THE AREA IS A GOOD METRIC

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Abstract. In the first part we extend the construction of the smooth normalcrossing divisors compactification of projectivized strata of abelian differentials given by Bainbridge, Chen, Gendron, Grushevsky and Möller to the case of \( k \)-differentials. Since the generalized construction is closely related to the original one, we mainly survey their results and justify the details that need to be adapted in the more general context.

In the second part we show that the flat area provides a canonical hermitian metric on the tautological bundle over the projectivized strata of finite area \( k \)-differentials which is good in the sense of Mumford. This result is useful in order to apply Chern-Weyl theory tools. It has already been used as an assumption in the work of Sauvaget for abelian differentials and will be used in a forthcoming paper of Chen, Möller and Sauvaget for quadratic differentials.

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1. Introduction

A natural invariant of a flat surface \((X, \omega)\) is the flat area \( \text{vol}(X, \omega) \), the area taken with respect to the form \(|\omega|\). As such, it defines a hermitian metric \( h \) on the tautological line bundle \( \mathcal{O}(\text{vol}) \) over the projectivized strata \( \mathbb{P}\Omega \mathcal{M}_g(\mu) \). This metric does not extend smoothly over the boundary, as the area of a flat surface tends to \( \infty \) when \( X \) acquires an infinite flat cylinder, i.e. when \( \omega \) acquires a simple pole. In Chern-Weyl theory applications, it suffices to show that the curvature form of the metric connection associated to the metric \( h \) represents the first Chern class of \( \mathcal{O}(\text{vol}) \) on a suitable compactification. This has been used as assumption by Sauvaget in [Sau18] for Masur-Veech volumes of the minimal strata of abelian differentials. While a workaround for this has been given in [CMSZ19], the computation of the volume of individual spin components in loc. cit. is still based on that assumption. Moreover, a forthcoming paper [CMS19] extends this line of thought to quadratic differentials and possibly to \( k \)-differentials. There, too, the volume of the canonical \( k \)-cover (see Section 2) provides a natural hermitian metric. Even for

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principal strata, where the Hodge bundle provides a smooth compactification, we do not see an easy route to prove the claim in the title, see the subtleties explained below. This paper consequently makes full use of the smooth compactification of strata of abelian differentials constructed in [BCGGM19b].

Given the applications in mind, the first part of this paper is a survey about the construction of the smooth compactification and the formal justification of the tempting claim that the construction extends to $k$-differentials, if the notions are appropriately adapted in the same way as [BCGGM19a] adapts [BCGGM18].

The compactification. Let $\mu = (m_1, \ldots, m_n)$ be a type of a meromorphic $k$-differential, i.e. $m_i$ are integers such that $\sum m_i = k(2g-2)$. We summarize the properties of our compactification $\Xi^k_{\mathcal{M}_{g,n}}(\mu)$ of the moduli space of $k$-differentials, the $n$ points being labeled throughout. The construction starts with the space $\Omega^k_{\mathcal{M}_{g,n}}(\mu)$ parameterizing $k$-differentials $q$ plus the choice of a $k$-th root $\omega$ of the pullback of $q$ to the canonical $k$-cover. The forgetful map makes $\Omega^k_{\mathcal{M}_{g,n}}(\mu) \to \Omega^k_{\mathcal{M}_{g,n}}(\mu)$ into an unramified cover of degree $k$.

Theorem 1.1. There exists a complex orbifold $\Xi^k_{\mathcal{M}_{g,n}}(\mu)$, the moduli space of multi-scale $k$-differentials, with the following properties.

i) The space $\Omega^k_{\mathcal{M}_{g,n}}(\mu)$ is dense in $\Xi^k_{\mathcal{M}_{g,n}}(\mu)$

ii) The boundary $D = \Xi^k_{\mathcal{M}_{g,n}}(\mu) \setminus \Omega^k_{\mathcal{M}_{g,n}}(\mu)$ is a normal crossing divisor.

iii) The rescaling action of $\mathbb{C}^*$ on $\Omega^k_{\mathcal{M}_{g,n}}(\mu)$ parameterizing $k$-differentials $q$ plus the choice of a $k$-th root $\omega$ of the pullback of $q$ to the canonical $k$-cover. The forgetful map makes $\Omega^k_{\mathcal{M}_{g,n}}(\mu) \to \Omega^k_{\mathcal{M}_{g,n}}(\mu)$ into an unramified cover of degree $k$.

iv) Via the canonical cover construction, the space $\Xi^k_{\mathcal{M}_{g,n}}(\mu)$ is embedded as suborbifold in the compactification $\Xi^k_{\mathcal{M}_{g,n}}(\mu)$ of the corresponding stratum of abelian differentials.

Here we only prove that $\Xi^k_{\mathcal{M}_{g,n}}(\mu)$ is a ‘moduli space’ in a very weak form, namely by exhibiting what its complex points correspond to, the multi-scale $k$-differentials introduced below. We leave it to the interested reader to adapt the functor from [BCGGM19a] to the context of $k$-differentials.

Besides the normal crossing boundary, the most relevant property for us is the existence of a convenient coordinate system, given by perturbed period coordinates. To introduce this, we first have to explain how to parameterize boundary points of $\Xi^k_{\mathcal{M}_{g,n}}(\mu)$.

Let $\Gamma = (V, H, E, g)$ be a stable graph, where $g$ is the genus assignment. A level graph is a stable graph together with a weak total order on the set of vertices. We usually specify the order using a level function $\ell$, usually normalized to take values in $0, -1, \ldots, -L$, with zero the top level. An enhanced level graph is a level graph together with an enhancement $\kappa: H \to \mathbb{Z}$ on the half-edges that specifies the number of prongs of the differential at the corresponding marked point, see Section 2.

Each of the levels of $\Gamma$ thus specifies a moduli space of $k$-differentials, the type being given by the enhancement. A collection of these differentials is called twisted differential and we call a twisted differential compatible with $\Gamma$ if it moreover satisfies the global $k$-residue-condition (GRC) from [BCGGM19a]. A multi-scale $k$-differential is a twisted differential compatible with $\Gamma$ up to projectivization of the lower levels, but together with the choice of an equivalence class of prong-matchings.
The details are given in Section 3 using the notion of level rotation torus. Leaving them aside, we can now describe the coordinates.

**Proposition 1.2.** In a neighborhood $U \subset \Xi_{\mathcal{M}_{g,n}}(\mu)$ of every point in the boundary stratum corresponding to an enhanced level graph $\Gamma$ with $L + 1$ levels there is an orbifold chart given by the perturbed period map

$$\text{PPer}: U \to \mathbb{C}^h \times \mathbb{C}^{L+1} \times \prod_{i=0}^{L-1} \mathbb{C}^{\dim E_{(i)}^{\text{grc}} - 1},$$

where $E_{(i)}^{\text{grc}}$ is some eigenspace in homology constrained by the GRC and where the corresponding coordinates are obtained by integrating perturbations of the twisted differential against these homology classes.

In this proposition, the first set of coordinates in $\mathbb{C}^h$ measure the opening of horizontal nodes and the second set in $\mathbb{C}^L$ measures the rescaling of the differentials on each level. Neither of them is a period, in fact they are exponentials, respectively roots, of periods. The statement about integration is intentionally vague, since we are not exactly integrating the (roots of) $k$-differentials parameterized by $U$, but its sum with a modification differential, as constructed in Section 3. Moreover, the path of integration is not between the zeros of those differentials but between neighboring points, thus the name 'perturbed'. Technically important is that these perturbations go to zero faster than the rescaling of the $k$-differential. The map PPer depends on many choices, however they are irrelevant for many local computations.

**Boundary divisors.** To a first approximation the boundary divisors, i.e. the irreducible components of the boundary $\Xi_{\mathcal{M}_{g,n}}(\mu) \setminus \Omega_{\mathcal{M}_{g,n}}(\mu)$, are given by graphs with one level and a single horizontal edge, and by graphs with two levels and no horizontal edge. However, in the setting of $k$-differentials the level graph does not specify the boundary divisor uniquely. In Section 2 we recall the notion of canonical $k$-cover, which is unique for $k$-differentials on smooth curves, but not in the stable case. An example for two different covers that give rise to different components of the boundary is given by [BCGGM19a, Figure 2]. In fact, the residue conditions are different in the two cases. Consequently, as second approximation the choice of a cyclic $k$-cover $\pi: \hat{\Gamma} \to \Gamma$ compatible with the canonical covers of the components (see Section 2) characterizes boundary components.

For a full description of the boundary, and also of its orbifold structure, we need the notion of prong-matchings and the definition of several groups associated with enhanced level graphs. A **prong-matching** at an edge $e$ is a cyclic-order preserving bijection of the $\kappa_e$ in- resp. outgoing prongs (separatrices) at the two ends of the node corresponding to $e$, see Section 3. To describe various group actions on prong-matchings, we view $\hat{\Gamma}$ as a graph with $L$ level *passages*, the first from level 0 to level $-1$, the second from level $-1$ to level $-2$ etc. The unit vector $e_i$ in the level rotation group $R_{\hat{\Gamma}} \cong \mathbb{Z}^L$ acts on the set of prong-matchings by shifting each edge crossing the $i$-th level passage by one counterclockwise turn. The relevance of this action stems from the level-wise rotation action by $(\mathbb{C}^\ast)^L$ on the level components of a twisted differential. Of particular importance is the subgroup $\text{Tw}_{\hat{\Gamma}}$ of $R_{\hat{\Gamma}}$ that fixes all prongs, the **twist group**.
Proposition 1.3. There is a bijection between covers \( \pi: \Gamma \to \Gamma \) of enhanced level graphs \( \Gamma \) of type \((g, n, k, \mu)\) and boundary strata \( D^\Gamma \) of the compactification \( \Xi_{\mathcal{M}_{g,n}}(\mu) \). Each \( D^\Gamma \) is a cover of the product of the level-wise projectivised moduli space of twisted differentials on \( \Gamma \) and its degree is the number of \( R^\Gamma \)-orbits of prong-matchings.

