Multivariable Adaptive Harmonic Steady-State Control for Rejection of Sinusoidal Disturbances Acting on an Unknown System

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Abstract—This paper presents an adaptive harmonic steady-state (AHSS) controller, which addresses the problem of rejecting sinusoids with known frequencies that act on a completely unknown multi-input multi-output linear time-invariant system. We analyze the stability and closed-loop performance of AHSS for single-input single-output systems. In this case, we show that AHSS asymptotically rejects disturbances.

I. INTRODUCTION

The rejection of sinusoidal disturbances is a fundamental control objective in many active noise and vibration control applications such as noise cancellation [1], helicopter vibration reduction [2], and active rotor balancing [3].

For an accurately modeled linear time-invariant (LTI) system, the internal-model principle can be used to design a feedback controller capable of rejecting sinusoidal disturbances of known frequencies [4]–[6]. In this case, disturbance rejection is accomplished by incorporating copies of the disturbance dynamics in the feedback loop.

If, on the other hand, an inaccurate model of the system is not available, but the open-loop dynamics are asymptotically stable, then adaptive feedback cancellation can be used to accomplish disturbance rejection [7], [8]. One approach for sinusoidal disturbance rejection is harmonic steady-state (HSS) control [9], which has been used for helicopter vibration reduction [2] and active rotor balancing [3]. To discuss HSS control, let \( G_{yu}(j\omega) \) denote the control-to-performance transfer function, and assume that there is a single known disturbance frequency \( \omega \). Then, HSS control requires an estimate of \( G_{yu}(j\omega) \). In the SISO case, the estimate of \( G_{yu}(j\omega) \), which is a single complex number, must have an angle within 90° of \( \angle G_{yu}(j\omega) \) to ensure closed-loop stability. In the MIMO case, closed-loop stability is ensured provided that the estimate of \( G_{yu}(j\omega) \) is sufficiently accurate. If there are multiple disturbance frequencies, then estimates are required at each frequency.

For certain applications \( G_{yu}(j\omega) \) can be difficult to estimate or subject to change. To address this uncertainty, online estimation methods have been combined with HSS control [10]–[12]. For example, a recursive-least-squares identifier is used in [10], [11] to estimate \( G_{yu}(j\omega) \) in real time; however, an external excitation signal, which degrades performance, is required to ensure stability.

In this paper, we present a new adaptive harmonic steady-state (AHSS) controller, which is effective for rejecting sinusoids with known frequencies that act on a completely unknown MIMO LTI system. We analyze the stability and closed-loop performance for SISO systems. We show that AHSS asymptotically rejects disturbances.

The new AHSS algorithm in this paper is a frequency-domain method, and all computations are with discrete Fourier transform (DFT) data. The AHSS algorithm including DFT is demonstrated on a simulation of an acoustic duct.

II. NOTATION

Let \( F \) be either \( \mathbb{R} \) or \( \mathbb{C} \). Let \( x(i) \) denote the \( i \)th element of \( x \in \mathbb{F}^n \), and let \( A_{i,j} \) denote the element in row \( i \) and column \( j \) of \( A \in \mathbb{F}^{m \times n} \). Let \( \| \cdot \| \) be the 2-norm on \( \mathbb{F}^n \).

Next, let \( \lambda^* \) denote the complex conjugate transpose of \( A \in \mathbb{F}^{m \times n} \), and define \( \| A \|_F \triangleq \sqrt{\text{tr} A^*A} \), which is the Frobenius norm of \( A \in \mathbb{F}^{m \times n} \).

Let \( \text{spec}(A) \triangleq \{ \lambda \in \mathbb{C} : \det(\lambda I - A) = 0 \} \) denote the spectrum of \( A \in \mathbb{F}^{m \times n} \), and let \( \lambda_{\max}(A) \) denote the maximum eigenvalue of \( A \in \mathbb{F}^{m \times n} \), which is Hermitian positive semidefinite. Let \( \angle \lambda \) denote the argument of \( \lambda \in \mathbb{C} \) defined on the interval \((−\pi, \pi]\) rad. Let \( \text{OLHP} \), \( \text{ORHP} \), and \( \text{CUD} \) denote the open-left-half plane, open-right-half plane, and closed unit disk in \( \mathbb{C} \), respectively. Define \( \mathbb{N} \triangleq \{0, 1, 2, \cdots \} \) and \( \mathbb{Z}^+ \triangleq \mathbb{N} \setminus \{0\} \).

