Understanding quantum effects from classical principles

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Abstract

Dynamics of a particle is formulated from classical principles that are amended by the uncertainty principle. Two best known quantum effects: interference and tunneling are discussed from these principles. It is shown that identical to quantum results are obtained by solving only classical equations of motion. Within the context of interference Aharonov-Bohm effect is solved as a local action of magnetic force on the particle. On the example of tunneling it is demonstrated how uncertainty principle amends traditional classical mechanics: it allows the momentum of the particle to change without the force being the cause of it.
I. Introduction

The phrase quantum effect was introduced into the language of physics to signify everything that cannot be understood in terms of traditional classical mechanics. It is used to emphasize deep division between the two views: one of the classical and the other of the quantum world. From the failures to explain them from classical principles a very important conclusion was reached: quantum dynamics is fundamental and classical is derived in the limit $\hbar \to 0$. This view can be relaxed somewhat by accepting that the approximation $\hbar \sim 0$ is sufficient for classical dynamics to be a suitable alternative to quantum, and it is called semi-classical. In this approximation one combines classical and quantum principles, but certain caution should be exercised because deciding when it is applicable is not sometimes clear. Quantum effects in this approximation are partially described, and this is the closest to what one can achieve with the classical principles.

Which are the quantum effects? In the first place this is interference and tunnelling, but there are many more: zero point energy, spin, discrete bound states, photo-electricity, ... to mention only few. As it was mentioned classical mechanics could not explain their physical origin from its principles, but the question is does the quantum mechanics? It does, but in saying so one thing should not be overlooked: in order to explain them the wave-particle dualism must be accepted without being able to rationally derive it from other principles. In fact the problem is not to comprehend either of the components: the concept of particle is well understood and so is the concept of the wave, but the union of the two. However, by accepting this logical system, the quantum mechanics, in which the wave-particle dualism is the postulate (and few others), one derives equation the solutions of which describe all the phenomena (not entirely, e.g. to describe the spontaneous emission one needs more elaborate theory). While there is no problem with the mathematical aspect of obtaining the correct result, there are problems with the physical understanding of them. However, the epithet ”physical” must be defined. Within the logical system of quantum mechanics ”physical” means that the essential features of the phenomena can be described by the properties of either the waves or the particles. In this respect the physics of interference is easily described by the wave-like nature of particles, but, for example, when it comes to the effect of tunneling there are problems. It cannot be explained from the particle-like nature of particles, but neither it
can be from the wave-like nature. In other words, there is no explanation how the waves get through the potential barrier, except that the solution of the wave equation predicts it.

Are there any alternative formulations of quantum mechanics that is based on the wave-particle dualism? This question inevitable rises the following one: why there is necessity for alternative formulations? The answer to the second is that alternative formulations offer a different viewpoint of physics, and as such they are very important, and the answer to the first is that there are. One alternative formulation is based on strictly abstract approach, in which the essential postulate is that there is observable-operator connection. Apart from that one there are other alternative formulations of quantum mechanics. The path integral method is the best known, in which instead of Schroedinger equation one postulates its integral equation form. The other is the Bohm’s method of quantum potentials, which in essence is not new formulation because classical trajectory equations are solved in the effective potential that is obtained from solution of Schroedinger equation. The third formulation is random classical mechanics, which departs from the usual approach to quantum mechanics by introducing the concept of probability into classical mechanics, however Schroedinger equation is retained indirectly by postulating imaginary diffusion coefficient for the probability. Characteristic of all the alternative formulations is that Schroedinger equation is postulated, in one form or the other, while classical mechanics plays no important role.

Alternative formulations are also known in classical mechanics, for example Lagrange or Hamilton formulations. They do not offer, in essence, anything different than the Newton equations of motion, but in many circumstances are more useful and emphasize different concepts in physics (e.g. the energy conservation law). However, one obvious formulation of classical mechanics had been entirely neglected. For centuries the basic ingredient of the scientific method was the concept of error, because no result of experiment is considered reliable if the error margin is not cited. Yet to the best of knowledge no reference work undertook to discuss the following question: given the error margins for initial conditions how this error propagates in time? This question is of utmost importance for theoretical predictions, because it can be shown (but not elaborated in details here) that the assumption of the precise initial conditions is academic in most circumstances. The meaning of this is that even the tiniest error, often in a relatively short time,
increases to such an extent that no reliable predictions could be made. Therefore instead of asking whereabouts of the particle if its initial conditions are known, more appropriate question is to ask for probability of whereabouts of the particle if the probability of its initial conditions is known. This shift in emphases means that instead of treating dynamics of a point in the phase space one should treat dynamics of a phase space density. In other words, the emphases is shifted from the Newton equations of motion to the Liouville equation. Again, the two formulations are equivalent, although analysis of the phase space density provides additional insight into the dynamics of the system.

Importance of formulating classical dynamics in terms of the Liouville equation is that the uncertainty principle can be imposed on its solutions, and this condition can be treated as the additional postulate in classical mechanics. It says that if the standard deviations for the coordinate and the momentum are $\Delta x$ and $\Delta p$, respectively, then

$$\Delta x \Delta p \geq c$$

where $c$ is a constant (determination of the constant, which has the value $c = \hbar/2$, is not discussed, but it can be done in the same way as from the black-body radiation law, by fitting theoretical predictions to the experimental data). By implementing the postulate it remains to find solution of the Liouville equation with that property. It is anticipated that quantum mechanical results will be obtained, and if this is the case then this approach could be treated as alternative formulation of quantum mechanics. The most important difference with the previous formulations is that the starting point is classical dynamics, which is formulated with the Liouville equation but amended with the uncertainty principle. The other formulations start, in one form or the other, by postulating Schroedinger equation, however, if the suggested formulation is correct Schroedinger equation should be derived. Therefore, this approach does not replace quantum mechanics but derives it from different principles that do not incorporate the wave-particle dualism. It should be pointed out that the uncertainty principle in quantum mechanics is the law, i.e. it is derived from more basic principles, e.g. the wave-particle dualism, while in this formulation it acquires the status of postulate.

The problem of implementing the uncertainty postulate into classical mechanics is purely a mathematical task. The main problem is to find a suitable parametrization of the phase space density that ensures that the
amendment is fulfilled at all times. The solution was demonstrated in several instances, but for the sake of completeness it will be described in the following section. Once this is done then the time evolution of the probability densities is obtained by solving the Liouville equation, but this essentially means solving Newton’s equations of motion. This is the essence of what it will be called the classical solution for dynamics of particles. In short, suitable parametrization of the phase space density that ensures the uncertainty principle, plus the Newton’s equations of motion, is classical dynamics. In contrast the traditional classical dynamics is based on the concept of trajectory and without the uncertainty principle. Quantum dynamics implies starting from the same initial conditions as in classical dynamics, but solving Schroedinger equation instead of Newton’s. As it will be shown classical dynamics describes two very important quantum effects: interference and tunneling (in this context the problem of the zero point energy will also be discussed).

II. Formulation of classical dynamics

The suggestion in Introduction of incorporating the uncertainty principle into classical mechanics seems contradiction with the concept of trajectory, the concept that is an integral part of traditional classical mechanics. This is indeed correct but it is no longer that if instead of deterministic view one assumes the probabilistic one. The arguments for this change were mentioned in Introduction. The shift of emphases in classical mechanics from the concept of trajectory to the concept of probability means that formally one replaces the Newton equations of motion

\[
\frac{d\vec{p}}{dt} = \vec{F} ; \quad \frac{d\vec{r}}{dt} = \frac{\vec{p}}{m}
\]

with the Liouville equation for the probability density in the phase space

\[
\frac{\partial \rho}{\partial t} + \frac{\vec{p}}{m} \cdot \nabla_{\vec{r}} \rho + \vec{F} \cdot \nabla_{\vec{p}} \rho = 0
\]

where \( m \) is mass of particle and \( \vec{F} \) is force acting on it. One step towards implementing the uncertainty principle is to change the meaning of the function \( \rho \). Instead of being treated as the probability density one should accept that it is a general, but real, function that satisfies the Liouville
equation. The reason for this change is the observation that \( \rho \) cannot be measured accurately because that implies accurate measurement of both the position and momentum of a particle, and this would violate the uncertainty principle. On the other hand, for the averages

\[
P(\vec{r}, t) = \int d^3p \ \rho(\vec{r}, \vec{p}, t) \quad Q(\vec{p}, t) = \int d^3r \ \rho(\vec{r}, \vec{p}, t)
\]

this restriction is not applicable because, for example, for the probability \( P(\vec{r}, t) \) to be measured it is not necessary to know the momentum. Therefore the phase space density (not the probability density) is treated as an auxiliary function that satisfies the Liouville equation, and whose initial value is obtained from the quantities such as (4), or from the probability current

\[
\vec{J}(\vec{r}, t) = \frac{1}{m} \int d^3p \ \vec{p} \ \rho(\vec{r}, \vec{p}, t)
\]

Therefore the quantities that have physical significance are the probabilities (4) and the probability current (5), and not the phase space density \( \rho \), although the time evolution of the former is derived from the latter.

