A class of renormalization group invariant scalar field cosmologies

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Abstract

We present a class of scalar field cosmologies with a dynamically evolving Newton parameter $G$ and cosmological term $\Lambda$. In particular, we discuss a class of solutions which are consistent with a renormalization group scaling for $G$ and $\Lambda$ near a fixed point. Moreover, we propose a modified action for gravity which includes the effective running of $G$ and $\Lambda$ near the fixed point. A proper understanding of the associated variational problem is obtained upon considering the four-dimensional gradient of the Newton parameter.
I. INTRODUCTION

The recent discovery that Einstein gravity is most probably renormalizable at a non-perturbative level [1, 2, 3, 4] has triggered many investigations on the possible consequences of these findings in cosmology. In [5], a cosmology of the Planck Era, valid immediately after the initial singularity, was discussed. In this model the Newton constant $G$ and the cosmological constant $\Lambda$ are dynamically coupled to the geometry by “improving” the Einstein equations with the renormalization group (hereafter RG) equations for Quantum Einstein Gravity [6]. This modified Einstein theory is not affected by the horizon and flatness problems of the cosmological standard model.

In [7], a similar framework has been extended to the study of the large scale dynamics of the Universe. In this case a solution of the “cosmic coincidence problem” [8] arises naturally without the introduction of a quintessence field, because the vacuum energy density $\rho_\Lambda \equiv \Lambda / 8\pi G$ is automatically adjusted so as to equal the matter energy density, i.e. $\Omega_\Lambda = \Omega_{\text{matter}} = 1/2$ [7]. We shall call the models discussed in [5, 7] as fixed point (hereafter FP) cosmologies, or equivalently, RG-invariant cosmologies.

In a nutshell, the renormalization group improvement consists in the modified Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -g_{\mu\nu}\Lambda(k) + 8\pi G(k)T_{\mu\nu}$$

(1)

where the Newton parameter $G$ and cosmological term $\Lambda$ are now dependent on the scale $k$, $k$ being the running cut-off of the renormalization group equation [6].

Gravitational theories with variable $G$ have been discussed in the context of the “induced-gravity” model [9] where the Newton constant is generated by means of a non-vanishing vacuum expectation value of a scalar field. However here the basic difference here is that the dynamical content of the theory is not determined by a dynamical rearrangement of the symmetry, but instead it is determined by the renormalization group approach applied to the (quantum) Einstein-Hilbert lagrangian. It is however interesting to notice the a dynamically evolving cosmological constant and gravitational interaction also appear in very general scalar-tensor cosmologies [10, 11].

This framework has been also applied in General Relativity in [12], in the dynamical context of a gravitational collapse, and in [13] for a Schwarzschild black hole. In cosmology, the dynamical evolution is instead determined by a set of renormalization group equations...
by means of the cut-off identification $k = k(t)$ which relates the energy scale of the running cutoff $k$ of the renormalization group, with the cosmic time $t$. In [7] it has been shown that, in a cosmological setting, the correct cutoff identification is $k \propto t^{-1}$; it is thus possible to determine $G(k(t))$ and $\Lambda(k(t))$ in Eq. (1) once a RG trajectory is determined. The aim of this paper is to extend the results discussed in [5, 7] to the case of a scalar field coupled to gravity.

Let us in fact assume that, besides the non-Gaussian fixed point discovered in [1] for pure gravity, the standard Gaussian fixed point is accessible in perturbation theory in the scalar sector (this is actually the case for a free scalar field as shown in [14], and it also emerges from the analysis of Ref. [15] for a self-interacting scalar theory). Then, a solution which is compatible with a possible RG trajectory for the scalar sector must predict a simple renormalizable potential for spin-0 particles. We thus show that there exists a class of solutions for the familiar $\phi^4$ renormalizable potential.

In addition, we also discuss a possible renormalization-group improvement at the level of the Einstein-Hilbert Lagrangian itself. In this case, solutions for a class of power-law self-interaction potentials are available only for some specific values of the quartic self-interaction coupling constants.

