Field theory of avalanche formation

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Self-organizing system is studied whose behavior is governed by field of an order parameter, a fluctuation amplitude of conjugate field and a couple of Grassmannian conjugated fields that define the entropy as a control parameter. Within the framework of self-consistent approach the dependencies of macro- and microscopic susceptibilities as well as memory and nonergodicity parameters are determined as a functions of the intensities of thermal and quenched disorders. Making use of the sandpile model shows that proposed scheme determines the conditions of avalanches formation in self-organized criticality phenomena.

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I. INTRODUCTION

Nowadays there is such an original situation in the theory of self-organizing systems. On the one hand, the synergetic concept has been developing successfully for more than twenty years. It allows us to explain the self-organization (ordering) of open system subjected to the environment disorder. On the other hand, the phenomena of the self-organized criticality such as an avalanche motion of the sand grains on inclined surface (sandpile model), intermittency in biological evolution, earthquakes and forest fire, pinning in the random medium etc. (see [1]) have been actively investigated for about ten years. However, in spite of the fact that the mission of both synergetics and theory of self-organized criticality is to explain the same phenomenon – the self-organization, they develop independently. It became so as within the framework of the synergetic approach the single statistical ensemble (formation of an avalanche) is investigated, whereas the examination of self-organized criticality models is reduced usually to study of evolution of hierarchical avalanche ensemble and is based on numerical methods and scaling representations.

Being the object of synergetics the formation of a single avalanche still remains an open question. It is evident that its reviewing is the only way to set connection between specified directions. Since phenomena of self-organized criticality are caused by avalanche ensemble behavior, and certain closed region of the state space corresponds to each avalanche, the standard formulation of the synergetic problem requires a nontrivial extension – while ordering self-organizing system we have to describe not only a symmetry breaking, but the ergodicity breaking that induces the clusterisation of phase space as well. In our paper a solution of this problem is suggested. In so doing we will investigate only the self-organization picture digressing from avalanche ensemble consideration (see [2]).

The first attempt of field description of an avalanche formation was undertaken within the framework of the one-parameter approach, based on nonlinear diffusion equation – thus, the feedback between open system and environment was not taken into account. Recently, two-parameter models were suggested, where the environment is represented either by control parameter, or conjugate field. According to avalanche-like subcritical mode is formed in case of adiabatic relationship of characteristic times and first-order transition mechanism. The approach of Ref. [10] allowed us to determine the critical indexes representing a scaling behavior of a self-organizing system within the framework of mean field approximation. Below we propose the generalized self-consistent scheme that is taken into account total number of the freedom degrees. Due to this we obtain not only mutually supplementary results but also the complete analytical description of an avalanche formation.

The suggested approach is based on synergetic generalization of the thermodynamic theory of phase transitions. The main feature of this theory is that within the conserved system (thermostat) a subsystem is segregated to represent a hydrodynamical mode whose amplitude is qualified as the order parameter, that determines the state of the subsystem. Here it is assumed that thermostat influences upon value of the order parameter both thermally – by varying of a control parameter, and immediately – by varying the field conjugated to the order parameter (in case of magnet the values , mean the magnetization, the magnetic field and the entropy). The distinctive peculiarity of the thermodynamic approach is that a postulation of the one-sided influence of thermostat on the ordering subsystem is declared, but not the reverse – the order parameter variation does not influence upon thermostat state parameters and . The synergetic approach considers the connection between an open subsystem and thermostat to be two-sided, so that the control parameter and conjugate field turn out to be the functions of order parameter. This connection manifests itself especially while the phase transition is described kinetically. So, the standard picture meeting the Landau-Khalatnikov dissipative dynamics is realized in adiabatic approximation, when the relaxation time of the order parameter is much longer that corresponding times for conjugate field and control parameter. To this end the making use of Lorentz system, primarily suggested in order to describe the turbulent airflows, happens to be rather convenient.

Our paper is organized as follows. In sections (basic expressions), (Lagrange formalism), and (Appendix) is shown that the Lorentz system corresponds to the simplest Lagrangian of the supersymmetric field with components giving an order parameter, a conjugate field, and entropy. It is rather important that in this case a combination of the Grassmannian components of superfield plays the role of control parameter in contradistinction to the usual field theory of a stochastic system, where they are an auxiliary variables which have no physical meaning. The fact that the variables meet the vector of supersymmetric space is a reflection of the self-consistent behavior of the synergetic system (in contrast to the statistical field scheme, where the superfield is only a convenient technical method). The study of superfield correlators is carried out in section As is known the components of such correlators are not independent in ergodic state – the presence of supersymmetry causes the fluctuation-dissipation theorem which connects stated components. If the quenched disorder is appeared, an ergodicity loosing happens that breaks the supersymmetry in its turn. This leads to appearance of singular additives to correlators that define memory and nonergodicity parameters and . The basic result of our work is the defining quantities dependencies on thermal and quenched disorders intensities. This allows us to find the conditions
of avalanches formation in the self-organized criticality mode appearance. The generic example of such a process is considered in section \[\text{VI}\] as the flow of sand on inclined surface. It is shown that this process can be represented by the Lorentz system, considering the horizontal and vertical components of the sand grain velocities to be the order parameter \(\eta\) and conjugate field \(h\), and the tangent of a surface inclination angle as the control parameter \(S\). Final section \[\text{V}\] is devoted to discussion of obtained results. It is shown that a parameter which determines the transition to time irreversible regime is given by ratio of a time of the quantum fluctuation to the one of order parameter. A critical value of effective interaction that bounds the domain of ordered state is found.

II. LORENTZ SYSTEM

To expound the microscopic scheme of the self-organization description let us study firstly the system consisting of Bozon and Fermion gases whose interaction is characterized by potential \(v\). Within the framework of secondary quantization Bozons are described by the \(b^+_l\) and \(b_l\) operators, satisfying the usual commutation relation: \([b_l, b^+_m] = \delta_l m\), where \(l, m\) are the site numbers. The two-level Fermion subsystem is represented by operators \(a^+_{\alpha l}, a_{\alpha l}, \alpha = 1, 2\), for which the anti-commutation relation \([a_{\alpha l}, a^+_{\beta m}] = \delta_l m \delta_{\alpha \beta}\) is fulfilled. The occupation numbers \(b^+_l b_l\) determine the Bozon distribution within \(k\)-representation that corresponds to the Fourier transform over lattice sites \(l\). To represent the Fermi subsystem we should introduce the operator \(d_l = a^+_{\alpha l} d_{\alpha l}\) determining the polarization with respect to the saturation over levels \(\alpha = 1, 2\), as an addition to the occupation numbers \(n_{\alpha l} = a^+_{\alpha l} a_{\alpha l}\). As a result, behavior of the system under consideration is defined by Dicke Hamiltonian

\[
H = \sum_k \left\{ (E_1 n_{k1} + E_2 n_{k2}) + \omega_k b^+_k b_k + \frac{i}{2} v (b^+_k d_k - d^+_k b_k) \right\},
\]  

where the \(k\)-representation is used, \(E_{1,2}\) are the Fermi levels energies, \(\omega_k\) represents the Bozon dispersion law and the imaginary unit before the interaction term \(v\) reflects the Hermitity property, the Planck constant is \(\hbar = 1\).

The Heisenberg equations of motion corresponding to Hamiltonian \([1]\) have the form

\[
\begin{align*}
\dot{b}_k &= -i \omega_k b_k + (v/2) d_k, \tag{2a} \\
\dot{d}_k &= -i \Delta d_k + (v/2) b_k (n_{k2} - n_{k1}), \tag{2b} \\
\dot{n}_{k1} &= (v/2) (b^+_k d_k + d^+_k b_k), \tag{2c} \\
\dot{n}_{k2} &= -(v/2) (b^+_k d_k + d^+_k b_k), \tag{2d}
\end{align*}
\]

where the dot stands for a derivative with respect to time and the quantity \(\Delta = E_2 - E_1\) is introduced. In case of resonance suggested the first terms in the right-hand sides of equations \([2a] , [2b]\) that contains frequencies \(\omega_k\) and \(\Delta\) may be eliminated by extracting the multipliers \(\exp(-i \omega_k t)\) and \(\exp(-i \Delta t)\) in the time dependencies \(b_k(t), d_k(t)\), respectively. On the other hand, if the dissipation is taken into account, these frequencies obtain imaginary additions \(-i/\tau_\eta, -i/\tau_h\) characterized by relaxation times \(\tau_\eta, \tau_h\) (here the conditions \(\text{Im} \omega_k < 0, \text{Im} \Delta < 0\) reflect the “causality” principle). As a result, equations \([2a] , [2b]\) get the dissipative terms \(-b_k/\tau_\eta, -d_k/\tau_h\), where \(\tau_\eta\) is the relaxation time of Bozon distribution and \(\tau_h\) is the Fermion polarization time. One can suppose that the dissipation influences onto the Fermi levels occupancies \(n_{k\alpha}(t)\) also. However, since the stationary values \(n_{k\alpha}^0 \neq 0\) (and in case of external pumping \(n_{k2}^0 > n_{k1}^0\) ) the dissipative terms in Eqs. \([2c] , [2d]\) have much complicated form: \(- (n_{k\alpha} - n_{k\alpha}^0)/\tau_S\), \(\alpha = 1, 2\), where \(\tau_S\) is the relaxation time of the Fermion distribution over level.

