Quandle Cocycle Quivers

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Abstract

We incorporate quandle cocycle information into the quandle coloring quivers we defined in [2] to define weighted directed graph-valued invariants of oriented links we call quandle cocycle quivers. This construction turns the quandle cocycle invariant into a small category, yielding a categorification of the quandle cocycle invariant. From these graphs we define several new link invariants including a 2-variable polynomial which specializes to the usual quandle cocycle invariant. Examples and computations are provided.

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1 Introduction

Quandles, algebraic structures whose axioms are derived from the Reidemeister moves, were introduced in [4, 5] and have been studied ever since. Associated to every oriented knot or link there exists a fundamental quandle, also called the knot quandle of the knot or link. The isomorphism class of the fundamental quandle of a knot determines the group system and hence the knot up to mirror image; in this sense the knot quandle is a complete invariant for knots, though not for split links or other generalizations such as virtual knots. The number of homomorphisms from a knot’s fundamental quandle to a finite coloring quandle, called the quandle counting invariant, is an integer-valued invariant of oriented knots and links.

The quandle counting invariant forms the basis for a class of computable knot and link invariants known as enhancements, including in particular quandle cocycle invariants. In [1], a theory of quandle cohomology was introduced. Quandle-colored link diagrams can be identified with certain elements of the second homology of the coloring quandle, and it naturally follows that elements of the second cohomology define invariants of quandle-colored isotopy. See [1, 3] etc. for more.

In [2], we introduced quandle coloring quivers, directed graph-value invariants of oriented knots and links associated to pairs \((X, S)\) consisting of a finite quandle \(X\) and a subset \(S\) of the ring of endomorphisms of \(X\). From these quivers we defined a new polynomial knot invariant, using this quiver structure to enhance the quandle counting invariant, called the in-degree polynomial. Since directed graphs can be interpreted as categories, this construction yielded a categorification of the quandle counting invariant.

In this paper we enhance quandle coloring quivers with quandle cocycles, obtaining quandle cocycle quivers. As in the previous case, this construction yields a categorification of the quandle cocycle invariant. From these quivers we derive new polynomial knot and link invariants, including a two-variable enhancement of the quandle 2-cocycle invariant which has both the quandle counting invariant and the quandle cocycle invariant as specializations but in general is a stronger invariant than either. The paper is organized as follows. In Section 2 we review quandles and quandle cohomology. In Section 3 we define quandle cocycle quivers and provide examples demonstrating the computation of the invariant and showing that the new polynomial invariant is a proper enhancement. We conclude in Section 4 with some questions for future research.

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2 Quandles and Quandle Cohomology

We begin with some preliminary definitions.

**Definition 1.** A set $X$ with a binary operation $\triangleright$ is a *quandle* if it satisfies

(i) for all $x \in X$, $x \triangleright x = x$

(ii) for all $y \in X$, the map $f_y : X \to X$ defined by $f_y(x) = x \triangleright y$ is a bijection, and

(iii) for all $x, y, z \in X$, $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

**Definition 2.** For an oriented link $L$, associate a label to each arc of $L$. At each crossing of $L$, assign a relation between arc labels as in the figure below.

In words, if we orient a crossing so that the overstrand (labeled $y$) points up, and we have arcs labeled $z$ and $x$ on the left and right respectively of the arc labeled $y$, then we have the crossing relation $z = x \triangleright y$.

The *fundamental quandle* $Q(L)$ is the quandle generated by the set of arc labels under the equivalence relations defined by crossing relations for each crossing of $L$.

**Remark 1.** The quandle axioms are defined in such a way that the fundamental quandle of an oriented link is invariant under Reidemeister moves, so $Q(L)$ is an invariant of $L$.

**Definition 3.** Let $X$ be a finite quandle and $L$ be an oriented link. Then a homomorphism $\phi : Q(L) \to X$ is called an $X$-coloring of $L$.

