A multi-partite entanglement measure and its holographic dual

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In this letter we define a natural generalization of the von Neumann entropy to multiple parties that is symmetric with respect to all the parties. We call this measure the multi-entropy. We show that for conformal field theories with holographic duals, the multi-entropy is computed by the area of an appropriate “soap-film” anchored on the boundary. We conjecture the quantum version of this prescription that takes into account the sub-leading corrections in $G_N$. As expected, the “quantum soap-film prescription” becomes the quantum extremal surface prescription in the case of two parties.

I. INTRODUCTION

In recent years, quantum information theoretic notions such as entanglement entropy [1][5] have been immensely useful in shedding light on some of the important questions pertaining to quantum gravity. In particular, it has helped understand holographic encoding of the gravitational Hilbert space and in turn the Black-hole entropy [7] [8] and the information paradox. We index the replicas by the subscript $i$. Let $\dim \mathcal{H}_a = d_a$ and $|\alpha_a\rangle, \alpha_a \in \{1, \ldots, d_a\}$ be its orthonormal basis. In $|\alpha_a\rangle$ basis, the state $|\Psi\rangle$ is given as

$$|\Psi\rangle = \sum_{\alpha_1=1}^{d_1} \cdots \sum_{\alpha_q=1}^{d_q} \psi_{\alpha_1 \ldots \alpha_q} |\alpha_1\rangle \otimes \cdots \otimes |\alpha_q\rangle. \quad (1)$$

A q-party entanglement measure is the information contained in the wavefunction $\psi$ (and its complex conjugate) that is invariant under “local unitary transformations”. They are products of unitary operations on the individual system $a$. Naturally, such a measure is obtained by contracting $\alpha_a$ indices for all $a$. Now $\alpha_a$ is a fundamental index on $\psi$ and an anti-fundamental index of $\bar{\psi}$. Hence, in order to construct invariants we need equal number of $\psi$’s and $\bar{\psi}$’s. We will call this number the replica number $n$. We index the replicas by the superscript $(i)$. In particular, the Hilbert space $\mathcal{H}_a$ of the $i$-th replica is denoted as $\mathcal{H}_a^{(i)}$ and the associated basis as $|\alpha_a^{(i)}\rangle$. The wavefunction of the $i$-th replica is then $\psi_{\alpha_1^{(i)} \ldots \alpha_q^{(i)}}$ and similarly its conjugate is $\bar{\psi}_{\alpha_1^{(i)} \ldots \alpha_q^{(i)}}$.

A. Bi-partite

Let us consider the following bi-partite entanglement measure with a fixed replica number $n$

$$Z_n = \text{Tr}(\rho^n), \quad \text{where} \quad \rho_{\alpha_1^{(i)} \alpha_2^{(i)}}^{(1)} \equiv \psi_{\alpha_1^{(i)} \alpha_2^{(i)}} \bar{\psi}_{\alpha_1^{(i)} \alpha_2^{(i)}}, \quad (2)$$

Here $\rho$ is the density matrix for party 1. With this notation, the squared norm of $|\Psi\rangle$ is $Z_1$. This measure is related to the familiar $n$-th Renyi entropy $S_n$ as

$$S_n = \frac{1}{1-n} \log(Z_n/Z_n^n). \quad (3)$$

Let us reformulate this measure using permutations acting on replicas. This will be useful while dealing with entanglement in quantum field theories using twist operators.

$$Z_n = (\psi_{\alpha_1^{(1)} \alpha_2^{(1)}} \psi_{\alpha_1^{(2)} \alpha_2^{(1)}} \ldots)(\bar{\psi}_{\alpha_1^{(1)}} \alpha_2^{(1)} \bar{\psi}_{\alpha_1^{(2)}} \alpha_2^{(2)} \ldots) \quad (4)$$

where $\sigma$ is a cyclic permutation element acting in the space of replicas $i = 1, \ldots, n$ of the second party i.e. $\sigma(\alpha_2^{(i)}) = \alpha_2^{(i+1)}$ with $\alpha_2^{(n+1)} \equiv \alpha_2^{(1)}$. Because this permutation acts only on the replica copies of party-2, we denote it as $\sigma_2$. A priori, it would seem that we should specify two permutation elements $(\sigma_1, \sigma_2)$, the first one acting on replicas of party-1 and the other acting on replicas of party-2 to get the most general index contraction, i.e.

