INJECTIVE OBJECTS IN THE CATEGORY OF FINITELY PRESENTED REPRESENTATIONS OF AN INTERVAL FINITE QUIVER

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Abstract. We characterize the indecomposable injective objects in the category of finitely presented representations of an interval finite quiver.

1. Introduction

Infinite quivers appear naturally in the covering theory of algebras; see such as [BG82, Gab81]. The category of finitely presented representations of certain locally finite quiver is studied in [RVdB02]. The result is used to classify the Noetherian Ext-finite hereditary abelian categories with Serre duality. More generally, the category of finitely presented representations of a strongly locally finite quiver is studied in [BLP13]. The result is used to study the bounded derived category of a finite dimensional algebra with radical square zero in [BL17].

Let $k$ be a field and $Q$ be an interval finite quiver (with infinite vertices). We denote by $\text{fp}(Q)$ the category of finitely presented representations of $Q$ over $k$. Previous to the study of its Auslander–Reiten quiver, a natural question is how about the injective objects.

For each vertex $a$, we denote by $I_a$ the corresponding indecomposable injective representation. Let $p$ be a left infinite path, i.e., an infinite sequence of arrows $\cdots \alpha_i \cdots \alpha_2 \alpha_1$ with $s(\alpha_{i+1}) = t(\alpha_i)$ for any $i \geq 1$. Denote by $[p]$ the equivalence class of left infinite paths containing $p$. Consider the indecomposable representation $Y_{[p]}$ introduced in [Jia, Section 5]. We have that if $Y_{[p]}$ lies in $\text{fp}(Q)$, then it is an indecomposable injective object; see Proposition 3.10.

Moreover, we classify the indecomposable injective objects in $\text{fp}(Q)$.

Main Theorem (see Theorem 3.11). Let $Q$ be an interval finite quiver. Assume $I$ is an indecomposable injective object in $\text{fp}(Q)$. Then either $I \simeq I_a$ for certain vertex $a$, or $I \simeq Y_{[p]}$ for certain left infinite path $p$.

Compared with [Jia Theorem 6.8], the difficulty here is to characterize when $I_a$ and $Y_{[p]}$ are finitely presented. The result strengthens a description of finite dimensional indecomposable injective objects in $\text{fp}(Q)$; see [BLP13 Proposition 1.16].

The paper is organized as follows. In Section 2, we recall some basic facts about quivers and representations. In Section 3, we study the injective objects in $\text{fp}(Q)$ and give the classification theorem. Some examples are given in Section 4.

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2. Quivers and representations

Let $k$ be a field, and $Q = (Q_0, Q_1)$ be a quiver, where $Q_0$ is the set of vertices and $Q_1$ is the set of arrows. For each arrow $\alpha: a \to b$, we denote by $s(\alpha) = a$ its source and by $t(\alpha) = b$ its target.

A path $p$ of length $l \geq 1$ is a sequence of arrows $\alpha_0 \cdots \alpha_{l-1}$ such that $s(\alpha_i) = t(\alpha_{i+1})$ for any $1 \leq i \leq l-1$. We set $s(p) = s(\alpha_0)$ and $t(p) = t(\alpha_l)$. We associate each vertex $a$ with a trivial path (of length 0) $e_a$ with $s(e_a) = a = t(e_a)$. A nontrivial path $p$ is called an oriented cycle if $s(p) = t(p)$. For any $a, b \in Q_0$, we denote by $Q(a, b)$ the set of paths $p$ from $a$ to $b$, i.e., $s(p) = a$ and $t(p) = b$.

If $Q(a, b) \neq \emptyset$, then $a$ is called a predecessor of $b$, and $b$ is called a successor of $a$. For $a \in Q_0$, we denote by $a^-$ the set of vertices $b$ with some arrow $b \to a$; by $a^+$ the set of vertices $b$ with some arrow $a \to b$.

A right infinite path $p$ is an infinite sequence of arrows $\alpha_0 \alpha_1 \cdots$ such that $s(\alpha_i) = t(\alpha_{i+1})$ for any $i \geq 1$. We set $t(p) = t(\alpha_0)$. Dually, a left infinite path $p$ is an infinite sequence of arrows $\cdots \alpha_n \alpha_{n+1}$ such that $s(\alpha_{i+1}) = t(\alpha_i)$ for any $i \geq 1$. We set $s(p) = s(\alpha_1)$. Here, we use the terminologies in Che15 Section 2.1. We mention that these are opposite to the corresponding notions in BLPT13 Section 1.

A representation $M = (M(a), M(\alpha))$ of $Q$ over $k$ means a collection of $k$-linear spaces $M(a)$ for every $a \in Q_0$, and a collection of $k$-linear maps $M(\alpha): M(a) \to M(b)$ for every arrow $\alpha: a \to b$. For each nontrivial path $p = \alpha_0 \alpha_1 \cdots \alpha_{l-1}$, we denote $M(p) = M(\alpha_0) \circ \cdots \circ M(\alpha_{l-2}) \circ M(\alpha_{l-1})$. For each $a \in Q_0$, we set $M(e_a) = 1_{M(a)}$. A morphism $f: M \to N$ of representations is a collection of $k$-linear maps $f_a: M(a) \to N(a)$ for every $a \in Q_0$, such that $f_b \circ M(\alpha) = N(\alpha) \circ f_a$ for any arrow $\alpha: a \to b$.

Let $\text{Rep}(Q)$ be the category of representation of $Q$ over $k$. We denote by $\text{Hom}(M, N)$ the set of morphism in $\text{Rep}(Q)$. It is well known that $\text{Rep}(Q)$ is a hereditary abelian category; see GR92 Section 8.2.

