Ground states of bi-harmonic equations with critical exponential growth involving constant and trapping potentials

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Abstract
In this paper, we first give a necessary and sufficient condition for the boundedness and the compactness of a class of nonlinear functionals in $H^2(\mathbb{R}^4)$ which are of their independent interests. (See Theorems 2.1 and 2.2.) Using this result and the principle of symmetric criticality, we can present a relationship between the existence of the nontrivial solutions to the semilinear bi-harmonic equation of the form

$$(-\Delta)^2u + \gamma u = f(u) \text{ in } \mathbb{R}^4$$

and the range of $\gamma \in \mathbb{R}^+$, where $f(s)$ is the general nonlinear term having the critical exponential growth at infinity. (See Theorem 2.7.) Though the existence of the nontrivial solutions for the bi-harmonic equation with the critical exponential growth has been studied in the literature, it seems that nothing is known so far about the existence of the ground-state solutions for this class of equations involving the trapping potential introduced by Rabinowitz (Z Angew Math Phys 43:27–42, 1992). Since the trapping potential is not necessarily symmetric, classical radial method cannot be applied to solve this problem. In order to overcome this difficulty, we first establish the existence of the ground-state solutions for the equation

$$(-\Delta)^2u + V(x)u = \lambda s \exp(2|s|^2) \text{ in } \mathbb{R}^4,$$

(0.1)

when $V(x)$ is a positive constant using the Fourier rearrangement and the Pohozaev identity. Then we will explore the relationship between the Nehari manifold and the corresponding limiting Nehari manifold to derive the existence of the ground state solutions for the Eq. (2.5) when $V(x)$ is the Rabinowitz type trapping potential, namely it satisfies

$$0 < \inf_{x \in \mathbb{R}^4} V(x) < \sup_{x \in \mathbb{R}^4} V(x) = \lim_{|x| \to +\infty} V(x).$$

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The same result and proof applies to the harmonic equation with the critical exponential growth involving the Rabinowitz type trapping potential in $\mathbb{R}^2$. (See Theorem 2.9.)

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### 1 Introduction

Let $\Omega$ be an open domain in $\mathbb{R}^n$. We will consider the following nonlinear partial differential equation with critical growth

$$(−Δ)^m u = f (u) \text{ in } \Omega \subset \mathbb{R}^n,$$

where $m = 1$ or $2$. Equation (1.1) have been extensively studied by many authors in bounded and unbounded domains.

In the case $n > 2m$, the subcritical and critical growth means that the nonlinearity cannot exceed the polynomial of degree $\frac{n + 2m}{n - 2m}$ by the Sobolev embedding. While in the case $n = 2m$, we say that $f (s)$ has critical exponential growth at infinity if there exists $\alpha_0 > 0$ such that

$$\lim_{|t| \to +\infty} \frac{|f (t)|}{\exp (\alpha t^2)} = \begin{cases} 0, & \text{for } \alpha > \alpha_0 \\ +\infty, & \text{for } \alpha < \alpha_0. \end{cases} \quad (1.2)$$

The critical exponential growth in the case $m = 1, \ n = 2$ is given by the Trudinger–Moser inequality [48,57]:

$$\sup_{u \in H_0^1(\Omega), \|\nabla u\|_{L^2(\Omega)} \leq 1} \int_{\Omega} e^{\alpha |u|^2} \, dx < \infty \iff \alpha \leq 4\pi, \quad (1.3)$$

and in the case $m = 2, \ n = 4$ is given by the Adams inequality [2]:

$$\sup_{u \in H_0^2(\Omega), \|\Delta u\|_{L^2(\Omega)} \leq 1} \int_{\Omega} e^{\alpha |u|^2} \, dx < \infty \iff \alpha \leq 32\pi^2. \quad (1.4)$$

The study of equation (1.1) with the critical exponential growth on bounded domain also involves a lack of compactness similar to the case $n > 2m$ at certain levels that the Palais–Smale compactness condition fails due to concentration phenomena. In order to overcome the possible failure of the Palais–Smale compactness condition, there is a common approach by using the Trudinger–Moser and Adams type inequalities (see [3,8,13,17,24] and references therein).

If $\Omega$ is the entire Euclidean space $\mathbb{R}^n$, the earlier study of the existence of solutions for equation (1.1) with the critical exponential growth can date back to the work of Atkinson and Peletier [4,5]. Indeed, the authors obtained the existence of ground state solutions for equation (1.1) by assuming that there exists some $y_0 > 0$ such that $g (t) = \log f (t)$ satisfies

$$g' (t) > 0, \ g'' (t) \geq 0,$$

for any $t \geq y_0$. This kind of growth condition allows us to take the nonlinearity $f (t) = (t^2 - t) \exp (t^2)$, which has critical exponential growth. In the literature, many authors have
considered the existence of solutions for equations of the form
\[ (-\Delta)^m u + V(x) u = f(u) \quad \text{in } \mathbb{R}^n, \]
where \( n = 2m \), the nonlinearity \( f(s) \) has critical exponential growth, and the potential \( V(x) \) is bounded away from zero. For the Eq. (1.4), the loss of compactness may be produced not only by the concentration phenomena but also by the vanishing phenomena! We will describe some of the relevant works below.

When \( V(x) \) is a coercive potential, that is,
\[ V(x) \geq V_0 > 0, \]
and additionally either \( \lim_{x \to \infty} V(x) = +\infty \) or \( \frac{1}{V} \in L^1(\mathbb{R}^n) \),
the existence and multiplicity results of Eq. (1.4) can be found in the papers [25,59,60] and the references therein. Their proofs depend crucially on the compact embeddings given by the coercive potential, and the vanishing phenomena can be ruled out.

When \( V(x) \) is the constant potential, i.e. \( V(x) = V_0 > 0 \), the natural space for a variational treatment of (1.4) is \( H^m(\mathbb{R}^n) \). It is well known that the embedding \( H^m(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n) \) is continuous but not compact, even in the radial case. In the case \( m = 2 \), the existence of nontrivial solutions for equation (1.4) was obtained by Chen et al. [14] (see also Lam et al. [29] for a proof of the subcritical Trudinger–Moser inequality and Sani [56] (see [54] or [5] for \( m = 1 \)) under the assumptions that for any \( p > 2 \),
\[ f(s) \geq \eta_0 s^{p-1}, \forall s \geq 0, \]
where \( \eta_0 \) is some constant depending on \( p \), and by Sani [56] (see [54] or [5] for \( m = 1 \)) under the assumption
\[ \lim_{|s| \to \infty} \frac{sf(s)}{\exp(32\pi^2 s^2)} \geq \beta_0 > 0. \]

In their proofs, the so-called Trudinger–Moser–Adams inequality in the whole \( \mathbb{R}^2 \) or \( \mathbb{R}^4 \) plays a crucial role. Now, let’s mention some of these inequalities. In 2000, Adachi–Tanaka [1] (see also O [19]) obtained a sharp Trudinger–Moser inequality on \( \mathbb{R}^n \):
\[ \sup_{u \in W^{1,n}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \Phi_n(\alpha |u|^{\frac{n}{n-1}}) dx \leq C(\alpha, n) \|u\|_n^p, \quad \text{iff } 0 < \alpha < \alpha_n, \]

\[ \Phi_n(t) := e^t - \sum_{i=0}^{n-2} \frac{t^i}{i!}. \]
Note that the inequality (1.7) has the subcritical form, that is \( \alpha < \alpha_n \). Later, in [53] and [37], Li and Ruf showed that the best exponent \( \alpha_n \) becomes admissible if the Dirichlet norm \( \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \) is replaced by Sobolev norm \( \int_{\mathbb{R}^n} \left( |u|^2 + |\nabla u|^2 \right) \, dx \). More precisely, they proved that
\[ \sup_{u \in W^{1,n}(\mathbb{R}^n)} \int_{\mathbb{R}^2} \Phi_n \left( \alpha |u|^{\frac{n}{n-1}} \right) dx < +\infty, \quad \text{iff } \alpha \leq \alpha_n. \]
Trudinger–Moser inequalities (1.7) and (1.8) in the Euclidean space. In fact, the equivalence and relationship between the suprema of critical and subcritical Trudinger–Moser inequalities have been established by Lam et al. [30] and such an equivalence has been used to establish the existence and non-existence of extremals for subcritical Trudinger–Moser inequalities on the entire space \( \mathbb{R}^n \) by the same authors [32].

In 1995, Ozawa [49] obtained the Adams inequality in Sobolev space \( W^{m - \frac{m}{2}, \infty}(\mathbb{R}^n) \) on the entire Euclidean space \( \mathbb{R}^n \) by using the restriction \( \|\Delta^\frac{m}{2} u\|_{m} \leq 1 \). However, with the argument in [23,49], one cannot obtain the best possible exponent \( \beta \) for this type of inequality. Sharp Adams inequality in the case of even order of derivatives was proved by Ruf and Sani [55] under the constraint

\[
\{ u \in W^{m, \frac{m}{2}} \| (I - \Delta)^\frac{m}{2} u \|_m \leq 1 \},
\]

when \( m \) is an even integer. When the order \( m \) of the derivatives is odd, a sharp Adams inequality was established by Lam and Lu [28]. A uniform proof was given for all orders \( m \) of derivatives including fractional orders of derivatives by Lam and Lu in [26] through a rearrangement-free argument.

The authors in [26] obtained the sharp Adams inequality under the Sobolev norm constraint: let \( \tau > 0 \),

\[
\sup_{u \in H^2(\mathbb{R}^4)} \int_{\mathbb{R}^4} \left( \exp(\beta |u(x)|^2) - 1 \right) dx \begin{cases} 
\leq C & \text{if } \beta \leq 32\pi^2, \\
+\infty & \text{if } \beta > 32\pi^2.
\end{cases}
\]  

(1.9)

In 2011, Ibrahim et al. [20] discovered a sharpened Trudinger–Moser inequality on \( \mathbb{R}^2 \)—the Trudinger–Moser inequality with the exact growth condition:

\[
\sup_{u \in H^1(\mathbb{R}^2)} \int_{\mathbb{R}^2} \frac{\exp(4\pi |u|^2) - 1}{(1 + |u|)^p} dx \leq C_p \int_{\mathbb{R}^2} |u|^2 dx \text{ iff } p \geq 2.
\]  

(1.10)

Later, (1.10) was extended to the general case \( n \geq 3 \) by Masmoudi and Sani [46] (see Lam et al. [31] for inequalities with exact growth under different norms) and to the framework of hyperbolic space by Lu and Tang [39]. It is interesting to notice that the Trudinger–Moser inequality with the exact growth condition can imply both the inequalities (1.7) and (1.8).

The Adams’ inequality with the exact growth condition was obtained by Masmoudi and Sani [45] in dimension 4:

\[
\sup_{u \in H^2(\mathbb{R}^4)} \int_{\mathbb{R}^4} \frac{\exp(32\pi |u|^2) - 1}{(1 + |u|)^p} dx \leq C_p \int_{\mathbb{R}^4} |u|^2 dx \text{ iff } p \geq 2,
\]  

(1.11)

and then established in any dimension \( n \geq 3 \) by Lu et al. [40] (see [47] for higher order case). Further improvement of Adams inequalities can also be found in recent work of Lu and Yang [43] where sharpened Hardy–Adams inequalities were established in \( \mathbb{R}^4 \) using Fourier analysis on hyperbolic spaces (see also the works of Li et al. [34] for higher even dimensions and [35] for all dimensions). These inequalities are the borderline cases of the higher order Hardy–Sobolev–Maz’ya inequalities established by Lu and Yang (see [42], [44]).

Based on the Trudinger–Moser inequality with the exact growth, Ibrahim et al. obtained a sufficient and necessary condition for compactness of general nonlinear functionals (see [46] for \( n \geq 3 \)). This sufficient and necessary condition is a strong tool to study the existence
of solutions for the semilinear equation under a very general assumption on the nonlinearity. Indeed, they consider the equations of the form:

$$-\Delta u + \gamma u = f(u) \quad \text{in } \mathbb{R}^2,$$

where $\gamma$ is a positive constant and $f(s)$ has the critical exponential growth at infinity. They establish the following result.

**Proposition 1.1** If $f$ satisfies $f(0) = 0$ and the conditions (1.2), (i) and (ii) (see Sect. 2), then there exists $\gamma^* \in (0, +\infty)$ such that for each $\gamma \in (0, \gamma^*)$, the equation admits a positive ground state solution.

The number $\gamma^*$ above is associated with the so called Trudinger–Moser ratio, and both the growth conditions (1.5) and (1.6) imply that the constant $\gamma$ appearing in (1.12) satisfies

$$\gamma < \gamma^*.$$

Their arguments depend crucially on the Pohozaev identity and Schwarz symmetrization argument.

Motivated by the results for Laplace equation plus a potential and with a nonlinear term of exponential growth, in this paper, we are interested in the existence of ground state solutions for the biharmonic equations

$$(-\Delta)^2 u + V(x)u = f(u) \quad \text{in } \mathbb{R}^4,$$

where the nonlinear term $f(s)$ has the critical exponential growth (1.2) at infinity and the potential $V(x)$ is bounded away from zero.

As far as we know, there are no related results about the existence of ground state solutions for the problem (1.13) by means of variational methods. In study of the problems involving the bi-harmonic operator, we will encounter many more difficulties than in the case for the Lapacian. For example, we cannot always rely on the maximum principle, and there is no Pólya–Szegö type inequality for the second order derivatives. Thus, we cannot use Schwarz symmetrization principle. Moreover, if $u$ belongs $H^2(\mathbb{R}^4)$, we cannot claim that $|u|, u^+$ or $u^-$ belong to $H^2(\mathbb{R}^4)$. Therefore, we cannot expect to obtain a positive solution.

## 2 The main results

In order to obtain the existence of solutions to the Eq. (1.13), we first establish the necessary and sufficient conditions for the boundedness and the compactness of general nonlinear functionals in $H^2(\mathbb{R}^4)$.

**Theorem 2.1** (Boundedness) Suppose that $g : \mathbb{R} \to [0, +\infty)$ is a Borel function and define

$$G(u) = \int_{\mathbb{R}^4} g(u)dx.$$

Then for any $K > 0$, the following conditions are equivalent

1. $$\lim_{|t| \to +\infty} |t|^2 \exp\left(-\frac{1}{K}|t|^2\right)g(t) < \infty,$$
2. $$\lim_{t \to 0} |t|^{-2}g(t) < \infty.$$
(2) There exists a constant $C_{g,K} > 0$ such that for any $u \in H^2(\mathbb{R}^4)$ satisfying $\int_{\mathbb{R}^4} |\Delta u|^2 \, dx \leq 32\pi^2 K$, there holds

$$\int_{\mathbb{R}^4} g(u) \, dx \leq C_{g,K} \int_{\mathbb{R}^4} |u|^2 \, dx.$$ 

**Theorem 2.2** (Compactness) Suppose that $g : \mathbb{R} \to [0, +\infty)$ is a continuous function and define

$$G(u) = \int_{\mathbb{R}^4} g(u) \, dx.$$ 

Then for any $K > 0$, the following conditions are equivalent

(3) $\lim_{t \to +\infty} \frac{1}{|t|^2} \exp(-\frac{1}{K}|t|^2) g(t) = 0$, $\lim_{t \to 0} \frac{1}{|t|^2} g(t) = 0$.

