Generalized Komar currents for vector fields

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Abstract

In this paper, on basis of three quadratic differential operators leaving the form degree of an arbitrary differential form unchanged, that is, the d’Alembertian operator and two combined ones from the Hodge coderivative and the exterior derivative, the usual Komar current for a Killing vector is formulated into another equivalent form. Then it is extended to more general currents in the absence of the linearity in the Killing vector field. Moreover, motivated by this equivalent of the usual Komar current, we put forward a conserved current corresponding to a generic vector with some constraint. Such a current can be generalized to the one with higher-order derivatives of the vector. The applications to some specific vector fields, such as the almost-Killing vectors, the conformal Killing vectors and the divergence-free vectors, are investigated. It is demonstrated that the above generalizations of the Killing vector can be uniformly described by a second-order derivative equation.

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1 Introduction

General relativity, together with its various modifications, possesses a diffeomorphic invariance in accordance with the requirement of the covariance principle for the theory itself. Ordinarily, such a symmetry is generated by vector fields, particularly by the Killing vectors. As a consequence, inspired with the famous Noether theorem, one can naturally anticipate that each vector field has the potential to bring into a conserved current, and further to give rise to a corresponding conserved charge.

In fact, a well-known significant example to demonstrate the above statement is the appearance of the so-called Komar conserved current for general relativistic spacetime manifolds \[1\], although it was found without the guidance of the standard Noether approach. By virtue of such a simple but useful current, the Komar integral was put forward. Till now, it has occupied an important position in the definition of the conserved charges in asymptotically flat spacetime. Nevertheless, when the Komar integral is carried out to calculate the conserved charges of spacetimes that do not behave as asymptotical flatness, it usually gives rise to the divergence problems. Not only that, it sometimes fails to reproduce the anticipated values for all the conserved charges even though they are finite \[2, 3, 4\]. Consequently, the usual Komar current, as well as the Komar integral, deserves to be modified. Actually, there has existed a lot of literature involved in its modifications and generalizations. For example, see the works \[5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17\].

According to the ordinary Komar current, one can observe that it merely consists of a second-order derivative term of the vector field. Apart from this, the conventional Komar current, as well as its various generalizations, is linear in the vector. In order to provide interesting insights into the Komar current corresponding to the Killing vector, we are going to take into account the non-linear generalization to the usual Komar current under the assumption that the current depends at most upon the second-order derivatives of the Killing vector and its linearity in the vector is lost. In terms of such a generalized current, it may be possible to build a meaningful formula for the conserved charges of gravity theories.

More generally, we may even go further, taking into account the construction of the conserved currents associated to a generic vector field (specifically, the conserved currents are restricted to depend only on the vector and the metric tensor, as well as their derivatives). As a matter of fact, each conserved current can be seen as a divergenceless vector. This gives us a clue that the conserved currents, as well as the vectors involved in them,
are able to be understood as 1-forms, which can be manipulated by all the operators within the framework of differential forms. Consequently, it is feasible to obtain various currents by letting the Hodge star operation $\star$ and the exterior derivative $d$, together with their combinations, act on the vector fields. However, in order to ensure that the form degree of the currents remains one, it is of great necessity to choose the operators that can leave the form degree of the 1-form vector fields unaltered. Really, as what has been illustrated in the work [18], for an arbitrary $p$–form $F$ in an $n$-dimensional spacetime manifold with a metric $g_{\mu\nu}$ having a Lorentzian signature $(-, +, +, \cdots)$, under the action of the generally-covariant d’Alembertian operator $\Box$, as well as the second-order operation $\delta d$ or $d\delta$ (the concrete forms for their action on $F$ can be found in the appendix [A]), the form degree of the $p$–form remains invariant. Here the codifferential (or the divergence operator) $\delta$ is defined through

$$\delta = (-1)^{np+n+1} \star d\star,$$

and its action on $F$ leads to $(\delta F)_{\mu_2\cdots\mu_p} = \nabla^{\mu_1} F_{\mu_1\cdots\mu_p}$ (for details on the operators $\delta$, $\delta d$ and $d\delta$, see [18] and references therein). Furthermore, it will be proved below that all the three quadratic differential operators $\Box$, $d\delta$ and $\delta d$ are the primary ones with the lowest differential order that can preserve the form degree of any differential form.

According to this, they may be the ideal candidates to generate conserved currents out of various vector fields. Therefore, we arrive at the question whether the three operators $\Box$, $d\delta$ and $\delta d$ can bring about some interesting conserved currents by means of their action on a Killing vector field or a generic one.

For the purpose of answering the above questions, under the condition that the vector fields (including the Killing vectors) together with the conserved currents are treated as 1-forms and the three objects $\Box$, $d\delta$ and $\delta d$ are chosen as the fundamental operators acting on them, we attempt to supply another way to find out the possible conserved currents associated with these vectors. The outcome will show that the usual Komar current can be generalized to the ones containing non-linear terms of the Killing vectors. Besides, the three operations together with their linear combination render us to conveniently put forward the conserved currents made up of quadratic or higher-order derivatives of any general vector field. Remarkably, by virtue of the operator constructed from the linear combination of the three elementary ones, the ordinary Komar current for the Killing vector is able to be written as a novel equivalent form, while the equations encompassing the second-order derivatives of the vector fields can be expressed as a unified formulation.

The remainder of the present paper goes as follows. In section 2 we are going to investigate various possible conserved currents associated with the Killing vector fields.
Subsequently, the usual Komar current will be generalized to the one with respect to an arbitrary vector by means of the three differential operations $\Box$, $\hat{d}$ and $d\hat{d}$. In order to demonstrate the meanings of the current, a couple of examples for some specific vector fields will be presented. In section 3 we shall pay attention to the extended currents that are dependent at most of second-order derivatives of a general vector field, as well as the currents consisting of higher-order derivative terms of the vector. The last section is our discussions and remarks. The main results of this paper will be summarized in appendix D.

2 Generalized Komar currents associated to vector fields

In the present section, with the help of all the three second-order differential operations $\Box$, $\hat{d}$ and $d\hat{d}$ (the introduction for them is given in the appendix A), it is of great convenience for us to put the ordinary Komar current for a Killing vector into another novel form. According to this, we shall put forward a more general conserved current that is made up of terms proportional to at most second-order derivatives of the Killing vector, as well as a conserved current with respect to arbitrary vector fields. As some examples, the currents associated with a couple of specific interesting vectors, such as the almost-Killing vectors, the conformal Killing vectors and the divergence-free vector fields, are deduced. It is worth noticing that it is postulated that the currents are merely dependent of the vector fields and metric tensors, as well as their derivatives, throughout this work.

2.1 Komar currents in terms of the operations $\Box$, $\hat{d}$ and $d\hat{d}$

As a beginning, for simplicity, we take into account the ordinary Komar current $J_K$ corresponding to a Killing vector $\xi^\mu$, which describes the infinitesimal isometries of an $n$-dimensional spacetime manifold equipped with a pseudo-Riemannian metric $g_{\mu\nu}$ and is determined by the well-known Killing equation $2\nabla_{(\mu}\xi_{\nu)} = 0$. The conserved Komar current is defined as

$$J_K = 2\nabla^\nu\nabla_{[\mu}\xi_{\nu]}dx^\mu = -\hat{d}\xi,$$

In accordance with the identity $\hat{\delta}^2 = 0$, it is easy to verify that the Hodge coderivative of $J_K$ identically vanishes ($\hat{\delta}J_K = 0$ or $\nabla_{\mu}J^K_{\mu} = 0$), independently of whether the metric tensor is on-shell or off-shell, even though $\xi^\mu$ is an arbitrary vector.
As what is shown in Eq. (2.1), the ordinary Komar conserved current can be completely constructed from the action of the operator $\hat{\delta}d$ on the Killing vector $\xi$. Apart from this operator, one may wonder whether the other two second-order operations $\Box$ and $d\hat{\delta}$ that also preserve the form degree could enter into the definition of $J_K$. Indeed, according to the property for the Killing vector

$$2\Box\xi - \chi d\hat{\delta}\xi - \hat{\delta}d\xi = 0,$$

(2.2)
together with the null divergence $\hat{\delta}\xi = 0$, it seems that a quite natural way to put the usual Komar current $J_K$ for the Killing vector into a novel form is to reexpress $J_K$ as the linear combination of the three second-order derivative 1-forms $\Box\xi$, $d\hat{\delta}\xi$ and $\hat{\delta}d\xi$. Doing so yields the identically conserved current

$$\tilde{J}_K = 2a_1\Box\xi - \chi d\hat{\delta}\xi + a_2\hat{\delta}d\xi.$$

(2.3)

Here and in what follows the coefficients $\chi$, $a_1$ and $a_2$ denote arbitrary constant parameters, but the latter two ones are constrained by $a_1 + a_2 = -1$ in order to make Eq. (2.3) coincide with the conventional form $J_K = -\hat{\delta}d\xi$ for the Killing vector. In addition to this, both the coefficients $a_1$ and $a_2$ can be chosen for agreement with the mass of the Schwarzschild black hole when $\tilde{J}_K$ is applied to define conserved charges like the Komar integral. For another special situation where $a_1 = \chi = 0$, the co-closed 1-form $\tilde{J}_K$ is able to be adopted as the conserved current corresponding to any vector field, such as the almost- and conformal Killing vectors. As a result, here we arrive at the aforementioned conclusion that each vector field can yield a conserved current. What is more, within the framework of four-dimensional general relativity, by means of Einstein’s gravitational field equation $R_{\mu\nu} - Rg_{\mu\nu}/2 = 8\pi G T_{\mu\nu}$ together with Eq. (C.1), the Komar current (2.3) coincides with the one presented in the work [19].

