ON SMOOTHNESS OF EXTREMIZERS OF THE TOMAS-STEIN INEQUALITY FOR $S^1$

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Abstract. We prove that the extremizers to the Tomas-Stein inequality for the one dimension sphere are smooth. This is achieved by studying the associated Euler-Lagrange equation.

1. Introduction

To understand the Fourier transform of functions on the Euclidean space, Stein [28] proposed the restriction problem. Let $d \geq 1$ be a fixed integer. Let $S$ be a smooth compact hypersurface with boundary in the space $\mathbb{R}^{d+1} = \mathbb{R} \times \mathbb{R}^d$ and $\sigma$ be the induced Lebesgue measure on $S$. Stein’s restriction problem asks for which $1 \leq p, q \leq \infty$ is the following estimate true,

$$\|\hat{f}\sigma\|_{L^q(\mathbb{R} \times \mathbb{R}^d)} \leq C_{p, q, d, S} \|f\|_{L^p(S)},$$

for all test functions, where $\hat{F}$ is the space time Fourier transform. It is not hard to see that $p, q$ satisfy the following necessary conditions

$$q > \frac{2(d + 1)}{d}, \quad \frac{d}{p'} \geq \frac{d + 2}{q},$$

where $p'$ is the conjugate exponent of $p$. This problem is related to several outstanding conjectures in harmonic analysis such as the Bochner-Riesz conjecture and the Kakeya conjecture; for the references, see for instance [4, 6, 16, 17, 29, 31].

Let $S = S^d$, the unit sphere in $\mathbb{R} \times \mathbb{R}^d$, and $\sigma$ be the surface measure. The Tomas-Stein inequality for the sphere is

$$\|\hat{f}\sigma\|_{L^{2+\frac{4}{d}}(\mathbb{R}^{d+1})} \leq \mathcal{R}_d \|f\|_{L^2(S^d)}$$

where $d \geq 1$, $\mathcal{R}_d$ denotes the optimal constant

$$\mathcal{R}_d = \sup_{f \in L^2, f \neq 0} \frac{\|\hat{f}\sigma\|_{L^{2+\frac{4}{d}}}}{\|f\|_{L^2(S^d)}}.$$

The Tomas-Stein inequality is a type of Fourier restriction inequality. Its variants, the Strichartz inequalities, are useful in the partial differential equations, see for instance [30]. In this note, we consider the extremal problem for the Tomas-Stein inequality for the one dimensional sphere $S^1$. The extremal problem includes the questions of proving existence and establishing characterization.

Key words and phrases. The Tomas-Stein inequality, extremizers, smoothness.
of extremizers. An extremizer \( f \) to the Tomas-Stein inequality (1) is a nonzero function \( f \in L^2 \) such that
\[
\| \hat{f}_\sigma \|_{L^2(S^{d-1})} = R_d \| f \|_{L^2(S^d)}.
\]

In this note, we specify the dimension \( d = 1 \) and write \( R = R_1 \). In [25], we have proved there exists an extremizer when \( d = 1 \). Here we establish the smoothness property of extremizers. The work [25] and this note follow roughly similar lines as in [11, 12]. In the previous work [11, 12], Christ and the author prove the existence of extremizers and established some characterization for the Tomas-Stein inequality (1) when \( d = 2 \). In this case, Foschi [15] settles down the problem by proving that constants are the only extremizers up to the complex modulation. In [9], for (1) when \( d = 1 \), Carneiro, Foschi, Silva and Thiele recently prove the same but a conditional result that constants are the only extremizers up to the complex modulation. This relies on the earlier work of Silva and Thiele [26] about the inequality of a 6-fold product of Bessel functions and the study of a functional equation of Cauchy-Pexider type on the sphere in Charalambides [10].

The work [11, 12, 25] are partly motivated by the recent progress of application of the concentration compactness method or the profile decompositions in critical dispersive partial differential equations, see for instance Bourgain [5], Colliander, Keel, Staffilani, Takaoka and Tao [13], Kenig and Merle [20, 21] for radial or general data.

The extremal problem for some related Fourier restriction inequalities have been intensively studied recently. The variant of (1) is the Strichartz inequality for the Schrödinger equation,
\[
\| e^{it\Delta} f \|_{L^{2+\frac{4}{d}}(\mathbb{R} \times \mathbb{R}^d)} \leq C \| f \|_{L^2(\mathbb{R}^d)}
\]
where \( e^{it\Delta} f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix \cdot \xi + i|\xi|^2 t} \hat{f}(\xi) \, d\xi \) and \( d \geq 1 \). This can be viewed as an estimate of the Fourier transform of a measure supported on the paraboloid in \( \mathbb{R} \times \mathbb{R}^d \). For \( d = 1 \) in (3), Kunze is the first to prove the existence of extremizers in [22] by an elaborate concentration-compactness argument. Foschi [14], Hundertmark and Zharnitsky [19] show that Gaussian functions are the only extremizers for (3) when \( d = 1, 2 \). Bennett, Bez, Carbery and Hundertmark [3] give a different proof of this fact by using the heat-flow method. In [8], Carneiro establishes some sharp Strichartz inequalities for the Schrödinger equation by generalizing the argument in [19]. When \( d \geq 3 \), we have proved the existence of extremizers by using the profile decompositions in [2]. For the wave equation, Bulut [7] has proved the existence of extremizers by using the profile decompositions in the spirit of [1].

