SMALL INJECTIVE RINGS

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ABSTRACT. Let $R$ be a ring, a right ideal $I$ of $R$ is called small if for every proper right ideal $K$ of $R$, $I + K \neq R$. A ring $R$ is called right small injective if every homomorphism from a small right ideal to $R_R$ can be extended to an $R$-homomorphism from $R_R$ to $R_R$. Properties of small injective rings are explored and several new characterizations are given for $QF$ rings and $PF$ rings, respectively.

1. Introduction

Throughout this paper rings are associative with identity. For a subset $X$ of a ring $R$, the left annihilator of $X$ in $R$ is $I(X) = \{ r \in R : rx = 0 \text{ for all } x \in X \}$. For any $a \in R$, we write $I(a)$ for $I(\{a\})$. Right annihilators are defined analogously. We write $J = J(R)$, $S_r$ and $S_l$ for the Jacobson radical, the right socle and the left socle of $R$, respectively. $I \subseteq_{ess} R_R$ means $I$ is an essential right ideal. $f = c \cdot (c \in R)$ means $f$ is a map multiplied by $c$ on the left side. For a right ideal $I$ of $R$, we write $I_n$ for the set of all $n \times 1$ matrices over $I$.

In this article, the definition of small injective ring is introduced. Several relations between small injectivity and other injectivities (self-injectivity, simple injectivity, $F$-injectivity and $FP$-injectivity) are given. A main theorem of Yousif and Zhou [13, Theorem 2.11] is greatly simplified and improved by Theorem 3.4. It is well known that a ring $R$ is quasi-Frobenius (or $QF$) if and only if $R$ is left or right artinian and left or right self-injective. $R$ is right $PF$ if it is a semiperfect, right self-injective ring with $S_r \subseteq_{ess} R_R$. Under small injective condition, we give some new characterizations of $QF$ rings and right

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2. Definitions and examples

Definition 2.1. A module $M_R$ is called small injective if every homomorphism from a small right ideal to $M_R$ can be extended to an $R$-homomorphism from $R_R$ to $M_R$. The left side can be defined similarly. A ring $R$ is called right small injective if it is small injective as a right $R$-module. $R$ is called small injective if it is left and right small injective.

Since $J$ is the sum of all small right (or left) ideals of a ring $R$, we have

Example 2.2. Every semiprimitive ring (that is $J=0$) is right and left small injective.

Proposition 2.3. The following are equivalent:
(1) $R$ is semiprimitive.
(2) Every right (or left) $R$-module is small injective.
(3) Every principal right (or left) ideal is small injective.

Proof. We only prove the right side, the left side is analogously. It is obvious that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Now we assume (3), if $J$ is nonzero, then there exists a nonzero small right ideal $xR$ which is small injective. It is clear that the inclusion map from $xR$ to $R_R$ is split. Thus $xR$ is a direct summand of $R$, which is a contradiction. $\square$
Proposition 2.4. A direct product of right $R$-modules $M = \prod M_i$ is small injective if and only if each $M_i$ is small injective.

Proof. By definition. □

Proposition 2.5. A direct product of rings $R = \prod_{i \in I} R_i$ is right small injective if and only if $R_i$ is right small injective, $\forall i \in I$.

