Entanglement revivals as a probe of scrambling in finite quantum systems

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The entanglement evolution after a quantum quench became one of the tools to distinguish integrable versus chaotic (non-integrable) quantum many-body dynamics. Following this line of thoughts, here we propose that the revivals in the entanglement entropy provide a finite-size diagnostic benchmark for the purpose. Indeed, integrable models display periodic revivals manifested in a dip in the block entanglement entropy in a finite system. On the other hand, in chaotic systems, initial correlations get dispersed in the global degrees of freedom (information scrambling) and such a dip is suppressed. We show that while for integrable systems the height of the dip of the entanglement of an interval of fixed length decays as a power law with the total system size, upon breaking integrability a much faster decay is observed, signalling strong scrambling. Our results are checked by exact numerical techniques in free-fermion and free-boson theories, and by time-dependent density matrix renormalisation group in interacting integrable and chaotic models.

I. INTRODUCTION

Integrable and chaotic many-body quantum systems are very different objects. The former have infinite classes of local charges constraining their dynamics; conversely, the latter have only few local integrals of motion. Integrable systems admit stable quasiparticles with infinite lifetime and elastic scattering between them [1]. When non-integrable models have some quasiparticles at all, they have a finite lifetime and inelastic scattering with particles production. One would then expect their unitary non-equilibrium dynamics (say following a quantum quench [2]) to be completely different [3]. Indeed, it is nowadays well established that while chaotic systems long time after a quench are described by the Gibbs (thermal) ensemble [4–11], an integrable system fails to thermalise and a statistical description of its asymptotic local properties requires a Generalised Gibbs Ensemble (GGE) [12–44], which takes into account all the local and quasilocal conserved quantities of the system [32–34]. However, there are many cases when the predictions from GGE and thermal ensembles are too close to each other. Furthermore, an integrable point constrains the dynamics of a relatively large neighbourhood in parameter space, giving rise to the phenomenon of prethermalisation, according to which only for extremely large times the thermal behaviour is attained, while at accessible times one observes a quasi-stationary state with quasi-integrable features [45–53].

These observations were among those that initiated the search for concepts that naturally take integrable and chaotic models apart. A very suggestive proposal concerns the scrambling of quantum information, hence of entanglement. Indeed, qualitatively we expect that in integrable systems the spreading of entanglement, due to the quasiparticles with a local-in-space structure, happens in a “localised” manner, i.e., local initial correlations are somewhat preserved. This is the essence of the quasiparticle picture for entanglement spreading [54–57]. This scenario must change in non-integrable models: the common lore is that inelastic processes and the quasiparticle decay should facilitate the delocalisation of the initial correlation in the global degrees of freedom of the system [58–67]. This idea originated in the context of the black-hole information paradox [68–70]. There the initial correlation between the interior and the exterior of the black hole gets dispersed in the global degrees of freedom of the radiation that remains after the black hole evaporates.

Unfortunately, probing the scrambling scenario in microscopic quantum many-body systems proved to be a daunting task. Several diagnostic tools have been devised, such as the tripartite mutual information [71, 72], out-of-time-ordered correlators [70], and operator space entanglement entanglement entropy [73–75]. An idea that is related to this paper is to diagnose scrambling from the time evolution of the mutual information between two disjoint intervals. Indeed, in integrable models, because of the infinite lifetime of the quasiparticles, the mutual information exhibits a peak at intermediate times, as first pointed out in conformal field theory (CFT) [54]. For chaotic systems, due to scattering and finite lifetime of quasiparticles, the peak should decay, signalling scrambling. This behaviour has been first proposed and observed in CFTs with large central charge [76–79] and better characterised for integrable systems with non-linear dispersion in Ref. [80].

It is of fundamental importance to devise further diagnostic tools to probe quantum information scrambling. Here, we propose and show that entanglement revivals in finite systems can be used for this purpose. The idea to use revival effects to distinguish between ergodic and non-ergodic systems has already been explored in Refs. [81, 82], but in different contexts. Nevertheless, the inspiration for our work came from recent results for maximally chaotic models...
in which the entanglement entropy shows no revivals at all in finite size \[83–85\], while in the integrable limit a perfect recurrence is observed. These results are the two black and white extremes of a more refined and grey structure which we investigate and characterise here.

Our main result is that it is possible to detect scrambling by monitoring the first entanglement revival. We consider a subsystem of fixed size \( \ell \) embedded in a system of finite size \( L \). We consider initial product states, i.e., with zero entanglement entropy. For time \( t \lesssim \ell \) the entropy increases linearly. At larger times \( \ell \ll t \lesssim L \) the entropy saturates. At the first revival time \( t = t_R \approx L \), the entanglement entropy decreases, and it exhibits a dip, which is followed by a later increase. As \( L \) increases the revival time \( t_R \approx L \) is shifted to longer times and the height of the dip decreases. We show that for integrable systems the dip decays algebraically in \( L \) (as \( L^{-1/2} \) within the quasiparticle picture), as we explicitly test in free-fermion, free-boson, and interacting integrable models. Remarkably, upon breaking integrability, the dip exhibits a much faster decay, signalling strong scrambling.

The paper is organised as follows. In Sec. II we introduce the scrambling of entanglement revivals in integrable models within the quasiparticle picture. In Secs. III and IV we work out explicit predictions and test them for free-fermion and free-boson models, respectively. In Sec. V we move to interacting integrable models focusing on the Heisenberg spin chain and a numerical study of its quench dynamics by means of time-dependent Density Matrix Renormalisation Group (tDMRG) techniques. In section VI we discuss the effects integrability breaking on the entanglement revivals. Our conclusions are in section VII.

II. SCRAMBLING AND ENTANGLEMENT REVIVALS IN INTEGRABLE MODELS

The typical protocol to drive a system out of equilibrium is the quantum quench \[2\]: an isolated system is initially prepared in a non-equilibrium pure state \( |\psi_0\rangle \) and then it is let evolve under the unitary dynamics governed by a Hamiltonian \( H \). Although the dynamics is unitary and the entire system remains in a pure state, at long times the reduced density matrix \( \rho_A \) of an arbitrary finite compact subsystem \( A \) of length \( \ell \) displays local thermodynamic equilibrium. The reduced density matrix \( \rho_A \) is defined as

\[
\rho_A \equiv \text{Tr}_B |\psi(t)\rangle \langle \psi(t)|, \tag{1}
\]

where the trace is over the degrees of freedom of the complement \( B \) of \( A \), and \( |\psi(t)\rangle \equiv e^{-iHt}|\psi_0\rangle \) is the time-dependent state of the system. Local thermal equilibrium means that \( \rho_A \) for large times equals the reduced density matrix of a statistical ensemble \[13\].

