SMALL AMPLITUDE SOLITARY WAVES IN THE
DIRAC–MAXWELL SYSTEM

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Abstract. We study nonlinear bound states, or solitary waves, in the Dirac–Maxwell system, proving the existence of solutions in which the Dirac wave function is of the form \( \phi(x, \omega)e^{-i\omega t} \), with \( \omega \in (-m, \omega_*) \) for some \( \omega_* > -m \). The solutions satisfy \( \|\phi(\cdot, \omega)\|_{L^2}^2 = O(\sqrt{m + \omega}) \) and \( \|\phi(\cdot, \omega)\|_{L^\infty} = O(m + \omega) \). The method of proof is an implicit function theorem argument based on the identification of the nonrelativistic limit as the ground state of the Choquard equation. This identification is in some ways unexpected on account of the repulsive nature of the electrostatic interaction between electrons, and arises as a manifestation of certain peculiarities (Klein paradox) which result from attempts to interpret the Dirac equation as a single particle quantum mechanical wave equation.

1. Introduction and results. The Dirac equation, which appeared in [14] just two years after the Schrödinger equation, is the correct Lorentz-invariant equation to describe particles with nonzero spin when relativistic effects cannot be ignored. The Dirac equation predicts accurately the energy levels of an electron in the Hydrogen atom, yielding relativistic corrections to the spectrum of the Schrödinger equation. Further higher order corrections arise on account of electromagnetic self-interactions, described mathematically by the Dirac–Maxwell Lagrangian, which aims to provide a self-consistent description of the dynamics of an electron interacting with its own electromagnetic field. The perturbative treatment of the Dirac–Maxwell system in the framework of second quantization allows computation of quantities such as the energy levels and scattering cross-sections, which have been compared successfully with experiment, although this quantum formalism does not provide the type of tangible description of particles and dynamical processes familiar from classical physics. Mathematically, the quantum theory (QED) has not been constructed, and indeed may not exist in the accepted analytical sense. In particular it is a curious fact that although the electron is the most stable elementary

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A particle known to physicists today, there is no mathematically precise formulation and proof of its existence and stability. This has resulted in an enduring interest in the classical Dirac–Maxwell system, both in the physics and mathematics literature. Regarding the former, the relevance of the classical equations of motion for QED has been widely debated. The prevalent view seems to be that the Dirac fermionic field does not have a direct meaning or limit in classical physics, and hence that the classical system is not really directly relevant to the world of observation. Nevertheless, there have been numerous attempts, by Dirac himself as well as by many others – see [13, 32, 22] and references therein – to construct localized solutions of the classical system, or some modification thereof, with the aim of obtaining a more cogent mathematical description of the electron (or other fundamental particles).

We consider the system of Dirac–Maxwell equations, where the electron, described by the standard “linear” Dirac equation, interacts with its own electromagnetic field which is in turn required to obey the Maxwell equations:

\[
\begin{aligned}
\gamma^\mu (i\partial_\mu - qA_\mu)\psi - m\psi &= 0, \\
\partial^\mu \partial_\mu A^\nu &= J^\nu, \quad \partial_\mu A^\mu &= 0,
\end{aligned}
\]

with the charge-current density \( J^\mu = (\rho, \mathbf{J}) \in \mathbb{R} \times \mathbb{R}^3 \) generated by the spinor field itself:

\[
J^\mu = q\bar{\psi}\gamma^\mu \psi, \quad 0 \leq \mu \leq 3.
\]

Above, \( \rho \) and \( \mathbf{J} \) are the charge and current, respectively. We denote \( \bar{\psi} = (\gamma^0 \psi^*)^* \) with \( \gamma^* \) the hermitian conjugate of \( \psi \in \mathbb{C}^4 \). The charge is denoted by \( q \) (so that for the electron \( q < 0 \)); the fine structure constant is the dimensionless coupling constant \( \alpha \equiv \frac{e^2}{\hbar c} \approx 1/137 \). We choose the units so that \( \hbar = c = 1 \). We have written the Maxwell equations using the Lorentz gauge condition \( \partial_\mu A^\mu = 0 \).

The Dirac \( \gamma \)-matrices satisfy the anticommutation relations

\[
\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad 0 \leq \mu, \nu \leq 3,
\]

with \( g^{\mu\nu} = \text{diag}[1, -1, -1, -1] \). The four-vector potential \( A^\mu \) has components \( (A^0, \mathbf{A}) \), with \( \mathbf{A} = \{A^j\}_{j=1}^3 \), so that the lower index version \( A_\mu = g_{\mu\nu}A^\nu \) has components \( (A^0, -\mathbf{A}) \) so \( A_0 = A^0 \). Following [4] and [5], we define the Dirac \( \gamma \)-matrices by

\[
\gamma^j = \begin{pmatrix}
0 & \sigma_j \\
-\sigma_j & 0
\end{pmatrix}, \quad \gamma^0 = \begin{pmatrix}
I_2 & 0 \\
0 & -I_2
\end{pmatrix},
\]

where \( I_2 \) is the \( 2 \times 2 \) unit matrix and \( \sigma_j \) are the Pauli matrices: \( \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \), \( \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). After introduction of a space-time splitting, the system (1.1) takes the form

\[
\begin{aligned}
\gamma^\mu (i\partial_\mu - qA_\mu)\psi - m\psi &= 0, \\
(\partial_\mu^2 - \Delta)A^0 &= q\psi^* \psi, \\
(\partial_\mu^2 - \Delta)\mathbf{A} &= q\psi^* \mathbf{\alpha}\psi.
\end{aligned}
\]

Above, \( \mathbf{\alpha} = (\alpha^1, \alpha^2, \alpha^3) \); \( \alpha^j \) and \( \beta \) are the \( 4 \times 4 \) Dirac matrices:

\[
\alpha^j = \begin{pmatrix}
0 & \sigma_j \\
\sigma_j & 0
\end{pmatrix}, \quad \beta = \begin{pmatrix}
I_2 & 0 \\
0 & -I_2
\end{pmatrix}.
\]
The $\alpha$-matrices and $\gamma$-matrices are related by
\[ \gamma^j = \beta \alpha^j, \quad 1 \leq j \leq 3; \quad \gamma^0 = \beta. \]

Numerical evidence for the existence of solitary wave solutions to the Dirac–Maxwell system (1.1) was obtained in [32] and then in [22], where it was suggested that such solutions are produced by the Coulomb repulsion from the negative part of the essential spectrum (the Klein paradox). The numerical results of [22] showed that the Dirac–Maxwell system has infinitely many families of solitary waves $\phi_N(x, \omega)e^{-i\omega t}$, $\omega \geq -m$. Here the nonnegative integer $N$ denotes the number of nodes of the positron component of the solution (number of zeros of the corresponding spherically symmetric solution to the Choquard equation; see §3). A variational proof of existence of solitary waves for $\omega \in (-m, 0)$ and with $N = 0$ first appeared in [15], and the generalization to handle $\omega \in (-m, m)$ is in [1].

In the present paper, we give a proof of existence of solitary wave solutions to the Dirac–Maxwell system based on the perturbation from the nonrelativistic limit and also obtain the precise asymptotics for the solution in this limit.

The solitary wave solution $(\phi(x)e^{-i\omega t}, A^\mu(x))$ satisfies the stationary system
\[ \omega \phi = \alpha \cdot (-i \nabla - qA)\phi + m\beta\phi + qA^0\phi, \quad -\Delta A^\mu = q\bar{\phi}\gamma^\mu\phi. \quad (1.6) \]

**Theorem 1.1.** There exists $\omega_* > -m$ such that for $\omega \in (-m, \omega_*)$ there is a solution to (1.6) of the form
\[ \phi(x, \omega) = \left[ \begin{array}{c} \epsilon^3 \Phi_1(x, \epsilon) \\ \epsilon^2 \Phi_2(x, \epsilon) \end{array} \right], \quad \epsilon = \sqrt{m^2 - \omega^2}, \]
with
\[ \Phi = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} \in C^\infty((0, \epsilon_*); H^2(\mathbb{R}^3; \mathbb{C}^2) \oplus H^2(\mathbb{R}^3; \mathbb{C}^2)), \quad \epsilon_* = \sqrt{m^2 - \omega_*^2}, \]
and with
\[ A^\mu \in C^\infty((0, \epsilon_*); \check{H}^1(\mathbb{R}^3, \mathbb{R}) \cap L^\infty(\mathbb{R}^3, \mathbb{R})), \quad 0 \leq \mu \leq 3. \]

Above, $\check{H}^1 = \check{H}^1(\mathbb{R}^3, \mathbb{R})$ is the homogeneous Dirichlet space of $L^6$ functions with the norm
\[ \|f\|_{\check{H}^1}^2 := \int_{\mathbb{R}^3} |\nabla f|^2 \, dx < \infty. \]

For small $\epsilon > 0$, one has
\[ \|\Phi_1 - \hat{\Phi}_1\|_{H^2} + \|\Phi_2 - \hat{\Phi}_2\|_{H^2} = O(\epsilon^2), \]
where $\hat{\Phi}_1(y), \hat{\Phi}_2(y)$ are of Schwartz class. The solutions can be chosen so that in the nonrelativistic limit $\epsilon = 0$ one has
\[ \hat{\Phi}_2(y) = \varphi_0(y)\mathbf{n}, \quad \hat{\Phi}_1(y) = \frac{i}{2m} \mathbf{\sigma} \cdot \nabla \varphi(y), \quad (1.7) \]
where $\mathbf{n} \in \mathbb{C}^2$, $|\mathbf{n}| = 1$, and $\varphi_0 \in \mathcal{S}(\mathbb{R}^3)$ is a strictly positive, spherically symmetric, strictly monotonically decaying (as a function of $|y|$) solution of Schwartz class to the Choquard equation
\[ -\frac{1}{2m} \varphi = -\frac{1}{2m} \Delta \varphi - \left( \frac{q^2}{4\pi} \ast |\varphi|^2 \right) \varphi, \quad \varphi(y) \in \mathbb{R}, \quad y \in \mathbb{R}^3. \quad (1.8) \]
The Dirac field $\phi(x, \omega)$ has exponential decay in $x$, while the electromagnetic potential satisfies

$$A^0(x, \omega) = q\frac{\|\phi\|^2_{L^2}}{4\pi |x|} + O((\langle x \rangle)^{-2}), \quad A(x, \omega) = O((\langle x \rangle)^{-2})$$

as $|x| \to +\infty$.

**Remark 1.** The existence of a positive spherically-symmetric solution $\varphi_0 \in \mathcal{S}(\mathbb{R}^3)$ to (1.8) was proved in [20].

To prove Theorem 1.1, we construct solitary wave solutions by deforming the solutions to the nonrelativistic limit (represented by the Choquard equation) via the implicit function theorem. Such a method was employed in [24, 16, 7] for the nonlinear Dirac equation and in [26, 30, 27] for Einstein–Dirac and Einstein–Dirac–Maxwell systems.

