A limit transition from the Heckman-Opdam hypergeometric functions to the Whittaker functions associated with root systems

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Abstract

We prove that the radial part of the class one Whittaker function on a split semisimple Lie group can be obtained as an appropriate limit of the Heckman-Opdam hypergeometric function.

Introduction

Among quantum integrable systems associated with root systems, there are three classes where there are well behaved joint eigenfunctions that are closely related with Lie theory. They are the trigonometric Calogero-Moser model, the rational CM model, and the Toda model. For the CM model, the eigenfunction is the Heckman-Opdam hypergeometric function and the Bessel function corresponding to trigonometric and rational cases respectively (cf. [12, 19, 20]). Among other eigenfunctions, they are up to constant multiples unique globally defined analytic functions. For special parameter they are the radial part of the spherical functions on a Riemannian symmetric space of the non-compact type and the Euclidean type respectively (cf. [14]). For the non-periodic Toda model, the eigenfunction is the class one Whittaker function defined by the Jacquet integral on a Riemannian symmetric space of the non-compact type (cf. [17, 11]). Among other eigenfunctions, it is up to a constant multiple unique eigenfunction of moderate growth.

On the other hand, there are two limit transitions between the Hamiltonians, one is from the trigonometric CM model to the rational CM model, and the other is from the trigonometric CM model to the Toda model (cf. [4, 16]). In rank one case, corresponding limit transitions are one from the Gauss hypergeometric function to the Bessel function,
and the other is from the Gauss hypergeometric function to the Macdonald function. In general, a limit transition for eigenfunctions in the former case was established by Ben Saïd-Ørsted [1] and de Jeu [4]. In this paper we establish a limit transition in the latter case. Namely we prove that a limit of the Heckman-Opdam hypergeometric function is the radial part of the Whittaker function on a split semisimple Lie group (Theorem 3). Similar result for functions on $Sp(2, \mathbb{R})$ was proved by Hirano-Ishii-Oda [15], which motivated the study of this paper.

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1 Preliminaries

1.1 The Heckman and Opdam hypergeometric function

In this subsection, we review on the Heckman-Opdam hypergeometric function associated with a root system. See [12] and [20] for details.

Let $\mathfrak{a}$ be a Euclidean space of dimension $n$. For $\alpha \in \mathfrak{a}^* \setminus \{0\}$ define

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}.$$ 

Let $R$ denote a reduced root system in $\mathfrak{a}^*$. Choose a positive system $R_+ \subset R$ and let $B$ denote the set of the simple roots. Let $W$ denote the Weyl group for $R$. For $\alpha \in R$ let $k_\alpha$ be a non-negative number such that $k_{w\alpha} = k_\alpha$ for all $w \in W$. We call $k : \alpha \mapsto k_\alpha$ a multiplicity function. We put

$$\rho(k) = \frac{1}{2} \sum_{\alpha \in R_+} k_\alpha \alpha.$$ 

We often identify $\mathfrak{a}^*$ with $\mathfrak{a}$.

Let $A = \exp \mathfrak{a}$ and

$$A_+ = \{ a \in A : \alpha(\log a) > 0 \text{ for all } \alpha \in R_+ \}.$$ 

Let $\{\xi_1, \ldots, \xi_n\}$ be an orthonormal basis of $\mathfrak{a}$. Define

$$L(k) = \sum_{i=1}^{n} \partial^2_{\xi_i} + \sum_{\alpha \in R_+} k_\alpha \frac{1 + e^{-\alpha}}{1 - e^{-\alpha}} \partial_\alpha.$$ 

(1)
There exist a commutative algebra \( \mathbb{D}(k) \) of differential operators containing \( L(k) \) and an isomorphism \( \gamma : \mathbb{D}(k) \to S(a)^W \).

Let \( Q \) be the \( \mathbb{Z} \)-span of \( R \) and \( Q_+ \) be the \( \mathbb{Z}_+ \)-span of \( R_+ \). There exists a solution \( \Phi(\lambda, k; a) \) for
\[
D \varphi = \gamma(D)(\lambda) \varphi
\]
of the form
\[
\Phi(\lambda, k; a) = \sum_{\mu \in Q_+} \Gamma_\mu(\lambda, k)e^{(\lambda - \rho(k) - \mu)(\log a)}, \quad \Gamma_0(\lambda, k) = 0.
\]
The coefficients \( \Gamma_\mu(\lambda, k) \) are determined by recurrence relations coming from \( L(k) \).

