The role of periodic orbits and bubbles of chaos during the transition to turbulence

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Starting with turbulence that explores a wide region in phase space, we discover several relative periodic orbits (RPOs) embedded within a subregion of the chaotic turbulent saddle. We also extract directly from simulation, several travelling waves (TWs). These TWs together with the RPOs are unstable states and are believed to provide the skeleton of the chaotic saddle. Earlier studies have shown that such invariant solutions can help to explain wall bounded shear flows, and a finite subset of them are expected to dominate the dynamics (Faisst & Eckhardt 2003; Pringle & Kerswell 2007; Hof et al. 2004). The introduction of symmetries is typically necessary to facilitate this approach. Applying only the shift-reflect symmetry, the geometry is less constrained than previous studies in pipe flow. A ‘long-period’ RPO is identified that is only very weakly repelling. Turbulent trajectories are found to frequently approach and frequently shadow this orbit. In addition the orbit characterises a resulting ‘bubble’ of chaos, itself a saddle, deep within the turbulent sea (Kreilos et al. 2014). We explicitly analyse the merger of the two saddles and show how it results in a considerable increase of the total lifetime. Both exits and entries to the bubble are observed, as the stable manifolds of the inner and outer saddles intertwine. We observe that the typical lifetime of the turbulence is influenced by switches between the inner and outer saddles, and is thereby dependent on whether or not it ‘shadows’ or ‘visits’ the vicinity of the long-period RPO (Cvitanović et al. 2014). These observations, along with comparisons of flow structures, show that RPOs play a significant role in structuring the dynamics of turbulence.

Key words:

1. Introduction

Several families of exact coherent structures, i.e. invariant sets of solutions, have been identified in plane Couette flow (Nagata 1990; Jiménez et al. 2005), plane Poiseuille flow (Waleffe 2001, 2003), square duct flow (Wedin et al. 2009; Okino et al. 2010) and the geometry discussed here, pipe flow (Wedin & Kerswell 2004; Pringle & Kerswell 2007; Faisst & Eckhardt 2003). In general, the state space is filled with a multitude of unstable invariant solutions that are explored by turbulent trajectories. At low flow rates the trajectory eventually escapes from the roller coaster ride through this neighbourhood and ends on the steady laminar attractor, turbulence is transient here and the turbulent

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neighbourhood corresponds to a chaotic saddle \cite{Kreilos2014}. In a circular pipe, the laminar Hagen–Poiseuille flow is linearly stable at all Reynolds numbers \( Re = DU/\nu \), where \( D \) is the pipe diameter, \( U \) the mean axial velocity and \( \nu \) the kinematic viscosity of the fluid. Several families of three-dimensional travelling wave (TW) solutions have been discovered \cite{FaissEtAl2003,WedinKerswell2004}, which represent the ‘simplest’ invariant solutions in pipe flow satisfying

\[
\mathbf{u}(r, \theta, z, t) = \mathbf{u}(r, \theta, z - ct),
\]

(1.1)

where \((r, \theta, z)\) are the usual cylindrical coordinates, \( \mathbf{u} = (u, v, w) \) are the corresponding velocity components, \( t \) the time and \( c \) the wave speed. The TW solutions originate from saddle-node bifurcations at a finite value of the Reynolds number. Mellibovsky and Eckhardt \cite{MellibovskyEckhardt2011} provide a fundamental study of TWs’ origins and their subsequent varied bifurcations. Periodic solutions on the other hand bifurcate classically in a Hopf bifurcation out of TWs. Orbits are important as they capture the dynamics. In a recent study of transition in Couette flow, Kreilos and Eckhardt \cite{KreilosEckhardt2012}, found that such orbits undergo a transition sequence to chaos. They followed the bifurcation of exact coherent states that undergo period-doubling cascades and end with a crisis bifurcation. Due to the strong advection of structures in pipe flow, only ‘relative’ periodic orbits (RPOs) are observed. These orbits include a streamwise translation with mean phase speed \( \overline{c} \). Of these, only few have been discovered so far \cite{DuguetEtAl2008,WillisEtAl2013,AvilaEtAl2013}. They are expected to capture the natural measure of turbulent flow \cite{CvitanovicGison2009,WillisEtAl2013}. Such RPOs satisfy

\[
\mathbf{u}(r, \theta, z, t) = \mathbf{u}(r, \theta, z - \overline{c}t, t + T),
\]

(1.2)

such that the motion appears \( T \)-periodic in a frame co-moving at speed \( \overline{c} \). The value \( \overline{c} \) is different for each RPO. Similar structures were also reported in plane Couette flow \cite{KawaharaKida2001,KawasakiSasa2005,Viswanath2007}. To date, the number of RPOs discovered in pipe flow is few and all searches were limited to subspaces with rotational symmetries \cite{MellibovskyEckhardt2011,WillisEtAl2013}. It is worth mentioning, however, that any solution found in a subspace are necessarily also solutions of the full space and hence represent physically consistent flow states. All RPOs discovered were embedded in regions of ‘lower’ energy (below turbulent levels) but it was speculated that RPOs at higher energy levels exist that underpin the dynamics of turbulence.

