The Trade-Offs of Private Prediction

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Abstract

Machine learning models leak information about their training data every time they reveal a prediction. This is problematic when the training data needs to remain private. Private prediction methods limit how much information about the training data is leaked by each prediction. Private prediction can also be achieved using models that are trained by private training methods. In private prediction, both private training and private prediction methods exhibit trade-offs between privacy, privacy failure probability, amount of training data, and inference budget. Although these trade-offs are theoretically well-understood, they have hardly been studied empirically. This paper presents the first empirical study into the trade-offs of private prediction. Our study sheds light on which methods are best suited for which learning setting. Perhaps surprisingly, we find private training methods outperform private prediction methods in a wide range of private prediction settings.

1 Introduction

Machine learning models are frequently trained on data that needs to remain private, even though the predictions produced by those models are revealed to the outside world. For example, a hotel recommendation model may be trained on hotel reservation data that its users should not have access to. Unless proper care is taken, a user of a hotel recommendation service may be able to extract hotel reservation data of other users from the recommendations they receive, which would result in a privacy violation \cite{7, 16, 37, 38, 44}. Such privacy violations may happen even when users do not have direct access to the model parameters but only to model predictions: several studies have demonstrated that it is possible to reconstruct model parameters from a series of model predictions \cite{6, 29, 39}.

The goal of private prediction is to prevent such privacy violations by limiting the amount of information about the training data that can be obtained from a series of model predictions \cite{11}. Private prediction methods \cite{4, 9, 11, 32} perturb the predictions of non-private models to obfuscate any information that can be inferred about the data used to trained those models. Private training methods \cite{1, 3, 8, 23, 31, 42, 40} can be used to obtain private predictions as well. In particular, private training guarantees that model parameters reveal little information about the training data. As a result, the predictions of privately trained models do not leak information about the training data either.

Both private prediction and private training methods exhibit a range of trade-offs when they are used in a private prediction setting. Specifically, there exist trade-offs between the accuracy of the predictions, the privacy level that can be guaranteed, the probability of a privacy failure, the number of predictions that is revealed publicly (i.e., the inference budget), and the amount of training data. While these trade-offs are theoretically well understood, little is known about them empirically. Prior empirical studies evaluate only private training methods and do not consider the private prediction setting \cite{20, 21}. This paper performs an empirical study into the trade-offs of private prediction. We aim to provide guidance to practitioners on which private prediction methods are most suitable for a given learning setting. Perhaps surprisingly, we find that private training methods offer a better privacy-accuracy trade-off than private prediction methods in many practical learning settings.

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2 Problem Statement

Consider a private machine learning model \( \phi(x; \theta) \) with parameters \( \theta \) that given a \( D \)-dimensional input vector, \( x \in \mathbb{R}^D \), produces a probability vector over \( C \) classes \( y \in \Delta^C \). Herein, \( \Delta^C \) represents the \((C-1)\)-dimensional probability simplex. The parameters \( \theta \) were obtained by fitting the model on a training set of \( N \) labeled examples, \( D = \{(x_1, y_1), \ldots, (x_N, y_N)\} \), that needs to remain private.

We consider three private training methods in this study: (1) the model sensitivity method, (2) loss perturbation, and (3) differentially private stochastic gradient descent (SGD).

We adopt a regularized empirical risk minimization framework in which we minimize:

\[
J(\theta; D) = \frac{1}{N} \sum_{n=1}^{N} \ell(\phi(x_n; \theta), y_n) + \lambda R(\theta)
\]

with respect to \( \theta \), where \( \ell(\cdot) \) is a loss function, \( R(\cdot) \) is a regularizer, and \( \lambda \geq 0 \) is a regularization parameter. Some of the private prediction methods that we study make one or more of the following assumptions to obtain the privacy guarantee in Equation 1:

1. The loss function \( \ell(\cdot) \) is strict convex, continuous, and differentiable everywhere.
2. The regularizer \( R(\theta) \) is 1-strongly convex, continuous, and differentiable everywhere w.r.t. \( \theta \).
3. The non-randomized model \( \phi'(\cdot) \) is linear in \( x \), that is, \( \phi'(x; \theta) = \theta^\top x \).
4. The loss function \( \ell(\cdot) \) is Lipschitz with a constant \( K \), that is, \( \|\nabla l\|_2 \leq K \).
5. The inputs are contained in the unit \( L_2 \) ball, that is, \( \|x\|_2 \leq 1 \) for all \( x \).

Detailed algorithms of all methods and proofs of all theorems are presented in the appendix.

3 Methods

This study performs an empirical analysis of both private training and private prediction methods. We adopt a regularized empirical risk minimization framework in which we minimize:

\[
J(\theta; D) = \frac{1}{N} \sum_{n=1}^{N} \ell(\phi(x_n; \theta), y_n) + \lambda R(\theta)
\]

3.1 Private Training

We consider three private training methods in this study: (1) the model sensitivity method, (2) loss perturbation, and (3) differentially private stochastic gradient descent (SGD).

Model sensitivity. A simple private training method is for the model owner to add noise to the model parameters in order to hide information about the training data that is captured in those parameters.
The model sensitivity method [8] constructs differentially private parameters $\theta$ by adding noise to the minimizer of $J(\cdot)$. Specifically, we extend [8] to multi-class linear models with $\theta \in \mathbb{R}^{D \times C}$ and sample the noise matrix $B$ from $p(B) \propto e^{-\beta\|B\|_F}$. To satisfy the property in Equation 1, we choose $\beta = \frac{N\lambda}{K}$. In practice, we adopt the multi-class logistic loss which has a Lipschitz constant $K = \sqrt{2}$.

Theorem 1. Given assumptions 1, 2, 3, 4, 5, the model sensitivity method is $(\epsilon,0)$-differentially private.

By changing $p(B)$ to a zero-mean isotropic Gaussian distribution with standard deviation $\sigma = \frac{2K\alpha}{N\lambda\sqrt{2\epsilon}}$ for an $\alpha$ that depends on $\epsilon$ and $\delta$, we can also obtain $(\epsilon,\delta)$-differentially private models for $\delta > 0$ [2].

Theorem 2. Given assumptions 1, 2, 3, 4, 5, the Gaussian model sensitivity method is $(\epsilon,\delta)$-differentially private for $\delta \in (0, 1)$.

Loss perturbation. Rather than perturbing the learned parameters after training, the model owner can instead randomly perturb the loss function that is minimized during training to obtain a differentially private model [8, 23]. We extend the loss perturbation method of [8, 23] to multi-class classification by minimizing the following randomly perturbed loss function:

$$J' (\theta; D) = \frac{1}{N} \sum_{n=1}^{N} \ell (\phi (x_n; \theta), y_n) + \frac{\lambda}{N} R (\theta) + \frac{1}{N} \text{tr} (B^T \theta) + \frac{1}{2} \rho \| \theta \|_F^2, \tag{3}$$

where $\text{tr}(\cdot)$ represents the trace of a square matrix. The noise matrix $B$ is sampled from $p(B) \propto e^{-\beta\|B\|_F}$. The scale parameter of the noise distribution $\beta = \frac{2\epsilon}{\sqrt{\lambda}}$ and the parameter $\rho$ that governs the additional $L_2$ regularization is set such that $\rho \geq \frac{2L}{\sqrt{C}}$. The constant $L$ is an upper bound on the eigenvalues of the Hessian of $\ell(\cdot)$, that is, $\lambda_{\max} (\nabla^2 \ell (\theta^T x, y)) \leq L$ for all $\theta, x$ and $y$. We use a multi-class logistic loss, which has a Hessian with eigenvalues bounded by $L = \frac{1}{2}$ (see appendix).

Theorem 3. Given a convex loss function $\ell(\cdot)$, a convex regularizer $R(\cdot)$ with continuous Hessians, assumptions 1, 2, 3, 4, and 5, and assuming that $\lambda_{\max} (\nabla^2 \ell) \leq L$, the loss perturbation method is $(\epsilon,0)$-differentially private.

By changing noise distribution $p(B)$ into a zero-mean isotropic Gaussian distribution with standard deviation $\sigma = \frac{\epsilon}{\sqrt{8\ln(2/\delta) + 4\epsilon}}$, we can also obtain $(\epsilon,\delta)$-differential privacy for $\delta > 0$ [23].

