Algebraic conditions and general solution to a system of quaternion tensor equations with applications

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Abstract: This paper investigates the necessary and sufficient algebraic conditions to a constrained system of Sylvester-type quaternion tensor equations. An explicit formula of the general solution regarding the Moore-Penrose inverses of some block given tensors is obtained. As an application of a particular case, we establish the solvability conditions and the general solution to a system of Sylvester-type quaternion tensor equations involving $\eta$-Hermitian unknowns. An algorithm with a numerical example is proposed to compute the general solution of the main system.

Keywords: Tensor, Moore-Penrose inverse, Quaternion, Tensor equation

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1. Introduction

A tensor is a multidimensional array. Specifically, a tensor is a generalization of a vector or matrix to higher dimensions \cite{5,10,12,21,34,35,40}. Tensors have applications in diverse areas such as machine learning, signal processing, biology, applied mechanics, data mining, pattern recognition, and numerical approaches algorithms for computing some generalized tensor and matrix equations \cite{1,18,22,24,25,28,31,33,37,51,54}. Hamilton \cite{19} was first presented the quaternion algebra over the real field $\mathbb{R}$

$$\mathbb{H} = \{d_0 + d_1 i + d_2 j + d_3 k : i^2 = j^2 = k^2 = ijk = -1, \ d_0, d_1, d_2, d_3 \in \mathbb{R}\}.$$ Quantum algebra is considered a non-commutative division ring. Quaternion, quaternion matrices, and quaternion tensors have applications in signal processing, color image processing, control theory, computer science, statistics and probability, quantum computing \cite{6,8,11,23,41,42}. Regularization of singular systems, computation of restricted singular value decomposition, and generalized systems of Sylvester-type matrix and tensor equations over the complex field $\mathbb{C}$ and the quaternion algebra $\mathbb{H}$ have been studied by many authors, see e.g. \cite{2,4,7,9,13,20,32,34,39,42,49,60,67,58}. Recently, Zhang and Kang \cite{59} propose the generalized modified Hermitian and skew-Hermitian splitting approach for computing the generalized Lyapunov equation:

$$AX + XA + \sum_{j=1}^{m} N_j X N_j^T + C = 0,$$
where $A, N_j \in \mathbb{C}^{n \times n}$ and $C = C^T$ are given matrices, $m \ll n$, $X \in \mathbb{C}^{n \times n}$ is the unknown matrix. However, here, we investigate the necessary and sufficient algebraic conditions for a two-sided four variable Sylvester-type linear tensor equation, and hence apply this equation to find the solvability conditions and the general solution to a constrained seven variables system of coupled tensor equations. The solvability conditions and the general solution of the Sylvester-type tensor equation:

$$A \ast_N X \ast_M B + C \ast_N Y \ast_M D = E$$  \hspace{1cm} (1.1)

was established in [21], where $A, B, C, D$ and $E$ are given tensors over $\mathbb{H}$. Equation (1.1) has an application in the discretization of higher dimension linear partial differential equations [20]. Here, we give a proper generalization of (1.1), namely,

$$A_1 \ast_N X_1 \ast_M B_1 + A_2 \ast_N X_2 \ast_M B_2 + A_2 \ast_N (C_1 \ast_N X_3 \ast_M D_3 + C_4 \ast_N W \ast_M D_4) \ast_M B_1 = E_1.$$  \hspace{1cm} (1.2)

Wang et al. [47] gave a comprehensive discussion to the following system of coupled two-sided Sylvester-type tensor equations:

$$\begin{cases}
A_1 \ast_N X_1 \ast_M B_1 = E_1, & A_2 \ast_N Y_1 \ast_M B_2 = E_2, \\
A_3 \ast_N Z = E_3, & Z \ast_M B_3 = E_4 \\
A_4 \ast_N X_1 \ast_M B_4 + C_1 \ast_N Z \ast_M D_4 = E_7, \\
A_5 \ast_N Y_2 \ast_M B_5 = E_8, \\
A_6 \ast_N W = E_3, & W \ast_M B_6 = E_9, \\
A_7 \ast_N X_2 \ast_M B_7 = E_4, \\
A_8 \ast_N X_3 \ast_M B_8 + A_9 \ast_N Y_1 \ast_M B_9 = E_9, \\
A_9 \ast_N (C_3 \ast_N X_3 \ast_M D_3 + C_4 \ast_N W \ast_M D_4) \ast_M B_1 = E_10.
\end{cases}$$  \hspace{1cm} (1.3)

They carried out the solvability conditions and the general solution in the Moore-Penrose inverses of some block given tensors. The quaternion system (1.3) considers as a proper extension of the tensor equation (1.1). We are motivated by wide applications of quaternion, quaternion matrices, quaternion tensors, even quaternion systems of Sylvester-type tensor equations. Equation (1.1) has an application in the discretization of higher dimension linear partial differential equations [20]. Here, we give a proper generalization of (1.1), namely,

$$A_1 \ast_N X_1 \ast_M B_1 = E_1, & A_2 \ast_N Y_1 \ast_M B_2 = E_2, \\
A_3 \ast_N Z = E_3, & Z \ast_M B_3 = E_4 \\
A_4 \ast_N X_1 \ast_M B_4 + C_1 \ast_N Z \ast_M D_4 = E_7, \\
A_5 \ast_N Y_2 \ast_M B_5 = E_8, \\
A_6 \ast_N W = E_3, & W \ast_M B_6 = E_9, \\
A_7 \ast_N X_2 \ast_M B_7 = E_4, \\
A_8 \ast_N X_3 \ast_M B_8 + A_9 \ast_N Y_1 \ast_M B_9 = E_9, \\
A_9 \ast_N (C_3 \ast_N X_3 \ast_M D_3 + C_4 \ast_N W \ast_M D_4) \ast_M B_1 = E_10.$$

(1.4)

As a particular case of (1.4), we derive the solvability conditions and the general solution to the system of Sylvester-type tensor equations:

$$\begin{cases}
A_1 \ast_N X_1 \ast_M B_1 = E_1, & A_2 \ast_N Y_1 \ast_M B_2 = E_2, \\
A_3 \ast_N W = E_3, & W \ast_M B_3 = E_4, \\
A_6 \ast_N X_1 \ast_M B_6 + A_9 \ast_N Y_1 \ast_M B_9 = E_9, \\
A_8 \ast_N Y_1 \ast_M B_8 + H_3 \ast_N Y_3 \ast_M J_3 = E_10.
\end{cases}$$  \hspace{1cm} (1.5)

Took et al. [43] define an $\eta$-Hermitian matrix, for $\eta \in \{i, j, k\}$, a quaternion square matrix $B$ over $\mathbb{H}$ is said to be an $\eta$-Hermitian matrix if $B^\eta = B$, where $B^\eta = -\eta B^\eta$. $\eta$-Hermitian matrices have applications in statistical signal processing and linear modeling [43][40]. As a direct implementation of the particular case (1.5), this study investigates the necessary and
sufficient algebraic conditions for the existence of a general solution to the following system of Sylvester-type tensor equations:

\[
\begin{align*}
A_1 * N X_3 * M A_1^T & = \mathcal{E}_1, \quad A_2 * N Y_3 * M A_2^T = \mathcal{E}_2, \quad A_3 * N W = \mathcal{E}_3, \\
A_6 * N A_1 + (A_8 * N A_1)^T + C_3 * N X_3 * M C_3^T & = \mathcal{E}_8, \\
A_8 * N Y_1 + (A_8 * N Y_1)^T + C_3 * N Y_3 * M C_3^T + C_4 * N W * M C_4^T & = \mathcal{E}_9,
\end{align*}
\]

(1.6)

where $X_3, Y_1, \text{ and } W$ are $\eta$-Hermitian unknowns and $\mathcal{E}_i = \mathcal{E}_i^T i \in \{1, 2, 9, 10\}$.

This paper is organized as follows. In Section 2, we recall some basic definitions and well-known results. Section 3 continues the algebraic solvability conditions and the general solution to (1.6). In Section 4, we investigate the particular case \[41\]. Consequently, we carry out the solvability conditions and the general solution to \[1.4\]. In Section 5, we summarized the results in giving the main conclusions.

2. Preliminaries

Throughout this paper, consider all tensors to be quaternion tensors. For convenience, we utilize the symbol $I(M)$ instead of $I_1 \times I_2 \times \ldots \times I_M$, for some positive integers $I_1, \ldots, I_M, M$. A tensor $P \in \mathbb{H}^{I_1 \times I_2 \times \ldots \times I_M \times J_1 \times J_2 \times \ldots \times J_N}$ can be written in the more straightforward form $P \in \mathbb{H}^{I(M) \times J(N)}$. A tensor $P \in \mathbb{H}^{I(N) \times J(N)}$ is called an even-order tensor. An even-order tensor $Q \in \mathbb{H}^{I(N) \times I(N)}$ is called an even-order square tensor. For a fixed element $q \in \mathbb{H}$, the symbol $\overline{q}$ stands for the conjugate of $q$. A quaternion tensor $P^* = (\overline{p_{i_1 \ldots i_N j_1 \ldots j_M}}) \in \mathbb{H}^{J(M) \times I(N)}$ calls the conjugate transpose of the tensor $P = (p_{i_1 \ldots i_N j_1 \ldots j_M}) \in \mathbb{H}^{I(N) \times J(M)}$. If $P = P^*$, then $P$ is called a quaternion Hermitian tensor.

Definition 2.1. [41] Let $A \in \mathbb{H}^{I(N) \times J(N)}$ and $B \in \mathbb{H}^{J(N) \times K(M)}$, then the Einstein product of $A$ and $B$ is denoted by $A * N B \in \mathbb{H}^{I(N) \times K(M)}$, where

\[
(A * N B)_{i_1 \ldots i_N k_1 \ldots k_M} = \sum_{j_1 \ldots j_N} a_{i_1 \ldots i_N j_1 \ldots j_N} b_{j_1 \ldots j_N k_1 \ldots k_M}.
\]

Moreover, $* N$ is associative over the set of all tensors with qualified order.

Definition 2.2. [41] An even order square tensor $D = (d_{i_1 \ldots i_N i_1 \ldots i_N}) \in \mathbb{H}^{I(N) \times I(N)}$ is called a diagonal tensor if $d_{i_1 \ldots i_N i_1 \ldots i_N} \neq 0$ and all its entries are zero. A diagonal tensor is said to be a unit tensor if $d_{i_1 \ldots i_N i_1 \ldots i_N} = 1$, which denotes by $I$.

Definition 2.3. [41] Let $A = (a_{i_1 \ldots i_N j_1 \ldots j_M}) \in \mathbb{H}^{I(N) \times J(M)}$, $B = (b_{i_1 \ldots i_N k_1 \ldots k_M}) \in \mathbb{H}^{I(N) \times K(M)}$, then the “row block tensor” of $A$ and $B$ is denoted by

\[
\begin{pmatrix} A & B \end{pmatrix} \in \mathbb{H}^{I(N) \times L(M)},
\]

where $L_s = J_s + K_s, s = 1, \ldots, M$ define as

\[
\begin{pmatrix} A & B \end{pmatrix}_{i_1 \ldots i_N l_1 \ldots l_M} = \begin{cases} a_{i_1 \ldots i_N l_1 \ldots l_M}, & \text{if } i_1 \ldots i_N \in [I_1] \ldots [I_N], l_1 \ldots l_M \in [J_1] \ldots [J_M], \\ b_{i_1 \ldots i_N l_1 \ldots l_M}, & \text{if } i_1 \ldots i_N \in [I_1] \ldots [I_N], l_1 \ldots l_M \in \Gamma_1 \ldots \Gamma_M, \\ 0, & \text{otherwise}, \end{cases}
\]

where $\Gamma_s = \{J_s + 1, \ldots, J_s + K_s\}, s = 1, \ldots, M$. For a given tensors $C = (c_{j_1 \ldots j_M i_1 \ldots i_N}) \in \mathbb{H}^{J(M) \times I(N)}$, $D = (d_{k_1 \ldots k_M i_1 \ldots i_N}) \in \mathbb{H}^{K(M) \times I(N)}$. The “column block tensor” of $C$ and $D$ is
denoted by
\[
\begin{pmatrix}
C \\
D
\end{pmatrix} \in \mathbb{H}^{L(M) \times I(N)},
\]
where \( L_s = J_s + K_s, \) \( s = 1, \ldots, M \) define as
\[
\begin{pmatrix}
C \\
D
\end{pmatrix}_{l_1 \ldots l_M i_1 \ldots i_N} = \begin{cases}
c_{l_1 \ldots l_M i_1 \ldots i_N}, & \text{if } l_1 \ldots l_M \in [J_1] \ldots [J_M], i_1 \ldots i_N \in [I_1] \ldots [I_N], \\
d_{l_1 \ldots l_M i_1 \ldots i_N}, & \text{if } l_1 \ldots l_M \in \Gamma_M, i_1 \ldots i_N \in [I_1] \ldots [I_N], \\
0, & \text{otherwise},
\end{cases}
\]
where \( \Gamma_s = \{ J_s + 1, \ldots, J_s + K_s \}, \) \( s = 1, \ldots, M. \)

**Proposition 2.1.**\([41]\) Let \( A \in \mathbb{H}^{I(P) \times K(N)} \) and \( B \in \mathbb{H}^{K(N) \times J(M)}, \) then

1. \( (A \ast_N B)^* = B^* \ast_N A^*, \)
2. \( \mathcal{I}_N \ast_N B = B, B \ast_M \mathcal{I}_M = B, \) where \( \mathcal{I}_N \in \mathbb{H}^{K(N) \times K(N)} \) and \( \mathcal{I}_M \in \mathbb{H}^{J(M) \times J(M)} \) are unit tensors.

