ON DIFFERENTIAL GRADED EILENBERG MOORE CONSTRUCTION

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INTRODUCTION

Monads are ubiquitous nowadays. In tensor triangulated geometry, for example, they were used by Balmer to characterize separated étale morphism of quasi-compact and quasi-separated schemes [Bal14]. Kleisli [Kle65] and Eilenberg and Moore [EM65] proved that any monad is a composition of an adjoint pair of functors. In one of his papers, Balmer [Bal11] asked the question when can a monad on a triangulated categories be written as a composition of an adjoint pair of exact functors. This question is difficult to answer while staying in the world of tensor triangulated categories. This paper gives a partial answer to this question when the triangulated category has a suitable enhancement (see theorem 2.30).

In the process we do the Eilenberg Moore construction over DG categories. The naive generalization of the definition of monad is too restrictive for applications. That was the motivation for defining weak monads (definition 2.21). This led to a reinterpretation of a construction done by Sosna [Sos12] and Elagin [Ela14] in terms of monads section 3.1.

One knows that Bousfield localization functors are monads by definition. We show that weak Bousfield localization functors correspond to Drinfeld quotients in lemma 3.5.

This paper is organized as follows. The first section recalls the Eilenberg Moore construction and reinterprets everything in the DG setting. We also introduce two strict 2 functors: \( H^0 \) and \( \underline{\text{pretr}} \). In the second section we discuss monads in enhancements and prove the main properties. Some applications and observations pertaining to the construction are given in section 3. In particular, interpretation of Sosna and Elagin’s construction, and an interpretation of Bousfield-type Drinfeld localization are discussed. Further, we indicate some applications to derived categories.

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1. EILENBERG MOORE CONSTRUCTION

1.1. Enriched monads. Most of the following is well known. The references are [Kel82], [EK66], [Koc70], [Koc71].
Let $\mathcal{V}$ be a symmetric monoidal category, consisting of a functor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, and an object $1_\mathcal{V}$ which acts as an identity for $\otimes$ satisfying certain compatibility conditions as in [Lev98, Part II, Chapter 1, page 375].

**Definition 1.1.** A $\mathcal{V}$ category $\mathcal{C}$ consists of the following data:

1. a collection of objects of $\mathcal{C}$;
2. for each pair of objects $A$ and $B$ in $\mathcal{C}$, an object $\text{Hom}_\mathcal{C}(A, B)$ of $\mathcal{V}$;
3. for each triple $A$, $B$ and $C$ of objects in $\mathcal{C}$, a “composition” morphism
   \[ \circ_{A, B, C} : \text{Hom}_\mathcal{C}(B, C) \otimes \text{Hom}_\mathcal{C}(A, B) \to \text{Hom}_\mathcal{C}(A, C) \]
   in $\mathcal{V}$; and
4. for each object $A$ in $\mathcal{C}$ a morphism $\text{id}_A : 1_\mathcal{C} \to \text{Hom}_\mathcal{C}(A, A)$

which satisfy axioms corresponding to associativity and $\text{id}_A$ being associated to the identity morphism.

A $\mathcal{V}$-functor $F : \mathcal{A} \to \mathcal{B}$ between two $\mathcal{V}$ categories is a map of objects together with morphisms $F(A, B) : \text{Hom}_\mathcal{A}(A, B) \to \text{Hom}_\mathcal{B}(F(A), F(B))$ which is compatible with composition and identity, for any pair of objects $A$ and $B$ in $\mathcal{A}$.

A $\mathcal{V}$-natural transformation $\theta : F \to G$ between two $\mathcal{V}$-functors $F, G : \mathcal{A} \to \mathcal{B}$ is a collection of maps $\theta(A) : 1_\mathcal{B} \to \text{Hom}_\mathcal{B}(F(A), G(A))$ for each object $A$ in $\mathcal{A}$ which is compatible with morphisms. To natural transformations $\rho$ and $\theta$ can be composed to give:

\[
\rho \circ \theta(A) : 1_\mathcal{B} \to 1_\mathcal{B} \otimes 1_\mathcal{B} \xrightarrow{\rho \otimes \theta} \text{Hom}_\mathcal{B}(G(A), H(A)) \otimes \text{Hom}_\mathcal{B}(F(A), G(A)) \xrightarrow{\circ_{F(A), G(A), H(A)}} \text{Hom}_\mathcal{B}(F(A), H(A)).
\]

**Definition 1.2.** Suppose $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ be two $\mathcal{V}$-functors between $\mathcal{V}$-categories. $F$ is said to be a left adjoint for $G$ if there is a $\mathcal{V}$-natural isomorphism

\[
a(A, B) : \text{Hom}_\mathcal{B}(FA, B) \to \text{Hom}_\mathcal{A}(A, GB)
\]

between functors from $\mathcal{A}^{op} \otimes \mathcal{V} \mathcal{B} \to \mathcal{V}$. For further details one can refer to [Kel05].

**Definition 1.3.** A monad $(M, \mu, \eta)$ on a $\mathcal{V}$-category $\mathcal{C}$ consists of an $\mathcal{V}$-endofunctor $M : \mathcal{C} \to \mathcal{C}$ and two natural transformations $\mu : M \circ M \to M$ and $\eta : 1_\mathcal{C} \to M$ such that the following two diagrams commute:

The following lemma is well known.

**Lemma 1.4.** Let $\mathcal{V}$ be a symmetric monoidal additive category. Let $\mathcal{C}$ be a $\mathcal{V}$-category. Then for every $\mathcal{V}$ monad, $M$, there exists a category $M$-mod, and two functors $F_M : \mathcal{C} \to M$-mod and $G_M : M$-mod $\to \mathcal{C}$, such that $M = G_M \circ F_M$. $G_M$ is right adjoint to $F_M$ and any other category $\mathcal{D}$ admitting functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{D}$, such that $G \circ F = M$, there is a unique functor $\Phi : \mathcal{D} \to M$-mod
such that the following diagram commutes.

\[
\begin{array}{ccc}
C & \xrightarrow{a} & \Phi \\
\downarrow & & \downarrow \\
D & \xleftarrow{\Phi} & M \text{-mod}
\end{array}
\]

\[
\begin{array}{ccc}
G & \xrightarrow{\alpha} & F \\
\downarrow & & \downarrow \\
F & \xleftarrow{\phi} & G_{M}
\end{array}
\]

\[
\begin{array}{ccc}
M_{\text{mod}} & \xrightarrow{\alpha^M} & M \\
\downarrow & & \downarrow \\
M & \xleftarrow{\phi} & \Phi
\end{array}
\]

**Proof.** The proof goes parallel to the Eilenberg Moore construction [EM65]. See for example, [Lin69]. □

1.2. DG Monads. Now we restrict to the case when the enriching category \( \mathcal{V} \) is the category of DG modules.

**Definition 1.5.** Let \( \mathcal{D} \) be a DG category. A DG-monad \((\mathcal{M}, \mu, \nu)\) is monad in a dg-module enhanced category. More precisely, \( \mathcal{M} : \mathcal{D} \to \mathcal{D} \) is a DG functor and \( \mu : \mathcal{M} \circ \mathcal{M} \to \mathcal{M} \) and \( \nu : \text{id} \to \mathcal{M} \) are DG natural transformations such that the following diagrams commute

\[
\begin{array}{ccc}
\mathcal{M}^3 & \xrightarrow{\mathcal{M}^2 \mu} & \mathcal{M}^2 \\
\downarrow & & \downarrow \\
\mathcal{M}^2 & \xrightarrow{\mu} & \mathcal{M}
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{M} \xrightarrow{\nu} \mathcal{M}^2 & \xrightarrow{\mathcal{M} \nu} & \mathcal{M} \\
\downarrow & & \downarrow \\
\mathcal{M} & \xrightarrow{\mu} & \mathcal{M}
\end{array}
\]

Mimicking the Eilenberg Moore construction, let us make the following definitions.

**Definition 1.6.** Let \( \mathcal{M} \) be a DG monad on a dg category \( \mathcal{C} \). \( \mathcal{M} \text{-mod} \) is the category whose objects are pairs \((x, \lambda)\) where \( x \) is an object of \( \mathcal{C} \) and \( \lambda : \mathcal{M} x \to x \) is a degree zero morphism satisfying the following commutative diagrams:

\[
\begin{array}{ccc}
\mathcal{M}^2 x & \xrightarrow{\mathcal{M} \lambda} & \mathcal{M} x \\
\downarrow & & \downarrow \\
\mathcal{M} x & \xrightarrow{\lambda} & x
\end{array}
\]

\[
\begin{array}{ccc}
x & \xrightarrow{\nu} & \mathcal{M} x \\
\downarrow & & \downarrow \\
x & \xrightarrow{\lambda} & x
\end{array}
\]

and \( d\lambda = 0 \). A morphism between \((x, \lambda)\) and \((x', \lambda')\) is a morphism \( \varphi : x \to x' \) such that the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{M} x & \xrightarrow{\mathcal{M} \varphi} & \mathcal{M} x' \\
\downarrow & & \downarrow \\
x & \xrightarrow{\varphi} & x'
\end{array}
\]

The enhanced Eilenberg Moore construction gives a DG category \( \mathcal{M} \text{-mod} \) automatically gets a DG structure.