The uncertainty principle requires that \( P(\vec{r}, t) \) and \( Q(\vec{p}, t) \) are related by the inequality (1), which puts a constraint on the possible phase space densities, the solutions of Liouville equation. The problem is, therefore, to select the family of functions with that requirement, which can be readily solved if certain rules from the Fourier analysis (for a reference see \[\text{12}\]) are recalled. According to these rules the probability densities are written as

\[
P(\vec{r}, t) = |f(\vec{r}, t)|^2 \quad Q(\vec{p}, t) = |g(\vec{p}, t)|^2
\]

and if the two functions are interrelated by

\[
f(\vec{r}, t) = \frac{1}{\sqrt{(2\pi\hbar)^3}} \int dp \ e^{i\vec{p} \cdot \vec{r}/\hbar} g(\vec{p}, t)
\]

then the inequality (1) is ensured. The relationships (3) and (7) are known from quantum mechanics, in fact they are integral part of it. It would appear therefore that in this way quantum mechanics is introduced through the "back door", however this is not correct. The mentioned relationships are a mathematical way of selecting those probabilities \( P(\vec{r}, t) \) and \( Q(\vec{p}, t) \) that obey the uncertainty principle, and before being used in quantum mechanics
they were known in the Fourier analysis. The same relationships are also used in the signal theory, although not for the probabilities but for the intensities of pulses in the time and frequency domains.

The next step is to find how the phase space density is related to the amplitude \( f \), because in this way the required constraint on the solutions of Liouville equation would be ensured. The relationship should be obtained by using the definitions (3), which are familiar rules for convolutions in Fourier analysis, and from that observation one obtains

\[
\rho(\vec{r}, \vec{p}, t) = \frac{1}{(\pi \hbar)^{3}} \int dq e^{2 i \vec{p} \cdot \vec{q}/\hbar} f^*(\vec{r} + \vec{q}, t) f(\vec{r} - \vec{q}, t) \tag{8}
\]

In order to prevent possible misunderstandings few comments about the function (8) are in order. It is known as the Wigner function, but it should not be considered here as the Wigner quasi-probability distribution i.e., the Weyl transform (up to constant) of a pure state density operator, because we did not introduced any quantum operator or state. Remember that \( f \) is only an auxiliary function used for the parametrization of the classical probability for the coordinate of the particle. The Wigner function is one of the many quasi-probability distribution functions invented to express quantum mechanical averages in the classical (phase space) manner, which is not the subject under discussion in the present work.

The phase space density should be solution of the Liouville equation (3), and if the constraint is the parametrization (8) then one derives the equation for \( f \). It is straightforward to show that for the polynomial potentials up to the second degree the equation that one derives for \( f \) is

\[
\imath \hbar \frac{\partial f}{\partial t} = -\frac{\hbar^2}{2m} \Delta f + V f \tag{9}
\]

in which Schrödinger equation is recognized. Therefore for the potentials of this kind the equation (9) is classical solution for the problem of implementing the uncertainty principle. This means that the problems such as free particle, charged particle in the homogeneous (in general a time varying) electric or magnetic field, harmonic oscillator, etc. are exactly described by the amended classical theory. However, it can be shown that the same conclusion is valid in general and the proof is based on observation that any potential can be divided-up into the segments of constant value. In each segment the classical solution that is based on solving (8) is in order, and
by conveniently adjusting the boundary conditions between segments one derives again (9), where now the potential $V$ is a general function. Consequently, free particle trajectories are used for the time evolution of the phase space density, a rather complicated procedure but in principle exact. However, if one is not interested in the solution in the phase space, only in the coordinate subspace, then it is sufficient to solve the equation (9), which is a much simpler task. The price, which is paid for this benefit, is the loss of information that the phase space provides, which will be demonstrated on the following examples.

It would appear that the steps that were taken here are in the reverse order as it is done when deriving the Liouville equation from Schrödinger equation, but in what way they are different is discussed in the Conclusion of this paper.

### III. Interference

The best-known quantum effect is interference that results from a particle having a choice to get through two slits. The setup is the following. In the y-z plane there is a screen with two circular slits that are centered on the y-axes at ± $y_0$, having the width $\Delta$. The screen ends at $x = 0$. The particle is sufficiently de-localized before the screen so that there is equal probability to enter either of the slits. In the slits, and just before exiting them, the probability of finding particle in the plane $x = 0$ is a sum of the form $P(x, y, z) = P_1(y, z)g_1(x) + P_2(y, z)g_2(x)$, where the index designates the probability centered at a particular hole. The probability $P_i(y, z)$ is not zero in a circle of the radius $\Delta$, and for simplicity it will be assumed to have the form of a Gaussian with that width. The probability $g_i(x)$ is determined essentially by the length of de-localization of the incoming particle along the x-axes, and for simplicity it will also be assumed to be Gaussian of the width $\Delta$. The average velocity of the particle in the x direction is $v_0$. The fact that the particle enters each slit with the same probability implies that the functional forms for $P_1(y, z)g_1(x)$ and $P_2(y, z)g_2(x)$ are the same. The moment when the maximum of the probability $P(x, y, z)$ is at $x = 0$ will be $t = 0$. Propagation of this probability density from this moment on will be as for a free particle, and the impact that the screen has on the motion in $x > 0$ will be neglected. Without considering the uncertainty principle the motion of the probability density can only be deduced if in addition to the
probability \( P(x, y, z) \) one also knows the velocity distribution of the particle. In the traditional classical mechanics this distribution is arbitrary, but with the uncertainty principle included it is no longer that. In fact the velocity distribution is not the main problem, it is the initial phase space density from which the initial conditions for classical trajectories are selected. The phase space density, if the uncertainty principle is included, is calculated from (8) where the function \( f \) is defined by (6). It is obvious that the definition (6) does not determine this function uniquely, because \( f \) is in general complex.

The phase of \( f \) is determined from the probability current (5), and if (8) is replaced for the phase space density it can be easily verified that

\[
P \nabla \arg(f) = \vec{J}
\]

By assuming that the probability current is known the phase is calculated as the line integral of the function \( \vec{J}/P \). In the example with the two slits the current in the y-z plane is zero, while in the x direction it is given by \( J = v_0 g(x) \).

