Brane-world Singularities

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Abstract

We study the behavior of spatially homogeneous brane-worlds close to the initial singularity in the presence of both local and nonlocal stresses. It is found that the singularity in these brane-worlds can be locally either isotropic or anisotropic. We then investigate the Weyl curvature conjecture, according to which some measure of the Weyl curvature is related to a gravitational entropy. In particular, we study the Weyl curvature conjecture on the brane with respect to the dimensionless ratio of the Weyl invariant and the Ricci square and the measure proposed by Grøn and Hervik. We also argue that the Weyl curvature conjecture should be formulated on brane (i.e., in the four-dimensional context).

1 Introduction

Recently there has been great interest in brane-world cosmological models\cite{1,2}, particularly in an attempt to understand the dynamics of the universe at early times. Brane-world models have a different qualitative behavior than their general-relativistic counterpart\cite{3,4,5,6}, especially at high energies when the energy density of the matter is larger than the brane tension and the behavior deviates significantly from the classical case. However, in spite of this interest, the initial singularity seems to be very little understood in these brane-world models. The earliest investigations of the initial singularity, which used only isotropic fluids as a source of matter, suggest a matter-dominated isotropic singularity\cite{3,7,8,9}. However, later work using anisotropic stresses indicate that the initial singularity need not necessarily be isotropic\cite{10}. These issues

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are also linked to whether or not the brane-worlds exhibit a chaotic singularity (see e.g. [11]).

There are a number of important issues that need to be addressed in early universe cosmology. Although inflation is a possible causal mechanism for homogenization and isotropization, there is a problem in that the initial conditions must be sufficiently smooth in order for inflation to subsequently take place [7]. This problem of initial conditions in inflation might be alleviated by a regime of chaotic mixing [12,13,14]; consequently the possibility of chaotic behaviour in brane-world models is of interest [5,11]. However, more importantly, an isotropic singularity in brane world cosmology might provide for the necessary sufficiently smooth initial conditions, and might in turn be explained by entropy arguments and the second law of thermodynamics [17]. The study of the initial singularity in cosmology may consequently be closely related to the notion of “gravitational entropy” and the arrow of time [15,16]. A more precise statement of this is the Weyl Curvature Conjecture (WCC) according to which the Weyl tensor should be related to the entropy of the gravitational field [17,18,19,20].

Therefore, in this paper we will investigate the WCC for brane-world models. In order to do this we have to carefully study the singularity of the brane-world models in the presence of general fluids with anisotropic stresses (such an investigation is lacking at present). An important conclusion from this investigation is, as we will show, that there exist two local past attractors for these brane-worlds; one isotropic past attractor and one anisotropic past attractor. For the isotropic singularity, the measure \( P \) defined by the ratio of the Weyl invariant and the Ricci square,

\[
P^2 = \frac{C^{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta}}{R^{\mu\nu} R_{\mu\nu}},
\]

(1)

decreases to zero as the singularity is approached, consistent with the WCC. The measure \( S \) [18,19], defined on the brane by

\[
S = \sqrt{h} P,
\]

(2)

behaves according to the WCC for both of these singularities.

We note that the WCC for brane-worlds involves the five-dimensional Weyl tensor; additional terms arising from the projection of the five-dimensional Weyl tensor onto the brane should be included in the brane-world scenario. However, we will argue that the WCC should be formulated in the four-dimensional context.

### 2 The equations of motion

We consider spatially homogeneous brane (or branes where the spatial derivatives can be neglected). The evolution equations on the brane are as follows [21,22]. The Friedmann equation,

\[
H^2 = \frac{\Lambda}{3} + \frac{1}{6} \sigma^{\mu\nu} \sigma_{\mu\nu} - \frac{1}{6} (3)^R + \frac{\kappa^2}{3} \rho + \frac{\kappa^2}{6 \lambda} \left[ \rho^2 - \frac{3}{2} \pi_{\mu\nu} \pi^{\mu\nu} \right] + \frac{2\mathcal{A}}{\kappa^2 \lambda},
\]