Given a DG category \( \mathcal{C} \), one can define an additive category \( H^0(\mathcal{C}) \), whose objects are the same as those of \( \mathcal{C} \), and

\[
\text{Hom}_{H^0(\mathcal{C})}(A, B) := H^0(\text{Hom}_\mathcal{C}(A, B)).
\]
In the following we study the strict 2 categories corresponding to categories of various categories. We refer to MacLane [ML98].

**Proposition 1.7.** $H^0$ induces a strict 2-functor from the category of DG categories to the category of pre-additive categories.

**Proof.** Consider the 2-functor $\mathcal{H}$ is defined as follows.

**On objects:** Let $\mathcal{D}$ be an object of $\mathcal{DGCat}$. $\mathcal{H}(\mathcal{D})$ is defined to be the preadditive category whose objects are the same as $\text{Ob}(\mathcal{D})$ and for objects $D, E \in \text{Ob}(\mathcal{D}) = \text{Ob}(\mathcal{D})$, $\text{Hom}_{\mathcal{H}(\mathcal{D})}(D, E) = H^0(\text{Hom}_{\mathcal{D}}(D, E))$. It is well known that this forms a pre-additive category.

**On 1-morphisms:** In $\mathcal{DGCat}$, the 1-morphisms are DG-functors. Given a DG-functor $F : \mathcal{D} \to \mathcal{E}$, we construct a functor $\mathcal{H}(F) : \mathcal{H}(\mathcal{D}) \to \mathcal{H}(\mathcal{E})$.

For any $D \in \text{Ob}(\mathcal{D}) = \text{Ob}(\mathcal{H}(\mathcal{D}))$, $\mathcal{H}(F)(D) = F(D)$. Suppose $f \in \text{Hom}_{\mathcal{H}(\mathcal{D})}(D, D')$ for $D, D' \in \text{Ob}(\mathcal{H}(\mathcal{D}))$. Since 

$$\text{Hom}_{\mathcal{H}(\mathcal{D})}(D, D') = H^0(\text{Hom}_{\mathcal{D}}(D, D')),$$

choose $\tilde{f} \in \text{Hom}_{\mathcal{D}}(D, D')$ such that $d\tilde{f} = 0$ and the image in $H^0(\text{Hom}_{\mathcal{D}}(D, D'))$ is $f$. One can define 

$$\mathcal{H}(F)(f) \in \text{Hom}_{\mathcal{H}(\mathcal{D})}(\mathcal{H}(F)(D), \mathcal{H}(F)(D')) = \text{Hom}_{\mathcal{H}(\mathcal{D})}(F(D), F(D')) = H^0(\text{Hom}_{\mathcal{E}}(F(D), F(D'))),$$

to be the class of $F(\tilde{f})$ in $H^0(\text{Hom}_{\mathcal{E}}(F(D), F(D')))$. Note that if $\tilde{f}$ and $\tilde{f}'$ both represent $f$, then $\tilde{f}' - \tilde{f} = dg$ for some $g$ and hence $F(\tilde{f}') - F(\tilde{f}) = F(dg) = dF(g)$. This proves that $\mathcal{H}(F)$ is well defined.

Now it is easy to check that $\mathcal{H}(\text{id}_D) = \text{id}_{\mathcal{H}(D)}$ and $\mathcal{H}(G \circ F) = \mathcal{H}(G) \circ \mathcal{H}(F)$.

**On 2-morphisms:** Two morphisms in a 2-category is a natural transformation. Suppose $\nu : F \to G$ be a natural transformation in $\mathcal{DGCat}$. Because of the assumption that $d\nu_D = 0$ for all objects $D$ in $\mathcal{DGCat}$, it makes sense to define $\mathcal{H}(\nu)_D := \text{class of } \nu_D : F(D) \to G(D)$ in $H^0(\text{Hom}_{\mathcal{E}}(F(D), G(D)))$.

It is routine to check that this construction behaves well with both horizontal and vertical compositions of natural transformations.

$\square$

**Notation 1.8.** By abuse of notation, we shall denote the two functor $\mathcal{H}$ by $H^0$.

Given a DG category $\mathcal{A}$, one can associate the category of twisted complexes [Dri04] (and also [BK90]), denoted by $\mathcal{A}^{pre}$, which is a pre-triangulated DG category.

**Proposition 1.9.** The construction $\mathcal{A} \mapsto \mathcal{A}^{pre}$ induces a strict 2 functor from $\mathcal{DGCat}$ to itself.

**Proof.** We define a two functor $\mathcal{P}$ by defining it on objects, functors and natural transformations.

**On objects:** $\mathcal{P}(\mathcal{A}) = \mathcal{A}^{pre}$

**On functors:** It is clear from the construction, using twisted complexes that a functor $F : \mathcal{A} \to \mathcal{B}$ lifts to a functor $\mathcal{P}(F) := F^{pre} : \mathcal{A}^{pre} \to \mathcal{B}^{pre}$.

**On natural transformations:** Let $F$ and $G$ be two functors from $\mathcal{A}$ to $\mathcal{B}$ and $\nu : F \to G$ be a natural transformation between them. Recall that $\mathcal{A}^{pre}$ consists of all twisted complexes: a formal finite collection of objects,
indexed by integers, and morphisms, indexed by pairs of integers, satisfying some constraints. Morphisms are \( \mathbb{Z} \times \mathbb{Z} \) (all but finitely many of the entries being 0) matrices of morphisms. \( F^{\text{pretr}} \) is constructing by applying \( F \) to each of the objects and morphisms; the constraints are preserved by the functoriality of \( F \). Thus one can define \( \Psi(\nu) : F^{\text{pretr}} \rightarrow G^{\text{pretr}} \) as follows.

For a twisted complex \( A = ((A_i)_{i \in \mathbb{Z}}, q_{ij}) \) in \( P(A) = A^{\text{pretr}} \) define \( P(\nu)(A) : P(F)(A) \rightarrow P(G)(A) \) as the diagonal matrix with \( \Psi(\nu)(A)_{ii} = \nu(A_i) \). It is easy to check that this defines a natural transformation.

\[ \square \]

\textbf{Remark 1.10.} Note a strict 2-functor between categories preserve equivalences. They take a pair of adjoint functors to a pair of adjoint functors. Also they take monads to monads. In later sections we use these fact for \( H^0 \) and \( (\ast)^{\text{pretr}} \).

\section{2. On a question of Balmer}

In this section, we give a recipe to construct triangulated subcategories of a given triangulated category \( \mathcal{C} \) which admits enhancements. The most common example of such categories are derived categories. The motivation for such a construction is a remark of Paul Balmer \cite[Remark 2.9]{Bal11}, which asks the following question. Suppose \( M : \mathcal{C} \rightarrow \mathcal{C} \) be a monad and that \( \mathcal{C} \) is pre-triangulated. Then, by Eilenberg and Moore \cite{EM65}, we can find an additive category \( \mathcal{D} \) such that \( M \) can be realized as a composition \( M = G \circ F \) of two adjoint functors \( F : \mathcal{C} \rightleftarrows \mathcal{D} : G \). Balmer asked when we can choose \( \mathcal{D} \) to be (pre-)triangulated such that \( F \) and \( G \) become exact.

Balmer answered this question under the assumptions that \( M \) is a stably separable monad. In this section, we explore some other ways of finding such adjoint pairs. More specifically, we shall give a construction of such an adjoint pair, when the triangulated category has an enhancement. We recall the definition of enhancements. For details we refer to \cite{LO10}.

We begin by observing that in certain cases it is enough to consider idempotent complete (Karoubian) triangulated categories.

\textbf{Lemma 2.1.} Given an exact functor \( F : \mathcal{C} \rightarrow \mathcal{D} \) between triangulated categories, one can extend the functor (in a canonical way) to \( F^\# : \mathcal{C}^\# \rightarrow \mathcal{D}^\# \).

\textbf{Proof.} This follows from the universal property of idempotent completion: Consider the composite functor \( \mathcal{C} \rightarrow \mathcal{D} \rightarrow \mathcal{D}^\# \). Being a functor to an idempotent complete triangulated category, this should factor via some unique \( F^\# : \mathcal{C}^\# \rightarrow \mathcal{D}^\# \). \cite{BS01}.