Recall that a subquiver $Q'$ of $Q$ is called full if any arrow $\alpha$ with $s(\alpha), t(\alpha) \in Q_0'$ lies in $Q'$. Let $M$ be a representation of $Q$. The support $\text{supp} M$ of $M$ is the full subquiver of $Q$ formed by vertices $a$ with $M(a) \neq 0$. The socle $\text{soc} M$ of $M$ is the subrepresentation such that $(\text{soc} M)(a) = \bigcap_{\alpha \in Q_1, s(\alpha) = a} \text{Ker} M(\alpha)$ for any vertex $a$. The radical $\text{rad} M$ of $M$ is the subrepresentation such that $(\text{rad} M)(a) = \sum_{\alpha \in Q_1, t(\alpha) = a} \text{Im} M(\alpha)$ for any vertex $a$.

We mention the following fact; see BLPT13 Lemma 1.1.

Lemma 2.1. If the support of a representation $M$ contains no left infinite paths, then $\text{soc} M$ is essential in $M$.

Proof. Let $N$ be a nonzero subrepresentation of $M$. Assume $x \in N(a)$ is nonzero for some vertex $a$. Since $\text{supp} M$ contains no left infinite paths, there exists some path $p$ in $\text{supp} M$ with $s(p) = a$ such that $N(p)(x) \neq 0$ and $N(\alpha p)(x) = 0$ for any arrow $\alpha$ in $Q$. Then $N(p)(x) \in (N \cap \text{soc} M)(t(p))$. It follows that $\text{soc} M$ is essential in $M$. □

Let $a$ be a vertex in $Q$. We define a representation $P_a$ as follows. For every vertex $b$, we let

$$P_a(b) = \bigoplus_{p \in Q(a, b)} kp.$$
For every arrow $\alpha : b \to b'$, we let
\[ P_a(\alpha) : P_a(b) \to P_a(b'), p \mapsto \alpha p. \]
Similarly, we define a representation $I_a$ as follows. For every vertex $b$, we let
\[ I_a(b) = \text{Hom}_k(\bigoplus_{p \in Q(b,a)} kp, k). \]

For every arrow $\alpha : b \to b'$, we let
\[ I_a(\alpha) : I_a(b) \to I_a(b'), f \mapsto (p \mapsto p\alpha). \]

The following result is well known; see [GR92, Section 3.7]. It shows that $P_a$ is a projective representation and $I_a$ is an injective representation in $\text{Rep}(Q)$.

**Lemma 2.2.** Let $M \in \text{Rep}(Q)$ and $a \in Q_0$.

1. The $k$-linear map
   \[ \eta_M : \text{Hom}(P_a, M) \to M(a), f \mapsto f(a)(e_a), \]
   is an isomorphism natural in $M$.
2. The $k$-linear map
   \[ \zeta_M : \text{Hom}(M, I_a) \to \text{Hom}_k(M(a), k), f \mapsto (x \mapsto f(a)(x)e_a), \]
   is an isomorphism natural in $M$.

**Proof.** (1) Consider the $k$-linear map
\[ \eta_M : M(a) \to \text{Hom}(P_a, M) \]
given by $(\eta'_M(x)b)(p) = M(p)(x)$ for any $x \in M(a)$, $b \in Q_0$ and $p \in Q(a,b)$. We observe that $\eta'_M \circ \eta_M = 1_{\text{Hom}(P_a, M)}$ and $\eta_M \circ \eta'_M = 1_{M(a)}$. Then $\eta_M$ is an isomorphism.

(2) Consider the $k$-linear map
\[ \zeta'_M : \text{Hom}_k(M(a), k) \to \text{Hom}(M, I_a) \]
given by $(\zeta'_M(f)x)(p) = f(M(p)(x))$ for any $f \in \text{Hom}_k(M(a), k)$, $b \in Q_0$, $x \in M(b)$ and $p \in Q(b,a)$. We observe that $\zeta'_M \circ \zeta_M = 1_{\text{Hom}(M, I_a)}$ and $\zeta_M \circ \zeta'_M = 1_{\text{Hom}_k(M(a), k)}$. It follows that $\zeta_M$ is an isomorphism.

An epimorphism $P \to M$ with projective $P$ is called a projective cover of $M$ if it is an essential epimorphism. A monomorphism $M \to I$ with injective $I$ is called an injective envelope of $M$ if it is an essential monomorphism. We mention that two injective envelopes of $M$ are isomorphic.

Given a collection $\mathcal{A}$ of representations, we denote by $\mathcal{A}$ the full subcategory of $\text{Rep}(Q)$ formed by direct summands of finite direct sums of representations in $\mathcal{A}$. We set $\text{proj}(Q) = \{P_a | a \in Q_0\}$ and $\text{inj}(Q) = \{I_a | a \in Q_0\}$.

A representation $M$ is called finitely generated if there exists some epimorphism $f : \bigoplus_{i=1}^n P_{a_i} \to M$, and is called finitely presented if moreover $\text{Ker} f$ is also finitely generated. We denote by $\text{fp}(Q)$ the subcategory of $\text{Rep}(Q)$ formed by finitely presented representations.

We have the following well-known fact.

**Proposition 2.3.** The category $\text{fp}(Q)$ is a hereditary abelian subcategory of $\text{Rep}(Q)$ closed under extensions.
Proof. Let $f : P \to P'$ be a morphism in $\text{proj}(Q)$. We observe that $\text{Im} f$ is projective since $\text{Rep}(Q)$ is hereditary. Then the induced exact sequence
\[ 0 \to \text{Ker} f \to P \to \text{Im} f \to 0 \]
splits. Therefore $\text{Ker} f \in \text{proj}(Q)$. It follows from [Aus66, Proposition 2.1] that $fp(Q)$ is abelian. We observe by the horseshoe lemma that $fp(Q)$ is closed under extensions in $\text{Rep}(Q)$. In particular, it is hereditary. □

We mention the following observations.

Lemma 2.4. Let $M$ be a finitely presented representation and $N$ be a finitely generated subrepresentation. Then $M/N$ is finitely presented.

