Orthogonal Polynomials of Askey-Wilson Type

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Abstract

We study two families of orthogonal polynomials. The first is a finite family related to the Askey–Wilson polynomials but the orthogonality is on \( \mathbb{R} \). A limiting case of this family is an infinite system of orthogonal polynomials whose moment problem is indeterminate. We provide several orthogonality measures for the infinite family and derive their Plancherel-Rotach asymptotics.

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1 Introduction

This paper outgrew from the first named author’s earlier paper [19] where some solutions to the Al-Salam–Chihara moment problem was found. We started with the weight function whose total mass was evaluated by Askey in [5] and we were led to Askey-Wilson polynomials with purely imaginary parameters, which are not necessarily pairs of complex conjugates. After a change of variable our polynomials form finite families of polynomials orthogonal on \( \mathbb{R} \). It turned out that Askey has already partially discovered this fact in [5]. The details are in Section 2. Section 2 also contains raising and lowering operators for the finite family of polynomials as well as the second order operator equation satisfied by them. The second order operator equation is of Sturm–Liouville type and are selfadjoint (symmetrical). Although the orthogonality holds for finitely many polynomials the polynomials are defined for all degrees. We determined the large degree asymptotics which shows that the zeros of the polynomials form a dense set in the segment connecting \( \pm i \).

When we further let one of the four parameters tend to zero, we have an infinite family of polynomials orthogonal on the imaginary axis with respect to infinitely many
probability measures. We identify one absolutely continuous measure and an infinite family of discrete measures of orthogonality. This is done in Section 3. In Section 4 we derive Plancherel–Rotach type asymptotics around the largest zero (soft edge) and beyond the largest zero (tail). We also develop the large degree asymptotics of the polynomials in the oscillatory range (bulk scaling). In addition, we develop a new type of asymptotics, where we let the parameters also tend to $\infty$ with $x$ around the largest zero. In this limit the leading terms of the asymptotics of the zeros, arranged from large to small, contains the zeros of the Ramanujan function. The Plancherel–Rotach asymptotics of the $q^{-1}$-Hermite polynomials, the Stieltjes–Wigert polynomials and the $q$-Laguerre polynomials are in [17], and [25]. The weight function given in Section 2 is not positive when the parameters are real. In Section 5 we treat the case of the finite family when the parameters are not real but are complex conjugates. This leads to positive weight functions.

This work extends the results of Ismail [19], where he studied the moment problem of the Al-Salam–Chihara polynomials for $q > 1$. The Al-Salam–Chihara polynomials first appeared in [2]. The $q > 1$ cases were first studied in [6]. We follow the treatments of the moment problem in [1], [28], and the spectral theory as in [30]. This work is a contribution to the study of specific moment problems. Many other moment problems have been studied over the years. Some references are [22], [8], [14], [12], [10], [13]. The most complete study is the $q^{-1}$-Hermite polynomials where theta functions made it possible to explicitly find, among other things, the $N$-extremal measures. References for orthogonal polynomials are [9], [26]. The operator equations derived in §2 extend the work of Ismail [16] on the $q^{-1}$-Hermite polynomials.

The Ramanujan, aka $q$-Airy function

$$A_q(z) = \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n} (-z)^n,$$

was introduced in [17], see also [18]. In many of our computations we shall use [15],

$$\left(aq^{-n}; q\right)_n = (q/a; q)_n (-a)^n q^{-\frac{n+1}{2}}.$$ 

2 A Finite Family of Orthogonal Polynomials

We shall use the notation

$$x = \frac{z - 1/z}{2}.$$ 

When we set $2x = z - 1/z$, then $z = x \pm \sqrt{x^2 + 1}$. We shall use the notation

$$z, \frac{1}{z} = x \pm \sqrt{x^2 + 1}, \quad \text{with} \quad |z| \leq |1/z|.$$
Set
\[(2.3) \quad u_n(x; a) = (-aq^{-n}z, aq^{-n}/z; q)_n.\]

Recall the Askey $q$-beta integral \[5, 15 \text{ Ex 6.10}\]
\[(2.4) \quad I(t_1, t_2, t_3, t_4) := \int_{\mathbb{R}} \frac{2z \prod_{j=1}^{4} (-t_j z, t_j/z; q)_{\infty}}{(-z^2, -q/z^2; q)_{\infty}} dx = -\log q (q; q)_{\infty} \frac{\prod_{1 \leq j < k \leq 4} (-t_j t_k/q; q)_{\infty}}{(t_1 t_2 t_3 t_4/q^4; q)_{\infty}},\]
which holds for $|t_1 t_2 t_3 t_4| < q^3$. Let
\[(2.5) \quad W(x, t) = \frac{2z \prod_{j=1}^{4} (-t_j z, t_j/z; q)_{\infty}}{(-z^2, -q/z^2; q)_{\infty}},\]
where $t = (t_1, t_2, t_3, t_4)$ and $x \in \mathbb{R}$ is defined by \[(2.1).\] The polynomials defined below in \[(2.7)\] were introduced by Askey in his seminal work \[5\] who proved the orthogonality relation in Theorem \[(2.1).\] We will include Askey’s proof because it will used to prove other orthogonality relations for the same polynomials. The proof is analogous to the attachment technique used by Askey and Wilson in \[7\] and others. The Askey-Wilson proof is also explained in \[18\].

**Theorem 2.1.** Given any $N \in \mathbb{N}$, let
\[(2.6) \quad t_1, t_2, t_3, t_4 \in \mathbb{R}, \quad |t_1 t_2 t_3 t_4 q^3| < q^{2N}.\]

Then the polynomials,
\[(2.7) \quad p_n(x, t) = (t_1/q)^n (-q^2/t_1 t_2, -q^2/t_1 t_3, -q^2/t_1 t_4; q)_n \times _4\phi_3 \left( \begin{array}{c} q^{-n}, q^{n+3}/t_1 t_2 t_3 t_4, -q/t_1 z, q z/t_1 \\ -q^2/t_1 t_2, -q^2/t_1 t_3, -q^2/t_1 t_4 \end{array} \right| q, q \right),\]
are orthogonal with respect to the normalized weight function,
\[(2.8) \quad w(x, t) = \frac{W(x, t)(t_1 t_2 t_3 t_4/q^3; q)_{\infty}}{(q; q)_{\infty} \log q^{-1} \prod_{1 \leq j < k \leq 4} (-t_j t_k/q; q)_{\infty}} = \frac{2z(t_1 t_2 t_3 t_4/q^3; q)_{\infty} \prod_{j=1}^{4} (-t_j z, t_j/z; q)_{\infty}}{(q, -z^2, -q/z^2; q)_{\infty} \log q^{-1} \prod_{1 \leq j < k \leq 4} (-t_j t_k/q; q)_{\infty}}\]
for $0 \leq n \leq N$ where $t = (t_1, t_2, t_3, t_4)$ and $x \in \mathbb{R}$ is defined by \[(2.1).\] Furthermore, the orthogonality relation is
\[(2.9) \quad \int_{\mathbb{R}} w(x, t) p_n(x, t) \overline{p_m(x, t)} dx = \frac{(-1)^n (1 - q^{n+3}/t_1 t_2 t_3 t_4) \prod_{1 \leq j < k \leq 4} (-q^2/t_j t_k; q)_n (q; q)_n}{(1 - q^{2n+3}/t_1 t_2 t_3 t_4) (q^4/t_1 t_2 t_3 t_4; q)_n} \delta_{m,n}.\]
Before proving Theorem 2.1 we next indicate the range of the parameters to ensure orthogonality. To determine the large \( z \) behavior of \( W \) we set \( z = q^{-m} \lambda \), where \( 1 < |\lambda| \leq 1/q \). For this \( x \) we have

\[
W(x; t) = \mathcal{O} \left( q^{-m} \prod_{j=1}^{4} \frac{(-t_j \lambda q^{-m}; q)_m}{(-\lambda^2 q^{-2m}; q)_{2m}} \right) = \mathcal{O} \left( (t_1 t_2 t_3 t_4 q^{-2})^m \right).
\]

For integrability we need \( W(x; t) \) to be \( \mathcal{O}(x^{-1-\epsilon}) \) for some positive \( \epsilon \). This happens if and only if \(|t_1 t_2 t_3 t_4 q^{-3}| < 1\). The moments \( \int_{\mathbb{R}} x^n W(x; t) dx \) exist for \( 0 \leq n \leq 2N \), if \(|t_1 t_2 t_3 t_4 q^{-3}| < q^{2N} \).

