Local Well-posedness for the Magnetohydrodynamics in the different two liquids case

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Abstract

We prove the local well-posedness for a two phase problem of magnetohydrodynamics with a sharp interface. The solution is obtained in the maximal regularity space: $H^1_2((0,T), L^q) \cap L^p((0,T), H^2_2)$ with $1 < p, q < \infty$ and $2/p + N/q < 1$, where $N$ is the space dimension.

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1 Introduction

We consider a two phase problem governing the motion of two incompressible electrically conducting capillary liquids separated by a sharp interface. The problem is formulated as follows: Let $\Omega$ be two reference domains in the $N$-dimensional Euclidean space $\mathbb{R}^N$ ($N \geq 2$). Assume that the boundary of each $\Omega_\pm$ consists of two connected components $\Gamma$ and $S_\pm$, where $\Gamma$ is the common boundary of $\Omega_\pm$. Throughout the paper, we assume that $\Gamma$ is a compact hypersurface of $C^3$ class, that $S_\pm$ are hypersurfaces of $C^2$ class, and that $\text{dist}(\Gamma, S_\pm) \geq d_\pm$ with some positive constants $d_\pm$, where the dist($A, B$) denotes the distance of any subsets $A$ and $B$ of $\mathbb{R}^N$ which is defined by setting $\text{dist}(A, B) = \inf \{ |x-y| \mid x \in A, y \in B \}$.

Let $\Omega = \Omega_+ \cup \Gamma \cup \Omega_-$ and $\Omega = \Omega_+ \cup \Omega_-$. The boundary of $\Omega$ is $S_+ \cup S_-$. We may consider the case that one of $S_\pm$ is an empty set, or that both of $S_\pm$ are empty sets. Let $\Gamma_t$ be an evolution of $\Gamma$ for time $t > 0$, which is assumed to be given by

$$\Gamma_t = \{ x = y + h(y, t) n(y) \mid y \in \Gamma \}$$

(1.1)

with an unknown function $h(y, t)$. We assume that $h|_{t=0} = h_0(y)$ is a given function. Let $\Omega_{t, \pm}$ be two connected components of $\Omega \setminus \Gamma_t$ such that the boundary of $\Omega_{t, \pm}$ consists of $\Gamma_t$ and $S_{t, \pm}$. Let $n_t$ be the unit outer normal to $\Gamma_t$ oriented from $\Omega_{t, +}$ into $\Omega_{t, -}$, and let $n_{t, \pm}$ be respective the unit outer normal to $S_{t, \pm}$. Given any functions, $v_{\pm}$, defined on $\Omega_{t, \pm}$, $v$ is defined by $v(x) = v_{\pm}(x)$ for $x \in \Omega_{t, \pm}$ for $t \geq 0$, where $\Omega_{0, \pm} = \Omega_{t, \pm}$. Moreover, what $v = v_{\pm}$ denotes that $v(x) = v_{\pm}(x)$ for $x \in \Omega_{t, +}$ and $v(x) = v_{\pm}(x)$ for $x \in \Omega_{t, -}$.

Let

$$[[v]](x_0) = \lim_{x \rightarrow x_0^{t, +}} v_{\pm}(x) - \lim_{x \rightarrow x_0^{t, -}} v_{\pm}(x)$$

for every point $x_0 \in \Gamma_t$, which is the jump quantity of $v$ across $\Gamma$. Let $\hat{\Omega} = \Omega_+ \cup \Omega_-$ and $\hat{\Omega} = \Omega_{t, +} \cup \Omega_{t, -}$.

The purpose of this paper is to prove the local well-posedness of the magnetohydrodynamical equations with interface condition in the different two liquids case, which are formulated by the set of the following equations:

$$\rho(\partial_t v + v \cdot \nabla v) - \text{Div}(T(v, p) + T_M(H)) = 0, \quad \text{div} v = 0 \quad \text{in} \quad \bigcup_{0 < t < T} \hat{\Omega}_t \times \{ t \},$$

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Here, \( v = v_\pm = (v_{\pm 1}(x,t), \ldots, v_{\pm N}(x,t))^T \) are the velocity vector fields, where \( M^T \) stands for the transposed \( M \), \( p = p_\pm(x,t) \) the pressure fields, and \( H = H_\pm = (H_{\pm 1}(x,t), \ldots, H_{\pm N}(x,t))^T \) the magnetic fields. The \( v, p, H \) and \( \Gamma_t \) are unknown, while \( v_0, H_0 \) and \( \rho_0 \) are prescribed \( N \)-component vectors. As for the remaining symbols, \( T(v, p) = v_\pm D(v_\pm) - p_\pm I \) is the viscous stress tensor, \( D(v_\pm) = \nabla v_\pm + (\nabla v_\pm)^T \) is the doubled deformation tensor whose \((i,j)\)th component is \( \partial_i v_{\pm j} + \partial_j v_{\pm i} \) and \( \partial_i = \partial/\partial x^i \), \( I \) the \( N \times N \) unit matrix, \( T_M(H) = T_M(H_\pm) = \mu_\pm \left( H_\pm \otimes H_\pm - \frac{1}{2} |H_\pm|^2 I \right) \) the magnetic stress tensor, \( \text{curl} v = \text{curl} v_\pm = \nabla v_\pm - (\nabla v_\pm)^T \) the doubled rotation tensor whose \((i,j)\)th component is \( \partial_i v_{\pm j} - \partial_j v_{\pm i} \), \( V_t \) the velocity of the evolution of \( \Gamma_t \) in the direction of \( n_\pm \), which is given by \( V_t = (\partial_t \rho) n_\pm \) in the case of \([11]\), and \( \mathcal{H}(\Gamma_t) N = 1 \) fold mean curvature of \( \Gamma_t \). In particular, in the three dimensional case, \( \partial_t \rho + \text{div} (\rho \mathbf{u}) = 0 \), \( \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mathbf{f} \) for any \( \mathbf{f} \) vector field, \( \partial_t \mathbf{u} = \frac{1}{ho} \left( -\nabla p + \mathbf{f} \right) \) in the three dimensional case.

\[
\begin{align*}
&\Delta v = -\text{div curl } v + \nabla \text{div } v, \quad \text{Div} (v \otimes H - H \otimes v) = \text{vd} \text{iv } H - \text{Hdiv } v + v \cdot \nabla H, \\
&\text{rot x rot } H = \text{Div} \text{curl } H, \quad \text{rot x v x H} = \text{Div} (v \otimes H - H \otimes v) \quad (3 \text{ dimensional case}),
\end{align*}
\]  

where \( \times \) is the exterior product in the three dimensional case. In particular, in the three dimensional case, the set of equations for the magnetic field in Eq. (1.2) are written by

\[
\partial_t H + \text{rot x (} \alpha^{-1} \text{rot } H - \mu v \times H) = 0, \quad \text{div } H = 0 \quad \text{in} \quad \bigcup_{0<t<T} \bar{\Omega}_t \times \{t\},
\]

\[
[[n_\pm \times \{\alpha^{-1} \text{rot } H - \mu v \times H\}]] = 0, \quad [[\mu H \cdot n_\pm]] = 0, \quad [[H - < H, n_\pm > n_\pm]] = 0 \quad \text{on} \quad \bigcup_{0<t<T} \Gamma_t \times \{t\}.
\]

This is a standard description, and so the set of equations for the magnetic field in Eq. (1.2) is the \( N \)-dimensional mathematical description for the magnetic fields with transmission conditions.

In the equilibrium state, \( v = 0, H = 0, \Gamma_t = \Gamma, \) and \( p \) is a constant state, and so we assume that

\[
p_0 = \sigma H(\Gamma).
\]  

In Eq. (1.2), there is one equation for the magnetic fields \( H_\pm \) too many, so that the equation instead of (1.2) is considered:

\[
\rho \partial_t v + v \cdot \nabla v - \text{div} \left( T(v, p) + T_M(H) \right) = 0, \quad \text{div } v = 0 \quad \text{in} \quad \bigcup_{0<t<T} \bar{\Omega}_t \times \{t\},
\]

\[
[[T(v, p) + T_M(H)]n_\pm] = \sigma H(\Gamma_t)n_\pm - p_0 n_\pm, \quad [[v]] = 0, \quad V_{\Gamma_t} = v_+ \cdot n \quad \text{on} \quad \bigcup_{0<t<T} \Gamma_t \times \{t\},
\]

\[
\partial_t H - \alpha^{-1} \Delta H - \text{Div } \mu (v \otimes H - H \otimes v) = 0 \quad \text{in} \quad \bigcup_{0<t<T} \bar{\Omega}_t \times \{t\},
\]

\[
[[\alpha^{-1} \text{curl } H - \mu (v \otimes H - H \otimes v)]n_\pm]] = 0, \quad [[\mu \text{div } H]] = 0 \quad \text{on} \quad \bigcup_{0<t<T} \Gamma_t \times \{t\},
\]
\[ [[\mu H \cdot n]] = 0, \quad [[H - < H, n>] = 0 \text{ on } \bigcup_{0 < t < T} \Gamma_t \times \{t\}, \]
\[ v_\pm = 0, \quad n_0 \cdot H_\pm = 0, \quad (\text{curl } H_\pm) n_\pm = 0 \text{ on } S_\pm \times (0, T), \]
\[ (v, H)|_{t=0} = (v_0, H_0) \text{ in } \Omega. \]

(1.5)

Namely, two equations: div $H_\pm = 0$ in $\Omega_\pm$ is replaced with one boundary condition: $[[\mu \text{div } H]] = 0$ on $\Gamma$. Frolova and Shibata [5] proved that in equations (1.5) if div $H = 0$ initially, then div $H = 0$ in $\Omega$ follows automatically for any $t > 0$ as long as solutions exist. Thus, the local wellposedness of equations (1.2) follows from that of equations (1.5) provided that the initial data $H_0$ satisfy the divergence condition: div $H_0 = 0$, which is a compatibility condition. This paper devotes to proving the local wellposedness of equations (1.5) in the maximal $L_p$-$L_q$ regularity framework under the assumption that $\rho_0$ is small enough, that is the interface $\Gamma_1$ is very close to the reference interface $\Gamma$ initially.

Since $\hat{\Omega}$ and $\Gamma$ are unknown, the set of equations in (1.5) is transformed to that of equations in $\hat{\Omega}$ and $\Gamma$ by the Hanzawa transform generated by $\rho$ (cf. Subsect. 2.4 below), and then the main result are stated in Subsect. 2.5 below.

The equations of magnetohydrodynamics (MHD) can be found in [1]. The solvability of MHD equations was first obtained by Padula and Solonnikov [11] in the case where $\Omega_{\text{ext}}$ is a vacuum region in the three dimensional Euclidean space $\mathbb{R}^3$. They proved the local well-posedness in the $L_2$ framework and used Sobolev-Slobodetskii spaces of fractional order. Later on, the global well-posedness was proved by Frolova [2] and Solonnikov and Frolova [20]. Moreover, the $L_p$ approach to the same problem was done by Solonnikov [18] [19] in [11]. by some technical reason, it was required that regularity class of the fluid is slightly higher than that of the magnetic field (cf. [11] p.331]). But, in this paper, we do not need this assumption, that is we can solve the problem in the same regularity classes for the fluid and magnetic field. The different point of this paper than in [11] appears in the iteration scheme (cf. (2.3) and (4.5)).

As a related topics, in [6, 7] and references therein Kacprzyk proved the local and global well-posedness of free boundary problem for the viscous non-homogeneous incompressible MHD in the case where an incompressible fluid is occupied in a domain $\Omega_{\text{ext}}$ bounded by a free surface $\Gamma_1$ subjected to an electromagnetic field generated in a domain $\Omega_{\text{int}}$ exterior to $\Omega_{\text{ext}}$ by some currents located on a fixed boundary $S_0$ of $\Omega_{\text{int}}$. In [6, 7], it is assumed that $S_0 = \emptyset$. On the free surface, $\Gamma_1$, free boundary condition without surface tension is not taken into account, the Lagrange transformation was applied, and so the viscous fluid part has one regularity higher than the electromagnetic fields part. An $L_2$ approach is applied and Sobolev-Slobodetskii spaces of fractional order are also used in [6, 7].

Finally, we explain some symbols used throughout the paper. We denote the set of all natural numbers, real numbers, and complex numbers by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$, respectively. Set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any multi-index $\kappa = (\kappa_1, \ldots, \kappa_N)$, $\kappa \in \mathbb{N}_0$, we set $\partial^\kappa = \partial_1^{\kappa_1} \cdots \partial_N^{\kappa_N}$, $|\kappa| = \kappa_1 + \cdots + \kappa_N$. For scalar $f$, and $N$-vector of functions, $g = (g_1, \ldots, g_N)$, we set $\nabla^N f = (\partial^\kappa f \mid |\kappa| = n) \text{ and } \nabla^N g = (\partial^\kappa g_j \mid |\kappa| = n, j = 1, \ldots, N)$. In particular, $\nabla f = f$, $\nabla^N g = g$, $\nabla^N f = f$, and $\nabla^N g = g$. For $1 \leq s \leq \infty$, $m \in \mathbb{N}_0$, $n \in \mathbb{R}$, and any domain $D \subset \mathbb{R}^N$, we denote by $L_{q}(D)$, $H_{q}^{n}(D)$, and $B_{q, p}^{s}(D)$ the standard Lebesgue, Sobolev, and Besov spaces, respectively, while $\| \cdot \|_{L_{q}(D)}$, $\| \cdot \|_{H_{q}^{n}(D)}$, and $\| \cdot \|_{B_{q, p}^{s}(D)}$ denote the norms of these spaces. We write $W_{q}^{n}(D) = B_{q, q}^{s}(D)$ and $H_{q}^{n}(D) = L_{q}(D)$. What $f = f_{\pm}$ means that $f(x) = f_{\pm}(x)$ for $x \in \Omega_{\pm}$. For $\mathcal{H} \in \{H_{q}^{n}, B_{q, p}^{s}\}$, the function spaces $\mathcal{H}(\hat{\Omega})$ ($\hat{\Omega} = \Omega_{\text{ext}} \cup \Omega_{\text{int}}$) are their norms defined by setting

\[
\mathcal{H}(\hat{\Omega}) = \{ f = f_{\pm} \mid f_{\pm} \in \mathcal{H}(\Omega_{\pm}) \}, \quad \| f \|_{\mathcal{H}(\hat{\Omega})} = \| f_{+} \|_{\mathcal{H}(\Omega_{\text{ext}})} + \| f_{-} \|_{\mathcal{H}(\Omega_{\text{int}})}.
\]

For any Banach space $X$ with the norm $\| \cdot \|_{X}$, $X^d$ denotes the product space defined by $\{ x = (x_1, \ldots, x_d) \mid x_j \in X \}$, while the norm of $X^d$ is simply written by $\| \cdot \|_{X}$, that is $\| x \|_{X} = \sum_{j=1}^{d} \| x_j \|_{X}$. For any time interval $(a, b)$, $L_{p}(a, b, X)$ and $H_{p}^{n}(a, b, X)$ denote the respective standard $X$-valued Lebesgue space and $X$-valued Sobolev space, while $\| \cdot \|_{L_{p}(a, b, X)}$ and $\| \cdot \|_{H_{p}^{n}(a, b, X)}$ denote their norms. Let $J$ and $F^{-1}$ be respective the Fourier transform and the Fourier inverse transform. Let $H_{p}^{n}(\mathbb{R}, X)$, $s > 0$, be the Bessel potential space of order $s$ defined by

\[
H_{p}^{n}(\mathbb{R}, X) = \{ f \in L_{p}(\mathbb{R}, X) \mid \| f \|_{H_{p}^{n}(\mathbb{R}, X)} = \| F^{-1}[(1 + |\tau|^2)^{1/4} F[f](\tau)] \|_{L_{p}(\mathbb{R}, X)} < \infty \}.
\]

For any $N$-vector of functions, $u = \{ u_1, \ldots, u_N \}$, sometimes $\nabla u$ is regarded as an $N \times N$-matrix of functions whose $(i, j)$th component is $\partial_j u_i$. For any $m$-vector $V = (v_1, \ldots, v_m)$ and $n$-vector $W = (w_1, \ldots, w_n)$, $V \otimes W$ denotes an $mn \times n$ matrix whose $(i, j)$th component is $V_i W_j$. For any $(mn \times N)$-matrix $A = (A_{ij, k} \mid i = 1, \ldots, m, j = 1, \ldots, n, k = 1, \ldots, N)$, $AV \otimes W$ denotes an $N$-column vector whose $k$th
component is the quantity: $\sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} \psi_i \psi_j$. Moreover, we define $AV \otimes W \otimes Z = (AV \otimes W) \otimes Z$. Inductively, we define $AV_1 \otimes \cdots \otimes V_n$ by setting $AV_1 \otimes \cdots \otimes V_{n-1} = (AV_1 \otimes \cdots \otimes V_{n-1}) \otimes V_n$ for $n \geq 4$.

Let $a \cdot b := a_1 b_1 + \cdots + a_n b_n$ for any $N$-vectors $a = (a_1, \ldots, a_N)$ and $b = (b_1, \ldots, b_N)$. For any $N$-vector $a$, let $a_\tau := a - <a, n> n$. For any two $N \times N$-matrices $A = (A_{ij})$ and $B = (B_{ij})$, the quantity $A : B$ is defined by $A : B = \sum_{i,j=1}^{N} A_{ij} B_{ji}$. For any domain $G$ with boundary $\partial G$, we set

$$(u, v)_G = \int_{G} u(x) \cdot \overline{v(x)} \, dx, \quad (u, v)_{\partial G} = \int_{\partial G} u \cdot \overline{v(x)} \, d\sigma,$$

where $\overline{v(x)}$ is the complex conjugate of $v(x)$ and $d\sigma$ denotes the surface element of $\partial G$. Given $1 < q < \infty$, let $q' = q/(q-1)$. Throughout the paper, the letter $C$ denotes generic constants and $C_{a,b,\cdots}$ the constant which depends on $a, b, \cdots$. The values of constants $C, C_{a,b,\cdots}$ may be changed from line to line.

## 2 Hanzawa transform and statement of main result

### 2.1 Hanzawa transform

Let $n$ be the unit normal to $\Gamma$ oriented from $\Omega_+$ into $\Omega_-$. Since $\Gamma_\epsilon$ is unknown, we assume that the $\Gamma_\epsilon$ is represented by \( [1, 1] \). Our task is to find not only $v, p$ and $H$ but also $h$. We know the existence of an $N$-vector, $\tilde{n}$, of $C^2$ functions defined on $\mathbb{R}^N$ such that

$$n = \tilde{n} \quad \text{on} \quad \Gamma, \quad \text{supp} \tilde{n} \subset U_{\Gamma}, \quad \|\tilde{n}\|_{H^2(\mathbb{R}^N)} \leq C$$  \hspace{1cm} (2.1)

with some constant $C$, where we have set $U_{\Gamma} = \bigcup_{x_0 \in \Gamma} \{x \in \mathbb{R}^N \mid \|x - x_0\| < \alpha\}$ with some constant $\alpha > 0$. We will construct $\tilde{n}$ in Subsec 2.3 below. We may assume that

$$\text{dist} (\text{supp} \tilde{n}, S_{\pm}) \geq 3d_{\pm}/4.$$  \hspace{1cm} (2.2)

In the following we write $\hat{\Omega} = \Omega_+ \cup \Omega_-$ and $\Omega = \hat{\Omega} \cup \Gamma$. Let $H_h$ be an extension function of $h$ such that $h = H_h$ on $\Gamma$. In fact, we take $H_h$ as a solution of the harmonic equation:

$$(-\Delta + \lambda_0)H_h = 0 \quad \text{in} \quad \hat{\Omega}, \quad H_h|_{\Gamma} = h$$  \hspace{1cm} (2.3)

with some large positive number $\lambda_0$ which guarantees the unique solvability of \( (2.2) \). In this case, if $h$ satisfies the regularity condition:

$$h \in H^1_p((0, T), W^{2-1/q}(\Gamma)) \cap L_p((0, T), W^{3-1/q}(\Gamma)), \quad (2.4)$$

then $H_h$ satisfies the regularity condition:

$$H_h \in H^1_p((0, T), H^2_q(\Omega)) \cap L_p((0, T), H^3_q(\Omega)),$$  \hspace{1cm} (2.5)

and possesses the estimate:

$$\|\partial^i H_h\|_{L_p((0, T), H^{2-i}_q(\Omega))} \leq C \|\partial^i h\|_{L_p((0, T), W^{3-i/q}(\Gamma))} \quad (i = 0, 1),$$

$$\|\partial^i H_h\|_{L_p((0, T), W^{3-i/q-1}(\Gamma))} \leq C \|\partial^i H_h\|_{L_p((0, T), H^{2-i}_q(\Omega))} \quad (i = 0, 1)$$  \hspace{1cm} (2.6)

for some constant $C > 0$.

To transform Eq. (1.5) to the equations on $\Omega$, we use Hanzawa transformation defined by

$$x = y + H_h(t, y)\tilde{n}(y):= \Xi_h(t, y).$$  \hspace{1cm} (2.7)

Let $\delta > 0$ be a small number such that

$$|\Xi_h(y_1, t) - \Xi_h(y_2, t)| \leq (1/2)|y_1 - y_2| \quad (2.8)$$

provided that

$$\sup_{0 < t < T} \|\nabla H_h(\cdot, t)\|_{L_{\infty}(\Omega)} \leq \delta.$$  \hspace{1cm} (2.9)

Here and in the following, we write $\nabla H_h = (\partial_k^\alpha H_h \mid |\alpha| \leq 1) = (H_h, \nabla H_h)$. From (2.6), the map $x = \Xi(y, t)$ is injective. And also, under suitable regularity condition on $H_h$, for example, $H_h \in C^{1+\alpha}$ for small $\delta$.  

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for each $t \in (0, T)$ with some small $\alpha > 0$, the map $x = \Xi_h(y, t)$ becomes an open and closed map, so that \{ $x = \Xi_h(y, t) \mid y \in \Omega$ \} = $\Omega$ because $x = \Xi_h(y, t)$ is an identity map on $\Omega \setminus U_\Gamma$. We assume that the initial surface $\Gamma_0$ is given by
\[ \Gamma_0 = \{ x = y + h_0(y) n \mid y \in \Gamma \} \]
with a given small function $h_0$. Let $H_{h_0}$ be an extension of $h_0$ which is given by a unique solution of equation (2.2), where $H_h$ and $h$ are replaced with $H_{h_0}$ and $h_0$, respectively.

Let $\Xi_{h_0} = y + H_{h_0} \tilde{n}$, and set
\[ u(y, t) = \mathbf{v}(\Xi_{h_0}^{-1}(y, t), t), \quad q(y, t) = p(\Xi_{h_0}^{-1}(y, t), t), \quad G(y, t) = H(\Xi_{h_0}^{-1}(y, t), t), \]
\[ \Gamma_\pm = \{ x = \Xi_h(y, t) \mid y \in \Gamma \setminus \Omega \}, \quad \Omega_{t, \pm} = \{ x = \Xi_h(y, t) \mid y \in \Omega_{t, \pm} \}, \]
\[ u_0(y) = \mathbf{v}_0(\Xi_{h_0}^{-1}(y)), \quad G_0(y) = H_0(\Xi_{h_0}^{-1}(y)). \]

(2.9)

Noting that $x = y$ near $S_\pm$, we have
\[ u_\pm = 0, \quad n_\pm \cdot G_\pm = 0, \quad (\text{curl } G_\pm) n_\pm = 0 \quad \text{on } S_\pm \times (0, T), \]
\[ (u, G, h)|_{t=0} = (v_0, H_0, h_0) \quad \text{in } \Omega \times \Gamma, \quad H_h|_{t=0} = h_0 \quad \text{on } \Gamma. \]

In what follows, we derive equations and interface conditions which $u$, $q$ and $G$ satisfy in $\Omega_{t, \pm}$ and on $\Gamma_t$.