We will not address the subtle question of connectivity of those \( D^\Gamma \).

The metric. We now return to our primary goal. The statement is about flat surfaces of finite area, so we suppose from now on that \( m_i > -k \). If \( \pi: \hat{\mathcal{X}} \to \mathcal{X} \) denotes the canonical covering associated with \((\mathcal{X}, q) \in \mathcal{M}_{g,n}(\mu)\) such that \( \pi^* q = \omega^k \), then the definition

\[
h(X, q) = \text{area}_{\hat{\mathcal{X}}} \omega = \frac{i}{2} \int_{\hat{\mathcal{X}}} \omega \wedge \bar{\omega}
\]

provides the tautological bundle \( O(-1) \) on \( P \mathcal{M}_{g,n}(\mu) \) with a hermitian metric. The moduli space \( \mathcal{P} \mathcal{OM}_{g,n}(\mu) \) has, besides the nice compactification \( \Xi_{\mathcal{M}_{g,n}}(\mu) \) discussed above, a highly singular compactification, the incidence variety compactification \( \mathcal{P} \Xi_{\mathcal{M}_{g,n}}(\mu) \) that has been studied in [BCGGM18] and [BCGGM19a]. It is the closure of \( \mathcal{P} \mathcal{OM}_{g,n}(\mu) \) inside the projectivized bundle of \( k \)-fold stable differentials twisted by the polar part of \( \mu \). This projectivized bundle provides an extension of the tautological bundle \( O(-1) \), whose restriction to the incidence variety compactification we denote by the same symbol.

There is a natural forgetful map \( \phi: \mathcal{P} \Xi_{\mathcal{M}_{g,n}}(\mu) \to \mathcal{P} \mathcal{OM}_{g,n}(\mu) \), which is an isomorphism restricted to \( \mathcal{P} \mathcal{OM}_{g,n}(\mu) \). The pullback of \( O(-1) \) thus provides an extension of the tautological bundle on \( \mathcal{P} \Xi_{\mathcal{M}_{g,n}}(\mu) \) that we still denote by the same symbol. It is this bundle whose Chern classes are relevant ([Sau18], [CMSZ19]) for computation of Masur-Veech volumes and Siegel-Veech constants. Our main theorem is:

Theorem 1.4. The metric \( h \) on \( O(-1) \) is a good metric in the sense of Mumford on the compactification \( \mathcal{P} \Xi_{\mathcal{M}_{g,n}}(\mu) \).

We recall the definition of a good metric in Section [4]. This property implies that \( c_1(O(-1)) = \frac{i}{2 \pi} [F_h] \) is represented by the curvature form \( F_h \) of the metric connection associated with \( h \).

In the case of only horizontal nodes the metric diverges as we approach the boundary. However in perturbed period coordinates coordinates the local calculation is essentially the calculation of Mumford for the special case of elliptic curves (times the number of horizontal nodes).

In the absence of horizontal nodes, the metric smoothly extends. This fits with the intuition that the area of the lower level surfaces goes to zero.

This behavior should be contrasted with the one of the flat area metric on the full Hodge bundle \( \mathcal{P} \Omega_{\mathcal{M}_g} \) for the principal stratum of abelian differentials. This compactification is smooth and also has a nice (plumbing) coordinate system. However consider a stable curve \((X, \omega)\) with two components, joined by two nodes and a differential that is zero on one component, while non-zero on the other. Arbitrarily small neighborhoods contain differentials supported on both components with non-zero residue at the nodes, and thus with infinite volume. As a conclusion, \( \text{vol}: \Omega_{\mathcal{M}_g} \to \mathbb{R} \cup \{\infty\} \) is not continuous and we thus avoid this space entirely here.
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2. Period coordinates and canonical covers of $k$-differentials

In this section we summarize well-known results about period coordinates, but also recall the period coordinates along the boundary strata of the incidence variety compactification from [BCGGM19a]. We start by recalling properties of the canonical $k$-cover.

Let $X$ be a Riemann surface and let $q$ be a meromorphic $k$-differential of type $\mu$. This datum defines (see e.g. [BCGGM19a Section 2.1]) a $k$-fold cover $\pi: \hat{X} \to X$ such that $\pi^* q = \omega^k$ is the $k$-power of an abelian differential. This differential is of type

$$\hat{\mu} := \left( \frac{\hat{m}_1, \ldots, \hat{m}_1}{\gcd(k,m_1)}, \frac{\hat{m}_2, \ldots, \hat{m}_2}{\gcd(k,m_2)}, \ldots, \frac{\hat{m}_n, \ldots, \hat{m}_n}{\gcd(k,m_n)} \right),$$

where $\hat{m}_i := \frac{k+m_i}{\gcd(k,m_i)} - 1$. We let $\hat{g} = g(\hat{X})$ and $\hat{n} = \sum \gcd(k,m_i)$. Note that $\hat{X}$ is disconnected, if $q$ is a $d$-th power of a $k/d$-differential for some $d > 1$.

We fix once and for all a $k$-th root of unity $\zeta$. The Deck group of $\pi$ contains a unique element $\tau$ such that $\tau^* \omega = \zeta \omega$. We fix this automorphism as well. We denote by $p = (p_1, \ldots, p_n)$ the tuple of marked points in $X$. Similarly, we denote by $\hat{p}$ the tuple of preimages of marked points in $\hat{X}$. The labeling is not canonical, even if we suppose that $\tau(z_i) = z_{i+1}$ within a set of preimages of a fixed point.

For the analogous statements about coverings in the stable case we first need to define twisted $k$-differentials and further preparation. An enhanced level graph for $k$-differentials is a level graph together with an enhancement map $\kappa: H \to \mathbb{Z}$ on the half-edges, satisfying the following properties:

i) If $h$ and $h'$ are paired to an edge, then $\kappa(h) + \kappa(h') = 0$.

ii) At a leg $h \in H \setminus E$ with order $m_i$, we impose that $\kappa(h) = m_i + k$.

iii) At each vertex $v \in V(\Gamma)$

$$k(2g(v) - 2) = \sum_{h \ni v} (\kappa(h) - k).$$

This completes the definition of twisted $k$-differentials and those compatible with a level graph $\Gamma$ given in the introduction. The global residue condition uses the fact that also a twisted differential $(X, q)$ defines canonical cyclic coverings $\pi: \hat{X} \to X$ such that $\pi^* q = \omega^k$ is the $k$-power of an abelian differential. Here however the edge identification are in general not uniquely determined by the requirement of a cyclic cover. Most of the local statement in the sequel depend on the specification of an identification that we record as a covering of graphs $\pi: \hat{\Gamma} \to \Gamma$.

We denote by $\mathcal{M}^k(\Gamma)$ the moduli space of twisted $k$-differential compatible with $\hat{\Gamma}$, suppressing the dependence on the initial type $\mu$. By the definition of the global residue condition and the main theorem of [BCGGM19a], all these differentials are smoothable to some $k$-differential on a smooth curve. Most relevant for us is the subvariety $\mathcal{M}^k_{\text{ma}}(\Gamma)$ parametrizing twisted $k$-differentials smoothable to a $k$-differential that is not a $k$-th power of an abelian differential. Points in $\mathcal{M}^k_{\text{ma}}(\Gamma)$ are usually denoted by $(X, q)$, or even more often by $(X, \omega)$, where $q = (q_i)_{i \in L(\Gamma)}$ is a
collection of \( k \)-differentials on the levels of \( X \), or alternatively \( \omega = (\omega_i)_{i \in L(\hat{\Gamma})} \) is a collection of one-forms on the cover \( \hat{X} \) of \( X \) induced by \( \hat{\Gamma} \to \Gamma \).

Next we define the subspaces of homology that we use for period coordinates. We fix some reference smooth surface \( \Sigma \) of genus \( g \) with \( n \) marked points that we partition as \( P \cup Z \) according to the poles of order \( \leq k \) among \( \mu \) and the 'zeros', i.e. with order \( > k \). We let \( \hat{\Sigma} \) be a model for the canonical covering surface, which is of genus \( \hat{g} \), and which comes with a map \( \hat{\sigma} : \hat{\Sigma} \to \Sigma \). We let \( \hat{P} \) and \( \hat{Z} \) be the preimages of \( P \) and \( Z \). They now correspond indeed to the zeros and poles of the type \( \hat{\mu} \).

\[
\begin{array}{ccc}
\hat{P}, \hat{Z} & \hat{\Sigma} & \hat{\Lambda}^\circ \supset \hat{\Lambda} \\
\downarrow \pi & \downarrow \pi & \downarrow \pi \\
P, Z & \Sigma & \Lambda^\circ \supset \Lambda
\end{array}
\]

If \( X \) is a stable curve and \( \pi \) a covering as above, we may find a multicurve \( \hat{\Lambda} \) in \( \hat{\Sigma} \) mapping under \( \pi \) to the multicurve \( \Lambda \) in \( \Sigma \), such that \( \hat{X} \) and \( X \) are obtained by pinching \( \hat{\Sigma} \) and \( \Sigma \) along \( \hat{\Lambda} \) and \( \Lambda \) respectively.

