CLASSICAL AND QUANTUM MECHANICS OF BLACK HOLES IN GENERIC 2-D DILATON GRAVITY

by

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ABSTRACT

A unified description is presented of the physical observables and thermo-
dynamic variables associated with black hole solutions in generic 2-D dilaton
gravity. The Dirac quantization of these theories is reviewed and an intrigu-
ing relationship between the entropy of the black hole and the WKB phase
of the corresponding physical wave functionals is revealed.

1Talk given by G. Kunstatter at the Conference on Heat Kernels and Quantum Gravity,
Winnipeg, August, 1994
1 Introduction

One of the most difficult and important goals of theoretical physics is to construct a mathematically and conceptually consistent theory of quantum gravity. The inadequacy of the usual semi-classical approximation, in which only matter fields are quantized, has recently been emphasized in the context of the so-called information loss paradox associated with the Hawking radiation of black holes [1]. There have been many attempts in recent years to avoid the pitfalls of the usual semi-classical approximation by considering models of quantum gravity in two spacetime dimensions. In the absence of matter, these models are exactly solvable both classically and quantum mechanically. The hope is that, in addition to providing useful conceptual and theoretical insights, in some cases they provide reasonable approximations to the quantum mechanical behaviour of physical black holes. One model in particular, (spherically symmetric gravity or SSG) is obtained by truncating all non-spherically symmetric modes in ordinary Einstein gravity in 3+1 dimensions [2]. Another model of current interest is string inspired 2-d dilaton gravity (SIG) which was recently used to investigate perturbative effects of backreaction on the end-point of gravitational collapse [3]. Finally, Jackiw-Teitelboim gravity (JT) [4], which was the first 2-D dilaton theory, can be obtained by imposing axial symmetry in 2+1 Einstein gravity. The latter has gained new importance in the present context due to the recent discovery [5] of black hole solutions in the 2+1 dimensional theory. These solutions have been used in an attempt to provide a statistical origin for black hole entropy [6].

The models mentioned above are all special cases of generic (1+1) dimensional dilaton gravity theories [7] in which a scalar field is non-minimally coupled to the spacetime curvature. As will be shown below, it is possible to provide a complete, unified treatment of all the models by adapting a suitable parametrization for the fields. In particular, we will be able to provide explicit representations for the classical observables for black holes (if they exist) in all models including the phase space variables, the surface gravity and the entropy. Moreover, it is possible to solve the constraints exactly in these models and write down physical quantum wavefunctions [8] which can be interpreted as a WKB approximation to the stationary states describing quantum black holes. One of the more intriguing results that emerges from
this analysis is the relationship between the imaginary part of the WKB phase and the classical thermodynamic entropy[9].

The paper is organized as follows. In Section 2 we present the action for generic 2-D dilaton gravity and discuss the space of the solutions. It turns out that the theories obey a generalized Birkhoff theorem[10]: all solutions have a timelike Killing vector, and can be parametrized by the value of a single coordinate invariant, conserved quantity(i.e the energy). Section 3 describes the classical thermodynamics for these models and shows how to calculate the surface gravity and entropy in arbitrary coordinates for the generic theory. Section 4 summarizes the canonical analysis for these theories, showing that the reduced phase space is two dimensional despite the fact that there exists only a one parameter family (up to space time diffeomorphism) of inequivalent solutions. The physical interpretation of these two phase space variables is also presented: as expected one is the energy while its conjugate is related to the Killing time separation between slices at spatial infinity. This interpretation was recently given for SSG by Kuchar[11]. Section 5 reviews the Dirac quantization of the models and shows the intriguing, generic relationship between the black hole entropy and the WKB phase. Section 6 closes with conclusions and prospects for future research.