### 2.2 Derivation of equations

In this subsection, we derive equations obtained by Hanzawa transformation: $x = y + H_h(y, t) \tilde{n}(y)$ from the first, second and third equations in Eq. (1.5). We assume that $H_h$ satisf/ies (2.8) with small positive number $\delta > 0$. We have
\[ \frac{\partial x}{\partial y} = I + \frac{\partial (H_h \tilde{n})}{\partial y} \]
and then, choosing $\delta > 0$ in (2.8) small enough, we see that there exists an $N \times N$ matrix, $V_0(K)$, of bounded real analytic functions def/ined on $U_h = \{ K \in \mathbb{R}^{N+1} \mid |K| \leq \delta \}$ with $V_0(0) = 0$ such that
\[ \frac{\partial y}{\partial x} = (\frac{\partial x}{\partial y})^{-1} = I + V_0(\nabla H_h). \]
(10.20)

Here and in the following, $K = (\kappa_0, \kappa_1, \ldots, \kappa_N)$ and $\kappa_0, \kappa_1, \ldots, \kappa_N$ are independent variables corresponding to $H_h$, $\partial H_h/\partial y_1, \ldots, \partial H_h/\partial y_N$, respectively. Let $V_{0ijk}(K)$ be the $(i, j)$ th component of $V_0(K)$, and then by the chain rule
\[ \frac{\partial}{\partial x_j} = \sum_{k=1}^N (\delta_{jk} + V_{0jk}(K)) \frac{\partial}{\partial y_k}, \quad \nabla_x = (I + V_0(K)) \nabla_y. \]
(11.21)

Since $V_{0jk}(0) = 0$, we may write $V_{0jk}(K) = \tilde{V}_{0jk}(K) K$ with
\[ \tilde{V}_{0jk}(K) K = \int_0^1 \frac{d}{d\theta} (V_{0jk}(\theta K)) \ d\theta. \]

In particular,
\begin{align*}
\text{curl}_{ij}(\mathbf{v}) &= \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} = \text{curl}_{ij}(\mathbf{u}) + V_{C_{ij}}(K) \nabla u, \\
D_{ij}(\mathbf{v}) &= \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} = D_{ij}(\mathbf{u}) + V_{D_{ij}}(K) \nabla u, \quad (2.12)
\end{align*}

with
\[ V_{C_{ij}}(K) \nabla u = \sum_{k=1}^N (V_{0jk}(K) \frac{\partial u_j}{\partial y_k} - V_{0ik}(K) \frac{\partial u_i}{\partial y_k}), \]
\[ V_{D_{ij}}(K) \nabla u = \sum_{k=1}^N (V_{0jk}(K) \frac{\partial u_i}{\partial y_k} + V_{0ik}(K) \frac{\partial u_j}{\partial y_k}). \]
Here and in the following, for an $N \times N$ matrix $A$, $A_{ij}$ denotes its $(i, j)$ th component and $(A_{ij})$ denotes an $N \times N$ matrix whose $(i, j)$ th component is $A_{ij}$. To obtain the first equation in (2.66) in Subsec 2.5 below, we make the pressure term linear. From $\nabla p = (I + \mathbf{V}_0(\mathbf{K}))\nabla q$ it follows that

$$\frac{\partial q}{\partial y_j} = \sum_{k=1}^{N} (\delta_{jk} + \frac{\partial x_k}{\partial y_j}) \frac{\partial p}{\partial x_k}.$$ 

Let $\mathbf{n} = (\tilde{n}_1, \ldots, \tilde{n}_N)^\top$, and then

$$\frac{\partial v_k}{\partial t} = \frac{\partial}{\partial t} u_i(y + H_k \mathbf{n}, t) + \sum_{j=1}^{N} \frac{\partial u_i}{\partial y_j} \frac{\partial H_k}{\partial t} \tilde{n}_j. \quad (2.13)$$

Thus, the first equation in (1.5) is transformed to

$$\frac{\partial q}{\partial y_m} = -\rho \sum_{i=1}^{N} (\delta_{mi} + \frac{\partial x_m}{\partial y_i})(\frac{\partial u_i}{\partial t} + \sum_{j=1}^{N} \frac{\partial u_i}{\partial y_j} \frac{\partial H_k}{\partial t} \tilde{n}_j + \sum_{j,k=1}^{N} u_j(\delta_{jk} + V_{oijk}(\nabla H_k)) \frac{\partial u_i}{\partial y_k})$$

$$+ \sum_{i,j,k=1}^{N} (\delta_{mi} + \frac{\partial x_m}{\partial y_i}) (\delta_{jk} + V_{oijk}(\nabla H_k)) \frac{\partial}{\partial y_k} \left( \nu(D_{mj}(u) + V_{D_{mj}}(\nabla H_k) \nabla u) + T_{M_{ij}}(G) \right)$$

$$= -\rho \partial t u_m + \sum_{k=1}^{N} \frac{\partial}{\partial y_k} (\nu D_{mk}(u)) \right) + f_{1m}(u, G, H_k)$$

with

$$f_{1m}(u, G, H_k) = (f_{11}(u, G, H_k), \ldots, f_{1N}(u, G, H_k))^\top,$$

we have

$$\rho \partial t u - \text{Div} \mathbf{T}(u, q) = f_1(u, G, H_k) \quad \text{in } \hat{\Omega} \times (0, T). \quad (2.15)$$

Since $V_{oijk}(0) = 0$ and $V_{D_{ij}}(0) = 0$, we may write

$$f_1(u, G, H_k) = f_0^1(\nabla H_k) \partial_t u + F_{H_k}^1(\nabla H_k) \partial_t H_k \nabla u + F_\nu^1(\nabla H_k) u \nabla u + F_\nu^2(\nabla H_k) u \nabla u + F_\nu^3(\nabla H_k) G \nabla G \quad (2.16)$$

where $f_0^1$ is a bounded function and $F_\nu^1(K)$ are some matrices of bounded analytic functions defined on $U_k$. Here and in the following, we write $\nabla H_k = (\partial_{y_k}^0 H_k \mid |a| \leq k)$ for $k \geq 2$ and $\nabla H_0 = (\partial_{y_k}^0 H_k \mid |a| \leq 1)$.

We next consider the divergence free condition: $\text{div} \mathbf{v} = 0$. By (2.11),

$$\text{div} \mathbf{v} = \sum_{j=1}^{N} \frac{\partial v_j}{\partial x_j} = \sum_{j,k=1}^{N} (\delta_{jk} + V_{oijk}(\nabla H_k)) \frac{\partial u_j}{\partial y_k} \quad (2.17)$$

Let $J = \det(\partial x/\partial y)$ and then, choosing $\delta > 0$ small enough in (2.8), we can write

$$J = 1 + J_0(\nabla H_k), \quad (2.18)$$

where $J_0(K)$ is a real analytic functions defined on $U_k$ such that $J_0(0) = 0$. Using this symbol, we have

$$(\text{div}_x \mathbf{v}, \varphi)_{\Omega_{\pm}} = - (\mathbf{v}_{\pm}, \nabla_x \varphi)_{\Omega_{\pm}} = - \sum_{j=1}^{N} (J u_{\pm,j}) \sum_{k=1}^{N} (\delta_{jk} + V_{oijk}(\nabla H_k)) \frac{\partial \varphi}{\partial y_k} |_{\Omega_{\pm}}$$

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\[
(\sum_{j,k=1}^{N} \frac{\partial}{\partial y_k} \{ J(\delta_{jk} + V_{0jk}(\nabla H_h)) u_{\pm j} \}, \varphi \} n_{\pm},
\]
so that
\[
\text{div } v_{\pm} = J^{-1} \sum_{j,k=1}^{N} \frac{\partial}{\partial y_k} \{ J(\delta_{jk} + V_{0jk}(\nabla H_h)) u_{\pm j} \},
\]
Combining (2.17), (2.19), and (2.21) yields that
\[
\text{div } u = g(u, H_h) = \text{div } g(u, H_h) \text{ in } \hat{\Omega} \times (0, T)
\]
with
\[
g(u, H_h) = \sum_{j=1}^{N} V_{0jk}(\nabla H_h) \frac{\partial u_{\pm j}}{\partial y_k} + J_0(\nabla H_h) \{ \text{div } u + \sum_{j,k=1}^{N} V_{0jk}(\nabla H_h) \frac{\partial u_{\pm j}}{\partial y_k} \},
\]
\[
g(u, H_h) = \sum_{j=1}^{N} \sum_{k=1}^{N} V_{0jk}(\nabla H_h) u_j + J_0(\nabla H_h) \{ \text{div } u + \sum_{j,k=1}^{N} V_{0jk}(\nabla H_h) \frac{\partial u_{\pm j}}{\partial y_k} \}.
\]
Since \( V_{0jk}(0) = J_0(0) = 0 \), we may write
\[
g(u, H_h) = G(\nabla H_h) \nabla u \otimes \nabla u, \quad g(u, H_h) = G_2(\nabla H_h) \nabla u \otimes u,
\]
where \( G_i(K) \) are some matrices of bounded analytic functions defined on \( U_5 \).

We next consider the third equation in Eq. (1.5). By (2.13),
\[
\mu \partial_t H = \mu \partial_t G + \mu \sum_{j,k=1}^{N} \tilde{n}_{ij} \partial \nabla G \partial H_h \frac{\partial t}{\partial t}.
\]
Moreover,
\[
\Delta = \sum_{j,k=1}^{N} \sum_{k=1}^{N} (\delta_{jk} + V_{0jk}(\nabla H_h)) \frac{\partial}{\partial y_k} \{ \sum_{\ell=1}^{N} (\delta_{j\ell} + V_{0j\ell}(\nabla H_h)) \frac{\partial}{\partial y_{\ell}} \}
\]
\[
= \sum_{j,k=1}^{N} \frac{\partial^2}{\partial y_j} + \sum_{\ell=1}^{N} \frac{\partial}{\partial y_{\ell}} (V_{0j\ell}(\nabla H_h) \frac{\partial}{\partial y_{\ell}}) + \sum_{\ell,k=1}^{N} V_{0jk}(\nabla H_h) \frac{\partial}{\partial y_k} ((\delta_{j\ell} + V_{0j\ell}(\nabla H_h)) \frac{\partial}{\partial y_{\ell}})
\]
\[
= \Delta + V_{\Delta 2}(\nabla H_h) \nabla \Delta + V_{\Delta 1}(\nabla H_h) \nabla
\]
with
\[
V_{\Delta 2}(\nabla H_h) \nabla \Delta = 2 \sum_{j,k=1}^{N} V_{0jk}(\nabla H_h) \frac{\partial^2}{\partial y_j} + \sum_{j,k,\ell=1}^{N} V_{0jk}(\nabla H_h) V_{0j\ell}(\nabla H_h) \frac{\partial^2}{\partial y_k \partial y_{\ell}},
\]
\[
V_{\Delta 1}(\nabla H_h) \nabla = \sum_{j,k=1}^{N} (\partial V_{0jk}(\nabla H_h) \frac{\partial}{\partial y_j} + \sum_{j,k,\ell=1}^{N} V_{0jk}(\nabla H_h) \frac{\partial V_{0j\ell}(\nabla H_h)}{\partial y_k} \frac{\partial}{\partial y_{\ell}}.
\]
Thus, setting
\[
f_2(u, G, H_h) = -\mu \sum_{j,k=1}^{N} \tilde{n}_{ij} \frac{\partial G}{\partial y_j} \frac{\partial H_h}{\partial t} + \alpha^{-1} V_{\Delta 2}(\nabla H_h) \nabla^2 G_m + \alpha^{-1} V_{\Delta 1}(\nabla H_h) \nabla G
\]
\[
+ \mu \sum_{j,k=1}^{N} (\delta_{jk} + V_{0jk}(K)) \frac{\partial}{\partial y_k} (u \otimes G - G \otimes u),
\]
we have
\[
\mu \partial_t G - \alpha^{-1} \Delta G = f_2(u, G, H_h) \text{ in } \hat{\Omega} \times (0, T).
\]
Since \( V_{0jk}(0) = 0 \), we may write
\[
f_2(u, G, H_h) = f_2(\nabla H_h) \nabla u \otimes \nabla G + f_2^2(\nabla H_h) \nabla^2 G + f_2^2(\nabla H_h) \nabla^2 H_h \otimes \nabla G
\]
\[
+ f_2^2(\nabla H_h) \nabla u \otimes \nabla G + f_2^2(\nabla H_h) u \otimes \nabla G.
\]
where \( f_2 \) is a bounded function and \( f_2^2(K) \) are some matrices of bounded analytic functions defined on \( U_5 \).
2.3 The unit outer normal and the Laplace Beltrami operator on $\Gamma_t$

Since $\Gamma$ is a compact hypersurface of $C^3$ class, we have the following proposition.

**Proposition 2.1.** For any constant $M_1 \in (0,1)$, there exist a finite number $n \in \mathbb{N}$, constants $M_2 > 0$, $d, d' \in (0,1)$, $N$-vectors of functions $\Phi^\ell \in C^3(\mathbb{R}^N)^N$, $n$ points $x^\ell \in \Gamma$ and two domains $\Omega_\pm$ such that the following assertions hold:

(i) The maps: $\mathbb{R}^N \ni x \mapsto \Phi^\ell(x) \in \mathbb{R}^N$ are bijective for $j \in \mathbb{N}$.

(ii) $\Omega = \bigcup_{j=1}^n \Phi^\ell(B_d) \cup \Omega_+ \cup \Omega_-$, $B_d = \Phi^\ell(B_d) \cup \Omega_\pm \subset \Phi^\ell(B_d \cap \mathbb{R}^N) \subset \Omega_\pm$ and $\Gamma \cap B_d(x^\ell) \subset \Phi^\ell(B_d \cap \mathbb{R}^N)$, where $B_d = \{x \in \mathbb{R}^N \mid |x| < d\}$, $B_d(x^\ell) = \{x \in \mathbb{R}^N \mid |x - x^\ell| < d\}$, $\mathbb{R}^N_+ = \{x = (x_1, \ldots, x_N) \mid x_N > 0\}$, and $\mathbb{R}^N_0 = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N = 0\}$.

(iii) There exist $n$ $C^\infty$ functions $\zeta^\ell$ such that $\text{supp} \zeta^\ell \subset B_d(x^\ell)$ and $\sum_{\ell=1}^n \zeta^\ell = 1$ on $\Gamma$.

(iv) $\nabla \Phi^\ell = A^\ell + B^\ell$, $\nabla (\Phi^\ell)^{-1} = A^\ell_{,-1} + B^\ell_{,-1}$ where $A^\ell$ are $n \times n$ constant orthogonal matrices and $B^\ell$ are $N \times N$ matrices of $C^3(\mathbb{R}^N)$ functions satisfying the conditions: $\|B^\ell\|_{L^\infty(\mathbb{R}^N)} \leq M_1$ and $\|\nabla B^\ell\|_{H^1(\mathbb{R}^N)} \leq M_2$ for $\ell = 1, \ldots, n$.

In what follows, we write $B_d(x^\ell)$ simply by $B^\ell$, and set $V_0 = B_d \cap \mathbb{R}^N$. In what follows, the index $\ell$ runs from 1 through $n$. Recall that $\Gamma \cap B^\ell \subset \Phi^\ell(V_0)$, $\sum_{\ell=1}^n \zeta^\ell = 1$ on $\Gamma$, and $\text{supp} \zeta^\ell \subset B^\ell \subset \Phi^\ell(B_d) \subset \Omega$.

Let

$$\tau_j(u) = \frac{\partial \Phi^\ell(u)}{\partial x_j} = A^\ell_j + B^\ell_j(u)$$

for $j = 1, \ldots, N$ and $u = (u_1, \ldots, u_N) \in \mathbb{R}^N$. By Proposition 2.1, $A^\ell_j$ are $N$-constan vectors and $B^\ell_j(u)$ are $N$-vector of functions such that

$$A^\ell_j \cdot A^\ell_k = \delta_{jk}, \quad \|B^\ell_j\|_{L^\infty(\mathbb{R}^N)} \leq M_1, \quad \|\nabla B^\ell_j\|_{H^1(\mathbb{R}^N)} \leq M_2$$

(2.26) where $\delta_{jk}$ are the Kronecker delta symbols defined by $\delta_{jj} = 1$ and $\delta_{jk} = 0$ for $j \neq k$. Notice that $\{\tau_j(u', 0)\}_{j=1}^{N-1}, u' = (u_1, \ldots, u_{N-1}, 0) \in V_0$, forms a basis of the tangent space of $\Gamma \cap B^\ell$. Let $g^\ell_{ij}(u) = \tau_i^\ell(u) \cdot \tau_j^\ell(u), G^\ell(u)$ an $N \times N$ matrix whose $(i, j)$ th component is $g^\ell_{ij}(u), g^\ell(u) = \sqrt{\det G^\ell(u)}, g^\ell_{ij}(u)$ the $(i, j)$ th component of $(G^\ell)^{-1}$, respectively. $G^\ell(u', 0)$ is a first fundamental matrix of the tangent space of $\Gamma \cap B^\ell$. By (2.26) there exist functions $\tilde{g}^\ell_{ij}(u), \tilde{g}^\ell_{ij}(u)$ and $\tilde{g}^\ell_{ij}(u)$ such that

$$g^\ell_{ij}(u) = \delta_{ij} + \tilde{g}^\ell_{ij}(u), \quad g^\ell(u) = 1 + \tilde{g}^\ell(u), \quad g^\ell_{ij}(u) = \delta_{ij} + \tilde{g}^\ell_{ij}(u),$$

$$\|(g^\ell_{ij}, \tilde{g}^\ell_{ij}, \tilde{g}^\ell_{ij})\|_{L^\infty(\mathbb{R}^N)} \leq CM_1, \quad \|\nabla (g^\ell_{ij}, \tilde{g}^\ell_{ij}, \tilde{g}^\ell_{ij})\|_{H^1(\mathbb{R}^N)} \leq CM_2.$$ 

(2.27)

Here and in the following the constant $C_{M_2}$ is a generic constant depending on $M_2$. Here and in the following, we may assume that $0 < M_1 < 1 \leq M_2$.

We now define an extension of $n$ to $\mathbb{R}^N$ satisfying (2.4). Let $\varphi_{i,j}(u) = \partial \Phi^\ell(u)/\partial x_j$ with $\Phi^\ell = (\Phi^\ell_1, \ldots, \Phi^\ell_N)^T$, and let $N^\ell_i(u)$ be an $N \times (N-1)$ be functions defined by setting

$$N^\ell_i(u) = (-1)^{i+N} \det \begin{pmatrix} \varphi_{1,1} & \cdots & \varphi_{1,N-1} \\ \vdots & \ddots & \vdots \\ \varphi_{i-1,1} & \cdots & \varphi_{i-1,N-1} \\ \varphi_{i+1,1} & \cdots & \varphi_{i+1,N-1} \\ \vdots & \ddots & \vdots \\ \varphi_{N,1} & \cdots & \varphi_{N,N-1} \end{pmatrix}$$

for $i = 1, \ldots, N - 1$, and set $N^\ell = (N^\ell_1, \ldots, N^\ell_N)^T$. Then, we have

$$< N^\ell_k, \frac{\partial \Phi^\ell}{\partial x_k} > = 0 \quad \text{for} \quad k = 1, \ldots, N - 1,$$

because

$$0 = \det \begin{pmatrix} \varphi_{1,1} & \cdots & \varphi_{1,N-1} & \varphi_{1,k} \\ \vdots & \ddots & \vdots & \vdots \\ \varphi_{N,1} & \cdots & \varphi_{N,N-1} & \varphi_{N,k} \end{pmatrix} = \sum_{j=1}^N N^\ell_j \varphi_{j,k} = < N^\ell_k, \frac{\partial \Phi^\ell}{\partial x_k} >$$
for $k = 1, \ldots, N - 1$. Let $\mathbf{n}^f = N^f / |N^f|$, and then
\[
< \mathbf{n}^f, \tau^f_j(u) >= 0 \quad \text{for } j = 1 \ldots N - 1 \text{ and } u \in \mathbb{R}^N.
\]

Moreover, by (2.26) $\|\nabla \mathbf{n}^f\|_{H^1_0(\mathbb{R}^N)} \leq C_{M_2}$ for some constant $C_{M_2}$ depending on $M_2$. Let
\[
\mathbf{n} = \sum_{\ell=1}^{n} \mathbf{c}^f \mathbf{n}^f \circ (\Phi^f)^{-1},
\]
and then $\mathbf{n}$ satisfies the properties given in (2.1).

Next, we give a representation formula of $\mathbf{n}_t$. Since $\Gamma_t \cap B^f$ is represented by $x = \Phi^f(u^*,0) + H_h(\Phi^f(u^*,0), t) \mathbf{n}(\Phi^f(u^*,0))$ for $(u^*,0) \in V_0$, setting $\bar{H}^f_h = H_h(\Phi^f(u), t)$, we define $\tau^f_t = (\tau^f_{11}(u), \ldots, \tau^f_{N-1})^\top$ by
\[
\tau^f_{ij}(u) = \frac{\partial}{\partial u_j} (\Phi^f(u) + \bar{H}^f_h(u,t) \mathbf{n}^f(u)).
\]

Notice that $\{\tau^f_{ij}(u^*,0)\}_{j=1}^{N-1}$ forms a basis of the tangent space $\Gamma_t$ locally. To obtain a formula of $\mathbf{n}_t$, we set $\mathbf{n}^f_t = a(\mathbf{n}^f + \sum_{j=1}^{N-1} \tau^f_{ij} j)$ and we decide $a$ and $b_j$ in such a way that $|\mathbf{n}^f_t| = 1$ and $< \mathbf{n}^f_t, \tau^f_t >= 0$. From $|\mathbf{n}^f_t|^2 = 1$ it follows that
\[
1 = a^2(\mathbf{n}^f + \sum_{j=1}^{N-1} b_j \tau^f_{ij}) \cdot (\mathbf{n}^f + \sum_{k=1}^{N-1} b_k \tau^f_{ik}) = a^2 (1 + \sum_{j,k=1}^{N-1} g^f_{jk}(u) b_j b_k),
\]
so that
\[
a = (1 + \sum_{j,k=1}^{N-1} g^f_{jk}(u) b_j b_k)^{-1/2}.
\]
Therefore, we have
\[
\mathbf{n}_t^\ell = (1 + < G^\ell (1 + (G^\ell)^{-1} L^\ell \tilde{H}_h^\ell)^{-1} (G^\ell)^{-1} \nabla^\ell \tilde{H}_h^\ell >^{-1/2})
\times (\mathbf{n} - < (1 + (G^\ell)^{-1} L^\ell \tilde{H}_h^\ell)^{-1} (G^\ell)^{-1} \nabla^\ell \tilde{H}_h^\ell >) >
= \mathbf{n} - < (G^\ell)^{-1} \nabla^\ell \tilde{H}_h^\ell > + O_t^2.
\] (2.31)

Since
\[
\frac{\partial \tilde{H}_h^\ell}{\partial u_j} = \sum_{k=1}^N \frac{\partial \Phi_k^\ell}{\partial u_j} \frac{\partial H_k}{\partial y_k} \circ \Phi^\ell
\]
setting
\[
< G^{-1} \nabla^\ell H_h, \tau > = \sum_{i=1}^n \zeta_i^\ell < (G^{-1})^{-1} \nabla^\ell \tilde{H}_h^\ell, \tau^\ell > + o(\Phi^\ell)^{-1},
\]
by (2.31) we see that there exists a matrix of functions, \( \mathbf{V}_n(y, K) \), defined on \( \mathbb{R}^N \times U_\delta \) such that
\[
\mathbf{n}_t = \mathbf{n} < G^{-1} \nabla^\ell H_h, \tau > + \mathbf{V}_n(\cdot, \tilde{H}_h) \tilde{H}_h \otimes \tilde{H}_h \text{ on } \Gamma
\] (2.32)
and \( \mathbf{V}_n(y, K) \) satisfies the following conditions: \( \supp \mathbf{V}_n(y, K) \subset U_\Gamma \) for any \( K \in U_\delta \), and
\[
\| \mathbf{V}_n \|_{L^\infty(\mathbb{R}^N \times U_\delta)} \leq C M_2,
\]
\[
| \nabla \mathbf{V}_n(y, \tilde{H}_h) | \leq C |\nabla^2 H_h |
\]
\[
| \nabla^2 \mathbf{V}_n(y, \tilde{H}_h) | \leq C \left( |\nabla^3 H_h | + |\nabla^2 H_h |^2 \right),
\]
\[
| \partial \mathbf{V}_n(y, \tilde{H}_h) | \leq C |\nabla \partial \tilde{H}_h |
\]
\[
| \nabla \partial \mathbf{V}_n(y, \tilde{H}_h) | \leq C \left( |\nabla^2 \partial \tilde{H}_h | + |\nabla \partial \tilde{H}_h | \right)
\]
provided that (2.8) holds with some small \( \delta > 0 \).

