Higher-order topological corner states induced solely by onsite potentials with mirror symmetry

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Higher-order topological insulators have triggered great interests because of exhibitions of non-trivial bulk topology on lower-dimensional boundaries like corners and hinges. While such interesting phases have been investigated in a plethora of systems by tuning staggered tunneling strength or manipulating existing topological phases, here we show that a higher-order topological phase can be driven solely by mirror-symmetric onsite potentials. We first introduce a simple chain model in one dimension that mimics the Su-Schrieffer-Heeger-like model. However, due to the lack of internal symmetries like chiral or particle-hole symmetry, the energies of the topological edge modes are not pinned at zero. Once the model is generalized to two dimensions, we observe the emergence of topological corner modes. These corner modes are intrinsic manifestation of non-trivial bulk band topology protected by mirror symmetry, and thus, they are robust against symmetry-preserved perturbations. Our study provides a concise proposal for realizing a class of higher-order topological insulators, which involves only tuning onsite energies. This can be easily accessible in experiments and provides a different playground for engineering topological corner modes.

I. INTRODUCTION

Since the discovery of quantized multipole insulators, higher-order (HO) topological phases and materials have attracted great interests because of their novel bulk-boundary correspondences [1–6]. In contrast to conventional (or first-order) topological phases, the topologically protected boundary states of HO topological phase exhibit lower dimensions. In other words, rth-order (codimension r ⩾ 2) topological phases in d dimensions host (d − r)-dimensional localized states, rather than d − 1 dimensional edge states. For example, in two-dimensional (2D) second-order topological insulators, the boundary states manifest as zero-dimensional (0D) corner states. A variety of candidates have been proposed to host HO topological phases [7–33].

Spatial symmetries enrich topological phases from an aspect differing from conventionally internal symmetries (i.e., particle-hole, time-reversal and chiral symmetries) [34]. These topological crystalline phases have been classified in a unified framework [35]. Later, crystalline symmetries have also been shown to play an important role in different types of HO topological insulators and superconductors [24, 36–40]. While there have been different approaches for realizing HO topological phases, most of them are driven from an existing non-trivial first-order topological phase or rely on extending some models similar to Su-Schrieffer-Heeger (SSH) model by adjusting staggered hopping. More recently, a handful of studies have shown that both first-order and HO topological insulators can be driven by tuning non-Hermitian effects including on-site gain and loss rates [41–43]. While non-Hermiticity has brought interesting aspects into topological phases of matter, they are harder to engineer and control in general. Thus, an interesting question naturally arises: whether existing first-order topological phase and non-Hermitian effects are necessary for driving HO topological phases?

In this paper, we answer the above question by enriching the family of HO topological insulators with a class of HO topological phases induced solely by real and mirror-symmetric onsite potentials. We first introduce a one-dimensional (1D) chain with mirror symmetry and show that a topological phase can be driven through only onsite energy difference between sublattices. Contrary to the usual SSH model and its derivatives, our model is protected by mirror symmetry, instead of chiral symmetry, and thus, the edge modes have non-zero energies. We then generalize the 1D model to a Wannier-type second-order topological insulator, which is driven from a Dirac semimetal by only onsite potentials. Such a topologically non-trivial phase can be characterized by Wannier centers. To show the flexibility of the proposed platform, we further present an intrinsic second-order topological insulator on a square lattice and characterize its topology using edge polarizations and quadrupole moment. Finally, to confirm the topological protection of this class of second-order topological insulators, we impose mirror-symmetry-preserved perturbations and show that the corner modes are robust up to a global energy shift.

This paper is organized as follows. In Sec. II, we introduce a 1D lattice with inversion symmetric onsite potentials, which serves as the base for our HO topological models. In Sec. III, we start with a honeycomb...
lattice with mirror-symmetric onsite potentials, and explore the HO topological phases. In Sec. IV, we turn to the study of a square lattice, and introduce edge polarizations and quadrupole moment to characterize the corresponding HO topological phase. Conclusions and discussions are drawn in Sec. V.