Proof. Let $\pi_i$ and $\iota_i$ be the $i$th projection and the $i$th inclusion canonically, $i \in I$. If $R$ is right small injective, for each $i$, suppose $f_i : T_i \to R_i$ is $R_i$-linear where $T_i$ is a right ideal of $R_i$. Then the map $0 \times \cdots \times T_i \times \cdots \times 0 \to 0 \times \cdots \times R_i \times \cdots \times 0$ given by $(0, \cdots, t_i, \cdots, 0) \mapsto (0, \cdots, f_i(t_i), \cdots, 0)$ is $R$-linear with $0 \times \cdots \times T_i \times \cdots \times 0$ a right ideal of $R$, so it has the form $c \cdot$ where $c \in R$. Thus $f_i = \pi_i(c) \cdot$. Conversely, let $\gamma : T \to R$ be $R$-linear, where $T$ is a right ideal of $R$. Write $T_i = \{ x \in R_i \mid \iota_i(x) \in T \}$, it is clear that $T_i$ is also a right ideal of $R_i$, $\forall i \in I$. Now define $\gamma_i : T_i \to R_i$ by $\gamma_i(x) = \pi_i(\gamma(\iota_i(x)))$, $x \in T_i, \forall i \in I$. Since $R_i$ is right small injective, $\gamma_i = c_i \cdot$, $\forall i \in I$. Thus for each $\bar{t} = \langle t_i \rangle \in T$, write $\gamma(\bar{t}) = \bar{s} = \langle s_i \rangle$. Since $T$ is a right ideal of $R$, $t_i \in T_i, \forall i \in I$. Thus $s_i = \pi_i(\bar{s} \cdot \iota_i(1_i)) = \pi_i(\gamma(\bar{t}) \cdot \iota_i(1_i)) = \pi_i(\bar{t} \cdot \iota_i(1_i)) = \pi_i(\gamma(\iota_i(t_i))) = \gamma_i(t_i) = c_i t_i$, whence $\bar{s} = \langle c_i \rangle \cdot \bar{t}$. So $R$ is right small injective. □

Example 2.6. Every right self-injective ring is right small injective. But the converse is not true. For example, the ring of integers $\mathbb{Z}$ is a semiprimitive ring but not a self-injective ring.

Example 2.7. The condition that $R$ is right small injective can not imply that $R$ is left small injective. In [10], Osofsky constructed a ring $R$ which is semiperfect, right self-injective, but not left self-injective. Then by Theorem 3.4, such ring is right small injective but not left small injective.

Proposition 2.8. Let $R$ be right small injective. If $e^2 = e \in R$ satisfies $ReR=R$, then $eRe$ is right small injective.
Proof. Let $S = eRe$ and $\theta : T \rightarrow S$ be an $S$-linear map, where $T$ is a small right ideal of $S$. Define $\bar{\theta} : TR \rightarrow R_R$ by $\bar{\theta}(\sum t_i r_i) = \sum \theta(t_i) r_i$, $t_i \in T$.

Now we prove that $\bar{\theta}$ is well defined. Let $\sum t_i r_i = 0$. If $r \in R$, we get $0 = \sum t_i r_i e_r = \sum t_i (e_r r e) \theta$, whence $0 = \sum \theta(t_i) (e_r r e) = \sum \theta(t_i) r_i r e$. Since $ReR = R$, $\sum \theta(t_i) r_i = 0$. So $\bar{\theta}$ is well defined. As $J(eRe) = eJe$, $TR$ is a small right ideal of $R$. Hence $\bar{\theta} = c\cdot$, where $c \in R$. Then $\forall t \in T$, $\theta(t) = e\theta(t) = e\bar{\theta}(t) = ect = (ec)et = (ece)t$. It follows that $\theta = (ece)\cdot$, as required. □

Remark 2.9. The condition that $ReR = R$ in the above proposition is necessary. For example ([3, Example 9]), let $R$ be the algebra of matrices, over a field $k$, of the form

$$
R = \begin{bmatrix}
a & x & 0 & 0 & 0 \\
0 & b & 0 & 0 & 0 \\
0 & 0 & c & y & 0 \\
0 & 0 & 0 & a & 0 \\
0 & 0 & 0 & 0 & b \\
0 & 0 & 0 & 0 & c
\end{bmatrix}.
$$

Set $e = e_{11} + e_{22} + e_{44} + e_{55}$, where $e_{ii}$ are matrix units. It is clear that $e$ is an idempotent of $R$ such that $ReR \neq R$. Then $R$ is right small injective, but $eRe$ is not right small injective.