(4) For any radially symmetric sequence $\{u_k\}_k \subseteq H^2(\mathbb{R}^4)$ satisfying $\int_{\mathbb{R}^4} |\Delta u|^2 \, dx \leq 32\pi^2 K$ and weakly converging to some $u$, we have that $G(u_k) \to G(u)$.

As an application, we study the following bi-harmonic equation with the constant potential,

$$(-\Delta)^2 u + \gamma u = f(u) \text{ in } \mathbb{R}^4,$$  \hspace{1cm} (2.1)

where the nonlinearity $f(t)$ is a continuous function on $\mathbb{R}$ satisfying (1.2), $f(0) = 0$ and the following properties:

(i) (Ambrosetti–Rabinowitz condition [48,50]) There exists $\mu > 2$ such that $0 < \mu F(t) = \mu \int_0^t f(s) \, ds$ for any $t \in \mathbb{R}$;

(ii) There exist $t_0$ and $M_0 > 0$ such that $F(t) \leq M_0 |f(t)|$ for any $|t| \geq t_0$.

(iii) $f(t) = o(t)$ as $t \to 0$.

**Remark 2.3** The condition (i) implies that $F(t) = o(t^2)$ as $t \to 0^+$. In fact, we can rewrite the condition as $\mu F(t) \leq t F'(t)$ to derive that $F(t) \leq t^\mu F(1)$ when $0 < t < 1$. However, since the critical point of Eq. (2.1) may not be positive, then in order to obtain the existence of non-trivial solution, we need to further assume that $f(t) = o(t)$ as $t \to 0$ which implies that $F(t) = o(t^2)$ as $t \to 0$ from the condition (i).

**Theorem 2.4** Assume that $f$ satisfies $f(0) = 0$ and the conditions (1.2), (i), (ii) and (iii), then there exists $\gamma^* \in (0, +\infty]$ such that for any $\gamma \in (0, \gamma^*)$, the Eq. (2.1) admits a non-trivial radial solution. Moreover, $\gamma^*$ is equal to the Adams’ ratio:

$$C^*_A = \sup \left\{ \frac{2}{\|u\|^2_2} \int_{\mathbb{R}^4} F(u) \, dx \mid u \in H^2_r(\mathbb{R}^4) \setminus \{0\}, \|\Delta u\|^2_2 \leq \frac{32\pi^2}{\alpha_0^2} \right\},$$

where $H^2_r(\mathbb{R}^4)$ is the collection of all radial functions in $H^2(\mathbb{R}^4)$. In particular, $\gamma^* = +\infty$ is equivalent to

$$\lim_{|t| \to +\infty} \frac{t^2 F(t)}{\exp(\alpha_0 t^2)} = +\infty.$$
Remark 2.5 If $F(t) = \frac{\exp(\alpha_0 t^2) - 1 - \alpha_0 t^2}{1 + |t|^p}$, obviously, $f(t) = F'(t)$ satisfies the conditions (1.2), (i), (ii) and (iii). By the Adams inequality with exact growth (1.11), we know that if $\theta < 2$, then $\gamma^* = +\infty$, and the Eq. (2.1) admits a non-trivial radial solution for any $\gamma > 0$. If $\theta \geq 2$, then $\gamma^* < +\infty$, and the Eq. (2.1) admits a non-trivial radial solution for any $\gamma \in (0, \gamma^*)$. Both the growth conditions of the nonlinearities used in [56] and [14] imply that the constant $\gamma$ appearing in Theorem 2.4 satisfies

$$\gamma < \gamma^*.$$ 

Corollary 2.6 Assume that $f$ satisfies $f(0) = 0$ and the conditions (1.2), (i), (ii), (iii) and $F(t)$ satisfies

$$\lim_{t \to \infty} t^2 F(t) = \exp(\alpha_0 t^2)$$

then for any $\gamma > 0$, the Eq. (2.1) admits a non-trivial radial solution.

Until now, whether the solutions obtained in Theorem 2.4 and Corollary 2.6 are ground state solutions is unknown. However, if the nonlinearity has the special form $f(t) = \lambda t \exp(2|t|^2)$, we can prove that the solutions obtained are ground-state solutions.

Theorem 2.7 For any $\gamma \in (0, +\infty)$, the equation

$$(-\Delta)^2 u + \gamma u = \lambda u \exp(2|u|^2)$$

admits a radial ground state solution if $\lambda \in (0, \gamma)$.

Based on the above Theorem, we can also obtain the existence of ground state solutions of the bi-harmonic equation with the Rabinowitz type potential, that is, the potential $V(x)$ is a continuous function satisfying

$$0 < \lambda < V_0 = \inf_{x \in \mathbb{R}^4} V(x) < \sup_{x \in \mathbb{R}^4} V(x) = \lim_{|x| \to \infty} V(x) = \gamma < +\infty.$$ \hspace{1cm} (2.2)

This kind of potential was first introduced by Rabinowitz [51].

Theorem 2.8 Assume that $V(x)$ is a continuous function satisfying (2.2), the equation

$$(-\Delta)^2 u + V(x) u = \lambda u \exp(2|u|^2)$$

admits a ground state solution which is not necessarily radial.

In the proof of Theorem 2.7, we cannot use the Schwarz symmetrization principle directly. In order to overcome this difficulty, we will apply the Fourier rearrangement proved by Lenzmann and Sok [33] to obtain a radially minimizing sequence for the infimum on the Pohozaev manifold. While in the proof of Theorem 2.8, we will exploit the relationship between the Nehari manifold and the corresponding limiting Nehari manifold.

Using Proposition 1.1 and carrying out the same proof procedure of Theorem 2.8, we can also obtain the existence of ground state solutions of the following Laplacian equation with the Rabinowitz type potential introduced in [51]:

$$- \Delta u + V(x) u = \lambda u \exp(|u|^2)$$

in $\mathbb{R}^2$. 

\hspace{1cm} (2.4)
Theorem 2.9  Assume that \( V(x) \) is a continuous function satisfying
\[
0 < \lambda < V_0 = \inf_{x \in \mathbb{R}^2} V(x) < \sup_{x \in \mathbb{R}^2} V(x) = \lim_{|x| \to \infty} V(x) = \gamma < +\infty,
\]
the Eq. (2.4) admits a ground state solution which is not necessarily radial.

As far as we know, the Rabinowitz type potentials are only involved in the study of equations with the subcritical polynomial growth (see e.g., [38, 51] and [58]). In the case of \( m = 1, n = 2 \), when we replace the operator \( -\Delta \) by \( -\varepsilon^2 \Delta \) in the above theorem when the nonlinear term has the exponential growth, the existence of semiclassical state \( u_\varepsilon \) was obtained by Alves and Figueiredo in \( \mathbb{R}^2 \) [7] if \( \varepsilon << 1 \). For other related work on the semiclassical state of nonlinear Schrödinger equations in the case of subcritical nonlinear polynomial growth, we just name a few among a vast literature, e.g., [9, 11, 18, 21, 41, 58], the book by Ambrosetti and Malchiodi [10] and many references therein. Nevertheless, as far as we are concerned, nothing is known if \( \varepsilon = 1 \) and the nonlinear term has the exponential growth. Theorem 2.9 appears to be the first existence result for equation with the critical exponential growth involving the Rabinowitz type trapping potential.

We remark that Theorems 2.7 and 2.8 also hold for the operator \(( -\Delta )^m + V \) in \( \mathbb{R}^{2m} \) for all \( m > 2 \) using the same argument with minimal modifications when the potential \( V \) is of Rabinowitz type. We also remark that in \( \mathbb{R}^2 \) that we have also proved in [16] the existence of ground-state solutions to the following Schrödinger equations with critical exponential growth:
\[
- \Delta u + V(x)u = f(u) \text{ in } \mathbb{R}^2,
\]
where \( V(x) \geq 0 \) and vanishes on some bounded domain of \( \mathbb{R}^2 \). The potential \( V \) is thus degenerate and does not satisfy the positive lower bound.

This paper is organized as follows. Section 3 is devoted to the proof of the necessary and sufficient conditions for the boundedness and the compactness of general nonlinear functionals in \( H^2(\mathbb{R}^4) \). In Sect. 4, we will prove the existence of non-trivial solutions of the equation (2.1) with the constant potential under a very general assumption on the nonlinearity. In Sect. 5, we prove the existence of ground state solutions for the bi-harmonic equation (2.1) with the constant potential when the nonlinearity has the special form \( f(s) = \lambda s \exp(2s^2) \). In Sect. 6, we prove the existence of ground state solutions for the bi-harmonic equation (2.1) with the Rabinowitz type potential.

Throughout this paper, the letter \( c \) always denotes some positive constant which may vary from line to line.

3 Necessary and sufficient conditions for the boundedness and compactness

In this section, we will give the necessary and sufficient conditions for the boundedness and the compactness of general nonlinear functionals which are of independent interests.

Proof of Theorems 2.1 and 2.2 Necessity of (1) in Theorem 2.1 and (3) in Theorem 2.2:

In order to prove the necessity of (1), we only need to verify that if (1) fails, then there exists a sequence \( \{u_k\}_k \subseteq H^2(\mathbb{R}^4) \) satisfying
\[
\int_{\mathbb{R}^4} |\Delta u_k|^2 dx \leq 32\pi^2 K, \quad \int_{\mathbb{R}^4} |u_k|^2 dx \to 0, \quad G(u_k) \to \infty.
\]
Similarly, in order to prove the necessity of (3), we only need to show that if (3) fails, then there exists a radially symmetric sequence \( \{u_k\}_k \subseteq H^2(\mathbb{R}^4) \) satisfying \( \| \Delta u_k \|_2^2 \leq 32\pi^2 K \) and weakly converging to 0, such that \( G(u_k) > \delta \) for some \( \delta > 0 \).

First, we consider the case that the conditions (1) and (3) fails at the origin. Assume that \( \{a_k\}_k \) and \( \{R_k\}_k \) are positive sequences satisfying \( \lim_{k \to \infty} a_k = 0 \) and \( \lim_{k \to \infty} R_k = \infty \).

Let \( \phi_k \subseteq H^2(\mathbb{R}^4) \) be a sequence of spherically symmetric functions given by

\[
\phi_k(x) = \begin{cases} 
  a_k, & \text{if } 0 \leq |x| \leq R_k, \\
  a_k(1 - R_k^2 - |x|^2 + 2R_k|x|), & \text{if } R_k < |x| \leq R_k + 1, \\
  \eta_k(x), & \text{if } R_k + 1 < |x| \leq R_k + 2, \\
  0, & \text{if } |x| > R_k + 2,
\end{cases}
\]

where \( \eta_k = -2a_k(|x| - R_k - 1)^3 + 4a_k(|x| - R_k - 1)^2 - 2a_k(|x| - R_k - 1) \). Obviously, \( \eta_k \) is a radial function on annulus \( B_{R_k+2}\setminus B_{R_k+1} \) satisfying the boundary condition

\[
\eta_k(x)|_{\partial B_{R_k+1}} = 0, \quad \frac{\partial \eta_k(x)}{\partial v}|_{\partial B_{R_k+1}} = -2a_k,
\]

and

\[
\eta_k(x)|_{\partial B_{R_k+2}} = 0, \quad \frac{\partial \eta_k(x)}{\partial v}|_{\partial B_{R_k+2}} = 0.
\]

Furthermore, careful calculations lead to

\[
\int_{B_{R_k+2}\setminus B_{R_k+1}} |\eta_k|^2 dx \lesssim a_k^2 R_k^3, \quad \int_{B_{R_k+2}\setminus B_{R_k+1}} |\Delta \eta_k|^2 dx \lesssim a_k^2 R_k^3.
\]

Hence we derive that there exists a constant \( c > 0 \) such that

\[
\int_{\mathbb{R}^4} |\phi_k|^2 dx \leq c a_k^2 R_k^3, \quad \int_{\mathbb{R}^4} |\Delta \phi_k|^2 dx \leq c a_k^2 R_k^3, \quad \text{and } G(\phi_k) \geq \frac{\omega_3}{4} g(a_k) R_k^4.
\]

If (1) is violated by \( \lim_{t \to 0} |r|^{-2} g(t) < \infty \), then there exists a sequence \( c_k \to \infty \) such that \( g(a_k) \geq c_k a_k^2 R_k^4 \). Let \( R_k = a_k^{-1/4} + a_k^{-1/2} c_k^{-1/8} \), then

\[
a_k^2 R_k^4 \lesssim a_k^{-1} + c_k^{-\frac{1}{4}} \to 0, \quad G(u_k) \geq \frac{\omega_3}{4} c_k a_k^2 R_k^4 \gtrsim c_k \to \infty.
\]

If (3) is violated by \( \lim_{t \to 0} |r|^{-2} g(t) > 0 \), then there exists \( \delta > 0 \) such that \( g(a_k) \geq \delta a_k^2 \).

Pick \( R_k = a_k^{-1/2} \), then \( a_k^2 R_k^4 = 1, a_k^2 R_k^3 \to 0 \) and

\[
G(u_k) \geq \frac{\omega_3}{4} g(a_k) R_k^4 \geq \frac{\omega_3}{4} \delta > 0.
\]

What left is to consider the case when the conditions (1) and (3) do not hold at infinity. Let \( \{b_k\}_k \subset \mathbb{R}^+, b_k \to \infty \), be such that

\[
\lim_{t \to +\infty} |t|^2 \exp \left( -\frac{1}{K} |t|^2 \right) g(t) = \lim_{k \to \infty} c_k,
\]

where

\[
c_k := b_k^2 \exp \left( -\frac{1}{K} b_k^2 \right) g(b_k).
\]
Set $R_k = \exp(-\frac{1}{K}b_k^2)$, then $c_k = b_k^2R_kg(b_k)$. Now, we consider the so-called Moser’s sequence $\{\psi_k\} \subseteq H^2(\mathbb{R}^4)$ consisting of spherically symmetric functions defined by

$$
\psi_k(x) = \begin{cases} 
    b_k - \frac{2K|x|^2}{R_k^4b_k} + \frac{2K}{b_k}, & \text{if } 0 \leq |x| \leq R_k^{1/4}, \\
    \frac{4K|\log|x||}{b_k^2}, & \text{if } R_k^{1/4} < |x| \leq 1, \\
    \eta_k, & \text{if } 1 < |x| \leq 2, \\
    0, & \text{if } |x| \geq 2,
\end{cases}
$$

where $\eta_k = -\frac{4K}{b_k^2}(|x| - 1)^3 + \frac{8K}{b_k^2}(|x| - 1)^2 - \frac{4K}{b_k^2}(|x| - 1)$. Obviously, $\eta_k$ is a radial function on annulus $B_2 \setminus B_1$ satisfying the boundary condition

$$
\eta_k(x)|_{\partial B_1} = 0, \quad \frac{\partial \eta_k(x)}{\partial \nu}|_{\partial B_1} = -\frac{4K}{b_k},
$$

and

$$
\eta_k(x)|_{\partial B_2} = 0, \quad \frac{\partial \eta_k(x)}{\partial \nu}|_{\partial B_2} = 0.
$$

