AUTOMATIC CONTINUITY AND UNIQUENESS OF POLISH SEMIGROUP TOPOLOGIES

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Abstract. If $S$ is a topological monoid such that every homomorphism from $S$ to a second countable topological monoid $T$ is continuous, then we say that $S$ has automatic continuity. In this paper, we show that many well-known, and extensively studied, monoids have automatic continuity with respect to some natural semigroup topology. Namely, the following have automatic continuity: the monoid $B\mathbb{N}$ of all binary relations on the natural numbers $\mathbb{N}$; the monoid $P\mathbb{N}$ of partial transformations; the full transformation monoid $\mathbb{N}^\mathbb{N}$; the monoid $\text{Inj}(\mathbb{N})$ of injective transformations on $\mathbb{N}$; the symmetric inverse monoid $I\mathbb{N}$; and the monoid $C(2^\mathbb{N})$ of continuous functions on the Cantor set $2^\mathbb{N}$.

Additionally, we show that: $B\mathbb{N}$ has no Polish semigroup topologies; $P\mathbb{N}$, $C(2^\mathbb{N})$, and the monoid $C([0,1]^N)$ of continuous functions on the Hilbert cube $[0,1]^\mathbb{N}$ each have a unique Polish semigroup topology; the monoid $\text{End}(\mathbb{Q}, \leq)$ of all order-endomorphisms of the rational numbers, and the monoid of endomorphisms of the countable random graph (the Rado graph) have at least one Polish semigroup topology; the monoid of self-embeddings of any Fraïssé limit which admits proper embeddings has at least two Polish semigroup topologies; $I\mathbb{N}$ has at least 3 Polish semigroup topologies, but a unique Polish inverse semigroup topology; and $\text{Inj}(\mathbb{N})$ and the monoid $\text{Surj}(\mathbb{N})$ of all surjective transformations of $\mathbb{N}$ have infinitely many distinct Polish semigroup topologies.

In proving the main results mentioned above we prove myriad ancillary results relating to the Zariski and Markov topologies on a semigroup; the minimum $T_1$ semitopological topology on a semigroup; the small index property for semigroups; and to the monoids mentioned above for sets of arbitrary cardinality.

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1. Motivation and background

If $S$ is a semigroup and $T$ is a topology on the set $S$, then $T$ is called a semigroup topology if the multiplication function from $S \times S$ (with the product topology) to $S$ is continuous. A semigroup together with a semigroup topology is referred to as a topological semigroup. In this paper, following [53], we are primarily concerned with the following type of question:

*When is a homomorphism $f : S \to T$ between topological semigroups necessarily continuous?*

More precisely, we ask if certain well-known topological semigroups $S$ have the property that every homomorphism from $S$ to a second countable topological semigroup $T$ is continuous; such a semigroup $S$ is said to have automatic continuity. A closely related question, that we also consider, is:

*Which semigroup topologies does a specific semigroup admit?*

A semigroup $S$ has automatic continuity with respect to at most one second countable topology on $S$. We are particularly interested in Polish semigroups. A topological space $X$ is Polish if it is completely metrizable and separable. A Polish semigroup is just a topological semigroup where the topology is Polish. In this article, we are interested in showing that certain well-known infinite semigroups and monoids have, or do not have, a unique Polish semigroup topology.

Both of the questions that are the focus of this paper originally arose in the context of groups; we will briefly, and probably incompletely, discuss the history of these questions for groups, and semigroups. A topological semigroup that happens to be a group is called a paratopological group, i.e. multiplication is continuous but inversion need not be. If $G$ is a paratopological group and inversion is also continuous, then $T$ is a group topology, and $G$ is a topological group.

The extensive history of problems of these two types for groups can be traced back to Cauchy, and Markov [38], respectively. Markov [38] asked whether there exists an infinite group whose only group topologies are the trivial and discrete topologies; such a group is called non-topologizable. Shelah [58] showed that a non-topologizable group exists assuming the continuum hypothesis; Hesse [22] showed that the assumption of the continuum hypothesis in Shelah’s construction can be avoided. Olshanski [47] showed that an infinite family of the Adian groups (constructed by Adian [1] as a counter-example to the Burnside problem) are non-topologizable; see also [12, Example 5.3.2] for an account of Olshanski’s argument. A more recent paper on this topic is [35].

Given that there exist groups where the only group topologies are trivial and discrete, it is natural to ask if there are groups that admit a unique non-trivial non-discrete group topology. Without some further assumptions on the topology, it does not appear that such groups are known to exist or not.

Polish groups are defined analogously to Polish semigroups. The fundamental results of R. M. Solovay [61] and S. Shelah [59] show that it is consistent with ZF without choice that any Polish group has a unique Polish group topology. On the other hand, the additive group of real numbers $\mathbb{R}$ is a Polish group with the usual topology on $\mathbb{R}$, as too is the additive group $\mathbb{R}^2$. The two groups $\mathbb{R}$ and $\mathbb{R}^2$ are isomorphic, since they are vector spaces of equal dimension over the rationals $\mathbb{Q}$, but are not homeomorphic, since $\mathbb{R}^2$ with any point removed is connected and $\mathbb{R}$ with any point removed is not.

In problem number 96 of the famous Scottish Book [39], Ulam asked if the symmetric group $\text{Sym}(\mathbb{N})$ on the natural numbers has a locally compact Polish group topology. The symmetric group $\text{Sym}(\mathbb{N})$ has a natural Polish topology, the subspace topology induced by the usual topology of the Baire space $\mathbb{N}^\mathbb{N}$. Ulam’s problem was answered in the negative by Gaughan [19], who also showed that every $T_1$ group topology on $\text{Sym}(\mathbb{N})$ contains the pointwise topology. It can be shown that if
$\mathcal{T}_1$ and $\mathcal{T}_2$ are Polish group topologies on the same group and $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then $\mathcal{T}_1 = \mathcal{T}_2$. It therefore follows from Gaughan [19] that $\text{Sym}(\mathbb{N})$ has a unique Polish group topology.

Many further examples of groups are known to have unique Polish group topologies: the groups of isometries of the Urysohn space and of the Urysohn sphere [55]; homeomorphism groups of a wide class of metric spaces such as any separable metric manifold [31] (including the Hilbert cube $[0,1]^\mathbb{N}$) or the Cantor set $2^\mathbb{N}$; the automorphism groups of many countable relational structures, in particular, Fraïssé limits, such as the rational numbers $\mathbb{Q}$ under the usual linear order, the countable random graph $R$ [25, 26], and the group of Lipschitz homeomorphisms of the Baire space [34]; Some further references on the uniqueness of Polish group topologies are [10, 18]. On the other hand, many groups have been shown to have no non-discrete Polish group topologies, for example: free groups [14]; the homeomorphism groups of the irrational and of the rational numbers and the group of Borel automorphisms of $\mathbb{R}$ [52]. Additional references include [11, 27, 28, 29, 30, 32, 54].

Several of the topological groups $G$ mentioned in the last paragraph satisfy a stronger property: that every homomorphism from $G$ to a separable (or some other topological condition) topological group $H$ is continuous. This notion is often referred to as the group $G$ having automatic continuity. Note that this notion of automatic continuity is superficially different from the notion introduced above: $G$ is required to have continuous inversion with the former but not the latter. We will show in Lemma 3.2 that the usual notion of automatic continuity for groups, found in the literature, coincides with the notion defined in this article when applied to a group.

The notion of automatic continuity is closely related to uniqueness of Polish group topologies. Recall that a Polish group topology on a given group $G$ cannot be properly contained in another Polish group topology on $G$. It follows that if $G$ is a Polish group such that every homomorphism from $G$ to a Polish group $H$ is continuous, then $G$ has a unique Polish group topology. Among many other interesting results in [34], it is shown that every homomorphism from a Polish group $G$ to a separable group $H$ is continuous if $G$ satisfies the following condition: there is a comeagre orbit of the action by conjugation of $G$ on $G^n$ for every $n \in \mathbb{N}$; such a group $G$ is said to have ample generics. Many permutation groups have ample generics: the symmetric group $\text{Sym}(\mathbb{N})$; the automorphism groups of the random graph [25, 26]; the free group on countably many generators [7]; some further references are [21, 56, 60]. Some groups do not have ample generics, for instance: $\text{Aut}(\mathbb{Q}, \leq)$ [36], and the homeomorphism group $\text{Homeo}(\mathbb{R})$ of the reals $\mathbb{R}$. The group $\text{Aut}([0,1], \lambda)$ was shown to have automatic continuity by Yaacov, Berenstein, and Melleray [66], as was the infinite-dimensional unitary or orthogonal group to a separable group; see Tsankov [63]; a further reference is [37]. In [54, Theorem 6.26], it is shown that the pointwise topology is the only non-trivial separable group topology on $\text{Sym}(\mathbb{N})$; thereby strengthening the result that the pointwise topology is the unique Polish group topology on $\text{Sym}(\mathbb{N})$.

Another notion that is implied by the notion of automatic continuity is the so-called small index property. A topological group $G$ has the small index property if every subgroup of index at most $\aleph_0$ is open; some authors assume this condition for subgroups of index less than $2^{\aleph_0}$. In Example 6.4, we give an example of a group satisfying the small index property but that does not have automatic continuity. A topological group $G$ has the small index property if and only if every homomorphism from $G$ to $\text{Sym}(\mathbb{N})$ is continuous. Many groups were shown to have the small index property before they were shown to have automatic continuity: $\text{Sym}(\mathbb{N})$ [50, 57, 13], the automorphism groups of countable vector spaces over finite fields [16]; $\text{Aut}(\mathbb{Q}, <)$ and the automorphism group of the countable atomless Boolean algebra [62]; the automorphism group of the random graph and all automorphism groups of $\omega$-categorical $\omega$-stable structures [25]; the automorphism groups of the Henson graphs [20].

The natural analogue of the small index property for a topological semigroup $S$ is that every right congruence $\rho$ on $S$ with countably many classes is open as a subset of $S \times S$ with the product topology; we refer to this property as the right small index property, the left small index property is defined analogously. In Proposition 9.5, we show that the notions of left and right small index property are distinct, by exhibiting an example which has one property but not the other. If a topological semigroup $S$ has automatic continuity, then $S$ has both left and right small index property (Corollary 6.3). In Proposition 6.1, we will show that a topological monoid $S$ has the
right small index property if and only if every homomorphism from \(S\) to \(\mathbb{N}^\mathbb{N}\) with the topology of pointwise convergence is continuous.

Another notion, considered in the literature for semigroups and monoids, is that of “automatic homeomorphism”, which is defined as follows. If \(M\) is a transformation monoid and \(C\) is a class of transformation monoids, then \(M\) is said to have automatic homeomorphism with respect to \(C\) if every isomorphism from \(M\) to a monoid \(N \in C\) is necessarily a homeomorphism. The topology in all cases is the subspace topology induced by the topology of pointwise convergence on \(X^X\) for an appropriate choice of countably infinite set \(X\).

For a submonoid \(N\) of \(X^X\) where \(X\) is countably infinite, let \(\sim\) be the least equivalence relation on \(X\) containing \((x, y) \in X \times X\) whenever there exists \(f, g \in N\) and \(z \in X\) such that \((x)f = z\) and \((y)g = z\). Examples of monoids having the property of “automatic homeomorphism” with respect to the class \(C\) of closed submonoids of \(X^X\) having finitely many \(\sim\)-classes, include the following from [48]:

- the monoid of non-decreasing functions on the rationals \(\mathbb{Q}\);
- the endomorphism monoid of the countable universal homogeneous poset;
- the monoid of non-expansive transformations of the rational Urysohn space; or the rational Urysohn sphere.

With respect to the class \(C\) of closed submonoids of \(\mathbb{N}^\mathbb{N}\) in [3, 5]:

- the monoid of order-endomorphisms or order-embeddings of the rational numbers \(\mathbb{Q}\);
- the monoid \(\text{Inj}(\mathbb{N})\) of injective functions on \(\mathbb{N}\);
- the monoid \(\mathbb{N}^\mathbb{N}\) of all functions on \(\mathbb{N}\);
- the monoid of self-embeddings of the countable random graph (the Rado graph);
- the endomorphism monoid of the countable random graph;
- the monoid of self-embeddings of the countable universal homogeneous digraph;

and also with respect to the class of endomorphism monoids of countable \(\omega\)-categorical structures in [5]:

- the endomorphism monoid of the countable universal homogeneous tournament;
- the monoid of self-embeddings of the countable universal homogeneous \(k\)-uniform hypergraphs.

Many of the results in [3, 5] are extended to the corresponding clones of polymorphisms (the precise definition of which lies outside the scope of this article). The notion of (left) small index property for transformation monoids is also considered in [5]. In [4], the authors give examples of two countable \(\omega\)-categorical structures, whose endomorphism monoids are isomorphic as abstract monoids, but where no isomorphism between them is a homeomorphism. Some further references are [49].

2. Definitions and main theorems

In this section, we will summarise the main theorems in this paper. In order to do this, we first introduce some notions related to topological semigroups, and the cast of semigroups appearing in the main theorems.

Suppose that \(S\) is a semigroup and \(\mathcal{T}\) is a topology on the set \(S\). Then we will say that \(\mathcal{T}\) is left semitopological for \(S\) if for every \(s \in S\) the function \(\lambda_s : S \to S\) defined by \((x)\lambda_s = sx\) is continuous. If every function \(\rho_s : S \to S\) defined by \((x)\rho_s = xs\) is continuous, then we say that \(\mathcal{T}\) is right semitopological for \(S\). If \(\mathcal{T}\) is left and right semitopological for \(S\), then we say that \(\mathcal{T}\) is semitopological for \(S\). Every semigroup topology is semitopological. We refer to a semigroup with a left semitopological topology as a left semitopological semigroup. Analogous definitions can be made for right semitopological and semitopological.

If \(\mathcal{T}_1\) and \(\mathcal{T}_2\) are topologies on a set \(X\), then the least topology containing \(\mathcal{T}_1\) and \(\mathcal{T}_2\) will be referred to as the topology generated by \(\mathcal{T}_1\) and \(\mathcal{T}_2\). If \(X\) and \(Y\) are topological spaces and \(\mathcal{B}\) is a sub-basis for \(Y\), then \(f : X \to Y\) is continuous if and only if \((B)f^{-1}\) is open for all \(B \in \mathcal{B}\). Hence if \(\mathcal{T}_1\) and \(\mathcal{T}_2\) are topological for \(S\), then so too is the topology generated by \(\mathcal{T}_1\) and \(\mathcal{T}_2\).
and $T_2$. The analogous statement holds if “topological” is replaced by “semitopological”, “right semitopological”, or “left semitopological”.

An inverse semigroup is a semigroup $S$ such that for every $x \in S$ there exists a unique $y \in S$ such that $xyx = x$ and $yx = y$; $y$ is usually denoted by $x^{-1}$. Inverse semigroup topologies and topological inverse semigroups are defined analogously to group topologies and topological groups. A semitopological group or inverse semigroup is a semitopological semigroup that happens to be a group or inverse semigroup, i.e., inversion is not assumed to be continuous.

The Fréchet-Markov topology of a semigroup $S$ is the intersection of all $T_1$ semigroup topologies on $S$. Similarly, the Hausdorff-Markov topology of a semigroup $S$ is the intersection of all Hausdorff semigroup topologies for $S$. The inverse Fréchet-Markov topology and inverse Hausdorff-Markov topologies of an inverse semigroup $S$ are similarly defined to the the intersections of all $T_1$ and Hausdorff inverse semigroup topologies on $S$ respectively. Clearly, the Fréchet-Markov topology on a semigroup is contained in the Hausdorff-Markov topology. The intersection of $T_1$ topologies is $T_1$ and the intersection of semigroup topologies is semitopological. Hence the Fréchet-Markov and Hausdorff-Markov topologies of a semigroup $S$ are $T_1$ and $S$ is semitopological with respect to both.

If $G$ is a topological group, then $T_0$ implies $T_{3\frac{1}{2}}$ and so, in particular, there is no distinction in the theory of topological groups between the notions of inverse Fréchet-Markov and inverse Hausdorff-Markov. In Propositions 4.6 and 4.7 we show that these notions are distinct in the context of topological semigroups.

By definition, the least $T_1$ topology that is semitopological for $S$ is contained in the Fréchet-Markov topology of $S$. These two topologies may coincide, for example, if $S$ is a semigroup of right zeros, then every topology on $S$ is a semigroup topology. Hence the least $T_1$ topology on a semigroup $S$ of right zeros is the cofinite topology, and the Hausdorff-Markov topology is the intersection of all the Hausdorff topologies on the set $S$, and it is straightforward to verify that this is also the cofinite topology.

If $S$ is a semigroup where the least $T_1$ topology $T$ with respect to which $S$ is semitopological happens to be a Hausdorff semigroup topology, then $T$ is the Hausdorff-Markov topology for $S$. For example, in [41, Theorem 3.1] it is shown that the least $T_1$ topology $T$ with respect to which $\mathbb{N}^n$ is semitopological, is the pointwise topology, which is also a Hausdorff semigroup topology on $\mathbb{N}^n$; see also Theorem 10.3. Hence $T$ coincides with the Fréchet-Markov and Hausdorff-Markov topologies on $\mathbb{N}^n$.

If $S$ is a semigroup then we use $S^1$ to denote the monoid obtained by adjoining an identity 1 to $S$. The Zariski topology on a semigroup $S$ is defined as the smallest topology on $S$ for which the elementary algebraic sets are closed. An elementary algebraic set is a set of the form

$$\{ s \in S : (s)\phi_1 = (s)\phi_2 \}$$

where $\phi_1, \phi_2 : S \to S$ are any functions such that $(s)\phi_1 = t_1 s t_2 s \cdots t_k s$, $k \geq 1$ for every $s \in S$ and for some fixed $t_1, \ldots, t_k \in S^1$ and similarly for $\phi_2$; see Section 4 for further details.

Let $X$ be any set. A binary relation on $X$ is just a subset of $X \times X$. We will denote the set of all binary relations on $X$ by $B_X$. If $f, g \in B_X$, then their composition $f \circ g$ is defined by

$$(x, y) \in f \circ g \quad \text{if} \quad (x, z) \in f \quad \text{and} \quad (z, y) \in g \quad \text{for some} \quad z \in X.$$

The set $B_X$ with composition of binary relations is the full binary relation monoid on $X$.

If $f \in B_X$, then we define the inverse of $f$ to be $f^{-1} = \{(y, x) : (x, y) \in f\}$. The relation $f^{-1}$ is not always an inverse for $f$, in the sense of inverse semigroups or groups, since $f \subseteq f f^{-1} f$, and this containment may be strict. If $\Sigma \subseteq X$ and $f \in B_X$, then the image of $\Sigma$ under $f$ is the set

$$(\Sigma) f = \{ y \in X : (x, y) \in f \quad \text{for some} \quad x \in \Sigma \}.$$

The domain and image of $f$ are $\text{dom}(f) = (X)f^{-1}$ and $\text{im}(f) = (X)f$.

In this paper functions are written to the right of their arguments, and composed from left to right. We will also consider the following natural subsemigroups of $B_X$: the partial transformation
monoid

\[ P_X = \{ f \in B_X : |\{(x)\} f| \leq 1 \text{ for all } x \in X \} \]

the full transformation monoid

\[ X^X = \{ f \in B_X : |\{(x)\} f| = 1 \text{ for all } x \in X \} \]

the symmetric inverse monoid

\[ I_X = \{ f \in X^X : |\{(x)\} f^{-1}| \leq 1 \text{ for all } x \in X \} \]

the monoid of injective functions

\[ \text{Inj}(X) = \{ f \in X^X : |\{(x)\} f^{-1}| \leq 1 \text{ for all } x \in X \} \]

the monoid of surjective functions

\[ \text{Surj}(X) = \{ f \in X^X : (X) f = X \} \]

These monoids have been extensively studied in the literature for semigroups when \( X \) is finite and infinite; see, for example, [15, 23, 42, 43, 44, 45] and the references therein.

If \( X \) is a topological space, then we denote by \( C(X) \) the monoid of continuous functions from \( X \) to \( X \). The Cantor set is \( 2^\mathbb{N} \) and the Hilbert cube is \( [0,1]^\mathbb{N} \) where \( [0,1] \) is the closed unit interval in \( \mathbb{R} \).

A relational structure is a set \( X \) together with a (possibly infinite) collection of finitary relations on \( X \). For example, the rational numbers \( \mathbb{Q} \) with the binary relation \( \leq \) is a relational structure. When the finitary relations on \( X \) are clear from the context, or do not require specific names, we will simply write that \( X \) is a relational structure, or refer to the finitary relations on \( X \) as a relational structure on \( X \).

If \( X \) is a relational structure, then the endomorphism monoid of \( X \) denoted \( \text{End}(X) \) is a closed submonoid of \( X^X \). The converse also holds, i.e. if \( M \) is a closed submonoid of \( X^X \), then there exists a relational structure on \( X \) such that \( M \) is its endomorphism monoid; see [9, Proposition 6.1] or [8, Theorem 5.8]. It follows that the endomorphism monoid on any relational structure on \( X \) is a Polish monoid with respect to the pointwise topology inherited from \( X^X \).

A self-embedding of a relational structure \( X \) is an isomorphism from \( X \) to a substructure of \( X \). We denote by \( \text{Emb}(X) \) the monoid of self-embeddings of a relational structure \( X \). It is straightforward to verify that \( \text{Emb}(X) \) is also a closed submonoid of \( \text{Inj}(X) \), \( \text{End}(X) \), and \( X^X \) with the pointwise topology.

A class \( \mathcal{K} \) of finite relational structures satisfying the following properties is called a Fraïssé class:

- **HP**: hereditary property: if \( A \) is a substructure of \( B \) and \( B \in \mathcal{K} \), then \( A \in \mathcal{K} \);
- **JEP**: joint embedding property: if \( A, B \in \mathcal{K} \), then there exists \( C \in \mathcal{K} \) such that both \( A \) and \( B \) are substructures of \( C \);
- **AEP**: amalgamated embedding property: if \( A, B, C \in \mathcal{K} \) and there are embeddings \( f_1 : A \rightarrow B \) and \( f_2 : A \rightarrow C \), then there exists \( D \in \mathcal{K} \) and embeddings \( g_1 : B \rightarrow D \) and \( g_2 : C \rightarrow D \) such that \( f_1 \circ g_1 = f_2 \circ g_2 \);

and \( \mathcal{K} \) contains countably many structures up to isomorphism and contains structures of arbitrarily large finite size. Examples of Fraïssé classes include: graphs, partial orders, linear orders, Boolean algebras, metric spaces with rational differences, and many more. Associated to every Fraïssé class \( \mathcal{K} \) is a Fraïssé limit which is the unique countable infinite structure such that: every finitely generated substructure is finite; any isomorphism between substructures extends to an automorphism; and the class of all finite structures that can be embedded into the Fraïssé limit equals \( \mathcal{K} \); see [17] and [24] for more details. Examples of Fraïssé limits include: the countably infinite random graph \([51]\); the linear order of rational numbers \( \mathbb{Q} \); the countable atomless boolean algebra; and the rational Urysohn space. These examples are the Fraïssé limits of the classes of all: finite graphs; finite linear orders; Boolean algebras; and finite metric spaces with rational distances, respectively.

The main theorems in this paper are summarised in Table 1. The theorems summarised in Table 1 encapsulate some versions of the main theorems later in the paper, given in terms of
the countably infinite set \( \mathbb{N} \). It should be noted that in the later sections, where we prove these theorems, there are versions for infinite sets of arbitrary cardinality. The versions of the main theorems for \( \mathbb{N} \) are often stronger than the corresponding theorems for arbitrary infinite sets \( X \), and are easier to state, which is why we only include the countably infinite case in this section.

This paper has four parts. In part 1, we discuss some properties of topologies on arbitrary semigroups. In particular, we prove some general statements about: automatic continuity in Section 3; the Zariski, Fréchet-Markov and Hausdorff-Markov topologies in Section 4; semigroups that can be embedded topologically as subsemigroups of the full transformation monoid \( \mathbb{N}^\mathbb{N} \) in Section 5; and the small index property in Section 6. In part 2, we address the two main questions of the paper for what we refer to as classical monoids. These are monoids that have been studied extensively in the literature of semigroups and monoids. In part 3, we consider monoids of continuous functions on two important topological spaces: the Hilbert cube and the Cantor space. In part 4, we consider monoids of endomorphisms and self-embeddings of some Fraïssé limits.

