Recurrence of Optimum for Training Weight and Activation Quantized Networks

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Abstract

Deep neural networks (DNNs) are quantized for efficient inference on resource-constrained platforms. However, training deep learning models with low-precision weights and activations involves a demanding optimization task, which calls for minimizing a stage-wise loss function subject to a discrete set-constraint. While numerous training methods have been proposed, existing studies for full quantization of DNNs are mostly empirical. From a theoretical point of view, we study practical techniques for overcoming the combinatorial nature of network quantization. Specifically, we investigate a simple yet powerful projected gradient-like algorithm for quantizing two-layer convolution networks, by repeatedly moving one step at float weights in the negative direction of a heuristic fake gradient of the loss function (so-called coarse gradient) evaluated at quantized weights. For the first time, we prove that under mild conditions, the sequence of quantized weights recurrently visit the global optimum of the discrete minimization problem for training fully quantized network. We also show numerical evidence of the recurrence phenomenon of weight evolution in training quantized deep networks.

1. Introduction

Deep neural networks (DNNs) have been profoundly transforming machine learning, in applications of computer vision, reinforcement learning, and natural language processing, and so on. While achieving human level or even superhuman performances, DNNs typically have tremendous number of weights with high resource consumption at inference time, which poses a challenge for their deployment on mobile devices used in our daily lives. To address this challenge, research efforts have been made to the quantizing weights and activations of DNNs while maintaining their performance. Quantization methods train DNNs with the weights and activation values being constrained to low-precision arithmetic rather than the conventional floating-point representation in full-precision. (Hubara et al., 2018; Zhou et al., 2016; Cai et al., 2017; Zhou et al., 2017; Louizos et al., 2019; Zhu et al., 2016), which offer the feasibility of running DNNs on CPUs rather than GPUs in real-time. For example, the XNOR-Net (Rastegari et al., 2016) with binary weights and activations sees 58× faster convolutional operations and 32× memory savings.

Training fully quantized DNN requires solving a challenging optimization problem with piecewise constant (and non-convex) training loss functions and a discrete set-constraint. That is, one considers the following constrained optimization problem for training quantized neural nets:

$$\min_w f(w) := E_{x \sim \rho(x)} [\ell(w; x)] \quad \text{subject to} \quad w \in Q$$

where $\ell(w; x)$ is the loss function for sample $x$, which is discrete-valued as non-linear activations are also quantized; $Q$ is the set of quantized weights. For general constrained minimization, the classical projected gradient descent (PGD):

$$w^{t+1} = \operatorname{proj}_Q (w^t - \eta_t E [\nabla_w \ell(w^t; x)])$$

is considered. Here $\operatorname{proj}_Q$ is the projection onto set $Q$ for quantizing float weights to ones at low bit-width, giving a weight quantization scheme. However, with quantized activations, the gradient of loss function $\nabla_w \ell(w; x)$ is almost everywhere (a.e.) zero, leaving the standard backpropagation and hence PGD inapplicable.

In this paper, we study the following iterative algorithm for training fully quantized networks

$$\begin{cases} y^{t+1} = y^t - \eta_t E [\nabla_w \ell(w^t; x)] \\ w^{t+1} = \operatorname{proj}_Q (y^{t+1}) \end{cases}$$ (QUANT)

where $\nabla_w \ell$ denotes some heuristic modification of the vanished $\nabla_w \ell$ based on the so-called straight-through estimator (STE) (Bengio et al., 2013; Hinton, 2012), rendering a valid search direction. Following (Yin et al., 2019), we shall refer to this fake ‘gradient’ induced by STE as coarse gradient throughout this paper. Compared with PGD which can be recast as the two-step iteration:

$$\begin{cases} y^{t+1} = w^t - \eta_t E [\nabla_w \ell(w^t; x)] \\ w^{t+1} = \operatorname{proj}_Q (y^{t+1}) \end{cases}$$ (PGD)
another key difference is that, in the gradient step, float weights $y^{t+1}$ is updated by perturbing $y^t$ instead of the current projection $w^t$.

1.1. Related works
For the best possible performance under quantization, the pre-trained full-precision networks need to be re-trained. In the regime of weight quantization, the BinaryConnect scheme:

$$\begin{align*}
    y^{t+1} &= y^t - \eta_t E[\nabla_w \ell(w^t; x)] \\
    w^{t+1} &= \text{proj}_Q(y^{t+1})
\end{align*}$$

was first proposed in (Courbariaux et al., 2015) for training DNNs with binary (1-bit) weights. It is similar to QUANT, but simply uses the standard gradient $\nabla_w \ell$ as the activation values were not quantized. The method was then extended to multi-bit weight quantization such as ternary weight networks (Li et al., 2016). On the theoretical side, (Li et al., 2017) analyzed the convergence of BinaryConnect scheme for weight quantization, and proved that $\{w^t\}$ converge to an error floor region of the optimal quantized weights under strong convexity and smoothness assumptions on $f$. Recently, (Lin et al., 2020) used an algorithm called “error feedback” for pruning networks (Han et al., 2015; Xiao et al., 2019). It is basically the same as BinaryConnect, except that the weight quantization step $\text{proj}_Q$ is replaced with weight pruning/thresholding which can also be viewed as a projection. The authors showed the convergence to a neighborhood of optimal solution under strong convexity and smoothness assumptions whose radius is $O(\sqrt{d})$ with $d$ being the number of model parameters. Moreover, it remains unclear whether the global optimum can actually be reached in this setting.

The idea of STE has been extensively used for efficiently handling discrete-valued functions arising in machine learning problems. A STE, used in the backward pass only, is a heuristic proxy that substitutes the a.e. zero derivative of discrete component composited in the loss function when computing the gradient under chain rule. Its applications include, but are not limited to, network quantization (Hubara et al., 2016; Cai et al., 2017; Zhou et al., 2016; Choi et al., 2018; Hubara et al., 2018; Uhlich et al., 2020; Blumenfeld et al., 2019), neural architecture search (Stamoulis et al., 2020), knowledge graphs (Xu & Li, 2019), discrete latent representations (Jang et al., 2017). For networks with binary activations (and real-valued weights), (Yin et al., 2019) showed that STE-based gradient (called coarse gradient) methods converge only when a proper STE like ReLU STE (Cai et al., 2017) is used. And they proved that the negation of the resulting coarse gradient points to a descent direction that makes the training loss decrease. For quantization of both weights and activations, (Hubara et al., 2016; 2018; Cai et al., 2017; Choi et al., 2018; Zhou et al., 2016) utilized QUANT scheme which is the combination of BinaryConnect and STE, and achieved state-of-the-art classification accuracies. Yet to our knowledge, no convergence results of QUANT have been established to date.

1.2. Main contributions
In this paper, we examine the quantization of one-hidden-layer networks with binary activation and binary or ternary weights using the QUANT algorithm. Surprisingly, the sequence of quantized weights $\{w^t\}$ generated by QUANT is generically divergent. Our key contributions are the first groundbreaking theoretical results on the dynamics of QUANT algorithm for learning fully quantized neural nets: (1) we prove the generic divergence if the teacher parameters are not in a quantized state, and give an explicit example of oscillatory divergence behavior (the sequence $\{w^t\}$ has period 3 and jumps between sub-optimal quantized states; see Example 1). (2) We explicitly point out, in the ternary case, the $n$ (out of $3^m - 1$) sub-optimal quantized states that $\{w^t\}$ could visit infinite many times; see Remark 1 and Lemma 8. (3) We prove that $\{w^t\}$ oscillates around the global optimum of quantization problem. Under conditions that teacher parameters and their quantized values are close enough (see Theorem 1), $\{w^t\}$ visits the quantized teacher parameters (the optimum) infinitely often (recurrence). Compared with theoretical results for BinaryConnect (Li et al., 2017; Lin et al., 2020), our analysis is more precise and in depth in order to overcome a biased gradient modification in QUANT based on straight-through estimator (STE) (Hinton, 2012; Bengio et al., 2013). Our result is stronger in that the recurrence behavior at global minimum holds without global convexity assumption of the loss function.

Organization. In section 2, we introduce the problem setup and present some useful preliminary results about the QUANT algorithm. In section 3, we summarize the main results regarding the recurrence behavior of QUANT algorithm. More technical details and sketch of proofs are presented in section 4.

2. Preliminaries
2.1. Problem Setup
We consider a one-hidden-layer model that outputs the prediction for an input $Z \in \mathbb{R}^{m \times n}$.

$$y(Z; w) := \sum_{i=1}^{m} v_i \sigma \left( Z_i^\top w \right) = v^\top \sigma \left( Z w \right)$$

where $Z_i^\top$ denotes the $i$-th row vector of $Z$; $w \in \mathbb{R}^n$ is the trainable weights in the first linear layer, and $v \in \mathbb{R}^m$ the weights in the second linear layer which are assumed to be known and fixed during the training process; the activation function $\sigma(x) = \mathbb{1}_{\{x > 0\}}$ is binary, acting component-wise
on the vector \(Zw\). The label is generated according to 
\[ y_Z := y(Z; w^*) \] 
for some unknown teacher (real-valued) parameters \(w^* \in \mathbb{R}^n\).