Now, let us introduce the macroscopic quantities:

\[
\begin{align*}
\eta_k &= \langle b^+_k \rangle = \langle b_k \rangle, \\
h_k &= \langle d_k \rangle = \langle d^+_k \rangle, \\
S_k &= \langle n_{k2} - n_{k1} \rangle, \\
S^{0}_k &= \langle n_{k2}^0 - n_{k1}^0 \rangle,
\end{align*}
\]
where the angular brackets mean thermodynamic averaging. Neglecting the correlation in distribution of particles over quantum states the Heisenberg equations (2), being contemplated by dissipative terms, result in the Lorentz system

\[ \tau_\eta \dot{\eta} = -\eta + A_\eta h, \]  
\[ \tau_h \dot{h} = -h + A_h \eta S, \]  
\[ \tau_S \dot{S} = (S^0 - S) - A_S \eta h. \]  

(4)

Here in terms of one-mode approximation the dependence on the wave vector \( \mathbf{k} \) is omitted and the constants defined by relationships \( 2A_\eta \equiv v \tau_\eta, 2A_h \equiv v \tau_h, A_S \equiv 2v \tau_S \) are introduced. Equations (4) contain following seven constants – pumping parameter \( S^0 \), three relaxation times \( \tau \) and three coupling constants \( A \). But as there are above relations between these last ones caused by interaction parameter \( v > 0 \), only five of them are independent. Since four of these fix the scales for quantities \( \eta, h, S, t \), so only the parameter of thermal disorder \( S^0 \) plays a substantial role and its value determines only the system behavior.

To analyze equations (4) we introduce the scales \( \eta_m, h_m, S_c \) defining ranges of the variation for the order parameter \( \eta \), the conjugate field \( h \) and the control parameter \( S \):

\[ \eta_m^{-2} \equiv A_h A_S = \tau_h \tau_S v^2, \quad h_m^{-1} \equiv A_\eta / \eta_m = (\tau_\eta / 2)(\tau_h \tau_S)^{1/2} v^2; \]
\[ S_c^{-1} \equiv A_h A_\eta = 2^{-2} \tau_\eta \tau_h v^2. \]  

(5)

Then, using magnitudes \( \eta, h, S \) normalized by \( \eta_m, h_m, S_c \) values, we results Eqs. (4) in the form:

\[ \tau_\eta \dot{\eta} = -\eta + h, \]  
\[ \tau_h \dot{h} = -h + \eta S, \]  
\[ \tau_S \dot{S} = (S^0 - S) - \eta h. \]  

(6)

In terms of adiabatic approximation \( \tau_h, \tau_S \ll \tau_\eta \) the left-hand sides of equations \( (6b), (6c) \) may be set equal to zero. Thus, we derive to a result

\[ h = S^0 \eta / (1 + \eta^2), \quad S = S^0 / (1 + \eta^2). \]  

(7)

With the order parameter growth in physical domain \( \eta \in [0, 1] \) the conjugate field increases and the control parameter decreases monotonically. If \( \eta > 1 \) the \( h(\eta) \) dependence is of decreasing shape, that corresponds to unstable state.

Inserting (7) into (6a) we obtain the Landau-Khalatnikov equation

\[ \tau_\eta \dot{\eta} = -\partial V / \partial \eta, \quad V \equiv \frac{1}{2} \left( \eta^2 - S^0 / (1 + \eta^2) \right). \]  

(8)

Its form is defined by synergetic potential \( V(\eta) \) having its minimum at point \( \eta_0 = (S^0 - 1)^{1/2} \). Hence it is seen that stationary value of the order parameter \( \eta_0 \neq 0 \) is realized under the thermal disorder conditions \( S^0 > 1 \) (\( S^0 > S_c \) in usual units). Thus, the magnitude \( S_c \) defined in last equality is the critical value of the control parameter. According to Eq. (7) in the stationary state one has \( h_0 = (S^0 - 1)^{1/2}, S_0 = 1 \). The last equality implies that despite supercritical value \( S^0 > 1 \) of thermal disorder the system relaxes so, that the stationary value of the control parameter \( S_0 = 1 \) is reduced to the critical one.

The mentioned relaxation is provided by the negative feedback of order parameter \( \eta \) and conjugate field \( h \) with control parameter \( S \) that is described by the last term of equation (6c). This feedback, displaying the Le Chatelier principle for the self-organizing system, compensates the \( S(\eta), h(\eta) \) thermostat’s state parameters increase which
takes place, when this feedback is absent. On the other hand, the positive feedback of quantities \( \eta \) and \( S \) with \( h \) in (11) is the reason for the self-organization. It is obvious that the stationary state \( \eta_0, h_0, S_0 \) can be realized only under the condition of inverse subsystem influence, characterized by order parameter \( \eta \), on the thermostat parameters \( h, S \).

It is worthwhile to note that inverting the signs of nonlinear terms in equations (6) causes the minus appearance in the right-hand side of the first equality (2), so that the susceptibility \( \chi = d\eta/dh \) becomes negative and this case does not meet the stable state.

Thus, the self-organization process takes place only if the negative feedback of order parameter \( \eta \) and field \( h \) with control parameter \( S \) and the positive feedback of \( \eta \) and \( S \) with \( h \) both exist. According to (13), (14) such a choice of signs is determined by the fact that the negative feedback provides falling-down of the control parameter \( S \) in the course of time, whereas the positive one insures the field \( h \) growth. Further we shall show that the value \( S \) is reduced to the entropy and its decrease reflects the non-conservation of the self-organizing system for which the second law of thermodynamics is not fulfilled. The crucial role of the increasing character of field \( h \) is stipulated by the fact that the linear equation (6a) for the order parameter \( \eta \) contains namely the field \( h \). As a result, influence of the field \( h \) on the velocity of \( \eta(t) \) increase, and on the self-organization process also, is direct, whereas the influence of control parameter is indirect.

The described scheme of self-organization meets the second-order phase transition. To describe the first-order transition one ought to set the relaxation time \( \tau_\eta \) of order parameter to be the function of its value \( \eta \) (15). Such a scheme of an avalanche formation is represented in (3).

### III. LAGRANGE FORMALISM

In the previous section we omitted, within the adiabatic approximation, the fluctuations of conjugate field \( h \) and control parameter \( S \) and, thus, we made it possible to reduce the Lorentz system (6) to the Landau-Khalatnikov equation (8). To form the Lagrange formalism one should accomplish the reverse transition supposing a fluctuation source \( \zeta \) appearance in (8). If the nonhomogeneity is considered, the generic expression is reduced to the Langevine equation (see [16] for example)

\[
\dot{\eta}(r,t) - \nabla^2 \eta(r,t) = f(r,t) + \zeta(r,t),
\]

where \( r \) is the coordinate measured in units of the correlation lengths \( \xi \) and \( t \) is the time related to the scale \( \tau_\eta \), the force \( f = -V_0'(\eta) \), \( V_0' \equiv \partial V_0/\partial \eta \) is defined by dependence \( V_0(\eta) \) for the bare potential related to fluctuations intensity \( T \). The term \( -\nabla^2 \eta \) in the left-hand side of Eq. (9) takes into account the spatial nonhomogeneity within the framework of Ginzburg-Landau model. The expression (9) is valid for nonconserved order parameter, otherwise terms \( -\nabla^2 \eta, f \) obtain the additional operator \( -\nabla^2 \) [16]. The fluctuational term is normalized by the white noise conditions

\[
\langle \zeta(r,t) \rangle = 0, \quad \langle \zeta(r,t)\zeta(r',t') \rangle = T\delta(r - r')\delta(t - t'),
\]

which correspond to averaging over the Gaussian distribution with dispersion \( T \).

