Longitudinal bulk strain solitons in a hyperelastic rod with quadratic and cubic nonlinearities

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Dedicated to V.E. Zakharov on the occasion of his 80th birthday.

Abstract

We study long nonlinear longitudinal bulk strain waves in a hyperelastic rod of circular cross section within the scope of the general weakly-nonlinear elasticity leading to a model with quadratic and cubic nonlinearities. We systematically derive the extended Boussinesq and Korteweg - de Vries - type equations and construct a family of approximate weakly-nonlinear soliton solutions with the help of near-identity transformations. These solutions are compared with the results of direct numerical simulations of the original nonlinear problem formulation, showing excellent agreement within the range of their asymptotic validity (waves of small amplitude) and extending their relevance beyond it (to the waves of moderate amplitude) as a very good initial guess. In particular, we were able to observe a stably propagating "table-top" soliton.

1 Introduction

Solitons have been a subject of a huge body of theoretical and experimental research in such areas as fluids and nonlinear optics, largely because of the compact form of the governing equations and availability of a large amount of experimental and observational data (see [1, 2, 3, 4] and references therein). In contrast to that, the studies of solitary waves in solids is a relatively recent area of research, generally requiring greater efforts because of the complexity and great variability of the properties of solids reflected in their constitutive relations, as well as significant experimental challenges (for example, [5, 6, 7, 8, 9, 10, 11] and references therein). Considerable progress has been made in the studies of bulk strain solitons in hyperelastic rods, starting with the works of G.A. Nariboli and A. Sedov [12] and L.A. Ostrovsky and A.M. Sutin [13], and significantly advanced by A.M. Samsonov and his group (see [14, 15, 16, 17, 18, 19] and references therein). Theoretical studies were based on the Boussinesq and Korteweg-de Vries-type models developed within the scope of the weakly-nonlinear elasticity theory (Murnaghan’s 5 constant model for elastic energy [20]), with differing degree of rigour. A systematic asymptotic analysis has been developed by H.-H. Dai and X. Fan [21] (although a systematic derivation of a Boussinesq-type equation was developed later, by F.E. Garbuzov et al.
K.R. Khunusdinova et al. [23], within the scope of nonlinear elasticity and lattice modelling, respectively. In [22] the derivations within the scope of the general weakly-nonlinear elasticity theory have been simplified and generalised to include surface loading and longitudinal pre-stretch, resulting in the Boussinesq- and forced Boussinesq-type models. The Boussinesq-type models have been used to study, in particular, the scattering of long longitudinal bulk strain solitary waves by delamination (see [24, 25, 26], and for related experiments see [27, 28]).

In the present paper we aim to study elastic solitons of both small and moderate amplitude, and therefore we extend the derivation of nonlinear two-directional long wave models for longitudinal waves to hyperelastic materials described by the 9 constant model for the energy of the elastic deformation including cubic and quartic terms. We account for both geometrical and physical sources of nonlinearity and develop a systematic asymptotic analysis. The derivations are performed using symbolic computations with MATHEMATICA [29]. We then derive a uni-directional extended Korteweg - de Vries (KdV) - type model and study its solitary wave solutions both analytically, with the help of near-identity transformations [30, 31] (see also the review [32] and references therein) and direct numerical simulations of the original problem formulation.

2 Problem formulation

We consider a rod of circular cross section with the radius \( R \) and use cylindrical coordinates \((x, r, \varphi)\) with the axial coordinate \( x \), radial coordinate \( r \) and angular coordinate \( \varphi \). We use the Lagrangian description and denote the displacement vector by \( U = (U, V, W) \), where \( U \) is the axial displacement, \( V \) is the radial displacement and \( W \) is the torsion.

![Figure 1: Rod of circular cross section.](image)

We use the fourth-order Landau-Lifshits constitutive relation [33] for the energy of the elastic deformation, which can be written as follows:

\[
\Pi = \frac{\lambda}{2} (\text{tr} \, \varepsilon)^2 + \mu \text{tr} \, \varepsilon^2 + \frac{A}{3} \text{tr} \, \varepsilon^3 + B \text{tr} \, \varepsilon \text{tr} \, \varepsilon^2 + \frac{C}{3} (\text{tr} \, \varepsilon)^3 \\
+ D \text{tr} \, \varepsilon \text{tr} \, \varepsilon^2 + F (\text{tr} \, \varepsilon)^2 \text{tr} \, \varepsilon^2 + G (\text{tr} \, \varepsilon^2)^2 + H (\text{tr} \, \varepsilon)^4, \tag{1}
\]

where \( \varepsilon = (\nabla U^T + \nabla U + \nabla U^T \cdot \nabla U) / 2 \) is the Cauchy-Green strain tensor. This is equivalent to the Murnaghan 9 constant model [7]:

\[
\Pi = \frac{\lambda + 2\mu}{2} I_1^2 - 2\mu I_2 + \frac{l + 2m}{3} I_3^3 - 2m I_1 I_2 + n I_3 + \nu_1 I_1^4 + \nu_2 I_1^2 I_2 + \nu_3 I_1 I_3 + \nu_4 I_2^2, \tag{2}
\]

where \( I_1 = \text{tr} \, \varepsilon, \ I_2 = [(\text{tr} \, \varepsilon)^2 - \text{tr} \, \varepsilon^2] / 2, \ I_3 = \text{det} \, \varepsilon \) and

\[
l = B + C, \ m = A/2 + B, \ n = A, \ \nu_1 = D + F + G + H, \ \nu_2 = -(2F + 3D + 4G), \ \nu_3 = 3D, \ \nu_4 = 4G. \tag{3}
\]
We now consider an exact reduction of the full equations of motion describing solutions with no torsion, and where the longitudinal and transverse displacements $U$ and $V$ are independent of $\phi$:

$$U = U(x, r, t), \quad V = V(x, r, t), \quad W = 0.$$  \hspace{1cm} (4)

