ON SIMPLE CHARACTERISATIONS OF SHEFFER $\psi$-POLYNOMIALS AND RELATED PROPOSITIONS OF THE CALCULUS OF SEQUENCES

Summary

A “Calculus of Sequences” had started by the 1936 publication of Ward suggesting the possible range for extensions of operator calculus of Rota-Mullin, considered by several authors and after Ward. Because of convenience we shall call the Wards calculus of sequences in its afterwards elaborated form – a $\psi$-calculus. The notation used by Ward, Viskov, Markowsky and Roman is accommodated in conformity with Rota’s way of exposition. In this manner $\psi$-calculus becomes in parts almost automatic extension of finite operator calculus. The $\psi$-extension relies upon the notion of $\partial_{\psi}$-shift invariance of operators. At the same time this calculus is an example of the algebraization of the analysis – here restricted to the algebra formal series. The efficiency of the notation used is further exemplified among others by easy proving of some Sheffer $\psi$-polynomials characterisation theorems as well as Spectral Theorem. $\psi$-calculus results may be extended to Markowsky “$Q$-umbral calculus”, where $Q$ stands for a generalised difference operator (not necessarily $\partial_{\psi}$-shift invariant) i.e. the one lowering the degree of any polynomial by one.

Contents

1. Introduction. – 2. Primary definitions, notation and the general observations. – 3. The general picture of $\text{End}(P)$. – 4. Characterisations of Sheffer $\psi$-polynomials and related propositions. – 5. Miscellaneous remarks and indications of several applications.

1. Introduction

Already since seventies of the past century it had been realised that the notions and methods of Rota-Mullin finite operator calculus (started in [1] and developed in [2, 3] – see also [4–6]) might be extended to the use of any polynomial sequences $\{p_n\}_{n=0}^{\infty}$ ($\deg p_n = n$) instead of those of binomial type only. The foundations of such an extension were led in 1975 by Viskov in [7] and then developed further in [8].
A. K. Kwaśniewski
– with very substantial reference to Boas and Buck [9, 10] – however without reference to Ward paper [11].

The decisive contribution to this conviction (on the possibility to develop umbral calculus for any polynomial sequences \( \{p_n\}_0^\infty \) instead of those of binomial type only) we owe to Markowsky [12] who apparently was not acquainted with Viskov papers [7, 8] at that time and neither [7] nor Ward paper [11] are quoted in [12]. Markowsky paper dealt with – as we would call it now [13] – generalised Sheffer polynomials.

One may learn much more from later works on subsequent and recent progress in extending the operator methods onto “calculus of sequences”. The reader is referred to the up-date position [14] by Loeb. This is extensive and exhaustive source of references and the survey of widespread developments over past decades. With such enriched experience – any way – we are led back to the Ward conception of calculus of sequences as the source.

In [15, 16] special type generalised Sheffer polynomials are called Sheffer \( \psi \)-polynomials in order to accommodate the author’s Rota-oriented notation with the notation used by Viskov in [7] on one side and notation of Rota in [3] on the other side. This (hopefully well aimed) notation enables to adopt from [3, 12] formulations and methods for proving majority of “\( \psi \)-propositions” or “\( Q \)-propositions” (see definitions 2.5. and 2.6.) in the general case of operator difference calculus taking care of and related to any polynomial sequences \( \{p_n\}_0^\infty \) instead of those of binomial type only or \( \psi \)-binomial type only. As an example of such presentation characterisation of Sheffer \( \psi \)-polynomials theorems and Spectral Theorem [16] are given among others. The note is organised as follows. Firstly the presentation of general picture of algebra of linear operators on polynomial algebra is accomplished. Then characterisations of Sheffer \( \psi \)-polynomials and related propositions of the calculus of sequences follow. At the end link is given to specific formulation of \( q \)-umbral calculus by Cigler [17] and Kirschenhofer [18]. This formulation might be related – as noted in [15] – to the so-called quantum groups [19]. The relevant \( q \)-extensions of what is now called sometimes – Rota-Mullin calculus – we owe also to earlier authors such as Al.-Salam, Carlitz (see Chihara [20, 21]), Goldmann, Rota, Andrews, Ihrig, Ismail, Verna and others. For the corresponding references see [22] and Roman’s book [23].

There – in chapter 6 one finds also the first principal results on “nonclassical umbral calculi”. (It seems so that the author of [19] was not acquainted with Viskov and Markowsky papers at that time). In [23] Roman refers to Ward’s papers including [11], which was not the case in Roman’s early publications on the subject. Meanwhile in [11] Ward proposes a generalisation of a large portion of calculus of finite differences almost in the spirit of later finite operator calculus. Ward accurately and relevantly called this generalisation – the calculus of sequences. (As a matter of fact – the \( q \)-calculus of sequences is already present as an example in Ward’s publication [11]). Here we shall deal with the past century further development of the finite operator calculus formulation of this calculus of sequences. To this end we choose as a motto for activities of the type to be presented – a quotation from Roman’s book (Chapter 6, p. 162 in [23] – with notation and reference changed into the one used in this note): “Let \( n! \) be a sequence of nonzero constants. If \( n! \) is replaced by \( c_n \) throughout the preceding theory, then virtually all of the results remain true, mutatis mutandis. In this way each sequence \( c_n \) gives rise to a distinct umbral calculus.
Actually, Ward [11] seems to have been the first to suggest such a generalisation (of the calculus of finite differences) in 1936, but the idea remained relatively undeveloped until quite recently, perhaps due to a feeling that it was mainly generalisation for its own sake. Our purpose here is to indicate that this is not the case.

2. Primary definitions, notation and the general observations

In the following we shall consider the algebra $P$ of polynomials $P = F[x]$ over the field $F$ of characteristic zero. All operators or functionals studied here are to be understood as linear operators or functionals on $P$. It shall be easy to see that they are always well defined. Whenever we say polynomial sequence $\{p_n\}_{n \geq 0}$ we shall always mean such sequence $\{p_n\}_{n \geq 0}$ of polynomials that $\deg p_n = n$.

Let $s = \{s_n\}_{n \geq 1}$ be an arbitrary sequence of numbers such that $s_n \neq 0$, $n \in N$. $s$-binomial coefficients are then defined with help of the generalised factorial $n_s! = s_1 s_2 s_3 \ldots s_n$ and $n^k_s = n_s(n - 1)_s(n - k + 1)_s$ in a usual way as $\binom{n}{k} = \frac{n^k_s}{k^s!}$. Consider $\mathfrak{S} – the family of functions’ sequences (in conformity with Viskov notation) such that

\[ \mathfrak{S} = \{ \psi; R \supset [a, b]; q \in [a, b]; \psi(q) : Z \to F; \psi_0(q) = 1; \psi_n(q) \neq 0; \psi_{-n}(q) = 0; n \in N \} \]

Following Roman [23–25] we shall call $\psi = \{\psi_n(q)\}_{n \geq 0}$; $\psi_n(q) \neq 0$; $n \geq 0$ and $\psi_0(q) = 1$ an admissible sequence. Let now $n_\psi$ – (in conformity with Viskov notation) – denotes $[16, 15]$

\[ n_\psi \equiv n_{\psi - 1}(q)\psi_n(q). \]

Then we have for the $\psi$-factorial $n_\psi! \equiv n_{\psi - 1}(q) \equiv n_\psi(n - 1)_\psi(n - 2)_\psi(n - 3)_\psi2_\psi1_\psi; 0_\psi! = 1$ and $\binom{n}{k}_\psi = \frac{n^k_s}{k^s!}$ for $\psi$-binomial coefficients where $n^k_s = n_\psi(n - 1)_\psi(n - 2)_\psi \ldots(n - k + 1)_\psi$ while $\exp_\psi \{y\} = \sum_{k=0}^{\infty} \frac{y^k_s}{k^s!}$ defines the $\psi$-exponential series. In this note we need not to specify any additional conditions on $\psi_n(q)$’s which would lead us to specifications of the $\psi$-calculus as for example the $q$-umbral calculus which is obtained with the following choice of an admissible $\psi$:

\[ \psi_n(q) = \frac{1}{R(q^n)!}; R(x) = \frac{1 - x}{1 - q}, \]

Definition 2.1. Let $\psi$ be admissible. Let $\partial_\psi$ be the linear operator lowering degree of polynomials by one defined according to $\partial_\psi x^n = n_{\psi} x^{n - 1}, n \geq 0$. Then $\partial_\psi$ is called the $\psi$-derivative.

Remark 2.1. The choice $\psi_n(q) = [R(q^n)!]^{-1}$ and $R(x) = \frac{1 - x}{1 - q}$ results in the well known $q$-factorial $n_q! = n_q(n - 1)_q!; 1_q! = 0_q! = 1$ while the $\psi$-derivative $\partial_\psi$ becomes now the Jackson’s derivative $[16, 15] \partial_q$: \[ (\partial_q \varphi)(x) = \frac{\varphi(x) - \varphi(qx)}{(1 - q)x}. \]
Example 2.1. [11] Let \( \{n_\psi\}_{n \geq 0} \) be the solution of a linear recurrence of \( r \)-th order for which its characteristic polynomials has \( r \) distinct roots \( \{\alpha_k\}_r^n \). Then the general solution reads: \( n_\psi = \beta_1 \alpha_1^n + \beta_2 \alpha_2^n + ... + \beta_r \alpha_r^n \) and one now should impose the two conditions: \( \sum_{k=1}^r \beta_k = 0 \) (because \( 0_\psi = 0 \)) and \( \sum_{k=1}^r \beta_k \alpha_k = 1 \) (because \( 1_\psi = 1 \)). Naturally for any formal series \( F(x) = \sum_{k=0}^\infty f_k x^k \), \( \partial_\psi F(x) = \sum_{k=0}^\infty k_\psi f_k x^{k-1} \) therefore \( \partial_\psi F(x) = \frac{1}{r} \sum_{k=1}^r \beta_k F(\alpha_k x) \). For \( r = 2 \) and \( \alpha_1 = q, \alpha_2 = 1, \beta_1 = (q-1)^{-1}, \beta_2 = -\beta_1 \) the \( \psi \)-derivative \( \partial_\psi \) becomes Jackson’s derivative i.e. \( \partial_\psi = \partial_q \) [11, 16]. Another specific choice is the Fibonacci sequence

\[
F_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} - \frac{1 - \sqrt{5}}{2} \right); \quad n \geq 0.
\]

Then \( \psi \)-derivative becomes “Fibonacci” derivative i.e. \( \partial_\psi = \partial_F \) with \( n_\psi \equiv n_F \equiv F_n \).

