Distribution of coherence in multipartite systems under entropic coherence measure

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The distribution of coherence in multipartite systems is one of the fundamental problems in the resource theory of coherence. To quantify the coherence in multipartite systems more precisely, we introduce new coherence measures, incoherent-quantum (IQ) coherence measures, on bipartite systems by the max- and min-relative entropies and provide the operational interpretation in certain subchannel discrimination problem. By introducing the smooth max- and min-relative entropies of incoherent-quantum (IQ) coherence on bipartite systems, we exhibit the distribution of coherence in multipartite systems: the total coherence is lower bounded by the sum of local coherence and genuine multipartite entanglement. Besides, we find the monogamy relationship for coherence on multipartite systems by incoherent-quantum (IQ) coherence measures. Thus, the IQ coherence measures introduced here truly capture the non-sharability of quantumness of coherence in multipartite context.

I. INTRODUCTION

The key feature of quantumness in a single system can be captured by quantum coherence, stemming from the superposition principle in quantum mechanics. Quantum coherence, as one of the most primitive quantum resource, plays a crucial role in a variety of applications ranging from thermodynamics [1, 2] to metrology [3]. Recently, the resource theory of coherence has attracted much attention [4–10]. There are other notable resource theories including quantum entanglement [11], asymmetry [12–18], thermodynamics [19], and steering [20], among which entanglement is the most famous one and can be used as a basic resource for various quantum information processing protocols such as superdense coding [21], remote state preparation [22, 23] and quantum teleportation [24].

In a resource theory, there are two basic elements: free states and free operations. The free states in the resource theory of coherence are called incoherent states, which are defined as the diagonal states in a given reference basis \( \{ | i \rangle \}_{i=0}^{d-1} \) for a d-dimensional system. The set of incoherent states is denoted by \( \mathcal{I} \). Any quantum state can be mapped to an incoherent state by the full dephasing operation \( \Delta(\rho) = \sum_{i=0}^{d-1} (| i \rangle \langle i |) \rho (| i \rangle \langle i |) \). However, there is still no general consensus on the set of free operations in the resource theory of coherence. Here, we take the incoherent operations (IO) as the free operations, where an operation \( \Lambda \) is called an incoherent operation (IO) if there exists a set of Kraus operators \( \{ K_i \} \) of \( \Lambda \) such that \( K_i \mathcal{I} K_i^\dagger \subseteq \mathcal{I} \) for each \( i \) [4]. To quantify the amount of coherence in the states, several operational coherence measures have been proposed, namely, the relative entropy of coherence [4], the \( l_1 \) norm of coherence [4], the max-relative entropy of coherence [25, 26] and the robustness of coherence [27]. These coherence measures provide the lucid quantitative and operational description of coherence.

The distribution of quantum correlations in multipartite systems is one of the fundamental properties distinguishing quantum correlations from the classical ones, as quantum correlations cannot be shared freely by the subsystems. For example, for any pure tripartite state, if Alice and Bob share a maximally entangled state, then neither Alice nor Bob can be entangled with Charlie, which is dubbed as the monogamy of entanglement [28–31]. Besides, the monogamy of coherence has been investigated in Refs. [32, 33], where it has been shown that the monogamy of coherence for relative entropy of coherence does not hold in general. The distribution of coherence in bipartite and multipartite systems has also been investigated in Ref. [34] and [35], respectively. In [35], the following trade-off relation in multipartite systems has been demonstrated:

\[
C \leq C_I + C_L,
\]

where \( C \) is the total coherence of the whole system, \( C_I \) is called intrinsic coherence which captures the coherence between different subsystems, \( C_L \) is called local coherence which describes the coherence located on each subsystem, and all these three coherence measures are defined by some distance measure which is required to satisfy some conditions, such as triangle inequality. However, it seems that none of the coherence measures defined by \( l_1 \) norm, relative entropy or Jensen-Shannon divergence meets the requirements in Ref. [35], as the existence of triangle inequality for relative entropy and Jensen-Shannon divergence is still unknown, while the \( l_1 \) norm is superadditive for product states, for which a coherence measure is required to be subadditive so that the local coherence will be upper bounded by the sum of the coherence in each subsystem [35]. Thus, a rigorous characterization of the distribution of coherence in multipartite system is imperative and of paramount importance.

Here, we investigate the distribution of coherence in multipartite system in terms of the max- and min-relative entropies. The well-known conditional and unconditional max- and min-entropies [36, 37] can be derived from the max- and min-rela-
tive entropies. Max- and min-relative entropies have also been used to define entanglement monotones and their operational significance in manipulation of entanglement has also been provided in Refs. [38–41]. Coherence measures based on the max- and min-relative entropies have been introduced in Refs. [25, 26], where the operational interpretations have also been provided in Ref. [25]. In this letter, incoherent-quantum (IQ) coherence measures on bipartite systems are introduced in terms of the max- and min-relative entropies, which capture the maximal advantage of bipartite states in certain subchannel discrimination problems. By introducing the smooth max- and min-relative entropies of IQ coherence measures on bipartite systems, we find that the total coherence of a multipartite state is lower bounded by the sum of local coherence in each subsystem and the genuine multipartite entanglement in the multipartite system. Moreover, we obtain the monogamy of coherence in terms of IQ coherence measure. Therefore, the IQ coherence measures introduced here truly capture the non-sharability of quantumness.

II. MAIN RESULT

Let $\mathcal{H}$ be a $d$-dimensional Hilbert space and $\mathcal{D}(\mathcal{H})$ the set of density operators acting on $\mathcal{H}$. The max-relative entropy of coherence for a given state $\rho \in \mathcal{D}(\mathcal{H})$ is defined as

$$C_{\max}(\rho) = \min_{\sigma \in \mathcal{E}} D_{\max}(\rho||\sigma),$$

where max-relative entropy $D_{\max}$ [38, 39] is defined as

$$D_{\max}(\rho||\sigma) = \min \{ \lambda \in \mathbb{R}_+ : \rho \leq 2^\lambda \sigma \}.$$

The coherence measure $C_{\max}$ has been proved to play an crucial role in some quantum information processing tasks in Ref. [25]. For multipartite state $\rho \in \mathcal{D}(\mathcal{H}^\otimes N)$, $C_{\max}(\rho) = \min_{\sigma_N \in \mathcal{I}_{1:2,\ldots,N}} D_{\max}(\rho||\sigma_N)$, where the incoherent states $I_{1:2,\ldots,N}$ is the set of incoherent states $\mathcal{D}(\mathcal{H}^\otimes N)$ and the state $\sigma_N \in I_{1:2,\ldots,N}$ has the following form

$$\sigma_N = \sum_i p_i \sigma_{i,1} \otimes \ldots \otimes \sigma_{i,N},$$

with all $\sigma_{i,k}$ being diagonal in the local basis.

To quantify the coherence in multipartite system more precisely, let us introduce the following coherence measure on bipartite system, which is called max-relative entropy of incoherent-quantum (IQ) coherence. For a bipartite state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, $\rho_{AB} = \sum_{i,k} \rho_{i,k} \sigma_{i,k}^A \otimes \sigma_{i,k}^B$, where $\sigma_{i,k}^A$ is incoherent, $\tau_{i,k}^B \in \mathcal{D}(\mathcal{H}_B)$.

As a coherence measure on bipartite systems, $C_{\max}^{AB}$ satisfies the following properties: (i) positivity, $C_{\max}^{AB}(\rho) \geq 0$ and $C_{\max}^{AB}(\rho) = 0$ iff $\rho \in IQ$; (ii) monotonicity under incoherent operation $\Lambda^A_{IO}$ on A side, that is, $C_{\max}^{AB}(\Lambda^A_{IO} \otimes I_B(\rho_{AB})) \leq C_{\max}^{AB}(\rho_{AB})$; (iii) strong monotonicity under incoherent operation on A side, that is, for incoherent operation $\Lambda^A_{IO}(\cdot) = \sum_i K_i^A(\cdot)K_i^A$ with $K_i^A \otimes I_BK_i^A \subset I$, $\sum_ip_iC_{\max}^{AB}(\tilde{\rho}_i) \leq C_{\max}^{AB}(\rho)$, where $p_i = \text{Tr}[K_i^A \rho_{AB} K_i^A]$ and $\tilde{\rho}_i = K_i^A \rho_{AB} K_i^A/p_i$; (iv) monotonicity under quantum operation $\Lambda^B$ on B side, $C_{\max}^{AB}(\Lambda^B(\rho_{AB})) \leq C_{\max}^{AB}(\rho_{AB})$; (v) quasi-convexity, for $\rho_{AB} = \sum_i p_i \rho_{i}$, $C_{\max}^{AB}(\rho_{AB}) \leq \max_i C_{\max}^{AB}(\rho_i)$.

