Is room-temperature superconductivity with phonons possible?

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Abstract. By recognizing the vital importance of two-hole Cooper pairs (CPs) in addition to the usual two-electron ones in a strongly-interacting many-electron system, the concept of CPs was re-examined with striking conclusions: namely, they are gapped and linearly-dispersive resonances with a finite lifetime—but provided the ideal-gas Fermi sea is replaced by a BCS-correlated unperturbed ground-state “sea.” Based on this, Bose-Einstein condensation (BEC) theory has been generalized to include not boson-boson interactions (also neglected in BCS theory) but rather boson-fermion (BF) interaction vertices reminiscent of the Fröhlich electron-phonon interaction in metals. Instead of phonons, the bosons in the generalized BEC (GBEC) theory are now both particle and hole CPs. Unlike BCS theory, the GBEC model is not a mean-field theory restricted to weak-coupling as it can be diagonalized exactly. It reproduces the BCS condensation energy exactly for any coupling, and each kind of CP is responsible for only half the condensation energy. The GBEC theory reduces to all the old known statistical theories as special cases—including the so-called “BCS-Bose crossover” picture which in turn generalizes BCS theory by not assuming that the interelectronic chemical potential equals the Fermi energy. Indeed, a BCS condensate is precisely the weak-coupling limit of a GBE condensate with equal numbers of both types of CPs. With feasible Cooper/BCS model interelectronic interaction parameter values, and even without BF interactions, the GBEC theory yields transition temperatures [including room-temperature superconductivity (RTSC)] substantially higher than the BCS ceiling of around 45K, without relying on non-phonon dynamics involving excitons, plasmons, magnons or otherwise purely-electronic mechanisms. The results are expected to shed light on the experimental search for RTSC.

Keywords: Bose-Einstein condensation; room-temperature superconductivity

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INTRODUCTION

Boson-fermion (BF) models of superconductivity (SC) as a Bose-Einstein condensation (BEC) go back to the mid-1950’s [1]-[4], pre-dating even the BCS-Bogoliubov theory [5]-[7]. Although BCS theory only contemplates the presence of “Cooper correlations” of single-particle states, BF models [1]-[4],[8]-[14] posit the existence of actual bosonic CPs. In spite of the central role played by CPs in both low- and high-\(T_c\) superconductivity, however, they are poorly understood. The fundamental drawback of early [1]-[4] BF models, which took two-electron (2e) bosons as analogous to diatomic molecules in a classical atom-molecule gas mixture, is the notorious absence of an electron energy gap \(\Delta (T)\). “Gapless” models are useful in locating transition temperatures if approached from above, i.e., \(T > T_c\). Even so, we are not aware of any calculations with the early BF models attempting to reproduce any empirical \(T_c\) values. The gap first began to ap-
pear in later BF models [8]-[14]. With two [11][12] exceptions, however, all BF models neglect the effect of hole CPs included on an equal footing with electron CPs to give a GBEC theory consisting of both bosonic CP species coexisting with unpaired electrons in a ternary BF model. Although magnetic-flux-quantization measurements have established the presence of pair charge carriers in both conventional [15]-[16] as well as cuprate [17] superconductors, no experiment has yet been done to our knowledge [18] that distinguishes between electron and hole CPs.

The “ordinary” CP problem [19] for two distinct interfermion interactions (the δ-well [20][21] or the Cooper/BCS model [5][19] interactions) neglects the effect of two-hole (2h) CPs treated on an equal footing with 2e-CPs—as Green’s functions [22], on the other hand, can naturally ensure. However, a crucial confirmed result [12] is that the BCS condensate is a very particular BE condensate with equal numbers of 2e- and 2h-CPs, each contributing to one-half the condensation energy [23]. This was already evident, though not fully appreciated, from the perfect symmetry about ε=µ, the electron chemical potential, of the well-known Bogoliubov \( v^2(\varepsilon) \) and \( u^2(\varepsilon) \) coefficients, where \( \varepsilon = \hbar^2k^2/2m \) is the electron energy and \( m \) its effective mass. The GBEC theory (also appropriately viewed elsewhere as a “complete boson-fermion model,” or CBFM) “unifies” [13] both BCS and ordinary BEC theories as special cases, and predicts substantially higher \( T_c \)’s than BCS theory with the same Cooper/BCS model interaction that mimics electron-phonon dynamics.

