EXTENSION DIMENSION FOR PARACOMPACT SPACES

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Dedicated to Jed Keesling on the occasion of his sixtieth birthday.

ABSTRACT. We prove existence of extension dimension for paracompact spaces. Here is the main result of the paper:

Theorem. Suppose $X$ is a paracompact space. There is a CW complex $K$ such that

a. $K$ is an absolute extensor of $X$ up to homotopy,

b. If a CW complex $L$ is an absolute extensor of $X$ up to homotopy, then $L$ is an absolute extensor of $Y$ up to homotopy of any paracompact space $Y$ such that $K$ is an absolute extensor of $Y$ up to homotopy.

The proof is based on the following simple result (see 1.6).

Theorem. Suppose $X$ be a paracompact space and $f : A \to Y$ is a map from a closed subset $A$ of $X$ to a space $Y$. $f$ extends over $X$ if $Y$ is the union of a family $\{Y_s\}_{s \in S}$ of its subspaces with the following properties:

a. Each $Y_s$ is an absolute extensor of $X$,

b. For any two elements $s$ and $t$ of $S$ there is $u \in S$ such that $Y_s \cup Y_t \subset Y_u$,

c. $A = \bigcup_{s \in S} \text{Int}_A(f^{-1}(Y_s))$.

That result implies a few well-known theorems of classical theory of retracts which makes it of interest in its own.

1. INTRODUCTION

A. Dranishnikov [Dr] introduced the concept of extension dimension for compact Hausdorff spaces as a generalization of both covering dimension and cohomological dimension.

1.1. Definition. Suppose $X$ is a compact Hausdorff space. A CW complex $K$ is called the extension dimension of $X$ if the following two conditions are satisfied:

a. $K$ is an absolute extensor of $X$,

b. If a CW complex $L$ is an absolute extensor of $X$, then $L$ is an absolute extensor of $Y$ for any compact Hausdorff space $Y$ such that $K$ is an absolute extensor of $Y$.

The meaning of Definition 1.1 is that extension dimension of $X$ is the minimal element of a subclass in a certain order on the class of all CW complexes. Namely, one can define $K \leq L$ if $C_K \subset C_L$, where $C_M$ is the class of all compact Hausdorff spaces $X$ such that

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$M \in AE(X)$. Now, $K$ is the extension dimension of $X$ if it is the minimal element among all $L$ such that $X \in C_L$.

One can ponder the existence of extension dimension for other classes of topological spaces. This was done by A.Dranishnikov and J.Dydak in [D-D$_1$] for separable metrizable spaces, and by I.Ivanšić and L.Rubin in [I-R] for metrizable spaces. However, the proofs in [D-D$_1$] and [I-R] are quite complicated. The author believes that, for a theory to be successful, its foundations should be fairly simple. The purpose of this paper is to provide quite an elementary proof of the existence of extension dimension for paracompact spaces.

One of the main ideas of extension theory is to investigate spaces by mapping them (or their subspaces) to spaces $K$ with good local properties. Traditionally, the spaces one wants to investigate are metrizable or compact Hausdorff. That tradition is the result of a natural evolution: euclidean spaces, their subspaces, their compactifications. Also, two classes of spaces with good local properties emerged; CW complexes and ANRs (absolute neighborhood retracts of metrizable spaces). Those two classes are known to be identical up to homotopy but as of now we do not know of a single class which could be used in their place. Is there a natural class of spaces which naturally combines metrizable spaces and compact Hausdorff spaces? The problem is that ANRs do not have to be absolute neighborhood extenders of compact Hausdorff spaces. One could bypass that problem by considering only maps $f : A \to K$ on closed subsets $A$ of $X$ which are $G_\delta$-subsets of $X$. Since being closed and a $G_\delta$ subset of a normal space $X$ is equivalent to be a zero subset (i.e., a set of the form $\alpha^{-1}(0)$ for some continuous $\alpha : X \to [0,1]$), let us formulate the corresponding variation of the concept of absolute extensor.

1.2. Definition. $Y \in AE_0(X)$ ($Y \in ANE_0(X)$, respectively) means that all maps $f : A \to Y$ extend over $X$ (over a neighborhood of $A$ in $X$, respectively) provided $A$ is a zero subset of $X$.

It is known that, if $K$ is an ANR and $X$ is paracompact space, then $K \in ANE_0(X)$. However, if $K$ is a CW complex the analogous statement is false. Indeed, van Douwen and Pol [D-P] constructed the strongest possible counterexample. In their case (see section 3 of [D-P]) $K$ is the cone over infinite discrete CW complex and $A$ is a closed subspace of a countable paracompact space $X$.

To avoid problems with extending maps to CW complexes over neighborhoods of closed subsets of paracompact spaces the papers [D-D$_1$] and [I-R] create subclasses of paracompact spaces. In [D-D$_1$] cw-spaces are defined as paracompact $k$-spaces $X$ such that any contractible CW complex $K$ is an absolute extensor of $X$. In [I-R] dd-spaces are defined.

In this paper the difficulty is avoided by switching the focus from extending maps to extending maps up to homotopy which may seem to be a more difficult task. However, there is a special class of generic maps to CW complexes (called locally compact maps) for which the two extension problems are equivalent. As a result we obtain three possible interpretations of extension dimension for paracompact spaces:

1.3. Theorem. Suppose $X$ is a paracompact space. There is a CW complex $K$ (called the extension dimension of $X$) such that
a. $K$ is an absolute extensor of $X$ up to homotopy,
b. If a CW complex $L$ is an absolute extensor of $X$ up to homotopy, then $L$ is an absolute extensor of $Y$ up to homotopy of any paracompact space $Y$ such that $K$ is an absolute extensor of $Y$ up to homotopy.

1.4. Theorem. Suppose $X$ is a paracompact space. There is a simplicial complex $K$ such that
a. $|K|_m$ is an absolute extensor of $X$ and is complete,

b. If a complete ANR $L$ is an absolute extensor of $X$, then $L$ is an absolute extensor any paracompact space $Y$ such that $|K|_m$ is an absolute extensor of $Y$.

1.5. Theorem. Suppose $X$ is a paracompact space. There is a simplicial complex $K$ such that

a. $|K|_m \in AE_0(X)$,

b. If $L \in AE_0(X)$ is an ANR, then $L \in AE_0(Y)$ for any paracompact space $Y$ such that $|K|_m \in AE_0(Y)$.

Let us start with a general, yet simple, result which is at the core of our approach to extension dimension theory.

1.6. Theorem. Suppose $X$ be a paracompact space and $f : A \to Y$ is a map from a closed subset $A$ of $X$ to a space $Y$. $f$ extends over $X$ if $Y$ is the union of a family $\{Y_s\}_{s \in S}$ of its subspaces with the following properties:

a. Each $Y_s$ is an absolute extensor of $X$,

b. For any two elements $s$ and $t$ of $S$ there is $u \in S$ such that $Y_s \cup Y_t \subset Y_u$,

c. $A = \bigcup_{s \in S} \text{Int}_A(f^{-1}(Y_s))$.