Furthermore, careful calculations lead to

$$
\int_{B_2 \setminus B_1} |\eta_k|^2 dx \lesssim \frac{K^2}{b_k^4}, \quad \int_{B_2 \setminus B_1} |\Delta \eta_k|^2 dx \lesssim \frac{K^2}{b_k^4}.
$$

Hence there exists a constant $c > 0$ such that

$$
\int_{\mathbb{R}^4} |\psi_k|^2 dx \leq \frac{cK^2}{b_k^2}, \quad \int_{\mathbb{R}^4} |\Delta \psi_k|^2 dx = 32\pi^2 K + O\left(\frac{1}{b_k^2}\right),
$$

and

$$
G(\psi_k) \geq \int_{K_k^{1/4}} g(\psi_k) \geq \frac{\omega_3}{4}g(b_k)R_k = \frac{\omega_3 c_k}{4b_k^2}.
$$

Define a new sequence $\{u_k\} \subseteq H^2(\mathbb{R}^4)$ by $u_k(x) = \psi_k(x/S_k)$, then

$$
\int_{\mathbb{R}^4} |u_k|^2 dx \leq \frac{cS_k^4K^2}{b_k^2}, \quad \int_{\mathbb{R}^4} |\Delta u_k|^2 dx = 32\pi^2 K + O\left(\frac{1}{b_k^2}\right)
$$

and

$$
G(u_k) = S_k^4G(\psi_k) \geq \frac{\omega_3 S_k^4 c_k}{4b_k^2}.
$$

Assume that the condition (1) does not hold at infinity, namely

$$
\lim_{|t| \to +\infty} |t|^2 \exp\left(-\frac{1}{K}|t|^2\right)g(t) = \lim_{k \to \infty} c_k = \infty.
$$

Set $S_k^4 = b_k^2c_k^{-1/2}$, there holds

$$
\int_{\mathbb{R}^4} |u_k|^2 dx \leq \frac{cS_k^4K^2}{b_k^2} \to 0, \quad \text{and } G(u_k) \geq \frac{\omega_3 S_k^4 c_k}{4b_k^2} \to \infty.
$$

Assume that the condition (3) fails at infinity, namely, there exists some $\delta > 0$ such that

$$
\lim_{|t| \to +\infty} |t|^2 \exp\left(-\frac{1}{K}|t|^2\right)g(t) = \lim_{k \to \infty} c_k = \delta > 0.
$$
Set $S^4_k = b_k^2$, we can easily verify that $u_k \to 0$ a.e. $\mathbb{R}^4$, and
\[
\int_{\mathbb{R}^4} |u_k|^2 dx \leq cK^2, \int_{\mathbb{R}^4} |\Delta u_k|^2 dx = 32\pi^2 K + O(\frac{1}{b_k^2}).
\]
Moreover, we also have $u_k \to 0$ a.e. $\mathbb{R}^4$ and
\[
G(u_k) \geq \frac{\omega_3}{4} \frac{S^4_k \delta}{b_k^2} = \frac{\omega_3 \delta}{4} > 0.
\]
This accomplishes the proof of the necessity of (1) and (3).

**Sufficiency of (1) and (2):**
We first prove that (1) can imply (2). Define a new Borel measurable function $\tilde{g}(t)$ by
\[
\tilde{g}(t) = g((32\pi^2 K)^{1/2} t).
\]
Obviously,
\[
\lim_{|t| \to +\infty} |t|^2 \exp(-32\pi^2 |t|^2) \tilde{g}(t) < \infty, \quad \text{and} \quad \lim_{t \to 0} |t|^{-2} \tilde{g}(t) < \infty.
\]
By the Adams' inequality (1.11) with the exact growth in $\mathbb{R}^4$, we derive that
\[
\int_{\mathbb{R}^4} \tilde{g}(u) dx \leq c \int_{\mathbb{R}^4} \Phi(32\pi^2 |u|^2) (1 + |u|)^{1/2} dx \leq c \int_{\mathbb{R}^4} |u|^2 dx
\]
Let $v = (32\pi^2 K)^{-1/2} u$. Then for any $u \in H^2(\mathbb{R}^4)$ satisfying $\|\Delta u\|_2^2 \leq 32\pi^2 K$, there holds
\[
\int_{\mathbb{R}^4} g(u) dx = \int_{\mathbb{R}^4} \tilde{g}(v) dx \leq c \int_{\mathbb{R}^4} |v|^2 dx \leq c \int_{\mathbb{R}^4} |u|^2 dx.
\]
Now, we turn to prove the sufficiency of (3). Let $g : \mathbb{R} \to [0, +\infty)$ be a continuous function satisfying
\[
\lim_{|t| \to +\infty} |t|^2 \exp\left(-\frac{|t|^2}{K}\right) g(t) = 0 \quad (3.1)
\]
and
\[
\lim_{t \to 0} |t|^{-2} g(t) = 0. \quad (3.2)
\]
For any radially symmetric sequence $\{u_k\}_k$ satisfying $\|\Delta u\|_2^2 \leq 32\pi^2 K$, and weakly converging to $u$, we will verify that
\[
\lim_{k \to \infty} G(u_k) - G(u) = \lim_{k \to \infty} \int_{\mathbb{R}^4} (g(u_k) - g(u)) dx = 0.
\]
Note that $\{u_k\}_k$ is a radial sequence in $H^2(\mathbb{R}^4)$, then
\[
|u_k(r)|^2 \leq \int_r^{+\infty} 2|u_k(s)||u_k'(s)| ds
\]
\[
\leq cr^{-3} \int_r^{+\infty} |u_k(s)s^{3/2}||u_k'(s)s^{3/2}| ds
\]
\[
\leq cr^{-3} \left( \int_{\mathbb{R}^4} |\nabla u_k|^2 dx \right)^{1/2} \left( \int_{\mathbb{R}^4} |u_k|^2 dx \right)^{1/2}.
\]
Hence $u_k(r) \to 0$ as $r \to \infty$ uniformly with respect to $k$. This together with (3.1) yields that for any $\varepsilon > 0$, there exists $R > 0$ such that
\[
\int_{\mathbb{R}^4 \setminus B_R} g(u_k) \, dx \leq \varepsilon \int_{\mathbb{R}^4} |u_k|^2 \, dx \leq c\varepsilon, \quad \int_{\mathbb{R}^4 \setminus B_R} g(u) \, dx \leq \varepsilon.
\]
(3.3)

On the other hand, through (3.2) we derive that for any $\varepsilon > 0$, there exists $L > 0$ independent of $k$ such that
\[
\int_{|u_k| > L} g(u_k) \, dx \leq c\varepsilon \int_{|u| > L} \frac{\exp(-\frac{1}{K} |u_k|^2)}{|u|^2} \, dx
\]
and
\[
\int_{|u| > L} g(u) \, dx \leq c\varepsilon \int_{|u| > L} \frac{\exp(-\frac{1}{K} |u|^2)}{|u|^2} \, dx.
\]

In view of (1.11), we derive that
\[
\int_{|u_k| > L} g(u_k) \, dx \leq c\varepsilon \int_{\mathbb{R}^4} |u_k|^2 \, dx \leq c\varepsilon, \quad \int_{|u| > L} g(u) \, dx \leq \varepsilon.
\]
(3.4)

Combining (3.3) and (3.4), one can get
\[
\lim_{k \to \infty} |G(u_k) - G(u)| \leq \left( \int_{\mathbb{R}^4 \setminus B_R} + \int_{B_R} \right) |g(u_k) - g(u)| \, dx
\]
\[
\leq c\varepsilon + \lim_{k \to \infty} \left( \int_{|u_k| > L} g(u_k) \, dx + \int_{|u| > L} g(u) \, dx \right)
\]
\[
+ \lim_{k \to \infty} \left( \int_{|u_k| \leq L, |x| \leq R} g(u_k) \, dx - \int_{|u| \leq L, |x| \leq R} g(u) \, dx \right)
\]
\[
\leq c\varepsilon + \lim_{k \to \infty} \left( \int_{|u_k| \leq L, |x| \leq R} g(u_k) \, dx - \int_{|u| \leq L, |x| \leq R} g(u) \, dx \right)
\]
\[
\leq c\varepsilon,
\]
where we have used the Lebesgue dominated convergence theorem in the last step. Then the proof is finished.

\[\square\]

4 Existence of non-trivial solutions for semilinear bi-harmonic equations

In this section, we consider the nontrivial solutions of semilinear bi-harmonic equation (2.1). We will employ the compactness result obtained in Theorem 2.1 and the principle of symmetric criticality to prove that Eq. (2.1) has a nontrivial radial solution under the assumption that the nonlinearity $f(t)$ satisfies mild conditions (i), (ii), (iii) and (1.2).

The natural functional associated to a variational approach to problem (2.1) is
\[
I_\gamma (u) = \frac{1}{2} \left( \|\Delta u\|_2^2 + \gamma \|u\|_2^2 \right) - \int_{\mathbb{R}^4} F(u) \, dx, \quad \forall u \in H^2(\mathbb{R}^4).
\]

Obviously, $I_\gamma \in C^1(H^2(\mathbb{R}^4), \mathbb{R})$ with
\[
I_\gamma'(u)v = \int_{\mathbb{R}^4} (\Delta u \Delta v + \gamma uv) \, dx - \int_{\mathbb{R}^4} f(u)v \, dx, \quad \forall u, v \in H^2(\mathbb{R}^4).
\]
Our goal is to prove the existence of non-trivial solutions of the equation (2.1). According to the principle of symmetric criticality, we only need to verify that \( u \) is a critical point restricted to the space \( H^2_0(\mathbb{R}^4) \). Motivated by the Pohozaev identity for Eq. (2.1), we introduce the functional

\[
G_\gamma(u) = \gamma \| u \|_2^2 - 2 \int_{\mathbb{R}^4} F(u) dx
\]

and the constrained minimization problem

\[
A_\gamma = \inf \left\{ \frac{1}{2} \| \Delta u \|_2^2 \mid u \in H^2_0(\mathbb{R}^4) \setminus \{0\}, \ G_\gamma(u) = 0 \right\}
\]

(4.1)

Set \( \mathcal{P}_r = \{ u \in H^2_0(\mathbb{R}^4) \setminus \{0\}, \ G_\gamma(u) = 0 \} \). We claim that \( \mathcal{P}_r \) is not empty. In fact, let \( u_0 \in H^2_0(\mathbb{R}^4) \) be positive and compactly supported in bounded domain \( \Omega \). Define

\[
h(s) := G_\gamma(su_0) = \gamma s^2 \| u_0 \|_2^2 - 2 \int_{\mathbb{R}^4} F(su_0) dx, \ \forall s > 0.
\]

\( \mathcal{P}_r \) being not empty is a direct result of the fact: \( h(s) > 0 \) for \( s > 0 \) small enough and \( h(s) < 0 \) for \( s > 0 \) sufficiently. Hence in order to obtain the desired result, it is suffices to show that \( h(s) > 0 \) for \( s > 0 \) small enough and \( h(s) < 0 \) for \( s > 0 \) sufficiently.

**Proof** We first prove that \( h(s) > 0 \) for \( s > 0 \) small enough. Since \( F(t) \) satisfies the condition (i), we can deduce from Remark 2.3 that \( F(t) \leq t^\mu \) when \( 0 < t < 1 \). On the other hand, from condition (1.2) and (ii), we know that there exists \( \rho_0 \) such that \( |F(t)| \leq M |f(t)| \leq M t^\mu e^{\rho_0 t^2} \) for \( t \) sufficiently large. Combining the estimate of \( F(t) \) for small \( t \) and big \( t \), we see that there exist two positive constant \( C_1 \) and \( C_2 \) such that \( F(t) \leq C_1 t^\mu + C_2 t^\mu e^{\rho_0 t^2} \). Using this estimate, we can write

\[
h(s) = \gamma s^2 \| u_0 \|_2^2 - 2 \int_{\mathbb{R}^4} F(su_0) dx \geq \gamma s^2 \| u_0 \|_2^2 - 2 s^\mu \int_{\mathbb{R}^4} (C_1 s^\mu + C_2 s^\mu e^{\rho_0 (su_0)^2}),
\]

(4.2)

which implies that \( h(s) > 0 \) for small \( s > 0 \) since \( \mu > 2 \).

Next, we prove that \( h(s) < 0 \) for \( s > 0 \) sufficiently large. We can rewrite the condition (ii) as \( \mu F(t) \leq t F'(t) \) to derive that \( F(t) \geq t^\mu F(1) \) when \( t > 1 \). Then it follows there exists \( C_3 > 0 \) such that \( F(t) \geq t^\mu F(1) - C_3 \). Noticing that \( u_0 \) is compactly supported in bounded domain \( \Omega \), we can write

\[
h(s) = \gamma s^2 \| u_0 \|_2^2 - 2 \int_{\Omega} F(su_0) dx
\]

\[
\leq \gamma s^2 \| u_0 \|_2^2 - 2 F(1)s^\mu \int_{\Omega} u_0^d dx + 2 C_3 |\Omega|,
\]

(4.3)

which implies that \( h(s) \) is negative for sufficiently large \( s > 0 \).

**Lemma 4.1** There exists a minimizing sequence \( \{u_k\}_k \in \mathcal{P}_r \) satisfying \( \| u_k \|_2 = 1 \) for \( A_\gamma \).

**Proof** Assume that \( \{u_k\}_k \) is a minimizing sequence for \( A_\gamma \), that is, \( u_k \in \mathcal{P}_r \) satisfying

\[
\lim_{k \to \infty} \frac{1}{2} \| \Delta u_k \|_2^2 = A_\gamma.
\]
Let \( \tilde{v}_k = u_k(\|u_k\|_2^{1/2} \cdot x) \), simple computations lead to \( \|\tilde{v}_k\|_2 = 1 \), \( \tilde{v}_k \in \mathcal{P}_r \) and \( \|\Delta \tilde{v}_k\|_2 = \|\Delta \tilde{v}_k\|_2 \). This accomplishes the proof of Lemma 4.1.

If the infimum \( A_\gamma \) is attained, then the minimizer \( u \in H_0^2(\mathbb{R}^4) \) under a suitable change of scale is a ground state solution of (2.1) constrained to the space \( H_0^2(\mathbb{R}^4) \). In fact, if \( u \) is a minimizer for \( A_\gamma \), then there exists a Lagrange multiplier \( \theta \in \mathbb{R} \) such that

\[
\Delta^2 u + \gamma u - f(u) = \theta(2\gamma u - 2f(u)) \quad \text{in} \quad \mathbb{R}^4
\]

namely,

\[
\Delta^2 u = (2\theta - 1)(\gamma u - f(u)) \quad \text{in} \quad \mathbb{R}^4.
\]

Recalling that \( u \in \mathcal{P}_r \), we have

\[
\int_{\mathbb{R}^4} (\gamma u - f(u))u \, dx = \gamma \|u\|_2^2 - 2 \int_{\mathbb{R}^4} F(u) \, dx + \int_{\mathbb{R}^4} (2F(u) - uf(u)) \, dx = - \int_{\mathbb{R}^4} (uf(u) - 2F(u)) \, dx < 0,
\]

as a consequence of (i). Moreover,

\[
\int_{\mathbb{R}^4} \Delta^2 u \cdot u \, dx = \int_{\mathbb{R}^4} |\Delta u|^2 \, dx > 0,
\]

hence \( 2\theta - 1 < 0 \). Therefore

\[
\tilde{u}(x) = u \left( \frac{x}{(1 - 2\theta)^{1/2}} \right) \quad \text{for a.e.} \ x \in \mathbb{R}^4
\]

is a non-trivial solution of (2.1) constrained to the space \( H_0^2(\mathbb{R}^4) \). According to the principle of symmetric criticality, then \( u \) is a non-trivial solution of (2.1).