One motivation for presenting here a new existence proof for the Dirac–Maxwell solitary waves is to realize mathematically the physical intuition sketched in [22], which explains the existence of these bound state solutions in terms of the Klein paradox (see e.g. [5, §3.3]). Moreover, once one knows that the excited eigenstates of the Choquard equation are nondegenerate (currently this nondegeneracy is established only for the ground state, $N = 0$ [19]), our argument will yield the existence of excited solitary wave solutions in the Dirac–Maxwell system, extending the results of [15] to $N \geq 1$ (see Remark 4 below); as mentioned in that article, the variational methods are hard to generalize to prove the existence of multiple solitary waves for each $\omega$ in the Dirac–Maxwell system (although such a multiplicity result has been obtained in [15] for the Dirac–Klein–Gordon system).

Another motivation for the bifurcation approach is that having the nonrelativistic (or small-amplitude) asymptotics of solitary waves is the first step towards analyzing their stability. Indeed, the physical significance of solitary waves requires not only existence but also stability, and it is to be hoped that the type of detailed information about the solutions which is a consequence of the existence proof in this article, but does not seem to be so easily accessible from the original variational constructions, will be helpful in future stability analysis (see Remark 6 below). In this context we mention some recent stability results for the nonlinear Dirac equation. Numerical results suggest that the nonlinear Dirac equation with scalar-type self-interaction, known as the Soler model, possesses solitary waves which are spectrally stable, that is, the linearization at the solitary wave has purely imaginary spectrum. Numerics indicate that all solitary waves in the cubic Soler model in one spatial dimension (known as Gross–Neveu model) are spectrally stable, except perhaps for $\omega$ very close to $\omega = 0$ [2]. Numerically, for the Soler model with cubic nonlinearity in the two-dimensional case, there exists $\omega_0 \in (0, m)$ such that solitary waves with $\omega \in (\omega_0, m)$ are spectrally stable, and the same is expected in the three-dimensional case with $\omega \lesssim \omega_* \approx 0.936m$ [12]. More general results on the spectral stability (that is, absence of linear instability) in the nonlinear Dirac equation have been obtained in [3] (possibility of bifurcations of nonzero-real-part eigenvalues from the origin) and in [6] (possibility of bifurcations of nonzero-real-part eigenvalues from the essential spectrum). The spectral stability of small amplitude solitary waves (that is, solitary waves in the nonrelativistic limit) in the charge-subcritical and charge-critical cases, when the nonlinear term is $|\psi^* \beta \psi|^k \beta \psi$ with $k \leq 2/n$, where $n \geq 1$ is the spatial dimension (under the technical assumption that $k > k_n$, with some $k_n \in (0, 2/n))$ was proved in [8]. The linear instability of small amplitude solitary
The asymptotic stability of solitary waves in the one-dimensional Soler model with respect to “radially symmetric” perturbations has been proved in [10].

Here is the plan of the paper. We give the heuristics and expected nonrelativistic scaling in §2. The Choquard equation, which is the nonrelativistic limit of the Dirac–Maxwell system, is considered in §3. In §4, we complete the proof of existence of solitary waves via the implicit function theorem.

2. Heuristics on the nonrelativistic limit. The small amplitude waves constructed in Theorem 1.1 are best understood physically in terms of the non-relativistic limit. Since we have set the speed of light and other physical constants equal to one, the relevant small parameter is the excitation energy (or frequency) as compared to the mass $m$. To develop some preliminary intuition regarding the non-relativistic limit, following [22], we neglect the magnetic field described by the vector-potential $A$, getting

$$i\partial_t \psi = -i\alpha \cdot \nabla \psi + \frac{\beta}{m} \psi + qA^0 \psi, \quad (\partial_t^2 - \Delta)A^0 = q\psi^\star \psi.$$  

We consider a family of solitary waves

$$\psi(x, t) = \phi(x, \omega)e^{-i\omega t}, \quad \omega \in \mathbb{R},$$

with $\phi(x, \omega) = \left[\phi_1(x, \omega) \phi_2(x, \omega)\right] \in \mathbb{C}^2$, where $\phi_1(x, \omega), \phi_2(x, \omega) \in \mathbb{C}$ and $A^0(x, \omega) \in \mathbb{R}$. Then $\phi_1, \phi_2,$ and $A^0$ satisfy

$$\begin{cases}
(\omega - m)\phi_1 = -i\sigma \cdot \nabla \phi_2 + qA^0 \phi_1, \\
(m + \omega)\phi_2 = -i\sigma \cdot \nabla \phi_1 + qA^0 \phi_2, \\
-\Delta A^0 = q(\phi_1^\star \phi_1 + \phi_2^\star \phi_2),
\end{cases} \quad (2.1)$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, the vector formed from the Pauli matrices. Let us try to find small amplitude solitary waves with $\omega \gtrsim -m$. Then $A^0$ is small and $-2m\phi_1 \approx -i\sigma \cdot \nabla \phi_2$,

$$-(m + \omega)\phi_2 \approx -\frac{1}{2m} \Delta \phi_2 - qA^0 \phi_2, \quad -\Delta A^0 \approx q(\phi_1^\star \phi_1 + \phi_2^\star \phi_2). \quad (2.2)$$

We notice from the above system that the effect of the electromagnetic interaction is now attractive; this is because we are analyzing states which bifurcate from the negative energy spectrum. It can easily be seen that if the same reasoning as above is applied when $\omega \lesssim m$, then it leads to an equation with a repulsive interaction, as is normal in electrostatics. This generation of an effective attraction out of negative energy states is one of a number of curious phenomena which arise from attempts to treat the Dirac equation as a single particle wave equation, collectively referred to under the label “Klein paradox” [28].

Let $c^2 = m^2 - \omega^2$, $0 < \epsilon \ll m$; then (2.2) suggests the following scaling:

$$y = \epsilon x, \quad \partial_y = \epsilon \partial_x, \quad A^0(x, \omega) = c^2 A^0(\epsilon x, \epsilon),$$

$$\phi_1(x, \omega) = c^2 \phi_1(\epsilon x, \epsilon), \quad \phi_2(x, \omega) = c^2 \phi_2(\epsilon x, \epsilon). \quad (2.3)$$

Note that while $\phi_j$ and $A^0$ depend on $\omega$ and $x$, it is convenient to consider the scaled functions $A^0$ and $\Phi_j$ as functions of $y = \epsilon x$ and $\epsilon = \sqrt{m^2 - \omega^2}$. In the limit
\( \epsilon \to 0 \), denoting

\[
\hat{\phi} = \lim_{\epsilon \to 0} \Phi, \quad \hat{A}^0 = \lim_{\epsilon \to 0} A^0,
\]

we arrive at the system

\[
\begin{cases}
-2m\hat{\phi}_1 = -i\sigma \cdot \nabla_y \hat{\phi}_2, \\
-\frac{1}{2m} \hat{\phi}_2 = -i\sigma \cdot \nabla_y \hat{\phi}_1 + q\hat{A}^0 \hat{\phi}_2, \\
-\Delta_y \hat{A}^0 = q\hat{\phi}_2^* \hat{\phi}_2,
\end{cases}
\quad (2.4)
\]

which can be rewritten as the following equation for \( \hat{\phi}_2(y) \) and \( \hat{A}^0(y) \) only:

\[
-\frac{1}{2m} \Delta_y \hat{\phi}_2 = -\frac{1}{2m} \Delta_y \hat{\phi}_2 - q\hat{A}^0 \hat{\phi}_2, \quad -\Delta_y \hat{A}^0 = q\hat{\phi}_2^* \hat{\phi}_2,
\quad (2.5)
\]

with the understanding that \( \hat{\phi}_1(y) \) is then obtained from the first equation of (2.4).

**Remark 2.** Regarding self-consistency of this approximation: one can check that, when using the scaling (2.3), the magnetic field vanishes to higher order in the limit \( \epsilon \to 0 \), in agreement with [22]. Indeed, \( A = -\Delta^{-1}J \), where \( J = q\psi^* \alpha \psi = O(\epsilon^3) \), hence \( A = -\Delta^{-1}J = O(\epsilon^3) \). The second equation from (2.1) would then take the form

\[
(m + \omega)\phi_2 = -i\sigma \cdot \nabla \phi_1 - qA \cdot \sigma \phi_1 + qA^0 \phi_2,
\]

where \( A \cdot \sigma \phi_1 = O(\epsilon^6) \) while other terms are \( O(\epsilon^4) \). Thus the approximation is at least formally self-consistent; the analysis in \( \S 4 \) makes this rigorous.

**Remark 3.** Regarding the symmetry: while it is clear that radial symmetry of both \( \phi_1 \) and \( \phi_2 \) is inconsistent with (2.4), solutions of the form given in [32],

\[
\phi^I(x) = \begin{bmatrix} g(r) 1 \\ if(r) \cos \theta \\ i f(r) \sin \theta \end{bmatrix}, \quad \phi^{II}(x) = \begin{bmatrix} f(r) \begin{bmatrix} \cos \theta \\ e^{i\phi} \sin \theta \end{bmatrix} \\ ig(r) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix},
\quad (2.6)
\]

are permitted in principle, suggesting that in the non-relativistic limit \( \hat{\phi}_2 \) could be radial, or, to be more precise, of the form \( \hat{\phi}_2(y) = \varphi(y) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbb{C}^2 \), where the spherically symmetric function \( \varphi(y) \in \mathbb{C}, \ y \in \mathbb{R}^3 \) is to satisfy

\[
-\frac{1}{2m} \varphi = -\frac{1}{2m} \Delta_y \varphi - q\hat{A}^0 \varphi, \quad -\Delta_y \hat{A}^0 = |\varphi|^2.
\quad (2.7)
\]

The starting point for our perturbative construction of solitary wave solutions to (1.4) is indeed a radial solution of (2.7), although the exact form of these solitary waves has to be modified from (2.6) when the effect of the magnetic field \( B = \nabla \times A \) is included; see [22, \S 5]. The method of proof we employ does not require any particular symmetry class of the solitary wave.