Proof. Define $U_s = (X - A) \cup \text{Int}_A(f^{-1}(Y_s))$ for each $s \in S$. Each $U_s$ is an open subset of $X$ and $X = \bigcup_{s \in S} U_s$. Since $X$ is paracompact, there is a locally finite partition of unity $\{g_s\}_{s \in S}$ on $X$ such that $g_s^{-1}(0,1] \subset U_s$ for each $s \in S$ (see [En], Theorem 5.1.9 and its proof). For all finite subsets $T$ of $S$ define $B_T = \{x \in X \mid g_s(x) > 0 \Rightarrow s \in T\}$. We plan to create, for all finite subsets $T$ of $S$, elements $a(T)$ of $S$ and maps $f_T : B_T \to Y_{a(T)}$ so that the following conditions are satisfied:

1. $Y_{a(F)} \subset Y_{a(T)}$ for each $F \subset T$,

2. $f_T|B_F = f_F$ for each $F \subset T$,

3. $f_T|A \cap B_T = f|A \cap B_T$.

This is going to be accomplished by induction on the number of elements of $T$. For one-element sets $T = \{s\}$ we simplify notation to $T = s$. Notice that $B_s = g_s^{-1}(1)$ for each $s \in S$. $\{B_s\}_{s \in S}$ is a discrete family and $f(A \cap B_s) \subset Y_s$ for each $s \in S$. Therefore we can extend each $f|A \cap B_s$ to $f_s : B_s \to Y_s$ and we put $a(s) = s$. Suppose $f_T$ and $a(T)$ exist for all $T$ with cardinality at most $n$. Given $T$ containing exactly $n + 1$ elements, pick $s \in S$ so that $Y_s$ contains all of $Y_{a(F)}$ with $F$ being a proper subset of $T$. Put $a(T) = s$. All of $f_T$, $F$ a proper subset of $T$, can be pasted together and produce a map $h$ on a closed subset $B$ of $B_T$ with values in $Y_s$ and extending $f$ on $A \cap B$. Since $f(A \cap B_T) \subset Y_s$, $h$ extends over $B_T$ and produces $f_T : B_T \to Y_{a(T)}$ with the desired properties.

Since $B_T \cap B_F = B_{T \cap F}$, all $f_T$ can be pasted together to produce a function $f' : X \to Y$ which is an extension of $f$. Any point $x \in X$ has a neighborhood $U$ which intersects only finitely many of $g_s^{-1}(0,1]$ which means that there is a finite set $T$ such that $U \subset B_T$. As $f'|B_T$ is continuous, so is $f'|U$ which completes the proof. $\Box$

Before applying 1.6 let us recall a canonical method from [Dy2] of converting results about absolute extensors to theorems about absolute neighborhood extensors. This is done by using the so-called covariant cones. For any space $P$ its covariant cone $\text{Cone}(P)$ is $P \times I/P \times \{1\}$ with the topology induced by open sets in $P \times [0,1)$ and a basis of neighborhoods of the vertex $v = P \times \{1\}/P \times \{1\}$ being $P \times [t,1)/P \times \{1\}$, $t \in [0,1)$. In [Dy2] (see Theorem 2.9) it is shown that if $P$ is Hausdorff, contains at least two points, and is an absolute neighborhood extensor of a space $M$, then $\text{Cone}(P)$ is an absolute extensor of $M$. Given a simplicial complex $K$ and an absolute extensor $L$ of $X$, $\text{Cone}(L)$ is an absolute extensor of $\text{Cone}(X)$. If $\{K_s\}_{s \in S}$ is a simplicial complex of absolute extensors of $X$ such that $|K|_m \in \bigcup_{s \in S} \text{Int}_A(f^{-1}(Y_s))$, then $\bigcup_{s \in S} K_s$ is an absolute extensor of $\bigcup_{s \in S} Y_s$. This is the general result which is at the core of our approach to extension dimension theory.
of $M$. Notice that, in case of normal spaces $M$, the proof of 2.9 in [Dy$_2$] applies to all spaces $P$ as the assumption of $P$ being Hausdorff and containing at least two points was used only to deduce that $M$ is normal.

1.7. Corollary. Suppose $X$ be a paracompact space and $f : A \to Y$ is a map from a closed subset $A$ of $X$ to a space $Y$. $f$ extends over a neighborhood of $A$ in $X$ if $Y$ is the union of a family $\{Y_s\}_{s \in S}$ of its subspaces with the following properties:

a. Each $Y_s$ is an absolute neighborhood extensor of $X$,

b. For any two elements $s$ and $t$ of $S$ there is $u \in S$ such that $Y_s \cup Y_t \subset Y_u$,

c. $A = \bigcup_{s \in S} \text{Int}_A(f^{-1}(Y_s))$.

Proof. Let $Z = \text{Cone}(Y)$ with vertex $v$ and $Z_s = \text{Cone}(Y_s)$ for each $s \in S$. Therefore, $f$ considered as a map from $A$ to $Z$ satisfies hypotheses of Theorem 1.6 and extends over $X$. Let $g : X \to Z$ be an extension of $f$ and let $U = g^{-1}(Z - \{v\})$. There is a retraction $r : Z - \{v\} \to Y$ which means that the composition of $g|U$ and $r$ produces an extension $f' : U \to Y$ of $f$. □

The strength of 1.7 is that it implies two well-known results from the theory of retracts and its proof is much simpler than those of original results. The first one is a theorem first proved by Dugundji [Du] (and independently by Kodama [Ko]) for the special case of simplicial complexes with the CW topology. In full generality it follows from a result of Cauty [Ca] that each CW complexes $K$ can be embedded in a polyhedron with CW topology in such a way that there is a retraction $r : U \to K$ from a neighborhood $U$ of $K$.

1.8. Corollary (Cauty-Dugundji-Kodama). CW complexes are absolute neighborhood extensors of metrizable spaces.

Proof. Finite subcomplexes of a CW complex $K$ form a family closed under finite sums, each of them is an absolute neighborhood extensor of normal spaces, and any map $f : A \to K$ from a first countable space has the property that each point $x \in A$ has a neighborhood $U$ such that $f(U)$ is contained in a finite subcomplex of $K$ (see [Dy$_2$], Corollary 4.5). Thus, 1.7 applies. □

The second one is a result of Hanner as proved in [Hu] in quite a complicated way on eleven pages (see Theorem 17.1 on pp. 68–79).

1.9. Corollary (Hanner). Suppose $X$ is a paracompact space. If a Hausdorff space $Y$ is a union of open subsets $U$ which are absolute neighborhood extensors of $X$, then $Y$ is an absolute neighborhood extensor of $X$.

Proof. The family of all open subsets of $Y$ which are absolute neighborhood extensors of $X$ is closed under finite unions (see [Hu], Theorem 8.2), so 1.7 applies. □

The author would like to thank Sergey Antonyan for asking questions about existence of a simple proof of Cauty-Dugundji-Kodama Theorem 1.8, and to Ivan Ivanišić for help with sorting out the issues related to CW complexes and ANE for paracompact spaces. Antonyan’s question stemmed from [AEM], where a proof of 1.8 is given which is simpler than the original one. Also, it is mentioned in [AEM] that our approach, when applied to the equivariant case, is of interest and offers simplifications similar to those in the non-equivariant case.
2. Locally compact maps

The simplicity of 1.6-1.7 and their applications made the author think that one should attempt to build extension theory based on 1.6. Since our interest is mostly in maps to CW complexes, the proof of 1.8 suggests that we need to concentrate on maps such that every point has a neighborhood whose image is contained in a finite subcomplex. A generalization to arbitrary spaces is obvious:

2.1. Definition. A map \( f : X \to Y \) is called \textbf{locally compact} if for every element \( x \in X \) there is a neighborhood \( U \) in \( X \) such that \( f(U) \) is contained in a compact subset of \( Y \).

