FREE QUASI-SYMMETRIC FUNCTIONS OF ARBITRARY LEVEL

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Abstract. We introduce analogues of the Hopf algebra of Free quasi-symmetric functions with bases labelled by colored permutations. As applications, we recover in a simple way the descent algebras associated with wreath products \( \Gamma \wr S_n \) and the corresponding generalizations of quasi-symmetric functions. Finally, we obtain Hopf algebras of colored parking functions, colored non-crossing partitions and parking functions of type \( B \).

1. Introduction

The Hopf algebra of Free quasi-symmetric functions \( FQSym \) is a certain algebra of noncommutative polynomials associated with the sequence \((S_n)_{n \geq 0}\) of all symmetric groups. It is connected by Hopf homomorphisms to several other important algebras associated with the same sequence of groups: Free symmetric functions (or coplactic algebra) \( FSym \), Non-commutative symmetric functions (or descent algebras) \( Sym \), Quasi-Symmetric functions \( QSym \), Symmetric functions \( Sym \), and also, Planar binary trees \( PBT \), Matrix quasi-symmetric functions \( MQSym \), Parking functions \( PQSym \), and so on.

Among the many possible interpretations of \( Sym \), we can mention the identification as the representation ring of the tower of algebras

\[
\mathbb{C} \to \mathbb{C} S_1 \to \mathbb{C} S_2 \to \cdots \to \mathbb{C} S_n \to \cdots,
\]

that is

\[
Sym \simeq \bigoplus_{n \geq 0} R(\mathbb{C} S_n),
\]

where \( R(\mathbb{C} S_n) \) is the vector space spanned by isomorphism classes of irreducible representations of \( S_n \), the ring operations being induced by direct sum and outer tensor product of representations [13].

Another important interpretation of \( Sym \) is as the support of Fock space representations of various infinite dimensional Lie algebras, in particular as the level 1 irreducible highest weight representations of \( \widehat{gl}_\infty \) (the infinite rank Kac-Moody algebra of type \( A_\infty \), with Dynkin diagram \( \mathbb{Z} \), see [5]).

The analogous level \( l \) representations of this algebra can also be naturally realized with products of \( l \) copies of \( Sym \), or as symmetric functions in \( l \) independent sets of variables

\[
(Sym)^{\odot l} \simeq Sym(X_0; \ldots ; X_{l-1}) =: Sym^{(l)},
\]

and these algebras are themselves the representation rings of wreath product towers \((\Gamma \wr S_n)_{n \geq 0}, \Gamma \) being a group with \( l \) conjugacy classes [13] (see also [26, 25]).
We shall therefore call for short $\text{Sym}(X_0; \ldots; X_{l-1})$ the algebra of symmetric functions of level $l$. Our aim is to associate with $\text{Sym}^{(l)}$ analogues of the various Hopf algebras mentioned at the beginning of this introduction.

We shall start with a level $l$ analogue of $\text{FQSym}$, whose bases are labelled by $l$-colored permutations. Imitating the embedding of $\text{Sym}$ in $\text{FQSym}$, we obtain a Hopf subalgebra of level $l$ called $\text{Sym}^{(l)}$, which turns out to be dual to Poirier’s quasi-symmetric functions, and whose homogenous components can be endowed with an internal product, providing an analogue of Solomon’s descent algebras for wreath products.

The Mantaci-Reutenauer descent algebra arises as a natural Hopf subalgebra of $\text{Sym}^{(l)}$ and its dual is computed in a straightforward way by means of an appropriate Cauchy formula.

Finally, we introduce a Hopf algebra of colored parking functions, and use it to define Hopf algebras structures on parking functions and non-crossing partitions of type $B$.

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2. Free quasi-symmetric functions of level $l$

2.1. $l$-colored standardization. We shall start with an $l$-colored alphabet

\begin{equation}
A = A^0 \sqcup A^1 \sqcup \cdots \sqcup A^{l-1},
\end{equation}

such that all $A^i$ are of the same cardinality $N$, which will be assumed to be infinite in the sequel. Let $C$ be the alphabet $\{c_0, \ldots, c_{l-1}\}$ and $B$ be the auxiliary ordered alphabet $\{1, 2, \ldots, N\}$ (the letter $C$ stands for colors and $B$ for basic) so that $A$ can be identified to the cartesian product $B \times C$:

\begin{equation}
A \simeq B \times C = \{(b, c), b \in B, c \in C\}.
\end{equation}

Let $w$ be a word in $A$, represented as $(v, u)$ with $v \in B^*$ and $u \in C^*$. Then the colored standardized word $\text{Std}(w)$ of $w$ is

\begin{equation}
\text{Std}(w) := (\text{Std}(v), u),
\end{equation}

where $\text{Std}(v)$ is the usual standardization on words.

