Spatial Decay Bounds for the Brinkman Fluid Equations in Double-Diffusive Convection

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Abstract: In this paper, we consider the Brinkman equations pipe flow, which includes the salinity and the temperature. Assuming that the fluid satisfies nonlinear boundary conditions at the finite end of the cylinder, using the symmetry of differential inequalities and the energy analysis methods, we establish the exponential decay estimates for homogeneous Brinkman equations. That is to prove that the solutions of the equation decay exponentially with the distance from the finite end of the cylinder. To make the estimate of decay explicit, the bound for the total energy is also derived.

Keywords: spatial decay estimates; Brinkman equations; Saint-Venant principle

1. Introduction

The Brinkman equations are one of the most important models in fluid mechanics. This model are mainly used to describe flow in a porous medium. For more details, one can refer to Nield and Bejan [1] and Straughan [2]. In the present paper, we define the Brinkman flow depending on the salinity and the temperature in a semi-infinite cylindrical pipe and derive the spatial decay properties. When the homogeneous initial-boundary conditions are applied on the lateral surface of the cylinder, We prove that the solutions of Brinkman equations decays exponentially with spatial variable.

In fact, the Brinkman equations have been studied by many papers in the literature. For example, Straughan [2] considered the mathematical properties of Brinkman equations as well as Darcy and Forchheimer equations, and stated how these equations describe the flow of porous media. Ames and Payne [3] studied the structural stability for the solutions to the viscoelasticity in an ill-posed problem. Franchi and Straughan [4] proved the structural stability for the solutions to the Brinkman equations in porous media in a bounded region. More relevant results one can see [5–10]. Paper [11] studied the double diffusive convection in porous medium and obtained the structural stability for the solutions. The continuous dependence for a thermal convection model with temperature-dependent solubility can be found in [12]. For more recent work about continuous dependence, one may refer to [13–19].

In this paper, let \( R \) be a semi-infinite cylinder and \( \partial R \) represents the boundary of \( R \). \( D \) denotes the cross section of the cylinder with the smooth boundary \( \partial D \) (see Figure 1).

In this paper, we also use the following notations

\[
R_z = \{(x_1, x_2, x_3) | (x_1, x_2) \in D, \quad x_3 > z \geq 0\},
\]

\[
D_z = \{(x_1, x_2, x_3) | (x_1, x_2) \in D, \quad x_3 = z \geq 0\},
\]

where \( z \) is a point along the \( x_3 \) axis. Clearly, \( R_0 = R \) and \( D_0 = D \). Letting \( u, T, C \) and \( p \) denote the fluid velocity, temperature, salt concentration and pressure, respectively.
Figure 1. Cylindrical pipe.

The Brinkman equations we study can be written as

\[
\frac{\partial u_i}{\partial t} = \nu \Delta u_i - k_1 u_i - p, \quad in \quad R \times \{t \geq 0\}, \quad (1)
\]

\[
\frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} = k_2 \Delta T, \quad in \quad R \times \{t \geq 0\}, \quad (2)
\]

\[
\frac{\partial C}{\partial t} + u_i \frac{\partial C}{\partial x_i} = k_3 \Delta C + \sigma \Delta T, \quad in \quad R \times \{t \geq 0\}, \quad (3)
\]

\[
u, \sigma > 0 \] denote the Brinkman coefficient, and the Soret coefficient, respectively. \( k_1, k_2, k_3 > 0 \). Without losing generality, we take them equal to 1. \( \Delta \) is the Laplacian operator. \( g_i(x) \) and \( h_i(x) \) are gravity field, which are given functions. We suppose that (1)–(4) have the following initial-boundary conditions

\[
u_i = 0, \quad T = C = 0, \quad on \quad \partial D \times \{t \geq 0\}, \quad (5)
\]

\[
u_i = 0, \quad T = C = 0, \quad on \quad R \times \{t = 0\}. \quad (6)
\]

\[
u_i = f_i(x_1, x_2, t), \quad T = F(x_1, x_2, t), \quad C = G(x_1, x_2, t), \quad on \quad D_0 \times \{t \geq 0\}, \quad (7)
\]

\[
u_i, \nu_i, \nu_i, T, T, C, C, p = o(x_3^{-1}) \ \text{uniformly in} \ x_1, x_2, t, \ \text{as} \ x_3 \to \infty. \quad (8)
\]

In (1)–(8) and in the following, the usual summation convention is employed with repeated Latin subscripts summed from 1 to 3 and repeat Greek subscript summed from 1 to 2. The comma is used to indicate partial differentiation, i.e., \( \nu_{ij} = \sum_{i,j=1}^3 \left( \frac{\partial \nu_i}{\partial x^j} \right)^2 \), \( \varphi_{\alpha,\beta} = \sum_{\alpha,\beta=1}^2 \left( \frac{\partial \varphi_2}{\partial x^\beta} \right)^2 \).

The purpose of this paper is to consider the spatial decay properties of the equations (1)–(8) in a semi-infinite cylindrical pipe by using the symmetry of differential inequalities, that is, to prove that the solutions of the equations decay exponentially with the distance from the finite end of the cylindrical pipe.

In Section 2, some auxiliary inequalities are presented. We establish some useful lemmas in Section 3. The spatial exponential decay estimate for the solution is established in Section 4. Finally, in Section 5 we derive the bounds for the total energies.

### 2. Auxiliary Results

In this paper, we will use some inequalities in the following sections. Thus, we firstly list them as follows.
Lemma 1. Let $D$ be a plane domain $D$ with the smooth boundary $\partial D$. If $w = 0$ on $\partial D$, then
\[
\int_D w \cdot w_{,\alpha} dA \geq \lambda_1 \int_{R_z} w^2 dx, \tag{9}
\]
where $\lambda_1$ is the smallest eigenvalue of the problem
\[
\Delta \phi + \lambda \phi = 0 \quad \text{in } D,
\]
\[
\phi = 0 \quad \text{on } \partial D.
\]
Many papers have studied this inequality, e.g., one may see [21,22].

A representation theorem will be also used in next sections. We write this theorem as

Lemma 2. Let $D$ be a plane Lipschitz bound region and $w$ be a differential function in $D$ which satisfies $\int_D \nabla w \cdot \nabla \phi = 0$, then there exists a vector function $\varphi_{\alpha}(x_1, x_2)$ such that
\[
\varphi_{\alpha,\alpha} = w \quad \text{in } D,
\]
\[
\varphi_\alpha = 0 \quad \text{on } \partial D,
\]
and a positive constant $\Lambda$ depending only on the geometry of $D$ such that
\[
\int_D \varphi_{\alpha,\beta} \varphi_{\alpha,\beta} dA \leq \Lambda \int_D \varphi_{\alpha,\alpha}^2 dA. \tag{10}
\]

The Lemma 2 was proofed by Babuška and Aziz [23] and Horgan and Wheeler [24] have used the Lemma 1 to viscous flow problems. The explicit upper bound of $\Lambda$ can be found in Horgan and Payne [25]. In this paper, this Lemma 2 is used to eliminate the pressure function difference terms $p$, since we can prove that $u_3$ satisfy the hypothesis of this Lemma 2 later.

If $w \in C^1_0(D)$ and $w \in C^1_0(R)$, the following Sobolev inequalities hold
\[
\int_D w^4 dA \leq \frac{1}{2} \left[ \int_D w^2 dA \right] \left[ \int_D w_{,\alpha} w_{,\alpha} dA \right], \tag{11}
\]
\[
\int_{R_z} w^6 dx \leq \Omega \left[ \int_{R_z} w_{,\alpha} w_{,\alpha} dA \right]^3. \tag{12}
\]
For (11), we assume that $w \to 0$ as $x_3 \to \infty$. Payne [26] has given the derivation of (12). For a special case of the results one can see [27,28]. They have obtained the optimal value of $\Omega$
\[
\Omega = \frac{1}{27} \left( \frac{3}{4} \right)^4.
\]

In the following, we also use the following lemma.

Lemma 3. If $w \in C^1(R_z)$, $w_{,\alpha} \big|_{\partial D} = 0$ and $w_{,\alpha} \to 0$ as $x_3 \to \infty$, then
\[
\int_{D_z} \left( w_{,\alpha} w_{,\alpha} \right)^2 dA \leq 4\sqrt{\Omega} \left[ \int_{R_z} w_{,\alpha} w_{,\alpha} dA \right]^2. \tag{13}
\]

We will also use the following lemmas which were derived in [29].
Lemma 4. Let that the function $\varphi$ is the solution of the problem

\[
\Delta \varphi = 0 \quad \text{in} \quad R_z,
\]
\[
\frac{\partial \varphi}{\partial n} = 0 \quad \text{on} \quad \partial D_z,
\]
\[
\frac{\partial \varphi}{\partial n} = g \quad \text{in} \quad D_z,
\]

where $\int_{D_z} g dA = 0$. Then

\[
\int_{D_z} \partial_\alpha \varphi dA = \int_{D_z} g^2 dA,
\]
\[
\int_{R_z} \partial_\alpha \varphi dA = \frac{1}{\sqrt{h}} \int_{D_z} g^2 dA,
\]

3. Some Useful Lemmas

In this section, we derive some useful lemmas which will be used in next section. First, we define a weighted energy expression

\[
E(z, t) = k \int_{R_z} (\xi - z) u_{i,j} \varphi_{ij} \, dx \, dy + \nu \int_{R_z} (\xi - z) u_{i,j} u_{i,j} \, dx \, dy
\]
\[
+ \rho_1 \int_{R_z} \int (\xi - z) T_{ij} T_{ij} \, dx \, dy + \rho_2 \int_{R_z} \int (\xi - z) C_{ij} C_{ij} \, dx \, dy
\]
\[
= E_1(z, t) + E_2(z, t) + E_3(z, t) + E_4(z, t),
\]

where $k, \rho_1, \rho_2$ are positive parameters and $\xi > z > 0$.