$$(\psi_{\alpha_1^{(1)} \alpha_2^{(1)}} \psi_{\alpha_1^{(2)} \alpha_2^{(2)}} \ldots)(\bar{\psi}_{\alpha_1^{(1)}} \alpha_2^{(1)} \bar{\psi}_{\alpha_1^{(2)}} \alpha_2^{(2)} \ldots) \quad (5)$$

However, we can always relabel replicas of $\bar{\psi}$ so that one of the $\sigma$’s, say $\sigma_1$, is brought to id form. In other words, $(\sigma_1, \sigma_2)$ and $(\sigma_1 g, \sigma_2 g), g \in S_n$ yield the same measure. We express this as an equivalence relation

$$(\sigma_1, \sigma_2) \sim (\sigma_1, \sigma_2) \quad (6)$$
There is yet another equivalence, namely under simultaneous relabeling of replica copies of both \(\psi\)'s and \(\bar{\psi}\)'s. Such a relabeling leads to conjugation of both \(\sigma_a\) by some \(h \in S_n\). We express this as the equivalence relation

\[
(\sigma_1, \sigma_2) \sim h^{-1} (\sigma_1, \sigma_2) h.
\]

(7)

If we use equation (6) to set \(\sigma_1 = \text{id}\), then thanks to equation (7), the bi-partite entanglement measure depends only on the conjugacy class of \(\sigma_2\). Conjugacy class in \(S_n\) is specified by cycle decomposition \(\{p_k\}\) where \(p_k\) is the number of \(k\)-cycles such that \(\sum_k k p_k = n\). For this class, it is easy to see that the resulting measure (3) is

\[
\prod_k \left(\text{Tr}(\rho^k)\right)^{p_k} = \prod_k \left(Z_k\right)^{p_k}. \tag{8}
\]

This shows that the ring of bi-partite measures is generated by \(Z_k\)’s.

\section*{B. Multi-partite}

Following this discussion, we can write a general multi-partite measure by giving permutation elements \((\sigma_1, \ldots, \sigma_q)\) with equivalence relations

\[
(\sigma_1, \ldots, \sigma_q) \sim (\sigma_1, \ldots, \sigma_q) \cdot g. \tag{9}
\]

\[
(\sigma_1, \ldots, \sigma_q) \sim h^{-1} (\sigma_1, \ldots, \sigma_q) h. \tag{10}
\]

Unlike bi-partite case, where Renyi entropies are the only independent measures of entanglement, for the case of three or higher number of parties, the number of measures increase exponentially or faster with the replica number \(n\). In this letter, we will not concern ourselves with general \(q\)-party measures but rather focus on a particular family of measures labeled by an integer and that is symmetric under the exchange of all the parties.

As in the bi-partite case, equation (9) can be used to set \(\sigma_1 = \text{id}\). But this “gauge fixing” obscures the symmetry between all the parties as it treats party-1 differently from others. So we will not use the gauge freedom (9) and (10) just yet and specify all \(\sigma_a\)’s. We will do so with \(n^q\) replicas. We index the replicas with a \(q\)-dimensional index vector \((i_1, \ldots, i_q)\) where each \(i_a = \{1, \ldots, n\}\). The permutation element \(\sigma_a\) is defined as the cyclic element acting only on \(i_a\):

\[
\sigma_a \cdot (\ldots, i_a, \ldots) = (\ldots, i_a + 1, \ldots), \quad a = 1, \ldots, q. \tag{11}
\]

As an element of the permutation group \(S_n^q\), its conjugacy class is \(p_n = n^{q-1}\) with all other \(p_k = 0\). It is also clear that this measure is symmetric in all the \(q\) parties.