Proof. By [3, Example 9], $R$ is a QF ring and $eRe$ is not a QF ring. Since $R$ is QF, $R$ is right small injective, and $eRe$ is artinian. If $eRe$ is right small injective, then $eRe$ is QF by Theorem 3.8, which is a contradiction. □

Question 2.10. Is right small injectivity a Morita invariant?

The method in the proof of the following theorem is owing to [5, Theorem 1]

Theorem 2.11. The following are equivalent for a ring $R$ and an integer $n \geq 1$:

1. $M_n(R)$ is right small injective.
2. For each right $R$-submodule $T$ of $J_n$, every $R$-linear map $\gamma : T \rightarrow R$ can be extended to $R_n \rightarrow R$. 


(3) For each right $R$-submodule $T$ of $J_n$, every $R$-linear map $\gamma : T \to R_n$ can be extended to $R_n \to R_n$.

Proof. We prove for $n=2$, the others are analogous. It is well known that $J(M_n(R)) = M_n(J), \forall n \geq 1$.

(1) $\Rightarrow$ (2). Given $\gamma : T \to R$ where $T \subseteq J_2$, consider the small right ideal $\bar{T} = [T \; T] = \{ [\alpha \; \beta] | \alpha, \beta \in T \}$ of $M_2(R)$. The map $\bar{\gamma} : \bar{T} \to M_2(R)$ defined by $\bar{\gamma}([\alpha \; \beta]) = \begin{bmatrix} \gamma(\alpha) & \gamma(\beta) \\ 0 & 0 \end{bmatrix}$ is $M_2(R)$-linear. By (1) we have $\bar{\gamma} = C \cdot$ where $C \in M_2(R)$, so $\gamma = \bar{C} \cdot$ where $\bar{C}$ is the first row of $C$. Hence $\gamma$ can be extended from $R_2$ to $R$.

(2) $\Rightarrow$ (3). Given (2), consider $\gamma : T \to R_2$ where $T \subseteq J_2$. Let $\pi_i : R_2 \to R$ be the $i$th projection, then (2) provides an $R$-linear map $\gamma_i : R_2 \to R$ extending $\pi_i \circ \gamma$, $i = 1, 2$. Thus $\bar{\gamma} : R_2 \to R_2$ extends $\gamma$ where $\bar{\gamma}(\bar{x}) = [\gamma_1(\bar{x}) \; \gamma_2(\bar{x})]^T$ for all $\bar{x} \in R_2$.

(3) $\Rightarrow$ (1). Write $S = M_2(R)$, consider $\gamma : T \to S_T$ where $T$ is a small right ideal of $S$. Then $T = [T_0 \; T_0]$ where $T_0 = \{ \bar{x} \in J_2 \mid [\bar{x} \; 0] \in T \}$ is a right $R$-submodule of $J_2$. Moreover, the $S$-linearity shows that $\gamma[\bar{x} \; 0] = [\bar{y} \; 0]$ for some $\bar{y} \in R_2$, and writing $\bar{y} = \gamma_0(\bar{x})$ yields an $R$-linear map $\gamma_0 : T_0 \to R_2$ such that $\gamma[\bar{x} \; 0] = [\gamma_0(\bar{x}) \; 0]$ for all $\bar{x} \in T_0$. Then $\gamma_0$ extends to an $R$-linear map $\bar{\gamma} : R_2 \to R_2$ by (3). Hence $\gamma_0 = C \cdot$ for some $C \in S$. If $[\bar{x} \; \bar{y}] \in T$ it follows that $\gamma([\bar{x} \; \bar{y}]) = \gamma([\bar{x} \; 0] + [\bar{y} \; 0] = [\gamma_0(\bar{x}) \; 0] + [\gamma_0(\bar{y}) \; 0] = [C\bar{x} \; C\bar{y}] = C[\bar{x} \; \bar{y}]$, which shows $\gamma = C \cdot$. \hfill $\square$

A ring $R$ is called right mininjective if every $R$-homomorphism from a minimal right ideal of $R$ to $R_R$ can be extended from $R_R$ to $R_R$. $R$ is called left minannihilator if every minimal left ideal is a left annihilator.