The proof of above properties can be given in the same way as that of $C_{\max}$ in Ref. [25]. For any bipartite state $\rho_{AB}$ with $\rho_A = \text{Tr}_B[\rho_{AB}]$, $C_{\max}^{AB}(\rho_{AB}) \geq C_{\max}(\rho_A)$, which comes directly from the property (iv). If the subsystem $B$ is trivial, i.e., $\dim \mathcal{H}_B = 1$, then $C_{\max}$ reduces to $C_{\max}$ on subsystem $A$.

Maximum advantage achievable in subchannel discrimination with the assistance of a quantum memory.– In the following, we investigate the information processing task: subchannel discrimination problem, which provides an operational interpretation of $C_{\max}^{AB}$. Subchannel discrimination is an important information task and it tells us which branch of the evolution a quantum system should go [43].

A subchannel $\mathcal{E}$ is defined to be a linear completely positive and trace non-increasing map, and if the subchannel $\mathcal{E}$ is also trace preserving, then $\mathcal{E}$ is called a channel. An instrument $\mathcal{J} = \{ E_k \}_{k}$ for a channel $\mathcal{E}$ is a collection of subchannels $E_k$ such that $\mathcal{E} = \sum_k E_k$. An incoherent instrument $\mathcal{J}$ for an IO $\mathcal{E}$ is a collection of subchannels $\{ E_k \}_{k}$ such that $\mathcal{E} = \sum_k E_k$ [25].

Given a bipartite state $\rho_{AB}$ and an instrument $\mathcal{J}_A = \{ E^A_k \}_{k}$ for a quantum channel $\mathcal{E}^A$ on part A, consider the joint positive operator valued measure (POVM) $\{ M_k^{AB} \}_{k}$ on $\mathcal{B}$ with $M_k^{AB} \geq 0$ and $\sum_k M_k^{AB} = I_{AB}$. The probability of successfully discriminating subchannels in $\mathcal{J}_A$ by joint POVM $\{ M_k^{AB} \}_{k}$ is given by

$$p_{\text{succ}}(\mathcal{J}_A, \{ M_k^{AB} \}_{k}, \rho_{AB}) = \sum_k \text{Tr}[E^A_k(\rho_{AB})M_k^{AB}].$$

And the optimal probability of successfully discriminating subchannels in $\mathcal{J}_A$ over all joint POVM is given by

$$p_{\text{succ}}(\mathcal{J}_A, \rho_{AB}) = \max_{\{ M_k^{AB} \}_{k}} \sum_k \text{Tr}[E^A_k(\rho_{AB})M_k^{AB}].$$

If the input states are restricted to be incoherent on A’s side, i.e., IQ states, then the optimal probability over all IQ states is

$$p_{\text{IQ succ}}(\mathcal{J}_A) = \max_{\sigma_{A,B} \in IQ} p_{\text{succ}}(\mathcal{J}_A, \sigma_{A,B}).$$

Theorem 1. Given a bipartite state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, it holds that

$$2^C_{\max}^{AB}(\rho_{AB}) = \max_{\sigma_A} p_{\text{succ}}(\mathcal{J}_A^A, \rho_{AB}) \frac{p_{\text{succ}}(\mathcal{J}_A^A, \rho_{AB})}{p_{\text{succ}}(\mathcal{J}_A^A)}. \tag{4}$$
where the maximization is taken over all the incoherent instrument $\mathcal{J}_A'$ on part $A$.

The proof of Theorem 1 is presented in Appendix A. This result illustrates that the maximal advantage of bipartite states in such subchannels discrimination problem can be exactly captured by $C_{\text{max}}^{A|B}$, which also provides an operational interpretation of $C_{\text{max}}^{A|B}$. As for any bipartite state $\rho_{AB}$ with reduced state $\rho_A$ on subsystem $A$, $C_{\text{max}}^{A|B}(\rho_{AB}) \geq C_{\text{max}}^{A|B}(\rho_A)$, this means that the success probability of discriminating sub-channels on part $A$ can be improved with the assistance of a quantum memory $B$. (See Fig. 1)

In Ref. [25], the min-relative entropy of coherence $C_{\text{min}}$ has also been defined,

$$C_{\text{min}}(\rho) = \min_{\sigma \in \mathcal{I}} D_{\text{min}}(\rho||\sigma),$$

where the min-relative entropy $D_{\text{min}}$ [38, 39] is defined as

$$D_{\text{min}}(\rho||\sigma) := -\log \text{Tr}[\Pi_\rho \sigma],$$

with $\Pi_\rho$ denoting the projector onto the support $\text{supp}[\rho]$ of $\rho$. Here, we define the min-relative entropy of IQ coherence on bipartite states,

$$C_{\text{min}}^{A|B}(\rho_{AB}) := \min_{\sigma_{A|B} \in \mathcal{I}_Q} D_{\text{min}}(\rho_{AB}||\sigma_{A|B}).$$

(5)

Moreover, the relative entropy of IQ coherence measure $C_{\text{r}}^{A|B}$ has also been defined in Ref. [9],

$$C_{\text{r}}^{A|B}(\rho_{AB}) := \min_{\sigma_{A|B} \in \mathcal{I}_Q} S(\rho_{AB}||\sigma_{A|B}),$$

where $C_{\text{r}}^{A|B}$ plays an important role in the assisted distillation of coherence [9, 42]. Since $D_{\text{min}}(\rho||\sigma) \leq S(\rho||\sigma) \leq D_{\text{max}}(\rho||\sigma)$ for any quantum states $\rho$ and $\sigma$ [38], we have the following relationship,

$$C_{\text{min}}^{A|B}(\rho_{AB}) \leq C_{\text{r}}^{A|B}(\rho_{AB}) \leq C_{\text{max}}^{A|B}(\rho_{AB}).$$

(6)

Let us introduce the $\epsilon$-smooth max- and min-relative entropy of IQ coherence as follows,

$$C_{\text{max}}^{A|B,\epsilon}(\rho_{AB}) := \min_{\rho'_{AB} \in \mathcal{B}_\epsilon(\rho_{AB})} C_{\text{max}}^{A|B}(\rho'_{AB}),$$

$$C_{\text{min}}^{A|B,\epsilon}(\rho_{AB}) := \min_{\rho'_{AB} \in \mathcal{B}_\epsilon(\rho_{AB})} \max_{0 \leq O_{AB} \leq I_{AB}} \min_{\sigma_{A|B} \in \mathcal{I}_Q} \log \text{Tr}[O_{AB}\sigma_{A|B}],$$

(7)

(8)

where $B_\epsilon(\rho) := \{ \rho' \geq 0 : \|\rho' - \rho\| \leq \epsilon, \text{Tr}[\rho'] \leq \text{Tr}[\rho] \}$ and $I_{AB}$ denotes the identity on $\mathcal{H}_A \otimes \mathcal{H}_B$. By the smooth max- and min-relative entropy of IQ coherence, the equivalence between $C_{\text{max}}^{A|B}$, $C_{\text{min}}^{A|B}$ and $C_{\text{r}}^{A|B}$ in the asymptotic limit can be also obtained,

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} C_{\text{max}}^{A^n|B^n,\epsilon}(\rho_{AB}^{\otimes n}) = C_{\text{r}}^{A|B}(\rho_{AB}).$$

(9)

The proof of this result is presented in Appendix B.