\textbf{Remark 2.2.} Note that, objects of \( \mathcal{C}^\# \) are of the form \((x, \pi)\) where \( \pi \) is an idempotent element of \( \text{Hom}_{\mathcal{C}}(x, x) \), and the morphisms are those morphisms which commute with the idempotents. \( F^\# \) described above satisfies \( F^\#(x, \pi) = (F(x), F(\pi)) \).

\textbf{Lemma 2.3.} Given an exact monad \( (M, \mu, \eta) \) on a triangulated category \( \mathcal{C} \), there is a monad \( M^\# \) on \( \mathcal{C}^\# \) which canonically extends \( M \).

\textbf{Proof.} Consider the extension \( M^\# \) as in lemma 2.1. One can define \( \mu^\# \) and \( \eta^\# \) object-wise as follows. For \((x, \pi) \) in \( \mathcal{C}^\# \), \( \mu^\#_{(x, \pi)} = \mu_x : (M^2 x, M^2 \pi) \rightarrow (M x, M \pi) \); and \( \eta^\#_{(x, \pi)} = \eta(x) \). It is easy to check that \((M^\#, \mu^\#, \eta^\#) \) is a monad on \( \mathcal{C}^\# \). \( \square \)
Proposition 2.4. Suppose $\mathcal{C}$ be a triangulated category and $\iota : \mathcal{C} \to \mathcal{C}^#$ be its idempotent completion. Suppose $F : \mathcal{C}^# \to \mathcal{D}' : G$ be an exact adjunction where $\mathcal{D}'$ is triangulated. Let $\mathcal{D}$ be the full subcategory of $\mathcal{D}'$ consisting of all objects $d$ such that $Gd$ is in the image of $\iota$. Then the following holds

1. $\mathcal{D}$ is triangulated,
2. $F$ and $G$ induce an adjoint pair of exact functors $F' : \mathcal{C} \rightleftarrows \mathcal{D} : G'$.
3. Let $(M, \mu, \eta)$ be a monad on $\mathcal{C}$ and $(M^#, \mu^#, \eta^#)$ be the extension to $\mathcal{C}^#$. Then if $\mathcal{D}' = M^# \cdot \text{mod}$, $F = F_M$ and $G = G_M$, then $\mathcal{D} = M \cdot \text{mod}$.

Proof. An full subcategory of a triangulated subcategory is triangulated if and only if it is closed under taking shifts and cones.

1. Let $d \in \mathcal{D}$. Then $d[n]$ is an object of $\mathcal{D}'$. Now $G(d[n]) = G(d)[n]$ is an object of the triangulated category $\mathcal{C}$. Thus, $d[n]$ belongs to $\mathcal{D}$. Now let $f : d \to d'$ be a morphism in $\mathcal{D}$. $\text{Cone}(f)$ is an object in $\mathcal{D}'$. Since $G$ is exact, and $\mathcal{C}$ is triangulated, we have the following diagram of distinguished triangles in $\mathcal{C}$:

$$
\begin{array}{ccc}
G(d) & \xrightarrow{G(f)} & G(d') \\
\downarrow & & \downarrow \\
G(d) & \xrightarrow{G(f)} & \text{Cone}(G(f))
\end{array}
$$

and the dotted arrow is an isomorphism. Thus $G(\text{Cone}(f))$ belongs to $\mathcal{C}$ and hence $\text{Cone}(f)$ belongs to $\mathcal{D}$. Thus $\mathcal{D}$ is triangulated.

2. Consider the diagram of functors

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\iota} & \mathcal{C}^# \\
\downarrow G & & \downarrow F \\
\mathcal{D} & \xrightarrow{\iota} & \mathcal{D}'
\end{array}
$$

Define $F' = F \circ \iota$, and note $G$ restricts to a functor $G' : \mathcal{D} \to \mathcal{C}$, by definition of $\mathcal{D}$. Using the fact that $(F, G)$ is an adjunction, it is easy to check that $(F', G')$ is an adjunction. Since $F$, $G$ and $\iota$ are exact, so are $F'$ and $G'$.

3. Note that the canonical functor $M \cdot \text{mod} \to M^# \cdot \text{mod}$ is fully faithful. We have to show that all objects $(x, \lambda)$ of $M^# \cdot \text{mod}$ such that $G_M((x, \lambda)) = x$ belongs to $\mathcal{C}$, are actually come from some object in $M \cdot \text{mod}$. But that is clear as in that case $\lambda$ will be a morphism in $\mathcal{C}$.

Corollary 2.5. If $\mathcal{C}$ is a triangulated category and if $M$ is a stably separable monad, then $M \cdot \text{mod}$ is triangulated such that $F_M$ and $G_M$ are exact.

Proof. Recall that a monad $M$ is separable if $\mu : M^2 \to M$ admits a section $\sigma : M \to M^2$ such that $M\mu \circ \sigma M = \sigma \circ \mu = \mu M \circ M \sigma$. One can extend $\sigma$ (objectwise) to $\sigma^#$ as above and one can see easily that it is a natural transformation from $M^#$ to $M^{#2}$ and satisfies the above relations. Thus if $M$ is separable, so is $M^#$.

Now by [Bal11, Corollary 4.3], $M^# \cdot \text{mod}$ is triangulated so that $F_{M^#}$ and $G_{M^#}$ are exact. Thus by the above proposition so is $M \cdot \text{mod}$; and $F_M$ and $G_M$ are exact.

\[ \square \]
Now we turn to techniques coming from DG categories.

**Definition 2.6.** An enhancement (resp. strong enhancement) of a triangulated category $\mathcal{C}$ is a pair $(\mathcal{E}, \epsilon)$ where $\mathcal{E}$ is a pretriangulated (resp. strongly pretriangulated) DG category and $\epsilon : H^0(\mathcal{E}) \to \mathcal{C}$ is an equivalence of triangulated categories.

**Definition 2.7.** Suppose $\mathcal{C}$ be a triangulated category which admits a DG enhancement $(\mathcal{E}, \epsilon)$. Also suppose $M : \mathcal{C} \to \mathcal{C}$ be a monad. A pre-exact DG monad $M : \mathcal{C} \to \mathcal{C}$ is said to lift $M$, if $H^0(M)$ is exact and the following diagram commutes:

\[
\begin{array}{ccc}
H^0(\mathcal{E}) & \xrightarrow{H^0(M)} & H^0(\mathcal{E}) \\
\epsilon & \downarrow & \epsilon \\
\mathcal{C} & \xrightarrow{M} & \mathcal{C}.
\end{array}
\]

Recall that strongly pretriangulated means that any object in $\mathcal{E}^{\text{pretr}}$ is DG isomorphic to an object of $\mathcal{E}$; and every every closed morphism $f$ in $\mathcal{E}$ of degree 0, the object Cone($f$) in $\mathcal{E}^{\text{pretr}}$ is DG isomorphic to an object in $\mathcal{E}$.

**Lemma 2.8.** If $\mathcal{E}$ is a strongly pretriangulated DG category, then $\mathcal{M}-\text{mod}$ is a pretriangulated DG category. Thus, $H^0(\mathcal{M}-\text{mod})$ is a triangulated category.

**Proof.** The graded structure on Hom$_{\mathcal{M}-\text{mod}}(x, y)$ is induced by that on Hom$_{\mathcal{E}}(x, y)$. If $\varphi \in$ Hom$_{\mathcal{M}-\text{mod}}(x, y)$, $d\varphi$ is easily seen to satisfy the conditions for being a morphism in $\mathcal{M}-\text{mod}$. Thus, it is a DG category.

Now we shall show that it is strongly pretriangulated. First note that for any object $(x, \lambda)$ in $\mathcal{M}-\text{mod}$, one can take $x, \lambda[1]$ to be $(x[1], \lambda[1])$, where $x[1]$ is an object in $\mathcal{E}$, DG isomorphic to $x[1]$ in $\mathcal{E}^{\text{pretr}}$. Also, for $\varphi : (x, \lambda) \to (x', \lambda')$ in $\mathcal{M}-\text{mod}$, define Cone($\varphi$) = (Cone$_{\mathcal{E}}$($\varphi$), $\lambda_c$) where Cone$_{\mathcal{E}}$($\varphi$) is the cone of $\varphi$ in $\mathcal{E}$. $\lambda_c$ is the unique functorial morphism between the cones (see [Toë11, section 5.1]):

\[
\begin{array}{ccc}
Mx & \xrightarrow{\varphi} & Mx' \\
\lambda & \downarrow & \lambda' \\
x & \xrightarrow{\varphi} & x'
\end{array}
\]

\[
\begin{array}{ccc}
\xrightarrow{\text{Cone}_{\mathcal{E}}(\varphi)} & \xrightarrow{\text{Cone}_{\mathcal{E}}(M\varphi)} \\
\lambda_c & \downarrow & \lambda
\end{array}
\]

\[
\begin{array}{ccc}
\xrightarrow{\text{Cone}_{\mathcal{E}}(\varphi)}
\end{array}
\]

**Remark 2.9.** In the above lemma, we need $\mathcal{E}$ to be strongly pretriangulated. For example, consider [Ela14, example 6.4]. In this example, even shifts do not exist.

