The diffusive SIS model in a low-risk or high-risk domain: Spreading or vanishing of the disease*

Kwang Ik Kim\textsuperscript{a}, Zhigui Lin\textsuperscript{b} and Huaping Zhu\textsuperscript{c}

\textsuperscript{a}Department of Mathematics, Pohang University of Science and Technology, Pohang, 790-784, Republic of Korea
\textsuperscript{b}School of Mathematical Science, Yangzhou University, Yangzhou 225002, China
\textsuperscript{c}Department of Mathematics and Statistics, York University, Toronto, ON, M3J 1P3, Canada

Abstract. A SIS reaction-diffusion model is proposed and investigated to understand the impact of spatial heterogeneity of environment on the persistence and eradication of an infectious disease. The free boundary is introduced to model the contact transmission at the spreading front of the disease. The behavior of positive solutions to a reaction-diffusion system in a radially symmetric domain are discussed. The basic reproduction number $R_0^F(t)$ associated with the diseases in the spatial setting is introduced for this reaction-diffusion SIS model with the free boundary, we prove that the fast diffusion and small initial infected domain are benefit for the control of the spatial spread of the disease. Sufficient conditions for the disease to be eradicated or to spread are also given, our result shows that the disease will spread to the whole area if there exists a $t_0 \geq 0$ such that $R_0^F(t_0) \geq 1$, that is, if the spreading domain is high-risk at some time, the disease will continue to spread till the whole area is infected; while if $R_0^F(0) < 1$, the disease may be vanishing or keep spreading depends on the initial number of the infective individuals.

Keywords: Reaction-diffusion systems; spatial SIS model; free boundary; basic reproduction number; spreading

1 Introduction

Mathematical models have been made to investigate the transmission of infectious diseases and the asymptotic profiles of the steady states of the diseases (see [2, 3, 4, 22, 34]).

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For classical compartmental epidemic models for infectious diseases described by ordinary differential systems, it is common that the so-called basic reproduction number decides whether the disease will be endemic [5, 30]. It is also common that for vector-borne diseases, backward bifurcation may occur in the compartmental models which reveals that besides the basic reproduction number, the endemic also depends on the initial sizes of the involving individuals (see [17, 31] and references therein). In recent years, spatial diffusion and environmental heterogeneity have been recognized as important factors to affect the persistence and eradication of diseases such as measles, tuberculosis and flu, etc., especially for vector-borne diseases, such as malaria, dengue, West Nile virus etc. More importantly, it is the spatial transmission and environmental heterogeneity that decide the speed and pattern of the spatial spread of infectious diseases. In this case, the usual basic reproduction number will not be enough to describe the disease transmission dynamics, especially to reflect the spatial features of the spread in the region considered. Therefore, it is essential to investigate the role of diffusion on the transmission and control of diseases in a heterogeneous environment.

To understand the dynamics of disease transmission in a spatially heterogeneous environment, an SIS epidemic reaction-diffusion model has been proposed by Allen, Bolker, Lou and Nevi in [1], and the model is described by the following coupled parabolic system:

\[
\begin{align*}
S_t - d_S \Delta S &= -\frac{\beta(x)SI}{S+I} + \gamma(x)I, \quad x \in \Omega, \quad t > 0, \\
I_t - d_I \Delta I &= \frac{\beta(x)SI}{S+I} - \gamma(x)I, \quad x \in \Omega, \quad t > 0
\end{align*}
\] 

(1.1)

with homogeneous Neumann boundary condition

\[
\frac{\partial S}{\partial \eta} = \frac{\partial I}{\partial \eta} = 0, \quad x \in \partial \Omega, \quad t > 0,
\]

where \( S(x,t) \) and \( I(x,t) \) represent the density of susceptible and infected individuals at location \( x \) and time \( t \), respectively, the positive constants \( d_S \) and \( d_I \) denote the corresponding diffusion rates for the susceptible and infected populations, \( \beta(x) \) and \( \gamma(x) \) are positive Hölder continuous functions, which account for spatial dependent rates of disease contact transmission and disease recovery at \( x \), respectively. The term \( \frac{\beta(x)SI}{S+I} \) is the standard incidence of disease, since the term \( \frac{SI}{S+I} \) is a Lipschitz continuous function of \( S \) and \( I \) in the open first quadrant, it can be extended to define to the entire first quadrant by defining it to be zero when either \( S = 0 \) or \( I = 0 \).

Letting \( N = S + I \), adding two equations in (1.1) and then integrating over \( \Omega \) yields that \( \frac{\partial}{\partial t} \int_{\Omega} N(x,t)dx = 0 \) for \( t > 0 \), this means that the total population size remains a constant, and recovered individuals become susceptible after survival from the infectious of the disease.

As in [1], we say that \( x \) is a **high-risk site** if the local disease transmission rate \( \beta(x) \) is greater than the local disease recovery rate \( \gamma(x) \). An **low-risk site** is defined in a similar manner. The habitat \( \Omega \) is characterized as **high-risk** (or **low-risk**) if the spatial average \( \left( \frac{1}{|\Omega|} \int_{\Omega} \beta(x)dx \right) \) of the transmission rate is greater than (or less than) the spatial average \( \left( \frac{1}{|\Omega|} \int_{\Omega} \gamma(x)dx \right) \) of the recovery rate, respectively.
To characterize the dynamics of the transmission of the disease, the authors in [1] introduced the basic reproduction number $R^N_0$ (we use $R^N_0$ for Neumann boundary condition to compare with $R^D_0$ for Dirichlet boundary condition defined later) by

$$R^N_0 = R^N_0(\Omega) = \sup_{\phi \in H^1(\Omega), \phi \neq 0} \left\{ \frac{\int_{\Omega} \beta \phi^2}{\int_{\Omega} d_I |\nabla \phi|^2 + \gamma \phi^2} \right\}. \quad (1.2)$$

They showed that if $R^N_0 < 1$, the population density $(S(t, \cdot), I(t, \cdot))$ converges to a unique disease free equilibrium $(S_0, 0)$, while if $R^N_0 > 1$, there exists a unique positive endemic equilibrium $(S^*, I^*)$. In addition, global stability of endemic steady state for some particular cases and particularly the asymptotical profiles of the endemic steady states as the diffusion coefficient for susceptible individuals is sufficiently small are given.

In some recent work [25, 26, 27, 28], Peng et al. further investigated the asymptotic behavior and global stability of the endemic equilibrium for system (1.1) subject to the Neumann boundary conditions, and provided much understanding of the impacts of large and small diffusion rates of the susceptible and infected population on the persistence and extinction of the disease.

For the SIS reaction-diffusion model (1.1) with Dirichlet boundary conditions

$$S = I = 0, \quad x \in \partial \Omega, \quad t > 0,$$

adding the two equations in (1.1), and integrating over $\Omega$ one yields that $\frac{\partial}{\partial t} \int_{\Omega} N(x, t) dx \leq 0$ for $t > 0$, it follows that the total population is decreasing and $\int_{\Omega} N(x, t) dx \to 0$ as $t \to \infty$. To avoid the loss of population in the boundary and the diffusion process, Huang et al. [18] added an additional growth term in the first equation of (1.1) and studied the global dynamics of the corresponding problem.

If $d_S = d_I = d$ and let $N = S + I$, then the system (1.1) becomes

$$\begin{cases}
N_t - d \Delta N = 0, & x \in \Omega, \quad t > 0, \\
I_t - d \Delta I = (\beta(x) - \gamma(x))I - \frac{\beta(x)}{N} I^2, & x \in \Omega, \quad t > 0.
\end{cases} \quad (1.3)$$

It must be pointed out that the solutions of system (1.3) subject to the Neumann or Dirichlet boundary conditions are always positive for any $t > 0$ no matter what the non-negative nontrivial initial data is given. It means that the disease spreads and becomes endemic to the whole area immediately even the infection is limited in a small region at the beginning. It does not reflect the reality that disease always spreads gradually from an endemic region to spread further to an larger area in terms of spatial spread.

Recently the free boundary has been introduced in many areas, especially the well-known Stefan condition has been used to describe the interaction and spreading process at the boundary. For example, it was used to describe the melting of ice in contact with water [29], in the modeling of oxygen in muscles [11], in a model of wound healing [10] and in the dynamics of a single species population [12, 13, 14, 15, 19, 20, 23, 24, 32, 33, 35]. There is a vast literature on the Stefan problem, and some important recent theoretical advances can be found in [6].

