On the distribution of barriers in the spin glasses

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We discuss a general formalism that allows study of transitions over barriers in spin glasses with long-range interactions that contain large but finite number, \( N \), of spins. We apply this formalism to the Sherrington-Kirkpatrick model with finite \( N \) and derive equations for the dynamical order parameters which allow "instanton" solutions describing transitions over the barriers separating metastable states. Specifically, we study these equations for a glass state that was obtained in a slow cooling process ending a little below \( T_c \), and show that these equations allow "instanton" solutions which erase the response of the glass to the perturbations applied during the slow cooling process. The corresponding action of these solutions gives the energy of the barriers, we find that it scales as \( \tau^0 \) where \( \tau \) is the reduced temperature.

The most prominent feature of a glass state of matter is the existence of an extensive number of metastable states separated by large energy barriers. This feature is reproduced in many infinite range models which allow a mean field treatment; in the framework of this mean field approach the properties of these states have been studied extensively over the last 20 years and a very detailed picture has emerged. Originally, the models studied were thought to describe only systems with frozen disorder, like spin glasses, but more recently it was realized that similar methods can be applied to frustrated systems without quenched disorder that may be viewed as analogues of ordinary glasses.

Empirically, the existence of the extensive number of states is revealed in a very slow dynamics of the glass phase that is referred to as creep or ageing; this slow dynamics is usually attributed to transitions between different metastable states. Although, as a result of this extensive theoretical work mentioned above, the properties of the individual states are well understood, transitions between these states have been deprived of due attention. In this Letter we propose an analytical approach to address this problem; we apply it to the Sherrington-Kirkpatrick model and derive integral equations which describe the transitions between metastable states. We were not able to solve these equations analytically but we can show numerically that they admit a solution with expected qualitative properties. Our results have two applications: firstly, we hope that most qualitative features of the barrier distribution found in the infinite range models will hold in the finite range physical glasses, and secondly, it seems possible to test the predictions of the infinite range models on purpose built Josephson arrays; in the latter systems the number of effective "spins", \( N \), is large \( (N \sim 100 - 300) \) but not infinite and our results should be directly applicable.

Specifically, we shall consider a dynamical version of the Sherrington-Kirkpatrick model with \( N \) soft spins that is characterized by the equations of motion

\[
\Gamma_0^{-1} \partial_t S_i(t) = \frac{\delta (\beta H)}{\delta S_i(t)} + \xi(t) \tag{1}
\]

\[
H = - \sum_{i,j} J_{ij} S_i S_j - \sum_i h_i S_i + \sum_i V(S_i) \tag{2}
\]

Here \( J_{ij} \) is a matrix of quenched random Gaussian couplings normalized by \( \langle J_{ij}^2 \rangle = 1/N \), \( h_i \) is an external magnetic field acting on spin \( i \), \( \xi(t) \) is random thermal noise with correlator \( \langle \xi(t) \xi(t') \rangle = 2\Gamma_0^{-1} \delta_{ij} \delta(t - t') \) and \( V(S) \) is a one spin potential that keeps \( \langle S_i^2 \rangle = 1 \); its exact form is irrelevant, but for example one can take \( V(S) = r_0 (S^2 - 1)^2 \), with \( r_0 \gg T \).

It is well established that the low temperature state of this model is completely characterized by two order parameters: the correlation function, \( D(t_1, t_2) = \langle S_i(t_1) S_i(t_2) \rangle \), and the response function, \( G(t_1, t_2) = \langle \delta S_i(t_1) \rangle / \langle \delta h_i(t_2) \rangle \). In the low temperature state both these functions acquire a part which is non-decaying in time; a particular feature of the glass phase is the appearance of a non-decaying contribution to the response function \( G(t_1, t_2) \), showing that a perturbation applied in the distant past (at time \( t_2 \)) continues to affect the present (at time \( t_1 \)). Furthermore, different sample histories leading to the same final temperature and magnetic field would produce different order parameters, \( D(t_1, t_2) \) and \( G(t_1, t_2) \): this is what one would expect in a system that can be trapped in any of many metastable states and where the state where it is trapped is determined by the thermal history. So, in order to specify the glass state one needs to specify the history of the sample that led to this state. In the following we shall consider the systems prepared by the slow cooling process in which the temperature is varied slowly compared to the spin flip rate, \( \Gamma_0 \). In this regime it is convenient to separate the correlation and response functions into “fast” and “slow” parts: \( D(t_1, t_2) = D_f(t_1, t_2) + q(t_1, t_2) \) and \( G(t_1, t_2) = G_f(t_1, t_2) + \Delta(t_1, t_2) \). The order parameters...
characterizing the state of the model at the end of a
monotonic slow cooling process ending at $T(t_1) = T_1$
are well known if the reduced temperature is small,
$(T_c - T)/T_c = \tau \ll 1$:

$$q(t_1,t_2) = \min(\tau_1, \tau_2) + O(\tau^2)$$
$$\Delta(t_1,t_2) = \theta(t_1 - t_2)2\tau_2 \left(\frac{d\tau}{dt}\right)_{t_2}$$
$$\delta q_t \equiv \tau_{t-} - q_{t, t} = -\tau_i^2$$

(3)

Here and in the following we denote $\tau_1 \equiv \tau(t_1)$. This form
of $\Delta(t_1,t_2)$ shows that any perturbation applied during
the cooling process has everlasting effects; this is natu-
ral since qualitatively each state continuously subdivides
during the cooling process and no transitions between
these states are possible within the framework of this ap-
proach since qualitatively each state continuously subdivides
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these states are possible within the framework of this ap-
proach which becomes exact at $N \to \infty$ and $T \to 0$.