Recall ([BCGGM19a, Section 2]) that the moduli space \( \Omega^k M_{g,n}(\mu) \) (and thus also \( \Omega^+_k M_{g,n}(\mu) \)) is locally modelled on the \( \omega \)-periods of the eigenspace

\[
E(\hat{\Sigma} \setminus \hat{P}, \hat{Z}) = H_1(\hat{\Sigma} \setminus \hat{P}, \hat{Z}, \mathbb{C})_{\tau = \zeta}.
\]

Similarly, we can describe local coordinates for the components of a twisted differential on a stable curve \( X \) with enhanced level graph \( \Gamma \) (not yet imposing full compatibility, i.e. the GRC). Let \( \Lambda^\circ \) be an open thickening of \( \Lambda \). We let \( \Lambda^\pm \) be the upper and lower boundaries of \( \Lambda^\circ \). The level structure on \( \Gamma \) organizes \( \Sigma \setminus \Lambda \) into levels \( \Sigma(i) \) and we denote the adjacent poles, zeros and boundaries \( \Lambda^\pm \) with the subscript \((i)\). All the notation apply with a hat to the corresponding objects on the \( k \)-cover. The level-\( i \) component of the twisted differential is thus modelled on

\[
E(i) = H_1(\Sigma(i) \setminus \{\hat{P}_{(i)} \cup \hat{\Lambda}^\circ_{(i)} \cup \hat{\Lambda}^+_{(i)} \cup \hat{Z}_{(i)}, \mathbb{C})_{\tau = \zeta}.
\]

We can now restate the main dimension estimate in the proof of [BCGGM19a, Theorem 6.2].

**Proposition 2.1.** Twisted differentials compatible with an enhanced level graph are locally modelled on the \( \omega \)-periods of \( \prod_{i=0}^{-L} E^\text{grc}(i) \), where \( E^\text{grc}(i) \subseteq E(i) \) is the subspace at level \( i \) cut out by the global residue condition. The dimensions of these subspaces

\[
\sum_{i=0}^{-L} \dim_{\mathbb{C}} E^\text{grc}(i) = \dim_{\mathbb{C}} \Omega^k M_{g,n} - h
\]

add up to the total dimension of the moduli space of \( k \)-differentials of type \( \mu \) minus the number of horizontal edges \( h \).

3. Construction of \( \Xi^k M_{g,n}(\mu) \)

In this section we recall the main technical tools from [BCGGM19b], construct the compactification and eventually prove Theorem 1.1. The definitions in Section 3.3 and Section 3.4 are direct adaptation of the abelian case by working on the canonical \( k \)-covers. Avoiding the discussion of Teichmüller spaces means omission of that aspect but also a simplification of notations. In the remaining sections we have to ensure at some places that constructions can be performed \( \tau \)-equivariantly.
3.1. Degeneration, undegeneration. We describe here two types of maps between level graphs $\Gamma$ that encode the degeneration of curves, together with the compatible maps between the coverings graphs $\hat{\Gamma}$ that form part of the degeneration datum. In fact, it is easier to first describe the inverse process of undegeneration that encodes all the $k$-differentials in a neighborhood of a given degenerate $k$-differentials.

Let $\pi: \hat{\Gamma} \to \Gamma$ by a cyclic $k$-covering of enhanced level graphs with $L + 1$ normalized levels. For any subset $I \subset \{-1, \ldots, -L\}$, to be memorized as the the level passages that remain, we define the vertical undegeneration $\delta_I$ as the following contraction of certain vertical edges. An edge $e$ is contracted by $\delta_I$ if and only if the levels $e^-, e^- + 1, \ldots, e^+ - 1$ all belong the the complement of $I$. Here the symbol $e^-$ denotes both the point in $\hat{X}$ at the bottom end of the edge $e$ and its level. The meaning should be clear from the context. Similarly, $e^+$ refers to the top end. This edge contraction is performed simultaneously on the domain and range of $\pi$ and induces a cyclic $k$-covering $\delta_I(\hat{\Gamma}) \to \delta_I(\Gamma)$ that we abbreviate as $\delta_I(\pi)$. We write $\delta_I(j)$ for the image of the $j$-th level under $\delta_I$.

Moreover, we define for any subset $E_0 \subset E^{\text{hor}}$ of the horizontal edges of $\Gamma$ the horizontal undegeneration $\delta_{E_0}$, to be the edge contraction that contracts precisely the edges in $E_0$ in $\Gamma$. Contracting simultaneously also on the $\pi$-preimages of $E_0$ in $\hat{\Gamma}$, we obtain a new cyclic $k$-covering $\delta_{E_0}(\pi): \delta_{E_0}(\hat{\Gamma}) \to \delta_{E_0}(\Gamma)$.

A general undegeneration is a pair $\delta = (\delta_I, \delta_{E_0})$, defined by composing a horizontal and a vertical undegeneration in either order. A degeneration is the inverse of an undegeneration. We write $\hat{\Gamma}' \to \hat{\Gamma}$ for a general degeneration of level graphs and $\delta^{\text{ver}}$ and $\delta^{\text{hor}}$ for the two constituents of an undegeneration $\delta$.

3.2. Prong-matchings as extra structure on twisted differentials. We start with the definition of prong-matchings and the welded surfaces constructed from these. Given a differential $\omega$ on $X$ that has been put in standard from $z^\kappa dz/z$ if $\kappa \geq 0$ or $(z^\kappa - r)dz/z$ if $\kappa \leq -2$, the prongs are the $|\kappa|$ tangent vectors $e^{2\pi i j/|\kappa|} \partial_z$ for $\kappa > 0$ and $-e^{2\pi i j/|\kappa|} \partial_z$ for $j = 0, \ldots, |\kappa| - 1$. At simple poles, prongs are not defined.

We now get back to a twisted differential $(\hat{X}, \hat{\omega}, \hat{\Gamma})$. Define a local prong-matching $\sigma_e$ at the vertical edge $e$ of $\hat{\Gamma}$ to be a cyclic order-reversing bijection between the prongs at the upper and lower end of $e$. A global prong-matching is a collection $\sigma = (\sigma_e)_{e \in E(\hat{\Gamma})}$ of local prong-matchings that is equivariant with respect to the action of $\tau$ permuting the edges and multiplying the local coordinates $z$ by $\zeta$.

A global prong-matching $\sigma$ on $\hat{X}$ gives an almost-smooth surface $\hat{X}_\sigma$, i.e. a smooth surface except for nodes corresponding to the horizontal nodes of $\hat{X}$, constructed by the following procedure of welding. Take the partial normalization of $\hat{X}$ separating branches at vertical nodes and perform the real oriented blowup of each pair of preimages. Then identify the boundary circles isometrically so as to identify boundary points that are paired by the prong-matching. More details of the construction can be found in Section 4 of [ECGGM19b], see also [ACG11]. (We only use subscripts $\sigma$ to denote weldings here and suppress the overline used in loc. cit. to avoid double decorations.) Horizontal nodes remain untouched in the welding procedure.

On almost-smooth surfaces any good arc $\gamma$, i.e. any arc transversal to the seams created by welding, has a well-defined turning number, that we denote by $\rho(\gamma)$. 

Adding the information of prong-matching to points in $\mathcal{M}^k(\hat{\Gamma})$ will give us a finite covering space. We will next construct this covering in detail for the subset $\mathcal{M}^k_{\text{na}}(\hat{\Gamma})$ of twisted differentials admitting a non-abelian smoothing, since we will be exclusively concerned with that space.

We define the set $\mathcal{M}^k_{\text{pm}}(\hat{\Gamma})$ to be tuples $(X, \omega, \sigma)$ consisting of a point $(X, \omega) \in \mathcal{M}^k_{\text{na}}(\hat{\Gamma})$ together with a prong-matching $\sigma$. There is an obvious notion of parallel transport of prong-matchings that allows to lift inclusions of contractible open sets $U \to \mathcal{M}^k_{\text{na}}(\hat{\Gamma})$ uniquely to maps $U \to \mathcal{M}^k_{\text{pm}}(\hat{\Gamma})$. Requiring that these lifts are holomorphic local homeomorphism provides $\mathcal{M}^k_{\text{pm}}(\hat{\Gamma})$ with a complex structure so that $\mathcal{M}^k_{\text{pm}}(\hat{\Gamma}) \to \mathcal{M}^k_{\text{na}}(\hat{\Gamma})$ is a covering map.

3.3. The level rotation torus. Our compactification combines the geometry of moduli spaces of $k$-differentials of lower complexity and aspects of a toroidal compactification. The torus action for the latter is given by the level rotation torus that we now define.