A key question is how entanglement spreads in out-of-equilibrium many-body system, both theoretically and experimentally, thanks to recent progress on the direct measurement of entanglement dynamics with cold atoms and trapped ions \[86–92\]. One of the most useful entanglement measures is the von Neumann (entanglement) entropy \[93\]

\[
S_{\ell} \equiv - \text{Tr} \rho_A \ln \rho_A. \tag{2}
\]

Here we are interested in the out-of-equilibrium dynamics of \( S_{\ell} \) after a quantum quench. Our main result is that the dynamics of \( S_{\ell} \) in finite-size systems depends dramatically on whether the Hamiltonian is integrable or chaotic.

To understand why this is the case, let us first briefly describe the quasiparticle picture for the entanglement spreading which is applicable to generic integrable models (first introduced in Ref. \[54\] in the context of CFT). According to this picture, the initial state acts as a source of quasiparticle excitations which are produced in pairs and uniformly in space. After being created, the quasiparticles move ballistically through the system with opposite velocities. Only quasiparticles created at the same point in space are entangled and while they move far apart they carry entanglement and correlation in the system. A pair contributes to the entanglement entropy at time \( t \) only if one particle of the pair is in \( A \) (the interval of length \( \ell \)) and its partner in \( B \). Keeping track of the linear trajectories of the particles, it is easy to conclude \[54, 55\]

\[
S(t) = \sum_n \left[ 2t \int dk \nu_n(k) s_n(k) + \ell \int dk s_n(k) \right]. \tag{3}
\]

Here the sum is over the species of particles \( n \) whose number depends on the model, \( k \) represents their quasimomentum (rapidity), \( \nu_n(k) \) is their velocity, and \( s_n(k) \) their contribution to the entanglement entropy. (Often we will work with a single species of quasiparticle omitting the sum over \( n \) and the subscripts). The quasiparticle prediction (3) for the entanglement entropy holds true in the space-time scaling limit, i.e. \( t, \ell \to \infty \) with the ratio \( t/\ell \) fixed. When a maximum quasiparticle velocity \( \nu_M \) exists (e.g., as a consequence of the Lieb-Robinson bound \[94\]), Eq. (3) predicts that for \( t \leq \ell/(2\nu_M) \), \( S_t \) grows linearly in time. Conversely, for \( t \gg \ell/(2\nu_M) \), only the second term survives and the entanglement is extensive in the subsystem size, i.e., \( S_{\ell} \propto \ell \). In order to give predictive power to Eq. (3), one
should fix the values of $v_n(k)$ and $s_n(k)$: the former are the group velocities of the excitations around the stationary state [55, 56, 95] and the latter are the thermodynamic entropy densities of the GGE [55, 56]. The validity of Eq. (3) has been carefully tested both analytically and numerically in free-fermion and free-boson models [54, 96–108] and in many interacting integrable models [55, 56, 109–113]. The mechanism for the entanglement evolution in chaotic systems is different, not as well understood as in integrable models and with many peculiar features. Nevertheless, the entanglement entropy grows linearly up to a time extensive in subsystem size [83–85, 114–121], exactly as in integrable systems.

Interestingly, Eq. (3) has been generalised to capture the non-equilibrium dynamics in many different physical situations and to other physical quantities. For example we mention quenches from inhomogeneous initial states [122–129] and states with more complicated quasiparticle structure than simple uncorrelated pairs [130–132]. Furthermore, the picture has been adapted to describe the steady-state Rényi entropies [133–136] and to the dynamics of the logarithmic negativity [137–139]. Very recently, it has been shown that by using the quasiparticle picture it is also possible to study the fate of the entanglement in free-fermion systems in the presence of dissipation [140]. An aspect that is of much importance for our analysis is the behaviour of the mutual information between two disjoint intervals [80]. In integrable systems, the mutual information exhibits an algebraic decay with the distance between the intervals, that within the quasiparticle picture has a decay exponent equal to 1/2. Away from the scaling limit, the power-law behaviour persists, but with a larger (and model-dependent) exponent. For non-integrable models, a much faster decay, compatible with an exponential, is observed.

We recall that in Eq. (3) we assume that subsystem $A$ of length $\ell$ is embedded in an infinite system. The result is different for a finite system of total length $L$ in which, as we are going to show, the entanglement entropy exhibits revivals (we focus on periodic boundary conditions, but other boundary conditions lead just to minor adjustments as long as they are compatible with integrability; indeed we will later turn to open boundary conditions too). We assume that $L$ is large enough so that quasiparticles are well-defined; this implies that the space-time scaling limit generalises as $t, \ell, L \to \infty$ with both $t/\ell$ and $\ell/L$ fixed (or, equivalently, $t/L$). At this point, to study the revivals we need to modify Eq. (3), simply accounting for the quasiparticles trajectories in a periodic system.

Let us start by describing what happens when all quasiparticles have the same velocity $v$ regardless of momentum, as it is the case, e.g., in CFT. The entropy first grows linearly up to $\ell/(2v)$. Then, it exhibits a plateau. Up to this time, the entanglement behaves like in an infinite system. The plateau terminates at $t = (L - \ell)/(2v)$ when the first quasiparticles produced at one boundary of the subsystem re-enter into it from the other edge after having turned around the circle. After this time, more and more quasiparticles will re-enter the subsystem producing a drop of the entropy that lasts until the *revival time* $t_R = L/(2v)$, when the dynamics restart exactly like if the system was at $t = 0$. And so on in a periodic fashion in time. This entropy dynamics is shown in Fig. 1 as a full line.

