Zeta Functions of Projective Toric Hypersurfaces over Finite Fields

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Abstract

I give a formula for the zeta function of a projective toric hypersurface over a finite field and estimate its Newton polygon. As an application this formula allows us to compute the exact number of rational points on the families of Calabi-Yau manifolds in Mirror Symmetry.

1 Introduction

Let $\mathbb{F}_q$ be a finite field of $q$ elements where $q$ is a power of a prime $p$. Let $f = \sum_{j \in J} a_j x^j \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be a Laurent polynomial over $\mathbb{F}_q$, where the sum is over a finite subset $J$ of $\mathbb{Z}^n$. We assume $a_j \neq 0$ for all $j \in J$. Let $U_f$ be the affine hypersurface in the $n$-torus $\mathbb{G}_m^n := (\mathbb{F}_q^*)^n$ defined by $\{f = 0\}$. We can construct a projective toric variety $\mathbb{P}_\Delta$ from the polytope $\Delta(f)$, the convex hull in $\mathbb{R}^n$ of the lattice points which occur as exponents of the monomials of $f$ together with the origin. Assume $\dim \Delta(f) = n$. According to the theory of toric varieties, $\mathbb{P}_\Delta$ is a compactification of the $n$-dimensional torus $\mathbb{G}_m^n$ by algebraic tori $\mathbb{G}_m^{\dim \sigma}$ of smaller dimensions. Each $\mathbb{G}_m^{\dim \sigma}$ corresponds to a nonempty face $\sigma \subseteq \Delta(f)$ [4].

This paper gives a formula for the zeta function of the projective closure $\overline{U_f}$ of the affine toric hypersurface $U_f$ in $\mathbb{P}_\Delta$ over a finite field $\mathbb{F}_q$ when $\Delta(f)$ is a simplex. This result extends Dwork’s work on a smooth, projective hypersurface in $\mathbb{P}^n$.

It is natural to begin with the zeta function on an affine hypersurface $U_f$ on $\mathbb{G}_m^n$ defined by $f$, and extend via the usual toric decomposition of $\mathbb{P}_\Delta$ to the zeta function of the toric projective closure $\overline{U_f}$. The zeta function of the toric projective closure of...
an affine toric hypersurface is determined by the product of the $L$-functions associated to each face $\sigma$ of $\triangle(f)$.

Our methods are $p$-adic and are based on the work of Dwork [5], [6]. The main theorem (Theorem 4.7) is the extension of Dwork’s cohomology theory from smooth, projective hypersurfaces in characteristic $p$ to $\triangle$-regular, toric projective hypersurfaces with simplex $\triangle(f)$. Given a Laurent polynomial $f$ on $\mathbb{G}^n_m$, we construct a complex of a $p$-adic Banach space $B$ on which a Frobenius operator $\alpha$ acts. In fact, the Banach space $B$ is generated by the monomials of $x_0 f$. The exponents of the monomials of $B$ can be identified with the lattice points in the monoid $M(x_0 f)$ generated by the vertices of $1 \times \triangle(f)$ and the origin in $\mathbb{R}^{n+1}$. Dwork trace formula implies that $L(x_0 f, t)^{(-1)^n}$ is the alternating product of the characteristic polynomial of the Frobenius operator $\alpha$ which acts on the $p$-adic Koszul complex induced by $\triangle(f)$. We have the same result for each face $\sigma$ of $\triangle(f)$.

The main observation is that when $\triangle(f)$ is a simplex, there is a bijection between nonempty faces of $\triangle(f)$ and nonempty subsets of the vertices of $\triangle(f)$. The product of the $L$-functions is the alternating product of the characteristic polynomials of Frobenius operators corresponding to each nonempty face. An inclusion-exclusion argument shows that the alternating product is determined by the subcomplex corresponding to the interior cone of the monoid $M(x_0 f)$. Passing to homology and using the acyclicity of $\triangle$-regularity, we have

$$Z(U_f, qt) = \frac{P(t)^{(-1)^n}}{(1 - qt) \cdots (1 - q^n t)}$$

where $P(t)$ is a polynomial of degree $(-1)^{n+1} \sum_{\sigma \subseteq \triangle(f)} (-1)^{\dim \sigma} (\dim \sigma)! \text{Vol}(\sigma)$.

As a consequence, we are able to bound below the Newton polygon of $P(t)$ by its Hodge polygon and their endpoints coincide. If $\triangle(f)$ is not a simplex, we do not expect that equation (1) holds in general.

Equation (1) can be used to compute the number of rational points on a mirror pair of Calabi-Yau hypersurfaces in arithmetic mirror symmetry when $\triangle(f)$ is a reflexive simplex.

This paper is organized as follows. In section 2 and 3, we recall some basic technique developed from Dwork [5], [6], Adolphson and Sperber [1], [2]. In section 4, we prove an analogous formula of the zeta function of projective toric hypersurfaces for simplex case. In section 5, we give a sharp lower bound for Newton polygon of the
zeta function. In section 6, we compute the zeta functions of the families of Calabi-Yau hypersurfaces.

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2 \textit{$p$-adic theory}

Let $p$ be a prime number, $\mathbb{Q}_p$ the field of $p$-adic numbers and $\mathbb{Z}_p$ the ring of $p$-adic integers. Let $\Omega$ be the completion of an algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{F}_q$ be the finite field of $q = p^a$ elements, $\mathbb{F}_q^k$ the extension of $\mathbb{F}_q$ of degree $k$, and denote by $K$ the unramified extension of $\mathbb{Q}_p$ in $\Omega$ of degree $a$. Let $\pi \in \Omega$ satisfy $\pi^p - 1 = -p$. Then $\Omega_1 = \mathbb{Q}_p(\pi)$ is a totally ramified extension of $\mathbb{Q}_p$ of degree $p^{a-1}$, in fact, $\Omega_1 = \mathbb{Q}_p(\zeta_p)$, where $\zeta_p$ is a primitive $p$-th root of unity. Let $\Omega_0 = K(\pi)$ be the compositum of $\Omega_1$ and $K$. Then $\Omega_0$ is an unramified extension of $\Omega_1$ of degree $a$. The residue class fields of $\Omega_0$ and $K$ are both $\mathbb{F}_q$, and the residue fields of $\Omega_1$ and $\mathbb{Q}_p$ are both $\mathbb{F}_p$. The Frobenius automorphism $x \mapsto x^p$ of $Gal(\mathbb{F}_q/\mathbb{F}_p)$ lifts to a generator $\tau$ of $Gal(\Omega_0/\Omega_1)(\cong Gal(K/\mathbb{Q}_p))$ which is extended to $\Omega_0$ by requiring $\tau(\pi) = \pi$. If $\zeta$ is a $(q-1)$-th root of unity in $\Omega_0$, then $\tau(\zeta) = \zeta^p$. Denote by “ord” the additive valuation on $\Omega$ normalized by $\text{ord} p = 1$, and denote by “ord$_q$” the additive valuation normalized by $\text{ord}_q q = 1$.

Let $f$ be a Laurent polynomial on the torus $\mathbb{G}_m^n := (\mathbb{F}_q^*)^n$ and write

$$f = \sum_{j \in J} a_j x^j \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}],$$

where the sum is over a finite subset $J$ of $\mathbb{Z}^n$. We may assume $a_j \neq 0$ for all $j \in J$. Each Laurent polynomial $f$ defines an affine toric hypersurface

$$U_f = \{(x_1, \ldots, x_n) \in \mathbb{G}_m^n | f(x_1, \ldots, x_n) = 0\}$$

in $\mathbb{G}_m^n$.

Define the polytope $\Delta := \Delta(f)$ of $f$ to be the convex hull in $\mathbb{R}^n$ of the lattice points which occur as exponents of the monomials of $f$ together with the origin. We can construct a toric variety $\mathbb{P}_\Delta$ from the polytope $\Delta(f)$ as follows:
\textbf{Definition.} The polytope ring $S_\Delta$ of a polytope $\Delta$ is defined by

$$S_\Delta = \mathbb{F}_q[x_0^r x^m : m = (m_1, \ldots, m_n) \in r \Delta \cap \mathbb{Z}^n, r \geq 0],$$

the subalgebra of $\mathbb{F}_q[x_0, x_1^\pm 1, \ldots, x_n^\pm 1]$ generated as a $\mathbb{F}_q$-vector space by 1 and all monomials $x_0^r x^m = x_0^r x_1^{m_1} \cdots x_n^{m_n}$ where $r \in \mathbb{N}$ such that the rational point $m/r = (m_1/r, \ldots, m_n/r)$ belongs to $\Delta$.

The standard grading of the polynomial ring $\mathbb{F}_q[x_0, x_1^\pm 1, \ldots, x_n^\pm 1]$ in $x_0$ induces the grading of $S_\Delta$:

$$\text{deg}(x_0^r x^m) = r$$

and the graded commutative $\mathbb{F}_q$-algebra

$$S_\Delta = \bigoplus_{r \geq 0} S^r_\Delta,$$

where $S^r_\Delta$ is a finite dimensional $\mathbb{F}_q$-vector space with the basis $x_0^r x^m$ ($m \in r \Delta \cap \mathbb{Z}^n$).

