Non-spherical Harish-Chandra Fourier transforms on real reductive groups

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Abstract. The Harish-Chandra Fourier transform, \( f \mapsto \mathcal{H}f \), is a linear topological algebra isomorphism of the spherical (Schwartz) convolution algebra \( \mathcal{C}^p(G//K) \) (where \( K \) is a maximal compact subgroup of any arbitrarily chosen group \( G \) in the Harish-Chandra class and \( 0 < p \leq 2 \)) onto the (Schwartz) multiplication algebra \( \mathcal{Z}(\mathfrak{F}^\epsilon) \) (of \( \mathfrak{w} \)-invariant members of \( \mathcal{Z}(\mathfrak{F}) \), with \( \epsilon = (2/p) - 1 \)). This is the well-known Trombi-Varadarajan theorem for spherical functions on the real reductive group, \( G \). Even though \( \mathcal{C}^p(G//K) \) is a closed subalgebra of \( \mathcal{C}^p(G) \), a similar theorem has not however been successfully proved for the full Schwartz convolution algebra \( \mathcal{C}^p(G) \) except; for \( \mathcal{C}^p(G/K) \) (whose method is essentially that of Trombi-Varadarajan, as shown by M. Eguchi); for few specific examples of groups (notably \( G = SL(2,\mathbb{R}) \)) and; for some notable values of \( p \) (with restrictions on \( G \) and/or on members of \( \mathcal{C}^p(G) \)). In this paper, we construct an appropriate image of the Harish-Chandra Fourier transform for the full Schwartz convolution algebra \( \mathcal{C}^p(G) \), without any restriction on any of \( G, p \) and members of \( \mathcal{C}^p(G) \). Our proof, that the Harish-Chandra Fourier transform, \( f \mapsto \mathcal{H}f \), is a linear topological algebra isomorphism on \( \mathcal{C}^p(G) \), equally shows that its image \( \mathcal{C}^p(\hat{G}) \) can be nicely decomposed, that the full invariant harmonic analysis is available and implies that the definition of the Harish-Chandra Fourier transform may now be extended to include all \( p \)-tempered distributions on \( G \) and to the zero-Schwartz spaces.

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§1. Introduction.

Let $G$ be a reductive group in the Harish-Chandra class where $C^p(G)$ is the Harish-Chandra-type Schwartz algebra on $G$, $0 < p \leq 2$, with $C^2(G) =: C(G)$. It is known that $C^\infty_c(G)$ is dense in $C^p(G)$, with continuous inclusion. The image of $C^p(G)$ under the (Harish-Chandra) Fourier transform on $G$ has been a pre-occupation of harmonic analysts since Harish-Chandra defined $C(G)$ leading to the emergence of Arthur’s thesis [1a], where the Fourier image of $C(G)$ was characterized for connected non-compact semisimple Lie groups of real rank one. Thereafter Eguchi [3a.] removed the restriction of the real rank and considered non-compact real semisimple $G$ with only one conjugacy class of Cartan subgroups as well as the Fourier image of $C^p(G/K)$ in [3b.], while Barker [2.] considered $C^p(SL(2, \mathbb{R}))$ as well as the zero-Schwartz space $C^0(SL(2, \mathbb{R}))$.

The complete $p = 2$ story for any real reductive $G$ is contained in Arthur [1b,c]. The most successful general result along the general case of $p$ is the well-known Trombi-Varadarajan Theorem [11.] which characterized the image of $C^p(G//K)$, $0 < p \leq 2$, for a maximal compact subgroup $K$ of a connected semisimple Lie group $G$ as a (Schwartz) multiplication algebra $\hat{\mathcal{Z}}(\mathfrak{g}^\vee)$ (of $\mathfrak{t}$–invariant members of $\mathcal{Z}(\mathfrak{g}^\vee)$, with $\epsilon = (2/p) - 1$; thus subsuming the works of Ehrepreis and Mautner [5.] and Helgason [7.]. However the characterization of the image of $C^p(G)$ for reductive groups $G$ in the Harish-Chandra class has not yet been achieved due to failure of the method of generalizing from the real rank one case (successfully employed in [1b.,c.], [3a.] and [10c.]) or from the spherical case (considered in [11.]).

This paper contains the full computation of the image of $C^p(G)$ for reductive groups $G$ under the Harish-Chandra Fourier transform. It is organised as follows. The next section contains detailed preliminary matters concerning the structure of $G$, its spherical functions and the Harish-Chandra-type Schwartz algebras, $C^p(G)$. This section contains the most significant results on the spherical Harish-Chandra Fourier transform of these Schwartz algebras (Theorems 2.2 and 2.3) and also considered the system of differential equations satisfied by spherical functions, with a computation given for any real rank one $G$.

Our main results are contained in §3 where a Schwartz algebra containing $\hat{\mathcal{Z}}(\mathfrak{g}^\vee)$ was constructed and we prove the full non-spherical Harish-Chandra Fourier transforms of $C^p(G)$ on real reductive groups $G$ (Theorem 3.8) show-
ing, at the same time, that its image $\mathcal{C}^p(\hat{G})$ has nice decompositions (Corollaries 3.9 and 3.10; which confirm the real reason for the ease of transition of results from [11.] to [3b.]) consisting of the Trombi-Varadarajan image, $\tilde{\mathcal{Z}}(\mathfrak{g}^\ast)$, at its center. These decompositions save us the need to make endless asymptotic estimates in our analysis, showing that all such estimates have been subsumed in the Trombi-Varadarajan image $\tilde{\mathcal{Z}}(\mathfrak{g}^\ast)$ (which enters the analysis naturally). The implication of this is that the fact that the spectrum in the theory for $G/K$ as computed in [3b.] is still pure imaginary is now shown to be mainly due to the contribution of the spherical case $G/\mathbb{K}$ to the symmetric space $G/K$ (and not just carried over to the case of $G/K$, as posited in [6.], p. 355). Indeed, the said decompositions of $\mathcal{C}^p(\hat{G})$ give natural and direct paths to and from $\tilde{\mathcal{Z}}(\mathfrak{f}^\ast)$, as already evident from the results of [3b.]. We then show how the Trombi-Varadarajan theorem (Corollary 3.11) could be recovered from our perspective. The Fourier transform of tempered distributions is thereafter extended to all of $\mathcal{C}^p(\hat{G})$ (Theorem 3.14). We also lift the results of [10d.] to give a full invariant harmonic analysis on $G$ (Theorems 3.18 and 3.20) with a proof of Rao-Varadarajan theorem for $\mathcal{C}^p(G)$ (Theorem 3.21). Basic results on the zero-Schwartz space $\mathcal{C}^0(G)$ were considered (Theorem 3.22) at the end of this section.

An application of our techniques is given in §4 to (what we call) spherical convolutions, $g_{\lambda, A}$, using the Harish-Chandra expansion of eigenfunctions on $G$, thus leading to the Harish-Chandra Fourier transforms of a distinguished convolution subalgebra of $\mathcal{C}^p(G/K)$ (Theorem 4.3) which contains the spherical part, $\mathcal{C}^p(G/\mathbb{K})$, of $\mathcal{C}^p(G)$. Further results on the structure of (canonical) wave-packets on $G$ and the considerations of the full Bochner theorem shall be the subjects of future endeavours.

§2. Structure of the Schwartz algebras on $G$.

Let $G$ be a group in the Harish-Chandra class. That is $G$ is a locally compact group with the properties that $G$ is reductive, with Lie algebra $\mathfrak{g}$, $[G : G^0] < \infty$, where $G^0$ is the connected component of $G$ containing the identity, in which the analytic subgroup, $G_1$, of $G$ defined by $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$ is closed in $G$ and of finite center and in which, if $G_C$ is the adjoint group of $\mathfrak{g}_C$, then $\text{Ad}(G) \subset G_C$. Such a group $G$ is endowed with a Cartan involution, $\theta$, whose fixed points form a maximal compact subgroup, $K$, of $G$ [6]. $K$ meets all connected components of $G$, in particular $K \cap G^0 \neq \phi$. Let $\mathfrak{t}$ denote the Lie algebra of $K$. 
We denote the universal enveloping algebra of $\mathfrak{g}_C$ by $U(\mathfrak{g}_C)$, whose members may be viewed either as left or right invariant differential operators on $G$. We shall write $f(x; a)$ for the left action $(af)(x)$ and $f(a; x)$ for the right action $(fa)(x)$ of $U(\mathfrak{g}_C)$ on functions $f$ on $G$. Let $C(G)$ represents the space of $C^\infty$-functions $f$ on $G$ for which

$$\sup_{x \in G} | f(b; x; a) | \Xi^{-1}(x)(1 + \sigma(x))^r < \infty,$$

for $a, b \in U(\mathfrak{g}_C)$ and $r > 0$. Here $\Xi$ and $\sigma$ are well-known elementary spherical functions defined below on $G$. $C(G)$ is known to be a Schwartz algebra under convolution while $C(G//K)$, consisting of the spherical members of $C(G)$, is a closed commutative subalgebra. $C^\infty_c(G)$ is densely contained in $C(G)$, with continuous inclusion.

Let $\hat{G}$ represent the set of equivalence classes of irreducible unitary representations of $G$. If $G_1$ is non-compact then the support of the Plancherel measure does not exhaust $\hat{G}$. We write $\hat{G}_t$ for this support, which generally contains a discrete part, $\hat{G}_d (\neq \emptyset$, if $\text{rank}(G) = \text{rank}(K))$, and a continuous part, $\hat{G}_t \setminus \hat{G}_d (\neq \emptyset, \text{always})$.

