On the satisfiability problem for a 4-level quantified syllogistic and some applications to modal logic (extended version)

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Abstract. We introduce a multi-sorted stratified syllogistic, called $4LQS^R$, admitting variables of four sorts and a restricted form of quantification over variables of the first three sorts, and prove that it has a solvable satisfiability problem by showing that it enjoys a small model property. Then, we consider the fragments $(4LQS^R)^h$ of $4LQS^R$, consisting of $4LQS^R$-formulae whose quantifier prefixes have length bounded by $h \geq 2$ and satisfying certain syntactic constraints, and prove that each of them has an NP-complete satisfiability problem. Finally we show that the modal logic K45 can be expressed in $(4LQS^R)^3$.

1. Introduction

Most of the decidability results in computable set theory concern one-sorted multi-level syllogistics, namely collections of formulae admitting variables of one sort only, which range over the von Neumann universe of sets (see [8,10] for a thorough account of the state-of-art until 2001). Only a few stratified syllogistics, where variables of different sorts are allowed, have been investigated, despite the fact that in many fields of computer science and mathematics often one has to deal with multi-sorted languages.\footnote{The locutions ‘multi-level syllogistic’ and ‘stratified syllogistic’ were chosen by Jack Schwartz to name many decidable fragments of computable set theory because he saw them as generalizations of Aristotelian syllogistics.}
For instance, in modal logics, one has to consider entities of different types, namely worlds, formulae, and accessibility relations.

In [13] an efficient decision procedure was presented for the satisfiability of the Two-Level Syllogistic language ($2LS$). $2LS$ has variables of two sorts and admits propositional connectives together with the basic set-theoretic operators $\cup$, $\cap$, and $\setminus$. Then, in [4], it was shown that the extension of $2LS$ with the singleton operator and the Cartesian product operator is decidable. Tarski’s and Presburger’s arithmetics extended with sets have been analyzed in [6]. Subsequently, in [5], a three-sorted language $3LSSPU$ (Three-Level Syllogistic with Singleton, Powerset and general Union) has been proved decidable. Recently, in [9], it was shown that the language $3LQS^R$ (Three-Level Quantified Syllogistic with Restricted quantifiers) has a decidable satisfiability problem. $3LQS^R$ admits variables of three sorts and a restricted form of quantification. Its vocabulary contains only the predicate symbols $=$ and $\in$. In spite of that, $3LQS^R$ allows one to express several constructs of set theory. Among them, the most comprehensive one is the set-formation operator, which in turn enables one to express other operators like the powerset operator, the singleton operator, and so on. In [9] it is also shown that the modal logic $S5$ can be expressed in a fragment of $3LQS^R$, whose satisfiability problem is NP-complete.

In this paper we present a decidability result for the satisfiability problem of the set-theoretic language $4LQS^R$ (Four-Level Quantified Syllogistic with Restricted quantifiers). $4LQS^R$ is an extension of $3LQS^R$ admitting variables of four sorts and a restricted form of quantification over variables of the first three sorts. In addition to the predicate symbols $=$ and $\in$, its vocabulary contains also the pairing operator $\langle \cdot, \cdot \rangle$.

We will prove that the theory $4LQS^R$ enjoys a small model property by showing how one can extract, out of a given model satisfying a $4LQS^R$-formula $\psi$, another model of $\psi$ but of bounded finite cardinality. The construction of the finite model extends the decision algorithm described in [9]. Concerning complexity issues, we will show that the satisfiability problem for each of the fragments $(4LQS^R)^h$ of $4LQS^R$, whose formulae are restricted to have their quantifier prefixes of length at most $h \geq 2$ and must satisfy certain additional syntactic constraints to be seen later, is NP-complete.

In addition to the modal logic $S5$, already expressible in the language $3LQS^R$, it turns out that in $4LQS^R$ one can also formalize several properties of binary relations (needed to define accessibility relations of well-known modal logics) and some Boolean operations over relations and the inverse operation over binary relations. We will also show that the modal logic $K45$ can be formalized in the fragment $(4LQS^R)^3$. As is well-known, the satisfiability problem for $K45$ is NP-complete; thus our alternative decision procedure for $K45$ can be considered optimal in terms of its computational complexity.

### 2. The language $4LQS^R$

Before defining the language $4LQS^R$ of our interest, it is convenient to present the syntax and the semantics of a more general, unrestricted four-level quantified fragment, denoted $4LQS$. Subsequently, we will introduce suitable restrictions over the formulae of $4LQS$ to characterize the sublanguage $4LQS^R$.

#### 2.1. The unrestricted language $4LQS$

**Syntax of $4LQS$.** The four-level quantified language $4LQS$ involves the four collections $\mathcal{V}_0$, $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{V}_3$ of variables. Each $\mathcal{V}_i$ contains variables of sort $i$, denoted by $X^i, Y^i, Z^i, \ldots$. When we refer to
variables of sort 0 we prefer to write \( x, y, z, \ldots \) instead of \( X^0, Y^0, Z^0, \ldots \). In addition to the variables in \( \mathcal{V}_2 \), terms of sort 2 include also pair terms of the form \( \langle x, y \rangle \), for \( x, y \in \mathcal{V}_0 \).

4LQS quantifier-free atomic formulae are classified as:

**level 0:** \( x = y, \ x \in X^1 \), for \( x, y \in \mathcal{V}_0, X^1 \in \mathcal{V}_1 \);

**level 1:** \( X^1 = Y^1, \ X^1 \in X^2 \), for \( X^1, Y^1 \in \mathcal{V}_1, X^2 \in \mathcal{V}_2 \);

**level 2:** \( T^2 = U^2, \ T^2 \in X^3 \), where \( T^2 \) and \( U^2 \) are terms of sort 2 and \( X^3 \in \mathcal{V}_3 \).

4LQS purely universal formulae are classified as:

**level 1:** \( (\forall z_1) \ldots (\forall z_n) \varphi_0 \), where \( \varphi_0 \) is any propositional combination of quantifier-free atomic formulae and \( z_1, \ldots, z_n \) are variables of sort 0;

**level 2:** \( (\forall Z_1^1) \ldots (\forall Z_m^1) \varphi_1 \), where \( \varphi_1 \) is any propositional combination of quantifier-free atomic formulae and of purely universal formulae of level 1, and \( Z_1^1, \ldots, Z_m^1 \in \mathcal{V}_1 \);

**level 3:** \( (\forall Z_1^2) \ldots (\forall Z_p^2) \varphi_2 \), where \( \varphi_2 \) is any propositional combination of quantifier-free atomic formulae and of purely universal formulae of levels 1 and 2, and \( Z_1^2, \ldots, Z_p^2 \in \mathcal{V}_2 \).

Finally, the formulae of 4LQS are all the propositional combinations of quantifier-free atomic formulae of levels 0, 1, 2, and of purely universal formulae of levels 1, 2, 3.

Next we introduce some notions that will be useful in the rest of the paper. Let \( \varphi \) be a 4LQS-formula. We can assume, without loss of generality, that \( \varphi \) contains as propositional connectives only ‘\( \neg \)’, ‘\( \vee \)’, and ‘\( \wedge \)’. Further, let \( S_\varphi \) be the syntax tree for \( \varphi \) (see \[12\] for a precise definition), and let \( \nu \) be a node of \( S_\varphi \). We say that a 4LQS-formula \( \psi \) occurs within \( \varphi \) at position \( \nu \) if the subtree of \( S_\varphi \) rooted at \( \nu \) is identical to \( S_\psi \). In this case we refer to \( \nu \) as an occurrence of \( \psi \) in \( \varphi \) and to the path from the root of \( S_\varphi \) to \( \nu \) as its occurrence path. An occurrence of a 4LQS-formula \( \psi \) within a 4LQS-formula \( \varphi \) is positive if its occurrence path deprived of its last node contains an even number of nodes labelled by a 4LQS-formula of type \( \neg \chi \). Otherwise, the occurrence is said to be negative.

**Semantics of 4LQS.** A 4LQS-interpretation is a pair \( \mathcal{M} = (D, M) \), where \( D \) is any nonempty collection of objects, called the domain or universe of \( \mathcal{M} \), and \( M \) is an assignment over the variables of 4LQS such that

- \( Mx \in D \), for each \( x \in \mathcal{V}_0 \);
- \( MX^1 \in \text{pow}(D) \), for each \( X^1 \in \mathcal{V}_1 \);
- \( MX^2 \in \text{pow}([\text{pow}(D)]) \), for each \( X^2 \in \mathcal{V}_2 \);
- \( MX^3 \in \text{pow}(\text{pow}(\text{pow}(D))) \), for each \( X^3 \in \mathcal{V}_3 \)

\[12\] We recall that, for any set \( s \), \( \text{pow}(s) \) denotes the powerset of \( s \), i.e., the collection of all subsets of \( s \).
We assume that pair terms are interpreted *à la* Kuratowski, and therefore we put
\[ M(x, y) =_{\text{def}} \{ \{ Mx \}, \{ Mx, My \} \}. \]

The introduction of a pairing operator in the language turned out to be very useful in view of the applications in Section 4. Moreover, even if many pairing operations are available (see for instance [14]), Kuratowski’s style of encoding ordered pairs results to be quite simple, at least for our purposes.

Let
- \( M = (D, M) \) be a 4LQS-interpretation,
- \( x_1, \ldots, x_n \in \mathcal{V}_0 \)
- \( X_1^1, \ldots, X_m^1 \in \mathcal{V}_1 \)
- \( X_1^2, \ldots, X_p^2 \in \mathcal{V}_2 \)
- \( u_1, \ldots, u_n \in D \)
- \( U_1^1, \ldots, U_1^m \in \text{pow}(D) \)
- \( U_1^2, \ldots, U_1^p \in \text{pow}(\text{pow}(D)) \)

By \( M[x_1/u_1, \ldots, x_n/u_n, X_1^1/U_1^1, \ldots, X_m^1/U_1^m, X_1^2/U_1^2, \ldots, X_p^2/U_1^p] \), we denote the interpretation \( M' = (D, M') \) such that \( M' x_i = u_i \) for \( i = 1, \ldots, n \), \( M' X_k^1 = U_k^1 \) for \( j = 1, \ldots, m \), \( M' X_k^2 = U_k^2 \) for \( k = 1, \ldots, p \), and which otherwise coincides with \( M \) on all remaining variables. Throughout the paper we use the abbreviations: \( M^z \) for \( M[z_1/u_1, \ldots, z_n/u_n] \), \( M^{Z_1} \) for \( M[Z_1^1/U_1^1, \ldots, Z_m^1/U_m^1] \), and \( M^{Z_2} \) for \( M[Z_1^2/U_1^2, \ldots, Z_p^2/U_p^2] \).

Let \( \varphi \) be a 4LQS-formula and let \( M = (D, M) \) be a 4LQS-interpretation. The notion of satisfiability of \( \varphi \) by \( M \) (denoted by \( M \models \varphi \)) is defined inductively over the structure of \( \varphi \). Quantifier-free atomic formulae are interpreted in the standard way according to the usual meaning of the predicates ‘\( = \)’ and ‘\( \in \)’, and purely universal formulae are evaluated as follows:

1. \( M \models (\forall z_1) \ldots (\forall z_n) \varphi_0 \) iff \( M[z_1/u_1, \ldots, z_n/u_n] \models \varphi_0 \), for all \( u_1, \ldots, u_n \in D \);
2. \( M \models (\forall Z_1^1) \ldots (\forall Z_m^1) \varphi_1 \) iff \( M[Z_1^1/U_1^1, \ldots, Z_m^1/U_m^1] \models \varphi_1 \), for all \( U_1^1, \ldots, U_m^1 \in \text{pow}(D) \);
3. \( M \models (\forall Z_1^2) \ldots (\forall Z_p^2) \varphi_2 \) iff \( M[Z_1^2/U_1^2, \ldots, Z_p^2/U_p^2] \models \varphi_2 \), for all \( U_1^2, \ldots, U_p^2 \in \text{pow}(\text{pow}(D)) \).

Finally, evaluation of compound formulae follows the standard rules of propositional logic. If \( M \models \varphi \), i.e. \( M \) satisfies \( \varphi \), then \( M \) is said to be a 4LQS-model for \( \varphi \). A 4LQS-formula is said to be satisfiable if it has a 4LQS-model. A 4LQS-formula is valid if it is satisfied by all 4LQS-interpretations.

### 2.2. Characterizing 4LQS\(^R\)

4LQS\(^R\) is the subcollection of the formulae \( \psi \) of 4LQS for which the following restrictions hold.
Restr. I. For every purely universal formula \((\forall Z^1_1)\ldots(\forall Z^1_m)\varphi_1\) of level 2 occurring in \(\psi\) and every purely universal formula \((\forall z_1)\ldots(\forall z_n)\varphi_0\) of level 1 occurring negatively in \(\varphi_1\), \(\varphi_0\) is a propositional combination of level 0 quantifier-free atomic formulae and the condition
\[
\neg \varphi_0 \rightarrow \bigwedge_{i=1}^n \bigwedge_{j=1}^m z_i \in Z^1_j
\]
is a valid 4LQS-formula (in this case we say that the formula \((\forall z_1)\ldots(\forall z_n)\varphi_0\) is linked to the variables \(Z^1_1,\ldots,Z^1_m\)).