**Proposition 2.10.** Suppose $M : \mathcal{C} \to \mathcal{C}$ is an exact monad on the triangulated category $\mathcal{C}$. Suppose $(\mathcal{E}, \epsilon)$ is a strong enhancement and that there exists a pre-exact DG monad $\mathcal{M} : \mathcal{E} \to \mathcal{C}$ which lifts $M$. Then there exists a triangulated category $D^{\text{tr}}$ and two functors: $G^{\text{tr}} : D^{\text{tr}} \to \mathcal{C}$ a left adjoint to $F^{\text{tr}} : \mathcal{C} \to D^{\text{tr}}$, such that $M = G^{\text{tr}} \circ F^{\text{tr}}$. $D^{\text{tr}}$ is triangulated and $F^{\text{tr}}$ and $G^{\text{tr}}$ are exact.

**Proof.** Consider the Eilenberg Moore construction on $\mathcal{M}-\text{mod}$. This gives us the adjunction $F_{\mathcal{M}} : \mathcal{E} \xleftarrow{\text{Eilenberg Moore}} \mathcal{M}-\text{mod} : G_{\mathcal{M}}$. Note in this case, by construction, $F_{\mathcal{M}}$ and $G_{\mathcal{M}}$ are exact functors. Since $H^0$ is a strict 2-functor, taking $H^0$ gives an adjunction: $F^{\text{tr}} : \mathcal{C} \xleftarrow{\text{Eilenberg Moore}} D^{\text{tr}} : G^{\text{tr}}$ where $D^{\text{tr}} := \mathcal{M}-\text{mod}$ is triangulated, and $F^{\text{tr}}$ and $G^{\text{tr}}$ are exact.
In the above discussion, we assumed the lift \((C, \epsilon)\) is strongly pretriangulated. One can weaken that assumption.

**Proposition 2.11.** Let \(M : C \to C\) be an exact monad on the triangulated category \(C\). If there exists an enhancement \((C, \epsilon)\) with a pre-exact DG monad \(M\) which lifts \(M\), then there exists a triangulated category \(D^{tr}\) and two adjoint functors \(F^{tr} : C \to D^{tr} : G^{tr}\) such that \(M = G^{tr} \circ F^{tr}\) and furthermore, \(G^{tr}\) and \(F^{tr}\) are exact.

**Proof.** Since both \(H^0\) and \(\mathcal{A} \mapsto \mathcal{A}^{pretr}\) are strict two functors, so is \(\mathcal{A} \mapsto \mathcal{A}^{tr}\). Hence, the same argument as above applied to \(C^{pretr}\) gives an adjunction. \(\square\)

In what follows, we shall reconsider the question, when \(M\)-mod for an exact monad on a triangulated category is itself triangulated. This time one can also ask if such a triangulation admits an enhancement.

**Proposition 2.12.** Suppose \(C\) is a triangulated category with an exact monad \((M, \mu, \eta)\). Assume that \(M\) has a lift \(M^{perf}\), a monad in an enhancement \((C, \epsilon)\). Let \(C^#\) be the idempotent completion of \(C\) and suppose \(M^#\) is the exact monad extending \(M\). Let \(M^{perf}\) be a lift of \(M\) to \(C^{perf}\). Further suppose \(M^{#}\)-mod has an enhancement \(E\), which admits an exact adjoint pair \(\tilde{F} : C^{perf} \to E : \tilde{G}^{←}\) satisfying \(\tilde{G}\tilde{F} = M^{perf}\). Then \(M\)-mod admits a triangulated structure with respect to which \(F_M\) and \(G_M\) become exact. Furthermore, \(M\)-mod admits an enhancement.

**Remark 2.13.** This proof is inspired by Elagin \[Ela14\]. In the proof, we give an description of the enhancement of \(M\)-mod in terms of \(E\).

**Remark 2.14.** Note that if \(M\) is separable, then so is \(M^{perf}\) and we can take \(E = M^{perf}\)-mod.

**Proof of Proposition 2.12.** Define \(\mathcal{D}_M\) to the full subcategory of \(E\) consisting of all objects \(e\) such that \(Ge\) which is an object of \(E^{perf}\) is quasi-isomorphic to the image of an object of \(\mathcal{E}\).

Being a full subcategory, \(\mathcal{D}_M\) already has a DG structure. On the other hand, given an object \(q\) in \(\mathcal{D}_M\) and \(m \in \mathbb{Z}\), suppose \(\tilde{G}q\) is quasi-isomorphic to \(c\) in \(\mathcal{E}\). Since \(\mathcal{E}\) is pretriangulated, \(q[m]\) is an object of \(\mathcal{E}\) and \(\tilde{G}(q[m]) = (\tilde{G}(q))[m]\) as an object of \(\mathcal{E}^{perf}\) is then quasi-isomorphic to \(c[m]\) and hence \(q[m]\) belongs to \(\mathcal{D}_M\).

Now let \(\varphi : q \to q'\) be a morphism in \(\mathcal{D}_M\). Then the cone of this map \(\text{Cone}(\varphi)\) belongs to \(\mathcal{E}\). Now we have a diagram, like in the triangulated case:

\[
\begin{array}{c}
\tilde{G}q \\ \downarrow \\
\tilde{G}q' \\
\end{array}
\begin{array}{c}
\tilde{G}\varphi \\
\downarrow \\
\tilde{G}\varphi' \\
\end{array}
\begin{array}{c}
\tilde{G}(\text{Cone}(\varphi)) \\
\downarrow \\
\text{Cone}(\tilde{G}\varphi) \\
\end{array}
\begin{array}{c}
\uparrow \\
\uparrow \\
\uparrow \\
\end{array}
\]

proving that the dotted arrow is an isomorphism and hence \(\tilde{G}\text{Cone}(\varphi)\) belongs to \(\mathcal{E}\). Thus \(\mathcal{D}_M\) is closed under taking shifts and cones. Thus it is pretriangulated.

Now we show that \(H^0(\mathcal{D}_M) = M\)-mod. In particular, this will prove that \(M\)-mod is triangulated and admits an enhancement: \(\mathcal{D}_M\).
Consider the diagram

\[
\begin{array}{ccc}
  H^0(\mathcal{E}) & \xrightarrow{\Phi} & M^\# \text{-mod} \\
  \downarrow \Phi & & \uparrow \text{universal} \\
  H^0(\mathcal{D}_M) & \xrightarrow{\cong} & M \text{-mod} \\
  \downarrow \cong & & \uparrow \tau \\
  \mathcal{D}_M & & M \text{-mod}
\end{array}
\]

\[
\begin{array}{ccc}
  \mathcal{D}_M & \xrightarrow{\cong} & M \text{-mod} \\
  \downarrow & & \uparrow \tau \\
  \mathcal{D}_M & & M \text{-mod}
\end{array}
\]

where \(\mathcal{D}_M\)-mod is the closure of \(M\)-mod in \(M^\#\)-mod under isomorphisms; thus \(\tau\) is an equivalence. \(\Phi\) is an equivalence by definition. The hooked arrow are all fully faithful. Thus \(\Psi\) is fully faithful. \(\Psi\) is essentially surjective by construction. Thus \(\Psi\) is an equivalence. Therefore, one can construct an equivalence \(\Gamma\).

We make the following definition for later use.

**Definition 2.15.** Suppose \(\mathcal{C}\) and \(\mathcal{E}\) are two pretriangulated DG categories which admit a pair of adjoint DG functors \(F : \mathcal{C} \overset{\sigma}{\longrightarrow} \mathcal{E} : G\). Then define \(\mathcal{D}(\mathcal{C}, \mathcal{E})\) to be the category of all objects \(e\) in \(\mathcal{E}\) such that \(Ge\) is quasi-isomorphic to an object in \(\mathcal{C}\).

We shall see that the above recovers Elagin’s Theorem 6.9(1) and (2). We shall prove another proposition which will allow us to recover the other two parts of the theorem.