Now we use the gauge freedom (9) with \(g = \sigma_1^{-1}\). This gives us an equivalent set of permutation elements

\[
(\text{id}, \sigma_2 \sigma_1^{-1}, \ldots, \sigma_q \sigma_1^{-1}). \tag{12}
\]

Note that the action of any of these permutations on \((i_1, \ldots, i_q)\) keeps the sum \(\sum_a i_a\) invariant. As a result, their action on \(n^q\) replicas splits into \(n\) orbits of \(n^{q-1}\) elements with \(\sum_a i_a = \text{constant}(\text{mod } n)\). Each orbit gives rise to the same invariant of \(\psi\)'s and \(\bar{\psi}\)'s. It is convenient to work with a single orbit, say with \(\sum_a i_a = 0\text{(mod } n)\). For this orbit, we can use the relabeling freedom (10) to set the replica index \(i_1\) to 1. With this gauge fixing, we get a convenient presentation of the rest of the permutation elements \((\sigma_2 \sigma_1^{-1}, \ldots, \sigma_q \sigma_1^{-1}) \equiv (\sigma_2, \ldots, \sigma_q)\).

\[
\sigma_a \cdot (\ldots, i_a, \ldots) = (\ldots, i_a + 1, \ldots), \quad a = 2, \ldots, q. \tag{13}
\]

We have denoted the gauge fixed permutation elements as \(\tilde{\sigma}_a\) to avoid confusion with un-gauge fixed permutation elements \(\sigma_a\). The index set in equation (13) starts from \(i_2\) because \(i_1\) has already been set to 1. We denote the measure defined by \(\tilde{\sigma}_a\) elements in (13) as \(Z^{(q)}_n\) and define Renyi multi-entropy \(S^{(q)}_n\) as

\[
S^{(q)}_n = \frac{1}{1 - n} \log \left(\frac{Z^{(q)}_n}{Z^{(q)}_1 n^{q-1}}\right). \tag{14}
\]

Here the factor \((Z^{(q)}_1 n^{q-1})\) serves to normalize the state. This is the family of \(q\)-party measure that we are interested in. Let us pause for a moment to consider properties of the set \(\{\tilde{\sigma}_a : a = 2, \ldots, q\}\).

\begin{itemize}
  \item \(\tilde{\sigma}_a\)’s have the same equivalence class given by the cycle \(p_n = n^{q-2}\) with all other \(p_k = 0\).
  \item \(\tilde{\sigma}_a^{-1} \tilde{\sigma}_b\) for \(a \neq b\) also have the same equivalence class.
  \item Together, \(\tilde{\sigma}_a\)’s generate the subgroup \(\mathbb{Z}_n^q\) of the permutation group.
\end{itemize}

These observations will be important in the future discussions.

In a way, fixing \(\sigma_1 = 1\) has a very natural interpretation. If we consider a single \(\psi\) and a single \(\bar{\psi}\) with \(i_1\) index contraction, we get the density matrix on \(H_2 \otimes \cdots \otimes H_q\). This density matrix has the index structure

\[
\rho_{\alpha_2 \cdots \alpha_q}^{\alpha_1 (i_2 + \cdots + i_q)}. \tag{15}
\]

This means the gauge fixed permutations \(\tilde{\sigma}_a\)’s describe index contractions of this density matrix. Let us see this with two examples, for \(q = 2\),

\[
Z^{(2)}_n = \sum_{i_2} \rho_{\alpha_2}^{\alpha_1 (i_2 + \cdots + i_q)} = \sum \rho_{\alpha_2}^{\alpha_1 (i_2 + \cdots + i_q)} \rho_{\alpha_2}^{\alpha_2 (i_1 + \cdots + i_q)} \cdots \rho_{\alpha_2}^{\alpha_1 (i_2 + \cdots + i_n)}. \tag{16}
\]

Here sum is over all repeated indices. With \(\tilde{\sigma}_2\) being a cyclic element as specified by equation (13), this is nothing but \(Z_n\) defined earlier. So we have showed \(S^{(2)}_n = S_n\). Graphically, denoting \(\rho\) as in figure [x], we get the graphical representation of \(S^{(2)}_n\) as a circular lattice of \(\rho\)’s of length \(n\) as in figure [x]. Each directed link in this lattice represents index contraction from a fundamental index to an anti-fundamental index. The advantage of this graphical notation is that it admits straightforward extension.
to higher number of parties. For instance, in the case of 3-parties, we denote the density matrix as in figure 2. Then $Z_n^{(3)}$ is the index contraction given by the toric lattice of $\rho$’s with both sides being of length $n$. This is shown in figure 2. Similarly, $Z_n^{(q)}$ is the index contraction given by $(q - 1)$-dimensional toric lattice with all sides being of length $n$.