For emerging and re-emerging infectious diseases, usually the spread of the disease starts at a source location and spread over areas where contact transmission occurs.
For example, West Nile virus, a kind of mosquito-borne virus arrived and caused an encephalitis outbreak in New York city in 1999 [3]. With mosquito as vector and bird as amplification host, West Nile virus has kept spreading and become established in the North America continent [9]. For infectious diseases like West Nile virus, it is essential to understand how the disease is spreading spatially over further to larger area to cause endemic, to determine the condition for the virus to spread spatially, to predict the spatial spread for the purpose of prevention and control.

As one preliminary study, we will focus on the changing of the infected domain and consider an SIS epidemic model with the free boundary to describe the spreading frontier of the disease. Spatial diffusion and environmental heterogeneity are two very complex aspect of the spread of the infectious diseases. For simplicity, we assume the region or environment is radially symmetric. We will investigate the behavior of the positive solution \((N(r,t), I(r,t); h(t))\) with \(r = |x|, x \in \mathbb{R}^n\) to the following problem

\[
\begin{aligned}
N_t - d\Delta N &= 0, & 0 < r, t > 0, \\
I_t - d\Delta I &= (\beta(r) - \gamma(r))I - \frac{\beta(r)}{N}I^2, & 0 < r < h(t), t > 0, \\
I(r,t) &= 0, & r \geq h(t), t > 0, \\
N_r(0,t) &= I_r(0,t) = 0, & t > 0, \\
h'(t) &= -\mu I_r(h(t),t), & h(0) = h_0 > 0, t > 0,
\end{aligned}
\tag{1.4}
\]

where \(n \geq 1, \nabla w = w_r + \frac{n-1}{r} w_r, r = h(t)\) is the moving boundary to be defined, \(h_0, d, \mu\) are positive constants, and the initial function \(N_0(r)\) is nonnegative and satisfies

\[
N_0 \in C^2([0, +\infty)), \quad N'_0(0) = 0, \quad N_0(r) > 0, \quad N_0(r) \to N_0^* \text{ as } r \to \infty.
\tag{1.5}
\]

Here we assume that the population density far away from the infected domain is almost a constant \(N_0^*\). For the initial distribution of the infected class \(I_0(r)\), it is nonnegative and satisfies

\[
I_0 \in C^2[0, h_0], \quad I'_0(0) = I_0(r) = 0, \quad r \in [h_0, +\infty) \text{ and } 0 < I_0(r), \quad r \in [0, h_0),
\tag{1.6}
\]

where the condition (1.6) indicates that at the beginning, the infected exists in the area with \(r \in [0, h_0]\), but for the area \(r \geq h_0\), no infected happens yet. Therefore, the model means that beyond the free boundary \(r = h(t)\), there is only susceptible, no infectious individuals.

The equation governing the free boundary, the spread front, \(h'(t) = -\mu u_r(h(t),t)\), is a special case of the well-known Stefan condition, which has been established in [23] for the diffusive populations. The positive constant \(\mu\) measures the ability of the infected transmit and diffuse towards the new area.

Different from the usual compartmental models and reaction-diffusion models with Dirichlet boundary conditions, it is natural that the basic reproduction number for the disease transmission modeled by the free boundary conditions will be time dependent. For the reaction-diffusion models with free boundary conditions, we will define the basic reproduction number based on the definition for Dirichlet boundary conditions, and
use the basic reproduction number to characterize the dynamics of the temporal and spatial spread of the disease. As a preliminary study, we will consider the case when the domain is radially symmetric and focus to describe when the diseases can be vanishing (eradicated) or can spread to become endemic further over the domain.

The remainder of this paper is organized as follows. In the next section, the global existence and uniqueness of the solution to (1.4) are proved by using a contraction mapping theorem, comparison principle is also employed. Section 3 is devoted to develop the basic reproduction numbers and their properties. Sufficient conditions for the disease to vanish is given in Section 4. Section 5 deals with the case and conditions for the disease to spread and become endemic. Finally, we give a brief discussion in Section 6.

2 Existence and uniqueness

In this section, we first prove the following local existence and uniqueness result by the contraction mapping theorem. We then use suitable estimates to show that the solution is well defined for all \( t > 0 \).

**Theorem 2.1** For any given \((N_0, I_0)\) satisfying (1.5), (1.6), and any \( \alpha \in (0, 1) \), there is a \( T > 0 \) such that problem (1.4) admits a unique solution 
\[
(N, I; h) \in C^{1+\alpha,1+\alpha/2}(D_T) \times C^{1+\alpha,1+\alpha/2}(D_T) \times C^{1+\alpha/2}(0, T]);
\]

moreover,
\[
\|N\|_{C^{1+\alpha,1+\alpha/2}(D_T)} + \|I\|_{C^{1+\alpha,1+\alpha/2}(D_T)} + |h|_{C^{1+\alpha/2}(0, T)} \leq C,
\]

where \( D_T = \{(r,t) \in \mathbb{R}^2 : r \in [0, h(t)], t \in [0, T]\} \) and \( D_T = \{(r,t) \in \mathbb{R}^2 : r \in [0, h(t)], t \in [0, T]\} \), \( C \) and \( T \) only depend on \( h_0, \alpha, \|N_0\|_{C^2([0, \infty])} \) and \( \|I_0\|_{C^2([0, h_0])} \).

**Proof:** As in [10], we first straighten the free boundary. Let \( \xi(s) \) be a function in \( C^3[0, \infty) \) satisfying
\[
\xi(s) = 1 \text{ if } |s - h_0| < \frac{h_0}{8}, \quad \xi(s) = 0 \text{ if } |s - h_0| > \frac{h_0}{2}, \quad |\xi'(s)| < \frac{5}{h_0} \text{ for all } s.
\]

Consider the transformation
\[
(y, t) \rightarrow (x, t), \text{ where } x = y + \xi(|y|)(h(t) - h_0 y/|y|), \quad y \in \mathbb{R}^n,
\]

which leads to the transformation
\[
(s, t) \rightarrow (r, t), \text{ with } r = s + \xi(s)(h(t) - h_0), \quad 0 \leq s < \infty.
\]

As long as
\[
|h(t) - h_0| \leq \frac{h_0}{8},
\]
the above transformation $x \rightarrow y$ is a diffeomorphism from $R^n$ onto $R^n$ and the transformation $s \rightarrow r$ is also a diffeomorphism from $[0, +\infty)$ onto $[0, +\infty)$. Moreover, it changes the free boundary $r = h(t)$ to the line $s = h_0$. Now, direct calculations show that

$$
\frac{\partial s}{\partial r} = \frac{1}{1 + \xi'(s)(h(t) - h_0)} \equiv \sqrt{A(h(t), s)},
$$

$$
\frac{\partial^2 s}{\partial r^2} = - \frac{\xi''(s)(h(t) - h_0)}{[1 + \xi'(s)(h(t) - h_0)]^3} \equiv B(h(t), s),
$$

$$
- \frac{1}{h'(t) \frac{\partial s}{\partial t}} = \frac{\xi(s)}{1 + \xi'(s)(h(t) - h_0)} \equiv C(h(t), s),
$$

$$
\frac{(n-1)\sqrt{A}}{s + \xi'(s)(h(t) - h_0)} \equiv D(h(t), s).
$$

If we set

$$
N(r, t) = N(s + \xi(s)(h(t) - h_0), t) = u(s, t),
$$

$$
I(r, t) = I(s + \xi(s)(h(t) - h_0), t) = v(s, t),
$$

then the free boundary problem (1.4) becomes

$$
\begin{cases}
  u_t - Ad\Delta_s u - (Bd + h'C + Dd)u_s = 0, & 0 < s, t > 0, \\
  v_t - Ad\Delta_s v - (Bd + h'C + Dd)v_s = (\beta - \gamma)v - \frac{2}{u}v^2, & 0 < s < h_0, t > 0, \\
  u_s(0, t) = v_s(0, t) = 0, & t > 0, \\
  v(s, t) = 0, & s \geq h_0, t > 0, \\
  h'(t) = -\mu v_s(h_0, t), h(0) = h_0, & t > 0, \\
  u(s, 0) = u_0(s), v(s, 0) = v_0(s), & 0 \leq s,
\end{cases}
$$

(2.2)

where $A = A(h(t), s)$, $B = B(h(t), s)$, $C = C(h(t), s)$, $D = D(h(t), s)$ and $u_0 = N_0, v_0 = I_0$.

