Differentiable perturbations of
Ornstein-Uhlenbeck operators

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Abstract

We prove an extension theorem for a small perturbation of the
Ornstein-Uhlenbeck operator \((L, D(L))\) in the space of all uniformly
continuous and bounded functions \(f : H \to \mathbb{R}\), where \(H\) is a separable
Hilbert space. We consider a perturbation of the form \(N_0\varphi = L\varphi + \langle D\varphi, F \rangle\) where \(F : H \to H\) is bounded and Fréchet differen-
tiable with uniformly continuous and bounded differential. Hence, we
prove that \(N_0\) is \(m\)-dissipative and its closure in \(C_b(H)\) coincides with
the infinitesimal generator of a diffusion semigroup associated to a
stochastic differential equation in \(H\).

Key words: Ornstein-Uhlenbeck semigroup; \(m\)-dissipative operator;
Lumer-Phillips theorem.

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1 Introduction and setting of the problem

Let \(H\) be a separable Hilbert space endowed with scalar product \(\langle \cdot, \cdot \rangle\) and
norm \(|\cdot|\). We shall always identify \(H\) with its topological dual space \(H^*\).
\(L(H)\) is the Banach space of all the linear and continuous maps in \(H\), endowed
with the usual norm \(|\cdot|_{L(H)}\). With \(C_b(H)\) (resp. \(C_b(H; H)\)) we denote the
Banach space of all uniformly continuous and bounded functions \(f : H \to \mathbb{R}\)
(resp. \(f : H \to H\)), endowed with the supremum norm \(|\cdot|\) (resp. \(|\cdot|_0\)).
We also denote by $C^1_b(H)$ (resp. $C^1_b(H;H)$) the space of all $f \in C_b(H)$ (resp. $f : H \to H$) that are Fréchet differentiable with differential in $C_b(H;H)$ (resp. with uniformly continuous and bounded differential $Df : H \to \mathcal{L}(H)$).

We assume the following

**Hypothesis 1.1.** (i) $A : D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous semigroup $(e^{tA})_{t \geq 0}$ of type $G(1, \omega)$, i.e. there exists $\omega \in \mathbb{R}$ such that

$$\|e^{tA}\|_{\mathcal{L}(H)} \leq e^{\omega t}, \quad t \geq 0; \tag{1}$$

(ii) $Q \in \mathcal{L}(H)$ is self adjoint and positive;

(iii) For any $t > 0$ the linear operator $Q_t$, defined as

$$Q_t x = \int_0^t e^{sA} Q e^{sA^*} x ds, \quad x \in H, \quad t \geq 0, \tag{2}$$

is of trace class.

(iv) $F \in C^1_b(H;H)$, and $K = \sup_{x \in H} \|DF(x)\|_{\mathcal{L}(H)}$.

It is well known (see, for instance, [44, 4]) that thanks to conditions (i)–(iii) it is possible to define the so called Ornstein-Uhlenbeck (OU) semigroup $(R_t)_{t \geq 0}$ in $C_b(H)$ by the formula

$$R_t \varphi(x) = \int_0^t \varphi(e^{sA} x + y) N_{Q_t}(dy), \quad x \in H, \tag{3}$$

where $N_{Q_t}$ is the Gaussian measure on $H$ of mean 0 and covariance operator $Q_t$ (see [44, 4]). It turns out that the OU semigroup is not a strongly continuous semigroup in $C_b(H)$ but it is a weakly continuous semigroup (see [77, 7]) and a $\pi$-semigroup (see [77, 7]). However, it is possible to define its infinitesimal generator in the weaker sense

$$D(L) = \left\{ \varphi \in C^*_b(H) : \exists g \in C_b(H), \lim_{t \to 0^+} \frac{R_t \varphi(x) - \varphi(x)}{t} = g(x), \quad x \in H, \quad \sup_{t \in (0, 1)} \left\| \frac{R_t \varphi - \varphi}{t} \right\| < \infty \right\} \tag{4}$$

$$L \varphi(x) = \lim_{t \to 0^+} \frac{R_t \varphi(x) - \varphi(x)}{t}, \quad \varphi \in D(L), \quad x \in H.$$
We are interested in the operator \((N_0, D(N_0))\) defined by

\[ N_0 \varphi = L \varphi + F \varphi, \quad \varphi \in D(N_0) = D(L) \cap C^1_b(H), \]

where

\[ F \varphi(x) = \langle D \varphi(x), F(x) \rangle. \]