In this note, we will prove that solutions to the following generalized Euler-Lagrange equation, which the extremizers satisfy, are smooth. The equation to the inequality (1) is
\[
f \sigma * f \sigma * f \sigma * \tilde{f} \sigma (x) = \lambda f(x), \text{ for almost everywhere } x \in S^1,
\]
where \( \lambda = R^6 \| f \|_{L^2}^4 \) and \( \tilde{f}(x) = \bar{f}(-x) \), \( \bar{f} \) denotes the complex conjugate of \( f \). The main result is the following.

**Theorem 1.1.** Any solution to the Euler-Lagrange equation (4) is smooth on \( S^1 \).

The proof of this theorem follows roughly the similar lines as in [12], where we prove that the extremizers for the Tomas-Stein inequality for the two dimensional sphere are smooth. The first step is to show that solutions to the generalized Euler-Lagrange equation gain some regularity depending on the critical points themselves; the second step is a bootstrap argument upgrading
the regularity to infinity, see Section 3. The difficulty is that there is no useful formula for the convolution $\sigma * \sigma * \sigma * \sigma * \sigma$, see also [9, Section 2].

This paper is organized as follows. In Section 2, we set up some notations. In Section 3, we give the main argument showing that the extremizers to (1) are smooth.

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2. Notation

For $s \geq 0$, $H^s = H^s(S^1)$ denotes the usual Sobolev space of functions having $s \geq 0$ derivatives in $L^2$. We also write $H^0 = L^2$. Consider the action of the group $O(2)$ of all rotations of $\mathbb{R}^2$ acting on $S^1$. This action gives rise in a natural way to actions on functions by

$$\Theta(f) = f \circ \Theta$$

and on finite Borel measures on $\mathbb{R}^2$ by $\Theta_*(\mu)(E) = \mu(\Theta(E))$. The extension satisfies the basic identity

$$\Theta_*(\mu * \nu) = \Theta_*(\mu) * \Theta_*(\nu).$$

Let $\{X_j : j = 1, 2\}$ be two $C^\infty$ vector fields on $S^1$ which generate rotations about the two coordinate axes; thus $\exp(tX_j)$ is obtained by rotating $x \in \mathbb{R}^2$ by $t$ radians about the $j$-th coordinate axis. These two vector fields span the one dimension tangent space to $S^1$ at each of its points. So $H^1(S^1)$ is equal to the set of all $f \in L^2(S^1)$ for all $X_j(f) \in L^2(S^1)$ for all indices $j \in \{1, 2\}$.

For $\alpha \in (0, 1)$, we denote by $\Lambda^\alpha$ the space of all Hölder continuous functions of order $\alpha$ on $S^1$, with norm

$$|f|_{\Lambda^\alpha} = \|f\|_{C^0} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$ 

For $\alpha \in (0, 1)$, $\Lambda^\alpha$ equals the set of all continuous functions $f$ for which there exists $C < \infty$ such that

$$|\exp(tX_j)f(x) - f(x)| \leq C|t|^\alpha$$

for all $t \in \mathbb{R}$ and $x \in S^1$ for $j = 1, 2$, with a corresponding equivalence of norms. We denote by $\text{Lip}(S^1)$ the space of all Lipschitz continuous functions from $S^1$ to $\mathbb{C}$, equipped with the norm

$$|f|_{\text{Lip}(S^1)} = \|f\|_{C^0} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$ 

For $0 \leq s \notin \mathbb{Z}$, we write $s = k + \alpha$, where $k \in \mathbb{Z}$ and $\alpha \in (0, 1)$. For $s \in (0, 1)$, we define $\mathcal{H}^s$ to be the set of all $f \in L^2(S^1)$ for which

$$|f|_{\mathcal{H}^s} = \|f\|_{L^2(S^1)} + \sum_{j=1}^2 \sup_{0 < |t| \leq 1} \frac{\|\exp(tX_j)f - f\|_{L^2(S^1)}}{|t|^s}.$$
is finite; for $s = 0$, we define $\mathcal{H}^0 = L^2$. For $s = k + \alpha$ with $k \in \mathbb{Z}^+$ and $\alpha \in (0, 1)$, $\mathcal{H}^s$ is the set of all $f \in L^2(S^1)$ for which

$$
\|f\|_{\mathcal{H}^s} = \|f\|_{L^2(S^1)} + \sum_{j=1}^{2} \sum_{0 < |t| \leq 1} \sup_{Y} \|Yf \circ \exp(tX_j) - Yf\|_{L^2(S^1)}
$$

is finite, where $Y$ ranges over the finite set of all compositions $X_{i_1} \circ X_{i_2} \circ \cdots \circ X_{i_m}$ with $0 \leq m \leq k$ factors. Here $f = Yf$, where $Y$ has zero factors. The mapping $f \mapsto \Theta(f) = f \circ \Theta$ maps $\mathcal{H}^s$ isometrically to $\mathcal{H}^s$, uniformly for all $\Theta \in O(2)$. For any $0 < t < s$, it is not hard to see that $\mathcal{H}^s$ is contained in the Sobolev space $H^t$, and

$$
\|f\|_{H^t} \leq C(s, t)\|f\|_{\mathcal{H}^s}
$$

for all $f \in \mathcal{H}^s$, see for instance [27, Chapter 5]. It is easy to see that,

$$
C^\infty \subset \Lambda^s \subset \mathcal{H}^s \subset H^s \subset L^2,
$$

where $s > 0$. This chain of inclusions shows that $C^\infty$ is dense in any larger space.

3. The Proof

In this section, we prove Theorem 1.1. We first show that solutions to the generalized Euler-Lagrange equation (1) gain some regularity, see Lemma 3.4. Then we upgrade the regularity to infinity, see Lemma 3.5. We begin with a trivial interpolation result.