To construct the Lagrangian corresponding to the Langevine equation (9), let us use the standard field scheme [14] based on a generating functional

\[
Z\{\eta(r,t)\} = \left\langle \prod_{(r,t)} \delta(\dot{\eta} - \nabla^2 \eta - f - \zeta) \right. \det \frac{\delta \zeta}{\delta \eta} \left. \right\rangle
\]

being the generalization of the partition function. Herein the continual product of \( \delta \)-functions takes into account that the condition (14) ought to be satisfied with all values \( r, t \) and the determinant represents the Jacobian of \( \zeta \) to \( \eta \) transition. We apply the Fourier transform for \( \delta \)-function leading to appearance of the field \( \varphi(r,t) \) and introduce the Grassmannian conjugate fields \( \bar{\psi}(r,t), \bar{\psi}'(r,t) \) into operation for integrated representation of the determinant. Thus, the equality (14) assumes the canonical form

\[
Z\{\eta\} = \int P\{\eta, \varphi, \bar{\psi}, \psi\} D\varphi D\bar{\psi} D\psi, \quad P \propto e^{-S}, \quad S \equiv \int L(\eta, \varphi, \bar{\psi}, \psi) dr dt,
\]

where the Lagrangian

\[
L = \varphi(\dot{\varphi} - \nabla^2 \varphi) + \bar{\psi}(\dot{\bar{\psi}} - \nabla^2 \bar{\psi}) - \varphi^2/2 + \varphi V_0'(\eta) + V_0''(\eta) \bar{\psi}\psi
\]

(13)
is measured in noise intensity $T$ units. The form of the corresponding Euler equations is as follows:

$$\dot{\eta} - \nabla^2 \eta = -V_0'(\eta) + \varphi,$$

(14a)

$$\dot{\varphi} + \nabla^2 \varphi = V_0''(\eta)\varphi + V_0'''(\eta)\bar{\psi}\psi,$$

(14b)

$$\dot{\psi} - \nabla^2 \psi = -V_0''(\eta)\psi,$$

(14c)

$$\dot{\bar{\psi}} + \nabla^2 \bar{\psi} = V_0'''(\eta)\bar{\psi}.$$

(14d)

The first one can be reduced to the Langevne equation (9) by replacing the field $\varphi$ by stochastic component $\zeta$. Since this equation corresponds to the maximum of a probability $P$ distribution in (12), $\varphi$ represents the amplitude of the most probable fluctuation of the field conjugated to the order parameter $\eta$ (the force $f$ is the average value of this field). Obviously, the conditions $\langle \zeta \rangle = 0, \varphi \neq 0$ imply that during the self-organization the bare Gaussian distribution transforms from unimodal into bimodal form with its maximums at points $\pm \varphi$. The signs distribution in front of the gradient terms in Eqs. (14) is quite remarkable: albeit the standard combination inherent in relaxation processes like diffusion is realized for components $\eta, \psi$, but the fields $\varphi, \bar{\psi}$ contain nonhomogeneity terms with opposite signs that means the autocatalitical increase of those components. It is shown in Appendix how the field equations (14) are reduced to the Lorentz system (6).

The most elegant method to represent developed field scheme is to incorporate the components $\eta, \psi, \bar{\psi}$ and the generalized force $\phi \equiv -\delta V_0(\eta)/\delta \eta$ into supersymmetrical field

$$\Phi = \eta + \bar{\psi}\chi + \bar{\chi}\psi + \bar{\chi}\chi\phi,$$

(15)

where Grassmannian coordinates $\chi, \bar{\chi}$ posses the same anti-commutation properties as the $\psi, \bar{\psi}$ fields. To represent the Lagrangian (13) supersymmetricaly one should first of all replace the bare potential $V_0(\eta)$ by renormalized one $\tilde{V}(\eta)$, presented in motion equation (A.7), and get rid of gradient addends, committing transform to the variational derivatives $\tilde{V}'(\eta) \equiv \delta \tilde{V}(\eta)/\delta \eta = \partial \tilde{V}(\eta)/\partial \eta - \nabla^2 \eta, \tilde{V}(\eta) \equiv \int \tilde{V}(\eta) d\mathbf{r}$. Then, expressing fluctuation amplitude $\varphi$ in terms of generalized force $\phi$ according to equality (A.9), we derive to the following form of the Lagrangian (13):

$$L = \left(\dot{\eta}^2/2 + \dot{\bar{\psi}}\psi - \phi^2/2\right) + \left(-\tilde{V}'(\eta)\phi + \bar{\psi}\tilde{V}''(\eta)\psi\right) + \tilde{V}'(\eta)\eta.$$

(16)

The last addend can be omitted as the total time derivative of $\tilde{V}(\eta)$ and in superfield representation (15) the Lagrangian (16) assumes the canonical form:

$$L = \int \Lambda(\Phi) d\bar{\chi} d\chi,$$

$$\Lambda \equiv (1/2)\Phi DD\Phi + \tilde{V}(\Phi).$$

(17)

Here the kinetic superenergy of the kernel $\Lambda$ corresponds to the first bracket of expression (16) and the potential superenergy $\tilde{V}(\Phi)$ – to the second one. The equalities (A.3), (A.7), and (A.8) within the framework of the $\Phi^4$-model give

$$\tilde{V} = -\frac{1}{2} \Phi^2 + \frac{3}{12} \Phi^4,$$

(18)

where the anharmonicity parameter $w > 0$ emerges because in contrast to the order parameter $\eta$ (see (3)) the superfield (13) cannot be scaled by the only magnitude $\eta_m$. The supersymmetry group generators have the form

1) It is shown in Appendix that the renormalization is caused by self-consistency of the superfield components (15).
\[ D = \frac{\partial}{\partial \chi} + \chi \frac{\partial}{\partial t}, \quad \bar{D} = \frac{\partial}{\partial \bar{\chi}} + \bar{\chi} \frac{\partial}{\partial t}. \]  

The superequations of motion ensuing from the superaction extremum condition \( S\{\Phi(z)\} = \int \Lambda(\Phi(z))dz, \quad z \equiv \{r,t,\bar{\chi},\chi\} \) read:

\[
(1/2)[\bar{D}, D]\Phi + \tilde{V}'(\Phi) = 0, \quad (1/2)[\bar{D}, D]\Phi \equiv \phi + \tilde{\psi} \chi + \bar{\chi} \tilde{\psi}. \tag{20}
\]

Projecting Eqs. (20) onto basis vectors 1, \( \bar{\chi}, \chi, \bar{\chi} \chi \) of the superspace, we arrive at equations

\[
\bar{\eta} = -\tilde{V}'\{\eta\} \phi + \tilde{V}''\{\eta\} \tilde{\psi}, \tag{21a}
\]

\[
\phi = -\tilde{V}'\{\eta\}, \tag{21b}
\]

\[
\psi = -\tilde{V}''\{\eta\} \psi, \tag{21c}
\]

\[
\tilde{\psi} = \tilde{V}''\{\eta\} \tilde{\psi}. \tag{21d}
\]

The last of this set can be obtained from (14e), (14f) by replacing \( V_0 \) by \( \tilde{V} \), and the relationship (21b) gives the definition of the force \( \phi \). Equation (21a) is obtained by time differentiating of equation (14a) and substituting the derivatives \( \bar{\eta}, \tilde{\phi} \) from (13), (14d) into resultant expression. Thus, systems (14), (21) turn out to be equivalent with accuracy to the bare potential \( V_0(\eta) \) renormalization. However, though the equations of the first set are symmetrical concerning the time derivative order, in (21) this symmetry is broken due to the transformation the fluctuation \( \varphi \) to the generalized force \( \phi \). Comparing Lagrangians (13) and (16) shows the fact that the above transformation provides the standard bilinear form of superlagrangian with respect to operators \( \bar{D} \) and \( D \). Easy to show that, if the gauge condition \( D \Phi = 0 \) is satisfied the Grassmanian fields \( \psi, \tilde{\psi} \) are suppressed and the single combination \( \chi \bar{\chi} \) replaces the pair of conjugate ones \( \bar{\chi}, \chi \). In this case the kinetic superenergy is linear relatively to the supergroup generator, moreover both pairs – \( \eta, \phi \) and \( \eta, \varphi \) are allowable to be used as superfield components [17]. However, as is shown in Appendix, at the description of the self-organization the behaviour of the Grassmanian fields represents entropy and, consequently, is essential. Therefore the gauge \( D \Phi = 0 \) is not fulfilled, and one should prefer making use of the generalized force \( \phi \) to the fluctuation amplitude \( \varphi \).