Theorem 4. Given a convex loss $\ell(\cdot)$, a convex regularizer $R(\cdot)$ with continuous Hessians, assumptions 1, 2, 3, 4, and 5, and assuming that $\lambda_{\max} (\nabla^2 \ell) \leq L$, the Gaussian loss perturbation method is $(\epsilon,\delta)$-differentially private.

Differentially private SGD (DP-SGD). Rather than post hoc perturbation of the final model parameters or pre hoc perturbation of the loss, the DP-SGD method limits the influence of individual examples in parameter updates [11]. DP-SGD draws a batch $B$ of training examples uniformly at
random and computes the gradient $g_n = \frac{\partial J(\theta; \{(x_n, y_n)\})}{\partial \theta}$ for each example in the batch. It clips the resulting per-example gradients to have a norm bound of $\nu$, aggregates the clipped gradients over the batch, and adds Gaussian noise to obtain a private, approximate parameter gradient $\tilde{g}$:

$$\tilde{g} = \frac{1}{|B|} \left[ \sum_{n \in \{1, \ldots, |B|\}} \left( \frac{g_n}{\max(1, \|g_n\|_2)} \right) + \mathcal{N}(0, \sigma^2 \nu^2 I) \right],$$

where $\sigma^2 \nu^2$ is the variance of the Gaussian, $0$ is an appropriately sized vector of zeros, and $I$ an appropriately sized identity matrix. The resulting $\tilde{g}$ is used to perform the parameter update. Akin to the model sensitivity method, choosing $\sigma = \frac{1}{2} \sqrt{\text{ln}(1.25/\delta)}$ makes each $\tilde{g}$ vector $(\epsilon, \delta)$-differentially private with respect to the examples in batch $B$. The privacy amplification theorem [22] states that $\tilde{g}$ is $(q\epsilon, q\delta)$-differentially private with respect to examples from $D$, with $q = |B|/N$.

Privacy guarantees on the parameters learnt by DP-SGD after $M$ parameter updates can be obtained via a "moments accountant". We use the moments accountant of [51], which uses an analysis based on Rényi differential privacy [30] to compute the noise scale $\sigma$ required for a given privacy loss $\epsilon$, privacy failure probability $\delta$, number of parameter updates $M$, batch size $|B|$, and training set size $N$.

### 3.2 Private Prediction

We consider two private prediction methods in this study: (1) the prediction sensitivity method and (2) the subsample-and-aggregate method.

**Prediction sensitivity.** The prediction sensitivity method adds noise to the logits, $\hat{Y} = \phi'(\hat{x}; \theta)$, predicted by the model. Specifically, it returns $\hat{Y} + b$, where the noise vector $b \in \mathbb{R}^C$ is sampled from $p(b) \propto e^{-\beta \|b\|_2}$. Given an inference budget $B$, we use standard composition [12] and set $\beta = \frac{N \lambda \epsilon}{2KB}$.

**Theorem 5.** Given assumptions [7] [2] [3] [4] and [5] the prediction sensitivity method is $(\epsilon, 0)$-differentially private.

We can obtain $(\epsilon, \delta)$-differentially private predictions for $\delta > 0$ by sampling $b$ from a zero-mean isotropic Gaussian distribution [2]. Assuming $\delta > 0$ also allows the use of the advanced composition theorem [13] [15]. We set the standard deviation to $\sigma = \frac{2K \alpha^*}{\sqrt{N}}$, where $\alpha^*$ depends on per-sample privacy values $\epsilon^*$ and $\delta^*$ that are found using a search algorithm (see appendix for details).

**Theorem 6.** Given assumptions [7] [2] [3] [4] and [5] the Gaussian prediction sensitivity method is $(\epsilon, \delta)$-differentially private for $\delta \in (0, 1)$.

When the budget $B$ is small, the $\sigma$ obtained with standard composition can be smaller than that obtained with advanced composition. In experiments, we always choose the smaller $\sigma$ of the two.

**Subsample-and-aggregate.** Subsample-and-aggregate methods train $T$ models on $T$ subsets of $D$, and perform differentially private aggregation of the predictions produced by the resulting...
We first evaluate the performance of linear models on the MNIST handwritten digit dataset [26]. We evaluate the five private prediction methods in experiments with linear models on the MNIST dataset (4.2) and convolutional networks on the CIFAR-10 dataset (4.3). Code reproducing our experiments is available from https://github.com/facebookresearch/private_prediction.

4 Experiments

We evaluate the five private prediction methods in experiments with linear models on the MNIST [26] dataset (4.2) and convolutional networks on the CIFAR-10 [24] dataset (4.3). Code reproducing our experiments is available from https://github.com/facebookresearch/private_prediction.

4.1 Experimental Setup

In all experiments, we choose the loss function \( \ell \) to be the multi-class logistic loss. When training linear models, we minimize the loss using L-BFGS [45] with a line search that checks the Wolfe conditions [41] for all methods except the DP-SGD method. We trained all models using \( L_2 \)-regularization except when using the DP-SGD or subsample-and-aggregate methods. (We note that these two methods have implicit regularization via gradient noise and ensembling, respectively.) We selected regularization parameter \( \lambda \) via cross-validation in a range between \( 10^{-2} \) and \( 5 \cdot 10^{-1} \). We choose the clip value \( v \) in DP-SGD by cross-validating over a range from \( 5 \cdot 10^{-4} \) to \( 5 \cdot 10^{-1} \). For simplicity, we ignore the privacy leakage due to cross-validation and hyperparameter tuning in our analysis. Unless noted otherwise, we perform subsample-and-aggregate with \( T = 256 \) models.

To evaluate the quality of our models, we measure their classification accuracy on the test set. Because private prediction methods are inherently noisy, we repeat every experiment 100 times using the same train-test split. We report the average accuracy corresponding standard deviation in all result plots.

4.2 Results: Linear Models

We first evaluate the performance of linear models on the MNIST handwritten digit dataset [26].
Accuracy as a function of privacy loss. Figure 1 displays the accuracy of the models on the test set (higher is better) as a function of the privacy loss $\epsilon$ (lower values imply more privacy) for an inference budget of $B = 100$. The results are presented for two values of the privacy failure probability: $\delta = 0$ in Figure 1a and $\delta = 10^{-5}$ in Figure 1b (DP-SGD does not support $\delta = 0$). Recall that most methods support both $\delta = 0$ and $\delta > 0$ by choosing different noise distributions or a different privacy analysis.

Figure 1 shows that the subsample-and-aggregate method outperforms the other methods for most values of $\epsilon$ in the $\delta = 0$ regime, but is less competitive when $\delta = 10^{-5}$ (higher probability of privacy failure). Loss perturbation performs best for small values of $\epsilon$ (much privacy) when $\delta = 0$. Model sensitivity underperforms the other two methods when $\delta = 0$ but is slightly more competitive when $\delta = 10^{-5}$ due to the use of a Gaussian noise distribution. DP-SGD outperforms all other methods when $\delta = 10^{-5}$. Prediction sensitivity performs poorly across the board.

Accuracy as a function of inference budget. Figure 1 shows experimental results for a particular choice of inference budget ($B = 100$). In private prediction methods, the amount of privacy provided changes when this budget changes. To investigate this trade-off, Figure 2 displays accuracy as a function of the inference budget, $B$, for a privacy loss setting of $\epsilon = 1$. As before, we show results for privacy failure probability $\delta = 0$ (Figure 2a) and $\delta = 10^{-5}$ (Figure 2b). In the plots, private training methods are horizontal lines because the amount of information they leak does not depend on $B$.

The results show that the subsample-and-aggregate method outperforms other private prediction methods in the low-budget regime when $\delta = 0$. However, this advantage vanishes when the number of inferences that needs to be supported, $B$, grows. Specifically, loss perturbation outperforms subsample-and-aggregate at $B \gtrsim 80$ predictions. The advantage also disappears when a small privacy failure probability is allowed. For $\delta = 10^{-5}$, two private training methods (loss perturbation and DP-SGD) perform at least as good as subsample-and-aggregate for all inference-budget values. We also note that both loss perturbation and model sensitivity benefit greatly from making $\delta$ non-zero.

Accuracy as a function of privacy failure probability. Figure 3 studies the effect of varying the privacy failure probability, $\delta$, for a privacy loss of $\epsilon = 1$ and inference budget $B = 100$. The results show that loss perturbation, DP-SGD, and model sensitivity work well even at very small values of $\delta$. By contrast, prediction sensitivity appears to require unacceptably high values of $\delta$ to obtain acceptable accuracy. The subsample-and-aggregate method benefits the most from increasing $\delta$.

