CMB anisotropies at second-order II: analytical approach

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Abstract. We provide an analytical approach to the second-order cosmic microwave background (CMB) anisotropies generated by the non-linear dynamics taking place at last scattering. We study the acoustic oscillations of the photon–baryon fluid in the tight coupling limit and we extend at second order the Meszaros effect. We allow for a generic set of initial conditions due to primordial non-Gaussianity and we compute all of the additional contributions arising at recombination. Our results are useful to provide the full second-order radiation transfer function at all scales necessary for establishing the level of non-Gaussianity in the CMB.

Keywords: CMBR theory, cosmological perturbation theory, inflation

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1. Introduction

Cosmological inflation [1] has become the dominant paradigm to understand the initial conditions for the CMB anisotropies and structure formation. In the inflationary picture, the primordial cosmological perturbations are created from quantum fluctuations ‘redshifted’ out of the horizon during an early period of accelerated expansion of the universe, where they remain ‘frozen’. They are observable as temperature anisotropies in the CMB. This picture has recently received further spectacular confirmation by the Wilkinson Microwave Anisotropy Probe (WMAP) three year set of data [2]. Since the observed cosmological perturbations are of the order of $10^{-5}$, one might think that first-order perturbation theory will be adequate for all comparisons with observations. However, this might not be the case. Present [2] and future [3] experiments may be sensitive to the non-linearities of the cosmological perturbations at the level of second- or higher-order perturbation theory. The detection of these non-linearities through the non-Gaussianity (NG) in the CMB [4] has become one of the primary experimental targets.

A possible source of NG could be primordial in origin, being specific to a particular mechanism for the generation of the cosmological perturbations. This is what makes a positive detection of NG so relevant: it might help in discriminating among competing scenarios which otherwise might be indistinguishable. Indeed, various models of inflation, firmly rooted in modern particle physics theory, predict a significant amount of primordial NG generated either during or immediately after inflation when the comoving curvature perturbation becomes constant on superhorizon scales [4]. While single-field [5] and two(multi)-field [6] models of inflation generically predict a tiny level of NG, ‘curvaton-type models’, in which a significant contribution to the curvature perturbation is generated after the end of slow-roll inflation by the perturbation in a field which has a negligible effect on inflation, may predict a high level of NG [7,8]. Alternatives to the curvaton model are those models characterized by the curvature perturbation being generated by an inhomogeneity in the decay rate [9,10], the mass [11] or the interaction rate [12] of the particles responsible for the reheating after inflation. In that case, the reheating can be the first one (caused by the scalar field(s) responsible for the energy density during inflation) or alternatively the particle species causing the reheating can be a fermion [13]. Other opportunities for generating the curvature perturbation occur at the end of inflation [14], during preheating [15], and at a phase transition producing cosmic strings [16].

On the other hand there exist many sources of NG in the CMB anisotropies beyond the primordial ones, it is essential to characterize these in order to distinguish them from a possible primordial signal. For example, statistics such as the bispectrum and the trispectrum of the CMB can be used to assess the level of primordial NG on various cosmological scales and to discriminate it from the one induced by secondary anisotropies and systematic effects [4], [17]–[19]. In this case, a positive detection of a primordial
NG in the CMB at some level might therefore confirm and/or rule out a whole class of mechanisms by which the cosmological perturbations have been generated.

Therefore, it is of fundamental importance to provide accurate theoretical predictions of all of the possible non-linear effects contributing to the overall NG in the CMB anisotropies. At second order in the perturbation theory one should provide a full prediction for the second-order radiation transfer function. A first step towards this goal has been taken in [20] (see also [21]), where the full second-order radiation transfer function for the CMB anisotropies on large angular scales in a flat universe filled with matter and cosmological constant was computed, including the second-order generalization of the Sachs–Wolfe effect, both the early and late integrated Sachs–Wolfe (ISW) effects and the contribution of the second-order tensor modes. These effects are due to gravity. In [22], we presented the computation of the full system of Boltzmann equations at second order describing the evolution of the photon, baryon and CDM fluids, neglecting polarization. In this way, we accounted also for the small-scale effects due to the collision terms. The equations we derived allowed us to follow the time evolution of the CMB anisotropies at second order at all angular scales from the early epoch, when the cosmological perturbations were generated, to the present through the recombination era. Therefore, reference [22] sets the stage for the computation of the full second-order radiation transfer function at all scales and for a generic set of initial conditions specifying the level of primordial non-Gaussianity.

At second order, one can see that there are many sources of NG in the CMB anisotropies, beyond the primordial one. The most relevant sources are the so-called secondary anisotropies, which arise after the last scattering epoch. The so-called scattering secondaries include the thermal Sunyaev–Zel’dovich effect, where hot electrons in clusters transfer energy to the CMB photons, the kinetic Sunyaev–Zel’dovich effect, produced by the bulk motion of the electrons in clusters, the Ostriker–Vishniac effect, produced by bulk motions modulated by linear density perturbations, and effects due to reionization processes. Gravitational secondaries are effects mediated by gravity and include the change in energy of photons when the gravitational potential is time-dependent or the gravitational lensing effect. Secondaries that result from a time-dependent potential are the ISW produced mainly on large scales when the dark energy at late times becomes dominant and the potential starts to decay, or the Rees–Sciama effect, produced during the matter-dominated epoch at second order and by the time evolution of the potential on non-linear scales. Gravitational lensing which causes the deflection of the photons’ path from the last scattering to us, does not create anisotropies, it only modifies existing ones. All of these secondary effects are most significant on small angular scales (except for the ISW effect). Moreover, the three-point function arising from the correlation of the gravitational lensing effect and the ISW effect generated by the matter distribution along the line of sight [23, 24] and the Sunyaev–Zel’dovich effect [25] are large and detectable by Planck [26]. Of course, on small angular scales, fully non-linear calculations of specific effects such as Sunyaev–Zel’dovich, gravitational lensing, etc, would provide a more accurate estimate of the resulting CMB anisotropy, however, as long as the leading contribution to second-order statistics such as the bispectrum is concerned, second-order perturbation theory suffices.

In this paper, we will focus on another relevant source of NG: the non-linear effects operating at the recombination epoch. The dynamics at recombination is quite
involved because all of the non-linearities in the evolution of the baryon–photon fluid at recombination and the ones coming from general relativity should be accounted for. The present paper can be considered as an application of the equations found in [22] and it offers an analytical study at second order of this complicated dynamics. This allows us to account for those effects that at the last scattering surface produce a non-Gaussian contribution to the CMB anisotropies that add to the primordial one. Such a contribution is relevant because it represents a major part of the second-order radiation transfer function, which must be determined in order to have complete control of both the primordial and non-primordial parts of NG in the CMB anisotropies and to gain from the theoretical side the same level of precision that could be reached experimentally in the near future [4].

In order to achieve this goal, we have considered the Boltzmann equations derived in [22] at second order describing the evolution of the photon, baryon and CDM fluids, and we have manipulated them further under the assumption of tight coupling between photons and baryons. This leads to the generalization at second order of the equations for the photon energy density and velocity perturbations which govern the acoustic oscillations of the photon–baryon fluid for modes that are inside the horizon at recombination. The evolution is that of a damped harmonic oscillator, with a source term which is given by the gravitational potentials generated by the different species. An interesting result is that, unlike the linear case, at second order the quadrupole moment of the photons is not suppressed in the tight coupling limit and it must be taken into account. We also find that the second-order CMB anisotropies generated at last scattering do not reduce only to the energy density and velocity perturbations of the photons evaluated at recombination, but a number of second-order corrections due to gravity at last scattering arise from the Boltzmann equations of [22]. We compute them when we decompose the CMB anisotropies in multipole moments.

The bulk of the paper deals with the computation of the analytical solutions for the acoustic oscillations of the photon–baryon fluid at second order. These solutions are derived adopting some simplifications which are also standard for an analytical treatment of the linear CMB anisotropies, and which nonetheless allow us to catch most of the physics at recombination. One of these simplifications is to study separately two limiting regimes: intermediate scales which enter the horizon in between the equality epoch ($\eta_{\text{eq}}$) and the recombination epoch ($\eta_r$), with $\eta_r^{-1} \ll k \ll \eta_{\text{eq}}^{-1}$, and shortwave perturbations, with $k \gg \eta_{\text{eq}}^{-1}$, which enter the horizon before the equality epoch. An alternative approach could be to derive a semianalytical solution by using the fits of [28] for the linear gravitational potentials. Otherwise a full numerical evaluation can be performed [29] using the set of Boltzmann equations of [22]. However, in this paper our main concern is to provide a simple estimate of the quantitative behaviour of the non-linear evolution taking place at recombination, offering at the same time all the tools for a more accurate computation. Note that the case $k \gg \eta_{\text{eq}}^{-1}$ has been treated in two steps. First, we just assume a radiation-dominated universe, and then we give a better analytical solution by solving the evolution of the perturbations from the equality epoch onwards taking into account that the dark matter perturbations around the equality epoch tend to dominate the second-order gravitational potentials. As a byproduct, this last step provides the Meszaros effect at second order. In deriving the analytical solutions, we have accurately accounted for the initial conditions set on superhorizon scales by the primordial non-Gaussianity. In fact,
the primordial contribution is always transferred linearly, while the real new contribution to the radiation transfer function is given by all of the additional terms provided in the source functions of the equations. Let us stress here that the analyses of the CMB bispectrum performed so far, as for example in [2, 26, 30], adopt just the linear radiation transfer function (unless the bispectrum originated by specific secondary effects, such as Rees–Sciama or Sunyaev–Zel’dovich effects, are considered).

Since the paper serves different purposes and achieves different goals, we summarize them as follows:

- We compute the second-order CMB anisotropies generated by non-linearities at recombination which will add to the primordial non-Gaussianity.
- We provide analytical solutions for acoustic oscillations of the photon–baryon fluid at second order in the tight coupling limit starting from the Boltzmann equations derived in [22].
- We compute the evolution of the CDM density perturbations (and the gravitational potentials) accounting for those modes that enter the horizon during the radiation-dominated epoch. This allows us to determine the second-order transfer function for the density perturbations, and in particular the generalization at second order of the Meszaros effect.
- We provide the multipole moments of the CMB anisotropies and hence we are able to compute that part of the second-order radiation transfer function that corresponds to small-scale effects at recombination, thus complementing the results of [20].

The paper is organized as follows. In section 2, we report the Boltzmann equations for the photons derived in [22]. In section 3, we recall how to treat them in the tight coupling limit at linear order in the perturbations and the standard way to get the analytical solutions for the photon–baryon fluid. In section 4, we derive the equations for the second-order energy density and velocity perturbations of photons in the tight coupling limit. A subsection is devoted to computing the second-order quadrupole moment of the photons. Section 5 deals with the expansion of CMB anisotropies in multipole moments and with the computation of those contributions to the CMB anisotropies which are generated at recombination. In section 6, we derive the analytical solutions describing the acoustic oscillations at second order of the photon–baryon fluid. We derive these solutions accounting for the primordial non-Gaussianity, and in the two regimes $\eta^{-1} \ll k \ll \eta_{\text{eq}}^{-1}$ (section 7) and $k \gg \eta_{\text{eq}}^{-1}$ (section 8). In section 9, we compute at first and second order the evolution on subhorizon scales of the density perturbations of CDM, thus arriving at a generalization of the Meszaros effect. This result also allows us to give a refined prediction for the CMB anisotropies in the case $k \gg \eta_{\text{eq}}^{-1}$. Section 10 contains our conclusions, and we also provide some appendices which mainly treat the gravity perturbations, and where we provide the generic solutions for the evolution of the second-order gravitational potentials for a radiation- or matter-dominated universe.

2. The Boltzmann equations

In this section, we report the Boltzmann equations derived in [22], while the goal of sections 3 and 4 is to derive the moments of the Boltzmann equations for photons in
the limit when the photons are tightly coupled to the the baryons (the electron–proton system) due to Compton scattering. We will first review briefly the standard computation at linear order and then derive the equations at second order in the perturbations, pointing out some interesting differences with respect to the linear case. The starting point is the Boltzmann equation at first and second order [22]

\[
\frac{\partial \Delta_i^{(1)}}{\partial \eta} + n^i \frac{\partial \Delta_i^{(1)}}{\partial x^i} + 4 \frac{\partial \Phi_i^{(1)}}{\partial x^i} n^i - 4 \frac{\partial \Psi_i^{(1)}}{\partial \eta} = -\tau' \left[ \Delta_i^{(0)} + \frac{1}{2} \Delta_i^{(1)} P_2(\hat{v} \cdot \mathbf{n}) - \Delta_i^{(1)} + 4 \mathbf{v} \cdot \mathbf{n} \right],
\]

(2.1)

and at second order

\[
\frac{1}{2} \frac{d}{d\eta} \left[ \Delta^{(2)} + 4 \Phi^{(2)} \right] + \frac{d}{d\eta} \left[ \Delta^{(1)} + 4 \Phi^{(1)} \right] - 4 \Delta^{(1)} \left( \Psi^{(1)'} - \Phi_i^{(1)} n^i \right) - 2 \frac{\partial}{\partial \eta} \left( \Psi^{(2)} + \Phi^{(2)} \right) + 4 \frac{\partial \omega_i^{(1)}}{\partial \eta} n^i + 2 \frac{\partial \chi_{ij}^{(1)}}{\partial \eta} n^i n^j = -\frac{\tau'}{2} \left[ \Delta_{00}^{(2)} - \Delta^{(2)} - \frac{1}{2} \sum_{m=-2}^{2} \sqrt{4\pi} \Delta_{2m}^{(2)} Y_{2m}(\mathbf{n}) \right] + 2 (\delta_i^{(1)} + \Phi_i^{(1)}) \left( \Delta_0^{(1)} + \frac{1}{2} \Delta_2^{(1)} P_2(\hat{v} \cdot \mathbf{n}) - \Delta_i^{(1)} + 4 \mathbf{v} \cdot \mathbf{n} \right) + 4 \mathbf{v}^{(2)} \cdot \mathbf{n} + 2 (\mathbf{v} \cdot \mathbf{n}) \left[ \Delta^{(1)} + 3 \Delta_0^{(1)} - \Delta_2^{(1)} \left( 1 - \frac{5}{2} P_2(\hat{v} \cdot \mathbf{n}) \right) \right] - \nu \Delta_1^{(1)} (4 + 2 P_2(\hat{v} \cdot \mathbf{n})) + 14 (\mathbf{v} \cdot \mathbf{n})^2 - 2 \nu^2.
\]

(2.2)

Let us recall some definitions of the quantities appearing in equations (2.1) and (2.2). \( \Phi = \Phi^{(1)} + \Phi^{(2)}/2 \) and \( \Psi = \Psi^{(1)} + \Psi^{(2)}/2 \) are the gravitational potentials in the Poisson gauge, while \( \omega_i \) and \( \chi_{ij} \) are the second-order vector and tensor perturbations of the metric according to equation (A.1). The photon temperature anisotropies are given by

\[
\Delta^{(i)}(x^i, n^i, \tau) = \int \frac{dp}{p^3} f^{(i)},
\]

(2.3)

which represents the photon fractional energy perturbation (in a given direction) being the integral of the photon distribution function \( f = f^{(1)} + f^{(2)}/2 \) over the photon momentum magnitude \( p \) (\( p^i = pn^i \)). The angular dependence of the photon anisotropies \( \Delta \) can be expanded as

\[
\Delta^{(i)}(x, n) = \sum_{\ell} \sum_{m=-\ell}^{\ell} \Delta_{\ell m}^{(i)}(x)(-i)^{\ell} \sqrt{\frac{4\pi}{2\ell + 1}} Y_{\ell m}(n),
\]

(2.4)

with

\[
\Delta_{\ell m}^{(i)} = (-i)^{-\ell} \sqrt{\frac{2\ell + 1}{4\pi}} \int d\Omega \Delta^{(i)} Y_{\ell m}^{\star}(n),
\]

(2.5)

where we warn the reader that the superscript stands by the order of the perturbation, while the subscripts indicate the order of the multipoles. At first order, one can drop the dependence on \( m \), setting \( m = 0 \) so that \( \Delta_{\ell m}^{(1)} = (-i)^{-\ell}(2\ell + 1)\delta_{00} \Delta_{\ell}^{(1)} \). It is understood that on the left-hand side of equation (2.2) one has to pick up for the total time derivatives.
only those terms which contribute to second order. Thus, we have to take (see [22])

\[
\frac{1}{2} \frac{d}{d\eta} \left[ \Delta^{(2)} + 4\Phi^{(2)} \right] + \frac{d}{d\eta} \left[ \Delta^{(1)} + 4\Phi^{(1)} \right] \bigg|^{(2)} = \frac{1}{2} \left( \frac{\partial}{\partial\eta} + n^i \frac{\partial}{\partial x^i} \right) \left( \Delta^{(2)} + 4\Phi^{(2)} \right) + n^i (\Phi^{(1)} + \Psi^{(1)})
\]

\[
+ \Psi^{(1)} \partial_i (\Delta^{(1)} + 4\Phi^{(1)}) + \left[ (\Phi^{(1)}_j + \Psi^{(1)}_j) n^i n^j - (\Phi^i + \Psi^i) \right] \frac{\partial \Delta^{(1)}}{\partial n^i}. \tag{2.6}
\]

Note that we can write \( \partial \Delta^{(1)} / \partial n^i = \left( \partial \Delta^{(1)} / \partial x^i \right) \left( \partial x^i / \partial n^i \right) = \left( \partial \Delta^{(1)} / \partial x^i \right) (\eta - \eta_i). \)

In equation (2.2), \( \delta_e^{(1)} \) is the relative energy density perturbation of the electrons. These are in turn strongly coupled with protons (p) via Coulomb interactions, such that the density contrasts and the velocities are driven to a common value \( \delta_e = \delta_p \equiv \delta_b \) and \( \mathbf{v}_e = \mathbf{v}_b \equiv \mathbf{v} \) for what can then be called the baryon fluid. Finally,

\[
\tau' = -\bar{n}_e \sigma_T a, \tag{2.7}
\]

is the differential optical depth for the Compton scatterings between photons and free electrons, where \( \sigma_T \) is the Thomson cross section, \( a \) is the scale factor, and \( \bar{n}_e \) is the mean density of free electrons. The tightly coupled limit corresponds to the Compton interaction rate much bigger than the expansion of the universe, \( \tau' / H \gg 1 \) (or \( \tau \gg 1 \)).

### 3. Linear Boltzmann equations

The first two moments of the photon Boltzmann equations are obtained by integrating equation (2.1) over \( d\Omega_n / 4\pi \) and \( d\Omega_n n^i / 4\pi \) respectively and they lead to the density and velocity continuity equations

\[
\Delta^{(1)}_{00} + \frac{4}{3} \partial_i v^{(1)i} - 4\Phi^{(1)} = 0, \tag{3.1}
\]

\[
v^{(1)i\gamma} + \frac{4}{3} \partial_i \Pi^{(1)ij} + \frac{1}{3} \Delta^{(1)i} + \Phi^{(1)i} = -\tau' \left( v^{(1)i} - v^{(1)\gamma} \right). \tag{3.2}
\]

Here we recall that \( \delta_e^{(1)} = \Delta^{(1)}_{00} = \int d\Omega \Delta^{(1)}/4\pi \) and that the photon velocity is given by [22]

\[
\frac{4}{3} v^{(1)i} = \int \frac{d\Omega}{4\pi} \Delta^{(1)n^i}. \tag{3.3}
\]

\( \Pi^{ij} \) is the quadrupole moment of the photons defined as

\[
\Pi^{ij} = \int \frac{d\Omega}{4\pi} \left( n^i n^j - \frac{1}{3} \delta^{ij} \right) \left( \Delta^{(1)} + \frac{\Delta^{(2)}}{2} \right). \tag{3.4}
\]

The two equations above are complemented by the momentum continuity equation for baryons, which can be conveniently written as

\[
v^{(1)i} = v^{(1)i\gamma} + R \frac{1}{\tau'} \left[ v^{(1)i\nu} + H v^{(1)i} + \Phi^{(1)i} \right], \tag{3.5}
\]

where we have introduced the baryon–photon ratio

\[
R = \frac{3}{4} \frac{\rho_b}{\rho_\gamma}. \tag{3.6}
\]
Equation (3.5) is in a form ready for a consistent expansion in the small quantity \( \tau^{-1} \) which can be performed in the tight coupling limit. By first taking \( v^{(1)i} = v^{(1)i}_\gamma \) at zero order and then using this relation in the left-hand side of equation (3.5), one obtains

\[
v^{(1)i} - v^{(1)i}_\gamma = \frac{R}{\tau} \left[ v^{(1)i}_\gamma + \mathcal{H} v^{(1)i}_\gamma + \Phi^{(1)i} \right].
\]  

(3.7)

Such an expression for the difference of velocities can be used in equation (3.2) to give the evolution equation for the photon velocity in the tightly coupled limit

\[
v^{(1)i}_\gamma + \mathcal{H} \frac{R}{1+R} v^{(1)i}_\gamma + \frac{1}{41+R} \Delta^{(1)i}_{00} + \Phi^{(1)i} = 0.
\]  

(3.8)

Note that in equation (3.8) we are neglecting the quadrupole of the photon distribution \( \Pi^{(1)ij} \) (and all the higher moments) since it is well known that at linear order such moment(s) are suppressed in the tight coupling limit by (successive powers of) \( 1/\tau \) with respect to the first two moments, the photon energy density and velocity. Equations (3.1) and (3.8) are the master equations which govern the photon–baryon fluid acoustic oscillations before the epoch of recombination when photons and baryons are tightly coupled by Compton scattering.

In fact, one can combine these two equations to get a single second-order differential equation for the photon energy density perturbations \( \Delta^{(1)00} \). Deriving equation (3.1) with respect to conformal time and using equation (3.8) to replace \( \partial_t v^{(1)i}_\gamma \) yields

\[
\left( \Delta^{(1)\nu}_{00} - 4 \Psi^{(1)\nu} \right) + \mathcal{H} \frac{R}{1+R} \left( \Delta^{(1)\nu}_{00} - 4 \Psi^{(1)\nu} \right) - c_s^2 \nabla^2 \left( \Delta^{(1)}_{00} - 4 \Psi^{(1)} \right) = \frac{4}{3} \nabla^2 \left( \Phi^{(1)} + \frac{\Psi^{(1)}}{1+R} \right),
\]  

(3.9)

where we have introduced the photon–baryon fluid sound of speed \( c_s = 1/\sqrt{3(1+R)} \). In fact, in order to solve equation (3.9) one needs to know the evolution of the gravitational potentials. We will come back later to the discussion of the solution of equation (3.9).

A useful relation we will use in the following is obtained by considering the continuity equation for the baryon density perturbation. By perturbing at first-order equation (6.22) of [22], we obtain

\[
\delta^{(1)}_{b\nu} + v^{i}_{\nu} - 3\Psi^{(1)\nu} = 0.
\]  

(3.10)

Subtracting equation (3.10) from equation (3.1) gives

\[
\Delta^{(1)\nu}_{00} - \frac{4}{3} \delta^{(1)}_{b\nu} + \frac{4}{3} (v^{(1)i}_\gamma - v^{(1)i}),i = 0,
\]  

(3.11)

which implies that at lowest order in the tight coupling approximation

\[
\Delta^{(1)}_{00} = \frac{4}{3} \delta^{(1)}_{b}.
\]  

(3.12)

for adiabatic perturbations.
In this section, we briefly recall how to obtain at linear order the solutions of the Boltzmann equations (3.9). These correspond to the acoustic oscillations of the photon–baryon fluid for modes which are within the horizon at the time of recombination. It is well known that, in the variable $(\Delta^{(1)}_{00} - 4\Psi^{(1)})$, the solution can be written as [28,31]

$$[1 + R(\eta)]^{1/4} (\Delta^{(1)}_{00} - 4\Psi^{(1)}) = A \cos[kr_s(\eta)] + B \sin[kr_s(\eta)]$$

$$\quad - 4 \frac{k}{\sqrt{3}} \int_0^\eta d\eta' [1 + R(\eta')]^{3/4} \left( \Phi^{(1)}(\eta') + \frac{\Psi^{(1)}(\eta')}{1 + R} \right) \sin[k(r_s(\eta) - r_s(\eta'))]$$

(3.13)

where the sound horizon is given by

$$r_s(\eta) = \int_0^\eta d\eta' c_s(\eta'),$$

(3.14)

with the ratio $R$ defined in equation (3.6). The first line of equation (3.13) corresponds to the solutions of the homogeneous equation, while the remaining integral corresponds to a particular solution of equation (3.13). The constants $A$ and $B$ must be fixed according to the initial conditions.

