DP color functions versus chromatic polynomials (II)

Meiqiao Zhang  F  |  Fengming Dong  F

National Institute of Education, Nanyang Technological University, Singapore, Singapore

Correspondence
Fengming Dong, National Institute of Education, Nanyang Technological University, Singapore, Singapore.
Email: fengming.dong@nie.edu.sg and donggraph@163.com

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Abstract
For any connected graph $G$, let $P(G, m)$ and $P_{DP}(G, m)$ denote the chromatic polynomial and Dvořák and Postle (DP) color function of $G$, respectively. It is known that $P_{DP}(G, m) \leq P(G, m)$ holds for every positive integer $m$. Let $DP_\approx$ (resp., $DP_<$) be the set of graphs $G$ for which there exists an integer $M$ such that $P_{DP}(G, m) = P(G, m)$ (resp., $P_{DP}(G, m) < P(G, m)$) holds for all integers $m \geq M$. Determining the sets $DP_\approx$ and $DP_<$ is an important open problem on the DP color function. For any edge set $E_0$ of $G$, let $E_\ell(G_0)$ be the size of a shortest cycle $C$ in $G$ such that $\cap_{e \in E_0} E(C) \setminus E_0$ is odd if such a cycle exists, and $E_\ell(G_0) = \infty$ otherwise. We denote $E_\ell(G_0)$ as $E_\ell(e)$ if $E_0 = \{e\}$. In this paper, we prove that if $G$ has a spanning tree $T$ such that $E_\ell(e)$ is odd for each $e \in E(G) \setminus E(T)$, the edges in $E(G) \setminus E(T)$ can be labeled as $e_1, e_2, \ldots, e_q$ with $E_\ell(e_i) \leq E_\ell(e_{i+1})$ for all $1 \leq i \leq q - 1$ and each edge $e_i$ is contained in a cycle $C_i$ of size $E_\ell(e_i)$ with $E(C_i) \subseteq E(T) \cup \{e_j : 1 \leq j \leq i\}$, then $G$ is a graph in $DP_\approx$. As a direct application, all plane near-triangulations and complete multipartite graphs with at least three partite sets belong to $DP_\approx$. We also show that if $E^*$ is a set of edges in $G$ such that $E_\ell(G(E^*))$ is even and $E^*$ satisfies certain conditions, then $G$ belongs to $DP_<$. In particular, if $E_\ell(G(E^*)) = 4$, where $E^*$ is a set of edges between two disjoint vertex subsets of $G$, then $G$ belongs to $DP_<$. Both results extend known ones by Dong and Yang.
1 | INTRODUCTION

1.1 | Proper coloring, list coloring, and DP coloring

In this article, we consider simple graphs only. For any graph \( G \), let \( V(G) \) and \( E(G) \) be the vertex set and edge set of \( G \), respectively. For any two disjoint subsets \( V_1 \) and \( V_2 \) of \( V(G) \), let \( E_G(V_1, V_2) \) be the set of the edges \( uv \in E(G) \), where \( u \in V_1 \) and \( v \in V_2 \). Let \( d_G(v_1, v_2) \) denote the length of a shortest path in \( G \) that connects a vertex in \( V_1 \) to a vertex in \( V_2 \). In particular, write \( d_G(V_1, V_2) \) as \( d_G(v_1, v_2) \) if \( V_1 = \{v_1\} \) and \( V_2 = \{v_2\} \). For any nonempty subset \( V_0 \) of \( V(G) \), let \( G[V_0] \) denote the subgraph of \( G \) induced by \( V_0 \). For any \( A \subset E(G) \), let \( G(A) \) be the spanning subgraph of \( G \) with edge set \( A \), denote by \( c(A) \) the number of components of \( G(A) \), and let \( G - A = G(E(G) \setminus A) \). If \( A \neq \emptyset \), then let \( V(A) \) be the set of vertices in \( G \) that are incident with at least one edge in \( A \), and let \( G[A] \) be the subgraph of \( G \) with vertex set \( V(A) \) and edge set \( A \). For any \( u \in V(G) \), let \( N_G(u) \) (or simply \( N(u) \)) be the set of neighbors of \( u \) in \( G \), and let \( G - u = G[V(G) \setminus \{u\}] \).

Denote the set of positive integers by \( \mathbb{N} \). For any \( m \in \mathbb{N} \), let \( [m] = \{1, \ldots, m\} \). For any graph \( G \), a proper coloring of \( G \) is a mapping \( c : V(G) \rightarrow \mathbb{N} \), such that \( c(u) \neq c(v) \) for all \( uv \in E(G) \). For any positive integer \( m \), a proper \( m \)-coloring of \( G \) is a proper coloring \( c \) with \( c(v) \in [m] \) for all \( v \in V(G) \). Then, the chromatic polynomial \( P(G, m) \) of \( G \) is a function that counts the number of proper \( m \)-colorings of \( G \) for each \( m \in \mathbb{N} \). The chromatic polynomial was originally designed by Birkhoff in [1] as a tool to attack the Four-Color Conjecture, but later gained unique research significance because of its elegant properties, see [2, 3, 12, 13] for reference.

To generalize proper coloring, Vizing [15] and Erdős, Rubin, and Taylor [8] independently introduced the notion of list coloring. For any graph \( G \), a list assignment \( L \) of \( G \) is a mapping from \( V(G) \) to the power set of \( \mathbb{N} \), and an \( L \)-coloring of \( G \) is a proper coloring \( c \) with \( c(v) \in L(v) \) for all \( v \in V(G) \). Denote the number of \( L \)-colorings of \( G \) by \( P(G, L) \).

We say \( L \) is an \( m \)-list assignment of \( G \) if \( |L(v)| = m \) holds for all \( v \in V(G) \). Then the list color function \( P_l(G, m) \) of \( G \) counts the minimum value of \( P(G, L) \) among all \( m \)-list assignments \( L \) of \( G \) for each \( m \in \mathbb{N} \). Obviously, \( P_l(G, m) \leq P(G, m) \) holds for each \( m \in \mathbb{N} \). It is now known that \( P_l(G, m) = P(G, m) \) holds whenever \( m \geq |E(G)| - 1 \) (see [5]). While the list color function of some graph might not be a polynomial [6], the list color function \( P_l(G, m) \) of any graph \( G \) inherits all the nice properties of the chromatic polynomial of \( G \) when \( m \) is sufficiently large. See [14, 16] for some studies on list color functions.

To make breakthroughs in list coloring, Dvořák and Postle [7] recently defined correspondence coloring, or DP-coloring. The formal definition is as follows.

For any graph \( G \), a cover of \( G \) is an ordered pair \( \mathcal{H} = (L, H) \), where \( H \) is a graph and \( L \) is a mapping from \( V(G) \) to the power set of \( V(H) \) satisfying the conditions below:

- \( V(H) \) is partitioned into \( |V(G)| \) subsets \( \{L(u) : u \in V(G)\} \),
- \( \mathcal{H} \) is a spanning subgraph of \( G \),
- for any nonempty subset \( V_1 \) of \( V(G) \), let \( \mathcal{H}_1 \) be the subgraph of \( \mathcal{H} \) induced by \( V_1 \), and let \( \mathcal{H}_1 \) be the set of neighbors of \( v \) in \( \mathcal{H} \) for all \( v \in V(G) \).
• for every \( u \in V(G) \), \( H[L(u)] \) is a complete graph,
• if \( u \) and \( v \) are not adjacent in \( G \), then \( E_H(L(u), L(v)) = \emptyset \), and
• for each edge \( uv \in E(G) \), \( E_H(L(u), L(v)) \) is a matching.

For any cover \( \mathcal{H} = (L, H) \) of \( G \), \( \mathcal{H} \) is \( m \)-fold if \( |L(v)| = m \) for all \( v \in V(G) \), and \( \mathcal{H} \) is full if \( |E_H(L(u), L(v))| = |L(u)| = |L(v)| \) holds for each edge \( uv \in E(G) \). An \( \mathcal{H} \)-coloring of \( G \) is an independent set \( I \) in \( H \) with \( |I| = |V(G)| \). Obviously, any \( \mathcal{H} \)-coloring \( I \) of \( G \) has the property that \( I \cap L(v) = 1 \) for each \( v \in V(G) \). Denote the number of \( \mathcal{H} \)-colorings of \( G \) by \( P(G, \mathcal{H}) \).

The DP color function \( P_{DP}(G, m) \) of \( G \), introduced by Kaul and Mudrock [10] in 2019, counts the minimum value of \( P(G, \mathcal{H}) \) among all \( m \)-fold covers \( \mathcal{H} \) of \( G \) for each \( m \in \mathbb{N} \). Note that \( P_{DP}(G, m) \leq P_i(G, m) \) holds for each \( m \in \mathbb{N} \). Therefore, for each \( m \in \mathbb{N} \),

\[
P_{DP}(G, m) \leq P_i(G, m) \leq P(G, m).
\]

(1)

It is known that the two inequalities in (1) can hold with equality simultaneously. For example, Kaul and Mudrock [10] proved that \( P_{DP}(G, m) = P(G, m) \) holds for all \( m \in \mathbb{N} \) when \( G \) is a chordal graph. However, unlike list color functions, not all DP color functions become the same as chromatic polynomials. In [10], it was shown that for any graph \( G \) whose girth is even, there exists an \( N \in \mathbb{N} \), such that \( P_{DP}(G, m) < P(G, m) \) for all integers \( m \geq N \). Therefore, a simple characterization of the two sets \( DP_\geq \) and \( DP_\leq \) becomes a research focus in the study on DP color functions, where

• \( DP_\geq \) is the set of graphs \( G \) for which there exists an integer \( M \) such that \( P_{DP}(G, m) = P(G, m) \) holds for all integers \( m \geq M \), and
• \( DP_\leq \) is the set of graphs \( G \) for which there exists an integer \( M \) such that \( P_{DP}(G, m) < P(G, m) \) holds for all integers \( m \geq M \).