By using the divergence theorem and Equations (1) and (4), we obtain

\[
E_1(z, t) = k \int_{R_z} (\xi - z) u_{i,j} \left[ v \Delta u_i - u_i - p_j + g_i T + h_i C \right] \, dx \, dy
\]
\[
= kv \int_{R_z} u_{i,j} u_{i,j} \, dx \, dy + k \int_{R_z} u_{i,j} p \, dx \, dy
\]
\[
+ k \int_{R_z} (\xi - z) u_{i,j} g_j T_{ij} \, dx \, dy + k \int_{R_z} (\xi - z) u_{i,j} h_i C_{ij} \, dx \, dy
\]
\[
- kv \int_{R_z} (\xi - z) u_{i,j} u_{i,j} \bigg|_{\eta = t} - k \int_{R_z} (\xi - z) u_{i,j} u_{i,j} \bigg|_{\eta = t}
\]
\[
\leq \frac{4}{i} A_i
\]
\[
- \frac{1}{2} k \int_{R_z} (\xi - z) u_{i,j} u_{i,j} \bigg|_{\eta = t} - \frac{1}{2} k \int_{R_z} (\xi - z) u_{i,j} u_{i,j} \bigg|_{\eta = t}.
\]

Using the Schwarz inequality, the arithmetic geometric mean inequality and (9), we can obtain

\[
A_1 \leq kv \left[ \int_{R_z} u_{i,j} u_{i,j} \, dx \, dy \right]^{\frac{1}{2}} \left[ \int_{R_z} u_{i,j} u_{i,j} \, dx \, dy \right]^{\frac{1}{2}}
\]
\[
\leq \frac{\sqrt{k}}{2} \left[ \int_{R_z} u_{i,j} u_{i,j} \, dx \, dy + \nu \int_{R_z} u_{i,j} u_{i,j} \, dx \, dy \right],
\]
\[
A_3 \leq \frac{k}{2} \left[ \int_{R_z} (\xi - z) u_{i,j} u_{i,j} \, dx \, dy \right]^{\frac{1}{2}} \left[ \int_{R_z} (\xi - z) T_{ij} T_{ij} \, dx \, dy \right]^{\frac{1}{2}}
\]
\[
\leq \frac{E_1}{2} \left[ \int_{R_z} (\xi - z) u_{i,j} u_{i,j} \, dx \, dy + \frac{k \lambda_1}{2} \int_{R_z} (\xi - z) T_{ij} T_{ij} \, dx \, dy \right],
\]
and

\[ A_4 \leq \frac{\varepsilon_2}{2} k \int_0^t \int_{R^2} (\xi - z) u_{1,\eta} u_{1,\eta} dxdy + \frac{k \varepsilon_2^2}{2 \lambda_1 \varepsilon_2} \int_0^t \int_{R^2} (\xi - z) C_{a,\nu} C_{a,\nu} dxdy, \]  

(21)

where \( \varepsilon_1, \varepsilon_2 > 0 \) will be determined later and

\[ \delta_1^2 = \max_D (g, g_i), \quad \delta_2^2 = \max_D (h, h_i), \]  

(22)

We note that for any \( z^* > 0 \), using (4) and (5),

\[ \int_{D_z} u_{3,\eta} dA = \int_{D_z^*} u_{3,\eta} dA - \int_{z}^{z^*} \int_{D_z} u_{3,3,\eta} dAd\xi \]

\[ = \int_{D_z^*} u_{3,\eta} dA + \int_{z}^{z^*} \int_{D_z} u_{a,a,\eta} dAd\xi \]

\[ = \int_{D_z} u_{3,\eta} dA. \]

Since

\[ \int_{D_0} f_{3,\eta} dA = 0, \quad t \geq 0, \]  

(23)

then,

\[ \int_{D_z} u_{3,\eta} dA = 0. \]

Under this assumption, using Lemma 2, there exist vector functions \( (\varphi_1, \varphi_2) \) such that

\[ \varphi_{a,\alpha} = u_{3,\eta} \quad \text{in} \quad D, \quad \varphi_{a} = 0 \quad \text{on} \quad \partial D. \]  

(24)

Hence we have

\[ A_2 = k \int_0^t \int_{R^2} \varphi_{a,\alpha} p dxd\eta = -k \int_0^t \int_{R^2} \varphi_{a,\alpha} p dxd\eta \]

\[ = k \int_0^t \int_{R^2} \varphi_{a} [u_{a,\eta} - v\Delta u_{a} + u_{a} - g_{a} T - h_{a} C] dxd\eta \]

\[ = k \int_0^t \int_{R^2} \varphi_{a} u_{a,\eta} dxd\eta + ku \int_0^t \int_{R^2} \varphi_{a,\alpha} u_{a,\alpha} dxd\eta \]

\[ + kv \int_0^t \int_{D_z} \varphi_{a} u_{a,3} dxd\eta + k \int_0^t \int_{R^2} \varphi_{a} u_{a} dxd\eta \]

\[ - k \int_0^t \int_{R^2} \varphi_{a} g_{a} T dxd\eta - k \int_0^t \int_{R^2} \varphi_{a} h_{a} C dxd\eta \]

\[ = A_{21} + A_{22} + A_{23} + A_{24} + A_{25} + A_{26}. \]  

(25)
Using the Schwarz, Poincaré and the AG mean inequalities, (9) and (10), we can obtain

\[ A_{21} \leq \left( \int_0^t \int_{R_2} \varphi_m \varphi_3 d\eta \right)^{\frac{1}{2}} \left( \int_0^t \int_{R_2} u_{m,\eta} u_{m,\eta} d\eta \right)^{\frac{1}{2}} \]
\[ \leq \frac{1}{\sqrt{\Lambda_1}} \left( \int_0^t \int_{R_2} \varphi_m \varphi_3 \varphi_\alpha \varphi_\beta d\eta \right)^{\frac{1}{2}} \left( \int_0^t \int_{R_2} u_{m,\eta} u_{m,\eta} d\eta \right)^{\frac{1}{2}} \]
\[ \leq \frac{\Lambda_1^\frac{1}{2}}{\sqrt{\Lambda_1}} \left( \int_0^t \int_{R_2} \varphi_3 \varphi_\alpha \varphi_\beta d\eta \right)^{\frac{1}{2}} \left( \int_0^t \int_{R_2} u_{m,\eta} u_{m,\eta} d\eta \right)^{\frac{1}{2}} \]
\[ \leq \frac{k\Lambda_1^\frac{1}{2}}{2\sqrt{\Lambda_1}} \int_0^t \int_{R_3} u_{i,\eta,\eta} d\eta, \quad (26) \]

\[ A_{22} \leq kv \left( \int_0^t \int_{R_2} \varphi_m \varphi_3 d\eta \right)^{\frac{1}{2}} \left( \int_0^t \int_{R_2} u_{m,\eta} u_{m,\eta} d\eta \right)^{\frac{1}{2}} \]
\[ \leq kv \Lambda_1^\frac{1}{2} \left( \int_0^t \int_{R_2} \varphi_3 \varphi_\alpha \varphi_\beta d\eta \right)^{\frac{1}{2}} \left( \int_0^t \int_{R_2} u_{m,\eta} u_{m,\eta} d\eta \right)^{\frac{1}{2}} \]
\[ \leq \frac{kv\Lambda_1^\frac{1}{2}}{2} \int_0^t \int_{R_3} \varphi_3 d\eta + \frac{kv\Lambda_1^\frac{1}{2}}{2} \int_0^t \int_{R_3} u_{m,\eta} u_{m,\eta} d\eta, \quad (27) \]

\[ A_{23} \leq kv \left( \int_0^t \int_{D_2} \varphi_3 \varphi_3 d\eta \right)^{\frac{1}{2}} \left( \int_0^t \int_{D_2} u_{m,\eta} u_{m,\eta} d\eta \right)^{\frac{1}{2}} \]
\[ \leq \frac{kv\Lambda_1^\frac{1}{2}}{\sqrt{\Lambda_1}} \left( \int_0^t \int_{D_2} \varphi_3 \varphi_\alpha \varphi_\beta d\eta \right)^{\frac{1}{2}} \left( \int_0^t \int_{D_2} u_{m,\eta} u_{m,\eta} d\eta \right)^{\frac{1}{2}} \]
\[ \leq \frac{kv\Lambda_1^\frac{1}{2}}{2\sqrt{\Lambda_1}} \int_0^t \int_{D_2} \varphi_3 d\eta + \frac{kv\Lambda_1^\frac{1}{2}}{2\sqrt{\Lambda_1}} \int_0^t \int_{D_2} u_{m,\eta} u_{m,\eta} d\eta, \quad (28) \]