We next represent \( \Delta_{\Gamma_\ell} \). Let \( G_t = (g_{ij}) \) be the first fundamental form and set \( g_t = \sqrt{\det(G_t)} \) and \( G_t^{-1} = (g^\ell_{ij}) \). Then, \( \Delta_{\Gamma_\ell} \) is given by setting
\[
\Delta_{\Gamma_\ell} f = \frac{1}{g_t} \sum_{i,j=1}^{N-1} (g_{ij}) \frac{\partial f}{\partial u_j}
\] on \( V_0 \).
(2.33)

Since \( < \tilde{\mathbf{n}}^\ell, \partial \tilde{\mathbf{n}}^\ell / \partial u_j > = 0 \) and \( \partial \Phi^\ell / \partial u_j, \tilde{\mathbf{n}} > = < \tau^\ell, \tilde{\mathbf{n}} > = 0 \), in view of (2.29), setting
\[
\alpha^\ell_{ij} = < \tau^\ell, \frac{\partial \tilde{\mathbf{n}}^\ell}{\partial u_j} > + < \tau^\ell, \frac{\partial \tilde{\mathbf{n}}^\ell}{\partial u_i} >, \quad \beta^\ell_{ij} = < \frac{\partial \tilde{\mathbf{n}}^\ell}{\partial u_j}, \frac{\partial \tilde{\mathbf{n}}^\ell}{\partial u_i} >,
\]
we have
\[
g^\ell_{ij} = < \tau^\ell, \partial \tilde{\mathbf{n}}^\ell > = g^\ell_{ij} + \alpha^\ell_{ij} \tilde{H}_h^\ell + \beta^\ell_{ij} (\tilde{H}_h^\ell)^2 + \frac{\partial \tilde{H}_h^\ell}{\partial u_j} + \frac{\partial \tilde{H}_h^\ell}{\partial u_j} > .
\]
Notice that \( \alpha^\ell_{ij} \) and \( \beta^\ell_{ij} \) are all bounded \( C^2 \) functions. Here, what a function, \( f \), is bounded \( C^2 \) means that \( f \) is a \( C^2 \) function and \( f \) and its derivatives up to order 2 are all bounded. Let \( g^\ell_t = \sqrt{\det(g^\ell_{ij})} \) and \( (G^\ell_t)^{-1} = (g^\ell_{ij}) \), and then by (2.8) with small \( \delta > 0 \) and (2.24), we have the representation formulas:
\[
g^\ell_{ij} = g^\ell + c^\ell_{ij}(u) \tilde{H}_h^\ell + O_t^2, \quad \frac{1}{g^\ell_t} = \frac{1}{g^\ell} + \gamma^\ell_{ij}(u) \tilde{H}_h^\ell + O_t^2, \quad g^\ell_{ij} = g^\ell + \gamma^\ell_{ij}(u) \tilde{H}_h^\ell + O_t^2,
\]
where \( \gamma^\ell_{ij}(u) \), \( \gamma^\ell_{ij}(u) \) and \( \gamma^\ell_{ij}(u) \) are some bounded \( C^2 \) functions defined on \( \mathbb{R}^N \). In view of (2.33), setting
\[
V^\ell_{\Delta ij} = \gamma^\ell_{ij}(u) \tilde{H}_h^\ell + O_t^2,
\]
\[
V^\ell_2 = \sum_{i=1}^{N-1} \left( \frac{\partial}{\partial u_i} (\gamma^\ell_{ij}(u) \tilde{H}_h^\ell) + \frac{\partial}{\partial u_i} O_t^2 + \frac{1}{g^\ell_t} (\partial_i (\gamma^\ell_{ij}(u) \tilde{H}_h^\ell) + \partial_i O_t^2)
\]
\[
+ (\gamma^\ell_{ij}(u) \tilde{H}_h^\ell + O_t^2) (\partial_i g^\ell + \partial_i (\gamma^\ell_{ij}(u) \tilde{H}_h^\ell) + \partial_i O_t^2) \right) \]
we have
\[
\Delta_{\Gamma_\ell} = \Delta_{\Gamma} + \hat{\Delta}_{\Gamma_\ell} \text{ on } \Gamma \cap B^\ell,
\] (2.35)
where $\hat{\Delta}_T$ is an operator defined by setting

$$
\Delta_T f = \sum_{\ell=1}^n \zeta^\ell \left( \sum_{i,j=1}^{N-1} V_{\Delta i j}^\ell \frac{\partial^2 f(\Phi^\ell(u',0))}{\partial u_i \partial u_j} + \sum_{j=1}^{N-1} V_{\Delta j}^\ell \frac{\partial f(\Phi^\ell(u',0))}{\partial u_j} \right) \circ (\Phi^\ell)^{-1}.
$$

We finally derive a formula for the surface tension. Recall that $H(\Gamma_t)u_t = \Delta_T x$ for $x \in \Gamma_t$. For $x \in \Gamma_t$, $x$ is represented by $x = \Phi^\ell(u',0) + \tilde{H}^\ell_t(u',0)t\tilde{n}^\ell(u',0)$ locally. By (2.35) and (2.36), we have

$$
<H(\Gamma_t)u_t, n> = <\Delta_T(y + H_h)n, n> + \sum_{\ell=1}^n \zeta^\ell \left( \sum_{i,j=1}^{N-1} V_{\Delta i j}^\ell \frac{\partial^2}{\partial u_i \partial u_j} (\Phi^\ell + \tilde{H}^\ell_t n^\ell) \cdot n^\ell \right) + \sum_{j=1}^{N-1} V_{\Delta j}^\ell \frac{\partial}{\partial u_j} (\Phi^\ell + \tilde{H}^\ell_t n^\ell) \cdot n^\ell > \nabla \Gamma_T n, \nabla \Gamma T n
$$

on $\Gamma$. Since $\Delta_T y = H_h(\Gamma) n$ for $y \in \Gamma$ and

$$
<\Delta_T n, n> = \sum_{\ell=1}^n \zeta^\ell \sum_{i,j=1}^{N-1} g_{ij}^\ell \frac{\partial^2 n^\ell}{\partial u_i \partial u_j} \cdot n^\ell > - \sum_{\ell=1}^n \zeta^\ell \sum_{i,j=1}^{N-1} g_{ij}^\ell \frac{\partial^2 n^\ell}{\partial u_i \partial u_j} \cdot \frac{\partial n^\ell}{\partial u_i} \cdot \frac{\partial n^\ell}{\partial u_j} = - <G^{-1} \nabla \Gamma_T n, \nabla \Gamma T n>
$$

as follows from $<\nabla n^\ell / \partial u_i, n^\ell > = 0$, we have

$$
<\Delta_T(y + H_h n), n> = H_h(\Gamma) + \Delta_T H_h - <G^{-1} \nabla \Gamma_T n, \nabla \Gamma_T n> > H_h.
$$

Moreover, by (2.30) and (2.34), we have

$$
< \nabla^T \nabla \Phi^\ell, \tilde{n}^\ell > = \gamma^\ell \frac{\partial^2}{\partial u_i \partial u_j} \Phi^\ell, \tilde{n}^\ell > \tilde{H}^\ell_t + O_T^2,
$$

$$
< \nabla^T \nabla (\tilde{H}^\ell_t n^\ell), \tilde{n}^\ell > = \nabla^T \nabla \tilde{H}^\ell_t n^\ell + \nabla^T \nabla \tilde{H}^\ell_t n^\ell > \tilde{H}^\ell_t
$$

$$
= \nabla^T \nabla \tilde{H}^\ell_t n^\ell - < \frac{\partial}{\partial u_i} \nabla^T \nabla \tilde{H}^\ell_t n^\ell > \tilde{H}^\ell_t < \frac{\partial}{\partial u_j} \nabla^T \nabla \tilde{H}^\ell_t n^\ell > \tilde{H}^\ell_t
$$

$$
< \nabla^T \nabla (\tilde{H}^\ell_t n^\ell), \tilde{n}^\ell > = \nabla^T \nabla \tilde{H}^\ell_t n^\ell.
$$

Combining these formulas gives that

$$
<H_h(\Gamma_t) n, n > = H_h(\Gamma) + \Delta_T H_h + a(y)H_h + \nabla v_s(y, \nabla H_h) \nabla H_h \otimes \nabla^2 H_h
$$

(2.37)

where $a(y)$ is a bounded $C^1$ function, and $v_s = v_s(y, K)$ are some matrices of functions defined on $\mathbb{R}^N \times U_5$ such that $\text{supp} v_s(y, K) \subset U_T$ for any $K \in U_5$, supp$_{t \in (0,T)} \| v_s(\cdot, \nabla H_h) \|_{L^\infty(\Omega)} \leq C_M$, $|\nabla v_s(y, \nabla H_h)| \leq C_M \nabla^2 H_h(y, t)$, $|\partial_t v_s(y, \nabla H_h)| \leq C_M |\nabla \partial_t H_h(y, t)|$.

(2.38)

provided that (2.8) holds with some small constant $\delta > 0$.

2.4 Derivation of transmission conditions and kinematic condition

We first consider the kinematic condition: $V_{\Gamma_t} = v_+ \cdot n_t$. Note that $v_+ = v_-$ on $\Gamma_t$. Since

$$
V_{\Gamma_t} = \frac{\partial x}{\partial t} \cdot n_t = \frac{\partial H_h}{\partial t} n \cdot n_t,
$$

it follows from (2.32) that

$$
\partial_t h + < \nabla h \perp n_+ > - u_+ \cdot n = < u_+ - \frac{\partial H_h}{\partial t} n, V_n(\cdot, \nabla H_h) \nabla H_h \otimes \nabla H_h > \nabla \Gamma_T h \perp \nabla ^\ell h + \nabla \Gamma_T \ell h > \cdot
$$

(2.39)

Here and in the following, we write

$$
< \nabla \Gamma_T h \perp n_+ > = \sum_{\ell=1}^n \zeta^\ell \left( \sum_{i,j=1}^{N-1} g_{ij}^\ell \frac{\partial h \circ \Phi^\ell}{\partial u_j} < \tilde{r}^\ell_i, u_+ \circ \Phi^\ell > \right).
$$
If we move $< \nabla T h, u_+ >$ to the right hand side in proving the local wellposedness by using a standard fixed point argument, we have to assume the smallness of initial velocity field $u_0$ as well as the smallness of initial height $h_0$. But, this is not satisfactory. We have to treat at least the large initial velocity case for the local well-posedness. To avoid the smallness assumption of initial velocity field, we use an idea due to Padula and Solonnikov [11]. Let $u_0 \in B_{2(1-1/p)}^0(\Omega)$ be an initial velocity field and set $u_0^\perp = u_0|_{\Omega_+}$. We know that $[[u_0]] = 0$ on $\Gamma$, which follows from the compatibility conditions. Let $u_0^\perp$ be an extension of $u_0^\perp$ to $\mathbb{R}^N$ such that $\tilde{u}_0^\perp = u_0^\perp$ in $\Omega_+$ and

$$\|u_0^\perp\|_{B_{2(1-1/p)}^2(\mathbb{R}^N)} \leq C\|u_0\|_{B_{2(1-1/p)}^2(\Omega_+)}.$$  \hfill (2.40)

Let

$$u_s = \frac{1}{\kappa} \int_0^s T_0(t)\tilde{u}_0^\perp \, ds$$

where $\{T_0(s)\}_{s \geq 0}$ is a $C^0$ analytic semigroup generated by $-\Delta + \lambda_0$ with large $\lambda_0$ in $\mathbb{R}^N$, that is

$$T_0(s)f = \mathcal{F}^{-1}[e^{-\kappa(|\xi|^2+\lambda_0)}\hat{f}(\xi)](x).$$

Here, $\hat{f}$ denotes the Fourier transform of $f$ and $\mathcal{F}^{-1}$ the inverse Fourier transform. We know that

$$\|T_0(\cdot)\tilde{u}_0^\perp L_p((0,\infty),H_{2}^1(\mathbb{R}^N)) + \|\partial_t T_0(\cdot)\tilde{u}_0^\perp L_p((0,\infty),L_q(\mathbb{R}^N)) + \|T_0(\cdot)\tilde{u}_0^\perp L_{\infty}(0,\infty),B_{2(1-1/p)}^0(\mathbb{R}^N)) \leq C\|u_0\|_{B_{2(1-1/p)}^2(\mathbb{R}^N)}.$$  \hfill (2.41)

which yields that

$$\|u_s\|_{B_{2(1-1/p)}^2(\mathbb{R}^N)} \leq C\|u_0^\perp\|_{B_{2(1-1/p)}^2(\Omega_+)}.$$ \hfill (2.42)

As a kinematic condition, we use the following equation:

$$\partial_t h + < \nabla T h, u_+ > - u \cdot n = d(u, H_h)$$ \hfill (2.43)

with

$$d(u, H_h) = < \nabla T h, u_+ > + < u - \partial H_h \frac{\partial H_h}{\partial t}, V_n, \nabla H_h, \nabla H_h \otimes \nabla H_h >.$$ \hfill (2.44)

Let $E_\perp$ be an the extension map acting on $u_\pm \in H_{2(1+\delta)}(\Omega_\pm)$ satisfying the properties: $E_\perp(u_\pm) \in H_{2(1+\delta)}(\Omega)$, $E_\perp(u_\pm) = u_\pm$ in $\Omega_\pm$,

$$(\partial_\alpha E_\perp(u_\pm))(x_0) = \lim_{x \to x_0, x \in \Omega_\pm} \partial_\alpha u_\pm(x)$$ \hfill (2.45)

for $x_0 \in \Gamma$ and $\alpha \in \mathbb{N}_0^N$ with $|\alpha| \leq 1$, and

$$\|E_\perp(u_\pm)\|_{H_{2(1+\delta)}(\Omega)} \leq C_{\delta, q}\|u_\pm\|_{H_{2(1+\delta)}(\Omega_\pm)}$$ \hfill (2.46)

for $t = 0, 1, 2$. Note that

$$[[\partial_\alpha^2 u]] = \partial_\alpha^2 E_\perp(u_+) - \partial_\alpha^2 E_\perp(u_-)$$ \hfill (2.47)

for $|\alpha| \leq 1$ on $\Gamma$. For the notational simplicity, we write

$$tr[u] = E_\perp(u_+) - E_\perp(u_-)$$ \hfill (2.48)

and then, we have

$$\partial_\alpha^2 tr[u]|_\Gamma = [[\partial_\alpha^2 u]]$$

$$\|tr[u]\|_{H_{2(1+\delta)}(\Omega)} \leq C\|u_+\|_{H_{2(1+\delta)}(\Omega)} + \|u_-\|_{H_{2(1+\delta)}(\Omega)} = C\|u\|_{H_{2(1+\delta)}(\Omega)}$$ \hfill (2.49)

for $i = 0, 1, 2$.

We next consider the interface conditions. First, we consider

$$[[\langle T(v, p) + T_M(H_h)\rangle n]] = \sigma H_h(\Gamma) n_t \quad \text{on } \Gamma_t.$$ \hfill (2.50)

Let

$$\Pi_0 d = d_0 < d, n_+ > n_+ \quad \Pi_0 d = d_0 < d, n_0 > n_0$$ \hfill (2.51)

The following lemma was given in Solonnikov [17].
Lemma 2.2. If $n_t \cdot n \neq 0$, then for arbitrary vector $d$, $d = 0$ is equivalent to

$$\Pi_0 \Pi_0 d = 0$$

and

$$n \cdot d = 0.$$  \hfill (2.52)

In view of Lemma 2.2, the interface condition (2.50) is equivalent to that the following two conditions hold:

$$\Pi_0 \Pi_0 (\nu (D(u) + V_D(K) \nabla u)) + T_M(G)) n_t = 0,$$  \hfill (2.53)

$$n \cdot (\nu (D(u) + V_D(K) \nabla u) - q I + T_M(G)) n_t - \sigma H_h (\Gamma) n_t = 0.$$  \hfill (2.54)

Here and hereafter, $V_D(K) \nabla u$ is the $N \times N$ matrix with $(i, j)$ components $V_{Dij}(K) \nabla u$ (cf. 2.12). Noting that $\Pi_0 \Pi_0 = \Pi_0$, we see that the condition (2.53) is written by

$$\Pi_0 [\nu D(u)] n = h'_1 (u, G, H_h)$$  \hfill (2.55)

with

$$h'_1 (u, G, H_h) = \Pi_0 (\Pi_0 - \Pi_1) [\nu (D(u))] n_t + \Pi_0 [\nu (D(u))] (n - n_t)$$

\hfill (2.56)

On the other hand, by (2.37) we see that Eq. (2.54) is written by

$$n \cdot [\nu (D(u) - q I)] n - \sigma (\Delta_f h + ah) = h_{1N} (u, G, H_h) + \sigma V_s (-, \nabla H_h) \nabla H_h \otimes \nabla^2 H_h.$$  \hfill (2.57)

with

$$h_{1N} (u, G, H_h) = (n \cdot n_t)^{-1} \{ n \cdot [\nu (D(u))] (n - n_t) - n \cdot [\nu V_D(K) \nabla u + T_M(G)] n_t \}.$$  \hfill (2.58)

In particular, setting

$$h_1 (u, G, H_h) = (h'_1 (u, G, H_h), h_{1N} (u, G, H_h) + \sigma V_s (-, \nabla H_h) \nabla H_h \otimes \nabla^2 H_h),$$

in view of (2.12), (2.32), and (2.49), we may write

$$h_1 (u, G, H_h) = V_1^h (-, \nabla H_h) \nabla H_h \otimes \nabla tr u + a(y) tr [G] \otimes tr [G] + V_2^h (-, \nabla H_h) \nabla H_h \otimes tr [G] \otimes tr [G]$$

$$+ V_3^h (-, \nabla H_h) \nabla H_h \otimes \nabla^2 H_h.$$  \hfill (2.59)

Here, $a(y)$ is an $N$-vector of bounded $C^2$ function, and $V_i^h (-, K)$ ($i = 1, 2$) are some matrices of functions defined on $R^N \times U_h$ satisfying the conditions: $\|V_i^h\|_{L_\infty (R^N \times U_h)} \leq C$, supp $V_i^h (y, K) \subset U_1$,

$$\|\nabla V_i^h (y, \nabla H_h)\| \leq C \nabla^2 H_h (y, t), \quad |\partial h V_i^h (y, \nabla H_h)| \leq C \nabla \partial h H_h (y, t).$$  \hfill (2.60)

provided that (2.5) holds with some somall $\delta > 0$.

From (2.12) we see that the interface condition: $[\alpha^{-1} \text{curl} H_h + \mu (v \otimes H_h - H_h \otimes v)] n_t = 0$ is written by

$$[\alpha^{-1} \text{curl} G] n = h_2 (u, G, H_h)$$  \hfill (2.61)

with

$$h_2 (u, G, H_h) = [\alpha^{-1} \text{curl} G](n - n_t) - [\alpha^{-1} V_C(K) \nabla u] n_t - [\mu (u \otimes G - G \otimes u)] n_t.$$  \hfill (2.62)

Here, $V_C(K) \nabla u$ is the $N \times N$ matrix with $(i, j)$ components quantities $V_{Cij}(K) \nabla u$ given in (2.12). In particular, in view of (2.12), (2.32), and (2.49), we may write

$$h_2 (u, G, H_h) = V_1^h (-, \nabla H_h) \nabla H_h \otimes \nabla tr u (E_u (u_+) + b(y) tr [u] \otimes tr [G])$$

$$+ V_2^h (-, \nabla H_h) \nabla H_h \otimes tr [u] \otimes tr [G].$$  \hfill (2.63)

Here, $b(y)$ is an $N$-vector of $C^2$ functions, and $V_i^h (-, K)$ ($i = 3, 4$) are some matrices of functions defined on $R^N \times U_h$ satisfying the same conditions as those stated in (2.60) provided that (2.5) holds with some somall $\delta > 0$.

From (2.17), we see that the interface condition: $[\mu \text{div} H_h] = 0$ is written by

$$[\mu \text{div} G] = h_3 (u, G, H_h)$$  \hfill (2.64)
which \( V_{0jk}(K) = \bar{V}_{0jk}(K)K \) are the symbols given in (2.11) and \( tr[u] = (tr[u]_1, \ldots, tr[u]_N) \). Finally, the interface conditions: \( [\mu \mathbf{h} \cdot \mathbf{n}] = 0 \) and \( [[\mathbf{h} - \mathbf{H}, \mathbf{n}] = 0 \) are written by
\[
[[\mu \mathbf{G} \cdot \mathbf{n}] = k_1(G, H_k), \quad [[\mathbf{G} - \mathbf{H}, \mathbf{n}] = k_2(G, H_k)
\]
with
\[
k_1(G, H_k) = [[\mu \mathbf{G} \cdot (\mathbf{n} - \mathbf{n}_i)], \quad k_2(G, H_k) = [[\mathbf{G} - \mathbf{H}, \mathbf{n} > \mathbf{n}]], [\mu \mathbf{G}, \mathbf{n} > \mathbf{n}] = k_2(G, H_k)
\]
In particular, in view of (2.32) and (2.39), we may write
\[
(k_1(G, H_k), k_2(G, H_k)) = \mathbf{V}_k^0(\cdot, \nabla \mathbf{H}_k) \nabla \mathbf{H}_k \otimes tr[G].
\]
Here, \( \mathbf{V}_k^0(\cdot, K) \) is some matrices of functions defined on \( \mathbb{R}^N \times U_\delta \) satisfying the same conditions as these stated in (2.60) provided that (2.8) holds with some small \( \delta > 0 \).

### 2.5 Statement of the local well-posedness theorem

Summing up the results obtained in subsections 2.2, 2.3, and 2.4, we have seen that equations (1.3) are transformed to the following equations:

\[
\begin{aligned}
\rho \partial_t u - \text{Div} \mathbf{T}(u, q) &= f_1(u, G, H_k), \quad \text{in } \hat{\Omega} \times (0, T), \\
\text{div} u &= g(u, H_k) = \text{div} g(u, H_k), \quad \text{in } \hat{\Omega} \times (0, T), \\
\partial_t h + \nabla \mathbf{h} \cdot \mathbf{u}_n - \mathbf{n} \cdot u &= d(u, H_k) \quad \text{on } \Gamma \times (0, T), \\
[[u]] = 0, \quad [[\mathbf{T}(u, q)]] - \sigma(\Delta_T h + ah) \mathbf{n} &= h_1(u, G, H_k), \quad \text{on } \Gamma \times (0, T), \\
\mu \partial_t \mathbf{G} - \alpha^{-1} \Delta \mathbf{G} &= f_2(u, G, H_k), \quad \text{in } \hat{\Omega} \times (0, T), \\
[[\alpha^{-1} \text{curl} \mathbf{G}]] \mathbf{n} &= h_2(u, G, H_k), \quad [[\mu \text{div} \mathbf{G}]] = h_3(u, G, H_k), \quad \text{on } \Gamma \times (0, T), \\
[[\mu \mathbf{G} \cdot \mathbf{n}] = k_1(G, H_k), \quad [[\mathbf{G} - \mathbf{H}, \mathbf{n} > \mathbf{n}]] = k_2(G, H_k), \quad \text{on } \Gamma \times (0, T), \\
\mathbf{u}_\pm = 0, \quad \mathbf{n}_\pm \cdot \mathbf{G}_\pm = 0, \quad (\text{curl} \mathbf{G}_\pm) \mathbf{n}_\pm = 0 \quad \text{on } S_\pm \times (0, T), \\
(u, G, h)|_{t=0} = (u_0, G_0, h_0) \quad \text{in } \hat{\Omega} \times \hat{\Gamma} \times \Gamma,
\end{aligned}
\]

where, \( H_k \) is a function satisfying Eq. (2.72) for \( h \).

The purpose of this paper is to prove the following local in time unique existence theorem.

**Theorem 2.3.** Let \( 2 < p < \infty, N < q < \infty \) and \( 2/p + N/q < 1 \) and \( B > 0 \). Assume that the condition (1.3) holds. Then, there exist a small number \( \epsilon \) and a small unique time \( T > 0 \) depending on \( B \) such that if initial data \( h_0 \in B_{q,p}^{1/p-1/4}(\Gamma) \) satisfies the smallness condition \( \|h_0\|_{B_{q,p}^{1/p-1/4}} \leq \epsilon \), and \( (v_0, H_0) \in B_{q,p}^{2(1-1/p)}(\hat{\Omega})^{2N} \) satisfies \( \|(v_0, H_0)\|_{B_{q,p}^{2(1-1/p)}(\hat{\Omega})} \leq B \) and the compatibility condition:

\[
\begin{aligned}
\text{div } v_0 &= 0 \quad \text{in } \hat{\Omega}, \\
\left[ i(\nu \mathbf{D}(v_0) + T_M(H_0) \mathbf{n}) \right]_\nu &= 0, \quad \left[ \mathbf{v}_0 \right] = 0 \quad \text{on } \Gamma, \\
\left[ [\alpha^{-1} \text{curl} H_0 + \mu (v_0 \otimes H_0 - H_0 \otimes v_0)] \mathbf{n} \right] &= 0, \quad \left[ [\mu H_0 \cdot \mathbf{n}] = 0, \quad \left[ H_0 - \mathbf{H}_0, \mathbf{n} > \mathbf{n} \right] = 0 \quad \text{on } \Gamma, \\
\mathbf{v}_{0\pm} &= 0, \quad \mathbf{n}_0 \cdot \mathbf{H}_{0\pm} = 0, \quad (\text{curl} \mathbf{H}_{0\pm}) \mathbf{n}_{0\pm} = 0 \quad \text{on } S_\pm,
\end{aligned}
\]

then, Eq. (2.66) admits unique solutions \( u, q, G, \) and \( h \) with

\[
\begin{aligned}
&u \in H^1_p((0, T), L^q(\hat{\Omega})) \cap L^p((0, T), H^2_q(\hat{\Omega})^N), \quad q \in L^p((0, T), H^1_q(\hat{\Omega})^N), \\
&G \in H^1_p((0, T), L^q(\hat{\Omega})) \cap L^p((0, T), H^2_q(\hat{\Omega})^N), \quad h \in H^1_p((0, T), W^{2-1/q}(\Gamma)^N) \cap L^p((0, T), W^{2-1/q}(\Gamma)), \\
&\|H_0\|_{L^\infty((0,T), H^2_q(\Gamma))} \leq \delta
\end{aligned}
\]

possessing the estimate:

\[
\begin{aligned}
&\|u, G\|_{L^p((0,T), H^2_q(\hat{\Omega}))} + \|\partial_t u, G\|_{L^p((0,T), L^q(\hat{\Omega}))} \\
&+ \|h\|_{L^p((0,T), W^{2-1/q}(\Gamma))} + \|\partial_t h\|_{L^p((0,T), W^{2-1/q}(\Gamma))} + \|\partial_{h\mu}\|_{L^\infty((0,T), W^{2-1/q}(\Gamma))} \leq f(B)
\end{aligned}
\]

Here, \( \delta \) is a constant appearing in (2.38) and \( f(B) \) is some polynomial of \( B \).
3 Linear Theory

Since the coupling of the velocity field and the magnetic field in (1.5) is semilinear, the linearized equations are decoupled. Namely, we consider the two linearized equations: one is the Stokes equations with transmission conditions on $\Gamma$ and non-slip conditions on $S_{\pm}$ and another is the system of the heat equations with transmission conditions on $\Gamma$ and the perfect wall conditions on $S_{\pm}$. In the following, we set $\Omega = \Omega_+\cup\Omega_-$ and $\Omega = \Omega_+\cup\Gamma$. And, we assume that $\Gamma$ is a compact hypersurface of $C^3$ class and that $S_{\pm}$ are hypersurfaces of $C^2$ class.