Recall that the standardization process sends a word $w$ of length $n$ to a permutation $\text{Std}(w) \in \mathfrak{S}_n$ called the standardized of $w$ defined as the permutation obtained by iteratively scanning $w$ from left to right, and labelling $1, 2, \ldots$ the occurrences of its smallest letter, then numbering the occurrences of the next one, and so on. Alternatively, $\text{Std}(w)$ is the permutation having the same inversions as $w$.

2.2. $\text{FQSym}^{(l)}$ and $\text{FQSym}^{(T)}$. A colored permutation is a pair $(\sigma, u)$, with $\sigma \in \mathfrak{S}_n$ and $u \in C^n$, the integer $n$ being the size of this permutation.

Definition 2.1. The dual free $l$-quasi-ribbon $G_{\sigma, u}$ labelled by a colored permutation $(\sigma, u)$ of size $n$ is the noncommutative polynomial

\begin{equation}
G_{\sigma, u} := \sum_{w \in A^n; \text{Std}(w) = (\sigma, u)} w \in \mathbb{Z}\langle A \rangle.
\end{equation}
Recall that the convolution of two permutations \( \sigma \) and \( \mu \) is the set \( \sigma \ast \mu \) (identified with the formal sum of its elements) of permutations \( \tau \) such that the standardized word of the \(|\sigma|\) first letters of \( \tau \) is \( \sigma \) and the standardized word of the remaining letters of \( \tau \) is \( \mu \) (see [21]).

**Theorem 2.2.** Let \((\sigma', u')\) and \((\sigma'', u'')\) be colored permutations. Then

\[
G_{\sigma', u'} G_{\sigma'', u''} = \sum_{\sigma \in \sigma' \ast \sigma''} G_{\sigma, u' \ast u''},
\]

where \(w_1 \cdot w_2\) is the word obtained by concatenating \(w_1\) and \(w_2\). Therefore, the dual free \(l\)-quasi-ribbons span a \(\mathbb{Z}\)-subalgebra of the free associative algebra.

Moreover, one defines a coproduct on the \(G\) functions by

\[
\Delta G_{\sigma, u} := \sum_{i=0}^{n} G_{(\sigma, u)[1,i]} \otimes G_{(\sigma, u)[i+1,n]},
\]

where \(n\) is the size of \(\sigma\) and \((\sigma, u)[a,b]\) is the standardized colored permutation of the pair \((\sigma', u')\) where \(\sigma'\) is the subword of \(\sigma\) containing the letters of the interval \([a, b]\), and \(u'\) the corresponding subword of \(u\).

For example,

\[
\Delta G_{3142, 2412} = 1 \otimes G_{3142, 2412} + G_{1,4} \otimes G_{231, 212} + G_{12, 42} \otimes G_{12, 21} + G_{312, 242} \otimes G_{1, 1} + G_{3142, 2412} \otimes 1.
\]

**Theorem 2.3.** The coproduct is an algebra homomorphism, so that \(FQSym^{(l)}\) is a graded bialgebra. Moreover, it is a Hopf algebra.

**Definition 2.4.** The free \(l\)-quasi-ribbon \(F_{\sigma, u}\) labelled by a colored permutation \((\sigma, u)\) is the noncommutative polynomial

\[
F_{\sigma, u} := G_{\sigma^{-1}, u^{-1} \sigma^{-1}},
\]

where the action of a permutation on the right of a word permutes the positions of the letters of the word.

For example,

\[
F_{3142, 2142} = G_{2413, 1422}.
\]

The product and coproduct of the \(F_{\sigma, u}\) can be easily described in terms of shifted shuffle and deconcatenation of colored permutations.

Let us define a scalar product on \(FQSym^{(l)}\) by

\[
\langle F_{\sigma, u}, G_{\sigma', u'} \rangle := \delta_{\sigma, \sigma'} \delta_{u, u'},
\]

where \(\delta\) is the Kronecker symbol.

