GENERALIZED ELLIPTIC FUNCTIONS AND THEIR APPLICATION TO A NONLINEAR EIGENVALUE PROBLEM WITH \( p \)-LAPLACIAN

SHINGO TAKEUCHI

Dedicated to Professor Yoshio Yamada on occasion of his 60\(^{th}\) birthday

Abstract. The Jacobian elliptic functions are generalized and applied to a nonlinear eigenvalue problem with \( p \)-Laplacian. The eigenvalue and the corresponding eigenfunction are represented in terms of common parameters, and a complete description of the spectra and a closed form representation of the corresponding eigenfunctions are obtained. As a by-product of the representation, it turns out that a kind of solution is also a solution of another eigenvalue problem with \( p/2 \)-Laplacian.

1. Introduction

In this paper we generalize the Jacobian elliptic functions and apply them to a nonlinear eigenvalue problem

\[
(\text{PE}_{pq}) \quad \begin{cases} 
(\phi_p(u'))' + \lambda \phi_q(u)(1 - |u|^q) = 0, & t \in (0, T), \\
u(0) = u(T) = 0,
\end{cases}
\]

where \( T, \lambda > 0 \), \( p, q > 1 \) and \( \phi_m(s) = |s|^{m-2}s \) (\( s \neq 0 \)), \( = 0 \) (\( s = 0 \)).

Problem \( (\text{PE}_{pq}) \) appears frequently in various articles as stationary problems. In particular, the equation for \( p = q = 2 \) is called, e.g., the Allen-Cahn equation, the Chafee-Infante equation [3], and a bistable reaction-diffusion equation with logistic effect. The equation for \( p = 2 < q \) is said to be a bistable reaction-diffusion equation with Allee effect. In case \( p = n \) and \( q = 2 \) with an \( n \)-dimensional domain, an equation of this type is known as the Euler-Lagrange equation of functional related to models introduced by Ginzburg and Landau for the study of phase transitions (cf. Problem 17 in [2]).

As to \( (\text{PE}_{pq}) \) for general \( p > 1 \), we have to mention the work [8] by Guedda and Véron [8]. They showed that if \( p = q > 1 \) then there exists a positive increasing sequence \( \{\lambda_n\} \) such that a pair of solutions \( \pm u_n \) of

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\[ \text{(PE}_{pq}\text{)} \] with \((n-1)\)-zeros \(z_j = jT/n \ (j = 1, 2, \ldots, n-1)\) bifurcates from the trivial solution at \(\lambda = \lambda_n\) and \(|u_n| \to 1\) uniformly on any compact set of \((0, T) \setminus \{z_1, z_2, \ldots, z_{n-1}\}\) as \(\lambda \to \infty\). Moreover, they proved that if \(p = q > 2\) then for each \(n \in \mathbb{N}\) there exists \(\Lambda_n > \lambda_n\) such that \(\lambda > \Lambda_n\) implies \(|u_n| = 1\) on flat cores \([z_{j-1} + \frac{T}{2n} (\frac{\lambda}{\lambda_n})^{1/p}, z_j - \frac{T}{2n} (\frac{\lambda}{\lambda_n})^{1/p}] \ (j = 1, 2, \ldots, n)\) of \(u_n\), where \(z_0 = 0\) and \(z_n = T\). This is a great contrast to case \(1 < p = q \leq 2\), where \(|u_n| < 1\) in \([0, T]\). Since the equation in \(\text{(PE}_{pq}\text{)}\) is autonomous, if \(u_n \ (n \geq 2)\) has flat cores, then there exists uncountable solution with \((n-1)\)-zeros near \(u_n\), which is produced by expanding and contracting the flat cores with preserving its total length \(T(1 - (\frac{\lambda}{\lambda_n})^{1/p})\). In this sense, the \(n\)-th branch \((\lambda, u_n)\) bifurcating from \((\lambda_n, 0)\) causes the second bifurcation at \((\Lambda_n, u_{\Lambda_n})\) for each \(n \geq 2\).

The phenomena of flat core in \([8]\) above was generalized to case \(p > 2\) and \(q > 1\) by the author and Yamada \([11]\). They also studied change in bifurcation depending on the relation between \(p\) and \(q\) (as far as the first bifurcation is concerned, their proof can be applied to case \(1 < p \leq 2\)), and showed that for each \(n \in \mathbb{N}\), if \(p > q\) then there exists a pair of solutions \(\pm u_n\) of \(\text{(PE}_{pq}\text{)}\) with \((n-1)\)-zeros for \(\lambda > 0\); if \(p = q\) then there exists \(\lambda_n > 0\) such that \(\text{(PE}_{pq}\text{)}\) has no solution with \((n-1)\)-zeros for \(\lambda \leq \lambda_n\) and \(\text{(PE}_{pq}\text{)}\) has a pair of solutions \(\pm u_n\) for \(\lambda > \lambda_n\) (the same result as \([8]\) ); if \(p < q\) then there exists \(\lambda_n^* > 0\) such that \(\text{(PE}_{pq}\text{)}\) has no solution with \((n-1)\)-zeros for \(\lambda < \lambda_n^*\) and \(\text{(PE}_{pq}\text{)}\) has a pair of solutions \(\pm u_n\) for \(\lambda = \lambda_n^*\) and \(\text{(PE}_{pq}\text{)}\) has two pairs of solutions \(\pm u_n, \pm v_n\) satisfying \(|u_n(t)| > |v_n(t)|\) with \(t \neq z_j \ (j = 0, 1, \ldots, n)\) for \(\lambda > \lambda_n^*\). In this sense, the point \((\lambda_n^*, u_{\lambda_n^*})\) causes the spontaneous bifurcation. In any case, each solution \(u_n\) has flat cores for sufficiently large \(\lambda\).

The purpose of this paper is to obtain a complete description of the spectra and a closed form representation of the corresponding eigenfunctions of \(\text{(PE}_{pq}\text{)}\), while the studies \([8]\) and \([11]\) above are done in the way of phase-plane analysis and no exact solution is given there.