All known invariant solutions other than the laminar flow are unstable at the Reynolds numbers for which turbulence is observed, but the dimensions of their unstable manifolds in phase space is typically low \cite{Kawahara2005,KerswellTutty2007,Waleffe2001,Viswanath2009}. Hence it is expected that they can be approached closely along their stable manifolds. The least unstable orbits are expected to be the most representative, and are the most likely to be extracted from simulation.

In this paper we isolate and link significant features of turbulent dynamics to several representative RPOs, including one with a much longer period time than previously computed orbits. We present results of invariant solutions for a ‘minimal’ practical set of symmetry, imposing only shift-reflect and no rotational symmetry. When the long-period RPO emerges it forms a localised bubble of chaos within the rest of turbulence. Shadowing, or ‘visits’ to this invariant solution significantly increase the turbulent lifetime. The resulting lifetime of a trajectory is dependent on the rate of switching, that is, we observe ‘entries’ and ‘exits’ from the bubble. In addition, the long-period RPO captures much of the qualitative features of turbulence, including streak break-up and repetitions in chaotic trajectories.
2. Numerics and Formulation

To find invariant states we first apply the symmetry reduction method of slices (Willis et al. 2013; Budanur et al. 2014) to pipe flow to obtain a quotient of the streamwise translation symmetry of turbulent flow states. Within the symmetry-reduced state space, all TWs reduce to equilibria and all RPOs reduce to periodic orbits. The method bypasses the difficulties of \( \bar{\varepsilon} \) being different for each RPO in an automatic manner, and permits much simpler identification and extraction of TWs and RPOs directly from turbulent chaotic trajectories. For simulations we use a hybrid spectral finite-difference code (Willis & Kerswell 2009) with 64 non-equispaced finite difference axial points, Fourier expansions evaluated on 48 azimuthal points and on 24 points per unit radius in the radial direction (idealised). The shift-reflect symmetry, carried by almost all known TWs, is applied,

\[
S : (u, v, w)(r, \theta, z, t) = (u, -v, w)(r, -\theta, z, t).
\]

(2.1)

For convergence and continuation of solutions we used a Newton-Krylov-hookstep algorithm (Viswanath 2007) with minor enhancements in adjustments to the norm (Willis et al. 2013). The relative residual of the RPOs is at least approximately \( 10^{-7} \) for the longest orbit, and is considerably less \( (\ll 10^{-8}) \) for the others. While we have not imposed further symmetries, some of the observed TW solutions are highly-symmetric N1 (Pringle et al. 2009), which also satisfy shift-and-rotate symmetry

\[
\Omega : (u, v, w)(r, \theta, z, t) = (u, v, w)(r, \theta + \pi, z - \pi/\alpha, t),
\]

(2.2)

where \( 2\pi/\alpha \ (\alpha = 1.25) \) is the wavelength. (See (Willis et al. 2013) Appendix regarding relationships between symmetries in pipe flow.) For the present work we fixed \( Re = 2300 \) and have chosen a domain of length \( 5D \), which is a compromise between the reduced computational expense of a smaller domain and the need for the pipe to be sufficiently long to accommodate turbulent dynamics. We note that it has recently been shown that the key features of localised TWs are easily encompassed within a domain of similar length in pipe flow (Chantry et al. 2014). For these parameters, turbulence is found to be transient with characteristic life-time \( t \approx 10^3 D/U \).