Now, we establish a relation between the attainability of \( A_\gamma \) and the Adams’ inequality with the exact growth (1.11). For this purpose, we introduce the Adams ratio

\[
C_A^L = \sup \left\{ \frac{2}{\|u\|_2^2} \int_{\mathbb{R}^4} F(u) \, dx \mid u \in H_0^2(\mathbb{R}^4) \setminus \{0\}, \|\Delta u\|_2^2 \leq L \right\}.
\]

The Adams threshold \( R(F) \) is given by

\[
R(F) = \sup \{L > 0 \mid C_A^L < +\infty\}.
\]

We denote by \( C_A^* = C_A^{R(F)} \) the ratio at the threshold \( R(F) \). By the growth condition (1.2) and (ii) of \( f(s) \), we obtain

\[
\lim_{|t| \to \infty} \exp(\alpha t^2) = \begin{cases} 0, & \text{if } \alpha > \alpha_0, \\ +\infty, & \text{if } \alpha < \alpha_0, \end{cases}
\]

and

\[
\lim_{t \to 0} \frac{F(t)}{t^2} = 0.
\]

Hence, thanks to Theorem 2.1, we derive \( R(F) = 32\pi^2/\alpha_0 \).

**Lemma 4.2** If \( A_\gamma < R(F)/2 \), then \( A_\gamma \) can be attained and \( A_\gamma = I_\gamma(u) \), where \( u \in H_0^2(\mathbb{R}^4) \) under a suitable change of scale is a nontrivial solution of equation (2.1) through the principle of symmetric criticality.
Proof Let $\{u_k\}_k$ be a radial minimizing sequence for $A_\gamma$, that is $u_k \in P$ satisfying
\[
\lim_{k \to \infty} \frac{1}{2} \|\Delta u_k\|_2^2 = A_\gamma \quad \text{and} \quad \|u_k\|_2^2 = 1.
\]
We also assume that $u_k \rightharpoonup u$ in $H^2(\mathbb{R}^4)$. We first prove that $A_\gamma > 0$. We argue this by contradiction. We assume that $A_\gamma = 0$, namely $\lim_{k \to \infty} \|\Delta u_k\|_2^2 = 0$, which implies that $u = 0$. Regarding
\[
\lim_{|t| \to +\infty} |t|^2 \exp(-\alpha |t|^2) F(t) = 0 \quad \text{for any } \alpha > \alpha_0, \quad \lim_{t \to 0} |t|^{-2} F(t) = 0,
\]
we derive that
\[
\lim_{k \to \infty} \int_{\mathbb{R}^4} F(u_k) dx = \int_{\mathbb{R}^4} F(u) dx
\]
through Theorem 2.2. On the other hand, since $u_k \in P$ and $\|u_k\|_2^2 = 1$, then
\[
0 < \gamma \lim_{k \to \infty} \|u_k\|_2^2 = 2 \lim_{k \to \infty} \int_{\mathbb{R}^4} F(u_k) dx = 2 \int_{\mathbb{R}^4} F(u) dx,
\]
which contradicts $u = 0$. This proves that $A_\gamma > 0$.

Now are in position to prove that if $A_\gamma < R(F)/2$, then $A_\gamma$ is attained. Under the assumption of Lemma 4.2, we have
\[
\lim_{k \to \infty} \|\Delta u_k\|_2^2 = 2 A_\gamma < R(F) = 32\pi^2/\alpha_0.
\]
Picking up $\frac{1}{K} > \alpha_0$ satisfying $\lim_{k \to \infty} \|\Delta u_k\|_2^2 \leq 32\pi^2 K$, then we derive that
\[
\lim_{|t| \to +\infty} |t|^2 \exp\left(-\frac{1}{K} |t|^2\right) F(t) = 0, \quad \lim_{t \to 0} |t|^{-2} F(t) = 0.
\]
It follows from Theorem 2.2 that
\[
\lim_{k \to \infty} \int_{\mathbb{R}^4} F(u_k) dx = \int_{\mathbb{R}^4} F(u) dx. \tag{4.5}
\]
Consequently,
\[
\gamma = \lim_{k \to \infty} \gamma \|u_k\|_2^2 = 2 \lim_{k \to \infty} \int_{\mathbb{R}^4} F(u_k) dx = 2 \int_{\mathbb{R}^4} F(u) dx
\]
and
\[
\frac{1}{2} \|\Delta u\|_2^2 \leq \lim_{k \to \infty} \frac{1}{2} \|\Delta u_k\|_2^2 = A_\gamma.
\]
In order to show $u$ is minimizer for $A_\gamma$, what left is to show that $G_{\gamma}(u) = 0$. Set
\[
h(t) = G_{\gamma}(tu) = \gamma \|tu\|_2^2 - \int_{\mathbb{R}^4} F(tu) dx.
\]
Obviously, in view of (4.5), we have
\[
G_{\gamma}(u) = \gamma \|u\|_2^2 - 2 \int_{\mathbb{R}^4} F(u) dx
\]
\[
\leq \lim_{k \to \infty} \left( \gamma \|u_k\|_2^2 - \int_{\mathbb{R}^4} F(u_k) dx \right) = \lim_{k \to \infty} G_{\gamma}(u_k) = 0.
\]
This implies $h(1) \leq 0$. From $\lim_{t \to 0} F(t) / t^2 = 0$, one can deduce that $h(t) > 0$ for $t > 0$ small enough. Consequently, there exists $s_0 \in (0, 1]$ such that $G_\gamma(s_0 u) = 0$. Then it follows that

$$A_\gamma \leq \frac{1}{2} \|\Delta s_0 u\|_2^2 = \frac{1}{2} s_0^2 \|\Delta u\|_2^2 \leq s_0^2 A_\gamma,$$

which proves that $s_0 = 1$ and $\frac{1}{2} \|\Delta u\|_2^2 = A_\gamma$. Then we accomplish the proof of Lemma 4.2.

Next, we show

**Lemma 4.3** The constrained minimization problem $A_\gamma$ associated to the functional $I_\gamma$ satisfies

$$A_\gamma < \frac{1}{2} R(F)$$

if and only if

$$\gamma < C_A^*.$$

**Proof** We first prove that if $A_\gamma < R(F)/2$, then $\gamma < C_A^*$. Obviously, if the $C_A^* = +\infty$, then $\gamma < C_A^*$ and the proof is complete. Therefore, without loss of generality, we may assume that $C_A^* < +\infty$. According to Lemma 4.2, we see that $A_\gamma$ could be achieved by a radial function $u \in \mathcal{P}_r$. Then according to the definition of the $A_\gamma$, we have $\|\Delta u\|_2^2 < 32 \pi^2 / \alpha_0$ and $\gamma \|u\|_2^2 = 2 \int_{\mathbb{R}^4} F(u) dx$. Define

$$g(s) = \frac{2}{s^2 \|u\|_2^2} \int_{\mathbb{R}^4} F(su) dx,$$

then $g(1) = \gamma$. Since $F$ satisfies the condition (i), then it is easy to see that $g(s)$ is monotone increasing. If we set $v = \frac{R(F)^{1/2}}{\|\Delta u\|_2} u$, then $\|\Delta v\|_2 = R(F)$ and

$$C_A^* \geq \frac{2}{\|v\|_2^2} \int_{\mathbb{R}^4} F(v) dx = g\left(\frac{R(F)^{1/2}}{\|\Delta u\|_2}\right) > g(1) = \gamma.$$

Next, it remains to verify that if $\gamma < C_A^*$, then $A_\gamma < R(F)/2$. We distinguish between the case $C_A^* < +\infty$ and $C_A^* = +\infty$.

In the case $C_A^* < +\infty$, since $\gamma < C_A^*$, then $\gamma < C_A^* - \varepsilon_0$ for some $\varepsilon_0 > 0$. It follows from the definition of $C_A^*$ that there exists some $u_0 \in H^2_0(\mathbb{R}^4)$ with $\|\Delta u_0\|_2^2 \leq R(F)$ satisfying

$$C_A^* - \varepsilon_0 < \frac{2}{\|u_0\|_2^2} \int_{\mathbb{R}^4} F(u_0) dx.$$

Consequently,

$$\gamma \|u_0\|_2^2 < 2 \int_{\mathbb{R}^4} F(u_0) dx,$$

namely $G_\gamma(u_0) < 0$. Let $h(s) = G_\gamma(s u_0)$ for $s > 0$. Since $h(1) < 0$ and $h(s) > 0$ for $s > 0$ small enough, then there exists $s_0 \in (0, 1)$ satisfying $h(s_0 u_0) = 0$. Therefore, we have $s_0 u_0 \in \mathcal{P}_r$ and

$$A_\gamma \leq \frac{1}{2} \|\Delta (s_0 u_0)\|_2^2 = \frac{1}{2} s_0^2 \|\Delta u_0\|_2^2 < \frac{1}{2} R(F).$$
In the case $C_A^* = +\infty$, for any $\gamma > 0$, there exists $u_0 \in H^2_r(\mathbb{R}^4)$ with $\|\Delta u_0\|_2^2 \leq R(F)$ satisfying
\[
\gamma \|u_0\|_2^2 < 2 \int_{\mathbb{R}^4} F(u_0)dx.
\]
Hence we can repeat the same arguments as case $C_A^* < +\infty$ to get the conclusion. \qed

\section{Existence of ground state solutions for bi-harmonic equations with the constant potential}

In this section, we will employ the Pohozaev manifold and Fourier rearrangement arguments to study the ground-states of the following semilinear bi-harmonic equation.

\[
(-\Delta)^2 u + \gamma u = \lambda u \exp(2|u|^2) \quad \text{in} \quad \mathbb{R}^4,
\]

where $\lambda$ is strictly smaller than the first eigenvalue of operator $(-\Delta)^2 + \gamma I$ in $\mathbb{R}^4$, namely
\[
\lambda < \inf_{u \in H^2(\mathbb{R}^4) \setminus \{0\}} \frac{\|\Delta u\|_2^2 + \gamma \|u\|_2^2}{\|u\|_2^2} = \gamma.
\]

The natural functional associated to a variational approach to problem (5.1) is
\[
I_\lambda(u) = \frac{1}{2} (\|\Delta u\|_2^2 + \gamma \|u\|_2^2) - \frac{\lambda}{4} \int_{\mathbb{R}^4} (\exp(2u^2) - 1) \, dx, \ \forall u \in H^2(\mathbb{R}^4).
\]

It is easy to obtain that $I_\lambda \in C^1(H^2(\mathbb{R}^4), \mathbb{R})$ with
\[
I_\lambda'(u)v = \int_{\mathbb{R}^4} (\Delta u \Delta v + \gamma uv) \, dx - \int_{\mathbb{R}^4} \lambda u \exp(2u^2) v \, dx, \ \forall u, v \in H^2(\mathbb{R}^4).
\]

We will prove that Eq. (5.1) has a radial ground-state solution for any $0 < \lambda < \gamma$. We recall that a solution $u$ of (5.1) is called a ground state if
\[
I_\lambda(u) \leq \inf \{I_\lambda(u) \mid u \neq 0, \ u \text{ is a weak solution of (5.1)}\}.
\]

Similar to the proof of Theorem 2.4, we introduce the Pohozaev functional
\[
G_\lambda(u) = \gamma \|u\|_2^2 - \frac{1}{2} \int_{\mathbb{R}^4} \lambda (\exp(2|u|^2) - 1) \, dx = (\gamma - \lambda) \|u\|_2^2 - \int_{\mathbb{R}^4} g_\lambda(u) \, dx,
\]

where $g_\lambda(t) = \frac{\lambda}{2} \left(\exp(2t^2) - 1 - 2t^2\right)$, and the constrained minimization problem
\[
A_\lambda = \inf \left\{ \frac{1}{2} \|\Delta u\|_2^2 \mid u \in H^2(\mathbb{R}^4) \setminus \{0\}, \ G_\lambda(u) = 0 \right\} = \inf \left\{ I_\lambda(u) \mid u \in H^2(\mathbb{R}^4) \setminus \{0\}, \ G_\lambda(u) = 0 \right\}.
\]

Set $\mathcal{P} = \{ u \in H^2(\mathbb{R}^4) \setminus \{0\}, \ G_\lambda(u) = 0 \}$, $\mathcal{P}$ is not empty by the same arguments of Sect. 4. Next, we will adapt the Fourier rearrangement method to show that there exists a radially minimizing sequence for $A_\lambda$. Such a Fourier rearrangement argument has also been used recently by Chen et al. \cite{15} to establish the existence of extremals for the subcritical Adams inequalities on the entire space.

\begin{lemma}
There exists a radially minimizing sequence $\{u_k\}_k$ satisfying $\|u_k\|_2^2 = 1$ for $A_\lambda$.
\end{lemma}
Proof Assume that \( \{u_k\}_k \) is a minimizing sequence for \( A_\lambda \), that is \( u_k \in \mathcal{P} \) satisfying

\[
\lim_{k \to \infty} \frac{1}{2} \| \Delta u_k \|_2^2 = A_\lambda.
\]

Denote by \( w_k = \mathcal{F}^{-1}\{\{\mathcal{F}(u_k)\}\}^* \) the Fourier rearrangement of \( u_k \), where \( \mathcal{F} \) is the Fourier transform on \( \mathbb{R}^4 \) (with its inverse \( \mathcal{F}^{-1} \)) and \( f^* \) stands for the Schwarz symmetrization of \( f \). Using the property of the Fourier rearrangement from [33], one can derive that

\[
\| \Delta w_k \|_2 \leq \| \Delta u_k \|_2, \quad \| w_k \|_2^2 = \| u_k \|_2^2.
\]

Then it follows that

\[
\int_{\mathbb{R}^4} (\exp(2w_k^2) - 1) \, dx \geq \int_{\mathbb{R}^4} (\exp(2u_k^2) - 1) \, dx.
\]

Then it follows that

\[
(\gamma - \lambda) \| w_k \|_2^2 = (\gamma - \lambda) \| u_k \|_2^2 = \int_{\mathbb{R}^4} g_\lambda(u_k) \, dx \leq \int_{\mathbb{R}^4} g_\lambda(w_k).
\]

Hence if we set

\[
\eta(t) = (\gamma - \lambda) \| tw_k \|_2^2 - \int_{\mathbb{R}^4} g_\lambda(tw_k),
\]

then \( \eta(1) \leq 0 \). On the other hand, one can easily see \( \eta(t) > 0 \) for \( t > 0 \) sufficiently small. Therefore, there exists \( t_k \in (0, 1) \) such that \( \eta(t_k) = 0 \), that is \( t_kw_k \in \mathcal{P} \). We obtain

\[
A_\lambda \leq I_\lambda(t_kw_k) = \frac{1}{2} \| \Delta(t_kw_k) \|_2^2 \leq \frac{1}{2} t_k^2 \| \Delta u_k \|_2^2 \leq I_\lambda(u_k).
\]

This implies that \( \{v_k\}_k := \{t_kw_k\}_k \) is a radial minimizing sequence for \( A_\lambda \). Let \( \tilde{v}_k = w_k (\|v_k\|_2^2 x) \), it is easy to check that \( \tilde{v}_k \) is a minimizing sequence for \( A_\lambda \) with \( \| \tilde{v}_k \|_2 = 1 \). This accomplishes the proof of Lemma 5.1. \( \square \)

Repeating the argument for (4.4), we can show that if the infimum \( A_\lambda \) is attained, then the minimizer \( u \in H^2_\mathcal{P}(\mathbb{R}^4) \) under a suitable change of scale is a ground state solution of (5.1).