We fit the described model with quantized weights \(w \in \mathcal{Q}\) and binary activation function \(\sigma(x) = \mathbb{1}_{\{x > 0\}}\) on the i.i.d. Gaussian data \(\{(Z, y_Z)\} \sim \mathcal{N}(0, I)\). In this paper, we will focus on the cases of binary and ternary weights. In the binary case, every quantized weight in \(w\) is either \(\alpha\) or \(-\alpha\) for some universal real-valued constant \(\alpha > 0\), or equivalently, \(\mathcal{Q} = \mathbb{R}_+ \times \{\pm 1\}^n\); this setup of binary weights is widely adopted in the literature; for example, (Rastegari et al., 2016). Similarly in the ternary case, we take \(\mathcal{Q} = \mathbb{R}_+ \times \{0, \pm 1\}^n\); see (Li et al., 2016; Yin et al., 2016) for examples.

We use the squared loss to measure the discrepancy between the model output and label:

\[
\ell(w; Z) := \frac{1}{2} (y(Z; w) - y_Z)^2
\]

\[= \frac{1}{2} (v^\top \sigma(Zw) - v^\top \sigma(Zw^*))^2.\]  

(4)

We cast the learning task as the following population loss minimization problem:

\[
\min_{w \in \mathbb{R}^n} f(w) := \mathbb{E}_{Z \sim \mathcal{N}(0, I)} \left[ \ell(w; Z) \right]
\]

subject to \(w \in \mathcal{Q}\),

where the sample loss function \(\ell(w; Z)\) is given in (4).

In the rest of the paper, we study the convergence behavior of QUANT described below in Algorithm 1 for solving optimization problem (5), in which \(\nabla f\) stands for an unusual gradient of \(f\) called coarse gradient (Yin et al., 2019), so as to side-step the vanished gradient issue. Since the loss function is scale-invariant, i.e., \(\ell(Z; w) = \ell(Z; c w)\) for any scalar \(c > 0\), without loss of generality, we assume that \(\|w^*\| = 1\) is unit-normed.

**Algorithm 1 QUANT algorithm for solving (5)**

**Input:** number of iterations \(T\), learning rate \(\eta_t\), weight bits \(b\).

**Initialize:** auxiliary real-valued weights \(y^0 \in \mathbb{R}^n\).

for \(t = 1\) to \(T\) do

\[y^t = y^{t-1} - \eta_t \nabla f(y^{t-1})\]

\[w^t = \text{proj}_Q(y^t)\]

end for

Throughout this paper we assume the following on the learning rate \(\eta_t > 0\):

1. \(\sum_{t=1}^{\infty} \eta_t = \infty\).
2. \(\eta_t\) is upper bounded by some positive constant \(\eta\).

2.2. Characterization of Optimal Solutions

To study the convergence of Algorithm 1, we first obtain the closed-form expression of the objective function for the optimization problem (4), which only depends on the angle between quantized weight vector \(w\) and the true weight vector \(w^*\). This helps us find the expression of global minimum to (1).

**Lemma 1.** Let \(w \neq 0\) be nonzero vector:

- the training loss in (5) is given by

\[
f(w) = \frac{\|v\|^2}{2\pi} \arccos \left( \frac{w^\top w^*}{\|w\|\|w^*\|} \right)
\]

- For any \(\delta > 0\), \(w = \delta \cdot \text{proj}_Q(w^*)\) is a global optimum of quantization problem (5).

The above result can be easily derived from Lemma 1 of (Yin et al., 2019), so we omit the proof. Lemma 1 states that the optimal quantized weights is just the projection of \(w^*\) onto \(\mathcal{Q}\), i.e., the direct quantization of teacher parameters \(w^*\). Note that the projection/quantization may not be unique, we refer to \(\text{proj}_Q(y)\) as any choice of the projection of \(y\) onto \(\mathcal{Q}\).

2.3. Coarse Gradient

In this part, we specify the coarse gradient \(\nabla f(w)\) in Algorithm 1. The standard back-propagation gives the gradient of \(\ell(w; Z)\) w.r.t. \(w\) by

\[
\nabla_w \ell(w; Z) = Z^\top (\sigma'(Zw) \circ v) \ell(w; Z).
\]

Note that \(\sigma'\) is zero a.e., which makes \(\nabla_w \ell(w; Z)\) inapplicable to the training. The sample coarse gradient w.r.t. \(w\) associated with the sample \((Z, y_Z)\) is given by replacing \(\sigma'\) with a surrogate derivative, known as straight-through estimator (STE) (Bengio et al., 2013; Yin et al., 2019). Here we consider the derivative of ReLU function \(\mu(x) = \max(x, 0)\) which is a widely used STE for quantization, namely, we modify the original gradient \(\nabla_w \ell(w; Z)\) as follows:

\[
\nabla_w \ell(w; Z) = Z^\top (\mu'(Zw) \circ v) \ell(w; Z).
\]

The coarse gradient induced by ReLU STE \(\mu'\) is just the expectation of \(\nabla_w \ell(w; Z)\) over \(Z \sim \mathcal{N}(0, I)\). We evaluate the coarse gradient \(\nabla f(w)\) used in Algorithm 1:

**Lemma 2.** The expected coarse gradient of \(\ell(w; Z)\) w.r.t. \(w\) is

\[
\nabla f(w) := \mathbb{E}_{Z \sim \mathcal{N}(0, I)} \left[ \nabla_w \ell(w; Z) \right]
\]

\[= \frac{\|v\|^2}{2\sqrt{2\pi}} \left( \frac{w}{\|w\|} - w^* \right).
\]

(6)

2.4. Weight Quantization Step

The following two lemmas give the closed-form formulas of the projection/quantization \(\text{proj}_Q(\cdot)\) in Algorithm 1 in the binary and ternary cases, respectively.
Figure 1. One-hidden-layer neural network. The first linear layer resembles a convolutional layer with each \( Z_i \) being a patch of size \( n \) and \( w \) being the shared weights or filter. The second linear layer serves as the classifier.

**Lemma 3 (Binary Case).** For any non-zero \( y \in \mathbb{R}^n \), the projection of \( y \) onto \( Q = \mathbb{R}_+ \times \{ \pm 1 \}^n \) is

\[
\text{proj}_Q(y) = \frac{\|y\|_1}{n} \text{sign}(y),
\]

where the sign function acts element-wise

\[
\text{sign}(y)_i = \begin{cases} 
1 & \text{if } y_i \geq 0 \\
-1 & \text{if } y_i < 0.
\end{cases}
\]

The above lemma is due to (Rastegari et al., 2016). In the ternary case, (Yin et al., 2016) gives the following result:

**Lemma 4 (Ternary Case).** For any non-zero \( y \in \mathbb{R}^n \), the projection of \( y \) on \( Q = \mathbb{R}_+ \times \{0, \pm 1\}^n \) is

\[
\text{proj}_Q(y) = \frac{\|y_{[-j^*]}\|_1}{\|y_{[-j^*]}\|_2} \text{sign}(y_{[-j^*]})
\]

where \( j^* = \arg\max_{1 \leq j \leq n} \|y_{[-j]}\|_2^2 \), and \( y_{[-j]} \in \mathbb{R}^n \) extracts the first \( j \) largest entries in magnitude of \( y \) and enforces 0 elsewhere. Here,

\[
\text{sign}(y)_i = \begin{cases} 
1 & \text{if } y_i > 0 \\
0 & \text{if } y_i = 0 \\
-1 & \text{if } y_i < 0.
\end{cases}
\]

**3. Main Results**

By Lemma 1, we assume, for the ease of presentation, that the iterates \( \{w^t\} \) are normalized, that is, we re-define \( w^t \) in Algorithm 1 by

\[
w^t = \text{proj}_Q(y^t) := \frac{\text{proj}_Q(y^t)}{\|\text{proj}_Q(y^t)\|}
\]

Our results extend trivially to the original QUANT without normalization as the value of \( f(w) \) does not depend on \( \|w\| \).

Furthermore, we denote by \( \text{proj}_Q(w^*) \) the normalization of the quantization/projection of \( w^* \), \( \text{proj}_Q(w^*) \), which is a global minimum according to Lemma 1. Our main results show that the optimum \( \text{proj}_Q(w^*) \) is recurrent as long as \( w^* \) is close to its normalized quantization.