The above discussion suggests that the system (2.7) determines the non-relativistic limit in the leading order. The system (2.7) describes a Schrödinger wave function with an attractive self-interaction determined by the Poisson equation. Since the sign of the interaction is attractive, (2.7) is often referred to as the stationary Newton–Schrödinger system. It is equivalent to a nonlocal equation for \( \varphi \) known as the Choquard equation, which is the subject of the next section.
3. The nonrelativistic limit: the Choquard equation. One arrives at the system (2.7) when looking for solitary wave solutions in the system
\[
\begin{align*}
  i\partial_t \psi &= -\frac{1}{2m} \Delta \psi - qV \psi, \\
  -\Delta V &= q\psi^* \psi,
\end{align*}
\]
(3.1)
This is the time-dependent Newton–Schrödinger system. If \((\varphi(x, \omega)e^{-i\omega t}, V(x, \omega))\) is a solitary wave solution, then \(\varphi\) and \(V\) satisfy the stationary system
\[
\omega \varphi = -\frac{1}{2m} \Delta \varphi - qV \varphi, \\
-\Delta V = q|\varphi|^2.
\]
(3.2)
We rewrite the system (3.1) in the non-local form, which is known as the Choquard equation [20]:
\[
\begin{align*}
  i\partial_t \psi &= -\frac{1}{2m} \Delta \psi + q^2 \Delta^{-1}(|\psi|^2) \psi, \\
  \psi(x, t) &\in \mathbb{C}, \quad x \in \mathbb{R}^3,
\end{align*}
\]
(3.3)
where \(\Delta^{-1}\) is the operator of convolution with \(-\frac{1}{4\pi|x|}\). The solitary waves are solutions of the form \(\psi(x, t) = \varphi(x, \omega)e^{-i\omega t}\), with \(\varphi\) satisfying the non-local scalar equation
\[
\omega \varphi = -\frac{1}{2m} \Delta \varphi + q^2 \Delta^{-1}(|\varphi|^2) \varphi.
\]
(3.4)
This suggests the following variational formulation for the problem: find critical points of
\[
E(\varphi) = \frac{1}{2m} \int_{\mathbb{R}^3} |\nabla \varphi|^2 \, dx - \frac{q^2}{8\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|\varphi(x)||\varphi(y)|^2}{|x-y|} \, dx \, dy,
\]
(3.5)
subject to the constraint \(Q(\varphi) = \text{const}\), with the charge functional defined by
\[
Q(\varphi) = \int_{\mathbb{R}^3} |\varphi(x)|^2 \, dx.
\]
(3.6)
This formulation is the basis of the existence and uniqueness proofs in the references which are summarized in the following theorem.

**Lemma 3.1** ([20, 21, 23]). For all \(\omega < 0\) and \(N \in \mathbb{Z}\), \(N \geq 0\), the equation (3.3) admits solitary wave solutions
\[
\psi(x, t) = \varphi_N(x, \omega)e^{-i\omega t}, \quad \lim_{|x|\to\infty} \varphi_N(x, \omega) = 0,
\]
with \(\varphi_N(x, \omega)\) a spherically symmetric solution of (3.4). These solutions differ by the number \(N\) of zeros (or nodes), of the profile functions \(\varphi_N(x, \omega)\), considered as functions of \(r = |x|\). The profile function \(\varphi_0\) with no zeros minimizes the value of the energy functional \(E(\varphi)\) amongst functions with fixed \(L^2\) norm, and is the unique (up to translation) positive \(H^1\) solution of (3.4); the corresponding solitary wave is called the ground state.

**Remark 4.** Together with the heuristics in the previous section, the above result suggests that for \(\omega\) sufficiently close to \(-m\) there might exist infinitely many families of solitary waves to the Dirac–Maxwell system, which differ by the number of nodes of the positronic component (two lower components of \(\varphi\)).

**Remark 5.** The \(\varphi(x, \omega)\) and \(V(x, \omega)\) for different values of \(\omega < 0\) can be scaled to produce a standard form as follows. Let \(\zeta > 0\) be such that \(\omega = -\zeta^2\) and write
\[ y = \zeta x, \; \varphi(x, \omega) = \zeta^2 u(\zeta x), \; \text{and} \; V(x, \omega) = \zeta^2 v(\zeta x). \] Then (3.2) is equivalent to the following system for \( u(y), v(y) \):

\[
-u = -\frac{1}{2m} \Delta_y u - qv u, \quad -\Delta_y v = q|u|^2. \tag{3.7}
\]

In the remainder of this section we summarize the properties of the linearized Choquard equation which follow from \([19]\) and are needed in \(\S4\). Consider a solution to the Choquard equation of the form

\[ \psi(x, t) = (\varphi_0(x) + R(x, t) + iS(x, t))e^{-is\omega t}, \]

with \( R(x, t), S(x, t) \) real-valued. The linearized equation for \( R, S \) is:

\[
\partial_t \begin{bmatrix} R \\ S \end{bmatrix} = \begin{bmatrix} 0 & L_0 \\ -L_1 & 0 \end{bmatrix} \begin{bmatrix} R \\ S \end{bmatrix},
\]

where

\[
L_0 = -\frac{1}{2m} \Delta - \omega_0 + \frac{q^2}{2} \Delta^{-1}(\varphi_0^2), \quad L_1 = L_0 + 2q^2 \Delta^{-1}(\varphi_0 \cdot \cdot)\varphi_0. \tag{3.9}
\]

Notice that \( L_1 = \frac{1}{2} (E''(\varphi_0) - \omega_0 Q''(\varphi_0)) \), with \( E(\varphi) \) from (3.5). Both \( L_0 \) and \( L_1 \) are unbounded operators \( L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \) which are self-adjoint with domain \( H^2(\mathbb{R}^3) \subset L^2(\mathbb{R}^3) \).

**Lemma 3.2.** The self-adjoint operator \( L_0 : H^2 \to L^2 \) is positive-definite, with \( 0 \in \sigma_d(L_0) \) a simple eigenvalue corresponding to a positive eigenfunction \( \varphi_0 \). The range of \( L_0 \) is \( \{\varphi_0\}^\perp \), the \( L^2 \)-orthogonal complement of the linear span of \( \varphi_0 \).

The self-adjoint operator \( L_1 : H^2 \to L^2 \) has exactly one negative eigenvalue, which we denote \(-\lambda_0\), and has a three-dimensional kernel \( \text{Ker} L_1 \) spanned by \( \{\partial_j \varphi_0\}_{j=1}^3 \). The range of \( L_1 \) is \( (\text{Ker} L_1)^\perp \), the \( L^2 \)-orthogonal complement of the linear span of the \( \{\partial_j \varphi_0\}_{j=1}^3 \).

**Proof.** Clearly \( L_0 \varphi_0 = 0 \); since \( \varphi_0 \) is positive, it follows that 0 is the lowest eigenvalue of \( L_0 \) (which is thus non-degenerate), with the rest of the spectrum separated from zero.

Now we focus on \( L_1 \); we proceed similarly to \([18, \text{Lemma 5.4.3}]\). The \( N = 0 \) ground state solution \( \varphi_0 \) to (3.4) is characterized in \([20]\) as the solution, unique up to translation and phase rotation, to the following minimization problem:

\[
E(\varphi_0) = I_{\mu} := \inf \{ E(\varphi); \; \varphi \in H^1(\mathbb{R}^3), \; \|\varphi\|^2_{L^2} = \mu \}, \tag{3.10}
\]

for certain \( \mu > 0 \); above, \( E(\varphi) \) is from (3.5). We claim that this implies that \( L_1 \geq 0 \) on \( \{\varphi_0\}^\perp \). Indeed, let \( \|v\|_{L^2} = \|\varphi_0\|_{L^2}, \; \langle v, \varphi_0 \rangle = 0 \). For \( s \in (-1, 1) \), define \( \varphi_s = (1 - s^2)^{1/2} \varphi_0 + sv \), so that \( Q(\varphi_s) = Q(\varphi_0) \). Calculating that \( \varphi_0 \big|_{s=0} = \varphi_0, \; \varphi_0 \big|_{s=0} = v, \; \varphi_0 \big|_{s=0} = -\varphi_0 \) we deduce from (3.10):

\[
0 \leq \partial_s^2 \big|_{s=0} E(\varphi_s) = \langle E'(\varphi_0), -\varphi_0 \rangle + \langle E''(\varphi_0)v, v \rangle \tag{3.11}
\]

\[
= -\omega_0 \langle Q'(\varphi_0), \varphi_0 \rangle + \langle E''(\varphi_0)v, v \rangle = \langle v, (E'' - \omega_0 Q'')v \rangle = \langle Q''v, v \rangle.
\]

establishing the claim. We took into account that \( \varphi_0 \) satisfies the stationary equation \( E'(\varphi_0) = \omega_0 Q(\varphi_0) \) and also that

\[
\langle Q'(\varphi_0), \varphi_0 \rangle = 2\|\varphi_0\|^2_{L^2} = 2\|v\|_{L^2}^2 = \langle Q'(v), v \rangle = \langle Q''v, v \rangle.
\]

So \( L_1 \) is non-negative on a codimension one subspace. On the other hand, since the integral kernel of \( \Delta^{-1} \) is strictly negative, while \( \varphi_0 \) is strictly positive and \( L_0 \varphi_0 = 0 \),
it follows that $\langle \varphi_0, L_1 \varphi_0 \rangle < 0$ so that there certainly exists one negative eigenvalue characterized as

$$-\Lambda_0 := \inf \{ \langle v, L_1 v \rangle; \|v\|_{L^2} = 1 \} < 0.$$ 

Let $\eta_0$ be the corresponding eigenfunction, $L_1 \eta_0 = -\Lambda_0 \eta_0$. To prove that $(-\Lambda_0, 0) \subset \rho(L_1)$, which is the resolvent set of $L_1$, consider the minimization problem

$$\inf \{ \langle v, L_1 v \rangle; \|v\|^2 = 1, \langle \eta_0, v \rangle = 0 \}.$$ 

(3.12)

The relation $L_0 \varphi_0 = 0$, together with translation invariance, implies that $L_1 \partial_j \varphi_0 = 0$. Moreover, it is proved in [19] that $\varphi_0$ is nondegenerate, in the sense that the kernel of $L_1$ is spanned by the $\partial_j \varphi_0$, $1 \leq j \leq 3$. Hence, by consideration of linear combinations of the eigenfunctions $\eta_0$ and $\partial_j \varphi_0$, we conclude that the value defined by (3.12) is $\leq 0$. In fact it must equal zero since if it were negative a simple compactness argument of the type appearing in [33, Proof of Proposition 2.9], based on the negativity of $\omega_0$, would imply the existence of a negative eigenvalue in the interval $(-\Lambda_0, 0)$ and with the corresponding eigenfunction $\eta_1$ orthogonal to $\eta_0$. But since $\eta_0, \eta_1$ would then be an orthogonal pair of eigenfunctions of $L_1$ with negative eigenvalues, there would necessarily exist some non-trivial linear combination of them having zero inner product with $\varphi_0$, contradicting the fact that $L_1$ is nonnegative on $\{ \varphi_0 \}^\perp$ (cf. (3.11)). \hfill $\Box$

We will also need the following bounds for the inverses of $L_0$ and $L_1$.

**Corollary 1.** $L_0^{-1}$ is a bounded operator $\{ \varphi_0 \}^\perp \cap L^2 \to H^2$, while $L_1^{-1}$ is a bounded operator $(\text{Ker } L_1)^\perp \cap L^2 \to H^2$. Also, in terms of the exponentially weighted Sobolev spaces $H^{s,\theta}(\mathbb{R}^3)$, with $s \in \mathbb{N}_0 = \{0, 1, 2, \ldots \}$ and $\theta \geq 0$, with the norms

$$\|u\|_{H^{s,\theta}} = \sum_{\alpha \in \mathbb{N}^3_0, |\alpha| \leq s} \|e^{\theta|x|} \partial_x^\alpha u\|_{L^2(dx)},$$

(3.13)

the mappings

$$L_0^{-1} : \{ \varphi_0 \}^\perp \cap H^{s,\theta} \to H^{s+2,\theta}$$

$$L_1^{-1} : (\text{Ker } L_1)^\perp \cap H^{s,\theta} \to H^{s+2,\theta}$$

are bounded for $\theta < |\omega_0|$.