The plan of this work is the following: in Sec. II we introduce the basic equations and present the scalar field solution. In Sec. III we discuss the RG improvement of the Einstein-Hilbert Lagrangian. Section IV is devoted to the conclusions.

II. THE MODEL

We now introduce the basic equations of the FP cosmologies for a scalar field matter component. Let us recall that the effective energy density and pressure of a generic scalar field read:

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi),$$

(2)

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi),$$

(3)
respectively. In term of $\rho_\phi$ and $p_\phi$ the coupled system of RG improved evolution equations read

\[
\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{1}{3} \Lambda + \frac{8\pi}{3} G \rho_\phi, \tag{4a}
\]

\[
\ddot{\phi} + 3\frac{\dot{a}}{a} \dot{\phi} + V'(\phi) = 0, \tag{4b}
\]

\[
\dot{\Lambda} + 8\pi G \rho_\phi = 0, \tag{4c}
\]

\[
G(t) \equiv G(k(t)), \quad \Lambda(t) \equiv \Lambda(k(t)), \tag{4d}
\]

Eq. (4a) is the improved Friedmann equation, Eq. (4b) is the Klein-Gordon equation, Eq. (4c) follows from the Bianchi identities, and Eqs. (4d) are determined from the renormalization group equations once the cutoff identification $k = k(t)$ is given. We define the vacuum energy density $\rho_\Lambda$, the total energy density $\rho_{\text{tot}}$ and the critical energy density $\rho_{\text{crit}}$ according to

\[
\rho_\Lambda(t) \equiv \frac{\Lambda(t)}{8\pi G(t)}, \tag{5}
\]

\[
\rho_{\text{tot}}(t) \equiv \rho_\phi + \rho_\Lambda, \tag{6}
\]

\[
\rho_{\text{crit}}(t) \equiv \frac{3}{8\pi G(t)} \left(\frac{\dot{a}}{a}\right)^2, \tag{7}
\]

with $H \equiv \dot{a}/a$. Hence we may rewrite the improved Friedmann equation (4a) in the form

\[
\frac{\dot{a}^2 + K}{a^2} = \frac{8\pi}{3} G(t) \rho_{\text{tot}}. \tag{8}
\]

We refer the various energy densities to the critical density (7):

\[
\Omega_\phi \equiv \frac{\rho}{\rho_{\text{crit}}}, \quad \Omega_\Lambda \equiv \frac{\rho_\Lambda}{\rho_{\text{crit}}}, \tag{9}
\]

\[
\Omega_{\text{tot}} = \Omega_\phi + \Omega_\Lambda \equiv \frac{\rho_{\text{tot}}}{\rho_{\text{crit}}}. \tag{10}
\]

It follows from these definitions that the Friedmann equation (8) becomes

\[
K = \dot{a}^2 \left[\Omega_{\text{tot}} - 1\right]. \tag{11}
\]

For a spatially flat universe ($K = 0$) we need $\rho_{\text{tot}} = \rho_{\text{crit}}$, as in standard cosmology. In the following we shall discuss only the $K = 0$ case. In order to solve the system (4) we consider the first three equations in (4) without the RG equations (4d). While in general (4) can be solved once $V(\phi)$ is given, we shall see that the perfect fluid ansatz $p_\phi = w \rho_\phi$, $w$ being a constant, is equivalent to assume a class of power-law potentials $V(\phi) \propto \phi^m$. 
We first consider the first three equations in (4) without the RG equations (4d), and then we determine the solutions consistent with a given RG trajectory. The potential can be written as

\[ V(\phi) = \frac{1}{2} \dot{\phi}^2 \left( \frac{1-w}{1+w} \right), \]  

which shows that the value \( w = -1 \) should be ruled out, as we will do from now on. By substitution in the Klein–Gordon equation (4b) we readily obtain

\[ \rho_\phi = \frac{1}{1+w} \phi^2 \equiv \frac{\mathcal{M}}{8\pi a^{3(1+w)}}, \] 