Proof. Let $f : P \to M$ be an epimorphism with $P \in \text{proj}(Q)$. Then $\text{Ker} f$ is finitely generated. Denote by $g$ the composition of $f$ and the canonical surjection $M \to M/N$. Consider the following commutative diagram.
\[
\begin{array}{cccccc}
0 & \to & \text{Ker} f & \to & P & \to & M/N & \to & 0 \\
& & h & \downarrow f & & \downarrow & & \downarrow & \\
0 & \to & N & \to & M & \to & M/N & \to & 0 \\
\end{array}
\]
We observe that the left square is a pushout and also a pullback. Then $h$ is an epimorphism and $\text{Ker} h \simeq \text{Ker} f$. In particular, $\text{Ker} h$ is finitely generated. Consider the exact sequence
\[ 0 \to \text{Ker} h \to \text{Ker} g \xrightarrow{h} N \to 0. \]
It follows that $\text{Ker} g$ is finitely generated. Then $M/N$ is finitely presented. □

The injective objects in $fp(Q)$ satisfy the following property.

Lemma 2.5. Let $I$ be an injective object in $fp(Q)$, and let $a \in Q_0$. Assume $p_i$ is a path from $a$ to $b_i$ for $1 \leq i \leq n$ such that $p_i$ is not of the form $u p_j$ with $u \in Q(b_j, b_i)$ for any $j \neq i$. Then the $k$-linear map
\[ \begin{pmatrix} I(p_1) \\ \vdots \\ I(p_n) \end{pmatrix} : I(a) \to \bigoplus_{i=1}^n I(b_i) \]
is a surjection.

We mention that one can consider the special case that $p_i : a \to b_i$ for $1 \leq i \leq n$ are pairwise different arrows.

Proof. We may assume each $b_i$ lies in $\text{supp} I$. For any $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \bigoplus_{i=1}^n I(b_i)$, we will find some pre-image under the given $k$-linear map.

For any $1 \leq i \leq n$, we consider the subrepresentation $M$ of $P_a \oplus I$ generated by $p_i - x_1 \in (P_a \oplus I)(b_i)$ and $p_j \in P_a(b_j)$ for all $j \neq i$. It follows from Lemma 2.4 that $(P_a \oplus I)/M$ is finitely presented.
We claim that \( M \cap I = 0 \). Indeed, let \( c \) be a vertex in \( \text{supp} M \). If \( Q(b_i, c) = \emptyset \), then \( M(c) \subseteq P_a(c) \) and hence \( M(c) \cap I(c) = 0 \). If \( Q(b_i, c) \neq \emptyset \), every \( y \in (M \cap I)(c) \) can be expressed as

\[
y = \sum_{i} \lambda_{iI}(P_a \oplus I)(u_{iI})(p_i - x_i) + \sum_{1 \leq j \leq n} \lambda_{jlj} P_a(u_{jlj})(p_j)
\]

\[
= \sum_{1 \leq j \leq n} \lambda_{jlj} u_{jlj} p_j - \sum_{i} \lambda_{iI} I(u_{iI})(x_i),
\]

for some \( \lambda_{jlj} \in k \) and pairwise different paths \( u_{jlj} \in Q(b_j, c) \). We observe that these \( u_{jlj}, p_j \) are pairwise different. Since \( y \in I(c) \) and \( P_a \cap I = 0 \), then every \( \lambda_{jlj} = 0 \). In particular, \( y = 0 \). We then obtain \( M(c) \cap I(c) = 0 \). It follows that \( M \cap I = 0 \).

We observe that the composition

\[
g: I \longrightarrow P_a \oplus I \longrightarrow (P_a \oplus I)/M
\]

is a monomorphism. Since \( I \) is an injective object, then there exists some morphism \( h: (P_a \oplus I)/M \rightarrow I \) such that \( h \circ g = 1_I \).

For any \( 1 \leq j \leq n \), we have that

\[
((P_a \oplus I)/M)(p_j)(\overline{y}) = \overline{x_i} = \begin{cases} x_i, & \text{if } j = i, \\ 0, & \text{if } j \neq i. \end{cases}
\]

Here, \( \overline{x}_i \) is the image of \( x_i \) in \((P_a \oplus I)/M\). Consider the commutative diagram

\[
\begin{array}{ccc}
I(a) & \xrightarrow{g_a} & ((P_a \oplus I)/M)(a) \\
\downarrow h_a & & \downarrow (h_a \circ g_a) \\
I(p_j) & \xrightarrow{g_{b_j}} & ((P_a \oplus I)/M)(p_j) \\
\downarrow h_{b_j} & & \downarrow (h_{b_j} \circ g_{b_j}) \\
I(b_j) & \xleftarrow{h_{b_j}} & ((P_a \oplus I)/M)(b_j).
\end{array}
\]

We observe that \( h_{b_i}(\overline{x}_i) = (h_{b_i} \circ g_{b_i})(x_i) = x_i \). Let \( y_i = h_a(\overline{y}) \). Then

\[
I(p_j)(y_i) = \begin{cases} x_i, & \text{if } j = i, \\ 0, & \text{if } j \neq i. \end{cases}
\]

Consider \( \sum_{i=1}^{n} y_i \in I(a) \). We have that

\[
\begin{pmatrix}
I(p_1) \\
\vdots \\
I(p_n)
\end{pmatrix}
\begin{pmatrix}
\sum_{i=1}^{n} y_i
\end{pmatrix}
= \begin{pmatrix}
I(p_1)(\sum_{i=1}^{n} y_i) \\
\vdots \\
I(p_n)(\sum_{i=1}^{n} y_i)
\end{pmatrix}
= \begin{pmatrix}
x_1 \\
\vdots \\
x_n
\end{pmatrix}.
\]

Then the result follows. \( \square \)

The following fact is a direct consequence.

**Corollary 2.6.** The support of an injective object in \( \text{fp}(Q) \) is closed under predecessors.

**Proof.** Let \( I \) be an injective object in \( \text{fp}(Q) \). Assume \( a \) is a vertex in \( \text{supp} I \) and \( b \in a^- \). We can choose some arrow \( \alpha: b \rightarrow a \). Lemma 2.3 implies that \( I(\alpha) \) is a surjection. In particular, the vertex \( b \) lies in \( \text{supp} I \). Then the result follows. \( \square \)
Recall that $Q$ is called \textit{interval finite} if $Q(a, b)$ is finite for any $a, b \in Q_0$. A quiver is called \textit{top finite} if there exist finitely many vertices of which every vertex is a successor, and is called \textit{socle finite} if there exist finitely many vertices of which every vertex is a predecessor.