We conclude with a few remarks on the stability of solitary waves to the Choquard equation. By Remark 5 we know the $\omega$-dependence of a localized solution $\varphi(x, \omega) e^{-i\omega t}$ to (3.3): one has $\varphi(x, \omega) = \zeta^2 u(\zeta|x|)$, where $\zeta = \sqrt{-\omega}$. From this we can obtain how the charge depends on the frequency $\omega < 0$:

$$Q(\omega) = \int_{\mathbb{R}^3} |\varphi(x, \omega)|^2 \, dx = \zeta^4 \int_{\mathbb{R}^3} |\zeta u(x)|^2 \, dx = \zeta \int_{\mathbb{R}^3} |u(y)|^2 \, dy = |\omega|^2 \int_{\mathbb{R}^3} |u(y)|^2 \, dy.$$ 

It follows that for all negative frequencies one has $\frac{d}{d\omega} Q(\omega) < 0$. By the Vakhitov–Kolokolov stability criterion [31], this leads us to expect the spectral stability of no-node solitary waves (the ground states) in the Choquard equation.

**Proposition 1.** The ground state solitary wave $\varphi_0(x) e^{-i\omega_0 t}$ of the Choquard equation (3.3) is spectrally stable.

**Proof.** To determine the point spectrum of $JL = \begin{bmatrix} 0 & L_0 \\ -L_1 & 0 \end{bmatrix}$ observe that if $\begin{bmatrix} R \\ S \end{bmatrix}$ is an eigenfunction corresponding to the eigenvalue $\lambda \in \mathbb{C}$, then $-\lambda^2 R = L_0 L_1 R$. If $\lambda \neq 0$, then one concludes that $R$ is orthogonal to $\text{Ker } L_0$, which is the linear span
We record the following formulae for the functional derivatives:

\[-\lambda^2 (R, L_0^{-1}R) = \langle R, L_1 R \rangle ,\]

which implies that \( \lambda^2 \in \mathbb{R} \). Moreover, since we already argued that (3.12) equals zero, one has \( \lambda^2 \leq 0 \), which yields \( \sigma_d(JL) \subset i\mathbb{R} \) and hence the absence of exponentially growing modes at the linearized level. Let us mention that the (nonlinear) orbital stability of the ground state solitary wave solution to the Choquard equation was proved in [9].

**Remark 6.** In view of [11, 8], one expects that the spectral stability or linear instability of small amplitude solitary waves is directly related to the spectral stability or linear instability of the corresponding nonrelativistic limit, which for Dirac–Maxwell is given by the Choquard equation. We hope that this may provide a route to understanding stability of small solitary waves solutions for the Dirac–Maxwell system.

4. **Proof of existence of solitary waves in Dirac–Maxwell system.** In this section, we complete the proof of Theorem 1.1. It is obtained as a consequence of Proposition 2 after the application of a rescaling motivated by the discussion in §3.

We write \( \phi(x, \omega) = \begin{bmatrix} \phi_1(x, \omega) \\ \phi_2(x, \omega) \end{bmatrix} \in \mathbb{C}^4 \), where for \( j = 1, 2 \) the \( \phi_j \in \mathbb{C}^2 \) are the components of \( \phi \) in the range of the projection operators \( \Pi_1 = \frac{1}{2}(1 + \beta) \), and \( \Pi_2 = \frac{1}{2}(1 - \beta) \) (under obvious isomorphisms of these subspaces with \( \mathbb{C}^2 \)). The components \( \phi_1 \) (resp. \( \phi_2 \)) are sometimes referred to as the electronic (resp. positronic) components, although strictly speaking this terminology should only be used after second quantization. Applying \( \Pi_1 \) and \( \Pi_2 \) to (1.6), we have:

\[
\begin{aligned}
\omega \phi_1 &= \sigma \cdot (-i \nabla - qA) \phi_1 + m \phi_1 + qA^0 \phi_1, \\
\omega \phi_2 &= \sigma \cdot (-i \nabla - qA) \phi_2 - m \phi_2 + qA^0 \phi_2, \\
-\Delta A^0 &= q(\phi_1^* \phi_1 + \phi_2^* \phi_2), \\
-\Delta A &= q \phi^* \alpha \phi = q(\phi_1^* \sigma \phi_2 + \phi_2^* \sigma \phi_1).
\end{aligned}
\]

We write (4.3) as

\[
A^0 = qN \ast (\phi_1^* \phi_1 + \phi_2^* \phi_2), \quad A = qN \ast (\phi_1^* \sigma \phi_2 + \phi_2^* \sigma \phi_1),
\]

and regard the potentials \( A^0 \) and \( A = \{A^j\}_{j=1}^3 \) as non-local functionals of \( \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \).

Above,

\[
N(x) = \frac{1}{4\pi|x|}, \quad x \in \mathbb{R}^3 \setminus \{0\},
\]

is the Newtonian potential. In abstract terms, the equations are of the form \( \omega Q' = \mathcal{E}' \) where the charge functional is

\[
Q(\phi) = \int_{\mathbb{R}^3} \phi^*(x) \phi(x) \, dx
\]

(cf. (3.6)), and, regarding \( A^0, A \) as non-local functionals (4.4) of \( \phi \), the Hamiltonian \( \mathcal{E}(\phi) \) is given by

\[
\mathcal{E}(\phi) = \int \left( -i \phi^* \alpha \cdot \nabla \phi + m \phi^* \beta \phi + q \left( A^0 \phi^* \phi - A \cdot (\phi^* \alpha \phi) \right) \right) \, dx.
\]

We record the following formulae for the functional derivatives:

\[
\frac{\delta Q}{\delta \phi(x)} = \phi^*(x), \quad \frac{\delta \mathcal{E}}{\delta \phi(x)} = (\alpha \cdot (-i \nabla - qA) \phi + m \beta \phi + qA^0 \phi^*)^\dagger(x);
\]
\[
\frac{\delta Q}{\delta \phi^*(x)} = \phi(x), \quad \frac{\delta E}{\delta \phi^*(x)} = (\alpha \cdot (-i\nabla - qA)\phi + m\beta\phi + qA^0\phi)(x).
\]

If, say, \(E\) has a directional derivative at \(\phi \in H^1(\mathbb{R}^3; \mathbb{C}^4)\) along the direction \(f \in \mathcal{S}(\mathbb{R}^3; \mathbb{C}^4)\), then
\[
\frac{d}{ds} \bigg|_{s=0} E(\phi + sf) = \langle E'(\phi), f \rangle = \int \left( \frac{\delta E}{\delta \phi(x)} f(x) + f^*(x) \frac{\delta E}{\delta \phi^*(x)} \right) dx. \tag{4.8}
\]

This integral extends to define a bounded linear map on \(L^2(\mathbb{R}^3; \mathbb{C}^4)\) which we continue to write as \(f \mapsto \langle E'(\phi), f \rangle\), and refer to as a directional derivative.

In accordance with the heuristics in \(\S 2\) we introduce functions \(\Phi_1(y, \epsilon), \Phi_2(y, \epsilon) \in \mathbb{C}^2\) and \(A^\mu(y, \epsilon)\) by the following scaling relations:
\[
\phi_1(x, \omega) = \epsilon^3 \Phi_1(\omega x, \epsilon), \quad \phi_2(x, \omega) = \epsilon^2 \Phi_2(\omega x, \epsilon),
\]
\[
A^0(x, \omega) = \epsilon^2 A^0(\omega x, \epsilon), \quad A^j(x, \omega) = \epsilon^3 A^j(\omega x, \epsilon), \tag{4.9}
\]
where \(\epsilon \in (0, m)\) and \(\omega \in (-m, 0)\) are related by \(\omega = -\sqrt{m^2 - \epsilon^2}\). Then, writing \(\nabla_y\) for the gradient with respect to \(y = \omega x\), \(1 \leq j \leq 3\), the system \((4.1)–(4.3)\) can be written as follows:
\[
-2m\Phi_1 + i\sigma \cdot \nabla_y \Phi_2 - \epsilon^2 qA^0\Phi_1 = -(m + \omega)\Phi_1 - \epsilon^2 qA^0\Phi_2, \tag{4.10}
\]
\[
\frac{1}{2m}\Phi_2 + i\sigma \cdot \nabla_y \Phi_1 - \epsilon^2 qA^0\Phi_2 = \left(\frac{1}{2m} - \frac{1}{m - \omega}\right)\Phi_2 - \epsilon^2 qA \cdot \Phi_1, \tag{4.11}
\]
\[
A^0 = qN * (\Phi_2^*\Phi_2 + \epsilon^2 \Phi_1^*\Phi_1), \quad A^j = qN * (\Phi_2^*\Phi_2 + \Phi_1^*\Phi_1). \tag{4.12}
\]

Recall that \(\varphi_0 \in \mathcal{S}(\mathbb{R}^3)\) is the ground state solution to the stationary Choquard equation \((3.4)\) with \(\omega_0 = -\frac{1}{2m} + \frac{1}{2m} \Delta \phi_0 - q^2 (N * \varphi_0^2) \phi_0 = 0. \tag{4.13}\)

That is, \(\varphi_0(y)\) is a strictly positive, spherically symmetric, smooth, strictly monotonically decaying (as a function of \(|y|\)) function of Schwartz class. As discussed in the previous section, such a solution exists by \([20]\). Using \(\varphi_0\), we can produce a solution to \((4.10)–(4.12)\) in the nonrelativistic limit \(\epsilon = 0\):
\[
\hat{\Phi}(y) = \begin{bmatrix} \hat{\Phi}_1(y) \\ \hat{\Phi}_2(y) \end{bmatrix} \in \mathbb{C}^4, \tag{4.14}
\]
with \(\hat{\Phi}_2(y) = \varphi_0(y) \begin{bmatrix} 1 \\ 0 \end{bmatrix}\) and \(\hat{\Phi}_1(y) = \frac{i}{2m} \sigma \cdot \nabla_y \hat{\Phi}_2(y)\):
\[
\hat{A}^0(y) = qN * \varphi_0^2, \quad \hat{A}^1(y) = -\frac{q}{m} N * \varphi_0 \varphi_0^2, \quad \hat{A}^2(y) = +\frac{q}{m} N * \varphi_0 \partial_1 \varphi_0, \quad \hat{A}^3(y) = 0 \tag{4.15}\.
\]

The symmetry of this configuration is axial, with the magnetic field along the z axis of symmetry.