Proposition 2.12. If $R$ is right small injective, then

(1) $R$ is right mininjective.
(2) $l(I \cap L) = l(I) + l(L)$, for any small right ideals $I$ and $L$ of $R$. 


(3) Every principal small left ideal of \( R \) is a left annihilator, so \( R \) is left minannihilator.

(4) \( l(bR \cap r(a)) = l(b) + Ra, \forall b \in R, a \in J. \)

\textbf{Proof.} For (1), since every minimal one-sided ideal of \( R \) is either nilpotent or a direct summand of \( R \) (see \[ 5, (10.22) \] Brauer’s Lemma), each right small injective ring is right mininjective. But the converse is not true (see Example 2.13).

(2) and (3) by the similar proof of \[ 1, \text{Lemma 30.9} \].

(4) see \[ 13, \text{Lemma 1.2} \]. \( \square \)

\textbf{Example 2.13.} (The Björk Example \[ 9, \text{Example 2.5} \]) A right mininjective ring may not be right small injective. Let \( F \) be a field and assume that \( a \mapsto \bar{a} \) is an isomorphism \( F \mapsto \bar{F} \subseteq F \), where the subfield \( \bar{F} \neq F \). Let \( R \) denote the left vector space on basis \( \{1, t\} \), and make \( R \) into an \( F \)-algebra by defining \( t^2 = 0 \) and \( ta = \bar{a}t \) for all \( a \in F \). Then \( R \) is a right mininjective ring but not a right small injective ring.

\textit{Proof.} It is mentioned in \[ 9, \text{Example 2.5} \] that \( R \) is semiprimary, right mininjective but not left mininjective. If \( R \) is right small injective, then it is right self-injective by Theorem 3.4. Thus it is a right \( PF \) ring, which shows \( R \) is left mininjective by \[ 9, \text{Theorem 5.57} \], a contradiction. \( \square \)

A ring \( R \) is called right Kasch if every simple right \( R \)-module can embed into \( R_R \).

\textbf{Proposition 2.14.} If \( R \) is right small injective and right Kasch, then

(1) \( rl(I) = I \) for every small right ideal \( I \) of \( R \).

(2) The map \( \theta : T \rightarrow l(T) \) from the set of maximal right ideals \( T \) of \( R \) to the set of minimal left ideals of \( R \) is a bijection. And the inverse map is given by \( K \mapsto r(K) \), where \( K \) is a minimal left ideal of \( R \).

(3) For \( k \in R \), \( R_k \) is minimal if and only if \( kR \) is minimal, in particular \( S_r = S_l \).
Proof. (1) is by \[13\] Lemma 2.4 (3)].
(2) is informed by \[9\] Theorem 2.32 (b)] and Proposition 2.12 (3).
For (3), if R_k is minimal, then r(k) is maximal by (2) which shows kR is also minimal. Conversely, if kR is minimal, then R_k is minimal by \[9\] Theorem 2.21 (a)].

\[\square\]

3. SOME RELATIONS BETWEEN SMALL INJECTIVITY AND OTHER INJECTIVITIES

Lemma 3.1. Let R be a semilocal ring and I a right ideal of R, then every homomorphism from a right ideal to I can be extended to an endomorphism of R_R if and only if every homomorphism from a small right ideal to I can be extended to an endomorphism of R_R.

Proof. (i) “ \[\implies\] ” is obvious.
(ii) “ \[\iff\] ” Let f be a homomorphism from a right ideal K of R to I. Since R is semilocal, there exists a right ideal L of R such that K + L = R and K \cap L \subseteq J(see \[9\] Corollary 3.2)]. Thus K \cap L is small and there exists an endomorphism g of R_R such that g(x) = f(x), \forall x \in K \cap L. Define F: R_R \rightarrow R_R such that for any x = k + l, k \in K, l \in L, F(x) = f(k) + g(l).