So far it is still unknown if there exists a graph \( G \) such that \( G \notin DP_\geq \) and \( G \notin DP_\leq \). Thus, a characterization of the graphs in \( DP_\geq \) or \( DP_\leq \) does not necessarily guarantee a characterization of the graphs in the other set.

In this paper, we shall introduce our new results on \( DP_\geq \) and \( DP_\leq \).

1.2 | Known results

Throughout this paper, we need only to consider connected graphs because for any disconnected graph \( G \) with components \( G_1, \ldots, G_k \),

\[
P_{DP}(G, m) = \prod_{i=1}^{k} P_{DP}(G_i, m).
\]

(2)

In this subsection, we introduce the known graphs contained in sets \( DP_\geq \) and \( DP_\leq \), respectively.

For any graph \( G \), let \( \mathcal{H}_{G,m} = (L_{G,m}, H_{G,m}) \) be the special full \( m \)-fold cover of \( G \) such that \( L_{G,m}(u) = \{(u, i) : i \in [m]\} \) for each \( u \in V(G) \) and \( E_{H_{G,m}}(L_{G,m}(u), L_{G,m}(v)) = \{(u, i)(v, i) : i \in [m]\} \) for each edge \( uv \in E(G) \). Obviously, \( P(G, \mathcal{H}_{G,m}) = P(G, m) \) for all \( m \in \mathbb{N} \). Let \( DP^* \) denote the set of graphs \( G \) for which there
exists $M \in \mathbb{N}$ such that for every $m$-fold cover $\mathcal{H} = (L, H)$ of $G$, if $H \not\cong H_{G,m}$, then $P(G, \mathcal{H}) > P(G, m)$ holds for all integers $m \geq M$. Clearly, $DP^* \subseteq DP_\infty$, but whether $DP^* = DP_\geq$ or not is currently unknown.

On the one hand, Mudrock and Thomason [11] showed that each graph with a dominating vertex belongs to $DP_\infty$. Actually they proved that each graph with a dominating vertex belongs to $DP^*$. Dong and Yang [4] then extended their conclusion to a large set of connected graphs (see Theorem 1.1).

For any edge $e$ in $G$, we define the girth of $e$ to be the size of a shortest cycle in $G$ that contains $e$, and denote it by $\ell_G(e)$. In particular, if $e$ is a cut edge, then define $\ell_G(e) = \infty$. Further, let $C_G(e)$ denote the set of cycles in $G$ that contains $e$ and has size $\ell_G(e)$. Clearly, $C_G(e) = \emptyset$ if $\ell_G(e) = \infty$.

**Theorem 1.1** (Dong and Yang [4]). Let $G$ be a graph with a spanning tree $T$. If for each edge $e$ in $E(G) \setminus E(T)$, $\ell_G(e)$ is odd and there exists $C \in C_G(e)$ such that $\ell_G(e') < \ell_G(e)$ for each $e' \in E(C) \setminus (E(T) \cup \{e\})$, then $G \in DP^*$ and hence $G \in DP_\infty$.

On the other hand, many graphs belonging to $DP_\infty$ were found. Kaul and Mudrock [10] discovered the fact that for any graph $G$ with an edge $e$, if $P(G - e, m) < mP(G, m)/(m - 1)$, then $P_{DP}(G, m) < P(G, m)$ holds, which implies that every graph whose girth is even belongs to $DP_\infty$. The latter conclusion was extended to the following one.

**Theorem 1.2** (Dong and Yang [4]). Graph $G$ belongs to $DP_\infty$ if $G$ contains an edge whose girth is even.

An edge gluing of two vertex disjoint graphs is a graph obtained by taking one edge from each of the two graphs and identifying the two edges as the same. Note that it has been proved in [4, 9] that the edge gluings of any graph in $DP_\infty$ and an arbitrary graph also belong to the set $DP_\infty$. Thus, if graph $G$ is obtained by edge-gluing 3-cycles to every edge of some graph in $DP_\infty$, then $G$ is a graph in $DP_\infty$ that does not satisfy the requirement in Theorem 1.1. Therefore, Theorem 1.2 cannot be a characterization of all the graphs in $DP_\infty$ (also shown in [4]).

### 1.3 New results

In this article, we will further extend Theorems 1.1 and 1.2. We first give the definition of a set of graphs.

A graph $G$ is called $DP$-good if $G$ has a spanning tree $T$ and a labeling $e_1, \ldots, e_q$ of the edges in $E(G) \setminus E(T)$, where $q = |E(G)| - |E(T)|$, such that $\ell_G(e_1) \leq \cdots \leq \ell_G(e_q)$ and for each $i \in [q]$, $\ell_G(e_i)$ is odd and $E(C_i) \subseteq E(T) \cup \{e_t : t \in [i]\}$ holds for some $C_i \in C_G(e_i)$. Obviously, the $q$ cycles $C_1, \ldots, C_q$ are pairwise distinct.

It is clear that any graph satisfying the condition in Theorem 1.1 is $DP$-good, but the converse statement may not be true. For example, the graph $G$ shown in Figure 1 is $DP$-good, but $G$ does not satisfy the requirement in Theorem 1.1. Assume that $T$ is a spanning tree of $G$ such that the condition in Theorem 1.1 is satisfied. Then, for any two edges $f_1, f_2 \in E(G) \setminus E(T)$ with $\ell_G(f_1) = \ell_G(f_2)$, there exist cycles $C_1 \in C_G(f_1)$ and $C_2 \in C_G(f_2)$ such that $f_i \not\in C_{3-i}$ for $i = 1$ or 2. It follows that $T$ must contain exactly two edges of each 3-cycle in $G$, as each edge with girth 3 is contained in exactly one 3-cycle. As a result, edge $e$ and at least one edge in $\{f, h\}$...
are not contained in \( T \). But, as \( \ell_G(e) = \ell_G(f) = \ell_G(h) = 5 \) and \( e \) is contained in the only cycle in \( C_G(f) \) (\( = C_G(h) \)), the requirement in Theorem 1.1 cannot be met, a contradiction. Hence DP-good graphs indeed form a larger set.

Moreover, the following theorem shows that each DP-good graph belongs to \( DP^* \).

**Theorem 1.3.** Every DP-good graph is in \( DP^* \).

As an immediate consequence of Theorem 1.3, Corollary 1.4 suggests that many special sets of graphs are contained in \( DP^* \), such as chordal graphs, complete multipartite graphs with at least three partite sets, and plane near-triangulations.

**Corollary 1.4.** Let \( G \) be a graph with vertex set \( \{v_i : i = 0, 1, ..., n\} \), where \( n \geq 1 \). If for each \( i \in [n] \), the set \( N(v_i) \cap \{v_j : 0 \leq j \leq i - 1\} \) is not empty and the subgraph of \( G \) induced by this vertex set is connected, then \( G \) is in \( DP^* \).

On the other hand, to extend Theorem 1.2, we shall first generalize the definition of the girth of an edge to the girth of an edge set. Given any subset \( E_0 \) of \( E(G) \), let \( C_G(E_0) \) be the set of the shortest cycles \( C \) in \( G \) such that \( E(C) \cap E_0 \) is odd (i.e., \( |E(C) \cap E_0| \) is odd and \( |E(C)\vert \leq |E(C')\vert \) holds for each cycle \( C' \) in \( G \) whenever \( E(C) \cap E_0 \) is odd). Then the girth of \( E_0 \), denoted by \( \ell_G(E_0) \), is defined to be the size of any cycle in \( C_G(E_0) \) if this set is nonempty, and \( \ell_G(E_0) = \infty \) otherwise. Obviously, \( \ell_G(E_0) < \infty \) if and only if \( G \) contains a cycle \( C \) such that \( |E(C) \cap E_0| \) is odd, and if \( E_0 = \{e\} \), then \( C_G(\{e\}) = C_G(e) \) and \( \ell_G(\{e\}) = \ell_G(e) \).

Let \( E^* \) be a set of edges in \( G \). Assume that each edge \( e \) in \( E^* \) is assigned a direction \( \vec{e} \) and only the edges in \( E^* \) are assigned directions. Let \( \overrightarrow{E^*} \) be the set of directed edges \( \vec{e} \) for all \( e \in E^* \). Then for any cycle \( C \) in \( G \), we say the directed edges in \( \overrightarrow{E^*} \) are balanced on \( C \) if \( |E(C) \cap E^*| \) is even and exactly half of the edges in \( E(C) \cap E^* \) are oriented clockwise along \( C \), and unbalanced otherwise. Obviously, the directed edges of \( \overrightarrow{E^*} \) are balanced on \( C \) when \( E(C) \cap E^* = \emptyset \), and unbalanced on \( C \) if \( |E(C) \cap E^*| \) is odd. Examples of cycles on which directed edges of \( \overrightarrow{E^*} \) are balanced or unbalanced are shown in Figure 2A,B, respectively, where \( E(C) \cap E^* = \{e_1, e_2, e_3, e_4\} \).

We now introduce the second main result in this article.
Theorem 1.5. Let $G$ be a connected graph and $E^*$ be a set of edges in $G$. Assume that

(i) $\ell_G(E^*)$ is even; and
(ii) there exists a way to assign a direction $\vec{e}$ for each edge $e \in E^*$ such that the directed edges in $\vec{E}^* = \{\vec{e} : e \in E^*\}$ are balanced on each cycle $C$ of $G$ with $|E(C)| < \ell_G(E^*)$.

Then $P(G, m) - P_{DP}(G, m) \geq \Omega \left( m^{V(G) - \ell_G(E^*) + 1} \right)$ holds, and hence $G \in DP_c$.

The following Corollary 1.6 of Theorem 1.5 further introduces a set of graphs in $DP_c$, which includes the graphs determined by Theorem 1.2.