Let $\Delta(x_0f) \subseteq \mathbb{R}^{n+1}$ be the convex hull of the origin in $\mathbb{R}^{n+1}$ and $(1, v)$ where $v \in \Delta(f)$. Define the monoid $M(x_0f) \subseteq \mathbb{Z}^{n+1}$ to be the lattice points generated by $\Delta(x_0f)$. It can be identified with the exponents of the monomials of the polytope ring $S_\Delta$.

We now can associate with the polytope $\Delta \subseteq \mathbb{R}^n$ the projective algebraic variety $\mathbb{P}_\Delta$.

\textbf{Definition.} The algebraic variety

$$\mathbb{P}_\Delta = \text{Proj}(S_\Delta)$$

is called the toric variety associated with polytope $\Delta$.

Given a face $\sigma$ of $\Delta(f)$, we define $\text{dim} \sigma$ to be the dimension of the smallest subspace of $\mathbb{R}^n$ containing $\sigma$. Assume $\text{dim} \Delta(f) = n$. Define $\text{Vol}(f)$ to be the volume of $\Delta(f)$ with respect to Lebesgue measure on $\mathbb{R}^n$.

Toric variety $\mathbb{P}_\Delta$ is a compactification of the $n$-dimensional torus $\mathbb{G}_m^n$ by algebraic tori $\mathbb{G}_m^{\text{dim } \sigma}$ of smaller dimensions. Each $\mathbb{G}_m^{\text{dim } \sigma}$ corresponds to a nonempty face $\sigma \subseteq \Delta$ of the polytope $\Delta$ [4]. That is, we have the so called toric decomposition

$$\mathbb{P}_\Delta = \bigsqcup_{\emptyset \neq \sigma \subseteq \Delta} \mathbb{G}_m^{\text{dim } \sigma}.$$
Let $U_f$ be the affine hypersurface in $\mathbb{G}_m^n$ defined by $f$. Define $\overline{U_f} := \text{Proj}(S_\Delta/(x_0f))$. It is the projective closure of $U_f$ in $\mathbb{P}_\Delta$ defined over $\mathbb{F}_q$. For any face $\sigma \subseteq \Delta$, we obtain an affine hypersurface $U_{f_\sigma} = \overline{U_f} \cap \mathbb{G}_m^{\dim \sigma}$ in the algebraic torus $\mathbb{G}_m^{\dim \sigma}$ defined by $f_\sigma = \sum_{j \in \sigma \cap J} a_j x_j^j$. Since $\overline{U_f}$ lies in $\mathbb{P}_\Delta$, we still have the toric decomposition $U_f = \bigcup_{\emptyset \neq \sigma \subseteq \Delta} U_{f_\sigma}$.

Let $V$ be a variety over $\mathbb{F}_q$, $V(\mathbb{F}_q)$ be the set of $\mathbb{F}_q$-rational points of $V$ and $N_k(V)$ be its cardinality. The zeta function of $V$ is defined to be

$$Z(V/\mathbb{F}_q, t) = \exp \left( \sum_{k=1}^{\infty} N_k(V) \frac{t^k}{k} \right).$$

Let $\Psi : \mathbb{F}_q \rightarrow \mathbb{Q}(\zeta_p)$ be a nontrivial additive character of $\mathbb{F}_q$. Define the exponential sum of $x_0f$ to be

$$S^*_k(x_0f) = \sum_{(x_0,x) \in (\mathbb{F}_q^*)^{n+1}} \Psi \circ \text{Tr}_{\mathbb{F}_q^k/\mathbb{F}_q}(x_0f(x)) \in \mathbb{Q}(\zeta_p),$$

and the associated $L$-function of $x_0f$ to be

$$L^*(x_0f, t) = \exp \left( \sum_{k=1}^{\infty} S^*_k(x_0f) \frac{t^k}{k} \right) \in \mathbb{Q}(\zeta_p)[[t]].$$

By theorem of Dwork-Bombieri-Grothendieck, $L^*(x_0f, t)$ is a rational function of $t$.

**Lemma 2.1.** Let $f \in \mathbb{F}_q^k[x_1^\pm 1, \ldots, x_n^\pm 1]$, $U_f$ the variety defined by $f = 0$ in the torus $\mathbb{G}_m^n$ and $N_k^*(U_f)$ the number of $\mathbb{F}_q$-rational points of $U_f$. Then

$$\sum_{(x_0,x) \in (\mathbb{F}_q^*)^{n+1}} \Psi \circ \text{Tr}_{\mathbb{F}_q^k/\mathbb{F}_q}(x_0f(x)) = q^k N_k^*(U_f) - N_k(\mathbb{G}_m^n)$$

where $x = (x_1, \ldots, x_n)$.

**Proof.** For any $x \in (\mathbb{F}_q^*)^n$, we have

$$\sum_{x_0 \in \mathbb{F}_q^k} \Psi \circ \text{Tr}_{\mathbb{F}_q^k/\mathbb{F}_q}(x_0f(x)) = \begin{cases} q^k & \text{if } f(x) = 0 \text{ (i.e., } x \in U_f), \\ 0 & \text{otherwise}. \end{cases}$$


This is a standard result from the theory of additive character sums. Thus
\[
\sum_{(x_0, x) \in \mathbb{F}_{q^k} \times (\mathbb{F}_{q^k}^*)^n} \Psi \circ \text{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(x_0 f(x)) = q^k N_k^*(U_f).
\]

In fact, \(\sum_{(x_0, x) \in \mathbb{F}_{q^k} \times (\mathbb{F}_{q^k}^*)^n} \Psi \circ \text{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(x_0 f(x))\) is a counting function, counting how many \(\mathbb{F}_{q^k}\)-rational points of \(U_f\). Hence
\[
\sum_{(x_0, x) \in (\mathbb{F}_{q^k}^*)^{n+1}} \Psi \circ \text{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(x_0 f(x)) = q^k N_k^*(U_f) - (q^k - 1)^n
\]
\[
= q^k N_k^*(U_f) - N_k(\mathbb{G}_m^n).
\]

Hence, we have
\[
Z(U_f, qt) = Z(\mathbb{G}_m^n, t) L^*(x_0 f, t).
\]

Applying the above lemma to each face \(\sigma \subseteq \Delta(f)\), we also have the analogous equations
\[
Z(U_f, \sigma, qt) = Z(\mathbb{G}_m^{\dim \sigma}, t) L^*(x_0 f, \sigma, t)
\]
where \(L^*(x_0 f, \sigma, t)\) denotes the corresponding \(L\)-function with respect to \(x_0 f, \sigma\).

Since \(\overline{U_f} = \bigsqcup_{\emptyset \neq \sigma \subseteq \Delta} U_f, \sigma\), \(N_k(\overline{U_f}) = \sum_{\emptyset \neq \sigma \subseteq \Delta} N_k(U_f, \sigma)\). We have
\[
Z(\overline{U_f}, qt) = \prod_{\emptyset \neq \sigma \subseteq \Delta} Z(U_f, \sigma, qt) = \prod_{\emptyset \neq \sigma \subseteq \Delta} Z(\mathbb{G}_m^{\dim \sigma}, t) \prod_{\emptyset \neq \sigma \subseteq \Delta} L^*(x_0 f, \sigma, t).
\]
Thus it suffices to consider \(\prod_{\emptyset \neq \sigma \subseteq \Delta} L^*(x_0 f, \sigma, t)\).

Let \(E(t)\) be the Artin-Hasse exponential series:
\[
E(t) = \exp \left( \sum_{i=0}^{\infty} \frac{t^{p^i}}{p^i} \right) \in (\mathbb{Z}_p \cap \mathbb{Q})[[t]]
\]

Let \(\gamma \in \Omega_1\) be a root of \(\sum_{i=0}^{\infty} \frac{p^i}{p^i} = 0\) satisfying \(\text{ord} \gamma = \frac{1}{(p-1)}\), and consider
\[
\theta(t) = E(\gamma t) = \sum_{i=0}^{\infty} \lambda_i t^i \in \Omega_1[[t]].
\]
The series $\theta(t)$ is a splitting function in Dwork’s terminology \[5\]. In particular, its coefficients satisfy

$$\text{ord} \lambda_i \geq \frac{i}{p-1}, \lambda_i \in \Omega_1.$$ 

Let $O_0$ be the ring of integers of $\Omega_0$. Consider the following spaces of $p$-adic functions (where $b \in \mathbb{R}, b \geq 0, c \in \mathbb{R}$):

$$L(b, c) = \left\{ \sum_{(r, m) \in M(x_0 f)} A_{r,m} x_0^r x_0^m | A_{r,m} \in \Omega_0, \text{ord} A_{r,m} \geq br + c \right\},$$

$$L(b) = \bigcup_{c \in \mathbb{R}} L(b, c),$$

$$B(O_0) = \left\{ \sum_{(r, m) \in M(x_0 f)} A_{r,m} \gamma^r x_0^r x_0^m | A_{r,m} \in O_0, A_{r,m} \to 0 \text{ as } r \to \infty \right\},$$

$$B = \left\{ \sum_{(r, m) \in M(x_0 f)} A_{r,m} \gamma^r x_0^r x_0^m | A_{r,m} \in \Omega_0, A_{r,m} \to 0 \text{ as } r \to \infty \right\}.$$

Observe that if $b > 1/(p-1)$ then $L(b) \subseteq B \subseteq L(1/(p-1))$. If in addition $c \geq 0$, then $L(b, c) \subseteq B(O_0)$. Define a norm on $B$ as follows: If $\xi = \sum_{(r, m) \in M(x_0 f)} A_{r,m} \gamma^r x_0^r x_0^m \in B$, then define

$$\|\xi\| = \sup_{(r, m) \in M(x_0 f)} |A_{r,m}|.$$

Note that $B(O_0)$ is a flat, separated, complete $O_0$-module (\[8\], P.91).