Restr. II. For every purely universal formula \((\forall Z^2_1)\ldots(\forall Z^2_p)\varphi_2\) of level 3 occurring in \(\psi\)
- every purely universal formula of level 1 occurring negatively in \(\varphi_2\) and not occurring in a purely universal formula of level 2, is only allowed to be of the form
\[
(\forall z_1)\ldots(\forall z_n)\neg\left(\bigwedge_{i=1}^n \bigwedge_{j=1}^n (z_i, z_j) = Y^2_{ij}\right), \text{ where } Y^2_{ij} \in V_2, \text{ for } i, j = 1, \ldots, n;
\]
- purely universal formulae \((\forall Z^1_1)\ldots(\forall Z^1_m)\varphi_1\) of level 2 may occur only positively in \(\varphi_2\).

Restriction II is similar to the one described in [9]. In particular, following [9], we recall that condition (1) guarantees that if a given interpretation assigns to \(z_1,\ldots,z_n\) elements of the domain that make \(\varphi_0\) false, then such elements must be contained in the intersection of the sets assigned to \(Z^1_1,\ldots,Z^1_m\). This fact is needed in the proof of statement (ii) of Lemma 3.5 to make sure that satisfiability is preserved in a suitable finite submodel (details, however, are not reported here and can be found in [9]).

Through several examples, in [9] it is argued that condition (1) is not particularly restrictive. Indeed, to establish whether a given 4LQS-formula is a 4LQS\(^R\)-formula, since condition (1) is a 2LS-formula, its validity can be checked using the decision procedure in [13], as 4LQS is a conservative extension of 2LS. In addition, in many cases of interest, condition (1) is just an instance of the simple propositional tautology \(\neg(A \rightarrow B) \rightarrow A\), and thus its validity can be established just by inspection.

Restriction II has been introduced to be able to express binary relations and several operations on them while keeping simple, at the same time, the decision procedure presented in Section 3.2.

Finally, we observe that though the semantics of 4LQS\(^R\) plainly coincides with that of 4LQS, in what follows we prefer to refer to 4LQS-interpretations of 4LQS\(^R\)-formulae as 4LQS\(^R\)-interpretations.

3. The satisfiability problem for 4LQS\(^R\)-formulae

We will solve the satisfiability problem for 4LQS\(^R\), i.e. the problem of establishing for any given formula of 4LQS\(^R\) whether it is satisfiable or not, as follows:

(i) firstly, we will show how to reduce effectively the satisfiability problem for 4LQS\(^R\)-formulae to the satisfiability problem for normalized 4LQS\(^R\)-conjunctions (these will be defined shortly);

(ii) secondly, we will prove that the collection of normalized 4LQS\(^R\)-conjunctions enjoys a small model property.

From (i) and (ii), the solvability of the satisfiability problem for 4LQS\(^R\) follows immediately. Additionally, by further elaborating on point (i), it could easily be shown that indeed the whole collection of 4LQS\(^R\)-formulae enjoys a small model property.
3.1. Normalized $4LQS^R$-conjunctions

Let $\psi$ be a formula of $4LQS^R$ and let $\psi_{DNF}$ be a disjunctive normal form of $\psi$. Then $\psi$ is satisfiable if and only if at least one of the disjuncts of $\psi_{DNF}$ is satisfiable. We recall that the disjuncts of $\psi_{DNF}$ are conjunctions of literals, namely atomic formulae or their negation. In view of the previous observations, we can suppose that our formula $\psi$ is a conjunction of level 0, 1, 2 quantifier-free literals and of level 1, 2, 3 quantified literals. In addition, we can also assume that no variable occurs both bound and free in $\psi$ and that distinct occurrences of quantifiers bind distinct variables.

For decidability purposes, negative quantified conjuncts occurring in $\psi$ can be eliminated as follows. Let $\mathcal{M} = (D, M)$ be a model for $\psi$, and let $¬(\forall z_1)\ldots(\forall z_n)\varphi_0$ be a negative quantified literal of level 1 occurring in $\psi$. Since $\mathcal{M} \models ¬(\forall z_1)\ldots(\forall z_n)\varphi_0$ if and only if $\mathcal{M}[z_1/u_1,\ldots,z_n/u_n] \models ¬\varphi_0$, for some $u_1,\ldots,u_n \in D$, we can replace $¬(\forall z_1)\ldots(\forall z_n)\varphi_0$ in $\psi$ by $¬(\varphi_0)_{z_1′,\ldots,z_n′}$, where $z_1′,\ldots,z_n′$ are newly introduced variables of sort 0. Negative quantified literals of levels 2 and 3 can be dealt with much in the same way and hence, we can further assume that $\psi$ is a conjunction of literals of the following types:

1. quantifier-free literals of any level;
2. purely universal formulae of level 1;
3. purely universal formulae of level 2 and 3 satisfying Restrictions $\Box_1$ and $\Box_2$ given in Section 2.2 respectively.

We call these formulae normalized $4LQS^R$-conjunctions.

3.2. A small model property for normalized $4LQS^R$-conjunctions

In view of the above reductions, we can limit ourselves to consider the satisfiability problem for normalized $4LQS^R$-conjunctions only. Thus, let $\psi$ be a normalized $4LQS^R$-conjunction and assume that $\mathcal{M} = (D, M)$ is a model for $\psi$.

We show how to construct, out of the model $\mathcal{M}$, a finite $4LQS^R$-interpretation $\mathcal{M}^* = (D^*, M^*)$ which is a model of $\psi$ sufficiently rich to reconstruct any possible counter-example to the formula and such that the size of $D^*$ depends solely on the size of $\psi$. We will proceed as follows. First, in Section 3.2.1 we outline a procedure for the construction of a nonempty finite universe $D^* \subseteq D$. In Steps 1 to 3 $D^*$ is provided with enough elements to properly interpret quantifier-free atomic formulae. Cases involving variables of levels 2 and 3 are treated in Step 2 by introducing an additional set of new variables, $\forall_1^F$. Finally, in Step 4 $D^*$ is further enriched to take care of purely universal formulae of level 2. Then we show how to relativize $\mathcal{M}$ to $D^*$ according to Definition 3.1 below, thus defining a finite $4LQS^R$-interpretation $\mathcal{M}^* = (D^*, M^*)$. Finally, we prove that $\mathcal{M}^*$ satisfies $\psi$.

3.2.1. Construction of the universe $D^*$

Let us denote by $\forall_0^1, \forall_1^F$, and $\forall_2^1$ the collections of variables of sort 0, 1, and 2 occurring free in $\psi$, respectively. We construct $D^*$ according to the following steps:

**Step 1:** Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, where

3 Atomic formulae are quantified atomic formulae and purely universal formulae of any level.
Let \( \mathcal{F}_1 \) ‘distinguishes’ the set \( S = \{ MX^2 : X^2 \in \mathcal{V}'_2 \} \), in the sense that \( K \cap \mathcal{F}_1 \neq K' \cap \mathcal{F}_1 \) for every distinct \( K, K' \in S \). Such a set \( \mathcal{F}_1 \) can be constructed by the procedure \textit{Distinguish} described in [7]. As shown in [7], we can also assume that \( |\mathcal{F}_1| \leq |S| - 1 \).

\( \mathcal{F}_2 \) satisfies \( |MX^2 \cap \mathcal{F}_2| \geq \min(3, |MX^2|) \), for every \( X^2 \in \mathcal{V}'_2 \). Plainly, we can also assume that \( |\mathcal{F}_2| \leq 3 \cdot |\mathcal{V}'_2| \).

**Step 2:** Let \( \{ F_1, \ldots, F_k \} = \mathcal{F} \setminus \{ MX^1 : X^1 \in \mathcal{V}'_1 \} \) and let \( \mathcal{V}'_1 = \{ X_1^1, \ldots, X_k^1 \} \subseteq \mathcal{V}'_1 \) be such that \( \mathcal{V}'_1 \cap \mathcal{V}'_1 = \emptyset \) and \( \mathcal{V}'_1 \cap \mathcal{V}'_1 = \emptyset \), where \( \mathcal{V}'_1 \) is the collection of bound variables in \( \psi \). Let \( \mathcal{M} \) be the interpretation \( \mathcal{M}[X_1^1/F_1, \ldots, X_k^1/F_k] \). Since the variables in \( \mathcal{V}'_1 \) do not occur in \( \psi \) (neither free nor bound), their evaluation is immaterial for \( \psi \) and therefore, from now on, we identify \( \mathcal{M} \) and \( \mathcal{M} \).

**Step 3:** Let \( \Delta = \Delta_1 \cup \Delta_2 \), where

- \( \Delta_1 \) distinguishes the set \( T = \{ MX^1 : X^1 \in \mathcal{V}'_1 \cup \mathcal{V}'_1 \} \) and \( |\Delta_1| \leq |T| - 1 \) holds (cf. Step 1 above);
- \( \Delta_2 \) satisfies \( |J \cap \Delta_2| \geq \min(3, |J|) \), for every \( J \in \{ MX^1 : X^1 \in \mathcal{V}'_1 \cup \mathcal{V}'_1 \} \). Plainly, we can assume that \( |\Delta_2| \leq 3 \cdot |\mathcal{V}'_1 \cup \mathcal{V}'_1| \).

We initialize \( D^* \) by putting

\[
D^* := \{ Mx : x \in \mathcal{V}'_0 \} \cup \Delta.
\]

(*\( D^* \) will possibly be enlarged during the subsequent Step 4.*)

**Step 4:** Let \( \chi_1, \ldots, \chi_r \) be all the purely universal formulae of level 2 occurring in \( \psi \). To each conjunct \( \chi_i = (\forall Z^1_{i,h_1}) \ldots (\forall Z^1_{i,h_m}) \varphi_i \), we associate the collection \( \varphi_{i,k_1}, \ldots, \varphi_{i,k_{\ell_i}} \) of atomic formulae of the form \( (\forall z_1) \ldots (\forall z_n) \varphi_0 \) present in the matrix of \( \chi_i \), and call the variables \( Z^1_{i,h_1}, \ldots, Z^1_{i,h_m} \) the arguments of \( \varphi_{i,k_1, \ldots, \varphi_{i,k_{\ell_i}}} \).

Let us put

\[
\Phi = \text{def} \{ \varphi_{i,k_j} : 1 \leq j \leq \ell_i \text{ and } 1 \leq i \leq r \}.
\]

Then, for each \( \varphi \in \Phi \) of the form \( (\forall z_1) \ldots (\forall z_n) \varphi_0 \) having \( Z^1_{1,1}, \ldots, Z^1_{m,1} \) as arguments, and for each ordered \( m \)-tuple \( (X^1_{h_1}, \ldots, X^1_{h_m}) \) of variables in \( \mathcal{V}'_1 \cup \mathcal{V}'_1 \), if \( M(\varphi_0)_{X^1_{h_1}, \ldots, X^1_{h_m}} = \text{false} \) we insert in \( D^* \) elements \( u_1, \ldots, u_n \in D \) such that

\[
M[z_1/u_1, \ldots, z_n/u_n](\varphi_0)_{X^1_{h_1}, \ldots, X^1_{h_m}} = \text{false},
\]

otherwise we leave \( D^* \) unchanged.

Next, we calculate a bound to the size of \( D^* \). Since \( |\mathcal{F}_1| \leq |S| - 1 \leq |\mathcal{V}'_2| - 1 \) and \( |\mathcal{F}_2| \leq 3|\mathcal{V}'_2| \) (cf. Step 1 above), we plainly have \( |\mathcal{F}| \leq 4|\mathcal{V}'_2| - 1 \). Analogously, just after Step 3, we have \( |\Delta| \leq 4(|\mathcal{V}'_1| + 4|\mathcal{V}'_2| - 1) - 1 \) and \( |D^*| \leq |\mathcal{V}'_0| + 4|\mathcal{V}'_1| + 16|\mathcal{V}'_2| - 5 \). Finally, after Step 4, if we let \( L_m \) denote the maximal length of the quantifier prefix of any purely universal formula of level 2 occurring in
We introduce now the notion of relativized interpretations.

In this section, we will define, out of the model \( \mathcal{M} = (D, M) \) of our normalized 4LQS\( R \)-conjunction \( \psi \), a finite interpretation \( \mathcal{M}' = (D', M') \) of bounded size, which also satisfies \( \psi \).

**Definition 3.1.** Let \( \mathcal{M} = (D, M), D^*, V'_1, V'_2, V''_2 \) be as above, and let \( d^* \in D^* \). The relativized interpretation \( \mathcal{M}' = \text{Rel}(\mathcal{M}, D^*, d^*, V'_1, V'_2, V''_2) \) of \( \mathcal{M} \) with respect to \( D^*, d^*, V'_1, V'_2, V''_2 \) is the 4LQS\( R \)-interpretation \( (D^*, M^*) \) such that

\[
M^*x = \begin{cases} 
Mx, & \text{if } Mx \in D^* \\
 d^*, & \text{otherwise},
\end{cases}
\]

\[
M^*X^1 = M^1 \cap D^*,
\]

\[
M^*X^2 = ((M^2 \cap \text{pow}(D^*)) \setminus \{M^*X^1 : X^1 \in (V'_1 \cup V''_2)\}) \\
\quad \cup \{M^*X^1 : X^1 \in (V'_1 \cup V''_2), M^1 \in M^2\},
\]

\[
M^*X^3 = ((M^3 \cap \text{pow}(D^*)) \setminus \{M^*X^2 : X^2 \in V''_2\}) \\
\quad \cup \{M^*X^2 : X^2 \in V''_2, M^2 \in M^3\}.
\]

Concerning \( M^*X^2 \) and \( M^*X^3 \), we observe that they have been defined in such a way that all the membership relations between variables of \( \psi \) of sorts 2 and 3 are the same in both the interpretations \( \mathcal{M} \) and \( \mathcal{M}' \). This fact will be proved in the next section.