**Part 1. Topologies on arbitrary semigroups**

### 3. Automatic Continuity

In this section we prove some results about automatic continuity for arbitrary semigroups.

An anti-homomorphism from a semigroup \( S \) to a semigroup \( T \) is a function \( \phi : S \to T \) such that \( (st)\phi = (t)\phi (s)\phi \) for all \( s, t \in S \). An anti-automorphism of a semigroup \( S \) is a bijective anti-homomorphism from \( S \) to \( S \).

**Proposition 3.1.** Let \( S \) be a semigroup and let \( T \) be a semigroup topology on \( S \). If \( \phi : S \to S \) is an automorphism or anti-automorphism, then

(i) \( (T)\phi = \{ (U)\phi : U \in T \} \) is a semigroup topology on \( S \);

(ii) if \( \mathcal{B} \) is a sub-base for \( T \), then \( \{ (B)\phi : B \in \mathcal{B} \} \) is a sub-base for \( (T)\phi \).

**Proof.** We will prove the proposition in the case when \( \phi \) is an anti-automorphism. The other case is even more straightforward.

(i). Let \( U \in T \) be arbitrary. If \( (s)\phi, (t)\phi \in S \) are such that \( (s)\phi (t)\phi \in (U)\phi \), then, since \( \phi \) is an anti-automorphism, \( (s)\phi (t)\phi = (ts)\phi \in (U)\phi \) and so \( ts \in U \). Since \( T \) is a semigroup topology, there exist open neighbourhoods \( V_s, V_t \in T \) of \( s \) and \( t \), respectively, such that \( V_t V_s \subseteq U \).
Thus $(V_s)\phi$ and $(V_t)\phi$ are open neighbourhoods of $(s)\phi$ and $(t)\phi$ under $(T)\phi$ and $(V_s)\phi (V_t)\phi = (V_t V_s)\phi \subseteq (U)\phi$.

(ii). By definition, $\{ (B) : B \in \mathcal{B} \} \subseteq (T)\phi$. Furthermore, $\phi$ is a bijection, so the unions and (finite) intersections of images are images of unions and (finite) intersections.

Lemma 3.2. Let $G$ be a semitopological group. Then $G$ has automatic continuity if and only if every homomorphism from $G$ to a second countable topological group is continuous.

Proof. ($\Leftarrow$) Let $S$ be a second countable topological semigroup and let $\phi : G \to S$ be a homomorphism. Then $H := (G)\phi$ is a subgroup of $S$ and the induced topology on $H$ is a second countable paratopological group topology for $H$; we denote this topology by $T$. If $T^{-1} := \{ U^{-1} : U \in T \}$, then, by Proposition 3.1, $T^{-1}$ is a semigroup topology on $H$ and, since $T^{-1}$ is homeomorphic to $T$, it is second countable.

It follows that the topology $T'$ generated by $T$ and $T^{-1}$ is also a second countable paratopological group topology for $H$. Since the topology $T'$ is generated by an inverse closed collection of sets, it follows that the inverse operation of $H$ is continuous under this topology, and thus $(H, T')$ is a second countable topological group. Since $\phi$ is a homomorphism, by the assumption of this implication $\phi$ is continuous with respect to $T'$ and the topology on $G$. If $U \in T$, then $U \in T'$ and so $(U)\phi^{-1}$ is open in $G$ by the continuity of $\phi : G \to (H, T')$.

($\Rightarrow$) This is immediate as second countable topological groups are second countable topological semigroups.

 Lemma 3.3. A topological semigroup $S$ has automatic continuity if and only if for all second countable topological semigroups $T$, all anti-homomorphisms $\phi : S \to T$ are continuous.

Proof. We show the right implication, the left implication is similar. Let $S$ and $T$ be as hypothesised. Let $T'$ and $\psi$ be such that $\psi : T \to T'$ is a homomorphism. We define a binary operation on $T'$ by $(s, t) \mapsto ((t)\psi^{-1}(s)\psi^{-1})\psi$. This operation makes $T'$ a topological semigroup and $\psi$ an anti-isomorphism. It follows that the map $\phi \psi$ is a homomorphism and is thus continuous. Therefore $\phi \psi \psi^{-1} = \phi$ is also continuous as required.

In the literature, see, for example, [34, 54, 53], a topological group $G$ is said to have automatic continuity if every homomorphism from $G$ to a separable group is continuous. Lemma 3.2 implies that if a particular group $G$ has automatic continuity in the sense usually meant in the literature, then $G$ also has automatic continuity in the sense we use it here. For example, the following groups have automatic continuity: the symmetric group $\text{Sym}(\mathbb{N})$ with the pointwise topology; and the group of homeomorphisms $H(2^\mathbb{N})$ of the Cantor set $2^\mathbb{N}$; see [34] and [54].

We require the notion of a Borel measurable function between topological spaces. Recall that a $\sigma$-algebra on a set $X$ is a collection of subsets of $X$ containing $\emptyset$ and which is closed under complements and countable unions (and hence also closed under countable intersections). If $X$ is a topological space, then a set $B$ is Borel if it belongs to the least $\sigma$-algebra containing the open sets in $X$. If $X$ and $Y$ are topological spaces, then $f : X \to Y$ is Borel measurable if the pre-image of every Borel set is Borel.

Proposition 3.4 (cf. Theorem 9.10 and Proposition 11.5 in [33]). If $G$ and $H$ are Polish semitopological groups and $f : G \to H$ is a Borel measurable homomorphism, then $f$ is continuous.

Proposition 3.5 (cf. Corollary 15.2 in [33]). If $X$ and $Y$ are Polish spaces and $f : X \to Y$ is a Borel measurable bijection, then $f^{-1}$ is Borel measurable also.

If $S$ is a semitopological semigroup and $A$ is a subset of $S$, then we say that $S$ satisfies property $X$ with respect to $A$ if the following holds:

$X$: if for every $s \in S$ there exists $f_s, g_s \in S$ and $a_s \in A$ such that $s = f_s a_s g_s$ and for every basic open neighbourhood $B$ of $a_s$ there exists an open neighbourhood $U$ of $s$ such that $U \subseteq f_s(B \cap A)g_s$. 

Lemma 3.6. Let $S$ be a semigroup, let $T$ be a topology with respect to which $S$ is semitopological, and let $A \subseteq S$. If $S$ has property $X$ with respect to $A$, then the following hold:

(i) if $T$ is a semitopological semigroup and $\phi : S \to T$ is a homomorphism such that $\phi|_A$ is continuous, then $\phi$ is continuous;
(ii) if $T'$ is a topology with respect to which $S$ is semitopological and $T'$ induces the same topology on $A$ as $T$, then $T'$ is contained in $T$;
(iii) if $T$ is Polish and $A$ is a Polish subgroup of $S$, then $T$ is maximal among the Polish topologies with respect to which $S$ is semitopological;
(iv) If $A$ is a semigroup which has automatic continuity, then the semigroup $S$ has automatic continuity.

Proof. (i). We denote the topology on $T$ by $T'$. We will show that $\phi$ is continuous at an arbitrary $s \in S$. Suppose that $U \in T'$ is an open neighbourhood of $(s)\phi$. Then, by property $X$, there are $f_s, g_s \in S$ and $t_s \in A$ such that $s = f_s t_s g_s$. Since $\phi$ is a homomorphism, $(s)\phi = (f_s)\phi (t_s)\phi (g_s)\phi$ and so

$$(t_s)\phi \in (U)(\lambda(f_s) \circ \rho(g_s))^{-1},$$

which we will denote by $V$. In particular, since $T$ is semitopological, $V$ is an open neighbourhood of $t_s$ in $T'$ and $((f_s)\phi V((g_s)\phi) \subseteq U$. Since $\phi|_T$ is continuous, $(V)\phi^{-1} \cap A$ is open in the subspace topology on $T$ induced by $T$. Hence $(V)\phi^{-1} \cap A = W \cap A$ for some $W \in T$. Since $W$ is an open neighbourhood of $t_s$ in $T$, there exists a basic open neighbourhood $B$ of $t_s$ in $T$ such that $B \subseteq W$. By the assumptions of the theorem, there exists an open neighbourhood $B'$ of $s$ such that $B' \subseteq f_s (B \cap A) g_s$. Then

$$(B')\phi \subseteq (f_s (B \cap A) g_s) \phi \subseteq (f_s (W \cap A) g_s) \phi = (f_s ((V)\phi^{-1} \cap T) g_s) \phi \subseteq (f_s ((V)\phi^{-1}) g_s) \phi \subseteq ((f_s)\phi V((g_s)\phi) \subseteq U,$$

and so $\phi$ is continuous at $s$.

(ii). Let $T'$ be a semitopological semigroup topology for $S$ that induces the same subspace topology as $T$ on $A$. Then the restriction of the identity homomorphism $id : (S, T) \to (S, T')$ to $A$ is continuous. Thus $id : (S, T) \to (S, T')$ is continuous by part (i) and so $T' \subseteq T$, as required.

(iii). Suppose that $T'$ is a Polish semitopological semigroup topology on $S$ and that $T \subseteq T'$. We will show that $T' \subseteq T$, and so $T$ is maximal. As in the previous part, it suffices to show that the restriction $id|_A$ of the identity function $id : (S, T) \to (S, T')$ is continuous.

Since $A$ is a Polish subspace of $S$ with respect to $T$, it follows that $T$ is $G_\delta$ in $T$, and so $T$ is $G_\delta$ in $T'$ also. Hence $A$ is a Polish subspace of $S$ with respect to $T'$. Since $id|_A^{-1}$ is a Borel measurable bijection between Polish spaces, it follows from Proposition 3.5 that $id|_A$ is a Borel measurable function between $(S, T)$ and $(S, T')$. Therefore, by Proposition 3.4, $id|_A$ is continuous.

(iv). Suppose that $T$ is a second countable topological semigroup and $\phi : S \to T$ is a homomorphism. It follows that $\phi|_A$ is a homomorphism from $A$ to $T$ which is therefore continuous by automatic continuity. It follows from part (i) that $\phi$ is continuous. □

4. The Zariski, Fréchet-Markov, and Hausdorff-Markov topologies

In this section we prove some results about the Zariski, Fréchet-Markov, and Hausdorff-Markov topologies, and their relationships to each other, for arbitrary semigroups.

Recall that the Zariski topology on a group $G$ is defined as the topology with sub-basis consisting of the sets

$$\{g \in G : (g)\phi \neq 1_G\}$$

where $1_G$ is the identity of $G$ and $\phi : G \to G$ is any function such that $(g)\phi = h_1 g^{i_1} h_2 g^{i_2} \cdots h_k g^{i_k}$ for every $g \in G$ and for some fixed $h_1, \ldots, h_k \in G$ and $i_1, \ldots, i_k \in \{-1, 1\}$. The Zariski topology has been extensively studied in the literature of topological groups.
We introduce the notion of the Zariski topology on a semigroup. Analogous to the definition for
groups, the Zariski topology on a semigroup $S$ is defined as the topology with sub-basis consisting of
\[
\{ s \in S : (s)\phi_1 \neq (s)\phi_2 \}
\]
where $\phi_1, \phi_2 : S \to S$ are any functions such that $(s)\phi_1 = t_1s_2s \cdots t_k$, $k \geq 1$ for every $s \in S$ and
for some fixed $t_1, \ldots, t_k \in S^1$, and similarly for $\phi_2$. The inverse Zariski topology for an inverse
semigroup is defined analogously but the $\phi$ functions are permitted to include inverse symbols
as they where in the case of groups. The inverse Zariski topology on a group coincides with the
traditional notion of the Zariski topology on a group.

It is not known whether the semigroup version of the Zariski topology applied to a group is
necessarily equal to the group version of the Zariski topology applied to the same group.

**Proposition 4.1.** The Zariski topology on any semigroup $S$ is contained in every Hausdorff
semigroup topology on $S$.

**Proof.** Recall that if $X$ is a Hausdorff topological space and $f, g : X \to X$ are continuous
functions, then the set
\[
\{ x \in X : (x)f = (x)g \}
\]
is closed in $X$. Let $T$ denote a Hausdorff semigroup topology on $S$. Let $\phi_1 : S \to S$ be from the
definition of a sub-basic open set for the Zariski topology. In other words, $(s)\phi_1 = t_1s_2s \cdots t_k$, $k \geq 1$ for every $s \in S$ and
for some fixed $t_1, \ldots, t_k \in S^1$. If $m = 2k$, then the function $\psi_1 : S \to S^m$ defined by
\[
(s)\psi_1 = (t_1, s, t_2, s, t_3, \ldots, t_k, s)
\]
is continuous in every coordinate, and hence is continuous with respect to $T$. The function $\phi_1$ is
then the composite of $\psi_1$ and the multiplication function from $S^m$ to $S$, and is hence continuous with respect to $T$. The function $\phi_2$ is continuous by an analogous argument. It follows that every
sub-basic open set $\{ s \in S : (s)\phi_1 \neq (s)\phi_2 \}$ for the Zariski topology is open in $T$. \hfill $\Box$

**Proposition 4.2.** The Zariski topology is semitopological on any semigroup.

**Proof.** Let $S$ be any semigroup, and let $l \in S$ be arbitrary. We will show that $\lambda_y : S \to S$ defined by $(x)\lambda_y = yx$ is continuous. Suppose that $\phi_1, \phi_2 : S \to S$ are defined by
\[
(s)\phi_1 = t_1s_2s \cdots t_k
\]
\[
(s)\phi_2 = u_1s_2u_1 \cdots u_l
\]
for some $k, l \geq 1$, for every $s \in S$, and for some fixed $t_1, \ldots, t_k, u_1, \ldots, u_l \in S^1$. Then
\[
\{ s \in S : (s)\phi_1 \neq (s)\phi_2 \} \lambda_y^{-1} = \{ x \in S : (yx)\phi_1 \neq (yx)\phi_2 \} = \{ x \in S : t_1(yx) \cdots t_k(yx) \neq u_1(yx) \cdots u_l(yx) \}.
\]
If
\[
(x)\phi_1' = t_1(yx)t_2(yx) \cdots t_k(yx)
\]
\[
(x)\phi_2' = u_1(yx)u_2(yx) \cdots u_1(yx),
\]
then it is clear that $\{ x \in S : (x)\phi_1' \neq (x)\phi_2' \}$ is open and so $S$ is left semitopological with respect to
the Zariski topology.

The proof that $S$ is right semitopological with respect to the Zariski topology is dual. \hfill $\Box$

See Figure 2 for the Hassé diagram of the containment of the Hausdorff-Markov, Fréchet-Markov, Zariski, and minimal $T_1$ topology that is semitopological for a given semigroup.

**Proposition 4.3.** Let $S$ be a semigroup and let $T$ be a topology semitopological for $S$. If $\phi : S \to S$ is an automorphism or anti-automorphism, then $(T)\phi = \{ (U)\phi : U \in T \}$ is semitopological for $S$. 
Inverse Hausdorff-Markov

Inverse Zariski

Hausdorff-Markov

Inverse Fréchet-Markov

Zariski

Fréchet-Markov

(Inverse) minimal $T_1$ semitopological

Figure 2. The Hassé diagram of the containment of the Hausdorff-Markov, Fréchet-Markov, Zariski, Inverse Hausdorff-Markov, Inverse Fréchet-Markov, Inverse Zariski, and Inverse minimal $T_1$ topology that is semitopological for a given semigroup. (We show that the Inverse minimal $T_1$ topology is equal to the Inverse minimal $T_1$ topology) It is known that all the shown containments can potentially be strict, we do not know if any more can be added.

Proof. We will prove the proposition in the case when $\phi$ is an anti-automorphism. The other case is even more straightforward. We will show that $(T)\phi$ is right semitopological, left semi-topological follows by symmetry. Let $U \in \mathcal{T}$ be arbitrary. If $(s)\phi, (t)\phi \in S$ are such that $(s)\phi (t)\phi \in (U)\phi$, then, since $\phi$ is an anti-automorphism, $(s)\phi (t)\phi = (ts)\phi \in (U)\phi$ and so $ts \in U$. Since $\mathcal{T}$ is left semitopological for $S$, there exists an open neighbourhood $V_s \in \mathcal{T}$ of $s$, such that $tV_s \subseteq U$. Thus $(V_s)\phi$ is an open neighbourhood of $(s)\phi$ under $(T)\phi$ and $(V_s)\phi (t)\phi = (tV_s)\phi \subseteq (U)\phi$. □

Proposition 4.4. Every automorphism, or anti-automorphism, of a semigroup $S$ is continuous with respect to each of the Zariski, Fréchet-Markov, and Hausdorff-Markov topologies for $S$, as well as the minimal $T_1$ topology semitopological for $S$.

Proof. We first consider the Zariski topology. It suffices to show that the image of an elementary algebraic set under a semigroup automorphism or anti-automorphism is elementary algebraic. Let

$$F = \{ s \in S : t_1st_2s \cdots t_k s = u_1su_2s \cdots u_l s \}$$
be an elementary algebraic set. Let \( \phi : S \rightarrow S \) be an anti-automorphism. Then

\[
F\phi = \{(s \phi) \in S : t_1 s t_2 s \cdots t_k s = u_1 s u_2 s \cdots u_l s\}
\]

\[
= \{s \in S : t_1 (s \phi^{-1}) t_2 (s \phi^{-1}) \cdots t_k (s \phi^{-1}) = u_1 (s \phi^{-1}) u_2 (s \phi^{-1}) \cdots u_l (s \phi^{-1})\}
\]

\[
= \{s \in S : (t_1 (s \phi^{-1}) t_2 (s \phi^{-1}) \cdots t_k (s \phi^{-1}) \phi = u_1 (s \phi^{-1}) u_2 (s \phi^{-1}) \cdots u_l (s \phi^{-1}) \phi\}
\]

\[
= \{s \in S : st_k os \cdots t_2 os t_1 \phi = su_1 os \cdots u_2 os u_1 \phi\},
\]

which is elementary algebraic. Similarly if \( \phi \) is an automorphism of \( S \), then \( F\phi \) will also be elementary algebraic.

Let \( \phi \) be an automorphism or anti-automorphism of \( S \). If \( T \) is any \( T_1 \) semigroup topology on \( S \), then, by Proposition 3.1, \( (T) \phi \) is also a \( T_1 \) semigroup topology for \( S \). If \( \mathcal{T} \) is the collection of all \( T_1 \) semigroup topologies on \( S \), then

\[
\left( \bigcap_{T \in \mathcal{T}} T \right) \phi = \bigcap_{T \in \mathcal{T}} (T \phi) = \bigcap_{T \in \mathcal{T}} T
\]

and thus \( \phi \) is continuous with respect to the Fréchet-Markov topology, which equals \( \bigcap_{T \in \mathcal{T}} T \). The proof for the Hausdorff-Markov topology and the minimal \( T_1 \) topology semitopological for \( Sp \) are similar (the latter using Proposition 4.3).

**Corollary 4.5.** If \( S \) is an inverse semigroup, then inversion is continuous in each of the Zariski, Fréchet-Markov, and Hausdorff-Markov topologies on \( S \), as well as the minimal \( T_1 \) topology semitopological for \( S \).

Recall that if a topological group is \( T_0 \), then it is \( T_{3\frac{1}{2}} \). On the other hand, every topology is a semigroup topology, and so no such implication holds for topological semigroups, in general. It is natural to ask if there is any implication among separation conditions for certain classes of semigroup, such as the inverse semigroups. Examples 4.7 and 4.6 show that there is no such implication for inverse semigroups either, and at the same time that the Fréchet-Markov and Hausdorff-Markov topologies are not always equal for (inverse) semigroups.

Recall that if \( \leq \) is a total order on a set \( X \), then the order topology on \( X \) is the topology with sub-basis consisting of the sets \( \{y \in X : x < y\} \) and \( \{y \in X : x > y\} \). If \( X \) is any set with a total order, then \( X \) with the operation max is a commutative inverse semigroup.

**Proposition 4.6.** Let \( X \) be an infinite set, let \( \leq \) be a total order on \( X \), and endow \( X \) with the structure of a semigroup by taking multiplication to be max. Then the Hausdorff-Markov topology and the Zariski topology on \( X \) coincide with the order topology on \( X \).

**Proof.** The elementary algebraic sets for the Zariski topology are precisely \( V_{a,b} := \{x \in X : \max\{x, a\} = \max\{x, b\}\} \) and \( W_{a,b} := \{x \in X : \max\{x, a\} = b\} \) where \( a, b \in X \) are arbitrary. Note that \( V_{a,b} = V_{b,a} \).

If \( a, b \in X \) are arbitrary, then the following hold:

1. If \( a = b \), then \( V_{a,b} = X \) and \( W_{a,b} = \{x \in X : x \leq a\} \);
2. If \( a < b \), then \( V_{a,b} = \{x \in X : x \geq b\} \) and \( W_{a,b} = \{b\} \);
3. If \( a > b \), then \( V_{a,b} = V_{b,a} = \{x \in X : x \geq a\} \) and \( W_{a,b} = \emptyset \).

Hence the complements of the sets \( V_{a,b} \) and \( W_{a,b} \) are sub-basic open sets for the order topology on \( X \) and are a basis for the Zariski topology, and so these two topologies coincide.

The order topology is also a Hausdorff semigroup topology for \( X \). It is routine to verify that \( X \) is Hausdorff. Suppose that \( \max\{a, b\} = b \in U \) for some sub-basic open \( U \). If \( a, b \in U \), then \( UU = U \). If \( a \notin U \), then \( U = \{y \in X : x > y\} \) for some \( x \), and so setting \( V = \{y \in X : y < b\} \), then \( VU \subseteq U \). Finally, \( \mathcal{T} \), being equal to the Zariski topology, is contained in the Hausdorff-Markov topology, which is contained in \( \mathcal{T} \), and so \( \mathcal{T} \) is the Hausdorff-Markov topology also.

**Proposition 4.7.** Let \( X \) be an infinite set, let \( \leq \) be a total order on \( X \), and endow \( X \) with the structure of a semigroup by taking multiplication to be max. Then the sets of the form

\[
B_{x,U} := \{y \in U : y > x\}
\]
where $U$ is a cofinite subset of $X$ and $x \in X^1$ is arbitrary, form a basis for the Fréchet-Markov topology on $X$.

Proof. Let $T$ be the topology on $X$ with the sets $B_{z,U}$ as a basis. Note that $B_{1,U} = U$ for all $U$ where $1 \in X^1$ is the adjoined identity. Hence the cofinite sets are basic open in $T$, and $T$ contains the cofinite topology, and it is $T_1$. Let $S$ be a $T_1$ semigroup topology for $X$. We show that $T \subseteq S$. Since $S$ is $T_1$, the singletons are closed in $S$ and hence

$$(X \setminus \{x\})\lambda^{-1}_x = \{y \in X : \max(x,y) \neq x\} = \{y \in X : y > x\} \in S$$

for all $x \in X$. If $x \in X$ is arbitrary and $U$ is a cofinite subset of $X$, then

$$B_{z,U} = \{y \in X : y > x\} \cap U,$$

and so $T \subseteq S$. It follows that $T$ is contained in the Fréchet-Markov topology for $X$, and hence it suffices to show that $T$ contains the Fréchet-Markov topology. We will show that $T$ is a semigroup topology for $X$.

Let $B_{z,U} \in T$ be an arbitrary basic open set and let $a, b \in X$ be such that $\max(a, b) \in B_{z,U}$. We show that there are open neighbourhoods $U_a, U_b$ of $a$ and $b$ respectively, such that $U_a U_b \subseteq B_{z,U}$. If $a = b$ then we choose $U_a = U_b = B_{a,U}$. Otherwise assume without loss of generality that $a < b$. Let $U_b := B_{\max(a,s),U}$ and $U_a := U \cup \{a\}$. If $a' \in U_a$ and $b' \in U_b$ then either $\max\{a', b'\} = b' \in U_b \subseteq B_{s,U}$ or $a' > b' > \max\{a, s\}$, in which case $\max\{a', b'\} = a' \in B_{\max\{a,s\},U} \subseteq B_{z,U}$ as required.

The topology defined in Proposition 4.7 can be distinct from the topology defined in Proposition 4.6. For example, if $X = \mathbb{Z}$, then every open set in the Fréchet-Markov topology is unbounded above, whereas some of the basic open sets in the Hausdorff-Markov topology are bounded above by definition. It follows that Fréchet-Markov topology for a semigroup can be strictly contained in its Hausdorff-Markov topology, even if we only consider commutative inverse semigroups.