In this way the initial conditions are determined and the function \( f \) is

\[
f(x, y, z) = \left[ \sqrt{P_1(y, z)} + \sqrt{P_2(y, z)} \right] \sqrt{g(x)} \, e^{iv_0 x}
\]

where from now on it will be assumed that \( m = \hbar = 1 \) (the square root of a sum of two functions is equal to the sum of the square roots of these functions only if they do not overlap). For a particular example of the Gaussian probabilities the function \( f \) is (non-essential factors are omitted for convenience)

\[
f(x, y, z) = \left[ e^{-\frac{1}{2\Delta^2}(y-y_0)^2} + e^{-\frac{1}{2\Delta^2}(y+y_0)^2} \right] \, e^{-\frac{1}{2\Delta^2}(x^2+y^2)+iv_0 x}
\]

from which the initial phase space density is

\[
\rho_0(x, y, z, v_x, v_y, v_z) = \left[ e^{-\frac{1}{2\Delta^2}(y-y_0)^2} + e^{-\frac{1}{2\Delta^2}(y+y_0)^2} + 2 \cos(2vy_0) \right] e^{-\frac{1}{2\Delta^2}(x^2+y^2)}
\]

At any time later, and if the particle is free, the phase space density is

\[
\rho(\vec{r}, \vec{v}, t) = \rho_0(\vec{r} - \vec{v} t, \vec{v})
\]

from which the probability \( P(\vec{r}, t) \) is

\[
P(\vec{r}, t) = \int d^3 v \, \rho(\vec{r}, \vec{v}, t) = \frac{1}{(\Delta^4 + t^2)^{3/2}} \left[ \cosh \left( \frac{2\Delta^2 y v y_0}{\Delta^4 + t^2} \right) + \cos \left( \frac{2tv y_0}{\Delta^4 + t^2} \right) \right] e^{-\frac{\Delta^2[(x-tv_0)^2+y^2+z^2]}{\Delta^4+t^2}}
\]
Figure 1: Typical interference pattern (left figure) from two slits that is observed on the screen along the x-axes. The two slits are along the y-axes. The pattern is shifted (right figure) if the localized magnetic field is placed in between the two slits.

The screen where the probability density is measured is at \( x = X \) and as a function of the z-y coordinates its typical form is shown in Figure 1 (left pattern).

The parameters where chosen arbitrarily, for the demonstration purpose only, and their values are: \( y_0 = 1000, \Delta = 100, \) and \( v_0 = 1 \). The screen is located at the distance \( X = 10^5 \), and the time at which the probability is observed is \( t = 10^5 \). Typical interference pattern is obtained, which is the same as if the particle is treated as a wave. However, in the treatment here the interference pattern is obtained by propagating the phase space density by classical trajectories, the equation (11), and by the classical rules of probability addition. This is made possible by having additional term in the initial phase space density (the third term in (11)), besides those that correspond to the typical classical probability densities that are centered around the slits. The interference term in the phase space density, as the additional term can be called, has two distinctive features: one is that it
has both positive and negative values, and the other that it is centered at a totally "non-physical" place, in between the two slits. The first feature is essential if by the classical rules of addition of probabilities one can describe the oscillatory structure of the probability density on the screen at $x = X$. The negative values of the phase space density rule out the possibility to attach to it physical significance of the probability density. However, as it was mentioned, this "non-physical" character of the phase space density is explained by impossibility to measure it in experiment.

While one could accept the possibility to work with the non-positive phase space densities, the location of the interference term in between the two slits rises at least two important questions. One is if it has physical significance, because its location would imply that it is only a "mathematical trick" by which the correct result is obtained. The other question is why is it placed in a region where it does not overlap with the space where the particle is certainly located, around the slits?

The physicality of the interference term of the phase space density can be tested, by applying the force on the particle that is only localized in the region between the two slits. For example this can be homogenous magnetic field that is localized in a tube of the radius smaller than $y_0$, centered at $y = 0$ and oriented in the $z$ direction. In the traditional view the phase space density has zero value in this region and hence the force would not have any effect on the pattern on the screen at $x = X$. However, the phase space density that is in accordance with the uncertainty principle has a non-zero value around $y = 0$ (the interference term), therefore all trajectories that originate there are affected by the magnetic field (it is tacitly assumed that the particle is charged). They will be affected for a time $t = T$ until they exit the magnetic tube, and for simplicity it will be assumed that this time is independent of the initial conditions of trajectory. The equation for these trajectories is (for simplicity the mass of particle is assumed to be unity)

$$d^2 t \vec{r} = \vec{v} \times \vec{H}$$

and for the assumed magnetic field, i.e. $\vec{H} = h_0 \hat{z}$, the solution is known exactly. After time $T$ the trajectory exits the magnetic field, and after that it goes as if no force in applied on the particle. It can be assumed that the magnetic field acts for a short time, meaning that $Th_0 \ll 1$, in which case the trajectory is

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \vec{v}_0 \times \vec{H} \left( t - \frac{1}{2} T \right)$$

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These trajectories are used to propagate the phase space density, but only the interference part, because the trajectories that originate around the two slits are not affected by the magnetic force. The time evolution of the interference part is

\[
\rho^{(\text{int})}(\vec{r}, \vec{v}, t) = \rho_0^{(\text{int})} \left[ \vec{r} - \vec{v}t - \vec{v} \times \vec{H} \left( t + \frac{1}{2}T \right) , \vec{v} + \vec{v} \times \vec{H} \right]
\]

where \( \rho_0^{(\text{int})} \) is the third term in (10). The probability \( P(\vec{r}, t) \) is now obtained by integrating the phase space density over the momentum (velocity) variables, which is finally

\[
P(\vec{r}, t) = \frac{1}{(\Delta^4 + t^2)^{3/2}} \left[ \cosh \left( \frac{2\Delta^2y y_0}{\Delta^4 + t^2} \right) + \cos \left( \frac{2ty + h_0 T^2x}{\Delta^4 + t^2} y_0 \right) e^{-\frac{\Delta^2y^2 h_0 y_0}{\Delta^4 + t^2}} \right]
\]

The effect of the localized magnetic field is real, despite the fact that it only affects what appears to be "non-physical" part of the phase space density. It is manifested essentially as the shift of the interference pattern, as it is shown in Figure 1 (right pattern), where the parameters are the same as before, but in addition \( T = 1000 \) and \( h_0 = 0.0002 \). Therefore, if the effect is confirmed then one can indeed argue that the interference part of the phase space density is physically real.

The effect is known as the Aharonov-Bohm effect and its main point was to show that the concept of potential is more physical than the concept of force. This force-potential dilemma is historic, but it was always thought that the latter is just a convenient mathematical simplification of treating electromagnetic field. In quantum mechanics it is the vector potential that enters the Schroedinger equation explicitly, and not the magnetic force, and based on the Aharonov-Bohm effect it was argued that with the concept of field one needs a non-local theory to explain it. Namely, the magnetic field in this effect is confined in a localized region in space while the vector potential is spread all over it, including both slits. Therefore the latter has local effect on the charged particle, while the magnetic force does not have. However, by starting from the classical principles the force was reinstated as a legitimate concept, because it was shown that the effect can be explained as a local event on the interference term. It could be argued that the problem of non-locality with the concept of field is replaced by the problem of non-locality.
in the phase space density. After all, there is always a question (the second mentioned earlier) why in the phase space there is a contribution that does not have any direct relationship to where the particle indeed is, i.e. around one of the slits? In fact the question is not so much why is it there (it is there because of the uncertainty principle), but how it comes to be there?

In order to answer this question one needs to understand the meaning of the initial conditions for the phase space density, and how they are formed. For the two slit problem the initial probability was single centered, i.e. a wide distribution that overlaps both slits. If the walls of the screen are very thin, and for the particle they are infinitely high potential barrier, then almost instantaneously the initial single centered distribution splits into two, well separated ones. This means that before overlapping with the screen the phase space density is single centered around $y = 0$ and $p_y = 0$ (the other degrees of freedom are not essential for discussion), but after exiting the slits the phase space density is non-zero around three centers. For the two, centered around $y = \pm y_0$, one can easily give arguments why they are there. However, there exists the third, around $y = 0$ (the interference term), which is a surprise because it is behind the infinite wall with respect to the original phase space density and it appears instantaneously. The choice is now either to accept this fact, but then one should accept the view that this classical approach is non-local theory, or to argue that formation of the interference term takes time. Intuitively the latter is the more acceptable view, but in order to prove it one would really need to work with the relativistic theory, in which correlation effects (and this is what one talks about in the formation of the interference term) cannot travel faster than the speed of light. In other words, sudden appearance of the interference term is nothing but an artifact of non-relativistic theory, where the signals can travel at arbitrary speed.

Solving the relativistic two slit experiment is quite demanding, but without the loss of generality one can treat simpler but analogous one dimensional problem. At $t = 0$ a probability distribution is formed around $y = 0$ in such a way that its momentum distribution contains two disjoined components, one with the average momentum $-m_0$ and the other with $m_0$. It is expected that the probability in the coordinate $y$ would split into two components traveling in the opposite directions (provided that $m_0$ is larger than the width of the momentum probability). After certain time one would have two disjoined probability densities, of the sort as in the two slits experiment, the only difference being that the two probabilities travel in the opposite directions.
By applying a force on each probability distribution one can reduce their average momentum to zero, in which case this would be precisely the initial conditions for two slits (one neglects finer details that make this statement not entirely correct). This problem can be treated by relativistic mechanics, in which case one observes how the phase space density is formed.