III. PROBLEM FORMULATION

Consider the system

\[
\dot{x}(t) = Ax(t) + Bu(t) + D_1d(t),
\]

\[
y(t) = Cx(t) + Du(t) + D_2d(t),
\]

where \( t \geq 0 \), \( x(t) \in \mathbb{R}^n \) is the state, \( x(0) = x_0 \in \mathbb{R}^n \) is the initial condition, \( u(t) \in \mathbb{R}^m \) is the control, \( y(t) \in \mathbb{R}^p \) is the measured performance, \( d(t) \in \mathbb{R}^q \) is the unmeasured disturbance, and \( A \in \mathbb{R}^{n \times n} \) is asymptotically stable. Define the transfer functions \( G_{yu}(s) \triangleq C(sI - A)^{-1}D_1 + D_2 \), and \( G_{ud}(s) \triangleq C(sI - A)^{-1}D_1 + D_2 \).

Our objective is to design a control \( u \) that reduces or even eliminates the effect of the disturbance \( d \) on the performance \( y \). We seek to design a control that relies on no model information of (1) and (2), and requires knowledge of only the disturbance frequencies \( \omega_1, \cdots, \omega_q \).

For simplicity, we focus on the case where \( d \) is the single-tone disturbance \( d(t) = d_c \cos \omega t + d_s \sin \omega t \). However, the
adaptive controller presented in this paper generalizes to the case where $d$ consists of multiple tones. We address multiple tones in Example 3.

For the moment, assume that $G_{yy}$, $G_{yd}$, $d_c$, and $d$ are known, and consider the harmonic control $u(t) = u_c \cos \omega t + u_s \sin \omega t$, where $u_c$, $u_s \in \mathbb{R}^m$. Define $\hat{u} \triangleq u_c - p u_s$, which is the value at frequency $\omega$ of the DFT obtained from a sampling of $u$. The HSS performance of (1) and (2) with control $\hat{u}$ is

$$y_{\text{hss}}(t, \hat{u}) \triangleq \text{Re} \left( M_s \hat{u} + \hat{d} \right) \cos \omega t - \text{Im} \left( M_s \hat{u} + \hat{d} \right) \sin \omega t,$$

where $M_s \triangleq G_{yy}(j \omega) \in \mathbb{C}^{\ell \times m}$ and $\hat{d} \triangleq G_{yd}(j \omega)(d_c - p u_s) \in \mathbb{C}^\ell$. The HSS performance $y_{\text{hss}}$ is the steady-state response of $y$, that is, $\lim_{t \to \infty} [y_{\text{hss}}(t, \hat{u}) - y(t)] = 0$ [13, Chap. 12.12]. Consider the cost function

$$J(\hat{u}) \triangleq \lim_{t \to \infty} \frac{1}{T} \int_0^T y_{\text{hss}}(\tau, \hat{u}) d\tau,$$

which is the average power of $y_{\text{hss}}$. Define

$$\hat{y}_{\text{hss}}(\hat{u}) \triangleq M_s \hat{u} + \hat{d},$$

which is the value at frequency $\omega$ of the DFT obtained from a sampling of $y_{\text{hss}}$. It follows from (3)–(5) that $J(\hat{u}) = \frac{1}{2} \hat{y}_{\text{hss}}(\hat{u}) \hat{y}_{\text{hss}}(\hat{u})$. The following result provides an expression for an open-loop control $\hat{u} = u_\sigma$ that minimizes $J$. The proof is omitted due to space limitations.

**Theorem 1.** Consider the cost function (4), and assume rank $M_s = \min\{\ell, m\}$. Then, the following statements hold:

i) Assume $\ell > m$, and define $u_\sigma \triangleq -(M_s^* M_s)^{-1} M_s^* \hat{d}$. Then,

$$y_{\text{hss}}(u_\sigma) = \left( I_\ell - M_s (M_s^* M_s)^{-1} M_s^* \right) \hat{d},$$

where $I_\ell$ is the identity matrix of size $\ell$. If $\hat{d} \not\in \text{Im}(M_s^* M_s)^{-1}$, then $y_{\text{hss}}(u_\sigma) \not\in \text{Im}(M_s^* M_s)^{-1}$, and for all $\hat{u} \in \mathbb{C}^m \setminus \{u_\sigma\}$, $J(u_\sigma) < J(\hat{u})$.

ii) Assume $\ell = m$, and define $u_\sigma \triangleq -M_s^{-1} \hat{d}$. Then,

$$y_{\text{hss}}(u_\sigma) = 0, J(u_\sigma) = 0,$$ and for all $\hat{u} \in \mathbb{C}^m \setminus \{u_\sigma\}$, $J(u_\sigma) < J(\hat{u})$.

iii) Assume $\ell < m$, and let $u_\sigma \in \{-M_s^* (M_s^* M_s)^{-1} \hat{d} + (I_m - M_s^* (M_s^* M_s)^{-1} M_s^*) v : v \in \mathbb{C}^m\}$. Then,$y_{\text{hss}}(u_\sigma) = 0$ and $J(u_\sigma) = 0$.