**Example 1.** Let $X$ be the quandle defined by the operation table

|   | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 3 | 4 | 2 |
| 2 | 4 | 2 | 1 | 3 |
| 3 | 2 | 4 | 3 | 1 |
| 4 | 3 | 1 | 2 | 4 |

and $L$ be the figure-8 knot as drawn below, with arc labels $a, b, c, d$. Then

$$Q(L) = \langle a, b, c, d \mid b = a \triangleright d = d \triangleright c, c = a \triangleright b = d \triangleright a \rangle,$$

and $\phi : Q(L) \to X$ that maps $a \mapsto 1, b \mapsto 2, c \mapsto 3, d \mapsto 4$ is an $X$-coloring of $L$, which we may visualize with the picture below.
The quandle axioms are the conditions required to ensure that for a given quandle coloring of a diagram on one side of a move, there is a unique quandle coloring of the diagram on the other side of the move which agrees with the original coloring outside the neighborhood of the move. From this, we obtain the following standard result:

**Theorem 1.** Let $X$ be a finite quandle. The number of colorings of an oriented knot or link diagram is an integer-valued invariant of ambient isotopy.

This theorem also follows from the observation that the set of quandle colorings of a knot or link diagram can be identified with the set $\text{Hom}(Q(L), X)$ of quandle homomorphisms from the fundamental quandle of the knot or link $L$ to the coloring quandle $X$. See [3] for more.

**Definition 4.** Let $X$ be a finite quandle, $A$ an abelian group and $C_{n}^{R}(X; A) = A[X_{n}]$, the set of $A$-linear combinations of ordered $n$-tuples of $X$. Let $C_{n}^{D}(X; A)$ be the subgroup generated by elements $(x_{1}, \ldots, x_{n})$ with $x_{j} = x_{j+1}$ for some $j$, and let $C_{n}^{Q}(X; A) = C_{n}^{R}(X; A)/C_{n}^{D}(X; A)$. Define $\partial_{n} : C_{n}^{R}(X; A) \rightarrow C_{n-1}^{R}(X; A)$ by setting

$$\partial_{n}(\bar{x}) = \sum_{k=1}^{n} (-1)^{k} (\partial_{n}^{0}(\bar{x}) - \partial_{n}^{1}(\bar{x}))$$

where we have

$$\partial_{n}^{0}(x_{1}, \ldots, x_{n}) = (x_{1}, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n})$$

$$\partial_{n}^{1}(x_{1}, \ldots, x_{n}) = (x_{1} \triangleright x_{k}, \ldots, x_{k-1} \triangleright x_{k}, x_{k+1}, \ldots, x_{n})$$

and extending linearly. Since $\partial_{n}(C_{n}^{D}(X; A)) \subset C_{n-1}^{D}(X; A)$, $\partial_{n}$ induces $\partial_{n}^{Q} : C_{n}^{Q}(X; A) \rightarrow C_{n-1}^{Q}(X; A)$. Then the $n$th quandle homology of $X$ is

$$H_{n}^{Q}(X) = \text{Ker}(\partial_{n}^{Q})/\text{Im}(\partial_{n+1}^{Q}).$$

Dualizing, we have $C_{n}^{n}(X; A) = \text{Hom}(C_{n}^{n}(X; A), A)$, $\delta_{n} : C_{n}^{n}(X) \rightarrow C_{n+1}^{n}(X; A)$ defined by

$$(\delta^{n}f)(x_{1}, \ldots, x_{n}) = f\partial_{n+1}(x_{1}, \ldots, x_{n+1})$$

and $n$th quandle cohomology of $X$ given by

$$H_{n}^{Q}(X) = \text{Ker}(\delta_{Q}^{n})/\text{Im}(\delta_{Q}^{n+1}).$$

An element of $\text{Ker}(\delta_{Q}^{n})$ is called a quandle $n$-cocycle and an element of $\text{Im}(\delta_{Q}^{n+1})$ is called a quandle $n$-coboundary.