The relativistic phase space density, from which Dirac equation is derived, in single dimension is:

\[
\rho(y, t, p, p_0) = \int dq \ dq_0 \left[ f^*(y + q, t + q_0) f(y - q, t - q_0) + g^*(y + q, t + q_0) g(y - q, t - q_0) \right] e^{2i pq - 2i pq_0} 
\]

where it was assumed that only positive energy components are present. At \( t = 0 \) the phase space density is

\[
\rho_0(y, p, p_0) = \int dq \ dq_0 \left[ f_0^*(y + q, q_0) f_0(y - q, -q_0) + g_0^*(y + q, q_0) g_0(y - q, -q_0) \right] e^{2i pq - 2i pq_0} 
\]

where the functions \( f_0 \) and \( g_0 \) are defined in the momentum space as

\[
f_0(y, q_0) = \int dk \ A(k) e^{iky - iq_0 e(k)} \quad ; \quad g_0(y, q_0) = \int dk \ w(k) A(k) e^{iky - iq_0 e(k)}
\]

\( A(k) \) is the momentum space amplitude for the initial conditions (the units are \( m = c = h = 1 \), \( w(k) = k/[1 + e(k)] \) and \( e(k) = \sqrt{1 + k^2} \). By evaluating the integrals in the variables \( q \) and \( q_0 \) the initial phase space density is (non-essential factors are omitted)

\[
\rho_0(y, p, p_0) = \int dk \ \delta \left[ 2p_0 - e(p - k) - e(p + k) \right] [1 + w(p - k)w(p + k)] A^*(p - k) A(p + k) e^{2iky}
\]

and at any later time the phase space density is

\[
\rho(y, t, p, p_0) = \rho_0(y - \frac{p}{p_0} t, p, p_0)
\]

\[
= \int dk \ \delta \left[ 2p_0 - e(p - k) - e(p + k) \right] [1 + w(p - k)w(p + k)] A^*(p - k) A(p + k) e^{2ik(y - \frac{p}{p_0} t)}
\]
The phase space density in only the coordinate-momentum variables is obtained by integrating in the variable \( p_0 \) (the fourth component of the four-momentum), in which case

\[
\rho(y, p, t) = \int dk \left[ 1 + w(p - k)w(p + k) \right] A^*(p - k)A(p + k) e^{2i ky - ie(p+k)t + ie(p-k)t}
\]

(12)

For the momentum amplitude it is now assumed to have the form

\[
A(p) = e^{-(p-m_0)^2/2\Delta^2} + e^{-(p+m_0)^2/2\Delta^2}
\]

where \( m_0 \gg \Delta \) (the two distribution do not overlap). When in the phase space (12) the product of the amplitudes is evaluated one gets three terms that are centered around \( y = 0 \): two terms that are centered around \( p = \pm m_0 \) and one around \( p = 0 \). Between all three contributions there is no overlap (or it is negligible). The first two contribute to the phase space density that is centered around \( y = \pm m_0 \) and \( p = \pm m_0 \), which is analogous to the phase space density centered around \( y = \pm y_0 \) in the two slit setup. These two contributions recede, each traveling at the speed \( m_0 \) (in the non-relativistic limit when \( m_0 \ll 1 \)) or at nearly the speed of light (in the relativistic limit when \( m_0 \gg 1 \)). They are of no interest for what is the intention to show. The third term, which is analogous to the interference term, should be analyzed in details. In the phase space its contribution is

\[
\rho_{int}(y, p, t) = e^{-p^2/2\Delta^2} \int dk \left( 1 + w(p - k)w(p + k) \right) \left[ e^{-(k-m_0)^2/2\Delta^2} + e^{-(k+m_0)^2/2\Delta^2} \right]
\]

\( e^{2iky - ie(p+k)t + ie(p-k)t} \)

If \( m_0 \) is small (and so is \( \Delta \) by assumption) then the function \( w(k) \) is small and can be neglected. Also one can write \( e(k) \approx 1 + \frac{1}{2}k^2 \), in which case the interference term in the phase space density is

\[
\rho_{int}(y, p, t) = e^{-p^2/2\Delta^2 - \Delta^2(x-pt)^2} \cos \left[ 2m_0(x - pt) \right]
\]

Its typical form is shown in Figure 2 for the parameters \( m_0 = 0.05 \) and \( \Delta = 0.01 \), and for two time instants: \( t = 0 \) (left figure) and \( t = 5000 \) (right figure). The main feature of the interference term is that its modulus is independent of time and \( m_0 \). Furthermore it changes its shape in unison with the rate at which the two main peaks in the phase space density separate from each other.
Figure 2: Interference term in the initial phase space density (left figure) for the probability that is localized around $y=0$ on the $y$-axes and if it has two isolated maxima along the momentum axes. After the initial instant the probability on the $y$-axes splits into two, and in the non-relativistic dynamics the interference term in the phase space density adapts its shape instantaneously to the new configuration (right figure).
On the other hand if \( m_0 \) is large (but the width \( \Delta \) is again small), and because most of contribution to the interference term comes from \( k = \pm m_0 \), one can write \( e(k) \approx |k| + \frac{1}{2|k|} \) and \( 1 + w(p - k)w(p + k) \sim 2m_0^{-1} \). This means that the interference term diminishes in the limit \( m_0 \to \infty \), and it is approximately

\[
\rho_{\text{int}}(y, p, t) \sim \frac{1}{m_0} e^{-\frac{\Delta^2}{2\Delta^2 x^2}} \cos \left[ 2m_0 \left( x - \frac{p}{m_0} t \right) \right]
\]

which is valid in the time interval \( t \ll m_0^3 \Delta^{-3} \). For longer times it diminishes as \( t^{-1} \). Besides being small contribution in the overall phase space density the interference term has additional feature that indicates its dependence on the time it takes the correlation to have effect on it (the two receding peaks travel at nearly the speed of light). The exponential term is "frozen" meaning that it is time independent, and the oscillatory term is changing but with a great time lag, in fact it is nearly constant also. If the two receding peaks are stopped by, say, a potential step, then the situation would be similar to the two slit problem. However, at that instant the interference part of the phase space density would be very small in amplitude and not having the adequate shape. By stopping the two peaks the phase space density would redistribute itself in order to match this situation but it is obvious that process would take some time. This indicates that the interference term in the two slits setup does not form itself instantaneously but in reality it takes some time, and therefore its source is in a physical process.

IV. Zero point energy

Among the quantum effects is the so-called zero point energy, which is the lowest possible of all stationary states of a particle in a potential (ground state energy). There are several reasons why it is in this group of effects, but the one with the greatest weight comes from the interpretation of the energy of the stationary states. According to the standard approach to the quantum-classical relationship the correspondence between the quantum and classical stationary states is only possible in the limit \( \hbar \to 0 \), or equivalently for large quantum numbers (Bohr's principle of correspondence). As a consequence, it can be shown relatively easily from the WKB approximation that the energy of stationary states have the following interpretation. If position of a particle is random then the modulus of its momentum is not, it is determined from the
energy conservation law. In other words, if $E_0$ is the energy of a stationary state then the phase space density should parametrize, in the classical limit, as

$$\rho(\vec{r}, \vec{p}) = \delta \left( \frac{p^2}{2m} + V - E_0 \right) = \frac{1}{\sqrt{2m(E_0 - V)}} \delta(p - p') \quad (13)$$

Indeed this limit is approached, on average, for the stationary states with large quantum numbers. This means that for the ground state there is not even approximate agreement between this classical limit and the quantum ground state, as shown in Figure 3. Classical probability curve (dotted line) has singularity at the points where momentum of the particle is zero. Large portion of the quantum probability, however, is outside these classical bounds, and the standard interpretation is that this is due to tunneling, and therefore classical interpretation is not possible.

