WEAK INTERACTIONS IN A BACKGROUND OF A
UNIFORM MAGNETIC FIELD. A MATHEMATICAL
MODEL FOR THE INVERSE $\beta$ DECAY.I.

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Abstract. In this paper we define a mathematical model for the
inverse $\beta$ decay in a uniform magnetic field. With this model we
associate a Hamiltonian with cutoffs in an appropriate Fock space.
No infrared regularization is assumed. The Hamiltonian is self-
adjoint and has a ground state. We study its essential spectrum
and determine its spectrum. Conditions for uniqueness of ground
state are given. The coupling constant is supposed sufficiently
small.

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1. Introduction.

A supernova is initiated by the collapse of a stellar core which leads to the formation of a protoneutron star which may be formed with strong magnetic fields typically of order $10^{16}$ Gauss. It turns out that the protoneutron star leads to the formation of a neutron star in a very short time during which almost all the gravitational binding energy of the protoneutron star is emitted in neutrinos and antineutrinos of each type. Neutron stars have strong magnetic fields of order $10^{12}$ Gauss. Thus neutrinos interactions are of great importance because of their capacity to serve as mediators for the transport and loss of energy and the following processes, the so-called "Urca" ones or inverse $\beta$ decays in Physics,

\begin{align}
\nu_e + n &\rightleftharpoons e^- + p \\
\overline{\nu}_e + p &\rightleftharpoons e^+ + n
\end{align}

(1.1) \hspace{1cm} (1.2)

play an essential role in those phenomena and they are associated with the $\beta$ decay

\begin{equation}
\nu \to p + e^+ + \overline{\nu}_e
\end{equation}

(1.3)

Here $e^-$ (resp. $e^+$) is an electron (resp. a positron). $p$ is a proton and $n$ a neutron. $\nu_e$ and $\overline{\nu}_e$ are the neutrino and the antineutrino associated with the electron.

See [8], [9], [12], [15] and references therein.

Due to the large magnetic field strengths involved it is quite fundamental to study the processes (1.1) and (1.2) in the presence of magnetic fields. These realistic fields may be very complicated in their structure but we assume these fields to be uniform which is a very good hypothesis because the range of the weak interactions is very short. Our aim is to study the processes (1.1) and (1.2) in a background of a uniform magnetic field.

Throughout this work we restrict ourselves to the study of processes (1.1), the study of processes (1.2) and (1.3) would be quite similar.

The advantage of a uniform magnetic field is that, in presence of this field, Dirac equation can be exactly solved. We can then quantize the corresponding field by using the canonical formalism and use the Fermi’s Hamiltonian for the $\beta$ decay in order to study the processes (1.1).

Throughout this paper we choose the units such that $c = \hbar = 1$. 

References
In this paper we consider a mathematical model for the process (1.1) in a uniform magnetic field based on a Fock space for electrons, protons, neutrons and neutrinos and on a Hamiltonian with cutoffs suggested by the Fermi’s Hamiltonian for the $\beta$ decay. The Fock space will also involve the antiparticles of the electrons and of the protons. No infrared regularization is assumed. We neglect the anomalous magnetic moments of the particles. Relativistic invariance is broken.

We study the essential spectrum of the Hamiltonian and the existence of ground states. The spectrum of the Hamiltonian is identical to its essential spectrum. We also get conditions in order to obtain uniqueness of ground states. Every result is obtained for a sufficiently small coupling constant.

In an another paper we shall study the scattering theory and the absolutely continuous spectrum of the Hamiltonian.

The paper is organized as follows. In the next two sections we quantize the Dirac fields for electrons, protons and their antiparticles in a uniform magnetic field. In the third section we quantize the Dirac fields for free neutrons, neutrinos and their antiparticles in helicity formalism. The self-adjoint Hamiltonian of the model is defined in the fourth section. We then study the existence of ground states and the essential spectrum. We conclude by giving conditions under which we get uniqueness of ground states for the Hamiltonian.

2. The quantization of the Dirac fields for the electrons and the protons in a uniform magnetic field.

In this paper we assume that the uniform classical background magnetic field in $\mathbb{R}^3$ is along the $x^3$-direction of the coordinate axis. There are several choices of gauge vector potential giving rise to a magnetic field of magnitude $B > 0$ along the $x^3$-direction. In this paper we choose the following vector potential $A(x) = (A^\mu(x), \mu = 0, 1, 2, 3$, where

$$ (2.1) \quad A^0(x) = A^2(x) = A^3(x) = 0, \quad A^1(x) = -x^2 B $$

Here $x = (x^1, x^2, x^3)$ in $\mathbb{R}^3$.

We recall that we neglect the anomalous magnetic moments of the particles of spin $\frac{1}{2}$.

The Dirac equation for a particle of spin $\frac{1}{2}$ with mass $m > 0$ and charge $e$ in a uniform magnetic field of magnitude $B > 0$ along the $x^3$-direction with the choice of the gauge (2.1) and by neglecting its anomalous magnetic moment is given by

$$ (2.2) \quad H_D(e) = \alpha \cdot \left( \frac{1}{i} \nabla - eA \right) + \beta m, $$
acting in the Hilbert space $L^2(\mathbb{R}^3, \mathbb{C}^4)$.

The scalar product in $L^2(\mathbb{R}^3, \mathbb{C}^4)$ is given by

$$\langle f, g \rangle = \sum_{j=1}^{4} \int_{\mathbb{R}^3} \overline{f(x)} g(x) d^3x$$

We refer to [24] for a discussion of the Dirac operator.

Here $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\beta$ are the Dirac matrices in the standard form:

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad i = 1, 2, 3,$$

where $\sigma_i$ are the usual Pauli matrices.

By [24, thm 4.3] $H_D(e)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$. The spectrum of $H_D(e)$ is equal to

$$\text{spec}(H_D(e)) = (-\infty, -m] \cup [m, \infty)$$

The spectrum of $H_D(e)$ is absolutely continuous and its multiplicity is not uniform. There is a countable set of thresholds, denoted by $S$, where

$$S = \{ -s_n, s_n; n \in \mathbb{N} \}$$

with $s_n = \sqrt{m^2 + 2n|e|B}$. See[17].

We consider a spectral representation of $H_D(e)$ based on a complete set of generalized eigenfunctions of the continuous spectrum of $H_D(e)$. Those generalized eigenfunctions are well known. See[20]. In view of (2.1) we use the computation of the generalized eigenfunctions given by [19] and [7]. See also [15] and references therein.

Let $(p^1, p^3)$ be the conjugate variables of $(x^1, x^3)$. By the Fourier transform in $\mathbb{R}^2$ we easily get

$$L^2(\mathbb{R}^3, \mathbb{C}^4) \simeq \int_{\mathbb{R}^2} L^2(\mathbb{R}, \mathbb{C}^4) dp^1 dp^3.$$

and

$$H_D(e) \simeq \int_{\mathbb{R}^2} H_D(e; p^1, p^3) dp^1 dp^3.$$

where

$$H_D(e; p^1, p^3) = \begin{pmatrix} m\sigma_0, & \sigma_1(p^1 - ex^2B) - i\sigma_2 \frac{d}{dx^2} + p^3\sigma_3 \\ \sigma_1(p^1 - ex^2B) - i\sigma_2 \frac{d}{dx^2} + p^3\sigma_3, & -m\sigma_0 \end{pmatrix}$$

Here $\sigma_0$ is the $2 \times 2$ unit matrix.

$H_D(e; p^1, p^3)$ is the reduced Dirac operator associated to $(e; p^1, p^3)$. 
$H_D(e; p^1, p^3)$ is essentially self-adjoint on $C^\infty_0(\mathbb{R}, \mathbb{C}^4)$ and has a pure point spectrum which is symmetrical with respect to the origin.

Set

\begin{equation}
E_n(p^3)^2 = m^2 + (p^3)^2 + 2n|e|B, \quad n \geq 0
\end{equation}

The positive spectrum of $H_D(e; p^1, p^3)$ is the set of eigenvalues $(E_n(p^3))_{n \geq 0}$ and the negative spectrum is the set of eigenvalues $(-E_n(p^3))_{n \geq 0}$. $E_0(p^3)$ and $-E_0(p^3)$ are simple eigenvalues and the multiplicity of $E_n(p^3)$ and $-E_n(p^3)$ is equal to 2 for $n \geq 1$.

Throughout this work $e$ will be the positive unit of charge taken to be equal to the proton charge.

We now give the eigenfunctions of $H_D(e; p^1, p^3)$ both for the electrons and for the protons. The eigenfunctions are labelled by $n \in \mathbb{N}, (p_1, p_2) \in \mathbb{R}^2$ and $s = \pm 1$. $n \in \mathbb{N}$ labels the nth Landau level.

### 2.1. Eigenfunctions of the reduced Dirac operator for the electrons.

We now compute the eigenfunctions of $H_D(-e; p^1, p^3)$ with $m = m_e$ where $m_e$ is the mass of the electron.

$E_n^{(e)}(p^3)$ and $-E_n^{(e)}(p^3)$ will denote the eigenvalues of $H_D(-e; p^1, p^3)$ for the electrons. We have $E_n^{(e)}(p^3)^2 = m_e^2 + (p^3)^2 + 2neB, n \geq 0$.

#### 2.1.1. Eigenfunctions of the electrons for positive eigenvalues.

For $n \geq 1$ $E_n^{(e)}(p^3)$ is of multiplicity two corresponding to $s = \pm 1$ and $E_0^{(e)}(p^3)$ is multiplicity one corresponding to $s = -1$.

Let $U_{\pm 1}^{(e)}(x^2, n, p^1, p^3)$ denote the eigenfunctions associated to $s = \pm 1$. For $s = 1$ and $n \geq 1$ we have

\begin{equation}
U_{\pm 1}^{(e)}(x^2, n, p^1, p^3) = \left( \frac{E_n^{(e)}(p^3) + m_e}{2E_n^{(e)}(p^3)} \right)^{\frac{1}{2}} \begin{pmatrix} I_{n-1}(\xi) \\ \frac{p^3}{E_n^{(e)}(p^3) + m_e} I_{n-1}(\xi) \\ -\sqrt{2neB} \frac{E_n^{(e)}(p^3) + m_e}{E_n^{(e)}(p^3)} I_n(\xi) \end{pmatrix}
\end{equation}

where

\begin{equation}
\xi = \sqrt{eB(x^2 - \frac{p^1}{eB})}
\end{equation}

\begin{equation}
I_n(\xi) = \left( \frac{\sqrt{eB}}{n!2^n \sqrt{\pi}} \right)^{\frac{1}{2}} \exp(-\xi^2/2) H_n(\xi).
\end{equation}
Here $H_n(\xi)$ is the Hermite polynomial of order $n$ and we define

\begin{equation}
I_{-1}(\xi) = 0
\end{equation}

For $n = 0$ we set

\begin{equation}
U_{+1}(x^2, 0, p^1, p^3) = 0
\end{equation}

For $s = -1$ and $n \geq 0$ we have

\begin{equation}
U_{-1}(x^2, n, p^1, p^3) = \left( \frac{E_n^{(e)}(p^3) + m_e}{2 E_n^{(e)}(p^3)} \right)^{\frac{1}{2}} \begin{pmatrix}
0 \\
\frac{I_n(\xi)}{\sqrt{2n_e B} E_n^{(e)}(p^3) + m_e} I_{n-1}(\xi) \\
\frac{p^3}{E_n^{(e)}(p^3) + m_e} I_n(\xi) \\
0
\end{pmatrix}
\end{equation}

Note that

\[
\int dx^2 U_s^{(e)}(x^2, n, p^1, p^3) U_{s'}^{(e)}(x^2, n, p^1, p^3) = \delta_{ss'}
\]

where $\dagger$ is the adjoint in $\mathbb{C}^4$.

2.1.2. Eigenfunctions of the electrons for negative eigenvalues.

Let $V_{\pm 1}(x^2, n, p^1, p^3)$ denote the eigenfunctions associated with the eigenvalue $-E_n^{(op)}(p^3)$ and with $s = \pm 1$.

For $s = 1$ and $n \geq 1$ we have

\begin{equation}
V_{+1}(x^2, n, p^1, p^3) = \left( \frac{E_n^{(e)}(p^3) + m_e}{2 E_n^{(e)}(p^3)} \right)^{\frac{1}{2}} \begin{pmatrix}
- \frac{p^3}{E_n^{(e)}(p^3) + m_e} I_{n-1}(\xi) \\
\frac{\sqrt{2n_e B} E_n^{(e)}(p^3) + m_e}{E_n^{(e)}(p^3) + m_e} I_n(\xi) \\
I_{n-1}(\xi) \\
0
\end{pmatrix}
\end{equation}

and for $n = 0$ we set

\begin{equation}
V_{+1}(x^2, 0, p^1, p^3) = 0
\end{equation}

For $s = -1$ and $n \geq 0$ we have

\begin{equation}
V_{-1}(x^2, n, p^1, p^3) = \left( \frac{E_n^{(e)}(p^3) + m_e}{2 E_n^{(e)}(p^3)} \right)^{\frac{1}{2}} \begin{pmatrix}
\frac{\sqrt{2n_e B} E_n^{(e)}(p^3) + m_e}{E_n^{(e)}(p^3) + m_e} I_n(\xi) \\
\frac{p^3}{E_n^{(e)}(p^3) + m_e} I_{n-1}(\xi) \\
I_n(\xi) \\
0
\end{pmatrix}
\end{equation}
We have
\[ \int \text{d}x^2 V_s^{(e)}(x^2, n, p^1, p^3)\hat{V}_{s'}^{(e)}(x^2, n, p^1, p^3) = \delta_{ss'} \]
where \( \hat{\cdot} \) is the adjoint in \( \mathbb{C}^4 \).

