Towards an algorithmisation of the Dirac constraint formalism

Gerdt V 1, Khvedelidze A 1, 2, 3, Palii Yu 1, 4

1 Laboratory of Information Technologies, Joint Institute for Nuclear Research, Dubna, 141980, Russia, [gerdt, palii]@jinr.ru
2 Department of Theoretical Physics, A.M. Razmadze Mathematical Institute, Tbilisi, GE-0193, Georgia
3 School of Mathematics and Statistics, University of Plymouth, Plymouth, PL4 8AA, United Kingdom, akhvedelidze@plymouth.ac.uk
4 Institute of Applied Physics, Moldova Academy of Sciences, Chisinau, MD-2028, Republic of Moldova

Abstract

Central issues of the Dirac constraint formalism are discussed in relation to the algorithmic methods of commutative algebra based on the Gröbner basis techniques. For a wide class of finite dimensional polynomial degenerate Lagrangian systems, we describe an algorithmic scheme of computation of the complete set of constraints, their separation into subsets of first and second class constraints as well as the construction of a generator of local symmetry transformations. The proposed scheme is exemplified by considering the so-called light-cone Yang-Mills mechanics with an SU(2) gauge structure group.

Keywords: constrained Hamiltonian dynamics, commutative algebra, Gröbner basis.

1 Work supported by NEST-Adventure contract 5006 (GIFT).
1 Introduction

Lagrangians used for the description of fundamental particles, such as electrons and photons, as well as quarks and gluons have a degenerate Hessian functions. This rather unusual property, compared to standard Lagrangian mechanical models, profoundly modifies the whole mathematical description of classical evolution. It demands the physical interpretation of constrained variables (e.g. longitudinal components of the electromagnetic potential) and also requires the generalisation of the canonical quantisation scheme. From the mathematical point of view the new element of the Hamiltonian description of a degenerate Lagrangian system is the involution analysis of the differential equations of motion. Its pivotal ingredients in the generalized Hamiltonian dynamics [1]-[5] are realised in the form of the Dirac scheme to determine constraints. This is related to [6, 7, 8] the formal theory of differential equations [9]. The process of the determining all the integrability conditions that can not be derived using only the algebraic operations with the existing differential equations is just the “reproduction” of constraints in the Dirac formalism. Having a complete set of constraints we are able to identify the set of “truly” dynamical evolution of the physical observables and perform the subsequent quantization.

Effective completion to involution of systems of differential equations needed in field theories represents a very complicated challenge requiring sophisticated computer-algebraic methods [10]. Similarly the generalized Hamiltonian formalism also needs an efficient algorithmisation and implementation in a proper computer algebra software.

In the present paper we apply the most universal algorithmic tool of commutative algebra, the Gröbner bases [11], as the main algorithmic ingredient of the generalized Hamiltonian dynamics for degenerate mechanical models with polynomial Lagrangians. In [12] it was already suggested to use the Gröbner bases for the computation and separation of constrains for such models. The underlying Dirac-Gröbner algorithm is based on the facility of the Gröbner bases method to manipulate with a polynomial in the phase variables modulo constraint manifold, and, in particular, to check whether the polynomial vanishes on the manifold. In the present note we propose some further algorithmic improvements and extensions aiming at the computational realization of the Hamiltonian reduction of degenerate mechanical system possessing local symmetries.

It should be noticed that constructive ideas of the involution analysis of differential equations combined with those from the Gröbner bases technique have culminated in the concept of involutive bases [13] as a special type of Gröbner bases providing the efficient involutive algorithms [14] for construction of the involutive as well as the reduced Gröbner bases.

The plan of this paper is as follows. We start (Section 2) with a brief description of the main issues in the Dirac constraint formalism that should be put into an algorithmic form suitable for effective calculations. In Section 3 the ways to achieve this goal for finite-dimensional mechanical systems with polynomial Lagrangians are described. Then (Section 4) we consider the so-called light-cone $SU(n)$ Yang-Mills mechanics as an in-
Towards an algorithmisation of the Dirac constraint formalism

Interesting example of constrained model for which the first algorithmic issue of the Dirac formalism, namely, construction of the primary constraints, can be performed for arbitrary $n$. The remaining algorithmic issues of the Dirac formalism are illustrated in Section 4 for this model specified to the simplest nontrivial structure group $SU(2)$. Finally, in Section 5 some conclusions are presented.

2 The issues requiring algorithmisation

Here we sketch briefly the basic notions and definitions from the Dirac constraint formalism for a finite dimensional degenerate Lagrangian system and make a list of the main procedures requiring an algorithmic reformulation.

Consider an $n$-dimensional mechanical system whose configuration space is $\mathbb{R}^n$ and the Lagrangian $L(q, \dot{q})$ is defined on a tangent space as a function of the coordinates $q := q_1, q_2, \ldots, q_n$ and velocities $\dot{q} := \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$.