**Proof of Theorem 2.1** Let

\[
p_n(x, t) = \sum_{k=0}^{n} \frac{(q^{-n}; q)_k}{(q; q)_k} a_{n,k} u_k(x; t_1).
\]

where \( a_{n,k} \) are to be determined. We now consider the integral

\[
I_{m,k} := \int_{\mathbb{R}} W(x; t) u_k(x; t_1) u_m(x; t_2) \, dx, \quad 0 \leq m, k \leq N.
\]

It is clear that \( I_{m,k} = I(t_1 q^{-k}, t_2 q^{-m}, t_3, t_4) \), hence for \( 0 \leq n, m \leq N \),

\[
\int_{\mathbb{R}} W(x; t) p_n(x, t) u_m(x; t_2) dx = I(t_1, t_2 q^{-m}, t_3, t_4) \sum_{k=0}^{n} \frac{(q^{-n}; q)_k}{(q; q)_k} a_{n,k} \frac{(-t_1 t_2 q^{-m-k-1}, -t_1 t_3 q^{-k-1}, -t_1 t_4 q^{-k-1}; q)_k}{(t_1 t_2 t_3 t_4 q^{-m-k-3}; q)_k}
\]

\[
= I(t_1, t_2 q^{-m}, t_3, t_4) \sum_{k=0}^{n} \frac{(q^{-n}; q)_k}{(q; q)_k} \frac{(-q^{m+2}/t_1 t_2, -q^{2}/t_1 t_3, -q^{2}/t_1 t_4; q)_k}{(q^{m+4}/t_1 t_2 t_3 t_4; q)_k} t_1^{2k} q^{-k(k+1)} (-1)^k a_{n,k}.
\]

We now choose

\[
a_{n,k} = C_n \frac{(q^{n+3}/t_1 t_2 t_3 t_4; q)_k}{(-q^{2}/t_1 t_2, -q^{2}/t_1 t_3, -q^{2}/t_1 t_4; q)_k} q^{k(k+2)} (-1)^k t_1^{-2k},
\]

and conclude that

\[
\int_{\mathbb{R}} W(x; t) p_n(x, t) u_m(x; t_2) dx = C_n I(t_1, t_2 q^{-m}, t_3, t_4) \Phi(\theta) \left( \begin{array}{c} q^{-n}, q^{n+3}/t_1 t_2 t_3 t_4, -q^{m+2}/t_1 t_2 \\ q^{m+4}/t_1 t_2 t_3 t_4, -q^{2}/t_1 t_2 \end{array} \right) (q, q)
\]

\[
= C_n I(t_1, t_2 q^{-m}, t_3, t_4) \frac{(q^{m+1-n}, q^2/t_3 t_4; q)_n}{(q^{m+4}/t_1 t_2 t_3 t_4, q^{-n-1} t_1 t_2; q)_n},
\]

4
which clearly vanishes for \( m < n \). We choose \( C_n \) to be the factor in front of the \( 4^q \phi_3 \) in (2.7). It is clear that the integral in (2.9) equals

\[
t_n^q \left( \frac{q^{-n}, q^{n+3}/t_1 t_2 t_3 t_4; q_n}{(q; q)_n} \right) \int_{\mathbb{R}} p_n(x; t) W(x; t)(-q/t_1 z, q z/t_1; q)_n dx
\]

\[
= t_n^q \left( \frac{q^{-n}, q^{n+3}/t_1 t_2 t_3 t_4; q_n}{(q; q)_n} \right) \int_{\mathbb{R}} p_n(x; t) W(x; t)(-q/t_2 z, q z/t_2; q)_n dx
\]

\[
= \frac{(-1)^n (1 - q^{n+3}/t_1 t_2 t_3 t_4) \prod_{1 \leq j < k \leq 4} (-q^2/t_j t_k; q)_n (q; q)_n}{(1 - q^{2n+3}/t_1 t_2 t_3 t_4)(q^4/t_1 t_2 t_3 t_4; q)_n} I(t_1, t_2, t_3, t_4).
\]

This completes the proof. \( \square \)

It must be noted that the weight function \( W \) is not positive on \( \mathbb{R} \) if the parameters \( t_1, t_2, t_3, t_4 \) are real and distinct but \( W \) is positive when the parameters form a pair of complex conjugates.

The orthogonality measure of the \( q^{-1} \)-Hermite polynomials is not unique. Askey [4] identified the weight function, [18]

\[
w_A(x) := \frac{-2z/\log q}{(q, -z^2, -q/z^2; q)_\infty}, \quad x = (z - 1/z)/2,
\]

for the \( q^{-1} \)-Hermite polynomials, here \( \int_{\mathbb{R}} w_A(x) dx = 1 \). Ismail and Masson [21] proved that the \( q^{-1} \)-Hermite polynomials are orthogonal with respect to a family of discrete measures supported at the sequences of points \( \{ x_n(\alpha) : -\infty < n < +\infty \} \) with the masses \( m_n(\alpha) \) at \( x_n(\alpha) \), where

\[
x_n(\alpha) = (q^{-n}/\alpha - \alpha q^n)/2, \quad m_n(\alpha) = \frac{\alpha^4 q^{n(2n-1)}(1 + \alpha^2 q^{2n})}{(-\alpha^2, -q/\alpha^2, q; q)_\infty},
\]

with \( \alpha \in (q, 1) \), see more details in [18] and [21]. These measure are the only measures which make the polynomials dense in their weighted \( L_2 \) spaces. They are normalized to have total mass 1.

We denote the measure in (2.11) by \( \mu_\alpha \) and define a measure \( \mu \) by

\[
\mu(x) = \frac{(t_1 t_2 t_3 t_4/q^3; q)_\infty \prod_{j=1}^4 (-t_j z, t_j/z; q)_\infty}{\prod_{1 \leq j < k \leq 4} (-t_j t_k/q; q)_\infty} \mu_\alpha(x),
\]

with \( x = (z - 1/z)/2 \). Then under the conditions in Theorem 2.1 we have the orthogonality relation

\[
\int_{\mathbb{R}} p_n(x, t) p_m(x, t) d\mu(x)
\]

\[
= \frac{(-1)^n (1 - q^{n+3}/t_1 t_2 t_3 t_4) \prod_{1 \leq j < k \leq 4} (-q^2/t_j t_k; q)_n (q; q)_n}{(1 - q^{2n+3}/t_1 t_2 t_3 t_4)(q^4/t_1 t_2 t_3 t_4; q)_n} \delta_{m,n}.
\]

5
The proof is the same as the proof of Theorem 2.1 once we evaluate the total mass of \( \mu \). The total mass of \( \mu \) is evaluated using the \( 6 \psi_6 \) summation theorem as indicated in [21]. We note that the only difference between (2.13) and (2.9) is that the normalized Askey measure \( w_A \) is replaced by \( \mu_\alpha \).

It may of interest to explain why \( \mu_\alpha \) is a discrete version of the Askey weight function \( w_A \). The parameterization used is \( x_n(\alpha) = (q^{-n}/\alpha - \alpha q^n)/2 \), so that

\[
\frac{dx}{\alpha} = \frac{q^{-n}/\alpha + \alpha q^n}{2}(- \log q)dn,
\]

and we interpret \( dn \) as the mesh used which is 1 in this case. At the same time, we find the Askey weight function in (2.10) has the property as follow:

\[
- \log q \ w_A(x_n(\alpha)) = \frac{2q^{-n}/\alpha}{(q, -q^{-2n}/\alpha^2, -q^{2n+1}\alpha^2; q)_\infty} = \frac{2q^{-n}(-q\alpha^2; q)_{2n}}{\alpha(-q^{-2n}/\alpha^2; q)_{2n}(q, -1/\alpha^2, -q\alpha^2; q)_\infty} = \frac{2\alpha^4n+1q^{2n^2}}{(-q/\alpha^2, -\alpha^2; q)_\infty}.
\]

Hence

\[
(2.14) \quad m_n(\alpha) = w_A(x_n(\alpha))dx_n(\alpha).
\]

It implies that any \( N \)-extremal measure is essentially equal to the value of the Askey weight functions multiplied by \( dx \) calculated at the mass points.

Recall that the Askey-Wilson polynomials [7], [15], [18] are defined by

\[
(2.15) \quad AW_n(\cos \theta, t) = (t_1 t_2, t_1 t_3, t_1 t_4; q)_n
\times \phi_3 \left( \begin{array}{c} q^{-n}, q^{n-1}t_1 t_2 t_3 t_4, t_1 e^{i\theta}, t_1 e^{-i\theta} \\ t_1 t_2, t_1 t_3, t_1 t_4 \end{array} \right) q, q.
\]

We note that the algebraic properties of the polynomials \( \{p_n(x; t)\} \) follow from the corresponding properties of \( \{AW_n(x; t)\} \) by using the simple rules

\[
(2.16) \quad t_j \rightarrow iq/t_j, 1 \leq j \leq 4, \quad e^{i\theta} \rightarrow -iz, \quad AW_n(\cos \theta, t) \rightarrow i^n p_n(x; t).
\]

For example the Ismail–Wilson generating function for the Askey-Wilson polynomials in [23], which is reproduced in [15], [18], implies the generating function

\[
(2.17) \quad \sum_{n=0}^{\infty} \frac{p_n(x, t)t^n}{(q, -q^2/t_1 t_2, -q^2/t_3 t_4; q)_n} = 2\phi_1 \left( \begin{array}{c} qz/t_1, qz/t_2, -q^2/t_1 t_2 \\ -q^2/t_3 t_4 \end{array} \right) q, t/z \phi_1 \left( \begin{array}{c} -q/z t_3, -q/z t_4, -q^2/t_3 t_4 \\ -q^2/t_3 t_4 \end{array} \right) q, tz.
\]

The symmetry of \( p_n(x, t) \) under \( t_j \leftrightarrow t_k, 1 \leq j, k \leq 4 \) gives additional generating functions.
The connection relation for our polynomials follow from the corresponding results for the Askey–Wilson polynomials in [7], [15]. In particular

\[ p_n(x, s) = \sum_{k=0}^{n} c_{k,n}(s, t) p_k(x, t), \]

where

\[ t = t_1, t_2, t_3, t_4 \]

and

\[ s = s_1, s_2, s_3, s_4, s_4 = t_4. \]

then by the connection coefficient problem for the Askey–Wilson problem in [15] we get

\[ c_{k,n}(s, t) = \frac{(-q^2/s_1t_4, -q^2/s_2t_4, -q^2/s_3t_4; q)_n}{(-q^2/s_1t_4, -q^2/s_2t_4, -q^2/s_3t_4, q, q^{k+3}/t_1t_2t_3t_4; q)_k} \frac{(q^{k-n}/t_4)^{k-n}}{n!} \times \phi_4 \left( \begin{array}{c} q^{k-n}, q^{n+k+3}/s_1s_2s_3t_4, -q^{k+2}/t_1t_4, -q^{k+2}/t_2t_4, -q^{k+2}/t_3t_4 \\ q^{2k+1}/t_1t_2t_3t_4, -q^{k+2}/s_1t_4, -q^{k+2}/s_2t_4, -q^{k+2}/s_3t_4 \end{array} \mid q, q \right). \]