2 Action and Space of Solutions

The most general action functional depending on the metric tensor $g_{\mu\nu}$ and a scalar field $\phi$ in two spacetime dimensions, such that it contains at most second derivatives of the fields can be written[7]:

$$S[g,\phi] = \frac{1}{2G} \int d^2x \sqrt{-g} \left( \frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \frac{1}{l^2} \tilde{V}(\phi) + D(\phi) R \right). \quad (1)$$

In the above, $G$ is the (dimensionless) 2-d gravitational constant and $D(\phi)$ and $\tilde{V}(\phi)$ are arbitrary functions of the scalar field. This action is a generalization of what occurs naturally if one restricts to only spherically symmetric modes in 3+1 Einstein gravity. To see this consider the spherically symmetric metric

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + e^{-2\phi(x)}d\Omega^2, \quad (2)$$
where the $x^\mu$ are coordinates on a two dimensional spacetime $M_2$ with metric $g_{\mu\nu}(x)$ and $d\Omega^2$ is the line element of the 2-sphere with area $4\pi$. The spherically symmetric vacuum Einstein solutions are the stationary points of the dimensionally reduced action functional:

$$\tilde{I}[g, \phi] = \frac{1}{2G} \int_{M_2} d^2x \sqrt{-g} e^{-2\phi} \left( R(g) + 2g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + 2e^{2\phi} \right).$$ (3)

which is of the same form as Eq. (1) above. As first discussed in [7] and shown explicitly in [8], one can eliminate the kinetic term for the scalar by reparametrizing the fields:

$$g_{\mu\nu} \rightarrow h_{\mu\nu} = \Omega^2(\phi) g_{\mu\nu},$$ (4)

$$\phi \rightarrow \tau = D(\phi),$$ (5)

with $\Omega^2 = \exp \frac{1}{2} \int d\phi \frac{d\phi}{(dD/d\phi)}$. This leads to an action functional of the form:

$$I[h, \tau] = \frac{1}{2G} \int_{M^2} d^2x \sqrt{-h} \left( \tau R(h) + \frac{1}{l^2} V(\tau) \right).$$ (6)

where $V(\tau)$ is an arbitrary function of the scalar field $\tau$. For spherically symmetric gravity, $V(\tau) = 1/\sqrt{2\tau}$, while $V(\tau) = \tau$ for JT, and $V(\tau) = 1$ for SIG.

The equations of motion take the simple form:

$$R = -\frac{1}{l^2} \frac{dV}{d\tau},$$ (7)

and

$$\nabla_\mu \nabla_\nu \tau - \frac{1}{2l^2} g_{\mu\nu} V = 0.$$ (8)

As shown in [10] the theory obeys a generalized Birkhoff’s theorem which states that up to spacetime diffeomorphisms there is a one parameter family of solutions for each choice of potential. The parameter is a coordinate-invariant constant of integration which can be expressed in covariant form as follows:

$$E = \frac{1}{2l}(\nabla \tau)^2 + J(\tau).$$ (9)

It will be shown in Section 4 that $E$ can be interpreted as the energy of the solution, as expected.
The solutions take a particularly simple form in what we call the Schwarzschild gauge (since the relationship to SSG is manifest). Choosing

\[ \tau = x/l, g_{tx} = 0, \]  

the most general solution is:

\[ ds^2 = -(J(x/l) - 2lE)dt^2 + (J(x/l) - 2lE)^{-1}dx^2, \]  

where \( J'(\tau) = V(\tau) \)

The solutions as written above clearly have a Killing vector, which can also be written in covariant form:

\[ k^\mu = \eta^\mu\nu \tau, \nu. \]  

In the above \( \eta^{\mu\nu} = -\eta^{\nu\mu} = \frac{1}{\sqrt{-g}} \epsilon^{\mu\nu} \) is the antisymmetric tensor. The constant \( l \) is required to make the Killing vector components dimensionless. It can easily be verified that Eq. (8) implies that \( k^\mu \) satisfies the Killing equation \( \nabla(\mu k^\nu) = 0 \) on shell. Moreover, it is clear that \( \tau,\mu k^\mu = 0 \) identically, so that the scalar field is also invariant along the Killing directions. Note that

\[ |k|^2 = -l^2 |\nabla \tau|^2 = 2lE - J(\tau). \]  