**Lemma 3.1.** For $0 < \beta < \alpha$,

$$
\|f\|_{\mathcal{H}^\beta} \leq C\|f\|^{1-\frac{\beta}{\alpha}}_{\mathcal{H}^\alpha}\|f\|^{\frac{\beta}{\alpha}}_{H^\alpha} \sim \|f\|^{1-\frac{\beta}{\alpha}}_{L^2}\|f\|^{\frac{\beta}{\alpha}}_{H^\alpha}.
$$

*Proof.* The inequality follows from

$$
\|f\|_{L^2} \leq \|f\|^{1-\frac{\beta}{\alpha}}_{L^2}\|f\|^{\frac{\beta}{\alpha}}_{H^\alpha}
$$

and for $0 < |t| \leq 1$,

$$
\frac{\|\exp(tX_j)f - f\|_{L^2}}{|t|^{\beta}} \leq \exp(tX_j)f - f \|^{1-\frac{\beta}{\alpha}}_{L^2} \left( \frac{\|\exp(tX_j)f - f\|_{L^2}}{|t|^{\alpha}} \right)^{\frac{\beta}{\alpha}} \leq C\|f\|^{1-\frac{\beta}{\alpha}}_{L^2}\|f\|^{\frac{\beta}{\alpha}}_{H^\alpha}.
$$

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**Lemma 3.2.** Let $\mu = \sigma * \sigma * \sigma * \sigma * \sigma$. Then $\|\mu\|_{L^\infty(\{|x| < 5\})} \leq C$ for some constant $C > 0$. Similarly we have $\|\sigma * \sigma * \sigma * \sigma\|_{L^\infty(\{|x| < 4\})} < C$ for some constant $C > 0$.

The proof uses the $L^1 \to L^\infty$ Hausdorff-Young inequality and the decay estimate of the sphere measure, $|\sigma(x)| \leq |x|^{-1/2}$, for sufficiently large $|x| > 0$. We omit it here.

**Lemma 3.3.** Suppose that $f_i \in \mathcal{H}^s$ for $i = 1, \cdots, 5$ and $s \geq 0$. Then

$$
\|f_1 \sigma * f_2 \sigma * f_3 \sigma * f_4 \sigma * f_5 \sigma\|_{\mathcal{H}^s} \leq C \prod_{i=1}^{5} \|f_i\|_{\mathcal{H}^s}.
$$
\textbf{Proof.} By the Cauchy-Schwarz inequality,

\begin{equation}
|f_1 \sigma \cdots \cdot f_5 \sigma(x)| \leq \left( |f_1|^2 \sigma \cdots \cdot |f_5|^2 \sigma(x) \right)^{1/2} \left( \sigma \cdots \cdot \sigma(x) \right)^{1/2}.
\end{equation}

If we integrate both sides, by Lemma 3.2,

\begin{equation}
\|f_1 \sigma \cdots \cdot f_5 \sigma \|^2_{L^2} \leq \int_{S^1} \left( |f_1|^2 \sigma \cdots \cdot |f_5|^2 \sigma(x) \right) |\sigma \cdots \cdot \sigma(x)| \, d\sigma
\end{equation}

\begin{equation}
\leq \sup_{x \in S^1} |\sigma \cdots \cdot \sigma(x)| \int_{S^1} |f_1|^2 \sigma \cdots \cdot |f_5|^2 \sigma(x) \, d\sigma
\end{equation}

\begin{equation}
\leq C \prod_{i=1}^5 \|f_i\|_{L^2}^2.
\end{equation}

This proves the lemma when \( s = 0 \).

Let \( s > 0 \). For \( 0 < |t| \leq 1 \), we just need to prove

\begin{equation}
\frac{\| \exp(t X_j) (f_1 \sigma \cdots \cdot f_5 \sigma) - f_1 \sigma \cdots \cdot f_5 \sigma \|^2_{L^2}}{|t|^{2s}}
\end{equation}

\begin{equation}
= \int_{S^1} \frac{\left( |f_1 \sigma \cdots \cdot f_5 \sigma| \circ \exp(t X_j)(y) - (f_1 \sigma \cdots \cdot f_5 \sigma)(y) \right)^2}{|t|^{2s}} \, d\sigma(y)
\end{equation}

\begin{equation}
\leq C \prod_{i=1}^5 \|f_i\|_{H^s}^2.
\end{equation}

We compute that, for \( j = 1, 2 \) and \( 0 < |t| \leq 1 \),

\begin{equation}
(f_1 \sigma \cdots \cdot f_5 \sigma) \circ \exp(t X_j) - (f_1 \sigma \cdots \cdot f_5 \sigma)
\end{equation}

\begin{equation}
= (f_1 \circ \exp(t X_j) - f_1) \sigma \circ (\exp(t X_j) f_2) \sigma \circ (\exp(t X_j) f_3) \sigma \circ (\exp(t X_j) f_4) \sigma \circ (\exp(t X_j) f_5) \sigma +
\end{equation}

\begin{equation}
+ (\exp(t X_j) f_1) \sigma \circ (f_2 \circ \exp(t X_j) - f_2) \sigma \circ (\exp(t X_j) f_3) \sigma \circ (\exp(t X_j) f_4) \sigma \circ (\exp(t X_j) f_5) \sigma +
\end{equation}

\begin{equation}
+ \cdots + (\exp(t X_j) f_1) \cdots (\exp(t X_j) f_5) \sigma \circ (\exp(t X_j) f_4) \sigma \circ (\exp(t X_j) f_1) \sigma \circ (f_5 \circ \exp(t X_j) - f_5).
\end{equation}

