SOME UNCONDITIONAL REGULARITY RESULTS FOR THE ISENTROPIC EULER EQUATIONS WITH $\gamma = 3$

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Abstract. In this paper, we study the regularity properties of bounded entropy solutions to the isentropic Euler equations with $\gamma = 3$. First, we use a blow-up technique to obtain a new trace theorem for all such solutions. Second, we use a modified De Giorgi type iteration on the kinetic formulation to show a new partial regularity result on the Riemann invariants. We are able to conclude that in fact for any bounded entropy solution $u$, the density $\rho$ is almost everywhere upper semicontinuous away from vacuum. To our knowledge, this is the first example of a nonlinear hyperbolic system where generic $L^\infty$ initial data give rise to bounded entropy solutions with a form of classical regularity. This provides one example that $2 \times 2$ hyperbolic systems can possess some of the more striking regularizing effects known to hold generically in the genuinely nonlinear, multidimensional scalar setting. While we are not able to use our regularity results to show unconditional uniqueness, the results substantially lower the likelihood that current methods of convex integration can be used in this setting.

1. Introduction

In this paper, we study the properties of bounded entropy solutions $u = (\rho, m)$ to the isentropic Euler equations with adiabatic index $\gamma = 3$:

\begin{align*}
\begin{cases}
\rho_t + m_x = 0 \\
m_t + \left( \frac{m^2}{\rho} + \frac{\rho^3}{3} \right)_x = 0,
\end{cases}
\end{align*}

where $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ are time and space, respectively, and $(\rho, m)$ are the unknown mass and momentum densities. We consider solutions $(\rho, m)$ which satisfy (1.1) in the sense of distributions and are entropic for all entropy, entropy-flux pairs. More precisely, we require that for any pair $(\eta, q)$, such that $\eta \in C(\mathbb{R}^+ \times \mathbb{R})$ is convex, $q \in C(\mathbb{R}^+ \times \mathbb{R})$, and $q' = \eta' f'$, where $f(\rho, m) = (m^2/\rho, \rho^3/3)^t$ is the flux associated to (1.1), $u$ satisfies the entropy inequality,

\begin{align*}
\eta(u)_t + q(u)_x \leq 0,
\end{align*}

in the sense of distributions. Throughout, we will work with solutions which are globally bounded, in the sense that $\rho, \frac{m}{\rho} \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ and $\rho, m \in L^\infty(\mathbb{R}^+; L^1)$, so that both the density and velocity are bounded.

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and \( u \) has finite energy. We recall that the global-in-time existence of such solutions to (1.1) for general initial datum \( u_0 \in L^1 \cap L^\infty \) was established by Tartar using the method of compensated compactness [39]. We study two closely related regularity properties for such bounded solutions to (1.1), namely, trace properties and local BV-like structure.

1.1. Trace Properties. In the case of scalar conservation laws with a convex flux, it is known the Kružkov entropy solution regularizes from \( L^\infty \) into \( BV \) [27, 33]. However, for multidimensional scalar conservation laws, it is well known that solutions may not be \( BV \), but still display striking regularization properties (see [31, 34, 28, 41, 9, 10, 14, 36] and references therein). One of the earliest examples of this phenomenon is the fractional Sobolev regularization implied by the kinetic formulation introduced by [31]. Thereafter, the kinetic formulation was used by [41] to show a one-sided trace property for solutions, which was later strengthened in [34, 28]. In particular, this implies that solutions have well-defined boundary values along lower dimensional hyper-surfaces, and these boundary values are obtained in a strong topology.

On the other hand, for \( 2 \times 2 \) systems, not only is regularization unknown for a generic system, but we lack even specific examples of systems known to possess regularizing effects, excepting the fractional Sobolev regularization for (1.1) implied by the kinetic formulation introduced in [32]. We are aware of exactly one trace theorem: in [40], Vasseur shows any bounded solution \( u \) to (1.1) satisfies \( u \in C(0,T;L^1_{loc}) \), which we interpret as a two-sided trace property in time. With the increasingly evident role spatial traces play in the uniqueness and stability theory (see the discussion of uniform traces below), it is rather unsatisfactory that there is not currently an analogue of the more general traces considered in [41] for the system (1.1). The first main goal of this paper is to rectify this gap in the literature.

In order to state our result, we introduce the following particular notion of a one-sided trace, used frequently below.

**Definition 1** (Strong Trace Property). We say a function \( u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^n \), \( u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}) \) is strong trace class if for any \( h : [0,T] \to \mathbb{R} \) Lipschitz, there are \( u^+, u^- : [0,T] \to \mathbb{R}^n \) such that \( u^+, u^- \in L^\infty([0,T]) \) and

\[
\lim_{k \to \infty} \text{ess sup}_{y \in (0,1/k)} \int_0^T |u(t,h(t)+y) - u^+(t)| + |u(t,h(t)-y) - u^-(t)| \, dt = 0.
\]

With this definition in hand, we are ready to state our first main result:

**Theorem 1.1.** Suppose \( u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^2 \) is an entropy solution to (1.1) such that \( u = (\rho, m) \) with \( \|\rho\|_{L^\infty} + \|m/\rho\|_{L^\infty} + \|u\|_{L^\infty L^1} < \infty \). Then, \( u \) has strong traces in the sense of Definition 1.

**Remark 1.** Since strong traces are obtained in a strongly in an \( L^1 \) sense, nonlinear functions of a solution \( u \) to (1.1) also have strong traces. For example, we have enough regularity to make sense of the characteristic fields \( \lambda_i(u^\pm) \) as well the Rankine-Hugoniot condition.

**Remark 2.** Of course, as \( u, u^+, u^- \) are all bounded, one may interpolate the strong trace as in Definition 1 with the \( L^\infty \) bound and put a power \( p \) inside the trace condition for any \( 1 \leq p < \infty \).

Combining Theorem 1.1 with a result of [30], we obtain the following corollary:

**Corollary 1.** Suppose \( u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^2 \) is an entropy solution to (1.1) such that \( u = (\rho, m) \) with \( \|\rho\|_{L^\infty} + \|m/\rho\|_{L^\infty} + \|u\|_{L^\infty L^1} < \infty \). Then, for any \( h : [0,T] \to \mathbb{R} \) a Lipschitz curve, for almost every \( t \in [0,T] \), either \( u^+(t) = u^-(t) \) or \( (u^-(t), u^+(t)) \) is an admissible shock with speed \( \dot{h}(t) \).

This corollary is interesting because it translates some a priori regularity into a tool that can be used unconditionally in the study of solutions to (1.1). The proof of the corollary follows immediately from [30, Lemma 6], which for the reader’s convenience is proved in the appendix.

Let us introduce one more trace property, which we refer to as the uniform trace property to distinguish it from strong traces. The following property is never used below, but is introduced formally to provide motivation for our later results.
Definition 2 (Uniform Trace Property). We say a function $u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^n$, $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ is uniform trace class if for any $h : [0, T] \to \mathbb{R}^n$ Lipschitz, there are $u^+, u^- : [0, T] \to \mathbb{R}^n$ such that $u^+, u^- \in L^\infty([0, T])$ and

$$\lim_{k \to \infty} \text{ess sup}_{y \in (0, 1/k)} \int_0^T |u(t, h(t) + y) - u^+(t)| + |u(t, h(t) - y) - u^-(t)| \ dt = 0. \tag{1.4}$$

A recent theorem found as [8, Theorem 1.3] and proved in [8, 18] employs the uniform trace property for a general class of fluxes for $2 \times 2$ systems that includes the isentropic Euler equations for any adiabatic index $\gamma$. The result states for any small $BV$ initial datum $u_0$, if $u$ and $v$ are bounded entropy solutions with initial datum $u_0$ and $u$ and $v$ satisfy the uniform trace property in the sense of Definition 2, then $u = v$. This theorem improves several preceding uniqueness results ([6, 5, 4]), yet still only provides conditional uniqueness for small $BV$ initial datum, because we do not know whether the uniform trace property holds for general $L^\infty$ solutions. This leads to two natural questions which motivate the remainder of our results: First, does special structure of \((1.1)\) let us prove uniform traces for entropy solutions? Second, can we make the uniqueness theorem work using only strong traces? We are only able to provide partial answers to both of these questions.

1.2. Fine Structure of Solutions. We seek first to improve the regularity results of Theorem 1.1 and bridge the gap between strong and uniform traces in the sense of Definitions 1 and 2. In the context of multidimensional scalar equations, where solutions need not regularize to $BV$, there have been several attempts to improve the trace conditions by showing $BV$-like properties for solutions. For instance, in [14] the authors characterize blow-up limits and prove for a given entropy solution, there is an $H^{n-1}$-rectifiable jump set $\mathcal{J}$ outside of which the solution is $VMO$. Further improvements in this direction have been made (see for example [2, 3] and references therein), with a particularly notable recent contribution by Silvestre. In [36], Silvestre shows that outside of $\mathcal{J}$, the solution is continuous in the sense that each point $x \notin \mathcal{J}$ is a Lebesgue point at which all blow-ups converge in $L^\infty_{loc}$.

Our main result in this direction mimics the multidimensional scalar case and studies some fine structure of entropy solutions to \((1.1)\). This necessarily involves the study of solutions at single points and thus we fix appropriate versions of the functions we study. Let us first recall that for the $\gamma = 3$ isentropic Euler equations as written in \((1.1)\), the Riemann invariants are the characteristic fields, and are given by

$$\lambda_1(\rho, m) = \frac{m}{\rho} - \frac{\rho}{2} \quad \text{and} \quad \lambda_2(\rho, m) = \frac{m}{\rho} + \frac{\rho}{2}. \tag{1.5}$$

Next, for $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, we define the following upper (resp. lower) semicontinuous envelopes of $g$ via

$$\overline{g}(t, x) := \lim_{r \to 0^+} \text{ess sup}_{(\tau, y) \in B_r(t, x)} g(\tau, y) \quad \text{and} \quad \underline{g}(t, x) := \lim_{(\tau, y) \in B_r(t, x)} \text{ess inf} g(\tau, y). \tag{1.6}$$

Note that for $g \in L^\infty_{loc}$, $\overline{g}$ is always upper semicontinuous while $g$ is always lower semicontinuous. For completeness, we also fix a specific version of each $g$, namely, let us pick

$$\hat{g}(t, x) := \limsup_{r \to 0^+} \frac{1}{|B_r(t, x)|} \int_{B_r(t, x)} g(\tau, y) \ d\tau dy. \tag{1.7}$$

Finally, let us recall that a point $(t, x)$ is a point of $VMO$ for a function $g$ if

$$\lim_{r \to 0^+} \frac{1}{|B_r(t, x)|} \int_{B_r(t, x)} g(\tau, y) - \hat{g}(t, x) \ d\tau dy = 0. \tag{1.8}$$

Note that points of $VMO$ are slightly more general than Lebesgue points, where one further requires that the averages $\hat{f}_{B_r(t, x)} g$ converge as $r \to 0^+$. We are now ready to state our second main result:

**Theorem 1.2.** Suppose $u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^2$ is an entropy solution to \((1.1)\) such that $u = (\rho, m)$ with $\|\rho\|_{L^\infty} + \|m/\rho\|_{L^\infty} < \infty$. Then, for any $(t, x)$ a point of $VMO$ for $\lambda_1(u)$ and $\lambda_2(u)$ with $\rho \neq 0$, $(t, x)$ is a Lebesgue point of $\rho$, $m$, $\lambda_1(u)$, and $\lambda_2(u)$. Moreover,

$$\hat{\rho}(t, x) = \overline{\rho}(t, x), \quad \hat{\lambda}_1(t, x) = \overline{\lambda}_1(t, x), \quad \hat{\lambda}_2(t, x) = \overline{\lambda}_1(t, x), \tag{1.9}$$
and

\begin{equation}
\dot{m}(t,x) = \begin{cases} 
m(t,x) & \text{if } (t,x) \in \{\Lambda_1 \geq 0\} 
\frac{m(t,x)}{\rho(t,x)} & \text{if } (t,x) \in \{\Lambda_2 \leq 0\}.
\end{cases}
\end{equation}

In particular, each of \(\rho, -\lambda_1(u),\) and \(\lambda_2(u)\) are upper semicontinuous almost everywhere on the open set \(\{\rho \neq 0\}\) and \(m\) is almost everywhere semicontinuous on the set \(\{\rho \neq 0\text{ and either }\Lambda_1 \geq 0\text{ or }\Lambda_2 \leq 0\}\).

**Remark 3.** Fundamentally, Theorem 1.2 is a rigorous formulation of the claim that for a non-vacuum solution \(u\), the density, \(\rho\), and the Riemann invariants, \(\lambda_i(u)\) for \(i = 1, 2\), are each semicontinuous almost everywhere. Moreover, for a large number of solutions we can even regain semicontinuity of the momentum, \(m\). Note that we obtain a form of classical regularity for the conserved quantities. To our knowledge, this is the only example of a system where we can show the production of any classical regularity similar to what can be shown in general for nonlinear multi-dimensional scalar conservation laws. In particular, it is unclear whether this property is generic for \(2 \times 2\) systems, specific to the \(\gamma = 3\) Euler system, or specific to systems with sufficiently many entropies.

**Remark 4.** Note, Theorem 1.2 only works far from vacuum. This stands in stark contrast to the scalar case in [36], where there is no problem with vacuum. Our problem with vacuum seems to stem from the limitations of the kinetic formulation of (1.1) which is central to the proof of both Theorem 1.1 and 1.2. The kinetic formulation loses meaning at vacuum states, since it is derived from entropies that vanish at vacuum. Therefore, for example, the kinetic formulation has trouble differentiating two vacuum states where nearby, the velocity \(u\) is drastically different. To circumvent this problem, one might attempt to use the relative entropy method developed in [12, 15] which often is able to overcome the difficulties posed by vacuum. However, merging the two methods seems difficult given the low level of regularity present.

**Remark 5.** In Theorem 1.2, we have no explicit description of the set of \(VMO\) points of an entropy solution \(u\). Instead, we rely upon Lebesgue’s Differentiation Theorem say that almost every \((t, x)\) is a Lebesgue point. In [14], the jump set \(J\) is explicitly defined in terms of the entropy dissipation measure and it is shown that each \((t, x) \notin J\) is a point of \(VMO\), using a Liouville theorem ([14, Proposition 6]) for blow-ups at such points. We expect one might be able to show a similar result here, and even prove the analogous statement that \(J\) is codimension 1 rectifiable. However, even the Liouville theorem appears to be much more subtle than that for the multidimensional scalar case and these problems are left for future work.

### 1.3. Generalized Characteristics

Our final result concerns the existence of generalized characteristics, i.e., solutions to the differential equation \(\dot{h}(t) = \lambda_i(u(t, h(t)))\), where due to a lack of regularity on the right hand side, solutions are understood in the sense of Filippov. In general, we are able to construct sub or super solutions to the generalized characteristic equation, using the regularity gained during the proof of Theorem 1.2.

