Structures of small closed non-orientable 3-manifold triangulations

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September 15, 2005*

Abstract

A census is presented of all closed non-orientable 3-manifold triangulations formed from at most seven tetrahedra satisfying the additional constraints of minimality and $\mathbb{P}^2$-irreducibility. The eight different 3-manifolds represented by these 41 different triangulations are identified and described in detail, with particular attention paid to the recurring combinatorial structures that are shared amongst the different triangulations. Using these recurring structures, the resulting triangulations are generalised to infinite families that allow similar triangulations of additional 3-manifolds to be formed. Algorithms and techniques used in constructing the census are included.

1 Introduction

It is useful when studying 3-manifold topology to have a complete reference of all 3-manifold triangulations satisfying some broad set of constraints. Examples include Callahan, Hildebrand and Weeks’ census of cusped hyperbolic 3-manifold triangulations formed from at most seven tetrahedra [7] and Matveev’s census of closed orientable triangulations formed from at most six tetrahedra [19].

Such references provide an excellent pool of examples for testing hypotheses and searching for triangulations that satisfy unusual properties. In addition they offer insight into the structures of minimal 3-manifold triangulations. In fact, since very few sufficient conditions for minimality are currently known, censuses play an important role in proving the minimality of small triangulations.

Much recent progress has been made in enumerating closed orientable 3-manifolds and their triangulations. Matveev presents a census of closed orientable triangulations formed from at most six tetrahedra [19], extended to seven tetrahedra by Ovchinnikov. Martelli and Petronio form a census of closed orientable 3-manifolds formed from up to nine tetrahedra [17], although they are primarily concerned with the 3-manifolds and their geometric structures and so the triangulations themselves are not listed. More recently their census has been extended to ten tetrahedra by Martelli [16].

Less progress has been made regarding closed non-orientable triangulations. Amendola and Martelli present a census of closed non-orientable 3-manifolds formed from up to six tetrahedra [1], a particularly interesting census because it is constructed without the assistance of a computer. Again these authors are primarily concerned with the 3-manifolds and their geometric structures, and so not all triangulations of these 3-manifolds are obtained.

Here we extend the closed non-orientable census of Amendola and Martelli to seven tetrahedra, and in addition we enumerate the many different triangulations of these 3-manifolds instead of just the 3-manifolds themselves. Furthermore we examine the combinatorial structures of these triangulations in detail, highlighting common constructions that recur throughout the census triangulations.

Independently of this work, Amendola and Martelli have also announced an extension of their theoretical census to seven tetrahedra [2]. As in their six-tetrahedron census they concentrate only

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*The initial version of this paper was released in November 2003. The update from September 2005 corrects an off-by-one error in the formulae of Theorems 4.6 and 4.9. It also includes general proofreading, particularly in the discussion of layered solid tori.
on the resulting 3-manifolds and not their different triangulations, but again the non-computational nature of their work is remarkable.

As with the previous closed censuses described above, we consider only triangulations satisfying the following constraints.

- **Closed**: The triangulation is of a closed 3-manifold. In particular it has no boundary faces, and each vertex link is a 2-sphere.
- **$\mathbb{P}^2$-irreducible**: The underlying 3-manifold has no embedded two-sided projective planes, and furthermore every embedded 2-sphere bounds a ball.
- **Minimal**: The underlying 3-manifold cannot be triangulated using strictly fewer tetrahedra.

Requiring triangulations to be $\mathbb{P}^2$-irreducible and minimal keeps the number of triangulations down to manageable levels, focussing only upon the simplest triangulations of the simplest 3-manifolds (from which more complex 3-manifolds can be constructed). Minimal triangulations prove to be particularly useful for studying the 3-manifolds that they represent, since they are frequently well structured as seen in both Matveev’s census [19] and the results presented here.

For the $\leq 7$-tetrahedron non-orientable census presented in this paper, a brief summary of results is presented in Table 1. Each triangulation is counted once up to isomorphism, i.e., a relabelling of the tetrahedra within the triangulation and their individual faces. It is worth noting that the number of triangulations is significantly larger than the number of 3-manifolds, since most 3-manifolds in the census can be realised by several different minimal triangulations.

| Tetrahedra | 3-Manifolds | Triangulations |
|------------|-------------|----------------|
| $\leq 5$   | 0           | 0              |
| 6          | 5           | 24             |
| 7          | 3           | 17             |
| **Total**  | **8**       | **41**         |

Table 1: Summary of closed non-orientable census results

The final 41 triangulations are found to be remarkably similar in their construction. By identifying these similarities we construct a small handful of infinite parameterised families of 3-manifold triangulations that encompass 38 of these 41 triangulations. The remaining three triangulations are all six-tetrahedron triangulations and might well be small exceptional cases that do not generalise at all — an extension of this census to higher numbers of tetrahedra should offer further insight.

In Section 2 we describe the method by which this census was constructed. The remainder of this paper is devoted to presenting the census results and describing in detail the combinatorial structures of the various triangulations. Section 3 describes the construction of thin $I$-bundles and layered solid tori, which are parameterised building blocks that recur frequently throughout the census triangulations. In Section 4 we combine these building blocks to form our infinite families of 3-manifold triangulations, as well as describing the three exceptional triangulations from the census that these families do not cover. Finally Section 5 closes with a full listing of the 41 triangulations found in the census, using the constructions of Sections 3 and 4 to simplify their descriptions and identify the underlying 3-manifolds.

All of the computational work was performed using Regina, a computer program that performs a variety of different calculations and procedures in 3-manifold topology [3, 6]. The program Regina, its source code and accompanying documentation are freely available from [http://regina.sourceforge.net/](http://regina.sourceforge.net/).

Special thanks must go to J. Hyam Rubinstein for many helpful discussions throughout the course of this research. The author would also like to thank the Australian Research Council, RMIT University and the University of Melbourne for their support.

## 2 Constructing the Census

As with most of the prior censuses listed in Section 1, this census is based upon a computer search through the possible triangulations that can be formed from various numbers of tetrahedra. At
the heart of this computer search is a procedure that generates triangulations from \( n \) tetrahedra using all possible identifications of tetrahedron faces under all possible rotations and reflections. Note that in practice this search can be refined using combinatorial techniques to avoid many isomorphic duplicates of triangulations.

Recall from Section 1 that we require only triangulations that are closed, non-orientable, minimal and \( P^2 \)-irreducible. Some of these properties are easily incorporated into the generation procedure described above. For instance, it is straightforward to adjust the generation procedure so that only non-orientable triangulations with no boundary faces are produced (we simply ensure that every tetrahedron face is matched with a partner, and we keep track of the orientability of each tetrahedron as we go). Other properties such as minimality and \( P^2 \)-irreducibility are less easily dealt with, and in many cases must be evaluated after potential triangulations have already been constructed.

Thus we can decompose the construction of a census into the two stages of generation and analysis as follows.

**Algorithm 2.1 (Census Construction)** Let \( n \) be some positive integer. A census of all closed non-orientable minimal \( P^2 \)-irreducible triangulations formed from \( n \) tetrahedra can be constructed in the following fashion.

1. Generate a set of triangulations that is guaranteed to include everything that should appear in the census. For instance, we might generate all closed non-orientable 3-manifold triangulations formed from \( n \) tetrahedra without regard for minimality or \( P^2 \)-irreducibility. In practice the set of triangulations that we generate is more complicated, as seen in Section 2.1 below.

   It should be ensured at this stage that no two triangulations in the generated set are isomorphic duplicates.

2. For each triangulation generated in step 1 determine whether it satisfies the full set of census constraints and if so then include it in the final results. For instance, in the example above we would need to test each triangulation for minimality and \( P^2 \)-irreducibility.

We proceed to examine in detail the individual steps of the census algorithm in Sections 2.1 and 2.2 below.

### 2.1 Generation of Triangulations

It can be observed that a triangulation formed from \( n \) tetrahedra can be uniquely determined by the following information:

- A *face pairing*, i.e., a partition of the \( 4n \) tetrahedron faces into \( 2n \) pairs indicating which tetrahedron faces are to be identified with which others;

- A list of the \( 2n \) rotations and/or reflections used to perform each of these \( 2n \) face identifications.

A first approach to step 1 of Algorithm 2.1 i.e., the generation of triangulations, could thus be as follows.

1. Enumerate all possible face pairings.

2. For each face pairing, try all possible \( 6^{2n} \) combinations of rotations and reflections for the \( 2n \) face identifications. Record each triangulation thus produced.

Note that there is no guarantee that the triangulations produced are actually triangulations of 3-manifolds. They are however guaranteed to have no boundary faces, and by keeping track of the orientation of each tetrahedron as we go it is straightforward to ensure non-orientability.

In practice this generation is exceptionally slow; we see already that step 2 is exponential in the number of tetrahedra. This accounts for the limited extent of current census results described in Section 1. By exploiting the fact that we are interested only in closed non-orientable minimal \( P^2 \)-irreducible triangulations of 3-manifolds, we can use a variety of methods to improve the running time of the census algorithm. These methods include the following.
Several results relating to face pairings are presented in [5], which allow many of the face pairings to be tossed away in step 1 and which use properties of the remaining face pairings to provide significant improvements to step 2. For the six-tetrahedron non-orientable census these results eliminate over 98% of the running time of the census generation.

Further improvements can be made by modifying step 4 to avoid low degree edges, a technique used successfully in earlier hyperbolic censuses [7, 11] and closed orientable censuses [17, 19]. Details of how this technique is applied to a closed non-orientable census can be found in [5].

The issue of isomorphism must also be dealt with. Isomorphic duplicates of face pairings are avoided by selecting a canonical representation for each face pairing from amongst all possible relabellings. For instance, the canonical representation might be the lexicographically smallest relabelling when written in some standard format.

Any face pairing that is not in its canonical representation can therefore be ignored. Furthermore, the generation of face pairings can be streamlined to toss away partially constructed face pairings that will clearly not have this property. Isomorphic duplicates of entire triangulations are dealt with in a similar way.