The equations of motion take the form

$$\rho \frac{\partial^2 U(x, r, t)}{\partial t^2} - \frac{\partial P_{xx}}{\partial x} - \frac{\partial P_{xr}}{\partial r} - \frac{P_{xr}}{r} = 0,$$  \hspace{1cm} (5)

$$\rho \frac{\partial^2 V(x, r, t)}{\partial t^2} - \frac{\partial P_{rx}}{\partial x} - \frac{\partial P_{rr}}{\partial r} - \frac{P_{rr} - P_{\phi \phi}}{r} = 0,$$  \hspace{1cm} (6)

while the third equation is identically satisfied. Here, $P_{\alpha \beta}$ denotes components of the first Piola-Kirchhoff stress tensor

$$\underline{P} = (I + \nabla U) \cdot \frac{\partial \Pi}{\partial E},$$  \hspace{1cm} (7)

where $I$ is the identity tensor.

We assume that the rod is not subjected to any external loading, i.e. the stress has to vanish at the surface of the rod

$$P_{rr} = P_{xr} = 0 \quad \text{at} \quad r = R.$$  \hspace{1cm} (8)

Since the component $P_{r\phi} \equiv 0$, the third boundary condition $P_{r\phi} = 0$ at $r = R$ is identically satisfied.

We consider longitudinal waves in a symmetric rod, hence we add symmetry conditions which require the longitudinal displacement to be an even function of $r$ and the radial displacement to be an odd function of $r$ (e.g., [22]).

### 3 Extended Boussinesq-type equation

We extend the approach developed in our previous paper [22]. We look for a solution of the problem in the form of power series expansions of the displacements in the radial coordinate:

$$U(x, r, t) = U_0(x, t) + r^2U_2(x, t) + r^4U_4(x, t) + r^6U_6(x, t) + \ldots,$$  \hspace{1cm} (9)

$$V(x, r, t) = rV_1(x, t) + r^3V_3(x, t) + r^5V_5(x, t) + r^7V_7(x, t) + \ldots,$$  \hspace{1cm} (10)

which follow from the symmetry conditions. We consider the waves of small amplitude and large length compared to the radius of the rod. Hence we non-dimensionalise the variables as follows:

$$\tilde{t} = \frac{t}{L/c}, \quad \tilde{x} = \frac{x}{L}, \quad \tilde{r} = \frac{r}{\varepsilon L}, \quad \tilde{U} = \frac{U}{\varepsilon L}, \quad \tilde{V} = \frac{V}{\varepsilon \delta L},$$  \hspace{1cm} (11)

which yields $\tilde{U}_n = \frac{L^nU_n}{\varepsilon L}$, $\tilde{V}_n = \frac{L^nV_n}{\varepsilon L}$ for $n \geq 0$, assuming that $L$ is the characteristic wavelength, $c$ is the linear wave speed, $E$ is the Young modulus, $\varepsilon$ is the small amplitude parameter (characterising the longitudinal strain), and $\delta = \frac{R}{L}$ is the second small parameter (long wavelength parameter). Here, the tilde denotes dimensionless variables and tractions. In the following we will use expressions for the
Young modulus and the Poisson ratio in terms of the Lame coefficients:

\[ E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \quad \nu = \frac{\lambda}{2(\lambda + \mu)}. \]  

(12)

Then, the expansions (9) and (10) take the form

\[ \tilde{U}(\tilde{x}, \tilde{r}, \tilde{t}) = \tilde{U}_0(\tilde{x}, \tilde{t}) + \tilde{r}^3 \tilde{U}_2(\tilde{x}, \tilde{t}) + \tilde{r}^4 \tilde{U}_4(\tilde{x}, \tilde{t}) + O(\tilde{r}^6), \]  

(13)

\[ \tilde{V}(\tilde{x}, \tilde{r}, \tilde{t}) = \tilde{r} \tilde{V}_1(\tilde{x}, \tilde{t}) + \tilde{r}^3 \tilde{V}_3(\tilde{x}, \tilde{t}) + \tilde{r}^5 \tilde{V}_5(\tilde{x}, \tilde{t}) + O(\tilde{r}^7). \]  

(14)

In what follows we omit the tildes.