For the description and representation, we first recall that the Jacobian elliptic function \(sn(t, k)\) with modulus \(k \in (0, 1)\) satisfies

\[
(1.1) \quad u'' + u(1 + k^2 - 2k^2u^2) = 0. 
\]

(e.g., Example 4 of p. 516 in the book \([13]\) of Whittaker and Watson). Eq. (1.1) reminds that the solution of nonlinear eigenvalue problem \(\text{(PE}_{pq}\text{)}\) with \(p = q = 2\) can be represented explicitly by using \(sn(t, k)\). Indeed, for any given \(k \in (0, 1)\), the set of eigenvalues of \(\text{(PE}_{22}\text{)}\) is given...
by

\[
\lambda_n(k) = (1 + k^2) \left( \frac{2nK(k)}{T} \right)^2
\]

for each \( n \in \mathbb{N} \), with corresponding eigenfunctions \( \pm u_{n,k} \), where

\[
u_{n,k}(t) = \sqrt{\frac{2k^2}{1 + k^2}} \text{sn} \left( \frac{2nK(k)}{T} t, k \right)
\]

and \( K(k) \) is the complete elliptic integral of the first kind

\[
K(k) = \int_0^1 \frac{ds}{\sqrt{(1 - s^2)(1 - k^2 s^2)}},
\]

(cf. Section 2 in [1]). Conversely, all nontrivial solutions are given by Eqs. (1.2) and (1.3), and in particular, it follows from Eq. (1.3) that all solutions satisfy \( |u| < 1 \).

In our study on \((\text{PE}_{pq})\), after the fashion of Jacobi’s \( \text{sn}(t, k) \), we introduce a new transcendental function \( \text{sn}_{pq}(t, k) \) with modulus \( k \in [0, 1) \). This satisfies

\[
(\phi_p'(u'))' + \frac{q}{p^*} \phi_q(u)(1 + k^q - 2k^q |u|^q) = 0,
\]

where \( p^* := p/(p - 1) \). Using \( \text{sn}_{pq}(t, k) \), we can obtain a complete description of the set of eigenvalues and the corresponding eigenfunctions of \((\text{PE}_{pq})\) as Eqs. (1.2) and (1.3) with

\[
K_{pq}(k) = \int_0^1 \frac{ds}{\sqrt{(1 - s^q)(1 - k^q s^q)}}.
\]

It is important that \( K_{pq}(k) \) converges to \( K_{p^*,q}(0) \) as \( k \to 1 - 0 \) if and only if \( p > 2 \). Indeed,

\[
\lim_{k \to 1 - 0} K_{pq}(k) = \int_0^1 \frac{ds}{(1 - s^q)^{\frac{2}{q}}} = K_{p^*,q}(0).
\]

Similarly, \( \text{sn}_{pq}(t, k) \) converges to \( \text{sn}_{p^*,q}(t, 0) \) as \( k \to 1 - 0 \). These convergent properties yield the existence of special solutions, not necessarily \( |u| < 1 \), and we can really construct the solutions of \((\text{PE}_{pq})\) with flat cores. Moreover, \( \text{sn}_{p^*,q}(t, 0) \) satisfies Eq. (1.4) with \( k = 0 \) and \( p \) replaced by \( p/2 \) as well as Eq. (1.3) with \( k = 1 \). Thus, we obtain the following (curious) property: a kind of solution of \((\text{PE}_{pq})\) is also a solution of the nonlinear eigenvalue problem with \( p/2 \)-Laplacian

\[
\begin{aligned}
&(\phi_p'(u'))' + \lambda \phi_q(u) = 0, \quad t \in (0, T), \\
u(0) = u(T) = 0.
\end{aligned}
\]
This paper is organized as follows. In Section 2, we introduce a generalized trigonometric function \( \sin_{pq}(t) \) given by Drábek and Manásevich [7] and define a new transcendental function \( \text{sn}_{pq}(t, k) \), which is a generalization of the Jacobian elliptic function \( \text{sn}(t, k) \) and an extension of \( \sin_{pq}(t) \) as \( \text{sn}_{pq}(t, 0) = \sin_{pq}(t) \). In Section 3, we apply them to nonlinear eigenvalue problems, particularly to the problem considered in [8] and [11], and obtain complete descriptions of the set of eigenvalues and the corresponding eigenfunctions.

2. Transcendental Functions

2.1. Generalized trigonometric functions. Generalized trigonometric functions were introduced by Drábek and Manásevich [7] (see also [6]). For \( \sigma \in [0, 1] \), we define (in a slightly different way from [7])

\[
\text{arcsin}_{pq}(\sigma) := \int_0^\sigma \frac{ds}{(1 - s^q)^{\frac{1}{p}}},
\]

where \( p > 1, q > 0 \). Letting \( s = z^{1/q} \), we have

\[
\text{arcsin}_{pq}(\sigma) = \frac{1}{q} \int_0^{\sigma^q} z^{\frac{1}{q} - 1} (1 - z)^{-\frac{1}{p}} \, dz = \frac{1}{q} \tilde{B} \left( \frac{1}{q}, \frac{1}{p^*}, \sigma^q \right),
\]

where \( \tilde{B}(s, t, u) \) denotes the incomplete beta function

\[
\tilde{B}(s, t, u) = \int_0^u z^{s-1} (1 - z)^{t-1} \, dz.
\]

We define the constant \( \pi_{pq} \) as

\[
\pi_{pq} := 2 \text{arcsin}_{pq}(1) = \frac{2}{q} \tilde{B} \left( \frac{1}{q}, \frac{1}{p^*} \right),
\]

where \( B(s, t) \) denotes the beta function

\[
B(s, t) = \tilde{B}(s, t, 1) = \int_0^1 z^{s-1} (1 - z)^{t-1} \, dz.
\]

We have that \( \text{arcsin}_{pq} : [0, 1] \to [0, \pi_{pq}/2] \), and is strictly increasing. Let us denote its inverse by \( \text{sin}_{pq} \). Then, \( \text{sin}_{pq} : [0, \pi_{pq}/2] \to [0, 1] \) and is strictly increasing. We extend \( \text{sin}_{pq} \) to all \( \mathbb{R} \) (and still denote this extension by \( \text{sin}_{pq} \)) in the following form: for \( t \in [\pi_{pq}/2, \pi_{pq}] \), we set \( \text{sin}_{pq}(t) := \text{sin}_{pq}(\pi_{pq} - t) \), then for \( t \in [-\pi_{pq}, 0] \), we define \( \text{sin}_{pq}(t) := -\text{sin}_{pq}(-t) \), and finally we extend \( \text{sin}_{pq} \) to all \( \mathbb{R} \) as a \( 2\pi_{pq} \) periodic function.