The quite exceptional property of \( \left( \begin{array}{c} n \\ k \end{array} \right)_\psi \equiv \frac{n_k!}{k_\psi f_k!} \) are integer numbers (see chapter 6, exercises 86 in [26]) in the case of “Fibonomial” coefficients.

Definition 2.2. Let \( E^\psi(\partial_\psi) \equiv \exp_\psi \{y \partial_\psi\} = \sum_{k=0}^\infty \frac{y^k \partial_\psi^k}{k_\psi f_k!} \). \( E^\psi(\partial_\psi) \) is called the generalised translation operator.

Note 2.1. [16, 15] \( E^\psi(\partial_\psi) f(x) = f(x + \psi a); (x + \psi a)^n = E^n(\partial_\psi) x^n; E^n(\partial_\psi) f = \sum_{n \neq 0} \frac{a^n}{n!} \partial_\psi^n f; \) and in general \( (x + \psi a)^n \neq (x + \psi a)^{n-1} (x + \psi a) \). Note also that in general \( (1 + \psi (-1))^{2n+1} \neq 0 \); \( n \geq 0 \) though \( (1 + \psi (-1))^{2n} = 0; \) \( n \geq 1 \). We learn from [11] the following.

Note 2.2. \( \exp_\psi (x + \psi y) \equiv \exp_\psi \{x\} \exp_\psi \{y\} \) while in general \( \exp_\psi \{x + y\} \neq \exp_\psi \{x\} \exp_\psi \{y\} \). Possible consequent utilisation of the identity \( \exp_\psi (x + \psi y) \equiv \exp_\psi \{x\} \exp_\psi \{y\} \) is quite encouraging. It leads among others to “\( \psi \)-trigonometry” either \( \psi \)-elliptic or \( \psi \)-hyperbolic via introducing \( \cos_\psi, \sin_\psi \) [11], \( \cosh_\psi, \sinh_\psi \) or in general \( \psi \)-hyperbolic functions of \( m \)-th order \( \{h_\psi^m\}_j \) defined according to \[27, 28\]

\[
R \ni a \rightarrow h_\psi^m(a) = \frac{1}{m} \sum_{k \in Z_m} \omega^{-kj} \exp_\psi (\omega^k a); \quad j \in Z_m, \omega = \exp \left( \frac{2 \pi i}{m} \right),
\]

where \( 1 < m \in N \) and \( Z_m = \{0, 1, ..., m - 1\} \). However note that elements of \( R \) (or any other field of characteristic zero chosen) are subjected to \( \psi \)-addition and \( \psi \)-subtraction [11, 16] and that \( (x + \psi a)^n \neq (x + \psi a)^{n-1}(x + \psi a) \) in general.

In the following the notion of \( \partial_\psi \)-shift invariance of operators and the notion of a polynomial sequence \( \{p_n\}_\infty \) is of \( \psi \)-binomial type are to play the crucial role.

Definition 2.3. Let us denote by \( \text{End}(P) \) the algebra of all linear operators acting on the algebra \( P \) of polynomials. Let \( \sum_\psi = \{T \in \text{End}(P): \forall \alpha \in F; [T; E^\alpha(\partial_\psi)] = 0 \} \). Then \( \sum_\psi \) is a commutative subalgebra of \( \text{End} \) (\( P \)) of \( F \)-linear operators. We shall call these operators \( T: \partial_\psi \)-shift invariant operators.
Definition 2.4. A polynomial sequence \( \{p_n\}_{n=0}^{\infty} \) is of \( \psi \)-binomial type if it satisfies the recurrence

\[
E^\psi (\partial_\psi) p_n(x) \equiv p_n(x + \psi y) \equiv \sum_{k=0}^{n} \binom{n}{k}_\psi p_k(x)p_{n-k}(y).
\]

Let us express now two characterisations [7] of polynomial sequences of \( \psi \)-binomial type in appealing Rota-oriented notation as a matter of this appealingness illustration.

Illustration 2.1

1. (Proposition 8 in [7]) A polynomial sequence \( \{p_n\}_{n=0}^{\infty} \) is Sheffer \( \psi \)-polynomial if and only if its “\( \psi \)-generating function” is of the form:

\[
\sum_{n \geq 0} \psi_n p_n(x)z^n = A(z) \exp_\psi (xg(z));
\]

\[
\psi(z) = \sum_{n \geq 0} \psi_n z^n; \quad \psi_n \neq 0; \quad n = 0, 1, 2, ...
\]

where \( A(z), g(z)/z \) are formal series with constant terms different from zero.

2. Also in [7] (Proposition 4) Viskov have proved that polynomial sequence \( \{p_n\}_{n=0}^{\infty} \) is of \( \psi \)-binomial type if and only if its “\( \psi \)-generating function” is of the form:

\[
\sum_{n \geq 0} \psi_n p_n(x)z^n = \exp_\psi (xg(z))
\]

for formal series \( g \) inverse to appropriate formal series. Note that for \( \psi_n(q) = [nq]^{-1} \) in (2.2) \( \psi(z) = \exp_q \{z\} \) and “\( \exp_q \) generating function” (2.3) takes the “\( q \)-umbral” form

\[
\sum_{n \geq 0} \frac{z^n}{n^q} p_n(x) = \exp_q (xg(z))
\]

Polynomial sequences of \( \psi \)-binomial type [2, 3] are known to correspond in one-to-one manner to special generalised differential operators \( Q \), namely to those \( Q = Q(\partial_\psi) \) which are \( \partial_\psi \)-shift invariant operators [7, 16, 15]. We shall deal in this note mostly with this special case i.e. with the so-called \( \psi \)-umbral calculus [16, 15]. However before to proceed let us deliver an elementary, basic information referring to the general case of “\( Q \)-umbral calculus” – as we call it – started by Markowsky in [12].

Definition 2.5. Let \( P = F[x] \). Let \( Q \) be a linear map \( Q : P \to P \) such that \( \forall \ p \in P \ \deg(Qp) = (\deg p) - 1 \) \( (\deg p = -1 \text{ means } p = \text{const} = 0) \). Then \( Q \) is called a generalised difference-tial operator [12] or Gel’fond-Leontiev [7] operator. The algebra of all linear operators commuting with the generalised difference-tial operator \( Q \) is denoted by \( \sum_Q \).
Right from the above definitions we infer that the following holds.

**Observation 2.1.** Let $Q$ be as in Definition 2.5. Let $Qx^n = \sum_{k=1}^{n} b_{n,k}x^{n-k}$ where $b_{n,1} \neq 0$. Without loose of generality take $b_{1,1} = 1$. Then $\exists \{q_k\}_{q \geq 2} \subset F$ and $\exists$ admissible $\psi$ such that

$$Q = \partial_\psi + \sum_{k \geq 2} q_k \partial_\psi^k$$

(2.5)

if and only if the following condition is fulfilled

$$b_{n,k} = \binom{n}{k} \psi b_{k,n}; \quad n \geq k \geq 1, \quad b_{n,1} \neq 0, \quad b_{1,1} = 1.$$

(2.5')

If $\{q_k\}_{q \geq 2}$ and an admissible $\psi$ exist then these are unique.

**Notation 2.1.** In the case (2.5') is true we shall write: $Q = Q(\partial_\psi)$.

**Remark 2.2.** Note that operators of the (2.5) form constitute a group under superposition of formal power series (compare with the formula (S) in [13]). Of course not all generalised difference-tial operators satisfy (2.5) i.e. are series just only in corresponding $\psi$-derivative $\partial_\psi$ (see Proposition 3.1). For example [14] let $Q = \frac{1}{2} D D x D - \frac{4}{3} D^3$. Then $Qx^n = \frac{1}{2} n^2 x^{n-1} - \frac{4}{3} n^3 x^{n-3}$. Therefore according to Observation 2.1 $\psi = \frac{1}{2} n^2$ and $\pm$ admissible $\psi$ such that $Q = Q(\partial_\psi)$.

**Observation 2.2.** From theorem 3.1 in [12] – we infer that generalised differential operators give rise to subalgebras $\sum_\psi$ of linear maps (plus zero map of course) commuting with a given generalised diifference-tial operator $Q$. The intersection of two different algebras $\sum_\psi$ and $\sum_\varphi$ is just zero map added.

The importance of the above Observation 2.2 as well as the definition below may be further fully appreciated in the context of the Theorem 2.1 and the Proposition 3.1 to come.

**Definition 2.6.** Let $\{p_n\}_{n \geq 0}$ be the normal polynomial sequence [12] i.e. $p_0(x) = 1$ and $p_n(0) = 0; \quad n \geq 1$. Then we call it the $\psi$-basic sequence of the generalised difference-tial operator $Q$ if in addition $Qp_n = n_\psi p_{n-1}$. We shall then call $Q$ in short the $\psi$-difference-tial operator. If $\{s_n\}_{n \geq 0}$ is such a polynomial sequence that $Qs_n = n_\psi s_{n-1}$, where $Q$ is the $\psi$-difference-tial operator then we call $\{s_n\}_{n \geq 0}$ the generalised Sheffer $Q$-$\psi$-sequence [13]. Parallely we define a linear map $\hat{x}_Q : P \rightarrow P$ such that $\hat{x}_Q p_n = \frac{(n+1)}{(n+1)_\psi} p_{n+1}, \quad n \geq 0$. We call $\hat{x}_Q$ the operator dual to $Q$. For $Q = Q(\partial_\psi) = \partial_\psi$ we write for short $\hat{x}_Q \equiv \hat{x}_\psi$.

