Background Symmetries In
Orbifolds With Discrete Wilson Lines.

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ABSTRACT

Target space symmetries are studied for orbifold compactified string theories containing Wilson line background fields. The symmetries determined are for those moduli which contribute to the string loop threshold corrections of the gauge coupling constants. The groups found are subgroups of the modular group \( \text{PSL}(2, \mathbb{Z}) \) and depend on the choice of discrete Wilson lines and the shape of the underlying six-dimensional lattice.

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1. Introduction

The compactified string background can be described by a two-dimensional conformal field theory with a central charge equal the number of the compactified dimensions. Of all known conformal field theories, orbifold models offer phenomenologically promising string compactified backgrounds [1-4]. Orbifolds are characterized by a set of moduli which parametrize, locally, the string background and correspond to the truly marginal deformations of the underlying conformal field theory [5, 7]. These moduli appear as massless scalars in the low-energy effective action with flat potentials to all orders in string perturbation theory. A fundamental difference between compactified string theories and those of Kaluza-Klein is the fact that the former theories have a novel symmetry known as duality [5, 7, 13,14, 15, 16, 17, 18]. This symmetry generalizes the well known $R \rightarrow 1/2R$ symmetry for circle compactification with $R$ being the radius of the circle. Target space duality symmetry restricts the form of the low-energy effective action and in particular any possible non-perturbative superpotential for the moduli [6].

The duality group for two-dimensional toroidal compactification is given by two copies of the modular group $PSL(2, Z)$ acting on the two complex moduli, $T$ and $U$ describing the two-dimensional target space [7] via the fractional linear transformations:

$$U \rightarrow \frac{aU + b}{cU + d}, \quad T \rightarrow \frac{a'T + b'}{c'T + d'},$$

where $ad - bc = 1$ and $a'd' - b'c' = 1$. Note that the $U$-symmetry is not of stringy origin and is also a symmetry of the Kaluza-Klein compactification.

For the two-dimensional $\mathbb{Z}_N$ orbifold, $(N \neq 2)$, the $U$ modulus is frozen, i.e., its value is fixed to a constant phase factor and the duality group associated with the complex $T$ modulus is always the modular group $PSL(2, Z)$. Clearly, in the realistic cases, the modular group can be realized as a duality group for each complex modulus associated with the three complex planes of the six-dimensional
orbifold provided that the orbifold lattice is decomposable in the following form
\[ \Lambda_6 = \Lambda_2 \oplus \Lambda_2 \oplus \Lambda_2. \]

Whilst the duality group associated with the complete spectrum of orbifold models is rather complicated in general, a somewhat simpler task is to study the duality symmetries of the twisted sectors contributing to the moduli dependent threshold corrections. Moreover, a knowledge of the latter will be important in any investigation of mechanisms that fix the values of the various moduli. In ref. [8] the string-loop threshold corrections to the gauge coupling constants [8, 9, 10, 11, 12, 28] arising from the twisted sectors with invariant planes of the orbifold were calculated. The models of [8] are such that the six-dimensional lattice \( \Lambda_6 \) of the orbifold is decomposable into a direct sum of a two-dimensional and a four-dimensional sub-lattices, \( i.e., \Lambda_6 = \Lambda_2 \oplus \Lambda_4 \), with the twist-invariant plane lying in \( \Lambda_2 \). These moduli-dependent threshold corrections turn out to be invariant under the modular group. This is not surprising since the modular group leaves the spectrum of twisted states contributing to the threshold corrections invariant. However there are many orbifold models with lattices that do not admit the above simplifying decomposition [26, 22]. For the more general choice of lattices it was found that the duality group for the moduli appearing in the threshold corrections to the gauge couplings [23, 24, 25] is, in some cases, a congruent subgroup of the modular group. The congruent groups \( \Gamma_0(n) \) and \( \Gamma^0(n) \) are represented by the following set of matrices

\[
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}; \quad ad - bc = 1,
\]

with \( c = 0 \pmod{n} \) and \( b = 0 \pmod{n} \), respectively.

In the presence of discrete Wilson lines background, the duality group is known to be broken [28-31]. In [31] the symmetries of the moduli relevant to string loop threshold corrections in orbifold theories with Wilson lines were considered with the assumption that the orbifold lattice \( \Lambda_6 = \Lambda_2 \oplus \Lambda_4 \), with the invariant plane
of the twist lying in $\Lambda_2$. It is our purpose to generalize the results of [31] to include other choices of the orbifold lattice. We organize this work as follows. First we review the one-dimensional compactification, \textit{i.e.}, compactification of a string coordinate on the circle. It is also shown that discrete Wilson lines background can break the $R \to 1/2R$ duality symmetry. In section 2, the analysis is extended to the more realistic toroidal and orbifold compactification in the absence of Wilson lines. In section 3 the duality symmetry for the two-dimensional case is considered. This analysis will be relevant to the study of duality symmetries of the twisted sectors contributing to the threshold corrections in the six-dimensional case. In the remaining sections, the duality symmetries for the moduli contributing to the threshold correction in orbifold models with Wilson lines background are presented. It is found that the duality group is in most cases given by a subgroup of the modular group $PSL(2, Z)$. The constraints satisfied by the parameters of the modular group depend on the choice of the Wilson line background and the shape of the underlying six dimensional lattice.

2. Circular Compactification

Consider a string coordinate compactified on a circle of radius $R$, the worldsheet action for the compact coordinate is given by

$$S = \frac{1}{2\pi} \int d\sigma d\tau \eta^{\alpha\beta} \partial_\alpha X \partial_\beta X,$$

(2.1)

where $(\sigma, \tau)$ and $\eta^{\alpha\beta}$ are the coordinates and the metric of the worldsheet. If the string is parametrized by $\sigma \in [0, 2\pi]$, then periodicity on the circle implies

$$X(\sigma + 2\pi) = X(\sigma) + 2\pi n R$$

(2.2)

where $n$, the winding number, is an integer representing the number of times the string wrap around the circle.
The compact coordinate $X$ satisfies a free wave equation with the boundary condition (2.2), therefore its solution splits into a left and right moving parts given by

\[
X_R = x_R - \frac{1}{2} p_R (\sigma - \tau) + i \sum_{k \neq 0} \frac{1}{k} \alpha_k e^{ik(\sigma - \tau)}
\]

\[
X_L = x_L - \frac{1}{2} p_L (\sigma + \tau) + i \sum_{k \neq 0} \frac{1}{k} \tilde{\alpha}_k e^{-ik(\sigma + \tau)},
\]

(2.3)

where $\alpha_k$ and $\tilde{\alpha}_k$ are the oscillators, and the left and right moving momenta are given by*

\[
p_L = \frac{m}{2R} + nR, \quad p_R = \frac{m}{2R} - nR.
\]

(2.4)

The $m$ dependent term is simply a consequence of quantum mechanics, while that of $n$ is of stringy nature and is due to the fact that strings are extended objects and can wrap around the circle.