For the first term in (11), by the Cauchy-Schwarz inequality,

\begin{equation}
(f_1 \circ \exp(t X_j) - f_1) \sigma \circ (\exp(t X_j) f_2) \sigma \circ (\exp(t X_j) f_3) \sigma \circ (\exp(t X_j) f_4) \sigma \circ (\exp(t X_j) f_5) \sigma
\end{equation}

\begin{equation}
\lesssim \left( |f_1 \circ \exp(t X_j) - f_1|^2 \sigma \circ |\exp(t X_j) f_2|^2 \sigma \circ |\exp(t X_j) f_3|^2 \sigma \circ |\exp(t X_j) f_4|^2 \sigma \circ |\exp(t X_j) f_5|^2 \sigma \right)^{1/2} \times (\sigma \cdots \cdot \sigma)^{1/2}.
\end{equation}

Applying the same reasoning to other terms in (12) and going back to (10), we see the claim in Lemma 3.3 for \( s > 0 \) is proved. Thus we finish the proof of Lemma 3.3. \( \square \)

Next we show that solutions to the Euler-Lagrange equation (4) gain some regularity.

\textbf{Lemma 3.4.} Suppose that \( f \in L^2 \) satisfies the Euler-Lagrange equation (4). Then \( f \in H^s \) for some \( s > 0 \). In particular, \( f \in H^1 \) for all \( 0 \leq t < s \).

\textbf{Proof.} We follow the proof of [12] Lemma 2.2. For any \( \epsilon > 0 \), we decompose \( f \) such that \( g_\epsilon \in C^\infty \) and \( \phi_\epsilon \in C^\infty \).
Recall that
\[ \| \phi_\epsilon \|_{\mathcal{H}^s} = \| \phi_\epsilon \|_{L^2} + \sum_{j=1}^{2} \sup_{0 < |t| \leq 1} \left\| \frac{\exp(tX_j) f - f}{|t|^s} \right\|_{L^2(S^1)}, \]
and
\[ \| \phi_\epsilon \|_{\Lambda^s} = \| \phi_\epsilon \|_{L^\infty} + \sup_{x \neq y} \frac{\| \phi_\epsilon(x) - \phi_\epsilon(y) \|}{|x - y|^s}. \]
Then since \( \phi_\epsilon \in C^\infty \),
\[ \| \phi_\epsilon \|_{\mathcal{H}^s} \leq C \| \phi_\epsilon \|_{\Lambda^s} < C_\epsilon < \infty. \]
We remark that this bound depends on \( \epsilon \).

From the Euler-Lagrange equation (4) and \( f = \phi_\epsilon + g_\epsilon \), we have
\[ g_\epsilon = \mathcal{L}(\phi_\epsilon, g_\epsilon) + \mathcal{N}(\phi_\epsilon, g_\epsilon), \]
where \( \mathcal{L} \) is linear in \( g_\epsilon \) and \( \mathcal{N} \) is nonlinear in \( g_\epsilon \). More precisely,
\[ \mathcal{L} = -\phi_\epsilon + \phi_\epsilon \sigma \ast \phi_\epsilon \sigma \ast \tilde{\phi}_\epsilon \sigma \ast \hat{\phi}_\epsilon \sigma + \]
\[ + 2 \phi_\epsilon \sigma \ast \phi_\epsilon \sigma \ast \tilde{\phi}_\epsilon \sigma \ast \hat{\phi}_\epsilon \sigma + 3 \phi_\epsilon \sigma \ast \phi_\epsilon \sigma \ast \tilde{\phi}_\epsilon \sigma \ast \hat{\phi}_\epsilon \sigma \ast g_\epsilon \sigma; \]
and
\[ \mathcal{N} = g_\epsilon \sigma \ast g_\epsilon \sigma \ast \tilde{g}_\epsilon \sigma \ast \hat{g}_\epsilon \sigma \]
\[ + 3 g_\epsilon \sigma \ast g_\epsilon \sigma \ast \tilde{g}_\epsilon \sigma \ast \hat{g}_\epsilon \sigma \ast \phi_\epsilon \sigma + 2 g_\epsilon \sigma \ast g_\epsilon \sigma \ast g_\epsilon \sigma \ast \tilde{g}_\epsilon \sigma \ast \hat{\phi}_\epsilon \sigma + \]
\[ + 3 g_\epsilon \sigma \ast \tilde{g}_\epsilon \sigma \ast \tilde{g}_\epsilon \sigma \ast \phi_\epsilon \sigma \ast \tilde{\phi}_\epsilon \sigma \ast \hat{\phi}_\epsilon \sigma + g_\epsilon \sigma \ast g_\epsilon \sigma \ast \tilde{\phi}_\epsilon \sigma \ast \hat{\phi}_\epsilon \sigma \ast \tilde{\phi}_\epsilon \sigma \ast \phi_\epsilon \sigma \ast \phi_\epsilon \sigma + \]
\[ + 6 g_\epsilon \sigma \ast \tilde{g}_\epsilon \sigma \ast \phi_\epsilon \sigma \ast \phi_\epsilon \sigma \ast \tilde{\phi}_\epsilon \sigma \ast \phi_\epsilon \sigma \ast \phi_\epsilon \sigma + 3 g_\epsilon \sigma \ast g_\epsilon \sigma \ast \phi_\epsilon \sigma \ast \phi_\epsilon \sigma \ast \tilde{\phi}_\epsilon \sigma \ast \phi_\epsilon \sigma \ast \phi_\epsilon \sigma. \]
For any \( \alpha > 0 \),
\[ \| \mathcal{L}(\phi_\epsilon, g_\epsilon) \|_{\Lambda^\alpha} \leq \| \phi_\epsilon \|_{\Lambda^\alpha} + C \| \phi_\epsilon \|_{\Lambda^\alpha}^5 + C \| \phi_\epsilon \|_{\Lambda^\alpha} \| g_\epsilon \|_{L^2} \]
Since \( \| \phi_\epsilon \|_{\Lambda^\alpha} < C_\epsilon < \infty \) and \( \| g_\epsilon \|_{L^2} \leq \| f \|_{L^2} \),
\[ \| \mathcal{L}(\phi_\epsilon, g_\epsilon) \|_{\Lambda^\alpha} < C_\epsilon < \infty. \]
Together with \( \| \mathcal{L}(\phi_\epsilon, g_\epsilon) \|_{\mathcal{H}^\alpha} \leq \| \mathcal{L}(\phi_\epsilon, g_\epsilon) \|_{\Lambda^\alpha} \), this implies
\[ \| \mathcal{L}(\phi_\epsilon, g_\epsilon) \|_{\mathcal{H}^\alpha} \leq C_\epsilon < \infty. \]
On the other hand, by Lemma 3.3,
\[ \| \mathcal{N}(\phi_\epsilon, g_\epsilon) \|_{\mathcal{H}^\alpha} \lesssim \| g_\epsilon \|_{\mathcal{H}^\alpha}^5 + \| g_\epsilon \|_{\mathcal{H}^\alpha} \| \phi_\epsilon \|_{\mathcal{H}^\alpha} + \| g_\epsilon \|_{\mathcal{H}^\alpha}^3 \| \phi_\epsilon \|_{\mathcal{H}^\alpha}^2 + \| g_\epsilon \|_{\mathcal{H}^\alpha}^2 \| \phi_\epsilon \|_{\mathcal{H}^\alpha}^3 \]
\[ \lesssim \epsilon^5 + \epsilon^4 + \epsilon^3 + \epsilon^2 \lesssim \epsilon^2, \]
as \( \| g_\epsilon \|_{\mathcal{H}^\alpha} \sim \| g_\epsilon \|_{L^2} \leq \epsilon \) and \( \| \phi_\epsilon \|_{\mathcal{H}^\alpha} \sim \| \phi_\epsilon \|_{L^2} \leq 1 \). By the triangle inequality we have
\[ \| \mathcal{L}(\phi_\epsilon, g_\epsilon) \|_{\mathcal{H}^\alpha} \leq C_\epsilon. \]
Choosing \( \epsilon \) sufficiently small, and interpolating between (17) and (19), we see that there exists \( s(\epsilon) \) depending on \( \epsilon \) such that
\[ \| \mathcal{L}(\phi_\epsilon, g_\epsilon) \|_{\mathcal{H}^{s(\epsilon)}} \lesssim \epsilon^\frac{1}{2}. \]
From the two bounds \( \|\phi_\epsilon\|_{\mathcal{H}^0} \leq 1 \) and \( \|\phi_\epsilon\|_{\mathcal{H}^0} < C(\epsilon) < \infty \), again choosing \( s(\epsilon) \) sufficiently small, we see that