**Theorem 2.** Consider the closed-loop system (9), which consists of (7) and (8). Assume that $\Lambda \subset \text{ORHP}$, and assume that $\rho$ satisfies

$$0 < \rho < \min_{\lambda \in \Lambda} \frac{2\text{Re} \lambda}{|\lambda|^2}.$$ Then, for all $u_0 \in \mathbb{C}^m$, $u_{\infty} \triangleq \lim_{k \to \infty} u_k$ exists and $y_{\infty} \triangleq \lim_{k \to \infty} y_k$ exists. Furthermore, for all $u_0 \in \mathbb{C}^m$, the following statements hold:

i) If $\ell > m$, then $u_{\infty} = -(M_s^* M_s)^{-1} M_s^* \hat{d}$ and $y_{\infty} = [I_\ell - M_s(M_s^* M_s)^{-1} M_s^*] \hat{d}$.

ii) If $\ell = m$, then $u_{\infty} = -M_s^{-1} \hat{d}$ and $y_{\infty} = 0$.

iii) If $\ell < m$, then $u_{\infty} = u_0 - M_s^* (M_s^* M_s)^{-1} (M_s u_0 + \hat{d})$ and $y_{\infty} = 0$.

Theorem 2 relies on the condition that $\Lambda \subset \text{ORHP}$. This condition depends on the estimate $M_s$ of $M_s$. In the SISO case, $\Lambda \subset \text{ORHP}$ if and only if $M_s$ is within $90^\circ$ of $M_s$, that is, $|\angle(M_s/M_s)| < \frac{\pi}{4}$. In this case, (10) is satisfied by a sufficiently small $\rho > 0$.

If $M_s = M_s$, then $\Lambda \subset \text{ORHP}$. In this case, (10) is satisfied if $\rho > 2/\lambda_{\text{max}}(M_s^* M_s)$.

If $\Lambda \cap \text{OLHP}$ is not empty, then for all $\rho > 0$, $I_\ell - \rho M_s^* M_s$ has at least one eigenvalue outside the CUD. In this case, (9) implies that $y_k$ diverges.

**V. ADAPTIVE HARMONIC STEADY-STATE CONTROL**

In this section, we present AHSS control, which does not require any information regarding $M_s$. Let $\mu \in (0, 1]$, $\nu_1 > 0$, and $u_0 \in \mathbb{C}^m$, and for all $k \in \mathbb{N}$, consider the control

$$u_{k+1} = u_k - \frac{\mu}{\nu_1 + \|M_k\|_F} M_k^* y_{k+1},$$

where $M_k \in \mathbb{C}^{\ell \times m}$ is an estimate of $M_s$ obtained from the adaptive law presented below. Note that (11) is reminiscent of the HSS control (8) except the fixed estimate $M_s$ is replaced by the adaptive estimate $M_k$, and the fixed gain $\rho$ is replaced by the $M_k$-dependent gain $\mu/\sqrt{\nu_1 + \|M_k\|_F^2}$.

To determine the adaptive law for $M_k$, consider the cost function $\mathcal{J} : \mathbb{R}^{\ell \times m} \times \mathbb{R}^{\ell \times m} \to [0, \infty)$ defined by

$$\mathcal{J}(M_t, M_t) \triangleq \frac{1}{2} \| (M_t + \mu M_t) (u_k - u_{k-1}) - (y_{k+1} - y_k) \|^2.$$ Note that $\mathcal{J}(\text{Re} M_s, \text{Im} M_s) = 0$, that is, $M_s$ minimizes $\mathcal{J}$.

