Analysis of point-contact models of the bounce of a hard spinning ball on a compliant frictional surface.

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Abstract

Inspired by the turf-ball interaction in golf, this paper seeks to understand the bounce of a ball that can be modelled as a rigid sphere and the surface as supplying an elasto-plastic contact force in addition to Coulomb friction. A general formulation is proposed that models the finite time interval of bounce from touch-down to lift-off. Key to the analysis is understanding transitions between slip and roll during the bounce. Starting from the rigid-body limit with an energetic or Poisson coefficient of restitution, it is shown that slip reversal during the contact phase cannot be captured in this case, which result generalises to the case of pure normal compliance. Yet, the introduction of linear tangential stiffness and damping, does enable slip reversal. This result is extended to general weakly nonlinear normal and tangential compliance. An analysis using Filippov theory of piecewise-smooth systems leads to an argument in a natural limit that lift-off while rolling is non-generic and that almost all trajectories that lift off, do so under slip conditions. Moreover, there is a codimension-one surface in the space of incoming velocity and spin which divides balls that lift off with backspin from those that lift off with topspin. The results are compared with recent experimental measurements on golf ball bounce and the theory is shown to capture the main features of the data.

Keywords: Coulomb friction; Piecewise-smooth dynamical systems; Impact mechanics.

1 Introduction

Golf is a highly technical sport with large amounts of sponsorship and prize money. Yet, until relatively recently there have been relatively few academic studies on the dynamics of a golf ball.

Those studies which do exist tend to focus on club-ball interaction and on the aerodynamic properties of the ball in flight. Perhaps the most influential work is the paper by Quintavalla [31], which provides a practical, parametrised model for the flight of a golf ball. Backed by a physical principles, the model relies on relatively few input parameters, such as lift and drag coefficients. The ease with which these can be estimated from experimental campaigns, have made the model one of the most well-known and commonly used in the field, both by manufacturers and legislators of the game of golf, and has enabled this mathematical model to define industry standards.

Following its flight however, a golf ball will bounce, perhaps several times, then typically roll, before coming to rest. Surprisingly little has been established for this bounce and roll phase. One of the difficulties is that while the launch conditions of a golf shot are typically well controlled, those of its landing are not. There are many variables both related to the surface and viscoelastic properties of the turf, as well the spin and velocity of the ball upon first landing.

Current literature on ball bounce tends to focus on cases where a deformable ball bounces on a rigid surface – see for example [16, 8, 10, 13, 20]. The results of such studies are more applicable to sports such as tennis, cricket or football, but less is known for the case of an almost rigid ball bouncing on a compliant surface, the type of bounce most commonly observed in golf.
One of the most comprehensive studies of golf ball-turf interaction was that of Haake \cite{21}, with the aim of identifying apparatus for a quantitative classification of golf turf. A number of models of bounce were also considered as part of that investigation, involving nonsmooth transitions between contact and non-contact phases, and also between slip and stiction. However, the models contained a wide range of parameters that would be hard to measure in the field, thus limiting their predictive power. A limited experimental data set was available at the time and as such the model never became widely verified and used.

Based on the same data set as Haake, a new study by Penner \cite{29} introduced an idea where the multiple forces acting on the ball’s surface during a compliant bounce are summed to a single force acting at a point at an angle. Penner thus suggested that a compliant bounce can be modelled as a rigid bounce (as defined in \cite{16}) against an inclined surface. This appeared to agree with the limited data set available, but appears to be too simplistic for a wider range of initial conditions.

Another study looking at golf ball bounce was presented by Cross in \cite{14} and models the bounce using a generalisation of the concept of Newtonian rigid impact to include a tangential coefficient of restitution (see also \cite{32}). Such a phenomenological coefficient can be fit to data and is said to allow for backwards bounce, but there is little physics incorporated into such a parameter and it cannot be used to predict \textit{a priori} what happens at the transition between roll and slip occur during a bounce. Once again, the theoretical result is backed by a limited set of experimental data.

Although each of the studies considered and explained a wide range of behaviours seen in golf (such as spin reversal) the lack of experimental validation or sound physical background leaves a gap in the applicability of the models. We thus seek a model that can be physically valid, can be described using a minimal number of parameters, and can be verified experimentally.

The key issue seems to be to understand possible transitions between slip and roll during the bounce. Such an approach requires the definition of a dry friction model; which can be a painful process to match to data – a central problem in the field of tribology \cite{3}. Common dry friction models include the simple Coulomb model, or its generalisation to models such as those due to Stribeck \cite{22}, LuGre \cite{17}, Dahl \cite{15}. The state of the art seems to be so-called rate-and-state friction which includes effects of slow creep and pre-slip through an auxiliary state variable that tries to capture the deformation of surface asperities; see e.g. \cite{30} and references therein. Fitting such models’ parameters to data can be crucial during sustained contact motion. However, in this work, we shall stick to the simple Coulomb model, as what seems to be important is understanding the nature of the transition between slip and stick (roll) during a rapid bounce. Also, we shall allow for a change in the nature of the surface during the bounce through the tangential compliance of the surface, rather than through more complex friction laws. We shall also allow for quite general forms of coupling between tangential and normal compliance, as in \cite{23} under certain natural scaling hypotheses on the relative size of normal and tangential forces.

Figure 1: A combination of frames from a recorded high-speed video of a ball bounce.
Modelling processes with rigid bounce and Coulomb friction can be a complex process and is known to lead to several different inconsistencies that are loosely termed the Painlevé paradox [11, 26, 28]. However, such paradoxes only occur for slender objects (where there is a sufficiently large coupling between normal and tangential degrees of freedom at contact) and do not come into play in this work. As we shall see, though, there are a range of finite-time singularities that are inherent in nonsmooth frictional contact problems associated with transitions between stick and slip, as are inherent in this work. These can be understood in terms of the theory of Filippov systems, see [18, 5, 25] and references therein.

In this paper we analyse the behaviour of a spinning uniform sphere as it bounces off a generic nonlinear compliant surface. Although applicable to a wide range of problems, our physical constraints and intuition will be directed here by the example of a golf ball bouncing. The key question we want to address is whether, under parsimonious assumptions about the nature of the surface presented in detail in Sec. 4, equations (30)-(34), a ball that enters a bounce spinning and subsequently grips the surface and enters a roll phase during the bounce, will lift off rolling or will lift off with reverse spin.

The rest of the paper is outlined as follows. We begin in Section 2 with a summary of the problem and present some experimental observations the details of which will be presented elsewhere. We show how a purely rigid-bounce model cannot account for what is observed, in which the ball is found to lift off with either topspin or backspin, but rarely, if ever, in a state of rolling. An improved model is presented in Section 3 which captures the finite time of contact and allows for linear visco-elastic behaviour in both the normal and tangential degrees of freedom. It is shown that such a model gives the correct qualitative behaviour. Section 4 then presents a generalised visco-elastic model, that allows for nonlinearity and coupling between normal and tangential degrees of freedom. We analyse the critical transition between topspin and backspin at lift-off and show that this represents a so-called two-fold singularity within the Filippov systems. We conclude that, in the natural limit of a small ratio between tangential and normal stiffness, the lift-off with rolling represents the codimension-one manifold of trajectories that passes through the singularity. Moreover, this manifold separates two open sets of initial conditions that enter a rolling state at some stage during the bounce, yet lift off with topspin, or with backspin. Section 5 contains conclusions and discusses open problems.

## 2 Preliminaries

Throughout this paper, we consider an isotopic rigid sphere of uniform density that is free to rotate about an axis through its centre and is moving in a plane that is perpendicular to its axis of rotation. The coordinates of the centre of mass of the ball will be denoted by \((x, y)\) and we denote its angular speed as \(\omega\). It is assumed for simplicity that there is a single point of contact between the ball and a compliant surface. The contact force acting on the ball can be decomposed into the normal and tangential components, denoted by \(\lambda_N\) and \(\lambda_T\) respectively. See Figure 2 for an illustration.

