Towards Auditing Unsupervised Learning Algorithms and Human Processes For Fairness

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Abstract. Existing work on fairness typically focuses on making known machine learning algorithms fairer. Fair variants of classification, clustering, outlier detection and other styles of algorithms exist. However, an understudied area is the topic of auditing an algorithm’s output to determine fairness. Existing work has explored the two group classification problem for binary protected status variables using standard definitions of statistical parity. Here we build upon the area of auditing by exploring the multi-group setting under more complex definitions of fairness.

Keywords: Classification · Auditing · Fairness · Combinatorial Optimization · Complexity

1 Introduction and Motivation

The AI community has made tremendous progress towards making algorithms fairer. Fairness has been studied in the context of many major ML tasks such as clustering, classification, ranking, embedding and anomaly detection. The area of fairness and ML algorithms can be divided loosely into three categories. The first category explores pre-processing data to make existing algorithms fairer. The Fairlets approach of [4] is perhaps the most well known example of pre-processing data so that k-means and k-median algorithms are guaranteed to produce fair classes (i.e., clusters). The second category that adds fairness rules into algorithms is perhaps the most popular area. Fairness rules have been added to clustering [14], classification [5], outlier detection [20] and ranking [1]. The third, and perhaps the most understudied category, is post-processing the results of algorithms. This work has two main sub-areas: (i) post-processing to make the output of algorithms fairer [7] and (ii) auditing the output of an algorithm [12] [6] to determine if it is fair (or not). Our work falls into this second sub-area. We view the algorithm/human-process as dividing people into classes (e.g., outlier/inlier, classes, category etc). We use the term “group” to refer to a protected status group, which in our work can be a complex definition across multiple protected status variables (PSVs).
Auditing is particularly important as it allows verification that an algorithm’s or human process’s output is fair. The latter is particularly understudied as human processes are particularly complex. For example, we explore (Section 6.2) the topic of auditing the fairness of California’s 53 electoral districts along 13 protected statuses, many of them taking multiple values. Existing work on auditing has only studied outlier detection [6] and classification [12]; though this work is useful, it is limited in several key ways. Notably, it is limited to the two class setting, binary protected status and most importantly unweighted settings as a measure of fairness. These settings are useful in selection problems such as job interviews or decision problems such as predicting recidivism where decisions are binary. However, many settings do not match this situation. Consider a credit card company that divides its customer base into $k$ classes and offers each class a different loyalty bonus. The classic two-class auditing work [12] does not fit this setting and cannot be made to fit this setting by repeating it with a one versus the rest group application. Our second measure of fairness (called “utility weighted”) studies this situation, and we observe that it is possible for a set of classes to be fair when ignoring weights but unfair when considering weights. Finally, consider our study in Section 6.3 where we audit news sources for fairness with respect to coverage of different protected status individuals. There, we are interested in ensuring equal coverage between protected status groups and not on a single protected status group. We study this in our third measure of fairness (called “pairwise equality”). Our contributions are as follows.

1. We formulate the search for unfairness as a combinatorial optimization problem and establish its computational intractability (Theorem 1), leading to a test that cannot be easily side-stepped.
2. We search for three types of unfairness:
   (a) Count-based unfairness, which has been studied by the community as statistical parity.
   (b) A novel utility weighted unfairness which allows the benefit/utility of some classes to be more than others.
   (c) A new pairwise unfairness which finds unfairness between two groups (i.e., PSV combinations) of individuals.
3. For all three formulations, our methods allow finding unfairness across multiple PSV values, a topic rarely covered by the literature so far.
4. Our experiments consider detecting unfairness in classes generated by algorithms as well as those created by human processes (e.g., congressional districts of California (see Section 6.2) and news articles grouped by source media (see Section 6.3)).

**Organization.** We begin by overviewing our method at a high level. We then provide details of our count-based unfairness test and show that it is computationally intractable. We extend that formulation to a utility based setting and then to a utility based settings that searches for unfairness over all classes. We then present experimental results, related work and conclude.
2 High Level Overview of Our Approach

Our approach to identify unfairness involves searching for protected status variable (PSV) combinations that are under-represented. We begin with a basic formulation that is similar to the classical count-based methods introduced by others [4] and then introduce new types of unfairness that we believe are interesting and useful. Our work can be seen as a framework for searching for unfairness.

**How we detect unfairness.** Our work searches for over/under-represented PSV combinations denoted by \( x \) (which represent groups of individuals). To tie our work back to classic set cover formulations [9] in theoretical computer science, we formulate our work as searching for a minimum number of occurrences of a disjunction of PSVs (e.g., Male \( \lor \) Young) that is over-represented in a class compared to the other classes (e.g., in the rest of the population). By DeMorgan’s law [15], this can also be seen as identifying an under-represented group corresponding to a conjunction of PSVs (e.g., Female \( \land \) Elderly). We search across all PSV combinations (groups of people) to find examples of unfairness. If no such PSV combination is returned, then we conclude that the division of people into classes is fair. A domain expert can determine whether the type of unfairness found is acceptable (or interesting), and our formulations can be run again to explicitly avoid finding such examples of unfairness.

**Types of unfairness considered.** We formulate three types of unfairness as outlined in Table 1 but others are possible in our framework:

1. **Count-based.** This applies a rule similar to the traditional definition of statistical parity [12]; it requires that the count of instances satisfying a PSV combination \( x \) (normalized by the class size) in a class is nearly the same as the proportion of the PSV count in the rest of the population. This definition of fairness says that a division is unfair if any class violates this rule.

2. **Utility weighted.** The above classic definition of statistical parity assumes each class is equally important/desirable. The credit card example discussed in the introduction does not meet this assumption. To address it, we introduce a novel count-based fairness that associates a utility/benefit with each class. Here, rather than just counting how many of the group \( x \) appears in a class, we perform a weighted count given the utility values for each class and compare this against a random allocation of the group across classes. Our optimization problem solves for these utility values (within bounds chosen by a domain expert).

3. **Pairwise equality.** Both types of fairness mentioned above identify a single PSV group \( (x) \) that is being treated unfairly. Here we introduce a new type of fairness that instead looks for unfairness between two PSV combinations \( x \) and \( w \) (i.e., two groups of people).