In order to give an analytical solution that catches most of the physics underlying equation (3.13) and which remains at the same time very simple to treat, we will make some simplifications following [32,33]. First, for simplicity, we are going to neglect the ratio $R$ wherever it appears, except in the arguments of the varying cosines and sines, where we will treat $R = R_*$ as a constant evaluated at the time of recombination. In this way, we keep track of a damping of the photon velocity amplitude with respect to the case $R = 0$ which prevents the acoustic peaks in the power spectrum from disappearing. Treating $R$ as a constant is justified by the fact that for modes within the horizon the timescale of the oscillations is much shorter than the timescale on which $R$ varies. If $R$ is a constant, the sound speed is just a constant

$$c_s = \frac{1}{\sqrt{3(1 + R_*)}},$$

(3.15)

and the sound horizon is simply $r_s(\eta) = c_s \eta$.

Second, we are going to solve for the evolutions of the perturbations in two well-distinguished limiting regimes. One regime is for those perturbations which enter the Hubble radius when matter is the dominant component, that is, at times much bigger than the equality epoch with $k \ll k_{eq} \sim \eta_{eq}^{-1}$, where $k_{eq}$ is the wavenumber of the Hubble radius at the equality epoch. The other regime is for those perturbations with much smaller wavelengths which enter the Hubble radius when the universe is still radiation dominated, that is, perturbations with wavenumbers $k \gg k_{eq} \sim \eta_{eq}^{-1}$. In fact, we are interested in perturbation modes which are within the horizon by the time of recombination $\eta_*$. Therefore, we will further suppose that $\eta_* \gg \eta_{eq}$ in order to study such modes in the first regime. Even though $\eta_* \gg \eta_{eq}$ is not the real case, it allows us to give some analytical solutions.

Before solving for these two regimes, let us fix the initial conditions which are taken on large scales deep in the radiation-dominated era (for $\eta \to 0$). During this epoch, for
adiabatic perturbations, the gravitational potentials remain constant on large scales (we are neglecting anisotropic stresses so that $\Phi^{(1)} \simeq \Psi^{(1)}$) and from the $(0-0)$-component of Einstein equations

$$\Phi^{(1)}(0) = -\frac{1}{2}\Delta^{(1)}_{00}(0).$$

(3.16)

On the other hand, from the energy continuity equation (3.1) on large scales

$$\Delta^{(1)}_{00} - 4\Psi^{(1)} = \text{const.},$$

(3.17)

from which the constant is fixed to be $\text{const.} = -6\Phi^{(1)}(0)$ and thus we find $B = 0$ and $A = -6\Phi^{(1)}(0)$.

With our simplifications, equation (3.13) then reads

$$\Delta^{(1)}_{00} = -6\Phi^{(1)}(0) \cos(\omega_0 \eta) - 8\sqrt{3} \int_0^\eta d\eta' \Phi^{(1)}(\eta') \sin[\omega_0(\eta - \eta')],$$

(3.18)

where $\omega_0 = kc_s$.

### 3.2. Perturbation modes with $k \ll k_{eq}$

This regime corresponds to perturbation modes which enter the Hubble radius when the universe is matter dominated at times $\eta \gg \eta_{eq}$. During matter domination, the gravitational potential remains constant (both on superhorizon and subhorizon scales), as one can see, for example, from equation (B.1), and its value is fixed to $\Phi^{(1)}(k, \eta) = \frac{9}{10}\Phi^{(1)}(0)$, where $\Phi^{(1)}(0)$ corresponds to the gravitational potential on large scales during the radiation-dominated epoch. Since we are interested in the photon anisotropies around the time of recombination, when matter is dominating, we can perform the integral appearing in equation (3.13) by taking the gravitational potential equal to its value during matter domination so that it is easily computed

$$2 \int_0^\eta d\eta' \Phi^{(1)}(\eta') \sin[\omega_0(\eta - \eta')] = \frac{18}{10} \frac{\Phi^{(1)}(0)}{\omega_0} (1 - \cos(\omega_0 \eta)).$$

(3.19)

Thus, equation (3.18) gives

$$\Delta^{(1)}_{00} - 4\Psi^{(1)} = \frac{9}{5} \Phi^{(1)}(0) \cos(\omega_0 \eta) - \frac{36}{5} \Phi^{(1)}(0).$$

(3.20)

The baryon–photon fluid velocity can then be obtained as $\partial_i v^{(1)i}_\gamma = -3(\Delta^{(1)}_{00} - 4\Phi^{(1)}(0)/4)$ from equation (3.1). In Fourier space

$$ik_i v^{(1)i}_\gamma = \frac{9}{10} \Phi^{(1)}(0) \sin(\omega_0 \eta)c_s,$$

(3.21)

where we use the convention $\partial_i v^{(1)i}_\gamma \rightarrow ik_i v^{(1)i}_\gamma(k)$ or equivalently

$$v^{(1)i}_\gamma = -\frac{i k_i}{k} \frac{9}{10} \Phi^{(1)}(0) \sin(\omega_0 \eta)c_s,$$

(3.22)

since the linear velocity is irrotational.
3.3. Perturbation modes with $k \gg k_{\text{eq}}$

This regime corresponds to perturbation modes which enter the Hubble radius when the universe is still radiation dominated at times $\eta \ll \eta_{\text{eq}}$. In this case, an approximate analytical solution for the evolution of the perturbations can be obtained by considering the gravitational potential for a pure radiation-dominated epoch, given by equation (B.16). For the integral in equation (3.18) we thus find

$$\int_{0}^{\eta} \Phi^{(1)}(\eta') \sin[\omega_{0}(\eta - \eta')] = -\frac{3}{2\omega_{0}} \cos(\omega_{0}\eta),$$

(3.23)

where we have kept only the dominant contribution oscillating in time, while neglecting terms which decay in time. The solution (3.18) becomes

$$\Delta_{00}^{(1)} - 4\Psi^{(1)} = 6\Phi^{(1)}(0) \cos(\omega_{0}\eta),$$

(3.24)

and the velocity is given by

$$v_{\gamma}^{(1)i} = -i\frac{k^i}{k} \Phi^{(1)}(0) \sin(\omega_{0}\eta)c_s.$$  

(3.25)

Note that the solutions (3.24) and (3.25) are in fact correct only when radiation is dominating. Indeed, between the epoch of equality and recombination, matter will start to dominate. We will account for such a period and its consequences on the CMB anisotropy evolution in a separate section showing that some corrections must be properly taken into account. However, for the time being we will continue discussing the case $k \gg k_{\text{eq}}$ just by adopting the gravitational potential for a radiation-dominated epoch, since it can be considered a first useful approximation in order to give the main quantitative features.

Before moving to the study of the tightly coupled solutions for the second-order CMB anisotropies, we want to recover the solutions (3.24) and (3.25) in an alternative way, which will be particularly useful for the second-order case. Instead of solving the second-order differential equation (3.9) for $\Delta_{00}^{(2)}$, we use directly the energy continuity equation (3.1). The reason for this is that for the case of radiation domination, the gravitational potential (B.16) at late times decays as $\eta^{-2}$, being approximated by

$$\Phi^{(1)}_k \simeq -3\Phi^{(1)}(0) \frac{\cos(\omega_{0}\eta)}{(k\eta/\sqrt{3})^2}.$$  

(3.26)

Note that in the expression for the gravitational potential (B.16), we account for the sound speed of the photon–baryon fluid, and as usual we keep it only in the argument of the sines and cosines.

On the other hand, from the $(0 - i)$-component of Einstein equation (B.17), we find that

$$v_{\gamma}^{(1)i} \simeq -\frac{1}{2H^2} \Phi^{(1)i} \equiv -i\frac{9}{2} \Psi^{(1)}(0) \frac{k^i}{k} \sin(\omega_{0}\eta)c_s,$$

(3.27)

and its divergence

$$\partial_{\eta} v_{\gamma}^{(1)i} \simeq -\frac{1}{2H^2} \partial_{\eta} \Phi^{(1)i} \equiv \frac{9}{2} \Psi^{(1)}(0) k \sin(\omega_{0}\eta)c_s,$$

(3.28)

where the second equalities are written in Fourier space and we have kept only the dominant terms at late time scaling like $\sin(\omega_{0}\eta)$. Note that we recover the same result of
potential to find the Boltzmann equations which will consist of first-order squared terms. The main points here is to compute the source term on the right-hand side of the moments of the equations at second order will have the same form as for the linear case, one of the photon–baryon fluid at second order. While we already know that the left-hand side, we use \( \partial \) to simplify equation (4.1). In the tightly coupled limit from equations (3.10) and (3.12). On the other hand, in the second-order photon energy density perturbations \( \Delta \). We integrate equation (3.29) using the late time expression (3.26) for the gravitational potential to find \( \Delta_{00}^{(1)} = 6 \Phi^{(1)}(0) \cos(\omega_0 \eta) \). The result in equation (3.30) agrees with the previous result (3.24) since the gravitational potential can be neglected at late times.

4. Second-order Boltzmann equations in the tightly coupled limit

Let us now treat, in a similar way, the photon Boltzmann equations at second order in the cosmological perturbations exploiting the regime of tight coupling between the photons and the baryons to find the governing equations for the acoustic oscillations of the photon–baryon fluid at second order. While we already know that the left-hand side of the equations at second order will have the same form as for the linear case, one of the main points here is to compute the source term on the right-hand side of the moments of the Boltzmann equations which will consist of first-order squared terms.

4.1. Energy continuity equation

Let us start by integrating equation (2.2) over \( d\Omega_n/4\pi \) to get the evolution equation for the second-order photon energy density perturbations \( \Delta_{00}^{(2)} \)

\[
\Delta_{00}^{(2)} + \frac{4}{3} \partial_i \nu_{\gamma}^{(2)i} + \frac{8}{3} \partial_i \left( \Delta_{00}^{(1)} \nu_{\gamma}^{(1)i} \right) - 4 \Psi^{(2)} \nu_{\gamma}^{(1)i} - \frac{8}{3} (\Phi^{(1)} + \Psi^{(1)}) \partial_i \nu_{\gamma}^{(1)i} \\
+ 2(\eta - \eta_i)(\Phi^{(1)} + \Psi^{(1)})_j \partial_j \Pi^{(1)ij} - \frac{4}{3} (\Phi^{(1)} + \Psi^{(1)})_i \Delta_{00}^{(1)} (\eta - \eta_i) \\
- 8 \Psi^{(1)} \Delta_{00}^{(1)} + \frac{32}{3} \Phi^{(1)}_i \nu_{\gamma}^{(1)i} = -8 \tau' \nu_{\gamma}^{(1)} (\nu^{(1)i} - \nu_{\gamma}^{(1)i}),
\]

where we have used the explicit definition for the second-order velocity of the photons [22]

\[
\frac{4 \nu_{\gamma}^{(2)i}}{3} = \frac{1}{2} \int d\Omega \frac{\Delta^{(2)}}{4\pi} n^i - \frac{4}{3} \delta^{(1)}(n^{(1)i}).
\]

We can now make use of the tight coupling expansion to simplify equation (4.1). In the left-hand side, we use \( \partial_i v_{\gamma}^{(1)j} = \partial_i v_{\gamma}^{(1)i} = 3\Psi^{(1)} - \delta^{(1)}_i = 3\Psi^{(1)} - 3\Delta^{(1)}/4 \) obtained in the tightly coupled limit from equations (3.10) and (3.12). On the other hand, in the right-hand side of equation (4.1), one can write

\[
(\nu^{(1)i} - \nu_{\gamma}^{(1)i}) = \frac{R}{\tau'} (v_{\gamma}^{(1)i} + \mathcal{H} v_{\gamma}^{(1)i} + \Phi^{(1)i}) = \frac{R}{\tau'} \left( \frac{\mathcal{H}}{1 + R} v_{\gamma}^{(1)i} - \frac{1}{4} \frac{\Delta_{00}^{(1)i}}{1 + R} \right),
\]

by using equation (3.7) and the evolution equation for the photon velocity (3.8). We thus arrive at the following equation

\[
\Delta_{00}^{(2)} + \frac{4}{3} \partial_i \nu_{\gamma}^{(2)i} - 4 \Psi^{(2)} = S_{\Delta},
\]

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where the source term is given by

\[ S_\Delta = \left( \Delta^{(1)}_{00} \right)' - 2(\Phi^{(1)} + \Psi^{(1)})(4\Phi^{(1)}' - \Delta^{(1)}_{00}') - \frac{8}{3} v^{(1)}_\gamma (\Delta^{(1)}_{00} + 4\Phi^{(1)})i + \frac{4}{3}(\eta - \eta_i)(\Phi^{(1)} + \Psi^{(1)})^i \Delta^{(1)}_{00,i} - \frac{8}{3} R \left( \frac{\mathcal{H}}{1 + R} v^{(1)}_\gamma - \frac{1}{4} \frac{v^{(1)}_\gamma \Delta^{(1)}_{00,i}}{1 + R} \right). \]  

(4.5)

### 4.2. Velocity continuity equation

We now derive the second moment of the Boltzmann equation (2.2) and then we take its tight coupling limit. The integration of equation (2.2) over \( d\Omega_i \) yields the continuity equation for the photon velocity

\[ \frac{4}{3} v^{(2)i} + \frac{1}{2} \partial_j \Pi^{(2)ji} + \frac{1}{3} \Delta^{(2)i}_{00} + \frac{2}{3} \Phi^{(2),i} + \frac{4}{3} \omega^i = - \frac{4}{3} \left( \Delta^{(1)i}_{00} \right)' + \frac{16}{3} \tilde{\Psi}^{(1)i} \tilde{v}^{(1)i} - 4 \Phi^{(1),i} \Pi^{(1)ji} - \frac{4}{3} \Phi^{(1),i} \Delta^{(1)i}_{00} - \frac{4}{3} \Phi^{(1),i} (\Phi^{(1)} + \Psi^{(1)}) - (\Phi^{(1)} + \Psi^{(1)}) \delta j v^{(1)i} \]

\[ -(\eta - \eta_i)(\Phi^{(1)} + \Psi^{(1)}) \delta j \int \frac{d\Omega}{4\pi} \left( \eta^i n^k - \frac{1}{3} \delta^i k \right) \Delta^{(1)n} - \frac{\tau'}{2} \left( \frac{4}{3} v^{(2)i} - \tilde{v}^{(2)i} \right) + \frac{8}{3} (\gamma^{(1)i} + \Phi^{(1)} + \Delta^{(1)}_{00}) (v^{(1)i} - \tilde{v}^{(1)i}) + 2 \delta^{(1)i} \Pi^{(1)ji} \right]. 

(4.6)

The difference between the second-order baryon and photon velocities \( (v^{(2)i} - \tilde{v}^{(2)i}) \) appearing in equation (4.6) is obtained from the baryon continuity equation which can be written as (see [22])

\[ v^{(2)i} = v^{(2)i}_\gamma + \frac{R}{\tau'} [(v^{(2)i}) + \mathcal{H} v^{(2)i} + 2\omega^i + 2\mathcal{H} \omega^i + \Phi^{(2),i}] - 2 \Phi^{(1),i} v^{(1)i} + \partial_i v^{(1)2} \]

\[ + 2 \Phi^{(1),i} (\Phi^{(1)} + \Psi^{(1)}) - \frac{3}{2} v^{(1)i} \Pi^{(1)ji} - 2(\Delta^{(1)i}_{00} + \Phi^{(1)}) (v^{(1)i} - \tilde{v}^{(1)i}) \]  

(4.7)

We want now to reduce equation (4.6) in the tightly coupled limit. We first insert expression (4.7) into equation (4.6). Note that the last three terms in equation (4.7) will cancel out. On the other hand, in the tight coupling limit expansion, one can set \( v^{(1)i}_\gamma = v^{(1)i}_\gamma \) and \( v^{(2)i} = v^{(2)i}_\gamma \) in the remaining terms on the right-hand side of equation (4.7).

Thus, equation (4.6) becomes

\[ (v^{(2)i}_\gamma + 2\omega^i)' + \mathcal{H} \frac{R}{1 + R} (v^{(2)i}_\gamma + 2\omega^i) + \frac{1}{4} \frac{\Delta^{(2)i}_{00}}{1 + R} + \Phi^{(2),i} = - \frac{3}{4(1 + R)} \partial_j \Pi^{(2)ji} - \frac{2}{1 + R} \left( \Delta^{(1)i}_{00} \right)' + \frac{8}{1 + R} \tilde{\Psi}^{(1)i} \tilde{v}^{(1)i} - \frac{2}{1 + R} \Phi^{(1),i} \Delta^{(1)i}_{00} \]

\[ - \frac{2}{1 + R} \Phi^{(1),i} (\Phi^{(1)} + \Psi^{(1)}) + \frac{4}{3(1 + R)} (\eta - \eta_i)(\Phi^{(1)} + \Psi^{(1)}) \delta j v^{(1)i} \]

\[ + 2 \frac{R}{1 + R} \tilde{\Psi}^{(1)i} \tilde{v}^{(1)i} - \frac{1}{2(1 + R)} (\Phi^{(1)} + \Psi^{(1)}) \Delta^{(1)i}_{00} - \frac{R}{1 + R} \delta^i b \]

\[ - \frac{2}{1 + R} \left( \Phi^{(1)} + \Psi^{(1)} \right) \delta^i b (v^{(1)i} - \tilde{v}^{(1)i}) \]  

(4.8)
where in the tightly coupled limit we are neglecting the first-order quadrupole (and higher-order moments) of the photon distribution since it is suppressed by $1/\tau$ with respect to the other terms. Next, for the term such as $\tau^i\Phi^j_s(\nu^{(1)i} - \nu^{(1)j}_\gamma)$, we use the relation previously derived in equation (4.3) with $\delta^{(1)}_b = 3\Delta^{(1)}_{00}/4$ and we use the first-order tight coupling equations (3.1) and (3.8) in order to further simplify equation (4.8). We finally obtain

$$
\nu^{(2)ij} + \mathcal{H} \frac{R}{1 + R} \nu^{(2)ij} + \frac{1}{4(1 + R)} + \Phi^{(2)i} = S^i_V,
$$

(4.9)

where

$$S^i_V = -\frac{3}{4(1 + R)} \partial_j \Pi^{(2)ji}_\gamma - 2\omega^i - 2\mathcal{H} \frac{R}{1 + R} \omega^i + 2 \frac{\mathcal{H} R}{(1 + R)^2} \Delta^{(1)i}_0 \nu^{(1)i}_\gamma + \frac{1}{4(1 + R)^2} \left( \Delta^{(1)2}_0 \right)^i
$$

$$+ \frac{8}{3(1 + R)} \Pi^{(1)i}_\gamma \partial_j \nu^{(1)ij}_\gamma + 2 \frac{R}{1 + R} \Psi^{(1)i} \nu^{(1)i} - 2(\Phi^{(1)} + \Psi^{(1)}) \Phi^{(1)i} + \frac{1}{2(1 + R)} (\Phi^{(1)} + \Psi^{(1)}) \Delta^{(1)i}_0
$$

$$- \frac{4}{3(1 + R)} (\eta - \eta_i)(\Phi^{(1)} + \Psi^{(1)}) \nu^{(1)i} \partial_j \nu^{(1)ij}$$

$$- \frac{R}{1 + R} \nu^{(2)i} + \frac{3}{21 + R} \Delta^{(1)i}_0 \left( \frac{\mathcal{H} R}{1 + R} \nu^{(1)i}_\gamma - \frac{1}{41 + R} \right) .
$$

(4.10)

We have spent some time giving the details of the computation for the photon Boltzmann equations at second order in the perturbations. As a summary of the results obtained so far, we refer the reader to equations (3.1)–(3.8) and equations (4.4)–(4.9) as our master equations which we will solve in the next sections. In particular, equation (4.9) is the second-order counterpart of equation (3.8) for the photon velocity in the tight coupling regime. Note that there are two important differences with respect to the linear case. One is that, in equation (4.9), there will be a contribution not only from scalar perturbations but also from vector modes which, at second order, are inevitably generated as non-linear combinations of first-order scalar perturbations. In particular, we have included the vector metric perturbations $\omega^i$ in the source term. Second, and most important, we have also kept in the source term the second-order quadrupole of the photon distribution $\Pi^{(2)ij}_\gamma$. At linear order, we can neglect it together with higher-order moments of the photons since they turn out to be suppressed with respect to the first two moments in the tight coupling limit by increasing powers of $1/\tau$. However, in the next section, we will show that at second order this does not hold anymore, as the photon quadrupole is no longer suppressed.

Finally, following the same steps that lead to equation (3.9) at linear order, we can derive a similar equation for the second-order photon energy density perturbation $\Delta^{(2)}_{00}$ which now will be characterized by the source terms $S_\Delta$ and $S^i_V$: 

$$
\left( \Delta^{(2)\nu}_0 - 4\Psi^{(2)\nu} \right) + \mathcal{H} \frac{R}{1 + R} \left( \Delta^{(2)\nu}_0 - 4\Psi^{(2)\nu} \right) - \gamma^2 \nabla^2 \left( \Delta^{(2)}_{00} - 4\Psi^{(2)} \right)
$$

$$= \frac{4}{3} \nabla^2 \left( \Phi^{(2)} + \Psi^{(2)} \right) + S^i_V + \mathcal{H} \frac{R}{1 + R} S_\Delta - \frac{4}{3} \partial_i S^i_V .
$$

(4.11)
4.3. Second-order quadrupole moment of the photons in the tight coupling limit

Let us now consider the quadrupole moment of the photon distribution defined in equation (3.4) and show that at second order it cannot be neglected in the tightly coupled limit, unlike for the linear case. We first integrate the right-hand side of equation (2.2) over $d\Omega_n (\ell n^j - \delta^{ij}/3)/4\pi$ and then we set it to be vanishing in the limit of tight coupling.