Accuracy as a function of training set size. Accuracy and privacy are also influenced by the number of training examples, $N$. Figure 4 studies this influence for privacy $(\epsilon, \delta) = (1, 0)$ and inference budget $B = 100$ on a dataset of up to one million digit images that were generated using InfiMNIST [27]. (We do not include DP-SGD because it does not support $\delta = 0$.) The results show that subsample-and-aggregate is more suitable for settings in which few training examples are available (small $N$) because of the regularizing effect of averaging predictions over $T$ models. However, it benefits less from increasing the training set size when $T$ remains fixed (recall that $T = 256$). When $N$ is large, loss perturbation and model sensitivity perform well because the scale of the noise that needs to be added to obtain a certain value of $\epsilon$ decreases as $N$ increases.

Figure 5: Test accuracy on MNIST dataset for privacy $(\epsilon, \delta) = (1, 0)$ and inference budget $B = 100$.
Accuracy as a function of number of dimensions. Accuracy and privacy may also vary with model capacity, which in linear models can be influenced via the number of input dimensions, $D$. Figure 5a presents the results of experiments in which we varied $D$ by applying PCA on the digit images. The figure shows that loss perturbation and model sensitivity are more competitive when the model has low capacity to memorize examples ($D$ is small), because this implies less parameter noise is needed.

Accuracy as a function of number of classes. Figure 5b investigates accuracy as a function of the number of classes, $C$. In this experiment, a digit classification problem with $C' < C$ classes only considers the first $C'$ digits in the MNIST dataset (digits 0 to $C' - 1$). Unsurprisingly, the accuracy of all methods decreases as the number of classes increases due to the increase in the problem’s Bayes error. The relative ranking of the methods appears to be largely independent of the number of classes.

Accuracy as a function of hyperparameters. Figure 6 studies the effect of: (1) the $L_2$-regularization parameter, $\lambda$, in loss perturbation and (2) the number of models, $T$, in subsample-and-aggregate. The results show that models overfit to the perturbation when $\lambda$ is too small, but underfit when $\lambda$ is too large. The optimal value for $\lambda$ depends on $\epsilon$. Similarly, larger values of $T$ lead to higher accuracies but accuracy deteriorates for very large values of $T$ due to overfitting. We note that increasing $T$ also increases computational requirements for training and inference, which grow as $O(T)$.

4.3 Results: Convolutional Networks

We also evaluated the DP-SGD and subsample-and-aggregate methods on the CIFAR-10 dataset using a ResNet-20 model (with “type A” blocks) as $\phi(\cdot)$. To facilitate the computation of per-example gradients in DP-SGD, we replaced batch normalization by group normalization. A non-private version of the resulting model achieves a test accuracy of 86.6% on CIFAR-10. All convolutional networks are trained using SGD with a batch size of 600, with an initial learning rate 0.1. In all experiments, the learning rate was divided by 10 thrice after equally spaced numbers of epochs. We use standard data augmentation during training: $\nu\phi$, random horizontal flipping and random crop resizing. The $T = 16$ models used in subsample-and-aggregate were trained for 500 epochs. Because privacy loss increases with number of epochs in DP-SGD, we trained our DP-SGD models for only 100 epochs. As before, we set the clip value $\nu$ in DP-SGD via cross-validation.

Figure 7 shows the accuracy of the resulting models as a function of privacy loss, $\epsilon$, for inference budget $B = 10$ (Figure 7a) and as a function of inference budget, $B$, for privacy loss $\epsilon = 1$ (Figure 7b). All results are for a privacy failure probability of $\delta = 10^{-5}$. The results in the figure show that the subsample-and-aggregate method outperforms DP-SGD training in some regimes: in particular, when $B$ is small and $\epsilon$ is large. In most other situations, however, DP-SGD appears to be the best method to obtain private predictions from a ResNet-20: for privacy loss $\epsilon = 1$, it surpasses subsample-and-aggregate for $B \gtrapprox 5$ predictions. In terms of accuracy, both methods lead to substantial losses in accuracy compared to a non-private model: even at a relatively large privacy loss of $\epsilon = 5$, the best private model underperforms its non-private counterpart by at least 30%.

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2 Other private prediction methods cannot be used here because deep networks violate assumptions 1 and 3.
5 Discussion

The results of our experiments demonstrate that there exist a range of complex trade-offs in private prediction. We summarize these trade-offs in Table 1. The table provides an overview of how privacy, accuracy, and computation are affected by increases in variables such as the number of training examples, the inference budget, the number of input dimensions, the amount of regularization, etc. We hope that this overview provides practitioners some guidance on the trade-offs of private prediction.

A perhaps surprising result of our study is that private training methods are often a better alternative for private prediction than private prediction methods. Methods such as subsample-and-aggregate appear to be most competitive for small inference budgets (tens or hundreds of test examples) and for small training sets, but tend to be surpassed by private training methods otherwise. We surmise that the main reason for this observation is that inference-budget accounting is fairly naive: the privacy guarantees assume that privacy loss grows linearly ($\delta = 0$) or with the square root ($\delta > 0$) of the number of inferences, which may be overly pessimistic. Inference-budget accounting may be improved, e.g., using accounting mechanisms akin to those in the PATE family of algorithms [34, 36].

Another way in which private prediction methods may be improved is by making stronger assumptions about the models with which they are used. In particular, the subsample-and-aggregate method is the only method that treats models as a pure black box. This is an advantage in the sense that it provides flexibility compared to private training methods that support only a small family of models: for example, subsample-and-aggregate could also be used to make private predictions for decision trees. But the flexibility of subsample-and-aggregate is also a weakness because it may cause the privacy guarantees to be too pessimistic. Indeed, it may be possible to improve private prediction methods for specific models by making additional assumptions about those models (linearity, smoothness, etc.) or by adapting the models to make them more suitable for private training or prediction [35].

Table 1: Effect of six hyperparameters (columns) on privacy, accuracy, and computational properties (rows): $\nearrow$ indicates that a property value goes up with the hyperparameter, $\searrow$ that it goes down, $\nwarrow$ that it goes up and then down, and $\cdots$ that it remains unchanged. Effects for method-independent hyperparameters are in the left columns: the number of training examples $N$, the number of classes $C$, and the number of dimensions $D$. Effects for method-specific hyperparameters are in the right columns: the inference budget $B$, the number of models $T$, and the $L_2$-regularization parameter $\lambda$.

| N | C | D | B | T | $\lambda$ |
|---|---|---|---|---|---|
| Privacy loss ($\epsilon$) | $\nearrow$ | $\nearrow$ | $\nearrow$ | $\nearrow$ | $\nearrow$ |
| Privacy failure probability ($\delta$) | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\cdots$ |
| Accuracy | $\nearrow$ | $\nearrow$ | $\nearrow$ | $\nearrow$ | $\nearrow$ |
| Training time | $\nearrow$ | $\nearrow$ | $\nearrow$ | $\cdots$ | $\cdots$ |
| Inference time | $\ldots$ | $\ldots$ | $\ldots$ | $\cdots$ | $\cdots$ |
Broader Impact

To date, there are a large number of machine-learning systems that are trained on data that needs to remain private or confidential but that expose predictions to the outside world. Examples of such systems include cloud services, content-ranking systems, and recommendation services. In controlled settings, experiments have shown that it may be possible to extract private information from such predictions. Yet, the machine-learning community currently has limited understanding of how much private information can be leaked from such services and how this information leakage can be limited. Over the past decade, a range of methods has been developed that aim to limit information leakage but these methods have primarily been studied theoretically. Indeed, little is known about their empirical performance in terms of the inevitable privacy-accuracy trade-off. This work helps to improve that understanding. As such, we believe this study helps anyone whose data is used to train modern machine-learning systems. In particular, our study can help the developers of those systems to better understand the privacy risks in their systems and may also help them find better operating points on the privacy-accuracy spectrum.

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A Preliminaries

As a reminder, we denote by \( \phi(x; \theta) \) a machine learning model with parameters \( \theta \) that given a \( D \)-dimensional input vector, \( x \in \mathbb{R}^D \), produces a probability vector over \( C \) classes \( y \in \Delta^C \), where \( \Delta^C \) represents the \( (C-1) \)-dimensional probability simplex. The parameters \( \theta \) are obtained by fitting the model on a training set of \( N \) labeled examples, \( \mathcal{D} = \{ (x_1, y_1), \ldots, (x_N, y_N) \} \).