The sets \( \left( U^{(e)}_{\pm}(., n, p^1, p^3) \right)_{(n,p^1,p^3)} \) and \( \left( V^{(e)}_{\pm}(., n, p^1, p^3) \right)_{(n,p^1,p^3)} \) of vectors in \( L^2(\mathbb{R}, \mathbb{C}^4) \) are an orthonormal basis of \( L^2(\mathbb{R}, \mathbb{C}^4) \).

This yields for \( \Psi(x) \) in \( L^2(\mathbb{R}^3, \mathbb{C}^4) \)

\[
(2.17) \\
\Psi(x) = \frac{1}{2\pi} \sum_{s=\pm 1} \text{L.i.m} \left( \sum_{n \geq 0} \int_{\mathbb{R}^2} \text{d}p_1 \text{d}p_3 e^{(p_1 x^1 + p_3 x^3)} \right) \left( c_s^{(e)}(n, p^1, p^3) U_{s}^{(e)}(x^2, n, p^1, p^3) + d_s^{(e)}(n, p^1, p^3) V_{s}^{(e)}(x^2, n, p^1, p^3) \right).
\]

where \( c_s^{(e)}(0, p^1, p^3) = d_s^{(e)}(0, p^1, p^3) = 0 \)

Let \( \tilde{\Psi}(x^2; p^1, p^3) \) be the Fourier transform of \( \Psi(.) \) with respect to \( x^1 \) and \( x^3 \):

\[ \tilde{\Psi}(x^2; p^1, p^3) = \text{L.i.m} \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-(p_1 x^1 + p_3 x^3)} \Psi(x^1, x^2, x^3) \text{d}x^1 \text{d}x^3. \]

We have

\[
(2.18) \\
c_s^{(e)}(n, p^1, p^3) = \int_{\mathbb{R}} U_{s}^{(e)}(x^2, n, p^1, p^3) \hat{\Psi}(x^2; p^1, p^3) \text{d}x^2 \\
d_s^{(e)}(n, p^1, p^3) = \int_{\mathbb{R}} V_{s}^{(e)}(x^2, n, p^1, p^3) \hat{\Psi}(x^2; p^1, p^3) \text{d}x^2.
\]

The complex coefficients \( c_s^{(e)}(n, p^1, p^3) \) and \( d_s^{(e)}(n, p^1, p^3) \) satisfy

\[
(2.19) \\
\| \Psi(.) \|_{L^2(\mathbb{R}^3, \mathbb{C}^4)}^2 \\
\sum_{s=\pm 1} \sum_{n \geq 0} \int \left( |c_s^{(e)}(n, p^1, p^3)|^2 + |d_s^{(e)}(n, p^1, p^3)|^2 \right) \text{d}p_1 \text{d}p_3 < \infty.
\]

2.2. Eigenfunctions of the reduced Dirac operator for the protons.

We now compute the eigenfunctions of \( H_D(e; p^1, p^3) \) with \( m = m_p \).

\( E_n^{(p)}(p^3) \) and \( -E_n^{(p)}(p^3) \) denote the eigenvalues of \( H_D(e, p^1, p^3) \) for the proton. We have \( E_n^{(p)}(p^3)^2 = m_p^2 + (p^3)^2 + 2neB, \ n \geq 0 \).
2.2.1. **Eigenfunctions of the proton for positive eigenvalues.**

For \( n \geq 1 \) \( E_n^{(p)}(p^3) \) is of multiplicity two corresponding to \( s = \pm 1 \) and \( E_0^{(p)}(p^3) \) is multiplicity one corresponding to \( s = 1 \).

Let \( U_{\pm 1}^{(p)}(x^2, n, p^1, p^3) \) denote the eigenfunctions associated with the eigenvalue \( E_n^{(p)}(p^3) \) and with \( s = \pm 1 \).

For \( s = 1 \) and \( n \geq 0 \) we have

\[
U_{+1}^{(p)}(x^2, n, p^1, p^3) = \left( \frac{E_n^{(p)}(p^3) + m_e}{2E_n^{(p)}(p^3)} \right)^{\frac{1}{2}} \begin{pmatrix}
I_n(\tilde{\xi}) \\
0 \\
\frac{m^3}{E_n^{(p)}(p^3) + m_p} I_n(\tilde{\xi}) \\
\frac{\sqrt{2}neB}{E_n^{(p)}(p^3) + m_p} I_{n-1}(\tilde{\xi})
\end{pmatrix}
\]

where

\[
\tilde{\xi} = \sqrt{eB} \left( x^2 + \frac{p^1}{eB} \right)
\]

\[
I_{-1}(\tilde{\xi}) = 0.
\]

For \( s = -1 \) and \( n \geq 1 \) we have

\[
U_{-1}^{(p)}(x^2, n, p^1, p^3) = \left( \frac{E_n^{(p)}(p^3) + m_p}{2E_n^{(p)}(p^3)} \right)^{\frac{1}{2}} \begin{pmatrix}
0 \\
I_n(\tilde{\xi}) \\
\frac{\sqrt{2}neB}{E_n^{(p)}(p^3) + m_p} I_{n+1}(\tilde{\xi}) \\
\frac{m^3}{E_n^{(p)}(p^3) + m_p} I_{n-1}(\tilde{\xi})
\end{pmatrix}
\]

For \( n = 0 \) we set

\[
U_{-1}^{(p)}(x^2, 0, p^1, p^3) = 0.
\]

Note that

\[
\int dx^2 U_s^{(p)}(x^2, n, p^1, p^3) \dagger U_{s'}^{(p)}(x^2, n, p^1, p^3) = \delta_{ss'}
\]

where \( \dagger \) is the adjoint in \( \mathbb{C}^4 \).

2.2.2. **Eigenfunctions of the positron for positive eigenvalues.**

The generalized eigenfunctions for the positron, denoted by \( U_{\pm 1}^{(p_o)}(x^2, n, p^1, p^3) \), are obtained from \( U_{\pm 1}^{(p)}(x^2, n, p^1, p^3) \) by substituting the mass of the electron \( m_e \) for \( m_p \). The associated eigenvalues are denoted by \( E_n^{(p_o)}(p^3) \).
2.2.3. *Eigenfunctions of the proton for negative eigenvalues.*

Let $V_{\pm l}^{(p)}(x^2, n, p^1, p^3)$ denote the eigenfunctions associated with the eigenvalue $-E_{n}^{(p)}(p^3)$ and with $s = \pm 1$.

For $s = 1$ and $n \geq 0$ we have

\begin{equation}
\tag{2.23}
V_{+1}^{(p)}(x^2, n, p^1, p^3) = \left( \frac{E_{n}^{(p)}(p^3) + m_p}{2E_{n}^{(p)}(p^3)} \right)^{\frac{1}{2}} \begin{pmatrix}
\frac{p^3}{E_{n}^{(p)}(p^3) + m_p} I_{n-1}(\tilde{\xi}) \\
- \frac{\sqrt{2neB}}{E_{n}^{(p)}(p^3) + m_p} I_{n-1}(\tilde{\xi}) \\
0 \\
I_{n-1}(\tilde{\xi})
\end{pmatrix}
\end{equation}

For $s = -1$ and $n \geq 1$ we have

\begin{equation}
\tag{2.24}
V_{-1}^{(p)}(x^2, n, p^1, p^3) = \left( \frac{E_{n}^{(p)}(p^3) + m_p}{2E_{n}^{(p)}(p^3)} \right)^{\frac{1}{2}} \begin{pmatrix}
- \frac{\sqrt{2neB}}{E_{n}^{(p)}(p^3) + m_p} I_{n}(\tilde{\xi}) \\
\frac{\sqrt{2neB}}{E_{n}^{(p)}(p^3) + m_p} I_{n-1}(\tilde{\xi}) \\
0 \\
I_{n-1}(\tilde{\xi})
\end{pmatrix}
\end{equation}

and for $n = 0$ we set

\begin{equation}
\tag{2.25}
V_{-1}^{(p)}(x^2, 0, p^1, p^3) = 0
\end{equation}

Note that

\[ \int dx^2 V_{s}^{(p)}(x^2, n, p^1, p^3) V_{s'}^{(p)}(x^2, n, p^1, p^3) = \delta_{ss'} \]

where $\dagger$ is the adjoint in $\mathbb{C}^4$.

The sets $(U_{\pm 1}^{(p)}(\ldots, n, p^1, p^3))_{(n,p^3,p^3)}$ and $(V_{\pm 1}^{(p)}(\ldots, n, p^1, p^3))_{(n,p^3,p^3)}$ of vectors in $L^2(\mathbb{R}, \mathbb{C}^4)$ are an orthonormal basis of $L^2(\mathbb{R}, \mathbb{C}^4)$.

This yields for $\Psi(x)$ in $L^2(\mathbb{R}^3, \mathbb{C}^4)$

\begin{equation}
\tag{2.26}
\Psi(x) = \frac{1}{2\pi} \sum_{s = \pm 1} \text{L.i.m} \left( \sum_{n \geq 0} \int_{\mathbb{R}^2} dp^1 dp^3 e^{ip^1 x^1 + ip^3 x^3} \right)
\begin{pmatrix}
\left(c_{s}^{(p)}(n, p^1, p^3)U_{s}^{(p)}(x^2, n, p^1, p^3) + d_{s}^{(p)}(n, p^1, p^3)V_{s}^{(p)}(x^2, n, p^1, p^3) \right)
\end{pmatrix}
\end{equation}

where $c_{-1}(0, p^1, p^3) = d_{-1}(0, p^1, p^3) = 0$
The complex coefficients \( c_s^{(p)}(n, p^1, p^3) \) and \( d_s^{(p)}(n, p^1, p^3) \) satisfy

\[
\|\Psi(.)\|_{L^2(\mathbb{R}^3, \mathbb{C}^s)}^2 = \sum_{s=\pm 1} \sum_{n \geq 0} \int \left( |c_s^{(p)}(n, p^1, p^3)|^2 + |d_s^{(p)}(n, p^1, p^3)|^2 \right) dp^1 dp^3 < \infty.
\]

We have

\[
\begin{align*}
c_s^{(p)}(n, p^1, p^3) &= \int_{\mathbb{R}} U_s^{(p)}(x^2, n, p^1, p^3) \hat{\Psi}(x^2; p^1, p^3) dx^2 \\
d_s^{(p)}(n, p^1, p^3) &= \int_{\mathbb{R}} V_s^{(p)}(x^2, n, p^1, p^3) \hat{\Psi}(x^2; p^1, p^3) dx^2.
\end{align*}
\]

2.2.4. **Eigenfunctions of the positron for negative eigenvalues.**

The generalized eigenfunctions for the positron, associated with the eigenvalues \(-E_n^{(po)}(p^3)\) and denoted by \( V_{\pm 1}^{(po)}(x^2, n, p^1, p^3) \), are obtained from \( V_{\pm 1}^{(p)}(x^2, n, p^1, p^3) \) by substituting the mass of the electron \( m_e \) for \( m_p \).

2.3. **Fock spaces for electrons, positrons, protons and antiprotons in a uniform magnetic field.**

It follows from section 2.1 and 2.2 that \((s, n, p^1, p^3)\) are the quantum variables for the electrons, the positrons, the protons and the antiprotons in a uniform magnetic field.

Let \( \xi_1 = (s, n, p^1_p, p^3_e) \) be the quantum variables of an electron and of a positron and let \( \xi_2 = (s, n, p^1_p, p^3_p) \) be the quantum variables of a proton and of an antiproton.

We set \( \Gamma_1 = \{-1, 1\} \times \mathbb{N} \times \mathbb{R}^2 \) for the configuration space for both the electrons, the positrons, the protons and the antiprotons. \( L^2(\Gamma_1) \) is the Hilbert space associated to each species of fermions.

We have, by (2.17), (2.18), (2.19), (2.26), (2.27) and (2.28),

\[
L^2(\Gamma_1) = l^2(L^2(\mathbb{R}^2)) \oplus l^2(L^2(\mathbb{R}^2)).
\]

Let \( \mathcal{F}(e) \) and \( \mathcal{F}(po) \) denote the Fock spaces for the electrons and the positrons respectively. \( \mathcal{F}(p) \) and \( \mathcal{F}(ap) \) denote the Fock spaces for the protons and the antiprotons respectively.

We have

\[
\mathcal{F}(e) = \mathcal{F}(po) = \mathcal{F}(p) = \mathcal{F}(ap) = \bigoplus_{n=0}^{\infty} \bigotimes_{a} L^2(\Gamma_1)
\]

\( \bigotimes_{a} L^2(\Gamma_1) \) is the antisymmetric \( n \)-th tensor power of \( L^2(\Gamma_1) \).
\[ \Omega_{(\alpha)} = (1, 0, 0, 0, \ldots) \text{ is the vacuum state in } \mathfrak{H}_{(\alpha)} \text{ for } \alpha = e, po, p, ap. \]

We shall use the notations

\[ \int_{\Gamma_1} d\xi_1 = \sum_{s=\pm 1} \sum_{n \geq 0} \int_{\mathbb{R}^2} dp^3_e \]

\[ \int_{\Gamma_2} d\xi_2 = \sum_{s=\pm 1} \sum_{n \geq 0} \int_{\mathbb{R}^2} dp^3_p. \]

\[ b_\epsilon(\xi_j) \ (\text{resp. } b_\epsilon^*(\xi_j)) \text{ are the annihilation (resp.creation) operators if } \epsilon = + \text{ for the electron when } j = 1 \text{ and for the proton when } j = 2. \]

\[ b_\epsilon(\xi_j) \ (\text{resp. } b_\epsilon^*(\xi_j)) \text{ are the annihilation (resp.creation) operators if } \epsilon = - \text{ for the positron when } j = 1 \text{ and for the antiproton when } j = 2. \]

The operators \( b_\epsilon(\xi_j) \) and \( b_\epsilon^*(\xi_j) \) fulfill the usual anticommutation relations (CAR), see e.g. [25].