Lemma 5.2 If \( A_\lambda < 8\pi^2 \), then \( A_\lambda \) is attained and \( A_\lambda = I_\lambda(u) \), where \( u \in H^2_\mathcal{P}(\mathbb{R}^4) \) under a suitable change of scale is a ground-state solution of equation (5.1).

Proof Recall the definition of \( A_\lambda \):

\[
A_\lambda = \inf \left\{ \frac{1}{2} \| \Delta u \|_2^2 \mid u \in H^2(\mathbb{R}^4) \setminus \{0\}, \ G_\lambda(u) = 0 \right\},
\]

where \( G_\lambda(u) = \gamma \| u \|_2^2 - \frac{1}{2} \int_{\mathbb{R}^4} \lambda (\exp(2|u|^2) - 1) \, dx = (\gamma - \lambda) \| u \|_2^2 - \int_{\mathbb{R}^4} g_\lambda(u) \, dx \) and \( \lambda < \gamma \). The proof follows from the same argument of Lemma 4.2 with the \( \gamma \| u \|_2^2 - \int_{\mathbb{R}^4} F(u) \, dx \) replaced by \( (\gamma - \lambda) \| u \|_2^2 - \int_{\mathbb{R}^4} g_\lambda(u) \, dx \) and \( R(F)/2 \) replaced by \( R(g)/2 = 8\pi^2 \). \( \square \)

Next, we prove

Lemma 5.3 The constrained minimization problem \( A_\lambda \) is actually strictly smaller than \( 8\pi^2 \).

Proof Note that the Adams ratio for \( g_\lambda(u) \) is \( +\infty \), hence there exists \( u_0 \in H^2(\mathbb{R}^4) \) such that

\[
(\gamma - \lambda) \leq \frac{1}{\| u_0 \|_2^2} \int_{\mathbb{R}^4} g_\lambda(u_0) \, dx, \quad \| \Delta u_0 \|_2^2 \leq 16\pi^2.
\]
thus, we have \( G_{\lambda}(u_0) = (\gamma - \lambda)\|u_0\|^2_2 - \int_{\mathbb{R}^4} g_{\lambda}(u_0) dx < 0 \). Then there exists \( s_0 \in (0, 1) \) such that \( s_0 u_0 \in \mathcal{P} \), which yields that
\[
A_{\lambda} \leq \frac{1}{2} \|\Delta(s_0 u_0)\|^2_2 = \frac{1}{2} s_0^2 \|\Delta u_0\|^2_2 \leq 8\pi^2 s_0^2 < 8\pi^2.
\]
Then the lemma is proved.

\[\square\]

6 Existence of ground state solutions for bi-harmonic equations with the Rabinowitz type potential

In this section, we are concerned with the ground states of the following quasilinear bi-harmonic equation with the Rabinowitz type potential

\[
(-\Delta)^2 u + V(x) u = \lambda \exp(2u^2) u, \tag{6.1}
\]

where \( \lambda \) and \( V(x) \) satisfy
\[
0 < \lambda < V_0 = \inf_{x \in \mathbb{R}^4} V(x) < \sup_{x \in \mathbb{R}^4} V(x) = \lim_{|x| \to \infty} V(x) = \gamma.
\]

The associated functional and Nehari Manifold are
\[
I_V(u) = \frac{1}{2} \int_{\mathbb{R}^4} (|\Delta u|^2 + V(x) |u|^2) \, dx - \frac{\lambda}{4} \int_{\mathbb{R}^4} (\exp(2u^2) - 1) \, dx
\]
and
\[
N_V = \{ u \in H^2(\mathbb{R}^4) \mid u \neq 0, N_V(u) = 0 \},
\]
respectively, where
\[
N_V(u) = \int_{\mathbb{R}^4} (|\Delta u|^2 + V(x) |u|^2) \, dx - \lambda \int_{\mathbb{R}^4} \exp(2u^2) u^2 dx.
\]

In order to study the Eq. (6.1), we introduce the following limiting equation

\[
(-\Delta)^2 u + \gamma u = \lambda \exp(2u^2) u. \tag{6.2}
\]

The corresponding functional and Nehari Manifold associated with (6.2) is

\[
I_\infty(u) = \frac{1}{2} \int_{\mathbb{R}^4} (|\Delta u|^2 + \gamma |u|^2) \, dx - \frac{\lambda}{4} \int_{\mathbb{R}^4} (\exp(2u^2) - 1) \, dx
\]
and
\[
N_\infty = \{ u \in H^2(\mathbb{R}^4) \mid u \neq 0, N_\infty(u) = 0 \},
\]
where
\[
N_\infty(u) = \int_{\mathbb{R}^4} (|\Delta u|^2 + \gamma |u|^2) \, dx - \lambda \int_{\mathbb{R}^4} \exp(2u^2) u^2 dx.
\]
One can easily verify that if $u \in \mathcal{N}_V$, then
\[ I_V(u) = \frac{\lambda}{4} \int_{\mathbb{R}^4} \left( \exp \left( 2u^2 - (\exp(2u^2) - 1) \right) \right) dx, \]
and if $u \in \mathcal{N}_\infty$, then
\[ I_\infty(u) = \frac{\lambda}{4} \int_{\mathbb{R}^4} \left( \exp \left( 2u^2 - (\exp(2u^2) - 1) \right) \right) dx. \]

**Lemma 6.1** For any $u \in H^2(\mathbb{R}^4) \setminus \{0\}$, there exist uniquely positive $t_u$ and $\tilde{t}_u$ such that $t_u u \in \mathcal{N}_V$ and $\tilde{t}_u u \in \mathcal{N}_\infty$.

**Proof** For any $u \in H^2(\mathbb{R}^4)$, we have
\[ N_V(tu) = t^2 \int_{\mathbb{R}^4} (|\Delta u|^2 + (V(x) - \lambda)|u|^2) dx - \lambda \int_{\mathbb{R}^4} (\exp(2t^2u^2) - 1) t^2 u^2 dx \]
\[ = t^2 \left( \int_{\mathbb{R}^4} (|\Delta u|^2 + (V(x) - \lambda)|u|^2) dx - \lambda \int_{\mathbb{R}^4} (\exp(2t^2u^2) - 1) t^2 u^2 dx \right) \]
Obviously, from the expression of $N_V(tu)$, it is not hard to find that
\[ N_V(tu) \leq t^2 \int_{\mathbb{R}^4} (|\Delta u|^2 + (V(x) - \lambda)|u|^2) dx - 2\lambda t^4 \int_{\mathbb{R}^4} u^4 dx, \]
which implies that $N_V(tu) < 0$ for sufficiently large $t$. On the other hand, obviously, we also have
\[ N_V(tu) \geq t^2 \int_{\mathbb{R}^4} (|\Delta u|^2 + (V(x) - \lambda)|u|^2) dx - 2\lambda t^4 \int_{\mathbb{R}^4} u^4 \exp(2t^2u^2) dx, \]
which implies that $N_V(tu) > 0$ for small $t$. Hence, there exists a $t_u > 0$ such that $t_u u \in \mathcal{N}_V$. As for the uniqueness of $t_u$, we notice
\[ N(tu) = t^2 \left( \int_{\mathbb{R}^4} (|\Delta u|^2 + (V(x) - \lambda)|u|^2) dx - \lambda \int_{\mathbb{R}^4} (\exp(2t^2u^2) - 1) u^2 dx \right). \]
Hence $N(tu) = 0$ is equivalent to the
\[ \theta(t) = \left( \int_{\mathbb{R}^4} (|\Delta u|^2 + (V(x) - \lambda)|u|^2) dx - \lambda \int_{\mathbb{R}^4} (\exp(2t^2u^2) - 1) u^2 dx \right) = 0. \]
By the monotonicity of $\theta(t)$, we easily see that $\theta(t)$ has a unique zero point in $(0, +\infty)$ which implies that there exists a unique $t_u$ such that $N(t_u u) = 0$. The proof for $\mathcal{N}_\infty$ is similar. \(\Box\)

Set
\[ m_\infty = \inf \{ I_\infty(u), u \in \mathcal{N}_\infty \} \quad \text{and} \quad m_V = \inf \{ I_V(u), u \in \mathcal{N}_V \}. \]

From Theorem 2.7, we know that $m_\infty$ coincides with the level $A_{\mathcal{V}}$ studied in the previous Sect. 5, and is attained by some $w \in \mathcal{N}_\infty$.

**Lemma 6.2** There holds
\[ 0 < m_V < m_\infty. \] (6.3)
From Lemma 6.1, we know that for any \( u \in \mathcal{N}_V \subseteq H^2 \left( \mathbb{R}^4 \right) \setminus \{0\} \), there exists a unique positive \( \tilde{t}_u \) such that \( \tilde{t}_u u \in \mathcal{N}_\infty \). Now, we claim that \( \tilde{t}_u > 1 \) and must be bounded with respect to \( u \in \mathcal{N}_V \). Indeed, for any \( u \in \mathcal{N}_V \), by the assumption of \( V \) (x), we have

\[
\lambda \int_{\mathbb{R}^4} \exp \left( 2u^2 \right) u^2 \, dx = \int_{\mathbb{R}^4} (|\Delta u|^2 + V \, |u|^2) \, dx < \int_{\mathbb{R}^4} (|\Delta u|^2 + \gamma \, |u|^2) \, dx.
\]

On the other hand, \( \tilde{t}_u u \in \mathcal{N}_\infty \) implies that

\[
\int_{\mathbb{R}^4} (|\Delta u|^2 + \gamma \, |u|^2) \, dx = \lambda \int_{\mathbb{R}^4} \exp \left( 2\tilde{t}_u^2 u^2 \right) u^2 \, dx.
\]

Combining this two estimate, we derive that

\[
\lambda \int_{\mathbb{R}^4} \exp \left( 2u^2 \right) u^2 \, dx < \lambda \int_{\mathbb{R}^4} \exp \left( 2\tilde{t}_u^2 u^2 \right) u^2 \, dx,
\]

which implies \( \tilde{t}_u > 1 \). Now, we prove the boundedness of \( \tilde{t}_u \). By the assumption of \( V \) (x), there exists some constant \( c > 0 \) such that

\[
c\lambda \int_{\mathbb{R}^4} \exp \left( 2u^2 \right) u^2 \, dx = c \int_{\mathbb{R}^4} (|\Delta u|^2 + V \, |u|^2) \, dx \\
\geq \int_{\mathbb{R}^4} (|\Delta u|^2 + (\gamma - \lambda) \, |u|^2) \, dx \\
= \lambda \int_{\mathbb{R}^4} (\exp \left( 2\tilde{t}_u^2 u^2 \right) - 1) \, u^2 \, dx.
\]

(6.4)

Since \( \lambda \int_{\mathbb{R}^4} (\exp (2u^2) - 1) \, u^2 \, dx = \int_{\mathbb{R}^4} (|\Delta u|^2 + (V \, x - \lambda) \, |u|^2) \, dx \), we derive that

\[
\lambda \int_{\mathbb{R}^4} (\exp (2\tilde{t}_u^2 u^2) - 1) \, u^2 \, dx \geq \lambda \tilde{t}_u^2 \int_{\mathbb{R}^4} (\exp (2u^2) - 1) \, u^2 \, dx \\
= \tilde{t}_u^2 \int_{\mathbb{R}^4} (|\Delta u|^2 + (V \, x - \lambda) \, |u|^2) \, dx.
\]

(6.5)

Combining (6.4) and (6.5), we conclude that

\[
c \int_{\mathbb{R}^4} (|\Delta u|^2 + V \, |u|^2) \, dx = c\lambda \int_{\mathbb{R}^4} \exp \left( 2u^2 \right) u^2 \, dx \\
\geq \lambda \int_{\mathbb{R}^4} (\exp \left( 2\tilde{t}_u^2 u^2 \right) - 1) \, u^2 \, dx \\
\geq \tilde{t}_u^2 \int_{\mathbb{R}^4} (|\Delta u|^2 + (V \, x - \lambda) \, |u|^2) \, dx.
\]

Therefore, \( \tilde{t}_u \) must be bounded.

To show that \( m_V < m_\infty \), it is enough to find \( u \) satisfying \( u \in \mathcal{N}_V \) such that \( I_V (u) < m_\infty \). From Theorem 2.7, we know that \( m_\infty \) is attained by some \( w \in \mathcal{N}_\infty \). By the definition of \( V \) (x), it is easy to check that

\[
\int_{\mathbb{R}^4} |\Delta w|^2 + V(x) |w|^2 \, dx < \int_{\mathbb{R}^4} |\Delta w|^2 + \gamma |w|^2 \, dx = \lambda \int_{\mathbb{R}^4} \exp \left( 2w^2 \right) w^2 \, dx.
\]

Hence there exists \( t \in (0, 1) \) such that

\[
\int_{\mathbb{R}^4} |\Delta (tw)|^2 + V(x) |tw|^2 \, dx = \int_{\mathbb{R}^4} \exp \left( 2(tw)^2 \right) (tw)^2 \, dx.
\]
which implies that \( u = tw \in \mathcal{N}_V \). Then it follows that
\[
m_V \leq I_V(tw) = \frac{\lambda}{4} \int_{\mathbb{R}^4} \left( \exp \left(2(tw)^2\right) - \exp \left(2(tw)^2\right) - 1 \right) dx
\]
\[
< \frac{\lambda}{4} \int_{\mathbb{R}^4} \left( \exp \left(2w^2\right) - \exp \left(2w^2\right) - 1 \right) dx = m_\infty.
\]

Next, we show \( m_V > 0 \). We prove this by contradiction. Assume that there exists some sequence \( u_k \in \mathcal{N}_V \) such that \( I_V(u_k) \to 0 \), that is,
\[
\frac{\lambda}{4} \int_{\mathbb{R}^4} \left( \exp \left(2u_k^2\right) - \exp \left(2u_k^2\right) - 1 \right) dx \to 0.
\]

Observe that for any \( t > 0 \), there exists some \( c \in (0, \frac{1}{2}] \) such that
\[
c \left( \exp \left(2t^2\right) - \frac{3}{4} \right) \leq \exp \left(2t^2\right) - \exp \left(2t^2\right) + 1
\]
\[
= \left( \frac{2t^2}{2} + \left( \frac{1}{2} - \frac{1}{3} \right) \right) (2t^2)^3 + \left( \frac{1}{3} - \frac{1}{4} \right) (2t^2)^4 + \cdots
\]
\[
+ \left( \frac{1}{(n-1)!} - \frac{1}{n!} \right) (2t^2)^n + \cdots \leq \exp \left(2t^2\right) - 2t^2.
\]

