The First Eigenvalue of $P$-Manifolds

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Abstract

Antonio Ros gave a lower bound for the first eigenvalue $\lambda_1$ of $\Delta$ of a $P$-manifold $(M, g)$ in terms of the lower bound on the Ricci curvature $Ric_M$ and asked what happened when this lower bound was achieved. In this paper we look into this question and show that there are strong implications on the geometry and topology of the underlying manifold. In particular we show that in case of spheres or real projective spaces we have isometry with the standard metric. In other cases, with some additional hypothesis, we again show isometry with standard models.

1 Introduction

Let $(M, g)$ be a compact Riemannian manifold, $\Delta$ the Laplacian of $(M, g)$ and $Spec(M, g) := \{\lambda_1 < \lambda_2 < \cdots\}$ the spectrum of $\Delta$ of $(M, g)$.

It is an important problem in geometry to find lower bounds for the eigenvalues of $\Delta$ of $(M, g)$ in terms of the given geometric data and characterize those Riemannian manifolds $(M, g)$ for which these lower bounds are attained. Lichnerowicz proved in [1] that if $(M, g)$ is a complete Riemannian manifold of dimension $n \geq 2$ with Ricci curvature $Ric_M \geq l$, then the first eigenvalue $\lambda_1$ satisfies the inequality $\lambda_1 \geq \frac{n}{n-1}l$. Later Obata proved in [2] that the equality is attained only for the round sphere of radius $\frac{1}{l}$. Considering this problem for $P$-manifolds, Antonio Ros proved in [3].

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that if \((M, g)\) is a \(P_{2\pi}\)-manifold of dimension \(n \geq 2\) with Ricci curvature \(\text{Ric}_M \geq l\), then the first eigenvalue \(\lambda_1\) satisfies the inequality \(\lambda_1 \geq \frac{1}{3}(2l+n+2)\) and the equality is attained iff for any first eigenfunction \(f\) we have that \(f(\gamma_u(t)) = A_u \cos t + B_u \sin t + C_u\) for \(u \in UM\). He further remarked that in view of Obata’s theorem this can happen only for a small class of manifolds.

In this paper we substantiate his claim by proving

**Theorem 1** Let \((M, g)\) be a \(P_{2\pi}\)-manifold of dimension \(n \geq 2\) with Ricci curvature \(\text{Ric}_M \geq l\) and \(\lambda_1 = \frac{1}{3}(2l+n+2)\). Then

1. (a) \(\lambda_1 = \frac{k(m+1)}{2} = \lambda_1(M)\) and \(l = \text{Ric}_M\) where \(M\) is a compact rank-1 symmetric space (CROSS) of dimension \(n = km\) with sectional curvature \(\frac{1}{4} \leq \text{K}_M \leq 1\) and \(k = 1, 2, 4, 8\) or \(n\) is degree of the generator of \(H^*(M, \mathbb{Q}) = H^*(\overline{M}, \mathbb{Q})\) and \(H^*(M, \mathbb{Z}_2) = H^*(\overline{M}, \mathbb{Z}_2)\).

(b) If \(k \geq 2\) then \(M\) is simply connected and the integral cohomology ring of \(M\) is same as that of \(\overline{M}\).

2. If \(k = 1\) then \((M, g)\) is isometric to \(\mathbb{R}P^n\) with constant sectional curvature \(\frac{1}{4}\).

3. If \(k = n\) then \((M, g)\) is isometric to \(S^n\) with constant sectional curvature 1. (Lichnerowicz-Obata theorem)

4. If \(k = 2, 4\) or \(8\) and if there is a first eigenfunction \(f\) without saddle points then \((M, g)\) is isometric to \(\overline{M}\) of dimension \(km\).

The main step in the proof of theorem 1 is the following

**Theorem 2** Let \((M, g)\) be a \(P_{2\pi}\)-manifold of dimension \(n \geq 2\) and \(\lambda\) be an eigenvalue of \(\Delta\) with an eigenfunction \(f\) such that \(f(\gamma_u(t)) = A_u \cos t + B_u \sin t + C_u\) for \(u \in UM\). Then \(\lambda = \frac{k(m+1)}{2} = \lambda_1(M)\) where \(M\) is as in theorem 1.

We refer to [2] and [3] for definitions, basic tools and results used in this paper.
2 Preliminaries

In this section we study the topology of critical sets of the function $f$ of the form $f(\gamma_u(t)) = A_u \cos t + B_u \sin t + C_u$ for $u \in UM$ on a $P \pi$-manifold $(M, g)$.

**Definition:** Let $(M, g)$ a complete Riemannian manifold. A subset $B \subseteq M$ is called *totally $a$-convex* if for any pair of points $a_1, a_2 \in B$ and any geodesic $\gamma : [0, r] \to M$ with $\gamma(0) = a_1$ and $\gamma(r) = a_2$ and $r < a$ then $\gamma([0, r]) \subseteq B$.

(See [4]).

**Theorem 3** Let $(M, g)$ be a $P \pi$ manifold and $f \in C^\infty(M)$ be such that $f(\gamma_u(t)) = A_u \cos t + B_u \sin t + C_u$ for $u \in UM$. Then

1. For each critical value $\alpha$ of the function $f$, the set $D_\alpha := \{ x \in M : f(x) = \alpha \& \nabla f(x) = 0 \}$ is a totally $\pi$-convex, totally geodesic submanifold of $(M, g)$.

2. $d(D_\alpha, D_\beta) = \pi$ for $\alpha \neq \beta$.