**Theorem 2.5.** For any \(U, V, W \in FQSym^{(l)}\),

\[
\langle \Delta U, V \otimes W \rangle = \langle U, VW \rangle,
\]

so that, \(FQSym^{(l)}\) is self-dual: the map \(F_{\sigma, u} \mapsto G_{\sigma, u}^*\) is an isomorphism from \(FQSym^{(l)}\) to its graded dual.
Note 2.6. Let \( \phi \) be any bijection from \( C \) to \( C \), extended to words by concatenation. Then if one defines the free \( l \)-quasi-ribbon as
\[
F_{\sigma,u} := G_{\sigma^{-1}, \phi(u) \sigma^{-1}},
\]
the previous theorems remain valid since one only permutes the labels of the basis \( (F_{\sigma,u}) \).

Moreover, if \( C \) has a group structure, the colored permutations \( (\sigma,u) \in S_n \times C^n \) can be interpreted as elements of the semi-direct product \( H_n := S_n \rtimes C^n \) with multiplication rule
\[
(\sigma; c_1, \ldots, c_n) \cdot (\tau; d_1, \ldots, d_n) := (\sigma \tau; c_{\tau(1)} d_1, \ldots, c_{\tau(n)} d_n).
\]
In this case, one can choose \( \phi(\gamma) := \gamma^{-1} \) and define the scalar product as before, so that the adjoint basis of the \( (G_h) \) becomes \( F_h := G_{h^{-1}} \). In the sequel, we will be mainly interested in the case \( C := \mathbb{Z}/l\mathbb{Z} \), and we will indeed make that choice for \( \phi \).

2.3. Algebraic structure. Recall that a permutation \( \sigma \) of size \( n \) is connected if, for any \( i < n \), the set \( \{ \sigma(1), \ldots, \sigma(i) \} \) is different from \( \{1, \ldots, i\} \).

We denote by \( C \) the set of connected permutations, and by \( c_n := |C_n| \) the number of such permutations in \( S_n \). For later reference, we recall that the generating series of \( c_n \) is
\[
c(t) := \sum_{n \geq 1} c_n t^n = 1 - \left( \sum_{n \geq 0} n! t^n \right)^{-1} = t + t^2 + 3 t^3 + 13 t^4 + 71 t^5 + 461 t^6 + O(t^7).
\]

Let the connected colored permutations be the \((\sigma,u)\) with \( \sigma \) connected and \( u \) arbitrary. Their generating series is given by \( c(lt) \).

It follows from \([3]\) that \( FQSym^{(l)} \) is free over the set \( G_{\sigma,u} \) (or \( F_{\sigma,u} \)), where \( (\sigma,u) \) is connected.

Since \( FQSym^{(l)} \) is self-dual, it is also cofree.

2.4. Primitive elements. Let \( L^{(l)} \) be the primitive Lie algebra of \( FQSym^{(l)} \). Since \( \Delta \) is not cocommutative, \( FQSym^{(l)} \) cannot be the universal enveloping algebra of \( L^{(l)} \). But since it is cofree, it is, according to \([12]\), the universal enveloping dipterous algebra of its primitive part \( L^{(l)} \). Let \( d_n = \dim L^{(l)}_n \).

Recall that the shifted concatenation \( w \bullet w' \) of two elements \( w \) and \( w' \) of \( \mathbb{N}^* \), is the word obtained by concatenating to \( w \) the word obtained by shifting all letters of \( w' \) by the length of \( w \). We extend it to colored permutations by simply concatenating the colors and concatenating with shift the permutations. Let \( G^{\sigma,u} \) be the multiplicative basis defined by \( G^{\sigma,u} = G_{\sigma_1, u_1} \cdots G_{\sigma_r, u_r} \) where \( (\sigma,u) = (\sigma_1, u_1) \cdot \cdots \cdot (\sigma_r, u_r) \) is the unique maximal factorization of \( (\sigma,u) \in S_n \times C^n \) into connected colored permutations.

Proposition 2.7. Let \( V_{\sigma,u} \) be the adjoint basis of \( G^{\sigma,u} \). Then, the family \( (V_{\alpha,u})_{\alpha \in C} \) is a basis of \( L^{(l)} \). In particular, we have \( d_n = l^n c_n \).