2.2 Analysis of Triangulations

For each triangulation $T$ that is constructed in step 1 of Algorithm 2.1 we must still determine whether $T$ is in fact a closed non-orientable minimal $P^2$-irreducible triangulation of a 3-manifold. It is straightforward to test whether $T$ is indeed a 3-manifold triangulation, and the generation described in Section 2.1 is already tweaked to ensure closedness and non-orientability.

It remains then to determine whether each triangulation is minimal and $P^2$-irreducible. These properties are somewhat more difficult to test for. No general test for minimality is currently known. Furthermore, current tests for reducibility involve either cutting along embedded 2-spheres (see for instance [14]) or crushing embedded 2-spheres to a point (see [11]). Cutting along 2-spheres produces very large triangulations that make such algorithms too slow for practice, and it is difficult to generalise the crushing algorithms to the non-orientable case due to a number of additional complications that arise.

We can however use a variety of alternative techniques to at least answer the questions of minimality and $P^2$-irreducibility for some triangulations. These techniques are outlined in Sections 2.2.1 to 2.2.4 below. Section 2.2.5 concludes with a discussion of the success of these techniques when applied to the particular $\leq 7$-tetrahedron non-orientable census under consideration.

2.2.1 Elementary Moves

It is often possible to make a modification local to a few tetrahedra within a triangulation that preserves the underlying 3-manifold, with no knowledge whatsoever of the global triangulation structure or of any properties of the 3-manifold. Such local modifications are referred to as elementary moves.

If a sequence of elementary moves can be found that reduces the number of tetrahedra in a triangulation, then since these moves preserve the underlying 3-manifold it follows that the original triangulation cannot be minimal.

Alternatively, if a sequence of elementary moves can be found that links two census triangulations with the same number of tetrahedra, it follows that any deductions regarding the minimality or $P^2$-irreducibility of one triangulation can be simultaneously applied to the other.

The individual elementary moves that were used for this census are described below. These moves are not new, and many were implemented in 1999 by David Letscher in his computer program Normal. Analogous techniques using local modifications of special spines have been used by Matveev [19] and Martelli and Petronio [17].

Note that for each of these elementary moves, it is relatively straightforward to test whether the move may be made. No properties are assumed of the underlying triangulation, and so these moves may be safely applied to triangulations with boundary faces or ideal vertices (vertices with higher genus links).

**Lemma 2.2 (Pachner Moves)** If a triangulation contains a non-boundary edge of degree three that belongs to three distinct tetrahedra then the following move can be made.

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These three tetrahedra (adjacent along our edge of degree three) are replaced with a pair of tetrahedra adjacent along a single face, as illustrated in Figure 1. This is called a 3-2 Pachner move and is one of the bistellar operations considered in [21]. This move preserves the underlying 3-manifold and reduces the number of tetrahedra in its triangulation by one.

Likewise, if a triangulation contains a non-boundary face belonging to two distinct tetrahedra then this move can be applied in reverse. This is called a 2-3 Pachner move and again preserves the underlying 3-manifold, this time increasing the number of tetrahedra by one.

Proof Since the modifications for each move take place entirely within the boundary of the illustrated polyhedron and since none of the vertex or edge links on the boundary are changed, it is clear that the underlying 3-manifold is preserved.

Lemma 2.3 (4-4 Move) If a triangulation contains a non-boundary edge of degree four that belongs to four distinct tetrahedra then the following move can be made.

These four tetrahedra are replaced with four different tetrahedra meeting along a new edge of degree four, running perpendicular to the old edge of degree four. This is called a 4-4 move and is illustrated in Figure 2. This move preserves both the underlying 3-manifold and the number of tetrahedra.

Proof Again the modifications take place entirely within the illustrated polyhedron, with no changes to the boundary vertex and edge links. Therefore the underlying 3-manifold is preserved. Note that a 4-4 move is simply a 2-3 Pachner move followed by a 3-2 Pachner move.

Lemma 2.4 (2-0 Vertex Move) Let \( v \) be an internal vertex of degree two in a triangulation. Assume that the two tetrahedra meeting \( v \) are distinct, and that these two tetrahedra meet along three different faces as illustrated in the left hand diagram of Figure 3. Assume that the remaining faces of each tetrahedron, i.e., the two faces opposite \( v \), are distinct and are not both boundary faces (though one of them may be a boundary face).

Then these two tetrahedra may be flattened to a single face as illustrated in the right hand diagram of Figure 3. This is called a 2-0 vertex move. This move preserves the underlying 3-manifold and reduces the number of tetrahedra in its triangulation by two.

Proof Since the two faces opposite \( v \) are distinct and are not both boundary faces, there is some third tetrahedron \( \Delta \) adjacent to one of these faces as illustrated in the left hand diagram of Figure 3.
The result of performing the 2-0 vertex move is illustrated in the right hand diagram of Figure 4. The modifications all take place within the boundary of the illustrated polyhedron and none of the vertex or edge links on the boundary of this polyhedron are changed. Thus the 2-0 vertex move preserves the underlying 3-manifold.

**Lemma 2.5 (2-0 Edge Move)** Let $e$ be a non-boundary edge of degree two in a triangulation, as illustrated in the left hand diagram of Figure 5. Assume that the two tetrahedra meeting $e$ are distinct. Assume also that the edges opposite $e$ in each tetrahedron, labelled $g$ and $h$ in the diagram, are distinct and are not both boundary edges (though one of them may be a boundary edge).

Consider now the four faces in the diagram that do not contain edge $e$ (these are the faces on either side of edges $g$ and $h$). Label these faces $G_1$, $G_2$, $H_1$ and $H_2$ as illustrated, so that edge $g$ lies between faces $G_1$ and $G_2$ and edge $h$ lies between faces $H_1$ and $H_2$. Assume that faces $G_1$ and $H_1$ are distinct and that faces $G_2$ and $H_2$ are distinct. Assume that all four of these faces are not identified in pairs (though we may have two of these faces identified, such as $G_1$ and $G_2$). Assume that we do not have a situation in which two of these faces are identified with each other and the remaining two are boundary faces.

Then these two tetrahedra may be flattened to a pair of faces, as illustrated in the right hand diagram of Figure 5. This is called a 2-0 edge move. This move preserves the underlying 3-manifold and reduces the number of tetrahedra in its triangulation by two.

**Proof** Consider the disc bounded by edges $g$ and $h$ that slices through our two tetrahedra. Since $g$ and $h$ are distinct and are not both boundary edges, we may crush this disc to a single edge without changing the underlying 3-manifold. After this operation we are left with two triangular pillows joined along a single edge as illustrated in the bottom left hand diagram of Figure 5. We may then retriangulate each of these pillows using two tetrahedra as illustrated in the bottom right hand diagram of Figure 5.

The conditions placed upon faces $G_1$, $G_2$, $H_1$ and $H_2$ allow us to use Lemma 2.4 to perform a 2-0 vertex move upon each pillow, thereby flattening each pillow to a face. This procedure is
Lemma 2.6 (2-1 Edge Move) Let $e$ be a non-boundary edge of degree one in a triangulation, as illustrated in the left hand diagram of Figure 7. Label the single tetrahedron containing $e$ as $\Delta$, label the endpoints of $e$ as $A$ and $B$ and label the remaining two vertices of $\Delta$ as $C$ and $D$.

Assume that face $CAD$ (the upper face of $\Delta$) is not a boundary face and let $\Delta'$ be the tetrahedron adjacent along this face. Assume that $\Delta$ and $\Delta'$ are distinct tetrahedra and label the remaining vertex of $\Delta'$ as $E$. Assume that edges $CE$ and $DE$ of tetrahedron $\Delta'$ are distinct and label them $g$ and $h$ respectively. Assume that edges $g$ and $h$ are not both boundary edges (though one of them may be a boundary edge).

Then the two tetrahedra $\Delta$ and $\Delta'$ may be merged into a single tetrahedron, with the region between edges $g$ and $h$ and vertex $A$ flattened to a single face. This operation is called a 2-1 edge move and is illustrated in the right hand diagram of Figure 7. This move preserves the underlying 3-manifold and reduces the number of tetrahedra in its triangulation by one.

Proof We employ a strategy similar to that used in the proof of Lemma 2.5. Consider the disc bounded by edges $g$ and $h$ that slices through both tetrahedra $\Delta$ and $\Delta'$. Since edges $g$ and $h$ are distinct and are not both boundary we can crush this disc to a single edge without changing the underlying 3-manifold. This reduces the region between edges $g$ and $h$ and vertex $A$ to a triangular pillow as illustrated in the central diagram of Figure 8. The pillow is retriangulated using two tetrahedra and the region between edges $g$ and $h$ and vertex $B$ is retriangulated using a single tetrahedron with a new internal edge of degree one.

Focusing our attention upon the triangular pillow and its interior vertex, we can use the constraints upon edges $g$ and $h$ to establish the conditions of Lemma 2.4. This allows us to perform a 2-0 vertex move upon the pillow, flattening it to a single face as illustrated in the right hand diagram of Figure 8. This completes the 2-1 edge move with no changes in the underlying 3-manifold and an overall reduction of one tetrahedron.
2.2.2 Normal Surfaces

The theory of normal surfaces offers a powerful tool in the development of algorithms in 3-manifold topology. Since our use of normal surfaces does not extend beyond this small section, we refer the reader to Hemion [8] for a detailed overview of normal surface theory in an algorithmic context.

When constructing a census of 3-manifold triangulations, an enumeration of the vertex embedded normal surfaces of a triangulation can help establish whether or not that triangulation is $\mathbb{P}^2$-irreducible. In particular, we can call upon the following results.

**Lemma 2.7** Let $T$ be a non-orientable 3-manifold triangulation. If a projective plane appears amongst the vertex embedded normal surfaces of $T$ then $T$ is not $\mathbb{P}^2$-irreducible.