In order to describe the maps \(\Phi \mapsto A^\mu\) precisely, we recall (see e.g. \([25]\)) that a homogeneous polynomial of degree \(n\) which maps \(\Phi \in E\) to \(\mathbb{P}(\Phi) \in F\), from a Banach space \(E\) to a Banach space \(F\), is a mapping of the form \(\mathbb{P}(\Phi) = A(\Phi, \ldots, \Phi)\) where \(A\) is a bounded \(n\)-linear symmetric map \(E \times \cdots \times E \to F\). A polynomial is a

\[^1\text{Recall that } * \text{ is Hermitian conjugate, so for example } f^* \text{ and } \frac{\delta E}{\delta \phi^*(x)} \text{ are, respectively, row and column vectors pointwise, so that the integrand is a scalar.}\]
Lemma 4.1. 1. Let \( \Phi = \left[ \Phi_1 \Phi_2 \right] \in H^1(\mathbb{R}^3, \mathbb{C}^4) \). Then \( A^\mu \) defined by (4.12) satisfy \( A^\mu \in L^\infty(\mathbb{R}^3) \), \( 0 \leq \mu \leq 3 \). Furthermore the mappings \( \Phi \mapsto A \) are degree 2 polynomial mappings \( H^1(\mathbb{R}^3, \mathbb{C}^4) \to L^\infty(\mathbb{R}^3) \), and similarly \( (\Phi, \epsilon) \mapsto A^0 \) is a polynomial mapping \( H^1(\mathbb{R}^3, \mathbb{C}^4) \times \mathbb{R} \to L^\infty(\mathbb{R}^3) \).

2. The formulae (4.12) also define mappings

\[
H^1(\mathbb{R}^3, \mathbb{C}^4) \to \dot{H}^1(\mathbb{R}^3), \quad \Phi \mapsto A
\]
and

\[
H^1(\mathbb{R}^3, \mathbb{C}^4) \times \mathbb{R} \to \dot{H}^1(\mathbb{R}^3), \quad (\Phi, \epsilon) \mapsto A^0
\]
which are polynomial mappings into the homogeneous Dirichlet space \( \dot{H}^1(\mathbb{R}^3) \).

3. Let \( \Phi = \left[ \Phi_1 \Phi_2 \right] \in H^2(\mathbb{R}^3, \mathbb{C}^4) \). Differentiation of (4.12) gives mappings \( \Phi \mapsto \nabla A \) and \( (\Phi, \epsilon) \mapsto \nabla A^0 \) which are polynomial mappings

\[
H^2(\mathbb{R}^3, \mathbb{C}^4) \to L^\infty(\mathbb{R}^3) \quad \text{and} \quad H^2(\mathbb{R}^3, \mathbb{C}^4) \times \mathbb{R} \to L^\infty(\mathbb{R}^3),
\]
respectively.

Proof. (1) The functions \( A^\mu \) defined by (4.12) are of the form \( N * h \) with \( h := fg \), where \( f, g \in H^1(\mathbb{R}^3) \). Due to the Sobolev embedding \( H^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3) \), the mapping \( (f, g) \mapsto h = fg \) is a continuous bilinear map \( H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \to L^1(\mathbb{R}^3) \cap L^6(\mathbb{R}^3) \). Also \( N * h = (\chi_{B_1} N) * h + (1 - \chi_{B_1}) N * h \) where \( B_1 \) is the unit ball in \( \mathbb{R}^3 \) and \( \chi_{B_1} \) is its characteristic function. It follows from the Hölder inequality that

\[
\|N * h\|_{L^\infty} \leq \|(\chi_{B_1} N)\|_{L^6} \|h\|_{L^3} + \|(1 - \chi_{B_1}) N\|_{L^\infty} \|h\|_{L^3};
\]

so that the mapping \( L^1 \cap L^3 \ni h \mapsto N * h \in L^\infty \) is a continuous linear map. It follows that the composition \( (f, g) \mapsto N * (fg) \) is a polynomial mapping \( H^1 \times H^1 \to L^\infty \).

To prove (2), we recall that by the Riesz representation theorem the linear operator \( (-\Delta)^{-1} = N \) is bounded \( L^6(\mathbb{R}^3) \to H^1(\mathbb{R}^3) \) since \( L^6(\mathbb{R}^3) \subset (H^1)' \). The result therefore follows from the fact that (continuing with the same notation) the mapping \( (f, g) \mapsto h = fg \) is a continuous bilinear map

\[
H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \to L^1(\mathbb{R}^3) \cap L^6(\mathbb{R}^3) \subset L^6(\mathbb{R}^3).
\]

The statement (3) is proved by noting that a similar structure holds for the differentiated versions of formulae (4.12) by the Leibniz rule, and so the same proof works. \( \square \)

Let

\[
X = H^2(\mathbb{R}^3, \mathbb{C}^2) \oplus H^2(\mathbb{R}^3, \mathbb{C}^2), \quad Y = H^1(\mathbb{R}^3, \mathbb{C}^2) \oplus H^1(\mathbb{R}^3, \mathbb{C}^2), \quad \text{(4.16)}
\]
and define the corresponding exponentially weighted spaces, using the norms introduced in Corollary 1:

\[
\begin{aligned}
X^\theta &= H^{2, \theta}(\mathbb{R}^3, \mathbb{C}^2) \oplus H^{2, \theta}(\mathbb{R}^3, \mathbb{C}^2), \\
Y^\theta &= H^{1, \theta}(\mathbb{R}^3, \mathbb{C}^2) \oplus H^{1, \theta}(\mathbb{R}^3, \mathbb{C}^2),
\end{aligned} \quad \theta \geq 0. \quad \text{(4.17)}
\]
The case \( \theta = 0 \) reduces to the standard Sobolev norms.
Introducing the notation
\[ P = -i\sigma \cdot \nabla_y, \tag{4.18} \]
we rewrite (4.10), (4.11) as the equation \( F = 0 \), where (for small nonnegative \( \theta \))
\[ F : X^\theta \times (-m, +m) \rightarrow Y^\theta, \tag{4.19} \]
\[ F : (\Phi, \epsilon) \mapsto \begin{bmatrix} 2m\Phi_1 + P\Phi_2 + \epsilon^2 q\Lambda^0\Phi_1 - (m + \omega)\Phi_1 - \epsilon^2 q\Lambda \cdot \sigma \Phi_2 \\ -\frac{1}{2m}\Phi_2 + P\Phi_1 + q\Lambda^0\Phi_2 + \left( \frac{1}{2m} - \frac{1}{m - \omega} \right)\Phi_2 - \epsilon^2 q\Lambda \cdot \sigma \Phi_1 \end{bmatrix}. \]

Above, \( \omega = -\sqrt{m^2 - \epsilon^2} \). As before, we regard the \( \Lambda^\mu = (\Lambda^0, \Lambda) \), \( \Lambda = \{ \Lambda^j \}_{j=1}^3 \), as non-local functionals \( \Lambda^\mu = \Lambda^\mu(\Phi, \epsilon) \) determined by (4.12). With this understood, the entire system (4.1)–(4.3) is encapsulated in the equation \( F(\Phi, \epsilon) = 0 \) for \( \Phi = [\Phi_1 \Phi_2] \) only. We note that in terms of the functionals \( Q \) and \( E \) defined by (4.6), (4.7), one has
\[ F(\Phi, \epsilon) = \begin{bmatrix} \epsilon^{-3} & 0 \\ 0 & \epsilon^{-4} \end{bmatrix} (E' - \omega Q') \begin{bmatrix} \epsilon^3 \Phi_1 \\ \epsilon^2 \Phi_2 \end{bmatrix}. \]

The nonrelativistic limit \( \hat{\Phi} \) satisfies \( F(\hat{\Phi}, 0) = 0 \) (cf. (4.14), (4.15)), so that to obtain solutions for small \( \epsilon \) by the implicit function theorem it is necessary to compute the derivative of \( F(\Phi, \epsilon) \) at the point \( (\Phi, 0) \). This is determined by the set of directional derivatives. Define \( e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), and let \( g \in H^1(\mathbb{R}^3, \mathbb{C}^2) \). To compute the directional derivatives, first note that \( \Lambda^j \) drops out on putting \( \epsilon = 0 \), and then note further that by (4.12) only the derivative of \( \Lambda^0 \) at \( (\Phi, \epsilon) = (\hat{\Phi}, 0) \) with respect to \( \Phi_2 \) is nonzero, with derivative given by
\[ \frac{d}{dt} \Lambda^0 \left( \begin{bmatrix} \hat{\Phi}_1 \\ \hat{\Phi}_2 + tg \end{bmatrix}, \epsilon \right)_{t=0, \epsilon=0} = 2qN * (\varphi_0 \text{Re}(e_1, g)\cdot \epsilon_c^2), \]
with the Newtonian potential from (4.5), where \( (v, w)_{\mathbb{C}^2} = \bar{v}_1 w_1 + \bar{v}_2 w_2 \) is the complex sesquilinear inner product of \( v, w \in \mathbb{C}^2 \). We deduce that for \( \mathbb{C}^2 \)-valued Schwartz functions \( U \) and \( V \),
\[ \frac{d}{dt} F \left( \begin{bmatrix} \hat{\Phi}_1 + tU \\ \hat{\Phi}_2 + tV \end{bmatrix}, \epsilon \right)_{t=0, \epsilon=0} = M \begin{bmatrix} U \\ V \end{bmatrix}, \]
where
\[ M = \begin{bmatrix} 2m \\ -\frac{1}{2m} + q\Lambda^0 + 2q^2\varphi_0 e_1 N * (\varphi_0 \text{Re}(e_1, \cdot)\cdot \epsilon_c^2) \end{bmatrix}. \tag{4.20} \]
and \( P = -i\sigma \cdot \nabla_y \) was introduced in (4.18). Thus the derivative of \( F \) at the non-relativistic limit point \( (\hat{\Phi}, 0) \) is the linear map \( D F(\hat{\Phi}, 0) \) given by the matrix \( M \). This is a differential operator, which we consider as an unbounded operator on \( L^2(\mathbb{R}^3; \mathbb{C}^2) \oplus L^2(\mathbb{R}^3; \mathbb{C}^2) \).

**Lemma 4.2.**
1. The map \( M : \begin{bmatrix} U \\ V \end{bmatrix} \mapsto \begin{bmatrix} F \end{bmatrix} \) is a Hermitian operator with domain \( X \) (cf. (4.16)).
2. For small nonnegative \( \theta \), the mapping \( M \) is continuous from \( X^\theta \) into \( Y^\theta \) (cf. (4.17)).
3. The kernel of $M$ is given by
\[
\text{Ker } M = \left\{ \left( -\frac{P V}{2m}, V \right) : V = (a \cdot \nabla \varphi_0 + ib \varphi_0) e_1 + c \varphi_0 e_2, \ (a, b, c) \in \mathbb{R}^3 \times \mathbb{R} \times \mathbb{C} \right\}.
\]

4. The range of $M : \begin{bmatrix} U \\ V \end{bmatrix} \mapsto \begin{bmatrix} F \\ G \end{bmatrix}$ is closed in the topology of $Y$ and is given by
\[
\text{Range } M = (\text{Ker } M)^\perp = \left\{ \begin{bmatrix} F \\ G \end{bmatrix} \in Y : \text{Re} \left( \frac{P F}{2m} - G \right)_1 \in (\text{Ker } L_1)^\perp, \right. \\
\left. \text{Im} \left( \frac{P F}{2m} - G \right)_1 \in (\text{Ker } L_0)^\perp, \right. \\
\left. \left( \frac{P F}{2m} - G \right)_2 \in (\text{Ker } L_0)^\perp \right\},
\]
where $\perp$ is the orthogonal complement with respect to the inner product in $L^2 \oplus L^2$.