Both the form of the solution in Schwarzschild gauge, and the magnitude of the Killing vector suggest that the dilaton theories admit black holes providing that there exists at least one curve in spacetime given by \( \tau(x,t) = \tau_0 = \text{constant} \), such that \( J(\tau_0) = 2lE \). In addition, \( J(\tau) \) must be monotonic (in \( \tau \)) in a neighbourhood of \( \tau_0 \). These conditions are satisfied for SSG, JT and SIG, as well as for a large class of additional theories[12]. For example in SSG, for which \( V(\tau) = 1/\sqrt{2\tau} \), the static solution for the metric in our parametrization is related to the usual Schwarzschild solution by the conformal reparametrization \( ds^2 = \sqrt{2\tau}ds^2_{\text{Schwarz}} \). In terms of the coordinate \( r = l\sqrt{2\tau} \), the metric Eq.(11) takes the form:

\[ h_{\mu\nu}dx^\mu dx^\nu = \frac{r}{l} \left\{ -(1 - 2m/r)dt^2 + (1 - 2m/r)^{-1} \right\} dr^2, \]  

where the mass \( m = l^2E \). Finally, \( (k^\mu) = (1,0) \) and \( |k|^2 = (2m - r)/l \).

For JT, we define \( \tau = x/l \), so that the solution takes the form:

\[ h_{\mu\nu}dx^\mu dx^\nu = \left\{ -(\frac{x^2}{l^2} - 2lE)dt^2 + (\frac{x^2}{l^2} - 2lE)^{-1} \right\} dx^2, \]  

For JT, we define \( \tau = x/l \), so that the solution takes the form:

\[ h_{\mu\nu}dx^\mu dx^\nu = \left\{ -(\frac{x^2}{l^2} - 2lE)dt^2 + (\frac{x^2}{l^2} - 2lE)^{-1} \right\} dx^2, \]  

For JT, we define \( \tau = x/l \), so that the solution takes the form:
The black hole in this case is the 2-D projection of the BTZ solution \([5]\) in 2+1 gravity: it is not asymptotically flat, because the metric describes a spacetime that is everywhere locally de-Sitter.

Finally, for the string inspired theory, with \(\tau = x/l\) as above:

\[
h_{\mu\nu} dx^\mu dx^\nu = \left\{ -(\frac{x}{l} - 2lE) dt^2 + \left(\frac{x}{l} - 2lE\right)^{-1} \right\} dr^2,
\]

What is remarkable is that the metric in the above solution is completely flat in the parametrization we have chosen. This is a consequence of the fact that in 2 spacetime dimensions all metrics are conformally flat: the reparametrization Eq.(4) has in this case transformed the physical metric to the flat metric in Eq.(16). As shown below, however, the information about the event horizon and associated thermal properties is still encoded in the solution, as long as one considers both the scalar field and the metric.

Note that at this stage we are discussing only local properties of the solutions. To establish rigorously the existence of an event horizon one needs to study global properties. In the three theories of immediate interest, these global properties do support the existence of event horizons. With a slight abuse of notation we will generically denote the curve defined by \(J(\tau_0) = 2lE\) as the event horizon, since this is the curve along which the Killing vector is null.

### 3 Thermodynamical Properties

Given the possible existence of black holes and event horizons, it is straightforward to use the explicit expression for the Killing vector to calculate the surface gravity and entropy of a generic 2-D black hole. The surface gravity \(\kappa\) is determined by the following expression, evaluated at the event horizon\([13]\):

\[
\kappa^2 = -\frac{1}{2} \nabla^\mu k^\nu \nabla_\mu k_\nu.
\]

Using Eq.(12) for \(k^\mu\) and the field equations Eq.(8) one obtains:

\[
\kappa = \frac{1}{2l} V(\tau_0),
\]
where \( V(\tau_0) \) is the potential evaluated at \( \tau = \tau_0 \) (i.e. on the event horizon). The sign in Eq. (18) was chosen to yield a positive surface gravity for positive energy. Note that \( \tau_0 \) is given implicitly as a function of the energy \( q \) by requiring \(|k|^2 = 0\) in Eq. (11).