II. 1D SUPERLATTICE WITH MIRROR-SYMMETRIC POTENTIALS

For simplicity, we consider a 1D superlattice consisting of two sublattice sites with different onsite potentials $V_a$ and $V_b$, while inter- and intra-site coupling $t$ is uniform. A minimal non-trivial model with mirror (inversion) symmetry has a unit cell $(V_a, V_b, V_a)$. Here for better demonstration, we use a configuration of $(V_a, V_b, V_b, V_a)$, as shown in Fig. 1 (a), and the corresponding system Hamiltonian in momentum space reads

$$h(k) = \frac{t}{2}(1 + \cos k) \sigma_x \tau_x + \frac{t}{2}(1 - \cos k) \sigma_y \tau_y + \sigma_0 \tau_0 + \frac{t}{2} \sin k (\sigma_y \tau_x + \sigma_x \tau_y) + V_- \sigma_z \tau_z + V_+ \sigma_0 \tau_0 \tag{1}$$

under the basis $\psi_k = (\hat{a}_{1,k}, \hat{a}_{2,k}, \hat{a}_{3,k}, \hat{a}_{4,k})^T$, where $V_\pm = (V_a + V_b)/2$, $\sigma_0$ and $\tau_0$ are identity matrices. The Hamiltonian $h(k)$ preserves mirror symmetry with $M h(k) M^{-1} = h(-k)$, where $M = \sigma_x \tau_x$. In the following, we set $V_+ = 0$, i.e., $V_a = -V_b = V > 0$ without loss of generality since $V_a I$ only shifts the energy bands globally, but doesn’t change the topological number of each band because the eigenvectors remain invariant. The four energy bands are given by

$$E_{\pm,+} = \pm \sqrt{2t^2 + V^2 + \sqrt{2t^2 (t^2 + 2V^2 + t^2 \cos k)}}$$
$$E_{\pm,-} = \pm \sqrt{2t^2 + V^2 - \sqrt{2t^2 (t^2 + 2V^2 + t^2 \cos k)}}$$

When $V = 0$, there are four gapless energy bands as shown in Fig. 1 (b). The bands $E_{+,+}$ ($E_{+,+}$) and $E_{+,+}$ ($E_{+,+}$) touch at momentum point $k = \pi$, and the bands $E_{+,+}$ and $E_{+,+}$ touch at $k = 0$ with linear dispersions. As $V$ increases, the bands $E_{+,+}$ ($E_{+,+}$) and $E_{+,+}$ ($E_{+,+}$) are separated with an energy gap

$$\Delta E_{gL} = \sqrt{t^2 + (t + V)^2} - \sqrt{t^2 + (t - V)^2}, \text{ at } k = \pi,$$

and there opens an energy gap $\Delta E_{g2} = \sqrt{t^2 + V^2} - t$ between $E_{+,+}$ and $E_{+,+}$ at $k = 0$ (see Fig. 1 (c)).

Since the 1D superlattice preserves mirror symmetry, each energy band would contribute a quantized topological invariant. We define the topological invariant as

$$\eta = -\frac{1}{\pi} \oint A_k dk_x \tag{2}$$

with $A_k = -i \langle u_{n,k} | \partial_k | u_{n,k} \rangle$. After calculations, we find that $\eta = 0, 0, 1, -1$ for bands $E_{+,+}$, $E_{+,+}$, $E_{+,+}$ and $E_{+,+}$, respectively. The topological invariants can also be equivalently defined by $\eta = -\lfloor m(0) - m(\pi) \rfloor/2$, where $m(0)$ and $m(\pi)$ are parities at momentum points $k = 0$ and $\pi$, respectively. In Fig. 1 (d), we denote even and odd parity as "+" and "-", respectively. We can also obtain consistent topological invariants for each band with that from Eq. (2).

To present the bulk-boundary correspondence, we calculate energies for a lattice under open boundary conditions. The energy level distributions are shown in Fig. 1 (d). It shows that two states emerge in the gap between the third band and fourth band from bottom to top. The inset of Fig. 1 (d) presents the particle density for these two states, which shows that these two in-gap states are localized at two ends of lattices, similar to the celebrated SSH model. This behavior confirms the bulk-boundary correspondence for this topological system.

