Error Analysis Of Symmetric Linear/Bilinear Partially Penalized Immersed Finite Element Methods For Helmholtz Interface Problems

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Abstract

This article presents an error analysis of the symmetric linear/bilinear partially penalized immersed finite element (PPIFE) methods for interface problems of Helmholtz equations. Under the assumption that the exact solution possesses a usual piecewise $H^2$ regularity, the optimal error bounds for the PPIFE solutions are derived in an energy norm and the usual $L^2$ norm provided that the mesh size is sufficiently small. A numerical example is conducted to validate the theoretical conclusions.

1 Introduction

This article is about the error analysis for the linear and bilinear partially penalized immersed finite element (PPIFE) methods developed in [29] for solving interface boundary value problems of the Helmholtz equation [5, 23] that is posed in a bounded domain $\Omega \subseteq \mathbb{R}^2$: find $u(X)$ that satisfies the Helmholtz equation and the first-order absorbing boundary condition:

\begin{align}
-\nabla \cdot (\beta \nabla u) - k^2 u &= f, \quad \text{in } \Omega^- \cup \Omega^+, \\
\beta \frac{\partial u}{\partial n_\Omega} + iku &= g, \quad \text{on } \partial \Omega,
\end{align}

(together with the jump conditions across the interface [4, 5, 9, 22, 23]:

\begin{align}
[u]_\Gamma := u^-|_\Gamma - u^+|_\Gamma &= 0, \\
[\beta \nabla u \cdot n]|_\Gamma := \beta^- \nabla u^- \cdot n|_\Gamma - \beta^+ \nabla u^+ \cdot n|_\Gamma &= 0,
\end{align}

where the domain $\Omega \subseteq \mathbb{R}^2$ is divided by an interface curve $\Gamma$ into two subdomains $\Omega^-$ and $\Omega^+$, with $\overline{\Omega} = \Omega^- \cup \Omega^+ \cup \Gamma$, $u^s = u|_{\Omega^s}$, $s = \pm$ and $n$ is the unit normal vector to the interface $\Gamma$, $k$ is the wave number, $i = \sqrt{-1}$, $n_\Omega$ is the unit outward normal vector to $\partial \Omega$, and the coefficient $\beta$ is a piecewise positive constant function such that

\begin{equation}
\beta(X) = \begin{cases} 
\beta^- & \text{for } X \in \Omega^-, \\
\beta^+ & \text{for } X \in \Omega^+.
\end{cases}
\end{equation}

The Helmholtz boundary value problems without interface have been widely studied in the context of finite element methods, including classic finite element methods [2, 3, 19, 20] and interior penalty Galerkin

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These finite element methods can also be used to solve Helmholtz interface problems with body-fitting meshes; however, their efficiency might be impeded for some applications where the locations and geometries of the interface keep changing because these conventional finite element methods require the meshes to be generated repeatedly according to the evolving interface configurations. To alleviate the burden of re-meshing, numerical methods based on interface-independent meshes have been developed, such as immersed interface methods (IIM) and cut finite element methods (CutFEMs). We refer the readers to [34, 35, 36] for applying those methods to Helmholtz interface problems.

The PPIFE methods developed in [29] are a class of finite element methods for solving Helmholtz interface problems with interface-independent meshes. Extensive numerical experiments reported in [29] clearly demonstrate the optimal convergence of these PPIFE methods, and this observation motivates us to carry out related error analysis to theoretically confirm their optimal convergence.

The error analysis of traditional finite element methods for solving Helmholtz problems can be found in [1, 20, 30]. Error analysis of traditional finite element methods for Helmholtz interface problems can be found in recent publications [8, 15] where the stability and the optimal error bounds for the finite element solution were proved. We note that Schatz’s argument is an important technique in the error analysis of traditional finite element methods for Helmholtz problems, with or without interface.

In this article, we also follow the framework of Schatz’s argument to carry out the error analysis for the linear and bilinear symmetric PPIFE methods developed in [29] for Helmholtz interface problems with a Robin boundary condition. We utilize the coercivity and continuity of the bilinear form corresponding to the elliptic operator [14, 28] to establish Gårding’s inequality and continuity of the bilinear form in these PPIFE methods which are key ingredients in Schatz’s argument. We also derive a special trace inequality that is valid for IFE functions which are not $H^1$ functions in general. In particular, under suitable assumptions about the regularity of the exact solution and the mesh size, we are able to establish the optimal error bounds in both an energy norm and the standard $L^2$ norm for these PPIFE methods for solving the Helmholtz interface problems.

The layout of the article is as follows: Section 2 introduces the notations and assumptions to be used in this article, and a symmetric PPIFE method for the Helmholtz interface problem (1.1) is described. In Section 3, optimal error bounds are derived for this PPIFE method in both the energy norm and $L^2$ norm. A numerical example is given in Section 4 to validate the theoretical results in Section 3.