Arguably the most important measure of bi-partite entanglement, the von-Neumann entropy $S$, is obtained as a limit

$$S = \lim_{n \to 1} S_n. \quad (16)$$

The Renyi multi-entropy $S_n^{(q)}$ is defined such that the limit

$$S^{(q)} = \lim_{n \to 1} S_n^{(q)}, \quad (17)$$
called the multi-entropy, has a number of nice properties.

1. It is symmetric in all the parties.
2. For $q = 2$, it reduces to the von-Neumann entropy.
3. It admits a convenient holographic description.

Properties 1 and 2 are obvious from the discussion so far. In the rest of the paper, we will discuss multi-entropy for holographic theories and describe what we exactly mean by property 3. The multi-entropy can also be computed from $Z_n^{(q)}$ using the formula

$$S^{(q)} = -\partial_q \log \left( \frac{Z_n^{(q)}}{(Z_1^{(q)})^{n-1}} \right) |_{n=1}. \quad (18)$$

We will use this formula and equation (17) interchangeably.

C. Example

Although the multi-entropy is amenable to holographic computation as we will see shortly, it is difficult to compute for finite dimensional quantum systems. This is because multi-entropy is defined only via the replica trick and, in general, it is difficult to compute $Z_n^{(q)}$ as an analytic function of $n$. However, for a special class of states that we call generalized GHZ states, the multi-entropy is readily computed. Consider $|\Psi\rangle \in \otimes_{a=1}^{n} \mathcal{H}_a$ with dim $\mathcal{H}_a = d$. The generalized GHZ state is defined as

$$|\Psi\rangle_{\text{GHZ}} = \sum_i \lambda_i |i\rangle \otimes \ldots \otimes |i\rangle. \quad (19)$$

Here $|i\rangle$ is an orthonormal basis in $\mathcal{H}_a$ for all $a$. The state is normalized so $\sum_i |\lambda_i|^2 = 1$. It is easy to compute $Z_n^{(q)}$ because the same index $i$ runs through all the contractions. It contributes $|\lambda_i|^{2n^{q-1}}$. Then the q-Renyi entropy is

$$S_n^{(q)} = \frac{1}{1 - q} \log \left( \sum_i |\lambda_i|^{2n^{q-1}} \right). \quad (20)$$

Taking the $n \to 1$ limit,

$$S^{(q)} = (1 - q) \sum_i |\lambda_i|^2 \log |\lambda_i|^2. \quad (21)$$

A bi-partite state can always be taken to the generalized GHZ form via Schmidt decomposition. Then $|\lambda_i|^2$ are the eigenvalues of Schmidt matrix. It is clear that $S^{(q)}$ agrees with the von-Neumann entropy for $q = 2$. We believe that it is extremely important to develop techniques to compute multi-entropy for general states to understand its quantum information theoretic properties.

III. MULTI-ENTROPY FROM HOLOGRAPHY

In this section, we will discuss computation of multi-entropy in a $D$-dimensional conformal field theory $\mathcal{T}$. Let the state $|\Psi\rangle$ be defined on a time symmetric Cauchy slice $\mathcal{R}$ of a $D$-manifold $\mathcal{M}$. It is given by a Euclidean path integral on the half-space $\mathcal{M}_\phi$ such that $\partial \mathcal{M}_\phi = \mathcal{R}$. The dual bra $\langle \Psi |$ is constructed by Euclidean path integral on the other half $\mathcal{M}_\bar{\phi}$. It is obtained from $\mathcal{M}_\phi$ by reflecting across $\mathcal{R}$. The squared norm of $\Psi$ is the partition function $Z_{\mathcal{M}}$ on $\mathcal{M}$. Let us decompose $\mathcal{R}$ into $q$ number of disjoint
regions $\mathcal{R}_a$, such that $\bigcup_a \mathcal{R}_a = \mathcal{R}$. Let the Hilbert space on region $\mathcal{R}_a$ be $\mathcal{H}_a$. We are interested in computing multi-entropy of the state $|\Psi\rangle$ under the decomposition $\otimes_a \mathcal{H}_a$. For theories that admit a weakly coupled gravity dual, this problem can be addressed holographically.