Now let us consider the stochastic differential equation in \(H\)

\[
\begin{aligned}
  dX(t) &= (AX(t) + F(X(t)))dt + Q^{1/2}dW(t) \quad t > 0, \\
  X(0) &= x, \quad x \in H,
\end{aligned}
\]

(5)

where \((W(t))_{t \geq 0}\) is a cylindrical Wiener process defined on a stochastic basis \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\). Since \(F \in C^1_b(H; H)\), problem (5) has a unique mild solution \((X(t, x))_{t \geq 0, x \in H}\) (see [44, 4]), that is for any \(x \in H\) the process \((X(\cdot, x))_{t \geq 0}\) is adapted to the filtration \((\mathcal{F}_t)_{t \geq 0}\) and it is continuous in mean square, i.e.

\[
\lim_{t \to s} \mathbb{E} \left[ |X(t, x) - X(s, x)|^2 \right] = 0, \quad \forall s \geq 0.
\]

This allows us to define a transition semigroup \((P_t)_{t \geq 0}\) in \(C_b(H)\), by setting

\[ P_t \varphi(x) = \mathbb{E} \left[ \varphi(X(t, x)) \right], \quad t \geq 0, \quad \varphi \in C_b(H), x \in H. \]

The semigroup \((P_t)_{t \geq 0}\) is not strongly continuous in \(C_b(H)\). However, it is a \(\pi\)-semigroup, and we can define its infinitesimal generator \((N, D(N))\) in the same way as for the OU semigroup

\[
\begin{aligned}
  D(N) &= \left\{ \varphi \in C_b(H) : \exists g \in C_b(H), \lim_{t \to 0^+} \frac{P_t \varphi(x) - \varphi(x)}{t} = g(x), \\
  &\quad x \in H, \sup_{t \in (0, 1)} \left\| \frac{P_t \varphi - \varphi}{t} \right\| < \infty \right\} \\
  N \varphi(x) &= \lim_{t \to 0^+} \frac{P_t \varphi(x) - \varphi(x)}{t}, \quad \varphi \in D(N), x \in H.
\end{aligned}
\]

The main result of this paper is the following

**Theorem 1.2.** Let us assume that Hypothesis 1.1 holds. Then, the operator \((N_0, D(N_0))\) defined by \(D(N_0) = D(L) \cap C^1_b(H)\) and \(N_0 \varphi = L \varphi + F \varphi, \forall \varphi \in D(N_0)\), is \(m\)-dissipative in \(C_b(H)\) and its closure is the operator \((N, D(N))\).

In [22, 2], it is proved that Theorem 1.2 holds with \(F \in C^1_b(H; H)\), that is \(F\) is Fréchet differentiable and its differential \(DF : H \to \mathcal{L}(H)\) is Lipschitz continuous.
Perturbations of OU operators as been the object of several papers (see, for instance, [22, 233, 355, 588, 8]). Frequently, additional assumptions are taken on the OU operator in order to have $D(L) \subset C^1_b(H)$.