Corollary 1.6. Let $G$ be a connected graph and let $E^* \subseteq E_G(V_1, V_2)$, where $V_1$ and $V_2$ are disjoint subsets of $V(G)$ with $V_1 \cup V_2 \neq V(G)$. Assume that

(i) $\ell_G(E^*)$ is even; and
(ii) for each cycle $C$ in $G$ with $|E(C) \cap E^*| > 0$, either $|E(C)| \geq \ell_G(E^*)$ or no component of the subgraph $C - (E(C) \cap E^*)$ is a $(v_1, v_2)$-path for some $v_1 \in V_1$ and $v_2 \in V_2$.

Then $P(G, m) - P_{DP}(G, m) \geq \Omega \left( m^{V(G) - \ell_G(E^*) + 1} \right)$ holds, and hence $G \in DP_c$.

Note that a special case of Corollary 1.6 happens when $|\ell_G(E^*)| = 4$. In this case, condition (ii) is redundant. Further, the next corollary of Corollary 1.6 can be applied more easily.

Corollary 1.7. Let $G$ be a connected graph and let $E^* = E_G(V_1, V_2)$, where $V_1$ and $V_2$ are disjoint subsets of $V(G)$ with $V_1 \cup V_2 \neq V(G)$. If there exists $C \in C_G(E^*)$ such that $|E(C) \cap E^*| = 1$, then $P(G, m) - P_{DP}(G, m) \geq \Omega \left( m^{V(G) - \ell_G(E^*) + 1} \right)$ holds, and hence $G \in DP_c$.

It can be verified that neither of the two graphs in Figure 3 contains an edge whose girth is even, but by Corollary 1.7, both of them belong to $DP_c$ by taking $V_1 = \{u_i : i = 1, 2, 3\}$ and $V_2 = \{v_i : i = 1, 2, 3\}$.

We will first introduce some notation and fundamental results on $m$-fold covers in Section 2. Then, the proofs of Theorem 1.3, Corollary 1.4, Theorem 1.5, and Corollary 1.6 will be given in Sections 3 and 4. Finally, in Section 5, we will apply Corollary 1.6 to determine some sets of plane graphs belonging to $DP_c$.
2
NOTATION AND PRELIMINARY FACTS ON AN
m-FOLD COVER

In this section, we introduce some notation and preliminary facts on an m-fold cover which will be applied in the proofs of Theorems 1.3 and 1.5.

Let \( G \) be a graph. By the definition, \( P_{DP}(G, m) \) is actually equal to the minimum value of \( P(G, \mathcal{H}) \) over all the full m-fold covers \( \mathcal{H} = (L_{G,m}, H) \) of \( G \). In what follows, we assume that \( \mathcal{H} = (L, H) \) is a full m-fold cover of \( G \) with \( L = L_{G,m} \).

For any edge \( e = uv \) in \( E(G) \), let
\[
X_e(G, \mathcal{H}) = E_H(L(u), L(v)) \setminus \{(u, i)(v, i) : i \in [m]\}
\]
and
\[
Y_e(G, \mathcal{H}) = \{i \in [m] : (u, i)(v, j) \in X_e(G, \mathcal{H})\} = \{i \in [m] : (u, j)(v, i) \in X_e(G, \mathcal{H})\}.
\]

Obviously, \( X_e(G, \mathcal{H}) \) and \( Y_e(G, \mathcal{H}) \) are equal if and only if \( e \) is horizontal with respect to \( \mathcal{H} \) and \( \mathcal{H} = (L, H) \) is a full m-fold cover of \( G \) because we can rename the vertices in \( L(u) \) for every vertex \( u \in V(G) \) to guarantee that \( E_H(L(u), L(v)) = \{(u, i)(v, i) : i \in [m]\} \) holds whenever \( uv \in E(T) \), during which the structure of graph \( H \) remains unchanged.

Let \( \mathcal{S}(\mathcal{H}) \) (simply \( \mathcal{S} \)) be the set of subsets \( S \) of \( V(H) \) with \( |S \cap L(v)| = 1 \) for each \( v \in V(G) \). Clearly, \( |S| = |V(G)| \) for each \( S \in \mathcal{S} \). For each \( U \subseteq V(G) \), let \( \mathcal{S}_U \) be the set of subsets \( S \) of \( V(H) \) such that \( |S \cap L(v)| = 1 \) for each \( v \in U \) and \( S \cap L(v) = \emptyset \) for each \( v \in V(G) \setminus U \).\footnote{FIGURE 3 Two graphs in \( DP_{<} \).} Clearly, \( |S| = |U| \) for each \( S \in \mathcal{S}_U \), and \( \mathcal{S} = \mathcal{S}_U \) when \( U = V(G) \).

For any subgraph \( G_0 \) of \( G \), let \( H_{G_0} \) be the subgraph of \( H \) with vertex set \( \cup_{u \in V(G_0)} L(u) \) and edge set \( \cup_{uv \in E(G_0)} E_H(L(u), L(v)) \). Let \( \mathcal{G}_{H}(G_0) \) be the set of graphs \( H_{G_0}[S] \) (i.e., the subgraph of \( H_{G_0} \) induced by \( S \)), where \( S \in \mathcal{S}_{V(G_0)} \), such that \( H_{G_0}[S] \cong G_0 \). Note that \( \mathcal{G}_{H}(G_0) \) may be empty, and \( H_{G_0}[S] \) is the induced subgraph \( H[S] \) whenever \( G_0 \) is an induced subgraph of \( G \). For each \( j \in [m] \), let \( S_j(G_0) = \{(v, j) : v \in V(G_0)\} \) and write \( H_{G_0}[S_j(G_0)] \) as \( H_j[G_0] \).

For each edge \( e = uv \in E(G) \), let \( \mathcal{S}_e \) be the set of \( S \in \mathcal{S} \) such that the two vertices in \( S \cap (L(u) \cup L(v)) \) are adjacent in \( H \). For each \( A \subseteq E(G) \), let \( \mathcal{S}_A = \bigcap_{e \in A} \mathcal{S}_e \). Then, by the inclusion–exclusion principle,
\[ P(G, \mathcal{H}) = \sum_{A \subseteq E(G)} (-1)^{|A|} |\mathcal{A}_A|, \]

which generalizes a well-known property of the chromatic polynomial that
\[ P(G, m) = \sum_{A \subseteq E(G)} (-1)^{|A|} m^{c(A)}. \]

For any graph \( F \), let \( B(F) \) be the set of bridges (i.e., cut edges) in \( F \), and let \( \tilde{B}(F) = E(F) \setminus B(F) \). Write \( \tilde{B}(G(A)) \) as \( \tilde{B}(A) \) for any \( A \subseteq E(G) \). The following properties hold, as proved in [4].

(i) For any \( A \subseteq E(G) \), if \( G_1, G_2, \ldots, G_{c(A)} \) are the components of \( G(A) \), then
\[ |\mathcal{A}_A| = \prod_{i=1}^{c(A)} |\mathcal{H}(G_i)|. \]

(ii) For any connected subgraph \( G_0 \) of \( G \), we have \( |\mathcal{G}_H(G_0)| \leq m \), where the equality holds if \( \tilde{B}(G_0) \cap S_G(\mathcal{H}) = \emptyset \) (i.e., \( \tilde{B}(G_0) \) does not contain sloping edges with respect to \( \mathcal{H} \)).

(iii) By Facts (i) and (ii), for each \( A \subseteq E(G) \), we have \( |\mathcal{A}_A| \leq m^{c(A)} \), where the equality holds if \( \tilde{B}(A) \cap S_G(\mathcal{H}) = \emptyset \).

(iv) Let \( \mathcal{E}(\mathcal{H}) \) (or simply \( \mathcal{E} \)) be the set of subsets \( A \) of \( E(G) \) such that \( \tilde{B}(A) \) contains at least one sloping edge with respect to \( \mathcal{H} \). Then Fact (iii) implies that
\[ P(G, \mathcal{H}) - P(G, m) = \sum_{A \in \mathcal{E}} (-1)^{|A|} (|\mathcal{A}_A| - m^{c(A)}). \]

(v) Fact (iii) also implies that for any \( k \in [n] \),
\[ \sum_{A \in \mathcal{E}} \sum_{c(A)=k} (-1)^{|A|} (|\mathcal{A}_A| - m^{c(A)}) \geq \sum_{A \in \mathcal{E}, c(A)=k} (|\mathcal{A}_A| - m^k) \]
and
\[ \sum_{A \in \mathcal{E}} \sum_{c(A)=k} (-1)^{|A|} (|\mathcal{A}_A| - m^{c(A)}) \leq \sum_{A \in \mathcal{E}, c(A)=k} (m^k - |\mathcal{A}_A|). \]

(vi) For any \( A \in \mathcal{E} \) and any sloping edge \( e \) in \( \tilde{B}(A) \), let \( G_1 \) be the component of \( G(A) \) containing \( e \). Then \( V(G_1) \geq c_G(e) \) and \( c(A) \leq |V(G)| - c_G(e) + 1 \), and \( A = c_G(e) \) whenever \( c(A) = |V(G)| - c_G(e) + 1 \).
Now we give the proof of Theorem 1.3.

Proof of Theorem 1.3. We need only to prove that there exists an $M \in \mathbb{N}$, such that whenever $m \geq M$, $P(G, \mathcal{H}) > P(G, m)$ holds for every full $m$-fold cover $\mathcal{H} = (L, H)$ of $G$ with $L = L_{G,m}$ and $H \not\equiv H_{G,m}$.

Suppose $n = |V(G)|$. As $G$ is DP-good, $G$ has a spanning tree $T$ and an edge labeling $e_1, ..., e_q$ of the edges in $E(G) \setminus E(T)$, such that $\ell_G(e_1) \leq \cdots \leq \ell_G(e_q)$ and for all $i \in [q]$, $\ell_G(e_i)$ is odd and $E(C_i) \not\subseteq E(T) \cup \{e_1, ..., e_i\}$ for some $C_i \in C_G(e_i)$.