**SIGNIFICANCE OF THE COOPER INSTABILITY**

A Bethe-Salpeter (BS) many-body equation (in the ladder approximation) treating both 2e and 2h pairs on an equal footing reveals that, while the CP problem [based on an ideal Fermi gas (IFG) ground state (the usual “Fermi sea”)] does not possess energy solutions with a real part, it does so when the IFG ground state is replaced by the BCS one. This is equivalent to starting from an unperturbed Hamiltonian that is the BCS ground state instead of the pure-kinetic-energy operator corresponding to the IFG. The remaining Hamiltonian terms are then assumed amenable to a perturbation treatment. As a result: i) CPs based not on the IFG-sea but on the BCS ground state survive through a nontrivial solution as “generalized” or “moving” CPs which are positive energy resonances with an imaginary energy term leading to finite-lifetime effects; ii) as in the “ordinary” CP problem, their dispersion relation in leading order in the total (or center-of-mass) momentum (CMM) \( \hbar K = \hbar (k_1 + k_2) \) is also linear (originally reported without proof in Ref. [25], p.33) rather than the quadratic \( \hbar^2K^2=2\Omega m \) of a composite boson (e.g., a deuteron) of mass \( 2m \) moving not in the Fermi sea but in vacuum; and iii) this latter “moving CP” solution, though often confused with it, is physically distinct from another more common trivial solution sometimes called the Anderson-Bogoliubov-Higgs (ABH) [6][7] collective excitation. The ABH mode is also linear in leading order and goes over into the IFG ordinary sound mode in zero coupling. All this occurs in 1D [28], 2D [29] as well as in the 3D study outlined earlier in Ref. [30]. In this section we focus on 2D because of its interest [31] for quasi-2D high-\( T_c \) cuprate superconductors. In general, the results will be crucial for BEC scenarios employing BF models of superconductivity, not only in exactly 2D as
with the Berezinskii-Kosterlitz-Thouless (BKT) \( [32] [33] \) transition, but also down to \((1 + \varepsilon)D\) which characterize the quasi-1D organo-metallic (Bechgaard salt) \([34] [35] [36] [37]\), and most recently multi-walled carbon nanotube \([37]\), SCs.

If \( \mathbf{K} = \mathbf{k} + \mathbf{k}_2 \) is the CMM and \( \mathbf{k} = \frac{1}{2} (\mathbf{k}_1 + \mathbf{k}_2) \) the relative momentum wavevectors of the 2e bound state, and \( \varepsilon_k \) the genuine Cooper instability is really \( (1) \) so that the original CP picture \( (1) \) in 2D \( \lambda \) with the Berezinskii-Kosterlitz-Thouless (BKT) \([32] [33]\) transition, but also down to \((1 + \varepsilon)D\) which characterize the quasi-1D organo-metallic (Bechgaard salt) \([34] [35] [36] [37]\), and most recently multi-walled carbon nanotube \([37]\), SCs.

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The first expression is exact in 2D and a very good approximation otherwise if \( \hbar \omega_D \) \( E_F \) where \( \omega_D \) is the Debye frequency. The sometimes misnamed “Cooper instability” \( (2) \) merely represents a negative-energy, stationary-state (i.e., infinite-lifetime) bound pair. We suggest, however, that unlike the apparent negative-but-real-\( \varepsilon_0 \) “instability” \( (2) \) the genuine Cooper instability is really \( (1) \) so that the original CP picture is meaningless if 2e- and 2h-CPs are treated on an equal footing, as consistency demands, since it leads to a purely-imaginary eigenvalue \( \varepsilon_0 \).

However, a BS treatment of pairs referred not to the IFG sea but to a BCS-correlated ground state “sea” vindicates the CP concept in terms of a new nontrivial solution. This is tantamount to starting not from the IFG unperturbed Hamiltonian but from the BCS one. Its physical justification is reinforced through the recovery of three expected results: a) the (trivial) ABH sound mode solution; b) the BCS \( T = 0 \) gap equation; and c) finite-lifetime effects of a “moving-CP” nontrivial solution in either 2D \([29]\) or 3D \([30]\). Thus, the IFG Green function \( G_0 (\mathbf{k}_1 ; \mathbf{E}_1) \) is replaced by the BCS one, say, \( G_0 (\mathbf{k}_1 ; \mathbf{E}_1) \) that now refers to an energy \( E_1 \) \( E_1 \) \( \xi_k = \xi_k + \Delta^2 \) with \( \xi_k \) \( \hbar^2 k^2 = m_\Delta \) \( E_\Delta \) and \( \Delta \) the \( T = 0 \) fermionic gap. It also contains the Bogoliubov functions \([24]\) \( v_k^2 = \frac{1}{2} \left( 1 + \xi_k = E_k \right) \) and \( u_k^2 = \frac{1}{2} \left( 1 - \xi_k = E_k \right) \). There are then two solutions to the BS equations. A trivial solution is the ABH energy eigenvalue \( \varepsilon_K \), which when Taylor-expanded about \( K = 0 \) gives for small \( \lambda \) in 2D

\[
\varepsilon_K = \frac{\hbar v_F}{2} K + O (K^2) + o (\lambda) ;
\]

where \( o (\lambda) \) denote interference interaction correction terms that vanish as \( \lambda ! 0 \). Note that the leading term is just the ordinary sound mode in an IFG with sound speed \( v_F = \sqrt{\frac{\lambda}{m}} \) in \( d \) dimensions, as determined straightforwardly from standard thermodynamic
formulae. Secondly, there is a nontrivial moving CP solution of the BCS-correlated-sea-based BS treatment, which is entirely new and leads to the pair energy \( \varepsilon_K \) which in 2D is [29]

\[
\varepsilon_K = 2\Delta + \frac{\lambda}{2\pi} \hbar v_F K + \frac{1}{9} \frac{\hbar v_F}{k_D} e^{1 - \lambda} K^2 - i \frac{\lambda}{\pi} \frac{\hbar v_F}{k_D} K^2 + \frac{1}{12} \frac{\hbar v_F}{k_D} e^{1 - \lambda} K^2 + O(K^3): \quad (4)
\]