\[ A_{24} \leq k \left( \int_0^t \int_{R_3} \varphi_m \varphi_3 d\eta \right)^{\frac{1}{2}} \left( \int_0^t \int_{R_3} u_{m,\eta} u_{m,\eta} d\eta \right)^{\frac{1}{2}} \]
\[ \leq \frac{k\Lambda_1^\frac{1}{2}}{\Lambda_1} \left( \int_0^t \int_{R_3} \varphi_3 \varphi_\alpha \varphi_\beta d\eta \right)^{\frac{1}{2}} \left( \int_0^t \int_{R_3} u_{m,\eta} u_{m,\eta} d\eta \right)^{\frac{1}{2}} \]
\[ \leq \frac{k\Lambda_1^\frac{1}{2}}{2\Lambda_1} \int_0^t \int_{R_3} \varphi_3 d\eta + \frac{k\Lambda_1^\frac{1}{2}}{2\Lambda_1} \int_0^t \int_{R_3} u_{m,\eta} u_{m,\eta} d\eta, \quad (29) \]

\[ A_{25} \leq k \left( \int_0^t \int_{R_3} \varphi_m \varphi_3 d\eta \right)^{\frac{1}{2}} \left( \int_0^t \int_{R_3} \varphi_3 T_\alpha T_\beta d\eta \right)^{\frac{1}{2}} \]
\[ \leq \frac{k\delta_1 \Lambda_1^\frac{1}{2}}{\Lambda_1} \left( \int_0^t \int_{R_3} \varphi_3 \varphi_\alpha \varphi_\beta d\eta \right)^{\frac{1}{2}} \left( \int_0^t \int_{R_3} T_\alpha T_\beta d\eta \right)^{\frac{1}{2}} \]
\[ \leq \frac{k\delta_1 \Lambda_1^\frac{1}{2}}{2\Lambda_1} \int_0^t \int_{R_3} \varphi_3 d\eta + \frac{k\delta_1 \Lambda_1^\frac{1}{2}}{2\Lambda_1} \int_0^t \int_{R_3} T_\alpha T_\beta d\eta, \quad (30) \]

\[ A_{26} \leq \frac{k\delta_2 \Lambda_1^\frac{1}{2}}{2\Lambda_1} \int_0^t \int_{R_3} \varphi_3 d\eta + \frac{k\delta_2 \Lambda_1^\frac{1}{2}}{2\Lambda_1} \int_0^t \int_{R_3} C_i C_i d\eta. \quad (31) \]

Inserting (26)–(31) into (25), then (19)–(21) and (25) into (18), and choosing \( \varepsilon_1 = \varepsilon_2 = \frac{1}{2} \), we obtain the following lemma.
Lemma 5. Let \( u, T, C, p \) be solutions of Equations (1)–(8) with \( g, h \in L_\infty(R \times \{ t > 0 \}) \) and \( \int_D f_3 dA = 0 \). Then
\[
E_1(z, t) + kv \int_{R_z} (\xi - z)u_{ij}u_{ij}dx \bigg|_{\eta = t} + k \int_{R_z} (\xi - z)u_{ij}u_{ij}dx \bigg|_{\eta = t}
\leq a_1 k \int_{R_z} \int_{R_z} u_{ij}u_{ij}dxd\eta + a_2 v \int_{R_z} \int_{R_z} u_{ij}u_{ij}dxd\eta + \frac{k\delta_1 \Lambda^2}{\lambda_1} \int_0^t \int_{R_z} C_{\alpha} C_{\alpha} dxd\eta + \frac{2k\delta_2 \Lambda^2}{\lambda_1} \int_0^t \int_{R_z} (\xi - z)C_{\alpha} C_{\alpha} dxd\eta,
\]
where
\[
a_1 = \sqrt{kv} + \frac{\Lambda^2}{\sqrt{\lambda_1}} + v\Lambda^2 + \frac{2\delta_1 \Lambda^2}{\lambda_1} + \frac{2\delta_2 \Lambda^2}{\lambda_1},
\]
\[
a_2 = \frac{\sqrt{kv}}{2} + \frac{k\Lambda^2}{2} + \frac{k\Lambda^2}{2v\sqrt{\lambda_1}}.
\]
Similar to Lemma 5, for \( E_2(z, t) \) we can obtain the following lemma.

Lemma 6. Let \( u, T, C, p \) be solutions of Equations (1)–(8) with \( g, h \in L_\infty(R \times \{ t > 0 \}) \) and \( \int_D f_3 dA = 0 \). Then
\[
E_2(z, t) + \frac{1}{2} \int_{R_z} (\xi - z)u_{ij}u_{ij}dx \bigg|_{\eta = t} + \frac{1}{2} \int_{R_z} (\xi - z)u_{ij}u_{ij}dx \bigg|_{\eta = t}
\leq a_3 \int_{R_z} \int_{R_z} u_{ij}u_{ij}dxd\eta + \frac{v}{2} \int_{R_z} \int_{R_z} u_{ij}u_{ij}dxd\eta + \frac{\Lambda^2}{2\lambda_1} \int_0^t \int_{R_z} u_{ij}u_{ij}dxd\eta + \frac{\delta_1 \Lambda^2}{2\lambda_1} \int_0^t \int_{R_z} T_{\alpha} T_{\alpha} dxd\eta + \frac{\delta_2 \Lambda^2}{2\lambda_1} \int_0^t \int_{R_z} C_{\alpha} C_{\alpha} dxd\eta + \frac{2\delta_3 \Lambda^2}{2\lambda_1 \epsilon_3} \int_0^t \int_{R_z} (\xi - z)C_{\alpha} C_{\alpha} dxd\eta,
\]
where
\[
a_3 = \frac{v}{2} + \frac{\Lambda^2}{2\lambda_1} + \frac{v\Lambda^2}{2\sqrt{\lambda_1}} + \frac{v\Lambda^2}{2\lambda_1} + \frac{k\Lambda^2}{2\lambda_1} + \frac{\delta_1 \Lambda^2}{2\lambda_1} + \frac{\delta_2 \Lambda^2}{2\lambda_1}.
\]
Proof. By the divergence theorem and Equations (1)–(8), we have
\[
E_2(z,t) = -\int_0^t \int_{R_z} (\xi - z) u_i u_i dxd\eta - \frac{1}{2} \int_{R_z} (\xi - z) u_i u_i|_{\eta=t} \\
- \nu \int_0^t \int_{R_z} u_i u_i dxd\eta + \int_0^t \int_{R_z} (\xi - z) u_i g_i Tdxd\eta \\
+ \int_0^t \int_{R_z} (\xi - z) u_i h_i C dxd\eta + \int_0^t u_3 p dxd\eta \\
\leq -\int_0^t \int_{R_z} (\xi - z) u_i u_i dxd\eta - \frac{1}{2} \int_{R_z} (\xi - z) u_i u_i|_{\eta=t} \\
+ \sum_{i=1}^4 B_i.
\]

Using the Schwarz inequality, the Poincaré inequality and the AG mean inequality, we can obtain
\[
B_1 \leq \nu \left[ \int_0^t \int_{R_z} u_i u_i dxd\eta \right]^{\frac{1}{2}} \\
\leq \nu \left[ \int_0^t \int_{R_z} u_i u_i dxd\eta + \int_0^t \int_{R_z} u_i u_i dxd\eta \right].
\]

Similar to (20) and (21), we have for \(B_2\) and \(B_3\)
\[
B_2 \leq \frac{\epsilon_3}{2} \int_0^t \int_{R_z} (\xi - z) u_i u_i dxd\eta + \frac{\epsilon_3^2}{2\lambda_1 \epsilon_3} \int_0^t \int_{R_z} (\xi - z) T_{,\alpha} T_{,\alpha} dxd\eta,
\]
and
\[
B_3 \leq \frac{\epsilon_4}{2} \int_0^t \int_{R_z} (\xi - z) u_i u_i dxd\eta + \frac{\epsilon_4^2}{2\lambda_2 \epsilon_2} \int_0^t \int_{R_z} (\xi - z) C_{,\alpha} C_{,\alpha} dxd\eta,
\]
where \(\epsilon_3, \epsilon_4\) are positive constants.