We end this section with a problem, that we do not know the solution to: is the Fréchet-Markov topology always contained in the Zariski topology? What is the Zariski topology of the symmetric inverse monoid?

5. Subsemigroups of $\mathbb{N}^\mathbb{N}$

It is well-known that a topological group $G$ is topologically isomorphic to a subgroup of the symmetric group $\text{Sym}(\mathbb{N})$ if and only if $G$ has a countable neighbourhood basis of the identity consisting of open subgroups. An analogous result for topological monoids and $\mathbb{N}^\mathbb{N}$, and some related algebraic structures, can be found in [6]; see also Theorem 5.2.

The following theorem is similar to [41, Theorem 3.1], and if a subsemigroup of $\mathbb{N}^\mathbb{N}$ satisfies the hypothesis of [41, Theorem 3.1], then it satisfies the conditions in Theorem 5.1 also.

**Theorem 5.1.** Let $X$ be an infinite set, and let $S$ be a subsemigroup of $X^X$ such that $S$ contains all of the constant transformations, and for every $x \in X$ there exists $f_x \in S$ such that $(x)f_x^{-1} = \{x\}$ and $(X)f_x$ is finite. If $S$ is a topology which is semitopological for $S$, then the following are equivalent:

(i) $S$ is Hausdorff;
(ii) $S$ is $T_1$;
(iii) $\{f \in S : (a,b) \in f\}$ is open with respect to $S$ for all $a,b \in X$;
(iv) $\{f \in S : (a,b) \in f\}$ is closed with respect to $S$ for all $a,b \in X$.

**Proof.** (ii) $\Rightarrow$ (iii) It suffices to show that every set of the form

$$\{g \in S : (i)g = j\}$$

for any $i, j \in X$ is open. Suppose that $i, j \in X$, and $g, h \in S$ are such that $h$ is the constant transformation with value $i$, and $g$ is arbitrary. Then $(i)g \neq j$ if and only if $hgf_j$ is constant with value belonging to $(X)f_j \setminus \{j\}$. Since $(X)f_j$ is finite by assumption, it follows that $\{g \in S : (i)g \neq j\}$ is the preimage under left multiplication by $h$ and right multiplication by $f_j$ of the
finite set consisting of those constant transformations whose image belongs to \((X)f_j \setminus \{j\}\). Hence \(\{g \in S : (i)g \neq j\}\) is closed.

The implications (i) \(\Rightarrow\) (ii), (iii) \(\Rightarrow\) (i), (iii) \(\Rightarrow\) (iv), and (iv) \(\Rightarrow\) (ii) all follow from the proofs of the analogous statements in [41, Theorem 3.1].

It is well-known that a \(T_0\) topological group \(G\) is a topologically isomorphic to a subgroup of \(\text{Sym}(\mathbb{N})\) if and only if \(G\) has a countable neighbourhood basis of the identity \(1_G\) consisting of countable index open subgroups. This latter condition is referred to as \(G\) being non-archimedean by some authors; see, for example, [6, Theorem 1]. In [6, Theorem 2], an analogous characterisation is given of those topological monoids that are topologically isomorphic to a closed submonoid of \(\mathbb{N}^\mathbb{N}\). In the next theorem, we prove an analogue of this result for \(T_0\) left semitopological semigroups and \(\mathbb{N}^\mathbb{N}\).

If \(X\) is a countable dense subset of \(\mathbb{N}^\mathbb{N}\), then the least submonoid \(\langle X \rangle\) of \(\mathbb{N}^\mathbb{N}\) is countable and dense also. It follows that \(\langle X \rangle\) is not \(G_\delta\), and hence not Polish. It follows that \(\langle X \rangle\) is not topologically isomorphic to any closed submonoid of \(\mathbb{N}^\mathbb{N}\), but it is obviously topologically isomorphic to a submonoid of \(\mathbb{N}^\mathbb{N}\).

Even though the statement of Theorem 5.2 is superficially different from that of [6, Theorem 2], the proof of Theorem 5.2 is essentially contained in the proof of [6, Theorem 2]. We include Theorem 5.2 because we will use it in Section 11 and we include the proof for the sake of completeness.

**Theorem 5.2** (cf. Theorem 2 in [6]). If \(S\) is a \(T_0\) left semitopological semigroup, then the following are equivalent:

(i) there is a sequence \(\{\rho_i : i \in \mathbb{N}\}\) of right congruences of \(S\), each having countably many classes, such that \(\{m/\rho_i : m \in S, i \in \mathbb{N}\}\) is a subbasis for \(S\);

(ii) \(S\) is topologically isomorphic to a subsemigroup of \(\mathbb{N}^\mathbb{N}\) (with the pointwise topology and right actions).

**Proof.** (ii) \(\Rightarrow\) (i). Assume without loss of generality that \(S\) is a subsemigroup of \(\mathbb{N}^\mathbb{N}\). If we define \(\rho_i = \{(f, g) : (i)f = (i)g\}\) for every \(i \in \mathbb{N}\), then clearly every \(\rho_i\) is a right congruence with the properties given in (i).

(i) \(\Rightarrow\) (ii). We give \(S^1\) the disjoint union topology of \(S\) and \(\{1\}\). If \(\rho\) is any right congruence on \(S\), then \(\rho \cup \{(1, 1)\}\) is a congruence on \(S^1\). It follows that \(S^1\) satisfies the hypothesis of (i). Let \(\{\rho_i : i \in \mathbb{N}\}\) be a sequence of right congruences of \(S^1\), each having countably many classes, such that \(\{m/\rho_i : m \in S^1, i \in \mathbb{N}\}\) is a subbasis for \(S^1\).

By assumption, \(\Omega = \{m/\rho_i : m \in S^1, i \in \mathbb{N}\}\) is countable, and we will show that \(S^1\) is topologically isomorphic to a subsemigroup of \(\Omega^\mathbb{N}\) with the pointwise topology, from which it will follow that \(S\) is also topologically isomorphic to a subsemigroup of \(\Omega^\mathbb{N}\). We define \(\phi : S^1 \to \Omega^\mathbb{N}\) such that \((m)\phi \in \Omega^\mathbb{N}\) is defined by

\[(n/\rho_i)(m)\phi = (nm)/\rho_i.\]

It is sufficient to show that \(\phi\) is a well-defined, injective, continuous, homomorphism such that if \(U\) is open in \(S^1\), then \((U)\phi\) is open in \((S^1)\phi\) with the subspace topology.

If \((a, b) \in \rho_i\) for some \(i \in \mathbb{N}\) and \(m \in S^1\), then \((a/\rho_i)(m)\phi = (am)/\rho_i = (bm)/\rho_i = (b/\rho_i)(m)\phi\), since \(\rho_i\) is a right congruence. It follows that \(\phi\) is well-defined.

Suppose that \(m, n \in S^1\) are such that \((m)\phi = (n)\phi\). It follows that \(m/\rho_i = (1/\rho_i)(m)\phi = (1/\rho_i)(n)\phi = n/\rho_i\) for all \(i \in \mathbb{N}\). In other words, \((m, n) \in \rho_i\) for all \(i \in \mathbb{N}\). But since \(\{m/\rho_i : m \in S^1, i \in \mathbb{N}\}\) is a subbasis for \(S^1\), and \(S^1\) is \(T_0\) it follows that \(\bigcap_{i \in \mathbb{N}} \rho_i = \Delta_{S^1}\). Hence \(m = n\) and \(\phi\) is injective.

Let \(a, m, n \in S^1\) and \(i \in \mathbb{N}\) be arbitrary. Then

\[ ((a/\rho_i)(m)\phi(n)\phi = (amm)/\rho_i = (a/\rho_i)(mn)\phi\]

and so \(\phi\) is a homomorphism.
To show that $\phi$ is continuous it is sufficient to show that the preimage under $\phi$ of every subbasic open set in $\Omega^1$ is open in $S^1$. Let $\mathcal{S}$ be the subbasis for $\Omega^1$ consisting of the sets

$$[\alpha, \beta] = \{ f \in \Omega : (\alpha)f = \beta \}$$

where $\alpha, \beta \in \Omega$. Suppose that $[\alpha, \beta] \in \mathcal{S}$. Then $\alpha = a/\rho_i$ and $\beta = b/\rho_j$ for some $a, b \in S^1$ and $i, j \in \mathbb{N}$. If $[a/\rho_i, b/\rho_j] \cap (S^1)\phi \neq \emptyset$, then $i = j$ by the definition of $\phi$. Hence we may suppose that without loss of generality that $\alpha = a/\rho_i$ and $\beta = b/\rho_i$. It follows that

$$[\alpha, \beta]^{-1} = \{ m \in S^1 : (\alpha)m\phi = \beta \} = \{ m \in S^1 : (a/\rho_i)(m/\rho_i) = b/\rho_i \} = \{ m \in S^1 : (am/\rho_i) = b/\rho_i \} = \{ m \in S^1 : am \in b/\rho_i \} = (b/\rho_i)\lambda^{-1}.$$

Since $S^1$ is left semitopological, $\lambda_a$ is continuous and since $b/\rho_i$ is open, so too is $[\alpha, \beta]^{-1}$. Hence $\phi$ is continuous.

If $m \in S^1$ and $i \in \mathbb{N}$ are arbitrary, then

$$(m/\rho_i)\phi = \{(n) : n \in S^1, m/\rho_i = n/\rho_i \} = \{(n) : n \in S^1, (1/\rho_i)(n)\phi = m/\rho_i \} = [1/\rho_i, m/\rho_i].$$

Hence every subbasic open set in $S^1$ is mapped to a subbasic open set in $(S^1)\phi$ and so $\phi$ is open.

The symmetric inverse monoid $I_\mathbb{N}$ is the natural analogue of the full transformation monoid $\mathbb{N}^\mathbb{N}$, in that every countable inverse monoid can be embedded in $I_\mathbb{N}$. In Theorem 9.4, we will show that, again analogous to the case of $\mathbb{N}^\mathbb{N}$, that $I_\mathbb{N}$ has a unique Polish inverse semigroup topology (multiplication is jointly continuous and inversion is also continuous). It is natural to want a characterisation of those $T_0$ inverse semitopological semigroups that can be embedded in $I_\mathbb{N}$ with its unique Polish inverse semigroup topology. Unfortunately, the authors were not able to formulate such an analogue, and so we leave this as an open problem.

**Question 5.3.** Characterise those $T_0$ inverse semitopological semigroups that can be embedded into $I_\mathbb{N}$ with its unique Polish inverse semigroup topology (see Theorem 9.4 for the definition of this topology).

We end this section by noting that not every Polish semigroup of cardinality at most $2^{\aleph_0}$ can embedded in $\mathbb{N}^\mathbb{N}$. For example, the reals $\mathbb{R}$ under addition form a connected Polish group, but $\mathbb{N}^\mathbb{N}$ is totally disconnected, and so $(\mathbb{R}, +)$ cannot be topology embedded in $\mathbb{N}^\mathbb{N}$.

It is an open question as to whether or not every countable Polish semigroup can be topologically embedded in $\mathbb{N}^\mathbb{N}$.

## 6. The Small Index Property

A topological semigroup $S$ is said to have the right small index property if every right congruence on $S$ with countably many classes is open. Similarly $S$ is said to have the left small index property if every left congruence on $S$ with countably many classes is open. These notions are notably equivalent when we consider only groups.

**Proposition 6.1.** A topological monoid $M$ has the right small index property if and only if every homomorphism from $M$ to $\mathbb{N}^\mathbb{N}$ with the pointwise topology is continuous.

**Proof.** First note that an equivalence relation on a topological space is open if and only if all of its equivalence classes are open.

We start by assuming that $M$ has the right small index property. Let $\phi : M \to \mathbb{N}^\mathbb{N}$ be a semigroup homomorphism. Recall that the pointwise topology on $\mathbb{N}^\mathbb{N}$ has a sub-basis consisting of...
the sets \( U_{i,j} = \{ f \in \mathbb{N}^\mathbb{N} : (i) f = j \} \) over all \( i, j \in \mathbb{N}. \) Note that \( U_{i,j} \) is an equivalence class under the right congruence

\[ \rho := \{ (f, g) \in \mathbb{N}^\mathbb{N} : (i) f = (i) g \}. \]

Since \( \phi \) is a homomorphism, the relation \( \rho' := \{ (f, g) \in M : (f \phi, g \phi) \in \rho \} \) on \( M \) is a right congruence on the monoid \( M \) and \( \rho' \) has at most as many classes as \( \rho. \) Since \( M \) has the right small index property, \( \rho' \) is open. The preimage of \( U_{i,j} \) under \( \phi \) is an equivalence class of \( \rho' \) and hence open. Since \( U_{i,j} \) is an arbitrary sub-basic open set, it follows that \( \phi \) is continuous.

Now suppose that every semigroup homomorphism \( \phi : M \to \mathbb{N}^\mathbb{N} \) is continuous. Let \( \rho \) be a right congruence on \( M \) with countable many classes. We define a homomorphism \( \phi : M \to (M/\rho)^{(M/\rho)} \) by

\[ (g/\rho)((f/\rho) \phi) = g f/\rho. \]

Giving \((M/\rho)^{(M/\rho)} \) the pointwise topology it is topologically isomorphic to \( \mathbb{N}^\mathbb{N} \) and thus \( \phi \) is continuous with respect to these topologies. Let \( m/\rho \) be an arbitrary equivalence class of \( \rho. \) It suffices to show that \( m/\rho \) is an open subset of \( M. \) The set \( \{(f/\rho) \phi : (1_{M/\rho})(f/\rho) = m/\rho \} \) is open in \((M/\rho) \phi \) under the subspace topology inherited from \( M/\rho^{M/\rho} \) and

\[ \{(f/\rho) \phi : (1_{M/\rho})(f/\rho) = m/\rho \}^{-1} = \{(f/\rho) \phi : f/\rho = m/\rho \}^{-1} = m/\rho. \]

Since \( \phi \) is continuous, it follows that \( m/\rho \) is indeed open as required. \( \square \)

By symmetry, we obtain the following corollary to Proposition 6.1.

**Corollary 6.2.** A topological monoid \( M \) has the left small index property if and only if every homomorphism from \( M \) to \( \mathbb{N}^\mathbb{N} \) with left actions is continuous.

The following corollary follows straight from the definition of automatic continuity, Proposition 6.1, and Corollary 6.2.

**Corollary 6.3.** If a topological monoid has automatic continuity then it has both the left and right small index properties.

The converse of Corollary 6.3 is not true. We give an example which demonstrates that the small index property is strictly weaker than automatic continuity even in the case of abelian Hausdorff topological groups.

**Example 6.4.** Define the topology \( \tau \) on the group of real numbers \( \mathbb{R} \) under addition by choosing the cosets of all countable index subgroups as a subbasis.

This subbasis is actually a basis as the intersection of finitely many subgroups of countable index is another subgroup of countable index. Moreover, \( \tau \) is a group topology since for every countable index subgroup \( G \) of \( \mathbb{R} \) we have \((G + x) + (G + y) \subseteq (G + x + y) \) and \(- (G + x) = (G - x) \) for all \( x, y \in \mathbb{R}. \) By its definition, \( \tau \) clearly gives \((\mathbb{R}, +) \) the small index property.

Since \( \tau \) is a group topology, to show that \( \tau \) is Hausdorff, it suffices to show that \( \tau \) is \( T_0. \) We will show that for all \( x \in \mathbb{R}\setminus\{0\} \) there is a countable index subgroup of \( \mathbb{R} \) which does not contain \( x. \) Let \( x \in \mathbb{R}\setminus\{0\}. \) By Zorn’s Lemma we can extend the set \( \{x\} \) to a basis \( B \) for \( \mathbb{R} \) as a vector space over \( \mathbb{Q}. \) Let \( G \) be the subspace spanned by \( B\setminus\{x\}. \) As \( B \) is linearly independent, \( x \not\in G. \) But as \( G \) is a codimension 1 subspace of a vector space over a countable field, it follows that \( G \) has countable index as required.

To see that \((\mathbb{R}, +) \) under \( \tau \) does not have automatic continuity, we will show that the identity map from \( \mathbb{R} \) under \( \tau \) to \( \mathbb{R} \) with the standard topology is not continuous. The interval \((-1, 1)\) is not open in \( \tau \) since all open sets in \( \tau \) contain a translation of a non-trivial group, and are thus unbounded.

In Proposition 9.5 we give an example of a topological semigroup which has the right small index property but not the left small index property, so in particular it also does not have automatic continuity. Moreover, this shows that the right and left small index properties are not equivalent in general.
Part 2. Classical monoids

7. The full binary relation monoid $B_X$

When looking for a topology on $B_X$ it is natural to consider its subsemigroups $X^X$ and $\text{Sym}(X)$, since these possess a unique Hausdorff second countable semigroup topology; the pointwise topology. A sub-basis for the pointwise topology on $X^X$ is the collection of sets:

$$\{ f \in X^X : (x)f = y \}$$

for every $x, y \in X$. In the next theorem, we give a natural extension of this sub-basis to $B_X$.

**Theorem 7.1.** Let $X$ be an infinite set and let

$$U_{x,y} = \{ f \in B_X : (x,y) \in f \}$$

for all $x, y \in X$. If $B_1$ is the topology on $B_X$ with sub-basis $\{U_{x,y} : x, y \in X\}$, then

(i) $B_1$ is a semigroup topology for $B_X$ and inversion of binary relations $f \mapsto f^{-1}$ is continuous;
(ii) $B_1$ is $T_0$ but not $T_1$, and the subspace topology induced by $B_1$ on the symmetric inverse monoid $I_X$ is not $T_1$;
(iii) the topological semigroup $B_X$ with topology $B_1$ has property $X$ with respect to $\text{Sym}(X)$;
(iv) every topology that is semitopological for $B_X$ and that induces the pointwise topology on $\text{Sym}(X)$ is contained in $B_1$;
(v) if $X$ is countable, then $B_X$ has automatic continuity with respect to $B_1$.

**Proof.** Note that a basis for $\mathcal{P}$ is given by the collection of sets $\{h \in B_X : f \subseteq h\}$ where $f \in B_X$ is finite, since such sets are precisely the finite intersections of sub-basic elements of $B_1$.

(i). Let $x, y \in X$ and $f, g \in B_X$ be such that $fg \in U_{x,y}$, i.e. $(x,y) \in fg$. Then there exists $z \in X$ such that $(x,z) \in f$ and $(z,y) \in g$. Conversely, for every $f', g' \in B_X$ with $(x,z) \in f'$ and $(z,y) \in g'$ we have that $(x,y) \in f'g'$. In other words, $f \in U_{x,z}$, $g \in U_{z,y}$ and

$$U_{x,y} = U_{x,z} \cap U_{z,y} \subseteq U_{x,y}.$$

Hence multiplication is continuous.

The map $f \mapsto f^{-1}$ is a homeomorphism since $(U_{x,y})^{-1} = U_{y,x}$.

(ii). If $f, g \in B_X$ are distinct, then either $f \not\subseteq g$ or $g \not\subseteq f$. Without loss of generality assume $f \not\subseteq g$. Then there exists $(x,y) \in X \times X$ such that $(x,y) \in f \setminus g$ and so $f \in U_{x,y}$ but $g \notin U_{x,y}$. Hence $\mathcal{P}$ is $T_0$.

On the other hand, suppose that $f, g \in I_X$ and $g \subseteq f$. Then every sub-basic open set $U_{x,y}$ containing $g$ also contains $f$. Thus every open set containing $g$ contains $f$ and so $\mathcal{P}$ is not $T_1$ (even when restricted to $I_X$).

(iii). Let $Y \subseteq X$ be such that $|Y| = |X \setminus Y| = |X|$. Enumerate $X$ as $X = \{x_i : i \in |X|\}$ and let $\{X_i : i \in |X|\}$ be a partition of $X \setminus Y$ such that $|X_i| = |X|$ for every $i \in |X|$. Define $f \in B_X$ by

$$f = \{(x_i, y) \in X \times X : i \in |X| \text{ and } y \in X_i \}.$$ 

Let $s \in B_X$ be arbitrary. A binary relation $t \in B_X$ satisfies $ftf^{-1} = s$ if and only if $t$ has the following property:

(1) for all $i, j \in |X| : (X_i \times X_j) \cap t \neq \emptyset$ if and only if $(x_i, x_j) \in s$.

Since $|Y| = |X_i| = |X|$, there exists $t_i \in \text{Sym}(X)$ satisfying (1).

Suppose that $V$ is a basic open neighbourhood of $t_s$. Then there exist finite $k \in B_X$ such that $V = \{h \in B_X : k \subseteq h\}$. If $U := \{h \in B_X : fkkf^{-1} \subseteq h\}$, then $U$ is open since $fkkf^{-1}$ is finite. We show that $s \in U \subseteq fVf^{-1}$. As $k \subseteq t_s$ and $ft_s = s$ it is clear that $s \in U$. It remains to show that

$$U \subseteq fVf^{-1}.$$ 

Let $u \in U$ be arbitrary. As in (1) we need only find $t \in V \cap \text{Sym}(X)$ with

for all $i, j \in |X| : (X_i \times X_j) \cap t \neq \emptyset$ if and only if $(x_i, x_j) \in u.$
If \(i, j \in |X|\) and \((X_i \times X_j) \cap k \neq \emptyset\), then \((x_i, x_j) \in f_k f^{-1} \subseteq u\). Therefore, as \(k\) is finite, we can extend \(k\) to an element \(t\) of \(V \cap \text{Sym}(X)\) with the desired property. So \(u = f t f^{-1} \in f (V \cap \text{Sym}(X)) f^{-1}\) and
\[
U \subseteq f (V \cap \text{Sym}(X)) f^{-1},
\]
as required.

(iii). The sub-basis for \(B_1\) induces the usual sub-basis for the pointwise topology on \(\text{Sym}(X)\), and so the topology on \(\text{Sym}(X)\) induced by \(B_1\) is the pointwise topology. Since \(B_X\) has property \(X\) with respect to \(\text{Sym}(X)\), by part (iii), it follows from Lemma 3.6(ii) that if \(T\) is a topology that is semitopological for \(B_X\), then \(T\) is contained in \(B_1\).

(iv). This follows from part (iii), Lemma 3.6(iv), and the automatic continuity of \(\text{Sym}(X)\).

As an immediate consequence of Theorem 7.1, there is no \(T_1\) topology that is semitopological for \(B_X\) and that induces the pointwise topology on \(\text{Sym}(X)\).

If instead of trying to extend the pointwise topology, we look for the weakest \(T_1\) topology that is semitopological for \(B_X\), then we obtain the following theorem.

**Theorem 7.2.** Let \(X\) be an infinite set and let \(B_2\) be the topology on \(B_X\) generated by the sets
\[
U_{x,y} = \{ h \in B_X : (x, y) \in h \} \quad \text{and} \quad V_{Y,Z} = \{ h \in B_X : (Y) h \subseteq Z \}
\]
for all \(x, y \in X\) and \(Y, Z \subseteq X\). Then

(i) \(B_2\) is a Hausdorff semigroup topology for \(B_X\) and inversion of binary relations \(f \mapsto f^{-1}\) is continuous;

(ii) every \(T_1\) topology that is semitopological for \(B_X\) contains \(B_2\);

(iii) \(B_2\) is not contained in any second countable topology;

(iv) \(B_2\) strictly contains \(B_1\);

(v) if \(X\) is countable, then \(B_2\) is the Fréchet-Markov, Hausdorff-Markov, and Zariski topologies for \(B_X\).

**Proof.** (i). If \(f, g \in B_X\) are distinct, then without loss of generality, there exist \(x, y \in X\) such that \((x, y) \in f\) but \((x, y) \notin g\). Then \(f \in U_{x,y}\) and \(g \in V_{\{x\}, X \setminus \{y\}}\). Since these two open sets are disjoint, \(B_2\) is Hausdorff.

The topology generated by the sets \(U_{x,y}\) is just \(P\) as defined in Theorem 7.1. Since \(P\) is a semigroup topology, it suffices to show that the pre-images under multiplication and inversion of the sub-basic sets \(V_{Y,Z}\) are open.