When $V < 0$, the topological invariants for bands $E_{+,+}$, $E_{+,+}$, $E_{+,+}$ and $E_{+,+}$ would be changed to $1, -1, 0, 0$, and two localized states emerge in the gap between the first and second band from bottom to top. In this sense $V = 0$ is a critical topological phase transition point for each energy band.

III. 2D HONEYCOMB LATTICE WITH MIRROR-SYMMETRIC POTENTIALS

We consider a graphene lattice with mirror-symmetric onsite potentials $(V_a, V_b, V_b, V_a)$ along $x$, as shown in

![FIG. 1: (a) Illustration of 1D lattice with onsite potentials $V_a$ and $V_b$. (b) and (c) Energy spectra at different onsite potentials with $V = 0$ and $V = 0.3$. (d) Eigenenergies for 1D chain with 100 sites and $V = 4.0$ under open-boundary conditions. $n$ denotes eigenstate index. The inset showcases the particle density versus site index for states indicated by blue and red dots in (d).]
The single-particle Hamiltonian is written as \( \hat{H} = \hat{H}_0 + \hat{H}_p \). The first term on r.h.s. reads \( \hat{H}_0 = -t \sum_{(i_m,j_n)} \left( \hat{a}_{i_m} \hat{a}_{j_n} + h.c. \right) \) with \( t \) the coupling between nearest-neighbor sites \( (i_m,j_n) \), and the mirror-symmetric potential is described by \( \hat{H}_p = \sum_{m,i} V_m \hat{a}_{i_m}^{\dagger} \hat{a}_{i_m} \) with \( V_{m=1,4} = V_a \) and \( V_{m=2,3} = V_b \) representing the mirror symmetric onsite potentials. The operator \( \hat{a}_{i_m}^{\dagger} \) (\( \hat{a}_{i_m} \)) creates (annihilates) a mode at site \( i_m \). The mirror-symmetric onsite potentials enlarge the unit-cell of the usual honeycomb lattice. Each unit-cell consists of four sublattices. The Bravais vectors are now described by \( \mathbf{a}_1 = (3,0) \) and \( \mathbf{a}_2 = (0,\sqrt{3}) \), as shown in Fig. 2. The first Brillouin zone (BZ) decreases correspondingly. Here we have set the lattice spacing of the honeycomb lattice to be unit. The total Hamiltonian \( \hat{H} \) in momentum space can be written as \( \hat{H} = \sum_{k} \psi_k \hat{h}(k) \psi_k^\dagger \) under the basis \( \psi_k = (\psi_{i_1}, \psi_{i_2}, \psi_{i_3}, \psi_{i_4})^T \) with

\[
\hat{h}(k) = \begin{pmatrix}
\alpha_{k} \sigma_0 \tau_x & -t \sin k_x \sigma_0 \tau_y + \beta_k \sigma_0 \tau_x & \gamma_k \sigma_y \tau_y & 
+ \frac{t}{2} \sin k_x \sigma_x \tau_x + V_\sigma \sigma_0 \tau_x + V_\tau \tau_x + \tau_x \n\end{pmatrix}
\]

where \( \alpha_k = t \left[ 1 + \cos \left( \sqrt{3} k_y \right) \right] \), \( \beta_k = \frac{1}{2} t \left( 1 + \cos k_x \right) \), \( \gamma_k = \frac{1}{2} t \left( 1 - \cos k_x \right) \), \( k_x = \sqrt{3} k_y \), \( k_3 = k \cdot (\mathbf{a}_1 + \mathbf{a}_2) = 3k_x + \sqrt{3} k_y \), and \( I = \sigma_0 \tau_0 \).

When \( V_a = V_b \), the Hamiltonian \( \hat{H} \) preserves inversion symmetry \( \mathcal{P} = \sigma_x \tau_x \mathcal{U} \), time-reversal symmetry \( \mathcal{T} = \mathcal{U} \), and \( C_3 \) rotation symmetry, where \( \mathcal{U} = \text{diag}(e^{i k_x - i k_y}, e^{-i k_x - i k_y}, e^{i k_x + i k_y}, e^{-i k_x + i k_y}) \) is a diagonal unitary matrix and \( \mathcal{K} \) is a conjugation operator. These symmetries lead to the system hosting two locally and globally stable Dirac points with geometric phases \( \pm \pi \).