When \( 0 < p \leq 1 \), we also define \( \text{arcsin}_{pq} \) as Eq. (2.1) for \( \sigma \in [0, 1) \). We have that \( \text{arcsin}_{pq} : [0, 1) \to [0, \infty) \), and is strictly increasing. Let us denote its inverse by \( \text{sin}_{pq} \). Then, \( \text{sin}_{pq} : [0, \infty) \to [0, 1) \) and is
strictly increasing. We extend \( \sin_{pq} \) to all \( \mathbb{R} \) as \( \sin_{pq}(t) := -\sin_{pq}(-t) \) for \( t \in (-\infty, 0] \) and still denote this extension by \( \sin_{pq} \).

**Remark 2.1.** We immediately find that \( \sin_{22}(t) = \sin(t) \) and \( \pi_{22} = \pi \) from the properties of the beta function. Moreover, \( \sin_{pp}(t) = \sin_p(t) \) and \( \pi_{pp} = \frac{2\pi}{p\sin\pi_p} \), where \( \sin_p \) and \( \pi_p \) are the generalized sine function and its half-period, respectively, appearing in [4], [5] and [6].

We define for \( t \in [0, \pi_{pq}/2] \) (in case \( 0 < p \leq 1 \), for \( t \in [0, \infty) \))

\[
\cos_{pq}(t) := (1 - \sin_{pq}^2(t))^{1/p},
\]

then we obtain

\[
\cos_{pq}^p(t) + \sin_{pq}^q(t) = 1,
\]

\[
\frac{d}{dt} \sin_{pq}(t) = \cos_{pq}(t).
\]

**Proposition 2.1.** For \( p, q > 1 \), \( \sin_{pq} \) satisfies for all \( \mathbb{R} \)

\[
(\phi_p(u'))' + \frac{q}{p^*} \phi_q(u) = 0.
\]

**Proof.** For \( t \in (0, \pi_{pq}/2) \) we have

\[
(\phi_p(u'))' = (\phi_p(\cos_{pq}(t)))' \]

\[
= \left((1 - \sin_{pq}^q(t))^{1/p_*}\right)'
\]

\[
= \frac{1}{p_*} (1 - \sin_{pq}^q(t))^{-\frac{1}{p_*}} \cdot (-q \sin_{pq}^{-1}(t)) \cdot \cos_{pq}(t)
\]

\[
= -\frac{q}{p_*} \phi_q(u).
\]

By symmetry of \( \sin_{pq} \), Eq. (2.2) holds true for \( t \neq t_n := n\pi_{pq}/2, n \in \mathbb{Z} \). Since \( \lim_{t \to t_n} (\phi_p(u'))' \) exists, \( \phi_p(u') \) is differentiable also at \( t = t_n \) and satisfies Eq. (2.2) for all \( \mathbb{R} \) in the classical sense. \( \square \)

2.2. **Generalized Jacobian elliptic functions.** We shall introduce new transcendental functions, which generalize the Jacobian elliptic functions. For \( \sigma \in [0, 1] \) and \( k \in [0, 1) \), we define

\[
(2.3) \quad \arcsn_{pq}(\sigma) = \arcsn_{pq}(\sigma, k) := \int_0^\sigma \frac{ds}{\sqrt{(1 - s^q)(1 - k^q s^q)}},
\]

where \( p > 1, q > 0 \). We define the constant \( K_{pq}(k) \) as

\[
K_{pq} = K_{pq}(k) := \arcsn_{pq}(1, k) = \int_0^1 \frac{ds}{\sqrt{(1 - s^q)(1 - k^q s^q)}}.
\]
We have that $\text{arcsn}_{pq} : [0, 1] \to [0, K_{pq}]$, and is strictly increasing. Let us denote its inverse by $\text{sn}_{pq} (\cdot) = \text{sn}_{pq} (\cdot, k)$. Then, $\text{sn}_{pq} : [0, K_{pq}] \to [0, 1]$ and is strictly increasing. We extend $\text{sn}_{pq}$ to all $\mathbb{R}$ (and still denote this extension by $\text{sn}_{pq}$) in the following form: for $t \in [K_{pq}, 2K_{pq}]$, we set $\text{sn}_{pq} (t) := \text{sn}_{pq} (2K_{pq} - t)$, then for $t \in [-2K_{pq}, 0]$, we define $\text{sn}_{pq} (t) := -\text{sn}_{pq} (-t)$, and finally we extend $\text{sn}_{pq}$ to all $\mathbb{R}$ as a $4K_{pq}$ periodic function.

When $0 < p \leq 1$, we also define $\text{arcsn}_{pq}$ as Eq. (2.3) for $\sigma \in [0, 1)$. We have that $\text{arcsn}_{pq} : [0, 1) \to [0, \infty)$, and is strictly increasing. Let us denote its inverse by $\text{sn}_{pq} (\cdot) = \text{sn}_{pq} (\cdot, k)$. Then, $\text{sn}_{pq} : [0, \infty) \to [0, 1)$ and is strictly increasing. We extend $\text{sn}_{pq}$ to all $\mathbb{R}$ as $\text{sn}_{pq} (t) := -\text{sn}_{pq} (-t)$ for $t \in (-\infty, 0]$ and still denote this extension by $\text{sn}_{pq}$.

The following proposition is crucial to our study.

**Proposition 2.2.** For $p, q > 0$, $K_{pq}$ is continuous and strictly increasing in $[0, 1)$, $2K_{pq}(0) = \pi_{pq}$ and $\text{sn}_{pq}(t, 0) = \sin_{pq} (t)$. Moreover,

$$
\lim_{k \to 1 - 0} 2K_{pq}(k) = \begin{cases} 
\pi_{\frac{p}{q}} & \text{if } p > 2, \\
\infty & \text{if } 0 < p \leq 2,
\end{cases}
$$

$$
\lim_{k \to 1 - 0} \text{sn}_{pq}(t, k) = \sin_{\frac{p}{q}} (t).
$$

**Proof.** The first half is trivial from the definitions of $K_{pq}$ and $\text{sn}_{pq}$. If $p > 2$, then the monotone convergence theorem of Beppo Levi gives

$$
\lim_{k \to 1 - 0} 2K_{pq}(k) = 2 \int_0^1 \frac{ds}{(1 - s^q)^{\frac{p}{q}}} = 2 \arcsin_{\frac{p}{q}} (1) = \pi_{\frac{p}{q}}.
$$

If $0 < p \leq 2$, then $2K_{pq}(k)$ diverges to $\infty$ as $k \to 1 - 0$ by Fatou’s lemma.