**Importance of searching across multiple PSVs.** In all three types of unfairness, we search for combinations/groups of PSVs that cause unfairness. This is critical as a set of classes may be fair at the individual PSV level but not when considering multiple PSVs. For example, the fraction of Females receiving a job offer may be fair (equals the fraction of females in the population) as could be
Table 1. The high level unfairness tests of a given division of instances into classes addressed by our combinatorial optimization problems. Symbols x and w represent subsets of PSVs. In our formulations, $\gamma = \beta - \alpha$ is the disparity gap set by a domain expert. For each type of unfairness, we have indicated the definition that specifies the corresponding optimization problem as a mathematical program.

| Name          | Unfairness Detected                                           | Test for Unfairness                                                                 |
|---------------|---------------------------------------------------------------|-------------------------------------------------------------------------------------|
| Count         | The count of the group x is under-represented in class i.    | $\exists x, i : P(x|\neg C_i) - P(x|C_i) \geq (\beta - \alpha) = \gamma$ Formulation in Problem 1. Proposition 1 in the supplement shows that this formulation is similar to classic disparate impact calculations ($P(x|C_i) \approx P(x)$). |
| Utility weighted | The weighted count of the group x in the current class division is under-represented compared to a random allocation of group members to classes. | $\exists U, x : \sum_k U_k | C_k | P(x|C_k) U_k \leq (N_x/K) \sum_k U_k - \gamma$ s.t. $a_k \leq U_k \leq b_k \forall k$, where $N_x$ is the number of instances covered by x in the population (see Lemma 1 and Problem 2). |
| Pairwise Equality | For two groups x and w, their weighted counts are substantially different, with x having less utility. | $\exists U, x, w : \sum_k U_k | C_k | P(w|C_k) - \sum_k U_k | C_k | P(x|C_k) \geq \gamma$ s.t. $a_k \leq U_k \leq b_k \forall k$ and $x^T w = 0$ (see Problem 3). |

the case for Married individuals, yet no Females $\land$ Married individuals may receive a job offer. Thus, in combination, there is unfairness.

**Importance of the hardness of our search problem.** Our work defines a combinatorial problem of searching for unfairness. Suppose each person is represented by $m$ binary PSVs. Then there are $2^m$ “types” or “groups” of people, and we must determine whether any combination of them is treated unfairly. It is tempting to say that such a search problem is obviously intractable; however, many problems with exponentially large search spaces have polynomial time algorithms (e.g., 2SAT, the Satisfiability problem in which each clause has at most two literals [17]). We demonstrate the difficulty of developing efficient algorithms for our search formulations by showing that our basic search problem (i.e., testing for count-based unfairness) is computationally intractable (Theorem 1). This is an important property for the following reason: if detecting unfairness is computationally hard, it means that making a result fairer by post processing is also computationally hard. In other words, if an algorithm produces a classification $\Pi$ into some number of classes and our optimization formulation finds an example of unfairness, then one cannot easily move around a few points to obtain another classification $\Pi'$ which is fair, even if it is known why $\Pi$ is unfair! Anecdotally, this is because even if we know a PSV combination that makes $\Pi$ unfair, when we fix it, we may introduce other combinations that cause unfairness. For certain fairness measures, this can be done efficiently in the single PSV case [7] but not for the case of multiple PSVs.
3 A Formulation for Count-Based Group Unfairness

| Variable | Meaning |
|----------|---------|
| $P, m$ | The set and the number of PSVs (i.e., $m = |P|)$.
| $x, w$ | Binary selection vectors for the PSVs for explanations using disjunctions. (Each vector represents a subset of $P$.) |
| $T, O, C_i$ | The set of instances in a target class, other class and the $i^{th}$ class respectively. (We also use $r$ to denote $|T|$ and $t_k$ to denote $|O_k|$.) |
| $y^j_k, z^j_k$ | Indicator variables for the $j^{th}$ instance in class $k$. The value $y^j_k (z^j_k)$ is 1 iff the $j^{th}$ instance in class $k$ is covered by $x (w)$.
| $U_1, U_2, \ldots, U_K$ | Utility (benefit) values associated with classes $C_1, C_2, \ldots, C_K$ respectively. |
| $\alpha, \beta$ | Bounds on coverage, with $\alpha < \beta$. The value $\gamma = \beta - \alpha$ is the tolerance to unfairness. |
| $k, K$ | An index to classes and the total number of classes respectively. |
| $a_k, b_k$ | Lower and upper bounds on the utility of the $k^{th}$ class, $1 \leq k \leq K$. |

Table 2. List of variables used in the mathematical programming formulations developed in the paper.

We first outline our test of unfairness for one class (the target class) which is repeated $K$ times (where $K$ is the number of classes) with each class taking a turn at being the target class. It is important to understand that our test is formulated as a search problem with the aim of finding a simplest example of unfairness; if there is no solution for this problems for all classes, this means the classification is fair. The notation used in the paper is summarized in Table 2.

**High-level description.** The objectives of our optimization problems is shown diagrammatically in Figure 1. The figure shows $K$ Venn diagrams (one for each class), and the coverage of the explanation ($x$) with respect to the PSVs is denoted by a black dashed rectangle. Coverage here means that an instance $\eta$ in that class is covered by $x$; a formal definition of this notion of coverage is as follows.

**Definition 1.** Let $C$ be a class and let vector $x$ represent a subset of (binary valued) PSVs. The set of instances in $C$ covered by $x$ includes each instance $\eta$ in $C$ such that at least one PSV in $x$ has the value 1 in the instance $\eta$.

**Example:** Suppose we have three binary PSVs, namely \{Female, LowIncome, Married\} and $x = (1, 1, 0)$. Thus, the selection vector $x$ represents the group/subset of individuals given by \{Female $\lor$ LowIncome\}; the vector $x$ covers any instance that represents a woman or a person whose income is considered low (or both).

The objective of our optimization problem is to find a simplest explanation ($x$) such that there is a class $C_i$ where $x$ is under-represented. The extent of over

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3 We use “simplest” to mean a vector $x$ with the smallest number of PSVs.
(or under) representation is specified through a parameter $\gamma$, where $0 < \gamma < 1$, chosen by a domain expert which we refer to as the disparity gap.

![Diagram of optimization problem](image)

**Fig. 1.** A diagrammatic overview of our optimization problem to find an explanation (denoted by $x$) in terms of the PSVs that is under-represented in one class than the others. The value $\gamma = \beta - \alpha$ is the disparity gap, which is a probability for count-based unfairness and a numerical value for utility-based unfairness.

### An integer linear program (ILP) for detecting unfairness in one class.

We now show how the unfairness detection problem mentioned above can be expressed as an ILP. Table 2 shows the notation used in our formulation.