We have the following observation.

\textbf{Lemma 2.7.} A top finite interval finite quiver contains no right infinite paths; a socle finite interval finite quiver contains no left infinite paths.

\textit{Proof.} Let $Q$ be a top finite interval finite quiver. Then there exist some vertices $b_1, b_2, \ldots, b_n$ such that any vertex is a successor of some $b_i$. Assume $Q$ contains a right infinite path $\alpha_1 \alpha_2 \cdots \alpha_j \cdots$. For each $j \geq 0$, we set $a_j = t(\alpha_{j+1})$. Since $Q(b_i, a_0)$ is finite, there exists some nonnegative integer $Z_i$ such that $Q(b_i, a_j) = \emptyset$ for any $j \geq Z_i$. Let $Z = \max_{1 \leq i \leq n} Z_i$. Then $a_Z$ is not a successor of any $b_i$, which is a contradiction. It follows that $Q$ contains no right infinite paths.

Similarly, a socle finite interval finite quiver contains no left infinite paths. \hfill \Box

3. \textsc{Finitely presented representations}

Let $k$ be a field and $Q$ be an interval finite quiver.

Recall that a representation $M$ is called \textit{pointwise finite} if $M(a)$ is finite dimensional for any vertex $a$, and is called \textit{finite dimensional} if moreover $\operatorname{supp} M$ contains only finitely many vertices.

We mention the following fact.

\textbf{Lemma 3.1.} The abelian category $\text{fp}(Q)$ is Hom-finite Krull–Schmidt, and every object is pointwise finite.

\textit{Proof.} Assume $\bigoplus_{i=1}^m P_{a_i} \rightarrow M$ is an epimorphism. We observe that each $P_{a_i}$ is pointwise finite, since $Q$ is interval finite. Then so is $M$.

Moreover, assume $\bigoplus_{j=1}^n P_{b_j} \rightarrow N$ is an epimorphism. Since $Q$ is interval finite, then $\operatorname{Hom}(\bigoplus_{i=1}^m P_{a_i}, \bigoplus_{j=1}^n P_{b_j})$ is finite dimensional. Consider the injections

\[
\operatorname{Hom}(M, N) \hookrightarrow \operatorname{Hom}(\bigoplus_{i=1}^m P_{a_i}, N) \hookrightarrow \operatorname{Hom}(\bigoplus_{i=1}^m P_{a_i}, \bigoplus_{j=1}^n P_{b_j}).
\]

It follows that $\operatorname{Hom}(M, N)$ is finite dimensional. Then the abelian category $\text{fp}(Q)$ is Hom-finite, and hence is Krull–Schmidt. \hfill \Box

We need the following properties of finitely presented representations.

\textbf{Lemma 3.2.} Let $M$ be a finitely presented representation.

1. $\supp M$ is top finite.
2. $\bigcup_{a \in \supp M} a^+ \setminus \supp M$ is finite.

\textit{Proof.} (1) Assume $f : \bigoplus_{i=1}^m P_{a_i} \rightarrow M$ is an epimorphism. Then every vertex in $\supp M$ is a successor of some $a_i$. In other words, $\supp M$ is top finite.

(2) Denote $\Delta = \bigcup_{a \in \supp M} a^+ \setminus \supp M$. We observe that $\ker f / \operatorname{rad} \ker f$ is semisimple and $(\ker f / \operatorname{rad} \ker f)(b) \neq 0$ for any $b \in \Delta$. If $\Delta$ is not finite, then $\ker f / \operatorname{rad} \ker f$ is not finitely generated, which is a contradiction. It follows that $\Delta$ is finite. \hfill \Box

\textbf{Corollary 3.3.} A finite dimensional representation $M$ is finitely presented if and only if $a^+$ is finite for any vertex $a$ in $\supp M$. 
Proof. For the necessary, we assume $M$ is finitely presented and $a$ is a vertex in supp $M$. Lemma 3.2 implies that $a^+ \setminus \text{supp} M$ is finite. Since supp $M$ contains only finitely many vertices, then $a^+ \cap \text{supp} M$ is finite. It follows that $a^+$ is finite.

For the sufficiency, we assume $a^+$ is finite for any vertex $a$ in supp $M$. Since $M$ is finite dimensional, there exists some epimorphism $f: P \to M$ with $P \in \text{proj}(Q)$.

Consider the subrepresentation $N$ of $P$ generated by $P(b)$, where $b$ runs over $\bigcup_{a \in \text{supp} M} a^+ \setminus \text{supp} M$. Then $N$ is contained in Ker $f$. We observe by Lemma 3.1 each $P(b)$ is finite dimensional. Then $N$ is finitely generated.

Consider the factor module Ker $f/N$. Its support is contained in supp $M$. Then it is finite dimensional and hence is finitely generated. Consider the exact sequence

$$0 \to N \to \text{Ker } f \to \text{Ker } f/N \to 0.$$ 

It follows that Ker $f$ is finitely generated, and then $M$ is finitely presented. □

Corollary 3.4. Let $a$ be a vertex. Then $I_a$ is finitely presented if and only if $a$ admits only finite predecessors $b$ and each $b^+$ is finite.

Proof. We observe that $I_a$ is finitely generated if and only if supp $I$ contains only finitely many vertices, since $Q$ is interval finite. The vertices in supp $I$ are precisely predecessors of $a$. Then the result follows from Corollary 3.3. □

The support of an injective object in fp($Q$) satisfies the following conditions.

Lemma 3.5. Let $I$ be an injective object in fp($Q$).

(1) $a^- \cup a^+$ is finite for any vertex $a$ in supp $I$.

(2) If supp $I$ contains no left infinite paths, then it contains only finitely many vertices.