\textbf{Remark.} It is easy to show that \( f : X \to Y \) is locally compact if and only if for any compact subset \( Z \) of \( X \) there is a neighborhood \( U \) of \( Z \) in \( X \) such that \( f(U) \) is contained in a compact subset of \( Y \).

Let us point out that, in the case of maps to simplicial complexes with the weak topology, the concept of locally compact map corresponds to the concept of locally finite partition of unity. In 2.2 and in the remainder of the paper we follow the notation of [M-S], where \( |L|_w \) is the body of a simplicial complex \( L \) equipped with the weak topology, and \( |L|_m \) is the body of a simplicial complex \( L \) equipped with the metric topology.

2.2. Proposition. Let \( L \) be a simplicial complex. A map \( f : X \to |L|_w \) is locally compact if and only if the corresponding partition of unity on \( X \) is locally finite.

\textbf{Proof.} Let \( V \) be the set of vertices of \( L \). The partition of unity corresponding to \( f \) is the set of maps \( f_v : X \to I \) (those are the barycentric coordinates of \( f(x) \) according to the terminology of [M-S]) so that \( f(x) = \sum_{v \in V} f_v(x) \cdot v \). \( \{f_v\}_{v \in V} \) being locally finite means that each point \( x \in X \) has a neighborhood \( U \) such that only finitely many \( f_v \) are non-zero on \( U \). That is the same as saying that \( f(U) \) is contained in a finite subcomplex of \( |L|_w \). \( \square \)

The remainder of this section is devoted to the homotopy theory of locally compact maps. We start with a few elementary observations.

2.3. Proposition. Suppose \( f : X \to Y \) and \( g : Y \to Z \) are maps. If \( f \) or \( g \) is locally compact, then \( g \circ f \) is locally compact.

\textbf{Proof.} Suppose \( x \in X \). If there is a neighborhood \( U \) of \( x \) in \( X \) such that \( f(U) \) is contained in a compact subset \( C \) of \( Y \), then \( gf(U) \) is contained in \( g(C) \) which is compact. If \( f(x) \) is contained in a neighborhood \( V \) in \( Y \) such that \( g(V) \) is contained in a compact subset \( C \) of \( Z \), then we put \( U = g^{-1}(V) \) and notice that \( gf(U) \) is contained in \( C \). \( \square \)

2.4. Proposition. Suppose \( X \) is the union of a locally finite family \( \{X_s\}_{s \in S} \) consisting of closed sets. Let \( f : X \to Y \) be a map. If \( f|X_s \) is locally compact for each \( s \in S \), then \( f \) is locally compact.

\textbf{Proof.} Suppose \( x \in X \). If \( x \in X_s \) for some \( s \in S \), we pick a neighborhood \( U_s \) of \( x \) in \( X \) such that \( f(U_s \cap X_s) \) is contained in a compact subset \( C_s \) of \( Y \). Let \( T \) be a finite subset of \( S \) such that \( x \in X_s \) if and only if \( s \in T \). Let \( W = X - \bigcup_{s \in S-T} X_s \), and put \( U = W \cap \bigcap_{s \in T} U_s \).

Obviously, \( U \) is a neighborhood of \( x \) in \( X \). It remains to show that \( f(U) \subset \bigcup_{s \in T} C_s \) which follows from \( U \subset \bigcup_{s \in T} U_s \cap X_s \). \( \square \)
2.5. Proposition. If \( f_i : X_i \to Y_i \) is locally compact for \( i = 1, 2 \), then \( f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2 \) is locally compact.

Proof. Suppose \((x_1, x_2) \in X_1 \times X_2\). Pick a neighborhood \( U_i \) of \( x_i \) in \( X_i \) such that \( f_i(U_i) \) is contained in a compact subset \( C_i \) of \( Y_i, i = 1, 2 \). Notice that \((f_1 \times f_2)(U_1 \times U_2) \subset C_1 \times C_2 \) and \( C_1 \times C_2 \) is compact. \( \square \)

Our next two results show that locally compact maps are prevalent, up to homotopy, among maps to CW complexes.

2.6. Proposition. If \( X \) is homotopy equivalent to a CW complex, then \( \text{id}_X : X \to X \) is homotopic to a locally compact map.

Proof. First consider \( X = |L|_w \), where \( L \) is a simplicial complex. \( X \) is paracompact and open stars \( \{St(v, L)\}, v \) is a vertex of \( L \), form an open cover of \( X \). Therefore we can find a locally finite partition of unity \( \{g_v\} \) on \( X \) so that \( g_v^{-1}(0, 1] \subset St(v, L) \) for each \( v \) (see [En], Lemma 5.1.8, Theorem 5.1.9 and its proof). That partition of unity induces a locally compact map \( g : X \to X \) with the property that if \( x \) belongs to a simplex \( \Delta \), then \( g(x) \in \Delta \). The function \( H : X \times I \to X \) defined by \( H(x, t) = (1 - t) \cdot x + t \cdot g(x) \) is continuous on \( \Delta \times I \) for each simplex \( \Delta \) which means that \( H \) is continuous. Thus, \( H \) is a homotopy joining \( \text{id}_X \) and \( g \).

If \( X \) is homotopy equivalent to a CW complex, then we can find maps \( u : X \to Y = |L|_w \) and \( d : |L|_w \to X \) such that \( d \circ u \) is homotopic to the identity \( \text{id}_X \) (see [M-S]). Let \( h : Y \to Y \) be a locally compact map homotopic to \( \text{id}_Y \). Put \( g = d \circ h \circ u \). Notice that \( g \) is a locally compact map (use 2.3) homotopic to \( \text{id}_X \). \( \square \)

2.7. Corollary. Suppose \( Y \) is a space such that \( \text{id}_Y : Y \to Y \) is homotopic to a locally compact map. If \( f : X \to Y \) is a map such that \( f|A \) is locally compact for some closed subset \( A \) of \( X \), then there is a homotopy \( H : X \times I \to Y \) starting at \( f \) such that \( H|A \times I \cup Y \times \{1\} \) is locally compact.

Proof. Let \( G : Y \times I \to Y \) be a homotopy joining \( \text{id}_Y \) and a locally compact map. Define \( H \) as \( G \circ (f \times \text{id}_I) \). \( H \) starts at \( f \), \( H|X \times \{1\} \) is the composition of \( f \) and a locally compact map, and \( H|A \times I \) is the composition of \( f \times \text{id}_I \) and \( \text{id}_A \times I \) (which is a locally compact map by 2.4) and \( H|A \times I \). By 2.3 and 2.4, \( H|A \times I \cup Y \times \{1\} \) is locally compact. \( \square \)

Our strategy from now on is to replace every map by a homotopic locally compact map. That calls for obvious generalizations of well-known concepts which will be useful in simplifying the exposition.

2.8. Definition. Suppose \( X \) is a space and \( K \) is a CW complex. \( K \in \text{AE}_{lc}(X) \) means that any locally compact map \( f : A \to K \) on a closed subset \( A \) of \( X \) extends to a locally compact map \( f' : X \to K \).

We are now ready for an analog of 1.6 which will be our main tool in presenting the extension theory of paracompact spaces.