Let us observe that $[Q, \hat{x}_Q] = id$ therefore triples of operators $\{Q, \hat{x}_Q, id\}$ provide us with a continuous family of generators of GHW in – as we call it – $Q$-representation of Graves-Heisenberg-Weyl algebra. In the following we shall restrict mostly to special case of generalised differential operators $Q$, namely to those $Q = Q(\partial_\psi)$ which are $\partial_\psi$-shift invariant operators [7, 8, 16]. Let us then start with appropriate Leibnitz $\psi$-rules for corresponding $\psi$-derivatives recalling at first that admissible sequence is such a sequence $\psi = \{\psi(q)\}_{n \geq 0}$ that $\psi_n(q) \neq 0; \quad n \geq 0$ and $\psi_0(q) = 1$. Then we have the following obvious observation. If $\psi = \{\psi_n(q)\}_{n \geq 0}$ and $\varphi = \{\varphi_n(q)\}_{n \geq 0}$ are two admissible sequences then $[\partial_\psi, \partial_\varphi] = 0$ if $\psi = \varphi$. For
any $\psi$-derivative an appropriate Leibnitz $\psi$-rules come true. Namely it is easy to see
that the following Leibnitz $\psi$-rules hold for any formal series $f$ and $g$ (see Remark
2.1):
for $\partial_q$: \[
\partial_q(f \cdot g) = (\partial_q f) \cdot g + (\hat{Q} f) \cdot (\partial_q g);
\]
for $\partial_R = R(q\hat{Q})\partial_0$:
\[
\partial_R(f \cdot g)(z) = R(q\hat{Q})\{(\partial_0 f)(z) \cdot g(z) + f(0)(\partial_q g)(z)\}
\]
where – note – $R(q\hat{Q})x^{n-1} = n_R x^{n-1}$; ($n_\psi = n_R = n_R(q) = R(q^n)$) and finally
for $\partial_\psi = \hat{n}_\psi \partial_0$:
\[
\partial_\psi(f \cdot g)(z) = \hat{n}_\psi \{(\partial_0 f)(z) \cdot g(z) + f(0)(\partial_q g)(z)\}
\]
where $\hat{n}_\psi x^{n-1} = n_\psi x^{n-1}; \ n \geq 1$.

Example 2.2. [13, 16] Let $Q(\partial_\psi) = D\hat{D}D$, where $\hat{D} f(x) = xf(x)$ and $D = d/dx$.
Then $\psi = \{(n^2)!\}^{-1}_{n \geq 0}$ and $Q = \partial_\psi$. Let $Q(\partial_\psi) = R(q\hat{Q})\partial_0 \equiv \partial_R$. Then
$\psi = \{(R(q^n))!\}^{-1}_{n \geq 0}$ and $Q = \partial_\psi \equiv \partial_R$.

Naturally with the choice $\psi_n(q) = [R(q^n)]!^{-1}$ and $R(x) = \frac{1-x}{1-q}$ the $\psi$-derivative $\partial_\psi$
becomes the Jackson’s derivative [16, 15] $\partial_q$: \[
(\partial_q \psi)(x) = \frac{1-q^2}{(1-q)} \partial_\psi \varphi(x).
\]
\]

The form equivalent to (2.6) form of Bernoulli-Taylor expansion one may find in Acta Eruditorum
from November 1694 [30] under the name “series universalissima”. (Taylor’s
expansion was presented in his “Methodus incrementorum directa et inversa” in
1715 – edited in London.) The example of $\psi$-derivative: $Q(\partial_\psi) = R(q\hat{Q})\partial_0 \equiv \partial_R$ i.e. $\psi = \{(R(q^n))!\}^{-1}_{n \geq 0}$, one may find in [31] where an advanced theory of general quantum coherent states is being developed. The operator $R(q\hat{Q})\partial_0$ is not recognised in
[31] as an example of $\psi$-derivative. Another very important example of $\psi$-derivative
one may find in [32] where all sequences of binomial type with persistent roots are
classified. There $Q = \partial_\psi = 2[2D\hat{D}D - D]$ defines - what is called by the authors of
[32] - the hyperbolic umbral calculus. Here $\psi = \{(2n(2n-1))!\}^{-1}_{n \geq 0}$.

Naturally, choosing any invertible formal series $S = S(\partial_\psi)$ in powers of $\partial_\psi$ one
arrives at infinity of examples of Appel $\psi$-sequences $s_\psi(x) = S^{-1}(\partial_\psi)x^n; \ n \geq 0$. We
are now in a position to define the basic objects of “$\psi$-umbral calculus” [16, 15].

Definition 2.7. Let $Q(\partial_\psi) : P \to P$; the linear operator $Q(\partial_\psi)$ is a $\partial_\psi$-delta operator
if
a) $Q(\partial_\psi)$ is $\partial_\psi$-shift invariant;  b) $Q(\partial_\psi)(id) = const \neq 0$.

The strictly related notion is that of the $\partial_\psi$-basic polynomial sequence:

Definition 2.8. Let $Q(\partial_\psi) : P \to P$; be the $\partial_\psi$-delta operator. A polynomial
sequence $\{p_n\}_{n \geq 0}; \ \deg p_n = n$ such that: 1) $p_0(x) = 1$;  2) $p_n(0) = 0$; $n > 0$;
3) $Q(\partial_\psi)p_n = \hat{n}_\psi p_{n-1}$ is called the $\partial_\psi$-basic polynomial sequence of the $\partial_\psi$-delta operator $Q(\partial_\psi)$.
Note 2.1. Let $\Phi(x; \lambda) = \sum_{n \geq 0} \frac{n!}{\psi_n(x)}$ denotes the $\psi$-exponential generating function of the $\partial_{\psi}$-basic polynomial sequence $\{p_n\}_{n \geq 0}$ of the $\partial_{\psi}$-delta operator $Q \equiv Q(\partial_{\psi})$ and let $\Phi(0; \lambda) = 1$. Then $Q\Phi(x; \lambda) = \lambda \Phi(x; \lambda)$ and $\Phi$ is the unique solution of this eigenvalue problem. In view of the Observation 2.2 we affirm that there exists such an admissible sequence $\varphi$ that $\Phi(x; \lambda) = \exp_{\varphi}[\lambda x]$.

The notation and naming established by Definitions 2.7 and 2.8 serve the target to preserve and to broaden simplicity of Rota’s finite operator calculus also in its extended “$\psi$-unbral calculus” case [16, 15]. As a matter of illustration of such notation efficiency let us quote after [16] the important Theorem 2.1 which might be proved using the fact that $\forall Q(\partial_{\psi}) \exists!$ invertible $S \in \sum_{\psi}$ such that $Q(\partial_{\psi}) = \partial_{\psi}S$.

(For Theorem 2.1 see also Theorem 4.3 in [12] – which holds for operators introduced by the definitions 2.5 and 2.6.)

Theorem 2.1. (Lagrange and Rodrigues $\psi$-formulas)

Let $\{p_n(x)\}_{n=0}^\infty$ be $\partial_{\psi}$-basic polynomial sequence of the $\partial_{\psi}$-delta operator $Q(\partial_{\psi})$.

Let $Q(\partial_{\psi}) = \partial_{\psi}S$. Then for $n > 0$:

1. $p_n(x) = Q(\partial_{\psi})^{-1}S^{-n+1}x^n$;
2. $p_n(x) = S^{-n}x^n - \frac{n}{n!}(S-n)^1x^{n-1}$;
3. $p_n(x) = \frac{n!}{n!}x_{\psi}S^{-n}x^{n-1}$;
4. $p_n(x) = \frac{n!}{n!}x_{\psi}(Q(\partial_{\psi})^{-1}p_{n-1}(x))$ (Rodrigues $\psi$-formula)

For the proof one uses typical properties of the Pincherle $\psi$-derivative as well as $x_{\psi}$ operator [16] defined below.

Definition 2.9. (compare with (17) in [7]) The Pincherle $\psi$-derivative i.e. the linear map $': \sum_{\psi} \to \sum_{\psi}$ is given by $T' = T\hat{x}_{\psi} - \hat{x}_{\psi}T_{\partial_{\psi}} = [T, \hat{x}_{\psi}]$ where the linear map $\hat{x}_{\psi}: P \to P$ is defined in the basis $\{x^n\}_{n \geq 0}$ as follows

$$\hat{x}_{\psi}x^n = \frac{\psi_{n+1}(q)(n+1)}{\psi_n(q)}x^{n+1} = \frac{(n+1)}{(n+1)}x^{n+1}, \quad n \geq 0.$$

Observation 2.3. The triples $\{\partial_{\psi}, x_{\psi}, \text{id}\}$ for any admissible $\psi$ constitute set of generators of the $\psi$-labelled representations of Graves-Heisenberg-Weyl (GHW) algebra [33–35]. Namely, as easily seen $[\partial_{\psi}, x_{\psi}] = \text{id}$.