**Theorem 1.** Consider the setup of quantization problem (5). Let \( Q \) be either \( \mathbb{R}_+ \times \{\pm 1\}^n \) (binary case) or \( \mathbb{R}_+ \times \{0, \pm 1\}^n \) (ternary case). There exists constant \( \epsilon > 0 \) that depends on the weight bit-width and dimension \( n \) only, such that for any \( w^* \) with

\[
0 < \|w^* - \text{proj}_Q(w^*)\| < \epsilon,
\]

we have \( w^t = \text{proj}_Q(w^*) \) for infinitely many \( t \) values, where \( \{w^t\} \) is the sequence generated by Algorithm 1 with any initializaton.

Intuitively, ternary weights should work better than binary weights. The following remark confirms this intuition by showing that the number of points where \( w^t \) visits infinitely many times is limited.

**Remark 1.** In the ternary case, we can further prove that the sequence \( \{w^t\} \) generated by Algorithm 1 has at most \( n \) sub-sequential limits.

**4. Proof Sketch**

On one hand, the binary case is rather simple. We show that part of the coordinates is stable while others have oscillating sign. We further prove that the set of oscillating coordinates is not empty as long as \( w^* \notin Q = \mathbb{R}_+ \times \{\pm 1\}^n \) is not quantized.

On the other hand, the proof of the ternary case follows the following steps. Our first step shows the sequence \( y^t \) generated by Algorithm 1 is bounded away from the origin for all but finitely many \( t \) values. Then, our second step
shows each coordinate of \( y^t \) is of the same sign of \( w^* \) for all but finitely many \( t \) values. This forces \( y^t \) to stay in the same orthant to which \( w^* \) belongs. As a matter of fact, an \( n \)-dimensional space has in total \( 2^n \) orthants, which means \( y^t \) can only stay in a small region near \( w^* \). After that, our third step furthermore cuts the orthant into \( n! \) congruent cones and argue \( y^t \) must stay in the same cone where \( w^* \) is for all but finitely many \( t \) values. In the last step, we prove the ternary case of Theorem 1, which asserts that as long as the underlying true parameter \( w^* \) is close to quantized state \( Q = \mathbb{R}_+ \times \{0, \pm 1\}^n \), i.e., any vertex of the cone it belongs to, the optimum is guaranteed to be recurrent.

4.1. Binary Weight

In view of Lemmas 1 and 3, we have that the normalized optimum of (5) is \( \frac{1}{\sqrt{n}} \text{sign}(w^*) \). The Lemma below shows that some coordinates of \( w^* \) generated by Algorithm 1 have oscillating signs.

**Proposition 1.** Let \( w^t \) be any infinite sequence generated by Algorithm 1. If \( |w^*_j| < \frac{1}{\sqrt{n}} \), then there exist infinitely many \( t_1 \) and \( t_2 \) such that \( w^t_1 = \frac{1}{\sqrt{n}} \) and \( w^t_2 = -\frac{1}{\sqrt{n}} \).

The above lemma clearly implies that \( w^t \) does not converge, as long as \( w^* \notin Q \).

**Corollary 1.** If \( w^* \notin Q \), then any sequence \( \{w^t\} \) generated by Algorithm 1 does not converge.

Since Algorithm 1 does not have a limit unless the weights in the network are already quantized, we ask a natural question: Can we guarantee the optimum to be visited infinitely many times? The general answer is no. We have the following example demonstrating that the optimum may never be achieved. We refer the reader of the following example to the appendix.

**Example 1.** Let \( w^* = \left( \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \) so that the best the optimum \( \text{proj}_Q w^* = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \). Let \( \eta_t = \eta, \lambda = \frac{\eta \| w \|^2}{6 \sqrt{2} \pi} \) and

\[
\begin{align*}
y^0_1 &\in (-\lambda, 0) \\
y^0_2 &\in (0, \lambda) \\
y^0_3 &\in (\lambda, 2\lambda) \\
y^0_4 &\in (0, \infty)
\end{align*}
\]

the sequence \( \{w^t\} \) generated by Algorithm 1 with initialization \( y^0 \) satisfies \( w^{t+1} = w^t \) and \( w^t \neq \text{proj}_Q w^* \) for all \( t \).

In the following, we give a sufficient condition for the optimum to be recurrent. The condition requires \( w^* \) to be close to \( Q \). The following result is for the the binary case of Theorem 1.

**Theorem 1 (Binary Case).** If the optimum

\[
\hat{w} := \text{proj}_Q(w^*) = \frac{1}{\sqrt{n}} \text{sign}(w^*)
\]

of (5) satisfies

\[
0 < \sum_{|w^*_j|<\frac{1}{\sqrt{n}}} |w^*_j - \hat{w}_j| < \frac{2}{\sqrt{n}}
\]

then there exist infinitely many \( t \) values for any sequence \( \{w^t\} \) generated by Algorithm 1 such that \( w^t = \text{proj}_Q(w^*) \).

4.2. Ternary Weights

The first result shows that \( w^t \) generated by Algorithm 1 is generally divergent, and it converges only when the true parameters \( w^* \in Q = \mathbb{R}_+ \times \{0, \pm 1\}^n \).

**Proposition 2 (Ternary Case).** Let \( \{w^t\} \) be any sequence generated by Algorithm 1. If \( w^* \notin Q = \mathbb{R}_+ \times \{0, \pm 1\}^n \), then \( \{w^t\} \) is not a convergent sequence.

In what follows, we detail the proof of convergence behavior of Algorithm 1.

Our first step is to rule out an exceptional case that the direction of \( y^t \) changes significantly in only one iteration. As shown in Lemma 2, the coarse gradient is bounded by a constant depending only on the fixed weight vector \( w \). So it suffices to show that \( \|y^t\| \) is bounded away from zero for all but finitely many \( t \) values.

**Lemma 5.** Let \( \{y^t\} \) be any auxiliary sequence generated by Algorithm 1. If \( w^* \notin Q \), then \( \|y^t\|_1 \) converges to infinity as \( t \) increases.

Lemma 5 shows that for any positive constant \( c > 0 \), we have \( \|y^t\|_1 > c \) for all but finitely many \( t \) values.

Since Lemma 5 guarantees that the direction of \( y^t \) will not change significantly, we cut down the region that \( y^t \) can belong to in two steps. To describe our first cut down, we need the following definition to make our statement precise.

**Definition 1.** For any \( x \in \mathbb{R}^n \), we define the orhant of \( x \) as

\[
O(x) := \{y \in \mathbb{R}^n : \text{sign}(y) = \text{sign}(x)\}
\]

where \( \text{sign}(\cdot) \) acts coordinate-wise. Furthermore, we say \( O(x) \) is regular if any coordinate of \( x \) is not zero.

We state some basic properties of the defined orhant.

**Proposition 3.** For any \( x, y \in \mathbb{R}^n \), the following statements are true:

1. Either \( O(x) = O(y) \) or \( O(x) \cap O(y) = \emptyset \).
2. \( x \in O(x) \).
Moreover, we say

**Proposition 4.** For any \( x \) and any \( y \), the cone of \( y \) is regular if \( O(x) \) is regular and any \( |y| \neq |x| \) for all \( j \neq i \).

**Definition 2.** For any non-zero vector \( x \in \mathbb{R}^n \), we define the cone of \( x \) to be

\[
\text{Cone}(x) := \left\{ y \in O(x) : \text{sign}(|y_j| - |y_i|) = \text{sign}(|x_j| - |x_i|) \text{ for } \forall i, j \in [n] \right\}.
\]

Moreover, we say \( \text{Cone}(x) \) is regular if \( O(x) \) is regular and any \( |x_j| \neq |x_i| \) for all \( j \neq i \).

**Proposition 4.** For any \( x, y \in \mathbb{R}^n \), the following statements are true:

1. Either \( \text{Cone}(x) = \text{Cone}(y) \) or \( \text{Cone}(x) \cap \text{Cone}(y) = \emptyset \).
2. \( x \in \text{Cone}(x) \).
3. If \( y \in \text{Cone}(x) \), then \( \text{Cone}(y) = \text{Cone}(x) \).
4. \( \cup_{y \in O(x)} \text{Cone}(y) = O(x) \).
5. Any regular orthant contains \( n! \) regular cones.

**Lemma 6.** Let \( \{y^t\} \) be any auxiliary real-valued sequence generated by Algorithm 1. If \( w^* \not\in \mathbb{Q} \), then any subsequential limit of \( y^t := \frac{y^t}{\|y^t\|} \) belongs to the closure of \( O(w^*) \). Furthermore, if \( O(w^*) \) is regular, then \( y^t \) lies in \( O(w^*) \) for all but finitely many \( t \) values.

The auxiliary weight vector \( y^t \) can only stay in a small region around \( w^* \) for large \( t \) values.

**Definition 3.** For any point \( x \in \mathbb{R}^n \), assume \( (j_1, j_2, \cdots, j_n) \) is a permutation of \([n]\) such that

\[
|x_{j_1}| \geq |x_{j_2}| \geq \cdots \geq |x_{j_n}|
\]

We define the set of vertexes of \( x \) to be

\[
\Lambda(x) := \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{k} \text{sign}(x_{j_i}) e_{j_i} : x_{j_{k+1}} \neq x_{j_k} \text{ are nonzeros} \right\}.
\]

Below are some basic facts about connection between vertexes and cones.