3.1 Two phase problem for the Stokes equations

This subsection is devoted to presenting the $L_p$-$L_q$ maximal regularity for the two phase problem of the Stokes equations with transmission conditions given as follows:

\[
\begin{cases}
\rho_0 u - \text{Div}(u, q) = f_1 & \text{in } \hat{\Omega} \times (0,T), \\
\text{div } u = g = \text{div } g & \text{in } \hat{\Omega} \times (0,T), \\
\partial_t h + \nabla G \cdot w_\kappa > -n \cdot u = d & \text{on } \hat{\Gamma} \times (0,T), \\
[u] = 0, \quad [[T(u, q)n]] - \sigma(ah + \Delta h) = h & \text{on } \hat{\Gamma} \times (0,T), \\
|u| = 0, \quad \mathbf{u} = 0 & \text{on } S_{\pm} \times (0,T), \\
(u, h)|_{t=0} = (u_0, h_0) & \text{in } \hat{\Omega} \times \Gamma.
\end{cases}
\] (3.1)

An assumption for equations (3.1) is the following:

(a.1) $a$ is a bounded $C^4$ functions defined on $\Omega$.

(a.2) $w_\kappa$ is a family of $N$-vector of functions defined on $\Gamma$ for $\kappa \in (0, 1)$ such that

\[
|w_\kappa(x)| \leq m_1, \quad |w_\kappa(x) - w_\kappa(y)| \leq m_1 |x - y|^b \text{ for any } x, y \in \Gamma, \quad \|w_\kappa\|_{W^{2-1/q}_q(\Gamma)} \leq m_2 \kappa^{-c}.
\]

Here, $m_1, m_2, b$ and $c$ are some positive constants and $r \in (N, \infty)$.

Theorem 3.1. Let $1 < p < \infty, 1 < q \leq r, 2/p + 1/q \neq 1, 2,$ and $T > 0$. Assume that the assumptions (a.1) and (a.2) are satisfied. Then, there exists $\gamma_0 > 0$ such that the following assertion holds: Let $u_0 \in L^{2p(1-1/p)}(\hat{\Omega}),$ and $h_0 \in L^{2p(1-1/p)}(\hat{\Gamma}).$ Let $f, g, h = (h', h_N),$ and $d$ appearing in the right-hand side of Eq. (3.1) be given functions satisfying the following conditions:

\[
f \in L_p((0,T), L_q(\hat{\Omega}))^N, \quad e^{-\gamma t} g \in L_p((\mathbb{R}, H^{1/2}_p(\hat{\Omega})) \cap H^{1/2}_p(\mathbb{R}, L_q(\hat{\Omega}))), \quad e^{-\gamma t} h \in L_p((\mathbb{R}, H^{1/2}_p(\hat{\Omega}))) \cap H^{1/2}_p((\mathbb{R}, L_q(\hat{\Omega}))),
\]

for any $\gamma \geq \gamma_0$ with some $\gamma_0$. Assume that $u_0, g,$ and $h$ satisfy the following compatibility conditions:

\[
\begin{align*}
\text{div } u_0 &= g|_{t=0} \quad \text{on } \hat{\Omega}, & \quad & \text{provided } 2/p + 1/q < 1, \\
[[\nu D(u_0)n]]_{t=0} &= h|_{t=0} \quad \text{on } \hat{\Gamma}, & \quad & \text{provided } 2/p + 1/q < 2, \\
[u_0] &= 0 \quad \text{on } \hat{\Gamma}, & \quad & \text{provided } 2/p + 1/q < 2,
\end{align*}
\]

where $d_{\gamma} = d - \gamma d, n > n.$ Then, Eq. (3.1) admit unique solutions $u, q,$ and $h$ with

\[
\begin{align*}
u_0 \in L_p((0,T), H^2_q(\hat{\Omega})), \quad & \kappa \in L_p((0,T), H^2_q(\hat{\Omega})), \quad q \in L_p((0,T), H^2_q(\hat{\Omega})), \\
\rho_0 \in L_p((0,T), W^{2-1/q}_q(\hat{\Omega})), \quad & \rho_0 \in L_p((0,T), W^{2-1/q}_q(\hat{\Omega})), \quad d \in L_p((0,T), W^{2-1/q}_q(\hat{\Gamma})),
\end{align*}
\]

possessing the estimates:

\[
\begin{align*}
&\|\partial_t u\|_{L_p((0,T), L_q(\hat{\Omega})))} + \|u\|_{L_p((0,T), H^{2}_q(\hat{\Omega})))} + \|\rho_0\|_{L_p((0,T), W^{2-1/q}_q(\hat{\Omega})))} + \|h\|_{L_p((0,T), W^{2-1/q}_q(\hat{\Gamma})))} \\
&\leq C e^{-\gamma \gamma_0} \|u_0\|_{B^{2(1-1/p)}_p(\hat{\Omega})} + \kappa^{-c} \|h_0\|_{B^{2(1-1/p)}_p(\hat{\Omega})} + \|f\|_{L_p((0,T), L_q(\hat{\Omega})))} + \|g\|_{L_p((0,T), L_q(\hat{\Gamma})))} + \|h\|_{L_p((0,T), L_q(\hat{\Omega})))} \\
&\quad + \|e^{-\gamma t} g\|_{L_p((0,T), L_q(\hat{\Omega})))} + \|e^{-\gamma t} h\|_{L_p(\mathbb{R}, L_q(\hat{\Omega})))} + \|e^{-\gamma t} h\|_{L_p((0,T), L_q(\hat{\Omega})))} + \|d\|_{L_p((0,T), L_q(\hat{\Omega})))}
\end{align*}
\]

for any $\gamma \geq \gamma_0$ with some constant $C > 0$ independent of $\gamma.$
Remark 3.2. (1) Theorem 3.3 has been proved in Shibata and Saito [12]. And the reason why we assume that $\Gamma$ is a compact in this paper is that the weak Neumann problem is uniquely solvable. Namely, if we consider the weak Neumann problem:

$$\langle \rho^{-1} \nabla u, \nabla \varphi \rangle_{\Omega} = \langle f, \nabla \varphi \rangle_{\Omega} \quad \text{for any } \varphi \in \mathcal{H}^1_{q'}(\Omega)$$

(3.5)

where we have set

$$\mathcal{H}^1_{q'}(\Omega) = \{ \varphi \in L_{q'}(\Omega) \mid \nabla \varphi \in L_{q'}(\Omega) \}, \quad q' = q/(q-1),$$

then for any $f \in L_{q'}(\Omega)^N$, problem (3.5) admits a unique solution $u \in \mathcal{H}^1_{q'}(\Omega)$ satisfying the estimate:

$$\|\nabla u\|_{L_q(\Omega)} \leq C\|f\|_{L_{q'}(\Omega)}$$

with some constant $C > 0$. If $\Gamma$ is unbounded, then in general we have to assume that the weak Neumann problem is uniquely solvable, except for a few cases like $\Gamma$ is flat, that is $\Gamma = \{ x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N = 0 \}$, or $\Gamma$ is asymptotically flat.

3.2 Two phase problem for the linear electro-magnetic field equations

This subsection is devoted to presenting the $L_p$-$L_q$ maximal regularity for the linear electro-magnetic field equations. The problem is formulated by the following equations:

$$\begin{cases}
\mu \partial_t \mathbf{H} - \alpha^{-1} \Delta \mathbf{H} = f & \text{in } \Omega \times (0, \infty), \\
[\alpha^{-1} \text{curl} \mathbf{H}] \cdot \mathbf{n} = \mathbf{h}' & \text{on } \Gamma \times (0, \infty), \\
[\mathbf{H} - \mathbf{H} \cdot \mathbf{n}] \cdot \mathbf{n} = \mathbf{k}' & \text{on } \Gamma \times (0, \infty), \\
\mathbf{n}_\pm \cdot \mathbf{H}_\pm = 0, \quad (\text{curl} \mathbf{H})_\pm = 0 & \text{on } S_\pm \times (0, \infty), \\
\mathbf{H}|_{t=0} = \mathbf{H}_0 & \text{in } \Omega.
\end{cases}$$

(3.6)

Theorem 3.3. Let $1 < p, q < \infty$, $2/p + 1/q \neq 1, 2$, and $T > 0$. Then, there exists a $\gamma_0$ such that the following assertion holds: Let $\mathbf{H}_0 \in L^2((1/p))\Omega)$ and let $f, \mathbf{h} = (\mathbf{h}', h_N)$, and $\mathbf{k} = (\mathbf{k}', k_N)$ be given functions appearing in the right-hand side of Eq. (3.6) and satisfying the following conditions:

$$\mathbf{f} \in L_p((0, T), L_q(\hat{\Omega})^N), \quad e^{-\gamma T} \mathbf{h} \in L_p(\mathbb{R}, H^1_{\alpha}(\hat{\Omega})^N) \cap H^{1/2}_p(\mathbb{R}, L_q(\hat{\Omega})^N),$$

$$e^{-\gamma T} \mathbf{k} \in L_p(\mathbb{R}, H^2_{\alpha}(\hat{\Omega})^N) \cap H^{1/2}_p(\mathbb{R}, L_q(\hat{\Omega})^N)$$

for any $\gamma \geq \gamma_0$. Assume that $\mathbf{f}$, $\mathbf{h}$ and $\mathbf{k}$ satisfy the following compatibility conditions:

$$[[\alpha^{-1} \text{curl} \mathbf{H}_0]] \mathbf{n} = \mathbf{h}'|_{t=0}, \quad [[\text{div} \mathbf{H}_0]] = h_N|_{t=0} \quad \text{on } \Gamma, \quad [[\text{curl} \mathbf{H}_0]] \mathbf{n}_\pm = 0 \quad \text{on } S_\pm$$

(3.7)

provided $2/p + 1/q < 1$;

$$[[\mathbf{H}_0 - \mathbf{H}_0 \cdot \mathbf{n}]] \cdot \mathbf{n} = \mathbf{k}'|_{t=0}, \quad [[\mu \mathbf{H}_0 \cdot \mathbf{n}]] = k_N|_{t=0} \quad \text{on } \Gamma, \quad \mathbf{n}_\pm \cdot \mathbf{H}_0 \pm = 0 \quad \text{on } S_\pm$$

(3.8)

provided $2/p + 1/q < 2$. Then, problem (3.6) admits a unique solution $\mathbf{H}$ with

$$\mathbf{H} \in L_p((0, T), H^2_{\alpha}(\hat{\Omega})^N) \cap H^1_p((0, T), L_q(\hat{\Omega})^N)$$

possessing the estimate:

$$\|\partial_t \mathbf{H}\|_{L_p((0, T), L_q(\Omega))} + \|\mathbf{H}\|_{L_p((0, T), H^2_{\alpha}(\Omega))} \leq Ce^{\gamma T} \left\{ \|\mathbf{H}_0\|_{H^{2(1-1/p)}_{\alpha}(\Omega)} + \|\mathbf{f}\|_{L_p(\mathbb{R}, L_q(\Omega))} \right\}$$

for any $\gamma \geq \gamma_0$ with some constant $C > 0$ independent of $\gamma$.

Remark 3.4. Theorem 3.3 was proved by Frolova and Shibata [5] under the assumption that $\Omega$ is a uniformly $C^3$ domain. Of course, if $\Gamma$ is a compact hypersurface of $C^3$ class, then $\Omega$ is a uniform $C^3$ domain.
4 Estimates of non-linear terms

First of all, we give an iteration scheme to prove Theorem 2.3 by the Banach fixed point theorem. Given \( h \) satisfying (2.2), let \( H_h \) be a unique solution of equation (2.2) satisfying (2.4) and (2.5). Let \( U_T \) be an underlying space defined by setting

\[
U_T = \{(u, G, h) \mid (u, G) \in L^p((0,T), \tilde{H}^2(\Omega)^2) \cap L^p((0,T), \tilde{H}^2(\Omega)^2),
\]

\[
h \in L^p((0,T), W^{2-1/q}(\Gamma)) \cap H^1((0,T), W^{2-1/q}(\Gamma)), \tag{4.1}
\]

\((u, G, h)|_{t=0} = (u_0, G_0, h_0) \in \Omega \times \Omega \times \Gamma, \quad E_T(u, G, h) \leq L, \quad \|H_h\|_{L^\infty((0,T), H^1(\Omega))} \leq \delta,\]

where we have set

\[
E_T(u, G, h) = E^1_T(u) + E^1_T(G) + E^2_T(h) + \|h\|_{L^\infty((0,T), W^2-1/q(\Gamma))},
\]

\[
E^1_T(w) = \|w\|_{L^p((0,T), H^1(\Omega))} + \|\partial_t w\|_{L^p((0,T), L^q(\Omega))} \quad w \in \{u, G\}, \tag{4.2}
\]

\[
E^2_T(h) = \|h\|_{L^p((0,T), W^{2-1/q}(\Gamma))} + \|\partial_t h\|_{L^p((0,T), W^{2-1/q}(\Gamma))}.
\]

For initial data \( u_0, G_0, \) and \( h_0, \) we assume that

\[
\|u_0\|_{H^{2(1-\rho)/q}(\Omega)} \leq B, \quad \|G_0\|_{H^{2(1-\rho)/q}(\Omega)} \leq B, \quad \|h_0\|_{H^{2(1-\rho)/q}(\Gamma)} \leq \epsilon. \tag{4.3}
\]

Here, \( B \) is a given positive number. Since we mainly consider the case where \( u_0 \) and \( G_0 \) are large, we may assume that \( B > 1 \) in the following. And we shall choose \( L > 0 \) large enough and \( \epsilon > 0 \) small enough, and so we may assume that \( 0 < \epsilon < 1 < \epsilon \). Given \((u, G, h) \in U_T\), let \((v, q, \rho)\) be solutions of the equations:

\[
\begin{cases}
\rho \partial_t v - \text{Div} \mathbf{T}(v, q) = f_1(u, G, H_h), & \text{in } \Omega \times (0, T), \\
\text{Div} v = g(u, H_h) = \text{Div} g(u, H_h) & \text{in } \Omega \times (0, T), \\
\partial_t \rho < \nabla \Gamma^\rho \perp u, > -\mathbf{n} \cdot v = d(u, H_h) & \text{on } \Gamma \times (0, T), \\
||v|| = 0, \quad [\mathbf{T}(v, q)n] - \sigma(\Delta \rho + a\rho)\mathbf{n} = h_1(u, G, H_h), & \text{on } \Gamma \times (0, T), \\
\mathbf{v}_\perp = 0 & \text{on } \Omega \times (0, T), \\
(v, \rho)|_{t=0} = (u_0, h_0) & \text{in } \Omega \times \Gamma.
\end{cases} \tag{4.4}
\]

And, let \( H \) be a solution of the equations:

\[
\mu \partial_t H - \alpha^{-1} \Delta H = f_2(u, G, H_h), \quad \text{in } \Omega \times (0, T),
\]

\[
[[\alpha^{-1} \text{curl} H]\mathbf{n}] = h_2(u, G, H_h), \quad [[\text{Div} H]\mathbf{n}] = h_3(u, G, H_h) \quad \text{on } \Gamma \times (0, T),
\]

\[
[[\mu \mathbf{H} \cdot \mathbf{n}] = k_1(G, H_p), \quad [[H - H, \mathbf{n} > \mathbf{n}]] = k_2(G, H_p) \quad \text{on } \Omega \times (0, T),
\]

\[
\mathbf{n}_\perp \cdot H_\perp = 0, \quad (\text{curl} H_\perp)n_\perp = 0 \quad \text{on } \Omega \times (0, T),
\]

\[
H|_{t=0} = G_0 \quad \text{in } \tilde{\Omega}.
\]

Notice that to define \( H \) we use not only \( H_h \) but also \( H_p \), unlike Padula and Solonnikov [11] to avoid their technical assumption that the velocity field is slightly regular than the magnetic field.

In this section, we shall show the estimates of the nonlinear terms appearing in the right sides of equations (4.2) and (4.5). Since \((u, G, h) \in U_T\), we have

\[
E_T(u, G, h) \leq L, \tag{4.6}
\]

\[
\|H_h\|_{L^\infty((0,T), H^1(\Omega))} \leq \delta. \tag{4.7}
\]

Below, we assume that \( 2 < p < \infty \), \( N < q < \infty \) and \( 2/p + N/q < 1 \). We use the following inequalities which follows from Sobolev’s inequality:

\[
\|f\|_{L^\infty(\Omega)} \leq C\|f\|_{H^1(\Omega)},
\]

\[
\|fg\|_{H^1(\Omega)} \leq C(\|f\|_{H^1(\Omega)}\|g\|_{H^1(\Omega)}),
\]

\[
\|fg\|_{H^2(\Omega)} \leq C(\|f\|_{H^2(\Omega)}\|g\|_{H^1(\Omega)} + \|f\|_{H^2(\Omega)}\|g\|_{H^2(\Omega)}),
\]

\[
\|fg\|_{L^{2-1/q}(\Gamma)} \leq C(\|f\|_{W^{2-1/q}(\Gamma)}\|g\|_{W^{2-1/q}(\Gamma)}) + \|f\|_{L^\infty(\Omega)}\|g\|_{L^{2-1/q}(\Gamma)}).
\]
For any \( C^k \) function, \( f(u) \), defined for \(|u| \leq \sigma \), we consider a composite function \( f(u(x)) \), and then for \( N < q < \infty \), we have

\[
\|\nabla (f(u)v)\|_{L_q(\Omega)} \leq C\|f, f'\|_{L_\infty(1 + \|\nabla u\|_{L_q(\Omega)})}}\|v\|_{H^1_q(\Omega)}\|w\|_{H^1_q(\Omega)};
\]

\[
\|\nabla^2 (f(u)v)\|_{L_q(\Omega)} \leq C\|f, f', f''\|_{L_\infty(1 + \|\nabla u\|_{H^1_q(\Omega)}^{1/k}\|v\|_{H^1_q(\Omega)} + \|v\|_{H^1_q(\Omega)})\|\nabla w\|_{H^1_q(\Omega)} (4.9)
\]

provided that \( \|u\|_{L_\infty(\Omega)} \leq \sigma \). We use the following estimate of the time trace proved by a real interpolation theorem:

\[
\|w\|_{L_\infty(0,T)B_{p,q}^{\infty,1}(\Omega)} \leq C\|w\|_{L_p(0,T)B_{p,q}^{\infty,1/2}(\Omega)} + \|\partial_t w\|_{L_p((0,T),B_{p,q}^{\infty,1/2}(\Omega))} \leq C(B + L) (4.10)
\]

for \( w \in \{u, G\} \),

\[
\|h\|_{L_\infty(0,T)B_{\infty,q}^{1/2,1}(\Gamma)} \leq C\|\rho_0\|_{B_{\infty,q}^{1/2,1}(\Gamma)} + \|\partial_t h\|_{L_p((0,T)B_{\infty,q}^{1/2,1}(\Gamma))} \leq CL, . (4.11)
\]

And we have

\[
\|h\|_{L_\infty(0,T)W^{2-1/q}_q(\Gamma)} \leq \|\rho_0\|_{W^{2-1/q}_q(\Gamma)} + T^{1/\tilde{p}}\|\partial_t h\|_{L_p((0,T)W^{2-1/q}_q(\Gamma))} \leq \epsilon + T^{1/\tilde{p}}L. (4.12)
\]

In what follows, we assume that \( 0 < \epsilon = T = \kappa < 1 \) and \( 1 \leq B, L \). In particular, \( \epsilon + LT^{1/\tilde{p}} \leq T + LT^{1/\tilde{p}} \leq 2LT^{1/\tilde{p}} \). In what follows, we assume that \( LT^{1/\tilde{p}} \leq 1 \), and so by (4.12),

\[
\|h\|_{L_\infty(0,T)W^{2-1/q}_q(\Gamma)} \leq LT^{1/\tilde{p}} \quad \|h\|_{L_\infty(0,T)W^{2-1/q}_q(\Gamma)} \leq 1. (4.13)
\]

We first estimate \( f_1(u, G, H_k) \). In view of (2.41), we may write

\[
f_1(u, G, H_k) = V_{f_1}(\cdot, \nabla H_k)(\nabla H_k \otimes (\partial_t u, \nabla^2 u) + \partial_t H_k \otimes \nabla u + u \otimes \nabla u + \nabla^2 H_k \otimes \nabla u + G \otimes \nabla G), (4.14)
\]

where \( V_{f_1}(y, K) \) is a matrix of bounded functions defined on \( \Omega \times \{K \in \mathbb{R}^{N+1} | |K| \leq \delta \} \). Applying (2.23), (2.24), and (4.18), we have

\[
\|f_1(u, G, H_k)\|_{L_q(\Omega)} \leq C\|\partial_t \nabla u, \nabla^2 u\|_{L_q(\Omega)} + \|\partial_t H_k, \nabla^2 H_k\|_{L_q(\Omega)} + \|\nabla u\|_{H^1_q(\Omega)} + \|\nabla^2 H_k\|_{H^1_q(\Omega)} + \|G\|_{H^1_q(\Omega)}^2.
\]

Here and in the following, we write

\[
\|f\|_{L_p((a,b))} = \left( \int_a^b |f(t)|^p \, dt \right)^{1/p}.
\]

For a maximal regularity term, \( f \), and a lower order term, \( g \), we estimate

\[
\|f g\|_{L_p(0,T)} \leq \|f\|_{L_p(0,T)} \|g\|_{L_\infty(0,T)}.
\]

And only in the lower order term, \( g \), case, we estimate

\[
\|g\|_{L_p(0,T)} \leq T^{1/p}\|g\|_{L_\infty(0,T)}.
\]

Thus, using (2.23), we have

\[
\|f_1(u, G, H_k)\|_{L_p((0,T),L_q(\Omega))} \leq C\|\partial_t \nabla u, \nabla^2 u\|_{L_q(\Omega)} + \|\nabla u\|_{L_p((0,T),H^1_q(\Omega))} + \|G\|_{H^1_q(\Omega)}^2 (4.15).
\]
By (2.5), (4.6), (4.10), (4.11), and (4.13), we have
\[
\|w\|_{L^\infty((0,T),H^s_0(\Omega))} \leq C(B + L) \quad \text{for } w \in \{u, G\},
\]
\[
\|\partial_tw\|_{L^p((0,T),L^q(\Omega))} + \|w\|_{L^p((0,T),H^s_0(\Omega))} \leq L \quad \text{for } w \in \{u, G\},
\]
\[
\|\partial_t h\|_{L^\infty((0,T),W^{1-1/q}_q(\Gamma))} \leq C E + (u, G, h) \leq CL,
\]
\[
\|\partial_t h\|_{L^p((0,T),W^{2-1/q}_q(\Gamma))} + \|h\|_{L^p((0,T),W^{3-1/q}_q(\Gamma))} \leq L.
\]

Thus, by (4.13) and (4.16)
\[
\|f_1(u, G, H_k)\|_{L^p((0,T),L^q(\Omega))} \leq C\{T^{1/p'} L^2 + T^{1/p}(L + B)^2\}.
\]

Since $1/p' > 1/p$ as follows from $1 < p' = p/(p - 1) < 2 < p < \infty$, we have
\[
\|f_1(u, G, H_k)\|_{L^p((0,T),L^q(\Omega))} \leq CT^{1/p}(B + L)^2.
\]

We next estimate $d(u, H_k)$ given in (2.14). We shall prove that
\[
\|d(u, H_k)\|_{L^\infty((0,T),W^{1-1/q}_q(\Gamma))} \leq CL(B + L)T^{1/p'};
\]
\[
\|d(u, H_k)\|_{L^p((0,T),W^{2-1/q}_q(\Gamma))} \leq C_s L^2 (B + L) T^{2/(1 + s)},
\]
where $s$ is a constant for which $s \in (0, 1 - 2/p)$. Here and in the following, $C_s$ is a generic constant depending on $s$, whose value may change from line to line.

In fact, by (2.5), (4.7), (4.8), and (4.9),
\[
\|d(u, H_k)\|_{W^{1-1/q}_q(\Gamma)} \leq C\{\|h\|_{W^{2-1/q}_q(\Gamma)}\|\|u + u_k\|_{H^1_0(\Omega)}
\]
\[
+ (1 + \|h\|_{W^{2-1/q}_q(\Gamma)})\|\|\|u\|_{H^1_0(\Omega)} + \|\partial_t h\|_{W^{1-1/q}_q(\Gamma)}\| u - u_k\|_{H^2_0(\Omega)}
\]
\[
+ \|h\|_{W^{3-1/q}_q(\Gamma)}\|\|u\|_{H^1_0(\Omega)} + \|\partial_t h\|_{W^{1-1/q}_q(\Gamma)}\| u - u_k\|_{H^2_0(\Omega)}
\]
\[
+ (\|\|h\|_{H^1_0(\Omega)} + \|\partial_t h\|_{W^{2-1/q}_q(\Gamma)}\| h\|_{W^{2-1/q}_q(\Gamma)} + (\|\|h\|_{H^1_0(\Omega)} + \|\partial_t h\|_{W^{2-1/q}_q(\Gamma)}\| h\|_{W^{2-1/q}_q(\Gamma)}).
\]

By (2.12), we have
\[
\|u + u_k\|_{L^\infty((0,T),H^1_0(\Omega))} \leq C(\|u\|_{L^\infty((0,T),H^1_0(\Omega))} + \|u_0\|_{L^\infty((0,T),H^2_{0,p}(1-1/p)(\Omega)}
\]
\[
\leq C(L + B),
\]
and so by (4.13) and (4.16)
\[
\|d(u, H_k)\|_{L^\infty((0,T),W^{1-1/q}_q(\Gamma))} \leq C\{T^{1/p'} L(B + L) + (1 + T^{1/p'} L)(T^{1/p'} L)^2(B + L)\}
\]
\[
\leq CT^{1/p'} L(B + L),
\]
which shows the first inequality in (4.18).