As in \([3]\), we conjecture that \( L^{(l)} \) is free.
3. Non-commutative symmetric functions of level $l$

Following McMahon [14], we define an $l$-partite number $n$ as a column vector in $\mathbb{N}^l$, and a vector composition of $n$ of weight $|n| := \sum_i n_i$ and length $m$ as a $l \times m$ matrix $I$ of nonnegative integers, with row sums vector $n$ and no zero column.

For example,

$$I = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 3 & 1 & 1 \\ 4 & 2 & 1 & 3 \end{pmatrix}$$

is a vector composition (or a 3-composition, for short) of the 3-partite number $\begin{pmatrix} 4 & 5 \\ 10 \end{pmatrix}$ of weight 19 and length 4.

For each $n \in \mathbb{N}^l$ of weight $|n| = n$, we define a level $l$ complete homogeneous symmetric function as

$$S_n := \sum_{u; |u| = n} G_{1\cdots n,u}.$$

It is the sum of all possible colorings of the identity permutation with $n_i$ occurrences of color $i$ for each $i$.

Let $\text{Sym}^{(l)}$ be the subalgebra of $\text{FQSym}^{(l)}$ generated by the $S_n$ (with the convention $S_0 = 1$). The Hilbert series of $\text{Sym}^{(l)}$ is easily found to be

$$S_l(t) := \sum_n \dim \text{Sym}^{(l)}_n t^n = \frac{(1-t)^l}{2(1-t)^l - 1}.$$

**Theorem 3.1.** $\text{Sym}^{(l)}$ is free over the set $\{S_n; |n| > 0\}$. Moreover, $\text{Sym}^{(l)}$ is a Hopf subalgebra of $\text{FQSym}^{(l)}$.

The coproduct of the generators is given by

$$\Delta S_n = \sum_{i+j=n} S_i \otimes S_j,$$

where the sum $i+j$ is taken in the space $\mathbb{N}^l$. In particular, $\text{Sym}^{(l)}$ is cocommutative.

We can therefore introduce the basis of products of level $l$ complete function, labelled by $l$-compositions

$$S^I = S_{i_1} \cdots S_{i_m},$$

where $i_1, \ldots, i_m$ are the columns of $I$.

**Theorem 3.2.** If $C$ has a group structure, $\text{Sym}^{(l)}_n$ becomes a subalgebra of $\mathbb{C}[C \wr S_n]$ under the identification $G_h \mapsto h$.

This provides an analogue of Solomon’s descent algebra for the wreath product $C \wr S_n$. The proof amounts to check that the Patras descent algebra of a graded bialgebra [17] can be adapted to $\mathbb{N}^l$-graded bialgebras.
As in the case \( l = 1 \), we define the \textit{internal product} \( * \) as being opposite to the law induced by the group algebra. It can be computed by the following splitting formula, which is a straightforward generalization of the level 1 version.

**Proposition 3.3.** Let \( \mu_r : (\text{Sym}^{(l)})^\otimes r \to \text{Sym}^{(l)} \) be the product map. Let \( \Delta^{(r)} : (\text{Sym}^{(l)}) \to (\text{Sym}^{(l)})^\otimes r \) be the \( r \)-fold coproduct, and \( *_r \) be the extension of the internal product to \( (\text{Sym}^{(l)})^\otimes r \). Then, for \( F_1, \ldots, F_r, \) and \( G \in \text{Sym}^{(l)} \),

\[
(F_1 \cdots F_r) * G = \mu_r[(F_1 \otimes \cdots \otimes F_r) *_r \Delta^{(r)} G].
\]

The group law of \( C \) is needed only for the evaluation of the product of one-part complete functions \( S_m * S_n \).

**Example 3.4.** With \( l = 2 \) and \( C = \mathbb{Z}/2\mathbb{Z} \),

\[
S\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} * S\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \mu_2 \left[ \left( S\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) \otimes S\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] _{\ll} \Delta S\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \\
= \left( S\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) \otimes S\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \left( S\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) \otimes S\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \\
= S\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + S\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + S\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.
\]

Recall that the underlying colored alphabet \( A \) can be seen as \( A^0 \sqcup \cdots \sqcup A^{l-1} \), with \( A^i = \{a^{(i)}_j, j \geq 1\} \). Let \( \mathbf{x} = (x^{(0)}, \ldots, x^{(l-1)}) \), where the \( x^{(i)} \) are \( l \) commuting variables. In terms of \( A \), the generating function of the complete functions can be written as

\[
\sigma_{\mathbf{x}}(A) = \prod_{i \geq 0} \left( 1 - \sum_{0 \leq j \leq l-1} x^{(j)} a^{(j)}_i \right)^{-1} = \sum_{n} S_n(A)\mathbf{x}^n,
\]

where \( \mathbf{x}^n = (x^{(0)})^{n_0} \cdots (x^{(l-1)})^{n_{l-1}} \).

