Particle Collision near 1+1 Dimensional Horava-Lifshitz Black Holes

M. Halilsoy and A. Ovgun
Physics Department, Eastern Mediterranean University, Famagusta, Northern Cyprus, Mersin 10, Turkey.

The unbounded center-of-mass (CM) energy of colliding particles near horizon of a black hole emerges even in 1+1- dimensional Hořava-Lifshitz gravity. The latter has imprints of renormalizable quantum gravity characteristics in accordance with simple power counting. The result obtained is valid also for a 1- dimensional Compton process between a massive/massless Hawking photon emanating from the black hole and an in falling massless/massive particle.

I. INTRODUCTION

It is known that in spacetime dimensions less than four gravity has no life of its own unless supplemented by external sources. With that addition we can have lower dimensional gravity and we can talk of black holes, wormholes, geodesics, lensing effect etc. in analogy with the higher dimensions. One effect that attracted much interest in recent times is the process of particle collisions near the horizon of black holes due to Banados, Silk and West [1] which came to be known as the BSW effect. This problem arose as a result of imitating the rather expensive venture of high energy particle collisions in the laboratory. From curiosity the natural question arises: is there a natural laboratory (a particle accelerator) in our cosmos that we may extract information/energy in a cheaper way? This automatically drew attentions to the strong gravity regions such as near horizon of black holes. Rotating black holes host greater energy reservoir due to their angular momenta and attentions naturally focussed therein first [2, 3]. In case the black hole is not spinning there are enough reasons yet to consider the collision process in the near horizon geometry of black holes.

The same idea can be tested in lowest dimensional black holes as well. One considers the radial geodesics and upon energy-momentum conservation in the center-of-mass (CM) frame the near horizon limit is checked whether the energy is bounded/unbounded. Our aim in this study is to consider black hole solutions in 1+1- dimensional Hořava-Lifshitz (HL) gravity [4] and check the BSW effect in such reduced dimensional theory. For a number of reasons HL gravity is promising as a candidate for a renormalizable quantum gravity physics as the inhomogenous scaling properties of time and space components as follows

\[ S = \frac{M_{Pl}^2}{2} \int d^3x dt \sqrt{g} \left( K_{ij} K^{ij} + \lambda K^2 + V(\phi) \right) \] (2)

where \( K_{ij} \) is the extrinsic curvature tensor with trace \( K \) and Planck mass \( M_{Pl} \). \( V(\phi) \) stands for the potential function of a scalar field \( \phi \), and \( \lambda \) is a constant (\( \lambda > 1 \)). Reduction from 3+1- D to 1+1- D results in the action [4].

II. 1+1-D HL BLACK HOLES

HL formalism in 3+1-D makes use of the ADM splitting of time and space components as follows

\[ ds^2 = -N^2 dt^2 + g_{ij} \left( dx^i + N^i dt \right) \left( dx^j + N^j dt \right) \] (1)

where \( N(t) \) and \( N^i \) are the lapse and shift functions, respectively. The action of this theory is
\[ S = \int dt dx \left( \frac{1}{2} \eta N^2 a_1^2 + \alpha N^2 \phi^2 - V(\phi) \right) \quad (3) \]

where \( \eta \) = constant, \( \alpha \) = constant, \( a_1 = (\ln N) \tau \) in which a ‘prime’ denotes \( \frac{d}{d\tau} \). We note that the first term in \( S \) is inherited from the geometric part of the action while the other two terms are from the scalar field source. For simplicity we have set also \( M_{Pl} = 1 \).

It has been shown in \([4]\) that by variational principle a general class of solutions is obtained as follows

\[ N(x)^2 = 2C_2 + \frac{A}{\eta} x^2 - 2C_1 x + \frac{B}{\eta x} + \frac{C}{3\eta x^2} \quad (4) \]

in which \( C_2, A, C_1, B \) and \( C \) are integration constants.