When \( V_a \neq V_b \), \( C_3 \) symmetry would be broken and Dirac points may be gapped. However, in this case the model also preserves mirror symmetries, i.e., \( \mathcal{M}_{x/y} h (k_{x/y}) \mathcal{M}_{x/y}^{\dagger} = h (-k_{x/y}) \) with \( \mathcal{M}_x = \sigma_x \tau_x \mathcal{U} \) and \( \mathcal{M}_y = \sigma_0 \tau_0 \mathcal{U} \). Without loss of generality, we again set \( V_a = - V_b = V \) in the following. The four energy bands in momentum space are solved as

\[
E_{\pm, +}(k) = \pm \sqrt{V^2 + t^2 \left( 3 + 2 \cos \sqrt{3} k_y \right) + 2 \sqrt{\alpha_k}}
\]

\[
E_{\pm, -}(k) = \pm \sqrt{V^2 + t^2 \left( 3 + 2 \cos \sqrt{3} k_y \right) - 2 \sqrt{\alpha_k}}
\]

where \( \alpha_k = t^2 \left( t^2 + V^2 + t^2 \beta_k \right) \) with \( \beta_k = 3 \cos k_x + (1 + \cos 3k_x) \cos \sqrt{3} k_y \).

Compared to the conventional graphene model with \( V = 0 \), the energy bands are folded and Dirac points shift to \( \mathbf{K} = (0, 2\sqrt{3}\pi/9) \) and \( \mathbf{K}' = (-2\sqrt{3}\pi/9) \) as plotted in Fig. 3(a1). As the potential \( V \) increases, two Dirac points remain massless while they approach each other in momentum space because the local stability is protected by \( \mathcal{P} \) and \( \mathcal{T} \) symmetries. They merge at the time-reversal-invariant point \( (0, \pi/\sqrt{3}) \) at \( V = V_c = t \), as shown in Fig. 3(b1). If \( V \) increases further, an energy gap opens as shown in Fig. 3(c). In the following, we will showcase the topological nature for each energy band.

A. Topological bands and topological invariants

From above symmetry analysis, both \( \mathcal{PT} \) symmetry and mirror symmetry are respected along \( x \) direction. The mirror symmetry \( \mathcal{M}_x \) guarantees the non-trivial quantization of polarization along the \( x \) direction. To present the polarization as the bulk property, we construct a Wilson loop operator \( \mathcal{W}_{x,k} \) in the \( x \) direction, where \( k \) represents the base point of the loop. We define the Bloch wave function of the occupied energy bands with negative energies as \( |u_{m,k} \rangle \), where \( \hat{H}(k) |u_{m,k} \rangle = E_m(k) |u_{m,k} \rangle \) with normalization condition \( \langle u_{m,k} | u_{n,k} \rangle = \delta_{m,n} \delta_{k,0} \). The Wilson loop operator is described by \( \mathcal{W}_{x,k} = \mathcal{F}_{x,k} + \mathcal{N}_x \mathcal{D}_{x,k} \cdots \mathcal{F}_{x,k+k_\Delta} \mathcal{F}_{x,k} \), where the elements for \( \mathcal{F}_{x,k} \) are defined by \( \left[ \mathcal{F}_{x,k} \right]_{m,n} = \langle u_{m+k_\Delta, n+k_\Delta} | u_{m,k} \rangle \delta_{k_\Delta, 2\pi} \). The Wilson loop invariant at each \( k_y \) is then defined by \( \eta_{x}(k_y) = -\frac{1}{\pi} \text{Tr} \left( \int \mathcal{A}_k dk_x \right) \), (4)