This realization gives rise to a Cauchy formula (see \[10\] for the \( l = 1 \) case), which in turn allows one to identify the dual of \( \text{Sym}^{(l)} \) with an algebra introduced by S. Poirier in \[18\].

4. \textbf{Quasi-symmetric functions of level \( l \)}

4.1. \textbf{Cauchy formula of level \( l \).} Let \( X = X^0 \sqcup \cdots \sqcup X^{l-1} \), where \( X^i = \{x^{(i)}_j, j \geq 1\} \) be an \( l \)-colored alphabet of commutative variables, also commuting with \( A \). Imitating the level 1 case (see \[3\]), we define the Cauchy kernel

\[
K(X, A) = \prod_{j \geq 1} \sigma_{(x^{(0)}_j, \ldots, x^{(l-1)}_j)}(A).
\]
Expanding on the basis $S^I$ of $\text{Sym}^{(l)}$, we get as coefficients what can be called the level $l$ monomial quasi-symmetric functions $M_l(X)$

\begin{equation}
K(X, A) = \sum_I M_l(X) S^I(A),
\end{equation}

defined by

\begin{equation}
M_l(X) = \sum_{j_1 < \cdots < j_m} x_{j_1}^{i_1} \cdots x_{j_m}^{i_m},
\end{equation}

with $I = (i_1, \ldots, i_m)$.

These last functions form a basis of a subalgebra $QS\text{ym}^{(l)}$ of $\mathbb{C}[X]$, which we shall call the algebra of quasi-symmetric functions of level $l$.

### 4.2. Poirier’s Quasi-symmetric functions.

The functions $M_l(X)$ can be recognized as a basis of one of the algebras introduced by Poirier: the $M_l$ coincide with the $M_{(C,v)}$ defined in [13], p. 324, formula (1), up to indexation.

Following Poirier, we introduce the level $l$ quasi-ribbon functions by summing over an order on $l$-compositions: an $l$-composition $C$ is finer than $C'$, and we write $C \leq C'$, if $C'$ can be obtained by repeatedly summing up two consecutive columns of $C$ such that no non-zero element of the left one is strictly below a non-zero element of the right one.

This order can be described in a much easier and natural way if one recodes an $l$-composition $I$ as a pair of words, the first one $d(I)$ being the set of sums of the elements of the first $k$ columns of $I$, the second one $c(I)$ being obtained by concatenating the words $i_{k,j}$ while reading of $I$ by columns, from top to bottom and from left to right. For example, the $3$-composition of Equation (17) satisfies

\begin{equation}
d(I) = \{5, 10, 14, 19\} \quad \text{and} \quad c(I) = 13333\ 22233\ 1123\ 12333.
\end{equation}

Moreover, this recoding is a bijection if the two words $d(I)$ and $c(I)$ are such that the descent set of $c(I)$ is a subset of $d(I)$. The order previously defined on $l$-compositions is in this context the inclusion order on sets $d$: $(d',c) \leq (d,c)$ iff $d' \subseteq d$.

It allows us to define the level $l$ quasi-ribbon functions $F_1$ by

\begin{equation}
F_1 = \sum_{I \leq \Gamma} M_I.
\end{equation}

Notice that this last description of the order $\leq$ is reminiscent of the order $\leq'$ on descent sets used in the context of quasi-symmetric functions and non-commutative symmetric functions: more precisely, since it does not modify the word $c(I)$, the order $\leq$ restricted to $l$-compositions of weight $n$ amounts for $l^n$ copies of the order $\leq'$. The computation of its Möbius function is therefore straightforward.