3. Relative Periodic Orbits

The classical physical quantities, input energy \( I = V^{-1} \oint dS [n \cdot u] p \), dissipation \( D = ||\nabla \times u||_2^2 / Re \) and the kinetic energy \( E = ||u||_2^2 / 2 \), where \( || \cdot || \) corresponds to a root-mean-square value, are often used for phase space representation. Figure 1 presents projections of several travelling waves and the relative periodic orbits. For notation we follow the works (Kerswell & Tutty 2007; Pringle et al. 2009). Highly-symmetric TWs with both shift-reflect and shift-rotate symmetry \( (S, \Omega) \) are named ‘N1’, and states with only shift-reflect symmetry \( (S) \) are called ‘S1’. We identify different RPOs with subscript corresponding to their period. (Note that these numbers refer to \( Re = 2300 \) and change with \( Re \) (Willis et al. 2013).) Due to energy balance, all TWs, time-averages of RPOs and a sufficiently long time-average of a turbulent flow must lie on the diagonal \( D = I \) (see Fig. 1(a)). While the lower branch travelling waves N1L and S1L (see Fig. 1(a)) are far from the turbulent flow, the TWs, N1M2 and N1U appear to be in the core of the turbulent region (Although the classes ‘N’ and ‘S’ are well known most of ‘1-fold’ solutions observed here are new.) Notably, all RPOs extracted from turbulent trajectories also lie within this region (in particular near the upper branch TW of N1U) while all short RPOs are enclosed by the long-period RPO\( _{72,001} \) in the energy plot (see Fig. 1(c)). Extraction of these orbits from simulation implies that they are among the least unstable, and are
Figure 1. Rate of energy input from the background pressure gradient $I$ (external power to maintain constant flux) versus (a) the dissipation rate $D$ and (b) the energy $E$ for the invariant solutions of TWs and RPOs, together with two turbulent orbits. (All quantities are normalised by their laminar counterparts.) Light gray presents a typical turbulent orbit while dark gray indicates orbits that shadow RPO$_{72.001}$ and enter and exit the bubble. (c) and inset in (a) show an expanded view of the region near N1M2 and N1U where all here discovered RPOs are embedded. Note, due to visibility a full relaminarisation trajectory is only given for one simulation.
Periodic orbits and bubbles of chaos

Figure 2. Dynamics of pipe flow for several RPOs and two turbulent runs. For the latter, one (green (dark gray) dashed) hangs around the long-period RPO_{72,001} (stuck in inner saddle) while the other shows a typical visit of the RPO_{72,001} orbit (here about two periods; typically about two to five periods) before it relaminarizes. (a) Energy $E_{3d}$ versus time $t[D/U]$ for each RPO. The inset shows the periodic oscillation of all RPOs discussed in this letter. (b) Friction of RPOs as indicated. One cycle of each period is shown red (light gray) solid, with continuation black dashed.

therefore expected to be important and potentially representative of turbulent flow. In particular the long-period RPO_{72,001} is found to be dynamically relevant. Trajectories that happen to approach the orbit tend to stay close to it and encircle it (see dark regions in Fig. 1) for extended periods. This results in an overall increase in lifetimes. All RPOs discussed here lie in the core region, associated with higher dissipation, and underline their potential role as organizing centers of the turbulent dynamics. In particular our RPOs are located near the solutions of upper branch TWs, rather than the lower branch TWs, the latter of which are more closely linked to transition.

Figure 2 illustrates the close correspondence between the dynamics of turbulent runs and those of RPO_{72,001}. Monitoring the energy $E_{3d} = ||u - \bar{u}||^2/2$ (top panel), where $\bar{u}$ is the axially averaged flow, and friction $\Lambda = 2DG/\rho U^2$ (where $G$ is the mean pressure gradient along the pipe and the density) (bottom panel) against time $t$ we show and example trajectory (green, dashed) that stays around the long-period RPO_{72,001} for a very long time, and another (gray) that shows a more typical ‘visit’ to RPO_{72,001}, here for approximately two periods. The group of periodic orbits is shown in the inset.
Figure 3. (a) Relaminarisation times of turbulent flows at $Re = 2300$ as a function of initial conditions (times in $D/U$) separated if its trajectories visit (▲) and non-visit (□) the vicinity of RPO$_{72.001}$ orbit (horizontal dashed lines present averaged lifetimes) and initials near that orbit (♦) which get stuck in the inner saddle. (b) Corresponding survivor function $S(t) = \exp[(t-t_0)/\tau_{true}]$, with $\tau_{true} = 1/r\sum_{i=1}^r t_i + (n-r)t_r$ (lifetime sample of size $n$ with truncation after $r$ decays) (Avila et al. 2010). The attractor of RPO$_{72.001}$ cause an offset to ‘nearby’ initial conditions that decay after longer times. ‘Switching’ refers to changes between inner and outer saddle. For comparison, results for $Re = 2250$ and $Re = 2450$ are also shown, which are below and above the range existing where RPO$_{72.001}$ exists ($2292 \lessapprox Re(RPO_{72.001}) \lessapprox 2423$).
neighbourhood of the orbit $RPO_{72.001}$ spend very long time within this subregion (labelled ‘stuck in inner saddle’), but are otherwise uncorrelated. While their distribution follows that of a memoryless process, their long period results in distribution that is almost the superposition of an exponential and a constant, similar to the observation in (Kreilos et al. 2014), suggesting that they are associated with a further saddle within the inner saddle (or another ‘bubble’ within the bubble). More typically, all other ICs lead to trajectories that visit the bubble near $RPO_{72.001}$ for two to five periods, and the extended lifetime is dependent on the rate of ‘switching’, or ‘entries’ and ‘exits’ from the bubble.