**Proposition 5.** For any \( x, y \in \mathbb{R}^n \) let \( k := |\Lambda(x)| \), the following statements are true:

1. \( 0 \leq k \leq n \).
2. \( \Lambda(x) \) is empty if and only if \( x = 0 \).
3. \( \Lambda(x) \) is a subset of the boundary of \( \text{Cone}(x) \).
4. \( \text{Cone}(x) = \text{Cone}(y) \) if and only if \( \Lambda(x) = \Lambda(y) \).
5. \( \text{proj}_Q(x) \in \Lambda(x) \).
6. \( y \) lies in \( \text{Cone}(x) \) if and only if there exists \( k \) positive numbers \( \{\mu_z(y) : z \in \Lambda(x)\} \) such that

\[
y = \sum_{z \in \Lambda(x)} \mu_z(y) z.
\]
Figure 3. Evolution of signs of weight filters in the last training epoch (or 600 iterations) of LeNet-5. Each of the six $25 \times 200$ blocks corresponds to evolution of the $5 \times 5$ convolutional filter over 200 iterations. **Top three rows:** Binary weights over the last 600 iterations of training, red/blue for sign values $1/-1$. **Bottom three rows:** Ternary weights over the last 600 iterations of training, red/green/blue for sign values $1/0/-1$.

7. $y$ lies in the closure of $\text{Cone}(\{x\})$ if and only if there exists $k$ non-negative numbers $\{\mu_z(y) : z \in \Lambda(\{x\})\}$ such that

$$y = \sum_{z \in \Lambda(\{x\})} \mu_z(y)z.$$ 

8.

$$\bigcup_{x \in \mathbb{R}^n} \Lambda(x) = \{x \in \mathbb{Q} : \|x\| = 1\}.$$

**Lemma 8.** Let $\{w^t\}$ be the sequence generated by Algorithm 1. If $w^* \notin \mathbb{Q} = \mathbb{R}_+ \times \{0, \pm 1\}^n$, then $w^t \in \Lambda(w^*)$ for all but finitely many $t$ values.

The following result is the ternary case of Theorem 1 stated in section 3.

**Theorem 1 (Ternary Case).** Let $\{z_j\}_{j=1}^k = \Lambda(w^*)$ where $z_1 = \hat{\text{proj}}_\mathbb{Q} w^*$ is the optimum and $w^* = \sum_{j=1}^k \lambda_j z_j$. If $0 < \sum_{j=2}^k \lambda_j < 1$, we have $w^t = \hat{\text{proj}}_\mathbb{Q} w^*$ for infinitely many $t$ values, where $w^t$ is any infinite sequence generated by Algorithm 1 with any initialization.

Intuitively, the parameter $\lambda_j$ in Theorem 1 stands for the proportion of time that $\{w^t\}$ stays at $z_j$. For instance, if $\lambda_j \approx 1$, then most of $\{w^t\}$ stay at $z_j$ so that the oscillation has a longer ‘period’ and is harder to observe. On the contrary, if all $\lambda_j$’s are almost the same then $\{w^t\}$ behaves like uniform distribution and oscillation becomes more obvious. Beside $\lambda_j$’s, a smaller learning rate can render $y^t$ moves slower which can also slow down the oscillation. Although
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there are ways to stabilize the training process, both our theorem and the experiments in the next section suggests the oscillation behavior is inevitable.

5. Experiments

In this section, we implement QUANT algorithm on both synthetic data and MNIST/CIFAR image data. Our goals are (1) to validate our theoretical findings and (2) to show the appearance of the oscillation behavior in more complicated setups. With that said, we emphasize that we did not extensively tune the hyper-parameters or use ad-hoc tricks to achieve the best possible validation accuracy. More comprehensive experimental results for QUANT-based approaches can be found in, for examples, (Cai et al., 2017; Choi et al., 2018; Hubara et al., 2018; Zhou et al., 2016). Here we report the validation accuracies on MNIST and CIFAR-10 for fully quantized networks in Table 1. For both synthetic and image data sets, we observed the oscillation behavior.

5.1. Synthetic Data

We take $m = 4$, $n = 8$ in (3) and construct $v \sim N(0, I_m)$ and $w^* \sim N(0, I_n)$ be random vectors. For each run, we fix $v$ and $w^*$ and train the neural network (3) by algorithm (1) for 200 iterations with a learning rate being 0.1. Fig. 2 show the evolution of binary/ternary weight of $w^t$ in the last 100 iterations. Each block of size $8 \times 100$ corresponds to the evolution of $w^t$ during the 100 iterations. The (quantized) global minimum $\text{proj}_Q w^*$ for each run is shown on the right side of the corresponding subplot in Fig. 2.

5.2. MNIST

We train LeNet-5 with binary/ternary weights and 4-bit activations using QUANT algorithm. For deep networks, the (quantized) global optimum is generally unknown, we instead show the oscillating behavior around local optimum. Note that Fig. 4 shows the training loss no longer drop significantly during the last 30 epochs (50 in total). This suggests the network parameters have reached a local valley.

5.3. CIFAR-10

We repeat the experiments on CIFAR-10 (Krizhevsky, 2009) with ResNet-20/VGG-11. We train ResNet-20 (He et al., 2016)/VGG-11 (Simonyan & Zisserman, 2014) with binary/ternary weights and 4-bits activation using QUANT for 200 epochs. We refer to the appendix for some figures that show similar oscillation behavior. Towards the end of training, although there has been no noticeable decay of training loss, we can still see the oscillating signs of the weights.

6. Concluding Remarks

We studied the convergence behavior of widely used QUANT algorithm (Hubara et al., 2016; Cai et al., 2017; Choi et al., 2018; Zhou et al., 2016) for the quantization of one-hidden-layer networks. We showed that the sequence of quantized weights $\{w^t\}$ generated by QUANT is generically divergent if the teacher parameters are not in a quantized state, and constructed an explicit example of oscil-

Table 1. Validation Accuracy of LeNet-5 on MNIST and ResNet-20/VGG-11 on CIFAR-10.

|                  | float | binary | ternary |
|------------------|-------|--------|---------|
| LeNet-5          | 99.37 | 99.33  | 99.34   |
| ResNet-20        | 92.33 | 89.42  | 90.86   |
| VGG-11           | 92.15 | 89.47  | 90.91   |

However, Fig. 3 shows the iterating sequence of model parameters still have oscillating signs towards the end of training.

Fig. 3 shows the evolution of the quantized weights of one convolution filter in the first convolution layer during the last 600 iterations. To visualize the weights, each quantized filter is reshaped into a 25-dimensional column vector. Each block (3 in a group) of size $25 \times 200$ corresponds to the evolution of the one filter during 200 iterations. As we can see from these two figures, a proportion of the weights do not converge to a limit but rather have oscillating signs.
latory divergence behavior. Under conditions that teacher parameters and their quantized values are close enough, we proved the recurrence of QUANT algorithm at the global minimum.