3. The function $f$ has only finitely many critical values.

**Proof of theorem 3:** Let $x \in M$. Then $f(\gamma_u(t)) = A_u \cos t + B_u \sin t + C_u$ for every $u \in U_xM$, the unit sphere in $T_xM$. If $x$ is a critical point of the function $f$, then, since $\nabla f(x) = 0$, we have that

$$B_u = \frac{d}{dt} |_{t=0} f(\gamma_u(0)) = \langle \nabla f(x), \gamma_u'(0) \rangle = 0$$

Therefore $f(\gamma_u(t)) = A_u \cos t + C_u$ for every $u \in U_xM$ and $x$ a critical point of the function $f$.

**Proof of 1:** Let $x, y \in D_\alpha$ and $\gamma_u$ be a geodesic joining $x$ and $y$ such that $\gamma_u(0) = x$ and $\gamma_u(r) = y$ for some $r \in \mathbb{R}^+$. Since $f(x) = f(y) = \alpha$ and $f(\gamma_u(t)) = A_u \cos t + C_u$, we have that $A_u + C_u = A_u \cos r + C_u$. Hence $A_u = 0$ if $r < 2\pi$. This shows that $f(\gamma_u(t)) = \alpha$ for all $t \in [0, r]$ and hence $\gamma([0, r]) \subseteq D_\alpha$. This proves that $D_\alpha$ is totally $2\pi$ convex.

To show that $D_\alpha$ is totally geodesic let us start with a $u \in UD_\alpha$, the unit tangent bundle of $D_\alpha$. Then, since $0 = \nabla^2 f(u, u) = -A_u$, we have that $f(\gamma_u(t)) = \alpha$, a constant. Hence $\gamma_u \subseteq D_\alpha$. Since $(M, g)$ is a $P \pi$-manifold,
this remains true even if $u$ is a unit tangent vector based at a boundary point of $D_\alpha$. Hence $D_\alpha$ is a totally geodesic submanifold of $(M, g)$.

**Proof of 2:** Let $\alpha$ and $\beta$ be two critical values of the function $f$ such that $\alpha \neq \beta$. Let $x \in D_\alpha$, $y \in D_\beta$ with $d(x, y) = t_0$ for some $t_0 \in \mathbb{R}^+$ and $\gamma_u$ be a geodesic segment such that $\gamma_u(0) = x$ and $\gamma_u(t_0) = y$. Then

$$f(\gamma_u(t)) = A_u \cos t + C_u$$

$$-A_u \sin t_0 = \frac{d}{dt} \big|_{t=t_0} f(\gamma_u(t))$$

$$= <\nabla f(y), \gamma'_u(t_0)>$$

$$= 0$$

This can happen only if $t_0 = \pi$. This proves that $d(D_\alpha, D_\beta) = \pi$ for $\alpha \neq \beta$.

**Proof of 3:** It is obvious as the critical submanifolds are constant distance apart.

### 2.1

Since the function $f$ has only finitely many critical values, we denote these critical values by $\max(f) = \alpha_1, \alpha_2, \cdots, \alpha_p = \min(f)$ and we denote by $D_i$ the critical submanifold $\{x \in M : f(x) = \alpha_i, \nabla f(x) = 0\}$.

Let $x \in D_{\max} = \{x \in M : f(x) = \max(f)\}$. Then $-\nabla^2 f(x)$ is positive semi-definite for each $x \in D_{\max}$. Therefore we can write the eigenvalues of $-\nabla^2 f(x)$ as $\mu_p(x) > \mu_{p-1}(x) > \cdots > \mu_2(x) > \mu_1(x) = 0$ where each $\mu_i(x)$ is a function on $D_{\max}$ for $1 \leq i \leq p(x)$ and $p(x) \in \{1, 2, \cdots, n\}$.

For each $i$, we denote by $E_{\mu_i(x)}$, the $\mu_i(x)$-eigenspace of $-\nabla^2 f(x)$, by $U_{\mu_i(x)}$ the unit sphere in $E_{\mu_i(x)}$ and by $S_{\mu_i(x)}(0, r)$ the sphere of radius $r$ in $E_{\mu_i(x)}$.

Let $u \in U_{\mu_i(x)}$. Then $\max(f) = A_u + C_u$ and $\mu_i(x) = -\nabla^2 f(u, u) = A_u$. Therefore $A_u$ and hence $C_u = \max(f) - A_u$ are constants on $U_{\mu_i(x)}$. Now we define $S(\mu_i(x), r) := \exp_x(S_{\mu_i(x)}(0, r))$, the exponential image of the sphere $S_{\mu_i(x)}(0, r)$ of radius $r$. Since the function $f$ is constant on $S(\mu_i(x), r)$ for each $r$, we have that $\nabla f$ is normal to $S(\mu_i(x), r)$. Therefore the geodesics $\gamma_u$ are integral curves of $\nabla f$ for $u \in U_{\mu_i(x)}$ and $\nabla f = -\mu_i(x) \sin t \partial_t$. From this it follows that $\nabla f(y) = 0$ for $y \in D_i(x) := S(\mu_i(x), \pi)$. Further $D_i(x) = \{y \in M : f(y) = \max(f) - 2\mu_i(x) \& \nabla f(y) = 0\}$. This can be seen as follows: Let $y \in D_i(x)$. Then

$$f(y) = f(\gamma_u(\pi))$$
\[
A_u + C_u = -2A_u + \max(f) = \max(f) - 2\mu_i(x)
\]

By theorem 3(1), each \(D_i(x)\) is a totally \(2\pi\)-convex, totally geodesic submanifold of \((M, g)\).