where \( \mathcal{M} \) is an integration constant. By substituting into Eq. (4a) we derive the following power-law solutions:

\[ a(t) = \left[ \frac{1}{2} \frac{3(1+w)^2}{2(n+2)} \mathcal{M} C \right]^{1/(3+3w)} \left[ \frac{4(n+2)}{12\pi (1+w) C n^2} \right]^{1/2} t^{-(n+2)/(3+3w)}, \]  

\[ \phi(t) = \left( \frac{4(n+2)}{12\pi (1+w) C n^2} \right)^{1/2} t^{-n/2}, \]  

\[ G(t) = C t^n, \]  

\[ \Lambda(t) = \frac{n(n+2)}{3(1+w)^2} \frac{1}{t^2}, \]

where \( C \) is a constant and \( n \) is a positive integer. For example, writing \( a(t) = \alpha t^\alpha, \Lambda = \beta t^{-2} \) and expressing \( G \) as in (14c), Eq. (4a) yields, for \( K = 0 \), a first-degree algebraic equation for \( \alpha \), which is solved by \( \alpha = \frac{(n+2)}{3(1+w)} \). Equation (13) is then integrated to get the result (14b). As anticipated, the potential is also a power law, i.e.

\[ V(\phi) = \frac{1 - w}{2 + 2w} \left( \frac{12\pi (w+1) C}{n+2} \right) \frac{2}{n} \left( \frac{n}{2} \right)^{2(n+2)/n} \phi^{2(n+2)/n}. \] 

The RG equations (4d) have not been used so far. What is the correct RG trajectory for a self-interacting scalar field coupled with gravity? Let us consider the RG-trajectory discussed in the introduction, where in the deep UV region we must have the non-Gaussian fixed point \([1, 2, 3, 4]\) in the gravitational sector, and the Gaussian one in the scalar field sector. In this case, since the renormalized trajectory ends at \((\lambda_*, g_*)\), the dimensionful quantities must run as

\[ G(k) = g_*/k^2, \quad \Lambda(k) = \lambda_* k^2 \] 

where \( g_*, \lambda_* \) are the dimensionless coupling \( g(k) \) and \( \lambda(k) \), respectively, at the ultraviolet non-Gaussian fixed point \( k \to \infty \). The numerical values have been obtained in the analysis of \([14, 15]\) and read \( g_* \approx 0.31, \lambda_* \approx 0.35 \) approximately.
The next step is to determine $k$ as a function of $t$. In [5] it was shown that the correct cutoff identification is given by

$$k(t) = \frac{\xi}{t}. \quad (17)$$

Therefore, we see from (16) and from (17) that $G = g* \xi^{-2} t^2$ and $\Lambda = \lambda* \xi^2 t^{-2}$, therefore we must choose $n = 2$ in (14) and $\xi^2 = 8/(1 + w)^2 \lambda*$ in (17). At last the following renormalization group invariant (or fixed-point) solution is obtained:

$$a(t) = \left[ \left( \frac{3}{8} \right)^2 (1 + w)^4 g* \lambda* \mathcal{M} \right]^{1/(3 + 3w)} t^{4/(3 + 3w)}, \quad (18a)$$

$$\phi(t) = \left( \frac{8}{9\pi(1 + w)^3 g* \lambda*} \right)^{1/2} \frac{1}{t}, \quad (18b)$$

$$G(t) = \frac{3}{8} (1 + w)^2 g* \lambda* t^2, \quad (18c)$$

$$\Lambda(t) = \frac{8}{3(1 + w)^2} \frac{1}{t^2}. \quad (18d)$$

The solution, as far as $a(t)$, $G(t)$ and $\Lambda(t)$ are concerned, is basically the same as what already discussed in [3, 7] but in this case the potential reads

$$V(\phi) = \frac{9\pi}{16} (1 - w)(1 + w)^2 g* \lambda* \phi^4, \quad (19)$$

which is the standard renormalizable quartic self-interacting potential for a massless scalar theory. The role of $w$ is now clear: it allows a convenient parametrization of the solution in terms of the parameter $w$ instead of the self-interaction coupling constant in the potential. It in fact measures the self-coupling strength $9(1 - w)(1 + w)^2 g* \lambda*/16$: for $w = 1$ (stiff matter equation of state) $V = 0$ and $\phi$ is a free field, while for $0 < w < 1$, $\phi$ is an interacting field. For $w > 1$ the theory is not bounded from below.