In the following we aim to bound the privacy loss on the training set \( \mathcal{D} \) given a budget of \( B \) queries, \( \mathcal{Q} = \{ \hat{x}_1, \ldots, \hat{x}_B \} \), to the model \( \phi(\cdot) \).

We adopt a regularized empirical risk minimization framework in which we minimize:

\[
J(\theta; \mathcal{D}) = \frac{1}{N} \sum_{n=1}^{N} \ell(\phi'(x_n; \theta), y_n) + \lambda R(\theta).
\]

with respect to \( \theta \), where \( \ell(\cdot) \) is a loss function, \( R(\cdot) \) is a regularizer, and \( \lambda \geq 0 \) is a regularization parameter. Some of the methods make one or more of the following assumptions:

1. The loss function \( \ell(\cdot) \) is strictly convex, continuous, and differentiable everywhere.
2. The regularizer \( R(\cdot) \) is 1-strongly convex, continuous, and differentiable everywhere w.r.t. \( \theta \).
3. The model \( \phi'(\cdot) \) is linear, i.e., \( \phi'(x; \theta) = \theta^T x \).
4. The loss function \( \ell(\cdot) \) is Lipschitz with a constant \( K \), i.e., \( \| \nabla l \|_2 \leq K \).
5. The inputs are contained in the unit \( L_2 \) ball, i.e., \( \| x \|_2 \leq 1 \) for all \( x \).

In the following, we denote by \( \| \mathbf{A} \|_F \) the Frobenius norm of the matrix \( \mathbf{A} \), i.e.:

\[
\| \mathbf{A} \|_F = \left( \sum_{i,j} A_{ij}^2 \right)^{1/2}.
\]

We use Lemma 1 in the privacy proofs for the model and prediction sensitivity methods.

**Lemma 1.** Given assumptions \( 1 \), \( 2 \), \( 3 \), \( 4 \) and \( 5 \) and letting \( \theta_1 = \arg \min_{\theta} J(\theta; \mathcal{D}) \) and \( \theta_2 = \arg \min_{\theta} J(\theta; \mathcal{D}') \), the sensitivity is bounded as \( \| \theta_1 - \theta_2 \|_F \leq \frac{2K}{N} \).

**Proof.** To bound the sensitivity, let \( G(\theta) = J(\theta; \mathcal{D}) \) and \( g(\theta) = J(\theta; \mathcal{D}') - J(\theta; \mathcal{D}) \). Note that \( \theta_1 = \arg \min_{\theta} G(\theta) \) and \( \theta_2 = \arg \min_{\theta} G(\theta) + g(\theta) \). Given Assumptions \( 1 \) and \( 2 \) we can apply Lemma 7 of \( 8 \) to get:

\[
\| \theta_1 - \theta_2 \|_F \leq \frac{1}{N} \max_{\theta} \| \nabla g(\theta) \|_F.
\]

Next we bound \( \| \nabla g(\theta) \|_F \). Let \( x, y \) and \( x', y' \) be the differing examples in \( \mathcal{D} \) and \( \mathcal{D}' \). Using assumption \( 3 \) we have:

\[
g(\theta) = \frac{1}{N} (\ell(\theta^T x, y) - \ell(\theta^T x', y')).
\]

Thus:

\[
\nabla g(\theta) = \frac{1}{N} (\nabla \ell(\theta^T x, y)x^T - \nabla \ell(\theta^T x', y')x'^T),
\]

and:

\[
\| \nabla g(\theta) \|_F \\
= \frac{1}{N} \| \nabla \ell(\theta^T x, y)x^T - \nabla \ell(\theta^T x', y')x'^T \|_F \\
\leq \frac{1}{N} \left( \| \nabla \ell(\theta^T x, y)x^T \|_F + \| \nabla \ell(\theta^T x', y')x'^T \|_F \right) \\
\leq \frac{1}{N} \left( \| \nabla \ell(\theta^T x, y) \|_2 \| x \|_2 + \| \nabla \ell(\theta^T x', y') \|_2 \| x' \|_2 \right) \\
\leq \frac{1}{N} K (\| x \|_2 + \| x' \|_2) \\
\leq \frac{2K}{N}.
\]
We use the triangle inequality in the third step, the fact that \( \|uv^\top\|_F = \|u\|_2\|v\|_2 \) in the fourth step, and assumptions 4 and 5 in steps five and six respectively. Thus we have:

\[
\|\nabla g(\theta)\|_F \leq \frac{2K}{N}.
\]  

(10)

Combining Equations 7 and 10 yields:

\[
\|\theta_1 - \theta_2\|_F \leq \frac{2K}{N^\lambda},
\]  

(11)

completing the proof.

For the perturbations used by the sensitivity methods, we sample noise from the distribution:

\[
p(B) = \frac{1}{z}e^{-\beta \|B\|_F},
\]  

(12)

where \( z \) is a normalizing constant. When \( \delta > 0 \) we use a zero-mean isotropic Gaussian distribution with a standard deviation of \( \sigma \):

\[
p(B) = \frac{1}{z}e^{-\frac{1}{2\sigma^2}\|B\|_F^2},
\]  

(13)

where \( z \) is a normalizing constant.

### B Private Training

#### B.1 Model Sensitivity

We generalize the sensitivity method of [8] to the setting of multi-class classification. The multi-class model sensitivity method achieves \( \epsilon \)-differential privacy with respect to the dataset \( D \) when releasing the model parameters \( \theta \). In this case, the budget of test examples is infinite since the model itself is differentially private. The model sensitivity method is given in Algorithm 1.

**Algorithm 1** Multi-class model sensitivity.

**Inputs:** Privacy parameter \( \epsilon \), regularization parameter \( \lambda \), and dataset \( D \).

**Output:** \( \theta_{\text{priv}} \), the \( \epsilon \)-differentially private classifier.

Sample \( B \) from the distribution in Equation 12 with \( \beta = \frac{N\lambda}{2K} \).

Compute \( \theta_{\text{priv}} = \arg\min_{\theta} J(\theta; D) + B \).

**Theorem 1.** Given assumptions 1, 2, 3, 4, and 5, the model sensitivity method in Algorithm 1 is \( \epsilon \)-differentially private.

**Proof.** We bound the privacy loss in terms of the sensitivity of the minimizer of \( J(\cdot) \) (Equation 5) in the Frobenius norm and then apply Lemma 1. We denote by \( A \) the application of differential privacy mechanism, i.e. a run of Algorithm 1.

For all datasets \( D \) and \( D' \) which differ by one example, we have:

\[
\log \frac{p(A(D) = \theta)}{p(A(D') = \theta)} = \log \frac{e^{-\beta \|\theta_1 + B\|_F}}{e^{-\beta \|\theta_2 + B\|_F}} = \beta (\|\theta_2 + B\|_F - \|\theta_1 + B\|_F) \leq \beta \|\theta_1 - \theta_2\|_F,
\]

where we use the triangle inequality, and the fact that: \( \theta_1 = \arg\min \theta J(\theta; D) \) and \( \theta_2 = \arg\min \theta J(\theta; D') \). Thus:

\[
\log \frac{p(A(D) = \theta)}{p(A(D') = \theta)} \leq \beta \|\theta_1 - \theta_2\|_F.
\]  

(14)
We apply Theorem 8 of [2] which states that for any 

\[ \log \frac{p(A(D) = \theta)}{p(A(D') = \theta)} \leq \beta \|\theta_1 - \theta_2\|_F \leq \frac{\beta 2K}{N\lambda}. \]  

(15)

Thus if we choose \( \beta = \frac{N\lambda}{2K} \), we achieve \( \epsilon \)-differentially private.

The model sensitivity method in Algorithm 1 is \( (\epsilon, \delta) \)-differentially private with \( \delta = 0 \). By letting \( \delta > 0 \), we can use noise from a Gaussian distribution.

Algorithm 2 Analytic Gaussian mechanism [2].

**Inputs:** Privacy parameters \( \epsilon \) and \( \delta \).

**Output:** Scalar \( \alpha \) used to compute the variance of the Gaussian noise distribution.

Let \( \delta_0 = \Phi(0) - e^\epsilon \Phi(-\sqrt{2\epsilon}) \).

if \( \delta \geq \delta_0 \) then

Define \( B^*_+(v) = \Phi(\sqrt{\epsilon v}) - e^\epsilon \Phi(-\sqrt{\epsilon(v + 2)}) \).