## 2 Notations and PPIFE Methods

Without loss of generality, we assume that $\Omega \subset \mathbb{R}^2$ is a polygonal domain. We let $\mathcal{T}_h$ be a triangular or a rectangular mesh of $\Omega$ whose set of nodes and set of edges are $\mathcal{N}_h$ and $\mathcal{E}_h$, respectively. We assume the mesh $\mathcal{T}_h$ is independent of the interface $\Gamma$; hence, some elements of $\mathcal{T}_h$ will intersect with $\Gamma$ which will be called interface elements. We use $\mathcal{T}^i_h$ to denote the set of interface elements and use $\mathcal{T}^n_h$ for the set of non-interface elements. Similarly, we let $\mathcal{E}^i_h$ and $\mathcal{E}^n_h$ be the set of interface edges and the set of non-interface edges, respectively. In addition, we use $\mathcal{E}_h^i$, $\mathcal{E}_h^n$ and $\mathcal{E}_h^I$ for the set of interior edges, the set of interior interface edges and the set of interior non-interface edges, respectively.

For each element $T \in \mathcal{T}_h$, we define its index set as $\mathcal{I}_T = \{1, 2, 3\}$ when $T$ is triangular, but $\mathcal{I}_T = \{1, 2, 3, 4\}$ when $T$ is rectangular. Let $\psi_{j,T}, \ j \in \mathcal{I}_T$ be the standard linear or bilinear Lagrangian shape functions on $T \in \mathcal{T}_h$ such that

$$\psi_{j,T}(A_i) = \delta_{ij}, \ \forall i, j \in \mathcal{I}_T. \quad (2.1)$$

We can use these shape functions to generate the following real polynomial spaces:

$$\bar{\mathcal{P}}(T) = \text{Span}\{\psi_{j,T}, \ j \in \mathcal{I}_T\} \quad \text{or} \quad \bar{\mathcal{Q}}(T) = \text{Span}\{\psi_{j,T}, \ j \in \mathcal{I}_T\}, \ \forall T \in \mathcal{T}_h, \quad (2.2)$$
depending on whether $T$ is triangular or rectangular. The related complex polynomial spaces are:

$$P(T) = \{v = v_1 + iv_2 : v_1, v_2 \in \tilde{P}(T)\} \quad \text{or} \quad Q(T) = \{v = v_1 + iv_2 : v_1, v_2 \in \tilde{Q}(T)\}, \quad \forall T \in \mathcal{T}_h. \quad (2.3)$$

Throughout this article, without stating otherwise, the notation “Span” represents spanning in the real number field.

Then, on every non-interface element $T \in \mathcal{T}_h^n$, the local real IFE space is the usual local real finite element space

$$\tilde{S}_h(T) = \tilde{P}(T) \quad \text{or} \quad \tilde{S}_h(T) = \tilde{Q}(T), \quad \forall T \in \mathcal{T}_h^n, \quad (2.4)$$

and its complex counterpart is

$$S_h(T) = P(T) \quad \text{or} \quad S_h(T) = Q(T), \quad \forall T \in \mathcal{T}_h^n. \quad (2.5)$$

In order to describe local IFE spaces on interface elements, we adopt the following standard assumptions on the mesh $\mathcal{T}_h$ of IFE spaces \cite{16, 17, 18}:

(H1) The interface $\Gamma$ cannot intersect an edge of any element at more than two points unless the edge is part of $\Gamma$.

(H2) If $\Gamma$ intersects the boundary of an element at two points, these intersection points must be on different edges of this element.

(H3) The interface $\Gamma$ is a piecewise $C^2$ function, and for every interface element $T \in \mathcal{T}_h^i$, $\Gamma \cap T$ is $C^2$.

![Figure 2.1: left: triangular interface element; right: rectangular interface element.](image)

Also, without loss of generality, we assume that $\Gamma \cap \partial \Omega = \emptyset$; hence, we can further assume that $\Gamma$ does not intersect with any boundary element because this assumption can be easily fulfilled when the mesh size is fine enough. To be specific, element $T \in \mathcal{T}_h$ is a boundary element provided that $|(\partial T) \cap (\partial \Omega)| \neq 0$, and we will use $\mathcal{T}_h^b$ to denote the collection of boundary elements of $\mathcal{T}_h$ from now on.

On the interface elements, we will use piecewise linear or bilinear polynomial as IFE shape functions \cite{16, 17, 26}. For a typical interface element $T \in \mathcal{T}_h$ with vertices $A_i, i \in \mathcal{I}_T$, the interface partitions its index set $\mathcal{I}_T$ into $\mathcal{I}_T^- = \{A_i : A_i \in T^-\}$ and $\mathcal{I}_T^+ = \{A_i : A_i \in T^+\}$. Furthermore, let $D$ and $E$ be the points where the interface $\Gamma$ intersects with the edges of $T$, as shown in Figure 2.1. Let $l$ be the line passing through $D$, $E$ with the normal vector $\hat{n} = (\hat{n}_x, \hat{n}_y)$. This line $l$ partitions $T$ into two subelements
A linear IFE function $\phi_T(x, y)$ on a triangular interface element $T$ is a piecewise linear polynomial in the following form \cite{26}:

$$\phi_T(x, y) = \begin{cases} 
\phi_T^-(x, y) = a^-x + b^-y + c^- & \text{if } (x, y) \in T^-, \\
\phi_T^+(x, y) = a^+x + b^+y + c^+ & \text{if } (x, y) \in T^+,
\end{cases} \quad (2.6)$$