In order to prove Theorem 1.2 we develop a technique introduced in [22, 2]. The idea is the following: since $F \in C^1_b(H; H)$, there exists a unique solution $\eta(\cdot, x)$ of the abstract Cauchy problem

$$\begin{cases}
\frac{d}{d\varepsilon}\eta(\varepsilon, x) = F(\eta(\varepsilon, x)), & \varepsilon > 0, \\
\eta(0, x) = x, & x \in H.
\end{cases}$$

Then, for any $\varepsilon > 0$ we define the operators $F_\varepsilon : C_b(H) \to C_b(H)$ and $N_\varepsilon : D(N_\varepsilon) \subset C_b(H) \to C_b(H)$ by setting

$$F_\varepsilon \varphi(x) = \frac{1}{\varepsilon} \left( \varphi(\eta(\varepsilon, x)) - \varphi(x) \right),$$

$$D(N_\varepsilon) = D(L) \cap C^1_b(H),$$

$$N_\varepsilon \varphi = L \varphi + F_\varepsilon \varphi, \quad \varphi \in D(N_\varepsilon).$$

By an approximation argument, we are able to prove that the operator $(N_0, D(N_0))$ is $m$-dissipative in $C_b(H)$. Then, by the Lumer-Phillips theorem, it will follow that the closure of $(N_0, D(N_0))$ coincides with the operator $(N, D(N))$.

### 1.1 Properties of $F_\varepsilon$

The following lemma collects some useful properties of $\eta$.

**Lemma 1.3.** The following estimates hold

$$|\eta(t, x)| \leq e\|F\|_{0t}|x|; \quad (6)$$

$$|\eta(t, x) - \eta(t, y)| \leq e^{Kt}|x - y|; \quad (7)$$

$$|\eta(t, x) - x| \leq c\|F\|_{0t} \quad (8)$$

$$\|\eta_x(t, x)\|_{L(H)} \leq e^{Kt} \quad (9)$$

$$\|\eta_x(t, x) - \eta_x(t, y)\|_{L(H)} \leq e^{Kt} \theta_{DF}(e^{Kt}|x - y|), \quad (10)$$

where $\theta_{DF} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is the modulus of continuity of $DF$. 


Proof. (6), (8), (9) have been proved in [22, Lemma 2.1]. (7). We have 

\[ |\eta(t, x) - \eta(t, y)| \]
\[ \leq |x - y| + \int_0^t |F(\eta(s, x)) - F(\eta(s, y))| ds \]
\[ \leq K \int_0^t |\eta(s, x) - \eta(s, y)| ds. \]

Then (7) follows by Gronwall’s Lemma. (10). Let \( x, y, h \in H \) and set 

\[ r^h(t) = \eta_x(t, x) \cdot h - \eta_x(t, y) \cdot h = p^h(t, x) - p^h(t, y), \]

where \( p^h(t, x) = \eta_x(t, x) \cdot h \) and \( p^h(t, y) = \eta_x(t, y) \cdot h \). Then \( r^h(t) \) fulfills the following equation

\[
\begin{aligned}
\frac{d}{dt} r^h(t) &= DF(\eta(t, x)) r^h(t) + [DF(\eta(t, x)) - DF(\eta(t, y))] p^h(t, x), \quad t > 0 \\
r^h(0) &= 0.
\end{aligned}
\]

Since \( |DF(\eta(t, x)) r^h(t)| \leq K |r^h(t)| \) it follows that \( r^h(t) \) is bounded by

\[ |r^h(t)| \leq \int_0^t e^{K(t-s)} \|DF(\eta(s, x)) - DF(\eta(s, y))\|_{\mathcal{L}(H)} |p^h(s, x)| ds. \]

By taking into account that \( DF : H \to \mathcal{L}(H; H) \) is uniformly continuous and bounded, we denote by \( \theta_{DF} \) the modulus of continuity of \( DF \). Hence, by (7), (9) we have

\[ |r^h(t)| \leq \int_0^t e^{K_s} \theta_{DF}(|\eta(s, x) - \eta(s, y)|) ds |h| \]
\[ \leq e^{Kt} \theta_{DF}(e^{Kt} |x - y|) |h| \]