**Proposition 2.2.**\([41]\) Consider the tensors \( (A \ B) \) and \( (C \ D) \) be given in (2.1) and (2.2), respectively. For a given tensor \( G \in \mathbb{H}^{I(N) \times I(N)}, \) we have that

1. \( G \ast_N (A \ B) = (G \ast_N A \ G \ast_N B) \in \mathbb{H}^{I(N) \times L(M)}, \)
2. \( (C \ D)^* \ast_N G = (C \ast_N G \ D \ast_N G) \in \mathbb{H}^{L(M) \times I(N)}, \)
3. \( (A \ B) \ast_M (C \ D) = A \ast_M C + B \ast_M D \in \mathbb{H}^{I(N) \times I(N)} \)

**Definition 2.4.**\([27]\) Let \( D \in \mathbb{H}^{I(N) \times J(N)} \), then the Moore-Penrose inverse of \( D \) is the unique tensor \( D^\dagger \in \mathbb{H}^{J(N) \times I(N)} \) satisfies:

1. \( D \ast_N D^\dagger \ast_N D = D, \)
2. \( D^\dagger \ast_N D \ast_N D^\dagger = D^\dagger, \)
3. \( (D^\dagger)^* = (D^\dagger)\ast, \)
4. \( (D^\dagger)\ast_N D = D^\dagger \ast_N D. \)

where \( R_D = I - D \ast_N D^\dagger \) and \( L_D = I - D^\dagger \ast_N D \) denote the projections along \( D. \)

**Definition 2.5.**\([27]\) Let \( \eta \) be an element in the quaternion algebra basis \( \{i, j, k\}. \) A tensor \( D \in \mathbb{H}^{I(N) \times I(N)} \) is said to be \( \eta \)-Hermitian if \( D = D^\eta, \) where \( D^\eta = -\eta D^\ast \eta. \)

**Proposition 2.3.**\([27]\) Let \( D \in \mathbb{H}^{I(N) \times I(N)}, \) then we have that

1. \( L_D \ast_N D^\dagger = D \ast_N L_D = 0, \)
2. \( (D^\ast)^\dagger = (D^\dagger)^\ast, \)
3. \( (L_D)^\ast = R_D N^\ast, \)
4. \( (D^\dagger)^\ast_N D^\dagger = D^\dagger \ast_N (D^\ast)^\dagger, \)

**Lemma 2.4.**\([27]\) Let \( A \in \mathbb{H}^{I(N) \times J(N)}, B \in \mathbb{H}^{K(M) \times L(M)}, C \in \mathbb{H}^{I(N) \times G(N)}, D \in \mathbb{H}^{H(M) \times L(M)} \) and \( E \in \mathbb{H}^{I(N) \times L(M)}. \) Set
\[
\mathcal{P} = R_A \ast_N C, \quad \mathcal{Q} = D \ast_M L_B, \quad S = C \ast_N L_D.
\]
Then (1.1) is solvable if and only if
\[ R_P * N R_A * N E = 0, \quad E * M L_B * M L_Q = 0, \]
\[ R_A * N E * M L_D = 0, \quad R_C * N E * M L_B = 0. \]

In that case, the general solution to (1.1) can be expressed as follows:
\[ X = A^t * N E * M B^t - A^t * N C * N D^t * N E * M B^t - A^t * N S * N C^t * N E * M Q^t \]
\[ * M D * M B^t - A^t * N S * N U_2 * M R_Q * M D * M B^t + L_A * N U_4 + U_5 * M R_B, \]
\[ \gamma = P^t * N E * M D^t + S^t * N E * M Q^t + L_P * N L_S * N U_1 + L_P * N U_2 \]
\[ * M R_Q + U_5 * M R_D, \]

where \( U_1, U_2, U_3, U_4 \) and \( U_5 \) are arbitrary tensors with suitable orders.

**Lemma 2.5.** Consider the system of tensor equations (1.3), where
\[ A_1 \in \mathbb{H}^{I(N) \times J(N)}, \quad A_2 \in \mathbb{H}^{I(N) \times Q(N)}, \quad A_3 \in \mathbb{H}^{I(N) \times P(N)}, \quad A_4 \in \mathbb{H}^{I(N) \times J(N)}, \]
\[ A_5 \in \mathbb{H}^{I(N) \times Q(N)}, \quad B_1 \in \mathbb{H}^{L(M) \times K(M)}, \quad B_2 \in \mathbb{H}^{S(M) \times K(M)}, \quad B_3 \in \mathbb{H}^{I(M) \times K(M)}, \]
\[ B_4 \in \mathbb{H}^{L(M) \times K(M)}, \quad B_5 \in \mathbb{H}^{S(M) \times K(M)}, \quad C_4 \in \mathbb{H}^{I(N) \times P(N)}, \quad C_5 \in \mathbb{H}^{I(N) \times P(N)}, \]
\[ D_3 \in \mathbb{H}^{K(M) \times K(M)}, \quad D_5 \in \mathbb{H}^{K(M) \times K(M)}, \quad \mathcal{P} \in \mathbb{H}^{I(N) \times K(M)}, \quad \mathcal{E}_i \in \mathbb{H}^{I(N) \times K(M)} \]

\( (i = 1, 4) \) are given tensors over \( \mathbb{H} \). Set
\[ A_6 = C_4 * N L_{A_5}, \quad B_6 = R_{B_5} * M D_4, \quad A_7 = C_5 * N L_{A_5}, \quad B_7 = R_{B_5} * M D_5, \]
\[ \mathcal{G} = \mathcal{P} - C_4 * N A_3^t * N E_3 * M D_4 - C_4 * N L_{A_5} * N E_4 * M B_3^t * M D_4, \]
\[ \mathcal{F} = Q - C_5 * N A_3^t * N E_3 * M D_5 - C_5 * N L_{A_5} * N E_4 * M B_3^t * M D_5, \]
\[ M_1 = R_{A_4} * M A_6, \quad N_1 = B_6 * M L_{B_4}, \quad S_1 = A_6 * N L_{M_1}, \]
\[ M_2 = R_{A_5} * M A_7, \quad N_2 = B_7 * M L_{B_5}, \quad S_2 = A_7 * N L_{M_2}, \]
\[ A_{11} = \left( L_{M_1} * N L_{S_1} \right), \quad B_{11} = \left( R_{B_6} \right), \quad A = R_{A_{11}} * N L_{M_1}, \]
\[ B = R_{N_1} * M L_{B_{11}}, \quad C = R_{A_{11}} * N L_{M_2}, \quad D = R_{N_2} * M L_{B_{11}}, \quad S = C * N L_{M_4}, \]
\[ \mathcal{E} = R_{A_{11}} * N \mathcal{E}_{11} * M L_{B_{11}}, \quad M = R_{A} * N \mathcal{C}, \quad N = D * M L_{B_5}, \quad C_{22} = A_4^t * N S_1 \]
\[ A_{22} = \left( L_{A_4} \right), \quad B_{22} = \left( R_{B_7} \right), \quad D_{22} = R_{N_1} * M B_6 * N B_4^t, \]
\[ A_{33} = R_{A_{22}} * N C_{22}, \quad B_{33} = D_{22} * M L_{B_{22}}, \quad E_{33} = R_{A_{22}} * N \mathcal{E}_{22} * M L_{B_{22}}, \]
\[ A_{44} = \left( L_{A_2} \right), \quad B_{44} = \left( R_{B_7} \right), \quad C_{44} = A_4^t * N S_2, \]
\[ D_{44} = R_{N_2} * M B_7 * N B_4^t, \quad E_{11} = M_{12}^t * N F * M B_7^t + S_1^t * N S_2 * N A_4^t * N F \]
\[ * M N_2^t - M_1^t * N G * M A_4^t - S_1^t * N S_1 * N A_5^t * N G * M N_1^t, \]
\[ E_{22} = A_4^t * N \mathcal{G} * M B_7 - A_4^t * N \mathcal{E}_1 * M B_7 - A_4^t * N S_1 * N A_5^t * N \mathcal{G} * M N_1 \]
\[ * M B_5 * M B_4^t - A_4^t * N A_6 * N M_1^t * N \mathcal{G} * M B_4^t, \]
\[ A_{11} = \mathcal{G} * M B_7 - A_4^t * N \mathcal{E}_1 * M B_7 - A_4^t * N S_1 * N A_5^t * N \mathcal{G} * M N_1 \]
\[ * M B_5 * M B_4^t - A_4^t * N A_6 * N M_1^t * N \mathcal{G} * M B_4^t, \]
\[
E_{44} = A_{1}^{+} * N \mathcal{F} * M B_{3}^{1} - A_{2}^{+} * N \mathcal{E}_{2} * M B_{2}^{1} - A_{3}^{+} * N S_{2} * N A_{1}^{+} * N \mathcal{F} * M N_{2} * M B_{3}^{1}

E_{66} = A_{33}^{+} * N \mathcal{E}_{33} * M B_{33}^{1} - A_{1}^{+} * N \mathcal{E} * M B^{\dagger} + A_{1}^{+} * N S \mathcal{N}^{1} * M D^{\dagger} + A_{1}^{+} * N \mathcal{C} * N \mathcal{M}^{1} * N \mathcal{E} * M B^{\dagger},
\]

\[
A_{55} = \mathcal{R}_{A_{44}} * N C_{44}, \quad A_{55} = D_{44} * M L_{B_{44}}, \quad E_{55} = \mathcal{R}_{A_{44}} * N E_{44} * M L_{B_{44}},
\]
\[
A_{66} = (L_{A} \quad L_{A_{33}}), \quad B_{66} = \left(\begin{array}{c}
\mathcal{R}_{B} \\
\mathcal{R}_{B_{33}}
\end{array}\right), \quad D_{66} = \mathcal{R}_{N} * M D * M B^{\dagger},
\]

\[
A_{77} = \mathcal{R}_{A_{66}} * N C_{66}, \quad B_{77} = D_{66} * M L_{B_{66}}, \quad \mathcal{E}_{77} = \mathcal{R}_{A_{66}} * N E_{66} * M L_{B_{66}},
\]
\[
A_{88} = (L_{M} * N \mathcal{L} \quad \mathcal{L}_{A_{33}}), \quad B_{88} = \left(\begin{array}{c}
\mathcal{R}_{D} \\
\mathcal{R}_{B_{33}}
\end{array}\right), \quad C_{88} = \mathcal{L}_{M}, \quad D_{88} = \mathcal{R}_{N},
\]
\[
E_{88} = A_{55}^{+} * N \mathcal{E}_{55} * M B_{55}^{1} - M^{1} * N \mathcal{E} * M B^{\dagger} - S^{\dagger} * N S \mathcal{N}^{1} * N \mathcal{E} * M N^{\dagger},
\]

Then (13) is solvable if and only if

\[
\mathcal{R}_{A_{4}} * N E_{3} = 0, \quad \mathcal{E}_{4} * M L_{B_{3}} = 0, \quad A_{3} * N E_{3} = \mathcal{E}_{4} * M B_{3},
\]

\[
\mathcal{R}_{A_{4}} * N \mathcal{G} * M L_{B_{4}} = 0, \quad \mathcal{R}_{A_{6}} * N \mathcal{G} * M L_{B_{6}} = 0, \quad \mathcal{R}_{S_{1}} * N \mathcal{R}_{A_{4}} * N \mathcal{G} = 0,
\]

\[
\mathcal{G} * M L_{B_{3}} * M L_{B_{3}} = 0, \quad \mathcal{R}_{A_{6}} * N \mathcal{F} * M L_{B_{5}} = 0, \quad \mathcal{R}_{A_{6}} * N \mathcal{F} * M L_{B_{5}} = 0,
\]

\[
\mathcal{R}_{S_{2}} * N \mathcal{R}_{A_{6}} * N \mathcal{F} = 0, \quad \mathcal{F} * M L_{B_{3}} * M L_{B_{3}} = 0, \quad \mathcal{R}_{M} * N \mathcal{R}_{A_{6}} * N \mathcal{E} = 0,
\]

\[
\mathcal{R}_{A} * N \mathcal{E} * M L_{D} = 0, \quad \mathcal{E} * M L_{B} * M L_{N} = 0, \quad \mathcal{R}_{C} * N \mathcal{E} * M L_{B} = 0,
\]

\[
\mathcal{R}_{A_{4}} * N E_{1} = 0, \quad E_{1} * M L_{B_{3}} = 0, \quad \mathcal{R}_{A_{2}} * N E_{2} = 0, \quad E_{2} * M L_{B_{3}} = 0,
\]

\[
\mathcal{R}_{A_{6}} * N E_{33} = 0, \quad \mathcal{E}_{33} * M L_{B_{33}} = 0, \quad \mathcal{R}_{S_{5}} * N \mathcal{E}_{55} = 0, \quad \mathcal{E}_{55} * M L_{B_{55}} = 0,
\]

\[
\mathcal{R}_{A_{77}} * N E_{77} = 0, \quad E_{77} * M L_{B_{77}} = 0, \quad \mathcal{R}_{A_{66}} * N \mathcal{E}_{66} = 0, \quad \mathcal{E}_{66} * M L_{B_{66}} = 0,
\]

\[
\mathcal{R}_{A} * N \tilde{\mathcal{E}} * M L_{\tilde{B}} = 0.
\]

Under these circumstances, the general solution to (13) can be expressed as follows:

\[
\mathcal{X} = A_{2}^{+} * N \mathcal{E}_{1} * M B_{1}^{\dagger} + L_{A_{1}} * N \mathcal{U}_{1} + U_{1} * M R_{B_{1}},
\]

\[
\mathcal{Y} = A_{2}^{+} * N \mathcal{E}_{2} * M B_{2}^{\dagger} + L_{A_{2}} * N \mathcal{U}_{2} + U_{2} * M R_{B_{2}},
\]

\[
\mathcal{Z} = A_{3}^{+} * N \mathcal{E}_{3} + L_{A_{3}} * N \mathcal{E}_{4} * M B_{3}^{\dagger} + L_{A_{3}} * N \mathcal{W} * M R_{B_{3}},
\]

where

\[
\mathcal{U}_{1} = \left(\begin{array}{c}
I \\
0
\end{array}\right) * N \left(A_{22}^{+} * N (\mathcal{E}_{22} - \mathcal{C}_{22} * N \mathcal{V}_{2} * M \mathcal{D}_{22}) - A_{22}^{+} * N \mathcal{H}_{12} * M B_{22}
\right.

+ \mathcal{L}_{A_{22}} * N \mathcal{H}_{11}),
\]

\[
\mathcal{U}_{2} = (\mathcal{R}_{A_{22}} * N (\mathcal{E}_{22} - \mathcal{C}_{22} * N \mathcal{V}_{2} * M \mathcal{D}_{22}) * M B_{22}^{\dagger} + A_{22} * N A_{22}^{+} * N \mathcal{H}_{12}
\]

\[
\mathcal{H}_{13} * M R_{B_{22}} * M \left(\begin{array}{c}
I \\
0
\end{array}\right),
\]
\[ \mathcal{U}_3 = \left( \begin{array}{cc} \mathbf{I} & 0 \end{array} \right) * N (A_{44}^t * N (E_{44} - C_{44} * N T_2 * M D_{44}) - A_{44}^t * N \mathcal{H}_{22} * M B_{44} \\
+ \mathcal{L}_{A_{44}} * N H_{21}) , \]

\[ \mathcal{U}_4 = (\mathcal{R}_{A_{44}} * N (E_{44} - C_{44} * N T_2 * M D_{44}) * M B_{44}^t + A_{44} * N A_{44}^t * N \mathcal{H}_{22} \\
\mathcal{H}_{23} * M \mathcal{R}_{B_{44}}) * M \left( \begin{array}{c} \mathbf{I} \\ 0 \end{array} \right) , \]

\[ \mathcal{W} = M_{1}^t * N \mathcal{G} * M B_{1}^t + S_{1}^t * N S_{1} * N A_{1}^t * N \mathcal{G} * M N_{1}^t - L_{M_1} * N L_{S_1} \\
* N \mathcal{V}_1 + L_{M_1} * N \mathcal{V}_2 * M \mathcal{R}_{N_1} + \mathcal{V}_3 * M \mathcal{R}_{B_6} , \]

\[ \mathcal{V}_1 = \left( \begin{array}{cc} \mathbf{I} & 0 \end{array} \right) * N (A_{11}^t * N (E_{11} - L_{M_1} * N \nu_2 * M \mathcal{R}_{N_1} - L_{M_2} * N T_2 * M \mathcal{R}_{N_2}) \\
+ W_1 * M B_{11} + L_{A_{11}} * N \mathcal{W}_2) , \]

\[ \mathcal{V}_2 = A_{1}^t * N \mathcal{E} * M B_{1}^t - A_{1}^t * N S * N C_{1}^t * N \mathcal{E} * M N_{1}^t * M \mathcal{D} * M B_{1}^t - A_{1}^t * N \]
\[ \mathcal{C} * N M_{1}^t * N \mathcal{E} * M B_{1}^t + A_{1}^t * N S * N W_{4} * M \mathcal{R}_{N} * M \mathcal{D} * M B_{1}^t \\
+ L_{A} * N \mathcal{W}_5 + W_0 * M \mathcal{R}_{B} , \]