If the analysis starts from the Liouville equation (3) then the stationary solutions are obtained by requiring that $\partial_t \rho = 0$, which has a general solution in the form $\rho = F \left( \frac{p^2}{2m} + V \right)$, where $F$ is any function that has finite norm. Therefore there is infinite number of stationary states, but one particular is

$$\rho(x, p) = e^{-a \left( \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \right)} \quad (14)$$

for, say, a one dimensional harmonic oscillator. The constant $a$ is arbitrary and can be fixed by requiring that the average energy of the oscillator is equal to the energy of the ground state of the quantum oscillator, i.e.

$$< E > = \int dp \frac{p^2}{2m} Q(p) + \int dx \frac{1}{2}m\omega^2 x^2 P(x) = E_0$$

The result for the probability density $P(x)$ is identical curve as in Figure 3 for the quantum solution.

The major departure from the standard classical analysis is in the interpretation of what the energy of the stationary states represents. The difference is summarized in the two expressions for the phase space densities, (13) and (14). Thus according to the solution that is based on the Liouville equation, both the coordinate and momentum of particle are chosen randomly, according to the prescribed phase space distribution. For each pair of these points in the phase space the appropriate energy of the particle does not coincide with the energy of the ground state, but on average it is equal to it. Therefore, the points in Figure 3 that appear to be in the classical forbidden
Figure 3: Probability for the ground state of the harmonic oscillator, if its energy is $E_0$. Quantum calculation is shown by the curve $P_{qv}(x)$ and the WKB by $P_{WKB}(x)$. The latter is treated in the standard interpretation as the classical limit of quantum dynamics. The probability beyond the classical turning points is interpreted as tunneling.
region, and hence being interpreted as tunneling, are in fact manifestation of entirely classical effect. The points that are considering tunneling express probability of finding particle with the energy that is larger than \( E_0 \), which according to (14) is not zero. In the phase space this is manifestly clear but from the solution of the Schroedinger equation (9) it is not, because one works with the function that is the average over all momentum part of the phase space. This in essence is the meaning of the comment at the end of the previous section.

Previous analysis was entirely classical, and the identity with the quantum result is accidental because the choice of the function \( F \) was arbitrary, however, the analysis explains the physical content of the zero point energy. The only contribution of the uncertainty principle is to select from the functions \( F \) only those that satisfy it, and as it turns out there is only one.

V. Tunneling

The effect of tunneling is one of the most intriguing quantum effects, but its proper understanding requires careful analysis. As it has already been indicated what is considered to be the tunneling effect in the case of the zero point energy it is in fact a classical effect. It is manifestation of the phase space density when it is averaged over the momentum subspace. However, that is not a typical tunneling effect, more appropriately it is described in a scattering of particle on a potential barrier. Discussion of this example starts by considering a much simpler one, which is scattering of a particle from an infinitely high potential barrier.

In this example the particle moves in the space on the negative x axes, and the barrier is positioned at \( x = 0 \). If initial probability density \( P_0(x) \) is given, and the initial probability current \( j_0(x) \), then the problem to find time evolution of the phase space density \( \rho(x, p, t) \), and from that the probability density \( P(x, t) \), is well defined. There are several reasons why the problem cannot be solved in the same way as for the interference on two slits, which was described in the previous section. One is that the potential is not of the parabolic type, which is a necessary condition that the equation (8) is derived from the parametrization (8). Derivation of the equation (8) does not appear a necessary condition for solving time evolution of the phase space density because this is done by propagating trajectories in a potential. The weakness of this argument is that by simple solution of this kind the uncertainty prin-
ciple may not be satisfied at arbitrary times, although at the initial instant it is by the parametrization (8). Therefore to ensure that this is the case the function \( f \) in the initial (8) should incorporate all the information about the potential, including the possible boundary conditions, and this is achieved by solving the equation (9). This argument appears non-physical because there is no reason why at the initial instant the particle, if it is well localized away from the barrier, should "know" of its existence. The same argument was applied for the existence of the interference term in the two slits experiment, and the answer was that this is an artifact of the non-relativistic theory, where infinite velocities are possible. At the initial instant one can ensure that the particle does not have "contact" with the barrier, but due to the infinite dispersion of momenta at any short time after the particle will be everywhere in the space, and therefore will know of the barrier. It can be shown, but not elaborated here, that if the relativistic theory is used, with the restriction that the probability density is zero outside certain boundary, then indeed the particle does not know of the existence of the barrier until this boundary reaches it. Because of that the boundary propagates as in the traditional classical mechanics, i.e. no quantum effects are observed. Therefore the initial (8) should be determined by using time independent solutions of the equation (9) as the basis in which \( P_0(x) \) is represented. However, there is a problem, as mentioned earlier, the equation (9) cannot be derived for the potentials other than harmonic. There is solution to this difficulty for potentials of the step-like character, and the infinite barrier is of this kind, which is to replace potential with the boundary condition. In other words, one can neglect the potential barrier and treat the particle as being free on the whole x axes. The barrier is mimicked by imposing the boundary condition on the probability and the probability current at \( x = 0 \) by demanding that at each instant they should be equal to zero. The net effect is the same, although the physics of the problem is not. The difference is that physics demands that for \( x > 0 \) the probability \( P(x, t) \) is zero, while from the imposed boundary condition this is not necessarily the case. In other words, the problem is solved mathematically formally, but the physical content should be extracted.

The initial function \( f \), from which the initial phase space density is calculated, is therefore obtained from a linear combination (again the units are used in which \( m = \hbar = 1 \))

\[
f(x) = \frac{1}{2\pi} \int dk \ A(k) \ (e^{ikx} - e^{-ikx})
\]
where the amplitude $A(k)$ should be determined from $P_0(x)$ and $j_0(x)$. The plane waves $e^\pm ikx$ are solutions of the stationary equation (9) and their combination ensures that the proper boundary condition at $x = 0$ is satisfied. For simplicity the initial conditions are

\[
P_0(x) = e^{-\frac{(x-x_0)^2}{\Delta^2}}; \quad j_0(x) = p_0 \, P_0(x)
\]

where $x_0$ is negative and chosen so that $|x_0| \gg \Delta$. It can be shown that (constant pre-factors are omitted for convenience)

\[
A(k) = e^{-(k-p_0)^2 \Delta^2/2 - ikx_0}
\]

in which case the initial phase space density is

\[
\rho_0(x, p) = \int_{-\infty}^{\infty} dq \, e^{2ipq} \, f^*(x + q) f(x - q)
\]

\[
= e^{-\frac{(x-x_0)^2}{\Delta^2} - (p-p_0)^2 \Delta^2} + e^{-\frac{(x+x_0)^2}{\Delta^2} - (p+p_0)^2 \Delta^2} - 2 \cos [2(p_0 x - px_0)] e^{-\frac{x^2}{\Delta^2} - p^2 \Delta^2}
\]

which consists of three terms: one centered around $x_0$ and $p_0$, the other around $-x_0$ and $-p_0$, and the third around $x = p = 0$ which represents the interference term. The structure of the phase space density is very similar to the one in the problem with two slits. However, the interest here is not to discuss the effect of interference but to note one important property of the momentum distribution, which has important repercussions for understanding the tunneling effect.

At any later time the phase space density is given by (11), and the probability for the coordinates is obtained from

\[
P(x, t) = \int_{-\infty}^{\infty} dp \, \rho_0(x - pt, p)
\]

which is not zero for $x > 0$, but for $x < 0$ it coincides with the true solution of the problem. The physical solution is therefore obtained by disregarding the probability for $x > 0$. However, the momentum distribution is obtained by integrating the physical solution for $f$, which is zero for $x > 0$, but the phase space density (17) was derived under the assumption that this function extends over the whole x axes. Therefore, by formally calculating the integral

\[
Q(p, t) = \int_{-\infty}^{0} dx \, \rho_0(x - pt, p)
\]
to obtain the momentum distribution is not a legitimate procedure. One needs to extract $f(x, t)$ from the probability $P(x, t)$ and the probability current $j(x, t)$, and then obtain $Q(p, t)$ from (6) and (7). This is the price one pays by replacing the potential barrier with the boundary condition. For the case under discussion the function $f(x, t)$ is relatively easily extracted, and the result is

$$f(x, t) = \frac{1}{\sqrt{\Delta^2 + it}} \left[ e^{-\frac{\Delta^2(x-x_0-p_0t)^2 + 2i(\Delta^2 p_0-x_0 t)x}{2(\Delta^2 + it)}} - e^{-\frac{\Delta^2(x+x_0+p_0t)^2 - 2i(\Delta^2 p_0-x_0 t)x}{2(\Delta^2 + it)}} \right]$$

The momentum function is

$$g(p, t) = \int_{-\infty}^{0} dx \ f(x, t)e^{-ipx}$$

which is not given explicitly because it is a rather lengthy expression, instead a typical probability $Q(p, t)$ (solid line) is shown in Figure 4 for four typical times: initial instant (a), just before the maximum of the Gaussian probability reaches the barrier ($t_0 = |x_0|/p_0$) (b), just after this instant (c) and long after that (d). Its shape is not what one would expect from the intuitive reasoning, and which is based on the fundamental law of classical mechanics that any change in momentum is caused by force.