Proof. Lemma 3.2 implies that supp $I$ is top finite. Then there exist some vertices $b_1, b_2, \ldots, b_n$ such that any vertex in supp $I$ is a successor of some $b_i$.

(1) We observe that supp $I$ is closed under predecessors; see Corollary 2.6. Then any vertex in $a^-$ is a successor of some $b_i$. If $a^-$ is infinite, then at least one $Q(b_i, a)$ is infinite, which is a contradiction. It follows that $a^-$ is finite.

We observe that $a^+ \cap \text{supp } I$ is finite. Indeed, otherwise Lemma 2.4 implies that $I(a)$ is not finite dimensional, which is a contradiction. Therefore, $a^+$ is finite. It follows that $a^- \cup a^+$ is finite.

(2) Assume the vertices in supp $I$ is infinite. Then there exists some $b_i$ whose successors contained in supp $I$ is infinite. Denote it by $a_0$. Since $a_0^+ \cap \text{supp } I$ is finite, then there exists some $a_1 \in a_0^+ \cap \text{supp } I$ whose successors contained in supp $I$ is infinite. Choose some arrow $\alpha_1: a_0 \to a_1$.

By induction, we obtain vertices $a_i$ and arrows $\alpha_{i+1}: a_i \to a_{i+1}$ for $i \geq 0$ in supp $M$. This is a contradiction, since $\cdots \alpha_i \cdots \alpha_2 \alpha_1$ is a left infinite path in supp $M$.

It follows that supp $M$ contains only finitely many vertices. □

For an injective object in fp($Q$) whose support contains no left infinite paths, we have the following characterization.

Proposition 3.6. Let $I$ be an injective object in fp($Q$) that supp $I$ contains no left infinite paths. Then

$$I \simeq \bigoplus_{a \in Q_0} f^\oplus_{a \in \text{dim}(\text{soc } I)(a)}.$$
Proof: It follows from Lemma 3.3 that supp $I$ contains only finitely many vertices. Let $J = \bigoplus_{a \in Q_0} I_a^{\dim(soc I(a))}$. This is a finite direct sum, since the vertices in supp $I$ are finite and $I$ is pointwise finite.

For any vertex $a$ in supp $I$, its predecessors are also contained in supp $I$; see Corollary 2.6. It follows that supp $J$ is a subquiver of supp $I$. Then soc $I$ and $J$ are finite dimensional. Corollary 3.3 implies that they are finitely presented. We observe by Lemma 2.4 that the inclusion soc $I \subseteq I$ and the injection soc $I \to J$ are injective envelopes in fp$(Q)$. Then the result follows.

For an injective object in fp$(Q)$ whose support contains some left infinite paths, we mention the following facts.

Lemma 3.7. Let $I$ be an injective object in fp$(Q)$, whose support contains left infinite paths. Denote by $\Delta$ the set of left infinite paths $p$ with $s(p)^- \cap$ supp $I = \emptyset$. Then $\Delta$ is finite, and every left infinite path $p$ in supp $I$ admits some $u$ such that $pu \in \Delta$.

Proof. Lemma 3.2 implies that supp $I$ is top finite. Assume vertices $b_1, b_2, \ldots, b_n$ satisfy that any vertex in supp $I$ is a successor of some $b_i$. We can assume each $b_i^- \cap$ supp $I = \emptyset$. Then the source of any left infinite path in $\Delta$ is some $b_i$.

Let $p$ be a left infinite path in supp $I$. We observe that $s(p)$ is a successor of some $b_i$. Choose some $u \in Q(b_i, s(p))$. Then $pu \in \Delta$.

Assume $\Delta$ is infinite. Denote $Z = \max_{1 \leq i \leq n} \dim I(b_i)$. Then there exist paths $u_j$ for $1 \leq j \leq nZ + 1$ with $s(u_j)$ being some $b_i$ such that each $u_j$ is not the of the form $vu_j$ for any $j' \neq j$. We observe that at least one $1 \leq i \leq n$ such that the number of $u_j$ with $s(u_j) = b_i$ is greater than $Z$. Then Lemma 2.5 implies that $\dim I(b_i) > Z$, which is a contradiction. It follows that $\Delta$ is finite.

Lemma 3.8. Let $I$ be an injective object in fp$(Q)$, whose support contains some left infinite path $\cdots \alpha_i \cdots \alpha_2 \alpha_1$. Set $a_i = s(\alpha_{i+1})$ for any $i \geq 0$. Then there exists some nonnegative integer $Z$ such that $a_i^+ = \{a_{i+1}\}$, $a_i^- = \{a_i\}$ and $I(\alpha_{i+1})$ is a bijection for any $i \geq Z$.

Proof. We observe by Lemma 3.2 that supp $I$ is top finite and there exists some nonnegative integer $Z_1$ such that $a_i^+$ is contained in supp $I$ for any $i \geq Z_1$.

It follows from Lemma 2.5 that $\dim I(a_i) \geq \dim I(a_{i+1})$ for any $i \geq 0$. Then there exists some nonnegative integer $Z_2 \geq Z_1$ such that $\dim I(a_i) = \dim I(a_{i+1})$ for any $i \geq Z_2$. Since $I(\alpha_{i+1})$ is a surjection by Lemma 2.5 then it is a bijection.

We claim that $a_i^+ = \{a_{i+1}\}$ and $Q(a_i, a_{i+1}) = \{a_{i+1}\}$ for any $i \geq Z_2$. Indeed, otherwise there exist some arrow $\beta: a_i \to b$ in supp $I$ with $i \geq Z_2$ and $\beta \neq \alpha_{i+1}$. Then Lemma 2.5 implies that $\dim I(a_i) \geq \dim I(a_{i+1}) + \dim I(b) > \dim I(a_{i+1})$, which is a contradiction.