In the introduction we defined the twist group to be the (full rank) subgroup $T_{\hat{\Gamma}}$ of the level rotation group $R_{\hat{\Gamma}} \cong \mathbb{Z}^L$ that fixes all prongs. The (reduced) level rotation torus $T_{\hat{\Gamma}}^r$ is the quotient

$$T_{\hat{\Gamma}}^r = \mathbb{C}^L / T_{\hat{\Gamma}}.$$

(Here reduced refers to the fact that $T_{\hat{\Gamma}}$ does not rotate the top level. We will introduce this action separately for projectivization and usually drop the adjective 'reduced'.) The level rotation torus can also be characterized ([BCGGM19b, Proposition 5.4]) as the connected component of the identity of the subtorus

$$\left\{ ((r_i, \rho_e))_{i,e} \in (\mathbb{C}^*)^L \times (\mathbb{C}^*)^{E(\hat{\Gamma})} \mid r_{e^-} \cdots r_{e^+ - 1} = \rho_e^{\epsilon_e} \text{ for all } e \in E(\hat{\Gamma}) \right\}.$$  

(2)

This characterization makes the reason for introducing $T_{\hat{\Gamma}}$ apparent, as there is a natural action of the level rotation torus on $\mathcal{M}^k_{\text{pm}}(\hat{\Gamma})$ given by

$$T_{\hat{\Gamma}}^r \times \mathcal{M}^k_{\text{pm}}(\hat{\Gamma}) \to \mathcal{M}^k_{\text{pm}}(\hat{\Gamma})$$

$$(r_i, \rho_e) \ast (X, (\omega_{(i)}), (\sigma_e)) = \left( X, \left( r_i \cdots r_{-1} \omega_{(i)} \right), (\rho_e \ast \sigma_e) \right)$$

(3)

where $\rho_e \ast \sigma_e$ is the prong-matching $\sigma_e$ post-composed with the rotation by $\arg(\rho_e)$ (if the full Dehn twist around $e$ corresponds to angle $2\pi$, equivalently by the rotation by $\kappa \arg(\rho_e)$ for the angle in the flat metric). We alert the reader that this action uses the 'triangular' basis, where the $i$-th component of $T_{\hat{\Gamma}}^r$ rotates the $i$-th level and all level below it by the amount $r_i$.

The toric variety associated with the level rotation torus will not be smooth in general. To obtain orbifold charts we define first the simple twist group $T_{\hat{\Gamma}}^s \subseteq T_{\hat{\Gamma}}$ as the twist group elements generated by rotations of one level at a time, i.e.

$$T_{\hat{\Gamma}}^s = \bigoplus_{i=1}^L T_{\delta_i(\hat{\Gamma})}.$$

We can now define the (reduced) simple level rotation torus as

$$T_{\hat{\Gamma}}^s = \mathbb{C}^L / T_{\hat{\Gamma}}^s.$$  

(4)

In order to describe the action of these tori we will need the integers

$$a_i = \text{lcm}_{e \in E(\hat{\Gamma})} k_e, \quad m_{e,i} = a_i / k_e$$

(5)
for \( i = -1, \ldots, -L \) and \( e \in E(\hat{\Gamma}) \). Now Proposition 5.4 in loc. cit. moreover shows that there is an identification \( T^s_{\hat{\Gamma}} \cong (\mathbb{C}^*)^L \) such that the projection \( T^s_{\hat{\Gamma}} \rightarrow T_{\hat{\Gamma}} \) is given in coordinates by

\[
(s_i) \mapsto (r_i, \rho_e) = \left( s_i^{a_i}, \prod_{i=e^-}^{e^+ - 1} s_i^{a_i/k_e} \right)
\]  

(6)

The composition of this parametrizations \( (6) \) of \( T^s_{\hat{\Gamma}} \) by \( T^s_\Gamma \) with the action \( (3) \) gives an action of \( s = (s_i) \in T^s_{\hat{\Gamma}} \) on welded surfaces and we denote the image of \( \hat{X}_\sigma \) under the action of \( s \) by \( \hat{X}_{s, \sigma} \).

3.4. The compactification as topological space. We start with the definition of \( \Xi_{g,n}(\mu) \) as a set. For each k-cyclic covering \( \pi : \hat{\Gamma} \rightarrow \Gamma \) we define the boundary stratum \( \Omega^k B^\pi_{\hat{\Gamma}} = \mathbb{M}_{pn}(\hat{\Gamma})/T_{\hat{\Gamma}} \) and we define the set

\[
\Xi_{g,n}(\mu) = \bigcup_{\pi: \hat{\Gamma} \rightarrow \Gamma} \Omega^k B^\pi_{\hat{\Gamma}}.
\]  

(7)

This union includes \( \Omega^k_{+n} \mathcal{M}_{g,n} \) for \( \pi \) being the trivial covering of a point to a point. Points of \( \Xi_{g,n}(\mu) \) are called multi-scale k-differentials, i.e. the preceding definition completes the specification of the equivalence relation stated in the introduction. Points of \( \Xi_{g,n}(\mu) \) are thus given by a tuple \((X, p, \hat{\Gamma}, \omega, \sigma)\) where \( \omega = (\omega(i))_{i=0}^{-L} \) is a tuple indexed according to the levels. We often write just \((X, \omega)\) or \((X, \omega, \hat{\Gamma})\). The equivalence classes are given by the orbits of the action \( (3) \) on \((\omega, \sigma)\).

We now provide this space with a topology by exhibiting all converging sequences. The basic idea is the conformal topology on \( \mathcal{M}_g \) where sequences converge if there is an exhaustion of the complement of nodes and punctures and conformal maps of the exhaustion to neighboring surfaces, see (b) below. For multiscale differentials we require moreover the convergences of the differentials as in (c) after a rescaling, where the magnitude of rescaling is compatible with the level structure, see (a) and (c). Since the conformal topology only requires the comparison maps to be diffeomorphisms near the nodes, which can twist arbitrarily, we need to add (d) to avoid constructing a non-Hausdorff space. In the sequel we write \( \hat{X}_{\sigma_n} \) for \((\hat{X}_n)_{\sigma_n} \) in a sequence of welded surfaces.

We say that a sequence \((\hat{X}_n, \omega_n, \hat{\Gamma}_n)\) converges to \((\hat{X}, \omega, \hat{\Gamma})\), if there exist representatives of all the equivalence classes (that we denote by the same letters), a sequence \( \varepsilon_n \rightarrow 0 \) and a sequence \( s_n = (s_{n,i})_{i=-1}^{-L} \in (\mathbb{C}^*)^L \) of tuples such that:

(a) For sufficiently large \( n \) there is an undegeneration \( \delta_n = (\delta_n^\text{ver}, \delta_n^\text{hor}) \) with \( \delta_n^\text{ver}(\hat{\Gamma}_n) = \Gamma \).
(b) For sufficiently large \( n \) there is an almost-diffeomorphism \( g_n : \hat{X}_{s_n, \sigma} \rightarrow \hat{X}_{\sigma_n} \) that is conformal on the \( \epsilon_n \)-thick part of \((\hat{X}, \hat{p})\) and that respects the marked points, up to relabeling in \( \pi \)-fibers.
(c) The restriction of \( \prod_{j=1}^{i-1} s_{n,j}^\text{ver} \cdot g_n^i(\omega_n) \) to the \( \epsilon_n \)-thick part of the level \( i \) subsurface of \((\hat{X}, \hat{p})\) converges uniformly to \( \omega(i) \).
(d) For any \( i, j \in L(\hat{\Gamma}) \) with \( i > j \), and any subsequence along which \( \delta_n^{\text{ver}}(i) = \delta_n^{\text{ver}}(j) \), we have
\[
\lim_{n \to \infty} \prod_{k=j}^{i-1} |s_{n,k}|^{-a_k} = 0.
\]

(e) The almost-diffeomorphism \( g_n \) are nearly turning-number preserving, i.e., for every good arc \( \gamma \) in \( \hat{X}_\sigma \), the difference \( \rho(g_n \circ F_{s_n} \circ \gamma) - \rho(F_{s_n} \circ \gamma) \) of turning numbers converges to zero, where \( F_{s_n} \) is the fractional Dehn twist around the edge \( e \) by the angle \( \prod_{j=1}^{i} \alpha_{s_{n,j}}^{\sigma_e} \).

This topology is exactly the topology of the compactification of the moduli spaces \( \Omega M_g(\hat{\mu}) \) in [BCGGM19b] restricted to the subspace of \( k \)-cyclic covers. Note that the inclusion of \( \hat{\Gamma} \) into the datum of a multi-scale \( k \)-differential implies that even boundary points have canonically determined \( k \)-covers. We may thus view
\[
\Xi_{k} M_{g,n}(\hat{\mu}) \subset \Xi_{\hat{\mu}} \hat{\Gamma},
\]
with the subspace topology.

**Proposition 3.1.** The moduli space \( \Xi_{k} M_{g,n}(\mu) \) is a Hausdorff topological space and its projectivization \( \mathbb{P}\Xi_{k} M_{g,n}(\mu) \) is a compact Hausdorff space.

**Proof.** This follows from the definition of the subspace topology, the fact that being a \( k \)-cover is a closed condition and [BCGGM19a, Theorem 9.4 and Proposition 14.2]. Alternatively, those proofs can be adapted directly to the current situation without Teichmüller markings. \( \square \)

### 3.5. Model differentials and modification differentials

In order to provide \( \Xi_{k} M_{g,n}(\mu) \) with a complex structure we use a local model space that automatically has a complex structure (as a finite cover of a product of spaces of non-zero \( k \)-differentials). The degeneration of differentials on lower components is emulated in the model space by vanishing of auxiliary parameters \( t_i \).

The action \( (3) \) of the simple level rotation torus \( T_{s_{\hat{\Gamma}}} \) makes \( \mathbb{M}_{k,\text{pm}}(\hat{\Gamma}) \) into a principal \( (\mathbb{C}^*)^L \)-bundle over the 'simple' version of the boundary stratum \( \Omega^k B_{s\hat{\Gamma}}^* = \mathbb{M}_{k,\text{pm}}(\hat{\Gamma})/T_{s_{\hat{\Gamma}}} \) and we define the (compactified) simple model domain \( \mathbb{M}_{k,\text{pm}}(\hat{\Gamma})^s \) to be the associated \( \mathbb{C}^L \)-bundle over \( \Omega^k B_{s\hat{\Gamma}}^* \).