The last property is proved as follows. By the symmetry of $\text{sn}_{pq} (\cdot, k)$, we may assume $t > 0$. Suppose $p > 2$ and that there exist $t_0, \varepsilon > 0$ and $\{k_j\}$ such that $k_j \to 1$ as $j \to \infty$ and

$$
|\sigma_{k_j} - \sin_{\frac{p}{q}} (t_0)| \geq \varepsilon,
$$

where $\sigma_{k_j} = \text{sn}_{pq} (t_0, k_j)$. Let $n \in \mathbb{Z}$ be the number satisfying $t_0 \in I_n := \left[n\pi_{\frac{p}{q}} / 2, (n + 1)\pi_{\frac{p}{q}} / 2 \right)$ and $\hat{j} \in \mathbb{N}$ a large number satisfying $t_0 \in I_n (k_j) := [nK_{pq}(k_j), (n + 1)K_{pq}(k_j)]$. We write $\text{sn}_{pq}^{(n)} (\cdot, k_j)$ as $\text{sn}_{pq} (\cdot, k_j)$ on $I_n (k_j)$ and $\text{sn}_{pq}^{(n)} (\cdot)$ as $\text{sn}_{pq} (\cdot)$ on $I_n$. Now, since $\sigma_{k_j}$ is bounded, we can choose a subsequence $\{k_{j'}\}$ of $\{k_j\}$ such that $\sigma_{k_{j'}} \to \sigma$ for some $\sigma \in [-1, 1]$ as $j' \to \infty$. Thus, as $j' \to \infty$

$$
t_0 = nK_{pq}(k_{j'}) + \text{arcsn}_{pq} (\sigma_{k_{j'}}) \to \frac{n\pi_{\frac{p}{q}}}{2} + \arcsin_{\frac{p}{q}} (\sigma),
$$
and hence \( \sigma = \sin_{\frac{1}{2}, q}(t_0) \), which contradicts (2.4). The proof to case \( 0 < p \leq 2 \) is similar and we omit it. \( \square \)

**Remark 2.2.** In case \( p > 2 \), \( 2K_{pq}(k) \) and \( \text{sn}_{pq}(\cdot, k) \) converge to the finite value \( \pi_{\frac{1}{2}, q} \) and to the finite-periodic function \( \sin_{\frac{1}{2}, q} \) as \( k \to 1 - 0 \), respectively. This is quite different from case \( p = 2 \), where \( 2K_{2q}(k) \) diverges to \( \infty \) and \( \text{sn}_{22}(t, k) \) converges to the monotone increasing function \( \sin_{\frac{1}{2}}(t) = \tanh(t) \) as \( k \to 1 - 0 \).

We define for \( t \in [0, K_{pq}] \) (in case \( 0 < p \leq 1 \), for \( t \in [0, \infty) \))

\[
\begin{align*}
\text{cn}_{pq}(t) &:= (1 - \text{sn}_{pq}^q(t))^{\frac{1}{p}}, \\
\text{dn}_{pq}(t) &:= (1 - k^q \text{sn}_{pq}^q(t))^{\frac{1}{p}},
\end{align*}
\]

then we obtain

\[
\begin{align*}
\text{cn}_{pq}^p(t) + \text{sn}_{pq}^q(t) &= 1, \\
\frac{d}{dt} \text{sn}_{pq}(t) &= \text{cn}_{pq}(t) \text{dn}_{pq}(t).
\end{align*}
\]

**Proposition 2.3.** For \( p, q > 1 \), \( \text{sn}_{pq} \) satisfies for all \( \mathbb{R} \)

\[
(2.5) \quad (\phi_p(u'))' + \frac{q}{p^*} \phi_q(u)(1 + k^q - 2k^q|u|^q) = 0,
\]

which includes Eq. (2.2) as case \( k = 0 \).

**Proof.** For \( t \in (0, K_{pq}(k)) \) we have

\[
(\phi_p(u'))' = (\phi_p(\text{cn}_{pq}(t) \text{dn}_{pq}(t)))'
\]

\[
\begin{align*}
&= \left(\frac{1}{p^*}((1 - \text{sn}_{pq}^q(t))(1 - k^q \text{sn}_{pq}^q(t)))^{\frac{1}{p}}\right)' \\
&= \frac{1}{p^*}((1 - \text{sn}_{pq}^q(t))(1 - k^q \text{sn}_{pq}^q(t)))^{-\frac{1}{p}} \\
&\quad \times (-q \text{sn}_{pq}^{-1}(t) \cdot (1 + k^q - 2k^q \text{sin}_{pq}^q(t))) \cdot \text{cn}_{pq}(t) \text{dn}_{pq}(t) \\
&= -\frac{q}{p^*} \phi_q(u)(1 + k^q - 2k^q|u|^q).
\end{align*}
\]

By symmetry of \( \text{sn}_{pq} \), Eq. (2.5) holds true for \( t \neq t_n := nK_{pq}(k), n \in \mathbb{Z} \). Since \( \lim_{t \to t_n} (\phi_p(u'))' \) exists, \( \phi_p(u') \) is differentiable also at \( t = t_n \) and satisfies Eq. (2.5) for all \( \mathbb{R} \) in the classical sense. \( \square \)

**Remark 2.3.** Letting \( s = \text{sn}_{pq}(t) \) in Eq. (2.3), we have

\[
\text{arcsn}_{pq} \left( \sigma, k \right) = \int_0^{\text{arcsn}_{pq}(\sigma)} \frac{dt}{\sqrt{1 - k^q \text{sn}_{pq}^q(t)}}.
\]
We define the amplitude function \( \text{am}_{pq}(\cdot,k) : [0,K_{pq}(k)] \to [0,\pi_{pq}/2] \) by
\[
t = \int_{0}^{\text{am}_{pq}(t,k)} \frac{d\theta}{\sqrt{1-k^2 \sin^2\theta(\theta)}},
\]
thus \( \text{sn}_{pq} \) is represented by \( \sin_{pq} \) as
\[
\text{sn}_{pq}(t,k) = \sin_{pq}(\text{am}_{pq}(t,k)).
\]

3. Applications

3.1. The \((p,q)\)-eigenvalue problem. Let \( T, \lambda > 0 \) and \( p, q > 1 \). We consider the nonlinear eigenvalue problem
\[
(\text{E}_{pq}) \quad \begin{cases}
(\phi_p(u'))' + \lambda \phi_q(u) = 0, & t \in (0, T), \\
u(0) = u(T) = 0.
\end{cases}
\]
Problem \((\text{E}_{pq})\) has been studied by many authors. In particular, in paper [10] of ˆOtani, the existence of infinitely many multi-node solutions was proved by using subdifferential operators method and phase-plane analysis combined with symmetry properties of the solutions. After that, Drábek and Manásevich [7] provided explicit forms of the whole spectrum and the corresponding eigenfunctions for \((\text{E}_{pq})\) (see also [6]). We follow [7] to understand completely the set of all solutions of \((\text{E}_{pq})\).