Observation 2.4. In view of the Observation 2.3 the general Leibnitz rule in $\psi$-representation of Graves-Heisenberg-Weyl algebra may be written (compare with 2.2.2 Proposition in [35]) as follows

$$\partial_{\psi}^{m} x_{\psi}^{n} = \sum_{k \geq 0} \left( \begin{array}{c} n \\ k \end{array} \right) \left( \begin{array}{c} m \\ k \end{array} \right) k! x_{\psi}^{m-k} \partial_{\psi}^{n-k}. \quad (2.7)$$

One derives the above $\psi$-Leibnitz rule from $\psi$-Heisenberg-Weyl exponential commutation rules exactly the same way as in $\{D, \hat{x}, \text{id}\}$ GHW representation – (compare with 2.2.1 Proposition in [35]). $\psi$-Heisenberg-Weyl exponential commutation relations read:

$$\exp(t\partial_{\psi}) \exp(a\hat{x}_{\psi}) = \exp(at) \exp(a\hat{x}_{\psi}) \exp(t\partial_{\psi}). \quad (2.8)$$
In the sequel we shall discover another, related to (2.6) and heuristically appealing form of a $\partial_\psi$-difference-ization rule for a specific new $*_\psi$ product of functions or formal series. To this end let us introduce this pertinent $\psi$-multiplication $*_\psi$ of functions or formal series as specified below.

**Notation 2.2.**

\[
x*_\psi x^n = \hat{x}_\psi(x^n) = \frac{(n+1)}{(n+1)!}x^{n+1}; \quad n \geq 0 \quad \text{hence} \quad x*_\psi 1 = 1\_xTx
\]

\[
x^n*_\psi x = \hat{x}_\psi^n(x) = \frac{(n+1)}{(n+1)!}x^{n+1}; \quad n \geq 0 \quad \text{hence} \quad 1*_\psi x = 1\_xTx \text{ therefore} \quad x*_\psi \alpha
\]

1 = \alpha1*_\psi x = n*_\psi \alpha = \alpha*_\psi x = \alpha1\_xTx \quad \forall x, \alpha \in \otimes; \quad f(x)*\psi x^n = f(\hat{x}_\psi)x^n.

For $k \neq n$, $x^n*_\psi x^k \neq x^k*_\psi x^n$ as well as $x^n*_\psi x^k \neq x^{n+k} - \text{ in general i.e. for arbitrary admissible } \psi; \text{ compare this with } (x + \alpha a)^n \neq (x + \psi a)^{n-1}(x + \psi a)$.

In order to facilitate in the future formulation of observations accounted for on the basis of $\psi$-calculus representation of GHW algebra we shall use what follows.

**Definition 2.10.** With Notation 2.2. adapted let us define the $*_\psi$ powers of $x$ according to

\[
x^n*_\psi \equiv x*_\psi x^{(n-1)} = \hat{x}_\psi \left(\frac{(n-1)}{(n-1)!}x^{n-1}\right) = x*_\psi x*_\psi \ldots *_\psi x = \frac{n!}{n\psi!}x^n; \quad n \geq 0.
\]

Note that $x^n*_\psi x^k*_\psi = \frac{n!}{n\psi!}x^{(n+k)}*\psi \neq x^k*_\psi x^n*_\psi = \frac{k!}{k\psi!}x^{(n+k)}*\psi$ for $k \neq n$ and $x^0*_\psi = 1$. This noncommutative $\psi$-product $*_\psi$ is devised so as to ensure the observations below.

**Observation 2.5.**

a) $\partial_\psi x^n* = n\_x(n-1)*; \quad n \geq 0$

b) $\exp_\psi[\alpha x]* = \exp_\psi[\alpha \hat{x}_\psi]* 1$

c) $\exp_\psi[\alpha x]* \{\exp_\psi[\beta \hat{x}_\psi]* 1\} = \exp_\psi[\{\alpha + \beta \}|\hat{x}_\psi] * 1$

d) $\partial_\psi \left(x^k*_\psi x^n*\right) = (\_Dx^k)*x^n* + x^k*_\psi \left(\partial_\psi x^n*\right)$

e) $\partial_\psi (f*_\psi g) = (\_Df)*\psi g + f*_\psi(\partial_\psi g); \quad f, g - \text{formal series}$

f) $f(\hat{x}_\psi)g(\hat{x}_\psi)1 = f(x)*\psi \tilde{g}(x); \quad \tilde{g}(x) = g(\hat{x}_\psi)1$.

Now the consequences of Leibnitz rule e) for difference-ization of the product are easily feasible. For example the Poisson $\psi$-process distribution $\pi_m(x) = \frac{1}{N(\lambda, x)p_m(x)}; \sum_{m \geq 0} \pi_m(x) = 1$ is determined by

\[
p_m(x) = \frac{(\lambda x)^m}{m!}*_\psi \exp_\psi[-\lambda x] \tag{2.9}
\]

which is the (up to a factor) unique solution of its corresponding $\partial_\psi$-difference equations system

\[
\partial_\psi p_m(x) + \lambda p_m(x) = \lambda p_m-1(x)m > 0; \quad \partial_\psi p_0(x) = -\lambda p_0(x) \tag{2.10}
\]

Naturally $N(\lambda, x) = \exp[\lambda x]*\psi \exp_\psi[-\lambda x]$.

As announced – the rules of $\psi$-product $*_\psi$ are accounted for – as a matter of fact – on the basis of $\psi$-calculus representation of GHW algebra. Indeed, it is enough to consult Observation 2.3 and to introduce $\psi$-Pincherle derivation $\hat{\partial}_\psi$ of series in
powers of the symbol $\hat{x}_\psi$ as below. Then the correspondence between generic relative formulas turns out evident.

**Observation 2.6.** Let $\hat{\partial}_\psi \equiv \frac{\partial}{\partial x_\psi}$ be defined according to $\hat{\partial}_\psi f(\hat{x}_\psi) = [\partial_\psi, f(\hat{x}_\psi)]$. Then

$$\hat{\partial}_\psi \hat{x}_\psi^n = n\hat{x}_\psi^{n-1}; \quad n \geq 0 \quad \text{and} \quad \hat{\partial}_\psi \hat{x}_\psi^n 1 = \partial_\psi x_\psi^n \quad \text{hence} \quad \left[\hat{\partial}_\psi, f(\hat{x}_\psi)\right] 1 = \partial_\psi f(x);$$

where $f$ is a formal series in powers $\hat{x}_\psi$ of or equivalently in $\ast_\psi$ powers of $x$.

As an example of application note how the solution of (2.10) is obtained from the obvious solution $\pi_m(\hat{x}_\psi)$ of the $\hat{\partial}_\psi$-Pincherle differential equation (2.11) formulated within G-H-W algebra generated by $\{\partial_\psi, \hat{x}_\psi, \text{id}\}$

$$\hat{\partial}_\psi \pi_m(\hat{x}_\psi) + \lambda \pi_m(\hat{x}_\psi) = \lambda \pi_{m-1}(\hat{x}_\psi)m > 0; \quad \partial_\psi \pi_0(\hat{x}_\psi) = -\lambda \pi_0(\hat{x}_\psi) \quad (2.11)$$

Namely: due to Observation 2.5 f) $p_m(x) = \pi_m(\hat{x}_\psi) 1$, where

$$\pi_m(\hat{x}_\psi) = \frac{(\lambda \hat{x}_\psi)^m}{m!} \exp_{\hat{x}_\psi} [-\lambda \hat{x}_\psi]. \quad (2.12)$$

### 3. The general picture of $\text{End}(P)$

In the following we shall consider the algebra $\text{End}(P)$ of linear operators on linear space $P$ of polynomials $P = F[x]$ over the field $F$ of characteristic zero.

Before we pass to characterisations of Sheffer $\psi$-polynomials and related propositions we shall draw an overview picture of the situation with underlying the possibility to develop umbral calculus for any polynomial sequences $\{p_n\}_0^\infty$ instead of those of traditional binomial type only. At first note that in 1901 it was proved [36] that every linear operator mapping $P$ into $P$ may be represented as infinite series in operators $\hat{x}$ and $D$. In 1986 the authors of [37] supplied an explicit expression for such series in most general case of polynomials in one variable (for many variables see [38]). Thus according to Proposition 1 from [37] one has:

**Proposition 3.1.** Let $\hat{Q}$ be a linear operator that reduces by one the degree of each polynomial. Let $\{q_n(\hat{x})\}_{n \geq 0}$ be an arbitrary sequence of polynomials in the operator $\hat{x}$. Then $\hat{T} = \sum_{n \geq 0} q_n(\hat{x})\hat{Q}^n$ defines a linear operator that maps polynomials into polynomials. Conversely, if $\hat{T}$ is linear operator that maps polynomials into polynomials then there exists a unique expansion of the form $\hat{T} = \sum_{n \geq 0} q_n(\hat{x})\hat{Q}^n$.

It is also a rather matter of an easy exercise (see Note 2.1.) to prove [37]:

**Proposition 3.2.** Let $\hat{Q}$ be a linear operator that reduces by one the degree of each polynomial. Let $\{q_n(\hat{x})\}_{n \geq 0}$ be an arbitrary sequence of polynomials in the operator $\hat{x}$. Let a linear operator that maps polynomials into polynomials be given by $\hat{T} = \sum_{n \geq 0} q_n(\hat{x})\hat{Q}^n$. Let $P(x; \lambda) = \sum_{n \geq 0} q_n(x)\lambda^n$ be the indicator of $\hat{T}$. Then there exists a unique formal series $\Phi(x; \lambda)$; $\Phi(0; \lambda) = 1$ such that $\hat{Q}\Phi(x; \lambda) = \lambda \Phi(x; \lambda)$ and $P(x; \lambda) = \Phi(x; \lambda)^{-1}\hat{T}\Phi(x; \lambda)$.
Example 3.1. \( \partial_x \exp_q \{ \lambda x \} = \lambda \exp_q \{ \lambda x \}; \quad \exp_q [ x ]_{x=0} = 1 \). (*)

Hence for indicator of \( \hat{T} \); \( \hat{T} = \sum_{n \geq 0} q_n ( \hat{x} ) \partial^n_q \), we have

\[
P(x; \lambda) = \left[ \exp_q \{ \lambda x \} \right]^{-1} \hat{T} \exp_q \{ \lambda x \}. \tag{**}
\]

After choosing \( \psi_q ( q ) = [ n_q ]^{-1} \) we get \( \exp_q \{ x \} = \exp_q \{ x \} \). In this connection note that \( \exp_0 ( x ) = \frac{1}{1 - x} \) and \( \exp(x) \) are mutual limit deformations. Therefore corresponding specifications of (*) such as \( \exp_q ( \lambda x ) = \frac{1}{1 - \lambda x} \) or \( \exp(\lambda x) \) lead to corresponding specifications of (**) for divided difference operator \( \partial_0 \) and \( D \) operator including special cases from [37].