**Proof** If the projective plane described above is two-sided, it follows immediately from the definition of $\mathbb{P}^2$-irreducibility that $T$ cannot be $\mathbb{P}^2$-irreducible. If the projective plane is one-sided on the other hand, it follows that a regular neighbourhood of this projective plane forms an $\mathbb{R}P^3$ connected sum component of the underlying 3-manifold. Since $\mathbb{R}P^3$ is orientable but $T$ is not, there must be more than one connected sum component and so $T$ cannot be irreducible (and therefore $T$ cannot be $\mathbb{P}^2$-irreducible).

**Lemma 2.8** Let $T$ be a 3-manifold triangulation. If the vertex embedded normal surfaces of $T$ contain no projective planes and no 2-spheres aside from the trivial vertex linking 2-spheres, then $T$ is $\mathbb{P}^2$-irreducible.

**Proof** Assume that $T$ is not $\mathbb{P}^2$-irreducible. Then either $T$ contains an embedded essential 2-sphere (an embedded 2-sphere that does not bound a ball) or $T$ contains an embedded two-sided projective plane.

It is proven by Kneser [15] and Schubert [22] that any triangulation containing an embedded essential 2-sphere contains an embedded essential normal 2-sphere. An analogous argument shows that if a triangulation contains an embedded projective plane then it contains an embedded normal projective plane. Therefore if $T$ is not $\mathbb{P}^2$-irreducible then it must contain an embedded normal surface whose Euler characteristic is strictly positive.

This normal surface $S$ can be expressed as $S = \lambda_1 V_1 + \ldots + \lambda_k V_k$ where each $V_i$ is a vertex embedded normal surface and each $\lambda_i > 0$. Since Euler characteristic is a linear function of normal surfaces, it follows that $\chi(S) = \lambda_1 \chi(V_1) + \ldots + \lambda_k \chi(V_k)$ and in particular that $\chi(V_i) > 0$ for some $i$. Thus some vertex embedded normal surface $V_i$ is either a 2-sphere or a projective plane.

Finally we observe that $V_i$ cannot be a vertex linking 2-sphere since a vertex link alone is not essential and the sum of a vertex link with any other normal surface is disconnected.

Note that the only case in which we cannot conclude whether or not a triangulation is $\mathbb{P}^2$-irreducible is the case in which its vertex embedded normal surfaces include a non-vertex-linking 2-sphere but no projective planes.

2.2.3 Special Subcomplexes

There are some particular subcomplexes whose presence within a triangulation indicates that the triangulation cannot be $\mathbb{P}^2$-irreducible. Two such subcomplexes, the pillow 2-sphere and the snapped 2-sphere, are described in detail below.
**Definition 2.9 (Pillow 2-Sphere)** A pillow 2-sphere is a 2-sphere formed from two faces of a triangulation. These two faces must be joined along all three edges of each face. Furthermore, these three edges must be distinct. The formation of a pillow 2-sphere is illustrated in Figure 9.

![Figure 9: Forming a pillow 2-sphere](image)

**Lemma 2.10** If a 3-manifold triangulation contains a pillow 2-sphere then this triangulation cannot be both minimal and \( P^2 \)-irreducible.

**Proof** This result is proven in [5]. The proof essentially involves converting the pillow 2-sphere to an embedded 2-sphere, which must bound a ball if the triangulation is to be \( P^2 \)-irreducible. The ball is removed and replaced with a simpler structure and the triangulation is then simplified. The result is a triangulation of the same 3-manifold using fewer than the original number of tetrahedra, showing the original triangulation to be non-minimal. 

**Definition 2.11 (Snapped 2-Sphere)** A snapped 2-sphere is a 2-sphere formed using two tetrahedra \( \Delta_1 \) and \( \Delta_2 \) as follows. Each tetrahedron \( \Delta_i \) is folded upon itself to form an edge \( e_i \) of degree one, as illustrated in Figure 10. Let \( f_i \) denote the edge opposite \( e_i \) in tetrahedron \( \Delta_i \), so that \( f_i \) bounds a disc \( \delta_i \) slicing through the midpoint of edge \( e_i \). Discs \( \delta_1 \) and \( \delta_2 \) are shaded in the diagram.

![Figure 10: Forming a snapped 2-sphere](image)

Edges \( f_1 \) and \( f_2 \) are then identified in the triangulation. As a result discs \( \delta_1 \) and \( \delta_2 \) are joined at their boundaries to form a 2-sphere within the triangulation, referred to as a snapped 2-sphere.

**Lemma 2.12** If a 3-manifold triangulation contains a snapped 2-sphere then this triangulation cannot be both minimal and \( P^2 \)-irreducible.

**Proof** Let \( T \) be a triangulation of the 3-manifold \( M \) and assume that \( T \) is \( P^2 \)-irreducible. Furthermore, assume that \( T \) contains a snapped 2-sphere as described in Definition 2.11.

Note that a snapped 2-sphere is an embedded 2-sphere, since it contains only one vertex and one edge and therefore has no possible points of self-intersection. As with all 2-spheres, it is also two-sided. We can thus slice triangulation \( T \) along this 2-sphere to obtain two spherical boundary components, each formed from a copy of the two discs \( \delta_1 \) and \( \delta_2 \).
We can now effectively cap each of these boundary components with a ball by identifying its two boundary discs together, producing the new triangulation $T'$ as illustrated in Figure 11. Note that $T$ and $T'$ contain precisely the same number of tetrahedra.

Since $T$ is $\mathbb{P}^2$-irreducible, the snapped 2-sphere must be separating in $T$ and so triangulation $T'$ consists of two disconnected components. Moreover, these two components represent a connected sum decomposition of $M$. Again by $\mathbb{P}^2$-irreducibility, it follows that one of these components is a 3-sphere and the other is a new triangulation of the original 3-manifold $M$. Since this new triangulation of $M$ contains strictly fewer tetrahedra than $T$, it follows that the original triangulation $T$ cannot be minimal.

2.2.4 Invariant Analysis

Once a triangulation is known to be $\mathbb{P}^2$-irreducible, invariant analysis can be used to prove its minimality. This technique requires the census to be constructed according to increasing numbers of tetrahedra, i.e., all 1-tetrahedron triangulations should be generated and analysed, then all 2-tetrahedron triangulations, all 3-tetrahedron triangulations and so on.

The key observation is that if some $\mathbb{P}^2$-irreducible triangulation $T$ is non-minimal, then a triangulation of the same 3-manifold must appear in an earlier section of the census formed from fewer tetrahedra. Thus, if some collection of 3-manifold invariants can be found that together distinguish the underlying 3-manifold of $T$ from any of the 3-manifolds constructed in earlier sections of the census, it follows that $T$ is a minimal $\mathbb{P}^2$-irreducible triangulation and is thus eligible to appear in our final list of census results.

The invariants that were used for this particular census include homology groups, fundamental group and the quantum invariants of Turaev and Viro [23]. The Turaev-Viro invariants, used also by Matveev [20], have proven exceptionally useful in practice for distinguishing 3-manifolds in both orientable and non-orientable settings.

2.2.5 Results

For the $\leq 7$-tetrahedron non-orientable census described in this paper, the techniques discussed in Sections 2.2.1 to 2.2.4 prove sufficient to determine precisely which triangulations are minimal and $\mathbb{P}^2$-irreducible. Furthermore, a combination of elementary moves and invariant analysis allows these minimal $\mathbb{P}^2$-irreducible triangulations to be grouped into equivalence classes according to which triangulations have identical underlying 3-manifolds. In this way we obtain a final census of 41 triangulations representing eight different 3-manifolds.

We proceed now to Section 3 which paves the way for a combinatorial analysis of these 41 census triangulations.
3 Common Structures

In order to make the census triangulations easier to both visualise and analyse, we decompose these triangulations into a variety of building blocks. Ideally such building blocks should be large enough that they significantly simplify the representation and analysis of the triangulations containing them, yet small enough that they can be frequently reused throughout the census.

This idea of describing triangulations using medium-sized building blocks has been used previously for the orientable case. Matveev describes a few orientable building blocks [19] and Martelli and Petronio describe a more numerous set of smaller orientable building blocks called bricks [17].

An examination of the non-orientable triangulations of this census shows a remarkable consistency of combinatorial structure. We therefore need only two types of building block: the thin \(I\)-bundle and the layered solid torus.

3.1 Thin \(I\)-Bundles

A thin \(I\)-bundle is essentially a triangulation of an \(I\)-bundle that has a thickness of only one tetrahedron between its two parallel boundaries. Thin \(I\)-bundles play an important role in the construction of minimal non-orientable triangulations and appear within all of the triangulations described in this census.

A thin \(I\)-bundle over a surface \(S\) is built upon a decomposition of \(S\) into triangles and quadrilaterals. Each triangle or quadrilateral of \(S\) corresponds to a single tetrahedron of the overall \(I\)-bundle.

We thus begin by discussing triangle and quadrilateral decompositions of surfaces and the properties that we require of such decompositions.

**Definition 3.1 (Well-Balanced Decomposition)** Let \(S\) be some closed surface. A well-balanced decomposition of \(S\) is a decomposition of \(S\) into triangles and quadrilaterals satisfying the following properties.

1. Every vertex of the decomposition meets an even number of quadrilateral corners.
2. If the quadrilaterals are removed then the surface breaks into a disconnected collection of triangulated discs.
3. There are no cycles of quadrilaterals. That is, any path formed by walking through a series of quadrilaterals, always entering and exiting by opposite sides, must eventually run into a triangle.

To clarify condition 3, Figure 12 illustrates some arrangements of quadrilaterals that contain cycles as described above. In each diagram the offending cycle is marked by a dotted line. Figure 13, on the other hand illustrates arrangements of quadrilaterals that do not contain cycles and so are perfectly acceptable within a well-balanced decomposition.