It will be convenient to find first the solution to the initial value problem
\[
(3.1) \quad \begin{cases}
(\phi_p(u'))' + \lambda \phi_q(u) = 0, \\
u(0) = 0, \ u'(0) = \alpha,
\end{cases}
\]
where without loss of generality we may assume \( \alpha > 0 \).

Let \( u \) be a solution to Eq. \((3.1)\) and let \( t(\alpha) \) be the first zero point of \( u'(t) \). On interval \((0,t(\alpha))\), \( u \) satisfies \( u(t) > 0 \) and \( u'(t) > 0 \), and thus
\[
\frac{u'(t)^p}{p^*} + \lambda \frac{u(t)^q}{q} = \lambda \frac{R^q}{q} = \frac{\alpha^p}{p^*},
\]
where \( R = u(t(\alpha)) > 0 \). Solving for \( u' \) and integrating, we find
\[
\left( \frac{q}{\lambda p^*} \right)^{\frac{1}{p'}} \int_{0}^{t} \frac{u'(s)}{(R^q - u(s)^q)^{\frac{1}{p'}}} \, ds = t,
\]
which after a change of variable can be written as
\[
t = \left( \frac{q}{\lambda p^*} \right)^{\frac{1}{p'}} \frac{1}{R^{\frac{q}{p'-1}}} \int_{0}^{\frac{u(t)}{R}} \frac{ds}{(1-s^q)^{\frac{1}{p'}}} = \left( \frac{q}{\lambda p^*} \right)^{\frac{1}{p'}} \frac{1}{R^{\frac{q}{p'-1}}} \arcsin_{pq}\left( \frac{u(t)}{R} \right).
\]
Thus we obtain the solution to Eq. (3.1) can be written as

\[ u(t) = R \sin \left( \frac{\lambda^p}{q} \right)^{\frac{1}{p}} R^{q-1} t, \]

where \( R = \left( \frac{q}{\lambda^p} \right)^{\frac{1}{p}} \alpha^\frac{1}{q} \).

**Theorem 3.1.** All nontrivial solutions of \((E_{pq})\) are given as follows. For any given \( R > 0 \), the set of eigenvalues of \((E_{pq})\) is given by

\[ \lambda_n(R) = \frac{q}{p} \left( \frac{n \pi pq}{T} \right)^p R^{q-1}, \]

for each \( n \in \mathbb{N} \), with corresponding eigenfunctions \( \pm u_{n,R} \), where

\[ u_{n,R}(t) = R \sin \left( \frac{n \pi pq}{T} t \right). \]

**Proof.** For given \( R > 0 \), by imposing that \( u \) in Eq. (3.2) satisfies the boundary conditions in \((E_{pq})\), we obtain that \( \lambda \) is an eigenvalue of \((E_{pq})\) if and only if

\[ \left( \frac{\lambda^p}{q} \right)^{\frac{1}{p}} R^{q-1} T = n \pi pq, \quad n \in \mathbb{N}, \]

and hence Eq. (3.3) follows. Expression (3.4) for the eigenfunctions follows directly from Eq. (3.2). \( \square \)

### 3.2. A perturbed \((p,q)\)-eigenvalue problem.

Let \( T, \lambda > 0 \) and \( p, q > 1 \). We consider the nonlinear eigenvalue problem

\[ \text{(PE}_{pq}\text{)} \quad \begin{cases} (\phi_p(u'))' + \lambda \phi_q(u)(1 - |u|^q) = 0, & t \in (0, T), \\ u(0) = u(T) = 0. \end{cases} \]

Problem \( \text{(PE}_{pq}\text{)} \) has been studied by Berger and Fraenkel \[1\] and Chafee and Infante \[4\] (\( p = q = 2 \)), Wang and Kazarinoff \[12\] and Korman, Li and Ouyang \[9\] (\( p = 2 < q \)), Guedda and Véron \[8\] (\( p = q > 1 \)), and Takeuchi and Yamada \[11\] (\( p > 2, q > 1 \)). However, there is no study providing explicit forms of the whole spectrum and the corresponding eigenfunctions for \( \text{(PE}_{pq}\text{)} \).

As we have done for \((E_{pq})\), it will be convenient to find first the solution to the initial value problem

\[ \begin{cases} (\phi_p(u'))' + \lambda \phi_q(u)(1 - |u|^q) = 0, \\ u(0) = 0, \ u'(0) = \alpha, \end{cases} \]

where without loss of generality we may assume \( \alpha > 0 \).
Let \( u \) be a solution to Eq. (3.5) and let \( t(\alpha) \) be the first zero point of \( u'(t) \). On interval \((0, t(\alpha))\), \( u \) satisfies \( u(t) > 0 \) and \( u'(t) > 0 \), and thus
\[
\frac{u'(t)^p}{p^*} + \frac{\lambda F(u)}{q} = \lambda \frac{F(R)}{q} = \alpha^p,
\]
where \( F(s) = s^q - \frac{1}{2}s^{2q} \) and \( R = u(t(\alpha)) \). Since we are interested in functions satisfying the boundary condition of \((\text{PE}_{pq})\), it suffices to assume \( 0 < R \leq 1 \), which means \(|u| \leq 1\). Moreover, we restrict to \( 0 < R < 1 \) and concentrate solutions satisfying \(|u| < 1\) for a while.