The formula (2.3) indicates that the conventional Komar conserved current is merely built out of the second-derivative terms of one Killing vector. However, more generally, if assumed that the conserved current comprises at most but not just the second-order derivatives of the Killing vector and the linearity in this vector is abandoned so that it is permitted to appear many times in each term, we are interested in finding the possible generalization of the Komar current (2.3).

To find out a satisfactory prescription under the above mentioned postulate, it is of great convenience to treat both the current and the Killing vector field as 1-forms. As what...
has been demonstrated in the appendix A, □, \( d\delta \) and \( \delta d \) are all the three elementary second-order covariant derivative operators preserving the form degree of an arbitrary differential form. Together with the help of Eq. (B.1), thence one finds that the aimed conserved current \( \tilde{J}_K \) has to take the general form

\[
\tilde{J}_K = \hat{J}_K + \bar{J}_K + \lambda \xi ,
\]

with the components of the 1-form \( \bar{J}_K \) presented by

\[
(\bar{J}_K)_{\mu} = \lambda_1 h_1 \nabla_\mu \xi^2 + \lambda_2 h_2 \xi_\nu (\nabla^\nu \xi^\nu) \nabla \rho \xi^\mu + \lambda_3 h_3 \nabla^\nu \nabla \mu \xi^\nu + \bar{U} \xi^\mu .
\]

Here the scalar \( \bar{U} \) is given by

\[
\bar{U} = \lambda_1 h_4 \xi^\sigma \square \xi_\sigma + \lambda_5 h_5 (\nabla^\nu \xi^2) \nabla_\nu \xi^2 + \lambda_6 h_6 (\nabla_\rho \xi_\sigma) \nabla^\rho \xi^\sigma + \lambda_7 h_7 R .
\]

In Eqs. (2.4), (2.5) and (2.6), \( \lambda = h(\xi^2) \), \( \xi^2 = \xi^\sigma \xi_\sigma \), \( h_i = h_i(\xi^2) \) and \( \lambda_i \)'s \((1 \leq i \leq 7)\) are restricted to constant parameters in response to the assumption that \( \hat{J}_K \) is solely associated to the variables \( g_{\mu \nu} \) \((g^{\mu \nu})\) and \( \xi^\mu \). By virtue of the properties for Killing vectors, which are presented in the appendix C, the codifferential of the 1-form \( \bar{J}_K \) is read off as

\[
\delta \bar{J}_K = \lambda_1 \nabla^\mu (h_1 \nabla_\mu \xi^2) + \frac{1}{2} \left( \lambda_2 h_2 + 2 \lambda_3 \frac{dh_3}{d(\xi^2)} \right) R_{\rho \sigma} \xi^\rho \nabla^\sigma \xi^2 .
\]

Specially, for a certain Killing vector fulfilling \( \xi^2 = Const \), it is able to ensure that \( \delta \bar{J}_K = 0 \). Nevertheless, in order to guarantee that the divergence of the current \( \bar{J}_K \) identically vanishes for any Killing vector within an arbitrary spacetime manifold, in addition to that \( \delta \bar{J}_K = 0 \) and \( \xi^\mu \nabla_\mu \lambda = 0 \) hold identically, \( \bar{J}_K \) is necessarily divergenceless. This demands that

\[
\lambda_1 = 0 , \quad \lambda_3 \frac{dh_3}{d(\xi^2)} = - \frac{1}{2} \lambda_2 h_2 .
\]

In Eq. (2.8), for example, when \( \lambda_3 \neq 0 \) and

\[
h_2 = \sum_{i=0}^{\infty} a_i (\xi^2)^i ,
\]

the solutions for the second equation are

\[
h_3 = - \frac{\lambda_2}{2 \lambda_3} \sum_{i=0}^{\infty} \frac{a_i}{i+1} (\xi^2)^{i+1} + C .
\]

Here and henceforth, the quantity \( C \), together with all the coefficients \( a_i \)'s, stands for arbitrary constant parameters. More generally, in the absence of the requirement that \( \lambda \) is
dependent of at most second-order derivatives, one interesting solution for the constraint equation $\xi^\mu \nabla_\mu \lambda = 0$ is the one

$$\lambda = \sum_i (a_i F_i + b_i W_i) + \sum_{i,j} k_{ij} \tilde{F}_i \tilde{W}_j.$$  \hspace{1cm} (2.11)

Here and in what follows the coefficients $b_i$'s and $k_{ij}$'s denote constant parameters. The scalars $\{F_i, i = 0, 1, \ldots\}$ and $\{W_i, i = 0, 1, \ldots\}$ in Eq. (2.11) are defined through

$$F_i = f_i^0(g)f_i^1(\mathcal{R})f_i^2(\nabla \mathcal{R})f_i^3(\nabla \nabla \mathcal{R}) \cdots,$$

$$W_i = w_i^0(g)w_i^1(\xi)w_i^2(\nabla \xi)w_i^3(\nabla \nabla \xi) \cdots,$$  \hspace{1cm} (2.12)

while the scalars $\tilde{F}_i$'s and $\tilde{W}_i$'s take the forms similar to $F_i$'s and $W_i$'s respectively. Notice that $\mathcal{R}$ in Eq. (2.12) denotes the standard Riemann tensor $R_{\rho\sigma\mu\nu}$ and $f_i^k(X) = 1$ ($w_i^k(X) = 1$) when such a term is absent, so it could be set $F_0 = W_0 = 1$. For instance, assumed that $i = 1$, the Ricci scalar $R$ could be written as $R = f_i^0f_i^1$ with $f_0^0(g) = g^{\rho\mu}g^{\sigma\nu}$ and $f_1^1(\mathcal{R}) = R_{\rho\sigma\mu\nu}$. What is more, if $i = 3$, the $\xi^\sigma \Box^\sigma$ part in the $\lambda_4$ term of Eq. (2.3) could be expressed as $w_3^0w_3^1w_3^2$, where $w_3^0 = g_{\rho\sigma}g^{\nu\mu}$, $w_3^1 = \xi^\rho$ and $w_3^2 = \nabla_\mu \nabla_\nu \xi^\sigma$. According to Eq. (C.10), it is shown that the Lie derivatives of $F_i$'s and $W_i$'s along the Killing vector $\xi^\mu$ disappear, giving rise to that $\xi^\mu \nabla_\mu \lambda = 0$ holds identically.

Therefore, we are able to put forward a generalized conserved current $\hat{\mathcal{J}}_K$ for any Killing vector, irrespective of whether the metric is on-shell or off-shell. $\hat{\mathcal{J}}_K$ depends at most upon the second-order derivatives of the Killing vector $\xi^\mu$ and is expressed as

$$\hat{\mathcal{J}}_K = \tilde{\mathcal{J}}_K + h(\xi^2)\xi + \bar{\mathcal{J}}_K \big|_{\lambda_1 = 0}.$$  \hspace{1cm} (2.13)

Here it is worth noting that the 1-form $\hat{\mathcal{J}}_K$ is constrained by Eq. (2.8). The current $\hat{\mathcal{J}}_K$ mainly differs from the usual Komar current $\tilde{\mathcal{J}}_K$ by non-linear terms of the Killing vector. As a direct result of $\hat{\mathcal{J}}_K$, it is allowed to single out the conserved current $\hat{\mathcal{J}}_K + R\xi = 2G_{\mu\nu}\xi^\nu dx^\mu$ with $a_1 + a_2 = -1$. In fact, the quantity $\bar{U}$ can be covered by the scalars $F_i$'s and $W_i$'s given by Eq. (2.12). Consequently, a straightforward generalization of Eq. (2.13) leads to the more general conserved current

$$\mathcal{J}_{GK} = \hat{\mathcal{J}}_K + (\lambda + C)\xi.$$  \hspace{1cm} (2.14)

Notice that $\lambda$ is presented by Eq. (2.11) in the above equation and $h(\xi^2)$ in Eq. (2.13) has been incorporated into $W_i$. What is more, if the spacetime is Ricci-flat, $\hat{\mathcal{J}}_K$ acquires the
following form
\[ \mathbf{J}_K \big|_{R_{\mu\nu}=0} = \mathbf{J}_K \big|_{\lambda_1=0} \cdot \tag{2.15} \]
In accordance with Eq. (2.13), if the number of the Killing vector in any term of the conserved currents is demanded to be one, one can find that \( \mathbf{\hat{J}}_K \) differs from \( \mathbf{J}_K \) by the terms proportional to the Killing vector, that is,
\[ \mathbf{\hat{J}}_K \to (a_1 + a_2)\hat{\delta}d\xi + (b_1 + b_2R)\xi \cdot \tag{2.16} \]
As a result, regardless of the terms \( b_1\xi \) and \( b_2R\xi \), under the condition that the conserved current is permitted to rely at most upon second-order derivatives of a Killing vector, such a current has to take the form proportional to the co-closed 1-form \( \hat{\delta}d\xi \). To this point, it could be concluded that the ordinary Komar conserved current is “unique”.