(3)
the shear propagation equations,
\[
\dot{\sigma}_{\mu\nu} + \Theta \sigma_{\mu\nu} = \kappa^2 \pi_{\mu\nu} - (3)\mathcal{R}_{(\mu\nu)}
\]
\[+
\frac{\kappa^2}{2\lambda} \left[ -(\rho + 3p)\pi_{\mu\nu} + \pi_{\alpha(\mu}\pi^{\alpha}_{\nu)} \right] + \frac{6}{\kappa^2\lambda} \mathcal{P}_{\mu\nu},
\] (4)
Raychaudhuri’s equation (\( \Theta = 3H \)),
\[
\dot{\Theta} + \frac{1}{3} \Theta^2 + \sigma^{\mu\nu} \sigma_{\mu\nu} + \frac{1}{2} \kappa^2 (\rho + 3p) - \Lambda = - \frac{1}{2\lambda\kappa^2} \left[ \kappa^4 (2\rho^2 + 3\rho p) + 12\mathcal{U} \right],
\] (5)
the dark energy propagation equation,
\[
\mathcal{U} + \frac{4}{3} \Theta \mathcal{U} + \sigma^{\mu\nu} \mathcal{P}_{\mu\nu} = \frac{\kappa^4}{12} \left[ 3\pi^{\mu\nu} \pi_{\mu\nu} + 3(\rho + p)\sigma^{\mu\nu} \pi_{\mu\nu} + \Theta \pi^{\mu\nu} \pi_{\mu\nu} - \sigma^{\mu\nu} \pi_{\alpha\mu} \pi^{\alpha}_{\nu} \right].
\] (6)
Here, \( H \) is the Hubble parameter; \( \sigma_{\mu\nu} \) is the shear tensor; \( (3)\mathcal{R}_{\mu\nu} \) is the 3-curvature; \( \rho \) the energy-density; \( p \) the isotropic pressure; \( \pi_{\mu\nu} \) the anisotropic stress tensor; \( \mathcal{U} \) and \( \mathcal{P}_{\mu\nu} \) are the nonlocal dark energy and the bulk graviton stress tensor, respectively. The nonlocal energy-flux \( \mathcal{Q}_{\mu} \) has to vanish due to the shear constraint equation. Note that there are no propagation equations for the nonlocal bulk graviton stress tensor, \( \mathcal{P}_{\mu\nu} \).

For a FRW brane near the initial singularity, the Friedmann equation gives
\[
H^2 \propto \rho^2,
\] (7)
and thus \( H, \rho \propto t^{-1} \). The three-volume, \( v \), goes as \( v \propto t^{1/\gamma} \). We will primarily be interested in spatially homogeneous and spatially flat branes. The FRW universe corresponds to an equilibrium point of the equations of motion when written as a dynamical system, and hence may serve as a local past attractor. The behavior of the brane-worlds close to these equilibrium points, which have the important property that \( H \propto t^{-1} \), are reduced to linear algebra by linearizing the set of equations with respect to the equilibrium point.

In the following we will assume that the brane-world contain two types of fluids; an isotropic fluid obeying the barotropic equation of state \( p_i = (\gamma - 1)\rho_i \), and a radiation type of fluid causing the anisotropic stresses \( \pi_{\mu\nu} \). The fluids separately obey the energy-momentum conservation equations
\[
\dot{\rho}_i + \gamma \Theta \rho_i = 0,
\]
\[
\dot{\rho}_r + \frac{4}{3} \Theta \rho_r + \sigma^{\mu\nu} \pi_{\mu\nu} = 0.
\] (8)

3 Isotropic or anisotropic initial singularity?

In an earlier paper [10] it was shown that the initial singularity for a magnetic brane-world could be either locally isotropic or anisotropic. The question we address here is: What happens in the case with more general types of stresses? Will
the same behavior persist? Will there exist one anisotropic and one isotropic past attractor for more general brane-worlds?

In contrast to the general-relativistic models, the brane-worlds will in general be matter-dominated as we approach the initial singularity. The question whether the singularity is anisotropic or isotropic reduces to the question of which fluid dominates at the initial singularity. If the anisotropic fluid dominates initially, the fluid will typically drive the initial singularity to an anisotropic state. If, on the other hand, the isotropic fluid dominates, the universe will be driven to an isotropic FRW state.

3.1 Isotropic singularity

Let us first assume that the initial singularity is isotropic; hence we have a FRW singularity with $H, \rho_i \propto t^{-1}$ so that the three-volume expands as a power-law $v \propto t^{1/\gamma}$. Moreover, isotropy implies that the anisotropic stresses and fluid have to be small in the sense that $\pi_{\mu\nu}, \sigma_{\mu\nu}, \rho_r \ll \rho_i$.