**Proposition 1.1.** Suppose \(u\) is an entropy solution to (1.1) such that \(u = (\rho, m)\) with \(\|\rho\|_{L^\infty} + \|m/\rho\|_{L^\infty} + \|u\|_{L^\infty \cap L^2} < \infty\) satisfying \(\text{ess inf}_{t,x} \rho(t,x) > 0\). Then, for any \(x_0 \in \mathbb{R}\), and \(i = 1, 2\), there are \(h_i : \mathbb{R}^+ \to \mathbb{R}\) Lipschitz, such that \(h_i(0) = x_0\) and for almost every \(t \in \mathbb{R}^+\),

\begin{equation}
\left\{ \begin{array}{ll}
\lambda_i(u^+) \leq \dot{h}_i(t) \leq \text{ess sup}_{w \in \text{range}(u)} \lambda_i(w) & \text{if } i = 1 \\
\text{ess inf}_{w \in \text{range}(u)} \lambda_i(w) \leq \dot{h}_i(t) \leq \lambda_i(u^+) & \text{if } i = 2.
\end{array} \right.
\end{equation}

Moreover, for almost every \(t\), either \(u^+(t) = u^-(t)\) or \((u^+(t), u^-(t))\) is an entropic shock with speed \(\dot{h}_i(t)\).

**Remark 6.** Generalized characteristics were used by DiPerna in [15] to show a weak/strong uniqueness result for a single shock. In particular, because DiPerna works with \(BV_{loc}\) solutions, generalized characteristics exist for free from the theory of Filippov (contained in [17]). This allows DiPerna to compare the fixed shock wave and weak solution using the relative entropy method, which is \(L^2\)-based. More recently, the stability and uniqueness results of the \(\alpha\)-contraction theory (see [29, 30, 35, 24, 25, 26]) culminating in the result of [18, 8] described above all rely essentially on the ability to solve ODEs where the right hand side is a nonlinear function of a weak solution \(u\). In general, the least restrictive assumptions on the class of weak solutions necessary for stability seems to be exactly uniform traces in the sense of Definition 2. Proposition 1.1 should be seen in this context as an attempt to bypass uniform traces and instead use Theorem 1.1 and
Theorem 1.2 to gain unconditional stability among the class of $L^\infty$ entropy solutions to (1.1). However, the solutions constructed in the sense of (1.11) seem to be too weak to obtain a significant uniqueness result.

**Remark 7.** While we are not able to use Proposition 1.1 to obtain uniqueness of small $BV$ solutions to (1.1), Theorems 1.1 and 1.2 still nearly rule out the possibility of convex integration of (1.1). In the $L^\infty$ framework described here, considering a 1d shock as 2d planar shock and convex integrating in 2d typically yields solutions which fail to have any strong trace property, let alone are semicontinuous (see [11] for more details). We do not rule out the possibility of convex integration in 1d for (1.1), but our results seem that either convex integration of entropy solutions to (1.1) is likely impossible.

1.4. **Proof Overview.** In Section 2, we begin by introducing the kinetic formulation of [32] and the additional tools we gain from it in the form of averaging lemmas, which originate in the work of [20] with roots in the earlier works [21, 1].

In Section 3, we use the kinetic formulation to prove Theorem 1.1. The proof employs the same particular structure of the $\gamma = 3$ isentropic Euler equation and similar techniques as the proof of Vasseur’s result. However, there are two new challenges in the proof. Vasseur’s theorem deals with two-sided traces along flat space-like curves. By contrast, Theorem 1.1 with one-sided traces along Lipschitz time-like curves. As (1.1) and its kinetic formulation are not symmetric in $t$ and $x$, one cannot simply interchange their roles. In addition, some care has to be taken when straightening the time-like curves to ensure one has sufficient regularity to perform all the desired arguments and that additional terms associated to the geometry of the curve do not affect the proof.

In Section 4, we use the kinetic formulation to prove a new partial regularity result for the Riemann invariants $\lambda_1(u)$ and $\lambda_2(u)$:

**Proposition 1.2.** Suppose $u = (\rho, m)$ is an entropy solution to (1.1) with $\|\rho\|_{L^\infty(B_2)} + \|m/\rho\|_{L^\infty(B_2)} \leq \Gamma$, $\|u\|_{L^\infty(B_2)} < \infty$, and $\text{ess inf}_{(t,x) \in B_2} \rho(t, x) \geq M > 0$. Suppose $\overline{\pi} = (\overline{\rho}, \overline{\pi})$ is a fixed non-vacuum state. Then, there is an $\varepsilon_0 = \varepsilon_0(\Gamma, M) > 0$, $\alpha \in (0, 1)$, and $\tilde{C} = \tilde{C}(\Gamma, M)$ such that for any $0 < \varepsilon < \varepsilon_0$,

\[
\int_{B_2} (\lambda_1(\overline{\pi}) - \lambda_1(u))_+ \, dxdt < \varepsilon \quad \text{implies} \quad \text{ess sup}_{(t,x) \in B_1} (\lambda_1(\overline{\pi}) - \lambda_1(u(t, x)))_+ < \tilde{C} \varepsilon^\alpha
\]

\[
\int_{B_2} (\lambda_2(u) - \lambda_2(\overline{\pi}))_+ \, dxdt < \varepsilon \quad \text{implies} \quad \text{ess sup}_{(t,x) \in B_1} (\lambda_2(u(t, x)) - \lambda_2(\overline{\pi}))_+ < \tilde{C} \varepsilon^\alpha.
\]

This result implies that if the $L^1$-oscillation below (resp. above) a fixed threshold is sufficiently small in a ball, then the $L^\infty$-oscillation below (resp. above) the threshold is controlled quantitatively on a smaller ball. The above result is the main new ingredient in the proof of Theorem 1.2 and involves a new application of the method of De Giorgi to conservation laws. The method of De Giorgi was introduced in [13] for the study of elliptic and parabolic equations where a gain of integrability from a Cacciopoli-type inequality is exploited. However, recently the method has seen novel applications in the areas of kinetic theory [23, 22, 19] and elsewhere (for example, see [37, 7] and the references therein), and most relevantly for us, in the work of Silvestre on scalar conservation laws [36]. Unlike in the scalar case, we are unable to work directly on the entropy solution $u$, but instead are forced to work on the Riemann invariants. Even then, we are still only able to obtain one-sided results. Yet, because of the algebraic identities,

\[
\rho = \lambda_2(u) - \lambda_1(u) \quad \text{and} \quad m = \frac{[\lambda_2(u)]^2 - [\lambda_1(u)]^2}{2},
\]

we are still able to leverage the one-sided results to gain information about the original conserved quantities. Throughout Section 4, we attempt to highlight when and why these differences with the scalar case arise.

In Section 5, we use our partial regularity result, Proposition 1.2, to prove Theorem 1.2. In addition, we construct sub-solutions (resp. super-solutions) to the generalized characteristic equation to prove Proposition 1.1.

2. **Preliminaries**

2.1. **Kinetic Formulation and Averaging Lemmas.** Here we begin by recalling that the kinetic formulation of scalar conservation laws introduced in [31] has an analogue for the isentropic Euler system introduced
in [32]. For \( \gamma = 3 \), the associated kinetic formulation is purely kinetic and has the particularly simple form:

\[
\begin{align*}
\partial_t f + v \partial_x f &= -\partial_v \rho, \\
 f(t, x, v) &= \chi_{[a(t, x), b(t, x)]}(v) \quad \text{for almost every } (t, x), \\
\text{supp}(f(t, x)) &\subseteq [-L, L],
\end{align*}
\]

(2.1)

where \( f(t, x, v) \) is our unknown, which has been augmented with the velocity variable \( v \in \mathbb{R} \), and \( \rho \) is a finite (non-negative) measure. From the kinetic equation, we recover (1.1) via the relations

\[
\rho(t, x) = \int_{-L}^{L} f(t, x, v) \, dv \quad \text{and} \quad m(t, x) = \int_{-L}^{L} v f(t, x, v) \, dv,
\]

(2.2)

which imply, in fact, \( a(t, x) = \lambda_1(t, x) \) and \( b(t, x) = \lambda_2(t, x) \). In this form, the following is known:

**Theorem 2.1** (From [32]). Let \((\rho, m)\) be a fixed entropy solution to (1.1) satisfying \( \rho, \frac{m}{\rho} \in L^\infty \) and \( \|
u\|_{L^\infty} < \infty \). Then, there is a unique pair \((f, \mu)\) so that \( f \in L^1 \cap L^\infty \), \( \mu \) is a finite measure, \( f \) and \( \mu \) solve (2.1), and \( f \) is related to \((\rho, m)\) via (2.2).

Next, we have the following averaging lemma, from which we will gain compactness of sequences of solutions to (2.1) in Section 3:

**Lemma 2.1** ([31, Theorem B]). Let \( 1 < p \leq 2 \) and say \( f_k \) is a sequence of distributional solutions to the equation

\[
\partial_t f_k + a(v) \partial_x f_k = (1 - \Delta_{x,t})^{1/2} (1 - \partial_v) r/2 g_k,
\]

(2.3)

for some function \( a \in C^\infty(\mathbb{R}) \), some \( r > 0 \), and some sequence \( g_k \). If \( \{g_k\} \) is compact in \( L^p([0, T] \times \mathbb{R} \times \mathbb{R}) \), \( \{f_k\} \) is bounded in \( L^p_{\text{loc}}([0, T] \times \mathbb{R} \times \mathbb{R}) \), and a satisfies

\[
\{(t, x)| t^2 + x^2 a(v) = 0 \},
\]

(2.4)

then for any \( \psi \in L^q(\mathbb{R}) \) for \( q = \frac{p}{p-1} \), \( \int_{\mathbb{R}} f_k(\cdot, \cdot, v) \psi(v) \, dv \) is compact in \( L^p_{\text{loc}}([0, T] \times \mathbb{R}) \).

Finally, we will also need quantitative averaging lemmas, which will be used in Section 4 to gain integrability. For a comprehensive study of averaging lemmas, we refer to [16], however, the specific one we need is:

**Lemma 2.2** ([38, Averaging Lemma 2.1]). Suppose \( f \in L^2 \) is a solution to

\[
\partial_t f + v \partial_x f = \partial_v \mu_2 + \partial_x \mu_1 + \mu_0 + g,
\]

(2.5)

where \( \mu_0, \mu_1, \mu_2 \in \mathcal{M} \), \( g \in L^1 \). Then, for \( \theta \in (0, 1/7), r = \frac{7}{\theta}, \) and for any \( \psi \in C^\infty_c([-2, 2]) \) a bump function,

\[
\left\| \int f(t, x, v) \psi(v) \, dv \right\|_{W^{\theta,r}_{\text{loc}}} \lesssim \|f\|_{L^2}^{6/7} (\|g\|_{L^1} + \|\mu_0\|_{TV} + \|\mu_1\|_{TV} + \|\mu_2\|_{TV})^{1/7}.
\]

(2.6)

In this section, we prove the following proposition, which immediately implies Theorem 1.1.

**Proposition 3.1.** For each \( h : [0, T] \rightarrow \mathbb{R} \) Lipschitz and \( f \in L^1 \cap L^\infty \) a solution to (2.1), there are functions \( f^+(t, v) \) and \( f^-(t, v) \) such that \( f^+(t, v) = \chi_{[a^+(t), b^+(t)]}(v) \) and \( f^-(t, v) = \chi_{[a^-(t), b^-(t)]}(v) \) and

\[
\lim_{n \to \infty} \text{ess sup}_{0 < y < \frac{1}{n}} \int_{[0,T] \times \mathbb{R}} |f(t, h(t) + y, v) - f^+(t, v)| \, dv \, dt = 0
\]

(3.1)

and

\[
\lim_{n \to \infty} \text{ess sup}_{0 < y < \frac{1}{n}} \int_{[0,T] \times \mathbb{R}} |f(t, h(t) - y, v) - f^-(t, v)| \, dv \, dt = 0.
\]

The proof is structured as in [40]. We prove that solutions to (2.1) have traces in a weak topology, in particular, we obtain our \( f^+, f^- \). Next, we blow-up around a fixed \((t, h(t))\) and use Lemma 2.1 to conclude strong compactness of the blow-up family. Finally, we identify the strong limit as the weak traces \( f^\pm \), which implies that the limit is in the desired topology.

**Step 1: Weak traces**
We begin by showing $f$ has weak traces along a fixed Lipschitz curve $h : [0, T] \to \mathbb{R}$. The main goal of this step is to prove that there exist bounded functions $f^+, f^- : [0, T] \times \mathbb{R} \to [0, 1]$, such that $f$ obtains $f^+$ and $f^-$ in the sense of distributions as $h$ is approached from the right and left, respectively. Results on weak traces in this setting date back to [9]. However, it is unclear whether results in their regular deformable boundary framework directly imply the traces we want. Instead, we have the following proposition:

**Proposition 3.2.** Say $f \in L^1 \cap L^\infty$ is a solution to (2.1). Then, there exists a full measure set, $\Omega \subset \mathbb{R}$, and functions, $f^+, f^- \in L^1 \cap L^\infty([0, T] \times \mathbb{R})$, with $0 \leq f^- \leq 1$, such that for any test function $\Phi \in C^\infty_c([0, T] \times \mathbb{R})$,

\[
\lim_{y \to 0^+, y \in \Omega} \int f(t, h(t) + y, v) \Phi(t, v) \, dt dv = \int f^+(t, v) \Phi(t, v) \, dt dv \\
\lim_{y \to 0^-, y \in \Omega} \int f(t, h(t) + y, v) \Phi(t, v) \, dt dv = \int f^-(t, v) \Phi(t, v) \, dt dv.
\]

(3.2)

Let us begin with the following partial result, which says that the map $y \mapsto (v - h(t)) f(t, h(t) + y, v)$ has left and right limits, in the sense of distributions. More precisely, we have:

**Lemma 3.1.** Suppose $f \in L^1 \cap L^\infty$ is a solution to (2.1). Then, there is a set $\Omega \subset \mathbb{R}$ of full measure and functions, $g^\pm \in L^\infty([0, T] \times \mathbb{R})$ such that for any $\Phi \in L^1([0, T] \times \mathbb{R})$ with essentially compact support,

\[
\lim_{y \to 0^\pm, y \in \Omega} H_\Phi(z) = \lim_{y \to 0^\pm, y \in \Omega} \int [v - h(t)] f(t, h(t) + y, v) \Phi(t, v) \, dt dv = \int g^\pm(t, v) \Phi(t, v) \, dt dv.
\]

(3.3)

**Proof.** We prove only (3.3) for right limits. The proof for left limits is identical. First, fix $\Phi \in C^\infty_c([0, T] \times \mathbb{R})$ and $\psi \in C^\infty_c(\mathbb{R})$ with $\|\psi\|_{L^\infty} \leq 1$ and take $\Psi(t, x, v) = \Phi(t, v) \psi(x - h(t))$. Since $\Psi$ is compactly supported and Lipschitz in $t$ and smooth in $x$ and $v$, $\Phi$ is an admissible test function for (2.1). It follows that

\[
\int \partial_v \Phi(t, v) \psi(x - h(t)) f(t, x, v) \, dv dt dx + [v - h(t)] \Phi(t, v) \psi'(x - h(t)) f(t, x, v) \, dv dt dx = \int \partial_v \Phi(t, v) \psi(x - h(t)) \, d\mu(v, t, x).
\]