Find \( v^* = \sup \{v \in \mathbb{R}_{\geq 0} : B^*_+(v) \leq \delta\} \).

Let \( \alpha = \sqrt{1 + v^*/2} - \sqrt{\epsilon^*/2} \).

else

Define \( B^-_+(u) = \Phi(-\sqrt{\epsilon u}) - e^\epsilon \Phi(-\sqrt{\epsilon(u + 2)}) \).

Find \( u^* = \inf \{u \in \mathbb{R}_{\geq 0} : B^-_+(u) \leq \delta\} \).

end if

We use the analytic Gaussian mechanism of [2], which results in better privacy parameters at the same noise scale than the standard Gaussian mechanism [13]. We rely on Algorithm 2, a subroutine of Algorithm 1 from [2], in order to compute the \( \alpha \) required by the analytic Gaussian mechanism. Following [2], we note that \( B^*_+(v) \) and \( B^-_+(u) \) are monotonic functions and use binary search to find the value of \( v^* \) and \( u^* \) in Algorithm 2 up to arbitrary precision.

Algorithm 3 Multi-class Gaussian model sensitivity.

**Inputs:** Privacy parameters \( \epsilon \) and \( \delta \), regularization parameter \( \lambda \), and dataset \( D \).

**Output:** \( \theta_{\text{priv}} \), the \( (\epsilon, \delta) \)-differentially private classifier.

Compute \( \alpha \) using Algorithm 2

Sample \( B \) from the distribution in Equation 13 with \( \sigma = \frac{2K\alpha}{N\lambda\sqrt{2\epsilon}} \).

Compute \( \theta_{\text{priv}} = \arg \min_{\theta} J(\theta; D) + B \).

Theorem 2. Given assumptions 1, 2, 3, 4 and 5 the Gaussian model sensitivity method in Algorithm 3 is \( (\epsilon, \delta) \)-differentially private for \( \delta \in (0, 1) \).

**Proof.** From Lemma 1 we have a bound on the sensitivity of \( \theta(D) = \arg \min_{\theta'} J(\theta'; D) \):

\[ \max_{D, D'} \|\theta(D) - \theta(D')\|_F \leq \frac{2K}{N\lambda}. \]  

(16)

We apply Theorem 8 of [2] which states that for any \( \epsilon \geq 0 \) and \( \delta \in (0, 1) \), the Gaussian mechanism with standard deviation \( \sigma \) provides \( (\epsilon, \delta) \)-differential privacy if and only if:

\[ \Phi\left(\frac{\Delta_2 f}{2\sigma} - \epsilon\sigma\right) - e^{\epsilon\sigma} \Phi\left(-\frac{\Delta_2 f}{2\sigma} + \epsilon\sigma\right) \leq \delta, \]  

(17)

where \( \Delta_2 f \) is the \( L_2 \) sensitivity of the function \( f(D) \), and \( \Phi(\cdot) \) is the cumulative distribution function (CDF) of the standard univariate Gaussian distribution.

Combining the upper bound on \( \Delta_2 f \) from Equation 16 with Theorem 9 of [2] shows that the Gaussian model sensitivity method in Algorithm 3 is \( (\epsilon, \delta) \)-differentially private. \( \Box \)
\[ J(\theta; D) = \frac{1}{N} \sum_{n=1}^{N} \ell(\phi'(x_n; \theta), y_n) + \frac{\lambda}{N} R(\theta), \quad \text{(18)} \]

We require additional notation to state and prove the privacy theorem of Algorithm 4. First, we let vec(A) denote the vectorization of the matrix A constructed by stacking the rows of A into a vector. If A ∈ R^{m,n} then vec(A) ∈ R^{m,n}. Second, we rely on the standard Kronecker product which we denote by the symbol ⊗. For two vectors we have u ⊗ v = vec(uv^T). We let λ_{max}(A) denote the maximum eigenvalue of a symmetric matrix A.

**Theorem 3.** Given a convex loss \( \ell(\cdot) \), a convex regularizer \( R(\cdot) \) with continuous Hessians, assumptions [3][4] and [5] and assuming that \( \lambda_{\text{max}}(\nabla^2 \ell) \leq L \), the loss perturbation method in Algorithm 4 is \( \epsilon \)-differentially private.

**Proof.** The proof structure here closely follows that of Theorem 9 in [8] and Theorem 2 in [23] with the appropriate generalizations to account for the multi-class setting.

Let \( w = \text{vec}(\theta) \) and \( b = \text{vec}(B) \). We aim to bound the ratio of the two densities:

\[ \frac{g(\theta_{\text{priv}} | D)}{g(\theta_{\text{priv}} | D')} = \frac{g(w_{\text{priv}} | D)}{g(w_{\text{priv}} | D')} \tag{19} \]

By the same argument used in the proof of Theorem 9 of [8], there is a bijection from \( w_{\text{priv}} \) to \( b \) for every \( D \). Hence the ratio in Equation (19) can be written as:

\[ \frac{g(w_{\text{priv}} | D)}{g(w_{\text{priv}} | D')} = \frac{\mu(b | D)}{\mu(b' | D')} \cdot \frac{|\det(J(w_{\text{priv}} \rightarrow b | D))|^{-1}}{|\det(J(w_{\text{priv}} \rightarrow b' | D'))|^{-1}}, \tag{20} \]

where \( \mu(\cdot) \) is the conditional noise density given dataset \( D \) and \( \mu(w_{\text{priv}} \rightarrow b | D) \) is the Jacobian of the mapping from \( w_{\text{priv}} \) to \( b \) for a given \( D \).

We bound each term in Equation (20) starting with a bound on the ratio of the determinants of the Jacobians.

For any \( w_{\text{priv}} \) and \( D \), we can solve for the noise term \( b \) added to the objective in Algorithm 4:

\[ b = -\lambda \nabla R(w_{\text{priv}}) - \sum_{i=1}^{N} \nabla \ell \otimes x_i - \rho w_{\text{priv}}, \tag{21} \]

where we use the short-hand \( \nabla_{\alpha} \ell(a) = \nabla \ell \). By assumption [3] we have \( \nabla_{\theta} \ell = \nabla \ell x^\top \) and hence \( \nabla_{w} \ell = \nabla \ell \otimes x \).

We can differentiate Equation (21) to get:

\[ \nabla b = -\lambda \nabla^2 R(w_{\text{priv}}) - \sum_{i=1}^{N} \nabla^2 \ell \otimes (x_i x_i^\top) - \rho I_{DC}, \tag{22} \]
where \( I_{DC} \) is the \( DC \)-dimensional identity matrix.

Without loss of generality, let \( D' \) contain one more example than \( D \) which we denote by \( x', y' \), i.e. \( D' = D \cup \{ (x', y') \} \). We define the matrices \( A \) and \( E \) as:

\[
A = \lambda \nabla^2 R(w_{\text{priv}}) + \sum_{i=1}^{N} \nabla^2 \ell (x_i, x_i^T) + \rho I_{DC},
\]

and:

\[
E = \nabla^2 \ell \otimes (x'x'^T).
\]

Thus:

\[
\frac{|\det(J(w_{\text{priv}} \rightarrow b' | D'))|}{|\det(J(w_{\text{priv}} \rightarrow b | D))|} = \frac{|\det(A + E)|}{|\det(A)|}.
\]

Given that \( \nabla^2 \ell \in \mathbb{R}^{C \times C} \) we know that \( \text{Rank}(\nabla^2 \ell) \leq C \). Since \( \text{Rank}(xx^T) = 1 \), we have that \( \text{Rank}(\nabla^2 \ell \otimes (xx^T)) \leq C \) [18], and thus \( \text{Rank}(E) \leq C \). Thus we can apply Lemma 2 to bound the ratio:

\[
\frac{|\det(A + E)|}{|\det(A)|} \leq (1 + \lambda_{\max}(A^{-1}E))^C.
\]

Given that \( R(\cdot) \) and \( \ell(\cdot) \) are convex, the eigenvalues of \( A \) are at least \( \rho \), and thus \( \lambda_{\max}(A^{-1}E) \leq \frac{1}{\rho} \lambda_{\max}(E) \). Given the assumption that \( \lambda_{\max}(\nabla^2 \ell) \leq L \) and by assumption \( \lambda_{\max}(xx^T) \leq 1 \), we have that \( \lambda_{\max}(\nabla^2 \ell \otimes (xx^T)) \leq L \) [18], thus \( \lambda_{\max}(E) \leq L \). Combining these observations with Equations 25 and 26 we have a bound on the ratio:

\[
\frac{|\det(J(w_{\text{priv}} \rightarrow b' | D'))|}{|\det(J(w_{\text{priv}} \rightarrow b | D))|} \leq \left(1 + \frac{L}{\rho}\right)^C.
\]