On a rectangular interface element $T$, a bilinear IFE function $\phi_T(x, y)$ is a piecewise bilinear polynomial in the following form \cite{17}:

$$\phi_T(x, y) = \begin{cases} 
\phi_T^-(x, y) = a^-x + b^-y + c^- + d^-xy & \text{if } (x, y) \in T^-, \\
\phi_T^+(x, y) = a^+x + b^+y + c^+ + d^+xy & \text{if } (x, y) \in T^+,
\end{cases} \quad (2.7)$$

It has been proven \cite{16, 17, 26} that IFE shape functions $\phi_{i,T}(x, y), i \in I_T$ in the form of \eqref{2.6} or \eqref{2.7} can be uniquely constructed such that

$$\phi_{i,T}(A_j) = \delta_{ij}, \quad \forall i, j \in I_T. \quad (2.8)$$

Then, we define the local real IFE space on the interface element $T \in T^i_h$ as

$$\tilde{S}_h(T) = \text{Span}\{\phi_{i,T}, i \in I_T\}, \quad \forall T \in T^i_h, \quad (2.9)$$

and its complex counterpart is

$$S_h(T) = \{v = v_1 + iv_2 : v_1, v_2 \in \tilde{S}_h(T)\}, \quad \forall T \in T^i_h. \quad (2.10)$$

Using the real local IFE spaces in \eqref{2.4} and \eqref{2.9}, we define the global real IFE space as follows:

$$\tilde{S}_h(\Omega) = \left\{ v \in L^2(\Omega) : v|_T \in \tilde{S}_h(T), \quad \forall T \in T_h, \text{v is continuous at each } A \in N_h \right\}, \quad (2.11)$$

and its complex counterpart is defined with the local complex IFE spaces in \eqref{2.5} and \eqref{2.10}:

$$S_h(\Omega) = \left\{ v \in L^2(\Omega) : v|_T \in S_h(T), \quad \forall T \in T_h, \text{v is continuous at each } A \in N_h \right\}. \quad (2.12)$$

In the discussion below, we will use the following function spaces. For $\tilde{\Omega} \subseteq \Omega$, let $H^p(\tilde{\Omega}), p \geq 0$ be the standard Sobolev space on $\tilde{\Omega}$ with the norm $\| \cdot \|_{p, \tilde{\Omega}}$ and semi-norm $| \cdot |_{p, \tilde{\Omega}}$. Furthermore, if $\Omega^s = \tilde{\Omega} \cap \Omega^s \neq \emptyset$, $s = \pm$, we let

$$PH^2(\Omega) = \{ u : u|_{\Omega^\pm} \in H^2(\Omega^s), \ s = \pm ; \ [u] = 0, \ [\beta \nabla u \cdot n_T] = 0 \text{ on } \Gamma \cap \tilde{\Omega}, \}$$

equipped with the following norms and semi-norms:

$$\| \cdot \|_{2, \tilde{\Omega}} = \sqrt{\| \cdot \|^2_{2, \Omega^-} + \| \cdot \|^2_{2, \Omega^+}}, \quad | \cdot |_{2, \tilde{\Omega}} = \sqrt{| \cdot |^2_{2, \Omega^-} + | \cdot |^2_{2, \Omega^+}}.$$
Now, we present the PPIFE methods developed in [29] for the Helmholtz interface problem in order to analyze them. The description of these PPIFE methods relies on the following space defined according to the mesh \( T_h \):

\[
V_h(\Omega) = \{ v \in L^2(\Omega) : v|_T \in H^1(T), \nabla v \cdot n_{\partial T} \in L^2(\partial T) \quad \forall T \in T_h, \quad [v]_e = 0 \quad \forall e \in \mathcal{E}_h \},
\]

(2.13)

and it can be easily verified that \( S_h(\Omega) \subset V_h(\Omega) \). We will also employ the following standard notations for penalty terms on edges of the mesh \( T_h \): on each \( e \in \mathcal{E}_h \) shared by two elements \( T_1^e \) and \( T_2^e \), let

\[
[v]_e = v|_{T_1^e} - v|_{T_2^e}, \quad \text{and} \quad \{v\}_e = \frac{1}{2} (v|_{T_1^e} + v|_{T_2^e}), \quad \forall v \in V_h(\Omega).
\]

(2.14)

According to [29], the Helmholtz interface problem (1.1) has a weak form as follows:

\[
b_h(u, v) = L_f(v) + (g, v)_{\partial \Omega}, \quad \forall v \in V_h(\Omega),
\]

(2.15)

in which the bilinear form \( b_h(\cdot, \cdot) \) is such that

\[
b_h(u, v) = a_h(u, v) + ik(u, v)_{\partial \Omega} - k^2(u, v)_{\Omega}, \quad \forall u, v \in V_h(\Omega),
\]