Define the complex gradient

$$\nabla \mathcal{J}(M_t, M_t) \triangleq \frac{\partial \mathcal{J}(M_t, M_t)}{\partial M_t} + j \frac{\partial \mathcal{J}(M_t, M_t)}{\partial \text{Re} M_t},$$

where $\partial \mathcal{J}(M_t, M_t)/\partial M_t$ is the derivative of $\mathcal{J}$ with respect to $M_t$.
\[ M_k = M_{k-1} - \frac{\gamma}{\nu_2 \mu^2 + (\nu_1 + \|M_{k-1}\|^2_k \|u_k - u_{k-1}\|_2^2) \|y_k - y_{k-1}\|^2_k} \left( M_{k-1}(u_k - u_{k-1}) - (y_k - y_{k-1}) \right) \\times (u_k - u_{k-1})^\ast. \]  

(15)

Thus, the AHSS control is given by \((11), (14), \text{and} (15)\). The control architecture is shown in Fig. 1. All AHSS computations are performed using complex DFT signals. At time \(kT_s\), the control \(u\) is updated using (6) and the complex signal \(u_k\). Note that \(u_k\) is calculated using \(y_k\), which is the DFT of \(y\) at frequency \(\omega\) sampled over the interval \([\{k-1\}T_s, kT_s]\), which corresponds to the time between the \(k-1\) and \(k\) steps.

The update period \(T_s\) must be sufficiently large such that the harmonic steady-state assumption \(y_{k+1} \approx \hat{y}_{\text{bas}}(u_k)\) is valid. Numerical testing suggests that \(T_s\) should be at least as large as the setting time associated with the slowest mode of \(A\), that is, \(T_s > 4/|\zeta_\omega|\), where \(\zeta_\omega\) and \(\omega_\omega\) are the damping ratio and natural frequency of the slowest mode of \(A\).

The AHSS controller parameters are \(\mu \in (0, 1], \gamma \in (0, 1], \nu_1 > 0, \text{and} \nu_2 > 0\). The gains \(\mu\) and \(\gamma\) influence the step size of the \(u_k\) and \(M_k\) update equations, respectively. The gain \(\nu_1\) and \(\nu_2\) influence the normalization of the \(u_k\) and \(M_k\) update equations, respectively.

VI. STABILITY ANALYSIS

The following result provides stability properties of the estimator (15). The proof follows from direct computation and is omitted due to space limitation.

**Proposition 1.** Consider the open-loop system (7), and the AHSS control (11), (14), and (15), where \(\mu \in (0, 1], \gamma \in (0, 1], \nu_1 > 0, \text{and} \nu_2 > 0\). Then, for all \(u_0 \in \mathbb{C}\) and \(M_0 \in \mathbb{C}\{0\}\), the estimate \(M_k\) is bounded, and for all \(k \in \mathbb{Z}_+^\ast\),

\[ \|M_k - M_\ast\|_F^2 - \|M_{k-1} - M_\ast\|^2_F \leq \frac{\nu_2}{\nu_2^2 + (\nu_1 + \|M_{k-1}\|^2_k \|u_k - u_{k-1}\|_2^2) \|y_k - y_{k-1}\|^2_k} \left( M_{k-1}(u_k - u_{k-1}) - (y_k - y_{k-1}) \right) \times (u_k - u_{k-1})^\ast. \]

(16)

Proposition 1 implies that \(\|M_k - M_\ast\|_F^2\) is nonincreasing.

We now analyze closed-loop performance under the assumption that the open-loop system is SISO. Define \(u_k \triangleq -d/M_\ast\), which exists because \(M_\ast \neq 0\). Note that if \(u_k \equiv u_\ast\), then \(y_k \equiv \hat{y}_{\text{bas}}\). Next, (7) implies that \(y_{k+1} = M_\ast u_k + d = M_\ast u_k + y_k - M_\ast u_{k-1}\), and substituting (11) yields

\[ y_{k+1} = y_k - \mu M_\ast \hat{y}_{\text{bas}}(u_k) - \frac{\nu_1}{\nu_2} \left( M_{k-1} u_k - M_{k-1} y_k \right), \]

(17)

where \(k \in \mathbb{Z}_+^\ast\) and \(y_1 = M_\ast u_0 + d\). Furthermore, (14) and (15) can be written as

\[ M_k = M_{k-1} - \frac{\gamma}{\nu_2} \left( M_{k-1} \ast u_k - M_{k-1} y_k \right) \]

which is the direction of the maximum rate of change of \(\beta\) with respect to \(M_t + jM_i\) [14]. Let \(M_0 \in \mathbb{C}\{0\}\), \(\gamma \in (0, 1]\), and \(\nu_2 > 0\), and for all \(k \in \mathbb{Z}_+^\ast\), consider the adaptive law

\[ M_k \triangleq M_{k-1} - \eta \nabla \beta(M_{k-1}, \text{Im} M_{k-1}), \]