Here, the upper and lower signs refer to 2e- and 2h-CPs, respectively, and \( k_D \) \( \omega_i=\omega_F \) with \( \omega_D \) the Debye frequency. A linear dispersion in leading order again appears, but now associated with the bosonic moving CP. From [41] the positive-energy \( \gamma_0 \) 2p-CP resonance has an energy width \( \Gamma_K \) and a lifetime \( \tau_K = h=2\Gamma_K = h=2 (\lambda=\pi) \hbar v_F K + (\phi v_F=12k_D)e^{1 - \lambda} K^2 \) that diverges at \( K = 0 \), falling to zero as \( K \) increases. Thus, “faster” moving CPs are shorter-lived and eventually break up, while “non-moving” \( K = 0 \) ones are in infinite-lifetime stationary states.

**GENERALIZED BEC THEORY OF SUPERCONDUCTORS**

The GBEC theory [11, 12] is described in \( d \) dimensions by the Hamiltonian \( H = H_0 + H_{int} \). The unperturbed Hamiltonian \( H_0 \) corresponds to a non-Fermi-liquid “normal” state which is an ideal (i.e., noninteracting) ternary gas mixture of unpaired fermions and both types of CPs, two-electron (2e) and two-hole (2h), namely

\[
H_0 = \sum_{\mathbf{k}_1,\mathbf{p}} \varepsilon_{\mathbf{k}_1} \, a_{\mathbf{k}_1,\mathbf{p}}^{+} a_{\mathbf{k}_1,\mathbf{p}} + \sum_{\mathbf{K}} E_{\mathbf{K}} b_{\mathbf{K}}^{+} b_{\mathbf{K}} + \sum_{\mathbf{K}} E_{\mathbf{K}} c_{\mathbf{K}}^{+} c_{\mathbf{K}} \quad (5)
\]

where as before \( \mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2 \) is the CP CMM wavevector while \( \varepsilon_{\mathbf{k}_1} = h^2 k_1^2 = 2m, \) e.g., are the single-electron, and \( E_{\mathbf{K}} \) the 2e-/2h-CP phenomenological, energies. Here \( a_{\mathbf{k}_1,\mathbf{p}}^{+} (a_{\mathbf{k}_1,\mathbf{p}}) \) creation (annihilation) operators for fermions. Similarly \( b_{\mathbf{K}}^{+} (b_{\mathbf{K}}) \) and \( c_{\mathbf{K}}^{+} (c_{\mathbf{K}}) \) for 2e- and 2h-CP bosons, respectively—although we do not attempt to construct them starting from the fermion operators. Two-hole CPs are postulated to be distinct and kinematically independent from 2e-CPs, all of which provides a ternary BF gas mixture. This postulate is firmly grounded on the experiments [15]-[17] cited before. The operator \( H_0 \) then has diagonal form and its exact eigenstates can be numerated by the sets of occupation numbers \( \varepsilon :: n_{\mathbf{k}_1,\mathbf{p}} :: N_{\mathbf{K}} :: M_{\mathbf{K}} :: \mathcal{J} \). The occupation numbers \( n_{\mathbf{k}_1,\mathbf{p}} \) of one-fermion states each take on only the two values 0 and 1, while those of the one-boson momentum-\( \mathbf{K} \) states of 2p-CPs \( N_{\mathbf{K}} \), and of 2h-CPs \( M_{\mathbf{K}} \), take on all values \( 0; 1; 2; \ldots \infty \).

The exact eigenstates of the Hamiltonian \( H_0 \) are then

\[
n :: n_{\mathbf{k}_1,\mathbf{p}} :: N_{\mathbf{K}} :: M_{\mathbf{K}} :: \mathcal{J} = \prod_{\mathbf{k}_1,\mathbf{p}} (a_{\mathbf{k}_1,\mathbf{p}}^{+})^{n_{\mathbf{k}_1,\mathbf{p}}} \prod_{\mathbf{K}} \frac{1}{N_{\mathbf{K}}!} b_{\mathbf{K}}^{+} N_{\mathbf{K}} \prod_{\mathbf{K}} \frac{1}{M_{\mathbf{K}}!} c_{\mathbf{K}}^{+} M_{\mathbf{K}} \, \mathcal{J} O \quad (6)
\]

where \( \mathcal{J} O \) is the vacuum state for fermions and simultaneously for 2e-CP and 2h-CP creation and annihilation operators. Specifically,

\[
a_{\mathbf{k}_1,\mathbf{p}} \, \mathcal{J} O, \quad b_{\mathbf{K}} \, \mathcal{J} O, \quad c_{\mathbf{K}} \, \mathcal{J} O = 0 : \quad (7)
\]
If $N_{op}$ is the number operator of the total number of electrons, charge conservation implies that

$$N_{op} = \sum_{k_{1}, \pi_{1}} a_{k_{1}, \pi_{1}}^{+} a_{k_{1}, \pi_{1}} + 2 \sum_{K} b_{K}^{+} b_{K} + 2 \sum_{K} c_{K}^{+} c_{K} \quad : : : : \tag{8}$$