IV. CORRELATION TECHNICS

Let us introduce the supersymmetrical correlator

\[
C(z,z') \equiv \langle \Phi(z)\Phi(z') \rangle, \quad z \equiv \{r,t,\bar{\chi},\chi\}. \tag{22}
\]

According to the motion equation (20) its bare component \( C^{(0)}(z,z') \) meeting the potential \( \tilde{V}^{(0)} = (1 - \sigma)\Phi^2/2 \) satisfies equality

\[
L_{k\omega}(\chi)C^{(0)}_{k\omega}(\chi',\chi) = \delta(\chi,\chi'), \quad L \equiv (1 - \sigma) + (1/2)[\bar{D}, D], \tag{23}
\]

where the transition to the time-spatial Fourier transforms is made and the supersymmetrical \( \delta \)-function \( \delta(\chi,\chi') = -(\bar{\chi} - \bar{\chi}')(\chi - \chi') \) is introduced. Taking into consideration definitions (19) and expressions (1/4)[\bar{D}, D] = -\omega^2, we obtain

\[
C^{(0)}(\chi,\chi') = \frac{1 - (\bar{\chi}\chi + \bar{\chi}'\chi') + [(1 - \sigma) + i\omega]\bar{\chi}\chi' + [(1 - \sigma) - i\omega]\bar{\chi}'\chi - \omega^2 \bar{\chi}\chi \bar{\chi}'\chi'}{(1 - \sigma)^2 + \omega^2}, \tag{24}
\]

where index \( \omega \) and spatial dispersion were omitted for brevity. Equation (24) is an expansion in basis components
\[
\begin{align*}
   B_0 &= -\bar{\chi}\chi, & B_1 &= -\bar{\chi}'\chi', \\
   T &= 1, & T_1 &= \bar{\chi}\chi' \\
   F_0 &= \bar{\chi}'\chi, & F_1 &= \bar{\chi}'\chi'
\end{align*}
\]  

(25)

whose functional product satisfies the following multiplication rules: \(B_0^2 = B_0, B_1^2 = B_1, F_0^2 = F_0, F_1^2 = F_1, B_0T_1 = T_1, B_1T = T, TB_0 = T, TT_1 = B_1, T_1B_1 = T_1, T_1T = B_0,\) and the other multiplicands are equal to zero. Thus, \(B_{0,1}, T, T_1, F_{0,1}\) form the closed basis, and it is convenient to expand supercorrelator \(\mathbb{G}\) in these components:

\[
\mathbb{C} = g_+ B_0 + g_- B_1 + ST + s T_1 + G_+ F_0 + G_- F_1.
\]

(26)

Insertion \((15)\) into \((22)\) derives to the coefficients

\[
\begin{align*}
   g_+ &= -\langle \phi\eta \rangle, & g_- &= -\langle \eta\phi \rangle; \\
   S &= \langle \eta^2 \rangle, & s &= -\langle |\phi|^2 \rangle; \\
   G_+ &= -\langle \psi\bar{\psi} \rangle, & G_- &= \langle \bar{\psi}\psi \rangle,
\end{align*}
\]

(27)

where the fact is considered that according to \((A.9)\) the field \(\phi\) is exclusively imaginary. Thus, the magnitudes \(g_\pm\) are reduced to advanced and retarded response functions of the order parameter \(\eta\) to field \(\phi\) action; \(S\) and \(s\) are the autocorrelators of order parameter \(\eta\) and field \(\phi\), and the functions \(G_\pm\) determine the correlation of Grassmannian conjugated fields \(\bar{\psi}, \psi\). The statements

\[
\begin{align*}
   g_+^{(0)} &= S^{(0)} = [(1-\sigma)^2 + \omega^2]^{-1}, & s^{(0)} &= -\omega^2[(1-\sigma)^2 + \omega^2]^{-1}, \\
   G_+^{(0)} &= [(1-\sigma) + i\omega]^{-1}, & G_-^{(0)} &= [(1-\sigma) - i\omega]^{-1}
\end{align*}
\]

(28)

are valid for the bare supercorrelator \((24)\). Hence, according to \((A.9)\) the relation \(\langle \eta\phi \rangle_o = G_-^{(0)}\) is deduced that is a special case of the Ward identity \((14)\). It means that the correlator of the Grassmannian fields is reduced to response function of the order parameter \(\eta\) to the fluctuation \(\varphi\).

Expansion \((26)\) allows us to treat supercorrelator \((22)\) as the vector of the direct product of superspaces. Since the Fermi components \(F_{0,1}\) do not couple with the Bose ones \(B_{0,1}, T, T_1\), we may use more compact basis, passing on from the field \(\phi\) to the fluctuation \(\varphi\). To doing so, we will neglect the correlation of the self-consistent field \(\phi\) whose structure factor is \(s \sim \omega^2\), when \(\omega \to 0\). Moreover, we will replace the response functions \(g_\pm\) to field \(\phi\) by corresponding functions \(G_\pm\) for the fluctuation \(\varphi\). Then the expansion \((24)\) takes such a compact form

\[
\mathbb{C} = G_+ \mathbb{A} + G_- \mathbb{B} + ST,
\]

(29)

where the basis operators \(\mathbb{A} \equiv B_0 + F_0, \mathbb{B} \equiv B_1 + F_1\) are introduced that satisfy the multiplication rules: \(\mathbb{A}^2 = \mathbb{A}, \mathbb{B}^2 = \mathbb{B}, \mathbb{B} \mathbb{T} = \mathbb{T}, \mathbb{T} \mathbb{A} = \mathbb{T}\) (other multiplicands are equal to zero).

In the issue the behavior of self-organizing system is described by the Lagrangian

\[
L = (\varphi\dot{\psi} + \bar{\psi}\dot{\chi} - \varphi^2/2) + \langle \bar{V} \{\eta\} \varphi + \bar{\psi} \bar{V}' \{\eta\} \bar{\psi} \rangle.
\]

(30)

We introduce the quenched disorder

\[
p^2 = \frac{\langle \phi(\mathbf{r}) - \phi \rangle^2 - (\Delta\varphi)^2}{(\Delta\varphi)^2}.
\]

(31)

Its magnitude characterizes the field random scattering \(\phi(\mathbf{r})\) (the dash in \((31)\) stands for averaging over coordinate \(\mathbf{r}\), the \((\Delta\varphi)^2 \equiv |\varphi_{\omega=0}|^2\) is the square-mean fluctuation). If the quenched disorder is brought in the action, meeting the Lagrangian \((30)\), component squared in \(\varphi\) fluctuation takes the form

\[
-\frac{1}{2} \int |\varphi_{\omega}|^2 \frac{d\omega}{2\pi} - \frac{p^2}{2} \int \delta(\omega)|\varphi_{\omega}|^2 d\omega.
\]

(32)

Here we have neglected an integration over \(\mathbf{r}\) and passed to Fourier transform over frequency \(\omega\). At the equilibrium disorder the field scatter \(\phi(\mathbf{r})\) is reduced to the square-mean fluctuation \((\Delta\varphi)^2\) so, that \(p = 0\) and expression \((22)\) has the canonical form \(-1/2 \int \varphi^2 d\mathbf{r}\). In the case of quenching one has \(p > 0\) and second addend in \((32)\) leads
to renormalization of the bare supercorrelator \( \langle 24 \rangle \) whose component \( S^{(0)} \) gets the multiplier \( 1 + 2\pi p^2\delta(\omega) \) in \( \langle 28 \rangle \). Respectively, we find the operator \( L \) in the motion equation \( \langle 23 \rangle \) as follows:

\[
L = L_+ \mathbf{A} + L_- \mathbf{B} + LT; \quad L_\pm = (1 - \sigma) \pm i\omega, \quad L = - \left[ 1 + 2\pi p^2\delta(\omega) \right].
\]  (33)

To obtain equation that defines the supercorrelator \( \langle 22 \rangle \) one should multiply \( \langle 20 \rangle \) by \( \Phi(z') \) and average the result over distribution \( P\{\Phi\} \) from \( \langle 14 \rangle \). Thus, we get the Dyson superequation

\[
C^{-1} = L - \Sigma,
\]  (34)

where within the framework of \( \Phi^4 \)-model \( \langle 18 \rangle \) the self-energy superoperator \( \Sigma \) is defined by equality

\[
\Sigma(z, z') = (2/3)w^2(1 + 3\sigma)^2(C(z, z'))^3, \quad z = \{r, t, \bar{x}, \chi\}.
\]  (35)

Herein the \( w > 0 \) is the anharmonicity parameter related to temperature \( T \) and the condition \( \int C(z, z)dz = 0 \) is taken into account that follows from Eqs. \( \langle 13 \rangle \) and \( \langle 23 \rangle \).