Let $m = |P|$ be the number of PSVs. We search for a subset of PSVs as given by the binary indicator vector $x$. For convenience, let $t_k = |O_k|$, $1 \leq k \leq K - 1$. We compute the fraction of instances in $T$ (i.e., the target class) and $O_1, \ldots, O_{K-1}$ (i.e., the other $K - 1$ classes) that are covered by $x$. To do this through an ILP, we represent each class $O_k$ as an $m \times |O_k|$ Boolean matrix, where each column represents a data point. The column vector for the $j$th data point in $O_k$, denoted by $O_{j \kappa}$, gives the 0/1 values of the $m$ PSVs for that point. Similarly, the target class $T$ is considered as an $m \times |T|$ matrix and its $i$th column is denoted by $T_i$.

To compute the fraction of instances in $T$ covered by $x$, we introduce binary variables $z_1, z_2, \ldots, z_r$, where $r = |T|$. We ensure that $z_i = 1$ iff the vector $x$ covers the $i$th point in $T$. Thus, $\sum_{i=1}^{r} z_i$ gives the number of points in $T$ covered by $x$. We want $x$ to cover at most $\alpha$ fraction of the points in $T$.

Similarly, for each class $O_k$ ($1 \leq k \leq K - 1$), we use $t_k = |O_k|$ additional 0/1 variables, denoted by $y_{k1}, y_{k2}, \ldots, y_{kt_k}$; here, variable $y_{kj}$ corresponds to the $j$th point in class $O_k$. We create constraints so that $y_{kj} = 1$ if a chosen vector $x$ covers the $j$th point in $O_k$, $1 \leq j \leq t_k$. Hence, $\sum_{j=1}^{t_k} y_{kj}$ gives the number of points of $O_k$ covered by the vector $x$. We create constraints to ensure that at least $\beta$ fraction of points in each of the classes $O_1, \ldots, O_{K-1}$ are covered by $x$.

If we set $\alpha = 0.5\beta$ and a solution to our optimization problem is found, it means that $x$ contains a PSV combination that matches a subset of people that are under-represented in $T$ and over-represented in all of the classes $O_1, \ldots, O_{K-1}$ by a factor of 2. Conversely, if no solution is found, then no such unfairness exists (given the requirements set by $\alpha$ and $\beta$). The ILP to achieve this is given below.

**Problem 1 Unfairness Detection In a Single Class (UDSC) Problem.** Formally, a decision version of this problem can be expressed as follows:
∃x : P(x|T) ≤ α, P(x|¬T) ≥ β, α < β.

Below, we specify an ILP formulation that focuses on finding a shortest explanation of unfairness.

Objective : argmin_x ||x||
satisfying the following constraints:
(1) For each class O_k (1 ≤ k ≤ K − 1), with |O_k| = t_k, the constraints are:
   \[ y_j^k \leq x^T O_j^k \quad \text{and} \quad m y_j^k \geq x^T O_j^k, \quad 1 \leq j \leq t_k. \]
(2) For the target class T, the constraints are:
   \[ z_i \leq x^T T_i \quad \text{and} \quad m z_i \geq x^T T_i, \quad 1 \leq i \leq |T|. \]
(3) The set of fairness-related constraints, with \( t_k = |O_k| \) and \( r = |T| \), are:
   \[ \sum_{j=1}^{t_k} y_j^k \geq \beta |O_k|, \quad 1 \leq k \leq K - 1 \quad \text{and} \quad \sum_{i=1}^{r} z_i \leq \alpha |T|. \]
(4) All the variables in x and all the auxiliary variables \( y_j^k \) (1 ≤ k ≤ K − 1, 1 ≤ j ≤ |O_k|) and \( z_i \) (1 ≤ i ≤ |T|) take on values from \{0, 1\}.

Notes:
1. We use ||x|| to denote the number of variables in x which are set to 1. Thus, this formulation tries to find a smallest explanation of unfairness (if one exists).
2. Let us consider the set of constraints (1) above. The constraint \( y_j^k \leq x^T O_j^k \) ensures that if x does not cover the \( j^{th} \) instance in \( O_k \), the variable \( y_j^k \) is forced to be 0. On the other hand, if the \( j^{th} \) instance in \( O_k \) is covered by x, the constraint \( m y_j^k \geq x^T O_j^k \) ensures that \( y_j^k \) is set to 1. Similar considerations apply to the constraints specified in (2).
3. The set of constraints (3) above on the summations involving y and z variables have the size of the respective classes on the right hand side to ensure that α and β can be interpreted as probabilities.

4 Extensions to Utility Based Classification

Previously our search for unfairness merely counted the number of individuals to determine unfairness. This is appropriate when there are multiple actions with the same or similar utility/benefit. But if the utilities \( (U_1, U_2, \ldots, U_K) \) of being in the different classes can vary, then there is even more opportunity for unfairness. Classes with different utilities arise when each group corresponding to a class is treated differently. For example, a credit card company classifying customers’ records may wish to give very different benefits/rewards to each class. Our work here tries to identify whether such rewards/utilities (within given bounds) yield unfairness. As before, if no solution exists, then the classification is deemed fair.

We divide our work on this topic into two types: (i) utility weighted unfairness and (ii) pairwise utility unfairness. For the former, we take our previous formulation but weight each class by its utility and compare it to expected utility. For
the latter, we create an optimization problem that attempts to find two different PSV combinations (denoted by \( x \) and \( w \)) whose expected utility difference across all classes is greater than a given threshold.

4.1 Utility Weighted Unfairness

In the formulation for unfairness search given in Problem 1, we implicitly gave each class/action an equal weight. Here we allow these weights (which we call “utilities”) to become part of the search problem for unfairness. Our formulation here can return both an example of unfairness (denoted by \( x \)) as before and also the utilities of the classes that cause the unfairness. Going back to our example with credit card customers, bounds on these utility values (denoted by \( a \) and \( b \)) can be given by a domain expert in accordance with the range of rewards that are say fiscally appropriate for an organization. Since our previous formulation is just a special case of this version with utilities, this formulation can identify unfairness which cannot be detected by count-based formulations. We present an example in Section B.2 of the supplement to point out that there are classifications where the count-based approach doesn’t detect unfairness, but the utility weighted approach reveals possible unfairness.

Our formulation now optimizes over additional variables for the utilities \( (U = \{U_1, \ldots, U_K\}) \). To present this formulation, we begin with a lemma that gives an expression for the expected total utility of the instances covered by a PSV combination \( x \) when such instances are distributed uniformly randomly across the \( K \) classes.