\[ \mathcal{V}_3 = (\mathcal{R}_{A_{11}} * N (E_{11} - L_{M_1} * N \nu_2 * M \mathcal{R}_{N_1} - L_{M_2} * N T_2 * M \mathcal{R}_{N_2}) \\
* M B_{11}^t - A_{11} * N W_1 - W_3 * M \mathcal{R}_{B_{11}}) * M \left( \begin{array}{c} \mathbf{I} \\ 0 \end{array} \right) , \]

\[ T_2 = M_{1}^t * N \mathcal{E} * M D_{1}^t + S_{1}^t * N S * N C_{1}^t * N \mathcal{E} * M N_{1}^t * M \mathcal{L} * N \mathcal{L}_{S} * N W_{7} \\
+ L_{M} * N W_{4} * M \mathcal{R}_{N} + W_{8} * M \mathcal{R}_{D} , \]

\[ W_4 = A_{17}^t * N \mathcal{E}_{17} * M B_{17}^t - L_{A_{17}} * N \mathcal{Q}_1 - Q_{2} * M \mathcal{R}_{B_{17}} , \]

\[ W_5 = \left( \begin{array}{cc} \mathbf{I} & 0 \end{array} \right) * N (A_{66}^t * N (E_{66} - C_{66} * N W_4 * M D_{66}) - A_{66}^t * N \mathcal{H}_{32} * M B_{66} \\
+ L_{A_{66}} * N \mathcal{H}_{31}) , \]

\[ W_6 = (\mathcal{R}_{A_{66}} * N (E_{66} - C_{66} * N W_4 * M D_{66}) * M B_{66}^t + A_{66} * N A_{66}^t * N \mathcal{H}_{32} \\
\mathcal{H}_{33} * M \mathcal{R}_{B_{66}}) * M \left( \begin{array}{c} \mathbf{I} \\ 0 \end{array} \right) , \]

\[ W_7 = \left( \begin{array}{cc} \mathbf{I} & 0 \end{array} \right) * N (A_{88}^t * N (E_{88} - C_{88} * N W_4 * M D_{88}) - A_{88}^t * N \mathcal{H}_{42} \\
* M B_{88} + L_{A_{88}} * N \mathcal{H}_{41}) , \]

\[ W_8 = (\mathcal{R}_{A_{88}} * N (E_{88} - C_{88} * N W_4 * M D_{88}) * M B_{88}^t + A_{88} * N A_{88}^t * N \mathcal{H}_{42} \\
\mathcal{H}_{43} * M \mathcal{R}_{B_{88}}) * M \left( \begin{array}{c} \mathbf{I} \\ 0 \end{array} \right) , \]

\[ Q_1 = \left( \begin{array}{cc} \mathbf{I} & 0 \end{array} \right) * N (\tilde{A}_{\mathcal{I}} * N \tilde{E} - \tilde{A}_{\mathcal{I}} * N K_2 * M \tilde{B} + L_{A} * N \tilde{K}_1) , \]

\[ Q_2 = (\mathcal{R}_{\mathcal{I}} * N \tilde{E} * M B_{\mathcal{I}}^t + \tilde{A} * N \tilde{A}^t * N K_2 + K_{3} * M \mathcal{R}_{\tilde{B}}) * M \left( \begin{array}{c} \mathbf{I} \\ 0 \end{array} \right) . \]

where \( W_i , K_i , H_{jk} (i,k = 1,3, j = 1,4) \) are arbitrary with suitable orders.

3. Algebraic solvability conditions and general solution to (1.5)

In the following Proposition, we provide a proper extension of the tensor equation (1.1), which plays an essential role in the proof-findings process. Precisely, we derive the solvability conditions and the general solution to (1.2)
Proposition 3.1. Let $A_1 \in \mathbb{H}^{I(N) \times J(N)}$, $A_2 \in \mathbb{H}^{I(N) \times G(N)}$, $B_1 \in \mathbb{H}^{K(M) \times L(M)}$, $B_2 \in \mathbb{H}^{H(M) \times L(M)}$, $C_3 \in \mathbb{H}^{G(N) \times Q(N)}$, $C_4 \in \mathbb{H}^{G(N) \times T(N)}$, $D_3 \in \mathbb{H}^{S(M) \times K(M)}$, $D_4 \in \mathbb{H}^{P(M) \times K(M)}$ and $E_1 \in \mathbb{H}^{I(N) \times L(M)}$ be given. Set
\begin{align}
\mathcal{M}_1 &= R_{A_1} \ast N A_2, \quad N_1 = B_2 \ast M L_{B_1}, \quad S_1 = A_2 \ast N L_{M_1}, \quad \hat{A}_1 = \mathcal{M}_1 \ast N C_3, \quad (3.1) \\
\hat{A}_2 &= \mathcal{M}_1 \ast N C_4, \quad \hat{B}_1 = D_3 \ast M B_1 \ast M L_{B_2}, \quad \hat{B}_2 = D_4 \ast M B_1 \ast M L_{B_2}, \quad (3.2) \\
\tilde{\mathcal{M}}_1 &= R_{\hat{A}_1} \ast N \tilde{A}_2, \quad \tilde{N}_1 = \tilde{B}_2 \ast M L_{\tilde{B}_1}, \quad \tilde{S}_1 = \hat{A}_2 \ast N L_{\tilde{M}_1}, \quad \tilde{E}_1 = R_{\hat{A}_1} \ast N E_1 \ast M L_{B_2}, \quad (3.3) \\
\ast M L_{B_2}, \quad \tilde{E}_1 = \hat{E}_1 - A_2 \ast N (C_3 \ast N \chi_3 \ast M D_3 + C_4 \ast N W \ast M D_4) \ast M B_1. \quad (3.4)
\end{align}

Then the following statements are equivalent:

(1) \( (3.2) \) is solvable.

(2) The conditions
\[
R_{\mathcal{M}_1} \ast N R_{A_1} \ast N E_1 = 0, \quad E_1 \ast M \mathcal{L}_{B_1} \ast M \mathcal{L}_{N_1} = 0, \quad R_{A_2} \ast N E_1 \ast M \mathcal{L}_{B_1} = 0
\]
are satisfying and there exist quaternion tensors $\chi_3$ and $W$ satisfy
\[
\hat{A}_1 \ast N \chi_3 \ast M \hat{B}_1 + \hat{A}_2 \ast N W \ast M \hat{B}_2 = \hat{E}_1.
\]

In that case, the general solution to \( (3.2) \) can be expressed as follows:
\[
\chi_1 = A_1^T \ast N \hat{E}_1 \ast M B_1^T - A_1^T \ast N A_2 \ast N \mathcal{M}_1^T \ast N \hat{E}_1 \ast M B_1^T - A_1^T \ast N S_1 \ast N A_2^T
\]
\[
\ast N \hat{E}_1 \ast M \mathcal{N}_1^T \ast M B_2 \ast M B_1^T - A_1^T \ast N S_1 \ast N U_2 \ast M R_{N_1} \ast M B_2 \ast M B_1^T
\]
\[
+ \mathcal{L}_{A_1} \ast N U_4 + \tilde{U}_5 \ast M R_{B_1},
\]
\[
\chi_2 = M_1^T \ast N \hat{E}_1 \ast M B_2^T + S_1^T \ast N S_1 \ast N A_2^T \ast N \hat{E}_1 \ast M \mathcal{N}_1^T + \mathcal{L}_{M_1} \ast N \mathcal{L}_{S_1}
\]
\[
\ast N U_4 + \tilde{U}_5 \ast M R_{N_1} \ast M U_6 \ast M R_{B_2},
\]
\[
\chi_3 = \tilde{A}_1^T \ast N \hat{E}_1 \ast M \tilde{B}_1^T - \tilde{A}_1^T \ast N \tilde{A}_2 \ast N \tilde{M}_1^T \ast N \hat{E}_1 \ast M \tilde{B}_1^T - \tilde{A}_1^T \ast N \tilde{S}_1 \ast N \tilde{A}_2^T
\]
\[
\ast N \hat{E}_1 \ast M \mathcal{N}_1^T \ast M \tilde{B}_2 \ast M \tilde{B}_1^T - \tilde{A}_1^T \ast N \tilde{S}_1 \ast N \tilde{U}_2 \ast M R_{\tilde{N}_1} \ast M \tilde{B}_2 \ast M \tilde{B}_1^T
\]
\[
+ \mathcal{L}_{\tilde{A}_1} \ast N \tilde{U}_4 + \tilde{U}_5 \ast M R_{\tilde{B}_1},
\]
\[
W = \tilde{A}_1^T \ast N \hat{E}_1 \ast M \hat{B}_2^T + \tilde{S}_1^T \ast N \tilde{S}_1 \ast N \tilde{A}_2^T \ast N \hat{E}_1 \ast M \mathcal{N}_1^T + \mathcal{L}_{\tilde{N}_1} \ast N \mathcal{L}_{\tilde{S}_1}
\]
\[
\ast N \tilde{U}_1 + \mathcal{L}_{\tilde{X}_1} \ast N \tilde{U}_2 \ast M R_{\tilde{X}_1} + \tilde{U}_3 \ast M R_{\tilde{B}_2},
\]
where $U_i, \tilde{U}_i \ (i = 1, 5)$ are arbitrary tensors with suitable orders.

Proof. (1) $\implies$ (2) We first, rewrite the tensor equation \( (3.2) \) in the form
\[
A_1 \ast N \chi_1 \ast M B_1 + A_2 \ast N \chi_2 \ast M B_2 = \hat{E}_1,
\]
where $\hat{E}_1$ gives by \( (3.4) \). By utilizing Lemma 2.4 we have that \( (3.6) \) is solvable if and only if there exist quaternion tensors $\chi_3$ and $W$ satisfy the following conditions:
\begin{align}
R_{\mathcal{M}_1} \ast N R_{A_1} \ast N \hat{E}_1 = 0, \quad \hat{E}_1 \ast M \mathcal{L}_{B_1} \ast M \mathcal{L}_{N_1} = 0, \quad R_{A_2} \ast N \hat{E}_1 \ast M \mathcal{L}_{B_1} = 0, \quad (3.7) \\
R_{\tilde{A}_1} \ast N \hat{E}_1 \ast M \mathcal{L}_{B_2} = 0. \quad (3.8)
\end{align}
In that case, the general solution to (3.10) can be expressed as

\[ X_1 = A_1^1 * N \hat{\mathbf{e}}_1 * M B_1^1 - A_1^1 * N A_2 * N M_1^1 * N \hat{\mathbf{e}}_1 * M B_1^1 - A_1^1 * N S_1 * N A_2^1 * N \mathbf{e}_1 * M N_1^1 * M B_1^1 + \mathcal{L}_{A_1} * N U_1 + U_6 * M \mathcal{R}_{B_1}, \]

\[ X_2 = M_1^1 * N \hat{\mathbf{e}}_1 * M B_2^1 + S_1^1 * N S_1 * N A_2^1 * N \mathbf{e}_1 * M N_1^1 * M \mathcal{L}_{S_1} + \mathcal{L}_{M_1} * N \mathcal{L}_{S_1} \]

The conditions (3.10) are satisfied if and only if the conditions

\[ \mathcal{R}_{M_1} * N \mathcal{R}_{A_1} * N \mathcal{E}_1 = 0, \mathcal{E}_1 * M \mathcal{L}_{B_1} * M \mathcal{L}_{S_1} = 0, \mathcal{R}_{A_2} * N \mathcal{E}_1 * M \mathcal{L}_{B_1} = 0 \] (3.10)

are satisfied, respectively. It is evident that the condition (3.8) satisfies if and only if there exist quaternion tensors \( \mathcal{X}_3 \) and \( \mathcal{W} \) satisfy

\[ \mathcal{A}_1 * N \mathcal{X}_3 * M \mathcal{B}_1 + \mathcal{A}_2 * N \mathcal{W} * M \mathcal{B}_2 = \hat{\mathbf{e}}_1. \] (3.11)

(2) \( \iff \) (3) By applying Lemma 2.1, we have that (3.11) is solvable if and only if

\[ \mathcal{R}_{\mathcal{X}_1} * N \mathcal{R}_{\mathcal{A}_1} * N \hat{\mathbf{e}}_1 = 0, \mathcal{E}_1 * M \mathcal{L}_{B_1} * M \mathcal{L}_{\mathcal{X}_1} = 0, \]

\[ \mathcal{R}_{\mathcal{A}_1} * N \hat{\mathbf{e}}_1 * M \mathcal{L}_{B_2} = 0, \mathcal{R}_{\mathcal{A}_2} * N \hat{\mathbf{e}}_1 * M \mathcal{L}_{B_1} = 0. \] (3.12)

In that case, the general solution to (3.11) can be expressed as

\[ X_3 = A_1^1 * N \hat{\mathbf{e}}_1 * M B_1^1 - A_1^1 * N \hat{\mathbf{e}}_1 * M B_1^1 - A_1^1 * N S_1 * N \hat{\mathbf{e}}_1 * M B_1^1 + \mathcal{L}_{A_1} * N \hat{\mathbf{u}}_1 + \hat{\mathbf{u}}_6 * M \mathcal{R}_{B_1}, \]

\[ W = M_1^1 * N \hat{\mathbf{e}}_1 * M B_2^1 + S_1^1 * N S_1 * N \hat{\mathbf{e}}_1 * M N_1^1 * M \mathcal{L}_{S_1} + \mathcal{L}_{M_1} * N \mathcal{L}_{S_1} \]

\[ \mathcal{R}_{M_1} * N \mathcal{R}_{A_1} * N \mathcal{E}_1 = 0, \mathcal{E}_1 * M \mathcal{L}_{B_1} * M \mathcal{L}_{S_1} = 0, \mathcal{R}_{A_2} * N \mathcal{E}_1 * M \mathcal{L}_{B_1} = 0. \] (3.13)

\[ \mathcal{A}_1 * N \mathcal{X}_3 * M + \mathcal{X}_2 * M \mathcal{B}_2 + \mathcal{C}_3 * N \mathcal{X}_3 * M \mathcal{D}_3 + \mathcal{C}_4 * N \mathcal{W} * M \mathcal{D}_4 = \hat{\mathbf{e}}_1. \]

**Corollary 3.2.*** Set \( \mathcal{A}_2 = \mathcal{B}_1 = I \), in (1.2), we can obtain the solvability conditions and the general solution to the following tensor equation:

\[ \mathcal{A}_1 * N \mathcal{X}_3 * M + \mathcal{X}_2 * M \mathcal{B}_2 + \mathcal{C}_3 * N \mathcal{X}_3 * M \mathcal{D}_3 + \mathcal{C}_4 * N \mathcal{W} * M \mathcal{D}_4 = \hat{\mathbf{e}}_1. \]