If $\mathfrak{p} = \{X \in \mathfrak{g} : \theta X = -X\}$ then $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$. Choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ with algebraic dual $\mathfrak{a}^*$ and set $A = \exp\mathfrak{a}$. For every $\lambda \in \mathfrak{a}^*$ put

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : [H, X] = \lambda(H)X, \forall H \in \mathfrak{a}\},$$

and call $\lambda$ a restricted root of $(\mathfrak{g}, \mathfrak{a})$ whenever $\mathfrak{g}_\lambda \neq \{0\}$. Denote by $\mathfrak{a}'$ the open subset of $\mathfrak{a}$ where all restricted roots are $\neq 0$, and call its connected components the Weyl chambers. Let $\mathfrak{a}^+$ be one of the Weyl chambers, define the restricted root $\lambda$ positive whenever it is positive on $\mathfrak{a}^+$ and denote by $\Delta^+$ the set of all restricted positive roots. We then have the Iwasawa decomposition $G = KAN$, where $N$ is the analytic subgroup of $G$ corresponding to $\mathfrak{n} = \sum_{\lambda \in \Delta^+} \mathfrak{g}_\lambda$, and the polar decomposition $G = K \cdot \text{cl}(A^+) \cdot K$, with $A^+ = \exp\mathfrak{a}^+$, and $\text{cl}(A^+)$ denoting the closure of $A^+$.

If we set $M = \{k \in K : \text{Ad}(k)H = H, H \in \mathfrak{a}\}$ and $M' = \{k \in K : \text{Ad}(k)a \subset \mathfrak{a}\}$ and call them the centralizer and normalizer of $\mathfrak{a}$ in $K$, respectively, then: (i) $M$ and $M'$ are compact and have the same Lie algebra and (ii) the factor $\mathfrak{m} = M'\backslash M$ is a finite group called the Weyl group. $\mathfrak{m}$ acts on $\mathfrak{a}_C^*$ as a group of linear transformations by the requirement

$$(s\lambda)(H) = \lambda(s^{-1}H),$$

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$H \in \mathfrak{a}$, $s \in \mathfrak{w}$, $\lambda \in \mathfrak{a}_c^*$, the complexification of $\mathfrak{a}^*$. We then have the Bruhat decomposition

$$G = \bigsqcup_{s \in \mathfrak{w}} (Bm_sB)$$

where $B = MAN$ is a closed subgroup of $G$ and $m_s \in M'$ is the representative of $s$ (i.e., $s = m_sM$).

Some of the most important functions on $G$ are the spherical functions which we now discuss as follows. A non-zero continuous function $\varphi$ on $G$ shall be called (elementary or zonal) spherical function whenever

(i) $\varphi(e) = 1$,

(ii) $\varphi \in C(G//K) := \{g \in C(G) : g(k_1xk_2) = g(x), k_1, k_2 \in K, x \in G\}$

and (iii) $f \ast \varphi = (f \ast \varphi)(e) \cdot \varphi$ for every $f \in C_c(G//K)$. This leads to the existence of a homomorphism $\lambda : C_c(G//K) \to \mathbb{C}$ given as $\lambda(f) = (f \ast \varphi)(e)$. This definition of an elementary spherical function is equivalent to the functional relation

$$\int_K \varphi(xky)dk = \varphi(x)\varphi(y),$$

$x, y \in G$. It has been shown by Harish-Chandra [6.] that elementary spherical functions on $G$ can be parametrized by members of $\mathfrak{a}_c^*$. Indeed every elementary spherical function on $G$ is of the form

$$\varphi_\lambda(x) = \int_K e^{(i\lambda-p)H(xk)}dk, \ \lambda \in \mathfrak{a}_c^*,$$

$$\rho = \frac{1}{2} \sum_{\lambda \in \Delta^+} m_\lambda \cdot \lambda,$$

where $m_\lambda = dim(\mathfrak{g}_\lambda)$, and that $\varphi_\lambda = \varphi_\mu$ iff $\lambda = s\mu$ for some $s \in \mathfrak{w}$. Some of the well-known properties are $\varphi_{-\lambda}(x^{-1}) = \varphi_\lambda(x)$, $\varphi_{-\lambda}(x) = \varphi_\lambda(x)$, $\lambda \in \mathfrak{a}_c^*$, $x \in G$, and if $\Omega$ is the Casimir operator on $G$ then $\Omega \varphi_\lambda = -(\langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle)\varphi_\lambda$, where $\lambda \in \mathfrak{a}_c^*$ and $\langle \lambda, \mu \rangle := tr(adH_\lambda adH_\mu)$ for elements $H_\lambda, H_\mu \in \mathfrak{a}$. The elements $H_\lambda, H_\mu \in \mathfrak{a}$ are uniquely defined by the requirement that $\lambda(H) = tr(adH_\lambda adH_\mu)$ and $\mu(H) = tr(adH adH_\mu)$ for every $H \in \mathfrak{a}$ ([6.], Propositions 3.1.4, 3.2.1, 3.2.2 and Theorem 3.2.3).

Clearly $\Omega \varphi_0 = 0$.

Let

$$\varphi_0(x) := \int_K e^{(-\rho(H(xk)))}dk$$

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be denoted as $\Xi(x)$ and define $\sigma : G \to \mathbb{C}$ as $\sigma(x) = \|X\|$ for every $x = k \exp X \in G$, $k \in K$, $X \in \mathfrak{a}$, where $\| \cdot \|$ is a norm on the finite-dimensional space $\mathfrak{a}$. These two functions are zeroth elementary spherical functions on $G$ and there exist numbers $c, d$ such that $1 \leq \Xi(a)e^{\rho(a)} \leq c(1 + \sigma(a))^d$. Also there exists $r_0 > 0$ such that $c_0 =: \int_G \Xi(x)^2(1 + \sigma(x))^mdx < \infty$ ([6.], p. 254).

For each $0 \leq p \leq 2$ define $\mathcal{C}^p(G)$ to be the set consisting of functions $f$ in $C^\infty(G)$ for which

$$\mu_{g_1,g_2;m}^p(f) := \sup_G |f(g_1; x; g_2)|\Xi(x)^{-2/p}(1 + \sigma(x))^m < \infty$$

where $g_1, g_2 \in \mathcal{U}(g_C)$, the universal enveloping algebra of $g_C$, $m \in \mathbb{Z}^+$, $x \in G$, $f(x; g_2) := \frac{d}{dt}|_{t=0} f(x \cdot (\exp tg_2))$ and $f(g_1; x) := \frac{d}{dt}|_{t=0} f((\exp tg_1) \cdot x)$.

We call $\mathcal{C}^p(G)$ the Schwartz-type space on $G$ for each $0 < p \leq 2$ and note that $\mathcal{C}^2(G)$ is the earlier Harish-Chandra space $\mathcal{C}(G)$ of rapidly decreasing functions on $G$. The inclusions

$$C^\infty_c(G) \subset \bigcap_{0<p<2} \mathcal{C}^p(G) \subset \mathcal{C}^p(G) \subset L^p(G)$$

are continuous and with dense images. It also follows that $\mathcal{C}^p(G) \subseteq \mathcal{C}^q(G)$ whenever $0 \leq p \leq q \leq 2$. Each $\mathcal{C}^p(G)$ is closed under involution and the convolution, *. Indeed $\mathcal{C}^p(G)$ is a Fréchet algebra ([12c.], p. 357) and the relation $\mathcal{C}^p(G) \ast \mathcal{C}^q(G) \subset \mathcal{C}^p(G)$ holds for all $p \geq q$ with $\frac{1}{p} + \frac{1}{q} = 1$; [3c.], Theorem 5.1. We endow $\mathcal{C}^p(G//K)$ with the relative topology as a subset of $\mathcal{C}^p(G)$.

We shall say a function $f$ on $G$ satisfies a general strong inequality if for any $r \geq 0$ there is a constant $c_r > 0$ such that

$$|f(y)| \leq c_r \Xi(y^{-1}x)(1 + \sigma(y^{-1}x))^{-r} \quad \forall \ x, y \in G.$$ 

We observe that if $x = e$ then, using the fact that $\Xi(y^{-1}) = \Xi(y)$ and $\sigma(y^{-1}) = \sigma(y)$, $\forall \ y \in G$, such a function satisfies

$$|f(y)| \leq c_r \Xi(y^{-1})(1 + \sigma(y^{-1}))^{-r} = c_r \Xi(y)(1 + \sigma(y))^{-r}, \quad \forall \ y \in G,$$

showing that a function on $G$ which satisfies a general strong inequality satisfies in particular a strong inequality (in the classical sense of Harish-Chandra, [12c.]). Members of $\mathcal{C}(G)$ are those functions $f$ on $G$ for which $f(g_1; \cdots; g_2)$ satisfies the strong inequality, for all $g_1, g_2 \in \mathcal{U}(g_C)$. We may then

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define \( C_x(G) \) to be those functions \( f \) on \( G \) for which \( f(g_1 \cdots g_2) \) satisfies the general strong inequality, for all \( g_1, g_2 \in U(g_C) \) and a fixed \( x \in G \). It is clear that \( C_x(G) = C(G) \) and that \( \bigcup_{x \in G} C_x(G) \), which contains \( C(G) \), may be given an inductive limit topology.

**Proposition 2.1.** \( \bigcup_{x \in G} C_x(G) \) is a Schwartz algebra. \( \square \)

The algebra \( \bigcup_{x \in G} C_x(G) \) is worthy of an independent study. See [9b.].

For any measurable function \( f \) on \( G \) we define the Harish-Chandra Fourier transform \( f \mapsto H(f) \) as

\[
(Hf)(\lambda) := \int_{AN} f(an)e^{-(\lambda + \rho)(\log a)} \, d adn, \quad \lambda \in \mathfrak{f}_1;
\]

[12a.], p. 364.

It is known (see [6.]) that for \( f, g \in L^1(G) \) we have:

(i.) \( H(f \ast g) = H(f) \cdot H(g) \) on \( \mathfrak{f}_1 \) whenever \( f \) (or \( g \)) is right - (or left-) \( K \)-invariant;

(ii.) \( H(f^*)(\varphi) = \overline{H(f)(\varphi^*)} \), \( \varphi \in \mathfrak{f}_1 \); hence \( H(f^*) = \overline{H(f)} \) on \( \mathcal{P} \) : and, if we define \( f^#(g) := \int_{K \times K} f(k_1 x k_2) \, dk_1 dk_2, x \in G \), then

(iii.) \( H(f^#) = H(f) \) on \( \mathfrak{f}_1 \), where \( \mathfrak{f}_1 \) is the set of all bounded spherical functions and \( \mathcal{P} \) is the subset of all positive-definite spherical functions.

In order to know the image of the Harish-Chandra Fourier transform when restricted to \( C^p(G//K) \) we need the following tube-spaces that are central to the statement of the well-known result of Trombi and Varadarajan [11.] (Theorem 2.2 below).