\[
(21) \quad \|\phi_\epsilon\|_{\mathcal{H}^s(\epsilon)} < \epsilon^{-1/5}.
\]

Next we use the argument of Picard’s iteration to show that \( f \) will gain some regularity. Fixing the small \( \epsilon > 0 \) above, we know that \( g_\epsilon \in L^2 \) and \( \phi_\epsilon \in C^\infty \). Define the iteration mapping and the ball in \( \mathcal{H}^s(\epsilon) \),

\[
(22) \quad \mathcal{L}_\epsilon(h) = \mathcal{L}(\phi_\epsilon, g_\epsilon) + \mathcal{N}(\phi_\epsilon, h),
\]

\[
B = B(\mathcal{L}(\phi_\epsilon, g_\epsilon), \epsilon^3).
\]

In the following two steps, we show that \( \mathcal{L}_\epsilon \) is a contraction map on \( B \). The first step is to show that \( \mathcal{L}_\epsilon \) maps \( B \) to itself. The second step is to show that \( \mathcal{L}_\epsilon \) Lipschitz with the Lipschitz constant strictly less than 1.

**Step 1.** For any \( h \in B \), by the triangle inequality and \( (20) \),

\[
(23) \quad \|h\|_{\mathcal{H}^s(\epsilon)} \leq \|h - \mathcal{L}(\phi_\epsilon, g_\epsilon)\|_{\mathcal{H}^s(\epsilon)} + \|\mathcal{L}(\phi_\epsilon, g_\epsilon)\|_{\mathcal{H}^s(\epsilon)} \lesssim \epsilon^{3/4} + \epsilon^{7/8} \lesssim \epsilon^{3/4}.
\]

Then similarly as in proving \( (18) \), by \( (21) \),

\[
(24) \quad \|\mathcal{N}(\phi_\epsilon, h)\|_{\mathcal{H}^s(\epsilon)} \lesssim \epsilon^{3/4} + \epsilon^{3/4} \epsilon^{3/8} + \epsilon^{3/4} \epsilon^{-1/8} + \epsilon^{3/4} \epsilon^{-1/2} + \epsilon^{3/4} \epsilon^{-1/4} \lesssim \epsilon^{3/4}/10.
\]

Then for \( h \in B \),

\[
(25) \quad \|\mathcal{L}_\epsilon(h) - \mathcal{L}(\phi_\epsilon, g_\epsilon)\|_{\mathcal{H}^s(\epsilon)} = \|\mathcal{N}(\phi_\epsilon, h)\|_{\mathcal{H}^s(\epsilon)} \lesssim \epsilon^{3/8}.
\]

This proves that \( \mathcal{L}_\epsilon \) is a map from \( B \) to \( B \).