To prove the second inequality in (4.18), we use the estimates:
\[
\|u - u_k\|_{L^p((0,T),H^2_0(\Omega))} \leq C(L + B),
\]
\[
\|u - u_k\|_{L^\infty((0,T),H^2_0(\Omega))} \leq C_s T^{2/(1 + s)} (L + B).
\]

Here, $s$ is a fixed constant for which $0 < s < 1 - 2/p$. In fact, by (2.12) and (4.3)
\[
\|u - u_k\|_{H^2_0(\Omega)} \leq C(||u(\cdot, t)||_{H^2(\Omega)} + \kappa^{-1/p} B),
\]
and so by (4.6) and (4.16) we have
\[
\|u - u_k\|_{L^p((0,T),H^2_0(\Omega))} \leq C(L + T^{1/p} \kappa^{-1/p} B) \leq C(L + B),
\]

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because we have taken $\kappa = T$. This shows the first inequality in (4.21). For $t \in (0, T)$ by (2.41), (4.3) and (4.6)

$$
\|u - u_k\|_{L^q(\Omega_T)} \leq \|u - u_0\|_{L^q(\Omega_T)} + \|u - u_k\|_{L^q(\Omega_T)}
$$

$$
\leq \int_0^T \|u(t)\|_{L^q(\Omega_T)} dt + \frac{1}{\kappa} \int_0^\kappa \|T_0(s)\tilde{u}_0^+ - u_0\|_{L^q(\Omega_T)} ds
$$

$$
\leq T^{1/p'} L + \frac{1}{\kappa} \int_0^\kappa \left( \int_0^s \|\partial_r T_0(r)\tilde{u}_0^+\|_{L^q(\Omega_T)} dr \right) ds
$$

$$
\leq T^{1/p'} L + C \kappa^{1/p'} B \leq C(L + B)T^{1/p'} .
$$

By real interpolation,

$$
\|u - u_k\|_{H^1(\Omega_T)} \leq C_s\|u - u_k\|_{L^q(\Omega_T)}^{1/(1+s)} \|u - u_k\|_{W^{1,q}(\Omega_T)}^{1/(1+s)} \leq C_s\|u - u_k\|_{L^q(\Omega_T)}^{1/(1+s)} \|u - u_k\|_{B^{2(1-1/p)}_{q,p}(\Omega)}^{1/(1+s)}
$$

for any $s \in (0, 1/2)$, which, combined with (2.42) and (4.10), yields the second inequality in (4.21).

Applying (4.13), (4.16), and (4.21) to the second inequality in (4.19) yields that

$$
\|u_t\|_{L^p(0,T), W^{2-1/q}_{q,q}} \leq C(L(B + L))^{1/p'} + C_s T^{1/p'} L(L + B)
$$

$$
+ LLT^{1/p'} (L + B) + LLT^{1/p'} + (L + B)LLT^{1/p'}
$$

$$
\leq CT^{3/2} L^2 (L + B),
$$

which proves the second inequality in (4.15).

We now estimate $g(u, H_0)$, $g(u, H_0)$ and $h_1(u, G, H_0)$ given in (2.22) and (2.20) respectively. We have to extend them to the whole time line $\mathbb{R}^1$. For this purpose, we first define operators which have nice behaviour at infinity in time and whose initial values are $u_0 \in L^2(\Omega)$. Let $w_{0,\pm} \in B^{2(1-1/p)}_{q,p}(\mathbb{R}^N)$ be extensions of $\mathcal{E}_x(w_{0,\pm})$ to $\mathbb{R}^N$ such that

$$
w_{0,\pm} = \mathcal{E}_x(w_{0,\pm}) \quad \text{in} \quad \Omega, \quad \|w_{0,\pm}\|_{B^{2(1-1/p)}_{q,p}(\mathbb{R}^N)} \leq C\|w_{0}\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \leq CB.
$$

Let $\gamma_0$ be a large positive number appearing in Theorem 3.1 and Theorem 3.3 and we fix $\gamma_1$ in such a way that $\gamma_1 > \gamma_0$. Let $T_t(0)w_{0,\pm}$ be defined by setting

$$
T_t(0)w_{0,\pm} = e^{-\gamma_1 t L}w_{0,\pm} = \mathcal{F}^{-1}[e^{-\gamma t L + 2\gamma t L}\mathcal{F}[\tilde{w}_{0,\pm}](\xi)].
$$

In particular, $T_t(0)w_{0,\pm}$ in $\Omega_\pm$, $T_t(0)w_{0,\pm} = \mathcal{E}_x(w_{0,\pm})$ in $\Omega$, and

$$
\|e^{\gamma t L}T_t(0)w_{0,\pm}\|_{H^1((0,\infty),L^q(\Omega))} + \|e^{\gamma t L}T_t(0)w_{0,\pm}\|_{L^p((0,\infty),H^2_q(\Omega))} \leq CB . \tag{4.22}
$$

We also construct a similar operator for $H_0$. Let $W$, $P$, and $\Xi$ be solutions of the equations:

$$
\rho \partial_t W + \lambda_0 W - \text{div} T(W, P) = 0, \quad \text{div} W = 0 \quad \text{in} \quad \hat{\Omega} \times (0, \infty),
$$

$$
\partial_t \Xi + \lambda_0 \Xi - W \cdot n = 0 \quad \text{on} \quad \Gamma \times (0, \infty),
$$

$$
[[T(W, P)n]] - \sigma(\Delta \Xi)n = 0, \quad \|W\| = 0 \quad \text{on} \quad \Gamma \times (0, \infty),
$$

$$
W_{\pm} = 0 \quad \text{on} \quad S_{\pm} \times (0, \infty),
$$

$$
(W, \Xi)_{t=0} = (0, h_0) \quad \text{in} \quad \hat{\Omega} \times \Gamma.
$$

For large $\lambda_0 > 0$ we know the unique existence of $W$, $P$, and $\Xi$ with

$$
W \in L^p((0, \infty), L^q(\hat{\Omega})) \cap L^p((0, \infty), H^2_q(\hat{\Omega})),
$$

$$
\Xi \in H^1_p((0, \infty), W^{2-1/q}_q(\Gamma)) \cap L^p((0, \infty), W^{3-1/q}_q(\Gamma)).
$$

possessing the estimate:

$$
\|e^{\gamma t L} W \|_{H^1_p((0,\infty),L^q(\hat{\Omega}))} + \|e^{\gamma t L} W \|_{L^p((0,\infty),H^2_q(\hat{\Omega}))} + \|e^{\gamma t L} W \|_{L^p((0,\infty),W^{2(1-1/p)}_q(\hat{\Omega}))}
$$

$$
+ \|e^{\gamma t L} \Xi \|_{H^1_p((0,\infty),W^{2-1/q}_q(\Gamma))} + \|e^{\gamma t L} \Xi \|_{L^p((0,\infty),W^{3-1/q}_q(\Gamma))} + \|e^{\gamma t L} \Xi \|_{L^p((0,\infty),W^{2(1-1/p)}_q(\Gamma))}.
$$
Let $T_h(t)h_0 = H_{\Xi}(x, t)$, where $H_{\Xi}$ is a unique solution of (2.22) with $h = \Xi$, and then by (2.25) we have

$$E\|e^{\gamma t}T_h(h_0)|H^2_{\xi}(\omega) + |e^{\gamma t}T(h)_0|\|_{L^2_{\infty}(0, \infty)}h^2_{\xi}(\omega))$$
$$+ |e^{\gamma t}T(h)_0|\|_{L^2_{\infty}(0, \infty)}h^2_{\xi}(\omega)) + |e^{\gamma t}T(h)_0|\|_{L^2_{\infty}(0, \infty)}h^2_{\xi}(\omega)) \leq C\epsilon.$$  \hspace{1cm} (4.23)

In what follows, a generic constant $C$ depends on $\gamma_1$ when we use (4.22) and (4.23), but $\gamma_1$ is eventually fixed in such a way that the estimates given in Theorem 3.1 and Theorem 3.3 hold, and so we do not mention the dependence on $\gamma_1$.

Given function, $f(t)$, defined on $(0, T)$, an extension, $e_T[f]$, of $f$ is defined by setting

$$e_T[f] = \begin{cases} 0 & \text{for } t < 0, \\ f(t) & \text{for } 0 < t < T, \\ f(2T - t) & \text{for } T < t < 2T, \\ 0 & \text{for } t > 2T. \end{cases}$$

Obviously, $e_T[f] = f$ for $t \in (0, T)$ and $e_T[f]$ vanishes for $t \notin (0, 2T)$. Moreover, if $f|_{t=0} = 0$, then

$$\partial_t e_T[f] = \begin{cases} 0 & \text{for } t < 0, \\ \partial_t f(t) & \text{for } 0 < t < T, \\ -(\partial_t f)(2T - t) & \text{for } T < t < 2T, \\ 0 & \text{for } t > 2T. \end{cases}$$

If $f \in L^p((0, T), X)$ with some Banach space $X$ and $f|_{t=0} = 0$, then

$$\|e_T[f]\|_{L^p(\mathbb{R}, X)} \leq 2\|f\|_{L^p((0, T), X)} \quad (1 \leq p \leq \infty),$$
$$\|e_T[f]\|_{L^p(\mathbb{R}, X)} \leq 2T^{1/p}\|f\|_{L^p((0, T), X)} \quad (1 \leq p \leq \infty).$$

Moreover, if $f|_{t=0} = 0$, then $e_T[f](t) = \int_0^t \partial_t e_T[f] ds$, and so

$$\|e_T[f]\|_{L^p(\mathbb{R}, X)} \leq 2(2T)^{1/p'}\|f\|_{L^p((0, T), X)} \quad (1 < p < \infty, \ p' = p/(p - 1)),$$

because $e_T[f]$ vanishes for $t \notin (0, 2T)$.

Let $\psi \in C^\infty(\mathbb{R})$ which equals one for $t > -1$ and zero for $t < -2$. Under these preparations, for $w \in \{u, G\}$ and $H_h$, we define the preparations, $E_1[w_\pm], E_1[tr[w]],$ and $E_2[H_h]$ by letting

$$E_1[w_\pm] = e_T[w_\pm - T_h(t)w_0_\pm \pm \psi(t)T_h(|t|)w_0_\pm, + $$
$$E_1[tr[w]] = e_T[tr[w] - T_h(t)tr[w_0]], + $$
$$E_2[H_h] = e_T[H_h - T_h(t)h_0], + $$

Here, we have set $tr[w_0] = w_0_+ - w_0_-$. Notice that $w_\pm - T_h(t)w_0_\pm = 0$ for $t = 0, tr[w] - T_h(t)tr[w_0] = 0$ for $t = 0, H_h - T_h(t)h_0 = 0$ for $t = 0. Obviously,

$$E_1[u_\pm] = u_\pm, \ E_1[tr[w]] = tr[w], \ E_2[H_h] = H_h \text{ for } 0 < t < T.$$ \hspace{1cm} (4.26)

By (1.16), (1.22) and (1.23), we have

$$\|e^{-\gamma t}E_1[w]\|_{L^p(\mathbb{R}, H^2_{\xi}(\omega))} \leq C(\epsilon^{2(\gamma - \gamma_1)}B + L),$$
$$\|e^{-\gamma t}E_1[w]\|_{L^p(\mathbb{R}, H^2_{\xi}(\omega))} \leq C(\epsilon^{2(\gamma - \gamma_1)}B + C\epsilon(\epsilon + L)T^{\gamma_1 - \gamma_1}), \hspace{1cm} (4.27)$$
$$\|E_2[H_h]\|_{L^p(\mathbb{R}, H^2_{\xi}(\omega))} \leq C(\epsilon + L)T^{\gamma_1 - \gamma_1}, \hspace{1cm} (4.27)$$

where $w \in \{u, G, tr[u], tr[G]\}$. In fact, the first and third inequalities in (4.27) follow from (4.22), (4.23) and (4.3). To prove the second inequality in (4.23), we observe that

$$\|e_T[w - T_h(t)w_0]\|_{L^2(\omega)} \leq \int_0^t \|\partial_t e_T[w - T_h(t)w_0]\|_{L^2(\omega)} ds$$

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\[ \left\| e^s w - T_v(t) w_0 \right\|_{W^{1,s}_q(\Omega)} \leq C_s \left\| e^s w - T_v(t) w_0 \right\|_{H^2(\Omega)} \leq C_s (B + L), \]

for any \( s \in (0, 1 - 2/p) \). Thus, using the interpolation inequality: \( \left\| v \right\|_{H^1(\Omega)} \leq C \left\| v \right\|^{1/(1+s)}_{L_q(\Omega)} \left\| v \right\|^{1/(1+s)}_{W^{1,s}_q(\Omega)} \), we have the second inequality in \( 4.27 \). By \( 4.22 \) and \( 4.13 \),

\[ \left\| \psi(t) T_v(t) h_0 \right\|_{L_q(R, H^1_q(\Omega))} \leq C \epsilon, \]

\[ \left\| e^s T_v(t) h_0 \right\|_{L_q(R, H^1_q(\Omega))} \leq C \left( \left\| H^1_q(\Omega) \right\|_{L_q(0,T), H^2_q(\Omega))} + \left\| T^s h_0 \right\|_{L_q((0,\infty), H^2_q(\Omega))} \right) \leq C(\epsilon + T^{1/p} L), \]

and so we have the last inequality in \( 4.27 \).

Choosing \( \epsilon > 0 \) and \( T > 0 \) small enough in the last inequality in \( 4.27 \), we may assume that

\[ \sup_{t \in R} \left\| \mathcal{E}_2[H^1_q(\Omega)] \right\|_{L_q(\Omega)} \leq \delta. \]  

(4.28)

And also,

\[ \left\| \mathcal{E}_2[H^1_q(\Omega)] \right\|_{L_q(R, H^1_q(\Omega))} \leq CLT^{1/p}, \quad \left\| \mathcal{E}_2[H^1_q(\Omega)] \right\|_{L_q(R, H^1_q(\Omega))} \leq 1. \]  

(4.29)

To estimate \( H^1_q(\Omega) \), we use the following lemmata.

**Lemma 4.1.** Let \( 1 < p < \infty \) and \( N < q < \infty \). Let

\[ f \in L_\infty(R, H^1_q(\Omega)) \cap L_\infty(R, L_q(\Omega)), \quad g \in H^1_q(\Omega) \cap L_p(R, H^1_q(\Omega)). \]

Then, we have

\[ \|f\|_{H^1_q(R, L_q(\Omega))} + \|f\|_{L_p(R, H^1_q(\Omega))} \leq C \left( \|\partial_t f\|_{L_p(R, L_q(\Omega))} + \|f\|_{L_p(R, H^1_q(\Omega))} \right)^{1/2} \left( \|g\|_{H^1_q(R, L_q(\Omega))} + \|g\|_{L_p(R, H^1_q(\Omega))} \right). \]

**Proof.** To prove Lemma 4.1, we use the fact that

\[ H^1_q(\Omega) \cap L_p(R, H^1_q(\Omega)) = (L_p(R, L_q(\Omega)), H^1_q(R, L_q(\Omega))) \cap L_p(R, H^1_q(\Omega)), \]

where \( (\cdot, \cdot)_{1/2} \) denotes a complex interpolation functor of order 1/2. We have

\[ \|f\|_{H^1_q(R, L_q(\Omega))} + \|f\|_{L_p(R, H^1_q(\Omega))} \leq C \left( \|\partial_t f\|_{L_p(R, L_q(\Omega))} \right) \left( \|g\|_{H^1_q(R, L_q(\Omega))} + \|g\|_{L_p(R, H^1_q(\Omega))} \right). \]

Moreover,

\[ \|f\|_{L_p(R, L_q(\Omega))} \leq C \|f\|_{L_p(R, H^1_q(\Omega))} \|g\|_{L_p(R, L_q(\Omega))}. \]

Thus, by complex interpolation, we have

\[ \|f\|_{H^1_q(R, L_q(\Omega))} + \|f\|_{L_p(R, H^1_q(\Omega))} \leq C \left( \|\partial_t f\|_{L_p(R, L_q(\Omega))} + \|f\|_{L_p(R, H^1_q(\Omega))} \right)^{1/2} \left( \|g\|_{H^1_q(R, L_q(\Omega))} + \|g\|_{L_p(R, H^1_q(\Omega))} \right). \]

Moreover, we have

\[ \|f\|_{L_p(R, H^1_q(\Omega))} \leq C \|f\|_{L_p(R, H^1_q(\Omega))} \|g\|_{L_p(R, H^1_q(\Omega))}. \]

Thus, combining these two inequalities give the required estimate, which completes the proof of Lemma 4.1.

**Lemma 4.2.** Let \( 1 < p, q < \infty \). Then,

\[ H^1_q(R, L_q(\Omega)) \cap L_p(R, H^1_q(\Omega)) \subset H^1_q(R, H^1_q(\Omega)) \]

and

\[ \|u\|_{H^1_q(R, H^1_q(\Omega))} \leq C(\|u\|_{L_p(R, H^1_q(\Omega))} + \|\partial_t u\|_{L_p(R, L_q(\Omega))}). \]
Proof. For a proof, see Shibata [13] Proposition 1].

We now estimate \( \mathbf{h}_1(\mathbf{u}, \mathbf{G}, H_b) \). In view of (2.30), we define an extension of \( \mathbf{h}_1(\mathbf{u}, \mathbf{G}, H_b) \) to the whole time interval \( \mathbb{R} \) by setting \( \mathbf{h}_1(\mathbf{u}, \mathbf{G}, H_b) = A^1 + A^2 + A^3 \) with

\[
A^1 = \bar{V}_1(\cdot, \nabla E_2[H_\mathbf{b}] - \nabla E_1[H_\mathbf{b}] \otimes \nabla \mathbf{E}_t[tr[\mathbf{u}]]),
\]
\[
A^2 = a(\cdot) E_1[tr[\mathbf{G}]] \otimes E_1[tr[\mathbf{G}]] + \bar{V}_2(\cdot, \nabla E_2[H_\mathbf{b}] - \nabla E_2[H_\mathbf{b}] \otimes E_1[tr[\mathbf{G}]] \otimes E_1[tr[\mathbf{G}]],
\]
\[
A^3 = \bar{V}_3(\cdot, \nabla E_2[H_\mathbf{b}] - \nabla E_2[H_\mathbf{b}] \otimes \nabla \mathbf{E}_2[H_\mathbf{b}].
\]

(4.30)

Obviously, \( \mathbf{h}_1(\mathbf{u}, \mathbf{G}, H_b) = \mathbf{h}_1(\mathbf{u}, \mathbf{G}, H_b) \) for \( t \in (0, T) \). To estimate \( A^1 \), for notational simplicity we set \( \mathcal{V}^1 = \bar{V}_1(\cdot, \nabla E_2[H_\mathbf{b}] - \nabla E_2[H_\mathbf{b}] \). By (4.28) and (4.27),

\[
\| \partial_t \mathcal{V}^1 \|_{L_\infty((\mathbb{R}, H_\mathbf{b}^1(\Omega)))} \leq C \| \partial_t E_1[H_\mathbf{b}] \|_{L_\infty((\mathbb{R}, H_\mathbf{b}^1(\Omega)))} \leq CL,
\]
\[
\| \mathcal{V}^1 \|_{L_\infty((\mathbb{R}, H_\mathbf{b}^1(\Omega)))} \leq C \| E_1[H_\mathbf{b}] \|_{L_\infty((\mathbb{R}, H_\mathbf{b}^1(\Omega)))} \leq CL_1^{1/p'},
\]

and so, we have

\[
(\| \partial_t \mathcal{V}^1 \|_{L_\infty((\mathbb{R}, L_\mathbf{p}^1(\Omega)))}, \| \mathcal{V}^1 \|_{L_\infty((\mathbb{R}, H_\mathbf{b}^1(\Omega)))})^{1/2} \| \mathcal{V}^1 \|_{L_\infty((\mathbb{R}, H_\mathbf{b}^1(\Omega)))} \leq CL^{1/(2p')}
\]

Thus, by (4.27), (4.31), Lemma 4.3 and Lemma 4.2, we have

\[
\| e^{-\gamma t} A^1 \|_{L_\mathbf{p}^{1/2}(\mathbb{R}, L_\mathbf{q}^1(\Omega))} + \| e^{-\gamma t} A^2 \|_{L_\mathbf{p}^{1/2}(\mathbb{R}, L_\mathbf{q}^1(\Omega))} \leq CL^{1/(2p')} \| e^{-\gamma t} \nabla E_1[tr[\mathbf{u}]] \|_{L_\mathbf{p}^{1/2}(\mathbb{R}, L_\mathbf{q}^1(\Omega))} + \| e^{-\gamma t} \nabla E_1[tr[\mathbf{u}]] \|_{L_\mathbf{p}^{1/2}(\mathbb{R}, L_\mathbf{q}^1(\Omega))}
\]

\[
\leq CT^{1/(2p')} L(e^{2(\gamma - \gamma_1)} B + L).
\]

Since

\[
\| e^{-\gamma t} \nabla \mathcal{V}_2[H_\mathbf{b}] \|_{L_\mathbf{p}^{1/2}(\mathbb{R}, L_\mathbf{q}^1(\Omega))} \leq \| e^{-\gamma t} \nabla E_2[H_\mathbf{b}] \|_{L_\mathbf{p}^{1/2}(\mathbb{R}, L_\mathbf{q}^1(\Omega))} \leq C(e^{2(\gamma - \gamma_1)} \| \mathbf{e} \| + L);
\]
\[
\| e^{-\gamma t} \nabla \mathcal{V}_2[H_\mathbf{b}] \|_{L_\mathbf{p}^{1/2}(\mathbb{R}, L_\mathbf{q}^1(\Omega))} \leq C\| e^{-\gamma t} \nabla E_2[H_\mathbf{b}] \|_{L_\mathbf{p}^{1/2}(\mathbb{R}, L_\mathbf{q}^1(\Omega))} \leq C(e^{2(\gamma - \gamma_1)} \| \mathbf{e} \| + L)
\]

as follows from (1.6), the third formula of (1.25) and (1.23), employing the same argument as in proving (1.32), we have

\[
\| e^{-\gamma t} A^3 \|_{L_\mathbf{p}^{1/2}(\mathbb{R}, L_\mathbf{q}^1(\Omega))} \leq C T^{1/(2p')} L(e^{2(\gamma - \gamma_1)} \| \mathbf{e} \| + L).
\]

We now estimate \( A^2 \). For this purpose we use the following estimate which follows from complex interpolation theory:

\[
\| \mathbf{f} \|_{L_\mathbf{p}^{1/2}(\mathbb{R}, L_\mathbf{q}^1(\Omega))} \leq C \| \mathbf{f} \|^{1/2}_{L_\mathbf{p}^{1/2}(\mathbb{R}, L_\mathbf{q}^1(\Omega))} \| \mathbf{f} \|^{1/2}_{L_\mathbf{p}(\mathbb{R}, L_\mathbf{q}^1(\Omega))}.
\]

(4.34)

Let

\[
A_1^2 = E_1[tr[\mathbf{G}]] \otimes \mathbf{E}_1[tr[\mathbf{G}]], \quad A_2^2 = \bar{V}_3(\cdot, \nabla E_2[H_\mathbf{b}] - \nabla E_2[H_\mathbf{b}] \otimes A_1^2.
\]

We further divide \( A_2^2 \) into \( A_2^2 = \sum_{j=1}^4 A_{2j}^2 \) with \( A_2^2 = A_{11}^2 + A_{12}^2 + A_{21}^2 + A_{22}^2 \), where

\[
A_{11}^2 = A_1 \otimes A_1, \quad A_{12}^2 = A_1 \otimes A_2, \quad A_{21}^2 = A_2 \otimes A_1, \quad A_{22}^2 = A_2 \otimes A_2,
\]
\[
A_1 = \psi(t) T_\mathbf{e}(|t|) tr[\mathbf{G}_0], \quad A_2 = e_T[tr[\mathbf{G}] - T_\mathbf{e}(t) tr[\mathbf{G}_0]].
\]

Using (1.8), we have

\[
\| e^{-\gamma t} A_{1i}^2 \|_{L_\mathbf{p}^{1/2}(\mathbb{R}, L_\mathbf{q}^1(\Omega))} \leq C \| e^{-\gamma t} A_1 \|_{L_\mathbf{p}^{1/2}(\mathbb{R}, L_\mathbf{q}^1(\Omega))} \| A_1 \|_{L_\mathbf{p}^{1/2}(\mathbb{R}, L_\mathbf{q}^1(\Omega))} \quad (i = 0, 1),
\]
\[
\| e^{-\gamma t} A_{2i}^2 \|_{L_\mathbf{p}^{1/2}(\mathbb{R}, L_\mathbf{q}^1(\Omega))} \leq C \| A_1 \|_{L_\mathbf{p}^{1/2}(\mathbb{R}, L_\mathbf{q}^1(\Omega))} \| A_2 \|_{L_\mathbf{p}^{1/2}(\mathbb{R}, L_\mathbf{q}^1(\Omega))} + \| A_1 \|_{L_\mathbf{p}^{1/2}(\mathbb{R}, L_\mathbf{q}^1(\Omega))} \| A_2 \|_{L_\mathbf{p}^{1/2}(\mathbb{R}, L_\mathbf{q}^1(\Omega))},
\]

(4.35)

\[
\| e^{-\gamma t} A_{2i}^2 \|_{L_\mathbf{p}(\mathbb{R}, L_\mathbf{q}^1(\Omega))} \leq C \| A_1 \|_{L_\mathbf{p}^{1/2}(\mathbb{R}, L_\mathbf{q}^1(\Omega))} \| A_2 \|_{L_\mathbf{p}(\mathbb{R}, L_\mathbf{q}^1(\Omega))},
\]
\[
\| e^{-\gamma t} A_{2i}^2 \|_{L_\infty(\mathbb{R}, L_\mathbf{q}^1(\Omega))} \leq C \| A_2 \|_{L_\mathbf{p}^{1/2}(\mathbb{R}, L_\mathbf{q}^1(\Omega))} \| A_2 \|_{L_\infty(\mathbb{R}, L_\mathbf{q}^1(\Omega))} \quad (i = 0, 1)
\]

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where we have set \( H_p^0 = L_p \) and used the fact that \( |e^{-\gamma t}A_2| \leq |A_2| \), which follows from \( A_2 = 0 \) for \( t \not\in (0, 2T) \).