The integration involves computation of various pieces. For clarity, we will consider each of them separately. The term $\Delta^{(2)}_{(00)}$ does not contribute. For the third term, we can write, from equation (2.4)

$$-\frac{1}{2} \sum_{m=-2}^{m=2} \frac{\sqrt{4\pi}}{3^{3/2}} \Delta^{(2)}_{2m} Y_{2m} = \frac{\Delta^{(2)}}{10} - \frac{1}{10} \sum_{\ell \neq 2} \sum_{m=-\ell}^{\ell} \Delta^{(2)}_{\ell m} (-1)^\ell \sqrt{\frac{4\pi}{2\ell + 1}} Y_{\ell m}, \quad (4.12)$$

so that the integral just brings $\Pi^{(2)ij}_{(0)} / 10$, since the only contribution in equation (4.12) comes from $\Delta^{(2)} / 10$ with all the other terms vanishing. The following nontrivial integral is

$$\int \frac{d\Omega}{4\pi} \left( n^i n^j - \frac{1}{3} \delta^{ij} \right) \Delta^{(1)}_2 P_2 (\hat{v} \cdot \mathbf{n}) = \hat{v}_k \hat{v}_l \int \frac{d\Omega}{4\pi} \left( n^i n^j - \frac{1}{3} \delta^{ij} \right) \Delta^{(1)}_2 \left( \frac{3}{2} n^k n^l - \frac{1}{2} \right) \Delta^{(1)}_2 = \Delta^{(1)}_2 \left( \hat{v}^i \hat{v}^j - \frac{1}{3} \delta^{ij} \right), \quad (4.13)$$

where the baryon velocity appearing in $P_2 (\hat{v} \cdot \mathbf{n})$ is first order and we make use of the following relations

$$\int d\Omega \ n^i = \int d\Omega n^i n^j n^k = 0, \quad \int \frac{d\Omega}{4\pi} n^i n^j = \frac{1}{3} \delta^{ij} ;$$

$$\int \frac{d\Omega}{4\pi} n^i n^j n^k n^l = \frac{1}{15} (\delta^{ij} \delta^{kl} + \delta^{ik} \delta^{lj} + \delta^{il} \delta^{kj}). \quad (4.14)$$

The integrals of $\delta^{(1)}_e \Delta^{(1)}_0$, $\Delta^{(1)}_e (\mathbf{v} \cdot \mathbf{n})$ and $\mathbf{v}^{(2)} \cdot \mathbf{n}$ vanish and

$$\nu \Delta^{(1)}_1 \int \frac{d\Omega}{4\pi} P_2 (\hat{v} \cdot \mathbf{n}) \left( n^i n^j - \frac{1}{3} \delta^{ij} \right) = \frac{1}{5} \nu \Delta^{(1)}_1 \left( \hat{v}^i \hat{v}^j - \frac{1}{3} \delta^{ij} \right) = \frac{4}{15} \left( \nu^i v^j - \frac{1}{3} \delta^{ij} v^2 \right), \quad (4.15)$$

where in the last step, we take $\Delta^{(1)}_1 = 4\nu/3$ in the tight coupling limit. Similarly, the integral of $14 (\mathbf{v} \cdot \mathbf{n})^2$ brings

$$14 \nu^k v^\ell \int \frac{d\Omega}{4\pi} n_k n_\ell \left( n^i n^j - \frac{1}{3} \delta^{ij} \right) = \frac{28}{15} \left( \nu^i v^j - \frac{1}{3} \delta^{ij} v^2 \right). \quad (4.16)$$

The integral of $2 (\mathbf{v} \cdot \mathbf{n}) \Delta^{(1)}$ can be performed by expanding the linear anisotropies as $\Delta^{(1)} = \sum_\ell (2\ell + 1) \Delta^{(1)}_\ell P_\ell (\hat{v} \cdot \mathbf{n})$. We thus find

$$2 \nu^k \hat{v}^m \int \frac{d\Omega}{4\pi} n_k \left( n^i n^j - \frac{1}{3} \delta^{ij} \right) n_m \Delta^{(1)}_\ell + O (\ell > 2) = \frac{16}{15} \left( \nu^i v^j - \frac{1}{3} \delta^{ij} v^2 \right), \quad (4.17)$$

where we have used equation (4.14) and $O (\ell > 2)$ indicates all of the integrals coming from the multipoles $\ell > 2$ in the expansion (for $\ell = 0$ and 2 they vanish.) In fact, we have
dropped the $O(\ell > 2)$ since they are proportional to first-order photon moments $\ell > 2$ which turn out to be suppressed in the tight coupling limit. Finally, the term proportional to $(\mathbf{v} \cdot \mathbf{n})\Delta_2^{(1)}(1 - P_2(\mathbf{v} \cdot \mathbf{n})/5$ gives a vanishing contribution.

Collecting all of the various pieces, we find that the third moment of the right-hand side of equation (2.2) is given by

$$-rac{\tau'}{2} \left[ -\Pi_{\gamma}^{(2)ij} + \frac{1}{10} \Pi_{\gamma}^{(2)ij} + 2\delta_{e}^{1}\left(-\Pi_{\gamma}^{(1)ij} + \frac{1}{10} \Delta_2^{(1)} \left(\hat{v}^i\hat{v}^j - \frac{1}{3} \delta^{ij}\right)\right) + \frac{12}{5} \left(\hat{v}^i\hat{v}^j - \frac{1}{3} \delta^{ij}v^2\right)\right].$$

(4.18)

Therefore, in the limit of tight coupling, when the interaction rate is very high, the second-order quadrupole moment is given by

$$\Pi_{\gamma}^{(2)ij} \simeq \frac{8}{3} \left(\hat{v}^i\hat{v}^j - \frac{1}{3} \delta^{ij}v^2\right),$$

(4.19)

by setting equation (4.18) to be vanishing (the term multiplying $\delta_{e}^{(1)}$ goes to zero in the tight coupling limit since it just comes from the first-order collision term). At linear order, one would simply get the term $9\tau'\Pi_{\gamma}^{(1)ij}/10$ implying that, in the limit of a high scattering rate $\tau'$, $\Pi_{\gamma}^{(1)ij}$ goes to zero. However, at second order, the quadrupole is not suppressed in the tight coupling limit because it turns out to be sourced by the linear velocity squared.

### 5. Second-order CMB anisotropies generated at recombination

The previous equations allow us to follow the evolution of the monopole and dipole of CMB photons at recombination. As at linear order, they will appear in the expression for the CMB anisotropies today $\Delta^{(2)}(k, \mathbf{n}, \eta_0)$ together with various integrated effects. Our focus now will be to obtain an expression for the second-order CMB anisotropies today $\Delta^{(2)}(k, \mathbf{n}, \eta_0)$ from which we can extract all of those contributions generated specifically at recombination due to the non-linear dynamics of the photon–baryon fluid. This expression will not only relate the moments $\Delta_{em}^{(2)}$ today to the second-order monopole and dipole at recombination as it happens at linear order, but one has to properly account also for additional first-order squared contributions. Let us see how to achieve this goal in some details.

As in linear theory (see e.g. [33, 34]), it is possible to write down an integral solution of the photon Boltzmann equation (2.2) in Fourier space. Following the standard procedure for linear perturbations, we write

$$\Delta^{(2)\nu} + ik\mu\Delta^{(2)} = e^{-ik\mu+\tau} \frac{d}{d\eta} \left[i\Delta^{(2)} e^{ik\mu\eta-\tau}\right] = S(k, \mathbf{n}, \eta)$$

(5.1)

in order to derive a solution of the form

$$\Delta^{(2)}(k, \mathbf{n}, \eta_0) = \int_0^{\eta_0} d\eta S(k, \mathbf{n}, \eta)e^{ik\mu(\eta-\eta_0)}e^{-\tau}.$$  

(5.2)

Here $\mu = \cos \vartheta = \hat{k} \cdot \mathbf{n}$ is the polar angle of the photon momentum in a coordinate system such that $\mathbf{e}_3 = \hat{k}$. At second order, the source term has been computed in [22] and can
be read off equations (2.2) and (2.6) to be
\[
S = -\tau'\Delta_{00}^{(2)} - 4n^i\Phi_{,j}^{(2)} + 4\Psi^{(2)} - 8\Delta^{(1)}(\Psi^{(1)} - n^i\Phi_{,i}^{(1)}) - 2n^i(\Phi^{(1)} + \Psi^{(1)})(\Delta^{(1)} + 4\Phi^{(1)})_j
\]
\[= -2\left[(\Phi^{(1)} + \Psi^{(1)})_j n^i n^j - (\Phi^{(1)} + \Psi^{(1)})^t\right] \frac{\partial \Delta^{(1)}}{\partial n^t} - 8\omega^t n^t - 4\chi_{ij} n^i n^j
\]
\[= \tau'\left[-\frac{1}{2} \sum_{m=2}^{2} \frac{\sqrt{4\pi}}{\sqrt{5/2}} \Delta_{2m}^{(2)} Y_{2m}(n) + 2\delta^{(1)}(n) \Delta^{(1)} - \Delta^{(1)}
\]
\[+ 4v \cdot n + \frac{1}{2} \Delta_{0}^{(1)} P_2(\hat{\mathbf{v}} \cdot \mathbf{n})) + 4\Psi^{(2)} \cdot \mathbf{n}
\]
\[+ 2(\mathbf{v} \cdot \mathbf{n}) \left[\Delta^{(1)} + 3\Delta_0^{(1) - \Delta_2^{(1)}} (1 - \frac{2}{\tau} P_2(\hat{\mathbf{v}} \cdot \mathbf{n}))\right]
\]
\[- v\Delta_1^{(1)} (4 + 2P_2(\hat{\mathbf{v}} \cdot \mathbf{n})) + 14(\mathbf{v} \cdot \mathbf{n})^2 - 2v^2 \right].
\] (5.3)

The key point here is to isolate all of those terms that multiply the differential optical depth \(\tau'\). The reason for this is that in this case, in the integral (5.2), one recognizes the visibility function \(g(\eta) = -e^{-\tau'\tau}\) which is sharply peaked at the time of recombination and whose integral over time is normalized to unity. Thus, for these terms the integral just reduces to the remaining integrand (apart from the visibility function) evaluated at recombination. The standard example that one encounters also at linear order is given by the first term appearing in the source \(S\), equation (5.3), that is \(-\tau'\Delta_{00}^{(2)}\). The contribution of this term to the integral (5.2) just reduces to
\[
\Delta^{(2)}(\mathbf{k}, \mathbf{n}, \eta_0) = \int_0^{\eta_0} \frac{d\eta}{e^{\int_k^{\mathbf{i}(\eta_i, -\eta_i)} e^{-\tau'}(\eta')\Delta_{00}^{(2)}(\eta') \Delta_{00}^{(2)}(\eta)},
\] (5.4)

where \(\eta_0\) is the epoch of recombination and, in the multipole decomposition (2.5), equation (5.4) brings the standard result
\[
\Delta^{(2)}(\eta_0) \propto \Delta^{(2)}(\eta_0) j\ell(k(\eta_0 - \eta_0)),
\] (5.5)

having used the Legendre expansion \(e^{ikx} = \sum_{\ell}(i)_{\ell}(2\ell + 1)j\ell(kx) P_\ell(\hat{\mathbf{k}} \cdot \hat{\mathbf{x}})\). In equation (5.5), the monopole at recombination is found by solving the Boltzmann equations equations (4.4)–(4.9) derived in the tight coupling limit.

Looking at equation (5.3), we recognize immediately the terms which multiply explicitly \(\tau'\) (the first one discussed in the example above and the last four lines of equation (5.3)). However, it is easy to realize from the standard procedure adopted at the linear order that such terms are not the only ones. This is clear by focusing, as an example, on the term \(-4n^i\Phi_{,i}^{(1)}\) in the source \(S\) which appears in the same form also at linear order. In Fourier space, one can replace the angle \(\mu\) with a time derivative and thus this term gives rise to [33, 34]
\[
-4ik \int_0^{\eta_0} \frac{d\eta}{e^{\int_k^{\mathbf{i}(\eta_i, -\eta_i)} e^{-\tau'}\mu\Phi^{(2)}} = -4 \int_0^{\eta_0} \frac{d\eta}{e^{\int_k^{\mathbf{i}(\eta_i, -\eta_i)}} (e^{ik\mu(\eta - \eta_0)})
\]
\[= 4 \int_0^{\eta_0} \frac{d\eta}{e^{ik\mu(\eta - \eta_0)}} (\Phi^{(2)}) - \tau'\Phi^{(2)}
\] (5.6)

where, in the last step, we have integrated by parts. In equation (5.6), the time derivative of the gravitational potential contributes to the integrated Sachs–Wolfe effect, but also
a \tau' results implying that we also have to evaluate \Phi^{(2)} at recombination. Thus, in the following, we look for those terms in the source (5.3) which give rise to a \tau' factor in the same way as for \(-4n^i\Phi^{(2)}_i\). In particular, let us consider the combination in equation (5.3)

\[ C \equiv 8\Delta^{(1)}(\Psi^{(1)} - n^i\Phi^{(1)}_i) - 2n^i(\Phi^{(1)} + \Psi^{(1)})(\Delta^{(1)} + 4\Phi^{(1)})_i \]

\[ = 8\Delta^{(1)}(\Psi^{(1)} - 8n^i(\Delta^{(1)}\Phi^{(1)})_i + 4\Phi^{(1)}n^i\Delta^{(1)} - 8n^i(\Phi^{(1)})^2)_i, \]

(5.7)

where for simplicity we are setting \Phi^{(1)} \simeq \Psi^{(1)}. We already recognize terms of the form \(n^i\partial_i(\cdot)\). Moreover, we can use the Boltzmann equation (2.1) to replace \(n^i\Delta^{(1)}\) in equation (5.7). This gives

\[ C = 8\Delta^{(1)}\Psi^{(1)} - 4\Phi^{(1)}\Delta^{(1)} - 16\Psi^{(1)}\Psi^{(1)} - 8n^i(\Delta^{(1)}\Phi^{(1)})_i - 16n^i(\Phi^{(1)})^2_i \]

\[ - 4\tau'\Psi^{(1)}_1 \left[ \Delta^{(1)}_0 - \Delta^{(1)} + 4\mathbf{v} \cdot \mathbf{n} + \frac{1}{2}\Delta^{(2)}_2 P_2(\mathbf{v} \cdot \mathbf{n}) \right]. \]

(5.8)

In fact, we will not be interested for our purposes in the first three terms of equation (5.8), since they will not contribute to the anisotropies generated at recombination.

Therefore, as a result of equations (5.3), (5.6) and (5.8), we can rewrite the source term (5.3) as

\[ S = S_s + S' \]

(5.9)

where

\[ S_s = -\tau' \left[ \Delta^{(2)}_{00} + 4\Phi^{(2)} - \frac{1}{2} \sum_{m=2}^2 \frac{\sqrt{4\pi}}{5^{5/2}} \Delta^{(2)}_{2m} Y_{2m}(\mathbf{n}) \right. \]

\[ + 2\delta_n^{(1)} (\Delta^{(1)}_0 - \Delta^{(1)} + 4\mathbf{v} \cdot \mathbf{n} + \frac{1}{2}\Delta^{(2)}_2 P_2(\mathbf{v} \cdot \mathbf{n}) + 4\mathbf{v}^{(2)} \cdot \mathbf{n} \]

\[ + 2(\mathbf{v} \cdot \mathbf{n}) (\Delta^{(1)} + 3\Delta^{(1)}_0 - \Delta^{(1)}_2 (1 - \frac{1}{2}P_2(\mathbf{v} \cdot \mathbf{n})) \]

\[ - v\Delta^{(1)}_1 (4 + 2P_2(\mathbf{v} \cdot \mathbf{n})) + 14(\mathbf{v} \cdot \mathbf{n})^2 - 2v^2 + 8\Delta^{(1)} \Phi^{(1)} \]

\[ + 16\Phi^{(1)} + 4\Psi^{(1)} \left[ \Delta^{(1)}_0 - \Delta^{(1)} + 4\mathbf{v} \cdot \mathbf{n} + \frac{1}{2}\Delta^{(2)}_2 P_2(\mathbf{v} \cdot \mathbf{n}) \right] \]

(5.10)

and

\[ S' = 4(\Phi^{(2)} + \Psi^{(2)})' - 8\omega' n^i - 4\chi'_{ij} n^i n^j - 2 \left[ (\Phi^{(1)} + \Psi^{(1)})_j n^i n^j - (\Phi^{(1)} + \Psi^{(1)})^i \right] \frac{\partial \Delta^{(1)}}{\partial n^i} \]

\[ + 8(\Delta^{(1)} \Phi^{(1)})' + 8\Delta^{(1)} \Psi^{(1)} - 4\Psi^{(1)} \Delta^{(1)} + 16\Psi^{(1)} \Psi^{(1)}. \]

(5.11)

In equation (5.9), \(S_s\) contains the contribution to the second-order CMB anisotropies created on the last scattering surface at recombination, while \(S'\) includes all of those effects which are integrated in time from the last scattering surface up to now, including the second-order integrated Sachs–Wolfe effect and the second-order lensing effect. Since the main concern of this paper is the CMB anisotropies generated at last scattering, from now on we will focus only on the contribution from the last scattering surface \(S_s\).
5.1. Multipole moment decomposition

The expression of the photon moments $\Delta_{\ell m}^{(2)}$ can be obtained from equation (2.5). Such a decomposition can be achieved by first expanding the source term $S$ as

\[
S(k, n, \eta) = \sum_{\ell} \sum_{m=-\ell}^{\ell} S_{\ell m}(k, \eta)(-i)^{\ell} \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell m}(n),
\]

and then taking into account the additional angular dependence in the exponential of equation (5.2) by recalling that

\[
e^{i k \cdot \mathbf{x}} = \sum_{\ell} (i)\ell (2\ell + 1) j_{\ell}(k x) P_{\ell}(k \cdot \hat{x}).
\]

Thus, the angular integral (2.5) just reduces to compute the expansion coefficients of the source term

\[
\Delta_{\ell m}^{(2)}(k, \eta_0) = (-1)^{-m}(i)^{-\ell}(2\ell + 1) \int_{0}^{\eta_0} d\eta \, e^{-\tau(\eta)}
\]

\[
\times \sum_{\ell_2} \sum_{m_2=0}^{\ell_2} (-i)^{\ell_2} S_{\ell_2 m_2}(k) \sum_{\ell'} j_{\ell_1}(k(\eta - \eta_0))
\]

\[
\times \left( \begin{array}{ccc} \ell_1 & \ell_2 & \ell \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} \ell_1 & \ell_2 & \ell \\ 0 & m_2 & -m \end{array} \right),
\]

where the Wigner 3- $j$ symbols appear because of the Gaunt integrals

\[
G_{\ell_1 \ell_2 \ell_3}^{m_1 m_2 m_3} = \sqrt{\frac{(2\ell_1 + 1)(2\ell_2 + 1)(2\ell_3 + 1)}{4\pi}} \left( \begin{array}{ccc} \ell_1 & \ell_2 & \ell \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} \ell_1 & \ell_2 & \ell \\ m_1 & m_2 & m_3 \end{array} \right).
\]

The observed anisotropies generated at the last scattering surface come from the source term $S_\star$ containing a $-\tau^2$ factor: this allows us to solve the time integral in equation (5.14) by evaluating the integrand at $\eta = \eta_0$, given that the visibility function $g(\eta) = -\tau^2 e^{-\tau}$ is peaked at the time of recombination.

6. Tightly coupled solutions for the second-order perturbations

In this section, we will solve the tightly coupled limit of the Boltzmann equations (4.4) and (4.6) at second order in perturbation theory. We will proceed as for the linear case, focusing on the two limiting cases of perturbation modes entering the horizon respectively much before and much after the time of equality. The solution of equation (4.11) can be written as

\[
[1 + R(\eta)]^{1/4}(\Delta_{00}^{(2)} - 4\Psi^{(2)}) = A \cos[kr_s(\eta)] + B \sin[kr_s(\eta)]
\]

\[
- \frac{4}{\sqrt{3}} \int_{0}^{\eta} d\eta' [1 + R(\eta')]^{3/4} \left( \Phi^{(2)}(\eta') + \frac{\Psi^{(2)}(\eta')}{1 + R} \right) \sin[k(r_s(\eta) - r_s(\eta'))]
\]

\[
+ \frac{\sqrt{3}}{k} \int_{0}^{\eta} d\eta' [1 + R(\eta')]^{3/4} \left( S_\Delta + \frac{\mathcal{H} R}{1 + R} S_\Delta - \frac{4}{3} k_i S_i \right) \sin[k(r_s(\eta) - r_s(\eta'))],
\]

(6.1)
where the source terms are given in equation (4.5) and (4.10). Note that we can write $S_{\Delta} + (HR/(1 + R))S_{\Delta} = (S_{\Delta}(1 + R))^\prime/1 + R$ so that we can perform an integration by parts in equation (6.1) leading to

$$
[1 + R(\eta)]^{1/4}(\Delta_{00}^{(2)} - 4\Psi^{(2)}) = A\cos[kr_s(\eta)] + B\sin[kr_s(\eta)] - \frac{\sqrt{3}}{k}S_{\Delta}(0)\sin[kr_s(\eta)]
$$

$$- 4\frac{k}{\sqrt{3}}\int_0^\eta d\eta' \left[1 + R(\eta')\right]^{3/4} \left(\Phi^{(2)}(\eta') + \frac{\Psi^{(2)}(\eta')}{1 + R}\right) \sin[k(r_s(\eta) - r_s(\eta'))]
$$

$$+ \int_0^\eta d\eta' S_{\Delta}(\eta') (1 + R(\eta'))^{1/4} \cos[k(r_s(\eta) - r_s(\eta'))]
$$

$$- 4\frac{ik_s}{\sqrt{3}}\int_0^\eta d\eta' S_{\Delta}^\prime(\eta') (1 + R(\eta'))^{3/4} \sin[k(r_s(\eta) - r_s(\eta'))]
$$

$$+ \frac{\sqrt{3}}{4k}\int_0^\eta d\eta' S_{\Delta}(\eta') (1 + R(\eta'))^{-1/4} R(\eta') \sin[k(r_s(\eta) - r_s(\eta'))].
$$

(6.2)

6.1. Setting the initial conditions: primordial non-Gaussianity

The integration constants $A$ and $B$ are fixed according to the initial conditions for the second-order cosmological perturbations. These refer to the values of the perturbations on superhorizon scales deep in the radiation-dominated period. We will consider the case of initial adiabatic perturbations, for which there exist some useful conserved quantities on large scales which as such directly carry the information about the initial conditions.

In the standard single-field inflationary model, the first seeds of density fluctuations are generated on superhorizon scales from the fluctuations of a scalar field, the inflaton [1]. Recently many other scenarios have been proposed as alternative mechanisms to generate such primordial seeds. These include, for example, the curvaton [35] and the inhomogeneous reheating scenarios [10], where essentially the first density fluctuations are produced through the fluctuations of a scalar field other than the inflaton. In order to follow the evolution on superhorizon scales of the density fluctuations coming from the various mechanisms, we use the curvature perturbation of uniform density hypersurfaces $\zeta = \zeta^{(1)} + \zeta^{(2)}/2 + \cdots$, where $\zeta^{(1)} = -\Psi^{(1)} - \mathcal{H}\delta^{(1)}/\rho^\prime$ and the expression for $\zeta^{(2)}$ is given by [36]

$$
\zeta^{(2)} = -\Psi^{(2)} - \mathcal{H}\frac{\delta^{(2)}\rho}{\rho^\prime} + \Delta\zeta^{(2)},
$$

(6.3)

with

$$
\Delta\zeta^{(2)} = 2\mathcal{H}\frac{\delta^{(1)\prime}\rho^\prime}{\rho^\prime} + 2\frac{\delta^{(1)\prime}\rho}{\rho^\prime}\left(\Psi^{(1)\prime} + 2\mathcal{H}\Psi^{(1)}\right)
$$

$$- \left(\frac{\delta^{(1)\prime}\rho}{\rho^\prime}\right)^2\left(\frac{\mathcal{H}\rho^\prime}{\rho^\prime} - \mathcal{H}^\prime - 2\mathcal{H}^2\right) + 2\Psi^{(1)2}.
$$

(6.4)

The crucial point is that the gauge-invariant curvature perturbation $\zeta$ remains constant on superhorizon scales after it has been generated during a primordial epoch and possible isocurvature perturbations are no longer present. Therefore, we may set the initial conditions at the time when $\zeta$ becomes constant. In particular, $\zeta^{(2)}$ provides the
necessary information about the ‘primordial’ level of non-Gaussianity generated either during inflation, as in the standard scenario, or immediately after it, as in the curvaton scenario. Different scenarios are characterized by different values of $\zeta^{(2)}$. For example, in the standard single-field inflationary model $\zeta^{(2)} = 2(\zeta^{(1)})^2 + \mathcal{O}(\epsilon, \eta)$ [5, 37], where $\epsilon$ and $\eta$ are the standard slow-roll parameters [1]. In general, we may parametrize the primordial non-Gaussianity level in terms of the conserved curvature perturbation as in [38]

$$\zeta^{(2)} = 2a_{NL} (\zeta^{(1)})^2,$$

(6.5)

where the parameter $a_{NL}$ depends on the physics of a given scenario. For example, in the standard scenario $a_{NL} \approx 1$, while in the curvaton case $a_{NL} = (3/4r) - r/2$, where $r \approx (\rho_{\gamma}/\rho)_{D}$ is the relative curvaton contribution to the total energy density at curvaton decay [4, 8]. In the minimal picture for the inhomogeneous reheating scenario, $a_{NL} = 1/4$. For other scenarios, we refer the reader to reference [4]. One of the best tools to detect or constrain the primordial large-scale non-Gaussianity is through the analysis of the CMB anisotropies, for example, by studying the bispectrum [4]. In that case, the standard procedure is to introduce the non-linearity parameter $f_{NL}$ characterizing non-Gaussianity in the large-scale temperature anisotropies [4, 26, 30]. To give the feeling of the resulting size of $f_{NL}$ when $|a_{NL}| \gg 1$, $f_{NL} \approx 5a_{NL}/3$ (see [4, 38]).