Since \( \rho \geq \frac{2LC}{\epsilon} \), we have:

\[
\left(1 + \frac{L}{\rho}\right)^C \leq \left(1 + \frac{\epsilon}{2C}\right)^C \leq e^{\epsilon/2},
\]

using the identity \( 1 + x \leq e^x \) for all \( x \). Thus we have a final bound on the ratio:

\[
\frac{|\det(J(w_{\text{priv}} \rightarrow b' | D'))|}{|\det(J(w_{\text{priv}} \rightarrow b | D))|} \leq e^{\epsilon/2}.
\]

Next, we bound the ratio \( \frac{\mu(b | D)}{\mu(b' | D')} \). From Equation 21 we have:

\[
b' = \nabla \ell \otimes x'.
\]

Thus:

\[
\|b'\|^2 - \|b\|^2 \leq \|b' - b\|^2 = \|\nabla \ell \otimes x'\|^2 \leq \|\nabla \ell\|_2 \|x'\|_2 \leq K,
\]

where we use the triangle inequality in the first step, the fact that \( \|u \otimes v\|_2 = ||uv^T||_F = \|u\|_2 \|v\|_2 \) in the third step, and assumptions 4 and 5 in the fifth step. Following the argument in the proof of Theorem 9 of [8], we have:

\[
\frac{\mu(b | D)}{\mu(b' | D')} \leq e^{\frac{\epsilon}{2\pi}(\|b\|^2 - \|b'\|^2)}.
\]

Combining Equations 31 and 32 we have:

\[
\frac{\mu(b | D)}{\mu(b' | D')} \leq e^{\epsilon/2}.
\]

Finally, combining Equations 20, 29 and 33 yields:

\[
g(w_{\text{priv}} | D) \frac{\mu(b | D)}{\mu(b' | D')} \cdot \left| \frac{|\det(J(w_{\text{priv}} \rightarrow b | D))|}{|\det(J(w_{\text{priv}} \rightarrow b' | D'))|} \right|^{-1} \leq e^{\epsilon/2} \leq e',
\]

completing the proof.
Algorithm 5 Multi-class Gaussian loss perturbation.

**Inputs:** Privacy parameters $\epsilon$ and $\delta$, and dataset $D$.
**Output:** $\theta_{\text{priv}}$, the $(\epsilon, \delta)$-differentially private classifier.

Set $\rho \geq \frac{2LC}{\epsilon}$.

Sample $B$ from the distribution in Equation 13 with $
abla \epsilon \sqrt{8 \log(2/\delta) + 4}$.

Compute $\theta_{\text{priv}} = \arg\min_\theta J(\theta, D) + \frac{1}{N} \text{tr}(B^T \theta) + \frac{\rho}{2N} \|\theta\|_F^2$.

We can use noise from a Gaussian distribution by letting $\delta > 0$ [23]. We generalize the approach of [23] to the multi-class setting.

**Theorem 4.** Given a convex loss $\ell(\cdot)$, a convex regularizer $R(\cdot)$ with continuous Hessians, assumptions 3, 4, and 5, and assuming that $\lambda_{\max}(\nabla^2 \ell) \leq L$, the loss perturbation method in Algorithm 5 is $(\epsilon, \delta)$-differentially private.

**Proof.** Let $w = \text{vec}(\theta)$ and $b = \text{vec}(B)$. We aim to bound the ratio of the two densities:

$$
\frac{g(\theta_{\text{priv}} | D)}{g(\theta_{\text{priv}} | D')} = \frac{g(w_{\text{priv}} | D)}{g(w_{\text{priv}} | D')},
$$

with probability at least $1 - \delta$.

As in Theorem 3 there is a bijection from $w_{\text{priv}}$ to $b$ for every $D$. Hence the ratio in Equation 35 can be written as:

$$
\frac{g(w_{\text{priv}} | D)}{g(w_{\text{priv}} | D')} = \mu(b | D) \cdot |\det(\nabla w_{\text{priv}} \to b | D)|^{-1} \cdot \mu(b' | D') \cdot |\det(\nabla w_{\text{priv}} \to b' | D')|^{-1},
$$

where $\mu(\cdot)$ is the conditional noise density given dataset $D$ and $\nabla w_{\text{priv}} \to b | D$ is the Jacobian of the mapping from $w_{\text{priv}}$ to $b$ for a given $D$.

From Equation 29 in the proof of Theorem 3 we have that:

$$
|\det(\nabla w_{\text{priv}} \to b | D)|^{-1} \cdot |\det(\nabla w_{\text{priv}} \to b' | D')|^{-1} \leq e^{\epsilon/2}.
$$

Following the same arguments as the proof of Theorem 2 in [23], we have with at least probability $1 - \delta$:

$$
\frac{\mu(b | D)}{\mu(b' | D')} \leq e^{\frac{\epsilon}{2\pi} \left(\beta K \sqrt{8 \log(2/\delta) + K^2}\right)}.
$$

If we choose $\beta \geq \frac{K}{\epsilon} \sqrt{8 \log(2/\delta) + 4K}$ then:

$$
\frac{\mu(b | D)}{\mu(b' | D')} \leq e^{\epsilon/2}.
$$

Combining Equations 36, 37, and 39 yields:

$$
\frac{g(w_{\text{priv}} | D)}{g(w_{\text{priv}} | D')} \leq e^{\epsilon},
$$

with probability at least $1 - \delta$, completing the proof.

**Lemma 2.** Given symmetric and positive semidefinite matrices, $A$ and $E$, if $A$ is full rank, and $E$ has rank at most $C$, then:

$$
\frac{\det(A + E)}{\det A} \leq (1 + \lambda_{\max}(A^{-1}E))^C,
$$

where $\lambda_{\max}(A^{-1}E)$ is the maximum eigenvalue of $A^{-1}E$. 

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Proof.

\[
\frac{\det(A + E)}{\det A} = \det(I + A^{-1}E) \\
= \prod_{c=1}^{C} (1 + \lambda_c(A^{-1}E)) \\
\leq (1 + \lambda_{\text{max}}(A^{-1}E))^C.
\]

In the second step we use the facts that \(\lambda_i(I + A^{-1}E) = 1 + \lambda_i(A^{-1}E)\) and that \(A^{-1}E\) has rank at most \(C\) and hence has at most \(C\) nonzero eigenvalues. \(\square\)

C Private Prediction

C.1 Prediction Sensitivity

The prediction sensitivity method protects the privacy of the underlying training dataset, \(D\), by adding a unique noise vector to the logit predictions, \(\phi'(\hat{x}; \theta)\), for each test example \(\hat{x}\).

The prediction sensitivity method is given in Algorithm 6. The algorithm accepts as input a budget \(B\) of test examples and guarantees \(\epsilon\)-differential privacy of \(D\) until the budget is exhausted after which the privacy of \(D\) degrades.

Algorithm 6 Multi-class prediction sensitivity.

Inputs: Privacy parameter \(\epsilon\), non-private model \(\phi'(. ; \theta)\), test example \(\hat{x}\), and budget \(B\).
Output: \(\hat{y}_{\text{priv}}\), the prediction which is \(\epsilon\)-differentially private with respect to the training dataset \(D\).

Sample \(b\) from the distribution in Equation 12 with \(\beta = \frac{N\lambda}{2KB}\).
Compute \(\hat{y}_{\text{priv}} = \phi'(\hat{x}; \theta) + b\).

Theorem 5. Given assumptions 1, 2, 3, 4, and 5, the prediction sensitivity method in Algorithm 6 is \(\epsilon\)-differentially private.

Proof. The proof proceeds the same way as that of Theorem 1. First, we bound the privacy loss in terms of the sensitivity of the minimizer of \(J(\cdot)\) in the Frobenius norm and then we apply Lemma 1.