Empirically, we expect that at finite $N$ these transitions
become allowed the response function $\Delta(t_1,t_2) = \theta(t_1 - t_2)2\tau_2 \left(\frac{d\tau}{dt}\right)_{t_2}$
decays and eventually the memory of a perturbation applied at
a past time is lost. Furthermore, we expect that barriers
formed between states with larger overlap are generally
smaller than the barriers between states with small over-
lap and that transitions between the former are there-
more likely to occur. States with large overlaps are
formed at later stages of a cooling process as a subdivi-
sion of the ancestor state, so we expect that the lowest
barriers would correspond to the transitions which de-
stroy only the memory of a recent past.

In fact, as we shall show below, a theory in terms
of only $q$ and $\Delta$ is insufficient to describe transitions
over the barriers and we need to introduce one more or-
der function. We explain the general formalism using
the simple example of a particle in a potential $V(x) = v x^2(1 - x)$
which is in the metastable state around $x = 0$
at low temperatures $T \ll v$ but might escape to $x = \infty$
with probability $\exp(-V_{max}/T)$. All correlation func-
tions in this warm-up problem can be obtained from the
dynamic equations of motion

$$\frac{dx}{dt} = -\frac{dV}{dx} + \zeta(t), \quad \langle \zeta(t)\zeta(t') \rangle = 2T \delta(t - t')$$

(4)

or, equivalently, from the generating functional

$$Z = \int D\hat{x} \exp(-A\{\hat{x}\}) S \hat{x} \hat{x}$$

$$A\{\hat{x}\} = \int \left[ \hat{x}\left(\frac{dx}{dt} + \frac{dV}{dx} + T\hat{x}^2 \right) dt \right]$$

(5)

where $S\{\hat{x}\} = \det \left(\frac{dx}{dt} + \frac{d^2 V}{dx^2}\right)$ is a Jacobean whose
effects one can ignore to get the results with only an ex-
ponential accuracy and $A_h\{x, \hat{x}\}$ are source terms, e.g. in
order to get correlators of $x$ one can use $A_h\{x, \hat{x}\} = hx\hat{x}$,
then $\langle x^k \rangle = \frac{d^k\langle x \rangle}{dt^k}$. At small temperatures the functional
integral in (5) is dominated by the saddle point solutions.
Varying the action $A\{x\}$ with respect to $x$ and $\hat{x}$ we get

$$\frac{dx}{dt} - \frac{dV}{dx} \hat{x} = 0$$
$$\frac{d\hat{x}}{dt} + \frac{d^2 V}{dx^2} \hat{x} = 2iT \hat{x}$$

(7)

These equations admit two dramatically different solutions.
In the first solution $\hat{x} = 0$, $\frac{d\hat{x}}{dt} = -\frac{dV}{dx}$ correspond-
ing to a particle that slides down the potential hill prac-
tically unaffected by the thermal noise; this solution has
zero action. The second solution is $iT \hat{x} = \frac{d\hat{x}}{dt}$, $\frac{d\hat{x}}{dt} = \frac{dV}{dx}$,
corresponding to the particle going up the hill instead of
down; it can be obtained from the first solution by in-
verting the sign of the time so that the response of this
particle to an external field is anti-causal. Within the
saddle point approximation the path of the particle es-
caping the metastable state at $x = 0$ consists of the latter
solution for $0 < x < x_{max}$ and the former for $x > x_{max}$
(here $x_{max}$ is the point where potential has a maximum);
the action associated with this path is $A = \beta \delta V$, so it
has probability $\exp(-\beta \delta V)$, reproducing, as it should,
the Boltzmann formula. Note that the general statement
that the correlator $(\hat{x}\hat{x}) = 0$ does not prevent one from
having this “instanton” solution with a finite conjugate
field. Note also that once the solutions with this field
non-zero are allowed, the action corresponding to these
solutions is no longer zero but this does not contradict
the general statement that $Z \equiv 1$ in the absence of source
terms in (6) because Jacobean contribution cancels the
contribution of these solutions in the absence of source
terms. As a more detailed study [3] shows in the presence
of the source terms Jacobean contribution does not af-
fect the leading exponential factor and that therefore the
simple analysis (like the one above) that ignores source
terms and the effects of Jacobean gives correct result for
correlators. In what follows we shall assume that this
conclusion is also true for the glass problem and ignore
the effects of Jacobean and of the source terms.