By (4.22), (4.24), (4.6), and (4.16), we have

\[
\|e^{-\gamma t}A_1\|_{H^1_p(\mathbb{R}, L^2(\Omega))} \leq C e^{2(\gamma - \gamma_1) B}; \quad \|A_1\|_{H^1_p(\mathbb{R}, L^2(\Omega))} \leq CB; \quad \|A_1\|_{L^\infty(\mathbb{R}, H^1_0(\Omega))} \leq CB; \\
\|A_2\|_{H^1_p(\mathbb{R}, L^2(\Omega))} \leq C(L + B); \quad \|A_2\|_{L^\infty(\mathbb{R}, H^1_0(\Omega))} \leq C(L + B); \\
\|A_2\|_{L_p(\mathbb{R}, L^2(\Omega))} \leq C T^{1/p} (\|tr[G]\|_{L^\infty((0,T), L^2(\Omega))} + \|T_v(\cdot)tr[G_0]\|_{L^\infty((0,T), H^1(\Omega))}) \leq C(L + B) T^{1/p}.
\]

(4.36)

Notice that \( A_2^0 \) and \( A_2^{11} \) have the same estimate. In view of (4.34), combining estimates in (4.38) and (4.30) gives that

\[
\|e^{-\gamma t}A_1^2\|_{H^{1/2}_p(\mathbb{R}, L^2(\Omega))} \leq C (e^{2(\gamma - \gamma_1) B^2} + (B(L + B))^{1/2}(B(L + B) T^{1/p})^{1/2} + (B + L)^2 T^{1/(2p)}) \\
\leq C (e^{2(\gamma - \gamma_1) B^2} + (B + L)^2 T^{1/(2p)}).
\]

(4.37)

And also, by (4.3)

\[
\|e^{-\gamma t}A_1^2\|_{L_p(\mathbb{R}, H^1_0(\Omega))} \leq C \{\|e^{-\gamma t}A_1\|_{L_p(\mathbb{R}, L^2(\Omega))}\|A_1\|_{L^\infty(\mathbb{R}, H^1_0(\Omega))} + 2\|A_1\|_{L^\infty(\mathbb{R}, H^1_0(\Omega))}\|A_2\|_{L_p(\mathbb{R}, H^1_0(\Omega))} \\
+ \|A_2\|_{L_p(\mathbb{R}, H^1_0(\Omega))}\|\bar{A}_2\|_{L^\infty(\mathbb{R}, H^1_0(\Omega))}\}.
\]

By (4.22), (4.24), (4.6), and (4.16) we have

\[
\|e^{-\gamma t}A_1\|_{L_p(\mathbb{R}, H^1_0(\Omega))} \leq C e^{2(\gamma - \gamma_1) B}; \quad \|A_2\|_{L_p(\mathbb{R}, H^1_0(\Omega))} \leq T^{1/p}\|A_2\|_{L^\infty((0,T), H^1_0(\Omega))} \\
\leq T^{1/p} (\|tr[G]\|_{L^\infty((0,T), H^1_0(\Omega))} + \|T_v(\cdot)tr[G_0]\|_{L^\infty((0,T), H^1_0(\Omega))} \leq C T^{1/p}(L + B).
\]

(4.38)

Using (4.36) and (4.38), we have

\[
\|e^{-\gamma t}A_1^2\|_{L_p(\mathbb{R}, H^1_0(\Omega))} \leq C (e^{2(\gamma - \gamma_1) B^2} + (L + B)^2 T^{1/p}).
\]

(4.39)

Moreover, by Lemma (4.1) and (4.31), we have

\[
\|e^{-\gamma t}A_2^0\|_{H^{1/2}_p(\mathbb{R}, L^2(\Omega))} + \|A_2\|_{L_p(\mathbb{R}, H^1_0(\Omega))} \leq C L T^{1/(2p')} (\|A_1^2\|_{H^{1/2}_p(\mathbb{R}, L^2(\Omega))} + \|A_2^2\|_{L_p(\mathbb{R}, H^1_0(\Omega))}),
\]

which, combined with (4.37) and (4.39), gives that

\[
\|e^{-\gamma t}A_2^0\|_{H^{1/2}_p(\mathbb{R}, L^2(\Omega))} + \|e^{-\gamma t}A_2^2\|_{L_p(\mathbb{R}, H^1_0(\Omega))} \\
\leq C (e^{2(\gamma - \gamma_1) B^2} + (L + B)^2 T^{1/(2p')} + L T^{1/(2p')} (e^{2(\gamma - \gamma_1) B^2} + (L + B)^2 T^{1/(2p')})) \\
\leq C (e^{2(\gamma - \gamma_1) B^2} + L(B^2 + L^2) T^{1/(2p')}),
\]

(4.40)

where we have used the facts: \( 1/(2p') < 1/(2p) \), \( e^{2(\gamma - \gamma_1)} < B e^{2(\gamma - \gamma_1)} \), and \( L \leq L^2 \).

Combining (4.32), (4.33), and (4.40) yields that

\[
\|\tilde{h}_1(u, G, H, h)\|_{H^{1/2}_p(\mathbb{R}, L^2(\Omega))} + \|\tilde{h}_1(u, G, H, h)\|_{L_p(\mathbb{R}, H^1_0(\Omega))} \\
\leq C (e^{2(\gamma - \gamma_1) B^2} + L(B^2 + L^2) + e^{2(\gamma - \gamma_1) B}) T^{1/(2p')}),
\]

(4.41)

where we have used the facts: \( 1/(2p') < 1/(2p) \), \( e^{2(\gamma - \gamma_1)} < B e^{2(\gamma - \gamma_1)} \), and \( L \leq L^2 \).

We finally consider \( g(u, H, h) \) and \( \tilde{g}(u, H) \). In view of (2.22), we set

\[
\tilde{g}(u, H, h) = G_1(\nabla E_2[H]) \nabla E_2[H] \otimes \nabla E_1[u], \quad \tilde{g}(u, H) = G_2(\nabla E_2[H]) \nabla E_2[H] \otimes E_1[u].
\]

(4.42)

Here, we have set \( E_t[u] = E_t[u] \) for \( x \in \Omega_\pm \). Obviously,

\[
\tilde{g}(u, H, h) = g(u, H, h), \quad \tilde{g}(u, H) = g(u, H) \quad \text{for } t \in (0, T)
\]
and \( \text{div} \tilde{g}(u, H_k) = \tilde{g}(u, H_k) \) as follows from (4.10), (2.21) and (2.21). Employing the same argument as that in proving (4.32), we have
\[
\|e^{-\gamma t} \tilde{g}(u, H_k)\|_{L^p(\mathbb{R}, L^2(\Omega))} + \|e^{-\gamma t} \tilde{g}(u, H_k)\|_{L^p(\mathbb{R}, H^1_0(\Omega))} \leq CT^{1/(2p')} (L + e^{2(\gamma - \gamma_1)B}). \tag{4.43}
\]

By (4.28) we have
\[
\|\partial_t \tilde{g}(u, H_k)\|_{L^2(\Omega)} \leq C\{\|\nabla E_2[H_k]\|_{L^2(\Omega)} \|\partial_t E_1[u]\|_{L^2(\Omega)} + \|\partial_t \nabla E_2[H_k]\|_{L^2(\Omega)} \|E_1[v]\|_{H^1_0(\Omega)}\}
\]
Thus, using (4.24), (4.13), (4.11), (4.13), (4.22), (4.23), and (4.28), we have
\[
\|e^{-\gamma t} \partial_t \tilde{g}(u, H_k)\|_{L^p(\mathbb{R}, L^2(\Omega))} \\
\leq C\{H_k\|_{L^p((0,T), H^1_0(\Omega))} + \|T_k(\cdot) h_0\|_{L^p((0,\infty), H^1_0(\Omega))}\} \\
\times \{\|\partial_t u\|_{L^p((0,T), L^2(\Omega))} + \|e^{-\gamma T_k(\cdot) u_0}\|_{H^1_0((-2,\infty), H^1_0(\Omega))}\} \\
+ (T^{1/p}) \|\partial_t H_k\|_{L^p((0,T), H^1_0(\Omega))} + \|\partial_t T_k(\cdot) h_0\|_{L^p((0,\infty), H^1_0(\Omega))}\} \tag{4.44}
\]
\[
\times \{\|u\|_{L^p((0,T), H^1_0(\Omega))} + \|e^{-\gamma T_k(\cdot) u_0}\|_{L^p((-2,\infty), H^1_0(\Omega))}\} \\
\leq C\{e + LT^{1/p}(L + e^{2(\gamma - \gamma_1)B}) + (T^{1/p}L + e)(L + e^{2(\gamma - \gamma_1)B})\} \\
\leq CL(L + e^{2(\gamma - \gamma_1)B})T^{1/p}.
\]

We now apply Theorem 5.1 to equations (4.4) and use the estimate in Theorem 5.1 with \( \gamma = \gamma_1 \). And then, assuming that \( 1 \leq B \leq L \), noting that \( s/(p'(1 + s)) < 1/(2p') < 1/(2p') \) and using (4.17), (4.18), (4.11), we have
\[
E_1^2(v) + \|\nabla q\|_{L^p((0,T), L^2(\Omega))} + E_2^2(\rho) \\
\leq C(1 + \gamma_1^{1/2})e^{\gamma_1 T^{1-1/p}} \{B^2 + T^{-1/p} \epsilon + L^3 T^{1/(p+1)}\}. \tag{4.45}
\]

Here and in the following \( s \in (0, 1 - 2/p) \) and \( \gamma_1 \) are fixed, and so we do not take care of the dependance of constants on \( s \) and \( \gamma_1 \).

By the third equation of (4.24), (4.22), and (4.23), we have
\[
\|\partial_t \rho\|_{L^p((0,T), W^{-1,1/2}(\Gamma))} \leq C\{\|\rho\|_{L^p((0,T), W^{-1,1/2}(\Gamma))} B + \|\nabla \rho\|_{L^p((0,T), H^1_0(\Omega))} + L(B)T^{1/p'}\} \\
\leq C\{\|\rho\|_{L^p((0,T), W^{-1,1/2}(\Gamma))} B + \|\nabla \rho\|_{L^p((0,T), H^1_0(\Omega))} + L(B)T^{1/p'}\} \\
\leq C\{B + E_1^2(v) + E_2^2(\rho) + L(B)T^{1/p'}\},
\]
where we used the facts that \( \epsilon B \leq B \) and \( T^{1/p}B \leq T^{1/p}L \leq 1 \). which, combined with (4.43), gives that
\[
E_1^2(v) + E_2^2(\rho) + \|\partial_t \rho\|_{L^p((0,T), W^{-1,1/2}(\Gamma))} \\
\leq C\{(1 + \gamma_1^{1/2})e^{\gamma_1 T^{1-1/p}} \{B^2 + T^{-1/p} \epsilon + L^3 T^{1/(p+1)}\} + B + L(B)T^{1/p'}\}.
\]

Noting that \( 0 < T = \epsilon < 1 \) and \( T^{-1/p} \epsilon = T^{-1/p} < 1 < B^2 \), we have
\[
E_1^2(v) + E_2^2(\rho) + \|\partial_t \rho\|_{L^p((0,T), W^{-1,1/2}(\Gamma))} \leq M_1(B^2 + L^3 T^{1/(p+1)}) \tag{4.46}
\]
for some positive constant \( M_1 \) depending on \( s \) and \( \gamma_1 \) provided that \( 0 < T < \epsilon = \kappa = T, T^{1/p'}L \leq 1, \) and \( L > B \geq 1 \).

We now estimate \( H \) by using Theorem 5.3 with the \( \gamma_1 \) given above. Let \( f_2(u, G, H_k) \) be a nonlinear term given in (2.23). Recalling the formula in (2.25) and employing the same argument as that in proving (4.17), we have
\[
\|f_2(u, G, H_k)\|_{L^p((0,T), L^2(\Omega))} \leq CT^{1/p}(L + B)^2. \tag{4.47}
\]

We next consider \( h_2(u, G, H_k) \) and \( h_3(u, G, H_k) \) given in (2.61) and in (2.63), respectively. Let \( \tilde{h}_2(u, G, H_k) \) and \( \tilde{h}_3(u, G, H_k) \) be their extension to \( \mathbb{R} \) with respect to \( t \) defined by setting
\[
\tilde{h}_2(u, G, H_k) = V_{h_k}^N(\nabla E_2[H_k])\nabla E_2[H_k] \otimes \nabla E_1[tr[u]] + b(y) \mathcal{E}_1[tr[u]] \otimes \mathcal{E}_1[tr[G]] \\
+ V_{h_k}^N(\cdot, \nabla E_2[H_k])\nabla E_2[H_k] \otimes \mathcal{E}_1[tr[u]] \otimes \mathcal{E}_1[tr[G]] \\
+ \tilde{h}_3(u, G, H_k) = \mu \sum_{j=1}^N \tilde{V}_{y_{jk}}(\nabla E_2[H_k])\nabla E_2[H_k] \otimes \frac{\partial}{\partial y_{jk}} \mathcal{E}_1[tr[u]], \tag{4.48}
\]

where \( V_{h_k}^N(\cdot) \) and \( \tilde{V}_{y_{jk}}(\cdot) \) are defined by (4.31) and (2.70), respectively.
Employing the same argument as in proving (4.32), we have
\[ \|e^{-\gamma t}(\tilde{h}_2(u, G, H_k), \tilde{h}_3(u, G, H_k))\|_{H_{\alpha}^2(\mathbb{R}, e^{\gamma t}B)} + \|e^{-\gamma t}(\tilde{h}_2(u, G, H_k), \tilde{h}_3(u, G, H_k))\|_{L_{\rho}(\mathbb{R}, H_{\alpha}^2(\Omega))} \leq C T^{1/2(p')}(e^{2(\gamma - \gamma_1)}B + L). \] (4.49)

We finally consider \(k_1(G, H_\rho)\) and \(k_2(G, H_\rho)\) given in (2.63). In view of (4.46), choosing \(L\) so large that \(M_1 B^2 < L/2\) and \(T\) so small that \(M_1 L^3 T^{\alpha/2(p' + 1)} \leq L/2\), we have
\[ E_2^T(\rho) + \|\partial_t \rho\|_{L_{\rho}(0, T), W_{\alpha}^{1 - 1/\gamma}(\Gamma)} \leq L. \] (4.50)

In particular, we have
\[ \|E_2[H_\rho]\|_{L_{\rho}(\mathbb{R}, H_{\alpha}^2(\Omega))} \leq C(\epsilon + LT^{1/2'}). \] (4.51)

Thus, choosing \(\epsilon = T\) so small, we may also assume that
\[ \sup_{t \in \mathbb{R}} \|E_2[H_\rho](\cdot, t)\|_{H_{\alpha}^2(\Omega)} \leq \delta. \] (4.52)

And also, we may assume that
\[ \|E_2[H_\rho]\|_{L_{\rho}(\mathbb{R}, H_{\alpha}^2(\Omega))} \leq LT^{1/2'}, \quad \|E_2[H_\rho]\|_{L_{\rho}(\mathbb{R}, H_{\alpha}^2(\Omega))} \leq 1. \] (4.53)

In view of (2.65), we define the extensions of \(k_1(G, H_\rho)\) and \(k_2(G, H_\rho)\) by setting
\[ (\tilde{k}_1(G, H_\rho), \tilde{k}_2(G, H_\rho)) = V_k^2(\gamma \nabla E_2[H_\rho]) \nabla E_2[H_\rho] \otimes E_1[\kappa[G]]. \] (4.54)

Obviously, \((\tilde{k}_1(G, H_\rho), \tilde{k}_2(G, H_\rho)) = (k_1(G, H_\rho), k_2(G, H_\rho))\) for \(t \in (0, T)\). By (3.39) and (4.52), we have
\[ \|\tilde{k}_1(G, H_\rho), \tilde{k}_2(G, H_\rho)\|_{L_{\rho}(\mathbb{R}, H_{\alpha}^2(\Omega))} \leq C \|\tilde{E}_2[H_\rho]\|_{L_{\rho}(\mathbb{R}, H_{\alpha}^2(\Omega))} \leq L^2(1 + \|\nabla E_2[H_\rho]\|_{L_{\rho}(\mathbb{R}, H_{\alpha}^2(\Omega))}) \|\tilde{E}_2[H_\rho]\|_{L_{\rho}(\mathbb{R}, H_{\alpha}^2(\Omega))}. \] (4.55)

By (4.22), (4.24), (4.27), and (4.50), we have
\[ \|e^{-\gamma t}(\tilde{k}_1(G, H_\rho), \tilde{k}_2(G, H_\rho))\|_{L_{\rho}(\mathbb{R}, H_{\alpha}^2(\Omega))} \leq C \|\tilde{E}_2[H_\rho]\|_{L_{\rho}(\mathbb{R}, H_{\alpha}^2(\Omega))} \leq L^2(1 + \|\nabla E_2[H_\rho]\|_{L_{\rho}(\mathbb{R}, H_{\alpha}^2(\Omega))}) \|\tilde{E}_2[H_\rho]\|_{L_{\rho}(\mathbb{R}, H_{\alpha}^2(\Omega))}. \] (4.56)

Applying the estimate in Theorem 3.3 with \(\gamma = \gamma_1\) to equations (4.5) and using (4.47), (4.49), (4.53), and (4.50), we have
\[ E_2^T(\rho) \leq C e^{\gamma_1 T} \{ T^1\rho \rho_{L_{\rho(\mathbb{R}, H_{\alpha}^2(\Omega)))}}^2 + \rho_{L_{\rho(\mathbb{R}, H_{\alpha}^2(\Omega)))}}^2 \} + L^2(L + B)T^{3/2(p')} \] (4.57)

which, combined with (4.55), yields that
\[ E_2^T(\rho) + \|\partial_t \rho\|_{L_{\rho}(0, T), W_{\alpha}^{1 - 1/\gamma}(\Gamma)} + E_2^T(\mathbf{H}) \leq M_1 (B^2 + L^3 T^{3/2(p')}) + C e^{\gamma_1 T} \{ B(1 + M_1 B^2 + M_1 L^3 T^{3/2(p')}) + L^2(1 + B) T^{3/2(p')} \} \]
\[ \leq M_1 (B^2 + L^3 T^{3/2(p')}) + C e^{\gamma_1 T} \{ B(1 + M_1 B^2) + (M_1 L^3 + C e^{\gamma_1 T} M_1 L^3 + L^2(1 + B)) T^{3/2(p')} \} \]
provided that \(0 < \epsilon = T = \kappa < 1, L > 1, B > 1\). Choosing \(L > 0\) so large that \(L/2 \geq M_1 B^2 + C e^{\gamma_1 T} (1 + M_1 B^2)\) and \(T > 0\) so small that \(L/2 \geq \{ M_1 L^3 + C e^{\gamma_1 T} M_1 L^3 + L^2(1 + B)\} T^{3/2(p')}\), and setting \(L = f(B) = 2(M_1 B^2 + C e^{\gamma_1 T} (1 + M_1 B^2))\), we see that \(E_T(v, H, \rho) \leq L\). If we define a map \(\Phi\) by \(\Phi(u, G, h) = (v, H, \rho)\), then, \(\Phi\) maps \(U_T\) into itself.
5 Estimates of the difference of nonlinear terms and completion of the proof of Theorem 2.3

Let \((u_i, G_i, h_i) \in U_T (i = 1, 2)\). In this section mainly we shall estimate \(E_T(v_1 - v_2, H_1 - H_2, \rho_1 - \rho_2)\) with \((v_i, H_i, \rho_i) = \Phi(u_i, G_i, h_i) (i = 1, 2)\) and then we shall prove that \(\Phi\) is a contraction map on \(U_T\) with a suitable constant. For notational simplicity, we set
\[
\begin{align*}
\tilde{v} &= v_1 - v_2, \quad \tilde{H} = H_1 - H_2, \quad \tilde{\rho} = \rho_1 - \rho_2, \quad F_1 = f_1(u_1, G_1, H_{h_1}) - f_1(u_2, G_2, H_{h_2}), \\
g &= g(u_1, H_{h_1}) - g(u_2, H_{h_2}), \quad \tilde{g} = g(u_1, H_{h_1}) - g(u_2, H_{h_2}), \quad D = d(u_1, H_{h_1}) - d(u_2, H_{h_2}), \\
H_1 &= h_1(u_1, G_1, H_{h_1}) - h_1(u_2, G_2, H_{h_2}), \quad F_2 = f_2(u_1, G_1, H_{h_1}) - f_2(u_2, G_2, H_{h_2}), \\
H_2 &= h_2(u_1, G_1, H_{h_1}) - h_2(u_2, G_2, H_{h_2}), \quad H_3 = h_3(u_1, G_1, H_{h_1}) - h_3(u_2, G_2, H_{h_2}), \\
K_1 &= k_1(G_1, H_{h_1}) - k_1(G_2, H_{h_2}), \quad K_2 = k_2(G_1, H_{h_1}) - k_2(G_2, H_{h_2}).
\end{align*}
\]

And then, \(\tilde{v}\) and \(\tilde{\rho}\) satisfy the following equations with some pressure term \(Q\):
\[
\begin{align*}
\partial_t \tilde{v} - \text{Div} T(\tilde{v}, Q) &= F_1, \quad \text{in } \bar{\Omega} \times (0, T), \\
\text{Div } \tilde{v} &= \text{div } g = \text{div } \tilde{g}, \quad \text{in } \bar{\Omega} \times (0, T), \\
\partial_t \tilde{\rho} + \langle \nabla \tilde{\rho} \rangle \cdot u > &\cdot \tilde{v} = D, \quad \text{on } \Gamma \times (0, T), \\
[[\tilde{v}]] &= 0, \quad [[T(\tilde{v}, Q)n]] - \sigma(\Delta_{\Gamma} \tilde{\rho} + a \tilde{\rho})n = H_1, \quad \text{on } \Gamma \times (0, T), \\
\tilde{v}_\pm &= 0, \quad \text{on } S_{\pm} \times (0, T), \\
(\tilde{v}, \tilde{\rho})|_{t=0} &= (0, 0) \quad \text{in } \bar{\Omega} \times \Gamma.
\end{align*}
\]

And \(H\) satisfies the following equations:
\[
\begin{align*}
\mu \partial_t \tilde{H} - \alpha^{-1} \Delta \tilde{H} &= F_2, \quad \text{in } \bar{\Omega} \times (0, T), \\
[[\alpha^{-1} \text{curl } \tilde{H}]]n &= H_2, \quad [[\text{div } \tilde{H}]] = H_3, \quad \text{on } \Gamma \times (0, T), \\
[[\mu \tilde{H} \cdot n]] = K_1, \quad [[\tilde{H} - \tilde{H} \cdot n > n]] &= K_2, \quad \text{on } \Gamma \times (0, T), \\
n_\pm \cdot \tilde{H}_\pm &= 0, \quad (\text{curl } \tilde{H}_\pm) n_\pm &= 0, \quad \text{on } S_\pm \times (0, T), \\
H|_{t=0} &= 0 \quad \text{in } \bar{\Omega}.
\end{align*}
\]