Our next task is to identify raising and lowering operators for our polynomials. The Askey–Wilson operator \( D_q \) and the averaging operator \( A_q \) are defined by

\[ (D_q f)(x) = \frac{\tilde{f}(q^{1/2}x) - \tilde{f}(q^{-1/2}x)}{(q^{1/2} - q^{-1/2})(z+1)/2}, \]

\[ (A_q f)(x) = \frac{1}{2} \left[ \tilde{f}(q^{1/2}x) + \tilde{f}(q^{-1/2}x) \right]. \]

A calculation gives

\[ D_q(-a/z, a/z; q)_k = -2a \frac{1 - q^k}{1 - q^{-1}} (-aq^{1/2}/z, aq^{1/2}z; q)_{k-1} \]

Therefore

\[ D_q p_n(x, t) = \frac{2(1 - q^n)(1 - q^{n+3}/t_1t_2t_3t_4)}{(1 - q)q^{(n-1)/2}} p_{n-1}(x, q^{-1/2}t). \]

We can also establish the raising operator relation

\[ \frac{1}{w(x; q^{1/2}t)} D_q w(x; t)p_n(x, t) = \frac{2q^{-n/2}(1 - q^2/t_1t_2t_3t_4)(1 - q^3/t_1t_2t_3t_4)}{(1 - q) \prod_{1 \leq j < k \leq 4}(1 + q/t_jt_k)} p_{n+1}(x, q^{1/2}t). \]
Proof. It readily follows that
\[
\frac{1}{W(x; q^{1/2}t)} D_q(W(x; t)(-q/t_1 z, qz/t_1; q)_k)
= \frac{2t q^{1/2}}{(1-q)} [(1 - t_1 t_2 t_3 t_4/q^{k+2})(-q^{1/2}/t_1 z, q^{1/2}z/t_1; q)_{k+1}
+ t_1 t_2 t_3 t_4/q^{k+2}(1 + q^{k+1}/t_1 t_2)(1 + q^{k+1}/t_1 t_3)(1 + q^{k+1}/t_1 t_4)(-q^{1/2}/t_1 z, q^{1/2}z/t_1; q)_k].
\]
This and (2.7) establish the desired relation. 

Combining (2.24) and (2.25) leads to the \(q\)-Sturm-Liouville equation
\[
\frac{1}{w(x; t)} D_q[w(x, q^{-1/2} t) D_q p_n(x; t|q)] = \frac{4q^{1-n}(1 - q^n)}{(1-q)^2} \times \frac{(1 - q^4/t_1 t_2 t_3 t_4)(1 - q^5/t_1 t_2 t_3 t_4)(1 - q^{n+3}/t_1 t_2 t_3 t_4)}{\prod_{1 \leq j < k \leq 4}(-q^2/t_j t_k)} p_n(x; t|q).
\]

By iterating (2.25) we derive the Rodrigues type formula
\[
\frac{1}{w(x; t)} D_q^n w(x; q^{-n/2} t)
= q^{-n(n-1)/4} \left( \frac{2}{1-q} \right)^n \frac{(q^4/t_1 t_2 t_3 t_4; q)_{2n}}{\prod_{1 \leq j < k \leq 4}(-q^2/t_j t_k; q)_n} p_n(x; t|q).
\]

A recursion relation for our polynomials follows from the recurrence relation of the Askey-Wilson polynomials \([3, 15, 18]\). Indeed we find that
\[
2xp_n(x, t) = A_n p_{n+1}(x, t) + B_n p_n(x, t) + C_n p_{n-1}(x, t), \quad n \geq 0,
\]
and
\[
x = \frac{z - 1/z}{2}, \quad p_{-1}(x, t) = 0, \quad p_0(x, t) = 1,
\]
where
\[
A_n = \frac{1 - q^{n+3}/t_1 t_2 t_3 t_4}{(1 - q^{2n+3}/t_1 t_2 t_3 t_4)(1 - q^{2n+4}/t_1 t_2 t_3 t_4)},
\]
\[
C_n = \frac{(1 - q^n) \prod_{1 \leq j < k \leq 4} (1 + q^{n+1}/t_j t_k)}{(1 - q^{2n+2}/t_1 t_2 t_3 t_4)(1 - q^{2n+3}/t_1 t_2 t_3 t_4)},
\]
and
\[
B_n = \frac{t_1}{q} - \frac{q}{t_1} - \frac{t_1}{q} A_n \prod_{j=2}^4 (1 + q^{n+2}/t_1 t_j) - \frac{q C_n}{t_1 \prod_{j=2}^4 (1 + q^{n+1}/t_1 t_j)}.
\]

Note that \(A_n\) and \(C_n\) are clearly symmetric in all parameters. What is not clear but is nevertheless true is that \(B_n\) is also symmetric in \(t_1, t_2, t_3, t_4\).
3 A Finite Family With Positive Weight Function

In this section we shall treat the case

\[ (3.1) \quad t_1 = t_2, \quad t_3 = t_4, \quad \exists t_1 \neq 0 \quad \text{and} \quad \exists t_3 \neq 0, \]

which implies \( w(x, t) > 0 \). As we saw in §2 the moments \( \int_{\mathbb{R}} x^m w(x)dx, m = 0, 1, \ldots, N \) exist when \( |t_1 t_2 t_3 t_4| < q^{2N+3} \).

The Sears transformation. (III.15) in [15] shows that \( p_n \) is symmetric in the parameters \( t_1, t_2, t_3, t_4 \). On the other hand it is clear that \( p_n \) is a polynomial in \( \frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3}, \frac{1}{t_4} \), hence it is also a polynomial in \( \frac{1}{t_1} \). This means that \( p_n \) is a polynomial in the elementary symmetric functions of \( \frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3}, \frac{1}{t_4} \), with real coefficients. Therefore \( p_n \) is a real polynomial of \( x \) when \( t_1, t_2, t_3, t_4 \) are chosen as in (3.1).

Let \( \sigma_j \) be the elementary symmetric functions of \( t_1, t_2, t_3, t_4 \), that is

\[ (3.2) \quad \sigma_1 = \sum_{j=1}^{4} t_j, \quad \sigma_2 = \sum_{1 \leq j < k \leq 4} t_j t_k, \]

\[ \sigma_3 = \sum_{1 \leq i < j < k \leq 4} t_1 t_j t_k, \quad \sigma_4 = t_1 t_2 t_3 t_4. \]

It is straightforward to show that the weight function satisfies the divided difference equations

\[ (3.3) \quad \frac{D_q w(x; q^{-1/2}t)}{w(x; t)} = \frac{2q(1 - \sigma_4/q^4)(1 - \sigma_4/q^5)}{(q - 1) \prod_{1 \leq j < k \leq 4} (1 + t_j t_k/q^2)} \times [2(1 - \sigma_4/q^4)x - \sigma_1/q - \sigma_3/q^3] \]

and

\[ (3.4) \quad \frac{A_q w(x; q^{-1/2}t)}{w(x; t)} = \frac{q^{1/2}(1 - \sigma_4/q^4)(1 - \sigma_4/q^5)}{\prod_{1 \leq j < k \leq 4} (1 + t_j t_k/q^2)} \times [(2x^2 + 1)(1 + \sigma_4/q^4) + x(\sigma_3/q^3 - \sigma_1/q) + \sigma_2/q^2]. \]

The following lemma is the analogue of integration by parts, [16].

**Lemma 3.1.** Let \( f \) and \( g \) with \( \int_{\mathbb{R}} f(z)\tilde{g}(q^{1/2}z) \frac{dz}{z} \) and \( \int_{\mathbb{R}} f(z)\tilde{g}(q^{-1/2}z) \frac{dz}{z} \) existed. We have

\[ (3.5) \quad \int_{\mathbb{R}} (D_q f)(x)g(x)dx = - \int_{\mathbb{R}} (D_q g)(x)f(x)dx. \]

When we apply Theorem 21.6.3 in [18], and use the above lemma, Lemma 3.1, we establish the following theorem.
Theorem 3.2. We assume that $g(x; t_1, t_2, t_3, t_4)$ is a positive function and its moments $\int_\mathbb{R} x^n g(x; t_1, t_2, t_3, t_4) dx$ exist for some non-negative integers $n$ with $0 \leq n \leq 2N$ when $|t_1 t_2 t_3 t_4| < q^{2N+3}$. If $g(x; t)$ satisfies the following equations

$$
\frac{D_q g(x; q^{-1/2}t)}{g(x; t)} = \frac{2q(1-\sigma_4/q^4)(1-\sigma_4/q^5)}{(q-1) \prod_{1 \leq j < k \leq 4} (1 + t_j t_k/q^2)} \times [2(1-\sigma_4/q^4)x - \sigma_1/q - \sigma_3/q^3]
$$

(3.6)

$$
\frac{A_q g(x; q^{-1/2}t)}{g(x; t)} = \frac{q^{1/2}(1-\sigma_4/q^4)(1-\sigma_4/q^5)}{\prod_{1 \leq j < k \leq 4} (1 + t_j t_k/q^2)} \times [(2x^2 + 1)(1 + s_4/q^4) + x(s_3/q^3 - \sigma_1/q) + s_2/q^2].
$$

then $g(x; t)$ is a weight function for polynomials $\{p_n(x; t_1, t_2, t_3, t_4 | q)\}_{n=0}^N$.