Proposition 1.4. For any \( \varphi \in C^1_b(H) \) we have

\[
\begin{aligned}
\lim_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon \varphi &= \mathcal{F} \varphi \quad \text{in } C^1_b(H). \\
\|\mathcal{F}_\varepsilon \varphi\| &\leq \|D \varphi\|_0 \|F\|_0.
\end{aligned}
\]
Proof. For all $\varphi \in C^1_b(H)$ we have
\[
\mathcal{F}_\varepsilon \varphi(x) - \mathcal{F} \varphi(x) = \frac{1}{\varepsilon} \int_0^\varepsilon \langle D\varphi(\eta(s,x)) - D\varphi(x), F(\eta(s,x)) \rangle \, ds \\
+ \frac{1}{\varepsilon} \int_0^\varepsilon \langle D\varphi(x), F(\eta(s,x)) - F(x) \rangle \, ds.
\]
Then by (8) we have
\[
|\mathcal{F}_\varepsilon \varphi(x) - \mathcal{F} \varphi(x)| \leq \frac{1}{\varepsilon} \int_0^\varepsilon (|\theta_{D\varphi}||\eta(s,x) - x|)\|F\|_0 + \|D\varphi\|_0 K|\eta(s,x) - x|) \, ds \\
\leq \frac{1}{\varepsilon} \int_0^\varepsilon (\theta_{D\varphi}||F||_0 s)|\eta(s,x)| + \|D\varphi\|_0 K||F||_0 s) \, ds \\
\leq (\theta_{D\varphi}||F||_0 \varepsilon) + \|D\varphi\|_0 K\varepsilon) \|F||_0
\]
where $\theta_{D\varphi}$ is the modulus of continuity of $D\varphi$. This yields (11). Moreover, we have
\[
\mathcal{F}_\varepsilon \varphi(x) = \frac{1}{\varepsilon} \int_0^\varepsilon \langle D\varphi(\eta(s,x)), F(\eta(s,x)) \rangle \, ds
\]
that implies (12). \hfill \Box

1.2  $m$-dissipativity of $N$.

Given $\varepsilon > 0$ we introduce the following approximating operator
\[
N_\varepsilon = L + \mathcal{F}_\varepsilon, \ D(N_\varepsilon) = D(L) \cap C^1_b(H).
\]
We have

Proposition 1.5. $N_\varepsilon$ is a $m$-dissipative operator in $C_b(H)$ for any $\varepsilon > 0$. Moreover, for any $f \in C^1_b(H)$ and any $\lambda > \omega + (e^{\varepsilon K} - 1)/\varepsilon$ the operator
\[
R(\lambda, N_\varepsilon) = (1 - T_\lambda)^{-1} R \left( \lambda + \frac{1}{\varepsilon}, L \right),
\]
where $T_\lambda : C_b(H) \to C_b(H)$ is defined by
\[
T_\lambda \psi(x) = R \left( \lambda + \frac{1}{\varepsilon}, L \right) \left[ \frac{1}{\varepsilon} \psi(\eta(\varepsilon, x)) \right], \quad x \in H, \ \psi \in C_b(H)
\]
maps $C^1_b(H)$ into $D(L) \cap C^1_b(H)$ and
\[
\|D R(\lambda, N_\varepsilon) f\|_0 \leq \frac{1}{\lambda - \omega} \frac{1}{e^{\varepsilon K - \varepsilon - 1} \varepsilon} \|D f\|_0.
\]
Proof. Let \( \varepsilon > 0, \lambda > 0, f \in C_b(H) \). The equation

\[
\lambda \varphi_{\varepsilon} - L\varphi_{\varepsilon} - \mathcal{F}(\varphi_{\varepsilon}) = f
\]

is equivalent to

\[
\left( \lambda + \frac{1}{\varepsilon} \right) \varphi_{\varepsilon} - L\varphi_{\varepsilon} - \mathcal{F}(\varphi_{\varepsilon}) = f + \frac{1}{\varepsilon} \varphi_{\varepsilon}(\eta(\varepsilon, \cdot))
\]

and to

\[
\varphi_{\varepsilon} = R \left( \lambda + \frac{1}{\varepsilon}, L \right) f + T_\lambda \varphi_{\varepsilon}.
\]

(15)

Since, as we can easily see, for any \( \lambda > 0 \)

\[
\|T_\lambda \psi\| \leq \frac{1}{1 + \lambda \varepsilon} \|\psi\|, \quad \forall \psi \in C_b(H),
\]