**Definition 2.16.** Suppose \(F : \mathcal{C} \overset{\epsilon}{\longrightarrow} \mathcal{D} : G\) be an adjoint pair of exact functors between two triangulated categories. Suppose \((\mathcal{C}, \epsilon)\) be an enhancement of \(\mathcal{C}\). A lift of the adjoint pair \((F, G)\) is a another adjoint pair \(\tilde{F} : \mathcal{C} \overset{\tilde{\epsilon}}{\longrightarrow} \mathcal{D} : \tilde{G}\), where \((\mathcal{D}, \tilde{\epsilon})\) is an enhancement of \(\mathcal{D}\) and we have commutative diagrams

\[
\begin{array}{ccc}
  H^0(\mathcal{C}) & \xrightarrow{H^0(\tilde{F})} & H^0(\mathcal{D}) \\
  \downarrow \epsilon & & \downarrow \tilde{\epsilon} \\
  \mathcal{C} & \xrightarrow{F} & \mathcal{D}
\end{array}
\quad
\begin{array}{ccc}
  H^0(\mathcal{C}) & \xrightarrow{H^0(\tilde{G})} & H^0(\mathcal{D}) \\
  \downarrow \epsilon & & \downarrow \tilde{\epsilon} \\
  \mathcal{C} & \xrightarrow{G} & \mathcal{D}
\end{array}
\]

**Definition 2.17.** Suppose \(F : \mathcal{C} \overset{C}{\longrightarrow} \mathcal{D} : G\) and \(F' : \mathcal{C}' \overset{C'}{\longrightarrow} \mathcal{D}' : G'\) be two adjoint pairs of exact functors between pretriangulated DG categories. A morphism between them consists of two functors \(C : \mathcal{C} \to \mathcal{C}'\) and \(D : \mathcal{D} \to \mathcal{D}'\) such that the following two diagrams commute

\[
\begin{array}{ccc}
  \mathcal{C} & \xrightarrow{C} & \mathcal{C}' \\
  \downarrow F & & \downarrow F' \\
  \mathcal{D} & \xrightarrow{D} & \mathcal{D}'
\end{array}
\quad
\begin{array}{ccc}
  \mathcal{C} & \xrightarrow{C} & \mathcal{C}' \\
  \downarrow G & & \downarrow G' \\
  \mathcal{D} & \xrightarrow{D} & \mathcal{D}'
\end{array}
\]

The same definition can be given for triangulated categories.
Lemma 2.18. Suppose $C : \mathcal{C} \to \mathcal{C}'$ and $E : \mathcal{E} \to \mathcal{E}'$ are two functors which induce a morphism $\varphi$ of the two adjoint pairs: $F : \mathcal{C}^{\text{perf}} \to \mathcal{E}$ and $F' : \mathcal{C}'^{\text{perf}} \to \mathcal{E}' : G'$, where all the categories involved are pretriangulated DG categories. Then one has a DG functor $\mathcal{D}(\varphi) : \mathcal{D}(\mathcal{C}, \mathcal{E}) \to \mathcal{D}(\mathcal{C}', \mathcal{E}')$.

Proof. Follows easily from chasing the diagram

and the definition of $\mathcal{D}$ to see that $E$ restricts to a functor, which we call $\mathcal{D}(\varphi)$. □

Proposition 2.19. Suppose $C$ be a triangulated category and let $M$ be an exact monad on $C$. Consider the setup and notation as in proposition 2.12. Suppose we have another triangulated category $C'$ with an exact monad $M'$. Consider the setup as in proposition 2.12 for $M'$ to get $M'$, $C'$ etc. (add primes to the names in the set up for $M$ to get the names of the objects in set up for $M'$). Suppose we have functors $C : \mathcal{C} \to \mathcal{C}'$ and $E : \mathcal{E} \to \mathcal{E}'$ which induce a morphism of adjoint pairs $\varphi$ from $F : \mathcal{C}^{\text{perf}} \to \mathcal{E} : G$ to $F' : \mathcal{C}'^{\text{perf}} \to \mathcal{E}' : G'$. Then the DG functor $\mathcal{D}(\varphi)$ makes the following diagram commute up to isomorphism

Moreover if $E$ and $C$ are quasi-isomorphisms, then so is $\mathcal{D}(\varphi)$.

Proof. This follows by chasing the following diagram

□

2.1. Weak monads on DG categories. The question of representing an exact monad on a triangulated category by an adjoint pair of exact functors can also be considered in the light of the following.
Definition 2.20. Suppose $F$ and $G$ are two DG functors from $\mathcal{C}$ to $\mathcal{D}$. A weak natural transformation $\alpha : F \to G$ is a collection of morphisms of degree 0 $\{\alpha_c : Fc \to Gc\}_{c \in \text{Ob}(\mathcal{C})}$ such that $d\alpha_c = 0$ and $c \mapsto H^0(\alpha_c)$ gives a natural transformation $H^0(\alpha) : H^0(F) \to H^0(G)$. A weak natural transformation $\alpha$ is said to be a weak natural isomorphism if $H^0(\alpha)$ is a natural isomorphism.

Definition 2.21. Suppose $\mathcal{C}$ is a dg category. A weak monad on $\mathcal{C}$ is a triple $(\mathcal{M}, \mu, \eta)$ where $\mathcal{M} : \mathcal{C} \to \mathcal{C}$ is a DG endofunctor and we have weak natural transformations $\mu : \mathcal{M}^2 \to \mathcal{M}$ and $\eta : \text{id} \to \mathcal{M}$ such that $(H^0(\mathcal{M}), H^0(\mu), H^0(\eta))$ forms a monad (in the classical sense) on $H^0(\mathcal{C})$.

Definition 2.22. Two DG functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are said to be weak adjoint if we have an adjunction $H^0(F) : H^0(\mathcal{C}) \longrightarrow H^0(\mathcal{D}) : H^0(G)$.

Lemma 2.23. Suppose $(F,G)$ be a pair of weak adjoint DG functors, then for each object $c$ in $\mathcal{C}$ we have a collection of morphisms $\epsilon_c : FGc \to c$ and $\eta_c : c \to GFc$ which form weak natural transformations.

Definition 2.24. Let $(\mathcal{M}, \mu, \eta)$ be a weak monad on $\mathcal{C}$. An $\mathcal{M}$ module is a pair $(x, \lambda)$ consisting of an object $x$ in $\mathcal{C}$ and a morphism $\lambda : \mathcal{M}x \to x$ of degree zero such that $d\lambda = 0$ and $\lambda \circ M\lambda = \lambda \circ \mu_x$ is quasi-isomorphic to 0. A morphism $\varphi : (x, \lambda) \to (y, \tau)$ between two weak monads is an element $\varphi \in \text{Hom}_\mathcal{C}(x, y)$ such that $\tau \circ M\varphi - \varphi \circ \lambda$ is quasi-isomorphic to zero. It is easy to see that $\mathcal{M}$ modules and these morphisms for a category which we shall denote by $\mathcal{M}$-hmod.

Proposition 2.25. Note that $\mathcal{M}$-hmod automatically get a DG structure. Furthermore, $H^0(\mathcal{M}$-hmod) is equivalent to $M$-mod.

Proof. To see that $\mathcal{M}$-hmod has a DG structure, the only thing we need to check is that given a morphism $\varphi$, $d\varphi$ also belongs to the category $\mathcal{M}$-hmod. But,

$$
\tau \circ M(d\varphi) - d\varphi \circ \lambda = \tau \circ dM(\varphi) - d\varphi \circ \lambda
$$

$$
= d(\tau \circ M\varphi - \varphi \circ \lambda) = d\tau \circ M(\varphi) \pm \varphi \circ d\lambda
$$

$$
= d(\tau \circ M\varphi - \varphi \circ \lambda). = 0
$$

and thus $d\varphi$ is also a morphism of $\mathcal{M}$ modules.

Note that there is an obvious functor $H^0(\mathcal{M}$-hmod) $\to$ $M$-mod which takes $(x, \lambda)$ to $x, [\lambda]$ where $[\lambda]$ is the class of $\lambda$ up to boundary. This functor is by construction essentially surjective. Also it is clear from the definition of morphism that this functor is fully faithful. Hence we get the required equivalence. \qed

Remark 2.26. Recall that by Balmer [Bal11 theorem 4.1] and corollary [2.5] if $M$ is separable $M$-mod is triangulated. From this it follows that, if $\mathcal{M}$ is a weak monad on a pre-triangulated DG category $\mathcal{C}$, such that $H^0(\mathcal{M})$ is separable, then $\mathcal{M}$-hmod is also a pre-triangulated DG category.