Let $\mathcal{H} = (L, H)$ be a full $m$-fold cover of $G$ with $L = L_{G,m}$ and $H \not\equiv H_{G,m}$. We can further assume that all the edges in $E(T)$ are horizontal with respect to $\mathcal{H}$ and $S_G(\mathcal{H}) \neq \emptyset$. Then, every sloping edge $e$ in $G$ with respect to $\mathcal{H}$ is of odd girth as $S_G(\mathcal{H}) \subseteq E(G) \setminus E(T)$.

In the following, write $X_e(G, \mathcal{H})$ and $Y_e(G, \mathcal{H})$ simply as $X_e$ and $Y_e$ for any edge $e \in E(G)$. Let $r = \min\{\ell_G(e) : e \in S_G(\mathcal{H})\}$, and let $e_{k_1}, ..., e_{k_t}$ be all the sloping edges in $G$ with $\ell_G(e_{k_i}) = r$, where $k_1 < k_2 < \cdots < k_t$. Then $r$ is odd, $r \geq 3$ and $1 \leq t \leq q$. Let $\mathcal{X}_r = \bigcup_{i=1}^{t} X_{e_{k_i}}$. Then $|\mathcal{X}_r| = \sum_{i=1}^{t} |X_{e_{k_i}}| \geq 1$.

Recall that the cycles $C_1, ..., C_q$ are pairwise distinct. We first prove three claims.

Claim 1. $\sum_{i=1}^{t} |G_{\mathcal{H}}(C_{k_i})| \leq mt - |\bigcup_{i=1}^{t} Y_{e_{k_i}}|$, that is, $\sum_{i=1}^{t} (m - |G_{\mathcal{H}}(C_{k_i})|) \geq |\bigcup_{i=1}^{t} Y_{e_{k_i}}|$.

Proof. It suffices to prove the two facts below:

(i) $|G_{\mathcal{H}}(C_{k_i})| = m - |Y_{e_{k_i}}|$; and
(ii) if $t \geq 2$, then $|G_{\mathcal{H}}(C_{k_{p+1}})| \leq m - |Y_{e_{k_{p+1}}} \setminus (\bigcup_{i=1}^{p} Y_{e_{k_i}})|$ holds for any $p \in [t - 1]$. Since $E(C_{k_i}) \not\subseteq E(T) \cup \{e_1, ..., e_{k_i}\}$ and $\ell_G(e_1) \leq \cdots \leq \ell_G(e_{k_i}) = r$, $C_{k_i}$ contains exactly one sloping edge $e_{k_i}$. Thus for any $j \in [m]$, $H_j[C_{k_i} - \{e_{k_i}\}] \cong C_{k_i} - \{e_{k_i}\}$, and $H_j[C_{k_i}] \cong C_{k_i}$ if and only if $j \notin Y_{e_{k_i}}$. Hence Fact (i) holds.

Assume that $t \geq 2$. Then similarly, for any $p \in [t - 1]$, all the edges in $E(C_{k_{p+1}}) \setminus \{e_{k_i}, e_{k_2}, ..., e_{k_{p+1}}\}$ are horizontal as $E(C_{k_{p+1}}) \not\subseteq E(T) \cup \{e_1, e_2, ..., e_{k_{p+1}}\}$ and $\ell_G(e_1) \leq \cdots \leq \ell_G(e_{k_{p+1}}) = r$. Let $j \in Y_{e_{k_{p+1}}} \setminus (\bigcup_{i=1}^{p} Y_{e_{k_i}})$. Then, $H_j[C_{k_{p+1}} - \{e_{k_{p+1}}\}] \cong C_{k_{p+1}} - \{e_{k_{p+1}}\}$ but $H_j[C_{k_{p+1}}] \not\cong C_{k_{p+1}}$. Hence Fact (ii) holds and Claim 1 follows.

Claim 2. The following inequality holds:

$$\sum_{i=1}^{t} \left( m^{n-2} + 1 - |G_{E(C_{k_i})}| \right) \geq \frac{|\mathcal{X}_r|}{q} m^{n-r}. \quad (9)$$

Proof. Since $|Y_e| = |X_e|$ for every edge $e \in E(G)$, we have
\[
\left| \bigcup_{i=1}^{t} Y_{e_i} \right| \geq \max_{i \in [t]} |X_{e_i}| \geq \frac{1}{t} \sum_{i=1}^{t} |X_{e_i}| = \frac{1}{t} |\mathcal{F}| \geq \frac{1}{q} |\mathcal{F}|. \tag{10}
\]

Then, by (5) and Claim 1,
\[
\sum_{i=1}^{t} \left( m^{n-r+1} - |\mathcal{F}_{E(C_k)}| \right) = \sum_{i=1}^{t} \left( m^{n-r+1} - m^{n-r}|\mathcal{F}_{E(C_k)}| \right) \geq \left| \bigcup_{i=1}^{t} Y_{e_i} \right| m^{n-r} \geq \frac{|\mathcal{F}|}{q} m^{n-r}. \tag{11}
\]

**Claim 3.** The following inequality holds:
\[
\sum_{A \in \mathcal{E}} (-1)^{|A|} (|\mathcal{F}_A| - m^c(A)) \geq \frac{|\mathcal{F}|}{q} m^{n-r}. \tag{12}
\]

**Proof.** Recall that for any \( A \in \mathcal{E} \), \( \tilde{B}(A) \) contains a sloping edge \( e \), where \( \ell_G(e) \geq r \). Thus, by (vi) in Section 2, \( \ell_G(e) = r = |A| \) holds whenever \( c(A) = n - r + 1 \). Therefore,
\[
\sum_{c(A)=n-r+1} (-1)^{|A|} (|\mathcal{F}_A| - m^c(A)) = \sum_{A \in \mathcal{E}, |A|=r, c(A)=n-r+1} (-1)^{|A|} (|\mathcal{F}_A| - m^c(A)) = \sum_{A \in \mathcal{E}, |A|=r} (m^c(A) - |\mathcal{F}_A|), \tag{13}
\]

where the last equality holds as \( r \) is odd.

By (iii) in Section 2, \( m^c(A) \geq |\mathcal{F}_A| \) for any \( A \subseteq E(G) \). Then, since \( E(C_k) \in \mathcal{E}, |E(C_k)| = r \) and \( c(E(C_k)) = n - r + 1 \) for each \( i \in [t] \), by (14) and Claim 2, we have
\[
\sum_{c(A)=n-r+1} (-1)^{|A|} (|\mathcal{F}_A| - m^c(A)) \geq \sum_{i=1}^{t} \left( m^{n-r+1} - |\mathcal{F}_{E(C_k)}| \right) \geq \frac{|\mathcal{F}|}{q} m^{n-r}. \tag{14}
\]

The rest of the proof is basically the same as in the proof of Theorem 1.1 given in [4]. For completeness, we restate the proofs of Claims 4–7 here with slight changes.

**Claim 4.** For any subgraph \( G_0 \) of \( G \), if \( \ell_G(e) \leq r \) for each sloping edge \( e \) in \( G_0 \), then \( |\mathcal{F}_{\tilde{H}(G_0)}| \geq m - |\mathcal{F}_r| \).
Proof. Since $\ell_G(e) \leq r$ for each sloping edge $e$ in $G_0$, each sloping edge in $G_0$ belongs to $\{e_1, \ldots, e_k\}$. Thus, for every $j \in [m] \setminus \bigcup_{i=1}^{t} Y_{e_i}$, $H_j[G_0] \cong G_0$ holds, implying that

$$
\mathcal{G}_H(G_0) \geq m - \left| \bigcup_{i=1}^{t} Y_{e_i} \right| \geq m - \sum_{i=1}^{t} |Y_{e_i}| = m - \sum_{i=1}^{t} |X_{e_i}| = m - |\mathcal{X}|. 
$$

Hence Claim 4 holds.

Claim 5. For any $A \in \mathcal{E}$ with $c(A) = n - r$, we have $|\mathcal{F}_A| \geq (m - |\mathcal{X}|)m^{n-r-1}$.

Proof. Since $A \in \mathcal{E}, \tilde{B}(A)$ contains a sloping edge $e$ with $\ell_G(e) \geq r$. Thus, by (vi) in Section 2, $G(A)$ has a component $G_0$ with $e \in E(G_0)$ and $|V(G_0)| \leq r$. Moreover, as $c(A) = n - r$, $|V(G_0)| \leq r + 1$ holds, and for any other component $G'$ of $G(A)$, $G'$ is either an isolated vertex or an edge, and thus $|\mathcal{G}_H(G')| = m$. Hence by (5), it suffices to prove that $|\mathcal{G}_H(G_0)| \geq m - |\mathcal{X}|$.

If $G_0$ is 2-connected, then for every edge $e \in E(G_0), e \in \tilde{B}(A)$, and thus $\ell_G(e) \leq |V(G_0)| \leq r + 1$ by (vi) in Section 2. Further, for each sloping edge $e$ in $G_0$, $\ell_G(e) \leq r$ as $r + 1$ is even and $\ell_G(e)$ is odd. Hence $|\mathcal{G}_H(G_0)| \geq m - |\mathcal{X}|$ holds by Claim 4.

Otherwise, $G_0$ has exactly two blocks as $G_0$ contains a cycle $C$ with $|V(C)| \geq r$. Then, it is clear that the two blocks of $G_0$ are $G_0[V(C)]$ and an edge $f$, where $|V(C)| = r$ and $|\mathcal{G}_H(G_0[f])| = m$. As $G_0[V(C)]$ is 2-connected, for every edge $e \in E(G_0[V(C)])$, $\ell_G(e) \leq |V(C)| \leq r$ holds, and thus $|\mathcal{G}_H(G_0[V(C)])| \geq m - |\mathcal{X}|$ follows from Claim 4. Consequently, $|\mathcal{G}_H(G_0)| \geq m - |\mathcal{X}|$ and Claim 5 holds.