![Figure 2: Setting of the notation used.](image-url)

We shall consider different models for the visco-elastic compliance of the surface, starting from the
Table 1: Range of initial conditions used in the experimental campaigns. For nondimensionalisation of data see the main text.

|                | Astroturf          | Real turf          |
|----------------|--------------------|--------------------|
|                | Dimensional range  | Dimensionless range| Dimensional range  | Dimensionless range|
| \( \dot{x}_0 \) | (1.53, 38.6) [m s\(^{-1}\)] | (71.1, 1810) | (0.0171, 36.8) [m s\(^{-1}\)] | (0.801, 1720) |
| \( \dot{y}_0 \) | (−36.4, −4.61) [m s\(^{-1}\)] | (−1710, −216) | (−33.0, −2.14) [m s\(^{-1}\)] | (−1550, −101) |
| \( \omega_0 \) | (−4550, 1040) [rpm] | (−477, 1090) | (−3880, 1090) [rpm] | (−407, 1140) |

simplest models of finite-time impact with friction \cite{33}, to those that allow for a finite time interval of contact from impact to lift off, under assumptions about the visco-elastic properties of the surface. To establish some notation, we shall choose an origin of co-ordinates so that at the moment of impact, the ball is at the origin, so that \((x_0, y_0) = (0, 0)\) and is travelling with initial horizontal velocity \(\dot{x}_0\) and vertical velocity \(\dot{y}_0 < 0\). We shall henceforth choose a length scale such that the ball has unit radius \(R = 1\). Similarly, the spin in the plane of motion at impact is denoted by \(\omega_0\), which can either be positive (counter-clockwise rotation, or \text{backspin}) or negative (clockwise rotation, or \text{topspin}). This quantity will always be expressed in rad s\(^{-1}\). In this study the ball is modelled as a rigid sphere.

2.1 Experimental observations

The main motivation for this paper is the recent set of experimental measurements obtained on golf ball bounce, the details of which are presented in the companion paper \cite{7}. Using a bespoke launcher, golf balls with a wide variety of incoming velocities and spins were launched at two different surfaces: an artificial turf and a well-maintained teeing turf. The results were captured using high-speed photography at frame rates of up to 10,000 frames per second. Figure 1 shows an example field of view. Automatic image processing enabled accurate measurement of time series of velocities and spin within the plane of the ball’s motion. In all cases it was found that the ball remained in contact with the surface for a finite amount of time, during which clear deformation of the ground could be seen, by noting that the position of the top of the ball fell and then rose. The range of initial conditions covered by the experimental campaign can be seen in Table 1, and its visual representation is given in Figure 3.

![Figure 3](image-url)

Figure 3: An overview of the span of the initial condition space. Note that data is dimensionless here \((R = 1)\). (a) Initial conditions obtained for the testing of artificial turf; (b) Initial conditions obtained for the teeing turf.

Figure 4 presents just one aspect of the data, namely the relation between the incoming and outgoing relative tangential velocity \(H(t)\) between the ball and the surface at the start and the end of the bounce. The results are presented using the notation of Sec. 4 below, where (recalling that we rescale length so that
We have that

\[ H(t) = \dot{x}(t) + \omega(t). \]

(1)

We also use the notation \( H_0 = \dot{x}_0 + \omega_0 \) to denote the tangential velocity component at the moment of impact and \( H_F = \dot{x}_F + \omega_F \) at the moment of lift off.

![Figure 4: Scatter of the experimental results of tangential velocity at lift off versus tangential velocity at first contact; (a) using artificial turf, (b) using fairway turf.](image)

We observe that for both the synthetic and actual turf, the data falls roughly into two classes. For one class, the tangential velocity at lift off is large and positive. Further examination of the data reveal that these cases correspond to trajectories that land with a large amount of backspin (that is, a large positive angular velocity), slip throughout the bounce impact, and lift off with a reduced amount of backspin. Another somewhat larger class of samples, appear to lift off with small negative relative velocity. Analysis of the trajectories show that these trajectories typically enter with a small amount of backspin, no spin or topspin but lift off with a small amount of topspin. In particular, there is little evidence of balls lifting off rolling, that is with the relative tangential velocity being zero. Examples of the raw time-series data for each of the two cases are illustrated in Figure 5. Note how the changes during the bounce are not recorded, as the ball is not fully visible in that phase due its immersion within the surface.

![Figure 5: Changes in the (non-dimensional) velocities before and after the bounce. (a) High tangential velocity at lift off. (b) Negative tangential velocity at lift off. All data gathered from the experimental session using teeing turf. Colours blue, red, orange and green represent vertical, horizontal, angular and tangential velocities respectively.](image)
2.2 Rigid impact with friction

Let us first look at the case of a purely rigid bounce, that is where the rigid ball bounces off a rigid surface subject to Coulomb friction. During the bounce normal force due to the contact, gravity and tangential force due to Coulomb friction act on the body, yielding the equations of motion

\[
\ddot{x} = \lambda_T, \quad \ddot{y} = \lambda_N - g, \quad \dot{\omega} = \frac{5}{2} \lambda_T. \quad (2)
\]

Many authors have considered impact problems of this type and results typically only differ by the assumptions made about how to capture energy loss through a coefficient of friction; see e.g. [9] and references therein.

Here we follow the approach used in [33] where the impact is assumed to last an infinitesimal amount of time and the contact forces are assumed to be $\delta$-function-like distributions. An important part of the process is to model transitions between roll and slip during the infinitesimal time of impact. This can lead to subtle effects if the impacting body is slender [27]. In the case of sphere, however, the implementation of an analogue to the single degree-of-freedom coefficient of restitution is straightforward and most methods lead to the same answer [4]. In particular here we assume, given the landing normal velocity $\dot{y}_0$, that lift-off is deemed to occur when the ball reaches a normal velocity $\dot{y}_F = -r \dot{y}_0$, where $r$ is called a kinematic coefficient of restitution. For simplicity, we assume $r$ is constant (although note the empirical studies [29, 32] which suggests golf turf may be modelled by a velocity-dependent coefficient of restitution). We are concerned about the dynamics of the point of contact, at which we can write the tangential and normal velocities as

\[
v = [v_Y, v_N]^T = [\dot{x} + \omega, \dot{y}]^T,\]

so that

\[
\frac{dv}{dt} = \begin{bmatrix} \frac{7}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_T \\ \lambda_N \end{bmatrix} = m^{-1} \lambda, \quad (4)
\]

where $\lambda$ is the vector of contact forces and $m^{-1}$ can be thought of as the inverse of the mass matrix in the “coordinate system” of $(v_T, v_N)$.