If \( \lambda \in a^*_C \) satisfies condition,
\[
(2\lambda + \mu, \mu) \neq 0 \text{ for all } \mu \in Q \setminus \{0\},
\]
then \( \{\Phi(w\lambda, k; a) : w \in W\} \) forms a basis of solution space of (2) on \( A_+ \).

Define
\[
\tilde{c}(\lambda, k) = \prod_{\alpha \in R_+} \frac{\Gamma((\lambda, \alpha^\vee))}{\Gamma((\lambda, \alpha^\vee) + k_\alpha)}
\]
and
\[
c(\lambda, k) = \frac{\tilde{c}(\lambda, k)}{\tilde{c}(\rho(k), k)}.
\]
Define
\[
F(\lambda, k; a) = \sum_{w \in W} c(w\lambda, k) \Phi(w\lambda, k; a).
\]
The function \( F \) is called the Heckman-Opdam hypergeometric function for the root system \( R \). It is well behaved compared to \( \Phi \).

**Theorem 1 (Heckman-Opdam)** \( F(\lambda, k; a) \) is a unique \( W \)-invariant solution for (2) that is analytic in \( a \in A \), holomorphic in \( \lambda \in a^*_C \), and
\[
F(w\lambda, k; a) = F(\lambda, k; a) \quad (w \in W),
\]
\[
F(\lambda, k; wa) = F(\lambda, k; a) \quad (w \in W).
\]

### 1.2 Notation on Lie groups

Let \( G \) be a normal real form of a complex semisimple Lie group and \( K \) a maximal compact subgroup. Let \( \mathfrak{g} \) and \( \mathfrak{k} \) be the Lie algebras of \( G \) and \( K \) respectively. Let \( \theta \) denote the corresponding Cartan involution of \( \mathfrak{g} \) and \( \mathfrak{g} = \mathfrak{k} + \mathfrak{p} \) be the decomposition into \( \pm 1 \) eigenspaces of \( \theta \). Equip the inner product \( ( , ) \) on \( \mathfrak{g} \) given by \( (X, Y) = -B(X, \theta Y) \) \((X, Y \in \mathfrak{g})\) where \( B( , ) \) is the Killing form on \( \mathfrak{g} \).
Fix a maximal abelian subspace $a$ of $p$. Let $\Sigma = \Sigma(\mathfrak{g}, a)$ denote the set of the restricted roots. Fix a positive system $\Sigma_+$ and let $\Pi$ denote the set of the simple roots. Notice that each root space has dimension 1, because we assume that $G$ is split. Put $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma_+} \alpha$. Let $W$ denote the Weyl group of $\Sigma$. It is isomorphic to $N_K(a)/Z_K(a)$, where $N_K(a)$ (resp. $Z_K(a)$) is the normalizer (resp. centralizer) of $a$ in $K$.

Let $n$ be the sum of root spaces for the positive roots. Put $N = \exp n$ and $A = \exp a$. Then we have the Iwasawa decomposition $G = NAK = KAN$.

**Remark 1** In the previous section, we adopt notation of Heckman and Opdam. Relations between notation in the previous section and this section are given by

\[ R = 2\Sigma, \quad R_+ = 2\Sigma_+, \quad B = 2\Pi, \quad k_{2\beta} = \frac{1}{2} m_\beta = \frac{1}{2} (\beta \in \Sigma). \]  

$L(k)$ is the radial part of the Laplace-Beltrami operator $L_{G/K}$ on $G/K$, $\mathbb{D}(k)$ consists of the radial parts of invariant differential operators on $G/K$ with respect to the Cartan decomposition $G = KAK$, $c(\lambda, k)$ is the Harish-Chandra $c$-function, and the hypergeometric function $F(\lambda, k; a)$ is the radial part of the spherical function on $G/K$.

### 1.3 The Whittaker function on a semisimple Lie group

In this subsection, we review on the class one Whittaker functions on a split semisimple Lie group following Hashizume [11].