The vertex operators of the conformal field theory of the compact coordinate $X$ have the following moduli-dependent scaling and spin

\[
H = \frac{1}{2} p_L^2 + \frac{1}{2} p_R^2 = \frac{m^2}{4R^2} + n^2 R^2,
\]

\[
S = \frac{1}{2} p_L^2 - \frac{1}{2} p_R^2 = mn.
\]

(2.5)

Clearly, the spectrum is invariant under the transformation

\[
R \leftrightarrow \frac{1}{2R}; \quad m \leftrightarrow n.
\]

(2.6)

This is the simplest form of duality symmetry in compactified string theories.

* for string slope parameter $\alpha'$ taken to be 1/2
Now consider the circular compactification in the presence of a discrete Wilson line background. In this case, the left and right moving momenta take the form

\[
P_L = \left( \frac{m - \frac{1}{2} A^t CA n - A^t Cl}{2R} + Rn, \quad 1 + An \right) \equiv (p_L, \quad \tilde{p}_L),
\]
\[
P_R = \left( \frac{m - \frac{1}{2} A^t CA n - A^t Cl}{2R} - Rn, \quad 0 \right) \equiv (p_R, \quad 0)
\]

where \( l \) is the momentum on the \( E_8 \times E_8' \) lattice, \( A \) is the discrete Wilson line background fields and \( C \) is the Cartan metric for the \( E_8 \times E_8' \) lattice.

In the presence of Wilson lines, the vertex operators have the following moduli-dependent scaling and spin

\[
H = \frac{(m - \frac{1}{2} A^t CA n - A^t Cl)^2}{4R^2} + n^2 R^2 + \frac{1}{2} \tilde{p}_L^t C \tilde{p}_L
\]
\[
S = (m - \frac{1}{2} A^t CA n - A^t Cl)n + \frac{1}{2} \tilde{p}_L^t C \tilde{p}_L,
\]

where \(^t \) denotes matrix transpose. Note that we have included the \( R \)-independent term \( \frac{1}{2} \tilde{p}_L^t C \tilde{p}_L \) because it depends on the winding number.

We look for duality symmetries that act only the components of \( p_L \) and \( p_R \) associated with the orbicircle, leaving the internal \( \tilde{p}_L \) momentum and the discrete Wilson line \( A \) invariant. An orbicircle of radius \( R \) is constructed from a circle of the same radius, by modding out by a \( (\mathbb{Z}_2\text{-valued}) \) reflection symmetry. Consider the following duality symmetry,

\[
R \leftrightarrow \frac{1}{2\rho R}; \quad (m - \frac{1}{2} A^t CA n - A^t Cl) \leftrightarrow \frac{1}{\rho} n.
\]

This can be a symmetry of the theory provided the quantum numbers transform
as integers. This places the following constraints on the parameter $\rho$

$$\rho \mathbf{A} \in \mathbb{Z},$$

$$\frac{\rho}{2}(\mathbf{A}^t \mathbf{C} \mathbf{A}) \in \mathbb{Z},$$

$$\frac{\rho}{4}(\mathbf{A}^t \mathbf{C} \mathbf{A})(\mathbf{A}^t \mathbf{C} \mathbf{A}) + (\mathbf{A}^t \mathbf{C} \mathbf{A}) + \frac{1}{\rho} \in \mathbb{Z}, \tag{2.10}$$

$$\frac{\rho}{2}(\mathbf{A}^t \mathbf{C} \mathbf{A}) \mathbf{A} + \mathbf{A} \in \mathbb{Z},$$

$$\rho \mathbf{A} \mathbf{A}^t \mathbf{C} \in \mathbb{Z}.$$ 

The discrete Wilson lines $\mathbf{A}$ must satisfy constraints which are equivalent to preserving world-sheet modular invariance $^*[30, 27]$

$$\mathbf{A}(\mathbf{I} - \mathbf{Q}) \in \mathbb{Z}$$

$$\frac{1}{2} \mathbf{A}^t \mathbf{C} \mathbf{A}(\mathbf{I} - \mathbf{Q}) + \frac{1}{2} (\mathbf{I} - \mathbf{Q}^*) \mathbf{A}^t \mathbf{C} \mathbf{A} \in \mathbb{Z} \tag{2.11}$$

For the orbicircle (2.11) implies that $\mathbf{A}^t \mathbf{C} \mathbf{A}$ is either zero or half-integer valued. However the constraints (2.10) can only be solved for $\mathbf{A} = \mathbf{0}$ and $\rho = 1$ which corresponds to the to the usual $R \rightarrow 1/2R$ duality symmetry. This demonstrates that discrete Wilson lines can break the stringy duality symmetry.

### 3. Toroidal Compactification

In this section, the target space duality symmetry of string compactified on a $d$-dimensional torus $\mathbf{T}^d$ [19,20] is reviewed. The worldsheet action for the compact coordinates, in the lattice basis, is given as

$$S_{torus} = \frac{1}{2\pi} \int d\sigma d\tau \eta^{\alpha\beta} \left( G_{ij} \partial_\alpha X^i \partial_\beta X^j + \epsilon^{\alpha\beta} B_{ij} \partial_\alpha X^i \partial_\beta X^j \right) \tag{3.1}$$

where the metric $G_{ij}$ is defined as the scalar product of the basis vectors $e_i$, $i = 1, \ldots, d$, of the lattice $\Lambda_d$ generating the torus $\mathbf{T}^d = \frac{R^d}{\Lambda_d}$ and $B_{ij}$ is the antisymmetric tensor. Together they describe the $d^2$-dimensional moduli space of

$^* \mathbf{Q}^* = \mathbf{Q}^{(-1)}$
toroidal compactification. Notice that in the action $S_{torus}$ the coordinates $X^i$ denote the components of the internal dimensions in the lattice basis, i.e., $X^\mu = e^\mu_i X^i$ where $\mu$ is the internal space index, therefore on the torus we have $X^i \equiv X^i + 2\pi n^i$.

In matrix notation, the left and right moving momenta can be written as [29]

$$p_L = \frac{m}{2} + (G - B)n, \quad p_R = \frac{m}{2} - (G + B)n,$$

(3.2)

where $n$ and $m$, the windings and the momenta respectively, are $d$-dimensional integer valued vectors, while $G$ and $B$ are $d \times d$ matrices representing the background metric and antisymmetric tensor. The moduli-dependent part of the scaling dimension and the spin of the vertex operators are given by

$$H = \frac{1}{2}(p^t_L G^{-1} p_L + p^t_R G^{-1} p_R) = \frac{1}{2} u^t \Xi u,$$

$$S = \frac{1}{2}(p^t_L G^{-1} p_L - p^t_R G^{-1} p_R) = \frac{1}{2} u^t \eta u,$$

(3.3)

where

$$u = \begin{pmatrix} n \\ m \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix}, \quad \Xi = \begin{pmatrix} 2(G - B)G^{-1}(G + B) & BG^{-1} \\ -G^{-1}B & \frac{1}{2}G^{-1} \end{pmatrix}.$$

(3.4)

Discrete target space duality symmetries are all those integer-valued linear transformations of the quantum numbers leaving the spectrum invariant. Denote these linear transformations by $\Omega$ and write [30]

$$\Omega : u \rightarrow S\Omega(u) = \Omega^{-1}u.$$

(3.5)