2.9. Theorem. Suppose a CW complex \( K \) is the union of a family \( \{K_s\}_{s \in S} \) of its subcomplexes so that for any two elements \( s \) and \( t \) of \( S \) there is \( u \in S \) with \( K_s \cup K_t \subset K_u \). Let \( X \) be a paracompact space. If, for each \( s \in S \), there is \( t \in S \) so that any locally compact map \( f : A \to K_s \) from a closed subset \( A \) of \( X \) extends to a locally compact map \( f' : X \to K_t \), then \( K \in \text{AE}_{lc}(X) \).

Proof. Suppose \( f : A \to K \) is a locally compact map, where \( A \) is a closed subset of \( X \). Given \( x \in A \) there is a neighborhood \( U \) of \( x \) in \( A \) so that \( f(U) \) is contained in a compact subset \( \tilde{K} \) of \( K \). If \( x \in A \) is a closed subset of \( X \), given \( x \in A \) there is a neighborhood \( U \) of \( x \) in \( A \) so that \( f(U) \) is contained in a compact subset \( \tilde{K} \) of \( K \).
ii. locally compact maps $f$.

Define $U_s = (X - A) \cup \text{Int}_{A}(f^{-1}(K_s))$ for each $s \in S$. Each $U_s$ is an open subset of $X$ and $X = \bigcup_{s \in S} U_s$. Since $X$ is paracompact, there is a locally finite partition of unity $\{g_s\}_{s \in S}$ on $X$ such that $g_s^{-1}(0,1] \subset U_s$ for each $s \in S$ (see [En], Lemma 5.1.8, Theorem 5.1.9 and its proof). For all finite subsets $T$ of $S$ define $B_T = \{x \in X \mid g_s(x) > 0 \implies s \in T\}$. We plan to create, for all finite subsets $T$ of $S$, the objects

i. elements $a(T), b(T)$ of $S$,

ii. locally compact maps $f_T : B_T \to K_{b(T)}$

so that the following conditions are satisfied:

1. $K_{a(F)} \subset K_{a(T)}$ for each $F \subset T$,

2. $K_{b(F)} \subset K_{b(T)}$ for each $F \subset T$,

3. any locally compact map $h : D \to K_{a(T)}$ on a closed subset $D$ of $X$ extends to a locally compact map $h' : X \to K_{b(T)}$,

4. $f_T|B_F = e_F$ for each $F \subset T$,

5. $f_T|A \cap B_T = f|A \cap B_T$.

This is going to be accomplished by induction on the number of elements of $T$. For one-element sets $T = \{s\}$ we simplify notation to $T = s$. Notice that $B_s = g_s^{-1}(1)$ for each $s \in S$. $\{B_s\}_{s \in S}$ is a discrete family and $f(A \cap B_s) \subset K_s$ for each $s \in S$. We put $a(s) = s$ and we find $t = b(s)$ so that any locally compact map $h : D \to K_s$ on a closed subset $D$ of $X$ extends to a locally compact map $h' : X \to K_t$. Therefore we can extend each $f|A \cap B_s$ to a locally compact $f_s : B_s \to K_t$. Suppose $f_T, a(T)$, and $b(T)$ exist for all $T$ with cardinality at most $n$. Given $T$ containing exactly $n + 1$ elements pick $s \in S$ so that $K_s$ contains all of $K_{b(F)}$ with $F$ being a proper subset of $T$. Put $a(T) = s$. We find $t = b(T)$ so that any locally compact map $h : D \to K_s$ on a closed subset $D$ of $X$ extends to a locally compact map $h' : X \to K_t$. All of $f_T, F$ a proper subset of $T$, can be pasted together and produce a locally compact (see 2.4) map $h$ on a closed subset $B$ of $B_T$ with values in $K_s$ and extending $f$ on $A \cap B$. Since $f(A \cap B_T) \subset K_s$, $h$ extends over $B_T$ and produces $f_T : B_T \to K_{b(T)}$ with the desired properties.

Since $B_T \cap B_F = B_{T \cap F}$, all $f_T$ can be pasted together to produce a function $f' : X \to K$ which is an extension of $f$. Any point $x \in X$ has a neighborhood $U$ which intersects only finitely many of $g_s^{-1}(0,1]$ which means that there is a finite set $T$ such that $U \subset B_T$. As $f'|B_T$ is locally compact, so is $f'|U$ which completes the proof.

2.10. Corollary. If $X$ is a paracompact space and $K$ is a contractible CW complex, then $K \in AE_{lc}(X)$.

Proof. Consider the cone $\text{Cone}(K)$ of $K$ with the weak topology. The family of cones of finite subcomplexes of $K$ forms a family satisfying hypotheses of 2.9. Since $K$ is a retract of its cone, $K \in AE_{lc}(X)$.

Our next result says that CW complexes are absolute neighborhood extensors of paracompact spaces if the class of locally compact maps is considered (notice that it does not make sense to talk about category of locally compact maps as identity $id_X : X \to X$ is locally compact if and only if $X$ is locally compact).

2.11. Corollary. If $X$ is a paracompact space, $K$ is a CW complex, and $f : A \to K$ is a locally compact map on a closed subset $A$ of $X$, then there exists a locally compact extension $f' : U \to K$ of $f$ over a neighborhood $U$ of $A$ in $X$. 


Proof. By 2.10 any locally compact map \( f : A \to K \), \( A \) closed in \( X \), extends to a locally compact \( g : X \to \text{Cone}(K) \). Let \( v \) be the vertex of \( \text{Cone}(K) \). Put \( U = g^{-1}(\text{Cone}(K) - \{v\}) \), \( r : \text{Cone}(K) - \{v\} \to K \) the canonical retraction, and \( f' = r \circ (g|U) \). \( \square \)

We will also need a Homotopy Extension Theorem for locally compact maps.

2.12. Corollary. Suppose \( X \) is a paracompact space, \( A \) is a closed subset of \( X \), and \( K \) is a CW complex. If \( H : A \times I \cup X \times \{0\} \to K \) is a locally compact map, then it extends to a locally compact \( H' : X \times I \to K \).

Proof. By 2.11 there is an open neighborhood \( V \) of \( A \times I \cup X \times \{0\} \) in \( X \times I \) and a locally compact extension \( G : V \to K \) of \( H \). Find a neighborhood \( U \) of \( A \) in \( X \) such that \( U \times I \subset V \) and pick a map \( a : X \to I \) such that \( a(A) \subset \{1\} \) and \( a(X - U) \subset \{0\} \). Notice that \( r : X \times I \to U \times I \cup X \times \{0\} \) defined by \( r(x,t) = (x,t \cdot r(x)) \) is continuous and is identity on \( A \times I \cup X \times \{0\} \). Therefore the composition \( H' = G \circ r \) is locally compact and extends \( H \). \( \square \)

Now we can reduce the question of extending a locally compact map to the question of extending it up to homotopy to an arbitrary, not necessarily locally compact, map.

2.13. Corollary. Suppose \( X \) is a paracompact space, \( A \) is a closed subset of \( X \), \( K \) is a CW complex, and \( f : A \to K \) is a locally compact map. The following conditions are equivalent:

a. \( f \) extends to a locally compact map \( f' : X \to K \).

b. \( f \) extends up to homotopy to a map \( f' : X \to K \).

Proof. \( a) \) is a special case of \( b) \).