Now, we are ready to investigate the coherence distribution in multiparticle systems. For convenience, we denote coherence measure $C^{A|B}(\rho_{AB}) := C^{A|B}(\rho_{AB}), C^{A}(\rho_{AB}) := C^{A}(\rho_{AB}), C^{r}(A|B) := C^{r}(A|B), C^{r}(A) := C^{r}(A)$. Although coherence is defined to capture the quantumness in a single system, collective coherence between different subsystems needs to be considered in multiparticle systems. To quantify the collective coherence between different subsystems, the local coherence needs to be omitted. Thus the minimization over the incoherent states $\mathcal{I}_1:2:...:N$ needs to be relaxed to the separable states $S_{1:2:...:N}$ [35], where $\tau_N \in S_{1:2:...:N}$ has the form $\tau_N = \sum_{i \in D} \tau_{i,N}^{(1)} \otimes \tau_{i,N}^{(2)} \otimes ... \otimes \tau_{i,N}^{(N)}$ with $\tau_{i,N}^{(k)} \in \mathcal{D}(\mathcal{H}_k)$. For an $N$-partite state $\rho_{A_1...A_N}$, the max-relative entropy of collective coherence is defined by

$$E_{\text{max}}^{A_1:...:A_N}(\rho_{A_1...A_N}) := \min_{\tau_N \in S_{1:2:...:N}} D_{\text{max}}(\rho_{A_1...A_N}||\tau_N),$$

which is the genuine multipartite entanglement among $A_1, ... , A_N$ in terms of the max-relative entropy. The multi-partite entanglement measure $E^{A_1:...:A_N}(\rho_{A_1...A_N})$ is denoted by $E(A_1 : : : A_N)$ for short in the following context.

Since the max-relative entropy fulfills the triangle inequality, i.e., $D_{\text{max}}(\rho||\sigma) \leq D_{\text{max}}(\rho||\tau) + D_{\text{max}}(\tau||\sigma)$, we have the following relation for any $N$-partite state $\rho_{A_1...A_N}$ according to [35],

$$C_{\text{max}}(A_1 ... A_N) \leq E_{\text{max}}(A_1 : : : A_N) + C_{\text{max}}(\rho_{A_1...A_N}),$$

which is the optimal separable states in $S_{1:2:...:N}$ such that $E_{\text{max}}(A_1 : : : A_N) = D_{\text{max}}(\rho_{A_1...A_N}||\sigma_{\text{min},N})$, and the coherence in the state $\sigma_{\text{min},N}$ is called “local coherence” in Ref. [35]. Besides, $C_{\text{max}}$ is subadditive for product states, i.e., $C_{\text{max}}(\rho_1 \otimes \rho_2) \leq C_{\text{max}}(\rho_1) + C_{\text{max}}(\rho_2)$. That is, $C_{\text{max}}$ satisfies all the requirements except for the symmetry in Ref. [35]. However, the relation between $C(\rho_{A_k})$ and the coherence $\sum_k C(\rho_{A_k})$ is still unclear, where $\rho_{A_k}$ is the reduced state of the $k$-th subsystem. We adopt the $C(\rho_{A_k})$ (or $C(A_k)$) to be the local coherence on the $k$-th subsystem and concentrate on the relation among the total coherence of multiparticle state $C(A_1 ... A_N)$, the local coherence $\{ C(A_k) \}_k$ and the genuine multipartite entanglement $E(A_1 : : : A_N)$.
Distribution of coherence in bipartite systems.— Let us begin with bipartite systems, for which we have the following result for any quantum state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$,

$$C_r(AB) \geq C_r(A|B) + C_r(B).$$

(10)

The proof of this result is based on the following fact,

$$C^\epsilon_{\text{max}}(AB) \geq C^\epsilon_{\text{max}}(A|B) + C^\epsilon_{\text{min}}(B),$$

where $\epsilon > 0$, $\epsilon' = \epsilon + 2\sqrt{\epsilon}$, $C^\epsilon_{\text{max}}$ and $C^\epsilon_{\text{min}}$ are smooth max- and min-relative entropy of coherence defined in Refs. [39, 40]. The details of the proof is presented in Appendix C. As $C_r(A|B) \geq C_r(A)$, the relation (10) is tighter than the known result $C_r(AB) \geq C_r(A) + C_r(B)$ in bipartite systems. This is because $C_r(A|B)$ (or $C_{\text{max}}(A|B)$) contains not only the local coherence on part A, but also the non-local correlation between A and B. In fact, $C_r(A|B) = C_r(A) + S(\rho_A) + \sum_i p_i S(\rho_{B|i}) - S(\rho_{AB}) \geq C_r(A) + \delta_{A\rightarrow B}$, where $p_i = (|i\rangle\langle i|)_A, \rho_{B|i} = (|i\rangle\langle i|)_B$ and $\delta_{A\rightarrow B} = S(\rho_A) - S(\rho_{AB}) + \min_{\{\pi^A\}} \sum_i p_i S(\rho_{B|i})$ is the quantum discord between A and B [44] with $\{\pi^A\}$ being the von Neumann measurements on part A. Thus, it is easy to get the relation $C_r(AB) \geq C_r(A) + C_r(B) + \delta_{A\rightarrow B}$. Similar result can also be obtained under the exchange of labels A and B. Therefore, we obtain the following relation for the distribution of relative entropy of coherence in bipartite systems,

$$C_r(AB) \geq C_r(A) + C_r(B) + \max \{\delta_{A\rightarrow B}, \delta_{B\rightarrow A}\},$$

(11)

where $\delta_{A\rightarrow B}$ and $\delta_{B\rightarrow A}$ are the corresponding quantum discord of bipartite state $\rho_{AB}$.

Distribution of coherence in multipartite systems.— In multipartite systems, the total coherence of the whole system also contains the nonlocal correlations between the subsystems.

Theorem 2. Given an N-partite state $\rho_{A_1A_2...A_N} \in \mathcal{D}(\otimes_{i=1}^{N=1} \mathcal{H}_{A_i})$ with $N \geq 2$, the total coherence, local coherence and genuine multipartite entanglement of this state have the following relationship,

$$C_r(A_1A_2...A_N) \geq E^\infty_r(A_1 : A_2 : ... : A_N) + \sum_{i=1}^{N} C_r(A_i),$$

(12)

where $E^\infty_r$ is the regularized relative entropy of entanglement defined by $E^\infty_r(A_1 : A_2 : ... : A_N) = \lim_{n \rightarrow \infty} \frac{1}{n} E_r(\rho_{A_1A_2...A_n})$ with $E_r(\rho_{A_1A_2...A_n}) = \min_{\tau_N \in \mathcal{S}(A_1 : A_2 : ... : A_N)} \|\tau_N - |\tau_N|\|_1$ [45–47], the minimization goes over all separable states $\tau_N \in \mathcal{S}(A_1 : A_2 : ... : A_N)$.

To prove Theorem 2, we only need to prove the case $N = 3$, which depends on the following relation,

$$C^\epsilon_{\text{max}}(ABC) \geq E^\epsilon_{\text{max}}(A : B : C) + C^\epsilon_{\text{max}}(A) + C^\epsilon_{\text{min}}(B) + C^\epsilon_{\text{min}}(C),$$

where $\epsilon_1 = \epsilon + 2\sqrt{\epsilon}$, $\epsilon_2 = \epsilon_1 + 2\sqrt{\epsilon_1}$ and $E^\epsilon_{\text{max}}$ is the smooth max-relative-entanglement defined in Refs. [39, 40]. The details of the proof for Theorem 2 is presented in Appendix C. This theorem illustrates that the total coherence in multipartite system contains not only the local coherence in each subsystem, but also the genuine multipartite entanglement among the multipartite systems, where the multipartite entanglement quantifies the collective coherence among these subsystems.