In this paper, we are particularly interested in quandle 2-cocycles, which are maps $\phi : A[X \times X] \rightarrow A$ for an abelian group $A$ (usually $\mathbb{Z}_{n}$ for us). These can be written as linear combinations of elementary functions $\chi_{i,j} : X \times X \rightarrow A$ where

$$\chi_{i,j}(x_{1}, x_{2}) = \begin{cases} 1, & \text{for } i = x_{1}, j = x_{2} \\ 0, & \text{otherwise.} \end{cases}$$

**Remark 2.** A quick computation will show that $\phi$ is a quandle 2-cocycle if and only if $\phi$ satisfies the condition that

$$\phi(x, y) + \phi(x \triangleright y, z) = \phi(x, z) + \phi(x \triangleright z, y \triangleright z)$$

for all $x, y, z \in X$ and $\phi(x, x) = 0$ for all $x \in X$. 
**Definition 5.** Let $L$ be an oriented link, $X$ be a finite quandle, and $\phi$ a quandle 2-cocycle. Let $v$ be an $X$-coloring of $L$. For an $X$-colored crossing $c$ of $L$, we define $\phi(c)$ by the following based on the whether the crossing is positively or negatively oriented:

In the illustration above, $x$ and $y$ are the elements of $X$ determined by the coloring $v$. Then we define $\phi(v)$ to be

$$\phi(v) = \sum_{c \in C} \phi(c),$$

where $C$ is the set of crossings in $L$.

**Remark 3.** It can be shown that the evaluation of $\phi$ on a quandle colored link satisfying the condition described in Remark 2 is equivalent to $\phi$ being invariant under a quandle-colored Reidemeister type III move. In fact, $\phi$ is an invariant of quandle-colored links. See [1, 3] for more.

**Definition 6.** Let $X$ be a finite quandle, $L$ an oriented link and $\phi \in H^2_Q(X; \mathbb{A})$. Then the polynomial

$$\Phi^\phi_X(L) = \sum_{v \in \mathcal{C}(L,X)} s^{\phi(v)}$$

where $\mathcal{C}(L,X)$ is the set of $X$-colorings of a diagram of $L$ and $\phi(v)$ is the Boltzmann weight of the $X$-colored diagram $v$ is the quandle 2-cocycle invariant of $L$. See [1, 2] for more.

### 3 Coloring Quivers and Cocycle Quivers

Let $L$ be an oriented link diagram. In [2] we defined the quandle coloring quiver invariant $Q^X_{\phi}(L)$ in the following way: given a finite quandle $X$ and set $S \subset \text{Hom}(X,X)$ of quandle endomorphisms, we make a directed graph with a vertex for each $X$-coloring of $L$ and a directed edge from $v_j$ to $v_k$ whenever $v_k = f(v_j)$ in the sense that each arc color in $v_k$ is obtained from the corresponding arc color in $v_j$ by applying $f$ for some $f \in S$.

**Example 2.** The links $L7n1$ and $L7n2$ have isomorphic quandle coloring quivers with respect to the quandle and endomorphism.
\[
\begin{array}{cccc}
\triangleright & 1 & 2 & 3 & 4 \\
1 & 1 & 1 & 1 & 1 \\
2 & 4 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 \\
4 & 2 & 4 & 4 & 4 \\
\end{array}
\]

\[f(1) = 4, f(2) = f(3) = f(4) = 3\]

namely

**Definition 7.** Let \(L\) be an oriented link, \(X\) be a finite quandle, \(S\) a set of quandle endomorphisms from \(X\) to \(X\), and \(\phi\) a 2-cocycle in \(C^2\mathcal{Q}(X)\). Then the *quandle cocycle quiver* \(Q^S_\phi X(L)\) is the directed graph with vertices corresponding to \(X\)-colorings of \(L\), edges from \(v_j\) to \(v_k\) whenever \(v_k = f(v_j)\) for some \(f \in S\), and weights \(\phi(v_j)\) at each vertex. When \(S = \{f\}\) is a singleton we will write \(f\) instead of \(\{f\}\) for simplicity.

**Example 3.** The links in example 2 are not distinguished by their coloring quivers with respect to the given quandle and endomorphism; however, the quandle cocycle quiver with cocycle

\[\phi = \chi_{1,2} + 2\chi_{1,3} + \chi_{1,4} + 2\chi_{2,1} + 3\chi_{3,2} + 3\chi_{3,4} + \chi_{4,1} \in C^2\mathcal{Q}(X; \mathbb{Z}_4)\]

does distinguish the links.