where \( \mathcal{A}_k \) is a non-Abelian Berry connection with \( (\mathcal{A}_k)_{mn} = -i \langle u_{m,k} | \partial_{k_n} u_{n,k} \rangle \). Following similar steps, the topological invariant \( \eta_{y}(k_z) \) could be obtained at each \( k_z \). Finally, the topological invariant, namely the Wannier center of Wannier bands is defined as \( (\eta'_{x}, \eta'_{y}) \) with \( \eta'_{x/y} = \frac{1}{2\pi} \int_{\mathbb{T}} \eta (k_{x/y}) \, dk_{x/y} \), where the reciprocal vectors are \( b_x = \pi/3 \) and \( b_y = \pi/\sqrt{3} \).
When $V = 0$, $\hat{H}$ becomes a conventional graphene model. The topological invariant for the band $E_{-,-}$ is $\eta_{x,-,-}(k_y) = 1$ if $-\pi/\sqrt{3} < k_y < -2\sqrt{3}\pi/9$ or $2\sqrt{3}\pi/9 < k_y < \pi/\sqrt{3}$ and $\eta_{x,E_{-,-}}(k_y) = 0$ otherwise. The topological invariant for the band $E_{+,-}$ is $\eta_{x,E_{+,-}}(k_y) = 0$ if $-\pi/\sqrt{3} < k_y < -2\sqrt{3}\pi/9$ or $2\sqrt{3}\pi/9 < k_y < \pi/\sqrt{3}$ and $\eta_{x,E_{+,-}}(k_y) = 1$ otherwise. The topological invariants for the bands $E_{+,+}$ and $E_{-,-}$ are $\eta_{x,E_{+,+}}(k_y) = 0$ for any $k_y$, as plotted in Fig. 3(a2). For $V = V_{c_1} = t$, the energy gap closes at momentum lines $k = (k_x, \pi/\sqrt{3})$ with any $k_x$, as presented in Fig. 3 (b1). The topological invariants for each band has been shown in Fig. 3 (b1), where $\eta_{x,E_{+,+}}(\pm\pi/\sqrt{3})$ and $\eta_{x,E_{-,-}}(\pm\pi/\sqrt{3})$ are not well defined indicated by the dashed lines. When $V > V_{c_1}$, the energy gap opens between bands $E_{+,+}$ and $E_{-,-}$. We find $\eta_{x,E_{+,+}}(k_y) = -1, \eta_{x,E_{+,+}}(k_y) = 1, \eta_{x,E_{-,-}}(k_y) = 0$, and $\eta_{x,E_{-,-}}(k_y) = 0$ for any $k_y$ [see Fig. 3(c2)]. Namely, when $V > V_{c_1}$, the Wannier centers of the bands $E_{+,+}$, $E_{-,-}$, and $E_{+,+}$ are $\eta_{y,E_{+,+}} = 1/2$, $\eta_{y,E_{-,-}} = 0$, $\eta_{y,E_{+,+}} = 0$, respectively.

Therefore, if $V > V_{c_1}$, the total Wannier center of the lowest three Wannier bands is quantized to a non-trivial value $(1/2,0)$ with lowest three energy bands $(E_{+,+}$, $E_{-,-}$, and $E_{+,+}$) being occupied. We dubbed this phase as second-order topological phase A (SOTA). Similar cases happen when $V < V_{c_2} = -t$, where we observe that the bands $E_{+,+}$ and $E_{+,+}$ become topologically trivial, while bands $E_{-,-}$ and $E_{-,-}$ are topologically non-trivial. These lead to that the Wannier center becomes $(1/2,0)$ if the bands $E_{-,-}$ is occupied. This phase is referred to second-order topological phase B (SOTB). In summary, the topological phase diagram is shown in Fig. 3(d).

Here, we would like to point out that the quantization of topological invariant $\eta_y$ is guaranteed by mirror symmetry along the $x$ direction. It is robust against weak mirror-symmetric perturbations as long as the corresponding energy gap doesn’t close.

**B. Topological corner modes and interface modes**

Consider a sample shown in Fig. 4(a). We set the parameter $V > V_{c_1}$ so that the system is in a topological phase SOTA with the Wannier center quantized to $(1/2,0)$. First take the case $V = 2.0$ as an example and its numeric results are plotted in Fig. 4(a) and (c). Two degenerate modes indicated by the blue dots emerge in the energy gap as shown in Fig. 4(c). The corresponding particle density distributions have been shown in Fig. 4(a). It presents that the two degenerate modes are located at two horizontal corners of the given sample. Fig. 4(b) shows the case $V = 5.5$, in which the corner modes become more localized as the strength of potential increases. Similarly, we observe that the topological corner modes also exist when $V < V_{c_2}$ in topological phase SOTB.