To preserve (3.3), $\Omega$ should satisfy the condition $\Omega^t \eta \Omega = \eta$ which means that $\Omega$ is an element of $O(d, d, Z)$. Also, the moduli get transformed as

$$\Xi \rightarrow \Omega^t \Xi \Omega.$$

(3.6)

The transformation (3.6) contains the action of the duality group on the moduli. Obviously, some $\Omega$ transformations do not mix windings and momenta quantum numbers, such symmetries are also present in Kaluza-Klein compactification.
The generalization of the above results to the case of orbifolds without Wilson lines background is straightforward [29, 30]. The orbifold is defined as the quotient of the torus by a group of automorphisms, the point group $P$ of the lattice which is normally taken to be a cyclic group. This group has the following action on the quantum numbers [29, 30]

$$u \rightarrow u' = Ru,$$ \hspace{1cm} (3.7)

where $R$ is given by the matrix

$$R = \begin{pmatrix} Q & 0 \\ 0 & Q^* \end{pmatrix},$$ \hspace{1cm} (3.8)

and the matrix $Q$ is defined as

$$\theta_{\mu}^\nu e_i^\nu \rightarrow e_j^\nu Q_{ji} \hspace{1cm} (3.9)$$

where $\theta_{\mu}^\nu$ is the matrix corresponding to the action of the generator of the point group on the six-dimensional internal space. For the point group to be a lattice automorphism, the background fields (moduli) must satisfy [21]

$$Q'GQ = G, \quad Q'BQ = B.$$ \hspace{1cm} (3.10)

Finally the target space symmetries of the untwisted sector of the orbifold are those of the torus commuting with the twist matrix $R$. More generally, the symmetries are those satisfying $\Omega R - R^k \Omega = 0$, $k = 1, \ldots N$, where $N$ is the order of the twist [30]. For six-dimensional orbifolds one defines the twist as $\theta = (\zeta_1, \zeta_2, \zeta_3)$ where the notation is such that the action of $\theta$ in the complex basis is $(e^{2\pi i \zeta_1}, e^{2\pi i \zeta_2}, e^{2\pi i \zeta_3})$. If the action of the point group generated by $\theta$ has no invariant planes, then the twisted sectors of the theory are independent of the moduli. This means that the twisted states have no windings nor momenta. However, if a point group element leaves a particular complex plane invariant then the corresponding
twisted sector states will have non-vanishing windings and momenta and thus their
scaling dimensions will depend on the moduli of the unrotated plane. Let $\theta^k$ be a
group element which leaves a particular complex plane invariant, then the twisted
states winding and momentum will satisfy

$$Q^k n = n; \quad Q^{*k} m = m.$$  \hspace{1cm} (3.11)

The target space symmetries of the twisted sectors for $\mathbb{Z}_N$ Coxeter orbifolds have
been considered in [24].

4. Two-Dimensional Toroidal Compactification.

In this section, the case of two-dimensional toroidal compactification is consid-
ered. The relevance of this case will become clear in what follows. It is convenient
to group the four real degrees of freedom of $G_{ij}$ and $B_{ij} = b_{ij}$, parametrizing
the background of two-dimensional compactification into two complex moduli [7] defined as

$$T = T_1 + i T_2 = 2 \left( b + i \sqrt{\det G} \right),$$
$$U = U_1 + i U_2 = \frac{1}{G_{11}} \left( G_{12} + i \sqrt{\det G} \right).$$  \hspace{1cm} (4.1)

Clearly, in terms of these moduli, the metric is given by

$$G = \frac{T_2}{2U_2} \begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix}$$  \hspace{1cm} (4.2)

and the moduli-dependent scale and the spin of the vertex operators of the under-
lying conformal field theory take the following form

$$H = \frac{1}{T_2 U_2} |TU n_2 + Tn_1 - Um_1 + m_2|^2$$
$$S = m_1 n_1 + m_2 n_2$$  \hspace{1cm} (4.3)

where $n_1$, $n_2$, $m_1$ and $m_2$ are integers denoting the windings and momenta. The
spectrum is invariant (in addition to the symmetries $T \leftrightarrow U$, $T \leftrightarrow -\bar{T}$, $U \leftrightarrow -\bar{U}$)
under two copies of the modular group $PSL(2, Z)$ acting on the moduli as

$$U \rightarrow \frac{aU + b}{cU + d}, \quad T \rightarrow \frac{a'T + b'}{c'T + d'}, \quad (4.4)$$

where $ad - bc = 1$ and $a'd' - b'c' = 1$. These transformations act on the quantum numbers as follows

$$u \rightarrow \begin{pmatrix} M & 0 \\ 0 & (M^*) \end{pmatrix} u, \quad u \rightarrow \begin{pmatrix} d'I_2 & -c'L \\ b'L & a'I_2 \end{pmatrix} u, \quad (4.5)$$

where

$$M = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (4.6)$$

In the presence of a $Z_2$-twist both moduli $T$ and $U$ are compatible with the point group and the two-dimensional $Z_2$-orbifold has the same duality symmetries as for the torus. However for other point groups, only the $T$ moduli survives the twist and the duality group is given by $PSL(2, Z)$ acting on $T$ (also $T \leftrightarrow -\bar{T}$ is a symmetry of the spectrum) for all two-dimensional $Z_N$ orbifold.

5. Duality Symmetries with Discrete Wilson Lines: $\Lambda_6 = \Lambda_2 + \Lambda_4$

The moduli-dependent threshold corrections to the gauge coupling constants have contributions from those twisted sectors of the orbifold model in which the point group element (the twist) leave a particular complex plane of the three complex planes of the orbifold invariant. Those sectors are known as the $\mathcal{N} = 2$ sectors because they possess two space-time supersymmetries [8,9, 10]. It is the duality symmetries of the spectrum of these sectors which we wish to study here. Twisted sectors are only sensitive to the geometry of their invariant planes, in other words, their states have conformal dimensions depending on the moduli $T$ and $U$ associated with the unrotated plane.
In six-dimensional orbifold compactification without Wilson lines and with lattices decomposable as $\Lambda_6 = \Lambda_2 + \Lambda_4$ and where a certain twist leaves a plane invariant lying in $\Lambda_2$, the corresponding twisted sector have states with windings and momenta taking values only in $\Lambda_2$ and its dual $\Lambda_2^*$. The moduli-dependent scaling and spin of these states is thus given as in (4.3). Therefore the spectrum is invariant under the action of $PSL(2, \mathbb{Z}) \times PSL(2, \mathbb{Z})$ one for each complex moduli.