Let \( C_\rho \) be the closed convex hull of the (finite) set \( \{ s_\rho : s \in \mathfrak{w} \} \) in \( \mathfrak{a}^* \), i.e.,

\[
C_\rho = \left\{ \sum_{i=1}^{n} \lambda_i (s_i \rho) : \lambda_i \geq 0, \ \sum_{i=1}^{n} \lambda_i = 1, \ s_i \in \mathfrak{w} \right\}
\]

where we recall that, for every \( H \in \mathfrak{a} \), \( (s_\rho)(H) = \frac{1}{2} \sum_{\lambda \in \Delta^+} m_\lambda \cdot \lambda(s^{-1}H) \).

Now for each \( \epsilon > 0 \) set \( \mathfrak{f}^\epsilon = \mathfrak{a}^* + i \epsilon C_\rho \). Each \( \mathfrak{f}^\epsilon \) is convex in \( \mathfrak{a}_C^\epsilon \) and

\[
\text{int}(\mathfrak{f}^\epsilon) = \bigcup_{0 < \epsilon' < \epsilon} \mathfrak{f}^{\epsilon'}
\]
Let us define $Z(\mathfrak{f}^\epsilon) = S(a^*)$ and, for each $\epsilon > 0$, let $Z(\mathfrak{f}^\epsilon)$ be the space of all $\mathbb{C}$-valued functions $\Phi$ such that (i) $\Phi$ is defined and holomorphic on $\text{int}(\mathfrak{f}^\epsilon)$, and (ii) each holomorphic differential operator $D$ with polynomial coefficients we have $\sup_{\text{int}(\mathfrak{f}^\epsilon)} |D\Phi| < \infty$. The space $Z(\mathfrak{f}^\epsilon)$ is converted to a Fréchet algebra by equipping it with the topology generated by the collection, $\| \cdot \|_{Z(\mathfrak{f}^\epsilon)}$, of seminorms given by $\| \Phi \|_{Z(\mathfrak{f}^\epsilon)} := \sup_{\text{int}(\mathfrak{f}^\epsilon)} |D\Phi|$. It is known that $D\Phi$ extends to a continuous function on all of $\mathfrak{f}^\epsilon$ ([11.], pp. 278 – 279). An appropriate subalgebra of $Z(\mathfrak{f}^\epsilon)$ for our purpose is the closed subalgebra $\overline{Z}(\mathfrak{f}^\epsilon)$ consisting of $\mathfrak{m}$-invariant elements of $Z(\mathfrak{f}^\epsilon)$, $\epsilon \geq 0$.

2.2 Theorem (Trombi-Varadarajan [11.]). Let $0 < p \leq 2$ and set $\epsilon = (2/p) - 1$. Then the Harish-Chandra Fourier transform $f \mapsto \mathcal{H}f$ is a linear topological algebra isomorphism of $C^p(G/K)$ onto $\mathcal{Z}(\mathfrak{f}^\epsilon)$. □

For the Schwartz algebras $C^p(G/K)$ a larger image than $\mathcal{Z}(\mathfrak{f}^\epsilon)$ is required under the Harish-Chandra Fourier transform. Following Eguchi M. and Kowata A. [4.] and Eguchi M. [3b.] we define the space $\mathcal{Z}(K/M \times \mathfrak{f}^\epsilon)$ as the space of all $\mathfrak{m}$-invariant $C^\infty$ complex-valued functions $F$ on $K/M \times \mathfrak{f}^\epsilon$ which satisfy the following conditions:

(i) for any $k \in K$, the function $\lambda \mapsto F(kM : \lambda)$ extends holomorphically to $\text{int}(\mathfrak{f}^\epsilon)$;
(ii) for any $m \in \mathbb{Z}^+$, $v \in S(\mathfrak{f})$,

$$\zeta^\epsilon_{\nu/m}(F) := \sup_{(kM : \lambda) \in K/M \times \text{int}(\mathfrak{f}^\epsilon)} \left| F(kM : \lambda; \partial(v)|(1 + |\lambda|)^m < \infty. \right.$$  

The seminorms $\zeta^\epsilon_{\nu/m}$ restrict on $\mathcal{Z}(\mathfrak{f}^\epsilon)$ to the earlier Trombi-Varadarajan seminorms, $\| \cdot \|_{Z(\mathfrak{f})}$, and convert $\mathcal{Z}(K/M \times \mathfrak{f}^\epsilon)$ into a Fréchet space. Indeed, $\mathcal{Z}(\mathfrak{f}^\epsilon) \subset \mathcal{Z}(K/M \times \mathfrak{f}^\epsilon)$, as a closed subspace.

We define the map $C^p(G/K) \to \mathcal{Z}(K/M \times \mathfrak{f}^\epsilon) : f \mapsto \mathcal{H}(f)$ now as

$$\mathcal{H}(f)(kM : \lambda) = \int_{AN} f(kan)e^{(-\lambda + \rho)(\log a)} \, dan, \; k \in K, \; \lambda \in \mathfrak{f}^\epsilon,$$

referring to it as the symmetric Harish-Chandra Fourier transform. A very important improvement on Theorem 2.2 is the following.

2.3 Theorem (Eguchi [3b.]). Let $0 < p \leq 2$ and set $\epsilon = (2/p) - 1$. Then the Harish-Chandra Fourier transform $f \mapsto \mathcal{H}f$ is a linear topological algebra isomorphism of $C^p(G/K)$ onto $\mathcal{Z}(K/M \times \mathfrak{f}^\epsilon)$. □
Our first main result, given as Theorem 3.8, contains Theorems 2.2 and 2.3 as special cases.

The polar decomposition of $G$ implies that every $K-$biinvariant function on $G$ is completely determined by its restriction to $A^+$. An example of such a function is the \textit{(zonal) spherical function}, $\varphi_{\lambda}, \lambda \in a_c^*$, on $G$. If we denote the restriction of $\varphi_{\lambda}$ to $A^+$ as $\tilde{\varphi}_{\lambda}$, then the following system of differential equations hold:

$$\tilde{q}\tilde{\varphi}_{\lambda} = \gamma(g)(\lambda)\tilde{\varphi}_{\lambda},$$

where $q \in \mathfrak{Q}(g_C)(:= U(g_C)^K = \text{centralizer of } K \text{ in } U(g_C))$, $\gamma := \gamma_{g/a}$ is the \textit{Harish-Chandra homomorphism} of $\mathfrak{Q}(g_C)$ onto $U(g_C)\mathfrak{w}$, the $\mathfrak{w}-$ invariant subspace of $U(g_C)$, with $\mathfrak{w}$ denoting the \textit{Weyl group} of the pair $(g,a)$, $tU(g_C) \cap \mathfrak{Q}(g_C)$ is the kernel of $\gamma$ and $\tilde{q}$ is the restriction of $q$ to $A^+$. Since

$$\tilde{q}\cdot f = \tilde{q}\cdot \tilde{f},$$

for every $f \in C^\infty(G//K)$ we conclude that $\tilde{q}$ is the \textit{radial component} of $q$. We define $q \in \mathfrak{Q}(g_C)$ to be \textit{spherical} whenever $q = \tilde{q}$.

The above system of differential equations have been extensively used by Harish-Chandra in the investigation of the nature of the spherical functions, $\varphi_{\lambda}$, their asymptotic expansions and their contributions to the Schwartz algebras on $G$. The history of this investigation dated back to the 1950's with the two-volume work of Harish-Chandra (See [6., p. 190]), which still attracts the strength of twenty-first century mathematicians (See [6.] and [9a.]). Other functions on $G$ satisfying different interesting transformations under members of $\mathfrak{Q}(g_C)$ have also been studied in the light of the approach taken by Harish-Chandra. We refer to [6.] and the references cited in it for further discussion.

Now if $G$ is a semisimple Lie group with real rank 1 then it is known (See [11.]) that the above system of differential equations can be replaced with

$$\delta'(\omega)\cdot \varphi_{\lambda} = \gamma(\omega)(\lambda)\cdot \varphi_{\lambda},$$

where $\omega$ is the \textit{Casimir operator} of $G$ and $\delta'(\omega)$ denotes the radial component of the differential operator, $\delta'(\omega)$, associated with $\omega$. If we load the structure of $G$, as a real rank 1 semisimple Lie group, into the last equation it becomes

$$\left(\frac{d^2}{dt^2} + \{(p + q) \coth t + q \tanh t\}\frac{d}{dt}\right)f_{\lambda} = \left(\lambda^2 - \frac{(p + 2q)^2}{4}\right)f_{\lambda},$$

for every function $f_{\lambda} \in C^\infty(G//K)$. We define $\lambda \in a_c^*$ to be \textit{spherical} whenever $f_{\lambda} = \delta'(\omega)f_{\lambda}$.
where \( p = n(\alpha), \ q = n(2\alpha), \ f_\lambda(t) := \varphi_\lambda(\exp tH_0) \) and \( H_0 \) is chosen in \( \mathfrak{a} \) such that \( \alpha(H_0) = 1 \) (See [12b.], p. 190 for the case of \( G = SL(2, \mathbb{R}) \)). Setting \( z = -(\sinh t)^2 \) transforms the above ordinary differential equation to the hypergeometric equation

\[
(z(z - 1) \frac{d^2}{dz^2} + ((a + b + 1)z - c) \frac{d}{dt} + ab)g_\lambda = 0,
\]

where \( g_\lambda(z) = \Phi_\lambda(t) \), \( z < 0 \), \( a = \frac{p+2q+2\lambda}{2} \), \( b = \frac{p+2q}{2} \), and \( c = \frac{p+q+1}{2} \), whose solution is from here given by the well-known Gauss hypergeometric function, \( F(a, b, c : z) \), defined as

\[
F(a, b, c : z) = \sum_{k=0}^{\infty} \frac{(a)_{k}(b)_{k}}{(c)_{k}} \frac{z^{k}}{k!},
\]

\(| z | < 1. \) (6., p. 136). It then follows that

\[
\varphi_\lambda(\exp tH_0) = F(a, b, c : z)
\]

(with \( z = -(\sinh t)^2 \)), and we conclude that the spherical functions on real rank 1 semisimple Lie groups are essentially the hypergeometric function. In other words, the hypergeometric functions form the spherical functions on any real rank 1 semisimple Lie group.