The rest of the proof follows from the contraction mapping theorem together with standard $L^p$ theory and the Sobolev imbedding theorem [21], we then omit it here, see Theorem 2.1 in [14] for details. \qed

To show that the local solution obtained in Theorem 2.1 can be extended to all $t > 0$, we need the following estimate.

**Lemma 2.2** Let $(N, I; h)$ be a bounded solution to problem (1.4) defined for $t \in (0, T_0)$ for some $T_0 \in (0, +\infty]$. Then there exist constants $C_1$ and $C_2$ independent of $T_0$ such that

$$
0 < N(r, t) \leq C_1 \text{ for } 0 \leq r < +\infty, \; t \in (0, T_0),
$$

$$
0 < I(r, t) \leq C_2 \text{ for } 0 \leq r < h(t), \; t \in (0, T_0).
$$

**Proof:** It is easy to see that $N \geq 0$ and $I \geq 0$ in $[0, +\infty) \times [0, T_0)$ as long as the solution exists.

Using the strong maximum principle to the equations in $[0, h(t)] \times [0, T_0)$, we immediately obtain

$$
I(r, t) > 0 \text{ for } 0 \leq r < h(t), 0 < t < T_0.
$$
Since $N(r, t)$ satisfies
\[
\begin{aligned}
N_t - d\Delta N &= 0, \quad 0 < r, \ t > 0, \\
N'(0,t) &= 0, \quad t > 0, \\
N(r,0) &= N_0(r) \geq 0, \quad 0 \leq r.
\end{aligned}
\]
We have $N(r, t) \leq ||N_0(r)||_{L^\infty[0, +\infty)}$ by using the Phragman-Lindelöf principle. Similarly, $I(r, t)$ satisfies
\[
\begin{aligned}
I_t - d\Delta I &= (\beta(r) - \gamma(r))I - \frac{\beta(r)}{N}I^2, \quad 0 < r < h(t), \ t > 0, \\
I'(0,t) &= I(r, t) = 0, \quad r = h(t), \ t > 0 \\
I(r,0) &= I_0(r), \quad 0 \leq r \leq h_0
\end{aligned}
\]
It follows from the maximum principle that $I \leq \max\{||N_0(r)||_{L^\infty[0, +\infty)}, ||I_0(r)||_{L^\infty[0, h_0]}\}$ in $[0, h(t)] \times [0, T_0]$.

The next lemma shows that the free boundary for problem (1.4) is strictly monotone increasing.

**Lemma 2.3** Let $(N, I; h)$ be a solution to problem (1.4) defined for $t \in (0, T_0)$ for some $T_0 \in (0, +\infty]$. Then there exists a constant $C_3$ independent of $T_0$ such that
\[
0 < h'(t) \leq C_3 \text{ for } t \in (0, T_0).
\]

**Proof:** Using the strong maximum principle to the equation of $I$ gives that
\[
I_r(h(t), t) < 0 \text{ for } 0 < t < T_0.
\]
Hence $h'(t) > 0$ for $t \in (0, T_0)$ by using the free boundary condition in (1.4).

It remains to show that $h'(t) \leq C_3$ for $t \in (0, +\infty)$ and some $C_3$. The proof is similar as that of Lemma 2.2 in [14] with $C_3 = 2MC_2\mu$ and
\[
M = \max \left\{ \sqrt{\frac{\beta}{2d}}, \frac{4||I_0||_{C^1([0, h_0])}}{3C_2} \right\}, \quad \overline{\beta} = \max_{[0, h_0]} \beta(r),
\]
we omit it here.\[\square\]

**Theorem 2.4** The solution of problem (1.4) exists and is unique for all $t \in (0, \infty)$.

**Proof:** It follows from the uniqueness that there is a number $T_{\max}$ such that $[0, T_{\max})$ is the maximal time interval in which the solution exists. Now we prove that $T_{\max} = \infty$ by the contradiction argument. Assume that $T_{\max} < \infty$. By Lemmas 2.2 and 2.3, there exist $C_1, C_2$ and $C_3$ independent of $T_{\max}$ such that for $t \in [0, T_{\max})$ and $r \in [0, h(t)]$,
\[
0 \leq N(r, t) \leq C_1, \ (r, t) \in [0, +\infty) \times [0, T_{\max}),
\]
\[
0 \leq I(r, t) \leq C_2, \ (r, t) \in [0, h(t)] \times [0, T_{\max}),
\]
and
\[ h_0 \leq h(t) \leq h_0 + C_3 t, \quad 0 \leq h'(t) \leq C_3, \quad t \in [0, T_{\text{max}}). \]

We now fix \( \delta_0 \in (0, T_{\text{max}}) \) and \( M > T_{\text{max}}. \) By standard parabolic regularity, we can find \( C_4 > 0 \) depending only on \( \delta_0, M, C_1, C_2 \) and \( C_3 \) such that
\[
||N(\cdot, t)||_{C^2([0, \infty)}, \quad ||I(\cdot, t)||_{C^2([0,h(t)])} \leq C_4
\]
for \( t \in [\delta_0, T_{\text{max}}). \) It then follows from the proof of Theorem 2.1 that there exists a \( \tau > 0 \) depending only on \( C_i (i = 1, 2, 3, 4) \) such that the solution of problem (1.4) with initial time \( T_{\text{max}} - \tau/2 \) can be extended uniquely to the time \( T_{\text{max}} - \tau/2 + \tau. \) But this contradicts the assumption. The proof is complete. \( \square \)

**Remark 2.1** It follows from the uniqueness of the solution to (1.4) and some standard compactness argument that the unique solution \((N, I, h)\) depends continuously on the parameters appearing in (1.4). This fact will be used in the sections below.

In what follows, we shall exhibit the comparison principle.

**Lemma 2.5** *(The Comparison Principle)* Assume that \( T \in (0, \infty), \) \( \mathbf{h}, \mathbf{h} \in C^1([0,T]), \)
\( \mathbf{T}(r,t) \in C([0,\mathbf{h}(t)] \times [0,T]) \cap C^2( (0,\mathbf{h}(t)) \times (0,T) ), \)
\( \mathbf{N}(t,r), \mathbf{N}(t,r) \in C([0,\infty) \times [0,T]) \cap C^2((0,\infty) \times (0,T)) \) and
\[
\begin{cases}
N_r - d\Delta N \leq 0 \leq N_t - d\Delta N, \\
T_t - d\Delta T \geq (\beta(r) - \gamma(r))T - \frac{\beta(r)}{\Delta}T^2, & 0 < r < \mathbf{h}(t), 0 < t \leq T, \\
I_t - d\Delta I \leq (\beta(r) - \gamma(r))I - \frac{\beta(r)}{\Delta}I^2, & 0 < r \leq \mathbf{h}(t), 0 < t \leq T, \\
N_r(0,t) = I_r(0,t) = 0, \quad T(r,t) = 0, & \mathbf{h}(t) \leq r, 0 < t \leq T, \\
N(0,0) = I(0,0) = 0, \quad T_r(0,0) = 0, \quad \mathbf{h}(t) \leq r, 0 < t \leq T, \\
\mathbf{h}'(t) \leq -\mu \mathbf{I}_r(h(t), t), \quad \mathbf{h}''(t) \geq -\mu \mathbf{I}_r(h(t), t), & 0 < t \leq T, \\
\mathbf{h}(0) \leq h_0 \leq \overline{\mathbf{h}}(0), \quad \mathbf{I}(0,0) \leq \mathbf{I}(r,0), \quad \mathbf{N}(r,0) \leq \mathbf{N}(r,0), & 0 \leq r \leq h_0,
\end{cases}
\]
then the bounded solution \((N, I, h)\) to the free boundary problem (1.4) satisfies
\[
h(t) \leq \overline{\mathbf{h}}(t) \text{ in } (0,T], \quad N(r,t) \leq \mathbf{N}(r,t), \quad I(r,t) \leq \mathbf{I}(r,t) \text{ for } (r,t) \in (0,\infty) \times (0,T],
\]
\[
h(t) \geq \underline{\mathbf{h}}(t) \text{ in } (0,T], \quad N(r,t) \geq \underline{\mathbf{N}}(r,t), \quad I(r,t) \geq \underline{\mathbf{I}}(r,t) \text{ for } (r,t) \in (0,\infty) \times (0,T].
\]

**Proof:** Since \((\mathbf{N}, \mathbf{T}, \mathbf{h})\) and \((\mathbf{N}, \mathbf{I}, \mathbf{h})\) are independent, we only need to prove that \(N \leq \mathbf{N},\)
\(I \leq \mathbf{T}\) and \(h \leq \mathbf{h};\) the result for \((\mathbf{N}, \mathbf{I}, \mathbf{h})\) can be proved in a similar way.