Now, we consider the distribution of entanglement (or collective coherence) in multipartite system. Although the relative entropy of entanglement and its regularized version do not have the monogamy relation in general [31], the genuine multipartite entanglement for any tripartite state $\rho_{ABC} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$ can be decomposed into bipartite entanglement as follows,

$$E^\infty_r(A : B : C) \geq E^\infty_r(A : BC) + E^\infty_r(B : C),$$

(13)

$$E^\infty_r(A : B : C) \geq E^\infty_r(A : B) + E^\infty_r(B : C).$$

(14)

The proof of this result is presented in Appendix C. Note that the relation (13) is also true for any N-partite systems, i.e.,

$$E^\infty_r(A_1 : ... : A_{N-1} : A_N) \geq E^\infty_r(A_1 : ... : A_{N-1}A_N) + E^\infty_r(A_{N-1} : A_N).$$

Thus, that the genuine N-partite entanglement can be decomposed into the $(N - 1)$-partite entanglement and bipartite entanglement. Moreover, since the relation (14) holds under the exchange of labels A, B and C, we have the following relation for the distribution of entanglement in tripartite systems:

$$E^\infty_r(A : B : C) \geq \frac{2}{3} [E^\infty_r(A : B) + E^\infty_r(A : C) + E^\infty_r(B : C)].$$

Monogamy relation for IQ coherence measures.— It has been shown in Refs. [32, 33] that, for relative entropy of coherence $C_r$, there exists some tripartite $\rho_{ABC}$ such that

$$C_r(ABC) \leq C_r(AB) + C_r(AC).$$

(15)

That is, the monogamy relation for $C_r$ does not hold in general. There are several reasons behind the failure of monogamy relation for $C_r$. One is the following fact: the right hand side of (15) contains two copies of local coherence $C_r(A)$, while the left hand side only contains one copy of $C_r(A)$. If the parts $B$ and $C$ are weakly correlated ( e.g. $\rho_{ABC} = \rho_{A_1B} \otimes \rho_{A_2C}$ with $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} = \mathcal{H}_A$), then the only one more copy of $C_r(A)$ will result in the failure of monogamy. Thus, to circumvent this problem, we define the monogamy of coherence for an $N + 1$-part state $\rho_{A_1...A_NB}$ in terms of the IQ coherence measure $C_{r|AB}$ as follows,

$$M = \sum_{k=1}^{N} C_r(A_k | B) - C_r(A_1...A_N | B).$$

(16)

It is monogamous for $M \leq 0$ and polynomial monogamous for $M > 0$. Here, we obtain the following monogamy of coherence in $N + 1$-partite systems.

Theorem 3. For any $N+1$-partite state $\rho_{A_1...A_NB}$, it holds that

$$C_r(A_1A_2...A_N | B) \geq \sum_{i=1}^{N} C_r(A_i | B).$$

(17)
The proof of Theorem 3 is presented in Appendix D. As $C_r(AB|C) \geq C_r(AB) + E_r^\infty (AB : C) \geq C_r(A) + C_r(B) + E_r^\infty (A : B) + E_r^\infty (AB : C)$, i.e., $C_r(AB|C)$ contains the nonlocal correlation between $A$ and $B$, we consider the relation between the quantity $C_r(AB|C) - C_r(A|C) - C_r(B|C)$ and the nonlocal correlation between $A$ and $B$. For any tripartite state $\rho_{ABC}$, the following relationship holds,

$$C_r(AB|C) \leq C_r(A|C) + C_r(B|C) + I(A : B|C)_{\rho}, \quad (18)$$

where $I(A : B|C)_{\rho} := S(\rho_{AC}) + S(\rho_{BC}) - S(\rho_{C}) - S(\rho_{ABC})$ is the conditional mutual information of $\rho_{ABC}$ which quantifies the correlation between subsystems $A$ and $B$ with respect to $C$ and can be used to define the squashed entanglement [48–50] (See the proof in Appendix D).

One of the basic properties of the relative entropy of entanglement $E_r$ distinguishing from other entanglement measures is the non-lockability [51], that is, the loss of entanglement is proportional to the number of qubits traced out when some part of the whole system is discarded, where this relation can be improved for the regularized relative entropy of entanglement $E_r^\infty$ [50] as follows,

$$E_r^\infty (A : BC) - E_r^\infty (A : C) \leq I(A : B|C)_{\rho}. \quad (18)$$

For the relative entropy of IQ coherence measure, the conditional mutual information $I(A : B|C)_{\rho}$ also provides an upper bound for the loss of coherence after some subsystem is discarded. That is, for any tripartite state $\rho_{ABC}$,

$$C_r(A|BC) - C_r(A|C) \leq I(A : B|C)_{\rho}, \quad (19)$$

which can be regarded as the non-lockability of relative entropy of IQ coherence measure. The proof of (19) is presented in Appendix D.

### III. CONCLUSION

Understanding the distribution of quantum coherence in multipartite systems is of fundamental importance. We have investigated the distribution of coherence in multipartite systems by introducing incoherent-quantum (IQ) coherence measures on bipartite systems in terms of the max- and min-relative entropies. It has been found that the max-relative entropy of IQ coherence characterizes maximal advantage of the bipartite in certain subchannel discrimination problems. Moreover, it has been shown that the total coherence of a multipartite system is lower bounded by the sum of local coherence and the genuine multipartite entanglement. From the IQ coherence measures, we have obtained the monogamy relation of coherence in multipartite systems.

Our results reveal the distribution of quantum coherence in multipartite systems, which substantially advance the understanding of the physical law that governs the distribution of quantum correlations in multipartite systems and pave the way for the further researches in this direction. This will also have deep implications in quantum information processing, quantum biology, quantum thermodynamics and other related areas of physics as well.

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Given a bipartite quantum state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$. $C_{\sigma_{AB}}^{\max}(\rho_{AB})$ can be expressed as

$$2C_{\sigma_{AB}}^{\max}(\rho_{AB}) = \max_{\Delta_A \otimes \mathbb{I}_B(\tau_{AB})'=\mathbb{I}_{AB}} \text{Tr}[\rho_{AB} \tau_{AB}], \quad (A1)$$

Proof: Due to the definition of $C_{\sigma_{AB}}^{\max}$, we have

$$2C_{\sigma_{AB}}^{\max}(\rho_{AB}) = \min_{\Delta \otimes \mathbb{I}_B(\sigma_{AB}) \geq \rho_{AB}} \text{Tr}[\sigma_{AB}].$$

Thus, to prove the result, we only need to prove

$$\min_{\Delta_A \otimes \mathbb{I}_B(\sigma_{AB}) \geq \rho_{AB}} \text{Tr}[\sigma_{AB}] = \max_{\tau_{AB} \geq 0} \text{Tr}[\rho_{AB} \tau_{AB}],$$

which is due to the definition of $C_{\sigma_{AB}}^{\max}$. The result is established.

First, it is easy to see that

$$\max_{\Delta_A \otimes \mathbb{I}_B(\tau_{AB})'=\mathbb{I}_{AB}} \text{Tr}[\rho_{AB} \tau_{AB}] = \max_{\tau_{AB} \geq 0} \text{Tr}[\rho_{AB} \tau_{AB}],$$

as for any positive operator $\tau_{AB} \geq 0$ with $\Delta_A(\tau_{AB}) \leq \mathbb{I}_{AB}$, we can always choose $\tau_{AB}' = \tau_{AB} + \mathbb{I}_{AB} - \Delta_A \otimes \mathbb{I}_B(\tau_{AB}) \geq 0$, then $\Delta_A \otimes \mathbb{I}_B(\tau_{AB}) = \mathbb{I}_{AB}$ and $\text{Tr}[\rho_{AB} \tau_{AB}'] \geq \text{Tr}[\rho_{AB} \tau_{AB}]$.