If $\mu$ is their chemical potential, the exact eigenvalues of $H_0 + \mu N_{op}$ are

$$E_{n_{K}, \pi_{1}, k_{1}} = \langle E_{+} (0) | 2 \mu | n_{K} \rangle + 2 \mu E_{0} | M_{0} + \sum_{k_{2}, \pi_{2}} (\epsilon_{k_{2}} - \mu) n_{k_{2}, \pi_{2}} \rangle + \sum_{K \neq 0} \langle E_{+} (K) | 2 \mu | n_{K} \rangle + \sum_{k_{2}, \pi_{2}} 2 \mu E_{0} | M_{K} \rangle \quad : : : : \tag{9}$$

The interaction Hamiltonian $H_{int}$ part of $H = H_{0} + H_{int}$ consists of four distinct BF interaction vertices each with two-fermion/one-boson creation or annihilation operators, depicting how unpaired electrons (subindex +) or holes (subindex -) combine to form the 2e- and 2h-CP intrinsic wavefunctions, respectively, in the relative coordinate of the two fermions. The GBEC Hamiltonian $H^{BCS} + H_{int}^{BCS}$ is very different from the well-known BCS Hamiltonian,

$$H^{BCS} + H_{int}^{BCS} = \sum_{k_{1}, \pi_{1}} \epsilon_{k_{1}} a_{k_{1}, \pi_{1}}^{+} a_{k_{1}, \pi_{1}} + \sum_{k_{1}, \pi_{1}} V_{k_{1}, \pi_{1}} a_{k_{1}, \pi_{1}}^{+} a_{k_{1}, \pi_{1}}^{+} a_{k_{1}, \pi_{1}} \quad : : : : \tag{11}$$

with, say, the Cooper/BCS model interaction, with $V > 0$;

$$V_{k_{1}, \pi_{1}} = \begin{cases} V = E_{l}^{d} & \text{if } \mu < \epsilon_{k_{1}} \epsilon_{l_{1}} < \mu + \hbar \omega_{D}, \\
0 & \text{otherwise} \end{cases} \quad : : : : \tag{12}$$

In Refs. [11][12] the energy form factors $f (k)$ for the GBEC theory are taken as

$$f (k) = \begin{cases} f = \frac{1}{2} [E (0) + \delta \epsilon] & \text{if } \frac{1}{2} [E (0) + \delta \epsilon] < \epsilon < \frac{1}{2} [E (0) + \delta \epsilon], \\
0 & \text{otherwise} \end{cases} \quad : : : : \tag{13}$$

in order for it to reduce properly to BCS theory, with [12] in the limit to be explained below, provided one identifies $f$ and $\delta \epsilon$ with $2V \hbar \omega_{D}$ and $\hbar \omega_{D}$, respectively. One then introduces the quantities $E_f$ and $\delta \epsilon$ as new phenomenological dynamical energy parameters (in addition to the positive BF vertex coupling parameter $f$) that replace the previous such $E (0)$, through the definitions

$$E_f = \frac{1}{4} \langle E_{+} (0) + E (0) \rangle \quad \text{and} \quad \delta \epsilon = \frac{1}{2} \langle E_{+} (0) - E (0) \rangle \quad : : : : \tag{14}$$
where $E \ (0)$ are the (empirically unknown) zero-CMM energies of the 2e- and 2h-CPs, respectively. The quantity $E_f$ is available as a possibly convenient energy scale. It is not to be confused with the Fermi energy $E_F = \frac{1}{2}mv_F^2 \ h^2 k_F^2 = 2m \ k_B T_F$ where $T_F$ is the Fermi temperature. If $n = \frac{2 \pi}{\beta n_f}$ is the total number-density of charge-carrier electrons, then $n = k_F^2 = 2 \pi$ in 2D and $= k_F^3 = 3 \pi^2$ in 3D. Thus, the Fermi energy $E_F$ equals $\pi h^2 n f = m \ L^d$ and $\frac{\phi^2}{2m} (\beta \ h^2 n f)^{2d/3}$ in 3D, while $E_f$ equals $\pi h^2 n_f = m \ L^d$ and $\frac{\phi^2}{2m} (\beta \ h^2 n_f)^{2d/3}$ in 3D, i.e., is the same as $E_F$ with $n$ replaced by $n_f$ which will serve as a convenient density scale. The quantities $E_f$ and $E_F$ coincide only when perfect 2e/2h-CP symmetry holds, i.e., when $n = n_f$.