For instance, in the framework of the Einstein gravity theory described by the Einstein-Hilbert Lagrangian
\[ L_{gr} = \sqrt{-g}(R - 2\Lambda), \tag{2.17} \]
where \( \Lambda \) is the cosmological constant, according to Eq. (2.16), a conserved current can be proposed as
\[ \mathbf{J}_{EH} = -\hat{\delta}d\xi + \frac{2nC\Lambda}{n-2}\xi. \tag{2.18} \]

2.2 The extended Komar currents associated to generic vector fields

We move on to consider the generalization for the Komar current (2.3) to the one with respect to an arbitrary vector \( V^\mu \) in \( n \)-dimensional spacetime. For the sake of convenience, the symmetrization and anti-symmetrization of the covariant derivative of this vector are supposed to take the respective forms
\[ 2\nabla_{(\mu}V_{\nu)} = \Phi_{\mu\nu}, \quad 2\nabla_{[\mu}V_{\nu]} = \Psi_{\mu\nu}, \tag{2.19} \]
Obviously, the tensor \( \Phi_{\mu\nu} \) (\( \Psi_{\mu\nu} \)) is (anti-)symmetric under the interchange of the two spacetime indices. The contraction \( \Phi \) between \( \Phi_{\mu\nu} \) and the metric tensor is denoted by \( \Phi = g^{\mu\nu}\Phi_{\mu\nu} = 2\nabla^\mu V_\mu \). In accordance with the notation of differential forms, the divergence of \( \Phi_{\mu\nu} \) is given by
\[ \nabla^\mu \Phi_{\mu\nu} dx^\nu = 2\Box V - \hat{\delta}dV, \tag{2.20} \]
with the help of Eq. (A.2), while the divergence of \( \Psi_{\mu\nu} \) is \( \nabla^\mu \Psi_{\mu\nu} dx^\nu = \hat{\delta}dV \). One can check that the first-order covariant derivatives of \( V^\mu \) can be completely expressed through
the three quantities $\Phi$, $\Phi_{\mu\nu}$ and $\Psi_{\mu\nu}$ attributed to the decomposition relations $2\nabla_\mu V_\nu = \Phi_{\mu\nu} + \Psi_{\mu\nu}$ and $2\nabla_\nu V_\mu = \Phi_{\mu\nu} - \Psi_{\mu\nu}$. According to this, it will be seen below that they are potential candidates to play the roles of the basic ingredients in the construction for the conserved current with respect to the vector $V^\mu$.

As the matter of fact, inspired by the form (2.3) for the usual Komar current, we assume that a conserved current $J_V$ ($\delta J_V = 0$) associated with the vector $V^\mu$ is a linear combination of the vectors $X^\mu_{(i)}$’s with weights $k_i$’s, namely,

$$ J^\mu_V = \sum_{i=1} k_i X^\mu_{(i)}. \quad (2.21) $$

Throughout the present work, the scalars $k_i$’s stand for arbitrary constant parameters. As a straightforward extension of the conventional Komar current, it is natural to impose the requirements that the number of $V^\mu$ in each $X^\mu_{(i)}$ is restricted to one and $X^\mu_{(i)}$’s are all the elements that can generate an arbitrary second-order derivative vector field of $V^\mu$. Hence the former ensures that the conserved current is linear in the vector field. As a consequence of Eq. (A.7), $X^\mu_{(i)}$’s have to be the quantities $\nabla^\mu \Phi_{\mu\nu}$, $\nabla_\nu \Phi$ and $\nabla^\mu \Psi_{\mu\nu}$. This implies that the conserved current $J_V$ can be further expressed as the following general form:

$$ J_V = k_1 (\nabla^\mu \Phi_{\mu\nu}) dx^\nu + k_2 d\Phi + (k_1 + k_3) (\nabla^\mu \Psi_{\mu\nu}) dx^\nu 
= 2k_1 \Box V + 2k_2 \delta V + k_3 \hat{\delta} dV, \quad (2.22) $$

as long as $V$ obeys the constraint

$$ k_1 \nabla^\mu \nabla_\mu \Phi_{\mu\nu} + k_2 \Box \Phi = 0. \quad (2.23) $$

Apparently, two simple cases of this constraint are the ones in which $k_1 = 0 = k_2$ for arbitrary vectors and $\Phi_{\mu\nu} = 0$ for the Killing vectors. Noteworthily, apart from the form (2.23), by means of Eq. (A.8), we are able to reformulate the constraint condition as the following form

$$ 2(k_1 + k_2) \Box \Phi + k_1 (V^\mu \nabla_\mu R + R_{\mu\nu} \Phi^{\mu\nu}) = 0. \quad (2.24) $$

Eq. (2.24) directly leads to the conclusion that the 1-form $J_V$ is co-closed for any vector $V^\mu$ fulfilling the condition $\Box \nabla_\mu V^\mu = 0$ when the spacetime is Ricci-flat. Enlightened by the structure of $J_V$, it is reasonable for us to introduce the most general quadratic operation $P(k_1, k_2, k_3)$ through the linear combination of the operators $\Box$, $d\delta$ and $\hat{\delta} d$, which
automatically leaves the form degree of any differential form unchanged and is expressed as

$$P(k_1, k_2, k_3) = k_1 \Box + k_2 d\delta + k_3 \delta d.$$  \hspace{1cm} (2.25)

By making use of such an operation, we are able to rewrite the current $J_V$ as $J_V = P(2k_1, 2k_2, k_3)V$. What is more, if both $k_1$ and $k_2$ are permitted to be functions of coordinates, it is worth noting that the condition (2.23) should be modified as

$$(\nabla^\mu k_1)(\nabla^\nu \Phi_{\mu\nu} - \nabla^\nu \Psi_{\mu\nu}) + k_1 \nabla^\mu \nabla^\nu \Phi_{\mu\nu} + k_2 \Box \Phi + (\nabla^\mu k_2)\nabla_\mu \Phi = 0.$$ \hspace{1cm} (2.26)

According to the second equality in Eq. (2.22), it is observed that $J_V$ covers the usual Komar current $J_K$ as the simplest case since $\Phi_{\mu\nu} = 0$ for the Killing vector $V = \xi$. More generally, when the vector $V^\mu$ is the so-called homothetic vector field (or homothety), namely, $\Phi_{\mu\nu} = C g_{\mu\nu}$ with the arbitrary constant factor $C$, the current $J_V$ turns to $J_V = (k_1 + k_3) \delta dV$. For another situation where the spacetime is Ricci-flat, namely, $R_{\mu\nu} = 0$, Eq. (A.1) enables us to obtain the current $J_V = (2k_1 + k_3) \delta dV$ for arbitrary $V^\mu$ when $k_2 = -k_1$. In fact, such a case can be covered by the one with $k_1 = k_2 = 0$.

Lastly, motivated by Eq. (2.2), we take into consideration of the current corresponding to the vector obeying the equation

$$P(2, \chi, \lambda)V = Y(V),$$ \hspace{1cm} (2.27)

where $Y(V)$ is an arbitrary 1-form with respect to the vector $V^\mu$ and the metric. In some sense, Eq. (2.27) can be regarded as the extension of Eq. (2.2) for the Killing vector field or the more general semi-Killing vector field [20] defined through the two constraints $2 \Box V = \delta dV$ and $\delta V = 0$, and it will be shown below that this equation also covers the ones associated to the almost- and conformal Killing vectors. As a specific example for Eq. (2.27), here we present the equation $P(2, 0, -1)V = \kappa V$ with a scalar $\kappa$. It has been demonstrated that an approximate Killing vector field might be defined via solving an eigenvalue problem associated to this equation [27, 28]. In a rather simple situation where $Y = 0$ and $\chi = \lambda = -2$, Eq. (A.1) enables us to simplify Eq. (2.27) as $R_{\mu\nu} V^\nu = 0$. What is more, according to Eq. (2.27), one may propose another generalization to the Killing vector field $\zeta$ in the absence of the requirement to preserve the linearity in the vector field, defined through

$$P(2, \chi, -1)\zeta = \lambda_1 L_\zeta \zeta + \lambda_2 \zeta \delta \zeta.$$ \hspace{1cm} (2.28)
Here $\mathcal{L}_\zeta \zeta = 2 \zeta^\nu \nabla_{(\mu} \zeta_{\nu)} dx^\mu$ and $\mathcal{L}_\zeta$ denotes the Lie derivative along the vector $\zeta$.