Let us denote by $t = t_0$ the time at which the impact is initiated. The subscript 0 will denote evaluation of other quantities at the time impulse

\[
\begin{equation}
\frac{dv}{dt} = m^{-1} \lambda + O(\varepsilon).
\end{equation}
\]

We now follow the idea introduced by Stronge in [33], where we replace our independent time variable by the new one, namely the normal impulse

\[
n = \int \lambda_N dt,
\]

which is a strictly increasing function of $\tilde{t}$. Dropping the tilde (for simplicity of notation) and ignoring terms $O(\varepsilon)$ we now have, with respect to the new independent variable,

\[
\frac{dv}{dn} = \frac{1}{\lambda_N} m^{-1} \lambda = \begin{bmatrix} \frac{7}{2} \lambda_T \\ \frac{2}{\lambda_N} \lambda_N \end{bmatrix}.\]

During the slipping phase of the motion $\lambda_T = -\text{sign}(v_T) \mu \lambda_N$, where $\mu$ is the coefficient of friction. Substituting this into Equation (7) and rearranging the derivatives to eliminate $n$ we have that during the slipping phase of the rigid bounce

\[
\frac{dv_T}{dv_N} = \pm \frac{7}{2} \mu. \quad (8)
\]

However, defining the tangential force during the roll needs a bit more care. We note that during the roll $v_T = \dot{x} + \omega = 0$ and thus it must also be true that $\ddot{x} + \dot{\omega} = \lambda_T + \frac{5}{2} \lambda_T = \frac{7}{2} \lambda_T = 0$. This in turns requires
\( \lambda_T = 0 \) during the roll, which implies that once the ball enters rolling, it may never leave it, and during the roll. Combined with Equation (7) we have for the rolling stage of the bounce

\[
\frac{dv_T}{dv_N} = 0. 
\] (9)

With these equations it is straightforward to work out conditions under which \( v_N \) and \( v_T \) vary, which can be expressed graphically using the method presented in [27], see in Figure 6. In particular, we distinguish between three cases in the space of initial conditions \((\dot{x}_0, \dot{y}_0, \omega_0)\):

**Case I:** the ball will slip through the impact (case \( I_{\pm} \)) if

\[
\left| \frac{\dot{x}_0 + \omega_0}{\dot{y}_0} \right| > \frac{7}{2}\mu (1 + r);
\]

**Case II:** the ball will enter the roll in the restitution phase (case \( II_{\pm} \)) if

\[
\frac{7}{2}\mu < \left| \frac{\dot{x}_0 + \omega_0}{\dot{y}_0} \right| < \frac{7}{2}\mu (1 + r);
\]

**Case III:** the ball will enter the roll in the compression phase (case \( III_{\pm} \)) if

\[
\left| \frac{\dot{x}_0 + \omega_0}{\dot{y}_0} \right| < \frac{7}{2}\mu.
\]

Note that none of the cases allows the ball to enter slipping once it has begun to roll during the bounce. This contravenes what we saw in the experimental data. Specifically, if we consider balls entering bounce with \( v_T > 0 \) then, this theory allows lift-off from bounce to occur either in forward slip (\( v_T > 0 \)) or in roll (\( v_T = 0 \)); there is no initial condition that leads to ‘spin reversal’ during bounce. (Note though that this observation that this does not actually preclude a ‘backwards bounce’ because with high backspin a ball that has \( v_T = 0 \) would lift off with \( \dot{x} < 0 \)).

Note that the absence of spin reversal also occurs if we introduce normal compliance into the theory, so that the bounce occupies a short \( O(\varepsilon) \) period of time, while keeping the Coulomb friction assumption in the tangential direction. For example, a straightforward calculation (see [6] for details) can be performed assuming a Kelvin-Voigt-like model of a spring and damper connected in parallel. Assuming the stiffness of the spring to be \( 1/\varepsilon \) and the damping parameter of the dashpot to be \( d/\varepsilon \), where \( d \) is a damping ratio, then one obtains the same conclusion about the absence of spin reversal. In particular, in the limit \( \varepsilon \to 0 \), with \( d < \frac{1}{2} \) (to assure the underdamped solution), one recovers the rigid bounce theory, with the coefficient of restitution

\[
r \sim \exp \left[ -\frac{\pi}{2} d + \frac{1}{2} d^2 + O(d^3) \right] + O(\varepsilon). \] (10)

More generally, it is clear that in order to observe spin reversal in a simple point-contact model, a more complex model of the mechanics in the tangential degree of freedom is required.

### 3 Linear elasto-plastic model

Let us now consider a model where the ball remains rigid as before, but compliance in both horizontal and vertical directions can be observed. As depicted in Figure 7 the elasto-plastic behaviour is assumed to follow one of a spring and damper connected in parallel, known in the literature as the Kelvin-Voigt model, see e.g. [19]. The dynamics between the normal and vertical directions is decoupled, and it is assumed that the ball is in contact with the surface at a single point only, where the normal and tangential forces again follow the Coulomb friction law. The equations of motion for such a system can be written in dimensionless form

\[
\ddot{x} + \frac{2d_1}{\varepsilon_1} \dot{x} + \frac{1}{\varepsilon_1^2} x = \lambda_T,
\]
\[
\ddot{y} + \frac{2d_2}{\varepsilon_2} \dot{y} + \frac{1}{\varepsilon_2^2} y = -g,
\]
\[
\dot{\omega} = \frac{5}{2} \lambda_T.
\] (11)
The start of the impact is denoted with solid black dot, and the end is denoted with a white rhombus. We differentiate between cases of slip throughout the impact (I±) and slip followed by stick in the restitution (II±) or compression (III±) phase. Here \( g \) is the acceleration due to gravity, \( \varepsilon_{1,2} \) are the stiffness ratios in horizontal and vertical directions and \( d_{1,2} \) are the damping ratios in the respective directions. Note the scaling that \( \varepsilon_{1,2} \ll 1 \) is consistent with a time scale in which the bounce take places over a rapid time scale, and passing to the limit \( \varepsilon_{1,2} \to 0 \) will result in a rigid model with a horizontal and vertical coefficients of restitution (determined by the damping ratios \( d_{1,2} \), provided that \( d_{1,2} < 1 \) so that the system is underdamped. In particular, we will assume \( d_2 < 1 \), which will insure that a ball coming into contact with the ground with \( \dot{y}_0 < 0 \) will lift-off at some later time with \( \dot{y}_F > 0 \).

Specifically, we can readily compute that the ball will remain in contact with the compliant surface for as long as

\[
\lambda_N = -\frac{2d_2}{\varepsilon_2} \dot{y} - \frac{1}{\varepsilon_2^2} y - g > 0
\]

and will lift off when \( \lambda_N = 0 \)

Returning to the physical motivation of golf-ball turf interaction, in addition to the scaling that \( \varepsilon_{1,2} \ll 1 \), we shall additionally assume that \( \varepsilon_1 \ll \varepsilon_2 \). Such a scaling implies that there is far greater compliance in the normal direction than the tangential. That this is a natural assumption can be understood by briefly considering instead the motion that would occur if \( \varepsilon_1 = \varepsilon_2 \), so that the stiffness experience is isotropic. In that case, in the absence of spin or damping, and assuming \( \dot{x} \) a ball with inbound angle \( \phi = \arctan(\dot{x}, \dot{y}) \)
would move along a straight line in the \((x,y)\)-plane and would lift-off with angle \(\phi + \pi\). This is not how turf behaves. Instead, observational data suggest that such a ball would lift with outbound angle close to \(-\phi\).

With such scaling, it is useful to introduce a new time rescaling

\[
\tau = \varepsilon_2^{-1} t = \mathcal{O}(1)
\]

and a dimensionless parameter

\[
\eta = \varepsilon_2/\varepsilon_1 \ll 1.
\]

The equations of motion can now be written in the dimensionless form

\[
X'' + 2d_1\eta X' + \eta^2 X = \Lambda_T, \\
Y'' + 2d_2 Y' + Y = -\varepsilon_2^2 g, \\
\Omega' = \frac{5}{2} \Lambda_T,
\]

where \(X(\tau) = x(t)\), \(Y(\tau) = y(t)\), \(\Omega(\tau) = \omega(t)\), \(\Lambda_T = \varepsilon_2^{-2} \lambda_T\), and a prime represents differentiation with respect to \(\tau\). We also define \(\Lambda_N = \varepsilon_2^{-2} \lambda_N\), that is

\[
\Lambda_N = -2d_2 Y' - Y - \varepsilon_2^2 g > 0.
\]

The Coulomb friction law (in the new variables) dictates that for \(|\Lambda_T| = \mu \Lambda_N\) the ball slips on the surface. However, greater care is needed for the rolling motion, that is when \(|\Lambda_T| = \mu \Lambda_N\).