Let $\hat{f} = \sum_{j \in J} \hat{a}_j x^j \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be the Teichmüller lifting of $f$; i.e., $(\hat{a}_j)^q = \hat{a}_j$ and the reduction of $\hat{f}$ mod $p$ is $f$. Let

$$F(f, x_0, x) = \prod_{j \in J} \theta(\hat{a}_j x_0 x^j),$$

$$F_a(f, x_0, x) = \prod_{i=0}^{a-1} F^{\tau^i}(f, x_0^{p^i}, x^{p^i})$$

where the map $\tau$ acts coefficient-wise on the power series $F(f, x_0, x)$. The estimate $\text{ord} \lambda_i \geq i/(p-1)$ implies $F(f, x_0, x)$ and $F_a(f, x_0, x)$ are well-defined as formal Laurent
series in \(x_0, x_1^\pm, \ldots, x_n^\pm\) with coefficients in \(\Omega_0\); in fact
\[
F(f, x_0, x) \in L\left(\frac{1}{p-1}, 0\right), \quad F_a(f, x_0, x) \in L\left(\frac{p}{q(p-1)}, 0\right).
\]

An easy calculation implies that \(L(b)\) is a ring.

Define an operator \(\psi\) on the formal Laurent series by
\[
\psi\left(\sum_{(r,m) \in M(x_0 f)} A_{r,m}x_0^r x^m\right) = \sum_{(r,m) \in M(x_0 f)} A_{pr,pm}x_0^r x^m.
\]
where \(pm\) denotes the \(n\)-tuple \((pm_1, \ldots, pm_n)\). This is the map on the power series which ignores about all \(x_0^r x^m\)-terms for which \(p \nmid (r, m)\) and replaces \(x_0^r x^m\) by \(x_0^{r/p} x^{m/p}\) in the terms for which \(p | (r, m)\). Here \(p | (r, m)\) means that \(p\) divides all of the entries in the integer vector \((r, m)\). One immediately deduces that \(\psi(L(b, c)) \subseteq L(pb, c)\).

Let \(\iota : L(p/(p-1)) \hookrightarrow L(p/q(p-1))\) be the canonical injection and denote by \(\alpha\) the composition
\[
L\left(\frac{p}{p-1}\right) \xrightarrow{\iota} L\left(\frac{p}{q(p-1)}\right) \xrightarrow{F_a(f, x_0, x)} L\left(\frac{p}{q(p-1)}\right) \xrightarrow{\psi^a} L\left(\frac{p}{p-1}\right),
\]
where the middle arrow means “multiplication by \(F_a(f, x_0, x)\)”. It follows that the operator \(\alpha = \psi^a \circ F_a(f, x_0, x)\) is an \(\Omega_0\)-linear endomorphism of the space \(B\) and \(L(b)\) for \(0 \leq b \leq p/(p-1)\). Furthermore, the operator \(\alpha_0 = \tau^{-1} \circ \psi \circ F(f, x_0, x)\) is an \(\Omega_1\)-linear endomorphism of \(B\) and \(L(b)\) for \(0 \leq b \leq p/(p-1)\) and is an \(\Omega_0\)-semilinear endomorphism of those spaces. It follows from Serre [9] that the operators \(\alpha^k\) and \(\alpha_0^k\), acting on \(B\) and \(L(b)\) for \(0 < b \leq p/(p-1)\), have well-defined traces. In addition, the Fredholm determinant \(\det(I - t\alpha)\) and \(\det(I - t\alpha_0)\) are \(p\)-adically entire (i.e., convergent for all \(t \in \Omega\)). Dwork trace formula asserts that
\[
S_k^\alpha(x_0 f) = (q - 1)^{n+1} \text{Tr}(\alpha^k), \quad (2)
\]
where \(\alpha\) acts either on \(B\) or on some \(L(b)\), \(0 < b \leq p/(p-1)\). The non-trivial additive character \(\Psi\) implicit on the left-hand side of (2) is given by
\[
\Psi(t) = \theta(1)^{\text{Tr}_{\mathbb{F}_q/F_p}(t)}
\]
for \(t \in \mathbb{F}_q\) (by [5], Lemma 4.1, \(\theta(1)\) is a primitive \(p\)-th root of unity). Equivalently, one can define an operator \(\delta\) on formal power series with constant term 1 by \(g(t)^\delta = g(t)/g(qt)\), and then (2) takes the form
\[
L^\delta(x_0 f, t)^{-1} = \det(I - t\alpha)^{\delta^{n+1}} \quad (3)
\]

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via the relationship \( \det(I - t\alpha) = \exp(-\sum_{k=1}^{\infty} \text{Tr}(\alpha^k)t^k/k) \).

### 3 Dwork Cohomology

Let \( f \) be a Laurent polynomial. Define the new polynomials \( F(x_0, x) = x_0f(x) - 1 \) and \( F_i(x_0, x) = x_i \frac{\partial}{\partial x_i} F(x_0, x) \) for \( 0 \leq i \leq n \).

**Definition.** A Laurent polynomial \( f \) is called \( \triangle \)-regular if \( \triangle(f) = \triangle \) and for every \( l \)-dimensional face \( \sigma \subseteq \triangle \) (\( l > 0 \)) the polynomial equations

\[
f_\sigma(x) = x_1 \frac{\partial f_\sigma}{\partial x_1} = \ldots = x_n \frac{\partial f_\sigma}{\partial x_n} = 0
\]

have no common solution in \( G^n_m \).

For \( 0 \leq l \leq n \), the sum

\[
\bigoplus_{|T|=l} S_\triangle e_T
\]

runs over all subsets \( T \subseteq \{0, \ldots, n\} \) of cardinality \( l \). If \( T = \{i_1, \ldots, i_l\} \) with \( i_1 < \ldots < i_l \), then \( e_T \) is the abbreviated notation for \( e_{i_1} \wedge \ldots \wedge e_{i_l} \) (the \( \{e_i\}_{0 \leq i \leq n} \) being a set of formal symbols). The boundary map \( \partial_l : \bigoplus_{|T|=l} S_\triangle e_T \to \bigoplus_{|T|=l-1} S_\triangle e_T \) is given by:

\[
\partial_l(\xi e_T) = \sum_{j=1}^{l} (-1)^{j-1} F_j \xi e_{T - \{i_j\}}
\]

where \( \xi \in S_\triangle \) and \( T = \{i_1, \ldots, i_l\} \) with \( i_1 < \ldots < i_l \). We have the Koszul complex \( K_*(f) \) on the elements \( F_0, \ldots, F_n \in S_\triangle \):

\[
K_*(f) : 0 \to \bigoplus_{|T|=n+1} S_\triangle e_T \xrightarrow{\partial_{n+1}} \ldots \to \bigoplus_{|T|=1} S_\triangle e_T \xrightarrow{\partial_1} S_\triangle \to 0.
\]

**Theorem 3.1.** ([3], Theorem 4.8) Let \( f \) be a Laurent polynomial and \( F = x_0f(x) - 1 \). Then the following conditions are equivalent:
1. \( f \) is \( \triangle \)-regular;
(ii) $F$ is $\Delta$-regular;
(iii) the elements \{\(F_0, F_1, \ldots, F_n\)\} give rise to a regular sequence in \(S_\Delta\);
(iv) the homology groups \(H_i(f)\) of the Koszul complex \(K_*(f)\) are zero for positive \(i\);
(v) the dimension of \(H_0(f)\) is \((n + 1)!\text{Vol}(x_0f)\).

Let \(\hat{F}(x_0, x)\) be the Teichmüller lifting of \(F(x_0, x)\). Define \(\hat{F}_i = x_i \partial \hat{F} / \partial x_i\) and \(\gamma_l = \sum_{i=0}^l \gamma^{p^i} / p^i\), which by definition of \(\gamma\) satisfies
\[
\text{ord}_p \gamma_l \geq \frac{p^{l+1}}{p - 1} - l - 1.
\]

For \(i = 0, \ldots, n\), define differential operators \(\hat{D}_i\) by
\[
\hat{D}_i = E_i + \hat{H}_i,
\]
where \(E_i = x_i \partial / \partial x_i\) and
\[
\hat{H}_i(x_0, x) = \sum_{l=0}^{\infty} \gamma^{p^l} \hat{F}_i^{p^l} (x_0^{p^l}, x^{p^l}) \in L \left( \frac{p}{p - 1}, -1 \right).
\]

The \(\hat{H}_i\) and \(\hat{D}_i\) operate on \(B\) and on \(L(b)\) for \(b \leq p/(p - 1)\). One verifies that they commute with one another. Furthermore, by [5] equation (4.35),
\[
\alpha \circ \hat{D}_i = q \hat{D}_i \circ \alpha \tag{4}
\]
for \(i = 0, \ldots, n\) as operators on \(B\) or on \(L(b), 0 < b \leq p/(p - 1)\).