Substituting (13) and (14) into the equations of motion (5) and (6) we obtain

\[
\begin{align*}
\rho c^2 U_{0tt} - (\lambda + 2\mu) U_{0xx} - 2(\lambda + \mu) V_{1x} - 4\mu U_2 + \Phi_{1,1}\varepsilon + \Phi_{1,2}\varepsilon^2 \\
+ [\rho c^2 U_{2tt} - (\lambda + 2\mu) U_{2xx} - 4(\lambda + \mu) V_{3x} - 16\mu U_4 + \Phi_{1,3}\varepsilon] r^2 \\
+ [\rho c^2 U_{4tt} - (\lambda + 2\mu) V_{5xx} - 6(\lambda + \mu) U_{6x} - 36\mu U_6] r^4 + O(\varepsilon^3, \varepsilon^2 r^2, \varepsilon r^4, r^6) &= 0, \\
\rho c^2 V_{1tt} - \mu V_{1xx} - 2(\lambda + \mu) U_{2x} - 8(\lambda + 2\mu) V_3 + \Phi_{2,1}\varepsilon + \Phi_{2,2}\varepsilon^2 \\
- [\rho c^2 V_{3tt} - \mu V_{3xx} - 4(\lambda + \mu) U_{4x} - 24(\lambda + 2\mu) V_5 + \Phi_{2,3}\varepsilon] r^2 \\
- [\rho c^2 V_{5tt} - \mu V_{5xx} - 6(\lambda + \mu) U_{6x} - 48(\lambda + 2\mu) V_7] r^4 + O(\varepsilon^3, \varepsilon^2 r^2, \varepsilon r^4, r^6) &= 0.
\end{align*}
\]

(15)

(16)

Here, the subscripts \( x \) and \( t \) denote partial derivatives and \( \Phi_{1,1}, \Phi_{1,2}, \Phi_{1,3} \) denote all nonlinear terms with the coefficients \( \varepsilon, \varepsilon^2 \) and \( \varepsilon r^2 \), respectively. The functions \( U_2, V_3, U_4, V_5, U_6 \) can be obtained using the power series expansions in \( \varepsilon \):

\[ U_2 = U_2^{(0)} + \varepsilon U_2^{(1)} + \varepsilon^2 U_2^{(2)} + \ldots \]

Equating to zero the coefficients at different powers of \( \varepsilon \) and \( r \) in (15) and (16) results in

\[
\begin{align*}
U_2 &= \frac{1}{4\mu} \left[ \rho c^2 U_{0tt} - (\lambda + 2\mu) U_{0xx} - 2(\lambda + \mu) V_{1x} + \varepsilon U_2^{(1)}(x,t) + \varepsilon^2 U_2^{(2)}(x,t) + O(\varepsilon^3), \\
V_3 &= \frac{1}{8(\lambda + 2\mu)} \left[ \rho c^2 V_{1tt} - 2(\lambda + \mu) U_{2x} - \mu V_{1xx} + \varepsilon V_3^{(1)}(x,t) + \varepsilon^2 V_3^{(2)}(x,t) + O(\varepsilon^3), \\
U_4 &= \frac{1}{16\mu} \left[ \rho c^2 U_{2tt} - (\lambda + 2\mu) U_{2xx} - 4(\lambda + \mu) V_{3x} + \varepsilon U_4^{(1)}(x,t) + O(\varepsilon^2), \\
V_5 &= \frac{1}{24(\lambda + 2\mu)} \left[ \rho c^2 V_{3tt} - 4(\lambda + \mu) U_{4x} - \mu V_{3xx} + \varepsilon V_5^{(1)}(x,t) + O(\varepsilon^2), \\
U_6 &= \frac{1}{36\mu} \left[ \rho c^2 U_{4tt} - (\lambda + 2\mu) U_{4xx} - 6(\lambda + \mu) V_{5x} + O(\varepsilon). \right]
\end{align*}
\]

(17)

(18)

(19)

(20)

(21)

The expressions for the functions \( U_2^{(1)}, U_2^{(2)}, V_3^{(1)}, V_3^{(2)}, U_4^{(1)}, V_5^{(1)} \) are cumbersome and are not shown here. Next, substituting the functions \( U_2, V_3, U_4, V_5, U_6 \) into the boundary conditions (8) we obtain the equations

\[
\begin{align*}
2(\lambda + \mu) V_1 + \lambda U_{0x} + \varepsilon \Psi_{1,1} + \varepsilon^2 \Psi_{1,2} + \delta^2 \left[ d_1 U_{0xx} + \rho c^2 d_2 U_{0xtt} + \rho c^2 d_3 V_{1tt} + d_4 V_{1xx} \right] \\
+ \delta^4 \left[ (d_5 V_{1xx} + \rho c^2 d_6 V_{1tt})_{xx} + \rho^2 c^4 (d_7 U_{0xx} + d_8 V_{1xx})_{ttt} + (d_9 U_{0xx} + \rho c^2 d_{10} U_{0tt} + O(\varepsilon^3, \varepsilon^2 r^2, \varepsilon r^4, r^6)) = 0,
\end{align*}
\]