If the anharmonicity is omitted, \( \Sigma = 0 \), the components \( \langle 28 \rangle \), diverging at the point of transition into self-organization state (\( \sigma = 1 \)), are obtained from Eq. \( \langle 34 \rangle \). Thus, the supersymmetrical field approach allows us to reproduce the main result following from the Lorentz system \( \langle 6 \rangle \) by means of the linear approximation. In the general case the self-energy superfunction should be expanded similar the supercorrelator \( \langle 24 \rangle \):

\[
\Sigma = \Sigma_+ \mathbf{A} + \Sigma_- \mathbf{B} + \Sigma \mathbf{T}.
\]  (36)

Then, according to Eq. \( \langle 33 \rangle \) the superequation \( \langle 34 \rangle \) is reduced to components

\[
G_\pm^{-1} = [(1 - \sigma) \pm i\omega] - \Sigma_\pm,
\]  (37a)

\[
S = [1 + 2\pi p^2\delta(\omega) + \Sigma]G_+G_-.
\]  (37b)

The explicit form of the expansion coefficients \( \langle 36 \rangle \) is given by expression \( \langle 35 \rangle \). In accordance with \( \langle 18 \rangle \) the supercorrelators multiplication should be understood in usual meaning, not in functional one: \( (T(\chi, \chi'))^2 = T(\chi, \chi'), \ A(\chi, \chi')T(\chi, \chi') = T(\chi, \chi')A(\chi, \chi'), \ B(\chi, \chi')T(\chi, \chi') = T(\chi, \chi')B(\chi, \chi') = B(\chi, \chi'), \) and the other multiplicands are equal to zero. As a result, for the spatially homogeneous case we obtain from \( \langle 32 \rangle \) the following

\[
\Sigma_\pm(t) = 2w^2(1 + 3\sigma)^2S^2(t)G_\pm(t),
\]  (38a)

\[
\Sigma(t) = (2/3)w^2(1 + 3\sigma)^2S^3(t).
\]  (38b)

At insertion of these expressions into the Dyson equation \( \langle 37 \rangle \) we will be in need of their frequency representation that contains convolutions. To avoid such a difficulty let us use the fluctuation-dissipation theorem \( \langle 14 \rangle \)

\[
S(\omega') = (2/\omega')\text{Im}G_\pm(\omega'), \quad \Sigma(\omega') = (2/\omega')\text{Im}\Sigma_\pm(\omega'),
\]  (39)

where \( \omega' \) is the real frequency. Using the spectral representation of the complex frequency \( \omega \) and integrating equalities \( \langle 39 \rangle \), we find

\[
S(t \to 0) = G_\pm(\omega \to 0), \quad \Sigma(t \to 0) = \Sigma_\pm(\omega \to 0).
\]  (40)

Since the \( G_\pm(\omega \to 0) \) gives \( \chi \) in the hydrodynamical limit \( \omega \to 0 \), we have

\[
S(t \to 0) = \chi \equiv G_\pm(\omega \to 0),
\]  (41a)

\[
\Sigma_\pm(\omega \to 0) = (2/3)w^2(1 + 3\sigma)^2\chi^3.
\]  (41b)
where the expression (38) is used in (44). In contrast to (19) here the self-energy components $\Sigma_{\pm}$ contain only the second order of the anharmonicity $\omega$.

The equations (37), (38), and (41) describe the behavior of the self-organizing system completely. Particularly, they represent not only the ordering phenomena but the effects of ergodicity breaking and memory appearance as well. These effects manifest themselves in elongation of correlators $G_{\pm}$, (42), (43b) into the Dyson equation (37b), within $\pm \chi \equiv \chi$ and (41) with index 0 and set self-energy superfunction $\Sigma_{\pm}$-representation we obtain the following relations:

\[G_{-}(\omega) = -\Delta + G_{-0}(\omega) , \quad S(t) = q + S_{0}(t)\] (42)

at the expanse of the Edwards-Anderson memory parameter $q = \langle q(q)\eta(0) \rangle$ and irreversible response $\Delta = \chi - \chi_{0}$ that is equal to difference of microscopic susceptibility $\chi \equiv G_{-}(\omega = 0)$ and macroscopic magnitude $\chi_{0} \equiv G_{-}(\omega = 0)$.

Now, let us insert the elongated correlators (42) into expressions (38). Then the renormalized components of the self-energy superfunction read

\[\Sigma_{\pm}(t) = 2w^{2}(1 + 3\sigma)^{2}q^{2}(-\Delta + G_{\pm0}(t)) + \Sigma_{\pm0}(t),\]
\[\Sigma_{\pm0}(t) \equiv 2w^{2}(1 + 3\sigma)^{2}S_{0}(t)G_{\pm0}(t)(2q + S_{0}(t)),\] (43a)

\[\Sigma(t) = (2/3)w^{2}(1 + 3\sigma)^{2}q^{2}(q + 3S_{0}(t)) + \Sigma_{0}(t),\]
\[\Sigma_{0}(t) \equiv (2/3)w^{2}(1 + 3\sigma)^{2}S_{0}^{2}(t)(3q + S_{0}(t)).\] (43b)

Here the nonlinear terms with respect to the correlators $G_{\pm0}$, $S_{0}$ are gathered in addends $\Sigma_{\pm0}$, $\Sigma_{0}$, in the second equality of (43), the relationship (112) is considered, and in equations (131) the addends containing $S_{0}\Delta \approx 0$ are omitted. If the memory is absent the first addends of $\Sigma_{\pm}(t)$, $\Sigma(t)$ vanish. Inserting the Fourier transforms of equations (32), (33) into the Dyson equation (37b), within $\omega$-representation we obtain the following relations:

\[q_{0} [1 - (2/3)w^{2}(1 + 3\sigma)^{2}\chi_{0}^{2}q_{0}^{2}] = p^{2}\chi_{0}^{2},\] (44)
\[S_{0} = \frac{(1 + \Sigma_{0})G_{+} + G_{-}}{1 - 2w^{2}(1 + 3\sigma)^{2}q^{2}G_{+}G_{-}}.\] (45)

The first one corresponds to the $\delta$-like addend caused by the presence of memory, when $\omega = 0$, and the second one meets the frequencies $\omega \to 0$. When $\omega = 0$ the characteristic combination $G_{+}G_{-}$ goes to $G_{+}G_{-} = \chi_{0}^{2}$ and the pole of the structure factor (13)

\[2w^{2}(1 + 3\sigma)^{2}q_{0}^{2} = \chi_{0}^{2}\] (46)

determines the point of ergodicity breaking $\sigma^{e}$, where $\chi_{0} = \chi$, $q_{0} = q$.

Now let us insert the Fourier transform of expression (13a) into equation (37a). With the help of equality (113) we obtain the following equation

\[G_{-1}^{-1} + 2w^{2}(1 + 3\sigma)^{2}q^{2}G_{-} = \left[(1 - \sigma) - i\omega\right] - 2w^{2}(1 + 3\sigma)^{2}\chi^{2}(q + \chi/3)\] (47)

for the retarded Green function in the hydrodynamical limit $\omega \to 0$, where the relationship (112) is taken into account. Hence the expression springs out for the microscopical susceptibility $\chi \equiv G_{-}(\omega \to 0)$

\[1 - (1 - \sigma)\chi + (2/3)w^{2}(1 + 3\sigma)^{2}\chi \left[(\chi + q)^{3} - q^{3}\right] = 0.\] (48)

The behavior of self-organizing system with quenched disorder is determined completely by equations (14), (44), and (48). Herewith one should contradistinguish the macroscopical magnitudes $q_{0}$, $\chi_{0}$ and microscopical ones $q$, $\chi$.

---

2) Let us point out the reverse sign of the irreversible response $\Delta$ as compared with the definition given for thermodynamical systems, where the ordering corresponds to low values of the noise intensity (temperature).

3) The unified function $G_{-}(\omega)$ can be used to define susceptibilities $\chi_{0}$, $\chi$ considering that magnitudes $\chi_{0} \equiv G_{-}(\omega = 0)$ and $\chi \equiv G_{-}(\omega \to 0)$ correspond to equilibrium and out-of-equilibrium values. In so doing we should equip all correlators in Eqs. (40) and (41) with index 0 and set $\omega = 0$. 