We have to estimate the nonlinear terms appearing in the right side of equations \((5.1)\) and \((5.5)\). We start with estimating \(F_1\). As was written in \((4.14)\), we write
\[
f_1(u, G, H_h) = V_{f_1}(\nabla H_h) f_3(u, G, H_h)
\]
with
\[
f_3(u, G, H_h) = \nabla H_h \otimes (\partial_t u, \nabla^2 u) + \partial_t H_h \otimes \nabla u + u \otimes \nabla u + \nabla^2 H_h \otimes \nabla u + G \otimes \nabla G.
\]
And then we can write \(F_1\) as follows:
\[
F_1 = (V_{f_1}(\nabla H_{h_1}) - V_{f_1}(\nabla H_{h_2})) f_3(u_1, G_1, H_{h_1}) + V_{f_1}(\nabla H_{h_2}) (f_3(u_1, G_1, H_{h_1}) - f_3(u_2, G_2, H_{h_2}));
\]
\[
= \nabla (H_{h_1} - H_{h_2}) \otimes \partial_t u + \nabla^2 u_1 + \partial_t H_h \otimes \nabla u + u \otimes \nabla u + \nabla^2 H_h \otimes \nabla u + G \otimes \nabla G.
\]
Since we may write
\[
V_{f_1}(\nabla H_{h_1}) - V_{f_1}(\nabla H_{h_2}) = \int_0^1 (d_K V_{f_1}(\nabla H_{h_2} + \theta \nabla (H_{h_1} - H_{h_2})) d\theta (\nabla (H_{h_1} - H_{h_2})).
\]
where \(d_K V_{f_1} f\) is the derivative of \(V_{f_1} (K)\) with respect to \(K\), noting that \(H_{h_1} - H_{h_2} = 0\) for \(t = 0\) and using \((4.14)\) and \((4.17)\), we have
\[
||V_{f_1}^L(\nabla H_{h_1}) - V_{f_1}^L(\nabla H_{h_2})||_{L_\infty((0,T), H^1(\Omega))} \leq T^{1/p'} \|\partial_t h_1 - \partial_t h_2\|_{L_p((0,T), H^2(\Omega))}.
\]
Since \(f_3(u, G, H_h)\) satisfies the estimate \((4.17)\), replacing \(h, u, G\) with \(h_1, u_1, G_1\), we have
\[
||f_3(u_1, G_1, H_{h_1})||_{L_p((0,T), L^p(\Omega))} \leq C(T^{1/p}(L + B)^2 + (\epsilon + T^{1/p'}L)L) \leq C T^{1/p}(L + B)^2.
\]
By (4.8) and (5.3), we have
\[
\| f_1(u_1, G_1, H_1, t) - f_2(u_2, G_2, H_2, t) \|_{L_p(\Omega)} 
\]
\[
\leq C\| h_1 - h_2 \|_{W^{2-1/q(\Omega)}} \| (\partial_t u_1, \nabla^2 u_1) \|_{L_p(\Omega)} + h_2 \| W^{2-1/q(\Gamma)} \| (\| \partial_t (u_1 - u_2) \| + \nabla^2 (u_1 - u_2)) \|_{L_p(\Omega)} 
\]
\[
+ \| \partial_t (h_1 - h_2) \|_{W^{2-1/q(\Gamma)}} \| u_1 \|_{L_p(\Omega)} + h_1 \| h_2 \|_{W^{2-1/q(\Gamma)}} \| u_1 \|_{L_p(\Omega)} 
\]
\[
+ \| u_1 - u_2 \|_{H^2(\Omega)} + \| (G_1, G_2) \|_{H^2(\Omega)} \| (G_1 - G_2) \|_{H^2(\Omega)}. 
\]
Since
\[
\| h_1 - h_2 \|_{L_p((0,T), W^{2-1/q(\Gamma)})} \leq T^{1/p} \| \partial_t (h_1 - h_2) \|_{L_p((0,T), W^{2-1/q(\Gamma)})}, 
\]
noting that \( u_1 - u_2 = 0 \) and \( G_1 - G_2 = 0 \) at \( t = 0 \), by (5.13), (5.16), and (5.10), we have
\[
\| f_1(u_1, G_1, H_1, t) - f_2(u_2, G_2, H_2, t) \|_{L_p((0,T), L_q(\Omega))} 
\]
\[
\leq C\| T^{1/p} \| \partial_t (h_1 - h_2) \|_{L_p((0,T), W^{2-1/q(\Gamma)}))} + LT^{1/p} E_T^2(u_1 - u_2) 
\]
\[
+ T^{1/p} \| \partial_t (h_1 - h_2) \|_{L_p((0,T), W^{2-1/q(\Gamma)})) (B + L) + LE_T^2(u_1 - u_2) + (B + L) E_T^2(u_1 - u_2) 
\]
\[
+ T^{1/p} \| \partial_t (h_1 - h_2) \|_{L_p((0,T), W^{2-1/q(\Gamma)})) (B + L) E_T^2(G_1 - G_2), 
\]
which, combined with (5.3) and (5.5), leads to
\[
\| F_1 \|_{L_p((0,T), L_q(\Omega))} \leq CT^{1/p}(L + B) E_T(u_1 - u_2, G_1 - G_2, h_1 - h_2). 
\]
where we have the estimate: \( T^{1/p} T^{1/p}(L + B)^2 \leq 2(T^{1/p} L) T^{1/p}(L + B) \leq 2T^{1/p}(L + B) \), which follows from \( B \leq L \) and \( T^{1/p} L \leq 1 \).

We next consider the difference \( D \). In view of (5.41), we write
\[
D = \langle \nabla T(H_{h_1} - H_{h_2}) \nabla u_1 - u_2 \rangle + \langle \nabla T H_{h_2} \nabla u_1 - u_2 \rangle 
\]
\[
+ \langle u_1 - u_2 - \partial_t (h_1 - h_2) \nabla u_1 - u_2 \rangle 
\]
\[
\langle \nabla T H_{h_2} \nabla u_1 - u_2 \rangle 
\]
\[
+ \langle u_2 - \partial_t (h_1 - h_2) \nabla u_1 - u_2 \rangle 
\]
\[
\langle \nabla T H_{h_2} \nabla u_1 - u_2 \rangle, 
\]
where we have set \( V_n(\cdot, K) K = \tilde{V}_n(\cdot, K) \). We have
\[
\| D \|_{L_p((0,T), W^{2-1/q(\Gamma)})} \leq C(L + B) T^{1/p} E_T(u_1 - u_2, G_1 - G_2, h_1 - h_2); 
\]
\[
\| D \|_{L_p((0,T), W^{2-1/q(\Gamma)})} \leq C(L + B) T^{1/p} E_T(u_1 - u_2, G_1 - G_2, h_1 - h_2). 
\]
In fact, noting that the difference \( \tilde{V}_n(\cdot, \nabla H_{h_1}) - \tilde{V}_n(\cdot, \nabla H_{h_2}) \) has the similar formula to that in (5.3), by (2.5), (1.7), (1.8), and (1.9), we have
\[
\| D \|_{W^{2-1/q(\Gamma)}} \leq C\| h_1 - h_2 \|_{W^{2-1/q(\Gamma)}} \| u_1 - u_2 \|_{H^2(\Omega)} + h_2 \| W^{2-1/q(\Gamma)} \| \| u_1 - u_2 \|_{H^2(\Omega)} 
\]
\[
+ \| u_1 - u_2 \|_{H^2(\Omega)} + \| \partial_t (h_1 - h_2) \|_{W^{2-1/q(\Gamma)}} (1 + \| h_1 \|_{W^{2-1/q(\Gamma)}}) \| h_1 \|_{W^{2-1/q(\Gamma)}} 
\]
\[
+ \| u_2 \|_{H^2(\Omega)} + \| \partial_t (h_2) \|_{W^{2-1/q(\Gamma)}} (1 + \| h_2 \|_{W^{2-1/q(\Gamma)}}) \| h_2 \|_{W^{2-1/q(\Gamma)}} 
\]
\[
\times \| h_1 - h_2 \|_{W^{2-1/q(\Gamma)}} \| h_2 \|_{W^{2-1/q(\Gamma)}} \| h_1 - h_2 \|_{W^{2-1/q(\Gamma)}}. 
\]
Thus, by (4.10), (4.13), (4.21), and (5.5) we have
\[
\| D \|_{L_p((0,T), W^{2-1/q(\Gamma)})} 
\]
\[
\leq C\{ T^{1/p} E_T^2(h_1 - h_2) T^{1/p} (L + B) + LT^{1/p} E_T^2(u_1 - u_2) 
\]
\[
+ LT^{1/p} (E_T(u_1 - u_2) + \| \partial_t (h_1 - h_2) \|_{L_p((0,T), W^{2-1/q(\Gamma)})}) + (L + B) T^{1/p} (\epsilon + LT^{1/p} E_T^2(h_1 - h_2)), 
\]
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which leads to the first inequality in (5.8), because $T^{\frac{1}{p+1+\gamma}} < 1$ as follows from $0 < T < 1$.

By (2.26), (4.17), (4.8), and (4.9), we have

$$
\|D\|_{W^{1+\gamma}_p(\Omega)} \leq C\left(\|h_1 - h_2\|_{W^{\frac{1}{p+1+\gamma}}_p(\Omega_\alpha)} + \|h_1 - h_2\|_{W^{\frac{1}{p+1+\gamma}}_p(\Omega_\alpha)} \right)
\|u_1 - u_2\|_{H^1_0(\Omega_\alpha)}.
$$

Since

$$
\|u_1 - u_2\|_{H^1_0(\Omega_\alpha)} \leq C_s T^{\frac{1}{p+1+\gamma}} \|\partial_h(u_1 - u_2)\|_{L^p((0,T),L^q(\Omega_\alpha))} \leq C_s T^{\frac{1}{p+1+\gamma}} E_T(u_1 - u_2)
$$

by (4.18), (4.10), and (4.21) we have

$$
\|D\|_{L^p((0,T),W^{\frac{1}{p+1+\gamma}}_p(\Omega))}
\leq C\left(\|h_1 - h_2\|_{L^p((0,T),W^{\frac{1}{p+1+\gamma}}_p(\Omega_\alpha))} + T^{1/p'} \|\partial_{h}(h_1 - h_2)\|_{L^p((0,T),W^{\frac{1}{p+1+\gamma}}_p(\Omega))} (B + L)
\right.
\left.+ LT^{\frac{1}{p+1+\gamma}} E_T(u_1 - u_2) + LT^{1/p'} \|u_1 - u_2\|_{L^p((0,T),H^1_0(\Omega_\alpha))}
\right.
\left.+ \|u_1 - u_2\|_{L^p((0,T),H^1_0(\Omega_\alpha))} \|\partial_{h}(h_1 - h_2)\|_{L^p((0,T),W^{\frac{1}{p+1+\gamma}}_p(\Omega))} L^2 T^{1/p'}
\right.
\left.+ LT^{1/p'} \|\partial_{h}(h_1 - h_2)\|_{L^p((0,T),W^{\frac{1}{p+1+\gamma}}_p(\Omega))} (B + L) \|h_1 - h_2\|_{L^p((0,T),W^{\frac{1}{p+1+\gamma}}_p(\Omega))} L^2 T^{1/p'}
\right.
\left.+ T^{1/p'} \|\partial_{h}(h_1 - h_2)\|_{L^p((0,T),W^{\frac{1}{p+1+\gamma}}_p(\Omega))} L + LT^{1/p'} \|\partial_{h}(h_1 - h_2)\|_{L^p((0,T),W^{\frac{1}{p+1+\gamma}}_p(\Omega))}
\right.
\left.+ (L + B)LT^{1/p'} \|\partial_{h}(h_1 - h_2)\|_{L^p((0,T),W^{\frac{1}{p+1+\gamma}}_p(\Omega))} + LT^{1/p'} \|h_1 - h_2\|_{L^p((0,T),W^{\frac{1}{p+1+\gamma}}_p(\Omega))}
\right.
\left.+ LT^{1/p'} \|h_1 - h_2\|_{L^p((0,T),W^{\frac{1}{p+1+\gamma}}_p(\Omega))}
\right),
$$

which yields (5.9).

We next consider $H_4$. In view of (4.30), we set

$$
\tilde{H}_4 = H_4^1 + a(y)H_4^2 + H_4^3 + H_4^4
$$

with

$$
H_4^1 = (\tilde{\nabla}_h^*(\cdot, \tilde{\nabla}E_2[H_n]) - \tilde{H}_h^*(\cdot, \tilde{\nabla}E_2[H_n])) \otimes \nabla E_1[tr[u_1]]
\left.+ \tilde{\nabla}_h^*(\cdot, \tilde{\nabla}E_2[H_n]) \nabla E_2[H_n] \otimes \nabla (E_1[tr[u_1]] - E_1[tr[u_2]])
\right),
$$

$$
H_4^2 = (E_1[tr[G_1]] - E_1[tr[G_2]]) \otimes E_1[tr[G_1]]
\left.+ E_1[tr[G_2]] \otimes (E_1[tr[G_1]] - E_1[tr[G_2]])
\right),
$$

$$
H_4^3 = (\tilde{\nabla}_h^*(\cdot, \tilde{\nabla}E_2[H_n]) - \tilde{H}_h^*(\cdot, \tilde{\nabla}E_2[H_n])) \otimes E_1[tr[G_1]] \otimes E_1[tr[G_1]]
$$

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\[ + \mathbf{V}_h^\ell(\cdot, \nabla E_2[H_{h_2}]) \nabla E_2[H_{h_2}] \otimes \mathcal{H}_1^\ell, \]

\[ \mathcal{H}_1^\ell = (\nabla \cdot (\nabla E_2[H_{h_1}]) - \nabla^2 E_2[H_{h_1}]) \otimes \nabla^2 E_2[H_{h_1}] + \nabla \cdot (\nabla E_2[H_{h_2}]) \nabla E_2[H_{h_2}] \otimes \nabla^2 (E_2[H_{h_1}] - E_2[H_{h_2}]), \]

where we have set

\[ \mathbf{V}_h^\ell(\cdot, \mathbf{K}) = \mathbf{V}_h^\ell(\cdot, \mathbf{K}) \mathbf{K}, \quad \tilde{\mathbf{V}}_h^\ell(\cdot, \mathbf{K}) = \mathbf{V}_h^\ell(\cdot, \mathbf{K}) \mathbf{K}, \quad \tilde{\mathbf{V}}_s(\cdot, \mathbf{K}) = \mathbf{V}_s(\cdot, \mathbf{K}) \mathbf{K}. \]

We see that \( \tilde{\mathcal{H}}_1 \) is defined for \( t \in \mathbb{R} \) and \( \tilde{\mathcal{H}}_1 = \mathcal{H}_1 \) for \( t \in (0, T) \). Writing

\[ \tilde{\mathbf{V}}_h^\ell(\cdot, \nabla E_2[H_{h_1}]) - \tilde{\mathbf{V}}_h^\ell(\cdot, \nabla E_2[H_{h_2}]) = \int_0^1 (d_K \tilde{\mathbf{V}}_h^\ell)(\cdot, \nabla E_2[H_{h_2}] + \theta \nabla (E_2[H_{h_1}] - E_2[H_{h_2}])) \, d\theta \]

\[ \times \nabla (E_2[H_{h_1}] - E_2[H_{h_2}]), \]

and since \( E_2[H_{h_1}] - E_2[H_{h_2}] = c_T[H_{h_1} - H_{h_2}] \), by \textbf{(5.1)}, \textbf{(1.3)}, \textbf{(3.24)}, and \textbf{(5.20)}, and \textbf{(5.6)}, we have

\[
\|e^{-\gamma t} \partial_t (\nabla E_2[H_{h_1}]) - \nabla E_2[H_{h_2}]\|_{L^\infty((0,T),W^{2-1/q}_q)} \leq C \left( \|\partial_t \nabla E_2[H_{h_1}]\|_{L^\infty((0,T),W^{2-1/q}_q)} + \|\nabla E_2[H_{h_2}]\|_{L^\infty((0,T),W^{2-1/q}_q)} \right).
\]

(5.11)

And also, by \textbf{(1.20)} and \textbf{(5.6)}, we have

\[
\|e^{-\gamma t} (\tilde{\mathbf{V}}_h^\ell(\cdot, \nabla E_2(H_{h_1})) - \tilde{\mathbf{V}}_h^\ell(\cdot, \nabla E_2(H_{h_2}))\|_{L^\infty((0,T),W^{2-1/q}_q)} \leq C \left( \|\nabla E_2[H_{h_1}]\|_{L^\infty((0,T),W^{2-1/q}_q)} + \|\nabla E_2[H_{h_2}]\|_{L^\infty((0,T),W^{2-1/q}_q)} \right).
\]

(5.12)

By Lemma \textbf{4.2} and \textbf{(1.20)}, we have

\[
\|\nabla \mathcal{E}_1[\nabla E_2(H_{h_1})] - \tilde{\mathcal{E}}_1(\cdot, \nabla E_2[H_{h_2}])\|_{L^\infty((0,T),W^{2-1/q}_q)} \leq C(B + L).
\]

Thus, setting

\[ \mathcal{H}_1^{\ell+1} = (\tilde{\mathbf{V}}_h^\ell(\cdot, \nabla E_2(H_{h_1})) - \tilde{\mathbf{V}}_h^\ell(\cdot, \nabla E_2[H_{h_2}])) \otimes \nabla \mathcal{E}_1[\nabla E_2(H_{h_1})], \]

by Lemma \textbf{4.3} we have

\[
\|e^{-\gamma t} \mathcal{H}_1^{\ell+1}\|_{H^{1/2}(\mathbb{R},L^q(\Omega))} \leq C(B + L)T^{1/q} \left( \|\partial_t(h_1 - h_2)\|_{L^\infty((0,T),W^{2-1/q}_q)} + \|\partial_t(h_1 - h_2)\|_{L^\infty((0,T),W^{2-1/q}_q)} \right).
\]

(5.13)

Noticing that \( \mathcal{E}_1[\nabla E_2(H_{h_1})] - \mathcal{E}_1[\nabla E_2(H_{h_2})] = CV_T[\nabla E_2(H_{h_1}) - \nabla E_2(H_{h_2})] \), by \textbf{(2.19)} and Lemma \textbf{4.2}, we have

\[
\|e^{-\gamma t} \nabla \mathcal{E}_1[\nabla E_2(H_{h_1})] - \mathcal{E}_1[\nabla E_2(H_{h_2})]\|_{H^{1/2}(\mathbb{R},L^q(\Omega))} \leq C \left( \|\partial_t(h_1 - h_2)\|_{L^\infty((0,T),W^{2-1/q}_q)} + \|\partial_t(h_1 - h_2)\|_{L^\infty((0,T),W^{2-1/q}_q)} \right).
\]

Thus, setting \( \mathcal{H}_1^{\ell+2} = \tilde{\mathbf{V}}_h^\ell(\cdot, \nabla E_2[H_{h_1}]) \nabla E_2[H_{h_2}] \otimes \nabla (\mathcal{E}_1[\nabla E_2(H_{h_1})] - \mathcal{E}_1[\nabla E_2(H_{h_2})]), \) by \textbf{(5.3)} and Lemma \textbf{4.3} we have

\[
\|e^{-\gamma t} \mathcal{H}_1^{\ell+2}\|_{H^{1/2}(\mathbb{R},L^q(\Omega))} \leq C \left( B + L \right) \left( \|\nabla E_2[H_{h_1}]\|_{L^\infty((0,T),H^{2}_q(\Omega))} + \|\nabla E_2[H_{h_2}]\|_{L^\infty((0,T),H^{2}_q(\Omega))} \right).
\]

(5.14)
We next consider $\mathcal{H}_1$. Since $\mathcal{E}_1[tr[G_1]] - \mathcal{E}_2[tr[G_2]] = e_T[tr[G_1] - tr[G_2]]$, we have
\[
\|e^{-\gamma t}(\mathcal{E}_1[tr[G_1]] - \mathcal{E}_1[tr[G_2]])\|_{H^2_p(L^q(\Omega))} \leq C\|G_1 - G_2\|_{H^2_p((0,T),L^q(\Omega))};
\]
\[
\|e^{-\gamma t}(\mathcal{E}_1[tr[G_1]] - \mathcal{E}_1[tr[G_2]])\|_{L^p(L^q(\Omega))} \leq CT^{1/p}\|G_1 - G_2\|_{L^p((0,T),H^2_p(\Omega))} + \|\partial_t(G_1 - G_2)\|_{L^p((0,T),L^q(\Omega))};
\]
\[
\|e^{-\gamma t}(\mathcal{E}_1[tr[G_1]] - \mathcal{E}_1[tr[G_2]])\|_{L^\infty(\Omega)} \leq \|G_1 - G_2\|_{L^\infty((0,T),H^2_p(\Omega))};
\]
which, combined with (5.17), yields that
\[
\|G_1 - G_2\|_{L^{p/(1+\varepsilon)}(L^q(\Omega))} \leq \|G_1 - G_2\|_{L^1((0,T),W^{1,2}_q(\Omega))};
\]
which, combined with (5.19), yields that
\[
\|e^{-\gamma t}H_1^2\|_{H^{p/2}_1(L^q(\Omega))} \leq C(L + B)T^{1/(2p)}E_1^1(G_1 - G_2).
\]

On the other hand, we have
\[
\|\mathcal{E}_1[tr[G_1]]\|_{H^2_p(L^q(\Omega))} \leq C(L + B);
\]
\[
\|\mathcal{E}_1[tr[G_1]]\|_{L^p(L^q(\Omega))} \leq C(L + B);
\]
\[
\|\mathcal{E}_1[tr[G_1]]\|_{L^\infty(\Omega)} \leq C(L + B)
\]
for $i = 1, 2$, and therefore by (4.34) we have
\[
\|e^{-\gamma t}H_1^2\|_{H^{p/2}_1(L^q(\Omega))} \leq C(L + B)T^{1/(2p)}E_1^1(G_1 - G_2).
\]

And also, by (4.8), (5.16) and (4.10), we have
\[
\|e^{-\gamma t}H_2^2\|_{L^p(\Omega)} \leq C(L + B)\|G_1 - G_2\|_{L^p((0,T),H^2_p(\Omega))};
\]
\[
\|e^{-\gamma t}H_2^2\|_{L^1(\Omega)} \leq C(L + B)\|G_1 - G_2\|_{L^1((0,T),H^2_p(\Omega))};
\]
which, combined with (5.17), yields that
\[
\|e^{-\gamma t}H_1^2\|_{H^{p/2}_1(L^q(\Omega))} + \|e^{-\gamma t}H_2^2\|_{L^p(\Omega)} \leq C(L + B)T^{1/(2p)}E_1^1(G_1 - G_2).
\]

Since
\[
\|\mathcal{E}_1[tr[G_1]] \otimes \mathcal{E}_1[tr[G_1]]\|_{H^{p/2}_1(L^q(\Omega))} + \|\mathcal{E}_1[tr[G_1]] \otimes \mathcal{E}_1[tr[G_1]]\|_{L^p(\Omega)} \leq C(L + B)^2,
\]
by Lemma 4.11, (5.11), and (5.12), we have
\[
\|e^{-\gamma t}H_1^{31}\|_{H^{p/2}_1(L^q(\Omega))} \leq CL(L + B)^2T^{1/p}\|(\partial_t(h_1 - h_2))\|_{L^p((0,T),W^{2,1}_q(\Gamma))} + \|\partial_t(h_1 - h_2)\|_{L^\infty((0,T),W^{1,1}_q(\Gamma))}.
\]
By (2.5), (4.29), (4.27), and (5.6), we have
\[
\|e^{-\gamma t}H_1^{31}\|_{L^p(\Omega)} \leq C(1 + \|\nabla \mathcal{E}_2[H_{h_1}]\|_{L^\infty(\Omega)} + \|\nabla \mathcal{E}_2[H_{h_2}]\|_{L^\infty(\Omega)}) \times \|\nabla e_T[H_{h_1} - H_{h_2}]\|_{L^p((0,T),H^2_p(\Omega))}\|\mathcal{E}_1[tr[G_1]]\|_{L^\infty(\Omega)}^2\)
\]
\[
\leq C(L + B)^2T^{1/p}\|\partial_t(h_1 - h_2)\|_{L^p((0,T),W^{2,1}_q(\Gamma))},
\]
which, combined with (5.19), yields that
\[
\|e^{-\gamma t}H_1^{31}\|_{H^{p/2}_1(L^q(\Omega))} + \|e^{-\gamma t}H_2^{31}\|_{L^p(\Omega)} \leq C(L + B)^2T^{1/p}\|\partial_t(h_1 - h_2)\|_{L^p((0,T),W^{2,1}_q(\Gamma))} + \|\partial_t(h_1 - h_2)\|_{L^\infty((0,T),W^{1,1}_q(\Gamma))}.
\]

Setting $H_1^{32} = \nabla \mathcal{E}_2[H_{h_1}a] \nabla \mathcal{E}_2[H_{h_2}] \otimes H_1$, by Lemma 4.11, (5.31), and (5.18), we have
\[
\|e^{-\gamma t}H_1^{32}\|_{H^{p/2}_1(L^q(\Omega))} + \|e^{-\gamma t}H_2^{32}\|_{L^p(\Omega)} \leq C(L + B)^2T^{1/2}E_1^1(G_1 - G_2),
\]
where we have used $1/p + 1/p' = 1$, which, combined with (5.20), yields that
\[
\|e^{-\gamma H^3_1}\|_{L^p_0(\mathbb{R},L^q_1(\Omega))} + \|e^{-\gamma H^1_1}\|_{L^p_0(\mathbb{R},H^1_1(\Omega))} \leq C L (L + B)^2 T^{1/p'} E_T^1 (G_1 - G_2), \tag{5.21}
\]

Since
\[
\|
\begin{align*}
\nabla^2 \mathcal{E}_1[H_h] & \|_{H^{1/2}_0(\mathbb{R},L^q_1(\Omega))} + \|\nabla^2 \mathcal{E}_1[H_h] & \|_{L^p_0(\mathbb{R},H^1_1(\Omega))} \\
\n\leq C & \left( \|h_1 & \|_{H^{1/2}_0((0,T),W^{2-1/q}_\infty(\Gamma))} + \|h_1 & \|_{L^p_0((0,T),H^{2-1/q}_\infty(\Gamma))} \\
+ & \|T_h(\cdot) & \|_{H^{1/2}_0((0,T),H^1_1(\Omega))} + \|T_h(\cdot) & \|_{L^p_0((0,T),H^1_1(\Omega))} \\
\leq C & (L + e) \leq 2CL;
\end{align*}
\]
\[
\|e^{-\gamma \nabla^2 \mathcal{E}_1[H_h] - \mathcal{E}_2[H_h]}\|_{H^{1/2}_0(\mathbb{R},L^q_1(\Omega))} + \|\nabla^2 \mathcal{E}_1[H_h] - \mathcal{E}_2[H_h] & \|_{L^p_0(\mathbb{R},H^1_1(\Omega))} \\
\leq C & (\|h_1 - h_2 & \|_{H^{1/2}_0((0,T),W^{2-1/q}_\infty(\Gamma))} + \|h_1 - h_2 & \|_{L^p_0((0,T),W^{2-1/q}_\infty(\Gamma))}),
\]
by Lemma 4.11, 4.31, 5.11, and 5.12, we have
\[
\|e^{-\gamma H^4_1}\|_{H^{1/2}_0(\mathbb{R},L^q_1(\Omega))} + \|e^{-\gamma H^1_1}\|_{L^p_0(\mathbb{R},H^1_1(\Omega))} \\
\leq C L^2 T^{1/(2p')} (E_T^2 (h_1 - h_2) + \|\partial_t (h_1 - h_2) & \|_{L^\infty((0,T),W^{2-1/q}_\infty(\Gamma))}),
\]
which, combined with (5.14), (5.18), and (5.21), yields that
\[
\|e^{-\gamma H^1_1}\|_{H^{1/2}_0(\mathbb{R},L^q_1(\Omega))} + \|e^{-\gamma H^4_1}\|_{L^p_0(\mathbb{R},H^1_1(\Omega))} \\
\leq C (B + L) T^{1/(2p')} E_T (u_1 - u_2, G_1 - G_2, h_1 - h_2), \tag{5.22}
\]
where we have used the fact that $1/p < 1/p'$.