Let $\psi$ be a unitary character of $N$. We denote the differential character of $n$ to $\sqrt{-1}\mathbb{R}$ by the same letter $\psi$. Let $C^\infty_\psi(G/K)$ denote the space of $C^\infty$-functions on $G$ satisfying $u(ngk) = \psi(n)u(g)$ for all $n \in N$, $g \in G$, and $k \in K$. By the Iwasawa decomposition the values of $u \in C^\infty_\psi(G/K)$ are completely determined by $u|_A$. Let $\mathbb{D}(G/K)$ denote the commutative algebra of left $G$-invariant differential operators on $G/K$ and $\chi_\lambda : \mathbb{D}(G/K) \to \mathbb{C}$ the Harish-Chandra homomorphism. Let $\mathcal{A}_\psi(G/K, \mathcal{M}_\lambda)$ be the subspace of $C^\infty_\psi(G/K)$ defined by

\[ \mathcal{A}_\psi(G/K, \mathcal{M}_\lambda) = \{ u \in C^\infty_\psi(G/K) : Du = \chi_\lambda(D)u \text{ for all } D \in \mathbb{D}(G/K) \}. \]

Notice that $C^\infty_\psi(G/K, \mathcal{M}_\lambda)$ consists of real analytic functions, because $L_{G/K}$ is an elliptic differential operator.

For $\beta \in \Pi$ let $X_\beta \in \mathfrak{g}_\beta$ be a unit root vector. For $\alpha \in B = 2\Pi$ put $l_\alpha = -\sqrt{-1}\psi(X_{\alpha/2})$. For $u \in \mathcal{A}_\psi(G/K, \mathcal{M}_\lambda)$, $\varphi = e^{-\rho}u|_A$ satisfies

\[ \left( \sum_{i=1}^n \frac{\partial^2}{\xi_i^2} - 2 \sum_{\alpha \in B} l_\alpha^2 e^\alpha \right) \varphi = (\lambda, \lambda)\varphi. \]
There exists a solution $\Psi_T(\lambda, \psi, a)$ for equation (7) of the form

$$
\Psi_T(\lambda, \psi, a) = a^\lambda \sum_{\mu \in \mathcal{Q}^+} b_\mu(\lambda) a^\mu, \quad b_0(\lambda) = 0.
$$

(8)

Moreover, extending function $u(a) = e^a \Psi_T(\lambda, \psi, a)$ on $A$ to $G$ so that $u \in C_\psi^\infty(G/K)$, it is also a joint eigenfunction of $D(G/K)$ and belongs to $\mathcal{A}_\psi(G/K, \mathcal{M}_\lambda)$. If $\lambda \in a_C^*$ satisfies condition (4), then $\{e^a \Psi_T(w\lambda, \psi, a) : w \in W\}$ forms a basis of $\mathcal{A}_\psi(G/K, \mathcal{M}_\lambda)|_A$ (cf. [11, Corollary 5.3, Theorem 5.4]).

For $\lambda \in a_C^*$ define function $1_\lambda$ on $G$ by

$$
1_\lambda(nak) = a^{\lambda + \rho} \quad (n \in N, a \in A, k \in K).
$$

For $g \in G$ let $H(g)$ denote the element of $\mathfrak{a}$ defined by $g \in K \exp H(g)N$. We normalize the Haar measure $dn$ and $\bar{d}n$ on $N$ and $\bar{N} = \theta N$ by

$$
\theta(dn) = d\bar{n}, \quad \int_{\bar{N}} e^{-2\rho(H(n))} d\bar{n} = 1,
$$

(cf. [14, Ch. IV, §6]). Define

$$
W(\lambda, \psi; g) = \int_N 1_\lambda(\bar{w}_0^{-1}ng)\psi(n)^{-1}dn.
$$

(9)

Here $\bar{w}_0$ is a representative in $N_K(\mathfrak{a})$ of the longest element $w_0 \in W$. For $\lambda \in a_C^*$ with $\text{Re} \lambda > 0$ ($\forall \alpha \in \Sigma_+$) the class one Jacquet integral $W(\lambda, \psi; g)$ converges absolutely and uniformly and belongs to $\mathcal{A}_\psi(G/K, \mathcal{M}_\lambda)$. Moreover $W(\lambda, \psi; g)$ can be continued to a meromorphic function of $\lambda \in a_C^*$ as an element of $\mathcal{A}_\psi(G/K, \mathcal{M}_\lambda)$ (cf. [11, Theorem 6.6]). We call this meromorphic continuation of $W(\lambda, \psi; a)$ the Whittaker function.