To check the stability of the isotropic singularity we first investigate the stability of the ratio of $\rho_r/\rho_i$. For the isotropic singularity to be stable for this combination of fluids, we must have $\rho_r/\rho_i \to 0$ as $t \to 0$. Using eq. (8), we get the following requirement for the isotropic singularity to be past stable (using also that $\pi_{\mu\nu}, \sigma_{\mu\nu}, \rho_r \ll \rho_i$)

$$\gamma > \frac{4}{3}.$$  

(9)

It is quite easy to understand this requirement intuitively. Close to isotropy the anisotropic fluid behaves as isotropic radiation. If the isotropic fluid has $\gamma < 4/3$, then the anisotropic fluid will be dominant initially. This would, however, drive the universe into an anisotropic state and hence the isotropic FRW would be unstable. If, on the other hand, the isotropic fluid is stiffer than radiation, then the isotropic fluid dominates initially and the isotropic singularity is stable. Close to the isotropic singularity the shear will behave as a $\gamma = 1$ fluid and will therefore also be suppressed near the initial singularity. We note that $\gamma > 4/3$ for scalar field matter close to the singularity (which acts effectively as stiff matter with $\gamma_{\text{eff}} = 2$).

The only thing we have assumed in the above heuristic analysis is that the dominant term in the Friedmann equation is the $\rho_i^2$-term. The dark-energy $U$ behaves close to isotropy as classical radiation and will therefore be sub-dominant to the anisotropic radiation fluid (which come with quadratic terms in the Friedmann equation). The nonlocal stresses $P_{\mu\nu}$ have the same source as the dark-energy (they both are different components of the Weyl tensor in the bulk) and hence one should expect that the nonlocal stresses are sub-dominant as well. The above stability analysis is therefore quite general and encompasses many different types of brane-world scenarios.

To find the asymptotic behavior of the shear we have to specialize to a certain type of anisotropic fluid. We assume that the anisotropic stresses are of the form $\pi_{\mu\nu} = C_{\mu\nu}\rho_r$, where $C_{\mu\nu}$ is a matrix of constants (or slowly varying functions). This choice includes the magnetic field as a special case and may give us a useful indication of the asymptotic behavior of the shear in the presence of stresses.

Consider the isotropic singularity with $\gamma > 4/3$. Assuming that $P_{\mu\nu}$ behaves approximately as the dark-energy term we can neglect the nonlocal stresses on
the right-hand-side of the shear propagation equations. To lowest order the equations are
\[ \dot{\sigma}_{\mu
u} + \Theta \sigma_{\mu
u} \propto \rho_i \rho_r \propto t^{-1 - \frac{4}{3} \gamma}. \tag{10} \]
This equation can be integrated to give to lowest order
\[ \sigma_{\mu\nu} = \mathcal{O} \left( t^{-\frac{4}{3} \gamma} \right) . \tag{11} \]

### 3.2 Anisotropic singularity

This case is more difficult than the isotropic one since the anisotropic singularity is almost entirely determined by the anisotropic matter. Also, the nonlocal stresses come into play and are difficult to deal with in full generality. The main reason for this is that from a brane point of view, there are no evolution equations for \( P_{\mu\nu} \). In principle, \( P_{\mu\nu} \) should arise as a solution to the full five-dimensional theory, but we shall assume \( P_{\mu\nu} = 0 \) in the specific example (equilibrium point) discussed below. However, by a continuity argument we can expand the results to \( P_{\mu\nu} = UD_{\mu\nu} \), where \( D_{\mu\nu} \) is sufficiently small.

We start out with the assumption that there exists an anisotropic equilibrium point in the past. We can then give a criterion that must be fulfilled if the equilibrium point is past stable. For an important class in which there exists such an equilibrium point, we shall show that the equilibrium point is past stable if and only if this criterion is satisfied.