Let us now introduce [16, 15] an important operator \( \hat{\imath}_Q ( \partial_q ) \) dual to \( Q(\partial_q) \).

Definition 3.1. Let \( \{ p_n \}_{n \geq 0} \) be the \( \partial_q \)-basic polynomial sequence of the \( \partial_q \)-delta operator \( Q(\partial_q) \). A linear map \( \hat{\imath}_Q ( \partial_q ) : P \rightarrow \mathbb{P} \); \( \hat{\imath}_Q ( \partial_q ) p_n = \frac{n + 1}{(n + 1)!} p^{n+1}_n ; n \geq 0 \) is called the operator dual to \( Q(\partial_q) \). (Note: for \( Q = id \) we have \( \hat{\imath}_Q ( \partial_q ) \equiv \hat{\imath}_\partial_q \equiv \hat{\imath}_\psi \).

Comment 3.1. Dual in the above sense corresponds to adjoint in \( \psi \)-umbral calculus language of linear functionals’ umbral algebra (compare with Proposition 1.1.21 in [18]).

It is obvious that the following holds:

Proposition 3.3. Let \( \{ q_n ( \hat{\imath}_Q ( \partial_q ) ) \}_{n \geq 0} \) be an arbitrary sequence of polynomials in the operator \( \hat{\imath}_Q ( \partial_q ) \). Then \( T = \sum_{n \geq 0} q_n ( \hat{\imath}_Q ( \partial_q ) ) Q(\partial_q)^n \) defines a linear operator that maps polynomials into polynomials. Conversely, if \( T \) is linear operator that maps polynomials into polynomials then there exists a unique expansion of the form

\[
T = \sum_{n \geq 0} q_n ( \hat{\imath}_Q ( \partial_q ) ) Q(\partial_q)^n \tag{3.1}
\]

Comment 3.2. The pair \( Q(\partial_q), \hat{\imath}_Q ( \partial_q ) \) of dual operators is expected to play a role in the description of quantum-like processes apart from the \( q \)-case now vastly exploited [19, 15, 16]. Naturally the Proposition 3.2 for \( Q(\partial_q) \) and \( \hat{\imath}_Q ( \partial_q ) \) dual operators is also valid.

Summing up: (see Definitions 2.5 and 2.6) We have the following picture for \( \text{End} ( P ) \) – the algebra of all linear operators acting on the algebra \( P \) of polynomials: \( Q(P) \subseteq \text{End} ( P ) \) and of course \( Q(P) \neq \text{End} ( P ) \) where \( Q(P) \) is the linear space of all \( \psi \)-difference-tial operators \( Q(\partial_q) \) (including \( \partial_q \)-delta operators \( Q(\partial_q) \) and plus zero map of course).

\( Q(P) \) breaks up into sum [12] of subsets \( \sum_L \cap Q(P)^* ; L \in Q(P) \) according to commutativity of these generalised differential operators \( L \) where \( \sum_L = \{ T \in \text{End} ( P ) ; [ T, L ] = 0 ; L \in Q(P) \} \). Any one dimensional subspace of \( \text{End} ( P ) \) spanned by arbitrary operator \( \hat{\imath} \in \text{End} ( P ) \) raising by one the degree of all polynomials has the empty intersection with \( Q(P)^* = Q(P)/\{ 0 \} \). Naturally to each \( \sum_L \) subalgebra i.e. to each \( Q \) operator there corresponds its dual operator \( \hat{\imath}_Q \); \( \hat{\imath}_Q \notin \sum_Q \) and both \( Q \) and \( \hat{\imath}_Q \) operators are sufficient to build up the whole algebra \( \text{End}(P) \) according to unique representation given by (3.1). With every such dual pair \( \{ Q, \hat{\imath}_Q \} \)
including \(Q(\partial_\psi)\) and \(\hat{x}_Q(\partial_\psi)\) which include \(Q = \text{id.}\) case i.e. \(\partial_\psi\) and \(\hat{x}_\partial_\psi \equiv \hat{x}_\psi\) we have for any admissible \(\psi\) the general picture statement.

**General statement:**

\[
\text{End}(P) = \{[\partial_\psi, \hat{x}_\psi]\} = \{[Q(\partial_\psi), \hat{x}_Q(\partial_\psi)]\} = \{[Q, \hat{x}_Q]\}
\]

i.e. the algebra \(\text{End}(P)\) is generated by any dual pair \(\{Q, \hat{x}_Q\}\) and in particular \(\text{End}(P)\) is generated by \(\{Q(\partial_\psi), \hat{x}_Q(\partial_\psi)\}\) including any dual pair \(\{\partial_\psi, \hat{x}_\psi\}\) determined by any choice of admissible sequence \(\psi\).

As a matter of fact and in another words: we have bijective correspondences between different commutation classes of operators from \(\text{End}(P)\); different abelian subalgebras \(\sum Q_i\); distinct \(\psi\)-representations of GHW algebras; different \(\psi\)-representations of the reduced incidence algebra \(R(L(S))\) – isomorphic to the algebra \(\Phi_{\psi}\) of \(\psi\)-exponential formal power series \(16\) and finally – distinct \(Q\)-umbral or \(\psi\)-umbral calculi \([7, 8, 12, 13, 32, 16]\). (Recall: \(R(L(S))\) is the reduced incidence algebra of \(L(S)\) where \(L(S) = \{A; A \subset S; |A| < \infty\}\); \(S\) is countable and \((L(S); \subset)\) is partially ordered set ordered by inclusion \([3, 16]\)).

This is the way the Rota’s devise has been carried into effect. The devise “much is the iteration of the few” \([3]\) – much of the properties of literally all polynomial sequences – as well as GHW algebra representations – is the application of few basic principles of the \(Q\)-umbral or in particular \(\psi\)-umbral difference operator calculus “\(\psi\)-integration” included.

**\(\psi\)-Integration Remark:** Recall: \(\partial_0 x^n = x^{n-1}\). \(\partial_0\) is identical with divided difference operator. \(\partial_0\) is identical with \(\partial_\psi\) for \(\psi = \{\psi(q)_n\}_{n \geq 0}; \psi(q)_n = 1; n \geq 0\). Let \(Qf(x) = f(qx)\). Recall also that to the “\(\partial_Q\) difference-ization” there corresponds also \(q\)-integration \([24–26]\) which is a right inverse operation to “\(q\)-difference-ization”.

Namely

\[
F(z) \equiv \left( \int_q \varphi \right)(z) := (1 - q)z \sum_{k=0}^{\infty} \varphi(q^kz) q^k
\]

(3.2)

i.e.

\[
F(z) \equiv \left( \int_q \varphi \right)(z) = (1 - q)z \left( \sum_{k=0}^{\infty} q^k \hat{Q}^k \varphi \right)(z) = \left( (1 - q)z \frac{1}{1 - qQ} \right)(z)
\]

(3.3)

Of course

\[
\partial_q \circ \int_q = \text{id}
\]

(3.4)

as

\[
\frac{1 - qQ}{1 - q} \partial_0 \left( (1 - q)z \frac{1}{1 - qQ} \right) = \text{id}.
\]

(3.5)

Naturally (3.5) might serve to define a right inverse operation to “\(q\)-difference-ization” \((\partial_q \varphi)(x) = \frac{1 - qQ}{1 - q} \partial_0 \varphi(x)\) and consequently the “\(q\)-integration” as represented
by (3.2) and (3.3). As it is well known the definite $q$-integral is a numerical approximation of the definite integral obtained in the $q \to 1$ limit. Following the $q$-case example we introduce now an $R$-integration (consult Remark 2.1).

$$\int_R x^n = \left(\hat{x} \frac{1}{R(qQ)}\right) x^n = \frac{1}{R(q^n+1)} x^{n+1}; \quad n \geq 0 \quad (3.6)$$

Of course $\partial_R \circ \int_R = \text{id}$ as

$$R(q\hat{Q}) \partial_0 \left( \hat{x} \frac{1}{R(qQ)} \right) = \text{id}. \quad (3.7)$$

Let us then finally introduce the analogous representation for $\partial_\psi$ difference-ization

$$\partial_\psi = \hat{n}_\psi \partial_0; \quad \hat{n}_\psi x^{n-1} = n_\psi x^{n-1}; \quad n \geq 1. \quad (3.8)$$

Then

$$\int_\psi x^n = \left(\hat{x} \frac{1}{n_\psi}\right) x^n = \frac{1}{(n+1)_\psi} x^{n+1}; \quad n \geq 0 \quad (3.9)$$

The Section Closing Remark: The picture that emerges discloses the fact that – any $\psi$-representation of finite operator calculus or equivalently – any $\psi$-representation of GHW algebra makes up an example of the algebraization of the analysis – naturally when constrained to the algebra of polynomials. We did restricted all our considerations to the algebra $P$ of polynomials or formal series. Therefore the distinction in-between difference and differentiation operators disappears. All linear operators on $P$ are both difference and differentiation operators if the degree of differentiation or difference operator is unlimited. For example

$$\frac{d}{dx} = \sum_{k \geq 1} \frac{d_k}{k!} \Delta_k \quad \text{where} \quad d_k = \left[ \frac{d}{dx} x^k \right]_{x=0} = (-1)^{k-1} (k-1)!$$

or

$$\Delta = \sum_{n \geq 1} \frac{\delta_n}{n!} \frac{d^n}{dx^n}$$

where $\delta_n = \left[ \Delta x^n \right]_{x=0} = 1$. Thus the difference and differential operators and equations are treated on the same footing.