First assume that \(h_0 < \overline{\mathbf{h}}(0)\). We claim that \(h(t) < \overline{\mathbf{h}}(t)\) for all \(t \in (0,T]\). By contradiction, if our claim does not hold, then we can find a first \(t^* \leq T\) such that \(h(t) < \overline{\mathbf{h}}(t)\) for \(t \in (0,t^*)\) and \(h(t^*) = \overline{\mathbf{h}}(t^*)\). It follows that
\[
h'(t^*) \geq \overline{\mathbf{h}}'(t^*).
\]
We now show that \((N, I) \leq (\overline{N}, \overline{T})\) in \([0, t^*] \times [0, \infty)\). Letting \(U = (\overline{N} - N)e^{-\kappa t}\) and \(V = (\overline{T} - I)e^{-\kappa t}\) yields

\[
\begin{aligned}
U_t - d\Delta U &\geq -KU, & 0 < r, 0 < t \leq t^*, \\
V_t - d\Delta V &\geq b_{21}U + (-K + b_{22})V, & 0 < r < h(t), 0 < t \leq t^*, \\
U_r(0, t) = V_r(0, t) = 0, V(r, t) \geq 0, & h(t) \leq r, 0 < t \leq t^*, \\
U(0, r) \geq 0, V(0, r) \geq 0, & 0 \leq r,
\end{aligned}
\]

where

\[
b_{21} = \frac{\beta(r)I^2}{N N} \geq 0, \quad b_{22} = (\beta(r) - \gamma(r)) - \frac{\beta(r)(\overline{T} + I)}{N},
\]

\(K\) is sufficiently large such that \(K \geq 1 + b_{21} + |b_{22}|\) in \([0, t^*] \times [0, \infty)\).

Noting that \((0, \infty)\) is unbounded and the second inequality of (2.4) holds partly, we cannot use maximum principle directly, next we prove that for any \(l > h(t^*)\),

\[
U(r, t) \geq -\frac{\tilde{M}(r^2 + 2dnt)}{l^2} \quad \text{and} \quad V(r, t) \geq -\frac{\tilde{M}(r^2 + 2dnt)}{l^2}
\]

in \([0, l] \times [0, t^*]\), where \(\tilde{M}\) is the upper bound of \(\overline{N}\) and \(N\) in \([0, +\infty) \times [0, t^*]\).

Set \(\overline{U}(r, t) = U + \frac{\tilde{M}(r^2 + 2dnt)}{l^2}\) and \(\overline{V}(r, t) = V + \frac{\tilde{M}(r^2 + 2dnt)}{l^2}\), then \((\overline{U}, \overline{V})\) satisfies

\[
\begin{aligned}
\overline{U}_t - d\Delta \overline{U} &\geq -K\overline{U}, & 0 < r < l, 0 < t \leq t^*, \\
\overline{V}_t - d\Delta \overline{V} &\geq b_{21}\overline{U} + (-K + b_{22})\overline{V}, & 0 < r < h(t), 0 < t \leq t^*, \\
\overline{U}_r(0, t) = \overline{V}_r(0, t) = 0, \overline{V}(r, t) \geq 0, & h(t) \leq r \leq l, 0 < t \leq t^*, \\
\overline{V}(l, t) = \overline{V}(l, t) + \frac{\tilde{M}(l^2 + 2dnt)}{l^2} > 0, & 0 < t \leq t^*, \\
\overline{U}(0, r) \geq 0, \overline{V}(0, r) \geq 0, & 0 \leq r \leq l.
\end{aligned}
\]

We now prove that \(\min\{\min_{[0, l] \times [0, t^*]} \overline{U}, \min_{[0, l] \times [0, t^*]} \overline{V}\} := \tau \geq 0\). If fact, if \(\tau < 0\), then there exists \((r_0, t_0) \in (0, h(t)) \times (0, t^*)\) such that \(\overline{U}(r_0, t_0) = \tau < 0\), or there exists \((r_1, t_1) \in (0, l) \times (0, t^*)\) such that \(\overline{V}(r_1, t_1) = \tau < 0\). For the former case, \((\overline{U}_t - d\Delta \overline{U})(r_0, t_0) \leq 0\), but

\[-K\overline{U}(r_0, t_0) \geq -K\tau > 0.\]

For the latter case, \((\overline{V}_t - d\Delta \overline{V})(r_1, t_1) \leq 0\), but

\([b_{21}\overline{U} + (-K + b_{22})\overline{V}](r_1, t_1) \geq (-K + |b_{22}|)\tau + b_{21}\tau \geq -\tau > 0.\]

Both are impossible. Therefore \(\tau \geq 0\), that is \(\overline{U} \geq 0\) and \(\overline{V} \geq 0\) in \([0, l] \times [0, t^*]\), which implies that

\[
U(r, t) \geq -\frac{\tilde{M}(r^2 + 2dnt)}{l^2}, \quad V(r, t) \geq -\frac{\tilde{M}(r^2 + 2dnt)}{l^2}
\]

for \(0 \leq r \leq l, 0 \leq t \leq t^*\). Taking \(l \to \infty\) yields that \(U(r, t) \geq 0\) and \(V(r, t) \geq 0\) in \([0, \infty) \times [0, t^*]\), therefore \(N \leq \overline{N}\) and \(I \leq \overline{T}\) in \([0, \infty) \times [0, t^*]\).

We now compare \(I\) and \(\overline{T}\) over the bounded region

\[
\Omega_{t^*} := \{(r, t) \in \mathbb{R}^{n+1} : 0 \leq r < h(t), 0 < t \leq t^*\}.
\]
Since \( Z(r,t) := \overline{T}(r,t) - I(r,t) \) satisfies
\[
Z_t - d\Delta Z \geq b_{21}(\overline{N} - N) + b_{22}Z, \quad 0 < r < h(t), \quad 0 < t \leq t^*.
\]
Using the strong maximum principle yields \( Z(r,t) > 0 \) in \( \Omega_t \), which together with \( Z(h(t^*),t^*) = 0 \) gives that \( Z_r(h(t^*),t^*) < 0 \), we then deduce that \( h'(t^*) < \overline{h}(t^*) \). But this contradicts (2.3). This proves our claim that \( h(t) < \overline{h}(t) \) for all \( t \in (0,T] \). We may now apply the above procedure over \([0,\infty) \times [0,T]\) to conclude that \( N \leq \overline{N} \) and \( I \leq \overline{T} \) in \([0,\infty) \times [0,T] \).

If \( h_0 = \overline{h}_0 \), we use approximation. For small \( \epsilon > 0 \), let \((N_\epsilon,I_\epsilon,h_\epsilon)\) denote the unique solution of (1.4) with \( h_0 \) replaced by \( h_0(1-\epsilon) \). Since the unique solution of (1.4) depends continuously on the parameters in (1.4), as \( \epsilon \to 0 \), \((N_\epsilon,I_\epsilon,h_\epsilon)\) converges to \((N,I,h)\), the unique solution of (1.4). The desired result then follows by letting \( \epsilon \to 0 \) in the inequalities \( N_\epsilon \leq \overline{N} \), \( I_\epsilon \leq \overline{T} \) and \( h_\epsilon < \overline{h} \).

Since \( \frac{\beta(r)}{\mu} \overline{T}^2 \geq 0 \), we then have the following comparison principle, which is similar to Lemma 3.5 in [14].