The replica trick [11] involves working with the tensor product theory $T^\otimes n^{-1}$ on $\mathcal{M}$. This theory has the discrete symmetry $S_n$. It admits co-dimension 2 twist defects labeled by elements of this permutation group. For every pair of regions $(\mathcal{R}_a, \mathcal{R}_b)$ that share a boundary, we insert the twist operator $O_{a^{-1}_b}$ on the common boundary. The measure $Z_n^{(q)}$ is then given by the correlation function of these twist operators. Let us denote the resulting replicated manifold as $\mathcal{M}_n$. The correlation function of twist operators is the partition function $Z_{\mathcal{M}_n}$ on $\mathcal{M}_n$. Using (14) we have

$$S_n^q = \frac{1}{1-n} \log(Z_{\mathcal{M}_n}/(Z_{\mathcal{M}})^{n^{-1}}).$$

Following [12], we will proceed to analyze this problem holographically. Let $\mathcal{B}_n$ be dominant the gravity solutions such that $\partial \mathcal{B}_n = \mathcal{M}_n$. The manifold $\mathcal{B}_n$ is a smooth manifold e.g. in the case of Einstein gravity with negative cosmological constant, it is of constant negative curvature. The holographic dictionary gives

$$\log Z_{\mathcal{M}_n} = -S_{\text{grav}}(\mathcal{B}_n).$$

Here $S_{\text{grav}}(\mathcal{X})$ is the gravitational action evaluated on the solution $\mathcal{X}$. For $n = 1$, this gives $\log Z_{\mathcal{M}} = -S_{\text{grav}}(\mathcal{B}), \mathcal{B} \equiv B_1$.

The background fields on the manifold $\mathcal{M}_n$ enjoy a replica symmetry. This is the symmetry generated by the permutation elements $\sigma_a^{-1}$ associated to all the twist operators. As remarked earlier, this group is $S_n$. Following [12], we will assume that the dominant bulk solution $\mathcal{B}_n$ enjoys this symmetry. The solution $\mathcal{B}_n$ consists of co-dimension 2 loci that are invariant under certain subgroups of the replica symmetry group. Some of these loci, called “external”, are anchored at the fixed points on the boundary (these are locations of twist operator insertions on $\mathcal{M}$) while the rest are “internal”. Let us denote the loci that are anchored at the fixed points corresponding to the twist operator $O_{a}$ as $\mathcal{L}_{a}$. They are invariant under the $\mathbb{Z}_n$ subgroup generated by $g$. Generically, two $\mathcal{L}'s$ can merge to form a different $\mathcal{L}$. Merging obeys the algebra $\mathcal{L}_{g_1} \times \mathcal{L}_{g_2} \rightarrow \mathcal{L}_{g_1 g_2}$. The internal loci come about because of such merging. Below we will assume that

1. Every fixed point locus is of the form $\mathcal{L}_{\sigma_a^{-1} \sigma_b}$.

We will justify this assumption shortly. Thanks to the special property of our permutation elements $\sigma_a$ stated below equation (14), this means that any locus is invariant under some $\mathbb{Z}_n$ subgroup of the replica symmetry. Every locus $\mathcal{L}_{g}$ appears in groups of $n^{q-2}$. This is because, the orbit of the action of replica group $\mathbb{Z}_n^\otimes q^{-1}$ consists of $n^{q-2}$ elements as its stabilizer is $\mathbb{Z}_n$. For $q = 2$, the number of elements in the orbit is 1.