The conserved value of the curvature perturbation $\zeta$ allows us to set the initial conditions for the metric and matter perturbations accounting for the primordial contributions. At linear order during the radiation-dominated epoch and on large scales $\zeta^{(1)} = -2\Psi^{(1)}/3$. On the other hand, after some calculations, one can easily compute $\Delta \zeta^{(2)}$ for a radiation-dominated epoch

$$\Delta \zeta^{(2)} = \frac{7}{2}(\Psi^{(1)})^2,$$

(6.6)

where in equation (6.4), one uses that on large scales $\delta^{(1)}_{\rho/\rho_{\gamma}} = -2\Psi^{(1)}$ and the energy continuity equation $\delta^{(1)}_{\rho/\rho_{\gamma}} + 4\mathcal{H}\delta^{(1)}_{\rho_{\gamma}} - 4\Psi^{(1)\prime}\rho_{\gamma} = 0$. Therefore, we find

$$\zeta^{(2)} = -\Psi^{(2)} + \frac{\Delta_{00}^{(2)}}{4} + \frac{7}{2}\Psi^{(1)2}(0),$$

(6.7)

where we are evaluating the quantities in the large-scale limit for $\eta \to 0$. Using the parametrization (6.5) at the initial times, the quantity $\Delta_{00}^{(2)} - 4\Psi^{(2)}$ is given by

$$\Delta_{00}^{(2)} - 4\Psi^{(2)} = 2(9a_{NL} - 7)\Psi^{(1)2}(0).$$

(6.8)

Since for adiabatic perturbations such a quantity is conserved on superhorizon scales, it follows that the constant $B = 0$ and $A = 2(9a_{NL} - 7)\Psi^{(1)2}(0)$.

Equations (6.1) and (6.2) are analytical expressions describing the acoustic oscillations of the photon–baryon fluid induced at second order for perturbation modes within the horizon at recombination. In the following, we will adopt similar simplifications already used for the linear case in order to provide some analytical solutions. In particular, if in equation (6.2) we treat $R$ as a constant we can write, using the initial conditions

$$\zeta^{(2)} = 2a_{NL} (\zeta^{(1)})^2,$$

(6.5)
determined above,

\[
(\Delta_{00}^{(2)} - 4\Psi^{(2)}) = 2(9a_{NL} - 7)\Psi^{(1)2}(0) \cos[kr_s(\eta)] - \frac{\sqrt{3}}{k}S_\Delta(0) \sin[kr_s(\eta)] \\
- \frac{4}{3} \int_0^\eta d\eta' (\Phi^{(2)}(\eta') + \Psi^{(2)}(\eta')) \sin[k(r_s(\eta) - r_s(\eta'))] \\
+ \int_0^\eta d\eta' S_\Delta(\eta') \cos[k(r_s(\eta) - r_s(\eta'))] \\
- \frac{4}{3} \frac{ik_i}{k_c} \int_0^\eta d\eta' S_{V_i}(\eta') \sin[k(r_s(\eta) - r_s(\eta'))].
\]

(6.9)

Note that we have also dropped the occurrence of \( R \) in \( \Phi^{(2)} + \Psi^{(2)}/R \).

### 7. Perturbation modes with \( k \gg k_{eq} \)

In order to study the contribution to the second-order CMB anisotropies coming from perturbation modes that enter the horizon during the radiation-dominated epoch, we will assume that the second-order gravitational potentials are the ones of a pure radiation-dominated universe throughout the evolution. Though not strictly correct, this approximation will give us the basic picture of the acoustic oscillations for the baryon–photon fluid occurring for these modes. Also for the second-order case, in section 9 we will provide the appropriate corrections accounting for the transition from radiation to matter domination, which is indeed (almost) achieved by the recombination epoch. Before moving to the details, a note of caution is in order here. At second order in the perturbations all the relevant quantities are expressed as convolutions of linear perturbations, leading to mode–mode mixing. In some cases, in our treatment for a given regime under analysis \((k \gg k_{eq} \text{ or } k \ll k_{eq})\), we use for the first-order perturbations the solutions corresponding to that particular regime, while the mode–mode mixing would require us to consider in the convolutions (where one is integrating over all the wavenumbers) a more general expression for the first-order perturbations (which analytically does not exist anyway).

For the computation of the CMB bispectrum this would be equivalent to considering just some specific scales, i.e. all of the three scales involved in the bispectrum should correspond approximately to wavenumbers \( k \gg k_{eq} \text{ or } k \ll k_{eq} \), and not a combination of the two regimes (a step towards the evaluation of the three-point correlation function has been taken in [27], where it was computed in the so-called squeezed triangle limit, where one mode has a wavelength much larger than the other two and is outside the horizon).

Having learned that at linear order the regime \( k \gg k_{eq} \) can be solved in the alternative way described by equations (3.27) and (3.29), we adopt the same procedure at second order: we will use equation (4.4) where we can neglect the gravitational potential term \( \Psi^{(2)'} \). The reason for this is again that also the second-order gravitational potentials decay at late times as \( \eta^{-2} \), while the second-order velocity \( v^{(2)i}_g \) oscillates in time. Let us now see that in detail.

The evolution equation for the gravitational potential \( \Psi^{(2)} \) is given by equation (B.18) and is characterized by the source term \( S_\gamma \), equation (B.19). In particular, the source term contains the second-order quadrupole moment of the photons \( \Pi_{ij}^{(2)} \). We saw in section 4.3 that at second order the quadrupole moment is not suppressed in the
tight coupling limit, being fed by the non-linear combination of the first-order velocities, equation (4.19). For the perturbation modes we are considering here, the velocity at late times is oscillating, being given by equation (3.27) in Fourier space. Since the linear gravitational potential (3.26) decays in time and for a radiation-dominated period $\mathcal{H} = 1/\eta$, it is easy to check that the dominant contribution at late times to the source term $S_\gamma$ simply reduces to

$$S_\gamma \simeq \frac{3}{2} \mathcal{H}^2 \frac{\partial \partial^i \Pi^{(2)}}{\eta^2} j \xi = \frac{F(k_1, k_2, k)}{\eta^2} C \Psi^{(1)}_{k_1}(0) \Psi^{(1)}_{k_2}(0) \sin(k_1 c_s \eta) \sin(k_2 c_s \eta),$$

(7.1)

where we have used equation (3.27),

$$C = \frac{9}{c_s^2 k_1 k_2},$$

(7.2)

and the sound speed is $c_s = 1/\sqrt{3(1 + R)}$. Before proceeding further, let us explain the notation that we are using. The equivalence symbol will be used to indicate that we are evaluating the expression in Fourier space. At second order in perturbation theory most of the Fourier transforms reduce to some convolutions. We will not indicate these convolutions explicitly but just through their kernel. For example, in equation (7.1) by $F(k_1, k_2, k)$ we actually indicate the convolution operator

$$F = \frac{1}{2\pi^3} \int d^3 k_1 d^3 k_2 \delta^{(1)}(k_1 + k_2 - k) F(k_1, k_2, k).$$

(7.3)

In the specific case of equation (7.1), the kernel is given by

$$F(k_1, k_2, k) = \frac{(k \cdot k_1)(k_1 \cdot k)}{k^2} - \frac{1}{3} k_1 \cdot k_2.$$

(7.4)

The choice of these conventions is due not only for simplicity and to keep our expressions shorter, but also because at the end we will be interested in the bispectrum of the CMB anisotropies generated at recombination, and the relevant expressions entering the bispectrum are just the kernels of the convolution integrals.

Having determined the leading contribution to the source term at late times, we can now solve the evolution equation (B.18). Since the source term scales like $\eta^{-2}$, it is useful to introduce the rescaled variable $\chi = \eta^2 \Psi^{(2)}$. Equation (B.18) then reads

$$\chi'' + \left(k^2 c_s^2 - \frac{2}{\eta^2}\right) \chi = \eta^2 S_\gamma.$$

(7.5)

For perturbation modes which are subhorizon with $k\eta \gg 1$, the solution of the homogeneous equation is given by

$$\chi_{\text{hom}} = A \cos(kc_s \eta) + B \sin(c_s k \eta),$$

(7.6)

from which we can build the general solution

$$\chi = \chi_{\text{hom}} + \chi_+ \int_0^\eta d\eta' \frac{\chi-(\eta')}{W(\eta')} S_\gamma(\eta') - \chi_- \int_0^\eta d\eta' \frac{\chi+(\eta')}{W(\eta')} S_\gamma(\eta'),$$

(7.7)

where $W = -k c_s$ is the Wronskian, and $\chi_+ = \cos(kc_s \eta)$, $\chi_- = \sin(c_s k \eta)$. Using equation (7.1), the integrals involve products of sines and cosines which can be performed
giving
\[ \chi = \chi_{\text{hom}} \frac{FC}{c_s^2} \Psi_{k_1}^{(1)}(0) \Psi_{k_2}^{(1)}(0) \]
\[ \times \left[ \frac{2k_1 k_2 \cos(k_1 c_s \eta) \cos(k_2 c_s \eta) - 2k_1 k_2 \cos(k_c \eta) + (k_1^2 + k_2^2 - k^2) \sin(k_1 c_s \eta) \sin(k_2 c_s \eta)}{k_1^4 + k_2^4 + k^4 - 2k_1^2 k_2^2 - 2k_1^2 k^2 - 2k_2^2 k^2} \right]. \]
\[ (7.8) \]

Thus, the gravitational potential \( \Psi^{(2)} \) at late times is given by
\[ \Psi_k^{(2)}(\eta) = -3\Psi^{(2)}(0) \frac{\cos(kc_s \eta)}{(kc_s \eta)^2} \frac{FC}{\eta^2 c_s^2} \Psi_{k_1}^{(1)}(0) \Psi_{k_2}^{(1)}(0) \]
\[ \times \left[ \frac{2k_1 k_2 \cos(k_1 c_s \eta) \cos(k_2 c_s \eta) - 2k_1 k_2 \cos(k_c \eta) + (k_1^2 + k_2^2 - k^2) \sin(k_1 c_s \eta) \sin(k_2 c_s \eta)}{k_1^4 + k_2^4 + k^4 - 2k_1^2 k_2^2 - 2k_1^2 k^2 - 2k_2^2 k^2} \right]. \]
\[ (7.9) \]

where we have set the integration constant \( B = 0 \) and \( A = -3\Psi^{(2)}(0)/(kc_s)^2 \) in order to match the homogeneous solution at late times which has the same form as equation (3.26). Here, \( \Psi^{(2)}(0) \) is the initial condition for the gravitational potential taken on large scales deep in the radiation-dominated era which will be determined in section 7.2.

Equation (7.9) shows the result that we anticipated: also at second order the gravitational potential varies in time oscillating with an amplitude that decays as \( \eta^{-2} \). Let us then take the divergence of the \((i-0)\) Einstein equation (A.10) expanded at second order
\[ \partial_i \left[ \frac{1}{2} \partial^i \Psi^{(2)\prime} + \frac{\mathcal{H}}{2} \partial^i \Phi^{(2)} + 2 \Psi^{(1)} \partial^i \Psi^{(1)\prime} + 2 \mathcal{H} \Psi^{(1)} \partial^i \Phi^{(1)} - \Psi^{(1)\prime} \partial^i \Phi^{(1)} \right] \]
\[ = -2 \mathcal{H}^2 \partial_i \left[ \frac{1}{2} \psi^{(2)\prime} + (\Phi^{(1)} + \Psi^{(1)}) v^{(1)i} + \Delta^{(1)\prime} v^{(1)i} \right], \]
\[ (7.10) \]

which, using the first-order \((i-0)\) Einstein equation (B.17) and \( \Phi^{(1)} \simeq \Psi^{(1)} \), reduces to
\[ \partial_i \left[ \frac{1}{2} \partial^i \Psi^{(2)\prime} + \frac{\mathcal{H}}{2} \partial^i \Phi^{(2)} - \Psi^{(1)\prime} \partial^i \Phi^{(1)} \right] = -2 \mathcal{H}^2 \partial_i \left[ \frac{1}{2} \psi^{(2)\prime} + \Delta^{(1)\prime} v^{(1)i} \right]. \]
\[ (7.11) \]

Since \( \Psi^{(1)} \) during a radiation-dominant period is given by equation (3.26) at late times, it is easy to see that \( (\Psi^{(1)\prime} \partial^i \Psi^{(1)}) \) will be oscillating and decaying as \( \eta^{-4} \) and thus can be neglected with respect to \( \Psi^{(2)\prime} \), which oscillates with an amplitude decaying as \( \eta^{-2} \). Also \( \mathcal{H} \Phi^{(2)} \) turns out to be subdominant. Recall that \( \Phi^{(2)} = \Psi^{(2)} - Q^{(2)} \) (see equation (A.12)) and \( Q^{(2)} \) is dominated by the second-order quadrupole of the photons in equation (B.19), so that \( \Phi^{(2)} \) scales like \( \Psi^{(2)} \) but there is the additional damping factor of the Hubble rate \( \mathcal{H} = 1/\eta \). Thus, the dominant terms give
\[ \partial_i v^{(2)\prime} \sim -\frac{1}{2 \mathcal{H}^2} \nabla^2 \Psi^{(2)\prime} - 2 \partial_i (\Delta^{(1)\prime} v^{(1)i}). \]
\[ (7.12) \]

Equation (7.12) is the equivalent of equation (3.28) and it allows us to proceed further in a similar way as for the linear case by using the results found so far, equations (7.9) and (7.12), in the energy continuity equation (4.4). In equation (4.4), the first- and second-order gravitational potentials can be neglected with respect to the remaining terms given by \( \Delta^{(1)} \) and \( v^{(1)i} \) which oscillate in time. Thus, replacing the divergence of the second-
order velocity by the expression (7.12), equation (4.4) becomes
\[ \Delta_{00}^{(2)\gamma} = \frac{2}{3H^2} \nabla^2 \Psi^{(2)\gamma} + \frac{8}{3} \partial_i v_i^{(1)\gamma} \Delta_{00}^{(1)\gamma} + \left( \Delta_{00}^{(1)\gamma} \right)', \] (7.13)
which, using the first-order equation (3.1), further simplifies to
\[ \Delta_{00}^{(2)\gamma} = \frac{2}{3H^2} \nabla^2 \Psi^{(2)\gamma}, \] (7.14)
where we have kept only the dominant terms at late times.

The gravitational potential \( \Psi^{(2)} \) is given in equation (7.9), so the integration of equation (7.14) gives
\[ \Delta_{00}^{(2)} = 6\Psi^{(2)}(0) \cos(kc_\gamma \eta) + \frac{2FC}{3\pi^2} \Psi^{(1)}(0) \Psi^{(1)}(0) k^2 \]
\[ \times \left[ \frac{2k_1 k_2 \cos(k_1 c_\eta) \cos(k_2 c_\eta) - 2k_1 k_2 \cos(k c_\eta) + (k_1^2 + k_2^2 - k^2) \sin(k_1 c_\eta) \sin(k_2 c_\eta)}{k_1^2 + k_2^2 + k^4 - 2k_1 k_2^2 - 2k_1^2 k^2 - 2k_2^2 k^2} \right]. \] (7.15)

Needless to say, modes for \( k \gg k_D \), where \( k_D \) indicates the usual damping length, are supposed to be multiplied by an exponential \( e^{-(k/k_D)^2} \) (see, e.g. [33]).

### 7.1. Vector perturbations

So far we have discussed only scalar perturbations. However, at second order in perturbation theory an unavoidable prediction is that vector (and tensor) perturbation modes are also produced dynamically as non-linear combinations of first-order scalar perturbations. In particular, note that the second-order velocity appearing in equation (5.10), giving rise to a second-order Doppler effect at last scattering, will contain a scalar and a vector (divergence free) part. Equation (7.12) provides the scalar component of the second-order velocity. We now derive an expression for the velocity that also includes the vector contribution.

The (second-order) vector metric perturbation \( \omega^i \) when radiation dominates can be obtained from equation (B.22)
\[ -\frac{1}{2} \nabla^2 \omega^i + 3H^2 \omega^i = -4H^2 \left( \delta_j^i - \frac{\partial_j \partial^i}{\nabla^2} \right) \left( \frac{v^{(2)\gamma}_i}{2} + \Delta_{00}^{(1)\gamma} v^{(1)\gamma}_i \right), \] (7.16)
where we have dropped the gravitational potentials \( \Psi^{(1)} \simeq \Phi^{(1)} \) which are subdominant at late times. On the other hand, from the velocity continuity equation (4.9) we get
\[ v^{(2)\gamma}_i + \frac{1}{4} \Delta_{00}^{(2)\gamma,i} = \frac{1}{4} \left( \Delta_{00}^{(2)\gamma} \right) + \frac{8}{3} v^{(1)\gamma}_j v^{(1)\gamma}_j - 2\omega^i - \frac{3}{4} \partial_k \Pi^{(2)\gamma ki}, \] (7.17)
eglecting the term proportional to \( R \) and the decaying gravitational potentials. Using the tight coupling equations at first order (3.1) and (3.2), and integrating over time, one finds
\[ v^{(2)\gamma}_i + 2(v^{(1)\gamma}_i \Delta_{00}^{(1)}) = -2\omega^i - \frac{1}{4} \int d\gamma' \Delta_{00}^{(2)\gamma,i} - \frac{3}{4} \int d\gamma' \partial_k \Pi^{(2)\gamma ki}. \] (7.18)
We can thus plug equation (7.18) into equation (7.16) to find that at late times (for $k\eta \gg 1$)
\[
\nabla^2 \omega^i = -3H^2 \left( \delta_j^i - \frac{\partial^i \partial_j}{\nabla^2} \right) \int d\eta' \partial_k \Pi^{(2)kj}_\gamma.
\]
(7.19)

We will come later to the explicit expression for the term on the right-hand side of equation (7.19). Here, it is enough to note that the second-order quadrupole oscillates in time and thus $\omega^i$ will decay in time as $H^2 = 1/\eta^2$. This shows that $\omega^i$ in equation (7.18) can in fact be neglected with respect to the other terms giving
\[
v^{(2)i}_\gamma = -2(v^{(1)i}_\gamma \Delta^{(1)00}_{00} - \frac{1}{3} \int d\eta' \Delta^{(2)i}_{00} - \frac{2}{3} \int d\eta' \partial_k \Pi^{(2)ki}_\gamma).
\]
(7.20)

It can be useful to compute the combination on the right-hand side of equation (7.19) $(\delta^i_j - \partial^i \partial_j / \nabla^2) \partial_k \Pi^{(2)kj}_\gamma$. The second-order quadrupole moment of the photons in the tightly coupled limit is given by equation (4.19), and
\[
\partial_k \Pi^{(2)kj}_\gamma = \frac{8}{3} \left[ \partial_k (v^k v^j) - 2v^k \partial^j v_k \right] = \frac{8}{3} \left[ v^j \partial_k v^k - v^k \partial^j v_k \right],
\]
(7.21)

where in the last step, we have used the fact that the linear velocity is the gradient of a scalar perturbation. We thus find
\[
\left( \delta^i_j - \frac{\partial^i \partial_j}{\nabla^2} \right) \partial_k \Pi^{(2)kj}_\gamma = \frac{8}{3} \left( v^i \partial_k v^k - v^k \partial^i v_k \right)
\]
\[
- \frac{8}{3} \frac{\partial^i}{\nabla^2} \left[ (\partial_k v^k)^2 + v^j \partial_j \partial_k v^k - \partial_j v^k \partial^j v_k - v^k \nabla^2 v_k \right].
\]
(7.22)

Note that if we split the quadrupole moment into scalar, vector (divergence-free) and tensor (divergence-free and traceless) parts as
\[
\Pi^{(2)kj}_\gamma = \Pi^{(2)kj}_\gamma + \Pi^{(2)kj}_\gamma + \Pi^{(2)kj}_\gamma + \Pi^{(2)kj}_\gamma,
\]
then it turns out that
\[
\left( \delta^i_j - \frac{\partial^i \partial_j}{\nabla^2} \right) \partial_k \Pi^{(2)kj}_\gamma = \nabla^2 \Pi^{(2)i}_\gamma,
\]
(7.24)

where $\Pi^{(2)i}_\gamma$ is the vector part of the quadrupole moment. Therefore, one can rewrite equation (7.19) as
\[
\omega^i = -3H^2 \int d\eta' \Pi^{(2)i}_\gamma.
\]
(7.25)

7.2. Initial conditions for the second-order gravitational potentials

In order to complete the study of the CMB anisotropies at second order for modes $k \gg k_{\text{eq}}$, we have to specify the initial conditions $\Psi^{(2)}(0)$ appearing in equation (7.15). These are set on superhorizon scales deep in the standard radiation-dominated epoch (for $\eta \rightarrow 0$) by exploiting the conservation in time of the curvature perturbation $\zeta$. On superhorizon scales, $\zeta^{(2)}$ is given by equation (6.7) during the radiation-dominated epoch and, using
the \((0-0)\)-Einstein equation in the large-scale limit \(\Delta_{00}^{(2)} = -2\Phi^{(2)} + 4\Phi^{(1)2}\), we find
\[
\zeta^{(2)} = -\frac{3}{2} \Psi^{(2)}(0) - \frac{1}{2} \left( \Phi^{(2)}(0) - \Psi^{(2)}(0) \right) + \frac{9}{2} \Psi^{(1)2}(0). \tag{7.26}
\]

The conserved value of \(\zeta^{(2)}\) is parametrized by \(\zeta^{(2)} = 2a_\text{NL}\zeta^{(1)2}\), where, as explained in section 6, the parameter \(a_\text{NL}\) specifies the level of primordial non-Gaussianity depending on the particular scenario for the generation of the cosmological perturbations. On the other hand, at second order the gravitational potentials differ according to equation (A.12), which for superhorizon modes during radiation domination gives
\[
\Phi^{(2)}(0) - \Psi^{(2)}(0) = -Q^{(2)}(0), \tag{7.27}
\]
where
\[
Q^{(2)}(0) = -2\nabla^{-2} \partial_k \Phi^{(1)}(0) \partial^k \Phi^{(1)}(0) + 6 \frac{\partial_i \partial^i}{\nabla^4} \left( \partial^i \Phi^{(1)}(0) \partial_j \Phi^{(1)}(0) \right) + \frac{9}{2} \mathcal{H}^2 \frac{\partial_i \partial^i}{\nabla^4} \Pi^{(2)ij}_{ij}, \tag{7.28}
\]
where we are evaluating equation (B.20) in the limit \(k\eta \ll 1\). The gravitational potential (B.16) just reduces to the constant \(\Phi^{(1)}(0)\), while the contribution from the second-order quadrupole moment in this limit reads
\[
\frac{9}{2} \mathcal{H}^2 \frac{\partial_i \partial^i}{\nabla^4} \Pi^{(2)ij}_{ij} = \frac{9}{2\eta^2} \left( \frac{8}{3} \delta^i_j v^i v^j - \frac{1}{3} \delta^i_j v^2 \right) = -3F \frac{C}{k^2\eta^2} \Psi^{(1)}_{k_1}(0) \Psi^{(1)}_{k_2}(0) \sin(k_1c_s\eta) \sin(k_2c_s\eta) \to \frac{27}{k^2} F \Psi^{(1)}_{k_1}(0) \Psi^{(1)}_{k_2}(0), \tag{7.29}
\]
where \(F\) and \(C\) are defined in equations (7.3) and (7.2). Therefore, we find that in Fourier space
\[
Q^{(2)}(0) = 33 F k^2 \Psi^{(1)}_{k_1}(0) \Psi^{(1)}_{k_2}(0), \tag{7.30}
\]
and from equation (7.26) we read off the initial condition as (convolution products are understood)
\[
\Psi^{(2)}(0) = \left[ -3(a_\text{NL} - 1) + 11 \frac{F(k_1, k_2, k)}{k^2} \right] \Psi^{(1)}_{k_1}(0) \Psi^{(1)}_{k_2}(0). \tag{7.31}
\]

### 7.3. Multipole moments

In this section, we give the expression for the CMB multipole moments observed today which are due to the perturbations of the photons at the last scattering surface. Therefore, we make use of equation (5.14) where we consider only the part \(S_*\) of the source term. As explained in section 5, \(S_*\) contains a \(-\tau'\) factor which reduces the time integral in equation (5.14) by evaluating the integrand at \(\eta = \eta_*\) given that the visibility function \(g(\eta) = -\tau'e^{-\tau} \) is peaked at the time of recombination. Therefore, we evaluate \(S_*\) at recombination in the tightly coupled limit for the modes \(k \gg k_\text{eq}\) using the previous results and we decompose it according to equation (5.12).
First, we use the solution for the photon–baryon fluid of equation (7.20) in equation (5.10) to find

\[
S_\ast(\eta) = -\tau\left[ \Delta_{00}^{(2)} - \frac{1}{2}\sum_{m=-2}^{2} \frac{\sqrt{4\pi}}{2^{3/2}} \Delta_{0m}^{(2)} Y_{2m}(n) - n \cdot \nabla \int d\eta' \Delta_{00}^{(2)} \right. \\
\left. - 3n \cdot \nabla \int d\eta' \Pi_{(2)}^{ij}(n) - 2(v \cdot n)\Delta_{00}^{(1)} + 2(v \cdot n)\Delta^{(1)} \right. \\
\left. - v\Delta_{1}^{(1)}(4 + 2P_{2}(v \cdot n)) + 14(v \cdot n)^2 - 2v^2 \right]. \tag{7.32}
\]

Note that in equation (5.10) we have neglected all the terms depending on the gravitational potentials since they decay in time, the terms proportional to the linear dipole which is suppressed in the tight coupling limit, and the terms proportional to \((\Delta_{00}^{(1)} - \Delta^{(1)} + 4v \cdot n)\) which is suppressed being just the first-order collision term.