\( b) \implies a) \). Suppose \( f : A \to K \) is a locally compact map and \( g : X \to K \) is a map such that \( g|A \) is homotopic to \( f \). Let \( H : A \times I \cup X \times \{1\} \to K \) be a map such that \( H(x,0) = f(x) \) for \( x \in A \) and \( H(x,1) = g(x) \) for \( x \in X \). 2.7 says that \( H \) is homotopic to a locally compact map \( H' \) in such a way that the homotopy from \( H \) to \( H' \) is locally compact on \( A \times \{0\} \). Concatenating \( H' \) with that homotopy produces a locally compact \( H'' : A \times I \cup X \times \{1\} \to K \) such that \( H''(x,0) = f(x) \) for \( x \in X \). By 2.12, \( H'' \) extends over \( X \times I \) which gives a locally compact extension of \( f \) over \( X \).

2.14. Definition. \( K \) is an absolute extensor up to homotopy of \( X \) if every map \( f : A \to K \), \( A \) closed in \( X \), extends over \( X \) up to homotopy.

2.13 means that, if \( X \) is paracompact and \( K \) is a CW complex, then \( K \in AE_{lc}(X) \) is equivalent to \( K \) being an absolute extensor of \( X \) up to homotopy. Our next result relates the concept of being an absolute extensor up to homotopy to the concept of being an absolute extensor in case of simplicial complexes.

2.15. Theorem. Suppose \( X \) is a paracompact space and \( K \) is a space. Consider the following conditions:

a. \( K \) is an absolute extensor of \( X \) up to homotopy.

b. \( K \in AE_0(X) \).

c. \( K \) is an absolute extensor of \( X \).

If \( K \) is an ANR for metrizable spaces, then Conditions \( a) \) and \( b) \) are equivalent. If \( K \) is complete ANR for metrizable spaces, then all three conditions are equivalent.

Proof. Assume \( K \) is an ANR for metrizable spaces.

\( a) \implies b) \). Suppose \( f : A \to K \) is a map, where \( A \) is a zero subset of \( X \). Since \( f \) extends over \( X \) up to homotopy, there is \( H : A \times I \cup X \times \{1\} \to K \) such that \( H(x,0) = f(x) \) for \( x \in X \) and \( H(x,1) = g(x) \) for \( x \in A \). By 2.12, \( H'' \) extends over \( X \times I \) which gives a locally compact extension of \( f \). Therefore, \( f \) extends up to homotopy to a map \( f' \). Since \( K \) is an ANR, \( f' \) extends to \( K \). Therefore, \( f \) extends to \( K \).

\( b) \implies c) \). Suppose \( f : A \to K \) is a map, \( A \) a zero subset of \( X \), and \( f \) extends to \( K \). By 2.12, \( f \) extends up to homotopy to \( K \).

\( c) \implies a) \). Suppose \( f : A \to K \) is a map, \( A \) a zero subset of \( X \), and \( f \) extends to \( K \) up to homotopy. By 2.12, \( f \) extends up to homotopy to \( K \). Therefore, \( f \) extends to \( K \).
x ∈ A. Notice that A × I ∪ X × {1} is a zero subset of X × I. Therefore we can find a map a : X × I → I such that A × I ∪ X × {1} = a^{-1}(0). Notice that K can be considered as a subset of some Banach space E. E is an absolute extensor of all paracompact spaces (see [Hu], Theorem 16.1b on p.63), so there is an extension G : X × I → E of H. Consider the subset K × {0} ∪ E × (0, 1] of E × I. Since K is an absolute neighborhood extensor of all metrizable spaces, there is a retraction r : U → K × {0} from a neighborhood U of K × {0} in K × {0} ∪ E × (0, 1]. Define F : X × I → K × {0} ∪ E × (0, 1] by G'(x, t) = (F(x, t), a(x, t)). U = F^{-1}(U) is a neighborhood of A × I ∪ X × {1} is a closed subset of X × I and r ∘ F is an extension of H over V. Therefore H extends over X × I which implies that f extends over X.

b) ⇒ a). Suppose f : A → K is a map from a closed subset of X. Since K is homotopy equivalent to a CW complex, 2.6-2.7 and 2.11 imply that there is a neighborhood U of A in X and a homotopy extension f' : U → K of f. Choose a map a : X → I such that a(A) ⊂ {0} and a(X − U) ⊂ {1}. Let B = a^{-1}(0). B is a zero subset of X. Since B ⊂ U, f'|B extends over X which proves that K is an absolute extensor of X up to homotopy.

Assume K is a complete ANR for metrizable spaces. Obviously, Condition c) is stronger than Condition b).

b) ⇒ c). Consider K as a subset of a Banach space E. Suppose f : A → K is a map from a closed subset of X. Since E is an absolute extensor of X, there is an extension F : X → E of f. Since K is a Gδ subset of E, F^{-1}(K) is a Gδ subset of X containing A. Therefore there is a zero subset B of X such that A ⊂ B ⊂ F^{-1}(K). Now, F|B extends over X which proves that K is an absolute extensor of X.

3. Extension dimension for paracompact spaces

The purpose of this section is to prove existence of extension dimension for paracompact spaces. It follows the same line of reasoning as in [Dr] for compact spaces or in [D-D1] for separable metrizable spaces. The difference is that 2.9 allows for a significant simplification of the argument.

3.1. Proposition. Suppose X is a paracompact space and \{K_s\}_{s ∈ S} is a family of pointed CW complexes. If each K_s is an absolute extensor of X up to homotopy, then the wedge K = ∨_{s ∈ S} K_s is an absolute extensor of X up to homotopy.

Proof. Let K_T = ∨_{s ∈ T} K_s for every finite subset T of S. K_T ∈ AE_{lc}(X) for all T implies K ∈ AE_{lc}(X) by 2.9. □

3.2. Proposition. Suppose X is a paracompact space and K ∈ AE_{lc}(X) is a CW complex. Let n be the density of X and let m be a cardinal number greater than or equal to max(2^n, 2^{ℵ_0}). For any subcomplex L of K containing at most m cells there is a subcomplex L' containing L such that

a. L' contains at most m cells,

b. Any locally compact map f : A → L, A closed in X, has a locally compact extension f' : X → L'.

Proof. Let Y be a dense subset of X with cardinality equal to n. Pick a point ∞ not belonging to K. List all functions from Y to L ∪ {∞}. There are at most m^n = m such functions. Keep only those functions g so that for some open set U_g there is a locally compact u_g : cl(U_g) → L so that g(x) = u_g(x) for x ∈ cl(U_g) ∩ Y and g(x) = ∞ for x ∈ X − cl(U_g). Pick an extension h_0 : X → K of u_g. The image h_0(X) contains at most m^n = m elements.
m cells, so by adding all of them we create a subcomplex $L'$ of $L$ containing at most $m$ cells.

Any locally compact $f : A \to L$ extends over an open neighborhood $U$ of $A$ in $X$. Let $f_1 : U \to L$ be such extension which is locally compact. Pick a neighborhood $V$ of $A$ in $X$ whose closure is contained in $U$. Let $g : Y \to L \cup \{\infty\}$ be defined by $g(x) = f_1(x)$ if $x \in Y \cap cl(V)$, $g(x) = \infty$ if $x \in Y - cl(V)$. The function $g$ has a locally compact map $h_g : X \to K$ and $cl(U_g) \cap Y$ must be equal to $cl(V) \cap Y$. Therefore $cl(U_g) = cl(V)$ and $h_g|A = f$. Thus, $f$ extends to a locally compact map from $X$ to $L'$. □

3.3. Corollary. Suppose $X$ is a paracompact space and $K \in AE_{lc}(X)$ is a CW complex. Let $n$ be the density of $X$ and let $m$ be a cardinal number greater than or equal to $\max(2^n, 2^{\aleph_0})$. For any subcomplex $L$ of $K$ containing at most $m$ cells there is a subcomplex $L'$ containing $L$ such that $L'$ contains at most $m$ cells and $L' \in AE_{lc}(X)$.