The interaction Hamiltonian (10) can be further simplified by dropping all $K \neq 0$ terms. This is also done in BCS theory but in the full BCS Hamiltonian (11). However, in the GBEC theory these terms are retained in (5) so that $H_0$ describes a non-Fermi-liquid normal state. Following Bogoliubov [39], the boson operators $b^*_{n,0} b_{n,0}$ and $c^*_n c_0$ remaining in (10) are then replaced by the “c-numbers” $N_0 (T)$ and $M_0 (T)$, respectively, where $N_0$ and $M_0$ are the number of $K = 0$ 2e- and 2h-CPs. The full GBEC Hamiltonian $H$ thus becomes bilinear in the fermion operators $a^*_{k,n}$, $a_{k,n}$ and is diagonalizable via a Bogoliubov transformation exactly without assuming weak coupling as in BCS theory. One constructs the grand potential $\Omega$ for the full GBEC Hamiltonian as

$$\Omega (T; L^d; \mu; N_0; M_0) = k_B T \ln \left[ \text{Tr} \ \exp \left\{ -\beta H = \mu N_0 \right\} \right]$$

(15)

where “Tr” stands for “trace” and $1 = k_B T$: Minimizing with respect to $N_0$ (the number of zero-CMM 2e-CPs) and $M_0$ (the same for 2h-CPs), while simultaneously fixing the total number $N$ of electrons via the electron chemical potential $\mu$, determines an equilibrium state of the system with volume $L^d$ and temperature $T$. One thus requires that

$$\frac{\partial \Omega}{\partial N_0} = 0; \quad \frac{\partial \Omega}{\partial M_0} = 0 \quad \text{and} \quad \frac{\partial \Omega}{\partial \mu} = N :$$

(16)

Here $N$ evidently includes both paired and unpaired CP electrons. Some algebra then leads to the three coupled integral Eqs. (7)-(9) of Ref. [11]. The relation between the fermion spectrum $E (\epsilon)$ and fermion energy gap $\Delta (\epsilon)$ turns out to be of the BCS form

$$E (\epsilon) = \frac{\epsilon}{\mu} + \Delta^2 (\epsilon) \quad \text{but where} \quad \Delta (\epsilon) = \frac{\beta}{\mu} \ m_0 f_+ (\epsilon) + \frac{\beta}{m_0} f_0 (\epsilon) :$$

(17)

This last expression for the gap $\Delta (\epsilon)$ implies a simple $T$-dependence rooted in the 2e-CP $n_0 (T) = \frac{\pi}{\beta n_f} L^d$ and 2h-CP $m_0 (T) = \frac{\pi}{\beta n_f} L^d$ number densities of BE-condensed bosons, i.e.,

$$\Delta (T) = \frac{\beta}{n_0 (T) f_+ (\epsilon)} + \frac{\beta}{m_0 (T) f_0 (\epsilon)} :$$

(18)

This generalizes the relation between BCS and BEC order parameters first reported in Ref. [8]. Self-consistent (at worst, numerical) solution of the aforementioned three coupled equations then yields the three thermodynamic variables of the GBEC theory

$$n_0 (T; n; \mu); \quad m_0 (T; n; \mu) \quad \text{and} \quad \mu (T; n) :$$

(19)
Most significantly, the three GBEC theory equations contain the key equations of five different statistical theories as special cases; for a detailed review see Ref. [14]. Figure 1 illustrates this in a flowchart. Perfect 2e/2h CP symmetry signifies equal number of 2e- and 2h-CPs, i.e., \( n_B(T) = m_B(T) \) as well as \( n_0(T) = m_0(T) \). With Eqs. (28) and (29) below imply that \( E_f \) coincides with \( \mu \), and the GBEC theory then reduces to:

i) the gap and number equations of the *BCS-Bose crossover picture* for the BCS model interaction—if the BCS parameters \( V \) and Debye energy \( \hbar \omega_D \) are identified with the BF interaction Hamiltonian \( H_{int} \) parameters \( f^2 = 2\delta \epsilon \) and \( \delta \epsilon \), respectively. The crossover picture for unknowns \( \Delta(T) \) and \( \mu(T) \) is now supplemented by the key expression relating BCS with BEC precisely, namely

\[
\Delta(T) = f^D \frac{n_0(T)}{m_0(T)} = f^D \frac{n_0(T)}{m_0(T)};
\]

(20)

The crossover picture is associated with many authors beginning in 1967 with Friedel and coauthors [40] and then given mayor impetus by Eagles [41] who in turn introduced the BEC mechanism into the picture; for reviews see Refs. [42][43]. However, it is widely unrecognized to be a very modest improvement, at least for the Cooper/BCS model interaction, over BCS theory *per se* since, e.g., an unphysically large \( \lambda \) of about
8 is required to bring $\mu(T_c) = E_F$ in 2D down from 1.00 to 0.998; indeed, $T_c$-values differ very slightly [23] in 2D between the crossover and BCS theories all the way up to $\lambda \approx 5$ when the Fermi surface originally pinned at $\mu$ disappears by becoming negative so that the model interaction breaks down. In fact, room-temperature superconductivity is predicted [23] by BCS theory in 2D for a material with $T_F = 10^3 \, K$, but only for $\lambda$ values somewhat larger than 10; these are still too unphysical as they exceed the Migdal [46] threshold of $\lambda > \frac{1}{2}$ for ionic-lattice instability. If one imposes that $\mu(T_c) = E_F$ exactly, as follows from the number equation for weak BF coupling $f$, the crossover picture is well-known to reduce to:

ii) *ordinary BCS theory* which is characterized by a *single* equation, the gap equation for any $T$. Thus, the BCS condensate is precisely a BE condensate whenever both $n_B(T) = m_B(T)$ and $n_0(T) = m_0(T)$ and the BF coupling $f$ is small. The condensation energies of the GBEC and BCS theories coincide exactly for all coupling.