**Definition 8.** Let \(L\) be a link, \(X\) a finite quandle, \(S \subset \text{Hom}(X, X)\) and \(\phi \in C^2\mathcal{Q}(X; A)\). We define the *quiver enhanced cocycle polynomial* to be the polynomial

\[\Phi^S_\phi X(L) = \sum_{e \in E(Q^S_X(L))} s^{\phi(v_j)} t^{\phi(v_k)}\]

where the edge \(e\) is directed from vertex \(v_j\) to vertex \(v_k\) in the quandle coloring quiver \(Q^S_X(L)\).
Remark 4. Since each directed edge contributes its \(s^{\phi(v_j)}t^{\phi(v_k)}\) value to the polynomial \(\Phi^S_{X,\phi}(L)\) independently, if we regard the quandle coloring quiver \(Q_{X,S}(L)\) as the union of the quivers \(Q_{X,f}(L)\) for endomorphisms \(f \in S\), then the polynomial can be separated into a sum of the cocycle quiver polynomials for each individual endomorphism:

\[
\Phi^S_{X,\phi}(L) = \sum_{f \in S} \Phi^f_{X,\phi}(L).
\]

It follows that evaluating \(\Phi^S_{X,\phi}(L)\) at \(t = 1\) yields \(|S|\Phi^\phi_X(L)\). In particular, when \(|S| = 1\), \(\Phi^S_{X,\phi}(L)\) evaluates at \(t = 1\) to the classical quandle 2-cocycle invariant as defined in [1] (see e.g. Example 8 in [1] and note that our \(s\) is their \(t\)).

Similarly, if \(f\) is the identity endomorphism, then \(\Phi^f_{X,\phi}(L)\) is the quandle cocycle invariant evaluated at \(st\).

Example 4. In [2] Example 6 we gave an example of two links \(L6a1\) and \(L6a5\) which have the same counting invariant value \(\Phi^Z_X(L) = 16\) with respect to the quandle \(X\) with operation table

\[
\begin{array}{cccccc}
\triangleright & 1 & 2 & 3 & 4 & 5 \\
1 & 1 & 3 & 1 & 3 & 1 \\
2 & 4 & 2 & 4 & 2 & 4 \\
3 & 3 & 1 & 3 & 1 & 3 \\
4 & 2 & 4 & 2 & 4 & 4 \\
\end{array}
\]

but are distinguished by the isomorphism class of their quandle coloring quivers \(Q^f_X(L)\) where \(f : X \to X\) is the endomorphism given by \(f(1) = f(3) = 4\) and \(f(2) = f(4) = 2\). (The published version incorrectly lists this as “\(f(1) = 4, f(1) = 2, f(1) = 4, f(1) = 2\).”) We note that for any coboundary \(\phi\), the Boltzmann weights are all 0 and the counting invariant \(\Phi^S_X(L) = 16\) is equal to the quiver enhanced polynomial, so this example also shows that the Boltzmann weight enhanced quiver is not determined by the quiver enhanced polynomial.

The links in example 3 are distinguished by their quandle cocycle quivers, but they are already distinguished by their quandle cocycle invariants with respect to the given quandle and cocycle. The next example shows that the quandle cocycle quiver can distinguish knots which have the same quandle cocycle invariant.

Example 5. Let \(X\) be the quandle defined by operation table

\[
\begin{array}{ccccccc}
\triangleright & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 3 & 2 & 5 & 4 & 1 \\
2 & 3 & 2 & 1 & 6 & 2 & 4 \\
3 & 2 & 1 & 3 & 3 & 6 & 5 \\
4 & 5 & 6 & 4 & 4 & 1 & 2 \\
5 & 4 & 5 & 6 & 1 & 5 & 3 \\
6 & 6 & 4 & 5 & 2 & 3 & 6. \\
\end{array}
\]