Proof. Put $L_1 = L$. Create, using 3.2, an increasing sequence of subcomplexes $L_n$ such that
a. $L_n$ contains at most $m$ cells,

b. Any locally compact map $f : A \to L_n$, $A$ closed in $X$, has a locally compact extension $f' : X \to L_{n+1}$.

Apply 2.9 to the family $\{L_n\}_{n \geq 1}$ and conclude that $L' = \bigcup_{n=1}^{\infty} L_n$ has the desired properties. □

3.4. Proof of 1.3.

Let $n$ be the density of $X$ and let $m$ be the cardinal number equal to $\max(2^n, 2^{\aleph_0})$. Pick a set of CW complexes containing at most $m$ cells so that any CW complex containing at most $m$ cells is listed there up to homeomorphism. Eliminate from that set CW complexes which are not absolute extensors of $X$ up to homotopy. Let $\{K_s\}_{s \in S}$ be the resulting set and put $K = \bigvee_{s \in S} K_s$. By 3.1 $K$ is an absolute extensor of $X$ up to homotopy. Suppose $L$ is a CW complex which is an absolute extensor of $X$ up to homotopy. We can express $L$ as the union of $\{L_t\}_{t \in T}$ of a partially ordered family of subcomplexes of $L$ such that each $L_t$ is homeomorphic to one of $K_s$ (see 3.3). If $K \in AE_{lc}(Y)$, then $K_s \in AE_{lc}(Y)$ for each $s \in S$ which implies $L \in AE_{lc}(Y)$ by 2.9. □

In practice one likes to be able to deal with absolute extensors rather than absolute extensors up to homotopy. We are able to produce the extension dimension of paracompact spaces by replacing CW complexes by complete simplicial complexes with the metric topology.

3.5. Proposition. For every CW complex $K$ there is a simplicial complex $L$ such that $|L|_m$ is complete, is homotopy equivalent to $K$, and the following two conditions are equivalent for any paracompact space $X$:

a. $K$ is an absolute extensor of $X$ up to homotopy.
b. $|L|_m \in AE(X)$.

Proof. Find a simplicial complex $M$ such that $|M|_m$ is homotopy equivalent to $K$ (see [M-S]). Triangulate $\bigcup_{n=1}^{\infty} |M^{(n)}|_m \times [n, \infty)$ as $|L|_m$ for some simplicial complex $L$. Clearly, $|L|_m$ is homotopy equivalent to $K$. Suppose it is an absolute extensor of $X$ up to homotopy. Notice that $L$ does not contain any full infinite subcomplex. Therefore $|L|_m$ is complete and 2.15 implies that $|L|_m$ is an absolute extensor of $X$. □
3.6. Proofs of 1.4 and 1.5.

By 1.3 there is a CW complex $K'$ such that
1. $K'$ is an absolute extensor of $X$ up to homotopy,
2. If a CW complex $L$ is an absolute extensor of $X$ up to homotopy, then $L$ is an absolute
tensor of $Y$ up to homotopy of any paracompact space $Y$ such that $K'$ is an absolute
tensor of $Y$ up to homotopy.

Pick a simplicial complex $K$ such that $|K|_m$ is complete, is of the same homotopy type as $K'$, and $|K|_m \in AE(X)$ (see 3.5).

Suppose $L$ is a complete ANR such that $L \in AE(X)$. Choose a CW complex $L'$ of the
same homotopy type as $L$. Suppose $Y$ is a paracompact space such that $|K|_m \in AE(Y)$. Now $K'$ is an absolute tensor of $Y$ up to homotopy and $L'$ is an absolute tensor of $X$ up to homotopy. Therefore $L'$ is an absolute tensor of $Y$ up to homotopy. Since $L$ is homotopy equivalent to $L'$, $L$ is an absolute tensor of $Y$ up to homotopy. By 2.15, $L \in AE(Y)$.

Suppose $L$ is an ANR such that $L \in AE_0(X)$. By 2.15, $L$ is an absolute tensor of $X$ up to homotopy. Choose a CW complex $L'$ of the same homotopy type as $L$. Suppose $Y$ is a paracompact space such that $|K|_m \in AE(Y)$. Now $K'$ is an absolute tensor of $Y$ up to homotopy and $L'$ is an absolute tensor of $X$ up to homotopy. Therefore $L'$ is an absolute tensor of $Y$ up to homotopy. Since $L$ is homotopy equivalent to $L'$, $L$ is an absolute tensor of $Y$ up to homotopy. By 2.15, $L \in AE_0(Y)$.

The Duality Theorem of Dranishnikov [Dr] says that each CW complex is equal to the
extension dimension of some compact Hausdorff space in the sense of Definition 1.1. It is
natural to ask if the same is true in the category of paracompact spaces.

3.7. Problem. Suppose $K$ is a CW complex. Is there a paracompact space $X$ so that $K$
is the extension dimension of $X$?

An obvious approach to solve 3.7 is to produce a compact space for $K$ as in [Dr]. The
remainder of this section is devoted to explaining why this approach fails by showing
paracompact spaces whose extension dimension is not the same as of a compact space.

3.8. Definition. If $K$ and $L$ are CW complexes, then $K \leq L$ means $L$ is an absolute
tensor up to homotopy of any paracompact space $X$ such that $K$ is an absolute tensor of $X$. This leads to an equivalence relation $\sim$ on the category of all CW complexes.

For any paracompact space $X$, ext–dim$(X)$ stands for its extension dimension in the
sense of 1.3 and is unique up to equivalence $\sim$. Now, for any paracompact spaces $X$ and
$Y$, $X \leq Y$ means ext–dim$(X) \leq$ ext–dim$(Y)$ and introduces a partial order on the class
of all paracompact spaces.

Let us present a view of the Stone–Čech compactification from the point of absolute
tensors.

3.9. Proposition. In the class of normal spaces let $X \leq_f Y$ mean that any finite CW
complex $K$ which is an absolute tensor of $Y$ must also be an absolute tensor of $X$. Suppose $X$
is a normal space. The class $\{Y \mid Y \leq_f X$ and $Y$ is compact$\}$ has $\beta(X)$ as its
maximum. Moreover, $X \leq_f \beta(X)$.

Proof. 3.9 is well-known in the form: $X$ and $\beta(X)$ have the same compact absolute
tensors. Let us sketch a proof for the sake of completeness. Suppose $K \in AE(\beta(X))$. Any map $f : A \to K, A$ closed in $X$ extends over $\beta(A)$ which is a closed subset of $\beta(X)$.
Therefore \( f \) extends over \( \beta(X) \) and \( K \in AE(X) \). Suppose \( K \in AE(X) \) and \( f : A \to K \) is a map, \( A \) closed in \( \beta(X) \). We can extend \( f \) over a closed neighborhood \( B \) of \( A \) in \( \beta(X) \).

Let \( g : X \to K \) be an extension of \( f|B \cap X \). Since \( K \) is compact, \( g \) extends over \( \beta(X) \). Let \( h : \beta(X) \to K \) be such extension. As \( h \) and \( g \) coincide on \( \text{Int}(B) \cap X \), they must coincide on \( \text{Int}(B) \). In particular, \( h \) is an extension of \( f \). \( \square \)

Here is an extension theory analog of the Stone-Čech compactification.