![FIG. 3: (a1)-(c1) Energy spectra at different onsite potentials. (a2)-(c2) Topological invariants for the energy band $E_{-,+}$ (red lines) and the energy band $E_{+,+}$ (blue lines) corresponding to (a1)-(c1). The strength for each case is (a1) and (a2) $V = 0$, (b1) and (b2) $V = 1.0$, (c1) and (c2) $V = 1.5$. (d) Phase diagram versus $V/t$. Other parameters are set to be $t = 1$, $b_x = \pi/3$, and $b_y = \pi/\sqrt{3}$.](image)

So far, we have focused on the special case with $V_a = -V_b$. We remarked that the quantization of a non-trivial topological invariant (Wannier center) is guaranteed by the mirror symmetry. It means that it is also respected when $V_a \neq V_b$ since $V_+ = (V_a + V_b)/2$ only shifts energies globally and $V_- = (V_a - V_b)/2$ determines the wave functions. Therefore, it is expected that the system is non-trivial when $|V_-| > V_{c_1}$, as long as the energy gap between nearby bands remains open. Here, we also would like to remark that while the mirror-symmetric potential perturbations may shift energies of corner modes, the Wannier center of the system is invariant, and the corner modes remain localized. This is different from conventional HO topological system where the energies of corner modes are usually pinned at zero.

As discussed above, a graphene model with appropriate mirror-symmetric potentials $V$ is a HO topological insulator characterized by Wannier center $(1/2,0)$. In the following, we consider two graphene sheets separated by a domain wall as sketched in Fig. 5(a). Here the translation symmetry of graphene lattice is broken along
FIG. 4: (a) Spatial density distribution of the corner modes with $V = 2$ indicated by blue dot in (c) and the radii of the pink disk is proportional to local density. (b) Similar to (a) but plotted with a different onsite potential $V = 5.5$ indicated by red dot in (c), demonstrating the corner modes become more localized as the strength of onsite potential increases. (c) Eigenspectrum versus potential $V$. The red lines denote two-fold degenerate corner modes. Common parameter is set to be $t = 1$.

$x$ direction, but the translation symmetry is respected along $y$. In the following, we take $k_y$ as a system parameter and treat $\hat{H}(k_y)$ as a quasi-one dimensional chain. The Hamiltonian $\hat{H}$ is then written as

$$\hat{H} = \sum_{k_y} \hat{H}(k_y) = \sum_{k_y} \hat{H}_0(k_y) + \hat{H}_p(k_y),$$

with

$$\hat{H}_0(k_y) = -t \sum_{i_x} \hat{a}^\dagger_{i_x,k_y} \hat{a}_{i_x,k_y} + \epsilon_{k_y} \hat{a}^\dagger_{i_x,k_y} \hat{a}_{i_x,k_y} + \hat{a}^\dagger_{i_x,k_y} \hat{a}^\dagger_{i_x,k_y} \hat{a}_{i_x,k_y},$$

$$\hat{H}_p(k_y) = \sum_{m,i_x} V_m \gamma_m \hat{a}^\dagger_{m,i_x,k_y} \hat{a}_{m,i_x,k_y}$$

where $\epsilon_{k_y} = t \left(1 + e^{i \sqrt{3} k_y}\right)$ and the domain wall structure is given by $\gamma_{m=1,4} = -\gamma_{m=2,3} = 1$ (left-hand side of the domain wall) and $\gamma_{m=1,4} = -\gamma_{m=2,3} = -1$ (right-hand side), as depicted in Fig. 5(a).

The energy spectra of the system can be derived from $\hat{H}(k_y) |u(k_y)\rangle = E(k_y) |u(k_y)\rangle$. We set the strength of appropriate potentials so that both graphene sheets are in different topological phases, i.e., the left and right ones are in SOTA and SOTB, respectively. Through numerical calculations, we obtain energy spectra of graphene sheets as shown in Fig. 5 (b), and observe localized states at the interface, i.e., the states with positive and negative energies localizing at the left-hand and right-hand side of the domain wall, respectively, as plotted in Fig. 5 (c) and (d). This confirms that two different topological phases (SOTA and SOTB) indeed exhibit different topological properties and such a setup can be used as the experimental setup of photonic higher-order topological insulators in graphene lattices.