The Lagrangian system is called a regular one if the rank $r := \text{rank} \| H_{ij} \|_2$ of the corresponding Hessian function $H_{ij} := \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$ is maximal ($r = n$). In this case the Euler-Lagrange equations
\begin{equation}
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0, \quad 1 \leq i \leq n
\end{equation}
rewritten explicitly as
\begin{equation}
H_{ij} \ddot{q}_j + \frac{\partial^2 L}{\partial q_j \partial \dot{q}_i} \dot{q}_j - \frac{\partial L}{\partial q_i} = 0
\end{equation}
can be resolved with respect to the accelerations ($\ddot{q}$) and there is no hidden constraints. Otherwise, if $r < n$, the Euler-Lagrange equations are degenerate or singular. In this case not all differential equations (1) are of second order, namely there are $n - r$ independent equations, Lagrangian constraints, containing only coordinates and velocities. Passing to the Hamiltonian description via a Legendre transformation
\begin{equation}
p_i := \frac{\partial L}{\partial \dot{q}_i}
\end{equation}
the degeneracy of the Hessian manifests itself in the existence of $n - r$ relations between coordinates and momenta, the primary constraints
\begin{equation}
\phi^{(1)}_{\alpha}(p, q) = 0, \quad 1 \leq \alpha \leq n - r.
\end{equation}
Equations (3) define the so-called primary constraints subset $\Sigma_1$. This definition is implicit and therefore the first algorithmisation topic is:

**Issue I:** Find all primary constraints describing the subset $\Sigma_1$.

From (3) the dynamics is constrained by the set $\Sigma_1$ and by the Dirac prescription is governed by the total Hamiltonian
\begin{equation}
H_T := H_C + U_{\alpha} \phi^{(1)}_{\alpha},
\end{equation}
where

\begin{equation}
H_C := \sum_{i=1}^{n} \frac{p_i^2}{2} - L(q, \dot{q}), \quad U_{\alpha} := \frac{\partial U}{\partial \phi^{(1)}_{\alpha}}
\end{equation}
which differs from the canonical Hamiltonian $H_C(p,q) = p_i q_i - L$ by a linear combination of the primary constraints with the Lagrange multipliers $U_α$.

The next step is to analyze the dynamical requirement that classical trajectories remain in $Σ_1$ during evolution,

$$\dot{φ}_α^{(1)} = \{H_T, φ_α^{(1)}\} \overset{Σ_1}{=} 0.$$  \hfill (5)

In (5) the evolutinal changes are generated by the canonical Poisson brackets with the total Hamiltonian (4) and the abbreviation $\overset{Σ_1}{=} \text{stands for a week equality}$, i.e., the right-hand side of (5) vanishes modulo the constraints.

The consistency condition (5), unless it is satisfied identically, may lead either to a contradiction or to a determination of the Lagrange multipliers $U_α$ or to new constraints. The former case indicates that the given Hamiltonian system is inconsistent.

In the latter case when (5) is not satisfied identically and is independent of the multipliers $U_α$ the left-hand side of (5) defines the new constraints. Otherwise, if the left-hand side depends on some Lagrange multipliers $U_α$ the consistency condition determines these multipliers, and, therefore, the constraints set is not enlarged by new constraints. The subsequent iteration of this consistency check ends up with the complete set of constraints and/or determination of some/or all Lagrange multipliers.

The number of Lagrange multipliers $U_α$ which can be found is determined by the so-called Poisson bracket matrix

$$M_{αβ} := \{φ_α, φ_β\},$$  \hfill (6)

where $Σ$ denotes the subset of a phase space defined by the all constraints including primary $φ_α^{(1)}$, secondary $φ_α^{(2)}$, ternary $φ_α^{(3)}$, etc., constraints, $Φ := (φ_α^{(1)}, φ_α^{(2)}, \ldots , φ_α^{(k)})$

$$Σ : \quad φ_α(p,q) = 0, \quad 1 ≤ α ≤ k.$$  \hfill (7)

The co-rank $s := k - \text{rank}(M)$ of matrix $M$ represent the number of first-class constraints $ψ_1, ψ_2, \ldots, ψ_s$, linear combinations of constraints $φ_α$

$$ψ_α(p,q) = \sum_β c_αβ(p,q) \phi_β,$$  \hfill (8)

whose Poisson brackets are weakly zero

$$\{ψ_α(p,q), ψ_β(p,q)\} \overset{Σ}{=} 0 \quad 1 ≤ α, β ≤ s.$$  \hfill (9)

The remaining functionally independent constraints form the subset of second-class constraints.

This method of constraints determination in the Dirac formalism represents the particular form of completion of the initial Hamiltonian equations to involution; the generated constraints are nothing else than the integrability conditions [6, 7, 8].

Therefore the second algorithmisation challenge can be formulated as

**Issue II:** Determine all integrability conditions and perform their separation into first and second class conditions.
First-class constraints play a very special role in the Hamiltonian description: they provide the basis for \textit{generators of local symmetry transformations}. The knowledge of a local symmetry transformation is important because according to physical requirement the physical observables are singlets under the gauge symmetry transformations.

So the next important algorithmisation problem is

\textbf{Issue III: Construct the generator of local symmetry transformation and find the basis for singlet observables.}

The last problem has direct impact on the process of \textit{Hamiltonian reduction}, that is a formulation of a new Hamiltonian system with a reduced number of degrees of freedom but equivalent to the initial degenerate one \cite{2,16,17}. The presence of \(s\) first-class constraints and \(r := k – s\) second-class constraints guarantees the possibility of local reformulation of the initial \(2n\) dimensional Hamiltonian system as a \(2n – 2s – r\) dimensional reduced Hamiltonian system (cf. \cite{7}).