(13)

where

\[ \eta = \frac{\gamma (\nu_1 + \|M_{k-1}\|^2_k \|u_k - u_{k-1}\|_2^2) \|y_k - y_{k-1}\|^2_k}{\nu_2 \mu^2 + (\nu_1 + \|M_{k-1}\|^2_k \|u_k - u_{k-1}\|_2^2) \|y_k - y_{k-1}\|^2_k}. \]

Using (12)–(14), it follows that, for all \(k \in \mathbb{N}\),

\[ M_k = M_{k-1} - \eta k \left(M_{k-1}(u_k - u_{k-1}) - (y_k - y_{k-1}) \right) \times (u_k - u_{k-1})^\ast. \]

VII. NUMERICAL EXAMPLES

Consider the acoustic duct of length \(L = 2\) m shown in Fig. 2, where all measurements are from the left end of the duct. A disturbance speaker is at \(\xi_d = 0.95\) m, while 2 control speakers are at \(\xi_{\phi_1} = 0.4\) m and \(\xi_{\phi_2} = 1.25\) m. All speakers have cross-sectional area \(A_s = 0.0025\) m\(^2\). The equation for the acoustic duct is

\[ \frac{\partial^2 \rho(p(t))}{\partial t^2} = \frac{\partial^2 p(t)}{\partial \xi^2} + \rho_0 \hat{v}_1(t) \delta(\xi - \xi_{\phi_1}) + \rho_0 \hat{v}_2(t) \delta(\xi - \xi_{\phi_2}) + \rho_0 \hat{d}(t) \delta(\xi - \xi_d) \]

where \(p(t)\) is the acoustic pressure, \(\delta\) is the Dirac delta, \(c = 343\) m/s is the phase speed of the acoustic wave, \(\hat{v}_1\) and \(\hat{v}_2\) are the speaker cone velocities of the control speakers, \(d\) is the speaker cone velocity of the disturbance speaker, and \(\rho_0 = 1.21\) kg/m\(^3\) is the equilibrium density of air at room conditions. See [15] for more details.

Using separation of variables and retaining modes \(r\) modes, the solution \(p(\xi, t)\) is approximated by \(p(\xi, t) = \sum_{r=0}^r q_i(t) V_i(\xi), \) where for \(i = 1, \ldots, r, V_i(\xi) \triangleq c \sqrt{2/L} \sin \pi \xi/L, \) and \(q_i\) satisfies the differential equation (1), where

\[ x(t) = \left[ \begin{array}{c} \int_0^t q_1(\sigma) d\sigma \\ \cdots \\ \int_0^t q_r(\sigma) d\sigma \\ q_r(t) \end{array} \right], \]

\[ A = \text{diag} \left( \left[ \begin{array}{cccc} -\alpha & 0 & \cdots & 0 \\ -\beta & -\alpha & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\beta & 0 & \cdots & -\alpha \end{array} \right] \right), \]

\[ B = \frac{\rho_0}{A_s} \left[ \begin{array}{c} V_1(\xi_{\phi_1}) \\ \cdots \\ V_r(\xi_{\phi_2}) \\ V_r(\xi_d) \end{array} \right]^T, \]

\[ D_1 = \frac{\rho_0}{A_s} \left[ \begin{array}{c} V_1(\xi_d) \\ \cdots \\ V_r(\xi_d) \end{array} \right]^T, \]
and for \( i = 1, \ldots, r \), \( \omega_{n_i} \triangleq i \pi c/L \) is the natural frequency of the \( i \)th mode, and \( \zeta_i = 0.2 \) is the assumed damping ratio of the \( i \)th mode.