**Lemma 1.** Let \( U_k \) denote the utility assigned to class \( k \), \( 1 \leq k \leq K \). Suppose the instances covered by a PSV combination \( x \) are distributed uniformly randomly over the \( K \) classes. Then the total expected utility of the instances covered by \( x \) is \( (N_x/K)\sum_{k=1}^{K} U_k \), where \( N_x \) is the number of instances covered by \( x \) in the population.

**Proof:** See supplement.

This above expression for the expected total utility of the instances covered by \( x \) was used in the second row of Table 1.

**Problem 2 Utility Weighted-Unfairness Detection.** From Lemma 1 and Table 1, the decision version of this problem can be expressed formally as follows.

\[
\exists U, x: \quad (\sum_{k} |C_k| P(x|C_k) U_k) \leq \alpha, \quad (N_x/K)\sum_{k} U_k \geq \beta, \quad \alpha < \beta \quad \text{and} \quad a_k \leq U_k \leq b_k, \quad 1 \leq k \leq K.
\]

We present an example in Section B.2 of the supplement to show that for a given classification, while count-based formulation (Problem 1) may not reveal unfairness, our utility-based formulation (Problem 2) can reveal unfairness.

We now present an integer program for Problem 2. First, we specify the variables used in the formulation.
(a) To be consistent with the notation used in Problem 1, we use $O_1, O_2, \ldots, O_K$ to denote the matrix representation of the $K$ classes. Note that the matrix representation of $O_k$ is of size $m \times t_k$, where $t_k = |O_k|, 1 \leq k \leq K$. As before, we use $O_k^j$ to denote the $j^{th}$ column (i.e., instance) of $O_k$. We introduce $t_k$ $\{0, 1\}$-valued variables $y_k^1, y_k^2, \ldots, y_k^k$ associated with $O_k, 1 \leq k \leq K$. The significance of these variables is the same as that in Problem 1.

(b) We use $F$ to denote the matrix representation of the population. Note that the matrix representation of $F$ is of size $m \times n$, where $n$ is the size of the population. We use $F^i$ to denote the $i^{th}$ column (i.e., instance) of $F$. We introduce $n \{0, 1\}$-valued variables $z_1, z_2, \ldots, z_n$ associated with $F$. Variable $z_i$ is used to check whether a PSV combination $x$ covers the $i^{th}$ instance of the population. (Thus, the significance of these variables is the same as that of the target class in Problem 1. Further, $\sum_{i=1}^{n} z_i$ gives the number of instances in the population covered by $x$.)

(c) We have variables $U_1, U_2, \ldots, U_K$ to represent the utilities of the $K$ classes.

We are now ready to specify the objective and constraints of the integer program for Utility-Weighted Unfairness Detection. The objective is \( \text{argmin}_{U, x} \| x \| \) and the constraints are as follows.

1. For each class $O_k (1 \leq k \leq K)$, with $|O_k| = t_k$, the constraints are:
   \[
   y_k^j \leq x^T O_k^j \quad \text{and} \quad m y_k^j \geq x^T O_k^j, \quad 1 \leq j \leq t_k.
   \]
   These constraints ensure that the variable $y_k^j$ is set to 1 if $x$ covers the $j^{th}$ instance in class $O_k$, and to 0 otherwise ($1 \leq j \leq t_k$ and $1 \leq k \leq K$).

2. For the population $F$, the constraints are:
   \[
   z_i \leq x^T F^i, \quad \text{and} \quad m z_i \geq x^T F^i, \quad 1 \leq i \leq n.
   \]
   These constraints ensure that the variable $z_i$ is set to 1 if $x$ covers the $i^{th}$ instance in the population $F$ and to 0 otherwise ($1 \leq i \leq n$).

3. The set of fairness-related constraints, with $t_k = |O_k|$ and $n$ being the size of the population are:
   \[
   \sum_{k=1}^{K} (U_k \sum_{j=1}^{t_k} y_k^j) \leq \alpha \quad \text{and} \quad (\sum_{i=1}^{n} z_i) \times (\sum_{k=1}^{K} U_k)/K \geq \beta.
   \]
   The first constraint above uses the total utility of the instances covered by $x$ in the given classification. The second constraint above uses the expected total utility of the instances covered by $x$ in the population when these instances are distributed randomly over the $K$ classes. (This constraint uses Lemma 1.)

4. Bounds on utility values: $a_k \leq U_k \leq b_k, \quad 1 \leq k \leq K$.

5. All the variables in $x$ and all the auxiliary variables $y_k^j (1 \leq k \leq K, \quad 1 \leq j \leq |O_k|)$ and $z_i (1 \leq i \leq n)$ take on values from $\{0, 1\}$.

Note: As this is a more complex search problem, the formulation uses non-linear constraints. In particular, constraints in (3) above are non-linear. As before, the values of $\alpha$ and $\beta$ are chosen by a domain expert depending on the desired disparity gap $\gamma$. 

4.2 Pairwise Utility Unfairness

Here we explore the extension of our earlier formulations to allow aggregation across multiple classes. Instead of testing whether there exists a subset of people (denoted again by $x$) who are under-represented in one class compared to the rest, we search for two groups of people, denoted by $x$ and $w$, whose expected utility when summed up over all classes differs by a value that is at or beyond a specified tolerance level.

To achieve this, we use variables $y^j_k$ for $x$ (and $z^j_k$ for $w$) to encode whether instance $j$ in class $k$ is covered by $x$ ($w$). These indicator variables are then summed and multiplied by the utility of each class and a constraint is imposed on the difference that is not tolerable using a chosen disparity threshold $\gamma$. (Recall that our optimization problems are tests of unfairness.) To achieve this, $\alpha_k$ and $\beta_k$ are the utility of instances in class $C_k$ covered by $x$ and $w$ respectively. The final constraint places a lower bound $\gamma$ on the sum of their difference.

**Problem 3 (Pairwise Utility Unfairness Detection.)** A formal statement of the decision version of this problem is as follows:

$$
\exists U, x, w : \sum_k U_k |C_k| P(x|C_k) - \sum_k U_k |C_k| P(w|C_k) \geq (\beta - \alpha) = \gamma \text{ s.t. } a_k \leq U_k \leq b_k \forall k \text{ and } x^T w = 0.
$$

An integer program for finding a shortest explanation of unfairness is as follows. The objective here is $\text{argmax}_{U, x, w} ||x - w||$ and the constraints are as follows.

(i) For each class $C_k$ ($1 \leq k \leq K$), with $|C_k| = t_k$, the constraints are as follows.