The line element is

\[ ds^2 = -N(x)^2 dt^2 + \frac{dx^2}{N(x)^2} \]

with the scalar field

\[ \phi(x) = \ln \sqrt{2C_2 + \frac{A}{\eta} x^2 - 2C_1 x + \frac{B}{\eta x} + \frac{C}{3\eta x^2}} \quad (5) \]

Note that the associated potential is

\[ V(\phi(x)) = A + \frac{B}{x^3} + \frac{C}{x^4} \]

and the Ricci scalar is calculated as

\[ R = -\frac{2}{\eta} \left( A + \frac{B}{x^3} + \frac{C}{x^4} \right) \]

In the case of \( C_2 = 1/2, B = -2M, \eta = 1 \) and \( A = C = C_1 = 0 \), it gives a Schwarzschild-like solution;

\[ N(x)^2 = 1 - \frac{2M}{x} \quad (6) \]

On the other hand, the choice of the parameters, for \( C_2 = 1/2, B = -2M, C = 3Q^2, \eta = 1 \) and \( A = C_1 = 0 \) gives a Reissner–Nordström-like solution.

\[ N(x)^2 = 1 - \frac{2M}{x} + \frac{Q^2}{x^2} \quad (7) \]

The new black hole solution which is derived by Bazeia et. al. \([3]\) is found by taking \( C_1 \neq 0, C_2 \neq 0, B \neq 0 \) and \( A = C = 0 \)

\[ N(x)^2 = 2C_2 - 2C_1 x + \frac{B}{\eta x}. \quad (8) \]

This solution develops the following horizons

\[ x_h^\pm = \frac{C_2}{2C_1} \pm \sqrt{\Delta}, \quad \Delta = \frac{C_2^2}{4C_1^2} + \frac{B}{2\eta C_1}. \quad (9) \]

As \( \Delta = 0 \) they degenerate, i.e., \( x_h^+ = x_h^- \).

The Hawking temperature is given in terms of the outer \( (x_h^+) \) horizon as follows

\[ T_H = \frac{1}{4\pi} \left. \frac{(N(x)^2)'}{x=x_h^+} \right|_{x=x_h^+}. \quad (10) \]

For the special case \( C_2 = 0, C_1 = -M \) and \( B = -2M \) the horizons are independent of the mass \( M \):

\[ x_h^\pm = \pm \frac{1}{\sqrt{\eta}} (\eta > 0) \quad (11) \]

The temperature is then given simply by

\[ T_H = \frac{M}{\pi}. \quad (12) \]

This is a typical relation between the Hawking temperature and the mass of black holes in \( 1+1 \) dimensions \([23]\).

### III. CM ENERGY OF PARTICLE COLLISION NEAR THE HORIZON OF THE 1+1 -D HL BLACK HOLE

Here we will derive the equations of motion of an uncharged massive test particle by using the method of geodesic Lagrangian. Such equations can be derived from the Lagrangian equation,

\[ \mathcal{L} = -\frac{1}{2} \left[ -N(x)^2 \left( \frac{dt}{d\tau} \right)^2 + \frac{1}{N(x)^2} \left( \frac{dx}{d\tau} \right)^2 \right] \quad (13) \]

Here, \( \tau \) is the proper time for time-like geodesics (or massive particles) The canonical momenta calculated as

\[ p_t = \frac{d\mathcal{L}}{dt} = N(x)^2 \dot{t} \quad (14) \]

\[ p_x = \frac{d\mathcal{L}}{dx} = -\frac{x}{N(x)^2} \quad (15) \]

The 1+1- D HL black hole have only one Killing vector \( \partial_t \). Hence, there is only one conserved quantity along the motion of the particle which can be labeled as \( E \). From eq. \((13)\), \( E \) is related to \( N(x)^2 \) as,

\[ p_t = \frac{d\mathcal{L}}{dt} = N(x)^2 \dot{t} = E \quad (16) \]
Hence,
\[ i = \frac{E}{N(x)^2} \]  
\[ \text{(17)} \]

The two-velocity of the particles are given by \( u^\mu = \frac{dx^\mu}{dt} \). We have already obtained \( u^i \) in the above derivation. To find \( u^x = \dot{x} \), the normalization condition for time-like particles, \( u^\mu u_\mu = -1 \) can be used as,
\[ g_{xx}(u^x)^2 + g_{xx}(u^x)^2 = -1 \]  
\[ \text{(18)} \]

By substituting \( u^x \) to eq.(18), one can obtain \( u^x \),
\[ (u^x)^2 = \dot{x}^2 = -V_{eff} \]  
\[ \text{(19)} \]

where \( V_{eff} \) is the effective potential for the motion, given by
\[ V_{eff} = N(x)^2 - E^2. \]  
\[ \text{(20)} \]