### 3.1 Defining rolling; a Filippov formulation

To consider what happens during rolling, it is useful to model the system as a Filippov System \([5, 18, 25]\). That is, with \(\Lambda_T\) well defined for the cases of slip, we note that the equations (15) can be written in the form

\[
p' = \begin{cases} F_1(p) & \text{if } H(p) > 0, \\ F_2(p) & \text{if } H(p) < 0, \end{cases}
\]

where \(p\) is the vector of state variables. Specifically, we have

\[
p = [X, X', Y, Y', \Omega]^	op
\]

and the smooth function

\[
H(p) = X' + \Omega.
\]

In 1988 Filippov \([18]\) proposed a consistent method of defining dynamics along the surface of discontinuity \(\{H(p) = 0\}\) in which a new vector field is introduced

\[
F_s(p) = (1 - \alpha) F_1(p) + \alpha F_2(p) \quad \text{when } H(p) = 0,
\]

where \(\alpha \in [0,1]\). That is, the vector field along the discontinuity surface is the unique convex linear combination of the two neighbouring vector fields, such that the resulting vector lies along the discontinuity. The vector field \(F_s\) in our system would define rolling, but confusingly, in deference to control applications, in Filippov systems \(F_s\) is termed the sliding vector field.

A ‘sliding’ trajectory is allowed to leave the surface of discontinuity when either \(\alpha = 0\) (therefore leaving into the region \(H(p) > 0\) where vector field \(F_1\) applies) or when \(\alpha = 1\) (where \(F_2\) applies). At the same time, \(\alpha\) can be explicitly calculated during the sliding motion by making that the assumption that the vector field must be orthogonal to \(\nabla H\). Thus, we have

\[
\alpha(p) = \frac{F_1(p) \cdot \nabla H}{(F_1(p) - F_2(p)) \cdot \nabla H}.
\]

9
Figure 8: Definition of the sliding dynamics — the discontinuity surface is denoted here with a plane \( H(p) = 0 \), the dynamics according to the two vector fields \( F_1 \) and \( F_2 \) are denoted with red and green arrows (above and below the discontinuity surface). Their convex linear combination lies along the dashed line, and we chose the unique solution that also aligns with the sliding surface.

### 3.2 Analysis of roll, slip and lift-off transitions

Having now defined the dynamics fully to be

\[
p'(p) = \begin{cases} 
   F_1(p) & \text{if } H(p) > 0, \\
   F_s(p) & \text{if } H(p) = 0, \\
   F_2(p) & \text{if } H(p) < 0,
\end{cases}
\]

(22)

with \( F_s \) specified in Equation (20), we can consider different cases of dynamics during the bounce. We represent possible geometries in Figure 9. In it we note that a ball can either slip throughout the bounce (either forward slip with \( H(p) > 0 \) or negative slip with \( H(p) < 0 \)) or can at some time \( t \) begin to roll (that is, the dynamics evolves along the discontinuity set where \( H(p) = 0 \)).

Let us now direct our attention to the latter of the presented cases, that is when the ball is rolling and in particular, we shall study how exiting such state is possible. As discussed before, exiting will be dependent on the value of the function \( \alpha(p) \), which for our system is explicitly calculated to be

\[
\alpha(p) = \frac{7\mu (2d_2Y' + Y) - 4d_1\eta X' - \eta^2 X}{7\mu (2d_2Y' + Y)} + \mathcal{O}(\varepsilon^2).
\]

(23)

We now analyse the transitions between rolling and slipping.

Suppose, for definiteness, that a trajectory along the discontinuity surface \( H(p) = 0 \) reaches a point \( p_1 \) where \( F_1(p_1) \cdot \nabla H = 0 \). That means that the sliding field definition is equal to the vector field \( F_1 \) at this point. Moreover, as dictated by (21), the trajectory should leave the surface into the half-space \( H(p_1) > 0 \).

Such a trajectory leaving the discontinuity surface must be tangent to \( H(p_1) = 0 \), therefore we have

\[
F_1(p_1) \cdot \nabla H = -2d_1\eta X' - \eta^2 X + \frac{7}{2} \mu [2d_2Y' + Y + \varepsilon^2g] = 0.
\]

(24)

Notice that the term \( 2d_2Y' + Y + \varepsilon^2g \) is in fact the \(-\Lambda_N\), given by (16) and thus remain negative throughout the bounce. On the other hand, in order to balance the first two terms on the right-hand side of (24), at \( p_1 \) we must have that \( \Lambda_N = \mathcal{O}(\eta) \ll 1 \).

Given this information, let us now consider the possible signs of the state variables at \( p_1 \). The bounce is initiated at \( Y = 0 \) and \( Y \leq 0 \) throughout the bounce. Suppose that at \( Y' < 0 \) at \( p_1 \), so that we are in the downwards (compression) phase of the bounce. Then since \( \Lambda_N = \mathcal{O}(\eta) \) and \( Y < 0 \) it must follow that both \( Y, Y' = \mathcal{O}(\eta) \) at \( p_1 \). \( Y \) being small means we must be close to the time of initiation of bounce. However, initially \( Y'(0) = \mathcal{O}(1) \) which gives a contradiction. Thus, we must have that \( Y' > 0 \) at \( p_1 \). That is, the transition from roll into slip can only happen during the upward (restitution) phase of the bounce.
Figure 9: Possible trajectories during the generic elasto-plastic bounce (in reduced geometry). The grey plane represents the lift off condition, the dotted plane is the codimension-1 discontinuity, the solid black curves represent surfaces where $F_1 \cdot \nabla = 0$ and $F_2 \cdot \nabla = 0$, which corresponds to the surfaces where a trajectory leaves the surface of discontinuity and enters either $F_1$ or $F_2$ (respectively). Point $s$ is an intersection of the two switching surfaces (known as a two-fold bifurcation point [24]) and lies on the lift off plane as well.

Red trajectory (a): The ball goes through the bounce with positive slip only ($H(p) > 0$). The bounce is terminated by the intersection with the grey plane, which represents the lift off condition.

Blue trajectory (b): The ball goes through the bounce with negative slip only ($H(p) < 0$). The bounce is terminated by the intersection with the grey plane, which represents the lift off condition.

Green trajectories (c-d): The ball enters the bounce with positive or negative slip (respectively) and eventually reaches the system discontinuity by entering roll ($H(p) = 0$). The bounce dynamics continue along the surface of discontinuity and are to be determined in the next sections.
Next, let us compute the second directional derivative at such a point $p_1$. We obtain

\[
\begin{align*}
(F_1(p_1) \cdot \nabla)^2 H &= \frac{7}{2} \mu Y'' + (7d_2 + 2d_1 \eta) \left[ -2d_2 Y'' - Y - \varepsilon_2^2 g \right] + \mathcal{O}(\eta^2) \\
&= \frac{7}{2} \mu Y'' + (7d_2 + 2d_1 \eta) \Lambda_N + \mathcal{O}(\eta^2).
\end{align*}
\]

Now, the above argument shows that both of the terms on the right-hand side of (25) are positive. Thus $(F_1(p_1) \cdot \nabla)^2 H > 0$, which confirms that the ball will leave the rolling region consistently into the $F_1$ region. 

We can similarly analyse the case where the ball transitions from rolling to slipping via the sliding flow becoming tangent to the vector field $F_2$ at some point $p = p_2$. We now have

\[
F_2(p_2) \cdot \nabla H = -2d_1 \eta X' - \eta^2 X - \frac{7}{2} \mu \left[ 2d_2 Y' + Y + \varepsilon_2^2 g \right] = 0,
\]

As before, we see that such a point can only occur if $\Lambda_N = \mathcal{O}(\eta)$. A similar argument shows that $Y'' > 0$ at any such point $p_2$. The argument then follows as before. The second directional derivative at the switching point will thus be negative, once again confirming consistency of Filippov’s theory.