Now we will show that the functions \(\mu_i(x)\)'s are all constant functions on \(D_{\max}\) in the following

**Lemma 1**

1. For each \(i \in \{1, 2, \cdots, p(x)\}\) the function \(\mu_i(x)\) is constant on \(D_{\max}\).

2. The function \(p(x)\) is constant on \(D_{\max}\).

**Proof:** For each \(x \in D_{\max}\), \(\mu_{p(x)}\) is the largest eigenvalue of \(-\nabla^2 f(x)\). Since the geodesics \(\gamma_u\) are all integral curves of \(\nabla f\) for \(u \in U_{\mu_p(x)}\) they will flow towards \(D_{\min} := \{y \in M : f(y) = \min(f)\}\) and they will reach \(D_{\min}\) at time \(\pi\). Hence we must have \(\max(f) - 2\mu_{p(x)} = \max(f) - 2\mu_{p(y)}\) for \(x, y \in D_{\max}\). This proves that \(\mu_{p(x)}\) is a constant function on \(D_{\max}\).

Now we will prove the following

**Sublemma 1** For each critical value \(\alpha\) of the function \(f\), the submanifold \(D_\alpha\) coincides with \(D_i(x)\) for every \(x \in D_{\max}\) and for some \(i \in \{1, 2, \cdots, p(x)\}\).

**Proof:** Let \(y \in D_\alpha\) and \(x \in D_{\max}\). Then \(d(x, y) = d(D_\alpha, D_{\max}) = \pi\). Since \(D_\alpha\) and \(D_{\max}\) are totally geodesic submanifolds of \((M, g)\), any geodesic segment \(\gamma_u\) joining \(\gamma_u(0) = x\) and \(\gamma_u(\pi) = y\) will meet both \(D_\alpha\) and \(D_{\max}\) orthogonally.

Let \(u_\theta := \cos \theta w + \sin \theta v\) be a curve in \(U_xM\) such that \(w \in E_{\mu_p(x)}\), \(v \in E_{\perp}\) and \(u_{\theta_0} = u\) for some \(\theta_0\). Then \(f(\gamma_{u_\theta}(t)) = A_{u_\theta} \cos t + C_{u_\theta}\) and

\[
A_{u_\theta} = -\nabla^2 f(u_\theta, u_\theta) = -\cos^2 \theta \nabla^2 f(w, w) - \sin^2 \theta \nabla^2 f(v, v) - 2 \sin \theta \cos \theta < \nabla^2 f(w), v > = \cos^2 \theta A_w + \sin^2 \theta A_v + 2 \sin \theta \cos \theta A_w < w, v > = \cos^2 \theta A_w + \sin^2 \theta A_v
\]

The third step in the above equation follows from the fact that \(w\) is an eigenvector of \(-\nabla^2 f\) and the last step follows since \(< w, v > = 0\).
Since $\theta_0$ is a critical point of the function $\theta \mapsto f(\gamma_{u_\theta}(\pi))$, we have that
\[
\frac{d}{d\theta}|_{\theta=\theta_0} f(\gamma_{u_\theta}(\pi)) = 0.
\]
Further, since
\[
f(\gamma_{u_\theta}(\pi)) = -A_{u_\theta} + C_{u_\theta}
\]
\[
= -2A_{u_\theta} + \max(f)
\]
\[
= \max(f) - 2(\cos^2 \theta A_w + \sin^2 \theta A_v)
\]
we have that
\[
0 = \frac{d}{d\theta}|_{\theta=\theta_0} f(\gamma_{u_\theta}(\pi))
\]
\[
= 2 \sin 2\theta_0 (A_v - A_w)
\]
Since $-\nabla^2 f(w, w) = A_w$ is the largest eigenvalue, we have that $A_v - A_w \neq 0$. Hence $\sin 2\theta_0 = 0$. i.e., $\theta_0 = \frac{\pi}{2}$. This shows that $u \in E_{\mu(p(x))}^\perp$. Now from min – max principle it follows that $\alpha = \mu_i(x)$ for all $x \in D_{\text{max}}$ and for some $i \in \{1, 2, \ldots, p(x)\}$. This completes the proof of the sublemma.

From the sublemma it follows that

1. Each eigenvalue $\mu_i(x)$ is a constant function on $D_{\text{max}}$ and the number of distinct eigenvalues of $-\nabla^2 f$ are constant on $D_{\text{max}}$. Hence $p(x) = p$, a constant independent of the point $x \in D_{\text{max}}$.

2. The only critical values of the function $f$ are $\max(f) - 2\mu_i$ where $1 \leq i \leq p$ and $\mu_i$’s are the eigenvalues of $-\nabla^2 f$ on $D_{\text{max}}$.

This proves the lemma.

Therefore $D_i := \{y \in M : f(y) = \max(f) - 2\mu_i \& \nabla f(y) = 0\}$ are the only critical submanifolds of the function $f$ with critical values $\alpha_i = \max(f) - 2\mu_i$.

Note that $D_1 = D_{\text{max}}$ and $D_p = D_{\text{min}}$.

For each eigenvalue $\mu$ of $-\nabla^2 f$ on $D_i$, we denote by $E_{\mu}$, the $\mu$-eigensubspace of $-\nabla^2 f$ and by $S_{\mu}(0, \pi)$ the sphere of radius $\pi$ in $E_{\mu}$.

Then we have the following

**Proposition 1**

1. The eigenvalues of $-\nabla^2 f$ on $D_i$ are $\{\mu_{ij} := \mu_j - \mu_i : 1 \leq j \leq p\}$.

2. For each $x \in D_i$, the map
\[
\exp_x|_{S_{\mu_{ij}}(0, \pi)} : S_{\mu_{ij}}(0, \pi) \to D_j
\]
is a fibration for each $j \neq i$. 

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3. Each $D_j$ is either

(a) an integral cohomology CROSS and the degree of the generator of $H^*(D_j, \mathbb{Z})$ is $k = 2, 4, 8$ or $n$, or

(b) diffeomorphic to $\mathbb{R}P^{d_{ij} - 1}$ where $d_{ij} = \dim E_{\mu_{ij}}$.