For any $s \in \mathbb{N}$ with $s \leq n - r$, let $\phi_s$ be the number of subsets $A \subseteq E(G)$ such that $c(A) = s, G(A)$ is not a forest, and $|A|$ is even.

Claim 6. The following inequality holds:

$$
\sum_{A \in \mathcal{E} \atop c(A) = n - r} (-1)^{|A|}(|\mathcal{F}_A| - m^{c(A)}) \geq -\phi_{n-r} |\mathcal{X}|m^{n-r-1}.
$$

Proof. By (7) and Claim 5,

$$
\sum_{A \in \mathcal{E} \atop c(A) = n - r} (-1)^{|A|}(|\mathcal{F}_A| - m^{c(A)}) \geq \sum_{A \in \mathcal{E} \atop c(A) = n - r \atop |A| \text{ is even}} (-1)^{|A|}m^{n-r-1} \geq -\phi_{n-r} |\mathcal{X}|m^{n-r-1}.
$$

Claim 7. For each $s \in [n - r - 1]$, we have
\[ \sum_{A \in \mathcal{E}, c(A) = s} (-1)^{|A|} (|A| - m^{c(A)}) \geq -\phi_s m^s. \quad (18) \]

**Proof.** By (7),

\[ \sum_{A \in \mathcal{E}, c(A) = s} (-1)^{|A|} (|A| - m^{c(A)}) \geq \sum_{A \in \mathcal{E}, c(A) = s} (|A| - m^s) \]
\[ \geq \sum_{A \in \mathcal{E}, c(A) = s, |A| \text{ is even}} (-m^s) \geq -\phi_s m^s. \quad (19) \]

Now we are going to prove the main result by recalling (6) that

\[ P(G, \mathcal{H}) - P(G, m) = \sum_{A \in \mathcal{E}} (-1)^{|A|} (|A| - m^{c(A)}). \quad (20) \]

By (vi) in Section 2 and Claims 3, 6, and 7, we have

\[ P(G, \mathcal{H}) - P(G, m) = \sum_{s=1}^{n-r+1} \sum_{A \in \mathcal{E}, c(A) = s} (-1)^{|A|} (|A| - m^{c(A)}) \]
\[ \geq \frac{|\mathcal{X}_s|}{q} m^{n-r} - \phi_{n-r} |\mathcal{X}_s| m^{n-r-1} \]
\[ - \sum_{s=1}^{n-r-1} \phi_s m^s \]
\[ \geq \frac{1}{q} m^{n-r} - \phi_{n-r} m^{n-r-1} - \sum_{s=1}^{n-r-1} \phi_s m^s, \quad (21) \]

where the last inequality holds when \( m \geq q\phi_{n-r} \). As \( q, \phi_1, ..., \phi_{n-r} \) are independent of the value of \( m \), there exists \( M_r \in \mathbb{N} \), such that \( P(G, \mathcal{H}) - P(G, m) > 0 \) for all \( m \geq M_r \). Let \( M = \max \{ M_r : 3 \leq r \leq n, r \text{ is odd} \} \). Then the result is proven. \( \square \)

The proof of Corollary 1.4 is given below.

**Proof of Corollary 1.4.** By Theorem 1.3, it suffices to prove the statement that \( G \) has a spanning tree \( T \) and a labeling \( e_1, e_2, ..., e_q \) of the edges in \( E(G) \setminus E(T) \) such that for each \( j \in [q], e_j \) is contained in a 3-cycle \( C_j \) of \( G \) with \( E(C_j) \subseteq E(T) \cup \{ e_t : t \in [j] \} \).

For the convenience of writing, let \( \Pi \) denote the above statement. We will prove statement \( \Pi \) by induction on \( n \). It holds obviously when \( n = 1 \) by the given condition.

Assume that statement \( \Pi \) holds whenever \( n < N \), where \( N \geq 2 \). Now consider the case that \( n = N \). By the given condition in Corollary 1.4, \( G \) has a vertex \( v_N \) such that \( G - v_N \) satisfies the given condition in the corollary and the subgraph \( G[N(v_N)] \) is connected.

By the inductive assumption, statement \( \Pi \) holds for \( G - v_N \). Let \( G' \) denote \( G - v_N \). Thus, \( G' \) has a spanning tree \( T' \) and a labeling \( e_1, e_2, ..., e_q' \) of the edges in \( E(G') \setminus E(T') \)
such that for each \( j \in [q'] \), \( e_j \) is contained in a 3-cycle \( C_j \) of \( G \) with \( E(C_j) \subseteq E(T') \cup \{ e_t : t \in [j] \} \).

Now let \( T \) be a spanning tree of \( G \) with \( E(T) = E(T') \cup \{ f \} \), where \( f = v_nv_w \) is any edge in \( G \) incident with \( v_n \). Since \( G[N(v_n)] \) is connected, \( G[N(v_n)] \) has a spanning tree, denoted by \( T_0 \). Then, the edges in \( E(G) \setminus \{ E(G') \cup \{ f \} \} \) can be ordered as a sequence \( e_{q'+1}, ..., e_q \), where \( q = q' + d_G(v_n) - 1 \) (i.e., \( |E(G) \setminus |E(T)| \) and \( e_j = v_nv_{s_j} \) for each \( j \in [q] \setminus [q'] \), such that \( d_{T_0}(v_w, v_{s_j}) \leq d_{T_0}(v_w, v_{s_{j+1}}) \) holds for all \( j \in [q - 1] \setminus [q'] \). An example of tree \( T_0 \) is shown in Figure 4.

Then it follows that for each \( j \in [q] \setminus [q'] \), \( e_j \) is contained in a 3-cycle \( C_j \) of \( G \) with \( E(C_j) \subseteq E(T) \cup \{ e_t : t \in [j] \} \). Thus, statement \( \Pi \) also holds for \( G \), and the result follows.

\[ \square \]

4 | PROOF OF THEOREM 1.5

We shall prove Theorem 1.5 in this section.

**Proof of Theorem 1.5.** Assume \( |V(G)| = n \) and \( E^* = \{ e_1, ..., e_k \} \), where \( k \geq 1 \).

If \( k = 1 \), then \( \ell_G(e_1) \) is even, and the result follows from Theorem 1.2 directly.

In the following, we assume that \( k \geq 2 \). Let \( r := \ell_G(E^*) \). By condition (ii) in Theorem 1.5, there exists a way to assign an orientation such that the directed edges in \( \overrightarrow{E^*} = \{ e_i : i \in [k] \} \) are balanced on every cycle \( C \) in \( G \) with \( E(C) < r \), where \( \overrightarrow{e_i} \) is the directed edge \((u_i, v_i)\) with tail \( u_i \) for each \( i \in [k] \).

For any positive integer \( m \), let \( \mathcal{H} = (L, H) \) be the \( m \)-fold cover of \( G \) defined below:

- \( L = L_{G,m} \);
- \( E_H(L(x), L(y)) = \{(x,i)(y,i) : i \in [m]\} \) for every edge \((x,y) \in E(G) \setminus E^* \); and
- \( E_H(L(u_i), L(v_i)) = \{(u_i,q)(v_i,q+1) : q \in [m-1] \} \cup \{(u_i,m)(v_i,1) \} \) for every edge \( e_i = u_iv_i \in E^* \).

Clearly, \( S_G(\mathcal{H}) = E^* \) (i.e., only edges in \( E^* \) are sloping in \( G \) with respect to \( \mathcal{H} \)).

An induced cycle of \( G \) is a cycle in \( G \) that is induced by some subset of \( V(G) \). We first analyze the structure of connected subgraphs \( G_0 \) of \( G \) with \( |V(G_0)| \leq r \) by several claims.

**Claim 1.** Let \( C \) be a cycle in \( G \). If \( |V(C)| \leq r \) and \( |E(C) \cap E^*| \) is odd, then \( C \) is an induced cycle of \( G \) with \( |V(C)| = r \).

![Figure 4](image-url) An example of spanning tree \( T_0 \) of \( G[N(v_t)] \).
Proof. By Condition (i) in Theorem 1.5, $|V(C)| = r$ trivially holds.

Suppose that there exists $e \in E(G) \setminus E(C)$ such that $e$ joins two vertices in $V(C)$. Then, $G$ contains a cycle $C'$ such that $V(C') \subseteq V(C)$, $|V(C')| < |V(C)| = r$ and $E(C') \cap E^\pi$ is odd, a contradiction to the definition of $r$.

Hence $G[V(C)] = C$ and Claim 1 holds. $\blacksquare$

Claim 2. Let $G_0 = (V_0, E_0)$ be a 2-connected subgraph of $G$. If $|V_0| \leq r$ and $|E^* \cap E_0| = 1$, then $|V_0| = r$ and $G_0$ is an induced cycle of $G$.

Proof. Since $G_0$ is 2-connected, $G_0$ contains a cycle $C$ with $|E(C) \cap E^\pi| = 1$, where $|V(C)| \leq |V_0| \leq r$. By Claim 1, $C$ is an induced cycle of $G$ with $|V(C)| = r$, which implies that $|V_0| = r$, $V_0 = V(C)$, and $G_0$ is $C$. Hence Claim 2 holds. $\blacksquare$

Claim 3. Let $G_0 = (V_0, E_0)$ be a connected subgraph of $G$ with $|V_0| \leq r$. If $|E^* \cap E_0| \geq 2$, then $G_0 - (E^* \cap E_0)$ is disconnected.