In this particular example the force is of a special kind, it only changes the sign of the momentum, and therefore the width of $Q(p, t)$ should not change in time because the modulus of the momentum is constant. The expected probability is the following: in the beginning particle moves towards the barrier and its momentum distribution is centered around $p_0$, and after long time the particle moves away from the barrier and its momentum distribution is centered around $-p_0$. These two probabilities should be the mirror images of each other. At any other time, in particular around $t = t_0$, the function $Q(p, t)$ is a combination of the two extreme cases. This is indeed the case if $Q(p, t)$ is calculated from the phase space distribution in which the interference term in (14) is neglected (from the traditional classical mechanics). The resulting probability is shown in Figure 4 by the dotted line, which deviates considerably from the probability when the uncertainty principle is implemented. The essence of the difference is in the change of the modulus of the momentum, which in traditional classical explanation it is attributed to the action of a force. However, there is no such force only the infinite barrier. In fact the momentum distribution changes dramatically between
Figure 4: Various stages of scattering on the infinite barrier as observed in the momentum space. At initial instant (a) the momentum of the particle is centered around a positive value. Just before (b) and after (c) the maximum of the probability on the coordinate axes reaches the barrier the momentum distribution widens. The dotted line is the momentum distribution from the traditional classical calculation. Long after the collision (d) the momentum distribution is centered around a negative value, and it is mirror image of the initial (a).
the initial instant and \( t \sim t_0 \), which is shown by calculating \( g(p, t) \) for large \( p \). In this limit (the non-essential factors are omitted)

\[
g(p, t) \sim \frac{1}{p^2} e^{-\frac{\Delta^2 (x_0 + p t)^2}{2(\Delta^4 + t^2)}}
\]

which means that the modulus of the function changes from \( g(p, t) = \exp(-p^2 \Delta^2/2) \) to \( g(p, t) \sim p^{-2} \) between these two instants. The widening effect of the momentum distribution has no source in the dynamics, because it is entirely the consequence of the change in the width of the probability in the coordinate. In other words, as the width of the probability in the coordinate of the particle changes the distribution of the momenta also changes, meaning that no classical dynamics can explain this effect because it is not caused by the action of a force.

The widening effect of the momentum distribution can be tested by assuming that the barrier is not infinitely high, say it has the value \( V_0 \). One consequence of the finite height is that all the phase space density for which \( p > \sqrt{2V_0} \) would ”leak” into the half space \( x > 0 \) and will manifest itself as the non-zero probability \( P(x, t) \). However, for a very high barrier the estimate of this probability, under the assumption that there is no effect due to the widening of the momentum distribution, gives

\[
P(x, t) \sim e^{-2V_0 \Delta^2}
\]

which is negligible small. Therefore, the prediction is, which is based on considering only the initial distribution of momenta, that the probability for particle to get over the barrier is negligible. On the other hand, if widening is taken into account then the estimate of the probability \( P(x, t) \) is obtained by first calculating the function \( f(x, t) \) from

\[
f(x, t) = \int dp \ e^{ipx} g(p, t) \sim e^{-\frac{\Delta^2 (x_0 + p t)^2}{2(\Delta^4 + t^2)}} \int dk \ \frac{e^{ikx}}{k^2 + 2V_0}
\]

where \( p \) was replaced by \( \sqrt{k^2 + 2V_0} \) so that it is explicitly taken into account that \( p > 2V_0 \). By evaluating the integral one gets the estimate

\[
P(x, t) \sim e^{-\frac{\Delta^2 (x_0 + p t)^2}{(\Delta^4 + t^2)}} e^{-2x \sqrt{2V_0}}
\]

which has two important features. One, the probability for over the barrier transmission is incomparable larger than the estimate based only on the
initial distribution of momenta, however, this happens when $t \sim |x_0/p_0|$. Second, the probability decays exponentially for increasing $x$, but there is no time dependence of it, except in the factor that indicates arrival of the incident probability. In other words, the probability for over the barrier transmission spreads instantaneously in the whole $x > 0$ half space. This, again, as an artifact of the non-relativistic dynamics, which can be shown by the relativistic treatment of this problem (because of its rather lengthy treatment the details are not given here).

The effect for over the barrier transmission, and the exponential dependence of its probability with the coordinate, can be tested by making the potential zero at some distance $x = \delta > 0$. If there is relatively large probability to observe the particle in the space $x > \delta$ then that would be direct test of the uncertainty principle, because this is the only way to explain its appearance. It would be called a paradox, and it is called the tunneling effect, because the only conclusion from the initial conditions is that such events are not possible (or with the negligible probability), and classical dynamics cannot account for such a large dispersion of momenta. However, it will be shown that once uncertainty principle is implemented the tunneling effect has classical explanation as the over the barrier transmission.

If the effect of tunneling is manifestation of the uncertainty principle, and not dynamics of the particle, then the question is whether there is any meaning in saying that the solutions of (time dependent) Schroedinger equation can be obtained by solving classical equations of motion. The answer is affirmative because inability to describe the change in the momentum distribution, which is due to the uncertainty principle, is replaced by the unusual initial phase space distribution. The meaning of this will be demonstrated on the more exact calculation for the over the barrier transmission, i.e. the tunneling probability on a step potential. Scattering on the step potential has been analyzed in details, and it was showed that classical and quantum calculations for the probability $P(x,t)$ produce identical results. Therefore the analysis of this example will not be analyzed in details, only the essentials points in the part that is relevant for this discussion.

The idea is, as discussed previously, to replace the potential by the boundary condition, but for the step potential its implementation is somewhat more elaborate than in the case of the infinite barrier. Complication is caused by the fact that the phase space density is not confined to the space $x < 0$, but it is also transmitted into the space $x > 0$. Because of that it is necessary
to analyze two separate sets of trajectories: one in the zero potential and
the other in the potential \( V_0 \). This means that if the same idea as for the
infinite barrier is used then two separate phase space densities should be de-
efined: one when the potential is zero \( \rho_1 \) and the other \( \rho_2 \) when it has the
value \( V_0 \). Both are defined on the whole \( x \) axes, but the phase space density
\( \rho_1(x, p, t) \) is only meaningful in the space \( x < 0 \) while \( \rho_2(x, p, t) \) in the space
\( x > 0 \). The initial conditions are set on the negative \( x \) axes, which means
that the phase space density \( \rho_1(x, p, t) \) is defined from them. On the other
hand, the phase space density \( \rho_2(x, p, t) \) does not have direct relationship to
them, only indirectly through the boundary condition at \( x = 0 \) for the two
quantities: the probability densities \( P_1(x, t) \) and \( P_2(x, t) \), and the probability
currents \( j_1(x, t) \) and \( j_2(x, t) \) that are defined for the phase space densities
\( \rho_1(x, p, t) \) and \( \rho_2(x, p, t) \), respectively. At the boundary it is required that
\( P_1(x, t) = P_2(x, t) \) and \( j_1(x, t) = j_2(x, t) \).