**Corollary 2.6** Assume that \( T \in (0,\infty) \), \( \overline{h} \in C^1([0,T]) \), \( \overline{T}(t,r) \in C([0,\overline{h}(t)] \times [0,T]) \cap C^2,1((0,\overline{h}(t)) \times (0,T]) \), and
\[
\begin{cases}
    T_t - d\Delta T \geq (\beta(r) - \gamma(r))T, & 0 < r < \overline{h}(t), \quad 0 < t \leq T, \\
    T_r(0,t) = 0, \quad T_r(t) = 0, & \overline{h}(t) \leq r, \quad 0 < t \leq T, \\
    h'(t) \geq -\mu \overline{T}_r(h(t),t), & 0 < t \leq T, \\
    h_0 \leq \overline{h}(0), & 0 \leq r \leq h_0, \\
    I_0(r) \leq \overline{T}(r,0), & 0 \leq r \leq \overline{h}_0,
\end{cases}
\]
then the solution \((N,I,h)\) to the free boundary problem (1.4) satisfies
\[
h(t) \leq \overline{h}(t) \text{ in } (0,T], \quad I(r,t) \leq \overline{T}(r,t) \text{ for } (r,t) \in (0,\infty) \times (0,T].
\]

We next fix \( N_0, \mu, \beta(r), \gamma(r) \), let \( I_0 = \sigma \phi(r) \) and examine the dependence of the solution on \( \sigma \), and we write \((N^\sigma,I^\sigma,h^\sigma)\) to emphasize this dependence. As a corollary of Lemma 2.5, we have the following monotonicity:

**Corollary 2.7** Let \( I_0 = \sigma \phi(r) \). For fixed \( N_0, \mu, \phi(r), \beta(r) \) and \( \gamma(r) \). If \( \sigma_1 \leq \sigma_2 \). Then \( I^{\sigma_1}(r,t) \leq I^{\sigma_2}(r,t) \) in \([0,h^{\sigma_1}(t)] \times (0,\infty) \), \( N^{\sigma_1}(r,t) \leq N^{\sigma_2}(r,t) \) in \([0,h^{\sigma_1}(t)] \times (0,\infty) \) and \( h^{\sigma_1}(t) \leq h^{\sigma_2}(t) \) in \((0,\infty) \).

## 3 Basic reproduction numbers

In this section, we first present the basic reproduction number and its properties and implications for the reaction-diffusion system (1.3) with Dirichlet boundary condition, and then discuss the basic reproduction number for the free boundary problem (1.4).
Let us introduced the basic reproduction number $R_0^D$ by

$$R_0^D = R_0^D(\Omega) = \sup_{\phi \in H^1_0(\Omega), \phi \neq 0} \left\{ \frac{\int_{\Omega} \beta \phi^2}{\int_{\Omega} d |\nabla \phi|^2 + \gamma \phi^2} \right\},$$

the following result was proved in [18] (Lemma 2.3):

**Lemma 3.1** $1 - R_0^D$ has the same sign as $\lambda_0$, where $\lambda_0$ is the principle eigenvalue of the problem

$$\begin{cases} -d \Delta \psi = \beta(x) \psi - \gamma(x) \psi + \lambda \psi, & x \in \Omega, \\ \psi(x) = 0, & x \in \partial \Omega. \end{cases}$$  \tag{3.1}$$

With the above defined reproduction number, we have

**Theorem 3.2** The following assertions hold.

(a) $R_0^D$ is a positive and monotone decreasing function of $d$;

(b) $R_0^D \to \max_{x \in \Omega} \frac{\beta(x)}{\gamma(x)}$ as $d \to 0$;

(c) $R_0^D \to 0$ as $d \to \infty$;

(d) There exists a threshold value $d^*_1 \in [0, \infty)$ such that $R_0 > 1$ for $d < d^*_1$ and $R_0 < 1$ for $d > d^*_1$. If all sites in the domain are lower-risk ($\beta(x) \leq \gamma(x)$ for $x \in \Omega$), we have $R_0^D < 1$ for all $d > 0$;

(e) Let $B_h$ be a ball in $\mathbb{R}^n$ with the radius $h$. Then $R_0^D(B_h)$ is strictly monotone increasing function of $h$, that is if $h_1 < h_2$, then $R_0(B_{h_1}) < R_0(B_{h_2})$. Moreover, $R_0^D(B_h) \to \sup_{r \in [0, \infty)} \frac{\beta_r}{\gamma(r)}$ as $h \to \infty$;

(f) If $\Omega = B_{h_0}$, $\beta(r) \equiv \beta$ and $\gamma(r) \equiv \gamma$, then

$$R_0^D = \frac{\beta}{nd\left(\frac{\pi}{2h_0}\right)^2 + \gamma}.$$ 

**Proof:** The proof of part (a), (b) and (d) are similar to that of Theorem 2 in [1]. The threshold value in part (d) can be described in the following manner:

$$d^*_1 = \sup \left\{ \frac{1}{\int_{\Omega} \frac{\beta - \gamma}{\beta - \gamma} \phi^2}{\int_{\Omega} \phi^2} : \phi \in W^{1,2}_0(\Omega) \text{ and } \int_{\Omega} (\beta - \gamma)\phi^2 > 0 \right\}.$$

It is easy to see that if $\beta(x) \leq \gamma(x)$ for $x \in \Omega$, then $d^*_1 = 0$.

Next, let’s first established part (f). It is well-known fact that the principle eigenvalue of the problem

$$\begin{cases} -d \Delta \psi = \lambda \psi, & x \in B_{h_0}, \\ \psi(x) = 0, & x \in \partial B_{h_0} \end{cases}$$

is $nd\left(\frac{\pi}{2h_0}\right)^2$, the desired result follows if $\beta$ and $\gamma$ are constants.
The proof of part (e) is similar to that of Corollary 2.3 in [7].

It remains to established part (c). Now we show that \( R_0^D \to 0 \) as \( d \to \infty \). In fact, if it is not true, there exists a positive \( a > 0 \) such that \( R_0^D \geq a \) for any \( d > 0 \) since \( R_0^D \) is monotone decreasing function of \( d \). It is a well-known fact that there exists a positive function \( \phi(x) \in C^2(\Omega) \) such that \( \| \phi \|_{L^\infty} = 1 \) and

\[
\begin{cases}
  -d \Delta \phi + \gamma \phi = \frac{\beta}{R_0^D} \phi, & x \in \Omega, \\
  \phi(x) = 0, & x \in \partial \Omega.
\end{cases}
\]

Dividing both sides of the above equation by \( d \) yields

\[-\Delta \phi + \frac{\gamma \phi}{d} = \frac{\beta}{R_0^D} \phi.\]

Since \( \frac{\gamma}{d} \to 0 \) and \( \frac{\beta}{R_0^D d} \to 0 \) as \( d \to \infty \), it follows from elliptic regularity that \( \phi \to \overline{\phi} \) in \( C(\Omega) \) as \( d \to \infty \) for some positive function \( \overline{\phi} \) satisfying

\[-\Delta \overline{\phi} = 0 \text{ in } \Omega, \quad \overline{\phi} = 0 \text{ on } \partial \Omega,
\]

we then have \( \overline{\phi} \equiv 0 \) in \( \Omega \), which leads to a contradiction. \( \square \)

Noticing that the domain for the free boundary problem (1.4) is changing with \( t \), so the basic reproduction number is not a constant and should be changing. Now we introduced the basic reproduction number \( R_0^F(t) \) for the free boundary problem (1.4) by

\[ R_0^F(t) := R_0^D(B_{h(t)}) = \sup_{\phi \in H_0^1(B_{h(t)}), \phi \neq 0} \left\{ \frac{\int_{B_{h(t)}} \beta \phi^2}{\int_{B_{h(t)}} |\nabla \phi|^2 + \gamma \phi^2} \right\}, \]

it follows from Lemma 2.3 and Theorem 3.2 that

**Theorem 3.3** \( R_0^F(t) \) is strictly monotone increasing function of \( t \), that is if \( t_1 < t_2 \), then \( R_0^F(t_1) < R_0^F(t_2) \). Moreover, if \( h(t) \to \infty \) as \( t \to \infty \), then \( R_0^F(t) \to \sup_{r \in [0, \infty)} \frac{\beta(r)}{\gamma(r)} \) as \( t \to \infty \);

**Remark 3.1** It follows from the above defined reproduction number that \( R_0^D(\Omega) > 1 \) for the high-risk domain \( \Omega \) and \( R_0^D(\Omega) < 1 \) for the low-risk domain provided that the diffusion rate \( d \) of the infected individuals is small.

4 Disease vanishing

Since \( N(r, t) \) satisfies

\[
\begin{cases}
  N_t - d \Delta N = 0, & t > 0, \quad r > 0, \\
  N_r(0, t) = 0, & t > 0, \\
  N(r, 0) > 0, \quad N(r, 0) \to N^*, & r > 0,
\end{cases}
\]

(4.1)
it is easy to see that \( \lim_{t \to +\infty} N(r,t) \to N^* \) uniformly in any bounded subset of \([0, \infty)\).