Solving for \( u' \) and integrating, we find
\[
\left(\frac{q}{\lambda p^*}\right)^{\frac{1}{p}} \int_0^t \frac{u'(s)}{\sqrt{F(R) - F(u(s))}} ds = t,
\]
which after a change of variable can be written as
\[
(3.6) \quad t = \left(\frac{q}{\lambda p^*}\right)^{\frac{1}{p}} \int_0^{\frac{u(t)}{R}} \frac{R}{\sqrt{F(R) - F(Rs)}} ds.
\]
It is easy to verify that
\[
F(R) - F(Rs) = F(R)(1 - s^q) \left(1 - \frac{R^q}{2 - R^q} s^q\right),
\]
and hence
\[
t = \left(\frac{q}{\lambda p^*}\right)^{\frac{1}{p}} \frac{R}{F(R)^{\frac{1}{p}}} \int_0^{\frac{u(t)}{R}} \frac{ds}{\sqrt{(1 - s^q)(1 - k^q s^q)}} \left(k^q := \frac{R^q}{2 - R^q}\right)
\]
\[
\quad = \left(\frac{q}{\lambda p^*}\right)^{\frac{1}{p}} \frac{R}{F(R)^{\frac{1}{p}}} \arcsn_{pq} \left(\frac{u(t)}{R}, k\right).
\]
Then we obtain that the solution to Eq. (3.5) can be written as
\[
(3.7) \quad u(t) = R \arcsn_{pq} \left(\left(\frac{\lambda p^*}{q}\right)^{\frac{1}{p}} \frac{F(R)^{\frac{1}{p}}}{R} t, k\right),
\]
where
\[
(3.8) \quad k = \left(\frac{R^q}{2 - R^q}\right)^{\frac{1}{q}},
\]
\[
R = \left(\frac{q}{\lambda p^*}\right)^{\frac{1}{p}} \alpha^p \left(\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{2q}{\lambda p^*}}\right)^{-\frac{1}{q}}.
\]
We first observe the structure of the set of all nontrivial solutions of \((\text{PE}_{pq})\) satisfying \(|u| < 1\).
Theorem 3.2 \[(u | < 1) . \] All nontrivial solutions for \( p \in (1, 2) \) and all nontrivial solutions with \( |u| < 1 \) for \( p > 2 \) are given as follows. For any given \( k \in (0, 1) \), the set of eigenvalues of \((\text{PE}_{pq})\) is given by

\[
\lambda_n(k) = \frac{q}{p^q} (1 + k^q) \left( \frac{2k^q}{1 + k^q} \right)^{-1} \left( \frac{2nK_{pq}(k)}{T} \right)^p
\]

for each \( n \in \mathbb{N} \), with corresponding eigenfunctions \( \pm u_{n,k} \), where

\[
u_{n,k}(t) = \left( \frac{2k^q}{1 + k^q} \right)^{\frac{1}{q}} \sin_{pq} \left( \frac{2nK_{pq}(k)}{T} t, k \right).
\]

Proof. For \( k \in (0, 1) \) given, we impose that Function \((3.7)\) with \( R \in (0, 1) \) decided from Eq. \((3.8)\) satisfies the boundary conditions in \((\text{PE}_{pq})\). Then, we obtain that \( \lambda \) is an eigenvalue of \((\text{PE}_{pq})\) if and only if

\[
\left( \frac{\lambda^p}{q} \right)^{\frac{1}{p}} \frac{F(R)^{\frac{1}{p}}}{R} T = 2nK_{pq}(k), \quad n \in \mathbb{N}.
\]

From Eq. \((3.7)\) again we have

\[
\frac{F(R)^{\frac{1}{p}}}{R} = \left( \frac{2k^q}{1 + k^q} \right)^{-\frac{1}{p} - \frac{1}{q}} (1 + k^q)^{-\frac{1}{p}},
\]

and hence we obtain Eq. \((3.9)\). Expression \((3.10)\) for the eigenfunctions follows then directly from Eq. \((3.7)\).

It remains to show that no other nontrivial solution of \((\text{PE}_{pq})\) is obtained when \( 1 < p \leq 2 \). Assume the contrary. Then there exist \( t_0 > 0 \) and a nontrivial solution \( u \) of \((\text{PE}_{pq})\) with \( R = u(t_0) = 1 \). However, the right-hand side of Eq. \((3.6)\) with \( t = t_0 \) diverges because \( \sqrt{\frac{F(1) - F(s)}{(1 - s^q)^{\frac{2}{p}}}} \) as \( s \to 1 - 0 \). Thus, \( t_* = \infty \), which is a contradiction. □

Next we find solutions of \((\text{PE}_{pq})\) with \( |u| \leq 1 \), except the solutions given by Theorem 3.2. From Proposition 2.2, one of solutions of Eq. \((3.3)\) is obtained by \( k \to 1 - 0 \) in Eq. \((3.7)\) with Eq. \((3.8)\), namely

\[
u(t) = \sin_{\frac{\pi}{2q}} \left( \left( \frac{\lambda^p}{2q} \right)^{\frac{1}{p}} t \right).
\]

Now we assume \( p > 2 \) and take a number \( t_* \) as \( \left( \frac{\lambda^p}{2q} \right)^{\frac{1}{p}} t_* = \pi_{\frac{\pi}{2q}} \), then \( u \) attains 1 at \( t = t_* \) (note that it is impossible to obtain such a solution when \( 1 < p \leq 2 \)). Using this \( u \), we can make the other solutions of Eq. \((3.3)\) as follows. In the phase-plane, the orbit \((u(t), u'(t))\) arrives at the equilibrium point \((1, 0)\) at \( t = t_* \) and can stay there for any finite time \( \tau \) before it begins to leave there. Then, the interval \([t_*, t_* + \tau]\) is a
flat core of the solution. Similarly, there is the other equilibrium point 
\((-1, 0)\), where the orbit can stay, and the solution has another flat core 
of any finite length. Thus we have solutions of Eq. (3.5) attaining ±1 
with any number of flat cores.

**Theorem 3.3** (\(|u| \leq 1\)). *Let \(p > 2\), then all nontrivial solutions are 
given as follows, in addition to Theorem 3.2. For any given \(\tau \in [0, T]\), 
the set of eigenvalues of \((\text{PE}_{pq})\) is given by*

\[
\Lambda_n(\tau) = \frac{2q}{p^*} \left( \frac{n\pi q}{T - \tau} \right)^p
\]

*for each \(n \in \mathbb{N}\), with corresponding eigenfunctions \(\pm u_{n,\{\tau_i\}}\), where 
\(u_{n,\{\tau_i\}}\) is any function given as follows: for any \(\{\tau_i\}_{i=1}^n\) with \(\tau_i \geq 0\) 
and \(\sum_{i=1}^n \tau_i = \tau\)*

\[
(3.11)
\]

\[
\begin{align*}
  u_{n,\{\tau_i\}}(t) &= \begin{cases} 
  (-1)^{j-1} \sin \frac{q}{2} \left( \frac{n\pi q}{T - \tau} \frac{T - T_j - t}{T - \tau} \right) & \text{if } T_{j-1} \leq t \leq T_{j-1} + \frac{T_{j-1} - T_j}{2n}, \\
  (-1)^{j-1} & \text{if } T_{j-1} + \frac{T_{j-1} - T_j}{2n} \leq t \leq T_{j-1} + \frac{T_{j-1} - T_j}{2n}, \\
  (-1)^{j-1} \sin \frac{q}{2} \left( \frac{n\pi q}{T - \tau} (T_{j-1} - t) \right) & \text{if } T_{j-1} + \frac{T_{j-1} - T_j}{2n} \leq t \leq T_{j-1}, \\
  \end{cases}
  \end{align*}
\]