(22)
\[\rho c^2 U_{0tt} - 2\lambda V_{1x} - (\lambda + 2\mu) U_{0xx} + \varepsilon \Psi_{2,1} + \varepsilon^2 \Psi_{2,2} + \varepsilon \delta^2 \Psi_{2,3} + \delta^2 \left[\varepsilon_1 U_{0xxxx} + \rho^2 c^2 \varepsilon_2 U_{0utt} + \rho^2 \varepsilon_3 U_{0xxxt} + \varepsilon_4 V_{1xxx} + \rho^2 \varepsilon_5 V_{1xtt}\right] + \delta^4 \left[\varepsilon_6 V_{1xx} + \rho^2 \varepsilon_7 V_{1tt}\right] + \rho^2 \varepsilon_8 \left(U_{0xx} + \varepsilon^2 \varepsilon_9 U_{0xx} + \rho^2 \varepsilon_{10} U_{0tt}\right)_{utt} + (\varepsilon_{11} U_{0xx} - \rho^2 \varepsilon_{12} U_{0tt})_{xxxx} + O(\varepsilon^3, \varepsilon^2 \delta^2, \varepsilon^4, \delta^6) = 0.\]

Here the coefficients \(d_i, e_i\) depend on the Lame elastic moduli, and \(\Psi_{i,j}\) denote nonlinear terms. Elimination of the function \(V_1\) from the equations (22) and (23) can be done by expanding it into the power series in \(\varepsilon\) and \(\delta^2\). Unknown terms in this expansion can be found by equating to zero the coefficients of \(\varepsilon, \delta^2, \varepsilon^2, \delta^4\) and \(\varepsilon \delta^2\) in (22):

\[V_1(x, t) = -\frac{\lambda}{2(\lambda + \mu)} U_0 x + \varepsilon f(x, t) + \delta^2 g(x, t) + \varepsilon^2 \tilde{f}(x, t) + \delta^4 \tilde{g}(x, t) + \varepsilon \delta^2 \tilde{h}(x, t) + O(\varepsilon^3, \varepsilon^2 \delta^2, \varepsilon^4, \delta^6).\]

Here we do not show the expressions for the functions \(f, g, \tilde{f}, \tilde{g}, \tilde{h}\), for brevity. Then, the substitution of \(V_1\) into (23) results in the following equation for \(U_0\):

\[U_{0tt} - U_{0xx} + \varepsilon \left[\frac{\beta_1}{E} U_{0xx}^2 + \alpha_1 U_{0ttt} + \alpha_2 U_{0xxtt} + \alpha_3 U_{0xxxx}\right] + \varepsilon^2 \left[\frac{\beta_2}{E^2} \varepsilon_3 U_{0xxxt} + \alpha_4 U_{0xxxx} + \alpha_5 U_{0xxttt} + \alpha_6 U_{0xxxxxt} + \alpha_7 U_{0xxxxxxx} + \tilde{J}_0(U_0)\right] + O(\varepsilon^3, \varepsilon^2 \delta^2, \varepsilon^4, \delta^6) = 0,\]

where the coefficients \(\alpha_i, \beta_i\) and the nonlinear function \(\tilde{J}_0\) are given in Appendix A. Then, assuming the balance between the nonlinear and dispersive terms \(\varepsilon \sim \delta^2\), and truncating this equation, we obtain an extended Boussinesq-type equation:

\[U_{0tt} - U_{0xx} + \varepsilon \left[\frac{\beta_1}{E} U_{0xx}^2 + \alpha_1 U_{0ttt} + \alpha_2 U_{0xxtt} + \alpha_3 U_{0xxxx}\right] + \varepsilon^2 \left[\frac{\beta_2}{E^2} \varepsilon_3 U_{0xxxt} + \alpha_4 U_{0xxxx} + \alpha_5 U_{0xxttt} + \alpha_6 U_{0xxxxxt} + \alpha_7 U_{0xxxxxxx} + \tilde{J}_0(U_0)\right] = 0.\]

The equation (26) can be rewritten in a simpler asymptotically equivalent form

\[U_{0tt} - U_{0xx} + \varepsilon \left[\frac{\beta_1}{E} U_{0xx}^2 + q_1 U_{0xxxt}\right] + \varepsilon^2 \left[\frac{\beta_2}{E^2} U_{0xxxt}^2 + \frac{q_2 - 2\gamma_1 - 4\gamma_4}{E} U_{0xx} U_{0xxx}\right] + \frac{2\gamma_1 + 4\gamma_4}{E} U_{0xx} U_{0xxx} + \frac{q_3}{E} U_0 x U_{0xxxxx} + q_4 U_{0xxxxxt} = 0,\]

where

\[q_1 = \alpha_1 + \alpha_2 + \alpha_3, \quad q_2 = 3\gamma_1 + 2\gamma_2 + 3\gamma_3 + 6\gamma_4 + \gamma_5 + 6\gamma_6 + 2\gamma_7, \quad q_3 = \gamma_1 + \gamma_3 + 2\gamma_4 + \gamma_5 + 2\gamma_6, \quad q_4 = \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7.\]

The dimensional form of the equation (27) is given by

\[\rho \left[\frac{1}{c^2} U_{0xx} + \frac{\beta_1}{\rho} U_{0xx}^2 + q_1 R^2 U_{0xxxt} + \frac{\beta_2}{\rho E} U_{0xxxt}^2 + \frac{(q_2 - 2\gamma_1 - 4\gamma_4) R^2}{\rho} U_{0xx} U_{0xxx}\right] + \frac{(2\gamma_1 + 4\gamma_4) R^2}{E} U_{0xx} U_{0xxx} + \frac{q_3 R^2}{\rho} U_0 x U_{0xxxxx} + q_4 R^4 U_{0xxxxxt} = 0.\]
A particular case of this equation has been considered in [34, 35, 36].