In general and for any \( G \) of arbitrary real rank we always have the Harish-Chandra series expansion for \( \varphi_\lambda \) given as

\[
\varphi_\lambda(h) = \sum_{s \in \mathfrak{d}} c(s\lambda) \left( e^{(s\lambda - \rho)(\log h)} + \sum_{\mu \in L^+} a_{\mu}(s\lambda)e^{(s\lambda - \rho - \mu)(\log h)} \right),
\]

valid for all \( h \in A^+ \), some \( \lambda \) and \( L^+ \) as in [6]. The function \( c \) is the well-known germ of harmonic analysis called Harish-Chandra \( c \)-function.

§3. Harish-Chandra Fourier transform on \( \mathcal{C}_c(G) \).

We denote the set of equivalence classes of the necessarily finite-dimensional irreducible representations of \( K \) by \( \mathcal{E}(K) \) whose character is \( \chi_0 \), for every \( \mathfrak{d} \in \mathcal{E}(K) \). The class functions \( \xi_0 : K \rightarrow \mathbb{C} \) defined as \( \xi_0(k) := \dim(\mathfrak{d})\chi_0(k^{-1}) \) are idempotents (i.e., \( \xi_0 * \xi_0 = \xi_0 \) with \( \xi_{\mathfrak{d}_1} * \xi_{\mathfrak{d}_2} = 0 \) whenever \( \mathfrak{d}_1 \neq \mathfrak{d}_2 \)). Choosing \( \pi \) to be any representation of \( K \) (which may be the restriction to
\( K \) of a representation of \( G \) in a complete Hausdorff locally convex space, \( V \), a continuous projection operator on \( V \) may be given as the image of \( \xi_0 \) under \( \pi \). That is,

\[
E_{\pi,0} := \pi(\xi_0) = \int_K \xi_0(k)\pi(k)dk = \text{dim}(\mathfrak{d}) \int_K \chi_0(k^{-1})\pi(k)dk
\]

(Here \( \int_K dk = 1 \)) Idempotency of \( \xi_0 \) assures that \( E_{\pi,0} \) is indeed a projection on \( V \) (since \( E_{\pi,0}^2 = E_{\pi,0} \) and \( E_{\pi,0}E_{\pi,0} = 0 \) whenever \( \mathfrak{d}_1 \neq \mathfrak{d}_2 \)) and that its range, written as \( V_0(= E_{\pi,0}(V)) \) is a closed linear subspace of \( V \) consisting mainly of members of \( V \) which transform according to \( \mathfrak{d} \); [12b.], p. 109.

The closed linear subspace \( V_0 \) above becomes familiar when \( V = C^p(G) \) under the usual regular representation. In this case the left and right regular representations are denoted as \( l \) and \( r \) given as \( (l(x)f)(y) = f(x^{-1}y) \) and \( (r(x)f)(y) = f(yx) \), respectively; \( x, y \in G \), \( f \in C^p(G) \); and it may be computed that for any \( \mathfrak{d} \in \mathcal{E}(K) \), \( E_{l,0} = l(\xi_0) \) and \( E_{r,0} = r(\xi_0) \) are the respective operators of left and right convolutions by the measure \( \xi_0 dk \) and \( \overline{\xi}_0 dk = \xi_0 dk \), respectively. Here \( \overline{\mathfrak{d}} \) is the class contragredient to \( \mathfrak{d} \). We therefore have a representation \( l \times r \) of \( G \times G \) on \( C(G) \) given as \( ((l \times r)(x, y)f)(z) = f(x^{-1}yz) \), \( x, y, z \in G \) and the corresponding projection \( E_{l \times r, (\mathfrak{d}_1 \times \mathfrak{d}_2)} = (l \times r)\xi_{(\mathfrak{d}_1 \times \mathfrak{d}_2)} \), which from the above remarks could be computed as

\[
E_{l \times r, (\mathfrak{d}_1 \times \mathfrak{d}_2)}f = \xi_{\mathfrak{d}_1} * f * \xi_{\mathfrak{d}_2},
\]

\( f \in C^p(G) \).

We now choose \( \mathfrak{d} \in \mathcal{E}(K) \). The image of \( C^p(G) \) under \( E_{l \times r, (\mathfrak{d} \times \overline{\mathfrak{d}})} \) is the closed subalgebra of \( C^p(G) \) denoted as \( C^p_0(G) \) and is exactly given as

\[
C^p_0(G) = \xi_0 * C^p(G) * \xi_0 = \{ \xi_0 * f * \xi_0 : f \in C^p(G) \};
\]

[6.], p. 11. Thus the members of \( C^p_0(G) \) are those of \( C^p(G) \) which may be written as \( \xi_0 * f * \xi_0 \) for some \( f \in C^p(G) \). That is, every \( g \in C^p_0(G) \) is given as \( g = \xi_0 * f * \xi_0 \), with \( f \in C^p(G) \). We shall henceforth write members \( g \) of \( C^p_0(G) \) as \( g_{\mathfrak{d},f} \), for some \( f \in C^p(G) \).

**Lemma 3.1** Let \( \mathfrak{d} \), \( \xi_0 \) and \( C^p_0(G) \) be as above. Then every \( g_{\mathfrak{d},f} \in C^p_0(G) \) satisfies the transformation \( \xi_0 * g_{\mathfrak{d},f} * \xi_0 = g_{\mathfrak{d},f} \).

**Proof.** We know by definition that every \( g_{\mathfrak{d},f} \in C^p_0(G) \) is of the form \( g_{\mathfrak{d},f} = \xi_0 * f * \xi_0 \); so that \( \xi_0 * g_{\mathfrak{d},f} * \xi_0 = \xi_0 * (\xi_0 * f * \xi_0) * \xi_0 = \xi_0^2 * f * \xi_0^2 = \xi_0 * f * \xi_0 = g_{\mathfrak{d},f} \). \( \square \)
It then means that members of the closed linear subspace $C_\theta^p(G)$ are the $(\mathfrak{d}, \mathfrak{F})$—spherical functions in $C^p(G)$, while the sphericalization operator $E_{\mathfrak{d}x, r, \mathfrak{F}}$ is the continuous projection $C^p(G) \to C_\theta^p(G)$. For the trivial representation $\mathfrak{d} = 1$ of $K$ we shall write $C_{\mathfrak{d}=1}^p(G)$ as $C^p(G/K)$. Here the general transformation $\xi_0 * g_{\kappa_0} * \xi_0 = g_{\kappa_0}(\xi_0)$ (for each $\mathfrak{d} \in \mathcal{E}(K)$, $f \in C^p(G)$), which is now $\xi_1 * \xi_0 = \xi_1 = g_{\kappa_0}$, becomes $(\xi_1 \ast g_{\kappa_0} * \xi_0)(x) = g_{\kappa_0}(x)$, $x \in G$; leading to the familiar expression $g_{\kappa_0}(k_1 r k_2) = g_{\kappa_0}(x)$, $k_1, k_2 \in K$, $x \in G$, for the $K$—biinvariance of spherical functions.

**Lemma 3.2** Let $0 < p \leq 2$ and $f \in C^p(G)$. Then $g_{\kappa_0} \in C^p(G/K)$.

**Proof.** The above remarks shows that $g_{\kappa_0}$ is a spherical function on $G$. If we now extend the definition of the function $\xi_0$ to all of $G$ by requiring that $\xi_0(kan) = e^{-\langle \lambda + \rho \rangle \rho \mu a}[\xi_0(k)$ with $\mathfrak{d} = 1$ and note that $C^p(G)$ is a convolution algebra (Theorem 5.1 of [3c]), we have the result. □

The last Lemma may be proved for the larger closed subalgebra $C^p(G/K)$ of $C^p(G)$ by the consideration of members of the closed subalgebra defined as $E_{\mathfrak{d}x, \mathfrak{F}}(C^p(G)) = C^p(G) * \xi_0$. The situation above may be completely extended to involve the idempotents $\xi_F$ defined for any finite subset $F$ of $\mathcal{E}(K)$. In this case we set $\xi_F = \sum_{\mathfrak{d} \in F} \xi_0$ in order to have $E_{\mathfrak{d}x, r, \mathfrak{F}}$ and the closed linear subspace $C_{\mathfrak{F}}^p(G) = \xi_F * C^p(G) * \xi_F$.

The surjectivity of the map $E_{\mathfrak{d}x, r, \mathfrak{F}} : C^p(G) \to C_{\mathfrak{F}}^p(G)$ assures that every Schwartz $(\mathfrak{d}, \mathfrak{F})$—spherical function on $G$ is in $C_{\mathfrak{F}}^p(G)$. Hence, for any $f \in C^p(G)$ the integral $\int_G g_{\kappa_0}(x) \varphi_{\kappa_0}(x) dx$ converges absolutely and uniformly for all $\lambda \in \mathfrak{F}_I$, ([10c.], p. 110, Lemma 3) and is continuous as a function on $\mathfrak{F}_{\text{I}}$, ([6.], p. 262). Indeed by Theorem 2.2, the Harish-Chandra Fourier transform $\mathcal{H} : C_1^p(G) \to \mathcal{Z}(\mathfrak{F})$ is a linear topological algebra isomorphism ([6.], p. 354) and for $f \in C_1^p(G)$, the function $f \to \mathcal{H}f$ is holomorphic on $\text{int}(\mathfrak{F})$. This means that the inverses $(\mathcal{H}g_{\kappa_0})^{-1}$ and $(\mathcal{H} \xi_0)^{-1}$ exist as functions on (at least) $\text{int}(\mathfrak{F})$.

We actually have more than this, as contained in the following Lemma which gives an insight into the necessity of Eguchi’s space $\mathcal{Z}(K/M \times \mathfrak{F})$ over the Trombi-Varadarajan space $\mathcal{Z}(\mathfrak{F})$, in the passage from $C^p(G/K)$ (of Theorem 2.2) to $C^p(G//K)$ (of Theorem 2.3). In what follows we shall denote the spherical Harish-Chandra Fourier transform of [11], the symmetric Harish-Chandra Fourier transform of [3b.] and the general Harish-Chandra Fourier transform of [1c.] by the same symbol $\mathcal{H}$, since it will be clear which of the three is in use at any given place.