(3.4)

Changing variables so that $y = x - h(t)$,

\[
\int [v - h(t)] \Phi(t, v) \psi(y) f(t, h(t) + y, v) \, dt dv dy = -\int \partial_v \Phi(t, v) \psi(y) f(t, h(t) + y, v) \, dt dv dy + \int \partial_v \Phi(t, v) \psi(x - h(t)) \, d\mu(v, t, x).
\]

(3.5)

Defining $H_\Phi(y) := \int [v - h(t)] f(t, h(t) + y, v) \Phi(t, v) \, dt dv$, since $\|\psi\|_{L^\infty}, \|f\|_{L^\infty} \leq 1$ and $\mu$ is a finite measure, we obtain

\[
\int \psi(y) H_\Phi(y) \, dy \leq C(\mu, \Phi) \|\psi\|_{L^\infty} \leq C(\mu, \Phi).
\]

(3.6)

By a standard characterization of BV functions, this implies $H_\Phi(z)$ is of bounded variation and has left and right essential limits for any $y$ and any $\Phi \in C^\infty_c([0, T] \times \mathbb{R})$. Now, fix $D \subset C^\infty_c([0, T] \times \mathbb{R})$ a countable dense subset of $L^1([0, T] \times \mathbb{R})$. Since $D$ is countable, there is a full measure subset of $\Omega$, still denoted $\Omega$, such that

\[
\lim_{y \to 0^+, y \in \Omega} H_\Phi(y) \quad \text{converges for any } \Phi \in D.
\]

(3.7)

Second, we note the functions $\{ (t, v) \mapsto [v - h(t)] f(t, h(t) + y, v) \mid y \in \mathbb{R} \}$ are uniformly bounded in $L^\infty([0, T] \times \mathbb{R})$. Thus, there is a sequence $y_n \to y^+$ with $y_n \in \Omega$ and function $\tilde{g}^+ \in L^\infty([0, T] \times \mathbb{R})$ such that for every $\Phi \in L^1([0, T] \times \mathbb{R})$,

\[
\lim_{n \to \infty} H_\Phi(y_n) = g^+(t, v) \Phi(t, v) \, dt dv, \quad \text{for every } \Phi \in L^1([0, T] \times \mathbb{R}).
\]

(3.8)

Thus, (3.7) and (3.8) together imply

\[
\lim_{y \to 0^+, y \in \Omega} H_\Phi(y) = g^+(t, v) \Phi(t, v) \, dt dv, \quad \text{for every } \Phi \in D.
\]

(3.9)
Third, we will conclude (3.9) holds for any \( \Phi \in L^1([0, T] \times \mathbb{R}) \) with essentially compact support. Since \( f \) is uniformly bounded and \( h \) is Lipschitz, we have the uniform (in \( y \)) bound,

\[
|H_\Phi(y)| \leq \|f\|_{L^\infty([0, T] \times \mathbb{R})} \|v - \hat{h}\|_{L^\infty(supp(\Phi))} \|\Phi\|_{L^1}.
\]

Therefore, a density argument using (3.9), \( D \) is a dense subset of \( L^1([0, T] \times \mathbb{R}) \) is dense, and (3.10), the proof is complete.

We are now ready to prove Proposition 3.2, essentially by showing \( f(t, v) = \frac{g(t, v)}{v - \hat{h}(t)} \).

**Proof of Proposition 3.2.** Again, we prove only (3.2) for right limits. First, we fix \( \Omega \) from Lemma 3.1 and note that the functions \( \{y \mapsto f(t, h(t) + y, v) \mid y \in \mathbb{R}\} \) are uniformly bounded in \( L^\infty([0, T] \times \mathbb{R}) \). Thus, it follows that there is a sequence \( y_n \to 0^+ \) with \( y_n \in \Omega \) and a function \( (t, v) \mapsto f^+(t, v) \), for which \( f(t, h(t) + y_n, v) \overset{n}{\to} f^+(t, v) \) in \( L^\infty([0, T] \times \mathbb{R}) \). Second, for each \( \varepsilon > 0 \), define the family of domains

\[
A_\varepsilon := \left\{(t, v) \mid 0 \leq t \leq T, \hat{h}(t) \text{ exists, and } |\hat{h}(t) - v| \leq \varepsilon \right\}.
\]

By Rademacher’s theorem and Fubini’s theorem, \( A_\varepsilon \) is measurable with measure \( |A_\varepsilon| = 2\varepsilon T \). We note that if \( \Phi \in L^1([0, T] \times \mathbb{R}) \) has essentially compact support \( K \) contained in \([0, T] \times [-L, L] \) with \( K \cap \Omega = 0 \) for some \( \varepsilon > 0 \), applying Lemma 3.1 to \( \frac{\Phi(t, v)}{h(t) - v} \) and using \( f(t, h(t) + y_n, v) \overset{n}{\to} f^+(t, v) \) in \( L^\infty([0, T] \times \mathbb{R}) \),

\[
\lim_{y \to 0^+, y \in \Omega} \int f(t, h(t) + y, v) \Phi(t, v) \ dtdv = \int f^+(t, v) \Phi(t, v) \ dtdv.
\]

In other words, for any \( L > 0 \) and \( \varepsilon > 0 \), \( f(t, h(t) + y, v) \overset{n}{\to} f^+(t, v) \) in \( L^\infty([0, T] \times [-L, L] \setminus A_\varepsilon) \). Third, fix \( \Phi \in L^1([0, T] \times \mathbb{R}) \). Then, take a sequence \( \varepsilon_n \to 0^+ \) and define \( \Phi_n(t, v) = \chi_{[0, T] \times [-n, n]}(1 - \chi_{A_{\varepsilon_n}}(t, v)) \Phi(t, v) \). Then,

\[
\lim_{y \to 0^+, y \in \Omega} \int \Phi_n(t, v) f(t, h(t) + y, v) \ dtdv = \int \Phi_n(t, v) f^+(t, v) \ dtdv, \quad \text{for each } n \in \mathbb{N}.
\]

Moreover, since \( |A_{\varepsilon_n}| = 2\varepsilon_n T, \Phi_n \to \Phi \) in \( L^1([0, T] \times \mathbb{R}) \). Thus, for \( z_1, z_2 > 0 \),

\[
\left| \int \Phi(t, v) [f(t, h(t) + z_1, v) - f(t, h(t) + z_2)] \ dtdv \right| \\
\leq 2\|\Phi - \Phi_n\|_{L^1} + \int \Phi_n(t, v) [f(t, h(t) + z_1, v) - f(t, h(t) + z_2)] \ dtdv.
\]

Now, for any \( \eta > 0 \), pick \( n \) sufficiently large so that \( \|\Phi - \Phi_n\|_{L^1} < \eta/2 \). Then, since for a fixed \( n \), the second term is Cauchy, the left hand side is bounded by \( \eta \) provided that \( z_1, z_2 \) are sufficiently small. Combined with \( f(t, h(t) + y_n, v) \overset{n}{\to} f^+(t, v) \) in \( L^\infty([0, T] \times \mathbb{R}) \), we obtain

\[
\lim_{y \to 0^+, y \in \Omega} \int \Phi(t, v) f(t, h(t) + y, v) \ dtdv = \int \Phi(t, v) f^+(t, v) \ dtdv, \quad \text{for each } \Phi \in L^1([0, T] \times \mathbb{R})
\]

and we conclude \( f(t, h(t) + y, v) \overset{n}{\to} f^+(t, v) \) in \( L^\infty([0, T] \times \mathbb{R}) \).

In particular, this means that \( f(t, h(t) + y, v) \) obtains its trace functions \( f^\pm(t, v) \) in the sense of distributions in \( t, v \) and weak star in \( L^\infty([0, T] \times \mathbb{R}) \). Because \( f \in L^1 \), this also implies that \( f(t, h(t) + y, v) \) obtains its trace functions weakly in \( L^1([0, T] \times \mathbb{R}) \). By lower semicontinuity of norms with respect to weak convergence, \( f^\pm \in L^1 \cap L^\infty \) with \( |f^\pm| \leq 1 \). Finally, since \( f f^\pm \Phi \ dtdv \geq 0 \) for any \( \Phi(t, v) > 0 \), \( f^\pm \geq 0 \) almost everywhere.

**Step 2:** Strong convergence of blow-ups

In this step, we will show that there is a sequence of scales \( \eta_n \to 0^+ \), so that blowing up \( f \) around a point \( (t, h(t)) \) at scales \( \eta_n \), we obtain a strong \( L^1 \) limit, \( f^\infty \), simultaneously for almost every point \( (t, h(t)) \). Moreover, our strong limit \( f^\infty \) will solve (2.1) with \( \mu = 0 \), except possibly along the plane \( \{y = 0\} \). The proof consists of using averaging lemmas for the kinetic formulation (2.1) to gain compactness for averages of \( f \). Since \( f \) is already weakly compact and is almost everywhere the characteristic function of an interval in \( v \), this will be sufficient to gain strong compactness.
\textbf{Definition 3} (Blow-up along a curve). For \( f \) a solution to \((2.1)\) with entropy dissipation measure \( \mu, t \to h(t) \) a fixed Lipschitz curve, and \((t, h(t))\) a fixed point on \( h \), we define the blow-up \((f_\eta, \mu_\eta)\) for \( \eta \) sufficiently small (depending on \( t \)) by

\begin{equation}
\begin{split}
f_\eta(\tau, y, v) &:= f(t + \eta \tau, h(t + \eta \tau) + \eta y, v) \\
\mu_\eta(\tau, y, v) &:= \eta \mu(t + \eta \tau, h(t + \eta \tau) + \eta y, v).
\end{split}
\end{equation}

Note that the domain of \( f_\eta \) is \([-\frac{T}{\eta}, \frac{T}{\eta}] \times \mathbb{R} \times \mathbb{R} \), which converges to \( \mathbb{R} \times \mathbb{R} \times \mathbb{R} \) as \( \eta \to 0^+ \). We define also the following sets for \( \tilde{T} > 0 \) arbitrary, \( D_\tilde{T} = [-\tilde{T}, \tilde{T}] \times [-1, 1] \times [-L, L] \) and \( D_\tilde{T}^+ = D_\tilde{T} \cap \{ y > 0 \} \) and \( D_\tilde{T}^- = D_\tilde{T} \cap \{ y < 0 \} \), where \( L \) is a fixed bound on the support of \( f(t, x, \cdot) \). Definition \((3.16)\) is meant in the sense of distributions so that, by scaling, \( f_\eta, \mu_\eta \) almost satisfy \((2.1)\) in the microscopic \((\tau, y, v)\) variables. Finally, fix \( J_\eta \) as the Lipschitz map

\[ J_\eta(\tau, y, v) := (\tau \eta + t, y \eta + h(t + \tau \eta), v), \]

so that \( \mu_\eta(A) = \frac{1}{\eta} \mu(J_\eta(A)) \) for any \( A \subset [-\frac{\tilde{T}}{\eta}, \frac{\tilde{T}}{\eta}] \times \mathbb{R} \times \mathbb{R} \).

\textbf{Proposition 3.3.} There is a sequence \( \eta_n \to 0 \), such that for almost every \( t \in [0, T] \), for \( f_n = f_{\eta_n} \), there exist \( f_\infty^+ \) and \( f_\infty^- \), characteristic functions of intervals (in \( v \)) for almost every \((\tau, y)\), such that for each \( \tilde{T} > 0 \),

\begin{equation}
\begin{split}
\lim_{n \to \infty} \int_{D_\tilde{T}^+} |f_\infty^+(\tau, y, v) - f_n(\tau, y, v)| = 0 \\
\lim_{n \to \infty} \int_{D_\tilde{T}^-} |f_\infty^-(\tau, y, v) - f_n(\tau, y, v)| = 0.
\end{split}
\end{equation}

Furthermore, the limit functions \( f_\infty^\pm \) satisfy the equations

\begin{equation}
\partial_\tau f_\infty^\pm + (v - \dot{h}(t)) \partial_y f_\infty^\pm = 0 \quad \text{for } (\tau, y, v) \in \bigcup_{\tilde{T} > 0} D_\tilde{T}^\pm.
\end{equation}

First, we have the following lemma concerning the structure of traces for the rescaled solutions:

\textbf{Lemma 3.2.} For \( f \in L^1 \) a weak solution to \((2.1)\), the rescaled solutions \( f_\eta \) satisfy

\begin{equation}
\partial_\tau f_\eta + (v - \dot{h}(t + \eta \tau)) \partial_y f_\eta = -\partial_{\nu \cdot v} \mu_\eta.
\end{equation}

Moreover, \( f_n \) has weak traces along the line \( y = 0 \) given by \( f_\eta^+(\tau, v) = f^+(t + \eta \tau, v) \). Furthermore, for any sequence \( \eta_n \to 0^+ \), there is a subsequence still denoted \( \eta_n \) such that for almost every \( t \in [0, T] \), the rescaled solutions \( f_n = f_{\eta_n} \) satisfy

\begin{equation}
\lim_{n \to \infty} \| f_\eta^+(\tau, v) - f^+(t, v) \|_{L^1_{\text{loc}}(\mathbb{R} \times [-L, L])} = 0.
\end{equation}

\textbf{Proof.} The blow-ups \( \{ f_\eta \} \) satisfy \((3.19)\) because \( f \) satisfies \((2.1)\) in the sense of distributions. We prove the convergence \((3.20)\) only for the right-hand trace \( f^+ \). We first note that for \( \varphi \in C^\infty_c((0, \frac{T}{\eta}) \times (-\infty, \infty)) \), Proposition \(3.2\) combined with the rescaling \((3.16)\) guarantees that if \( f_\eta^+ \) denotes the right trace of \( f \) along \( h(t) \), for a fixed \( \eta > 0 \),

\begin{equation}
\lim_{y \to 0^+, \varphi \in C^\infty_c} \int_{\eta \in \Omega} \left[ f_\eta(\tau, y, v) - f^+(t + \eta \tau, v) \right] \varphi(\tau, v) \, d\tau dv = 0.
\end{equation}

Thus, we conclude \( f_\eta \) has weak traces \( f_\eta^+(\tau, v) \) along the line \( y = 0 \) and \( f_\eta^+(\tau, v) = f^+(t + \eta \tau, v) \). The convergence in \((3.20)\) then follows from the continuity of translations on \( L^1 \). Namely, for \( y \) and \( \eta \in \mathbb{R} \) fixed, let us define

\begin{equation}
F_\eta(t) := \int_{-\tilde{T}}^{\tilde{T}} \int_{-L}^{L} f^+(t + \eta \tau, v) - f^+(t, v) \, d\tau dv.
\end{equation}

Then, \( F_{\eta_n} \to 0 \) in \( L^1([0, T]) \) for each \( \tilde{T} > 0 \). By a diagonalization argument, there is a subsequence, still denoted \( \eta_n \), such that \( F_{\eta_n}(t) \to 0 \) for each \( \tilde{T} > 0 \) and for Lebesgue almost every \( t \), as desired. \hfill \blacksquare
Next, we have the following lemma, which shows that we may find a sequence of scales along which the entropy dissipation measure, $\mu$, disappears as we blow up:

**Lemma 3.3.** Suppose $\mu$ is a finite measure. Then, there is a sequence $\eta_n$ such that for almost every $t$, $\mu_{\eta_n} \left( \frac{D^+_T}{T} \right) \to 0$ for all $T > 0$.