For any \(Y \subseteq X\) and \(f, g \in B_X\), we clearly have \(f \in V_{Y,(Y)f}\) and \(g \in V_{(Y)f,(Y)fg}\). Moreover, if \(f' \in V_{Y,(Y)f}\) and \(g' \in V_{(Y)f,(Y)fg}\), then \((Y)f'g' \subseteq (Y)fg' \subseteq (Y)fg\). Thus if \(fg \in V_{Y,Z}\), then \(f'g' \in V_{Y,Z}\) and so multiplication is continuous.

Since
\[
(V_{Y,Z})^{-1} = V_{X \setminus Z, X \setminus Y},
\]
it follows that \(f \mapsto f^{-1}\) is also continuous under \(B_2\).

(ii). Suppose that \(B_2\) is a \(T_1\) topology under which \(B_X\) is semitopological. We will show that the sub-basic open sets of \(B_2\) are open in \(B_2'\) and so, in particular, \(B_2 \subseteq T'\).

For any \(Y, Z \subseteq X\) and \(f \in B_X\) consider the product
\[
(Y \times Y) \circ f \circ ((X \setminus Z) \times (X \setminus Z)) = g.
\]
Either \((Y)f \subseteq Z\) and \(g = \emptyset\), or \((Y)f \not\subseteq Z\) and \(g = Y \times (X \setminus Z)\). Thus
\[
\{(Y \times (X \setminus Z))\} \left(\lambda_{Y \times Y} \circ \rho_{(X \setminus Z) \times (X \setminus Z)}\right)^{-1} = \{f \in B_X : (Y)f \subseteq Z\} = B_X \setminus V_{Y,Z}
\]
and
\[
\{\emptyset\} \left(\lambda_{Y \times Y} \circ \rho_{(X \setminus Z) \times (X \setminus Z)}\right)^{-1} = \{f \in B_X : (Y)f \not\subseteq Z\} = V_{Y,Z}
\]
as the continuous pre-images of finite sets, are closed in \(B_2'\). Hence \(V_{Y,Z}\) and \(B_X \setminus V_{Y,Z}\) are open in \(B_2'\). If \(x, y \in X\), we may let \(Y = \{x\}\) and \(Z = X \setminus \{y\}\), and then
\[
B_X \setminus V_{Y,Z} = \{f \in B_X : (Y)f \not\subseteq Z\} = \{f \in B_X : (x, y) \notin f\} = U_{x,y}
\]
and so \( U_{x,y} \) is clopen in \( B_2 \) also.

(iii). If \( X \) is uncountable, then the collection

\[
\{ V_{X,\{x\}} \cap U_{x,x} : x \in X \} \subseteq B_2
\]

consists of pairwise disjoint open sets and is uncountable. Hence, if \( X \) is uncountable, then \( B_2 \) is not contained in any second countable topology.

Suppose that \( X \) is countable and that \( \{ X_i : i \in I \} \) is a family of subsets of \( X \) with cardinality \( 2^{\aleph_0} \) such that \( X_i \not\subseteq X_j \) whenever \( i \neq j \). For every \( i \in I \), choose \( f_i \in B_X \) such that \((X)f_i = X_i\). If \( \mathcal{U} \) is a basis any topology containing \( B_2 \), then, for every \( i \in I \), there exists \( U_i \in \mathcal{U} \) such that \( f_i \in U_i \subseteq V_{X,X_i} \). If \( f_i \in U_j \subseteq V_{X,X_j} \), then \( X_i = (X)f_i \subseteq X_j \) and so \( i = j \). In other words, \( i \neq j \) implies that \( f_i \not\subseteq U_j \). Thus \( |\mathcal{U}| \geq |I| = 2^{\aleph_0} \) and so no topology containing \( B_2 \) is second countable.

(iv). The subspace topology on \( \text{Sym}(X) \) induced by \( B_1 \) is the pointwise topology. On the other hand, if \( Y \) and \( Z \) are infinite coinfinite sets, then \( V_{Y,Z} \cap \text{Sym}(X) \) is not open in the pointwise topology on \( \text{Sym}(X) \).

(v). Since the minimal \( T_1 \) topology that is semitopological for \( B_X \) coincides with the Hausdorff-Markov topology, by parts (i) and (ii), it follows that the Fréchet-Markov, Hausdorff-Markov and Zariski topologies all equal \( B_2 \); see Figure 2.

We obtain the following corollary to Theorem 7.2(iii).

**Corollary 7.3.** Let \( X \) be an infinite set. Then no second countable \( T_1 \) topology is semitopological for \( B_X \). In particular, \( B_X \) possesses no Polish semigroup topologies.

8. **The partial transformation monoid \( P_X \)**

Recall that the partial transformation monoid \( P_X \) consists of all partial functions on \( X \)

\[
P_X = \{ f \in B_X : |(\{x\})f| \leq 1 \ \text{for all} \ x \in X \}.
\]

A natural way of defining a semigroup topology on \( P_X \) is to embed \( P_X \) into the full transformation monoid \( X^X \), and use the subspace topology induced by the pointwise topology on \( X^X \). We will show that this topology is simultaneously the weakest \( T_1 \) semigroup topology on \( P_X \) and the finest extension of the pointwise topology of \( \text{Sym}(X) \) to \( P_X \).

Roughly speaking, the natural way of embedding \( P_X \) into \( X^X \) is to add a new element \( \bullet \) to \( X \) that will represent “not defined”. More precisely, if \( X \) is a set, \( \bullet \not\in X \), and \( Y = X \cup \{ \bullet \} \), then the function \( \phi : P_X \rightarrow Y^Y \) defined by

\[
(x)(f)\phi = \begin{cases} (x)f & \text{if } x \in \text{dom}(f) \\ \bullet & \text{if } x \not\in \text{dom}(f) \end{cases}
\]

is an embedding. Note that, in particular, if \( g = (f)\phi \), then \( (\bullet)g = \bullet \). We will refer to \( \phi \) as the natural embedding of \( P_X \) into \( Y^Y \).

**Theorem 8.1.** Let \( X \) be an infinite set and let \( \mathcal{P} \) be the topology on \( P_X \) generated by the sets

\[
U_{x,y} = \{ h \in P_X : (x,y) \in h \} \quad \text{and} \quad W_x = \{ h \in P_X : x \not\in \text{dom}(h) \}
\]

for all \( x, y \in X \). Then the following hold:

(i) the topology \( \mathcal{P} \) is the subspace topology on \( P_X \) induced by the pointwise topology on \( Y^Y \) and the natural embedding \( \phi : P_X \rightarrow Y^Y \) defined in (2);

(ii) \( \mathcal{P} \) is a Hausdorff semigroup topology for \( P_X \);

(iii) if \( \mathcal{S} \) is a topology that is \( T_1 \) and semitopological for \( P_X \), then \( \mathcal{S} \) contains \( \mathcal{P} \);

(iv) \( \mathcal{P} \) is the Hausdorff-Markov, Fréchet-Markov, and Zariski topologies for \( P_X \);

(v) \( P_X \) has property \( X \) with respect to \( \mathcal{P} \) and \( \text{Sym}(X) \);

(vi) if \( \mathcal{S} \) is a topology that is semitopological for \( P_X \) and \( \mathcal{S} \) induces the pointwise topology on \( \text{Sym}(X) \), then \( \mathcal{S} \) is contained in \( \mathcal{P} \);

(vii) if \( X \) is countable, then \( P_X \) has automatic continuity with respect to \( \mathcal{P} \);
(viii) the topology $\mathcal{P}$ is the unique $T_1$ topology that induces the pointwise topology on $\text{Sym}(X)$ and that is semitopological for $P_X$;

(ix) if $X$ is countable, then $\mathcal{P}$ is the unique Polish topology that is semitopological for $P_X$;

(x) if $X$ is countable, then $\mathcal{P}$ is the unique $T_1$ second countable semigroup topology for $P_X$.

Proof. (i). The image of $P_X$ under the natural embedding $\phi$ defined in (2) is the set

$$\{ h \in Y^Y : (\bullet)h = \bullet \}.$$ 

The pointwise topology on $Y^Y$ is generated by the sets $A_{x,y} = \{ f \in Y^Y : (x,y) \in f \}$ for all $x,y \in Y$. Hence the topology induced on $(P_X)\phi$ is generated by the sets $A_{x,y} \cap (P_X)\phi$. Note that $A_{x,\bullet} \cap (P_X)\phi = (P_X)\phi$ and $A_{\bullet,y} \cap (P_X)\phi = \emptyset$ for all $y \in X = Y \setminus \{ \bullet \}$. If $x,y \in X$, then

$$A_{x,y} \cap (P_X)\phi = \{ h \in (P_X)\phi : (x,y) \in h \} = (U_{x,y})\phi$$

and

$$A_{x,\bullet} \cap (P_X)\phi = \{ h \in (P_X)\phi : (x,\bullet) \in h \} = (W_x)\phi.$$

Hence $\phi$ is a homeomorphism between $\mathcal{P}$ and the topology generated by $A_{x,y} \cap (P_X)\phi$.

(ii). Since $Y^Y$ is a Hausdorff topological semigroup under the pointwise topology, it follows from part (i) that $P_X$ is a Hausdorff topological semigroup under $\mathcal{P}$.

(iii). Let $x,y \in X$. We will show that the sub-basic open sets of $\mathcal{P}$ are open in $\mathcal{S}$. If $f \in P_X$ is arbitrary, then $\{ \{x\},\{y\}\} \cap X \subseteq \{ f \in P_X : x \neq f(y) \}$ is an open neighbourhood of $x$ if and only if $x \neq f(y)$.

$$P_X \setminus U_{x,y} = \{ f \in P_X : (x,y) \notin f \} = (\emptyset) \cup (\lambda_{\{x,y\}} \cup \rho_{\{y\}})^{-1}$$

is the continuous pre-image of the finite set $\{ \emptyset \}$ and hence closed in $\mathcal{S}$. Hence $U_{x,y}$ is open in $\mathcal{S}$ for any $x,y \in X$.

Similarly, if $c_x \in X^X \subseteq P_X$ is the constant function with image $\{ x \}$, then $c_x f c_x = c_x$ if and only if $x \in \text{dom}(f)$. Hence $\Phi \setminus W_x = \{ f \in P_X : x \in \text{dom}(f) \}$ is closed in $\mathcal{S}$, and $W_x$ is open in $\mathcal{S}$ for all $x \in X$.

(iv). This follows immediately from parts (ii), (iii) and Figure 2.

(v). Let $f \in X^X \subseteq P_X$ be an injective function such that $|X \setminus \text{im}(f)| = |X|$ and let $g \in P_X$ be such that $|X \setminus \text{dom}(g)| = |X|$ and $|\{x\}g^{-1}| = |X|$ for every $x \in X$.

Fix $s \in P_X$. Then $ftg = s$ for some $t \in P_X$ if and only if

$$\{ x \} f t \in \{ (x)s \} g^{-1} \quad \text{if} \quad x \in \text{dom}(s) \quad \text{and} \quad \{ x \} f t \in \text{dom}(g) \quad \text{if} \quad x \in X \setminus \text{dom}(s)$$

for all $x \in X$. Since $Xf, X \setminus \text{im}(f), X \setminus \text{dom}(g)$, and every $\{x\}g^{-1}$ have cardinality $|X|$, there exists $t_s \in \text{Sym}(X)$ satisfying (3) for all $x \in X$ and so $f t_s g = s$.

If $k \in P_X$ is finite, and $Z$ is a finite subset of $X$, then

$$R_{k,Z} = \{ h \in P_X : k \subseteq h \text{ and } Z \cap \text{dom}(h) = \emptyset \} = \bigcap_{x \in \text{dom}(k)} U_{x,(x)k} \cap \bigcap_{z \in Z} W_z.$$ 

It follows that the collection $\mathcal{B}$ of all the sets $R_{k,Z}$ forms a basis for $\mathcal{P}$. Let $B \in \mathcal{B}$ be a basic open neighbourhood of $t_s$. Since $t_s$ is a permutation, $\text{dom}(t_s) = X$ and so $B = \{ h \in P_X : k \subseteq h \}$ for some finite, injective $k \in P_X$. Since $s = f k g \cup f(t_s) \setminus k g$, if $x \in \text{dom}(s)$, then either $x \in \text{dom}(f k g)$ or $x \in \text{dom}(f(t_s) \setminus k g)$. In the former case, $x \notin (X \setminus \text{dom}(g))k^{-1}f^{-1}$, and in the latter, $x \notin (\text{dom}(k)f^{-1}) \cap (X \setminus \text{dom}(g))k^{-1}f^{-1}$. By the contrapositive, if $x \in (\text{dom}(k)f^{-1}) \cap (X \setminus \text{dom}(g))k^{-1}f^{-1}$, then $x \notin \text{dom}(s)$. Thus, since $k$ is finite,

$$U := R_{f k g,Z} = \{ h \in P_X : f k g \subseteq h \text{ and } Z \cap \text{dom}(h) = \emptyset \},$$

where $Z = (\text{dom}(k)f^{-1}) \cap ((X \setminus \text{dom}(g))k^{-1}f^{-1})$, is an open neighbourhood of $s$.

It suffices to show that $U \subseteq f(B \cap \text{Sym}(X))g$. Let $u \in U$. We will prove that there exists $t_u \in B \cap \text{Sym}(X)$ such that $u = f t_u g$.

If $z \in \text{dom}(k) \cap \{\{x\}u^{-1}\}f$ for some $x \in X$, then $f k g \subseteq u$ implies that $(z)k \in \{\{x\}g^{-1} \cup X \setminus \text{dom}(g))$. We will show that $(z)k \in \{\{x\}g^{-1}$ for some $z \in \text{dom}(k) \cap \{\{x\}u^{-1}\}f$, there exists $z' \in \text{dom}(k)f^{-1} \cap \{\{x\}u^{-1}\} \subseteq \text{dom}(u)$ such that $(z')f = z$. Since $(\text{dom}(k)f^{-1}) \cap ((X \setminus \text{dom}(g))k^{-1}f^{-1}) \cap \{\{x\}u^{-1}\}$
dom(u) = ∅, z′ \not\in (X \setminus \text{dom}(g))k^{-1} f^{-1} and so (z)k = (z′)fk \not\in X \setminus \text{dom}(g). We have shown that for all x \in X
\[ z \in \text{dom}(k) \cap (\{(x)\}u^{-1})f \] implies that (z)k \in (\{(x)\})g^{-1}.
Thus we may define t_z \in P_X such that \text{dom}(t_z) = (\{(x)\})u^{-1} f, \text{im}(t_z) \subseteq (\{(x)\})g^{-1} \setminus (X \setminus \{(x)\})u^{-1} k, and (z)t_z = (z)k. Since \|\{(x)\}u^{-1} f\| \leq \|\{(x)\})g^{-1}\| = |X| and k is injective and finite, we may choose t_z \in P_X to be injective.
If z \in \text{dom}(k) \cap (X \setminus \text{dom}(u))f, then we will show that (z)k \not\in \text{dom}(g). Suppose to the contrary that (z)k \in \text{dom}(g). Then there exists z′ \in X \setminus \text{dom}(u) such that z = (z′)f \in \text{dom}(k) and (z′)fk \in \text{dom}(g). This implies that z′ \in \text{dom}(fk) \subseteq \text{dom}(u), a contradiction.
Therefore we may define t_\bullet \in P_X such that \text{dom}(t_\bullet) = (X \setminus \text{dom}(u))f, \text{im}(t_\bullet) \subseteq X \setminus \text{dom}(g), \text{im}(t_\bullet) \cap (X \setminus \text{dom}(u))k = \emptyset, and
\[ z \in \text{dom}(k) \cap (X \setminus \text{dom}(u))f \Rightarrow (z)t_\bullet = (z)k. \]
In particular, since k is injective, we may choose t_\bullet to be injective also, and since k is finite, we can choose t_\bullet so that
\[ |(X \setminus \text{dom}(g)) \setminus (X \setminus \text{dom}(u))f|t_\bullet| = |X|. \]
Let t : X \setminus \text{im}(f) \longrightarrow X \cup_{y \in Y} \text{im}(t_y) be any bijection such that (z)t = (z)k for all z \in \text{dom}(t).
Note that if x, y \in Y and x \neq y, then \text{dom}(t_x) \cap \text{dom}(t_y) = \emptyset and so
\[ t_u := t \cup \bigcup_{y \in Y} t_y \in P_X. \]
Since \bigcup_{y \in Y} \text{dom}(t_y) = \text{im}(f), \text{dom}(t_u) = X. Since t and every t_y, y \in Y, is injective, and \text{im}(t_x) \cap \text{im}(t_y) = \emptyset if x \neq y, t_u is injective also. In particular, t_u \in \text{Sym}(X). By construction, k \subseteq t_u and so t_u \in \text{Sym}(X) \cap B. Finally, we will show that
\[ ft_u g = u. \]
Suppose that x \in X is arbitrary. Then either: x \in (\{(y)\})u^{-1} for some y \in \text{im}(u); or x \not\in \text{dom}(u). In the first case, (x)f \in (\{(y)\})u^{-1} f = \text{dom}(t_y) and so, by the definition of t_y, (x)f t_y \in (\{(y)\})g^{-1}. Therefore, in the first case, (x)f t_u g = (x)f t_y g = y = (x)u. In the second case, (x)f \in (X \setminus \text{dom}(u))f = \text{dom}(t_\bullet) and by the definition of t_\bullet, (x)f t_\bullet \not\in \text{dom}(g). Thus x \not\in \text{dom}(ft_u g) and so neither (x)f t_u g nor (x)u is defined.

(vi). Suppose that S is a topology that is semitopological for P_X and that S the pointwise topology on \text{Sym}(X). Since W_z \cap X^X = \{ h \in P_X : z \not\in \text{dom}(h) \} \cap X^X = \emptyset, the topology induced by \mathcal{P} on \text{Sym}(X) is just the pointwise topology. Hence S and \mathcal{P} induce the same topology on \text{Sym}(X). Since P_X has property X with respect to \mathcal{P} and \text{Sym}(X), it follows that S \subseteq \mathcal{P} by Lemma 3.6(ii).

(vii). This follows immediately from (v), Lemma 3.6(iv) and the automatic continuity of \text{Sym}(X).

(viii). This follows from (iii) and (vi).

(ix). Let X be countable. By part (i), P_X is homeomorphic to its image under \phi. It is easy to see that (P_X)\phi is a closed subset of Y^\gamma under the pointwise topology. Thus P_X under \mathcal{P} is homeomorphic to a closed subspace of a Polish space and is hence Polish.
Suppose that S is a Polish topology that is semitopological for P_X. Then S is T_1 and so \mathcal{P} \subseteq S by part (iii). Since \mathcal{P} is Polish and \text{Sym}(X) is a Polish subgroup of P_X, it follows from Lemma 3.6(iii) that S \subseteq \mathcal{P}. Hence S = \mathcal{P}, as required.

(x). Suppose that S is a T_1 second countable semigroup topology for P_X. By part (iii), \mathcal{P} is contained in S. By part (vii), applied to the identity function from (P_X, \mathcal{P}) to (P_X, S), S is contained in \mathcal{P} also. \[ \square \]

It is natural to ask how the topology \mathcal{P} from Theorem 8.1 relates to the semigroup topologies on P_X induced by those on B_X given in Theorems 7.1 and 7.2.
Proposition 8.2. If $X$ is an infinite set and $B_1$ and $B_2$ are the topologies on the full binary relation monoid $B_X$ from Theorems 7.1 and 7.2, respectively, then the subspace topology on $P_X$ induced by $B_1$ is strictly contained in $\mathcal{P}$ and $\mathcal{P}$ is strictly contained in the subspace topology on $P_X$ induced by $B_2$.

Proof. Clearly from the definitions of $B_1$ and $\mathcal{P}$, the subspace topology induced by $B_1$ on $P_X$ is contained in $\mathcal{P}$. This containment is strict because $\mathcal{P}$ is $T_1$ but $B_1$ is not $T_1$ on $I_X \subseteq P_X$ by Theorem 7.1(ii).

The second containment follows since
\[
\{h \in P_X : x \notin \text{dom}(h)\} = \{h \in B_X : \{(x)\}h \subseteq \emptyset\} \cap P_X \quad \text{and} \quad \{h \in B_X : \{(x)\}h \subseteq \emptyset\} \in B_2
\]
for all $x \in \Omega$. The topology $\mathcal{P}$ induces the pointwise topology on $\text{Sym}(X)$, but $B_2$ does not by the proof of Theorem 7.1(iv). \hfill \qed

9. The symmetric inverse monoid $I_X$

Recall that the symmetric inverse monoid
\[
I_X = \{f \in X^X : |\{(x)\}f^{-1}| \leq 1 \text{ for all } x \in X\};
\]
As its name suggests, $I_X$ is an inverse semigroup with group of units $\text{Sym}(X)$. In fact, $I_X$ is the analogue of $\text{Sym}(X)$ for inverse semigroups: every inverse semigroup is isomorphic to an inverse subsemigroup of $I_X$ for some $X$.

We will now construct the coarsest $T_1$ topologies under which $I_X$ is a semitopological semigroup, a topological semigroup and a topological inverse semigroup, respectively.

Theorem 9.1. Let $X$ be an infinite set and let $I_1$ denote the topology on $I_X$ generated by the sets
\[
U_{x,y} = \{h \in I_X : (x,y) \in h\} \quad \text{and} \quad V_{x,y} = \{h \in I_X : (x,y) \notin h\}
\]
for all $x, y \in X$. Then the following hold:

(i) the topology $I_1$ is Hausdorff, semitopological for $I_X$ and inversion is continuous;
(ii) $I_1$ is the least $T_1$ topology that is semitopological for $I_X$;
(iii) if $X$ is countable, then $I_1$ is Polish.

Proof. (i). If $f, g \in I_X$ and $f \neq g$, then without loss of generality there exist $x, y \in X$ such that
\[
(x,y) \in f \text{ but } (x,y) \notin g.
\]
Then $f \in U_{x,y}$ and $g \in V_{x,y}$, and so $I_1$ is Hausdorff.

The sets $U_{x,y}$ generate the subspace topology on $I_X$ induced by the topology $B_1$ on $B_X$ defined in Theorem 7.1. Hence, by Theorem 7.1(i), the subspace topology induced by $B_1$ on $I_X$ is a semigroup topology where inversion is continuous also. Hence it suffices consider the sub-basic open sets $V_{x,y}, x, y \in X$.

Suppose that $x, y \in X$ are fixed. If $\iota : I_X \to I_X$ is defined by $(f)\iota = f^{-1}$, then $(V_{x,y})\iota = \{h \in I_X : (y,x) \notin h\} = V_{y,x}$ is open in $I_1$, and hence $\iota$ is continuous. Suppose that $f \in I_X$ is arbitrary. If $x \notin \text{dom}(f)$, then $(x,y) \notin fg$ for all $g \in I_X$. Hence $(V_{x,y})\lambda_f^{-1} = I_X$ is open. If $x \in \text{dom}(f)$ and $fg \in V_{x,y}$ for some $g \in I_X$. Then $(x)f, y) \notin g$. Thus $g \in V_{x,y} \subseteq (V_{x,y})\lambda_f^{-1}$ and so $(V_{x,y})\lambda_f^{-1}$ is open. Hence $\lambda_f$ is continuous for every $f \in I_X$. Since $\rho_f = \iota\lambda_{f^{-1}}\iota$ is a composition of continuous functions, $\rho_f$ is continuous also.

(ii). Let $S$ be a $T_1$ topology that is semitopological for $I_X$. Let $x, y \in X$ and let $f \in I_X$ be arbitrary. Then
\[
\{x,y\} \circ f \circ \{(x,y)\} = \emptyset \quad \text{if and only if} \quad f \in V_{x,y}
\]
and
\[
\{x,y\} \circ f \circ \{(x,y)\} = \{(x,y)\} \quad \text{if and only if} \quad f \in U_{x,y}.
\]
Since $S$ is $T_1$, the singletons $\{\emptyset\}$ and $\{(x,y)\}$ are closed in $S$. Thus their respective pre-images $V_{x,y}$ and $U_{x,y}$ under multiplication by $\lambda_f(x,y) \circ \rho((x,y))$ are closed in $S$. Hence they are both open in $S$, as they are mutual complements. Thus $S_1 \subseteq S$, as required.

(iii). Without loss of generality, suppose that $X = \mathbb{N} = \{0,1,2,\ldots\}$. The given sub-basis for $I_1$ is countable, and so $I_1$ is second-countable and hence separable. To show that $I_1$ is Polish, we will find a complete metric on $I_N$ that induces $I_1$. 