A similar generating functional procedure with ex-
tremal analysis is conveniently employed to discuss the
spin glass models of Eqs. (1) and (2). By analogy with
above simple example we expect that saddle point solu-
tions corresponding to transitions over the barriers ex-
ist also in the glass model if the fields conjugate to the
variables $D(t_1,t_2)$ and $G(t_1,t_2)$ are allowed to acquire
non-zero values. So, we have to repeat the derivation
of the equations for the order parameters without mak-
ing the usual assumption that the variables conjugated
to $D(t_1,t_2)$ and $G(t_1,t_2)$ are zero. The first steps of the
derivation are not changed; these steps are explained well
in the literature [7][8][10], so we only recall them here.
The dynamics (1) is reproduced by the generating func-
tional $Z = \int D\hat{S} D\hat{x} D\hat{\chi} \exp(-A_B\{S, \hat{S}\} - A_F\{\hat{\chi}, \chi\})$
where $\hat{S}$, $\hat{S}$ and $A_B\{S, \hat{S}\}$ are analogous of $x$, $\hat{x}$ and
$A\{x, \hat{x}\}$ and additional integrals over fermionic fields $\chi, \hat{\chi}$
reproduce the Jacobean. To simplify the notations it is
very convenient to introduce, following Kurchan [10], su-
percoordinates $\theta$ and $\bar{\theta}$ that allow one to write the action
in terms of the superfield $\phi(\theta, \bar{\theta}) \equiv S + \theta \bar{\chi} + \bar{\theta} \chi + \theta \bar{\theta} \hat{S}$.
In these compact notations the full action $A\{\phi\}$ can be represented as a sum of a single spin part, $A_0$, and interacting part, $A_{\text{int}} = i \frac{1}{4} \sum_{i,j} J_{ij} \phi_i \phi_j$, the latter being the only term containing the couplings $J_{ij}$. This action is linear in the coupling $J_{ij}$, allowing one to average the generating functional over the Gaussian distribution of $J_{ij}$. This results in a four ‘spin’ interaction which one can decouple by a symmetric superfield $Q_{\tau_1, \tau_2}$ (here and below we use notation $T$ to denote the set of variables $(t, \theta, \bar{\theta})$. As can be shown in the illustrative example discussed in the last paragraph, we assume that the fermionic contribution can be ignored in a study to leading (exponential) order, so we retain only Bose components of the order parameter field $Q_{\tau_1, \tau_2}$:

$$Q^{(B)}_{\tau_1, \tau_2} = D_{t_1, t_2} + G_{t_1, t_2} \bar{\theta}_2 \theta_2 + \tilde{G}_{t_1, t_2} \bar{\theta}_1 \theta_1 +$$

$$D_{t_1, t_2} \theta_1 \theta_2 \theta_2$$

(8)

The symmetry $Q_{\tau_1, \tau_2} = Q_{\tau_2, \tau_1}$ implies that functions $D_{t_1, t_2}$ and $\tilde{D}_{t_1, t_2}$ are symmetric and that $G_{t_1, t_2} = G_{t_2, t_1}$, so $\tilde{G}_{t_1, t_2}$ is a backward response. The next technical step is to integrate out the spin variables which are now decoupled and get the effective action of the field $Q(\tau_1, T_2)$:

$$Z = \int DQ \exp(-NA\{Q\})$$

(9)

$$A = \int \frac{1}{4} \beta_1 Q^2_{\tau_1, \tau_2} \beta_{t_1} d\tau_1 dT_2 + A_S\{Q\}$$

(10)

where

$$A_S\{Q\} = -\ln \left[ \int D\phi \exp(-A_0\{\phi\}) \right]$$

$$+ \frac{1}{2} \beta_1 Q_{\tau_1, \tau_2} \beta_{t_1} Q_{\bar{\phi}^2}(\bar{T}_1)(T_2) d\tau_1 dT_2$$

(12)

is the effective action of $Q$ generated by decoupled spin degrees of freedom, $\phi$. The large factor $N$ in front of the action in Eq. (1) allows one to look only for the saddle point solutions which are obtained by varying the action with respect to $Q$ or, more explicitly, with respect to its components $\tilde{D}$, $D$ and $G$. The solution with zero action corresponds to the usual choice $\tilde{D}_{t_1, t_2} = 0, G_{t_2, t_1} = 0$ at $t_1 > t_2$; but neither of these relations necessarily holds for the saddle point solution with non-zero action that we are looking for. As we shall show explicitly below, if these conditions are imposed the usual solutions for the response functions of the glassy state are recovered. We shall of course need an explicit form of $A_S\{Q\}$; in the general case this action is very complicated, so to simplify we shall consider only the vicinity of the transition temperature ($\tau \ll 1$) and keep only the terms which affect the long time dynamics at such temperatures. Furthermore, we shall keep only the leading terms in the expansion in the conjugate order parameters $\tilde{D}_{t_1, t_2}$ and $G_{t_1, t_2}$ because, as we shall show below, transitions over small barriers (which, of course, dominate) involve only small values of these quantities.