On the other hand, for no 2h-CPs present the GBEC theory reduces [11] also to:

iii) the *BEC BF model* in 3D of Friedberg and T.D. Lee [9, 10] characterized by the relation $\Delta(T) = f n_0(T)$ first reported in Ref. [8]; but lacking 2h-CPs this model cannot be fully related to BCS theory. When $f = 0$ it reduces to:

iv) the *ideal BF model (IBFM)* of Refs. [44, 45] that predicts nonzero 2e-CP BEC $T_c$’s even in 2D. The “gapless” IBFM cannot describe the superconducting phase. But considered as a model for the *normal state* it should provide feasible $T_c$’s as singularities within a BE scenario that are approached from above $T_c$, and this is indeed [45] the case. Finally, in the limit of no unpaired electrons this model in 3D reduces to:

v) the familiar $T_c$-formula of ordinary BEC in 3D but where the boson number-density is temperature dependent.

The vastly more general GBEC theory has been applied in both 2D and 3D and gives sizeable enhancements in $T_c$’s over BCS theory for moderate departures from perfect 2e/2h-pair symmetry. This is attained for the *same* Cooper/BCS interaction model (coupling strength $\lambda$ and cutoff $\hbar \omega_D$) parameter values often used in conventional SCs. The three coupled equations of the GBEC theory that determine the $d$-dimensional BE-condensate number-densities $n_0(T)$ and $m_0(T)$ of 2e- and 2h-CPs, respectively, as well as the electron chemical potential $\mu(T)$, were first solved numerically [12] around the BCS point of the $T_c$ vs. $n$ phase diagram. At $n=n_f = 1$ one has perfect 2e/2h-CP symmetry; the plain $n_f$ can be seen to be the number density $n_f(T)$ of unpaired but BCS-correlated electrons when $\Delta = 0$ and $T = 0$; whenever $\mu = E_f$. In general

$$
n_f(T) = \sum_{k}^{\infty} d\varepsilon N(\varepsilon) \left[ 1 + \frac{\varepsilon - \mu}{E(\varepsilon)} \tanh \frac{1}{2} \beta E(\varepsilon) \right]_0^{\infty} \tag{21}
$$

with $E(\varepsilon)$ defined in (17). This expression is *precisely* the BCS expression for the electron number density

$$
n = \sum_{k} v_k^2(T) \tag{22}
$$

where $v_k(T)$ is the temperature-dependent Bogoliubov function. Alongside two gap-like equations involving $n_0(T)$ and $m_0(T)$, the third, or “complete” number, equation
explicitly reads
\[ n_f (T') + 2n_0 (T') + 2n_{B+} (T') + 2m_0 (T') + 2m_{B+} (T') = n \tag{23} \]
with \( m_{B+} (T) \), e.g., being precisely the number of “pre-formed” \( K > 0 \) 2h-CPs, and \( n_{B+} (T) \) the same for 2e-CPs. Besides the normal phase consisting of the ideal BF ternary gas described by \( H_0 \), three different stable BEC phases emerge: two pure phases consisting of a 2e-CP BEC and a 2h-CP BEC, as well as a mixed phase consisting of both types of BECs in varying proportions. For a half-and-half mixed phase all the boson number-density terms in (23) cancel out and the BCS number equation \( n_f (T') = n \) is recovered.

We shall focus on the linear dispersion that occurs in leading order in \( K \) for “ordinary” CPs in a Fermi sea as well as for “generalized” CPs in a BCS-correlated state. For the latter, the boson excitation energy \( \varepsilon \) to be used has a leading term in the many-body Bethe-Salpeter (BS) CP dispersion relation given by \( \varepsilon ' (\lambda = 4) h v_F K \) in 3D \[30\], as part of an expansion similar to (4) in 2D. As before, \( \lambda V N (E_F) \) where \( N (E_F) \) is the electron DOS (for one spin) at the Fermi surface. Note that \( \varepsilon \) is no longer the quadratic \( h^2 K^2 = 2 (2m) \) often assumed \[1\]-[4], \[9\]-[12], \[43\] and associated with a composite boson of mass \( 2m \) moving not in the Fermi sea but in vacuum.