Now in the presence of a quantized Wilson line background, the twisted states have the left and right moving momenta, $P_L$ and $P_R$ given, in the lattice basis, by [29, 30]

$$P_L = \left( \frac{m}{2} + (G - B - \frac{1}{4} A^t CA)n - \frac{1}{2} A^t Cl, \quad 1 + An \right),$$
$$P_R = \left( \frac{m}{2} - (G + B + \frac{1}{4} A^t CA)n - \frac{1}{2} A^t Cl, \quad 0 \right) , \quad (5.1)$$

where $l$ is the momentum on the $E_8 \times E_8'$ lattice, $G$, $B$ and $A$ are the metric, antisymmetric tensor and Wilson line background fields, $C$ is the Cartan metric for the $E_8 \times E_8'$ lattice. Let $\theta$ be the generator of the point group of the orbifold and suppose that the $\theta^k$-twisted sector leave invariant the plane lying in $\Lambda_2$. The action of the point group on the quantum numbers can be written

$$u \to u' = Ru \quad (5.2)$$

where

$$u = \begin{pmatrix} n \\ m \\ l \end{pmatrix} \quad (5.3)$$

and

$$R = \begin{pmatrix} Q & 0 & 0 \\ \alpha & Q^* & (1 - Q^*) A^t C \\ A(I - Q) & 0 & I \end{pmatrix} \quad (5.4)$$

with

$$\alpha = \frac{1}{2} A^t CA(I - Q) + \frac{1}{2}(I - Q^*) A^t CA. \quad (5.5)$$
Note that the choice of Wilson lines must satisfy the constraints (2.11) and hence the entries of $\mathbf{R}$ are all integers [30]. The states in the $\theta^k$-twisted sector have windings and momenta satisfying the condition

$$\mathbf{R}^k \mathbf{u} = \mathbf{u}. \quad (5.6)$$

The solution of (5.6) is

$$(I - Q^k)n = 0$$

$$(I - Q^*k)(m - \frac{1}{2} A^t C An - A^t Cl) = 0. \quad (5.7)$$

Assume that the fixed plane of $\theta^k$ is the first complex plane, then $Q^k$ is block diagonal with the $2 \times 2$ identity matrix as its leading block. If we consider the new variables $n$ and $\hat{m} = m - \frac{1}{2} A^t C An - A^t Cl$, then $n$ and $\hat{m}$ can only take non-zero values in their first two components. Define

$$u_\perp = \begin{pmatrix} n \\ \hat{m} \\ \hat{l} \end{pmatrix} \quad (5.8)$$

where $\hat{l} = l + An$. This basis diagonalizes the action of the point group element $\theta$,

$$\theta : \quad u_\perp \rightarrow u'_\perp = \mathbf{R}_\perp u_\perp, \quad (5.9)$$

where

$$\mathbf{R}_\perp = \begin{pmatrix} Q & 0 & 0 \\ 0 & Q^* & 0 \\ 0 & 0 & I \end{pmatrix}. \quad (5.10)$$

In terms of the new variables, the left and right momenta $\mathbf{P}_L$ and $\mathbf{P}_R$ take the
form

\[
\begin{align*}
P_L & \equiv \left( \frac{\hat{m}_0}{2} + (G_\perp - B_\perp)n_0, \quad \hat{l}_0 \right) \\
P_R & \equiv \left( \frac{\hat{m}_0}{2} - (G_\perp + B_\perp)n_0, \quad 0 \right) \\
\hat{m}_0 & = m_0 - \frac{1}{2} A_{\perp}^t C A_{\perp} n_0 - A_{\perp}^t C l \\
\hat{l}_0 & = 1 + A_{\perp} n_0
\end{align*}
\]

(5.11)

where \( G_\perp \) and \( B_\perp \) are the moduli associated with the first complex plane. The quantity \( A_\perp \) is an \( 8 \times 2 \) matrix whose elements are the first two columns of \( A \) also \( n_0 \) and \( \hat{m}_0 \) are understood to be 2-dimensional vectors whose elements are the first two components (the only non-vanishing) of \( n \) and \( \hat{m} \) respectively. In this case the scaling dimension of the fields and their spin is

\[
\begin{align*}
H &= \frac{1}{2} u_\perp^{t} \Xi_\perp u_\perp + \frac{1}{2} \hat{l}_0^{t} C \hat{l}_0 \\
S &= \frac{1}{2} u_\perp^{t} \eta u_\perp + \frac{1}{2} \hat{l}_0^{t} C \hat{l}_0
\end{align*}
\]

(5.12)

Here

\[
\Xi_\perp = \begin{pmatrix}
2(G_\perp - B_\perp)G_\perp^{-1}(G_\perp + B_\perp) & B_\perp G_\perp^{-1} \\
-G_\perp^{-1}B_\perp & \frac{1}{2} G_\perp^{-1}
\end{pmatrix}, \quad u_\perp = \begin{pmatrix}
n_0 \\
\hat{m}_0 \\
\hat{l}_0
\end{pmatrix}
\]

(5.13)

Thus apart from the additional term coming from the internal gauge lattice, the above expressions for \( H \) and \( S \), when expressed in terms of the complex moduli \( T \) and \( U \), are identical to those of (4.3) but with modified quantum numbers.

We define the duality symmetries as those leaving \( \hat{l}_0 \) and the discrete Wilson lines invariant\(^*\). Using the results of the previous section, it can be easily seen that \( H \) and \( S \) (5.12) are invariant, in the first instance, under the transformations

\(^*\) It should be remarked that this definition is in contrast to the situation where continuous Wilson lines are present, since the latter do transform under duality symmetries [32].
$PSL(2,Z) \times PSL(2,Z)$ acting on $T$ and $U$ as in (4.4), where $T$ and $U$ are constructed out of $G_\perp$ and $B_\perp$ using Eq.(4.1). However there will be a constraint on the parameters $\{a, b, c, d, a', b', c', d'\}$ arising from the fact the quantum numbers should transform as integers. Under the transformations (4.4), the modified quantum numbers transform as follows,

$$
\begin{align*}
\mathbf{u}_\perp &\to \begin{pmatrix} M & 0 & 0 \\ 0 & (M^*) & 0 \\ 0 & 0 & I \end{pmatrix} \mathbf{u}_\perp, \\
\mathbf{u}_\perp &\to \begin{pmatrix} d'I_2 & -c'L & 0 \\ b'L & a'I_2 & 0 \\ 0 & 0 & I \end{pmatrix} \mathbf{u}_\perp,
\end{align*}
$$

(5.14)

where $M$ and $L$ are given by (4.6).

In order for the quantum number $n_0$, $m_0$ and $l$ to transform as integers, the following constraints are obtained

$$
A_\perp(I - M) \in Z,
$$

$$
\frac{1}{2} A_\perp^t C A_\perp (I - M) + \frac{1}{2} (I - M^*) A_\perp^t C A_\perp \in Z
$$

(5.15)

for the $U$-duality transformation parameters, and

$$
c'A_\perp \in Z,
$$

$$
\frac{c'}{2} A_\perp^t C A_\perp \in Z,
$$

$$
c'A_\perp L A_\perp^t C \in Z,
$$

$$
(1 - d') A_\perp - \frac{c'}{2} A_\perp L A_\perp^t C A_\perp \in Z,
$$

(5.16)

$$
(1 - d') C A_\perp + \frac{c'}{2} C A_\perp L A_\perp^t C A_\perp \in Z,
$$

$$
(1 - \frac{a'}{2} - \frac{d'}{2}) A_\perp^t L A_\perp - \frac{c'}{4} A_\perp^t C A_\perp L A_\perp^t C A_\perp \in Z.
$$

For those of the $T$-duality. The constraints (5.15) and (5.16) therefore break the $PSL(2,Z)$ duality symmetry associated with both the complex moduli $T$ and $U$ down to a subgroup.
6. Duality Symmetries with Discrete Wilson Lines: General $\Lambda_6$.