The construction directly implies:

**Proposition 3.2.** The compactified simple model domain \( \mathbb{M}_{k,\text{pm}}(\hat{\Gamma})^s \) is smooth with normal crossing boundary divisor. If \( t_i \) is a coordinate on \( \mathbb{C}^L \), then the boundary divisor \( D_i = \{ t_i = 0 \} \) corresponds to model differentials compatible with \( \delta_i(\hat{\Gamma}) \) and its degenerations.

The space \( \mathbb{M}_{k,\text{pm}}(\hat{\Gamma}) \), being just a GRC-subspace in a product of moduli spaces, obviously comes with a universal family of curves and \( k \)-differentials that we can pull back to \( \mathbb{M}_{k,\text{pm}}(\hat{\Gamma}) \). Since the level rotation torus only acts on differentials and prong-matchings, not on the curve, the universal curve descends to a family of curves \( f: \hat{\cal{X}} \to \mathbb{M}_{k,\text{pm}}(\Gamma)^s \). Over small enough open sets \( W \subset \mathbb{M}_{k,\text{pm}}(\Gamma)^s \) (even at the boundary!) we can fix a scale of the \( T_{s_{\hat{\Gamma}}} \)-orbits and work with a collection \( \eta = (\eta_i)_{i=0}^{L} \) of families of differentials, not identically zero on any component of any fiber. From now on with stick to the convention that \( \eta \) denotes (families of) model differentials.
(that come with degeneration parameters \(t_i\)) whereas \(\omega\) denotes (families of) multi-scale differentials (that may become zero on components of special fibers). We alert the reader of the inevitable notation problem that for trivial families (i.e. just a single surface \(\tilde{X}\)) multi-scale-\(k\)-differentials are equivalence classes of prong-matched twisted differentials, thus both denoted by \(\omega\), and model-differentials are the same objects, albeit denoted by \(\eta\). It is only in degenerating families that the difference becomes apparent.

Note that the boundary of the compactified simple model domain comes with a natural stratification given by the subset of \([-1, \ldots, -L]\) of the \(t_i\) that are zero.

Modification differentials will be used for plumbing and also for perturbed period coordinates on charts of \(M^k_{pm}(\Gamma)^s\). In the sequel we check that the setup of \[\text{BCGGM19b}\] Section 11] works in the \(k\)-equivariant setup. We define

\[
\mathbf{t} \ast \eta = (t_{[i]} \cdot \eta(i))_{i \in L(\tilde{\Gamma})} = (t_{[i]-1}^{i-1} \cdots t_{[i]}^{i-1} \cdot \eta(i))_{i \in L(\tilde{\Gamma})},
\]

for \(\mathbf{t} = (t_{-1}, \ldots, t_{-L}) \in (\mathbb{C}^*)^L\).

**Definition 3.3.** A equivariant family of modifying differentials over \(W\) equipped with the universal differential \(\mathbf{t} \ast \eta\) is a family of meromorphic differentials \(\xi = (\xi(i))_{i=0}^L\) on \(f: \tilde{X} \to W\), such that

(i) the equivariance \(\tau^* \xi = \zeta \cdot \xi\) holds,

(ii) the differentials \(\xi(i)\) are holomorphic, except for possible simple poles along both horizontal and vertical nodal sections, and except for marked poles,

(iii) the component \(\xi(-L)\) vanishes identically and moreover \(\xi_i\) is divisible by \(t_{[i]-1}\) for each \(i = -1, \ldots, -L + 1\), and

(iv) the sum \(\mathbf{t} \ast \eta + \xi\) has opposite residues at every node.

**Proposition 3.4.** The universal family \(f: \tilde{X} \to W\) equipped with the universal differential \(\mathbf{t} \ast \eta\) admits an equivariant family of modifying differentials.

**Proof.** The proof of \[\text{BCGGM19b}\] Proposition 11.3] works in the situation where the edges of \(\tilde{\Gamma}\) are images of the pinched multicurve \(\Lambda\) via a family of markings \(\Sigma \to \tilde{X}\) by a reference surface \(\Sigma\). Choosing \(W\) contractible, we may assume that we have such a marking here as well.

The proof in loc. cit. starts by taking the subspace \(V = \langle \lambda \in \Lambda \rangle_Q\) and the subspace \(V_P\) spanned by the loops around the marked poles inside \(H_1(\tilde{X} \setminus \hat{P}, Q)\). We define \(V_N = V + V_P\). The proof proceeds by searching for a complementary subspace \(V_C\) (i.e. with \(V_N \cap V_C = 0\)) such that the projection \(p(V')\) to \(H_1(\tilde{X}, Q)\) is a Lagrangian subspace, where \(V' = V_N + V_C\). The proof then constructs \(\xi = (\xi_i)\) for each \(w \in W\) from assignments \(\rho_i : V_i \to \mathbb{C}\) determined by the periods of the fiber \(\eta_w\) on subspaces \(V_i\) of \(V + V_P\) generated by multicurves associated to edges whose lower level is below \(i\). Relevant here is that those \(\xi\) satisfy all properties of Definition 3.3] except possibly the equivariance in (i). Moreover, \(\xi\) depends uniquely on an extension \(\rho'_i\) of \(\rho_i\), that we may chose to be zero on \(V_C\).

If we can find a subspace \(V_C\) which is \(\tau\)-invariant, then the extended residue assignment \(\rho'_i\) is \(\tau\)-equivariant (with \(\tau\) acting by multiplication by \(\zeta\) on the range) and thus \(\xi\) satisfies (i). To find such a \(V_C\), we enlarge \(V_C\) and thus \(V' = V_C + V_N\) step by step, staying \(\tau\)-invariant at each step, until \(p(V')\) is a Lagrangian subspace. If at some step \(V'\) is \(\tau\)-invariant, but \(p(V')\) is strictly contained in a Lagrangian
subspace, we may find an element $\gamma$ that pairs trivially with $p(V')$. But then $\tau^i(\gamma)$ also pairs trivially with $p(V')$ for all $i$ and we add it to $V_C$ the span of all $\tau^i(\gamma)$.

3.6. The perturbed period map. Periods give local coordinates on $\mathcal{M}^k_{pm}(\bar{\Gamma})$ and thus on $\mathcal{M}^k_{pm}(\hat{\Gamma})$. Together with the tuple of degeneration parameters $t$ and deprived of one coordinate per level to fix the scale of projectivization they give local coordinates of $\mathcal{M}^k_{pm}(\hat{\Gamma})^s$. We introduce some perturbation of these coordinates here and show that this still gives local coordinates. The reason for this procedure is that the perturbed period coordinates can still be used after plumbing, see Section 3.7. Together with horizontal plumbing parameters it will provide coordinates on an orbifold chart of $\Xi(\bar{\mathcal{M}})_{g,n}(\mu)$. Except for the use of appropriate eigenspaces this is exactly [BCGGM19b, Section 11].

Near the marked point $e^+$ corresponding to the upper end (say on level $i = i(e^+)$) of each of the vertical nodes, choose an auxiliary section $s^+_i : W \to \hat{\mathcal{X}}$ such that

$$\int_{e^+} s^+_i(w) \eta(i) = \text{const},$$

where the constant is sufficiently small, depending on $W$, and constrained by the plumbing construction later. Near each zero marked $z_j$ of $\eta$ (say on level $i = i(z_j)$) choose an auxiliary section $s_j : W \to \hat{\mathcal{X}}$ that coincides with the barycenter of the zeros of $\eta(i) + t^{-1}_{[i]} \cdot \xi(i)$ that result from the deformation of $z_j$. We let $\gamma_{i,j}$ for $i = 0, \ldots, -L$ and $j = 1, \ldots, \dim E^\text{grc}_{(i)}$ be a basis of the subspaces $E^\text{grc}_{(i)}$ of homology. Since the contribution of each level to the twisted differentials compatible with a level graph is positive-dimensional (by the rescaling of the differential), the definition of periods coordinates along the boundary in Proposition 2.1 implies that for each $i$ there exists some $j$ such that $\int_{\gamma_{i,j}} \eta(i) \neq 0$. We use this to fix the scale of the projectivization and assume that the periods for $j = 1$ are normalized on each level, i.e. $\int_{\gamma_{i,j}} \eta(i) = 1$.

The $i$-th level component of the perturbed period map is now given by

$$\text{PPer}_i : \begin{cases} W &\to \mathbb{C}^\dim E^\text{grc}_{(i)} - 1 + \delta_{i,0}, \\ ([\hat{\mathcal{X}}, \eta, t]) &\mapsto \left(\int_{\gamma_{i,j}} \eta(i) + t^{-1}_{[i]} \cdot \xi(i)\right)^{\dim E^\text{grc}_{(i)}}_{j=2-\delta_{i,0}}, \end{cases}$$

where $\delta_{i,0}$ is Kronecker’s delta and where the integrals are to be interpreted starting and ending at the nearby points determined by the sections $s^+_i$ and $s_j$ rather than the true zeros of $\eta$. The reason for this technical step is that those nearby points are still present after the surfaces has been plumbed (‘Step 2’ below).