(2.16)

where \((\cdot, \cdot)_{\partial \Omega}\) and \((\cdot, \cdot)_{\Omega}\) are the standard \( L^2 \) inner products of the two involved functions on \( \partial \Omega \) and \( \Omega \), respectively, \( a_h(\cdot, \cdot) : V_h(\Omega) \times V_h(\Omega) \to \mathbb{C} \) is a bilinear form defined as

\[
a_h(u, v) = \sum_{T \in T_h} \int_T \beta \nabla u \cdot \nabla v dX - \sum_{e \in \mathcal{E}_h} \int_e \{ \beta \nabla u \cdot n_e \} e[v]_e ds
\]

\[
- \sum_{e \in \mathcal{E}_h} \int_e \{ \beta \nabla \tau \cdot n_e \} e[u]_e ds + i \sum_{e \in \mathcal{E}_h} \sigma_e^{0} |e| \int_e \frac{\sigma_e^{0}}{|e|} [u]_e ds, \quad \forall u, v \in V_h(\Omega),
\]

(2.17)

and \( L_f(\cdot) : V_h(\Omega) \to \mathbb{C} \) is a linear form defined as

\[
L_f(v) = \int_{\Omega} f \nabla v dX = (f, v)_{\Omega}, \quad \forall v \in V_h(\Omega).
\]

(2.18)

As usual, the weak form (2.15) and the fact that \( S_h(\Omega) \subset V_h(\Omega) \) lead to the symmetric PPIFE methods for the Helmholtz interface problem (1.1): find \( u_h \in S_h(\Omega) \), such that

\[
b_h(u_h, v_h) = L_f(v_h) + (g, v_h)_{\partial \Omega}, \quad \forall v_h \in S_h(\Omega).
\]

(2.19)

### 3 Error Analysis of Symmetric PPIFE Methods

The error estimation to be presented for the symmetric PPIFE methods described by (2.19) will use the following three energy norms and the broken \( H^1 \) norm for functions \( v \in V_h(\Omega) \):

\[
\|v\|_h^2 = \sum_{T \in T_h} \int_T \beta \|\nabla v\|^2 dX + \sum_{e \in \mathcal{E}_h} \sigma_e^{0} \int_e \|e|^{-1/2}[v]\|^2 ds,
\]

(3.1)

\[
\|v\|^2_h = \|v\|_h^2 + \sum_{e \in \mathcal{E}_h} (\sigma_e^{0})^{-1} \int_e \|e|^{1/2}\{\beta \nabla v \cdot n_e\}\|^2 ds,
\]

(3.2)
\[ \|v\|_{H^1}^2 = \|v\|_H^2 + k^2\|v\|_{L^2(\Omega)}^2, \quad (3.3) \]
\[ \|v\|_{1,\Omega}^2 = \sum_{T \in T_h} \|v\|_{H^1,T}^2, \quad |v|_{1,\Omega}^2 = \sum_{T \in T_h} |v|_{1,T}^2. \quad (3.4) \]

First we make the following assumption on the regularity of the exact solution

**Assumption 3.1.** Assume that the exact solution \( u \) to the interface problem (7.7) is in \( PH^2(\Omega) \) and the following estimate holds for some constant \( C \):

\[ \|u\|_{2,\Omega} \leq C(k + k^{-1})(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}). \quad (3.5) \]

Assumption 3.1 can be satisfied when the \( \partial\Omega \) and \( \Gamma \) are sufficiently smooth, see [25, 30] and [31] for more details. Next we recall a standard estimate for the trace of a \( H^1 \) function on \( \partial\Omega \) in the following lemma.

**Lemma 3.1.** Assume that \( v \in H^1(\Omega) \), then there exists a constant \( C \) such that

\[ \|v\|_{L^2(\partial\Omega)}^2 \leq C\|v\|_{L^2(\Omega)}^2(\|v\|_{L^2(\Omega)} + |v|_{1,\Omega}). \quad (3.6) \]

**Proof.** This result is given in (4.37) in [25]. \( \square \)

For each function \( v \in PH^2(\Omega) \oplus S_h(\Omega) \), we let \( J_h v \) be its interpolation in the standard continuous (i.e., \( H^1 \)) linear or bilinear finite element space defined on the same mesh \( T_h \) such that

\[ J_h v|_T = J_{h,T} v, \quad \text{with} \quad J_{h,T} v(X) = \sum_{i \in I_T} v(A_i) \psi_i(T)(X), \quad \forall X \in T, \quad \forall T \in T_h. \quad (3.7) \]

Upper bounds of \( J_h v \) are given in the following lemma.