The details of deriving the phase space densities are omitted, because
that was shown elsewhere. The analysis is relatively complicated for the
review, and so only the final result will be cited, for the particular case of
interest: the tunneling. The initial phase space density \( \rho_2^0(x, p) \) is given by

\[
\rho_2^0(x, p) = 16 \text{ Re} \left\{ \int dk A(k) A^* (k_p^-) \frac{k [2p - K(k)]}{[k + K(k)] [k + K(k^-)]} e^{2ix[K(k)-p]} \right\}
\]

where \( K(k) = \sqrt{k^2 - 2V_0} \). The variable \( k_p^- \) is defined as

\[
k_p^- = \pm \left[ K(k) - 2p + i\sqrt{2V_0} \right]^{1/2} \left[ K(k) - 2p - i\sqrt{2V_0} \right]^{1/2}
\]

where the sign is selected from the requirement that \( \text{Im}[K(k^-)] < 0 \). The integration path is in the upper half of the complex \( k \)-plane, which has import-
ant feature to avoid two Riemann cuts that are defined there, and preferably
it should go through the stationary point of the phase of the integrand (this is
particularly important for the study of tunneling). For the specific case when
\( \sqrt{2V_0} \gg p_0 \) the initial phase space density is approximately (the non-essential
factors are again omitted)

\[
\rho_2^0(x, p) \sim e^{-2x\sqrt{2V_0}} \text{ Re} \left[ e^{-2ixp} \int dk A(k) A^* (k_p^-) k \right]
\]

which appears not to make physical sense: it is unbounded on the negative \( x \)
axes. This is very unpleasant because at later time the phase space density
is
\[ \rho_2(x, p, t) = \rho_2^0(x - pt, p) \sim e^{-2(x - pt)\sqrt{2V_0}} \operatorname{Re} \left[ e^{-2i(x-pt)p} \int dk A(k)A^*(k^-) k \right] \]
and in the space \( x > 0 \), where by definition it is meaningful, its amplitude increases without bounds in the limit \( t \to \infty \). Therefore it could be rejected as non-physical. However, it should be recalled that it is the probability \( P_2(x, t) \) that has physical meaning, and this should be finite for all time. In other words, from the physics of the problem it should follow that
\[
\lim_{t \to \infty} P_2(x, t) = \lim_{t \to \infty} \int dp \rho_2^0(x - pt, p) = 0 \tag{19}
\]
For the amplitude (16), this can be explicitly proved by evaluating two integrals, in the variables \( k \) and \( p \), by the stationary phase method, where the phase of the integrand is
\[
\vartheta(p, k) = -\frac{1}{2}(k_p^- - p_0)^2 \Delta^2 + ik_p^- x_0 - \frac{1}{2}(k - p_0)^2 \Delta^2 - ikx_0 - 2(x - pt) \left( \sqrt{2V_0} + ip \right)
\]
The set
\[
\partial_p \vartheta(p, k) = 0 \quad ; \quad \partial_k \vartheta(p, k) = 0
\]
defines the stationary points, which are obtained by first making the replacement \( p = r/\Delta^2 \sqrt{2V_0} \) and then in the equations retain the leading terms in the powers of \( V_0 \). The solution of this approximate set of equations is (the details are not shown, because obtaining it is straightforward but relatively lengthy)
\[
k_{st} = \frac{p_0 \Delta^2 - ix_0}{\Delta^2 + it} \quad ; \quad p_{st} = \Delta^2 \left[ -p_0 \Delta^4 + tx_0 \right] \frac{x_0 + tp_0}{\sqrt{2V_0} (\Delta^4 + t^2)^{3/2}}
\]
The probability \( P_2(x, t) \) is therefore
\[
P_2(x, t) \sim \operatorname{Re} \left[ e^{\vartheta(p_{st}, k_{st})} \right]
\]
which, it can be shown, is equal to (18), and also to the solution from the quantum treatment. The limit \( t \to \infty \) is finite, but not zero, however it is very small. The fact that this limit is not zero is an artifact of the choice for the initial probability, which will be discussed shortly. Therefore, despite the
fact that the phase space density increases without bounds in the space $x < 0$ the probability $P_2(x,t)$ has all prerogatives to be physically acceptable.

The stationary value $p_{st}$ for the momentum plays the role of the average momentum $p_0$ for the free particle, but in this case it is measured with respect to the potential $V_0$, and its initial value is $p_{st} = -\frac{p_0 x_0}{\sqrt{V_0}}$, i.e. it is very small. One confirms this by calculating the probability current $j_2(x,t)$ in the space $x > 0$, and from the definition (3) the velocity of the particle is $v_{tunn} = j_2(x,t)/P_2(x,t)$. This calculation produces the identity $v_{tunn} = p_{st}$. Therefore, if one can talk about the *tunneling velocity* of the particle then this would be $v_{tunn}$, and it follows that it is very small.

As it was shown classical mechanics describes tunneling but not because it is result of dynamics, but because of the specific initial phase space density that results from implementing the uncertainty principle. From this phase space density, and by using solutions of classical equations of motion, one describes tunneling effect, and the result is the same as by solving time dependent Schroedinger equation. One aspect of this solution is quite intriguing, and needs to be understood properly. Time dependence of the tunneling probability (18) has a very specific form: time variation of the probability at $x = 0$ is instantaneously transmitted to all points $x > 0$. This means that at some point $x = \delta$ the time variation of the probability is identical with that at the point $x = 0$. If the potential is cut at $x = \delta$ then propagation of the probability in the space $x > \delta$ is the same as for a free particle. In fact it has the same time dependence as if one chooses for it the initial $P_0(x)$, and as if there is no gap. The only difference is that the amplitude of this probability is scaled by the factor $e^{-2\Delta x_0\sqrt{2V_0}}$. The net effect is that the probability travels from the space $x < 0$ to $x > \delta$ as if there is no gap, i.e. as if the tunneling velocity is infinite. This finding directly contradicts what had been shown before, that the tunneling velocity is very small. The controversy can be resolved by noting one important physical aspect of tunneling: the interval within which the tunneling probability is significant is of the order $x_{tunn} \sim (2V_0)^{-1/2}$. This means that the time it takes for the particle to travel this distance, at the velocity $v_{tunn}$, is

$$t_{tunn} = \frac{x_{tunn}}{v_{tunn}} \sim \frac{(\Delta^4 + t^2)^2}{\Delta^2 \left(-p_0 \Delta^4 + t x_0\right) \left(x_0 + tp_0\right)} \sim -\frac{\Delta^2}{p_0 x_0}$$

which is independent of $V_0$. In the last step time dependence was neglected. Physical circumstances require that $|x_0| \gg \Delta$, and also that during the time
of scattering the shape of the probability is nearly constant, which means that $t \ll \Delta^2$. From the characteristic collision time $t \sim |x_0|/p_0$ it follows that $p_0 = M x_0 / \Delta^2$, where $M$ is a large number. The tunneling time is then

$$t_{\text{tunn}} \sim \left( \frac{\Delta}{x_0} \right)^2 \frac{\Delta^2}{M}$$

which is very short compared with the characteristic time variation of the probability for the free particle. Indeed, if the tunneling time is multiplied by $p_0$ then the distance that the free probability travels during this interval is

$$x = p_0 t_{\text{tunn}} \sim x_0 \left( \frac{\Delta}{x_0} \right)^2 = \Delta \frac{\Delta}{x_0}$$

which is small. Therefore, the reason why the probability has time dependence (18) is that the tunneling velocity, although in absolute magnitude is very small, is sufficiently large so that any change at one end is transmitted "instantaneously" to the other.

There are other questions in connection with the tunneling effect, e.g. how the specific form of the spatial dependence (18) is formed, but they cannot be answered without considering more precise theoretical model. In particular to answer these questions the initial probability (15) is not adequate, because it would be more accurate to work with the one that is strictly zero outside certain boundary. In connection with this one should also use relativistic theory, because any cutoff in the probability on the x axes makes distribution of momenta very wide, which also includes relativistic values, i.e. for a given momentum $p$ the velocity of particle $p/\sqrt{1+p^2}$ is nearly the speed of light. However, considering these issues would require more extensive discussion, but results would not contribute in an essential way to the understanding of the tunneling effect.