Proposition 3.5. The perturbed period map

$$\text{PPer}^\text{MD} : W \to \mathbb{C}^L \times \prod_{i=0}^{-L} \mathbb{C}^{\dim E^\text{grc}_{(i)} - 1 + \delta_{i,0}}, \quad ([\hat{\mathcal{X}}, \eta, t]) \mapsto \left(t : \prod_{i=0}^{-L} \text{PPer}_i([\hat{\mathcal{X}}, \eta, t])\right)$$

is open and locally injective on a neighborhood of the most degenerate boundary stratum $W_A = \cap_{i=1}^L D_i$ in the compactified model domain $\mathcal{M}^k_{pm}(\bar{\Gamma})^s$.

We will write $(t, w) = \text{PPer}^\text{MD}([\hat{\mathcal{X}}, \eta, t])$.

Proof. As in [BCGGM19b, Proposition 11.6] it suffices to show that the derivative is surjective, by dimension comparison. For the tangent directions to the boundary
this follows from Proposition 2.1 (and the fact that we have projectivized the lower levels). For the transverse direction this follows since the $t_i$ are the local coordinates of the $\mathbb{C}^r$-bundles used to construct the compactifications.

The reader should keep in mind, that in the model domain with its equisingular family of curves horizontal nodes are untouched. They enter in Proposition 1.2 only after plumbing horizontal nodes, see below.

3.7. The complex structure and the proof of Theorem 1.1

The outline of the proof of Theorem 1.1 consists of the following steps.

1) Construct locally covers $U^s \to U$ for small open sets $U \subset \Xi_{\mathbb{C}^r}^{\mu}(\mathbb{C}^r)$ that will be used as orbifold charts.

2) Perform a plumbing construction on the pullback of the universal family $f: \hat{\mathcal{X}} \to \mathbb{M}_{\mu}(\hat{\Gamma})^s$ to small open sets $W$ and via the second projection to $W \times \Delta^h_\mathbb{C}$ obtain a family $\mathcal{Y} \to W \times \Delta^h_\mathbb{C}$ together with a family of differentials.

3) Use the moduli properties of the strata of $\Xi_{\mathbb{C}^r}^{\mu}(\mathbb{C}^r)$ to obtain moduli maps $\Omega\text{Pl} : W \times \Delta^h_\mathbb{C} \to U^s$ for appropriately chosen target set $U$, defined stratum by stratum.

4) Show that $\Omega\text{Pl}$ is a homeomorphism near a central point $P \times (0, \ldots, 0) \in W \times \Delta^h_\mathbb{C}$ and thus provide charts there.

The charts constructed in this way depend on many choices, in the construction of the modification differential and the parameters for plumbing. However, since there the $\tau$-equivariance needs to be respected and since we need this in the next section. The technical Step 3) and Step 4) proceed exactly as in [BCGGM19b].

Step 1. In order to provide $\Xi_{\mathbb{C}^r}^{\mu}(\mathbb{C}^r)$ with a complex structure we consider the neighborhood $U$ of a point $(X, \omega, \hat{\Gamma})$ that we may assume to be at the boundary, say for level graph $\hat{\Gamma}$. (The following description assumes that $(X, \omega)$ has no auto-morphisms. In general we should start from an orbifold chart, and add the extra orbifold structure described below.) The compactified simple model domain is a $K = T_{\mathbb{C}^r}/T_{\mathbb{C}^r}^\mu$-cover of the (in general) singular space that we would get by compactifying the $T_{\mathbb{C}^r}$-quotient of $\mathbb{M}_{\mu}(\hat{\Gamma})$. Consequently, we have to pass locally near $(X, \omega, \hat{\Gamma})$ to a $K$-cover of $U$. We define this cover $U^s$ as follows. Define the auxiliary simple boundary stratum to be $\Omega^k \mathcal{B}^{\mu}_{\mathbb{C}^r} = \Omega^k \mathcal{B}^{\mu}_{\mathbb{C}^r}/T_{\mathbb{C}^r}^\mu$. As a set

$$U^s = \left\{ (X', \omega', \hat{\Gamma}') \in \bigcup_{\hat{\Gamma}, \ldots, \hat{\Gamma}} \Omega^k \mathcal{B}^{\mu}_{\mathbb{C}^r} : \varphi((X', \omega', \hat{\Gamma}')) \in U \right\},$$

where $\varphi$ is induced by the quotient maps $\Omega^k \mathcal{B}^{\mu}_{\mathbb{C}^r} \to \Omega^k \mathcal{B}^{\mu}_{\mathbb{C}^r}$. We provide $U^s$ with a topology where convergence is formally given exactly by the same conditions as for $\Xi_{\mathbb{C}^r}^{\mu}(\mathbb{C}^r)$ in Section 3.3, but where now the 'existence of representatives of the equivalence classes' is up to the torus $T_{\mathbb{C}^r}^\mu$ rather than the quotient torus $T_{\mathbb{C}^r}$.

Step 2. To start the plumbing construction we first define the plumbing fixture for each vertical edge $e \in E(\hat{\Gamma})$ to be the degenerating family of annuli

$$\mathbb{V}_e = \left\{ (w, t, u, v) \in W \times \Delta^2_\mathbb{C} : uv = \prod_{i-e}^{+1} l_i^m_{i-1} \right\},$$

(12)
that only depends on the $t$-part of the perturbed period coordinates $(t,w)$ of $W$. We equip $\mathbb{V}_e$ with the family of differentials

$$
\Omega_e = \left( t_{[e]} \cdot u^{\kappa_e - 1} - \frac{r'_e}{u} \right) du = \left( -t_{[e]} \cdot v^{-\kappa_e - 1} + \frac{r'_e}{v} \right) dv,
$$

where $t_{[1]} = t_1^{a_1} \cdots t_1^{a_1}$ and where $r'_e = r'_e(w,t)$ are the residues of the universal family over model domain. Inside the plumbing fixture we define the gluing annuli $A^{\pm}_e$ by $\delta/R < |u| < \delta$ and $\delta/R < |v| < \delta$ respectively. The sizes $\delta$, $R$ and the size of the neighborhood $W$ will be determined in terms of the geometry of the universal family, to ensure for example that plumbing annuli are not overlapping.

Suppose we only have vertical nodes. The plumbing construction proceeds bottom up. Near each of the nodes of bottom level we put the family of differentials $\eta_{(-L)}$ in standard form $(v^{-\kappa_e - 1} + \frac{\kappa}{\pi})dv$ so that after rescaling with $t_{[e]}$ it can be glued to $\Omega_e$ for $r'_e = t_{[e]}r_e$. That such a normal form exists in families is the content of \cite[Theorem 3.3]{BCGGM19}. The functions $r'_e$ determine the modification differential $\xi_{(-L+2)}$ as the proof of Proposition 3.3 shows, see \cite[Corollary 11.4]{BCGGM19}. We will thus put $t_{[e+\Gamma]}\eta_{(-L)} + \xi_{(-L+1)}$ in standard form near $e^+$ using the normal form on the deformation of an annulus \cite[Theorem 4.5]{BCGGM19} or \cite[Theorem 12.2]{BCGGM19}) and this glues with the form \eqref{13} on the upper end of the annulus. Iterating the procedure allows to plumb the collection of families of one-forms

$$
t \ast \eta + \xi = \left( t_{[1]} \cdot \eta_{(i)} + \xi_{(i)} \right)_{i=0}^{-L}
$$

on the equisingular family of curves $X \to W$ to a family of one-forms $\omega$ on a degenerating family of curves $Y \to W$.

In the preceding construction we have neglected so far that the choice of the normal form is unique only up multiplication by a $\kappa_e$-th root of unity. The pruning that is part of the datum of the universal family over the model domain determines this choice. Many more details, using reference sections to make the construction rigorous, are given in \cite[Section 12]{BCGGM19}.

The whole construction can obviously performed $\tau$-equivariantly, since the modification differential is $\tau$-equivariant and since the sizes of the neighborhoods and plumbing annuli are determined by the rates of degeneration of $t \ast \eta + \xi$, i.e. by $\tau$-equivariant data.

Finally, we investigate horizontal nodes of $\hat{\Gamma}$, that come in $\tau$-orbits of length $k$ and that we thus label as $e_1^{(a)}, \ldots, e_k^{(a)}$ for $0 \leq a < k$. We parameterize the plumbing by additional plumbing parameters $x = (x_1, \ldots, x_h) \in \Delta_h$ and define the (horizontal) plumbing fixture to be

$$
\mathbb{W}_j = \left\{ (w,t,x,u,v) \in W \times \Delta_h \times \Delta_2^3 : uv = x_j \right\},
$$

independently of the upper label $a$ of $e_j^{(a)}$, equipped with the family of holomorphic one-forms

$$
\Omega_j = -r'_{e_j}^+ (w,t)du/u = r'_{e_j}^- (w,t)dv/v,
$$

where $\pm r'_{e_j}^\pm (w,t)$ is the residue of $t \ast \eta + \xi$ at the $j$-th horizontal node. Here the gluing happens along annuli $B_j^{\pm}$ by $\delta/R < |u| < \delta$ and $\delta/R < |v| < \delta$. 


Step 3. The existence of moduli maps on each stratum of the simple model domain to $\mathbb{S}_{g,n}(\mu)$ is immediate from the construction of $\mathbb{S}_{g,n}(\mu)$ as union of strata $\Omega^k\Gamma = \mathfrak{M}^k\Gamma / \Gamma$ and the property of $\mathfrak{M}^k\Gamma$ as moduli space of $k$-differentials. We let $U$ be the range of the union of these maps. The map factors through $U^*$ since both this space and the simple model domain are defined as $T^*\Gamma$-equivalence classes. [BCGGM19b Section 2] provides more details.