**3.10. Theorem.** Suppose \( X \) is a paracompact space. The class \( \{ Y \mid Y \leq X \text{ and } Y \text{ is compact} \} \) has a maximum \( X' \). There are separable metrizable spaces \( X \) such that \( X' < X \).

**Proof.** Let \( K \) be the extension dimension of \( X \). Let \( X' \) be a compact Hausdorff space such that \( K \in AE(X') \) and \( L \in AE(X') \), \( L \) a CW complex, implies \( L \in AE(Y) \) for any compact Hausdorff space \( Y \) such that \( K \in AE(Y) \) (see [Dr]). Since \( K \in AE(X') \), \( X' \leq X \).

If \( Y \leq X \) for some compact Hausdorff space \( Y \), then it simply means \( K \in AE(Y) \). To prove \( Y \leq X' \) consider \( M = \text{ext–dim}(X') \). We need \( M \in AE(Y) \) which follows from the way \( X' \) was chosen.

In [D-D₂], Theorem 4.7, it is shown that if \( G \) is a countable abelian group, and \( A_p \) is the ring of \( p \)-adic integers for some prime number \( p \), then there is a separable space \( X \) of dimension 2 such that \( \dim_G X \neq \dim_{A_p} X \). Consider \( G \) to be \( Z \) localized at \( p \) (all rational numbers with denominators relatively prime to \( p \)). Now, \( \text{ext–dim}(X') = \text{ext–dim}(X) \) implies \( \dim_G X' \neq \dim_{A_p} X' \) which is impossible for compact spaces (see [Ku]). \( \square \)

### 4. Union theorem for paracompact spaces

In this section we prove the Union Theorem for paracompact spaces, thus demonstrating that our extension theory of paracompact spaces is quite natural.

To make sure that the approach in [Dy₁] works we need the following result.

**4.1. Lemma.** Suppose \( A \) is a subset of a hereditarily paracompact space \( X \). Any map \( f : A \to K \) from \( A \) to a CW complex \( K \) extends up to homotopy over a neighborhood of \( A \) in \( X \).

**Proof.** It suffices to consider the case of \( f \) being locally compact and \( K = |L|_w \) for some simplicial complex \( L \). Let \( \{U_s\}_{s \in S} \) be a family of open sets in \( X \) such that \( A \subset U = \bigcup_{s \in S} U_s \) and \( f(A \cap U_s) \) is contained in a compact subset of \( K \) for each \( s \in S \). Pick a locally finite partition \( \{g_s\}_{s \in S} \) on \( U \) (\( U \) is a paracompact space) such that \( g_s^{-1}(0, 1] \subset U_s \) for each \( s \in S \). \{g_s\}_{s \in S} \) may be viewed as a locally compact map \( g : U \to |L'|_w \), where \( L' \) is the full simplicial complex with the same vertices as \( L \). Notice that \( g|A \) is homotopic to \( f \) as maps to \( |L|_w \). Pick a locally compact map \( h : |L'|_w \to |L|_w \) homotopic to identity and extend it over a neighborhood \( V \) of \( |L|_w \) in \( |L'|_w \). Now, the composition of \( g^{-1}(V) \to V \to |L|_w \) extends \( f \) up to homotopy. \( \square \)

**4.2. Lemma.** Suppose \( A \) is an \( F_\sigma \)-subset of a paracompact space \( X \). If \( K \) is a CW complex which is an absolute extensor of \( X \) up to homotopy, then \( K \) is an absolute extensor of \( A \) up to homotopy.

**Proof.** \( A \) is paracompact by 5.1.28 of [En]. Suppose \( A = \bigcup_{i=1}^{\infty} B_n \), where \( B_n \) is a closed subset of \( X \) for each \( n \). We may assume that \( B_n \subset B_{n+1} \) for each \( n \). Suppose \( C \) is a closed subset of \( A \). Pick a closed subset \( D \) of \( X \) such that \( C = D \cap A \). Suppose \( f : C \to K \) is a locally compact map to a CW complex. Extend \( f \) over a closed neighborhood \( C_1 \) of \( C \) in \( A \), then use the fact that \( K \in AE(K(B_n)) \) to extend it over \( C_1 \cup B_n \). The resulting map
map $f_1 : C_1 \cup B_1 \to K$ is locally compact by 2.4. Suppose we have a locally compact map $f_n : C_n \cup B_n$ such that $C_n$ is a closed neighborhood of $C_{n-1} \cup B_{n-1}$ in $A$. Extend it over a closed neighborhood $C_{n+1}$ of $C_n \cup B_n$ and use the fact that $K \in \overline{AE}_{lc}(B_{n+1})$ to extend it over $C_{n+1} \cup B_{n+1}$. The resulting map $f_{n+1} : C_{n+1} \cup B_{n+1} \to K$ is locally compact by 2.4. The direct limit $f'$ of maps $f_n$ is an extension of $f$ and is locally compact. Indeed, given $x \in A$ we find the smallest $n$ such that $x \in C_n \cup B_n$. $f'(x)$ equals $f_n(x)$. Since $C_{n+1}$ is a closed neighborhood of $C_n \cup B_n$ and $f_{n+1}$ is locally compact, there is a neighborhood $U$ of $x$ in $A$ such that $f_{n+1}(U) = f'(U)$ is contained in a compact subset of $K$. \hfill \Box

4.3. Theorem. Suppose $X$ is a hereditarily paracompact space. Let $K$ and $L$ be CW complexes. If $K$ is an absolute extensor of $A \subset X$ up to homotopy and $L$ is an absolute extensor of $B \subset X$ up to homotopy, then the join $K \star L$ is an absolute extensor of $A \cup B$ up to homotopy.

Proof. It suffices to consider $X = A \cup B$. We may assume that both $K$ and $L$ are simplicial complexes equipped with CW topology, $K = |K'|_w$ and $L = |L'|_w$. We will be working with locally compact maps which are ideal for the following reason: if $f : Y \to |M|_m$ is a map such that every $y \in Y$ has a neighborhood $U$ with $f(U)$ contained in a finite subcomplex of $|M|_m$, then $f$ considered as a function from $Y$ to $|M|_w$ is continuous.

Suppose $C$ is a closed subset of $A \cup B$ and $f : C \to K \star L$ is a locally compact map. Notice that $f$ defines two closed, disjoint subsets $C_K = f^{-1}(K)$, $C_L = f^{-1}(L)$ of $C$ and locally compact maps $f_K : C - C_L \to K$, $f_L : C - C_K \to L$, $\alpha : C \to [0,1]$ such that:

1. $\alpha^{-1}(0) = C_K$, $\alpha^{-1}(1) = C_L$,
2. $f(x) = (1 - \alpha(x)) \cdot f_K(x) + \alpha(x) \cdot f_L(x)$ for all $x \in C$.

Indeed, each point $x$ of a simplicial complex $M$ can be uniquely written as $x = \sum_{v \in M^{(0)}} \phi_v(x) \cdot v$, where $M^{(0)}$ is the set of vertices of $M$ ($\{\phi_v(x)\}$ are called barycentric coordinates of $x$). We define $\alpha(x)$ as $\sum_{v \in L^{(0)}} \phi_v(f(x))$, $f_K(x)$ is defined as $\sum_{v \in K^{(0)}} \phi_v(f(x)) \cdot v / (1 - \alpha(x))$ and $f_L(x)$ is defined as $\sum_{v \in L^{(0)}} \phi_v(f(x)) \cdot v / (\alpha(x))$.