Moreover, one can directly obtain the $F_1$ as the commutative image of certain $F_{\sigma,u}$: any pair $(\sigma,u)$ such that $\sigma$ has descent set $d(I)$ and $u = c(I)$ will do.
5. The Mantaci-Reutenauer algebra

Let $e_i$ be the canonical basis of $\mathbb{N}^l$. For $n \geq 1$, let

$$S_n^{(i)} = S_n e_i \in \mathsf{Sym}^{(l)},$$

be the monochromatic complete symmetric functions.

**Proposition 5.1.** The $S_n^{(i)}$ generate a Hopf-subalgebra $\mathsf{MR}^{(l)}$ of $\mathsf{Sym}^{(l)}$, which is isomorphic to the Mantaci-Reutenauer descent algebra of the wreath products $\mathsf{S}_n^{(l)} = (\mathbb{Z}/l\mathbb{Z}) \wr \mathfrak{S}_n$.

It follows in particular that $\mathsf{MR}^{(l)}$ is stable under the composition product induced by the group structure of $\mathsf{S}_n^{(l)}$. The bases of $\mathsf{MR}^{(l)}$ are labelled by colored compositions (see below).

The duality is easily worked out by means of the appropriate Cauchy kernel. The generating function of the complete functions is

$$\sigma_{MR}^X(A) := 1 + \sum_{j=0}^{l-1} \sum_{n \geq 1} S_n^{(j)} \cdot (x^{(j)})^n,$$

and the Cauchy kernel is as usual

$$K_{MR}(X, A) = \prod_{i \geq 1} \sigma_{x_i}^{MR}(A) = \sum_{(I, u)} M_{(I, u)}(X) S^{(I, u)}(A),$$

where $(I, u)$ runs over colored compositions $(I, u) = ((i_1, \ldots, i_m), (u_1, \ldots, u_m))$ that is, pairs formed with a composition and a color vector of the same length. The $M_{I, u}$ are called the monochromatic monomial quasi-symmetric functions and satisfy

$$M_{(I, u)}(X) = \sum_{j_1 < \cdots < j_m} (x_{j_1}^{(u_1)})^{i_1} \cdots (x_{j_m}^{(u_m)})^{i_m}.$$

**Proposition 5.2.** The $M_{(I, u)}$ span a subalgebra of $\mathbb{C}[[X]]$ which can be identified with the graded dual of $\mathsf{MR}^{(l)}$ through the pairing

$$\langle M_{(I, u)}, S^{(J, v)} \rangle = \delta_{I, J} \delta_{u, v},$$

where $\delta$ is the Kronecker symbol.

Note that this algebra can also be obtained by assuming the relations

$$x_i^{(p)} x_i^{(q)} = 0, \text{ for } p \neq q,$$

on the variables of $\mathsf{QSym}^{(l)}$.

Baumann and Hohlweg have another construction of the dual of $\mathsf{MR}^{(l)}$ [2] (implicitly defined in [13], Lemma 11).
6. Level \( l \) parking quasi-symmetric functions

6.1. Usual parking functions. Recall that a parking function on \([n] = \{1, 2, \ldots, n\}\) is a word \(a = a_1a_2 \cdots a_n\) of length \(n\) on \([n]\) whose nondecreasing rearrangement \(a^+ = a'_1a'_2 \cdots a'_n\) satisfies \(a'_i \leq i\) for all \(i\). Let \(\text{PF}_n\) be the set of such words. It is well-known that \(|\text{PF}_n| = (n+1)^n - 1\).

Gessel introduced in 1997 (see [24]) the notion of prime parking function. One says that \(a\) has a breakpoint at \(b\) if \(|\{a_i \leq b\}| = b\). The set of all breakpoints of \(a\) is denoted by \(\text{BP}(a)\). Then, \(a \in \text{PF}_n\) is prime if \(\text{BP}(a) = \{n\}\).

Let \(\text{PPF}_n \subset \text{PF}_n\) be the set of prime parking functions on \([n]\). It can easily be shown that \(|\text{PPF}_n| = (n-1)^n - 1\) (see [24]).

We will finally need one last notion: \(a\) has a match at \(b\) if \(|\{a_i < b\}| = b - 1\) and \(|\{a_i \leq b\}| \geq b\). The set of all matches of \(a\) is denoted by \(\text{Ma}(a)\).

We will now define generalizations of the usual parking functions to any level in such a way that they build up a Hopf algebra in the same way as in [16].