For the decomposition of \(S_\ast\) in multipole moments, \(\Delta_{00}^{(2)}\) just gives \(\Delta_{00}^{(2)} \delta_{\ell 0} \delta_{m 0}\). Similar terms, which do not carry angular dependence in equation (5.10), are \(-2v^2\) and \(-4v\Delta_{1}^{(1)}\).

Note that in the limit of tight coupling, we can use \(\Delta_{1}^{(1)} = 4v/3\). For the terms which are quadratic in the velocities, it is convenient to write

\[
14(v \cdot n)^2 - 2v^2 - 4v\Delta_{1}^{(1)} = 14(n^i n^j - \frac{1}{3}\delta^{ij})v_i v_j - \frac{8}{3}v^2. \tag{7.33}
\]

The term \(14(n^i n^j - \frac{1}{3}\delta^{ij})v_i v_j\) can be decomposed with multipoles given by (in Fourier space)

\[
-14(-i)^{\ell} \sqrt{\frac{2\ell + 1}{4\pi}} \left( \frac{4\pi}{3} \right)^2 v(k_1)v(k_2)(-1)^{-m} \\
\times \sum_{m_1, m_2} Y_{1m_1}^{*}(k_1)Y_{1m_2}^{*}(k_2)\mathcal{G}_{11\ell}^{m_1 m_2 - m} - \frac{1}{3}\delta_{\ell 0} \delta_{m 0}, \tag{7.34}
\]

where in Fourier space, for the first-order velocity, we use the convention

\[
\mathbf{v}(k_1) = \hat{v}(k_1)\hat{k}_1, \tag{7.35}
\]

and the convolution products are implicitly assumed in a similar way as in equation (7.3).

In order to derive equation (7.34) and the following expressions, we use the addition theorem of the spherical harmonics

\[
P_{\ell}(\hat{k} \cdot \mathbf{n}) = \frac{4\pi}{2\ell + 1} \sum_{m=-\ell}^{\ell} Y_{\ell m}^{*}(\hat{k})Y_{\ell m}(\mathbf{n}). \tag{7.36}
\]

Note that the term \(-4v\Delta_{1}^{(1)}P_{2}(v \cdot n)\) can be written as \(-4(n^i n^j - \delta^{ij})v_i v_j\) of the same type as that in equation (7.34). The multipoles of \((-2\Delta_{00}^{(1)}v \cdot n)\) are \(2\sqrt{(4\pi/3)\Delta_{00}^{(1)}(k_2)v(k_2)Y_{1m}^{*}(k_1)\delta_{\ell 1}}\) using the same rules as above.

Let us now consider the term 2\((v \cdot n)\Delta^{(1)}\). From equation (2.4), we can write

\[
\Delta^{(1)} \simeq \Delta_{00}^{(1)} + 3\sqrt{\frac{4\pi}{3}} \Delta_{1}^{(1)} Y_{10}, \tag{7.37}
\]
where we are neglecting higher-order multipoles in the tight coupling limit. Therefore, the multipoles of $2(\mathbf{v} \cdot \mathbf{n}) \Delta^{(1)}$ are given by

$$2i(-i)^{-\ell} \sqrt{\frac{2\ell+1}{4\pi}} v(k_1) \frac{4\pi}{3} \sum_{m_1=-1}^{1} Y^{*}\ell_{m_1}(k_1) \left[ \Delta^{(1)}_{00}(k_2) \delta_{\ell \ell'} \delta_{m_1 m_0} + \Delta^{(1)}_1(k_2) \delta_{\ell \ell'} \left( \frac{4\pi}{3} (-1)^{-m} \mathcal{g}^m_{1\ell m} \right) \right].$$

(7.38)

Note that for $\ell = 0$ and $m = 0$, equation (7.38) gives $8v^2/3$ which then will cancel the second term on the right-hand side of equation (7.34). This can be accounted for by simply neglecting such a term in equation (7.34) and writing equation (7.38) by specifying $\ell \neq 0, m \neq 0$.

The term $-\mathbf{n} \cdot \nabla \int d\eta' \Delta^{(2)}_{00}$ has the expansion coefficient

$$\delta_{\ell \ell'} \delta_{m_0 k} \int^\eta d\eta' \Delta^{(2)}_{00}(\eta').$$

(7.39)

Finally, the expansion coefficient for the second term in equation (7.32), $[-\sum_{m=-2} \sqrt{4\pi} \Delta^{(2)}_{2m}(\mathbf{n})/(2\pi)^2]$ reduces to $\Delta^{(2)}_{2m} \delta_{2\ell}/10$, while the term $-3\mathbf{n} \cdot \nabla \int d\eta' \tilde{\Gamma}^{(2)}_{ij}$ has the expansion coefficient

$$8i \sqrt{\frac{4\pi}{3}} \left[ -k_2 + \mathbf{v}(k_1) \cdot \mathbf{k}_2 \right] \int d\eta' v(k_1)v(k_2)Y_{m}(\hat{k}_1) \delta_{\ell \ell'},$$

(7.40)

where we have used equation (4.19).

Collecting all of the previous results, we find

$$S_{\mathbf{x}\ell m} = -r' \left\{ (-i)^{-\ell} \sqrt{\frac{2\ell+1}{4\pi}} \left[ \sqrt{4\pi} \Delta^{(2)}_{00} \delta_{\ell 0} \delta_{m 0} + i8\pi \sqrt{\frac{4\pi}{3}} v(k_1) \Delta^{(1)}_{1}(k_2) (-1)^{-m} \mathcal{g}^m_{1\ell} \delta_{\ell \ell'} \left( \frac{4\pi}{3} \right) \left[ v(k_1)v(k_2) (-1)^{-m} \sum_{m_1, m_2} Y^{*}\ell_{1m_1}(k_1) Y^{*}\ell_{2m_2}(k_2) \mathcal{g}^m_{1\ell m_1} \mathcal{g}^m_{1\ell m_2} \right] + \delta_{\ell \ell'} \delta_{m_0 k} \int^\eta d\eta' \Delta^{(2)}_{00}(\eta') + 8 \sqrt{\frac{4\pi}{3}} \left[ -k_2 + \mathbf{v}(k_1) \cdot \mathbf{k}_2 \right] \int d\eta' v(k_1)v(k_2)Y_{m}(\hat{k}_1) \delta_{\ell \ell'} + \delta_{\ell \ell'} \frac{\Delta^{(2)}_{2m}}{10} \right\}.$$

(7.41)

Equation (7.41) is all we need to get the multipole moments today given by equation (5.14).

8. Perturbation modes with $k \ll k_{\text{eq}}$

Let us consider the photon perturbations which enter the horizon between the equality epoch and the recombination epoch, with wavelengths $\eta^{-1} < k < \eta_{\text{eq}}^{-1}$. In fact, in order to find some analytical solutions, we will assume that by the time of recombination the universe is matter dominated $\eta_{\text{eq}} \ll \eta_s$. In this case, the gravitational potentials are

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sourced by the dark matter component and their evolution is given in section B.1. At linear order, the gravitational potentials remain constant in time, while at second order they are given by equation (B.4). In turn, the gravitational potentials act as an external force on the CMB photons as in equation (4.11) describing the CMB energy density evolution in the tightly coupled regime.

For the regime of interest, it proves convenient to use the solution of equation (4.11) found in (6.9). The source functions $S_\Delta$ and $S_i^t$ are given by equations (4.5) and (4.10), respectively. In particular, $S_\Delta$ at early times—$S_\Delta(0)$ appearing in equation (6.9)—vanishes. For a matter-dominated period

$$S_\Delta(R = 0) = \left( \Delta^{(1)}_{00} \right)' - \frac{16}{3} \Psi^{(1)} \partial_i v_\gamma^{(1)i} + \frac{16}{3} \left( v_\gamma^{(2)} \right)' + \frac{8}{3} (\eta - \eta_i) \partial^j \Psi^{(1)} \partial_i \Delta^{(1)}_{00},$$

(8.1)

where we have used the linear evolution equations (3.1) and (3.8) with $\Phi^{(1)} = \Psi^{(1)}$, and

$$S_i^t(R = 0) = \frac{8}{3} v_\gamma^{(1)i} \partial_i v_\gamma^{(1)j} + \frac{\partial^j}{\partial^i} \Delta^{(1)2}_{00} - 2 \partial^j \Psi^{(1)2} - \Psi^{(1)} \partial^j \Delta^{(1)}_{00} + \frac{8}{3} (\eta - \eta_i) \partial^j \Psi^{(1)} \partial_i v_\gamma^{(1)j} - 2 \omega'' - \frac{3}{4} \partial^j \Pi^{(2)ij}.$$  

(8.2)

As at linear order, we are evaluating all of our expressions in the limit $R = 3\rho_0/4\rho_\gamma \to 0$, while retaining a non-vanishing and constant value for $R$ in the expression for the photon–baryon fluid sound speed entering in the sines and cosines, equation (3.15). Using the linear solutions (3.20) and (3.22) for the energy density and velocity of photons, the source functions in Fourier space read

$$S_\Delta(R = 0) = \left[ -2 \left( \frac{6}{5} \right)^2 k_2 c_\omega \cos(k_1 c_\omega \eta) \sin(k_2 c_\omega \eta) ight.$$

$$\left. + \frac{108}{25} k_2 c_\omega \sin(k_2 c_\omega \eta) - \frac{32}{3} \left( \frac{9}{10} \right)^2 \frac{k_1}{k_2} \cdot \vec{k}_2 c_\omega^3 \sin(k_1 c_\omega \eta) \cos(k_2 c_\omega \eta) ight.$$

$$\left. - \frac{12}{5} (\eta - \eta_i) k_1 \cdot \vec{k}_2 \left( \frac{6}{5} \cos(k_2 c_\omega \eta) - \frac{18}{5} \right) \right] \Psi^{(1)}_{k_1}(0) \Psi^{(1)}_{k_2}(0),$$

(8.3)

and

$$S_i^t(R = 0) = \left[ -i \frac{2}{3} \left( \frac{9}{10} \right)^2 \frac{k_1}{k_2} \vec{k}_2 \sin(k_1 c_\omega \eta) \sin(k_2 c_\omega \eta) ight.$$

$$\left. + i \frac{k_2^i}{4} \left( \frac{6}{5} \cos(k_1 c_\omega \eta) - \frac{18}{5} \right) \left( \frac{6}{5} \cos(k_2 c_\omega \eta) - \frac{18}{5} \right) ight.$$

$$\left. - 2ik_2^i \left( \frac{9}{10} \right)^2 - i \frac{9}{10} k_2 \left( \frac{6}{5} \cos(k_2 c_\omega \eta) - \frac{18}{5} \right) - 2 \omega'' ight.$$

$$\left. + i \frac{8}{3} \left( \frac{9}{10} \right)^2 c_\omega (\eta - \eta_i) k_1 \cdot \vec{k}_2 \frac{k_1}{k_2} \sin(k_2 c_\omega \eta) ight.$$

$$\left. + i \frac{2}{3} \left( \frac{9}{10} \right)^2 \frac{k_2}{k_1} \vec{k}_1 \sin(k_1 c_\omega \eta) \sin(k_2 c_\omega \eta) \right] \Psi^{(1)}_{k_1}(0) \Psi^{(1)}_{k_2}(0).$$

(8.4)

In $S_i^t$, we have used expression (4.19) for the second-order quadrupole moment $\Pi^{(2)ij}_\gamma$ of the photons in the tight coupling limit, with the velocity $v^{(1)} = v^{(1)}_\gamma$. Note that, for the modes crossing the horizon at $\eta > \eta_{eq}$, we have expressed the gravitational potential
during the matter-dominated period in terms of the initial value on superhorizon scales deep in the radiation-dominated epoch as $\Psi^{(1)} = 9\Psi^{(1)}(0)/10$.

As for the second-order gravitational potentials, we have to compute the combination $\Phi^{(2)} + \Psi^{(2)}$ appearing in equation (6.9). The gravitational potential $\Psi^{(2)}$ is given by equation (B.4), while $\Phi^{(2)}$ is given by

$$\Phi^{(2)} = \Psi^{(2)} - Q^{(2)},$$

according to the relation (A.12), where for a matter-dominated period

$$Q^{(2)} = 5\nabla^{-4}\partial_i\partial_j(\partial^i\Psi^{(1)}\partial^j\Psi^{(1)}) - \frac{5}{3}\nabla^{-2}(\partial_k\Psi^{(1)}\partial^k\Psi^{(1)}).$$

We thus find

$$\Phi^{(2)} + \Psi^{(2)} = 2\Psi^{(2)}_m(0) - \frac{1}{7}\left(\partial_k\Psi^{(1)}\partial^k\Psi^{(1)} - \frac{10}{3}\nabla^{-2}\partial_i\partial^j(\partial^i\Psi^{(1)}\partial_j\Psi^{(1)})\right)\eta^2
- 5\nabla^{-4}\partial_i\partial^j(\partial^i\Psi^{(1)}\partial_j\Psi^{(1)}) + \frac{5}{3}\nabla^{-2}(\partial_k\Psi^{(1)}\partial^k\Psi^{(1)}),$$

which in Fourier space reads

$$\Phi^{(2)} + \Psi^{(2)} = 2\Psi^{(2)}_m(0) + \left[\frac{1}{7}G(k_1, k_2, k)\eta^2 - \frac{5}{k^2}F(k_1, k_2, k)\right] \left(\frac{9}{10}\right)^2 \Psi^{(1)}(0)\Psi^{(1)}_k(0),$$

where the kernels of the convolutions are given by equation (7.4) and

$$G(k_1, k_2, k) = k_1 \cdot k_2 - \frac{10(k \cdot k_1)(k \cdot k_2)}{3k^2}.$$ (8.9)

In equation (8.7), $\Psi^{(2)}_m(0)$ is the initial condition for the gravitational potential fixed at some time $\eta_h > \eta_{eq}$. For the regime of interest, it corresponds to the value of the gravitational potential on superhorizon scales during the matter-dominated epoch.

We are now able to compute the integrals entering in the solution (6.9). The one involving the second-order gravitational potentials is straightforward to compute

$$-\frac{4}{3}\int_{0}^{\eta} d\eta' \int_{0}^{\eta'} d\eta'' \left(\Phi^{(2)} + \Psi^{(2)}\right) \sin[kc_s(\eta - \eta'')].$$

$$= -\frac{8}{3c_s^2} \left(1 - \cos(kc_s\eta)\right)\Psi^{(2)}_m(0) - \frac{4}{3c_s} \left[-\frac{5}{k^2}F(k_1, k_2, k)\right] \left(\frac{1}{k^2}\right)^{\frac{1}{10}} \Psi^{(1)}(0)\Psi^{(1)}_k(0).
+ \frac{1}{7k^2c_s^2}G(k_1, k_2, k) \left(-2 + (kc_s\eta)^2 + 2\cos(kc_s\eta)\right) \right] \left(\frac{9}{10}\right)^2 \Psi^{(1)}(0)\Psi^{(1)}_k(0).$$

For the two remaining integrals, in the following we will show only the terms that in the final expression for $\Delta^{(2)}_{10}$ and the second-order velocity $\nu^{(2)}_i$ give the dominant contributions for $k\eta \gg 1$, even though we have performed a full computation. The integral over the source function $S_\Delta$ yields

$$\int_{0}^{\eta} d\eta' \int_{0}^{\eta'} d\eta'' \sin[kc_s(\eta - \eta'')] = \frac{9}{5} \left[\frac{12}{5} \frac{k_1 \cdot k_2}{c_s^2k^2} + \frac{4}{5c_s} \frac{k_1 \cdot k_2}{k^2 - k_2^2}\eta \right] \sin(k_2c_s\eta) + (1 \leftrightarrow 2) \Psi^{(1)}(0)\Psi^{(1)}_k(0),$$

$$\int_{0}^{\eta} d\eta' \int_{0}^{\eta'} d\eta'' \cos[kc_s(\eta - \eta'')] = \frac{9}{5} \left[\frac{12}{5} \frac{k_1 \cdot k_2}{c_s^2k^2} + \frac{4}{5c_s} \frac{k_1 \cdot k_2}{k^2 - k_2^2}\eta \right] \cos(k_2c_s\eta) + (1 \leftrightarrow 2) \Psi^{(1)}(0)\Psi^{(1)}_k(0),$$
where (1 ↔ 2) stands for an exchange of indices. The terms that have been dropped in the expression (8.11) all vary in time as a cosine. However, we have written the first term because, upon integration over time, it will give a non-negligible contribution to the velocity $v_c^{(2)}$. For the last integral, we find

$$-\frac{4i}{3} \frac{k_1}{k c_s} \int_0^\eta d\eta' S_V^i \sin[k c_s(\eta - \eta')] = \left[ \frac{27}{25} \frac{2k \cdot k_2 + k^2}{k^2 c_s^2} \right. \right.
+ \frac{36}{25} \frac{(k_1 \cdot k_2)(k \cdot k_2)}{k_2(k^2 - k_2^2)} \eta \sin(k_2 c_s \eta) + (1 \leftrightarrow 2) \right] \Psi_{k_1}^{(1)}(0) \Psi_{k_2}^{(1)}(0),$$

(8.12)

where the terms that have been dropped are proportional to cosines.

From the general solution (6.9) and the expression (B.4) for the second-order gravitational potential $\Psi^{(2)}$, we thus obtain

$$\Delta_{00}^{(2)} = \left( 4 - \frac{8}{3c_s^2} \right) \Psi_m^{(2)}(0) + \left[ 2(9a_{NL} - 7)\Psi_{k_1}^{(1)}(0)\Psi_{k_2}^{(1)}(0) + \frac{8}{3c_s^2}\Psi_m^{(2)}(0) \right] \sin(k c_s \eta)
+ \frac{2}{7} \left( \frac{9}{10} \right)^2 \left( 1 - \frac{2}{3c_s^2} \right) G(k_1, k_2, k) \eta^2 \Psi_{k_1}^{(1)}(0) \Psi_{k_2}^{(1)}(0).$$

(8.13)

We warn the reader that in writing equation (8.13) we have kept all of those terms that contain the primordial non-Gaussianity parametrized by $a_{NL}$, and the terms which dominate at late times for $k \eta \gg 1$.

### 8.1. Initial conditions for the second-order gravitational potentials

The initial condition $\Psi_m^{(2)}(0)$ for the modes that cross the horizon after the equality epoch is fixed by the value of the gravitational potential on superhorizon scales during the matter-dominated epoch. To compute this value, we use the conservation on superhorizon scales of the curvature perturbation $\zeta^{(2)}$ defined in equation (6.3). For a matter-dominated period, the curvature perturbation on large scales turns out to be

$$\zeta^{(2)} = -\Psi_m^{(2)}(0) + \frac{1}{3} \frac{\delta^{(2)} \rho}{\rho_m} + \frac{38}{9} \Psi_m^{(1)}(0)^2,$$

(8.14)

where we used the energy continuity equation $\delta^{(1)} \rho_m + 3H \delta^{(1)} \rho_m - 3 \rho_m \Psi^{(1)} = 0$ and the (0 − 0) Einstein equation $\delta^{(1)} \rho_m / \rho_m = -2 \Psi^{(1)}$ in the superhorizon limit.

From the (0 − 0) Einstein equation on large scales $\delta^{(2)} \rho_m / \rho_m = -2 \Phi^{(2)} + 4 \Psi^{(1)}$ giving

$$\zeta^{(2)} = -\frac{\Phi_m^{(2)}(0)}{3} - \frac{\Phi_m^{(2)}(0)}{3} - \frac{\Phi_m^{(2)}(0)}{3} + \frac{38}{9} \Psi_m^{(1)}(0)^2.$$

(8.15)

The conserved value of $\zeta^{(2)}$ is parametrized as in equation (6.5), $\zeta^{(2)} = 2a_{NL} \zeta^{(1)} = (50a_{NL}/9) \Psi^{(1)}$, with $\zeta^{(1)} = -5 \Psi^{(1)} / 3$ on large scales after the equality epoch. At second order, the two gravitational potentials in a matter-dominated epoch differ according to equation (8.6), and using equation (8.15) we find

$$\Psi_m^{(2)}(0) = -\frac{27}{10} (a_{NL} - 1) \Psi^{(1)}(0)^2 + \left( \frac{7}{10} \right)^2 \left[ 2 \nabla^{-4} \partial_i \partial_j (\partial^i \Psi^{(1)}(0) \partial_j \Psi^{(1)}(0)) \right.
- \frac{2}{3} \nabla^{-2} (\partial_i \Psi^{(1)}(0) \partial_j \Psi^{(1)}(0)).$$

(8.16)
We have expressed the gravitational potential during the matter-dominated period $\Psi^{(1)}$ in terms of the initial value on superhorizon scales after the equality epoch as $\Psi^{(1)} = 9\Psi^{(1)}(0)/10$. In Fourier space, equation (8.16) becomes

$$
\Psi_m^{(2)}(0) = \left[ -\frac{27}{10}(a_{NL} - 1) + 2 \left( \frac{9}{10} \right)^2 \frac{F(k_1, k_2, k)}{k^2} \right] \Psi_k^{(1)}(0) \Psi_k^{(1)}(0),
$$

where $F$ is the kernel defined in equation (7.4).