We assume that there exists an anisotropic equilibrium point such that \( \pi_{\mu\nu}, \sigma_{\mu\nu} \propto t^{-1}, \) and \( \rho_i \ll \rho_r \). We also assume the asymptotic values of \( \rho_r \) and \( \Theta \) are \( \rho_r = r/t \) and \( \Theta = \theta_0/t \), respectively, where \( r \) and \( \theta_0 \) are constants. This implies that we have the asymptotic value
\[ \frac{\pi_{\mu\nu} \pi_{\mu\nu}}{\rho_r^2} \approx \text{constant} \equiv D \tag{12} \]

Using eq. (8) we get the following asymptotic relation
\[ -1 + \frac{4}{3} \theta_0 + Dr = 0. \tag{13} \]
Consider the ratio \( R = \rho_i/\rho_r \). For the equilibrium point to be past stable \( \dot{R} > 0 \), so using eqs. (8) and (13), we obtain
\[ \theta_0 \gamma < 1. \tag{14} \]
The constant \( \theta_0 \) is related to the volume expansion rate as \( v \propto t^{\theta_0} \), so eq. (14) gives an upper limit on the exponent on the power-law behavior of the volume close to the equilibrium point. Note that if \( \theta_0 < 1/2 \), then this inequality is automatically satisfied for all \( \gamma \in [0, 2] \).

So far we have only analysed the stability in the \( R \) direction. In order for the equilibrium point to be past stable, it has to be stable in the remaining variables as well. Let us now provide an example which is past stable in the other variables. The example is a two-parameter set of different matter configurations which interpolates between the magnetic brane-worlds and a brane-world with two isotropic fluids. Only eq. (14) then remains to be satisfied.
Given an anisotropic stress in the diagonal form
\[
\pi_{\mu\nu} = \rho_r \text{diag} \left( 0, x + \sqrt{3}y, x - \sqrt{3}y, -2x \right),
\]
where \(x, y\) are constants and we assume that the nonlocal stresses vanish, \(\mathcal{P}_{\mu\nu} = 0\). We will also assume that the dominant energy condition
\[
|p_j| = \left| \frac{\rho_r}{3} + \pi_{jj} \right| \leq \rho_r
\]
is satisfied for the fluid \(\rho_r\). This implies that \((x, y)\) are restricted to a compact subset of \(\mathbb{R}^2\).

We define the following constants
\[
\Delta = 2(x^2 + y^2) + x^3 - 3xy^2, \\
\Sigma^2 = (2x + x^2 - y^2)^2 + (2y - 2xy)^2, \\
L^2 = 2 \left[ 2 + 9(x^2 + y^2) \right] \Delta, \\
a = 36\Delta^2 + 8\Sigma^2 + 6L^2, \\
B = \sqrt{(a + L^2)^2 - 8a(\Sigma^2 + L^2) - L^2}.
\]

Then there exist a past equilibrium point given by the asymptotic values
\[
\Theta = \frac{3(a - B)\Delta}{4a}, \\
\sigma_+ = \frac{(2x + x^2 - y^2)B}{6\Delta a} \frac{1}{t}, \\
\sigma_- = \frac{(2y - 2xy)B}{6\Delta a} \frac{1}{t}, \\
\frac{\kappa^2}{\lambda} \rho = \frac{B(B + a)}{12\Delta a^2} \frac{1}{t^2}, \\
\rho_i = 0, \\
\frac{2\mu}{\kappa^2 \lambda} = \frac{9\Delta^2(a - B)^2 - 4B^2\Sigma^2 - 2\Delta B(B + a) \left[ 1 - 9(x^2 + y^2) \right]}{144\Delta^2 a^2} \frac{1}{t^2}.
\]

Here are \(\sigma_\pm\) related to \(\sigma_{\mu\nu}\) via
\[
\sigma_{\mu\nu} = \text{diag} \left( 0, \sigma_+ + \sqrt{3}\sigma_- , \sigma_+ - \sqrt{3}\sigma_- , -2\sigma_+ \right).
\]

It can be shown that these asymptotic solutions are past attractors within the subset \(\mathcal{P}_{\mu\nu} = 0\), provided that inequality (14) holds. Further, it can be shown that \(\theta_0 < 3/4\) for nonzero \(x, y\).