We note that, unlike [37], the equation obtained using a systematic asymptotic procedure contains not only the additional cubic nonlinear term, but also several other nonlinear and dispersive terms. The expression for the coefficient in front of the cubic nonlinearity in [37] has a typo. The correct formula can be found in Appendix A (given in terms of the Landau moduli, relations between the Landau and Murnaghan moduli can be found in (3)).

4 Extended Korteweg - de Vries - type equation and solitons

We introduce the characteristic variables \( \xi = x - ct \), \( \eta = x + ct \), \( \tau = \varepsilon t \) and expand the unknown function \( U_0 \) into the power series in \( \varepsilon \). In order to derive a uni-directional model we look for a solution in the form

\[
U_0(x, t) = U_0^{(0)}(\xi, \tau) + \varepsilon U_0^{(1)}(\xi, \tau) + \varepsilon^2 U_0^{(2)}(\xi, \tau) + \ldots,
\]

which can be justified by allowing the dependence of the higher-order terms on both characteristic variables and requiring the absence of secular terms in the asymptotic expansion (e.g. [4]). Substitution of \( U_0 \) into the equation (26) (one could also use (27)) yields

\[
\left( \frac{\beta_1}{2E} U_0^{(0)} \frac{\partial}{\partial \xi} - \frac{q_1}{2} U_0^{(0)} \frac{\partial}{\partial \xi^2} \right) \varepsilon + \left( \frac{\beta_1}{2E} U_0^{(0)} \frac{\partial}{\partial \xi^2} U_0^{(0)} - \frac{q_1}{2} U_0^{(0)} \frac{\partial}{\partial \xi^3} \right) \varepsilon + (2\alpha_1 + \alpha_2) U_0^{(0)} \\
- \frac{1}{2} \frac{\beta_2}{2E} U_0^{(0)} \frac{\partial}{\partial \tau} \left( U_0^{(0)} \frac{\partial}{\partial \xi^2} U_0^{(0)} - \frac{q_2}{2} U_0^{(0)} \frac{\partial}{\partial \xi^3} U_0^{(0)} - \frac{q_3}{2} U_0^{(0)} \frac{\partial}{\partial \xi^4} U_0^{(0)} - \frac{q_4}{2} U_0^{(0)} \frac{\partial}{\partial \xi^5} U_0^{(0)} \right) \right) + O(\varepsilon^2) = 0,
\]

where the coefficients \( q_1, q_2, q_3, q_4 \) are given in (28). The \( \tau \)-derivatives in the \( O(\varepsilon) \) terms in (31) can be eliminated using the asymptotic relation \( U_0^{(0)} = \frac{\beta_1}{2E} U_0^{(0)} + \frac{q_1}{2} U_0^{(0)} + O(\varepsilon) \). Then, introducing the new function \( u = U_0^{(0)} + \varepsilon U_0^{(1)} \) we obtain

\[
u_\tau - \frac{\beta_1}{2E} (u^2)_\xi - \frac{q_1}{2} u_{\xi\xi\xi} - \varepsilon \left[ \frac{3\beta_2 + \beta_1^2}{6E^2} (u^\alpha)_\xi + \frac{2q_2 + 3(q_1 - 4(\alpha_1 + \alpha_2))\beta_1}{4E} u_{\xi u_{\xi\xi}} \right. \\
+ \frac{q_3 + (q_1 - 2(\alpha_1 + \alpha_2))\beta_1}{2E} u_{\xi u_{\xi\xi\xi}} + \left( \frac{q_4}{2} - \frac{q_1(2\alpha_1 + \alpha_2)}{2} + \frac{q_2^2}{8} \right) u_{\xi u_{\xi\xi\xi\xi}} \right] + O(\varepsilon^2) = 0.
\]

The dimensional form of the equation (32) is given by

\[
\frac{1}{c} u_t - \frac{\beta_1}{2E} (u^2)_x - \frac{q_1 R^2}{2} u_{\xi\xi\xi} - \frac{3\beta_2 + \beta_1^2}{6E^2} (u^\alpha)_x = \left[ \frac{2q_2 + 3(q_1 - 4(\alpha_1 + \alpha_2))\beta_1 R^2}{4E} u_{\xi u_{\xi\xi}} \right. \\
- \left. \frac{q_3 + (q_1 - 2(\alpha_1 + \alpha_2))\beta_1 R^2}{2E} u_{\xi u_{\xi\xi\xi}} - \left( \frac{q_4}{2} - \frac{q_1(2\alpha_1 + \alpha_2)}{2} + \frac{q_2^2}{8} \right) R^4 u_{\xi u_{\xi\xi\xi\xi}} \right] = 0,
\]

where the dimensional \( \xi = x - ct \) and \( \varepsilon^2 = E/\rho \). Note that since \( U \) is a longitudinal displacement the function \( u \) can be treated as a longitudinal strain. The equation (32) has been derived and studied mainly in the context of waves in fluids (see [38, 39, 40, 41, 42, 43, 44, 45, 46] and references therein), and is often referred to as an extended Korteweg - de Vries (eKdV) equation. To the best of our knowledge, this is the first derivation of this equation in the context of waves in solids. Some other extended models have been obtained in [47, 48].