Step 4. To show that the resulting map $\Omega Pl: W \times \Delta^h \to U^*$ is continuous we have to invoke the definition of the topology on $U^*$ to show that the images of a converging sequence converges. This entails exhibiting the almost-diffeomorphisms $g_n$ with the properties (a)--(e). These $g_n$ construct level by level, bottom up, using conformal identifications of flat surfaces with the same periods ([BCGGM19b Theorem 2.7]), a $C^1$-quasi-conformal extension of these maps across the plumbing cylinder and the equivalence of the conformal and $C^1$-quasi-conformal topology on strata of abelian differentials ([BCGGM19b Section 3.5] together with the disc coordinates $x$ and locally injective. Openness amounts to showing that for any converging sequence converges. This entails exhibiting the almost-diffeomorphisms $g_n$ with the properties (a)--(e). These $g_n$ construct level by level, bottom up, using conformal identifications of flat surfaces with the same periods ([BCGGM19b Theorem 2.7]), a $C^1$-quasi-conformal extension of these maps across the plumbing cylinder and the equivalence of the conformal and $C^1$-quasi-conformal topology on strata of abelian differentials ([BCGGM19b Section 2]).

To show that $\Omega Pl$ is a homeomorphism we need to show that this map is open and locally injective. Openness amounts to showing that for any converging sequence in $U^*$, say converging to $(\hat{X}, \omega, \hat{\Gamma})$, we can eventually undo the plumbing construction and find $\Omega Pl$-preimages in the model domain $\mathfrak{M}^k\Gamma\Gamma$ claimed in iii) of Theorem 1.1. Local injectivity amounts to checking uniqueness of the previous unplumbing steps using perturbed period coordinates. See [BCGGM19b Section 12.5-12.7] for details on these steps.

The action of $C^*$ on the $k$-th root $\omega$ defines an action on the space $\Omega_{g,n}\mathcal{M}_{g,n}$ that is equivariant via $\lambda \mapsto \lambda^k$ with a $C^*$-action on $\Omega^k\mathcal{M}_{g,n}$. The quotients of both actions is the same space $\mathfrak{M}_{g,n}(\mu)$. We encourage the reader to revisit all the steps to check that the first action extends equivariantly to all auxiliary spaces, multiplying simultaneously all forms at all levels. The resulting quotient of $\mathfrak{M}_{g,n}(\mu)$ by $C^*$ is the compactification $\mathfrak{P}\mathfrak{M}_{g,n}(\mu)$ claimed in iii) of Theorem 1.1.

The proof of Proposition 1.2 is contained in these statements, since Proposition 1.2 together with the disc coordinates $x_j$ used in (15) gives local coordinates on $\mathfrak{M}^k\Gamma\Gamma^* \times \Delta^h$. Consequently, the perturbed period coordinates are given by

$$\begin{align*}
\text{PPer}: \quad U^* \xrightarrow{\Omega Pl^{-1}} W \times \Delta^h & \quad \to \mathbb{C}^h \times \mathbb{C}^{L+1} \times \prod_{i=0}^{L} \mathbb{C}^{\dim E^{F}_{1}(1)-1} \\
[\hat{X}, \omega] \xrightarrow{\Omega Pl^{-1}} [()\hat{X}, \eta, t, x)] & \quad \mapsto \left( x; t; \prod_{i=0}^{L} \text{PPer}_i(\hat{X}, \eta, t) \right)
\end{align*}$$

(17)
on open orbifold charts $U^*$ of $\mathfrak{P}\mathfrak{M}_{g,n}(\mu)$, using the inverse of the homeomorphism $\Omega Pl$ constructed in Step 3 and 4.

4. THE AREA FORM IS GOOD

Here we prove our main Theorem 1.4 and start with the definition of a good metric in the sense of [Mum77] on a smooth $r$-dimensional variety (or orbifold) $X$.

Suppose that $\overline{X}$ is the compactification of $X$ with a normal crossing boundary divisor $\partial X = \overline{X} \setminus X$. Let $\mathcal{L}$ be a line bundle on $\overline{X}$. A metric $h$ on $\mathcal{L}|X$ is good, if for each point $p \in \partial X$ there is a neighborhood $\Delta^r$ with coordinates such that
\[ \partial X = \{ \prod_{i=1}^{\ell} x_i = 0 \} \] and such that the function \( h_s = h(s, s) \) for a local generating section \( s \) of \( L \) has the following properties:

(i) There exist \( C > 0 \) and \( n \in \mathbb{N} \) such that \( |h_s| < C \left( \sum_{i=1}^{\ell} \log |x_i| \right)^{2n} \) and \( |h_s^{-1}| < C \left( \sum_{i=1}^{\ell} \log |x_i| \right)^{2n} \).

(ii) the connection one-form \( \partial \log h \) and the curvature two-form \( \overline{\partial} \partial \log h \) have Poincaré growth.

Here a \( p \)-form \( \eta \) is said to have Poincaré growth on \( \Delta' \) if for any choice of sections \( v_i \) of \( T_{\partial X}(\Delta') \) there is \( C \) such that

\[ |\eta(v_1, \ldots, v_p)|^2 \leq C \prod_{i=1}^{p} \omega_P(v_i, v_i) \]

holds for \( \omega_P \) the product of the Poincaré metrics \( |dx_i|^2/x_i^2 \log |x_i|^2 \) in the coordinates \( x_i \) for \( i \leq \ell \) and the euclidean metric in the other coordinates.

We place ourselves now in the setting of the theorem and rewrite the expression for the metric. Recall that now \( m_i > -\kappa \) and thus the sets \( P \) and \( \tilde{P} \) as defined in Section 2 are empty. Note that the total space of the line bundle \( \mathcal{O}(-1) \) defined in the introduction as the \( \varphi \)-pullback of \( \mathcal{O}(-1) \) from the incidence variety compactification, is nothing but the total space of the projection \( \mathbb{P}^k \mathcal{M}_{g,n}(\mu) \to \mathbb{P}^k \mathcal{M}_{g,n}(\mu) \).

In order to test for goodness we thus consider small open sets \( U \) in \( \mathbb{P}^k \mathcal{M}_{g,n}(\mu) \) and the growth of the area for a choice of scale of the family of differentials \( \omega \) over \( U \). Arguing inductively on \( k \), we may also suppose that we are dealing with primitive \( k \)-differentials, i.e. that the canonical \( k \)-cover is connected.

We start with the definition of the corresponding hermitian form. For a symplectic basis \( \alpha_1, \ldots, \alpha_{\tilde{g}}, \beta_1, \ldots, \beta_{\tilde{g}} \) of the absolute homology \( H_1(\tilde{\Sigma}, \mathbb{Z}) \) and for \( \omega, \eta \in H^1(\tilde{X}, \mathbb{C}) \) we define hermitian form

\[ \langle \omega, \eta \rangle = \frac{i}{2} \sum_{i=1}^{\tilde{g}} (\omega(\alpha_i) \overline{\eta(\beta_i)} - \omega(\beta_i) \overline{\eta(\alpha_i)}) \]  

(18)

with the abbreviations \( \omega(\alpha_i) = \int_{\alpha_i} \omega \) etc. By Riemann’s bilinear relations we can rewrite the metric defined in (14) using the hermitian form as

\[ h(X, q) = \langle \omega, \omega \rangle = \frac{i}{2} \sum_i (a_i \overline{b_i} - b_i \overline{a_i}), \]

where we introduce another abbreviation \( a_i = \omega(\alpha_i) \) and \( b_i = \omega(\beta_i) \), to be used if \( \omega \) is the only one-form that appears.

**Proof of Theorem 1.4.** We consider the metric near a boundary point \((\tilde{X}, \omega, \tilde{\Gamma}) \in \partial \mathbb{P}^k \mathcal{M}_{g,n}(\mu) \). As first case we **suppose that \( X \) has only one horizontal node that we moreover suppose to be non-separating.** Consequently, \( \tilde{X} \) has \( k \) nodes. We pick a convenient basis of \( H_1(\tilde{\Sigma}, \mathbb{Z}) \) on a smooth model \( \tilde{\Sigma} \) (connected by our standing primitivity assumption) that is pinched to \( \tilde{X} \). The \( k \) pinched curves \( \alpha_i \in \tilde{\Sigma} \) are linearly independent and form a \( \tau \)-orbit in homology. Next, we take the symplectic dual curves \( \beta_i \) with the intersection pairing \( \langle \alpha_j, \beta_i \rangle = \delta_{ij} \). Note that \( \beta_i \) is well-defined in a neighborhood of \((\tilde{X}, \omega, \tilde{\Gamma}) \) (only) up to adding an integer.
multiple of $\alpha_i$. We arbitrarily complement these elements by $\alpha_i, \beta_i \in H_1(\hat{\Sigma}, \mathbb{Z})$ for $i = k + 1, \ldots, \hat{g}$ to a symplectic basis.