For a domain-wall structure, we may also consider the general case $(V_a, V_b, V_b, V_a)$ with $V_a \neq -V_b$. The numeric calculations also demonstrate the existence of topological localized interface modes. Finally, we also consider mirror-symmetric perturbations and find that, although the energies of localized states and bulk states vary, the topological interface states always localize at the domain wall. In this sense, the topological interface states are robust and mirror-symmetry protected.
IV. 2D SQUARE LATTICE WITH MIRROR-SYMMETRIC POTENTIALS

We now consider a square lattice with mirror-symmetric potentials and each square plaquette enclosing a \(\pi\) flux, as shown in Fig. 6 (a). The Hamiltonian is written as

\[
h(k) = t\sigma_x^z\sigma_y^z\sigma_0^x + t\sigma_y^x\sigma_y^y\sigma_0^y + te^{-ik_x}\sigma_x^z\sigma_y^y\sigma_0^y + te^{-ik_y}\sigma_x^z\sigma_y^x\sigma_0^x + \text{h.c.,} + V\sigma_x^z\sigma_y^y\sigma_y^z,
\]

(7)

where \(t\) and \(V\) denote the coupling between nearest-neighbor sites and onsite potential, respectively. \(\sigma_x^z\) and \(\sigma_y^z\) are Pauli matrices acting on the degrees of freedom spanned along \(x\) and \(y\), respectively, while \(\sigma_0^x\) and \(\sigma_0^y\) are identity matrices. The ladder operator \(\sigma_x^z\) reads \(\sigma_x^z = (\sigma_x^z \pm i\sigma_y^z)/2\) and \(\sigma_y^z\) is defined similarly. The Hamiltonian preserves mirror symmetries along both \(x\) and \(y\) as \(\mathcal{M}_x \mathcal{H}(k_x, k_y) \mathcal{M}_x^{-1} = h(-k_x, k_y)\) and \(\mathcal{M}_y \mathcal{H}(k_x, k_y) \mathcal{M}_y^{-1} = h(k_x, -k_y)\), where \(\mathcal{M}_x = \sigma_x^z\sigma_y^y\sigma_y^x\) and \(\mathcal{M}_y = \sigma_0^x\sigma_0^y\sigma_y^y\). There are eight pairs of energy bands and each pair is doubly degenerate. When \(V = 0\), the two central energy bands touch at Dirac point \(\Gamma = (0,0)\). When \(V \neq 0\), an energy gap opens at \(\Gamma\) with the gap \(\Delta E_\gamma = 2(\sqrt{V^2 + 2U^2} - \sqrt{2t})\). In addition, there also opens a gap between the first two pairs and second-two pairs of bands form top to bottom as potential \(V\) increases, which implies that the topological phase transition may occur with opening the gap.

Consider a square sample with \(36 \times 36\) sites with \(V = 3.0\). We compute its eigenenergy level distributions as shown in Fig. 6(c). It showcases that there are four energy modes in the energy gap. After plotting the particle density distributions (see Fig. 6(b)), we find these modes are localized at four corners of the sample.

To characterize the topological properties of corner states, we compute the edge polarizations \(p^{edge,y}(p^{edge,x})\) using Wilson loops on a torus geometry where the lattice has open boundary along \(y\) (\(x\)) but periodic boundary along \(x\) (\(y\)). The polarization distribution along \(y\) is defined by [1, 4]

\[
p_x(i_y) = \frac{1}{N_x} \sum_{j,k_x,\alpha,n} \left| \langle u_{k_x}^n \rangle_{i_y,\alpha} \langle \nu_x^j \rangle_{n}^* \nu_x^j \right|^2.
\]

(8)