Using a continuity argument\(^1\) we can show that there also exist a past attractor for models with nonlocal anisotropic stresses of type \(\mathcal{P}_{\mu\nu} = \mu D_{\mu\nu}\) where \(D_{\mu\nu}\) is sufficiently small. Hence, there is a class of models with \(\mathcal{P}_{\mu\nu} \neq 0\) which have an anisotropic past attractor. How large this class is, and if this anisotropic past attractor exists for generic brane-worlds, needs further work.\(^2\)

\(^1\)The argument uses the fact that finding the equilibrium points of a dynamical system is basically a question of finding solutions to an algebraic set of equations. Also, the stability analysis for these equilibrium points depends on the eigenvalues of a set of linear equations which are continuous functions of the \(C_{\mu\nu}\) and \(D_{\mu\nu}\).

\(^2\)The paper considers brane dynamics with nonlocal stresses. However, the brane-world does not contain an additional anisotropic fluid and thus there are no anisotropic past attractors.
4 The Weyl Curvature Conjecture

We have seen how different brane-world scenarios have two different past attractors. Here we will explore both of these possibilities further and check whether the brane-worlds behave according to the WCC (see [18, 19] for a review).

For the isotropic singularity we need to know how the Weyl tensor approaches the asymptotic values. In the previous section we found that $\sigma_{\mu\nu} \propto t^{-4/3}\gamma$ (recall that $\gamma > 4/3$ in this case). The dominant term in the Weyl curvature invariant will therefore be $(\sigma_{\mu\nu})^2$. Hence,

$$
(C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta})_I = t^{-4}.O\left(t^{\frac{4}{3}\gamma}\right).
$$

(20)

The Ricci square behaves as $t^{-4}$, thus

$$
P_I = O\left(t^{\frac{4}{3}\gamma}(3\gamma-4)\right),
$$

(21)

$$
S_I = O\left(t^{\frac{4}{3}\gamma}(3\gamma-1)\right).
$$

Hence, the isotropic singularity behaves according to both the $P$-version and the $S$-version of the WCC in this case.

For the anisotropic singularity, the behavior is easier to establish. The Weyl curvature behaves according to

$$
(C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta})_A = t^{-4}.
$$

(22)

The Ricci tensor evolves similarly, so

$$
P_A = \text{constant},
$$

$$
S_A \propto t^{\theta_0}.
$$

(23)

Recall that $\theta_0$ obeys the bound $\theta_0 \gamma < 1$, and the explicit value of $\theta_0$ depends very much on the matter content. In all of the cases we have explored here, $\theta_0 < 3/4$.

We can conclude that the measure $S$ behaves according to the WCC for both the anisotropic and the isotropic singularity.

5 A four-dimensional or five-dimensional WCC?

So far we have considered only the four-dimensional version of the WCC, seen from the brane point of view. However, a question that arises for these brane-worlds is whether one should consider the WCC from a five-dimensional perspective or a four-dimensional perspective. Even a four-dimensional measure of “gravitational entropy” should have a part arising from the projection of the five-dimensional Weyl tensor onto the brane. Hence the gravitational entropy on the brane will have one part corresponding to the four-dimensional Weyl tensor, and a part from the remaining components of the projection of the five-dimensional Weyl tensor. These components are

$$
E_{\mu\nu} = \langle 5 \rangle C_{ABCD}n^A n^B g^C g^D,
$$

(24)

$$
B_{\mu\nu\alpha} = \langle 5 \rangle C_{ABCD}n^A g^C g^D g^B g^D,
$$

(25)

Although $P_A$ is asymptotically a constant, its derivative can be either sign asymptotically.

4It should be noted that all these entities should be defined in the bulk in the limit close to the brane. The Weyl tensor itself experiences a delta function singularity at the brane [24].
where $n^A$ is a unit vector orthogonal to the brane, and $g^A_{\mu}$ is the projection tensor onto the brane (see [25, 26]). These nonlocal terms give rise to “dark energies” and stresses that do not come from standard model particles. More explicitly, the tensor $E_{\mu\nu}$ can be decomposed as

$$E_{\mu\nu} = \frac{6}{\lambda\kappa^2} \left[ U \left( u_\mu u_\nu + \frac{1}{3} h_{\mu\nu} \right) + P_{\mu\nu} + 2Q_{(\mu}u_{\nu)} \right],$$

and thus is, up to a constant factor, formally equivalent to the energy-momentum tensor of a radiation fluid with energy-density $U$, anisotropic stress $P_{\mu\nu}$, and energy-flux $Q_{\mu}$. The entropy of particles is calculable from standard thermodynamics, and hence there should also correspond an entropy due to the dark energies. These terms are purely gravitational effects but behaves very much like ordinary matter when it comes to the evolution of four-dimensional space-time. Actually, the notion of gravitational entropy might even be more easily understandable for these terms than for the more mysterious Weyl tensor.