Lemma 2.27. We have a weak adjunction $F_M : \mathcal{C} \longrightarrow \mathcal{M}$-hmod $: G_M$ where $G_M$ is the forgetful functor and $F_M(x, \lambda) = (\mathcal{M}x, \mu_x)$. Further $\mathcal{M} = G_M \circ F_M$

Proof. Easy. \qed

Definition 2.28. One can define a full subcategory $\mathcal{M}$-hfree of $\mathcal{M}$-hmod to be the subcategory generated by image of $F_M$. 

Proposition 2.29. Suppose \( \mathcal{M} \) is a weak monad on a DG category \( \mathcal{C} \). Suppose \( F : \mathcal{C} \to \mathcal{D} : G \) be an weak adjoint pair of functors. Then there is a comparison functor \( K : \mathcal{D} \to \mathcal{M} \text{-hmod} \) which is unique up to a weak natural isomorphism. Furthermore, one also gets a unique (up to weak natural isomorphism) functor \( L : \mathcal{M} \text{-hfree} \to \mathcal{D} \). All these functors can be put together in the following diagram

```
\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{G} & \mathcal{C} \\
\downarrow{\mathcal{M} \text{-hfree}} & & \downarrow{\mathcal{M} \text{-hmod}} \\
\mathcal{D} & \xrightarrow{K} & \mathcal{M} \text{-hmod}
\end{array}
\]
```

where \( L \circ F_{\mathcal{M}} = F, G \circ L = G_{\mathcal{M}}, F_{\mathcal{M}} = K \circ F, \) and \( G = G_{\mathcal{M}} \circ K \).

Proof. The proof is a direct modification of that in Eilenberg and Moore [EM65] and Balmer [Bal11, proposition 2.8]. □

Theorem 2.30. Suppose \( \mathcal{C} \) be a triangulated category, and let \( (M, \mu, \eta) \) be a monad on \( \mathcal{C} \). If \( \mathcal{C} \) admits an enhancement \( (\mathcal{C}, \epsilon) \) and an endofunctor \( \mathcal{M} \) on \( \mathcal{C} \) such that \( H^0(\mathcal{M}) = M \). Then there exists a triangulated category \( \mathcal{D} \) and an exact adjunction \( F : \mathcal{C} \to \mathcal{D} : G \).

Proof. Note that by definition, we can choose degree 0 cycles \( \mu_x \) and \( \eta_x \) for each object \( x \in \mathcal{C} \) such that \( H^0(\mu_x) = \mu_x \) and \( H^0(\eta_x) = \eta_x \). It is now clear, by definitions, that the \( \mu_x \) and \( \eta_x \) will give weak natural transformations which will make \( (\mathcal{M}, \mu, \eta) \) into a weak monad. Now \( \mathcal{M} \text{-hmod} \) is a DG category admitting adjoint functors \( (F_M, G_M) \) as above. Consider \( \iota : \mathcal{M} \text{-hmod} \to \mathcal{M} \text{-hmod}^{\text{pretr}} \). Define \( F : \mathcal{C} \to \mathcal{M} \text{-hmod}^{\text{pretr}} \) to be \( \iota \circ F_M \). Also since \( \mathcal{C} \) is pretriangulated, \( G_M \) factors as \( G \circ \iota \). Now \( (F, G) \) forms an weak adjoint pair. □

2.2. Some examples. In the above proposition, we showed under certain conditions, an exact monad \( M \) on a triangulated category can be realized as from an exact adjoint pair to some triangulated category. A stronger question would be when does \( M \text{-mod} \) itself has a triangulation. In this subsection we see some examples where this may not be true.

Example 2.31. First we give an example where a monad, which is not separable, but whose category of modules is triangulated, and the corresponding adjoint functors are exact.

Consider a field \( k \) and let \( \ell \) be any extension of \( k \). Let \( C_{dg}(k) \) (resp. \( C_{dg}(\ell) \)) be the DG category of complexes of \( k \)-modules (resp. \( \ell \)-modules). Fix equivalences \( \epsilon_k : H^0(C_{dg}(k)) \to K(k \text{-mod}) \) and \( \epsilon_\ell : H^0(C_{dg}(\ell)) \to K(\ell \text{-mod}) \) where \( K(? \text{-mod}) \) is the homotopy category of complexes of modules over the corresponding field. Now consider the object \( L \) of \( K(k \text{-mod}) \), which is concentrated at degree 0 and \( L_0 = \ell \). It is clear that \( L \) is a ring object and \( M(\_):= L \otimes_{K(k \text{-mod})} \_ \) is a monad. Considering \( L \) as an object of \( C_{dg}(k) \), we see that \( \mathcal{M} \text{-mod} \) is equivalent to \( C_{dg}(\ell) \), and hence \( \mathcal{D}_{tr} \) is equivalent to \( K(\ell \text{-mod}) \). Now \( G_{tr} : K(\ell \text{-mod}) \to K(k \text{-mod}) \) is the forgetful functor, which is faithful. Thus \( \mathcal{D}_{tr} \) is equivalent to \( M \text{-proj} \) by [Bal11, proposition 2.10]. On the other hand, \( K(F \text{-mod}) \) is equivalent to the \( \mathcal{G}(F) \), where \( \mathcal{G}(F) \) is the category of graded modules over the field \( F \) (see, for example, [Ke94, section 1]). Now the action of \( M \) on \( K(k \text{-mod}) \cong \mathcal{G}(k) \) is given by the usual action on each component. Thus \( M \text{-mod} \) is equivalent to \( \mathcal{G}(\ell) \) and thus is equivalent to \( K(\ell \text{-mod}) \) which
is triangulated. Now $G_M$ was already faithful, and is exact with respect to this triangulation. Note that all the categories here are idempotent complete. Thus we get $M$-proj $\cong D_{tr} \cong M$-mod.

**Example 2.32.** Now we give an example in which $G_{tr}$ is not faithful, but we have an exact adjoint representing $M$. Let $f : X \to \text{Spec } k$ be the structure morphism for a $k$ projective variety $X$. Then the monad $M_f := Rf_* \circ f^* = \ast \otimes Rf_* \mathcal{O}_X$ admits an exact adjoint $(f^*, Rf_*)$ where $Rf_*$ is not faithful. However, in this case $M_f$-proj $\cong \text{mod } M_f$. We give an example of a monad $M$ which is represented by an exact adjoint pair, where the right adjoint is not faithful and $M$-proj is not equivalent to $M$-mod.

Consider a field $k$ and let $A := k[X]/(X^2)$. Let $\mathcal{D} = K(k$-mod) and $M(\ast) := A_{\ast} \otimes \ast$ where $A_{\ast}$ is the object in $K(k$-mod), which is concentrated on degree $0$ and $A_0 = A$. Define $C_{dg}(k)$ is an enhancement of $K(k$-mod) and $\mathcal{M}(\ast) := A_{\ast} \otimes \ast : C_{dg}(k) \to C_{dg}(k)$ is a lift of $M$. As before, it is easy to see that $\mathcal{M}$-mod is equivalent to $C_{dg}(A)$ and hence $D_{tr}$ is equivalent to $K(A$-mod). Now $G_{tr} : D_{tr} \to \mathcal{D}$ is the forgetful functor $G_{tr} : K(A$-mod) $\to K(k$-mod).

I claim that $G_{tr}$ is not faithful: Consider the object

$$B_{\ast} := \cdots \to \frac{k[X]}{(X^2)} \to \frac{k[X]}{(X^2)} \to \frac{k[X]}{(X^2)} \to \cdots$$

in $K(A$-mod) where each of the maps are just multiplication by $X$. One checks that if there were a homotopy between $\text{id}_{B_{\ast}}$ and $0$: if there was such a homotopy $s_i : k[X]/(X^2) \to k[X]/(X^2)$ such that $s_i(1) = a_i + b_iX$, then one can check that $(X \circ s_i + s_{i+1} \circ X)(1) = (b_i + b_{i+1})X$ which can never be $1 = (\text{id}_{B_{\ast}} - 0)(1)$. On the other hand, on $G_{tr}(B_{\ast})$ in $K(k$-mod) the identity map is homotopic to $0$ as $B_{\ast}$ is exact. Therefore $G_{tr}$ cannot be faithful.

Also in this case, one can check that $M$-mod is the same as $\mathcal{G}(A)$ (as before). If $M$-mod was triangulated, $\mathcal{G}(A)$ would be a triangulated, abelian variety and hence every short exact sequence would split. However in $\mathcal{G}(A)$, the sequence of modules concentrated at degree $0$:

$$0 \to k \xrightarrow{a \mapsto aX} \frac{k[X]}{(X^2)} \to k \to 0$$

is a short exact sequence which does not split. Therefore, $M$-mod is not triangulated.

Also in this example $M$-proj is not equivalent to $M$-mod as there are objects in $\mathcal{G}(A)$ (for example, $k$) which are not free $A$ modules.