**Proof:** Let $E_{\mu}(x)$ be an eigensubspace of $-\nabla^2 f(x)$ on $D_i$ for $\mu \neq 0$. Since $\exp_x(S_\mu(0, \pi))$ is a critical submanifold of the function $f$ it must be one of the $D_j$’s for $j \neq i$.

Let $u \in E_{\mu}(x)$ be a unit vector. Then $f(\gamma_u(0)) = A_u + C_u = \max(f) - 2\mu_i$ and $f(\gamma_u(\pi)) = -A_u + C_u = \max(f) - 2\mu_j$. Since $\mu = -\nabla^2 f(u, u) = A_u$, we have that

$$\max(f) - 2\mu_j = -A_u + C_u = -2A_u + A_u + C_u = -2\mu + \max(f) - 2\mu_i$$

Therefore $\mu = \mu_j - \mu_i$. This proves the first part of the lemma.

Let $x \in C_i$ and $D_{\mu_{ij}}(0, \pi) := \{v \in E_{\mu_{ij}}(x) : \|v\| \leq \pi\}$. Let $M_{ij}(x) := \exp_x(D_{\mu_{ij}}(0, \pi))$. Then each $M_{ij}(x)$ is a submanifold of $M$ and $M_{ij}(x)$ is also a Blaschke manifold at $x$ with totally geodesic cut-locus $D_j$. Hence it follows from [8] and [6] that $\exp_x|S_{\mu_{ij}}(0, \pi) : S_{\mu_{ij}}(0, \pi) \to D_j$ is a fibration for $j \neq i$.

Since $(M, g)$ is a $P$-manifold, the index of geodesics is a constant, say $(k - 1)$. From the fact that each $M_{ij}(x)$ is a Blaschke manifold at $x$, it follows that $k - 1 \in \{0, 1, 3, 7, n - 1\}$ and the dimension of the fibres is $k - 1$ for all these fibrations. We note that this fact can also be verified combinatorially.

If $k - 1$ is positive then each $M_{ij}(x)$ is simply connected integral cohomology CROSS and the degree of the generator of $H^*(M_{ij}(x), \mathbb{Z})$ is $k$. Since $D_j$ is the cut-locus of $x$ in $M_{ij}(x)$, it follows that $D_j$ is also a simply connected integral cohomology CROSS and the degree of the generator of $H^*(D_j, \mathbb{Z})$ is $k$.

Since each $M_{ij}(x)$ is a Blaschke manifold at $x \in D_i$, for any point $y \in D_j = \text{Cut}(x)$ in $M_{ij}(x)$, the order of conjugacy is atleast $\dim \Lambda(x, y)$ along any geodesic $\gamma$ with respect to $x$; here $\Lambda(x, y)$ is the link between $x$ and $y$. (See [2]). Therefore if the index $k - 1$ is zero, then $\dim \Lambda(x, y) = 0$ for every $y \in D_j$. Now since $x$ is not conjugate along any geodesic $\gamma$ from $y$ to $x$.
and $\Lambda(x, y)$ has only two elements, we see that $\exp_x|s_{ij}(0, \pi)\colon S_{ij}(0, \pi) \to D_j$ is a two sheeted covering. This proves that $D_j$ is diffeomorphic to $\mathbb{H}P^{d_j-1}$. This proves 3, 4 and the proposition completely.

3 Proof of theorem 2

Let $\lambda$ be an eigenvalue of $\Delta$ with an eigenfunction $f$ such that $f(\gamma_u(t)) = A_u \cos t + B_u \sin t + C_u$ for $u \in UM$. We know from theorem 3 that the function has only finitely many critical values say $\{\alpha_i : 1 \leq i \leq p\}$. Let $D_{\text{max}} = D_1, D_2, \ldots, D_p = D_{\text{min}}$ be the critical submanifolds of the function $f$ with critical values $\alpha_i$.

Let $\mu_p > \mu_{p-1} > \cdots > \mu_2 > \mu_1 = 0$ be the eigenvalues of $-\nabla^2 f$ on $D_{\text{max}}$. We saw in proposition 1 that for each $x \in D_{\text{max}}$, the map $\exp_x|s_{ij}(0, \pi)\colon S_{ij}(0, \pi) \to D_j$ is fibration with fibres of dimension $k - 1$. Therefore we can write $\dim E_{\mu_j} = kr_j$ for some non-negative integer $r_j \in \{1, 2, \ldots, n\}$. Hence $\dim D_j = k(r_j - 1)$.

We also saw in proposition 1 that the eigenvalues of $-\nabla^2 f$ on $D_i$ are $\{\mu_{ij} : \mu_j - \mu_i : 1 \leq j \leq p\}$ and $\exp|s_{j\mu_j}(0, \pi)\colon S_{\mu_j}(0, \pi) \to D_j$ is a fibration for $j \neq i$. In particular $\exp|s_{ij}(0, \pi)\colon S_{ij}(0, \pi) \to D_{\text{max}}$ is a fibration. Hence $\dim E_{\mu_{ij}} = \dim E_{\mu_j} = kr_j$ and $\dim E_{-\mu_i} = \dim D_{\text{max}} + k = k(r_1 + 1)$.