Assume vertices $b_1, b_2, \ldots, b_n$ satisfy that every vertex in supp $I$ is a successor of some $b_j$. We observe that $|Q(b_j, a_{i+1})| > |Q(b_j, a_i)|$ for any $i \geq 0$, and at least one $|Q(b_j, a_{i+1})| > |Q(b_j, a_i)|$ if $a_i^+ \neq \beta_i$. Then there exists some nonnegative integer $Z_3$ such that $a_i^+ = \{a_i\}$ for any $i \geq Z_3$. Indeed, otherwise there exists some $1 \leq j \leq n$ such that the sequences $\langle |Q(b_j, a_i)| \rangle_{i \geq 0}$ is unbounded. Then Lemma 2.5 implies that $I(b_i)$ is not finite dimensional, which is a contradiction.

Let $Z = \max \{Z_2, Z_3\}$. Then the result follows.

Following [Che15, Subsection 2.1], we define an equivalent relation on left infinite paths. Two left infinite paths $\cdots \alpha_i \cdots \alpha_2 \alpha_1$ and $\cdots \beta_i \cdots \beta_2 \beta_1$ are equivalent if
there exist some positive integers $m$ and $n$ such that
\[
\cdots \alpha_1 \cdots \alpha_m + \alpha_{m+1} \beta_n = \cdot \cdots \beta_i \cdots \beta_{n+1} \beta_n.
\]

Let $p$ be a left infinite path. We denote by $[p]$ the equivalence class containing $p$. We mention that $[p]$ is a set. For any vertex $a$, we denote by $[p]_a$ the subset of $[p]$ formed by left infinite paths $u$ with $s(u) = a$.

Following [Jia, Section 5] and [Che15 Subsection 3.1], we introduce a representation $Y_{[p]}$ as follows. For every vertex $a$, we let
\[
Y_{[p]}(a) = \text{Hom}_k(\bigoplus_{u \in [p]_a} ku, k).
\]
For every arrow $\alpha : a \to b$, we let
\[
Y_{[p]}(\alpha) : Y_{[p]}(a) \to Y_{[p]}(b), f \mapsto (u \mapsto u\alpha).
\]
We mention that these $Y_{[p]}$ are indecomposable and pairwise non-isomorphic; see the dual of [Jia Proposition 5.4].

Recall that a quiver is called uniformly interval finite, if there exists some positive integer $Z$ such that for any vertices $a$ and $b$, the number of paths $p$ from $a$ to $b$ is less than or equal to $Z$; see [Jia Definition 2.3].

We characterize when $Y_{[p]}$ is finitely presented.

**Lemma 3.9.** Let $p$ be a left infinite path. Then $Y_{[p]}$ is finitely presented if and only if $\text{supp} Y_{[p]}$ is top finite uniformly interval finite and $\bigcup_{a \in \text{supp} Y_{[p]}} a^+ \setminus \text{supp} Y_{[p]}$ is finite.

**Proof.** For the necessary, we assume $Y_{[p]}$ is finitely presented. It follows from Lemma 3.2 that $\text{supp} Y_{[p]}$ is top finite and $\bigcup_{a \in \text{supp} Y_{[p]}} a^+ \setminus \text{supp} Y_{[p]}$ is finite.

Assume vertices $b_1, b_2, \ldots, b_n$ satisfy that any vertex in $\text{supp} Y_{[p]}$ is a successor of some $b_i$. Let $Z = \max_{1 \leq i \leq n} \dim Y_{[p]}(b_i)$. For any vertices $a$ and $a'$ in $\text{supp} Y_{[p]}$, there exists some $Q(b_i, a) \neq 0$. Since $Q$ contains no oriented cycles, we have that
\[
Z \geq |[p]_{b_i}| \geq |Q(b_i, a')| \geq |Q(a, a')|.
\]
It follows that $\text{supp} Y_{[p]}$ is uniformly interval finite.

For the sufficiency, we assume $p = \cdots \alpha_1 \cdots \alpha_2 \cdot \alpha_1$. Set $a_i = s(\alpha_{i+1})$ for any $i \geq 0$. Assume vertices $b_1, b_2, \ldots, b_n$ satisfy that any vertex in $\text{supp} Y_{[p]}$ is a successor of some $b_i$. Since $\text{supp} Y_{[p]}$ is uniformly interval finite, then
\[
\{|Q(b_j, a_i)| \mid i \geq 0, 1 \leq j \leq n\}
\]
is bounded. We observe that $|Q(b_j, a_i)| \leq |Q(b_j, \alpha_{i+1})|$. Then there exists some nonnegative integer $Z_1$ such that $|Q(b_j, a_i)| = |Q(b_j, \alpha_{Z_1})|$ for any $i \geq Z_1$ and $1 \leq j \leq n$. In particular, $a_{i+1} = \{a_i\}$ and $Q(a_i, a_{i+1}) = \{a_{i+1}\}$ for any $i \geq Z_1$.

Since $\bigcup_{a \in \text{supp} Y_{[p]}} a^+ \setminus \text{supp} Y_{[p]}$ is finite, there exists some nonnegative integer $Z_2$ such that $a_{i+1}^+$ is contained in $\text{supp} Y_{[p]}$ for any $i \geq Z_2$. Let $Z = \max \{Z_1, Z_2\}$. Then $a_{i+1}^+ = \{a_i\}$, $a_i^+ = \{a_{i+1}\}$ and $Y_{[p]}(a_{i+1})$ is a bijection.

Consider the subrepresentation $N$ of $Y_{[p]}$ generated by $Y_{[p]}(a_Z)$. We observe that $N \simeq F_{a_Z} \oplus \dim Y_{[p]}(a_Z)$ and hence is finite presented. Moreover, $\text{supp}(Y_{[p]}\!//N)$ contains only finitely many vertices $b$ and $b^+$ is finite. Then $Y_{[p]}\!//N$ is finitely presented by Corollary 3.3. Consider the exact sequence
\[
0 \to N \to Y_{[p]} \to Y_{[p]}\!//N \to 0.
\]
It follows that $Y_{[p]}$ is finitely presented.

We can show that finitely presented $Y_{[p]}$ for some left infinite path $p$ is an indecomposable injective object in $\text{fp}(Q)$; compare the dual of [Jin, Proposition 6.2].