This quandle has endomorphism \(f(1) = f(6) = 2, f(2) = f(5) = 4\) and \(f(3) = f(4) = 6\) and cocycle

\[
\phi = 2\chi_{(1,2)} + 2\chi_{(1,3)} + 2\chi_{(1,4)} + 2\chi_{(1,5)} + \chi_{(2,3)} + 2\chi_{(2,4)} + \chi_{(3,2)} + 2\chi_{(3,5)} + 2\chi_{(4,2)} + \chi_{(4,5)} + 2\chi_{(5,3)} + \chi_{(5,4)} + \chi_{(6,2)} + \chi_{(6,3)} + \chi_{(6,4)} + \chi_{(6,5)}
\]

in \(C^2_Q(X;\mathbb{Z}_3)\). Then the knots \(6_1\) and \(7_7\) both have quandle cocycle polynomial

\[
\Phi^\phi_X(6_1) = 6 + 12s + 12s^2 = \Phi^\phi_X(7_7)
\]

but are distinguished by their quiver enhanced polynomials

\[
\Phi^{f,\phi}_X(6_1) = 6 + 12st + 12s^2t^2 \neq 6 + 12st^2 + 12s^2t = \Phi^{f,\phi}_X(7_7).
\]
Example 6. Continuing with the same quandle from example 5, we computed $\Phi^f_\chi(L)$ using our python code for the prime knots with up to eight crossings and prime links with up to seven crossings with the cocycle with $\mathbb{Z}_4$ coefficients

\[
\phi = \chi_1 + 3\chi_4 + 2\chi_{15} + 3\chi_{21} + 3\chi_{23} + 2\chi_{24} + \chi_{25} + \chi_{31} + 3\chi_{35} + 3\chi_{36} + \chi_{41} + \chi_{42} + 2\chi_{45} + 3\chi_{46} + 3\chi_{51} + \chi_{54} + \chi_{56} + 3\chi_{62} + 3\chi_{64} + \chi_{65}
\]

and three arbitrarily chosen endomorphisms (where we write $f$ by specifying $[f(1), \ldots, f(n)]$):

\[
\begin{align*}
f_1 &= [1, 2, 3, 2, 1] \\
f_2 &= [1, 5, 4, 3, 2, 6] \\
f_3 &= [3, 1, 2, 5, 6, 4]
\end{align*}
\]

The results are collected in the table. For simplicity we list only the nontrivial values; unlisted knots have $\Phi^f_\chi(L) = 6$.

| $L$   | $\Phi_{\chi}^{f_1}(L)$ | $\Phi_{\chi}^{f_2}(L)$ | $\Phi_{\chi}^{f_3}(L)$ |
|-------|-------------------------|-------------------------|-------------------------|
| 3_1   | 30                      | 30                      | 30                      |
| 6_1   | $12s^2t^2 + 4s^2 + 4t^2 + 10$ | $16s^2t^2 + 14$ | $16s^2t^2 + 14$ |
| 7_4   | 30                      | 30                      | 30                      |
| 7_7   | $12s^2t^2 + 4s^2 + 4t^2 + 10$ | $16s^2t^2 + 14$ | $16s^2t^2 + 14$ |
| 8_10  | 54                      | 54                      | 54                      |
| 8_11  | 54                      | 54                      | 54                      |
| 8_15  | 54                      | 54                      | 54                      |
| 8_18  | 48$s^2t^2 + 16s^2 + 16t^2 + 22$ | 64$s^2t^2 + 38$ | 40$s^2t^2 + 24s^2 + 24t^2 + 14$ |
| 8_19  | 54                      | 54                      | 54                      |
| 8_20  | 54                      | 54                      | 54                      |
| L2a1  | 12                      | 12                      | 12                      |
| L4a1  | 12                      | 12                      | 12                      |
| L5a1  | 12                      | 12                      | 12                      |
| L6a1  | 12                      | 12                      | 12                      |
| L6a2  | 12                      | 12                      | 12                      |
| L6a3  | 36                      | 36                      | 36                      |
| L6a4  | 24                      | 24                      | 24                      |
| L6a5  | 48                      | 48                      | 48                      |
| L6a1  | 24                      | 24                      | 24                      |
| L7a1  | 16$s^2t^2 + 8s^2 + 16t^2 + 20$ | 24$s^2t^2 + 36$ | 12$s^2t^2 + 12s^2 + 12t^2 + 24$ |
| L7a2  | 12                      | 12                      | 12                      |
| L7a3  | 12                      | 12                      | 12                      |
| L7a4  | 12                      | 12                      | 12                      |
| L7a5  | 12$s^2t^2 + 4s^2 + 4t^2 + 16$ | 16$s^2t^2 + 20$ | 10$s^2t^2 + 6s^2 + 6t^2 + 14$ |
| L7a6  | 12                      | 12                      | 12                      |
| L7a7  | 24                      | 24                      | 24                      |
| L7n1  | 12                      | 12                      | 12                      |
| L7n2  | 12                      | 12                      | 12                      |