**Lemma 3.2.** There exists a constant \( C \) such that the following hold for all \( v \in PH^2(\Omega) \oplus S_h(\Omega) \):

\[ \|J_h v\|_{L^2(\Omega)} \leq \|v\|_{L^2(\Omega)} + Ch|v|_{1,\Omega}, \quad (3.8) \]
\[ |J_h v|_{1,\Omega} \leq C|v|_{1,\Omega}. \quad (3.9) \]

**Proof.** Let \( v \in PH^2(\Omega) \oplus S_h(\Omega) \). Then \( v|_T \in H^2(T) \) on \( T \in T_h^n \) and \( v|_T \in PH^2(T) \oplus S_h(T) \) on \( T \in T_h^l \). Denote \( Y_i(t, X) = tA_i + (1-t)X, t \in [0, 1] \). By the first order Taylor expansion, we have

\[ v(A_i) = v(X) + \int_0^1 \nabla v(Y_i(t, X)) \cdot (A_i - X)dt. \quad (3.10) \]

Using (3.10) and the partition of unity of the linear and bilinear finite element shape functions on \( T \in T_h \), we have

\[ J_h v(X) = J_{h,T} v(X) = v(X) + \sum_{i \in I_T} \left( \int_0^1 \nabla v(Y_i(t, X)) \cdot (A_i - X)dt \right) \psi_i(T), \quad \forall X \in T. \quad (3.11) \]

Since there exists a constant \( C \) such that \( \|\psi_i\|_{L^\infty(T)} \leq C \) and \( \|A_i - X\| \leq Ch \), from (3.11), we have

\[ \|J_h v\|_{L^2(T)} \leq \|v\|_{L^2(T)} + C \left( \int_T \left( \sum_{i \in I_T} \int_0^1 \nabla v(Y_i(t, X)) \cdot (A_i - X)dt \right)^2 dX \right)^{1/2} \]
\[ \leq \|v\|_{L^2(T)} + Ch \int_0^1 \left( \sum_{i \in I_T} \int_T \|\nabla v(Y_i(t, X))\|^2 dX \right)^{1/2} dt \]
\[ \leq \|v\|_{L^2(T)} + Ch|v|_{1,T}. \quad (3.12) \]
Similarly, by \( \| \nabla \psi_{i,T} \|_{L^\infty(T)} \leq C h^{-1} \), we have
\[
\| \nabla J_h v \|_{L^2(T)} = \left\| \sum_{i \in I_T} v(A_i) \nabla \psi_{i,T}(X) \right\|_{L^2(T)}
\]
\[
= \left( \int_T \left( \sum_{i \in I_T} \int_0^1 \nabla v(Y_i(t,X)) (A_i - X) dt \nabla \psi_{i,T}(X) \right)^2 dX \right)^{1/2}
\]
\[
\leq C h^{-1} C h \| v \|_{1,T}
\]
\[
\leq C \| v \|_{1,T}.
\]

Then, summing (3.12) and (3.13) over all elements \( T \in \mathcal{T}_h \) leads to estimates (3.18) and (3.19), respectively.

Since the IFE space \( S_h(\Omega) \) is not a subspace of \( H^1(\Omega) \) in general [17, 26], the trace inequality cannot be applied to functions in \( PH^2(\Omega) \oplus S_h(\Omega) \), for which, nevertheless, we can derive a similar trace inequality as follows.

**Theorem 3.1.** There exists a constant \( C \) such that for every \( v \in PH^2(\Omega) \oplus S_h(\Omega) \) the following inequality holds:
\[
\| v \|_{L^2(\partial\Omega)}^2 \leq C \left( \| v \|_{L^2(\Omega)}^2 + h \| v \|_{1,\Omega} \right) \| v \|_{1,\Omega}.
\]

**Proof.** Let \( v \) be a function in \( PH^2(\Omega) \oplus S_h(\Omega) \) and let \( J_h v \) be its standard finite element interpolation described by (3.7), and we have
\[
\| v \|_{L^2(\partial\Omega)}^2 \leq 2 \left( \| v - J_h v \|_{L^2(\partial\Omega)}^2 + \| J_h v \|_{L^2(\partial\Omega)}^2 \right).
\]

We estimate the second term on the right hand side of (3.15) first. Since \( J_h v \) is in \( H^1(\Omega) \), by Lemma 3.1 and Lemma 3.2 we have
\[
\| J_h v \|_{L^2(\partial\Omega)}^2 \leq C \| J_h v \|_{L^2(\Omega)}^2 \| v \|_{L^2(\partial\Omega)}^2 + \| J_h v \|_{1,\Omega} \| v \|_{1,\Omega} + \| v \|_{1,\Omega} \| v \|_{1,\Omega}
\]
\[
\leq C \left( \| v \|_{L^2(\Omega)}^2 + h \| v \|_{1,\Omega} \right) \| v \|_{1,\Omega}.
\]

For the first term on the right hand side of (3.15), we note that \( v \in H^2(T) \) on \( T \in \mathcal{T}_h^b \) because of the assumption that the interface \( \Gamma \) does not touch boundary elements when \( h \) is small enough. Then, using the standard trace inequality on \( T \in \mathcal{T}_h^b \) and the approximation capability of finite element space, we have
\[
\| v - J_h v \|_{L^2(\partial\Omega)}^2 \leq \sum_{T \in \mathcal{T}_h^b} \| v - J_h v \|_{L^2(T)}^2
\]
\[
\leq C h^{-1} \sum_{T \in \mathcal{T}_h^b} (\| v - J_h v \|_{L^2(T)}^2 + h^2 \| \nabla (v - J_h v) \|_{L^2(T)}^2)
\]
\[
\leq C h^{-1} \sum_{T \in \mathcal{T}_h^b} (C h^2 \| v \|_{1,T}^2 + h \cdot C \| v \|_{1,T})
\]
\[
\leq C h \sum_{T \in \mathcal{T}_h^b} | v |_{1,T}^2
\]
\[
\leq C h \| v \|_{1,\Omega}^2.
\]

Finally, the inequality (3.13) follows from applying (3.16) and (3.17) to (3.15). \( \square \)
Theorem 3.2. There exists a constant $C$ such that the following estimate holds for every $u \in PH^2(\Omega)$:

$$
|||I_h u - u|||_H \leq C h ||u||_{2,\Omega}, \quad \forall u \in PH^2(\Omega),
$$

(3.19)

provided that $kh \leq C_0$ for some constant $C_0$.