![Figure 12: Arrangements of quadrilaterals that include cycles](image)

**Example 3.2** Figure 14 illustrates a handful of well-balanced decompositions of the torus. Each of these decompositions can be seen to satisfy all of the conditions of Definition 3.1.

Figure 15 however illustrates some decompositions of the torus that are not well-balanced. The first diagram illustrates a decomposition that breaks condition 1 of Definition 3.1; one of the offending vertices is marked with a black circle. The second diagram shows how this same decomposition breaks condition 3; two cycles of quadrilaterals are marked with dotted lines. The
Figure 13: Arrangements of quadrilaterals that do not include cycles

Figure 14: Well-balanced decompositions of the torus

Once we have obtained a well-balanced decomposition of a surface, we can flesh out its triangles and quadrilaterals into a full 3-manifold triangulation as follows.

Definition 3.3 (Enclosing Triangulation) Let $D$ be a well-balanced decomposition of a closed surface. The enclosing triangulation of $D$ is the unique triangulation formed as follows.

Each triangle or quadrilateral of $D$ is enclosed within its own tetrahedron as illustrated in Figure 16. The faces of these tetrahedra are then identified in the unique manner that causes these discs to be connected according to the decomposition $D$. Any tetrahedron faces that do not meet the discs of $D$ (i.e., the faces parallel to triangular discs) are left to become boundary faces of the triangulation.

As an example, Figure 17 illustrates two quadrilaterals $q$ and $q'$ placed within tetrahedra $\Delta$ and $\Delta'$ respectively. Suppose edges $e$ and $e'$ of these quadrilaterals are identified within the surface decomposition $D$. Then the tetrahedron faces $f$ and $f'$ must be identified as illustrated so that edges $e$ and $e'$ can be connected correctly.

Since each disc of the decomposition $D$ runs through the centre of a tetrahedron, we see that the enclosing triangulation simply thickens the 2-manifold decomposition $D$ into a 3-dimensional space. The original surface with its triangles and quadrilaterals in turn becomes a normal surface running through the centre of the enclosing triangulation.

Lemma 3.4 Let $D$ be a well-balanced decomposition of the closed surface $S$. Then the enclosing triangulation of $D$ is a 3-manifold triangulation representing an $I$-bundle (possibly twisted) over $S$.

Proof Let $T$ be the enclosing triangulation. Our first step is to prove that $T$ is in fact a triangulation of a 3-manifold.

It can be shown that each edge not meeting the central decomposition $D$ is in fact a boundary edge of the triangulation. This is clearly true of edges belonging to tetrahedra that enclose triangles of $D$, since faces parallel to triangles of $D$ become boundary faces of the overall triangulation. This is also true of edges belonging to tetrahedra that enclose quadrilaterals of $D$, since condition 2 of

Figure 15: Decompositions of the torus that are not well-balanced
Definition 3.1 (no cycles of quadrilaterals) ensures that each such edge is identified with an edge of one of the boundary faces previously described.

Each internal edge of $T$ therefore cuts through a vertex of the decomposition $D$. Condition 11 of Definition 3.1 (vertices meet an even number of quadrilateral corners) ensures that no internal edge of $T$ is identified with itself in reverse.

We turn now to the vertices of $T$. By observing how the tetrahedra are hooked together we see that the link of each vertex $v$ is of the form illustrated in Figure 18.

The dark shaded triangles around the outside correspond to tetrahedra enclosing triangles of $D$ in which $v$ belongs to a boundary face. This is illustrated in the left hand diagram of Figure 19. These dark shaded triangles provide the boundary edges of the vertex link.

The medium shaded triangles just inside this boundary correspond to tetrahedra enclosing quadrilaterals of $D$, as illustrated in the central diagram of Figure 19. Each of these medium shaded triangles meets the boundary at a single point, since we observed earlier that each edge of $T$ running parallel to a quadrilateral of $D$ is in fact a boundary edge.

The remaining pieces of the vertex link, corresponding to the light shaded region in the interior of Figure 18, are provided by tetrahedra enclosing triangles of $D$ in which $v$ lies opposite the boundary face. In these cases the pieces of the vertex link run parallel to the triangles of $D$ as illustrated in the right hand diagram of Figure 19. From condition 2 of Definition 3.1 (triangulated regions form discs in $D$) we see that these interior pieces combine to form a topological disc. Thus $v$ is a boundary vertex (i.e., the darker band forming the boundary of Figure 18 actually exists), and more importantly the entire link of vertex $v$ is a topological disc.

Thus our enclosing triangulation $T$ is indeed a triangulation of a 3-manifold. From the fact that the well-balanced decomposition of the surface $S$ runs through the centre of each tetrahedron, as well as the earlier observation that each vertex, edge and face not meeting this surface decomposition in fact forms part of the triangulation boundary, it is straightforward to see that the enclosing triangulation represents an $I$-bundle (possibly twisted) over $S$. 

Figure 16: Enclosing triangles and quadrilaterals within tetrahedra

![Figure 16](image16.png)

Figure 17: Constructing the enclosing triangulation

![Figure 17](image17.png)

Figure 18: A vertex link in an enclosing triangulation

![Figure 18](image18.png)
Definition 3.5 (Thin $I$-Bundle) A thin $I$-bundle over a closed surface $S$ is the enclosing triangulation of a well-balanced decomposition of $S$. This well-balanced decomposition is referred to as the central surface decomposition of the thin $I$-bundle.

We see then that Lemma 3.4 simply states that a thin $I$-bundle over a surface $S$ is what it claims to be, i.e., an actual triangulation of an $I$-bundle over $S$.

Before closing this section we present a method of marking a well-balanced decomposition that allows us to establish precisely how the corresponding $I$-bundle is twisted, if at all.

Definition 3.6 (Marked Decomposition) A well-balanced decomposition can be marked to illustrate how it is embedded within its enclosing triangulation. Markings consist of solid lines and dotted lines, representing features above and below the central surface respectively.

The different types of markings are illustrated in Figure 20. Each quadrilateral lies between two perpendicular boundary edges of the triangulation, one above and one below. These boundary edges are represented by a solid line and a dotted line as illustrated in the left hand diagram of Figure 20. Each triangle lies between a boundary vertex and a boundary face. If the boundary vertex lies above the triangle (and the boundary face below) then the triangle is marked with three solid lines as illustrated in the central diagram of Figure 20. If the boundary vertex lies below the triangle (and the boundary face above) then the triangle is marked with three dotted lines as illustrated in the right hand diagram.

![Figure 20: Marking quadrilaterals and triangles](image)

Since the edges and vertices of the $I$-bundle boundary all connect together, it follows that the markings must similarly connect together, with solid lines matched with solid lines and dotted lines matched with dotted lines. Following these markings across edge identifications therefore allows us to see if and where a region above the central surface moves through a twist to become a region below.

Example 3.7 Figure 21 illustrates three well-balanced decompositions of the torus with markings. In the first diagram we see that the $I$-bundle is twisted across the upper and lower edge identifications, since solid lines change to dotted lines across these identifications and vice versa. There is no twist however across the left and right edge identifications since the solid line does not change.

![Figure 21: Markings on well-balanced torus decompositions](image)
In the second diagram we see that the \( I \)-bundle is twisted across all of the outer edge identifications, with solid and dotted lines being exchanged in every case. In the third diagram we see that there are no twists at all, and that the \( I \)-bundle is in fact simply the product \( T^2 \times I \).

### 3.2 Layered Solid Tori

A key structure that appears frequently within both orientable and non-orientable minimal triangulations is the layered solid torus. Layered solid tori have been discussed in a variety of informal contexts by Jaco and Rubinstein. They appear in [12] and are treated thoroughly in [13]. Analogous constructs involving special spines of 3-manifolds can be found in [18] and [19]. The preliminary definitions presented here follow those given in [5].

In order to describe the construction of a layered solid torus we introduce the process of layering. Layering is a transformation that, when applied to a triangulation with boundary, does not change the underlying 3-manifold but does change the curves formed by the boundary edges of the triangulation.

**Definition 3.8 (Layering)** Let \( T \) be a triangulation with boundary and let \( e \) be one of its boundary edges. To *layer a tetrahedron on edge* \( e \), or just to *layer on edge* \( e \), is to take a new tetrahedron \( \Delta \), choose two of its faces and identify them with the two boundary faces on either side of \( e \) without twists. This procedure is illustrated in Figure 22.

![Figure 22: Layering a tetrahedron on a boundary edge](image)

Note that layering on a boundary edge does not change the underlying 3-manifold; the only effect is to thicken the boundary around edge \( e \). Moreover, once a layering has been performed, edge \( e \) is no longer a boundary edge but instead edge \( f \) (which in general represents a different curve on the boundary of the 3-manifold) has been added as a new boundary edge.

**Definition 3.9 (Layered Solid Torus)** A *standard layered solid torus* is a triangulation of a solid torus formed as follows. We begin with the one-triangle Möbius band illustrated in the left hand diagram of Figure 23, where the two edges marked \( e \) are identified according to the arrows and where \( g \) is a boundary edge. This Möbius band can be embedded in \( \mathbb{R}^3 \) as illustrated in the right hand diagram of Figure 23.

![Figure 23: A one-triangle Möbius band](image)

In this embedding our single triangular face has two sides, marked \( F \) and \( F' \) in the diagram. We make an initial layering upon edge \( e \) as illustrated in Figure 24 so that faces \( ABC \) and \( BCD \)
of the new tetrahedron are joined to sides $F$ and $F'$ respectively of the original triangular face. Although the initial Möbius band is not actually a 3-manifold triangulation, the layering procedure remains the same as described in Definition 3.8.

Since $F$ and $F'$ are in fact opposite sides of the same triangular face, we see that faces $ABC$ and $BCD$ become identified as illustrated in Figure 25. The result is the well-known one-tetrahedron triangulation of the solid torus. The identified faces $ABC$ and $BCD$ are shaded in the diagram.