5. The inverse of $M : \begin{bmatrix} U \\ V \end{bmatrix} \mapsto \begin{bmatrix} F \\ G \end{bmatrix}$ is given by
\[
U = \frac{1}{2m}(F - PV), \\
V = e_1 V_1 + e_2 V_2 \\
= \left( L^{-1}_1 \text{Re} \left( \frac{P F}{2m} - G \right)_1 + iL^{-1}_0 \text{Im} \left( \frac{P F}{2m} - G \right)_1 \right) e_1 + L^{-1}_0 \left( \frac{P F}{2m} - G \right)_2 e_2.
\]
where the definitions and properties of the operators $L_0, L_1$ are given in §3 (cf. (3.9)).

**Proof.** The proof depends on some properties of the linearized Choquard equation from [19] which are stated in §3. The fact in (1) that $M$ is Hermitian follows from the fact that $P$ is Hermitian. From Lemma 4.1 the assertion (2) is immediate from the properties of $N$ and the fact that $\varphi_0$ and its partial derivatives are smooth and exponentially decreasing. To prove (3), (4), and (5), we consider how to solve $M \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} F \\ G \end{bmatrix}$, i.e. the system
\[
M \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} 2mU + PV \\ PU - \frac{V}{2m} + qA^0 V + 2q^2 \varphi_0 e_1 N * (\varphi_0 \text{Re } V_1) \end{bmatrix} = \begin{bmatrix} F \\ G \end{bmatrix}.
\]
We first express $U$ in terms of $V$ by $U = \frac{1}{2m}(F - PV)$, and, writing $V = V_1 e_1 + V_2 e_2$,
\[
\frac{P F}{2m} + \frac{\Delta V}{2m} - \frac{V}{2m} + qA^0 V + 2q^2 \varphi_0 e_1 N * (\varphi_0 \text{Re } V_1) = G.
\]
Referring to the definitions of $L_0$ and $L_1$ in §3 (cf. (3.9)), with $\omega_0$ set equal to $-1/(2m)$, we arrive at the following equations:
\[
L_1 V_1 = \left( \frac{P F}{2m} - G \right)_1, \quad L_0 V_2 = \left( \frac{P F}{2m} - G \right)_2.
\]
(4.21)
It is useful here that the components with respect to $e_1$ and $e_2$ are decoupled. The identification of the kernel in (3) is then a specialization of this, given the information on $\text{Ker } L_0$ and $\text{Ker } L_1$ in §3, and also (4) is a consequence of the identification of the ranges of $L_0$ and $L_1$ given in §3 (cf. Lemma 3.2). \[\square\]
The existence statement in Theorem 1.1 now almost follows from using the implicit function theorem to solve \( F = 0 \). In order to handle the degeneracies arising from symmetries we use the following trick from [29], which we state as a lemma applying to functionals \( \mathcal{E} \) and \( Q \) defined on a general real Hilbert space \( H \). In the present paper the relevant choice is \( H = L^2(\mathbb{R}^3; \mathbb{C}^4) \), with the real \( L^2 \) inner product

\[
\langle \phi, \psi \rangle_{L^2} = \text{Re} \int_{\mathbb{R}^3} \phi^*(x) \psi(x) \, dx. \tag{4.22}
\]

**Lemma 4.3.** Let \( \{\xi^\alpha\}_{\alpha \in I} \) be a finite collection of elements of a real Hilbert space \( H \), indexed by \( I \), all lying in some subspace \( F \subset H \) with the property that \( \mathcal{E} \) and \( Q \) are differentiable along each direction \( f \in F \) with directional derivatives \( \langle Q', f \rangle \) and \( \langle \mathcal{E}', f \rangle \) for \( f \in F \). Assume further that the \( \{\xi^\alpha\} \) correspond to infinitesimal symmetries, in the sense that \( \langle Q', \xi^\alpha \rangle = 0 = \langle \mathcal{E}', \xi^\alpha \rangle \) for all \( \alpha \in I \). Let \( \phi \) satisfy

\[
\omega Q' - \mathcal{E}' - \sum_{\alpha \in I} a_\alpha \xi^\alpha = 0, \tag{4.23}
\]

for some set of numbers \( a_\alpha \in \mathbb{R} \). Then \( a_\alpha = 0 \forall \alpha \in I \) as long as the matrix \( \langle \xi^\alpha, \xi^\beta \rangle \) is nondegenerate.

**Proof.** Put \( f = \xi^\beta \) and make use of the assumptions, then \( \sum_{\alpha \in I} a_\alpha \langle \xi^\alpha, \xi^\beta \rangle = 0 \), which implies \( a_\alpha = 0 \forall \alpha \in I \) by the nondegeneracy of the matrix \( \langle \xi^\alpha, \xi^\beta \rangle \). \( \square \)

**Remark 7.** It follows from the proof that instead of (4.23) it is sufficient to assume that

\[
\left\langle \omega Q'(\phi) - \mathcal{E}'(\phi) - \sum_{\alpha \in I} a_\alpha \xi^\alpha, f \right\rangle = 0, \quad \forall f \in F.
\]

**Example.** For a simple example consider \( \psi : \mathbb{R} \rightarrow \mathbb{C} \) and \( Q = \frac{1}{2} \int |\nabla \psi|^2 + \frac{1}{p+1} |\psi|^{p+1} \, dx \) the symmetry of phase rotation corresponds to the infinitesimal symmetry \( \xi(\psi) = i\psi \), and it is easy to check that given an \( H^1 \) distributional solution of \( \omega Q' - \mathcal{E}' - a_0 \xi = 0 \), i.e. a weak solution of \( -\Delta \psi - |\psi|^p \psi = \omega \psi - i a_0 \psi \), for any \( a_0 \in \mathbb{R} \), one necessarily has \( a = 0 \). The same holds in higher dimensions as long as \( p \) is such that the equation holds as an equality in \( H^{-1} \).

**Remark 8.** The advantage of solving a more general equation with the unknown “multipliers” \( a_\alpha \) is that in an implicit function theorem setting, the multipliers can be varied to fill out the part of the cokernel corresponding to the symmetries. It is then shown after the fact that the multipliers are equal to zero. The choice of \( \xi^\alpha \) is determined by the symmetry group; in the case of Dirac–Maxwell the relevant group is the seven-dimensional group generated by translations, rotations and phase rotation. Thus the index set is \( \alpha \in \{1, \ldots, 7\} \) with the corresponding multipliers \( a_\alpha \) written in order as \( (a, b, a_0) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \). The infinitesimal versions of these actions give the following vector fields on the phase space \( H^1(\mathbb{R}^3; \mathbb{C}^4) \) ([5] or [28, §3.4]):

\[
\xi = \nabla \phi, \quad \eta = l \phi + \frac{i}{2} \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} \phi, \quad \zeta = i \phi, \tag{4.24}
\]

where \( l = \{\epsilon_{ijk} x_j \partial_k\}_{i=1}^3 \) is the standard angular momentum generator. The Lorentz invariance of the Dirac construction ensures that

\[
\begin{bmatrix} l + \frac{i}{2} \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}, -i \alpha \cdot \nabla \end{bmatrix} = 0. \tag{4.25}
\]
For example, let \( \phi \in H^{2,\theta}(\mathbb{R}^3; \mathbb{C}^4) \) for some \( \theta > 0 \); then, since the Hamiltonian density, i.e. the integrand in (4.7), is a scalar with respect to Euclidean transformations, we have

\[
\int_{\mathbb{R}^3} \left( -i\phi^* \alpha \nabla \tilde{\phi} + m\phi^* \beta \phi + \frac{q}{2} \left( \tilde{A}^0 \phi^* \tilde{\phi} - \tilde{A} \cdot (\phi^* \alpha \tilde{\phi}) \right) \right) dx
= \int_{\mathbb{R}^3} \left( -i\phi^* \alpha \nabla \phi + m\phi^* \beta \phi + \frac{q}{2} \left( A^0 \phi^* \phi - A \cdot (\phi^* \alpha \phi) \right) \right) dx,
\]

where \( \tilde{\phi}, \tilde{A}^\mu \) are obtained by the action of a spatial rotation on \( \phi, A^\mu \). Differentiation of this integral identity with respect to the parameter of rotation \( \eta = \{\eta_j\}_{j=1}^3 \) and use of (4.25) leads to

\[
\langle \mathcal{E}'(\phi), \eta_j \rangle = \int_{\mathbb{R}^3} \left( \eta_j^* (x) \frac{\delta \mathcal{E}}{\delta \phi^*(x)} + \frac{\delta \mathcal{E}}{\delta \phi(x)} \eta_j (x) \right) dx = 0, \quad 1 \leq j \leq 3. \tag{4.26}
\]

The same is true for translations and phase rotations (i.e., the case of \( \xi \) and \( \zeta \), respectively, in place of \( \eta \)). We call vector fields on the phase space such as \( \eta \) generalized infinitesimal symmetries if they are locally square integrable and satisfy (4.26) and \( \langle Q', \eta \rangle = 0 \) when \( \phi \in H^{1,\theta}(\mathbb{R}^3, \mathbb{C}^4) \) for some nonnegative \( \theta \).

**Example.** As an example of Lemma 4.3 for the case at hand, with \( \mathcal{E}, Q \) as in (4.6) and (4.7), assume that \( \phi \in H^1(\mathbb{R}^3; \mathbb{C}^4) \) is such that

\[
\omega Q'(\phi) - \mathcal{E}'(\phi) - a \cdot \xi + ia_0 \zeta = 0 \quad (\text{in } L^2),
\]

with \( (a_0, a) \in \mathbb{R} \times \mathbb{R}^3 \), and \( \xi, \zeta \) as in (4.24). Then in fact

\[
\omega Q'(\phi) - \mathcal{E}'(\phi) = 0 \quad (\text{in } L^2).
\]

In order to treat the rotational symmetry \( \eta \) a technical modification is needed on account of the linear growth at infinity of the coefficient in the angular momentum vector field \( I = \{\epsilon_{ijk} x_i \partial_3 \}_{i=1}^3 \), which potentially means that \( \eta \) might not be square integrable. The most efficient way to circumvent this issue seems to be to work in the exponentially weighted spaces \( H^{s,\theta} \) defined above. The following lemma, which is proved in exactly the same way as Lemma 4.3, gives a slightly more general setting than needed.