To bound \(B_4\) in (32), we also require that
\[
\int_D f_3 dA = 0.
\]

Then using the Lemma 2 in Section 2, there exist vector functions \((\tilde{\varphi}_1, \tilde{\varphi}_2)\) such that
\[
\tilde{\varphi}_{a,\alpha} = u_3, \quad \text{in } D, \quad \tilde{\varphi}_a = 0, \quad \text{on } \partial D.
\]

Therefore, we have
\[
B_4 = \int_0^t \int_{R_z} \tilde{\varphi}_{a,\alpha} p dxd\eta \\
= \int_0^t \int_{R_z} \tilde{\varphi}_{a,\alpha} - v \Delta u_a + u_a - g_a T - h_a C dxd\eta \\
= \int_0^t \int_{R_z} \tilde{\varphi}_{a,\alpha} + v \int_0^t \int_{R_z} \tilde{\varphi}_{a,\beta} u_{a,\beta} dxd\eta + v \int_0^t \int_{D_z} \tilde{\varphi}_{a,\alpha} u_a dAd\eta \\
+ \int_0^t \int_{R_z} \tilde{\varphi}_{a,\alpha} u_a dxd\eta - \int_0^t \int_{R_z} g_a T \tilde{\varphi}_a dxd\eta - \int_0^t \int_{R_z} h_a C \tilde{\varphi}_a dxd\eta \\
\leq \sum_{i=1}^6 B_{4i}.
\]
As the derivation of (26)–(32), we conclude that

\[
B_{41} \leq \left( \int_0^t \int_{R_z} \tilde{\varphi}_a \tilde{\varphi}_a dxd\eta \right)^{\frac{1}{2}} \left( \int_0^t \int_{R_z} u_{a,\eta}u_{a,\eta} dxd\eta \right)^{\frac{1}{2}} \\
\leq \frac{1}{\sqrt{\lambda_1}} \left( \int_0^t \int_{R_z} \tilde{\varphi}_a \tilde{\varphi}_a dxd\eta \right)^{\frac{1}{2}} \left( \int_0^t \int_{R_z} u_{a,\eta}u_{a,\eta} dxd\eta \right)^{\frac{1}{2}} \\
\leq \frac{\Lambda_1}{\sqrt{\lambda_1}} \left( \int_0^t \int_{R_z} u_1^2 dxd\eta \right)^{\frac{1}{2}} \left( \int_0^t \int_{R_z} u_{a,\eta}u_{a,\eta} dxd\eta \right)^{\frac{1}{2}} \\
\leq \frac{\Lambda_2}{2\lambda_1} \left[ \int_0^t \int_{R_z} u_{3,a} u_{3,a} dxd\eta + \int_0^t \int_{R_z} u_{i,j} u_{i,j} dxd\eta \right],
\]

(38)

\[
B_{42} \leq \nu \left( \int_0^t \int_{D_z} \tilde{\varphi}_a \tilde{\varphi}_a dxd\eta \right)^{\frac{1}{2}} \left( \int_0^t \int_{D_z} u_{a,\beta}u_{a,\beta} dxd\eta \right)^{\frac{1}{2}} \\
\leq \frac{\nu \Lambda_2}{\sqrt{\lambda_1}} \left( \int_0^t \int_{D_z} u_2^2 dxd\eta \right)^{\frac{1}{2}} \left( \int_0^t \int_{D_z} u_{a,\beta}u_{a,\beta} dxd\eta \right)^{\frac{1}{2}} \\
\leq \frac{\nu \Lambda_2}{2\lambda_1} \int_0^t \int_{D_z} u_{i,j} u_{i,j} dxd\eta,
\]

(39)

\[
B_{43} \leq \nu \left( \int_0^t \int_{D_z} \tilde{\varphi}_a \tilde{\varphi}_a dxd\eta \right)^{\frac{1}{2}} \left( \int_0^t \int_{D_z} u_{a,3} u_{a,3} dxd\eta \right)^{\frac{1}{2}} \\
\leq \frac{\nu \Lambda_2}{\sqrt{\lambda_1}} \left( \int_0^t \int_{D_z} u_3^2 dxd\eta \right)^{\frac{1}{2}} \left( \int_0^t \int_{D_z} u_{a,3} u_{a,3} dxd\eta \right)^{\frac{1}{2}} \\
\leq \frac{\nu \Lambda_2}{2\lambda_1} \int_0^t \int_{D_z} u_{i,j} u_{i,j} dxd\eta,
\]

(40)

\[
B_{44} \leq \left( \int_0^t \int_{R_z} \tilde{\varphi}_a \tilde{\varphi}_a dxd\eta \right)^{\frac{1}{2}} \left( \int_0^t \int_{R_z} u_4 u_4 dxd\eta \right)^{\frac{1}{2}} \\
\leq \frac{\Lambda_2}{\lambda_1} \int_0^t \int_{R_z} u_4 u_4 dxd\eta \\
\leq \frac{k \Lambda_1}{2\lambda_1} \int_0^t \int_{R_z} u_{i,a} u_{i,a} dxd\eta,
\]

(41)

\[
B_{45} \leq \left( \int_0^t \int_{R_z} \tilde{\varphi}_a \tilde{\varphi}_a dxd\eta \right)^{\frac{1}{2}} \left( \int_0^t \int_{R_z} \varphi_a \varphi_a T^2 dxd\eta \right)^{\frac{1}{2}} \\
\leq \frac{\delta_1 \Lambda_2}{\lambda_1} \left( \int_0^t \int_{R_z} u_{3,a} u_{3,a} dxd\eta \right)^{\frac{1}{2}} \left( \int_0^t \int_{R_z} T_{a} T_{a} dxd\eta \right)^{\frac{1}{2}} \\
\leq \frac{\delta_1 \Lambda_2}{2\lambda_1} \left[ \int_0^t \int_{R_z} u_{3,a} u_{3,a} dxd\eta + \int_0^t \int_{R_z} T_{a} T_{a} dxd\eta \right],
\]

(42)

\[
B_{46} \leq \frac{\delta_2 \Lambda_2}{2\lambda_1} \left[ \int_0^t \int_{R_z} u_{3,a} u_{3,a} dxd\eta + \int_0^t \int_{R_z} C_{a} C_{a} dxd\eta \right].
\]

(43)

Inserting (38)–(43) into (37), we obtain

\[
B_4 \leq \left[ \frac{\Lambda_2}{2\lambda_1} + \frac{\nu \Lambda_2}{2\lambda_1} \frac{\sqrt{\lambda_1}}{2\lambda_1} \frac{k \Lambda_1}{2\lambda_1} + \frac{\delta_1 \Lambda_2}{2\lambda_1} + \frac{\delta_2 \Lambda_2}{2\lambda_1} \right] \int_0^t \int_{R_z} u_{i,j} u_{i,j} dxd\eta \\
+ \frac{\Lambda_2}{2\lambda_1} \int_0^t \int_{R_z} u_{i,j} u_{i,j} dxd\eta + \frac{\delta_1 \Lambda_2}{2\lambda_1} \int_0^t \int_{R_z} T_{a} T_{a} dxd\eta \\
+ \frac{\delta_2 \Lambda_2}{2\lambda_1} \int_0^t \int_{R_z} C_{a} C_{a} dxd\eta.
\]

(44)
Inserting (33), (34), (35) and (44) into (32) and choosing \( \varepsilon_3 = \varepsilon_4 = \frac{1}{2} \), we can obtain Lemma 5.

Next we may bound \( E_3(z,t) \). First we let \( T_M \) denotes that the maximum of \( T \) by using the maximum principle in \( \mathbb{R} \), i.e.,

\[
T_M = \max_{D \times \{t \geq 0\}} F(x_1, x_2, t).
\]

Integrating by parts, using (3), (5), (6), (7) together with (9) and the AG mean inequality, we have

\[
E_3(z,t) = -\rho_1 \int_0^t \int_{D_3} TT_3dx \eta - \rho_1 \int_0^t \int_{D_3} T(T_\eta + u_3 T_\eta)dx \eta
\]

\[
= -\frac{\rho_1}{2} \int_{D_3} (\xi - z)T^2d\eta \bigg|_{\eta=t} + \frac{\rho_1}{2} \int_0^t \int_{D_3} T^2dAd\eta + \frac{\rho_1}{2} \int_0^t \int_{D_3} u_3 T^2dx \eta
\]

\[
\leq -\frac{\rho_1}{2} \int_{D_3} (\xi - z)T^2d\eta \bigg|_{\eta=t} + \frac{\rho_1}{2\lambda_1} \int_0^t \int_{D_3} T_\eta T_\eta dAd\eta
\]

\[
+ \frac{\rho_1 T_M}{2} \left( \int_0^t \int_{D_3} u_3^2dx \eta \right)^\frac{1}{2} \left( \int_0^t \int_{D_3} T^2dAd\eta \right)^\frac{1}{2}
\]

\[
\leq -\frac{\rho_1}{2} \int_{D_3} (\xi - z)T^2d\eta \bigg|_{\eta=t} + \frac{\rho_1}{2\lambda_1} \int_0^t \int_{D_3} T_\eta T_\eta dAd\eta
\]

\[
+ \frac{\rho_1 T_M}{4\lambda_1} \left( \int_0^t \int_{D_3} u_3^2dx \eta \right) + \int_0^t \int_{D_3} T T_\eta dx \eta.
\]

Using Equations (3)–(7) and integrating by parts, we obtain

\[
E_4(z,t) = -\rho_2 \int_0^t \int_{D_3} CC_3dx \eta - \rho_2 \int_0^t \int_{D_3} (\xi - z)C \left[ C_\eta + u_3 C_\eta - \sigma \Delta T \right] dx \eta
\]

\[
\leq \frac{\rho_2}{2} \int_0^t \int_{D_3} C^2dAd\eta - \frac{\rho_2}{2} \int_{D_3} (\xi - z)C^2d\eta \bigg|_{\eta=t} + \frac{\rho_2}{2} \int_0^t \int_{D_3} u_3 C^2dx \eta
\]