Therefore, the final fourth algorithmisation challenge we formulate here as

\textbf{Issue IV: Construct an equivalent unconstrained Hamiltonian system on the reduced phase space.}

\section{How the algorithm works}

Here we extend the main ideas of \cite{12} and describe the algorithmic basics that can be used to solve the problems stated in the previous section. In doing so, we restrict our consideration to an arbitrary dynamical system with finitely many degrees of freedom whose Lagrangian is a polynomial in coordinates and velocities with rational (possibly parametric) coefficients \(L(q, \dot{q}) \in \mathbb{Q}[q, \dot{q}].\)

\subsection{Primary constraints}

For degenerate systems the primary constraints \cite{3} are consequences of the polynomial relations \cite{2}. These relations generate the polynomial ideal in \(\mathbb{Q}[q, \dot{q}, p]\)

\[ I_{p,q,\dot{q}} \equiv \text{Id}(\bigcup_{i=1}^{n} \{p_{i} – \partial L/\partial \dot{q}_{i}\}) \subset \mathbb{Q}[p, q, \dot{q}]. \] (10)

Thereby, primary constraints \cite{3} belong to the radical \(\sqrt{I_{p,q}}\) of the elimination ideal

\[ I_{p,q} = I_{p,q,\dot{q}} \cap \mathbb{Q}[p, q]. \]

Correspondingly, for an appropriate term ordering which eliminates \(\dot{q}\), a Gröbner basis of \(I_{p,q}\) (denotation: \(GB(I_{p,q})\)) is given by \cite{11,18}

\[ GB(I_{p,q}) = GB(I_{p,q,\dot{q}}) \cap \mathbb{Q}[p, q]. \]

\footnote{Throughout this section we use some standard notions and definitions of commutative algebra (see, for example, \cite{18}).}
This means that construction of the Gröbner basis for the ideal \( (10) \) with omitting elements in the basis depending on velocities and then constructing of \( GB(\sqrt{I_{p,q}}) \) allows us to compute the set of primary constraints. If \( GB(\sqrt{I_{p,q}}) = \emptyset \) then the dynamical system is regular. Otherwise, the algebraically independent set \( \Phi_1 \) of primary constraints is the subset \( \Phi_1 \subset GB(\sqrt{I_{p,q}}) \) such that
\[
\forall \phi(p,q) \in \Phi_1 : \phi(p,q) \not\in \text{Id}(\Phi_1 \setminus \{\phi(p,q)\}).
\] Verification of (11) is algorithmically done by computing the following normal form: \( NF(\phi,GB(\text{Id}(\Phi_1 \setminus \{\phi\})) \). Therefore, all the computational steps described above admit full algorithmisation by means of Gröbner bases. In addition, the canonical Hamiltonian \( H_c(p,q) \) is computed as \( NF(p_iq_i - L,GB(I_{p,q},\dot{q})) \).

3.2 Complete set of constraints and their separation

The dynamical consequences (5) of a primary constraint can also be algorithmically analyzed by computing the normal form of the Poisson brackets of the primary constraint and the total Hamiltonian modulo \( GB(\sqrt{I_{p,q}}) \). Here the Lagrange multipliers \( U_\alpha \) in (4) are treated as time-dependent functions. If the non-vanishing normal form does not contain \( U_\alpha \), then it is nothing else than the secondary constraint. In this case the set of primary constraints is enlarged by the secondary constraint obtained and the process is iterated. At the end either the complete set \( \Phi \) of constraints (7) is constructed or inconsistency of the dynamical system is detected. The detection holds when the intermediate Gröbner basis, whose computation is a part of the iterative procedure, becomes \( \{1\} \).

To separate the set \( \Phi = \{\phi_1, \ldots, \phi_k\} \) into of first and second class constraints the Poisson bracket \( k \times k \) matrix \( M \) (6) is built. Its entries are computed as normal forms of the Poisson brackets of the constraints modulo a Gröbner basis of the ideal generated by set \( \Phi \).

To construct \( s := k - r \); where \( r = \text{rank}(M) \) first-class constraints as linear combinations (8) of constraints (7) satisfying (9) it suffices to find the basis \( P = \{p_1, \ldots, p_{k-r}\} \) of the null space (kernel) of the linear transformation defined by \( M \). Every vector \( p \in P \) generates the first-class constraint of form \( p_\alpha \phi_\alpha \).

Now consider the \( s \times k \) matrix \( (p_j)_\alpha \) composed of components of vectors in \( P \) and find a basis \( T := \{t_1, \ldots, t_r\} \) of the null space of the corresponding linear transformation. For every vector \( t \in T \) the second-class constraint is constructed as \( t_\alpha \phi_\alpha \).
Towards an algorithmisation of the Dirac constraint formalism

Thus the constraints separation can be done using linear algebra operations with the matrix \( M \) alone. Together with the Gröbner bases technique this implies full algorithmisation for computing the complete set of algebraically independent constraints and their separation (Issues I and II of Section 2).