(As before, the notation $C^j_k$ represents the $j^{th}$ column of the $m \times t_k$ Boolean matrix representing $C_k$.)

$$
\begin{align*}
y^j_k &\leq x^T C^j_k \quad \text{and} \quad m y^j_k \geq x^T C^j_k, \quad 1 \leq j \leq t_k \\
z^j_k &\leq w^T C^j_k \quad \text{and} \quad m z^j_k \geq w^T C^j_k, \quad 1 \leq j \leq t_k \\
\alpha_k &= U_k \sum_{j=1}^{t_k} y^j_k, \quad \beta_k = U_k \sum_{j=1}^{t_k} z^j_k
\end{align*}
$$

(ii) Other constraints:

$$
\sum_{k=1}^{K} \alpha_k - \sum_{k=1}^{K} \beta_k \geq \gamma, \quad a_k \leq U_k \leq b_k \quad (1 \leq k \leq K) \quad \text{and} \quad x^T w = 0.
$$

**Note:** The constraint $x^T w = 0$ above ensures that the sets of PSVs represented by $x$ and $w$ are disjoint. (For example, this prevents the possibility of a subset relationship between $x$ and $w$.)

5 Proof of Computational Intractability

This section can be skipped on first reading with the understanding that the underlying problem of searching for the simplest count based fairness is computationally intractable. That is, under a standard hypothesis in computational complexity [17], there can be no general purpose algorithm that finds $x$ efficiently. This is important as it points out the difficulty of efficiently modifying an existing unfair classification to create a classification that is fair.
To investigate the computational complexity of the UDSC problem (defined as Problem 1), we use the following decision version of the problem.

**Unfairness Detection in a Single Class** (UDSC)

**Given:** A collection of $K \geq 2$ pairwise disjoint classes $T, O_1, \ldots, O_{K-1}$ and a set $P = \{p_1, p_2, \ldots, p_m\}$ of $m$ PSVs, positive integers $\alpha$ and $\beta$, where $\alpha < \beta$.

**Question:** Is there a subset $P' \subseteq P$ such that $P'$ covers at most $\alpha$ instances of $T$ and at least $\beta$ instances in each of the other classes $O_1, O_2, \ldots, O_{K-1}$?

For simplicity in presenting the proof, we have used $\alpha$ and $\beta$ as integers in the above formulation. It is straightforward to express them as fractions of the population size. Unlike the ILP formulation, UDSC defined above is a decision problem; it does not require the minimizing the number of PSVs used in the explanation. Nevertheless, we have the following theorem.

**Theorem 1.** The UDSC problem is $NP$-complete even for two classes.

**Proof:** See supplement.

### 6 Experiments

We explore our three formulations to measure fairness from three different situations (clustering, human processes and classification). These serve to validate our formulations and also illustrate their use in practical situations.

1. **Count-Based Group Unfairness.** We evaluate the fairness of solutions produced by existing fair-by-design clustering algorithms. We observe not unexpectedly that focusing on a single PSV can produce unfairness with respect to other PSVs. This is a simple but necessary result to show the need for fairness across multiple PSVs.

2. **Utility Weighted Unfairness.** Here we search for examples of unfairness in the 53 congressional districts in California amongst multiple PSVs collected during the 2010 census. This is an example of identifying unfairness in a historical classes produced by humans.

3. **Pairwise Utility Group Fairness.** We explore a novel use of budgeting time to read articles from multiple sources so as not to get a biased perspective on a topic. These sources are created by a complex decision/classification process.

#### 6.1 Evaluating the Unfairness of Fair-By-Design Clustering Algorithms

We take the output of a classic (fairlet-based) fair-by-design clustering algorithm [2] which ensures fairness for just one PSV and then measure fairness across the remaining PSVs. Even though this is a simple experiment, we believe that it is necessary. We take the classic Adult Data set [8] studied by many fair clustering papers [3, 4, 7, 13, 18] which contains four PSVs (gender, education,
marital-status, occupation). We produce a fair clustering for just a single PSV (as the fairlets method allows) and then measure unfairness across the remaining three PSVs. In all experiments we use $K = 6$ as is typical with this data set. This is achieved by solving Problem 1 for each cluster in turn as the target, and if any solution is returned, the clustering is deemed unfair and the PSV combination causing the unfairness noted. If a PSV combination is found, we re-run the formulation in Problem 1 again with an additional orthogonality constraint to find a new PSV combination (example of unfairness) until no unfairness is discovered.

We set $\gamma$ to be 20% less than the median population probability (mean of two middle values) of all PSV combinations. The results shown in Table 3 indicate the need for measuring unfairness across multiple PSVs.

| PSV Balanced     | No. of Unfair Combinations in the Remaining PSVs |
|------------------|-----------------------------------------------|
| Gender (G)       | 5 (E, EM, EMO, G, OM)                         |
| Education (E)    | 3 (GO, GM, GMO)                               |
| Marital Status (M)| 5 (E, ED, G, GD, EGO)                       |
| Occupation (O)   | 3 (EM, MG, EMG)                               |

Table 3. Measuring the fairness of the output of classic fairness-by-design clustering algorithms [2] on the census/adult data set [8]. The algorithm balanced the PSV in the left column and we report the number and examples of unfairness found on the remaining three PSVs (maximum of 8). Unfairness is reported if there exists a clustering which contains an individual that is under-represented so as to cause disparate impact (20% discrepancy).

6.2 Evaluating Utility Based Unfairness for Census Data

The previous experiment inherently identified unfairness in a particular class by identifying if a group of individuals was greatly under-represented in one particular class compared to the remaining classes. However, such a fairness test ignores the utility of the classes as discussed in Section 4. Indeed it is possible our previous test can say a solution is fair but a utility weighted test say the opposite. (As mentioned earlier, an example to illustrate this appears in Section B.2.) Here, we consider the utility of the clusters ($U_1, \ldots, U_K$) when detecting unfairness. If a protected status (denoted by x) group’s weighted utility for the given set of classes is substantially different from the expected utility (over randomly created classes) then the classification is deemed unfair.