Let \(K_*(B, \{\hat{D}_i\}_{i=0}^n)\) be the Koszul complex on \(B\) formed from \(\hat{D}_0, \ldots, \hat{D}_n\), i.e., for \(0 \leq l \leq n\),
\[
K_l(B) = \bigoplus_{|T| = l} Be_T,
\]
where the sum runs over all subsets \(T \subseteq \{0, \ldots, n\}\) of cardinality \(l\). The boundary map \(\partial_l : K_l \to K_{l-1}\) is given by:
\[
\partial_l(\xi e_T) = \sum_{j=1}^l (-1)^{j-1} \hat{D}_{ij}(\xi)e_{T-\{i_j\}}
\]
where \(\xi \in B\) and \(T = \{i_1, \ldots, i_l\}\) with \(i_1 < \ldots < i_l\). Define an endomorphism \(\alpha_l : K_l \to K_l\) by
\[
\alpha_l = \bigoplus_{|T| = l} q^l \alpha.
\]
Equation (4) implies that $\alpha_*$ is a chain map on the Koszul complex $K_*$. 

\[
\begin{array}{ccccccc}
0 & \longrightarrow & K_{n+1} & \xrightarrow{\partial_{n+1}} & K_n & \xrightarrow{\partial_n} & \cdots & \xrightarrow{\partial_1} & K_0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \alpha_{n+1} & & \alpha_n & & \alpha_1 & & \alpha_0 & & \downarrow \\
0 & \longrightarrow & K_{n+1} & \xrightarrow{\partial_{n+1}} & K_n & \xrightarrow{\partial_n} & \cdots & \xrightarrow{\partial_1} & K_0 & \longrightarrow & 0 \\
\end{array}
\]

It then follows from (3) that 

\[
L^*(x_0 f, t)^{(-1)^n} = \prod_{l=0}^{n+1} \det(I - t\alpha_l|K_l)^{(-1)^l}.
\]

Passing to homology, we conclude that 

\[
L^*(x_0 f, t)^{(-1)^n} = \prod_{l=0}^{n+1} \det(I - tH_l(\alpha_*)|H_l(K_*))^{(-1)^l},
\]

where $H_l(\alpha_*)$ denotes the endomorphism of $H_l(K_*)$ induced by $\alpha_l$.

Some of the main results of [1] are summarized in the following theorem.

**Theorem 3.2.** Let $f \in F_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be $\triangle$-regular and $\dim \triangle(f) = n$. The complex $K_*(B, \{\widehat{D_i}\}_{i=0}^{n})$ is acyclic in positive dimension. In addition, $H_0(K_*(B, \{\widehat{D_i}\}_{i=0}^{n}))$ has dimension $(n+1)!{\text{Vol}}(x_0 f)$ and $H_0(\alpha_*)$ is invertible. Hence 

\[
L^*(x_0 f, t)^{(-1)^n} = \det(I - tH_0(\alpha_*)|H_0(K_*(B, \{\widehat{D_i}\}_{i=0}^{n}))) \in \mathbb{Z}[t]
\]

is a polynomial of degree $(n+1)!{\text{Vol}}(\triangle(f))$.

**Proof.** All statements follows immediately from Theorem 2.9, Theorem 2.18, Corollary 2.19, and Theorem 3.13 of [1].

\[\square\]

### 4 Simplex $\triangle(f)$

Suppose $H$ is a hyperplane in $\mathbb{R}^{n+1}$ passing through the origin: $\sum_{j=0}^{n} a_j x_j = 0$. Suppose in addition the $a_j$ are rational. We denote by $\widehat{D_H}$ the differential operator
on $B$ defined by $\tilde{D}_H = \sum_{j=0}^n a_j \tilde{D}_j$.

Suppose that $\triangle(f)$ is an $n$-dimensional simplex with vertices $\{v_0, \ldots, v_n\} \subseteq \mathbb{R}^n$. Then so is $\triangle(x_0f)$ with vertices $\{(0, \ldots, 0), (1, v_0), \ldots, (1, v_n)\} \subseteq \mathbb{R}^{n+1}$. Let $H_i$ be a hyperplane through the codimension-one faces of $\triangle(x_0f)$ that contain $\{(0, \ldots, 0), (1, v_0), \ldots, (1, v_i), \ldots, (1, v_n)\}$. Then the normal vectors of $H_0, \ldots, H_n$ form a basis in $\mathbb{R}^{n+1}$. We may assume the equation of each $H_i$ has the form

$$\sum_{j=0}^n a_{ij} x_j = 0,$$

(5)

where $a_{ij} \in \mathbb{Z}$ for all $i, j$ and $\{a_{ij}\}_{j=0}^n$ have greatest common divisor 1. The differential operators $\{\tilde{D}_H\}_{i=0}^n$ on $B$ commute with one another and $\alpha \circ \tilde{D}_H = q \tilde{D}_H \circ \alpha$.

**Lemma 4.1.** Let $f \in \mathbb{F}_q[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]$ be $\triangle$-regular and $F(x_0, x) = x_0 f(x) - 1$. Suppose $p \nmid \det(a_{ij})$ where the matrix $(a_{ij})$ is defined as in (5). Then Theorem 3.2 is valid with $\{\tilde{D}_i\}_{i=0}^n$ replaced by $\{\tilde{D}_H\}_{i=0}^n$.

**Proof.** Since for each face $\tau$ of $\triangle(x_0f)$,

$$\left(\sum_{j=0}^n a_{ij} x_j \partial F / \partial x_j\right)_{\tau} = \sum_{j=0}^n a_{ij} x_j \partial F_{\tau} / \partial x_j. $$

The hypothesis that $f$ is $\triangle$-regular and that $p \nmid \det(a_{ij})$ implies that for each face $\tau$ of $\triangle(x_0f)$, the polynomials $\left(\sum_{j=0}^n a_{ij} x_j \partial F / \partial x_j\right)_{\tau}, i = 0, \ldots, n$, have no common zero in $(\mathbb{F}_q^*)^{n+1}$. One can repeat the arguments of [1] with $x_i \partial F / \partial x_i$ replaced by $\sum_{j=0}^n a_{ij} x_j \partial F / \partial x_j$.

For $j = 0, \ldots, n$, we define $\Omega_0$-endomorphism $\theta_j : B \to B$ by

$$\theta_j \left(\sum_{(r,m) \in M(x_0f)} A_{r,m} \gamma^r x_0^r x_m^m\right) = \sum_{(r,m) \in M(x_0f) \cap H_j} A_{r,m} \gamma^r x_0^r x_m^m. $$

(6)

Note that $\theta_j$ is also a ring homomorphism of $B$ to itself.

**Lemma 4.2.** With the notation as above, we have

$$\tilde{D}_{H_j}(B) \subseteq \ker(\theta_j) \quad \text{for} \quad 0 \leq j \leq n$$

(7)

$$\theta_j^2 = \theta_j \quad \text{for} \quad 0 \leq j \leq n$$

(8)

$$\theta_j \circ \tilde{D}_H \circ \theta_j = \theta_j \circ \tilde{D}_H \quad \text{for} \quad 0 \leq i, j \leq n.$$

(9)
Proof. Let $\sum_{i=0}^{n} a_{ji}x_i = 0$ be the defining equation of $H_j$. Write $x^u = x_0^{u_0} \cdots x_n^{u_n}$ where $u \in M(x_0f)$. Then $\mathcal{D}_{H_j} = \sum_{i=0}^{n} a_{ji}(E_i + \widehat{H}_i(x_0, x))$. Consider $\theta_j \circ (\sum_{i=0}^{n} a_{ji}E_i)$, we have

$$\sum_{i=0}^{n} a_{ji}E_i(x^u) = \left( \sum_{i=0}^{n} a_{ji}u_i \right) x^u = 0 \text{ if } u \in H_j$$

$$\theta_j \left( \sum_{i=0}^{n} a_{ji}E_i(x^u) \right) = \left( \sum_{i=0}^{n} a_{ji}u_i \right) \theta_j(x^u) = 0 \text{ if } u \notin H_j.$$ Hence $\theta_j \circ (\sum_{i=0}^{n} a_{ji}E_i)(x^u) = 0$ for any $u \in M(x_0f)$.

In order to show $\theta_j \circ (\sum_{i=0}^{n} a_{ji}H_i(x_0, x)) \cdot (x^u) = 0$, it suffices to show $\theta_j \circ (\sum_{i=0}^{n} a_{ji}(x_i \frac{\partial}{\partial x_i} v^n)) \cdot (x^u) = 0$ where $x^v = x_0^{v_0} \cdots x_n^{v_n}$ is a monomial of $F$. Clearly,

$$\theta_j \circ \left( \sum_{i=0}^{n} a_{ji}(x_i \frac{\partial}{\partial x_i} v^n) \cdot x^u \right) = \theta_j \circ \left( \sum_{i=0}^{n} a_{ji}v_i x^n \cdot x^u \right) = \left( \sum_{i=0}^{n} a_{ji}v_i \right) \theta_j(x^{u+v}).$$

If $v \in H_j$, then $\sum_{i=0}^{n} a_{ji}v_i = 0$. If $v \notin H_j$, then $u + v \notin H_j$. Hence $\theta_j(x^{u+v}) = 0$. (7) is proved. (8) is clear.