Having obtained a 3-manifold triangulation of the solid torus, we finish the construction by performing some number of additional layerings upon boundary edges, one at a time. We may layer as many times we like or we may make no additional layerings at all. There are thus infinitely many different standard layered solid tori that can be constructed.

It is useful to consider the Möbius band on its own to be a degenerate layered solid torus containing zero tetrahedra. A non-standard layered solid torus can also be formed by making the initial layering upon edge $g$ of the Möbius band instead of edge $e$, although such structures are not considered here.

We can observe that each standard layered solid torus has two boundary faces and represents the same underlying 3-manifold, i.e., the solid torus. What distinguishes the different layered solid tori is the different patterns of curves that their boundary edges make upon the boundary torus.

**Definition 3.10 (Three-Parameter Torus Curves)** Let $T$ be a torus formed from two triangles as illustrated in Figure 26. Label one of these triangles $+$ and the other $−$. Select some ordering of the three edges and label these edges $e_1$, $e_2$ and $e_3$ accordingly.

Consider some oriented closed curve on this torus as illustrated in Figure 27. Using this curve we can assign a number to each edge $e_i$, this being the number of times the curve crosses edge $e_i$ from $+$ to $−$ minus the number of times it crosses edge $e_i$ from $−$ to $+$. If the numbers assigned to edges $e_1$, $e_2$ and $e_3$ are $p$, $q$ and $r$ respectively, we refer to our oriented curve as a $(p, q, r)$ curve. Thus, for instance, the curve illustrated in Figure 27 is a $(2, 3, −5)$ curve.

It is trivial to show that any $(p, q, r)$ curve satisfies $p+q+r = 0$. It is also straightforward to show that if the curve is embedded then either $p$, $q$ and $r$ are pairwise coprime or $(p, q, r) = (0, 0, 0)$. We can use three-parameter torus curves to categorise layered solid tori as follows.
Definition 3.11 (Layered Solid Torus Parameters) Let $L$ be a layered solid torus. Upon the two faces that form the boundary torus, draw the boundary of a meridinal disc of the underlying solid torus. Assign to this meridinal curve some arbitrary orientation and arbitrarily label the two boundary faces $+$ and $−$.

The meridinal curve then forms some $(p, q, r)$ curve on the boundary torus. $p$, $q$ and $r$ are said to be the parameters of the layered solid torus $L$, and $L$ is said to be a $(p, q, r)$ layered solid torus, denoted $\text{LST}(p, q, r)$.

Note that a $(p, q, r)$ layered solid torus is also a $(-p, -q, -r)$ layered solid torus. Note furthermore that the layered solid torus parameters are not ordered, so for instance a $(p, q, r)$ layered solid torus could also be written with parameters $(p, r, q)$ or $(q, r, p)$.

Example 3.12 A $(1, 2, -3)$ layered solid torus formed from one tetrahedron is illustrated in Figure 28. This is in fact the same layered solid torus illustrated in Figure 26 formed by a single layering upon edge $e$ of the Möbius band of Figure 25.

The back two faces of the tetrahedron are identified; specifically face $PQR$ is identified with face $QRS$. The meridinal curve is illustrated by the dotted line drawn upon the boundary faces, and its intersections with the edges of the boundary torus are marked.

Example 3.13 The degenerate layered solid torus with zero tetrahedra, i.e., the Möbius band, is an example of a $(1, 1, -2)$ layered solid torus. The meridinal curve is marked in Figure 29 with a dotted line. The three edges of the (degenerate) boundary torus are $g$ and the front and rear sides of $e$ (recalling that the Möbius band is embedded in $\mathbb{R}^3$). Once more the intersections of the meridinal curve with these boundary edges are marked with black circles in the diagram.
Recall from Definition 3.9 that a layered solid torus is built by layering upon boundary edges one at a time. We can trace the parameters of the layered solid torus as these successive layerings take place. The following result is relatively straightforward to prove by keeping track of intersections with boundary edges.

**Lemma 3.14** Layering on the edge with parameter $r$ in a $(p, q, r)$ layered solid torus produces a $(p, -q, q - p)$ layered solid torus.

Armed with this result we can present a general method for constructing a layered solid torus with a given set of parameters.

**Algorithm 3.15 (Layered Solid Torus Construction)** For any pairwise coprime integers $p$, $q$ and $r$ satisfying $p + q + r = 0$ and $\max(|p|, |q|, |r|) \geq 3$, the following algorithm can be used to construct a standard layered solid torus with parameters $(p, q, r)$. Without loss of generality, let $|r| \geq |p|, |q|$. Since $p$, $q$ and $r$ are pairwise coprime we see that in fact $|r| > |p|, |q|$, and since $p + q + r = 0$ we also see that $|p|, |q| > |q - p|$. If either $|p| \geq 3$ or $|q| \geq 3$ we can therefore use this same algorithm to construct a smaller LST$(p, -q, q - p)$ and then layer on the edge with parameter $|q - p|$ to obtain an LST$(p, q, r)$. Since $\max(|p|, |q|, |r|)$ decreases at each stage we are guaranteed that this recursion will eventually terminate.

If $|p|, |q| < 3$ then the only triples satisfying the given constraints are $(p, q, r) = \pm(1, 2, -3)$. In this case we use the one-tetrahedron LST$(1, 2, -3)$ illustrated in Example 3.12.

**Example 3.16** Suppose we wish to construct an LST$(3, 7, -10)$ using Algorithm 3.15. To form an LST$(3, 7, -10)$ we layer on edge 4 in an LST$(3, 4, -7)$. To form an LST$(3, 4, -7)$ we layer on edge 1 in an LST$(1, 3, -4)$. To form an LST$(1, 3, -4)$ we layer on edge 2 in an LST$(1, 2, -3)$, and an LST$(1, 2, -3)$ is given to us in Example 3.12. The resulting triangulation has four tetrahedra.

Algorithm 3.15 is in fact the optimal method for constructing a layered solid torus, as shown by the following result of Jaco and Rubinstein [13].

**Theorem 3.17** Let $p$, $q$ and $r$ be pairwise coprime integers for which $p + q + r = 0$ and $\max(|p|, |q|, |r|) \geq 3$. Then there is a unique LST$(p, q, r)$ up to isomorphism that uses the least possible number of tetrahedra, and this is the LST$(p, q, r)$ constructed by Algorithm 3.15.

Throughout the following sections, any reference to an LST$(p, q, r)$ is assumed to refer to this unique minimal LST$(p, q, r)$.

## 4 Families of Closed Triangulations

Having developed a set of medium-sized building blocks in Section 3, we can now combine these building blocks to form closed triangulations. In this section we present a number of families of closed triangulations, each of which represents an infinite class of triangulations sharing a common large-scale structure. It is seen in Section 5 that, with the exception of the three triangulations described in Section 4.5, the four families presented here together encompass all closed non-orientable minimal $\mathbb{P}^2$-irreducible triangulations formed from up to seven tetrahedra.
A categorisation of triangulations into infinite families as described above is certainly appealing. Large classes of triangulations may be studied simultaneously, and algorithms become available for generating triangulations of infinite classes of 3-manifolds.

Furthermore, when presented with an arbitrary triangulation of an unknown 3-manifold, having a rich collection of such families at our disposal increases the chance that we can identify this 3-manifold. Specifically, if we can manipulate the triangulation into a form that is recognised as a member of one of these families then the underlying 3-manifold can be subsequently established.

In the case of orientable 3-manifolds, several infinite parameterised families of triangulations are described in the literature [4, 17, 18, 19].

4.1 Notation

We begin by outlining some notation that is used to describe torus and Klein bottle bundles over the circle.

- \( T^2 \times I/A \) represents a torus bundle over the circle, where \( A \) is a unimodular \( 2 \times 2 \) matrix indicating the homeomorphism under which the torus \( T^2 \times 0 \) is identified with the torus \( T^2 \times 1 \). Note that this space is orientable or non-orientable according to whether the determinant of \( A \) is +1 or −1.

  More specifically, let \( \mu_0 \) and \( \lambda_0 \) be closed curves that together generate the fundamental group of the first torus and let \( \mu_1 \) and \( \lambda_1 \) be the curves parallel to these on the second torus. If

  \[
  A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},
  \]

  then the homeomorphism under which the two tori are identified maps curve \( \mu_0 \) to \( \mu_1^a \lambda_1^c \) and curve \( \lambda_0 \) to \( \mu_1^b \lambda_1^d \).

- \( K^2 \times I/A \) represents a Klein bottle bundle over the circle, where \( A \) is again a unimodular \( 2 \times 2 \) matrix indicating the homeomorphism under which the Klein bottle \( K^2 \times 0 \) is identified with the Klein bottle \( K^2 \times 1 \).

  Let \( \mu_0 \) and \( \lambda_0 \) be orientation-preserving and orientation-reversing closed curves respectively on the first Klein bottle that meet transversely in a single point. It is known that every element of the fundamental group of this Klein bottle can be represented as \( \mu^p \lambda^q \) for some unique pair of integers \( p \) and \( q \).

  Let \( \mu_1 \) and \( \lambda_1 \) be the corresponding curves on the second Klein bottle. If

  \[
  A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},
  \]

  then the homeomorphism under which the two Klein bottles are identified maps curve \( \mu_0 \) to \( \mu_1^a \lambda_1^c \) and curve \( \lambda_0 \) to \( \mu_1^b \lambda_1^d \). It is shown in [9] that every such matrix \( A \) must be of the form

  \[
  A = \begin{bmatrix} \pm 1 & b \\ 0 & \pm 1 \end{bmatrix}.
  \]

The notation described above is consistent with that used by Matveev for orientable 3-manifolds [19].

4.2 Layered Surface Bundles

Our first family of triangulations is the family of layered surface bundles, constructed from untwisted thin \( I \)-bundles as follows.