**Lemma 3.3.** The function $(\mathcal{H} \xi_0)^{-1}$ exists and lives on $K/M \times \text{int}(\mathfrak{F})$.

**Proof.** We first recall the extension of $\xi_0$ to all of $G(= KAN)$ by writing
\( \xi_\delta(kan) = e^{-(\lambda+\rho)(\log a)}\xi_\delta(k) \), so that

\[
(\mathcal{H}_{\xi_\delta})(\lambda) = \int_G \xi_\delta(x) \varphi_\lambda(x) dx
\]

\[
= \int_{KAN} \xi_\delta(kan) \varphi_\lambda(kan) e^{2\rho(\log a)} dk d\alpha
\]

\[
= \int_{KAN} e^{-(\lambda+\rho)(\log a)} \xi_\delta(k) \varphi_\lambda(an) e^{2\rho(\log a)} dk d\alpha
\]

\[
= \int_{AN} e^{-(\lambda+\rho)(\log a)} [\left( \int_K \xi_\delta(k) \varphi_\lambda(an) \right) d\alpha]
\]

which by (4.1.3) of [3b] is a function on \( K/M \times \text{int}(\mathfrak{g}^*) \). We then have the result, due to its holomorphy on \( \text{int}(\mathfrak{g}^*) \).

The situation of the last Lemma for \( d = 1 \) is instructive and may be considered separately. Indeed, the same conclusion as in Lemma 3.3 may be deduced for \( d = 1 \) via the structure of the class-1 representations corresponding to the elementary spherical functions \( \varphi_\lambda \) as follows.

**Lemma 3.4.** We always have that \((\mathcal{H}_{\xi_1})(\lambda) = (\mathcal{A}^\varphi_{-\lambda})\), where \( \mathcal{A} \) is the Abel transform and \( \sim \) is the Fourier transform on \( A \).

**Proof.** We already know from Lemma 3.3, that

\[
(\mathcal{H}_{\xi_1})(\lambda) = \int_{AN} e^{-\lambda(\log a)} \varphi_\lambda(an) d\alpha.
\]

Hence, \((\mathcal{H}_{\xi_1})(\lambda) = \int_A e^{\rho(\log a)} \int_N \varphi_\lambda(an) d\alpha = \int_A (\mathcal{A} \varphi_{-\lambda})(a) e^{-\lambda(\log a)} da = (\tilde{\mathcal{A}^\varphi}_{-\lambda}) \). Thus the inverse \((\mathcal{H}_{\xi_1})^{-1}(\lambda)\) is the inversion of the operation of finding the Abel transform of the elementary spherical functions \( \varphi_\lambda \) (represented as an integral on \( K/M \)), followed by the operation of finding its Fourier transform on \( A \). Thus as \( \lambda \in \mathfrak{g}_I \) is the parametrization of the class-1 representation \( \pi_{1,\lambda} \)

(a.) whose matrix coefficients, defined by the constant function 1, is \( \varphi_\lambda \) and (b.) which acts in \( L^2(K/M) \) ([6], p. 103 – 104), it follows that \((\mathcal{H}_{\xi_1})(\lambda) = (\pi_{1,\lambda}(\xi_1)1,1)\) is in \( L^2(K/M) \) and that the end result, \((\mathcal{H}_{\xi_1})^{-1}(\lambda)\) of the inverted operations may be realized as a member of \( L^2(K/M) \) for each \( \lambda \in \mathfrak{g}_I \). □

The above two Lemmas reveal that the passage from the space \( \tilde{\mathcal{Z}}(\mathfrak{g}^*) \) of Trombi-Varadarajan to the space \( \tilde{\mathcal{Z}}(K/M \times \mathfrak{g}^*) \) of Eguchi and the fact that
the spectrum in [3b.] is still pure imaginary (as known in [11.]) are natural and are solely contributed to by the class-1 principal series representations, $\pi_{1,\lambda}$. Due to the importance of these functions on $K/M$ (as seen in [3b.]) we shall now consider a candidate for the image of the whole of $C^p(G)$ under the non-spherical Harish-Chandra Fourier transform, $\mathcal{H}$.

**Definition 3.5.** Let $0 < p \leq 2$ and set $C^p(\widehat{G})$ as

$$C^p(\widehat{G}) := \{(\mathcal{H}\xi_1)^{-1} \cdot h \cdot (\mathcal{H}\xi_1)^{-1} : h \in \mathcal{Z}(\mathfrak{h}_0)\}.$$  □

Observe that each of the three factors of every member of $C^p(\widehat{G})$ is a function on $K/M \times \mathfrak{k}^\ast$, where we set $h(kM : \lambda) := h(\lambda)$, $\lambda \in \mathfrak{k}^\ast$. Thus every member of the above set $C^p(\widehat{G})$ may therefore be seen as a function on $K/M \times \mathfrak{k}^\ast \times K/M$ ([1c., p. 17]); so that $C^p(\widehat{G})$ may be thought of as the function-space of the form $\mathcal{Z}(K/M \times \mathfrak{k}^\ast \times K/M)$ (modelled on $\mathcal{Z}(\mathfrak{h}_0)$ and $\mathcal{Z}(K/M \times \mathfrak{k}^\ast)$). It is clear, from Lemma 3.3, that

$$\mathcal{Z}(\mathfrak{h}_0) \subset \mathcal{Z}(K/M \times \mathfrak{k}^\ast) \subset C^p(\widehat{G})$$

and that both $\mathcal{Z}(\mathfrak{h}_0)$ and $\mathcal{Z}(K/M \times \mathfrak{k}^\ast)$ are subspaces of $C^p(\widehat{G})$. We extend the seminorms on $\mathcal{Z}(K/M \times \mathfrak{k}^\ast)$ to all of $C^p(\widehat{G})$ by setting

$$\eta_{v,m}(F) := \sup_{(kM : \lambda \in K/M \times \text{int}(\mathfrak{k}^\ast) \times K/M, v \in S(\mathfrak{h}_0))} |F(kM : \lambda : kM; \partial(v))|(1 + |\lambda|)^m,$$

for $F \in C^p(\widehat{G})$, $v \in S(\mathfrak{h}_0)$. Observe that $\eta_{v,m}$ restricts to $\zeta_{v,m}$ on $\mathcal{Z}(K/M \times \mathfrak{k}^\ast)$. The following result is now immediate.

**Lemma 3.6.** $C^p(\widehat{G})$ is a Schwartz algebra. □

**Corollary 3.7.** $\mathcal{Z}(\mathfrak{h}_0)$ and $\mathcal{Z}(K/M \times \mathfrak{k}^\ast)$ are closed subalgebras of $C^p(\widehat{G})$.

**Proof.** Both $\mathcal{Z}(\mathfrak{h}_0)$ and $\mathcal{Z}(K/M \times \mathfrak{k}^\ast)$ are Schwartz subalgebras of $C^p(\widehat{G})$. □

The following is our first main result giving the full non-spherical image of $C^p(G)$ under the Harish-Chandra Fourier transform and may be compared with Theorems 2.2 and 2.3.

**Theorem 3.8.** (The Fundamental Theorem of Harmonic Analysis on $G$). Let $0 < p \leq 2$, then the Harish-Chandra Fourier transform, $\mathcal{H}$, sets up a linear topological algebra isomorphism $\mathcal{H} : C^p(G) \rightarrow C^p(\widehat{G})$.

**Proof.** Let $f \in C^p(G)$ be arbitrarily chosen, then (by Lemma 3.2) there exists $g_{1,f} \in C^p_1(G)$ given as $g_{1,f} = \xi_1 \ast f \ast \xi_1$, such that (by Theorem 2.2 we have) $\mathcal{H}g_{1,f} \in \mathcal{Z}(\mathfrak{h}_0)$. Hence, we have that

$$\mathcal{H}(g_{1,f}) = \mathcal{H}(\xi_1 \ast f \ast \xi_1) = \mathcal{H}(\xi_1) \cdot \mathcal{H}(f) \cdot \mathcal{H}(\xi_1),$$

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so that
\[ \mathcal{H}(f) = (\mathcal{H}\xi_1)^{-1} \cdot (\mathcal{H}g_{1,f}) \cdot (\mathcal{H}\xi_1)^{-1} \in C^p(\hat{G}). \]
That is, \( \mathcal{H}(C^p(G)) \subset C^p(\hat{G}) \).

Due to the detailed results of [6.], [10c.], [3b.] and [4.] on the already known properties of \( \mathcal{H} \) concerning its linearity, continuity, injectivity and homomorphism, each of these properties of \( \mathcal{H} \) now reduces to the same property for \( \mathcal{H}|_{C^p(G//K)} \), due to the direct dependence of members of \( \hat{G} \) on members of \( \hat{Z}(\mathfrak{g}^c) \) as now seen from Definition 3.5. For example, since \( \mathcal{H} \) is already shown in the above paragraph to map \( C^p(G) \) to \( C^p(\hat{G}) \), the injectivity of \( \mathcal{H} \) follows from the already known injectivity of \( \mathcal{H}|_{C^p(G//K)} \). We are only left to show that \( \mathcal{H} \) is surjective onto \( C^p(\hat{G}) \).

To this end, let \( j \in C^p(\hat{G}) \). That is, let \( j = (\mathcal{H}\xi_1)^{-1} \cdot h \cdot (\mathcal{H}\xi_1)^{-1} \) for some \( h \in \hat{Z}(\mathfrak{g}^c) \). Define \( f \) as the convolutions given as
\[ f = \mathcal{H}^{-1}((\mathcal{H}\xi_1)^{-1}) \ast \mathcal{H}^{-1}h \ast \mathcal{H}^{-1}((\mathcal{H}\xi_1)^{-1}). \]
Now \( \mathcal{H}^{-1}h \in C^p(G//K) \) (by the Trombi-Varadarajan theorem); hence we have that \( \mathcal{H}^{-1}h \in C^p(G//K) \subset C^p(G) \) and that \( \mathcal{H}^{-1}((\mathcal{H}\xi_1)^{-1}) \) is the Eguchi-pullback \( \hat{Z}(K/M \times \mathfrak{g}^c) \to C^p(G//K) \subset C^p(G) \), [3b.], p. 193.