We can use the explicit expression for $\Psi_m^{(2)}(0)$ in equation (8.13), still keeping only the terms that contain the primordial non-Gaussianity parametrized by $a_{NL}$, and the terms which dominate at late times for $k\eta \gg 1$ to find

$$
\Delta_{00}^{(2)} = \left[ \frac{27}{5}(a_{NL} - 1) - \frac{2}{5}(9a_{NL} - 19) \cos(k_c \eta) - \frac{2}{7} \left( \frac{9}{10} \right)^2 G(k_1, k_2, k) \eta^2 \right] \Psi_k^{(1)}(0) \Psi_k^{(1)}(0),
$$

where we have also used in equation (8.13) $c_s \simeq 1/\sqrt{3}$ (except in the argument of the cosine for the reason explained in section 3.1).

### 8.2. Second-order photon velocity perturbation

The second-order velocity of the photons can be obtained from equation (4.9) where, as usual, we drop $R$

$$
v^{(2)i}_\gamma \simeq \int_0^\eta d\eta' \left( S'_V - \partial^i \Phi^{(2)} - \frac{1}{4} \partial^i \Delta_{00}^{(2)} \right).
$$

The second-order gravitational potential in a matter-dominated universe can be obtained from equations (8.5), (8.6) and equation (B.4) as

$$
\Phi^{(2)} = \Psi_m^{(2)}(0) - \frac{1}{14} \left( \partial_k \Psi^{(1)} \partial^k \Psi^{(1)} - \frac{10}{3} \nabla^2 \Psi^{(1)} \partial^i \partial^j \Psi^{(1)} \right) \eta^2 - 5 \nabla^2 \partial^i \partial^j \Psi^{(1)} \partial^k \Psi^{(1)} + \frac{5}{3} \nabla^2 \partial^i \partial^j \partial^k \Psi^{(1)}.
$$

In Fourier space, this becomes

$$
\Phi^{(2)} = \Psi_m^{(2)}(0) + \left[ \frac{1}{14} G(k_1, k_2, k) \eta^2 - \frac{5}{k^2} F(k_1, k_2, k) \right] \left( \frac{9}{10} \right)^2 \Psi_k^{(1)}(0) \Psi_k^{(1)}(0),
$$

where the kernels of the convolutions are given by equations (7.4) and (8.9). The integral over $\Phi^{(2)}$ in equation (8.19) is then easily computed

$$
- \int_0^\eta d\eta' \partial^i \Phi^{(2)} \equiv -ik^i \left[ \Psi_m^{(2)}(0) \eta + \left( \frac{5}{12} G(k_1, k_2, k) \eta^3 - \frac{5}{k^2} F(k_1, k_2, k) \eta \right) \left( \frac{9}{10} \right)^2 \Psi_k^{(1)}(0) \Psi_k^{(1)}(0) \right],
$$

where, as usual, the equivalence symbol means that we are evaluating a given expression in Fourier space. For the integral over the source function $S'_V$, we use its expression in
Fourier space, equation (8.4), and the dominant terms for \( k \eta \gg 1 \) are

\[
\int_0^\eta d\eta' S_V^i \equiv \left( 2ik^i \left( \frac{9}{10} \right)^2 + i k^i_2 \frac{81}{25} + i \frac{8}{3} \left( \frac{9}{10} \right)^2 \frac{1}{k^4} (k_2^2 - k_1^2) \right) \eta \\
- \frac{i}{2c_s} \frac{8}{3} \left( \frac{9}{10} \right)^2 \frac{k_1 \cdot k_3 k_2^i}{k_2} \eta \cos(k_2 c_s \eta).
\]

(8.23)

Note that, in order to compute this integral, we must know the second-order vector metric perturbation \( \omega^i \). This is easily obtained for a matter-dominated universe from equation (B.5). Using equations (B.7) and (B.2) one finds

\[
\omega^i = -\frac{4}{3} \left( \frac{9}{10} \right)^2 \nabla^{-4} \partial_j \left[ \partial^i \nabla^2 \Psi^{(1)}(0) \partial_j \Psi^{(1)}(0) - \partial^j \nabla^2 \Psi^{(1)}(0) \partial^i \Psi^{(1)}(0) \right] \eta,
\]

(8.24)

giving rise to the third term in equation (8.23).

Finally, for the integral over \( \Delta_{00}^{(2)} \) some caution is needed. Since in the final expression for \( v_\gamma^{(2)i} \) the dominant terms at late times turn out to be proportional to \( \eta \), one has to use an expression for \( \Delta_{00}^{(2)} \) that keeps track of all those contributions that, upon integration, scale like \( \eta \). Thus, we must use the expression written in equation (8.13), plus equation (8.11) and equation (8.12), and some terms of equation (8.10) that have been previously neglected in equation (8.13). Then, we find for \( k \eta \gg 1 \)

\[
-\frac{1}{4} \int_0^\eta d\eta' \partial^i \Delta_{00}^{(2)} \equiv -\frac{ik^i}{4} \left[ -4\Psi_m^{(2)}(0) \eta + \left( 2(9a_{NL} - 7) \Psi_{k_1}^{(1)}(0) \Psi_{k_2}^{(1)}(0) + 8\Psi_m^{(2)}(0) \right) \frac{\sin(kc_s \eta)}{kc_s} \right] \\
- \frac{i}{4} \left[ - \frac{2}{21} \left( \frac{9}{10} \right)^2 G \eta^3 - \frac{361}{25 c_s^2 (k_2^2 - k_1^2) (k_2 + k \cdot k_2)} \right] \times \eta \frac{\cos(k_2 c_s \eta)}{k_2 c_s} + (1 \rightarrow 2) + \left( \frac{20}{3} \frac{9}{10} \right)^2 \frac{F}{k^2} + \frac{8}{21 c_s^4} \left( \frac{9}{10} \right)^2 \frac{G}{k^2} \\
+ \frac{18}{5} \frac{12}{k_2^2 c_s^2} + \frac{54}{25 c_s^2} \frac{k^2 + k \cdot (k_1 + k_2)}{k^2} \right] \Psi_{k_1}^{(1)}(0) \Psi_{k_2}^{(1)}(0).
\]

(8.25)

Using equations (8.22), (8.23) and (8.25) we get

\[
v_\gamma^{(2)i} = \left[ \frac{i k^i}{k_1 10 c_s} (9a_{NL} - 19) \sin(kc_s \eta) + \frac{i}{50 c_s} k_1 \cdot k_2 \left( \frac{2k^i}{k^2 - k_2^2} + k \cdot k_2 \right) \right] \times \eta \cos(k_2 c_s \eta) + (1 \rightarrow 2) + \left[ -\frac{2}{21 c_s^4} k^i \left( \frac{9}{10} \right)^2 \frac{G}{k^2} - i k^i \frac{54}{25 c_s^2} \frac{k_1 \cdot k_2}{k^2} \\
+ 2i \left( \frac{9}{10} \right)^2 k^i + \frac{81}{50 c_s} (k_2^i + k_1^i) - i \frac{27}{50 c_s^2} k^2 + k \cdot (k_1 + k_2) \right] \frac{k^i}{k^2} + \frac{i}{3} \frac{9}{10} \left( \frac{2k_2^2 - k_1^2}{k^2} \right) (k \cdot k_1 k_2^i - k \cdot k_2 k_1^i) \eta \Psi_{k_1}^{(1)}(0) \Psi_{k_2}^{(1)}(0).
\]

(8.26)

To obtain equation (8.26), we have also used the explicit expression (8.17) for \( \Psi_m^{(2)}(0) \) and we have kept the terms depending on \( a_{NL} \) parametrizing the primordial non-Gaussianity and the terms that dominate at late times for \( k \eta \gg 1 \).
8.3. Multipole moments

The expression for the multipole moments (5.14) due to the anisotropies generated at recombination are easily found. The multipole moments for the source term $S_\ast$, equation (5.10), can be computed similarly to equation (7.41), and in addition we keep those terms which are proportional to the gravitational potentials. We thus find

$$S_{\ast \ell m} = -\tau \left\{ ( -i )^{-\ell} \sqrt{\frac{2\ell + 1}{4\pi}} \left[ \sqrt{4\pi} \left( \Delta^{(2)}_{00} + 4\Phi^{(2)} + 16\Phi^{(1)} + 4\Psi^{(1)} \right) \delta_{\ell 0} \delta_{m 0} + 8i \frac{4\pi}{3} v(k_1)\Delta^{(1)}_{00}(k_2)Y^*_m(k_1)\delta_{\ell 1} 
\right. $$

$$ + \left. i8\sqrt{\frac{4\pi}{3}} v(k_1)\Delta^{(1)}_{01}(k_2)(1) - m \mathcal{G}^{m_1 0 - m}(\ell \neq 0; m \neq 0) 
\right. $$

$$ - 10 \left( \frac{4\pi}{3} \right)^2 v(k_1)v(k_2)(-1)^{-m} \sum_{m_1, m_2} Y^*_m(k_1)Y^*_m(k_2)\mathcal{G}^{m_1 m_2 - m}_{11\ell} \right] 
\right. $$

$$ + (8\Phi^{(1)} - 4\Psi^{(1)}) \Delta^{(1)}_{\ast \ell m} \pm 16(\Psi^{(1)} \mathbf{v})_m \delta_{\ell 1} \pm \mathbf{v}^{(2)}_m \delta_{\ell 1} + \delta_{\ell 2} \frac{\Delta^{(2)}_{\ast m}}{10} \right\}, \quad (8.27)$$

where the minus sign must be used when $m = 0$ and the plus sign when $m = \pm 1$. In equation (8.27), $v^{(2)}_m$ represents the scalar and vortical components of the velocity perturbation

$$\mathbf{v}^{(2)}(k) = i \mathbf{v}^{(2)}(k) \hat{k} + \sum_{m = \pm 1} v^{(2)}_m \mathbf{e}_2 \mp \frac{\mathbf{i} e_1}{\sqrt{2}}, \quad (8.28)$$

where $\mathbf{e}_i$ forms an orthonormal basis with $\hat{k}$ (and $v^{(2)}_0 \equiv v^{(2)}$). They can be easily obtained from equation (8.26). Moreover, for a generic quantity $f(x) \mathbf{v}$ we have indicated the corresponding scalar and vortical components with $(f \mathbf{v})_m$ and their explicit expression is easily found by projecting the Fourier modes of $f(x) \mathbf{v}$ along the $\hat{k} = \mathbf{e}_3$ and $(\mathbf{e}_2 \mp \mathbf{i} \mathbf{e}_1)$ directions

$$(f \mathbf{v})_m(k) = (\pm) \int \frac{d^3 k_1}{(2\pi)^3} v^{(1)}(k_1) f(k_2) Y^*_m(\hat{k}_1) \sqrt{\frac{4\pi}{3}}, \quad (8.29)$$

where the minus sign must be used for $m = -1, +1$ and the plus sign in correspondence of $m = 0$.

9. Perturbation modes with $k \gg k_{\text{eq}}$: improved analytical solutions

In section 7, we have computed the perturbations of the CBM photons at last scattering for the modes that cross the horizon at $\eta < \eta_{\text{eq}}$ under the approximation that the universe is radiation dominated. However, around the equality epoch, through recombination, the dark matter component will start to dominate. In this section, we will account for its contribution to the gravitational potential and for the resulting perturbations of the photons from the equality epoch onwards. This leads to a more realistic and accurate analytical solution for the acoustic oscillations of the photon–baryon fluid for the modes of interest.
The starting point is to consider the density perturbation in the dark matter component for subhorizon modes during the radiation-dominated epoch. Its value at the equality epoch will fix the magnitude of the gravitational potential at $\eta_{\text{eq}}$ and hence the initial conditions for the subsequent evolution of the photons fluctuations during the matter-dominated period. At linear order, the procedure is standard (see, e.g., [39] and [33]), and we will use a similar one at second order in the perturbations.

9.1. Subhorizon evolution of CDM perturbations for $\eta < \eta_{\text{eq}}$

From the energy and velocity continuity equations for CDM, it is possible to isolate an evolution equation for the density perturbation $\delta_d = \delta \rho_d / \bar{\rho}_d$, where the subscript $d$ stands for cold dark matter. In [22], we have obtained the Boltzmann equations up to second order for CDM. The number density of CDM evolves according to [22]

$$\frac{\partial n_d}{\partial \eta} + e^{\Phi + \Psi} \frac{\partial (v_{d,i} n_d)}{\partial x^i} + 3(\mathcal{H} - \Psi') n_d - 2e^{\Phi + \Psi} \Psi_k v_{d,k} n_d + e^{\Phi + \Psi} \Phi_k v_{d,k} n_d = 0. \quad (9.1)$$

At linear order, $n_d = \bar{n}_d + \delta^{(1)} n_d$ and one recovers the usual energy continuity equation

$$\delta^{(1)} n_d = -\Phi^{(1)} \nabla^2 \Phi^{(1)}, \quad (9.2)$$

where $\delta^{(1)} = \delta^{(1)} \rho_d / \bar{\rho}_d = \delta^{(1)} n_d / \bar{n}_d$. The CDM velocity at the same order of perturbation obeys [22]

$$v^{(1)}_d + \mathcal{H} v^{(1)}_d = -\Phi^{(1)}. \quad (9.3)$$

Perturbing $n_{\text{CDM}}$ up to second order, we find

$$\frac{\partial \delta^{(2)}}{\partial \eta} + v^{(2)}_{d,i} - 3 \Psi^{(2)} = -2(\Phi^{(1)} + \Psi^{(1)}) v^{(1)}_{d,i} - 2v^{(1)}_{d,i} \delta^{(1)} - 2v^{(1)}_{d,i} \phi^{(1)}_i + 6\Psi^{(1)} \delta^{(1)} - (4\Psi^{(1)} + 2\Phi^{(1)}) v^{(1)}_{d,k}. \quad (9.4)$$

The right-hand side of this equation can be further manipulated by using the linear equation (9.2) to replace $v^{(1)}_{d,i}$, yielding

$$\frac{\partial \delta^{(2)}}{\partial \eta} + v^{(2)}_{d,i} - 3 \Psi^{(2)} = 4\delta^{(1)} (\Psi^{(1)} - 6 v^{(1)}_{d,i} \Phi^{(1)} - 2v^{(1)}_{d,i} \delta^{(1)} - 2v^{(1)}_{d,i} \phi^{(1)}_i + 2\Psi^{(1)} v^{(1)}_{d,k}, \quad (9.5)$$

where we use $\Phi^{(1)} = \Psi^{(1)}$. In [22], the evolution equation for the second-order CDM velocity perturbation has already been obtained

$$v^{(2)}_{d,i} + \mathcal{H} v^{(2)}_{d,i} + 2\omega_{d,i} + 2\mathcal{H} \omega_{d,i} + \Phi^{(2)}_{d,i} = 2(\Psi^{(1)} v^{(1)}_{d,i} - 2v^{(1)}_{d,i} \delta^{(1)} - 4\Phi^{(1)} \phi^{(1)}_{d,i}. \quad (9.6)$$

At linear order, we can take the divergence of equation (9.3) and, using equation (9.2) to replace the velocity perturbation, we obtain a differential equation for the CDM density contrast

$$[a \left(3\Psi^{(1)} - \delta^{(1)} \right)]' = -a \nabla^2 \Phi^{(1)}, \quad (9.7)$$

which can be rewritten as

$$\delta^{(1)}_{d,i} + \mathcal{H} \delta^{(1)}_{d,i} = S^{(1)} \quad (9.8)$$
where

$$S^{(1)} = 3\Psi^{(1)\prime\prime} + 3\mathcal{H}\Psi^{(1)\prime} + \nabla^2 \Phi^{(1)}. \quad (9.9)$$

When the radiation is dominating, the gravitational potential is mainly due to the perturbations in the photons, and \(a(\eta) \propto \eta\). For subhorizon scales, equation (9.8) can be solved following the procedure introduced in [40]. Using the Green method, the general solution to equation (9.8) (in Fourier space) is given by

$$\delta^{(1)}(k, \eta) = C_1 + C_2 \ln(\eta) - \int_0^\eta d\eta' S^{(1)}(\eta') \eta'(\ln(k\eta') - \ln(k\eta)), \quad (9.10)$$

where the first two terms correspond to the solution of the homogeneous equation. At early times, the density contrast is constant with

$$\delta^{(1)}(0) = \frac{3}{4} \Delta^{(1)}(0) = -\frac{3}{2} \Phi_k^{(1)}(0), \quad (9.11)$$

having used the adiabaticity condition, and thus we fix the integration constant as

$$C_1 = -\frac{3}{2} \Phi_k^{(1)}(0)/2, \quad (9.12)$$

and \(C_2 = 0\). The gravitational potential during the radiation-dominated epoch is given by equation (B.16) and it starts to decay as a given mode enters the horizon. Therefore, the source term \(S^{(1)}\) behaves in a similar manner and this implies that the integrals over \(\eta'\) reach asymptotically a constant value. Once the mode has crossed the horizon, we can thus write the solution as

$$\delta^{(1)}(k, \eta) = A^{(1)} \Phi_k^{(1)}(0) \ln[B^{(1)} k\eta], \quad (9.13)$$

where the constants \(A^{(1)}\) and \(B^{(1)}\) are defined as

$$A^{(1)} \Phi_k^{(1)}(0) = \int_0^\infty d\eta' S^{(1)}(\eta') \eta', \quad (9.14)$$

and

$$A^{(1)} \Phi_k^{(0)} \ln[B^{(1)}] = -\frac{3}{2} \Phi_k^{(1)}(0) - \int_0^\infty d\eta' S^{(1)}(\eta') \eta' \ln(k\eta'). \quad (9.15)$$

The upper limit of the integrals can be taken to infinity because the main contribution comes from when \(k\eta \sim 1\), and once the mode has entered the horizon the result will change by a very small quantity. Using equation (B.16) to compute the source function \(S^{(1)}\), and performing the integrals in equations (9.14) and (9.15), one finds that \(A^{(1)} = -9.0\) and \(B^{(1)} \approx 0.62\). More accurate values found in [40] through a full numerical integration of the equations are \(A^{(1)} = -9.6\) and \(B^{(1)} = 0.44\).

Before moving to the second-order case, a useful quantity to compute is the CDM velocity in a radiation-dominated epoch. From equation (9.3), it is given by

$$v_d^{(1)i} - \frac{1}{a} \int_0^\eta d\eta' \Phi^{(1)}(a(\eta')) \equiv -3(ik^{(i)}) \Phi_k^{(1)}(0) \frac{k c_s \eta - \sin(kc_s \eta)}{k^3 c_s^2 \eta^2}, \quad (9.16)$$

where the last equality holds in Fourier space and we have used equation (B.16) (and the fact that \(a(\eta) \propto \eta\) when radiation dominates).
Combining equations (9.5) and (9.6) we get the analogue of equation (9.8) at second order in perturbation theory

$$\delta_d^{(2)''} + \mathcal{H}\delta_d^{(1)'} = S^{(2)},$$  \hspace{1cm} (9.17)

where the source function is

$$S^{(2)} = 3\Psi^{(2)'} + 3\mathcal{H}\Psi^{(2)'} + \nabla^2\Phi^{(2)} - 2\partial_i(\Psi^{(1)}v^{(1)i}_d) + \nabla^2v^{(1)2}_d + 2\nabla^2\Phi^{(1)2}$$

$$+ \left[ a\left( 4\delta^{(1)'}d_d^{(1)} - 6\left( \Psi^{(1)2}\right)' + \left( \delta^{(1)2}_d - 2v^{(1)i}\delta^{(1)i}_d + 2\Psi^{(1)k}\delta^{(1)k}_d \right) \right) \right].$$ \hspace{1cm} (9.18)

In fact, we write equation (9.17) in a more convenient way as

$$\delta_d^{(2)''} - 3\Psi^{(2)''} - s'_1 + \mathcal{H}(\delta^{(2)'}_d - 3\Psi^{(2)'} - s_1) = s_2,$$ \hspace{1cm} (9.19)

where, for simplicity, we have introduced the two functions

$$s_1 = 4\delta^{(1)'}d_d^{(1)} - 6\left( \Psi^{(1)2}\right)' + \left( \delta^{(1)2}_d - 2v^{(1)i}\delta^{(1)i}_d + 2\Psi^{(1)k}\delta^{(1)k}_d \right),$$ \hspace{1cm} (9.20)

and

$$s_2 = \nabla^2\Phi^{(2)} - 2\partial_i(\Psi^{(1)'}v^{(1)i}_d) + \nabla^2v^{(1)2}_d + 2\nabla^2\Phi^{(1)2}.$$ \hspace{1cm} (9.21)

In this way, we get an equation of the same form as (9.8) in the variable \([\delta^{(2)} - 3\Psi^{(2)} - \int_0^\eta d\eta' s_1(\eta')]\) with source \(s_2\) on the right-hand side. Its solution in Fourier space, therefore, is just as equation (9.10)

$$\delta_d^{(2)} - 3\Psi^{(2)} - \int_0^\eta d\eta' s_1(\eta') = C_1 + C_2 \ln(\eta) - \int_0^\eta d\eta' s_2(\eta') \eta' \left[ \ln(k\eta') - \ln(k\eta) \right].$$ \hspace{1cm} (9.22)

As we will see, equation (9.22) provides the generalization of the Meszaros effect at second order in perturbation theory.

### 9.2. Initial conditions

In the next two sections, we will compute explicitly the expression (9.22) for the second-order CDM density contrast on subhorizon scales during the radiation-dominated era. First, let us fix the constants \(C_1\) and \(C_2\) by appealing to the initial conditions. At \(\eta \to 0\), the left-hand side of equation (9.22) is constant, as can be checked using the results of section 7.2 and the condition of adiabaticity at second order (see, e.g., [4,41]) which relates the CDM density contrast at early times on superhorizon scales to the energy density fluctuations of photons by

$$\delta_d^{(2)}(0) = \frac{3}{4}\Delta^{(2)}_{00}(0) - \frac{1}{3} \left( \delta^{(1)}_d(0) \right)^2 = \frac{3}{4}\Delta^{(2)}_{00}(0) - \frac{3}{4} \left( \Phi^{(1)}(0) \right)^2,$$ \hspace{1cm} (9.23)

where in the last step we have used equation (9.11). Therefore, we can fix \(C_2 = 0\) and

$$C_1 = \delta_d^{(2)}(0) - 3\Psi^{(2)}(0).$$ \hspace{1cm} (9.24)

Equation (6.8) gives \(\Delta^{(2)}_{00}(0) - 4\Psi^{(2)}(0)\) in terms of the primordial non-Gaussianity parametrized by \(a_{NL}\), and the expression for \(\Psi^{(2)}(0)\) have already been computed in
Let us first focus on the integral
\[
\int \text{well approximated by the expression}
\]
compute the integrals we are interested in the limit of their expression for late times contrast on subhorizon scales during the radiation-dominated epoch. Therefore, once we
Let us recall that we are interested in the evolution of the CDM second-order density perturbations, the function \(s_1(\eta)\) can be written in a more convenient way as
\[
s_1(\eta) = -6\Psi^{(1)}v^{(1)i}d,i + \left(\delta^{(1)i}_d\right)' - 2v^{(1)i}d,1 + 2(\Psi^{(1)}v^{(1)k}d,k)
\]
and then
\[
\int_0^\eta d\eta' s_1(\eta') = \left(\delta^{(1)i}_d(\eta)\right)' - \left(\delta^{(1)i}_d(0)\right)' + \int_0^\eta d\eta' \left[-2v^{(1)i}d,1 + 2(\Psi^{(1)}v^{(1)k}d,k) - 6\Psi^{(1)}v^{(1)i}d,i\right].
\]
Equation (9.26) together with equation (8.17) allows us to compute the constant \(C_1\) as
\[
C_1 = \delta^{(2)}_d(0) - 3\Psi^{(2)}(0) = \left[\frac{27}{2}(a_{NL} - 1) + \frac{9}{4}\right] \Psi^{(1)}_{k_1}(0)\Psi^{(1)}_{k_2}(0).
\]

### 9.3. Computation of the integrals over the source functions

We now compute the integrals over the functions \(s_1\) and \(s_2\) appearing in equation (9.22). Let us first focus on the integral \(\int_0^\eta d\eta' s_1(\eta')\).