(16)

the operator \( T_\lambda \) is a contraction in \( C_b(H) \) and so equation (15) has a unique solution \( \varphi_{\varepsilon} \in C_b(H) \) done by \( \varphi_{\varepsilon} = R(\lambda, N_\varepsilon)f \). Moreover, by (13), (16) it holds

\[
\|\varphi_{\varepsilon}\| \leq \frac{1}{\lambda + \frac{1}{\varepsilon}} \left[ \|f\| + \frac{1}{\varepsilon} \|\varphi_{\varepsilon}\| \right].
\]

Consequently,

\[
\|\varphi_{\varepsilon}\| \leq \frac{1}{\lambda} \|f\|.
\]

Then, \( N_\varepsilon \) is \( m \)-dissipative. Now let \( f \in C^1_b(H) \). We recall that for any \( \lambda > 0, \psi \in C_b(H) \)

\[
R(\lambda, L)\psi(x) = \int_0^\infty e^{-\lambda t} R_t \psi(x) dt
\]

(17)

and that

\[
DR_t \psi(x) = \int_H e^{tA^*} D\psi(e^{tA} x + y) Q_t(dy).
\]

Hence, for any \( \lambda > \omega \)

\[
DR(\lambda, L)\psi(x) = \int_0^\infty \int_H e^{-\lambda t} e^{tA^*} D\psi(e^{tA} x + y) Q_t(dy) dt
\]

(18)

and so

\[
\|DR(\lambda, L)\psi\|_0 \leq \frac{1}{\lambda - \omega} \|D\psi\|_0
\]

(19)

Moreover, as it can be easily seen by (18), \( DR(\lambda, L)\psi \) is uniformly continuous. Then \( R(\lambda, L) : C_b^1(H) \rightarrow C^1_b(H) \). Now, in order to prove that \( T_\lambda : C_b^1(H) \rightarrow \)
$C_b^1(H)$ it is sufficient to show that $\psi(\eta(\varepsilon, x)) \in C_b^1(H)$, for any $\psi \in C_b^1(H)$. Indeed, by a standard computation, we have

$$D\psi(\eta(\varepsilon, \cdot))(x) = \eta^*_x(\varepsilon, x)D\psi(\eta(\varepsilon, x)), \quad x \in H.$$  

Consequently, by (7), (10) we have

$$|D\psi(\eta(\varepsilon, \cdot))(x) - D\psi(\eta(\varepsilon, \cdot))(\overline{x})| \leq \|\eta^*_x(\varepsilon, x) - \eta^*_x(\varepsilon, \overline{x})\|_{L(H)}|D\psi(\eta(\varepsilon, x))|$$

$$+ \|\eta^*_x(\varepsilon, \overline{x})\|_{L(H)}|D\psi(\eta(\varepsilon, x)) - D\psi(\eta(\varepsilon, \overline{x}))|$$

$$\leq e^{\varepsilon K}q_{DF}(e^{\varepsilon K}|x - \overline{x}|)|D\psi|_0 + e^{\varepsilon K}q_{DF}(|\eta(\varepsilon, x) - \eta(\varepsilon, \overline{x})|)$$

$$\leq e^{\varepsilon K}q_{DF}(e^{\varepsilon K}|x - \overline{x}|)\|D\psi\|_0 + e^{\varepsilon K}q_{DF}(e^{\varepsilon K}|x - \overline{x}|).$$

for any $x, \overline{x} \in H$. So, $DT_\lambda\psi(\cdot)$ is uniformly continuous. Now we prove that $T_\lambda$ is a contraction in $C_b^1(H)$. By (13), (17) we have

$$T_\lambda\psi(x) = \frac{1}{\varepsilon} \int_0^\infty e^{-(\lambda + \frac{1}{\varepsilon} t)}Q_t\psi(\eta(\varepsilon, \cdot))(x)dt$$

$$= \frac{1}{\varepsilon} \int_0^\infty \int_H e^{-(\lambda + \frac{1}{\varepsilon} t)}\eta^*_x(\varepsilon, tA_x + y)D\psi(\eta(\varepsilon, tA_x + y))N_{Q_t}(dy)dt$$