In addition, following the convention described in [25, Section 4.1] and [25, Section 4.2], we assume that the fermionic creation and annihilation operators of different species of particles anticommute (see [5] arXiv for explicit definitions). In our case this property will be verified by the creation and annihilation operators for the electrons, the protons, the neutrons, the neutrinos and their respective antiparticles.

Therefore the following anticommutation relations hold for \( j = 1, 2 \)

\[ \{ b_\epsilon(\xi_j), b_\epsilon^*(\xi_{j'}) \} = \delta_{ee'} \delta(\xi_j - \xi_{j'}), \]

\[ \{ b_\epsilon^*(\xi_1), b_\epsilon^*(\xi_2) \} = 0. \]

where \( \{ b, b' \} = bb' + b'b \) and \( b^* = b \) or \( b^* \).

Recall that for \( \varphi \in L^2(\Sigma_1) \), the operators

\[ b_{j,\epsilon}(\varphi) = \int_{\Gamma_1} b_\epsilon(\xi_j) \overline{\varphi(\xi_j)} d\xi_j. \]

\[ b_{j,\epsilon}^*(\varphi) = \int_{\Gamma_1} b_\epsilon^*(\xi_j) \varphi(\xi_j) d\xi_j. \]

are bounded operators on \( \mathfrak{H}_e \) and \( \mathfrak{H}_{po} \) for \( j = 1 \) and on \( \mathfrak{H}_p \) and \( \mathfrak{H}_{ap} \) for \( j = 2 \) satisfying

\[ \| b_{j,\epsilon}^*(\varphi) \| = \| \varphi \|_{L^2}, \]

2.4. Quantized Dirac fields for the electrons and the protons in a uniform magnetic field.

We now consider the canonical quantization of the two classical fields (2.17) and (2.26).
Recall that the charge conjugation operator $C$ is given, for every $\Psi(x)$, by

\[
(2.35) \quad C \begin{pmatrix}
\Psi_1(x) \\
\Psi_2(x) \\
\Psi_3(x) \\
\Psi_4(x)
\end{pmatrix} = \begin{pmatrix}
-\Psi_4^*(x) \\
-\Psi_3^*(x) \\
\Psi_2^*(x) \\
\Psi_1^*(x)
\end{pmatrix}
\]

Here $*$ is the complex conjugation.

Let $\Psi(\cdot)$ be locally in the domain of $H_D(e)$. We have

\[
(2.36) \quad H_D(-e)C\Psi = EC\Psi \quad \text{if} \quad H_D(e)\Psi = -E\Psi
\]

By (2.35) and (2.36) we obtain

\[
(2.37) \quad \begin{align*}
(CV_{+1}^{(e)}) (x^2, n, p_1^1, p_3^3) &= U_{-1}^{(po)} (x^2, n, -p_1^1, -p_3^3) \quad \text{for} \quad n \geq 1. \\
(CV_{-1}^{(e)}) (x^2, n, p_1^1, p_3^3) &= -U_{+1}^{(po)} (x^2, n, -p_1^1, -p_3^3) \quad \text{for} \quad n \geq 0. \\
(CV_{+1}^{(p)}) (x^2, n, p_1^1, p_3^3) &= U_{-1}^{(ap)} (x^2, n, -p_1^1, -p_3^3) \quad \text{for} \quad n \geq 0. \\
(CV_{-1}^{(p)}) (x^2, n, p_1^1, p_3^3) &= -U_{+1}^{(ap)} (x^2, n, -p_1^1, -p_3^3) \quad \text{for} \quad n \geq 1.
\end{align*}
\]

By (2.37) we set

\[
(2.38) \quad \begin{align*}
U^{(e)}(x^2, \xi_1) &= U_s^{(e)}(x^2, n, p_1^1, p_3^3) \quad \text{for} \quad \xi_1 = (s, n, p_1^1, p_3^3), n \geq 0. \\
W^{(e)}(x^2, \xi_1) &= V_{-1}^{(e)}(x^2, n, -p_1^1, -p_3^3) \quad \text{for} \quad \xi_1 = (1, n, p_1^1, p_3^3), n \geq 0. \\
W^{(e)}(x^2, \xi_1) &= V_{+1}^{(e)}(x^2, n, -p_1^1, -p_3^3) \quad \text{for} \quad \xi_1 = (-1, n, p_1^1, p_3^3), n \geq 0. \\
W^{(e)}(x^2, \xi_1) &= 0 \quad \text{for} \quad \xi_1 = (-1, 0, p_1^1, p_3^3).
\end{align*}
\]

By using (2.37) and (2.38) the symmetric of charge canonical quantization of the classical field (2.17) gives the following formal operator associated with the electron and denoted by $\Psi_{(e)}(x)$:

\[
(2.39) \quad \Psi_{(e)}(x) = \frac{1}{2\pi} \int d\xi_1 \left( e^{i(p_1^1 x^1 + p_3^3 x^3)} U^{(e)}(x^2, \xi_1)b_+(\xi_1) + e^{-i(p_1^1 x^1 + p_3^3 x^3)} W^{(e)}(x^2, \xi_1)b^*_-(\xi_1) \right).
\]

For a rigorous approach of the quantization see [11].

We further note that

\[
(2.40) \quad \{ \Psi_{(e)}(x), \Psi_{(e)}(x')^\dagger \} = \delta(x, x')
\]

See [7].
By (2.37) we now set

\[
U^{(p)}(x^2, \xi_2) = U_s^{(p)}(x^2, n, p_{p}^1, p_{p}^3) \text{ for } \xi_2 = (s, n, -p_{p}^1, -p_{p}^3), n \geq 0.
\]

\[
W^{(p)}(x^2, \xi_2) = V^{(p)}_{+1}(x^2, n, -p_{p}^1, -p_{p}^3) \text{ for } \xi_2 = (-1, n, p_{p}^1, p_{p}^3), n \geq 0.
\]

\[
W^{(p)}(x^2, \xi_2) = V^{(p)}_{-1}(x^2, n, -p_{p}^1, -p_{p}^3) \text{ for } \xi_2 = (1, n, p_{p}^1, p_{p}^3), n \geq 1.
\]

\[
W^{(p)}(x^2, \xi_2) = 0 \text{ when } \xi_2 = (1, 0, p_{p}^1, p_{p}^3).
\]

By using (2.37) and (2.41) the symmetric of charge canonical quantization of the classical field (2.26) gives the following formal operator associated to the proton and denoted by \( \Psi^{(p)}(x) \):

\[
\Psi^{(p)}(x) = \frac{1}{2\pi} \int d\xi_2 \left( e^{i(p_{p}^1 x^1 + p_{p}^3 x^3)} U^{(p)}(x^2, \xi_2) b_+(\xi_2) + e^{-i(p_{p}^1 x^1 + p_{p}^3 x^3)} W^{(p)}(x^2, \xi_2) b^*_-(\xi_2) \right).
\]

We further note that

\[
\{ \Psi^{(p)}(x), \Psi^{(p)}(x')^\dagger \} = \delta(x - x')
\]

See [7].

### 3. The Quantization of the Dirac Fields for the Neutrons and the Neutrinos in Helicity Formalism.

As stated in the introduction we neglect the magnetic moment of the neutrons. Therefore neutrons and neutrinos are purely neutral particles without any electromagnetic interaction. We suppose that the neutrinos and antineutrinos are massless as in the Standard Model.

The quantized Dirac fields for free massive and massless particles of spin \( \frac{1}{2} \) are well-known.

In this work we use the helicity formalism, for free particles. See, for example, [24] and [16].

The helicity formalism for particles is associated with a spectral representation of the set of commuting self adjoint operators \( (P, H^3) \). \( P = (P^1, P^2, P^3) \) are the generators of space-translations and \( H^3 \) is the helicity operator \( \frac{1}{2} \frac{P \cdot \Sigma}{|P|} \) where \( |P| = \sqrt{\sum_{i=1}^{3} (P^i)^2} \) and \( \Sigma = (\Sigma^1, \Sigma^2, \Sigma^3) \) with for \( j = 1, 2, 3 \)

\[
\Sigma^j = \begin{pmatrix}
\sigma_j & 0 \\
0 & \sigma_j
\end{pmatrix}
\]
3.1. The quantization of the Dirac field for the neutron in helicity formalism.

The Dirac equation for the neutron of mass $m_{ne}$ is given by

\begin{equation}
H_D = \alpha \cdot \frac{1}{i} \nabla + \beta m_{ne},
\end{equation}

acting in the Hilbert space $L^2(\mathbb{R}^3, \mathbb{C}^4)$.

It follows from the Fourier transform that

\begin{equation}
L^2(\mathbb{R}^3, \mathbb{C}^4) \simeq \int_{\mathbb{R}^3}^\oplus \mathbb{C}^4 d^3p.
\end{equation}

where

\begin{equation}
H_D(p) = \begin{pmatrix}
m_{ne}\sigma_0, & \sigma \cdot p \\
\sigma \cdot p, & -m_{ne}\sigma_0
\end{pmatrix}
\end{equation}

Here $\sigma_0$ is the $2 \times 2$ unit matrix, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and $p = (p^1, p^2, p^3)$ with $\sigma \cdot p = \sum_{j=1}^{3} \sigma_j p^j$.

$H_D(p)$ has two eigenvalues $E^{(ne)}(p)$ and $-E^{(ne)}(p)$ where $E^{(ne)}(p) = \sqrt{|p|^2 + m_{ne}^2}$.

The helicity, denoted by $H_3(p)$, is given by

\begin{equation}
H_3 = \frac{1}{2} \begin{pmatrix}
\frac{\sigma \cdot p}{|p|}, & 0 \\
0, & \frac{\sigma \cdot p}{|p|}
\end{pmatrix}
\end{equation}

$H_3(p)$ commutes with $H_D(p)$ and has two eigenvalues $\frac{1}{2}$ and $-\frac{1}{2}$.

Set (see[24, Appendix.1.F.])

\begin{equation}
h_+(p) = \frac{1}{\sqrt{2|p|(|p|-p^3)}} \begin{pmatrix} p^1 - ip^2 \\ |p| - p^3 \end{pmatrix}
\end{equation}

and

\begin{equation}
h_-(p) = \frac{1}{\sqrt{2|p|(|p|-p^3)}} \begin{pmatrix} p^3 - |p| \\ p^1 + ip^2 \end{pmatrix}
\end{equation}

For $|p| = p^3$ we set

\begin{equation}
h_+(p) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\end{equation}

and

\begin{equation}
h_-(p) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{equation}

We have $(\sigma \cdot p)h_\pm(p) = \pm|p|h_\pm(p)$. 
Let
\[
a_{\pm}(p) = \frac{1}{\sqrt{2}} \left( 1 \pm \frac{m_{ne}}{E^{(ne)}(p)} \right)^{1/2}
\]

The two eigenfunctions of the eigenvalue \(E^{(ne)}(p)\) associated with helicities \(\frac{1}{2}\) and \(-\frac{1}{2}\) are denoted by \(U^{(ne)}(p, \pm \frac{1}{2})\) and are given by
\[
U^{(ne)}(p, \pm \frac{1}{2}) = \left( a_{\pm}(p) h_{\pm}(p), \pm a_{\pm}(p) h_{\pm}(p) \right)
\]

Turning now to the two eigenfunctions for the eigenvalue \(-E^{(ne)}(p)\) we set
\[
\widetilde{h}_{+}(p) = \frac{1}{\sqrt{2|p|(|p| + p^3)}} \left( |p| + p^3 \right)^{1/2} \left( p^1 + ip^2 \right)
\]
\[
\widetilde{h}_{-}(p) = \frac{1}{\sqrt{2|p|(|p| + p^3)}} \left( -p^1 + ip^2 \right)
\]

For \(|p| = -p^3\) we set
\[
\widetilde{h}_{+}(p) = \left( 0, 1 \right)
\]
\[
\widetilde{h}_{-}(p) = \left( 1, 0 \right)
\]

Note there exists a constant \(\lambda(p)\) in \(\mathbb{C}\) such that \(|\lambda(p)| = 1\) and
\[
\widetilde{h}_{+}(p) = \lambda(p) h_{+}(p)
\]
\[
\widetilde{h}_{-}(p) = \lambda(p) h_{-}(p)
\]

where
\[
\lambda(p) = \frac{p^1 - ip^2}{|p|^2 - ip^2}
\]

The two eigenfunctions of the eigenvalue \(-E^{(ne)}(p)\) associated to helicities \(\frac{1}{2}\) and \(-\frac{1}{2}\) are denoted by \(V^{(ne)}(p, \pm \frac{1}{2})\) and are given by
\[
V^{(ne)}(p, \pm \frac{1}{2}) = \left( \mp a_{\pm}(p) \widetilde{h}_{\pm}(p), a_{\pm}(p) \widetilde{h}_{\pm}(p) \right)
\]

The four vectors \(U^{(ne)}(p, \pm \frac{1}{2})\) and \(V^{(ne)}(p, \pm \frac{1}{2})\) are an orthonormal basis of \(\mathbb{C}^4\).

\(U^{(ne)}(p, \pm \frac{1}{2}) e^{i(p \cdot x)}\) and \(V^{(ne)}(p, \pm \frac{1}{2}) e^{i(p \cdot x)}\) is a complete set of generalized eigenfunctions of \(H_D\) with positive and negative eigenvalues \(\pm E^{(ne)}(p)\).