Note that, using the linear equations (9.2) and (9.3) for the CDM density and velocity perturbations, the function \(s_1(\eta')\) can be written in a more convenient way as
\[
\int_0^\eta d\eta' s_1(\eta') = \left(\delta^{(1)i}_d(\eta)\right)' - \left(\delta^{(1)i}_d(0)\right)' + \int_0^\eta d\eta' \left[-2v^{(1)i}d,1 + 2(\Psi^{(1)}v^{(1)k}d,k) - 6\Psi^{(1)}v^{(1)i}d,i\right].
\]
In equation (9.29), all of the quantities are known being first-order perturbations: the linear gravitational potential \(\Psi^{(1)}\) for a radiation-dominated era is given in equation (B.16), the CDM velocity perturbation corresponds to equation (9.16) and the CDM density contrast is given by equation (9.13). Thus, the integral in equation (9.29) reads (in Fourier space)
\[
\int_0^\eta d\eta' \left[-3A^{(1)}(k_1 \cdot k_2) \frac{k_1 c_s \eta' - \sin(k_1 c_s \eta')}{k_1 c_s^3 \eta'^2} \ln(B^{(1)}k_2 \eta') + (9(k \cdot k_1) - 27k_1^2) \right.
\]
\[
\times \frac{k_1 c_s \eta' - \sin(k_1 c_s \eta')}{k_1 c_s^3 \eta'^2} \sin(k_1 c_s \eta') - k_2 c_s \eta' \cos(k_2 c_s \eta')}{k_2 c_s^3 \eta'^3} \left[\Psi^{(1)}_{k_1}(0)\Psi^{(1)}_{k_2}(0) \right].
\]
Let us recall that we are interested in the evolution of the CDM second-order density contrast on subhorizon scales during the radiation-dominated epoch. Therefore, once we compute the integrals we are interested in the limit of their expression for late times \((k \eta) \gg 1\). For the first contribution to equation (9.30), we find that at late times it is well approximated by the expression
\[
\int_0^\eta d\eta' \left[3A^{(1)}(k_1 \cdot k_2) \frac{k_1 c_s \eta' - \sin(k_1 c_s \eta')}{k_1 c_s^3 \eta'^2} \ln(B^{(1)}k_2 \eta')
\]
\[
\simeq 3A^{(1)}(k_1 \cdot k_2) \left[2.2 \left(\frac{1.2}{2} \ln(k_1 c_s \eta)\right)^2 + \ln(B^{(1)}k_2 \eta) \ln(k_1 c_s \eta)\right].
\]
We have also computed the remaining integral in equation (9.30), but it turns out to be negligible compared to equation (9.31). The reason for this is that the integrand oscillates with an amplitude decaying in time as $\eta^{-3}$, and thus it leads only to a constant (the argument is the same as that we used at linear order to compute the integrals in equation (9.10)). Thus, we can write
\[
\int_0^\eta d\eta' s_1(\eta') = \left( \delta^{(1)}_d(\eta) \right)^2 - \left( \delta^{(1)}_d(0) \right)^2 - 3A^{(1)}_d \frac{k_1 \cdot k_2}{k_1^2 c_s^2} \left[ 2.2 \left( -\frac{1.2}{2} [\ln(k_1 c_s \eta)]^2 + \ln(B^{(1)} k_2 \eta) \ln(k_1 c_s \eta) \right) \right]. \tag{9.32}
\]

We now compute the integrals over the function $s_2(\eta)$ given in equation (9.21). Since at late times $\phi^{(1)2} \sim 1/\eta^4$ and $(\Psi^{(1)} v^{(1)2}_d)_i \sim 1/\eta^3$, the main contribution to the integral will come from the two remaining terms, $\Phi^{(2)}$ and $v^{(1)2}_d$, whose amplitudes scale at late times as $1/\eta^2$
\[
s_2 \sim \nabla^2 \Phi^{(2)} + \nabla^2 v^{(1)2}_d. \tag{9.33}
\]

There are two integrals that we have to compute:
\[
\int_0^\eta d\eta' s_2(\eta') \eta' \ln(k\eta') \tag{9.34}
\]
and the one multiplying $\ln(k\eta)$
\[
\int_0^\eta d\eta' s_2(\eta') \eta'. \tag{9.35}
\]

Let us first consider the contributions from $\nabla^2 v^{(1)2}_d$. The second integral is easily computed using the expression (9.16) for the linear CDM velocity. We find that at late times, the dominant term is
\[
-\int_0^\eta d\eta' \nabla^2 v^{(1)2}_d \eta' \equiv - \left[ \frac{9}{c_s^4} k^2 \frac{k_1 \cdot k_2}{k_1^2 k_2^2} \ln(k\eta) \right] \Psi^{(1)}_{k_1}(0) \Psi^{(1)}_{k_2}(0) \quad (k\eta \gg 1). \tag{9.36}
\]

The first integral can be computed by making the following approximation: we split the integral between $0 < k\eta < 1$ and $k\eta > 1$, and for $0 < k\eta < 1$ we use the asymptotic expression
\[
v^{(1)2}_d \approx -\frac{1}{2} ik^2 \eta \Psi^{(1)}_{k}(0) \quad (k\eta \ll 1), \tag{9.37}
\]
while for $k\eta > 1$ we use the limit
\[
v^{(1)2}_d \approx -\frac{3}{c_s^2} k^2 \frac{k_1 \cdot k_2}{k_1^2 k_2^2} \Psi^{(1)}_{k}(0) \quad (k\eta \gg 1). \tag{9.38}
\]

The integral for $0 < k\eta < 1$ just gives a constant, while the integral for $k\eta > 1$ gives the dominant contribution at late times as being proportional to $[\ln(k\eta)]^2$, so that we can write
\[
\int_0^\eta d\eta' \nabla^2 v^{(1)2}_d \eta' \ln(k\eta') = \frac{9}{2c_s^4} k^2 \frac{k_1 \cdot k_2}{k_1^2 k_2^2} \left[ \ln(k\eta)]^2 \Psi^{(1)}_{k_1}(0) \Psi^{(1)}_{k_2}(0) \quad (k\eta \gg 1). \tag{9.39}
\]
As far as the contribution to the integrals (9.34) and (9.35) due to $\nabla^2 \Phi^{(2)}$ is concerned, we have only to keep track of the initial condition provided by the primordial non-Gaussianity. We have verified that all of the other terms give a negligible contribution. This is easy to understand: the integrand function on large scale is a constant, while at late times it oscillates with decreasing amplitudes as $\eta^{-2}$, and thus the integrals will tend asymptotically to a constant. We find that

$$\int_0^\eta d\eta' \nabla^2 \Phi^{(2)}(\eta') \simeq -9\Phi^{(2)}(0),$$

(9.40)

and

$$\int_0^\eta d\eta' \nabla^2 \Phi^{(2)}(\eta') \ln(k\eta') \simeq \left(-9 + 9\gamma - 9\frac{\ln 3}{2}\right) \Phi^{(2)}(0),$$

(9.41)

where $\gamma = 0.577\ldots$ is the Euler constant, and $\Phi^{(2)}(0)$ is given by equation (7.27).

Therefore, from equations (9.39), (9.36), and (9.40), (9.41) we find that for $k\eta \gg 1$,

$$\int_0^\eta d\eta' s_2(\eta') \eta' \left[\ln(k\eta') - \ln(k\eta)\right] = -\frac{9}{2} \left(\frac{k_1 \cdot k_2}{k_1^2 k_2^2}\right) \left[\ln(k\eta)\right]^2 \Psi^{(1)}_{k_1}(0) \Psi^{(1)}_{k_2}(0)$$

$$+ 9\Phi^{(2)}(0) \left(-9 + 9\gamma - 9\frac{\ln 3}{2}\right) \ln(k\eta) \Phi^{(2)}(0).$$

(9.42)

Let us collect the results of equations (9.27), (9.32) and (9.42) into equation (9.22). We find that for $k\eta \gg 1$,

$$\delta^{(2)}_{\delta}(k\eta \gg 1) = \left[-3(a_{NL} - 1)A_1 \ln(B_1 k\eta) + A_1^2 \ln(B_1 k_1 \eta) \ln(B_1 k_2 \eta)\right]$$

$$+ \left[\frac{3}{2} A_1 \frac{k_1 \cdot k_2}{k_1^2 k_2^2} 2.2 \left(-\frac{1.2}{2} \left[\ln(k_1 c_s \eta)\right]^2 + \ln(B_1 k_2 \eta) \ln(k_1 c_s \eta)\right) + (1 \leftrightarrow 2)\right]$$

$$+ \frac{9}{2} c_s^2 k_1^2 k_2^2 \left[\ln(k\eta)\right]^2 \Psi^{(1)}_{k_1}(0) \Psi^{(1)}_{k_2}(0).$$

(9.43)

Note that in equation (9.22) we have neglected $\Psi^{(2)}$, which decays on subhorizon scales during the radiation-dominated epoch (see equation (7.9)), and we have used equations (9.11) and (9.13). Equation (9.43) represents the second-order Meszaros effect: the CDM density contrast on small scales (inside the horizon) slowly grows starting from the initial conditions that, at second order, are set by the primordial non-Gaussianity parameter $a_{NL}$. As one could have guessed, the primordial non-Gaussianity is just transferred linearly. The other terms scale in time as a logarithm squared. We stress that the computation of these terms allows one to derive the full transfer function for the matter perturbations at second order accounting for the dominant second-order corrections. In the next section, we will use (9.43) to fix the initial conditions for the evolution on subhorizon scales of the photons density fluctuations $\delta^{(2)}_{\delta}(0\eta)$ after the equality epoch.

### 9.4. Computation of $\Delta^{(2)}_{0\eta}$ for $\eta > \eta_{eq}$ and modes crossing the horizon during the radiation epoch

In this section we derive the energy density perturbations $\Delta^{(2)}_{0\eta}$ of the photons during the matter-dominated epoch, for the modes that cross the horizon before equality. In
section 8, we have already solved the problem assuming matter domination for modes crossing the horizon after equality. Thus, it is sufficient to take the result (8.13) and replace the initial conditions
\[ \Delta_{00}^{(2)} = \left( 4 - \frac{8}{3c_s^2} \right) \Psi_m^{(2)}(0) + \left[ A + \frac{8}{3c_s^2} \Psi_m^{(2)}(0) \right] \cos(kc_s \eta) + B \sin(kc_s \eta) \]
\[ + \frac{2}{7} \left( 1 - \frac{2}{3c_s^2} \right) G(k_1,k_2,k) \eta^2 \Psi_k^{(1)} \Psi_k^{(1)} \]
\[ (9.44) \]
where we have restored the generic integration constants \( A \) and \( B \), \( \Psi_k^{(1)} \) is the linear gravitational potential (which is constant for the matter era) and \( \Psi_m^{(2)}(0) \) represents the initial condition for the second-order gravitational potential fixed at some time \( \eta_i > \eta_{eq} \). Equation (9.43) allows us to fix the proper initial conditions for the gravitational potentials on subhorizon scales (accounting for the fact that around the equality epoch they are mainly determined by the CDM density perturbations). At linear order, this is achieved by solving the equation for \( \delta_d^{(1)} \) which is obtained from equation (9.7) and the \((0 - 0)\)-Einstein equation which reads (see equation (A.9))
\[ 3 \mathcal{H} \Psi^{(1)'} + 3 \mathcal{H}^2 \Psi^{(1)} - \nabla^2 \Psi^{(1)} = -\frac{3}{2} \mathcal{H}^2 \left( \frac{\rho_d}{\rho} \delta_d^{(1)} + \frac{\rho_{\gamma}}{\rho} \Delta_{00}^{(1)} \right). \]
\[ (9.45) \]
On small scales, one neglects the time derivatives of the gravitational potential in equations (9.7) and (9.45) to obtain
\[ \delta_d^{(1)''} + \mathcal{H} \delta_d^{(1)'} = \frac{3}{2} \mathcal{H}^2 \delta_d^{(1)}, \]
\[ (9.46) \]
where we have also dropped the contribution to the gravitational potential from the radiation component. The solution of this equation is matched to the value that \( \delta_d^{(1)} \) has during the radiation-dominated epoch on subhorizon scales, equation (9.13), and one finds that for \( \eta \gg \eta_{eq} \) on subhorizon scales, the gravitational potential remains constant with
\[ \Psi_k^{(1)}(\eta > \eta_{eq}) = \frac{\ln(0.15k\eta_{eq})}{(0.27k\eta_{eq})^2} \Psi_k^{(1)}(0). \]
\[ (9.47) \]
We skip the details of the derivation of equation (9.47) since it is a standard computation that the reader can find, for example, in [33, 39]. Since around \( \eta_{eq} \) the dark matter begins to dominate, an approximation to the result (9.47) can be simply achieved by requiring that during matter domination the gravitational potential remains constant to a value determined by the density contrast (9.13) at the equality epoch
\[ \nabla^2 \Psi^{(1)}(\eta_{eq}) \simeq \frac{3}{2} \mathcal{H}^2 \delta_d^{(1)}(\eta_{eq}), \]
\[ (9.48) \]
from equation (9.45) on small scales, leading to
\[ \Psi_k^{(1)}(\eta > \eta_{eq}) \simeq -\frac{6}{(k\eta_{eq})^2} \delta_d^{(1)}(\eta_{eq}) = \frac{\ln(B_1k\eta_{eq})}{(0.13k\eta_{eq})^2} \Psi_k^{(1)}(0), \]
\[ (9.49) \]
where we used \( a(\eta) \propto \eta^2 \) during matter domination and equation (9.13) with \( A_1 = -9.6 \) and \( B_1 = 0.44 \).

At second order, we follow a similar approximation. The general solution for the evolution of the the second-order gravitational potential \( \Psi^{(2)} \) for \( \eta > \eta_{eq} \) is given by
We find that the horizon during the radiation epoch. The (0 - 0)-Einstein equation reads

\[3\mathcal{H}\Psi^{(2)^{'}} + 3\mathcal{H}^2\Phi^{(2)} - \nabla^2\Psi^{(2)} - 6\mathcal{H}^2\left(\Phi^{(1)}\right)^2 - 12\mathcal{H}\Phi^{(1)}\Psi^{(1)^{'}} - 3\left(\Psi^{(1)^{'}}\right)^2 + \partial_t\Psi^{(1)^{'}}\Phi^{(1)} - 4\Phi^{(1)}\nabla^2\Psi^{(1)} = -\frac{3}{2}\mathcal{H}^2\left(\frac{\rho_d^2}{\rho}\delta_d^{(2)} + \frac{\rho}{\rho}\Delta_{00}^{(2)}\right). \tag{9.50}\]

We fix the initial conditions with the matching at equality (neglecting the radiation component)

\[\nabla^2\Psi^{(2)} - \partial_t\Psi^{(1)^{'}}\Phi^{(1)} - 4\Phi^{(1)}\nabla^2\Psi^{(1)}|\eta_{eq} \cong \frac{3}{2}\mathcal{H}^2\delta_d^{(2)}|\eta_{eq}, \tag{9.51}\]

where for small scales we neglected the time derivatives in equation (9.50). Using equation (9.43) to evaluate \(\delta_d^{(2)}|\eta_{eq}\) and equation (9.47) to evaluate \(\Psi_{k1}^{(1)}(\eta_{eq})\) we find in Fourier space

\[\Psi^{(2)}(\eta_{eq}) = \left[-3(a_{NL} - 1)\frac{\ln(B_1k_{\eta_{eq}})}{(0.13\eta_{eq})^2} + \frac{(k_1 \cdot k_2)}{k^2} - 4\right]\frac{\ln(0.15k_1\eta_{eq})\ln(0.15k_2\eta_{eq})}{(0.27k_1\eta_{eq})^2} \frac{\ln(0.15k_1\eta_{eq})\ln(0.15k_2\eta_{eq})}{(0.27k_2\eta_{eq})^2} + A_1\ln(B_1k_{\eta_{eq}})\frac{\ln(B_1k_{2\eta_{eq}})}{(0.13\eta_{eq})^2} - \frac{27}{c_4^4k_1k_2}\frac{k_1 \cdot k_2}{k^2}\frac{[\ln(k_1\eta_{eq})]^2}{(0.13\eta_{eq})^2}\]

\[-\frac{3}{2}\frac{k_1 \cdot k_2}{c_s^2k_1^2k_2^2}\left[\frac{1.2}{2}\frac{[\ln(k_1\eta_{eq})]^2}{(0.13\eta_{eq})^2}\right] \Psi^{(1)}_{k_1}(0)\Psi^{(1)}_{k_2}(0). \tag{9.52}\]

In equation (9.44), the initial condition \(\Psi_{m}^{(2)}(0)\) is given by equation (9.52) and \(\Psi^{(1)}\) is given by equation (9.47). The integration constants can be fixed by comparing at \(\eta \simeq \eta_{eq}\) the oscillating part of equation (9.44) to the solution \(\Delta_{00}^{(2)}\) obtained for modes crossing the horizon before equality and for \(\eta < \eta_{eq}\), equation (7.15). Thus, for \(\eta \gg \eta_{eq}\) and \(k \gg \eta_{eq}^{-1}\), we find that

\[\Delta_{00}^{(2)} = -4\Psi^{(2)}(\eta_{eq}) + \tilde{A}\cos(kc_s\eta) - \frac{2\tilde{g}}{7}G(k_1, k_2, k)\eta^2\Psi^{(1)}_{k_1}(\eta_{eq})\Psi^{(1)}_{k_2}(\eta_{eq}), \tag{9.53}\]

where

\[\tilde{A} = 6\Psi^{(2)}(0) - \frac{6(k \cdot k_1)(k \cdot k_2)}{c_s^4k_1k_2}\cos(kc_s\eta_{eq})\left\{2k_1k_2\cos(k_1c_s\eta_{eq})\cos(k_2c_s\eta_{eq}) - 2k_1k_2\cos(kc_s\eta_{eq})ight\}

\[\times \left\{k_1^2 + k_2^2 - k_1^2\right\} + \left\{k_1^2 + k_2^2 + k_4^2 - 2k_1^2k_2^2 - 2k_1^2k_2^2 - 2k^2k_1^2\right\}^{-1}, \tag{9.54}\]

and \(\Psi^{(2)}(0)\) is given in equation (8.17).

10. Conclusions

In this paper, we have performed an analytical investigation of the second-order CMB anisotropies generated at recombination. In particular, we have provided analytical solutions for the acoustic oscillations of the photon–baryon fluid in the tight coupling limit. One of the steps of this computation requires us to generalize at second order the
Meszaros effect, describing the evolution of the CDM density perturbations on subhorizon scales. If on one hand we have kept track of the primordial non-Gaussian contribution, on the other the main effort has been put on the determination of all the additional second-order effects arising at recombination, which are a new potential source to the non-Gaussianity of the CMB anisotropies. They constitute the main core of the second-order radiation transfer function necessary to establish the level of non-Gaussianity in the CMB. Our results give a simplified estimate of the non-linear dynamics at recombination and serve as support for a numerical study of these effects which is under investigation [29] and which will provide a more accurate analysis.

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Appendix A. Einstein equations

In this appendix, we provide all of the necessary expressions to deal with the gravitational part of the problem we are interested in, that is, the second-order CMB anisotropies generated at recombination and the acoustic oscillations of the baryon–photon fluid. The first part of the appendix contains the expressions for the metric and Einstein tensor perturbed up to second order around a flat Friedmann–Robertson–Walker background, the energy momentum tensors for massless (photons) and massive particles (baryons and cold dark matter), and the relevant Einstein equations. The second part deals with the evolution equations and the solutions for the second-order gravitational potentials in the Poisson gauge. According to the regimes studied in sections 7 and 8, we have considered various epochs, in particular the radiation and the matter-dominated eras.

A.1. The metric tensor

We adopt the Poisson gauge which eliminates one scalar degree of freedom from the $g_{0i}$ component of the metric and one scalar and two vector degrees of freedom from $g_{ij}$. We will use a metric of the form

$$ds^2 = a^2(\eta) \left[ -e^{2\Phi}d\eta^2 + 2\omega_i \, dx^i \, d\eta + (e^{-2\Psi} \delta_{ij} + \chi_{ij}) \, dx^i \, dx^j \right], \quad (A.1)$$

where $a(\eta)$ is the scale factor as a function of the conformal time $\eta$, and $\omega_i$ and $\chi_{ij}$ are the vector and tensor perturbation modes respectively. Each metric perturbation can be expanded into a linear (first-order) and a second-order part, as for example, the gravitational potential $\Phi = \Phi^{(1)} + \Phi^{(2)}/2$. However, in the metric (A.1) the choice of the exponentials greatly helps in computing the relevant expressions, and thus we will always keep them where it is convenient. From equation (A.1), one recovers at linear order the well-known longitudinal gauge while at second order, one finds $\Phi^{(2)} = \phi^{(2)} - 2(\Phi^{(1)})^2$ and $\Psi^{(2)} = \psi^{(2)} + 2(\psi^{(1)})^2$ where $\phi^{(1)}$, $\psi^{(1)}$ and $\phi^{(2)}$, $\psi^{(2)}$ (with $\phi^{(1)} = \Phi^{(1)}$ and $\psi^{(1)} = \Psi^{(1)}$) are the first- and second-order gravitational potentials in the longitudinal (Poisson) gauge adopted in [4, 42] as far as scalar perturbations are concerned. For the vector and tensor perturbations, we will neglect linear vector modes since they are not produced in standard mechanisms for the generation of cosmological perturbations (as inflation), and we also...
neglect tensor modes at linear order, since they give a negligible contribution to second-order perturbations. Therefore, we take $\omega_i$ and $\chi_{ij}$ to be second-order vector and tensor perturbations of the metric.