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References
Bengio, Y., Léonard, N., and Courville, A. Estimating or propagating gradients through stochastic neurons for conditional computation. arXiv preprint arXiv:1308.3432, 2013.
Blumenfeld, Y., Gilboa, D., and Soudry, D. A mean field theory of quantized deep networks: The quantization-depth trade-off. In Advances in Neural Information Processing Systems, pp. 7036–7046, 2019.
Cai, Z., He, X., Sun, J., and Vasconcelos, N. Deep learning with low precision by half-wave gaussian quantization. In IEEE Conference on Computer Vision and Pattern Recognition, 2017.
Choi, J., Wang, Z., Venkataramani, S., Chuang, P. I.-J., Srinivasan, V., and Gopalakrishnan, K. Pact: Parameterized clipping activation for quantized neural networks. arXiv preprint arXiv:1805.06085, 2018.
Courbariaux, M., Bengio, Y., and David, J.-P. Binaryconnect: Training deep neural networks with binary weights during propagations. In Advances in Neural Information Processing Systems, pp. 3123–3131, 2015.
Han, S., Mao, H., and Dally, W. J. Deep compression: Compressing deep neural networks with pruning, trained quantization and huffman coding. arXiv preprint arXiv:1510.00149, 2015.
He, K., Zhang, X., Ren, S., and Sun, J. Deep residual learning for image recognition. In IEEE conference on computer vision and pattern recognition, pp. 770–778, 2016.
Hinton, G. Neural networks for machine learning. coursera. Coursera, video lectures, 2012.
Hubara, I., Courbariaux, M., Soudry, D., El-Yaniv, R., and Bengio, Y. Binarized neural networks: Training neural networks with weights and activations constrained to +1 or -1. arXiv preprint arXiv:1602.02830, 2016.
Hubara, I., Courbariaux, M., Soudry, D., El-Yaniv, R., and Bengio, Y. Quantized neural networks: Training neural networks with low precision weights and activations. Journal of Machine Learning Research, 18:1–30, 2018.
Jang, E., Gu, S., and Poole, B. Categorical reparameterization with gumbel-softmax. In International Conference on Learning Representations (ICLR), 2017.
Krizhevsky, A. Learning multiple layers of features from tiny images. Tech Report, 2009.
Li, F., Zhang, B., and Liu, B. Ternary weight networks. arXiv preprint arXiv:1605.04711, 2016.
Li, H., De, S., Xu, Z., Studer, C., Samet, H., and Goldstein, T. Training quantized nets: A deeper understanding. In Advances in Neural Information Processing Systems, pp. 5811–5821, 2017.
Lin, T., Stich, S. U., Barba, L., Dmitriev, D., and Jaggi, M. Dynamic model pruning with feedback. In International Conference on Learning Representations, 2020.
Louizos, C., Reisser, M., Blankevoort, T., Gavves, E., and Welling, M. Relaxed quantization for discretized neural networks. In International Conference on Learning Representations, 2019.
Rastegari, M., Ordonez, V., Redmon, J., and Farhadi, A. Xnor-net: Imagenet classification using binary convolutional neural networks. In European Conference on Computer Vision, pp. 525–542. Springer, 2016.
Simonyan, K. and Zisserman, A. Very deep convolutional networks for large-scale image recognition. arXiv preprint arXiv:1409.1556, 2014.
Stamoulis, D., Ding, R., Wang, D., Lymberopoulos, D., Priyantha, N. B., Liu, J., and Marculescu, D. Single-path mobile automl: Efficient convnet design and nas hyperparameter optimization. IEEE Journal of Selected Topics in Signal Processing, 2020.
Uhlich, S., Mauch, L., Cardinaux, F., Yoshiyama, K., García, J. A., Tiedemann, S., Kemp, T., and Nakamura, A. Mixed precision dnns: All you need is a good parametrization. In International Conference on Learning Representations (ICLR), 2020.
Xiao, X., Wang, Z., and Rajasekaran, S. Autoprune: Automatic network pruning by regularizing auxiliary parameters. In Advances in Neural Information Processing Systems, pp. 13681–13691, 2019.
Xu, C. and Li, R. Relation embedding with dihedral group in knowledge graph. Annual Conference of the Association for Computational Linguistics, 2019.
Yin, P., Zhang, S., Xin, J., and Qi, Y. Training ternary neural networks with exact proximal operator. ArXiv, abs/1612.06052, 2016.
Yin, P., Lyu, J., Zhang, S., Osher, S. J., Qi, Y., and Xin, J. Understanding straight-through estimator in training activation quantized neural nets. In International Conference on Learning Representations, 2019.
Zhou, A., Yao, A., Guo, Y., Xu, L., and Chen, Y. Incremental network quantization: Towards lossless CNNs with low-precision weights. arXiv preprint arXiv:1702.03044, 2017.
Zhou, S., Wu, Y., Ni, Z., Zhou, X., Wen, H., and Zou, Y. Dorefa-net: Training low bitwidth convolutional neural networks with low bitwidth gradients. arXiv preprint arXiv:1606.06160, 2016.
Zhu, C., Han, S., Mao, H., and Dally, W. J. Trained ternary quantization. arXiv preprint arXiv:1612.01064, 2016.
Appendix

**Lemma 2.** The expected coarse gradient of \(f(w; z)\) w.r.t. \(w\) is

\[
\tilde{\nabla} f(w) = \frac{||v||^2}{2\sqrt{2\pi}} \left( \frac{w}{||w||} - w^* \right).
\]

**Proof of Lemma 2.** (Yin et al., 2019) gives

\[
\tilde{\nabla} f(w) = \frac{||w^*||^2}{\sqrt{2\pi}} \left( \frac{w}{||w||} - \cos \left( \frac{\theta}{2} \right) \frac{w + w^*}{||w + w^*||} \right).
\]

Let \(\tilde{w} = \frac{w}{||w||}\), we can easily see from Fig. 5 that the coarse

\[
\text{Figure 5. 2-dim section of } \mathbb{R}^n \text{ spanned by } \tilde{w} \text{ and } w^*
\]

gradient can be further simplified as (6)

**Proposition 1.** Let \(\{w_t\}\) be any infinite sequence generated by Algorithm 1. If \(w_0 < \frac{1}{\sqrt{n}}\), then there exist infinitely many \(t_1\) and \(t_2\) values such that \(w_{t_1} > \frac{1}{\sqrt{n}}\) and \(w_{t_2} = -\frac{1}{\sqrt{n}}\).

**Proof of Lemma 1.** For notational simplicity, since \(\|w_j^*\| < \frac{1}{\sqrt{n}}\), we have

\[
\alpha := \frac{1}{\sqrt{n}} - w_j^* > 0 \text{ and } \beta := \frac{1}{\sqrt{n}} + w_j^* > 0.
\]

Using Lemma 3 in Algorithm 1, we see that

\[
y_j^{t+1} = y_j^t + \eta \frac{||v||^2}{2\sqrt{2\pi}} (w_j^* - w_j^t)
\]

\[
= y_j^t + \eta \frac{||v||^2}{2\sqrt{2\pi}} \left( w_j - \frac{1}{\sqrt{n}} \text{sign}(y_j^t) \right),
\]

and thus

\[
y_j^{t+1} = \begin{cases} y_j^t - \frac{\eta \frac{||v||^2}{2\sqrt{2\pi}}}{\alpha} & \text{if } y_j^t \geq 0 \\ y_j^t + \frac{\eta \frac{||v||^2}{2\sqrt{2\pi}}}{\beta} & \text{if } y_j^t < 0 \end{cases}
\]

Since \(y_j^t\) is bounded for each fixed \(t \geq 0\) and \(j \in [n]\), our desired result follows from our assumptions on learning rate \(\eta\).

**Corollary 1.** If \(w^* \notin \mathcal{Q}\), any sequence \(\{w^t\}\) generated by Algorithm 1 does not converge.

**Proof of Corollary 1.** Since \(w^* \notin \mathcal{Q}^1\), we know there must exist some \(j \in [n]\) such that \(|w_j^*| < \frac{1}{\sqrt{n}}\) and Proposition 1 gives our desired result.

**Example 1.** Let \(w^* = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\) so that the best the optimum \(\tilde{\text{proj}}_\mathcal{Q} w^* = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\). Let \(\eta = \eta = \frac{\eta ||v||^2}{6\sqrt{2\pi}}\) and

\[
\begin{align*}
&y_j^0 \in (-\lambda, 0) \\
y_j^1 \in (0, \lambda) \\
y_j^2 \in (\lambda, 2\lambda) \\
y_j^3 \in (0, \infty)
\end{align*}
\]

the sequence \(\{w^t\}\) generated by Algorithm 1 with initialization \(y^0\) satisfies \(w^{t+3} = w^t\) and \(w^t \neq \text{proj}_\mathcal{Q} w^*\) for all \(t\).

**Proof of Example 1.** In order to show the periodicity, it suffices to show \(w^{t+3} = w^t\). Note that \(\partial_w \tilde{f}(w) < 0\) we have \(y_j^t > 0\) for all \(t\) since \(w^0 > 0\). It follows that \(w_1^t = w_2^t = \frac{1}{2}\). Next, we would like to show the periodicity of \(w_j^t\) for \(j \in [3]\). Note that

\[
y_j^{t+1} = \begin{cases} y_j^t + 2\lambda & \text{if } y_j^t < 0 \\ y_j^t - \lambda & \text{if } y_j^t > 0 \end{cases}
\]

we choose \(w_j^t = \frac{1}{6} \) so that with

\[
\lambda = \frac{\eta ||v||^2}{6\sqrt{2\pi}}
\]

we have

\[
y_j^{t+1} = \begin{cases} y_j^t + 2\lambda & \text{if } y_j^t < 0 \\ y_j^t - \lambda & \text{if } y_j^t > 0 \end{cases}
\]

Hence, we have

\[
w^t = \begin{cases} (1, 1, 1, 1) & \text{if } t \equiv 0(\text{mod } 3) \\ (1, 2, 2, 2) & \text{if } t \equiv 1(\text{mod } 3) \\ (1, 1, 2, 2) & \text{if } t \equiv 2(\text{mod } 3) \end{cases}
\]

**Theorem 1** (Binary Case). If the optimum \(\tilde{w} := \tilde{\text{proj}}_\mathcal{Q} w^*\) of (5) satisfies

\[
\sum_{|w_j^*| < \frac{1}{\sqrt{n}}} |w_j^* - \tilde{w}_j| < \frac{2}{\sqrt{n}}
\]

then there exists infinitely many \(t\) values for any sequence \(\{w^t\}\) generated by Algorithm 1 such that \(w^t = \text{proj}_\mathcal{Q}(w^*)\).
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Proof of Theorem 1 on b = 1. Without loss of generality, we can assume \( w_j^* \geq 0 \) for all \( j \in [n] \) so that \( \bar{w}_j = \frac{1}{\sqrt{n}} \) for all \( j \).