3.3 Generator of local symmetry transformations

The local symmetries are generated by first-class constraints (cf. [5]) but the presence of the second-class constraints makes realization of the symmetry transformations very subtle. To overcome some of these difficulties one can effectively eliminate the second class constraints by changing the initial Poisson bracket to the new Dirac bracket defined as

\[
\{ f, g \}_D := \{ f, g \} - \{ f, \chi_\alpha \} C^{-1}_{\alpha\beta} \{ \chi_\beta, g \},
\]

where \( \chi_\alpha \) \((1 \leq \alpha \leq r)\) denotes the second-class constraints, and the invertible \( r \times r \) matrix \( C_{\alpha\beta} \) is defined as

\[
C_{\alpha\beta} := \sum \{ \chi_\alpha, \chi_\beta \}.
\]

Since for an arbitrary function \( f \) it follows that \( \{ f, \chi_\alpha \}_D = 0 \) the second-class constraints can be set to zero either before or after evaluating a Dirac bracket. This last evaluation, modulo the constraint functions, can be performed algorithmically exploiting the Gröbner bases. After elimination of all second-class constraints follow to the Dirac conjecture [1] the generator \( G \) of local transformations is expressed as a linear combination of all first-class constraints

\[
G = \sum_{\beta=1}^{k_1} \epsilon^{(1)}_{\beta} \phi^{(1)}_{\beta} + \sum_{\gamma=k_1+1}^{s} \epsilon^{(2)}_{\gamma} \phi^{(2)}_{\gamma},
\]

and its action on phase space coordinates \((q, p)\) is given now with the aid of the Dirac bracket

\[
\delta q_i = \{ G, q_i \}_D, \quad \delta p_i = \{ G, p_i \}_D.
\]

In (12) the coefficients \( \epsilon^{(1)}_{\beta} \) and \( \epsilon^{(2)}_{\gamma} \) are functions of time \( t \) and the first sum includes \( k_1 \) primary first-class constraints while the second sum contains the all remaining first-class constraints. Not all of the functions \( \epsilon \) in (12) are independent ones. Here we briefly state how following the method suggested in [19] one can extract the irreducible set of functions from the set of \( \epsilon \). The total time derivative of the gauge-symmetry generator (12) is given in terms of the Dirac bracket of \( G \) and the canonical Hamiltonian:

\[
\frac{dG}{dt} = \frac{\partial G}{\partial t} + \{ G, H_C \}_D.
\]

Since the set of first-class constraints is complete, the Dirac bracket in the right-hand side of (13) is

\[
\{ \phi_\mu, H_C \}_D = \rho_{\mu\nu} \phi_\nu.
\]
The functions $\rho_{\mu\nu}$ can be algorithmically computed by using the Gröbner bases method. To perform this computation one can use, for example, the extended Gröbner basis algorithm [20]. Given a set of polynomials $F = \{f_1, \ldots, f_m\} \subset \mathbb{Q}[p, q]$ generating the polynomial ideal $\text{Id}(F)$, this algorithm yields the explicit representation

$$g_i = h_{ij} f_j$$

of elements in the Gröbner basis $\{g_1, \ldots, g_n\}$ of this ideal in terms of the ideal generated by polynomials in $F$. Therefore, having computed a Gröbner basis for the ideal generated by the first-class constraints and the corresponding polynomial coefficients for the elements in the Gröbner basis as given in (15), the coefficients $\rho_{\mu\nu}$ are easily computed by reduction [11, 18, 20] of the Dirac bracket in (14) modulo the Gröbner basis expressed in terms of the first-class constraints $\phi_\nu$. Note that one can similarly compute the algebra of first-class constraints

$$\{\phi_\alpha, \phi_\beta\}_D = \varrho_{\alpha\beta\gamma}\phi_\gamma,$$

if the structure functions $\varrho_{\alpha\beta\gamma}$ are polynomials in $p, q$.

The generator of local transformation is conserved modulo the primary constraints

$$\frac{dG}{dt} \Sigma_1 = 0 \Rightarrow \varepsilon_\gamma^{(2)} \phi_\gamma^{(2)} + \varepsilon_\beta^{(1)} \rho_{\beta\gamma} \phi_\gamma^{(2)} + \varepsilon_\delta^{(2)} \rho_{\delta\gamma} \phi_\gamma^{(2)} \Sigma_1 = 0.$$

Since, by their construction, the constraints $\phi_\gamma^{(2)}$ do not vanish on the primary-constraint manifold $\Sigma_1$, the relations (16) represent the following system of differential equations on the gauge functions $\varepsilon_\gamma^{(1)}$ and $\varepsilon_\gamma^{(2)}$

$$\varepsilon_\gamma^{(2)} + \varepsilon_\beta^{(1)} \rho_{\beta\gamma} + \varepsilon_\delta^{(2)} \rho_{\delta\gamma} = 0, \quad (k_1 + 1 \leq \gamma \leq s),$$

where the index $\beta$ runs from 1 to $k_1$, $\gamma$ runs from $k_1 + 1$ to $s$ and the functions $\rho_{\mu\nu}$ are projected on to the subset $\Sigma_1$.

Since the differential system (17) is underdetermined, one can express the functions $\varepsilon_\beta^{(1)}$ in terms of arbitrary functions $\varepsilon_\gamma^{(2)}(t)$ and their derivatives [19]. Since this last procedure is algorithmic, this completes the algorithmic construction of the generator of the local symmetry transformation.