**VI. Conclusion**

The aim of the previous discussion was to show that the two basic quantum effects: interference and tunneling, can be explained and quantitatively described by formulating dynamics of a particle from the classical principles, with addition of the uncertainty principle. The results are identical to those obtained from quantum mechanics, and by that it is meant solutions of Schroedinger equation. The two approaches are different ways of seeing the
same effects, in many respects analogous to analyzing the motion of classical particles from either the Newton’s equations of motion or from the Lagrange principle of least action. Or, solving the harmonic oscillator problem starting from matrix mechanics or from the differential equation. The advantage of analyzing quantum effects from the classical principles is that one works in the phase space, and therefore sees the problem with additional degrees of freedom. In this respect more information is available about the system, which is lost if one only works in, say, the coordinate space. The last is characteristic of quantum mechanics, and although one can switch between the coordinate and momentum spaces, one never works in both at the same time. Crudely speaking, quantum mechanics works with the averaged quantities in one of the phase space coordinates and because of that one easily makes erroneous conclusions. This is best observed in the analysis of the zero point energy in one of the previous sections. From quantum mechanics one makes conclusion that relatively large portion of the probability for the zero point energy is due to tunneling, but in fact its shape is explained entirely classically.

There was attempt to overcome this drawback of quantum mechanics by formulating quantum phase space density, which would enable to study dynamics of the particle in all the phase space variables. The transform that extends quantum dynamics into the phase space is the Wigner function $w(\vec{r}, \vec{p}, t)$, which was mentioned in Section 2. For the wave function $\psi(\vec{r}, t)$ it is defined through the property

$$|\psi(\vec{r}, t)|^2 = \int d^3p \ w(\vec{r}, \vec{p}, t) \ ; \ |\phi(\vec{p}, t)|^2 = \int d^3r \ w(\vec{r}, \vec{p}, t)$$

where $\phi(\vec{p}, t)$ is the wave function in the momentum space. This, however, is the only connection with the phase space density that was used throughout the paper. Namely, the Wigner function, or the quantum phase space density, does not satisfy any simple equation, and definitely not the Liouville equation, except in the special case of the harmonic-type forces. The Wigner function is used as the mean to study the limit $\hbar \to 0$ of quantum mechanics, and as the proof that this is classical mechanics it is shown that Liouville equation is obtained. Based on such arguments the quantum phase space density was used in the study of the quantum-classical relationship. The quantum phase space density is mentioned in the context of this work because it is the base of a dilemma: is classical mechanics the limit of quantum when $\hbar \to 0$ or is quantum mechan-
ics derived from classical by taking into account the uncertainty principle? This point was discussed elsewhere\textsuperscript{16} and therefore will not be discussed here in details. The resumé of this discussion is that by strictly taking the limit $h \to 0$ in quantum mechanics one obtains classical but with a special property, in which the phase space density is parametrized as (13). On the other hand, by starting from classical mechanics one obtains correct result for, say, the ground state of harmonic oscillator.

The disadvantage of working in the phase space (or from classical principles) is that solving problems is more difficult, and often not straightforward. As discussed on the example of tunneling the essential new feature that the uncertainty principle introduces into dynamics is that the change in the momentum is not necessarily caused by the action of a force (it can be called \textit{non-dynamic effect}). Therefore, action of a force on the particle is not sufficient to reproduce quantum results, one needs to take into account that momentum changes from the other cause, the implementation of the uncertainty principle. The exception is harmonic force, for which it can be shown not to affect the phase space density in a way that would change the momentum distribution that cannot be explained by the force itself. This can be demonstrated by solving scattering problem on inverted parabolic potential $V(x) = -\frac{1}{2}wx^2$. Given initial conditions $x_i$ and $p_i$ for classical trajectory in this potential its time dependence is

$$x = x_i \cosh(\omega t) + \frac{p_i}{\omega} \sinh(\omega t) \quad ; \quad p = p_i \cosh(\omega t) + \omega x_i \sinh(\omega t)$$

and the phase space density $\rho(x, p, t)$, if it is initial $\rho_0(x, p)$, is

$$\rho(x, p, t) = \rho_0 \left[ x \cosh(\omega t) - \frac{p}{\omega} \sinh(\omega t), -p \cosh(\omega t) + \omega x \sinh(\omega t) \right]$$

From this phase space density one calculates the probability $P(x, t)$ from the definition (4). This is classical solution for arbitrary $\rho_0(x, p)$, and specifically if the initial conditions (15) are chosen then the result is

$$P(x, t) = \frac{1}{\sqrt{\pi \Delta_t^2}} e^{-(x-x_t)^2/\Delta_t^2}$$

where

$$\Delta_t = \Delta \left[ \cosh^2(\omega t) + \frac{\sinh^2(\omega t)}{\omega^2 \Delta^4} \right]^{1/2}$$

$$x_t = x_0 \cosh(\omega t) + \frac{p_0}{\omega} \sinh(\omega t)$$
The same result is obtained if the probability \( P(x, t) \) is calculated from quantum mechanics, i.e. by solving Schroedinger equation for the time evolution of the wave function. This result is independent of the particular choice of the initial conditions, but for the Gaussian type (15) it has convenient analytic form.

For other than harmonic potentials the contribution of the non-dynamic effects may be significant, but sometime negligible. However, the problem can be avoided by working with the step like potentials, as mentioned in Section 2, in which case the change in potential is replaced by the boundary conditions on the phase space density. By doing that one also includes the non-dynamic effects into account. The procedure is exact, but the result for the time evolution of the phase space density is relatively complicated. The solution simplifies considerably if one is only interested in the time evolution of the probability \( P(x, t) \), when it is sufficient to solve the equation (9), the procedure that is valid for any potential. That it is a correct one can be demonstrated on one example. It will be assumed that the potential is a delta function at the origin, i.e. \( V(x) = W_0 \delta(x) \), and the initial conditions are (15). The potential divides the space into \( x < 0 \) and \( x > 0 \), and in each one the particle moves in zero potential. Time evolution of the phase space density \( \rho^>(x, p, t) \) in \( x > 0 \) can be thought to originate from some initial \( \rho^<_0(x, p) \). That initial phase space density is obtained by requiring that at \( x = 0 \) the phase space density \( \rho^<_0(x, p, t) \) for the space \( x < 0 \) changes smoothly into \( \rho^>(x, p, t) \). The initial phase space density for \( \rho^<_0(x, p, t) \) is obtained from (15) by the same procedure as described in the section on tunneling, with a slight modification due to the fact that the phase space density penetrates into the space \( x > 0 \). The proper connection between the two phase space densities is ensured by the proper choice of the function \( f \) that enters the phase space density (8). It can be shown, without giving the details, that this function for the two spaces is

\[
 f^<(x) = \int dk \ A(k) \left[ e^{i k x} + R(k) e^{-i k x} \right] ; \quad f^>(x) = \int dk \ A(k) T(k) e^{i k x} 
\]

where the coefficients are

\[
 R(k) = -\frac{i W_0}{k + i W_0} ; \quad T(k) = \frac{k}{k + i W_0} 
\]

The phase space density \( \rho^>_0(x, p) \) has analytic form if \( W_0 \gg p_0 \), in which
case

\[ \rho_0^>(x, p) \sim \frac{1}{W_0^2} e^{-\Delta^2(p-p_0)^2 - \frac{1}{\Delta^4}(x-x_0)^2} \left[ 2(x-x_0)^2 + 2\Delta^4p^2 - \Delta^2 \right] \]

and its time evolution is \( \rho^>(x, p, t) = \rho_0^>(x - pt, p) \). The probability \( P(x, t) \) of finding the particle in the space \( x > 0 \) (the tunneling probability) is then (normalization is omitted)

\[ P(x, t) = \int dp \rho^>(x, p, t) = \frac{(x-x_0)^2 + \Delta^4p_0^2}{W_0^2(\Delta^4 + t^2)^{3/2}} e^{\frac{\Delta^2}{\Delta^4 + t^2}(x-x_0 - pt)^2} \]

which is exactly the same result as if the problem was solved by quantum mechanics.

In conclusion one can say that the alternative formulation of dynamics of particle, which is based on the classical principles with the amendment of the uncertainty principle, gives identical results as the original formulation in terms of the wave-particle dualism. In other words, Schroedinger equation is derived from these classical principles, which was confirm in the analysis of two quantum effects: interference and tunneling.
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