For ease of notation, we will often omit the reference to the element \( d^* \in D^* \) and write simply \( \text{Rel}(\mathcal{M}, D^*, V'_1, V'_2, V''_2) \) in place of \( \text{Rel}(\mathcal{M}, D^*, d^*, V'_1, V'_2, V''_2) \), when \( d^* \) is clear from the context.

The following useful properties are immediate consequences of the construction of \( D^* \), for any \( x, y \in V''_0, X^1, Y^1 \in V'_1, \) and \( X^2, Y^2 \in V''_2 \):

(A) if \( M^1 \neq M^1 \), then \( (M^1 \Delta M^1) \cap D^* \neq \emptyset \).

(B) if \( M^2 \neq M^2 \), there is a \( J \in (M^2 \Delta M^2) \cap \{M^1 : X^1 \in (V'_1 \cup V''_2)\} \) such that \( J \cap D^* \neq \emptyset \).

(C) if \( M(x, y) \neq M(x, y) \), there is a \( J \in (M^2 \Delta M(x, y)) \cap \{M^1 : X^1 \in (V'_1 \cup V''_2)\} \) such that \( J \cap D^* \neq \emptyset \), and if \( J \in M^2 \), then \( J \cap D^* \neq \{Mx, My\} \).

\[4\] We recall that for any sets \( s \) and \( t \), \( s \Delta t \) denotes the symmetric difference of \( s \) and \( t \), namely the set \( (s \setminus t) \cup (t \setminus s) \).
3.3. Soundness of the relativization

As above, let $\mathcal{M} = (D, \mathcal{M})$ be a $4LQS^R$-interpretation satisfying our given normalized $4LQS^R$-conjunction $\psi$, and let $D^*$, $V_1^F$, $V_2^F$, and $\mathcal{M}^*$ be defined as before. The main result of this section is Theorem 3.1 which states that if $\mathcal{M}$ satisfies $\psi$, then $\mathcal{M}^*$ satisfies $\psi$ as well. The proof of Theorem 3.1 exploits the technical Lemmas 3.1, 3.2, 3.3, 3.4, and 3.5 below. In particular, Lemma 3.1 states that $\mathcal{M}$ satisfies a quantifier-free atomic formula $\varphi$, fulfilling conditions (A), (B), and (C) above, if and only if $\mathcal{M}^*$ satisfies $\varphi$ too. Lemmas 3.2, 3.3, and 3.4 claim that suitably constructed variants of $\mathcal{M}^*$ and the small models resulting by applying the construction of Section 3.2 to the corresponding variants of $\mathcal{M}$ can be considered identical. Finally, Lemma 3.5 which follows from Lemmas 3.1, 3.2, 3.3, and 3.4 states that $\mathcal{M}^*$ satisfies all quantified conjuncts of $\psi$ which are satisfied by $\mathcal{M}$.

**Lemma 3.1.** The following statements hold:

(a) $\mathcal{M}^* \models x = y$ iff $\mathcal{M} \models x = y$, for all $x, y \in \mathcal{V}_0$ such that $Mx, My \in D^*$;

(b) $\mathcal{M}^* \models x \in X^1$ iff $\mathcal{M} \models x \in X^1$, for all $X^1 \in \mathcal{V}_1$ and $x \in \mathcal{V}_0$ such that $Mx \in D^*$;

(c) $\mathcal{M}^* \models X^1 = Y^1$ iff $\mathcal{M} \models X^1 = Y^1$, for all $X^1, Y^1 \in \mathcal{V}_1$ such that condition (A) holds;

(d) $\mathcal{M}^* \models X^1 \in X^2$ iff $\mathcal{M} \models X^1 \in X^2$, for all $X^1 \in (\mathcal{V}_1^F \cup \mathcal{V}_2^F)$, $X^2 \in \mathcal{V}_2$;

(e) $\mathcal{M}^* \models X^2 = Y^2$ iff $\mathcal{M} \models X^2 = Y^2$, for all $X^2, Y^2 \in \mathcal{V}_2$ such that condition (B) holds;

(f) $\mathcal{M}^* \models \langle x, y \rangle = X^2$ iff $\mathcal{M} \models \langle x, y \rangle = X^2$, for all $x, y \in \mathcal{V}_0$ such that $Mx, My \in D^*$ and $X^2 \in \mathcal{V}_2$ such that condition (C) holds;

(g) $\mathcal{M}^* \models \langle x, y \rangle \in X^3$ iff $\mathcal{M} \models \langle x, y \rangle \in X^3$, for all $x, y \in \mathcal{V}_0$ such that $Mx, My \in D^*$ and $X^2 \in \mathcal{V}_2$ such that condition (C) holds;

(h) $\mathcal{M}^* \models X^2 \in X^3$ iff $\mathcal{M} \models X^2 \in X^3$, for all $x, y \in \mathcal{V}_0$ such that $Mx, My \in D^*$ and $X^2 \in \mathcal{V}_2$ such that conditions (B) and (C) hold.

**Proof:**

(a) Let $x, y \in \mathcal{V}_0$ be such that $Mx, My \in D^*$. Then $M^*x = Mx$ and $M^*y = My$, so we have immediately that $\mathcal{M}^* \models x = y$ iff $\mathcal{M} \models x = y$.

(b) Let $X^1 \in \mathcal{V}_1$ and let $x \in \mathcal{V}_0$ be such that $Mx \in D^*$. Then $M^*x = Mx$, so that $M^*x \in M^*X^1$ iff $MX \in MX^1 \cap D^*$ iff $MZ \in MX^1$.

(c) If $MX^1 = MY^1$, then plainly $M^*X^1 = M^*Y^1$. On the other hand, if $MX^1 \neq MY^1$, then, by condition (A), $(MX^1 \triangle MY^1) \cap D^* \neq \emptyset$ and thus $M^*X^1 \neq M^*Y^1$.

(d) If $MX^1 \in MX^2$, then $M^*X^1 \in M^*X^2$. On the other hand, suppose by contradiction that $MX^1 \notin MX^2$ and $M^*X^1 \in M^*X^2$. Then, there must necessarily be a $Z^1 \in (\mathcal{V}_1^F \cup \mathcal{V}_2^F)$ such that $MZ^1 \in MX^2$, $MZ^1 \neq MX^1$, and $M^*X^1 = M^*Z^1$. Since $MZ^1 \neq MX^1$ and $(MZ^1 \triangle MX^1) \cap D^* \neq \emptyset$, by condition (A), we have $M^*X^1 \neq M^*Z^1$, which is a contradiction.
If $MX^2 = MY^2$, then $M^*X^2 = M^*Y^2$. On the other hand, if $MX^2 \neq MY^2$, by condition (3), there is a $J \in (MX^2 \triangle MY^2) \cap \{MX^1 : X^1 \in (V_1^t \cup V_1^p)\}$ such that $J \cap D^* \neq \emptyset$. Let $J = MX^1$, for some $X^1 \in (V_1^t \cup V_1^p)$, and suppose without loss of generality that $MX^1 \in MX^2$ and $MX^1 \notin MY^2$. Then, by (3), $M^*X^1 \in M^*X^2$ and $M^*X^1 \notin M^*Y^2$, and hence $M^*X^2 \neq M^*Y^2$.

If $M(x, y) = MX^2$, then $M^*(x, y) = M^*X^2$. If $M(x, y) \neq MX^2$, then there is a $J \in (MX^2 \triangle M(x, y)) \cap \{MX^1 : X^1 \in (V_1^t \cup V_1^p)\}$ satisfying the constraints of condition (3). Let $J = MX^1$, for some $X^1 \in (V_1^t \cup V_1^p)$, and suppose that $MX^1 \in MX^2$ and $MX^1 \notin M(x, y)$. Then $M^*X^1 \in M^*X^2$ and since $M^*X^1 \notin \{Mx, My\}$, it follows that $M^*X^1 \neq M^*(x, y)$. On the other hand, if $MX^1 \in M(x, y)$ and $MX^1 \notin MX^2$, then either $MX^1 = \{Mx\}$ or $MX^1 = \{My\}$. In both cases $MX^1 = M^*X^1$ and thus if $MX^1 \notin MX^2$, it plainly follows that $M^*X^1 \neq M^*X^2$.

Let $x, y \in V_0$ and $X^3 \in V_3$ be such that $M(x, y) \in MX^3$. Then $M^*(x, y) \in M^*X^3$. On the other hand, suppose by contradiction that $M(x, y) \notin MX^3$ and $M^*(x, y) \in M^*X^3$. Then, there must be an $X^2 \in V_2$ such that $M^*X^2 \in M^*X^3$, $M^*X^2 = M^*(x, y)$, and $MX^2 \neq M(x, y)$. But this is impossible by (1).

If $MX^2 \in MX^3$ then $M^*X^2 \in M^*X^3$. Now suppose by contradiction that $MX^2 \notin MX^3$ and that $M^*X^2 \notin M^*X^3$. Then, either there is a $Y^2 \in V_2$ such that $M^*X^2 \notin MY^2$ and $M^*X^2 = M^*Y^2$, which is impossible by (3), or there is a $x, y \in V_0$, $Mx, My \in D^*$, such that $MX^2 \neq M(x, y)$ and $M^*X^2 = M^*(x, y)$, but this is absurd by (1).

In view of the next technical lemmas, we introduce the following notations. Let $u_1, \ldots, u_n \in D^*$, $U_1, \ldots, U_m \in \text{pow}(D^*)$, and $U^1, \ldots, U^2_p \in \text{pow}(\text{pow}(D^*))$. Then we put

$$M^{*, z}_u = M^*[z_1/u_1, \ldots, z_n/u_n],$$
$$M^{*, Z^1}_u = M^*[Z_1^1/U_1, \ldots, Z_m^1/U_m],$$
$$M^{*, Z^2}_u = M^*[Z_1^2/U_1, \ldots, Z_p^2/U_p],$$

and also

$$M^{*, *} = \text{Rel}(M^*, D^*, V_1^t, V_1^p, V_2^t),$$
$$M^{Z_1^1, *}_u = \text{Rel}(M^{Z_1^1}, D^*, V_1^t \cup \{Z_1^1, \ldots, Z_m^1\}, V_1^p, V_2^t),$$
$$M^{Z_2^2, *}_u = \text{Rel}(M^{Z_2^2}, D^*, F^*, V_1^t, V_1^p, V_2^t \cup \{Z_1^2, \ldots, Z_p^2\}).$$

The next three lemmas claim that, under certain conditions, the following pairs of $4LQS^R$-interpretations $M^{*, z}_u$ and $M^{z, *}_u$, $M^{*, Z^1}_u$ and $M^{Z^1, *}_u$, and $M^{*, Z^2}_u$ and $M^{Z^2, *}_u$ can be identified, respectively.

**Lemma 3.2.** Let $u_1, \ldots, u_n \in D^*$, and let $z_1, \ldots, z_n \in V_0$. Then, the $4LQS^R$-interpretations $M^{*, z}_u$ and $M^{z, *}_u$ coincide.

**Proof:**

The proof of the lemma is carried out by showing that $M^{*, z}_u$ and $M^{z, *}_u$ agree over variables of all sorts.
• Let $x \in V_0$. Since $u_1, \ldots, u_n \in D^*$, the thesis follows immediately.

• Let $X^1 \in V_1$, then $M^{*,Z}X^1 = M^*X^1 = MX^1 \cap D^* = M^{Z,*}X^1$.

• Let $X^2 \in V_2$, then we have the following equalities:

\[ M^{*,Z}X^2 = M^*X^2 = \left( (MX^2 \cap \text{pow}(D^*)) \setminus \{ M^*X^2 : X^1 \in (V'_1 \cup V^F_1) \} \right) \]
\[ \cup \{ M^*X^1 : X^1 \in (V'_1 \cup V^F_1) \}, \quad MX^1 \in MX^2 \}
\[ = \left( (M^{Z,*}X^2 \cap \text{pow}(D^*)) \setminus \{ M^{Z,*}X^2 : X^2 \in V'_2 \} \right) \]
\[ \cup \{ M^{Z,*}X^1 : X^1 \in (V'_1 \cup V^F_1) \}, \quad M^{Z,*}X^1 \in M^{Z,*}X^2 \}
\[ = M^{Z,*}X^2. \]

• Let $X^3 \in V_3$, then the following holds:

\[ M^{*,Z}X^3 = M^*X^3 = \left( (MX^3 \cap \text{pow}(D^*)) \setminus \{ M^*X^2 : X^2 \in V'_2 \} \right) \]
\[ \cup \{ M^*X^2 : X^2 \in V'_2, MX^2 \in MX^3 \}
\[ = \left( (M^{Z,*}X^3 \cap \text{pow}(D^*)) \setminus \{ M^{Z,*}X^2 : X^2 \in V'_2 \} \right) \]
\[ \cup \{ M^{Z,*}X^2 : X^2 \in V'_2, M^{Z,*}X^2 \in M^{Z,*}X^3 \}
\[ = M^{Z,*}X^3. \]

\[ \Box \]

**Lemma 3.3.** Let $Z^1_1, \ldots, Z^1_m \in V_1 \setminus (V'_1 \cup V^F_1)$ and $U^1_1, \ldots, U^1_m \in \text{pow}(D^*) \setminus \{ M^*X^1 : X^1 \in (V'_1 \cup V^F_1) \}$. Then, the $4LQS^R$-interpretations $M^{*,Z^1}$ and $M^{Z^1,*}$ coincide.

**Proof:**

We prove the lemma by showing that $M^{*,Z^1}$ and $M^{Z^1,*}$ agree over variables of all sorts.

1. Clearly $M^{*,Z^1}X = M^*X = M^{Z^1,*}X$, for all individual variables $x \in V_0$.

2. Let $X^1 \in V_1$. If $X^1 \notin \{ Z^1_1, \ldots, Z^1_m \}$, then $M^{Z^1,*}X^1 = M^{Z^1}X^1 \cap D^* = MX^1 \cap D^* = M^*X^1 = M^{*,Z^1}X^1$.