Taking into account the different velocities $v(k)$ of the quasiparticles is now trivial: it is enough to sum/integrate

![FIG. 1. Schematic diagram of entanglement dynamics for finite-size integrable systems. The continuous line is the quasiparticle prediction for a model with exactly linear dispersion, such as a CFT. Note the perfect revival at $t_R$. The dashed-dotted line is the quasiparticle prediction for a model with a realistic nonlinear dispersion. For $L \to \infty$ the entropy saturates (dashed line), while at finite size (dot-dashed line) it shows a dip of height $\delta S$, which is the main quantity we consider. The precise definition of $\delta S$ is shown pictorially as the difference between the asymptotic result for infinite systems ($S(\infty)$, dot line) and the minimum of the entropy close to the first revival.](image-url)
over all possible quasi-momenta \( k \) the result for each mode. For the case of a single species of quasiparticles, one obtains

\[
S_{t}(t) = \int_{\{\frac{2\pi}{L}k\}<\frac{t}{\ell}} \frac{dk}{2\pi} s(k) L \left( \frac{2v(k)t}{L} \right) + \ell \int_{\frac{1}{2}\frac{2\pi}{L}s\{k\}<1-\frac{t}{\ell}} \frac{dk}{2\pi} s(k) + \int_{1-\frac{t}{\ell}}^{\frac{1}{2}\frac{2\pi}{L}s\{k\}<1-\frac{t}{\ell}} \frac{dk}{2\pi} s(k) L \left( 1 - \frac{2v(k)t}{L} \right),
\]

where \( \{ x \} \) denotes the fractional part of \( x \), e.g., \( \{1.76\} = 0.76 \). This result has been already reported in Ref. [141] (suggested by one of the present authors, as acknowledged there), but in a different context. The structure of Eq. (4) is the same as Eq. (3); \( s(k) \) is the thermodynamic entropy of the GGE that describes the steady state, and \( v(k) \) are velocities of the lowest-lying excitations around the corresponding thermodynamic macrostate. Taking into account the existence of more species trivially amounts to sum also over the different species, each with its own velocity and entropy density in momentum space, in full analogy to Eq. (3).

We now discuss the time evolution of \( S_{t} \) for fixed \( \ell, L \) as predicted by Eq. (4), for systems with nonlinear quasiparticles dispersion. The qualitative behavior is shown as a dashed-dotted line in Fig. 1. The entropy grows linearly up to \( \ell/(2v_{M}) \), where now \( v_{M} \) is the maximum quasiparticles velocity. Importantly, in a realistic quantum many-body system there is a continuum of excitations with different velocities. This implies that the “plateau” in Fig. 1 at \( \ell/(2v_{M}) \ll t \ll (L - \ell)/(2v_{M}) \) is not flat but shows a slow saturation in the limit \( L \to \infty \). Such slow saturation terminates at \( t = (L - \ell)/(2v_{M}) \). After the most important aspect for our purposes comes: at \( t = t_{R} = L/(2v_{M}) \) the entropy exhibits a partial revival, in contrast to the case with a single velocity when the revival is perfect. The reason for this partial revival is obvious: when the fastest quasiparticles do not entangle anymore \( A \) and \( B \) because the pairs are back to their original position, there are still many slow quasiparticles that are still contributing to the entropy.

The definition of \( \delta S \) is illustrated pictorially in Fig. 1.

We now discuss the behavior of \( \delta S \) in the framework of the quasiparticle picture. We consider a generic integrable quantum many-body systems, with a continuum of entangling quasiparticles. Since we focus on the first revival at \( t_{R} = L/(2v_{M}) \), we replace \( t \to t_{R} \) in Eq. (4), which yields

\[
S_{t}(t) = \int_{\{\frac{2\pi}{L}k\}<\frac{t}{\ell}} \frac{dk}{2\pi} s(k) L \left( \frac{v(k)t}{v_{M}} \right) + \ell \int_{\frac{1}{2}\frac{2\pi}{L}s\{k\}<1-\frac{t}{\ell}} \frac{dk}{2\pi} s(k) + \int_{1-\frac{t}{\ell}}^{\frac{1}{2}\frac{2\pi}{L}s\{k\}<1-\frac{t}{\ell}} \frac{dk}{2\pi} s(k) L \left( 1 - \frac{v(k)t}{v_{M}} \right).
\]

We now show that Eq. (6) implies that \( \delta S \) (cf. (5)) scales like \( \delta S \propto L^{-1/2} \) for \( L \to \infty \). In fact, in the limit \( L \to \infty \) (i.e. \( L \gg \ell \), the dip of the entanglement entropy, as discussed above, is dominated by the quasiparticles with \( v \approx v_{M} \) (slowest quasiparticles did not have time to go around the system). Therefore, we expand the velocity of quasiparticle \( v(k) \) around \( k_{M} \) (the momentum with maximum velocity, \( v_{k_{M}} = v_{M} \)) up to second order in \( k - k_{M} \) as

\[
\frac{v(k)}{v_{M}} = 1 - \frac{v''}{2v_{M}} (k_{M} - k)^2 + o(k_{M} - k)^3, \quad \text{with } v'' = \frac{\partial^2 v(k)}{\partial k^2} \bigg|_{k=k_{M}}.
\]

If there are more momenta with maximum velocity, we should just sum over them. We assume that the entanglement content of the quasiparticles \( s(k) \) is such that \( s(k) = s_{M} + o(1) \) with a nonzero \( s_{M} \equiv s_{k_{M}} \). Plugging these expansions in Eq. (6), we obtain

\[
S_{t}(t_{R}) = S_{t}(\infty) - \frac{1}{3\pi} \left( \frac{2v_{M}}{v''} \right)^{1/2} \left( \frac{\ell}{L} \right)^{1/2} \frac{L}{\ell} s_{M} + tO \left( \frac{\ell}{L} \right),
\]

where \( S_{t}(\infty) \) is the asymptotic value of entanglement entropy in the thermodynamic limit, cf. Eq. (5). Few comments are needed. First of all, we only considered one species of quasiparticles, but for the validity of Eq. (8) this is
unimportant: when there are more types of quasiparticles, these have well separated maximum velocities and so for large \( \ell \) and \( L \) only one matters and Eq. (8) remains valid. However, the different species of quasiparticles can give strong finite size effects, as we shall see. Second, we note that Eq. (8) has the same structure as the formula describing the decay of the mutual information peak between two far apart intervals that has been derived in Ref. [80] and indeed has the same physical origin. Finally, let us stress the main limitation of Eq. (8). We obtained it in the quasiparticle picture in the limit \( \ell/L \to 0 \). However, this is not fully justified because the quasiparticle picture is valid for \( \ell/L \) fixed and finite. Still, we can think of the limit in which \( \ell = aL \) with \( a \ll 1 \). Thus, we expect the quasiparticle picture to describe an intermediate regime between very small \( \ell \) (in which \( \ell/L \) is not of order 1) and the scaling regime with \( \ell \) of the same order of \( L \). We recall that this is exactly what it has been found for the mutual information peak in [80]. There is not a general approach to understand the scaling for very small \( \ell \) since it is expected to be model dependent; for example, from the analogous result of the mutual information peak [80], we would expect for free fermion the dip to decay as \( L^{-2/3} \) while for free bosons as \( L^{-1} \).