Studying solitary wave solutions of the derived equation directly is a complicated task, and therefore here we aim to reduce the eKdV equation (32) to the Gardner equation using direct and
inverse near-identity transformations of the form

\[
\dot{u} = u + \varepsilon (a_1 u_{\xi\xi} + a_2 u_{\xi\tau} + a_3 u_{\xi}) \int_{\xi_0}^{\xi} ud\xi, \quad (34)
\]

\[
u = \dot{u} - \varepsilon (a_1 \dot{u}_{\xi\xi} + a_2 \dot{u}_{\xi\tau} + a_3 \dot{u}_{\xi}) \int_{\xi_0}^{\xi} \dot{u}d\xi, \quad (35)
\]

up to \(O(\varepsilon^2)\) corrections. We note that the general near-identity transformations discussed in [30, 31, 32] contain also the \(\varepsilon a_4 u^2\) term which we do not use here since we wish to retain both the quadratic and cubic nonlinearities in the equation (32). It is well-known from the studies in fluids that the Gardner equation has a rich family of solitons reducing to KdV solitons in the case of small amplitude [43]. We would like to use these known solutions and to compare the related analytical solutions of our derived model with the results of direct numerical simulations of the original problem formulation. Various near-identity transformations have been used to study nonlinear waves in two- and three-layered fluids (e.g., [49, 50, 51, 52]). We note that in the context of solids there are 9 free parameters (constants characterising elastic properties of various materials), and generally there is more freedom in the choice of the coefficients of the equation (32) than in the known fluid contexts.

The appropriate choice of the coefficients allows us to eliminate all higher-order dispersive terms from (32):

\[
a_1 = \frac{1}{12} \left[ \frac{10q_1}{q_1} + q_1 + 2(2\alpha_1 + \alpha_2) + \frac{3(q_3 - q_2)}{\beta_1} \right],
\]

\[
a_2 = \frac{4q_4 - 4q_1(2\alpha_1 + \alpha_2) + q_1^2}{6q_1^2}, \quad a_3 = \frac{\beta_1 q_1(4\alpha_1 + 2\alpha_2 + q_1) + 3q_1 q_3 - 8\beta_1 q_4}{9Eq_1^2}.
\]

The resulting Gardner equation takes the form:

\[
\dot{u}_\tau - \frac{\beta_1}{2E} (\dot{u}^2)\xi - \frac{q_1}{2} \dot{u}_{\xi\xi\xi} - \varepsilon \frac{\beta_2}{2E^2} (\dot{u}^3)_{\xi} + O(\varepsilon^2) = 0, \quad (36)
\]

where \(\beta_2 = \beta_2 + (\beta_1^2 (1 - 2\alpha_2) - a_3 E \beta_1)/3\). The equation (36) has a family of solitary wave solutions parametrised by the amplitude parameter \(M\) (e.g. [43]):

\[
\dot{u}(\xi, t) = \frac{M}{1 + N \cosh K\theta}, \quad N = \sqrt{1 + \frac{3\varepsilon \beta_2 M}{2E}}, \quad K = \frac{\beta_1 M}{3Eq_1}, \quad v = -\frac{\beta_1 M}{6E}, \quad (37)
\]

where \(\theta = \xi - vt\). We use this solution to create an asymptotic solution of the originally derived extended KdV equation (32) with the accuracy up to \(O(\varepsilon^2)\) terms using the inverse near-identity transformation (35):

\[
u(\xi, t) = \frac{M}{1 + N \cosh K\theta} \left[ 1 - \frac{\varepsilon a_1 NK^2 [N(\cosh 2K\theta - 3) - 2 \cosh K\theta]}{2(1 + N \cosh K\theta)^2} \right.
\]

\[
+ \frac{\varepsilon NK \sinh K\theta}{1 + N \cosh K\theta} \left( -a_2 \xi v + \frac{2a_3 M \tanh \left( \sqrt{\frac{1-N}{1+N}} \tanh \frac{K\theta}{2} \right)}{\sqrt{1-N^2}} \right). \quad (38)
\]

We note that the term which contains \(\xi\) explicitly is not secular, because it is of the same order as \(\xi/\cosh K\xi\), which decays to 0 as \(\xi \to \infty\) (this term can be removed by a phase shift). The dimensional
form of the solutions (37) and (38), respectively, is as follows

\[
\hat{u}(\xi, t) = \frac{M}{1 + N \cosh K\theta}, \quad N = \sqrt{1 + \frac{3\beta_2 M}{2\beta_1 E}}, \quad K = \sqrt{\frac{\beta_1 M}{3E\eta_1 R^2}}, \quad v = -\frac{\beta_1 Mc}{E};
\]

\[
u(\xi, t) = \frac{M}{1 + N \cosh K\theta} \left[ 1 - \frac{a_1 N K^2 R^2}{2(1 + N \cosh K\theta)^2} \left[ N(cosh 2K\theta - 3) - 2 \cosh K\theta \right] \right] + \frac{NK \sinh K\theta}{1 + N \cosh K\theta} \left( -\frac{a_2 R^2 \xi v}{c} + \frac{2a_3 M \tanh \left( \sqrt{\frac{1-N}{1+N}} \tan K\theta \right)}{\sqrt{1-N^2}} \right) .
\]