Now we prove that F is well defined. If k_1 + l_1 = k_2 + l_2, k_i \in K, l_i \in L, i = 1, 2, then k_1 - k_2 = l_2 - l_1 \in K \cap L. Hence f(k_1 - k_2) = g(l_2 - l_1), which shows F(k_1 + l_1) = F(k_2 + l_2). Thus F is an endomorphism of R_R such that F|_K=f. \[\square\]

A right ideal I of R is said to lie over a summand of R_R if there exists a direct decomposition R_R = P_R \oplus Q_R with P \subseteq I and Q \cap I is small in R. In this case, I = P \oplus (Q \cap I).

Lemma 3.2. Let R be a ring and I a right ideal of R. If every (m-generated) right ideal lies over a summand of R_R, then every homomorphism from an (m-generated) small right ideal to I can be extended to an endomorphism of R if and only if every homomorphism from an (m-generated) right ideal to I can be extended to an endomorphism of R.
Proof. Let \( K \) be any \((m\text{-generated})\) right ideal of \( R \) and \( f \) a homomorphism from \( K \) to \( I \). Since \( K \) lies over a summand of \( R_R \), there exists an idempotent \( e^2 = e \in R \) such that \( K = eR \oplus L \), where \( L \subseteq J \) is an \((m\text{-generated})\) small right ideal. Now we show that \( K = eR \oplus (1-e)L \). It is obvious that \( eR + (1-e)L \) is a direct sum. If \( x \in eR \oplus L \), then \( x = a+b \), for some \( a \in eR, \ b \in L \). Thus \( x = a + eb + (1-e)b \in eR \oplus (1-e)L \). On the other hand, if \( y \in eR \oplus (1-e)L \), write \( y = c + (1-e)d \) where \( c \in eR, \ d \in L \). Hence \( y = c + (1-e)d = (c-ed) + d \in eR \oplus L \). As \( J \) is an ideal which is a small right ideal, so \( (1-e)L \subseteq J \) is also an \((m\text{-generated})\) small right ideal.

Since every homomorphism from an \((m\text{-generated})\) small right ideal to \( R_R \) can be extended to an endomorphism of \( R \), there exists a homomorphism \( g \) from \( R_R \) to \( R_R \) such that \( g|_{(1-e)L} = f|_{(1-e)L} \). Now we define \( F \) from \( R_R \) to \( R_R \) such that \( F(x) = f(ex) + g((1-e)x) \), whence \( F \) is a well defined homomorphism. Then for every \( x = a + b \in K = eR \oplus (1-e)L \) where \( a \in eR, \ b \in (1-e)L \), \( F(x) = f(ex) + g((1-e)x) = f(a) + g(b) = f(a) + f(b) = f(a+b) = f(x) \), which shows that \( F|_K = f \).

The converse is obvious. \( \square \)

Let \( I, \ K \) be two right ideals of \( R \) and \( m \geq 1 \). \( R \) is called right \((I, K)\)-\( m \)-injective (see [13]) if, for any \((m\text{-generated})\) right ideal \( U \subseteq I \) and any \( R \)-homomorphism from \( U_R \) to \( K_R \) can be extended from \( R_R \) to \( R_R \). A ring \( R \) is called right simple injective (right simple \( J \)-injective) if for any homomorphism from a right ideal (small right ideal) of \( R \) to \( R_R \) with simple image can be extended from \( R_R \) to \( R_R \). \( R \) is called right \((m\text{-injective})\) \( F \)-injective if every homomorphism from an \((m\text{-generated})\) finitely generated right ideal of \( R \) to \( R_R \) can be extended from \( R_R \) to \( R_R \). \( R \) is called right \((I, K)\)-\( FP \)-injective if, for any \( n \geq 1 \) and any finitely generated \( R \)-submodule \( N \) of \( I_n \), every \( R \)-homomorphism \( f : N \to K \) can be extended to an \( R \)-homomorphism \( g : R_n \to R \). \( R \) is right \( FP \)-injective if \( R \) is right \((R, R)\)-\( FP \)-injective.
Lemma 3.3. [13, Lemma 1.3] The following are equivalent for two right ideals $I$ and $K$ of $R$:

1. $R$ is right $(I,K)$-FP-injective.
2. $M_n(R)$ is right $(M_n(I), M_n(K))$-1-injective for every $n \geq 1$.