Comparing before and after the range where the long period orbit $RPO_{72.001}$ exists ($2292 \lesssim Re (RPO_{72.001}) \lesssim 2423$) in Fig. 3 (gray circles), it is observed that at slightly higher $Re = 2450$ the flow has inherited the longer lifetime associated with the bubble (compare with slope for the ‘switching’ case). The bubble has fully merged with the outer saddle so that the spuriously long lifetimes associated with the possible saddle within in the inner saddle, no longer feature.

Our simulations highlight that visits, or ‘switches’, to the ‘bubble’ near $RPO_{72.001}$ are typically of around two to five period times, i.e. ‘shadowing’ this orbit two to five times, see Fig. 2(a). Thus visiting the bubble strongly affects the turbulent lifetimes suggesting that the RPOs that structure this region play a key role in the evolution of the turbulent flow. Notably also, perturbations of the shorter RPOs results in a turbulent flow that also remain in this region and around $RPO_{72.001}$ for substantial times. This strongly suggests further ‘shadowing’ of the orbits, and that longer orbits may be constructed in terms of shorter ones.

Figure 4 depicts a full cycle of the temporal evolution of spatial structures for $RPO_{72.001}$ and a turbulent flow in its neighbourhood in dark region in Fig. 1. The vortex structures given by isosurfaces of streamwise vorticity $\omega_z$ (constant value for all snapshots of 60\% of its maximum) show significant variation over one period of $RPO_{72.001}$. While at the low energy point of the cycle (see point (a) in phase space plots) the shape is quite smooth with moderate vorticity. It significantly increases during the cycle at higher energy levels (b) (see also online available material), then structure breaks up (c). There is clear correspondence between the vortex structures of $RPO_{72.001}$ and the turbulent flows nearby. For both cases one observes the same formation sequence that changes from a streamwise vortex-dominant shape to the form of low-velocity streaks dominating the structure (see movie1.avi). This supports the notion that RPOs can encompass much of the structure and dynamics of turbulent flows.

4. Conclusion

In summary, applying a minimal set of symmetry we have discovered exact coherent solutions of the Navier-Stokes equations in pipe flow and extracted RPOs directly from turbulent velocity fields. We have discovered an RPO with long-period period of $72.001 \text{ }D/U$, much longer than previously discovered in pipe flow and comparable shear flows. The ‘bubble’ around the orbit is a chaotic saddle within the wider saddle of turbulence, and supports the idea that bubbles of chaos continue to have and influence after the crisis bifurcation (Kreilos & Eckhardt 2012). Here, the stable manifold of inner and outer saddles are observed to be highly intertwined, resulting in entry and exit to the bubble. At the initially chosen parameters we were able to observe the switching process clearly, while from lower to higher $Re$ we have calculated that there is an increase in the mean lifetime. Trajectories that visit the bubble have significantly longer characteristic lifetimes than those that do not appear to visit this region, and we have seen that the
Figure 4. Evolution cycle of the spatio-temporal structures of RPO$_{72.001}$ ($T = 72.001$) and a turbulent flow hanging around it (for about three periods of RPO$_{72.001}$) in the bubble. The phases for the visualisation are identified by the symbols (a),(b),(c) in the phase plane ($E, I$) (1) and the monitored energy $E_{3d}$ (2) (see also Figs. 1 and 2). Flow structures are visualised over one full cycle at three times with a constant interval about $24 \ t[D/U]$. Vortex structures are presented by isosurfaces of streamwise vorticity $\omega_z$ at $\pm 0.6 \max(\omega_z)$ (red (light gray) is positive and blue (dark gray) is negative) relative to the laminar flow for RPO$_{72.001}$ (left column) and nearby turbulent flow (right column) (From top to bottom they present snapshots of (a),(b) and the (c)). Color maps at right of each plot present corresponding cross sections (light (darker) indicate positive (negative) relative to the laminar flow.). See also online available material movie1.avi.
lifetime is dependent on the rate of switching (i.e. frequency of visits) to the bubble, rather than simply on the lifetime within the bubble itself. As a general picture, the state space is expected to be filled with a multitude of unstable invariant solutions, and turbulent trajectories visit the least unstable solutions. The long-period RPO and observations of shadowing manifests the perception that RPOs capture much of the natural measure of turbulent flows, within the subregion at least. Such RPOs are the most likely to be extracted directly from simulation. In future, the natural place to extend these ideas is to the larger domain, where localised structures such as puffs may exist. Work by Avila et al. (private communication [Results presented at EC565 meeting in Cargese 2014]) for lower Re but larger domains, appear to also be supportive. It is expected that the identification of RPOs will be a valuable tool in decoding natural characteristics of turbulence.

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