Recall that every natural number \( m \) is the set \( m = \{0, \ldots, m-1\} \). We define \( d : I_N \times I_N \to \mathbb{R} \) by

\[
d(f, g) = \frac{1}{m+1} \quad \text{where } m = \min \{ n \in \mathbb{N} : (n \times n) \cap f \neq (n \times n) \cap g \}
\]

if \( f \neq g \) and we define \( d(f, g) = 0 \) if \( f = g \). It is straightforward to show that \( d \) is a metric on \( I_N \). We will now show that \( d \) induces \( \mathcal{I}_1 \).

A basis for \( \mathcal{I}_1 \) is given by the finite intersections of the sub-basic sets, which is the collection of sets

\[
B_{f,g} = \{ h \in I_N : f \subseteq h \text{ and } g \cap h = \emptyset \}
\]

for all finite \( f, g \in I_N \). Let \( f \in I_N \) and \( m \in \mathbb{N} \) be arbitrary. Then the open ball of radius \( \frac{1}{m+1} \) around \( f \) is

\[
B \left( f, \frac{1}{m+1} \right) = \left\{ g \in I_N : d(f, g) < \frac{1}{m+1} \right\} = \{ g \in I_N : (m \times m) \cap f = (m \times m) \cap g \} = B_{(m \times m) \cap f,(m \times m) \setminus f}.
\]

Thus every open ball with respect to \( d \) is open in \( \mathcal{I}_1 \) and so the topology induced by \( d \) is contained in \( \mathcal{I}_1 \).

On the other hand, let \( B_{f,g} \) be an arbitrary non-empty basic open set of \( \mathcal{I}_1 \) and let \( h \in B_{f,g} \). Then \( f \subseteq h \) and \( g \cap h = \emptyset \). If \( m \) is the maximal element of \( \text{dom}(f) \cup \text{im}(f) \cup \text{dom}(g) \cup \text{im}(g) \), then \( h \in B(h,1/m+1) \subseteq B_{f,g} \). Since \( h \) was arbitrary, it follows that \( B_{f,g} \) is open in the topology induced by \( d \) and so the topology induced by \( d \) contains \( \mathcal{I}_1 \). Hence the two topologies are equal.

To see that \( d \) is complete, let \( f_0, f_1, f_2, \ldots \in I_N \) be a Cauchy sequence. If \( m \in \mathbb{N} \), then for all sufficiently large \( i, j \in \mathbb{N} \) we have that \( d(f_i, f_j) < \frac{1}{m+1} \) and so if \( 0 \leq x, y \leq m \), then \( (x, y) \in f_i \) if and only if \( (x, y) \in f_j \). In particular, any pair \( (x, y) \) is either contained in all \( f_i \) or eventually not contained in all \( f_i \). If \( f = \{ (x, y) \in \mathbb{N} \times \mathbb{N} : \text{there exists } N \in \mathbb{N} \text{ such that } (x, y) \in f_i \text{ for all } i \geq N \} \), then \( f \in I_N \) and \( (f_i)_{i \in \mathbb{N}} \) converges to \( f \) with respect to \( d \) and so \( d \) is complete.

We will show that there are precisely two minimal \( T_1 \) semigroup topologies on \( I_X \). The first is just the topology induced by the minimal \( T_1 \) semigroup topology on \( P_X \) (see Theorem 8.1) and the second consists of the inverses \( U^{-1} = \{ f^{-1} : f \in U \} \) of the open sets \( U \) of the first one.

**Theorem 9.2.** Let \( X \) be an infinite set, let \( \mathcal{I}_2 \) be the topology on the symmetric inverse monoid \( I_X \) generated by the collection of sets,

\[
U_{x,y} = \{ h \in I_X : (x, y) \in h \} \quad \text{and} \quad W_x = \{ h \in I_X : x \notin \text{dom}(h) \}
\]

and let \( \mathcal{I}_3 \) be the topology on \( I_X \) generated by the sets

\[
U_{x,y} = \{ h \in I_X : (x, y) \in h \} \quad \text{and} \quad W_x^{-1} = \{ h \in I_X : x \notin \text{im}(h) \}
\]

for all \( x, y \in X \). Then the following hold:

(i) \( (I_X, \mathcal{I}_2) \) and \( (I_X, \mathcal{I}_3) \) are homeomorphic, Hausdorff, topological semigroups;

(ii) every \( T_1 \) semigroup topology for \( I_X \) contains \( \mathcal{I}_2 \) or \( \mathcal{I}_3 \);

(iii) \( \mathcal{I}_2 \cap \mathcal{I}_3 \) is the Fréchet-Markov topology for \( I_X \);

(iv) \( \mathcal{I}_2 \cap \mathcal{I}_3 \) is the Hausdorff-Markov topology for \( I_X \);

(v) if \( X \) is countable, then \( \mathcal{I}_2 \) and \( \mathcal{I}_3 \) are Polish.

**Proof.** (i). The Hausdorff semigroup topology \( \mathcal{P} \) on \( P_X \) defined in Theorem 8.1 induces \( \mathcal{I}_2 \) on \( I_X \). Hence \( I_X \) is a Hausdorff topological semigroup under \( \mathcal{I}_2 \).

The map \( f \mapsto f^{-1} \) defines an anti-automorphism of \( I_X \). The images of the sub-basis for \( \mathcal{I}_2 \) under inversion give the sub-basis for \( \mathcal{I}_3 \). By Proposition 3.1, it follows that \( (I_X, \mathcal{I}_3) \) is a topological semigroup homeomorphic to \( (I_X, \mathcal{T}) \).
(ii). Let $S$ be any $T_1$ semigroup topology for $I_X$. By Theorem 9.1(ii), $S$ contains the topology $T_3$ with sub-basis
\[ U_{x,y} = \{ h \in I_X : (x, y) \in h \} \quad \text{and} \quad V_{x,y} = \{ h \in I_X : (x, y) \notin h \} \]
for all $x, y \in X$. It remains to show that either \( \{ h \in I_X : x \notin \text{dom}(x) \} \in S \) for all $x \in X$ or \( \{ h \in I_X : y \notin \text{im}(h) \} \in S \) for all $y \in X$.

For every $x \in X$, the set $V_{x,x} = \{ h \in I_X : (x, x) \notin h \}$ is an open neighbourhood of $\emptyset$ under $S$. Since $\emptyset \cdot \emptyset = \emptyset \in V_{x,x}$ and $S$ is a semigroup topology, there exists an open neighbourhood $U$ of $\emptyset$ such that $U \cdot U \subseteq V_{x,x}$. In other words, $(x, x) \notin u$ for any $u, v \in U$. If $z \in X$, then $z \notin \{ y \in X : (y, x) \notin u \}$ for all $u \in U$, and $z \notin \{ y \in X : (y, x) \notin u \}$ for all $u \in U$, then there exist $u, v \in U$ such that $(z, z) \in u$ and $(z, x) \in v$, and so $(x, x) \in uv$, a contradiction. Hence every $z \in X$ belongs to one or the other of the sets:
\[ \{ y \in X : (x, y) \notin u \} \quad \text{or} \quad \{ y \in X : (y, x) \notin u \} \]
and so one of these two sets has cardinality $|X|$.

Suppose \( \{ y \in X : (x, y) \notin u \} \) for all $u \in U$\}. By taking a subset if necessary, it follows that there exists $Y \subseteq X$ with $|Y| = |X \setminus Y|$ such that $(x, y) \notin u$ for all $y \in Y$ and $u \in U$. Let $p \in \text{Sym}(X)$ be any involution such that $(Y)p = X \setminus Y$, and so $(X \setminus Y)p = Y$. Then for any $u \in U$ and any $y \in X \setminus Y$ since $(y)p \in Y$ it follows that $(x, y)p \notin u$ and so $(x, y) \notin up$. Let $V = U \cup up$. Then $V$ is an open neighbourhood of $\emptyset$ and $x \notin \text{dom}(f)$ for all $f \in V$. Let $g \in I_X$ be arbitrary. If $x \in \text{dom}(g)$, then $x \in \text{dom}((x, x) \circ g)$ and so $(x, x) \circ g \notin V$. On the other hand, if $x \notin \text{dom}(g)$, then $\{(x, x)\} \circ g = \emptyset \in V$. Thus
\[ (V)^{\lambda_{x\setminus y=1}^{-1}}(x) = \{ g \in I_X : (x, x) \circ g \in V \} = \{ g \in I_X : x \notin \text{dom}(g) \} \]
is open in $S$. Since $x \in X$ was arbitrary, it follows that $T_2 \subseteq S$.

If \( \{ y \in X : (y, x) \notin u \} \) for all $u \in U$, then $T_3 \subseteq S$ by an analogous argument.

(iii) and (iv). These follow directly from parts (i), (ii) and Figure 2.

(v). It is shown in Theorem 8.1(ix) that the partial transformation monoid $P_X$ forms a Polish semigroup with the topology $\mathcal{P}$ defined in that theorem. The topology induced by $\mathcal{P}$ on $I_X$ is $T_2$. It is routine to verify that $I_X$ is a closed subset of $P_X$ under the topology $\mathcal{P}$, and so $T_2$ is Polish also. Thus $T_3$ is Polish as it is homeomorphic to $T_2$.

It is natural to ask if the Hausdorff-Markov topology $T_2 \cap T_3$ coincides with the minimal $T_1$ semitopological semigroup topology $T_1$ for $I_X$ as defined in Theorem 9.1. The next proposition says: no.

**Proposition 9.3.** $T_1 \subseteq T_2 \cap T_3$.

**Proof.** Let $T_1$ be the topology from Theorem 9.1. Let $x \in X$ be fixed. Let
\[ F := \{(y, x), (x, y) : y \in X\}. \]

We show that $F$ is closed in $T_2 \cap T_3$ but not in $T_1$.

We can express the complement of $F$ as:
\[ U := I_X \setminus F = \{ h \in I_X : x \notin \text{dom}(h) \} \cup \bigcup_{y \in X} \{ h \in I_X : (x, y) \in h \} \setminus \{(y, x), (x, y)\}. \]

As $T_2$ is $T_1$ this is a union of (disjoint) open sets. It follows that $U$ is open and hence $F$ is closed. Since $\{ f^{-1} : f \in F \} = F$ and $\{ V^{-1} : V \in T_2 \} = T_3$, it follows that $F$ is also closed in $T_3$.

It remains to show that $F$ is not closed in $T_1$. Suppose for a contradiction that $U \in T_1$. As $\emptyset \in U$, it follows that there exists a finite intersection of the sub-basic open sets $V_{x,y}$ containing $\emptyset$ and contained in $U$. Hence there exists a finite $f \in I_X$ such that $\emptyset \in \{ h \in I_X : h \cap f = \emptyset \} \subseteq U$. If $y \in X \setminus \text{dom}(f) \cup \text{im}(f)$, then $k = \{(y, x), (x, y)\} \in \{ h \in I_X : h \cap f = \emptyset \}$ and $k \in F$, a contradiction. \( \square \)
It is natural to ask for complete metrics on $I_X$ that induce $\mathcal{I}_2$ and $\mathcal{I}_3$ from Theorem 9.2. Such metrics can be defined using the natural embedding $\phi$ defined in (2):

\[
d_1(f, g) = \begin{cases} 
0 & \text{if } f = g \\
\frac{1}{m+1} & \text{if } f \neq g 
\end{cases}
\]

where $m = \min\{x \in \mathbb{N} : (x)(f) \neq (x)(g)\}$

and

\[
d_2(f, g) = \begin{cases} 
0 & \text{if } f = g \\
\frac{1}{m+1} & \text{if } f \neq g 
\end{cases}
\]

where $m = \min\{x \in \mathbb{N} : (x)(f^{-1}) \neq (x)(g^{-1})\}$.

We will now show that the topology generated by the union of the two minimal $T_1$ semigroup topologies $\mathcal{I}_2$ and $\mathcal{I}_3$ on $I_X$ is simultaneously the minimal $T_1$ inverse semigroup topology on $I_X$ as well as the finest topology inducing the pointwise topology on $\text{Sym}(X)$.

**Theorem 9.4.** Let $X$ be an infinite set, let $\mathcal{I}_4$ be the topology on the symmetric inverse monoid $I_X$ generated by the collection of sets,

$U_{x,y} = \{ h \in I_X : (x, y) \in h \}$, $W_x = \{ h \in I_X : x \notin \text{dom}(h) \}$, and $W_x^{-1} = \{ h \in I_X : x \notin \text{im}(h) \}$.

Then the following hold:

(i) the topology $\mathcal{I}_4$ is a Hausdorff inverse semigroup topology for $I_X$;

(ii) if $\mathcal{S}$ is a $T_1$ inverse semigroup topology for $I_X$, then $\mathcal{I}_4$ is contained in $\mathcal{S}$;

(iii) the Fréchet-Markov inverse semigroup topology for $I_X$ is $\mathcal{I}_4$;

(iv) the Hausdorff-Markov inverse semigroup topology for $I_X$ is $\mathcal{I}_4$;

(v) $I_X$ has property $X$ with respect to $\mathcal{I}_4$ and $\text{Sym}(X)$;

(vi) if $\mathcal{S}$ is a topology that is semitopological for $I_X$ and $\mathcal{S}$ induces the pointwise topology on $\text{Sym}(X)$, then $\mathcal{S}$ is contained in $\mathcal{I}_4$;

(vii) $\mathcal{I}_4$ is the unique $T_1$ inverse semigroup topology on $I_X$ inducing the pointwise topology on $\text{Sym}(X)$;

(viii) if $X$ is countable, then $I_X$ has automatic continuity with respect to $\mathcal{I}_4$;

(ix) if $X$ is countable, then $\mathcal{I}_4$ is the unique $T_1$ second-countable inverse semigroup topology on $I_X$;

(x) if $X$ is countable, then $\mathcal{I}_4$ is the unique Polish inverse semigroup topology on $I_X$.

**Proof.** The topology $\mathcal{I}_4$ is the topology generated by the union $\mathcal{I}_2 \cup \mathcal{I}_3$ defined in Theorem 9.2.

(i). Since $\mathcal{I}_4$ is generated by the Hausdorff semigroup topologies $\mathcal{I}_2$ and $\mathcal{I}_3$ (by Theorem 9.2(i)), it follows that $\mathcal{I}_4$ is a Hausdorff semigroup topology for $I_X$. If $U$ is any of the sub-basic open sets defining $\mathcal{I}_4$, then $U^{-1} = \{ f^{-1} : f \in U \}$ is also a sub-basic open set, and so $\mathcal{I}_4$ is an inverse semigroup topology.

(ii). Let $\mathcal{S}$ be any $T_1$ inverse semigroup topology for $I_X$. By Theorem 9.2(ii), $\mathcal{S}$ contains either $\mathcal{I}_2$ or $\mathcal{I}_3$. Since inversion is continuous with respect to $\mathcal{S}$ and $\{ U^{-1} : U \in \mathcal{I}_2 \} = \mathcal{I}_3$, it follows that $\mathcal{S}$ contains $\mathcal{I}_4$.

(iii). This follows immediately from (ii).

(iv). Since by (vii) the inverse Fréchet-Markov topology on $I_X$ is $\mathcal{I}_4$, and by (i), this topology is Hausdorff. Thus $\mathcal{I}_4$ is also the inverse Hausdorff-Markov topology.

(v). The only sub-basic open sets of $\mathcal{I}_4$ that have non-empty intersection with $\text{Sym}(X)$ are $U_{x,y} = \{ h \in I_X : (x, y) \in h \}$ where $x, y \in X$. Hence $\mathcal{I}_4$ induces the pointwise topology on $\text{Sym}(X)$.

Let $f \in I_X$ satisfy $\text{dom}(f) = X$ and $|X \setminus \text{im}(f)| = |X|$ and let $s \in I_X$ be arbitrary. Then $t \in \text{Sym}(X)$ satisfies $ftf^{-1} = s$ if and only if

$$(x, (x)s) \in ftf^{-1} \text{ for all } x \in \text{dom}(s) \text{ and } x \notin \text{dom}(ftf^{-1}) \text{ for all } x \in X \setminus \text{dom}(s),$$

which follows if and only if

$$(xf)t = (x)sf \text{ for all } x \in \text{dom}(s) \text{ and } (xf)t \in X \setminus \text{im}(f) \text{ for all } x \in X \setminus \text{dom}(s).$$
Since \(|\text{im}(f)| = |X \setminus \text{im}(f)|\) we can define such a \(t \in \text{Sym}(X)\) by
\[
(x)t = \begin{cases} 
(x)f^{-1}s f & \text{if } x \in \text{dom}(f) \\
(x)\phi_1 & \text{if } x \in (X \setminus \text{dom}(f)) \\
(x)\phi_2 & \text{if } x \notin \text{im}(f)
\end{cases}
\]
where \(\phi_1 : (X \setminus \text{dom}(s))f \to X \setminus \text{im}(f)\) is an injection with \(|(X \setminus \text{im}(f)) \setminus \text{im}(\phi_1)| = |X|\), and \(\phi_2 : X \setminus \text{im}(f) \to ((X \setminus \text{im}(f)) \setminus \text{im}(\phi_1))\) is a bijection.

If \(k \in I_X\) is finite, and \(Y\) and \(Z\) are finite subsets of \(X\), then
\[
R_{k,Y,Z} = \{ h \in I_X : k \subseteq h \text{ and } Y \cap \text{dom}(h) = Z \cap \text{im}(h) = \emptyset \} = \bigcap_{x \in \text{dom}(k)} U_{x,(x)k} \cap \bigcap_{y \in Y} W_y \cap \bigcap_{z \in Z} W_z^{-1}.
\]
Hence the collection \(\mathcal{B}\) of all such sets \(R_{k,Y,Z}\) forms a basis for \(I_4\). Let \(U := R_{k,Y,Z} \in \mathcal{B}\) be any such basic open neighbourhood of \(t\) where \(k, Y\), and \(Z\) are finite. Since \(\text{dom}(t) = \text{im}(t) = X\), it follows that \(Y = \emptyset\) and \(Z = \emptyset\). Hence
\[
f(U \cap \text{Sym}(X))f^{-1} = \{ fgf^{-1} : g \in \text{Sym}(X) \text{ and } k \subseteq g \}.
\]
If \(u = fgf^{-1} \in f(U \cap \text{Sym}(X))f^{-1}\), then
\[
\begin{align*}
(1) \ & fkf^{-1} \subseteq u; \\
(2) \ & (x)f^{-1} \notin \text{dom}(u) \text{ whenever } x \in \text{dom}(k) \text{ and } (x)k \notin \text{dom}(f^{-1}); \\
(3) \ & (x)f^{-1} \notin \text{im}(u) \text{ whenever } x \in \text{im}(k) \text{ and } (x)k^{-1} \notin \text{im}(f).
\end{align*}
\]
Conversely, if \(u \in I_X\) satisfies (1)-(3), then we can find \(g \in \text{Sym}(X)\) such that \(k \subseteq g\) and \(g\) satisfies condition (6) (where \(s\) and \(t\) are replaced by \(u\) and \(g\), respectively) as follows
\[
(x)g = \begin{cases} 
(x)f^{-1}uf & \text{if } x \in \text{dom}(u) \\
(x)\phi_1 & \text{if } x \in (X \setminus \text{dom}(u)) \\
(x)\phi_2 & \text{if } x \notin \text{im}(f)
\end{cases}
\]
where \(\phi_1 : (X \setminus \text{dom}(s))f \to (X \setminus \text{im}(f)) \setminus \{ (x)k : x \in X \setminus \text{dom}(f) \}\) is an injection which agrees with \(k\) when they are both defined and \(|(X \setminus \text{im}(f)) \setminus \text{im}(\phi_1)| = |X|\), and \(\phi_2 : X \setminus \text{im}(f) \to ((X \setminus \text{im}(u)f) \setminus \text{im}(\phi_1))\) is a bijection which agrees with \(k\) when they are both defined.

Thus \(u = fgf^{-1} \in f(U \cap \text{Sym}(X))f^{-1}\). Hence \(f(U \cap \text{Sym}(X))f^{-1}\) consists precisely of those \(s' \in I_X\) that satisfy (S1)-(S3). We define
\[
Y = \{(y)f^{-1} : y \in \text{dom}(k) \text{ and } (y)k \notin \text{dom}(f^{-1})\}
\]
and
\[
Z = \{(z)f^{-1} : z \in \text{im}(k) \text{ and } (z)k^{-1} \notin \text{dom}(f^{-1})\}.
\]
Since \(k\) is finite, so too are \(fkf^{-1}\), \(Y\), and \(Z\). Thus \(f(U \cap \text{Sym}(X))f^{-1} = R_{k,Y,Z}\) is open in \(I_4\).

The result now follows by Lemma 3.6 (ii). We will now show that \(f(U \cap \text{Sym}(X))f^{-1}\) is open in \(I_4\).

(vi). This follows from (v) together with Lemma 3.6(ii).

(vii). This follows straight from (ii) and (vi).

(viii). This follows from (v), Lemma 3.6(vi) and the automatic continuity of \(\text{Sym}(X)\).

(ix). If \(X\) is countable, then the sub-basis for \(I_4\) is countable also. Thus \(I_4\) is second-countable. On the other hand, if \(S\) is any \(I_1\) second countable, inverse semigroup topology for \(I_X\), then, by part (ii), \(I_4 \subseteq S\). By part (vii), we also have \(S \subseteq I_4\) and so \(S = I_4\), as required.

(x). Suppose that \(X = \mathbb{N}\). We only need to show that \(I_4\) is completely metrizable, since separability and uniqueness then follow from part (ix). Let \(\phi\) be the natural embedding defined in (2) of \(I_{N}\) into \(\mathbb{N} \cup \{\emptyset\}^\mathbb{N}\). We define the metric \(d\) on \(I_{N}\) by \(d(f,f) = 0\) and if \(f \neq g\), then \(d(f,g) = \frac{1}{m+1}\) where
\[
m = \min\{y \in \mathbb{N} : (y)f \phi \neq (y)g \phi \text{ or } (y)f^{-1} \phi \neq (y)g^{-1} \phi\}.
\]
It is routine to show that \(d\) is a metric on \(I_{N}\). (In fact, \(d\) is the maximum of the metrics \(d_1\) and \(d_2\) defined in (4) and (5).)
We will now show that the topology induced by $d$ is $\mathcal{I}_4$. As in the proof of part (v), the sets
$$R_{k,Y,Z} = \{ f \in I_N : k \subseteq f \text{ and } Y \cap \text{dom}(f) = Z \cap \text{im}(f) = \emptyset \}$$
where $f \in I_N$ is finite and $Y$ and $Z$ are finite subsets of $X$, form a basis $\mathcal{B}$ for $\mathcal{I}_4$.

For any $f \in I_N$ and $m \in \mathbb{N}$, we have that
$$B \left( f, \frac{1}{m + 1} \right) = \{ g \in I_N : (x)f \phi = (x)g \phi \text{ and } (x)f^{-1} \phi = (x)g^{-1} \phi \text{ for all } x \in m \}$$
$$= R_{k,Y,Z}$$
where $k = (f \cap (m \times N)) \cup (f \cap (N \times m)), Y = m \setminus \text{dom}(f)$, and $Z = m \setminus \text{im}(f)$. Hence every open ball under $d$ is open in $\mathcal{I}_4$ and so the topology induced by $d$ is contained in $\mathcal{I}_4$.

Suppose that $f \in I_X$ is finite, $Y, Z$ are finite subsets of $X$, and
$$M = \max (\text{dom}(f) \cup \text{im}(f) \cup Y \cup Z) \in \mathbb{N}.$$ 
If $g \in R_{f,Y,Z}$, then $B(g, 1/M) \subseteq R_{f,Y,Z}$ and so $R_{f,Y,Z}$ is open in the topology induced by $d$. We have shown that $\mathcal{I}_4$ coincides with the topology induced by $d$.

To show that $d$ is complete, suppose that $f_0, f_1, f_2, \ldots$ is a Cauchy sequence. For every $m \in \mathbb{N}$, there exists $M \in \mathbb{N}$ such that $d(f_i, f_j) < 1/m$. So if $x \leq m$, then \((x)f_i \phi = (x)f_j \phi \text{ and } (x)f_i^{-1} \phi = (x)f_j^{-1} \phi \text{ for all } i, j \geq M. \) In particular, the sequences \((x)f_0 \phi, (x)f_1 \phi, (x)f_2 \phi, \ldots) \text{ and } \((x)f_0^{-1} \phi, (x)f_1^{-1} \phi, (x)f_2^{-1} \phi, \ldots) \) are eventually constant with values $(x)F \text{ and } (x)F^{-1}$, respectively. This defines $F \in I_N$ and $f_0, f_1, f_2, \ldots \to F$. So the metric is complete.