**Proof.** We will prove the statement only for the right domains, i.e. $D^+_T$. First, using Fubini’s theorem to slice in $y$, we compute for a fixed $t$, $\eta$, and $\tilde{T}$ satisfying $T - \eta \tilde{T} > t > \eta \tilde{T}$,

\[
\mu_{\eta} \left( \frac{D^+_T}{T} \right) = \frac{1}{\eta} \mu \left( J_{\eta} \left( \frac{D^+_T}{T} \right) \right) = \frac{1}{\eta} \mu \left( \{ (t + \eta \tau, h(t + \eta \tau) + \eta y, v) \mid \tau \in [-\tilde{T}, \tilde{T}], y \in (0, 1), v \in [-L, L] \} \right) = \frac{1}{\eta} \int_{t-\eta \tilde{T}}^{t+\eta \tilde{T}} \mu_{\tau} \left( S_{\tau} \right) \, dt,
\]

(3.23)

where $\mu_{\tau}$ denotes $\mu$ restricted to the slice $\{ \tau \} \times \mathbb{R} \times [-L, L]$ and, similarly, $S_{\tau}$ denotes the slice in $\tau$, which is given via

\[
S_{\tau} := \{ \tau \} \times (h(\tau), h(\tau) + \eta] \times [-L, L].
\]

Thus, integrating in $t$, using Fubini’s theorem, and changing variables, yields

\[
\int_{\eta \tilde{T}}^{T - \eta \tilde{T}} \mu_{\eta} \left( \frac{D^+_T}{T} \right) \, dt = \frac{1}{\eta} \int_{-\eta \tilde{T}}^{\eta \tilde{T}} \int_{t-\eta \tilde{T}}^{t+\eta \tilde{T}} \mu_{\tau} \left( S_{\tau} \right) \, d\tau dt = \frac{1}{\eta} \int_{-\eta \tilde{T}}^{\eta \tilde{T}} \int_{t-\eta \tilde{T}}^{T - \eta \tilde{T}} \mu_{\tau + \tau} \left( S_{\tau + \tau} \right) \, d\tau d\tau = \frac{1}{\eta} \int_{-\eta \tilde{T}}^{\eta \tilde{T}} \int_{-\eta \tilde{T}}^{T - \eta \tilde{T} - \tau} \mu_{\tau} \left( S_{\tau} \right) \, d\tau dt
\]

(3.24)

\[
\leq \frac{1}{\eta} \int_{-\eta \tilde{T}}^{\eta \tilde{T}} \mu \left( \{ (t, h(t) + \eta y, v) \mid t \in [0, T], y \in (0, 1), v \in [-L, L] \} \right) \, d\tau = 2 \tilde{T} \mu \left( \{ (t, h(t) + \eta y, v) \mid t \in [0, T], y \in (0, 1), v \in [-L, L] \} \right).
\]

Note that the above integration in $t$ is only over $\tilde{T} < t < T$ (as $\eta \tilde{T}$ simply to ensure that rescaling of $\mu$, which we recall depends on $h : [0, T] \to \mathbb{R}$, is well-defined. Since the intersection over all $\eta > 0$ is empty and $\mu$ is a finite measure, the right hand side converges to 0. Therefore, $\chi_{[\eta \tilde{T}, T-\eta \tilde{T}]}(t) \mu_{\eta} \left( \frac{D^+_T}{T} \right) \to 0$ strongly in $L^1([0, T])$ for any $T > 0$. After extracting subsequences and performing a diagonalization argument in $\tilde{T}$, there is a sequence $\eta_n \to 0^+$ such that for almost every $t \in [0, T]$, $\mu_{\eta_n} \left( \frac{D^+_T}{T} \right) \to 0$ for each $T > 0$. \hfill \blacksquare

The next lemma is taken directly from [40], but proved in Section 6 for the reader’s convenience:

**Lemma 3.4 ([40, Lemma 1.1]).** Suppose that for each $n$, $f_n : \mathbb{R} \times \mathbb{R} \times [-L, L] \to [0, 1]$ is a characteristic function of an interval for almost every $(t, x)$. If $f_n \overset{\ast}{\rightarrow} f$ in $L^\infty$, $\int_{-L}^{L} f_n(t,x) \, dv \to \int_{-L}^{L} f(t,x) \, dv$, and $\int_{-L}^{L} v f_n(t,x) \, dv \to \int_{-L}^{L} v f(t,x) \, dv$, then $f_n \to f$ in $L^1_{\text{loc}}$ and for almost every $(t, x)$, $f(t,x)$ is the characteristic function of an interval.

We are now ready to proceed with the proof of Proposition 3.3:

**Proof of Proposition 3.3.** First, we use Lemma 3.3 to pick a subsequence of $\eta_n$ such that for almost every $t$,

\[
\lim_{n \to \infty} \mu_{\eta_n} \left( \frac{D^+_T}{T} \right) = \lim_{n \to \infty} \mu_{\eta_n} \left( \frac{D^-_T}{T} \right) = 0, \quad \text{for each } T > 0.
\]

In particular, we note

\[
\sup \{ \mu_{\eta_n}(D^-_T), \mu_{\eta_n}(D^+_T) \} \lesssim_T 1.
\]
Furthermore, defining $a(v) = v - \dot{h}(t)$, by Lemma 3.2, $f_n$ satisfy
\begin{equation}
(3.25) \quad \partial_t f_n + a(v)\partial_y f_n = \left(\dot{h}(t + \eta_n \tau) - \dot{h}(t)\right) \partial_y f_n - \partial_{vv} \mu_n.
\end{equation}
We wish to apply the compactness result of Lions, Perthame, and Tadmor in Lemma 2.1 to (3.25). First, since we have a particularly simple $a$, the non-degeneracy condition (2.4) is immediate. Second, we show
\begin{equation}
(3.26) \quad (1 + \Delta_{\tau,y})^{-1/2}(1 + \partial_{vv})^{-3/2} \left[\left(\dot{h}(t + \eta_n \tau) - \dot{h}(t)\right) \partial_y f_n - \partial_{vv} \mu_n\right] \rightarrow 0,
\end{equation}
where the convergence is in $L^1_{loc}$ as $n \to \infty$. By the Lebesgue differentiation theorem, for any $t$ a Lebesgue point of $\dot{h}(t)$,
\begin{equation}
(3.27) \quad \left\|\dot{h}(t + \eta_n \tau) - \dot{h}(t)\right\|_{L^2((-\bar{T},\bar{T}))} \rightarrow 0,
\end{equation}
for any $\bar{T} > 0$. Therefore, a simple duality argument using $\|f_n\|_{L^\infty} \leq 1$ implies
\begin{equation}
(3.28) \quad \left\|\left(\dot{h}(t + \eta_n \tau) - \dot{h}(t)\right) \partial_y f_n\right\|_{H^{-1}(D^\pm_T)} \rightarrow 0,
\end{equation}
for any fixed $t$ a Lebesgue point of $\dot{h}$ and any $\bar{T} > 0$. Now, since $(1 + \partial_{vv})^{-3/2}(1 + \Delta_{\tau,y})^{-1/2} : H^{-1} \left(D^\pm_T\right) \rightarrow L^2 \left(D^\pm_T\right)$ is a bounded linear operator, we conclude
\begin{equation}
(3.29) \quad \left\|(1 + \Delta_{\tau,y})^{-1/2}(1 + \partial_{vv})^{-3/2} \left(\dot{h}(t + \eta_n \tau) - \dot{h}(t)\right) \partial_y f_n\right\|_{L^p(D^\pm_T)} \rightarrow 0.
\end{equation}
On the other hand, from the Morrey’s inequality, we obtain the embedding $\mathcal{M}_{loc} \hookrightarrow W^{-\frac{1}{p},p}_{loc}$ for $s > \frac{3(p-1)}{p}$. Note that for $1 < p < \frac{6}{5}$, we have $\mathcal{M}_{loc} \hookrightarrow W^{-\frac{1}{p},p}_{loc}$. Therefore, repeating the above argument yields for each $\bar{T} > 0$,
\begin{equation}
(3.30) \quad \left\|(1 + \Delta_{\tau,y})^{-1/2}(1 + \partial_{vv})^{-3/2} \partial_{vv} \mu_n\right\|_{L^p(D^\pm_T)} \rightarrow 0.
\end{equation}
and we have established (3.26).

Thus, using $0 \leq f_n \leq 1$ we apply Banach-Alaoglu to obtain $f^\pm_{\infty}$ such that $f_n \overset{\ast}{\rightharpoonup} f^\pm_{\infty}$ in $L^\infty \left(D^\pm_T\right)$ and $f_n \overset{\ast}{\rightharpoonup} f^\pm_{\infty}$ in $L^\infty \left(D^{-\bar{T}}_T\right)$ for each $\bar{T} > 0$. Now, applying the averaging lemma result, Lemma 2.1, we are guaranteed that for any compactly supported $\psi \in L^\infty_c(\mathbb{R})$, $\int f_n(\cdot,\cdot,v)\psi(v) \, dv \rightarrow \int f^\pm_{\infty}(\cdot,\cdot,v)\psi(v) \, dv$ in $L^1(D^\pm_T)$ and similarly for $f^\pm_{\infty}$. In particular, taking $\psi(v) = \chi_{[-L,L]}(v)$ and $\psi(v) = v\chi_{[-L,L]}(v)$, Lemma 3.4 guarantees for almost every $(\tau,y)$, $f^\pm_{\infty}$ and $f^\pm_{\infty}$ are characteristic functions of intervals obtained strongly in $L^1 \left(D^\pm_T\right)$ and $L^1 \left(D^{-\bar{T}}_T\right)$, respectively, for each $\bar{T} > 0$.

Finally, we note that (3.26) guarantees that the limit functions $f^\pm_{\infty}$ satisfy the desired limiting equations, namely (3.18).

**Step 3: Rigidity**

We now use that along our particular sequence of blow-up scales, the entropy-dissipation measure vanishes to conclude that our limit, $f_{\infty}$, is constant on the two strips $D^\pm_T$ and $D^{-\bar{T}}_T$. Moreover, we will show that our limit is actually the weak trace and conclude by Proposition 3.3 that the weak traces $f^\pm(t,v)$ are characteristic functions of intervals (in $v$) for almost every $t$.

**Proposition 3.4.** There is a sequence of scales $\eta_n \to 0^+$ such that for almost every $t \in [0,T]$, there are functions $f^\pm_{\infty}(\tau,y,v) \in L^\infty(D_T)$ with
\begin{equation}
\lim_{n \to \infty} \left\|f_{\eta_n} - f^\pm_{\infty}\right\|_{L^1(D^\pm_T)} = 0,
\end{equation}
for each $\bar{T} > 0$. Furthermore, for almost every $t$, and almost every $(\tau,y,v) \in D^\pm_T$, $f^\pm(t,v) = f^\pm_{\infty}(\tau,y,v)$.
Proof. Pick $\eta_n \to 0^+$ given by Proposition 3.3. Then, up to extracting a subsequence still called $\eta_n$, both Proposition 3.3 and Lemma 3.2 hold along $\eta_n$. For $\varphi \in C_c^\infty([-T, T] \times [-L, L])$ and $y > 0$ we define the functional
\begin{equation}
(3.31) \quad g^\varphi_\eta(y) = \int \left[(v - \hat{h}(t + \eta_n \tau)) f_n(\tau, y, v) - (v - \hat{h}(t)) f^\varphi_\infty(\tau, y, v)\right] d\tau dv.
\end{equation}
Now, for $\psi \in C^1_c([0, 1])$ with $\|\psi\|_{L^\infty} \leq 1$, using $f_n$ solve (3.19), we have
\begin{equation}
(3.32) \quad \left|\int \psi'(y) g^\varphi_\eta(y) dy\right| \leq \int \left|\psi(y) \partial_\tau \varphi(\tau, v) \left[f^\varphi_\infty(\tau, y, v) - f_n(\tau, y, v)\right]\right| d\tau dv dy
+ \int |\psi(y) \partial_{\nu^\nu} \varphi(\tau, v)| \, d\mu_n(\tau, y, v).
\end{equation}
As $\mu_{\eta_n}$ is a finite measure for each $n$, $g^\varphi_\eta(y)$ is of bounded variation on $[0, 1]$, with variation bounded via
\begin{equation}
(3.33) \quad \text{Var}(g^\varphi_\eta) \leq \|\partial_\tau \varphi\|_{L^\infty} \|f^\varphi_\infty - f_{\eta_n}\|_{L^1(D^\varphi_\infty)} + \|\mu_{\eta_n}\|_{TV} \|\partial_{\nu^\nu} \varphi\|_{L^\infty}.
\end{equation}
From Proposition 3.3 $f_{\eta_n} \to f^\varphi_\infty$ in $L^1(D^\varphi_\infty)$ as $n \to \infty$ and $\mu_{\eta_n} \to 0$ in total variation by Lemma 3.3 as $n \to \infty$. It follows that $\text{Var}(g^\varphi_\eta)$ is bounded via (defined because $g^\varphi_\eta$ is BV),
\begin{equation}
(3.34) \quad g^\varphi(0^+) \leq g^\varphi(1) + \text{Var}(g^\varphi) \to 0.
\end{equation}
On the other hand, for Lebesgue points of $\hat{h}$, $h(t + \eta_n \tau) \to \hat{h}(t)$ in $L^1([\bar{T}, \bar{T}])$. Note also, Proposition 3.2 and Lemma 3.2 imply $f^\varphi_\infty$ and $f_n$ have a weak traces along the line $y = 0$. More specifically, $f^\varphi_n(\tau, v) = f^+(t + \eta_n \tau, v) \to f^+(t, v)$ in $L^1_{\text{loc}}(\tau, v)$ for almost every $t$, which implies
\begin{equation}
(3.35) \quad g^\varphi(0^+) \to \int (v - \hat{h}(t)) \left[f^+(t, v) - f^\varphi_\infty(\tau, 0^+, v)\right] \varphi(\tau, v) d\tau dv,
\end{equation}
where $f^\varphi_\infty(\tau, 0^+, v)$ denotes the right (weak) trace of $f^\varphi_\infty$ along $y = 0$. By the fundamental lemma of the calculus of variations,
\begin{equation}
(3.36) \quad \left(v - \hat{h}(t)\right) \left[f^+(t, v) - f^\varphi_\infty(\tau, 0^+, v)\right] = 0,
\end{equation}
for almost every $(\tau, v) \in [-\bar{T}, \bar{T}] \times [-L, L]$. Since $(v - \hat{h}(t)) \neq 0$ for almost all $(\tau, v)$, it follows that
\begin{equation}
(3.37) \quad f^+(t, v) = f^\varphi_\infty(\tau, 0^+, v),
\end{equation}
in the same sense. Because the limit equation for $f^\varphi_\infty$, namely (3.18), is a transport equation with constant initial data, $f^\varphi_\infty$ is constant. Namely,
\begin{equation}
(3.38) \quad f^\varphi_\infty(\tau, y, v) = f^\varphi_\infty\left(\tau + \frac{y}{v - \hat{h}(t)}, 0^+, v\right) = f^+(t, v),
\end{equation}
for each $y > 0$ and almost every $\tau, v$. \hfill \blacksquare