Then

$$DT_\lambda\psi(x) =$$

$$= \frac{1}{\varepsilon} \int_0^\infty \int_H e^{-(\lambda + \frac{1}{\varepsilon} t)}e^{tA}\eta^*_x(\varepsilon, tA_x + y)D\psi(\eta(\varepsilon, tA_x + y))N_{Q_t}(dy)dt$$

By (9) it follows

$$|DT_\lambda\psi(x)| \leq \frac{1}{\varepsilon} \int_0^\infty e^{-(\lambda + \frac{1}{\varepsilon} - \omega) t}e^{\varepsilon K}\|D\psi\|_0 dt$$

$$= e^{\varepsilon K} \frac{1}{1 + \varepsilon(\lambda - \omega)}\|D\psi\|_0.$$  

Therefore, for any $\lambda > \omega + (e^{\varepsilon K} - 1)/\varepsilon$ the linear operator $T_\lambda$ is a contraction in $C_b^1(H)$ and its resolvent satisfies

$$(1 - T_\lambda)^{-1}(C_b^1(H)) \subset C_b^1(H),$$

$$\|D(1 - T_\lambda)^{-1}\psi\|_0 \leq \frac{1}{1 - \frac{e^{\varepsilon K}}{1 + \varepsilon(\lambda - \omega)}}\|D\psi\|_0.$$  

(20)

This implies

$$R(\lambda, N_\varepsilon)(C_b^1(H)) = (1 - T_\lambda)^{-1}R\left(\lambda + \frac{1}{\varepsilon}, L\right)(C_b^1(H)) \subset C_b^1(H).$$

Then, $N_\varepsilon$ is $m$-dissipative. Finally, (14) follows by (19) and (20).
Lemma 1.6. The operator $N_0$ is dissipative in $C_b(H)$.

Proof. We have to prove that $\|\lambda \varphi - N_0 \varphi\| \geq \lambda \|\varphi\|$ for any $\varphi \in D(N_0)$, $\lambda > 0$. So, if $\varphi \in D(L) \cap C^1_b(H)$ and $\lambda > 0$ we set

$$\lambda \varphi - L \varphi - F \varphi = f.$$ 

then for any $\varepsilon > 0$ we have

$$\lambda \varphi - N_\varepsilon \varphi = f + F \varphi - F_\varepsilon \varphi.$$ 

It follows

$$\varphi = R(\lambda, N_\varepsilon)(f + F \varphi - F_\varepsilon \varphi)$$

and

$$\|\varphi\| \leq \frac{1}{\lambda}(\|f\| + \|F \varphi - F_\varepsilon \varphi\|)$$

Then by (11) it follows

$$\|\varphi\| \leq \frac{1}{\lambda} \|f\|.$$ 

Since $N_0$ is dissipative, its closure $\overline{N}_0$ is still dissipative (maybe it is multivalued). By the following theorem follows Theorem 1.2.

Theorem 1.7. $N_0$ is m-dissipative.

Proof. Let $f \in C^1_b(H)$, $\varepsilon \in (0, 1)$ and $\lambda > \omega + e^K - 1$. We denote by $\varphi_\varepsilon$ the solution of

$$\lambda \varphi_\varepsilon - N_\varepsilon \varphi_\varepsilon = f.$$ 

By Proposition 1.5 we have $\varphi_\varepsilon \in D(L) \cap C^1_b(H) = D(N_0)$, then $\varphi_\varepsilon$ is solution of

$$\lambda \varphi_\varepsilon - N_0 \varphi_\varepsilon = f + F_\varepsilon \varphi_\varepsilon - F \varphi_\varepsilon.$$ 