Below we shall use a diagram expansion to derive the full saddle point equations following from the action (10). However, before we embark on it, it is instructive to do a preliminary analytic calculation assuming that one can keep only quadratic terms in the action $A_0\{\phi\}$. The results of this calculation can not be used below $T_c$ because there non-linear terms are essential but this derivation is much simpler and gives correctly all terms of the full equations except one thereby providing a useful reference point. In this approximation all integrals over $\phi$ in (10) are Gaussian and can be performed explicitly, so $A_S$ becomes:

$$A_S = \text{Str} \ln(\hat{Q}_0^{-1} - \beta^2 \hat{Q})$$

(13)

where Str denotes integral over times and trace over supersymmetric variables, $\hat{Q}$ are understood as operators and $\hat{Q}_0 = (\phi(\bar{T}_1)(\phi(T_2)))$. Explicitly $\hat{Q}_0^{(B)} = D^0_{t_1, t_2} + G^0_{t_1, t_2} \bar{\theta}_2 \theta_2 + \tilde{G}^0_{t_1, t_2} \bar{\theta}_1 \theta_1$. Here $G_0$ and $D_0$ are single site spin response and correlation functions which contain all local effects. Inverting $Q_0$ we get $(\hat{Q}_0^{(B)})^{-1} = -\langle G_0^{-2} D_0 \rangle_{t_1, t_2} + \langle G_0^{-1} \rangle_{t_1, t_2} \bar{\theta}_2 \theta_2 + \langle G_0^{-1} \rangle_{t_1, t_2} \bar{\theta}_1 \theta_1$, next we insert it into (13) and rewrite it in a more explicit form:

$$A_S = \frac{1}{2} \ln \det \left| \frac{\beta^2 \bar{D}}{(G_0^{-1} - \beta^2 G)} \right|$$

(14)

Finally, as we shall see below, transitions over the barriers in the vicinity of $T_c$ require only small noise fields $\tilde{D}$; this allows us to simplify the action further by expanding in $\tilde{D}$: retaining only the leading and subleading terms in it we get the simplified expression for the full action (10):

$$A = \frac{1}{2} \beta^2 \text{Tr} \left[ G \bar{G} + D \bar{D} \right] + \text{Tr} \ln (G_0^{-1} - \beta^2 G) -$$

$$\frac{1}{2} \beta^2 \text{Tr} \left( G_0^{-1} - \beta^2 G \right)^{-1} \Pi \left[ (G_0^{-1} - \beta^2 G^{-1})^{-1} \right]^{-1} \bar{D}$$

(15)

where $G, D, \bar{D}$ and $\Pi = (G_0^{-2} D_0) + \beta^2 D$ are understood as operators so that, e.g. $\text{Tr} G \bar{G} = \int dt dt' G_{t_1, t_2} \bar{G}_{t_2, t_1}$. Below we shall get an equation for $\bar{D}$ by varying this action with respect to $\bar{D}$, so to get the leading and subleading terms in this equation we shall need to keep also terms up to quadratic order in $D$:}

$$A = \frac{1}{2} \beta^2 \text{Tr} \left[ G G + D \bar{D} \right] + \text{Tr} \ln (G_0^{-1} - \beta^2 G) -$$

$$\frac{1}{2} \beta^2 \text{Tr} \left( G_0^{-1} - \beta^2 G \right)^{-1} \Pi \left[ (G_0^{-1} - \beta^2 G) \right]^{-1} \bar{D} -$$

$$\frac{1}{4} \beta^4 \text{Tr} \left[ (G_0^{-1} - \beta^2 G)^{-1} \Pi \left[ (G_0^{-1} - \beta^2 G) \right]^{-1} \bar{D} \right]^2$$

Varying action (15) with respect to $G$, $\bar{D}$ and $D$ fields and keeping only the leading and subleading terms in $\bar{D}$ we get equations
\[ G = (G_0^{-1} - \beta^2 G)^{-1} + GIG^d \hat{D}G \]  
\[ D = GIG^d - GIG^d \hat{D}GIG^d \]  
\[ \hat{D} = G^d \hat{D}G - G^d \hat{D}GIG^d \hat{D}G \]

Here we have dropped the factors of $\beta$ in the subleading terms in $\hat{D}$, ignoring their dependence on $\tau = T_c - t$ because these terms are already small in $\tau$ as explained above; we simplified the second terms in all the equations noting that they are already of the next order in $\tau$ which allows us to substitute $[G_0^{-1} - \beta^2 G]^{-1} \approx G$ in these terms; finally, we have also simplified the second and the third equations using the first equation to express $(G_0^{-1} - \beta^2 G)^{-1}$ via $G$. These equations are analogues of the equations (1) for the toy model. At $\hat{D} = 0$ this system of equations reduces to the equations derived in (1) for the glass dynamics above $T_c$, leading to the response and correlation functions that satisfy fluctuation-dissipation theorem (FDT). As was shown in (1) the quadratic approximation to the local spin response is sufficient above $T_c$ so in this temperature range the equations (16-18) are correct. However, in this temperature range there are no barriers and so for our present purpose these equations are not very interesting either. To finish our discussion of the equations for $T > T_c$ we give the final formula for the action expressed via the solutions $G$, $\hat{D}$ and $D$ of the equations (16-18) which are simplified by making the same approximations as we did in deriving equations (16-18):

\[ A = \frac{1}{2} \beta^2 Tr \left[ GG + D \hat{D} \right] + Tr \ln (G_0^{-1} - \beta^2 G) - \frac{1}{2} \beta^2 Tr GIG^d \hat{D} + \frac{3}{4} \beta^4 Tr \left\{ GIG^d \hat{D} \right\}^2 \] 

Note that because we have already used the equations (16-18) to simplify the action the solutions of these equations do not necessarily minimize the action (19).