Step 4: Proof of Proposition 3.1

Fix $t \mapsto h(t)$ a Lipschitz path and $f$ a weak solution to (2.1). Then, there exist $f^+$ and $f^-$ weak traces for $f$ along $h(t)$ for which $0 \leq f^\pm \leq 1$, $f \in L^1 \cap L^\infty$. By Proposition 3.4, we know that $f^\pm(t, \cdot)$ are realized as the blow-up along a sequence of scales $\eta_n \to 0^+$. Thus, by Lemma 3.4, we conclude $f^\pm(t)$ are characteristic functions of intervals (in $v$) for almost every $t$. Now, Proposition 3.1 follows since weak convergence of characteristic functions to characteristic functions implies strong convergence. More precisely, by Proposition 3.2, we have for any $\varphi \in L^1([0, T] \times \mathbb{R} \times [-L, L])$, compactly supported,
\begin{equation}
(3.39) \quad \lim_{n \to \infty} \text{ess sup}_{0 < y < \pm T} \left|\int_0^T \left[f(t, h(t) + y, v) - f^+(t, v)\right] \varphi(t, v) \, dt dv\right| = 0.
\end{equation}
Finally, recall $f$ and $f^+$ are characteristic functions in $v$, so that taking $\varphi(t, v) = f^+(t, v)$ and $\varphi(t, v) = 1 - f^+(t, v)$ completes the proof.
4. Partial Regularity

In this section, we prove Proposition 1.2, which is a type of partial regularity theorem for the Riemann invariants of (1.1). The argument is inspired by the first lemma of De Giorgi introduced in [13] and adapted to the context of scalar conservation laws by [36, Theorem 6.1]. Note that for (1.1) the Riemann invariants appear to decouple and each solve Burger’s equation, where the two remain coupled only by the Rankine-Hugoniot jump condition. Thus, if it is known a priori that no shocks are present in a solution, one can apply Silvestre’s method directly to recover some form of continuity. However, when it is unknown whether shocks are present, we must necessarily consider both Riemann invariants $\lambda_1$ and $\lambda_2$ simultaneously, which leads to a different usage of the kinetic formulation and different results. We will attempt to illustrate exactly where the difficulties arise and why they result lead to a weaker result than in the scalar case.

Let us begin with an estimate on the total variation of $\mu$ on unbounded $v$-intervals using (2.1):

**Lemma 4.1.** Let $f \in L^1(B_2 \times \mathbb{R})$ be a weak solution of (2.1) with $\text{supp } f(t,x) \subset [-L, L]$. Then, for any $0 < r < R$, and any $a \in \mathbb{R}$,

$$
\mu(B_r \times (-\infty, a]) \lesssim_L (R-r)^{-1} \int_{B_r} \int_{-\infty}^a f \, dv \, dx \, dt \quad \text{and} \quad \mu(B_r \times [a, \infty)) \lesssim_L (R-r)^{-1} \int_{B_r} \int_a^L f \, dv \, dx \, dt
$$

**Remark 8.** Note that as a consequence of the above lemma, we recover that the support of $\mu$ is contained within $[-L, L]$, which is not immediately apparent from (2.1). Also, as expected, $\mu$ is not concentrated in velocity.

**Remark 9.** We are not able to show the corresponding estimates for $1 - f$, namely we **CANNOT** show

$$
\mu(B_r \times [a, b]) \lesssim_L (R-r)^{-1} \int_{B_r} \int_a^b (1 - f) \, dv \, dx \, dt.
$$

If we were able to show bounds of the form (4.2), we would be able to obtain weak-strong stability and uniqueness results for all entropy solutions to (1.1), regardless of size in $L^\infty$ and whether or not the strong solution is a shock. In particular, (4.2) would allow us to obtain partial regularity results directly for a weak solution $u$ to (1.1), not just on the characteristic fields $\lambda_1(u)$ and $\lambda_2(u)$. However, variants of (4.2) seem unlikely since even if $f \equiv 1$ on some region $A$, we need not have $\mu(A) = 0$; for example, take a shock wave $(u_L, u_R, \sigma_{LR})$, where $u_L$ and $u_R$ are chosen so that the corresponding $f$ is identically 1 for a portion of the wave front, while $\mu$ does not vanish.

**Proof.** We prove first the bounds for velocity intervals of the form $[a, \infty)$. Fix $b < a$, $\varphi \in C_c^\infty(\mathbb{R}^2)$, and $\zeta \in C_c^\infty(\mathbb{R})$ such that

$$
\begin{cases}
0 \leq \varphi(t, x) \leq 1, \\
\varphi(t, x) = 1 \quad \text{if } (t, x) \in B_r, \\
\varphi(t, x) = 0 \quad \text{if } (t, x) \notin B_R \
|\nabla \varphi| \leq \frac{C}{R-r}
\end{cases}
$$

and

$$
\begin{cases}
0 \leq \zeta(v) \leq 1, \\
\zeta(v) = 0 \quad \text{if } |v - b| > 2 \varepsilon^{-1} \text{ or } |v - b| < \varepsilon, \\
\zeta = 1 \quad \text{if } \varepsilon < |v - b| < \varepsilon^{-1}, \\
|v - b)\zeta'(v)| \leq C, \\
|v - b)^2\zeta''(v)| \leq C
\end{cases}
$$

Finally, we define

$$
\psi(v) = \begin{cases}
\frac{(v-b)^2}{2} & \text{if } v > b \\
0 & \text{otherwise}
\end{cases}
$$

We note $\psi \in C^1(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus \{b\})$ with $\psi''(v) = \chi_{(b, \infty)}(v)$. Thus, we test (2.1) with $\varphi(t, x)\psi(v)\zeta(v) \in C_c^\infty(\mathbb{R}^2)$ to obtain

$$
\int (\partial_t \varphi + \nu \partial_x \varphi) \psi(v)\zeta(v) f \, dx \, dt \, dv = \int (\psi\zeta)^'' \varphi \, d\mu,
$$

Now, we use

$$
(\psi\zeta)'' = \psi''(v)\zeta(v) + 2\psi'(v)\zeta'(v) + \psi(v)\zeta''(v),
$$

where the right hand side is bounded pointwise by a constant $C$ independent of $\varepsilon$ and $v$ and the right hand side converges pointwise everywhere to $\chi_{(b, \infty)}(v)$ as $\varepsilon \to 0^+$. Note, we do not know whether $\mu$ is concentrated
in velocity a priori, only that $\|\mu\|_{T^\varepsilon} < \infty$, so we are careful to ensure our bounds hold pointwise everywhere. Thus, by the Lebesgue dominated convergence theorem for the measure $\mu$, we have

$$ (4.4) \quad \mu(B_2 \times [a, \infty)) \leq \int \chi(b, \infty) \varphi \, d\mu = \int (\partial_t \varphi + v \partial_x \varphi) \psi(v) f \, dxdt \, dv \leq \frac{C(L)}{R - r} \int_{B_R} f \, dxdvt. $$

Finally, taking $b \uparrow a$, the Lebesgue dominated convergence yields the desired result for intervals of the form $[a, \infty)$. Redefining $\psi$ so that

$$ \psi(v) = \begin{cases} \frac{(v-b)^2}{2} & \text{if } v < b \\ 0 & \text{otherwise}, \end{cases} $$

the same argument yields the desired result for intervals of the form $(-\infty, a]$ as well.

The following estimate is critical for the proof of Proposition 1.2. The estimate encodes the gain of integrability coming from localizing from a larger space-time domain and larger range of velocities to a smaller space-time domain and smaller range of velocities.

**Lemma 4.2.** Say $f \in L^1(C_2 \times [-L, L])$ is a weak solution of (2.1) with $\text{supp } f(t, x) \subset [-L, L]$. Then, there is a $\theta_0$ (explicitly computable as $\frac{1}{2}$) such that for any $0 < r < R < 2$ and any $-L \leq l_2 \leq l_1 \leq L$, and any $0 < \alpha < \frac{1}{r}$,

$$ (4.5) \quad \|f\|_{L^1(C_2 \times [l_1, L])} \lesssim (l_1 - l_2)^{-2\theta_0} (R - r)^{-\theta_0} \|f\|_{L^1(B_R \times [l_2, L])} \left\{ \int_{l_1}^L f(t, x, v) > 0 \, dv \right\} \cap B_r. $$

$$ (4.6) \quad \|f\|_{L^1(C_2 \times [-L, l_2])} \lesssim (l_1 - l_2)^{-2\theta_0} (R - r)^{-\theta_0} \|f\|_{L^1(B_R \times [-L, l_1])} \left\{ \int_{-L}^L f(t, x, v) > 0 \, dv \right\} \cap B_r. $$

**Proof.** We will only prove the estimates for sets of the form $B_r \times [l_1, L]$ as the other estimates are obtained in an identical manner. For this proof, we let $z = (t, x) \in \mathbb{R}^2$ denote space-time. Thus, (2.1) reads as

$$ (4.7) \quad a(v) \cdot \nabla z f = -\partial_{vv} \mu, $$

where $a(v) = (1, v)$. Now, we fix $f \in L^1 \cap L^\infty$ a weak solution of (4.7) with $\text{supp } f(t, x) \subset [-L, L]$ and corresponding finite measure $\mu$. We also fix $\varphi \in C_c^\infty(\mathbb{R}^2)$ and $\psi \in C_c^\infty(\mathbb{R})$ satisfying

$$ \begin{aligned} &0 \leq \varphi(z) \leq 1 \\ &\varphi(z) = 1 \quad \text{if } |z| < r \\ &\varphi(z) = 0 \quad \text{if } |z| > R \\ &|\nabla \varphi| \leq \frac{C}{R - r} \quad \text{and} \quad |\nabla \psi| \leq \frac{C}{l_1 - l_2}. \end{aligned} $$

where $\tilde{R} = \frac{R + r}{2}$. We define the localized quantities

$$ g(z, v) := \psi(v) \varphi(z) f(z, v), \quad \mu_0(z, v) := - (\varphi(z) \psi''(v)) \mu(z, v), \quad \mu_1(z, v) := (2\varphi(z) \psi'(v)) \mu(z, v), \quad \mu_2(z, v) := - (\varphi(z) \psi(v)) \mu(z, v), $$

which solve a version of (4.7) with lower order terms, namely

$$ (4.8) \quad a(v) \cdot \nabla z g = \partial_{vv} \mu_2 + \partial_v \mu_1 + \mu_0 + \psi f [a(v) \cdot \nabla \varphi]. $$

Therefore, applying the quantitative averaging lemma 2.2 to $g$, we conclude for any $0 < \theta < \theta_0 = \frac{1}{4}$,

$$ (4.9) \quad \left\| g \right\|_{W^{\theta, \frac{3}{2} - \delta_0}} \leq \theta L \left\| g \right\|_{L^1(B_R \times [l_2, L])} \left\| \mu_2 \right\|_{TV} + \left\| \mu_1 \right\|_{TV} + \left\| \mu_0 \right\|_{TV} + \left\| \psi f (a \cdot \nabla \varphi) \right\|_{L^1}. $$

We upper bound each term on the right hand side, we first note $0 \leq \psi, \varphi, f \leq 1$, and therefore

$$ \left\| g \right\|_{L^2} \leq \left\| f \right\|_{L^2(B_R \times [l_2, L])}. $$

Second, we use $\|a \psi\|_{L^\infty([-L, L])} \leq L$ to obtain the bound

$$ (4.10) \quad \left\| \psi f (a \cdot \nabla \varphi) \right\|_{L^1} \leq \frac{CL}{R - r} \left\| f \right\|_{L^1(B_R \times [l_2, L])}. $$
Third, for each $i = 0, 1, 2$, we use $rac{\partial \psi}{\partial v} \leq \frac{C}{(l_1 - l_2)^i}$ and Lemma 4.1 to obtain
\begin{equation}
\|\mu_i\|_{TV} \leq \frac{C}{(l_1 - l_2)^i} \frac{\|\mu\|_{TV(B_R \times [l_2, L])}}{(l_2 - 2l_2)^i (R - R)^i} \frac{\|f\|_{L^1(B_R \times [l_2, L])}}{(l_2 - 2l_2)^i (R - R)^i}.
\end{equation}
We conclude
\begin{equation}
\left\| \int \varphi(z) \psi(v) f(z, v) \, dv \right\|_{W^{0, \frac{2}{q}}(\mathbb{R}^2)} \lesssim \frac{1}{(l_2 - l_1)^{2\theta_0} (R - r)^{\frac{\theta_0}{q}}} \|f\|_{L^1(B_R \times [l_2, L])}.
\end{equation}
Now, by the Sobolev embedding $W^{0, \frac{2}{q}}(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2)$ for $q = \frac{2}{1+(\theta_0 - \theta)}$, we lower bound the left hand side as
\begin{equation}
\left\| \int \varphi(z) \psi(v) f(z, v) \, dv \right\|_{W^{0, \frac{2}{q}}(\mathbb{R}^2)} \gtrsim_{q} \left\| \int_{l_1}^{L} \varphi(z) f(z, v) \, dv \right\|_{L^q}.
\end{equation}
Finally, we use $\varphi \leq 1$ and Hölder’s inequality, in the form
\begin{equation}
\int_{B_r} \int_{l_1}^{L} f(z, v) \, dv \, dz \leq \left\| f \right\|_{L^q} \left\{ z \in B_r \mid \int_{l_1}^{L} f(z, v) \, dv > 0 \right\}^{\alpha},
\end{equation}
where $\alpha = \frac{q-1}{q} = \frac{1-\theta}{2}$ which can be made arbitrarily close to $\frac{1}{2}$ by taking $\theta$ arbitrarily close to $\theta_0$, which completes the proof. \[\square\]

Next, we iterate the gain from Lemma 4.2 to prove the following lemma, which is the direct analogue of Proposition 1.2, stated for the kinetic equation (2.1):