**Step 2.** We take \( h_1, h_2 \in B \). Then by \( (23) \),

\[
(26) \quad \|h_1\|_{\mathcal{H}^s(\epsilon)} \lesssim \epsilon^{3/4}, \quad \text{and} \quad \|h_2\|_{\mathcal{H}^s(\epsilon)} \lesssim \epsilon^{3/4}.
\]

Note that by \( (21) \), \( \|\phi_\epsilon\|_{\mathcal{H}^s(\epsilon)} \leq \epsilon^{-1/4} \), then by Lemma \( 3.3 \)

\[
(27) \quad \mathcal{L}_\epsilon(h_2) - \mathcal{L}_\epsilon(h_1) = \mathcal{N}(\phi_\epsilon, h_2) - \mathcal{N}(\phi_\epsilon, h_1)
\]

\[
\lesssim \|h_2 - h_1\|_{\mathcal{H}^s(\epsilon)} \left( 5\epsilon^{3/4} + 5 \times 5\epsilon^{3/4} \epsilon^{3/8} + 10 \times 3\epsilon^{3/4} \epsilon^{-1/8} \epsilon^{3/4} + 10 \times 2\epsilon^{3/4} \epsilon^{-1/4} \epsilon^{3/4} \right).
\]

To conclude, if taking \( \epsilon \) sufficiently small,

\[
(28) \quad \|\mathcal{L}_\epsilon(h_2) - \mathcal{L}_\epsilon(h_1)\|_{\mathcal{H}^s(\epsilon)} \leq \alpha \|h_2 - h_1\|_{\mathcal{H}^s(\epsilon)}
\]

for some \( 0 < \alpha < 1 \). So \( \mathcal{L}_\epsilon \) is a contraction mapping on \( B \). Therefore there exists a unique \( h_\epsilon \in B \subset \mathcal{H}^s(\epsilon) \) such that

\[
(29) \quad h_\epsilon = \mathcal{L}_\epsilon(h_\epsilon) = \mathcal{L}(\phi_\epsilon, g_\epsilon) + \mathcal{N}(\phi_\epsilon, h_\epsilon).
\]

Since \( \mathcal{H}^s(\epsilon) \subset \mathcal{H}^0 = L^2 \), and in \( L^2 \) there holds

\[
\|g_\epsilon\|_{\mathcal{H}^s(\epsilon)} \leq \|\phi_\epsilon\|_{\mathcal{H}^s(\epsilon)} + \|g_\epsilon\|_{\mathcal{H}^s(\epsilon)} + \|h_\epsilon\|_{\mathcal{H}^s(\epsilon)}.
\]

Therefore \( h_\epsilon \) is in \( L^2 \). This upgrades \( g_\epsilon \in \mathcal{H}^s(\epsilon) \). It in turn shows that \( f \in \mathcal{H}^s(\epsilon) \). Note that \( s(\epsilon) \) depends on \( f \).

The second main ingredient is a bootstrap lemma.
Lemma 3.5. For any $\epsilon > 0$, there exists $\delta > 0$ such that for any $s \in [\epsilon, \infty) \setminus \mathbb{Z}$ and any function $f \in H^s(\mathbb{S}^1)$, then
\[
 f \ast f \ast f \ast f \ast f |_{S^1} \in H^t(\mathbb{S}^1)
\]
for all $t \in [0, s + \delta] \setminus \mathbb{Z}$.

This proof is similar to [12, Lemma 3.2]. It relies on the following proposition, which is in the same spirit as [12, Lemma 2.6].

Proposition 3.6. For any $\epsilon > 0$, there exists $\delta > 0$ such that
\[
 f_1 \ast f_2 \ast f_3 \ast f_4 \ast h \sigma \in H^\delta
\]
whenever $f_i \in H^\epsilon(\mathbb{S}^1)$, $1 \leq i \leq 4$, and $h \in H^0(\mathbb{S}^1)$, and
\[
 ||f_1 \ast f_2 \ast f_3 \ast f_4 \ast h \sigma||_{H^\delta} \leq C_\epsilon \prod_{j=1}^4 ||f_i||_{H^\epsilon} ||h||_{H^0}.
\]

Proof. Without loss of generality, we suppose that
\[
 ||f_i||_{H^\epsilon} = 1, \text{ for } 1 \leq i \leq 4, \text{ and } ||h||_{H^0} = 1.
\]

We divide the proof in the following 3 steps.

Step 1. We establish the inequality (31) under the hypothesis that $f_i \in \text{Lip}(\mathbb{S}^1)$, for $1 \leq i \leq 4$.

Let $F(x) = f_1 \ast f_2 \ast f_3 \ast f_4 \ast h \sigma(x)$. We claim that $F \in H^1(\mathbb{S}^1)$. Recall that $\mu = \sigma \ast \sigma \ast \sigma \ast \sigma \ast \sigma$.

\[
 |F(x) - F(y)| = \left| \int f_1(x - x_1 - x_2 - x_3 - x_4) f_2(x_1) f_3(x_2) f_4(x_3) h(x_4) d\mu \right|
\]
\[
 - \int f_1(y - x_1 - x_2 - x_3 - x_4) f_2(x_1) f_3(x_2) f_4(x_3) h(x_4) d\mu \right|
\]
\[
 \leq C |x - y| \int |f_2(x_1) f_3(x_2) f_4(x_3)| h(x_4) d\mu
\]
\[
 \leq C \left( \prod_{j=1}^4 ||f_i||_{\text{Lip}(\mathbb{S}^1)} \right) |x - y| \int |h(x_4)| d\mu.
\]