This together with \( N_V(u_k) = 0 \) implies that
\[
\int_{\mathbb{R}^4} \exp \left(2u_k^2\right) dx \to 0 \text{ and } \int_{\mathbb{R}^4} |\Delta u_k|^2 dx \to 0.
\]

Indeed, by \( I_V(u_k) \to 0 \) and (6.6), we can get
\[
\int_{\mathbb{R}^4} \left( \exp \left(2u_k^2\right) - \exp \left(2u_k^2\right) \right) dx \to 0.
\]

Since
\[
N_V(u_k) = \int_{\mathbb{R}^4} (|\Delta u|^2 + V(x) |u_k|^2) dx - \lambda \int_{\mathbb{R}^4} \exp \left(2u_k^2\right) u_k^2 dx = 0.
\]

then we have
\[
\int_{\mathbb{R}^4} (|\Delta u|^2 + (V(x) - \lambda) |u_k|^2) dx = \frac{\lambda}{2} \int_{\mathbb{R}^4} \left( \exp \left(2u_k^2\right) - \exp \left(2u_k^2\right) \right) dx.
\]

By (6.7) and the assumption of \( V(x) \), we can obtain
\[
\int_{\mathbb{R}^4} (|\Delta u|^2 + |u_k|^2) dx \to 0
\]
and
\[
\int_{\mathbb{R}^4} \exp \left(2u_k^2\right) u_k^2 dx \to 0.
\]

Pick \( \tilde{t}_{u_k} > 0 \) such that \( \tilde{t}_{u_k} u_k \in \mathcal{N}_\infty \). Since \( \tilde{t}_{u_k} \) is bounded, then \( \int_{\mathbb{R}^4} |\Delta \tilde{t}_{u_k} u_k|^2 dx \to 0 \). Hence, it follows that
\[
I_\infty \left( \tilde{t}_{u_k} u_k \right) = \frac{\lambda}{4} \int_{\mathbb{R}^4} \left( \exp \left(2\tilde{t}_{u_k}^2 u_k^2\right) - \exp \left(2\tilde{t}_{u_k}^2 u_k^2\right) - 1 \right) dx
\]
\[
\leq \frac{\lambda}{4} \int_{\mathbb{R}^4} \left( \exp \left(2\tilde{t}_{u_k}^2 u_k^2\right) - 1 \right) dx
\]
\[
\leq c \int_{\mathbb{R}^4} |u_k|^2 dx \to 0.
\]
where we have used the Adams inequality with the exact growth, i.e. Theorem 2.1 and fact that \( \int_{\mathbb{R}^4} |\Delta u_k|^2 \, dx \to 0 \).

On the other hand, we have

\[
I_\infty (\bar{t}_k u_k) = \frac{\lambda}{4} \int_{\mathbb{R}^4} \left( \exp \left( 2\bar{t}_k^2 u_k^2 \right) 2\bar{t}_k^2 u_k^2 - \left( \exp \left( 2\bar{t}_k^2 u_k^2 \right) - 1 \right) \right) \, dx \geq m_\infty.
\]

This is a contradiction. Therefore, \( m_V > 0 \).

We now consider a minimizing sequence \( \{u_k\}_k \subset \mathcal{N}_V \).

**Lemma 6.3** \( \{u_k\}_k \) is bounded in \( H^2(\mathbb{R}^4) \).

**Proof** From (6.6), we know

\[
\int_{\mathbb{R}^4} (|\Delta u_k|^2 + (V(x) - \lambda) |u_k|^2) \, dx = \frac{\lambda}{2} \int_{\mathbb{R}^4} (\exp (2u_k^2) - 1) 2u_k^2 \, dx \\
\leq \frac{c\lambda}{2} \left( \int_{\mathbb{R}^4} (\exp (2u_k^2) 2u_k^2 - \left( \exp (2u_k^2) - 1 \right)) \, dx \right) \\
\to 2cm_V.
\]

Then the proof is finished from the assumption of \( V(x) \).

Since the sequence \( \{u_k\}_k \) is bounded in \( H^2(\mathbb{R}^4) \), then up to a subsequence, there exists \( u \in H^2(\mathbb{R}^4) \), such that

- \( u_k \to u \) weakly in \( H^2(\mathbb{R}^4) \) and in \( L^p(\mathbb{R}^4) \), for any \( p > 1 \),
- \( u_k \to u \) in \( L^p_{\text{loc}}(\mathbb{R}^4) \),
- \( u_k \to u \), a.e.

By extracting a subsequence, if necessary, we define \( \beta, l \geq 0 \) as

\[
\beta = \lim_{k \to +\infty} \int_{\mathbb{R}^4} \left( \exp (2u_k^2) - 1 \right) u_k^2 \, dx \quad \text{and} \quad l = \int_{\mathbb{R}^4} (\exp (2u^2) - 1) u^2 \, dx.
\]

By the weak convergence, it is obvious that \( l \in [0, \beta] \).

**Lemma 6.4** There results \( \beta > 0 \).

**Proof** We argue this by contradiction. Assume that \( \beta = 0 \), then by (6.6), we have

\[
I_V (u_k) = \frac{\lambda}{4} \int_{\mathbb{R}^4} \left( \exp \left( 2u_k^2 \right) 2u_k^2 - \left( \exp \left( 2u_k^2 \right) - 1 \right) \right) \, dx \to 0,
\]

which contradicts (6.3).

**Lemma 6.5** The case \( l = 0 \) cannot occur.

**Proof** We prove this by contradiction. If \( l = 0 \), then \( u = 0 \), and \( u_k \to 0 \) in \( L^2_{\text{loc}}(\mathbb{R}^4) \). We first claim that:

\[
\lim_{k \to +\infty} \int_{\mathbb{R}^4} (\gamma - V(x)) |u_k|^2 \, dx = 0. \quad \text{(6.8)}
\]

For any fixed \( \varepsilon > 0 \), we take \( R_\varepsilon > 0 \) such that

\[
|\gamma - V(x)| \leq \varepsilon, \quad \text{for any } |x| > R_\varepsilon.
\]
Combining this and the boundedness of $u_k$ in $H^2(\mathbb{R}^4)$, we derive that
\[
\int_{\mathbb{R}^4} (\gamma - V(x)) |u_k|^2 \, dx = \int_{B_{R_k}} (\gamma - V(x)) |u_k|^2 \, dx + \int_{B_{R_k}^c} (\gamma - V(x)) |u_k|^2 \, dx \\
\leq c \int_{B_{R_k}} |u_k|^2 \, dx + M \varepsilon,
\]
where $M = \sup_k \int_{\mathbb{R}^4} |u_k|^2 \, dx$. This together with $u_k \to 0$ in $L^2_{loc}(\mathbb{R}^4)$ as $k \to \infty$ yields that
\[
\int_{\mathbb{R}^4} (\gamma - V(x)) |u_k|^2 \, dx \leq c \varepsilon,
\]
which implies (6.8) holds.

From the proof of Lemma 6.2, we know that there exists some bounded sequence $t_k \geq 1$ such that $t_k u_k \in \mathcal{N}_\infty$, that is,
\[
\int_{\mathbb{R}^4} (|\Delta u_k|^2 + (\gamma - \lambda) |u_k|^2) \, dx - \lambda \int_{\mathbb{R}^4} (\exp(2t_k^2 u_k^2) - 1) u_k^2 \, dx = 0. \tag{6.9}
\]
On the other hand, since $u_k \in \mathcal{N}_V$, then
\[
\int_{\mathbb{R}^4} (|\Delta u_k|^2 + (V(x) - \lambda) |u_k|^2) \, dx - \lambda \int_{\mathbb{R}^4} (\exp(2u_k^2) - 1) u_k^2 \, dx = 0. \tag{6.10}
\]
Combining (6.9) and (6.10), we get
\[
\lambda \int_{\mathbb{R}^4} (\exp(2t_k^2 u_k^2) - 1) u_k^2 \, dx - \lambda \int_{\mathbb{R}^4} (\exp(2u_k^2) - 1) u_k^2 \, dx \\
= \int_{\mathbb{R}^4} (|\Delta u_k|^2 + (\gamma - \lambda) |u_k|^2) \, dx - \int_{\mathbb{R}^4} (|\Delta u_k|^2 + (V(x) - \lambda) |u_k|^2) \, dx \\
= \int_{\mathbb{R}^4} (\gamma - V(x)) |u_k|^2 \, dx \to 0.
\]
Hence
\[
\int_{\mathbb{R}^4} (\exp(2t_k^2 u_k^2) - 1) u_k^2 \, dx = \int_{\mathbb{R}^4} (\exp(2u_k^2) - 1) u_k^2 \, dx + o_k(1). \tag{6.11}
\]

Next, we claim that $t_k \to t_0 = 1$ as $k \to \infty$. We prove this by contradiction. Assume that $t_0 > 1$. We carry out the proof in two cases.

**Case 1** There exists some $N \geq 2$ such that
\[
\lim_{k \to \infty} \int_{\mathbb{R}^4} u_k^{2N} \, dx > 0. \tag{6.12}
\]
Since $t_k \to t_0 > 1$, we can choose $k$ large enough such that $t_k > 0 \equiv t_0 - \left(\frac{1}{2} - \frac{1}{2}\right) > 1$, then by (6.12) we get
\[
\int_{\mathbb{R}^4} (\exp(2t_k^2 u_k^2) - 1) u_k^2 \, dx \geq \int_{\mathbb{R}^4} (\exp(2t_0^2 u_k^2) - 1) u_k^2 \, dx \\
> \int_{\mathbb{R}^4} (\exp(2u_k^2) - 1) u_k^2 \, dx > 0,
\]
which is a contradiction with (6.11).
Case 2 For any \( N \geq 2 \), there holds
\[
\lim_{k \to \infty} \int_{\mathbb{R}^4} u_k^{2N} \, dx \to 0.
\]
Since
\[
I_V(u_k) = \frac{\lambda}{4} \int_{\mathbb{R}^4} \left( \exp \left( 2u_k^2 \right) \frac{2u_k^2}{2} - \left( \exp \left( 2u_k^2 \right) - 1 \right) \right) \, dx
\]
\[
= \int_{\mathbb{R}^4} \left( \frac{2u_k^2}{2} + \left( \frac{1}{2} - \frac{1}{3!} \right) (2u_k^2)^3 + \left( \frac{1}{3!} - \frac{1}{4!} \right) (2u_k^2)^4 + \cdots \right.
\]
\[
+ \left( \frac{1}{(n-1)!} - \frac{1}{n!} \right) (2u_k^2)^n + \cdots \right) dx,
\]
by \( I_V(u_k) \to m_V > 0 \) and \( \lim_{k \to \infty} \int_{\mathbb{R}^4} u_k^{2N} \, dx \to 0 \), we know that the positive value of \( I_V(u_k) \) cannot be provided by any finite term in the expansion above. Also, from this expansion, we see that the coefficients of the expansion of \( I_V(u_k) \) convergence to the coefficients of
\[
\frac{\lambda}{2} \int_{\mathbb{R}^4} \exp \left( 2u_k^2 \right) u_k^2 \, dx,
\]
as \( n \to +\infty \). Thus, we have
\[
\frac{\lambda}{2} \int_{\mathbb{R}^4} \exp \left( 2u_k^2 \right) u_k^2 \, dx \to I_V(u_k) \to m_V \text{, as } k \to \infty,
\]
and then we conclude that
\[
\int_{\mathbb{R}^4} \left( \exp \left( 2t_k^2 u_k^2 \right) - 1 \right) u_k^2 \, dx \geq \int_{\mathbb{R}^4} \left( \exp \left( 2t_0^2 u_k^2 \right) - 1 \right) u_k^2 \, dx
\]
\[
\geq t_0^2 \int_{\mathbb{R}^4} \left( \exp \left( 2u_k^2 \right) - 1 \right) u_k^2 \, dx
\]
\[
\to t_0^2 \frac{2m_V}{\lambda} > \frac{2m_V}{\lambda}
\]
\[
= \lim_{k \to \infty} \int_{\mathbb{R}^4} \left( \exp \left( 2u_k^2 \right) - 1 \right) u_k^2 \, dx,
\]
which contradicts (6.11). Thus the claim is proved.

Now, by (6.8) we can obtain
\[
m_\infty \leq \lim_{k \to +\infty} I_\infty (t_k u_k) = \lim_{k \to +\infty} \left( I_V (t_k u_k) + \frac{1}{2} t_k^2 \int_{\mathbb{R}^4} (\gamma - V(x)) \left| u_k \right|^2 \, dx \right)
\]
\[
= \lim_{k \to +\infty} I_V (t_k u_k) \leq \lim_{k \to +\infty} t_k^2 I_V (u_k) = m_V
\]
which contradicts (6.3). This accomplishes the proof of Lemma 6.5. \( \square \)

Lemma 6.6 If \( l = \beta \), then \( u \in \mathcal{N}_V \) and \( I_V (u) = m_V \).

Proof If \( l = \beta \), then \( \int_{\mathbb{R}^4} \left( \exp (2u_k^2) - 1 \right) u_k^2 \, dx \to \int_{\mathbb{R}^4} \left( \exp (2u^2) - 1 \right) u^2 \, dx \) as \( k \to +\infty \). Then one can get
\[
\int_{\mathbb{R}^4} \left( |\Delta u|^2 + (V(x) - \lambda) |u|^2 \right) \, dx \leq \lim_{k \to \infty} \int_{\mathbb{R}^4} \left( |\Delta u_k|^2 + (V(x) - \lambda) |u_k|^2 \right) \, dx
\]
\begin{equation}
\lim_{k \to \infty} \lambda \int_{\mathbb{R}^4} (\exp(2u_k^2) - 1) u_k^2 dx = \lambda \int_{\mathbb{R}^4} (\exp(2u^2) - 1) u^2 dx.
\end{equation}

If the above equality holds, then \( u \in \mathcal{N}_V \), and the lemma is proved. Therefore, it remains to show that the case
\begin{equation}
\int_{\mathbb{R}^4} (|\Delta u|^2 + V(x)|u|^2) dx < \lambda \int_{\mathbb{R}^4} \exp(2u^2) u^2 dx
\end{equation}
cannot occur. In fact, if \((6.13)\) hold, we can take some \( t \in (0, 1) \) such that \( tu \in \mathcal{N}_V \). Indeed, let
\[ g(t) = \int_{\mathbb{R}^4} (|\Delta (tu)|^2 + (V(x) - \lambda)|tu|^2) dx - \lambda \int_{\mathbb{R}^4} (\exp(2(tu)^2)(tu)^2 - (tu)^2) dx. \]

Obviously, \( g(t) \) is positive for small \( t \). This together with \( g(1) < 0 \) implies that there exists \( t \in (0, 1) \) such that \( g(t) = 0 \), that is \( tu \in \mathcal{N}_V \). Therefore, we have
\begin{equation}
m_V \leq \frac{\lambda}{4} \int_{\mathbb{R}^4} (\exp(2t^2u^2) 2t^2u^2 - (\exp(2t^2u^2) - 1)) dx
\end{equation}
\begin{equation}
< \frac{\lambda}{4} \int_{\mathbb{R}^4} (\exp(2u^2) 2u^2 - (\exp(2u^2) - 1)) dx
\end{equation}
\begin{equation}
\leq \lim_{k \to \infty} \frac{\lambda}{4} \int_{\mathbb{R}^4} (\exp(2u_k^2) 2u_k^2 - (\exp(2u_k^2) - 1)) dx = m_V,
\end{equation}
which is a contradiction. \( \square \)