Firstly, if \( w_j^* > \frac{1}{\sqrt{n}} \), we know
\[
y_j^{t+1} = y_j^t + \eta \frac{||v||^2}{2 \sqrt{2\pi}} (w_j^* - w_j^t) \geq y_j^t + \eta \frac{||v||^2}{2 \sqrt{2\pi}} (w_j^* - \frac{1}{\sqrt{n}}),
\]
so that
\[
y_j^t \geq y_j^0 + \eta \frac{||v||^2}{2 \sqrt{2\pi}} \left( \sum_{s=0}^{t-1} \eta_s \right) (w_j^* - \frac{1}{\sqrt{n}})
\]
where the right hand side goes to infinity and thus \( y_j^t = \hat{w}_j \) for all but finitely many \( t \) values.

Secondly, if \( w_j^* = \frac{1}{\sqrt{n}} \), we know when \( y_j^t < 0 \):
\[
y_j^{t+1} = y_j^t + \eta \frac{||v||^2}{2 \sqrt{2\pi}} (w_j^* - w_j^t) = y_j^t + \eta \frac{||v||^2}{2 \sqrt{2\pi} \sqrt{n}}
\]
holds so that there must exist some \( t \) such that \( y_j^t > 0 \). Once \( y_j^t > 0 \) we have \( w_j^* = w_j^t \) so that \( y_j^{t+1} = y_j^t \) and hence \( y_j^t = \hat{w}_j \) for all but finitely many \( t \) values.

Third, if \( w_j^* < \frac{1}{\sqrt{n}} \), we have \( y_j^t \cdot \hat{O}_j \langle f(w^t) \rangle > 0 \) so that \( y_j^t \) is increasing when \( y_j^t < 0 \) and decreasing when \( y_j^t > 0 \). This tells us \( y_j^t \) is bounded uniformly in \( t \). Further more,
\[
y_j^t = y_j^0 + \eta \frac{||v||^2}{2 \sqrt{2\pi}} \left[ \sum_{s=0}^{t-1} \frac{1}{\sqrt{n}} \eta_s \right] (w_j^* - \frac{1}{\sqrt{n}}) + \left( \sum_{s=0}^{t-1} \eta_s \right) (w_j^* + \frac{1}{\sqrt{n}})
\]
For notation simplicity, we let
\[
\alpha_j = \frac{1}{\sqrt{n}} - w_j^* > 0 \quad \text{and} \quad \beta_j = w_j^* + \frac{1}{\sqrt{n}} > 0,
\]
\[
a_j^t = \frac{1}{t} \sum_{s=0}^{t-1} \frac{1}{\sqrt{n}} \eta_s \quad \text{and} \quad b_j^t = \frac{1}{t} \sum_{s=0}^{t-1} \eta_s.
\]
Now, we have
\[
\frac{y_j^t - y_j^0}{t} = \frac{||v||^2}{2 \sqrt{2\pi}} \left( -\alpha_j a_j^t + \beta_j b_j^t \right).
\]
Since \( y_j^t \) is bounded for all \( w_j^* < \frac{1}{\sqrt{n}} \), we let \( t \to \infty \) so that left hand side vanishes and
\[
\lim_{t \to \infty} \frac{b_j^t}{a_j^t + b_j^t} = \frac{\alpha_j}{\alpha_j + \beta_j}.
\]
By assumption, we have
\[
\lim_{t \to \infty} \sum_{j=1}^{n} \frac{b_j^t}{a_j^t + b_j^t} = \sum_{j=1}^{n} \frac{\alpha_j}{\alpha_j + \beta_j} < 1.
\]
Hence, we know
\[
\lim_{t \to \infty} \sum_{j=0}^{t-1} \frac{1}{\sqrt{n}} \eta_s \geq \lim_{t \to \infty} \left[ \left( 1 - \sum_{j=1}^{n} \frac{b_j^t}{a_j^t + b_j^t} \right) \eta_s \right] = \infty,
\]
where we used the assumption \( \sum_{t=0}^{\infty} \eta_t = \infty \). Now, the desired result follows.

Proposition 2 (Ternary Case). Let \( w^t \) be any sequence generated by Algorithm 1. If \( w^* \not\in Q \), then \( \{w^t\} \) is not a converging sequence.

Proof of Proposition 2. We prove by contradiction. Observe that \( Q \cap S^{n-1} \) is a finite set, we know \( w^t \) converges to \( w^\infty \) is equivalent to \( w^t = w^\infty \) for all but finitely many \( t \) values. Assume \( w^t = w^\infty \) for all but finitely many \( t \) values, we know there exists some \( T \geq 0 \) such that \( w^t = w^\infty \) for all \( t \geq T \). Thus,
\[
y^{T+1} = y^T - \sum_{s=0}^{t-1} \eta_{T+s} \hat{O}_j \langle f(w^{T+s}) \rangle
\]
\[
= y^T - \left( \sum_{s=0}^{t-1} \eta_{T+s} \right) \hat{O}_j \langle f(w^\infty) \rangle
\]
\[
= y^t + \left( \sum_{s=0}^{t-1} \eta_{T+s} \right) \frac{||v||^2}{2 \sqrt{2\pi}} (w^* - w^\infty).
\]
Now, we have
\[
\langle y^{T+1}, w^\infty \rangle = \langle y^T, w^\infty \rangle + \left( \sum_{s=0}^{t-1} \eta_{T+s} \right) \frac{||v||^2}{2 \sqrt{2\pi}} (w^* - w^\infty, w^\infty)
\]
where
\[
\langle w^* - w^\infty, w^\infty \rangle = \langle w^*, w^\infty \rangle - 1 < 0.
\]
Note that \( \sum_{s=0}^{\infty} \eta_{T+s} = \infty \), there exists some \( T_1(T) \), such that for all \( t > T_1(T) \), \( \langle y^t, w^\infty \rangle < 0 \).

This contradicts Lemma 4 and our desired result follows. \( \square \)

Lemma 5. Let \( \{y^t\} \) be any auxiliary sequence generated by Algorithm 1. \( w^* \not\in Q \), then \( \|y^t\|_1 \) converges to infinity as \( t \) increases.

Proof of Lemma 5. \( Q \cap S^{n-1} \) is a compact set because it is finite. Also, since \( Q \) is symmetric, \( w^* \not\in Q \) also implies \( -w^* \not\in Q \). It follows that
\[
\alpha := \inf_{w^* \in Q \cup S^{n-1}} \theta(w^*, w) \in (0, \pi).
\]
Hence, for any \( w \in Q \cup S^{n-1} \) we have
\[
\langle -\hat{O}_j \langle f(w) \rangle, w^* \rangle = \frac{||v||^2}{2 \sqrt{2\pi}} (w^* - w, w^*) \geq \frac{||v||^2}{2 \sqrt{2\pi}} (1 - \cos \alpha).
\]
Now, we know
\[
\langle y^T, w^* \rangle = \langle y^0, w^* \rangle + \sum_{t=0}^{T-1} \eta_t \langle -\hat{O}_j \langle f(w^t) \rangle, w^* \rangle
\]
\[
\geq \langle y^0, w^* \rangle + \left( \sum_{t=0}^{T-1} \eta_t \right) \frac{||v||^2}{2 \sqrt{2\pi}} (1 - \cos \alpha).
\]
Let \( T \to \infty \), we see that \( \lim_{T \to \infty} \|y^T\|_1 = \infty \) which is equivalent to \( \lim_{T \to \infty} \|y^T\|_1 = \infty \).