Proof. By Theorem 3.14 in [17], Theorem 3.7 in [26], and Theorem 4.2 in [14], it follows

$$
|||I_h u - u|||^2_H = ||I_h u - u||^2_H + k^2 ||I_h u - u||_{L^2(\Omega)},
$$

$$
\leq C h^2 ||u||^2_{2,\Omega} + C k^2 h^4 ||u||^2_{2,\Omega} \leq C h^2 ||u||^2_{2,\Omega},
$$

which proves (3.19). \(\square\)

We now proceed to the error estimation for the symmetric PPIFE methods described by (2.19), and we will follow Schatz’s argument [33]. We start from the Gårding’s inequality for $b_h(.,.)$ in the following lemma.

Lemma 3.3. There exist constants $C_1$ and $C_2$ such that the following inequality holds for $\sigma^0$ sufficiently large

$$
|b_h(v,v)| \geq C_1 \|v\|_H^2 - C_2 k^2\|v\|_{L^2(\Omega)}, \quad \forall v \in S_h(\Omega).
$$

(3.20)

Proof. First of all, we note that

$$
|b_h(v,v)| \geq \frac{1}{\sqrt{2}} \left( \text{Re}(b_h(v,v)) + \text{Im}(b_h(v,v)) \right)
$$

$$
= \frac{1}{\sqrt{2}} \left( \text{Re}(a_h(v,v)) + \text{Im}(a_h(v,v)) + k\|v\|^2_{L^2(\partial\Omega)} - k^2\|v\|^2_{L^2(\Omega)} \right). \tag{3.21}
$$

Next, we introduce the bilinear form $\tilde{a}_h(.,.) : V_h(\Omega) \times V_h(\Omega) \rightarrow \mathbb{C}$ such that

$$
\tilde{a}_h(u,v) = \tilde{a}_h(u,v) + \sum_{e \in E_h} \frac{\sigma_0^{e} |e|}{|e|} \int_e \bar{u} \overline{[n]}_e v ds, \quad \forall v \in V_h(\Omega).
$$

For each $v \in S_h(\Omega)$, we let $v = v_1 + iv_2$ with $v_1 = \text{Re}(v) \in \tilde{S}_h(\Omega)$ and $v_2 = \text{Im}(v) \in \tilde{S}_h(\Omega)$. Since $a_h(.,.)$ and $\tilde{a}_h(.,.)$ are both bilinear and symmetric, we have

$$
a_h(v,v) = a_h(v_1,v_1) + a_h(v_2,v_2).
$$

It follows that

$$
\text{Re}(a_h(v,v)) + \text{Im}(a_h(v,v)) = \tilde{a}_h(v_1,v_1) + \tilde{a}_h(v_2,v_2) + \sum_{e \in E_h} \frac{\sigma_0^{e} |e|}{|e|} \int_e [v_1]_e \overline{[n]}_e v ds + \sum_{e \in E_h} \frac{\sigma_0^{e} |e|}{|e|} \int_e [v_2]_e \overline{[n]}_e v ds. \tag{3.22}
$$
Because $v_1, v_2 \in \tilde{S}_h(\Omega)$, we can apply Theorem 4.3 in [14] to (3.22) so that there exists a constant $\kappa > 0$ such that
\[
\text{Re}(a_h(v, v)) + \text{Im}(a_h(v, v)) \geq \kappa(\|v_1\|_h^2 + \|v_2\|_h^2) = \kappa\|v\|_h^2.
\] (3.23)
Therefore, applying (3.23) to (3.21) we have
\[
|b_h(v, v)| \geq \frac{1}{\sqrt{2}} \left( \kappa\|v\|_h^2 + \kappa k^2\|v\|_{L^2(\Omega)}^2 - k^2(1 + \kappa)\|v\|_{L^2(\Omega)}^2 \right),
\]
\[
\geq \frac{1}{\sqrt{2}} \left( \kappa\|v\|_H^2 - k^2(1 + \kappa)\|v\|_{L^2(\Omega)}^2 \right),
\]
which proves (3.20).

The following lemma is about the continuity of the bilinear form $b_h(\cdot, \cdot)$.

**Lemma 3.4.** There exists a constant $C$ such that for every $y, v \in PH^2(\Omega) \oplus S_h(\Omega)$ the following inequality holds
\[
|b_h(y, v)| \leq C\|y\|_H\|v\|_H,
\] (3.24)
provided that $kh \leq C_0$ for some constant $C_0$.