4. Characterisations of Sheffer $\psi$-polynomials and related propositions

Let us now pass to characterisations of Sheffer $\psi$-polynomials and related propositions.

Definition 4.1. A polynomial sequence $\{s_n(x)\}_{n=0}^\infty$ is called the Sheffer $\psi$-sequence of the $\partial_\psi$-delta operator $Q(\partial_\psi)$ if (1) $s_0(x) = c \neq 0$; (2) $Q(\partial_\psi)s_n(x) = n_\psi s_{n-1}(x)$. 

On simple characterisations of Sheffer $\psi$-polynomials and... 57
The following characterisation of Sheffer $\psi$-sequence of the $\partial_\psi$-delta operator $Q(\partial_\psi)$ relates it to the unique pair of a $\partial_\psi$-basic sequence of $Q(\partial_\psi)$ and the corresponding $\partial_\psi$-shift invariant operator $S$ [16].

**Proposition 4.1.** Let $Q(\partial_\psi)$ be a $\partial_\psi$-delta operator with $\partial_\psi$-basic polynomial sequence $\{q_n(x)\}_{n=0}^\infty$. Then $\{s_n(x)\}_{n=0}^\infty$ is a sequence of Sheffer $\psi$-polynomials of $Q(\partial_\psi)$ if there exists an invertible $\partial_\psi$-shift invariant operator $S$ such that $s_n(x) = S^{-1}q_n(x)$. Sheffer $\psi$-sequence $\{s_n(x)\}_{n=0}^\infty$ labelled by $S$ is therefore referred to as the Sheffer $\psi$-sequence of $Q(\partial_\psi)$ relative to $S$. It is not difficult to find out that $S^{-1} = \sum_{k \geq 0} \frac{s_k(0)}{k!}Q(\partial_\psi)^k$, where $\{s_n(x)\}_{n=0}^\infty$ is the sequence of Sheffer $\psi$-sequence $Q(\partial_\psi)$ relative to $S$ [16].

**Remark 4.1** The Proposition 4.1 is also valid in the “$Q$-case” also i.e. for the $\psi$-difference-tial operators $Q$ and for the generalised Sheffer $\psi$-polynomials $\{s_n\}_{n \geq 0}$ (see definitions 2.5 and 2.6). In such a $Q$-case we shall say that $Q$-$\psi$-sequence $\{s_n(x)\}_{n=0}^\infty$ is the generalised Sheffer $\psi$-sequence of $Q$ relative to $S$.

With this in mind we arrive at the general conclusion.

**Conclusion 4.1.** The family of generalised Sheffer $\psi$-sequences $\{s_n(x)\}_{n=0}^\infty$ corresponding to a fixed $\psi$-difference-tial operator $Q$ i.e. the family of Sheffer $Q$-$\psi$-sequences (see the definitions 2.5 and 2.6) is labelled by the abelian group of all invertible operators $S \in \Sigma_Q$. The families of these generalised Sheffer $\psi$-polynomials are orbits of such groups.

$\psi$-calculus or $Q$-calculus in parts appears to be almost automatic extension of the classical operator calculus of Rota-Mullin [16]. Therefore – as a rule – we shall omit proofs of propositions and theorems stated. Proofs might be either found in [16, 15, 39, 40] or adopted from corresponding ones in [3] in most of relevant cases. Recall again that the general results of $\psi$-calculus presented here may be extended to Markowsky $Q$-calculus where $Q$ stands for a generalised difference operator i.e. the one lowering the degree of any polynomial by one.

As a matter of fact all statements of standard finite operator calculus of Rota are valid also in the case of $\psi$-extension under the almost automatic replacement of $\{D, x, \text{id}\}$ by their $\psi$-representation correspondent generators of GHW i.e. $\{\partial_\psi, \hat{x}_\psi, \text{id}\}$. In most general case of $Q$-umbral representation these are the $\{Q, \hat{x}_Q, \text{id}\}$ triples of operators to be used. Naturally any specification of admissible $\psi$ – for example the famous one defining $q$-calculus – has its own characteristic properties not pertaining to the standard case of Rota calculus realisation. Nevertheless the overall picture and system of statements depending only on GHW algebra is the same modulo some automatic replacements in formulas quoted in the sequel [16, 39]. Let us then pass to the presentation of characterisations of Sheffer $\psi$-sequences and related propositions in order to indicate the first main features of extended finite operator calculus. Apart from Sheffer $\psi$-sequences characterisation by Proposition 4.1 and Viskov theorems from section 2 – (see Illustration 2.1 – formulas (2.1–2.4)) the following propositions and theorems come true [16, 15, 39]. At first we restate the Sheffer $\psi$-sequences characterisation due to Viskov (see Illustration 2.1.) in our Rota-oriented further notation [16].
Proposition 4.2. Let $Q$ be a $\partial_\psi$-delta operator. Let $S$ be an invertible $\partial_\psi$-shift invariant operator. Let $Q = q(\partial_\psi)$ and $S = s(\partial_\psi)$. Let $q^{-1}(t)$ be the $\psi$-exponential formal power series inverse to $q(t)$. Then the $\psi$-exponential generating function of Sheffer $\psi$-sequence $\{s_n(x)_{n=0}^{\infty}\}$ of $Q$ relative to $S$ is given by

$$\sum_{k \geq 0} \frac{s_k(x)}{k!} z^k = s(q^{-1}(z)) \exp_{\psi} \{xq^{-1}(z)\}.$$

Naturally the corresponding extended versions of binomial and second binomial theorems hold and here we quote the second one as another Sheffer $\psi$-sequences characterisation.

Theorem 4.1. (Sheffer $\psi$-Binomial Characterisation Theorem)

Let $Q(\partial_\psi)$ be the $\partial_\psi$-delta operator with the $\partial_\psi$-basic polynomial sequence $\{q_n(x)\}_{n=0}^{\infty}$. Then $\{s_n(x)\}_{n=0}^{\infty}$ is the Sheffer $\psi$-sequence of $Q(\partial_\psi)$ relative to an invertible $\partial_\psi$-shift invariant operator $S$ if

$$s_n(x + \psi) = \sum_{k \geq 0} \left( \begin{array}{c} n \\ k \end{array} \right) \psi s_k(x) q_{n-k}(y).$$

There exist other characterisations of course and we are not going to present all of available ones. We finish our examples of such characterisation statements with another proposition characterising Sheffer $\psi$-sequences [3, 16].

Proposition 4.3. A sequence $\{s_n(x)\}_{n=0}^{\infty}$ is the Sheffer $\psi$-sequence of a $\partial_\psi$-delta operator $Q(\partial_\psi)$ with the $\partial_\psi$-basic polynomial sequence $\{q_n(x)\}_{n=0}^{\infty}$ if there exists such a $\partial_\psi$-delta operator $A$ (not necessarily associated with $\{s_n(x)\}_{n=0}^{\infty}$) and the sequence $\{c_n\}_{n=0}^{\infty}$ of constants such that

$$A s_n(x) = \sum_{k \geq 0} \left( \begin{array}{c} n \\ k \end{array} \right) s_k(x) c_{n-k} \quad n \geq 0.$$

We shall present now some propositions announced earlier which are related to Sheffer $\psi$-sequences. We have chosen for that the spectral theorem and few, main umbral propositions [3, 16, 40]. To start with let us note that the natural inner product may be associated with any $Q$-$\psi$-sequence $\{s_n(x)\}_{n=0}^{\infty}$ of a $\psi$-difference-tial operator $Q$ – relative to $S \in \Sigma_Q$. For that to see define the linear operator $W$: $s_n(x) \rightarrow x^n$.

Definition 4.2. Let $S$ be an operator the Sheffer $\psi$-sequence $\{s_n(x)\}_{n=0}^{\infty}$ of generalised differential operator $Q$ is related to. Let $W$ be the linear operator such that $W : s_n(x) \# x^n$. We then define the bilinear form

$$(f(x), g(x))_{Q,S} := ([Wf](Q)Sg(x)]_{x=0} \quad f, g \in P.$$

It is then easy to observe the important property of this bilinear form – now on the reals.

Observation 4.1. The bilinear form $(f(x), g(x))_{Q,S} := ([Wf](Q)Sg(x)]_{x=0} \quad f, g \in P$ is a positive definite inner product over reals if $n_\psi > 0$; $n \in N$. 

Proof: \((s_k(x), s_n(x))_{Q,S} = [Q^k S s_n(x)]_{x=0} = [Q^k q_n(x)]_{x=0} = \frac{n!}{n^k} q_{n-k}(0) = n! \delta_{nk}\)
where \(\{q_n(x)\}_{n=0}^\infty\) is the \(\psi\)-basic sequence of the generalised difference-tial operator \(Q\). Let now \(H = (P; (\, )_{Q,S})\). We shall denote by \((\, )_{Q,S}: H \times H \to R\) the scalar product associated with Sheffer \(\psi\)-sequence \(\{s_n(x)\}_{n=0}^\infty\). The unitary space \(H = (P; (\, )_{Q,S})\) is then completed to the unique Hilbert space \(\mathbb{H} = H\). If so then the following theorem is valid in \(\psi\)-extended and also in \(Q\)-extended and case of finite operator calculus.