When $Y = 0$ in Eq. (2.27), the conserved current (2.22) for the vector determined by this equation transforms into $J_V = (2k_2 - \chi k_1)\hat{\delta}V + (k_3 - \lambda k_1)\hat{d}V$, while the constraint (2.23) becomes $2k_2 - \chi k_1 \Box \Phi = 0$. Specially, when $\chi = 2k_2/k_1$, the resulting conserved current further becomes $J_V = (k_3 - \lambda k_1)\hat{d}V$. Apart from $J_V$, the current associated with such a vector field could be defined as

$$\begin{align*}
J_Y^V &= a_1(2\Box V + \chi d\hat{\delta} V - Y) + a_2\hat{d}V \\
&= (a_2 - \lambda a_1)\hat{d}V.
\end{align*}$$

(2.29)

It is apparently divergence-free and recovered by $J_V$. In parallel, considering the vector field satisfying the well-known Proca equation $P(0,0,1)V = \lambda V$ (the constant parameter $\lambda \neq 0$) or the more general one $P(0,\chi,\lambda)V = Y(V)$, we obtain the related identically conserved current $J_P^V = a_1V + a_2\hat{d}V$ or $\tilde{J}_V^Y = a_1(\chi d\hat{\delta} V - Y) + a_2\hat{d}V$.

### 2.3 Constructing conserved currents out of $J_V$

In order to illustrate the generalized Komar current $J_V$, within this subsection, we shall take into consideration of a couple of examples associated to several interesting specific vector fields. Particularly, we are going to concentrate on the vectors describing approximate symmetries, such as the almost- and conformal Killing vectors, together with the affine collineation vector, attributed to the fact that Killing vector fields corresponding to exact symmetries can not be generally found in many physically important spacetimes.

First, when $\Phi_{\mu\nu}$ in Eq. (2.19) is restricted to $\nabla^\mu \Phi_{\mu\nu} = 2(\chi - 1)\nabla^\mu \Phi$, or equivalently $2\nabla^\mu \nabla_{(\mu} V_{\nu)} = \chi \nabla_{\nu} \nabla_\mu V^\mu$, the vector $V^\mu$ coincides with the almost-Killing vector $\kappa^\mu$, defined through \([20, 21, 22]\)

$$\Box \kappa_\mu + R^\nu_\mu \kappa_\nu = (\chi - 1)\nabla_\mu (\nabla \cdot \kappa).$$

(2.31)

Here $\nabla \cdot \kappa = \nabla_\mu \kappa^\mu = \hat{\delta} \kappa$. The above equation can be regarded as the generalization of the divergence for the ordinary Killing equation (the special case where $\chi = 1$) and it is reexpressed as

$$P(2,-\chi,-1)\kappa = 0$$

(2.32)
by virtue of Eq. (A.1). It corresponds to the \( \lambda = -1 \) case of Eq. (2.27) and completely coincides with Eq. (2.22) for the Killing vector field under the condition that \( \hat{\delta}\kappa = 0 \). Provided that the almost-Killing vector fulfills \( \hat{\delta}d\hat{\delta}\kappa = 0 \), we have \( \hat{\delta}\Box\kappa = 0 = \Box \hat{\delta}\kappa \) and \( \delta\Omega(\kappa) = 0 \), where \( \Omega \) is given by Eq. (A.3). What is more, when \( V^\mu = \kappa^\mu \), the current \( J_V \) goes to

\[
J_{AKV} = (\chi k_1 + 2k_2)d\delta\kappa + (k_1 + k_3)\hat{\delta}d\kappa.
\]

Here we have to impose the constraint \( k_1 \chi + 2k_2 = 0 \) or \( \hat{\delta}d\hat{\delta}\kappa = 0 \) to guarantee that \( \hat{\delta}J_{AKV} = 0 \). Particularly, when \( k_1 = 1 \), \( k_2 = -\chi/2 \) and \( k_3 = -2 \), it covers the current \( J_{AKV}^\nu \) in [6] 22. To see this clearly, note that Eq. (A.8) is of great use.

Second, if \( V^\mu \) is a conformal Killing vector, which maintains the metric up to an arbitrary multiplicative factor \( \phi \) (the so-called conformal factor) and is defined by

\[
\Phi_{\mu\nu} = \phi g_{\mu\nu},
\]

by analogy with the situation for the almost-Killing vector, Eq. (2.27) becomes

\[
P(2, -2/n, -1)V = 0
\]

and the current \( J_V \) transforms into

\[
J_{CKV} = (k_1 + nk_2)d\phi + (k_1 + k_3)\hat{\delta}d\phi,
\]

provided that \( k_1 + nk_2 = 0 \) or \( \Box \phi = 0 \). Obviously, Eq. (2.36) for the conformal Killing vector is just a special case of Eq. (2.27) for the almost-Killing vector. Letting \( k_1 = 1 \), \( k_2 = -1/4 \) and \( k_3 = -2 \) in the above equation, one finds that \( J_{CKV}^\mu \) is equivalent to the four-dimensional current \( J_C^\mu \) associated to a conformal vector field in [6].

Third, for the affine collineation vectors satisfying \( \nabla_\rho \Phi_{\mu\nu} = 0 \), we gain \( 2\Box V = \hat{\delta}dV \) and \( d\Phi = 0 \). Thus the corresponding current \( J_{ACV} = (k_1 + k_3)\hat{\delta}dV \), taking the same form as the one with respect to the Killing vector.

Fourth, we take into account the conserved current for the vector being of the form

\[
\Phi_{\mu\nu} = 2B_{\mu\nu} = (2\nabla^\rho \nabla_\sigma + R^{\rho\sigma})C_{\mu\rho\sigma},
\]

where \( C_{\mu\rho\sigma} \) is the standard Weyl tensor, while \( B_{\mu\nu} \) denotes the traceless, symmetric and conserved Bach tensor, corresponding to \( g^{\mu\nu}B_{\mu\nu} = 0 \), \( B_{\mu\nu} = B_{\nu\mu} \) and \( \nabla^\nu B_{\mu\nu} = 0 \) respectively. For such a vector, the conserved current \( J_{BV} = (k_1 + k_3)\hat{\delta}dV \). Besides, for the Killing vector \( \xi^\mu \), another conserved current associated to the Bach tensor is \( J_B^\mu = k_1 B^{\mu\nu} \xi_\nu \).
Fifth, supposed that the vector $V^\mu$ can be expressed as the Hodge coderivative of some 2-form $\omega$ in analogy with the works [9, 10], that is, $V = \hat{\delta} \omega$ (here the divergenceless vector field $V^\mu$ is unnecessarily restricted to the Killing vector), the conserved current $J_V$ then turns into

$$J_\omega = 2k_1 (R^\rho_{\mu\nu} \nabla^\rho \omega_{\mu\nu}) dx^\mu + (2k_1 + k_3) \hat{\delta} d\hat{\delta} \omega,$$

with the help of Eq. (A.1). Here $\hat{\delta} J_\omega = 0$ holds identically only if $\nabla^\mu (R_{\mu\nu} \nabla^\rho \omega_{\rho\nu}) = 0$.

Particularly, in the framework of the $n$-dimensional Einstein gravity theory described by the conventional Einstein-Hilbert Lagrangian (2.17), by means of the equation of motion $R_{\mu\nu} = 2\Lambda g_{\mu\nu}/(n-2)$, we obtain the conserved current

$$J_{gr}\omega = 4k_1 \Lambda \hat{\delta} \omega + (2k_1 + k_3) \hat{\delta} d\hat{\delta} \omega.$$

(2.39)

When $V^\mu$ is just the Killing vector $\xi^\mu$, namely, $\xi = \hat{\delta} \omega$, the co-closed 1-form $J_{gr}\omega |_{2k_1=-1,k_3=0}$ is consistent with the conserved current (2.18), as well as the one given in [9, 10]. It has been demonstrated in [23] that $J_{gr}\omega |_{2k_1=-1,k_3=0}$ might provide a novel understanding on the thermodynamics of asymptotically anti-de Sitter (AdS) black holes. Besides, if $V^\mu$ is the solution of the Proca equation $\hat{\delta} dV = \lambda V$, yielding $\delta V = 0$, the 1-form field $J_{gr}\omega$ can be proposed as its corresponding conserved current.