4 Asymptotics

The Askey-Wilson polynomials also have an $sW_7$ representation \[15\]; which, in view of (2.16), yields the representation

(4.1) $p_n(x, t) = \frac{(-q^2/t_1 t_2, -q^2/t_1 t_3, -q^2/t_2 t_3, -q/t_4 z; q)_n}{(-q^3 z/t_1 t_2 t_3; q)_n} \times sW_7(-q^2 z/t_1 t_2 t_3; q z/t_1, q z/t_2, q z/t_3, q^{n+3}/t_1 t_2 t_3 t_4, q^{-n}; q, t_4/z)$

and we also have the representation

(4.2) $p_n(x, t) = \left( -\frac{q^2}{t_2 t_3}, -\frac{q^2}{t_2 t_4}, -\frac{q^2}{t_3 t_4}; q \right)_n \left( Q_n \left( -\frac{1}{z}, t \right) + Q_n \left( z, t \right) \right)$,

where

(4.3) $Q_n(w, t) = \left( -\frac{w q^{n+3}}{t_1 t_3 t_4}, -\frac{w q^{n+3}}{t_2 t_3 t_4}, -\frac{w q^{n+2}}{t_2 t_3}, -\frac{w q}{t_2}, -\frac{q}{w}, -\frac{q}{w}, -\frac{q}{w}; q \right)_\infty \times w^n sW_7 \left( \frac{w^2 q^{n+2}}{t_2 t_3}, -\frac{q^{n+2}}{t_2}, \frac{q}{w}, \frac{q}{w}, \frac{q}{w}; t_2 t_3, -t_1 w, -t_4 w; q, -\frac{q^{n+2}}{t_1 t_4} \right)$.

For each fixed $z \in \mathbb{C}$ the form (4.3) leads to

(4.4) $Q_n(w, t) \approx \frac{w^n B(w^{-1})}{\left( -\frac{q^2}{t_2 t_3}, -\frac{q^2}{t_2 t_4}, -\frac{q^2}{t_3 t_4}; q \right)_\infty}$, $n \to \infty$,

where

(4.5) $B(w) = \left( -\frac{q w}{t_1}, -\frac{q w}{t_2}, -\frac{q w}{t_3}, -\frac{q w}{t_4}; q \right)_\infty$. 

10
Let \( z_1, z_2 \) be the roots of \( 2x = z - 1/z \), with \( |z_1| \leq |z_2| \). It is easy to see that \( |z_1| = |z_2| \) if and only if \( x \) is purely imaginary and \( ix \in [-1,1] \). Moreover \( z_1 = z_2 \) if and only if \( x = \pm i \). It is clear that \( z_1z_2 = -1 \). Therefore

\[
(4.6) \quad p_n(x, t) = z_2^n \frac{(qz_1/t_1, qz_1/t_2, qz_1/t_3, qz_1/t_4; q)_\infty}{(-z_2^2; q)_\infty} [1 + o(1)],
\]

if \( ix \not\in [-1,1] \). If \( ix \in (-1,1) \) we let \( z_1 = ie^{i\theta}, z_2 = ie^{-i\theta} \), with \( \theta \in (0, \pi) \). Then

\[
(4.7) \quad p_n(x, t) = 2C(x) \cos(n\theta + \phi - n\pi/2) [1 + o(1)],
\]

where \( C(x) \geq 0 \), and

\[
(4.8) \quad C(x)e^{i\phi} = \frac{(qie^{-i\theta}/t_1, qie^{-i\theta}/t_2, qie^{-i\theta}/t_3, qie^{-i\theta}/t_4; q)_\infty}{(e^{-2i\theta}; q)_\infty}.
\]

We note that the above asymptotic formulas can also be derived from the generating function (2.17) using Darboux’s asymptotic method, [27].

When \( n \) is large we know that the polynomials are no longer orthogonal but (4.7) indicates that the polynomials have their zeros dense in the segment connecting \( \pm i \).

We now derive the large \( N \) asymptotics of \( p_N \), when \( t_1 \) and \( t_2 \) are of the form \( t_1q^N \) and \( t_2q^N \), respectively.

First, we recall the transformation [15]

\[
(4.9) \quad 4\phi_3 \left( \begin{array}{c} q^{-n}, a, b, c \\ d, e, f \end{array} \right| q, q) = \left( \frac{bc}{d} \right)^n \frac{(de/bc, df/bc; q)_n}{(e, f; q)_n} 4\phi_3 \left( \begin{array}{c} q^{-n}, a, d/b, d/c \\ d, de/bc, df/bc \end{array} \right| q, q).
\]

Let \( a = qz/t_1, b = -qt_1z, c = q^{n+3}/t_1t_2t_3t_4, d = -q^2/t_1t_3, e = -q^2/t_1t_2 \) and \( f = -q^2/t_1t_4 \), and then we have

\[
(4.10) \quad 4\phi_3 \left( \begin{array}{c} q^{-n}, q^{n+3}/t_1t_2t_3t_4, -q/t_1z, qz/t_1 \\ -q^2/t_1t_2, -q^2/t_1t_3, -q^2/t_1t_4 \end{array} \right| q, q) = (q^{n+2}/t_1t_2t_4z)^n \left( \frac{(-q^{-n}t_4z, -q^{-n}t_2z; q)_n}{(-q^2/t_1t_2, -q^2/t_1t_4; q)_n} \right) 4\phi_3 \left( \begin{array}{c} q^{-n}, -t_2t_4q^{-(n+1)}, qz/t_3, qz/t_1 \\ -q^2/t_1t_3, -q^2t_1t_4, q^{-n}zt_4, q^{-n}zt_2 \end{array} \right| q, q)
\]

Next, we define

\[
(4.11) \quad v_n = \frac{(-t_1t_3/q, -q/zt_2; q)_n}{(q, t_3/z; q)_n} 4\phi_3 \left( \begin{array}{c} q^{-n}, -t_2t_4/q, q^{1-n}z/t_3, qz/t_1 \\ -q^{2-n}/t_1t_3, -zt_2, -q^{-n}zt_2 \end{array} \right| q, q) = \sum_{k=0}^{n} (qz/t_1, -t_2t_4/q; q)_k (-t_1t_3/q, -q/zt_2; q)_{n-k} (t_1/t_2)^k.
\]
Therefore, we have

\[
(4.12) \quad 2\phi_1 \left( \frac{qz}{t_1}, -\frac{t_2t_4}{q} \middle| q, \frac{t_1t}{t_2} \right) 2\phi_1 \left( \frac{-q/zt_2}{t_3/z} \middle| q, t \right).
\]

We consider transform

\[
(4.13) \quad 2\phi_1(a, b; c; q, z) = \frac{(b, az; q)_{\infty}}{(c, z; q)_{\infty}} 2\phi_1(c/b, az; q, b)
\]

which leads to

\[
(4.14) \quad \sum_{n=0}^{\infty} v_n t^n = \frac{-t_1t_3/q, -t_2t_4/q, qzt/t_2, -tq/zt_2; q)_{\infty}}{(t_3/z, -zt_4, t, tt_1/t_2; q)_{\infty}} 2\phi_1 \left( \frac{qz/t_2, tt_1/t_2}{tq/t_2} \middle| q, -\frac{t_4t_2}{q} \right)
\times 2\phi_1 \left( \frac{qz/t_1, t}{-tq/zt_2} \middle| q, -\frac{t_4t_2}{q} \right)
\]

If we apply Darboux’s method, we get

\[
(4.15) \quad \lim_{n \to \infty} v_n = \frac{-t_1t_3/q, -t_2t_4/q; q)_{\infty}}{(t_3/z, -zt_4, q)_{\infty}} \left[ \frac{(qz/t_2, -q/zt_2; q)_{\infty}}{(q, t_1/t_2; q)_{\infty}} 2\phi_1 \left( \frac{qz/t_2, t_1/t_2}{qz/t_2} \middle| q, -\frac{t_2t_4}{q} \right) \right]
+ \left( \frac{t_1}{t_2} \right)^n \frac{(qz/t_1, -q/zt_1; q)_{\infty}}{(q, t_2/t_1; q)_{\infty}} 2\phi_1 \left( \frac{qz/t_1, t_2/t_1}{-q/t_1z} \middle| q, -\frac{t_1t_3}{q} \right).
\]

Note that

\[
(4.16) \quad p_n(x; t_1, t_2, t_3q^n, t_4q^n \mid q) = (q, t_3/z, -t_4z; q_n q^{-n^2}(q^2/t_1t_2t_4)^n v_n.
\]

Hence, we can establish the following formula

\[
\left( \frac{q^n t_1 t_2 t_3 t_4}{(q, t_1 t_3, -t_2 t_4; q)_{\infty}} \right)^n p_n(x; t_1, t_2, q^n t_3, q^n t_4 \mid q)
= \frac{t_2^n (qz/t_2, -q/zt_2; q)_{\infty}}{(q, t_1/t_2; q)_{\infty}} 2\phi_1 \left( \frac{qz/t_2, t_1/t_2}{qz/t_2} \middle| q, -\frac{t_2t_4}{q} \right)
+ \frac{t_1^n (qz/t_1, -q/zt_1; q)_{\infty}}{(q, t_2/t_1; q)_{\infty}} 2\phi_1 \left( \frac{qz/t_1, t_2/t_1}{-q/t_1z} \middle| q, -\frac{t_1t_3}{q} \right) \left[ 1 + o(1) \right]
\]