**Theorem 3.3.*** Consider the quaternion system of tensor equations (1.2), where

\[ \mathcal{A}_1 \in \mathbb{H}^I(N) \times J(N), \mathcal{A}_2 \in \mathbb{H}^I(N) \times Q(N), \mathcal{A}_3 \in \mathbb{H}^I(N) \times P(N), \mathcal{A}_4 \in \mathbb{H}^A(N) \times E(N), \]

\[ \mathcal{A}_5 \in \mathbb{H}^C(N) \times V(N), \mathcal{A}_6 \in \mathbb{H}^A(N) \times E(N), \mathcal{A}_7 \in \mathbb{H}^A(N) \times G(N), \mathcal{A}_8 \in \mathbb{H}^C(N) \times V(N), \]

\[ \mathcal{A}_9 \in \mathbb{H}^C(N) \times G(N), \mathcal{B}_1 \in \mathbb{H}^L(M) \times K(M), \mathcal{B}_2 \in \mathbb{H}^S(M) \times K(M), \mathcal{B}_3 \in \mathbb{H}^T(M) \times K(M), \]

\[ \mathcal{B}_4 \in \mathbb{H}^I(M) \times B(M), \mathcal{B}_5 \in \mathbb{H}^U(M) \times D(M), \mathcal{B}_6 \in \mathbb{H}^F(M) \times B(M), \mathcal{B}_7 \in \mathbb{H}^H(M) \times B(M), \]

\[ \mathcal{B}_8 \in \mathbb{H}^D(M) \times R(M), \mathcal{B}_9 \in \mathbb{H}^U(M) \times R(M), \mathcal{C}_3 \in \mathbb{H}^G(N) \times J(N), \mathcal{C}_4 \in \mathbb{H}^G(N) \times P(N), \]

\[ \mathcal{D}_3 \in \mathbb{H}^L(M) \times F(M), \mathcal{D}_4 \in \mathbb{H}^T(M) \times F(M), \mathcal{H}_3 \in \mathbb{H}^G(N) \times Q(N), \mathcal{H}_4 \in \mathbb{H}^C(N) \times P(N), \]

\[ \mathcal{J}_3 \in \mathbb{H}^E(M) \times D(M), \mathcal{J}_4 \in \mathbb{H}^I(M) \times D(M), \mathcal{E}_1 \in \mathbb{H}^I(N) \times K(M), \mathcal{E}_2 \in \mathbb{H}^I(N) \times K(M), \]

\[ \mathcal{E}_3 \in \mathbb{H}^C(N) \times D(M), \mathcal{E}_4 \in \mathbb{H}^P(N) \times K(M), \mathcal{E}_5 \in \mathbb{H}^A(N) \times F(M), \mathcal{E}_6 \in \mathbb{H}^G(N) \times B(M), \]

\[ \mathcal{E}_7 \in \mathbb{H}^C(N) \times D(M), \mathcal{E}_8 \in \mathbb{H}^C(N) \times D(M), \mathcal{E}_{10} \in \mathbb{H}^C(N) \times R(M), \]
are given tensors over $\mathbb{H}$. Set

\[ A_0 = A_0 \ast_N L_{A_0}, \quad \tilde{B}_7 = R_{B_4} \ast_M B_7, \]

\[ \tilde{\epsilon}_9 = \tilde{e}_9 - A_6 \ast_N A_1 \ast_N \tilde{e}_5 \ast_M B_6 - A_7 \ast_N \tilde{e}_6 \ast_M B_1^\dagger \ast_M B_7, \]

\[ M_{11} = R_{A_6} \ast_N A_7, \quad N_{11} = \tilde{B}_7 \ast_M L_{B_7}, \quad S_{11} = A_7 \ast_N L_{M_{11}}, \]

\[ \tilde{\epsilon}_1 = \tilde{e}_9 - A_7 \ast_N (C_3 \ast_N A_3 \ast_M D_3 + C_4 \ast_N W \ast_M D_4) \ast_M B_6, \]

\[ \tilde{A}_4 = M_{11} \ast_N C_3, \quad \tilde{C}_4 = M_{11} \ast_N C_4, \quad B_4 = D_3 \ast_M B_6 \ast_M L_{B_7}, \]

\[ \tilde{D}_4 = D_4 \ast_M B_6 \ast_M L_{B_7}, \quad \tilde{p} = R_{\tilde{A}_6} \ast_N \tilde{e}_9 \ast_M L_{B_7}, \]

\[ \tilde{A}_8 = A_8 \ast_N L_{A_0}, \quad \tilde{B}_9 = R_{B_5} \ast_M B_9, \]

\[ \tilde{\epsilon}_{10} = \tilde{e}_{10} - A_8 \ast_N A_1 \ast_N \tilde{e}_7 \ast_M B_8 - A_9 \ast_N \tilde{e}_8 \ast_M B_1^\dagger \ast_M B_9, \]

\[ M_{22} = R_{\tilde{A}_8} \ast_N A_9, \quad N_{22} = \tilde{B}_9 \ast_M L_{B_9}, \quad S_{22} = A_9 \ast_N L_{M_{22}}, \]

\[ \tilde{\epsilon}_2 = \tilde{e}_{10} - A_9 \ast_N (H_3 \ast_N A_3 \ast_M J_3 + H_4 \ast_N W \ast_M J_4) \ast_M B_8, \]

\[ \tilde{A}_9 = M_{22} \ast_N H_3, \quad \tilde{C}_5 = M_{22} \ast_N H_4, \quad \tilde{B}_9 = J_3 \ast_M B_8 \ast_M L_{B_9}, \]

\[ \tilde{D}_5 = J_4 \ast_M B_8 \ast_M L_{B_9}, \quad \tilde{Q} = R_{\tilde{A}_8} \ast_N \tilde{e}_{10} \ast_M L_{B_9}, \]

\[ C_6 = \tilde{C}_4 \ast_N L_{A_0}, \quad D_6 = R_{B_5} \ast_M \tilde{D}_4, \quad C_7 = \tilde{C}_5 \ast_N L_{A_3}, \quad D_7 = R_{B_4} \ast_M \tilde{D}_5, \]

\[ G = \tilde{p} - \tilde{C}_4 \ast_N A_1 \ast_N \tilde{e}_4 \ast_M \tilde{D}_4 - \tilde{C}_4 \ast_N L_{A_3} \ast_N \tilde{e}_4 \ast_M B_1^\dagger \ast_M \tilde{D}_4, \]

\[ F = \tilde{Q} - \tilde{C}_5 \ast_N A_1 \ast_N \tilde{e}_4 \ast_M \tilde{D}_5 - \tilde{C}_5 \ast_N L_{A_3} \ast_N \tilde{e}_4 \ast_M B_1^\dagger \ast_M \tilde{D}_5, \]

\[ M_1 = R_{\tilde{A}_8} \ast_M C_6, \quad N_1 = D_6 \ast_M L_{B_9}, \quad S_1 = C_6 \ast_N L_{M_{11}}, \]

\[ M_2 = R_{\tilde{A}_8} \ast_M C_7, \quad N_2 = D_7 \ast_M L_{B_9}, \quad S_2 = C_7 \ast_N L_{M_{12}}, \]

\[ A_{11} = (L_{M_{11}} \ast_N L_{S_1}) \ast_M (L_{M_{12}} \ast_N L_{S_2}), \quad B_{11} = \left( \begin{array}{c} R_{D_6} \\ R_{D_7} \end{array} \right), \]

\[ \epsilon_{11} = M_1 \ast_N F \ast_M D_7^\dagger + S_2 \ast_N S_2 \ast_N C_7 \ast_N F \ast_M N_2^\dagger - M_1 \ast_N G \ast_M D_6^\dagger - S_1 \ast_N S_1 \ast_N C_6 \ast_N G \ast_M N_1^\dagger, \quad A = R_{A_{11}} \ast_N L_{M_{11}}, \]

\[ B = R_{N_1} \ast_M L_{B_1}, \quad C = R_{A_{11}} \ast_N L_{M_{12}}, \quad D = R_{N_5} \ast_M L_{B_1}, \]

\[ \epsilon = R_{A_{11}} \ast_N \tilde{e}_{11} \ast_M L_{B_1}, \quad M = R_{A_4} \ast_N C, \quad N = D \ast_M L_{B_1}, \quad S = C \ast_N L_{M_{1}}, \]

\[ A_{22} = (L_{A_1} \ast M_{A_1}), \quad B_{22} = \left( \begin{array}{c} R_{B_1} \\ R_{B_5} \end{array} \right), \quad C_{22} = A_1^\dagger \ast_N S_1, \]

\[ D_{22} = R_{N_1} \ast_M D_6 \ast_N B_1^\dagger, \quad E_{22} = A_4^\dagger \ast N G \ast M B_1^\dagger - A_4 \ast N \epsilon_1 \ast M B_1^\dagger - A_4^\ast \ast N S_1 \ast N C_6 \ast N G \ast M N_1^\dagger, \]

\[ A_{33} = R_{A_{22}} \ast_N C_{22}, \quad B_{33} = D_{22} \ast_M L_{B_2}, \quad E_{33} = R_{A_{22}} \ast_N \tilde{e}_{22} \ast_M L_{B_2}, \]

\[ A_{44} = (L_{A_2} \ast M_{A_2}), \quad B_{44} = \left( \begin{array}{c} R_{B_2} \\ R_{B_5} \end{array} \right), \quad C_{44} = A_2^\dagger \ast_N S_2, \]

\[ D_{44} = R_{N_5} \ast_M D_7 \ast_N B_1^\dagger, \quad E_{44} = A_5^\dagger \ast N F \ast M B_1^\dagger - A_5 \ast N \epsilon_2 \ast M B_1^\dagger - A_5^\ast \ast N S_2 \ast N C_7 \ast N G \ast M N_2^\dagger, \]

\[ A_{55} = R_{A_{44}} \ast_N C_{44}, \quad B_{55} = D_{44} \ast_M L_{B_4}, \quad E_{55} = R_{A_{44}} \ast_N \tilde{e}_{44} \ast_M L_{B_4}, \]

\[ A_{66} = (L_A \ast M_{A_3}), \quad B_{66} = \left( \begin{array}{c} R_B \\ R_{B_5} \end{array} \right), \quad C_{66} = A_3^\dagger \ast_N S, \]
\[ D_{66} = R_{A} * M D * M B^1, \quad \varepsilon_{66} = A_{i}^{1} * N \varepsilon_{33} * M B_{33}^{1} - A_{i}^{1} * N \varepsilon_{4} * M B_{4}^{1} + A_{i}^{1} * N S_{4} * N \varepsilon_{33} * M N_{33}^{1} * M D * N B_{1}^{1} + A_{i}^{1} * N \varepsilon_{4} * M \varepsilon_{1} * M N_{1}^{1} * N \varepsilon_{33} * M B_{33}^{1}, \]
\[ A_{77} = R_{A_{66}} * N \varepsilon_{66} * M B_{66}^{*}, \quad \varepsilon_{77} = R_{A_{66}} * N \varepsilon_{66} * M B_{66}^{*}, \]  
\[ A_{88} = \left( L_{M} * N L_{S} L_{A_{55}} \right), \quad B_{88} = \left( R_{D} \right), \quad \varepsilon_{88} = L_{M}, \quad D_{88} = R_{N}, \]
\[ \varepsilon_{88} = A_{i}^{4} * N \varepsilon_{55} * M B_{55}^{1} - M_{i}^{1} * N \varepsilon_{4} * M B_{1}^{1} - S_{i}^{1} * N S_{4} * N \varepsilon_{33} * M N_{33}^{1}, \]
\[ A_{99} = R_{A_{88}} * N \varepsilon_{88} * M B_{88}^{*}, \quad B_{99} = D_{88} * M B_{88}^{*}, \quad \varepsilon_{99} = R_{A_{88}} * N \varepsilon_{88} * M B_{88}^{*}, \]
\[ \bar{A} = \left( L_{A_{77}} - L_{A_{99}} \right), \quad \bar{B} = \left( R_{B_{77}} \right), \]
\[ \bar{E} = A_{i}^{1} * N \varepsilon_{77} * M B_{77}^{1} - A_{j}^{4} * N \varepsilon_{99} * M B_{99}^{1}. \]

then the system \[ \text{(3.14)} \] is solvable if and only if
\[ R_{A_{4}} * N \varepsilon_{5} = 0, \quad E_{6} * M L_{B_{4}} = 0, \quad R_{A_{5}} * N \varepsilon_{7} = 0, \quad E_{8} * M L_{B_{5}} = 0, \]
\[ R_{A_{11}} * N R_{\tilde{A}_{6}} * N \tilde{E}_{6} = 0, \quad \tilde{E}_{6} * M L_{B_{4}} * M L_{N_{11}} = 0, \quad R_{A_{4}} * N \tilde{E}_{6}, * M L_{B_{6}} = 0, \]
\[ = 0, \quad R_{A_{42}} * N R_{\tilde{A}_{6}} * N \tilde{E}_{10} = 0, \quad \tilde{E}_{10} * M L_{B_{4}} * M L_{N_{22}} = 0, \quad R_{A_{5}} * N \tilde{E}_{10} = 0, \]
\[ * M L_{B_{5}} = 0, \quad R_{A_{4}} * N \varepsilon_{3} = 0, \quad E_{4} * M L_{B_{5}} = 0, \quad A_{3} * N \varepsilon_{3} = 0, \quad E_{4} * M B_{3}, \]
\[ R_{A_{4}} * N \tilde{E}_{6} * M L_{D_{4}} = 0, \quad R_{C_{6}} * N \tilde{E}_{6} * M L_{B_{4}} = 0, \quad R_{S_{1}} * N R_{\tilde{A}_{6}} * N \tilde{E}_{6} = 0, \]
\[ G * M L_{B_{4}} * M L_{N_{11}} = 0, \quad R_{A_{4}} * N F * M L_{D_{7}} = 0, \quad R_{C_{6}} * N F * M L_{B_{5}} = 0, \]
\[ R_{S_{2}} * N R_{\tilde{A}_{6}} * N F = 0, \quad F * M L_{B_{5}} * M L_{N_{22}} = 0, \quad R_{M} * N R_{A_{4}} * N \tilde{E}_{6} = 0, \]
\[ R_{A_{4}} * N \varepsilon_{1} * M L_{B_{5}} = 0, \quad R_{A_{2}} * N \varepsilon_{2} * M L_{B_{5}} = 0, \quad E_{2} * M B_{2} = 0, \]
\[ R_{A_{33}} * N \varepsilon_{33} = 0, \quad \varepsilon_{33} * M L_{B_{33}} = 0, \quad R_{A_{65}} * N \varepsilon_{55} = 0, \quad \varepsilon_{55} * M L_{B_{55}} = 0, \]
\[ R_{A_{77}} * N \varepsilon_{77} = 0, \quad \varepsilon_{77} * M L_{B_{77}} = 0, \quad R_{A_{99}} * N \varepsilon_{99} = 0, \quad \varepsilon_{99} * M L_{B_{99}} = 0, \]
\[ R_{A_{4}} * N \tilde{E}_{6} * M L_{B_{5}} = 0. \]