In the following, we consider the case \( 0 < l < \beta \). If \( 0 < l < \beta \), then \( u_k \to u \neq 0 \) weakly in \( H^2(\mathbb{R}^4) \). One can choose an increasing sequence \( \{R_j\} \to +\infty \) such that \( R_{j+1} > R_j + 1 \), and
\begin{equation}
\int_{B_{R_j}} (\exp(2u^2) - 1) u^2 dx = l + o_j(1),
\end{equation}
and
\begin{equation}
\int_{B_{R_j}^c} |u|^p dx = o_j(1),
\end{equation}
for any \( 1 \leq p < \infty \). We define
\[ C_j = B_{R_{j+1}} \setminus B_{R_j} = \{ x \in \mathbb{R}^4 | R_j \leq |x| < R_j + 1 \}. \]

**Lemma 6.7** For the \( C_j \) given above, we have
\begin{equation}
\int_{C_j} \exp(2u_k^2) u_k^2 dx = o_j(1)
\end{equation}
and
\begin{equation}
\int_{C_j} |\Delta u_k|^2 dx = o_j(1).
\end{equation}
**Lemma 6.8** There holds

\[
\sum_{i=1}^{\infty} \int_{C_{j_i}} \exp(2u_k^2) \, u_k^2 \, dx = \infty.
\]

However, on the other hand,

\[
\sum_{i=1}^{\infty} \int_{C_{j_i}} \exp(2u_k^2) \, u_k^2 \, dx \leq \int_{\mathbb{R}^4} \exp(2u_k^2) \, u_k^2 \, dx = \frac{1}{\lambda} \int_{\mathbb{R}^4} (|\Delta u_k|^2 + V(x) |u_k|^2) \, dx < \infty,
\]

which arrives at a contradiction. Similarly, one can also prove (6.16). \( \square \)

**Lemma 6.8** There holds

\[
\lim_{k \to \infty} \int_{\mathbb{R}^4} (\exp(2u_k^2) - 1 - 2u_k^2) \, dx = \int_{\mathbb{R}^4} (\exp(2u^2) - 1 - 2u^2) \, dx.
\]

**Proof** Let \( u_k^* \) and \( u^* \) be the symmetric decreasing rearrangements of \( u_k^* \) and \( u \). Using the radial lemma (see [22], Lemma 1.1, Chapter 6), we have

\[
|u_k^*(x)| \leq \frac{c}{|x|^2},
\]

for a.e. \( x \in \mathbb{R}^4 \). Now, we only need to prove

\[
\lim_{k \to \infty} \int_{\mathbb{R}^4} \left( \exp\left(2|u_k^*|^2\right) - 1 - 2|u_k^*|^2\right) \, dx = \int_{\mathbb{R}^4} \left( \exp\left(2|u^*|^2\right) - 1 - 2|u^*|^2\right) \, dx.
\]

For any \( R > 0 \), we first claim that

\[
\lim_{k \to \infty} \int_{\{|x| < R\}} \left( \exp\left(2|u_k^*|^2\right) - 1 - 2|u_k^*|^2\right) \, dx = \int_{\{|x| < R\}} \left( \exp\left(2|u^*|^2\right) - 1 - 2|u^*|^2\right) \, dx.
\]

Indeed, for any \( s > 0 \), we have

\[
\left| \int_{\{|x| < R\}} \left( \exp\left(2|u_k^*|^2\right) - 1 - 2|u_k^*|^2\right) \, dx - \int_{\{|x| < R\}} \left( \exp\left(2|u^*|^2\right) - 1 - 2|u^*|^2\right) \, dx \right|
\]

\[
\leq \left| \int_{\{|x| < R\} \cap \{u_k^* < s\}} \left( \exp\left(2|u_k^*|^2\right) - 1 - 2|u_k^*|^2\right) \, dx \right|
\]

\[
- \int_{\{|x| < R\} \cap \{u_k^* < s\}} \left( \exp\left(2|u^*|^2\right) - 1 - 2|u^*|^2\right) \, dx
\]

\[
+ \left| \int_{\{|x| < R\} \cap \{u_k^* \geq s\}} \left( \exp\left(2|u_k^*|^2\right) - 1 - 2|u_k^*|^2\right) \, dx \right|
\]

\[
- \int_{\{|x| < R\} \cap \{u_k^* \geq s\}} \left( \exp\left(2|u^*|^2\right) - 1 - 2|u^*|^2\right) \, dx
\]

\[
= I_{k,R,s} + I_{I_k,R,s}.
\]
A direct application of the dominated convergence theorem leads to $I_{k,R,s} \to 0$. For $I_{k,R,s}$, we have

$$
\int_{|x|<R\cap\{u_k^s \geq s\}} \left( \exp \left( 2|u_k^s|^2 \right) - 1 - 2|u_k^s|^2 \right) dx
\leq \frac{1}{s^2} \int_{\mathbb{R}^4 \cap \{u_k^s \geq s\}} |u_k^s|^2 \left( \exp \left( 2|u_k^s|^2 \right) - 1 - 2|u_k^s|^2 \right) dx
\leq \frac{1}{s^2} \int_{\mathbb{R}^4} u_k^2 \exp \left( 2u_k^2 \right) dx \to 0, \quad s \to \infty,
$$

where we have used the fact that $\int_{\mathbb{R}^4} u_k^2 \exp \left( 2u_k^2 \right) dx$ is bounded. Consequently, $I_{k,R,s} \to 0$, and the claim is finished.

On the other hand, by (6.17), we have

$$
\int_{|x|>R} \left( \exp \left( 2|u_k^s|^2 \right) - 1 - 2|u_k^s|^2 \right) dx \leq c \int_{|x|>R} |u_k^s|^4 dx \leq c \int_{R}^{+\infty} \frac{1}{r^3} dr \leq cR^{-2} \to 0,
$$
as $R \to \infty$. Similarly, we have

$$
\int_{|x|>R} \left( \exp \left( 2|u_k^s|^2 \right) - 1 - 2|u_k^s|^2 \right) dx \to 0,
$$
as $R \to \infty$. We finish the proof. \( \square \)

**Lemma 6.9** (Concentration compactness principle [14]) Assume that $0 < a < q(x) < b$ and $\{u_k\}$ is a sequence in $H^2(\mathbb{R}^4)$ such that $\int_{\mathbb{R}^4} (|\Delta u_k|^2 + q(x) |u_k|^2) dx = 1$ and $u_k \rightharpoonup u \neq 0$ in $H^2(\mathbb{R}^4)$. If

$$
0 < p < \frac{1}{\left( 1 - \int_{\mathbb{R}^4} (|\Delta u|^2 + q(x) |u|^2) dx \right)},
$$

then

$$
\sup_k \int_{\mathbb{R}^4} \left( \exp \left( 32\pi^2 p u_k^2 \right) - 1 \right) dx < \infty.
$$

**Lemma 6.10** If $\int_{\mathbb{R}^4} (|\Delta u|^2 + (V(x) - \lambda) |u|^2) dx > \lambda \int_{\mathbb{R}^4} (\exp (2u^2) - 1) u^2 dx$, then

$$
\lim_{k \to \infty} \int_{B_{Rj}} \left( \exp (2u_k^2) - 1 \right) u_k^2 dx = \int_{B_{Rj}} \left( \exp (2u^2) - 1 \right) u^2 dx
$$

provided that $j$ is large enough.

**Proof** Obviously,

$$
\lim_{k \to \infty} \int_{\mathbb{R}^4} (|\Delta u_k|^2 + (V(x) - \lambda) |u_k|^2) dx \geq \int_{\mathbb{R}^4} (|\Delta u|^2 + (V(x) - \lambda) |u|^2) dx
$$

through lower semi-continuity. We split the proof into two cases.

**Case 1** If

$$
\lim_{k \to \infty} \int_{\mathbb{R}^4} (|\Delta u_k|^2 + (V(x) - \lambda) |u_k|^2) dx = \int_{\mathbb{R}^4} (|\Delta u|^2 + (V(x) - \lambda) |u|^2) dx,
$$
then $u_k \to u$ in $H^2(\mathbb{R}^4)$. Using Hölder’s inequality, we know that for any $p > 1$,

$$\int_{\mathbb{R}^4} (\exp (2u_k^2) - 1)u_k^2 \, dx \leq c \int_{\mathbb{R}^4} (\exp (2pu_k^2) - 1)u_k^{2p} \, dx \leq c \left( \int_{\mathbb{R}^4} (\exp (4pu_k^2) - 1) \, dx \right)^{1/2} \left( \int_{\mathbb{R}^4} u_k^{4p} \, dx \right)^{1/2}.$$ 

Since

$$\int_{\mathbb{R}^4} (\exp (4pu_k^2) - 1) \, dx$$

$$= \int_{\mathbb{R}^4} (\exp (8pu_k^2) + 8pu_0^2) - 1) \, dx$$

$$= \int_{\mathbb{R}^4} (\exp (8pu_0^2) \exp (8pu_k^2) - 1) \, dx$$

$$= \int_{\mathbb{R}^4} (\exp (8pu_k^2) - 1) (\exp (8pu_k^2) - 1) \, dx +$$

$$\int_{\mathbb{R}^4} (\exp (8pu_k^2) - 1) \, dx + \int_{\mathbb{R}^4} (\exp (8pu_k^2) - 1) \, dx$$

$$\leq \left( \int_{\mathbb{R}^4} (\exp (16pu_0^2) - 1) \, dx \right)^{1/2} \left( \int_{\mathbb{R}^4} (\exp (16pu_k^2) - 1) \, dx \right)^{1/2}$$

$$+ \int_{\mathbb{R}^4} (\exp (8pu_0^2) - 1) \, dx + \int_{\mathbb{R}^4} (\exp (8pu_k^2) - 1) \, dx$$

$$< c,$$

where we have used the Adams inequality (1.9) and the fact $\|u_k - u_0\|_{H^2(\mathbb{R}^4)} \to 0$, as $k \to +\infty$. Therefore,

$$\int_{\mathbb{R}^4} (\exp (2u_k^2) - 1)u_k^2 \, dx < +\infty,$$

and then we get

$$\lim_{k \to +\infty} \int_{B_{R_j}} ((\exp (2u_k^2) - 1)u_k^2) \, dx = \int_{B_{R_j}} ((\exp (2u^2) - 1)u^2) \, dx.$$

Case 2 If $\lim_{k \to +\infty} \int_{\mathbb{R}^4} (|\Delta u_k|^2 + (V(x) - \lambda) |u_k|^2) \, dx > \int_{\mathbb{R}^4} (|\Delta u|^2 + (V(x) - \lambda) |u|^2) \, dx$, we set

$$v_0 = \lim_{k \to +\infty} \int_{\mathbb{R}^4} (|\Delta u_k|^2 + (V(x) - \lambda) |u_k|^2) \, dx.$$

We claim that there exists some $q > 1$ sufficiently closed to 1 such that

$$q \lim_{k \to +\infty} \int_{\mathbb{R}^4} (|\Delta u_k|^2 + (V(x) - \lambda) |u_k|^2) \, dx < \frac{16\pi^2}{1 - \int_{\mathbb{R}^4} (|\Delta v_0|^2 + (V(x) - \lambda) |v_0|^2) \, dx}.$$

(6.18)
Indeed, from Lemma 6.8 we have
\[
\int_{\mathbb{R}^4} (|\Delta u_k|^2 + (V(x) - \lambda) |u_k|^2) \, dx \cdot \left(1 - \int_{\mathbb{R}^4} (|\Delta v_0|^2 + (V(x) - \lambda) |v_0|^2) \, dx\right)
\leq \int_{\mathbb{R}^4} (|\Delta u_k|^2 + (V(x) - \lambda) |u_k|^2) \, dx - \int_{\mathbb{R}^4} (|\Delta u|^2 + (V(x) - \lambda) |u|^2) \, dx + o_k(1)
\leq \frac{\lambda}{2} \int_{\mathbb{R}^4} (\exp(2u_k^2) - 1 - 2u_k^2) \, dx - \frac{\lambda}{2} \int_{\mathbb{R}^4} (\exp(2u^2) - 1 - 2u^2) \, dx
+ 2I_V(u_k) - 2I_V(u) + o_k(1).
\]

Since \(\int_{\mathbb{R}^4} (|\Delta u|^2 + (V(x) - \lambda) |u|^2) \, dx = \lambda \int_{\mathbb{R}^4} (\exp(2u^2) - 1) u^2 \, dx\), then it follows that
\[
I_V(u) > \frac{\lambda}{4} \int_{\mathbb{R}^4} (\exp(2u^2) - 2u^2 - (\exp(2u^2) - 1)) \, dx > 0. \tag{6.19}
\]

Combining (6.19) and Lemma 5.3, we conclude that
\[
\lim_{k \to +\infty} I_V(u_k) = m_V < m_\infty < 8\pi^2.
\]

This proves the claim.

By Lemma 6.9, there exists some \(p_0 > 1\) such that
\[
\sup_k \int_{\mathbb{R}^4} (\exp(2u_k^2) - 1)^{p_0} \, dx < \infty,
\]
thus there exist some \(1 < \tilde{p}_0 < p_0\) and \(p\) close to 1, such that
\[
\int_{B_{R_j}} (\exp(2u_k^2) - 1) u_k^{2\tilde{p}_0} \, dx < c \int_{B_{R_j}} (\exp(2\tilde{p}_0 u_k^2) - 1) u_k^{2\tilde{p}_0} \, dx
\leq c \left(\int_{\mathbb{R}^4} (\exp(2p\tilde{p}_0 u_k^2) - 1) \, dx\right)^{1/p} \left(\int_{\mathbb{R}^4} u_k^{2p'\tilde{p}_0} \, dx\right)^{1/p'}
\leq c.
\]

where in the last inequality we have used the Sobolev inequality on \(H^2(\mathbb{R}^4)\). Therefore, we get
\[
\lim_{k \to \infty} \int_{B_{R_j}} (\exp(2u_k^2) - 1) u_k^2 \, dx = \int_{B_{R_j}} (\exp(2u^2) - 1) u^2 \, dx. \tag{6.20}
\]

From Lemma 6.7, since \(R_j + 1 < R_{j+1}\), we can extract a subsequence \(u_{k_j}\) such that for every \(j \in \mathbb{N}\),
\[
\int_{B_{R_j}} (\exp(2u_{k_j}^2) - 1) u_{k_j}^2 \, dx = \beta + o_j(1),
\]
and
\[
\int_{C_j} (\exp(2u_{k_j}^2) - 1) u_{k_j}^2 \, dx = o_j(1), \quad \int_{C_j} |\Delta u_{k_j}|^2 \, dx = o_j(1), \quad \int_{C_j} u_{k_j}^2 \, dx = o_j(1).
\]

We take \(\{u_{k_j}\}\) as a new minimizing sequence renaming it \(\{u_j\}_j\).
Now, for every \( j \), we define a function \( \psi_j \in C^\infty_c(\mathbb{R}^4) \) satisfying \( 0 \leq \psi_j(x) \leq 1 \), \( \psi_j(x) = 1 \) if \( |x| \leq R_j \), \( \psi_j(x) = 0 \) if \( |x| > R_j + 1 \), and \( |\nabla \psi_j(x)|, |\Delta \psi_j(x)| \leq c \) for every \( x \). We also define auxiliary functions

\[
\begin{align*}
 u'_j &= \psi_j u_j, \\
 u''_j &= (1 - \psi_j) u_j.
\end{align*}
\]

Obviously, we have \( u_j = u'_j + u''_j \) for every \( j \).