**Proof.** By the same arguments used for proving Theorem 4.4 in [14], we can show that there exists a constant $C$ such that
\[
|a_h(y, v)| \leq C\|y\|_h\|v\|_h.
\]
Since $\|y\|_H \geq k\|y\|_{L^2(\Omega)}$, then
\[
|b_h(y, v)| \leq C\|y\|_h\|v\|_h + k^2\|y\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + Ck\|y\|_{L^2(\partial \Omega)}\|v\|_{L^2(\partial \Omega)}
\]
\[
\leq C\|y\|_H\|v\|_H + \|y\|_H\|v\|_H + Ck\|y\|_{L^2(\partial \Omega)}\|v\|_{L^2(\partial \Omega)}.
\] (3.25)
For the third term on the right hand side of (3.25), applying Theorem 3.1 we have
\[
k^2\|y\|_{L^2(\partial \Omega)}\|v\|_{L^2(\partial \Omega)} \leq Ck^2(|y|_{L^2(\Omega)} + k|y|_{L^2(\Omega)} + k|y|_{L^2(\Omega)} + k|y|_{L^2(\Omega)})\|y\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}
\]
\[
= C(k|y|_{L^2(\Omega)} + kh|y|_{L^2(\Omega)} + kh|y|_{L^2(\Omega)} + kh|y|_{L^2(\Omega)})\|y\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}
\]
\[
\leq C\|y\|_H^2\|v\|_H^2.
\] (3.26)
Thus, applying (3.20) to (3.21) leads to (3.24). 

Following Schatz’s argument [33], we now derive a posteriori error estimate for the symmetric PPIFE solution in the following lemma.

**Lemma 3.5.** Let $u \in PH^2(\Omega)$ be the exact solution to the problem (1.1), and let $u_h$ be the solution produced by symmetric PPIFE method (2.19) with $\alpha_v^0$ large enough, then there exists a constant $C$ such that
\[
\|u - u_h\|_{L^2(\Omega)} \leq C(k + 1/k)h\|u - u_h\|_H,
\] (3.27)
provided that $kh \leq C_0$ for some constant $C_0$.

**Proof.** We define an auxiliary function $z \in PH^2(\Omega)$ as the solution to the problem (1.1), with $f$ replaced by $e = u - u_h$ and $g$ replaced by the zero function. In the weak form (2.15) for $z$, choosing $v = e$ as the test function yields
\[
\|e\|_{L^2(\Omega)}^2 = b_h(z, e).
\]
Let $I_h z$ be the interpolent of $z$ in IFE space $S_h(\Omega)$ defined by (3.18), it follows
\[
b_h(I_h z, e) = b_h(I_h z, u) - b_h(I_h z, u_h) = (f, I_h z)_{\Omega} + (g, I_h z)_{\partial \Omega} - (f, I_h z)_{\Omega} - (g, I_h z)_{\partial \Omega} = 0.
\]
Thus $b_h(z, e) = b_h(z - I_h z, e)$. Therefore, by Lemma 3.3, Theorem 3.2 and Assumption 3.1 we have
\[
\|e\|_{L^2(\Omega)}^2 = b_h(z - I_h z, e) \\
\leq C \|z - I_h z\|_{H^1} \|e\|_{H^1} \\
\leq Ch \|z\|_{2,\Omega} \|e\|_{H^1} \\
\leq C(k + 1/k)h \|e\|_{L^2(\Omega)} \|e\|_{H^1},
\]
which proves (3.27).

Now, we are ready to derive the optimal error bounds in both the energy norm and $L^2$ norm for the symmetric PPIFE methods described by (2.19).

**Theorem 3.3.** Under the conditions of Lemma 3.3, there exists a constant $C$ such that
\[
\|u - u_h\|_{H^1} \leq Ch \|u\|_{2,\Omega},
\]
provided that $(k^2 + 1)h$ is sufficiently small.

**Proof.** First, we assume that $(k^2 + 1)h$ is sufficiently small such that $kh \leq C_0$ for some constant $C_0$. Denote $e = u - u_h$, $e_h = u_h - I_h u$, then by Lemma 3.3 and Lemma 3.4 we have
\[
C_1 \|e_h\|_{H^1}^2 - C_2 k^2 \|e_h\|_{L^2(\Omega)}^2 \leq b_h(e_h, e_h) = b_h(u - I_h u, e_h) \leq C \|u - I_h u\|_{H^1} \|e_h\|_{H^1}.
\]
By the fact that $\|e_h\|_{H^1} \geq k \|e_h\|_{L^2(\Omega)}$, we then have
\[
\|e_h\|_{H^1}^2 \leq C \|u - I_h u\|_{H^1} \|e_h\|_{H^1} + Ck \|e_h\|_{L^2(\Omega)}^2 \\
\leq C \|u - I_h u\|_{H^1} \|e_h\|_{H^1} + C k \|e_h\|_{H^1} \|e_h\|_{L^2(\Omega)}.
\]
Therefore, using Theorem 3.2 we have
\[
\|e_h\|_{H^1} \leq C \|u - I_h u\|_{H^1} + C k \|e_h\|_{L^2(\Omega)} \\
\leq Ch \|u\|_{2,\Omega} + C k \|e\|_{L^2(\Omega)}.
\]
Furthermore, by Lemma 3.5 and the approximation capability of IFE spaces [17, 26], we have
\[
\|e_h\|_{H^1} \leq Ch \|u\|_{2,\Omega} + C k (k + 1/k)h \|e\|_{H^1} + Ch^2 \|u\|_{2,\Omega} \\
\leq Ch \|u\|_{2,\Omega} + C k (k + 1/k)h \|e\|_{H^1} + Ch^2 \|u\|_{2,\Omega},
\]
which proves (3.27) provided that $(k^2 + 1)h$ is sufficiently small. \qed
Remark 3.1. Resort to the idea in [22], if \( u_h \) is a PPIFE solution corresponding to \( u = 0 \), then from Theorem 3.3 it follows that \( u_h = 0 \) provided that \( h \) is sufficiently small guaranteeing \((k^2 + 1)h \) is sufficiently small. This implies that the linear system to solve \( u_h \) induced from the symmetric PPIFE scheme (2.19) is nonsingular; therefore, the PPIFE solution \( u_h \) defined by (2.19) exists and is unique.