California consists of 53 congressional districts (CDs). Each of them can be considered a class containing a subset of the 1700+ Zip Code Tabulation Areas (ZCTAs) [19] as shown in Figure 3 in the supplement (Section D.1). For each ZCTA, we have its assignment to a CD, population size and the fraction of its population having the following well known demographic attributes [10]:

...
Foreign born, Chinese, Black, Indian, Vietnamese, Filipino, White, 65 years+, Female, Japanese, American Indian, Native Hawaiian, Islander

We use this information to create a synthetic population of individuals who match the demographic information in each CD and then measure the fairness of the 53 CDs (classes). Each CD has a different median local property tax basis (per capita) which is used as the utility measure as it indicates a general quality of living given local taxes fund schools, local sports, parks and other important quality of living indicators. We use the formulation specified as Problem 2. If no solution is found for any of these problems, then the CDs are “fair” in that no PSV-combination defined group of people is allocated 20% less money than their expected utility if they were assigned randomly to the CDs. Our method discovers the simplest forms of unfairness and we repeat our experiment 100 times, each time adding an orthogonality constraint to not discover a previously found form of unfairness. We calculated the distribution of unfairness found in the 53 CDs and found that it is concentrated in the following districts: 13th-Oakland, 16th Fresno, 21st Hanford 24th-Santa Barbara, 37th-Los Angeles and 39th La Habra (see Figure 2). An overwhelming fraction of the unfairness explanations centered on race but not on country of birth or gender.

![Frequency Distribution of Congressional District Versus Examples of Unfairness Found](image)

**Fig. 2.** The distribution of the 100 shortest explanations/examples of unfairness across California’s 53 congressional districts. The x-axis refers to the congressional district and the y-axis indicates how often unfairness was found in the district.

### 6.3 Using Pairwise Utility-Based Fairness For Reading Times

Here we explore the situation of finding fairness between different protected status groups. This allows finding a new style of comparative unfairness in that group \( x \) is being given unfair (under-represented) treatment compared to another group \( w \). Consider the situation where you have a collection of \( r \) sources of documents with each document each on \( k \) different topics.
We use the Twitter Dataset of Health News [11] (the topics being the health of various types of individuals) which contains the classified Twitter feeds of the following 16 health news sites/sources.

bbchealth, cbchealth, cnnhealth, everydayhealth, foxhealth, gdnhealth, goodhealth, KaiserHealth, latimeshealth, msnhealthnews, NBChealth, nprhealth, nytimeshealth, reuters-health, usnewshealth, wsjhealth

An article may be on one or more of the following protected status topics {Gender, Handicapped, Poverty} and our aim is to get a balanced overview of each. Each news site contains many articles (see Table 4 in the supplementary material for an example).

Our third optimization formulation (Problem 3) can be used to search for a pair of under/over represented PSV combinations. The lower \((a_1, \ldots, a_k)\) and upper \((b_1, \ldots, b_k)\) bounds on the utility values \(U_1, \ldots, U_k\) can be set as the allowable time to spend on each news source. If no solution is found, then we can spend between \(a_k\) and \(b_k\) units of time on news source \(k\) (1 \(\leq k \leq 16\)) and get a balanced (fair) view of the overall topic. Conversely, if our formulation returns a solution, then we get over/under represented PSVs given by \(x\) and \(w\). We make two simplifying assumptions: each article in a source is randomly chosen and all articles take equal time to read.

In our experiment, we assume we have a total of 16 hours (i.e., 60 minutes per source) and set \(a_k = 50\) minutes and \(b_k = 70\) minutes, for 1 \(\leq k \leq 16\); more sophisticated bounds can be set depending on the size of each repository. We set \(\gamma\) to be 15, indicating that the time difference spent reading about any two PSV combination should not be greater than 15 minutes. After solving Problem 3 with the above parameters we find our optimization problem returned no solution. Hence, we conclude that spending between 50 and 70 minutes per news source won’t lead to a biased account of the healthcare topic, given the simplifying assumptions made earlier. If the problem had returned a solution, then \(x\) and \(w\) identify a pair of under/over represented PSV combinations (health topics).

7 Related Work

We discuss two areas of related work and discuss how they differ from our own work. The first of these areas is the work on fairness in classification and the second area is that of auditing classification algorithms.

**Fair-by-Design Clustering/Classification.** The fairness-by-design clustering/classification algorithms (e.g., [4]) measure fairness by calculating the balance of class \(i\) defined by \(B_i = \min\left(\frac{\#\text{Red}_i}{\#\text{Blue}_i}, \frac{\#\text{Blue}_i}{\#\text{Red}_i}\right)\), where \(\#\text{Red}_i\) (\(\#\text{Blue}_i\)) indicates the number of red (blue) instances in a class. (One can think of red and blue instances in a class as representing women and men respectively.) The fairness of a classification is then simply the minimal balance across the classes, that is, \(\min(B_1, \ldots, B_K)\). Optimizing this criterion is equivalent to requiring \(P(\text{Red} | C_i) \approx P(\text{Red}) \forall i\), that is, the probability of finding a red instance in a
class is equal the probability of finding a red instance in the population; a similar condition holds for blue instances as well [2].

Our work is fundamentally different in that: (i) we are testing for fairness where as this work generates fair classification, (ii) our tests involve multiple PSVs and iii) our work extends beyond simple count-based fairness.

**Auditing Classifiers.** The work on auditing classifiers [12] considers the application of a binary classifier to a data set and certify it is fair. The authors define fairness here with respect to two properties: (i) statistical parity and (ii) false positive group level fairness. Since our work is in the unsupervised setting, the second property is not applicable. To discuss the first property, we note that Kearns et al. [12] allow a user to specify groups of instances and require statistical parity to hold for each group. Let $Red$ denote one of the groups and let the target class be denoted by $C_+$. Then the statistical parity property can be viewed as requiring that $P(Red, C_+) = P(Red)P(C_+)$. This condition can be seen to imply (through simple algebraic manipulations) that $P(Red|C_+) = P(Red)$; hence, the condition is equivalent to the fair-by-design criterion for fairness discussed earlier. Proposition 1 (in the supplement) shows our count based formulation is equivalent to this measure of fairness. However, their work is for binary classification and focuses only on one class (the target class). Only our count based measure of fairness is related to this work. Our fairness measure requires that $\forall i, P(Red|C_i) \approx P(Red|\neg C_i)$, which for many classes of equal size simplifies to $P(Red|C_i) \approx P(Red) \forall i$, as $\neg C_i$ is nearly the population of instances. However, there are significant differences. Firstly, we measure fairness across all classes, not just one; most importantly, we check for fairness across all possible PSV combinations and not just for a given set of groups as in [12]. To the best of our knowledge, our work on utility-based fairness has not been studied in the literature.

8 Summary and Conclusions

Most work on fairness focuses on fair-by-design algorithms to produce fair output. Here, we take the alternative direction of testing whether the output of an algorithm is fair. We explore the topic of testing whether a given set of classes is unfair (given parameters set by domain experts) as a series of combinatorial optimization problems designed to search for unfairness.