The Hawking temperature for the generic black hole solution can be calculated heuristically by defining the Euclidean time \( t_E = it \) in Eq. (11) and then finding the periodicity condition on \( t_E \) that makes the solution everywhere regular. This is done by defining the coordinate \( R^2 := a(J(\tau_0) - 2lE) \) and choosing the constant \( a \) so that the spatial part of the metric goes to \( dR^2 \) at the event horizon \( \tau_0 \). A straightforward calculation gives \( a = |2l/V(\tau_0)| \), so that the Hawking temperature, which is the inverse of the period of \( t_E \), is:

\[
T_H = \frac{1}{2\pi} \frac{V(\tau_0)}{2l} = \frac{\kappa}{2\pi},
\]

as expected. Note that this calculation does not depend on the details of the model: it merely requires the existence of a horizon at which \( J(\tau_0) = 2lE \).

The entropy, \( S \), can now be determined by inspection of Eq. (13). In particular, if one varies the solution, but stays on the event horizon (i.e. at \( \tau = \tau_0 \)), one finds the variation of the energy to be:

\[
\delta E = \frac{1}{2lG} \delta J(\tau_0) = \frac{1}{2lG} V(\tau_0) \delta \tau_0 .
\]

Identifying the Hawking temperature and surface gravity derived above, we find that the first law of thermodynamics \( \delta E = T\delta S \) will be satisfied providing we identify the entropy to be

\[
S = \frac{2\pi}{G} \tau_0 .
\]

This expression agrees with Wald’s more general local geometric formulation for the entropy of a black hole. For SIG \( V(\tau) = 1/\sqrt{2\tau} \), and for the solution given above \( \tau_0 = 2m^2/l_p^2 \). Thus Eq. (21) gives the correct entropy for a spherically symmetric black hole. It also agrees with results for black holes in SIG and for the dimensionally reduced BTZ black hole obtained as a solution to JT.
4 Hamiltonian Analysis

We now review the Hamiltonian analysis of the general 1+1–dimensional theory\[8\]. Spacetime is split into a product: \( M_2 \simeq \Sigma \times R \) and the metric \( h_{\mu\nu} \) is given an ADM-like parameterization:\[17\].

\[
ds^2 = e^\alpha \left[ -\left( M^2 + N^2 \right) dt^2 + (dx + Mdt)^2 \right].
\] (22)

where \( \alpha, M \) and \( N \) are functions on spacetime \( M_2 \). We define the quantity \( \sigma \) by \( \sigma^2 := M^2 + N^2 \). Also, in the following, we denote by the overdot and prime, respectively, derivatives with respect to the time coordinate \( t \) and spatial coordinate \( x \).

The canonical momenta for the fields \( \{\alpha, \tau\} \) are respectively:

\[
\Pi_\alpha = \frac{1}{2G\sigma} (M\tau' - \dot{\tau}),
\] (23)

\[
\Pi_\tau = \frac{1}{2G\sigma} (-\dot{\alpha} + M\alpha' + 2M'),
\] (24)

The vanishing of the momenta canonically conjugate to \( M \) and \( \sigma \) yield the primary constraints for the system. By following the standard Dirac prescription\[18\], it can be shown\[10\] that the canonical Hamiltonian is, up to a surface term, a linear combination of first class constraints:

\[
H_0 = \frac{1}{2G} \int dx \left[ \left( \frac{\dot{\tau}}{\tau} \right) \mathcal{F} - \left( \frac{\sigma e^\alpha}{l\tau'} \right) \tilde{G} \right] dx .
\] (25)

where \( \dot{\tau} \) is to be considered as a function of the phase space coordinates and lapse and shift functions, given implicitly by Eq.(23). The constraints \( \mathcal{F} \) and \( \tilde{G} \) generate spatial diffeomorphism and translations along the Killing vector\[9\], respectively. They are:

\[
\mathcal{F} := \alpha'\Pi_\alpha + \tau'\Pi_\tau - 2\Pi'_{\alpha}
\] (26)

\[
\tilde{G} := (q[\alpha, \tau, \Pi_\alpha, \Pi_\tau])' \approx 0 ,
\] (27)

where we have defined the variable \( q[\alpha, \tau] \) as

\[
q[\alpha, \tau] := \frac{l}{2} \left[ e^{-\alpha} \left( 2G\Pi_{\alpha}^2 - (\tau')^2 \right) + l^{-2}J(\tau) \right].
\] (28)

8
The expression on the right-hand side above is nominally an implicit function of the spatial coordinate, but is constant on the constraint surface. Moreover, it is straightforward to show that \( q \) commutes with both constraints \( \mathcal{F}, \tilde{\mathcal{G}} \). Thus, the constant mode of \( q \) is a physical observable in the Dirac sense.