A ring $R$ is called semiregular if $R/J$ is von Neuman regular and idempotents lift modulo $J$.

Theorem 3.4. Let $R$ be a ring, we have

1. If $R$ is semilocal, then $R$ is right self-injective if and only if $R$ is right small injective.
2. If $R$ is semilocal, then $R$ is right simple injective if and only if $R$ is right simple $J$-injective.
3. If $R$ is semiregular, $I$ is a right ideal of $R$. Then $R$ is right $(J,I)$-m-injective if and only if $R$ is right $(R,I)$-m-injective. In particular, $R$ is right $(J,S_r)$-m-injective if and only if $R$ is right $(R,S_r)$-m-injective, $R$ is right $(J,R)$-m-injective if and only if $R$ is right $m$-injective, $R$ is right $F$-injective if and only if $R$ is right $(J,R)$-k-injective, $\forall k \geq 1$.
4. If $R$ is semiregular, then $R$ is right $(J,R)$-FP-injective if and only if $R$ is right FP-injective.

Proof. (1) and (2) by Lemma 3.1. For (3), since $R$ is semiregular, every finitely generated right (or left) ideal of $R$ lies over a summand of $R$ (see [7, Theorem 2.9]). Then (3) is clear by Lemma 3.2. Since semiregularity is a Morita invariant (see [7, Corollary 2.8]) and $J(M_n(R))=M_n(J)$, (4) is followed by (3) and Lemma 3.3.

Since a semiperfect ring is both semilocal and semiregular, we have

Corollary 3.5. Let $R$ be a semiperfect ring, we have

1. $R$ is right self-injective if and only if $R$ is right small injective.
2. $R$ is right simple injective if and only if $R$ is right simple $J$-injective.
3. Let $I$ be a right ideal of $R$. Then $R$ is right $(J,I)$-m-injective if and only
if $R$ is right $(R,I)$-$m$-injective. In particular, $R$ is right $(J,S_r)$-$m$-injective if and only if $R$ is right $(R,S_r)$-$m$-injective, $R$ is right $(J,R)$-$m$-injective if and only if $R$ is right $m$-injective, $R$ is right $F$-injective if and only if $R$ is right $(J,R)$-$k$-injective, $\forall k \geq 1$.

(4) $R$ is right $(J,R)$-FP-injective if and only if $R$ is right FP-injective.

**Corollary 3.6.** [13, Theorem 2.11] Let $R$ be a semiperfect ring with $S_r \subseteq \text{ess} R$.

1. If $R$ is right $(J,S_r)$-$(m+1)$-injective, then $R$ is right $(R,S_r)$-$m$-injective.
2. If $R$ is right $(J,R)$-$(m+1)$-injective, then $R$ is right $m$-injective.
3. If $R$ is right simple $J$-injective, then $R$ is right simple injective.
4. If every homomorphism from a small right ideal of $R$ to $R$ can be extended to an $R$-homomorphism from $R_R$ to $R_R$, then $R$ is right self-injective.

**Theorem 3.7.** The following are equivalent:

1. $R$ is right PF.
2. $R$ is a semilocal, right small injective ring with $S_r \subseteq \text{ess} R_R$.