Next, we prove that

$$\min_{\sigma_{AB} \geq 0} \text{Tr}[\sigma_{AB}] = \max_{\tau_{AB} \geq 0} \text{Tr}[\rho_{AB} \tau_{AB}]. \quad (A2)$$

The left side of equation (A2) can be expressed as the following semidefinite programming (SDP)

$$\min \text{Tr}[C_1 \sigma_{AB}],$$

s.t. $\Lambda(\sigma_{AB}) \geq C_2,$

$$\sigma_{AB} \geq 0,$$

where $C_1 = \mathbb{I}$, $C_2 = \rho_{AB}$ and $\Lambda = \Delta_A \otimes \mathbb{I}_B$. Then the dual SDP is given by

$$\max \text{Tr}[C_2 T_{AB}],$$

s.t. $\Lambda^\dagger(\tau_{AB}) \leq C_1,$

$$\tau_{AB} \geq 0.$$ 

That is,

$$\max \text{Tr}[\rho_{AB} T_{AB}],$$

s.t. $\Delta_A \otimes \mathbb{I}_B(\tau_{AB}) \leq \mathbb{I}_{AB},$

$$\tau_{AB} \geq 0.$$ 

Note that the dual is strictly feasible as we only need to choose $\sigma_{AB} = 2\lambda_{\max}(\rho_{AB}) \mathbb{I}_{AB}$, where $\lambda_{\max}(\rho_{AB})$ is the maximum eigenvalue of $\rho_{AB}$. Thus, strong duality holds, and the equation (A2) is proved.

\[\Box\]

Proof of Theorem 1. Due to the definition of $C_{\sigma_{AB}}^{\max}$, there exists an IQ state $\sigma_{AB}$ such that $\rho \leq 2C_{\sigma_{AB}}^{\max}(\rho_{AB}) \sigma_{AB}$. Thus, for any incoherent instrument $I^\dagger_A$ and joint POVM $\{M_{kAB}\}_k$, $\rho_{\text{succ}}(I^\dagger_A, \{M_{kAB}\}_k, \rho_{AB})$.

$$\rho_{\text{succ}}(I^\dagger_A, \{M_{kAB}\}_k, \rho_{AB}) \leq 2C_{\sigma_{AB}}^{\max}(\rho_{AB}) \rho_{\text{succ}}(I^\dagger_A, \{M_{kAB}\}_k, \sigma_{AB}),$$

that is,

$$\rho_{\text{succ}}(I^\dagger_A, \rho_{AB}) \leq 2C_{\sigma_{AB}}^{\max}(\rho_{AB}) P_{\text{succ}}(I^\dagger_A). \quad (A3)$$

Now, consider a special incoherent instrument such that the equality (A3) holds. Let us take the incoherent instrument

**Appendix A: Operational interpretation of $C_{\sigma_{AB}}^{\max}$**

**Lemma 4.** Given a bipartite quantum state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$. $C_{\sigma_{AB}}^{\max}(\rho_{AB})$ can be expressed as

$$2C_{\sigma_{AB}}^{\max}(\rho_{AB}) = \max_{\Delta_A \otimes \mathbb{I}_B(\tau_{AB})'=\mathbb{I}_{AB}} \text{Tr}[\rho_{AB} \tau_{AB}], \quad (A1)$$

Proof: Due to the definition of $C_{\sigma_{AB}}^{\max}$, we have $2C_{\sigma_{AB}}^{\max}(\rho_{AB}) = \min_{\Delta \otimes \mathbb{I}_B(\sigma_{AB}) \geq \rho_{AB}} \text{Tr}[\sigma_{AB}].$ Thus, to prove the result, we only need to prove

$$\min_{\Delta \otimes \mathbb{I}_B(\sigma_{AB}) \geq \rho_{AB}} \text{Tr}[\sigma_{AB}] = \max_{\tau_{AB} \geq 0} \text{Tr}[\rho_{AB} \tau_{AB}],$$

which is due to the definition of $C_{\sigma_{AB}}^{\max}$. The result is established.
\( \bar{\gamma}_A = \{ \bar{e}_A^j \}_{j=1}^d \) as \( \bar{e}_A^j (\cdot) = \frac{1}{d_A} U_A^j (\cdot) U_A^{j\dagger} \) with \( U_A^j = e^{i \frac{2\pi j}{d_A} \hat{H}_A} \) and \( H_A = \sum_j j |j \rangle \langle j |_A \). Then, for any IQ state \( \sigma_{A|B} \),

\[
p_{\text{succ}}(\bar{\gamma}_A, \{ M_k^{AB} \}, \sigma_{A|B}) = \sum_k \frac{1}{d_A} \text{Tr} \left[ U_A^j \sigma_{A|B} U_A^{j\dagger} M_k^{AB} \right]
\]

\[
= \frac{1}{d_A} \text{Tr} \left[ \sigma_{A|B} \sum_k U_A^{j\dagger} M_k^{AB} U_A^j \right]
\]

\[
= \frac{1}{d_A} \text{Tr} \left[ \Delta_A \otimes I_B (\sigma_{A|B}) \sum_k U_A^{j\dagger} M_k^{AB} U_A^j \right]
\]

\[
= \frac{1}{d_A} \text{Tr} \left[ \sigma_{AB} \Delta_A \otimes I_B (\sum_k U_A^{j\dagger} M_k^{AB} U_A^j) \right]
\]

\[
= \frac{1}{d_A} \text{Tr} \left[ \sigma_{AB} \Delta_A \otimes I_B \right] = \frac{1}{d_A},
\]

where the forth line comes from the fact that \( \Delta_A \otimes I_B (\sigma_{A|B}) = \sigma_{A|B} \) for any IQ state \( \sigma_{A|B} \), the sixth line comes from the fact that \( U_A^j \) is diagonal in the local basis \( \{ |j \rangle \}_{j=1}^d \) and 

\[
\Delta_A \otimes I_B (\sum_k U_A^{j\dagger} M_k^{AB} U_A^j) = \sum_k U_A^{j\dagger} \Delta_A \otimes I_B (M_k^{AB}) U_A^j
\]

\[
= \sum_k \Delta_A \otimes I_B (M_k^{AB})
\]

\[
= \Delta_A \otimes I_B (I_{AB}) = I_{AB}.
\]

Besides, according to Lemma 4, there exists an positive operator \( \tau_{AB} \) with \( \Delta_A \otimes I_B (\tau_{AB}) = I_{AB} \), such that \( 2^{C_{0}^{\text{max}}(\rho_{AB})} = \text{Tr} [\rho_{AB} \tau_{AB}] \). Define \( N_k^{AB} = \frac{1}{d_A} U_A^{j\dagger} \tau_{AB} U_A^j \), then \( N_k^{AB} \geq 0 \) and \( \sum_k N_k^{AB} = \frac{1}{d_A} \sum_k U_A^{j\dagger} \tau_{AB} U_A^j = \Delta_A \otimes I_B (\tau_{AB}) = I_{AB} \), that is, \( \{ N_k^{AB} \} \) is a joint POVM on A and B. Hence

\[
p_{\text{succ}}(\bar{\gamma}_A, \{ N_k^{AB} \}, \rho_{AB}) = \sum_k \frac{1}{d_A} \text{Tr} \left[ \rho_{AB} U_A^{j\dagger} N_k^{AB} \right]
\]

\[
= \frac{1}{d_A} \text{Tr} \left[ \rho_{AB} \tau_{AB} \right]
\]

\[
= \frac{1}{d_A} \text{Tr} \left[ \rho_{AB} \tau_{AB} \right]
\]

That is,

\[
\frac{p_{\text{succ}}(\bar{\gamma}_A, \rho_{AB})}{p_{\text{succ}}(\bar{\gamma}_A)} \geq 2^{C_{0}^{\text{max}}(\rho_{AB})}.
\]

\[\square\]

Appendix B: Properties of smooth IQ max- and min-relative entropies

Due to the definition of smooth max-relative entropy of IQ coherence measure, it can also be rewritten as

\[
C_{\text{max}}^{\alpha}(\rho_{AB}) = \min_{\rho_{AB} \in B_{\epsilon}(\rho_{AB})} \min_{\sigma_{A|B} \in IQ} D_{\text{max}}(\rho_{AB} || \sigma_{A|B}),
\]

where \( D_{\text{max}}(\rho || |\sigma) \) is the smooth max-relative entropy \([38–40]\) defined as

\[
D_{\text{max}}^{\alpha}(\rho || |\sigma) = \inf_{\rho' \in B_{\epsilon}(\rho)} D_{\text{min}}(\rho' || |\sigma),
\]

and \( B_{\epsilon}(\rho) := \{ \rho' \geq 0 : ||\rho' - \rho||_1 \leq \epsilon, \text{Tr} [\rho'] \leq \text{Tr} [\rho] \} \). The equivalence between \( C_{\text{max}}^{\alpha}(\rho_{AB}) \) and \( C_{\text{max}}^{\alpha}(\rho_{AB}) \) in the asymptotic limit is given in the following proposition.