We claim that $F_\varepsilon \varphi_\varepsilon - F \varphi_\varepsilon \to 0$ in $C_b(H)$ as $\varepsilon \to 0^+$. Indeed it holds

$$F_\varepsilon \varphi_\varepsilon(x) - F \varphi_\varepsilon(x) =$$

$$= \frac{1}{\varepsilon} \int_0^\varepsilon \left( (D \varphi_\varepsilon(\eta(s,x)), F(\eta(s,x))) + (D \varphi_\varepsilon(x), F(x)) \right) ds$$

$$= \frac{1}{\varepsilon} \int_0^\varepsilon \left( (D \varphi_\varepsilon(\eta(s,x)) - D \varphi_\varepsilon(x), F(\eta(s,x))) + (D \varphi_\varepsilon(x), F(\eta(s,x)) - F(x)) \right) ds.$$
Hence
\[ |\mathcal{F}_\varepsilon \varphi_\varepsilon(x) - \mathcal{F} \varphi_\varepsilon(x)| \leq \frac{1}{\varepsilon} \int_0^\varepsilon (|D\varphi_\varepsilon(\eta(s,x)) - D\varphi_\varepsilon(x)||F|_0 + |D\varphi_\varepsilon||F(\eta(s,x)) - F(x)|) \, ds \]

By (8) we have
\[ |F(\eta(s,x)) - F(x)| \leq K|\eta(s,x) - x| \leq K\|F\|_0 s \leq K\|F\|_0 \varepsilon. \]

Notice now that since \( \varphi_\varepsilon = R(\lambda, N_\varepsilon) f \) and \( \varepsilon \in (0,1) \), by (14) it follows
\[ \|D\varphi_\varepsilon\|_0 \leq c_1\|Df\|_0, \]
for all \( \varepsilon \in (0,1) \), where \( c_1 = (\lambda - \omega - K e^K)^{-1} \). This also implies
\[ |D\varphi_\varepsilon(\eta(s,x)) - D\varphi_\varepsilon(x)||_0 \leq c_1\|Df(\eta(s,x) + \cdot) - Df(x + \cdot)||_0 \]
\[ \leq c_1|\theta_{Df}(|\eta(s,x) - x|)| \leq c_1\theta_{Df}(\|F\|_0 \varepsilon), \]
where \( \theta_{Df} : \mathbb{R}^+ \to \mathbb{R}^+ \) is the modulus of continuity of \( Df \). So we find
\[ |\mathcal{F}_\varepsilon \varphi_\varepsilon(x) - \mathcal{F} \varphi_\varepsilon(x)| \leq \frac{1}{\varepsilon} \int_0^\varepsilon \theta_{Df}(\|F\|_0 \varepsilon) \, ds \]
Then \( \mathcal{F}_\varepsilon \varphi_\varepsilon - \mathcal{F} \varphi_\varepsilon \to 0 \) in \( C_b(H) \), as \( \varepsilon \to 0^+ \). Finally, we have obtained
\[ \lim_{\varepsilon \to 0^+} [\lambda \varphi_\varepsilon - N_0 \varphi_\varepsilon] = f \]
in \( C_b(H) \). Therefore the closure of the range of \( \lambda - N \) includes \( C^1_b(H) \), which is dense in \( C_b(H) \). So, since \( N_0 \) is dissipative, by the Lumer-Phillips theorem the closure \( N_0 \) of \( N_0 \) is \( m \)-dissipative.

**1.3 Proof of Theorem 1.2**

By Theorem 1.7 the operator \( N_0 \) is \( m \)-dissipative in \( C_b(H) \). It is also known that if \( \varphi \in D(L) \cap C^1_b(H) \), then \( N\varphi = L\varphi + \mathcal{F}\varphi \) (see, for instance, [66, 6]) and therefore \( (N, D(N)) \) is an extension of \( (N_0, D(N_0)) \). Finally, since the operator \( (N, D(N)) \) is closed (see Proposition 3.4 in [77, 7]), by the Lumer-Phillips theorem it follows that the closure of \( (N_0, D(N_0)) \) in \( C_b(H) \) coincides with \( (N, D(N)) \).
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