By the Schwarz and the AG mean inequalities, it follows that from (47)

\[
E_4(z,t) + \frac{\rho_2}{2} \int_{D_3} (\xi - z)C^2d\eta \bigg|_{\eta=t}
\]

\[
\leq \frac{\rho_2}{2\lambda_1} \int_0^t \int_{D_3} C^2dAd\eta + \frac{\sigma \rho_2}{2\sqrt{\lambda_1}} \int_0^t \int_{D_3} C_\eta C_\eta ddx \eta
\]

\[
+ \frac{\sigma \rho_2}{2\sqrt{\lambda_1}} \int_0^t \int_{D_3} T_3 T_3 dx \eta + \frac{\rho_2}{2} \int_0^t \int_{D_3} (\xi - z)C_3 C_3 dx \eta
\]

\[
+ \frac{\sigma \rho_2}{2\varepsilon_5} \int_0^t \int_{D_3} (\xi - z)T_\eta T_\eta dx \eta + \frac{\rho_2}{2} \int_0^t \int_{D_3} u_3 C^2dx \eta,
\]

for an arbitrary constant \( \varepsilon_5 > 0 \).
In order to bound the last term on the right of (48), using the Equations (9), (11) and (13), the Schwarz inequality and the AG mean inequality to obtain

\[
\frac{\rho_z^2}{2} \int_0^t \int_{\mathcal{R}_t} \rho_3 C^2 dx d\eta \leq \frac{\rho_z^2}{2} \int_0^t \left( \int_{\mathcal{R}_t} (\rho_3 C)^2 dx \right)^{\frac{1}{2}} d\eta
\]

where the bound for max \( \left\{ \int_{\mathcal{R}_t} C^2 dx \right\} \) will be derived later.

Inserting (49) back into (48), we have

\[
E_4(z, t) + \frac{\rho_z}{2} \int_{\mathcal{R}_t} (\xi - z) C^2 dx \bigg|_{\eta = t} \\
\leq \frac{\rho_z}{2} \int_{\mathcal{R}_t} \int_{\mathcal{D}_t} C_{\alpha} C_{\alpha} d\eta + \frac{\rho_z}{2} \int_{\mathcal{R}_t} \int_{\mathcal{D}_t} C_{i} C_{i} d\eta + \frac{\rho_z}{2} \int_{\mathcal{R}_t} T_3 T_3 d\eta
\]

\[
+ \frac{\rho_z \epsilon_6}{2} \int_{\mathcal{R}_t} (\xi - z) C_{\alpha} C_{\alpha} d\eta + \frac{\rho_z}{2} \int_{\mathcal{R}_t} (\xi - z) T_3 T_3 d\eta
\]

Combining (46) and (50), we obtain the following Lemma.

**Lemma 7.** Let \( u, T, C, \rho \) be solutions of Equations (1)–(8) with \( g, h \in L_\infty(R \times \{ t > 0 \}) \) and \( \int_{\mathcal{D}_t} 3 d\eta = 0 \). Then

\[
E_3(z, t) + E_4(z, t) \frac{1}{2} \int_{\mathcal{R}_t} (\xi - z) \left[ \rho_1 T^2 + \rho_2 C^2 \right] dx \bigg|_{\eta = t} \\
\leq \frac{\rho_1}{2} \int_{\mathcal{D}_t} T_{\alpha} T_{\alpha} d\eta + \frac{\rho_2}{2} \int_{\mathcal{D}_t} C_{\alpha} C_{\alpha} d\eta
\]

\[
+ \left[ \frac{\rho_2 \epsilon_6}{2} + \frac{\rho_1 T}{4\alpha_1} \right] \int_{\mathcal{R}_t} T_3 T_3 d\eta
\]

where \( \epsilon_6 \) is a positive constant. Next, we use Lemmas 5–7 to prove our main result.
4. Main Result

First, we introduce a new function

\[
\psi(z,t) = k \int_0^t \int_{R_e} (\zeta - z) u_{i,j} u_{i,j} dxd\eta + v \int_0^t \int_{R_e} (\zeta - z) u_{i,j} u_{i,j} dxd\eta \\
+ \rho_1 \int_0^t \int_{R_e} (\zeta - z) T_j T_j dxd\eta + \rho_2 \int_0^t \int_{R_e} (\zeta - z) C_j C_j dxd\eta \\
+ kv \int_{R_e} (\zeta - z) u_{i,j} u_{i,j} dx \big|_{\eta=t} + (k + \frac{1}{2}) \int_{R_e} (\zeta - z) u_{i,j} u_{i,j} dx \big|_{\eta=t} \\
+ \frac{1}{2} \int_0^t \int_{R_e} (\zeta - z) u_{i,j} u_{i,j} dxd\eta + \frac{1}{2} \int_{R_e} (\zeta - z) \left[ \rho_1 T^2 + \rho_2 C^2 \right] dxd\eta \bigg|_{\eta=t}
\]  

(51)

Using Lemmas 4–6 and in view of (51), we have

\[
\psi(z,t) \leq a_4 \int_0^t \int_{R_e} u_{i,j} u_{i,j} dxd\eta + a_5 \int_0^t \int_{R_e} u_{i,j} u_{i,j} dxd\eta \\
+ a_6 \int_0^t \int_{R_e} T_j T_j dxd\eta + \frac{v}{2} \int_0^t \int_{R_e} u_{i,j} u_{i,j} dxd\eta + a_7 \int_0^t \int_{R_e} C_j C_j dxd\eta \\
+ \frac{kvA^2}{2\sqrt{\lambda_1}} \int_0^t \int_{D_2} u^2 \xi_j dAd\eta + \frac{kvA^2}{2\sqrt{\lambda_1}} \int_0^t \int_{D_2} u_{a,3} u_{a,3} dAd\eta \\
+ \frac{\rho_1}{2\lambda_1} \int_0^t \int_{D_2} T_j T_j dAd\eta + \frac{\rho_2}{2\lambda_1} \int_0^t \int_{D_2} C_j C_j dAd\eta \\
+ \frac{2k_1^2}{\lambda_1} \int_0^t \int_{R_e} (\zeta - z) T_j T_j dxd\eta + \frac{2k_1^2}{\lambda_1} \int_0^t \int_{R_e} (\zeta - z) C_j C_j dxd\eta \\
+ \frac{\delta_1^2}{2\lambda_1 \epsilon_3} \int_0^t \int_{R_e} (\zeta - z) T_j T_j dxd\eta + \frac{\delta_1^2}{2\lambda_1 \epsilon_3} \int_0^t \int_{R_e} (\zeta - z) C_j C_j dxd\eta \\
+ \frac{\sigma_\rho \epsilon_6}{2} \int_0^t \int_{R_e} (\zeta - z) C_j C_j dxd\eta + \frac{\sigma_\rho \epsilon_6}{2} \int_0^t \int_{R_e} (\zeta - z) T_j T_j dxd\eta
\]

(52)

where

\[
a_4 = a_1 k + \frac{\Lambda^2}{2\lambda_1}, a_5 = a_2 v + a_3 + \frac{\rho_1 T_M}{4\lambda_1} + \frac{\rho_2 \Omega^2}{2\lambda_1^2} \max \left\{ \left( \int_{R_e} C^2 dx \right)^\frac{1}{2} \right\},
\]

\[
a_6 = \frac{k\delta_1 \Lambda^2}{\lambda_1} + \frac{\delta_1 \Lambda^2}{2\lambda_1}, a_7 = \frac{k\delta_1 \Lambda^2}{\lambda_1} + \frac{\delta_1 \Lambda^2}{2\lambda_1} + \frac{\sigma_\rho \epsilon_6}{2\lambda_1^2} \max \left\{ \left( \int_{R_e} C^2 dx \right)^\frac{1}{2} \right\}.
\]

Choosing $\epsilon_6 = \frac{1}{2\sigma_\rho}, \rho_2 = \frac{2k_1^2}{\lambda_1 \epsilon_6^2}, \rho_1 = \frac{4k_1^2}{\lambda_1^2}, a_7 = \frac{\sigma_\rho \epsilon_6}{2\lambda_1^2} + \frac{\delta_1 \Lambda^2}{2\lambda_1}$ and define

\[
\Psi(z,t) = k \int_0^t \int_{R_e} (\zeta - z) u_{i,j} u_{i,j} dxd\eta + v \int_0^t \int_{R_e} (\zeta - z) u_{i,j} u_{i,j} dxd\eta \\
+ \frac{1}{2} \rho_1 \int_0^t \int_{R_e} (\zeta - z) T_j T_j dxd\eta + \frac{1}{2} \rho_2 \int_0^t \int_{R_e} (\zeta - z) C_j C_j dxd\eta \\
+ kv \int_{R_e} (\zeta - z) u_{i,j} u_{i,j} dx \big|_{\eta=t} + (k + \frac{1}{2}) \int_{R_e} (\zeta - z) u_{i,j} u_{i,j} dx \big|_{\eta=t} \\
+ \frac{1}{2} \int_0^t \int_{R_e} (\zeta - z) u_{i,j} u_{i,j} dxd\eta + \frac{1}{2} \int_{R_e} (\zeta - z) \left[ \rho_1 T^2 + \rho_2 C^2 \right] dxd\eta \bigg|_{\eta=t}
\]

(53)
we can have from (52)

\[
\Psi(z,t) \leq a_4 \int_0^t \int_{R_z} u_{i,j} u_{i,j} dxd\eta + a_5 \int_0^t \int_{R_z} u_{i,j}u_{i,j} dxd\eta + a_6 \int_0^t \int_{R_z} T_{i}T_{i} dxd\eta + \frac{v}{2} \int_0^t \int_{R_z} u_{i,j} dxd\eta + a_7 \int_0^t \int_{R_z} C_{x}C_{x} dxd\eta
\]