Our first formulation tested for unfairness using a count-based definition of fairness which is similar to those measures for statistical parity although it measures fairness across multiple PSVs (see Proposition 1). However, these count based methods equate unfairness with under-representation in one class and hence inherently assume that being in one class is equally desirable as being in another. Using utilities to model the benefit of being in different classes allows the search for cost-sensitive unfairness across multiple classes which has not been studied in the fair classification literature. Our final formulation explores the important topic of finding pairs of protected status groups that are not being
treated equally. This is often how fairness is evaluated in challenging situations such as access to gifted and talented education (GATE) programs in schools.

If no solution exists to our optimization problems we deem the classification fair; otherwise, our methods return an explanation for why the classification is unfair. When a solution exists, the domain expert can determine if it is significant. Since our formulations lead to \( \text{NP} \)-hard problems, they cannot be easily sidestepped. This means that even if we say a classification is unfair and why it is unfair, an efficient algorithm to manipulate the existing classification to make it fair cannot exist under a standard hypothesis in computational complexity.

To demonstrate the usefulness of our formulations, we explored several new domains including testing for fairness in California’s 53 congressional districts and how to budget time across multiple reading sources (the classes) so as to obtain a non-biased (fair) view of a topic.

Acknowledgments: This work was supported in part by NSF Grants IIS-1908530 and IIS-1910306 titled: “Explaining Unsupervised Learning: Combinatorial Optimization Formulations, Methods and Applications”.

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Supplementary Material

A Additional Material for Section 2

We mentioned in Table 1 of Section 2 that our count-based unfairness is similar to the classic disparate impact calculation. Here, we provide a formal statement and proof of that statement.

**Proposition 1.** Suppose a set $S$ of $n$ instances is partitioned into $K$ nonempty classes $C_1, C_2, \ldots, C_K$. Further, suppose for a PSV combination $x$, $P(x \mid C_i) = P(x \mid \neg C_i)$ for each $i$, $1 \leq i \leq K$. Then $P(x \mid C_i) = P(x)$ for each $i$, $1 \leq i \leq K$.

**Proof:** Let $N(x)$ denote total number of instances of $S$ covered by the PSV combination $x$. Consider any class $C_i$ and let $N(C_i, x)$ denote the number of instances of $C_i$ covered by $x$. Thus,

$$P(x) = N(x)/n \quad \text{and} \quad P(x \mid C_i) = N(C_i, x)/|C_i|. \quad (1)$$

Now, we use the condition that $P(x \mid C_i) = P(x \mid \neg C_i)$. Note that

$$P(x \mid \neg C_i) = [N(x) - N(C_i, x)]/(n - |C_i|).$$

Thus, the condition $P(x \mid C_i) = P(x \mid \neg C_i)$ yields

$$N(C_i, x)/|C_i| = [N(x) - N(C_i, x)]/[n - |C_i|]. \quad (2)$$

Simplifying Equation (2), we get

$$N(x)/n = N(C_i, x)/|C_i|. \quad (3)$$

By inspecting Equations (1) and (3), it is seen that the left size of Equation (3) is equal to $P(x)$ and its right side is equal to $P(x \mid C_i)$. Thus, the proposition follows from Equation (3).

B Additional Material for Section 4

B.1 Statement and Proof of Lemma 1

**Statement of Lemma 1:** Let $U_k$ denote the utility assigned to class $k$, $1 \leq k \leq K$. Suppose the instances covered by a PSV combination $x$ are distributed uniformly randomly over the $K$ classes. Then the total expected utility of the instances covered by $x$ is $(N_x/K) \sum_{k=1}^{K} U_k$, where $N_x$ is the number of instances covered by $x$ in the population.
Proof: Let $\ell = N_X$ and $M = \{w_1, w_2, \ldots, w_{\ell}\}$ be the set of all instances in the population covered by $x$. Let $h_i$ be the random variable that gives the utility of $w_i$ when the instances in $M$ are distributed uniformly randomly across the $K$ classes, $1 \leq i \leq \ell$. Thus, the random variable $H = \sum_{i=1}^{\ell} h_i$ gives the total utility of the instances in $M$. By linearity of expectation \[16\], we have $E[H] = \sum_{i=1}^{\ell} E[h_i]$. To find $E[h_i]$, we note that the probability that $w_i$ gets assigned to any specific class $k$ is $1/K$ and the corresponding utility is $U_k$. Therefore, 

$$E[h_i] = \sum_{k=1}^{K} U_k / K$$

$$E[H] = \sum_{i=1}^{\ell} E[h_i] = (\ell/K) \sum_{k=1}^{K} U_k.$$  

Since $\ell = N_X$, the lemma follows. 

\[ \blacksquare \]

\section*{B.2 Example of Unfairness Using Utility-Weighted Unfairness}

We mentioned in Section 4 that while our count-based formulation (Problem1) may not reveal unfairness, the utility-based formulation (Problem 2) can reveal unfairness. Here, we present an example to illustrate this.

Example: Suppose $S$ is a set with 32 instances and suppose 8 instances of $S$ are covered by a PSV combination $x$. Thus, $P(x) = 8/32 = 1/4$ and $N_X = 8$. Assume further that $S$ is partitioned into two classes $C_1$ and $C_2$ such that the following conditions hold:

(i) $|C_1| = 24$ and 6 instances of $C_1$ are covered by $x$.
(ii) $|C_2| = 8$ and 2 instances of $C_1$ are covered by $x$.

We note that $P(x \mid C_1) = P(x \mid C_2) = 1/4$. In other words, for $i = 1, 2$, $P(x \mid C_i) = P(x \mid \neg C_i)$. Hence, by the formulation of count-based unfairness (Problem 1), this classification is fair.

Now, suppose we assign the utility value $U_1 = 1$ and $U_2 = 4$ for the two classes $C_1$ and $C_2$ respectively. For these utility values, the values of the two expressions used in the formulation of Problem 2 are as follows.

(i) The value of the expression $|C_1| P(x \mid C_1) U_1 + |C_2| P(x \mid C_2) U_2$ is given by

$$24 \times (1/4) \times 1 + 8 \times (1/4) \times 4 = 14.$$  

(ii) The value of the expression $(N_X/2)(U_1 + U_2)$ is given by $(8/2) \times (1+4) = 20$.

Thus, we have utility values $U_1 = 1$ and $U_2 = 4$ such that in the formulation of Problem 2, $\alpha = 14$, $\beta = 20$ and $\alpha < \beta$. Therefore, the utility-weighted fairness formulation points out a possible unfairness situation while the count-based formulation does not detect unfairness.