On the other hand, if $X^1 = Z^1_j$ for some $j \in \{1, \ldots, m\}$, we have

\[ M^{Z^1,*}Z^1_j = M^{Z^1}Z^1_j \cap D^* = U^1_j \cap D^* = U^1_j = M^{*,Z^1}Z^1_j. \]

3. Let $X^2 \in V_2$. Then we have

\[ M^{*,Z^1}X^2 = M^*X^2 = \left( (MX^2 \cap \text{pow}(D^*)) \setminus \{ M^*X^2 : X^1 \in (V'_1 \cup V^F_1) \} \right) \]
\[ \cup \{ M^*X^1 : X^1 \in (V'_1 \cup V^F_1) \}, \quad MX^1 \in MX^2 \}, \quad (3) \]

\[ M^{Z^1,*}X^2 = \left( (M^{Z^1}X^2 \cap \text{pow}(D^*)) \setminus \{ M^{Z^1,*}X^2 : X^1 \in ((V'_1 \cup V^F_1) \cup \{ Z^1_1, \ldots, Z^1_m \}) \} \right) \]
\[ \cup \{ M^{Z^1,*}X^1 : X^1 \in ((V'_1 \cup V^F_1) \cup \{ Z^1_1, \ldots, Z^1_m \}), \quad M^{Z^1}X^1 \in M^{Z^1}X^2 \}
\[ = \left( (MX^2 \cap \text{pow}(D^*)) \setminus \{ M^*X^1 : X^1 \in (V'_1 \cup V^F_1) \} \right) \]
\[ \cup \{ U^1_j : j = 1, \ldots, m \} \cup \{ M^*X^1 : X^1 \in (V'_1 \cup V^F_1), \quad MX^1 \in MX^2 \}
\[ \cup \{ U^1_j : j = 1, \ldots, m \} \cap MX^2 \}. \quad (4) \]
Lemma 3.4. Let $Z_1, \ldots, Z_2^2 \in \mathcal{V}_2 \setminus \mathcal{V}_0$ and $U_2, \ldots, U_2^2 \in \text{pow}(D^*) \setminus \{M^* X^2 : X^2 \in \mathcal{V}_2\}$. Then the $\mathcal{ALQS}^R$-interpretations $\mathcal{M}^{*,Z^2}$ and $\mathcal{M}^{Z^2,*}$ coincide.

Proof:
We show that $\mathcal{M}^{*,Z^2}$ and $\mathcal{M}^{Z^2,*}$ coincide by proving that they agree over variables of all sorts.

1. Plainly $M^{*,Z^2} x = M^* x = M^{Z^2,*} x$, for every $x \in \mathcal{V}_0$.
2. Let $X^1 \in \mathcal{V}_1$. Then $M^{*,Z^2} X^1 = M^* X^1 = M^{Z^2,*} X^1$. 

\[ P_1 = MX^2 \cap \text{pow}(D^*), \]
\[ P_2 = \{M^* X^1 : X^1 \in (V_1^* \cup V_1^F)\}, \]
\[ P_3 = \{U_j : j = 1, \ldots, m\}, \]
\[ P_4 = \{M^* X^1 : X^1 \in (V_1^* \cup V_1^F), MX^1 \in MX^2\}, \]
\[ P_5 = \{U_j : j = 1, \ldots, m\} \cap MX^2, \]

then by (3) and (4) can be rewritten as
\[ M^{*,Z^1} X^2 = (P_1 \setminus P_2) \cup P_4 \]
\[ M^{Z^1,*} X^2 = (P_1 \setminus (P_2 \cup P_3)) \cup P_4 \cup P_5. \]

Moreover, since, as can easily verified, we have
\[ P_2 \cap P_3 = \emptyset, \quad P_5 = P_1 \cap P_3, \quad \text{and} \quad P_4 \subseteq P_2, \]

then
\[ (P_1 \setminus P_2) \cup P_4 = (P_1 \setminus (P_2 \cup P_3)) \cup P_4 \cup (P_1 \cap P_3) \]
\[ = (P_1 \setminus (P_2 \cup P_3)) \cup P_4 \cup P_3. \]

Therefore, (5) and (6) readily imply $M^{*,Z^1} X^2 = M^{Z^1,*} X^2.$
3. Let $X^2 \in \mathcal{V}_2$ such that $X^2 \notin \{Z^2_1, \ldots, Z^2_p\}$. Then

$$M^{*,Z^2}X^2 = M^{*}[Z^2_2/U^2_1, \ldots, Z^2_p/U^2_p]X^2 = M^*X^2$$

and

$$M^{Z^2,*}X^2 = ((M^{Z^2}X^2 \cap \text{pow}(D^*)) \setminus \{M^{Z^2,*}X^1 : X^1 \in (\mathcal{V}'_1 \cup \mathcal{V}'_2')\})$$

$$\cup \{M^{Z^2,*}X^1 : X^1 \in (\mathcal{V}'_1 \cup \mathcal{V}'_2'), M^{Z^2}X^1 \in M^{Z^2}X^2\}$$

$$= ((M^*X^2 \cap \text{pow}(D^*)) \setminus \{M^*X^1 : X^1 \in (\mathcal{V}'_1 \cup \mathcal{V}'_2')\})$$

$$\cup \{M^*X^1 : X^1 \in (\mathcal{V}'_1 \cup \mathcal{V}'_2'), M^X^1 \in M^*X^2\}$$

$$= M^*X^2.$$

Since $M^{*,Z^2}X^2 = M^{Z^2,*}X^2$ the thesis follows, at least in the case in which $X^2 \notin \{Z^2_1, \ldots, Z^2_p\}$. On the other hand, if $X^2 \in \{Z^2_1, \ldots, Z^2_p\}$, say $X^2 = Z^2_j$, then $M^{*,Z^2}X^2 = U^2_j$, and

$$M^{Z^2,*}X^2 = ((M^{Z^2}X^2 \cap \text{pow}(D^*)) \setminus \{M^{Z^2,*}X^1 : X^1 \in (\mathcal{V}'_1 \cup \mathcal{V}'_2')\})$$

$$\cup \{M^{Z^2,*}X^1 : X^1 \in (\mathcal{V}'_1 \cup \mathcal{V}'_2'), M^{Z^2}X^1 \in M^{Z^2}X^2\}$$

$$= (U^2_j \setminus \{M^*X^1 : X^1 \in (\mathcal{V}'_1 \cup \mathcal{V}'_2')\})$$

$$\cup \{M^*X^1 : X^1 \in (\mathcal{V}'_1 \cup \mathcal{V}'_2'), M^X^1 \in U^2_j\}$$

$$= U^2_j.$$

Clearly the thesis follows also in this case.

4. Let $X^3 \in \mathcal{V}_3$. Then we have

$$M^{*,Z^2}X^3 = M^*X^3 = (\{M^X^3 \cap \text{pow}(\text{pow}(D^*))\} \setminus \{M^*X^2 : X^2 \in \mathcal{V}'_2\})$$

$$\cup \{M^*X^2 : X^2 \in \mathcal{V}'_2, M^X^2 \in M^{X^3}\}$$

(7)

$$M^{Z^2,*}X^3 = ((M^{Z^2}X^3 \cap \text{pow}(\text{pow}(D^*)) \setminus \{M^{Z^2,*}X^2 : X^2 \in \mathcal{V}'_2 \cup \{Z^2_1, \ldots, Z^2_p\}\})$$

$$\cup \{M^{Z^2,*}X^2 : X^2 \in \mathcal{V}'_2 \cup \{Z^2_1, \ldots, Z^2_p\}, M^{Z^2}X^2 \in M^{Z^2}X^3\}$$

$$= ((M^X^3 \cap \text{pow}(\text{pow}(D^*)) \setminus (\{M^*X^2 : X^2 \in \mathcal{V}'_2\} \cup \{U^2_j : j = 1, \ldots, p\}))$$

$$\cup \{M^*X^2 : X^2 \in \mathcal{V}'_2, M^X^2 \in M^{X^3}\}$$

$$\cup \{U^2_j : j = 1, \ldots, p\} \cap M^{X^3}.$$

(8)

By putting

$$P_1 = M^X^3 \cap \text{pow}(\text{pow}(D^*)),$$

$$P_2 = \{M^*X^2 : X^2 \in \mathcal{V}'_2\},$$

$$P_3 = \{U^2_j : j = 1, \ldots, p\},$$

$$P_4 = \{M^*X^2 : X^2 \in \mathcal{V}'_2, M^X^2 \in M^{X^3}\},$$

$$P_5 = \{U^2_j : j = 1, \ldots, p\} \cap M^{X^3}.$$
Proof:

(iii) if $\psi$ be conjuncts of $\psi$. Then, Lemma 3.1 yields $M^*,Z^3 = M^{Z^2,*}X^3$.

\[\square\]

The following lemma proves that satisfiability is preserved in the case of purely universal formulae.

Lemma 3.5. Let $(\forall z_1)\ldots(\forall z_n)\varphi_0$, $(\forall Z_1^1)\ldots(\forall Z_m^1)\varphi_1$, and $(\forall Z_1^2)\ldots(\forall Z_p^2)\varphi_2$ be conjuncts of $\psi$. Then

(i) if $\mathcal{M} \models (\forall z_1)\ldots(\forall z_n)\varphi_0$, then $\mathcal{M}^* \models (\forall z_1)\ldots(\forall z_n)\varphi_0$;

(ii) if $\mathcal{M} \models (\forall Z_1^1)\ldots(\forall Z_m^1)\varphi_1$, then $\mathcal{M}^* \models (\forall Z_1^1)\ldots(\forall Z_m^1)\varphi_1$;

(iii) if $\mathcal{M} \models (\forall Z_1^2)\ldots(\forall Z_p^2)\varphi_2$, then $\mathcal{M}^* \models (\forall Z_1^2)\ldots(\forall Z_p^2)\varphi_2$.

Proof:

(i) Assume by contradiction that there exist $u_1,\ldots,u_m \in D^*$ such that $\mathcal{M}^*,Z^2 \not\models \varphi_0$. Then, there

must be an atomic formula $\varphi'_0$ in $\varphi_0$ that is interpreted differently in $\mathcal{M}^*,Z^2$ and in $\mathcal{M}^*$. Recalling

that $\varphi_0$ is a propositional combination of quantifier-free atomic formulae of any level, let us first

suppose that $\varphi'_0$ is $X^2 = Y^2$ and, without loss of generality, assume that $\mathcal{M}^*,Z^2 \not\models X^2 = Y^2$.

Then $M^*,Z^2 \not\models M^{*,Z^2}X^2$, so that, by Lemma 3.2, $M^{*,Z^2}X^2 \not\models M^{*,Z^2}Y^2$. Then, Lemma 3.1 yields

$M^{*,Z^2}X^2 \not\models M^{*,Z^2}Y^2$, a contradiction. The other cases are proved in an analogous way.

(ii) This case can be proved much along the same lines as the proof of case (ii) of Lemma 4 in [9].

Here, one has to take care of the fact that $\varphi_1$ may contain purely universal formulae of level 1 occurring only positively in $\varphi_1$ and not satisfying Restriction I of Section 2.2. This is handled similarly to case (i) of this lemma. Another issue that has to be considered is the fact that the collection of relevant variables of sort 1 for $\psi$ are not just the variables occurring free in $\psi$, namely

the ones in $V'_1$, but also the variables in $V'_1^F$, introduced to denote the elements distinguishing the

sets $M^*,X^2$, for $X^2 \in V'_2$. Then, (7) and (8) can be respectively rewritten as

\[M^*,Z^2 X^3 = (P_1 \setminus P_2) \cup P_4 \quad (9)\]

\[M^{Z^2,*}X^3 = (P_1 \setminus (P_2 \cup P_3)) \cup P_4 \cup P_5. \quad (10)\]

Moreover, it is easy to verify that the following relations hold:

\[P_2 \cap P_3 = \emptyset, \quad P_5 = P_1 \cap P_3, \quad \text{and} \quad P_4 \subseteq P_2,\]

so that

\[(P_1 \setminus P_2) \cup P_4 = (P_1 \setminus (P_2 \cup P_3)) \cup P_4 \cup (P_1 \cap P_3) = (P_1 \setminus (P_2 \cup P_3)) \cup P_4 \cup P_5. \quad (11)\]

Therefore, in view of (9) and (10) above, (11) yields $M^*,Z^2 X^3 = M^{Z^2,*}X^3$.

\[\square\]
Assume, by way of contradiction, that $\mathcal{M} \models (\forall Z_1^2) \ldots (\forall Z_p^2) \varphi_2$, but $\mathcal{M}^* \not\models (\forall Z_1^2) \ldots (\forall Z_p^2) \varphi_2$. Hence there exist $U_1^2, \ldots, U_p^2 \in \text{pow}(\text{pow}(D^*))$ such that $\mathcal{M}^{*,Z^2} \not\models \varphi_2$.

Without loss of generality, assume that $U_i^2 = M^* X_i^2$, for $1 \leq i \leq k$ and where $X_1^2, \ldots, X_k^2 \in \mathcal{V}_2$, and that $U_j^2 \neq M^* X_j^2$, for $k + 1 \leq j \leq p$ and $X_j^2 \in \mathcal{V}_2$, for some $k \geq 0$.