From (53), we have

\[
\frac{\partial \Psi(z,t)}{\partial z} = k \int_0^t \int_{R_z} u_{i,j} u_{i,j} dxd\eta + v \int_0^t \int_{R_z} u_{i,j}u_{i,j} dxd\eta + \frac{1}{2} \rho_1 \int_0^t \int_{R_z} T_{i}T_{i} dxd\eta + \frac{1}{2} \rho_2 \int_0^t \int_{R_z} C_{x}C_{x} dxd\eta +kv \int_{R_z} u_{i,j} u_{i,j} dxd\eta |_{\eta=t} + (k + \frac{1}{2}) \int_{R_z} u_{i,j} dx |_{\eta=t}
\]

and

\[
\frac{\partial^2 \Psi(z,t)}{\partial z^2} = k \int_0^t \int_{D_z} u_{i,j} u_{i,j} dAd\eta + v \int_0^t \int_{D_z} u_{i,j}u_{i,j} dAd\eta + \frac{1}{2} \rho_1 \int_0^t \int_{D_z} T_{i}T_{i} dAd\eta + \frac{1}{2} \rho_2 \int_0^t \int_{D_z} C_{x}C_{x} dAd\eta +kv \int_{D_z} u_{i,j} u_{i,j} dAd\eta |_{\eta=t} + (k + \frac{1}{2}) \int_{D_z} u_{i,j} dA |_{\eta=t}
\]

Combining (54), (55) and (56), we have

Thus

\[
\Psi(z,t) \leq K_1 \left[ - \frac{\partial \Psi(z,t)}{\partial z} \right] + K_2 \frac{\partial^2 \Psi(z,t)}{\partial z^2},
\]

where

\[
K_1 = \max \left\{ \frac{a_4}{k}, \frac{a_5}{v}, \frac{a_6}{\rho_1}, \frac{a_7}{\rho_2} \right\},
\]

\[
K_2 = \max \left\{ \frac{\nu \Lambda_1^{\frac{1}{2}}}{2 \sqrt{\Lambda_1}}, \frac{\kappa \Lambda_2^{\frac{1}{2}}}{2 \sqrt{\Lambda_1}}, \frac{1}{2 \lambda_1} \right\}.
\]

Inequality (57) can be rewritten as

\[
\frac{\partial}{\partial z} \left\{ e^{-\ell_2 z} \left( \frac{\partial \Psi}{\partial z} + \ell_2 \Psi \right) \right\} \geq 0,
\]

where

\[
\ell_1 = \frac{K_1}{2K_2} + \frac{1}{2} \sqrt{\frac{K_1^2}{K_2^2} + \frac{4}{K_2^2}}, \quad \ell_2 = - \frac{K_1}{2K_2} + \frac{1}{2} \sqrt{\frac{K_1^2}{K_2^2} + \frac{4}{K_2^2}}.
\]
Integrating (58) from \( z \) to \( \infty \) leads to

\[
\frac{\partial \Psi}{\partial z} + \ell_2 \Psi \leq 0,
\]

and hence

\[
\Psi(z, t) \leq \Psi(0, t)e^{-\ell_2 z}. \tag{59}
\]

Combining (53) and (59), we can obtain the following theorem.

**Theorem 1.** Let \( u, T, C, p \) be solutions of Equations (1)–(8) with \( g, h \in L_\infty(\mathbb{R} \times \{ t > 0 \}) \) and \( \int_D f_3dA = 0 \). Then

\[
k \int_0^t \int_{R_c} (\xi - z)u_{i,j}u_{i,j}\psi d\psi + v \int_0^t \int_{R_c} (\xi - z)u_{i,j}u_{i,j}\psi d\psi + \frac{1}{2} \rho_1 \int_0^t \int_{R_c} (\xi - z)T_{i,j}T_{i,j}\psi d\psi + \frac{1}{2} \rho_2 \int_0^t \int_{R_c} (\xi - z)C_{i,j}C_{i,j}\psi d\psi + k \int_0^t \int_{R_c} (\xi - z)u_{i,j}u_{i,j}\psi d\psi \bigg|_{\eta = t} + \left( k + \frac{1}{2} \right) \int_0^t \int_{R_c} (\xi - z)u_{i,j}u_{i,j}\psi d\psi \bigg|_{\eta = t} + \frac{1}{2} \int_0^t \int_{R_c} (\xi - z)u_{i,j}u_{i,j}\psi d\psi + \frac{1}{2} \int_0^t \int_{R_c} (\xi - z)\left[ \rho_1 T^2 + \rho_2 C^2 \right]\psi d\psi \bigg|_{\eta = t}
\]

\[
\leq \Psi(0, t)e^{-\ell_2 z}. \tag{60}
\]

**Remark 1.** The result of Theorem 1 belongs to the study of Saint-Venant principle, which shows that the fluid decays exponentially with spatial variables on the cylinder.

**Remark 2.** Theorem 1 shows that the solutions of Equations (1)–(8) decays exponentially as \( z \to \infty \). To make the decay bound explicit, we have to derive the bounds for \( \Psi(0, t) \) and \( \max_t \int_R C^2 d\psi \) in next section.

5. **Bounds of \( \Psi(0, t) \) and \( \max_t \int_R C^2 d\psi \)**

From the previous section, we can see that \( a_5 \) involves the quantities \( \max_t \int_R C^2 d\psi \). To make our main result explicit, we have to derive bounds of \( \Psi(0, t) \) and \( \max_t \int_R C^2 d\psi \) in term of the physical parameters \( \sigma, \nu, g, h, \) the boundary data and so on. To do this, we begin with

\[
\int_0^t \int_R T_{i,j}T_{i,j}\psi d\psi = - \int_0^t \int_D F T_{i,j}\psi d\psi - \int_0^t \int_R T^2 d\psi \bigg|_{\eta = t}. \tag{61}
\]

Now we assume that \( S \) is a sufficiently smooth function satisfying the same initial and boundary conditions as \( T \).

Thus,

\[
\int_0^t \int_R T_{i,j}T_{i,j}\psi d\psi = - \int_0^t \int_D S T_{i,j}\psi d\psi - \int_0^t \int_R T^2 d\psi \bigg|_{\eta = t}
\]

\[
= \int_0^t \int_R S_{i,j}T_{i,j}\psi d\psi - \int_0^t \int_R (T - S)(T_{i,j} - u_iu_j)\psi d\psi
\]

\[
= \int_0^t \int_R S_{i,j}T_{i,j}\psi d\psi - \int_0^t \int_R T^2 dx \bigg|_{\eta = t} + \int_R TS dx \bigg|_{\eta = t}
\]

\[
- \int_0^t \int_R S_{i,j}T_{i,j}\psi d\psi - \int_0^t \int_R S_{i,j}u_iu_j\psi d\psi - \frac{1}{2} \int_0^t \int_D f F^2 dA \psi. \tag{62}
\]
Using the Schwarz and the arithmetic-geometric mean inequalities, we can obtain
\[
\int_R T^2 dx \bigg|_{\eta=t} + \int_0^t \int_R T_j T_i dx d\eta \leq \frac{1}{2} \int_R S^2 dx \bigg|_{\eta=t} - \frac{1}{2} \int_0^t \int_D fF^2 dAd\eta \\
+ \left( \frac{\epsilon_1}{2} \int_0^t \int_R T_j T_i dx d\eta + \frac{1}{2\epsilon_1} \int_0^t \int_R S_j S_i dx d\eta \right) \\
+ \left( \frac{\epsilon_2}{2\lambda_1} \int_0^t \int_R T_j T_i dx d\eta + \frac{1}{2\epsilon_2\lambda_1} \int_0^t \int_R S_j S_i dx d\eta \right) \\
+ \left( \frac{\epsilon_3 T_M}{2} \int_0^t \int_R u_i u_i dx d\eta + \frac{T_M}{2\epsilon_3} \int_0^t \int_R S_j S_i dx d\eta \right),
\]
where \( \epsilon_1, \epsilon_2, \epsilon_3 \) are positive constants. Choosing
\[
\epsilon_1 = \frac{1}{2}, \quad \epsilon_2 = \frac{\lambda_1}{2},
\]
we can obtain
\[
\int_R T^2 dx \bigg|_{\eta=t} + \int_0^t \int_R T_j T_i dx d\eta \leq \frac{\epsilon_3 T_M}{2} \int_0^t \int_R u_i u_i dx d\eta + \text{data}.
\]

Obviously, the data terms in (65) involve \( \frac{1}{2} \int_R S^2 dx \bigg|_{\eta=t}, \int_0^t \int_R S_j S_i dx d\eta, \int_0^t \int_R S_j S_i dx d\eta \) and \( -\frac{1}{2} \int_0^t \int_D fF^2 dAd\eta \). Similarly, we can bound \( \int_0^t \int_R C_j C_i dx d\eta \) as well as \( \max_i \int_R C^2 dx \).