In this section we consider the previous discussions for more general choices of orbifold lattices $\Lambda_6$. As an example, consider the Coxeter orbifold $Z_6 - II - a$, with the twist defined by $\theta = (2, 1, -3)/6$ and $\Lambda_6$ given by the $SU(6) \times SU(2)$ root lattice. This model has the first and third planes unrotated by the $\theta^3$ and $\theta^2$ twists, respectively. Let us consider the $\theta^2$ twisted sector. We want to determine the symmetry group for the moduli $(T_3, U_3)$ of the third complex plane, which leaves the spectrum of the $\theta^2$ twisted sectors invariant.

The matrix $Q$ defining the twist action on the quantum numbers is given by

$$Q = \begin{pmatrix}
0 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}. \quad (6.1)$$

The constant background fields compatible with the twist $\theta$ are given as

$$G = \begin{pmatrix}
 r^2 & x & l^2 & R^2 & l^2 & u^2 \\
x & r^2 & x & l^2 & R^2 & -u^2 \\
l^2 & x & r^2 & x & l^2 & u^2 \\
R^2 & l^2 & x & r^2 & x & -u^2 \\
l^2 & R^2 & l^2 & x & r^2 & u^2 \\
u^2 & -u^2 & u^2 & -u^2 & u^2 & y
\end{pmatrix}, \quad (6.2)$$

$$B = \begin{pmatrix}
0 & -\beta & -\delta & 0 & \delta & -\gamma \\
\beta & 0 & -\beta & -\delta & 0 & \gamma \\
\delta & \beta & 0 & -\beta & -\delta & -\gamma \\
0 & \delta & \beta & 0 & -\beta & \gamma \\
-\delta & 0 & \delta & \beta & 0 & -\gamma \\
\gamma & -\gamma & \gamma & -\gamma & \gamma & 0
\end{pmatrix}, \quad (6.3)$$
with $R^2 = -2l^2 - r^2 - 2x$.

In the $\theta^2$ twisted sector, the twisted states are characterized by the windings and momenta satisfying the condition

$$ (I - Q^2)n = 0, $$

$$ (I - Q'^2)(m - \frac{1}{2}A^tCAn - A^tCl) = 0. $$

and are given by

$$ n_{sol} = \begin{pmatrix} n_5 \\ 0 \\ n_5 \\ 0 \\ n_5 \\ n_6 \end{pmatrix}, \quad m_{sol} = \begin{pmatrix} \hat{m}_5 \\ -\hat{m}_5 \\ \hat{m}_5 \\ -\hat{m}_5 \\ \hat{m}_5 \\ \hat{m}_6 \end{pmatrix}. $$

The moduli-dependent scaling and spin $H_3$ and $S_3$ associated with the vertex operators creating the $\theta^2$ twisted states are given by

$$ H = \frac{1}{2} \bar{u}_\perp \Xi_\perp \bar{u}_\perp + \frac{1}{2} (l + A_\perp n_0)^t C(l + A_\perp n_0), $$

$$ S = \frac{1}{2} \bar{u}_\perp \eta \bar{u}_\perp + \frac{1}{2} (l + A_\perp n_0)^t C(l + A_\perp n_0), $$

where $\Xi_\perp$ is given by the expression in (5.13) and

$$ \bar{u}_\perp = \begin{pmatrix} n_5 \\ n_6 \\ 3\hat{m}_5 \\ \hat{m}_6 \end{pmatrix}, \quad \mathbf{G}_\perp = \begin{pmatrix} 6l^2 + 3r^2 & 3u^2 \\ 3u^2 & y \end{pmatrix}, \quad \mathbf{B}_\perp = \begin{pmatrix} 0 & -\gamma \\ \gamma & 0 \end{pmatrix}. $$

$(\mathbf{G}_\perp, \mathbf{B}_\perp)$ being the metric and the antisymmetric tensor corresponding to the third complex plane. Also, $\mathbf{A}_\perp$ is defined by
\( \mathbf{A}_{\text{sol}} = \mathbf{A} K_2 \)

\[
\begin{pmatrix}
  n_1 \\
  n_2 \\
  n_3 \\
  n_4 \\
  n_5 \\
  n_6
\end{pmatrix} \equiv \mathbf{A}_\perp \begin{pmatrix} n_5 \\ n_6 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix}
  0 & 0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

(6.7)

Note that one can write

\[
\begin{pmatrix} \hat{m}_5 \\ \hat{m}_6 \end{pmatrix} = \begin{pmatrix} m_5 \\ m_6 \end{pmatrix} - \frac{1}{2} \tilde{\mathbf{A}}_\perp \mathbf{C} \mathbf{A}_\perp \begin{pmatrix} n_5 \\ n_6 \end{pmatrix} - \tilde{\mathbf{A}}_\perp \mathbf{C},
\]

(6.8)

where

\[
\mathbf{K}_3^t \begin{pmatrix} \hat{m}_1 \\ \vdots \\ \hat{m}_6 \end{pmatrix} \equiv \begin{pmatrix}
  \hat{m}_5 \\
  \hat{m}_6 \\
  0 \\
  0 \\
  0 \\
  0
\end{pmatrix}, \quad \mathbf{K}_3^t = \begin{pmatrix}
  1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

(6.9)

\( \tilde{\mathbf{A}}_\perp \equiv \) first two rows of \( \mathbf{K}_3^t \mathbf{A}_t \).

We repeated the analysis for the Coxeter orbifolds listed in Table 1 in and in general it was found that the moduli-dependent scaling and spin of twisted states with invariant planes can always be written in the form

\[
H = \frac{1}{2} \mathbf{u}_\perp \mathbf{E}_\perp \mathbf{u}_\perp + \frac{1}{2} \left( \mathbf{I} + \mathbf{A}_\perp \mathbf{n}_0 \right)^t \mathbf{C} \left( \mathbf{I} + \mathbf{A}_\perp \mathbf{n}_0 \right),
\]

\[
S = \frac{1}{2} \mathbf{u}_\perp \eta \mathbf{u}_\perp + \frac{1}{2} \left( \mathbf{I} + \mathbf{A}_\perp \mathbf{n}_0 \right)^t \mathbf{C} \left( \mathbf{I} + \mathbf{A}_\perp \mathbf{n}_0 \right),
\]

(6.10)

where \( \mathbf{u}_\perp = \begin{pmatrix} \mathbf{n}_0 \\ \mathbf{K}_1 \hat{\mathbf{m}}_0 \end{pmatrix} \) is a four-vector containing the 2-dimensional vector \( \mathbf{n}_0 \) and \( \hat{\mathbf{m}}_0 \), the winding and momentum of the invariant plane, \( \mathbf{G}_\perp \) and \( \mathbf{B}_\perp \) are the metric
and antisymmetric tensors corresponding to the invariant complex plane and $K_1$ is a constant $2 \times 2$ integer-valued matrix. For example, in the $\mathbb{Z}_6 - II - a$ model considered above, the $\theta^2$ sector has

$$K_1 = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (6.11)