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10
(the first correspond to frequency $\omega = 0$ and the second to limit $\omega \to 0$). The distinctive feature of this hierarchy is that macroscopical values $q_0, \chi_0$ depend exclusively on quenched disorder intensity $p$ while the microscopical ones $q, \chi$ – on thermal disorder $\sigma$. Thereafter, to determine values $q_0, \chi_0$ the magnitude $\sigma$ should be considered to be equal to value $\sigma^c(p)$ on the line of ergodicity breaking, and the quenched disorder intensity takes the critical value $p^c(\sigma)$ for the defining of $q, \chi$. Hence, equations (44), (46) determine the macroscopical values $q_0, \chi_0$ and equation (48) defines microscopical ones $q$ and $\chi$. Herewith the external addition $\phi_{ext}$ to the self-consistent field $\phi$ (hereinafter $\phi_{ext} = 0$ is supposed), the dispersion $p$ of field $\phi$ fixing the intensity of the quenched disorder (51), and the parameter of the thermal disorder $\sigma \equiv S^0/S_2$ act as the state parameters.

Combining the equalities (44), (46), we get the expression for the macroscopical memory parameter

$$q_0 = (3/4)^{1/3}(1 + 3\sigma^c)^{-2/3}(p/w)^{2/3},$$

which increases with $p$ growth. Inserting Eq. (49) into Eq. (46) gives the macroscopical susceptibility

$$\chi_0 = 2^{1/6}3^{-1/3}w^{-1/3}(1 + 3\sigma^c)^{-1/3}p^{-2/3}$$

that decreases with $p$ growth.

Fixing the memory parameter in equation (48) by expression (compare it with (49))

$$q = (3/4)^{1/3}(1 + 3\sigma)^{-2/3}(p^c/w)^{2/3},$$

we find microscopical values $q, \chi$ dependence on $\sigma$ (Figs. 1, 2). The fact that, when the thermal disorder is small the functions $q(\sigma)$ and $\chi(\sigma)$ are two-valued is distinctive for these dependencies. Since the susceptibility should increase near the ordering point the values $q$ and $\chi$, shown in Figs. 1, 2 by the dotted lines, are nonstable. There is an abruptness at the ordering point $\sigma_c$, where $d\chi/d\sigma = \infty$. The growth of anharmonicity parameter $w$ leads to the reduction of both susceptibilities $\chi$ and $\chi_0$.

The ergodicity-breaking point is fixed by equation

$$3A + (A + 1)^3 = 1 + 2(1 - \sigma^c)p^{-2}, \quad A = 2^{5/6}3^{-2/3}(1 + 3\sigma^c)^{1/3}w^{1/3}p^{-4/3},$$

obtained from condition $\chi = \chi_0$ according to equalities (48)-(51). As is seen from Fig. 2 the quantity $\sigma_c$ that meets to the ordering point is defined by the maximum value $\sigma^c$ corresponding to the breaking of ergodicity. The dependencies $\sigma^c(p), \sigma_c(p)$ which represent the phase diagram of the self-organizing system are shown in Fig. 3. The disordered state region corresponding to small values of $\sigma$, and the position of its ‘dividing line’ does not depend on $p$, whereas the ergodicity region is bound by small values of $p$. Comparing Fig. 3a with 3b shows that the behavior of self-organizing system is rather sensitive to the anharmonicity parameter $w$ picked. The appropriate dependence $\sigma_c(w)$ for the thermal disorder parameter that corresponds to ordering, is adduced in Fig. 4. The $\sigma_c$ takes its maximum value $\sigma_c = 1$, when $w = 0$, and the growth of $w$ causes the $\sigma_c$ monotonous decrease until $\sigma_c = 0$ at $w_c = 0.064$. The maximum value $p_m \equiv p^c(\sigma = 0)$ of the quenched disorder at the ergodicity breaking line changes with the value $w$ according to dependence shown in Fig. 5. When $w$ is small, the quantity $p_m \propto w^{-1/2}$ increases infinitely and the ergodicity region disappears with growth of anharmonicity parameter above critical value $w_c$.

Nonergodicity parameter is defined by solution of Eqs. (48)-(51). At constant value of quenched disorder intensity $p$ (Fig. 6a) the three regimes are possible. At small $p$ macroscopic susceptibility $\chi_0$ exceeds microscopic quantity $\chi(\sigma)$ for every values of $\sigma$ (see Fig. 2), and system is always in nonergodic state. When parameter $p$ reaches the values which exceeds threshold $p_c$ meeting the condition $\chi_0(p) = \chi(\sigma_c)$, the nonergodicity parameter $\Delta = \chi - \chi_0$ takes the nonzero values only within the range $\sigma > \sigma^c$. Microscopic susceptibility $\chi$ for every values of $\sigma$ exceeds the macroscopic one $\chi_0$ starting from value $p_m$ that meets the condition $\chi_0(p) = \chi(\sigma = 0)$ and system is nonergodic always. At fixed value of thermal disorder $\sigma$ (Fig. 6b) the dependence $\Delta(p)$ is defined by infinite increase of macroscopic susceptibility $\chi_0 \sim p^{-2/3}$ in the region of weak quenched disorder $p \to 0$. As a result, nonergodicity parameter takes the nonzero values starting from critical value $p^c$, and increases at further growth of quenched disorder monotonically.

Analytical representation of $\Delta(\sigma, p)$ dependence is possible only near the line of ergodicity breaking $\sigma^c(p)$. Setting in Eq. (48) $0 < \sigma - \sigma^c \ll \sigma^c$, $\chi = \chi_0 + \Delta, \Delta \ll \chi$ in first order over small values $(\sigma - \sigma^c)/\sigma^c$, $\Delta/\chi$ in accordance with Eq. (50), we find

$$\Delta \sim p^{-2/3}$$

4) Albeit we review disordered state, we can consider that by parity of reasoning with the thermodynamical systems the microscopical susceptibility decreases from the maximum value $\chi(\sigma_c)$ down to zero within the ordered region. Thus, the ergodicity region of ordered state is bound by the dotted line like the one shown in Fig. 3.
\[ \Delta = B(p)(\sigma - \sigma^c), \quad B \equiv \left[ 1 + \frac{1}{2} \sigma^c \chi_0 - \frac{1}{6} \left( \frac{\chi_0}{q_0} \right)^2 \right] \left[ (1 - \sigma^c) - \frac{2}{\chi_0} + \frac{\chi_0}{6q_0^2} \right]^{-1}. \] (53)

The coefficient \( B(p) \) diverges at the point corresponding to divergence of derivative \( \partial \chi / \partial \sigma \) at \( \sigma = \sigma_c \) (see Fig. 2). At fixed value of thermal disorder it is necessary to make the expansion in \( q_0^c - q_0 \) in equality (54). Then taking into account the dependence (49) in linear approximation, we obtain

\[ \Delta = q_0^c \left[ (p/p^c)^{2/3} - 1 \right], \] (54)

where the critical value of memory parameter \( q_0^c \) meets the ergodicity breaking point \( p^c \).

### V. FLOW OF SAND GRAINS MOTION ON INCLINED SURFACE

The viscous flow of sand grains on plain trajectory \( y = y(x) \) will be shown to represent the simplest example of self-organized criticality. The time dependencies of horizontal \( \dot{x} = \partial x / \partial t \) and vertical \( \dot{y} = \partial y / \partial t \) velocity components, and slope of sand surface \( y' = \partial y / \partial x \) fix the behavior of a system. In the autonomous mode they satisfy the equations of Debye relaxation

\[ \frac{d\dot{x}}{dt} = -\frac{\dot{x}}{\tau_x}, \] (55a)

\[ \frac{d\dot{y}}{dt} = -\frac{\dot{y}}{\tau_y(0)}, \] (55b)

\[ \frac{dy'}{dt} = \frac{y_0' - y'}{\tau_S}, \] (55c)

where the \( \tau_x, \tau_y(0), \tau_S \) are corresponding relaxation times. In Eqs. (55) is supposed that the stationary state meets the sand grains being at rest (\( \dot{x} = \dot{y} = 0 \)), the slope \( y_0' \neq 0 \) plays a role of control parameter.