**Theorem 3.4.** Under the conditions of Theorem 3.3, there exists a constant \( C \), such that
\[
\| u - u_h \|_{L^2(\Omega)} \leq C(k + 1/k)h^2 \| u \|_{L^2(\Omega)}.
\]

Proof. The estimate (3.30) follows directly from Lemma 3.5 and Theorem 3.3.

### 4 A Numerical Example

In this section, we present a numerical example to validate the error estimates in Theorems 3.3 and 3.4. We note that [29] provides quite a few numerical examples to illustrate convergence features of the PPIFE methods developed there for solving the Helmholtz interface problems. However, the exact solutions in the examples presented in [29] have a regularity better than piecewise \( H^r \) with \( r > 2 \). Hence, it is interesting to see how the PPIFE solution converges when the exact solution only has piecewise \( H^2 \) regularity.

Specifically, let the domain \( \Omega = (-1, 1) \times (-1, 1) \) be separated by the circular interface \( \Gamma : x^2 + y^2 - r_0^2 = 0, \ r_0 = \pi/6.28 \) into two subdomains
\[
\Omega^- = \{ (x, y) : x^2 + y^2 < r_0^2 \}, \ \Omega^+ = \Omega \setminus \Omega^-.
\]

We generate a Cartesian triangular mesh \( T_h \) of \( \Omega \) by partitioning \( \Omega \) into \( N \times N \) congruent squares so that \( h = 2/N \), and then partitioning each square into two congruent triangles by its diagonal line. We let functions \( f \) and \( g \) in the interface problem (1.1) be generated with the following exact solution:
\[
u(x, y) = \begin{cases} \frac{2 + i}{\beta^+} r^\alpha, & (x, y) \in \Omega^-, \\ \frac{2 + i}{\beta^-} r^\alpha + \left( \frac{2 + i}{\beta^-} - \frac{2 + i}{\beta^+} \right) r_0^\alpha, & (x, y) \in \Omega^+, \end{cases}
\]
where \( \alpha = 1.5, \ r = \sqrt{x^2 + y^2} \). We choose \( \sigma_0 = 30 \max \{ \beta^-, \beta^+ \} \) for the parameter required in (2.17). It can be verified that, \( u \in PH^2(\Omega) \setminus PH^3(\Omega) \). Table 1 presents errors of the symmetric PPIFE solutions \( u_h \) generated on a sequence of uniform triangular meshes \( T_h \) of \( \Omega \) in a certain configuration of \( k, \beta^-, \beta^+ \).

The results demonstrate that, for fixed \( k \), the symmetric PPIFE solutions converge optimally in both semi-\( H^1 \) and \( L^2 \) norms to the exact solution \( u \in PH^2(\Omega) \setminus PH^3(\Omega) \), and this validates the theoretical results established in Theorem 3.3 and Theorem 3.4 in the previous section.

| \( N \) | \( \| u - u_h \|_{0, \Omega} \) | Rate | \( \| u - u_h \|_{1, \Omega} \) | Rate |
|---|---|---|---|---|
| 10 | 3.6019e-02 | NA | 5.2313e-01 | NA |
| 20 | 1.6412e-02 | 1.1340 | 2.5292e-01 | 1.0485 |
| 40 | 6.6539e-03 | 1.3025 | 1.1802e-01 | 1.0997 |
| 80 | 1.3425e-03 | 2.3092 | 5.1500e-02 | 1.1964 |
| 160 | 2.7744e-04 | 2.2747 | 2.4983e-02 | 1.0436 |
| 320 | 7.7328e-05 | 2.2381 | 1.2427e-02 | 1.0075 |
| 640 | 1.9455e-05 | 1.9909 | 6.1961e-03 | 1.0040 |
| 1280 | 4.7698e-06 | 2.0281 | 3.0947e-03 | 1.0015 |

Table 1: Errors of the PPIFE solution, \( k = 10, \beta^- = 1, \beta^+ = 10 \).
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