Firstly, we introduce a function \( H \):
\[
\frac{\partial H}{\partial t} + u_j H_j = \Delta H, \quad \text{in } R \times \{ t > 0 \},
\]
\[
H = 0, \quad \text{in } R \times \{ t = 0 \},
\]
\[
H = 0, \quad \text{on } \partial D \times \{ x_3 > 0 \} \times \{ t \geq 0 \},
\]
\[
H = G(x_1, x_2, t), \quad \text{on } D \times \{ t > 0 \}.
\]

Then we have
\[
(C - H)_j + u_i (C - H)_i = \Delta (C - H) + \sigma \Delta T, \quad \text{in } R \times \{ t > 0 \},
\]
\[
C - H = 0, \quad \text{in } R \times \{ t = 0 \},
\]
\[
C - H = 0, \quad \text{on } \partial D \times \{ x_3 > 0 \} \times \{ t \geq 0 \},
\]
\[
C - H = 0, \quad \text{on } D \times \{ t > 0 \}.
\]

By the triangle inequality, we obtain that
\[
\left( \int_0^t \int_R C_j C_i dx d\eta \right)^{\frac{1}{2}} \leq \left[ \int_0^t \int_R (C - H)_j (C - H)_i dx d\eta \right]^{\frac{1}{2}} + \left[ \int_0^t \int_R H_j H_i dx d\eta \right]^{\frac{1}{2}},
\]
and
\[
\left[ \max_i \int_R C^2 dx \right]^{\frac{1}{2}} \leq \left[ \max_i \int_R (C - H)^2 dx \right]^{\frac{1}{2}} + \left[ \max_i \int_R H^2 dx \right]^{\frac{1}{2}}.
\]

Then,
\[
\frac{1}{2} \int_R (C - H)^2 dx \bigg|_{\eta=t} + \int_0^t \int_R (C - H)_j (C - H)_i dx d\eta = -\sigma \int_0^t \int_R (C - H)_j T_i dx d\eta,
\]
which follows that
\[
\frac{1}{2} \int_R (C - H)^2 dx|_{u=\eta} + \int_0^t \int_R (C - H)_j(C - H)_j d\eta dx
\]
\[
\leq \sigma^2 \int_0^t \int_R T_jT_j d\eta dx
\]
\[
\leq \frac{\epsilon_5 T_0 \sigma^2}{2} \int_0^t \int_R u_i u_i d\eta dx + \text{data.}
\]  
(71)

Just as in the computation for $T$, we have the following inequality
\[
\frac{1}{2} \int_R H^2 dx|_{u=\eta} + \int_0^t \int_R H_jH_j d\eta dx \leq \epsilon_4 \int_0^t \int_R u_i u_i d\eta dx + \text{data.}
\]  
(72)

Thus,
\[
\frac{1}{2} \int_R C^2 dx|_{u=\eta} + \int_0^t \int_R C_jC_j d\eta dx \leq \epsilon_5 \int_0^t \int_R u_i u_i d\eta dx + \text{data},
\]  
(73)

where $\epsilon_5 > 0$ depends on $\epsilon_3, \epsilon_4$ and $\sigma$. Next we have to derive a bound for $\int_0^t \int_R u_i u_i d\eta dx$ in term of data. To do this, we define a function
\[
\omega_i = f_i e^{-\xi_1 z},
\]  
(74)

for some positive constant $\omega_i$. Then,
\[
\left[ \int_0^t \int_R u_i u_i d\eta dx \right]^2 \leq \left[ \int_0^t \int_R (u_i - \omega_i)_j(u_i - \omega_i)_j d\eta dx \right]^2 + \int_0^t \int_R \omega_i \omega_i d\eta dx.
\]

Obviously, we find that the last term of (75) is a data term. Now
\[
\nu \int_0^t \int_R (u_i - \omega_i)_j(u_i - \omega_i)_j d\eta dx
\]
\[
= - \int_0^t \int_R (u_i - \omega_i)_j \left[ (u_i - \omega_i) + p_j - g_jT - h_i C + \omega_i - \nu \Delta \omega_i \right] d\eta dx
\]  
(75)

or
\[
\frac{\nu}{2} \int_0^t \int_R (u_i - \omega_i)_j(u_i - \omega_i)_j d\eta dx
\]
\[
\leq - \int_0^t \int_R p\omega_i d\eta dx + \frac{\delta_1^2}{2} \int_0^t \int_R T^2 d\eta dx + \frac{\delta_2^2}{2} \int_0^t \int_R C_j C_j d\eta dx + \text{data.}
\]  
(76)

Noting that
\[
\omega_{i,i} = (f_n,a - \zeta_1 f_3)e^{-\xi_1 z} = 0,
\]  
(77)

in $R$ for $\zeta_1 = \frac{f_n,a}{f_3}$, we can rewrite (76) as
\[
\frac{\nu}{2} \int_0^t \int_R (u_i - \omega_i)_j(u_i - \omega_i)_j d\eta dx
\]
\[
\leq \frac{\delta_1^2}{2\lambda_1} \int_0^t \int_R T_jT_j d\eta dx + \frac{\delta_2^2}{2\lambda_1} \int_0^t \int_R C_j C_j d\eta dx + \text{data.}
\]  
(78)
Inserting (78) back into (75), we may have a bound of the form
\[
\int_0^1 \int_R u_{i,j} u_{i,j} dxdy \leq C_1 \int_0^1 \int_R T_j T_i dxdy + C_2 \int_0^1 \int_R C_j C_i dxdy + \text{data},
\] (79)
for computable $C_1$ and $C_2$. Combining (65) and (73) and by inequality (17), we have
\[
\int_0^1 \int_R u_{i,j} u_{i,j} dxdy \leq \frac{C_1 T M}{2 \lambda_1} \epsilon_3 \int_0^1 \int_R u_{i,j} u_{i,j} dxdy + \frac{C_2 \epsilon_5}{\lambda_1} \int_0^1 \int_R u_{i,j} u_{i,j} dxdy + \text{data}.
\] (80)
It is clear to see that
\[
\int_0^1 \int_R u_{i,j} u_{i,j} dxdy \leq \text{data},
\] (81)
for $\epsilon_3 = \frac{\Lambda_1}{\epsilon_3 T M}, \epsilon_5 = \frac{\Lambda_1}{\epsilon_5}$. From (65) and (73), we can obtain
\[
\max \int R T^2 dx \leq \text{data}, \quad \max \int R C^2 dx \leq \text{data},
\] (82)
and
\[
\int_0^1 \int_R T_j T_i dxdy \leq \text{data}, \quad \int_0^1 \int_R C_j C_i dxdy \leq \text{data}.
\] (83)
Next we seek bound for the total energy $\Psi(0, t)$. From (54) we can obtain for $\Psi(0, t)$
\[
\Psi(0, t) \leq a_4 \int_0^1 \int_R u_{i,j} u_{i,j} dxdy + \frac{\nu}{2} \int_0^1 \int_R u_{i,j} dxdy + b_1 \int_0^1 \int_D u_{a,3} u_{a,3} dAdy + \text{data}.
\] (84)
We are left to derive bounds for $\int_0^1 \int_R u_{i,j} u_{i,j} dxdy$ and $\int_0^1 \int_D u_{a,3} u_{a,3} dAdy$. Multiplying (1) with $u_{i,j}$ and integrating in the region $R \times [0, t]$, we have
\[
\int_0^1 \int_R u_{i,j} u_{i,j} dxdy = \int_0^1 \int_R u_{i,j} \left[ \nu \Delta u_i - u_i - p, j + g, i + h, C \right] dxdy,
\] (85)
which follows that
\[
\int_0^1 \int_R u_{i,j} u_{i,j} dxdy \leq -2\nu \int_0^1 \int_D u_{a,3} u_{a,3} dAdy + 2 \int_0^1 \int_D u_{3,\eta} p dAdy + \frac{\delta^2}{2 \lambda_1} \int_0^1 \int_R T_j T_i dxdy + \frac{\delta^2}{2 \lambda_1} \int_0^1 \int_R C_j C_i dxdy + \text{data} \leq -2\nu \int_0^1 \int_D u_{a,3} f_{a,\eta} p dAdy + 2 \int_0^1 \int_D f_{3,\eta} p dAdy + \text{data},
\] (86)
where we have used the fact $u_{3,3} = -u_{a,a} = -f_{a,a}$ on $D_0$ and (83), and $\epsilon_6$ is a positive constants. For the first term of (86), using the Schwarz and the AG mean inequalities we have
\[
-2\nu \int_0^1 \int_{D_0} f_{a,\eta} u_{a,3} dxdy \leq 2\nu \left( \int_0^1 \int_{D_0} u_{a,3} u_{a,3} dAdy \right)^{\frac{3}{2}} \left( \int_0^1 \int_{D_0} f_{a,\eta} f_{a,\eta} dAdy \right)^{\frac{1}{2}} \leq \int_0^1 \int_{D_0} u_{a,3} u_{a,3} dAdy + \text{data}.
\] (87)
To bound the second term on the right of (86), we define $\overline{p}$ to be the mean value of $p$ over $D_0$, i.e.,

$$\overline{p} = \frac{1}{|D_0|} \int_{D_0} p dA,$$

(88)

where $|D_0|$ is the measure of $D_0$. Since

$$\int_{D_0} f_{3,\eta} \overline{p} dA = \int_{D_0} f_{3,\eta} dA = 0,$$

(89)

we obtain

$$\int_{D_0} f_{3,\eta} p dA = \int_{D_0} f_{3,\eta} (p - \overline{p}) dA.$$

(90)

It follows by using Schwarz inequality that

$$\int_0^t \int_{D_0} f_{3,\eta} (p - \overline{p})^2 dAd\eta \leq \text{data} + \epsilon_6 \int_0^t \int_{D_0} (p - \overline{p})^2 dAd\eta,$$

(91)

where $\epsilon_6$ is a positive constant to be determined later.