In the current case the multi-scale differential case $\omega = (\omega_0) = (\eta_0)$ consists of a single one-form. Recall from Step 2 in Section 3.7 that points in a neighborhood of $\hat{\Sigma}, \omega, \hat{\Gamma}$ are obtained from surfaces $(\hat{X}', \eta') \in \Omega M_{\hat{g} - k, \hat{n} + 2k}(\hat{\mu}, (-1)^{2k})$ that admit an action by $\langle \tau \rangle \cong \mathbb{Z}/k$, by gluing in $k$ times each of the plumbing fixtures $W$ in a $\tau$-equivariant way, parameterized by a coordinate $x = (x) \in \Delta$ as in Step 2 above. By Proposition 1.2 and explicitly (17) the coordinates near the boundary point $x$ and the periods in the $\zeta_k$-eigenspace of $\eta'$. We denote by $\omega'$ the differential obtained from $\eta'$ after the plumbing construction. Notice that $\omega'$ is a holomorphic differential on the plumbed surfaces having all plumbing parameters $x_i$ different from zero. Our aim is to rewrite the area form, which is defined using $\omega'$ periods in the interior, in terms of the perturbed period coordinates, i.e. $x$ and $\eta'$ periods, which give charts near the boundary. We abbreviate $a_i = \omega'(\alpha_i)$ and $b_i = \omega'(\beta_i)$.

Next we decompose $\beta_j = \beta_j^X + \beta_j^\circ$ into the ‘exterior’ part $\beta_j^X$ outside the plumbing fixture and the part $\beta_j^\circ$ between the two seams of the plumbing fixture, as in Figure 1.

![Figure 1. Decomposing the $\beta_i$ into exterior and interior of the plumbing fixture](image)

The separation happens at fixed sections (of the universal family over the stratum $\Omega M_{\hat{g} - k, \hat{n} + 2k}(\hat{\mu}, (-1)^{2k})$) in the neighborhood of $(\hat{X}', \eta')$ in the plumbing annuli $B_j$, say at the points $u = \delta_0$ and $v = \delta_0$. Equation (16) simplifies in the one-level case to $\Omega_j = r_j\text{dv}/\text{v}$ where $r_j = \zeta_j^1 a_1/2\pi i = a_j/2\pi i$. We compute

$$b_j = \int_{\beta_j} \omega' = \int_{\beta_j^X} \eta' + \int_{\delta_0}^{x/\delta_0} \Omega_j = \int_{\beta_j^X} \eta' + r_j \text{log} x - 2 \text{log} \delta_0,$$

which is well-defined in $\mathbb{C} + r_j\mathbb{Z}$ because of the ambiguity of $\beta_j$. By definition of the area form and since $\omega'$ and $\eta'$ agree outside the plumbing fixture,

$$h(X, q) = \frac{i}{2} \sum_{j=1}^k (a_j \bar{b}_j - b_j \bar{a}_j) + \frac{i}{2} \sum_{j=k+1}^{\hat{g}} (a_j \bar{b}_j - b_j \bar{a}_j)$$

$$= C - k \cdot \pi |a_1|^2 \text{log} |x| + \frac{i}{2} \sum_{j=k+1}^{\hat{g}} (a_j \bar{b}_j - b_j \bar{a}_j)$$

$$= 19$$
is independent of the ambiguity of \( b_j \). Here \( C \) is some function that stems from the integration in the thick part and that is independent of \( x \). It is bounded on the neighborhood under consideration.

For a **general \( X \) that has only horizontal nodes** we arrive at a similar formula. For a separating node of \( X \) the \( k \) preimages are jointly separating. We take the core curves \( \alpha_j \) of \( k-1 \) of the preimages. The symplectically dual curves \( \beta_j \) are constructed crossing \( \alpha_j \) and returning via the \( k \)-th preimage. In the general case we choose a spanning tree of the dual graph of \( X \) and treat the corresponding (say \( h_s \)) nodes as separating to construct the appropriate symplectic base. The complement of the spanning tree consists of (say \( h_n = h - h_s \)) nodes that we treat as non-separating. We conclude that the metric has in the chosen symplectic basis the form

\[
h(X, q) = -\frac{\pi}{2} \sum_{i=1}^{h} k_i |a_i|^2 \log |x_i| + \frac{i}{2} \sum_{j=c+1}^{\tilde{g}} (a_j b_j - b_j \tilde{\alpha}_j) + C
\]

with \( C \) independent of the plumbing parameters \( x = (x_1, \ldots, x_h) \), where \( c = k h_n + (k-1) h_s \) and where \( k_i \in \{k-1, k\} \) depending on whether the node is non-separating or not.

Now we show that (20) indeed defines a good metric. Since we are on \( \mathbb{P} \tilde{\mathcal{M}}_{g,n}(\mu) \) we take a slice of the projectivization and assume \( a_1 = 1 \) in the sequel. The \( a_i \) appearing in the above formula are \( \omega' \)-periods and thus bounded in the neighborhood under consideration. The boundary \( \partial \mathbb{P} \tilde{\mathcal{M}}_{g,n}(\mu) \) is given by \( \{ \prod x_j = 0 \} \). This implies the property (i) of good, since the second summand is bounded and since the area is bounded from below.

With the second property in mind, we compute first

\[
\partial h = -\frac{\pi}{2} \sum_{i=1}^{h} k_i \left( \pi |x_i| da_i + |a_i|^2 \frac{dx_i}{2x_i} \right) + L
\]

for some differential form \( L \), a combination of \( da_j \) and \( db_j \) with coefficients that are linear in the \( \pi j \) and \( \tilde{b}_j \) for \( j = c+1, \ldots, \tilde{g} \), in particular they are bounded coefficients. To check Poincaré growth of the connection one-form \( \partial \log h = \partial h/h \) we remark that \( |h|^{-1} \) goes to zero like \( 1/\sum \log |x_i| \). Consequently the terms in \( |\partial h/h|^2 \) that stems from the second term of the sum in (21) are bounded above by a constant multiple of \( |dx_i|^2 / x_i^2 \), so the ratio is just bounded by the Poincaré metric on the \( i \)-th coordinate. The growth rate of the other terms is obviously smaller.

To check the growth of the curvature form we compute

\[
\overline{\partial \partial \log h} = \frac{\overline{\partial \partial h}}{h} + \frac{\overline{\partial h} \wedge \partial h}{h^2}.
\]

The second summand is a wedge product of two terms \( \partial h/h \) and \( \overline{\partial h}/h \) that are individually of Poincaré growth by the previous calculation, so their wedge product is of Poincaré growth. Since

\[
\overline{\partial \partial h} = -\frac{\pi}{2} \sum_{i=1}^{h} k_i \left( \log |x_i| da_i \wedge da_i + \frac{a_i}{2x_i^2} dx_i \wedge da_i + \frac{a_i}{2x_i} da_i \wedge dx_i \right) + B
\]

for some bounded function \( B \), it is easy to check that also the second part of the definition of good metric is satisfied.
We next treat **suppose that \( X \) has only vertical nodes.** A local orbifold chart is a neighborhood \( U \) of the point \((\hat{X}, \omega)\) in the model domain. Now \( \omega \) is a collection \( \omega_i \) of non-zero differential forms on the subsurface \( \hat{X}_{(i)} \) on the \( i \)-th level. The neighborhood of \((\hat{X}, \omega)\) consists of the plumbing construction applied to the differential forms \((\prod_{j=i}^{L-1} t_j a_j) \eta_j + \xi_j\) on the universal family over model domain restricted to the small neighborhood. Here \( t = (t_i)_{i=1}^{L-1} \) is the collection of 'opening-up' parameters in the polydisc and the positive integers \( a_j \) are determined by the enhanced level graph \( \Gamma \) via (5). The fiber over \( \eta = (\eta_j) \) over the central point \( \hat{X} \) agrees with \( \omega \) by construction.

From the modification differentials \( \xi = (\xi_j) \) we mainly have to retain that they tend to zero faster than \( t_i \), see Definition 3.3 iii).

Consider the fiber \( \hat{X}_{(i),u} \) over \( u \in U \) of the level-\( i \) subsurface over the model domain. Let \( E^+_{(i)} \) the edges connecting that surface to higher levels and \( E^-_{(i)} \) the edges connecting that surface to lower levels. Consider the subsurface where for each \( e \in E^+_{(i)} \) the interior of the plumbing annuli \( A^+_e \) (and thus the pole) has been removed. The \( \eta_{(i)} \)-area of this subsurface is bounded for \( u \in U \), since the areas of the plumbing cylinders \( V_e \) for \( e \in E^-_{(i)} \) tend to the area of a disc with metric \( z^m dz \) with \( m \geq 0 \). Consequently, for any sequence \((\hat{X}_n, \eta_n)\) of surfaces plumbed from \((\hat{X}_n, \eta_n)\) using the parameters \( t_n \) with the property \( t_i \to 0 \) for all \( i = -1, \ldots, -L \) and with \((\hat{X}_n, \eta_n) \to (\hat{X}, \omega) \) in the model domain, we get

\[
\text{area}_{\Omega}^{\text{Pl}}(\hat{X}_n, \eta_n, t_n) (\omega_n) = \sum_{i=0}^{-L} \text{area}_{\hat{X}_n} ((\prod_{j=1}^{i} t_j a_j) \eta_{i,n} + \xi_{i,n}) \to \text{area}_{\hat{X}} (\omega_0),
\]

which is finite.

In the **general case of \( X \) having of horizontal and vertical nodes**, we observe that the plumbing constructions happen on disjoint cusp neighborhoods and do not interact. The area in the general case is a sum over all levels as in (22) (with lower levels scaled by products of \( t_i \)) of expressions that involve the contribution of horizontal nodes as in (20). These can be dominated by growth Poincaré growth for the parameters corresponding to horizontal nodes. This completes the proof of Theorem 1.4. \( \square \)

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