The above described algorithmic procedures have been implemented as a Maple package (currently for Maple 10), and this package was used to perform the computations presented in the next section.

It is worth noting here that the remaining part of Issue III as well as Issue IV still require an algorithmisation.

4 Light-cone Yang-Mills mechanics

Now we discuss the application of the general scheme described above to a mechanical model originated from Yang-Mills gauge theory formulated on the light-cone under the assumption of spatial homogeneity of the gauge fields.
The standard action of Yang-Mills field theory with structure group $SU(n)$ in four-dimensional Minkowski space $M_4$, endowed with a metric $\eta$ is

$$S := \frac{1}{g^2_0} \int_{M_4} \text{tr} \ F \wedge *F,$$

where $g_0$ is a coupling constant and the $SU(n)$ algebra valued curvature two-form

$$F := dA + A \wedge A$$

is constructed from the connection one-form $A$. The connection and curvature, as Lie algebra valued quantities, are expressed in terms of the antihermitian algebra basis

$$A = A^a T^a, \quad F = F^a T^a, \quad a = 1, 2, \ldots, n^2 - 1.$$ 

The metric $\eta_{\alpha\beta}$ enters the action through the dual field strength tensor defined in accordance with the Hodge star operation

$$*F_{\mu\nu} := \frac{1}{2} \sqrt{\text{det}(\eta)} \epsilon_{\mu
u\alpha\beta} F^{\alpha\beta},$$

with totally antisymmetric tensor $\epsilon_{\mu
u\alpha\beta}$.

The light-cone version of the theory is formulated using the frame where two vectors $e_{\pm} := \frac{1}{\sqrt{2}} (e_0 \pm e_3)$ tangent to the light-cone are combined with the orthogonal pair $e_k$, $k = 1, 2$. The corresponding coordinates are usually called (see, e.g. [21]) light-cone coordinates $x^\mu = (x^+, x^-, x^\perp)$

$$x^\pm := \frac{1}{\sqrt{2}} (x^0 \pm x^3), \quad x^\perp := x^k, \quad k = 1, 2.$$ 

The non-zero components of the metric are $\eta_{++} = \eta_{-+} = -\eta_{11} = -\eta_{22} = 1$. The connection one-form in the light-cone basis is given as

$$A := A_+ dx^+ + A_- dx^- + A_k dx^k. \quad \text{(19)}$$

By definition, the Lagrangian of light-cone Yang-Mills mechanics follows from the corresponding Lagrangian of Yang-Mills theory if one supposes that the components of the connection one-form $A$ in (19) only depend on the light-cone “time variable” $x^+$

$$A_\pm = A_\pm (x^+), \quad A_k = A_k (x^+).$$

Substitution of this ansatz into the classical action (18) defines the Lagrangian of light-cone Yang-Mills mechanics

$$L := \frac{1}{2g^2} \left( F^a_{+-} F^a_{+-} + 2 F^a_{+k} F^a_{-k} - F^a_{12} F^a_{12} \right), \quad \text{(20)}$$

where $g$ is the “renormalized” coupling constant $g^2 = g^2_0 / (\text{Volume})$ and the light-cone components of the field-strength tensor are given by

$$F^a_{+-} := \frac{\partial A^a_+}{\partial x^+} + \Gamma^{abc} A^b_+ A^c, \quad \text{(16)}$$

and

$$F^a_{12} := \frac{\partial A^a}{\partial x^1} + \Gamma^{abc} A^b_1 A^c_2, \quad \text{(17)}$$

where $\Gamma^{abc}$ is the structure constant of the $SU(n)$ algebra.
\[ F_{+k} := \frac{\partial A_k}{\partial x^+} + \Gamma_{abc}^b A_+^c A_k^c, \]
\[ F_{-k} := \Gamma_{abc}^b A_+^c A_k^c, \]
\[ F_{ij} := \Gamma_{abc}^b A_+^c A_j^c, \quad i, j, k = 1, 2. \]

Therefore, (20) determines the \( SU(n) \) Yang-Mills light-cone mechanics as \( 4(n^2 - 1) \)-dimensional system with configuration coordinates \( A_\pm, A_k \) evolving with respect to the light-cone time \( \tau := x^+ \).

The Legendre transformation
\[ \pi^+_a := \frac{\partial L}{\partial A_+^a}, \quad \pi^-_a := \frac{\partial L}{\partial A_-^a} = \frac{1}{g^2} \left( A_+^a + \Gamma_{abc} A_+^b A_-^c \right), \]
\[ \pi^k_a := \frac{\partial L}{\partial A_k^a} = \frac{1}{g^2} \Gamma_{abc} A_-^b A_k^c \]
gives the canonical Hamiltonian
\[ H_C = \frac{g^2}{2} \pi^-_a \pi^-_a - \Gamma_{abc} A_+^b \left( A_\pm^a \pi^-_a + A_\pm^c \pi^k_a \right) + \frac{1}{2g^2} F_{12}^a F_{12}^a. \quad (21) \]

The non-vanishing Poisson brackets between the fundamental canonical variables are
\[ \{ A_\pm^a, \pi^\pm_b \} = \delta^b_a, \quad \{ A_\pm^a, \pi^k_b \} = \delta^k_a \delta^b_b. \]