Now we will compute the Laplacian of the function $f$ on $D_i$’s. Since $f$ is an eigenfunction of $\Delta$ with eigenvalue $\lambda$, for each $x \in D_{\text{max}}$

$$\lambda \max(f) = \Delta f(x) = \text{Tr}(-\nabla^2 f(x)) = k \sum_{i=1}^{p} r_i \mu_i$$

and for each $y \in D_j$

$$\lambda \alpha_j = \Delta f(y)$$

But we know that $\alpha_j = \max(f) - 2\mu_j$. Therefore

$$\lambda(\max(f) - 2\mu_j) = k(r_1 + 1)(\mu_1 - \mu_j) + k \sum_{i \geq 2} r_i (\mu_i - \mu_j)$$
\[ -k\mu_j + k \sum_{i=1}^{p} r_i (\mu_i - \mu_j) \]
\[ = -k\mu_j + k \sum_i r_i \mu_i - k\mu_j \sum_i r_i \]
\[ = -\frac{k(1 + \sum_i r_i)}{2} \mu_j + \lambda \max(f) \]

This proves that
\[ \lambda = \frac{k(m+1)}{2} \]

where \( m = \sum_i r_i \).

We know from Bott-Samelson theorem for \( P \)-manifolds that \( H^*(M, \mathcal{Q}) \) has exactly one generator. (See [1], [2]). From proposition 1 it follows that the degree of the generator is \( k \). Therefore \( \lambda = \frac{k(m+1)}{2} = \lambda_1(\overline{M}) \) where \( \overline{M} \) is a CROSS of dimension \( km \) with sectional curvature \( \frac{1}{4} \leq K_{\overline{M}} \leq 1 \) and \( H^*(M, \mathcal{Q}) = H^*(\overline{M}, \mathcal{Q}) \). This proves theorem 2.

### 4 Proof of theorem 1

By hypothesis \( \text{Ric}_M \geq l \) and \( \lambda_1 = \frac{4}{3}(2l + n + 2) \). Hence for any first eigenfunction \( f \) we have that \( f(\gamma_u(t)) = A_u \cos t + B_u \sin t + C_u \) for \( u \in UM \). (See [10]).

**Proof of 1a:** It follows from theorem 2 that \( \lambda_1 = \frac{k(m+1)}{2} \). Since \( \lambda_1 \) is also equal to \( \frac{4}{3}(2l + n + 2) \), we get that \( l = \frac{k(m-1)}{4} + (k-1) = \text{Ric}_{\overline{M}} \). Again from the proof of theorem 2 it follows that \( H^*(M, \mathcal{Q}) = H^*(\overline{M}, \mathcal{Q}) \) and also that \( H^*(M, Z_2) = H^*(\overline{M}, Z_2) \). This completes the proof of 1a.

**Proof of 1b:** If \( k \geq 2 \) then each \( D_i \) is a simply connected submanifold of \( M \) of dimension less than or equal to \( km - k \). If one of the \( D_i \) is of dimension \( km - k \), then there are only two critical submanifolds \( D_{\text{max}} \) and \( D_{\text{min}} \) of the function \( f \) and one of them is a point. Let us assume that \( D_{\text{min}} = \{p\} \). Then \( M \) is a Blaschke manifold at \( p \) with simply connected cut-locus \( D_{\text{max}} \). Hence \( M \) is simply connected.

Now we assume that all the critical submanifolds are of dimension less than or equal to \( km - 2k \). For each \( i \), the critical submanifold \( D_i \) is contained in \( M \cup_{j \neq i} D_j \) and \( D_i \) is a strong deformation retract of \( M \setminus \cup_{j \neq i} D_j \). Since the codimension of each \( D_j \) is greater than or equal to 4 we have that \( \pi_1(M) = \frac{2k}{k} \leq \lambda_{\overline{M}}(\overline{M}) \)
\[ \pi_1(M \setminus \cup_{j \neq i} D_j). \] Therefore \( \pi_1(D_i) = \pi_1(M \setminus \cup_{j \neq i} D_j) = \pi_1(M) \). This proves that \( M \) is simply connected.

We have seen in proposition 1 that each \( D_j \) is a simply connected integral cohomology CROSS and the degree of the generator of \( H^*(D_j, \mathbb{Z}) \) is \( k \). Further we are attaching only \( rk \) dimensional cells to each of these \( D_j \)'s. This proves that the integral cohomology ring of \( M \) is same as that of \( \tilde{M} \).

**Remark:** If the integral cohomology ring of \( M \) is same as that of the cohomology projective plane then the function can have at most three critical submanifolds \( D_{\text{max}} \) and \( D_{\text{min}} \) and one saddle. If there are three critical submanifolds then all of them are points; if there are only two critical submanifolds \( D_{\text{max}} \) and \( D_{\text{min}} \) again one of them is a point.

**Proof of 2:** Since \( k = 1 \) we have that \( \text{Ric}_M \geq \frac{n-1}{4} \). Let \( (\tilde{M}, \tilde{g}) \) be the universal cover of \((M, g)\) and \( \Pi : \tilde{M} \to M \) the covering map. Then, since \( \text{Ric}_M = \text{Ric}_\tilde{M} \), we have that \( \text{Ric}_\tilde{M} \geq \frac{n-1}{4} \). Hence by Bonnet-Myers theorem \( \text{diam}(\tilde{M}, \tilde{g}) \leq 2\pi \). Now we will show that \( \text{diam}(\tilde{M}, \tilde{g}) \geq 2\pi \). Then from the rigidity of Bonnet-Myers theorem it will follow that \((\tilde{M}, \tilde{g})\) is isometric to \( S^n \) with constant sectional curvature \( \frac{1}{4} \) and our proof will also show that \( M \) is isometric to \( \mathbb{R}P^n \) with sectional curvature \( \frac{1}{2} \).

Let us fix a point \( x_0 \in D_{\text{max}} \) and let \( \tilde{D}(0, \pi) \subseteq T_{x_0} \tilde{M} \) be the disk of radius \( \pi \) in \( T_{x_0} \tilde{M} \). Then \( \exp_{x_0} : D(0, \pi) \to M \) is a smooth on-to map. We identify the antipodal points in the boundary \( S(0, \pi) \) of \( D(0, \pi) \) and denote it by \( \mathbb{R}P^n := D(0, \pi)/<(u, -u) : u \in S(0, \pi)> \), the quotient space of this identification.