This yields for \(\Psi(x)\) in \(L^2(\mathbb{R}^3, \mathbb{C}^4)\)
\[ \Psi(x) = \left( \frac{1}{2\pi} \right)^{\frac{3}{2}} \sum_{\lambda = \pm \frac{1}{2}} \text{L.i.m.} \left( \int_{\mathbb{R}^3} d^3p e^{i(p \cdot x)} \right) \]

\[ \left( U^{(ne)}(p, \lambda) a(p, \lambda) + V^{(ne)}(p, \lambda) c(p, \lambda) \right). \]

with

\[ \| \Psi(.) \|^2_{L^2(\mathbb{R}^3, C^4)} = \sum_{\lambda = \pm \frac{1}{2}} \int_{\mathbb{R}^3} d^3p (|a(p, \lambda)|^2 + |c(p, \lambda)|^2) < \infty. \]

### 3.1.1. Fock space for the neutrons.

Let \( \xi_3 = (p, \lambda) \) be the quantum variables of a neutron and an antineutron where \( p \in \mathbb{R}^3 \) is the momentum and \( \lambda \in \{-\frac{1}{2}, \frac{1}{2}\} \) is the helicity.

We set \( \Gamma_2 = \mathbb{R}^3 \times \{-\frac{1}{2}, \frac{1}{2}\} \) for the configuration space of the neutron and the antineutron.

Let \( \mathcal{F}_{(ne)} \) and \( \mathcal{F}_{(ane)} \) denote the Fock spaces for the neutrons and the antineutrons respectively.

We have

\[ \mathcal{F}_{(ne)} = \mathcal{F}_{(ane)} = \bigoplus_{n=0}^{\infty} \bigotimes_{a} L^2(\Gamma_2) \]

\( \bigotimes_{a} L^2(\Gamma_2) \) is the antisymmetric \( n \)-th tensor power of \( L^2(\Gamma_2) \).

\( \Omega_{(\beta)} = (1, 0, 0, 0, \ldots) \) is the vacuum state in \( \mathcal{F}_{(\beta)} \) for \( \beta = ne, ane \).

In the sequel we shall use the notations

\[ \int_{\Gamma_2} d\xi_3 = \sum_{\lambda = \pm \frac{1}{2}} \int_{\mathbb{R}^3} d^3p \]

\( b_\epsilon(\xi_3) \) (resp.\( b_\epsilon^*(\xi_3) \)) is the annihilation (resp.creation)operator for the neutron if \( \epsilon = + \) and for the antineutrino if \( \epsilon = - \).

The operators \( b_\epsilon(\xi_3) \) and \( b_\epsilon^*(\xi_3) \) fulfil the usual anticommutation relations (CAR) and they anticommute with \( b_\gamma^*(\xi_j) \) for \( j = 1, 2 \) according to the convention described in [25, Section 4.1]. See [5] arXiv for explicit definitions.
Therefore the following anticommutation relations hold for $j = 1, 2$

\[
\{ b_{j+}(\xi_3), b_{j+}^*(\xi_3') \} = \delta_{ee'}\delta(\xi_3 - \xi_3') ,
\]

\[
\{ b_{j-}(\xi_3), b_{j-}^*(\xi_j) \} = 0 .
\]

(3.17)

Recall that for $\varphi \in L^2(\Gamma_2)$, the operators

\[
b_{3,\epsilon}(\varphi) = \int_{\Gamma_2} b_{\epsilon}(\xi_3)\varphi(\xi_3)d\xi_3 .
\]

\[
b_{3,\epsilon}^*(\varphi) = \int_{\Gamma_2} b_{\epsilon}^*(\xi_3)\varphi(\xi_3)d\xi_3 .
\]

(3.18)

are bounded operators on $\mathfrak{F}(n)$ and $\mathfrak{F}(an)$ satisfying

\[
\|b_{3,\epsilon}^*(\varphi)\| = \|\varphi\|_{L^2} .
\]

(3.19)

3.1.2. **Quantized Dirac Field for the neutron in helicity formalism.**

By (2.35) we get

\[
C(V^{(ne)}(p, \lambda)) = U^{(ne)}(-p, \lambda)
\]

(3.20)

Setting

\[
U^{(ne)}(p, \lambda) = U^{(ne)}(\xi_3)
\]

\[
V^{(ne)}(-p, \lambda) = W^{(ne)}(\xi_3).
\]

(3.21)

and applying the canonical quantization we obtain the following quantized Dirac field for the neutron:

\[
\Psi_{(ne)}(x) = \left(\frac{1}{2\pi}\right)^3 \int d\xi_3 \left(e^{i(p\cdot x)}U^{(ne)}(\xi_3)b_{+}(\xi_3) + e^{-i(p\cdot x)}W^{(ne)}(\xi_3)b_{-}^*(\xi_3)\right)
\]

(3.22)

3.2. **The quantization of the Dirac field for the neutrino.**

Throughout this work we suppose that the neutrinos we consider are those associated with the electrons.

The Dirac equation for the neutrino is given by

\[
H_D = \alpha \cdot \frac{1}{i} \nabla,
\]

(3.23)

acting in the Hilbert space $L^2(\mathbb{R}^3, \mathbb{C}^4)$.

By (3.3) it follows from the Fourier transform that

\[
H_D \simeq \int_{\mathbb{R}^3} H_D(p)d^3p
\]

(3.24)

where
\begin{align}
H_D(p) &= \begin{pmatrix}
0, & \sigma \cdot p \\
\sigma \cdot p, & 0
\end{pmatrix}
\tag{3.25}
\end{align}

\(H_D(p)\) has two eigenvalues \(E^{(\nu)}(p)\) and \(-E^{(\nu)}(p)\) where \(E^{(\nu)}(p) = |p|\).

The helicity given by
\[
\frac{1}{2} \gamma_5 = \frac{1}{2} \begin{pmatrix}
0 & I \\
I & 0
\end{pmatrix}
\]
commutes with \(H_D(p)\) and has two eigenvalues \(\frac{1}{2}\) and \(-\frac{1}{2}\).

The two eigenfunctions of the eigenvalue \(E^{(\nu)}(p)\) associated with helicities \(\frac{1}{2}\) and \(-\frac{1}{2}\) are denoted by \(U^{(\nu)}(p, \pm \frac{1}{2})\). The two eigenfunctions of the eigenvalue \(-E^{(\nu)}(p)\) associated with helicities \(\frac{1}{2}\) and \(-\frac{1}{2}\) are denoted by \(V^{(\nu)}(p, \pm \frac{1}{2})\). They are given by
\begin{align}
U^{(\nu)}(p, \pm \frac{1}{2}) &= \frac{1}{\sqrt{2}} \begin{pmatrix}
h_\pm(p) \\
\pm h_\pm(p)
\end{pmatrix}
\tag{3.26}
\end{align}

\[
V^{(\nu)}(p, \pm \frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix}
\mp \tilde{h}_\pm(p) \\
\tilde{h}_\pm(p)
\end{pmatrix}
\]

The four vectors \(U^{(\nu)}(p, \pm \frac{1}{2})\) and \(V^{(\nu)}(p, \pm \frac{1}{2})\) are an orthonormal basis in \(\mathbb{C}^4\).

In the following, according to the theory of neutrinos and antineutrinos (see [14]) and taking account of the charge conjugation, \(U^{(\nu)}(p, -\frac{1}{2})\) will be the eigenfunction of a neutrino with a momentum \(p\), an energy \(|p|\) and a helicity \(-\frac{1}{2}\). \(-V^{(\nu)}(-p, \frac{1}{2})\) will be the eigenfunctions of an antineutrino with a momentum \(p\), an energy \(|p|\) and a helicity \(\frac{1}{2}\).

Thus the classical field, denoted by \(\Phi(x)\) and associated with the neutrino and the antineutrino, is given by
\begin{align}
\Phi(x) &= \left( \frac{1}{2\pi} \right)^\frac{3}{2} \text{L.i.m.} \left( \int_{\mathbb{R}^3} d^3p \left( e^{i(p \cdot x)} U^{(\nu)}(p, -\frac{1}{2}) a(p, -\frac{1}{2}) - e^{-i(p \cdot x)} V^{(\nu)}(-p, \frac{1}{2}) c(p, \frac{1}{2}) \right) \right),
\tag{3.27}
\end{align}

with
\[
\| \Phi(.) \|_{L^2(\mathbb{R}^3, \mathbb{C}^4)}^2 = \int_{\mathbb{R}^3} d^3p \left( |a(p, -\frac{1}{2})|^2 + |c(p, \frac{1}{2})|^2 \right) < \infty.
\]
3.2.1. **Fock space for the neutrinos and the antineutrinos.**

Let \( \xi_4 = (p, -\frac{1}{2}) \) be the quantum variables of a neutrino where \( p \in \mathbb{R}^3 \) is the momentum and \(-\frac{1}{2}\) is its helicity. In the case of the antineutrino we set \( \tilde{\xi}_4 = (p, \frac{1}{2}) \) where \( p \in \mathbb{R}^3 \) and \( \frac{1}{2} \) is its helicity. Neutrinos are left-handed and antineutrinos are right-handed. See [14].

\( L^2(\mathbb{R}^3) \) is the Hilbert space of the states of the neutrinos and of the antineutrinos.

Let \( F(\nu) \) and \( F(\nu) \) denote the Fock spaces for the neutrinos and the antineutrinos respectively.

We have

\[
F(\nu) = F(\nu) = \bigoplus_{n=0}^{\infty} \bigotimes_n a_{L^2(\mathbb{R}^3)}
\]

\( \bigotimes_n a_{L^2(\mathbb{R}^3)} \) is the antisymmetric \( n \)-th tensor power of \( L^2(\mathbb{R}^3) \).

\( \Omega(\delta) = (1, 0, 0, 0, \ldots) \) is the vacuum state in \( F(\delta) \) for \( \delta = \nu, \bar{\nu} \).

In the sequel we shall use the notations

\[
\int_{\mathbb{R}^3} d\xi_4 = \int_{\mathbb{R}^3} d^3p \\
\int_{\mathbb{R}^3} d\tilde{\xi}_4 = \int_{\mathbb{R}^3} d^3p
\]

\( b_+ (\xi_4) \) (resp.\( b^*_+ (\xi_4) \)) is the annihilation (resp.creation) operators for the neutrino and \( b_- (\tilde{\xi}_4) \) (resp.\( b^*_- (\tilde{\xi}_4) \)) is the annihilation (resp.creation) operators for the antineutrino.

The operators \( b_+ (\xi_4) \), \( b^*_+ (\xi_4) \), \( b_- (\tilde{\xi}_4) \) and \( b^*_- (\tilde{\xi}_4) \) fulfil the usual anticommutation relations (CAR) and they anticommute with \( b^j_\epsilon (\xi_j) \) for \( j = 1, 2, 3 \) according the convention described in [25, Section 4.1]. See [5] arXiv for explicit definitions.

Therefore the following anticommutation relations hold for \( j = 1, 2, 3 \)

\[
\{ b_+ (\xi_4), b^*_+ (\xi'_4) \} = \delta (\xi_4 - \xi'_4) , \\
\{ b_- (\tilde{\xi}_4), b^*_- (\tilde{\xi}'_4) \} = \delta (\tilde{\xi}_4 - \tilde{\xi}'_4) , \\
\{ b^j_\epsilon (\xi_4), b^j_- (\xi'_4) \} = 0 , \\
\{ b^j_\epsilon (\xi_4), b^j_\epsilon (\xi_j) \} = \{ b^j_- (\tilde{\xi}_4), b^j_- (\xi_j) \} = 0.
\]
Recall that for $\varphi \in L^2(\mathbb{R}^3)$, the operators
\begin{align}
b_{4,+}(\varphi) &= \int_{\mathbb{R}^3} b_+(\xi_4)\varphi(\xi_4)d\xi_4. \\
b_{4,-}(\varphi) &= \int_{\mathbb{R}^3} b_-(\xi_4)\varphi(\xi_4)d\xi_4. \\
b_{4,+}^*(\varphi) &= \int_{\mathbb{R}^3} b_+^*(\xi_4)\varphi(\xi_4)d\xi_4. \\
b_{4,-}^*(\varphi) &= \int_{\mathbb{R}^3} b_-^*(\xi_4)\varphi(\xi_4)d\xi_4.
\end{align}
(3.31)

are bounded operators on $\mathfrak{F}(\nu)$ and $\mathfrak{F}(\overline{\nu})$ satisfying
\begin{align}
||b_{4,\epsilon}^*(\varphi)|| &= ||\varphi||_{L^2}.
\end{align}
(3.32)

where $\epsilon = \pm$.

3.2.2. **Quantized Dirac Field for the neutrino.**

$e^{i(p \cdot x)}U^{(\nu)}(p, -\frac{1}{2})$ and $e^{i(p \cdot x)}V^{(\nu)}(p, \frac{1}{2})$ are generalized eigenfunctions of $H_D$ for neutrinos with positive and negative eigenvalues $\pm E^{(\nu)}(p)$ respectively.

By (2.35) we get
\begin{align}
C(V^{(\nu)}(p, \frac{1}{2})) &= U^{(\nu)}(-p, \frac{1}{2}).
\end{align}
(3.33)

Setting
\begin{align}
U^{(\nu)}(p, -\frac{1}{2}) &= U^{(\nu)}(\xi_4) \\
V^{(\nu)}(-p, \frac{1}{2}) &= W^{(\overline{\nu})}(\tilde{\xi}_4).
\end{align}
(3.34)

and applying the canonical quantization we obtain the following quantized Dirac field for the neutrino:
\begin{align}
\Psi^{(\nu)}(x) &= \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \left(\int d\xi_4e^{i(p \cdot x)}U^{(\nu)}(\xi_4)b_+(\xi_4) \\
&\quad + \int d\tilde{\xi}_4e^{-i(p \cdot x)}W^{(\overline{\nu})}(\tilde{\xi}_4)b_+^*(\tilde{\xi}_4)\right).
\end{align}
(3.35)

4. **The Hamiltonian of the Model.**

The processes (1.1) and (1.2) are associated with the $\beta$ decay of the neutron (see [14],[15] and [26]).