Since $K \in \overline{AE}_{lc}(A - C_L)$ by 4.2, $f_K$ extends over $(C \cup A) - C_L$. To make sure that there is a locally compact extension we proceed as follows: first extend $f_K$ over a closed neighborhood $D$ of $C - C_L$ in $(C \cup A) - C_L$. Let $u : B \to K$ be a locally compact extension of $f_K|(C - C_L)$. Extend $u|B \cap (A - C_L)$ to a locally compact $v : A - C_L \to K$. Pasting $v$ and $f_K$ results in a locally compact map.

Consider a homotopy extension $g_K : U_A \to K$ of $f_K$ over a neighborhood $U_A$ of $(C \cup A) - C_L$ in $X - C_L$. Since $C - C_L$ is closed in $U_A$, we may assume that $g_K$ is an actual extension of $f_K : C - C_L \to K$ (see 2.13). Similarly, let $g_L : U_B \to L$ be an extension of $f_L$ over a neighborhood $U_B$ of $(C \cup B) - C_K$ in $X - C_K$. Notice that $X = U_A \cup U_B$. Let $\beta : X \to [0,1]$ be an extension of $\alpha$ such that $\beta(X - U_B) \subset \{0\}$ and $\beta(X - U_A) \subset \{1\}$. Define $f' : X \to K \star L$ by

\[ f'(x) = (1 - \beta(x)) \cdot g_K(x) + \beta(x) \cdot g_L(x) \quad \text{for all } x \in U_A \cap U_B, \]

\[ f'(x) = g_K(x) \quad \text{for all } x \in U_A - U_B, \]

and

\[ f'(x) = g_L(x) \quad \text{for all } x \in U_B - U_A. \]

Notice that $f'$ is an extension of $f$. Now, it suffices to prove that $f' : X \to |K' \star L'|_m$ is continuous. Indeed, as identity $|K' \star L'|_w \to |K' \star L'|_m$ is a homotopy equivalence it would certify the existence of an extension of $f : C \to |K' \star L'|_w$ up to homotopy which is all we need in view of 2.13.
To prove the continuity of \( f' : X \rightarrow |K' \ast L'|_m \) we need to show that \( \phi_v f' \) is continuous for all vertices \( v \) of \( K' \ast L' \) (see [M-S, Theorem 8 on p.301]). Without loss of generality we may assume that \( v \in K' \). Then,

\[
\phi_v f'(x) = (1 - \beta(x)) \cdot \phi_v g_K(x) \quad \text{for all } x \in U_A
\]

and

\[
\phi_v f'(x) = 0 \quad \text{for all } x \in U_B - U_A.
\]

Clearly, \( \phi_v f'|U_A \) is continuous. Suppose \( x_0 \in (U_B - U_A) \cap \text{cl}(U_A) \) and \( M > 0 \). Since \( \phi_v f'(x_0) = 0 \), it suffices to show existence of a neighborhood \( W \) of \( x_0 \) such that \( \phi_v f'(W) \subseteq [0, M) \). As \( \beta(x_0) = 1 \), there is a neighborhood \( W \) of \( x_0 \) so that \( \beta(W) \subseteq (1 - M, 1] \). If \( x \in W \cap (U_B - U_A) \), then \( \phi_v f'(x) = 0 \). If \( x \in W \cap U_A \), then \( \phi_v f'(x) = (1 - \beta(x)) \cdot \phi_v g_K(x) \leq 1 - \beta(x) < M \). \( \square \)

5. Spaces with all maps being locally compact

It is of interest to see which maps to CW complexes are locally compact.

5.1. Problem. Characterize all paracompact spaces \( X \) so that any map \( f : A \rightarrow K \), \( A \) closed in \( X \) and \( K \) a CW complex, is locally compact.

This section is devoted to partial answers to 5.1.

5.2. Proposition. Suppose \( f : X \rightarrow Y \) is a perfect map and \( X \) is a paracompact space. If every map from \( X \) to a CW complex is locally compact, then every map from \( Y \) to a CW complex is locally compact.

**Proof.** Suppose \( g : Y \rightarrow K \) is a map from \( Y \) to a CW complex. Let \( y_0 \in Y \). Since \( g \circ f \) is locally compact, for each \( x \in f^{-1}(y_0) \) there is a neighborhood \( U_x \) such that \( g f(U_x) \) is contained in a compact subset \( Z_x \) of \( K \). As \( f^{-1}(y_0) \) is compact, \( f^{-1}(y_0) \subset \bigcup U_x \) for some finite subset \( F \) of \( f^{-1}(y_0) \). Since \( f \) is closed there is a neighborhood \( U \) of \( y_0 \) in \( Y \) with \( f^{-1}(U) \subset \bigcup U_x \). Now \( g(U) = g f(f^{-1}(U)) \subset g f(\bigcup_{x \in F} U_x) \subset \bigcup_{x \in F} Z_x \) which proves that \( g \) is locally compact. \( \square \)

5.3. Proposition. Suppose \( A \) is a subset of \( X \) and has a countable basis of neighborhoods. If \( f : X \rightarrow K \) is a map to a CW complex such that \( f(A) \) is contained in a compact subset of \( K \), then there is a neighborhood \( U \) of \( A \) in \( X \) such that \( f(U) \) is contained in a compact subset of \( K \).

**Proof.** There is a finite subcomplex \( K_0 \) of \( K \) containing \( f(A) \). Choose a basis of neighborhoods \( \{U_n\}_{n \geq 1} \) of \( A \) in \( X \). Suppose none of \( f(U_n) \) is contained in a finite subcomplex of \( K \). Choose, by induction, elements \( w_n \in f(U_n) \) so that the smallest subcomplex of \( K \) containing \( K_0 \) and \( w_1, \ldots, w_{n-1} \) does not contain \( w_n \). The set \( C = \{w_i\}_{i \geq 1} \) is closed in \( K \) and misses \( K_0 \), so \( f^{-1}(C) \) is closed and misses \( A \). Pick \( m \) so that \( U_m \subset X - f^{-1}(C) \). Now \( w_m \in K - C \), a contradiction. \( \square \)

5.4. Corollary. If \( X \) is the union of its compact subsets which have a countable basis of neighborhoods, then any map from \( X \) to a CW complex is locally compact.

**Remark.** Hausdorff spaces \( X \) such that every point is contained in a compact subset \( Z \) with countable basis of neighborhoods are discussed in [En] (Exercise 3.1.E to section 1 of chapter 2) under the name of pointwise countable type. The class of such spaces is closed under perfect maps and inclusions.
contains locally compact spaces, first countable spaces, is closed under finite cartesian products, is hereditary with respect to closed subsets, and is hereditary with respect to Gδ-subsets (in particular, all topologically complete spaces belong to the class). It is also easy to show that if \( f : X \to Y \) is a perfect map and \( Y \) belongs to the class, than \( X \) belongs to the class.

**5.5. Problem.** Suppose \( X \) is a paracompact space such that any map from a closed subset \( A \) of \( X \) to a CW complex is locally compact. Let be \( Y \) a compact space. Is every map from a closed subset \( A \) of \( X \times Y \) to a CW complex locally compact?

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