The values for $K_1$ for some Coxeter orbifolds are summarized in Appendix A. Under the duality transformations (4.4), the quantum numbers $\tilde{u}_\perp$ transform as follows,

$$\tilde{u}_\perp = \begin{pmatrix} n_0 \\ K_1 \hat{m}_0 \\ \hat{l}_0 \end{pmatrix} \rightarrow \begin{pmatrix} M & 0 & 0 \\ 0 & (M^*) & 0 \\ 0 & 0 & I \end{pmatrix} \tilde{u}_\perp, \hspace{0.5cm} \tilde{u}_\perp \rightarrow \begin{pmatrix} d'I_2 & -c'L & 0 \\ b'L & a'I_2 & 0 \\ 0 & 0 & I \end{pmatrix} \tilde{u}_\perp.$$  \hspace{1cm} (6.12)

In order for the quantum numbers $n_0$, $m_0$ and $l$ to transform as integers, the following constraints must be satisfied,

$$A_\perp(I - M) \in \mathbb{Z},$$

$$K_1^{-1} M^* K_1 \in \mathbb{Z},$$

$$\tilde{A}_\perp^t CA_\perp - \frac{1}{2} \tilde{A}_\perp^t CA_\perp M - \frac{1}{2} K_1^{-1} M^* K_1 \tilde{A}_\perp^t CA_\perp \in \mathbb{Z},$$

$$(I - K_1^{-1} M^* K_1) \tilde{A}_\perp^t C \in \mathbb{Z},$$  \hspace{1cm} (6.13)

and

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\[ c' A \perp L K_1 \in Z \]
\[ c' A \perp L K_1 \tilde{A}_t \perp C \in Z, \]
\[ (1 - d') A \perp - \frac{c'}{2} A \perp L K_1 \tilde{A}_t \perp C A \perp \in Z, \]
\[ c' K_1 \tilde{A}_t \perp C \in Z \]
\[ \frac{c'}{2} K_1 \tilde{A}_t \perp C A \perp \in Z, \]
\[ \frac{c'}{2} \tilde{A}_t \perp C A \perp L K_1 \in Z, \]
\[ (1 - a' - \frac{d'}{2}) \tilde{A}_t \perp C A \perp L K_1 \tilde{A}_t \perp C A \perp + b' K_1^{-1} L \in Z. \]

The matrices \( K_2 \) and \( K_3 \) for the Coxeter orbifolds are listed in Appendix A.

Note that for the cases when \( K_1 = \beta I \), one can redefine the \( T \) modulus as \( \tilde{T} = \frac{T}{\beta} \) and in terms of the new moduli one obtains the constraints (5.15) and (5.16), but with \( A_t \perp \) replaced with \( \tilde{A}_t \perp \) as defined in (6.9) and with \( A_\perp \) defined by (6.7). The constraints (5.16) don’t involve the parameter \( b' \) and thus in terms of the redefined modulus \( \tilde{T} \), one obtains a larger symmetry group. Also, in the absence of Wilson lines but where \( K_1 \neq \beta I \), the duality group \( PSL(2, Z) \) is broken down to a subgroup. This has been demonstrated in [24], and we list in Table 2 these subgroups obtained for the orbifolds given in Table 1. As a concrete example, we go back to the \( Z_6 - \Pi - a \) orbifold and consider the following choice of Wilson line,

\[
A^t = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\] (6.15)

Using (6.13), (6.14), (6.11), (6.7) and (6.9) we found that the \( T_3 \)-duality symmetry for the \( \theta^2 \) sector is \( \Gamma^0(6) \) and the \( U_3 \)-duality is \( \Gamma_0(6) \).
For the $\theta^3$ sector, the value of $K_1$ is $2I$. In this case we define a new moduli $\tilde{T}_1 = \frac{T_1}{2}$ and employ the constraints (5.16) with $\tilde{A}^t_\perp$ and $A_\perp$ as defined by (6.9) and (6.7) respectively. However with the above choice of Wilson line, one finds that the values of $A_\perp$ and $\tilde{A}^t_\perp$ are both vanishing and therefore the Wilson line decouple completely from the theory as far as the duality symmetries are concerned. Thus the duality symmetry for the $\theta^3$ sector in terms of $\tilde{T}_1$ is $\text{PSL}(2, Z)$. More examples are listed in Table 3. In fact, this phenomenon of decoupling of the Wilson lines in particular twisted sectors occurs for more general choices of the Wilson lines than those listed in Table 3. For example, the most general allowed Wilson lines in the $Z_6-I\!I-b$ orbifold take the form

$$A^t = \begin{pmatrix} \frac{v}{3} \\ \frac{v}{3} \\ 0 \\ 0 \\ \frac{w}{2} \\ \frac{w}{2} \end{pmatrix}, \quad X = \frac{1}{2} \begin{pmatrix} 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -2 & 0 \\ -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \end{pmatrix}$$

(6.16)

where $v$ and $w$ are $E_8$ lattice vectors, (for simplicity we have only considered Wilson lines in $E_8$ not $E_8 \times E_8'$.) The entry $0$ in (6.16) denotes a zero vector of the lattice. The corresponding Wilson lines $A_\perp$ and $\tilde{A}^t_\perp$ in the $\theta^2$ twisted sector have the property that they depend on the components of the vector $w$ only. Similarly, in the $\theta^3$ sector $A_\perp$ and $\tilde{A}^t_\perp$ depend on the components of $v$ only. Thus if we choose any Wilson line with $v = 0$, $w \neq 0$ we would have decoupling in the $\theta^3$ sector, and hence a $\text{PSL}(2, Z)$ duality group. A similar phenomenon occurs for all the $Z_6$ orbifolds. When decoupling occurs in a twisted sector, one can explicitly compute the moduli dependent threshold correction coming from this sector, despite the fact that the Wilson lines do break the gauge symmetry.

In summary, we have studied the duality symmetries of orbifolds moduli as-
associated with invariant planes. These are relevant to threshold corrections to the
gauge coupling constants in the presence of discrete Wilson lines, which are not yet
known. The automorphic functions of the subgroups obtained should be related
to the expression of these corrections. The threshold corrections are essential in
the evaluation of any possible non-perturbative superpotential in addition to their
relevance to gauge couplings unification. They are also of fundamental importance
in the study of supersymmetry breaking. Our methods should be relevant to the
study of duality symmetries in the presence of continuous Wilson lines [32, 33]. We
hope to report on these questions in future publications.

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Table Captions

Table. 1. Non-decomposable $Z_N$ orbifolds. For the point group generator $\theta$, we display $(\zeta_1, \zeta_2, \zeta_3)$ such that the action of $\theta$ in the complex plane orthogonal basis is $(e^{2\pi i \zeta_1}, e^{2\pi i \zeta_2}, e^{2\pi i \zeta_3})$. The twisted sectors displayed are those with invariant complex planes.