We can now give an example to demonstrate that the right small index property is really distinct from the left small index property.

**Proposition 9.5.** The topological semigroup $(I_N, \mathcal{I}_2)$ has the right small index property but not the left small index property.

**Proof.** We start by showing that $(I_N, \mathcal{I}_2)$ does not have the left small index property. Let $x \in \mathbb{N}$ be fixed. We define a left congruence on $I_N$ by
$$\{(f, g) \in I_N : x \notin \text{im}(f) \cup \text{im}(g) \text{ or } (x \in \text{im}(f) \cap \text{im}(g) \text{ and } (x)f^{-1} = (x)g^{-1})\}.$$ 
The set $\{ f \in I_N : x \notin \text{im}(f) \}$ is a class of this left congruence. As this set is not open in $\mathcal{I}_2$ it follows that $(I_N, \mathcal{I}_2)$ does not have the left small index property.

We next show that $(I_N, \mathcal{I}_2)$ has the right small index property. Let $\rho$ be a right congruence on $I_N$ with countably many classes. By Theorem 9.4 (viii) together with Corollary 6.3, we have that $\rho$ is open with respect to $\mathcal{I}_4$. Let $f \in I_N$ be arbitrary. It suffices to show that $f/\rho \in \mathcal{I}_2$, so we need only find some open neighbourhood $U \in \mathcal{I}_2$ of $f$, for which $U \subseteq f/\rho$. Let
$$V = \{ g \in I_N : h \subseteq g, X \cap \text{dom}(g) = \emptyset, Y \cap \text{im}(g) = \emptyset \},$$
where $h \subseteq N \times N, X, Y \subseteq N$ are finite, be a basic open neighbourhood of $f$ in $\mathcal{I}_4$ such that $V \subseteq f/\rho$. We then define
$$U := \{ g \in I_N : h \subseteq g, X \cap \text{dom}(g) = \emptyset \}.$$ 
The set $U$ is an open neighbourhood of $f$ in $\mathcal{I}_2$. It therefore suffices to show that $U \subseteq f/\rho$. Let $g \in U$ be arbitrary. Let $k \in I_N$ be such that $k|_{\text{im}(h)}$ is the identity function, $\text{dom}(k) = N$ and $\text{im}(k) = N \setminus Y$. Since $\text{dom}(k) = N$, it follows that $kk^{-1}$ is the identity of $I_N$. Furthermore, $fk \in V \subseteq f/\rho$ and also $gk \in V \subseteq f/\rho$. So $gk/\rho = f/\rho = fk/\rho$ and hence
$$gk/\rho = fk/\rho \Rightarrow gk = fk \Rightarrow gk^{-1} = fk^{-1} \Rightarrow g/\rho = f/\rho,$$
as required. \qed

By Theorem 9.2(ii), every Polish semigroup topology on $I_N$ contains $\mathcal{I}_2$ or $\mathcal{I}_3$ and, by Theorem 9.4(vii), is contained in $\mathcal{I}_4$. It is therefore natural to ask the following question.

**Question 9.6.** Are $\mathcal{I}_2, \mathcal{I}_3$ and $\mathcal{I}_4$ the only Polish semigroup topologies on $I_N$?
10. The full transformation monoid $X^X$

In this section we strengthen the main results in [41]. Recall that a sub-basis for the pointwise topology on $X^X$ consists of the sets

$$U_{x,y} = \{ f \in X^X : (x)f = y \}$$

for all $x, y \in X$.

**Theorem 10.1.** If $X$ is an infinite set, then the topological semigroup $X^X$ has property $X$ with respect to $\text{Sym}(X)$.

**Proof.** Let $\phi : X \to X \times X$ be a bijection and let $\pi_2 : X \times X \to X$ be defined by $(x, y)\pi_2 = y$. We define functions $f, g \in X^X$ by

$$(x)f = (x, x)\phi^{-1} \quad \text{and} \quad (x)g = (x)\phi\pi_2.$$  

It suffices to show that for all $s \in X^X$ there is $t_s \in \text{Sym}(X)$ such that $s = ft_sg$ and for every open neighbourhood $B$ of $t_s$ there exists an open neighbourhood $U$ of $s$ with $U \subseteq f(B \cap \text{Sym}(X))g$. Let $s \in X^X$ and let $t \in \text{Sym}(X \times X)$ be any permutation such that

$$(x,y)t = (x, (x)s)$$

for all $x \in X$. We then define $t_s \in X^X$ to be $\phi t\phi^{-1}$. From the definitions of $f, g,$ and $t_s$,

$$(x)ft_sg = (x, x)\phi^{-1} \phi t\phi^{-1} \phi\pi_2 = (x, x)t\pi_2 = (x, (x)s)\pi_2 = (x)s$$

for all $x \in X$. Let $B$ be an open neighbourhood of $t_s$. Then there exist $x_1, x_2, \ldots, x_n \in X$ such that $t_s \in \bigcap_{i=1}^n U_{x_i, (x_i)t_s} \subseteq B$. We will show that

$$f(B \cap \text{Sym}(X))g \supseteq \{ k \in X^X : (x)k = (x)s \text{ for all } (x, x) \in \{ x_1, x_2, \ldots, x_n \}\phi \} =: V,$$

which is sufficient as this set is a neighbourhood of $s$. Let $k \in V$. Due to the restriction on $k$ we may find $p \in \text{Sym}(X \times X)$ such that

$$(x, x)p = (x, (x)k),$$

for all $x \in X$ and

$$(x_i)p = (x_i)t$$

for all $i$. We then define $t_k \in X^X$ to be $\phi p\phi^{-1}$. By the choice of $p$ it follows that $(x_i)t_k = (x_i)\phi p\phi^{-1} = (x_i)t\phi^{-1} = (x_i)t_s$ for every $i$, and so $t_k \in U_{x_i, (x_i)t_s}$ for every $i$. Thus $t_k \in B \cap \text{Sym}(X)$ and as before

$$(x)ft_kg = (x, x)\phi^{-1} \phi p\phi^{-1} \phi\pi_2 = (x, x)p\pi_2 = (x, (x)k)\pi_2 = (x)k.$$  

Hence $k = ft_kg \in f(B \cap \text{Sym}(X))g$ and so, since $k$ was arbitrary, $V \subseteq f(B \cap \text{Sym}(X))g$, as required.  \[\square\]

**Corollary 10.2.** If $X$ is an infinite set, then the pointwise topology is the only $T_1$ semigroup topology on $X^X$ which induces the pointwise topology on $\text{Sym}(X)$.

**Proof.** By Theorem 5.1 every $T_1$ topology on $X^X$ contains the pointwise topology. By Theorem 10.1 together with Lemma 3.6(ii) this is also the largest topology which induces the pointwise topology on $\text{Sym}(X)$.

\[\square\]

**Theorem 10.3.** The full transformation monoid $\mathbb{N}^\mathbb{N}$ with the pointwise topology has automatic continuity.

**Proof.** This follows from Theorem 10.1, Lemma 3.6(iv), and the automatic continuity of $\text{Sym}(\mathbb{N})$.  \[\square\]

**Corollary 10.4.** The full transformation monoid $\mathbb{N}^\mathbb{N}$ has a unique $T_1$ second countable semigroup topology.

**Proof.** By Theorem 10.3 every second countable semigroup topology for $\mathbb{N}^\mathbb{N}$ is contained in the pointwise topology. By Theorem 5.1 every $T_1$ topology on $\mathbb{N}^\mathbb{N}$ contains the pointwise topology.  \[\square\]
11. Injective and surjective transformations

We show that there is not a unique Polish semigroup topology on
\[ \text{Inj}(X) = \{ f \in X^X : f \text{ is injective} \}. \]

**Theorem 11.1.** Let \( X \) be an infinite set, let \( J_1 \) be the topology on \( \text{Inj}(X) \) generated by the pointwise topology and the sets
\[ \{ f \in \text{Inj}(X) : y \not\in \text{im}(f) \} \]
for every \( x \in X \), and let \( J_2 \) be the topology on \( \text{Inj}(X) \) generated by \( J_1 \) and the collection of sets
\[ \{ f \in \text{Inj}(X) : |X \setminus \text{im}(f)| = n \} \]
for every cardinal \( n \leq |X| \). Then the following hold:
\begin{enumerate}[(i)]
\item the pointwise topology, \( J_1 \), and \( J_2 \) are distinct semigroup topologies for \( \text{Inj}(X) \). Moreover, if \( X \) is countable, then these topologies are Polish;
\item if \( X \) is countable, then there are infinitely many distinct Polish semigroup topologies on \( \text{Inj}(X) \) containing the pointwise topology and contained in \( J_2 \);
\item \( \text{Inj}(X) \) has property \( X \) with respect to \( J_2 \) and \( \text{Sym}(X) \);
\item if \( S \) is any semigroup topology on \( \text{Inj}(X) \) and \( S \) induces the pointwise topology on \( \text{Sym}(X) \), then \( S \) is contained in \( J_2 \);
\item if \( X \) is countable, then \( \text{Inj}(X) \) has automatic continuity with respect to \( J_2 \);
\item the Zariski topology for \( \text{Inj}(X) \) is the pointwise topology.
\end{enumerate}

**Proof.** (i). The pointwise topology is a semigroup topology for \( X^X \), which is Polish for countable \( X \), and \( \text{Inj}(X) \) is closed in \( X^X \) under the pointwise topology, hence the pointwise topology is a semigroup topology for \( \text{Inj}(X) \), which is Polish when \( X \) is countable.

Similarly, \( J_1 \) is the topology that \( \text{Inj}(X) \) inherits as a subspace of the symmetric inverse monoid \( I_{NX} \) under the Polish semigroup topology \( J_4 \) defined in Theorem 9.4. As \( \text{Inj}(X) \) is closed in this topology, \( J_1 \) is a Polish semigroup topology for \( \text{Inj}(X) \) also.

To show that \( J_2 \) is a semigroup topology, note that
\[ |X \setminus \text{im}(fg)| = |X \setminus \text{im}(f)| + |X \setminus \text{im}(g)| \]
for all \( f, g \in \text{Inj}(X) \). So, if \( f, g \in \text{Inj}(X) \) are such that \( fg \in \{ h \in \text{Inj}(X) : |X \setminus \text{im}(h)| = n \} = U \), then \( |X \setminus \text{im}(f)| = a \) and \( |X \setminus \text{im}(g)| = b \) for some cardinals \( a, b \leq |X| \) such that \( a + b = n \). Thus \( f \in V = \{ h \in \text{Inj}(X) : |X \setminus \text{im}(h)| = a \} \) and \( g \in W = \{ h \in \text{Inj}(X) : |X \setminus \text{im}(h)| = b \} \) and \( VW \subseteq U \). Hence \( J_2 \) is a semigroup topology.

We now show that \( J_2 \) is Polish when \( X \) is countable. We may assume without loss of generality that \( X = \mathbb{N} \). We construct a sequence of Polish topologies \( \mathcal{S}_0 = J_1, S_1, \ldots \) such that
\[ F_n = \{ f \in \text{Inj}(\mathbb{N}) : |\mathbb{N} \setminus \text{im}(f)| = n \} \]
is closed in \( \mathcal{S}_n \) and open in \( \mathcal{S}_{n+1} \) for all \( n \in \mathbb{N} \). If \( n = 0 \), and \( f \not\in F_n \), then there exists \( y \in \mathbb{N} \setminus \text{im}(f) \), and so
\[ \{ f \in \text{Inj}(\mathbb{N}) : y \not\in \text{im}(f) \} \]
is an open neighbourhood of \( f \) contained in the complement of \( F_n \).

Suppose that we have defined \( S_1, \ldots, S_m \) with the properties above. Then we define \( S_{m+1} \) to be the topology generated by \( S_m \) and \( F_m \). It follows by [33, Lemma 13.2] that \( S_{m+1} \) is Polish, and
\[ \text{Inj}(\mathbb{N}) \setminus F_{m+1} = \{ f \in \text{Inj}(\mathbb{N}) : |\mathbb{N} \setminus \text{im}(f)| \neq m + 1 \} = \{ f \in \text{Inj}(\mathbb{N}) : |\mathbb{N} \setminus \text{im}(f)| > m + 1 \} \cup \{ f \in \text{Inj}(\mathbb{N}) : |\mathbb{N} \setminus \text{im}(f)| < m + 1 \}. \]
The set \( \{ f \in \text{Inj}(\mathbb{N}) : |\mathbb{N} \setminus \text{im}(f)| > m + 1 \} \) is open in \( J_1 \) and hence is open in \( S_{m+1} \). The set
\[ \{ f \in \text{Inj}(\mathbb{N}) : |\mathbb{N} \setminus \text{im}(f)| < m + 1 \} = \bigcup_{k=0}^{m} F_k \]
is also open by assumption. Hence the topology generated by \( \bigcup_{n=0}^{\infty} \mathcal{S}_n \) is Polish by [33, Lemma 13.3]. Finally, by a similar argument, \( F_{\mathcal{K}_n} \) is closed in the topology generated by \( \bigcup_{n=0}^{\infty} \mathcal{S}_n \), so by applying [33, Lemma 13.2] again, \( \mathcal{J}_2 \) is Polish.

Although it is not strictly necessary, we note that a complete metric on \( \text{Inj}(\mathbb{N}) \) that induces \( \mathcal{J}_2 \) is:

\[
d(f, g) = \begin{cases} 
0 & \text{if } f = g; \\
1 & \text{if } |\mathbb{N} \setminus \text{im}(f)| \neq |\mathbb{N} \setminus \text{im}(g)| \\
\frac{1}{m+1} & \text{otherwise}; 
\end{cases}
\]

where \( m = \min\{n \in \mathbb{N} : (n \times n) \cap f \neq (n \times n) \cap g\} \).

(ii). It suffices to show that the Polish semigroup topologies \( \mathcal{S}_0, \mathcal{S}_1, \ldots \) constructed in the proof of part (ii) are distinct. Suppose that \( n \in \mathbb{N} \) is arbitrary. We must show that \( F_n \) is not open in \( \mathcal{S}_n \). If \( f \in F_n \) is arbitrary and \( B \) is any basic open neighbourhood of \( f \), then

\[
B = \{g \in \text{Inj}(\mathbb{N}) : h \subseteq g \text{ and } \text{im}(g) \cap Y = \emptyset\}
\]

for finite \( h \in I_\mathbb{N} \) and finite \( Y \subseteq \mathbb{N} \). It is routine to verify that there exists \( g \in B \) such that \( g \notin F_n \), and hence \( F_n \) is not open in \( \mathcal{S}_n \).

(iii). Let \( s \in \text{Inj}(X) \) be arbitrary. We show that there exist \( f_s, g_s \in \text{Inj}(X) \) and \( a_s \in \text{Sym}(X) \) such that \( s = f_s a_s g_s \) and for every basic open neighbourhood \( B \) of \( a_s \) there exists an open neighbourhood \( U \) of \( s \) such that \( U \subseteq f_s(B \cap \text{Sym}(X))g_s \).

Let \( f_s = s, a_s = 1_{\text{Inj}(X)}, \) and \( g_s = 1_{\text{Inj}(X)} \) and let \( B \) be a basic open neighbourhood of \( a_s \). As \( a_s = 1_{\text{Inj}(X)} \) there is some finite (possibly empty) \( Y \subseteq X \) such that \( B \cap \text{Sym}(X) = \{h \in \text{Sym}(X) : (y, y) \in h \text{ for all } y \in Y\} \).

It is easy to see that \( h \in \text{Inj}(X) \) is an element of \( f_s(B \cap \text{Sym}(X))g_s \) if and only if

1. \( y \notin \text{im}(h) \) for all \( y \in Y \setminus \text{im}(f_s) \);
2. \( (y)s^{-1}, y) \in h \) for all \( y \in Y \cap \text{im}(f_s) \);
3. \( |X \setminus \text{im}(s)| = |X \setminus \text{im}(h)| \).

So \( f_s(B \cap \text{Sym}(X))g_s \) is an open neighbourhood of \( s \) as required.

(iv). This follows by part (iii) together with Lemma 3.6(ii).

(v). This follows by part (iii) together with Lemma 3.6(iv) and the automatic continuity of \( \text{Sym}(X) \).

(vi). If \( X \) is given the structure of a complete graph without loops, then \( \text{End}(X) = \text{Inj}(X) \), and this part of the theorem follows from Proposition 16.2.

We will show that there is not a unique Polish topology on the set \( \text{Surj}(X) \) of all surjective transformations of \( X \).

**Theorem 11.2.** Let \( X \) be an infinite set, let \( \mathcal{S}_1 \) denote the topology generated by the pointwise topology and the set \( \text{Sym}(X) \), and let \( \mathcal{S}_2 \) be the topology generated by pointwise topology together with the collection of sets

\[
U_{m,n} := \{f \in \text{Surj}(X) : |(n) f^{-1}| = m\}
\]

for all \( n \in X \) and cardinals \( m \leq |X| \). Then the following hold:

1. there is no continuous semigroup isomorphism from \( \text{Surj}(X) \) with the pointwise topology to \( \text{Surj}(X) \) with the topology \( \mathcal{S}_1 \);
2. the pointwise topology and \( \mathcal{S}_1 \) are distinct semigroup topologies on \( \text{Surj}(X) \), if \( X \) is countable then \( \mathcal{S}_2 \) is also a semigroup topology and all of these topologies are Polish;
3. if \( X \) is countable, then there are infinitely many distinct Polish semigroup topologies on \( \text{Surj}(X) \) containing the pointwise topology and contained in \( \mathcal{S}_2 \);
4. if \( X \) is countable, then \( \text{Surj}(X) \) with the topology \( \mathcal{S}_1 \) embeds into \( \mathbb{N}^\mathbb{N} \) with the pointwise topology.
Proof. (i). As $\text{Sym}(X)$ is the group of units of $\text{Surj}(X)$ it is preserved under any semigroup isomorphism. The group $\text{Sym}(X)$ is open in $S_1$ by definition but it is not open in the pointwise topology, thus any semigroup isomorphism will be discontinuous.

(ii). The pointwise topology is a semigroup topology for $X^X$ is it is also a semigroup topology for $\text{Surj}(X)$. If $X$ is countable then $\text{Surj}(X)$ is $G_\delta$ in $X^X$, it follows that the pointwise topology is a Polish semigroup topology on $\text{Surj}(X)$.

Since $\text{Inj}(X)$ is closed in $X^X$, it follows that $\text{Sym}(X) = \text{Inj}(X) \cap \text{Surj}(X)$ is closed in $\text{Surj}(X)$ with respect to the pointwise topology. It follows by [33, Lemma 13.2], that $S_1$ will be Polish for countable $X$, and it remains for us to show that $S_1$ is compatible with the multiplication in $\text{Surj}(X)$. It suffices to show that $\{ (f,g) : fg \in \text{Sym}(X) \}$ is open in $\text{Surj}(X) \times \text{Surj}(X)$. It is routine to verify that $\text{Surj}(X) \setminus \text{Sym}(X)$ is an ideal in $\text{Surj}(X)$. Hence if $f, g \in \text{Surj}(X)$ are such that $fg \in \text{Sym}(X)$, then $f, g \in \text{Sym}(X)$. Hence $\text{Sym}(X) \times \text{Sym}(X) = \{ (f,g) : fg \in \text{Sym}(X) \}$ is open, as required.

It remains to show that $S_2$ is a Polish semigroup topology when $|X| = \aleph_0$. We may assume without loss of generality that $X = \mathbb{N}$. First note that for all $f, g \in \text{Surj}(\mathbb{N})$ and $n \in \mathbb{N}$ we have that

$$|(n)\langle fg \rangle^{-1}| = \sum_{i \in \langle n \rangle g^{-1}} |(i)\langle f \rangle^{-1}|.$$  

If $fg \in U_{m,n}$, then there are three cases to consider:

1. If $m \in \mathbb{N}$, then  

$$fg \in \left( \bigcap_{i \in \langle n \rangle g^{-1}} \bigcup_{\langle i \rangle f^{-1} \subseteq 1} \bigcup_{\langle \langle n \rangle g^{-1} \rangle \cap \langle \langle n \rangle g^{-1} \rangle \cap \{h \in \text{Surj}(\mathbb{N}) : (i)h = (i)g\} \right) \subseteq U_{m,n}. $$

2. If $m = \aleph_0$ and $|(n)\langle g \rangle^{-1}| = m$, then  

$$fg \in \text{Surj}(X)U_{m,n} \subseteq U_{m,n}. $$

3. If $m = \aleph_0$ and $|(n)\langle g \rangle^{-1}| \neq \aleph_0$, then there is some $i \in \langle n \rangle g^{-1}$ such that $|(i)\langle f \rangle^{-1}| = \aleph_0$ and so  

$$fg \in U_{\aleph_0,i}\{h \in \text{Surj}(\mathbb{N}) : (i)h = n\} \subseteq U_{\aleph_0,n}. $$

It remains to show that $S_2$ is Polish. For all $m, n \in \mathbb{N}$ we define $T_{m,n}$ to be the topology generated by the pointwise topology together with the sets $U_{i,n}$ for $i \leq m$. We show by induction on $m$, that $T_{m,n}$ is Polish for all $m, n \in \mathbb{N}$.

By the definition of $\text{Surj}(\mathbb{N})$, $U_{0,n} = \emptyset$ for all $n \in \mathbb{N}$. It follows that, $T_{0,n}$ is the pointwise topology, which is Polish, for all $n \in \mathbb{N}$. This establishes the base case of the induction.

For the inductive step, suppose that $m > 0$ and $T_{m-1,n}$ is Polish for all $n \in \mathbb{N}$. The topology $T_{m,n}$ is generated by $T_{m-1,n}$ and $U_{m,n}$. To apply [33, Lemma 13.2] and conclude that $T_{m,n}$ is Polish, it suffices to show that $U_{m,n}$ is closed in $T_{m-1,n}$. By the definition of $U_{m,n}$,

$$\text{Surj}(\mathbb{N}) \setminus U_{m,n} = \{ f \in \text{Surj}(\mathbb{N}) : |(n)\langle f \rangle^{-1}| < m \} \cup \{ f \in \text{Surj}(\mathbb{N}) : |(n)\langle f \rangle^{-1}| > m \}. $$

The set $V = \{ f \in \text{Surj}(\mathbb{N}) : |(n)\langle f \rangle^{-1}| < m \}$ is the union of the open sets $\{ f \in \text{Surj}(\mathbb{N}) : |(n)\langle f \rangle^{-1}| = i \} = U_{i,n}$, for all $i < m$, and so $V$ is open in $T_{m-1,n}$. The set $W = \{ f \in \text{Surj}(\mathbb{N}) : |(n)\langle f \rangle^{-1}| > m \}$ is the union of the open sets

$$\{ f \in \text{Surj}(\mathbb{N}) : \langle y \rangle f = n \text{ for all } y \in Y \}$$

in the pointwise topology, for all finite subsets $Y$ of $\mathbb{N}$ with at least $m + 1$ elements, and so $W$ too is open. Therefore $\text{Surj}(\mathbb{N}) \setminus U_{m,n}$ is open and so $U_{m,n}$ is closed.

We have shown that $T_{m,n}$ is Polish for every $m, n \in \mathbb{N}$. For every $m \in \mathbb{N}$, we define $T_m$ to be the least topology containing $T_{m,n}$ for all $n \in \mathbb{N}$. By [33, Lemma 13.3], $T_m$ is Polish for all $m \in \mathbb{N}$. By considering only case (1) in the proof that $S_2$ is a semigroup topology, it follows that $T_m$ is also a semigroup topology for all $m \in \mathbb{N}$.
If $T$ is the least topology containing $T_m$ for all $m \in \mathbb{N}$, then, by [33, Lemma 13.3] again, $T$ is Polish. If $n \in \mathbb{N}$ is arbitrary, then
\[
\text{Surj}(\mathbb{N}) \setminus U_{\aleph_0,n} = \{ f \in \text{Surj}(\mathbb{N}) : |(n)f^{-1}| < \aleph_0 \}
\]
which is the union of all of the open sets $U_{i,n}, i \in \mathbb{N}$, as in the inductive step above, is open in $T$. Hence the topology generated by $T$ and any of the sets $U_{\aleph_0,n}, n \in \mathbb{N}$, is a Polish topology. The topology generated by the (countable) union of all such topologies is $S_2$, and so, by applying [33, Lemma 13.3] one last time, $S_2$ is Polish.