We now derive the full equations below $T_c$ where non-linearity of the action $A_0(\phi)$ becomes essential, employing a diagram technique. To construct this technique we note that the saddle point equations can be rewritten in the form $\hat{D}_{t_1,t_2} = \langle \hat{S}_{t_1} \hat{S}_{t_2} \rangle_S$, $D_{t_1,t_2} = \langle \hat{S}_{t_1} \hat{S}_{t_2} \rangle_S$ and $G_{t_1,t_2} = \langle \hat{S}_{t_1} \hat{S}_{t_2} \rangle_S$ where $\langle \ldots \rangle_S$ denotes the average over a single spin variable taken with the weight $\exp(-A_0(\phi) + \int Q_{T_1,T_2} \phi(T_1) \phi(T_2) dT_1 dT_2)$. We shall use a formal expansion in the term $\int Q_{T_1,T_2} \phi(T_1) \phi(T_2) dT_1 dT_2$ to derive equations for the correlation and response functions and then reconstruct the action that gives them; furthermore, as above, we shall keep only the leading and subleading terms in $\hat{D}$ in these equations.

The known form of the equations for $D_{t_1,t_2}$ and $G_{t_1,t_2} = 0$ in the case when $\hat{D}_{t_1,t_2} = 0$ and $D_{t_1,t_2} = 0$ allows us to reconstruct all the terms in the action $A$ that are linear in the conjugate fields. We shall go briefly over the derivation of these equations because later it will allow us more easily to explain how to augment these equations for non-zero conjugate fields and construct the final action.

We start with the equation for the response function $G$ at $\hat{D} = 0$. In this approximation $G$ satisfies the Dyson equation diagrammatically represented as shown in Figs 1a and Fig 1b, the first of these contributions sufficient to reproduce the correct dynamics above $T_c$ and equivalent to (10) at $\hat{D} = 0$ while the second is needed to reproduce replica symmetry breaking effects in thermodynamics and memory effects in the correct dynamical solution (1). Together they yield

\[ G = [G_0^{-1} - \beta G + 3y(D^2G)]^{-1}, \] 

which leads to the conventional equation for the dynamics below $T_c$ and eventually to the result (1). Here and below inversion should be understood as operator inversion, $\beta G$ stands for $\beta_1 G_{t_1,t_2} G_{t_3,t_4}$, $G_0$ is the bare local spin response function and we use parenthesis () to imply the usual (arithmetic) product of two functions as opposed to the operator product implied everywhere else. The exact form of $G_0^{-1}$ is not important, because we need only its low frequency asymptote for which we assume a general form $G_0(\omega)^{-1} = a - i\omega \Gamma$ where $a$ is the renormalized zero frequency response and $\Gamma$ is a renormalized relaxation rate; these renormalizations are due to high energy processes and are not singular. Finally, the coefficient $y$ describes the strength of a local four spin correlator, in the following we shall take the value $y = 2/3$ corresponding to the hard Ising spins (10, (10), but note that this value can be always reduced to unity by scaling the correlation functions in the final slow cooling equations, so it is not important for the qualitative properties of the solutions. The factor 3 in front of the subleading term in Eq. (20) is a combinatorial factor associated with the self energy diagram, it appears because two intermediate propagators in it are $D$s and only one is $G$; note that it does not appear in the corresponding diagram for $D$ where all intermediate propagators are $D$s. A non-zero conjugate field $\hat{D}$ gives a new contribution to the equation for $G$. Diagrammatically this is as shown in Fig. 1a, 1b (terms giving Eq. (20)) and Fig. 1c (new term). Combining and manipulating these terms leads to

\[ G = \hat{G} + GIG^d \hat{D}G \]  
\[ \hat{G} \equiv [G_0^{-1} - \beta G - 3y(D^2G)]^{-1} \]  
\[ \Pi \equiv G_0^{-2}D_0 + \beta D\beta + y(D^3) \]

where we have dropped factors of $\beta \approx 1$ in the terms subleading in $\hat{D}$ because the corrections are small in the vicinity of $T_c$ and have replaced $\hat{G}$ by $G$ in the second term because the difference between these is $O(\hat{D})$.