3 Various generalizations of the current $J_V$

In the present section, we are going to take into consideration of the generalizations of the conserved current $J_V$. We shall mainly concentrate on the extended current that is dependent of at most second-order derivatives of the vector field, as well as the one encompassing higher-order derivative terms of the vector.

First, if it is allowed that the current is able to admit the first-order covariant derivatives of $V^\mu$, the current $J_V$ may be generalized to the one

$$\tilde{J}_V = J_V + J_{V1},$$

(3.1)

in which the 1-form $J_{V1}$ is presented by

$$(J_{V1})_\mu = (b_1 \Phi g_{\mu\nu} + (b_2 + b_3) \Phi_{\mu\nu} + b_3 \Psi_{\mu\nu}) V^\nu$$

$$= b_1 \Phi V_\mu + b_2 \Phi_{\mu\nu} V^\nu + b_3 \nabla_\mu (V \cdot V).$$

(3.2)
Here $b_i (i = 1, 2, 3)$ are constant parameters or functions of the scalar $V \cdot V$. To make the coderivative of $\tilde{J}_V$ vanish, the vector $V^\mu$ has to obey that $\hat{\delta} J_{V1} = -\hat{\delta} J_V$. Besides, if the number of the vector $V^\mu$ in each term is not restricted to one, from a mathematical point of view, the most general 1-form that contains the terms of no higher than second order covariant derivative of the vector $V^\mu$ is

$$\tilde{J}_V = J_V + J_{V1} + J_{V2} + \lambda V,$$

where $J_{V2}$ is presented by Eq. (B.1) in the appendix B. $\tilde{J}_V$ is the conserved current associated with the vector satisfying the equation $\hat{\delta} \tilde{J}_V = 0$.

In some sense, the generalized current $\tilde{J}_V$ given by Eq. (3.3) can be applied to understand the Noether current corresponding to the diffeomorphism symmetry of a spacetime manifold, generated by an arbitrary vector field $V^\mu$, in the context of the Einstein gravity equipped with the Lagrangian (2.17). According to the standard Noether method, the conserved current associated with the diffeomorphism symmetry is given by $J_{NC} = -\hat{\delta} d V$ [21 25]. On the other hand, as what has been mentioned above, supposed that the current depends solely on the terms proportional to at most the second-order derivatives of a vector, the satisfactory conserved current for $V^\mu$ has to be $\tilde{J}_V$. However, since $\tilde{J}_V$ is necessarily divergence-free for arbitrary vectors, it is demanded that the 1-forms $J_{V1}$ and $J_{V2}$, as well as the parameter $\lambda$ together with the ones $k_1, k_2$ in $J_V$, disappear. This indicates that the current $\tilde{J}_V$ further becomes $\tilde{J}_V = k_3 \hat{\delta} d V$. In other words, the co-closed 1-form $k_3 \hat{\delta} d V$, which maintains the form degree of the arbitrary 1-form $V$ and has the lowest differential order, is the only covariant expression of the conserved current that is identically conserved and linear in any vector field. To this point, one is able to observe that the Noether current $J_{NC}$ has to contain the expression proportional to $\hat{\delta} d V$. The above discussions maybe give an explanation why Komar could succeed in finding out the simple expression $2\nabla_\nu \nabla[^\mu \xi^\nu]$ (here $\xi^\mu$ is an arbitrary vector field according to the notation in [1]) for the conserved current in [1] without the assistance of the standard Noether approach.

Next, if the conserved current, which is required to be linear in the vector $V^\mu$, is permitted to consist of all the terms proportional to $(2i)$-th-order (the integer $i$ is restricted to $1 \leq i \leq N$ for some given positive integer $N$) derivatives of this vector, the current $J_V$ can be straightforwardly generalized to the higher-derivative one $J_{HV}^{(2N)}$, taking the following
general form

\[ J_{HV}^{(2N)} = \sum_{i=1}^{N} J_{HV}^{(2i)} , \]

\[ J_{HV}^{(2i)} = \sum_{l} \alpha_{il} O_{il} (\Box, d, \hat{\delta}) V . \] (3.4)

In Eq. (3.4), \( \alpha_{il} \)'s are constant parameters, and \( l \) denotes an arbitrary non-zero permutation of the total \((2i - K)\) operators \( \Box, d, \) and \( \hat{\delta} \), where the integer \( K \) \((0 \leq K \leq i)\) represents the number of \( \Box \) in each operator \( O_{il} \). The number of \( \hat{\delta} \) or \( d \) is \((i - K)\) because the coderivative \( \hat{\delta} \) has to be paired with the exterior derivative \( d \). Besides, the combinations \( d^k \) and \( \hat{\delta}^k \) \((k \geq 2)\) are not allowed to exist due to the fact that \( \hat{\delta}^2 = d^2 = 0 \). Generally, the 1-forms \( O_{il} V \) have the forms \( \Box^i V, (d\hat{\delta})^i V \) and \( (\hat{\delta}d)^i V \), together with the following structures mixing the \((\Box, \hat{\delta}, d)\) operators

\[ \Box \cdots \hat{\delta} \cdots d \cdots V , \]
\[ d \cdots \Box \cdots \hat{\delta} \cdots V , \]
\[ \hat{\delta} \cdots \Box \cdots d \cdots V . \] (3.5)

For instance, when \( i = 1, 2 \), the 1-forms \( J_{HV}^{(2)} \) and \( J_{HV}^{(4)} \), consisting of second-order and fourth-order derivatives of the vector \( V \) respectively, can be expressed as

\[ J_{HV}^{(2)} = \alpha_{11} \Box V + \alpha_{12} d \hat{\delta} V + \alpha_{13} \hat{\delta} d V , \]
\[ J_{HV}^{(4)} = \alpha_{21} \Box^2 V + \alpha_{22} (d\hat{\delta})^2 V + \alpha_{23} (\hat{\delta}d)^2 V + \alpha_{24} \Box \hat{\delta} d V + \alpha_{25} \Box d \hat{\delta} V + \alpha_{26} d \Box \hat{\delta} V + \alpha_{27} \hat{\delta} d \Box V + \alpha_{28} \hat{\delta} d \Box V + \alpha_{29} \hat{\delta} d \Box V . \] (3.6)

In the above equation, \( J_{HV}^{(2)} \) is completely equivalent to \( J_{V} \) given by Eq. (2.22). Its combination with \( J_{HV}^{(4)} \) makes us write down the expression

\[ J_{HV}^{(4)} = J_{HV}^{(2)} + J_{HV}^{(4)} . \] (3.7)

For another example on the conserved currents consisting of the sixth-order derivatives of an arbitrary Killing vector, see the work [29].

As a special case of \( J_{HV}^{(2N)} \), we consider the following current generated by the \((\Box, d\hat{\delta}, \hat{\delta}d)\) operators, together with their various combinations

\[ J_{HV}^{(2N)} = \sum_{j} k_{j} f_{j} (\Box, d\hat{\delta}, \hat{\delta}d) V . \] (3.8)
Concretely, \( J^{(2N)}_{HV} \) can be written as

\[
J^{(2N)}_{HV} = \sum_{i=1}^{N} \left( J^{(2i)}_{HV} + \bar{J}^{(2i)}_{HV} \right).
\]  

\( (3.9) \)

Here the 1-forms \( \{ J^{(2i)}_{HV}, i = 1, \cdots, N \} \), which have to satisfy the constraint \( \bar{\delta} \sum_{i=1}^{N} J^{(2i)}_{HV} = 0 \) in order to guarantee the current \( J^{(2N)}_{HV} \) to be co-closed, is defined through

\[
J^{(2i)}_{HV} = \sum_{i_2,i_4,\cdots,j_0,j_1,\cdots} s^{j_0 j_1 \cdots} j_{i_2 i_4 \cdots} P^{j_0} \bar{V}^{j_1 j_2 \cdots} + \sum_{i_2,i_4,\cdots,j_0,j_1,\cdots} t^{j_0 j_1 \cdots} j_{i_2 i_4 \cdots} P^{j_0} \bar{V}^{j_1 j_2 \cdots}
\]

\( (3.10) \)

with the 1-forms \( \bar{V}^{j_1 j_2 \cdots} \) defined by

\[
\bar{V}^{j_1 j_2 \cdots} = (P^{j_i} P^{j_2} P^{j_3} P^{j_4} P^{j_5} P^{j_6} \cdots) \bar{V},
\]