Under these conditions, the general solution to \[ \text{(3.14)} \] can be expressed as:

\[ x_{1} = A_{i}^{1} * N \varepsilon_{5} + L_{A_{4}} * N \varphi_{11}, \]
\[ x_{2} = \varepsilon_{6} * M B_{1}^{1} + V_{22} * M \tilde{R}_{B_{4}}, \]
\[ y_{1} = A_{i}^{1} * N \varepsilon_{7} + L_{A_{5}} * N \varphi_{33}, \]
\[ y_{2} = \varepsilon_{8} * M B_{1}^{1} + V_{44} * M \tilde{R}_{B_{5}}, \]
\[ x_{3} = A_{i}^{1} * N \varepsilon_{1} * M B_{1}^{1} + L_{A_{4}} * N U_{1} + U_{2} * M \tilde{R}_{B_{1}}, \]
\[ y_{3} = A_{2}^{1} * N \varepsilon_{2} * M B_{1}^{1} + L_{A_{5}} * N U_{6} + U_{4} * M \tilde{R}_{B_{2}}, \]
\[ w = A_{i}^{1} * N \varepsilon_{3} + L_{A_{4}} * N \varphi_{4} * M B_{1}^{1} + L_{A_{5}} * N U_{6} * M \tilde{R}_{B_{1}}, \]

where

\[ \varphi_{11} = A_{i}^{1} * N \varepsilon_{1} * M B_{1}^{1} - A_{i}^{1} * N A_{7} * M \varepsilon_{11} * N \varepsilon_{1} * M B_{1}^{1} - A_{i}^{1} * N S_{11} * N \varepsilon_{1} * M B_{1}^{1} - A_{i}^{1} * N \varepsilon_{1} * M B_{1}^{1} * M T_{21} * M \tilde{R}_{N_{11}} * M \tilde{B}_{7} * M B_{1}^{1} + L_{A_{4}} * N \varphi_{11} + \varphi_{51} * M \tilde{R}_{B_{6}}, \]
\( V_{22} = M_{11}^{\dagger} * N \hat{\epsilon}_1 * M \vec{B}_1^\dagger + S_{11}^{\dagger} * N S_{11} * N A_1^{\dagger} * N \hat{\epsilon}_1 * M \vec{N}_{11}^\dagger + L_{M_{11}} * N \)
\( L_{S_{11}} * N T_{11} + L_{M_{11}} * N T_{21} * M R_{N_{11}} + T_{31} * M R_{B_{21}}, \) \( \tag{3.35b} \)
\( V_{33} = \vec{A}_3^{\dagger} * N \hat{\epsilon}_2 * M B_1 - \vec{A}_3 * N A_0 * N M_{12}^{\dagger} * N \hat{\epsilon}_2 * M B_1 - \vec{A}_3^{\dagger} * N S_{22} \)
\( * N A_0^{\dagger} * N \hat{\epsilon}_2 * M \vec{N}_{22}^{\dagger} * M \vec{B}_0 * M B_0 - \vec{A}_3 \) \( * N S_{22} * N J_{31} * M R_{N_{22}} * M \)
\( B_0 * M B_0 + L_{A_3} * N J_{41} + J_{51} * M R_{B_2}, \) \( \tag{3.35c} \)
\( V_{44} = M_{12}^{\dagger} * N \hat{\epsilon}_2 * M \vec{B}_0^\dagger + S_{22} * N S_{22} * N A_0 * N \hat{\epsilon}_2 * M \vec{N}_{22} + L_{M_{22}} * N \)
\( L_{S_{22}} * N J_{41} + L_{M_{32}} * N J_{31} * M R_{N_{22}} + J_{31} * M R_{B_2}, \) \( \tag{3.35d} \)
\( U_1 = (I \ 0) * N (A_{12}^{\dagger} * N (\epsilon_{22} - C_{22} * N V_2 * M D_{22}) - A_{12}^{\dagger} * N H_{12} * M B_{22} \)
\( + L_{A_{22}} * N H_{11}), \) \( \tag{3.35e} \)
\( U_2 = (R_{A_{22}} * N (\epsilon_{22} - C_{22} * N V_2 * M D_{22}) * M B_{22}^{\dagger} + A_{22} * N A_{12}^{\dagger} * N H_{12} \)
\( H_{12} * M R_{B_{22}}) * M (I \ 0), \) \( \tag{3.35f} \)
\( U_3 = (I \ 0) * N (A_{44}^{\dagger} * N (\epsilon_{44} - C_{44} * N T_2 * M D_{44}) - A_{44}^{\dagger} * N H_{22} * M B_{44} \)
\( + L_{A_{44}} * N H_{21}), \) \( \tag{3.35g} \)
\( U_4 = (R_{A_{44}} * N (\epsilon_{44} - C_{44} * N T_2 * M D_{44}) * M B_{44}^{\dagger} + A_{44} * N A_{44}^{\dagger} * N H_{22} \)
\( H_{23} * M R_{B_{44}}) * M (I \ 0), \) \( \tag{3.35h} \)
\( U_5 = M_{11}^{\dagger} * N G * M D_0^\dagger + S_1^{\dagger} * N S_1 * N C_0^{\dagger} * N G * M N_{11}^{\dagger} - L_{M_{11}} * N L_S \)
\( * N V_1 + L_{A_{11}} * N V_2 * M R_{N_1} + V_3 * M R_{D_0}, \) \( \tag{3.35i} \)
\( V_1 = (I \ 0) * N (A_{11}^{\dagger} * N (\epsilon_{11} - L_{M_{11}} * N V_2 * M R_{N_1} - L_{M_{12}} * N T_2 * M R_{N_2}) \)
\( + W_1 * M B_{11} + L_{A_{11}} * N W_2), \) \( \tag{3.35j} \)
\( V_2 = A_1^{\dagger} * N \hat{\epsilon}_1 * M B_1^\dagger - A_1 * N S * N C_1^\dagger * N \hat{\epsilon} * M N_{11}^\dagger * M D * M B_1^\dagger - A_1^{\dagger} * N \)
\( \hat{\epsilon} * M M_1^{\dagger} * N \hat{\epsilon} * M B_1^\dagger + A_1^{\dagger} * N S * N W_4 * M R_{N_1} * M D * M B_1^\dagger \)
\( + L_{A_1} * N W_5 + W_6 * M R_{B_1}, \) \( \tag{3.35k} \)
\( V_3 = (R_{A_{11}} * N (\epsilon_{11} - L_{M_{11}} * N V_2 * M R_{N_1} - L_{M_{12}} * N T_2 * M R_{N_2}) * M B_{11}^\dagger \)
\( - A_{11} * N W_1 - W_3 * M R_{B_{11}}) * M (I \ 0), \) \( \tag{3.35l} \)
\( T_2 = M_1^{\dagger} * N \hat{\epsilon} * M D_1^\dagger + S_1^{\dagger} * N S * N C_1^\dagger * N \hat{\epsilon} * M N_{11}^\dagger + L_{M_1} * N L_S \)
\( * N W_7 + L_{M_1} * N W_4 * M R_N + W_8 * M R_D, \) \( \tag{3.35m} \)
\( W_4 = A_1^{\dagger} * N \epsilon_{77} * M B_{77}^\dagger - L_{A_{77}} * N Q_1 - Q_2 * M R_{B_{77}}, \) \( \tag{3.35n} \)
\( W_5 = (I \ 0) * N (A_{66}^{\dagger} * N (\epsilon_{66} - C_{66} * N W_4 * M D_{66}) - A_{66}^{\dagger} * N H_{32} * M B_{66} \)
\( + L_{A_{66}} * N H_{31}), \) \( \tag{3.35o} \)
\( W_6 = (R_{A_{66}} * N (\epsilon_{66} - C_{66} * N W_4 * M D_{66}) * M B_{66}^\dagger + A_{66} * N A_{66}^{\dagger} * N H_{32} \)
\( H_{33} * M R_{B_{66}}) * M (I \ 0), \) \( \tag{3.35p} \)
\( W_7 = (I \ 0) * N (A_{88} * N (\epsilon_{88} - C_{88} * N W_4 * M D_{88}) - A_{88} * N H_{42} * M B_{88} \)
\( + L_{A_{88}} * N H_{41}), \) \( \tag{3.35q} \)
\( W_8 = (R_{A_{88}} * N (\epsilon_{88} - C_{88} * N W_4 * M D_{88}) * M B_{88}^\dagger + A_{88} * N A_{88}^{\dagger} * N H_{42} \)
\( H_{43} * M R_{B_{88}}) * M (I \ 0), \) \( \tag{3.35r} \)
If the conditions (3.16) are satisfied, respectively, in that case, the general solution expresses in quaternion tensor equations (3.36), (3.37), (3.38) and (3.39) are solvable respectively if and only if

\[ X_t = \text{tensors} (3.48) \text{ is solvable if and only if the conditions (3.17) are satisfied and there exist quaternion]}

\[ W, \ k, \ i \ (i, k = 1, 3, \ j = 1, 4), \ T_{11} \text{ and } J_{11} (l = 1, 5) \text{ are arbitrary tensors over } \mathbb{H}. \]

**Proof.** We, first, separate the system of tensor equations (1.4) into eight blocks:

\[ A_4 * N X_1 = \varepsilon_5, \quad A_2 * N B_4 = \varepsilon_6, \quad A_5 * N \varepsilon_1 = \varepsilon_7, \quad \varepsilon_2 * M B_6 = \varepsilon_8, \]

\[ A_1 * N X_3 * M B_1 = \varepsilon_1, \quad A_2 * N \varepsilon_3 * M B_2 = \varepsilon_2, \]

\[ A_3 * N W = \varepsilon_3, \quad W * M B_3 = \varepsilon_4, \]

\[ A_6 * N X_1 * M B_6 + A_7 * N X_2 * M B_7 + A_7 * N (C_3 * N X_3 * M D_3 + C_4 * N W * M D_4) * M B_6 = \varepsilon_9, \]

and

\[ A_8 * N \varepsilon_3 * M B_8 + A_9 * N \varepsilon_2 * M B_9 + A_9 * N (H_3 * N \varepsilon_3 * M J_3 + H_4 * N W * M J_4) * M B_8 = \varepsilon_{10}, \]

Our goal is to investigate the solvability conditions that make these eight groups have a solution, and hence we investigate an expression of this solution. Applying Lemma 2.4, we have that the quaternion tensor equations (3.36), (3.37), (3.38) and (3.39) are solvable respectively if and only if the conditions (3.16) are satisfied, respectively. In that case, the general solution expresses in the form

\[ X_1 = A_4^t * N \varepsilon_5 + \mathcal{L}_{A_4} * N \varepsilon_1, \]

\[ X_2 = \varepsilon_6 * M B_4^t + \mathcal{V}_{22} * M \mathcal{R}_{B_4}, \]

\[ \varepsilon_1 = A_3^t * N \varepsilon_7 + \mathcal{L}_{A_5} * N \varepsilon_3, \]

\[ \varepsilon_2 = \varepsilon_8 * M B_5^t + \mathcal{V}_{44} * M \mathcal{R}_{B_5}, \]

where \( \mathcal{V}_{ii} (i = 1, 4) \) are arbitrary tensors with qualified orders. Substitute expressions (3.44) and (3.45) into (3.42) yields:

\[ \hat{A}_6 * N \varepsilon_{11} * M B_6 + A_7 * N \varepsilon_2 * M \hat{B}_7 + A_7 * N (C_3 * N X_3 * M D_3 + C_4 * N W * M D_4) * M B_6 = \hat{\varepsilon}_9, \]

where \( \hat{A}_6, \hat{B}_7 \) and \( \hat{\varepsilon}_9 \) are given by (3.14a) and (3.14b). Utilizing Proposition 3.1, we have that (3.48) is solvable if and only if the conditions (3.17) are satisfied and there exist quaternion tensors \( X_3 \) and \( W \) satisfy the following equation:

\[ \hat{A}_4 * N X_3 * M \hat{B}_4 + \hat{C}_4 * N W * M \hat{D}_4 = \hat{P}, \]
where $\hat{A}_i, \hat{B}_i, \hat{C}_i, \hat{D}_i$, and $\hat{P}$ are defined in (3.14c)-(3.14f). In that case, the quaternion tensors $\mathcal{V}_{11}$ and $\mathcal{V}_{22}$ can be expressed as

$$
\mathcal{V}_{11} = \hat{A}_6^* \ast_N \hat{E}_1 \ast_M \hat{B}_6^* - \hat{A}_6^* \ast_N \hat{A}_7 \ast_N \mathcal{M}_{11}^\dagger \ast_N \hat{E}_1 \ast_M \hat{B}_6^* - \hat{A}_6^* \ast_N S_{11} \ast_N \hat{A}_7 \ast_N \mathcal{M}_{11}^\dagger \ast_N \hat{B}_7
$$

$$
\mathcal{V}_{22} = \mathcal{M}_{11} \ast_N \hat{E}_1 \ast_M \hat{B}_7^* + S_{11} \ast_N \hat{E}_1 \ast_M \hat{A}_7 \ast_N \mathcal{M}_{11}^\dagger \ast_N \hat{B}_7^* + \mathcal{L}_{11} \ast_N \mathcal{T}_{11} \ast_M \mathcal{R}_{N_{11}} \ast_M \hat{B}_7
$$

with $\mathcal{T}_{11} (l = 1, 5)$ are arbitrary tensors with appropriate sizes. Now, we summarize all previous processes in our proof. The system of Sylvester-type quaternion tensor equations (1.4) is solvable if and only if the conditions (3.16), (3.17) and (3.18) are satisfied and there exist quaternion tensors $\hat{X}_3$ and $\hat{W}$ verify the following system:

\[
\begin{cases}
A_1 \ast_N \hat{X}_3 \ast_M \mathcal{B}_1 = \hat{E}_1, \\
A_2 \ast_N \hat{Y}_3 \ast_M \mathcal{B}_2 = \hat{E}_2, \\
A_3 \ast_N \hat{W} = \hat{E}_3, \\
A_4 \ast_N \hat{X}_4 \ast_M \mathcal{B}_4 + \hat{C}_4 \ast_N \hat{W} \ast_M \hat{D}_4 = \hat{P}, \\
A_5 \ast_N \hat{Y}_3 \ast_M \hat{B}_5 + \hat{C}_5 \ast_N \hat{W} \ast_M \hat{D}_5 = \hat{Q}.
\end{cases}
\]

Finally, utilizing Lemma 2.5 we have that (3.51) is solvable if and only if the conditions defined by (3.19)-(3.27) are satisfying. In that case, $\hat{X}_3$, $\hat{Y}_3$ and $\hat{W}$ can expressed as

\[
\begin{align*}
\hat{X}_3 &= \hat{A}_4^\dagger \ast_N \hat{E}_1 \ast_M \hat{B}_4^* + \mathcal{L}_4 \ast_N \hat{U}_1 + \hat{U}_2 \ast_M \mathcal{R}_B, \\
\hat{Y}_3 &= \hat{A}_4^\dagger \ast_N \hat{E}_2 \ast_M \hat{B}_4^* + \mathcal{L}_4 \ast_N \hat{U}_A \ast_M \mathcal{R}_B, \\
\hat{W} &= \mathcal{A}_4 \ast_N \hat{E}_3 + \mathcal{L}_4 \ast_N \hat{E}_4 \ast_M \hat{B}_4^* + \mathcal{L}_4 \ast_N \hat{U}_4 \ast_M \mathcal{R}_B,
\end{align*}
\]

where $\hat{U}_i (i = 1, 5)$ are arbitrary tensors defined by (3.30a)-(3.30i). All computations can be run on MATLAB 2020b.
Algorithm 3.4. Calculate the general solution to (1.4).

1. Input the system of two-sided four coupled Sylvester-like quaternion tensor equations (1.4) with viable orders over \( \mathbb{H} \).
2. Compute all quaternion tensors, which appeared in (3.3a)–(3.3c).
3. Check whether the Moore-Penrose inverses conditions in Theorem 3.3 are satisfying or not. If not, return “The system (1.4) is inconsistent”.
4. Else compute the quaternion unknowns \( X_i, Y_i, W \), where \((i = 1, 3)\) by (3.28), (3.35).
5. Output the general solution of the system (1.4) is \( X_i, Y_i, W \).