The key question for final consideration is whether the ball can lift off (that is, reach the termination point of bounce) while rolling. For that to be the case, we would need $\Lambda_N = 0$ as the lift off condition, but thus also $\Lambda_T = 0$, which can be thought of as $F_s$ being tangent to $F_1$ and $F_2$ simultaneously. Considering the form of the function $\alpha(p)$ we would have that precisely at lift off we must simultaneously satisfy the two conditions

\[4d_1 \eta X' + \eta^2 X = 0 \quad \text{and} \quad 2d_2 Y' + Y + \varepsilon_2^2 g = 0,
\]

which imposes an additional constraint on the initial-value problem. Thus there will be at most a codimension-one surface in the space of initial conditions for which both conditions can be simultaneously satisfied and lift off with rolling would occur with probability zero. However, to make this statement more precise requires a proper analysis of the singularity where $F_s$ is simultaneously tangent to $F_1$ and $F_2$, because it is possible that the point satisfying (27) could be an attractor for sliding initial conditions. Such an analysis forms the subject of the next section.

Briefly though, Figure 10 shows a numerical verification that lift-off with roll seems to not be an attractor in this case. Here trajectory (A) with initial conditions $p_0 = [0.3543, -0.1603128, -0.1608, 3.4739, 0.1603128]^T$ is the trajectory which leaves the bounce within the friction cone, thus rolling. Initial conditions of trajectory (A) is perturbed slightly and gives rise to trajectories (B) and (C), which lift off on the boundary of the slipping cone, therefore slipping. (b) Tangential velocities of each of the trajectories. Note how trajectories (B) and (C) enter slipping motion (non-zero tangential velocity $H$) shortly before lift-off.

![Figure 10](image-url)

Figure 10: Numerical results for the system (15) for $\mu = 0.3$, with three different initial conditions; (a) projected onto the friction cone $(\lambda_T, \lambda_N)$ and as slip velocity $H$ as a function of time. See text for details.
Thus we have found precisely the opposite conclusion from the rigid impact model in Section 2.2, namely that an introduction of even minimal, compliance in the normal direction causes the ball to always lift off slipping. That is, despite a transition to roll during the bounce, with probability zero does does the ball lift off rolling. This conclusion appears to be consistent with the experimental data, but it would be useful to check whether it is an artefact of the specific linear Kelvin-Voigt model we have chosen.

4 A generalised elasto-plastic model

To test the generality of our findings to refinements in the surface model, we now consider a generalisation of the model form the previous section by posing equations of motion of the form

\[
\ddot{x} + u(x, \dot{x}, y, \dot{y}, \omega) \dot{x} + z(x, \dot{x}, y, \dot{y}, \omega) x = \lambda_T, \\
\ddot{y} + d(y, \dot{y}) \dot{y} + k(y, \dot{y}) y = -g, \\
\dot{\omega} = \frac{5}{2} \lambda_T,
\]

where \(u, z, d, k\) are general, possibly nonlinear, functions, \(g\) is the acceleration due to gravity and all other variables have the same meaning as before. Note this model allows for nonlinear, and velocity dependent stiffness and damping coefficients, as well dependence of the tangential mechanics on depth, spin and normal velocity. We will however impose some asymptotic constraints on the functions \(u, z, d\) and \(k\) in the course of our theory (see equations (30)-(34) below).

Under our assumption of a point of contact, we assume that the frictional contact between the ball and the surface can be modelled according to the Coulomb friction law, that is

\[|\lambda_T| = \mu \lambda_N = \mu (-d(y, \dot{y}) \dot{y} - k(y, \dot{y}) y - g)\]
during slipping and

\[|\lambda_T| < \mu \lambda_N = \mu (-d(y, \dot{y}) \dot{y} - k(y, \dot{y}) y - g)\]
when the ball is rolling (sticking), where \(\mu\) is the coefficient of friction.

The ball will remain in contact with the surface until normal forces due to the mechanism are balanced by those due to gravity, that is when

\[\lambda_N = -d(y, \dot{y}) \dot{y} - k(y, \dot{y}) y - g = 0\]  (29)

We will refer to (29) as the lift-off condition; the horizontal and vertical velocities at this instance will be denoted by \(\dot{x}_F\) and \(\dot{y}_F\) respectively, and the lift-off spin denoted by \(\omega_F\).

One can think of the form of (28) as a Kelvin-Voigt-like model, where springs and dampers, connected in parallel, dictating the behaviour of the sphere during the bounce in the vertical and horizontal direction independently. The major difference, between the classical Kelvin-Voigt model and the one used here is that the springs and dampers are allowed to be nonlinear and to feature coupling between normal and tangential degrees of freedom. An illustration of this setup can be seen in Figure 11.

We impose certain conditions on our functions \(u, z, d, k\) suggested by physical intuition. Firstly, we require the stiffness and damping in both vertical and horizontal directions to increase as the ball moves further into the bouncing surface, that is those values should increase as the vertical position \(y\) decreases:

\[u_y(x, \dot{x}, y, \dot{y}, \omega) < 0,\]
\[d_y(y, \dot{y}) < 0.\]  (30)

The motivation for this, is that in reality, point contact is an approximation to a Hertzian-like contact in which the true area of contact increases with depth \(-y\). In a similar fashion, we require that the normal force acting on the body increases with depth

\[\frac{\partial}{\partial y} \lambda_N = \frac{\partial}{\partial y} (-d(y, \dot{y}) \dot{y} - k(y, \dot{y}) y - g) < 0.\]  (31)
Given that the ball is landing with $\dot{x}_0 > 0$, in the horizontal direction we will require that the stiffness increases when the ball is moving to the right, and decreases when the ball reverses (moves to the left):
\[
\text{sign} \left( u_\dot{x}(x, \dot{x}, y, \dot{y}, \omega) \right) < 0 = \text{sign} \ (\dot{x}) .
\] (32)

Our analysis is motivated by the case of a rapid ball bounce, which we suppose to take $O(\varepsilon)$ amount of time, where $\varepsilon \ll 1$. We thus introduce a re-scaling of time
\[
\tau = \varepsilon^{-1} t
\] (33)
and with it, we denote state variables in the new variable as $X(\tau) = x(t), Y(\tau) = y(t), \Omega(\tau) = \omega(t)$. To ensure the appropriate balance of terms in (28) we now require
\[
k(y, \dot{y}) = \varepsilon^{-1} \kappa(Y, Y'),
d(y, \dot{y}) = \varepsilon^{-2} \delta(Y, Y'),
\] (34)
where $' = \frac{d}{d\tau}$ denotes the derivative with respect to the new time variable $\tau$, and $\kappa(Y, Y'), \delta(Y, Y') \sim O(1)$ are general functions.

At the same time, we anticipate a much greater stiffness in the horizontal direction than in the vertical section. One can think of this requirement as avoiding the case of a “jelly-like” surface, where the symmetry in vertical and horizontal direction would causes an outbound trajectory, in the absence of spin, to be the time-reverse of the incoming trajectory, irrespective of the angle of incidence with the surface. We will thus introduce a distinguished scaling so that the damping and stiffness function in the horizontal directions to be
\[
v(X, X', Y, Y', \Omega) = u(x, \dot{x}, y, \dot{y}, \omega) \sim O(1),
\]
\[
\zeta(X, X', Y, Y', \Omega) = z(x, \dot{x}, y, \dot{y}, \omega) \sim O(1).
\] (35)

Our final requirement on the rescaled function $\kappa$, $\zeta$, $\delta$ and $\upsilon$ is that their (partial) derivatives with respect to the state variables be at most of $O(1)$. That is we do not expect abrupt changes in the material constants with displacement or velocity. That is, taking $q$ to be any of the state variables $X, X', Y, Y', \Omega$ we have that
\[
\frac{\partial}{\partial q} \upsilon, \frac{\partial}{\partial q} \zeta, \frac{\partial}{\partial q} \delta, \frac{\partial}{\partial q} \kappa \sim O(1).
\] (36)

Following the rescaling and the introduction of new functions, the resulting dynamical system for modelling the bounce becomes
\[
X'' + \varepsilon \upsilon(X, X', Y, Y', \Omega) X' + \varepsilon^2 \zeta(X, X', Y, Y', \Omega) X = \Lambda_T,
\]
\[
Y'' + \delta(Y, Y') Y' + \kappa(Y, Y') Y = -\varepsilon^2 g,
\]
\[
\Omega' = \frac{5}{2} \Lambda_T,
\] (37)
where \( \Lambda_T = \varepsilon^2 \lambda_T \).