\( (3.11) \)

while the co-closed 1-forms \( \{ \bar{J}^{(2i)}_{HV}, i = 1, \cdots, N \} \) can be expressed as a similar form like the usual Komar current \( J_K \), that is,

\[
\bar{J}^{(2i)}_{HV} = -\bar{\delta} d\bar{V}^{(i)},
\]

\( (3.12) \)

by virtue of the 1-form \( \bar{V}^{(i)} \), given by

\[
\bar{V}^{(i)} = \sum_{i_2,i_4,\cdots,j_0,j_1,\cdots} u^{j_0 j_1 \cdots} j_{i_2 i_4 \cdots} P^{j_0-1} \bar{V}^{j_1 j_2 \cdots}
\]

\( (3.13) \)

In Eqs. \( (3.10) \) and \( (3.13) \), \( s^{j_0 j_1 \cdots} \), \( t^{j_0 j_1 \cdots} \), \( s^{j_0 j_1 \cdots} \) and \( t^{j_0 j_1 \cdots} \) refer to arbitrary constant parameters. The positive integers \( i_2, i_4, \cdots \) run over the numbers 2 and 3, the integer \( j_0 \geq 1 \), and the non-negative integers \( j_0, j_1, j_2, \cdots \) are constrained by \( j_0 + j_1 + j_2 + \cdots = i \). For convenience, we have used the notations

\[
P_1 = P(1,0,0), \quad P_2 = P(0,1,0), \quad P_3 = P(0,0,1),
\]

\( (3.14) \)

while \( P^0_1 = P^0_2 = P^0_3 \) denote identity operators. By the way, it should be pointed out that the two operators \( P_2 \) and \( P_3 \) have to be separated by \( P_1 \) attributed to the fact that \( P_2 P_3 = 0 = P_3 P_2 \). Apparently, the current \( J^{(2N)}_{HV} \) includes \( J^{(1)}_H \) in work [18] as a special case. When \( N = 1 \), \( \bar{V}^{j_1 j_2 \cdots} = V \) and it accordingly becomes \( J^{(2)}_{HV} = J_V \). In contrast with the ordinary Komar current \( J_K \), the co-closed 1-form \( \bar{J}^{(2i)}_{HV} \) is able to be viewed as the Noether current corresponding to the diffeomorphism symmetry generated by the vector field \( \bar{V}^{(i)} \), testifying the one-to-one correspondence between the vector field and the conserved current.
in another manner. Apart from this, it may be interpreted as the higher-order derivative correction to $J_K$. Therefore, $\tilde{J}^{(2)}_{HV}$ or $J^{(2N)}_{HV}$ could have the applicability in computing the conserved charges of higher-derivative extended gravity theories, such as the well-known Lovelock gravity, and the so-called (generalized) quasi-topological gravities.

As an example of Eq. (3.9), we specialize to the situation where $N = 2$. The 1-form $J^{(2N)}_{HV}$ accordingly turn into

$$J^{(4)}_{HV} = J_V + k_{11}P_1^2V + k_{22}P_2^2V + k_{33}P_3^2V$$
$$+ k_{12}P_1P_2V + k_{13}P_1P_3V$$
$$+ k_{21}P_2P_1V + k_{31}P_3P_1V.$$  \hspace{1cm} (3.15)

Supposed that $V$ is the Killing vector $\xi$, the commutation relation given by Eq. (A.6) enables us to write down

$$J^{(4)}_{HV}(\xi) = (k_1 + k_3)\hat{\delta}d\xi + \frac{1}{2}(k_{31} + 2k_{33})(\hat{\delta}d)^2\xi$$
$$+ (k_{11} + 2k_{13})\Box^2\xi,$$  \hspace{1cm} (3.16)

while $\mathcal{J}^{(4)}_{HV}(\xi) = J^{(4)}_{HV}(V \rightarrow \xi)$ can be expressed as [29]

$$\mathcal{J}^{(4)}_{HV}(\xi) = J^{(4)}_{HV}(\xi) + \alpha_{28}\hat{\delta}\Box d\xi.$$  \hspace{1cm} (3.17)

Apparently, the 1-forms $J^{(4)}_{HV}(\xi)$ and $\mathcal{J}^{(4)}_{HV}(\xi)$ are conserved under the condition that $(k_{11} + 2k_{13})\hat{\delta}\Box^2\xi = 0$. Within [29], it has been demonstrated that the surface integral with respect to the potential derived from $\mathcal{J}^{(4)}_{HV}(\xi)$ can be used to define the mass and angular momentum of Einstein gravity in asymptotically anti-de Sitter (AdS) spacetimes. If adding a co-closed 1-form $J^{(4)}_{R}(\xi)$ deduced from Eq. (2.14) to $\mathcal{J}^{(4)}_{HV}(\xi)$, we are able to get a more general identically conserved current depending at most upon the fourth-order derivatives of the Killing vector, that is,

$$\hat{J}^{(4)}_{HV}(\xi) = k_1\hat{\delta}d\xi + k_2(\hat{\delta}d)^2\xi + k_3\hat{\delta}\Box d\xi + \hat{\delta}X_{(2)} + J^{(4)}_{R}(\xi).$$  \hspace{1cm} (3.18)

In Eq. (3.18), the 2-form $X_{(2)}$ is defined by

$$X_{(2)}^{\mu\nu} = 2\left(\beta_1R^{\mu\nu}_{\rho\sigma} + \beta_2R^{\mu}_{[\rho}\delta^{\nu]}_{\sigma] + \beta_3R\delta^{[\mu}_{\rho}\delta^{\nu]}_{\sigma]}\right)\nabla^\rho\xi^\sigma,$$  \hspace{1cm} (3.19)

where $\beta_i$'s are arbitrary constant parameters. By utilizing Eqs. (A.1) and (A.3), $X_{(2)}$ is rewritten as

$$X_{(2)} = \beta_1d\hat{\delta}d\xi - \beta_1\Box d\xi + \beta_3Rd\xi$$
$$- (2\beta_1 + \beta_2)R^\mu_{\mu}\nabla_\nu\xi_\rho dx^\mu \wedge dx^\nu.$$  \hspace{1cm} (3.20)

17
particularly, for the einstein manifold with $R_{\mu \nu} = \lambda g_{\mu \nu}$, we have $X_{(2)} = \beta_1 d\delta d \xi - \beta_1 \Box d \xi + \lambda(2\beta_1 + \beta_2 + n\beta_3) d \xi$. the identically conserved $J^{(4)}_R(\xi)$ is defined by

$$J^{(4)}_R(\xi) = (a_0 + a_1 R + a_2 \Box R + a_3 R^2 + a_4 R_{\mu \nu} R^{\mu \nu} + a_5 R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma}) \xi.$$  

(3.21)

here the coefficient of $\xi$ in $J^{(4)}_R(\xi)$ is a special case of $\lambda$ given by eq. (2.11), and it can be understood as the lagrangian density of the gravity theories with fourth-order derivatives. thus the conserved current (3.18) may be applicable in the definition of the conserved charges for the no more than fourth-order derivative gravity theories, such as the einstein gravity, the einstein-weyl gravity and the einstein-gauss-bonnet gravity.

more generally, according to eqs. (2.12) and (3.12), a conserved current associated with no more than $(2N)$-th order derivatives of the killing vector $\xi$ can be presented by

$$\hat{J}^{(2N)}_{HV}(\xi) = \sum_{i=1}^{N} \hat{J}^{(2i)}_{HV}(V \rightarrow \xi) + \xi \sum_{j} a_j F_j.$$  

(3.22)

here the scalars $F_j$'s are demanded to be dependent at most of $(2N)$-th order derivatives of the metric tensor.

Furthermore, if the vector $V^\mu$ can not ensure the coderivatives of the 1-forms $J_V, \hat{J}_V, \hat{J}_V^{(2N)}$ and $\hat{J}^{(2N)}_{HV}$ disappear, like in [6], we can introduce a scalar field $\psi(x)$ to cancel out the non-vanishing divergence. for example, when $\hat{\delta} \sum_{i=1}^{N} \hat{J}^{(2i)}_{HV} \neq 0$, we are able to modify $\hat{J}^{(2N)}_{HV}$ as the conserved current $\hat{J}^{(2N)}_{GHV}$, presented by

$$\hat{J}^{(2N)}_{GHV} = \hat{J}^{(2N)}_{HV} + d\psi.$$  

(3.23)

taking a divergence, we see that the scalar $\psi$ has to fulfill the wave equation

$$\Box \psi + \hat{\delta} \sum_{i=1}^{N} \hat{J}^{(2i)}_{HV} = 0.$$  

(3.24)

more concretely, by letting $\psi = (\chi k_1 + 2k_2)\delta \kappa$, the 1-form $J_{AKV}$ given by eq. (2.33) can be put into the divergence-free form ($J_{AKV} - d\psi$).