Now, the two-velocities can be written as,
\[ u^i = \frac{E}{N(x)^2} \]  
\[ \text{(21)} \]
\[ u^x = \dot{x} = \sqrt{-V_{eff}} = \sqrt{N(x)^2 - E^2}. \]  
\[ \text{(22)} \]

We proceed now to present the CM energy of two particles with two-velocity \( u^\mu_1 \) and \( u^\mu_2 \). We will assume that both have rest mass \( m_0 = 1 \). The CM energy is given by,
\[ E_{cm} = \sqrt{2 \left( 1 - g_{\mu\nu}u^\mu_1 u^\nu_2 \right)} \]  
\[ \text{(23)} \]

so
\[ \frac{E_{cm}^2}{2} = \left( 1 + \frac{E_1 E_2}{N(x)^2} + \frac{\sqrt{E_1^2 - N(x)^2} \sqrt{E_2^2 - N(x)^2}}{N(x)^2} \right) \]  
\[ \text{(24)} \]

So, the lowest order term gives the CM energy of two particles as
\[ \frac{E_{cm}^2}{2} = 1 + \frac{E_1 E_2 - |E_1 E_2|}{N(x)^2} + \frac{(E_1 E_2)^2}{2 |E_1 E_2|} \]  
\[ \text{(25)} \]

There are two cases for this CM energy, when \( E_1 E_2 < 0 \), the CM energy is reduced to
\[ \frac{E_{cm}^2}{2} = 1 - 2 \frac{|E_1 E_2|}{N(x)^2} \]  
\[ \text{(26)} \]

which is unbounded for \( x \rightarrow x_h \).

On the other case, when \( E_1 E_2 > 0 \), the CM energy is independent from metric function, hence it gives always the finite energy.
\[ E_{cm}^2 = \frac{(E_1 + E_2)^2}{|E_1 E_2|} \]  
\[ \text{(27)} \]

So it should have \( E_1 E_2 < 0 \) to obtain an unbounded CM energy near to horizon of the HL black holes when we have the limiting value as \( x \rightarrow x_h \).

IV. SOME EXAMPLES

A. Schwarzchild-like Solution

In the case of \( C_2 = 1/2, B = -2M, \eta = 1 \) and \( A = C = C_1 = 0 \), it gives Schwarzchild-like solution where
\[ V(\phi(x)) = -\frac{2M}{x^3} \]  
\[ \text{(28)} \]

and
\[ N(x)^2 = 1 - \frac{2M}{x} \]  
\[ \text{(29)} \]

For the CM energy on the horizon, we have to compute the limiting value of eq.(24) as \( x \rightarrow x_h = 2M \), where is the horizon of the black hole. Setting \( E_1 E_2 < 0 \) as is, the CM energy near the event horizon for 1+1 D Schwarzchild BH is
\[ E_{cm}^2(x \rightarrow x_h) = \infty \]  
\[ \text{(30)} \]

From the case of \( E_1 E_2 > 0 \), it is shown that the CM energy is finite. This result for 4-D Schwarzchild Black hole is already calculated by Baushev [21]. When the location of particle 1 which has positive energy approaches the horizon, on the other hand the particle 2 escaping from the horizon with negative energy might give us the BSW effect \( E_{cm}^2 \rightarrow \infty \) so there is BSW effect for 1+1 Schwarzchild-like Solution when the condition \( E_1 E_2 < 0 \) is satisfied.

B. Reissner-Nordstrom-like solution

On the other hand, the choice of the parameters, for \( C_2 = 1/2, B = -2M, C = 3Q^2, \eta = 1 \) and \( A = C_1 = 0 \) gives the Reissner–Nordström-like solution.
\[ N(x)^2 = 1 - \frac{2M}{x} + \frac{Q^2}{x^2} \] (31)

and

\[ V(\phi(x)) = -\frac{2M}{x^3} + \frac{3Q^2}{x^4} \] (32)

so the CM energy is calculated by using the limiting value of eqn. (26)

\[ E_{cm}^2(x \to x_h) = \infty \] (33)

so there is a BSW effect.

C. The Non-Black Hole case

The simplest solution in [4] without scalar potential case is given as follows.

For \( C_1 = -M \), \( C_2 = -M \), \( \eta = 1 \), and \( A = B = C = 0 \) where \( V(\phi(x)) = 0 \) we have

\[ N(x)^2 = 2Mx - 1. \] (34)

This is not a black hole solution and is transformable to the Rindler metric in 1+1-D.