**Proposition 5.** Given a bipartite state \( \rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B) \), we have

\[
C_{\text{max}}^{\alpha}(\rho_{AB}) = \lim_{\epsilon \to 0} \frac{1}{n} C_{\text{max}}^{\alpha}(\rho_{AB}^{\otimes n}),
\]

**Proof.** The set of incoherent-quantum states \( I_{A^n} Q_{B^n} \) in \( D((\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n}) \) has the following form \( \sigma_{A^n|B^n} = \sum_i \rho_i^{A^n} \sigma_i^{A^n} \otimes \sigma_i^{B^n} \) with \( \sigma_i^{A^n} \) being incoherent and \( \tau_i^{B^n} \) full rank; (3) If \( \rho \in I_{A^n+1} Q_{B^n+1} \) then \( \text{Tr}_k [\rho] \in I_{A^n} Q_{B^n} \) for any \( k \in \{1, ..., n+1 \} \) where \( \text{Tr}_k \) means the partial trace on the kth \( \mathcal{H}_A \otimes \mathcal{H}_B \) in \( (\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n} \); (4) If \( \rho \in I_{A^n} Q_{B^n} \), and \( \tau \in I_{A^n} Q_{B^n} \), then \( \rho \otimes \tau \in I_{A^{n+1}+1} Q_{B^n+n} \); (5) Each \( I_{A^n} Q_{B^n} \) is permutation invariant, that is, for every state \( \rho \in I_{A^n} Q_{B^n} \) and \( \pi \in S_n \) is the symmetry group of order \( n \) and \( P_{\pi} \) is the representation of the element \( \pi \in S_n \) in the space \( (\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n} \) which is given by \( P_{\pi} (\psi_1 \otimes \cdots \otimes \psi_n) = \psi_{\pi^{-1}(1)} \otimes \cdots \otimes \psi_{\pi^{-1}(n)} \). Thus according to the generalized Quantum Stein Lemma \([52]\), we have

\[
\lim_{n \to \infty} \frac{1}{n} C_{\text{max}}^{\alpha}(\rho_{AB}^{\otimes n}) = \min_{\sigma_{A^n|B^n} \in I_{A^n} Q_{B^n}} \frac{S(\rho_{AB} || \sigma_{A^n|B^n})}{n}.
\]
Given a bipartite state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, we have
\[
\lim_{n \to \infty} \text{Tr} \left[ (I - M_{A^n B^n}) \rho_{AB}^{\otimes n} \right] = 0,
\]
and for every integer $n$ and every incoherent-quantum state $\sigma_{A^n B^n} \in \mathcal{I}_{A^n} \mathcal{Q}_{B^n}$,
\[
- \log \text{Tr} \left[ M_{A^n B^n} \sigma_{A^n B^n} \right] + \frac{1}{n} C^r_{A|B}(\rho_{AB}) \geq C^\epsilon_{A|B}(\rho_{AB}).
\]

Proof. Due to the fact that $C_{A|B}(\rho_{AB}) = \lim_{\epsilon \to 0} \min_{n \to \infty} \frac{1}{n} C_{A|B}(\rho_{AB})$ and $D_{A|B}' \leq D_{A|B}', \quad \lim_{\epsilon \to 0} \min_{n \to \infty} \frac{1}{n} C_{A|B}(\rho_{AB}) = C^r_{A|B}(\rho_{AB})$.

Next, we prove the converse direction,
\[
\lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} C_{\min} A^n B^n, \quad \epsilon(\rho_{AB}) \geq C^r_{A|B}(\rho_{AB}).
\]

According to Lemma 6, for any $\epsilon > 0$, there exists a sequence of POVMs $\{ M_{A^n B^n} \}$ such that for sufficient large integer $n$, $\text{Tr} \left[ \rho_{AB}^{\otimes n} M_{A^n B^n} \right] \geq 1 - \epsilon$. Hence
\[
C_{\min} A^n B^n, \quad \epsilon(\rho_{AB}) \geq \min_{\sigma_{A^n B^n} \in \mathcal{I}_{A^n} \mathcal{Q}_{B^n}} - \log \text{Tr} \left[ M_{A^n B^n} \sigma_{A^n B^n} \right] \geq n(C^r_{A|B}(\rho_{AB}) - \epsilon),
\]
where the last inequality comes from the direct part of Lemma 6. Therefore,
\[
\lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} C_{\min} A^n B^n, \quad \epsilon(\rho_{AB}) \geq C_{r}^r_{A|B}(\rho_{AB}).
\]

Appendix C: Details about the proof in the distribution of coherence

To prove the results in the distribution of coherence, the following lemmas is necessary.

Lemma 8. (Gentle operator lemma [53, 54]) Suppose $\rho$ is a subnormalized state with $\text{Tr}[\rho] \leq 1$, $\rho \geq 0$ and $M$ is an operator with $0 \leq M \leq I$, $\text{Tr}[\rho M] \geq \text{Tr}[|\rho|] - \epsilon$, then
\[
\left| \rho - \sqrt{M} \rho \sqrt{M} \right|_1 \leq 2\sqrt{\epsilon},
\]
where $\|A\|_1 = \text{Tr}[\sqrt{A^\dagger A}]$.

Lemma 9. Given a bipartite state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ and a parameter $\epsilon \geq 0$,
\[
C^\epsilon_{\max}(AB) \geq C^\epsilon_{\max}(A|B) + C^\epsilon_{\max}(B),
\]
where $\epsilon' = \epsilon + 2\sqrt{\epsilon}$, $C^\epsilon_{\max}$ and $C^\epsilon_{\min}$ are smooth max- and min-relative entropy of coherence defined in [25].
\[
C^\epsilon_{\max}(\rho) = \min_{\rho' \in B_{\epsilon}(\rho)} C^\epsilon_{\max}(\rho'),
\]"
min_{\sigma_B \in I_B} - \log Tr [T_B \sigma_B]. \) Thus, for any incoherent state \( \sigma_B \in I_B, 2^{-C_{\min}^e(\rho_B)} \geq Tr [T_B \sigma_B]. \)

Applying \( \sqrt{Tr(B)} \cdot \sqrt{Tr(B)} \) on the both sides of (C3), we get

\[
\lambda \sum_i p_i \sigma_{A,i} \otimes \sqrt{T_B \sigma_{B,i} \sqrt{T_B} \geq \sqrt{T_B} \rho_{AB} \sqrt{T_B}. \quad (C4)
\]

Then \( \bar{\sigma}_{AB} = \sum_i p_i \sigma_{A,i} \otimes \sqrt{T_B \sigma_{B,i} \sqrt{T_B} =: \mu_{\sigma_{AB}} \) with \( \sigma_{AB} \in I_A \cap Q_B \) and \( \mu = Tr [\bar{\sigma}_{AB}] = \sum_i p_i Tr [T_B \sigma_{B,i}] \leq 2^{-C_{\min}(B)}. \) Besides, \( \rho'_{AB} = \sqrt{T_B \rho_{AB} \sqrt{T_B} \in B_c(\rho_{AB}) \) with \( \epsilon' = \epsilon + 2\sqrt{c} \) as

\[
\left\| \sqrt{T_B \rho_{AB} \sqrt{T_B} - \rho_{AB} \right\|_1 \\
\leq \left\| \sqrt{T_B \rho_{AB} \sqrt{T_B} - \sqrt{T_B} \rho_{AB} \sqrt{T_B} \right\|_1 + \left\| \sqrt{T_B} \rho_{AB} \sqrt{T_B} - \rho_{AB} \right\|_1 \\
\leq \epsilon + 2\sqrt{c},
\]

where the last inequality comes from the fact that \( \bar{\rho}_{AB} \in B_c(\rho_{AB}) \) and the gentle operator lemma 8. Hence, \( \lambda \rho_{AB} \geq \rho'_{AB} \) with \( \sigma_{AB} \in I_A \cap Q_B \). Thus,

\[
C_{\max}^e(A|B) \leq C_{\max}^e(\rho'_{AB}) \\
\leq \log \lambda + \log \mu \\
\leq C_e(AB) - C_{\min}(B).
\]