In order for the variation of the above Hamiltonian to yield the correct equations of motion it is necessary to add the following surface term:

\[
H_{ADM} = \int d\Sigma \left( \left( \frac{\sigma e^\alpha}{l}\right) \frac{q}{G} \right). \tag{29}
\]

It is easy to verify that for solutions of the form Eq. (11), \( \sigma e^\alpha / l' = 1 \). Hence, \( H_{ADM} = q/G \) is the ADM energy, as expected.

In terms of the canonical momenta the magnitude of the Killing vector can be written as:

\[
| k |^2 = l^2 e^{-\alpha} \left[ (2G\Pi_\alpha)^2 - (\tau')^2 \right]. \tag{30}
\]

Thus the observable \( q \) is:

\[
q = \frac{1}{2l} \left( | k |^2 + J(\tau) \right) = E. \tag{31}
\]

This proves that the parameter \( E \) defined in Section 2 is one of the physical phase space observables in the theory, and that it is the energy of the solution as anticipated in Section 2.

The momentum conjugate to \( q \), is found by inspection to be:

\[
p := -\int_\Sigma dx \frac{2\Pi_\alpha e^\alpha}{(2G\Pi_\alpha)^2 - (\tau')^2}. \tag{32}
\]

It can easily be verified that the Poisson algebra for the fields and the momenta leads directly to \( \{q, p\} = 1 \). Moreover, under a general gauge transformation the change in \( p \) is the integral of a total divergence, which vanishes only if the test functions go to zero sufficiently rapidly at infinity. The value of \( p \) therefore depends on the global properties of the spacetime slicing. This is consistent with the generalized Birkhoff theorem which states that there is only one independent diffeomorphism invariant parameter characterizing the space of solutions.
The observable $p$ can be written in covariant form:

$$p = -2 \int_{\Sigma} dx^\mu \frac{k_\mu}{|k|^2}.$$  \hspace{1cm} (33)

This form of the expression shows that in static slicings $p = 0$. Moreover, the integrand of $p$ has a pole at the location of any event horizon in the model. Thus, analytic continuation is in general required to make the expression well defined, and may introduce an imaginary part to the observable $p$. For example, in spherically symmetric gravity, one can show that in Kruskal coordinates the observable $p$ integrated along a slice of constant Kruskal time $T$ takes the simple form:

$$p = \frac{2m}{G} \int dX \left[ \frac{1}{X - T} - \frac{1}{X + T} \right]$$
$$= \frac{2m}{G} \ln \left( \frac{X - T}{X + T} \right)_{X_i}^{X_f}. \hspace{1cm} (34)$$

$p$ is therefore precisely the difference in Schwarzschild times at the endpoints of the spatial slice. In this case there are simple poles at $X = \pm T$ (i.e. at $r = 2m$), so that for an eternal black hole, with suitable analytic continuation, $\text{Im} p = 2\pi m/G$. Although this potential imaginary piece is irrelevant classically for the Schwarzschild time, it may have some significance in the quantum theory in which $p$ is a physical phase space observable.