We now make use of the replica symmetry in the bulk to construct the orbifold $\mathcal{B}_n = B_n/\mathbb{Z}_n$. Due to symmetry, the classical gravitational actions on the two spaces are related as

$$S_{\text{grav}}(\mathcal{B}_n) = n^{q-1} S_{\text{grav}}(\mathcal{B}_n).$$

The orbifold $\mathcal{B}_n$ has a nice property that $\partial \mathcal{B}_n = \mathcal{M}$. A group of $n^{q-2}$ number of $\mathcal{L}_g$ become a single conical singularity of opening angle $2\pi/n$ in the orbifold $\mathcal{B}_n$. Let us denote this singularity as $\mathcal{L}_\hat{g}$. Let us denote the web created by these singularities as $\mathcal{W}$. Consider a co-dimension 1 slice $\mathcal{C} \subset \mathcal{B}_n$ that contains $\mathcal{W}$ and $\partial \mathcal{C} = \mathcal{R}$. There are multiple such slices and the precise choice doesn’t matter for the following discussion. Every singularity becomes a co-dimension 1 wall in $\mathcal{C}$ and the web $\mathcal{W}$ yields its chamber decomposition. As we move from $\mathcal{R}_a$ to $\mathcal{R}_b$ through $\mathcal{C}$, we must encounter at least one wall because the permutation elements $\sigma_a$ and $\sigma_b$ are different. At this stage we make an assumption about $\mathcal{W}$ that

2. There is no chamber which lies completely in the interior of $\mathcal{C}$.

As a result, we get a one-to-one map between the chambers and boundary regions. Let us denote the chamber adjacent to $\mathcal{R}_a$ as $\mathcal{C}_a$. It has the property $\partial \mathcal{C}_a \cap \mathcal{M} = \mathcal{R}_a$. The web $\mathcal{W}$ consists of only those walls that separate $\mathcal{C}_a$ and $\mathcal{C}_b$ for some $(a, b)$. Such a wall must be of the type $\mathcal{L}_{\sigma_a^{-1} \sigma_b}$. The parent $\mathcal{L} \subset \mathcal{B}_n$ must also be of the same type as well. This justifies our assumption.

To compute the $q$-Renyi entropy using equations (23) and (24), we need to evaluate the gravitational action on the orbifold solution $\mathcal{B}_n$. In what follows, we will specialize to the case of Einstein gravity. The orbifold solution is a smooth solution of Einstein’s equations with constant negative curvature except at $\mathcal{L}$ where it has a conical singularity of opening angle $2\pi/n$. There are multiple approaches to compute the action on this solution. In our opinion, the quickest approach is to use a cosmic brane to model the singularity. We add to the gravitational action, a brane action $S_{\text{br}}$ for a brane supported on $\mathcal{W}$ where

$$S_{\text{br}}^{(n)} = \frac{n-1}{4nG_N} \int d y^{D-1} \sqrt{h} = \frac{n-1}{4nG_N} A.$$
locri. To accommodate such a meeting we simply let the corresponding cosmic branes meet. A priori, one can add arbitrary terms in the brane action supported only at the meeting locus. However, as the brane is only an auxiliary object that engineers the conical singularity, we do not add any extra term to $S_{br}^{(n)}$.

Finding the solution to the theory $S_{grav} + S_{br}^{(n)}$ is still a daunting task (see [15–17] for computation of bi-partite Renyi entropy). But to compute multi-entropy, only the limit $n \to 1$ is relevant. As the only $n$ dependence appears in the coefficient in $S_{br}^{(n)}$, the solution can be analytically continued away from $n$ integer. Moreover, in the limit $n \to 1$, the tension of the brane goes to zero and the solution can be found in the probe limit as brane web configuration that extremizes $S_{br}$ in the fixed background $\mathcal{B}$. The solution obeys the equation of motion $\partial_\mu S_{grav} = -\partial_\mu S_{br}^{(n)} = -\frac{A}{4G_N}$. This, along with equations [17, 22] and (23), shows that the multi-entropy is given a simple formula

$$S^{(q)} = \frac{A(W)}{4G_N}$$

Here $A(W)$ is the area of the extremal brane web $W$ in $\mathcal{B}$. In case there are multiple solutions, we pick the one with minimum extremal area as that corresponds to the most dominant solution. The brane web $W$ obeys the topological conditions,

1. $W$ is anchored at the boundaries of all the regions $R_a$'s.