Hence, as \( f \) is now shown to be the convolutions of members of \( C^p(G) \) and \( C^p(G) \) is a convolution algebra, we conclude that
\[ f := \mathcal{H}^{-1}((\mathcal{H}\xi_1)^{-1}) \ast \mathcal{H}^{-1}h \ast \mathcal{H}^{-1}((\mathcal{H}\xi_1)^{-1}) \in C^p(G). \]
Finally, we have that \( \mathcal{H}(f) = (\mathcal{H}\mathcal{H}^{-1}((\mathcal{H}\xi_1)^{-1})) \cdot (\mathcal{H}\mathcal{H}^{-1}h) \cdot (\mathcal{H}\mathcal{H}^{-1}((\mathcal{H}\xi_1)^{-1})) = (\mathcal{H}\xi_1)^{-1} \cdot h \cdot (\mathcal{H}\xi_1)^{-1} = j \) as expected. The inverse map

\[ \mathcal{H}^{-1} : C^p(\hat{G}) \to C^p(G) \]
is continuous, being the continuous extension to \( C^p(\hat{G}) \) of the (continuous) map \( \mathcal{H}^{-1}|_{\hat{Z}(K/M \times \mathfrak{g}^c)} : \hat{Z}(K/M \times \mathfrak{g}^c) \to C^p(G//K) \) of Eguchi, [3b.]. □

**Corollary 3.9.** The algebra \( \mathcal{H}(C^p(G)) \cong C^p(\hat{G}) \) may also be seen as
\[ (\mathcal{H}\xi_1)^{-1} \cdot \hat{Z}(\mathfrak{g}^c) \cdot (\mathcal{H}\xi_1)^{-1}, \]
where \( (\mathcal{H}\xi_1)^{-1} \) denotes the fixed function given as \( (\mathcal{H}\xi_1)^{-1}(\lambda) \), for \( \lambda \in \mathfrak{g}^c \).

**Proof.** By Definition 3.5 and Theorem 3.8 we have that \( \mathcal{H}(C^p(G)) \cong C^p(\hat{G}) := \{(\mathcal{H}\xi_1)^{-1} \cdot h \cdot (\mathcal{H}\xi_1)^{-1} : h \in \hat{Z}(\mathfrak{g}^c)\} = (\mathcal{H}\xi_1)^{-1} \cdot \hat{Z}(\mathfrak{g}^c) \cdot (\mathcal{H}\xi_1)^{-1}. \) □

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Direct computations of members of $\mathcal{C}^p(\widehat{G})$ may also be embarked on. It is clear that the algebra $\mathcal{C}^p(\widehat{G})$ is still in its *bundled* form and that it would need to be further opened up than has been done in the decomposition contained in Corollary 3.9. Indeed, it has to be explicitly computed and understood for different examples of $G$ and its members parametrized, while already known cases are shown to be deduced from it. In particular, we are yet to give an explicit computation of the inverse map $H^{-1} : \mathcal{C}^p(\widehat{G}) \to \mathcal{C}^p(G)$ in the theory of wave-packets or discuss the explicit nature of contributions of the discrete and principal series of representations of $G$ to $\mathcal{C}^p(\widehat{G})$ or ask if there is still a split (as known for $\mathcal{C}^p(SL(2, \mathbb{R}))$ in [2] and for $\mathcal{C}^p(\widehat{G} : F)$ in [10d.]) into the discrete and principal parts at the level of $\mathcal{C}^p(\widehat{G})$ or consider other well known results on the spherical case for all of $\mathcal{C}^p(G)$.

However, Theorem 3.8 marks a significant attainment since Harish-Chandra defined the Schwartz space $\mathcal{C}(G) := \mathcal{C}^2(G)$ and should be seen as an harvest of known results. Our approach and attainment of the full Harish-Chandra transform in Theorem 3.8 also gives a fresh impetus to the practice of harmonic analysis in the tradition of Harish-Chandra. For example and with $\text{rank}(G) = \text{rank}(K)$, the image $\mathcal{Z}(\mathfrak{g}^e)$ is known to be decomposable into a discrete part $\mathcal{Z}_B(\mathfrak{g}^e)$ and principal part $\mathcal{Z}_H(\mathfrak{g}^e)$. That is,

$$\mathcal{Z}(\mathfrak{g}^e) = \mathcal{Z}_B(\mathfrak{g}^e) \times \mathcal{Z}_H(\mathfrak{g}^e)$$

(See [10c.], p. 109, [11.] and [12b.], p. 272 (Theorem 53)), where $B$ and $H$ are compact and non-compact Cartan subgroups of $G$. Thus, from Corollary 3.9, members $\mathfrak{g}$ of the Schwartz algebra $\mathcal{C}^p(\widehat{G})$ are pairs $\mathfrak{g} = (\mathfrak{g}_B, \mathfrak{g}_H)$ given as $\mathfrak{g}_B = (\mathcal{H}\xi_1)^{-1} \cdot h_B \cdot (\mathcal{H}\xi_1)^{-1}$ and $\mathfrak{g}_H = (\mathcal{H}\xi_1)^{-1} \cdot h_H \cdot (\mathcal{H}\xi_1)^{-1}$ (with $h_B \in \mathcal{Z}_B(\mathfrak{g}^e)$ and $h_H \in \mathcal{Z}_H(\mathfrak{g}^e)$) both of which are linearly related as given in [10c.], p. 109 (Definition 3). Hence, to answer one of the questions raised in the paragraph above, the image $\mathcal{C}^p(\widehat{G})$ of the non-spherical Harish-Chandra Fourier transform on $G$ has a decomposition $\mathcal{C}^p(\widehat{G}) \cong \mathcal{C}^p_B(\widehat{G}) \times \mathcal{C}^p_H(\widehat{G})$ where

$$\mathcal{C}^p_B(\widehat{G}) = \{ \mathfrak{g}_B : \mathfrak{g}_B = (\mathcal{H}\xi_1)^{-1} \cdot h_B \cdot (\mathcal{H}\xi_1)^{-1}, \ h_B \in \mathcal{Z}_B(\mathfrak{g}^e) \}$$

and

$$\mathcal{C}^p_H(\widehat{G}) = \{ \mathfrak{g}_H : \mathfrak{g}_H = (\mathcal{H}\xi_1)^{-1} \cdot h_H \cdot (\mathcal{H}\xi_1)^{-1}, \ h_H \in \mathcal{Z}_H(\mathfrak{g}^e) \}.$$
to any \( j = (\mathcal{H}\xi_1)^{-1} \cdot a \cdot (\mathcal{H}\xi_1)^{-1} \in \tilde{Z}(K/M \times \mathfrak{F}^c \times K/M) \) (with \( a \in \tilde{Z}(\mathfrak{F}^c) \)), is given as

\[
\psi_j = \mathcal{H}^{-1}((\mathcal{H}\xi_1)^{-1} \ast \mathcal{H}^{-1}((\mathcal{H}\xi_1) \cdot j \cdot (\mathcal{H}\xi_1)) \ast \mathcal{H}^{-1}((\mathcal{H}\xi_1)^{-1}).
\]

That is (by eliminating \( j \)),

\[
\psi_j = \mathcal{H}^{-1}((\mathcal{H}\xi_1)^{-1} \ast \mathcal{H}^{-1}a \ast \mathcal{H}^{-1}((\mathcal{H}\xi_1)^{-1});
\]

from which we have earlier seen (in the proof of Theorem 3.8) that

\[
\mathcal{H}(\psi_j) = j.
\]

We observe here that the general wave-packets \( \psi_j \) on \( G \) are expressible in terms of the (normalized) spherical wave-packets \( \mathcal{H}^{-1}a \) on \( G//K \) and that the general wave-packets \( \psi_j \) on \( G \) assumes a decomposition into a convolution of three wave-packets; namely \( \mathcal{H}^{-1}((\mathcal{H}\xi_1)^{-1}), \mathcal{H}^{-1}a \) and \( \mathcal{H}^{-1}((\mathcal{H}\xi_1)^{-1}) \), with the spherical wave-packets \( \mathcal{H}^{-1}a \) (with \( a \in \tilde{Z}(\mathfrak{F}^c) \)) at the center. The above split of \( \mathcal{C}^p(\hat{G}) \cong \mathcal{C}^p_B(\hat{G}) \times \mathcal{C}^p_H(\hat{G}) \) into discrete and principal parts is equally possible for the general wave-packets \( \psi_j \) and this also answered one of the questions raised in the first paragraph after Corollary 3.9. Details of these shall be considered in a forthcoming paper.

The next two Corollaries show how to recover Theorems 2.2 and 2.3 from Theorem 3.8 and also gives corresponding restricted form of the decomposition in Corollary 3.9 for Theorem 2.3.

**Corollary 3.10.** The Harish-Chandra Fourier transform of \( \mathcal{C}^p(G/K) \) has a decomposition into a product of a function on \( K/M \) with another function on \( \mathfrak{F}^c \). That is,

\[
\tilde{Z}(K/M \times \mathfrak{F}^c) = (\mathcal{H}\xi_1)^{-1} \cdot \tilde{Z}(\mathfrak{F}^c). \quad \square
\]

**Corollary 3.11** (Trombi-Varadarajan theorem). Let \( 0 < p \leq 2 \), then

\[
\mathcal{H}(\mathcal{C}^p(G//K)) \cong \tilde{Z}(\mathfrak{F}^c).
\]

**Proof.** We have from Lemma 3.1 that every \( f \in \mathcal{C}^p(G//K) \) satisfies \( \xi_1 \ast f \ast \xi_1 = f \). Hence, \( \mathcal{H}\xi_1 \cdot \mathcal{H}f \cdot \mathcal{H}\xi_1 = \mathcal{H}f \); so that

\[
\mathcal{H}f = (\mathcal{H}\xi_1)^{-1} \cdot \mathcal{H}f \cdot (\mathcal{H}\xi_1)^{-1} \in \mathcal{C}^p(\hat{G}),
\]
by Theorem 3.8. As $C^p(G//K)$ is a proper closed subalgebra of $C^p(G)$, the continuity of $H$ ([6.], p. 340) means that $Hf$ resides in a closed subalgebra of $C^p(\hat{G})$. Now combining the fact that $Hf \in \tilde{Z}(\mathfrak{g}^e)$ ([6.], Theorem 7.8.6) with Corollary 3.7 in the presence of Theorem 3.8 gives the result. \square

Analogous argument to Corollary 3.11 may also be given to prove the main result of [3b.]. The next result may be seen from [1c.], p. 17, [12a.], p. 364 and [3b.], p. 193.