**Theorem 4.2. (\(Q\)-calculus Spectral Theorem) [3, 16]**

Let \(\{s_n(x)\}_{n=0}^\infty\) be generalised Sheffer \(\psi\)-sequence of \(\psi\)-difference-tial operator \(Q\) with \(\psi\)-basic \(\{q_n(x)\}_{n=0}^\infty\) and relative to \(S \in \Sigma_Q\). Then there exists a unique essentially self-adjoint operator \(A_{Q,S}: \mathbb{H} \to \mathbb{H}\) given by \(A_{Q,S} = \sum_{k \geq 1} \frac{\psi^{\prime} q_k(x)}{(k-1)!} Q^k\) such that the spectrum of \(A_{Q,S}\) consists of \(n = 0, 1, 2, 3, \ldots\) where \(A_{Q,S} s_n(x) = ns_n(x)\). The quantities \(u_k\) and \(\hat{v}_k\) are calculated according to:

\[
    u_k = - \left[ (\log S) \hat{x}_\psi^{-1} q_k(x) \right]_{x=0} \quad \text{and} \quad \hat{v}_k(x) = \hat{x}_\psi \left[ \frac{d}{dx} q_k \right] (0).
\]

The linear map \(\hat{\cdot}: \Sigma_Q \to \Sigma_Q\) (as recalled by Definition 4.4) is the Pincherle \(Q\)-\(\psi\)-derivative. As for the main umbral propositions [3, 16, 40] we quote here just two right after a definition.

**Definition 4.3.** Let \(T : P \to P\) be a linear operator not necessarily an element of \(\Sigma_Q\). If there exist two \(\psi\)-basic sequences \(\{p_n\}_{n=0}^\infty\) and \(\{q_n\}_{n=0}^\infty\) of \(\psi\)-difference-tial operators from \(\Sigma_Q\) such that \(Tp_n = q_n\), \(n \geq 0\), then we shall call \(T\) the \(Q\)-\(\psi\)-umbral operator.

As in [3, 40] we may now prove that the following theorem holds.

**Theorem 4.3.** Let \(T\) be any \(Q\)-\(\psi\)-umbral operator. Then for \(S \in \Sigma_Q\) we have:

- a) the map \(S \to TST^{-1}\) is an automorphism of the algebra \(\Sigma_Q\),
- b) if \(L\) is any \(\psi\)-difference-tial operator then \(P = TL T^{-1}\) is a \(\psi\)-difference-tial operator,
- c) if \(S = s(Q)\) and \(L = L(Q)\) are \(\psi\)-exponential formal power series in \(Q\) then \(TST^{-1} = s(P)\) where \(P = TL(Q)T^{-1}\),
- d) the \(Q\)-\(\psi\)-umbral operator maps any Sheffer \(Q\)-\(\psi\)-sequence into a Sheffer \(Q\)-\(\psi\)-sequence.

In order to formulate the next important \(Q\)-\(\psi\)-umbral theorem we need the operator, which we shall call the Pincherle \(Q\)-\(\psi\)-derivative.

**Definition 4.4.** A linear map \(\hat{\cdot}: \Sigma_Q \to \Sigma_Q; \quad T' = T\hat{x}_Q - \hat{x}_Q T \equiv [T, \hat{x}_Q] \in \Sigma_Q\) is called the Pincherle \(Q\)-\(\psi\)-derivative.

As in [3, 40] we may now prove that the following theorem holds.

**Theorem 4.4.** Let \(U\) be a \(Q\)-\(\psi\)-umbral operator \(Uq_n(x) = n! \hat{x}_Q^n 1 = x^n\) and let \(\{q_n\}_{n \geq 0}\) be the \(\psi\)-basic sequences of \(Q\)-\(\psi\)-difference-tial operator \(L = L(Q)\). Then \(U' = \hat{x}_Q U(L(Q) - I)\).
5. Miscellaneous remarks and indications of several applications

A) As announced in the Introduction we shall present now a specific formulation of q-umbral calculus by Cigler [17] and Kirschenhofer [18]. This formulation might be related – as noted in [15] – to the so-called quantum groups [19]. Namely we shall indicate how one may formulate q-extension of polynomial sequence \{p_n\}_{n=0}^\infty of q-binomial type by

\[ p_n(A + B) \equiv \sum_{k \geq 0} \binom{n}{k}_q \ p_k(A)p_{n-k}(B) \quad \text{where} \quad [B, A]_q \equiv BA - qAB = 0. \quad (5.2) \]

A and B might be interpreted then as coordinates on quantum q-plane (see [19], Chapter 4). For example \( A = \hat{x} \) and \( B = yQ \) where \( Q\varphi(x) = \varphi(qx) \). If so then the following identification takes place:

\[ p_n(x + qy) \equiv E^y(\partial_q)p_n(x) = \sum_{k \geq 0} \binom{n}{k}_q \ p_k(x)p_{n-k}(y) = p_n \left( \hat{x} + y\hat{Q} \right) \quad (5.3) \]

Also q-Sheffer polynomials \( \{s_n(x)\}_{n=0}^{\infty} \) are defined equivalently (see 2.1.1 in [18]) by

\[ s_n(A + B) \equiv \sum_{k \geq 0} \binom{n}{k}_q \ s_k(A)p_{n-k}(B) \quad (5.4) \]

where \([B, A]_q \equiv BA - qAB = 0\) and \( \{p_n(x)\}_{n=0}^{\infty} \) of q-binomial type. For example \( A = \hat{x} \) and \( B = y\hat{Q} \) where \( \hat{Q}\varphi(x) = \varphi(qx) \). Then the following identification is also evident:

\[ s_n(x + qy) \equiv E^y(\partial_q)s_n(x) = \sum_{k \geq 0} \binom{n}{k}_q \ s_k(x) \ p_{n-k}(y) = s_n \left( \hat{x} + y\hat{Q} \right) \quad (5.5) \]

This means that one may formulate q-extended finite operator calculus with help of the “quantum q-plane” q-commuting variables \( A, B: AB - qBA \equiv [A, B]_q = 0 \). A natural question then arises: is there a \( \psi \)-analogue extension of quantum q-plane formulation? If one [15, 16] introduces the evident natural extension of q-commutator as done in (see below) then the answer given in [15, 16] is in negative. The above identifications of polynomial sequence \( \{p_n\}_{n=0}^{\infty} \) of q-binomial type and Sheffer q-sequences \( \{s_n(x)\}_{n=0}^{\infty} \) fail to be extended to the more general \( \psi \)-case. Therefore one can not formulate that way the \( \psi \)-extended finite operator calculus with help of the “quantum \( \psi \)-plane” \( \hat{q}_{\psi, Q} \)-commuting variables \( A, B: AB - \hat{q}_{\psi, Q}BA \equiv [A, B]_{\hat{q}_{\psi, Q}} = 0 \). The corresponding new objects above are defined accordingly [15].
\textbf{Definition 5.1.} Let $\{p_n\}_{n \geq 0}$ be the $\psi$-basic sequence of the $\psi$-difference-tial operator $Q$. Then the $\hat{q}_{\psi,Q}$-operator is a liner map
\[
\hat{q}_{\psi,Q} : P \rightarrow P; \quad \hat{q}_{\psi,Q}p_n = \frac{(n+1)\psi - 1}{n\psi}p_n; \quad n \geq 0.
\]
\textbf{Note 5.1.} For $Q(\partial_{\psi}) = \partial_{\psi}$ the natural notation is $\hat{q}_{\psi,\text{id}} \equiv \hat{q}_{\psi}$. For $\psi_n(q) = \frac{1}{n!(q^n)!}$ and
\[
R(x) = \frac{1-x}{1-q} \hat{q}_{\psi,Q} \equiv \hat{q}_{R,\text{id}} \equiv \hat{q}_R \equiv \hat{q}_{q,\text{id}} \equiv \hat{q}_q \equiv \hat{q}
\]
and
\[
\hat{q}_{\psi,Q}x^n = q^n x^n.
\]
\textbf{Definition 5.2.} Let $A$ and $B$ be linear operators acting on $P; A : P \rightarrow P; B : P \rightarrow P$. Then $AB - \hat{q}_{\psi,Q}BA \equiv [A,B]_{\hat{q}_{\psi,Q}}$ is called $\hat{q}_{\psi,Q}$-mutator of $A$ and $B$ operators.

Naturally $Q\hat{x}_Q - \hat{q}_{\psi,Q}\hat{x}_QQ \equiv [Q, \hat{x}_Q]_{\hat{q}_{\psi,Q}} = \text{id}$. This is easily verified in the $\psi$-basic sequence of the operator $Q$. In conclusion the case of $q$-extended finite operator calculus – or $q$-calculus in short – is fairly enough distinguished by the Cigler-Kirchenhofer approach and the quantum plane notion among the infinite variety of $Q,\psi$-umbral calculi.