We want finally to comment on another interesting aspect of Eq. (8). If we would expand for large \( L \) and at fixed \( \ell \), the quasiparticle prediction for the entanglement evolution (6) at any time away from the revivals, one would trivially obtain an analytic behaviour in \( L^{-1} \), because the velocity, away from the maximum has a linear term in \( k \). In this respect, the non-analytic scaling at the revival time is a straightforward consequence of expanding close to an extreme of the velocity (exactly like in Landau-Ginzburg approach, the mean field exponent \( 1/2 \) is caused by expanding close to the minimum of the free energy).

In the following section, we numerically verify Eq. (8) in free-fermion and free-boson models, as well as in interacting integrable models. We will show how for small \( \ell \) there is a crossover towards another power-law scaling. Finally, we will provide numerical evidence that in non-integrable models the dip of the entanglement revival decays faster than algebraically (likely exponentially).

### III. FREE FERMIONS

In this section focus on the \( XY \) chain that can be mapped to a free-fermion model. The \( XY \) spin chain with a transverse magnetic field has Hamiltonian

\[
H = -\sum_{j=1}^{L} \left[ \frac{1 + \gamma}{2} \hat{S}_j^x \hat{S}_{j+1}^x + \frac{1 - \gamma}{2} \hat{S}_j^y \hat{S}_{j+1}^y + h \hat{S}_j^z \right],
\]

(9)

where \( \hat{S}_i^a \) are spin-1/2 operators acting at site \( i \), \( \gamma \) is the anisotropy, \( h \) the transverse field, and we use periodic boundary condition. The Hamiltonian (9) can be diagonalized by a combination of Jordan-Wigner, Fourier transform, and Bogoliubov transformations, leading to the free fermion model [142]

\[
H = \sum_k \epsilon_k \hat{c}_k^\dagger \hat{c}_k, \quad \text{with} \quad \epsilon_k^2 = (h - \cos k)^2 + \gamma^2 \sin^2 k,
\]

(10)

and \( \hat{c}_k \) are standard spinless fermionic ladder operators. The quasiparticle velocity is \( v(k) = d\epsilon_k/dk \). Note that the quasiparticles’ velocities do not depend on the initial state because the system is non-interacting. This is not the case in the presence of interactions [55, 95].

Here we consider a quench of the magnetic field \( h \) and of \( \gamma \). Precisely, the system is initially prepared in the ground state of (9) with magnetic field \( h_0 \) and \( \gamma_0 \). Then, the parameters are suddenly changed as \( h_0 \to h \) and \( \gamma_0 \to \gamma \). The quench is parametrised in terms of the difference of the Bogoliubov angles of pre- and post-quench Hamiltonians, i.e. [54, 96, 143]

\[
\cos \Delta_k = \frac{h h_0 - \cos k (h + h_0) + \cos^2 k + \gamma \gamma_0 \sin^2 k}{\epsilon_0},
\]

(11)

where \( \epsilon_0, \epsilon \) stand for pre- and post-quench dispersion relations (see Eq. (10)).

The GGE built with local conservation laws for the Hamiltonian (9) is equivalent [22, 26] to the one built with mode occupation numbers \( \hat{n}_k = \hat{c}_k^\dagger \hat{c}_k \) and it is convenient to work with the latter. The GGE density matrix is then [22, 26]

\[
\rho_{\text{GGE}} = \frac{e^{-\sum_k \lambda_k \hat{n}_k}}{Z},
\]

(12)
where $Z = \text{Tr} e^{-\sum_k \lambda_k \hat{n}_k}$ ensures the normalization $\text{Tr} \rho_{\text{GGE}} = 1$. The Lagrange multipliers $\lambda_k$ are fixed by imposing that the expectation value of $\hat{n}_k$ in the initial state coincides with its GGE prediction

$$
\langle \hat{n}_k \rangle_{\text{GGE}} = -\frac{\partial}{\partial \lambda_k} \sum_p \ln(1 + e^{-\lambda_p}) = \frac{1}{1 + e^{\lambda_k}}.
$$

(13)

Now $\lambda_k$ is obtained requiring that $\langle \hat{n}_k \rangle_{\text{GGE}} = \langle \psi_0 | \hat{n}_k | \psi_0 \rangle = n_k$. The thermodynamic entropy of the GGE is the thermodynamic entropy obtained from the occupation $n_k$, which reads

$$
S_{\text{GGE}} = -\text{Tr} [\rho_{\text{GGE}} \ln \rho_{\text{GGE}}] = \sum_k -n_k \ln n_k - (1 - n_k) \ln(1 - n_k) = \sum_k s(k),
$$

(14)

where, $s(k) = -n_k \ln n_k - (1 - n_k) \ln(1 - n_k)$ is identified as the entropy contribution of the quasiparticle with momentum $k$.

In terms of (11), we have $n_k = \frac{1 + \cos \Delta_k}{2}$ and so the quasiparticles entanglement content $s(k)$ reads

$$
s(k) = -\frac{1 + \cos \Delta_k}{2} \ln \left( \frac{1 + \cos \Delta_k}{2} \right) - \frac{1 - \cos \Delta_k}{2} \ln \left( \frac{1 - \cos \Delta_k}{2} \right).
$$

(15)

The quasiparticle prediction for the entanglement entropy for a quench in the XY chain is obtained by plugging this value of $s(k)$ and $v(k) = v_M \sin k$ in Eq. (4). This time evolution (in the thermodynamic limit) has been also confirmed by ab initio approach both on the lattice [96] and in the field theory limit [144].

### A. A special case: The Néel quench in the XX chain

There is a special case of the quench above that we want to discuss separately because it will be important for the generalisation to interacting (both integrable and not) fermionic model. This is the time evolution starting from the Néel state (in the $z$ direction) and evolving with the isotropic XX Hamiltonian. This quench just amounts to take the limit $\hbar_0 \to -\infty$ and $\gamma \to 0$ in the above formulas, but we want to quickly discuss it here.

For $\gamma = h = 0$, the Jordan-Wigner transformation maps the Hamiltonian to the tight-binding model

$$
H = -\frac{1}{2} \sum_{i=1}^{L} \hat{c}_i^{\dagger} \hat{c}_{i+1} + \text{h.c},
$$

(16)

where, $\hat{c}_i$ and $\hat{c}_i^{\dagger}$ are local spinless fermionic annihilation and creation operators satisfying standard anti-commutation relations. The Hamiltonian (16) can be diagonalised in the momentum basis and can be written as

$$
H = \sum_k \epsilon_k \hat{c}_k^{\dagger} \hat{c}_k,
$$

(17)

where, $\epsilon_k = -\cos k$ (the velocity is $v(k) = d\epsilon_k/dk = \sin k$ with maximum $v_M = 1$ at $k_M = \pm \pi/2$). $\hat{c}_k^{\dagger}$ and $\hat{c}_k$ are fermionic creation and annihilation operators in momentum space and they are closely related to the ones in Eq. (10) for the XY chain.