**Lemma 4.3.** Fix $L > 0$ and $-L < a < b < L$. Then for any $f$ solving (2.1) with $\supp(f(t, x)) \subset [-L, L]$
with $\ess\inf_{(t,x) \in B_2} \int f(t, x) \, dx > M$, there is an $\varepsilon_0 = \varepsilon_0(L, M)$, $\bar{C} = \bar{C}(L, M) > 0$, and $\alpha \in (0, 1)$ such that, for any $0 < \varepsilon < \varepsilon_0$, setting $\eta = \bar{C} \varepsilon^{\alpha}$, if $f$ solves (2.1) with velocities supported in $[-L, L]$, then
\begin{enumerate}
\item \begin{equation}
\int_{B_2} \int_{-L}^{L} f(t, x, v) \, dv < \varepsilon \quad \text{implies} \quad f(t, x, v) = 0 \quad \text{for} \quad (t, x, v) \in B_1 \times [b + \eta, L]
\end{equation}
\item \begin{equation}
\int_{B_2} \int_{-L}^{a} f(t, x, v) \, dv < \varepsilon \quad \text{implies} \quad f(t, x, v) = 0 \quad \text{for} \quad (t, x, v) \in B_1 \times [-L, a - \eta]
\end{equation}
\end{enumerate}

**Proof.** Let us define the scales $r_k = 1 - 2^{-k}$, $B_k = B_{r_k}(0)$, and $l_k = \eta(1 - 2^{-k})$, for $\eta < \varepsilon_0$. Let us also fix $\theta = \theta_0$ from Lemma 4.2 and $\delta = \theta/2$. Then, we define our iteration quantities as
\begin{equation}
U_k = \int_{B_k} \int_{b + l_k}^{L} f(t, x, v) \, dv \, dx \, dt \quad \text{and} \quad V_k = \int_{B_k} \int_{-L}^{-l_k} f(t, x, v) \, dv \, dx \, dt.
\end{equation}
Our goal is to show that for an appropriate choices of $\eta$, $\varepsilon_0$, $U_k \to 0$ and $V_k \to 0$ provided $U_0, V_0 < \varepsilon < \varepsilon_0$. Now, using Lemma 4.2 and $U_k$, $V_k$ are decreasing, we obtain the following inequalities:
\begin{equation}
U_{k+1} \leq \frac{C U_k^{\frac{1+\theta}{2}}}{(l_{k+1} - l_k)^{2\theta}(r_{k+1} - r_k)^{\theta}} \left\{ \int_{b + l_{k+1}}^{L} f(t, x, v) \, dv > 0 \right\} \cap B_{k+1} \left[ \frac{1-\frac{\theta}{2}}{2} \right]
\end{equation}
\begin{equation}
V_{k+1} \leq \frac{C V_k^{\frac{1+\theta}{2}}}{(l_{k+1} - l_k)^{2\theta}(r_{k+1} - r_k)^{\theta}} \left\{ \int_{-l_{k+1}}^{a-l_k} f(t, x, v) \, dv > 0 \right\} \cap B_{k+1} \left[ \frac{1-\frac{\theta}{2}}{2} \right].
\end{equation}
Next, we estimate the measure of the sets appearing on the right hand sides. We note, if $\int_{b + l_{k+1}}^{L} f(t, x, v) \, dv > 0$, then because $\int_{-L}^{L} f \, dx > M$ on $B_2$ and $f(t, x, \cdot)$ is the characteristic function of an interval, we must have
\begin{equation}
\int_{b + l_k}^{L} f(t, x, v) \, dv > \min(M, l_{k+1} - l_k).
\end{equation}
Thus, by Chebychev’s inequality we have
\begin{equation}
\left\{ \int_{b + l_{k+1}}^{L} f(t, x, v) \, dv > 0 \right\} \cap B_{k+1} \leq \frac{U_k}{\min(M, l_{k+1} - l_k)} = \frac{U_k}{\min(M, \eta 2^{-k-1})}
\end{equation}
Using a similar argument for \( \int_{-L}^{a_{k+1}} f(t, x, v) \, dv \), we obtain

\[
U_{k+1} \leq \frac{CU_k^{1+rac{\theta}{\alpha} + \frac{1}{\eta^2}}}{(l_{k+1} - l_k)^{2\theta}(r_k - r_{k+1})^\theta \min(M, l_{k+1} - l_k)^{\frac{1}{\alpha}}}
\]

\[
V_{k+1} \leq \frac{CV_k^{1+rac{\theta}{\alpha} + \frac{1}{\eta^2}}}{(l_{k+1} - l_k)^{2\theta}(r_k - r_{k+1})^\theta \min(M, l_{k+1} - l_k)^{\frac{1}{\alpha}}}
\]

Now, if \( \eta < M \), we have \( \min(M, l_{k+1} - l_k) = l_{k+1} - l_k \) for each \( k \),

\[
U_{k+1} \leq \frac{CU_k^{1+rac{\theta}{\alpha} + \frac{1}{\eta^2}}}{(l_{k+1} - l_k)^{2\theta}(r_k - r_{k+1})^\theta \min(M, l_{k+1} - l_k)^{\frac{1}{\alpha}}} = \frac{C2^{(k+1)\left[\frac{1}{\eta^2} + \frac{1}{\eta^2}\right]}U_k^{1+\frac{\theta}{\alpha}}}{\eta^{\frac{2\theta}{\alpha} + \frac{\theta}{\eta^2}}}
\]

\[
V_{k+1} \leq \frac{CV_k^{1+rac{\theta}{\alpha} + \frac{1}{\eta^2}}}{(l_{k+1} - l_k)^{2\theta}(r_k - r_{k+1})^\theta \min(M, l_{k+1} - l_k)^{\frac{1}{\alpha}}} = \frac{C2^{(k+1)\left[\frac{1}{\eta^2} + \frac{1}{\eta^2}\right]}V_k^{1+\frac{\theta}{\alpha}}}{\eta^{\frac{2\theta}{\alpha} + \frac{\theta}{\eta^2}}}
\]

Finally, setting \( \alpha^{-1} = \frac{7\theta + 2}{\eta^2} \),

\[
\frac{U_{k+1}}{\eta^{\frac{1}{\alpha}}} \leq C2^{(k+1)\left[\frac{1}{\eta^2} + \frac{1}{\eta^2}\right]} \left( \frac{U_k}{\eta^{\frac{1}{\alpha}}} \right)^{1+\frac{\theta}{\alpha}} \quad \text{and} \quad \frac{V_{k+1}}{\eta^{\frac{1}{\alpha}}} \leq C2^{(k+1)\left[\frac{1}{\eta^2} + \frac{1}{\eta^2}\right]} \left( \frac{V_k}{\eta^{\frac{1}{\alpha}}} \right)^{1+\frac{\theta}{\alpha}}
\]

Therefore, (4.25) implies that there is a sufficiently large constant \( \tilde{C} \) such that if

\[
\eta^{-\frac{1}{\alpha}}(U_0) = \eta^{-\frac{1}{\alpha}}\|f\|_{L^1(B(2) \times [b, L])} < \eta^{-\frac{1}{\alpha}} \varepsilon \leq \frac{1}{C\varepsilon}
\]

and similarly for \( V_0 \), then

\[
\lim_{k \to \infty} U_k = \lim_{k \to \infty} V_k = 0.
\]

Choosing \( \eta = \tilde{C} \varepsilon^\alpha \) and \( \varepsilon_0 = (M/\tilde{C})^{1/\alpha} \) completes the proof.

We are now ready to prove Proposition 1.2:

**Proof of Proposition 1.2.** We first note that \( u = (\rho, m) \) satisfies \( \|\rho\|_{L^\infty(B_2)} + \|\rho/m\|_{L^\infty(B_2)} \leq \Gamma \) so the corresponding \( f \) solving (2.1), has a uniform velocity bound \( L = 2\Gamma \) on \( B_2 \). Now, we wish to apply Lemma 4.3 to this \( f \). Thus, we note \( \rho(t, x) \geq M \) for almost every \( (t, x) \in B_2 \) exactly implies \( \int f \, dv \geq M \). Also, since \( \overline{\rho} = (\overline{\gamma}, \overline{\pi}) \), is a constant solution to (1.1), the corresponding kinetic function is \( \chi_{[a, b]} \) for \( a = \overline{\gamma} - \frac{\overline{\pi}}{2} = \lambda_1(\overline{\pi}) \) and \( b = \overline{\gamma} + \frac{\overline{\pi}}{2} = \lambda_2(\overline{\pi}) \) constant. Since \( f \) is given via the formula,

\[
f(t, x, v) = \chi_{\left[\frac{\overline{\rho}}{2} - \frac{\overline{\pi}}{2} + \frac{1}{2}\right]}(v) = \chi_{\lambda_1(u), \lambda_2(v)}(v),
\]

we have

\[
\int_{B_2} (\lambda_1(\overline{\pi}) - \lambda_1(u))_+ = \int_{B_2} \int_{-L}^{a} f(t, x, v) \, dv \, dx \, dt \quad \text{and} \quad \int_{B_2} (\lambda_2(u) - \lambda_2(\overline{\pi}))_+ = \int_{B_2} \int_{b}^{L} f(t, x, v) \, dv \, dx \, dt.
\]

Therefore, for \( 0 < \varepsilon < \varepsilon_0 \), where \( \varepsilon_0 = \varepsilon_0(L, M) \) is from Lemma 4.3, we have

\[
\text{if} \quad \| (\lambda_1(\overline{\pi}) - \lambda_1(u))_+ \|_{L^1(B_2)} \leq \varepsilon \quad \text{then} \quad f(t, x, v) = 0, \ (t, x) \in B_1, \ v < a - \tilde{C} \varepsilon^\alpha,
\]

\[
\text{if} \quad \| (\lambda_2(u) - \lambda_2(\overline{\pi}))_+ \|_{L^1(B_2)} \leq \varepsilon \quad \text{then} \quad f(t, x, v) = 0, \ (t, x) \in B_1, \ v > b + \tilde{C} \varepsilon^\alpha.
\]

Thus, we compute pointwise for \( (t, x) \in B_1 \),

\[
(\lambda_1(u(t, x)) - \lambda_1(u(t, x)))_+ = \int_{-L}^{a} f(t, x, v) \, dv = \int_{a-\tilde{C} \varepsilon^\alpha}^{a} f(t, x, v) \, dv \leq \tilde{C} \varepsilon^\alpha
\]

\[
(\lambda_2(u(t, x)) - \lambda_2(u(t, x)))_+ = \int_{b}^{L} f(t, x, v) \, dv = \int_{b+\tilde{C} \varepsilon^\alpha}^{L} f(t, x, v) \, dv \leq \tilde{C} \varepsilon^\alpha,
\]

which completes the proof.
5. Applications of Partial Regularity

5.1. Semicontinuity: Proof of Theorem 1.2. In this section, we present our first application of our partial regularity result and prove Theorem 1.2. We begin by recalling several notations and definitions for versions of $L^\infty$ functions used in Section 1 to state Theorem 1.2. First, we recall that for $g \in L^{1}_{loc}(\mathbb{R}^n)$, we say that $x$ is a point of VMO for a function $g$, if

\begin{equation}
\lim_{r \to 0^{+}} \frac{1}{|B_r(x)|} \int_{B_r(x)} \left| g(y) - \frac{1}{|B_r(x)|} \int_{B_r(x)} g(z) \, dz \right| \, dy = 0.
\end{equation}

Note, that this is a slight relaxation of the notion of a Lebesgue point, which further requires $\int_{B_r(x)} g(z) \, dz \to g(x)$ as $r \to 0^+$. Second, we recall the upper and lower semicontinuous envelopes, which we denote by $\overline{g}$ and $\underline{g}$ are defined via

\begin{equation}
\overline{g}(x) = \lim_{r \to \infty} \text{ess sup}_{y \in B_r} g(y) \quad \text{and} \quad \underline{g}(x) = \lim_{r \to \infty} \text{ess inf}_{y \in B_r} g(y).
\end{equation}

One can check that for $g \in L^\infty$, $\overline{g}$ is upper semicontinuous and $g \leq \overline{g}$ almost everywhere and similarly $g$ is lower semicontinuous and $\underline{g} \leq g$ almost everywhere. Finally, we recall that we pick the following version for scalar functions we study:

\begin{equation}
\hat{g}(x) = \lim_{r \to \infty} \inf_{y \in B_r} g(y) \, dy.
\end{equation}

The Riemann Invariants

Now, we fix $u = (\rho, m)$ an entropy solution as in the statement of Theorem 1.2 and $(t, x)$ a point of VMO of $\lambda_1(t, x) := \lambda_1(u(t, x))$ and $\lambda_2(u(t, x)) := \lambda_2(u(t, x))$ such that $\rho(t, x) \neq 0$. We will prove $(t, x)$ is a Lebesgue point for $\lambda_1$ and $\lambda_2$ with value $\underline{\lambda}_1$ or $\overline{\lambda}_2$, respectively. To this end, we pick any sequence of scales $r_n \to 0^+$. Then, using $u \in L^\infty$, we conclude there is a subsequence $r_{n_k}$ along which $\int_{B_{r_{n_k}}(t, x)} \lambda_1(\tau, y) \, d\tau dy \to \Lambda$. Note, since $(t, x)$ is VMO for $\lambda_1$, along $r_{n_k}$ we have

\begin{equation}
\lim_{k \to \infty} \frac{1}{|B_{r_{n_k}}(t, x)|} \int_{B_{r_{n_k}}(t, x)} |\lambda_1(u(\tau, y)) - \Lambda| \, d\tau dy = 0.
\end{equation}

Consider the blow-up sequence $u_k(\tau, y) := u((t, x) + r_{n_k}(\tau, y))$. Note that by translation invariance and scaling, each blow-up $u_k$ is a solution to (1.1) satisfying $\|\rho_k\|_{L^\infty} + \|m_k\|_{L^\infty} \leq \Gamma$, where $\Gamma = \|u\|_{L^\infty}$. Moreover, $\rho_k \geq \rho / 2$ almost everywhere on $B_2$ for each $k$ sufficiently large and $\|\rho_k\|_{L^\infty(B_2)} \leq \frac{2\rho}{\Gamma}$ for sufficiently large $k$. Thus, for $k$ sufficiently large so $\|\lambda_1(u_k) - \Lambda\|_{L^1(B_2)} < \varepsilon_0$, $\varepsilon_0$ is as in the statement of Proposition 1.2, we conclude

\begin{equation}
\text{ess sup}_{B_{r_{n_k}}} [\Lambda - \lambda_1(u_k)]_+ \leq \|\Lambda - \lambda_1(u_k)\|_{L^1(B_2)} \to 0.
\end{equation}

Therefore, on one hand, we conclude

\begin{equation}
\Lambda \leq \underline{\lambda}_1(t, x).
\end{equation}

On the other hand, from the definition of $\Lambda$, we have

\begin{equation}
\Lambda = \lim_{k \to \infty} \int_{B_{r_{n_k}}(t, x)} \lambda_1(\tau, y) \, d\tau dy \geq \lim_{k \to \infty} \text{ess inf}_{(\tau, y) \in B_{r_{n_k}}(t, x)} \lambda_1(\tau, y) = \underline{\lambda}_1(t, x).
\end{equation}

From the uniqueness of the limit, we conclude $(t, x)$ is a Lebesgue point of $\lambda_1$ with value $\underline{\lambda}_1(t, x)$. The same argument for $\lambda_2$ yields $(t, x)$ is a Lebesgue point of $\lambda_2$ with value $\overline{\lambda}_2(t, x)$. The definitions of $\underline{\lambda}_1$ and $\overline{\lambda}_2$ then guarantee $\lambda_1(t, x) = \underline{\lambda}_1(t, x)$ and $\lambda_2(t, x) = \overline{\lambda}_2(t, x)$.