Since the motion along different directions is not independent we ought to add to Eq. (55a) the transverse component \( f = \dot{y}/\gamma \) of force caused by motion along \( y \) axes (\( \gamma \) is the kinetic coefficient). As a result, Eq. (55a) takes the form

\[ \tau_x \ddot{x} = -\dot{x} + a^{-1} \dot{y}, \] (56)

where \( a \equiv \gamma / \tau_x \). Note, that in accordance with diffusion equation \( \dot{y} = Dy'' \), where \( D \) is the diffusion coefficient, the stated force is proportional to curvature of sand surface:

\[ f = (D/\gamma)y'' \] (57)

In the stationary state, when \( \ddot{x} = 0 \), equation (56) solution yields the dependence \( y = ax + \text{const} \) for the tangent to sand surface. On the other side, here the friction force (57) becomes proportional to longitudinal component: \( f = \tau_x^{-1} \dot{x} \).

Taking into consideration the relationship (57) and obvious equality \( dy'/dt = \dot{y}' + y'' \dot{x} \), the expression (55c) assumes the form

\[ \tau_S \dot{y}' = (y_0' - y') - (\tau_S / D) \dot{y} \dot{x}. \] (58)

Analogously for the vertical component we obtain

\[ \tau_y \ddot{y} = -\dot{y} + \frac{\tau_y}{\tau_x} y' \dot{x}, \quad \frac{1}{\tau_y} \equiv \frac{1}{\tau_y(0)} \left( 1 + \frac{y_0' \tau_y(0)}{a \tau_x} \right). \] (59)

Here the nonlinear addends of higher order are omitted and relaxation time \( \tau_y \) is introduced that depends on stationary slope \( y_0' \).
Equations (58), (59), and (60) coincide with the Lorentz system (4), if the horizontal $\dot{x}$ and vertical $\dot{y}$ velocity components with respect to the order parameter $\eta$ and conjugate field $\bar{h}$, and the slope $x'$ to the control parameter $S$. Then the coupling constants of the system (4) take the form

$$
A_\eta = a^{-1}, \quad A_h = \tau_y / \tau_x, \quad A_S = \tau_S / D.
$$

(60)

Taking into account the relationships $A_\eta \propto v \tau_x$, $A_h \propto v \tau_y$, and $A_S \propto v \tau_S$ obtained from the comparison with the microscopic model (1), it is seen that the values of phenomenological parameters $\gamma \propto D \propto \tau_x \propto v^{-1}$ are fixed by microscopic one $v$. In the previous section the self-organized criticality has been shown to be realized, if $v$ exceeds the value $v_c$. It implies that avalanche formation on the sand surface takes place spontaneously, if shear viscosity being proportional to the relaxation time $\tau_x$, is bounded from above by critical value $v_c$. This condition is similar to transition criteria of viscous liquid into turbulent flow mode.

VI. CONCLUSIONS

As the above shows at fixed values of thermal and quenched disorder intensities $\sigma, \eta$, the behavior of self-organizing system is represented, on the one side, by order parameter field $\eta$ and fluctuation amplitude $\varphi$ of conjugate field, and on the other side – by the couple of Grassmannian conjugated fields $\psi, \bar{\psi}$. Within the microscopic representation the Boson and Fermion gases, interacting between themselves by means of potential $\eta$ in Hamiltonian (1), meet them. Since with transition to self-consistent field scheme the couple of three-tail vertexes, every meeting the $w$, forms the four-tail one whose anharmonicity parameter $w$ is in bare superpotential (18), it is possible to suppose that the relation $w = v^2$ is fulfilled. Then the expression (A.6) for the critical value of thermal disorder $S_c$, in which we should take into consideration the factor $w^{-1}$ (see after (18)), takes the form

$$
S_c = \frac{\varepsilon \tau_S}{2 \tau_\eta} \left( \frac{T}{v} \right)^2 \left( \frac{\xi}{a} \right)^2,
$$

(61)

where we pass to dimension quantities, $T$ is the noise intensity reduced to temperature for thermodynamic systems. The obtained $S_c(v)$ dependence has the same character as in the last formula (58). Identifying they, we find the expression for disagreement parameter $\varepsilon$ that provides the time irreversibility in energy balance equation (A.2):

$$
\varepsilon = c \left( \frac{\tau_0}{\tau_\eta} \right)^2; \quad c = \frac{2}{\pi^2} \left( \frac{a}{\xi} \right)^2 \frac{\tau_\eta^2}{\tau_\eta \tau_S}, \quad \tau_0 \equiv \frac{2\pi \hbar}{T}.
$$

(62)

Here the dimension units are used, $\hbar$ is the Planck constant. The coefficient $c$ is defined by ratio of adiabaticity parameter $(a/\xi)^2$, expressed in terms of spatial scales, to the corresponding value $\tau_\eta \tau_S / \tau_\eta^2$ in terms of characteristic times. It is possible to believe that $c$ is the constant, and disagreement parameter (12) is determined by square of ratio of quantum fluctuation time $\tau_0$ to macroscopic time $\tau_\eta$ of the order parameter change. Obviously, the condition $\varepsilon \ll 1$ is satisfied always.

The peculiarity of self-consistent supersymmetric scheme presented in sections IV, V is that it allows us to go out of framework of the adiabatic approximation. On conditions of its applicability, $v \ll v_c$, the critical value of thermal disorder is defined by equality (61). With growth of interaction parameter $v$ the complete suppression of disordered state takes place, when values $v$ overcomes a critical value $v_c = 0.252$. In this case, the effects of mutual influence of superfield components manifest themselves substantially, and the dependences type of represented in Figs. 1-6 would be used.

We have shown for the simplest example of sand flow on the inclined surface (section IV) that above scheme represents the avalanches formation in self-organized criticality phenomena. If can suppose that the effective potential form as usually the complicated landscape in system’s configuration space is similar to the spin glass. Therefore the theory of diffusion over nodes of hierarchical tree, formed by set of statistical ensembles of nonergodic system, would be used for the complete description of self-organized criticality. The above approach pretends to describe the conditions for single avalanche formation of lowest hierarchical level only. As is seen from Fig. 4 one can determine the two system types: at supercritical values of interaction parameter $v > v_c$ the ordering process realizes independently on external conditions; in opposite case $v < v_c$ the system passes into self-organization mode,

5) Obviously, the relationship $\gamma \propto D$ of kinetic and diffusion coefficients represents the Einstein expression.
if the thermal disorder intensity is above the critical value $\sigma_c$. Examination of the simplest sandpile model shows that on the macroscopic level the condition of spontaneous avalanches formation is similar to turbulence criteria.

Phase diagram of a system defines the total landscape pattern and allows to predict the self-organized criticality character as a function of external conditions. So, Fig. 3 shows that system is nonergodic if the quenched disorder intensity exceeds the maximum value $p_m$ fixed by value of $\nu$. According to dependence represented in Fig. 5 at supercritical value $\nu > \nu_c$ the self-organized criticality process does not requires the quenched disorder at all. It is worthwhile to note that numerical experiments Ref. [3], [4], where the avalanches intensities are stochastic values, meet the large values of quenched disorder.

VII. APPENDIX

To give the form of synergetic system (6) for the field equations (14) we multiply Eq. (14c) by $\bar{\psi}$ to left-hand side, and Eq. (14d) by $\psi$ to right-hand side, and add the results. Then for quantities

$$ S = \bar{\psi}\psi, \quad j = (\nabla \bar{\psi})\psi - \bar{\psi}\nabla \psi \quad (A.1) $$

we obtain the continuity equation $\dot{S} + \nabla j = 0$, that, obviously, expresses the entropy conservation law for conserved systems. Respectively, combinations (A.1) of Grassmannian fields determine the entropy $S$ and its current $j$. It is characteristic that Eqs. (14d), (14d) for Grassmannian conjugated fields $\psi, \bar{\psi}$ differ only by sign in front of the time derivative, so that dependencies $\dot{\psi}(t), \dot{\bar{\psi}}(t)$ coincide at it inversion. Namely this circumstance provides the entropy conservation condition albeit for each of the fields $\psi, \bar{\psi}$ this condition is not fulfilled: according to (14c) in the homogeneous case the quantity $\dot{\psi}(t)$ decreases exponentially with decrement $t^{-1} \int_0^t V_0''(\eta(t'))dt'$, and the conjugate field $\dot{\bar{\psi}}(t)$ increases with the equal value of increment. For the entropy $S \equiv \bar{\psi}\psi$ the pointed out processes are compensated, and magnitude of $S$ is conserved. As a result, the obtained continuity equation does not contain feedback with order parameter $\eta$.