The Néel state in fermionic basis is

$$
|\psi_0\rangle = \prod_{i=1}^{L/2} \hat{c}_{2i}^{\dagger} |0\rangle.
$$

(18)

It is straightforward to check that for the Néel state in the thermodynamic limit $n_k = 1/2$ for all $k$, which implies $s(k) = \ln 2$, as it can be also obtained from the limit of Eq. (11). This reflects the fact that all the nonzero overlaps between the Néel state and the eigenstates of the tight-binding chain are equal [145]. The quasiparticle prediction for the entanglement entropy after the Néel quench is obtained by replacing $s(k) = \ln 2$ and $v(k) = v_M \sin k$ in Eq. (4).

### B. Numerical study of the revivals

For free-fermion models the exact dynamics of the entanglement entropy is straightforwardly obtained from the time-dependent fermionic two-point correlations restricted to the subsystem $A$. This technique is very standard and we do not review it here, the interested reader can consult the extensive literature on the subject [146–148].
Let us first focus on the XX chain (the tight-binding model) after the quench from the Néel state. Our results are reported in Fig. 2. Fig. 2 (a) shows the dynamics of the entanglement entropy $S_\ell$ for a subsystem of $\ell$ sites embedded in a chain of size $L$. We plot the entropy density $S_\ell/\ell$ for $\ell = 200$ versus the rescaled time $2v_M t/L$, with $v_M$ the maximum velocity; the revival time is $t_R = L/(2v_M)$. The dashed lines are the exact numerical results for $L = 600, 800, 1000, 1500$. Several entanglement revivals are visible in the figure. The continuous lines are the predictions using the quasiparticle picture Eq. (3). They are in excellent agreement with the numerical results: in fact $\ell = 6$ is just a finite fraction of the total $L$ considered in the figure and we are safely working in the scaling limit $t, \ell, L \to \infty$ with fixed ratios.

Now, we study how the height of the dip at the first revival time is damped as $L$ increases. To quantitatively address this behaviour, we plot $\delta S$ versus $L$ in Fig. 2 (b). Here $\delta S$ is defined as in section II, cf. Eq. (5). We show $\delta S$ as a function of $L$ for fixed $\ell = 200$. It is evident that for the values of $L$ we considered (up to $10^4$) the quasiparticle picture works well, since the numerical value for $\delta S$ and the quasiparticle one stay almost on top of each other. Still, it is evident that as $L$ increases, some deviations between the two appear and a decay faster than $L^{-1/2}$ is starting. To highlight this crossover it is better to use smaller values of $\ell$ rather than increasing the one of $L$. Hence, in the inset of 2 (b) we report again the scaling of $\delta S$ but for $\ell = 6$ rather than 200. In it is clear that in this case the decay of $\delta S$ with $L$ is much faster and well described by the expected law $L^{-2/3}$. To summarise, Fig. 2 confirms both the quasiparticle decays prediction (8) with $\delta S \sim L^{-1/2}$ at intermediate values of $L$ and the crossover to $L^{-2/3}$ for larger $L$. We recall that the origin of the exponent 2/3 is related to the super-diffusion taking place on the light-cone of free fermion models related to Airy processes, see e.g. Refs. [127, 150–155] for similar results.

We end this section by briefly discussing numerical results for the quench in the $XY$ chain. We focus on the Ising case $\gamma = \gamma_0 = 1$. The quench parameters for the transverse fields are fixed as $h = 2, h_0 = 2.4$, but we checked that different values provide equivalent results. Figure 3 summarise our results for fixed $\ell = 100$ and $L = 500, 1000, 2000$. These values have been chosen to be sure to work within the scaling limit with large $\ell, L$ but with finite ratio. In the main panel we report $S_\ell/\ell$ as function of time: the agreement between the numerical data and the quasiparticle prediction is always excellent. Curiously, the first revival $t_R = L/2v_M$ is quantitatively more pronounced than in the previously considered Néel quench in the XX chain. In the inset of Fig. 3 we fix again $\ell = 100$ and study the scaling of the dip $\delta S$ as function of $L$ in the window $L \in [500, 6000]$. For these values of $L$, the numerical data are well captured by the quasiparticle prediction and scaling $\delta S \sim L^{-1/2}$. Also in this case, for the largest considered values of $L$ we observe a crossover towards a faster decay, that once again can be studied in more details. However, we do not perform this investigation here.
FIG. 3. Entanglement revivals in the XY chain. \( S_\ell / \ell \) versus rescaled time \( 2v_M t / L \) for different values of \( L \) after the quench from \((\gamma_0, h_0)\) to \((\gamma, h)\), with \( \gamma = \gamma_0 = 1 \) (Ising model), \( h = 2h_0 = 2.4 \) and \( \ell = 100 \). Solid lines correspond to the quasiparticle prediction. The inset shows the scaling of \( \delta S \) with \( L \) and the dashed line is a guide to the eye going as \( L^{-1/2} \).

IV. FREE BOSONS

In this section we move our attention to a bosonic system, namely the harmonic chain with Hamiltonian

\[
H = \frac{1}{2} \sum_{n=0}^{L-1} \left[ \pi_n^2 + m^2 \phi_n^2 + (\phi_{n+1} - \phi_n)^2 \right],
\]

with periodic boundary conditions. Eq. (19) defines a chain of \( L \) harmonic oscillators with frequency (mass) \( m \) and with nearest-neighbor quadratic interactions. Here \( \phi_n \) and \( \pi_n \) are the position and the momentum operators of the \( n \)-th oscillator, with equal-time commutation relations \([\phi_m, \pi_n] = i \delta_{nm}\) and \([\phi_n, \phi_m] = [\pi_n, \pi_m] = 0\). Eq. (19) is the lattice discretisation of a one-dimensional Klein-Gordon quantum field theory.

The Hamiltonian (19) can be diagonalised in the momentum basis and it can be written as

\[
H = \sum_k \epsilon_k \hat{b}_k^\dagger \hat{b}_k, \quad \text{where} \quad \epsilon_k^2 = m^2 + 2 \left( 1 - \cos k \right),
\]

and \( \hat{b}_k^\dagger (\hat{b}_k) \) is the bosonic creation (annihilation) operator.