**Lemma 4.4.** Let \( \phi \in H^{2,\theta}(\mathbb{R}^3; \mathbb{C}^4) \) for some \( \theta \geq 0 \), and assume there is a finite set \( \{\xi^\alpha\}_{\alpha \in I} \) of generalized infinitesimal symmetries, in the sense of Remark 8, which all lie in some subspace \( F \subset L^2_{\text{loc}}(\mathbb{R}^3; \mathbb{C}^4) \). Assume that \( \phi \) satisfies

\[
\omega Q'(\phi) - \mathcal{E}'(\phi) - \sum_{\alpha \in I} a_\alpha \xi^\alpha (x) = 0
\]

for some set of numbers \( a_\alpha \in \mathbb{R} \), and for some finite set \( \{\xi'^\alpha\}_{\alpha \in I} \) of elements of \( F' \), the dual space of \( F \). If the matrix with entries \( \langle \xi'^\alpha, \xi^\beta \rangle_{L^2} \), computed using the inner product (4.22), is nondegenerate then \( a_0 = 0 \) \( \forall \alpha \in I \).

In the case at hand, under the assumption \( \phi \in H^{2,\theta}(\mathbb{R}^3; \mathbb{C}^4) \) for some \( \theta > 0 \), all the vector fields in (4.24) are actually square integrable, but it is nevertheless necessary to introduce a spatial cut-off into the definition of the \( \xi'^\alpha \) for which the nondegeneracy assumption holds, see below. We are looking for \( \Phi(\epsilon) \) in the form

\[
\Phi(\epsilon) = \hat{\Phi} + \Psi(\epsilon), \quad \Psi(0) = 0. \tag{4.27}
\]
We use the same component notation as above: \( \hat{\Phi} = \begin{bmatrix} \hat{\Phi}_1 \\ \hat{\Phi}_2 \end{bmatrix} \in \mathbb{C}^4, \Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \in \mathbb{C}^4 \). To make use of Lemma 4.4 we will apply the implicit function theorem to the function

\[
\mathcal{G}_R(\Psi, a, b, \epsilon) = \mathcal{F}(\hat{\Phi} + \Psi, \epsilon) + \chi_R a \nabla_y \left[ \epsilon (\hat{\Phi}_1 + \Psi_1) \right]_y + \chi_R b \cdot \left( \left[ i \tilde{L} (\hat{\Phi}_1 + \Psi_1) + \frac{i}{2} \sigma (\hat{\Phi}_1 + \Psi_1) \right] \right).
\]

Here \( \chi_R(\cdot) = \chi(\cdot/R) \), where \( R \geq 1 \) and \( \chi \in C_0^\infty(\mathbb{R}^3) \) is a radially symmetric function which satisfies \( \chi(y) = 1 \) for \( |y| \leq 1 \) and \( \chi(y) = 0 \) for \( |y| > 2 \).

**Remark 9.** Referring to Remark 8, we have introduced a linear combination of the six infinitesimal symmetries corresponding to translation and rotation, but with a spatial cut-off enforced by multiplication by \( \chi_R \), replacing \( \xi, \eta \) by

\[
\xi_R = \chi_R \nabla \phi, \quad \eta_R = \chi_R \left( I + \frac{i}{2} \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} \right) \phi, \quad R \geq 1,
\]

respectively. (It is not necessary to also introduce a multiplier for phase rotation due to the presence of infinitesimal rotation around \( x_3 \)-axis). In terms of the original variables (cf. (4.24)):

\[
\mathcal{G}_R(\Psi, a, b, \epsilon) = \begin{bmatrix} \epsilon^{-3} & 0 \\ 0 & \epsilon^{-4} \end{bmatrix} \left( \mathcal{F}' - \omega Q' + c a \cdot \xi_R + \epsilon^2 b \cdot \eta_R \right).
\]

(29)

The idea is to solve \( \mathcal{G}_R = 0 \) for some fixed large \( R \gg 1 \) and then to show that this actually gives solutions to \( \mathcal{F} = 0 \) for \( \epsilon \) sufficiently small. The spatial cut-off ensures that \( \mathcal{G}_R \) is well-behaved on the Sobolev spaces \( H^{\theta, 0} \).

**Proposition 2.** There is \( \epsilon_*>0 \) such that for \( \epsilon \in (-\epsilon_*, \epsilon_*) \) there is a solution to (4.1)–(4.3), with \( \omega = -\sqrt{m^2 - \epsilon^2} \), given by the Ansatz (4.9) with \( \Phi(\epsilon) = \hat{\Phi} + \Psi(\epsilon) \) obtained from a \( C^{\infty} \)-function

\[
\Psi \in C^{\infty}((-\epsilon_*, \epsilon_*) \cap \mathbb{R}^3; C^4) \cap \text{Ker} \ M^\perp
\]

for small positive \( \theta \), and satisfying \( \Psi(0) = 0 \), and \( A^0 \in C^{\infty}((-\epsilon_*, \epsilon_*) \cap \mathbb{R}^3; \dot{H}^1 \cap L^\infty) \), \( A^j \in C^{\infty}((-\epsilon_*, \epsilon_*) \cap \mathbb{R}^3; \dot{H}^1 \cap L^\infty) \) given by (4.12):

\[
A^0 = \rho N * (\hat{\Phi}_1 \hat{\Phi}_1 + \epsilon^2 \hat{\Phi}_2 \hat{\Phi}_2), \quad A^j = \rho N * (\hat{\Phi}_1 \sigma \hat{\Phi}_2 + \hat{\Phi}_2 \sigma \hat{\Phi}_1),
\]

Above, \( \dot{H}^1 = \dot{H}^1(\mathbb{R}^3, \mathbb{R}) \) is the homogeneous Dirichlet space of \( L^6 \) functions with

\[
\| f \|_{H^1}^2 := \int_{\mathbb{R}^3} |\nabla f|^2 \, dx < \infty.
\]

One has

\[
\| \Phi(\epsilon) - \hat{\Phi} \|_{H^2} = O(\epsilon^2), \quad \epsilon \in (-\epsilon_*, \epsilon_*).
\]

(30)

The functions \( \Phi_1(y, \epsilon), \Phi_2(y, \epsilon) \) are even in \( \epsilon \).
Proof. The proof of existence of solutions to (4.1)–(4.3) is by the implicit function theorem and Lemma 4.3, perturbing from the nonrelativistic limit point \( F(\hat{\Phi}, 0) = 0 \).

To start, we claim that \( F \), as defined in (4.19), is a \( C^\infty \)-function
\[
F : X^\theta \times (-m, +m) \to Y^\theta, \quad \theta \geq 0.
\]

To prove this, we notice that the expression for \( F \) is manifestly smooth in \( \epsilon \) for \( \epsilon^2 < m^2 \), and its dependence on \( \Phi_j \) is built up from compositions of certain multilinear maps and linear operators; the structure of the expressions obtained after successive differentiation is the same. Referring to the specific formulae, the fact that these expressions are all \( C^\infty \) is an immediate consequence of Lemma 4.1 and the fact that multiplication gives continuous bilinear (and hence smooth) maps \( H^{1,\theta} \times H^{2,\theta} \to H^{1,\theta} \) and \( H^{2,\theta} \times H^{2,\theta} \to H^{2,\theta} \) (Moser inequalities) for \( \theta \geq 0 \). For example, consider the term
\[
A^0 \Phi_2 = qN * \left( \Phi_2^* \Phi_2 + \epsilon^2 \Phi_1^* \Phi_1 \right) \Phi_2,
\]
for \( \Phi_1, \Phi_2 \in H^2(\mathbb{R}^3, \mathbb{C}^2) \). By Lemma 4.1, both \( A^0 \) and \( \nabla A^0 \) are bounded in \( L^\infty \), and consequently since \( \Phi_2 \in H^{2,\theta} \), the product rule implies that \( A^0 \Phi_2 \) is bounded in \( H^{1,\theta} \). On the other hand, the mapping (4.31) is cubic and can be expressed in an obvious way as a composition of the embedding
\[
H^{2,\theta} \times (-m, +m) \to H^{2,\theta} \times H^{2,\theta} \times (-m, +m) \times (-m, +m),
\]
with a mapping into \( H^{1,\theta} \) which is both multilinear and bounded (by identical reasoning to that in the previous sentence). The composition is therefore smooth by the chain rule. Analogous reasoning for the other terms shows that \( F \) defines a smooth mapping \( X^\theta \times (-m, +m) \to Y^\theta \) as required.

Computing the derivatives of (4.28) at \( \epsilon = 0, \Psi = 0 \) and using the spherical symmetry of the ground state solution of (4.13), we see that the functions \( \{\partial_{\theta_j} G_R, \partial_{\phi^j} G_R; 1 \leq j \leq 3\} \) converge strongly in \( L^2(dy) \) as \( R \to +\infty \) to the basis for \( \text{Ker} \, M \) given in Lemma 4.2. This establishes that if \( R \) is sufficiently large (depending only on \( \phi_0 \)), then the derivative of \( G_R \) at \( \epsilon = 0, \Psi = 0, a = 0, b = 0 \) with respect to \( (\Psi, a, b) \) is a linear homeomorphism from \( ((\text{Ker} M)^\perp \cap X^\theta) \times \mathbb{R}^3 \times \mathbb{R}^3 \) onto \( Y^\theta \) for small positive \( \theta \).

It follows that for such \( R \) there is \( \epsilon_\ast > 0 \) such that there exist \( C^\infty \)-functions \( \epsilon \mapsto (\Psi(\epsilon), a(\epsilon), b(\epsilon)) \in X \times \mathbb{R}^3 \times \mathbb{R}^3 \), defined for \( \epsilon \in (-\epsilon_\ast, \epsilon_\ast) \), such that
\[
G_R(\Psi(\epsilon), a(\epsilon), b(\epsilon), \epsilon) = 0, \quad \Psi(\epsilon) \perp \text{Ker} \, M, \quad \epsilon \in (-\epsilon_\ast, \epsilon_\ast).
\]
(3.22)

(This latter condition serves to divide out by the action of the symmetry group, giving a local slice.) Referring to Lemma 4.4, to deduce that these in fact generate solutions of \( F = 0 \) for sufficiently small \( \epsilon > 0 \), it is sufficient to verify that \( a(\epsilon) = 0, b(\epsilon) = 0 \), which is in turn a consequence of the nondegeneracy of the appropriate matrix of inner products, scaled as above. This amounts to the need to verify nondegeneracy of the \( 6 \times 6 \) matrix
\[
\begin{bmatrix}
\langle \partial_{\phi^j} \phi, \chi_R \partial_{\phi^k} \phi \rangle & \langle \partial_{\phi^j} \phi, \chi_R (l_{k'} + \frac{i}{2} \Sigma_{k'}) \phi \rangle \\
\langle (l_{j'} + \frac{i}{2} \Sigma_{j'}) \phi, \chi_R \partial_{\phi^k} \phi \rangle & \langle (l_{j'} + \frac{i}{2} \Sigma_{j'}) \phi, \chi_R (l_{k'} + \frac{i}{2} \Sigma_{k'}) \phi \rangle
\end{bmatrix}
\]
for small \( \epsilon \). (In the matrix (4.33) the indices \( j, j', k, k' \) run between 1 and 3.)
Lemma 4.5. For fixed $R$ chosen sufficiently large, the matrix given by (4.33),
evaluated at $\phi(x, \omega) = \left[ \begin{array}{c} e^2(\hat{P}_1 + \Psi_1) \\ e^2(\hat{P}_2 + \Psi_2) \end{array} \right]_{y=\epsilon x}$, $\omega = -\sqrt{m^2 - \epsilon^2}$, is nondegenerate for sufficiently small $\epsilon > 0$.