The global variable $p$ has a natural physical interpretation in terms of the time separation at infinity of neighbouring spacelike surfaces which are asymptotically normal to the Killing vector field $k^\mu$. Let $U$ be the “triangular region” of spacetime bounded by spacelike surfaces $\Sigma_1, \Sigma_2$ and by a timelike surface $T$ at infinity tangent to $k^\mu$. By using Gauss’ law, it is possible to show that between two such surfaces, the difference in $p$ is

$$p_2 - p_1 = -\int_T d\theta,$$  \hspace{1cm} (35)

where $p_1, p_2$ are the values of $p$ on $\Sigma_1, \Sigma_2$, respectively, and $\theta$ is a parametrization of the timelike line $T$ such that the induced metric

$$h_{\theta\theta} := h_{\mu\nu} \frac{\partial x^\mu}{\partial \theta} \frac{\partial x^\nu}{\partial \theta} = |k|^2.$$
5 Dirac Quantization

We will now outline the procedure for the quantization of the generic theory in the functional Schrodinger representation. Following the Dirac prescription we first define a Hilbert space in terms of wave functionals $\psi[\alpha, \tau]$ of the configuration space coordinates $\alpha(x)$ and $\tau(x)$. The momenta conjugate to $\alpha$ and $\tau$ are represented as self-adjoint operators on this Hilbert space. If the Hilbert space measure is:

$$<\psi|\psi> = \int \prod_x d\alpha(x) d\tau(x) \mu[\alpha, \tau] \psi^*[\alpha, \tau] \psi[\alpha, \tau]$$

then

$$\hat{\Pi}_\alpha = -i\hbar \frac{\delta}{\delta \alpha(x)} + \frac{i\hbar}{2} \frac{\delta \ln(\mu[\alpha, \tau])}{\delta \alpha}$$

$$\hat{\Pi}_\tau = -i\hbar \frac{\delta}{\delta \tau(x)} + \frac{i\hbar}{2} \frac{\delta \ln(\mu[\alpha, \tau])}{\delta \alpha}$$

Physical states are those that are annihilated by the quantum operators corresponding to the constraints $\mathcal{F}$ and $\tilde{\mathcal{G}}$. Since the latter is quadratic in the momentum one has to find a suitable factor ordering that makes the operator self-adjoint with respect to the chosen functional measure. In the following we will consider only the lowest order WKB approximation, which is insensitive to the factor ordering and choice of measure. We can therefore set $\mu[\alpha, \tau] = 1$ without loss of generality. The extension of this analysis to higher order is currently under investigation.

We now focus on stationary states of the theory, which we define to be eigenstates of the energy operator $\hat{q}$ with constant eigenvalue, $E$:

$$\hat{q}\psi[\alpha, \tau] = E\psi[\alpha, \tau]$$

\footnote{Note that the semi-classical approach taken here is somewhat different than the one suggested in \textsuperscript{8} and \textsuperscript{20}, in which the constraints were solved classically for the momenta, and then imposed exactly as quantum mechanical constraints on the space of physical states. The WKB approximation has also recently been discussed in the present context by Lifschytz et al.}
that are annihilated by the diffeomorphism constraint:

\[ \hat{\mathcal{F}} \psi[\alpha, \tau] = 0 \]  

Note that the eigenvalue equation above is sufficient (although not necessary) for the state to satisfy the Hamiltonian constraint, since:

\[ \hat{\mathcal{G}} \psi[\alpha, \tau] = E' \psi[\alpha, \tau] = 0 \]  

Thus, the quantum constraints Eq.(39) and Eq.(40) yield physical states which satisfy:

\[ \hat{H} \psi[\alpha, \tau] = E \psi[\alpha, \tau] \]  

where \( \hat{H} \) is the quantized version of \( H := H_0 + H_{ADM} \) given in Section 4. These are therefore the analogue of "stationary states" in ordinary non-relativistic quantum mechanics.

To implement the WKB approximation, we assume that the wave functional can be written:

\[ \psi[\alpha, \tau] = \exp\frac{i}{\hbar}[S_0[\alpha, \tau] + \hbar S_1[\alpha, \tau] + ...] \]  

To lowest order in \( \hbar \) we find that \( S_0 \) must be invariant under spatial diffeomorphisms:

\[ \alpha \frac{\delta S_0}{\delta \alpha} + \tau \frac{\delta S_0}{\delta \tau} - 2 \left( \frac{\delta S_0}{\delta \alpha} \right)' = 0 \]  

and that it obeys the Hamilton Jacobi equation associated with the energy function \( q[\alpha, \tau] \) given in Eq.(28):

\[ e^{-\alpha} \left( 2G \left( \frac{\delta S_0}{\delta \alpha} \right)^2 - (\tau')^2 \right) + l^{-2} J(\tau) = \frac{2E}{l}. \]  