Let $\bar{\varphi}_2$ be the formula obtained by simultaneously substituting $Z_1^2, \ldots, Z_k^2$ with $X_1^2, \ldots, X_k^2$ in $\varphi_2$, and let $\mathcal{M}^{*,Z^2}_k = \mathcal{M}^*[Z_k^2/U_k^2, \ldots, Z_p^2/U_p^2]$. Further, let $\mathcal{M}Z^2_k$ be a $4LQSM^R$-interpretation differing from $\mathcal{M}^{*Z^2}_k$ only in the evaluation of $Z_1^2, \ldots, Z_k^2$, with $MZ'_k Z_k^2 = MX_k^2, \ldots, MZ'_k Z_k^2 = MX_k^2$.

We distinguish the following two cases:

**Case $k = p$:** If $k = p$, then $\mathcal{M}^{*,Z^2}_k$ and $\mathcal{M}^*$ coincide and a contradiction can be obtained by showing that the implications

$$\mathcal{M}^{*,Z^2} \not\models \varphi_2 \Rightarrow \mathcal{M}^* \not\models \varphi_2 \Rightarrow \mathcal{M} \not\models \varphi_2 \Rightarrow \mathcal{M}^{*Z^2} \not\models \varphi_2$$

hold, since these together with the fact that $\mathcal{M}^{*,Z^2} \not\models \varphi_2$ would yield $\mathcal{M} \not\models (\forall Z_1^2) \ldots (\forall Z_p^2) \varphi_2$, contradicting our initial hypothesis. The first implication, $\mathcal{M}^{*,Z^2} \not\models \varphi_2 \Rightarrow \mathcal{M}^* \not\models \varphi_2$, is plainly derived from the definition of $\bar{\varphi}_2$. The second one, $\mathcal{M}^* \not\models \varphi_2 \Rightarrow \mathcal{M} \not\models \varphi_2$, can be proved as follows. For every purely universal formula either of level 1 or of level 2, $\bar{\varphi}_2$, occurring only positively in $\bar{\varphi}_2$, it follows that $\mathcal{M}^* \not\models \varphi_2 \Rightarrow \mathcal{M} \not\models \varphi_2$ by reasoning as in case (i) or in case (ii) of the present lemma, respectively. For each other atomic formula $\bar{\varphi}_2'$ occurring in $\bar{\varphi}_2$ we have to show that $\mathcal{M}^*$ and $\mathcal{M}$ evaluate $\bar{\varphi}_2'$ in the same manner. If $\bar{\varphi}_2'$ is a quantifier-free atomic formula, the proof follows directly from Lemma 3.1. If $\bar{\varphi}_2'$ is an atomic formula of level 1, it can only be of type $(\forall z_1) \ldots (\forall z_n) - (\bigwedge_{i,j=1}^n \langle z_i, z_j \rangle = Y_{ij}^2)$, where $Y_{ij}^2$ is any variable in $\mathcal{V}_2$. Reasoning analogously to case (i) of the present lemma, it follows that $\mathcal{M} \models \varphi_2' \Rightarrow \mathcal{M}^* \models \varphi_2'$. Next, let us prove that $\mathcal{M}^* \models \varphi_2' \Rightarrow \mathcal{M} \models \varphi_2'$. Assume by contradiction that $\mathcal{M} \not\models \varphi_2'$. That is, $\mathcal{M} \not\models (\forall z_1) \ldots (\forall z_n) - (\bigwedge_{i,j=1}^n \langle z_i, z_j \rangle = Y_{ij}^2)$. Then, there are $u_1, \ldots, u_n \in D$ such that $\mathcal{M}[z_1/u_1, \ldots, z_n/u_n] \models \bigwedge_{i,j=1}^n \langle z_i, z_j \rangle = Y_{ij}^2$. By the construction in Section 3.2 all these $u_i$s are in $D^*$, $MY_{ij} = M^*Y_{ij}$ and thus we finally obtain that

$$\mathcal{M}^* \not\models (\forall z_1) \ldots (\forall z_n) - (\bigwedge_{i,j=1}^n \langle z_i, z_j \rangle = Y_{ij}^2),$$

contradicting our hypothesis.

Finally, $\mathcal{M} \not\models \varphi_2 \Rightarrow \mathcal{M}^{*Z^2} \not\models \varphi_2$, follows from the definition of $\bar{\varphi}_2$ and of $Z^2$.

**Case $k < p$:** In this case, the schema of the proof is analogous to the one in the previous case. However, since $\mathcal{M}^{*,Z^2}_k$ and $\mathcal{M}^*$ do not coincide, the single steps are carried out in a slightly different manner. Thus, for the sake of clarity we report below the details of the proof.

In order to obtain a contradiction we prove that the following implications hold

$$\mathcal{M}^{*,Z^2} \not\models \varphi_2 \Rightarrow \mathcal{M}^{*,Z^2}_k \not\models \varphi_2 \Rightarrow \mathcal{M}^{*Z^2}_k \not\models \varphi_2 \Rightarrow \mathcal{M}^{*Z^2} \not\models \varphi_2.$$


The first implication, $M^{*}Z^2 \not\models \varphi_2 \Rightarrow M^{*}Z^2_k \not\models \varphi_2$, can be immediately deduced from the definition of $\varphi_2$ and of $M^{*}Z^2_k$. The second implication, $M^{*}Z^2_k \not\models \varphi_2 \Rightarrow M^{Z^2}_k \not\models \varphi_2$, can be proved as shown next. If $\varphi'_2$ is a purely universal formula either of level 1 or of level 2 occurring only positively in $\varphi_2$, we have $M^{*}Z^2_k \not\models \varphi'_2$ and, since $M^{*}Z^2_k$ and $M^{Z^2}_k$ coincide (by Lemma 3.4), we obtain $M^{Z^2}_k \not\models \varphi'_2$. Then, reasoning as in case (i) (if $\varphi'_2$ is of level 1) or in case (ii) (if $\varphi'_2$ is of level 2) of the present lemma, it follows that $M^{Z^2}_k \not\models \varphi'_2$. If $\varphi'_2$ is a quantifier-free atomic formula occurring in $\varphi_2$, we prove that $\varphi'_2$ in $\varphi_2$ is interpreted in $M^{*}Z^2_k$ and in $M^{Z^2}_k$ in the same way, using Lemmas 3.4 and 3.1.

If $\varphi'_2$ is a purely universal formula of level 1, it must have the form

$$\forall Y_2(\forall z_1)\ldots(\forall z_n)\neg(\bigwedge_{i,j=1}^n(z_i, z_j) = Y^2_{ij}),$$

where $Y^2_{ij}$ is any variable in $V_2$. In this case the proof is carried out as shown next. Reasoning as in case (i), we have $M^{Z^2}_k \models \varphi'_2 \Rightarrow M^{Z^2}_k \models \varphi'_2$, and by Lemma 3.4, that $M^{*}Z^2_k \models \varphi'_2$. Proceeding as in the first case of this item of the present lemma, we obtain that $M^{Z^2}_k \models \varphi'_2$, and, by Lemma 3.4, that $M^{*}Z^2_k \models \varphi'_2$. Finally, the third implication, $M^{Z^2}_k \not\models \varphi_2 \Rightarrow M^{Z^2}_k \not\models \varphi_2$ follows directly from the definition of $\varphi_2$ and of $Z^2$.

We can now state and prove our main result.

**Theorem 3.1.** Let $M$ be a $4LQS^R$-interpretation satisfying a normalized $4LQS^R$-conjunction $\psi$. Then $M^* \models \psi$, where $M^*$ is the relativized interpretation of $M$ with respect to a domain $D^*$ satisfying (2).

**Proof:**
We only have to prove that $M^* \models \psi'$, for each conjunct $\psi'$ occurring in $\psi$. Each such $\psi'$ must be of one of the types (1)–(3) enumerated in Section 3.1. By applying either Lemma 3.1 or Lemma 3.5 to each $\psi'$ (according to its type) we obtain the thesis.

From the above reduction and relativization steps, the following result follows easily:

**Corollary 3.1.** The fragment $4LQS^R$ enjoys a small model property (and therefore it has a solvable satisfiability problem).

### 4. Expressiveness of the language $4LQS^R$

Much as shown in [9], the language $4LQS^R$ can express a restricted variant of the set-formation operator, which in turn allows one to express other significant set operators such as binary union, intersection, set difference, the singleton operator, the powerset operator (over subsets of the universe only), etc. More
specifically, atomic formulae of type \( X^i = \{ X^{i-1} : \varphi(X^{i-1}) \} \), for \( i = 1, 2, 3 \), can be expressed in \( 4LQS^R \) by the formulae

\[
(\forall X^{i-1})(X^{i-1} \in X^i \iff \varphi(X^{i-1}))
\]

provided that the syntactic constraints of \( 4LQS^R \) are satisfied.

Since \( 4LQS^R \) is a superlanguage of \( 3LQS^R \), the language \( 4LQS^R \) can express the syllogistic \( 2LS \) (cf. [13]) and the sublanguage \( 3LSSP \) of \( 3LSSPU \) not involving the set-theoretic construct of general union, since these are expressible in \( 3LQS^R \), as shown in [9]. We recall that \( 3LSSPU \) admits variables of three sorts and, besides the usual set-theoretical constructs, it involves the ‘singleton set’ operator \( \{ \cdot \} \), the powerset operator \( \text{pow} \), and the general union operator \( \cup_n \). \( 3LSSP \) can plainly be decided by the decision procedure presented in [5] for the whole fragment \( 3LSSPU \).

Among the other constructs of set theory which are expressible in the language \( 4LQS^R \) (cf. [9]), we cite:

- literals of the form \( X^2 = \text{pow}_{<h}(X^1) \), where \( \text{pow}_{<h}(X^1) \) denotes the collection of subsets of \( X^1 \) with less than \( h \) elements;
- the unordered Cartesian product \( X^2 = X^1 \times \ldots \times X^1 \), where \( X^1 \times \ldots \times X^1 \) denotes the collection \( \{ \{ x_1, \ldots, x_n \} : x_1 \in X^1, \ldots, x_n \in X^1 \} \); and
- literals of the form \( A = \text{pow}^*(X^1, \ldots, X^1) \), where \( \text{pow}^*(X^1, \ldots, X^1) \) is the variant of the powerset introduced in [3] which denotes the collection

\[
\{ Z : Z \subseteq \bigcup_{i=1}^n X^1_i \text{ and } Z \cap X^1_i \neq \emptyset, \text{ for all } 1 \leq i \leq n \}
\]

For instance, a literal of the form \( X^2 = \text{pow}_{<h}(X^1) \), with \( h \geq 2 \), can be expressed by the \( 4LQS^R \)-formula

\[
(\forall Y^1)(Y^1 \in X^2 \iff (\forall z)(z \in Y^1 \rightarrow z \in X^1) \land (\forall z_1) \ldots (\forall z_h)(\bigwedge_{i=1}^h z_i \in Y^1 \rightarrow \bigvee_{i,j=1}^h z_i = z_j))
\]

as can be easily verified.

4.1. Other applications of \( 4LQS^R \)

Within the \( 4LQS^R \) language it is also possible to define binary relations over elements of a domain together with several conditions on them which characterize accessibility relations of well-known modal logics. These formalizations are illustrated in Table [11]

Usual Boolean operations over relations can be defined as shown in Table [2]. The language \( 4LQS^R \) allows one also to express the inverse \( X^3_{R^2} \) of a given binary relation \( X^3_{R^3} \) (namely, to express the literal \( X^3_{R^2} = (X^3_{R^3})^{-1} \) by means of the \( 4LQS^R \)-formula \( (\forall z_1, z_2) (\langle z_1, z_2 \rangle \in X^3_{R^3} \iff \langle z_2, z_1 \rangle \in X^3_{R^2}) \).

In the next section we will present an application of the decision procedure for \( 4LQS^R \)-formulae to modal logic. For this purpose we introduce below a family \( \{ (4LQS^R)^h \}_{h \geq 2} \) of fragments of \( 4LQS^R \), each of which has an \( \text{NP} \)-complete satisfiability problem, and then show, in the next section, that
that the bound in (2) is quadratic in the size of formulae will allow us to deduce that

$$L \rightarrow M$$

and

$$M$$

that for completeness of the decision problem for the modal logic follows that \(\text{a satisfiable} LQS \rightarrow LQS\)

Asymmetric

Antisymmetric

Intransitive

Weakly-connected

Transitive

Binary relation

\((\forall Z^2)(Z^2 \in X^3_R \leftrightarrow -((z_1, z_2) = Z^2))\)

Reflexive

\((\forall z_1)((z_1, z_1) \in X^3_R)\)

Symmetric

\((\forall z_1, z_2)((z_1, z_2) \in X^3_R \rightarrow (z_2, z_1) \in X^3_R)\)

Transitive

\((\forall z_1, z_2, z_3)((z_1, z_2) \in X^3_R \land (z_2, z_3) \in X^3_R) \rightarrow (z_1, z_3) \in X^3_R)\)

Euclidean

\((\forall z_1, z_2, z_3)((z_1, z_2) \in X^3_R \land (z_1, z_3) \in X^3_R) \rightarrow (z_2, z_3) \in X^3_R)\)

Weakly-connected

\((\forall z_1, z_2, z_3)((z_1, z_2) \in X^3_R \land (z_1, z_3) \in X^3_R) \rightarrow (z_2, z_3) \in X^3_R)\)

Irreflexive

\((\forall z_1)-((z_1, z_1) \in X^3_R)\)

Intransitive

\((\forall z_1, z_2, z_3)((z_1, z_2) \in X^3_R \land (z_2, z_3) \in X^3_R) \rightarrow -((z_1, z_3) \in X^3_R)\)

Antisymmetric

\((\forall z_1, z_2)((z_1, z_2) \in X^3_R \land (z_2, z_1) \in X^3_R) \rightarrow (z_1 = z_2))\)

Asymmetric

\((\forall z_1, z_2)((z_1, z_2) \in X^3_R \rightarrow -((z_1, z_1) \in X^3_R))\)

Table 1. 4LQS^R formalization of conditions of accessibility relations

| Intersection       | \(X^3_R = X^3_{R_1} \cap X^3_{R_2}\) |
|--------------------|--------------------------------------|
| Union              | \(X^3_R = X^3_{R_1} \cup X^3_{R_2}\) |
| Complement         | \(X^3_{R_1} = X^3_{R_2}\)           |
| Set difference     | \(X^3_R = X^3_{R_1} \setminus X^3_{R_2}\) |
| Set inclusion      | \(X^3_{R_1} \subseteq X^3_{R_2}\)   |
| \((\forall Z^2)(Z^2 \in X^3_R \leftrightarrow (Z^2 \in X^3_{R_1} \land Z^2 \in X^3_{R_2}))\) |
| \((\forall Z^2)(Z^2 \in X^3_R \leftrightarrow (Z^2 \in X^3_{R_1} \lor Z^2 \in X^3_{R_2}))\) |
| \((\forall Z^2)(Z^2 \in X^3_{R_1} \leftrightarrow -Z^2 \in X^3_{R_2})\) |
| \((\forall Z^2)(Z^2 \in X^3_{R_1} \leftrightarrow (Z^2 \in X^3_{R_1} \land -Z^2 \in X^3_{R_2}))\) |
| \((\forall Z^2)(Z^2 \in X^3_{R_1} \leftrightarrow Z^2 \in X^3_{R_2})\) |

Table 2. 4LQS^R formalization of Boolean operations over relations

the modal logic K45 can be formalized in \((4LQS^R)^3\) in a succinct way, thus rediscovering the NP-completeness of the decision problem for K45 (cf. [15]).