### 3. Applications and observations

#### 3.1. Equivariant triangulated categories as category of modules.

In the following we recover a construction done by Sosna [Sos12] proposition 3.7] which was further improved by Elagin [Ela14].

Let $\mathcal{A}$ be a $k$-linear triangulated category and let $G$ be a finite group acting on $\mathcal{A}$ in the sense of Deligne. Define a monad:

$$M_G(A) = \oplus_{g \in G} g^* A, \quad M_G(f : A \to B) = \oplus g^*(f) : \oplus g^* A \to \oplus g^* B$$

where $g^*$ is the functor corresponding to $g \in G$. In this case, $\mu$ is given by projection and $\eta = \oplus g^*$.
Theorem 3.1. The monad $M_G$ is separable if and only if $|G|$ is invertible.

Proof. Note that $M_G$ can be written as a composition of two adjoint functors $(p_*, p^*)$ in the notation of Elagin [Ela14, Pg 8-9]. If $|G|$ is invertible, the counit $p_*p^* \to \text{id}_{A^G}$ has a section $\eta/|G|$ by [Ela14, eqn 3.2]. Thus $M_G$ is separable by [Bal11, remark 3.9]. Considering the description of counit, it is clear that $\eta$ can be the only inverse for the counit and hence the invertibility $|G|$ is necessary. □

Suppose $\mathcal{A}$ is idempotent complete and $|G|$ is invertible, Balmer [Bal11, theorem 4.1] implies that $M_{G, \text{-}\text{mod}}$ is triangulated. Now $M_{G, \text{-}\text{mod}}$ can be seen to be the category of $G$-equivariant objects $\mathcal{A}^G$ [Ela14, definition 3.4]. Thus $\mathcal{A}^G$ in this case is triangulated.

Now suppose, $\mathcal{A}$ is not idempotent complete $|G|$ is invertible. For brevity of notation, $F = p_*$, $G = p^*$. We have a diagram

$$
\begin{array}{ccc}
A & \xrightarrow{F} & A^G \\
G & \xrightarrow{G^*} & G^G \\
A^G & \xrightarrow{K} & (A^G)^G.
\end{array}
$$

Note that the adjoints $(F, G)$ extends to an adjoint $(F^#, G^#)$. Since $G$ is faithful, so is $G^#$. Note that one can also extend the monad $M_G$ to give monad $M^G_G$ on $A^#$. This gives us the adjoint pair $(F^#, G^#)$ and $M^G_G$ is separable. Now it is easy to see, following Balmer, that $(A^G)^G$ is triangulated (and $G'$ is also faithful). Since $G'$ is faithful, $(A^G)^G$ is equivalent to $M^G_G$-proj $\cong M^G_G$-mod $\cong (A^G)^G$. This is the unique triangulation which makes $G^#$ (or $G'$) exact. Thus for any triangulated $k$-linear category $\mathcal{A}$, $(A^G)^#\mathcal{A}$ is triangulated. Now using corollary 2.3 we conclude that $M_{G, \text{-}\text{mod}} \cong A^G$ is triangulated.

We can further ask if $A^G$ admits an enhancement. By proposition 2.12 and 2.19 one sees that if one considers an enhancement $\mathcal{E} = \mathcal{A}$ of $\mathcal{A}$ and take $\mathcal{E}' = (\mathcal{A}^G)^{\text{perf}}$, one recovers theorem 6.9 of Elagin. In particular, $H^0(\mathcal{E}(\mathcal{E}', \mathcal{E}))$ is equivalent to $\mathcal{A}^G$. Elagin represents $\mathcal{E}(\mathcal{E}', \mathcal{E})$ by $\mathfrak{D}_G(\mathcal{A})$.

3.2. Relation with Drinfeld quotients.

Definition 3.2. An DG endofunctor $L : \mathcal{C} \to \mathcal{C}$ is a weak Bousfield localization functor if there is a weak natural isomorphism $\eta : \text{id}_{\mathcal{C}} \to L$ such that $L\eta : L \to L^2$ is a weak natural isomorphism and $L\eta = \eta L$ as weak natural transformations.

Remark 3.3. Note that by definition $H^0(L)$ is a Bousfield localization functor. Now $L\eta$ is a weak natural isomorphism means $H^0(L\eta)$ admits an inverse, say $\mu : H^0(L)^2 \to H^0(L)$. We can take any collection of lifts $\{\mu_x : L^2x \to Lx\}_{x \in \text{Ob}(\mathcal{C})}$ such that $H^0(\mu_x) = \mu_x$ for every object $x$ in $\mathcal{C}$. $\{\mu_x : L^2x \to Lx\}_{x \in \text{Ob}(\mathcal{C})}$. Note that this makes $(L, \mu, \eta)$ into a weak monad. Further note that in this case $H^0(L)$ is a separable monad. One might define a weak monad to be separable if its $H^0$ is a separable monad. In this sense, $L$ is a separable weak monad.

Definition 3.4. A full DG subcategory $\mathcal{E}'$ of $\mathcal{C}$ is said to be weak admissible if the inclusion functor $\mathcal{E}' \hookrightarrow \mathcal{C}$ has a weak right adjoint.

Note that the above definition is stronger than the one given in [CT13, section A.11]. The definition given there is more suitable for working in the category $\text{Hqe}$ [CT13, section A.5].
Lemma 3.5. Let $\mathcal{C}$ be a pre-triangulated DG category and $L$ be an exact DG endofunctor on $\mathcal{C}$. Then the following are equivalent:

1. $L$ is a weak Bousfield localization functor.
2. If $\mathcal{J} := \ker L$, i.e. the full DG subcategory of $\mathcal{C}$ consisting of objects $c$ such that $Lc$ is quasi-isomorphic to $0$. Then $\mathcal{J}$ is a weak admissible subcategory.
3. The Drinfeld quotient $\mathcal{C} \mapsto \mathcal{C}/\mathcal{J}$ admits a weak right adjoint.

Proof. All the properties reduce to Krause [Kra10, proposition 4.9.1] after taking $H^0$. □

Remark 3.6. In the set up of the above lemma, as $L$ is a separable weak monad, $L$-mod is a pre-triangulated DG category (see remark 2.20); and is quasi-equivalent to $\mathcal{C}/\mathcal{J}$ (since after applying $H^0$ we reduce to ) and hence the Eilenberg Maclane construction, modified as in definition 2.24 recovers weak Bousfield localizations of pre-triangulated DG categories.

3.3. Implications for derived categories.

Example 3.7. Consider an abelian category $\mathcal{A}$. Let $\mathcal{C}^\#(\mathcal{A})$ be the additive category of complexes of objects from $\mathcal{A}$, $K^\#(\mathcal{A})$ be the corresponding homotopy category; and $C^\#_{dg}(\mathcal{A})$ be the dg category of complexes [Kel94] (# will denote $+$, $-$, $b$ or nothing, depending on whether we look at bounded above, bounded below, bounded or unbounded complexes). Suppose $(M, \mu, \eta)$ be an exact monad on $\mathcal{A}$. Then one have induced monads $M_{dg}$ on $C^\#_{dg}(\mathcal{A})$ and $M_{tr}$ on $K^\#(\mathcal{A})$.

Lemma 3.8. $M_{dg}$-mod is equivalent to $C^\#_{dg}(M$-mod), and hence $K^\#(M$-mod) is equivalent to $H^0(M_{dg}$-mod).

Proof. Recall that $H^0$ is a strict 2-functor by Proposition 1.7. It is also to verify that one has a functor from the category of additive categories to categories of chain complexes given by $\mathcal{A} \mapsto C^\#_{dg}(\mathcal{A})$. Thus the monad $(M, \mu, \eta)$ on $\mathcal{A}$ induces a monad $(C^\#_{dg}(M), C^\#_{dg}(\mu), C^\#_{dg}(\eta))$ on $C^\#_{dg}(\mathcal{A})$, which we denote by $M_{dg}$. Applying $H^0$ we get a monad $M_{tr}$ on $K^\#(\mathcal{A})$.

We first prove that there is an equivalence between $M_{dg}$-mod $C^\#_{dg}(M$-mod). Define

$$
\Psi: C^\#_{dg}(M$-mod) \to M_{dg}$-mod
by sending

\[ \begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
Mx^{i-1} & \rightarrow & \lambda^{i-1} & \rightarrow & x^{i-1} & \\
Md^{i-1} & \downarrow & d^{i-1} & & & \\
((x^*, x^*), d^*) & \rightarrow & Mx^i & \rightarrow & \lambda^i & \rightarrow & ((x^*, d^*), \lambda^*) \\
Md^i & \downarrow & d^i & & & \\
Mx^{i+1} & \rightarrow & \lambda^{i+1} & \rightarrow & x^{i+1} & \\
Md^{i+1} & \downarrow & d^{i+1} & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array} \]

It is easy to see that this functor is actually an equivalence of categories.