The Whittaker function $W(\lambda, \psi; g)$ is up to a constant multiple a unique element of $\mathcal{A}_\psi(G/K, \mathcal{M}_\lambda)$ that is of moderate growth (cf. [3, Theorem 9.1]).

The analytic properties of the integral (9) were studied by Jacquet [17], Schiffmann [21], Goodman-Wallach [7], and Hashizume [10, 11], etc. $W(\lambda, \psi; g)$ satisfies the following functional equation:

$$
W(\lambda, \psi; g) = M(w, \lambda, \psi)W(w\lambda, \psi; g) \quad (w \in W).
$$

(10)

Here the function $M(w, \lambda, \psi)$ is given by the product formula

$$
M(ww', \lambda, \psi) = M(w', \lambda, \psi)M(w, w'\lambda, \psi) \quad (w, w' \in W),
$$

(11)

$$
M(s_\alpha, \lambda, \psi) = \left( \frac{2l_\alpha^2}{(\alpha, \alpha)} \right)^{(\lambda, \alpha^\vee)} \frac{\Gamma(-(\lambda, \alpha^\vee) + 1/2)}{\Gamma((\lambda, \alpha^\vee) + 1/2)} \quad (\alpha \in B),
$$

(12)

where $s_\alpha$ denote the simple reflection corresponding to $\alpha \in B$. (cf. [11, (7.5)–(7.7)]. Notice again that $R = 2\Sigma$ and $G$ is split.)
Let $c(\lambda)$ denote the Harish-Chandra $c$-function for the split Lie group $G$, which is given by $c(\lambda) = c(\lambda, k)$ with $k_\alpha = 1/2$ for all $\alpha \in R$. That is

\[
c(\lambda) = \frac{\tilde{c}(\lambda)}{\tilde{c}(\rho)}, \quad \tilde{c}(\lambda) = \prod_{\alpha \in R^+} \frac{\Gamma((\lambda, \alpha^\vee))}{\Gamma((\lambda, \alpha^\vee) + \frac{1}{2})}.
\]

Hashizume [11, Theorem 7.8] expressed the Whittaker function $W(\lambda, \psi; a)$ as a linear combination of $\Psi_T(w\lambda, \psi; a)$ explicitly.

**Theorem 2 (Hashizume)** Let $\psi$ be a non-degenerate character of $N$ and assume that $\lambda \in a^*_C$ satisfies (4). Then

\[
W(\lambda, \psi; a) = a^{\rho} \sum_{w \in W} M(w_0w, \lambda, \psi)c(w_0w\lambda)\Psi_T(w\lambda, \psi; a) \quad (a \in A_+).
\]

(13)

2. Limit transition from the Heckman-Opdam hypergeometric function to the Whittaker function

2.1 Limit transition from the Calogero-Moser Hamiltonian to the Toda Hamiltonian

In this subsection, we review on the limit transition from the quantum trigonometric Calogero-Moser model to the Toda model.

Define a function $\delta(k) = \delta(k; a)$ by

\[
\delta(k)^{1/2} = \prod_{\alpha \in R^+} (e^{\frac{1}{2} k_\alpha} - e^{-\frac{1}{2} k_\alpha})^{k_\alpha}.
\]

We have

\[
\delta(k)^{1/2} \circ \{L(k) + (\rho(k), \rho(k))\} \circ \delta(k)^{-1/2} = \sum_{i=1}^n \delta^2 \xi_i + \sum_{\alpha \in R^+} \frac{k_\alpha(1 - k_\alpha)(\alpha, \alpha)}{4 \sinh^2 \frac{1}{2} \alpha}
\]

(14)

We denote the right hand side of (14) by $H_{CM}(k)$. It is the Hamiltonian for the trigonometric Calogero-Moser model.