Formulae in \((4LQS^R)^h\) must satisfy various syntactic constraints. First of all, all quantifier prefixes occurring in a formula in \((4LQS^R)^h\) must have their length bounded by the constant \(h\). Thus, given a satisfiable \((4LQS^R)^h\)-formula \(\varphi\) and a \(4LQS^R\)-model \(\mathcal{M} = (D, M)\) for it, from Theorem 3.1 it follows that \(\varphi\) is satisfied by the relativized interpretation \(\mathcal{M}^* = (D^*, M^*)\) of \(\mathcal{M}\) with respect to a domain \(D^*\) whose size is bounded by the expression in (2). But since in this case \(L_m \leq h\) and \(L_n \leq h\), where \(L_m\) and \(L_n\) are defined as in Step 4 of the construction of \(D^*\) (cf. Section 3.2.1), it follows that the bound in (2) is quadratic in the size of \(\varphi\). The remaining syntactic constraints on \((4LQS^R)^h\)-formulae will allow us to deduce that \(M^* X^2 \subseteq \text{pow}_{<h}(D^*)\), for any free variable \(X^2\) of sort 2 in \(\varphi\), and \(M^* X^3 \subseteq \text{pow}_{<h}(\text{pow}_{<h}(D^*))\), for any free variable \(X^3\) of sort 3 in \(\varphi\), so that the model \(\mathcal{M}^*\) can be guessed in nondeterministic polynomial time in the size of \(\varphi\), and one can check in deterministic polynomial time that \(\mathcal{M}^*\) actually satisfies \(\varphi\), proving that the satisfiability problem for \((4LQS^R)^h\)-formulae is in NP. As the satisfiability problem SAT for propositional logic can be readily reduced to that for \((4LQS^R)^h\)-formulae, the NP-completeness of the latter problem follows.
Definition 4.1. \((4LQS^R)^h\)-formulas)

Let \(\varphi\) be a \(4LQS^R\)-formula involving the designated free variables \(X^1_U\), \(X^2_{<h}\), and \(X^3_{<h}\) (of sort 1, 2, and 3, respectively). Let \(X^1_p, \ldots, X^2_p\) be the free variables of sort 2 occurring in \(\varphi\), distinct from \(X^2_{<h}\). Likewise, let \(X^3_1, \ldots, X^3_p\) be the free variables of sort 3 occurring in \(\varphi\), distinct from \(X^3_{<h}\). Then \(\varphi\) is a \((4LQS^R)^h\)-formula, with \(h \geq 2\), if it has the form (up to the order of the conjuncts)

\[\xi^1_U \land \xi^2_{<h} \land \xi^3_{<h} \land \psi^1 \land \ldots \land \psi^p \land \psi^3 \land \chi,\]

where

1. \(\xi^1_U \equiv \forall (z)(z \in X^1_U)\),
   i.e., \(X^1_U\) is the (nonempty) universe of discourse;

2. \(\xi^2_{<h} \equiv \forall Z^1 \left( Z^1 \in X^2_{<h} \rightarrow (\forall z_1) \ldots (\forall z_h) \left( \bigwedge_{i=1}^h z_i \in Z^1 \rightarrow \bigvee_{i,j=1,i<j} h z_i = z_j \right) \right),\)
   i.e., \(X^2_{<h} \subseteq \text{pow}_{<h}(X^1_U)\) (together with formula \(\xi^1_U\));

3. \(\xi^3_{<h} \equiv \forall Z^2 \left( Z^2 \in X^3_{<h} \rightarrow \left( (\forall Z^1)(Z^1 \in Z^2 \rightarrow Z^1 \in X^2_{<h}) \right. \right.
   \left. \land (\forall Z^1_1) \ldots (\forall Z^1_h) \left( \bigwedge_{i=1}^h Z^1_i \in Z^2 \rightarrow \bigvee_{i,j=1,i<j} h Z^1_i = Z^1_j \right) \right),\)
   i.e., \(X^3_{<h} \subseteq \text{pow}_{<h}(\text{pow}_{<h}(X^1_U))\) (together with formulae \(\xi^1_U\) and \(\xi^2_{<h}\));

4. either \(\psi^1_i \equiv \forall Z^1(Z^1 \in X^1_i \rightarrow Z^1 \in X^2_{<h})\) or \(\psi^2_i \equiv X^2_i \in X^3_{<h}\), for \(i = 1, \ldots, p\),
   so that, \(X^1_i \subseteq \text{pow}_{<h}(X^1_U)\), for \(i = 1, \ldots, p\) (together with formulae \(\xi^1_U\) and \(\xi^2_{<h}\));

5. \(\psi^3_j \equiv \forall Z^2(Z^2 \in X^3_j \rightarrow Z^2 \in X^3_{<h})\), for \(j = 1, \ldots, k\),
   i.e., \(X^3_j \subseteq \text{pow}_{<h}(\text{pow}_{<h}(X^1_U))\), for \(j = 1, \ldots, k\) (together with formulae \(\xi^1_U\), \(\xi^2_{<h}\), and \(\xi^3_{<h}\));

6. \(\chi\) is a propositional combination of
   (a) quantifier-free atomic formulae of any level,
   (b) purely universal formulae of level 1 of the form
       \((\forall z_1) \ldots (\forall z_n)\varphi_0\),
       with \(n \leq h\),
   (c) purely universal formulae of level 2 of the form
       \((\forall Z^1_1) \ldots (\forall Z^1_m)(((Z^1_1 \in X^2_{<h} \land \ldots \land Z^1_m \in X^2_{<h}) \rightarrow \varphi_1)\),
       where \(m \leq h\) and \(\varphi_1\) is a propositional combination of quantifier-free atomic formulae and of purely universal formulae of level 1 satisfying (6b) above,
   (d) purely universal formulae of level 3 of the form
       \((\forall Z^2_1) \ldots (\forall Z^2_p)(((Z^2_1 \in X^3_{<h} \land \ldots \land Z^2_p \in X^3_{<h}) \rightarrow \varphi_2)\),
       where \(p \leq h\) and \(\varphi_2\) is a propositional combination of quantifier-free atomic formulae, and of purely universal formulae of level 1 and of level 2 satisfying (6b) and (6c) above.
Having defined the fragments \((4LQS^R)^h\), for \(h \geq 2\), next we prove that each of them has an NP-complete satisfiability problem.

**Theorem 4.1.** The satisfiability problem for \((4LQS^R)^h\) is NP-complete, for any \(h \geq 2\).

**Proof:**
The satisfiability problem SAT for propositional logic can be readily reduced to the one for \((4LQS^R)^h\)-formulae, for any \(h \geq 2\), as follows. Given a formula \(\rho \in \text{SAT}\), we construct a quantifier-free \((4LQS^R)^h\)-formula \(\varphi_\rho\) by replacing each propositional letter \(P_i\) in \(\rho\) by the quantifier-free formula \(x_i \in X^1\), where \(X^1\) is a fixed variable of sort 1 and the \(x_i\)s are distinct variables of sort 0 in a one-one correspondence with the distinct propositional letters in \(\rho\). Plainly, \(\rho\) is propositionally satisfiable if and only if \(\varphi_\rho\) is satisfiable by a \(4LQS^R\)-model. Therefore the NP-hardness of the satisfiability problem for \((4LQS^R)^h\)-formulae follows.

To prove that our problem is in NP, we reason as follows. Let

\[
\varphi \equiv \xi_U^1 \land \xi_{<h}^2 \land \xi_{<h}^3 \land \psi_1^1 \land \ldots \land \psi_p^1 \land \psi_1^2 \land \ldots \land \psi_k^2 \land \chi
\]  

be a satisfiable \((4LQS^R)^h\)-formula, and let \(H_\varphi\) be a set of formulae constructed as follows. Initially, we put

\[
H_\varphi := \{\xi_U^1, \xi_{<h}^2, \xi_{<h}^3, \psi_1^1, \ldots, \psi_p^1, \psi_1^2, \ldots, \psi_k^2, \chi\}
\]

and then, we modify \(H_\varphi\) according to the following six rules, until no rule can be further applied:\(^5\)

R1: if \(\xi \equiv \neg \xi_1\) is in \(H_\varphi\), then \(H_\varphi = (H_\varphi \setminus \{\xi\}) \cup \{\xi_1\}\),

R2: if \(\xi \equiv \xi_1 \land \xi_2\) (resp., \(\xi \equiv \neg (\xi_1 \lor \xi_2)\)) is in \(H_\varphi\) (i.e., \(\xi\) is a conjunctive formula), then we put \(H_\varphi := (H_\varphi \setminus \{\xi\}) \cup \{\xi_1, \xi_2\}\) (resp., \(H_\varphi := (H_\varphi \setminus \{\xi\}) \cup \{\neg \xi_1, \neg \xi_2\}\)),

R3: if \(\xi \equiv \xi_1 \lor \xi_2\) (resp., \(\xi \equiv \neg (\xi_1 \land \xi_2)\)) is in \(H_\varphi\) (i.e., \(\xi\) is a disjunctive formula), then we choose a \(\xi_i, i \in \{1, 2\}\), such that \(H_\varphi \cup \{\xi_i\}\) (resp., \(H_\varphi \cup \{\neg \xi_i\}\)) is satisfiable and put \(H_\varphi := (H_\varphi \setminus \{\xi\}) \cup \{\xi_i\}\) (resp., \(H_\varphi := (H_\varphi \setminus \{\xi\}) \cup \{\neg \xi_i\}\)),

R4: if \(\xi \equiv \neg (\forall z_1) \ldots (\forall z_n)\varphi_0\) is in \(H_\varphi\), then \(H_\varphi := (H_\varphi \setminus \{\xi\}) \cup \{\neg (\varphi_0)_{\bar{z}_1} \ldots \bar{z}_n\}\), where \(\bar{z}_1, \ldots, \bar{z}_n\) are newly introduced variables of sort 0,

R5: if \(\xi \equiv \neg (\forall Z_1^1) \ldots (\forall Z_m^1)\varphi_1\) is in \(H_\varphi\), then \(H_\varphi := (H_\varphi \setminus \{\xi\}) \cup \{\neg (\varphi_1)_{\bar{Z}_1^1} \ldots \bar{Z}_m^1\}\), where \(\bar{Z}_1^1, \ldots, \bar{Z}_m^1\) are fresh variables of sort 1,

R6: if \(\xi \equiv \neg (\forall Z_1^2) \ldots (\forall Z_m^2)\varphi_2\) is in \(H_\varphi\), then \(H_\varphi := (H_\varphi \setminus \{\xi\}) \cup \{\neg (\varphi_2)_{\bar{Z}_1^2} \ldots \bar{Z}_m^2\}\), where \(\bar{Z}_1^2, \ldots, \bar{Z}_m^2\) are newly introduced variables of sort 2.

Plainly, the above construction terminates in \(O(|\varphi|)\) steps and if we put \(\psi \equiv \bigwedge_{\xi \in H_\varphi} \xi\), it turns out that

(a) \(\psi\) is a satisfiable \((4LQS^R)^h\)-formula,

(b) \(|\psi| = O(|\varphi|)\), and

\(^{5}\)We recall that an implication \(A \rightarrow B\) has to be regarded as a shorthand for the disjunction \(\neg A \lor B\).
(c) $\psi \rightarrow \forall \varphi$ is a valid $4LQS^R$-formula.

In view of (a)–(c) above, to prove that our problem is in $\text{NP}$, it is enough to construct in nondeterministic polynomial time a $4LQS^R$-interpretation and show that we can check in polynomial time that it actually satisfies $\psi$.

Let $\mathcal{M} = (D, M)$ be a $4LQS^R$-model for $\psi$ and let $\mathcal{M}^* = (D^*, M^*)$ be the relativized interpretation of $\mathcal{M}$ with respect to a domain $D^*$ satisfying (2), hence such that $|D^*| = O(|\psi|^{h+1})$, since $\psi$ is a $(4LQS^R)^h$-formula (cf. Theorem 3.1 and the construction described in Sections 3.2.1 and 3.2.2).