Proof. Clearly the dominant terms arise from the second (“large”) component $e^2(\hat{P}_2 + \Psi_2)$, giving rise to diagonal matrix elements which, referring to the block form in (4.33), are $O(\epsilon^4)$. Using $\|\Psi_j\|_{H^2} = O(\epsilon)$, we will deduce the result from nondegeneracy of the matrix with $\Psi_j$ set equal to zero and $R = +\infty$. To start with, using

$$
\epsilon^{-2} \phi = \left[ \begin{array}{c} \epsilon \hat{P}_2 \\ \Phi_2 \end{array} \right] + \left[ \begin{array}{c} \epsilon \Psi_1 \\ \Psi_2 \end{array} \right] \quad \text{and} \quad \Phi_2 = \left[ \begin{array}{c} \phi_0 \\ 0 \end{array} \right],
$$

we calculate the first diagonal term:

$$
e^{-4} \left( \partial_{y^i} \phi, \chi_R \partial_{y^k} \phi \right)_{L^2} = e^2 \left( \partial_{y^i} \phi, \left( -\frac{\Delta_y}{4m^2} \right) \partial_{y^k} \phi \right)_{L^2} + \left( \partial_{y^i} \phi, \partial_{y^k} \phi \right)_{L^2} + O(\epsilon) + o(1)
$$

$$
= \frac{\delta_{kk}}{3} \left( \varphi_0, (-\Delta_y) \varphi_0 \right)_{L^2} + O(\epsilon) + o(1),
$$

where we took into account the spherical symmetry of $\varphi_0$, which leads to

$$
\left( \partial_{y^i} \varphi_0, \partial_{y^i} \varphi_0 \right)_{L^2} = \frac{1}{3} \left( \varphi_0, (-\Delta_y) \varphi_0 \right)_{L^2}.
$$

The notation $o(1)$ indicates the error term which is independent of $\epsilon$ and has limit zero as $R \to +\infty$, and arises from the limit of convergent integrals such as

$$
\left( \partial_{y^i} \varphi_0, \chi_R \partial_{y^k} \varphi_0 \right)_{L^2} = \left( \partial_{y^i} \varphi_0, \partial_{y^k} \varphi_0 \right)_{L^2} + o(1).
$$

Next the off-diagonal terms are $O(\epsilon R) + o(1)$; indeed, using the same expression for $\epsilon^{-2} \phi$ as above, we compute:

$$
e^{-4} \left( \partial_{y^i} \phi, \chi_R (l_{k'}^j + \frac{i}{2} \Sigma_{k'}) \phi \right)_{L^2} = -\frac{e^2}{4m^2} \left( \partial_{y^i} \Phi, \left[ \begin{array}{c} \varphi_0 \\ 0 \end{array} \right] \right)_{L^2} + \left( \partial_{y^i} \left[ \begin{array}{c} \varphi_0 \\ 0 \end{array} \right] \right)_{L^2} + O(\epsilon R) + o(1).
$$

The first two terms are actually identically zero since $\varphi_0$ is spherically symmetric (so that by parity considerations it is $L^2$-orthogonal to all of its first partial derivatives, which are in turn orthogonal to all of the second partial derivatives). The $O(\epsilon R)$ error term arises from the bound $\|\chi_R \Psi_j\|_{L^2} \leq \text{const} \|\Psi_j\|_{H^2}$, etc.

Finally, for the second diagonal term:

$$
e^{-4} \left( l_{j^j} + \frac{i}{2} \Sigma_{j^j} \phi, \chi_R (l_{k'} + \frac{i}{2} \Sigma_{k'}) \phi \right)_{L^2} = \frac{\delta_{jj}}{4} \langle \varphi_0, \varphi_0 \rangle_{L^2} + O(\epsilon R^2) + o(1).
$$

(Recall that $\varphi_0$ is radial so that $l_j \varphi_0 = 0$ for each $j$.) The nondegeneracy of the matrix (4.33) for large fixed $R$ (again depending only on $\varphi_0$) and sufficiently small $\epsilon$ follows.

Returning to the proof of Proposition 2, the above implies that if $R$ is fixed sufficiently large then there is an interval $(-\epsilon_*, \epsilon_*)$ on which there is a solution
\( \Phi(y, \epsilon) \) of \( \mathcal{F}(\Phi, \epsilon) = 0 \). Now the implicit function theorem proves that this solution is \( C^\infty \) as a function of \( \epsilon \in (-\epsilon_*, \epsilon_*), \) and so

\[
\|\Phi(\epsilon) - \hat{\Phi}\|_{H^2} = O(\epsilon). \tag{4.34}
\]

To prove a stronger estimate (4.30), we take the derivative of (4.19) with respect to \( \epsilon \) at \( \epsilon = 0 \); this yields

\[
M \partial_\epsilon \hat{\Phi}_{|\epsilon=0} = 0,
\]

with \( M \) given by (4.20). Due to (4.27), one has \( \partial_\epsilon \hat{\Phi}_{|\epsilon=0} = \partial_\epsilon \hat{\Psi}_{|\epsilon=0} \); the requirement (4.32) leads to \( \partial_\epsilon \hat{\Phi}_{|\epsilon=0} = \partial_\epsilon \hat{\Psi}_{|\epsilon=0} = 0, \) and hence

\[
\|\Phi(\epsilon) - \hat{\Phi}\|_{H^2} = O(\epsilon^2).
\]

Finally, notice that since the explicit dependence of \( \mathcal{F} \) is on \( \epsilon^2 \), we have

\[
\mathcal{F}(\Phi_1(\epsilon), \Phi_2(\epsilon), -\epsilon) = \mathcal{F}(\Phi_1(\epsilon), \Phi_2(-\epsilon), +\epsilon) = 0 \tag{4.35}
\]

and hence \( \Phi_j(\epsilon) = \Phi_j(-\epsilon) \), since otherwise it would be possible to contradict the local uniqueness part of the conclusion of the implicit function theorem (applied to \( \mathcal{G}_R \) with \( a = 0, b = 0 \)). This completes the proof of Proposition 2 and thus of the existence part of Theorem 1.1. \( \square \)

**Remark 10.** The solutions of \( \mathcal{F}(\Phi, \epsilon) = 0 \) are obtained for both positive and negative \( \epsilon \) close to zero, but the \( \epsilon \) negative branch apparently gives rise to solutions of the Dirac–Maxwell system via (4.9) which are related to the positive branch as follows. By (4.9), the branch which corresponds to negative \( \epsilon \) has the form

\[
\tilde{\phi}(x, \omega) = \begin{bmatrix} \tilde{\phi}_1(x, \omega) \\ \tilde{\phi}_2(x, \omega) \end{bmatrix} = \begin{bmatrix} (-\epsilon)^3 \phi_1(-\epsilon x, -\epsilon) \\ (-\epsilon)^2 \phi_2(-\epsilon x, -\epsilon) \end{bmatrix} = \begin{bmatrix} -\epsilon^3 \phi_1(\epsilon x, \epsilon) \\ \epsilon^2 \phi_2(\epsilon x, \epsilon) \end{bmatrix},
\]

\[\omega = -\sqrt{m^2 - \epsilon^2}, \quad \epsilon \geq 0,\]

where we took into account that \( \phi_1(y, \epsilon), \phi_2(y, \epsilon) \) obtained from Proposition 2 are even in \( \epsilon \). Comparing to (4.9), we conclude that this branch is related to the \( \epsilon \) positive branch \( \phi(x, \omega) \) by

\[
\tilde{\phi}_1(x, \omega) = -\phi_1(-x, \omega), \quad \tilde{\phi}_2(x, \omega) = \phi_2(-x, \omega),
\]

so that \( \tilde{A}^0(x, \omega) = \tilde{A}^0(-x, \omega), \quad \tilde{A}(x, \omega) = -A(-x, \omega) \). Consequently, these two branches have the same magnetic field but opposite electric field (see [5, §2.3 and §5.4]).

**Remark 11.** We briefly consider the symmetry properties of the solitary wave solutions: in [32, §2], Wakano gives the Ansatz for the solitary waves in the cylindrical coordinates \( (\rho, z = r \cos \theta, \phi) \), from which symmetry properties can be deduced. For our situation the relevant Ansatz for the Dirac wave function is

\[
\phi(x) = \begin{bmatrix} \varphi_1(\rho, \theta) \\ \varphi_2(\rho, \theta) e^{i\Phi} \\ i \varphi_3(\rho, \theta) \\ i \varphi_4(\rho, \theta) e^{i\Phi} \end{bmatrix}. \tag{4.36}
\]

An alternative approach to the existence theorem would be to set the problem up and then apply the implicit function theorem entirely within this symmetry class. The uniqueness assertion of the implicit function theorem would then imply that the solutions so constructed agree with those obtained above from Proposition 2.
Remark 12. The solution obtained is actually a convergent power series in $\epsilon$ since all mappings involved are analytic and so the analytic implicit function theorem holds.

Lemma 4.6. There is $C < \infty$ such that

$$A^0(y) = \frac{q\|\Phi_2\|^2 + \epsilon^2\|\Phi_1\|^2}{4\pi|y|} + O(|\langle y \rangle|^{-2}), \quad |A(y)| \leq C|\langle y \rangle|^{-2}.$$ 

Proof. We just apply the multipole expansion [17] to (4.12). The integrands are quadratic in the components of the Dirac field $\phi \in H^{2,0}$, and hence have exponential decay

$$\sup_{y \in \mathbb{R}^3} e^{2\theta|y|} \left( |\Phi_2^* \Phi_2 + \epsilon^2 \Phi_1^* \Phi_1| + |\Phi_1^* \sigma \Phi_2 + \Phi_2^* \sigma \Phi_1| \right) < \infty.$$ 

This allows that $A^0$ has leading asymptotic behaviour given by the Coulomb law

$$A^0(y) = \frac{q\|\Phi_2\|^2 + \epsilon^2\|\Phi_1\|^2}{4\pi|y|} + O(|\langle y \rangle|^{-2})$$ 

as $|y| \to +\infty$. The vector potential $A$ however has no monopole component because the currents $J$ have zero integral when evaluated on any stationary solution which decays rapidly at spatial infinity. Indeed for a stationary solution the conservation law $\partial_\mu J^\mu = 0$ implies that $J$ is divergence-free, and hence:

$$0 = \partial_t \int_{\mathbb{R}^3} J^0 y^k \, dy = -\int_{\mathbb{R}^3} (\partial_j J^j) y^k \, dy = \int_{\mathbb{R}^3} J^k \, dy, \quad 1 \leq k \leq 3.$$ 

As a consequence, the multipole expansion implies that $A = O(|\langle y \rangle|^{-2})$. 

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