The equation (9) is satisfied by $x^u$ where $u \in H_j$. For $u \notin H_j$, $\theta_j \circ \mathcal{D}_{H_j} \circ \theta_j = 0$. Let $\sum_{j=0}^{n} a_{ij}x_j = 0$ be the defining equation of $H_i$. Since $\theta_j(\sum_{j=0}^{n} a_{ij}E_j)(x^u) = 0$. Clearly, the exponents of the monomials of $\widehat{H}_i(x_0, x) \cdot x^u \notin H_j$ for all $i$ as $u \notin H_j$. That means $\theta_j \widehat{H}_i(x_0, x)(x^u) = 0$ for all $i$. Hence $\theta_j \circ \mathcal{D}_{H_i} = 0$. □

Let $S = \{0, 1, \ldots, n\}$. For $A \subseteq S$, let $\theta_A$ be the composition of all $\theta_j$ for $j \in A$. Let $B_A = \theta_A(B)$. The maps $\mathcal{D}_{H_j}^{A} \overset{\text{def}}{=} \theta_A \circ \mathcal{D}_{H_j}$ for $j \in S$ are stable on $B_A$. Thus we may form the Koszul complex $K_*(B_A, \{\mathcal{D}_{H_j}^{A}\}_{j \in S-A})$ which we abbreviate by $K_*(A)$. Note that we ignore the differential operators $\mathcal{D}_{H_j}^{A}$ for $j \in A$ since by (7) they are identically zero on $B_A$. For each subset $A' \subseteq S - A$, we define $B_{A,A'} = \bigcap_{j \in A'} (\ker \theta_j | B_A)$ and a subcomplex $K_*(A, A')$ of $K_*(A)$ by setting for $0 \leq l \leq n + 1 - |A|$.

$$K_l(A, A') = \bigoplus_{T \subseteq S - A, |T| = l} \left( \bigcap_{j \in A' - T} (\ker \theta_j | B_A) \right) e_T, \quad (10)$$

where the sum runs over all subsets $T \subseteq S - A$ of cardinality $l$, and if $A' - T = \emptyset$ we take $\bigcap_{j \in A' - T} (\ker \theta_j | B_A)$ to mean $B_A$. It is straightforward to check that the
boundary maps are stable on the submodules, hence they define a subcomplex. Note that $K_*(A) = K_*(A, \emptyset)$ and that $K_*(\emptyset)$ is the complex $K_*(B, \{\widehat{D}_{H_i}\}_{i=0}^n)$.

We can define an action of Frobenius on these complexes. Since $\theta_j$ commutes with $\psi$ and with multiplication by $F_a(f, x_0, x)$, we may define an endomorphism

$$\alpha_A = \psi^a \circ \theta_A(F_a(f, x_0, x)) : B_A \to B_A$$

that satisfies

$$\alpha_A \circ D_{H_i}^A = q D_{H_i}^A \circ \alpha_A$$

for $i = 0, 1, \ldots, n$. As before, this induces an endomorphism $\alpha_A : K_*(A) \to K_*(A)$ by setting

$$\alpha_{A,l} = \bigoplus_{|T|=l} q^l \alpha_A : K_l(A) \to K_l(A).$$

It is straightforward to check that $\alpha_A$ induces endomorphism $\alpha_{A',A'}$ of the subcomplexes $K_*(A, A')$ for all $A' \subseteq S - A$, as well. We denote by $H_*(\alpha_A)$ and $H_*(\alpha_{A,A'})$ the endomorphisms of the associated homology groups.

We now explain the arithmetic significance of these complexes. For $f = \sum_{j \in J} a_j x^j \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $A \subseteq S$, define

$$f_A = \sum_{j \in J, (1,j) \in \bigcap_{i \in A} H_i} a_j x^j \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].$$

We denote the corresponding exponential sums on $\mathbb{G}_{m+1}^{n-|A|}$ by $S^*_k(x_0 f_A)$ and the corresponding $L$-function by $L^*(x_0 f_A, t)$.

In Section 3 we associate to $f$ a Koszul complex $K_*(B)$ with Frobenius operator $\alpha$ satisfying Theorem 3.2 when $f$ is $\triangle$-regular. The same construction can be applied to each $f_A$, and we denote the resulting Koszul complex and Frobenius operator by $K_*(B(x_0 f_A))$ and $\alpha_{x_0 f_A}$, respectively. Let $\widehat{D}_{A,i}$, $i = 0, \ldots, n$ be the corresponding differential operators. When $f$ is $\triangle$-regular, so are all the $f_A$, hence Theorem 3.2 holds for Koszul complex $K_*(B(x_0 f_A), \{\widehat{D}_{A,i}\}_{i=0}^n)$. In addition, if $p \nmid \det(a_{ij})$ the proof of Lemma 4.1 show that Theorem 3.2 holds for the Koszul complex $K_*(B(x_0 f_A), \{\widehat{D}_{A,H_i}\}_{i=0}^n)$, where

$$\widehat{D}_{A,H_i} \coloneqq \sum_{j=0}^n a_{ij} \widehat{D}_{A,j}.$$
Comparing $K_*(B(x_0f_A), \{\widehat{D}_{A,i}\}_{i=0}^n)$ with the complex $K_*(A)$ constructed in this section, one sees that $K_*(B(x_0f_A))$ and $K_*(A)$ are naturally identified as Banach spaces with action of Frobenius. Under this identification of spaces, the differential operator $\hat{D}_{A,H}^i$ is identified with $\hat{D}_{A,H}^i$ for $i = 0, \ldots, n$. Thus $K_*(B(x_0f_A), \{\widehat{D}_{A,i}\}_{i=0}^n)$ and $K_*(B_A, \{\widehat{D}_{A,i}^A\}_{i=0}^n)$ are isomorphic as complexes with action of Frobenius. The hypothesis that $\Delta(f)$ be simplex implies that $\dim \Delta(x_0f_A) = n + 1 - |A|$ for each $A \subseteq S$. Keeping in mind that $\hat{D}_{A,H}^i = 0$ (as operator on $B_A$) for $i \in A$, Theorem 3.2 applied to $x_0f_A$ gives:

**Lemma 4.3.** Let $f \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be $\Delta$-regular and suppose $\Delta(f)$ is an $n$-dimensional simplex and $p \nmid \det(a_{ij})$ where the matrix is defined as in (5). For all $A \subseteq S$, the complex $K_*(A)$ is acyclic in dimension $> 0$, $\dim H_0(K_*(A)) = (n + 1 - |A|)! \text{Vol}(x_0f_A)$, and $H_0(\alpha_A)$ is invertible. Define

$$P_A(t) = \det(I - t\alpha_A|B_A).$$

Then $\det(I - tH_0(\alpha_A)|H_0(K_*(A)))$ is a polynomial in $\mathbb{Z}[t]$ of degree $(n + 1 - |A|)! \text{Vol}(x_0f_A)$ and

$$L^*(x_0f_A, t)^{(1)^{n-|A|}} = P_A(t)^{(n+1-|A|)}.$$

Let $A \subseteq S$, $A' \subseteq S - A$, and $j \in A'$. By (8), the map $\theta_j : B_{A,A' - \{j\}} \to B_{A \cup \{j\}, A' - \{j\}}$ is surjective. The kernel of $\theta_j$ is naturally identified with $B_{A,A'}$. Hence there is a short exact sequence

$$0 \to B_{A,A'} \to B_{A,A' - \{j\}} \xrightarrow{\theta_j} B_{A \cup \{j\}, A' - \{j\}} \to 0. \quad (11)$$

The surjective map $\theta_j : B_{A,A' - \{j\}} \to B_{A \cup \{j\}, A' - \{j\}}$ induces a surjective map $\theta_{j,*}$ of complexes

$$\theta_{j,*} : K_*(A, A' - \{j\}) \to K_*(A \cup \{j\}, A' - \{j\})$$

with kernel $K_*(A, A')$. Hence there is a short exact sequence of complexes with Frobenius action

$$0 \to K_*(A, A') \to K_*(A, A' - \{j\}) \xrightarrow{\theta_{j,*}} K_*(A \cup \{j\}, A' - \{j\}) \to 0. \quad (12)$$
Let \( P_{A,A'}(t) = \det(I - t\alpha_{A,A'}|B_{A,A'}) \). Note that \( P_{A,\emptyset}(t) = P_A(t) \). The short exact sequence (11) gives

\[
P_{A,A'}(t) = \frac{P_{A,A'-(j)}(t)}{P_{A\cup\{j\},A'-(j)}(t)}.
\]

We summarize these results:

**Lemma 4.4.** Let \( f \in \mathbb{F}_q[x_1^\pm, \ldots, x_n^\pm] \) be \( \triangle \)-regular and suppose \( \triangle(f) \) is an \( n \)-dimensional simplex and \( p \nmid \det(a_{ij}) \) where the matrix \( (a_{ij}) \) is defined as in (5). For all \( A \subseteq S \) and \( A' \subseteq S - A \), we have

\[
P_{A,A'}(t)^{(-1)^{|A|}} = \prod_{A \subseteq C \subseteq A \cup A'} P_{C}(t)^{(-1)^{|C|}}.
\]