In view of the remarks just before Definition 4.1, to complete our proof it is enough to check that

- $M^*X^2 \subseteq \text{pow}_{<h}(D^*)$, for any free variable $X^2$ of sort 2 in $\psi$ (which entails that $|M^*X^2| = O(|D^*|^h)$),
- $M^*X^3 \subseteq \text{pow}_{<h}(\text{pow}_{<h}(D^*))$, for any free variable $X^3$ of sort 3 in $\psi$ (which entails that $|M^*X^3| = O(|D^*|^h)$), and
- $\mathcal{M}^* \models \psi$ can be verified in deterministic polynomial time.

To prove that $M^*X^2 \subseteq \text{pow}_{<h}(D^*)$, for any free variable $X^2$ in $\psi$, we reason as follows. Let $X^2$ be a variable of sort 2 occurring free in $\psi$. From Definition 3.1 we recall that

$$M^*X^2 = ((MX^2 \cap \text{pow}(D^*)) \setminus \{M^*X^1 : X^1 \in (V'_i \cup V'_F)\}) \cup \{M^*X^1 : X^1 \in (V'_i \cup V'_F), MX^1 \in MX^2\}. \quad (13)$$

Observe that

$$MX^2 \subseteq \text{pow}_{<h}(D). \quad (14)$$

Indeed, if the variable $X^2$ coincides with $X^2_{<h}$, then (14) follows from the fact that $\psi$ contains the conjunct $\xi^2_{<h}$. On the other hand, if $X^2$ is distinct from $X^2_{<h}$, then $\psi$ contains either the conjunct $(\forall Z^1)(Z^1 \in X^2 \rightarrow Z^1 \in X^2_{<h})$ or the conjunct $X^2 \in X^3_{<h}$. In the first case, $(\forall Z^1)(Z^1 \in X^2 \rightarrow Z^1 \in X^2_{<h})$ together with the conjunct $\xi^2_{<h}$, implies again (14). From (13) and (14), we get $M^*X^2 \subseteq \text{pow}_{<h}(D^*)$.

The other case is handled in a similar way.

Checking that $M^*X^3 \subseteq \text{pow}_{<h}(\text{pow}_{<h}(D^*))$, for any free variable $X^3$ of sort 3 in $\psi$, can be carried out much as was done for free variables of sort 2.

From what we have shown so far, it follows that in nondeterministic polynomial time one can construct

- the $(4LQS^R)^h$-formula $\psi$, as a result of applications of rules R1–R6 to the initial set $H_\varphi$ (corresponding to the input formula $\varphi$) until saturation is reached,
- the $4LQS^R$-interpretation $\mathcal{M}^* = (D^*, M^*)$ (of $\psi$).

By the soundness of rules R1–R6, it follows that the $4LQS^R$-formula $\psi \rightarrow \forall \varphi$ is valid. Thus, we obtain a succinct certificate of the satisfiability of $\varphi$ if we show that it is possible to check in polynomial time that $\mathcal{M}^* \models \psi$ holds. This is equivalent to show that we can check in polynomial time that $\mathcal{M}^* \models \xi$, for every conjunct $\xi$ of $\psi$. We distinguish the following cases.
ξ is a quantifier-free atomic formula: Since all variables in ξ are interpreted by \( \mathcal{M}^* \) with sets of polynomial size, the task of checking memberships and equalities among such sets can be performed in polynomial time.

ξ is a purely universal formula of level 1 \((∀z_1)\ldots(∀z_n)φ_0\), with \( n \leq h \): We have that \( \mathcal{M}^* \models (∀z_1)\ldots(∀z_n)φ_0 \) if and only if \( \mathcal{M}^*[z_1/u_1,\ldots,z_n/u_n] \models φ_0 \), for every \( u_1,\ldots,u_n \in D^* \).

From the previous case, for any \( u_1,\ldots,u_n \in D^* \), one can compute in polynomial time whether \( \mathcal{M}^*[z_1/u_1,\ldots,z_n/u_n] \models φ_0 \). Since the collection of such \( n \)-tuples \( u_1,\ldots,u_n \in D^* \) has polynomial size in \(|φ|\), it turns out that one can check that \( \mathcal{M}^* \models (∀z_1)\ldots(∀z_n)φ_0 \) in polynomial time.

ξ is a purely universal formula of level 2: If

\[
ξ \equiv ξ^2_{<h} \equiv (∀Z^1)((Z^1 \in X^2_{<h} \rightarrow ((∀z_1)\ldots(∀z_h)(\bigwedge_{i=1}^h z_i \in Z^1 \rightarrow (\bigvee_{i<j}^h z_i = z_j))))),
\]

in order to verify that \( \mathcal{M}^* \models ξ \), it is enough to check that \( \mathcal{M}^* X^2_{<h} \subseteq \text{pow}_{<h}(D^*) \), which can be clearly done in polynomial time.

If \( ξ \equiv (∀Z^1)(Z^1 \in X \rightarrow Z^1 \in X^2_{<h}) \), with \( X \) a free variable of sort 2, then in order to verify that \( \mathcal{M}^* \models ξ \) it is enough to check whether \( \mathcal{M}^* X \subseteq \mathcal{M}^* X^2_{<h} \), which again can be done in polynomial time.

Finally, if \( ξ \equiv (∀Z^1)\ldots(∀Z^m)(Z^1_1 \in X^2_{<h} \land \ldots \land Z^1_m \in X^2_{<h} \rightarrow φ_1) \) where \( m \leq h \) and \( φ_1 \) is a propositional combination of quantifier-free atomic formulae and of purely universal formulae of level 1 of the form \((∀z_1)\ldots(∀z_n)φ_0\), with \( n \leq h \) (cf. Definition 4.1(6c)), then \( \mathcal{M}^* \models ξ \) if and only if \( \mathcal{M}^*[Z^1_1/U^1_1,\ldots,Z^1_m/U^1_m] \models φ_1 \), for every \( U^1_1,\ldots,U^1_m \in M^* X^2_{<h} \).

Again, the latter task can be accomplished in polynomial time, since, in view of the previous two cases \( \mathcal{M}^*[Z^1_1/U^1_1,\ldots,Z^1_m/U^1_m] \models φ_1 \) can be checked in polynomial time, for each \( m \)-tuple \( U^1_1,\ldots,U^1_m \in M^* X^2_{<h} \), and the number of such \( m \)-tuples is polynomial.

ξ is a purely universal formula of level 3: This case can be handled much along the same lines of the previous case.

Summing up, we have shown that the satisfiability problem for \((4LQS^R)^h\)-formulae is in \( \text{NP} \). This, together with its \( \text{NP} \)-hardness, which was shown before, implies the \( \text{NP} \)-completeness of our problem.

\[ \square \]

In the next section we show how the fragment \( 4LQS^R \) can be used to formalize the modal logic \( K45 \).

4.2. Applying \( 4LQS^R \) to modal logic

The modal language \( L_M \) is based on a countably infinite set of propositional letters \( P = \{p_1,p_2,\ldots\} \), the classical propositional connectives ‘¬’, ‘∧’, and ‘∨’, the modal operators ‘□’, ‘◊’ (and the parentheses). \( L_M \) is the smallest set such that \( P \subseteq L_M \), and such that if \( φ,ψ \in L_M \), then \( ¬φ,φ \land ψ,φ \lor ψ,□φ,◊φ \in L_M \). Lower case letters like \( p \) denote elements of \( P \) and Greek letters like \( φ \) and \( ψ \) represent
formulas of $L_M$. Given a formula $\varphi$ of $L_M$, we indicate with $SubF(\varphi)$ the collection of the subformulæ of $\varphi$. The modal depth of a formula $\varphi$ is the maximum nesting depth of modalities occurring in $\varphi$. In the rest of the paper we also make use of the propositional connective ‘$\rightarrow$’ defined in terms of ‘$\neg$’ and ‘$\lor$’ as: $\varphi \rightarrow \psi \equiv \neg \varphi \lor \psi$.

A normal modal logic $N$ is any subset of $L_M$ which contains all the tautologies and the axiom

$$K: \ (p_1 \rightarrow p_2) \rightarrow (\Box p_1 \rightarrow \Box p_2),$$

and which is closed with respect to the following rules:

(Modus Ponens): if $\varphi, \varphi \rightarrow \psi \in N$, then $\psi \in N$,

(Necessitation): if $\varphi \in N$, then $\Box \varphi \in N$,

(Substitution): if $\varphi \in N$, then $s \varphi \in N$,

where $\varphi, \psi \in L_M$, and the formula $s \varphi$ is the result of uniformly substituting in $\varphi$ propositional letters with formulæ (the reader may consult a text on modal logic like [2] for more details).

A Kripke frame is a pair $\langle W, R \rangle$ such that $W$ is a nonempty set of possible worlds and $R$ is a binary relation on $W$ called accessibility relation. If $R(w, u)$ holds, we say that the world $u$ is accessible from the world $w$. A Kripke model is a triple $\langle W, R, h \rangle$, where $\langle W, R \rangle$ is a Kripke frame and $h$ is a function mapping propositional letters into subsets of $W$. Thus, $h(p)$ is the set of all the worlds in which $p$ is true.

Let $K = \langle W, R, h \rangle$ be a Kripke model and let $w$ be a world in $K$. Then, for every $p \in \mathcal{P}$ and for every $\varphi, \psi \in L_M$, the satisfaction relation $|=\,\text{is defined as follows:}$

- $K, w |= p$ iff $w \in h(p)$;
- $K, w |= \varphi \lor \psi$ iff $K, w |= \varphi$ or $K, w |= \psi$;
- $K, w |= \varphi \land \psi$ iff $K, w |= \varphi$ and $K, w |= \psi$;
- $K, w |= \neg \varphi$ iff $K, w \not|= \varphi$;
- $K, w |= \Box \varphi$ iff $K, w' |= \varphi$, for every $w' \in W$ such that $(w, w') \in R$;
- $K, w |= \Diamond \varphi$ iff there is a $w' \in W$ such that $(w, w') \in R$ and $K, w' |= \varphi$.

A formula $\varphi$ is said to be satisfied at $w$ in $K$ if $K, w |= \varphi$; $\varphi$ is said to be valid in $K$ (and we write $K |= \varphi$), if $K, w |= \varphi$, for every $w \in W$.

The smallest normal modal logic is $K$, which contains only the modal axiom $K$ and whose accessibility relation $R$ can be any binary relation. The other normal modal logics admit together with $K$ other modal axioms drawn from the ones in Table 3.

Translation of a normal modal logic into the $4LQS^R$ language is based on the semantics of propositional and modal operators. For any normal modal logic, the formalization of the semantics of modal operators depends on the axioms that characterize the logic.

In the case of the logic $K45$, whose decision problem has been shown to be NP-complete in [15], the modal formulæ $\Box \varphi$ and $\Diamond \varphi$ can be expressed in the $4LQS^R$ language and thus the logic $K45$ can be entirely translated into the $4LQS^R$ fragment. This is shown in what follows.
4.2.1. The logic $K45$

The normal modal logic $K45$ is obtained from the logic $K$ by adding to $K$ the axioms 4 and 5 listed in Table 3. Semantics of the modal operators $\Box$ and $\Diamond$ for the logic $K45$ can be described as follows. Given a formula $\varphi$ of $K45$ and a Kripke model $K = (W, R, h)$, we put:

- $K \models \Box \varphi$ if $K, v \models \varphi$, for every $v \in W$ s.t. there is a $w' \in W$ with $(w', v) \in R$,
- $K \models \Diamond \varphi$ if $K, v \models \varphi$, for some $v \in W$ s.t. there is a $w' \in W$ with $(w', v) \in R$.

This formulation allows one to express a formula $\varphi$ of $K45$ into the $4LQS^R$ fragment. In order to simplify the definition of the translation function $\tau_{K45}$ introduced below, we give the notion of the “empty formula”, to be denoted by $\Lambda$, and which will not be interpreted in any particular way. The only requirement on $\Lambda$ needed for the definitions to be given below is that $\Lambda \land \psi$ and $\psi \land \Lambda$ must be regarded as syntactic variations of $\psi$, for any $4LQS^R$-formula $\psi$.

Intuitively, the translation function $\tau_{K45}$ associates to each formula $\varphi$ of $K45$ a $4LQS^R$-formula defining a variable $X_\varphi$ of sort 1, which denotes the subset $W_\varphi$ of $W$ such that $K, w \models \varphi$ if and only if $w \in W_\varphi$, for every Kripke model $K = (W, R, h)$. We proceed as follows.