Even for the harmonic chain, the GGE density matrix can be constructed from the mode occupation numbers \( \hat{n}_k = \hat{b}_k^\dagger \hat{b}_k \) [16, 57], and it is given by

\[
\rho_{\text{GGE}} = e^{-\sum_k \lambda_k \hat{n}_k} / Z,
\]

where \( Z = \text{Tr} e^{-\sum_k \lambda_k \hat{n}_k} \) ensures the normalization \( \text{Tr} \rho_{\text{GGE}} = 1 \). The Lagrange multipliers \( \lambda_k \) are fixed by imposing that the expectation value of \( \hat{n}_k \) in the initial state coincides with its GGE prediction

\[
\langle \hat{n}_k \rangle_{\text{GGE}} = -\frac{\partial}{\partial \lambda_k} \sum_p \ln(1 - e^{-\lambda_p}) = \frac{1}{e^{\lambda_k} - 1}.
\]

Now \( \lambda_k \) can be obtained by imposing that the expectation value of \( \hat{n}_k \) in the initial state \( |\psi_0\rangle \) coincides with its GGE prediction, i.e., \( \langle \hat{n}_k \rangle_{\text{GGE}} = (\psi_0 | \hat{n}_k | \psi_0 \rangle = n_k \). The thermodynamic entropy is

\[
S_{\text{GGE}} = -\text{Tr} \left[ \rho_{\text{GGE}} \ln \rho_{\text{GGE}} \right] = \sum_k (1 + n_k) \ln(1 + n_k) - n_k \ln n_k = \sum_k s(k),
\]

where \( s(k) = (1 + n_k) \ln(1 + n_k) - n_k \ln n_k \) is identified as the entropy contribution of the quasiparticle with momentum \( k \).
FIG. 4. Entanglement revivals in the harmonic chain. $S/\ell$ versus rescaled time $2v_M t/L$. The data are for the mass quench (from $m_0 = 2$ to $m = 4$), where $\ell = 40$. Solid lines correspond to the quasiparticle prediction and dashed ones to the exact numerical data. The inset shows the height of the revival $\delta S$ as a function of $L$ for two different quenches from $m_0 = 2$, both for $\ell = 40$ and $\ell = 10$. While for $\ell = 40$ the power-law $\sim L^{-1/2}$ correctly captures our data, for $\ell = 10$ there is a clear crossover to the truly asymptotic scaling $L^{-1}$.

We consider the quantum quench in which the harmonic chain is initially prepared in the ground-state $|\psi_0\rangle$ of the Hamiltonian (19) with $m = m_0$, and at time $t = 0$ the mass is quenched to a different value $m \neq m_0$ [2, 16, 28]. There have been extensive studies focusing on the critical ($m \to 0$) and continuum limit [39, 40, 98, 139, 156]. The validity of the quasiparticle picture for the entanglement entropy has been tested in all these situations [56, 98, 139, 156]. Note that for the quench to the massless case, a subleading logarithmic correction due to the presence of zero mode appears [156] and for this reason we will stay away from the critical case in the numerical analysis.

We use the notation $\epsilon^0_k$ for the dispersion relation in the initial state and $\epsilon_k$ for the one for $t > 0$. The conserved occupation number $n_k$ for this quench is given by [16, 56, 57]

$$n_k = \langle \psi_0 | \hat{n}_k | \psi_0 \rangle = \langle \psi_0 | a_k^\dagger a_k | \psi_0 \rangle = \frac{1}{4} \left( \frac{\epsilon_k}{\epsilon_k^0} + \frac{\epsilon_k^0}{\epsilon_k} \right) - \frac{1}{2}.$$  \hspace{1cm} (24)

We are now ready to study the finite size behaviour of the entanglement entropy after a quench in the harmonic chain. The quasiparticle prediction for the entanglement entropy after this quench is given by Eq. (4) in which we use $s(k)$ given by (23) with (24) and $v(k) = \frac{d\epsilon_k}{dk}$ with $\epsilon_k$ in (20). Numerically, the entanglement dynamics is obtained (similarly to the case of free fermions) from the two-point correlation functions by use of standard techniques, as detailed e.g. in Refs. [139, 147–149]. In Fig. 4 we report the numerical data for the time evolution of the entanglement entropy after the quench of the oscillators’ mass from $m_0 = 2$ to $m = 4$ (other values of the masses away from the critical point provide equivalent results), for fixed $\ell = 40$ and $L = 800, 1200, 2000$ chosen to be safely within the scaling limit of large $\ell, L$ with their ratio finite. The numerical data are compared with the quasiparticle prediction (4) and the agreement is perfect.

We now turn to the study of the dip at the first revival for the same quench parameters as above. The scaling of $\delta S$ as obtained from the numerical data is investigated in the inset of Fig. 4 as a function of $L$ (for fixed $\ell = 40$). As expected, since with these parameters we are working in the scaling limit, we perfectly reproduce the scaling $\delta S \propto L^{-1/2}$. However, also for bosons, for larger $L$ we expect a crossover to a different behaviour. Employing the same logic as for fermions, rather than increasing the value of $L$, it is easier and more effective to reduce the one of $\ell$. Hence, in the inset of Fig. 4 we also report the data for $\ell = 10$ that very clearly shows for large $L$ the crossover to the truly asymptotic behaviour $L^{-1}$. This scaling $L^{-1}$ is just a consequence of the fact that, away from the critical point, the non-equilibrium correlation functions of the harmonic chain are analytic and no singular scaling can take place.
FIG. 5. Time evolution of the entanglement entropy after a quench from the Néel state to the XXZ Hamiltonian with $\Delta = 2$ and with open boundary conditions. Four values of $\ell = 3, 5, 7, 9$ are displayed and $L$ ranges in the interval $L \in [20, 38]$. Revivals are evident for the larger considered $\ell$, but their quantitative analysis is made difficult by the presence of spurious finite-size effects. Although we do not report the analysis here, the dip $\delta S$ at $\ell = 9$ is algebraic with exponent between 1 and 2.