4 Conclusions and remarks

by virtue of the quadratic differential operators $\Box, d\delta$ and $\delta d$, together with their linear combination $P(k_1, k_2, k_3)$ given by (2.25), we have generalized the usual komar current
to the ones for Killing vectors or generic vector fields. Starting with the Komar conserved current (2.3) expressed by the above operations, we have derived the most general conserved current (2.13) under the condition that the current without the linearity in the Killing vector field relies at most upon the second-order derivatives of this vector, as well as its generalization (2.14). Subsequently, the current (2.3) is extended to the one $J_{V}$ (2.22) for a generic vector field constrained by Eq. (2.23). As some examples to $J_{V}$, we have derived the conserved currents associated with a couple of interesting specific vector fields, such as the almost- and conformal Killing vectors, together with the affine collineation vector. Remarkably, in terms of the combined operator $P$, all the equations involved in second-order derivatives of these vectors can be written as the unified formulation (2.27). Furthermore, the operator $P$ assists us to conveniently generalize $J_{V}$ to the conserved currents $\mathcal{J}_{HV}^{(2N)}$ and $H_{HV}^{(2N)}$ given by Eqs. (3.24) and (3.9) respectively, which contain higher-order derivatives of a general vector field and might be used to defined the conserved charges of higher-derivative gravity theories. In order to understand the general conserved currents $\mathcal{J}_{HV}^{(2N)}$ and $H_{HV}^{(2N)}$, we propose that an effective approach is to compare it with the Noether current obtained through the standard Noether method.

Apparently, the conserved currents in the present work are constructed from the mathematical point of view. However, they have physical significance. Among them, one or more may be expected to possess some nice properties beyond the usual Komar current. For instance, although the conventional Komar current breaks down in the definition for the mass of asymptotically AdS black holes, in the work [29], it has been shown that it corrected by the fourth-order derivatives of the Killing vector can yield their physical mass. Therefore, in order to understand these currents, some research desired is to investigate the applications of the conserved currents in the definition for the conserved charges of various gravity theories, particularly of the higher-derivative gravities. Noteworthily, when these currents are applied to compute the charges of some specific gravity theory, one may encounter the problem how to fix the coefficients in them for the sake of finding out the appropriate conserved currents. Anyway, like in [29], we think that such conserved currents at least have to give rise to convergent charges that satisfy the first law of thermodynamics or match those via other typical methods. We look forward to the further investigations to answer all the mentioned problems.
A Introduction to the operators $\Box$, $\hat{\delta}d$ and $d\hat{\delta}$

Within the work [18], it has been demonstrated that both the two fundamental operators $\hat{\delta}d$ and $d\hat{\delta}$, constructed out of the combination of the Hodge star $\star$ and the exterior derivative $d$, can be employed in the computations of the second-order derivatives of an arbitrary $p$–form $F$ and maintain its form degree unchanged. Here the Hodge star operation is defined through $(\star F)_{\mu_1 \cdots \mu_q} = (p!)^{-1} F^{\nu_1 \cdots \nu_p} \epsilon_{\nu_1 \cdots \nu_p \mu_1 \cdots \mu_q}$ under its action on the $p$–form $F$, while the exterior differential of $F$ is presented by $(dF)_{\mu_0 \cdots \mu_p} = (p + 1) \nabla_{[\mu_0} F_{\mu_1 \cdots \mu_p]}$. In terms of the linear combination of $d\hat{\delta}$ and $\hat{\delta}d$, the action of the ordinary generally covariant d’Alembertian operator $\Box = g^\mu_\nu \nabla_\mu \nabla_\nu$ on $F$, which leaves the form degree unaltered as well, could be written as the so-called Weitzenböck identity [18]

$$\Box F = d\hat{\delta}F + \hat{\delta}dF - \Omega(F), \quad (A.1)$$

where in component notation the $p$-forms $\Box F$, $d\hat{\delta}F$, and $\hat{\delta}dF$ are given by

$$(\Box F)_{\mu_1 \cdots \mu_p} = \Box F_{\mu_1 \cdots \mu_p},$$

$$(d\hat{\delta}F)_{\mu_1 \cdots \mu_p} = p \nabla_{[\mu_1} \nabla^\nu F_{\nu]\mu_2 \cdots \mu_p]},$$

$$\hat{\delta}dF)_{\mu_1 \cdots \mu_p} = \Box F_{\mu_1 \cdots \mu_p} - p \nabla^\nu \nabla_{[\mu_1} F_{\nu] \mu_2 \cdots \mu_p]}, \quad (A.2)$$

respectively, while the component of $p$-form $\Omega(F)$ takes the form

$$(\Omega(F))_{\mu_1 \cdots \mu_p} = -p R^\rho_{[\mu_1} F_{\sigma]\mu_2 \cdots \mu_p]} + \frac{p(p - 1)}{2} R^\rho_{[\mu_1 \mu_2} F_{\sigma] \mu_3 \cdots \mu_p]}.$$ \quad (A.3)

Here $R_{\rho\sigma\mu\nu}$ denotes the standard Riemann–Christoffel tensor of the spacetime metric, defined through $2 \nabla_{[\mu} \nabla_{\nu]} V_\rho = R_{\mu\nu\rho\sigma} V^\sigma$. In terms of Eq. \[(A.2)\], we obtain $(\hat{\delta}d)(d\hat{\delta}) = (d\hat{\delta})(\hat{\delta}d) = 0$, as well as the commutation relations $[d\hat{\delta}, \Box] F = \Omega(d\hat{\delta}F) - d\hat{\delta}\Omega(F)$ and $[\hat{\delta}d, \Box] F = \Omega(\hat{\delta}dF) - \hat{\delta}d\Omega(F)$. Particularly, in the case where $F$ becomes the 1-form $V$, the former is concretely written as

$$[d\hat{\delta}, \Box] V = \frac{1}{2} (\nabla_\mu (V^\nu \nabla_\nu R) + 2 \nabla_\mu (R_{\rho\sigma} V^\nu \nabla^\sigma) - 2 R_{\mu\nu\sigma} \nabla^\nu \nabla^\sigma V_\sigma) dx^\mu, \quad (A.4)$$
and the latter turns into

$$\mathbf{V} = \left[ (\nabla^\sigma \nabla_\rho R_{\rho\mu\sigma\nu} - R^\rho_{\sigma\nu} R_{\rho\mu\sigma\nu} + R_\mu^\rho R_{\rho\nu} ) V^\nu + (\nabla^\nu R_{\rho\sigma\mu\nu} + \nabla^\nu R_{\nu\mu\rho\sigma} + \nabla_\sigma G_{\rho\mu} ) \nabla^\sigma V^\rho - R^\mu_{\sigma\nu} \nabla_\sigma V_\nu + R_{\mu\nu} \nabla^\nu \nabla^\sigma V_\sigma \right] dx^\mu. \quad (A.5)$$

When $\mathbf{V}$ is the Killing vector $\xi$, Eqs. (A.4) and (A.5) further become respectively

$$d \hat{\delta} \square \xi = \square d \hat{\delta} \xi = 0,$$

$$\hat{\delta} d \square \xi = \square \hat{\delta} d \xi - \square^2 \xi + \Omega(\square \xi). \quad (A.6)$$

Eq. (A.2) indicates that $\square$, $d \hat{\delta}$ and $\hat{\delta} d$ denote all the three elementary second-order covariant derivative operators that can preserve the form degree of an arbitrary $p$-form. To see this clearly, letting the covariant derivative operation $\nabla_\mu$ act on a $p$-rank antisymmetric tensor $F_{\mu_1 \cdots \mu_p}$ twice in succession, one finds that all the resultant $p$-rank antisymmetric tensors have to take the three types of forms:

$$\nabla_\nu \nabla_\nu F_{\mu_1 \cdots \mu_p},$$

$$\nabla_{[\mu_1} \nabla^{\nu} F_{\mu_2 \cdots \nu] \cdots \mu_p},$$

$$\nabla^{\nu} \nabla_{[\mu_1} F_{\mu_2 \cdots \nu] \cdots \mu_p}. \quad (A.7)$$

In the above equation, the first and second expressions are covered by the components of $\square F$ and $d \hat{\delta} F$ respectively, while the third one in the notation of differential forms is just the linear combination of the $p$-forms $\square F$ and $\hat{\delta} d F$. Consequently, as what has been shown in [18], instead of the operators $\{ \nabla^\nu \nabla_\nu, \nabla_\mu \nabla^{\nu}, \nabla^{\nu} \nabla_\mu \}$, the three ones $\{ \square, d \hat{\delta}, \hat{\delta} d \}$ can be always applied to act on antisymmetric tensors because of their own advantages in differential forms. Moreover, Eq. (A.7) obviously shows that all the three operations $\square$, $d \hat{\delta}$ and $\hat{\delta} d$ are the fundamental ones of the lowest differential order, which leave the form degree unchanged.