**Lemma 3.12.** The Harish-Chandra Fourier transform of every function $f \in C^p(G)$ may be computed (for $k_1, k_2 \in K$, $\lambda \in \mathfrak{g}_I$) as

$$(Hf)(k_1 M : \lambda : k_2 M) = (H\xi_1)^{-1} \cdot \int_{AN} f(an)e^{(\lambda + \rho)(\log a) \text{dadn}} \cdot (H\xi_1)^{-1}.$$  

In particular, $Hf$ is independent of $k_1, k_2 \in K$. \square

**Corollary 3.13.** For $k_1, k_2 \in K$, $\lambda \in \mathfrak{g}_I$ we have

$$(H\xi_1)(k_1 M : \lambda : k_2 M) = \int_{AN} e^{-2\lambda(\log a)} \text{dadn}.$$  

**Proof.** Compute using the expression

$$(Hf)(k_1 M : \lambda : k_2 M) = \int_{AN} f(an)e^{(\lambda + \rho)(\log a) \text{dadn}}. \square$$

The setting for the harmonic analysis of $C^p(G : F)$ in [10c.] is that of the space

$$\{(H\xi_F)^{-1} \cdot h \cdot (H\xi_F)^{-1} : h \in \tilde{Z}(\mathfrak{g}^e)\},$$  

for any finite $F \subset \mathcal{E}(K)$ and with the restriction on the $K-$type. We however note that, since this restriction on $K-$type is a slight generalization of and reduces to the condition of $K-$biinvariance of a spherical function, it follows that Trombi’s spaces $C^p(G : F)$ and $C^p(\hat{G} : F)$ are not that far from Trombi-Varadarajan’s spaces $C^p(G//K)$ and $\tilde{Z}(\mathfrak{g}^e)$ ([10b.], p. 291), respectively.

Now let $C^p(G)'$ and $C^p(\hat{G})'$ denote the respective topological dual spaces of $C^p(G)$ and $C^p(\hat{G})$, which are topological vector spaces in the weak topology ([3b.], p. 214 and [3c.]). A distribution on $G$ will be said to be $p-$tempered if it extends to a continuous linear functional from $C^p(G)$ to $\mathbb{C}$. A 2-$tempered$ distribution is simply called tempered. The precise meaning for the Harish-Chandra Fourier transform of a $p-$tempered distribution on $G$ is immediate from the following.
Theorem 3.14. The transpose $H' : C^p(\hat{G})' \to C^p(G)'$ of $H$ is a linear topological isomorphism. □

The full invariant harmonic analysis on $G$ is now attainable; See Coda in [12b.] and [10d.]. It is to be noted here that the results of [10d.] depends on and is therefore restricted by those results in [10c.], where the Harish-Chandra Fourier transform $f \mapsto Hf$ was considered only for $f \in C^p(G : F)$ ($\subset C^p(G)$). We shall however show here that in the presence of the above Theorem 3.8 (which is valid for all $f \in C^p(G)$) a full invariant harmonic analysis on $G$ may now be developed as follows, with proof essentially as in [10d.].

To this end, let $\theta_\pi$ denote the global characters of a quasi-simple admissible representation $\pi$ of $G$ and, for $f \in C^p(G)$, write $\hat{f}$ which is defined on $\hat{G}$ as

$$\hat{f}(\pi) = \theta_\pi(f) = \int_{G'} f(x^{-1}) \theta_\pi(x) dx$$

($G'$ being the regular set in $G$ on which $\theta_\pi$ is well-known to be analytic) and is termed the invariant Harish-Chandra Fourier transform of $f$. Since the global character $\theta_\pi$ is the distribution on $G$ given as $\theta_\pi(f) = \text{tr}(\int_G f(x) \pi(x) dx)$ we then have that

$$\hat{f}(\pi) = \text{tr}(\int_{G'} f(x) \pi(x) dx) = \text{tr}(Hf), \ f \in C^p(G).$$

It has been shown by Trombi [8c., d.] that the split into discrete and principal parts is to be expected even at the level of $C^p(\hat{G})$ (See also Theorems 60 and 65 in [12b.]). We therefore proceed as follows.

We denote the principal and discrete series of representations of $G$ by $\pi_{\sigma, \lambda}$ (with $\sigma \in \hat{M}$, $\lambda \in \text{int}(\mathfrak{f}^c)$) and $\pi_\omega$ (with $\omega \in \hat{G}_d$), respectively. The following results are well-known.

Lemma 3.15. ([10d.]) Let $0 < p \leq 2$ and $f \in C^p(G)$. Then

(i.) $\hat{f}(\sigma : \cdot)$ is an entire function on $\mathfrak{f}_c$ of exponential type;
(ii.) $\hat{f}(s\sigma : s\lambda) = \hat{f}(\sigma : \lambda)$ for all $s \in \mathfrak{w}$, $(\sigma, \lambda) \in \hat{M} \times \mathfrak{f}_c$;
(iii.) $\hat{f}(\sigma : \lambda) = 0$, if $\sigma \notin \hat{M}$ and $\hat{f}(\omega) = 0$, if $\omega \notin \hat{G}_d$.

Proof. It is sufficient to prove these results for $f \in C^\infty_c(G)$. See [10d.] and (3.3.9) of [6.]. □

The properties included in Lemma 3.15 suggest a candidate for the image of the invariant Harish-Chandra Fourier transform. Set

$$C^p(G) = \{\theta_{\sigma, \lambda} : \sigma \in \hat{M}, \lambda \in \text{int}(\mathfrak{f}_c^s)\} \bigcup \{\theta_\omega : \omega \in \hat{G}_d\} \bigcup \mathcal{B}_p$$
where \( \mathcal{B}_p \) is as in [10.d.] (See also [12.b., p. 285].)

**Definition 3.16** ([10.d.]) Let \( \mathcal{C}^p(C(G))_o \) denote the linear space of all complex-valued functions on \( C^p(G) \) such that for \( L \in \mathcal{C}^p(C(G))_o \) (and by denoting \( L(\theta_{\sigma,\lambda}) \) as \( L(\sigma, \lambda) \) and \( L(\theta_\omega) \) as \( L(\omega) \));

(i.) each \( L(\sigma : \cdot) \) is holomorphic on \( \text{int}(\tilde{\mathcal{F}}_c^e) \);

(ii.) \( L(s\sigma : s\lambda) = L(\sigma : \lambda) \) for all \( s \in \mathcal{W} \), \( (\sigma, \lambda) \in \tilde{M} \times \text{int}(\tilde{\mathcal{F}}_c^e) \);

(iii.) \( L(\sigma : \lambda) = 0 \), if \( \sigma \notin \tilde{M} \) and \( L(\omega) = 0 \), if \( \omega \notin \tilde{G}_d \);

(iv.) \( \sup_{\tilde{M} \times \text{int}(\tilde{\mathcal{F}}_c^e)} (1 + | \lambda |)^\alpha | L(\sigma : \lambda ; u) | =: \nu_{u,\alpha}^p(L) < \infty \), for all \( \alpha \in \mathbb{R} \), \( u \in S(\tilde{\mathcal{F}}_c) \).

Now let \( \mathcal{C}^p(C(G)) \) denote the subspace of functions \( L \in \mathcal{C}^p(C(G))_o \) such that

\[
L(\sigma ; t\zeta ; \partial^k(u)) = \sum_{\theta \in \mathcal{B}_p} c_p(\theta_{\sigma,t\zeta,k} : \theta)L(\theta),
\]

where \( \sigma \in \tilde{M} \), \( t \in \mathcal{W} \), \( \zeta \in V_p \), \( 0 \leq k \leq \theta_t(\zeta) - 1 \), \( u \in S(\tilde{\mathcal{F}}_c) \). The reader is referred to the remarks on p. 285 of [12.b.] following the definition of \( \tilde{f} \) for the motivation of and necessity for the set \( \mathcal{B}_p \) and the requirement on \( \tilde{L} \) given above in the case of \( SL(2,\mathbb{R}) \). We then give \( \mathcal{C}^p(C(G)) \) the topology generated by the family of seminorms \( \mu_{u,\alpha}^p \) given as

\[
\mu_{u,\alpha}^p(L) = \nu_{u,\alpha}^p(L) + \left( \sum_{\omega \in \tilde{G}_d} | L(\omega) |^2 \right)^{\frac{1}{2}},
\]

\( u \in S(\tilde{\mathcal{F}}_c) \), \( \alpha \in \mathbb{R} \).

**Lemma 3.17.** The family \( \mu_{u,\alpha}^p \) of seminorms convert \( \mathcal{C}^p(C(G)) \) into a Schwartz space. □

**Theorem 3.18.** The invariant Harish-Chandra Fourier transform \( f \mapsto \hat{f} \) is a linear topological algebra isomorphism \( \mathcal{C}^p(G) \rightarrow \mathcal{C}^p(C(G)) \).

**Proof.** The map \( tr \) is clearly continuous, using an argument analogous to Proposition 1 of [10.d., p. 235], while the surjectivity argument for \( f \mapsto \hat{f} \) in Theorem 1 of [10.d., p. 235] also holds for all \( C^p(G) \), now that we already have the full isomorphism \( \mathcal{H} : C^p(G) \cong C^p(\hat{G}) \) in Theorem 3.8. □

We now have the linear topological algebra isomorphisms

\[
\mathcal{C}^p(G) \cong \mathcal{C}^p(\hat{G}) \cong \mathcal{C}^p(C(G)) \text{ given by } f \mapsto \mathcal{H}f \mapsto \hat{f}.
\]

**Proposition 3.19.** Members of \( \mathcal{C}^p(C(G)) \) are all of the form

\[
L((\sigma ; \lambda) \oplus \omega) = tr \left[ ((\mathcal{H}_\xi_1)^{-1})^2 \cdot \int_{AN} f(an)e^{(\lambda + \rho)(\log a)} da \right],
\]

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for any $f \in C^p(G)$. More precisely, we have that $L(\sigma, \lambda) = tr\left[\begin{pmatrix}((\mathcal{H}\xi_1)^{-1})^2 \cdot h_H \end{pmatrix}\right]$, with $h_H \in \mathcal{Z}_H(\mathfrak{g}^t)$ and $L(\omega) = tr\left[\begin{pmatrix}((\mathcal{H}\xi_1)^{-1})^2 \cdot h_B \end{pmatrix}\right]$, with $h_B \in \mathcal{Z}_B(\mathfrak{g}^t)$.