Since \( M \) does not have conjugate points along the geodesics \( \gamma \) up to length \( \pi \), \( \exp_{x_0} \) : \( \mathbb{R}P^n \to M \) is a smooth on-to map of maximal rank. Therefore \( \exp_{x_0} : \mathbb{R}P^n \to M \) is a covering and the map \( \Pi : \tilde{M} \to M \) factors through \( \mathbb{R}P^n \). Since \( \exp_{x_0} : \mathbb{R}P^n \to M \) is a covering, we know that the map \( (\exp_{x_0})_* : \pi_1(\mathbb{R}P^n) \to \pi_1(M) \) is injective. Since the geodesic loops joining different critical levels are non-trivial in \( \mathbb{R}P^n \) they are non-trivial in \( \tilde{M} \) also.

Now let us fix one such geodesic loop \( \gamma \) of length \( 2\pi \) in \( \pi_1(M) \). Let \( \tilde{\gamma} \) be the lift of \( \gamma \) in \( \tilde{M} \) with \( \tilde{\gamma}(0) = x \) and \( \tilde{\gamma}(2\pi) = y \). We claim that \( d(x, y) = 2\pi \).

Suppose not. Then there exists a geodesic segment \( \tilde{\gamma}_1 \) of length less than \( 2\pi \) joining \( x \) and \( y \). Since \( \tilde{M} \) is simply connected there exists a homotopy between \( \tilde{\gamma} \) and \( \tilde{\gamma}_1 \) fixing the end points \( x \) and \( y \). This homotopy will go down to \( M \) to give a homotopy between \( \gamma \) and \( \gamma_1 = \Pi(\tilde{\gamma}_1) \). Since the geodesic \( \gamma_1 \) is based at a critical point and of length less than \( 2\pi \), it can be homotoped
to the base point along the integral curves of $\nabla f$. Hence $\gamma$ is also trivial in $\pi_1(M)$, a contradiction. Therefore $\text{diam}(\tilde{M}, \tilde{g}) \geq 2\pi$. This proves that $(\tilde{M}, \tilde{g})$ is isometric to $S^n$ with constant sectional curvature $\frac{1}{4}$ and $(M, g)$ is isometric to $\mathbb{R}P^n$ with constant sectional curvature $\frac{1}{4}$. This completes the proof.

**Proof of 3:** Since $k = n$ we have that $\text{Ric}_M \geq n - 1$. Further we also know $\text{diam}(M, g) = \pi$. Hence it follows from Bonnet-Myers theorem that $(M, g)$ is isometric to $S^n$ with constant sectional curvature $1$.

**Remark:** We note that, since $k = n$, the function $f$ has only two critical points, namely the maxima and minima and they are non-degenerate. Hence $f$ does not have saddle points.

### 4.1 Proof of 4

In this subsection we will assume that $\max(f)$ and $\min(f)$ are the only critical values of the function $f$. Hence $D_{\max}$ and $D_{\min}$ are the only critical submanifolds of the function $f$ in $(M, g)$. Therefore $-\nabla^2 f$ has only two eigenvalues on $D_{\max}$. By normalising the function $f$, we may assume that these two eigenvalues are 1 and 0. Hence we can write $f(\gamma_u(t)) = \cos t + C$ for $u \in (UD_{\max})^\bot$, the unit normal bundle of $D_{\max}$. Since the integral curves of $\nabla f$ are geodesics, the normal geodesic spheres around $D_{\max}$ are level sets of the function $f$.

Now we get bounds for $\nabla^2 f(u, u)$ for every $u \in UM$.

Let $S(t)$ be the geodesic sphere of radius $t$ around $D_{\max}$. Then $f(x) = \cos t + C$ for $x \in S(t)$ and $f(\gamma_u(t)) = A_u \cos t + B_u \sin t + C_u$ for $u \in U_x M$. Then $\gamma_u(0) \in S(t)$ and $\gamma_u(\pi) \in S(t_1)$ for some $t_1$ such that $0 \leq t_1 \leq \pi$. Since $A_u + C_u = \cos t + C$ and $-A_u + C_u = \cos t_1 + C$, we have that $A_u = \frac{1}{2} (\cos t - \cos t_1)$. Therefore

$$-\nabla^2 f(u, u) = \begin{cases} A_u \\ \frac{1}{2} (\cos t - \cos t_1) \end{cases}$$

Since $-1 \leq \cos t_1 \leq 1$, we get that

$$\frac{1 - \cos t}{2} \geq \nabla^2 f(u, u) \geq -\frac{1 + \cos t}{2}$$

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Having got these bounds for $\nabla^2 f$, we define two eigensubbundles of $\nabla^2 f$

$$E_{1-\cos t/2} := \{ E \in TM : \nabla^2 f(E) = \frac{1 - \cos t}{2} \}$$

$$E_{-1+\cos t/2} := \{ E \in TM : \nabla^2 f(E) = -\frac{1 + \cos t}{2} \}$$

Then we have the following

**Lemma 2**

1. The eigensubbundles $E_{1-\cos t/2}$ and $E_{-1+\cos t/2}$ of $\nabla^2 f$ are parallel along $\nabla f$.

2. $E_{1-\cos t/2}$ and $E_{-1+\cos t/2}$ are eigensubbundles of $R(\cdot, \nabla f)\nabla f$ with eigenvalue $\frac{1}{4} \| \nabla f \|^2$.