For the integral in the measure $\mu$, by the Cauchy-Schwarz inequality and Lemma 3.2,
\[
 \int |h(x_4)| d\mu = \int |h(x_4)| d\sigma(x - x_1 - x_2 - x_3 - x_4) d\sigma(x_1) d\sigma(x_2) d\sigma(x_3) d\sigma(x_4)
\]
\[
 \leq \left( \int |h(x_4)|^2 d\sigma(x - x_1 - x_2 - x_3 - x_4) d\sigma(x_1) d\sigma(x_2) d\sigma(x_3) d\sigma(x_4) \right)^{1/2} ||\mu||_{L^\infty}^{1/2}
\]
\[
 \leq C \left( \int_{\mathbb{S}^1} \left( \int_{(\mathbb{S}^1)^4} d\sigma(x - x_1 - x_2 - x_3 - x_4) d\sigma(x_1) d\sigma(x_2) d\sigma(x_3) d\sigma(x_4) \right) |h(x_4)|^2 d\sigma(x_4) \right)^{1/2}
\]
\[
 \leq C ||h||_{L^2(\mathbb{S}^1)} ||\sigma \ast \sigma \ast \sigma \ast \sigma||_{L^\infty}^{1/2} \leq C ||h||_{L^2(\mathbb{S}^1)}.
\]
Therefore we conclude that
\begin{equation}
|F(x) - F(y)| \leq C|x - y| \|h\|_{L^2(S^1)},
\end{equation}
where \( C = C \left( \prod_{j=1}^{4} \|f_i\|_{\text{Lip}(S^1)} \right) \). Thus for \( j = 1, 2 \), for \( 0 < |t| \leq 1 \),
\begin{equation}
\frac{\| \exp(tX_j)F - F \|_{L^2(S^1)}}{|t|} = \frac{\| \exp(tX_j)F - F \|_{L^2(S^1)}}{|t|} < \infty.
\end{equation}
This leads to
\begin{equation}
\sum_{j=1}^{2} \sup_{0 < |t| \leq 1} \frac{\| \exp(tX_j)F - F \|_{L^2(S^1)}}{|t|} < \infty.
\end{equation}
From Lemma \[3.3\] \( F \in L^2(S^1) \). So \( h \in \mathcal{H}^1(S^1) \).

**Step 2.** For any \( f \in \mathcal{H}^\varepsilon(S^1) \) and \( \eta > 0 \), there exists a decomposition that \( f = f^\delta + f^b \), where \( f^\delta \in \text{Lip}(S^1) \) and
\begin{align*}
\|f^b\|_{H^0} &\leq \eta \|f\|_{\mathcal{H}^\varepsilon}, \\
\|f^\delta\|_{\text{Lip}(S^1)} &\leq \eta^{-C(\varepsilon)} \|f\|_{\mathcal{H}^\varepsilon}, \\
\|f^\delta\|_{H^0} &\leq C \|f\|_{\mathcal{H}^\varepsilon},
\end{align*}
where \( C, C(\varepsilon) \) independent of \( f \). The existence of such decompositions follows from the inclusion that \( \mathcal{H}^\varepsilon \subset H^\tau \) for some \( \tau = \tau(\varepsilon) > 0 \), together with standard properties of \( H^\tau \). We perform such decompositions to each \( f_i, 1 \leq i \leq 4 \). **Step 1** implies that
\begin{equation}
f_1^\delta \sigma * f_2^\delta \sigma * f_3^\delta \sigma * f_4^\delta \sigma * h \sigma \in \mathcal{H}^1(S^1),
\end{equation}
with the operator norm \( O(\eta^{-4C(\varepsilon)}) \). On the other hand,
\begin{equation}
\|f_1^\delta \sigma * f_2^\delta \sigma * f_3^\delta \sigma * f_4^\delta \sigma * h \sigma\|_{L^2(S^1)} \leq C \prod_{j=1}^{4} \|f_i^\delta\|_{L^2(S^1)} \|h\|_{L^2} \leq C \eta^4.
\end{equation}
Similarly the contributions of the pairs \((f_i^\delta, f_j^\delta), 1 \leq i, j \leq 4\), belong to \( L^2(S^1) \) with norms \( O(\eta) \), since \( f_i^\delta \in H^0 \) is of \( O(1) \).

So far we have shown that for any \( \eta > 0 \), \( F = f_1 \sigma * f_2 \sigma * f_3 \sigma * f_4 \sigma * h \sigma \) can be decomposed as the sum of two functions
\begin{equation}
F = F_\eta + F^\eta,
\end{equation}
where \( F_\eta \in L^2 \) and \( \|F_\eta\|_{L^2} \leq \eta \), and \( F^\eta \in \mathcal{H}^1 \), and \( \|F^\eta\|_{\mathcal{H}^1} \leq C \eta^{-C(\varepsilon)} \). Then we claim that \( F \in \mathcal{H}^\delta \) for some \( \delta \) depending on \( \varepsilon \).