It is now interesting to compare the performance of the simple Gardner soliton (37) and the formula (38) for the solution of the original extended KdV equation both within the range of its formal asymptotic validity (i.e. waves of small amplitude), as well as the case when cubic and quadratic nonlinear terms become comparable ($\varepsilon \beta_2 \sim \beta_1$, waves of moderate amplitude), and also to check whether the formula (38) is a better approximation to the numerical solution of the original problem than the simple Gardner soliton (37). We note that, strictly speaking, the case of waves of moderate amplitude is beyond the range of validity of the asymptotic expansion, however it is interesting to test whether the asymptotic formula can still be useful in one way or another. We note that weakly-nonlinear solutions have been compared with the results of direct numerical simulations of the original problem formulations in several settings relevant to the oceanic studies (see [52, 53, 54, 55, 56, 57] and references therein). To the best of our knowledge there were no comparisons for solids.

We consider two hyperelastic materials with the elastic moduli given in Table 1. Here for brevity we study only solitons of negative polarity (solitons of compression), hence we choose the moduli A, B and C so that the coefficient of quadratic nonlinearity is negative. The Material 1 has a negative coefficient of cubic nonlinearity (GE− case), therefore the corresponding family of small-amplitude solitons contains “table-top” solitons. This coefficient for the Material 2 is positive (GE+ case). We note that $q_1 = -\nu^2/2 < 0$, hence the dispersive coefficient in the Gardner equation (36) is always positive. Examples of solitons in both materials are given in Figure 2.

| Young's m. | Poisson’s ratio, $\nu$ | Landau moduli, GPa | Density $\rho$, kg/m$^3$ | Coefficients |
|-----------|----------------------|----------------------|------------------|--------------|
| $E$, GPa  | $A$ | $B$ | $C$ | $D$ | $F$ | $G$ | $H$ | $\beta_1/E$ | $\beta_2/E^2$ |
| Material 1 | 5 | 0.34 | -5.85 | -2.93 | 1000 | 1 | 13.3 |
| Material 2 | 5 | 0.34 | -5.85 | 14.18 | 1000 | 1 | -13.3 |

Figure 2: Comparison of the Gardner solitons (39) and the eKdV asymptotic solutions (40) in dimensional variables for two materials: (a) Material 1, $M = -0.04$ (small amplitude) and $M = -0.049982$ (moderate amplitude); (b) Material 2, $M = -0.05$ (small amplitude) and $M = -0.2$ (moderate amplitude).
5 Numerical simulations

In order to compare the derived asymptotic solutions with the results of direct numerical simulations of the original problem formulation (5) – (8) we use a multidomain pseudospectral method [58]. A set of Legendre polynomials in both $x$ and $r$ variables are used for the spatial discretization of the problem: $U(x, r, t) = \sum_{n,m} \hat{U}_{nm}(t) \Phi_n(x) \Psi_m(r)$, where $U$ is the displacement vector. The multidomain method allows us to compute the solution relatively quickly on a fine mesh with 600-650 points in $x$, split into 20-25 domains, and 5 points in $r$ (the rod we consider is thin compared to the wavelength, hence we do not need a large number of points in the $r$ coordinate).

In Figures 3, 4, 5 the numerical results for the original problem formulation are obtained using the initial condition in the form of the eKdV asymptotic solution (38). The initial soliton and the KdV soliton are plotted for comparison. The data for the soliton’s velocity and shape is summarised in Figure 6. From these comparisons we can see that overall the solution (38) performs better than the solution of the KdV or Gardner equation, although all three solutions work very well for the case of waves of small amplitude, and the Gardner soliton is also a physically relevant approximation in the case of waves of moderate amplitude. Although the initial condition in the form of (38) has "horns" in the case of waves of moderate amplitude (Fig. 4), which seems to be physically irrelevant, this initial condition allowed us to very quickly generate and observe a "table-top" soliton of the original problem formulation. We noticed that in this case more energy of the initial wave was transferred to this "table-top" soliton and less energy was radiated away compared to the case of the Gardner initial soliton (37). From our experiments we can conclude that, at moderate amplitude, the constructed weakly-nonlinear solution can be used at least as a very good initial guess in order to generate moderately-nonlinear soliton solutions.

![Figure 3: Evolution of the initial soliton (38) in the Material 1 (GE– case). Nondimensional variables $\xi$, $\tau$. Amplitude of the initial nondimensional soliton equals 1; $\varepsilon = 0.005$ in the left plot, $\varepsilon = 0.027$ in the right plot (small amplitude).](image-url)
Figure 4: Evolution of the initial soliton (38) in the Material 1 (GE− case). Nondimensional variables $\xi$, $\tau$. Amplitude of the initial nondimensional soliton equals 1; $\varepsilon = 0.0499$ in the left plot, $\varepsilon = 0.0499825675$ in the right plot (moderate amplitude).