Other properties of the solution (18) have been extensively discussed in [3, 7] and we shall not repeat this discussion here. We simply point out that for the solution (18), we have $\Omega_\phi = \Omega_\lambda = 1/2$ at any time.

### III. IMPROVING THE ACTION

One of the striking properties of the renormalization group trajectory (16) is that the following relation holds:

$$\Lambda = \frac{g* \lambda*}{G}. \quad (20)$$
This fact has a deep meaning and is related with the possibility of reducing the number of coupling constants in a RG-invariant theory \cite{16}. What happens in our case is that near the fixed point it is always possible to consider $\Lambda = \Lambda(G)$ and the effective scaling is ruled only by $G$, for instance. This fact suggests that a more fundamental approach should consider $\Lambda$ as a function of $G$ from the beginning, perhaps at the level of the action itself.

Let us therefore consider the action $S = S_g + S_m$, where $S_m$ is the action for the matter field, and

$$S_g = \int_M d^4x \sqrt{-g} \left( \frac{R}{G} - \frac{2\Lambda(G)}{G} \right), \quad (21)$$

where $M$ is the portion of space-time we have access to. This is a well-defined starting point for promoting $G$ and $\Lambda$ to the role of dynamical variables in a fully covariant way. However, since the scalar curvature contains second derivatives of the metric and $G$ is no longer constant, some extra care is necessary to obtain a well-posed variational problem. Indeed, on denoting by $\Gamma^\alpha_{\mu\nu}$ the Christoffel symbols, and defining \cite{17}

$$w^\alpha \equiv g^{\mu\nu} \delta \Gamma^\alpha_{\mu\nu} - g^{\alpha\nu} \delta \Gamma^\mu_{\mu\nu}, \quad (22)$$

variation of $S_g$ yields

$$\delta S_g = \int_M \frac{1}{G} \left( \frac{R}{2} g^{\alpha\beta} - R^{\alpha\beta} \right) \delta g_{\alpha\beta} \sqrt{-g} d^4x + \int_M \left[ \frac{R}{G^2} + \frac{2\Lambda}{G} - \frac{2}{G} \frac{d\Lambda}{dG} \right] \delta G \sqrt{-g} d^4x$$

$$- \int_M \partial_{\alpha} \left( \frac{\sqrt{-g} w^\alpha}{G} \right) d^4x - \int_M \frac{G_{\alpha}}{G^2} \sqrt{-g} w^\alpha d^4x. \quad (23)$$

Thus, even upon choosing variations $\delta g_{\mu\nu}$ and $\delta \Gamma^\alpha_{\mu\nu}$ such that $w^\alpha$ vanishes on the boundary of $M$ \cite{17}, the variation of the action functional $S_g$ does not reduce to the first line of Eq. (23), because the fourth term on the right-hand side of Eq. (23) survives. We are therefore assuming that the gravitational part of the action is actually $\tilde{S}_g$ such that

$$\delta \tilde{S}_g = \delta S_g + \int_M \frac{G_{\alpha}}{G^2} \sqrt{-g} w^\alpha d^4x. \quad (24)$$