V. THE HEISENBERG XXZ CHAIN AS A PARADIGM OF INTEGRABLE MODEL

To investigate the effect of the interactions, here we consider the paradigm of integrable model, namely the spin-$1/2$ anisotropic Heisenberg chain (XXZ spin chain). The Hamiltonian is

$$\hat{H} = \sum_{i=1}^{L-1} \left[ \frac{1}{2} \left( \hat{S}^+_i \hat{S}^+_{i+1} + \hat{S}^-_i \hat{S}^-_{i+1} \right) + \Delta \left( \hat{S}^z_i \hat{S}^z_{i+1} - \frac{1}{4} \right) \right].$$

(25)

Here $\hat{S}^\alpha_i$ are spin-$1/2$ operators acting at site $i$ of the chain and $\Delta$ is the anisotropy parameter. For $\Delta = 0$, it reduces to the XX chain of section III A and it is mapped to free fermions. For $\Delta \neq 0$, the chain is genuinely interacting and it can be solved by the Bethe ansatz [157, 158].

The non-equilibrium dynamics starting from many initial states with low (mainly zero) entanglement has been considered in several manuscripts in the literature [29–33, 159–167], employing different techniques based on integrability. Here, for conciseness, we focus on the non-equilibrium dynamics of the entanglement entropy for a single quench, although we expect our results to be more general. Namely, we only consider the dynamics starting from (one of the two degenerate) Néel state

$$|\psi_0\rangle = |\uparrow\downarrow\uparrow\ldots\rangle.$$

(26)

The GGE describing the steady state after the quench from the Néel state can be constructed analytically using the Quench-Action method [159, 160]. Based on this solution for the Bethe ansatz root density, and exploiting Eq. (3), the quasiparticle prediction for the entanglement dynamics after the quench from the Néel state in the thermodynamic limit has been explicitly written down in Refs. [55, 56] for $\Delta \geq 1$. In Eq. (3) one just needs to interpret $s_n(\lambda)$ as the Yang-Yang entropy densities of the GGE and $v_n(\lambda)$ as the velocities of the excitations built on top of the excited state itself.

However, despite the integrability of the model, calculating the entanglement entropy, even at equilibrium [168], is a daunting task. Furthermore, the study of the time evolution exploiting integrability is extremely difficult even for simpler observables [169, 170]. Hence, in order to access the non-equilibrium dynamics of the entanglement entropy we have to resort to numerical simulations. Here we use tDMRG techniques [171–173]. Consequently, all the data for entanglement dynamics reported in this and in the following Section are obtained using the tDMRG algorithm, as implemented in the iTensor library [174]. We work with the maximum bond dimension $\chi = 2000$ and with a time step $\Delta t = 0.02$ for all simulations. We have checked the robustness of our results by performing simulations for different values of $\delta t$ and $\chi$. Furthermore, all of our results have been verified with exact diagonalisation for small system sizes.

The only minor drawback of tDMRG is that it works more effectively with open boundary conditions, rather than periodic ones. This is not a main problem for the revivals dynamics: they are always present in the entanglement
FIG. 6. Entanglement revivals in the Heisenberg chain for $\Delta < 1$ with tDMRG simulations for open boundary conditions. (a) $S_t/\ell$ versus rescaled time $t/L$ for $\Delta = 0.25$. Several values of $L$ are shown. The value of $\ell$ is fixed to $\ell = 4$. The inset shows the density of entanglement entropy $S_t/\ell$ in the plateau as a function of $L$. The dashed line is the extracted value for the steady state entropy. (b) The scaling of the dip $\delta S$ as a function of $L$ for $\Delta = 0, 0.25, 0.35, 0.5$. The dashed (dot-dashed) line is a guide to the eye going like $\sim L^{-1/2}$ ($L^{-2}$).

Dynamics but now they are due to quasiparticles bouncing back at the chain boundaries, rather than going around the full chain. This effect amounts to a minor modification in the semiclassical quasiparticles’ trajectories so that the revival occurs at $t_R = L/v_M$ and not at $t = L/(2v_M)$ as for a periodic chain.

At this point it is rather natural to start our analysis from the case $\Delta \geq 1$ for which an exact solution for the stationary state exists. This implies that we also know exactly the value of the maximal velocity and the stationary entropy, both important for the study of the dip of the revivals. The numerical time evolution of the entanglement entropy after a quench from the Néel state to the XXZ Hamiltonian is reported in Fig. 5 for $\Delta = 2$ (different values for all $\Delta \geq 1$ produces qualitatively equivalent results). With tDMRG we can access relatively small values of $\ell$ and $L$. In the figure we report $\ell = 3, 5, 7, 9$ (the data for even $\ell$ would be too close and make the figure difficult to read, but they are very similar) and we focus on $L$ in the interval $L \in [20, 38]$. With a large numerical effort it is possible to increase slightly these values, but for a more quantitative analysis, one needs to at least double both $\ell$ and $L$. In the figure, revivals are evident for $\ell = 7$ and $\ell = 9$, although they take place slightly after $L/v_M$. This is clearly due to the presence of many species of particles (called strings) which are not yet well separated for $\ell = 9$. The resulting finite size effects are huge and a quantitative analysis of the dip of the revival is unstable, although we know a priori the values of $t_R$ and $S_t(\infty)$ that enter in the definition (5) of $\delta S$. Although we do not present any analysis of the dip of the first revival, the data at both $\ell = 9$ and $\ell = 7$ are well compatible with an algebraic decay of $\delta S$ with an exponent which is between 1 and 2. This is different from the quasiparticle prediction 1/2, but we cannot conclude whether this is due to the oscillations of the data or to the fact that we are not yet in the scaling regime, although we tend to believe more to the second explanation.

Since the data for $\Delta > 1$ are quantitatively not convincing enough, it is rather natural to move our attention to the window $\Delta \in [0, 1)$. The main advantage is that we expect smaller finite size effects because (at rational values of $\Delta$) the number of species of quasiparticles is finite (often small) and so they should produce less interference effects and clearer revivals. The drawback is that there is not (yet) an exact Bethe ansatz solution for the dynamics from the Néel state (and any other one for what matters). Hence, we do not know exactly either the value of $S_t(\infty)$ or that of $t_R$ (because we do not know $v_M$). The time evolution of the entanglement entropy for $\Delta = 0.25$ is reported in Figure 6(a). We focus on $\ell = 4$ and $L$ in the range $L \in [20, 28]$. The reason of this small value of $\ell$ (compared to the study at $\Delta > 1$ in Fig. 5) is that the entanglement entropy grows much faster (its asymptotic density is more than the double than at $\Delta = 2$), limiting the performance of the tDMRG algorithm. However, even at this small value of $\ell$ we see a neat first revival with very small finite size effects, confirming our expectations. In this case, we do not know the asymptotic value of the entropy, but we can extract it in a very robust manner from the numerics since the plateau is very stable (as a difference with the case $\Delta > 1$). The determination $S_t(\infty)$ is reported for $\Delta = 0.25$ in the inset of Fig. 6(a). Curiously enough, the asymptotic value $S_t(\infty)$ depends very little on $\Delta$ in all the window $\Delta \in [0, 1)$ and it is very close to the value at $\Delta = 0$, i.e. $\ell \ln 2$ (but it is definitively different). In order to study the behaviour of the dip of $\delta S$ as function of $L$ (see again Fig. 1) for fixed $\ell$, we would need to know $v_M$ to extract it.
VI. INTERACTIONS THAT BREAK INTEGRABILITY