Proof. Suppose that $G_0 - (E^* \cap E_0)$ is connected. Let $e', e'' \in E^* \cap E_0$, and let $P$ be a path in $G_0 - (E^* \cap E_0)$ connecting the two end-vertices of $e'$. Consequently, the edge set $E(P) \cup \{e'\}$ forms a cycle $C$ in $G_0$ with $|V(C)| \leq r$ and $E(C) \cap E^* = \{e'\}$. By Claim 1, $C$ is an induced cycle with $|V(C)| = r$, implying that $G_0$ is $C$, a contradiction to the fact that $e'' \in E_0$. Hence Claim 3 holds. $\blacksquare$

Claim 4. Let $G_0 = (V_0, E_0)$ be a connected subgraph of $G$ with $|V_0| \leq r$. If $|E^* \cap E_0| \geq 2$, then no edge $e$ in $E^* \cap E_0$ joins two vertices in any component $G'$ of $G_0 - (E^* \cap E_0)$ (i.e., each component $G'$ of $G_0 - (E^* \cap E_0)$ is an induced subgraph of $G_0$).

Proof. Assume that $G'$ is a component of $G_0 - (E^* \cap E_0)$ and $e$ is an edge in $E^* \cap E_0$ that joins two vertices in $G'$. By Claim 3, $|V(G')| < |V_0|$. Then, $G' + e$ has a block containing $e$, say $G_1$, where $|V(G_1)| \leq |V(G')| < |V_0| \leq r$. But, as $E(G_1) \cap E^\pi = 1$, Claim 2 implies that $|V(G_1)| = r$, a contradiction. $\blacksquare$

Claim 5. Let $G_0 = (V_0, E_0)$ be a connected subgraph of $G$ with $|V_0| \leq r$. Assume that $\{U_1, U_2\}$ is a partition of $V_0$ such that $E_0 \cap E^* = E_{G_0}(U_1, U_2)$ and $G_0[U_i]$ is connected for both $i = 1, 2$. If $G_0$ is not a cycle of size $r$, then for all the edges $e_{i_1}, ..., e_{i_s} \in E_{G_0}(U_1, U_2)$, the vertices $u_{i_1}, ..., u_{i_s}$ must be in the same set $U_s$ for some $s \in \{1, 2\}$, as shown in Figure 5.

Proof. Let $E' = E_{G_0}(U_1, U_2)$. If $|E'| = 1$, then the result trivially holds.

In the following, assume that $|E'| \geq 2$. We need only to prove the two facts below on any two edges $e_{ip}, e_{iq}$ in $E'$:

(i) if there is a cycle $C$ in $G_0$ shorter than $r$ with $|E(C) \cap E^*| = \{e_{ip}, e_{iq}\}$, then $u_{ip}$ and $u_{iq}$ are contained in the same set $U_s$ for some $s \in \{1, 2\}$;

(ii) otherwise, there exists $e_{ij} \in E' \setminus \{e_{ip}, e_{iq}\}$, such that there is a cycle $C_1$ in $G_0$ shorter than $r$ with $E(C_1) \cap E^* = \{e_{ip}, e_{ij}\}$ and a cycle $C_2$ in $G_0$ shorter than $r$ with $E(C_2) \cap E^* = \{e_{ij}, e_{iq}\}$.
Since \( G_0[U_i] \) is connected for both \( i = 1, 2, \) \( G_0 \) has a cycle \( C \) with \( |E(C) \cap E^\bullet| = \{e_{ip}, e_{iq}\} \). If \( |V(C)| < r \), Condition (ii) in Theorem 1.5 indicates that \( \overrightarrow{e_p} = (u_{ip}, v_{ip}) \) and \( \overrightarrow{e_q} = (u_{iq}, v_{iq}) \) are balanced on \( C \), implying that \( u_{ip} \) and \( u_{iq} \) must be in the same set \( U_i \) for some \( s \in \{1, 2\} \). Fact (i) holds.

Now suppose that \( G_0 \) does not have a cycle \( C \) shorter than \( r \) with \( |E(C) \cap E^\bullet| = \{e_{ip}, e_{iq}\} \). Thus, \( |V(C)| = r \), implying that \( |V(C)| = |V(G_0)| \). As \( G_0 \) is not a cycle of size \( r \), there is an edge \( e \in E_0 \setminus E(C) \). Obviously, \( e \notin E(G_0[U_i]) \cup E(G_0[U_2]) \).

Otherwise, \( G_0 \) has a cycle \( C' \) shorter than \( r \) with \( |E(C') \cap E^\bullet| = \{e_{ip}, e_{iq}\} \), a contradiction. Thus, \( e \in E' = E_{G_0}(U_i, U_2) \). Assume that \( e = e_{ij} \in E' \setminus \{e_{ip}, e_{iq}\} \). Then, there are cycles \( C_1 \) and \( C_2 \) in \( C + e_{ij} \) such that \( E(C_1) \cap E^\bullet = \{e_{ip}, e_{ij}\} \) and \( |E(C_2) \cap E^\bullet| = \{e_{ij}, e_{iq}\} \). Note that both \( C_1 \) and \( C_2 \) are shorter than \( r \). Fact (ii) holds and Claim 5 follows.

\[ \text{Claim 6.} \quad \text{Let } G_0 = (V_0, E_0) \text{ be a connected subgraph of } G \text{ with } |V_0| \leq r \text{ and } |E_0 \cap E^\bullet| \geq 2. \text{ If } G_0 \text{ is not a cycle of size } r, \text{ then } |G_{ij}(G_0)| = m. \]

**Proof.** By Claim 3, we can assume that \( G_0 - (E_0 \cap E^\bullet) \) has \( s \geq 2 \) components \( G_1, G_2, \ldots, G_s \), where \( G_i = (V_i, E_i) \) for \( i \in [s] \). Then, Claim 4 implies that each \( G_i \) is an induced subgraph of \( G_0 \), that is, \( E_0 \cap E^\bullet = \bigcup_{1 \leq i < j \leq 3} E_{G_0}(V_i, V_j) \).

Let \( G' \) be the graph with vertex set \( V(G') = \{g_1, \ldots, g_s\} \) in which \( g_i g_j \) is an edge if and only if \( E_{G_0}(V_i, V_j) \neq \emptyset \). Let \( \overrightarrow{G'} \) be the digraph obtained from \( G' \) by converting each edge \( g_i g_j \) in \( G' \) into a directed edge whose tail is \( g_i \) if and only if \( u_q \in V_i \) for some edge \( e_q = u_q v_q \) in \( E_{G_0}(V_i, V_j) \). Note that the orientation of directed edges in \( \overrightarrow{G'} \) is well-defined due to the result in Claim 5. An example is shown in Figure 6B.

As \( G_0 \) is connected, \( G' \) is also connected. Let \( T \) be a spanning tree of \( G' \) with as many leaves as possible. Thus, \( T \) has at least \( \Delta(G') \) leaves, where \( \Delta(G') \) is the maximum degree of \( G' \). For each vertex \( g_i \) in \( G' \), there is a unique path, denoted by \( P_i \), in \( T \) from \( g_i \) to \( g_j \). Denote by \( \varphi_1(i) \) the number of those edges in \( P_i \) whose corresponding directed edges in \( \overrightarrow{G'} \) are along the direction of path \( P_i \) from \( g_i \) to \( g_j \), and denote by \( \varphi_2(i) \) the number of the remaining edges in \( P_i \). Thus, \( \varphi_1(i) + \varphi_2(i) = |E(P)| \).

Now, let \( \varphi(i) = \varphi_1(i) - \varphi_2(i) \) for each \( i \in [s] \). For the digraph \( \overrightarrow{G'} \) in Figure 6B, if \( T \) is the spanning tree of \( G' \) with edge set \( \{g_1 g_4, g_2 g_4, g_3 g_4, g_2 g_5, g_2 g_6\} \), then

\[
\varphi(1) = 0, \quad \varphi(2) = \varphi(3) = 2, \quad \varphi(4) = \varphi(6) = 1, \quad \varphi(5) = 3. \quad (22)
\]
We will complete the proof of this claim by proving three subclaims.

**Subclaim 6.1.** For any edge $g_i, g_j \in E(T)$, $\phi(j) = \phi(i) + 1$ whenever $(g_i, g_j)$ is the corresponding directed edge of $g_i, g_j$ in $\overrightarrow{G}$. 

Assume that $(g_i, g_j)$ is the corresponding directed edge of $g_i, g_j$ in $\overrightarrow{G}$. As $g_i, g_j \in E(T)$, either $g_i$ is on the path $P_j$, or $g_j$ is on the path $P_i$. If $g_i$ is on the path $P_j$, then $\phi_1(j) = \phi_1(i) + 1$ and $\phi_2(j) = \phi_2(i)$. If $g_j$ is on the path $P_i$, then $\phi_1(j) = \phi_1(i)$ and $\phi_2(j) = \phi_2(i) - 1$. Thus, Subclaim 6.1 follows in both cases.

For every $q \in [m]$, let $S_q$ be the set in $\mathcal{J}|_{V_0}$ defined as follows:

$$S_q = \bigcup_{i=1}^{s} \{(v, (q + \phi(i))(mod m)) : v \in V_i\}, \quad (23)$$

where $(v, 0) = (v, m)$ for all $v \in V_0$. Obviously, $\{S_1, ..., S_m\}$ is a partition of $V(H_{G_0})$.

**Subclaim 6.2.** If $\phi(j) = \phi(i) + 1$ holds for each directed edge $(g_i, g_j)$ in $\overrightarrow{G}$, then $H_{G_0}[S_q] \cong G_0$ for all $q \in [m]$, and hence Claim 6 holds.

Let $\phi$ be the bijection from $V_0$ to $S_q$ defined below: for any $v \in V_0 = \cup_{1 \leq i \leq s} V_i$, 

![Figure 6](image6.png)

**Figure 6** An example of $\overrightarrow{G}$. (A) $G_0$ and (B) $\overrightarrow{G'}$.

![Figure 7](image7.png)

**Figure 7** With a spanning tree $T$ consisting of dense edges, the value of $\phi(i)$ for each $i \in [6]$ is shown beside its vertex $g_i$. 

as given in Figure 7.

We will complete the proof of this claim by proving three subclaims.

**Subclaim 6.1.** For any edge $g_i, g_j \in E(T)$, $\phi(j) = \phi(i) + 1$ whenever $(g_i, g_j)$ is the corresponding directed edge of $g_i, g_j$ in $\overrightarrow{G}$. 