Two feedback microphones are in the duct at \( \xi_{\phi_1} = 0.3 \) m and \( \xi_{\phi_2} = 1.7 \) m, and they measure the acoustic pressures \( \phi_1(t) = p(\xi_{\phi_1}, t) \) and \( \phi_2(t) = p(\xi_{\phi_2}, t) \), respectively. Thus, for \( i = 1, 2 \), \( \phi_i(t) = C_i x(t) \), where 
\[
C_i = \frac{p_0}{V_1(\xi_{\phi_i})} \left[ 0 \ V_1(\xi_{\phi_i}) \cdots 0 \ V_r(\xi_{\phi_i}) \right].
\]
For all examples, \( r = 5 \) and \( x(0) = 0 \). The DFT is performed using a 1 kHz sampling frequency. The HSS and AHSS parameters \( \rho_1 \) for HSS and AHSS. The control is turned on after 1 s. Let \( y \) be the microphone
\[
y = y_1 = \frac{\phi_1}{\xi_{\phi_1}} \quad \text{and} \quad y = y_2 = \frac{\phi_2}{\xi_{\phi_2}}.
\]
For \( \xi \) of the \( \xi \) mode, and \( \xi \) respectively. Thus, for \( \xi \)
\[
M = [2 \cos \omega_{n_i} t, 2 \cos \omega_{n_i} t, \ldots, 2 \cos \omega_{n_i} t]'.
\]

**Example 1.** SISO \((m = \ell = 1)\). Let \( u = \psi_1 \), \( y = \phi_1 \), \( \psi_2 = 0 \), and \( d = \sin \omega_1 t + 2 \cos \omega_1 t \). First, consider the case where \( M_0 \) is within 90° of the \( M_0 \), specified, \( M_0 = 2e^{3M} M_0 \). Figure 3 shows \( y \) and \( u \) for HSS and AHSS. The control is turned on after 1 s. Both HSS and AHSS yield asymptotic disturbance rejection. Next, let \( M_0 = 2e^{3M} M_0 \) which is not within 90° of \( M_0 \). Figure 4 shows \( y \) and \( u \) for HSS and AHSS. In this case, \( y \) with HSS diverges, whereas \( y \) with AHSS converges to zero. Figure 5 shows the trajectory of the estimate \( M_k \), which moves toward \( M_k \). Proposition 1 states that \( |M_k - M_k| \) is nondecreasing; however, this result assumes that \( y \) reaches harmonic steady state. Figure 5 shows that \( M_k \) may increase slightly in practice but generally decreases.

**Example 2.** Single-input two-output \((m = 1 \) and \( \ell = 2)\). Let \( u = \psi_1, y = [\phi_1 \phi_2]'T, \psi_2 = 0 \), and \( d = \sin \omega_1 t + 2 \cos \omega_1 t \). First, consider the case where \( M_0 \) is selected such that \((10)\) is satisfied, specifically, \( M_0 = [1.5e^{M} (M_k)_{1,1}, 0.5e^{M} (M_k)_{2,1}]'T \). Note that the optimal control is \( u_\star = -1.66 + j0.98 \), which minimizes the average power \((4)\). Figure 6 shows \( y \) and \( u \) for HSS and AHSS. The control is turned on after 1 s. In this case, HSS and AHSS each yield \( u_k \to u_\star \) as \( k \to \infty \). Thus, \( \lim_{k \to \infty} ||y_k|| \) is minimized. Next, let \( M_0 = [1.5e^{M} (M_k)_{1,1}, 0.5e^{M} (M_k)_{2,1}]'T \), which does not satisfy \((10)\). Figure 6 shows \( y \) and \( u \) for HSS and AHSS. In this case, \( y \) diverges, whereas \( y \) with AHSS converges and \( u_k \to u_\star \) as \( k \to \infty \), which implies that \( \lim_{k \to \infty} ||y_k|| \) is minimized.

**Example 3.** MIMO \((m = 2 \) and \( \ell = 2)\) with a two-tone disturbance. Let \( u = [\psi_1 \psi_2]'T, y = [\phi_1 \phi_2]'T, \) and \( d = \sin \omega_1 t + \sin \omega_2 t + \cos \omega_1 t + \cos \omega_2 t \), which is a two-tone disturbance. Define \( M_{k,1} \triangleq G_{y1}(j\omega_1) \), and \( M_{k,2} \triangleq G_{y2}(j\omega_2) \). Since \( d \) has 2 tones, we use 2 copies of the HSS or AHSS algorithm—one copy at each disturbance...
Appendix A: Proof of Theorem 3