*where \(T_0 = 0\) and \(T_j = \frac{(T - \tau)j}{n} + \sum_{i=1}^j \tau_i\) for \(j = 1, 2, \ldots, n\).*

**Proof.** For each \(n \in \mathbb{N}\), it suffices to construct solutions with \((n - 1)\)-zeros. Let \(\tau \in [0, T]\). They are all generated by the eigenvalue and the 
corresponding eigenfunction of \((\text{PE}_{pq})\) with \(T\) replaced by \(T - \tau\)

\[
\Lambda_n(\tau) = \frac{2q}{p^*} \left( \frac{n\pi q}{T - \tau} \right)^p, \\
u_{n,\tau}(t) = \sin \frac{q}{2} \left( \frac{n\pi q}{T - \tau} t \right),
\]

which are obtained from Eqs. (3.9) and (3.10) with \(k \to 1 - 0\), respectively. In the phase-plane, the orbit \((u_{n,\tau}(t), u'_{n,\tau}(t))\) goes through 
the equilibrium points \((\pm 1, 0)\) in \(n\)-times without staying there as \(t\) 
increases from 0 to \(T - \tau\). Therefore, if the orbit stays the \(i\)-th equilibrium point for time \(\tau_i\), where \(\tau_1 + \tau_2 + \cdots + \tau_n = \tau\), then we can obtain 
Solution (3.11) with \(n\)-flat cores in \([0, T]\). \(\square\)

In Theorems 3.2 and 3.3 we give parameters \(k\) and \(\tau\) to obtain the 
eigenvalue and the corresponding eigenfunction of \((\text{PE}_{pq})\). Conversely, 
giving any \(\lambda > 0\), we can observe the set \(S_\lambda\) of all solutions of \((\text{PE}_{pq})\) 
by considering the inverses of \(\lambda_n\) and \(\Lambda_n\).
Theorem 3.4. Let \( p > 1 \) and \( q > 1 \).

Case \( p > q \). For any \( \lambda > 0 \) there exists a strictly decreasing positive sequence \( \{k_j\}_{j=1}^{\infty} \) such that \( k_j \to 0 \) as \( j \to \infty \) and
\[
S_\lambda = \{0\} \cup \bigcup_{j=1}^{\infty} \{\pm u_{j, k_j}\}.
\]

Case \( p = q \). If
\[
0 < \frac{q}{p^*} \left( \frac{n \pi q}{T} \right)^p < \lambda \leq \frac{q}{p^*} \left( \frac{(n + 1) \pi q}{T} \right)^p, \quad n \in \mathbb{N},
\]
then there exists a strictly decreasing positive sequence \( \{k_j\}_{j=1}^{n} \) such that
\[
S_\lambda = \{0\} \cup \bigcup_{j=1}^{n} \{\pm u_{j, k_j}\}.
\]

Case \( p < q \). There exists \( \lambda_1 > 0 \) such that if \( 0 < \lambda < \lambda_1 \), then \( S_\lambda = \{0\} \). If \( n^p \lambda_1 \leq \lambda < (n + 1)^p \lambda_1, \ n \in \mathbb{N}, \) then there exist a strictly decreasing positive sequence \( \{k_j\}_{j=1}^{n} \) and a strictly increasing positive sequence \( \{\ell_j\}_{j=1}^{n} \) such that \( k_j > \ell_j, \ j = 1, 2, \ldots, n - 1 \) and
\[
S_\lambda = \{0\} \cup \bigcup_{j=1}^{n} \{\pm u_{j, k_j}\} \cup \bigcup_{j=1}^{n} \{\pm u_{j, \ell_j}\},
\]
where \( u_{n, k_n} = u_{n, \ell_n} \) with \( k_n = \ell_n \) for \( \lambda = n^p \lambda_1 \) and \( |u_{n, k_n}| > |u_{n, \ell_n}| \) (\( t \neq jT/n, \ j = 1, 2, \ldots, n - 1 \)) with \( k_n > \ell_n \) otherwise.

In any case, each \( k_j, \ell_j \) is calculated by Eq. (3.9) for \( \lambda_j \), and the corresponding solution is given in Form (3.10).

When \( 1 < p \leq 2 \), we have \( k_j < 1 \). When \( p > 2 \), in addition, if
\[
\lambda \geq \frac{2q}{p^*} \left( \frac{m \pi q}{T} \right)^p, \quad m \in \mathbb{N},
\]
then for each \( j = 1, 2, \ldots, m \), the set \( \{\pm u_{j, k_j}\} \) above is replaced by \( \cup_{\{\tau_i\}} \{\pm u_{j, \{\tau_i\}}\} \), where \( \cup_{\{\tau_i\}} \) is the union for all \( \{\tau_i\}_{i=1}^{j} \) satisfying \( \tau_i \geq 0 \) and
\[
\sum_{i=1}^{j} \tau_i = T - j \pi q \left( \frac{2q}{\lambda p^*} \right)^{\frac{1}{p}}.
\]
The nontrivial solution \( u_{j, \{\tau_i\}} \) is given in Form (3.11).
Proof. First we assume $1 < p \leq 2$. In this case, we have already known that all nontrivial solutions of (PE$_{pq}$) are obtained by Theorem 3.2.

Now we fix $\lambda > 0$. We obtain that $\lambda$ is the $j$-th eigenvalue of (PE$_{pq}$) if and only if from Eq. (3.9) there exists $k \in (0, 1)$ such that

\[
(3.12) \quad \frac{T}{2j} \left( \frac{\lambda p^*}{q} \right)^{\frac{j}{p}} = (1 + k^q)^{\frac{j}{p}} \left( \frac{2k^q}{1 + k^q} \right)^{\frac{j - \frac{1}{p}}{q}} K_{pq}(k) =: \Phi(k).
\]

Case $p > q$. $\Phi(k)$ is strictly increasing in $(0, 1)$ and it follows from Proposition 2.2 that $\Phi(0) = 0$ and $\lim_{k \to 1^-} \Phi(k) = \infty$. Thus, there exists a unique $k = k_j(\lambda)$ satisfying Eq. (3.12). For $j$ and $k_j$, a unique solution $u_{j,k_j}$ of (PE$_{pq}$) is obtained by Eq. (3.10).