Recall that $R = 2\Sigma$ (Remark 1) and let $B = 2\Pi$ be the simple system of $R_+ = 2\Sigma_+$. We assume that $l_\alpha = 1$ for all $\alpha \in B$. That is we assume that $\psi$ is a special non-degenerate
unitary character of $N$. The left hand side of (7) gives the Hamiltonian for the quantum Toda model

$$H_T = \sum_{i=1}^{n} \frac{\partial^2 {\xi}_i}{\partial {\xi}_i^2} - 2 \sum_{\alpha \in B} e^\alpha. \quad (15)$$

Let $M$ be a positive real number. Define a positive multiplicity function $k_M$ by

$$k_M(\alpha)(k_M(\alpha) - 1) (\alpha, \alpha) = 2e^{2M}$$

and define $a_M \in A$ by

$$\log a_M = w_0 \log a + M \rho^\vee,$$

where $w_0$ is the longest element of $W$. Notice that

$$\rho^\vee = \frac{1}{2} \sum_{\beta \in \Sigma^+} \beta^\vee = \sum_{\alpha \in R^+} \alpha^\vee$$

is the Weyl vector of $\Sigma^\vee = 2R^\vee$ and $(\alpha, \rho^\vee) = 1$ for all $\alpha \in \Pi = 1/2 B$ (cf. Bourbaki [2, Ch VI Proposition 29]).

We shall consider limits of the hypergeometric function when $M \to \infty$. Taking a limit of $H_{CM}(k)$, we have the following lemma.

**Lemma 1** For any $\varphi \in C^\infty(A)$,

$$\lim_{M \to \infty} H_{CM}(k_M) \varphi(a_M) = H_T \varphi(a). \quad (16)$$

This limit procedure was proved by Inozemtsev [16] (see also [5, Section 7] and [18]).

### 2.2 Limit transition of eigenfunctions

Define

$$\Psi_{CM}(\lambda, k; a) = \delta(k; a)^{1/2} \Phi(\lambda, k; a).$$

By (3) and (14), $\varphi(a) = \Psi_{CM}(\lambda, k; a)$ is of the form

$$\Psi_{CM}(\lambda, k; a) = \sum_{\mu \in \Lambda} b_\mu(\lambda, k)e^{(\lambda-\mu)(\log a)}, \quad b_0(\lambda, k) = 1 \quad (17)$$

and it is a solution of

$$H_{CM}(k) \varphi = (\lambda, \lambda) \varphi. \quad (18)$$

On the other hand, as we have seen in subsection 1.3, there is a series solution $\varphi(a) = \Psi_T(\lambda; a)$ of

$$H_T \varphi = (\lambda, \lambda) \varphi.$$
Proposition 1 If $\lambda \in a_C^*$ satisfies condition (4), then
\[
\lim_{M \to \infty} e^{-(\lambda, \rho')M} \Psi_{CM}(\lambda, k; a_M) = \Psi_T(w_0\lambda; a) \quad (a \in A_+).
\] (19)
The convergence is uniform on each subchamber
\[
\{a \in A_+ : \alpha(\log a) > c > 0 (\alpha \in B)\},
\]
where $c > 0$ is arbitrary.

proof. The proof is an easy modification of the estimate of the Harish-Chandra series due to Gangolli [6] (see also Helgason [14, Ch IV §5]).

Substituting $k_M$ and $a_M$ to (17) and (18), we have
\[
\Psi_{CM}(\lambda, k; a_M) = e^{(\lambda, \rho')M} \sum_{\mu \in Q^+} \tilde{b}_\mu(\lambda, M)e^{(w_0\lambda + \mu)(\log a)}, \quad \tilde{b}_0(\lambda, M) = 1
\] (20)
and
\[
\left(\sum_{i=1}^n \partial_{\xi_i}^2 - 2 \sum_{\alpha \in R^+} e^{2M} \sum_{j=1}^{\infty} j e^{-j((\alpha, \rho')M + w_0\alpha(\log a))}\right)\Psi_{CM}(\lambda, k_M; a_M)
\]
\[
= (\lambda, \lambda)\Psi_{CM}(\lambda, k_M; a_M)
\] (21)
Substituting (20) to (21) gives the recurrence relation
\[
(2w_0\lambda + \mu, \mu)\tilde{b}_\mu(\lambda, M)
\]
\[
= 2 \sum_{\alpha \in R^+} \sum_{j \geq 1, \mu + jw_0\alpha \in Q^+} e^{(2- j(\alpha, \rho')M + w_0\alpha(\log a))} \tilde{b}_{\mu + jw_0\alpha}(\lambda, M).
\] (22)
Since $(\alpha, \rho') = 2$ for $\alpha \in B = 2\Pi$ and $(\alpha, \rho') \geq 4$ for $\alpha \in R^+ \setminus B$, recurrence relation (22) converges to
\[
(2w_0\lambda + \mu, \mu)\tilde{b}_\mu(\lambda, \infty) = 2 \sum_{\alpha \in B} \tilde{b}_{\mu - \alpha}(\lambda, \infty)
\] (23)
as $M \to \infty$. (23) is nothing but the recurrence relation for the coefficients in expansion (8) for $\Psi_T(w_0\lambda; a)$ (cf. [11, §4]).