\( \square \)

Lemma 9. Let \( w = \text{proj}_Q(y) \), then \( |y_j| < \frac{1}{\sqrt{n}} \|y\|_1 \) implies \( w_j = 0 \).
Proof of Lemma 9. Without loss of generality, we assume \( y_j \geq 0 \) for all \( i \in [n] \) and \( y_j < \frac{1}{\sqrt{n}} \| y \|_1 \) for a fixed \( j \in [n] \). Let \( \delta = \frac{1}{\sqrt{n}} \| y \|_1 \) and
\[
j^* = \arg \max_j \frac{\| y_j \|_1^2}{j}
\]
for any \( 1 \leq k \leq n - j^* \)
\[
\frac{\| y_j \|_1^2}{j^*} = \frac{\| y_j \|_1^2}{j^*} - \frac{\| y_j \|_1^2}{j^*} + \frac{\| y_j \|_1^2}{j^*} \\
\geq \frac{\| y_j \|_1^2}{j^*} - \frac{\| y_j \|_1^2}{j^*} + \frac{\| y_j \|_1^2}{j^*} \\
= k \left( \frac{\| y_j \|_1^2}{j^*} - j^* \delta \left( \frac{\| y_j \|_1^2}{j^*} + \frac{\| y_j \|_1^2}{j^*} \right) \right).
\]
With \( \tau = \frac{\| y_j \|_1^2}{n^2} \), we have
\[
k \left( \frac{\| y_j \|_1^2}{j^*} - j^* \delta \left( \frac{\| y_j \|_1^2}{j^*} + \frac{\| y_j \|_1^2}{j^*} \right) \right) \\
\geq k \left( \frac{\| y_j \|_1^2}{j^*} - k \delta \right)^2 - 2n \delta \left( \frac{\| y_j \|_1^2}{j^*} + \frac{\| y_j \|_1^2}{j^*} \right) \\
= (n \delta)^2 \left( \tau^2 - 4 \tau + 1 \right).
\]
Note that
\[
\tau = \frac{\| y_j \|_1^2}{n^2} \geq \frac{\| y \|_1^2}{n^2} \geq 4,
\]
we conclude that
\[
\frac{\| y_j \|_1^2}{j^*} \geq \frac{\| y_j \|_1^2}{j^*} + \frac{\| y_j \|_1^2}{j^*} \\
and hence \( j^* \leq j^* \). Now, Lemma 4 gives \( w_j \equiv 0 \).
\]

Lemma 10. Let \( \{ w^j \} \) and \( \{ y^j \} \) be the sequence and the auxiliary sequence generated by Algorithm 1. Assume \( w^* \notin Q \), the following statements hold:
- If \( w^* = 0 \), then \( y^j \) is bounded and \( w^j \) is bounded for all but finitely many \( t \) values.
- If \( w^* \neq 0 \), then \( \text{sign}(y^j) = \text{sign}(w^j) \) for all but finitely many \( t \) values.

Proof of Lemma 10. On the one hand, we consider the case \( w^* = 0 \), so that
\[
y^{j+1} = y^j + \eta \frac{\| v \|_2^2}{2\sqrt{2\pi}} (w^* - w^j) = y^j + \eta \frac{\| v \|_2^2}{2\sqrt{2\pi}} w^j.
\]
Note that Lemma 4 shows \( y^j \) and \( w^j \) are of the same sign if \( w^j \neq 0 \), we know \( y^j \) is bounded by \( C_j := \max \{ \| y^j \|, \eta \| w^j \|_2^2 \} \).
Moreover, Lemma 5 shows \( \| y^j \|_1 \geq 5nC_j \) for all but finitely many \( t \) values. Finally, we see from Lemma 9 that \( w^j = 0 \) for all but finitely many \( t \) values.

On the other hand, consider the case \( w^* \neq 0 \). Without loss of generality, we can assume \( w^j > 0 \). Note that whenever \( y^j \leq 0 \), we also have \( w^j \leq 0 \) so that
\[
y^{j+1} = y^j + \eta \frac{\| v \|_2^2}{2\sqrt{2\pi}} (w^* - w^j) \geq y^j + \eta \frac{\| v \|_2^2}{2\sqrt{2\pi}} w^j.
\]
From the above inequality, we see that \( y^j \) is increasing where the increment is bounded from below by \( \eta \frac{\| v \|_2^2}{2\sqrt{2\pi}} w^* \) so that \( \sum \eta^t = \infty \), so that there must exist some \( T_j > 0 \) such that \( y^j > 0 \). With Lemma 5, we can without loss of generality assume that \( \| y^j \|_1 \geq 5n \delta \) for all \( t \geq T_j \). For ease of notation, we let \( \delta = \eta \frac{\| v \|_2^2}{2\sqrt{2\pi}} \) so that \( \| y^j \|_1 \geq 5n \delta \) for all \( t \geq T_j \). We shall next prove that \( y^j \geq 0 \) for all \( t \geq T_j \). We prove by induction, assume \( y^j > 0 \) for some \( t > T_j \) and show \( y^{j+1} > 0 \).

1. If \( y^j > \delta \),
\[
y^{j+1} = y^j + \eta \frac{\| v \|_2^2}{2\sqrt{2\pi}} (w^* - w^j) \geq y^j - \delta > 0.
\]
2. If \( 0 < y^j \leq \delta \), since \( \| y^j \|_1 \geq 5n \delta \), Lemma 9 shows \( w_j = 0 \) so that
\[
y^{j+1} = y^j + \eta \frac{\| v \|_2^2}{2\sqrt{2\pi}} (w^* - w^{j}) = y^j + \frac{\eta \delta}{\eta} w^j > y^j > 0.
\]
Combining the above two cases, we get our desired result.

Lemma 6. Let \( \{ y^j \} \) be any auxiliary sequence generated by Algorithm 1. If \( w^* \notin Q \), then any sub-sequential limit of \( y^j := \frac{w_j}{\| w \|_1} \) belongs to the closure of \( O(w^*) \). Furthermore, if \( O(w^*) \) is regular, then \( y^j \) lies in \( O(w^*) \) for all but finitely many \( t \) values.

Proof of Lemma 6. By Lemma 10, we see that \( \text{sign}(y^j) = \text{sign}(w^j) \) for all \( w^* \neq 0 \). We only need to prove \( w^j \neq 0 \) implies \( \lim_{t \to \infty} y^j = 0 \). Indeed, by Lemma 10, we know that \( y^j \) is bounded by \( C_j \) while Lemma 5 tells us \( \| y^j \|_1 \) goes to infinity. Thus, \( \lim_{t \to \infty} y^j = \frac{w^j}{\| w \|_1} = 0 \).

Lemma 11. Let \( \{ w^* \} \) and \( \{ y^j \} \) be any sequence and auxiliary sequence generated by Algorithm 1. Assuming that \( w^* \notin Q^2 \), we have the following fact.

1. If \( |w^*| > |w^*| \) then \( |y^j| > |y^j| \) for all but finitely many \( t \) values.
2. If \( |w^*| = |w^*| \), then \( |y^j| - |y^j| \) is bounded and \( |w^j| = |w^j| \) for all but finitely many \( t \) values.
Proof of Lemma 11. Without loss of generality, we can assume \( w_j^\ast \geq w_{j+1}^\ast \geq \cdots \geq w_n^\ast \geq 0 \).

For the first statement, we only need to show that \( w_j^\ast > w_{j+1}^\ast \) implies \( y_j^\ast > y_{j+1}^\ast \) for all but finitely many \( t \) values. Note that whenever \( y_j^\ast < y_{j+1}^\ast \), then Lemma 4 implies \( w_j^\ast \leq w_{j+1}^\ast \), hence

\[
y_j^{t+1} - y_{j+1}^{t+1} = (y_j^t + \eta_t \|v\|^2 \|v\|_j) - (y_{j+1}^t + \eta_t \|v\|^2 \|v\|_i) \leq y_j^t - y_{j+1}^t + \eta_t \|v\|^2 \|v\|_j - \|v\|_i.
\]

Now that we know \( y_j^t - y_{j+1}^t \) is increasing as long as it is negative and \( \sum \eta_t = \infty \). Therefore, we conclude that there exist infinitely many \( t \) values such that \( y_j^t - y_{j+1}^t > 0 \). We can therefore assume \( y_j^t - y_{j+1}^t > 0 \), where \( T \) is the constant in Lemma 5 such that \( \|v\|^2 \|v\|_j \geq 5n \sqrt{2\pi} \) for all \( t \geq T \) where we set \( \epsilon = 2 \|w_j^\ast\|^2 \). Next, we would like to show \( y_j^t - y_{j+1}^t > 0 \) for all \( t \geq T \) by induction.

Next, assuming \( y_j^t - y_{j+1}^t > 0 \), we want to show \( y_j^{t+1} - y_{j+1}^{t+1} > 0 \).