Case $p = q$. $\Phi(k)$ is strictly increasing in $(0, 1)$ and it follows from Proposition 2.2 that $\Phi(0) = \pi_{pq}/2$ and $\lim_{k \to 1^-} \Phi(k) = \infty$. Thus, if

\[
T \left( \frac{\lambda p^*}{q} \right)^{\frac{j}{p}} > \frac{\pi_{pq}}{2},
\]

namely,

\[
\lambda > \frac{q}{p^*} \left( \frac{j \pi_{pq}}{T} \right)^{p},
\]

then there exists a unique $k = k_j(\lambda)$ satisfying Eq. (3.12). For $j$ and $k_j$, a unique solution $u_{j,k_j}$ of (PE$_{pq}$) is obtained by Eq. (3.10).

Case $p < q$. It is clear that $\lim_{k \to 0^+} \Phi(k) = \lim_{k \to 1^-} \Phi(k) = \infty$. Changing variable $r = \frac{k^q}{1 + k^q}$, we can write $\Phi$ as

\[
\Psi(r) = \int_0^1 \frac{(1 + s^q)^{\frac{j}{p}} - \frac{j}{q}}{(1 - s^q)^{\frac{j}{p}}} \psi((1 + s^q)r) \, ds, \quad r \in (0, 1/2),
\]

where $\psi(t) = t^{\frac{1}{p} - \frac{1}{q}}(1 - t)^{-\frac{1}{q}}$. It is easy to see that $\psi$ is convex in $(0, 1)$ because $\psi(t) > 0$ and

\[
(\log \psi(t))'' = \left( \frac{1}{p} - \frac{1}{q} \right) \frac{1}{r^2} + \frac{1}{p} \frac{1}{(1 - t)^2} > 0.
\]

Then, $\Psi$ is twice-differentiable in $(0, 1/2)$ and

\[
\Psi''(r) = \int_0^1 \frac{(1 + s^q)^{\frac{j}{p} + \frac{1}{q} + 2}}{(1 - s^q)^{\frac{j}{p}}} \psi''((1 + s^q)r) \, ds > 0.
\]

Thus, $\Psi$ is convex and there exists $k_* \in (0, 1)$ such that $\Phi(k_*)$ is the only one critical value, and hence the minimum of $\Phi$ in $(0, 1)$.
If
\[ \frac{T}{2j} \left( \frac{\lambda p^*}{q} \right)^{\frac{1}{p}} = \Phi(k_*), \]
namely,
\[ \lambda = j^p \lambda_1 := \frac{q}{p^*} \left( \frac{2j \Phi(k_*)}{T} \right)^p, \]
then \( k_* \) satisfies Eq. (3.12). For \( j \) and \( k_* \), a unique solution \( u_{j,k_*} \) of \((\text{PE}_{pq})\) is obtained by Eq. (3.10). Moreover, if
\[ \frac{T}{2j} \left( \frac{\lambda p^*}{q} \right)^{\frac{1}{p}} > \Phi(k_*), \]
namely, \( \lambda > j^p \lambda_1 \), then there exist \( k = k_j(\lambda) \) and \( \ell_j(\lambda) \) such that
\[ k_j(\lambda) = \Phi^{-1} \left( \frac{T}{2j} \left( \frac{\lambda p^*}{q} \right)^{\frac{1}{p}} \right) \in (k_*, 1), \]
\[ \ell_j(\lambda) = \Phi^{-1} \left( \frac{T}{2j} \left( \frac{\lambda p^*}{q} \right)^{\frac{1}{p}} \right) \in (0, k_*). \]

For \( j, k_j \) and \( \ell_j \), solutions \( u_{j,k_j} \) and \( u_{j,\ell_j} \) of \((\text{PE}_{pq})\) are obtained by Eq. (3.10).

Next, we assume \( p > 2 \). In any case, a similar proof as above with \( \lim_{k \to 1^{-}0} \Phi(k) = 2^{\frac{1}{p} - 1} \pi^{\frac{2}{p}}q \) instead of \( \lim_{k \to 1^{-}0} \Phi(k) = \infty \) gives that it is impossible to find \( k_m \in (0, 1) \) above satisfying Eq. (3.12), provided
\[ \lambda \geq \frac{2q}{p^*} \left( \frac{m \pi^{\frac{2}{p}}q}{T} \right)^p, \quad m \in \mathbb{N}. \]

Then, however, for each \( j = 1, 2, \ldots, m \), we can take \( \tau \in [0, T) \) such that
\[ \lambda = \frac{2q}{p^*} \left( \frac{j \pi^{\frac{2}{p}}q}{T - \tau} \right)^p, \]
and Theorem 3.3 yields the solutions \( u_{j,\{\tau_i\}} \), where \( \{\tau_i\}_{i=1}^j \) is any sequence satisfying that \( \tau_i \geq 0, \sum_{i=1}^j \tau_i = \tau \). \hfill \Box

It follows directly from Representation (3.11) of Theorem 3.3 that a kind of solution of \((\text{PE}_{pq})\) with \( p \)-Laplacian is also an eigenfunction of \((\text{E}_{\frac{2}{p},q})\) with \( p/2 \)-Laplacian.
Corollary 3.1. Let $p > 2$. For each $n \in \mathbb{N}$, any solution $u_{n,\{\tau_i\}}$ of $(PE_{pq})$ in Theorem 3.3 satisfies

$$
\left(\frac{\phi_p(u')}{}\right)' + \left(\frac{(p-2)q}{p}\left(\frac{n\pi_p q}{T-\tau}\right)\phi_q(u)\right) = 0
$$

in the intervals where $|u_{n,\{\tau_i\}}| < 1$, where $\tau = \sum_{i=1}^{n} \tau_i$. In particular, for each $n \in \mathbb{N}$, the solution $u_{n,\{0\}}$ of $(PE_{pq})$ with $\tau = 0$ is an eigenfunction of $(E_{p^2,q})$, that is,

$$
\left\{
\begin{aligned}
(\phi_p(u'))' + \left(\frac{(p-2)q}{p}\left(\frac{n\pi_p q}{T}\right)\phi_q(u)\right) &= 0, \quad t \in (0,T), \\
u(0) &= u(T) = 0.
\end{aligned}
\right.
$$

Moreover, $u_{n,\{0\}}$ is characterized by $u_{n,R}$ with $R = 1$ in Eq. (3.4) with $p$ replaced by $p/2$.

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DEPARTMENT OF GENERAL EDUCATION, KOGAKUIN UNIVERSITY, 2665-1 NAKANO, HACHIOJI, TOKYO 192-0015, JAPAN

*E-mail address: shingo@cc.kogakuin.ac.jp*