**Proof.** Use induction on \( |A'| \). For \( |A'| = 0 \), that means \( A' = \emptyset \) and \( C = A \).

\[
P_{A,\emptyset}(t)^{(-1)^{|A|}} = P_A(t)^{(-1)^{|A|}} = \prod_{A \subseteq C \subseteq A \cup \emptyset} P_{C}(t)^{(-1)^{|C|}}.
\]

Suppose \( A' \) is non-empty. For \( j \in A' \) by (13), we have

\[
P_{A,A'}(t)^{(-1)^{|A|}} = \frac{P_{A,A'-(j)}(t)^{(-1)^{|A|}}}{P_{A\cup\{j\},A'-(j)}(t)^{(-1)^{|A|}}}
= \left( P_{A,A'-(j)}(t)^{(-1)^{|A|}} \right) \left( P_{A\cup\{j\},A'-(j)}(t)^{(-1)^{|A|}} \right)
= \left( \prod_{A \subseteq C_1 \subseteq A \cup A' \text{ - \{j\}}} P_{C_1}(t)^{(-1)^{|C_1|}} \right) \left( \prod_{A \cup \{j\} \subseteq C_2 \subseteq A \cup A' \text{ - \{j\}}} P_{C_2}(t)^{(-1)^{|C_2|}} \right)
= \left( \prod_{A \subseteq C \subseteq A \cup A' \text{ - \{j\}}} P_{C}(t)^{(-1)^{|C|}} \right) \left( \prod_{A \subseteq C \subseteq A \cup A' \text{ - \{j\}}} P_{C}(t)^{(-1)^{|C|}} \right)
= \prod_{A \subseteq C \subseteq A \cup A'} P_{C}(t)^{(-1)^{|C|}}.
\]

\( \square \)
Hence we have the following lemma.

Lemma 4.5. Let \( f \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) be \( \Delta \)-regular and suppose \( \Delta(f) \) is an \( n \)-dimensional simplex and \( p \nmid \det(a_{ij}) \) where the matrix \( (a_{ij}) \) is defined as in (5). For all \( A \subseteq S \) and \( A' \subseteq S - A \), we have

\[
P_A(t) = \prod_{S - A' \supseteq A} P_{S - A', A'}(t)
\]

Proof. By Lemma 4.4 we have

\[
\prod_{S - A' \supseteq A} P_{S - A', A'}(t) = \prod_{S - A' \supseteq A \subseteq S - A' \subseteq C} P_C(t)(-1)^{|C| + |S - A'|}
\]

\[
= \prod_{A \subseteq C} \prod_{A \subseteq S - A' \subseteq C} P_C(t)(-1)^{|C| + |S - A'|} \left( \sum_{A \subseteq S - A' \subseteq C} (-1)^{|S - A'|} \right)
\]

\[
= \prod_{A \subseteq C} P_C(t)
\]

since

\[
\sum_{A \subseteq S - A' \subseteq C} (-1)^{|S - A'|} = \begin{cases} (-1)^{|A|} & \text{if } C = A, \\ 0 & \text{if } C \neq A. \end{cases}
\]

The geometric meaning of above can be interpreted as follows: \( B_A \) consists of monomials \( x_0^r x^m \) where \( (r, m) \) lies in the cone \( M(x_0 f_A) = \bigcap_{j \in A} H_j \). For \( A' \subseteq S - A \), \( B_{A,A'} \) consists of monomials \( x_0^r x^m \) where \( (r, m) \) lies in the cone \( M(x_0 f_A) \) but not in \( H_j, j \in A' \). In particular, \( B_{A,S - A} \) consists of monomials \( x_0^r x^m \) where \( (r, m) \) lies in the interior of the cone \( M(x_0 f_A) \). In the cone \( M(x_0 f_A) \), the map \( \theta_j : B_{A,A' - \{j\}} \to B_{A \cup \{j\}, A' - \{j\}} \) in (6) is a projection onto \( H_j \) which sends the monomials \( x_0^r x^m \) to 0 where \( (r, m) \in H_j \). That is why such monomials \( x_0^r x^m \in B_{A,A' - \{j\}} \), with \( (r, m) \notin H_j \), precisely lie in \( B_{A,A'} \) and have the exact sequence (11). Deleting the points \( (r, m) \) not in \( H_j \) where \( x_0^r x^m \in B_{A,A' - \{j\}} \) gives (13). Lemma 4.4 is followed by the inclusion and exclusion principle and Lemma 4.5 by the boundary decomposition theorem [10].

Lemma 4.6. Suppose that for all \( A \subseteq S \),

\[
H_i(K_s(A, \emptyset)) = 0 \text{ for } i > 0.
\]

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Then for all $A' \subseteq S - A$,

$$H_i(K_*(A, A')) = 0 \text{ for } i > 0.$$ 

In particular,

$$H_i(K_*(\emptyset, S)) = 0 \text{ for } i > 0.$$ 

Furthermore, there is an injection

$$H_0(K_*(\emptyset, S)) \hookrightarrow H_0(K_*(\emptyset, \emptyset)).$$

**Proof.** The proof is by induction on $|A'|$, the case $|A'| = 0$ being the hypothesis of the theorem. From (12) we get the long exact sequence

$$\cdots \rightarrow H_{i+1}(K_*(A \cup \{j\}, A' - \{j\})) \rightarrow H_i(K_*(A, A')) \rightarrow H_i(K_*(A, A' - \{j\})) \rightarrow \cdots$$

where $j \in A'$. By induction hypothesis, the two “outside” homology groups vanish for $i > 0$. Therefore $H_i(K_*(A, A')) = 0$ for $i > 0$ also. The exact homology sequence associated to (12) then reduces to the short exact sequence

$$0 \rightarrow H_0(K_*(A, A')) \rightarrow H_0(K_*(A, A' - \{j\})) \rightarrow H_0(K_*(A \cup \{j\}, A' - \{j\})) \rightarrow 0. \quad (14)$$

In particular, there is an injection $H_0(K_*(A, A')) \hookrightarrow H_0(K_*(A, A' - \{j\}))$. The existence of the injection asserted by the lemma then follows by induction on $|A'|$. 

**Theorem 4.7.** Let $f \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be $\triangle$-regular and $\overline{U_f}$ the toric projective closure defined by $f$. Suppose $\triangle(f)$ is an $n$-dimensional simplex and $p \nmid \det(a_{ij})$ where the matrix $(a_{ij})$ is defined as in (5). Then

$$Z(\overline{U_f}, qt) = \frac{\det (I - tH_0(\alpha_{\emptyset,S})|H_0(K_*(\emptyset, S)))^{(-1)^n}}{(1 - qt) \cdots (1 - q^n t)}$$
Proof. By Lemma 4.3 and Lemma 4.5, we have

\[
\prod_{A \subseteq S} L^*(x_0 f_A, t)^{(-1)^n} = \left( \prod_{A \subseteq S} L^*(x_0 f_A, t)^{(-1)^{|A|}} \right)^{(-1)^{|A|}} 
\]

\[
= \prod_{A \subseteq S} \left( P_A(t)^{\delta_{n+1-|A|}} \right)^{(-1)^{|A|}} 
\]

\[
= \prod_{A \subseteq S} \prod_{i=0}^{n+1-|A|} P_A(q^i t)^{(-1)^{|A|+\left(\frac{n+1-|A|}{i}\right)}} 
\]

\[
= \prod_{A \subseteq S} \prod_{i=0}^{n+1-|A|} P_{S-A',A'}(q^i t)^{(-1)^{|A|+\left(\frac{n+1-|A|}{i}\right)}} 
\]

\[
= \prod_{A \subseteq S} \prod_{i=0}^{n+1-|A|} P_{S-A',A'}(q^i t)^{(-1)^{|A|+\left(\frac{n+1-|A|}{i}\right)}} 
\]

\[
= \prod_{A \subseteq S} P_{S-A',A'}(q^i t)^{(-1)^{|A|+\left(\frac{n+1-|A|}{i}\right)}} 
\]

where \( A \) corresponds to the face \( \sigma = \bigcap_{i \in A} H_i \) of \( \Delta(x_0 f) \), while \( S - A' \supseteq A \) and \( S - A'' \supseteq A \) correspond to the subfaces \( \sigma' = \bigcap_{i \in S-A'} H_i \) and \( \sigma'' = \bigcap_{i \in S-A''} H_i \) of \( \sigma \). Since

\[
\sum_{A \subseteq (S-A') \cap (S-A'')} (-1)^{|A|} = \begin{cases} 1 & \text{if } (S - A') \cap (S - A'') = \emptyset, \\ 0 & \text{if } (S - A') \cap (S - A'') \neq \emptyset \end{cases} 
\]

\[
\prod_{A \subseteq S} L^*(x_0 f_A, t)^{(-1)^n} = \prod_{(S-A') \cap (S-A'')=\emptyset} P_{S-A',A'}(q^i t)^{(-1)^{|A''|}} 
\]

\[
= \prod_{A''} \prod_{A' \subseteq A''} P_{S-A',A'}(q^i t)^{(-1)^{|A''|}} 
\]

\[
= \prod_{A''} \prod_{A' \subseteq A''} \det \left( I - q^i t |A''| B_{S-A',A'} \right)^{(-1)^{|A''|}} 
\]

\[
= \prod_{A''} \det \left( I - t(\alpha_{S-A',A'} |A''| K_{|A''|} (S - A', A')) \right)^{(-1)^{|A''|}} 
\]