The five-dimensional Weyl curvature invariant can be written as

$$(5) C^{ABCD}C_{ABCD} = (4) C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta} + 6E_{\mu\nu}E_{\mu\nu} + 4B_{\mu\nu\alpha}.$$

The tensor $B_{\mu\nu\alpha}$ can be expressed in terms of the exterior curvature of the hypersurfaces as [26]

$$B_{\mu\nu\alpha} = 2
\begin{pmatrix}
\nabla_{[\mu}K_{\nu]\alpha] + \frac{2}{3} \left( \nabla_{[\beta}T_{\mu]} - \nabla_{[\mu}K_{\beta]} \right) g_{\nu]\alpha
\end{pmatrix}
$$

where the covariant derivative, $\nabla_{\mu}$, is the covariant derivative associated with the metric on the brane, $g_{\alpha\beta}$. Further, from the junction conditions and the $\mathbb{Z}_2$ symmetry, we can relate the exterior curvature close to the brane and the energy-momentum tensor on the brane,

$$K^+_{\mu\nu} = -K^-_{\mu\nu} = -\frac{1}{6}\kappa^2\lambda g_{\mu\nu} - \frac{1}{2}\kappa^2 \left( T_{\mu\nu} - \frac{1}{3} g_{\mu\nu}T \right).$$

In our case, where the brane is spatially homogeneous, the components $B_{\mu\nu\alpha}$ can be replaced with time-derivatives of the matter on the brane.

There are different versions of the WCC that can be investigated.

### 5.1 The WCC: Modified four-dimensional version

First, we will consider the four-dimensional version where we take into account the extra terms from the projection of the five-dimensional Weyl tensor. Only the four-dimensional versions of the tensors shall be used.

For the isotropic singularity, we will consider a FRW brane for which exact five-dimensional solutions are known. In this case,

$$(B^{\mu\nu\alpha}B_{\mu\nu\alpha})_{FRW} = 0,$$

using the junction conditions and energy-momentum conservation of the fluid on the brane. The term $E^{\mu\nu}E_{\mu\nu}$ is, in general, given by

$$E^{\mu\nu}E_{\mu\nu} = \frac{36}{\lambda^2\kappa^2} \left( \frac{4}{3} U^2 + P_{\mu\nu}P_{\mu\nu} - 2Q^\mu Q_{\mu} \right).$$
Hence, for the FRW brane\(^5\),

\[
(4) C^{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta} + 6 E^{\mu\nu} E_{\mu\nu} + 4 B^{\mu\nu\alpha\beta} B_{\mu\nu\alpha\beta} \propto t^2. \tag{32}
\]

For \(\gamma \geq 2/3\) the Ricci square on the brane diverges as \(\rho^2\), so

\[
P_{\text{FRW}} \propto t^{\frac{2}{3}\gamma - 2} \tag{33}
\]

\[
S_{\text{FRW}} \propto t^{\frac{1}{3\gamma - 1}}. \tag{34}
\]

For the anisotropic singularity, we get a similar behavior as in the pure brane case:

\[
\left( (5) C^{ABCD} C_{ABCD} \right)_A \propto t^{-4},
\]

\[
P_A = \text{constant},
\]

\[
S_A \propto t^{6\gamma - 1}. \tag{35}
\]

5.2 The WCC: The five-dimensional version

The five-dimensional curvature tensors experience \(\delta\)-function singularities at the location of the brane and thus care is needed in this case. Let us therefore consider the case where the volume under consideration is entirely in the bulk but very close to the brane.

The measure \( (5) P = (5) C^{ABCD} C_{ABCD} / (5) R^{MN} R_{MN} \) can be used to investigate the five-dimensional WCC in the bulk. The bulk, which is taken to be an Einstein space with negative curvature\(^6\), has \( (5) R_{AB} \propto \Lambda_B (5) g_{AB} \). On the brane, the five-dimensional Ricci tensor also experiences a \(\delta\)-function singularity. Hence, it is of crucial importance where one wants to consider the WCC.

If we consider \( (5) P \) in the bulk, close to the brane, we can use the known exact solution for the FRW brane.