Example 3.5. Assume that the fourth ordered tensors in (1.4) are given as

\[
A_1(;, 1, 1) = (k_{i+k})_1, A_1(:, 2, 1) = (5i_{100})_1, A_1(:, 1, 2) = (2-i_{02k})_1, \\
A_1(:, 2, 2) = (0_{j+0})_1, A_1(:, 1, 1) = (k_{2-k})_1, B_1(:, 1, 1) = (4j_{00})_1, \\
B_1(:, 2, 1) = (5_{j+0})_1, B_1(:, 1, 2) = (7_{j+0})_1, A_2(:, 1, 1) = (6_{k-0})_1, \\
A_2(:, 1, 1) = (8_{j+0})_1, A_2(:, 2, 1) = (9_{j+0})_1, B_2(:, 1, 1) = (10_{j+0})_1, \\
B_2(:, 1, 2) = (11_{j+0})_1, B_2(:, 2, 1) = (12_{j+0})_1, B_2(:, 2, 2) = (13_{j+0})_1, \\
B_3(:, 2, 1) = (14_{j+0})_1, B_3(:, 2, 2) = (15_{j+0})_1, A_3(:, 1, 1) = (16_{j+0})_1, \\
A_3(:, 1, 1) = (17_{j+0})_1, A_3(:, 2, 1) = (18_{j+0})_1, A_3(:, 2, 2) = (19_{j+0})_1, \\
B_3(:, 1, 1) = (20_{j+0})_1, B_3(:, 1, 2) = (21_{j+0})_1, B_3(:, 2, 1) = (22_{j+0})_1, \\
B_3(:, 2, 2) = (23_{j+0})_1, E_1(:, 1, 1) = (24_{j+0})_1, E_1(:, 2, 1) = (25_{j+0})_1, \\
E_1(:, 2, 2) = (26_{j+0})_1, E_2(:, 1, 1) = (27_{j+0})_1, E_2(:, 2, 1) = (28_{j+0})_1, \\
E_2(:, 2, 2) = (29_{j+0})_1, E_3(:, 1, 1) = (30_{j+0})_1, E_3(:, 1, 2) = (31_{j+0})_1, \\
E_3(:, 2, 1) = (32_{j+0})_1, E_3(:, 2, 2) = (33_{j+0})_1, E_4(:, 1, 1) = (34_{j+0})_1, \\
E_4(:, 1, 2) = (35_{j+0})_1, E_4(:, 2, 1) = (36_{j+0})_1, E_4(:, 2, 2) = (37_{j+0})_1, \\
E_5(:, 1, 1) = (38_{j+0})_1, E_5(:, 1, 2) = (39_{j+0})_1, E_5(:, 2, 1) = (40_{j+0})_1, \\
E_5(:, 2, 2) = (41_{j+0})_1.
\]
$$\mathcal{E}_6(\cdot; 1, 1) = \binom{j-2k}{0} i^{i+j-k} k^{i+2k}, \quad \mathcal{E}_6(\cdot; 2, 1) = \binom{-1-j-k}{0} 2i$$

$$\mathcal{E}_6(\cdot; 1, 2) = \binom{2i-j+k}{0} i^{i-j-k} k^{i+2k}, \quad \mathcal{E}_6(\cdot; 2, 2) = \binom{-2-j-i}{i} 1$$

$$\mathcal{E}_7(\cdot; 1, 1) = \binom{0}{0} 2j, \quad \mathcal{E}_7(\cdot; 1, 2) = \binom{2i-3+j}{0} 0, \quad \mathcal{E}_7(\cdot; 2, 1) = \binom{3i+3j-3k}{0} 3i$$

$$\mathcal{E}_7(\cdot; 2, 2) = \binom{4i-5}{0} 4, \quad \mathcal{E}_8(\cdot; 1, 1) = \binom{4j-2}{2k} 2$$

$$\mathcal{E}_8(\cdot; 1, 2) = \binom{-1+3i-j-3k}{0} i^{i-k} k^{-i-j} 1$$

$$\mathcal{E}_8(\cdot; 2, 2) = \binom{9-3k}{3i} 2i^{3j+k} 3i, \quad \mathcal{A}_0(\cdot; 1, 1) = \binom{0}{k} 2i, \quad \mathcal{A}_0(\cdot; 1, 2) = \binom{-1}{i-1} 0$$

$$\mathcal{A}_0(\cdot; 2, 1) = \binom{0}{1} 1, \quad \mathcal{A}_0(\cdot; 2, 2) = \binom{j+k}{0} 1, \quad \mathcal{B}_0(\cdot; 1, 2) = \binom{1}{2i-j} 0$$

$$\mathcal{B}_0(\cdot; 1, 1) = \binom{0}{0} 0, \quad \mathcal{B}_0(\cdot; 2, 1) = \binom{j+k}{0} 0, \quad \mathcal{B}_0(\cdot; 2, 2) = \binom{0}{0} 0$$

$$\mathcal{A}_7(\cdot; 1, 1) = \binom{0}{0} 0, \quad \mathcal{A}_7(\cdot; 1, 2) = \binom{j+k}{0} 0, \quad \mathcal{A}_7(\cdot; 2, 1) = \binom{j+i-k}{0} 0$$

$$\mathcal{A}_7(\cdot; 2, 2) = \binom{9-3k}{0} 3i$$

$$\mathcal{A}_9(\cdot; 1, 1) = \binom{2i}{0} 0, \quad \mathcal{A}_9(\cdot; 1, 2) = \binom{1-i-k}{0} 0, \quad \mathcal{A}_9(\cdot; 2, 1) = \binom{j+k}{0} 0$$

$$\mathcal{A}_9(\cdot; 2, 2) = \binom{1}{2i-j} 0$$

$$\mathcal{B}_9(\cdot; 1, 1) = \binom{0}{0} 0, \quad \mathcal{B}_9(\cdot; 1, 2) = \binom{1+j}{0} 0, \quad \mathcal{B}_9(\cdot; 2, 1) = \binom{j+i-k}{0} 0$$

$$\mathcal{B}_9(\cdot; 2, 2) = \binom{0}{0} 0$$

$$\mathcal{H}_3(\cdot; 1, 1) = \binom{2}{0} 0, \quad \mathcal{H}_3(\cdot; 1, 2) = \binom{0}{0} 0, \quad \mathcal{H}_3(\cdot; 2, 1) = \binom{3}{0} 0, \quad \mathcal{H}_3(\cdot; 2, 2) = \binom{0}{0} 0$$

$$\mathcal{J}_3(\cdot; 1, 1) = \binom{1}{0} 0, \quad \mathcal{J}_3(\cdot; 1, 2) = \binom{0}{0} 0, \quad \mathcal{J}_3(\cdot; 2, 1) = \binom{0}{0} 0, \quad \mathcal{J}_3(\cdot; 2, 2) = \binom{0}{0} 0$$

$$\mathcal{J}_4(\cdot; 1, 1) = \binom{1}{0} 0, \quad \mathcal{J}_4(\cdot; 1, 2) = \binom{0}{0} 0, \quad \mathcal{J}_4(\cdot; 2, 1) = \binom{0}{0} 0, \quad \mathcal{J}_4(\cdot; 2, 2) = \binom{0}{0} 0$$

$$\mathcal{J}_6(\cdot; 1, 1) = \binom{2}{0} 0, \quad \mathcal{J}_6(\cdot; 1, 2) = \binom{0}{0} 0, \quad \mathcal{J}_6(\cdot; 2, 1) = \binom{0}{0} 0, \quad \mathcal{J}_6(\cdot; 2, 2) = \binom{0}{0} 0$$

$$\mathcal{E}_9(\cdot; 1, 1) = \binom{2}{0} 0, \quad \mathcal{E}_9(\cdot; 1, 2) = \binom{0}{0} 0, \quad \mathcal{E}_9(\cdot; 2, 1) = \binom{0}{0} 0, \quad \mathcal{E}_9(\cdot; 2, 2) = \binom{0}{0} 0$$

$$\mathcal{E}_{10}(\cdot; 1, 1) = \binom{5}{0} 0, \quad \mathcal{E}_{10}(\cdot; 1, 2) = \binom{0}{0} 0, \quad \mathcal{E}_{10}(\cdot; 2, 1) = \binom{0}{0} 0, \quad \mathcal{E}_{10}(\cdot; 2, 2) = \binom{0}{0} 0$$
\[ \mathcal{E}_9(:,; 1, 2) = \begin{pmatrix} -6 - 8i - 3j & 15 + 10i - 7j + 7k \\ 8 + 8i - 2j + 14k & -7 - 10i + 2j + 17k \end{pmatrix}, \mathcal{E}_9(:,; 2, 2) = \begin{pmatrix} 4 + 4i + 6j + 2k - 9 + 3i + 3j + 2k \\ -6 + 2i - 3k + 3 + 5i + 15j - k \end{pmatrix}. \]

Direct computations yields
\[
\begin{align*}
\mathcal{R}_{A_4} & \ast \mathcal{E}_5 = 0, \quad \mathcal{E}_6 \ast _2 \mathcal{L}_{B_1} = 0, \quad \mathcal{R}_{A_5} \ast \mathcal{E}_7 = 0, \quad \mathcal{E}_8 \ast _2 \mathcal{L}_{B_2} = 0, \\
\mathcal{R}_{M_{11}} & \ast _2 \mathcal{R}_{\tilde{A}_6} \ast _2 \mathcal{E}_9 = 0, \quad \mathcal{E}_9 \ast _2 \mathcal{L}_{B_8} \ast _2 \mathcal{L}_{N_{11}} = 0, \quad \mathcal{R}_{A_7} \ast _2 \mathcal{E}_9 \ast _2 \mathcal{L}_{B_9} = 0, \\
\mathcal{R}_{M_{22}} & \ast _2 \mathcal{R}_{\tilde{A}_8} \ast _2 \mathcal{E}_{10} = 0, \quad \mathcal{E}_{10} \ast _2 \mathcal{L}_{B_8} \ast _2 \mathcal{L}_{N_{22}} = 0, \quad \mathcal{R}_{A_6} \ast _2 \mathcal{E}_{10} \ast _2 \mathcal{L}_{B_8} = 0, \\
\mathcal{R}_{A_3} & \ast _2 \mathcal{E}_3 = 0, \quad \mathcal{E}_4 \ast _2 \mathcal{L}_{B_1} = 0, \quad \mathcal{R}_{A_3} \ast _2 \mathcal{E}_3 = \mathcal{E}_4 \ast _2 \mathcal{B}_3, \\
\mathcal{R}_{\tilde{A}_3} & \ast _2 \mathcal{G} \ast _2 \mathcal{L}_{D_6} = 0, \quad \mathcal{R}_{C_6} \ast _2 \mathcal{G} \ast _2 \mathcal{L}_{B_4} = 0, \quad \mathcal{R}_{S_1} \ast _2 \mathcal{R}_{\tilde{A}_4} \ast _2 \mathcal{N} \mathcal{G} = 0, \\
\mathcal{G} \ast _2 \mathcal{L}_{B_4} & \ast _2 \mathcal{L}_{N_4} = 0, \quad \mathcal{R}_{\tilde{A}_5} \ast _2 \mathcal{F} \ast _2 \mathcal{L}_{D_7} = 0, \quad \mathcal{R}_{C_7} \ast _2 \mathcal{F} \ast _2 \mathcal{L}_{B_5} = 0, \\
\mathcal{R}_{S_2} & \ast _2 \mathcal{R}_{\tilde{A}_5} \ast _2 \mathcal{F} = 0, \quad \mathcal{F} \ast _2 \mathcal{L}_{B_5} \ast _2 \mathcal{L}_{N_2} = 0, \quad \mathcal{R}_{M_4} \ast _2 \mathcal{R}_{\tilde{A}_2} \ast _2 \mathcal{E} = 0, \\
\mathcal{R}_{A_2} & \ast _2 \mathcal{E} \ast _2 \mathcal{D} = 0, \quad \mathcal{E} \ast _2 \mathcal{L}_{B_2} \ast _2 \mathcal{L}_{N} = 0, \quad \mathcal{R}_{C} \ast _2 \mathcal{E} \ast _2 \mathcal{L}_{B} = 0, \\
\mathcal{R}_{A_4} & \ast _2 \mathcal{E}_1 = 0, \quad \mathcal{E}_1 \ast _2 \mathcal{L}_{B_1} = 0, \quad \mathcal{R}_{A_2} \ast _2 \mathcal{E}_2 = 0, \quad \mathcal{E}_2 \ast _2 \mathcal{B}_2 = 0, \\
\mathcal{R}_{A_{33}} & \ast _2 \mathcal{E}_{33} = 0, \quad \mathcal{E}_{33} \ast _2 \mathcal{L}_{B_{33}} = 0, \quad \mathcal{R}_{A_{55}} \ast _2 \mathcal{E}_{55} = 0, \quad \mathcal{E}_{55} \ast _2 \mathcal{L}_{B_{55}} = 0, \\
\mathcal{R}_{A_{77}} & \ast _2 \mathcal{E}_{77} = 0, \quad \mathcal{E}_{77} \ast _2 \mathcal{L}_{B_{77}} = 0, \quad \mathcal{R}_{A_{99}} \ast _2 \mathcal{E}_{99} = 0, \quad \mathcal{E}_{99} \ast _2 \mathcal{L}_{B_{99}} = 0, \\
\mathcal{R}_{\tilde{A}} & \ast _2 \mathcal{E} \ast _2 \mathcal{L}_{B} = 0.
\end{align*}
\]

Consequently, \( \mathcal{E}_3 \) is solvable. In that case, the general solution to \( \mathcal{E}_3 \) can be expressed as
\[
\begin{align*}
\mathcal{X}_3(:,; 1, 1) &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad \mathcal{X}_3(:,; 1, 2) = \begin{pmatrix} 0 & 2i \\ k & 0 \end{pmatrix}, \quad \mathcal{X}_3(:,; 2, 1) = \begin{pmatrix} -i & k \\ 0 & 1 \end{pmatrix}, \\
\mathcal{X}_3(:,; 2, 2) &= \begin{pmatrix} 1 & k \\ 0 & 0 \end{pmatrix}, \quad \mathcal{X}_3(:,; 2, 1) = \begin{pmatrix} -1 & -i \\ 0 & 0 \end{pmatrix}, \quad \mathcal{X}_3(:,; 1, 2) = \begin{pmatrix} -3i & 0 \\ 0 & 0 \end{pmatrix}, \\
\mathcal{Y}_3(:,; 1, 1) &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{Y}_3(:,; 1, 2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{Y}_3(:,; 2, 1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
\mathcal{Y}_3(:,; 2, 2) &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{Y}_3(:,; 1, 2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{Y}_3(:,; 2, 1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\end{align*}
\]

4. Some implementations of the main system

Remark 4.1. Set \( \mathcal{A}_4 = \mathcal{B}_4 = \mathcal{A}_5 = \mathcal{B}_5 = 0 \) in the system \( \mathcal{E}_3 \), we obtain the solvability conditions and the general solution for the system of tensor equations:
\[
\begin{align*}
\mathcal{A}_1 \ast _N \mathcal{X}_3 \ast _M \mathcal{B}_1 &= \mathcal{E}_1, \\
\mathcal{A}_2 \ast _N \mathcal{Y}_3 \ast _M \mathcal{B}_2 &= \mathcal{E}_2, \\
\mathcal{A}_3 \ast _N \mathcal{W} &= \mathcal{E}_3, \quad \mathcal{W} \ast _M \mathcal{B}_3 = \mathcal{E}_4, \\
\mathcal{A}_6 \ast _N \mathcal{X}_1 \ast _M \mathcal{B}_6 + \mathcal{A}_7 \ast _N \mathcal{X}_2 \ast _M \mathcal{B}_7 &= 0, \\
\mathcal{A}_7 \ast _N (\mathcal{C}_3 \ast _N \mathcal{X}_3 \ast _M \mathcal{D}_3 + \mathcal{C}_4 \ast _N \mathcal{W} \ast _M \mathcal{D}_4) \ast _M \mathcal{B}_6 &= \mathcal{E}_9, \\
\mathcal{A}_8 \ast _N \mathcal{Y}_1 \ast _M \mathcal{B}_8 + \mathcal{A}_9 \ast _N \mathcal{Y}_2 \ast _M \mathcal{B}_9 &= 0, \\
\mathcal{A}_9 \ast _N (\mathcal{H}_3 \ast _N \mathcal{Y}_3 \ast _M \mathcal{J}_3 + \mathcal{H}_4 \ast _N \mathcal{W} \ast _M \mathcal{J}_4) \ast _M \mathcal{B}_8 &= \mathcal{E}_{10}. 
\end{align*}
\]
Now, we consider the solvability conditions and the general solution to the system (1.4), as a particular case of (1.4).