For the CM energy on the horizon, we have to compute the limiting value of eq. (26) as \( x \to x_h = \frac{1}{2M} \), where lies the horizon.

After some calculations, we get the limiting value of eq. (26):

\[ E_{cm}^2(x \to x_h) = \infty \] (35)

D. The Extremal case of the Reissner-Nordstrom like black hole

For the extremal case we have with \( M = Q \), from eq. (31)

\[ N(x)^2 = \left( 1 - \frac{M}{x} \right)^2 \] (36)

so that it also gives the same answer from eq. (26) as

\[ E_{cm}^2(x \to x_h) = \infty. \] (37)

E. Specific New Black Hole Case

The new 3-parametric black hole solution given by Bazeia, Brito and Costa [4] is chosen as

\[ N(x)^2 = 2C_2 - 2C_1x + \frac{B}{\eta x} \] (38)

with the potential

\[ V(\phi(x)) = \frac{B}{x^3}. \] (39)

For the special case \( C_2 = 0, C_1 = -M \) and \( B = -2M \) we have

\[ N(x)^2 = 2Mx - \frac{2M}{\eta x} \] (40)

with suitable potential which is

\[ V(\phi(x)) = -\frac{2M}{x^3}. \] (41)

The CM energy of two colliding particles is calculated by taking the limiting values of eq. (26)

\[ E_{cm}^2(x \to x_h) = \infty \] (42)

Hence the BSW effect arises here as well.

F. Near Horizon Coordinates

We have explored the region near the horizon by replacing \( r \) by a coordinate \( \rho \). The proper distance from the horizon \( \rho \) :

\[ \rho = \int \sqrt{g_{xx}(x)}dx = \int_{x_h}^{x} \frac{1}{N(x')}dx' \] (43)

The first example is the Schwarzchild-like solution which is

\[ N(x)^2 = 1 - \frac{2M}{x} \]
so that proper distance is calculated as
\[
\rho = \int_{x_h}^{x} (1 - \frac{2M}{x})^{-\frac{1}{2}} dx
= \sqrt{x(x - 2M)} + 2MG \sinh^{-1}(\sqrt{x/2M} - 1).
\]  

The new metric is
\[
ds^2 = -(1 - \frac{2M}{x(\rho)}) dt^2 + d\rho^2
\]
where \( \rho \simeq 2\sqrt{2M(x - 2M)} \), gives approximately
\[
ds^2 \simeq -\frac{\rho^2}{(4M)^2} dt^2 + d\rho^2
\]
which is once more the Rindler line element.

The CM energy of two colliding particles is given by
\[
\frac{E_{cm}^2}{2m_0^2} = 1 + \frac{(4M)^2 \left( E_1 E_2 - \sqrt{E_1^2 - \frac{\rho^2}{(4M)^2}} \sqrt{E_2^2 - \frac{\rho^2}{(4M)^2}} \right)}{\rho^2}
\]
so that there is BSW effect for \( \rho \to 0 \).

V. HAWKING PHOTON VERSUS AN INFALLING PARTICLE

Hawking radiation is accepted as a reality in the world of black holes. The massless photon of such an emission can naturally scatter an infalling particle or vice versa. This phenomenon is analogous to a Compton scattering taking place in 1+1-dimensions. Null-geodesics for a photon can be described simply by
\[
\frac{dt}{d\lambda} = \frac{E_1}{N^2}
\]
\[
\frac{dx}{d\lambda} = \pm \sqrt{E_1^2 - N^2}
\]
where \( \lambda \) is an affine parameter and \( E_1 \) stands for the photon energy. Defining \( E_1 = \hbar \omega_0 \), where \( \omega_0 \) is the frequency (with the choice \( \hbar = 1 \)) we can parametrize energy of the photon by \( \omega_0 \) alone. The center-of-mass energy of a Hawking photon and the infalling particle can be taken now as
\[
E_{cm}^2 = -(p^\mu + k^\mu)^2
\]
in which \( p^\mu \) and \( k^\mu \) refer to the particle and photon, 2-momenta, respectively. This amounts to
\[
E_{cm}^2 = m^2 - 2m g_{\mu\nu} u^\mu k^\nu,
\]

since we have for the particle \( p^\mu = m \left( \frac{E_1}{N^2}, \sqrt{E_1^2 - N^2} \right) \) and for the photon \( k^\mu = \left( \frac{E_1}{N^2}, -E_1 \right) \). One obtains
\[
E_{cm}^2 = m^2 + \frac{2mE_1}{N^2} \left( E_2 + \sqrt{E_2^2 - N^2} \right).
\]