*Proof.* It is obvious that (1) implies (2). By Theorem 3.4, the ring satisfying (2) is right self-injective, which shows that idempotents can be lifted modulo $J$. Hence $R$ is semiperfect. Thus (2) implies (1). □

**Theorem 3.8.** The following are equivalent:

1. $R$ is QF;
2. $R$ is right (or left) perfect, right and left small injective;
3. $R$ is a semilocal and right small injective ring with ACC (or DCC) on right annihilators.
4. $R$ is a right small injective ring with ACC on right annihilators in which $S_r \subseteq \text{ess} R_R$.
5. $R$ is a semiregular and right small injective ring with ACC on right annihilators.
Proof. It is obvious that (1) ⇒ (2), (3), (4) and (5). It is proved in [12, Lemma 2.11] that a right mininjective ring with ACC on right annihilators in which $S_r \subseteq_{ess} R_R$ is semiprimary. Then the rings in (2)-(4) are all semilocal rings. Hence they are all right self-injective by Theorem 3.4. Thus (2), (3) and (4) are clear by [3, Theorem 2.3, Theorem 4.1(b)]. For (5), $R$ is right $F$-injective by Theorem 3.4. Then (5) is implied by [11, Corollary 3]. □

4. EXTENSIONS OF SMALL INJECTIVE RINGS

Given a ring $R$ and a bimodule $R_V$, the trivial extension of $R$ by $V$ is the ring $S = R \ltimes V = \{(r, v) : r \in R, v \in V\}$ with the usual addition and multiplication $(r, v)(r', v') = (rr', rv' + vr')$. In fact, $R \ltimes V$ is isomorphic to the ring of all matrices $\begin{bmatrix} r & v \\ 0 & r \end{bmatrix}$ where $r \in R$ and $v \in V$ with the usual matrix operations. For convenience, we let $I \ltimes V = \{(r, v) : r \in I, v \in V\}$ where $I$ is a subset of $R$. Clearly, $V$ is an ideal of $S$, $V^2 = 0$ and $S/V \cong R$. The Jacobson radical of $S$ is $J(R) \ltimes V$.

Proposition 4.1. Let $S = R \ltimes V$, where $R$ is a ring and $R_V$ a bimodule of $R$. If $S$ is right small injective, then $V$ is self-injective as a right $R$-module and $R = \text{End}_R V_R$ canonically.

Proof. Let $K_R$ be a right $R$-submodule of $V$ and $f$ a right $R$-homomorphism from $K$ to $V$. Then $\{ \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} : a \in K \}$ is a small right ideal of $S$. Now we define $g$ from $\{ \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} : a \in K \}$ to $S$ such that $g(\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & f(a) \\ 0 & 0 \end{bmatrix}$, $a \in K$. It is clear that $g$ is right $S$-linear. Since $S$ is right small injective, there exists $\begin{bmatrix} b & c \\ 0 & b \end{bmatrix} \in S$ satisfying $g(\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & f(a) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} b & c \\ 0 & b \end{bmatrix} \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & ba \\ 0 & 0 \end{bmatrix}$. Thus $f(a) = ba, \forall a \in K$, which shows that $V_R$ is right self-injective and $R = \text{End}_R V_R$ canonically. □

Corollary 4.2. Let $S = R \ltimes V$, where $R$ is a ring and $R_V$ a bimodule of $R$. If $V$ is injective as a right $R$-module. Then the following are equivalent:
(1) $S$ is right self-injective;
(2) $S$ is right small injective;
(3) $R = \text{End}_R V$ canonically;

Proof. By [2, Theorem 2] and the above proposition. \hfill \Box

Corollary 4.3. Let $R$ be a ring and $S = R \times R$. Then the following are equivalent:
(1) $R$ is right self-injective;
(2) $S$ is right self-injective;
(3) $S$ is right small injective;

Remark 4.4. In the above corollary, $R$ is right small injective can not implies that $S$ is right small injective. For example, let $S = \mathbb{Z} \times \mathbb{Z}$. Since $\mathbb{Z}$ is semiprimitive, $\mathbb{Z}$ is right small injective. But $S$ can not be right small injective. If not, by the above corollary, we have that $\mathbb{Z}$ is self-injective, which is a contradiction.