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where $S - A' \subseteq A''$ and $i = |A''|$. Passing to homology, we have

$$
\prod_{A \subseteq S} L^*(x_0f_A, t)^{(-1)^n} = \prod_{A''} \det \left( I - q^{|A''|} t H_{|A''|}(\alpha_{S - A', A'}) | H_{|A''|}(K_*(S - A', A')) \right)^{(-1)^{|A''|}}
$$

Since $f$ is $\triangle$-regular, so as $f_{S - A'}$ for all $A' \subseteq S$. By theorem 3.2, $H_{|A''|}(S - A', \emptyset) = 0$ for $|A''| > 0$. Lemma 4.6 implies that $H_{|A''|}(S - A', A') = 0$ for $|A''| > 0$. For $|A''| = 0$, that means $A'' = \emptyset$ and $A' = S$, we can conclude that

$$
\prod_{A \subseteq S} L^*(x_0f_A, t)^{(-1)^n} = \det \left( I - t H_0(\alpha_{\emptyset, S}) | H_0(K_*(\emptyset, S)) \right)
$$

The zeta function of $U_f$ can be expressed as

$$
Z(U_f, qt) = \prod_{\emptyset \neq \sigma \subseteq \triangle} Z(G_{m}^{\dim \sigma}, t) \prod_{\emptyset \neq \sigma \subseteq \triangle} L^*(x_0f_{\sigma}, t).
$$

Since

$$
P^n = \bigcup_{i=1}^{n+1} \left( G_{m}^{i-1} \right)^{\binom{n+1}{i}}
$$

where $i$ is the number of nonzero entries in the homogenous coordinate in $P^n$, we have

$$
\sum_{\emptyset \neq \sigma \subseteq \triangle} N_k(G_{m}^{\dim \sigma}) = \sum_{i=0}^{n} \binom{n+1}{i+1} (q^k - 1)^i = N_k(P^n).
$$

Then

$$
Z(U_f, qt) = Z(P^n, t) \prod_{A \subseteq S} L^*(x_0f_A, t)
\quad = \frac{L^*(x_0f_S, t)^{-1}}{(1 - t) \cdots (1 - q^nt)} \prod_{A \subseteq S} L^*(x_0f_A, t).
$$

Substitute $A = S$ into lemma 4.3, we have

$$
L^*(x_0f_S, t)^{-1} = P_S(t)
\quad = \det(I - t \alpha_{S}|K_*(S))
\quad = 1 - t
$$
where $\alpha_S$ is an identity endomorphism on $K_*(S)$. Hence
\[
Z(f,qt) = \frac{\prod_{A \subseteq S} L^*(x_0 f_A, t)}{(1-qt) \cdots (1-q^n t)} = \frac{\det(I - tH_0(\alpha_{\emptyset,S})|H_0(K_*(\emptyset, S)))^{(-1)^n}}{(1-qt) \cdots (1-q^n t)}.
\]

The mapping $H_0(\alpha_{\emptyset,S})$ is invertible as an endomorphism of $H_0(K_*(\emptyset, S)) = B_{\emptyset,S}/ \sum_{i=0}^n \widehat{D}_i B_{\emptyset,S-i}$. It is easily seen that $H_0(\alpha_{\emptyset,S})$ is compatible with the action of Frobenius.

**Lemma 4.8.** Let $f \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be $\triangle$-regular and suppose $\triangle(f)$ is an $n$-dimensional simplex and $p \nmid \det(a_{ij})$ where the matrix $(a_{ij})$ is defined as in (5). For all $A \subseteq S$ and $A' \subseteq S - A$, we have
\[
det(I - tH_0(\alpha_{A,A'})|H_0(K_*(A, A')))^{(-1)^{|A|}} = \prod_{A \subseteq C \subseteq A \cup A'} \det(I - tH_0(\alpha_C)|H_0(K_*(C)))^{(-1)^{|C|}}
\]

**Proof.** Use induction on $|A'|$ to the exact sequence (14). The proof is similar to the proof of lemma 4.4. \hfill \Box

Hence, $\det(I - tH_0(\alpha_{\emptyset,S})|H_0(K_*(\emptyset, S)))$ is a polynomial of degree
\[
\sum_{C \subseteq S} (-1)^{|C|}(n + 1 - |C|)!\Vol(x_0 f_C).
\]

5 \textit{\textbf{p-adic estimates}}

In this section, we give a sharp lower bound for the Newton polygon of the polynomial $\det(I - tH_0(\alpha_{\emptyset,S})|H_0(K_*(\emptyset, S)))$.

Let $\triangle(f)$ be an $n$-dimensional simplex. For $A \subseteq S$, the graded ring $S_{\triangle}$ induces the graded subring
\[
S_{\triangle,A} = \left\{ \sum_{(r,m) \in M(x_0 f)} A_{r,m} x_0^r x_m^{\epsilon} \in S_{\triangle} | A_{r,m} = 0 \text{ if } (r,m) \notin \bigcap_{i \in A} H_i \right\}
\]

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with the grading \( \deg(x_0^r x^m) = r \). Let \( S_{\Delta,A}^r \) be a finite dimensional \( \mathbb{F}_q \)-vector space with the basis \( x_0^r x^m \in S_{\Delta,A} \). Define \( F_{H_i}(x_0, x) = \sum_{j=0}^{n} a_{ij} F_j(x_0, x) \in S_{\Delta,A}^1 \) and \( F_{H_i}^A(x_0, x) = \theta_A \circ F_{H_i}(x_0, x) \in S_{\Delta,A}^1 \) where \( (a_{ij}) \) is defined as in (5).

**Lemma 5.1.** Let \( f \in \mathbb{F}_q[x_0^\pm 1, \ldots, x_n^\pm 1] \) be \( \Delta \)-regular and suppose \( \Delta(f) \) is an \( n \)-dimensional simplex and \( p \nmid \det(a_{ij}) \) where the matrix \( (a_{ij}) \) is defined as in (5). For all \( A \subseteq S \), the set \( \{ F_{H_i}^A(x_0, x) \}_{i \notin A} \) taken in any order forms a regular sequence on \( S_{\Delta,A} \), hence the Koszul complex \( \overline{K}_*(S_{\Delta,A}, \{ F_{H_i}^A(x_0, x) \}_{i \notin A}) \) is acyclic in positive dimension. Furthermore,

\[
\dim_{\mathbb{F}_q} H_0(\overline{K}_*(S_{\Delta,A}, \{ F_{H_i}^A(x_0, x) \}_{i \notin A})) = (n + 1 - |A|)!\text{Vol}(x_0 f_A).
\]

**Proof.** The hypotheses imply that \( x_0 f_A \) is \( \Delta \)-regular and \( \Delta(x_0 f_A) \) is also an \( (n + 1 - |A|) \)-dimensional simplex. The argument of Lemma 4.1 shows that one can simply repeat the proof of Theorem 3.1. \( \square \)

One defines \( \theta_j : S_{\Delta,A} \to S_{\Delta,A \cup \{j\}} \) as in (6). For \( A' \subseteq S - A \), this allows us to define, as in (10), subcomplexes \( \overline{K}_*(A, A') \) of the Koszul complex \( \overline{K}_*(S_{\Delta,A}, \{ F_{H_i}^A(x_0, x) \}_{i \notin A}) \). Proceeding as in the deduction of Lemma 4.8, we deduce:

**Lemma 5.2.** Let \( f \in \mathbb{F}_q[x_0^\pm 1, \ldots, x_n^\pm 1] \) be \( \Delta \)-regular and suppose \( \Delta(f) \) is an \( n \)-dimensional simplex and \( p \nmid \det(a_{ij}) \) where the matrix \( (a_{ij}) \) is defined as in (5). For all \( A \subseteq S \) and \( A' \subseteq S - A \), the complex \( \overline{K}_*(A, A') \) is acyclic in positive dimension. Furthermore,

\[
\dim_{\mathbb{F}_q} H_0(\overline{K}_*(\emptyset, S)) = \sum_{C \subseteq S} (-1)^{|C|} (n + 1 - |C|)!\text{Vol}(x_0 f_C).
\]

Let \( S_{\Delta,A,A'} = \bigcap_{j \in A'} (\ker \theta_j | S_{\Delta,A}) \). By Lemma 5.2,

\[
H_0(\overline{K}_*(\emptyset, S)) = S_{\Delta,\emptyset,S} / \sum_{i=0}^{n} F_{H_i}(x_0, x) S_{\Delta,\emptyset,S-\{i\}}
\]

is a finite dimensional graded ring. Let \( h_\Delta(k) \) be the dimension over \( \mathbb{F}_q \) of graded piece of degree \( k \) of \( H_0(\overline{K}_*(\emptyset, S)) \). It can be shown (as in [4], Lemma 2.9 and [7], Lemma 1.5) that \( h_\Delta(k) = 0 \) for \( i > n + 1 \). Define \( v(\Delta) = \dim_{\mathbb{F}_q} H_0(\overline{K}_*(\emptyset, S)) \). Then \( \sum_{k=0}^{n+1} h_\Delta(k) = v(\Delta) \).