**Proposition 3.10.** Let $p$ be a left infinite path such that $Y_{[p]}$ is finitely presented. Then $Y_{[p]}$ is an indecomposable injective object in $\text{fp}(Q)$.

**Proof.** Assume $p = \cdots \alpha_i \cdots \alpha_2 \alpha_1$. For each $i \geq 0$, we set $a_i = s(\alpha_i+1)$. Consider the morphism $\psi_{i+1}: I_{a_{i+1}} \to I_{a_i}$ given by $(\psi_{i+1})_b(f)(u) = f(\alpha_{i+1}u)$ for any $f \in I_{a_{i+1}}(b)$ and $u \in Q(b, a_i)$. We observe that $(I_{a_i})_{i \geq 0}$ forms an inverse system, and $Y_{[p]}$ is the inverse limit in $\text{Rep}(Q)$; see [Jin, Proposition 5.7].

Given any exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

in $\text{fp}(Q)$, it is also an exact sequence in $\text{Rep}(Q)$. Applying $\text{Hom}(-, I_{a_i})$, we obtain an exact sequence of inverse systems of $k$-linear spaces

$$0 \longrightarrow (\text{Hom}(N, I_{a_i})) \longrightarrow (\text{Hom}(M, I_{a_i})) \longrightarrow (\text{Hom}(L, I_{a_i})) \longrightarrow 0.$$  

Lemma 2.2 implies that $\text{Hom}(N, I_{a_i}) \simeq \text{Hom}_k(N(a_i), k)$. We observe by Lemma 3.1 that it is finite dimensional. Then $(\text{Hom}(N, I_{a_i}))$ satisfies Mittag–Leffler condition naturally. It follows from [Wei91, Proposition 3.5.7] the exact sequence

$$0 \longrightarrow \lim_{\rightarrow} \text{Hom}(N, I_{a_i}) \longrightarrow \lim_{\rightarrow} \text{Hom}(M, I_{a_i}) \longrightarrow \lim_{\rightarrow} \text{Hom}(L, I_{a_i}) \longrightarrow 0.$$  

For any $X \in \text{fp}(Q)$, there exist natural isomorphisms

$$\text{lim}_{\rightarrow} \text{Hom}(X, I_{a_i}) \simeq \text{Hom}(X, \lim_{\rightarrow} I_{a_i}) \simeq \text{Hom}(X, Y_{[p]}).$$

Then we obtain the exact sequence

$$0 \longrightarrow \text{Hom}(N, Y_{[p]}) \longrightarrow \text{Hom}(M, Y_{[p]}) \longrightarrow \text{Hom}(L, Y_{[p]}) \longrightarrow 0.$$  

It follows that $Y_{[p]}$ is an indecomposable injective object in $\text{fp}(Q)$. □

Now, we can classify the indecomposable injective objects in $\text{fp}(Q)$.

**Theorem 3.11.** Let $Q$ be an interval finite quiver. Assume $I$ is an indecomposable injective object in $\text{fp}(Q)$. Then either $I \simeq I_a$ where $a$ admits only finite predecessors $b$ and each $b^+$ is finite, or $I \simeq Y_{[p]}$ where $[p]$ is an equivalence class of left infinite paths such that $\text{supp} Y_{[p]}$ is top finite uniformly interval finite and $\bigcup_{a \in \text{supp} Y_{[p]} \backslash \text{supp} Y_{[p]}} a^+ \backslash \text{supp} Y_{[p]}$ is finite.

**Proof.** If $\text{supp} I$ contains no left infinite path, Proposition 3.6 implies that $I \simeq I_a$ for some vertex $a$. Corollary 6.1 implies that $a$ admits only finite predecessors $b$ and each $b^+$ is finite.

Now, we assume $\text{supp} I$ contains some left infinite path. Let $\Delta$ be the set of left infinite paths $p$ with $s(p) \cap \text{supp} I = \emptyset$. It follows from Lemma 3.7 that $\Delta$ is finite.

For every $p \in \Delta$, we assume $p = \cdots \alpha_{p,j} \cdots \alpha_{p,2} \alpha_{p,1}$. We let $a_{p,j} = s(\alpha_{p,j+1})$ for each $j \geq 0$. By Lemma 3.3 there exists some nonnegative integer $Z_p$ such that $a_{p,j} = \{a_{p,j+1}\}$, $a_{p,j+1} = \{a_{p,j}\}$ and $I(\alpha_{p,j+1})$ is a bijection for any $j \geq Z_p$.

Consider the subrepresentation $M$ of $I$ generated by $I(a_{p,Z_p+1})$ for all $p \in \Delta$. It follows from Lemma 2.4 that $I/M$ is finitely presented. Then it is an injective object in $\text{fp}(Q)$, since $\text{fp}(Q)$ is hereditary.

We observe by Lemma 3.7 that $\text{supp}(I/M)$ contains no left infinite paths. Indeed, assume $\cdots \alpha_i \cdots \alpha_2 \alpha_1$ is a left infinite path in $\text{supp}(I/M)$. It also lies in $\text{supp} I$. 

Then these $\alpha_i$ for $i$ large enough lie in $\text{supp } M$. Therefore, they do not lie in $\text{supp}(I/M)$, which is a contradiction.

It follows from Proposition 3.6 that

$$I/M \cong \bigoplus_{b \in Q_0} I_b^{\oplus \dim(\text{soc}(I/M))(b)}.$$

For any $p \in \Delta$, we observe that $(\text{soc}(I/M))(a_p, z_p) = I(a_p, z_p) \neq 0$ and $I(a_p, j)$ is a bijection for any $j \geq Z_p$. The previous isomorphism can be extended as

$$I \cong \left( \bigoplus_{b \in Q_0} I_b^{\oplus \dim(\text{soc } I)(b)} \right) \bigoplus_{p \in \Delta} Y_p^{\oplus \dim(I(a_p, z_p))}.$$

Since $I$ is indecomposable, then $\Delta$ contains only one left infinite path $p$ and $\dim(I(a_p, z_p)) = 1$. Then $\text{soc } I = 0$ and $I \cong Y_p$. It follows from Lemma 3.9 that $\text{supp } Y_p$ is top finite and uniformly interval finite, and $\bigcup_{a \in \text{supp } Y_p} a^+ \setminus \text{supp } Y_p$ is finite.