Let \(\theta_1 = \arg \min_\theta J(\theta; D)\) and \(\theta_2 = \arg \min_\theta J(\theta; D')\). For all datasets \(D\) and \(D'\) which differ by one example and all \(x\), we have:

\[
\log \frac{p(A(D, x) = y)}{p(A(D', x) = y)} = \log \frac{e^{-\beta |\theta_1^T x + b|_2}}{e^{-\beta |\theta_2^T x + b|_2}} = \beta \left( ||\theta_2^T x + b||_2 - ||\theta_1^T x + b||_2 \right) \\
\leq \beta ||\theta_1^T x - \theta_2^T x||_2 \\
\leq \beta ||(\theta_1 - \theta_2)^T x||_2,
\]

where we use the triangle inequality in the second to last step. We also have that:

\[
||(\theta_1 - \theta_2)^T x||_2 \leq ||\theta_1 - \theta_2||_F ||x||_2 \leq ||\theta_1 - \theta_2||_F,
\]

where we use Cauchy-Schwarz in the first inequality and assumption 5 in the second. Thus:

\[
\log \frac{p(A(D, x) = y)}{p(A(D', x) = y)} \leq \beta ||\theta_1 - \theta_2||_F.
\]

Combining Equation 43 with Lemma 1 yields:

\[
\log \frac{p(A(D, x) = y)}{p(A(D', x) = y)} \leq \beta ||\theta_1 - \theta_2||_F \leq \frac{\beta 2K}{N\lambda}.
\]
If we choose $\beta = \frac{N\lambda}{2K}$, we achieve $\epsilon$-differential privacy when releasing the predictions on a single example $x$. If we choose $\beta = \frac{N\lambda}{2KB}$ then by standard compositional arguments of differential privacy we achieve $\epsilon$-differential privacy with a budget of $B$ test queries.

Algorithm 7 Multi-class Gaussian prediction sensitivity.

**Inputs:** Privacy parameters $\epsilon$ and $\delta$, non-private model $\phi'()$, test example $x$, and budget $B$.

**Output:** $y_{\text{priv}}$, the prediction which is $(\epsilon, \delta)$-differentially private with respect to the training dataset $D$.

Let $\sigma' = \frac{2K\alpha}{N\sqrt{2\epsilon}}$ with $\alpha$ computed using Algorithm 2. $\epsilon^* = \epsilon / B$, and $\delta^* = \delta / B$.

Minimize $\sigma'' = \frac{2K\alpha}{N\sqrt{2\epsilon}}$ over $\delta' \in (0, \delta - B\delta^*)$ using linear search where:

\[ \epsilon^* = \sqrt{\frac{2B}{\delta'}} \left( \sqrt{\ln(1/\delta')} + \epsilon - \sqrt{\ln(1/\delta')} \right), \]

\[ \delta^* = (\delta - \delta') / B, \]

and where $\alpha$ is computed using Algorithm 2.

Sample $b$ from the distribution in Equation 13 with $\sigma = \min(\sigma', \sigma'')$.

Compute $y_{\text{priv}} = \phi'(\hat{x}; \theta) + b$.

Theorem 6. Given assumptions 7, 2, 3, 4, and 5, the Gaussian prediction sensitivity method in Algorithm 7 is $(\epsilon, \delta)$-differentially private for $\delta \in (0, 1)$.

**Proof.** As in the proof of Theorem 2 we bound the $L_2$ sensitivity and apply Theorem 9 of [2].

Let $\theta_1 = \arg \min_{\theta} J(\theta; D)$ and similarly $\theta_2 = \arg \min_{\theta} J(\theta; D')$. From Theorem 5 we have:

\[ \max_{D, D', x} \|\theta_1^T x - \theta_2^T x\|_2 \leq \frac{2K}{N\lambda}. \]  

(45)

Recall from our description of the Gaussian model sensitivity method that Theorem 8 of [2] states that the Gaussian mechanism with standard deviation $\sigma'$ is $(\epsilon^*, \delta^*)$-differentially private if and only if:

\[ \Phi \left( \frac{\Delta_2 f}{2\sigma'} - \frac{\epsilon^* \sigma'}{\sqrt{\Delta_2 f}} \right) - e^{\epsilon^*} \Phi \left( -\frac{\Delta_2 f}{2\sigma'} - \frac{e^* \sigma'}{\sqrt{\Delta_2 f}} \right) \leq \delta^*, \]  

(46)

where $\Delta_2 f$ is the $L_2$ sensitivity of $f(\cdot)$.

Combining Equation 45 with Theorem 9 of [2], we can obtain $(\epsilon^*, \delta^*)$-differential privacy on a single prediction by setting:

\[ \sigma'' = \frac{2K\alpha}{N\sqrt{2\epsilon}}. \]  

(47)

where $\alpha$ is defined in Algorithm 2.

Theorem 1.1 of [14] states that $B$ compositions of a $(\epsilon^*, \delta^*)$-differentially private mechanism satisfies $(\epsilon, \delta)$-differential privacy where:

\[ \epsilon = \sqrt{2B \ln(1/\delta')} \epsilon^* + B e^* (\epsilon^* - 1) / 2, \]  

(48)

and:

\[ \delta = \delta' + B\delta^*. \]  

(49)

We can solve for $\epsilon^*$ and $\delta^*$ in terms of $\epsilon$, $\delta$ and $\delta'$. Given that $\epsilon$ and $\delta$ are pre-specified, this leaves $\delta'$ to be determined.

Using the fact that $1 + x \leq e^x$ for all $x$, we have:

\[ \sqrt{2B \ln(1/\delta')} \epsilon^* + B (\epsilon^*)^2 / 2 \leq \sqrt{2B \ln(1/\delta')} \epsilon^* + B e^* (\epsilon^* - 1) / 2. \]  

(50)

This is a quadratic in $\epsilon^*$ which we can solve to obtain:

\[ \epsilon^* = \sqrt{2B \left( \sqrt{\ln(1/\delta')} + \epsilon - \sqrt{\ln(1/\delta')} \right)}. \]  

(51)

We also have:

\[ \delta^* = (\delta - \delta') / B. \]  

(52)
The algorithm will be $(\epsilon, \delta)$-differentially private with respect to $B$ predictions for any $\epsilon^*$ and $\delta^*$ which satisfy the above equations. Thus, for a given $\epsilon$, $\delta$ and $B$, we can choose $\delta'$ which minimizes $\sigma''$ in Equation 47 and achieve $(\epsilon, \delta)$-differential privacy with a budget $B$.

For small values of the budget $B$, the standard composition theorem of $(\epsilon, \delta)$-differential privacy (e.g., Theorem 3.16 of [10]) may actually lead to smaller standard deviations in the Gaussian noise distribution. Combining the standard composition theorem with Equation 45 and Theorem 9 of [2], we can obtain $(\epsilon, \delta)$-differential privacy on a single prediction by setting:

$$\sigma' = \frac{2K\alpha}{N\lambda\sqrt{2\epsilon^*}},$$

(53)

where $\alpha$ is obtained via Algorithm 2 with $\epsilon^* = \epsilon/B$ and $\delta^* \delta/B$.

Because both $\sigma'$ and $\sigma''$ provide the required differential privacy guarantee, so does the mechanism in Algorithm 7 that sets $\sigma = \min(\sigma', \sigma'')$.

C.2 Subsample-and-Aggregate

The subsample-and-aggregate approach of [11] splits the training dataset $D$ into $T$ disjoint subsets of size $|D|/T$. A set of $T$ models, $\{\phi'_1(\cdot; \theta_1), \ldots, \phi'_T(\cdot; \theta_T)\}$, are learned, one for each subset, where $\phi'_t(\cdot)$ outputs a one-hot vector of size $C$. To make a prediction, the classifiers are combined using a soft majority vote. For a given $x$, the class label $y$ is predicted with probability proportional to:

$$\exp (\beta : |\{t : t \in \{1, \ldots, T\}, \phi'_t(x; \theta_t) = y\})].$$

(54)

Privacy by classifying with probability proportional to an exponentiated utility function is known as the exponential mechanism [13] and is due to [28].

Algorithm 8 Subsample-and-aggregate.

**Inputs**: Privacy parameters $\epsilon$ and $\delta$, training dataset $D$, a number of models $T$, and a set of queries $Q = \{x_1, \ldots, x_B\}$.

**Output**: $\{y_1, \ldots, y_B\}$, the predictions which are $\epsilon$-differentially private with respect to the training dataset $D$.