### A.2. The connection coefficients

Let us give our definitions for the connection coefficients and their expressions for the metric (A.1). The number of spatial dimensions is $n = 3$. Greek indices $(\alpha, \beta, \ldots, \mu, \nu, \ldots)$ run from 0 to 3, while Latin indices $(a, b, \ldots, i, j, k, \ldots, m, n, \ldots)$ run from 1 to 3. The total spacetime metric $g_{\mu\nu}$ has the signature $(-, +, +, +)$. The connection coefficients are defined as

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\rho} \left( \frac{\partial g_{\rho\gamma}}{\partial x^\beta} + \frac{\partial g_{\rho\beta}}{\partial x^\gamma} - \frac{\partial g_{\beta\gamma}}{\partial x^\rho} \right).$$

(A.2)

The Riemann tensor is defined as

$$R^\alpha_{\beta\mu\nu} = \Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\beta\mu,\nu} + \Gamma^\alpha_{\lambda\mu} \Gamma^\lambda_{\beta\nu} - \Gamma^\alpha_{\lambda\nu} \Gamma^\lambda_{\beta\mu}. $$

(A.3)

The Ricci tensor is a contraction of the Riemann tensor

$$R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}, $$

(A.4)

and in terms of the connection coefficient it is given by

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\nu\alpha} + \Gamma^\alpha_{\sigma\alpha} \Gamma^\sigma_{\mu\nu} - \Gamma^\alpha_{\sigma\nu} \Gamma^\sigma_{\mu\alpha}. $$

(A.5)

The Ricci scalar is given by contracting the Ricci tensor

$$R = R^\mu_{\mu}. $$

(A.6)

The Einstein tensor is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. $$

(A.7)

For the connection coefficients, we find

$$\Gamma_{00}^0 = \mathcal{H} + \Phi', $$

$$\Gamma_{0i}^0 = \frac{\partial \Phi}{\partial x^i} + \mathcal{H} \omega_i, $$

$$\Gamma_{ij}^0 = \omega^i + \mathcal{H} \omega_j + e^{2\Psi+2\Phi} \frac{\partial \Phi}{\partial x^i}, $$

$$\Gamma_{ij}^0 = -\frac{1}{2} \left( \frac{\partial \omega_j}{\partial x^i} + \frac{\partial \omega_i}{\partial x^j} \right) + e^{-2\Psi-2\Phi} (\mathcal{H} - \Psi') \delta_{ij} + \frac{1}{2} \chi'_{ij} + \mathcal{H} \chi_{ij}, $$

$$\Gamma_{0j}^i = (\mathcal{H} - \Psi') \delta_{ij} + \frac{1}{2} \chi'_{ij} + \frac{1}{2} \left( \frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} \right), $$

$$\Gamma_{jk}^i = -\mathcal{H} \omega^i \delta_{jk} - \frac{\partial \Psi}{\partial x^k} \delta_{ij}^i - \frac{\partial \Psi}{\partial x^j} \delta_{ik}^i + \frac{\partial \Psi}{\partial x^i} \delta_{jk} + \frac{1}{2} \left( \frac{\partial \chi^i_j}{\partial x^k} + \frac{\partial \chi^i_k}{\partial x^j} + \frac{\partial \chi^i_j}{\partial x^i} \right). $$

(A.8)
A.3. Einstein tensor

The components of the Einstein tensor are

\[
G^0_0 = - \frac{e^{-2\phi}}{a^2} \left[ 3H^2 - 6H\Psi' + 3(\Psi')^2 - e^{2\phi} + 2\phi \right], \\
G^i_0 = \frac{2e^{2\phi}}{a^2} \left[ \partial^i \Psi' + (H - \Psi') \partial^i \Phi \right] - \frac{1}{2a^2} \nabla^2 \omega^i + \left( 4H^2 - 2\frac{a''}{a} \right) \frac{\omega^i}{a^2}, \\
G^i_j = \frac{1}{a^2} \left[ e^{-2\phi} \left( H^2 - 2\frac{a''}{a} - 2\Psi' \Psi' - 3(\Psi')^2 + 2H(\Phi' + 2\Psi') + 2\Psi'' \right) \right. \\
+ e^{2\phi} \left( \partial_k \Phi \partial^k \Phi + \nabla^2 \Phi - \nabla^2 \Psi \right) \right] \delta^j_i \\
+ \frac{e^{2\phi}}{a^2} \left( -\partial^i \Phi \partial^j \Phi - \partial^i \partial^j \Phi + \partial^i \partial^j \Phi + \partial^i \Phi \partial_j \Phi + \partial^i \Phi \partial_j \Psi - \partial^i \Psi \partial_j \Phi \right) \\
- \frac{H}{a^2} \left( \partial^i \omega^j + \partial_j \omega^i \right) - \frac{1}{2a^2} \left( \partial^i \omega^j + \partial_j \omega^i \right) + \frac{1}{a^2} \left( \omega^j \partial^i \omega_j + \frac{1}{2} \chi^j_i - \frac{1}{2} \nabla^2 \chi^j_i \right). 
\]

(A.11)

Taking the traceless part of equation (A.11), we find

\[
\Psi - \Phi = Q, 
\]

where \( Q \) is defined by

\[
\nabla^2 Q = -P + 3N, 
\]

with

\[
P \equiv P^i_i, 
\]

and

\[
P^i_j = \partial^i \Phi \partial_j \Psi + \frac{1}{2} \left( \partial^i \Phi \partial_j \Phi - \partial^i \Phi \partial_j \Psi \right) + 4\pi G_N a^2 e^{-2\phi} T^i_j, \\
\nabla^2 N = \partial_i \partial^i P^i_j. 
\]

(A.15)

The trace of equation (A.11) therefore gives

\[
e^{-2\phi} \left( H^2 - 2\frac{a''}{a} - 2\Psi' \Psi' - 3(\Psi')^2 + 2H(3\Psi' - Q') + 2\Psi'' \right) \\
+ \frac{e^{2\phi}}{3} \left( 2\partial_k \Phi \partial^k \Phi + \partial_k \Psi \partial^k \Psi - 2\partial_k \Phi \partial^k \Psi + 2(P - 3N) \right) = \frac{8\pi G_N}{3} a^2 T^k_k. 
\]

(A.16)

From equation (A.10), we may deduce an equation for \( \omega^i \)

\[
-\frac{1}{2} \nabla^2 \omega^i + \left( 4H^2 - 2\frac{a''}{a} \right) \omega^i \\
= - \left( \delta^i_j - \frac{\partial^i \partial_j}{\nabla^2} \right) \left( 2 \left( \partial^i \Psi' + (H - \Psi') \partial^i \Phi \right) - 8\pi G_N a^2 e^{-2\phi} T^i_0 \right). 
\]

(A.17)
A.4. Energy momentum tensors

A.4.1. Energy momentum tensor for photons. The energy momentum tensor for photons is defined as

\[ T_{\mu\nu} = \frac{2}{\sqrt{-g}} \int \frac{d^3 P}{(2\pi)^3} \frac{P^\mu P^\nu}{P^0} f, \]  

(A.18)

where \( g \) is the determinant of the metric (A.1) and \( f \) is the distribution function. We thus obtain

\[ T_{\gamma 0} = -\bar{\rho}_\gamma \left( 1 + \Delta^{(1)}_{00} + \frac{\Delta^{(2)}_{00}}{2} \right), \]  

(A.19)

\[ T_{\gamma i} = -\frac{4}{3} e^{\phi + \Phi} \bar{\rho}_\gamma \left( v^{(1)i}_\gamma + \frac{1}{2} v^{(2)i}_\gamma + \Delta^{(1)}_{00} v^{(1)i}_\gamma \right) + \frac{1}{3} \bar{\rho}_\gamma e^{\phi - \Phi} \omega^i, \]  

(A.20)

\[ T_{ij} = \bar{\rho}_\gamma \left( \Pi^i_{\gamma j} + \frac{1}{3} \delta^i_j \left( 1 + \Delta^{(1)}_{00} + \frac{\Delta^{(2)}_{00}}{2} \right) \right), \]  

(A.21)

where \( \bar{\rho}_\gamma \) is the background energy density of photons and

\[ \Pi^i_{\gamma j} = \int \frac{d\Omega}{4\pi} \left( n^i n^j - \frac{1}{3} \delta^i_j \right) \left( \Delta^{(1)} + \frac{\Delta^{(2)}_{00}}{2} \right) \]  

(A.22)

are the quadrupole moments of the photons.

A.4.2. Energy momentum tensor for massive particles. The energy momentum tensor for massive particles of mass \( m \), number density \( n \) and degrees of freedom \( g_d \)

\[ T_{\mu\nu} = \frac{g_d}{\sqrt{-g}} \int \frac{d^3 Q}{(2\pi)^3} \frac{Q^\mu Q_\nu}{Q^0} g_m, \]  

(A.23)

where \( g_m \) is the distribution function. We obtain

\[ T_{m 0} = -\rho_m = -\bar{\rho}_m \left( 1 + \delta^{(1)}_m + \frac{1}{2} \delta^{(2)}_m \right), \]  

(A.24)

\[ T_{m i} = -e^{\phi + \Phi} \rho_m v^i_m = -e^{\phi + \Phi} \bar{\rho}_m \left( v^{(1)i}_m + \frac{1}{2} v^{(2)i}_m + \delta^{(1)}_m v^{(1)i}_m \right), \]  

(A.25)

\[ T_{m j} = \rho_m \left( \delta^i_j \frac{T_{m i}}{m} + v^i_m v^j_m \right) = \bar{\rho}_m \left( \delta^i_j \frac{T_{m i}}{m} + v^{(1)i}_m v^{(1)}_{m j} \right), \]  

(A.26)

where \( \bar{\rho}_m \) is the background energy density of the massive particles and we have included the equilibrium temperature \( T_m \).
Appendix B. Solutions of Einstein equations in various eras

B.1. Matter-dominated era

During the phase in which the CDM is dominating the energy density of the universe, \( a \sim \eta^2 \) and we may use equation (A.16) to obtain an equation for the gravitational potential at first-order in perturbation theory (for which \( \Phi^{(1)} = \Psi^{(1)} \))

\[
\Phi^{(1)''} + 3 \mathcal{H} \Phi^{(1)'} = 0,
\]

which has two solutions \( \Phi^{(1)}_+ = \text{constant} \) and \( \Phi^{(1)}_- = \mathcal{H}/a^2 \). At the same order of perturbation theory, the CDM velocity can be read off from equation (A.10)

\[
v^{(1)i} = -\frac{2}{3 \mathcal{H}} \partial_i \Phi^{(1)}.
\]

At second order, using equations (A.16) and (A.13) and the fact that the first-order gravitational potential is constant, we find an equation for the gravitational potential at second order \( \Psi^{(2)} \)

\[
\Psi^{(2)''} + 3 \mathcal{H} \Psi^{(2)'} = S_m,
\]

\[
S_m = -\partial_k \Phi^{(1)} \partial^k \Phi^{(1)} + N = -\partial_k \Phi^{(1)} \partial^k \Phi^{(1)} + \frac{10}{3} \frac{\partial_i \partial^i}{\nabla^2} (\partial_i \Phi^{(1)} \partial_j \Phi^{(1)}) ,
\]

whose solution is

\[
\Psi^{(2)} = \Psi^{(2)}_m(0) + \Phi^{(1)}_+ \int_0^\eta d\eta' \frac{\Phi^{(1)}_+(\eta')}{W(\eta')} S_m(\eta') - \Phi^{(1)}_- \int_0^\eta d\eta' \frac{\Phi^{(1)}_- (\eta')}{W(\eta')} S_m(\eta')
\]

\[
= \Psi^{(2)}_m(0) - \frac{1}{14} \left( \partial_k \Phi^{(1)} \partial^k \Phi^{(1)} - \frac{10}{3} \frac{\partial_i \partial^i}{\nabla^2} \left( \partial_i \Phi^{(1)} \partial_j \Phi^{(1)} \right) \right) \eta^2 ,
\]

with \( W(\eta) = W_0/a^3 \) (\( a_0 = 1 \)) being the Wronskian obtained from the corresponding homogeneous equation. In equation (B.4), \( \Psi^{(2)}_m(0) \) represents the initial condition (taken conventionally at \( \eta \rightarrow 0 \)) deep in the matter-dominated phase.

From equation (A.17), we may compute the vector perturbation in the metric

\[
-\frac{1}{2} \nabla^2 \omega^i = 3 \mathcal{H}^2 \frac{1}{\nabla^2} \partial_j \left( \partial^j \delta^{(1)} v^{(1)j} - \partial^j \delta^{(1)} v^{(1)i} \right) ,
\]

where we have made use of the fact that the vector part of the CDM velocity satisfies the relation \( \left( \delta^i - \frac{\partial_i \partial^i}{\nabla^2} \right) v^{(2)i} = -\omega^i \).

The matter contrast \( \delta^{(1)} \) satisfies the first-order continuity equation

\[
\delta^{(1)'} = -\frac{\partial v^{(1)i}}{\partial x^i} = -\frac{2}{3 \mathcal{H}} \nabla^2 \Phi^{(1)} .
\]

Going to Fourier space, this implies that

\[
\delta^{(1)}_k = \delta^{(1)}_k(0) + \frac{k^2 \eta^2}{6} \Phi^{(1)} ,
\]

where \( \delta^{(1)}_k(0) \) is the initial condition in the matter-dominated period.
B.2. Universe filled by matter and a relativistic component

We extend the results above for the case of CDM and a relativistic component whose energy density will be indicated with $\rho_v$. At first order in perturbation theory, the trace of the $(i - j)$-component of Einstein equations, equation (A.16), yields

$$
\Psi^{(1)\nu} + 3 H \Psi^{(1)\nu} = \mathcal{H}Q^{(1)\nu} + \frac{1}{3} \nabla^2 Q^{(1)} + \frac{1}{2} \frac{\dot{\rho}_v}{\mathcal{H}^2} \Delta^{(1)\nu}_{\nu0}, \tag{B.8}
$$

From equation (A.10), the linear CDM velocity can be expressed as

$$
v^{(1)i}_m = - \frac{2}{3} \mathcal{H}^{-2} \frac{\dot{\rho}_T}{\rho_m} \left( \partial^i \Psi^{(1)\nu} + \mathcal{H} \partial^i \Phi^{(1)} \right) - \frac{8}{9} \frac{\dot{\rho}_v}{\mathcal{H}^2} \nu^{(1)i}_v. \tag{B.9}
$$

At second order using equation (A.16) and equation (A.13), we find

$$
\Psi^{(2)\nu} + 3 \mathcal{H} \Psi^{(2)\nu} = S_m,
$$

$$
S_m = 6 \left( \Psi^{(1)\nu} \right)^2 + 2 \Psi^{(1)\nu} \Psi^{(1)\nu} - \frac{1}{3} \left( 2 \partial_k \Phi^{(1)} \partial^k \Phi^{(1)} + \partial_k \Psi^{(1)} \partial^k \Psi^{(1)} - 2 \partial_k \Phi^{(1)} \partial^k \Psi^{(1)} \right)
+ \frac{4 \dot{\rho}_T}{9 \rho_m} \mathcal{H}^{-2} \left( \partial^i \Psi^{(1)\nu} + \mathcal{H} \partial^i \Phi^{(1)} \right)^2 + \left( \frac{8}{9} \right)^2 \mathcal{H}^2 \frac{\dot{\rho}_v}{\mathcal{H}^2} \nu^{(1)i} \nu^{(1)i}_v
+ \frac{36}{27} \frac{\dot{\rho}_v}{\rho_m} \left( \partial^i \Psi^{(1)\nu} + \mathcal{H} \partial^i \Phi^{(1)} \right) \nu^{(1)i}_v + \mathcal{H} \frac{\dot{\rho}_v}{\mathcal{H}^2} \Delta^{(2)\nu}_{\nu0}
+ \frac{1}{3} \nabla^2 Q^{(2)} + \mathcal{H} Q^{(2)} + \frac{4}{3} \left( \Phi^{(1)} + \Psi^{(1)} \right) \nabla^2 Q^{(1)} + 2 \mathcal{H} \frac{\dot{\rho}_v}{\mathcal{H}^2} \Phi^{(1)} \Delta^{(1)\nu}_{\nu0}, \tag{B.10}
$$

where at second order, we find

$$
\frac{1}{2} \nabla^2 Q^{(2)} = 3 \nabla^2 \partial^i \partial^j \left[ \partial^i \Phi^{(1)} \partial^j \Phi^{(1)} + \frac{1}{2} \left( \partial^i \Phi^{(1)} \partial_j \Phi^{(1)} - \partial^j \Phi^{(1)} \partial_i \Phi^{(1)} \right) - \frac{3}{2} \mathcal{H}^2 \frac{\dot{\rho}_v}{\mathcal{H}^2} \nu^{(1)i} \nu^{(1)i}_v \right]
+ \frac{3}{2} \mathcal{H}^2 \frac{\dot{\rho}_v}{\mathcal{H}^2} \left( \partial^i \Phi^{(1)} \partial_j \Phi^{(1)} \right) \Pi^{(1)ij}_{\nu} \nu^{(1)i}_v
- \frac{3}{2} \mathcal{H}^2 \frac{\dot{\rho}_v}{\mathcal{H}^2} \left( \partial^i \Phi^{(1)} \partial^j \Phi^{(1)} \right) \nu^{(1)i}_v
- \frac{1}{2} \left( \partial^i \Phi^{(1)} \partial^{(1)\nu} \partial^i \Psi^{(1)} \partial^{(1)\nu} \partial_j \Psi^{(1)} \right), \tag{B.11}
$$

where $v^2 \equiv v^{(1)k} v^{(1)i}_k$ and one has to use the expression (B.9).

For the second-order vector metric perturbation, we find

$$
- \frac{1}{2} \nabla^2 \omega^i = \left( 4 \mathcal{H}^2 - 2 \frac{\dot{\rho}_v}{\rho_T} \right) \omega^i = - \left( \delta^i_j \right) \omega^j - \partial^i \nabla^2 \omega_j
+ \frac{2}{2} \mathcal{H}^2 \frac{\dot{\rho}_v}{\mathcal{H}^2} v^{(2)ij}_v + 4 \mathcal{H}^2 \frac{\dot{\rho}_v}{\mathcal{H}^2} \nu^{(1)i}_v v^{(1)j}_v + \frac{3}{2} \mathcal{H}^2 \frac{\dot{\rho}_v}{\mathcal{H}^2} \nu^{(2)ij}_v + 3 \Delta^{(1)\nu}_{\nu0} \nu^{(1)i}_v
+ 4 \mathcal{H}^2 \left( \Phi^{(1)} - \Psi^{(1)} \right) \dot{\rho}_v v^{(1)i}_v + 3 \mathcal{H}^2 \frac{\dot{\rho}_v}{\mathcal{H}^2} \Phi^{(1)} v^{(1)i}_v + \mathcal{H}^2 \frac{\dot{\rho}_v}{\mathcal{H}^2} \omega^j. \tag{B.12}
$$
B.3. Radiation-dominated era

We consider a universe dominated by photons and massless neutrinos. The energy momentum tensor for massless neutrinos has the same form as that for the photons. During the phase in which radiation is dominating the energy density of the universe, $a \sim \eta$ and we may combine equations (A.9) and (A.16) to obtain an equation for the gravitational potential $\Psi^{(1)}$ at first order in perturbation theory

$$\Psi^{(1)''} + 4\mathcal{H}\Psi^{(1)'} - \frac{1}{3} \nabla^2 \Psi^{(1)} = \mathcal{H}Q^{(1)'} + \frac{1}{3} \nabla^2 Q^{(1)},$$

(B.13)

where the total anisotropy stress tensor is

$$\Pi^i_T = \rho_T \Pi^i_\gamma + \rho_\nu \Pi^i_\nu.$$

(B.14)

We may safely neglect the quadrupole and solve equation (B.13) setting $u_{\pm} = \Phi^{(1)}_{\pm} \eta$. Then equation (B.13) becomes, going to Fourier space,

$$u'' + \frac{2}{\eta} u' + \left( \frac{k^2}{3} - \frac{2}{\eta^2} \right) u = 0.$$

(B.15)

This is the spherical Bessel function of order 1 with solutions $u_+ = j_1(\eta k \sqrt{3})$, the spherical Neumann function, and $u_- = n_1(\eta k \sqrt{3})$, the spherical Bessel function. The latter blows up as $\eta$ gets very small and we discard it on the basis of the initial conditions. The final solution is therefore

$$\Phi^{(1)}_k = 3\Phi^{(1)(0)} \sin(\eta k \sqrt{3}) - (\eta k \sqrt{3}) \cos(\eta k \sqrt{3}),$$

(B.16)

where $\Phi^{(1)(0)}$ represents the initial condition deep in the radiation era.

At the same order of perturbation theory, the radiation velocity can be read off from equation (A.10)

$$v^{(1)i}_\gamma = -\frac{1}{2\mathcal{H}^2} \frac{(a\partial^i \Phi^{(1)})'}{a}.$$  

(B.17)

At second order, combining equations (A.9), (A.16), we find

$$\Psi^{(2)''} + 4\mathcal{H}\Psi^{(2)'} - \frac{1}{3} \nabla^2 \Psi^{(2)} = S_\gamma,$$

(B.18)

$$S_\gamma = 4 \left( \Psi^{(1)'} \right)^2 + 2\Phi^{(1)'} \Psi^{(1)'} + \frac{4}{3} (\Phi^{(1)} + \Psi^{(1)}) \nabla^2 \Psi^{(1)}$$

$$- \frac{2}{3} (\partial_k \Phi^{(1)} \partial^k \Phi^{(1)} + \partial_k \Psi^{(1)} \partial^k \Psi^{(1)} - \partial_k \Phi^{(1)} \partial^k \Psi^{(1)})$$

$$+ \mathcal{H}Q^{(2)'} + \frac{4}{3} \nabla^2 Q^{(2)} + \frac{4}{3} (\Phi^{(1)} + \Psi^{(1)}) \nabla^2 Q^{(1)},$$

(B.19)

$$\frac{1}{2} \nabla^2 Q^{(2)} = -\partial_k \Phi^{(1)} \partial^k \Psi^{(1)} - \frac{1}{2} (\partial_k \Phi^{(1)} \partial^k \Phi^{(1)} - \partial_k \Psi^{(1)} \partial^k \Psi^{(1)})$$

$$+ \frac{3}{2} \nabla^2 \partial^j \frac{\partial \Pi^{(2)i}}{\nabla^2} - \frac{9}{2} \mathcal{H}^2 \partial^j \frac{\partial \Pi^{(2)i}}{\nabla^2} \left( \Psi^{(1)i} \right),$$

(B.20)
whose solution is
\[ \Psi^{(2)} = \Psi_{\text{hom}}^{(2)} + \Phi_{1}^{(1)} \int_{0}^{\eta} d\eta' \frac{\Phi_{1}^{(1)}(\eta')}{W(\eta')} S_{\gamma}(\eta') - \Phi_{-1}^{(1)} \int_{0}^{\eta} d\eta' \frac{\Phi_{-1}^{(1)}(\eta')}{W(\eta')} S_{\gamma}(\eta'), \]

where \( W(\eta) = (a(0)/a)^{4} \) is the Wronskian, and \( \Psi_{\text{hom}}^{(2)} \) is the solution of the homogeneous equation.

The equation of motion for the vector metric perturbations reads
\[
-\frac{1}{2} \nabla^{2} \omega^{j} + 4 \mathcal{H}^{2} \omega^{j} = \left( \delta_{ij} - \frac{\partial^{i} \partial^{j}}{\mathcal{H}^{2}} \right) \left[ 2 \Psi^{(1)} \partial^{i} \Phi^{(1)} - 2 \mathcal{H}^{2} \left( \frac{\rho_{T}}{\rho_{T}} \nu_{i}^{(2)} + \frac{\bar{\rho}_{v}}{\bar{\rho}_{v}} \nu_{i}^{(2)} \bar{v}^{j} + 2 \frac{\bar{\rho}_{v}}{\rho_{T}} \Delta_{00}^{(1)} \nu_{i}^{(1)} + 2 \frac{\rho_{T}}{\bar{\rho}_{v}} \Delta_{00}^{(1)} \nu_{i}^{(1)} \nu_{i}^{(1)} \right) + 2 (\Phi^{(1)} - \Psi^{(1)} \frac{\bar{\rho}_{v}}{\rho_{T}} \nu_{i}^{(1)} \bar{v}^{j}) + \mathcal{H}^{2} \frac{\bar{\rho}_{v}}{\rho_{T}} \nu_{i}^{(1)} \bar{v}^{j} \right].
\] (B.22)

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