The $\beta$ decay process can be described by the well known four-fermion effective Hamiltonian for the interaction (see [15, 6.93])
WEAK INTERACTIONS IN AN UNIFORM MAGNETIC FIELD

(4.1) 
\[ H_{\text{int}} = \] 
\[ \frac{\tilde{G}}{\sqrt{2}} \int d^3x \left( \bar{\Psi}_{(p)}(x) \gamma^a (1 - g_A \gamma_5) \Psi_{(ne)}(x) \right) \left( \bar{\Psi}_{(e)}(x) \gamma^a \Psi_{(\nu)}(x) \right) \] 
\[ + \frac{\tilde{G}}{\sqrt{2}} \int d^3x \left( \bar{\Psi}_{(\nu)}(x) \gamma^a (1 - \gamma_5) \Psi_{(e)}(x) \right) \left( \bar{\Psi}_{(ne)}(x) \gamma^a (1 - g_A \gamma_5) \Psi_{(p)}(x) \right) \]

Here \( \gamma^a, \alpha = 0, 1, 2, 3 \) and \( \gamma_5 \) are the Dirac matrices in the standard representation. \( \Psi_{(\cdot)}(x) \) and \( \bar{\Psi}_{(\cdot)}(x) \) are the quantized Dirac fields for \( p, n, e \) and \( \nu \). \( \Psi_{(\cdot)}(x) = \Psi_{(\cdot)}(x) \gamma^0 \). \( \tilde{G} = G_F \cos \theta_c \), where \( G_F \) is the Fermi coupling constant with \( G_F \approx 1.16639(2) \times 10^{-5} \text{GeV}^{-2} \) and \( \theta_c \) is the Cabbibo angle with \( \cos \theta_c \approx 0.9751 \). Moreover \( g_A \approx 1.27 \). See [6].

The neutrino \( \nu \) is the neutrino associated to the electron and usually denoted by \( \nu_e \).

From now on we restrict ourselves to the study of processes (1.1). Antineutrons and antineutrinos will not be involved in our model.

4.1. The free Hamiltonian.

We set

\[ \mathfrak{F}^{(e)} = \mathfrak{F}_{(e)} \otimes \mathfrak{F}_{(po)} \]  
\[ \mathfrak{F}^{(p)} = \mathfrak{F}_{(p)} \otimes \mathfrak{F}_{(ap)} \]  
\[ \mathfrak{F}^{(ne)} = \mathfrak{F}_{(ne)} \]  
\[ \mathfrak{F}^{(\nu)} = \mathfrak{F}_{(\nu)} \]  
\[ \mathfrak{F} = \mathfrak{F}^{(e)} \otimes \mathfrak{F}^{(p)} \otimes \mathfrak{F}^{(ne)} \otimes \mathfrak{F}^{(\nu)} \]  

(4.2)

We set

\[ \omega(\xi_1) = E_n^{(e)}(p^3) \quad \text{for } \xi_1 = (s, n, p^1, p^3) \]  
\[ \omega(\xi_2) = E_n^{(p)}(p^3) \quad \text{for } \xi_2 = (s, n, p^1, p^3) \]  
\[ \omega(\xi_3) = \sqrt{|p|^2 + m_{ne}^2} \quad \text{for } \xi_3 = (p, \lambda) \]  
\[ \omega(\xi_4) = |p| \quad \text{for } \xi_4 = (p, -\frac{1}{2}) \]  

(4.3)

Let \( H_D^{(e)} \) (resp.\( H_D^{(p)}, H_D^{(ne)} \) and \( H_D^{(\nu)} \) ) the Dirac Hamiltonian for the electron (resp. the proton, the neutron and the neutrino).
The quantization of $H_{0,D}^{(e)}$, denoted by $H_{0,D}^{(e)}$ and acting on $\mathcal{F}^{(e)}$, is given by

$$H_{0,D}^{(e)} = \sum_{\epsilon = \pm} \int \omega(\xi_1) b_\epsilon^*(\xi_1) b_\epsilon(\xi_1) d\xi_1$$

Likewise the quantization of $H_{0,D}^{(p)}, H_{0,D}^{(ne)}$ and $H_{0,D}^{(v)}$, denoted by $H_{0,D}^{(p)}, H_{0,D}^{(ne)}$ and $H_{0,D}^{(v)}$ respectively, acting on $\mathcal{F}^{(p)}, \mathcal{F}^{(ne)}$ and $\mathcal{F}^{(v)}$ respectively, is given by

$$H_{0,D}^{(p)} = \sum_{\epsilon = \pm} \int \omega(\xi_2) b_\epsilon^*(\xi_2) b_\epsilon(\xi_2) d\xi_2$$

$$H_{0,D}^{(ne)} = \int \omega(\xi_3) b_+^*(\xi_3) b_+(\xi_3) d\xi_3$$

$$H_{0,D}^{(v)} = \int \omega(\xi_4) b_+^*(\xi_4) b_+(\xi_4) d\xi_4.$$
The free Hamiltonian for our model, denoted by $H_0$ and acting on $\mathfrak{F}$, is now given by

\[
H_0 = H^{(e)}_{0,D} \otimes 1 \otimes 1 \otimes 1 + 1 \otimes H^{(p)}_{0,D} \otimes 1 \otimes 1 + 1 \otimes 1 \otimes H^{(ne)}_{0,D} + 1 \otimes 1 \otimes 1 \otimes H^{(v)}_{0,D}.
\]

(4.9)

$H_0$ is essentially self-adjoint on $\mathcal{D} = \mathcal{D}^{(e)} \hat{\otimes} \mathcal{D}^{(p)} \hat{\otimes} \mathcal{D}^{(ne)} \hat{\otimes} \mathcal{D}^{(v)}$. Here $\hat{\otimes}$ is the algebraic tensor product.

$\text{spec}(H_0) = [0, \infty)$ and $\Omega$ is the eigenvector associated with the eigenvalue $\{0\}$ of $H_0$.

Let $S^e$ be the set of the thresholds of $H^{(e)}_{0,D}$:

$$S^{(e)} = \{ s^{(e)}_n ; n \in \mathbb{N} \}$$

with $s^{(e)}_n = \sqrt{m^2_e + 2neB}$.

Likewise let $S^p$ be the set of the thresholds of $H^{(p)}_{0,D}$:

$$S^{(p)} = \{ s^{(p)}_n ; n \in \mathbb{N} \}$$

with $s^{(p)}_n = \sqrt{m^2_p + 2neB}$.

Let $S^{ne}$ be the set of the thresholds of $H^{(ne)}_{0,D}$:

$$S^{(ne)} = \{ nm_{ne} ; n \in \mathbb{N}, \text{such that } n \geq 1 \}$$

Then

\[
\mathfrak{G} = S^{(e)} \cup S^{(p)} \cup S^{(ne)}
\]

(4.10)

is the set of the thresholds of $H_0$.

4.2. The Interaction.

By (4.1) let us now write down the formal interaction, denoted by $V_I$, involving the protons, the neutrons, the electrons and the neutrinos associated to the electrons together with their antiparticles in the Schrödinger representation for the process (1.1). We have

\[
V_I = V^{(1)}_I + V^{(2)}_I + V^{(3)}_I + V^{(2)}_I
\]

(4.11)

Set

\[
q = p_e + p_p
\]

(4.12)

\[
r = p_{ne} + p_\nu.
\]

After the integration with respect to $(x^1, x^3) V_I$ is given by
not be an eigenvector of the total Hamiltonian as expected in Physics. 

due to the $\delta U_{\text{smoother kernels}}$ in the Schrödinger representation. 

$\delta U_{\text{distributions and the ultraviolet cutoffs }}$. 

In the Fock space $V_{\text{distributions that occur in the (V(\nu))}}$. 

In order to get well defined operators in $\mathfrak{F}$ we have to substitute smoother kernels $F^{(\beta)}(\xi_2, \xi_3)$, $G^{(\beta)}(\xi_1, \xi_4)$, where $\beta = 1, 2$, both for the $\delta$-distributions and the ultraviolet cutoffs. 

We then obtain a new operator denoted by $H_1$ and defined as follows in the Schrödinger representation.

$$H_1 = H_1^{(1)} + H_1^{(2)} + H_1^{(3)} + H_1^{(4)}$$
with

\[ H_I^{(1)} = \int d\xi_1 d\xi_2 d\xi_3 d\xi_4 \left( \int dx^2 e^{ix^2_2r^2} \right) \left( \overline{U^{(p)}}(x^2, \xi_2)\gamma^\alpha(1 - g_A\gamma_5)U^{(ne)}(\xi_3) \right) \left( \overline{U^{(e)}}(x^2, \xi_1)\gamma_\alpha(1 - \gamma_5)U^{(ne)}(\xi_4) \right) \]

\[ F^{(1)}(\xi_2, \xi_3) G^{(1)}(\xi_1, \xi_4) b_+^*(\xi_1) b_+^*(\xi_2) b_+(\xi_3) b_+(\xi_4). \]

\[ H_I^{(2)} = \int d\xi_1 d\xi_2 d\xi_3 d\xi_4 \left( \int dx^2 e^{-ix^2_2r^2} \right) \left( \overline{U^{(ne)}}(\xi_4)\gamma_\alpha(1 - \gamma_5)U^{(e)}(x^2, \xi_1) \right) \left( \overline{U^{(ne)}}(\xi_3)\gamma^\alpha(1 - g_A\gamma_5)U^{(p)}(x^2, \xi_2) \right) \]

\[ F^{(1)}(\xi_2, \xi_3) G^{(1)}(\xi_1, \xi_4) b_+^*(\xi_1) b_+^*(\xi_2) b_+(\xi_3) b_+(\xi_4). \]

\[ H_I^{(3)} = \int d\xi_1 d\xi_2 d\xi_3 d\xi_4 \left( \int dx^2 e^{-ix^2_2r^2} \right) \left( \overline{U^{(ne)}}(\xi_4)\gamma_\alpha(1 - \gamma_5)W^{(e)}(x^2, \xi_1) \right) \left( \overline{U^{(ne)}}(\xi_3)\gamma^\alpha(1 - g_A\gamma_5)W^{(p)}(x^2, \xi_2) \right) \]

\[ F^{(2)}(\xi_2, \xi_3) G^{(2)}(\xi_1, \xi_4) b_+^*(\xi_1) b_+^*(\xi_2) b_+(\xi_3) b_+(\xi_4). \]

\[ H_I^{(4)} = \int d\xi_1 d\xi_2 d\xi_3 d\xi_4 \left( \int dx^2 e^{ix^2_2r^2} \right) \left( \overline{W^{(p)}}(x^2, \xi_2)\gamma^\alpha(1 - g_A\gamma_5)U^{(ne)}(\xi_3) \right) \left( \overline{W^{(e)}}(x^2, \xi_1)\gamma_\alpha(1 - \gamma_5)U^{(ne)}(\xi_4) \right) \]

\[ F^{(2)}(\xi_2, \xi_3) G^{(2)}(\xi_1, \xi_4) b_+(\xi_1) b_+(\xi_2) b_+(\xi_3) b_+(\xi_4). \]

The total Hamiltonian is then

\[ H = H_0 + gH_I \]

Here \( g \) is the real coupling constant.

We now give the hypothesis that the kernels \( F^{(1)}(\xi_1, \xi_2), G^{(1)}(\xi_1, \xi_2), F^{(2)}(\xi_1, \xi_2), G^{(2)}(\xi_1, \xi_2), \) \( \beta = 1, 2 \), and the coupling constant \( g \) have to satisfy in order to associate with the formal operator \( \hat{H} \) a well defined self-adjoint operator in \( \mathfrak{F} \).

Throughout this work we assume the following hypothesis.
Hypothesis 4.1. For $\beta = 1, 2$ we assume

\[ F^{(\beta)}(\xi_2, \xi_3) \in L^2(\Gamma_1 \times \Gamma_2) \]
\[ G^{(\beta)}(\xi_1, \xi_4) \in L^2(\Gamma_1 \times \mathbb{R}^3) \]

Let $\langle ., . \rangle_{\mathbb{C}^4}$ be the scalar product in $\mathbb{C}^4$. We have

\[ U^{(p)}(x^2, \xi_2)\gamma^\alpha(1 - g_A\gamma_5)U^{(ne)}(\xi_3) = \langle U^{(p)}(x^2, \xi_2), \gamma^\alpha(1 - g_A\gamma_5)U^{(ne)}(\xi_3) \rangle_{\mathbb{C}^4} \]
\[ U^{(e)}(x^2, \xi_1)\gamma^\alpha(1 - \gamma_5)U^{(ne)}(\xi_4) = \langle U^{(e)}(x^2, \xi_1), \gamma^\alpha(1 - \gamma_5)U^{(ne)}(\xi_4) \rangle_{\mathbb{C}^4} \]
\[ U^{(ne)}(\xi_3)\gamma^\alpha(1 - g_A\gamma_5)W^{(p)}(x^2, \xi_2) = \langle U^{(ne)}(\xi_3), \gamma^\alpha(1 - g_A\gamma_5)W^{(p)}(x^2, \xi_2) \rangle_{\mathbb{C}^4} \]

Set

\[ C_0 = \frac{1}{m_e} \left( \sup_{\alpha} \| \gamma^\alpha(1 - g_A\gamma_5) \| \right) \left( \sup_{\alpha} \| \gamma^\alpha(1 - \gamma_5) \| \right) \]

We then have

Proposition 4.2. For every $\Phi \in D(H_0)$ we obtain

\[ \| H_{I}^{(j)} \Phi \| \leq C_0 \| F^{(1)}(., .) \|_{L^2} \| G^{(1)}(., .) \|_{L^2} \| (H_0 + m_{ne}) \Phi \| \]

for $j = 1, 2$.

\[ \| H_{I}^{(j)} \Phi \| \leq C_0 \| F^{(2)}(., .) \|_{L^2} \| G^{(2)}(., .) \|_{L^2} \| (H_0 + m_{ne}) \Phi \| \]

for $j = 3, 4$.