Proof. Define $\tilde{M}_k \triangleq M_k - M_*, V_M(M_k) \triangleq |M_k|^2$, and $\Delta V_M(k) \triangleq V_M(M_k) - V_M(M_{k-1})$. It follows from Proposition 1 that for all $k \in \mathbb{Z}^+$

$$\Delta V_M(k) \leq -\frac{\gamma |M_{k-1}|^2 |y_k|^2 |\tilde{M}_{k-1}|^2}{\nu_2 + |M_{k-1}|^2 |y_k|^2}. \quad (19)$$

frequency. Let $M_{1,0}$ and $M_{2,0}$ denote the initial estimates of $M_{1,1}$ and $M_{2,2}$. First, consider the case where $M_{1,0}$ and $M_{2,0}$ are such that (10) is satisfied, specifically, $M_{1,0} = 0.6e^{j\pi} M_{1,1}$, and $M_{2,0} = 0.9e^{j\pi} M_{2,2}$. Figure 8 shows $y$ and $u$ for HSS and AHSS. The control is turned on after 1 s. Both HSS and AHSS yield asymptotic disturbance rejection. Next, consider the case where $M_{1,0} = 0.2e^{j\pi} M_{1,1}$, and $M_{2,0} = 0.6e^{j\pi} M_{2,2}$, which do not satisfy (10). Figure 9 shows $y$ and $u$ for HSS and AHSS. In this case, $y$ with HSS diverges, whereas $y$ with AHSS converges to zero. △

Fig. 6: For a single-input two-output plant satisfying (10), both HSS and AHSS minimize $\lim_{k \to \infty} ||y_k||$. Dashed lines show $\pm |u_0|$.

Fig. 7: For a single-input two-output plant that does not satisfy (10), the response $y$ with HSS diverges, whereas AHSS minimizes $\lim_{k \to \infty} ||y_k||$. Dashed lines show $\pm |u_0|$.

Fig. 8: For a MIMO plant that satisfies (10) with a 2-tone disturbance, both HSS and AHSS yield $u(t) \to 0$ as $t \to \infty$.

Fig. 9: For a MIMO plant that does not satisfy (10) with a 2-tone disturbance, the response $y$ with HSS diverges, whereas AHSS yields $y(t) \to 0$ as $t \to \infty$.

Next, define $V_\nu(y_k) \triangleq |y_k|^2$ and $\Delta V_\nu(k) \triangleq V_\nu(y_{k+1}) - V_\nu(y_k)$. Evaluating $\Delta V_\nu(k)$ along the trajectories of (17) yields

$$\Delta V_\nu(k) = -\frac{\nu_1 M_*^* M_{k-1}}{\nu_1 + |M_{k-1}|^2} \left(2Re M_* M_{k-1}^* - \frac{|M_*|^2 |M_{k-1}|^2}{\nu_1 + |M_{k-1}|^2}\right), \quad (20)$$

Note that $|\tilde{M}_{k-1}|^2 = |M_{k-1}|^2 + |M_*|^2 - 2Re M_* M_{k-1}^*$, and it follows from (20) that

$$\Delta V_\nu(k) = -\frac{\nu_1 |y_k|^2}{\nu_1 + |M_{k-1}|^2} \left(|\tilde{M}_{k-1}|^2 + |M_*|^2 - |\tilde{M}_{k-1}|^2\right).$$
\[
-\frac{\mu[M_k^2|M_{k-1}|^2]}{\nu_1 + |M_{k-1}|^2}.
\]

Define the Lyapunov function \( V(y_k, \bar{M}_{k-1}) \triangleq \ln(1 + aV_0(y_k) + bV_0(\bar{M}_{k-1})) \), where \( a \triangleq ([M_0] + 2|M_0|^2)/\nu_2 \), and \( b > 0 \) is provided later. Consider the Lyapunov difference
\[
\Delta V(k) \triangleq V(y_{k+1}, \bar{M}_k) - V(y_k, \bar{M}_{k-1}).
\]

Since for all \( x > 0 \), \( \ln x \leq x - 1 \), evaluating \( \Delta V \) along the trajectories of (17) and (18) yields
\[
\Delta V(k) = \ln \left( 1 + \frac{a\Delta V_0(k)}{1 + aV_0(y_k)} \right) + b\Delta V_M(k)
\]
\[
\leq \frac{a\Delta V_0(y_k)}{1 + aV_0(y_k)} + b\Delta V_M(k).
\]

Substituting (19) and (21) into (23) yields
\[
\Delta V(k) \leq -\frac{a\mu[y_k]^2 \left( [M_0^2 - \mu[M_0]^2|M_{k-1}|^2/\nu_1 + |M_{k-1}|^2 \right]}{(1 + a[y_k]^2)(\nu_1 + |M_{k-1}|^2)}
\]
\[
+ \frac{a\mu[y_k]^2|\bar{M}_{k-1}|^2}{1 + a[y_k]^2}(\nu_1 + |M_{k-1}|^2)
\]
\[
- b\gamma [M_{k-1}]^2y_k^2[M_{k-1}]^2/\nu_2 + a\nu_2[y_k]^2[M_{k-1}]^2/\nu_2,
\]
where \( c_1 \triangleq \frac{a\mu[M_0]^2}{\nu_1 + a[y_k]^2} > 0 \).