The FRW-brane can be embedded in an ambient Schwarzschild-AdS space\(^27\)

\[
ds^2 = -f(R)dt^2 + \frac{dR^2}{f(R)} + R^2 \gamma_{ij} dx^i dx^j, \quad f(R) = 1 - \frac{U}{R^2} - \frac{\Lambda_B}{6} R^2, \tag{36}
\]

where \(\gamma_{ij}\) is the 3-dimensional spatial metric encompassing the fifth dimension. For simplicity, we have also assumed the closed FRW model. The induced metric on the brane is

\[
ds^2 = -d\tau^2 + R(\tau)^2 \left[ \frac{d\tau^2}{1 - \tau^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right], \tag{37}
\]

where \(\tau\) is the proper time of the brane, and \(R(\tau)\) describes the motion of the brane; i.e. \(R(\tau)\) can be identified as the scale factor \(a(t)\). We can then easily find the Weyl curvature invariant:

\[
\left( (5) C^{ABCD} C_{ABCD} \right)_{\text{FRW,bulk}} \propto R^{-8}. \tag{38}
\]

\(^5\)For an anisotropic brane with shear degrees of freedom, in addition to this mode we get the usual shear modes from the four-dimensional Weyl invariant, \( (4) C^{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta} \), found earlier.

\(^6\)Recently a scalar field in the bulk has also been included, but these spacetimes have only been investigated at a perturbative level.
Hence,

\[
\left(\delta P\right)_{FRW,\text{bulk}} \propto R^{-4},
\]

\[
\left(\delta S\right)_{FRW,\text{bulk}} \propto R^{-1}.
\]

Thus, both measures diverge as we approach the initial singularity. For the anisotropic singularity, unfortunately there are not many exact solutions known to study the possible behavior. However, there are exact Kasner solutions [28], and we can use these solutions to obtain

\[
\left(\delta S\right)_{A,\text{bulk}} \rightarrow \infty, \quad \text{as} \quad t \rightarrow 0.
\]

Therefore we conclude that the measures investigated here diverge badly in the bulk as we approach the initial singularity. Thus the WCC as formulated above is not valid in the bulk.

This is perhaps not very surprising since the physics in the bulk is very different than the physics on the brane; indeed, the matter in the bulk need not even obey the weak energy condition. Moreover, matter is confined to the four-dimensional brane, and the physical energy conditions are only satisfied on the brane. Therefore, we conclude that the WCC should be formulated in the four-dimensional context.

### 6 Conclusion

In this paper we have presented a thorough investigation of the initial singularity in brane-world cosmological models. It was shown that for a class of spatially homogeneous brane-worlds with anisotropic stresses, both local and nonlocal, the brane-worlds could have either an isotropic singularity or an anisotropic singularity.

We also discussed the Weyl Curvature Conjecture, according to which the gravitational entropy is related to the Weyl curvature, in the context of the newly proposed brane-worlds. We found that the isotropic equilibrium point is typical in the brane-world scenario.

For both the isotropic and the anisotropic singularity the Weyl curvature invariant, \( C_{\alpha\beta\gamma\delta} \), the Weyl to Ricci ratio, \( P^2 = C_{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}/R^{\mu\nu}R_{\mu\nu} \), and the measure \( S = \sqrt{\Pi P} \) were investigated. The measure \( S \) behaved according to the WCC in both cases, while \( P \) showed generic increasing behavior only for the isotropic singularity. Hence \( P \) and \( S \) can be distinguished as measures of gravitational entropy.

However, for the brane-worlds a “gravitational entropy” may also have additional terms arising from the higher-dimensional bulk. The nonlocal terms, \( \mathcal{U} \) and \( \mathcal{P}_{\mu\nu} \), are projections of the five-dimensional Weyl tensor and behaves very much like a fluid upon the cosmological evolution of the universe. Hence, these terms are, in some sense, a direct manifestation of the Weyl tensor in terms of dark energies and stresses. These terms should therefore be incorporated into a gravitational entropy, one way or another. However, it seems as if the WCC can only be formulated on the brane where it can be seen upon as the “shadow” of the five-dimensional theory.
Finally, we remark that brane-worlds may serve as an interesting arena for studying the WCC. Actually, the brane-worlds may serve as a scenario were gravitational entropy and its relation to the Weyl tensor may be more easily understood.

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