Proof of Eq. (10). This result comes directly from the Lemma 9 and the following facts,

\[
\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{\epsilon} C_{\min} \leq C_{\max}^e(\rho_{\otimes n}) = C_r(\rho), \quad (C5)
\]

\[
\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{\epsilon} C^e(A|B) \leq C^e(A|B) = C_{\max}^e(\rho_{AB}), \quad (C6)
\]

where (C5) has been proved in Ref. [25]. \( \Box \)

Lemma 10. Given a bipartite \( \rho_{AB} \in D(H_A \otimes H_B) \) and a parameter \( \epsilon \geq 0 \), we have

\[
C_{\max}^e(A|B) \geq E_{\max}'(A : B) + C_{\min}(A), \quad (C7)
\]

where \( \epsilon' = \epsilon + 2\sqrt{c} \) and \( E_{\max}' \) is the smooth max-relative entropy of entanglement defined in [39, 40] as follows,

\[
E_{\max}'(A : B) := \min_{\rho'_{AB} \in B_c(\rho_{AB})} E_{\max}(\rho'_{AB}),
\]

with \( B_c(\rho) = \{ \rho' \geq 0 : \|\rho' - \rho\|_1 \leq \epsilon, Tr [\rho'] \leq Tr [\rho] \}. \)

Moreover, for any tripartite state \( \rho_{ABC} \in D(H_A \otimes H_B \otimes H_C) \), we have

\[
C_{\max}^e(AB|C) \geq E_{\max}'(A : B : C) + C_{\min}(A) + C_{\min}(B),
\]

where \( \epsilon'' = \epsilon + 2\sqrt{c} \).

Proof. The proof of the tripartite is similar to that of bipartite case. Hence, we only prove the bipartite case. Due to the definition of \( C_{\max}^e(A|B) \), there exists an optimal state \( \rho_{AB} \in B_c(\rho_{AB}) \) such that \( C^e_{\max}(A|B) = C_{\max}^e(\rho_{AB}). \) Let us take \( \lambda = 2C_{\max}^e(A|B) \), then there exists an incoherent-quantum state \( \sigma_{AB} = \sum_i p_i \sigma_{A,i} \otimes \tau_{B,i} \in I_A \cap Q_B \) with \( \sigma_{A,i} \) being incoherent and \( \tau_{B,i} \in \mathcal{D}(H) \) such that

\[
\lambda \sum_i p_i \sigma_{A,i} \otimes \tau_{B,i} \geq \rho_{AB}. \quad (C8)
\]

According to the definition of \( C_{\min} \), there exists a positive operator \( T_A \) with \( 0 \leq T_A \leq I_A \) and \( Tr [T_A \rho_A] \geq 1 - \epsilon \) such that \( C_{\min}(A) = \min_{\sigma_A \in I_A} - \log Tr [T_A \sigma_A]. \) Thus, for any incoherent state \( \sigma_A \in I_A \), \( C_{\min}^e(A) \leq - \log Tr [T_A \sigma_A] \), i.e.,

\[
2^{-C_{\min}(A)} \geq \frac{1}{\epsilon} C_{\min}^e(A),
\]

Applying \( \sqrt{T_A} (\cdot) \sqrt{T_A} \) on the both sides of (C8), one gets

\[
\lambda \sum_i p_i \sqrt{T_A} \sigma_{A,i} \sqrt{T_A} \otimes \tau_{B,i} \geq \sqrt{T_B} \rho_{AB} \sqrt{T_B}. \quad (C9)
\]

Then \( \bar{\sigma}_{AB} = \sum_i p_i \sqrt{T_A} \sigma_{A,i} \sqrt{T_A} \otimes \tau_{B,i} =: \mu_{\sigma_{AB}} \) with \( \sigma_{AB} \) being separable, i.e., \( \sigma_{AB} \in S_{AB} \) and \( \mu = Tr [\bar{\sigma}_{AB}] = \sum_i p_i Tr [T_A \sigma_{A,i}] \leq 2^{-C_{\min}(A)}. \) Besides, \( \rho'_{AB} = \sqrt{T_B} \rho_{AB} \sqrt{T_B} \in B_c(\rho_{AB}) \) with \( \epsilon' = \epsilon + 2\sqrt{c} \). That is, \( \lambda \rho_{AB} \geq \rho'_{AB} \) with \( \sigma_{AB} \in S_{AB} \). Thus,

\[
E_{\max}'(A : B) \leq E_{\max}(\rho'_{AB}) \\
\leq \log \lambda + \log \mu \\
\leq C_{\max}^e(A|B) - C_{\min}(A).
\]

Proof of Theorem 2. Here, we only need to prove the case \( N=3 \). Due to Lemmas 9 and 10, we have

\[
C_{\max}^e(ABC) \geq C_{\max}^e(A : B : C) + C_{\min}(A) + C_{\min}(B) + C_{\min}(C),
\]

where \( \epsilon_1 = \epsilon + 2\sqrt{c} \) and \( \epsilon_2 = \epsilon_1 + 2\sqrt{2\epsilon_1} \). It has been proved that \( E_{\max}' \) is equivalent to the regularized relative entropy of entanglement \( E_{\max}' \) in asymptotic limit [39, 40], that is,

\[
\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{\epsilon} E_{\max}'(\rho_{AB}) = E_{\max}'(A : B).
\]

Thus, combined with (C5), we obtain the theorem. \( \Box \)

Lemma 11. Given a tripartite state \( \rho_{ABC} \in D(H_A \otimes H_B \otimes H_C) \) and a parameter \( \epsilon \geq 0 \), we have

\[
E_{\max}'(A : B : C) \geq E_{\max}'(A : B) + E_{\min}(B : C),
\]

\[
E_{\max}(A : B : C) \geq E_{\max}(A : B) + E_{\min}(B : C),
\]

where \( \epsilon' = \epsilon + 2\sqrt{c} \) and \( E_{\min}' \) is the smooth min-relative entropy of entanglement defined in [39, 40] as follows,

\[
E_{\min}'(B : C) := \max_{\sigma_{\in S_{B,C}}} \min_{\sigma \in S_{B,C}} \log Tr [\sigma \sigma_{0}],
\]

with \( S_{B,C} \) being the set of separable states.
Proof. The proof of these two inequalities are similar, we only prove the first one. Due to the definition of $E_{\max}^e(A : B : C)$, there exists an optimal state $\tilde{\rho}_{ABC} \in B_e(\rho_{ABC})$ such that $E_{\max}^e(A : B : C) = E_{\max}^{ABC}(\tilde{\rho}_{ABC})$. Taking $\lambda = 2E_{\max}(AB,C)$, we have that there exists a separable state $\tau_{A:B:C} = \sum_i p_i \tau_{A,i} \otimes \tau_{B,i} \otimes \tau_{C,i} \in S_{A:B:C}$ such that

$$\lambda \sum_i p_i \tau_{A,i} \otimes \tau_{B,i} \otimes \tau_{C,i} \geq \tilde{\rho}_{ABC}. \quad (C10)$$

According to the definition of $E_{\max}^e(B : C)$, there exists an operator $T_{BC}$ with $0 \leq T_{BC} \leq I_{BC}$, $\text{Tr}[T_{BC} \rho_{BC}] \geq 1 - \epsilon$ such that $E_{\max}^e(B : C) = \min_{\tau_{B,C} \in S_{B,C}} - \log \text{Tr}[T_{BC} \tau_{B,C}]$. Thus, for any separable $\tau_{B,C} \in S_{B,C}, E_{\max}^e(B : C) \leq - \log \text{Tr}[T_{BC} \tau_{B,C}]$, i.e., $E_T^{\min}(B : C) \leq \text{Tr}[T_{BC} \tau_{B,C}]$.