**Proof:** Let $x \in D_{\text{max}}$ and $\gamma$ be a geodesic starting at $x$ such that $\gamma'(0) \in (UD_{\text{max}})^\perp$. Let $J$ be a Jacobi field along $\gamma$ describing the variation of the geodesic $\gamma$ such that $J(0) \in T D_{\text{max}}$ and $J(\pi) = 0$. We normalise $J$ such that $\| J'(\pi) \| = 1$. Then, since $J$ is a Jacobi field, $[J, \gamma'(t)] = 0$ along the geodesic $\gamma$. Further, since $\gamma'(t) = \frac{\nabla f}{\| \nabla f \|}$, we have that $\gamma'(t) = -\partial_t$. Hence

$$- < J', J > = \frac{1}{\| \nabla f \|} < \nabla_J \nabla f, J > \leq \frac{\| J \|^2}{\| \nabla f \|} \frac{1 - \cos t}{2}$$

$$< J', J > \geq -\frac{1}{2} \frac{\sin \frac{t}{2}}{\cos \frac{t}{2}}$$

The function $\frac{\| J \|^2}{\cos^4 \frac{t}{2}}$ is smooth and non-vanishing on $R$. Hence we can take the positive square root $\frac{\| J \|}{| \cos \frac{t}{2} |}$ of $\frac{\| J \|^2}{\cos^4 \frac{t}{2}}$ which is again smooth. Since the function $\cos t$ is an even function, $\cos \frac{t}{2}$ is positive on $(-\pi, \pi)$. Therefore from the last step of the above equation it follows that

$$\frac{d}{dt} \log \left( \frac{\| J \|}{\cos \frac{t}{2}} \right) \geq 0$$

on $(-\pi, \pi)$. Now since $(M, g)$ is a $P_{2\pi}$-manifold, we have that $J(t) = J(t+2\pi)$. Hence $\frac{\| J \|}{\cos \frac{t}{2}} |_{t=-\pi} = \frac{\| J \|}{\cos \frac{t}{2}} |_{t=\pi} = 2$. This proves that $\frac{\| J \|}{\cos \frac{t}{2}} = 2$ for $t \in [-\pi, \pi]$ and
equality must hold everywhere in the above inequalities. This proves that $J$ is an eigenvector field of $\nabla^2 f$ with eigenvalue $\frac{1 - \cos t}{2}$. Since $\|J\| = 2 \cos \frac{t}{2}$, we can write $J(t) = 2 \cos \frac{t}{2} E(t)$ where $E(t) \in E_{1 - \cos t}$ is a unit vector field along $\gamma$. Since $J$ is a Jacobi field along $\gamma$

$$J' = \nabla_J \gamma' = \frac{1}{\|\nabla f\|} \nabla_J \nabla f = \frac{1 - \cos t}{2} \|\nabla f\| J$$

$$= \frac{1 - \cos t}{2} \|\nabla f\|^2 \cos \frac{t}{2} E$$

on the other hand $J' = \sin \frac{t}{2} E + \cos \frac{t}{2} E'$. This shows that $E'$ is along the direction of the vector field $\dot{E}$. Since $E'$ is a unit vector field along $\gamma$, $E' \perp E$. Therefore $E' = 0$ along $\gamma$. This proves that $E_{1 - \cos t}$ is parallel along $\nabla f$.

Now by a similar argument we can show that the eigensubbundle $E_{-1 + \cos t}$ is also parallel along $\nabla f$ by using the inequality that $\nabla^2 f(u, u) \leq -\frac{1 + \cos t}{2}$. (For a proof see [3]). This completes the proof of the lemma 3(1).

Now we set out to prove lemma 3(2). Let $E \in E_{1 - \cos t}$ be a unit vector at $t = 0$ and $J$ be a Jacobi field describing the variation of a normal geodesic $\gamma$ starting $D_{\text{max}}$, such that $J(0) = 2E$. Then from what we have seen above $J(t) = 2 \cos \frac{t}{2} E(t); E(t)$ parallel along $\gamma$. Therefore

$$R(J, \gamma') \gamma' = -J'' = \frac{1}{4} J$$

and this proves that $E_{1 - \cos t}$ is eigensubbundle of $R(\cdot, \nabla f) \nabla f$ with eigenvalue $\frac{1}{4} \|\nabla f\|^2$ along $\nabla f$. The same arguments will prove that $E_{-1 + \cos t}$ is also an eigensubbundle of $R(\cdot, \nabla f) \nabla f$ with eigenvalue $\frac{1}{4} \|\nabla f\|^2$. This completes the proof of the lemma.

Let $\dim D_{\text{max}} = ka$ and $\dim D_{\text{min}} = kb$ for some non-negative integers $a$ and $b$. Then $\dim E_{1 - \cos t} = ka$ and $\dim E_{-1 + \cos t} = kb = k(m - a + 1)$.

Let $E_{-\cos t} := \left( E_{1 - \cos t} \oplus E_{-1 + \cos t} \right)^\perp$ be the orthogonal complement of $E_{1 - \cos t} \oplus E_{-1 + \cos t}$ in $\tilde{T} M$. Then we have the following
Lemma 3 \( E_{-\cos t} \) is an eigensubbundle of

1. \( \nabla^2 f \) with eigenvalue \(-\cos t\)

2. \( R(., \nabla f) \nabla f \) with eigenvalue \( \| \nabla f \|^2 \)