To switch on this feedback we ought to take into account the disagreement of right-hand sides of Eqs. (14c), (14d) that reflects the macroscopic time irreversibility. In this aim we introduce the coefficient $1 + \varepsilon$ in the right-hand side of Eq. (14d) defined by the disagreement parameter $\varepsilon \ll 1$ (its value is determined in section VII). Then in the right-hand side of continuity equation the term $-\varepsilon V_0''(\eta)S$ appears. In addition we take into consideration that self-organization process realizes only at stationary current $\dot{j}$ which leads to thermostat entropy $S$ increase with constant velocity $-\nabla j \equiv (\tau_\eta/\tau_S)S^0$ (at that the entropy of self-organizing system $\Delta S \equiv S^0 - S$ decreases naturally). As a result, entropy balance equation reads

$$ \dot{S} = (\tau_\eta/\tau_S)S^0 - \varepsilon V_0''(\eta)S. \quad (A.2) $$

From here in the stationary regime $\dot{S} = 0$, the equality $S = (\tau_\eta/\tau_S)S^0/V_0''(\eta)$ is obtained that is reduced to the form (7) for the bare potential

$$ V_0 = \eta^2/2 + \eta^4/12. \quad (A.3) $$

Now let us examine equation (14b) for the fluctuation field $\varphi(r, t)$. In contrast to conjugate field $h$ the fluctuational one $\varphi$ is the nonhomogeneous even in the stationary state. We use the approximation $\nabla^2 \varphi = (\xi/a)^2 \varphi$, where $a$ is the scale of stationary fluctuation variation, $\xi > a$ is the correlation length to be the scale of the coordinate $r$. Then in the stationary state $\dot{\varphi} = 0$ from Eq. (14b) the relationship follows

$$ \varphi = \frac{V_0'''(\eta)S}{(\xi/a)^2 V_0''(\eta)}, \quad (A.4) $$

showing that the values $\varphi, S$ are connected at expanse of anharmonicity $V_0'''(\eta) \neq 0$ of bare potential only. In addition the stability condition $\varphi > 0$ requires that the scale of nonhomogeneity does not exceed the value $(V_0'''(\eta))^{-1/2} \xi$. At much rigorous requirement $(a/\xi)^2 V_0''(\eta) \ll 1$ the addend $-V_0''(\eta)$ in the denominator may be omitted. The pointed out inequality means the adiabatic condition that, however, relates not the time but spatial scales.

In terms of adiabatic approximation Eq. (14b) for amplitude $\varphi(t)$ of most probable fluctuation takes the form

$$ \dot{\varphi} = - (\xi/a)^2 \varphi + V_0'''(\eta)S. \quad (A.5) $$

Taking into consideration $S(\eta)$ dependence in the stationary state, $\dot{\varphi} = 0$, we obtain $\varphi = (\sigma/2)V_0'''(\eta)/V_0''(\eta)$, $\sigma \equiv S^0/S_c$, where the characteristic value of control parameter is introduced
Using approximation (A.3), we see that found dependence \( \varphi(\eta) \) is reduced to (6).

Lastly, let us consider equation (14a) for the order parameter field \( \eta(r, t) \). Inserting there the dependence \( \varphi(\eta) \), we arrive at the Ginzburg-Landau-Khalatnikov equation

\[
\dot{\eta} - \nabla^2 \eta = -\frac{\partial V}{\partial \eta}, \quad \ddot{V}(\eta) \equiv V_0(\eta) - (\sigma/2)\ln V_0''(\eta) \tag{A.7}
\]

differing from (8) by gradient term appearance. For dependence (A.3) the synergetic potential assumes the form

\[
\ddot{V}(\eta) = V(\eta) + \eta^4/12, \quad V(\eta) \equiv (1/2)\left(\eta^2 - \sigma \ln (1 + \eta^2)\right) \tag{A.8}
\]

differing from (8) by the addend \( \eta^4/12 \). This distinction is caused by the circumstance that the potential \( V \) is defined at constant field \( h \), whereas \( \ddot{V} \) – at fluctuation amplitude \( \varphi \) fixed. In other words, the first potential is the field \( h \) function, whereas the second one depends on fluctuation amplitude \( \varphi \). The quantities \( h, \varphi \) represent the couple of conjugated stationary state parameters (like the volume and pressure in thermodynamics) and the synergetic potentials \( \ddot{V}(h) \), \( \ddot{V}(\varphi) \) are coupled by Legendre transformation \( \ddot{V} = V - h\varphi \). The state equation governing the \( h(\varphi) \) dependence follows from the condition \( h = -\frac{\partial \ddot{V}}{\partial \varphi} \). But it is made simpler to introduce the field \( \phi = -V_0'(\eta) + \nabla^2 \eta \equiv -\delta V_0(\eta)/\delta \eta \) that is reduced to the force \( f \) in Eq. (1) with accuracy to gradient addend. Then equation (14a) assumes the form

\[
\dot{\eta} = \phi + \varphi. \tag{A.9}
\]

Comparing Eqs. (A.9) and (6a), we find the important relationship

\[
h = \eta + (\phi + \varphi) \equiv (\eta - V_0'(\eta) + \nabla^2 \eta) + \varphi. \tag{A.10}
\]

In the stationary state (\( \dot{\eta} = 0 \)) the amplitude of the most probable fluctuation \( \varphi = -\phi \) coincides with generalized force with accuracy to sign, and field \( h = \eta \) is reduced to the order parameter. In the general case the disagreement \( \phi + \varphi \neq 0 \) results in variation of order parameter in the course of time and difference between the fields \( h, \varphi \) is conditioned by nonlinear component of generalized force \( \phi \). Note that obtained Eqs. (A.7), (A.3), (A.2) and the Lorentz equations (6) coincide in their mathematical structure only. So, the entropy balance equation (A.2) contains the negative feedback similar to equation (6), however it is expressed by addend \( -\varepsilon V_0''(\eta)S \) that is proportional to the entropy, whereas the corresponding term \( -A_2\eta\dot{h} \) does not contain entropy. The difference of addends \( V_0''(\eta)S \) in (A.3) and \( A_2\eta S \) in (1), meeting the positive feedback, is less essential (for bare potential (A.3) they are equal at all). Finally the field equation (A.7) for the order parameter differs from the Landau-Khalatnikov equation (8) by account of spatial nonhomogeneity only. As was noted, the physical reason for above distinctions is that the Lorentz equations contains the field \( h \) conjugate to order parameter, whereas the amplitude of most probable fluctuation \( \varphi \) is in the initial field equations (14). Since the \( h \) and \( \varphi \) play a role of conjugated parameters of system’s stationary state, then developed field formalism and Lorentz scheme are mutually supplementary approaches – the first one is used at fixed value of most probable fluctuation amplitude \( \varphi \), and the second one – at fixed field \( h \). Obviously, the second case is realized much naturally (to pass to this we should exploit the state equation (A.10)).

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CAPTIONS

to the paper ”Field theory of avalanches formation” by A.I. Olemsko i and A.V. Khomenko

Fig. 1. Dependence of microscopic memory parameter $q$ on thermal disorder intensity $\sigma$ at anharmonicity parameter $w = 0.01$ (dashed line meets the unstable state, $\sigma_c$ is the ordering point).

Fig. 2. Dependence of microscopic $\chi$ and macroscopic $\chi_0$ susceptibility on thermal disorder intensity $\sigma$ at anharmonicity parameter $w = 0.01$ (dashed and thin solid lines meet the unstable state; thick line – stable state; $\sigma_c$ is the point of ergodicity breaking, $\sigma_c$ is the ordering point).

Fig. 3. Phase diagram: (a) at $w = 0.01$; (b) at $w = 0.02$ ($O$ is the ordered phase, $D$ – the disordered phase; $E$ – ergodic, $N$ – nonergodic; the dashed line shows the precise boundary of ordered ergodic region).

Fig. 4. Dependence of thermal disorder intensity $\sigma_c$ corresponding to the ordering point on anharmonicity parameter $w$.

Fig. 5. Dependence of maximal intensity $p_m$ of quenched disorder at the boundary of ergodic region on anharmonicity parameter $w$.

Fig. 6. Dependence of nonergodicity parameter $\Delta$: (a) on thermal disorder intensity $\sigma$ (curves 1-6 meet the $p = 0.6; 0.8; 1.0; 1.4; 2.0; 4.0$, respectively); (b) on quenched disorder intensity $p$ (curves 1-5 meet $\sigma = 0.0; 0.1; 0.2; 0.3; 0.4$, respectively). The anharmonicity parameter $w = 0.01$. 


Fig. 2
Fig. 3
Figure 4
Fig. 5
Fig. 6