5 An Infinite Family of Orthogonal Polynomials

With the notation

\[
(5.1) \quad t = (t_1, t_2, t_3)
\]
we set

\[
V_n(x, \mathbf{t}|q) := V_n(x; t_1, t_2, t_3|q)
\]

and

\[
w(x, \mathbf{t}) = \frac{2z \prod_{j=1}^{3} (-t_j z, t_j/z; q)_\infty}{(q, -z^2, -q/z^2; q)_\infty \log q^{-1} \prod_{1 \leq j < k \leq 3} (-t_j t_k/q; q)_\infty}.
\]

It is clear that

\[
V_n(x; \mathbf{t}|q) = \lim_{t_4 \to 0} \frac{p_n(x, \mathbf{t})}{(-q^2/t_1 t_4, -q^2/t_1 t_2, -q^2/t_2 t_3; q)_n}.
\]

When \( t_1, t_2, t_3 > 0 \) the orthogonality relation (2.9) yields

\[
\int_{\mathbb{R}} V_n(x; \mathbf{t}|q)V_m(x; \mathbf{t}|q)d\mu(x; \mathbf{t}|q) = \frac{(q, -q^2/t_1 t_3; q)_n}{(-q^2/t_1 t_2, -q^2/t_2 t_3; q)_n} \left( \frac{t_3^2}{q^3} \right)^n \delta_{m,n},
\]

where \( x \) is as in (2.1) and

\[
d\mu(x; \mathbf{t}|q) = \frac{2z \prod_{j=1}^{3} (-t_j z, t_j/z; q)_\infty dx}{(q, -z^2, -q/z^2; q)_\infty \log q^{-1} \prod_{1 \leq j < k \leq 3} (-t_j t_k/q; q)_\infty}.
\]

Note that the measure defined above is a signed measure.

Next we record the raising and lowering operators for the polynomials \( \{V_n(x; \mathbf{t}|q)\} \) by taking the limit as \( t_4 \to 0+ \) of the corresponding formulas in §2. The result is

\[
\mathcal{D}_q V_n(x; \mathbf{t}|q) = \frac{-2q^{(n+3)/2}(1 - q^n)V_{n-1}(x; q^{-1/2}t_1, q^{-1/2}t_2, q^{-1/2}t_3|q)}{(1 - q)(1 + q^2/t_1 t_2)(1 + q^2/t_2 t_3)(t_2 t_3)},
\]

and

\[
\frac{1}{w(x; q^{1/2}\mathbf{t})} \mathcal{D}_q w(x; \mathbf{t}) V_n(x, \mathbf{t}) = \frac{2q^{3-n/2} V_{n+1}(x, q^{1/2}\mathbf{t})}{(1 - q)t_1^2 t_2 t_3 (1 + q/t_1 t_3)}.
\]

Therefore (5.7) leads to the following \( q \)-Sturm-Liouville operator equation

\[
\frac{1}{w(x, t_1, t_2, t_3, 0)} \mathcal{D}_q [w(x, q^{-1/2}t_1, q^{-1/2}t_2, q^{-1/2}t_3, 0) \mathcal{D}_q V_n(x; \mathbf{t}|q)] = \frac{-4q^n}{(1 - q)^2 t_1^2 t_2 t_3^2} \prod_{1 \leq j < k \leq 3} (1 + q^2/t_j t_k) V_n(x; \mathbf{t}|q).
\]

By iterating (5.7) we derive the Rodrigues-type formula

\[
\frac{1}{w(x; \mathbf{t})} \mathcal{D}_q w(x; q^{-n/2}\mathbf{t}) = q^{-n(n-1)/4} \left( \frac{2}{1 - q} \right)^n \left( \frac{q^4/t_1 t_2 t_3^4; q}_{-q^2/t_1 t_3, -q^2/t_2 t_4, -q^2/t_3 t_4; q} P_n(x; \mathbf{t}|q). \right.
\]
Let \( t_4 \to 0 \), we obtain another Rodrigues formula
\[
\frac{1}{w(x, t_1, t_2, t_3, 0)} D^n_q[w(x, q^{-n/2}t_1, q^{-n/2}t_2, q^{-n/2}t_3, 0)] = \\
\left( \frac{2}{1 - q} \right)^n \frac{q^{3n^2/4+17n/4}}{t_1^{2n}t_2t_3(-q^2/t_1t_3; q)_n V_n(x; t|q)}.
\]

(5.10)

We now prove an orthogonality relation for \( \{V_n(x; t_1, t_2, t_3)\} \). To prove the orthogonality relations, we need a lemma to compute the total mass at first.

**Lemma 5.1.**
\[
\int_{\mathbb{R}} (-qt_1z, qt_1/z, -qt_2z, qt_2/z, -qt_3z, qt_3/z; q)_\infty d\mu_\alpha(x) = (-qt_1t_2, -qt_1t_3, -qt_2t_3; q)_\infty \\
\text{where } d\mu_\alpha(x) \text{ is as in (2.11).}
\]

This is a special case of Theorem 3.5 in [19].

**Theorem 5.2.** The polynomials \( \{V_n(x, t)|q\} \) satisfy the orthogonality relation
\[
\int_{\mathbb{R}} V_n(x; t|q)V_m(x; t|q)(-qt_1z, qt_1/z, -qt_2z, qt_2/z, -qt_3z, qt_3/z; q)_\infty d\mu_\alpha(x) \\
= \frac{(q, -q^2/t_1t_3; q)_n}{(-q^2/t_1t_2, -q^2/t_2t_3; q)_n} (-qt_1t_2, -qt_1t_3, -qt_2t_3; q)_\infty \left( \frac{t_1^3}{q^2} \right)^n \delta_{m,n} \\
\text{where } d\mu_\alpha(x) \text{ is defined by (2.11).}
\]

**Proof.** From the discussion before the proof of Theorem ?? we conclude that
\[
w_A(x_n(\alpha)) = -\log q \frac{m_\alpha(\alpha)}{z_n(\alpha) + 1/z_n(\alpha)}.
\]

Consider the inner products
\[
\langle f, g \rangle_\alpha = \sum_{n=-\infty}^{\infty} \hat{f}(z_n(\alpha)) \hat{g}(z_n(\alpha)) (q^{-1/2} - q^{1/2}) (z_n(\alpha) + 1/z_n(\alpha)) / 2.
\]

(5.12)

We find \( z_{n+1}(\alpha) = q^{-1}z_n(\alpha) \). Therefore
\[
\langle D_q f, g \rangle_\alpha = - \sum_{n=-\infty}^{\infty} \hat{g}(z_n(\alpha)) \left[ \hat{f}(q^{1/2}z_n(\alpha)) - \hat{f}(q^{-1/2}z_n(\alpha)) \right]
\]
\[
= \sum_{n=-\infty}^{\infty} \hat{g}(z_n(\alpha)) \hat{f}(q^{-1/2}z_n(\alpha)) - \sum_{n=-\infty}^{\infty} \hat{g}(z_{n+1}(\alpha)) \hat{f}(q^{1/2}z_{n+1}(\alpha))
\]
\[
= \sum_{n=-\infty}^{\infty} \hat{g}(z_n(\alpha)) \hat{f}(z_n(\alphaq^{1/2})) - \sum_{n=-\infty}^{\infty} \hat{g}(q^{-1/2}z_n(\alphaq^{1/2})) \hat{f}(z_n(\alphaq^{1/2}))
\]
\[
= \sum_{n=-\infty}^{\infty} \hat{f}(z_n(\alphaq^{1/2})) [\hat{g}(q^{1/2}z_n(\alphaq^{1/2})) - \hat{g}(q^{-1/2}z_n(\alpha))] \\
= - \langle f, D_q g \rangle_{\alpha q^{1/2}}
\]
Let
\[ \lambda_n(t) = \frac{-4q^7}{(1-q)^2}t_1^2t_2^2t_3^2 \prod_{1 \leq j < k \leq 3}(1+q^2/t_j t_k). \]

We apply (5.8) and use (2.14), to find that \( \lambda_n \) times the left-side of (5.11) is equal to
\[
\int_{\mathbb{R}} \frac{1}{w(x; t_1, t_2, t_3)} \left( \frac{1}{w(x; t_1, t_2, t_3)} \right) V_m(x; t_1, t_2, t_3) \times D_q \left[ w(x; t_1 q^{-1/2}, t_2 q^{-1/2}, t_3 q^{-1/2}, 0) \right] \right] d\mu_\alpha(x)
\]
\[
eq \sum_{j=-\infty}^{\infty} V_m(x_j(\alpha); t_1, t_2, t_3) \frac{z_j(\alpha) + 1/z_j(\alpha)}{2}
\times D_q \left[ w(x; t_1 q^{-1/2}, t_2 q^{-1/2}, t_3 q^{-1/2}, 0) \right] \right]_{x=x_j(\alpha)}.
\]

Hence,
\[
\lambda_n \int_{\mathbb{R}} V_m(x; t_1, t_2, t_3) V_n(x; t_1, t_2, t_3) \left( -q t_1 z, q t_1 / z, -q t_2 z, q t_2 / z, -q t_3 z, q t_3 / z ; q \right) \infty d\mu_\alpha
\]
\[
= \frac{1}{q^{1/2} - q^{-1/2}} \left\langle D_q \left[ w(x; t_1 q^{-1/2}, t_2 q^{-1/2}, t_3 q^{-1/2}, 0) \right] D_q V_n(x; t_1, t_2, t_3) \right\rangle_{\alpha}
\]
\[
= - \frac{1}{q^{1/2} - q^{-1/2}} \left\langle w(x; t_1 q^{-1/2}, t_2 q^{-1/2}, t_3 q^{-1/2}, 0) D_q V_n(x; t_1, t_2, t_3) \right\rangle_{\alpha} q^{1/2}.
\]