Assume that $(g_i, g_j)$ is the corresponding directed edge of $g_i, g_j$ in $\overrightarrow{G}$. As $g_i, g_j \in E(T)$, either $g_i$ is on the path $P_j$, or $g_j$ is on the path $P_i$. If $g_i$ is on the path $P_j$, then $\phi_1(j) = \phi_1(i) + 1$ and $\phi_2(j) = \phi_2(i)$. If $g_j$ is on the path $P_i$, then $\phi_1(j) = \phi_1(i)$ and $\phi_2(j) = \phi_2(i) - 1$. Thus, Subclaim 6.1 follows in both cases.

For every $q \in [m]$, let $S_q$ be the set in $\mathcal{J}|_{V_0}$ defined as follows:

$$S_q = \bigcup_{i=1}^{s} \{(v, (q + \phi(i))(mod m)) : v \in V_i\}, \quad (23)$$

where $(v, 0) = (v, m)$ for all $v \in V_0$. Obviously, $\{S_1, ..., S_m\}$ is a partition of $V(H_{G_0})$.

**Subclaim 6.2.** If $\phi(j) = \phi(i) + 1$ holds for each directed edge $(g_i, g_j)$ in $\overrightarrow{G}$, then $H_{G_0}[S_q] \cong G_0$ for all $q \in [m]$, and hence Claim 6 holds.

Let $\phi$ be the bijection from $V_0$ to $S_q$ defined below: for any $v \in V_0 = \cup_{1 \leq i \leq s} V_i$,
\( \phi(v) = (v, (q + \varphi(i))(\text{mod } m)), \) if \( v \in V_i. \) \( \tag{24} \)

To show that \( H_{G_0}[S_q] \cong G_0, \) it suffices to prove that for each edge \( uv \in E(G_0), \phi(u) \) and \( \phi(v) \) are adjacent in \( H. \)

For any \( uv \in E_i, \) where \( 1 \leq i \leq s, \) we have \( uv \in E_0 \setminus E^*, \) implying that \( (u, (q + \varphi(i))(\text{mod } m)) \) and \( (v, (q + \varphi(i))(\text{mod } m)) \) are adjacent in \( H \) by the definition of \( H. \)

Now take any edge \( uv \in E_{G_0}(V_i, V_j) \subseteq E^*, \) where \( 1 \leq i, j \leq s. \) Without loss of generality, assume that \( (g_i, g_j) \) is a directed edge in \( G'. \) Then \( \varphi(j) = \varphi(i) + 1 \) by the given condition in the subclaim. By the definition of \( H, (u, (q + \varphi(i))(\text{mod } m)) \) and \( (v, (q + \varphi(j))(\text{mod } m)) \) are adjacent in \( H. \)

Hence \( H_{G_0}[S_q] \cong G_0 \) for each \( q \in [m], \) and the subclaim holds.

Subclaim 6.3. \( \varphi(j) = \varphi(i) + 1 \) holds for each directed edge \( (g_i, g_j) \) in \( G'. \)

Suppose that \( \varphi(j) \neq \varphi(i) + 1 \) for some directed edge \( (g_i, g_j) \) in \( G'. \) By Subclaim 6.1, \( g_i, g_j \in E(G') \setminus E(T). \) Let \( C' \) be the fundamental cycle of edge \( g_i g_j \) in \( G' \) with respect to spanning tree \( T. \) Assume that \( g_{j_1}, g_{j_2}, \ldots, g_{j_t} \) are the consecutive vertices on \( C', \) where \( t \geq 3, j_1 = i \) and \( j_t = j. \)

As \( G_i \) is connected for all \( i \in [s], \) we can choose a shortest cycle \( C \) in \( G_0 \) such that

\[
E(C) \cap E^* \subseteq \bigcup_{q=1}^{t} E_{G_0}(V_{j_q}, V_{j_{q+1}}), \quad |E(C) \cap E_{G_0}(V_{j_q}, V_{j_{q+1}})| = 1, \quad \forall q \in [t],
\]

where \( V_{j_{t+1}} = V_j. \) Thus, \( |E(C) \cap E^*| = t. \) Clearly, \( t \) is an even integer; otherwise, Claim 1 implies that \( G_0 \) is a cycle of size \( r, \) a contradiction.

Suppose that \( |V(C)| < r. \) By Condition (ii) in Theorem 1.5, the directed edges of \( E^* \) are balanced on \( C, \) implying that the directed edges in \( G' \) are balanced on \( C'. \) By counting the number of edges in \( C' \) that are oriented clockwise and counterclockwise along \( C' \) separately, we have \( \varphi_1(j_1) + \varphi_2(j_1) + 1 = \varphi_1(j_t) + \varphi_2(j_t), \) implying that \( \varphi(j_1) + 1 = \varphi(j_t) \) (i.e., \( \varphi(i) + 1 = \varphi(j) \)), a contradiction.

Thus, \( |V(C)| = r, \) and so \( V(C) = V_0. \) Therefore, \( t = s \) and \( T \) is a path in \( G' \) with \( |V(T)| = |V(G')| = s. \) Moreover, due to the choice of \( C, \) for each \( q \in [s], E_q \subseteq E(C) \) and \( |E_{G_0}(V_{j_q}, V_{j_{q+1}})| = 1, \) implying that \( E_{G_0}(V_{j_q}, V_{j_{q+1}}) \subseteq E(C). \)

If \( G' \) is a cycle, then \( G' = C'. \) The above conclusion implies that \( E_0 = E(C), \) and thus \( G_0 \) is a cycle of size \( r, \) a contradiction. Thus, \( G' \) is not a cycle, implying that \( \Delta(G') \geq 3. \) Then, \( T \) has at least three leaves, a contradiction to the fact that \( T \) is a path. Hence Subclaim 6.3 holds.

By Subclaims 6.2 and 6.3, \( G_0(G_0) = m \) and Claim 6 holds.

Claim 7. For any \( A \in \mathcal{E}, \) if either \( c(A) > n - r + 1 \) or \( c(A) = n - r + 1 \) and \( |A| \neq r, \) then \( |G_0| = m^{c(A)} \) holds.

**Proof.** As \( A \in \mathcal{E}, E^* \cap \overline{B}(A) \neq \emptyset. \) Then, by (i) and (ii) in Section 2, it suffices to prove that for every block \( G_0 = (V_0, E_0) \) of \( G(A) \) with \( E_0 \cap E^* \neq \emptyset, |G_0(G_0)| = m \) holds.

Suppose \( G_0 = (V_0, E_0) \) is a block of \( G(A) \) with \( E_0 \cap E^* \neq \emptyset \) and \( |G_0(G_0)| < m. \) As \( c(A) \geq n - r + 1, |V_0| \leq r. \) Then by Claims 2 and 6, \( |V_0| = |E_0| = r, \) implying that either \( c(A) < n - r + 1 \) or \( c(A) = n - r + 1 \) and \( |A| = r, \) a contradiction. Hence Claim 7 holds.
Claim 8. If \( m > k \), then \( \mathcal{G}_H(C) = \emptyset \) for any cycle \( C \) in \( G \) such that \( |E(C) \cap E^k| \) is odd.

Proof. Assume that \( |E(C) \cap E^k| = 2s + 1 \) for some integer \( s \geq 0 \) and \( z_1, z_2, \ldots, z_q \) are consecutive vertices in \( C \), where \( q \geq 3 \). Suppose that \( \mathcal{G}_H(C) \neq \emptyset \). Then, there exists a cycle \( C' \) in \( H \) with consecutive vertices \( (z_i, h_i), (z_2, h_2), \ldots, (z_q, h_q) \). By the definition of \( \mathcal{H} = (L, H), h_{i+1} - h_i \neq 0 \) if and only if \( z_i z_{i+1} \in E^* \), and \( h_{i+1} - h_i \in \{0, 1, -1, m - 1, 1 - m\} \) for all \( i \in [q] \), where \( h_{q+1} = h_1 \) and \( z_{q+1} = z_1 \). Thus, \( h_{i+1} - h_i \neq 0 \) holds for exactly \( 2s + 1 \) integers \( i \) in \( [q] \).

Assume that there are exactly \( t \) integers \( i \) in \( [q] \) such that \( h_{i+1} - h_i = \{m - 1, 1 - m\} \). Then, there are exactly \( (2s + 1 - t) \) integers \( i \) in \( [q] \) such that \( h_{i+1} - h_i \in \{1, -1\} \). It follows that

\[
0 = \sum_{i=1}^{q} (h_{i+1} - h_i) = t'(m - 1) + s' \times 1,
\]

where \( t' \) and \( s' \) are some integers with \( |t'| \leq t \) and \( |s'| \leq 2s + 1 - t \) such that both \( t - t' \) and \( (2s + 1 - t) - s' \) are even.

Suppose that \( t' \neq 0 \). Without loss of generality, assume that \( t' \geq 1 \). Then \( s' \geq -(2s + 1 - 1) = -2s \), and (26) implies that

\[
0 = t'(m - 1) + s' \geq (m - 1) - 2s \geq (m - 1) - (k - 1) \geq m - k > 0,
\]

a contradiction. Hence \( t' = 0 \), implying that \( t \) is even. As \( s' \geq (2s + 1 - t) \) is even, by (26),

\[
0 = s' = (s' - (2s + 1 - t)) + (2s - t + 1) \equiv 1 \pmod{2},
\]

a contradiction. Thus, Claim 8 holds.

Claim 9. The following inequality holds when \( m > k \):

\[
\sum_{A \in \mathcal{E}} (-1)^{|A|}(|\mathcal{S}_A| - m^{c(A)}) \leq -m^{n-r+1}.
\]

Proof. Let \( C_0 \) be any cycle in \( C^c_0(E^*) \). By Claim 8, \( |\mathcal{S}_{E(C_0)}| = |\mathcal{G}_H(C_0)| \neq 0 \) holds.