Figure 5: Evolution of the initial soliton (38) in Material 2 (GE+ case). Nondimensional variables $\xi$, $\tau$. Amplitude of the initial nondimensional soliton equals 1; $\varepsilon = 0.02$ in the left plot (small amplitude), $\varepsilon = 0.05$ in the right plot (moderate amplitude).
Figure 6: Relations between the soliton parameters. The left / right plots correspond to the GE− / GE+ case. All solitons are approximated by the function \( \frac{1}{M} + N \cosh K(x - vt) \). The dots on the solid curve indicate results obtained from a set of numerical simulations. The cubic spline interpolation was used to draw the numerical curves.
6 Conclusions

In this paper we derived the extended Boussinesq and Korteweg - de Vries equations describing long nonlinear longitudinal bulk strain waves in generic weakly-nonlinear hyperelastic materials with the accuracy up to an including the cubic terms in the equations. The extended Korteweg - de Vries equation was then reduced to the Gardner equation with the help of a near-identity transformation in order to make use of the known family of soliton solutions of this equation. The inverse near-identity transformation was used to obtain the solution of the derived extended Korteweg - de Vries equation. The solutions were compared with each other and with the results of direct numerical simulations of the original nonlinear problem formulation, showing very good agreement for the waves of small amplitude, but also reasonably extending their relevance to the waves of moderate amplitude. In particular, the weakly-nonlinear solution has allowed us to generate and observe a stably propagating moderately-nonlinear longitudinal “table-top” soliton.

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Appendix A. Coefficients

Here, the coefficients in the equation (25) are given in terms of the Young modulus $E$ and the Poisson ratio $\nu$ (instead of the Lame moduli $\lambda$ and $\mu$) and the third and fourth Landau moduli $A, B, C, D, E, F, G$:

\[
\alpha_1 = \alpha_3 = \frac{1 + \nu}{4}, \quad \alpha_2 = \frac{1 + \nu + \nu^2}{2},
\]

\[
\alpha_4 = \frac{(1 + \nu)^2}{48}, \quad \alpha_5 = -\frac{5 + 3\nu + 10\nu^3 - 4\nu^4 - 12\nu^5}{48(1 - \nu)},
\]

\[
\alpha_6 = \frac{14 + 5\nu + 16\nu^2 - 8\nu^3 - 24\nu^4}{96(1 - \nu)}, \quad \alpha_7 = \frac{6 + 13\nu + 14\nu^2 + 6\nu^3}{96(1 + \nu)},
\]

\[
\beta_1 = -\left(\frac{3}{2}E + A(1 - 2\nu^3) + 3B(1 + 2\nu^2)(1 - 2\nu) + C(1 - 2\nu^3)\right),
\]

\[
\beta_2 = 4(B + C)^2 - E\left(2A + 6B + 2C + 4(D + F + G + H) + \frac{E}{2}\right) + 4\nu(-5B^2 - 14BC - 9C^2 + E(3B + 3C + 2D + 4F + 8H)) + 4\nu^2(18B^2 + 44BC + 30C^2 + 2A(B + C) - E(3B + 6C + 6F + 4G + 24H)) + 4\nu^3(-32B^2 - 76BC - 40C^2 - 6AB - 10AC + E(A + 6B + 4C + 2D + 8F + 32H)) + 4\nu^4(A^2 + 28B^2 + 40BC + 12A(B + C) - 4E(D + 2F + G + 4H)) - 4\nu^5(A^2 + 4A(B - 2C) - 4(3B^2 + 20BC + 12C^2)) - 8\nu^6(A + 6B + 4C)^2
\]

\[
\bar{J}_0 = \frac{1}{E}\left[\gamma_1(U_0x_0)_{tt} + \gamma_2(U_0tt)_{xx} + \gamma_3(U_0x_0U_0tt)_{xx} + \gamma_4(U_02x)_{xtt} + \gamma_5(U_0x_0U_0xtt)_{x} + \gamma_6(U_02x)_{xxx} + \gamma_7(U_02xx)_{x} \right]
\]
\[ \gamma_1 = \frac{1 + \nu}{4} \left[ 2E + A(1 - \nu^2) + 2B(1 + \nu)(1 - 2\nu) \right], \quad \gamma_2 = \frac{\gamma_1}{2}, \quad \gamma_3 = -4\gamma_1, \]

\[ \gamma_4 = -\frac{1 + \nu}{8} \left[ E + A + 2B + 2\nu(B + 2C) - \nu^2(A + 20B + 24C - 2E) + 4\nu^3(A + 10B + 12C) 
- 8\nu^4(A + 6B + 4C) \right], \]

\[ \gamma_5 = -\frac{1 + \nu}{4} \left[ 5E + 3A + 10B + 4C - 2\nu(9B + 10C) + \nu^2(A + 12B + 24C + 2E) 
- 4\nu^3(A + 2B - 4C) - 8\nu^4(A + 6B + 4C) \right], \]

\[ \gamma_6 = -\frac{1}{8} \left[ 4E + 3A + 10B + 4C + \nu(3A - 12B - 20C + 4E) + \nu^2(A + 6B + 24C + 2E) 
- \nu^3(7A + 20B - 16C) - 8\nu^4(A + 6B + 4C) \right], \]

\[ \gamma_7 = -\frac{1}{8} \left[ 8E + 5A + 18B + 8C + \nu(A - 36B - 44C + 2E) + \nu^2(3A + 42B + 72C + 2E) 
- \nu^3(13A + 60B + 16C) - 8\nu^4(A + 6B + 4C) \right], \]

The relations between the Landau and Murnaghan moduli can be found in (3).

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