**Lemma 6.11** The following properties hold as \( j \to \infty \):

1. \( u'_j \to u \) weakly in \( H^2(\mathbb{R}^4) \), strongly in \( L^p(\mathbb{R}^4) \) for any \( 1 \leq p < \infty \), and \( u''_j \to 0 \) weakly in \( H^2(\mathbb{R}^4) \).

2. There results

\[
\begin{align*}
\int_{\mathbb{R}^4} \left( \exp \left( 2u_j^2 \right) - 1 \right) u_j^2 \, dx \\
= \int_{\mathbb{R}^4} \left( \exp \left( 2u'_j \right) - 1 \right) (u'_j)^2 \, dx + \int_{\mathbb{R}^4} \left( \exp \left( 2u''_j \right) - 1 \right) (u''_j)^2 \, dx + o_j(1) .
\end{align*}
\]

(6.21)

3. There results

\[
\begin{align*}
\int_{\mathbb{R}^4} \left( |\Delta u_j|^2 + V(x) |u_j|^2 \right) \, dx \\
= \int_{\mathbb{R}^4} \left( |\Delta u'_j|^2 + V(x) |u'_j|^2 \right) \, dx + \int_{\mathbb{R}^4} \left( |\Delta u''_j|^2 + V(x) |u''_j|^2 \right) \, dx + o_j(1) .
\end{align*}
\]

(6.22)

**Proof** The first property is obvious, by the definitions of \( u'_j \) and \( u''_j \). Now, we check the second equality. By (6.15), we derive that

\[
\begin{align*}
\int_{\mathbb{R}^4} \left( \exp \left( 2u'_j^2 \right) - 1 \right) u'_j^2 \, dx \\
= \int_{B_{R_j}} \left( \exp \left( 2u'_j^2 \right) - 1 \right) u'_j^2 \, dx \\
+ \int_{C_j} \left( \exp \left( 2u'_j^2 \right) - 1 \right) u'_j^2 \, dx + \int_{B_{R_j}^c + 1} \left( \exp \left( 2u'_j^2 \right) - 1 \right) u'_j^2 \, dx \\
= \int_{B_{R_j}} \left( \exp \left( 2(u'_j)^2 \right) - 1 \right) (u'_j)^2 \, dx + \int_{C_j} \left( \exp \left( 2u'_j^2 \right) - 1 \right) u'_j^2 \, dx \\
+ \int_{B_{R_j}^c + 1} \left( \exp \left( 2(u''_j)^2 \right) - 1 \right) (u''_j)^2 \, dx \\
= \int_{\mathbb{R}^4} \left( \exp \left( 2(u'_j)^2 \right) - 1 \right) (u'_j)^2 \, dx \\
+ \int_{\mathbb{R}^4} \left( \exp \left( 2(u''_j)^2 \right) - 1 \right) (u''_j)^2 \, dx + o_j(1) .
\end{align*}
\]
We now prove the third property. Since \( V(x) > 0 \), direct computation leads to

\[
\int_{\mathbb{R}^4} V(x) \left| u_j \right|^2 dx = \int_{B_{R_j}} V(x) \left| u_j \right|^2 dx + \int_{B_{R_j}^c} V(x) \left| u_j \right|^2 dx + \int_{C_j} V(x) \left| u_j \right|^2 dx
\]

\[
= \int_{B_{R_j}} V(x) \left| u_j \right|^2 dx + \int_{B_{R_j}^c} V(x) \left| u_j \right|^2 dx + o_j(1)
\]

We now only need to show that

\[
\int_{\mathbb{R}^4} |\Delta u_j|^2 dx = \int_{\mathbb{R}^4} |\Delta u_j'|^2 dx + \int_{\mathbb{R}^4} |\Delta u_j''|^2 dx + o_j(1).
\]

Observing

\[
\int_{\mathbb{R}^4} |\Delta u_j|^2 dx = \int_{\mathbb{R}^4} \left( |\Delta u_j'|^2 + |\Delta u_j''|^2 \right) dx
\]

\[
= \int_{\mathbb{R}^4} |\Delta u_j'|^2 dx + \int_{\mathbb{R}^4} |\Delta u_j''|^2 dx + \int_{\mathbb{R}^4} \Delta u_j' \cdot \Delta u_j'' dx,
\]

in order to obtain the desired result, we only need to verify that

\[
\int_{\mathbb{R}^4} \Delta u_j' \cdot \Delta u_j'' dx = o_j(1).
\]

We can write

\[
\int_{\mathbb{R}^4} \Delta u_k' \cdot \Delta u_k'' dx
\]

\[
= \int_{\mathbb{R}^4} \left( u_j \Delta \psi_j + \psi_j \Delta u_j + 2 \nabla u_j \nabla \psi_j \right) \cdot \left( u_j \Delta \left( 1 - \psi_j \right) + \left( 1 - \psi_j \right) \Delta u_j + 2 \nabla \left( 1 - \psi_j \right) \nabla u_j \right) dx
\]

\[
= \int_{\mathbb{R}^4} \left( u_j \Delta \psi_j + \psi_j \Delta u_j + 2 \nabla u_j \nabla \psi_j \right) \cdot \left( -u_j \Delta \psi_j + \left( 1 - \psi_j \right) \Delta u_j - 2 \nabla \psi_j \nabla u_j \right) dx
\]

\[
= \int_{\mathbb{R}^4} \left( -|u_j|^2 |\Delta \psi_j|^2 + \left( 1 - \psi_j \right) u_j \Delta \psi_j \Delta u_j - 2 u_j \Delta \psi_j \nabla \psi_j \nabla u_j \right) dx
\]

\[
+ \int_{\mathbb{R}^4} \left( -u_j \psi_j \Delta u_j \Delta \psi_j + \psi_j \left( 1 - \psi_j \right) |\Delta u_j|^2 - 2 \psi_j \nabla \psi_j \Delta u_j \nabla u_j \right) dx
\]

\[
+ \int_{\mathbb{R}^4} \left( -2 \nabla u_j \nabla \psi_j u_j \Delta \psi_j + 2 \nabla u_j \nabla \psi_j \left( 1 - \psi_j \right) \Delta u_j - 4 \nabla u_j \cdot \nabla \psi_j \right) dx
\]

\[
= I + II + III.
\]

For \( I \), we have

\[
|I| = \left| \int_{\mathbb{R}^4} \left( -|u_j|^2 |\Delta \psi_j|^2 + \left( 1 - \psi_j \right) u_j \Delta \psi_j \Delta u_j - 2 u_j \Delta \psi_j \nabla \psi_j \nabla u_j \right) dx \right|
\]

\[
\leq c \int_{\mathbb{R}^4} |u_j|^2 dx + \int_{\mathbb{R}^4} |u_j| |\Delta u_j| dx + c \int_{\mathbb{R}^4} |u_j \nabla u_j| dx
\]
Lemma 6.12 There holds
\[ \frac{\lambda}{4} \int_{\mathbb{R}^4} \left( \exp\left(2u^2\right) 2u^2 - \left(\exp\left(2u^2\right) - 1\right) \right) dx \leq m_V. \]

Proof Since \( u_j \to u \) weakly in \( H^2(\mathbb{R}^4) \), thus
\[ \frac{\lambda}{4} \int_{\mathbb{R}^4} \left( \exp\left(2u^2\right) 2u^2 - \left(\exp\left(2u^2\right) - 1\right) \right) dx \]
\[ \leq \frac{\lambda}{4} \lim_{j \to +\infty} \int_{\mathbb{R}^4} \left( \exp\left(2u_j^2\right) 2u_j^2 - \left(\exp\left(2u_j^2\right) - 1\right) \right) dx \]
\[ = m_V. \]

\[ \square \]
Lemma 6.13 \textit{It cannot be}
\begin{equation}
\int_{\mathbb{R}^4} (|\Delta u|^2 + V(x)|u|^2) \, dx < \lambda \int_{\mathbb{R}^4} \exp(2u^2) \, u^2 \, dx. \tag{6.27}
\end{equation}

\textbf{Proof} If (6.27) is true, then there exists some \( t \in (0, 1) \) such that \( t u \in \mathcal{N}_V \). Therefore, by Lemma 6.12, we have
\begin{equation}
m_V \leq I_V(t u) = \frac{\lambda}{4} \int_{\mathbb{R}^4} \left( \exp(2r^2u^2)2r^2u^2 - \left( \exp(2r^2u^2) - 1 \right) \right) \, dx
< \frac{\lambda}{4} \int_{\mathbb{R}^4} \left( \exp(2u^2)2u^2 - \left( \exp(2u^2) - 1 \right) \right) \, dx \leq m_V,
\end{equation}
which is a contradiction. \( \square \)

Lemma 6.14 \textit{It cannot be}
\begin{equation}
\int_{\mathbb{R}^4} (|\Delta u|^2 + V(x)|u|^2) \, dx > \lambda \int_{\mathbb{R}^4} \exp(2u^2) \, u^2 \, dx. \tag{6.28}
\end{equation}

\textbf{Proof} By Lemma 6.11, we get
\begin{align*}
\int_{\mathbb{R}^4} \left( |\Delta u_j'|^2 + (V(x) - \lambda) |u_j'|^2 \right) \, dx &+ \int_{\mathbb{R}^4} \left( |\Delta u_j''|^2 + (V(x) - \lambda) |u_j''|^2 \right) \, dx \\
&= \int_{\mathbb{R}^4} \left( |\Delta u_j|^2 + (V(x) - \lambda) |u_j|^2 \right) \, dx + o_j(1) = \lambda \int_{\mathbb{R}^4} \left( \exp(2u_j^2) - 1 \right) u_j^2 \, dx + o_j(1) \\
&= \lambda \int_{\mathbb{R}^4} \left( \exp\left(2\left(u_j'\right)^2\right) - 1 \right) \left(u_j'\right)^2 \, dx + \lambda \int_{\mathbb{R}^4} \left( \exp\left(2\left(u_j''\right)^2\right) - 1 \right) \left(u_j''\right)^2 \, dx + o_j(1).
\end{align*}

Assume for contradiction that (6.28) holds, then we can pick some \( \delta > 0 \) such that
\begin{equation}
\int_{\mathbb{R}^4} (|\Delta u|^2 + V(x)|u|^2) \, dx > \lambda \int_{\mathbb{R}^4} \exp(2u^2) \, u^2 \, dx + \delta. \tag{6.29}
\end{equation}

Since \( u_j' \to u \) weakly in \( H^2(\mathbb{R}^4) \), by (6.28) and Lemma 6.10, we have
\begin{align*}
\liminf_{j \to +\infty} \int_{\mathbb{R}^4} \left( |\Delta u_j'|^2 + (V(x) - \lambda) |u_j'|^2 \right) \, dx &\geq \int_{\mathbb{R}^4} \left( |\Delta u|^2 + (V(x) - \lambda) |u|^2 \right) \, dx \\
&> \lambda \int_{\mathbb{R}^4} \left( \exp(2u^2) - 1 \right) u^2 \, dx + \delta \\
&= \lambda \int_{\mathbb{R}^4} \left( \exp\left(2\left(u_j'\right)^2\right) - 1 \right) \left(u_j'\right)^2 \, dx + \delta + o_j(1).
\end{align*}

Hence, we have
\begin{equation}
\int_{\mathbb{R}^4} \left( |\Delta u_j''|^2 + V(x) |u_j''|^2 \right) \, dx < \lambda \int_{\mathbb{R}^4} \exp\left(2\left(u_j''\right)^2\right) \left(u_j''\right)^2 \, dx - \delta + o_j(1)
\end{equation}
for \( j \) large enough. Since \( u_j'' \to 0 \), weakly in \( H^2(\mathbb{R}^4) \), and arguing as (6.8) in Lemma 6.5, we can obtain
\begin{equation}
\lim_{j \to +\infty} \int_{\mathbb{R}^4} \left( |\Delta u_j'|^2 + V(x) |u_j'|^2 \right) \, dx = \int_{\mathbb{R}^4} \left( |\Delta u''|^2 + \gamma |u''|^2 \right) \, dx.
\end{equation}
Therefore, it follows that for \( j \) large enough, there holds
\[
\int_{\mathbb{R}^4} \left( \left| \Delta u_j^\prime \right|^2 + \gamma \left| u_j^\prime \right|^2 \right) \, dx < \lambda \int_{\mathbb{R}^4} \exp \left( 2 \left| u_j^\prime \right|^2 \right) \left( u_j^\prime \right)^2 \, dx - \delta + o_j(1).
\]
By the usual argument, we can find some \( t_j \in (0, 1) \) such that \( t_j u_j^\prime \in \mathcal{N}_\infty \), so we conclude that
\[
m_\infty \leq I_\infty \left( t_j u_j^\prime \right)
= \frac{\lambda}{4} \int_{\mathbb{R}^4} \left( \exp \left( 2 \left| t_j u_j^\prime \right|^2 \right) 2 \left| t_j u_j^\prime \right|^2 - \left( \exp \left( 2 \left| u_j^\prime \right|^2 \right) - 1 \right) \right) \, dx
\leq \frac{\lambda}{4} \int_{\mathbb{R}^4} \left( \exp \left( 2 \left| u_j^\prime \right|^2 \right) 2 \left| u_j^\prime \right|^2 - \left( \exp \left( 2 \left| u_j^\prime \right|^2 \right) - 1 \right) \right) \, dx
\leq \frac{\lambda}{4} \int_{\mathbb{R}^4} \left( \exp \left( 2 \left| u_j^\prime \right|^2 \right) 2 \left| u_j^\prime \right|^2 - \left( \exp \left( 2 \left| u_j^\prime \right|^2 \right) - 1 \right) \right) \, dx
+ \frac{\lambda}{4} \int_{\mathbb{R}^4} \left( \exp \left( 2 \left| u_j^\prime \right|^2 \right) 2 \left| u_j^\prime \right|^2 - \left( \exp \left( 2 \left| u_j^\prime \right|^2 \right) - 1 \right) \right) \, dx
\leq I_V \left( u_j \right) + o_j(1).
\]
Let \( j \to \infty \), we derive \( m_\infty \leq m_V \), which is a contradiction. This accomplishes the proof of Lemma 6.14.

End of the proof of Theorem 2.8 We can now conclude easily the proof of the main result of this section. The previous lemmas say that in any case the weak limit \( u \) satisfies \( u \neq 0 \) and \( N_V(u) = 0 \), then
\[
I_V \left( u \right) = \frac{\lambda}{4} \int_{\mathbb{R}^4} \left( \exp \left( 2 u^2 \right) 2 u^2 - \left( \exp \left( 2 u^2 \right) - 1 \right) \right) \, dx \leq m_V.
\]
So we have \( I_V(u) = m_V \) and \( u \) is a minimum point for \( I_V \) on \( \mathcal{N}_V \), hence \( u \) is a ground state solution of equation (2.3).

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