**Step 3.** Let \( 0 < |t| \leq 1 \) and \( \eta > 0 \) be a parameter to be determined. For \( F = F_\eta + F^\eta \), then
\begin{equation}
\| \exp(tX_j)F^\eta - F^\eta \|_{L^2(S^1)} \leq C |t| \|F^\eta\|_{\mathcal{H}^1} \leq C |t| \eta^{-C(\varepsilon)};
\end{equation}
and
\begin{equation}
\| \exp(tX_j)F_\eta - F_\eta \|_{L^2(S^1)} \leq 2 \|F_\eta\|_{L^2(S^1)} \leq 2 \eta.
\end{equation}
Then by the triangle inequality
\begin{equation}
\| \exp(tX_j)F - F \|_{L^2(S^1)} \leq C |t| \eta^{-C(\varepsilon)} + 2 \eta.
\end{equation}
Define \( \eta \) by \( C |t| \eta^{-C(\varepsilon)} = 2 \eta \), then

\[
\eta = \left( \frac{C |t|}{2} \right)^{\frac{1}{1+C(\varepsilon)}}.
\]

Therefore

\[
\| \exp(tX_j) F - F \|_{L^2(S^1)} \leq 4 \left( \frac{C}{2} \right)^{\frac{1}{1+C(\varepsilon)}} |t|^{\frac{1}{1+C(\varepsilon)}} = C |t|^{\frac{1}{1+C(\varepsilon)}}.
\]

Let \( \delta = (1 + C(\varepsilon))^{-1} \). This finishes the proof of Proposition 3.6.

Therefore from Lemma 3.4 and 3.5, the proof of Theorem 1.1 is complete.

REFERENCES

[1] H. Bahouri and P. Gérard. High frequency approximation of solutions to critical nonlinear wave equations. Amer. J. Math., 121(1):131–175, 1999.
[2] P. Bégout and A. Vargas. Mass concentration phenomena for the \( L^2 \)-critical nonlinear Schrödinger equation. Trans. Amer. Math. Soc., 359(11): 5257–5282, 2007.
[3] J. Bennett, N. Bez, A. Carbery, and D. Hundertmark. Heat-flow monotonicity of Strichartz norms. Analysis and PDE, Vol. 2, No. 2: 147–158, 2009.
[4] J. Bourgain. Besicovitch type maximal operators and applications to Fourier analysis. Geom. Funct. Anal., 1(2): 147–187, 1991.
[5] J. Bourgain. Global wellposedness of defocusing critical nonlinear Schrödinger equation in the radial case. J. Amer. Math. Soc., 12(1): 145–171, 1999.
[6] J. Bourgain, L. Guth. Bounds on oscillatory integral operators based on multilinear estimates. Geom. Funct. Anal., 21(6): 1239–1295, 2011.
[7] A. Bulut. Maximizers for the Strichartz inequalities for the Wave Equation. Differential and Integral Equations, 23: 1035-1072, 2010.
[8] E. Carneiro. A sharp inequality for the Strichartz norm. Int. Math. Res. Not. IMRN, (16): 3127–3145, 2009.
[9] E. Carneiro, D. Foschi, D. Silva and C. Thiele. A sharp trilinear inequality related to Fourier restriction on the circle. arXiv:1509.06674.
[10] M. Charalambides. On Restricting Cauchy-Pexider Equations to Submanifolds. Aequationes Math. 86: 231-253, 2013.
[11] M. Christ and S. Shao. Existence of extremals for a Fourier restriction inequality. Analysis and PDE. 5(2): 261–312, 2012.
[12] M. Christ and S. Shao. On the extremisers of an adjoint Fourier restriction inequality. Advance in Mathematics. 230 (3): 957–977, 2012.
[13] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in \( \mathbb{R}^3 \). Ann. of Math. (2), 167(3):767–865, 2008.
[14] D. Foschi. Maximizers for the Strichartz inequality. J. Eur. Math. Soc. (JEMS), 9(4): 739–774, 2007.
[15] D. Foschi. Global maximizers for the sphere adjoint Fourier restriction inequality. J. Funct. Anal. 268(3): 690–702, 2015.
[16] L. Guth. A restriction estimate using polynomial partitioning. Journal of the American Mathematical Society, 29(2): 371–413,2016.
[17] L. Guth. Restriction estimates using polynomial partitioning II. arXiv:1603.04250.
[18] D. Hundertmark and S. Shao. Analyticity of extremals to the Airy-Strichartz inequality. Bull. London Math. Soc. 44(2): 336-352, 2012.
[19] D. Hundertmark and V. Zharnitsky. On sharp Strichartz inequalities in low dimensions. Int. Math. Res. Not., pages Art. ID 34080, 18, 2006.
[20] C. Kenig and F. Merle. Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case. Invent. Math., 166(3): 645–676, 2006.
[21] C. Kenig and F. Merle. Global well-posedness, scattering and blow-up for the energy-critical focusing non-linear wave equation. Acta Math., 201(2): 147–212, 2008.
[22] M. Kunze. On the existence of a maximizer for the Strichartz inequality. Comm. Math. Phys., 243(1): 137–162, 2003.
[23] S. Shao. The linear profile decomposition for the Airy equation and the existence of maximizers for the Airy Strichartz inequality. *Anal. PDE*, 2(1): 83–117, 2009.

[24] S. Shao. Maximizers for the Strichartz and the Sobolev-Strichartz inequalities for the Schrödinger equation. *Electron. J. Differential Equations*, 3: 1-13, 2009.

[25] S. Shao. On existence of extremizers for the Tomas-Stein inequality for $S^1$. *Journal of Functional Analysis*, 270: 3996-4038, 2016.

[26] D. Silva and C. Thiele. Estimates for certain integrals of products of six Bessel functions. arXiv:1509.06309.

[27] E. Stein. Singular integrals and differentiability properties of functions. Princeton Mathematical Series, No. 30, Princeton, NJ, 1970.

[28] E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[29] T. Tao. A sharp bilinear restrictions estimate for paraboloids. *Geom. Funct. Anal.*, 13(6): 1359–1384, 2003.

[30] T. Tao. Nonlinear dispersive equations, local and global analysis. *CBMS Regional Conference Series in Mathematics*, 106, 2006.

[31] T. Wolff. A sharp bilinear cone restriction estimate. *Ann. of Math. (2)*, 153(3): 661–698, 2001.

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