CRITICAL TEMPERATURES IN 3D: GBEC WITH \( f = 0 \)

Fully equivalent to a “complete boson-fermion model” (CBFM), the GBEC theory with \( f = 0 \) becomes an “ideal boson-fermion model” (ICBFM), see Fig. 1. The ICBFM is completely described in \( d \) dimensions by the Hamiltonian \( H_0 \) defined by (5). One can construct its associated grand potential as
\[ \Omega_0 (T; L^d; \mu; N_0; M_0) = \frac{k_B T}{Z^d} \ln \left[ \text{Tr} \exp \left\{ \beta (H_0 - \mu N_{op}) \right\} \right] \tag{24} \]
with \( N_0 \) and \( M_0 \) as before the number of zero-CMM 2e- and 2h-CPs, respectively. One gets
\[ \Omega_0 (T; L^d; \mu; N_0; M_0) = \varepsilon_+ (0) + 2\mu N_0 + 2\mu E (0) M_0 \]
\[ 2k_B T B_d L^d \int_0^{Z^d} dK \ln \left[ 1 + \exp \left\{ \beta (\varepsilon - \mu) \right\} \right] \]
\[ + \ k_B T B_d L^d \int_0^{Z^d} dK \ln \left( 1 + \exp \left\{ \beta (\varepsilon - \mu) \right\} \right) \]
\[ 0^+ \]
\[ + \ k_B T B_d L^d \int_0^{Z^d} dK \ln \left( 1 + \exp \left\{ \beta (\varepsilon - \mu) \right\} \right) \]
\[ 0^+ \]
where \( B_d = 1 = \pi, 1 = 2\pi \text{ and } 1 = 2\pi^2 \text{ for } d = 1; 2 \text{ and } 3; \text{ respectively. An equilibrium thermodynamic state makes } \Omega_0 (T; L^d; \mu; N_0; M_0) \text{ stationary with respect to } N_0 \text{ and to} \]
$M_0$ and requires that the number density of electrons $n = \frac{N}{L^d}$ remain constant. Thus, one imposes that
\[
\frac{\partial \Omega_0}{\partial N_0} = 0; \quad \frac{\partial \Omega_0}{\partial M_0} = 0 \quad \text{and} \quad \frac{\partial \Omega_0}{\partial \mu} = N; \quad (26)
\]
This leads to the three relations
\[
[E_+, \langle 0 \rangle] 2\mu = 0; \quad [\bar{\mu} E, \langle 0 \rangle] = 0 \quad \text{and} \quad n_f (T) + 2n_B (T) 2n_B (T) = n \quad (27)
\]
with the latter being the “number equation” that ensures charge conservation in the ternary mixture. In the last relation
\[
n_B (T) n_0 (T) + B_d \int_{0+}^\infty dK K^d 1 \exp \{\beta [E_+, \langle 0 \rangle] 2\mu + \varepsilon_k\} g 1 \quad (28)
\]
while
\[
n_f (T) 2B_d \int_{0+}^\infty dK K^d 1 \exp \{\beta [\bar{\mu} E, \langle 0 \rangle + \varepsilon_k\} g 1 \quad (29)
\]
This last expression can be interpreted as the number density of unpaired electrons at any $T$ and $\varepsilon_k = \hbar^2 k^2 / 2m$ is the electron energy.

Consider the three equations (27) assuming only that $E_+, \langle 0 \rangle < E, \langle 0 \rangle$. In 3D, the third equation of (27) explicitly becomes
\[
n \bar{\varepsilon}_k = 3\pi^2 = \pi^2 \int_{0+}^\infty dK K^d 1 \exp \{\beta \varepsilon_k \mu\} g 1 \quad (30)
\]
This last expression can be interpreted as the number density of unpaired electrons at any $T$ and $\varepsilon_k = \hbar^2 k^2 / 2m$ is the electron energy.

Take first the limiting case $2\mu = E_+, \langle 0 \rangle$ when the first equation of (27) is satisfied but not the second. From (26) this implies that $M_0 (T)$ must vanish for all $T$, but that $N_0 (T) \neq 0$ at least for some $T$ so that a pure 2e-CP BEC phase may occur below a critical temperature $T_c$ (possibly zero) determined by $n_0 (T_c, n) = 0$. After substituting (14) in (31) with $\mu = E_+, \langle 0 \rangle = 2$, thus eliminating $E_+, \langle 0 \rangle$ in favor of $E_f \bar{\varepsilon} = 2 \hbar^2 / 2m (3\pi^2 n_f)^2$ and $\delta \varepsilon = \hbar \omega_D$, we obtain the dimensionless “working number equation” for the pure 2e-CP
BEC phase critical temperature

\[
1=3 = \int_{0}^{\infty} \tilde{k}^2 d\tilde{k} \left[ \exp \left( \tilde{k}^2 \right) \tilde{n}^{2=3} \nu=2 \right] \tilde{T}_c g + 1 \right] + \int_{0}^{\infty} \tilde{K}^2 d\tilde{K} \left[ \exp \left( \lambda \tilde{K} + 4\nu = 2\tilde{T}_c \right) \right] (32)
\]

where \( \tilde{k} = k_{\parallel}, \tilde{K} = K_{\parallel}, \tilde{n} = n_{\parallel}, \tilde{T}_c \), \( \tilde{c} = c, \) and \( \tilde{m} \) from (32) follows and the critical temperature for this phase is now determined by

\[
1 \leq \frac{\hbar}{\lambda} \nu = 2 \tilde{T}_c \quad (\text{for } n=2) \quad (35)
\]

This can now be solved for \( \tilde{T}_c \) as a function of \( \tilde{n} \) which is plotted as the dashed curve in Fig. 2 for \( \lambda = \frac{1}{4} \) and \( \nu = 0.005 \).