The Hessian of the Lagrangian system (20) is degenerate, \( \det \left| \frac{\partial^2 L}{\partial A \partial A} \right| = 0 \), and as a result there are primary constraints whose computation by the algorithm of Section 3.1 gives
\[ \varphi_a^{(1)} := \pi^+_a = 0, \]
\[ \chi^a_k := g^2 \pi^k_a + \Gamma_{abc} A_-^b A_k^c = 0. \quad (22) \]

The non-vanishing Poisson brackets between these constraints are
\[ \{ \chi_i^a, \chi_j^b \} = 2 g^2 \Gamma_{abc} A_-^c \delta_{ij}. \]

According to the Dirac prescription, the presence of primary constraints affects the dynamics of the degenerate system. Now the generic evolution is governed by the total Hamiltonian
\[ H_T := H_C + U_a(\tau) \varphi_a^{(1)} + V^a_k(\tau) \chi^a_k, \]
where the Lagrange multipliers \( U_a(\tau) \) and \( V^a_k(\tau) \) are unspecified functions of the light-cone time \( \tau \). Using this Hamiltonian the dynamical self-consistence of the primary constraints
may be checked. From the requirement of conservation of the primary constraints $\varphi^{(1)}_a$ it follows that
\[ 0 = \dot{\varphi}^{(1)}_a = \{\pi^+_a, H_T\} = f^{abc} \left( A^b_c \pi^-_c + A^a_k \pi^-_c \right). \] (24)
Therefore, there are three secondary constraints $\varphi^{(2)}_a$
\[ \varphi^{(2)}_a := f^{abc} \left( A^b_c - \pi^-_c + A^b c \right) = 0 \quad (25) \]
which obey the $SU(n)$ algebra
\[ \{\varphi^{(2)}_a, \varphi^{(2)}_b\} = f^{abc} \varphi^{(2)}_c. \]
The same procedure for the primary constraints $\chi^a_k$ gives the following self-consistency conditions
\[ 0 = \dot{\chi}^a_k = \{\chi^a_k, H_C\} - 2 g^2 f^{abc} V^b_k A^a_c. \]
A further issue, the identification of the first class constraints among the primary constraints $\chi^a_k$, depends on the rank of the structure group. Below we specify to the simplest special unitary group of rank one.

4.1 The $SU(2)$ structure group
Here we present the results of our computations performed for the case of $SU(2)$ algebra where the structure constants are given by the totally antisymmetric three dimensional Levi-Civita symbol, $f^{abc} = \epsilon^{abc}$. Constraints and their separation. Computation of the complete set of constraints, as described in Section 3.2, gives nine primary constraints $\varphi^{(1)}_a, \chi^a_k$ and three secondary constraints $\varphi^{(2)}_a$, in accordance with (22) and (25). Performing the separation of the primary constraints (23) we find two additional first-class constraints
\[ \psi_k := A^a_- \chi^a_k, \]
and four second class constraints
\[ \chi^{a \perp}_k := \chi^a_k - \frac{(A^b_- \chi^a_k) A^a_-}{(A^1_-)^2 + (A^2_-)^2 + (A^3_-)^2}. \]
The new first class constraints $\psi_i$ are abelian, $\{\psi_i, \psi_j\} = 0$, and also have zero Poisson brackets with all other constraints, while the second class constraints $\chi^{a \perp}_k$ have the following non-zero Poisson bracket relations
\[ \{\chi^{a \perp}_i, \chi^{b \perp}_j\} = 2 g^2 \epsilon^{abc} A^c_- \delta_{ij}, \]
\[ \{\varphi^{(2)}_a, \chi^{a \perp}_k\} = \epsilon^{abc} \chi^{c \perp}_k. \]
Summarizing, there are 8 functionally independent first-class constraints $\varphi^{(1)}_a, \psi_k, \varphi^{(2)}_a$ and 4 second-class constraints $\chi^{a \perp}_k$. 

Generator of local symmetry transformations. The presence of two first class constraints \( \psi_i \) raises the question of the existence of new local symmetries as well as the expected \( SU(2) \) gauge symmetry. To clarify this point we construct the corresponding generator of local symmetry transformation following Section 3.2. We start from the expression

\[
G = \sum_{a=1}^{3} \varepsilon_a^{(1)} \varphi_a^{(1)} + \sum_{i=1}^{2} \eta_i \psi_i + \sum_{a=1}^{3} \varepsilon_a^{(2)} \varphi_a^{(2)},
\]

(26)

with the eight light-cone time-dependent functions \( \varepsilon_a^{(1)}(\tau) \), \( \varepsilon_a^{(2)}(\tau) \) and \( \eta_i(\tau) \), then compute the functions \( \rho \) (see eq. (14)). Equation (16) reads now as

\[
(\varepsilon_a^{(2)} + \varepsilon_a^{(1)} - \eta_i A_i^a) \phi_a^{(2)} \sum \eta_i A_i^a = 0.
\]

Therefore expressing \( \varepsilon_a^{(1)} \) in terms of the functions \( \varepsilon_a^{(2)} \), the generator of local transformation takes the final form

\[
G = \left( -\varepsilon_a^{(2)} + \varepsilon_a^{(2)} \phi_a^{(1)} + \eta_i \psi_i + \varepsilon_a^{(2)} \phi_a^{(2)} \right).
\]

(27)

Analyzing the changes of the canonical coordinates \( A^a \) and \( \pi^a \) generated by (27) we find that the abelian subgroup of the 5-parameter local symmetry is in some sense “inherited” from the rigid conformal symmetry of initial Yang-Mills theory. But now, instead of the conformal symmetry, the light-cone \( SU(2) \) Yang-Mills mechanics has the \( SL(2,R) \) dynamical group of symmetry. Moreover, the group action is accompanied by the abelian transformations generated by two constraints \( \psi_i \). A detailed discussion of this symmetry realization will be given elsewhere.