(iii). By the proof of (ii), it suffices to show that $T_i$ and $T_j$ are distinct for all $i \neq j$. Furthermore, since $T_i \subseteq T_j$ whenever $i \leq j$, it suffices to show that $T_i \neq T_{i+1}$ for every $i \in \mathbb{N}$. We show that $U_{i+1,0}$ is not open in $T_i$. Suppose that $f \in U_{i+1,0}$ is arbitrary. Every basic open neighbourhood of $f$ in $T_i$ is of the form
\[
\bigcap_{(m,n) \in Z} U_{m,n} \cap \{ g \in \text{Surj}(\mathbb{N}) : h \subseteq g \} = \{ g \in \text{Surj}(\mathbb{N}) : h \subseteq g \text{ and } m = |(n)g^{-1}| \text{ for all } (m,n) \in Z \},
\]
where $h \in \mathcal{PN}$ and $Z \subseteq \{1,2,\ldots,i\} \times \mathbb{N}$ are finite. In particular, every basic open neighbourhood of $f$ contains many $g \in \text{Surj}(\mathbb{N})$ such that $|(0)g^{-1}| > i + 1$ and hence no basic open neighbourhood of $f$ is contained in $U_{i+1,0}$.

(iv). We note that the topology $S_1$ on $\text{Surj}(\mathbb{N})$ satisfies Theorem 5.2(i), and so it is possible to topologically embed $\text{Surj}(\mathbb{N})$ with the topology $S_1$ into $\mathbb{N}^\mathbb{N}$ with the pointwise topology. Suppose that $\sigma$ is the equivalence relation of $\text{Surj}(\mathbb{N})$ with classes $\text{Sym}(\mathbb{N})$ and $\text{Surj}(\mathbb{N}) \setminus \text{Sym}(\mathbb{N})$. Then $\sigma$ is a right congruence on $\text{Surj}(\mathbb{N})$ (even a two-sided congruence, but this is not important here). If $\{\rho_i : i \in \mathbb{N}\}$ is a sequence of right congruences on $\text{Surj}(\mathbb{N})$ such that $\{m/\rho_i : m \in \text{Surj}(\mathbb{N})\}$ is a sub-basis for the pointwise topology, then $\{m/\rho_i : m \in \text{Surj}(\mathbb{N})\} \cup \{\sigma\}$ satisfies part (i) of Theorem 5.2. \hfill $\square$

Part 3. Monoids of continuous functions

12. Preliminaries

If $X$ and $Y$ are topological spaces, then the compact-open topology on $C(X,Y)$ (the space of all continuous functions from $X$ to $Y$) is the topology generated by the sub-basis consisting of the sets:
\[
[K,U] = \{ f \in C(X,Y) : (K)f \subseteq U \},
\]
where $K \subseteq X$ is compact, and $U \subseteq Y$ is open.

If $X$ is compact and metrizable with compatible metric $d$, then $C(X,X) = C(X)$ is separable with respect to the compact-open topology, and the topology is induced by
\[
d_\infty(f,g) = \sup\{d((x)f,(x)g) : x \in X\};
\]
see [65, Proposition 1.3.3]. The space $X$ is complete since it is compact, and $d_\infty$ is therefore complete also, meaning that $C(X)$ with the compact-open topology is Polish.

**Theorem 12.1.** Let $X$ be a compact metrizable space. If $C(X)$ is a first countable Hausdorff topological semigroup with respect to some topology $T$ and $a : X \times C(X) \rightarrow X$ defined by
\[
(p,f)a = (p)f
\]
is continuous, then $T$ contains the compact-open topology.

**Proof.** It suffices to show that if $K$ is compact and $U$ is open in $X$, then the complement $F = \{ f \in C(X) : \exists k \in K, f(k) \notin U \}$ of $[K,U]$ is closed in $T$.

Suppose that $f_1,f_2,\ldots \in F$ converges to $f \in C(X)$ with respect to $T$. For every $i \in \mathbb{N}$, there exists $x_i \in K$ such that $(x_i)f_i \notin U$. Since $K$ is compact, it follows that $(x_i)_{i \in \mathbb{N}}$ contains a convergent subsequence $(x_{(i)j})_{i \in \mathbb{N}}$. If $x$ is the limit of this subsequence, then
\[
(x)f = (x,f)a = \left( \lim_{i \rightarrow \infty} x_{(i)y}, \lim_{i \rightarrow \infty} f_{(i)y} \right) a = \lim_{i \rightarrow \infty} (x_{(i)y},f_{(i)y})a = \lim_{i \rightarrow \infty} (x_{(i)y},f_{(i)y}) \in X \setminus U
\]
and so $f \in F$, and $F$ is closed in $T$, as required. \hfill $\square$
Lemma 12.2. Let $X$ be topological space and suppose that $C(X)$ is a first countable Hausdorff topological semigroup with respect to a topology $\mathcal{T}$. Then the set $F$ of constant functions in $C(X)$ is closed in $\mathcal{T}$.

Proof. It is routine to verify that $F$ is the set of right zeros in $C(X)$. If $f \in C(X)$ and $(k_i)_{i \in \mathbb{N}}$ is a sequence in $F$ that converges to $k \in C(X)$, then $fk = \lim_{i \to \infty} f k_i = \lim_{i \to \infty} k_i = k$ and so $k \in F$.

Throughout this part of the paper, we will consider various topologies on $C(X)$ starting with a fixed topology on $X$.

13. The Hilbert cube

In this section, we show that the monoid $C(Q)$ of continuous functions on the Hilbert cube $Q = [0,1]^N$ has a unique Polish semigroup topology. The full transformation monoids $X^X$ are universal in the sense that every semigroup can be embedded as a subsemigroup of some $X^X$. On the other hand, if $S$ is a Polish semigroup and $S$ can be embedded into $X^X$ for some set $X$, then it must satisfy the conditions in Theorem 5.2, and not every Polish semigroup satisfies these conditions. The same can be said for the symmetric groups: they are universal for groups, but not for Polish groups. However, there is a Polish group into which every other Polish group embeds as a topological group. To define this group, we require the notion of the compact-open topology, which we define just now.

Since $C(X)$ is Polish with the compact-open topology, if $X$ is compact and metrizable, it follows that $C(Q)$ is Polish with the compact-open topology. We will use the following metric on $Q$:

$$d((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i},$$

which is bounded above by 1, and so the corresponding $d_\infty$ is also bounded above by 1.

Since the compact-open topology is Polish on the group of homeomorphisms of any compact metric space (see, for example, [65, Proposition 1.3.3]), it follows that $H(Q)$ is a Polish subgroup of $C(Q)$ with the compact-open topology. In fact, it was shown by Kallman [31] that the compact-open topology is the unique Polish topology on $H(Q)$.

Uspenski˘ı’s Theorem [33, Theorem 9.18] and [64] states that every Polish group is isomorphic to a (necessarily closed) subgroup of the group $H(Q)$ of homeomorphisms of $Q$. A similar result holds for separable metrizable compactifiable semigroups, every such semigroup is topologically isomorphic to a subsemigroup of $C(Q)$; see [40, Theorem 5.2].

Question 13.1. Is there a Polish semigroup $S$ such that every Polish semigroup $T$ can be topologically embedded in $S$? In particular, does $C(Q)$ have this property.

We will show that the compact-open topology on $C(Q)$ is the unique Polish semigroup topology on $C(Q)$. We will invoke Theorem 12.1 to show that every Polish semigroup topology on $C(Q)$ contains the compact-open topology.

Proposition 13.2. Suppose that $C(Q)$ is a Polish semigroup with respect to some topology. Then the function $a : Q \times C(Q) \to Q$ defined by

$$(p, f)a = (p)f$$

is continuous.

Proof. Let $\mathcal{T}$ denote the given Polish semigroup topology on $C(Q)$, and let $d$ be any metric that induces the usual topology on $Q$. Suppose that $F$ denotes the set of constant functions in $C(Q)$ and that $\phi : F \to Q$ is the function that sends $f \in F$ to the unique value in $(Q)f$.

We will show that $\phi$ is a homeomorphism, which will allow us to conclude the proof as follows. If $\gamma : Q \times C(Q) \to C(Q) \times C(Q)$ is defined by $(p, f)\gamma = (p\phi^{-1}, f)$, then $\gamma$ is continuous since it is continuous in each coordinate, and if $M : C(Q) \times C(Q) \to C(Q)$ is the multiplication function on $C(Q)$, then $M$ is continuous also. Thus $(p, f)\gamma M\phi = (p\phi^{-1}, f)M\phi = (p\phi^{-1} \circ f)\phi$ and $(p)\phi^{-1} \in F$ is constant with value $p$, and so $p\phi^{-1} \circ f$ is constant with value $(p)f$, and, finally,
and Proposition \ref{prop:subspace_topology}. If \( U \) is an open set in \( Q \), it suffices to show that 
\[
(U)\phi^{-1} = \{ f \in (Q) : (Q)f = \{ q \}, \; q \in U \} \text{ is open in } F. 
\]
If \( f : Q \to Q \) is defined by 
\[
(q)f = (d(q, Q \setminus U), d(q, Q \setminus U), \ldots),
\]
then since \( f \) is continuous in every coordinate, \( f \) is continuous. Since \( C(Q) \) is \( T_1 \), \( F \) is \( T_1 \), and so the singleton set containing the constant function \( g : Q \to Q \) with value \((0, 0, \ldots)\) is closed. Hence \((g)\rho_f^{-1} \cap F \) is closed in \( F \), and so \( F \setminus ((g)\rho_f^{-1}) \) is open. If \( h \in F \setminus ((g)\rho_f^{-1}) \), then there exists \( q \in Q \) such that 
\[
(q)h \neq (0, 0, \ldots)
\]
and so \( (q)h \in U \) by the definition of \( f \). On the other hand, if \( h \in F \) is such that \((q)h \in U \), then 
\[
(q)h \neq (0, 0, \ldots, 0, 0),
\]
and so \((q)h \in U \) is open, as required.

Next, we show that \( \phi^{-1}|_{(0, 1)^N} : (0, 1)^N \to F \) is continuous also. Since \( F \) is closed in \( C(Q) \) with respect to \( T \), by Lemma \ref{lem:polish_closure}, it follows that \( F \) is Polish. Since \( \phi \) is continuous, \( \rho \) is Borel measurable, and so by Proposition \ref{prop:continuous_measurable}, \( \phi^{-1} \) is Borel measurable also. It suffices, by Proposition \ref{prop:polish_homeomorphism}, to endow \((0, 1)^N \) and \(((0, 1)^N)^{-1} \subseteq F \) with Polish semitopological group structures such that 
\[
\phi^{-1}|_{(0, 1)^N} : (0, 1)^N \to ((0, 1)^N)^{-1} \text{ is a homomorphism.}
\]
With the operation of component-wise addition, \( \mathbb{R}^N \) is an abelian topological group, and \( \mathbb{R}^N \) is homeomorphic to \((0, 1)^N \). Therefore we may endow \((0, 1)^N \) with the additive group structure of \( \mathbb{R}^N \) corresponding to an order-isomorphism between \( \mathbb{R} \) and \((0, 1) \). We define \( * \) on \((0, 1)^N \) by \( x * y = ((x)\phi + (y)\phi)\phi^{-1} \). It remains to show that \((0, 1)^N \) is continuous with \( * \) is right topological, left topological will then follow immediately since \( \mathbb{R}^N \) is abelian. Suppose that \( g = (g_1, g_2, \ldots) \in (0, 1)^N \) is arbitrary. Then \( \rho_g : (0, 1)^N \to (0, 1)^N \) is continuous since \( (0, 1)^N \) is a topological group. We extend \( \rho_g \) to \( \rho'_g : Q \to Q \) so that the \( i \)th coordinate of \((x_1, x_2, \ldots)\rho'_g \) is \( x_i + g_i \) if \( x_i \in (0, 1) \) and \( x_i = 0 \) or \( x_i = 1 \). Since \( \rho'_g \) is an order-isomorphism onto \([0, 1] \) in every component, it follows that \( \rho'_g \) is continuous in every component, and is hence continuous, i.e. \( \rho'_g \in C(Q) \). Thus \( x \mapsto x * \rho'_g \), where \( x \in C(Q) \) is continuous, and so, in particular, \( x \mapsto x * \rho'_g \) restricted to \( x \in ((0, 1)^N)^{-1} \) is also continuous on \((0, 1)^N \) with the subspace topology. If \( x \in ((0, 1)^N)^{-1} \) is arbitrary, then \( x \) is a constant function with value \((x_1, x_2, \ldots) \in (0, 1)^N \) and so 
\[
x * (g)\phi^{-1} = ((x)\phi + g)\phi^{-1} = (x)\rho_g \phi^{-1}
\]
by the definition of \( * \) and \( \rho_g \). Recall that \((x)\phi = (x_1, x_2, \ldots) \in (0, 1)^N \) by the definition of \( \phi \). But 
\[
(x)\rho_g \phi^{-1} = (x_1, x_2, \ldots)\rho_g \phi^{-1} = (x_1 + g_1, x_2 + g_2, \ldots)\phi^{-1}
\]
and \((x_1 + g_1, x_2 + g_2, \ldots)\phi^{-1} \) is the constant function from \( Q \) to \( Q \) with value \((x_1 + g_1, x_2 + g_2, \ldots) \), which is equal to \((x) \circ \rho_g \). Therefore 
\[
x * (g)\phi^{-1} = ((x)\phi + g)\phi^{-1} = (x)\rho_g \phi^{-1} = x * \rho_g = x * \rho'_g
\]
and since \( x \mapsto x * \rho'_g \) is continuous on \((0, 1)^N \) it follows that \((0, 1)^N \) is continuous with \( * \) is a right topological group.

To conclude the proof, we must show that \( \phi^{-1} : Q \to F \) is continuous. We define \( \theta : [0, 1] \to [1/4, 3/4] \) by \( (x)\theta = x/2 + 1/4 \) and \( \overline{\theta} \in C(Q) \) be the function that applies \( \theta \) in every coordinate of \( Q \). If \( \theta : [0, 1] \to [0, 1] \) is any continuous extension of \( \theta^{-1} \), and \( \overline{\theta'} \in C(Q) \) applies \( \theta' \) in every coordinate, then \( \theta \theta' = \text{id}_Q \) and \( \phi^{-1} = \overline{\theta} \phi^{-1}|_{(0, 1)^N} \rho_{\overline{\theta'}} \). Hence \( \phi^{-1} \) being the composition of the continuous functions \( \overline{\theta}, \phi^{-1}|_{(0, 1)^N} \), and \( \rho_{\overline{\theta'}} \) is itself continuous.

\begin{corollary}
Every Polish semigroup topology on the monoid of continuous functions on the Hilbert cube \( Q = [0, 1]^N \) contains the compact-open topology.
\end{corollary}

\begin{proof}
This follows immediately from Theorem \ref{thm:continuous_functions_topo} and Proposition \ref{prop:compact_open_topo}.
\end{proof}

To show that \( C(Q) \) has a unique Polish semigroup topology we require Theorem 5.4.2 from \cite{hopf}, which we state for the sake of completeness.
Proposition 13.4 (cf. Theorem 5.4.2 in [65]). Let $K_1$ and $K_2$ be compact subsets of $(0,1)^\mathbb{N}$ and let $k : K_1 \to K_2$ be a homeomorphism with $d_\infty(k, \text{id}_{K_1}) < \varepsilon$. Then there exists a homeomorphism $h_k \in H(Q)$ such that $d_\infty(h_k, \text{id}_Q) < \varepsilon$ and $h_k|_{K_1} = k$.

Theorem 13.5. The monoid $C(Q)$ of continuous functions on the Hilbert cube together with the compact-open topology has property $X$ with respect to its group of units $H(Q)$.

Proof. If $S$ is $C(Q)$ with the compact-open topology, $\mathcal{B}$ is the basis for the compact-open topology on $C(Q)$ consisting of open balls with respect to the $d_\infty$ metric on $C(Q)$, and $T$ is $H(Q)$, then, we show that there exist $f, g \in C(Q)$ and $t_s \in H(Q)$ such that $s = ft_sg$ and for every ball $B = B_{d_\infty}(t_s, \varepsilon) \in \mathcal{B}$ there exists $\delta > 0$ such that $B_{d_\infty}(s, \delta) \subseteq f(B \cap H(Q))g$.

Suppose that $s \in C(Q)$ is arbitrary. We define the required $f, g \in C(Q)$ via two continuous functions $f', g' : [0,1] \to [0,1]$ that are defined by:

$$(x)\ f' = \frac{2x + 1}{4}, \quad (x)\ g' = \begin{cases} 0 & \text{if } x \leq \frac{1}{4} \\ \frac{4x - 1}{2} & \text{if } \frac{1}{4} \leq x \leq \frac{3}{4} \\ 1 & \text{if } x \geq \frac{3}{4}. \end{cases}$$

Clearly $f'$ is continuous, and so too is $g'$ since it is piecewise continuous on finitely many closed intervals. Also $f'g'$ is the identity function $\text{id}_{[0,1]}$. We define $f, g \in C(Q)$ by applying $f'$ and $g'$ in certain coordinates, as follows:

$$(x_1, x_2, \ldots)f = ((x_1)f', (x_2)f', \ldots)$$

$$(x_1, x_2, \ldots)g = ((x_2)g', (x_4)g', \ldots)$$

$$(x_1, x_2, \ldots)h = ((x_1)g', (x_2)g', \ldots).$$

Since $f'g' = \text{id}_{[0,1]}$, it follows that $fh = \text{id}_Q$. Let $A = [1/4, 3/4]^\mathbb{N}$. Of course, $A$ is homeomorphic to $Q$, and $A$ is a compact subset of $(0,1)^\mathbb{N}$. To find $t_s \in H(Q)$ such that $ft_sg = s$, we show that there exists a homeomorphism $t'_s$ from $A$ into a subspace of $A$, that can be extended to a homeomorphism of $Q$ by Proposition 13.4. We denote by $\pi_i : Q \to [0,1]$ the $i$th projection of $Q$, that is, $(x_1, x_2, \ldots)\pi_i = x_i$

for any $(x_1, x_2, \ldots) \in Q$. If $\vec{x} = (x_1, x_2, \ldots) \in A$ and $k \in C(Q)$ is arbitrary, then we define $t'_k : A \to A$ by

$$(\vec{x})t'_k = (x_1, x_2, \ldots)t'_k = (x_1, (\vec{x})hk\pi_1 f', x_2, (\vec{x})hk\pi_2 f', \ldots).$$

From the definition of $t'_k$, if $\vec{x} \in Q$ is arbitrary, then

$$(\vec{x})t'_k g = (x_1, x_2, \ldots)ft'_k g$$

$$= ((x_1)f', (x_2)f', \ldots)t'_k g$$

$$= ((x_1)f', \vec{x} h k \pi_1 f', (x_2)f', \vec{x} h k \pi_2 f', \ldots)g$$

$$= ((x_1)f', \vec{x} k \pi_1 f', (x_2)f', \vec{x} k \pi_2 f', \ldots)g$$

$$= (\vec{x} k \pi_1 f', \vec{x} k \pi_2 f', \ldots)$$

$$= \vec{x} k$$

and so $k = ft'_k g$.

Since $t'_k$ is continuous in every component, $t'_k$ is continuous, and it is clearly a bijection onto its image. Since $A$ is compact, and every bijective continuous function between compact Hausdorff spaces is a homeomorphism, it follows that $t'_k$ is a homeomorphism for every $k \in C(Q)$. By our choice of metric for $Q$ in (8), $d_\infty(t'_s, \text{id}_A) < 2$. Hence by Proposition 13.4, we may extend $t'_s$ to $t_s \in H(Q)$.

If $\varepsilon \in \mathbb{R}, \varepsilon > 0$, is arbitrary, then we will show that

$$B_{d_\infty}(s, \varepsilon) \subseteq f(B_{d_\infty}(t_s, \varepsilon) \cap H(Q))g.$$
If \( a \in B_{d_s}(s, \varepsilon) \) and \( t'^*_s : A \rightarrow A \) is defined as in (9), then \( ft'_s g = a \). Since \( f' : [0, 1] \rightarrow [0, 1] \) is a contraction and \( d_\infty(a, s) < \varepsilon \), it follows that
\[
d_\infty(t'^*_a, t'_s) = \sup \{ d(\bar{x} t'^*_a, \bar{x} t'_s) : \bar{x} \in A \} = \sup \left\{ \sum_{i \in \mathbb{N}} \frac{\bar{x} h s \pi_i f' - \bar{x} h a \pi_i f'}{2^{2i}} : \bar{x} \in A \right\} \leq \sup \left\{ \sum_{i \in \mathbb{N}} \frac{|\bar{x} h s \pi_i - \bar{x} h a \pi_i|}{2^{2i}} : \bar{x} \in A \right\} = \sup \left\{ \sum_{i \in \mathbb{N}} \frac{|\bar{x} s \pi_i - \bar{x} a \pi_i|}{2^{2i}} : \bar{x} \in (A)h = Q \right\} \leq \sup \left\{ \sum_{i \in \mathbb{N}} \frac{|\bar{x} s \pi_i - \bar{x} a \pi_i|}{2^i} : \bar{x} \in Q \right\} = d_\infty(a, s) < \varepsilon.
\]
Therefore
\[
d_\infty((t'^*_s)^{-1} t'^*_a, \text{id}(A)t'_s) = d_\infty((t'^*_s)^{-1} t'^*_a, (t'_s)^{-1} t'_s)
\]
and, since \((t'_s)^{-1} : (A)t'_s \rightarrow A \) is a surjective function,
\[
d_\infty((t'^*_s)^{-1} t'^*_a, (t'_s)^{-1} t'_s) = d_\infty(t'^*_a, t'_s) < \varepsilon,
\]
and so, by Proposition 13.4, there exists \( \phi \in H(Q) \) extending \((t'_s)^{-1} t'^*_s \) such that \( d_\infty(\phi, \text{id}_Q) < \varepsilon \). It follows that \( t_s \phi \in B_{d_\infty}(t_s, \varepsilon) \) and \( ft_s \phi g = ft'_s \phi g = ft'_s g = a \) and so \( a \in f(B_{d_\infty}(t_s, \varepsilon) \cap H(Q) )g \).

**Theorem 13.6.** The compact-open topology is the unique Polish semigroup topology for the monoid of continuous functions on the Hilbert cube \([0, 1]^\mathbb{N}\).

**Proof.** By Corollary 13.3 every such topology contains the compact-open topology. It therefore suffices to show that no Polish semigroup topology for \( C(Q) \) contains the compact open topology. This follows from Theorem 13.5 together with Lemma 3.6 (iii).

### 14. The Cantor Space

In this section we show that the monoid of continuous functions \( C(2^\mathbb{N}) \) on the Cantor space \( 2^\mathbb{N} \) has a unique Polish semigroup topology, the compact-open topology. The proof is analogous to, but not the same as, the proof of the uniqueness of the Polish semigroup topology on the Hilbert cube given in the last section.

To prove that the compact-open topology is also a maximal Polish semigroup topology on \( 2^\mathbb{N} \) we require an analogue of Proposition 13.4. Unfortunately, we were not able to locate such an analogue in the literature, and so we provide our own, the proof of which is similar to the proof of Proposition 13.4 given in [65]. In the following proposition we require the \( d_\infty \) metric on \( C(2^\mathbb{N}) \) for which we ought to fix a metric on \( 2^\mathbb{N} \). Since \( 2^\mathbb{N} \) is a subset of the Hilbert cube, we can define the metric \( d \) on \( 2^\mathbb{N} \) to be the restriction of the metric on the Hilbert cube defined in (8). Recall that
\[
d((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i},
\]
where \((x_1, x_2, \ldots), (y_1, y_2, \ldots) \in 2^\mathbb{N} \). We also require a metric on \( 2^\mathbb{N} \times 2^\mathbb{N} \) which we define by
\[
\rho((a_1, a_2), (b_1, b_2)) = \max\{d(a_1, b_1), d(a_2, b_2)\},
\]
and the associated metric \( \rho_\infty \) as defined in (7). If \( n \in \mathbb{N}, n > 0 \), we denote the finite sequence of length \( n \) consisting solely of 0 by \( 0^n \), and we denote the infinite sequence consisting solely of the value 0 by \( 0^\infty \).

