^\lambda S^{\mu \nu}) \left( \frac{1}{2} \partial_\mu \frac{u_\nu}{T} + \frac{\omega_{\mu \nu}}{T} \right) . \]

In this case \( \partial_\mu S^{\mu \nu} \) is given by Eq. (20) with \( \Delta \) replaced by \( \Delta' \). Interestingly, \( S^{\mu \nu} \) is just the same as \( S^{\mu \nu}_{\text{Can}} \). Then, the constrains from the entropy principle amount to those in the canonical formulation from Eqs. (7), (8), and (9). In principle, alternatively, one may constrain \( S^{\mu \nu} \) directly from Eq. (21) employing the following tensor decomposition:

\[ S^{\mu \nu} = 2 \delta^{[\mu} u^{\nu]} - \epsilon^{\mu \nu \rho \sigma} u_\rho S_\sigma \]

with \( u \cdot s = 0 \). We have tried but this is a difficult task to constrain \( s^\mu \) and \( S^{\mu \nu} \) from the entropy principle due to the presence of derivatives. The difficulty seems to favor the canonical choice of \( S^{\mu \nu} \).

We emphasize that such a difference by the total derivative is irrelevant to bulk thermodynamics properties and a stringent condition of the local thermal equilibrium gives rise to the physical difference in the entropy current. We make a remark here; this total derivative shift is quite analogous to \( V^\mu \) in Refs. [36, 37]. There, the shift by \( V^\mu \) appears from the dynamical KMS condition in the effective field theory approach to hydrodynamics. It would be a very interesting future work to pursue a possible relationship. Our transformation from \( S^{\mu \nu} \) to \( S'^{\mu \nu} \) is actually analogous to the hydrodynamical treatment of the triangle anomaly in Ref. [38], where the EMT is also symmetric and some terms proportional to the vorticity are added to the entropy flow. We also emphasize that our observation is consistent with the claim made in Refs. [31, 32]. They found using the density operator that the canonical and the Belinfante EMTs are equivalent only in equilibrium but they are not in nonequilibrium systems [39]. In our analysis the pseudo-gauge transformation generates conserved EMTs and leads to different expressions for the entropy current. With those different expressions the physics is not equivalent once we take account of dissipative terms and impose the second law of thermodynamics, \( \partial_\mu S^{\mu \nu} \geq 0 \), for dynamics out of equilibrium.

**Physical interpretation of spin correction terms:** We have seen that we must introduce a modified entropy current and then the entropy principle supports the canonical results in Eqs. (8) and (9). Nevertheless, we emphasize that the Belinfante EMT should be physical and the spin corrections by Eq. (18) are physical as well. We must be, however, careful of the physical interpretation in relativistic hydrodynamics. The heat flow correction by \( \delta h^\mu \), for example, is not physical by itself.

In relativistic hydrodynamics \( u^\mu \) is not unique in general and one should make a choice of the frame; the common choice is the Landau frame or the energy frame. Then, in this frame, the heat flow is absent by construction. More specifically, we should impose the Landau condition for the relativistic hydrodynamics and choose the fluid velocity \( u^L_\mu \) to satisfy \( \Delta_{\rho \sigma} T_{\mu \nu} u_{L \mu} = 0 \), where “L” denotes the quantities in the Landau frame. We can introduce the fluid velocity, \( u^L_\mu \), as

\[ u^L_\mu = u^\mu + \frac{1}{e + p} (h^\mu + \delta h^\mu) . \]

We can also transfer the Belinfante EMT in Eq. (15) to the one in the Landau frame as

\[ T_{\mu \nu}^L = (e + \delta e) u^\mu_L u^\nu_L - (p + \delta p) \Delta_{\rho \sigma} T_{\mu \nu}^L + \pi_{\mu \nu} + \delta \pi_{\mu \nu} + O(\partial^2) \]

and there is no term corresponding to the heat flow.

In this frame with the fluid velocity given by Eq. (25) the heat flow is absent but the modified current remains finite, which reads:

\[ j_{L(1)}^\mu = \left( j_{(1)}^\mu - \frac{n}{e + p} h^\mu \right) + \delta j_{(1)}^\mu \]

with

\[ \delta j_{(1)}^\mu = - \frac{n}{e + p} \delta h^\mu . \]
form in Eq. (24). Namely, Eq. (24) gives $s^i = S^0$ and $S^i = \frac{1}{2} \epsilon^{ijk} S^{jk}$. The complete expression of $\delta h^k$ in Eq. (18) involves many terms, and we can simplify them by taking the nonrelativistic reduction of $u^k = (1, v)$ with $v \to 0$, while derivatives of $v$ are still kept nonvanishing. Then, in the three-vector representation, we find,

$$
\delta j^{(1)} = -\frac{n}{2(\varepsilon + p)} \left[ \nabla \times S + \dot{v} \times S \right] + (\nabla \cdot \dot{v}) s - 2g \cdot (\nabla \cdot \dot{v}) v + \ddot{s} .
$$

One may think that the overall sign is opposite to that in the quantum spin vorticity theory [35]. This difference is attributed to the frame choice. We are working in a frame comoving with the heat flow, and this reverses the overall direction of the induced current.

**Summary:** We formulated the spin hydrodynamics using the symmetric EMT which is commonly considered to be physical. The added terms satisfy an identity for the spin tensor which corresponds to the quantum spin vorticity principle. The equations of motion are equivalent, but we found that the entropy analysis makes an inequivalent deviation. The entropy current derived from the canonical formulation is different from the one from the symmetric EMT by a total derivative. Therefore, if we impose a constraint not globally but locally from the second law of thermodynamics, the pseudo-gauge transformation would lead to different physical contents in nonequivalent systems. With our formulation based on the symmetric EMT, we established a relation between the spin vorticity (i.e., the rotation of the spin) and the (electric) current, $\delta j \propto \nabla \times S$, in a hydrodynamical way. One may find a similar relation using the Dirac equation in quantum field theories, and our formula is more complete with fluid velocity terms. Applications to the heavy-ion phenomenology should deserve further investigations.

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