Hamiltonian reduction to unconstrained system. Now that we have the generator of local transformation, we can address the question of finding a set of suitable coordinates part of which represent the invariants of these transformations. Solving this problem will let us project our system onto the constraint manifold and thus determine the unconstrained Hamiltonian system. We refer for details to [15], and here present the set of corresponding singlet variables (as an example of the solution of the second part of (Issue III)). We also give a result of subsequent implementation of a Hamiltonian reduction (Issue IV) of the “redundant” degrees of freedom associated to the symmetries generated by constraints \( \varphi_a^{(1)}, \varphi_a^{(2)} \) and \( \psi_a \).

Let us pass to a matrix notation: the \( 3 \times 3 \) matrix \( A_{ab} \) whose entries of the first two columns are \( A^a_i \) and the third column is composed by the elements \( A^a_3 \). Now one can verify that the elimination of local degrees of freedom associated with the three constraints \( \varphi_a^{(2)} \) can be achieved by using the polar representation [22]

\[
A = OS
\]

where \( S \) is a positive definite \( 3 \times 3 \) symmetric matrix and the orthogonal matrix \( O \) is parameterized by three Euler angles. It turns out that these three angles represent the pure gauge degrees of freedom corresponding to the constraints \( \varphi_a^{(2)} \).
Towards an algorithmisation of the Dirac constraint formalism

To find the gauge degrees connected with the remaining two abelian constraints \( \psi_1, \psi_2 \) one can pass to a principal axes representation for the symmetric matrix \( S \)

\[
S = R^T \text{diag} (q_1, q_2, q_3) R
\]

with the orthogonal matrix \( R(\chi_1, \chi_2, \chi_3) \) given in terms of the Euler angles \((\chi_1, \chi_2, \chi_3)\). Now again it turns out that the two angles \( \chi_1 \) and \( \chi_2 \) are pure gauge degrees of freedom. Solving for the remaining second class constraints \( \chi^a \perp \) leads to an unconstrained system which represents a free particle or, considering the complex solutions to the second class constraints, to a more interesting model, the so-called conformal mechanics. In this case the diagonal variable \( q_1 \) and the angular variable \( \chi_3 \) together with the corresponding conjugate momenta \( p_1 \) and \( p_{\chi_3} \) are two unconstrained canonical pairs and their dynamics is governed by the reduced Hamiltonian

\[
H = \frac{g^2}{2} \left( p_1^2 + \frac{p_{\chi_3}^2}{4q_1} \right), \tag{28}
\]

which is a projection of the canonical Hamiltonian \((21)\) to the constraints shell. Finally, noting that \( p_{\chi_3} \) is a constant of motion, the Hamiltonian \((28)\) coincides with the Hamiltonian of conformal mechanics with the coupling constant \( p_{\chi_3}^2/4 \).

5 Concluding comments

In this paper we have raised several issues for a constrained mechanical systems which require computational realization. We described how using the Gröbner basis technique the computation and separation of the complete set of constraints as well as the construction of the local gauge transformations can be achieved in degenerate mechanical models whose Lagrangians are polynomials in coordinates and velocities. The remaining challenges, namely, the construction of a basis for singlet (gauge-invariant) variables as well as the subsequent Hamiltonian reduction still needs algorithmisation. However, a first step in this direction also has been performed. In systems with first-class constraints the configuration space should be factorized by the local symmetry group in order to find a gauge invariant basis. The infinitesimal structure of a local symmetry group is encoded in the generator of gauge transformations, and we have shown that its construction allows an effective algorithmisation.

As an example of the effectiveness of the proposed algorithms light-cone Yang-Mills mechanics with the \( SU(2) \) structure group was analysed in details: we determined and separated constraints, constructed a local invariance transformation and found the equivalent unconstrained Hamiltonian system.

For the \( SU(2) \) light-cone mechanics the computations with our implementation in Maple 10, which is an improved and extended version of that given in [12], takes about 1 minute on a machine with a 1.7 GHz processor. This uses the standard Gröbner package in the Maple library. Unfortunately, recent extensions of the Maple Gröbner bases facilities with
the packages Gb and Fgb developed by J.C. Faugère [23] do not improve on the standard package. Gb is slower for our problems while Fgb cannot deal with the parametric coefficients. For the same reason we cannot use our software GINV [24] to implement the involutive algorithms [14] for involutive or/and Gröbner bases. Manipulation with parametric coefficients is essential for the Dirac formalism due to the presence of physical parameters (e.g. masses, coupling constants) in the initial Lagrangian, the Lagrange multipliers in the total Hamiltonian (4) and the time-dependent functions in the generator (12) of local symmetry transformations.