\textbf{B) We end our exposition of the fundamentals of $\psi$-umbral calculus with indications of several examples of identities} resulting straightforwardly from the fact that dual pairs $\{Q, \hat{x}_Q\}$ − (and in particular $\{Q(\partial_{\psi}), \hat{x}_Q(\partial_{\psi})\}$ including any dual pair $\{\partial_{\psi}, \hat{x}_{\psi}\}$ determined by any choice of admissible sequence $\psi$) − provide us with distinct $Q,\psi$-representations of GHW algebra as seen from $\text{End}(P) = \{\{\partial_{\psi}, \hat{x}_{\psi}\}\} = \{\{Q(\partial_{\psi}), \hat{x}_Q(\partial_{\psi})\}\} = \{\{Q, \hat{x}_Q\}\}$.\n
1. Let $\{a_n(x)\}_{n \geq 0}$ be any Appel $\psi$-sequence. Then (compare (4) in [41]) we have
\[
\hat{x}_Q \sum_{m=0}^{n} a_m(Q) \frac{\hat{x}_Q^m}{m!} \equiv a_n(Q) \frac{\hat{x}_Q^{n+1}}{m!}.
\]
Also other identities from [41–42] apply specifically to dual pairs $\{Q, \hat{x}_Q\}$. For example:

2. Compare (6) and (7) in [42]
\[
(Q\hat{x}_Q Q)^n = Q^n \hat{x}_Q^n Q^n; \quad (\hat{x}_Q Q Q\hat{x}_Q)^n = \hat{x}_Q^n Q^n \hat{x}_Q^n.
\]

3. Similarly identities from [43] apply to dual pairs $\{Q, \hat{x}_Q\}$. For example one has for $\hat{A}^{2m-\psi} = \hat{A} \left( \hat{A} - (n-1)\text{id} \right) \left( \hat{A} - 2\text{id} \right) \ldots \left( \hat{A} - (n-1)\text{id} \right)$ (compare with (8) in [43]) the identity
\[
\hat{x}_Q^n Q^n \left[ f(\hat{x}_Q) \right] = (\hat{x}_Q Q^{2m-\psi} \left[ f(\hat{x}_Q) \right]) .
\]

4. Lagrange inversion formula and more general formula from the theorem due to Viskov (in [29] point 18°) also apply automatically to dual pairs $\{Q, \hat{x}_Q\}$ under evident replacements.
References

[1] R. Mullin and G. C. Rota, On the foundations of combinatorial theory, III. Theory of binomial enumeration, in: Graph Theory and Its Applications. Ed. by B. Harris, Academic Press, New York 1970, pp. 167–213.
[2] G. -C. Rota, D. Kahaner, and A. Odlyzko, On the foundations of combinatorial theory VII. Finite operator calculus, J. Math. Anal. Appl. 42 (1973), 684–760.
[3] G. -C. Rota, Finite Operator Calculus, Academic Press, Inc. 1975.
[4] J. P. Fillmore and S. G. Williamson, A linear algebra setting for the Rota-Mullin theory of polynomials of binomial type, Lin. Multilinear Alg. 1 (1973), 67–80.
[5] A. M. Garcia, An exposé of the Mullin-Rota theory of polynomials of binomial type, Lin. and Multilinear Alg. 1 (1973), 47–65.
[6] R. B. Brown, Sequences of functions of binomial type, Discrete Math. 6 (1973), 313–331.
[7] O. V. Viskov, Operator characterization of generalized Appel polynomials, Soviet Math. Dokl. 16 (1975), 1521–1524.
[8] O. V. Viskov, On the basis in the space of polynomials, Soviet Math. Dokl. 19 (1978), 250–253.
[9] R. P. Boas Jr. and R. C. Buck, Polynomials defined by generating relations, Amer. Math. Monthly 63 (1959), 626–632.
[10] R. P. Boas Jr. and R. C. Buck, Polynomial Expansions of Analytic Functions, 2nd ed., Springer, Berlin 1964.
[11] M. Ward, A calculus of sequences, Amer. J. Math. 58 (1936), 255–266.
[12] G. Markowsky, Differential operator and theory of binomial enumeration, Math. Anal. Appl. 63 (1978), 145–155.
[13] A. Di Bucchianico and D. Loeb, A simpler characterization of Sheffer polynomials, Stud. Appl. Math. 92 (1994), 1–15.
[14] D. E. Loeb, The World of Generating Functions and Umbral Calculus (LaBRI URA CNRS 1304) August 4, 1999 “Dedicated to Gian-Carlo Rota for his inspiration and friendship” http://www.tug.org/applications/tex4ht/tugmml/a14x.htm
[15] A. K. Kwaśniewski, On extended finite operator calculus of Rota and quantum groups, Integral Transforms and Special Functions 2 (2001), 333–340.
[16] A. K. Kwaśniewski, Towards $\psi$ extension of finite operator calculus of Rota, Rep. Math. Phys. 48 (2001), 305–342.
[17] J. Cigler, Operatormethoden für $q$-Identitäten, Monatsh. Math. 88 (1979), 87–105.
[18] P. Kirschenhofer, Binomialfolgen, Schefferfolgen und Faktorfolgen in den $q$-Analysis, Abt. II Oster. Akad. Wiss. Math. Naturw. Kl. 188 (1979), 263–315.
[19] Ch. Kassel, Quantum Groups, Springer-Verlag, New York 1995.
[20] T. S. Chihara, *Orthogonal polynomials with Brenke type generating functions*, Duke Math. J. 35 (1968), 505–518.

[21] T. S. Chihara, *The orthogonality of a class of Brenke polynomials*, Duke Math. J. (1971), 599–603.

[22] S. R. Roman, *More on the umbral calculus with emphasis on the q-umbral calculus*, J. Math. Anal. Appl. 107 (1985), 222–234.

[23] S. M. Roman, *The Umbral Calculus*, Academic Press 1984.

[24] S. M. Roman, *The theory of the umbral calculus I*, J. Math. Anal. Appl. 87 (1982), 58–115.

[25] S. M. Roman, *The theory of the umbral calculus II*, J. Math. Anal. Appl. 89 (1982), 290–314.

[26] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics. A Foundation of Computer Science*, Addison-Wesley Publishing Company 1994.

[27] A. K. Kwaśniewski, *Higher order recurrences for analytical functions of Tchebysheff type*, Advances in Applied Clifford Algebras 9 (1999), 41–54.

[28] W. Bajguz and A. K. Kwaśniewski, *On generalization of Lucas symmetric functions and Tchebysheff polynomials*, Integral Transforms and Special Functions 8 (1999), 165–174.

[29] O. V. Viskov, *Noncommutative approach to classical problems of analysis*, Trudy Matematicheskogo Instituta AN SSSR 177 (1986), 21–32.

[30] N. Ya. Sonin, *Riad Ivana Bernoulli*, Izv. Akad. Nauk 7 (1897), 337–353.

[31] A. Odzijewicz, *Quantum algebras and q-special functions related to coherent states maps of the disc*, Commun. Math. Phys. 192 (1998), 183–215.

[32] A. Di Bucchianico and D. Loeb, *Sequences of binomial type with persistent roots*, J. Math. Anal. Appl. 199 (1996), 39–58.

[33] C. Graves, *On the principles which regulate the interchange of symbols in certain symbolic equations*, Proc. Royal Irish Academy 6 (1853–1857), 144–152.

[34] O. V. Viskov, *On one result of George Boole*, Integral Transforms and Special Functions 1 no. 2 (1997), 2–7.

[35] P. Feinsilver and R. Schott, *Algebraic Structures and Operator Calculus*, vol. I. *Representations and Probability Theory*, Kluwer Academic Publishers, Dordrecht 1993.

[36] S. Pincherle and U. Analdi, *Le operazioni distributive e le loro applicazioni all'analisi*, ed. N. Zanichelli, Bologna 1901.

[37] S. G. Kurbanov and V. M. Maximov, *Mutual expansions of differential operators and divided difference operators*, Dokl. Akad. Nauk Uz SSR 4 (1986), 8–9.

[38] A. Di Bucchianico and D. Loeb, *Operator expansion in the derivative and multiplication x*, Integral Transforms and Special Functions 4 (1996), 49–68.
On simple characterisations of Sheffer $\psi$-polynomials and...

[39] A. K. Kwaśniewski, *Extended finite operator calculus – an example of algebraization of analysis*, Inst. Comp. Sci. UwB/Preprint 28/April/2001; Universitatis Iagellonicae Acta Mathematica, to appear.

[40] A. K. Kwaśniewski, *Some characteristic theorems for $\psi$-umbral operators* Inst. Comp. Sci. UwB/Preprint 27/April/2001; Integral Transforms and Special Functions (2002), in press.

[41] O. V. Viskov, *A noncommutative identity for Appel polynomials*, Matem. Zametki 64 (1998), 307–311.

[42] O. V. Viskov, *A commutative-like noncommutative identity*, Acta Sci. Math. (Szeged) 59 (1994), 585–590.

[43] O. V. Viskov and H. M. Srivastava, *New approach to certain identities involving differential operators*, J. Math. Anal. Appl. 186 (1994), 1–9.

Institute of Computer Science
Białystok University
Sosnowa 64, PL-15-887 Białystok
Poland

Presented by Julian Ławrynowicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on September 26, 2002

PROSTE CHARAKTERYZACJE $\psi$-WIELOMIANÓW SHEFFERA I WYNIKAJĄCE STĄD IMPLIKACJE DOTYCZĄCE RACHUNKU CIĄGÓW

**Streszczenie**

„Rachunek ciągów” zapoczątkowała publikacja Warda (1936) sugerująca możliwy zakres rozszerzeń rachunku operatorowego Roty-Mullina, rozważanych po Wardzie przez wielu autorów. Dla wygody opracowane później warianty rachunku ciągów Warda będziemy nazywali $\psi$-rachunkiem. Oznaczenia używane przez Warda, Viskova, Markowsky’ego i Romana są w niniejszej pracy zastosowane w uzgodnieniu z oznaczeniami Roty. W ten sposób $\psi$-rachunek staje się częściowo niemal automatycznym rozszerzeniem skończonego rachunku operatorów. $\psi$-rozszerzenie opiera się na pojęciu niezmienniczości operatorów względem $\partial_{\psi}$-przesunięcia. Jednocześnie omawiany rachunek jest przykładem algebraizacji analizy – tutaj algebraizacją ograniczonej do algebry szeregów formalnych. Dogodność użytych oznaczeń wyraża się między innymi przez możliwość łatwego udowodnienia pewnych twierdzeń o charakterystyce $\psi$-wielomianów Sheffera jak również twierdzenia spektralnego. Wyniki $\psi$-rachunku mogą być rozszerzone na „rachunek $Q$-umbralny Markowsky’ego”, gdzie $Q$ oznacza uogólniony operator różnicy (nie musi on być niezmienniczy względem $\partial_\psi$-przesunięcia), tj. operator obniżający stopień dowolnego wielomianu o jeden.