By (4.23),(4.24) and (4.25) the estimates (4.26) are examples of $N_x$ estimates(see [13]). The proof is similar to the one of [4, Proposition 3.7] and details are omitted.

Let $g_0 > 0$ be such that

\[ 2g_0C_0 \left( \sum_{\beta=1}^{2} \| F^{(\beta)}(., .) \|_{L^2} \| G^{(\beta)}(., .) \|_{L^2} \right) < 1 \]

We now have

Theorem 4.3. For any $g$ such that $|g| \leq g_0$, $H$ is a self-adjoint operator in $\mathfrak{F}$ with domain $D(H) = D(H_0)$ and is bounded from below. $H$ is essentially self-adjoint on any core of $H_0$.

Setting

\[ E = \inf \sigma(H) \]

we have for every $|g| \leq g_0$

\[ \sigma(H) = \sigma_{ess}(H) = [E, \infty) \]

with $E \leq 0$. 

Here $\sigma(H)$ is the spectrum of $H$ and $\sigma_{\text{ess}}(H)$ is the essential spectrum of $H$.

**Proof.** By Proposition 4.2 and (4.27) the proof of the self-adjointness of $H$ follows from the Kato-Rellich theorem.

We turn now to the essential spectrum. The result about the essential spectrum in the case of models involving bosons has been obtained by [10, theorem 4.1] and [2]. In the case of models involving fermions the result has been obtained by [23]. In our case involving only massive fermions and massless neutrinos we use the proof given by [23].

Thus we have to construct a Weyl sequence for $H$ and $E + \lambda$ with $\lambda > 0$.

Let $T$ be the self-adjoint multiplication operator in $L^2(\mathbb{R}^3)$ defined by $Tu(p_4) = |p_4|u(p_4)$. $T$ is the spectral representation of $H_{\nu}^{(\nu)}$ for the neutrinos of helicity $-\frac{1}{2}$ in the configuration space $L^2(\mathbb{R}^3)$. See (3.27).

Every $\lambda > 0$ belongs to the essential spectrum of $T$. Then there exists a Weyl sequence $(f_n)_{n \geq 1}$ for $T$ and $\lambda > 0$ such that

\begin{align*}
  f_n &\in D(T) \text{ for } n \geq 1. \\
  \|f_n\| &\leq 1 \text{ for } n \geq 1. \\
  w - \lim_{n \to \infty} f_n &= 0. \\
  \lim_{n \to \infty} (T - \lambda)f_n &= 0. \\
\end{align*}

(4.28)

Let

\begin{align*}
  f_n(\xi_4) &= f_n(p_4) \\
  b^+_{r,4}(f_n) &= \int b_+(\xi_4)\overline{f_n(\xi_4)}d\xi_4 \\
  b^*_{r,4}(f_n) &= \int b^*_+(\xi_4)f_n(\xi_4)d\xi_4. \\
\end{align*}

(4.29)

In the following we identify $b^*_{r,4}(f_n)$ with its obvious extension to $\mathcal{F}$. An easy computation shows that, for every $\Psi \in D(H)$,
\[
\left[ H_I^{(1)}, b^*_+(f_n) \right] \Psi = \\
\int \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \left( \int \mathrm{d}x^2 e^{-ix^2r^2} \right) \\
\left( U(p)(x^2, \xi_2) \gamma_\alpha (1 - g_A \gamma_5) U^{(ne)}(\xi_3) \right) F^{(1)}(\xi_2, \xi_3) \\
\left\langle d_n(x) G^{(1)}(\xi_1, \xi_4) U(p)(\xi_4) U^{(ne)}(\xi_4) \mathrm{d}\xi_4 \right\rangle C^4 \\
b_+^*(\xi_1) b^*_+ (\xi_2) b_+ (\xi_3) \Psi.
\]

(4.31) \quad \left[ H_I^{(1)}, b_+(f_n) \right] \Psi = 0

(4.32) \quad \left[ H_I^{(2)}, b_+(f_n) \right] \Psi = \\
- \int \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \left( \int \mathrm{d}x^2 e^{-ix^2r^2} \right) \\
\left( U^{(ne)}(\xi_3) \gamma_\alpha (1 - g_A \gamma_5) U(p)(x^2, \xi_2) \right) F^{(1)}(\xi_2, \xi_3) \\
\left\langle d_n(x) G^{(1)}(\xi_1, \xi_4) U^{(p)}(\xi_4) U^{(ne)}(\xi_4) \mathrm{d}\xi_4, \gamma_0 \gamma_\alpha (1 - g_A \gamma_5) U^{(c)}(x^2, \xi_1) \right\rangle C^4 \\
b_+^*(\xi_3) b_+ (\xi_2) b_+ (\xi_1) \Psi.

(4.33) \quad \left[ H_I^{(2)}, b^*_+(f_n) \right] \Psi = 0

(4.34) \quad \left[ H_I^{(3)}, b_+(f_n) \right] \Psi = \\
- \int \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \left( \int \mathrm{d}x^2 e^{-ix^2r^2} \right) \\
\left( U^{(ne)}(\xi_3) \gamma_\alpha (1 - g_A \gamma_5) W^{(p)}(x^2, \xi_2) \right) F^{(2)}(\xi_2, \xi_3) \\
\left\langle d_n(x) G^{(2)}(\xi_1, \xi_4) U^{(p)}(\xi_4) U^{(ne)}(\xi_4) \mathrm{d}\xi_4, \gamma_0 \gamma_\alpha (1 - g_A \gamma_5) W^{(c)}(x^2, \xi_1) \right\rangle C^4 \\
b_+^*(\xi_3) b^-_+ (\xi_2) b^-_+ (\xi_1) \Psi.
we get

\[ \left[ H_I^{(3)}, b_{+,4}^*(f_n) \right] \Psi = 0 \]

(4.36)

\[ \left[ H_I^{(4)}, b_{+,4}^*(f_n) \right] \Psi = \int d\xi_1 d\xi_2 d\xi_3 \left( \int d^2 x e^{-ix^2 r^2} \frac{W(\nu) (x^2, \xi_2) \gamma_\alpha (1 - g_A \gamma_5) U^{(\nu)} (\xi_3)}{\xi_2 \xi_3} \right) \]

\[ \left\langle W(\nu) (x^2, \xi_1), \gamma^0 \gamma_\alpha (1 - \gamma_5) \left( \int f_n (\xi_4) G(2) (\xi_1, \xi_4) U^{(\nu)} (\xi_4) d\xi_4 \right) \right\rangle_{\xi_4} \]

\[ b_+ (\xi_3) b_- (\xi_2) b_- (\xi_1) \Psi. \]

(4.37)

\[ \left[ H_I^{(4)}, b_{+,4} (f_n) \right] \Psi = 0 \]

Let \( P_H(.) \) be the spectral measure of \( H \). For any \( \epsilon > 0 \) the orthogonal projection \( P_H([E, E + \epsilon]) \) is different from zero because \( E \) belongs to \( \sigma(H) \).

Let \( \Phi \in \text{Ran}(P_H([E, E + \epsilon])) \) such that \( \| \Phi \| = 1 \). We set

\[ \Psi_{n, \epsilon} = (b_{+,4} (f_n) + b_{+,4}^* (f_n)) \Phi, \quad n \geq 1 \]

Let us show that there exists a subsequence of \( (\Psi_{n, \epsilon})_{n \geq 1, \epsilon > 0} \) which is a Weyl sequence for \( H \) and \( E + \lambda \) with \( \lambda > 0 \).

By Hypothesis 4.1, (4.30), (4.32), (4.34), (4.36) and the \( N_\tau \) estimates we get

(4.39)

\[
\sup \left( \left\| \left[ H_I^{(3)}, b_{+,4} (f_n) \right] \Psi \right\|, \left\| \left[ H_I^{(4)}, b_{+,4} (f_n) \right] \Psi \right\| \right) \\
\leq C_0 \| F^{(1)}(\epsilon) \|_{L^2(\Gamma_1, \Gamma_2)} \left( \int \left\| \int f_n (\xi_4) G^{(1)} (\xi_1, \xi_4) U^{(\nu)} (\xi_4) d\xi_4 \right\|_{C^4}^2 d\xi_1 \right)^{\frac{1}{2}} \| (H_0 + m_{ne})^{\frac{1}{2}} \Psi \|.
\]

\[
\sup \left( \left\| \left[ H_I^{(3)}, b_{+,4} (f_n) \right] \Psi \right\|, \left\| \left[ H_I^{(4)}, b_{+,4}^* (f_n) \right] \Psi \right\| \right) \\
\leq C_0 \| F^{(2)}(\epsilon) \|_{L^2(\Gamma_1, \Gamma_2)} \left( \int \left\| \int f_n (\xi_4) G^{(2)} (\xi_1, \xi_4) U^{(\nu)} (\xi_4) d\xi_4 \right\|_{C^4}^2 d\xi_1 \right)^{\frac{1}{2}} \| (H_0 + m_{ne})^{\frac{1}{2}} \Psi \|.
\]
Note that
\begin{equation}
\|\Psi_{n,\epsilon}\| = 1, \ n \geq 1
\end{equation}
Setting $T_{\nu} = T$ we have for every $\Psi \in D(H)$
\begin{equation}
\begin{aligned}
\left( H\Psi, \Psi_{n,\epsilon} \right) &= \\
\left( \Psi, (b_{+,4}(f_n) + b_{+,4}^*(f_n))H\Phi_\epsilon + (b_{+,4}^*(T f_n) - (b_{+,4}(T f_n))\Phi_\epsilon \\
+ g[H_I, (b_{+,4}(f_n) + b_{+,4}^*(f_n))]\Psi_\epsilon \right).
\end{aligned}
\end{equation}

See [10].

This yields
\begin{equation}
H\Psi_{n,\epsilon} =
\begin{aligned}
(b_{+,4}(f_n) + b_{+,4}^*(f_n))H\Phi_\epsilon + (b_{+,4}^*(T f_n) - (b_{+,4}(T f_n))\Phi_\epsilon \\
+ g[H_I, (b_{+,4}(f_n) + b_{+,4}^*(f_n))]\Psi_\epsilon
\end{aligned}
\end{equation}

and
\begin{equation}
\begin{aligned}
(H - E - \lambda)\Psi_{n,\epsilon} = \\
(b_{+,4}(f_n) + b_{+,4}^*(f_n))(H - E)\Psi_\epsilon \\
+ (b_{+,4}((T + \lambda)f_n) + b_{+,4}^*((T - \lambda)f_n))\Psi_\epsilon \\
+ g[H_I, (b_{+,4}(f_n) + b_{+,4}^*(f_n))]\Psi_\epsilon.
\end{aligned}
\end{equation}

By (3.19) this yields for $|g| \leq g_0$
\begin{equation}
\begin{aligned}
\| (H - E - \lambda)\Psi_{n,\epsilon} \| &\leq \\
2\epsilon + 2|\lambda||b_{+,4}(f_n)\Psi_\epsilon| + 2\|((T - \lambda)f_n)\| \\
+ |g||[H_I, b_{+,4}(f_n)]\Psi_\epsilon| + |g||[H_I, b_{+,4}^*(f_n)]\Psi_\epsilon|
\end{aligned}
\end{equation}

Let $\{g_k|k = 1, 2, 3, ....\}$ be an orthonormal basis of $L^2(\mathbb{R}^3)$ and consider
\begin{equation}
b_{+,4}(g_{k_1})b_{+,4}^*(g_{k_2})b_{+,4}(g_{k_3})......b_{+,4}^*(g_{k_m})\Omega_\nu \in \mathfrak{F}(\nu)
\end{equation}

where the indices can be assumed ordered $k_1 < .... < k_m$. Fock space vectors of this type form a basis of $\mathfrak{F}(\nu)$ (see [24]). By [23, Lemma 2.1] this yields for every $\epsilon > 0$
\begin{equation}
s - \lim_{n \rightarrow \infty} b_{+,4}(f_n)\Psi_\epsilon = 0,
\end{equation}
\begin{equation}
w - \lim_{n \rightarrow \infty} b_{+,4}^*(f_n)\Psi_\epsilon = 0.
\end{equation}
By (3.26) and Hypothesis 4.1 we have

\[
\lim_{n \to \infty} \left( \int \left| \int f_n(\xi_4)G^{(1)}(\xi_1, \xi_4)U^{(\nu)}(\xi_4)d\xi_4 \right|^2 d\xi_1 \right)^{\frac{1}{2}} = 0
\]

\[
\lim_{n \to \infty} \left( \int \left| \int f_n(\xi_4)G^{(2)}(\xi_1, \xi_4)U^{(\nu)}(\xi_4)d\xi_4 \right|^2 d\xi_1 \right)^{\frac{1}{2}} = 0.
\]

(4.47)

It follows from (4.28), (4.39), (4.44), (4.46) and (4.47) that for every \( \epsilon > 0 \)

\[
\limsup_{n \to \infty} \|(H - E - \lambda)\Psi_{n, \epsilon}\| \leq 2\epsilon
\]

This yields

\[
\lim_{\epsilon \to 0} \limsup_{n \to \infty} \|(H - E - \lambda)\Psi_{n, \epsilon}\| = 0
\]

(4.49)

In view of (4.49) there exists a subsequence \((\Psi_{n_j, \epsilon_j})_{j \geq 1}\) such that

\[
\lim_{j \to \infty} \|(H - E - \lambda)\Psi_{n_j, \epsilon_j}\| = 0
\]

(4.50)

Furthermore it follows from (4.46) that \( w - \lim_{j \to \infty} \Psi_{n_j, \epsilon_j} = 0 \).

The sequence \((\Psi_{n_j, \epsilon_j})_{j \geq 1}\) is a Weyl sequence for \( H \) and \( E + \lambda \) with \( \lambda > 0 \).