The Density

The main tool for analyzing $\rho$ is the algebraic identity, $\rho = \lambda_2(u) - \lambda_1(u)$. Thus, we immediately see that $(t, x)$ a point of VMO of $\lambda_1(u)$ and $\lambda_2(u)$ implies $(t, x)$ is a point of VMO for $\rho$. Also, since $(t, x)$ is a Lebesgue point for $\lambda_1$ and $\lambda_2$, we find

\begin{equation}
\hat{\rho} = \lim_{r \to 0^+} \int_{B_r(t, x)} \rho = \lim_{r \to 0^+} \int_{B_r(t, x)} (\lambda_2 - \lambda_1) = \overline{\lambda}_2(t, x) - \underline{\lambda}_1(t, x),
\end{equation}

where $\hat{\rho}$ is the essential supremum of $\rho$.
and \((t, x)\) is a Lebesgue point for \(\rho\). Next, a few basic inequalities imply

\[
\lim_{r \to 0^+} \int_{B_r(t, x)} \rho \leq \lim_{r \to 0^+} \esssup_{y \in B_r(t, x)} \rho(y) \leq \lim_{r \to 0^+} \left[ \esssup_{y \in B_r(t, x)} \lambda_2(y) - \essinf_{y \in B_r(t, x)} \lambda_1(y) \right] = \overline{\lambda}_2(t, x) - \underline{\lambda}_1(t, x).
\]

Thus, we conclude

\[
\rho(t, x) = \overline{\rho}(t, x) = \overline{\lambda}_2(t, x) - \underline{\lambda}_1(t, x).
\]

**The Momentum**

The main tool for analyzing \(m\) is the algebraic identity, \(m = \frac{\lambda_2 - \lambda_1}{2}\). Just as for the density, we see \((t, x)\) a point of \(VMO\) of \(\lambda^1\) and \(\lambda^2\) implies \((t, x)\) is a Lebesgue point for \(m\) with

\[
\hat{m} = \lim_{r \to 0^+} \int_{B_r(t, x)} m = \lim_{r \to 0^+} \frac{1}{2|B_r(t, x)|} \int_{B_r(t, x)} \lambda_2^2(y) - \lambda_1^2(y) \, dy = \frac{\overline{\lambda}_2(t, x) - \underline{\lambda}_1(t, x)}{2}.
\]

Now, we have two cases. First, if \(\overline{\lambda}_1(t, x) \geq \overline{\lambda}_1(t, x) \geq 0\), then

\[
\hat{m}(t, x) = \overline{m}(t, x).
\]

In this case, \(\hat{m}(t, x) = \overline{m}(t, x)\). Second, if \(\overline{\lambda}_1(t, x) \leq \overline{\lambda}_2(t, x) \leq 0\), then

\[
\hat{m}(t, x) = \underline{m}(t, x).
\]

In this case, \(\hat{m}(t, x) = \underline{m}(t, x)\).

**Semicontinuity Almost Everywhere**

Finally, we note that by Lebesgue’s Differentiation Theorem, almost every \((t, x) \in \mathbb{R}^+ \times \mathbb{R}\) is a point of \(VMO\) (and even Lebesgue point) of both \(\lambda_1\) and \(\lambda_2\). Therefore, the equalities

\[
\lambda_1 = \underline{\lambda}_1, \quad \lambda_2 = \overline{\lambda}_2, \quad \text{and} \quad \rho = \overline{\rho}
\]

hold almost everywhere on the open set \(\{\rho \neq 0\}\) and imply almost everywhere semicontinuity. Similarly, the equality

\[
m = \overline{m}
\]

holds almost everywhere on the set \(\{\underline{\lambda}_1 \geq 0\} \cap \{\rho \neq 0\}\) and the equality

\[
m = \underline{m}
\]

holds almost everywhere on the set \(\{\overline{\lambda}_2 \leq 0\} \cap \{\rho \neq 0\}\).

5.2. **Generalized Characteristics: Proof of Proposition 1.1.** In this section, we prove Proposition 1.1. In particular, we construct solutions (in a very weak sense) to the ODE

\[
\dot{h}(t) = V(u(t, h(t))),
\]

where \(V(u) = \min(\lambda_1(u), \sigma)\) or \(\max(\lambda_2(u), \sigma)\) for some fixed \(\sigma\) and \(u\) solves (1.1). In order to find solutions, we mollify the right hand side as \(V(u_\varepsilon)\) and deduce existence of solutions \(h_\varepsilon\) to the approximating problem and compactness of \(\{h_\varepsilon\}\) via the Picard-Lindelöf Theorem. However, due to the lack of regularity of \(V(u(t, \cdot))\), even strong compactness is insufficient to conclude the limit satisfies the desired ODE in an appropriate sense. Therefore, we first show that we may use the trace result of Theorem 1.1 to obtain some strong convergence in \(L^1\) of \(u_\varepsilon\) as we let \(\varepsilon \to 0^+\). Second, we show that we may use the partial regularity result of Proposition 1.2 to upgrade the convergence to an \(L^\infty\)-type convergence. Third, we use the deduced regularity to show that the limit curve \(h\) indeed satisfies the desired ODE, albeit in the very weak sense claimed in Proposition 1.1.

**Step 1: \(L^1\)-Convergence of Blow-ups**
We use the strong trace property of solutions \( u \) to (1.1) established in Section 3 to ensure blow-ups around points \((t, h(t))\) converge in \(L^1\). Note that instead of blowing up around points \((t, h(t))\) along the curve \(h(t)\) as done in Section 3, we will take a blow-up that preserves the equation (1.1).

**Lemma 5.1.** Suppose \( u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^2 \) with \( u \in L^\infty \) satisfies the strong trace property in the sense of Definition 1. Then, for any \( h : [0, T] \to \mathbb{R} \) Lipschitz and for any sequence \( \eta_k \to 0^+ \), there is a subsequence, still denoted \( \eta_k \), such that:

for almost every \( t \), the half-spaces \( H_t^\pm := \{(\tau, y) \mid y \leq \pm h(t)\tau\} \) are well-defined and for each compact \( K^\pm \subset H_t^\pm \),

\[
(5.18) \quad \lim_{k \to \infty} \int_{K^\pm} |u(t - \eta_k\tau, h(t) \pm \eta_k y) - u^\pm(t)| \, d\tau dy = 0.
\]

**Proof.** We will prove the statements only for the right trace. First, we note that the strong trace property of solutions \( u \) and continuity of translations on \( L^1 \) together imply for each \( T, Y > 0 \),

\[
(5.19) \quad \lim_{\eta \to 0^+} \int_0^T \int_{-T}^T \int_0^Y |u(t - \eta\tau, h(t - \eta\tau) + \eta y) - u^+(t - \eta\tau)| \, dy d\tau dt = 0.
\]

Next, we note

\[
(5.20) \quad \lim_{\eta \to 0^+} \int_0^T \int_{-T}^T |u^+(t - \eta\tau) - u^+(t)| \, d\tau dt = 0.
\]

Also, taking \( \gamma_\eta = \frac{h(t - \eta\tau) - h(t)}{\eta} \), \( U_1^\eta(t, \tau, y) = |u(t - \eta\tau, h(t - \eta\tau) + \eta y) - u^+(t)| \), and \( U_2^\eta(t, \tau, y) = |u(t - \eta\tau, h(t) + \eta y) - u^+(t)| \), by a change of variables and the fundamental theorem of calculus, we obtain the bound

\[
(5.21) \quad \left| \int_0^T \int_{-T}^T \left( \int_0^Y U_1^\eta(t, \tau, y) \, dy - \int_0^Y U_2^\eta(t, \tau, y) \, dy \right) \, d\tau dt \right|
\]

\[
\leq 4\|u\|_{L^\infty} \int_0^T \int_{-T}^T |\gamma_\eta + \dot{h}(t)\tau| \, d\tau dt
\]

\[
\leq 4\|u\|_{L^\infty} \int_0^T \int_{-T}^T \frac{1}{\eta} \int_{t - \eta\tau}^{t + \eta\tau} |\dot{h}(s) - \dot{h}(t)| \, ds d\tau dt \to 0,
\]

where the right-hand side converges to 0 as \( \eta \to 0^+ \) by the Lebesgue differentiation theorem. Combining (5.19), (5.20), and (5.21), we obtain for any \( T > 0, Y > 0 \),

\[
(5.22) \quad \lim_{\eta \to 0^+} \int_0^T \int_{H_t^+ \cap [-T, T] \times [-Y, Y]} |u(t - \eta\tau, h(t) + \eta y) - u^+(t)| \, dy d\tau dt = 0.
\]

Thus, for any sequence \( \eta \to 0^+ \), by a diagonalization argument, we can find a subsequence along which (5.18) holds.

**Step 2: Uniform Convergence of Blow-ups**

In this step, we combine the \( L^1 \)-convergence result of the previous step with the partial regularity result of Proposition 1.2 to obtain for almost every \( t \) the blow-ups \( u_k \) converge away from a shock. In particular, the information we are concerned with will be located near either \((0, -1)\) or \((0, 1)\) in the local \((\tau, y)\)-coordinates. Therefore, we show that at almost every \((t, h(t))\), we can find a scale \( \alpha(t) \) so that near \((0, \pm 1)\) we avoid the possible shock at \( y = -\tau h(t) \) and can apply Proposition 1.2 at the scale \( \alpha(t) \), yielding the following lemma:
Lemma 5.2. Suppose $u$ is an entropy solution to (1.1) with $u = (\rho, m)$, $\|\rho\|_{L^\infty} + |m/\rho|_{L^\infty} + \|u\|_{L^\infty} < \infty$, and $\rho > 0$ everywhere. Suppose further that $\sigma \in \mathbb{R}$ and $V_i : \mathcal{V}_i \to \mathbb{R}$ is defined via

$$V_i(w) = \begin{cases} 
\min(\lambda_i(w), \sigma) & \text{if } i = 1 \\
\max(\lambda_i(w), \sigma) & \text{if } i = 2 
\end{cases}$$

Then, for any fixed Lipschitz $h : [0, T] \to \mathbb{R}$, there is an $0 < \alpha(t) < 1$ sufficiently small such that

$$\lim_{\eta \to 0^+} \sup_{(\tau, y) \in B_\alpha(\eta)(0, 1)} (V_1(u^+(t)) - V_1(u(t + \eta \tau, h(t) \pm \eta y)))_+ = 0 \tag{5.23}$$

$$\lim_{\eta \to 0^+} \sup_{(\tau, y) \in B_\alpha(\eta)(0, 1)} (V_2(u(t + \eta \tau, h(t) \pm \eta y)) - V_2(u^+(t)))_+ = 0, \tag{5.24}$$

for almost every $t \in [0, T]$.

Proof. We prove only the case $i = 1$. First, by Lemma 5.1, we have a sequence of scales $\eta_k \to 0^+$ such that for almost every $t$, the half-spaces $H^\pm_t = \left\{ (\tau, y) \mid y \leq \pm \tau \dot{h}(t) \right\}$ are defined, $\{y > 0\} \subset H^+_t$, $\{y < 0\} \subset H^-_t$, and for which the blow-ups $u_k := u(t + \eta_k \tau, h(t) + \lambda_k y)$ satisfy $u_k \to u^+(t)$ in $L^1_{\text{loc}}(H^+_t)$. Note, by translation and scaling invariance, $u_k$ is a solution to (1.1) in the $(\tau, y)$ variables. Moreover, each $u_k$ satisfies the same global $L^\infty$ bounds and uniform lower bound,

$$\inf_{(\tau, y) \in B_1} \rho(\tau, y) = \inf_{(\tau, y) \in B_1} \rho(\dot{\tau}, h(t)) > 0.$$  

Second, set $\lambda_1, k = \lambda_1(u_k)$. Then, because $\lambda_1$ is continuous (as a function on state space), certainly $\lambda_{1, k} \to \lambda_1(u^+(t))$ in $L^1_{\text{loc}}(H^+_t)$ for almost every $t$. Therefore, for almost every $t$, we pick $\alpha = \alpha(t)$ sufficiently small so that the balls $B_{2\alpha}(0, \pm 1)$ is compactly contained in $H^\pm$. Applying Proposition 1.2, we conclude

$$\lim_{k \to \infty} \sup_{B_a(0, \pm 1)} (\lambda_1(u^+(t)) - \lambda_{1, k})_+ = 0. \tag{5.26}$$

Finally, we use the specific structure of $V_1$ to obtain (5.23). Fix $\varepsilon > 0$ and note that (5.26) implies for $k$ sufficiently large, $\lambda_{1, k} \geq \lambda_1(u^+(t)) - \varepsilon$ on $B_a(0, \pm 1)$. Now, since $\min(\cdot, \sigma)$ is increasing,

$$V_1(u_k) = \min(\lambda_{1, k}, \sigma) \geq \min(\lambda_1(u^+(t)) - \varepsilon, \sigma) \geq \min(\lambda_1(u^+(t)), \sigma) - \varepsilon = V_1(u^+(t)) - \varepsilon,$$

which concludes the proof of (5.23) for $i = 1$. The proof for $i = 2$ is identical. \hfill \blacksquare

**Step 3: Existence of Generalized Characteristics**

In this step, we use Lemma 5.2 from the preceding step to prove Proposition 1.1. We will prove the proposition only for the case $i = 1$, i.e. the case of generalized 1-characteristics. The proof for $i = 2$ is nearly identical. Recall, in the case of $i = 1$, we are attempting to solve the ODE

$$\begin{align*}
\dot{h}(t) &= V(u(t, h(t))) \\
\dot{h}(0) &= x_0,
\end{align*} \tag{5.27}$$

where $V = \min(\lambda_1, \sigma)$ for $\sigma = \sup_{t, x} \lambda_1(t, x)$.