Note that the above formula is symmetric in \( m \) and \( n \) and that \( \lambda_n \) is strictly monotonous in \( n \). Hence the integral (5.11) will vanish if \( m \neq n \). \( \int_{\mathbb{R}} d\mu(x) \) has been computed in Lemma 5.1. To calculate the \( \int_{\mathbb{R}} p_n^2(x) d\mu(x) \), we apply the well-known fact that if a sequence of orthogonal polynomial have the following the recurrence relations
\[ p_{n+1}(x) = [A_n x + B_n]p_n(x) - C_n p_{n-1}(x), \]
then their orthogonality relation is
\[ \int_{\mathbb{R}} p_n(x) p_m(x) d\mu(x) = \frac{\delta_{n,m} A_0}{A_n} C_1 \cdots C_n \int_{\mathbb{R}} d\mu(x). \]

We apply this fact and use the recurrence relation (5.13), to establish (5.11) up to the evaluation of the integral \( \int_{\mathbb{R}} d\mu(x) \). This is the complete proof. \( \square \)

For completeness we record the three term recurrence relation of the polynomials \( \{V_n(x; t|q)\} \) which follows as the limiting case \( t_4 \to 0+ \) of (2.28). The result is
\[ 2x V_n(x; t|q) = a_n V_{n+1}(x; t|q) + b_n V_n(x; t|q) + c_n V_{n-1}(x; t|q), \quad n \geq 0, \]
(5.13)
with
\[
x = \frac{z - \frac{1}{z}}{2}, \quad V_1(x; t|q) = 0, \quad V_0(x; t|q) = 1,
\]
where,
\[
a_n = -\frac{t_2 t_3}{q^{2n+2}} \left( 1 + q^{2n}/t_1 t_2 \right) \left( 1 + q^{2n}/t_2 t_3 \right),
\]
\[
b_n = \frac{t_1 + t_2 + t_3}{q^{n+1}} + \frac{t_1 t_2 t_3}{q^{2n+3}} (1 + q - q^{n+1}),
\]
\[
c_n = -\frac{t_2^2 t_3}{q^{2n+3}} (1 - q^n) (1 + q^{n+1}/t_1 t_3).
\]

The $V_n$'s are not symmetric but the following renormalization is symmetric in the parameters $t_1, t_2, t_3$. The symmetric form is
\[
\hat{V}_n(x, t|q) := \tilde{V}_n(x; t_1, t_2, t_3|q)
\]
\[
= \frac{1}{(-q^2/t_2 t_3; q)_n} {}_3\phi_2 \left( \begin{array}{c} -q^{-n}, -q/t_1 z, q z/t_1 \\ -q^2/t_1 t_3, -q^2/t_1 t_2 \end{array} \right| q, -q^{n+2}/t_2 t_3 \right).
\]

We now wish to study the connection coefficients for our polynomials. One can use the Rodrigues-type formula to derive the desired connection relation by following the proof of the corresponding result for the Askey-Wilson polynomials given in [24], see also [18].

**Theorem 5.3.** We have
\[
V_n(x, s) = \sum_{k=0}^{n} e_{k,n} (s, t) V_k(x, t),
\]
where
\[
t = t_1, \quad t_2, \quad t_3
\]
and
\[
s = s_1, \quad s_2, \quad s_3, \quad s_3 = t_3
\]
with
\[
e_{k,n} (s, t) = \frac{(-q^2/s_1 t_3, q; q)_n (-q^2/t_2 t_3, -q^2/t_1 t_2; q)_k (q; q)_{n-k}}{(-q^2/s_1 s_2; q)_n (-q^2/s_1 t_3, -q^2/s_2 t_3, q; q)_k (q; q)_{n-k}} \frac{t_2}{s_1 s_2}^k
\]
\[
\times {}_3\phi_2 \left( \begin{array}{c} q^{k-n}, -q^{k+2}/t_1 t_3, -q^{k+2}/t_2 t_3 \\ -q^{k+2}/s_1 t_3, -q^{k+2}/s_2 t_3 \end{array} \right| q, \frac{q^{n-k-1} t_1 t_2}{s_1 s_2} \right).
For completeness we record a generating function for the polynomials \( \{V_n\} \). We let \( t_4 \to 0 \) in the generating function (2.17) and conclude that

\[
2\phi_1 \left( \frac{qz/t_1, qz/t_3}{-q^2/t_1t_3} \mid q, -\frac{t}{z} \right) \frac{(-t/z; q)\infty}{(tt_3/q; q)\infty}
= \sum_{n=0}^{\infty} \frac{(-q^2/t_2t_3; q)_n}{(q; q)_n} V_n(x; t|q) \left( \frac{tt_3}{t_1} \right)^n,
\]

(5.21)

Using the symmetry of \( V_n \), in its parameters, see (5.16) we can write additional equivalent generating functions.

We conclude this section with a theorem which allows us to generate many additional weight functions for the polynomials \( \{V_n(x; t)\} \).

The product rule for \( D_q \) is

\[
(D_q f g)(x) = (A_q f)(x)(D_q g)(x) + (A_q g)(x)(D_q f)(x).
\]

(5.22)

As was pointed out in [19] and under the assumptions in Lemma 3.1 the eigenfunctions of

\[
\frac{1}{v(x)} D_q [v(x) D_q y] = \lambda y.
\]

(5.23)

corresponding to distinct eigenvalues are orthogonal with respect to \( v \) on \( \mathbb{R} \), if \( v(x) > 0 \) on \( \mathbb{R} \). The proof also uses (5.22).

**Theorem 5.4.** We assume that \( f(x; t_1, t_2, t_3) \) is a positive function and its moments \( \int_{\mathbb{R}} x^n f(x; t_1, t_2, t_3) dx \) exist for all non-negative integers \( n \). If \( f \) satisfies the following equations

\[
\frac{D_q f(x; q^{-1/2} t)}{f(x; t)} = \frac{2q(2x - t_1/q - t_2/q - t_3/q - t_1t_2t_3/q^3)}{(q - 1) \prod_{1 \leq j < k \leq 3} (1 + t_j t_k/q^2)} A_q f(x; q^{-1/2} t)
= \prod_{1 \leq j < k \leq 3} (1 + t_j t_k/q^2)
\times \left[ 2x^2 + 1 + x(t_1t_2t_3/q^3 - t_1/q - t_2/q - t_3/q) + (t_1t_2 + t_1t_3 + t_2t_3)/q^2 \right].
\]

(5.24)

then \( f \) is weight function for polynomials \( \{V_n(x; t | q)\}_{n=0}^{\infty} \).

The proof follows from Lemma 3.1 and (5.22). The details are identical to the the proof of the corresponding result in [19].
6 Pointwise and Plancherel-Rotach Asymptotics

A theorem of Ismail and Li [20], which is also stated as Theorem 7.2.7 in [18], implies that the largest zero of $V_n$ is $O(q^{-2n})$.

Since

$$
\sum_{k=0}^{n} \frac{(-q/t_1 z, qz/t_1)}{(q, -q^2/t_1 t_3, -q^2/t_1 t_2; q)_k} \left( \frac{q^2}{t_2 t_3} \right)^k
$$

Write $(q^{-n}; q)_k$ as $(-1)^k q^{(k)}(q; q)_n/(q; q)_{n-k}$, then apply Tannery’s theorem (the discrete analogue of the Lebesgue Dominated Convergence Theorem) to get,

$$
\lim_{n \to \infty} \frac{q^n V_n(x; t_1, t_2, t_3 | q)}{t_1^n} = \left( \frac{-q^2/t_1 t_3; q_\infty}{-q^2/t_2 t_3; q_\infty} \right)
$$

The limit is uniform on compact subsets of the complex plane. In particular, it implies that the entire function,

$$
2\phi_2 \left( \frac{-ae^\xi, ae^{-\xi}}{-ab, -ac} \middle| q, -bc \right), \quad a, b, c > 0
$$

has only real zeros.

In order to develop the Plancherel–Rotach asymptotics we set

$$
z = q^{-cn}/s, \quad x = (z - 1/z)/2.
$$

Write the $3\phi_2$ in the definition of $V_n$ in (5.2) as $\sum_{k=0}^{n}$. We shall use the identities

$$
(q^{-n}; q)_n = \frac{(q; q)_n}{(q; q)_{n-k}} (-1)^k q^{(k)-n_k}, \quad (A; q)_k = (q^{1-k}/A; q)_k A^k q^{(k)}.
$$

**Theorem 6.1.** Let $z_n(s) = q^{-cn}/s$, then $x_n(s) = (q^{-cn}/s - q^{cn}/s)/2$.

(a) If $c = 2$, the polynomial $V_n$ has the soft edge limiting behavior

$$
\lim_{n \to \infty} (-st_2 t_3)^n q^{-n^2-n} V_n(x_n(s); t_1, t_2, t_3)
$$

$$
= \frac{(q; q)_\infty}{(q; q)_{n-k}} \left( \sum_{k=0}^{\infty} (-1)^k q^{k^2-2k(s t_1 t_2 t_3)} \right)
$$

(b) If $c > 2$, then $V_n$ has the asymptotic property

$$
\lim_{n \to \infty} q^{(1-n^2-n)} (-st_2 t_3)^n V_n(x_n(s); t_1, t_2, t_3 | q) = \frac{1}{(q; q)_{n-k}}.
$$
(c) If $1 \leq c < 2$, we set $m = \left\lfloor \frac{cn}{2} \right\rfloor$ and $r = \left\{ \frac{cn}{2} \right\} = \frac{cn}{2} - m$. For fixed $r$ and as $n \to \infty$ we have

$$
\lim_{n \to \infty} (-st_2t_3)^m q^{n+c^2n^2/4-2m}m^{-n}V_n(x_n(s); t_1, t_2, t_3)
$$