Many other link invariants can be defined from quandle cocycle quivers. For instance, we can modify the incidence matrix of the graph incorporating the cocycle information:

Definition 9. Let $X$ be a quandle, $\phi \in H^2_Q(X; A)$ a quandle 2-cocycle with values in an abelian group $A$, and $S \subset \text{Hom}(X, X)$ a set of endomorphisms. Then for a oriented link $L$ and choice of numbering for
vertices and edges in $Q^f_X(L)$, we define a matrix $M_{Q^f_X(L)}$ whose entry in row $j$ column $k$ is

$$
\begin{cases}
-\phi(v_j) & v_j = \text{Source}(e_k) \\
\phi(v_j) & v_j = \text{Target}(e_k) \\
0 & \text{Else}
\end{cases}
$$

The matrix $M_{Q^f_X(L)}$ itself depends on our choice of numbering for vertices and edges, but we can obtain from it several link invariants including but not limited to:

- The rank of $M_{Q^f_X(L)}$,
- The isomorphism class of the linear transformation between $R$-modules determined by the matrix,
- The Smith normal form of the matrix when $R$ is a PID,
- The eigenvalues and characteristic polynomial when $M_{Q^f_X(L)}$ is a square matrix,
- The elementary ideals of $M_{Q^f_X(L)}$

and more.

**Example 7.** Let $L$ be the link $L_{4a1}$ (the $(4,2)$ torus link) and consider the quandle, endomorphism and cocycle

| $\mathcal{Q}$ | 1 | 2 | 3 |
|----------------|---|---|---|
| 1              | 1 | 2 | 2 |
| 2              | 2 | 2 | 1 |
| 3              | 3 | 3 | 3 |

$f = [1, 1, 2]$, $\phi = 2\chi_{13} + 3\chi_{23} + 4\chi_{31} + 4\chi_{32} \in H^2_Q(X; \mathbb{Z}_5)$.

Then we compute the matrix $M_{Q^f_X(L)} \in M_9(\mathbb{Z}_5)$ for the cocycle quiver:

Then for example, we obtain cocycle quiver characteristic polynomial value $(x + 3)^5x^4 \in \mathbb{Z}_5[x]$.

**Example 8.** Using the same quandle and cocycle as in example 7 with endomorphism $f = [2, 1, 3]$, the link $L_{7a3}$ has cocycle quiver and matrix

$$
\begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
$$
with characteristic polynomial \((x^2 + 3)^2x^5 \in \mathbb{Z}_5[x]\).

4 Questions

We conclude with a few questions and direction for future work.

As we saw in example 4, the cocycle polynomial does not determine the cocycle quiver. We are curious about which \(R\)-colored quivers are obtainable as quandle cocycle quivers of knots and links. For example, the out-degree of every vertex must me the same, namely \(|S|\), so not every quiver is eligible. A link with cocycle quiver

![Cocycle Quiver Diagram]

would have the same cocycle quiver polynomial as \(L7a3\) but be distinguished by the cocycle quiver itself, though we do not know of such a link.

To what extent is possible to reverse engineer link/quandle/endomorphism/cocycles to fit a particular \(R\)-colored quiver? What are necessary and sufficient conditions for a \(R\)-colored quiver to be the quandle cocycle quiver of a knot or link, and given such a quiver how can we construct the set of all links with the given \(R\)-colored quiver as \(\mathbb{Q}_{X,S}(L)\)?

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