4. Examples

Let $k$ be a field. We will give some examples.

**Example 4.1.** Assume $Q$ is the following quiver.

\[ \cdots \leftarrow \alpha_4 \leftarrow 3 \leftarrow \alpha_3 \leftarrow 2 \leftarrow \alpha_2 \leftarrow 1 \leftarrow \alpha_1 \leftarrow 0 \]

For each $n \geq 0$, we consider the representation $I_n$. We observe that the predecessors of $n$ are $i$ for $0 \leq i \leq n$, and $i^+$ is finite. Corollary 3.4 implies that $I_n$ is finitely presented.

Let $p = \cdots \alpha_i \cdots \alpha_2 \alpha_1$. Then $\text{supp } Y_p = Q$. We observe that $\text{supp } Y_p$ is top finite uniformly interval finite and $\bigcup_{a \in \text{supp } Y_p} a^+ \setminus \text{supp } Y_p$ is the empty set. Lemma 3.9 implies that $Y_p$ is finitely presented.

We observe by Theorem 3.11 that

$$\{I_n \mid n \geq 0\} \cup \{Y_p\}$$

is a complete set of indecomposable injective objects in $\text{fp}(Q)$.

**Example 4.2.** Assume $Q$ is the following quiver.

\[ \cdots \leftarrow \alpha_2 \leftarrow 1 \leftarrow \alpha_1 \leftarrow 0 \leftarrow \alpha_0 \leftarrow -1 \leftarrow \alpha_{-1} \leftarrow \cdots \]

For each integer $n$, we consider the representation $I_n$. We observe that all $i \leq n$ are predecessors of $n$. Corollary 3.4 implies that $I_n$ is not finitely presented.

Let $p = \cdots \alpha_i \cdots \alpha_2 \alpha_1$. Then $\text{supp } Y_p = Q$, which contains a right infinite path. Then it is not top finite by Lemma 2.7. It follows from Lemma 3.9 that $Y_p$ is not finitely presented.

We observe by Theorem 3.11 that $\text{fp}(Q)$ contains no nonzero injective objects.
Example 4.3. Assume $Q$ is the following quiver.

We mention that $Q$ is interval finite, but it is not locally finite (i.e., for any vertex $a$, the set of arrows $\alpha$ with $s(\alpha) = a$ or $t(\alpha) = a$ is finite).

For each $n \geq 0$, we consider the representation $I_n$. We observe that the set of predecessors of $n$ is $\{0 \leq i \leq n\}$, but $0^+$ is not finite. Then Corollary 3.4 implies that $I_n$ is not finitely presented.

Let $p = \cdots \beta_3 \beta_2$. Then $supp Y_{[p]} = Q$, which is not uniformly interval finite. Lemma 3.9 implies that $Y_{[p]}$ is not finitely presented.

We observe by Theorem 3.11 that $fp(Q)$ contains no nonzero injective objects.

Example 4.4. Assume $Q$ is the following quiver.

For each $n \geq 0$, we consider the representations $I_{a_n}$ and $I_{b_n}$. We observe that the set of predecessors of $a_n$ is $\{a_i|0 \leq i \leq n\}$, and the one of $b_n$ is $\{a_i|0 \leq i \leq n\} \cup \{b_i|0 \leq i \leq n\}$. Since each $a_i^+$ and $b_i^+$ are both finite, Corollary 3.4 implies that $I_{a_n}$ and $I_{b_n}$ are finitely presented.

Let $p = \cdots \alpha_i \cdots \alpha_2 \alpha_1$. Then $supp Y_{[p]}$ is the full subquiver of $Q$ formed by $a_i$ for all $i \geq 0$. We observe that $\bigcup_{n\in supp Y_{[p]}} a^+ \setminus supp Y_{[p]}$ contains all $b_i$ and then is infinite. Lemma 3.9 implies that $Y_{[p]}$ is not finitely presented.

Let $q = \cdots \beta_3 \cdots \beta_2$. Then $supp Y_{[q]} = Q$. Since $Q$ is not uniformly interval finite, then $Y_{[q]}$ is not finitely presented by Lemma 3.9.

We observe that $\{[p], [q]\}$ is the set of equivalence classes of left infinite paths. It follows from Theorem 3.11 that

$$\{I_{a_i}|i \geq 0\} \cup \{I_{b_i}|i \geq 0\}$$

is a complete set of indecomposable injective objects in $fp(Q)$.

Example 4.5. Assume $Q$ is the following quiver.

We observe that $a^+$ is finite for any $a \in Q_0$. Consider the representations $I_{a_i}$ for $i \geq 0$ and $I_{b_j}$ for $j \in \mathbb{Z}$. The set of predecessors of $a_i$ is finite and the one of $b_j$ is not. It follows from Corollary 3.4 that $I_{a_i}$ is finitely presented, while $I_{b_j}$ is not.

Let $p = \cdots \alpha_i \cdots \alpha_2 \alpha_1$. Then $supp Y_{[p]}$ is the full subquiver of $Q$ formed by $a_i$ for all $i \geq 0$. We observe that $supp Y_{[p]}$ is top finite uniformly interval finite, and
It follows from Lemma 3.9 that \( Y_p \) is finitely presented.

Let \( q = \cdots \beta_2 \beta_1 \). Then \( \text{supp} Y_p \) is the full subquiver of \( Q \) formed by \( a_0 \), \( a_1 \), and \( b_j \) for all \( j \in \mathbb{Z} \). Since \( Q \) is not top finite, then \( Y_q \) is not finitely presented by Lemma 3.9.

We observe that \( \{ [p], [q] \} \) is the set of equivalence classes of left infinite paths. It follows from Theorem 3.11 that

\[
\{ I_{a_i} | i \geq 0 \} \cup \{ Y_p \}
\]

is a complete set of indecomposable injective objects in \( \text{fp}(Q) \).

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