The equation for the correlation function $D_{t_1,t_2}$ in the leading approximation in $\hat{D}$ is shown in Fig. 1d; in analogy with the equation for the response function shown
in Fig 1a we keep here only the leading and subleading terms in the self energy. This equation together with the equation for $G$ yields the full system of equations for the dynamics of the spin glass below the transition temperature in the absence of transitions over the barriers; the adiabatic approximation to these equations was discussed in e.g. [15]. The graphical representation of the subleading term in $\bar{D}$ is shown in Fig. 1c; it is analogous to the subleading term in the equation (17) obtained in the quadratic approximation. Combining these terms we get

$$D = \bar{G}\pi \bar{G}^d + \bar{G} \pi G \bar{D} \bar{G} \pi \bar{G}^d$$  \hspace{1cm} (24)

Again, to the order required we may replace $\bar{G}$ by $G$ in the second term. A further simplification results from using (21) to express $\bar{G}$ via $G$ in the first term yielding

$$D = G\pi G^d - G\pi \bar{G}^d \bar{D} \bar{G} \pi G^d$$  \hspace{1cm} (25)

Finally, the equation for $\bar{D}$ follows from the summation of the diagrams shown in Fig. 1f and Fig. 1g; note that here we have two contributions to the subleading part that graphically differ by the directions of the arrows inside the self energy term:

$$\bar{D} = G^d \left[ \beta \bar{D} \beta + 3y(D^2 \bar{D}) + 6y(GG^d D) \right] G - G^d G \pi G^d \bar{D} \bar{G}$$  \hspace{1cm} (26)

Here, as above, we have simplified the final expression using equations (21) to express $\bar{G}$ via $G$ in the first term, replaced $\bar{G}$ by $G$ in the second term and neglected factors of $\beta \approx 1$ and terms proportional to $y \left[ (D^2 \bar{D}) + (GG^d D) \right] D$ because all these effects are small in the vicinity of $T_c$.

Equations (24, 25) form the full system of equations describing dynamics of the spin glass below $T_c$, it differs from the system (14,15) (that can be used above $T_c$) only by the terms proportional to $y$ which describe the strength of the local non-linear spin response. The action corresponding to the system of equations (24,25) is rather cumbersome, so we shall not write it here. Instead we shall further simplify these equations by assuming that all fields change adiabatically slowly, i.e. that the time scale at which they change is much longer than the spin flip time scale $\Gamma^{-1}$. In this approximation the fast part of the Green function can be replaced by a delta-function, so we get $G_{t_1,t_2} = (1-q)\delta(t_1-t_2) + \Delta_{t_1,t_2}$. We shall simplify these equations even more by keeping only the leading terms in the reduced temperature $\tau$ and using the fact that $\tau - q \approx \delta q = O(\tau^2)$. To prove this general statement consider, say, equation (23) at $t_1-t_2 \geq T_0^{-1}$ and keep only the terms of the order of $\tau^2$. The second term in this equation is at least of the order of $\tau^3$ and can be ignored completely. Moreover in the approximation one can replace $G_{t_1,t_2} \rightarrow (1-q)\delta(t_1-t_2)$ and $\Pi \rightarrow \beta D \beta$ in the first term. Collecting the remaining terms we get $(\tau - q) D_{t_1,t_2} = O(\tau^3)$, so the existence of a non-zero $D_{t_1,t_2} = \delta_{t_1,t_2} = O(\tau^2)$ implies that $\delta q = O(\tau^2)$. The remaining two equations are also satisfied to the order of $\tau^2$ if $(\tau - q) D_{t_1,t_2} = O(\tau^3)$, so one gets non-trivial equations keeping only the terms of the order of $\tau^4$:

$$\left( \delta q_{t_1} + \delta q_{t_2} + 3yq^2_{t_2,t_2} \right)\Delta_{t_1,t_2} + \int \left( \Delta_{t_2,t_2} + \int \delta q_{t_1,t_2} + q_{t_1,t_2} D_{t_1,t_2} dt + \int q_{t_1,t_2} \Delta_{t_1,t_2} + \int q_{t_1,t_2} \delta q_{t_1,t_2} + q_{t_1,t_2} D_{t_1,t_2} + 3q_{t_1,t_2} \Delta_{t_2,t_2} \Delta_{t_1,t_1} - \int \delta q_{t_1,t_2} \delta q_{t_1,t_2} \Delta_{t_2,t_1} - \int \Delta_{t_1,t_2} \delta q_{t_1,t_2} \Delta_{t_2,t_1} - \int \Delta_{t_1,t_2} \delta q_{t_1,t_2} \Delta_{t_2,t_1} - \int \Delta_{t_1,t_2} \delta q_{t_1,t_2} \Delta_{t_2,t_1} - \int \Delta_{t_1,t_2} \delta q_{t_1,t_2} \Delta_{t_2,t_1} \right) dt'' = 0$$  \hspace{1cm} (27)