Table. 2. Duality groups for the twisted sectors displayed in Table 1. $T_i$ and $U_i$ are the complex moduli associated with the invariant $i$-th plane.

Table. 3. This table contains explicit examples of duality groups for non-decomposable $Z_N$ Coxeter orbifolds, the second column in the table displays the non-vanishing components of a particular choice of Wilson line. Note that we have not included $Z_{12} - I - a$, as no Wilson lines are allowed in this example.
### TABLE 1

| Orbifold   | Point group generator | Twisted sector | Lattice                  |
|------------|-----------------------|----------------|--------------------------|
| $Z_4 - a$  | $(1, 1, -2)/4$        | $\theta^2$    | $SU(4) \times SU(4)$     |
| $Z_4 - b$  | $(1, 1, -2)/4$        | $\theta^2$    | $SU(4) \times SO(5) \times SU(2)$ |
| $Z_6 - II - a$ | $(2, 1, -3)/6$   | $\theta^2, \theta^3$ | $SU(6) \times SU(2)$ |
| $Z_6 - II - b$ | $(2, 1, -3)/6$   | $\theta^2, \theta^3$ | $SU(3) \times SO(8)$ |
| $Z_6 - II - c$ | $(2, 1, -3)/6$   | $\theta^2, \theta^3$ | $SU(3) \times SO(7) \times SU(2)$ |
| $Z_8 - II - a$ | $(1, 3, -4)/8$ | $\theta^2$    | $SU(2) \times SO(10)$   |
| $Z_{12} - I - a$ | $(1, -5, 4)/12$  | $\theta^3$    | $E_6$                    |

### TABLE 2

| Orbifold   | Duality group                  |
|------------|-------------------------------|
| $Z_4 - a$  | $\Gamma_{T_3/2}=\Gamma^0(2)$, $\Gamma_{U_3}=PSL(2, Z)$ |
| $Z_4 - b$  | $\Gamma_{T_5}=\Gamma^0(2)$, $\Gamma_{U_3}=\Gamma_0(2)$. |
| $Z_6 - II - a$ | $\Gamma_{T_3}=\Gamma^0(3)$, $\Gamma_{U_3}=\Gamma_0(3)$, $\Gamma_{T_1/2}=PSL(2, Z)$ |
| $Z_6 - II - b$ | $\Gamma_{T_5}=\Gamma^0(3)$, $\Gamma_{(U_3+2)}=\Gamma^0(3)$, $\Gamma_{T_1}=PSL(2, Z)$ |
| $Z_6 - II - c$ | $\Gamma_{T_3}=\Gamma^0(3)$, $\Gamma_{U_3}=\Gamma_0(3)$, $\Gamma_{T_1}=PSL(2, Z)$ |
| $Z_8 - II - a$ | $\Gamma_{T_3}=\Gamma^0(2)$, $\Gamma_{U_3}=\Gamma_0(2)$ |
| $Z_{12} - I - a$ | $\Gamma_{T_3/2}=PSL(2, Z)$ |
| Orbifold | Non-zero components of $A$ | Duality Symmetry |
|----------|--------------------------|------------------|
| $Z_4 - a$ | $A_{11} = -A_{31} =$ | $T_3$: $\Gamma^0(2)$, |
| | $A_{12} = -A_{32} =$ | $U_3$: $\Gamma^0(2)$ |
| | $A_{13} = -A_{33} = \frac{1}{2}$ | |
| $Z_4 - b$ | $A_{11} = -A_{31} =$ | $T_3$: $\Gamma^0(2)$, |
| | $A_{12} = -A_{32} =$ | $U_3$: $\Gamma_0(2)$ |
| | $A_{13} = -A_{33} = \frac{1}{2}$ | |
| $Z_6 - II - a$ | $A_{16} = -A_{36} = \frac{1}{2}$ | $T_3$: $\Gamma^0(6)$, |
| | | $U_3$: $\Gamma_0(6)$ |
| | | $T_1/2$: $PSL(2, Z)$ |
| $Z_6 - II - b$ | $A_{15} = -A_{35} =$ | $T_3$: $c = 0 \ (mod\ 2), \ b = 0 \ (mod\ 3)$, |
| | $A_{16} = -A_{36} = \frac{1}{2}$ | $U_3 + 2$: $\Gamma^0(6)$ |
| | | $T_1$: $PSL(2, Z)$ |
| $Z_6 - II - c$ | $A_{16} = -A_{36} = \frac{1}{2}$ | $T_3$: $c = 0 \ (mod\ 2), \ b = 0 \ (mod\ 3)$, |
| | | $U_3$: $\Gamma_0(6)$ |
| | | $T_1$: $PSL(2, Z)$ |
| $Z_8 - II - a$ | $A_{14} = -A_{34} =$ | $T_3$: $c = 0 \ (mod\ 2), \ b = 0 \ (mod\ 2)$, |
| | $A_{15} = -A_{35} =$ | $U_3$: $\Gamma_0(4)$ |
| | $A_{16} = -A_{36} = \frac{1}{2}$ | |
APPENDIX A

\[ Z_4 - a \]

The \( \theta^2 \) sector

\[ K_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \]

\[ K_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \]

\[ K_3^t = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]

\[ Z_4 - b \]

The \( \theta^2 \) sector

\[ K_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ K_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ K_3^t = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]

\[ Z_6 - II - a \]
The $\theta^2$ sector

\[ K_1 = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \]

\[ K_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \]

\[ K_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \]

The $\theta^3$ sector

\[ K_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \]

\[ K_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \]

\[ K_3^t = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \]

$\mathbb{Z}_6 - II - b$

The $\theta^2$ sector

\[ K_1 = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}, \]

\[ K_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}, \]

\[ K_3 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \]
The $\theta^3$ sector

\[
K_1 = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

\[
K_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
= K_3^t
\]

$Z_6 \cdot II \cdot c$

The $\theta^2$ sector

\[
K_1 = \begin{pmatrix}
3 & 0 \\
0 & 1
\end{pmatrix},
\]

\[
K_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
= K_3^t
\]

The $\theta^3$ sector

\[
K_1 = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

\[
K_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
= K_3^t.
\]
The $\theta^2$ sector

\[ K_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ K_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ K_3^t = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]

APPENDIX B

In this appendix we demonstrate that the modular subgroups obtained as duality symmetries listed in Table 3, are independent of the particular realization we choose for the windings and momenta that lie in an $\mathcal{N} = 2$ fixed plane. To illustrate this independence, we shall consider a particular example, namely $\mathbb{Z}_6 - II - b$ in the $\theta^2$ twisted sector; generalization to all other cases will then become apparent. We may represent the windings and momenta for $n$ and $\hat{m}$ of states in the $\theta^2$ twisted sector in the following ways,
\[
\mathbf{n}_{\text{sol}} = \begin{pmatrix}
0 \\
0 \\
n_5 + n_6 \\
0 \\
n_5 \\
n_6
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
n_3 \\
0 \\
n_3 - n_6 \\
n_6
\end{pmatrix}
\]  
(B.1)

\[
\hat{\mathbf{m}}_{\text{sol}} = \begin{pmatrix}
0 \\
0 \\
\hat{m}_5 + \hat{m}_6 \\
-\hat{m}_5 - \hat{m}_6 \\
\hat{m}_5 \\
\hat{m}_6
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
\hat{m}_3 \\
-\hat{m}_3 \\
\hat{m}_3 - \hat{m}_6 \\
\hat{m}_6
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
-\hat{m}_4 \\
\hat{m}_4 \\
-\hat{m}_4 - \hat{m}_6 \\
\hat{m}_6
\end{pmatrix}
\]