**Theorem 4.2.** Consider the system of tensor equations (1.4), where

\[
A_1 \in \mathbb{H}^{I(N) \times J(N)}, \quad A_2 \in \mathbb{H}^{I(N) \times Q(N)}, \quad A_3 \in \mathbb{H}^{I(N) \times P(N)}, \quad A_6 \in \mathbb{H}^{A(N) \times E(N)},
\]
\[
A_8 \in \mathbb{H}^{C(N) \times R(N)}, \quad B_1 \in \mathbb{H}^{L(M) \times K(M)}, \quad B_2 \in \mathbb{H}^{S(M) \times K(M)}, \quad B_3 \in \mathbb{H}^{T(M) \times K(M)},
\]
\[
B_7 \in \mathbb{H}^{H(N) \times B(N)}, \quad B_9 \in \mathbb{H}^{O(M) \times D(M)}, \quad C_3 \in \mathbb{H}^{I(N) \times J(N)}, \quad C_4 \in \mathbb{H}^{I(N) \times P(N)}, \quad
\]
\[
D_3 \in \mathbb{H}^{L(M) \times K(M)}, \quad D_4 \in \mathbb{H}^{T(M) \times K(M)}, \quad H_3 \in \mathbb{H}^{I(N) \times Q(N)}, \quad H_4 \in \mathbb{H}^{I(N) \times P(N)},
\]
\[
J_3 \in \mathbb{H}^{S(M) \times K(M)}, \quad J_4 \in \mathbb{H}^{T(M) \times K(M)}, \quad E_1 \in \mathbb{H}^{I(N) \times K(M)}, \quad E_2 \in \mathbb{H}^{I(N) \times K(M)},
\]
\[
E_3 \in \mathbb{H}^{I(N) \times T(M)}, \quad E_4 \in \mathbb{H}^{P(N) \times K(M)}, \quad E_9 \in \mathbb{H}^{A(N) \times B(M)}, \quad E_{10} \in \mathbb{H}^{C(N) \times D(M)}
\]

are given tensors over \( \mathbb{H} \). Set

\[
\hat{A}_4 = R_{A_6} *_N C_3, \quad \hat{C}_4 = R_{A_6} *_N C_4, \quad \hat{B}_4 = D_3 *_M L_{B_7},
\]
\[
\hat{D}_4 = D_4 *_M L_{B_7}, \quad \tilde{P} = R_{A_6} *_N E_0 *_M L_{B_7}, \quad \hat{A}_5 = R_{A_6} *_N H_3, \quad \hat{C}_5 = R_{A_6}
\]
\[
*_N H_4, \quad \hat{B}_6 = J_3 *_M L_{B_9}, \quad \hat{D}_5 = J_4 *_M L_{B_9}, \quad \hat{Q} = R_{A_6} *_N E_{10} *_M L_{B_9},
\]
\[
\hat{C}_6 = \hat{C}_5 *_N L_{A_3}, \quad \hat{D}_6 = R_{B_5} *_M \hat{D}_4, \quad \hat{C}_7 = \hat{C}_6 *_N L_{A_3}, \quad \hat{D}_7 = R_{B_5} *_M \hat{D}_5,
\]
\[
\hat{G} = \tilde{P} - \hat{C}_4 *_N A_3 *_M \hat{D}_4 - \hat{C}_4 *_N L_{A_3} *_N E_4 *_M B_1 *_M \hat{D}_4,
\]
\[
\hat{F} = \hat{Q} - \hat{C}_5 *_N A_3 *_N E_4 *_M \hat{D}_5 - \hat{C}_5 *_N L_{A_3} *_N E_4 *_M B_1 *_M \hat{D}_5,
\]
\[
M_1 = R_{\hat{A}_4} *_M \hat{C}_6, \quad N_1 = D_6 *_M L_{B_4}, \quad S_1 = C_6 *_N L_{M_1},
\]
\[
M_2 = R_{\hat{A}_4} *_M C_7, \quad N_2 = D_7 *_M L_{B_5}, \quad S_2 = C_7 *_N L_{M_2},
\]
\[
A_{11} = \left( L_{M_4} *_N S_3, \quad L_{M_2} *_N S_2 \right), \quad B_{11} = \left( \frac{R_{D_6}}{R_{D_7}} \right),
\]
\[
E_{11} = M_1 *_N F *_M D_1 + S_1 *_N S_2 *_N C_1 *_N F *_M N_2 - M_1 *_N G *_M D_1
\]
\[-S_1 *_N S_1 *_N C_1 *_N G *_M N_1, \quad A = R_{A_{11}} *_N L_{M_1},
\]
\[
B = R_{N_1} *_M L_{B_{11}}, \quad C = R_{A_{11}} *_N L_{M_2}, \quad D = R_{N_2} *_M L_{B_{11}}, \quad E = R_{A_{11}}
\]
\[*_N E_{11} *_M L_{B_{11}}, \quad M = R_{A_6} *_N C, \quad N = D *_M L_{B}, \quad S = C *_N L_{M},
\]
\[
A_{22} = \left( L_{A_5}, \quad L_{A_5} \right), \quad B_{22} = \left( \frac{R_{B_1}}{R_{B_4}} \right), \quad C_{22} = \hat{A}_1 *_N S_1, \quad D_{22} = R_{N_1} *_M
\]
\[D_6 *_N \hat{B}_1, \quad E_{22} = \hat{A}_1 *_N G *_M \hat{B}_1 - \hat{A}_1 *_N E_1 *_M B_1 - \hat{A}_1 *_N S_1 *_N C_0
\]
\[*_N G *_M N_1 *_M D_6 *_M \hat{B}_1 - \hat{A}_1 *_N C_0 *_N M_1 *_N G *_M \hat{B}_4, \quad A_{33} = R_{A_{22}} *_N C_{22}, \quad B_{33} = D_{22} *_M L_{B_{22}}, \quad E_{33} = R_{A_{22}} *_N E_{22} *_M L_{B_{22}},
\]
\[
A_{44} = \left( L_{A_5}, \quad L_{A_5} \right), \quad B_{44} = \left( \frac{R_{B_2}}{R_{B_1}} \right), \quad C_{44} = \hat{A}_3 *_N S_2, \quad D_{44} = R_{N_1} *_M D_7 *_N \hat{B}_2
\]
\[E_{44} = \hat{A}_3 *_N F *_M \hat{B}_2 - \hat{A}_3 *_N E_2 *_M B_2 - \hat{A}_3 *_N S_2
\]
\[*_N C_2 *_N F *_M N_2 *_M D_7 *_M \hat{B}_2 - \hat{A}_3 *_N C_2 *_N M_2 *_N F *_M \hat{B}_2, \quad A_{55} = R_{A_{44}} *_N C_{44}, \quad B_{55} = D_{44} *_M L_{B_{44}}, \quad E_{55} = R_{A_{44}} *_N E_{44} *_M L_{B_{44}},
\]
Apply Theorem 3.3, whenever $A_4 = A_5 = B_4 = B_5 = 0$ and $B_6 = B_8 = A_7 = A_9 = I$. 

Under these constraints, the general solution to (1.3) can be expressed as:

\[
\begin{align*}
\mathcal{X}_1 &= A_6^\dagger *N \mathcal{E}_1 - \tilde{\mathcal{T}}_{21} *M B_7 + L_{A_6} *N \mathcal{T}_{41}, \\
\mathcal{X}_2 &= R_{A_6} *N \mathcal{E}_1 *M B_7 + A_6 *N \tilde{\mathcal{T}}_{21} + \mathcal{T}_{31} *M R_{B_7}, \\
\mathcal{Y}_1 &= A_6^\dagger *N \mathcal{E}_2 - \tilde{\mathcal{T}}_{21} *M B_9 + L_{A_6} *N \mathcal{J}_{41}, \\
\mathcal{Y}_2 &= R_{A_6} *N \mathcal{E}_2 *M B_9 + A_8 *N \tilde{\mathcal{T}}_{21} + \mathcal{J}_{31} *M R_{B_9}, \\
\mathcal{X}_3 &= A_6^\dagger *N \mathcal{E}_3 *M B_7 + L_{A_6} *N U_4 + U_2 *M R_{B_7}, \\
\mathcal{Y}_3 &= A_6^\dagger *N \mathcal{E}_3 *M B_9 + L_{A_6} *N U_4 + U_4 *M R_{B_9}, \\
W &= A_3^\dagger *N \mathcal{E}_3 + U_3 *N \mathcal{E}_4 *M B_7 + L_{A_6} *N U_6 *M R_{B_7}, \\
\tilde{\mathcal{E}}_1 &= \mathcal{E}_5 - C_3 *N \mathcal{X}_3 *M D_5 - C_4 *N W *M D_4, \\
\tilde{\mathcal{E}}_2 &= \mathcal{E}_{10} - H_3 *N \mathcal{Y}_3 *M J_3 - H_4 *N W *M J_4, \\
\end{align*}
\]

and $U_i$ (i = 1, 5) are defined by (3.35i), (3.35j) with $W_{ij}$, $K_i$, $H_{jk}$ (i, k = 1, 3, j = 1, 4), $\mathcal{T}_{11}$, $\tilde{\mathcal{T}}_{21}$, $J_{11}$ and $\tilde{J}_{21}$ (l = 3, 4) are arbitrary tensors with suitable orders.

Proof. Apply Theorem 3.3 whenever $A_4 = A_5 = B_4 = B_5 = 0$ and $B_6 = B_8 = A_7 = A_9 = I$. 

19
Theorem 4.3. Consider the system of tensor equations (4.6), where

\[
\begin{align*}
A_1 &\in \mathbb{H}^{(I(N) \times J(N)), \ A_2 \in \mathbb{H}^{(I(N) \times Q(N)), \ A_3 \in \mathbb{H}^{(I(N) \times P(N)), \ A_6 \in \mathbb{H}^{(J(N) \times E(N)), \ A_8 \in \mathbb{H}^{(J(N) \times R(N)), \ C_3 \in \mathbb{H}^{(I(N) \times J(N)), \ C_4 \in \mathbb{H}^{(I(N) \times P(N)), \ H_3 \in \mathbb{H}^{(I(N) \times Q(N)), \ H_4 \in \mathbb{H}^{(I(N) \times P(N)), \ E_1 = E_i^* \in \mathbb{H}^{(I(N) \times I(N)), \ E_3 \in \mathbb{H}^{(I(N) \times P(N)), \ (i \in \{1, 2, 9, 10\})}}}
\end{align*}
\]

are given tensors over \( \mathbb{H} \). Set

\[
\begin{align*}
\mathcal{A}_4 &= \mathcal{R}_{A_4} \ast N C_4, \ \ \mathcal{C}_4 = \mathcal{R}_{A_6} \ast N C_4, \ \ \mathcal{P} = \mathcal{R}_{A_8} \ast N E_9 \ast M \ (\mathcal{R}_{A_6})^\eta, \\
\mathcal{A}_5 &= \mathcal{R}_{A_8} \ast N H_3, \ \ \mathcal{C}_5 = \mathcal{R}_{A_6} \ast N H_4, \ \ \mathcal{Q} = \mathcal{R}_{A_4} \ast N E_{10} \ast M \ (\mathcal{R}_{A_6})^\eta, \\
\mathcal{C}_6 &= \mathcal{C}_4 \ast N L_{A_5}, \ \ \mathcal{C}_7 = \mathcal{C}_5 \ast N L_{A_3}, \ \ \mathcal{G} = \mathcal{P} = \mathcal{C}_4 \ast N A_3^\dagger \ast N E_3 \ast N (\mathcal{C}_4)^\eta \\
&- \mathcal{C}_5 \ast N L_{A_3} \ast N E_7^\eta \ast N (A_3^\dagger)^\eta \ast N (\mathcal{C}_4)^\eta, \ \ \mathcal{F} = \mathcal{Q} = \mathcal{C}_5 \ast N A_3^\dagger \ast N E_3 \ast N (\mathcal{C}_5)^\eta, \\
\mathcal{M}_1 &= \mathcal{R}_{A_4} \ast N C_6, \\
\mathcal{S}_1 &= \mathcal{C}_6 \ast N L_{M_1}, \ \ \mathcal{M}_2 = \mathcal{R}_{A_8} \ast N C_7, \ \ \mathcal{S}_2 = \mathcal{C}_7 \ast N L_{M_2}, \\
A_{11} &= (\mathcal{L}_{M_4} \ast N L_{S_1} \ \ \mathcal{L}_{M_2} \ast N L_{S_2}), \ \ B_{11} = \begin{pmatrix} R_{C_6^\dagger} \\ R_{C_7^\dagger} \end{pmatrix}, \\
\mathcal{E}_{11} &= \mathcal{N}_1^\dagger \ast N F \ast N (C_1^\dagger)^\eta + S_1^\dagger \ast N S_2 \ast N C_1^\dagger \ast N F \ast N (A_1^\dagger)^\eta - \mathcal{N}_1^\dagger \\
&\ast N G \ast N (C_0^\dagger)^\eta - S_1^\dagger \ast N S_1 \ast N C_0^\dagger \ast N G \ast N (A_1^\dagger)^\eta, \ \ A = \mathcal{R}_{A_1} \ast N L_{M_1}, \\
B &= \mathcal{R}_{A_4} \ast N L_{B_1}, \ \ C = \mathcal{R}_{A_1} \ast N L_{M_2}, \ \ D = \mathcal{R}_{A_4} \ast N L_{B_1}, \ \ E = \mathcal{R}_{A_1} \ast N E_1 \ast N L_{B_1}, \ \ M = \mathcal{R}_A \ast N C, \ \ N = D \ast M \ast L_{B_1}, \ \ S = C \ast N L_{M_2}, \\
\mathcal{A}_{22} &= (\mathcal{L}_A, \mathcal{L}_{A}), \ \ \mathcal{C}_{22} = \tilde{A}_1^\dagger \ast N C_0 \ast N L_{M_4}, \ \ \mathcal{A}_{33} = \mathcal{R}_{A_2} \ast N C_{22}, \\
\mathcal{E}_{22} &= \tilde{A}_1^\dagger \ast N G \ast M \ast N (\tilde{A}_1^\dagger)^\eta - A_1 \ast N E_1 \ast M (A_1^\dagger)^\eta - \tilde{A}_1 \ast N S_1 \ast N C_0^\dagger \ast N G \\
&\ast M \ast M_0^\dagger \ast N (A_1^\dagger)^\eta - A_1 \ast N C_6 \ast N M_1^\dagger \ast N G \ast M \ast (A_1^\dagger)^\eta, \\
\mathcal{E}_{33} &= \mathcal{R}_{A_2} \ast N E_2 \ast N (\mathcal{R}_{A_2})^\eta, \ \ \mathcal{A}_{44} = (\mathcal{L}_A, \mathcal{L}_{A}), \ \ \mathcal{C}_{44} = \tilde{A}_1 \ast N C_7 \\
&\ast N L_{M_2}, \mathcal{E}_{44} = \tilde{A}_1 \ast N F \ast N (\tilde{A}_1^\dagger)^\eta - A_1 \ast N E_2 \ast N (A_1^\dagger)^\eta - \tilde{A}_1 \ast N S_2 \ast N C_1^\dagger \ast N F \\
&\ast N M_2^\dagger \ast N C_0^\dagger \ast N (\tilde{A}_1^\dagger)^\eta - \tilde{A}_1 \ast N C_7 \ast N M_2^\dagger \ast N F \ast M \ast (\tilde{A}_1^\dagger)^\eta, \\
\mathcal{A}_{55} &= \mathcal{R}_{A_4} \ast N C_{44}, \ \ \mathcal{E}_{55} = \mathcal{R}_{A_4} \ast N E_4 \ast M (\mathcal{R}_{A_4})^\eta, \ \ \mathcal{A}_{66} = (\mathcal{L}_A, \mathcal{L}_{A_3}), \\
\mathcal{B}_{66} &= \begin{pmatrix} R_B \\ R_{A_5^\dagger} \end{pmatrix}, \ \ \mathcal{C}_{66} = A_1 \ast N S, \ \ \mathcal{D}_{66} = \mathcal{R}_N \ast N D \ast N B_1^\dagger, \\
\mathcal{E}_{66} &= A_1^\dagger \ast N E_3 \ast N (A_1^\dagger)^\eta - A_1 \ast N E \ast N B_1^\dagger + A_1 \ast N S \ast N C_1 \ast N E \\
&\ast N M_1^\dagger \ast N D \ast N B_1^\dagger + A_1 \ast N C \ast N M_1^\dagger \ast N E \ast N B_1^\dagger, \\
\mathcal{A}_{77} &= \mathcal{R}_{A_6} \ast N C_{66}, \ \ \mathcal{B}_{77} = \mathcal{D}_{66} \ast N L_{B_{66}}, \ \ \mathcal{E}_{77} = \mathcal{R}_{A_6} \ast N E_6 \ast M \ast L_{B_{66}}, \\
\mathcal{A}_{88} &= (\mathcal{L}_M \ast N L_{S} \ \ \mathcal{L}_{A_5}), \ \ \mathcal{B}_{88} = \begin{pmatrix} R_D \\ R_{A_5^\dagger} \end{pmatrix}, \ \ \mathcal{C}_{88} = L_{M_4}, \ \ \mathcal{D}_{88} = \mathcal{R}_N, \\
\mathcal{E}_{88} &= A_1^\dagger \ast N E_5 \ast N (A_1^\dagger)^\eta - M_1 \ast N E \ast N B_1^\dagger - S_1 \ast N S \ast N C_1 \ast N E \ast M \\
&\ast N_1 \ast N A_9 = \mathcal{R}_{A_8} \ast N C_{88}, \ \ \mathcal{B}_{99} = \mathcal{D}_{88} \ast M \ast L_{B_{88}}, \ \
Then (1.5) is solvable if and only if