Partition $D$ into $T$ disjoint subsets $\{D_t : t = 1, \ldots, T\}$ of size $|D|/T$.

for all $D_t$ do

Train classifier $\phi'_t(\cdot; \theta_t)$ on $D_t$.

end for

if $\delta > 0$ then

Set $\beta = \max \left(\frac{\epsilon}{B}, \sqrt{2/B} \left(\sqrt{\ln(1/\delta)} + \epsilon - \sqrt{\ln(1/\delta)}\right)\right)$.

else

Set $\beta = \epsilon/B$.

end if

for $b = 1, \ldots, B$ do

Sample $\tilde{y}_b$ with probability proportional to Equation 54

end for

Theorem 7. The subsample-and-aggregate method in Algorithm 8 with $\beta = \epsilon/B$ is $\epsilon$-differentially private.

Proof. Consider any two datasets $D$ and $D'$ which differ by at most one example. Since the classifiers $\phi'_t(\cdot)$ are trained on disjoint subsets of $D$, at most one classifier can change its prediction for any given instance $x$ when we switch from training on $D$ to $D'$. If we let $\phi_t(\cdot; \theta)$ denote classifiers trained on $D$ and $\phi'_t(\cdot; \theta')$ denote classifiers trained on $D'$, we have:

$$\arg \max_{\tilde{D}, \tilde{D}', x, y} \left|\left|\{t : t \in \{1, \ldots, T\}, \phi_t(x; \theta_t) = y\} - \{t : t \in \{1, \ldots, T\}, \phi'_t(x; \theta'_t) = y\}\right|\right| \leq 1.$$  

(55)
In other words, the sensitivity of the majority vote is 1. For all adjacent datasets, $D$ and $D'$, all examples $x$, and all class labels $y$:

$$
\log p(A(x, D) = y) = \log \frac{\exp(\beta \cdot \{t : t \in \{1, \ldots, T\}, \phi'_i(x; \theta_i) = y\})}{\exp(\beta \cdot \{t : t \in \{1, \ldots, T\}, \phi'_i(x; \theta'_i) = y\})} \\
\leq \beta \cdot \{t : t \in \{1, \ldots, T\}, \phi'_i(x; \theta_i) = y\} - \beta \cdot \{t : t \in \{1, \ldots, T\}, \phi'_i(x; \theta'_i) = y\} \\
\leq \beta \{t : t \in \{1, \ldots, T\}, \phi'_i(x; \theta_i) = y\} - \{t : t \in \{1, \ldots, T\}, \phi'_i(x; \theta'_i) = y\} \\
\leq \beta,
$$

(58)

where we use Equation (55) in the second to last step.

Thus if we set $\beta = \epsilon / B$, by standard compositional arguments [12], we achieve $\epsilon$-differential privacy with a budget of $B$ test queries by predicting $y$ with probability proportional to Equation (54).

We can achieve a better scaling of $\beta$ with the budget $B$ by letting $\delta > 0$ and relying on the advanced composition theorem [15].

**Theorem 8.** The subsample-and-aggregate method in Algorithm 8 with $\beta = \max(\beta', \beta'')$, where $\beta' = \epsilon / B$ and $\beta'' = \sqrt{2B}\left(\sqrt{\ln(1/\delta)} + \epsilon - \sqrt{\ln(1/\delta)}\right)$, is $(\epsilon, \delta)$-differentially private.

**Proof.** From Theorem 7 the subsample-and-aggregate method with $B = 1$ is $(\beta, 0)$-differentially private.

Theorem 1.1 of [14] states that $B$ compositions of a $(\beta, 0)$-differentially private mechanism satisfies

$$
\left(\sqrt{2B\ln(1/\delta)}\beta + B\beta(e^\beta - 1)/2, \delta\right).
$$

We can solve for $\beta$ such that the subsample-and-aggregate algorithm with a budget of $B$ satisfies $(\epsilon, \delta)$-differential privacy. Using the fact that $1 + x \leq e^x$ for all $x$, we have:

$$
\sqrt{2B\ln(1/\delta)}\beta + B\beta^2/2 \leq \sqrt{2B\ln(1/\delta)}\beta + B\beta(e^\beta - 1)/2.
$$

(60)

Hence if $\sqrt{2B\ln(1/\delta)}\beta + B\beta^2/2 \leq \epsilon$ then the resulting algorithm will satisfy $(\epsilon, \delta)$-differential privacy. This is a quadratic in $\beta$ which we can solve to obtain:

$$
\beta = \sqrt{2/B}\left(\sqrt{\ln(1/\delta)} + \epsilon - \sqrt{\ln(1/\delta)}\right).
$$

(61)

If this value of $\beta$ is smaller than $\epsilon / B$, we can use Theorem 7 instead and set $\beta = \epsilon / B$.

**D Multi-class Logistic Loss**

In practice, we need to specify $\ell(\theta^T x, y)$ and bound $\|\nabla \ell\|_2$ for the model sensitivity and loss perturbation methods. The commonly used multi-class logistic loss is given by:

$$
\ell(a, y) = \sum_{i=1}^C a_i \log \frac{e^{y_i}}{\sum_{j=1}^C e^{a_j}},
$$

(62)

where, for linear models, $a = \theta^T x$.

**Theorem 9.** The Lipschitz constant of the multi-class logistic loss (Equation (62)) is $K = \sqrt{2}$.

**Proof.** Since $y \in \Delta^C$, we can write Equation (62) as:

$$
\ell(a, y) = y^\top a - \log Z,
$$

(63)

where $Z = \sum_{j=1}^C e^{a_j}$, which has a gradient given by:

$$
\nabla \ell = y - \frac{1}{Z} e^a.
$$

(64)
By Corollary 1, the maximum $L_2$ distance between any two points on the probability simplex $\Delta^C$:

$$\frac{\|\nabla \ell\|_2}{\|\nabla \ell\|_2} = \left\| \mathbf{y} - \frac{e^a}{\sum_{j=1}^C e^a_j} \right\|_2 \leq \sqrt{2}. \quad (65)$$

For the loss perturbation method, we also need to bound the eigenvalues and the rank of the Hessian of Equation 62 with respect to $a$.

The Hessian of the multi-class logistic loss with respect to $a$ is given by:

$$\nabla^2 \ell = \text{diag}(\mathbf{p}) - \mathbf{p} \mathbf{p}^\top, \quad (66)$$

where $\mathbf{p} = \frac{1}{Z} e^a \mathbf{e}$. Since $\nabla^2 \ell \in \mathbb{R}^{C \times C}$, the rank is at most $C$.

**Theorem 10.** The eigenvalues of the Hessian of the multi-class logistic loss are bounded by 0.5, i.e.:

$$\lambda_{\max}(\nabla^2 \ell(\theta, x, y)) \leq 0.5, \quad (67)$$

for all $\theta, x$ and $y$.

**Proof.** The Hessian of the multi-class logistic loss is given by Equation 66. We use the fact that the eigenvalues of a square matrix $A$ are contained in the union of the Gerschgorin discs constructed from the rows of $A$ [19]. The Gerschgorin disc of the $i$-th row of $A$ has a center at $A_{ii}$ and a radius of $\sum_{j \neq i} |A_{ij}|$. Hence, an upper bound on the $i$-th Gerschgorin disc of $\text{diag}(\mathbf{p}) - \mathbf{p} \mathbf{p}^\top$ is given by:

$$(p_i - p_i^2) + p_i \sum_{j \neq i}^C p_j = p_i(1 - p_i) + p_i(1 - p_i) \leq 0.25 + 0.25 = 0.5, \quad (68)$$

where we use the facts that $\sum_{j \neq i}^C p_j = 1 - p_i$ and $(1 - p_i)p_i \leq 0.25$ for $p_i \in [0, 1]$. Hence:

$$\lambda_{\max}(\nabla^2 \ell) \leq 0.5. \quad (69)$$

We rely on the following Corollary to bound the $L_2$ distance between two points on the probability simplex, $\Delta^C$.

**Corollary 1.** For any two points $\mathbf{u}$ and $\mathbf{v}$ on the $(C - 1)$-dimensional probability simplex their $L_2$ distance is no more than $\sqrt{2}$, i.e.:

$$\max_{\mathbf{u}, \mathbf{v} \in \Delta^C} \|\mathbf{u} - \mathbf{v}\|_2 \leq \sqrt{2}. \quad (70)$$

**Proof.** Bauer’s maximum principle states that a convex function on a convex set attains its maximum at an extreme point [25]. The $L_2$ norm is a convex function, the simplex $\Delta^C$ is a convex set. The extreme points of the simplex are when $\mathbf{u}$ and $\mathbf{v}$ are vertices which for $\Delta^C$ implies they are standard basis vectors. The $L_2$ distance between two standard basis vectors is $\sqrt{2}$. \qed