Next we derive the simplified expression for the action that corresponds to the adiabatic and $\tau \ll 1$ approximations used in deriving (27-29). First we note that the spin part of the action, $A_S$, is zero if $\bar{D} \equiv 0$, $\Delta_{t_1,t_2} \equiv 0$; this can most easily be verified by inspecting the diagrammatic expansion for the action. In this representation $A_S$ is given by the sum of closed diagrams which contain only $G$ lines or contain at least one $\bar{D}$ line so at least one of the of the lines in any closed diagram is either $\Delta_{t_1,t_2}$ or $\bar{D}$, therefore $A_S = 0$ if both these functions are zero. Further, variational derivatives of the action with respect to the functions $\bar{D}$ and $G$ give spin correlators, i.e. $-\frac{\delta A}{\delta \bar{D}} = \bar{G} \equiv \left[ G_0^{-1} - \beta G \beta - 3y(D^2 G) \right]^{-1}$ at $\bar{D} \equiv 0$ and $-\frac{\delta A}{\delta G} = \frac{1}{2} \left[ \bar{G} \pi G^d + \bar{G} \pi \bar{G}^d \bar{D} \bar{G} \pi \bar{G}^d \right]$ so these two equations completely determine it. To simplify the action in the vicinity of the transition temperature we note that the part of the action which is zeroth order in $\bar{D}$ necessarily contains at least one advanced response function, $G_A$. As discussed above, we expect that in the vicinity of $T_s$ such anomalous terms are small (because barriers are small) and so we can expand in $\bar{D}$. Keeping at most terms of the second order in $G_A$ and $\bar{D}$ we get

$$A_S = -\beta^2 Tr[GG_A G_R] - \frac{1}{2} \frac{1}{2} \left[ G_{AA} - \frac{1}{2} \frac{1}{2} \left[ \bar{G} \pi G^d \bar{D} \bar{G} \pi G^d \right] \right]^{-1}$$  \hspace{1cm} (30)

where in analogy with the derivation of the equations (22,24,26) we have neglected self energy corrections to the terms that are of the second order in $G_A$. The full action is $A = A_S + \beta^2 [Tr G_A G_R + \frac{1}{2} Tr D \bar{D}]$. We can simplify this expression by replacing $G_R$ and $D$ in the term $\beta^2 [Tr G_A G_R + \frac{1}{2} Tr D \bar{D}]$ by the r.h.s of the equations (22,24). Keeping only the terms of the second order in $G_A$ and $\bar{D}$ and retaining $\beta$ only in the leading terms we get
\[ A = \frac{1}{2} \text{Tr} [G_A G_R]^2 + \text{Tr} [G_A G_R \Pi G_R \tilde{D} G_R] - \frac{1}{4} \text{Tr} [C \Pi C^\dagger \tilde{D}]^2 \]

Finally we assume that \( G_A \) and \( \tilde{D} \) are adiabatically slow and keep only the leading terms in \( \tau \):

\[ A = \text{Tr} \left[ \Delta_A \Delta_A \Delta_R + \Delta_A \tilde{D} q - \frac{1}{4} q \tilde{D} q \tilde{D} \right] \quad (31) \]

where \( \Delta_A \equiv \Delta_{t_1 t_2} \theta(t_2 - t_1) \) (\( \Delta_R \equiv \Delta_{t_1 t_2} \theta(t_1 - t_2) \)) are advanced (retarded) parts of the adiabatic response.

For the numerical solution of the equations (27-29) it is convenient to regard them as the equations for the saddle point of the effective action

\[ A_{eff} = -\frac{1}{3} \int \Delta_{t_1 t_2} \Delta_{t_2 t_3} \Delta_{t_3 t_1} dt_1 dt_2 dt_3 - \frac{1}{2} \int (\delta q_{t_1} + \delta q_{t_2} + y q_{t_1 t_2}^2) q_{t_1 t_2} \tilde{D}_{t_1 t_2} dt_1 dt_2 - \frac{1}{2} \int (\delta q_{t_1} + \delta q_{t_2} + 3 y q_{t_1 t_2}^2) \Delta_{t_1 t_2} \Delta_{t_2 t_1} dt_1 dt_2 - \int q_{t_1 t_2} \tilde{D}_{t_1 t_2} \Delta_{t_3 t_4} \Delta_{t_4 t_3} dt_1 dt_2 dt_3 dt_4 + \frac{1}{4} \int q_{t_1 t_2} \Delta_{t_1 t_2} q_{t_3 t_4} \tilde{D}_{t_3 t_4} dt_1 dt_2 dt_3 dt_4 \]

Using the equations (27-29) and representing \( \Delta \) as a sum of the retarded and advanced parts we can simplify the action (22) and get the same result (11) as above.

Scaling analysis of the equations (27-29) shows that their solutions have the following scales: \( \Delta \sim \tau \frac{\tau}{\tau^2}, q \sim \tau \) and \( \tilde{D} \sim \tau^2 \frac{\tau}{\tau^2} \). Using these values to estimate the last (quadratic in \( \tilde{D} \)) term in (27) we see that it scales as \( \int \tau \left( \frac{\tau}{\tau^2} \right)^2 \tau^2 dt^4 \sim \tau^8 \) whereas the first two terms (and the action itself scale as \( \int \tau \left( \frac{\tau}{\tau^2} \right)^3 dt^3 \sim \tau^6 \) and \( \int \tau \left( \frac{\tau}{\tau^2} \right)^2 dt^3 \sim \tau^6 \)). These estimates allow us to neglect the last terms in the expressions (31) and (22) for the action and the last terms in the equations (28) and (24).