For $\mu \in Q^+$ we write $\mu = \sum_{\alpha \in B} n_\alpha \alpha$ and put $n(\mu) = \sum_{\alpha \in B} n_\alpha$. Choose a constant $c$ such that
\[
|2(w_0\lambda + \mu, \mu)| \geq c n(\mu)
\]
for all $\mu \in Q^+$. By (22) we have
\[
|\tilde{b}_\mu(\lambda, M)| \leq 2c^{-1} \sum_{\alpha \in R^+} \sum_{j \geq 1, \mu + jw_0\alpha \in Q^+} e^{(2- j(\alpha, \rho')M)j} |\tilde{b}_{\mu + jw_0\alpha}(\lambda, M)|
\]
\[
\leq 2c^{-1} \sum_{\alpha \in R^+} \sum_{j \geq 1, \mu + jw_0\alpha \in Q^+} j |\tilde{b}_{\mu + jw_0\alpha}(\lambda, M)|
\]
\[
\leq \frac{2c^{-1}}{1 - \mu \rho'} \sum_{\alpha \in R^+} \sum_{j \geq 1} j |\tilde{b}_{\mu + jw_0\alpha}(\lambda, M)|
\]
\[
\leq \frac{2c^{-1}}{1 - \mu \rho'} \sum_{\alpha \in R^+} \sum_{j \geq 1} j |\tilde{b}_{\mu + jw_0\alpha}(\lambda, M)|
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\leq \frac{2c^{-1}}{1 - \mu \rho'} \sum_{\alpha \in R^+} \sum_{j \geq 1} j |\tilde{b}_{\mu + jw_0\alpha}(\lambda, M)|
\]
\[
\leq \frac{2c^{-1}}{1 - \mu \rho'} \sum_{\alpha \in R^+} \sum_{j \geq 1} j |\tilde{b}_{\mu + jw_0\alpha}(\lambda, M)|
\]
for $M > 0$. We can prove in the same way as the proof of [14, Ch IV Lemma 5.3, Lemma 5.6] that there exists a constant $K_{a,t}$ such that

$$|\tilde{b}_\mu(\lambda, M)| \leq K_{\lambda,a} a^\mu$$

for all $\mu \in \Lambda$. This estimate shows the convergence of the series (20) and also guarantees the limit transition (19).

Now we state and prove our main result:

**Theorem 3** Assume that $\lambda \in a_+^\ast$ satisfies (4). Then

$$\lim_{M \to \infty} \delta(k; a_M)^{1/2} \tilde{c}(\rho(k), k) \prod_{\alpha \in R_+} \Gamma(k_M(\alpha)) F(\lambda, k_M; a_M)$$

$$= \tilde{c}(\rho) f(\lambda) a^{-\rho} W(\lambda, \psi; a),$$

(24)

where

$$f(\lambda) = \prod_{\alpha \in R_+} \left( \frac{(\alpha, \alpha)}{2} \right)^{-(\lambda, \alpha^\vee)/2} \Gamma((\lambda, \alpha^\vee) + \frac{1}{2})$$

(25)

and $\psi$ is a unitary character of $N$ defined by $l_\alpha = 1 (\alpha \in B)$.