Here \(N_x\) is the number of unit-cell along the \(x\) direction. \(\langle u_{k_x}^n \rangle_{i_y,\alpha}\) denotes the \((i_y,\alpha)\)-th component of occupied state \(\langle u_{k_x}^n \rangle\), where \(i_y\) and \(\alpha\) are the site index along \(y\) and sublattice degrees of freedom along \(x\), respectively. \(\langle \nu_x^j \rangle_{n}\) is the \(j\)th component of \(n\)th eigenvector corresponding to the Wannier values \(\nu_x^j\) of the Wannier Hamiltonian \(H_{W}\), \(-i \ln W_x\). \(W_x\) represents the Wilson loop operator, i.e., \(W_x = F_{x,k_x}(N_x - 1) + F_{x,k_x} + \cdots + F_{x,k_x+N_x-1}\). With \(F_{x,k_x}\)

Figure 6(d) presents the edge polarization \(p_x(i_y)\) versus site index \(i_y\) with \(V = 3\) and the occupied state number \(n_{occ} = 4n_y * 3/4\), where \(n_y\) is the site number along \(y\). Similar results are obtained on a torus geometry where the lattice has open boundary along \(x\) but periodic boundary along \(y\). These results implies that localized modes exist at the corners of the sample, consistent with the results in Fig. 6(b) and (d).

In addition, this topological phase can also be characterized by quadrupole momentum [30]. We consider \(N_o\) states are occupied represented as \(|\chi_l\rangle = \sum_{j\alpha} \phi_{l,j\alpha} |j\alpha\rangle\), where \(l = 1, ..., N_o\), and \(j\alpha\) denotes indices of unit-cells and sublattice sites with \(j = 1, ..., N_u\) and \(\alpha = 1, 2, ..., 16\). By arranging \(N_o\) occupied state columnwise, we construct a unitary matrix \(U\) with dimension \(16N_u \times N_o\).

The density of particles is defined by

\[
n = \frac{-i}{2\pi} \text{Tr} \ln U^{\dagger}OU.
\]

(9)

Here \(O\) is a diagonal \(16N_u \times 16N_u\) dimensional matrix with elements \(O_{4(j-1) + \alpha_1, 4(j-1) + \alpha_2} = \exp(\frac{1}{2i\pi x_{\alpha_1} y_{\alpha_2}}|N_x N_y\rangle\langle N_x N_y|)\), where \(N_x\) and \(N_y\) are...
for example, non-Hermitian effects or nonlinear interactions with periodic boundary conditions, we obtain $\delta_i = \delta_0 \kappa_i$, $\kappa_i \in [0, 1]$ is a random quantity, and $\delta_0 = \kappa_{N_x - i_x + 1, N_y - i_y + 1} \in [0, 1]$ is a random quantity for $i_x \leq N_x/2$ and $i_y \leq N_y/2$ (see Fig. 7(b)). We take a $36 \times 36$ square lattice as an example and numerically compute its energies, as shown in Fig. 7(a). We observe that the energies for four in-gap states acquire a finite energy shift. We showcase the density distribution of these four states in Fig. 7(b). It shows that they remain localized at four corners of the sample, which indicates the induced second-order topological phase with corner modes are robust.

We next consider the general random perturbation on onsite potentials represented as $H_{ap} = \sum_i \delta_i a_i a_i^\dagger$, where $\delta_i = \delta_0 \kappa_i$, $\kappa_i \in [0, 1]$ is a random number. Through numeric calculations we find four in-gap states may acquire different energies. It implies the corner states no longer have the same energies in the absence of mirror symmetries. However, these modes remain localized at corners as long as they are in the band-gap.

To summarize, the corner states are robust against mirror-symmetric perturbations, but they acquire some non-zero energies.

**Appendix A: Robustness of corner states against perturbations**

We consider two cases to show the robustness of corner states against perturbations. First we impose mirror-symmetric perturbation on onsite potentials described as $H_{ap} = \sum_i \eta_i \hat{a}_i^\dagger \hat{a}_i$, where $\eta_i = \eta_0 \kappa_{i_x, i_y}$, $\eta_0$ is the amplitude of the random potential, and $\kappa_{i_x, i_y} = \kappa_{N_x - i_x, N_y - i_y + 1} \in [0, 1]$ is a random quantity for $i_x \leq N_x/2$ and $i_y \leq N_y/2$.
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