This concludes the proof of theorem 4.3.

\[
\square
\]

5. Existence of a ground state for the Hamiltonian \( H \).

Set

\[
K(F, G) = \sum_{\beta=1}^{2} \|F^{(\beta)}(\cdot, \cdot)\|_{L^2} \|G^{(\beta)}(\cdot, \cdot)\|_{L^2}.
\]

(5.1)

\[
C = 2C_0.
\]

\[
B = 2m_e C_0.
\]

By (4.26) and (5.1) we get for every \( \psi \in D(H) \)

\[
\|H_I \psi\| \leq K(F, G) (C\|H_0 \psi\| + B\|\psi\|)
\]

(5.2)

In order to prove the existence of a ground state for the Hamiltonian \( H \) we shall make the following additional assumptions on the kernels \( G^{(\beta)}(\xi_1, \xi_4), \beta = 1, 2, \).

From now on \( p_4 \) is the momentum of the neutrino with helicity \(-1/2\).
Hypothesis 5.1. There exists a constant $\tilde{K}(G) > 0$ such that for $\beta = 1,2$ and $\sigma > 0$

\[ (i) \int_{\Gamma_1 \times \mathbb{R}^3} \frac{|G^{(\beta)}(\xi_1, \xi_4)|^2}{|p_4|^2} \, d\xi_1 d\xi_4 < \infty, \]

\[ (ii) \left( \int_{\Gamma_1 \times \{|p_4| \leq \sigma\}} |G^{(\beta)}(\xi_1, \xi_4)|^2 \, d\xi_1 d\xi_4 \right)^{\frac{1}{2}} \leq \tilde{K}(G) \sigma. \]

We have

Theorem 5.2. Assume that the kernels $F^{(\beta)}(\cdot, \cdot)$ and $G^{(\beta)}(\cdot, \cdot)$, $\beta = 1,2$, satisfy Hypothesis 4.1 and Hypothesis 5.1. Then there exists $g_1 \in (0, g_0]$ such that $H$ has a ground state for $g \leq g_1$.

5.0.1. Spectrum of the Hamiltonians with infrared cutoffs. In order to prove theorem 5.2 we first need to get an important result about the spectrum of the Hamiltonians with infrared cutoffs.

Let us first define the cutoff operators which are the Hamiltonians with infrared cutoff with respect to the momentum of the massless fermion.

For that purpose, let $\chi_0(\cdot) \in C^\infty(\mathbb{R}, [0,1])$ with $\chi_0 = 1$ on $(-\infty, 1]$ and $\chi_0 = 0$ on $[2, \infty]$. For $\sigma > 0$ and $p_4 \in \mathbb{R}^3$, we set

\[
\chi_\sigma(p_4) = \chi_0(|p_4|/\sigma), \quad \bar{\chi}_\sigma(p_4) = 1 - \chi_\sigma(p_4).
\]

The operator $H_{I,\sigma}$ is the interaction given by (4.17) associated with the kernels $F^{(3)}(\xi_2, \xi_3)\bar{\chi}_\sigma(p_4)G^{(3)}(\xi_1, \xi_4)$ instead of $F^{(3)}(\xi_2, \xi_3)G^{(3)}(\xi_1, \xi_4)$.

We then set

\[ (5.4) \quad H_\sigma = H_0 + gH_{I,\sigma}. \]

We now introduce

\[ (5.5) \quad \Gamma_{4,\sigma} = \mathbb{R}^3 \cap \{|p_4| < \sigma\}, \quad \Gamma_4^\sigma = \mathbb{R}^3 \cap \{|p_4| \geq \sigma\}, \]

\[ \mathfrak{F}_{4,\sigma} = \mathfrak{F}_{a}(L^2(\Gamma_{4,\sigma})) , \quad \mathfrak{F}_{4}^\sigma = \mathfrak{F}_{a}(L^2(\Gamma_4^\sigma)) . \]

$\mathfrak{F}_{4,\sigma} \otimes \mathfrak{F}_{4}^\sigma$ is the Fock space for the massless neutrino such that $\mathfrak{F}^{(\nu)} \simeq \mathfrak{F}_{4,\sigma} \otimes \mathfrak{F}_{4}^\sigma$.

Now, we set

\[ (5.6) \quad \mathfrak{F}^\sigma = \mathfrak{F}^{(e)} \otimes \mathfrak{F}^{(p)} \otimes \mathfrak{F}^{(ne)} \otimes \mathfrak{F}_{4}^\sigma \quad \text{and} \quad \mathfrak{F}_\sigma = \mathfrak{F}_{4,\sigma} \]

and we have

\[ (5.7) \quad \mathfrak{F} \simeq \mathfrak{F}^\sigma \otimes \mathfrak{F}_\sigma. \]

We further set
We identify $H^{4}_{0}$ with its obvious extension to $\mathcal{F}$. We let

$$H^{4}_{0,\sigma} = \int_{|p_{4}| < \sigma} |p_{4}|^{2} b_{4}^{*}(\xi_{4}) b_{4}(\xi_{4}) d\xi_{4}.$$  (5.9)

We identify $H^{4}_{0,\sigma}$ and $H^{4}_{0,\sigma}$ with their obvious extension to $\mathcal{F}_{\sigma}$ and $\mathcal{F}_{\sigma}$ respectively.

Then, on $\mathcal{F}_{\sigma} \otimes \mathcal{F}_{\sigma}$, we have

$$H^{4}_{\sigma} = H^{4}_{0,\sigma} \otimes 1_{\sigma} + 1_{\sigma} \otimes H^{4}_{0,\sigma}.$$  (5.10)

where $1_{\sigma}$ (resp. $1_{\sigma}$) is the identity operator on $\mathcal{F}_{\sigma}$ (resp. $\mathcal{F}_{\sigma}$).

Using the definitions

$$H^{\sigma} = H_{\sigma}|_{\mathcal{F}_{\sigma}} \quad \text{and} \quad H^{\sigma}_{0} = H_{0}|_{\mathcal{F}_{\sigma}},$$  (5.11)

we get

$$H^{\sigma} = H^{(e)}_{0,D} + H^{(p)}_{0,D} + H^{(ne)}_{0,D} + H^{4,\sigma}_{0} + g H_{1,\sigma} \quad \text{on} \quad \mathcal{F}_{\sigma},$$  (5.12)

and

$$H_{\sigma} = H^{\sigma} \otimes 1_{\sigma} + 1_{\sigma} \otimes H^{4}_{0,\sigma} \quad \text{on} \quad \mathcal{F}_{\sigma} \otimes \mathcal{F}_{\sigma}.$$  (5.13)

Now, for $\delta \in \mathbb{R}$ such that $0 < \delta < m_{3}$, we define the sequence $(\sigma_{n})_{n \geq 0}$ by

$$\sigma_{0} = 2m_{3} + 1,$$

$$\sigma_{1} = m_{3} - \frac{\delta}{2},$$

$$\sigma_{n+1} = \gamma \sigma_{n} \quad \text{for} \quad n \geq 1,$$

where

$$\gamma = 1 - \frac{\delta}{2m_{3} - \delta}. $$  (5.15)

For $n \geq 0$, we then define the cutoff operators on $\mathcal{F}^{n} = \mathcal{F}_{\sigma_{n}}$ by

$$H^{n} = H^{\sigma_{n}}, \quad H^{n}_{0} = H^{\sigma_{n}}_{0}.$$  (5.16)

We set, for $n \geq 0$,

$$E^{n} = \inf \sigma(H^{n}).$$  (5.17)
We also define the cutoff operators on $\mathfrak{F}$ by

$$H_n = H_{\sigma_n}, \quad H_{0,n} = H_{0,\sigma_n},$$

We set, for $n \geq 0$,

$$E_n = \inf \sigma(H_n).$$

Note that

$$E^n = E_n$$

One easily shows that, for $|g| \leq g_0$,

$$|E^n| = |E_n| \leq \frac{|g| K(F,G)B}{1 - g_0 K(F,G)C}$$

See [5, 3] for a proof.

We now set

$$\tilde{K}(F,G) = 2 \left( \sum_{\beta=1,2} \|F^{(\beta)}(\cdot, \cdot)\|_{L^2(\Gamma_1 \times \Gamma_1)} \right) \tilde{K}(G).$$

where $\tilde{K}(G)$ is the constant given in Hypothesis 5.2(ii)

$$\tilde{C} = \frac{C}{(1 - g_0 K(F,G)C)}$$

$$\tilde{B} = \frac{B}{(1 - g_0 K(F,G)C)^2}$$

$$\tilde{D}(F,G) = \max \left\{ \frac{4(2m_3 + 1)\gamma}{2m_3 - \delta}, \ 2 \right\} \tilde{K}(F,G)(2m_3\tilde{C} + \tilde{B})$$

Let $g_1^{(\delta)}$ be such that

$$0 < g_1^{(\delta)} < \min \left\{ 1, \frac{\gamma - \gamma^2}{3\tilde{D}(F,G)} \right\}. $$

and let

$$g_3 = \frac{1}{2\tilde{K}(F,G)(2\tilde{C} + \tilde{B})}$$

Setting

$$g_2^{(\delta)} = \inf \{g_3, g_1^{(\delta)} \}$$
and applying the same method as the one used for proving proposition 4.1 in [3] we finally get the following result for which we omit the details:

**Proposition 5.3.** Suppose that the kernels $F^{(\beta)}(\cdot, \cdot)$, $G^{(\beta)}(\cdot, \cdot)$, $\beta = 1, 2$, satisfy Hypothesis 4.1 and Hypothesis 5.1(ii). Then, for $|g| \leq g_2^{(\delta)}$, $E^n$ is a simple eigenvalue of $H^n$ for $n \geq 1$, and $H^n$ does not have spectrum in the interval $(E^n, E^n + (1 - 3g\frac{D(F,G)}{7})\sigma_n)$.

5.0.2. *Proof of theorem 5.2.*

**Proof.** In order to prove the existence of a ground state for $H$ we adapt the proof of theorem 3.3 in [5]. By Proposition 5.3 $H^n$ has a ground state, denoted by $\phi^n$, in $\mathcal{F}^n$ such that

$$H^n\phi^n = E^n\phi^n, \quad \phi^n \in \mathcal{D}(H^n), \quad \|\phi^n\| = 1, \quad n \geq 1.$$  

Therefore $H_n$ has a normalized ground state in $\mathcal{F}$, given by $\tilde{\phi}_n = \phi^n \otimes \Omega_n$, where $\Omega_n$ is the vacuum state in $\mathcal{F}_n$,

$$H_n\tilde{\phi}_n = E^n\tilde{\phi}_n, \quad \tilde{\phi}_n \in \mathcal{D}(H_n), \quad \|\tilde{\phi}_n\| = 1, \quad n \geq 1.$$

Let $H_{1,n}$ be the interaction $H_{1,\sigma_n}$. It follows from the pull-through formula that

$$H_0 + gH_{1,n}b_+(\xi_4)\tilde{\phi}_n = E_nb_+(\xi_4)\tilde{\phi}_n - \omega(\xi_4)b_+(\xi_4)\tilde{\phi}_n - (gV_n^1(\xi_4) + gV_n^2(\xi_4))\tilde{\phi}_n$$

where

$$V^{(1)}(\xi_4) = \int d\xi_1d\xi_2d\xi_3 \left( \int dx^2 e^{-ix^2r^2} \right. \frac{(U^{(c)}(\xi_4)\gamma_\alpha(1 - \gamma_5)U^{(e)}(x^2, \xi_1)) (U^{(nc)}(\xi_3)\gamma_\alpha(1 - gA\gamma_5)U^{(p)}(x^2, \xi_2))}{F^{(1)}(\xi_2, \xi_3) G^{(1)}(\xi_1, \xi_4) b^*_+(\xi_3)b_+(\xi_2)b_+(\xi_1)}.$$  

$$V^{(2)}(\xi_4) = \int d\xi_1d\xi_2d\xi_3 \left( \int dx^2 e^{-ix^2r^2} \right. \frac{(U^{(nc)}(\xi_4)\gamma_\alpha(1 - \gamma_5)W^{(c)}(x^2, \xi_1)) (U^{(nc)}(\xi_3)\gamma_\alpha(1 - gA\gamma_5)W^{(p)}(x^2, \xi_2))}{F^{(2)}(\xi_2, \xi_3) G^{(2)}(\xi_1, \xi_4) b^*_+(\xi_3)b^*_+(\xi_2)b^*_+(\xi_1)}.$$  

Thus, by (5.30), (5.31),(5.32) and (5.33), we obtain

$$H_n - E_n + \omega(\xi_4)) b_+(\xi_4)\tilde{\phi}_n = -g \left( V^{(1)}(\xi_4) + V^{(2)}(\xi_4) \right) \tilde{\phi}_n.$$
We have for \( \beta = 1, 2 \),
\[
\|V^{(\beta)}(\xi_4)\tilde{\phi}_n\| \leq C_0 \|F^{(\beta)}(\cdot, \cdot)\|_{L^2(\Gamma_1 \times \Gamma_1)} \|G^{(\beta)}(\cdot, \xi_4)\|_{L^2(\Gamma_1)} \times \|H_0 + m_{ne}\tilde{\phi}_n\|.
\]
(5.35)

The estimates (5.35) are examples of \( N_\tau \) estimates (see [13]). The proof is similar to the one of [4, Proposition 3.7] and details are omitted.