\(\mathbf{n}_{\text{sol}}\) and \(\hat{\mathbf{m}}_{\text{sol}}\) are solutions to the constraints (5.6) with \(k = 2\), and any pair of choices for \(\mathbf{n}_{\text{sol}}\) and \(\hat{\mathbf{m}}_{\text{sol}}\) should give rise to the same modular subgroup. To prove this consider a definite choice for \(\mathbf{n}_{\text{sol}}\) and \(\hat{\mathbf{m}}_{\text{sol}}\) e.g.,

\[
\mathbf{n}_{\text{sol}} = \begin{pmatrix}
0 \\
0 \\
n_5 + n_6 \\
0 \\
n_5 \\
n_6
\end{pmatrix}
= \begin{pmatrix}
n_1 \\
n_2 \\
n_3 \\
n_4 \\
n_5 \\
n_6
\end{pmatrix}
\]  
(B.2)

\[
\hat{\mathbf{m}}_{\text{sol}} = \begin{pmatrix}
0 \\
0 \\
\hat{m}_5 + \hat{m}_6 \\
-\hat{m}_5 - \hat{m}_6 \\
\hat{m}_5 \\
\hat{m}_6
\end{pmatrix}
= K_2^t
\]

Because of the choice of \(\hat{\mathbf{m}}_{\text{sol}}\) in (B.2), the corresponding projection matrix \(K_3^t\)
which was defined earlier in the text, is given by

\[
K_t^3 \begin{pmatrix} \hat{m}_1 \\ \hat{m}_2 \\ \hat{m}_3 \\ \hat{m}_4 \\ \hat{m}_5 \\ \hat{m}_6 \end{pmatrix} = \begin{pmatrix} \hat{m}_5 \\ \hat{m}_6 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (B.3)
\]

The modular subgroups corresponding to this choice of solutions will be determined by the ‘effective’ 2-dimensional Wilson lines \(A_\perp\) and \(\tilde{A}_\perp^t\) defined in terms of the 6-dimensional Wilson line and the matrices \(K_2\) and \(K_3^t\). Now consider a different choice of solutions given in (B.1), e.g.,

\[
n'_{\text{sol}} = \begin{pmatrix} 0 & \nu_3 \\ \nu_3 & 0 \\ 0 & \nu_6 \end{pmatrix}; \quad \hat{n}'_{\text{sol}} = \begin{pmatrix} 0 \\ \hat{m}_3 \\ -\hat{m}_3 \\ \hat{m}_3 - \hat{m}_6 \\ \hat{m}_6 \end{pmatrix}. \quad (B.4)
\]

Correspondingly, there will be new matrices \(K_2, K_3^t, A_\perp'\) and \(\tilde{A}_\perp^t\). It is straightforward to determine the connection between these two sets of Wilson lines,

\[
A_\perp = A_\perp' \mathcal{F} \quad \tilde{A}_\perp^t = \mathcal{F} \tilde{A}_\perp^t \quad (B.5)
\]

where in (B.5), \(\mathcal{F}\) is

\[
\mathcal{F} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (B.6)
\]

In general we would have found two different matrices \(\mathcal{G}\) and \(\mathcal{F}\) connecting \(A_\perp, \tilde{A}_\perp^t\) to the same quantities primed. With the choice of solutions in (B.2) and (B.4), \(\mathcal{G} = \mathcal{F}\).
We have to show now, that the constraints on $T$ and $U$ duality due to $A_\perp, \tilde{A}_\perp^t$ are the same as those due to $A_\perp', \tilde{A}_\perp'^t$ leading to the same modular subgroup. In the basis defined by $n_0 = \begin{pmatrix} n_5 \\ n_6 \end{pmatrix}$, $\hat{m}_0 = \begin{pmatrix} \hat{m}_5 \\ \hat{m}_6 \end{pmatrix}$ the transformations under $T$ and $U$ duality symmetries are

\begin{align*}
n_0 &\rightarrow d'n_0 - c'L K_1 \hat{m}_0, \quad K_1 \hat{m} \rightarrow b'L n_0 + a'K_1 \hat{m}_0 \\
n_0 &\rightarrow M n_0, \quad K_1 \hat{m}_0 \rightarrow M^* K_1 \hat{m}_0
\end{align*}

(B.7)

In the new basis $n_0' = \begin{pmatrix} n_3 \\ n_6 \end{pmatrix}$, $\hat{m}_0' = \begin{pmatrix} \hat{m}_3 \\ \hat{m}_6 \end{pmatrix}$ the same transformations become

\begin{align*}
n_0' &\rightarrow d'n_0' - c'F L K_1' \hat{m}_0' \\
K_1' \hat{m}_0' &\rightarrow b'L F^{-1} n_0' + a'K_1' \hat{m}_0' \\
n_0' &\rightarrow F M F^{-1} n_0' \\
K_1' \hat{m}_0' &\rightarrow M^* K_1' \hat{m}_0'.
\end{align*}

(B.8)

It is now straightforward to derive the constraints on $T$ and $U$ duality transformations in this new basis, with the Wilson lines $A_\perp'$ and $\tilde{A}_\perp'^t$ following the procedures outlined in section 6. As regards the constraints on $T$ and $U$ duality, one finds that conditions (6.13) and (6.14) are reproduced in terms of the Wilson lines $A_\perp$ and $\tilde{A}_\perp$ except that in various equations there is pre and/or post multiplication by either $F^{-1}$ or $F$. However, since $F^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ is itself integer-valued, it is clear that if constraints (6.13) and (6.14) are satisfied in the ‘old’ basis, they will remain so in the new. Hence, the modular subgroups that satisfy these conditions are independent of the two choices (B.2) and (B.4) for $n_{sol}$ and $\hat{m}_{sol}$. Furthermore, it is now apparent that for any pair of solutions $n_{sol}$ and $\hat{m}_{sol}$ for the $Z_6 - II - b$ orbifold given in (B.1) we will arrive at the same conclusion, because the corresponding matrices $F$, and $G$, together with their inverses, are always integer-valued. Generalization to all other Coxeter orbifolds follows directly. Since in any given orbifold $n_{sol}$ and $\hat{m}_{sol}$ satisfy the $\mathcal{N} = 2$ constraints (3.11) involving
the integer-valued matrix $Q$ and its powers, the corresponding matrices $F$ and $G$ will inevitably be integer-valued. That the same is true for $F^{-1}$ and $G^{-1}$ follows from the fact that $Q^{-1}$ is also integer-valued, since $\theta^{-1}$ is a lattice automorphism.

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