These two equations can be solved algebraically to yield:

\[ \frac{\delta S_0}{\delta \alpha} = Q[\alpha, \tau; q] \]  

and

\[ \frac{\delta S_0}{\delta \tau} = \frac{1}{Q[\alpha, \tau]} (2\tau'' - \tau' \alpha' - e^\alpha V(\tau)) \]
where
\[ Q := \sqrt{(\tau')^2 + e^\alpha \left( \frac{2E}{l} - \frac{J(\tau)}{l^2} \right)} \] (48)

The closure of the constraint algebra guarantees that these functional differential equations can be integrated. The solution, first given in [8] is

\[ S_0[\alpha, \tau; q] = \int dx \left[ Q + \frac{\tau'}{2} \ln \left( \frac{\tau' - Q}{\tau' + Q} \right) \right], \] (49)

As usual in the WKB method, the imaginary parts of the phase are related to quantum mechanical tunnelling into classically forbidden regions, which in this case occur for \( Q[\alpha, \tau] = 0 \). This follows from the fact that \( Q = \Pi_\alpha \) on the constraint surface (see Eq.(46) above). It is interesting to note that if we restrict to classically allowed regions, for which \( Q^2 \geq 0 \), then the phase \( S \) can still acquire an imaginary part from the logarithm when \((\tau')^2 - Q^2 \leq 0\). This is precisely the region where the Killing vector for the solution is spacelike (in a non-singular coordinate system for which \( e^\alpha \) is positive). Therefore for theories with an event horizon, the logarithm in Eq.(49) can be analytically continued so that

\[ ImS_0 = \frac{i\pi \tau_0}{2} = i\frac{S}{4}. \] (50)

The imaginary part of the WKB phase is therefore proportional to the entropy of the black hole. This is consistent with an earlier heuristic result obtained for spherically symmetric gravity[20] and suggests a relationship between black hole entropy, and quantum mechanical tunnelling processes.

\section{6 Conclusions}

By choosing a suitable parametrization, we have been able to present a unified treatment of the most general dilaton gravity theory, including a complete characterization of the classical space of solutions, phase space variables and

\footnote{In the case of Jackiw-Teitelboim gravity, this solution has recently been shown[21] to be quantum mechanically equivalent to the wave function obtained by quantizing the gauge theoretic version of that theory.}
thermodynamical quantities. We have also outlined the Dirac quantization of the theory in the functional Schrodinger representation, and calculated the WKB phase. In the process, an interesting connection emerged between the imaginary part of the phase of these wave functionals and the entropy of the corresponding black holes.

These results indicate that it may be possible to extract significant information concerning Hawking radiation, black hole entropy and perhaps even the endpoint of gravitational collapse from these models. Before this can be done, however, several issues should be investigated and clarified. It should be possible to go beyond leading order in the WKB approximation, defining a suitable Hilbert space measure and verifying the closure of the constraint algebra[22]. It is also necessary to investigate these models with matter, since Hawking radiation is not strictly possible in vacuum dilaton gravity due to the absence of radiative modes. Finally it should be possible to check whether one can provide a statistical mechanical interpretation of the black hole entropy in these simple models in terms of quantum states on the horizon, as done recently for the BTZ black holes by Carlip[6]. On the surface, the states in Carlip’s analysis appear to arise due to the presence of non-axially symmetric gauge modes, negating the possibility of a similar result in the dimensionally reduced case. However, since black holes in JT gravity have the same temperature and entropy as the 2+1 case, one might expect to be able to account for this entropy using similar techniques in both theories. These questions are currently under investigation.

Acknowledgements
The authors are grateful to A. Barvinsky, S. Carlip, V. Frolov, D. McManus, P. Sutton, G.A. Vilkovisky and D. Vincent for helpful discussions. This work was supported in part by the Natural Sciences and Engineering Research Council of Canada.

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