2. $W$ contains sub-webs that are homologous to all the regions $R_a$'s.

The second condition is the reformulation of the statement that between any two chambers $C_a$ and $C_b$ there must be at least one wall. As the solution minimizes the area subject to these conditions, it doesn’t allow any chamber that lies completely in the interior of $C$. This justifies our assumption [2].

The brane web in question is extremely familiar in $D = 4$. In the three dimensional Cauchy slice $\mathcal{C}$ it resembles a soap film in hyperbolic space anchored on a given “wire frame” at infinity. Specifying the anchor does not specify the soap film uniquely but when combined with the homology condition and the global minimum condition, it does. The close analogy to soap-films leads us to call our prescription, the soap-film prescription. To describe the soap-film it is convenient to introduce the label $\mathcal{L}^{(k)}$ for the special locus of co-dimension $k$. It is defined inductively as the meeting locus of $\mathcal{L}^{(k-1)}$ with $\mathcal{L}^{(2)} = \mathcal{L}$. For $D = 4$ it is known that

- Three $\mathcal{L}^{(2)}$'s meet at $\mathcal{L}^{(3)}$ at an angle $2\pi/3 = \cos^{-1}(-1/2)$.

- Four $\mathcal{L}^{(3)}$’s meet at $\mathcal{L}^{(4)}$ at an angle $\cos^{-1}(-1/3)$.

These are known as Plateau’s laws of soap-film. In dimension $D$ we expect $k + 1$ of $\mathcal{L}^{(k)}$’s to meet at $\mathcal{L}^{(k+1)}$ at an angle $\cos^{-1}(-1/k)$ and so on until we get to $\mathcal{L}^{(D)}$.

We have verified our proposal for 2D CFTs with large central charge. These calculations will appear in an accompanying paper [18].

IV. QUANTUM CORRECTIONS

Quantum correction to the multi-entropy can be found using the replica trick in the bulk. We consider $n^{th}$ replica copies of the orbifold $\mathcal{B}_a$ and insert the bulk twist operators $\mathcal{V}_q$ at $\mathcal{L}_q$. This has the effect of reversing the orbifold and give back the geometry $\mathcal{B}_a$. The partition function on the replicated manifold is then simply the partition function $Z_n^{(q)}$. This is the partition function that goes into the calculation of the multi-entropy. If the replica trick was performed on the bulk geometry $\mathcal{B}_a$ it would have given the bulk multi-entropy $S_{bulk}^{(q)}(W)$ corresponding to the chamber decomposition $C_a$ directly, however, because the replica trick was performed on $\mathcal{B}_a$, it is not obvious that what we get is $S_{bulk}^{(q)}(W)$. This situation is similar to the bi-partite case [19] (see also [20]). There, the difference between the two quantities is captured by changing the classical solution by $O(G_N)$ to account for the one loop expectation value of the stress tensor. This changes the area and hence the entanglement entropy by $O(1)$. We expect a similar formula to give the sub-leading correction to the multi-entropy

$$S^{(q)} = \frac{\langle \hat{A}(W) \rangle}{4G_N} + S_{bulk}^{(q)}(W) + \text{c.t.}$$

Here $\hat{A}(W)$ is the area operator of the soap film and c.t. are the counter-terms that render $S_{bulk}^{(q)}(W)$ finite.

Following [6, 21] we conjecture a formula that is valid to all orders in $1/G_N$ perturbation theory: Multi-entropy is given by the above formula but $W$ is not the ordinary area minimizing soap-film but rather the “quantum soap-film” i.e. the soap film that minimizes the combination

$$\frac{\langle \hat{A}(W) \rangle}{4G_N} + S_{bulk}^{(q)}(W).$$

In analogy with the bi-partite case, we call this prescription the quantum soap-film prescription.

For general gravitational theories, we expect that the multi-entropy is given by Wald-type corrections as computed in [6, 22] for bi-partite case.