The conditions introduced in (30), (31) and (32) can now be written in the form

\[
\begin{align*}
&v_Y(X, X', Y, Y', \Omega) < 0, \\
&\delta_Y(Y, Y') < 0, \\
&\text{sign} (v_X(X, X', Y, Y', \Omega)) = \text{sign} (X')
\end{align*}
\]

and

\[
\frac{\partial}{\partial Y} \Lambda_N = \frac{\partial}{\partial y} \left( -\delta(Y, Y') Y' - \kappa(Y, Y') Y - \varepsilon^2 g \right) < 0.
\]

The Coulomb friction law once again defines our dynamics for the slipping motion, where

\[
\Lambda_T = -\text{sign} (H(p)) \Lambda_N,
\]

where

\[
H(p) = X' + \Omega \quad \text{and} \quad \Lambda_N = - (\delta Y' + \kappa Y + \varepsilon^2 g),
\]

and \( p \) is once again the five-dimensional vector of dynamical coordinates defined by (18). Compared to the simple analysis in Section 3 additional consideration is needed for the rolling motion, when \( H(p) = X' + \Omega = 0 \). Precise study of the onset and loss of rolling motion and their implication for the overall dynamics will be our main interest in what follows.

### 4.1 The two-fold singularity

Let us consider again the possibility of lift off whilst rolling. Recall that at lift off \( \Lambda_N = 0 \) and as such the definition of friction dictates that we must also have \( \Lambda_T = 0 \). At the same time, to agree with the formulation as a Filippov System, the lift off point must coincide with the lower dimensional spaces where \( \alpha(p) = 0 \) and \( \alpha(p) = 1 \). Such point is denoted as \( s \) in Figure 9.

The point where the two switching surfaces (\( \alpha = 0 \) and \( \alpha = 1 \)) coincide is known in the literature as the two-fold singularity [24]. It is now of our interest to determine whether such point is attracting or not. Indeed, an attracting two-fold singularity (which we will from now on denote as point \( s \)), will mean that rolling trajectories will tend towards it, therefore reaching lift off with \( H(p) = 0 \). Alternatively, if \( s \) is not attracting, there will only be a finite, number of lower-dimensional trajectories tending to \( s \).

A full unfolding of two-fold singularity can be found in [24] with the formal results following from Theorems 6.1 & 6.2 in the aforementioned book. We look at the case of our interest only in this paper.

Let us consider an \( n \)-dimensional discontinuous system

\[
p' = \begin{cases} 
F_1(p) & \text{if } H(p) > 0, \\
F_2(p) & \text{if } H(p) < 0,
\end{cases}
\]

where \( H(p) = 0 \) is the discontinuous surface, along which \( p' = F_s(p) = (1 - \alpha(p)) F_1(p) + \alpha(p) F_2(p) \), with \( \alpha \) defined by Equation (21) as before.

Suppose the system possesses two switching surfaces \( \alpha(p) = 0 \) (equivalent with \((F_1 \cdot \nabla) H(p)\)) and \( \alpha(p) = 1 \) (equivalent with \((F_2 \cdot \nabla) H(p)\)) which intersect at a point \( s \).

We define new coordinates:

\[
\begin{align*}
&z_1 = \theta(p) H(p), \\
&z_2 = -\varphi(p) F_1 \cdot \nabla (\theta(p) H(p)), \\
&z_3 = \frac{F_2 \cdot \nabla (\theta(p) H(p))}{\varphi(p)},
\end{align*}
\]

where

\[
\theta(p) = \left| (F_1 \cdot \nabla)^2 H(p) (F_2 \cdot \nabla)^2 H(p) \right|^{-1/2}, \quad \varphi(p) = \left| \frac{F_2 \cdot \nabla^2 H(p)}{(F_1 \cdot \nabla)^2 H(p)} \right|^{1/4}.
\]

We note that \( z_1 = 0 \) is precisely the discontinuity surface of (4.1) with \( z_2 = 0 \) being equivalent to the switching surface \( \alpha = 0 \) and \( z_3 \) being equivalent to \( \alpha = 1 \). Thus, the two-fold singularity is precisely the
n - 3 dimensional manifold \( \{ (z = z_1, \ldots, z_n) : z_1 = z_2 = z_3 = 0 \} \), with \( z_4, \ldots, z_n \) defined to be any \( n - 3 \) dimensional coordinates orthogonal to \( z_1, z_2 \) and \( z_3 \).

In deriving the normal form of the dynamics we apply a time rescaling \( t \mapsto \varphi t \) when \( z_2 < 0 \) and \( t \mapsto t/\varphi \) when \( z_2 > 0 \). Since \( \varphi > 0 \) the phase portrait remains unchanged under this transformation.

With details and series expansions presented in the original source, one finds that in the neighbourhood of the two-fold singularity, under the transformations outlined before, the flow becomes

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2 \\
\dot{z}_3 \\
\dot{z}_4, \ldots, n
\end{bmatrix} = \begin{bmatrix}
-\sigma_1, \nu_1 \\
\sigma_2, \nu_2 \\
\sigma_3, \nu_3
\end{bmatrix} \quad \text{if} \quad z_1 > 0
\]
\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2 \\
\dot{z}_3 \\
\dot{z}_4, \ldots, n
\end{bmatrix} = \mathcal{O}(|z|), \quad \text{if} \quad z_1 < 0
\]

where

\[
\sigma_i = \text{sign} \left( (F_i \cdot \nabla)^2 H(s) \right) \quad \text{and}
\]

\[
\nu^1 = \frac{(F_1 \cdot \nabla) (F_2 \cdot \nabla) H}{\sqrt{(F_1 \cdot \nabla)^2 H \cdot (F_2 \cdot \nabla)^2 H}} \bigg|_{z=s}, \quad \nu^2 = -\frac{(F_2 \cdot \nabla) (F_1 \cdot \nabla) H}{\sqrt{(F_1 \cdot \nabla)^2 H \cdot (F_2 \cdot \nabla)^2 H}} \bigg|_{z=s}
\]

As discussed in [24] the quantities \( \nu_{1,2} \) characterise the local curvature of the flow, with \( \nu_1 \nu_2 \) quantifying the jump in the vector field \( F_{1,2} \) and the singularity.