The pure \( 2h \)-CP BEC phase comes from the limiting case \( 2\mu = E \quad (0) \) as then the second, but not the first, relation in (27) is satisfied, so that, again from (26), \( N_0 (T') = 0 \) for all \( T \) but \( M_0 (T') \neq 0 \) for some \( T \). A working number equation similar to but different from (32) follows and the critical temperature for this phase is now determined by \( m_0 (T_c;n) = 0 \). It eventually gives

\[
1=3 = \frac{\hbar}{\lambda} \nu = 2 \tilde{T}_c \quad (\text{for } n=2) \quad (35)
\]

No bounded solution of this equation for \( \tilde{T}_c \) when \( \lambda = 1=2 \) and \( \hbar \omega_D = E_F = 0 \) 0.005 was found. The BCS value from their formula \( T_c = T_F \quad \Gamma = 34 \nu \omega_D = E_F \exp (1=\tilde{\lambda}) \) is \( \lambda = 0.0008 \) (large black dot in Fig. 2); it lies within the range \( T_c = T_F \quad \Gamma = 3 \) 10 0.03 empirically found for conventional, elemental SCs. Empirical data [49] for “exotic” SCs, however, fall within the range \( T_c = T_F \quad \Gamma = 0.01 \quad 0.05 \). Thus, moderate departures from perfect 2e/2h-CP symmetry enable the ICBFM to access, unlike BCS theory, empirical \( T_c \) values for exotic SCs without abandoning electron-phonon dynamics.
FIGURE 2. Temperature $T$ (in units of $T_F$) vs. electron density $n$ (in units of $n_f$ as defined in text) phase diagram in 3D showing the critical temperature $T_c$ phase boundary for the pure $2e$-CP BEC phase (dashed curve) for $\lambda = 1=2$ with $\hbar\omega_D=E_F = 0.005$. The BCS $T_c= T_F' = 1.134 (\hbar\omega_D=E_F) \exp (-1=\lambda)$ gives $0.0008$ for these values of $\lambda$ and $\hbar\omega_D=E_F$; it is marked by a large dot. There is no pure $2h$-CP BEC phase solution for $n=n_f > 1$, nor a mixed phase with both types of BECs for any $n=n_f > 0$, as reported in Ref. [12] for $f \neq 0$ but for a quadratic CP dispersion. RTSC refers to room-temperature superconductivity in a material with $T_F = 10^3 K$.

Finally, for intermediate values of $\mu$, namely for $E_0 < 2\mu < E_+ (\mu)$; neither the first nor second equations of (26) are satisfied so that $n_0 (T) = 0 = m_0 (T)$ for all $T$; this implies no condensed phases whatsoever. Thus, the ICBFM (characterized by zero BF coupling $f$) contains no mixed phase in contrast to the CBFM where $f \neq 0$ (Ref. [12] for a quadratic dispersion). This case will be treated elsewhere with the correct linear dispersion.

CONCLUSIONS

Cooper pairs (CPs) are meaningless if referred to the ideal Fermi gas “sea” when hole pairs are included along with electron pairs, but survive as positive-energy, finite-
lifetime, plasmon-like objects with a linear (instead of quadratic) rise in total, or center-of-mass, momentum $K$ when referred to a BCS-correlated sea instead.

The new generalized BEC (GBEC) theory includes as limiting cases the following theories: i) BCS and ii) BCS-Bose "crossover," when the BE condensate consists of equal numbers of electron- and hole-pairs. It also contains: iii) the Friedberg-T.D. Lee BEC model, iv) the "ideal boson-fermion model" (IBFM), and v) ordinary BEC theory when there are no unpaired fermions. The BCS condensate is precisely a BE condensate of equal numbers of 2e/2h-pairs and weak coupling. Without abandoning electron-phonon dynamics the GBEC theory leads to 2-to-3 order-of-magnitude higher $T_c$'s—including room-temperature superconductivity. All this rests on four essential ingredients: 1) 2h-CPs cannot and must not be neglected in a fully self-consistent treatment in any many-fermion system, otherwise a spurious value of $T_c$ may result that corresponds not to a stable but rather to a metastable state; 2) CPs are bosons, even though BCS pairs not; 3) CPs are linearly-dispersive for small $K$; 4) to achieve higher $T_c$'s one must depart from the perfect 2e/-2h-CP symmetry of the BCS condensate which in fact is a BE condensate. Neglected in the GBEC theory thus far, however, are: a) $K > 0$ terms in the boson-fermion vertex interactions; b) boson-boson interactions (as also in BCS theory); c) a $T > 0$ Bethe-Salpeter CP treatment; d) different hole and electron effective masses; and e) ionic-lattice crystallinity effects which might initially be introduced via Van Hove singularities in the electronic DOS or via “bipolarons” instead of CPs.

Finally, at least two mysteries have surfaced here. In spite of neglecting boson-boson interactions between severely-overlapping CPs, why has the BCS theory been so successful in describing at least conventional SCs? Why are simple models (such as the BCS or the GBEC theories) quite able to be of any relevance whatsoever in such complex strongly-interacting many-electron systems like SCs?

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