On the one hand, if \( y_j^t - y_{j+1}^t \geq \epsilon \), we have

\[
y_j^{t+1} - y_{j+1}^{t+1} = (y_j^t - y_{j+1}^t) + \eta_t \|v\|^2 \|v\|_j - \eta_t \|v\|^2 \|v\|_i > \epsilon.
\]

On the other hand, if \( y_j^t - y_{j+1}^t < \epsilon \), we still have

\[
y_j^{t+1} - y_{j+1}^{t+1} > (y_j^t - y_{j+1}^t) - \eta_t \|v\|^2 \|v\|_j - \eta_t \|v\|^2 \|v\|_i > \epsilon.
\]

so that it suffices to show \( w_j^t = w_{j+1}^t \). From Lemma 4, we see that with

\[
j^\ast = \arg \max_{j \in \mathbb{N}} \frac{\|\|v\|\|_j}{1}.
\]

we only need to show \( j \neq j^\ast \). We prove by contradiction, assuming \( j = j^\ast \) so that \( w_j^t > 0 \) and \( w_{j+1}^t = 0 \). Lemma 9 shows \( y_j^t \geq \frac{1}{2} \|\|v\|\|_j \). Also, (7) gives

\[
\frac{\|\|v\|\|_j}{1} - 2(j - 1) \leq \frac{\|\|v\|\|_j}{j} = \left( \frac{\|\|v\|\|_j}{1} + y_j^t \right) \leq \left( \frac{\|\|v\|\|_j}{j} \right)^2.
\]

Simplifying the above inequality, we get

\[
\left( \frac{\|\|v\|\|_j}{1} \right)^2 - 2(j - 1) \left( \frac{\|\|v\|\|_j}{1} \right) - (j - 1) \leq 0.
\]

Left hand side is a quadratic function of \( \frac{\|\|v\|\|_j}{1} \), we know

\[
\frac{\|\|v\|\|_j}{1} \leq j - 1 + \sqrt{(j - 1)^2 - \frac{n - 1}{n - 1}} < 2n.
\]

We write equation (8) in a different way and get

\[
\frac{\|\|v\|\|_j}{1} \leq \left( \frac{\|\|v\|\|_j}{1} + y_j^t \right)^2.
\]

Now, we use \( j = j^\ast \) again, to get

\[
\frac{\|\|v\|\|_j}{1} \leq \frac{\|\|v\|\|_j}{1} + y_j^t.
\]

Rewriting the above inequality, we get

\[
\frac{\|\|v\|\|_j}{1} \leq \frac{\|\|v\|\|_j}{1} + 2(y_j^t + y_{j+1}^t).
\]

Combining (10) and (12), we get

\[
\frac{\|\|v\|\|_j}{1} \leq \frac{\|\|v\|\|_j}{1} + 2(y_j^t + y_{j+1}^t).
\]

Solving the above inequality, we get

\[
y_j^t - y_{j+1}^t \geq \frac{\|\|v\|\|_j}{1} + y_j^t - y_{j+1}^t \leq 2 \frac{\|\|v\|\|_j}{1} + (y_j^t)^2 \leq 0.
\]

Recalling that \( y_j^t \geq \frac{1}{2}(\frac{\|\|v\|\|_j}{1} \geq \sqrt{2\pi}) \), we have

\[
y_j^t - y_{j+1}^t > \frac{1}{2} \left( \frac{\|\|v\|\|_j}{1} \right)^2 > 0.
\]

This contradiction shows \( j \neq j^\ast \), and hence \( w_j^\ast = w_{j+1}^\ast \) and it follows that \( y_j^{t+1} > y_{j+1}^{t+1} \). Now, we have proved our first statement.

For the second statement, since \( w_j^\ast = w_i^\ast \), we have

\[
y_j^{t+1} - y_i^{t+1} = (y_j^t + \eta_t \|v\|^2 \|v\|_j) - (y_i^t + \eta_t \|v\|^2 \|v\|_i) \leq (y_j^t - y_i^t) - \eta \|v\|^2 \|v\|_j - \eta \|v\|^2 \|v\|_i.
\]

Hence, we know that \( |y_j^t - y_i^t| \) is bounded by

\[
C_{i,j} := \max \left\{ |y_j^0 - y_i^0|, \frac{\eta \|v\|^2 \|v\|_j}{\sqrt{2\pi}} \right\}.
\]
Without loss of generality, we can assume \( j < i \) and \( \min \{ y'_j, y'_i \} \geq 0 \) by Lemma 10. Recalling (14), we have \( w'_j \neq w'_i \) implying that
\[
|y'_j - y'_i| > \frac{\max \{ y'_j, y'_i \}^2}{2} \geq \frac{1}{2} \left( \frac{\|y'\|}{5n} \right)^2
\]
where the right hand side goes to infinity. This contradicts the boundedness of \( |y'_j - y'_i| \) if there are infinitely many \( t \) values such that \( w'_j \neq w'_i \).

\[\square\]

Lemma 7. Let \( \{ y^t \} \) be any auxiliary sequence generated by Algorithm 1. If \( w^* \not\in Q \), then any sub-sequence limit of \( \tilde{y}^t := \frac{y^t}{\|y^t\|} \) belongs to the closure of \( \text{Cone}(w^*) \). Moreover, if \( \text{Cone}(w^*) \) is regular, then \( \tilde{y}^t \in \text{Cone}(w^*) \) for all but finitely many \( t \) values.

Proof of Lemma 7. Note that we only need to show for any sub-sequence limit \( y \) of \( \tilde{y}^t \), we have \( \text{sign} \{ |y_j| - |y_i| \} = \text{sign} \{ |w'_j| - |w'_i| \} \). The first statement of Lemma 11 tells us that it is true for all sign \( \{ |w'_j| - |w'_i| \} \neq 0 \). Thus, it suffices to show that \( |w'_j| = |w'_i| \) implies \( |y_j| = |y_i| \).

Note that the second statement of Lemma 11 says that \( |y_j| - |y_i| \) is bounded by \( C_{t, j} \), while Lemma 5 gives \( \lim_{t \to \infty} \|y\| = \infty \). We see that
\[
|\tilde{y}_j| = \lim_{k \to \infty} \frac{|y^t_j|}{\|y^t_k\|} = \lim_{k \to \infty} \frac{|y^t_i|}{\|y^t_k\|} = |\tilde{y}_i|.
\]

\[\square\]

Lemma 8. Let \( \{ w^t \} \) be the sequence generated by Algorithm 1. If \( w^* \not\in Q \), then \( w^t \in \Lambda(w^*) \) for all but finitely many \( t \) values.

Proof of Lemma 8. First, by Proposition 4, \( y^t \in \text{Cone}(w^*) \) implies \( \text{Proj}_Q(y^t) \in \Lambda(w^*) \).

Second, let \( \partial \text{Cone}(w^*) = \overline{\text{Cone}(w^*)} - \text{Cone}(w^*) \). Now, a non-zero \( y^t \in \partial \text{Cone}(w^*) \) implies \( \text{Cone}(y^t) \subset \partial \text{Cone}(w^*) \), so that we also have \( \text{Proj}_Q(y^t) \in \Lambda(y^t) \subset \Lambda(w^*) \).

Third, by compactness of \( \overline{\text{Cone}(w^*)} \cap S^{n-1} \), we know there exists some \( \epsilon > 0 \) such that \( \tilde{y}^t := \frac{y^t}{\|y^t\|} \) lies in \( \epsilon \)-neighborhood of \( \text{Cone}(w^*) \cap S^{n-1} \) implying \( \tilde{y}^t \in \Lambda(w^*) \).

Finally, Lemma 7 suggests \( \tilde{y}^t \) lies in \( \varepsilon \)-neighborhood of \( \text{Cone}(w^*) \) for all but finitely many \( t \) values. We get our desired result.

\[\square\]

Theorem 1 (Ternary Case). Let \( \{ z_j \}_{j=1}^k = \Lambda(w^*) \) where \( z_1 = \text{Proj}_Q w^* \) is the optimum and \( w^* = \sum_{j=1}^k \lambda_j z_j \). If \( 0 < \sum_{j=2}^k \lambda_j < 1 \), we have \( w^t = \text{Proj}_Q w^* \) for infinite many \( t \) values, where \( w^t \) is any infinite sequence generated by Algorithm 1 with any initialization.

Proof of Theorem 1 (Ternary Case). Note that Lemma 7 suggests \( \tilde{y}^t = \frac{y^t}{\|y^t\|} \) lies in \( \varepsilon \)-neighborhood of \( \text{Cone}(w^*) \) for all but finitely
Recurrence of Optimum for Training Weight and Activation Quantized Networks

Figure 6. Training Loss of CIFAR-10. **Left:** Binary/Ternary weight ResNet-20. **Right:** Binary/Ternary weight VGG-11.

Figure 7. **Evolution of signs of weight filters in the last training epoch (or 600 iterations) of ResNet-20.** Each of the three $27 \times 200$ blocks corresponds to evolution of the $3 \times 3 \times 3$ convolutional filter over 200 iterations. Binary weights over the last 600 iterations of training, red/blue for sign values $1/-1$. 
Figure 8. Evolution of signs of weight filters in the last training epoch (or 600 iterations) of ResNet-20. Each of the three $27 \times 200$ blocks corresponds to evolution of the $3 \times 3 \times 3$ convolutional filter over 200 iterations. Ternary weights over the last 600 iterations of training, red/green/blue for sign values $1/0/-1$.

Figure 9. Evolution of signs of weight filters in the last training epoch (or 600 iterations) of VGG-11. Each of the three $27 \times 200$ blocks corresponds to evolution of the $3 \times 3 \times 3$ convolutional filter over 200 iterations. Binary weights over the last 600 iterations of training, red/blue for sign values $1/-1$. 
Figure 10. Evolution of signs of weight filters in the last training epoch (or 600 iterations) of VGG-11. Each of the three $27 \times 200$ blocks corresponds to evolution of the $3 \times 3 \times 3$ convolutional filter over 200 iterations. Ternary weights over the last 600 iterations of training, red/green/blue for sign values $1/0/-1$. 