**proof.** By the following formula for the Gamma function

$$\lim_{x \to \infty} \frac{\Gamma(\mu + x)}{\Gamma(x) x^\mu} = 1,$$

we have

$$\tilde{c}(\lambda, k_M) \sim f(\lambda) \tilde{c}(\lambda) \prod_{\alpha \in R_+} \frac{e^{-(\lambda, \alpha^\vee)/2}}{\Gamma(k_M(\alpha))}$$

(26)

as $M \to \infty$. By (5), Proposition 1, and (26), we have

$$\lim_{M \to \infty} \delta(k; a_M)^{1/2} \tilde{c}(\rho(k), k) \prod_{\alpha \in R_+} \Gamma(k_M(\alpha)) F(\lambda, k_M; a_M)$$

$$= \sum_{w \in W} f(w\lambda) \tilde{c}(w\lambda) \Psi_T(w_0w\lambda; a).$$

(27)

On the other hand, by Theorem 2, the right hand side of (24) is a linear combination of $\Psi_T(w\lambda; a) (w \in W)$, where the coefficient of $\Psi_T(w\lambda; a)$ is given by

$$d(w, \lambda) := f(\lambda) M(w_0w, \lambda, \psi) \tilde{c}(w_0w\lambda).$$

(28)

For $\beta \in R$, it follows from (11) and (12) that

$$d(w, s_\beta \lambda) = f(s_\beta \lambda) M(w_0w, s_\beta \lambda, \psi) \tilde{c}(w_0ws_\beta \lambda)$$

$$= f(s_\beta \lambda) M(s_\beta, \lambda, \psi)^{-1} M(w_0ws_\beta, \lambda, \psi) \tilde{c}(w_0ws_\beta \lambda)$$

$$= d(ws_\beta, \lambda).$$
The last equality follows from $s_\beta(B \setminus \{\beta\}) = B \setminus \{\beta\}$. Thus the right hand side of (24) is $W$-invariant with respect to $\lambda$. We have $d(w_0, \lambda) = f(\lambda)\tilde{c}(\lambda)$ and it coincides with the coefficient of $\Psi_T(\lambda, l; a)$ in the right hand side of (27), hence the result follows. \hfill \Box

**Example 1** For $R$ of type $A_1$

$$F(\lambda, k; a_t) = 2F_1(\frac{1}{2}(k - \lambda), \frac{1}{2}(k + \lambda); k + \frac{1}{2}; -\sinh^2 t),$$

where $2F_1(a, b, c; z)$ is the Gauss hypergeometric function. Theorem 3 reads

$$\lim_{k \to \infty} k^{-1/2} 2^{-k} \sinh^k(-t + M) F(\lambda, k; a_{-t+M}) = \frac{1}{\sqrt{\pi}} K_\lambda(e^t/2),$$

where $K_\lambda(z)$ is the Macdonald function.

**Remark 2** We restrict ourselves to split semisimple Lie groups, because the Hamiltonian (15) of the Toda lattice depends only on reducible root system. Class one Whittaker function given by the Jacquet integral for a non-split semisimple Lie group is a constant multiple of the Whittaker function for the split Lie group of the same indivisible restricted roots.

We can change parameters $l^2_\alpha (\alpha \in B)$ in the left hand side of (7) by making a shift of variables, as it was pointed out by [18, §2.1]. Let $\varpi_\alpha (\alpha \in B)$ denote the fundamental weights corresponding to $B$. If we put

$$\log a = \log a' - \sum_{\alpha \in B, \ l_\alpha \neq 0} \frac{2}{(\alpha, \alpha)} \varpi_\alpha \log l^2_\alpha,$$

then

$$\alpha(\log a) = \begin{cases} 
\alpha(\log a') - \log l^2_\alpha & (\alpha \in B, \ l_\alpha \neq 0) \\
\alpha(\log a') & (\alpha \in B, \ l_\alpha = 0)
\end{cases}$$

Thus in the new coordinates $\sum_{i=1}^n \partial^2_{\xi_i} - 2 \sum_{\alpha \in B} l^2_\alpha e^\alpha$ becomes $\sum_{i=1}^n \partial^2_{\xi_i} - 2 \sum_{\alpha \in B, \ l_\alpha \neq 0} e^\alpha$. Moreover, if $l_\alpha = 0$ for some $\alpha$, then the Whittaker function can be reduced to lower rank cases. This is the reason why we assume $l_\alpha = 1$ for all $\alpha \in B$.

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