On the other hand, it follows from Lemma 2.3 that \( r = h(t) \) is monotonic increasing and therefore there exists \( h_\infty \in (0, +\infty) \) such that \( \lim_{t \to +\infty} h(t) = h_\infty \). Therefore the spatial transmission of a disease depends on whether \( h_\infty = \infty \) and \( \lim_{t \to +\infty} ||I(\cdot,t)||_{C([0,h(t)])} = 0 \). We then have the following definitions:

**Definition 4.1** The disease is **vanishing** if \( h_\infty < \infty \) and \( \lim_{t \to +\infty} ||I(\cdot,t)||_{C([0,h(t)])} = 0 \), while the disease is **spreading** if \( h_\infty = \infty \) and \( \lim \sup_{t \to +\infty} ||I(\cdot,t)||_{C([0,h(t)])} > 0 \).

The next result shows that if \( h_\infty < \infty \), then vanishing happens.

**Lemma 4.1** If \( h_\infty < \infty \), then \( \lim_{t \to +\infty} ||I(\cdot,t)||_{C([0,h(t)])} = 0 \).

**Proof:** Assume \( \lim \sup_{t \to +\infty} ||I(\cdot,t)||_{C([0,h(t)])} = \delta > 0 \) by contradiction. Then there exists a sequence \((r_k, t_k)\) in \([0, h(t)) \times (0, \infty)\) such that \( I(r_k, t_k) \geq \delta/2 \) for all \( k \in \mathbb{N} \), and \( t_k \to \infty \) as \( k \to \infty \). Since that \( 0 \leq r_k < h(t) < h_\infty < \infty \), we then have that a subsequence of \( \{r_n\} \) converges to \( r_0 \in [0, h_\infty) \). Without loss of generality, we assume \( r_k \to r_0 \) as \( k \to \infty \).

Define \( N_k(r,t) = N(r, t_k + t) \) and \( I_k(r,t) = I(r, t_k + t) \) for \( r \in [0, h(t_k + t)) \), \( t \in (-t_k, \infty) \). It follows from the parabolic regularity that \( \{(N_k, I_k)\} \) has a subsequence \( \{(N_{k_i}, I_{k_i})\} \) such that \( (N_{k_i}, I_{k_i}) \to (\tilde{N}, \tilde{I}) \) as \( i \to \infty \) and \((\tilde{N}, \tilde{I})\) satisfies

\[
\begin{cases}
\tilde{N}_t - d\Delta \tilde{N} = 0, & 0 < r < h_{\infty}, \ t \in (-\infty, \infty), \\
\tilde{I}_t - d\Delta \tilde{I} = (\beta(r) - \gamma(r))\tilde{I} - \frac{\beta(r)}{N} \tilde{I}^2, & 0 < r < h_{\infty}, \ t \in (-\infty, \infty).
\end{cases}
\]

Note that \( \tilde{I}(r_0, 0) \geq \delta/2 \), therefore \( \tilde{I} > 0 \) in \([0, h_\infty) \times (-\infty, \infty)\). Recalling that \( (\beta(r) - \gamma(r)) - \frac{\beta(r)}{N} \tilde{I} \) is bounded by \( M := ||\beta(r) + \gamma(r)||_{L^\infty(0, \infty)} \). Using Hopf lemma to the equation \( \tilde{I}_t - d\Delta \tilde{I} \geq -M\tilde{I} \) at the point \((h_\infty, 0)\) yields that \( \tilde{I}(h_\infty, 0) \leq -\sigma_0 \) for some \( \sigma_0 > 0 \).

On the other hand, \( h(t) \) is increasing and bounded. Moreover, for any \( 0 < \alpha < 1 \), there exists a constant \( \tilde{C} \) depends on \( \alpha, h_0, ||I_0||_{C^{1+\alpha}[0,h_0]} \) and \( h_\infty \) such that

\[
||I||_{C^{1+\alpha,(1+\alpha)/2([0,h(t)] \times [0,\infty))}} + ||h||_{C^{1+\alpha/2([0,\infty))}} \leq \tilde{C}.
\]

In fact, let us straighten the free boundary in a way different from that in Theorem 2.1. Define

\[
s = \frac{h_0 r}{h(t)}, \ u(s,t) = N(r,t), \ v(s,t) = I(r,t).
\]

Direct calculations yield that

\[
I_t = v_t - \frac{h'(t)}{h(t)}sv_s, \ I_r = \frac{h_0}{h(t)}v_s, \ \Delta_s I = \frac{h_0^2}{h^2(t)}\Delta_s v.
\]

Then \( v(s,t) \) satisfies

\[
\begin{align*}
\begin{cases}
v_t - d\Delta_s v - \frac{h'(t)}{h(t)}sv_s = v(\beta(r) - \gamma(r) - \frac{\beta(r)}{\alpha}v), & 0 < s < h_0, \ t > 0, \\
v_s(0,t) = v(h_0, t) = 0, & t > 0, \\
v(s,0) = I_0(s) \geq 0, & 0 \leq s \leq h_0.
\end{cases}
\end{align*}
\]
This transformation changes the free boundary $r = h(t)$ to the fixed line $s = h_0$, at the expense of making the equation more complicated. It follows from Lemmas 2.2 and 2.3 that there exist $M_1$ and $M_2$ such that

$$
\|v(\beta(r) - \gamma(r) - \frac{\beta(r)}{u})v\|_{L^\infty} \leq M_1, \quad \|\frac{h'(t)}{h(t)}s\|_{L^\infty} \leq M_2.
$$

Applying standard $L^p$ theory and then the Sobolev imbedding theorem \cite{21}, we find that

$$
\|v\|_{C^{1+\alpha,(1+\alpha)/2}([0,h_0] \times [0,\infty))} \leq C_4,
$$

where $C_4$ is a constant depending on $\alpha$, $h_0$, $M_1$, $M_2$ and $\|I_0\|_{C^{1+\alpha}[0,h_0]}$. This immediately leads to (4.2).

Since $\|h\|_{C^{1+\alpha/2}([0,\infty))} \leq \tilde{C}$ and $h(t)$ is bounded, we then have $h'(t) \to 0$ as $t \to \infty$, that is, $I_r(h(t_k),t_k) \to 0$ as $t_k \to \infty$ by the free boundary condition. Moreover, the fact $\|I\|_{C^{1+\alpha,(1+\alpha)/2}([0,h(t)] \times [0,\infty))} \leq \tilde{C}$ gives that $I_r(h(t_k),t_k) = (I_k)_r(h(t_k),0) \to \tilde{I}_r(h_\infty,0)$ as $k \to \infty$, which leads to a contradiction to the fact that $\tilde{I}_r(h_\infty,0) \leq -\sigma_0 < 0$. Thus

$$
\lim_{t \to +\infty} \|I(\cdot,t)\|_{C([0,h(t)])} = 0.
$$

Now we give sufficient conditions so that the disease is vanishing.

**Theorem 4.2** Suppose $R^F_0(0)(:= R^D_0(B_{h_0})) < 1$. Then $h_\infty < \infty$ and

$$
\lim_{t \to +\infty} \|I(\cdot,t)\|_{C([0,h(t)])} = 0
$$

if $\|I_0(r)\|_{C([0,h(t)])}$ is sufficiently small.

**Proof:** We are going to construct a suitable upper solution for $I$ and then apply Corollary 2.6. Since that $R^D_0(B_{h_0}) < 1$, it follows from Lemma 3.1 that there is a $\lambda_0 > 0$ and $\psi(r) > 0$ in $[0,h_0)$ such that

$$
\begin{cases}
-d\Delta \psi = \beta(r)\psi - \gamma(r)\psi + \lambda_0 \psi, & 0 < r < h_0, \\
\psi(r) = 0, & r = h_0.
\end{cases}
$$

Therefore, there exists a small $\delta > 0$ such that

$$
\delta(1+\delta)^2 + [(1+\delta)^2 - 1]\tilde{\beta} \leq \lambda_0,
$$

where $\tilde{\beta} = \|\beta(r)\|_{L^\infty([0,\infty))}$.