Consideration of light-cone mechanics with $n \geq 3$ is under current study. Here we note only that a recent paper [25] on geodesic motion on the $SU(3)$ group provides us with a useful parametrization suitable for this investigation.

Acknowledgments

The authors are indebted to M. Lavelle, D. McMullan and D. Mladenov for helpful discussions concerning this work. The presented research was partially supported by grant No.04-01-00784 from the Russian Foundation for Basic Research and grant 5362.2006.2 from the Ministry of Education and Science of the Russian Federation.

References

[1] Dirac P.A.M.: Generalized Hamiltonian Dynamics, Canad. J. Math. 2, 129–148, 1950; Lectures on Quantum Mechanics, Belfer Graduate School of Science, Monographs Series, Yeshiva University, New York, 1964.
[2] Sundermeyer K.: Constrained Dynamics, Lecture Notes in Physics vol.169, Springer-Verlag, Berlin Heidelberg New York, 1982.
[3] Hanson E.J., Regge T. and Teitelboim C.: Constrained Hamiltonian Systems, Accademia Nazionale dei Lincei, 1976.
[4] Gitman D.M. and Tyutin I.V.: Quantization of Fields with Constraints, Springer-Verlag, Berlin Heidelberg, 1990.
[5] Henneaux M. and Teitelboim C.: Quantization of Gauge Systems, Princeton University Press, Princeton, New Jersey, 1992.
[6] Hurtley D.H., Tucker R.W. and Tuckey P.: Constrained Hamiltonian Dynamics and Exterior Differential Systems. J. Phys. A. 24, 5252–5265, 1991.
[7] Seiler W.M. and Tucker R.W.: Involution and Constrained Dynamics I: the Dirac Approach. J. Phys. A. 28, 4431–4451, 1995.
[8] Seiler W.M.: Involution and Constrained Dynamics. II: the Faddeev-Jackiw Approach, J. Phys. A. 28, 7315–7331, 1995.
[9] Pommaret J.F.: Partial Differential Equations and Group Theory. New Perspectives for Applications, Kluwer, Dordrecht, 1994.
[10] Calmet J., Hausdorf M. and Seiler W.M.: A Constructive Introduction to Involution, in: R.Akerkar (Ed.) International Symposium on Applications of Computer Algebra-ISACA 2000, Allied Publishers, New Delhi, 33–50, 2001.
[11] Buchberger B. and Winkler F. (Eds.): Gröbner Bases and Applications, Cambridge University Press, 1998.
[12] Gerdt V.P. and Gogilidze S.A.: Constrained Hamiltonian Systems and Gröbner Bases, in: V.G.Ganzha, E.W.Mayr and E.V.Vorozhtsov (Eds.), Computer Algebra in Scientific Computing, Springer-Verlag, Berlin, 138-146, 1999.
[13] Gerdt V.P., Blinkov Yu.A.: Involutive Bases of Polynomial Ideals. Math. Comp. Simul. 45, 519–542, 1998, arXiv:math.AC/9912027
Towards an algorithmisation of the Dirac constraint formalism

[14] Gerdt V.P.: Involutive Algorithms for Computing Gröbner Bases, in: S.Cojocaru, G.Pfister and V.Ufnarovski (Eds), Computational Commutative and Non-Commutative algebraic geometry, NATO Science Series, IOS Press, 199–225, 2005, arXiv:math.AC/0501111.  
[15] Gerdt V.P., Khvedelidze A.M. and Mladenov D.M.: Light-cone SU(2) Yang-Mills Theory and Conformal Mechanics, arXiv:hep-th/0210022.  
[16] Gogilidze S.A, Pervushin V.N. and Khvedelidze A.M.: Reduction in Systems with Local Symmetry, Phys. Part. Nucl. 30, 160–208, 1999.  
[17] Khvedelidze A.M.: On the Hamiltonian Formulation of Gauge Theories in Terms of Physical Variables, J. Math. Sci. 119, 513–555, 2004.  
[18] Cox D., Little J. and O’Shea D.: Ideals, Varieties and Algorithms. 2nd Edition, Springer-Verlag, New York, 1996.  
[19] Gogilidze S.A., Sanadze V.V., Surovtsev Y.S. and Tkebuchava F.G.: Local Symmetries in Systems with Constraints, J. Phys. A 27 6509–6524, 1994.  
[20] Becker T. and Weispfenning V.: Gröbner Bases. A Computational Approach to Commutative Algebra. Graduate Texts in Mathematics 141, Springer-Verlag, New York, 1993.  
[21] Brodsky S.J., Pauli H.-C. and Pinsky S.S.: Quantum Chromodynamics and Other Field Theories on the Light-cone, Phys. Rep. 301, 299–486, 1998.  
[22] Zelobenko D.R.: Compact Lie Groups and Their Representations. Translations of Mathematical Monographs, Vol. 40, AMS, 1978.  
[23] http://fgbrs.lip6.fr/salsa/Software/  
[24] http://invo.jinr.ru  
[25] Gerdt V., Horan R., Khvedelidze A., Lavelle M., McMullan D. and Palii Y.: On the Hamiltonian Reduction of Geodesic Motion on SU(3) to SU(3)/SU(2), J. Math. Phys. 47, 2006, arXiv:hep-th/0511245