For every propositional letter $p$, let $\tau_{K45}^1(p) = X^1_p$, with $X^1_p \in \mathcal{V}_1$, and let $\tau_{K45}^2 : K45 \rightarrow 4LQS^R$ be the function defined recursively as follows:

- $\tau_{K45}^2(\Lambda) = \Lambda$,
- $\tau_{K45}^2(\neg \varphi) = (\forall z)(z \in X^1_{\neg \varphi} \iff \neg (z \in X^1_\varphi)) \land \tau_{K45}^2(\varphi)$,
- $\tau_{K45}^2(\varphi \land \varphi_2) = (\forall z)(z \in X^1_{\varphi \land \varphi_2} \iff (z \in X^1_\varphi \land z \in X^1_{\varphi_2})) \land \tau_{K45}^2(\varphi) \land \tau_{K45}^2(\varphi_2)$,
- $\tau_{K45}^2(\varphi \lor \varphi_2) = (\forall z)(z \in X^1_{\varphi \lor \varphi_2} \iff (z \in X^1_\varphi \lor z \in X^1_{\varphi_2})) \land \tau_{K45}^2(\varphi) \land \tau_{K45}^2(\varphi_2)$,
- $\tau_{K45}^2(\Box \psi) = (\forall z_1)((\neg (\forall z_2)\neg((z_2, z_1) \in X^3_R)) \rightarrow z_1 \in X^1_\psi) \land (\forall z)(z \in X^1_\psi) \land \neg (z \in X^1_\psi)$
  \land \neg (\forall z_1)((\neg (\forall z_2)\neg((z_2, z_1) \in X^3_R)) \land (\forall z)(z \in X^1_\psi) \land \tau_{K45}^2(\psi)$,
- $\tau_{K45}^2(\Diamond \psi) = (\forall z_1)((\neg (\forall z_2)\neg((z_2, z_1) \in X^3_R)) \land (\forall z)(z \in X^1_\psi) \land (\forall z)(z \in X^1_\psi) \land (\forall z)(z \in X^1_\psi) \land \tau_{K45}^2(\psi)$.

Table 3. Axioms of normal modal logics

| Axiom | Schema | Condition on $R$ (see Table 1) |
|-------|--------|--------------------------------|
| T     | $\Box p \rightarrow p$ | Reflexive |
| 5     | $\Diamond p \rightarrow \Box \Diamond p$ | Euclidean |
| B     | $p \rightarrow \Box \Diamond p$ | Symmetric |
| 4     | $\Box p \rightarrow \Box \Box p$ | Transitive |
| D     | $\Box p \rightarrow \Diamond p$ | Serial: $(\forall w)(\exists u)R(w, u)$ |
where $\Lambda$ is the empty formula, $X^1_\varphi, X^1_{\varphi \land \varphi'}, X^1_{\varphi \lor \varphi'}, X^1_\varphi, X^1_{\varphi'}, \psi, X^1_\psi, \varphi', \varphi_1, \varphi_2 \in \mathcal{V}_1$, and $X^3_\varphi \in \mathcal{V}_3$.

Finally, for every $\varphi$ in $K45$, if $\varphi$ is a propositional letter in $P$ we put $\tau_{K45}(\varphi) = \tau_{K45}^1(\varphi)$, otherwise $\tau_{K45}(\varphi) = \tau_{K45}^2(\varphi)$. Next, by means of the following formulae, we characterize a variable $X^3_\psi$ of sort 3, intended to denote the accessibility relation $R$ of the logic $K45$:

- $\chi_1 = (\forall z_1)(\forall z_2)((z_1, z_2) \in X^3_{\varphi_2})$,
- $\chi_2 = (\forall z_2)((Z^2 \in X^3_{\varphi_2}) \rightarrow ((Z^2 \in X^3_{\varphi_1} \leftrightarrow (\forall z_1)(\forall z_2) - ((z_1, z_2) \in Z^2)))$,
- $\chi_3 = (\forall z_1, z_2, z_3)((z_1, z_2) \in X^3_{\varphi_1} \land (z_2, z_3) \in X^3_{\varphi_2}) \rightarrow (z_1, z_3) \in X^3_{\varphi_3}),$
- $\chi_4 = (\forall z_1, z_2, z_3)((z_1, z_2) \in X^3_{\varphi_1} \land (z_1, z_2) \in X^3_{\varphi_2}) \rightarrow (z_2, z_3) \in X^3_{\varphi_3}),$
- $\psi^2 = (\forall z_2)(Z^2 \in X^3_R \rightarrow Z^2 \in X^3_{\varphi_2}).$

Correctness of the translation is stated by the following lemma.

**Lemma 4.1.** For every formula $\varphi$ of the logic $\tau_{K45}$, $\varphi$ is satisfiable in a model $K = \langle W, R, h \rangle$ if and only if there is a $4LQS^R$ interpretation satisfying $x \in X^1_\varphi$.

**Proof:**

Let $\bar{w}$ be a world in $W$. We construct a $4LQS^R$ interpretation $\mathcal{M} = (W, M)$ as follows:

- $Mx = \bar{w}$,
- $MX^1_p = h(p)$, where $p$ is a propositional letter and $X^1_p = \tau_{K45}(p)$,
- $M \tau_{K45}(\psi) = \text{true}$, for every subformula $\psi$ of $\varphi$, distinct from a propositional letter.

To prove the lemma, it would be enough to show that $K, \bar{w} \models \varphi$ iff $M \models x \in X^1_\varphi$. However, it is more convenient to prove the following more general property:

Given a $w \in W$ and a $v \in \mathcal{V}_0$ such that $My = w$, we have

$$K, w \models \varphi \iff M \models y \in X^1_\varphi.$$

We proceed by structural induction on $\varphi$ by considering for simplicity only the relevant cases in which $\varphi = \square \psi$ and $\varphi = \diamond \psi$.

- Let $\varphi = \square \psi$ and assume that $K, w \models \square \psi$. Let $v$ be a world of $W$ such that $\langle u, v \rangle \in R$ for some $u \in W$, and let $x_1, x_2 \in \mathcal{V}_0$ be such that $v = Mx_1$ and $u = Mx_2$. We have that $K, v \models \psi$ and, by inductive hypothesis, $M \models x_1 \in X^1_\psi$. Since $M \models \tau_{K45}(\square \psi)$, then $M \models (\forall z_1)((\neg(\forall z_2) - (z_2, z_1) \in X^3_{\varphi_1}) \rightarrow (z_1, z_1) \in X^1_\psi \rightarrow (\forall z_1)(\forall z_2)(z_2, z_1) \in X^3_{\varphi_1}) \rightarrow (z_1, z_1) \in X^1_\psi \rightarrow (\forall z_1)(\forall z_2)(z_2, z_1) \in X^3_{\varphi_1}) \rightarrow (z_1, z_1) \in X^1_\psi).$ Hence $M[z_1/v, z_2/u, z/w] \models ((z_2, z_1) \in X^3_R \rightarrow z_1 \in X^1_\psi) \rightarrow z \in X^1_\psi$ and thus $M \models ((x_2, x_1) \in X^3_R \rightarrow x_1 \in X^1_\psi) \rightarrow y \in X^1_\psi.$ Since $M \models (x_2, x_1) \in X^3_R \rightarrow x_1 \in X^1_\psi$, by modus ponens we have the thesis. The thesis follows also in the case in which there is no $u$ such that $\langle u, v \rangle \in X^3_{\varphi_1}$. In fact, in that case $M \models (x_2, x_1) \in X^3_R \rightarrow x_1 \in X^1_\psi$ holds for any $x_2 \in \mathcal{V}_0$.

Consider next the case in which $K, w \not\models \square \psi$. Then, there must be a $v \in W$ such that $\langle u, v \rangle \in X^3_{\varphi_1}$, for some $u \in W$, and $K, v \not\models \psi$. Let $x_1, x_2 \in \mathcal{V}_0$ be such that $Mx_1 = v$ and $Mx_2 = u$. Then, by inductive hypothesis, $M \not\models x_1 \in X^1_\psi$. 

By definition of $M$, we have $M \models \neg((\forall z_1)\neg((\neg((\forall z_2)\neg((z_2, z_1) \in \mathcal{X}^3_R)) \land \neg(z_1 \in \mathcal{X}^1_\psi)) \rightarrow (\forall z)\neg(z \in \mathcal{X}^1_{\Box\psi})$. By the above instantiations and by the hypotheses, we have that $M \models (((x_2, x_1) \in \mathcal{X}^3_R) \land \neg(x_1 \in \mathcal{X}^1_\psi)) \rightarrow \neg(y \in \mathcal{X}^1_{\Box\psi})$ and $M \models ((x_2, x_1) \in \mathcal{X}^3_R) \land \neg(x_1 \in \mathcal{X}^1_\psi)$. Thus, by modus ponens, we obtain the thesis.

- Let $\varphi = \Diamond\psi$ and assume that $K, w \models \Diamond\psi$. Then there are $u, v \in \mathcal{W}$ such that $\langle u, v \rangle \in \mathcal{R}$ and $K, v \models \psi$. Let $x_1, x_2 \in \mathcal{V}_0$ be such that $Mx_1 = v$ and $Mx_2 = u$. Then, by inductive hypothesis, $M \models x_1 \in \mathcal{X}^1_\psi$. Since $M \models \tau_{K45}(\Diamond\psi)$, it follows that $M \models \neg((\forall z_1)\neg((\neg((\forall z_2)\neg((z_2, z_1) \in \mathcal{X}^3_R)) \land \neg(z_1 \in \mathcal{X}^1_\psi)) \rightarrow (\forall z)\neg(z \in \mathcal{X}^1_{\Box\psi})$. By the hypotheses and the variable instantiations above it follows that $M \models (((x_2, x_1) \in \mathcal{X}^3_R) \land x_1 \in \mathcal{X}^1_\psi) \rightarrow y \in \mathcal{X}^1_{\Box\psi}$ and $M \models ((x_2, x_1) \in \mathcal{X}^3_R) \land x_1 \in \mathcal{X}^1_\psi$. Finally, by an application of modus ponens the thesis follows.

On the other hand, if $K, w \not\models \Diamond\psi$, then for every $v \in \mathcal{W}$, either there is no $u \in \mathcal{W}$ such that $\langle u, v \rangle \in \mathcal{R}$, or $K, v \not\models \psi$. Let $x_1, x_2 \in \mathcal{V}_0$ be such that $Mx_1 = v$ and $Mx_2 = u$. If $K, v \not\models \psi$, by inductive hypothesis, we have that $M \not\models y \in \mathcal{X}^1_\psi$.

Since $M \models ((\forall z_1)(((\forall z_2)\neg((z_2, z_1) \in \mathcal{X}^3_R)) \lor \neg(z_1 \in \mathcal{X}^1_\psi)) \rightarrow (\forall z)\neg(z \in \mathcal{X}^1_{\Box\psi})$, by the hypotheses and by the variable instantiations above we get $M \models \neg((x_2, x_1) \in \mathcal{X}^3_R) \lor \neg(x_1 \in \mathcal{X}^1_\psi) \rightarrow \neg(y \in \mathcal{X}^1_{\Box\psi})$ and $M \models \neg((x_2, x_1) \in \mathcal{X}^3_R) \lor \neg(x_1 \in \mathcal{X}^1_\psi))$. Finally, by modus ponens we infer the thesis.

It can be easily verified that $\tau_{K45}(\varphi)$ is polynomial in the size of $\varphi$ and that its satisfiability can be checked in nondeterministic polynomial time since the formula

$$\xi^1_{\Box W} \land \xi^2_{\Box d} \land \xi^3_{\Box 3} \land \psi^2_1 \land (\chi_1 \land \chi_2 \land \chi_3 \land \chi_4 \land \tau_{K45}(\varphi))$$

belongs to $(4LQS^R)^3$. Thus, the decision algorithm for $4LQS^R$ we have presented and the translation function described above yield a nondeterministic polynomial decision procedure for testing the satisfiability of any formula $\varphi$ of K45.

5. Conclusions and future work

We have presented a decidability result for the satisfiability problem for the fragment $4LQS^R$ of multi-sorted stratified syllogistic embodying variables of four sorts and a restricted form of quantification. As the semantics of the modal formulae $\Box \varphi$ and $\Diamond \varphi$ in the modal logic $K45$ can be easily formalized in a fragment of $4LQS^R$, admitting a nondeterministic polynomial decision procedure, we obtained an alternative proof of the NP-completeness of $K45$. The results reported in the paper offer numerous hints of future work, some of which are discussed in what follows.

Recently, we have analyzed several fragments of elementary set theory. It will be interesting to ameliorate existing techniques to verify in a formal way the truth of expressivity results that for the moment we have only conjectured. Moreover, we plan to find complexity results for the fragments $3LQS^R$ (cf. [2]) and $4LQS^R$, and for some of their sublanguages like, for instance, the sublanguages of $4LQS^R$ characterized by the fact that quantifier prefixes have length bounded by a constant. According

$^{6}$\(\xi^1_{\Box W}\) is intended to characterize a nonempty set of possible worlds.
to the construction of Section 3.2.1 small models for formulae of these sublanguages have a finite domain $D^*$ that is polynomial in the size of the formula. However, their formulae are not subject to the syntactical constraints characterizing formulae of the $(4LQS^R)^h$ languages and allowing the satisfiability problem for the $(4LQS^R)^h$ fragments to be NP-complete.

As we mentioned in the Introduction, stratified syllogistics have been studied less than one sorted multi-level ones. Thus, a comparison of the results obtained in this paper with the results regarding one sorted multi-level set theoretic decidability is in order.

Formalizations of modal logics in set theory have already been provided within the framework of hyperset theory [11] and of weak set theories [11], without the extensionality and foundation axioms.

We intend to continue our study, started with [9], concerning the limits and possibilities of expressing modal, and more generally, non-classical logics in the context of stratified syllogistics. Currently, in the case of modal logics characterized by a liberal accessibility relation like $K$, we are not able to translate the modal formulae $\Box \varphi$ and $\Diamond \varphi$ in $4LQS^R$. We plan to verify if $4LQS^R$ allows one to express modal logics with nesting of modal operators of bounded length. We also intend to investigate extensions of $4LQS^R$ which allow one to express suitably constrained occurrences of the composition operator on binary relations and of the set-theoretic operator of general union. We expect that these extensions will make it possible to express all the normal modal logic systems and several multi-modal logics. Finally, since within $4LQS^R$ we are able to express Boolean operations on relations, we plan to investigate the possibility of translating fragments of Boolean modal logic and expressive description logics admitting boolean constructors over roles.

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