We begin by mollifying our ODE and using compactness arguments to construct our solution $h$. We begin by taking $\psi \in C^\infty_c(\mathbb{R})$ a fixed bump function satisfying $0 \leq \psi \leq 1$, $\int \psi = 1$, and $\text{supp}(\psi) \subset (0, 1)$. We then define the standard rescaling $\psi_\varepsilon(y) = \varepsilon^{-1} \psi(y/\varepsilon)$ and take $h^\varepsilon$ to be the global-in-time Lipschitz solution to the following approximating ODE:

$$\begin{align*}
\dot{h}^\varepsilon(t) &= \int_\mathbb{R} \int_\mathbb{R} V(u(t - \tau, h^\varepsilon(t) - y)) \psi_\varepsilon(y) \psi_\varepsilon(-\tau) \, dy \, d\tau, \\
\dot{h}^\varepsilon(0) &= x_0.
\end{align*} \tag{5.28}$$

Note, (5.28) immediately implies the uniform in $\varepsilon$ bound $\|\dot{h}^\varepsilon\|_{L^\infty} \leq \|V\|_{L^\infty}$. Therefore, combined with $h^\varepsilon(0) = x_0$, by the Banach-Alaoglu Theorem and Arzelà-Ascoli Theorem, there is a sequence $\varepsilon_k \to 0^+$ and a function $h : [0, T] \to \mathbb{R}$, such that $h^\varepsilon_k \to h$ uniformly and $h^\varepsilon_k \to h$ in $L^\infty$. In particular, from (5.28), we immediately deduce $h(0) = x_0$, $\|\dot{h}\|_{L^\infty} \leq \|V\|_{L^\infty}$, so that $h$ is Lipschitz. Because $h$ is Lipschitz, Corollary 1 allows us to conclude that for almost every $t$, either $u^+(t) = u^-(t)$ or $(u^+(t), u^-(t), \dot{h}(t))$ is an entropic shock.
In order to show the lower bound of (1.11), we pass to a subsequence of \( \varepsilon_k \) along which the conclusion of Lemma 5.2 holds. Now that we have constructed our shift \( h \), we fix some notation for the remainder of the section:

- First, we denote by \( u^+ = u^+(t) \) and \( u^- = u^-(t) \) the one-sided strong traces of \( u \) along our fixed Lipschitz \( h \).
- Second, we define \( V_{min}(t) = \min(V(u^+), V(u^-)) \).
- Third, we define \( \delta_k(t) = h_k(t) - h(t) \), which will track the convergence rate of the approximants locally in \( t \).

Thus, we have two sequences of scales, namely \( \varepsilon_k \), which is uniform in \( t \), and \( \delta_k \), which varies in \( t \). We now analyze two possible scenarios: first, when the convergence is fast, i.e. \( \delta_k(t) \leq \varepsilon_k \); and, second, when the convergence is slow, i.e. \( \varepsilon_k = o(\delta_k(t)) \).

First, we show (1.11), at a point \( t \), provided \( \delta_k := h_k(t) - h(t) \to 0 \) sufficiently quickly. To handle such points we need only the convergence provided by the strong trace via Lemma 5.1. Fix \( \alpha, K > 0 \) sufficiently large such that for \( k \geq K \), \( |\delta_k(t)| \leq \alpha \varepsilon_k \). Then, (5.28) combined with the properties of \( \psi_\varepsilon \) implies for sufficiently large \( k \),

\[
\left[ V_{min}(t) - \hat{h}_k(t) \right]_+ = \left[ \int_{\mathbb{R}} \int_{\mathbb{R}} (V_{min}(t) - V(u(t - \tau, h_k(t) - y))) \psi_\varepsilon_k(y) \psi_\varepsilon_k(-\tau) \, dy \, d\tau \right]_+ \\
\leq \frac{1}{\varepsilon_k^2} \int_{-\varepsilon_k}^{0} \int_{0}^{\varepsilon_k} \left[ V_{min}(t) - V(u(t - \tau, h_k(t) - y)) \right]_+ \, dy \, d\tau \\
\leq \frac{1}{\varepsilon_k^2} \int_{-\varepsilon_k}^{0} \int_{-\varepsilon_k}^{\varepsilon_k - \delta_k} \left[ V_{min}(t) - V(u(t - \tau, h(t) - y)) \right]_+ \, dy \, d\tau \\
= \int_{-1}^{0} \int_{-1}^{1+\alpha} \left[ V_{min}(t) - V(u(t - \varepsilon_k \tau, h(t) - \varepsilon_k y)) \right]_+ \, dy \, d\tau \\
\leq \int_{[-1,0] \times [-\alpha,1+\alpha] \cap H^0_+} \left[ V(u^-(t)) - V(u(t - \varepsilon_k \tau, h(t) - \varepsilon_k y)) \right]_+ \, dy \, d\tau \\
+ \int_{[-1,0] \times [-\alpha,1+\alpha] \cap H^1_-} \left[ V(u^+(t)) - V(u(t - \varepsilon_k \tau, h(t) - \varepsilon_k y)) \right]_+ \, dy \, d\tau.
\]

Now, because we chose \( \varepsilon_k \to 0 \) so that Lemma 5.1 holds, the right hand side converges to 0.

Second, we show (1.11), at a point \( t \), provided \( \delta_k := h_k(t) - h(t) \to 0 \) sufficiently slowly. To handle such points we need the \( L^\infty \) convergence provided by Lemma 5.2. Suppose, by possibly passing to a subsequence, that \( \delta_k(t) \) has a sign, say \( \delta_k(t) \geq 0 \). Then, a similar computation as in (5.29) yields

\[
\left[ V_{min}(t) - \hat{h}_k(t) \right]_+ \leq \frac{1}{\varepsilon_k^2} \int_{-\varepsilon_k}^{0} \int_{0}^{\varepsilon_k - \delta_k} \left[ V_{min}(t) - V(u(t - \tau, h(t) - y)) \right]_+ \, dy \, d\tau \\
= \frac{1}{\varepsilon_k^2} \int_{-\varepsilon_k}^{0} \int_{-\delta_k}^{\varepsilon_k - \delta_k} \left[ V_{min}(t) - V(u(t - \tau, h(t) - y)) \right]_+ \, dy \, d\tau \\
\leq \frac{\delta_k^2}{\varepsilon_k^2} \int_{-\varepsilon_k}^{0} \int_{-\delta_k}^{\varepsilon_k - \delta_k} \left[ V(u^-) - V(u(t - \delta_k \tau, h(t) - \delta_k y)) \right]_+ \, dy \, d\tau \\
\leq \sup_{(\tau,y) \in \left[ \frac{\delta_k}{\varepsilon_k}, 0 \right] \times \left[ -\delta_k, \frac{\delta_k}{\varepsilon_k} \right]} \left[ V(u^-) - V(u(t - \delta_k \tau, h(t) - \delta_k y)) \right]_+.
\]

Now, using \( \limsup_{k \to \infty} \frac{\delta_k}{\varepsilon_k} = 0 \), for \( k \) sufficiently large depending on \( t \), \( \frac{\delta_k}{\varepsilon_k} \leq \frac{\alpha(t)}{2} \), where \( \alpha(t) \) is from Lemma 5.2. Thus, Lemma 5.2 implies the right hand side of (5.30) converges to 0. A similar argument using \( u^+ \) in place of \( u^- \) works if \( \delta_k(t) \) has the opposite sign. Finally, if \( \delta_k \) does not have a sign, every subsequence has a further subsequence with a sign along which \( \lim_{k \to \infty} \left[ V_{min}(t) - \hat{h}_k(t) \right]_+ = 0 \).
Finally, we patch together the above arguments to obtain (1.11). Note that by passing to subsequences, together (5.29) and (5.30) imply for almost every \( t \in [0, T], \)

\[
\lim_{k \to \infty} \left[ V_{\min}(t) - \hat{h}_k \right]_+ = 0,
\]

for the full family \( \{h_k\} \). Thus, we use the dominated convergence theorem and the uniform bound \( \|\hat{h}_k\|_{L^\infty} \leq \|V\|_{L^\infty} \) to conclude for any \( \varphi \in C_c^\infty([0, T]) \) with \( \varphi \geq 0, \)

\[
\int_0^T \left( V_{\min}(t) - \hat{h}_k(t) \right) \varphi(t) \, dt \leq \int_0^T \left[ V_{\min}(t) - \hat{h}_k(t) \right]_+ \varphi(t) \, dt \to 0.
\]

Therefore, the pointwise convergence \( \hat{h}_k \xrightarrow{\ast} \hat{h} \) in \( L^\infty \) implies \( \hat{h}(t) \geq V_{\min}(t) \) in the sense of distributions and hence pointwise for almost every \( t \in [0, T] \).

We obtain the upper bound in (1.11) much more straightforwardly. We note \( \hat{h}_k \xrightarrow{\ast} \hat{h} \) in \( L^\infty([0, T]) \) and \( V \leq \text{ess sup}_{t,x} \lambda_1(t, x) \) implies that for any \( \varphi \in C_c^\infty([0, T]) \) with \( \varphi \geq 0, \)

\[
\int \varphi(\text{ess sup}_{t,x} \lambda_1(t, x) - \hat{h}) = \lim_{k \to \infty} \int \varphi(\text{ess sup}_{t,x} \lambda_1(t, x) - \hat{h}_k) \geq 0.
\]

Thus, \( \hat{h} \leq \text{ess sup}_{t,x} \lambda_1(t, x) \) in the sense of distributions and therefore pointwise.

6. Appendix

Here we prove the compactness lemma originally found as [40, Lemma 1.1]:

**Proof of Lemma 3.4.** We begin by denoting \( \rho_k = \int_{-L}^L f_k \, dv \) and \( m_k = \int_{-L}^L f_k \, dv \) so that \( \rho_k \to \rho \) and \( m_k \to m \) in \( L^1_{\text{loc}} \). Thus, extracting a subsequence, we have \( m_{nk} \to m \) and \( \rho_{nk} \to \rho \) converge pointwise almost everywhere. However, because \( f_{nk} \) is the characteristic function of an interval, the 0-th and 1-st moments (in \( v \)) of \( f_{nk} \) completely determine \( f_{nk} \) as

\[
(6.1) \quad f_{nk}(t, x, v) = \chi_{\left[ \frac{m_{nk} - \rho_{nk}}{\rho_{nk}}, \frac{m_{nk} + \rho_{nk}}{\rho_{nk}} \right]}(v).
\]

Therefore, the pointwise convergence is sufficient to conclude

\[
(6.2) \quad f_{nk} \to \chi_{\left[ \frac{m - \rho}{\rho}, \frac{m + \rho}{\rho} \right]}(v) := g(t, x, v).
\]

By Lebesgue dominated convergence, we conclude \( f_{nk} \to g \) in \( L^1_{\text{loc}} \). Now, since for any subsequence of \( f_n \), there is a further subsequence which converges to \( g \) in \( L^1_{\text{loc}} \), we conclude that the entire family \( \{f_n\} \) converges to \( g \) in \( L^1_{\text{loc}} \). Finally, because we also have \( f_n \xrightarrow{\ast} f \) in \( L^\infty \), we conclude \( f = g \) and we have demonstrated explicitly that \( f \) is a characteristic function of an interval (in \( v \)). \( \blacksquare \)

Here we provide a proof that solutions to a conservation law like (1.1) satisfy that if they have traces along a curve, then the trace is a shock or continuous. The result in the case of \( BV \) solutions is originally due to Dafermos, but we refer to [35] for a proof. This proof follows that of [30, Lemma 6]:

**Proof of Corollary 1.** Fix \( u \) a solution to (1.1), \( f \) the corresponding flux of (1.1), and \( h : [0, T] \to \mathbb{R} \) a Lipschitz curve. Pick \( \psi(x) \in C^\infty \) with \( 0 \leq \psi \leq 1 \) and \( \text{supp}(\psi) \subset (0, 1) \) and let \( \psi_\varepsilon(x) \) be the standard mollifier, \( \psi_\varepsilon(x) = \varepsilon^{-1} \psi \left( \frac{x}{\varepsilon} \right) \). Next, we pick \( \Psi_\varepsilon(x) = \int_x^{\infty} \psi_\varepsilon(y) - \psi_\varepsilon(-y) \). Then, \( \Psi_\varepsilon \) is smooth, compactly supported and by Fubini, converges to 0 in \( L^1(\mathbb{R}) \). Moreover, for any \( \varphi \in C_c^\infty([0, T]) \) and any \( v : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^m \) strong trace in the sense of Definition 1,

\[
(6.3) \quad \int_0^T \int_\mathbb{R} \varphi(t) (\psi_\varepsilon(x - h(t)) v(t, x) \, dx \, dt = \int_0^T \int_\mathbb{R} \varphi(t) \psi_\varepsilon(x) [v(t, h(t) + x) - v(t, h(t) - x)] \, dx \, dt = \int_0^T \int_\mathbb{R} \varphi(t) \psi_\varepsilon(x) [v(t, h(t) + \varepsilon x) - v(t, h(t) - \varepsilon x)] \, dx \, dt.
\]
Finally, the right hand side converges to
\[ \int_0^\infty [v^+(t) - v^-(t)] \varphi(t) \, dt \]
by the strong trace property. Therefore, testing the weak form of (1.1) with \( \varphi(t)\Psi(x - h(t)) \), to obtain
\[
0 = -\int_0^\infty \int_\mathbb{R} \partial_t \varphi(t) \Psi(x - h(t))u(t, x) \, dx \, dt \\
+ \int_0^\infty \int_\mathbb{R} \varphi(t)\Psi'(x - h(t))\dot{h}(t) \, dx \, dt \\
- \int_0^\infty \int_\mathbb{R} \varphi(t)\Psi'(x - h(t))f(u(t, x)) \, dx \, dt \\
:= I_1 + I_2 + I_3.
\]
We note first that because \( \Psi_\varepsilon \to 0 \) in \( L_1^1 \), \( I_1 \to 0 \). Second, because \( h \) is Lipschitz, \( \dot{h}(t)u(t, x) \) is strong trace in the sense of Definition 1 with traces \( \dot{h}(t)u^\pm(t) \). Therefore, (6.3) implies
\[
\lim_{\varepsilon \to 0^+} I_2 = \int_0^T \varphi(t)\dot{h}(t) [u^+(t) - u^-(t)] \, dt.
\]
Third, because \( f \) is continuous, \( f(u) \) is strong trace in the sense of Definition 1 by a simple compactness argument using the dominated convergence theorem. Then, (6.3) implies
\[
\lim_{\varepsilon \to 0^+} I_3 = \int_0^T \varphi(t) [f(u^+(t)) - f(u^-(t))] \, dt.
\]
Using the fundamental lemma of calculus of variations, we conclude the Rankine-Hugoniot jump condition,
\[
f(u^+(t)) - f(u^-(t)) = \dot{h}(t) [u^+(t) - u^-(t)] \quad \text{for almost every } t \in [0, T].
\]
A similar argument for a continuous entropy, entropy-flux pair \((\eta, q)\), yields the entropy inequality
\[
q(u^+(t)) - q(u^-(t)) \leq \dot{h}(t) [\eta(u^+(t)) - \eta(u^-(t))] \quad \text{for almost every } t \in [0, T].
\]
We therefore conclude for almost every \( t \), either \( u^+(t) = u^-(t) \) or \((u^-(t), u^+(t), \dot{h}(t))\) is an entropic shock. ■

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