We were not able to guess the ansatz for the analytical solution of the equations (27-29). Instead we show numerically that the solution that we expected on physical grounds indeed exists. As explained above, we expect that as the temperature is decreased a given metastable state continues to subdivide and that the barriers between these filial states are smaller than the barriers separating the ancestor states. Therefore we expected that a rare fluctuation (“instanton”) might take the system from one of these filial states to another leaving the ancestor state unchanged. In this case the memory of the perturbation applied at the times when these filial states were formed is completely lost, i.e. \( \Delta_{t_1 t_2} = 0 \) for \( t_2 > t_1 \). Indeed, we observe a solution with this property: in Fig 2a we show \( \Delta_{t_1 t_2} \) obtained numerically for the slow cooling process which leads to the final reduced temperature \( \tau_f = 0.1 \). This numerical solution clearly has the property that the memory of the latest perturbation is erased. Note also that the conjugate field corresponding to this solution is small: it is proportional to \( \tau \) and moreover seems to have a numerically small coefficient. This solution corresponds to the action \( S/N \approx 0.1 \tau^6 \). By construction this solution is not time-invariant, instead it is localized at the end of the cooling procedure. We expect that these equations also admit a solution of a different kind which occurs when the system is first slowly cooled down to the final temperature, \( \tau(t_f) = \tau_f \), and then kept at this temperature for a long time. This solution corresponds to a transition over the barrier at some \( t_1 \), in this solution \( \Delta_{t_1 t_2} = 0 \) for \( t_2 < t_1 < t_f \) for all \( t_2 < t_1 < t_f \) for \( t_1 < t_f \). Such a solution should correspond to the transition between the states on the lowest level of the hierarchy and erase only a memory of the events in the narrow time range \( (t_1, t_f) \) when this hierarchy was formed; from (31) it is clear that such solutions have very small action (suppressed by additional power of \( (\tau - \tau_f)^2 \)), so there is no lower bound on the energy of the barrier in this approach. This low bound might appear when terms of next order in \( 1/N \) are taken into account and it is quite probable that the numerical result (11) that the lowest barriers scale with \( 1/N^{1/4} \) is due to this mechanism.

In conclusion we have developed the formalism which allows one to study the distribution of barriers in a spin glass with a large but finite number of spins, \( N \). We considered the particular example of the Sherrington-Kirkpatrick model and derived analytically the equations for the dynamic order parameters of this model that admit transition over the barriers. We also showed numerically that these equations admit a non-trivial solution for which the energy of the barrier scales as \( N \tau^6 \). It remains an open question whether this solution is unique or in fact these solutions admit a whole family of solutions and the energy of the barriers has a broad distribution. It also remains to be investigated whether these results and general approach can be applied to the common physical situation of a spin glass where each spin has finite number of neighbours but the total number of spins is infinite.

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Fig. 1 Diagrams giving equations (21-26). Here the lines with the arrows in the same direction denote $G$, the lines with arrows pointing outwards denote $D$ and the lines with arrows pointing inwards denote $\hat{D}$. Thick lines are full (renormalized) correlation functions, a thin line with arrows in the same direction is a Green function $G_0$, a gray thick line is $\tilde{G}$, i.e. the response function calculated in the zeroth order in $\hat{D}$, a gray circle denotes $\beta$ and a gray diamond denotes a four spin vertex, $\sqrt{y}$. The geometric series for $\tilde{G}$ is shown in (a) and (b); this series sums to (22) which is equivalent to the well known [6–8] dynamical equations. First order corrections to $G$ are shown in (c); as is clear from this series a general diagram contains $0, 1, \ldots$ bare Green functions to the right of $D_0$ and $0, 1, \ldots$ between $\hat{D}$ and $D_0$ so the sum of these terms is $GG^{-1}_0 (G^{-1}_0 G)$ making the sum of all diagrams linear in $\hat{D}G\tilde{G}^1D\tilde{G}$ with $\Pi$ that is given by (23). In (d) we show the leading terms in the expansion for $D$, this series is similar to the one shown in (c) and gives $G\tilde{G}^1$. In (e) we show one term which gives the first order correction to $\hat{D}$; this term is proportional to $D^2$ and together with the analogous terms that are proportional to $D\hat{D}_0$ or $\hat{D}^2_0$ it gives $G\tilde{G}^1D\tilde{G}\tilde{G}$. Finally, in (f) and (g) we show the leading and subleading terms for $\hat{D}$; note that the leading term for this function does not contain a bare part and moreover its self-energy contains either $\hat{D}$ or two response functions with opposite directions ensuring that the usual solution (in which $\hat{D} = 0$ and response is purely retarded) is self-consistent.

Fig. 2 A numerical solution of the system of equations (27-29) for the slow cooling process leading to the final reduced temperature $\tau(t_f) = \tau_f = 0.1$. All displayed functions are scaled by $\tau_f$. Fig 2a shows $\Delta$, Fig 2b shows $D$ and Fig. 2c shows $\hat{D}$. From Fig 2a it is clear that in this solution a perturbation applied during the later part of the cooling process, $t_{tr} < t < t_f$ has no effect on the state at $t_f$, so this solution corresponds to the transition that erases memory of these times; we also observe that this transition has relatively little effect on the correlation function $D$ but results in the appearance of the ’noise’ field $\hat{D}$ localized around times $t \sim t_{tr}$.

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