Similarly as in \cite{14}, we set

$$
\sigma(t) = h_0(1 + \delta - \frac{\delta}{2}e^{-\delta t}), \quad t \geq 0,
$$

and

$$
w(t,x) = \varepsilon e^{-\delta t}\psi(rh_0/\sigma(t)), \quad 0 \leq r \leq \sigma(t), \quad t \geq 0.
$$

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Straightforward computations yield
\[ w_t - d\Delta w - (\beta(r) - \gamma(r))w \]
\[ = -\delta w - \varepsilon e^{-\delta t} \psi'(t) \frac{\sigma'(t)}{\sigma(t)} + \left( \frac{h_0}{\sigma(t)} \right)^2 (\beta - \gamma + \lambda_0)w - (\beta - \gamma)w \]
\[ \geq w[-\delta + \left( \frac{1}{(1 + \delta)^2} - 1 \right) \beta + \frac{1}{(1 + \delta)^2} \lambda_0] \geq 0 \]
for all \( t > 0 \) and \( 0 < r < \sigma(t) \). On the other hand, we have \( \sigma'(t) = h_0 \frac{\delta}{2} e^{-\delta t} \) and
\[ -w_r(t, \sigma(t)) = -\varepsilon \frac{h_0}{\sigma(t)} \psi'(h_0) e^{-\delta t}. \]
We now choose \( \varepsilon = -\frac{\delta h_0}{4 \mu \psi(h_0)} \), then we have
\[
\begin{cases}
  w_t - d\Delta w \geq (\beta(r) - \gamma(r))w, & 0 < r < \sigma(t), \ t > 0, \\
  w = 0, \quad \sigma'(t) > -\mu w_r, & r = \sigma(t), \ t > 0, \\
  \frac{\partial w}{\partial r}(0, t) = 0, & t > 0, \\
  \sigma(0) = (1 + \frac{\delta}{2})h_0 > h_0.
\end{cases}
\]
If \( ||I_0||_{L^\infty} \leq \varepsilon \psi(\frac{h_0}{1 + \frac{\delta}{2}}) \), then \( I_0(r) \leq \varepsilon \psi(\frac{h_0}{1 + \frac{\delta}{2}}) \leq w(r, 0) \) for \( r \in [0, h_0] \) since that \( h_0 < \sigma(0) = h_0(1 + \delta/2) \), we can apply Corollary 2.6 to conclude that \( h(t) \leq \sigma(t) \) and \( I(r, t) \leq w(r, t) \) for \( 0 \leq r \leq h(t) \) and \( t > 0 \). It follows that \( h_\infty \leq \lim_{t \to \infty} \sigma(t) = h_0(1 + \delta) < \infty \) and \( \lim_{t \to +\infty} ||I(\cdot, t)||_{C([0, h(t)])} = 0. \)

### 5 Disease spreading

In this section, we are going to give the sufficient conditions so that the disease is spreading. We first prove that if \( R_0^F(0) = R_0^D(B_{h_0}) \geq 1 \), the disease is spreading.

**Theorem 5.1** If \( R_0^F(0) \geq 1 \), then \( h_\infty = \infty \) and \( \lim_{t \to +\infty} ||I(\cdot, t)||_{C([0, h(t)])} > 0 \), that is, spreading happens.

**Proof:** We first consider the case that \( R_0^F(0) = R_0^D(B_{h_0}) > 1 \). In this case, we have that the eigenvalue problem
\[
\begin{cases}
  -d\Delta \psi = \beta(r)\psi - \gamma(r)\psi + \lambda_0 \psi, & x \in B_{h_0}, \\
  \psi(x) = 0, & x \in \partial B_{h_0}
\end{cases}
\]
(5.1)
admits a positive solution \( \psi(r) \) with \( ||\psi||_{L^\infty} = 1 \), where \( \lambda_0 \) is the principle eigenvalue. It follows from Lemma 3.1 that \( \lambda_0 < 0 \).

We are going to construct a suitable lower solutions to (5.4) and we define
\[
N(r, t) = N_0(0) = \inf_{r \geq 0} N_0(r) \geq 0, \quad 0 \leq r, \ t \geq 0,
\]
\[
I(r, t) = \begin{cases} \delta \psi(r), & 0 \leq r \leq h_0, \ t \geq 0, \\ 0, & r > h_0, \ t \geq 0, \end{cases}
\]
where \( \delta \) is sufficiently small such that \( 0 < \delta \leq \frac{N_0(-\lambda_0)}{\beta} \) and \( \delta \psi \leq I_0(x) \) in \([0, h_0]\).
Direct computations yield

\[
L - d_1 \Delta I - (\beta(r) - \gamma(r))I + \frac{\beta(r)}{N}I^2 = \delta\psi(r)[\lambda_0 + \frac{\beta(r)}{N}\delta\psi(r)] \\
\leq 0
\]

for all \( t > 0 \) and \( 0 < r < h_0 \). Then we have

\[
\begin{cases}
N_t - d_N \Delta N = 0, & 0 < r, \ t > 0, \\
L - d_1 \Delta L \leq (\beta(r) - \gamma(r))L - \frac{\beta(r)}{N}L^2, & 0 < r < h_0, \ t > 0, \\
N(0, t) = 0, \ L(0, t) = 0, & t > 0, \\
I(r, t) = 0, & r \geq h_0, \ t > 0, \\
0 = h_0 \leq -\mu L_0(h_0, t) = -\mu \delta \psi(h_0), & t > 0, \\
N(r, 0) \leq N_0(r), \ I(r, 0) \leq I_0(r), & r \geq 0.
\end{cases}
\]

Hence we can apply Lemma 2.5 to conclude that \( I(r, t) \geq I(r, t) \) in \([0, h_0] \times [0, \infty)\). It follows that \( \liminf_{t \to +\infty} ||I(\cdot, t)||_{C([0, h(t)])} \geq \delta\psi(0) > 0 \) and therefore \( h_\infty = +\infty \) by Lemma 4.1.

If \( R^F_0(0) = R^D_0(B_{h_0}) = 1 \). Then for any positive time \( t_0 \), we have \( h(t_0) > h_0 \) and \( R^D_0(B_{h(t_0)}) > R^D_0(B_{h_0}) = 1 \) by the monotonicity in Theorem 3.3. Replaced the initial time 0 by the positive time \( t_0 \), we then have \( h_\infty = +\infty \) as above. \( \square \)

**Remark 5.1** It follows from the above proof that spreading happens if there exists \( t_0 \geq 0 \) such that \( R^F_0(t_0) \geq 1 \).

Combing Lemma 4.1 and Theorem 5.1, we immediately obtain the following spreading-vanishing dichotomy:

**Theorem 5.2** Let \((N(r, t), I(r, t); h(t))\) be the solution of the free boundary problem (1.4). Then the following alternative holds:

Either

(i) Spreading: \( h_\infty = +\infty \) and \( \liminf_{t \to +\infty} ||I(\cdot, t)||_{C([0, h(t)])} > 0 \);

or

(ii) Vanishing: \( h_\infty \leq h^* \) with \( R^D_0(B_{h^*}) = 1 \) and \( \lim_{t \to +\infty} ||I(\cdot, t)||_{C([0, h(t)])} = 0 \).

**Proof:** In fact, if \( h_\infty \leq h^* \) with \( R^D_0(B_{h^*}) = 1 \), then \( \lim_{t \to +\infty} ||I(\cdot, t)||_{C([0, h(t)])} = 0 \) by Lemma 4.1, vanishing happens. If \( h_\infty > h^* \), then there exists \( T \) sufficiently large such that \( R^F_0(T) := R^D_0(B_{h(T)}) > 1 \) by the monotonicity in Theorem 3.2, then \( h_\infty = +\infty \) by Theorem 5.1 or Remark 5.1.

Theorem 4.2 shows if \( R^F_0(0) < 1 \), vanishing happens for small initial value of infected individuals, the next result shows that spreading happens for large value.
Theorem 5.3 Suppose that \( \{ r \in [0, \infty) : \beta(r) > \gamma(r) \} \) is nonempty and \( R_0^F(0) < 1 \). Then \( h_\infty = \infty \) and \( \liminf_{t \to +\infty} ||I(t, \cdot)||_{C([0, h(t)])} \neq 0 \) if \( ||J_0(r)||_{C([0, h(t)])} \) is sufficiently large.