**Definition 4.1 (Layered Surface Bundle)** A layered surface bundle is a triangulation of a closed surface bundle over the circle formed using the following construction. We begin with a thin \( I \)-bundle over a closed surface \( S \), as described in Definition 3.5. We require this thin \( I \)-bundle to have no twists, so that it has two parallel boundary components each representing the same surface \( S \).
We then have the option of layering additional tetrahedra upon these boundary surfaces in order to change the curves formed by the boundary edges; see Definition 3.8 for a detailed description of the layering process. Finally we identify the two boundary surfaces according to some homeomorphism, forming an \( S \)-bundle over the circle.

We proceed to describe in detail some particular classes of layered surface bundles that feature in the \( \leq 7 \)-tetrahedron non-orientable census.

**Definition 4.2** In Figure 30 we see five specific thin \( I \)-bundles. Triangulations \( T_6^1 \), \( T_6^2 \) and \( T_7 \) are thin \( I \)-bundles over the torus and triangulations \( K_6^1 \) and \( K_6^2 \) are thin \( I \)-bundles over the Klein bottle. Each of these \( I \)-bundles is untwisted and therefore has two parallel boundary components, each formed from two triangles.

![Figure 30: The untwisted thin I-bundles \( T_6^1 \), \( T_6^2 \), \( T_7 \), \( K_6^1 \) and \( K_6^2 \)](attachment:image)

The left hand portion of each diagram depicts the central surface decomposition of the thin \( I \)-bundle, complete with markings as described in Definition 4.2. The right hand portion illustrates the upper and lower boundary components of the thin \( I \)-bundle, marked in solid lines and dotted lines respectively. Note that these boundary components correspond to the triangles of the central surface decomposition, since each tetrahedron enclosing a triangle provides a single boundary face.

For each thin \( I \)-bundle we mark directed edges \( \alpha_1 \) and \( \beta_1 \) on the upper boundary component and \( \alpha_2 \) and \( \beta_2 \) on the lower boundary component. Note that in each case \( \alpha_1 \) and \( \beta_1 \) generate the fundamental group of the upper boundary surface and \( \alpha_2 \) and \( \beta_2 \) generate the fundamental group of the lower boundary surface.

Let \( p, q, r \) and \( s \) be integers and let \( \theta \) be one of the thin \( I \)-bundles depicted in Figure 30. We define \( B_{\theta[p,q;r,s]} \) to be the specific layered surface bundle obtained by identifying the upper and lower boundaries of \( \theta \) so that directed edge \( \alpha_1 \) maps to \( \alpha_2^p \beta_2^q \) and directed edge \( \beta_1 \) maps to \( \alpha_2^r \beta_2^s \).

Note that for some values of \( p, q, r \) and \( s \) this identification can be realised by an immediate mapping of the corresponding boundary faces. On the other hand, for some values of \( p, q, r \) and \( s \) an additional layering of tetrahedra is required so that \( \alpha_2^p \beta_2^q \), \( \alpha_2^r \beta_2^s \) and the corresponding diagonal actually appear as edges of the lower boundary surface.

Note also that for some values of \( p, q, r \) and \( s \) this construction is not possible since there is no homeomorphism identifying the upper and lower boundaries as required.
**Example 4.3** To illustrate this construction we present the triangulations $B_{T_7[1,1]_{1,0}}$ and $B_{T_7^2[-1,1]_{2,-1}}$.

The construction of $B_{T_7[1,1]_{1,0}}$ begins with the thin $I$-bundle $T_7$, depicted in Figure 31. Our task is to identify the upper and lower boundaries so that $\alpha_1$ and $\beta_1$ map to $\alpha_2\beta_2$ and $\alpha_2$ respectively.

![Figure 31: Constructing the layered surface bundle $B_{T_7[1,1]_{1,0}}$](image)

This can in fact be done using a direct identification, by mapping boundary face $RPS$ to $YZX$ and mapping boundary face $PSQ$ to $YWX$. The final triangulation $B_{T_7[1,1]_{1,0}}$ therefore has seven tetrahedra, with no additional layering taking place.

The construction of $B_{T_7^2[-1,1]_{2,-1}}$ is slightly more complex. Figure 32 illustrates the thin $I$-bundle $T_7^2$. Here we must identify the boundaries so that $\alpha_1$ and $\beta_1$ map to $\alpha_2^{-1}\beta_2$ and $\alpha_2^2\beta_2^{-1}$ respectively. Unfortunately this cannot be done using a direct identification of the boundaries since $\alpha_2^2\beta_2^{-1}$ does not appear as an edge of the lower boundary surface.

![Figure 32: Constructing the layered surface bundle $B_{T_7^2[-1,1]_{2,-1}}$](image)

We are therefore forced to layer a new tetrahedron onto the lower boundary. This additional tetrahedron, labelled $ABCD$, is layered upon edge $XZ$ so that faces $YZW$ and $CBD$ are identified and faces $WZX$ and $ACB$ are identified. As a result we obtain a new lower boundary edge $AD$ that indeed represents the curve $\alpha_2^2\beta_2^{-1}$.

We can thus complete the triangulation by identifying the new lower boundary with the original upper boundary, mapping face $PQR$ to $DBA$ and face $QRS$ to $DCA$. The final triangulation $B_{T_7^2[-1,1]_{2,-1}}$ has seven tetrahedra.

For any layered surface bundle of a form described in Definition 4.2 the underlying $3$-manifold can be identified using the following result.

**Theorem 4.4** For each set of integers $p$, $q$, $r$ and $s$ for which the corresponding triangulations can be constructed, the underlying $3$-manifolds of the layered surface bundles with parameters $p$, $q$, $r$ and $s$ are as follows.

- $B_{T_7[p,q]_{r,s}}$ and $B_{T_7^2[p,q]_{r,s}}$ are both triangulations of the space $T^2 \times I / \left\{ \begin{array}{c} p \times q \\ r \times s \end{array} \right\}$.
- $B_{T_7[p,q]_{r,s}}$ is a triangulation of the space $T^2 \times I / \left\{ \begin{array}{c} (p+q)(r+s) \\ q \times s \end{array} \right\}$.
- Assume that $p+r$ is odd, $|p-q|=|r-s|=1$ and $p-q+r-s=0$. Then triangulations $B_{K^2_{[p,q]_{r,s}}}$ and $B_{K^2_{[p,q]_{r,s}}}$ both represent the space $K^2 \times I / \left\{ \begin{array}{c} (p-r)(r-0) \\ s-r \end{array} \right\}$, where the symbol $\{x\}$ is defined to be $1$ if $x$ is odd and $0$ if $x$ is even.
It can be observed from Figure 30 that $\alpha_2$ is parallel to $\alpha_1$ and $\beta_2$ is parallel to $\beta_1$ within each of the $I$-bundles $T_6^1$, $T_6^2$, $K_6^1$ and $K_6^2$. Within the $I$-bundle $T_7$ we find that $\alpha_2$ is parallel to $\alpha_1$ and that $\beta_2$ is parallel to $\alpha_1\beta_1$.

Given these observations, it is a simple matter to convert the identification of the two boundary surfaces into the canonical form described in Section [1]1 and thus establish the above results.  ■

4.3 Plugged Thin $I$-Bundles

Plugged thin $I$-bundles are formed by attaching layered solid tori to twisted $I$-bundles over the torus. The resulting 3-manifolds are all Seifert fibred, where we allow Seifert fibred spaces over orbifolds as well as over surfaces. Details of the construction are as follows.

Definition 4.5 (Plugged Thin $I$-Bundle) A plugged thin $I$-bundle is a 3-manifold triangulation formed using the following construction. Begin with one of the thin $I$-bundles over the torus depicted in Figure 33. Note from the markings on the diagrams that each $I$-bundle is twisted and non-orientable, specifically with a twist as we wrap from top to bottom in each diagram and no twist as we wrap from left to right.

The four triangles of each central surface decomposition shown in Figure 33 correspond to the four boundary faces of each $I$-bundle. In each case these boundary faces combine to form a torus as illustrated in Figure 34. We observe that each of these torus boundaries is formed from two annuli, one on the left and one on the right. Our construction is then completed by attaching a layered solid torus to each of these annuli as illustrated in Figure 35.

Let the layered solid tori have parameters $\text{LST}(p_1, q_1, r_1)$ and $\text{LST}(p_2, q_2, r_2)$ as explained in Definition [3]3. Furthermore, let the layered solid torus edges with parameters $p_1$ be attached to the left and right edges of the $I$-bundle boundary and let the layered solid torus edges with parameters $q_1$ be attached to the top and bottom edges of the $I$-bundle boundary as shown in Figure 35. Then
the particular plugged thin I-bundle that has been constructed is denoted $H_{\theta[p_1,q_1,p_2,q_2]}$, where $\theta$ denotes the original thin I-bundle chosen from Figure 33.

Note that instead of attaching a standard layered solid torus, the two faces of an annulus may simply be identified with each other by attaching the 0-tetrahedron degenerate LST(2,−1,−1), i.e., a Möbius band. For brevity, if a pair $p_i, q_i$ is omitted from the symbolic name of a plugged thin I-bundle then this pair is assumed to be 2,−1. For instance, the plugged thin I-bundle $H_{T_6^3[3,−1][2,−1]}$ is in fact $H_{T_6^3[3,−1]}$.

Note that the triangulations $H_{\theta[p_1,q_1,p_2,q_2]}$ and $H_{\theta[p_2,q_2,p_1,q_1]}$ are isomorphic. This can be seen from the symmetries of the layered solid torus and of the thin I-bundles described in Figure 33.

The following result allows us to identify the underlying 3-manifold of a plugged thin I-bundle.