Let us estimate \( \|H_0\tilde{\phi}_n\| \). By (5.2) we get
\[
\|g\|_{H_{I,n}\tilde{\phi}_n} \leq |g|K(F, G) \left( C\|H_0\tilde{\phi}_n\| + B \right)
\]
and
\[
\|H_0\tilde{\phi}_n\| \leq |E_n| + |g|\|gH_{I,n}\tilde{\phi}_n\|
\]
and, by (5.21), we obtain
\[
\|H_0\tilde{\phi}_n\| \leq \frac{g_0K(F, G)B}{1 - g_0K(F, G)C} \left( 1 + \frac{1}{1 - g_0K(F, G)C} \right) = M
\]
(5.38)

By (5.38) \( \|H_0\tilde{\phi}_n\| \) is bounded uniformly in \( n \) and \( |g| \leq g_0 \) and by (5.34),(5.35) and (5.38) we get
\[
\|b_+(\xi_4)\tilde{\phi}_n\| \leq \frac{|g|C_0|P_4|}{\|P_4\|} \left( \sum_{\beta=1}^2 \|F^{(\beta)}(\cdot, \cdot)\|_{L^2} \|G^{(\beta)}(\cdot, \xi_4)\|_{L^2} \right) (M + m_{ne})
\]
uniformly with respect to \( n \).

By Hypothesis 5.1(i) and (5.39) there exists a constant \( C(F, G) > 0 \) such that
\[
\int \|b_+(\xi_4)\tilde{\phi}_n\|^2 d\xi_4 \leq C(F, G)^2g^2
\]
(5.40)

Since \( \|\tilde{\phi}_n\| = 1 \), there exists a subsequence \((n_k)_{k \geq 1}\), converging to \( \infty \) such that \((\tilde{\phi}_{n_k})_{k \geq 1}\) converges weakly to a state \( \tilde{\phi} \in \mathfrak{F} \). By adapting the proof of theorem 4.1 in [4, 1] it follows from (5.40) that there exists \( g_2 \) such that \( 0 < g_2 \leq g_2^{(\delta)} \) and \( \tilde{\phi} \neq 0 \) for any \( |g| \leq g_2 \). Thus \( \tilde{\phi} \) is a ground state of \( H \).

\( \Box \)
6. Uniqueness of a ground state of the Hamiltonian \( H \).

In order to obtain a result about the uniqueness of the ground states we need to introduce a new hypothesis.

Let

\[
K = \bigcup_{j=1}^{4} \{ p_4 = (p_1^j, p_2^j, p_3^j) \in \mathbb{R}^3 | p_4^j = 0 \}
\]

**Hypothesis 6.1.** We suppose that \( \frac{\partial}{\partial p_4} G^{(\beta)}(\ldots) \) and \( \frac{\partial^2}{\partial p_1^j \partial p_4} G^{(\beta)}(\ldots) \) belong to \( L^2(\Gamma_1) \times L^2_{\text{loc}}(\mathbb{R}^3 \setminus K) \) for \( \beta = 1, 2 \).

We then have

**Theorem 6.2.** Suppose that the kernels \( F^{(\beta)}(\ldots) \) and \( G^{(\beta)}(\ldots) \) satisfy hypothesis 4.1, 5.1 and 6.1 for \( \beta = 1, 2 \). Then there exists \( g_2 > 0 \) such that \( g_2 \leq g_1 \) and \( \dim(\text{Ker}(H - E)) = 1 \) for \( |g| \leq g_2 \).

**Proof.** In order to prove the uniqueness of the ground states we apply the method developed by F. Hiroshima. See [18]. For other applications of this method see [21] and [22].

Consider \( f(\ldots) \in C^2_0(\mathbb{R}^3 \setminus K) \) and \( \Phi, \Psi \in D(H) \).

By (4.31), (4.34), (4.35) and (4.37) we get

\[
(\Phi, [b_{+4}(f), H_I] \Psi) = \int f(\xi_4) (\Phi, T(\xi_4) \Psi) \, d\xi_4
\]

where

\[
T(\xi_4) = T^{(1)}(\xi_4) + T^{(2)}(\xi_4)
\]

with

\[
T^{(1)}(\xi_4) = \sum_{j=1}^{4} \int d\xi_1 d\xi_2 d\xi_3 \left( \int dx^2 e^{-ix^2 r^2} \right) \left( \overline{U^{(ue)}(\xi_3)} \gamma^\alpha (1 - g_A \gamma_5) U^{(p)}(x^2, \xi_2) \right) \overline{F^{(1)}(\xi_2, \xi_3)}
\]

\[
\overline{G^{(1)}(\xi_1, \xi_4) U^{(\nu)}(\xi_4)_j \left( \gamma^0 \gamma^\alpha (1 - \gamma_5) U^{(e)}(x^2, \xi_1) \right)_j}
\]

\[
b_+^* (\xi_3) b_+ (\xi_2) b_+ (\xi_1).
\]
\[ T^{(2)}(\xi_4) = \]
\[ \sum_{j=1}^{4} \int d\xi_1 d\xi_2 d\xi_3 \left( \int dx^2 e^{-ix^2 r^2} \right) \]
\[ \left( \overline{U^{(ne)}(\xi_3) \gamma^\alpha (1 - g_A \gamma_5) W^{(p)}(x^2, \xi_2)} \right) F^{(2)}(\xi_2, \xi_3) \]
\[ G^{(2)}(\xi_1, \xi_4) \overline{U^{(\nu)}(\xi_4)} \left( \gamma^0 \gamma^\alpha (1 - \gamma_5) W^{(e)}(x^2, \xi_1) \right) \]
\[ b^*_+(\xi_3) b^*_+(\xi_2) b^*_+(\xi_1). \]

We now have to prove that

\[ \int f(\xi_4) \left( \Phi, e^{-it(H-E+\omega(\xi_4))T(\xi_4)\Psi_g} \right) d\xi_4 \in L^1([0, \infty), dt) \]

where \( \Psi_g \) is a ground state of \( H \) such that \( \| \Psi_g \| = 1 \).

We have

\[ \int f(\xi_4) \left( \Phi, e^{-it(H-E+\omega(\xi_4)) T(\xi_4) \Psi_g} \right) d\xi_4 = \sum_{j=1}^{4} \left( I_j^{(1)}(t) + I_j^{(2)}(t) \right) \]

where

\[ I_j^{(1)}(t) = \]
\[ \left( e^{it(H-E)} \Phi, \int d\xi_1 d\xi_2 d\xi_3 \left( \int dx^2 e^{-ix^2 r^2} \right) \right) \]
\[ \left( U^{(ne)}(\xi_3) \gamma^\alpha (1 - g_A \gamma_5) \overline{U^{(p)}(x^2, \xi_2)} \right) F^{(1)}(\xi_2, \xi_3) \]
\[ \left( \int \overline{f(\xi_4)} e^{-it\omega(\xi_4)} G^{(1)}(\xi_1, \xi_4) \overline{U^{(\nu)}(\xi_4)} d\xi_4 \right) \left( \gamma^0 \gamma^\alpha (1 - \gamma_5) \overline{U^{(e)}(x^2, \xi_1)} \right) \]
\[ b^*_+(\xi_3) b^*_+(\xi_2) b^*_+(\xi_1) \Psi_g \right). \]

and
\[ I_j^{(2)}(t) = \left( e^{it(H-E)}\Phi, \int d\xi_1 d\xi_2 d\xi_3 \left( \int dx^2 e^{-ix^2r^2} \right) \right. \]

\[ \left( \bar{U}^{(ne)}(\xi_3)\gamma^\alpha(1 - g_A\gamma_5)W^{(p)}(x^2, \xi_2) \right) F^{(2)}(\xi_2, \xi_3) \]

\[ \left( \int \bar{f}(\xi_4)e^{-it\omega(\xi_4)}G^{(2)}(\xi_1, \xi_4)\bar{G}^{(2)}(\xi_4, \xi_3) d\xi_4 \right) \left( \gamma^0\gamma^\alpha(1 - \gamma_5)W^{(e)}(x^2, \xi_1) \right) \]

\[ b_+(\xi_3)b_+(\xi_2)b_+(\xi_1)\Psi_g \].

In (6.8) and (6.9) we now use for \( p_4 \in \mathbb{R}^3 \setminus K \) and for \( t \geq 1 \)

\[ e^{-it\omega(\xi_4)} = -\frac{1}{t^2} \left( \left| p_4 \right| \right) \left( \frac{\left| p_4 \right|}{p_4^2} \right) \frac{\partial^2}{\partial p_4^1 \partial p_4^2} e^{-it\omega(\xi_4)}. \]

By (6.10) and by integrating by parts \( I_j^{(1)}(t) \) is a finite sum of terms of the following form

\[ \frac{1}{t^2} \left( e^{it(H-E)}\Phi, \int d\xi_1 d\xi_2 d\xi_3 \left( \int dx^2 e^{-ix^2r^2} \right) \right. \]

\[ \left( \bar{U}^{(ne)}(\xi_3)\gamma^\alpha(1 - g_A\gamma_5)U^{(p)}(x^2, \xi_2) \right) F^{(1)}(\xi_2, \xi_3) \]

\[ \left( \int \bar{f}(\xi_4)e^{-it\omega(\xi_4)}G^{(1)}(\xi_1, \xi_4) d\xi_4 \right) \left( \gamma^0\gamma^\alpha(1 - \gamma_5)U^{(e)}(x^2, \xi_1) \right) \]

\[ b_+(\xi_3)b_+(\xi_2)b_+(\xi_1)\Psi_g \].

Likewise \( I_j^{(2)}(t) \) is a finite sum of terms of the following form

\[ \frac{1}{t^2} \left( e^{it(H-E)}\Phi, \int d\xi_1 d\xi_2 d\xi_3 \left( \int dx^2 e^{-ix^2r^2} \right) \right. \]

\[ \left( \bar{U}^{(ne)}(\xi_3)\gamma^\alpha(1 - g_A\gamma_5)W^{(p)}(x^2, \xi_2) \right) F^{(2)}(\xi_2, \xi_3) \]

\[ \left( \int \bar{f}(\xi_4)e^{-it\omega(\xi_4)}\bar{G}^{(2)}(\xi_1, \xi_4) d\xi_4 \right) \left( \gamma^0\gamma^\alpha(1 - \gamma_5)W^{(e)}(x^2, \xi_1) \right) \]

\[ b_+(\xi_3)b_+(\xi_2)b_+(\xi_1)\Psi_g \].
\( \tilde{f}_j(\cdot) \) depends on \( U^{(\nu)}(\cdot)_j \) and is equal to one of the following functions \( f(\cdot) U^{(\nu)}(\cdot)_j, \frac{\partial}{\partial p_1^j} \left( f(\cdot) U^{(\nu)}(\cdot)_j \right), \frac{\partial}{\partial p_2^j} \left( f(\cdot) U^{(\nu)}(\cdot)_j \right) \) and \( \frac{\partial^2}{\partial p_1^j \partial p_2^j} \left( f(\cdot) U^{(\nu)}(\cdot)_j \right) \) according to the integration by parts.

Similarly \( \tilde{G}^{(\beta)}(\cdot, \cdot) \) is one of the following functions \( G^{(\beta)}(\cdot, \cdot), \frac{\partial}{\partial p_1^j} G^{(\beta)}(\cdot, \cdot), \frac{\partial^2}{\partial p_1^j \partial p_2^j} G^{(\beta)}(\cdot, \cdot) \).

By Hypothesis 4.1, Hypothesis 6.1 and the \( N_\tau \) estimates it follows from (6.11) and (6.12) that there exists a constant \( C_1 > 0 \) such that, for \( j = 1, 2, 3, 4 \) and \( \beta = 1, 2 \), we have for \( t \geq 1 \)

\[
|I_j^{(\beta)}(t)| \leq \frac{C_1}{t^2} \| \tilde{f}_j(\cdot) \|_{L^2(\mathbb{R}^3)} \| F^{(\beta)}(\cdot, \cdot) \|_{L^2(\Gamma_1) \times L^2(\Gamma_1)} \times \| \chi_f(\cdot) \tilde{G}^{(\beta)}(\cdot, \cdot) \|_{L^2(\Gamma_1) \times L^2(\mathbb{R}^3)} \| (H_0 + m_{ne})^{\frac{1}{2}} \Psi_g \|,
\]

where \( p_4 \to \chi_f(p_4) \) is the characteristic function of the support of \( f(\cdot) \).

It follows from (5.35) and (6.13) that (6.6) is satisfied.

By Hypothesis 4.1, Hypothesis 5.1 and the \( N_\tau \) estimates it follows from (6.4) and (6.5) that, for any ground state \( \Psi_g \), we have

\[
\int_{\mathbb{R}^3} \| T(\xi_4) \Psi_g \|^2 d\xi_4 < \infty
\]

(6.14)

\[
\int_{\mathbb{R}^3} \| (H - E + \omega(\xi_4))^\dagger T(\xi_4) \Psi_g \|^2 d\xi_4 < \infty
\]

We now have

\[
\| (H_0 + m_{ne})^{\frac{1}{2}} \Psi_g \| \leq \| (H_0 + m_{ne}) \Psi_g \| \leq |E| + |g| \| H_1 \Psi_g \|.
\]

(6.15)

By (5.2) we get

\[
|g| \| H_1 \Psi_g \| \leq |g| K(F, G) (\| (H_0 + m_{ne}) \Psi_g \| + m_{ne} C + B)
\]

Recall that \( g_0 K(F, G) C < 1 \). By (6.15) and (6.16) we obtain

\[
\| (H_0 + m_{ne}) \Psi_g \| \leq \frac{1}{1 - g_0 K(F, G) C} (|E| + g_0 K(F, G) C m_{ne} + g_0 K(F, G) B).
\]

(6.17)

By Hypothesis 4.1, Hypothesis 5.1, the \( N_\tau \) estimates, (6.15) and (6.17) it follows that, for any ground state \( \Psi_g \), we get

\[
\lim_{g \to 0} \max_{\Psi_g} g^2 \int_{\mathbb{R}^3} \| (H - E + \omega(\xi_4))^{-1} T(\xi_4) \Psi_g \|^2 d\xi_4 = 0
\]

(6.18)
According to [18, Theorem 4.2] this yields that there exists $g_2 > 0$ such that $g_2 \leq g_1$ and $\dim(\text{Ker}(H-E))=1$ for $|g| \leq g_2$.
This concludes the proof of Theorem 6.2.

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