Additionally, for an arbitrary vector $V^\mu$, Eq. (A.1) yields the commutation relation between $\hat{\delta}$ and $\square$, that is,

$$\left[ \hat{\delta}, \square \right] \mathbf{V} = \frac{1}{2} V^\mu \nabla_\mu R + R_{\mu\nu} \nabla^{(\mu} V^{\nu)} \quad (A.8)$$

Apart from this, the more general commutation relation $\left[ \hat{\delta}, \square^m \right] \mathbf{V}$ has been presented in the work [18].

21
B The expression for $J_{V^2}$

We assume that the 1-form $J_{V^2}$ only depends on the terms proportional to the second-order derivatives of the vector $V^\mu$, and the number of $V^\mu$ in each term is permitted to be more than one. Hence, the components of the 1-form $J_{V^2}$ take the most general form

$$
(J_{V^2})_\mu = \lambda_{01} \nabla^\nu \Phi_{\mu\nu} + \lambda_{02} \nabla_\mu \Phi + \lambda_{03} \nabla^\nu \Psi_{\mu\nu} + V^\nu \left( \lambda_{11} \Phi_{\mu\nu} + \lambda_{12} \Phi \Psi_{\mu\nu} + \lambda_{13} \Phi_{\mu\rho} \Phi_{\rho\nu} + \lambda_{14} \Phi_{\mu\nu} \Psi_{\rho\mu} + \lambda_{15} \Phi_{\rho\mu} \Psi_{\mu\nu} + \lambda_{16} \Psi_{\nu} \Psi_{\mu\nu} \right)
$$

with the scalar $U$ defined by

$$
U = u_{01} \Phi^2 + u_{02} \Phi_{\rho\sigma} \Phi_{\rho\sigma} + u_{03} \Psi_{\rho\sigma} \Psi_{\rho\sigma} + V^\rho \left( u_{11} \nabla_\rho \Phi + u_{12} \nabla^\sigma \Phi_{\rho\sigma} + u_{13} \nabla^\sigma \Psi_{\rho\sigma} \right)
$$

In Eqs. (B.1) and (B.2), the scalars $\lambda_{ij} = \tilde{\lambda}_{ij} f_{ij}(V)$ and $u_{ij} = \tilde{u}_{ij} h_{ij}(V)$. Here $\tilde{\lambda}_{ij}$’s and $\tilde{u}_{ij}$’s stand for arbitrary constant parameters since it is assumed that the vector field and the metric tensor, together with their derivatives, are the only variables of the conserved currents throughout the present work.

C Some properties of Killing vectors

In this appendix, we shall provide some important properties of the Killing vector $\xi^\mu$, which are tightly relevant to our calculations.

In particular, when the differential form $F$ in Eqs. (A.1) and (A.2) is the 1-form Killing vector field $\xi$, according to both the equations, one obtains the well-known relationship

$$
\Box \xi = \Omega(\xi),
$$

or

$$
\Box \xi_\mu + R_{\mu\nu} \xi^\nu = 0. \tag{C.1}
$$

Additionally, as a special case where the 1-form $V$ in Eq. (A.8) is just the Killing vector field $\xi$, in terms of the identity $2 \Box \xi = \hat{\delta} d \xi$ from Eq. (2.20), giving rise to $\hat{\delta} \Box \xi = 0$, together with the divergenceless equation $\hat{\delta} \xi = 0$, Eq. (A.8) leads to

$$
\xi^\mu \nabla_\mu R = 0, \quad \nabla^\mu \left( \Box^2 \xi_\mu + R_{\mu\nu} R^{\rho\sigma} \xi_{\rho\sigma} \right) = 0, \tag{C.2}
$$
The first equation in Eq. (C.2) can be written as $L_\xi R = 0$. Here $L_\xi$ denotes the Lie derivative along the Killing vector $\xi^\mu$.

Letting the covariant derivative act on the Killing vector $\xi^\mu$ twice, one obtains the following equation

$$\nabla_\mu \nabla_\nu \xi_\rho = R_{\rho\mu\nu\sigma} \xi^\sigma .$$

(C.3)

This is attributed to the fact that $\nabla_\mu \xi_\nu = -\nabla_\nu \xi_\mu$ and $d^2 \xi = 0$ yields $\nabla_\mu \nabla_\nu \xi_\rho = 0$. Obviously, Eq. (C.1) can also arise from the contraction between the $\mu$ and $\nu$ indices in the above equation. In light of Eq. (C.3), one further arrives at the identity

$$\xi^\rho \left( \nabla_\mu \nabla_\nu \xi_\rho \right) \nabla_\mu \xi_\nu = 0 ,$$

(C.4)

or equivalently,

$$R_{\rho\mu\nu\sigma} \xi^\rho \nabla_\mu \xi_\nu = 0 .$$

(C.5)

Expanding the above equation yields

$$\xi_\rho \left( \nabla_\rho \xi_\sigma \right) \nabla_\sigma \xi_\rho = 0 .$$

(C.6)

Expanding the above equation yields

$$\xi_\rho \left( \nabla_\rho \xi_\sigma \right) \nabla_\sigma \xi_\rho = 0 .$$

(C.7)

giving rise to

$$\xi_\rho \left( \nabla_\rho \xi_\sigma \right) = 0 .$$

(C.8)

It is worth noticing that the above equation can be deduced instead from the one $\xi_\rho \nabla_\nu (\xi_\sigma \Box_\sigma) = -L_\xi \left( R_{\mu\nu\xi} \xi^\mu \xi^\nu \right)$ by virtue of Eq. (C.1), while the latter vanishes since $L_\xi R_{\mu\nu} = 0$ and $L_\xi \xi^\mu = 0$. Apart from this, by making use of the commutation relation $L_\xi \nabla_\mu = \nabla_\mu L_\xi$ between the Lie derivative along the Killing vector and the covariant derivative, we are able to prove that

$$\xi^\mu \nabla_\mu \left[ (\nabla_\nu \xi^2) (\nabla_\rho \xi^2) \right] = 2 (\nabla_\nu \xi^2) \nabla_\rho \xi^2 = 0 .$$

(C.9)

More generally, for the scalars $F_i$’s and $W_i$’s defined by Eq. (2.12), their Lie derivatives along the Killing vector are read off as

$$\xi^\mu \nabla_\mu F_i = F_i \left( L_\xi g \right) + F_i \left( L_\xi R \right) = 0 ,$$

$$\xi^\mu \nabla_\mu W_i = W_i \left( L_\xi g \right) + W_i \left( L_\xi R \right) = 0 .$$

(C.10)

Lastly, another useful identity associated with the Killing vector field is

$$\left( \nabla_\mu \xi^\nu \right) \left( \nabla_\rho \xi_\sigma \right) = 0 .$$

(C.11)
D Summarization of conserved currents

In this appendix, for convenience, we summarize our main results in this paper, which are the conserved currents presented by Table 1.

Table 1: Conserved currents with respect to vector fields

| Current        | In equation | Constraint               |
|----------------|-------------|--------------------------|
| $J_K, \dot{J}_K$ | (2.1), (2.3) | None                     |
| $\dot{J}_K, J_K$ | (2.4), (2.13) | Eq. (2.7)                |
| $J_V, J_V^\gamma$ | (2.22), (2.29) | Eqs. (2.23), (2.27)    |
| $J_{AKV}$      | (2.33)      | $(k_1 \chi + 2k_2)\delta d\delta \kappa = 0$ |
| $J_{CKV}$      | (2.36)      | $(k_1 + nk_2)\Box \phi = 0$ |
| $J^{\omega}$   | (2.39)      | None                     |
| $\dot{J}_V, J_V$ | (3.1), (3.3) | $\delta J_V, \delta \dot{J}_V = 0$ |
| $\mathcal{J}^{(2N)}_{HV}, \dot{\mathcal{J}}^{(2N)}_{HV}$ | (3.4) | $\delta \mathcal{J}^{(2N)}_{HV} = 0$ |
| $\mathcal{J}^{(2N)}_{HV}$ | (3.9) | $\delta \sum_{i=1}^N \dot{J}^{(2i)}_{HV} = 0$ |
| $\mathcal{J}^{(2i)}_{HV}, \dot{\mathcal{J}}^{(2i)}_{HV}$ | (3.10), (3.12) | $\delta \mathcal{J}^{(2i)}_{HV} = 0$ |
| $\mathcal{J}^{(4)}_{HV}(\xi)$ | (3.16), (3.17) | $(k_1 + 2k_13)\delta \Box^2 \xi = 0$ |
| $\dot{\mathcal{J}}^{(4)}_{HV}(\xi)$ | (3.18) | None                     |

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