To deal with the integral $\int_0^t \int_{D_0} (p - \overline{p})^2 dAd\eta$, we let an auxiliary function $\chi$ satisfying:

$$\Delta \chi = 0, \quad \frac{\partial \chi}{\partial n} = 0 \quad \text{on} \quad \partial D_0, \quad \frac{\partial \chi}{\partial n} = p - \overline{p}, \quad \text{in} \quad D_0.$$

(92)

From the definition of $\overline{p}$ in (88), it is clear that $\int_{D_0} (p - \overline{p}) dA = 0$. Thus, the necessary condition for the existence of a solution is satisfied and we compute

$$\int_0^t \int_{D_0} (p - \overline{p})^2 dAd\eta = \int_0^t \int_{\partial R} (p - \overline{p}) \frac{\partial \chi}{\partial n} dxd\eta = \int_0^t \int_{\partial R} \chi \frac{\partial (p - \overline{p})}{\partial n} dxd\eta
= \int_0^t \int_{\partial R} \chi \left[ -u_{i,\eta} + \nu u_{i,j,j} - u_i + g_i T + h_i C \right] dxd\eta.$$

(93)

Since

$$v \int_0^t \int_{\partial R} \chi i u_{i,j,j} dxd\eta = -v \int_0^t \int_D \chi i u_{i,j,j} dxd\eta - v \int_0^t \int_D \chi i u_{i,j} dxd\eta
= v \int_0^t \int_D \chi f_{3,\eta,\alpha} dAd\eta - v \int_0^t \int_D \chi u_{\alpha,\alpha} dAd\eta
+ v \int_0^t \int_D \chi u_{3,\alpha} dAd\eta + v \int_0^t \int_D \chi u_{\alpha,3,\alpha} dAd\eta
= -v \int_0^t \int_D \chi u_{\alpha,\alpha} dAd\eta + v \int_0^t \int_D \chi f_{3,\alpha} dAd\eta.$$

(94)
From (93), we can obtain
\[
\int_0^1 \int_{D_0} (p - \bar{p})^2 dAd\eta = \int_0^t \int_R \chi_j \left[ -u_{i,j} + \nu u_{i,j} - u_i + g_i T + h_i C \right] dxd\eta \\
\leq \left( \int_0^t \int_R \chi_i \chi_j dxd\eta \right)^{\frac{1}{2}} \left( \int_0^t \int_R u_{i,j} u_{i,j} dxd\eta \right)^{\frac{1}{2}} \\
+ v \left( \int_0^t \int_D \chi_a \chi_a dAd\eta \right)^{\frac{1}{2}} \left( \int_0^t \int_D u_{3,a} u_{3,a} dAd\eta \right)^{\frac{1}{2}} \\
+ \frac{1}{\sqrt{\lambda_1}} \left( \int_0^t \int_R \chi_i \chi_i dxd\eta \right)^{\frac{1}{2}} \left( \int_0^t \int R u_{i,j} u_{i,j} dxd\eta \right)^{\frac{1}{2}} \\
+ \frac{\delta_1}{\sqrt{\mu \lambda_1}} \left( \int_0^t \int R T_i T_j dxd\eta \right)^{\frac{1}{2}} + \frac{\delta_2}{\sqrt{\mu \lambda_1}} \left( \int_0^t \int C_i C_j dxd\eta \right)^{\frac{1}{2}}.
\]
(95)

Making use of (15), (16), (81) and (83) with \( g = p - \bar{p} \), we have
\[
\left[ \int_0^t \int_{D_0} (p - \bar{p})^2 dAd\eta \right]^{\frac{1}{2}} \\
\leq \left( \int_0^t \int_R u_{i,j} u_{i,j} dxd\eta \right)^{\frac{1}{2}} + v \left( \int_0^t \int_R u_{3,a} u_{3,a} dAd\eta \right)^{\frac{1}{2}} \\
+ \frac{1}{\sqrt{\lambda_1}} \left( \int_0^t \int_R \chi_i \chi_i dxd\eta \right)^{\frac{1}{2}} \left( \int_0^t \int R u_{i,j} u_{i,j} dxd\eta \right)^{\frac{1}{2}} \\
+ \frac{\delta_1}{\sqrt{\mu \lambda_1}} \left( \int_0^t \int R T_i T_j dxd\eta \right)^{\frac{1}{2}} + \frac{\delta_2}{\sqrt{\mu \lambda_1}} \left( \int_0^t \int C_i C_j dxd\eta \right)^{\frac{1}{2}},
\]
(96)

which follows that
\[
\int_0^t \int_{D_0} (p - \bar{p})^2 dAd\eta \leq data + c_3 \int_0^t \int_{D_0} u_{3,a} u_{3,a} dAd\eta + c_4 \int_0^t \int R u_{i,j} u_{i,j} dxd\eta.
\]
(97)

Obviously, from (97) we must establish a bound for the term \( \int_0^t \int_{D_0} u_{3,a} u_{3,a} dAd\eta \). To do this, we begin with the identity
\[
\int_0^t \int_R u_{i,j} \left[ \nu u_{i,j} - u_i - p_{j,i} - u_{i,j} + g_i T + h_i C \right] dxd\eta = 0.
\]
(98)

Integrating (98) by parts, we can have
\[
- v \int_0^t \int_{D_0} u_{3,a} u_{3,a} dAd\eta + v \int_0^t \int_R u_{i,j} u_{i,j} u_{i,j} dxd\eta + \int_0^t \int_R u_{3,a} u_{3,a} dAd\eta \\
+ \int_0^t \int R u_{3,a} p_{j,i} dxd\eta + \int_0^t \int R u_{3,a} u_{i,j} dxd\eta + \int_0^t \int R u_{3,a} g_i T dxd\eta \\
+ \int_0^t \int R u_{3,a} h_i C dxd\eta = 0,
\]
(99)

which follows that
\[
\int_0^t \int_{D_0} u_{3,a} u_{3,a} dAd\eta \leq data + \int_0^t \int_{D_0} u_{3,a} p dAd\eta + c_7 \int_0^t \int R u_{i,j} u_{i,j} dxd\eta.
\]
(100)

where \( c_7 \) is a positive constant.
As the derivation of (91), for the term \[ \int_0^t \int_{D_0} u_{a,3}pdAd\eta \] we can obtain
\[ \int_0^t \int_{D_0} u_{a,3}pdAd\eta \leq data + \epsilon_8 \int_0^t \int_{D_0} (p-\overline{p})^2dAd\eta, \] (101)
where \( \epsilon_8 \) is a positive constant.

Combining (97), (100) and (101), we have
\[ (1 - \epsilon_8 c_3) \int_0^t \int_{D_0} (p-\overline{p})^2dxd\eta \leq data + c_3 \epsilon_7 \int_0^t \int_R u_{i,\eta}u_{i,\eta}dxd\eta. \] (102)

Combing (86), (87), (91) and (100), we obtain
\[ (1 - \epsilon_7) \int_0^t \int_R u_{i,\eta}u_{i,\eta}dxd\eta \leq data + (\epsilon_6 + \epsilon_7) \int_0^t \int_{D_0} (p-\overline{p})^2dAd\eta. \] (103)

Choosing \( \epsilon_7 \) and \( \epsilon_8 \) small enough such that \( 1 - \epsilon_8 c_3 > 0 \) and \( 1 - \epsilon_7 > 0 \), from (102) and (103) we can obtain
\[ \int_0^t \int_R u_{i,\eta}u_{i,\eta}dxd\eta \leq data, \] (104)
and
\[ \int_0^t \int_{D_0} (p-\overline{p})^2dAd\eta \leq data. \] (105)

Inserting (101) back into (100), we obtain
\[ \int_0^t \int_{D_0} u_{a,3}u_{a,3}dxd\eta \leq data + \epsilon_8 \int_0^t \int_{D_0} (p-\overline{p})^2dAd\eta + \epsilon_7 \int_0^t \int_R u_{i,\eta}u_{i,\eta}dxd\eta. \] (106)

In light of (104) and (105), we have
\[ \int_0^t \int_{D_0} u_{a,3}u_{a,3}dxd\eta \leq data. \] (107)

Recalling (84) and using (104) and (107), we obtain
\[ \Psi(0,t) \leq data, \] (108)
which is to say that we have bounded the total energy.

### 6. Conclusions

In this paper, we consider the spatial decay bounds for the Brinkman equations in double-diffusive convection in a semi-infinite pipe. Using the results of this paper, we can continue to study the continuous dependence of the solution on the parameters in the system of equations. In addition, Using the results of this paper, we can continue to study the continuous dependence of the solution on the parameters in the system of equations. This research can refer to the method of [30,31]. In addition, if Equation (1) is replaced by a nonlinear problem (e.g., Forchheimer equations), it will be a more interesting topic.

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