Obviously, \( E(C_0) \) is in \( \mathcal{E} \) with \( |E(C_0)| = r \) and \( c(E(C_0)) = n - r + 1 \). Then, due to Claim 7, the fact that \( r \) is even, and (iii) in Section 2, we have

\[
\sum_{A \in \mathcal{E}} (-1)^{|A|}(|\mathcal{S}_A| - m^{c(A)}) = \sum_{A \in \mathcal{E}, |A| = r} (-1)^{|A|}(|\mathcal{S}_A| - m^{c(A)})
\]

\[
= \sum_{A \in \mathcal{E}, |A| = r} (|\mathcal{S}_A| - m^{c(A)}) \leq |\mathcal{S}_{E(C_0)}| - m^{n-r+1} = -m^{n-r+1}.
\]
For any $s \in \mathbb{N}$ with $s \leq n - r$, let $\phi_s$ be the number of subsets $A \subseteq E(G)$ such that $c(A) = s$, $G(A)$ is not a forest, and $|A|$ is odd.

**Claim 10.** For each $s \in [n - r]$, the following inequality holds:

$$
\sum_{A \in \mathcal{E}, c(A) = s} (-1)^{|A|}(|\mathcal{I}_A| - m^c(A)) \leq \phi_s m^s.
$$

(31)

**Proof.** By (8),

$$
\sum_{A \in \mathcal{E}, c(A) = s} (-1)^{|A|}(|\mathcal{I}_A| - m^c(A)) \leq \sum_{A \in \mathcal{E}, c(A) = s} (m^s - |\mathcal{I}_A|)
$$

$$
\leq \sum_{A \in \mathcal{E}, c(A) = s} m^s
$$

(32)

$$
\leq \phi_s m^s.
$$

\[\Box\]

Now, by (6) and Claims 7, 9, and 10, we have

$$
P(G, \mathcal{H}) - P(G, m) = \sum_{s=1}^{n-r+1} \sum_{A \in \mathcal{E}, c(A) = s} (-1)^{|A|}(|\mathcal{I}_A| - m^s)
$$

$$
\leq -m^{n-r+1} + \sum_{s=1}^{n-r} \phi_s m^s,
$$

(33)

where the inequality holds when $m > k$. As $k, \phi_1, ..., \phi_{n-r}$ are independent of the value of $m$, $P(G, m) - P(G, \mathcal{H}) \geq \Omega(m^{n-r+1})$. Hence the result is proven. \[\Box\]

We shall conclude this section by proving Corollary 1.6.

**Proof of Corollary 1.6.** Let $E^* = \{e_1, e_2, ..., e_k\} \subseteq E_G(V_1, V_2)$, where $e_i = u_i v_i$ with $u_i \in V_1$ and $v_i \in V_2$ for all $i \in [k]$. For each $i \in [k]$, let $\overrightarrow{e_i}$ be the directed edge $(u_i, v_i)$, and let $\overrightarrow{E^*} = \{\overrightarrow{e_i} : i \in [k]\}$. By Theorem 1.5, it suffices to verify that $\overrightarrow{E^*}$ is balanced on every cycle $C$ of $G$ with $|E(C)| < \ell_G(E^*)$.

Let $C$ be any cycle of $G$ such that $|E(C)| < \ell_G(E^*)$ and $|E(C) \cap E^*|$ is positive. By the definition of $\ell_G(E^*)$, $|E(C) \cap E^*| = 2r$ for some positive integer $r$, where $2r \leq k$. Without loss of generality, assume that $E(C) \cap E^* = \{e_i : i \in [2r]\}$.

Let $P$ be any minimal path of $C$ that contains exactly two edges in $E(C) \cap E^*$, say $e_i$ and $e_j$. Obviously, by the minimality of $P$, $e_i$, and $e_j$ must be the two edges incident with two end-vertices of $P$. Then, the consecutive vertices on $P$ cannot appear in any one of the following orders:

$$
u_i, u_i, ..., u_j, v_j \text{ or } v_i, u_i, ..., v_j, u_j.
$$

(34)
Otherwise, some component of \( C = (E(C) \cap E^*) \) is either a \((v_i, u_j)\)-path or a \((u_i, v_j)\)-path in \( G - E^* \), contradicting the given condition. Thus, the consecutive vertices on \( P \) must appear in one of the following orders:

\[
u_i, v_i, ..., v_j, u_j \quad \text{or} \quad v_i, u_i, ..., u_j, v_j.
\]  

(35)

Since \( |E(C) \cap E^*| = 2r \), by the definition of directed edges in \( E^* \), the above conclusion implies that the directed edges of \( E^* \) are balanced on \( C \).

The result then follows from Theorem 1.5. \( \square \)

## 5 | PLANE GRAPHS

By Corollary 1.4, every plane near-triangulation is DP-good and thus belongs to \( DP^* \). In the following, we consider those plane graphs \( G \) in which at least two faces are not bounded by 3-cycles. We will first show that such a plane graph \( G \) may belong to \( DP^* \) if some face of \( G \) is bounded by an even cycle.

### Corollary 5.1

Let \( G \) be any 2-connected plane graph in which each 3-cycle is the boundary of some face of \( G \). If at least two faces of \( G \) are not bounded by 3-cycles and one of them is bounded by a 4-cycle, then \( G \in DP^* \).

**Proof.** We can choose a shortest sequence of faces \( F_0, F_1, ..., F_t \) in \( G \), where \( t \geq 1 \), \( F_0 \) is bounded by a 4-cycle and \( F_t \) is bounded by more than 3 edges, such that \( F_i \) is bounded by a 3-cycle for each \( i \in [t-1] \), and faces \( F_{i-1} \) and \( F_i \) share an edge \( e_i \) on their boundaries for each \( i \in [t] \). An example of the subgraph consisting of vertices and edges on boundaries of faces \( F_0, F_1, ..., F_t \) is shown in Figure 8, where \( t = 8 \).

If \( t = 1 \), then \( |E(G) e_1| = 4 \) and thus \( G \in DP^* \) by Theorem 1.2. Now assume that \( t \geq 2 \). As \( F_1, F_2, ..., F_{t-1} \) are all bounded by 3-cycles, \( e_i \) and \( e_{i+1} \) have a common end-vertex for each \( i \in [t-1] \). Thus, \( e_i \) can be written as \( e_i = u_i v_i \) for all \( i \in [t] \) such that either \( u_i = u_{i+1} \) (i.e., \( u_i \) and \( u_{i+1} \) are the same vertex) or \( v_i = v_{i+1} \) for all \( i \in [t-1] \). Let \( V_1 = \{ u_i : i \in [t] \} \) and \( V_2 = \{ v_i : i \in [t] \} \). Then \( E^* = \{ e_i : i \in [t] \} \subseteq E_G(V_1, V_2) \).

As \( F_0 \) is bounded by a 4-cycle, \( G \) has a 4-cycle \( C \) with \( E(C) \cap E^* = 1 \). But, as each 3-cycle in \( G \) must be the boundary of some face of \( G \), there is no 3-cycle \( C \) in \( G \) with \( |E(C) \cap E^*| = 1 \). As the dual edges of the edges in \( E^* \) actually form a shortest path connecting vertices \( F_0^* \) and \( F_t^* \) in the dual plane graph \( G^* \) of \( G \), there is no 3-cycle \( C \) in \( G \).

![Figure 8](image_url)

**Figure 8** Subgraph consisting of vertices and edges on the boundaries of faces \( F_0, F_1, ..., F_8 \).
with \(|E(C) \cap E^*| = 3\). Therefore, \(\ell_G(E^*) = 4\). Thus, by Corollary 1.6, \(G \in DP_\leq\), and the result holds.

It is not difficult to generalize Corollary 5.1 as stated below.

**Corollary 5.2.** Let \(G\) be any 2-connected plane graph. If \(t \geq 1\) and \(F_0, F_1, \ldots, F_t\) are faces in \(G\) that satisfy the following conditions, then \(G \in DP_\leq\):

(i) only \(F_0\) and \(F_t\) are not bounded by 3-cycles, \(F_0\) is bounded by an \(r\)-cycle, where \(r\) is even, and \(F_t\) is bounded by a cycle of length at least \(r\);

(ii) for each \(i \in [t]\), faces \(F_{i-1}\) and \(F_i\) share an edge \(e_i\) on their boundaries; and

(iii) for \(E^* = \{e_i : i \in [t]\}\), if \(C\) is a cycle in \(G\) with \(|E(C) \cap E^*| \neq \emptyset\), then \(|E(C)| \geq r\) holds whenever either \(F_0\) or \(F_t\) is within cycle \(C\).

By Corollaries 1.6 and 5.1, it is interesting to notice that many graphs in \(DP_\leq\) have their structures with a doughnut shape, as shown in Figure 9, where \(E^* \subseteq E_G(V_1, V_2)\) for two disjoint vertex sets \(V_1\) and \(V_2\), and \(C\) is a shortest cycle in \(G\) such that \(|E(C) \cap E^*|\) is odd and \(|E(C)|\) is even.

However, for some other plane graphs that also look like doughnuts, we still do not know whether they belong to \(DP_\geq\) or \(DP_\leq\). For example, for a 2-connected plane graph \(G\) that is not a near-triangulation, if \(\ell_G(e) = 3\) for all \(e \in E(G)\), and those faces in \(G\) not bounded by 3-cycles have, respectively, \(q_1, q_2, \ldots, q_t\) edges on their boundaries, where \(4 \leq q_1 \leq q_2 \leq \cdots \leq q_t\), and \(q_i\) is even whenever \(q_i < q_j\) and \(q_i\) is even, then it is unknown whether \(G\) belongs to \(DP_\leq\) or \(DP_\geq\). For the particular case that \(q_i\) is odd for all \(i \in [t - 1]\), we suspect that \(G\) belongs to \(DP_\geq\).

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