Proof: We will construct a vector \((U, V, h)\) such that
\[
U(r, t) = N_0 = \inf_{r \geq 0} N_0, \quad h(t) = \sqrt{t + \delta}, \quad 0 \leq r, t \geq 0,
\]
\[
V(r, t) = \begin{cases}
\frac{M}{(t+\delta)^{r}} \frac{\phi(\sqrt{t+\delta})}{\phi'(1)}, & 0 \leq r \leq \sqrt{t+\delta}, \quad t \geq 0, \\
0, & r > \sqrt{t+\delta}, \quad t \geq 0,
\end{cases}
\]
where \( \delta, M, T_0, k \) are chosen in the following way:
\[
0 < \delta \leq \min\{1, 2\}, \quad R_0^D(B_{\sqrt{T_0}}) > 1,
\]
\[
k > \mu_0 + (T_0 + 1)||\gamma + \frac{\beta(r)I}{N_0}||_\infty, \quad -2\mu M \phi'(h_0) > (T_0 + 1)^k.
\]
Here we have used the fact that \( R_0^D(B_h) \to \sup_{r \in [0, \infty)} \frac{\beta(r)}{\gamma(r)} > 1 \) as \( h \to \infty \).

Direct computations yield
\[
V_t - d_t \Delta V - (\beta(r) - \gamma(r))V + \frac{\beta(r)}{U} V^2
\]
\[
= -\frac{M}{(t+\delta)^{r+1}}[d \Delta \phi + \frac{r}{2\sqrt{t+\delta}} \phi' + (k + (\beta - \gamma - \frac{\beta}{U})(t + \delta)) \phi]
\]
\[
\leq -\frac{M}{(t+\delta)^{r+1}}[d \Delta \phi + \frac{1}{2} \phi' + \mu_0 \phi]
\]
\[
= 0
\]
for all \( 0 < t \leq T_0 \) and \( 0 < r < h \).
\[
h'(t) + \mu V_r(h, t) = \frac{1}{2\sqrt{t+\delta}} + \frac{\mu M}{(t+\delta)^{k+1/2}} \phi'(1) < 0.
\]
Then we have
\[
\begin{cases}
U_t - d_N \Delta U = 0, & 0 < r, 0 < t \leq T_0, \\
V_t - d_I \Delta V \leq (\beta(r) - \gamma(r))V - \frac{\beta(r) \gamma(r)}{2} V^2, & 0 < r < h, 0 < t \leq T_0, \\
U(0, t) = 0 = V(0, t), & 0 < t \leq T_0, \\
V(r, t) = 0, & r \geq h, 0 < t \leq T_0, \\
\frac{\partial h}{\partial t} < -\mu V(\mathbb{R}, 0), & 0 < t \leq T_0, \\
\frac{\partial U}{\partial t} < N_0(r), & r \geq 0.
\end{cases}
\]

If \(V(r, 0) = \frac{M}{\tau} \phi(\frac{r}{\sqrt{\delta}}) < I_0(r)\) in \([0, \sqrt{\delta}]\). Then using Lemma 2.5 yields that \(h(t) \geq h(t)\) in \([0, T_0]\). In particular, \(h(T_0) \geq h(T_0) = \sqrt{T_0 + \delta} \geq \sqrt{T_0}\). Noting that \(R_0^B(B_{\sqrt{T_0}}) > 1\), we then have \(h \infty = +\infty\) by Theorem 5.1.

**Remark 5.2** The assumption \(\{r \in [0, \infty) : \beta(r) > \gamma(r)\}\) is nonempty means that there exists at least one high-risk site. In this case, the disease can spread to the whole area even if \(R_0^F(0) < 1\).

**Theorem 5.4** (Sharp threshold) Suppose that \(\{r \in [0, \infty) : \beta(r) > \gamma(r)\}\) is nonempty. Fixed \(N_0, h_0\). Let \((N, I; h)\) be a solution of (1.4) with \(I_0 = \sigma \phi(r)\) for some \(\sigma > 0\). Then there exists \(\sigma^* = \sigma^*(\phi) \in [0, \infty)\) such that spreading happens when \(\sigma > \sigma^*\), and vanishing happens when \(0 < \sigma < \sigma^*\).

**Proof:** It follows from Theorem 5.1 that spreading always happens if \(R_0^F(0) \geq 1\). Hence in this case we have \(\sigma^*(\phi) = 0\) for any \(\phi\).

For the remaining case \(R_0^F(0) < 1\). Define
\[
\sigma^* := \sup\{\sigma_0 : h \infty(\sigma \phi) < \infty \text{ for } \sigma \in (0, \sigma_0]\}.
\]

By Lemma 4.2, we see that in this case vanishing happens for all small \(\sigma > 0\), therefore, \(\sigma^* \in (0, \infty]\). On the other hand, it follows from Theorem 5.2 that in this case spreading happens for all big \(\sigma\). Therefore \(\sigma^* \in (0, \infty]\), and spreading happens when \(\sigma > \sigma^*\), vanishing happens when \(0 < \sigma < \sigma^*\) by Corollary 2.7.

We claim that vanishing happens when \(\sigma = \sigma^*\). Otherwise \(h \infty = \infty\) for \(\sigma = \sigma^*\). Since \(R_0^F(t) \to \sup_{r \in [0, \infty]} \frac{\beta(r)}{\gamma(\tau)} > 1\) as \(t \to \infty\), therefore there exists \(T_0 > 0\) such that \(R_0^F(T_0) := R_0^D(B_{h(T_0)}) > 1\). By the continuous dependence of \((N, I, h)\) on its initial values, we can find \(\epsilon > 0\) small so that the solution of (1.4) with \(I_0 = (\sigma^* - \epsilon) \phi(r)\), denoted by \((N_\epsilon, I_\epsilon, h_\epsilon)\) satisfies \(R_0^D(B_{h_\epsilon(T_0)}) > 1\). This implies that spreading happens to \((N_\epsilon, I_\epsilon, h_\epsilon)\), which contradicts the definition of \(\mu^*\). The proof is complete.

### 6 Discussion

In this paper, we have considered a spatial SIS epidemic model describing the spatial transmission of diseases and examined the dynamical behavior of the population \((N, I)\)
with spreading front \( r = h(t) \) defined by (1.4). We have obtained some analytic results about the asymptotic properties of the spatial spread of infectious diseases.

The basic reproduction numbers \( R_0^D \) and \( R_0^F(t) \) are introduced for the system with Dirichlet boundary condition and the system with the free boundary, respectively. It is proved that if \( R_0^F(t_0) \geq 1 \) for some \( t_0 \geq 0 \), spreading always happens or the disease will become endemic (Theorem 5.1 and Remark 5.1). If \( R_0^F(0) < 1 \), vanishing of the spreading of the disease happens provided that the initial value of the infected individuals \( I_0 \) is sufficiently small (Theorem 4.2) and spreading happens provided that \( I_0 \) is large and there at least exists one high-risk site (Theorem 5.2).

One of the main contributions of this work is the development and analysis of the basic reproduction numbers. We now have four basic reproduction numbers: \( R_0 \) used for the ODE system, \( R_0^N \) (see (1.2)) defined for the diffusive system with Neumann boundary condition, \( R_0^D \) and \( R_0^F(t) \) defined in this paper. They are all closely related, \( R_0 \) is actually equals to \( R_0^N \) with \( \Omega \) replaced by the whole space \( \mathbb{R}^n \), \( R_0^F(t) \) is \( R_0^D \) with \( \Omega \) replaced by the changing space \( B_{h(t)} \). However, they are different, \( R_0 \), \( R_0^N(t) \) and \( R_0^D \) are all constants, while \( R_0^F(t) \) depends on time \( t \), the temporal dependence of the basic reproduction number is a intrinsic characteristic of the spreading over a changing domain. It follows from the definition of \( R_0^D \) (or \( R_0^F(0) \)) that it is decreasing with respect to \( d \) and increasing with respect to \( h_0 \), so fast diffusion and small initial infected size are in favor of the disease to vanish, or prevention and control, the latter implies that early control is better to prevent the outbreak of the disease to spread over larger area.

Through the preliminary study in this paper, it is reasonable to conclude that (1.4) are promising alternatives to (1.1) with Neumann boundary condition and the modified system with Dirichlet boundary condition in [15] for the temporal and spatial modeling of disease spreading, and there are still more work to do to understand the dynamics of the model. Among many of the unknowns, it is more interesting to study the asymptotic spreading speed when spreading happens? The pioneer study in [14] on the asymptotic spreading speed of the free boundary problem in a logistic model will allow us to estimate the speed of the spreading front of the disease. We keep it as a future work when use West Nile virus as a concrete example.

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