\section*{C Additional Material for Section 5}

\subsection*{C.1 Statement and Proof of Theorem 1}

\textbf{Statement of Theorem 1:} The UDSC problem is \textbf{NP}-complete even for two classes.
**Proof:** It is easy to see that UDSC is in NP since given a subset \( P' \) of PSVs one can efficiently check that \( P' \) covers at most \( \alpha \) instances in \( T \) and at least \( \beta \) instances in each of the other classes.

To prove \( \mathsf{NP} \)-hardness, we use a reduction from the **Minimum Set Cover** (MSC) problem: given a universe \( U = \{u_1, u_2, \ldots, u_n\} \), a collection \( S = \{S_1, S_2, \ldots, S_m\} \), where each \( S_j \) is a subset of \( U \) (1 \( \leq \) \( j \) \( \leq \) \( m \)) and an integer \( r \leq m \), is there is a subcollection \( S' \) of \( S \) such that \( |S'| \leq r \) and the union of the sets in \( S' \) is equal to \( U \)? It is well known that MSC is \( \mathsf{NP} \)-complete even when \( r < n \) [9].

The reduction from MSC to UDSC is as follows. This reduction produces two classes, namely a target class \( T \) and a class \( \mathcal{O} \).

**Intuitive idea behind the reduction:** The target class \( T \) contains objects corresponding to the sets in the MSC problem. The other class \( \mathcal{O} \) contains objects corresponding to the universe in the MSC problem. Each set in the MSC problem also represents a PSV. The reduction specifies that the chosen combination of PSVs must cover at most \( r \) objects from \( T \) (to ensure the upper bound on the size of the solution to MSC) and all \( n \) objects in \( \mathcal{O} \) (to ensure that a set collection that covers all the elements of \( U \) can be obtained from the chosen PSV combination).

The details of the reduction are as follows.

1. The set of PSVs \( P = \{p_1, p_2, \ldots, p_m\} \) is in one-to-one correspondence with the collection \( S = \{S_1, S_2, \ldots, S_m\} \).
2. We set \( \beta = |U| = n \). The class \( \mathcal{O} = \{a_1, a_2, \ldots, a_n\} \) with \( n \) instances is in one-to-one correspondence with the universe \( U = \{u_1, u_2, \ldots, u_n\} \).
3. Suppose the element \( u_i, \ 1 \leq i \leq n, \) appears in subsets \( S_{i_1}, S_{i_2}, \ldots, S_{i_t} \) for some \( t \geq 1 \). Then, for the instance \( a_i \in \mathcal{O}, \ 1 \leq i \leq n, \) the PSVs \( p_{i_1}, p_{i_2}, \ldots, p_{i_t} \) have the value 1 and the remaining PSVs have value 0.
4. We set \( \alpha = r \) where \( r \) is the bound on the number of sets in the MSC instance. The target class \( T = \{b_1, b_2, \ldots, b_m\} \) has \( m \) instances. Since \( r < n \) in the MSC problem, we satisfy the constraint that \( \alpha < \beta \) in the UDSC problem.
5. For each instance \( b_j \in T, \ 1 \leq j \leq m \), the PSV \( p_j \) has the value 1 and the other PSVs have the value 0.

This completes our polynomial time reduction. We will now prove that there is a solution to the UDSC problem if there is a solution to the MSC problem.

Suppose \( S' = \{S_{j_1}, S_{j_2}, \ldots, S_{j_u}\} \), where \( \ell \leq r \), is a solution to the MSC problem. We first show that the subset \( P' = \{p_{j_1}, p_{j_2}, \ldots, p_{j_u}\} \) covers \( \beta = n \) instances in \( \mathcal{O} \). To see this, consider any instance \( a_i \in \mathcal{O} \). Since \( S' \) is a solution to MSC, there is a set \( S_{j_i} \in S' \) that covers the element \( u_i \in U \) corresponding to \( a_i \). By our construction, the PSV \( p_{j_i} \) has the value 1 for \( a_i \) and therefore \( P' \) covers \( a_i \). Further, \( P' \) covers \( \ell \leq r = \alpha \) instances in \( T \) since each PSV in \( P' \) covers exactly one instance in \( T \). Thus, \( P' \) is a solution to the UDSC problem.

Suppose \( P'' = \{p_{j_1}, p_{j_2}, \ldots, p_{j_u}\} \) is a solution to the UDSC problem. If \( |P'| = \ell > r = \alpha \), then again \( P'' \) would cover \( \alpha + 1 \) or more instances in \( T \). Therefore, \( |P'| = \ell \leq r = \alpha \). Let \( S' = \{S_{j_1}, S_{j_2}, \ldots, S_{j_u}\} \) be the subcollection of \( S \) constructed from \( P' \). To see that \( S' \) forms a solution to MSC, consider any...
element $u_i \in U$. Since $P'$ is a solution to UDSC, there is a PSV, say $p_{j_y} \in P'$, that covers $a_i \in \varnothing$, the instance corresponding to $u_i \in U$. By our construction of $S'$, the element $u_i$ is covered by the set $S_{j_y} \in S'$. Thus, $S'$ forms a solution to the MSC problem, and this completes our proof of Theorem 1.

\section*{D Additional Material for Section 6}

\subsection*{D.1 The Congressional Districts in California, USA}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{california_districts.png}
\caption{The 53 California congressional districts (the classes) and the 1700+ ZCTA (Zip Code Tabulated Areas) that comprise them (the instances).}
\end{figure}
### D.2 Fair Reading Sources

| Date          | Title                                                                 | URL                           |
|---------------|-----------------------------------------------------------------------|--------------------------------|
| Wed Apr 01    | Blood test for Down’s syndrome hailed                                | [http://bbc.in/1BO3eWQ](http://bbc.in/1BO3eWQ) [http://bbc.in/1ChTANp](http://bbc.in/1ChTANp) |
| Wed Apr 08    | New approach against HIV ‘promising’                                 | [http://bbc.in/1E6jAjt](http://bbc.in/1E6jAjt) |
| Thu Apr 09    | Breast cancer risk test devised                                       | [http://bbc.in/1CimpJF](http://bbc.in/1CimpJF) |
| Tue Apr 07    | Why strenuous runs may not be so bad after all                        | [http://bbc.in/1Ceq0Y7](http://bbc.in/1Ceq0Y7) |
| Mon Apr 06    | VIDEO: Health surcharge for non-EU patients                           | [http://bbc.in/1C5Mlbk](http://bbc.in/1C5Mlbk) |

**Table 4.** This table shows a few examples of articles on health from the BBC health care website. The first article would be tagged as being on handicapped, the third about women and the fifth about poverty. Note that the tagging is based on an article’s content and not its title.