**Proof:** First we note that \( \dim (E_{-\cos t} \oplus E_{1+\cos t}) = k(m-1) \). Therefore the dimension of \( E_{-\cos t} \) is \( k \). Let us choose an orthonormal basis \( E_2, E_3, \ldots, E_k \) of \( E_{-\cos t}, E_{k+1}, E_{k+2}, \ldots, E_{ka} \) of \( E_{1-\cos t} \) and \( E_{ka+1}, E_{ka+2}, \ldots, E_{km} \) of \( E_{1+\cos t} \). Then

\[
\sum_{i=2}^{k} < R(E_i, \nabla f) \nabla f, E_i > = \text{Ric}_M(\nabla f, \nabla f) - \sum_{j=k+1}^{kn} < R(E_j, \nabla f) \nabla f, E_j >
\]

\[
= \left[ \frac{k(m-1)}{4} + (k-1) \right] \| \nabla f \|^2 - \frac{k(m-1)}{4} \| \nabla f \|^2
\]

\[
= (k-1) \| \nabla f \|^2
\]

Now, for \( 2 \leq i \leq k \), we define the vector fields \( W_i = \sin t E_i(t) \), where each \( E_i \) is a parallel vector field along \( \gamma \) such that \( E_i(0) = E_i \). Then from the Index lemma, it follows that \( 0 \leq I(W_i, W_i) = \int_0^\pi (\langle W_i', W_i' \rangle > - R(W_i, \gamma') \gamma', W_i >) \). Therefore

\[
0 \leq \sum_{i=2}^{k} I(W_i, W_i)
\]

\[
= \sum_{i=2}^{k} \int_0^\pi \cos^2 t < E_i, E_i > - \sin^2 t K(E_i, \gamma')
\]

\[
= (k-1) \int_0^\pi (\cos^2 t - \sin^2 t)
\]

\[
= 0
\]

Hence \( W_i = \sin t E_i(t) \) are Jacobi fields along \( \gamma \) for \( 2 \leq i \leq k \). Now it can be easily verified that \( E_{-\cos t} \) is an eigensubbundle of \( \nabla^2 f \) with eigenvalue \(-\cos t\) and also an eigensubbundle of \( R(., \gamma') \gamma' \) with eigenvalue 1. This completes the proof of the lemma.

**An interesting Remark:** When \( k = 2 \), we don’t need the condition on \( \text{Ric}_M \) to show that \( E_{-\cos t} \) is an eigensubbundle of \( \nabla^2 f \) with eigenvalue \(-\cos t\) and also an eigensubbundle of \( R(., \gamma') \gamma' \) with eigenvalue 1. We give the proof below.
Let \( x \in D_{\text{max}} \). Then
\[
\Delta f(x) = \frac{k(m+1)}{2} f(x)
\]
\[
\Delta f(x) = \frac{k(m+1)}{2} (1 + C)
\]

Therefore
\[
k(m+1) (1 + C) = Tr(-\nabla^2 f(x))
\]
\[
= -Tr(\nabla^2 f(x) \left| E_{\frac{1+\cos t}{2}} \right) - Tr(\nabla^2 f(x) \left| E_{-\cos t} \right)
\]
\[
= k(m - a)
\]

Hence \( C = \frac{m-(2a+1)}{m+1} \).

Now let \( p \in M \). Then \( f(p) = \cos t + C \) for some \( t \) and
\[
k(m+1) \frac{\cos t + C}{2} = Tr(-\nabla^2 f(p))
\]
\[
= -\mu_1 - \mu_2 - Tr(\nabla^2 f(p) \left| E_{\frac{1+\cos t}{2}} \right)
\]
\[
- Tr(\nabla^2 f(p) \left| E_{\frac{1-\cos t}{2}} \right)
\]
\[
= \cos t - \mu_2 - ka(\frac{1 - \cos t}{2})
\]
\[
+k(m - (a + 1))(\frac{1 + \cos t}{2})
\]

Hence by substituting the value \( \frac{m-(2a+1)}{m+1} \) for \( C \) we get that \( \mu_2 = -\cos t \).

This completes the proof.

An important consequence of lemma 3 is that, for each \( x \in D_{\text{max}} \) the map \( \exp_x : S(0, \pi) \to D_{\text{min}} \) and for each \( y \in D_{\text{min}} \) the map \( \exp_y : S(0, \pi) \to D_{\text{max}} \) are great sphere fibrations; here \( S(0, \pi) \) denotes the normal sphere of radius \( \pi \) at the corresponding points. Now we prove the following

**Lemma 4** For every \( x \in D_{\text{max}} \), the map
\[
\exp_x : S(0, \pi) \to D_{\text{min}}
\]
and for every \( x \in D_{\text{min}} \), the map
\[
\exp_x : S(0, \pi) \to D_{\text{max}}
\]
are congruent to Hopf fibrations.
Proof: See [4] and [9].

Proof of 4: Let us fix a $\mathcal{P}^a(k) \subseteq \mathcal{P}^m(k)$. We denote by $(TD_{\text{max}})^\perp$, the normal bundle of $D_{\text{max}}$ and by $(T\mathcal{P}^a(k))^\perp$, the normal bundle of $\mathcal{P}^a(k)$. Since the map $\exp_x : S(0, \pi) \to D_{\text{min}}$ is congruent to Hopf fibration for each $x \in D_{\text{max}}$ there is a fibre preserving isometry $I : (TD_{\text{max}})^\perp \to (T\mathcal{P}^a(k))^\perp$. Using this isometry we define a map

$$\Phi : M \setminus D_{\text{min}} \to \mathcal{P}^n(k)$$

as follows: For every $q \in M \setminus D_{\text{min}}$ there is a unique $x \in D_{\text{max}}$ and a unique geodesic segment joining $x$ and $q$ and we define $\Phi(q) := \exp \circ I \circ \exp^{-1}(q)$. This map carries the geodesics orthogonal to $D_{\text{max}}$ to geodesics orthogonal to $\mathcal{P}^a(k)$ and matches the normal geodesic spheres around $D_{\text{max}}$. To complete the proof we only have to show that $d\Phi$ preserves the length of the Jacobi fields along these normal geodesics. This follows from [3]. Hence the proof of theorem 1(4).

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