**Proof.** We combine the remarks following Corollary 3.9 with Lemma 3.12. □

We shall refer to a map $m$ of functions on $G$ as being invariant whenever $m(f^y) = m(f)$, where $f^y(x) = f(y^{-1}xy)$, $x, y \in G$. The following result is the reason for the term invariant harmonic analysis.

**Theorem 3.20.** The invariant Harish-Chandra Fourier transform $f \mapsto \hat{f}$ is invariant in the above sense.

**Proof.** We set $y_1 = y$ and $y_2 = y^{-1}$ in the formula on p. (1.6) of [1c.], to get

$$(\mathcal{H}f^y)(\lambda) = \pi_{\sigma,\lambda}(y) \cdot \mathcal{H}f \cdot \pi_{\sigma,\lambda}(y^{-1}),$$

for all $(\sigma, \lambda) \in \widehat{M} \times int(\mathfrak{g}^t)$, $f \in C^p(G)$. Hence, we have $\hat{f}^y = tr(\mathcal{H}f^y) = tr(\pi_{\sigma,\lambda}(y) \cdot \mathcal{H}f \cdot \pi_{\sigma,\lambda}(y^{-1})) = tr(\pi_{\sigma,\lambda}(y) \cdot \pi_{\sigma,\lambda}(y^{-1}) \cdot \mathcal{H}f) = tr(\mathcal{H}f) = \hat{f}$. □

Denote the kernel of the map $tr$ in $C^p(G)$ by $K^p(G)$. It is known that $K^2(SL(2, \mathbb{R}))$ is the closure of the span of the commutators in $C^2(SL(2, \mathbb{R}))$; [12b], Theorem 65, p. 289. Then $C^p(C(G))/K^p(G)$ is a commutative Fréchet algebra with operation induced from the convolution on $C^p(C(G))$. The duo of Theorem 3.18 and and the Fundamental Theorem of homomorphisms imply that

$$C^p(G)/K^p(G) \cong C^p(C(G)).$$

See also Theorem 66 of [12b]. The following is a generalization of Rao-Varadarajan theorem on $C^2(SL(2, \mathbb{R}))$ to all of $C^2(G)$.

**Theorem 3.21.** $f \in K^p(G)$ if and only if $\theta(f) = 0$, for every $\theta \in C^p(G)$. A $p$-tempered distribution on $G$ is invariant if and only if it vanishes on $K^p(G)$.

**Proof.** Our argument is as in [12b], p. 289. □

A further realization of members of $C^p(C(G))$ than in Proposition 3.19 and an explicit computation of the members of $K^p(G)$ are desirable. A way forward is in the computation of $(\mathcal{H}\xi_1)^{-1}$. Let us now consider the zero-Schwartz spaces, $C^0(G)$ and $C^0(\widehat{G})$.

We define

$$C^0(G) = \bigcap_{0 < p \leq 2} C^p(G),$$

from which it follows that

$$C^\infty_c(G) \subset C^0(G) \subset \cdots \subset C^{p_1}(G) \subset C^{p_2}(G) \subset \cdots$$

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with $0 < p_1 \leq p_2 \leq 2$. We topologize $C^0(G)$ with the projective limit topology for this intersection. Each of these subspaces is a Fréchet algebra under convolution (See [2.], and the references contained in section 19), $C_c^\infty$ is dense in $C^0(G)$ with continuous inclusion. This shows that (both of) the (invariant and non-invariant) Harish-Chandra Fourier transform $\mathcal{H}$ on $C^p(G)$ may be restricted to $C^0(G)$. We also define $C^0(\hat{G})$ in a completely analogous manner as done for $C^0(\hat{G})$. Hence, $\mathcal{H}$ of Theorem 3.8 restricts to $C^0(G)$ and we have $\mathcal{H} : C^0(G) \to C^0(\hat{G})$.

The spaces $C^0(G)$ and $C^0(\hat{G})$ may also be topologized by means of other seminorms, instead of the projective limit topologies for their corresponding intersections ([2.], p. 99 and p. 102). We however have the following result in any of the said equivalent topologies.

**Theorem 3.22.** The Harish-Chandra Fourier transform $\mathcal{H}$ sets up a linear topological algebra isomorphism $\mathcal{H} : C^0(G) \to C^0(\hat{G})$.

**Proof.** We simply take $\mathcal{H} : C^0(G) \to C^0(\hat{G})$ as the restriction of the linear topological algebra isomorphism in Theorem 3.8. □

§4. The example of spherical convolutions; $g_{\lambda,A} := \tilde{f} \ast \varphi_\lambda$.

Let $f \in C^p(G)$ and define $\tilde{f}$ by $\tilde{f} = f_{|A^+}$, then $\tilde{f} \in C^p(A^+)$. [6.], p. 255. Hence,

$$\delta'(q)g_{\lambda,A} = \tilde{f} \ast (\delta'(q)\varphi_\lambda) = \gamma(q)(\lambda)(\tilde{f} \ast \varphi_\lambda) = \gamma(q)(\lambda)(g_{\lambda,A}).$$

□

This Lemma shows that $g_{\lambda,A}$ is an eigenfunction on $A^+$. Hence, $g_{\lambda,A}$ has a Harish-Chandra series expansion on $A^+$ which is

$$g_{\lambda,A}(h) = \sum_{s \in \mathbb{W}} c(s\lambda) \left( e^{(s\lambda - \rho)(\log h)} + \sum_{\mu \in L^+} a_\mu(s\lambda)e^{(s\lambda - \rho - \mu)(\log h)} \right),$$

valid for all $h \in A^+$, and some $\lambda, L^+$ in [6.], with the $c-$function now given as $c(s\lambda) = \sum_{1 \leq i \leq w} \gamma^s(\lambda)g_{\lambda,A}(h_0; u_i^s)$ where $w = |\mathbb{W}|$, $s \in \mathbb{W}$ in which $\gamma^s(\lambda)$ are
the entries of the inverse matrix of the invertible \( w \times w \) matrix \( (\gamma_{si}(\lambda))_{1 \leq i \leq w} \) (with \( \gamma_{si}(\lambda) = \Phi(s\lambda : h_o : u'_i) \), a basis for \( A(A^+ : \chi_{\lambda}) \), \( h_o \in A^+ \) and each \( u'_i \) is the radial component of each \( u_i \in S(\mathcal{F}) \).

In order to then compute \( \mathcal{H}(C^p(A^+)) \) we shall employ the methods of §3. to prove the following.

**Lemma 4.2.** \( \mathcal{H}\tilde{f} = (\mathcal{H}g_{\lambda,A}) \cdot (\mathcal{H}\varphi_{\lambda})^{-1} \) for \( \lambda \in \mathfrak{F}_I \).

**Proof.** \( \mathcal{H}g_{\lambda,A} = \mathcal{H}(\tilde{f} \ast \varphi_{\lambda}) = (\mathcal{H}\tilde{f}) \cdot (\mathcal{H}\varphi_{\lambda}) \). \( \square \)

This reveals that the non-spherical Harish-Chandra Fourier of \( \tilde{f} \) is given in terms of \( \mathcal{H}g_{\lambda,A} \) and \( (\mathcal{H}\varphi_{\lambda})^{-1} \) in which both \( g_{\lambda,A} \) and \( \varphi_{\lambda} \) are elementary spherical functions.

**Proposition 4.3.** \( \mathcal{H}(C^p(A^+)) = \{ (\mathcal{H}g_{\lambda,A}) \cdot (\mathcal{H}\varphi_{\lambda})^{-1} : \lambda \in \mathfrak{F}_I \} \). \( \square \)

Explicit computations for both \( \mathcal{H}g_{\lambda,A} \) and \( (\mathcal{H}\varphi_{\lambda})^{-1} \) via the consideration of their Harish-Chandra series expansions give concrete members of \( \mathcal{H}(C^p(A^+)) \). Here \( \mathcal{H} \) is the spherical Harish-Chandra Fourier transform given as

\[
(\mathcal{H}f)(\nu) = \int_G f(x) \varphi_{-\nu}(x) dx = \int_{Kc\ell(A^+) \cdot K} f(ka_t k) \varphi_{-\nu}(a_t) J(a_t) dkdtdk,
\]

with \( J(a_t) = \prod_{\lambda \in \Delta^+} [\sinh \lambda(H)]^{\dim g_\lambda} \) ([6.], p. 73). This is however straightforward.

We believe that the explicit computation of the function \( (\mathcal{H}\xi_1)^{-1}(\lambda) \), with \( \lambda \in \mathfrak{F}^e \), is necessary in order to pave way for further research along our perspective. This will however be taken up in another paper. Presently, we have that

\[
(\mathcal{H}\xi_1)(\lambda) = \int_{AN} e^{(-\lambda+\rho)(\log a)} \varphi_{\lambda}(an) dan = \int_{AN} e^{(-\lambda+\rho)(\log a)} \left[ \sum_{s \in \mathfrak{D}} c(s\lambda) \left( e^{(s\lambda-\rho)(\log(\alpha a))} + \sum_{\mu \in L^+} a_\mu(s\lambda)e^{(s\lambda-\rho-\mu)(\log(\alpha a))} \right) \right] dan
\]

\[
= \sum_{s \in \mathfrak{D}} c(s\lambda) \left( \int_{AN} e^{(s\lambda-\rho)(\log(\alpha a))} dan + \sum_{\mu \in L^+} a_\mu(s\lambda) \int_{AN} e^{(s\lambda-\rho-\mu)(\log(\alpha a))} dan \right)
\]

(since \( \log(\alpha a) = \log(a) \)) whose inverse is required in Theorem 3.8.
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