(6.6)

$$
= \frac{q^2q^{3+2r}}{(q, -q^2/t_1t_2, -q^2/t_2t_3; q)_\infty}.
$$

Proof of (a). First we find that

$$
q^nV_n(x; t_1, t_2, t_3|q) \frac{(-q^2/t_2t_3; q)_n}{t_1^n (-q^2/t_1t_3; q)_n}
$$

(6.7)

$$
= \sum_{k=0}^n \frac{(-q/t_1z, qz/t_1; q)_k}{(q, -q^2/t_1t_3, -q^2/t_1t_2; q)_k} \left( \frac{q^2}{t_2t_3} \right)^k (q^{-n}; q)_k (-q^n)_k
$$

$$
= \sum_{k=0}^n \frac{(-q/t_1z, t_1/zq^k; q)_k}{(q, -q^2/t_1t_3, -q^2/t_1t_2; q)_k} \left( \frac{q^2z}{t_1t_2t_3} \right)^k (q; q)_n (q^2)_k (-1)_k.
$$

We replace $z$ by $z_n = q^{-2n}/s$ and conclude that the right-hand side of the above formula is equal to

$$
q^{-n^2}(q; q)_n \sum_{k=0}^n \frac{(-q^{1+2n}/s/t_1, t_1q^{2n-k}; q)_k}{(q, -q^2/t_1t_3, -q^2/t_1t_2; q)_k} \left( \frac{q^2}{st_1t_2t_3} \right)^k (q; q)_n (q^{n-k})^2 (q; q)_{n-k}.
$$

Replace $k$ by $n - k$ in the above sum to change it to

$$
q^{-n^2}(q; q)_n \sum_{k=0}^n \frac{(-q^{1+2n}/s/t_1, t_1q^{n+k}; q)_{n-k}}{(q, -q^2/t_1t_3, -q^2/t_1t_2; q)_{n-k}} \left( \frac{q^2}{st_1t_2t_3} \right)^{n-k} (q; q)_k.
$$

As $n \to \infty$, we use Tannery’s theorem to establish part (a) of the theorem.

Proof of (b). First we apply (6.7) and replace $z$ by $q^{-cn}/s$. The result is

$$
q^nV_n(x_n(s); t_1, t_2, t_3|q) \frac{(-q^2/t_2t_3; q)_n}{t_1^n (-q^2/t_1t_3; q)_n}
$$

$$
= \sum_{k=0}^n \frac{(-q^{1+cn}/s/t_1, t_1s/q^{1-c}; q)_k}{(q, -q^2/t_1t_3, -q^2/t_1t_2; q)_k} \left( \frac{q^{2-cn}}{st_1t_2t_3} \right)^k (q; q)_n (q^{k^2})(-1)_k
$$

When we interchange $k$ and $n - k$, the above expression becomes

$$
\left( \frac{q^{(1-c)n+2}}{-st_1t_2t_3} \right)^n \sum_{k=0}^n \frac{(-q^{1+cn}/s/t_1, t_1s/q^{1-c}; q)_{n-k}}{(q, -q^2/t_1t_3, -q^2/t_1t_2; q)_{n-k}}
$$

$$
\times \left( -st_1t_2t_3 \right)^k (q; q)_n (q^{k^2-2k+(c-2)nk}),
$$

Let $n \to \infty$, and use Tannery’s theorem, establish (6.5).
Proof of (c). First, we write the sum in (5.2) as \( \sum_{k=0}^m + \sum_{k=m+1}^n \). Next, replace \( k \) by \( m - k \) in the first sum, and replace \( k \) by \( k + m + 1 \) in the second sum. This leads to

\[
q^{n-c^2} V_n(x_n(s); t_1, t_2, t_3|q) (-q^2/t_2t_3; q)_n
= \sum_{k=0}^m \frac{(-q^{1+c} s/t_1, t_1 s/q^{m-k-cn}; q)_{m-k}}{(q, -q^2/t_1 t_3; q)_{m-k}} \frac{(-q^2)}{(q)_{n-k-m}} \frac{(q; q)_n}{(q; q)_{n+k-m}} q^{(k+r)^2}
+ \sum_{k=0}^{n-m-1} \frac{(-q^{1+c} s/t_1, t_1 s/q^{k+m+1-cn}; q)_{k+m+1}}{(q, -q^2/t_1 t_3; q)_{k+m+1}} \frac{(-q^2)}{(q; q)_{n-k-m}} \frac{(q; q)_n}{(q; q)_{n-m-k-1}} q^{(k+1-r)^2}.
\]

For fixed \( r \), then we apply Tannery’s theorem as \( n \to \infty \) and use the Jacobi triple product identity. The result is

\[
\lim_{n \to \infty} (-st_2 t_3)^m q^{n+c^2 n^2/4-2m} t_1^{m-n} V_n(x_n(s); t_1, t_2, t_3)
= \frac{1}{(q, -q^2/t_1 t_2, -q^2/t_2 t_3; q)_\infty} \sum_{k=0}^\infty (-1)^k \left[ q^{(k+r)^2} \left( \frac{q^2}{st_1 t_2 t_3} \right)^{-k} - q^{(k+1-r)^2} \left( \frac{q^2}{st_1 t_2 t_3} \right)^{k+1} \right]
= \frac{1}{(q, -q^2/t_1 t_2, -q^2/t_2 t_3; q)_\infty} \sum_{k=-\infty}^\infty (-1)^k q^{(k+r)^2} \left( \frac{q^2}{st_1 t_2 t_3} \right)^k
= q^{2 \left( \frac{q^{3+2r}}{st_1 t_2 t_3}, \frac{q^{-(1+2r)}}{st_1 t_2 t_3}; q^2 \right)_{\infty}}.
\]

This complete the proof. \( \square \)

In the rest of the section we consider cases where at least one parameter depends on the degree \( n \) and \( n \to \infty \). For any \( \alpha > 0 \), let

\[
(6.8) \quad t_1 \to t_1 q^{-n\alpha}, \quad x_1(n) = \frac{q^{-n\alpha} z - q^{n\alpha} / z}{2}.
\]

Since,

\[
\frac{q^{n^2\alpha+n}}{t_1^n} (-q^2/t_2 t_3; q)_n V_n(x_1(n); t_1 q^{-n\alpha}, t_2, t_3|q)
= 3 \phi_2 \left( \begin{array}{c}
q^{-n}, -q^{2n\alpha}/t_1 z, q z/t_1 \\
-q^{2+n\alpha}/t_1 t_3, -q^{2+n\alpha}/t_1 t_2
\end{array} \right| q, -q^{n+2}_{t_2 t_3},
\]

then,

\[
(6.9) \quad \lim_{n \to \infty} \frac{q^{n^2\alpha+n}}{t_1^n} V_n(x_1(n); t_1 q^{-n\alpha}, t_2, t_3|q) = \frac{(-q^3 z/t_1 t_2 t_3; q)_\infty}{(-q^2/t_2 t_3; q^2)_\infty}.
\]
Theorem 6.2. Let

\[(6.10)\quad \alpha > \beta > 0, \gamma, \delta > 0, \gamma + \delta = \alpha - \beta = 1\]

and

\[(6.11)\quad z, t_1, t_2, t_3 \in \mathbb{C}, \quad z \cdot t_1 \cdot t_2 \cdot t_3 \neq 0.\]

Then there exists a positive number

\[(6.12)\quad 0 < \eta < \min \{1, (\alpha + \beta), (\beta + \delta), (\beta + \gamma)\}\]

such that

\[(6.13)\quad V_n(x_2(n); t_1 q^{-\beta}, t_2 q^{-\gamma}, t_3 q^{-\delta}|q) = \frac{t_n^{n}}{q^{n^2(n+\beta)}} \left( A_q \left( \frac{q^2 z}{t_1 t_2 t_3} \right) + O(q^n) \right)\]

as \(n \to \infty.\)

Proof. Let

\[t_1 \to t_1 q^{-\beta}, \quad t_2 \to t_2 q^{-\gamma}, \quad t_3 \to t_3 q^{-\delta}, \quad x_2(n) = \frac{q^{-\alpha} z - q^{\alpha}/z}{2}.\]