We have established in the previous sections the validity of the quasiparticle picture Eq. (4) to describe the entanglement revivals. Moreover, we showed that the dip in the entanglement entropy at the revival time is damped as $S_t(t_R)\sim L^{-1/2}$ in the scaling regime (cf. (8)) for free-fermion and free-boson models. For interacting integrable models it is difficult to access the scaling regime, but we can anyhow show that the decays of the dip is algebraic with an exponent between 1 and 2 (these two extremes are shown as guides to the eyes in the plots). We could also extract effective exponents for the decay, but we do not find their values significative for these small value of $\ell$ and $L$.

We stress that upon increasing the values of $\ell$ and $L$, we expect the data to crossover to the quasiparticle picture results with a behaviour of the dip going like $L^{-1/2}$, also in the presence of interactions for any $\Delta$. However, unlike for free models, here we are restricted to the system size $L \sim 30 - 40$ due to the limitation of available numerical techniques. Hence, the quasiparticle regime cannot be accessed. In fact, the prediction for the time evolution from the quasiparticle picture in Fig. 5 together with the numerical data for $\Delta > 1$, the agreement would be satisfactory only at short time, as already shown in Refs. [55, 56]. Anyhow, in spite of the many drawbacks of this quantitative analysis, we can conclude, without doubts, that the decay of the dip of the revival is algebraic with an exponent between 1/2 and 2 for all values of $\Delta$. Such a conclusion also reinforces the same result found for $\Delta > 1$ but with less stable data.

As an example of a chaotic model, we consider a non-integrable perturbation added to the XXZ Hamiltonian (25), in such a way to have an integrable limit. Among the various possibilities to break integrability, we opt for the addition of a longitudinal magnetic field $h_x$ modifying the Hamiltonian as

$$\hat{H} = \sum_{i=1}^{L-1} \frac{1}{2} \left( \hat{S}_{i+1}^+ \hat{S}^-_{i} + \hat{S}^+_{i} \hat{S}^-_{i+1} \right) + \Delta \left( \hat{S}_{i+1}^z \hat{S}^-_{i+1} - \frac{1}{4} \right) + h_x \sum_i \hat{S}_i^x. \tag{27}$$

The Hamiltonian (25) is non-integrable except at the isotropic point $\Delta = 1$.

As for the integrable XXZ chain, we use tDMRG simulation to access the entanglement dynamics following a quench governed by the Hamiltonian (27). Again we focus on the dynamics starting from the Néel state and we only consider $\Delta = 0.5$ which is far enough from the integrable point at $\Delta = 1$. Figure 7 shows the entanglement evolution...
for different values of $h_x$ for fixed $\ell = 4$ and various $L$ (only (b) panel). It is clear from Fig. 7(a) that, at fixed $L$, the revival fades away as the integrability breaking parameter $h_x$ is increased. However, to perform a quantitative analysis of the dip $\delta S$, we need first to numerically extract $S_\ell(\infty)$. Unlike the integrable case (see Fig. 6), here we observe clear finite-size effects. In the inset of Fig. 7(a), we show the maximum of $S_\ell(\infty)/\ell$ as a function of $1/L$. The data suggest to fit $S_\ell^M/\ell$ with a linear form $S_\ell^M/\ell = S_\ell(\infty)/\ell + a/L$ (where $S_\ell(\infty)$ and $a$ are fitting parameters). Interestingly, we observe that $S_\ell(\infty)/\ell \approx 0.67$, which is close to the maximum entropy density ln 2, which is also the value at $\Delta = h_x = 0$. Since the model is not integrable, $t_R$ is not well defined, so in the definition of $\delta S$ we can use the minimum of $S_\ell$ close to the revival, as we did for the integrable XXZ spin-chain. Anyhow it is very difficult to evaluate this minimum for $h_x \neq 0$ as soon as $L$ is moderately large. For example in Fig. 7 (b) we report the time evolution of the entanglement entropy at fixed $h_x = 0.2$, but for different values of $L$. We can observe a small revival only for $L = 12$, but for larger $L$ the decay is too fast to identify a clear minimum in the entropy. If one attempts an analysis at this not well-identified dip, the decay is much faster that any power-law, likely exponential, but could be even faster. Our findings match well the results found in Ref. [80] from the scaling of the peak of the mutual information.

VII. CONCLUSIONS

In this work we investigated the revivals of the entanglement entropy of an interval of fixed length after a quantum quench in finite-size systems. Our main result is that both in integrable and in non-integrable systems the strength of the revival is damped as a function of the system size. However, while the damping is algebraic for integrable systems, it is much faster in chaotic ones (strong scrambling). Within the quasiparticle picture for the entanglement spreading of integrable models, the exponent of the power-law decay is $1/2$. However, we provide compelling evidence that this exponent describes only an intermediate regime for $\ell = aL$ with $a \ll 1$. For very small $\ell$, there is a crossover toward a truly asymptotic regime with a model dependent exponent which is larger than $1/2$ (e.g., it is equal to 2/3 for the XX chain and to 1 for gapped free bosons). Our results suggest that the entanglement revivals provide a useful tool to diagnose scrambling in quantum many-body systems.

There are several interesting directions for future work. An simple generalisation of our work would be to understand the scaling of entanglement revivals in higher-dimensional free models, for which the same techniques used here trivially apply. A more difficult generalisation would be to understand what happens for Rényi entropies. Indeed, while for integrable models the quasiparticle picture is expected to work (with some troubles though, see Refs. [133–136]), for chaotic systems with a conserved charge there are effects of diffusion also at intermediate times [175–178] and it is unclear how they could affect the revivals. Another natural question concerns whether revivals can survive to the addition of some quenched disorder and, if yes, how they disappear as the system size grows. Finally, in this work we focused on entanglement revivals after a global quantum quench. However, revivals are expected also in different non-equilibrium protocols such as, just to quote two examples, the domain-wall quench [179–181] and geometric quenches [182, 183]. It would be interesting to understand the damping of entanglement revivals also in these situations.

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