With the new variables, along the discontinuity surface \( z_1 = 0 \) we define (as before)

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2 \\
\dot{z}_3 \\
\dot{z}_4, \ldots, n
\end{bmatrix} = (1 - \alpha) \begin{bmatrix}
-\sigma_2, \nu_1 \\
\sigma_1, \nu_2, \sigma_2
\end{bmatrix} + \alpha \begin{bmatrix}
\nu_2, -\sigma_1, \nu_1 \\
\sigma_2, \nu_1
\end{bmatrix} \begin{bmatrix}
0 \\
0
\end{bmatrix}, \quad \text{if} \quad z_1 < 0
\]

where \( \alpha \) is solved so that \( \dot{z}_1 = z_1 = 0 \). To the leading order the dynamics in (47) are independent of \( z_4, \ldots, n \). Therefore, the dynamics in \( z_2 \) and \( z_3 \) can be separated and, with \( \alpha \) evaluated explicitly, we have the two-dimensional system

\[
\begin{bmatrix}
\dot{z}_2 \\
\dot{z}_3
\end{bmatrix} = \frac{1}{z_2 + z_3} \begin{bmatrix}
\nu_2 & -\sigma_1 \\
\sigma_2 & \nu_1
\end{bmatrix} \begin{bmatrix}
z_2 \\
z_3
\end{bmatrix}. \quad \text{(48)}
\]

The two switching surfaces (or folds) \( z_2 \) and \( z_3 \) will divide the vector field along the discontinuity \( z_1 = 0 \) into four regions

\[
\begin{align*}
D_{\text{att.}} &= \{ z : z_1 = 0; z_2, z_3 > 0 \}, \\
D_{\text{rep.}} &= \{ z : z_1 = 0; z_2, z_3 < 0 \}, \\
D_{C^1} &= \{ z : z_1 = 0; z_2 < 0 < z_3 \}, \\
D_{C^2} &= \{ z : z_1 = 0; z_3 < 0 < z_2 \}.
\end{align*}
\]

With such definition, the vector field outside of the discontinuity \( z_1 = 0 \) is directed into \( D_{\text{att.}} \), away from \( D_{\text{rep.}} \), and crosses \( D_{C^1} \) in the direction of increasing \( z_1 \), and crosses \( D_{C^2} \) in the direction of decreasing \( z_1 \).

According to [25], three types of two-fold singularity can be distinguished based on the nature of the flow outside of the discontinuity surface, specifically the nature of the grazing tangencies at the sliding region; see Fig[12] Each of these cases are analysed in detail, here we analyse only the case that is relevant to the ball-bounce problem, namely the visible case.

### 4.2 Visible two-fold singularity

When \( \sigma_1 > 0 > \sigma_2 \) then \( s \) is the intersection of visible folds, and we call such point a visible two-fold. Trajectories away from the discontinuity are presented in Figure[12a]. In this case (48) becomes

\[
\begin{bmatrix}
\dot{z}_2 \\
\dot{z}_3
\end{bmatrix} = \frac{1}{z_2 + z_3} \begin{bmatrix}
\nu_2 & -1 \\
-1 & \nu_1
\end{bmatrix} \begin{bmatrix}
z_2 \\
z_3
\end{bmatrix}. \quad \text{(50)}
\]

Understanding the nature of the fixed point \( (0, 0) \) is a straightforward, albeit laborious, exercise in algebra. Detailed derivation can be found in [23 Chapter 13.4], we give the simple summary here.
Figure 12: Classification of two-fold singularities, depending on the flow outside of the discontinuity surface. (a) Visible two-fold (interpreted for the case of a ball bounce, with the lift-off surface indicated). Here, red trajectories correspond to flow with $H > 0$ and green trajectories with $H < 0$. (b,c) Adaptions to the above figure to deal with the (b) invisible two-fold and (c) the visible-invisible two-fold. Figure redrawn based on images in [21].
The linearised system \[ (\nabla \cdot F_1)^2 H(s) = \frac{7}{2} \mu \frac{\partial}{\partial Y'} (\Lambda_N) Y' + O(\varepsilon), \]

where we have used that \( \Lambda_N(s) = 0 \) together with the constraint on the size of the derivatives \[36\]. Noting that \( s \) can only be reached in the restitution phase (i.e. when \( Y' > 0 \)) we see that \( (\nabla \cdot F_1)^2 H(s) > 0 \), confirming thus that a trajectory leaving the rolling surface \( H = 0 \) through the switching region \( \alpha = 0 \) will indeed enter the \( F_1 \) vector field as expected. A similar direct calculation shows that

\[ (\nabla \cdot F_2)^2 H(s) = \frac{7}{2} \mu \frac{\partial}{\partial Y'} (\Lambda_N) Y' + O(\varepsilon), \]

which in turns confirms a consistent behaviour when leaving the rolling surface \( H = 0 \) through the switching surface \( \alpha = 1 \).

Recalling that the classification of the two-fold singularity depends on the quantities \( \sigma_{1,2} \) (as defined by

Figure 13: Phase portraits for the visible two-fold singularity. (a) case where \( \nu_1 \nu_2 < 1 \) or when \( \nu_{1,2} < 0 \) and \( \nu_1 \nu_2 > 1 \). Reproduced from \[24\].
Equation (45) from Equations (51) and (52) we find that
\[ \sigma_1 = \text{sign} \left( (\nabla \cdot F_1)^2 H(s) \right) \]
\[ = \text{sign} \left( -\frac{7\mu}{2} \partial_y (\Lambda_N) Y' + \mathcal{O}(\varepsilon) \right) = 1 \]  
\[ \sigma_2 = \text{sign} \left( (\nabla \cdot F_2)^2 H(s) \right) \]
\[ = \text{sign} \left( \frac{7\mu}{2} \partial_y (\Lambda_N) Y' + \mathcal{O}(\varepsilon) \right) = -1. \]  

We thus have \( \sigma_1 > 0 > \sigma_2 \), which indeed confirms from (28) that this is an is a visible two-fold singularity.

Let us now determine \( \nu_1, \nu_2 \). From (46) we have that
\[ \nu_1 \nu_2 = 1 + 20 \frac{\partial^{(v)}(v)}{\partial Y} \frac{X'}{2} Y' g \varepsilon^3 + \mathcal{O}(\varepsilon^4). \]  

By our assumption (39) the denominator in (54) is negative. Furthermore, the condition introduced in (38) dictates that \( \frac{\partial^{(v)}(v)}{\partial Y} X' > 0 \), therefore yielding \( \nu_1 \nu_2 < 1 \), and thus we obtain a phase portrait shown in Figure 13a. This is the case where \( s \) is a non-attracting point, with only a single trajectory passing through it within the normal form.

Interpreting our result with respect to the ball bounce, we note that the ball that enters rolling during the bounce (governed by the aforementioned conditions) can only leave with a roll along a codimension-one manifold within the space of initial conditions. Thus, with probability 1, the ball must leave slipping.

4.3 Rigid bounce revisited

Let us briefly return to the case of the rigid bounce studied in 2.2. We concluded the studies in that section with a result, where a ball that entered rolling during the bounce could no longer enter the slip. One could think that this contradicts our main result here — but the case is more subtle than that.

The rigid bounce is a limiting case of the model studied in this paper, where the tangential compliance tends to zero; \( \varepsilon \to 0 \). This, in turn, means that the quantity \( \nu_1 \nu_2 = 1 \), and hence the normal form of the visible two-fold singularity given by (50) is based on a singular matrix. Classification is thus more convoluted than that for the general case, but has already been studied in detail in [25, chapter 13] and is known as the diabolo bifurcation.

Rather than studying the singularity for that case in detail, one can notice interesting geometries about that case. Since the lack of tangential compliance affects the tangential force during the roll, the Filippov formulation should lead to an “equal” contribution from the two vector fields \( F_1 \) and \( F_2 \). Explicit calculation of the function \( \alpha(p) \) as given by Equation (21) shows \( \alpha = 1/2 \) throughout the roll, and thus \( F_s(p) = (F_1(p) + F_2(p))/2 \).

The geometry on the friction cone is thus quite simple — all trajectories that enter the discontinuity (roll) region land precisely on the attracting trajectory that leads to a lift off with roll. In other words — that attracting trajectory is the only trajectory allowed in the rolling region for the case of the rigid bounce under Filippov’s formulation.