Now since \( H^0 \) is a strict 2-functor, we get that \( H^0(\Psi) \) gives an equivalence between \( \mathcal{K}^{\#}(M \text{-mod}) \) and \( H^0(M_{dg}\text{-mod}) \).

\[ \square \]

In the case when \( M_{tr} \) is separable, Balmer proposition 4.1 \cite{Balmer11} will imply that \( M_{tr}\text{-mod} \) is equivalent to \( H^0(M_{dg}\text{-mod}) \), and hence in this particular case, \( M_{tr}\text{-mod} \) will be equivalent to \( \mathcal{K}^{\#}(M \text{-mod}) \).

This is true in some geometric situations which we describe below.

**Corollary 3.9.** Let \( X \) be a quasi-projective scheme over some field \( k \). Then the following are true.

1. Suppose \( G \) is a finite group acting on \( X \), such that \(|G|, \text{char } k\) = 1. Let us denote by \( D^b_G(X) \) the bounded derived category of the abelian category of \( G \)-equivariant sheaves. Note that if \( G \) acts on \( X \), it induces an action on \( D^b(X) \). Let \( D^b(X)^G \) be the category described above. Then

\[ D^b_G(X) \cong D^b(X)^G. \]

2. Suppose \( \alpha \) is an element of the Brauer group \( Br(X) \) corresponding to the Azumaya algebra \( A_\alpha \). Then

\[ D^b(X, \alpha) \cong A_\alpha \text{-mod}_{D^b(X)}, \]

where we (by abuse of notation) denote the monad \( M(\_ := A_\alpha \otimes \_ ) \) by \( A_\alpha \). See \cite{Caldero02} section 4 for definitions and properties of twisted derived categories.

**Proof.** Since \( X \) has enough locally free sheaves, we have

\[ D^b(X) \cong \mathcal{K}^-(\text{lfrr}(X)) \]

where \( \text{lfrr}(X) \) is the additive category of all locally free, finite rank sheaves over \( X \). Let us denote the locally free finite rank \( G \) equivariant sheaves by \( \text{lfrr}_G(X) \). Then it is easy to see that

\[ D^b_G(X) \cong \mathcal{K}^-(\text{lfrr}_G(X)). \]
Consider the monad $M_G$ defined in section 3.1. $M_G$ is a monad on $lfr(X)$ such that $M_G$-mod is $lfr(G(X))$. Also $M_G$ is separable, by 3.1. If we denote the lift of $M_G$ to $K^-(lfr(X))$ by $M_G^\wedge$, the above discussion implies, $M_G$-mod $\simeq K^-(M_G$-mod), which is the same as

$$D^b(X)^G \cong D^b_G(X)$$

which proves \(1\)

To prove \(2\) note that

$$D^b(X, \alpha) \cong K^-(lfr_\alpha(X)),$$

where $lfr_\alpha(X)$ is the category of locally free, finite rank $\alpha$-twisted coherent sheaves. Further, $lfr_\alpha(X) \cong A_\alpha$-mod$_{lfr(X)}$. Since $A_\alpha$ is separable, the result follows.

\(\square\)

Example 3.10. Specializing to $A = \mathcal{O}_X$-$\mathcal{M}$-mod. Then any sheaf of rings give a monad. Now any ring object in $K(A)$ will give us a weak monad in $C_{dg}(A)$.

Example 3.11. Let $D(A)$ be a derived category of an abelian category $A$ with tensor structure such that every object in has K-flat resolution (see [Spa88] for definition). Let $R$ be a ring object in $D(A)$. Then $M(\omega) := R \otimes \omega$ gives a monad on $D(A)$. Let $K$-flat be the DG category of h-flat objects. $K$-flat is an enhancement of $D(A)$. Let $\mathcal{R}$ be a $K$-flat replacement of $R$. Then $M(\omega) := \mathcal{R} \otimes \omega$ defines a weak monad on $K$-flat.

For example, let $f : X \to Y$ morphism of schemes. Then we have an adjunction $L_f^* : Dqc(Y) \xrightarrow{\sim} Dqc(X) : R_f\ast$. Define $M_f := R_f\ast \circ L_f^* : Dqc(Y)$ which is a monad. Note that $M_f(\omega) \cong R_f\ast \mathcal{O}_X \otimes L_\omega$ (by adjunction formula), and hence $M_f$ is realized as tensoring with the ring object $R_f\ast \mathcal{O}_X$.

Consider the enhancement $K$-flat of $Dqc(Y)$. Replacing $R_f\ast \mathcal{O}_X$ by a K-flat resolution $\mathcal{R}$, we get a weak monad $M_f(\omega) := \mathcal{R} \otimes \omega$ on $K$-flat.

Example 3.12. Now we consider an example related to Drinfeld quotients.

Definition 3.13. Let $\mathcal{C}$ be a DG category. Suppose $(M_1, \mu_1, \eta_1)$ and $(M_2, \mu_2, \eta_2)$ are two monads on $\mathcal{C}$. We say that the two monads are compatible if there exist natural isomorphisms of degree zero

\[
\begin{align*}
M_1M_2 & \simeq M_2M_1, \\
M_1\mu_2 & \simeq \mu_2M_1, \\
M_1\eta_2 & \simeq \eta_2M_1, \\
M_2\mu_1 & \simeq \mu_1M_2, \\
M_2\eta_1 & \simeq \eta_1M_2.
\end{align*}
\]

To simplify notation, we suppress precise reference to canonical isomorphisms but use it, when necessary, in the proof.

Using above definition we can get a base change type of result for the Eilenberg-Moore construction.

Proposition 3.14. Let $\mathcal{C}$ be a DG category with two compatible monads $M_1$ and $M_2$.

1. $(M_2M_1, \mu_2 \ast \mu_1, \eta_2 \ast \eta_1)$ is a monad on $\mathcal{D}$ where $\ast$ is a vertical composition of natural transformations.

2. $M_2$ (resp. $M_1$) induces a monad, say $\tilde{M}_2$ (resp. $\tilde{M}_1$, on $M_1$-mod (resp. $M_2$-mod).

3. $\tilde{M}_2$-mod $\simeq M_1M_2$-mod, $\tilde{M}_1$-mod $\simeq M_2M_1$-mod.
Proof. To prove the first assertion we use the 2-category structure to get all the compatible diagrams. Now to get the induced monad we observe that the definition of compatibility of monads induces the endofunctor
\[ \bar{M}_2 : M_1\text{-mod} \to M_1\text{-mod}; (x, \lambda) \mapsto (M_2(x), \lambda M_2(x)) \]
where \( \lambda M_2(x) \) is composition of \( M_2(\lambda x) \) with canonical isomorphism between \( M_1 M_2 \) and \( M_2 M_1 \). Now using other two conditions in the definition of compatibility gives induced monad structure on \( \bar{M}_2 \). Finally to get the equivalence we define the functor
\[ \Psi : \bar{M}_2\text{-mod} \to M_1 M_2\text{-mod}; ((x, \lambda), \alpha) \mapsto (x, \alpha M_2(\lambda)), f \mapsto f \]
with quasi-inverse given by \( (x, \beta) \mapsto ((x, \beta M_2(\eta_1^1)), \beta M_1(\eta_2^1)), f \mapsto f \). □

Remark 3.15. If we have to commutative monoids on a tensor category then we get compatible monads and above result recovers bi-module category.

Corollary 3.16. If \( A \) is a K-flat ring object in \( C(X) \) then \( D(X)^{tr} \simeq D(A) \). In particular, if \( A \) is a flat algebra over a ring \( R \) then \( D^b(R)^{tr} \simeq D^b(A) \).

Proof. Here we use the Drinfeld quotient \( C_{dg}(X)/L \) as an enhancement of \( D(X) \) where \( L \) is the full subcategory of acyclic complexes. Since \( A \) is K-flat, by definition, it induces an endofunctor on quotient category and gives a monad. Now using above result the module category is equivalent to Drinfeld quotient of \( C_{dg}(A\text{-mod}) \) by acyclic complexes. We get following commutative diagram of adjoint pairs

\[
\begin{array}{ccc}
C_{dg}(X) & \cong & C_{dg}(A\text{-mod}) \\
\downarrow & & \downarrow \\
C_{dg}(X)/L & \cong & C_{dg}(A\text{-mod})/L \\
\end{array}
\]

Further to prove the second assertion we can work with the bounded complexes as enhancement. □

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