The content of our postulate is non-trivial, since the two variations do not differ by the integral of a total derivative, as is clear from (22) and (24). As far as we know, such a crucial point had not been previously appreciated in the literature. The explicit construction of $\tilde{S}_g$ itself is more easily obtained upon using an Arnowitt–Deser–Misner space-time foliation. On using the standard notation for induced metric $h_{ij}$, extrinsic curvature $K_{ij}$, lapse $N$ and shift $N^i$ \cite{18} one can show that the action (here $K \equiv h^{ij} K_{ij}$, $h \equiv \det h_{ij}$)

$$\tilde{S}_g \equiv \int_M \frac{(R - 2\Lambda)}{G} \sqrt{-g} d^4x + 2 \int_M (K \sqrt{h})_{ij} \sqrt{h} d^4x - 2 \int_M \frac{f^i}{G} d^4x, \quad (25)$$
where \( f^i \equiv K \sqrt{h} N^i - \sqrt{h} h^{ij} N_j \), reduces to
\[
\tilde{S}_g = \int_M \frac{N \sqrt{h}}{G} \left( K_{ij} K^{ij} - K^2 + (3) R - 2\Lambda \right) d^4 x,
\]
(26)
where \((3) R\) is the scalar curvature of the spacelike hypersurfaces which foliate the space-time manifold when the \( \mathbb{R} \times \Sigma \) topology is assumed. The action (25) is the 3 + 1 realization of an action fulfilling the condition (24), as can be seen upon using the Leibniz rule to re-express
\[
\frac{1}{G} (K \sqrt{h})_\partial = \frac{G_\partial}{G^2} K \sqrt{h} + \left( \frac{K \sqrt{h}}{G} \right)_\partial,
\]
\[
\frac{1}{G} f^i_{,i} = \frac{G_i}{G^2} f^i + \left( \frac{f^i}{G} \right)_i.
\]
On the other hand, the identity (26) shows that \( \tilde{S}_g \) is eventually cast in the desired form suitable for calculus of variations, which only involves the induced metric and its first derivatives, as well as the undifferentiated Newton parameter.

At this stage, variation of \( \tilde{S}_g \) with respect to \( g_{\mu \nu} \) leads to Eqs. (4) and variation with respect to \( G \) gives an additional constraint equation (see also [19]):
\[
-\frac{R}{G} + 2\Lambda - 2 \frac{d\Lambda}{dG} = 0.
\]
(27)
This equation, jointly with Eq. (14) and the field equations yields
\[
2\Lambda = 8\pi G (\rho + 3p) = 8\pi G \rho (1 + 3w).
\]
(28)
Such a formula is a new equation with respect to the analysis in Ref. [7], and expresses a restriction which only holds if the potential is renormalizable. By inserting the general solution (14) in (28) we have
\[
n = (1 + 3w).
\]
(29)
In particular, for the case of interest \( n = 2 \), and hence \( w = 1/3 \), leading in turn to the renormalizable potential
\[
V(\phi) = \frac{2\pi}{3} g_\ast \lambda \phi^4.
\]
(30)
The relevant property of this solution is that the effective strength of the interaction self-coupling is determined entirely by the fixed point values \( g_\ast \) and \( \lambda_\ast \). For a free scalar field \( g_\ast \lambda_\ast \approx 0.11 \) [14], and this value does not change in a significant way in the interacting case [15]. Loop corrections are then expected to be small and the leading tree-level form of the potential (30) holds. We can thus regard the cosmology (18) with \( w = 1/3 \) as an exact solution of the modified Einstein action \( \tilde{S}_g + S_m \) which is consistent with a RG flow near the non-Gaussian fixed point in the gravitational sector and the Gaussian one in the matter sector.
IV. CONCLUSION

We have presented a class of power-law cosmologies with variable $G$ and $\Lambda$ in the case of a scalar field matter component, Eq. (14) (cf. important previous work in Ref. 20 on scalar fields coupled to gravity within the framework of renormalization group equations). We have then extended the FP cosmology presented in 5 by including the RG evolution Eq. (16) in the general solution (14). Last, we have presented a new RG-improvement at the level of the action which picks out a specific self-interaction strength value for the scalar field potential. The scalar solution (18) with $w = 1/3$ can, at best, be considered only a toy model of the initial state of Universe. However, it may be helpful in understanding a more complete framework where the dynamical evolution of the gravitational field and the matter field near the initial singularity is consistent with RG scaling law of the renormalized theory near a fixed point.

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