5 Conclusion

Inspired by empirical observations on the experimental data from [77], we used Filippov theory to analyse ball bounce within generalised point contact models that have both normal and tangential compliance. Under physically realistic scaling laws, we show that a rigid spinning sphere will typically lift off with some non-zero relative tangential velocity. The case of rolling lift off forms the boundary case between forward and backward slipping cases, and therefore should not be observed in practice. This is contrary to a much more well studied and well understood problem of a rigid ball bouncing off a rigid surface. We have identified the underlying principle of the slip and roll transition, which has not been considered in detail before, and a careful understanding of that will be paramount to future modelling.
We are now in possession of what is thought to be the largest data set on the given problem, however fitting the models to the data remains to be an issue. Indeed, the focus of our analysis, that is the discontinuity, makes it difficult to find a structured and widely applicable ways of fitting the models to the data.

With appropriate measurements, is easy to decide whether the observed bounce trajectory entered roll at some point or not. One of the most challenging tasks of the modelling will thus be to solve that problem in the initial condition space. That is, given the data, future work will be aimed at identifying the lower dimensional divide of the space which will identify the initial condition that will lead to the bounce with slip only and those that will enter gripping at some point.

The aim of identifying such key features is to allow for more precise modelling of the compliant surface, in particular in the game of golf. A particular basis of nonlinear functions can be selected and thus fitted with appropriate parameters using the data available. Avoiding high-dimensional problems or finite element solutions remains an open problem.

A part of the problem at hand is identifying parameters together with their physical meaning and the ways to measure them. The golf industry currently tends to quantify the ground’s firmness and stiffness using either USGA TruFirm (Turf-thumper) device or the Clegg Hammer – the measurements performed by these are centred around measuring the deceleration of a free-falling object. There is some evidence, however, showing that behaviour of a ball on two turfs with similar measurements can be significantly different, thus suggesting that important properties are not captured by either of the tools [2].

Some immediate extensions of the problem considered are evident and present an interesting challenge. One could possibly resolve more complex contacts such as Hertzian using the current formulation therefore extending the current study to the models of elastic half spaces [3 21]. A further idea to explore would also involve a side spin, that is a spin off. This could present a generalisation of the problem to many other disciplines, applicable to a further ball-sports but also a wider set of impact-problems in general. Both of the aforementioned extensions would require the extension of modelling friction in 3D, a problem that presents a far greater order of complexity and requires a more considerate approach, whether through the extension of the current models or the regularisation of such in higher dimensions – see e.g. [1, 12].

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References

[1] M. Antali and G. Stepan. Nonsmooth analysis of three-dimensional slipping and rolling in the presence of dry friction. Nonlinear Dynamics, 97:1799–1817, 2019.

[2] Stepbern Baker, Andrew Owen, and Anthony Woollacott. An assessment of the USGA Turf–Thumper and comparison with impact measurements made with two Clegg impact hammers. Sports Turf Research Institute, 2005.

[3] J.R. Barber. Contact Mechanics. Springer, 2018.

[4] J.A. Batlle. On Newton’s and Poisson’s rules of percussive dynamics. J. Appl. Mech., 60:376–381, 1993.

[5] M. Di Bernardo, C.J. Budd, A.R. Champneys, and P. Kowalczyk. Piecewise-smooth dynamical systems: theory and applications, volume 163. Springer Science & Business Media, 2008.

[6] S.W. Biber. Investigation into the golf ball and turf interaction – a study on friction and impact problems, 2022. Phd Thesis. In preparation.
[7] S.W. Biber, K. Jones, A.R. Champneys, and R. Szalai. Measurements and linearized models for golf ball bounce, 2022. In preparation.

[8] M.R. Brake. An analytical elastic-perfectly plastic contact model. *International Journal of Solids and Structures*, 49(22):3129–3141, 2012.

[9] B. Brogliato. *Nonsmooth Mechanics: Models, Dynamics and Control*. Springer, 3rd edition, 2018.

[10] M.J. Carré, D.M. James, and S.J. Haake. Impact of a non-homogeneous sphere on a rigid surface. *Proceedings of the Institution of Mechanical Engineers, Part C: Journal of mechanical engineering science*, 218(3):273–281, 2004.

[11] A.R. Champneys and P.L. Varkonyi. The Painlevé paradox in contact mechanics. *IMA J. App. Math.*, 81:538–588, 2016.

[12] Noah Cheesman, S John Hogan, and Kristian Uldall Kristiansen. The geometry of the Painlevé paradox. *SIAM Journal on Applied Dynamical Systems*, 21(3):1798–1831, 2022.

[13] L.P. Cordingley. Advanced modelling of surface impacts from hollow sports balls. *Ph.D. Thesis, Loughborough University*, 2002.

[14] R. Cross. Backward bounce of a spinning ball. *European Journal of Physics*, 39(4):045007, 2018.

[15] Phil R Dahl. A solid friction model. Technical report, Aerospace Corp El Segundo Ca, 1968.

[16] C.B Daish. *The physics of ball games*. Hodder and Stoughton, London, 1981.

[17] C Canudas De Wit, Henrik Olsson, Karl Johan Astrom, and Pablo Lischinsky. A new model for control of systems with friction. *IEEE Transactions on automatic control*, 40(3):419–425, 1995.

[18] A.F. Filippov. *Differential equations with discontinuous righthand sides: control systems*, volume 18. Springer Science & Business Media, 1988.

[19] W.N. Findley and F.A. Davis. *Creep and relaxation of nonlinear viscoelastic materials*. Courier corporation, 2013.

[20] H. Ghaednia, X. Wang, S. Saha, Y. Xu, A. Sharma, and R.L. Jackson. A review of elastic–plastic contact mechanics. *Applied Mechanics Reviews*, 69(6), 2017.

[21] S.J. Haake. An apparatus for measuring the physical properties of golf turf and their application in the field. *Ph.D. Thesis, The University of Aston in Birmingham*, 1989.

[22] DP Hess and A Soom. Friction at a lubricated line contact operating at oscillating sliding velocities. 1990.

[23] N. Hoffmann and L. Gaul. Effects of damping on mode-coupling instability in friction induced oscillations. *Z. Angew. Mat. Mech.*, 83:524–534, 2003.

[24] M.R. Jeffrey. An update on that singularity. In *Extended Abstracts of the CRM, Spring 2016*, pages 107–112. Springer, 2017.

[25] M.R. Jeffrey. *Hidden dynamics*. Springer, 2018.

[26] K.U. Kristiansen and S.J. Hogan. Le canarde de Painlevé. *SIAM J. App. Dyn. Sys.*, 17:859–908, 2018.

[27] A. Nordmark, H. Dankowicz, and A.R. Champneys. Discontinuity-induced bifurcations in systems with impacts and friction: Discontinuities in the impact law. *Int. J. Nonlinear Mechanics*, 44:1011–1023, 2009.

[28] A. Nordmark, P.L. Varkonyi, and A.R. Champneys. Dynamics beyond dynamic jam; unfolding the Painlevé paradox singularity. *SIAM J. App. Dynamical Sys.*, 17:1267–1309, 2018.
[29] A.R. Penner. The run of a golf ball. *Canadian Journal of Physics*, 80(8):931–940, 2002.

[30] T. Putelat, J.H.P. Dawes, and A.R. Champneys. A phase-plane analysis of localized frictional waves. *Proc. Roy. Soc. Lond. A*, 473:Art. no. 20160606, 2017.

[31] S.J. Quintavalla. A generally applicable model for the aerodynamic behavior of golf balls, 2002. Science and Golf IV, Routledge.

[32] W.-J. Roh and C.-W. Lee. Golf ball landing, bounce and roll on turf. *Procedia Engineering*, 2:3237–3242, 2010.

[33] W.J. Stronge. *Impact mechanics*. Cambridge University Press, 2000.