**Theorem 4.6** Let $p_1$ and $q_1$ be coprime integers and let $p_2$ and $q_2$ be coprime integers, where $p_1 \neq 0$ and $p_2 \neq 0$. Then the underlying 3-manifolds of the plugged thin I-bundles with parameters $p_1, q_1, p_2$ and $q_2$ are as follows.1

- $H_{T_6^3[p_1,q_1,p_2,q_2]}$, $H_{T_6^3[p_1,q_1,p_2,q_2]}$ and $H_{T_6^3[p_1,q_1,p_2,q_2]}$ are each triangulations of the Seifert fibred space SFS $(\mathbb{R}P^2 : (p_1, q_1), (p_2, p_2 + q_2))$.
- $H_{T_6^3[p_1,q_1,p_2,q_2]}$ is a triangulation of the Seifert fibred space SFS $(\mathbb{R}P^2 : (p_1, q_1), (p_2, q_2))$, where the orbifold $D$ is a disc with reflector boundary.

**Proof** Consider the boundary torus of each of the thin I-bundles $T_6^3$, $T_6^3$, $T_6^3$ and $T_6^3$ as seen in Figure 33. We fill each of these boundary tori with circular fibres running parallel to the left and right sides as illustrated in Figure 33.

![Figure 36: Circular fibres on the boundary tori of thin I-bundles](image)

Our aim is to find compatible fibrations of the thin I-bundles. Each of the I-bundles $T_6^3$, $T_6^3$ and $T_6^3$ can be realised as a trivial Seifert fibred space over the Möbius band. The I-bundle $T_6^3$ on the other hand can be realised as a trivial Seifert fibred space over an orbifold, where this base orbifold is an annulus with one reflector boundary. In all cases the fibration of the I-bundle is compatible with the fibration of the boundary torus.

Attaching our two layered solid tori then completes the fibrations, filling each base space with a disc and introducing exceptional fibres with parameters $(p_1, q_1)$ and $(p_2, p_2 + q_2)$. After normalising the Seifert invariants, the resulting 3-manifolds can be expressed as SFS $(\mathbb{R}P^2 : (p_1, q_1), (p_2, p_2 + q_2))$ and SFS $(\mathbb{R}P^2 : (p_1, q_1), (p_2, q_2))$ as claimed.

### 4.4 Plugged Thick I-Bundles

Plugged thick I-bundles are similar to the plugged thin I-bundles of Section 4.3 except that instead of attaching layered solid tori directly to thin I-bundles we first wrap the thin I-bundles with an additional padding of tetrahedra. As with plugged thin I-bundles, the resulting 3-manifolds are all Seifert fibred.

We begin by presenting four triangulations of a twisted I-bundle over the torus, each of which has two vertices on the boundary.

**Definition 4.7 (Two-Vertex I-Bundles $\tilde{T}_3^1, \ldots, \tilde{T}_3^2$)** Let $\tilde{T}_3^1$, $\tilde{T}_3^2$ and $\tilde{T}_3^3$ be the thin I-bundles over the torus depicted in Figure 34. Note that $\tilde{T}_3^1$ and $\tilde{T}_3^3$ are in fact the same triangulation presented in different ways. From the markings we see that each I-bundle is twisted and non-orientable, specifically with a twist as we wrap from top to bottom and no twist as we wrap from left to right. The boundary torus of each I-bundle is illustrated alongside each diagram.

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1In a previous version of this paper, the first 3-manifold was incorrectly given as SFS $(\mathbb{R}P^2 : (p_1, q_1), (p_2, q_2))$. 

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We see that $\tilde{T}_5^1$ already has two boundary vertices. For $\tilde{T}_3^3$ and $\tilde{T}_5^3$ we modify the boundary by attaching a square pyramid formed from two tetrahedra. The apex of this pyramid becomes the second boundary vertex.

Figure 38 shows three new $I$-bundles $\tilde{T}_5^2$, $\tilde{T}_5^3$ and $\tilde{T}_5^4$ obtained in this fashion. To construct $\tilde{T}_5^2$ and $\tilde{T}_5^3$ we attach a pyramid to the boundary of $\tilde{T}_3^3$; triangulation $\tilde{T}_3^3$ differs in that the base of the pyramid wraps around the upper and lower edges of the diagram. To construct $\tilde{T}_5^4$ we attach a pyramid to the boundary of $\tilde{T}_2^3$. The new two-vertex boundary tori are depicted on the right hand side of Figure 38.

We see then that each of the $I$-bundles $\tilde{T}_5^1$, $\tilde{T}_5^2$, $\tilde{T}_5^3$ and $\tilde{T}_5^4$ is formed from five tetrahedra and has a two-vertex torus boundary.

Having constructed the $I$-bundles $\tilde{T}_3^1$, $\tilde{T}_3^2$, $\tilde{T}_5^3$ and $\tilde{T}_5^4$, we proceed to define a plugged thick $I$-bundle as follows.

**Definition 4.8 (Plugged Thick $I$-Bundle)** A plugged thick $I$-bundle is a 3-manifold triangulation formed using the following construction. Beginning with one of the two-vertex twisted $I$-bundles $\tilde{T}_3^1$, $\tilde{T}_5^2$, $\tilde{T}_5^3$ or $\tilde{T}_5^4$, we layer a single tetrahedron onto a specific edge of the boundary torus. This layering must form a new boundary edge running vertically from top to bottom; Figure 39 shows where this layering occurs for each of the $I$-bundles $\tilde{T}_3^1$, $\tilde{T}_5^2$, $\tilde{T}_5^3$ and $\tilde{T}_5^4$.

Specifically, for $\tilde{T}_3^1$, $\tilde{T}_5^2$ and $\tilde{T}_5^4$ the new tetrahedron is layered upon the top and bottom edges of the diagram. For $\tilde{T}_5^3$ the new tetrahedron is layered upon the main diagonal. The resulting $I$-bundles are labelled $\tilde{T}_3^1'$, $\tilde{T}_5^2'$, $\tilde{T}_5^3'$ and $\tilde{T}_5^4'$ respectively, and their new torus boundaries are shown on the right hand side of Figure 39.
Figure 39: Layering a tetrahedron to form twisted \( I \)-bundles \( \tilde{T}_1' \), \( \tilde{T}_2' \), \( \tilde{T}_3' \) and \( \tilde{T}_4' \)

At last we find ourselves in familiar territory. The boundary tori of the new six-tetrahedron \( I \)-bundles \( \tilde{T}_1' \), \( \tilde{T}_2' \), \( \tilde{T}_3' \) and \( \tilde{T}_4' \) are each formed from two annuli, one on the left and one on the right. As with the plugged thin \( I \)-bundles of the previous section, we complete the construction by attaching a layered solid torus to each annulus as illustrated in Figure 40.

Figure 40: Attaching layered solid tori to the torus boundary

Let the layered solid tori have parameters \( \mathrm{LST}(p_1, q_1, r_1) \) and \( \mathrm{LST}(p_2, q_2, r_2) \), where the edges with parameters \( p_i \) are attached to the left and right edges of the \( I \)-bundle boundary and where the edges with parameters \( q_i \) are attached to the top and bottom edges of the \( I \)-bundle boundary. Let \( \theta \) be \( \tilde{T}_1' \), \( \tilde{T}_2' \), \( \tilde{T}_3' \) or \( \tilde{T}_4' \) according to which of the two-vertex twisted \( I \)-bundles was selected at the beginning of the construction. Then the specific plugged thick \( I \)-bundle that has been constructed is denoted \( K_{\theta|p_1, q_1|p_2, q_2} \).

Again the two faces of an annulus may simply be identified with each other by attaching the degenerate \( \mathrm{LST}(2, -1, -1) \), i.e., a Möbius band. If a pair \( p_i, q_i \) is omitted from the symbolic name of a plugged thick \( I \)-bundle then this pair is once more assumed to be \( 2, -1 \). As an example, the plugged thick \( I \)-bundle \( K_{\tilde{T}_3' | 3, -1} \) is in reality \( K_{\tilde{T}_3' | 2, -1} \).

The identification of the underlying 3-manifold of a plugged thick \( I \)-bundle is similar to that for a plugged thin \( I \)-bundle as seen in the following result.

**Theorem 4.9** Let \( p_1 \) and \( q_1 \) be coprime integers and let \( p_2 \) and \( q_2 \) be coprime integers, where \( p_1 \neq 0 \) and \( p_2 \neq 0 \). Then the underlying 3-manifolds of the plugged thick \( I \)-bundles with parameters \( p_1, q_1, p_2 \) and \( q_2 \) are as follows.\(^2\)

\(^2\)In a previous version of this paper, these 3-manifolds were incorrectly given as \( \mathrm{SFS}(\mathbb{R}P^2 : (p_1, q_1), (p_2, q_2)) \) and
• \( K_{T_1} | p_1, q_1; p_2, q_2 \), \( K_{T_2} | p_1, q_1; p_2, q_2 \) and \( K_{T_3} | p_1, q_1; p_2, q_2 \) are each triangulations of the Seifert fibred space SFS \( \mathbb{RP}^2 : (p_1, q_1), (p_2, p_2 + q_2) \).

• \( K_{T_4} | p_1, q_1; p_2, q_2 \) is a triangulation of the Seifert fibred space SFS \( \bar{D} : (p_1, q_1), (p_2, -q_2) \), where the orbifold \( \bar{D} \) is a disc with reflector boundary.

**Proof** The proof is almost identical to the proof of Theorem 4.6. Once more we fill the boundary tori of \( \tilde{T}_1', \tilde{T}_2', \tilde{T}_3' \), and \( \tilde{T}_4' \) with circular fibres that run parallel to the left and right sides as illustrated in Figure 41.

![Circular fibres on the boundary tori of thick I-bundles](image.png)

Compatible fibrations of the interior I-bundles and the exterior layered solid tori are found as before, resulting in the 3-manifolds listed above.  

### 4.5 Exceptional Triangulations

Three triangulations appear in the \( \leq 7 \)-tetrahedron non-orientable census that do not fit neatly into any of the families described thus far. Each of these triangulations consists of precisely six tetrahedra and is formed using a variant of a previous construction. Specific details of these three triangulations are as follows.