To show that \((0, M_0)\) is a Lyapunov stable equilibrium, define \( \mathcal{D} \triangleq \{ x \in \mathbb{C} : |x| < [M_0]/2 \} \), and note that for all \((y_k, \bar{M}_{k-1}) \in \mathcal{D}, |M_{k-1}| \geq |M_0|/2 \).

\[
\Delta V(k) \leq -\frac{c_1[y_k]^2}{1 + a[y_k]^2} + a\mu[y_k]^2|\bar{M}_{k-1}|^2/\nu_2(1 + a[y_k]^2)
\]
\[
- b\gamma [M_{k-1}]^2y_k^2[M_{k-1}]^2/\nu_2 + a\nu_2[y_k]^2[M_{k-1}]^2/\nu_2,
\]
(25)

where \( c_1 \triangleq \frac{a\mu[M_0]^2}{\nu_1 + a[y_k]^2} > 0 \).

To show that \((0, M_0)\) is a Lyapunov stable equilibrium, define \( \mathcal{D} \triangleq \{ x \in \mathbb{C} : |x| < [M_0]/2 \} \), and note that for all \((y_k, \bar{M}_{k-1}) \in \mathcal{D}, |M_{k-1}| \geq |M_0|/2 \).

\[
\Delta V(k) \leq \frac{\kappa [M_{k-1}^2|M_{k-1}|^2]}{\nu_1 + |M_{k-1}|^2} \bar{M}_{k-1},
\]
which has the solution \( \bar{M}_k = \beta_k M_0 \), where \( \beta_k \triangleq \prod_{i=0}^{k-1} (1 - \frac{\gamma [M_i^2|M_{i+1}|^2]}{\nu_2 + |M_{i+1}|^2[M_{i+1}]^2}) \).

Thus, for all \( k \in \mathbb{Z}^+ \),
\[
|M_k|^2 = |M_k + M_0|^2 = |M_0\beta_k + M_0|^2 = |M_0|^2\beta_k^2 + 2\Re(\bar{M}_k M_0^*) \beta_k + |M_0|^2.
\]

Since \( f(x) \triangleq |\bar{M}_0|^2x^2 + 2\Re(\bar{M}_0 M_0^*)x + |M_0|^2 \) is quadratic and positive definite in \( x \), it follows that \( f \) is minimized at \(-\Re(\bar{M}_0 M_0^*)/|M_0|^2\). Thus, for all \( k \in \mathbb{N} \),
\[
|M_k|^2 \geq \frac{|M_0|^2 - \Re(\bar{M}_0 M_0^*)^2}{|M_0|^2} = c_2^2.
\]

Thus, for all \( k \in \mathbb{N}, |M_k| > c_2 \). Let \( b \triangleq \frac{\nu_1}{\nu_2 + c_2^2} \), and it follows from (25) that
\[
\Delta V(k) \leq -\frac{c_1[y_k]^2}{1 + a[y_k]^2} + \frac{a\mu[y_k]^2|\bar{M}_{k-1}|^2}{\nu_1 + a[y_k]^2}.
\]

Next, since \( V \) is positive-definite, and for all \( k \in \mathbb{N} \), \( \Delta V(k) \) is nonpositive, it follows from (22) and (27) that \( 0 \leq \lim_{k \to \infty} \sum_{i=1}^{k} \frac{c_1[y_i]^2}{\nu_1 + a[y_i]^2} \leq -\lim_{k \to \infty} \sum_{i=1}^{k} \Delta V(i) = V(y_1, \bar{M}_0) - \lim_{k \to \infty} \sum_{i=1}^{k} \Delta V(i) = V(y_1, \bar{M}_0) = 0 \), where the upper and lower bounds imply that all the limits exist. Thus, \( \lim_{k \to \infty} c_1[y_k]^2/\nu_1 + a[y_k]^2 = 0 \). Since in addition, \( \nu_1 + a[y_k]^2 \) is a positive-definite function of \( y_k \), it follows that \( \lim_{k \to \infty} y_k = 0 \). Furthermore, (7) implies that \( \lim_{k \to \infty} u_k = \lim_{k \to \infty} (y_{k+1} - d)/M_0 = u_* \). \( \square \)

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