Now applying $\sqrt{T_{BC}}(\cdot)\sqrt{T_{BC}}$ on the both sides of (D3), one obtains

$$\lambda \sum_i p_i \sqrt{T_{BC}} \tau_{A,i} \otimes \tau_{B,i} \otimes \tau_{C,i} \sqrt{T_{BC}} \geq \sqrt{T_{BC}} \tilde{\rho}_{ABC} \sqrt{T_{BC}}.$$

Then $\tilde{\tau}_{ABC} = \sum_i p_i \sqrt{T_{BC}} \tau_{A,i} \otimes \tau_{B,i} \otimes \tau_{C,i} \sqrt{T_{BC}} =: \mu \tau_{A:BC}$, where $\tau_{A:BC}$ is separable with respect to the partition $\tau_{A : BC}$, i.e., $\tau_{A:BC} \in S_{A:BC}$ and $\mu = \text{Tr}[\bar{T}_{ABC}] = \sum_i p_i \text{Tr}[T_{BC} \tau_{B,i} \otimes \tau_{C,i}] \leq 2E_{\max}^e(B : C)$. Besides, $\rho'_{ABC} = \sqrt{T_{BC}} \tilde{\rho}_{ABC} \sqrt{T_{BC}} \in B_e(\rho_{ABC})$ with $\epsilon' = \epsilon + 2\sqrt{\epsilon}$. That is, $\lambda \mu \tau_{A:BC} \geq \rho'_{ABC}$ with $\tau_{A:BC} \in S_{A:BC}$. Thus,

$$E_{\max}^{e'}(A : BC) \leq E_{\max}^{ABC}(\rho'_{ABC})$$

$$\leq \log \lambda + \log \mu$$

$$\leq E_{\max}^e(A : B : C) - E_{\max}^e(B : C).$$

Proof of Eqs. (13) and (14). This result comes directly from Lemma 11 and the equivalence between $E_{\max}^e$ and $E_{\max}^e$ in the asymptotic limit.

Appendix D: Monogamy of coherence

Lemma 12. Given a tripartite state $\rho_{ABC} \in D(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$, one has

$$C_{\max}^{e}(AB|C) \geq C_{\max}^{e'}(AB|C) + C_{\min}^{e'}(B|C), \quad (D1)$$

$$C_{\max}^{e}(AB|C) \geq C_{\max}^{e'}(A|C) + C_{\min}^{e'}(B|C), \quad (D2)$$

where $\epsilon' = \epsilon + 2\sqrt{\epsilon}$.

Proof. The proof of these two inequalities is similar, we only prove the second one. Due to the definition of $C_{\max}^{e}(AB|C)$, there exists an optimal state $\tilde{\rho}_{ABC} \in B_e(\rho_{ABC})$ such that $C_{\max}^{e}(AB|C) = C_{\max}^{e'}(\rho_{ABC})$. Let us take $\lambda = 2C_{\max}^{e}(AB|C)$, then there exists an incoherent-quantum state $\sigma_{ABC} = \sum_i p_i \sigma_{A,i} \otimes \sigma_{B,i} \otimes \sigma_{C,i} \in I_{ABC}$ with $\sigma_{A,i}, \sigma_{B,i}$ being incoherent such that

$$\lambda \sum_i p_i \sigma_{A,i} \otimes \sigma_{B,i} \otimes \sigma_{C,i} \geq \tilde{\rho}_{ABC}. \quad (D3)$$

According to the definition of $C_{\min}^{e}(B|C)$, there exists a positive operator $T_{BC}$ with $0 \leq T_{BC} \leq I_{BC}$, $\text{Tr}[T_{BC} \rho_{BC}] \geq 1 - \epsilon$ such that $C_{\min}^{e}(B|C) = \min_{\rho_{BC} \in I_{BC}} -\log \text{Tr}[T_{BC} \rho_{BC}]$. Thus, for any incoherent-quantum state $\sigma_{BC} \in I_{BC}$, $C_{\min}^{e}(B|C) \leq -\log \text{Tr}[T_{BC} \sigma_{BC}]$, i.e., $C_{\min}^{e}(B|C) \leq -\log \text{Tr}[T_{BC} \sigma_{BC}]$.

Applying $\sqrt{T_{BC}}(\cdot)\sqrt{T_{BC}}$ on the both sides of (D3) and take a partial trace on part B, we get

$$\lambda \sum_i p_i \text{Tr}_B \left[\sqrt{T_{BC}} \sigma_{A,i} \otimes \sigma_{B,i} \otimes \sigma_{C,i} \sqrt{T_{BC}}\right]$$

$$\geq \text{Tr}_B \left[\sqrt{T_{BC}} \tilde{\rho}_{ABC} \sqrt{T_{BC}}\right].$$

Then $\sigma_{AC} = \sum_i p_i \text{Tr}_B \left[\sqrt{T_{BC}} \sigma_{A,i} \otimes \sigma_{B,i} \otimes \sigma_{C,i} \sqrt{T_{BC}}\right] =: \mu \sigma_{AC}$, where $\sigma_{AC}$ is an incoherent-quantum state, i.e., $\sigma_{AC} \in I_{AC}$ and $\mu = \text{Tr}[\tilde{\rho}_{AC}] = \sum_i p_i \text{Tr}[T_{BC} \sigma_{B,i} \otimes \sigma_{C,i}] \leq 2C_{\min}^{e}(B|C)$. Besides, $\rho'_{AC} = \text{Tr}_B \left[\sqrt{T_{BC}} \tilde{\rho}_{ABC} \sqrt{T_{BC}}\right] = B_e(\rho_{AC})$ with $\epsilon' = \epsilon + 2\sqrt{\epsilon}$, as $\|\rho_{AC} - \rho_{AC}\| \leq \|\sqrt{T_{BC}} \tilde{\rho}_{ABC} \sqrt{T_{BC}} - \rho_{AC}\| \leq \epsilon'$. That is, $\lambda \mu \sigma_{AC} \geq \rho'_{AC}$ with $\sigma_{AC} \in I_{AC}$...

Proof of Theorem 3. These results come directly from Lemma 12 and the equivalence between $C_{\max}^{A|B,E}$ and $C_{\max}^{A|B}$ in the asymptotic limit.

In the data processing, there is a chain rule for von Neumann entropy, $S(AB|C) = S(A|BC) + S(B|C)$, where the condition entropy $S(A|B) =: S(\rho_{AB}) - S(\rho_{B})$ with $\rho_{B}$ being the reduced state on subsystem B. There is also a chain rule for $C_r$ as follows: for any tripartite $\rho_{ABC} \in D(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$, it holds that

$$C_r(AB|C) \geq C_r(A|BC) + C_r(B|C), \quad (D4)$$

which results from Lemma 12 by the equivalence between $C_{\max}^{A|B,E}$ and $C_{\max}^{A|B}$ in the asymptotic limit.

Proof of Equation (18). The result comes directly from the definition of coherence measure $C_r$. Due to the definitions, $C_r(AB|C), C_r(A|C)$ and $C_r(B|C)$ can be written as

$$C_r(AB|C) = S(\Delta_A \otimes \Delta_B(\rho_{ABC})) - S(\rho_{ABC});$$

$$C_r(A|C) = S(\Delta_A(\rho_{AC})) - S(\rho_{AC});$$

$$C_r(B|C) = S(\Delta_B(\rho_{BC})) - S(\rho_{BC});$$
where $\rho_{AC}$ and $\rho_{BC}$ are the corresponding reduced states of $\rho_{ABC}$. Thus

$$ C_r(AB|C) - C_r(A|C) - C_r(B|C) $$

$$ = S(\Delta_A \otimes \Delta_B(\rho_{ABC})) $$

$$ - S(\rho_{ABC}) - [S(\Delta_A(\rho_{AC})) - S(\rho_{AC})] $$

$$ - [S(\Delta_B(\rho_{BC})) - S(\rho_{BC})] $$

$$ = [S(\rho_{AC}) + S(\rho_{BC}) - S(\rho_{C}) - S(\rho_{ABC})] $$

$$ - [S(\Delta_A(\rho_{AC})) + S(\Delta_B(\rho_{BC})) $$

$$ - S(\rho_{C}) - S(\Delta_A \otimes \Delta_B(\rho_{ABC}))] $$

$$ = I(A : B|C)_\rho - I(A : B|C)_{\Delta_A \otimes \Delta_B(\rho)} $$

$$ \leq I(A : B|C)_\rho, $$

where the last inequality comes from the fact that $I(A : B|C) \geq 0$. □

**Proof of Equation (19).** This comes directly from (D4) and Equation (18) in the main context. □