We now consider $g$ and $G$. In view of (4.42), we set
\[
\tilde{g} = G_1 (\nabla \mathcal{E}_1[H_h], \nabla \mathcal{E}_1[u_1]) - G_1 (\nabla \mathcal{E}_2[H_h], \nabla \mathcal{E}_2[u_1]) \otimes \nabla \mathcal{E}_1[u_2], \tag{5.23}
\]
\[
\tilde{G} = G_2 (\nabla \mathcal{E}_1[H_h], \nabla \mathcal{E}_1[u_1], \nabla \mathcal{E}_2[H_h], \nabla \mathcal{E}_2[u_1]) - G_2 (\nabla \mathcal{E}_2[H_h], \nabla \mathcal{E}_2[H_h], \nabla \mathcal{E}_1[u_2]).
\]

And then, $\tilde{g}$ and $\tilde{G}$ are defined for $t \in \mathbb{R}$ and $g = \tilde{g}$ and $G = \tilde{G}$ for $t \in (0, T)$. Employing the same argument as in proving (5.14), we have
\[
\|e^{-\gamma \tilde{g}}\|_{H^{1/2}_0(\mathbb{R},L^q_1(\Omega))} + \|e^{-\gamma \tilde{G}}\|_{L^p_0(\mathbb{R},H^1_1(\Omega))} \\
\leq C (B + L) T^{1/(2p')} E_T (u_1 - u_2, G_1 - G_2, h_1 - h_2).
\]

To estimate $\tilde{G}$, we write $\tilde{G} = G_1 + G_2$ with
\[
G_1 = (\tilde{G}_2 (\nabla \mathcal{E}_1[H_h], \nabla \mathcal{E}_2[H_h]) \otimes \mathcal{E}_1[u_1], \tag{5.24}
\]
\[
G_2 = G_2 (\nabla \mathcal{E}_1[H_h], \nabla \mathcal{E}_2[H_h]) \otimes (\mathcal{E}_1[u_1] - \nabla \mathcal{E}_2[u_2]),
\]
where we have set $\tilde{G}_2 (K) = \tilde{G}_2 (K) K$. To estimate $\partial_t G_1$, we write
\[
\partial_t G_1 = \int_0^1 (d_K \tilde{G}_2 (\nabla \mathcal{E}_2[H_h] + \theta \nabla \mathcal{E}_2[H_h]) - \mathcal{E}_2[H_h]) \nabla (\mathcal{E}_2[H_h] - \nabla \mathcal{E}_2[H_h]) \otimes \partial_t \mathcal{E}_1[u_1] \\
+ \left( \int_0^1 (d_K \tilde{G}_2 (\nabla \mathcal{E}_2[H_h] + \theta \nabla \mathcal{E}_2[H_h]) \nabla \partial_t (\mathcal{E}_2[H_h] - \nabla \mathcal{E}_2[H_h]) \otimes \partial_t \mathcal{E}_1[u_1] \\
+ \left( \int_0^1 (d_K \tilde{G}_2 (\nabla \mathcal{E}_2[H_h] + \theta \nabla \mathcal{E}_2[H_h] - \mathcal{E}_2[H_h]) \nabla (\mathcal{E}_2[H_h] - \nabla \mathcal{E}_2[H_h]) \right) \partial_t ((1 - \theta) \nabla \mathcal{E}_2[H_h] + \theta \nabla \mathcal{E}_2[H_h]) d\theta \\
\otimes \nabla \mathcal{E}_2[H_h] - \mathcal{E}_2[H_h]) \otimes \mathcal{E}_1[u_1].
\]

By (4.28), (4.3), (4.3), (4.27), and (5.6), we have
\[
\|e^{-\gamma \partial_t G_1}\|_{L^p_0(\mathbb{R},L^q_1(\Omega))} \\
\leq C (\|h_1 - h_2 & \|_{L^\infty((0,T),W^{2-1/q}_\infty(\Gamma))} \|\partial_t \mathcal{E}_1[u_1] & \|_{L^p_0(\mathbb{R},L^q_1(\Omega))})
\]

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\[ T^{1/p} \| \partial_t (h_1 - h_2) \|_{L_\infty((0,T), W_w^{2-1/s}(T))} \| E_1[u_1] \|_{L_\infty((0,T), H_1^s(\Omega))} \]
\[ + T^{1/p} \| \partial_t E_2[H_1] \|_{L_\infty((0,T), W_w^{2-1/s}(T))} + \| \partial_t E_2[H_2] \|_{L_\infty((0,T), W_w^{2-1/s}(T))} \]
\[ \times \| h_1 - h_2 \|_{L_\infty((0,T), W_w^{2-1/s}(T))} \| E_1[u_1] \|_{L_\infty((0,T), H_1^s(\Omega))} \]
\[ \leq C(T^{1/p} \| \partial_t (h_1 - h_2) \|_{L_\infty((0,T), W_w^{2-1/s}(T))} (L + B) \]
\[ + TL \| \partial_t (h_1 - h_2) \|_{L_\infty((0,T), W_w^{2-1/s}(T))} (L + B) \]
\[ + T^{1/p} \| \partial_t (h_1 - h_2) \|_{L_\infty((0,T), W_w^{2-1/s}(T))} (L + B) \]
\[ \leq CT^{1/p} (L + B) (\| \partial_t (h_1 - h_2) \|_{L_\infty((0,T), W_w^{2-1/s}(T))} + \| \partial_t (h_1 - h_2) \|_{L_\infty((0,T), W_w^{2-1/s}(T))}) \]
(5.24)

where we have used \( T^{1/p'} L \leq 1 \). Since \( E_1[u_1] - E_2[u_2] = e_T[u_1 - u_2] \), writing
\[ \partial_t G_2 = \partial_t E_2[H_1] = \partial_t E_2[H_2] + \frac{\partial_t e_T[u_1 - u_2]}{e_T[u_1 - u_2] - \rho_1 G_2}, \]
by (4.28), (4.29), (4.27), and (5.10)
\[ \| e^{-\gamma t} \partial_t G_2 \|_{L_\infty((0,T), L_q(\Omega))} \leq C(T^{1/p} \| \partial_t (u_1 - u_2) \|_{L_\infty((0,T), L_q(\Omega))} + \| \partial_t E_2[H_2] \|_{L_\infty((0,T), W_w^{2-1/s}(T))} \| u_1 - u_2 \|_{L_\infty((0,T), L_q(\Omega))} \]
\[ \leq C(T^{1/p'} \| \partial_t (u_1 - u_2) \|_{L_\infty((0,T), L_q(\Omega))} + T^{1/p'} L E_T[u_1 - u_2], \]
which, combined with (5.24), yields that
\[ \| e^{-\gamma t} \partial_t G \|_{L_\infty((0,T), L_q(\Omega))} \leq C(L + B) T^{1/p} L E_T[u_1 - u_2, G_1 - G_2, h_1 - h_2]. \]
(5.25)

Applying Theorem 5.3 to equations (5.3) and using (5.7), (5.9), (5.22), (5.23), and (5.25), we have
\[ E_T[v_1 - v_2] + E_T^2(\rho_1 - \rho_2) \leq C(1 + \gamma_1^{1/2} e^\gamma L^3 T^{1/p'} L E_T[u_1 - u_2, G_1 - G_2, h_1 - h_2]. \]
(5.26)

provided that \( LT^{1/p'} \leq 1 \), \( 0 < T = \kappa = \epsilon < 1 \), and \( L > B \geq 1 \).

Moreover, by the third equation of (5.3), (5.8), and (2.12), we have
\[ \| \partial_t (\rho_1 - \rho_2) \|_{L_\infty((0,T), W_w^{2-1/s}(T))} \leq C(\| \rho_1 - \rho_2 \|_{L_\infty((0,T), W_w^{2-1/s}(T))} + \| v_1 - v_2 \|_{L_\infty((0,T), H_1^s(\Omega))} \]
\[ + T^{1/p'} (L + B) E_T[u_1 - u_2, G_1 - G_2, h_1 - h_2] \]
\[ \leq C(B T^{1/p'} \| \partial_t (\rho_1 - \rho_2) \|_{L_\infty((0,T), W_w^{2-1/s}(T))} + \| v_1 - v_2 \|_{L_\infty((0,T), H_1^s(\Omega))} \]
\[ + T^{1/p'} (L + B) E_T[u_1 - u_2, G_1 - G_2, h_1 - h_2] \]
which, combined with (5.26) and \( BT^{1/p'} \leq 1 \), yields that
\[ E_T^2(v_1 - v_2) + E_T^2(\rho_1 - \rho_2) \leq C(1 + \gamma_1^{1/2} e^\gamma L^3 T^{1/p'} L E_T[u_1 - u_2, G_1 - G_2, h_1 - h_2]. \]
(5.27)

with some constant \( M_2 \) depending on \( s \in (0, 1 - 2/p) \) and \( \gamma_1 > 0 \) provided that \( LT^{1/p'} \leq 1 \), \( 1 \leq B \leq L \), and \( 0 < T = \kappa = \epsilon < 1 \).

We now consider \( \mathbf{H} = \mathbf{H}_1 - \mathbf{H}_2 \). We first consider \( F_2 \). In view of (2.23), we may write
\[ f_2(u, G, H) = V^h(\nabla H) f_4(u, G, H) \]
with
\[ f_4(u, G, H) = (\nabla G \otimes \partial_t H + \nabla H \otimes \nabla^2 G + \nabla^2 H \otimes \nabla G + \nabla u \otimes G + u \otimes \nabla G, \]
where \( V^h(K) \) is some matrix of smooth funtions of \( K \) for \( |K| < \delta \). And then, employing the same argument as in proving (5.7), we have
\[ \| F_2 \|_{L_\infty((0,T), L_q(\Omega))} \leq CT^{1/p} (L + B) E_T[u_1 - u_2, G_1 - G_2, h_1 - h_2]. \]
(5.28)
provided that $T^{1/p' L} \leq 1$, $1 < B \leq L$, and $0 < T = \epsilon < 1$.

Concerning $\mathcal{H}_2$ and $\mathcal{H}_3$, in view of (4.48), we define $\tilde{\mathcal{H}}_2$ and $\tilde{\mathcal{H}}_3$ by setting $\tilde{\mathcal{H}}_2 = B_1 + b(y)B_2 + B_3$ with

\[ B_1 = (\tilde{\mathbf{V}}^1_h(\cdot, \nabla \mathbf{E}_2[H_{\rho_1}]) - \tilde{\mathbf{V}}^1_h(\cdot, \nabla \mathbf{E}_2[H_{\rho_2}])), \nabla \mathbf{E}_1[tr[u_1]] + \nabla \mathbf{E}_1[tr[u_2]]); \]
\[ B_2 = (\mathbf{E}_1[tr[u_1]] - \mathbf{E}_1[tr[u_2]])) \nabla \mathbf{E}_1[tr[G_1]] + \mathbf{E}_1[tr[u_1]] \nabla \mathbf{E}_1[tr[G_1]] - \mathbf{E}_1[tr[G_2]]); \]
\[ B_3 = (\tilde{\mathbf{V}}^4_h(\cdot, \nabla \mathbf{E}_2[H_{\rho_1}]) - \tilde{\mathbf{V}}^4_h(\cdot, \nabla \mathbf{E}_2[H_{\rho_2}])) \mathbf{E}_1[tr[u_1]] \nabla \mathbf{E}_1[tr[G_1]] + \nabla \mathbf{E}_1[tr[G_1]] \nabla \mathbf{E}_1[tr[G_2]]; \]

where we have set

\[ \tilde{\mathbf{V}}^4_h(\cdot, \mathbf{K}) = \mathbf{V}^4_h(\cdot, \mathbf{K}), \quad \tilde{\mathbf{V}}^4_h(\cdot, \mathbf{K}) = \mathbf{V}^4_h(\cdot, \mathbf{K}); \]

and

\[ \tilde{\mathcal{H}}_3 = -\mu \sum_{j,k=1}^{N} (\tilde{V}_{0j}(\mathbf{K}) - \tilde{V}_{0j}(\mathbf{K}))[\nabla \mathbf{E}_2[tr[u_1]] - \nabla \mathbf{E}_2[tr[u_2]]]; \]

Obviously, $\tilde{\mathcal{H}}_i$ are defined for $t \in \mathbb{R}$, and $\tilde{\mathcal{H}}_i = \mathcal{H}_i$ for $t \in (0, T)$ for $i = 3, 4$. Employing the same argument as in proving (5.14) and (5.22), we have

\[ \|e^{-\gamma t} \tilde{\mathcal{H}}_3\|_H^{1/2}(\mathcal{L}_x(\omega)) \leq C L(B + L)^{1/(2p')} E_T(u_1 - u_2, G_1 - G_2, h_1 - h_2); \]

\[ \|e^{-\gamma t} \tilde{\mathcal{H}}_2\|_H^{1/2}(\mathcal{L}_x(\omega)) \leq C L(B + L)^{1/(2p')} E_T(u_1 - u_2, G_1 - G_2, h_1 - h_2); \]

provided that $T^{1/p' L} \leq 1$ and $0 < \epsilon = T < 1, 1 \leq L, \text{ and } 1 \leq B$.

We finally consider $\mathcal{K}_1$ and $\mathcal{K}_2$. As was mentioned in (4.28), we may assume that

\[ \text{sup}_{t \in \mathbb{R}} \|\mathbf{E}_2[tr[H_{\rho_1}]]\|_{H^{1/2}(\omega)} \leq \delta \quad (i = 1, 2). \]

In view of (4.51), we set $\tilde{\mathcal{K}}_1 = \tilde{\mathcal{K}}_1 + \tilde{\mathcal{K}}_2$ with

\[ \tilde{\mathcal{K}}_1 = (\tilde{\mathbf{V}}^4_h(\cdot, \nabla \mathbf{E}_2[H_{\rho_1}]) - \tilde{\mathbf{V}}^4_h(\cdot, \nabla \mathbf{E}_2[H_{\rho_2}])) \nabla \mathbf{E}_1[tr[G_1]]; \]

\[ \tilde{\mathcal{K}}_2 = \mathbf{V}^4_h(\cdot, \nabla \mathbf{E}_2[H_{\rho_1}]) \nabla \mathbf{E}_2[H_{\rho_2}] \nabla \mathbf{E}_2[H_{\rho_2}] \nabla \mathbf{E}_2[H_{\rho_2}]; \]

where we have set $\tilde{\mathbf{V}}^4_h(\cdot, \mathbf{K}) = \mathbf{V}^4_h(\cdot, \mathbf{K})$. Obviously, $\tilde{\mathcal{K}}$ is defined for $t \in \mathbb{R}$ and $\tilde{\mathcal{K}} = (\mathcal{K}_1, \mathcal{K}_2)$ for $t \in (0, T)$. To estimate $\mathcal{K}_1$, we write

\[ \tilde{\mathbf{V}}^4_h(\cdot, \nabla \mathbf{E}_2[H_{\rho_1}]) - \tilde{\mathbf{V}}^4_h(\cdot, \nabla \mathbf{E}_2[H_{\rho_2}]) \]

\[ = \int_0^1 (d^r_k \tilde{\mathbf{V}}^4_h(\cdot, \nabla \mathbf{E}_2[H_{\rho_1}]) + \theta(\nabla \mathbf{E}_2[H_{\rho_1}]) - \nabla \mathbf{E}_2[H_{\rho_2}])] d\theta \nabla \mathbf{E}_2[H_{\rho_1} - H_{\rho_2}]; \]

and then by (4.9) we have

\[ \|\tilde{\mathbf{K}}_1\|_{H^{1/2}(\omega)} \leq C \|\mathbf{E}_1[tr[G_1]]\|_{H^{1/2}(\omega)} \|\mathbf{E}_2[H_{\rho_1}] - H_{\rho_2}]\|_{H^{1/2}(\omega)} \|\nabla \mathbf{E}_2[H_{\rho_1}] - H_{\rho_2}]\|_{H^{1/2}(\omega)} \]

\[ + (\|\nabla \mathbf{E}_2[H_{\rho_1}]\|_{H^{1/2}(\omega)} + \|\nabla \mathbf{E}_2[H_{\rho_2}]\|_{H^{1/2}(\omega)})(1 + \|\nabla \mathbf{E}_2[H_{\rho_1}]\|_{H^{1/2}(\omega)} + \|\nabla \mathbf{E}_2[H_{\rho_2}]\|_{H^{1/2}(\omega)}) \times \|\mathbf{E}_1[tr[H_{\rho_1}] - H_{\rho_2}]\|_{H^{1/2}(\omega)}. \]

Noting that $\mathbf{E}_1[tr[H_{\rho_1}] - H_{\rho_2}]$ vanishes for $t \notin (0, 2T)$, we have

\[ \|\nabla (H_{\rho_1} - H_{\rho_2})\|_{L^\infty(\mathbb{R}, H^{1/2}(\omega))} \leq CT^{1/p'} \|\theta(\rho_1 - \rho_2)\|_{L^p(0,T), W^{1/2 - 1/q}(\omega)}; \]
Thus, by (4.27) and (5.39),
\[
\left\| e^{-\gamma t} \mathcal{K}_1 \right\|_{L_p(R, H^s_v(\Omega))} \leq C \left\{ T^{1/p'} \left( L + e^{2(\gamma - \gamma_1)} B \right) \| \partial_t (p_1 - p_2) \|_{L_p((0,T), W^{2-1/q}_v(\Gamma))} + \left( B e^{2(\gamma - \gamma_1)} + (L + B) T^{1/p} \right) \| p_1 - p_2 \|_{L_p((0,T), W^{2-1/q}_v(\Gamma))} + L (L + B) T^{1/p'} \| \partial_t (p_1 - p_2) \|_{L_p((0,T), W^{2-1/q}_v(\Gamma))} \right\}
\]
\[
+ \left( (L + B) T^{1/p} \right) \| p_1 - p_2 \|_{L_p((0,T), W^{2-1/q}_v(\Gamma))} \}
\]
\[
\leq C \left( e^{2(\gamma - \gamma_1)} B + L (L + B) T^{1/p} \mathcal{E}_T^2(p_1 - p_2) \right).
\]

Using (4.3), we have
\[
\| \mathcal{K}_2 \|_{H^s_v(\Omega)} \leq C \left\{ \| \nabla \mathcal{E}_2[H_{p_1}] \|_{H^s_v(\Omega)} \| e_T[tr[G_1] - tr[G_2]] \|_{H^s_v(\Omega)} + \| \nabla \mathcal{E}_2[H_{p_2}] \|_{H^s_v(\Omega)} \| e_T[tr[G_1] - tr[G_2]] \|_{H^s_v(\Omega)} + \| \nabla \mathcal{E}_2[H_{p_1}] \|_{H^s_v(\Omega)} (1 + \| \nabla \mathcal{E}_2[H_{p_2}] \|_{H^s_v(\Omega)}) \| \nabla \mathcal{E}_2[H_{p_2}] \|_{H^s_v(\Omega)} \| e_T[tr[G_1] - tr[G_2]] \|_{H^s_v(\Omega)} \right\}
\]

Employing the same argument as in (5.15), we have
\[
\| tr[G_1] - tr[G_2] \|_{L_p((0,T), H^s_v(\Omega))} \leq C T^{1/p} \mathcal{E}_T^2(G_1 - G_2)
\]
for some $s \in (0, 1 - 2/p)$, and so have
\[
\left\| e^{-\gamma t} \mathcal{K}_2 \right\|_{L_p(R, H^s_v(\Omega))} \leq C \left\{ L T^{1/p} \mathcal{E}_T^2(G_1 - G_2) + L T^{1/p'} \mathcal{E}_T^2(G_1 - G_2) \right\}
\]
\[
\leq C T^{1/p} \mathcal{E}_T^2(G_1 - G_2).
\]

By (4.8) and (5.9), we have
\[
\| \partial_t \mathcal{K}_1 \|_{L_p(R, L_q(\Omega))} \leq C \left\{ (\| \partial_t \nabla \mathcal{E}_2[H_{p_1}] \|_{L_q(\Omega)} + \| \partial_t \nabla \mathcal{E}_2[H_{p_2}] \|_{L_q(\Omega)}) \| \nabla e_T[H_{p_1} - H_{p_2}] \|_{H^s_v(\Omega)} \| \nabla e_T[G_1] \|_{H^s_v(\Omega)} \right\}
\]
\[
+ \| \partial_t \nabla e_T[H_{p_1} - H_{p_2}] \|_{L_q(\Omega)} \| \nabla e_T[G_1] \|_{H^s_v(\Omega)} \| \partial_t \nabla e_T[G_1] \|_{L_q(\Omega)} \right\}
\]
\[
\leq C \left\{ (\| \partial_t \nabla \mathcal{E}_2[H_{p_1}] \|_{L_q(\Omega)} + \| \partial_t \nabla \mathcal{E}_2[H_{p_2}] \|_{L_q(\Omega)}) \| \nabla e_T[H_{p_1} - H_{p_2}] \|_{H^s_v(\Omega)} \| \nabla e_T[G_1] \|_{H^s_v(\Omega)} \right\}
\]
\[
+ \| \partial_t \nabla \mathcal{E}_2[H_{p_1}] \|_{L_q(\Omega)} \| \nabla e_T[G_1] \|_{H^s_v(\Omega)} \| \partial_t \nabla e_T[G_1] \|_{L_q(\Omega)} \right\}
\]

Thus, we have
\[
\left\| e^{-\gamma t} \partial_t \mathcal{K}_1 \right\|_{L_p(R, L_q(\Omega))} + \left\| e^{-\gamma t} \partial_t \mathcal{K}_2 \right\|_{L_p(R, L_q(\Omega))} \leq C \left\{ T \| \partial_t (p_1 - p_2) \|_{L_p((0,T), W^{2-1/q}_v(\Gamma))} \right\} \left\{ B + L + T^{1/p'} \| \partial_t (p_1 - p_2) \|_{L_p((0,T), W^{2-1/q}_v(\Gamma))} \right\}
\]
\[
+ T^{1/p'} \| \partial_t (p_1 - p_2) \|_{L_p((0,T), W^{2-1/q}_v(\Gamma))} \left\{ B + L + T^{1/p} L T^{1/p} \mathcal{E}_T^2(G_1 - G_2) \right\}
\]
\[
+ T^{1/p} L T^{1/p} \mathcal{E}_T^2(G_1 - G_2) + L T^{1/p} \| \partial_t (p_1 - p_2) \|_{L_p((0,T), L_q(\Omega))} \left\{ B + L + T^{1/p} \mathcal{E}_T^2(G_1 - G_2) \right\}
\]
\[
\leq C T^{1/p} (L + B) \mathcal{E}_T^2(G_1 - G_2) + E_T^2(G_1 - G_2)
\]
where we have set $E_T^2(p_1 - p_2) = E_T^2(p_1 - p_2) + \| \partial_t (p_1 - p_2) \|_{L_p((0,T), W^{2-1/q}_v(\Gamma))}$. Putting these inequalities together gives that
\[
\left\| e^{-\gamma t} \mathcal{K} \right\|_{L_p(R, H^s_v(\Omega))} \leq C \left\{ e^{2(\gamma - \gamma_1)} B + L (L + B) T^{1/p} \mathcal{E}_T^2(p_1 - p_2) \right\}
\]
\[
+ C (L + B) T^{1/p} \mathcal{E}_T^2(G_1 - G_2).
\]
with
\[ N_T(L, B) = (Ce^{\gamma_1}(B + L(B + L)T^{\frac{1}{2p}}) + 1)M_2L^3T^{\frac{1}{2p}} + Ce^{\gamma_1}L(B + L)^{2T^{1/(2p)}}. \]

Thus, choosing \( T \) so small that \( N_T(L, B) \leq 1/2 \), we see that the \( \Phi \) is a contraction map from \( U_T \) into itself, and so there is a unique fixed point \((u, G, h) \in U_T \) of the map \( \Phi \). This \((u, G, h) \) solves equations uniquely and possessing the properties mentioned in Theorem 2.3. This completes the proof of Theorem 2.3.

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