If $R$ and $S$ are rings and $_RV_S$ is a bimodule, the formal triangular matrix ring of $R$ and $S$ by $V$ is the ring $U = \begin{bmatrix} R & V \\ 0 & S \end{bmatrix}$ of all matrices $\begin{bmatrix} r & v \\ 0 & s \end{bmatrix}$ where $r \in R, s \in S, v \in V$ with the usual matrix operations. It is clear that $J(U) = \begin{bmatrix} J(R) & V \\ 0 & J(S) \end{bmatrix}$.

Proposition 4.5. Assume that $U = \begin{bmatrix} R & V \\ 0 & S \end{bmatrix}$ is right small injective. Then:
(1) $S$ is right small injective.
(2) For every right $S$-submodule $K$ of $V_S$, $\text{Hom}_S(K, S) = 0$.
(3) If $\gamma : K_S \rightarrow V_S$ is $S$-linear, where $K$ is an $S$-submodule of $V_S$, then $\gamma = r \cdot$ for some $r \in R$.
(4) $X_S \subseteq \begin{bmatrix} V_S \\ J(S) \end{bmatrix}$ and $\theta : X_S \rightarrow \begin{bmatrix} V_S \\ S \end{bmatrix}$ is $S$-linear, then $\theta = C \cdot$ for some $C \in U$. 
Proof. (1). If $\gamma : T \to S$, where $T$ is a small right ideal of $S$ and $\gamma$ is $S$-linear. Then $\bar{T} = \begin{bmatrix} 0 & 0 \\ 0 & T \end{bmatrix}$ is a small right ideal of $U$ and $\bar{\gamma} : \bar{T} \to U$ is $S$-linear by defining $\bar{\gamma}(\begin{bmatrix} 0 & 0 & 0 \\ t \end{bmatrix}) = \begin{bmatrix} 0 & 0 \\ 0 & \gamma(t) \end{bmatrix}$. Hence $\bar{\gamma} = \begin{bmatrix} r & v \\ 0 & s \end{bmatrix}$, for some $r \in R, s \in S, v \in V$. Thus $\gamma = s \cdot$

(2). Let $\gamma : K_S \to S$, where $K_S$ is a right $S$-submodule of $V_S$. Then $\bar{K} = \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}$ is a small right ideal of $U$ and $\bar{\gamma} : \bar{K} \to U$ is $U$-linear by defining $\bar{\gamma}(\begin{bmatrix} 0 & k \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & 0 \\ 0 & \gamma(k) \end{bmatrix}$. Hence $\bar{\gamma} = \begin{bmatrix} r & v \\ 0 & s \end{bmatrix}$, for some $r \in R, s \in S, v \in V$. So $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \gamma(k) \end{bmatrix} = \begin{bmatrix} r & v \\ 0 & s \end{bmatrix} \begin{bmatrix} 0 & k \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & rk \\ 0 & 0 \end{bmatrix}$, which implies $\gamma = 0$.

(3) If $\gamma$ is as given, then $\bar{\gamma} : \bar{K} \to U$ is $U$-linear if we define $\bar{\gamma}(\begin{bmatrix} 0 & k \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & \gamma(k) \\ 0 & 0 \end{bmatrix}$. Hence $\bar{\gamma} = \begin{bmatrix} r & v \\ 0 & s \end{bmatrix}$, for some $r \in R, s \in S, v \in V$. Hence $\bar{\gamma} = r \cdot$

(4). Write $[0 \ X] = \{ \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} | \begin{bmatrix} v \\ s \end{bmatrix} \in X \}$. Then $[0 \ X]$ is a small right ideal of $U$ and $\bar{\theta} : [0 \ X] \to U$ is $U$-linear by defining $\bar{\theta} \begin{bmatrix} 0 & v \\ 0 & s \end{bmatrix} = \begin{bmatrix} 0 & \theta(\begin{bmatrix} v \\ s \end{bmatrix}) \end{bmatrix}$. Thus $\bar{\theta} = C \cdot$ for some $C \in U$, and so is $\theta = C \cdot$

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