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[1] P. Calabrese and J. L. Cardy, Entanglement entropy and quantum field theory, J. Stat. Mech. 0406, P06002 (2004) [arXiv:hep-th/0405152].

[2] S. Ryu and T. Takayanagi, Holographic derivation of entanglement entropy from AdS/CFT, Phys. Rev. Lett. 96, 181602 (2006) [arXiv:hep-th/0603001].

[3] V. E. Hubeny, M. Rangamani, and T. Takayanagi, A Covariant holographic entanglement entropy proposal, JHEP 07, 062 [arXiv:0705.0016 [hep-th]].

[4] H. Casini, M. Huerta, and R. C. Myers, Towards a derivation of holographic entanglement entropy, JHEP 05, 036 [arXiv:1102.0440 [hep-th]].

[5] X. Dong, Holographic Entanglement Entropy for General Higher Derivative Gravity, JHEP 01, 044 [arXiv:1310.5713 [hep-th]].

[6] N. Engelhardt and A. C. Wall, Quantum Extremal Surfaces: Holographic Entanglement Entropy beyond the Classical Regime, JHEP 01, 073 [arXiv:1408.3203 [hep-th]].

[7] J. D. Bekenstein, Black holes and entropy, Phys. Rev. D 7, 2333 (1973).

[8] S. W. Hawking, Particle Creation by Black Holes, Commun. Math. Phys. 43, 199 (1975).

[9] S. Dutta and T. Faulkner, A canonical purification for the entanglement wedge cross-section, JHEP 03, 178 [arXiv:1905.00577 [hep-th]].

[10] P. Calabrese, J. Cardy, and E. Tonni, Entanglement negativity in quantum field theory, Phys. Rev. Lett. 109, 130502 (2012) [arXiv:1206.3092 [cond-mat.stat-mech]].

[11] Y. Kusuki, J. Kudler-Flam, and S. Ryu, Derivation of holographic negativity in AdS$_3$/CFT$_2$, Phys. Rev. Lett. 123, 131603 (2019) [arXiv:1907.07824 [hep-th]].

[12] A. Lewkowycz and J. Maldacena, Generalized gravitational entropy, JHEP 08, 090 [arXiv:1304.4926 [hep-th]].

[13] W. G. Unruh, G. Hayward, W. Israel, and D. Mcmanus, Cosmic string loops are straight, Phys. Rev. Lett. 62, 2897 (1989).

[14] B. Boisseau, C. Charmousis, and B. Linet, Dynamics of a selfgravitating thin cosmic string, Phys. Rev. D 55, 616 (1997) [arXiv:gr-qc/9607029].

[15] M. Headrick, Entanglement Renyi entropies in holographic theories, Phys. Rev. D 82, 126010 (2010) [arXiv:1006.0047 [hep-th]].

[16] L.-Y. Hung, R. C. Myers, M. Smolkin, and A. Yale, Holographic Calculations of Renyi Entropy, JHEP 12, 047 [arXiv:1110.1084 [hep-th]].

[17] X. Dong, The Gravity Dual of Renyi Entropy, Nature Commun. 7, 12472 (2016) [arXiv:1601.06788 [hep-th]].

[18] A. Gadde, V. Krishna, and T. Sharma, Multi-partite entanglement measures and their holographic duals, To appear.

[19] T. Faulkner, A. Lewkowycz, and J. Maldacena, Quantum corrections to holographic entanglement entropy, JHEP 11, 074 [arXiv:1307.2892 [hep-th]].

[20] T. Barrella, X. Dong, S. A. Hartnoll, and V. L. Martin, Holographic entanglement beyond classical gravity, JHEP 09, 109 [arXiv:1306.4682 [hep-th]].

[21] X. Dong and A. Lewkowycz, Entropy, Extremality, Euclidean Variations, and the Equations of Motion, JHEP 01, 081 [arXiv:1705.08453 [hep-th]].

[22] L.-Y. Hung, R. C. Myers, and M. Smolkin, On Holographic Entanglement Entropy and Higher Curvature Gravity, JHEP 04, 025 [arXiv:1101.5813 [hep-th]].