Let \( f \) be \( \Delta \)-regular over \( \mathbb{F}_q \), \( \Delta(f) \) an \( n \)-dimensional simplex and \( p \nmid \det(a_{ij}) \) where the matrix \( (a_{ij}) \) is defined as in (5). Define the Hodge polygon \( \text{HP}(\Delta) \) in \( \mathbb{R}^2 \) to be the
polygon with the vertices $(0, 0)$ and \((\sum_{k=0}^{m} h_{\Delta}(k), \sum_{k=0}^{m} k h_{\Delta}(k))\) for \(m = 0, \ldots, n + 1\). It can be shown (\cite{2}, Theorem 4.8) that \(h(k) = h(n + 1 - k)\). Then
\[
\sum_{i=0}^{n+1} k h(k) = \frac{n + 1}{2} v(\Delta).
\]

Write
\[
det \left( I - t H_0(\alpha(\emptyset, S)) | H_0(K_\ast(\emptyset, S)) \right) = \sum_{m=0}^{v(\Delta)} A_m t^m,
\]
where \(A_0 = 1\) and \(A_m \in \mathbb{Z}\). Define the Newton polygon \(NP(f)\) of \(\sum_{m=0}^{v(\Delta)} A_m t^m\) to be the lower convex closure in \(\mathbb{R}^2\) of the points \((m, \text{ord}_q(A_m))\) for \(m = 0, \ldots, v(\Delta)\).

**Theorem 5.3.** Suppose \(f \in \mathbb{F}_q[x_\pm 1, \ldots, x_\pm 1]\) is \(\Delta\)-regular, \(\Delta(f)\) is an \(n\)-dimensional simplex and \(p \nmid \det(a_{ij})\) where the matrix \((a_{ij})\) is defined as in (5). Then the Newton polygon \(NP(f)\) lies above the Hodge polygon \(HP(\Delta)\) and their endpoints coincide.

**Proof.** This is proved by the method used in (\cite{11}, Theorem 3.10). The coincidence of the endpoints, which was not proved in [1], follows as in (\cite{6}, paragraph preceding Theorem 7.1).

\[
\square
\]

**Remark 5.4.** Let \(\mathcal{M}_p(\Delta)\) be the moduli space of all \(\Delta\)-regular, Laurent polynomials over \(\mathbb{F}_p\) with \(\Delta(f) = \Delta\) and \(\mathcal{H}_p(\Delta)\) the moduli space of those \(f \in \mathcal{M}_p(\Delta)\) such that its Newton polygon coincides with its Hodge polygon. It can be shown \([10]\) and \([11]\) that there is an integer \(D^*(\Delta)\) depending only on \(\Delta\) such that if \(p\) is a large prime and \(p \equiv 1 \pmod{D^*(\Delta)}\), then the Newton polygon of \(\det \left( I - t H_0(\alpha(\emptyset, S)) | H_0(K_\ast(\emptyset, S)) \right)\) coincides generically with the Hodge polygon \(HP(\Delta)\), i.e., for such \(p\), \(\mathcal{H}_p(\Delta)\) is a Zariski dense, open subset of \(\mathcal{M}_p(\Delta)\).

### 6 Calabi-Yau Hypersurfaces

Let \(n \geq 2\) be a positive integer. We consider the universal family of Calabi-Yau complex hypersurfaces of degree \(n + 1\) in the projective space \(\mathbb{P}^n\). Its mirror family is a one-parameter family of toric hypersurfaces. To construct the mirror family, we consider the one-parameter subfamily \(X_\lambda\) of complex projective hypersurfaces of
degree $n+1$ in $\mathbb{P}^n$ defined by 

$$f(x_1, \ldots, x_{n+1}) = x_1^{n+1} + \cdots + x_{n+1}^{n+1} + \lambda x_1 \cdots x_{n+1} = 0,$$

where $\lambda \in \mathbb{C}$ is the parameter. The variety $X_\lambda$ is a Calabi-Yau manifold when $X_\lambda$ is smooth. Let $\mu_{n+1}$ denote the group of $(n+1)$-th roots of unity. Let

$$G = \{(\zeta_1, \ldots, \zeta_{n+1})|\zeta_1^{n+1} = 1, \zeta_1 \cdots \zeta_{n+1} = 1\}/\mu_{n+1} \cong (\mathbb{Z}/(n+1)\mathbb{Z})^{n-1},$$

where $\mu_{n+1}$ is embedded in $G$ via the diagonal embedding. The finite group $G$ acts on $X_\lambda$ by

$$(\zeta_1, \ldots, \zeta_{n+1})(x_1, \ldots, x_{n+1}) = (\zeta_1 x_1, \ldots, \zeta_{n+1} x_{n+1}).$$

The quotient $X_\lambda/G$ is a projective toric hypersurface $Y_\lambda$ in the toric variety $\mathbb{P}_\Delta$, where $\Delta$ is the simplex in $\mathbb{R}^n$ with vertices $\{e_1, \ldots, e_n, -(e_1 + \cdots + e_n)\}$ and the $e_i$’s are the standard coordinate vectors in $\mathbb{R}^n$. Explicitly, the variety $Y_\lambda$ is the projective closure in $\mathbb{P}_\Delta$ of the affine toric hypersurface in $\mathbb{G}^n_m$ defined by

$$g(x_1, \ldots, x_n) = x_1 + \cdots + x_n + \frac{1}{x_1 \cdots x_n} + \lambda = 0.$$

We are interested in the number of $\mathbb{F}_q$-rational points on $Y_\lambda$.

The toric variety $\mathbb{P}_\Delta$ has the following disjoint decomposition:

$$\mathbb{P}_\Delta = \bigsqcup_{\sigma \in \Delta} \mathbb{G}_{m}^{\dim \sigma},$$

where $\sigma$ runs over all non-empty faces of $\Delta$. Accordingly, the projective toric hypersurface $Y_\lambda$ has the corresponding disjoint decomposition

$$Y_\lambda = \bigsqcup_{\sigma \in \Delta} Y_{\lambda, \sigma}, \ Y_{\lambda, \sigma} = Y_\lambda \cap \mathbb{G}_{m}^{\dim \sigma}.$$

For $\sigma = \Delta$, the subvariety $Y_{\lambda, \Delta}$ is simply the affine toric hypersurface defined by $g = 0$ in $\mathbb{G}_m^n$. For zero-dimensional $\sigma$, $Y_{\lambda, \sigma}$ is empty. For a face $\sigma$ with $1 \leq \dim \sigma \leq n-1$, one checks that $Y_{\lambda, \sigma}$ is isomorphic to the affine toric hypersurface in $\mathbb{G}_m^{\dim \sigma}$ defined by

$$1 + x_1 + \cdots + x_{\dim \sigma} = 0.$$

For such a $\sigma$, the inclusion-exclusion principle shows that

$$\#Y_{\lambda, \sigma}(\mathbb{F}_q) = q^{\dim \sigma - 1} - \binom{\dim \sigma}{1} q^{\dim \sigma - 2} + \cdots + (-1)^{\dim \sigma - 1} \binom{\dim \sigma}{\dim \sigma - 1}.$$
Thus,
\[
#Y_{\lambda,\sigma}(\mathbb{F}_q) = \frac{1}{q} \left( (q - 1)^{\dim \sigma} + (-1)^{\dim \sigma + 1} \right).
\]
This formula holds even for zero dimensional \(\sigma\) as both sides would then be zero.

Putting these calculations together, we deduce that
\[
#Y_{\lambda}(\mathbb{F}_q) = #Y_{\lambda,\triangle}(\mathbb{F}_q) - \frac{1}{q} \sum_{\sigma \in \triangle} \frac{1}{q} (q - 1)^{\dim \sigma} + (-1)^{\dim \sigma + 1},
\]
where \(\sigma\) runs over all non-empty faces of \(\triangle\) including \(\triangle\) itself. Since \(\triangle\) is a simplex, one computes that
\[
\sum_{\sigma \in \triangle} \frac{1}{q} (q - 1)^{\dim \sigma} + (-1)^{\dim \sigma + 1} = \frac{q^{n+1} - 1}{q - 1} + (-1) = \frac{q(q^n - 1)}{q - 1}.
\]
This implies that
\[
#Y_{\lambda}(\mathbb{F}_q) = #Y_{\lambda,\triangle}(\mathbb{F}_q) - \frac{1}{q} \frac{q^n - 1}{q - 1}.
\]
This equality holds for all \(\lambda \in \mathbb{F}_q\), including the case \(\lambda = 0\).

One of the basic problems in arithmetic mirror symmetry is to compare the number of rational points on a mirror pair \((X_{\lambda}, Y_{\lambda})\) of Calabi-Yau manifolds. Theorem 3.7 then gives us
\[
#Y_{\lambda}(\mathbb{F}_q) = \frac{q^n - 1}{q - 1} + \frac{(-1)^n \text{Tr}(H_0(\alpha(0,S))|H_0(K_*(0,S)))}{q}.
\]
It can be shown that \(#X_{\lambda}(\mathbb{F}_q) \equiv #Y_{\lambda}(\mathbb{F}_q) \mod q\) [12].

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