In the near horizon limit this reduces to
\[
E_{cm}^2 = m^2 + \frac{2mE_1}{N^2} \left( E_2 + |E_2| - \frac{N^2}{2|E_2|} \right).
\]
Note that for \( E_2 < 0 \) we have \( E_{cm}^2 \) given by
\[
E_{cm}^2 = m^2 \left( 1 - \frac{E_1}{m |E_2|} \right)
\]
which is finite and therefore is not of interest. On the other hand for \( E_2 > 0 \) we obtain an unbounded \( E_{cm}^2 \).

VI. CONCLUSION

Particle collision problem is considered near the horizon of 1+1- dimensional Hořava-Lifshitz (HL) black holes. Our aim is to show that the BSW effect which arises in higher dimensional black holes applies also in the 1+1- D. The theory we adapted is not general relativity but instead the recently popular HL gravity. We employed the class of 5- parametric black hole solutions found recently [4]. The class has particular limits of flat Rindler, Schwarzschild and Reissner-Nordstrom like solutions. For each case we have calculated the center-of-mass (CM) energy of the particles and shown that the energy can grow unbounded. In other words the strong gravity near the event horizon effects the collision process with unlimited source to turn it into a natural accelerator. The model we use applies also to the case of a photon/particle collision with similar characteristics. Finally, we must admit that absence of rotational effects in 1+1- D confines the problem to the level of a toy model.
[1] M. Banados, J. Silk and S. M. West, Phys. Rev. Lett. 103, 111102 (2009).
[2] Ted Jacobson, Thomas P. Sotiriou, Phys. Rev. Lett. 104 (2010) 021101.
[3] Kayll Lake, Phys. Rev. Lett. 104 (2010) 211102.
[4] D. Bazeia, F. A. Brito and F. G. Costa, Phys. Rev. D 91, (2015) 044026.
[5] Horava, Petr, Phys. Rev. D 79 (2009) 084008.
[6] R. Arnowitt, S. Deser, C. W. Misner, Gen Relativ Gravit 40 (2008) 1997.
[7] E. Berti, V. Cardoso, L. Gualtieri, F. Pretorius and U. Sperhake, Phys. Rev. Lett. 103, 239001 (2009).
[8] M. Banados, B. Hassanain, J. Silk and S. M. West, Phys. Rev. D 83, 023004 (2011).
[9] T. Jacobson and T. P. Sotiriou, Phys. Rev. Lett. 104, 021101 (2010).
[10] O. B. Zaslavskii, JETP Lett. 92, 571 (2010).
[11] S. W. Wei, Y. X. Liu, H. T. Li and F. W. Chen, J. High Energy Phys. 12 (2010) 066.
[12] O. B. Zaslavskii, Phys. Rev. D. 86 (2012) 124039.
[13] O. B. Zaslavskii, Phys. Rev. D. 88 (2013) 104016.
[14] V. Tanatarov and O. B. Zaslavskii, Phys. Rev. D. 88 (2013) 064036.
[15] N. Tsukamoto and C. Bambi, Phys. Rev. D. 91 (2015) 084013.
[16] O. B. Zaslavskii, Phys. Rev. D. 90 (2014) 107503.
[17] S. G. Ghosh, P. Sheoran and M. Amir, Phys. Rev. D. 90 (2014) 103006.
[18] A. Galajinsky, Phys. Rev. D. 88 (2013) 027505.
[19] V. P. Frolov, Phys. Rev. D. 85 (2012) 024020.
[20] A. Al Zahrani, V. P. Frolov and A. Shoom, Phys. Rev. D. 87 (2013) 084043.
[21] A. N. Baushev, Int. J. Mod. Phy. D. 18 (2009) 1195-1203.
[22] M. Hallsoy, O. Gurtug and S. Habib Mazharimousavi, Gen. Rel. Grav. 45 (2013) 2363-2381.
[23] S. W. Hawking, Phys. Rev. D. 13 (1976) 191.