**Definition 4.10 (Triangulations \( E_{6,1} \) and \( E_{6,2} \))** Triangulations \( E_{6,1} \) and \( E_{6,2} \) use a construction similar to the plugged thin I-bundles discussed in Section 4.3. Let \( \tilde{T}_6^1 \) and \( \tilde{T}_6^2 \) be the thin I-bundles over the torus depicted in the upper section of Figure 42 (these are the same \( \tilde{T}_6^1 \) and \( \tilde{T}_6^2 \) as used in Definition 4.5). Each of these thin I-bundles has a torus boundary, illustrated in the lower portion of the figure. Triangulations \( E_{6,1} \) and \( E_{6,2} \) are formed from \( \tilde{T}_6^1 \) and \( \tilde{T}_6^2 \) respectively by identifying the faces of their torus boundaries as follows.

![Thin I-bundles used in the construction of \( E_{6,1} \) and \( E_{6,2} \)](image.png)

Figure 42 illustrates the construction of \( E_{6,1} \). Beginning with \( \tilde{T}_6^1 \), we identify the boundary face \( ADB \) with \( EBC \) and we identify the boundary face \( DBE \) with \( EFC \). The resulting edge identifications are shown on the right hand side of the diagram.

The construction of \( E_{6,2} \) is illustrated in Figure 141. This time we begin with \( \tilde{T}_6^2 \), identifying the boundary face \( AEB \) with \( ECF \) and identifying the boundary face \( DAE \) with \( BEC \). Again the resulting edge identifications are shown.

**Definition 4.11 (Triangulation \( E_{6,3} \))** Triangulation \( E_{6,3} \) is formed from a pair of three-tetrahedron thin I-bundles as follows. Let \( \tilde{T}_3^1 \) be the thin I-bundle over the torus depicted in Figure 140.
Figure 43: Constructing exceptional triangulation $E_{6,1}$

Figure 44: Constructing exceptional triangulation $E_{6,2}$

Figure 45: The thin $I$-bundle used in the construction of $E_{6,3}$

The construction of $E_{6,3}$ involves taking two copies of $\tilde{T}_3^1$ and identifying their boundary tori according to a particular homeomorphism. This identification is illustrated in Figure 45. Specifically, face $ABD$ is identified with $XZW$ and face $ACD$ is identified with $ZWy$. The resulting edge identifications are shown in the diagram.

Each of the exceptional triangulations $E_{6,1}$, $E_{6,2}$ and $E_{6,3}$ can be converted using the elementary moves of Section 2.2.1 into a layered surface bundle, at which point the underlying 3-manifold can be identified using Theorem 4.4. A list of the resulting 3-manifolds can be found in Section 5.2.

5 Census Results

We conclude with a presentation of all closed non-orientable minimal $\mathbb{P}^2$-irreducible triangulations formed from at most seven tetrahedra. Recall from Section 1 that this list contains 41 distinct triangulations, together representing just eight different 3-manifolds.

Section 5.1 lists these 41 triangulations according to method of construction and Section 5.2 groups them according to their underlying 3-manifolds. Note that almost all of the 3-manifolds found in the census allow for more than one minimal triangulation. In such cases every minimal triangulation is presented.

A Regina data file containing all of the census triangulations listed here can be downloaded from the Regina website.

5.1 Triangulations

We present here the 41 census triangulations ordered first by number of tetrahedra and then by method of construction. Table 2 shows how these 41 triangulations are distributed amongst the different families described in Section 4. Note that the figures in the six-tetrahedron column sum to 25 whereas the total is listed as 24; this is because one of the six-tetrahedron triangulations can be viewed as both a layered torus bundle and a layered Klein bottle bundle.
The seven-tetrahedron closed non-orientable minimal \( \mathbb{P}^2 \)-irreducible triangulations leaving 24 distinct triangulations in our list.

Layered torus bundles

Layered Klein bottle bundles

Plugged thin I-bundles

Plugged thick I-bundles

Exceptional triangulations

| Tetrahedra                  | 1–5 | 6  | 7  | Total |
|-----------------------------|-----|----|----|-------|
| Layered torus bundles       | 0   | 6  | 4  | 10    |
| Layered Klein bottle bundles| 0   | 8  | 0  | 8     |
| Plugged thin I-bundles      | 0   | 4  | 6  | 10    |
| Plugged thick I-bundles     | 0   | 4  | 7  | 11    |
| Exceptional triangulations  | 0   | 3  | 0  | 3     |
| **Total**                   | 0   | 24 | 17 | 41    |

Table 2: Frequencies of triangulations from different families

Since there are no closed non-orientable minimal \( \mathbb{P}^2 \)-irreducible triangulations formed from five tetrahedra or fewer, we use six tetrahedra as the starting point for our detailed enumeration.

### 5.1.1 Six Tetrahedra

The six-tetrahedron closed non-orientable minimal \( \mathbb{P}^2 \)-irreducible triangulations are as follows.

- The layered torus bundles \( B_{T^6_4}[-1.0] \), \( B_{T^6_4}[0,1] \), \( B_{T^6_4}[1,1] \), \( B_{T^6_4}[1,0] \), \( B_{T^6_4}[1,1,1] \), and \( B_{T^6_4}[1,0,0] \) as described by Definition 4.2, where one of these layered torus bundles is isomorphic to a layered Klein bottle bundle as discussed below;
- The plugged thin I-bundles \( H_{T^6_4} \), \( H_{T^6_4} \), \( H_{T^6_4} \), and \( H_{T^6_4} \) as described by Definition 4.5;
- The plugged thick I-bundles \( K_{T^6_4} \), \( K_{T^6_4} \), \( K_{T^6_4} \), and \( K_{T^6_4} \) as described by Definition 4.8;
- The exceptional triangulations \( E_{6,1} \), \( E_{6,2} \), and \( E_{6,3} \) as described in Section 4.3.

Although 25 triangulations are named above, two of these are isomorphic. Specifically, the layered torus bundle \( B_{T^6_4}[0,1] \) is isomorphic to the layered Klein bottle bundle \( B_{K^6_4}[0,0,0,0] \), leaving 24 distinct triangulations in our list.

### 5.1.2 Seven Tetrahedra

The seven-tetrahedron closed non-orientable minimal \( \mathbb{P}^2 \)-irreducible triangulations are as follows.

- The layered torus bundles \( B_{T^7_4}[-1,1,2] \), \( B_{T^7_4}[0,1] \), \( B_{T^7_4}[-1,1] \), \( B_{T^7_4}[1,0] \) and \( B_{T^7_4}[1,1] \) as described by Definition 4.1;
- The plugged thin I-bundles \( H_{T^7_4} \), \( H_{T^7_4} \), \( H_{T^7_4} \), and \( H_{T^7_4} \) as described by Definition 4.3;
- The plugged thick I-bundles \( K_{T^7_4} \), \( K_{T^7_4} \), \( K_{T^7_4} \), \( K_{T^7_4} \), \( K_{T^7_4} \), \( K_{T^7_4} \), \( K_{T^7_4} \), \( K_{T^7_4} \), and \( K_{T^7_4} \) as described by Definition 4.8.
5.2 3-Manifolds

We close with a table of all closed non-orientable $\mathbb{P}^2$-irreducible 3-manifolds formed from seven tetrahedra or fewer. These 3-manifolds are listed in Table 3 along with their first homology groups and minimal triangulations. Recall from Section 4 that the orbifold $\bar{D}$ is a disc with reflector boundary.

| $\Delta$ | 3-Manifold | Triangulations | Homology |
|----------|------------|----------------|----------|
| 6        | $T^2 \times I/\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ | $B_{T^2_0}[-1,1][0,0]$ | $\mathbb{Z}$ |
|          | $T^2 \times I/\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ | $B_{T^2_0}[-1,0][0,1]$, $B_{T^2_0}[0,-1][0,1]$, $B_{T^2_0}[0,-1][1,0]$, $E_{6,3}$ | $\mathbb{Z} \oplus \mathbb{Z}$ |
|          | $K^2 \times S^1$ | $B_{K^2_0}[-1,0][0,1]$, $B_{K^2_0}[0,0][0,1]$ | $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ |
|          | $K^2 \times I/\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ | $B_{K^2_0}[-1,0][0,1]$, $B_{K^2_0}[0,0][0,1]$, $H_{K^2_0}$, $H_{K^2_0}$, $H_{K^2_0}$, $K_{K^2_0}$, $K_{K^2_0}$, $K_{K^2_0}$, $E_{6,2}$ | $\mathbb{Z} \oplus \mathbb{Z}$ |
|          | $K^2 \times I/\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ | $B_{K^2_0}[-1,0][0,1]$, $B_{K^2_0}[0,0][0,1]$, $H_{K^2_0}$, $K_{K^2_0}$, $E_{6,3}$ | $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ |
| 7        | $T^2 \times I/\begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}$ | $B_{T^2_0}[-1,1][2,-1]$, $B_{T^2_0}[0,-1][2,-1]$, $B_{T^2_0}[-1,1][1,0]$, $B_{T^2_0}[0,-1][1,0]$ | $\mathbb{Z} \oplus \mathbb{Z}$ |
|          | SFS $(\mathbb{R}P^2 : (2, 1) (3, 1))$ | $H_{T^2_0}$, $H_{T^2_0}$, $H_{T^2_0}$, $H_{T^2_0}$, $K_{T^2_0}$, $K_{T^2_0}$, $K_{T^2_0}$, $K_{T^2_0}$, $K_{T^2_0}$, $K_{T^2_0}$, $K_{T^2_0}$, $E_{6,2}$ | $\mathbb{Z}$ |
|          | SFS $(\bar{D} : (2, 1) (3, 1))$ | $H_{T^2_0}$, $K_{T^2_0}$, $K_{T^2_0}$, $K_{T^2_0}$, $E_{6,3}$ | $\mathbb{Z} \oplus \mathbb{Z}$ |

Table 3: All eight 3-manifolds and their 41 minimal triangulations

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