Then,

\[
V_n(x_2(n); t_1 q^{-\beta}, t_2 q^{-\gamma}, t_3 q^{-\delta}|q) = 3 \phi_2 \left( \frac{q^{-n}, -q^{1+n(\alpha+\beta)}/t_1 z, q^{-n(\gamma+\delta)}/t_1}{-q^{2+n(\beta+\gamma)}/t_1 t_3, -q^{2+n(\beta+\gamma)}/t_1 t_2} \right)
\]

\[= \sum_{k=0}^{n} \frac{(-q^2/t_1 t_3)^k (q^{-n}, q^{-1+n}/t_1 z; q)_n}{(q; q)_k} q^{2k^n} \left( \frac{-q^{1+n(\alpha+\beta)}/t_1 z; q}{k} \right)\]

\[= \sum_{k=0}^{n} \frac{q^{k^2} (q; q)_n}{(q; q)_k} \left( \frac{(t_1/z; q)_n}{(t_1/z; q)_n} \frac{(q^{-2} z/t_1 t_2 t_3)^k (-q^{1+n(\alpha+\beta)}/t_1 z; q)_k}{(q; q)_k (q; q)_{n-k}} (t_1/z; q)_{n-k} (q^{2+n(\beta+\gamma)}/t_1 t_3, -q^{2+n(\beta+\gamma)}/t_1 t_2; q)_{n-k} \right)\]

\[= S_1(n) + S_2(n),\]

where

\[S_1(n) = \sum_{k=0}^{\left\lfloor \sqrt{n} \right\rfloor} \frac{q^{k^2} (q; q)_n (t_1/z; q)_n (q^{-2} z/t_1 t_2 t_3)^k (-q^{1+n(\alpha+\beta)}/t_1 z; q)_k}{(q; q)_k (q; q)_{n-k} (t_1/z; q)_{n-k} (-q^{2+n(\beta+\gamma)}/t_1 t_3, -q^{2+n(\beta+\gamma)}/t_1 t_2; q)_{n-k}}\]

and

\[S_2(n) = \sum_{k=\left\lceil \sqrt{n} \right\rceil+1}^{n} \frac{q^{k^2} (q; q)_n (t_1/z; q)_n (q^{-2} z/t_1 t_2 t_3)^k (-q^{1+n(\alpha+\beta)}/t_1 z; q)_k}{(q; q)_k (q; q)_{n-k} (t_1/z; q)_{n-k} (-q^{2+n(\beta+\gamma)}/t_1 t_3, -q^{2+n(\beta+\gamma)}/t_1 t_2; q)_{n-k}}.\]
Since
\[
\frac{(q; q)_n}{(q; q)_{n-k} (t_1/z; q)_{n-k}} = 1 + O \left( q^{n-\sqrt{n}} \right), \quad n \to \infty
\]
and
\[
\frac{(-q^{1+n(\alpha+\beta)/t_1 z}; q)_k}{(-q^{2+n(\beta+\delta)/t_1 t_3}, -q^{2+n(\gamma+\delta)/t_1 t_2})_k} = \frac{(-q^{1+n(\alpha+\beta)/t_1 z}; q)_{\infty}}{(-q^{1+n(\alpha+\beta)+k/t_1 z}; q)_{\infty}} \\
\times \frac{(-q^{2+n(\beta+\delta)/t_1 t_3}; q)_{\infty} (-q^{2+n(\gamma+\delta)/t_1 t_2}; q)_{\infty}}{(-q^{2+n(\beta+\delta)/t_1 t_3}; q)_{\infty} (-q^{2+n(\beta+\gamma)/t_1 t_2}; q)_{\infty}} = 1 + O \left( q^{n-\sqrt{n}} \right), \quad n \to \infty,
\]
then,
\[
S_1(n) = \sum_{k=0}^{\lfloor \sqrt{n} \rfloor} q^{k^2} \frac{(-q^{2 z} z)^k}{(q; q)_k (t_1 t_2 t_3)} + O \left( q^{\eta n} \right)
\]
\[
= A_q \left( \frac{q^{2 z}}{t_1 t_2 t_3} \right) - \sum_{k=\lfloor \sqrt{n} \rfloor + 1}^{n} q^{k^2} \frac{(-q^{2 z} z)^k}{(q; q)_k (t_1 t_2 t_3)} + O \left( q^{\eta n} \right)
\]
\[
= A_q \left( \frac{q^{2 z}}{t_1 t_2 t_3} \right) + O \left( q^{\eta n} \right), \quad n \to \infty,
\]
where
\[
0 < \eta < \min \{1, (\alpha + \beta), (\beta + \delta), (\beta + \gamma)\}.
\]
It is clear that
\[
S_1(n) = O \left( q^{\eta n} \right), \quad \frac{(-q^{2+n(\beta+\delta)/t_1 t_3}; q)_n}{(-q^{2+n/t_2 t_3}; q)_n} = O \left( q^{\eta n} \right), \quad n \to \infty,
\]
and Theorem 6.2 is obtained by combining (6.14), (6.15) and (6.16).

**Theorem 6.3.** For any
\[
0 < \alpha < 1, \quad w \notin \left\{ q^{-\left(2n-1\right)/2} | n \in \mathbb{Z} \right\} \cup \{0\}
\]
we have
\[
\left( \frac{\sqrt{q}}{i} \right)^n V_n \left( \frac{w + w^{-1}}{2}; q^{\frac{1}{2}}, q^{-\alpha}, q^{-\alpha} \big| q \right) = \theta_4 \left( w; q^\frac{1}{2} \right) \left\{ 1 + O \left( q^{\eta n} \alpha \right) \right\},
\]
as \( n \to \infty \), where
\[
\theta_4(w; q) = (q^2, q/w, qw; q^2)_\infty = \sum_{n=-\infty}^\infty (-1)^n q^{n^2} w^n.
\]
Proof. First we observe that

\[
V_n(x; t|q) = \frac{t^n_n}{q^n} (-q^2/t_1 t_3; q)_{n-3} \phi_2 \left( \begin{array}{c}
q^{-n}, -q/t_1 z, qz/t_1 \\
-q^2/t_1 t_3, -q^2/t_1 t_2
\end{array} \right| q, -q^{n+2}/t_2 t_3)
\]

\[
= \frac{t^n_n}{q^n} (-q^2/t_1 t_3; q)_{n-1} \sum_{k=0}^{n} \left( q^{-n}, -q/t_1 z, qz/t_1 \right) \phi_k \left( \begin{array}{c}
q^{n+2}/t_2 t_3
\end{array} \right)
\]

\[
= \frac{t^n_n}{q^n} (-q^2/t_1 t_3; q)_{n-1} \sum_{k=0}^{n} \left( q^2+k^2/2 \right) \phi_k \left( \begin{array}{c}
q^{n+2}/t_2 t_3
\end{array} \right)
\]

\[
	imes \frac{(-q/t_1 z, qz/t_1; q)_\infty}{(-q^{k+1}/t_1 z, q^{k+1}z/t_1; q)_\infty}
\]

\[
= \frac{t^n_n}{q^n} (-q^2/t_1 t_3; q)_{n-1} \sum_{k=0}^{n} \left( q^2+k^2/2 \right) \phi_k \left( \begin{array}{c}
q^{n+2}/t_2 t_3
\end{array} \right)
\]

\[
	imes \frac{(-q/t_1 z, qz/t_1; q)_\infty}{(-q^{k+1}/t_1 z, q^{k+1}z/t_1; q)_\infty}
\]

(6.20) \quad t_2 = t_3 = q^{-n_\alpha}, \quad 0 < \alpha < 1

and

(6.21) \quad z \notin \{-q^n/t_1|n \in \mathbb{N}\} \cup \{t_1q^{-n}|n \in \mathbb{N}\}.

As \( n \to \infty \), since

\[
\sum_{0 \leq k \leq \sqrt{n}} \frac{q^{k^2/2+k^2/2}}{(q; q)_k (q; q)_{n-k}} \left( \frac{q}{t_2 t_3} \right)^k \frac{(-q^{2+k}/t_1 t_3, -q^{2+k}/t_1 t_2; q)_\infty}{(-q^{k+1}/t_1 z, q^{k+1}z/t_1; q)_\infty}
\]

\[
= \sum_{0 \leq k \leq \sqrt{n}} \frac{q^{k^2/2+3k^2/2+2n_\alpha}}{(q; q)_k (q; q)_{n-k}} \left( -q^{2+2k+n_\alpha}/t_1, -q^{2+2k+n_\alpha}/t_1; q)_\infty
\]

\[
= \left( 1 + O \left( q^{n-\sqrt{n}} \right) \right) \sum_{0 \leq k \leq \sqrt{n}} \frac{q^{k^2/2+3k^2/2+2n_\alpha}}{(q; q)_k (q; q)_{n-k}} \left( -q^{k+1}/t_1 z, q^{k+1}z/t_1; q)_\infty
\]

\[
= \left( 1 + O \left( q^{n-\sqrt{n}} \right) \right) \left( 1 + O \left( q^{n_\alpha} \right) \right) = 1 + O \left( q^{n_\alpha} \right),
\]

and

\[
\sum_{\sqrt{n} < k \leq n} \frac{q^{k^2/2+k^2/2}}{(q; q)_k (q; q)_{n-k}} \left( \frac{q}{t_2 t_3} \right)^k \frac{(-q^{2+k}/t_1 t_3, -q^{2+k}/t_1 t_2; q)_\infty}{(-q^{k+1}/t_1 z, q^{k+1}z/t_1; q)_\infty} = O \left( q^{n_\alpha} \right),
\]
then,

\[
(6.22) \quad \sum_{k=0}^{n} \frac{q^{k^2/2+k/2}}{(q; q)_k (q; q)_{n-k}} \left( \frac{q}{t_2 t_3} \right)^k \frac{(-q^{2+k}/t_1 t_3, -q^{2+k}/t_1 t_2; q)_\infty}{(-q^{k+1}/t_1 z, q^{k+1}z/t_1; q)_\infty} = 1 + O(q^{n\alpha}).
\]

This together with

\[
(6.23) \quad \frac{(q, -q^2/t_1 t_3; q)_n}{(-q^2/t_2 t_3; q)_n} = \frac{(-q^{2+n\alpha}/t_1; q)_n}{(-q^{2+n\alpha}/t_2; q)_n} = (q; q)_\infty (1 + O(q^{n\alpha}))
\]

and

\[
(6.24) \quad \frac{1}{(-q^2/t_1 t_3, -q^2/t_1 t_2; q)_\infty} = \frac{1}{(-q^{2+n\alpha}/t_1, -q^{2+n\alpha}/t_1; q)_\infty} = 1 + O(q^{n\alpha})
\]

gives

\[
(6.25) \quad \left( \frac{q}{t_1} \right)^n V_n(x; t|q) = (q, -q/t_1 z, qz/t_1; q)_\infty \{1 + O(q^{n\alpha})\}
\]
as \(n \to \infty\). Theorem 6.3 is proved by taking \(t_1 = i\sqrt{q}\) and \(z = iw\).

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