6.2. Colored parking functions. Let \(l\) be an integer, representing the number of allowed colors. A colored parking function of level \(l\) and size \(n\) is a pair composed of a parking function of length \(n\) and a word on \([l]\) of length \(l\).

Since there is no restriction on the coloring, it is obvious that there are \(l^n(n+1)^{n-1}\) colored parking functions of level \(l\) and size \(n\).

It is known that the convolution of two parking functions contains only parking functions, so one easily builds as in [16] an algebra \(\text{PQSym}^{(l)}\) indexed by colored parking functions:

\[
G_{(a',u')}G_{(a'',u'')} = \sum_{a \in a' \ast a''} G_{(a,u \cdot u')}.
\]

Moreover, one defines a coproduct on the \(G\) functions by

\[
\Delta G_{(a,u)} = \sum_{i \in \text{BP}(a)} G_{(a,u)_{[1,i]}} \otimes G_{(a,u)_{[i+1,n]}}
\]

where \(n\) is the size of \(a\) and \((a,u)_{[a,b]}\) is the parkized colored parking function of the pair \((a',u')\) where \(a'\) is the subword of \(a\) containing the letters of the interval \([a,b]\), and \(u'\) the corresponding subword of \(u\).

**Theorem 6.1.** The coproduct is an algebra homomorphism, so that \(\text{PQSym}^{(l)}\) is a graded bialgebra. Moreover, it is a Hopf algebra.

6.3. Parking functions of type \(B\). In [20], Reiner defined non-crossing partitions of type \(B\) by analogy to the classical case. In our context, he defined the level 2 case. It allowed him to derive, by analogy with a simple representation theoretical result, a definition of parking functions of type \(B\) as the words on \([n]\) of size \(n\).

We shall build another set of words, also enumerated by \(n^n\) that sheds light on the relation between type-A and type-B parking functions and provides a natural Hopf algebra structure on the latter.
First, fix two colors 0 < 1. We say that a pair of words \((a, u)\) composed of a parking function and a binary colored word is a level 2 parking function if

- the only elements of \(a\) that can have color 1 are the matches of \(a\).
- for all element of \(a\) of color 1, all letters equal to it and to its left also have color 1,
- all elements of \(a\) have at least once the color 0.

For example, there are 27 level 2 parking functions of size 3: there are the 16 usual ones all with full color 0, and the eleven new elements

\[(111, 100), (111, 110), (112, 100), (121, 100), (211, 010),\]
\[(113, 100), (131, 100), (311, 010), (122, 010), (212, 100), (221, 100).\]  

The first time the first rule applies is with \(n = 4\), where one has to discard the words \((1122, 0010)\) and \((1122, 1010)\) since 2 is not a match of 1122. On the other hand, both words \((1133, 0010)\) and \((1133, 1010)\) are \(B_4\)-parking functions since 1 and 3 are matches of 1133.

**Theorem 6.2.** The restriction of \(PQSym^{(2)}\) to the \(G\) elements indexed by level 2 parking functions is a Hopf subalgebra of \(PQSym^{(2)}\).

### 6.4. Non-crossing partitions of type \(B\).

Remark that in the level 1 case, non-crossing partitions are in bijection with non-decreasing parking functions. To extend this correspondence to type \(B\), let us start with a non-decreasing parking function \(b\) (with no color). We factor it into the maximal shifted concatenation of prime non-decreasing parking functions, and we choose a color, here 0 or 1, for each factor. We obtain in this way \(\binom{2n}{n}\) words \(\pi\), which can be identified with type \(B\) non-crossing partitions.

Let

\[P^\pi = \sum_{a \sim \pi} F_a\]

where \(\sim\) denotes equality up to rearrangement of the letters. Then,

**Theorem 6.3.** The \(P^\pi\), where \(\pi\) runs over the above set of non-decreasing signed parking functions, form the basis of a cocommutative Hopf subalgebra of \(PQSym^{(2)}\).

All this can be extended to higher levels in a straightforward way: allow each prime non-decreasing parking function to choose any color among \(l\) and use the factorization as above. Since non-decreasing parking functions are in bijection with Dyck words, the choice can be described as: each block of a Dyck word with no return-to-zero, chooses one color among \(l\). In this version, the generating series is obviously given by

\[\frac{1}{1 - l^{1-\sqrt{1-4l}}},\]

For \(l = 3\), we obtain the sequence A007854 of [22].
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