**Proposition 14.1.** Suppose that \( A \) and \( B \) are closed subsets of \( 2^\mathbb{N} \) such that there exists a homeomorphism \( \phi : A \rightarrow B \) such that \( d_\infty(\phi, \text{id}_A) < \varepsilon \). Then there exists a homeomorphism \( \phi' : 2^\mathbb{N} \times 2^\mathbb{N} \rightarrow 2^\mathbb{N} \times 2^\mathbb{N} \) such that \((a, 0^n)\phi' = (a\phi, 0^n) \) for all \( a \in A \), and \( \rho_\infty(\phi', \text{id}_{2^\mathbb{N} \times 2^\mathbb{N}}) < \varepsilon \).
Theorem 14.2. The monoid $C(2^N)$ of continuous functions on the Cantor set together with the compact-open topology has property $X$ with respect to its group of units $H(2^N)$.

Proof. Similarly to the proof of Theorem 13.6, we will show that for every $s \in C(2^N)$, there exist $f, g \in C(2^N)$ and $t_s \in H(2^N)$ such that $s = ft_sg$ and for every ball $B = B_{d_\infty}(t_s, \varepsilon)$ there exists $\delta > 0$ such that $B_{d_{\infty}}(s, \delta) \subseteq f(B \cap H(2^N))g$.

Let $\Phi : 2^N \rightarrow 2^N \times 2^N$ be a homeomorphism. If $\varepsilon > 0$, then since $\Phi$ is homeomorphism between compact metric spaces, $\Phi$ and $\Phi^{-1}$ are uniformly continuous, and so there exists $r(\varepsilon) \in \mathbb{R}$ such that $r(\varepsilon) > 0$ and the following conditions hold:

(i) $d(a, b) < r(\varepsilon)$ implies that $\rho((a)\Phi, (b)\Phi) < \varepsilon$ for any $a, b \in 2^N$;

(ii) $\rho(a, b) < r(\varepsilon)$ implies that $d((a)\Phi^{-1}, (b)\Phi^{-1}) < \varepsilon$ for all $a, b \in 2^N \times 2^N$.

Let $s \in C(2^N)$ be arbitrary, and let $f, g \in C(2^N)$ be defined by:

(10) $(x)f = ((x, x)\Phi^{-1}, 0^\infty)\Phi^{-1}$

(11) $(x)g = (x)\Phi \pi_1 \Phi \pi_1$. 

Proof. It is well-known that if $F$ is a non-empty closed subset of the Cantor space, then there exists a continuous function $g_F : 2^N \rightarrow F$ such that $(x)g_F = x$ for all $x \in 2^N$ (see, for example, [2, Lemma 3.59]).

Let $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, be fixed, and let $n \in \mathbb{N}$ be such that $\sum_{i=n}^{\infty} \frac{1}{2^i} < \varepsilon$. We will view $2^N \times 2^N$ as $2^N \times (2^N \times 2^N)$ in the obvious way, but continue using the metric $\rho$. We define $\phi_1, \phi_2, \phi_3 : 2^N \times 2^N \rightarrow 2^N \times 2^N$ by

$$(a, (b, c))\phi_1 = (a, (b, c + a))$$

$$(a, (b, c))\phi_2 = (a - (c)g_A + (c)g_A\phi, (b, c))$$

$$(a, (b, c))\phi_3 = (a, (b, c - (a)g_B\phi^{-1})).$$

It is clear that $\phi_1, \phi_2, \phi_3$ are homeomorphisms.

The required homeomorphism $\phi' : 2^N \times 2^N \rightarrow 2^N \times 2^N$ is $\phi' = \phi_1\phi_2\phi_3$ since

$$\begin{align*}
(a, 0^\infty)\phi_1\phi_2\phi_3 &= (a, (0^\infty, a))
\phi_2\phi_3 \\
&= (a - (a)g_A + (a)g_A\phi, (0^\infty, a))
\phi_3 \\
&= ((a)\phi, (0^\infty, a)) \\
&= ((a)\phi, (0^\infty, a) - (a)g_B\phi^{-1})) \\
&= ((a)\phi, 0^\infty).
\end{align*}$$

It remains to show that $\rho_\infty(\phi', \text{id}_{2^N \times 2^N}) = \sup\{\rho((x, y)\phi', (x, y)) : x, y \in 2^N\} < \varepsilon$. By definition,

$$\rho((x, y)\phi', (x, y)) = \max\{d((x, y)\phi' \pi_1, x), d(((x, y)\phi' \pi_2, y)\},$$

where $\pi_1, \pi_2 : 2^N \times 2^N \rightarrow 2^N$ are the projection functions. By the assumption on $n$, and the fact that in the definitions of $\phi_1, \phi_2, \phi_3$ the first $n$ terms in the second component are not altered, it follows that

$$d((x, y)\phi' \pi_2, y) < \sum_{i=n}^{\infty} \frac{1}{2^i} < \varepsilon.$$

Since $\phi_1$ and $\phi_3$ do not change the first coordinate of their arguments, it suffices to check that $d(a, (a, (b, c))\phi_2\pi_1) < \varepsilon$ for all $(a, (b, c)) \in 2^N \times 2^N$. If $(a, (b, c)) \in 2^N \times 2^N$ is arbitrary, then $\lambda_{(c)g_A - a} : 2^N \rightarrow 2^N$ is an isometry, with respect to component-wise addition modulo 2 and the metric $d$, and so

$$
\begin{align*}
d(a, (a, (b, c))\phi_2\pi_1) &= d(a, a - (c)g_A + (c)g_A\phi) \\
&= d((c)g_A - a, (c)g_A - a + a - (c)g_A + (c)g_A\phi) \\
&< \varepsilon,
\end{align*}
$$

since $d_\infty(\phi, \text{id}_{A}) < \varepsilon$. 

$\square$
Note that $fg$ is the identity on $2^\mathbb{N}$. Next, we will show that there exists $t_s \in H(2^\mathbb{N})$ such that $s = ft_s g$.

Let $s', \Delta : 2^\mathbb{N} \to 2^\mathbb{N} \times 2^\mathbb{N}$ be defined by

$$ (x)s' = ((x)s, x) $$

and $(x)\Delta = (x, x)$. Clearly $\Delta$ is a homeomorphism from $2^\mathbb{N}$ to $(2^\mathbb{N})\Delta$, and, $s'$ is injective and continuous, since it is continuous in each component, and the inverse of $s'$ is also continuous because $(x)s, x \mapsto x$ is a projection. Hence $s'$ is a homeomorphism between $2^\mathbb{N}$ and $(2^\mathbb{N})s'$. If $\phi = \Phi \Delta^{-1} s' \Phi^{-1}$ and $x \in 2^\mathbb{N}$ is arbitrary, then

$$ (x, x) \Phi^{-1} \phi = (x, x) \Phi^{-1} \Phi \Delta^{-1} s' \Phi^{-1} = (x)s' \Phi^{-1} = ((x)s, x) \Phi^{-1} $$

and so $\phi$ is a homeomorphism from $A = (2^\mathbb{N})\Delta \Phi^{-1} = \{(x, x) \Phi^{-1}: x \in 2^\mathbb{N}\}$ to $B = (A)\phi$. By Proposition 14.1, there exists a homeomorphism $\zeta : 2^\mathbb{N} \times 2^\mathbb{N} \to 2^\mathbb{N} \times 2^\mathbb{N}$ such that

$$ ((x, x) \Phi^{-1}, 0^\infty) \zeta = ((x, x) \Phi^{-1} \phi, 0^\infty) $$

for all $x \in 2^\mathbb{N}$.

We will show that the required homeomorphism of $2^\mathbb{N}$ is

$$ t_s = \Phi \zeta \Phi^{-1}. $$

If $x \in 2^\mathbb{N}$ is arbitrary, then

$$ (x)f t_s g = ((x, x) \Phi^{-1}, 0^\infty) \Phi^{-1} t_s g $$

by (10)

$$ = ((x, x) \Phi^{-1}, 0^\infty) \zeta \Phi^{-1} g $$

by (15)

$$ = ((x, x) \Phi^{-1}, 0^\infty) \Phi^{-1} g $$

by (14)

$$ = ((x, x) \Phi^{-1} \phi, 0^\infty) \Phi^{-1} \Phi \pi_1 \Phi \pi_1 $$

by (11)

$$ = (x, x) \Phi^{-1} \phi \Phi \pi_1 $$

$$ = (x, x) s, x \Phi^{-1} \Phi \pi_1 $$

by (13)

$$ = (x)s. $$

Suppose that $\varepsilon > 0$ is arbitrary. It remains to show that there exists $\delta > 0$ such that $B_{d_\infty}(s, \delta) \subseteq f(B_{d_\infty}(t_s, \varepsilon) \cap H(2^\mathbb{N}))g$. We set $\delta = r(\varepsilon \cdot r(x))$. Let $u \in B_{d_\infty}(s, \delta)$. We define $u' : 2^\mathbb{N} \to 2^\mathbb{N} \times 2^\mathbb{N}$ by $(x)u' = ((x)u, x)$ for all $x \in 2^\mathbb{N}$, and, similar to the proof above for $s'$, $u'$ is a homeomorphism from $2^\mathbb{N}$ to $(2^\mathbb{N})u'$. Then $\rho_\infty(u', s') = d_\infty(u, s) < \delta$. Hence

$$ \rho_\infty(\id_{(2^\mathbb{N})s'}, (s')^{-1} u') = \rho_\infty((s')^{-1} s', (s')^{-1} u') = \rho_\infty(u', s') = d_\infty(u, s) < \delta $$

since left multiplication by $(s')^{-1}$ is an isometry. It follows that

$$ d_\infty(\Phi \id_{(2^\mathbb{N})s'}, \Phi (s')^{-1} u' \Phi^{-1}) < r(\varepsilon) $$

by (i) and (ii), from the second paragraph of this proof.

By Proposition 14.1 applied to the homeomorphism $\Phi (s')^{-1} u' \Phi^{-1} : (2^\mathbb{N})s' \Phi^{-1} \to (2^\mathbb{N})u' \Phi^{-1}$, there exists $\gamma \in H(2^\mathbb{N} \times 2^\mathbb{N})$ such that

$$ (a, 0^\infty) \gamma = (a \Phi (s')^{-1} u' \Phi^{-1}, 0^\infty) $$

and $\rho_\infty(\gamma, \id_{2^\mathbb{N} \times 2^\mathbb{N}}) < r(\varepsilon)$. Since left multiplication by $\Phi^{-1} t_s \Phi$ is an isometry of $C(2^\mathbb{N} \times 2^\mathbb{N})$,

$$ \rho_\infty(\Phi^{-1} t_s \Phi \gamma, \Phi^{-1} t_s \Phi) = \rho_\infty(\gamma, \id_{2^\mathbb{N} \times 2^\mathbb{N}}) < r(\varepsilon) $$

and so

$$ d_\infty(t_s \Phi \gamma \Phi^{-1}, t_s) = d_\infty(\Phi \Phi^{-1} t_s \Phi \gamma \Phi^{-1}, \Phi \Phi^{-1} t_s \Phi^{-1}) < \varepsilon.$$
Hence \( f(t_s \Phi \gamma \Phi^{-1}) \in f((B_{d_{\infty}}(t_s, \varepsilon) \cap H(2^N)))g \). We conclude the proof by showing that \( f(t_s \Phi \gamma \Phi^{-1}) = u \). Suppose that \( x \in 2^N \) is arbitrary. Then

\[
(x)f(t_s \Phi \gamma \Phi^{-1})g = ((x, x) \Phi^{-1}, 0^\infty) \Phi^{-1}t_s \Phi \gamma \Phi^{-1}g
\]

by (10)

\[
= ((x, x) \Phi^{-1}, 0^\infty) \zeta \gamma \Phi^{-1}g
\]

by (15)

\[
= ((x, x) \Phi^{-1} \phi, 0^\infty) \Phi^{-1}g
\]

by (14)

\[
= ((x, x) \Phi^{-1}, 0^\infty) \Phi^{-1}g
\]

by (13)

\[
= (((x)s, x) \Phi^{-1}(s')^{-1}u' \Phi^{-1}, 0^\infty) \Phi^{-1}g
\]

by (16)

\[
= (((x)s, x)(s')^{-1}u' \Phi^{-1}, 0^\infty) \Phi^{-1}g
\]

by (12)

\[
= (((x)u, x) \Phi^{-1}, 0^\infty) \Phi^{-1}g
\]

by (11)

\[
= (((x)u, x) \Phi^{-1}, 0^\infty) \Phi^{-1}\Phi_{\pi_1} \Phi_{\pi_1}
\]

by (11)

\[
= (x, x) \Phi^{-1} \Phi_{\pi_1}
\]

\[
= (x, u) \Phi^{-1}
\]

\[
\square
\]

**Theorem 14.3.** The monoid of continuous functions on the Cantor set \( 2^N \) together with the compact-open topology has automatic continuity.

**Proof.** This follows from Theorem 14.2, Lemma 3.6, and the automatic continuity of the group of homeomorphisms of the Cantor set. \( \square \)

**Proposition 14.4.** Suppose that \( C(2^N) \) is a second countable Hausdorff semigroup with respect to some topology. Then the function \( a : 2^N \times C(2^N) \to 2^N \) defined by

\[
(p, f)a = (p)f
\]

is continuous.

**Proof.** This proof is similar to the proof of Proposition 13.2, and we proceed using the same notation as in the proof of that proposition: \( T \) denotes the given Polish semigroup topology on \( C(2^N) \); \( d \) is any metric that induces the usual topology on \( 2^N \); \( F \) is the set of constant functions in \( C(2^N) \); and \( \phi : F \to 2^N \) is the function that sends \( f \in F \) to the unique value in \( (2^N)f \). Furthermore, we suppose that \( \mathcal{B} \) is any basis for \( 2^N \) consisting of clopen sets and such that \( 2^N \not\in \mathcal{B} \).

Similarly to the proof of Proposition 13.2, it suffices to show that \( \phi \) is a homeomorphism.

To show that \( \phi \) is continuous, suppose that \( U \in \mathcal{B} \). It suffices to show that \((U)\phi^{-1} = \{ f \in F : (2^N)f = \{ q \}, q \in U \} \) is open in \( F \). If \( y \not\in U \), then we define \( f : 2^N \to 2^N \) such that

\[
(x)f = \begin{cases} 
  x & \text{if } x \in U \\
  y & \text{if } x \not\in U.
\end{cases}
\]

Since \( U \) and \( 2^N \setminus U \) are open, and \( f \) is continuous with respect to both sets, it follows that \( f \) is continuous. Since \( C(2^N) \) is \( T_1 \), \( F \) is \( T_1 \), and so the singleton set containing the constant function \( g : 2^N \to 2^N \) with value \( y \) is closed. Hence \( \{ g \}\rho^{-1}_f \cap F \) is closed in \( F \), and so \( F \setminus \{ g \}\rho^{-1}_f \) is open in \( F \). If \( h \in F \setminus \{ g \}\rho^{-1}_f \), then there exists \( x \in 2^N \) such that

\[
(x)hf \neq y
\]

and so \( (x)h \in U \) by the definition of \( f \). On the other hand, if \( h \in F \) is such that \( (x)h \in U \), then \( (x)hf \neq y \), and so \( F \setminus \{ (g)\rho^{-1}_f \} = (U)\phi^{-1} = \{ f \in F : (2^N)f = \{ q \}, q \in U \} \) is open, as required.

Next, we show that \( \phi^{-1} : 2^N \to F \) is continuous also. Note that the topology induced on \( F \) by the compact open topology is precisely the standard topology on \( 2^N \) (when viewing an element of \( F \) as it’s image under \( \phi \)). It therefore suffices to show that the topology induced on \( F \) by \( T \) is contained in the topology induced on \( F \) by the compact open topology. This follows immediately from Theorem 14.3. \( \square \)

As an immediate corollary of the last proposition together with Theorem 12.1, we obtain the following.
Corollary 14.5. Every second countable Hausdorff semigroup topology on the monoid of continuous functions on the Cantor space $2^\mathbb{N}$ contains the compact-open topology.

Corollary 14.6. The compact-open topology is the unique second countable Hausdorff semigroup topology on the monoid of continuous functions on the Cantor set.

Proof. By Theorem 14.5 every such topology contains the compact-open topology and by Theorem 14.3 every such topology is contained in the compact-open topology.

Part 4. Endomorphism monoids of Fraïssé limits

15. Self-embedding monoids

We can use the distinct Polish topologies on $\text{Inj}(\mathbb{N})$ to show that the self-embedding monoids of many Fraïssé limits do not have unique Polish topologies.

Corollary 15.1. Let $X$ be a Fraïssé limit such that $\text{Emb}(X) \neq \text{Aut}(X)$. Then $\text{Emb}(X)$ has at least two distinct Polish semigroup topologies.

Proof. By Theorem 11.1(i), the pointwise topology and the topology $\mathcal{J}_1$ are Polish semigroup topologies on $\text{Inj}(X)$. Since $\text{Emb}(X)$ is closed, it follows that these two topologies are also Polish semigroup topologies on $\text{Emb}(X)$. The group $\text{Aut}(X)$ is dense in $\text{Emb}(X)$ with the pointwise topology. Hence, by the assumption that $\text{Emb}(X) \neq \text{Aut}(X)$, $\text{Aut}(X)$ is not closed in the pointwise topology, but $\text{Aut}(X) = \text{Sym}(X) \cap \text{Emb}(X)$ is closed in the topology induced by $\mathcal{J}_1$. Therefore these two topologies are distinct.

16. Endomorphism monoids

It was shown in [54] that that any homomorphism from $\text{Aut}(\mathbb{Q}, \leq)$ into a separable group is continuous. It follows that $\text{Aut}(\mathbb{Q}, \leq)$ has a unique Polish group topology. In this section, we consider $T_1$ semigroup topologies on $\text{End}(\mathbb{Q}, \leq)$. Since $\text{End}(\mathbb{Q}, \leq)$ is a closed subsemigroup of $\mathbb{Q}^\mathbb{Q}$, endowed with the pointwise topology, it follows that the pointwise topology on $\text{End}(\mathbb{Q}, \leq)$ is a Polish semigroup topology.

Theorem 16.1. Every $T_1$ semitopological semigroup topology on $\text{End}(\mathbb{Q}, \leq)$ contains the pointwise topology.

Proof. Since the constant transformations on $\mathbb{Q}$ are contained in $\text{End}(\mathbb{Q}, \leq)$ and given $x \in \mathbb{Q}$, there exist $f_x \in \text{End}(\mathbb{Q}, \leq)$ such that $(x)f_x^{-1} = \{x\}$ and $(\mathbb{Q})f_x$ is finite. For example, $f_x \in \text{End}(\mathbb{Q}, \leq)$ might be defined by

$$(y)f_x = \begin{cases} 
  x - 1 & \text{if } y < x \\
  x & \text{if } y = x \\
  x + 1 & \text{if } y > x.
\end{cases}$$

The result then follows by Theorem 5.1.

Since $\text{Aut}(\mathbb{Q}, \leq)$ is a $G_δ$ subset of $\text{End}(\mathbb{Q}, \leq)$, it follows from [46] that every Polish semigroup topology on $\text{End}(\mathbb{Q}, \leq)$ induced the pointwise topology on $\text{Aut}(\mathbb{Q}, \leq)$.

Next, we consider endomorphism monoids of infinite graphs. We are primarily concerned with the countable random graph $\mathcal{R}$, but by proving the following more general result, which also proves Theorem 11.1(v).

Proposition 16.2. If $\Delta$ denotes an infinite graph without loops where every vertex is contained in a complete induced subgraph of size $|\Delta|$, then the Zariski topology on $\text{End}(\Delta)$ equals the pointwise topology.

Proof. The Zariski topology on $\text{End}(\Delta)$ is contained in every Hausdorff semigroup topology on $\text{End}(\Delta)$, by Proposition 4.1, and so the Zariski topology is contained in the pointwise topology.

For the converse, it suffices to show that if $u$ and $v$ are nodes in $\Delta$, then

$$\{f \in \text{End}(\Delta) : (u)f = v\},$$
being an arbitrary sub-basic open set in the pointwise topology, is open in the Zariski topology.
By assumption, there exists a complete induced subgraph $\Gamma$ in $\Delta$ with $|\Gamma|=|\Delta|$. Let $w_1$ and $w_2$ be
distinct nodes in $\Gamma$ and let $\pi: \Delta \setminus \{v\} \to \Gamma \setminus \{w_1, w_2\}$ be any bijection. We define $f_1, f_2: \Delta \to \Gamma$
so that
\[(x)f_i = \begin{cases} (x)\pi & \text{if } x \neq v \\ w_i & \text{if } x = v \end{cases}
\]
for $i = 1, 2$. Since $\Gamma$ is complete, $f_1, f_2 \in \text{End}(\Delta)$. The set
\[U = \{ s \in \text{End}(\Delta) : (s)\rho_{f_1} \neq (s)\rho_{f_2} \} = \{ s \in \text{End}(\Delta) : v \in (\Delta)s \}
\]
is open in the Zariski topology. Every node in $\Delta$ is contained in an complete induced subgraph
of size $|\Delta|$, we denote by $\Gamma'$ such an induced subgraph of $\Delta$ containing the node $u$. If $\Gamma' \setminus \{u\}$ is
partitioned into complete graphs $\Sigma_1$ and $\Sigma_2$ with $|\Sigma_1| = |\Sigma_2| = |\Delta|$, then we set $l_1: \Delta \to \Sigma_1 \cup \{u\}$
and $l_2: \Delta \to \Sigma_2 \cup \{u\}$ to be any bijections. Since $\Sigma_1 \cup \{u\}$ is a complete graph, it follows that
$l_1 \in \text{End}(\Delta)$, and similarly, $l_2 \in \text{End}(\Delta)$. By Proposition 4.2, $\lambda_1, \lambda_2: \text{End}(\Delta) \to \text{End}(\Delta)$ are
continuous with respect to the Zariski topology, and so
\[V = (U)\lambda_1^{-1} \cap (U)\lambda_2^{-1}
\]
\[= \{ f \in \text{End}(\Delta) : l_1f \in U \text{ and } l_2f \in U \}
\]
\[= \{ f \in \text{End}(\Delta) : v \in (\Delta)l_1f \text{ and } v \in (\Delta)l_2f \}
\]
\[= \{ f \in \text{End}(\Delta) : v \in (\Sigma_1 \cup \{u\})f \cap (\Sigma_2 \cup \{u\})f \}
\]
is open in the Zariski topology. If $f \in V$, then there exist $u_1 \in \Sigma_1 \cup \{u\}$ and $u_2 \in \Sigma_2 \cup \{u\}$ such
that $(u_1)f = (u_2)f = v$. Since $f$ is an endomorphism and $\Sigma_1 \cup \Sigma_2 \cup \{u\} = \Gamma'$ is a complete graph,
it follows that $u_1 = u_2$. But the only node in both $\Sigma_1 \cup \{u\}$ and $\Sigma_2 \cup \{u\}$ is $u$, and so $(u)f = v$.
In particular, $V = \{ f \in \text{End}(\Delta) : (u)f = v \}$ is open in the Zariski topology, and we have shown that the pointwise topology is contained in the Zariski topology.

It was shown in [34] that a Polish group $G$ with ample generics has the property that every homomorphism $\phi$ from $G$ to a separable group $H$ is continuous. It is also shown in [26, 25] that if $R$
denotes the countable random graph, then $\text{Aut}(R)$ has ample generics. It follows that $\text{Aut}(R)$
has a unique Polish group topology. Since $R$ is a relational structure, it follows that the pointwise
topology on $\text{End}(R)$ is a Polish semigroup topology on $\text{End}(R)$.

**Theorem 16.3.** If $R$ denotes the countable random graph, then the Zariski topology on $\text{End}(R)$
equals the pointwise topology.

**Proof.** The random graph $R$ satisfies the hypothesis of Proposition 16.2.

**Corollary 16.4.** If $R$ denotes the countable random graph, then the pointwise topology is contained
in any Hausdorff semigroup topology on $\text{End}(R)$

**Proof.** This follows immediately from Proposition 4.1 and Theorem 16.3.

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