\[ R_{A_3} * N E_3 = 0, \quad A_3 * N E_3 = (A_3 * N E_3)^\dagger, \quad R_{A_1} * N E_1 = 0, \]
\[ R_{A_2} * N E_2 = 0, \quad R_{A_4} * N \hat{G} * N \mathcal{E}_6^\dagger * = 0, \quad R_{c_6} * N \hat{G} * M \mathcal{L}_{\hat{A}_4}^\dagger * = 0, \]
\[ R_{S_1} * N R_{A_6} * N \hat{G} = 0, \quad \hat{G} * M \mathcal{L}_{A_6}^\dagger * * N \mathcal{M}_{A_6}^\dagger = 0, \]
\[ R_{A_5} * N \hat{F} * M \mathcal{L}_{C_5}^\dagger = 0, \quad R_{C_5} * N \hat{F} * N \mathcal{L}_{\hat{A}_5}^\dagger = 0, \quad R_{S_2} * N R_{A_5} * N \hat{F} = 0, \]
\[ F * N \mathcal{L}_{\hat{A}_5}^\dagger * * N \mathcal{M}_{A_5}^\dagger = 0, \quad R_{M} * N R_{A} * N \hat{E} = 0, \quad R_{A} * N \mathcal{E} * M \mathcal{L}_D = 0, \]
\[ \hat{E} * M \mathcal{L}_B * M \mathcal{L}_N = 0, \quad R_{C} * N \hat{E} * * M \mathcal{L}_B = 0, \quad R_{A_3} * N E_{33} = 0, \]
\[ E_{33} * M \mathcal{L}_{B_{33}} = 0, \quad R_{A_{55}} * N E_{55} = 0, \quad E_{55} * M \mathcal{L}_{B_{55}} = 0, \quad R_{A_{77}} * N E_{77} = 0, \]
\[ E_{77} * M \mathcal{L}_{B_{77}} = 0, \quad R_{A_{99}} * N E_{99} = 0, \quad E_{99} * M \mathcal{L}_{B_{99}} = 0, \]
\[ R_{A} * N \hat{E} * M \mathcal{L}_B = 0. \]

Under these conditions, the general solution to (1.4) can be expressed as:

\[
(X_1, Y_1, X_3, Y_3, W) = \frac{1}{2} \left( X_{11} + A_{12}^{\dagger}, Y_{11} + A_{12}^{\dagger}, X_{33} + A_{33}^{\dagger}, Y_{33} + A_{33}^{\dagger}, W_1 + W_1^{\dagger} \right)
\]

(4.4)

where

\[
X_{11} = A_{1}^{\dagger} * N \hat{E}_1 - \hat{T}_{21} * M A_{1}^{\dagger} + \mathcal{L}_{A_6} * N \mathcal{T}_{41},
\]
\[
X_{12} = R_{A_6} * N \hat{E}_1 * M A_6^{\dagger} + A_6 * N \hat{T}_{21} + \mathcal{T}_{31} * M R_{A_6}^\dagger, \]
\[
Y_{11} = A_{2}^{\dagger} * N \hat{E}_2 - \hat{T}_{21} * M A_2^{\dagger} + \mathcal{L}_{A_6} * N \mathcal{T}_{41},
\]
\[
Y_{12} = R_{A_6} * N \hat{E}_2 * M A_6^{\dagger} + A_8 * N \hat{T}_{21} + \mathcal{T}_{31} * M R_{A_6}^\dagger, \]
\[
X_{33} = A_{3}^{\dagger} * N \mathcal{E}_1 * M (A_1)^{\dagger} \mathcal{E}_1 + \mathcal{L}_{A_1} * N U_1 + U_2 * M R_{A_1}^\dagger, \]
\[
Y_{33} = A_{4}^{\dagger} * N \mathcal{E}_2 * M (A_2)^{\dagger} \mathcal{E}_2 + \mathcal{L}_{A_3} * U_3 + U_4 * M R_{A_3}^\dagger, \]
\[
W_1 = A_{3}^{\dagger} * N \mathcal{E}_3 + \mathcal{L}_{A_3} * N \mathcal{E}_3^{\dagger} * N (A_1)^{\dagger} \mathcal{E}_3 + \mathcal{L}_{A_3} * N U_6 * M R_{A_3}^\dagger, \]
\[
\hat{E}_1 = E_9 - C_3 * N X_3 + C_3^{\dagger} - C_4 * N W * M C_4^\dagger, \]
\[
\hat{E}_2 = E_{10} - H_3 * N \mathcal{Y}_{3} * M H_3^\dagger - H_4 * N W * M H_4^\dagger, \]
\[
U_1 = \left( I \quad 0 \right) * N (A_{22} * N (E_{22} - C_{22} * N \mathcal{V}_2 * N C_{22}^\dagger)) - A_{12}^{\dagger} * N H_{12} * N A_{12}^\dagger,
\]
\[
+ \mathcal{L}_{A_{22}} * N \mathcal{H}_{11}, \]
\[
U_2 = (R_{A_{22}} * N (E_{22} - C_{22} * N \mathcal{V}_2 * M C_{22}^\dagger)) * N (A_{22}^{\dagger})^\dagger + A_{22} * N A_{22}^{\dagger}
\]
\[
+ N H_{12} * H_{13} * M R_{A_{22}^{\dagger}} * M \left( I \quad 0 \right), \]
\[
U_3 = \left( I \quad 0 \right) * N (A_{44} * N (E_{44} - C_{44} * N \mathcal{T}_2 * N C_{44}^\dagger)) - A_{44} * N H_{22} * N A_{44}^\dagger,
\]
\[
+ \mathcal{L}_{A_{44}} * N \mathcal{H}_{21}. \]
Now, we show that the formula (4.4) can be a solution to system (1.6). It is known that $H$ is a solution to (1.6), then it is evident that

\[
(I \ 0) \ H = (4.4) \text{ and } (3.35t) \text{ with } W_1, K_i \text{ and } \mathcal{H}_{jk} (i, k = 1, 3, \ j = 1, 4). \]

**Proof.** Consider the following system of tensor equations:

\[
\begin{aligned}
A_3 \ast N W_1 &= \mathcal{E}_3, \quad W_1 \ast N A_3^{\ast} = \mathcal{E}_3^{\ast}, \\
A_1 \ast N A_3 \ast N A_1^{\ast} &= \mathcal{E}_1, \quad A_2 \ast N Y_{33} \ast N A_2^{\ast} &= \mathcal{E}_2, \\
A_6 \ast N X_{11} + X_{12} \ast N A_6^{\ast} + C_3 \ast N X_{33} \ast N C_3^{\ast} + C_4 \ast N W_1 \ast N C_4^{\ast} &= \mathcal{E}_9, \\
A_8 \ast N Y_{11} + Y_{12} \ast N A_8^{\ast} + H_3 \ast N Y_{33} \ast N H_3^{\ast} + H_4 \ast N W_1 \ast N H_4^{\ast} &= \mathcal{E}_{10}.
\end{aligned}
\]  

(4.5)

First, we show that (1.6) is solvable if and only if (4.5) is solvable. Claim that $(X_1, Y_1, X_3, Y_3, W)$ is a solution to (1.6), then it is evident that $(X_{11}, X_{12}, Y_{11}, Y_{12}, X_{33}, Y_{33}, W_1) = (X_1, X_1^{\ast}, Y_1, Y_1^{\ast}, X_3, Y_3, W)$ is a solution to (4.5). Conversely, if $(X_{11}, X_{12}, Y_{11}, Y_{12}, X_{33}, Y_{33}, W_1)$ is a solution to (4.5). Now, we show that the formula (4.4) can be a solution to system (1.6). It is known that $X_3, Y_3,$ and $W$ are $\eta$-Hermitian tensors. By Applying (1.4) on (1.6) yields:

\[
A_1 \ast N X_3 \ast N A_1^{\ast} = A_1 \ast N \left( \frac{X_{33} + X_{33}^{\ast}}{2} \right) \ast N A_1^{\ast} \\
= \frac{1}{2} A_1 \ast N X_{33} \ast N A_1^{\ast} + \frac{1}{2} \left( A_1 \ast N X_{33} \ast N A_1^{\ast} \right)^{\ast} = \mathcal{E}_1.
\]

Similarly, it can be verified that

\[
A_2 \ast N Y_3 \ast N A_2^{\ast} = \mathcal{E}_2, \\
A_3 \ast N W = A_3 \ast N \left( \frac{W_1 + W_1^{\ast}}{2} \right) = \frac{1}{2} A_3 \ast N W_1 + \frac{1}{2} \left( W_1 \ast N A_3^{\ast} \right)^{\ast} = \mathcal{E}_3, \\
A_6 \ast N X_1 + (A_6 \ast N X_1)^{\ast} + C_3 \ast N X_3 \ast N C_3^{\ast} + C_4 \ast N W \ast N C_4^{\ast} \\
= A_6 \ast N \left( \frac{X_{11} + X_{12}^{\ast}}{2} \right) + \left( A_6 \ast N \left( \frac{X_{11} + X_{12}^{\ast}}{2} \right) \right)^{\ast} \\
C_3 \ast N \left( \frac{X_{33} + X_{33}^{\ast}}{2} \right) \ast N C_3^{\ast} + C_4 \ast N \left( \frac{W_1 + W_1^{\ast}}{2} \right) \ast N C_4^{\ast} \\
= \frac{1}{2} \left[ A_6 \ast N X_{11} + X_{12} \ast N A_6^{\ast} + C_3 \ast N X_{33} \ast N C_3^{\ast} + C_4 \ast N W_1 \ast N C_4^{\ast} \right] \\
+ \frac{1}{2} \left[ A_6 \ast N X_{11} + X_{12} \ast N A_6^{\ast} + C_3 \ast N X_{33} \ast N C_3^{\ast} + C_4 \ast N W_1 \ast N C_4^{\ast} \right]^{\ast} = \mathcal{E}_9.
\]

Similarly, it can be found that

\[
A_8 \ast N Y_1 + (A_8 \ast N Y_1)^{\ast} + H_3 \ast N Y_3 \ast M H_3^{\ast} + H_4 \ast N W \ast M H_4^{\ast} = \mathcal{E}_{10}.
\]
Therefore, (4.4) is a solution to (1.6). Consequently, apply Theorem 4.2 on (4.5), we can establish the solvability conditions and the general solution to (1.6).

5. Conclusion

We derive the necessary and sufficient algebraic conditions for the existence of a solution to (1.4) in Theorem 3.3. We established an explicit formula of the general solution in terms of the Moore-Penrose inverses of some block-given tensors. An algorithm with a numerical example is investigated to compute the general solution to (1.4). As a particular case of (1.4), we discuss the solvability conditions and the general solution to (1.5) in Theorem 4.2. As an implementation of Theorem 4.2, we carry out the solvability conditions and an expression of the general solution to (1.6), whenever $X_3, Y_3$ and $W$ are $\eta$-Hermitian tensors. All results are valid over an arbitrary division ring.

As a consequence the main findings in Section 3, we infer the solvability constraints of the two-sided linear matrix equation $A_1X_1B_1 + A_2X_2B_2 + A_3(C_3X_3D_3 + C_4WD_4)B_1 = E_1$ can be characterized by Moore-Penrose inverses of some provided matrices and rank equalities. Consequently, we can obtain the solvability constraints and the general solution to the following system of two-sided and coupled matrix equations:

$$\begin{cases}
A_6X_1B_6 + A_7X_2B_7 + A_7(C_3X_3D_3 + C_4WD_4)B_6 = E_9, \\
A_8Y_1B_8 + A_9Y_2B_9 + A_9(H_3Y_3J_3 + H_4WD_4)J_8 = E_{10}.
\end{cases}$$

with respect to

$$\begin{cases}
A_1X_1B_1 = E_1, & A_2Y_3B_2 = E_2, \\
A_4X_1 = E_5, & X_2B_4 = E_6, \\
A_5Y_1 = E_7, & Y_2B_5 = E_8, \\
A_3W = E_3, & WB_3 = E_4.
\end{cases}$$

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