Dynamical structure of Pure Lovelock gravity

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Abstract

We study dynamical structure of Pure Lovelock gravity in spacetime dimensions higher than four using the Hamiltonian formalism. The action consists of cosmological constant and a single higher-order polynomial in the Riemann tensor. Similarly to Einstein-Hilbert action, it possesses a unique constant curvature vacuum and charged black hole solutions. We analyze physical degrees of freedom and local symmetries in this theory. In contrast to the Einstein-Hilbert case, a number of degrees of freedom depends on the background and can vary from zero to the maximal value carried by the Lovelock theory.

1 Introduction

Lovelock-Lanczos gravity \cite{1,2} is a natural generalization of General Relativity to higher dimensions. It provides the most general gravity action yielding the second order field equations in the metric $g_{\mu\nu}(x)$. In a $(d+1)$-dimensional spacetime, the action is given by

$$I[g] = \int d^{d+1}x \sum_{k=0}^{[d/2]} \alpha_k L_k.$$  \hspace{1cm} (1.1)
Each term in the sum is characterized by the coupling constant $\alpha_k$ multiplied by the dimensionally continued Euler density $L_k$ of order $k$ in the curvature,

$$L_k = \frac{1}{2k} \sqrt{-g} \delta^{\mu_1 \ldots \mu_{2k}}_{\nu_1 \ldots \nu_{2k}} R^\nu_1^\mu_2 \cdots R^\nu_{2k-1}_k^\mu_{2k}. \quad (1.2)$$

Here $R^\alpha_{\beta\mu
u}$ is the Riemann curvature tensor and $\delta^{\mu_1 \ldots \mu_{2k}}_{\nu_1 \ldots \nu_{2k}}$ is the totally antisymmetric generalized Kronecker delta of order $k$ defined as the determinant of the $k \times k$ matrix $[\delta^{\mu_1}_{\nu_1} \delta^{\mu_2}_{\nu_2} \cdots \delta^{\mu_k}_{\nu_k}]$. This kind of action, polynomial in curvature, is of significant interest in theoretical physics because it describes a wide class of models. It has been shown in Refs. [3, 4] that, for arbitrary constants $\alpha_k$, a degeneracy may appear in the space of solutions because the metric is not fully fixed by the field equations. For instance, if the action has non-unique degenerate vacua, then the temporal component $g_{tt}$ of any static spherically symmetric ansatz remains arbitrary [5]. This problem can be avoided by a special choice of the coefficients $\alpha_k$. The most simple example is given by the Einstein-Hilbert (EH) term alone, which has the unique Minkowski vacuum. Presence of the positive or negative cosmological constant term makes the theory to have the unique de Sitter (dS) or anti-de Sitter (AdS) vacuum, respectively.

Another way to fix the coefficients $\alpha_k$ is to have a unique vacuum in the theory but degenerated, which leads to Chern-Simons gravity in odd dimensions and Born-Infeld gravity in even dimensions [6]. In those theories all couplings are expressed only in terms of the gravitational interaction and the cosmological constant. Also, choosing the coefficients up to a certain order $k = 1, \ldots, \lfloor d/2 \rfloor \equiv N$ leads to a family of non-equivalent theories whose black hole solutions were studied in [7] and also in [8] for the maximal case with $k = N$.

Recently, there has been suggested another possibility, where instead of the full Lovelock series only two terms in the sum are considered in the action: the cosmological constant and a polynomial in the curvature of order $p$. These Pure Lovelock (PL) gravities [9] remarkably admit non-degenerate vacua in even dimensions, while in odd dimensions they have a unique non-degenerate dS and AdS vacuum. Their black hole solutions are asymptotically indistinguishable from the ones appearing in General Relativity [5]. That is even though the action and equations of motion are free of the linear Einstein-Hilbert term. This similar asymptotic behavior of two theories seems to extend also to the level of the dynamics and a number of physical degrees of freedom in the bulk.

The properties of PL gravity have been discussed in the literature. Stability of PL black holes has been analyzed in [10]. Application of gauge/gravity duality to phase transitions in quantum field theories dual to Pure Gauss-Bonnet AdS gravity were studied in Ref. [12]. It can be shown that in any dimension $d + 1$ there is a special power $p$ such that the black hole entropy behaves as in any particular lower dimension. In case of the maximum power, $p = N$, such as five-dimensional Pure Gauss-Bonnet action, they exhibit a peculiar thermodynamical behavior [5, 11], where temperature and entropy bear the same relation to horizon radius as in the case for 3D and 4D dimensions, respectively. Thermodynamical parameters are thus universal in terms of horizon radius for all odd $D = 2N + 1$ and even $D = 2N + 2$ dimensions.

Dynamical aspects of PL theory were analyzed in Ref. [13] in terms of analogs of the Riemann and Weyl tensors for $N$th order PL gravity. It turns out that it is possible to define an $N$th order Riemann curvature with the property that trace of its Bianchi derivative yields the same divergence free (analogue of Einstein tensor) second rank tensor as the one obtained by the corresponding Lovelock polynomial action. Thus, one can obtain the gravitational equations for
PL gravity [14] [15] in the same way as one does for the Einstein equations from the Bianchi identity. However, there is one crucial difference, which is that the second Bianchi identity (i.e., vanishing of Bianchi derivative) is only satisfied by the Riemann tensor and not by its $N$th order analogue. The former has therefore a direct link to the metric, while for the latter this relation is more involved. What yields the divergence-free tensor is vanishing of the trace of Bianchi derivative, and not necessarily derivative itself. From this perspective, PL gravity could be seen as kinematic, which means that the $N$th order Riemann tensor is entirely given in terms of the corresponding Ricci tensor in all critical odd $D = 2N + 1$ dimensions, and it becomes dynamic in the even $D = 2N + 2$ dimensions. This might uncover a universal feature of gravitational dynamics in all critical odd and even dimensions, making it drastically different in critical odd dimensions. More precisely, the PL vacuum is flat with respect to $N$th order Riemann tensor, but not relative to Riemann tensor. This suggests that there are no dynamical degrees of freedom in the critical odd dimensions relative to the former but that may not be the case for the latter.

On the other hand it has been argued in Ref. [16] that the metric Lovelock theory should have the same number of degrees of freedom as the higher-dimensional Einstein-Hilbert gravity, namely $D(D - 3)/2$. This is different than expected from our previous discussion, which suggested fewer physical fields. However, a number of degrees of freedom can change with the backgrounds. For example, Lovelock-Chern-Simons gravity has different number of degrees of freedom in different sectors of the phase space [17] [18]. Due to non-linearity of the theory, the symplectic matrix might have different rank depending on the background [19] causing more symmetries and less degrees of freedom in some of them, what was explicitly demonstrated in Chern-Simons supergravity [20]. It can also happen that the constraints become functionally depended in certain symmetric backgrounds [21].

We wish therefore to provide a detailed analysis of the dynamical structure of PL theory by explicitly performing Hamiltonian analysis and exploring until what extent it is similar to General Relativity, and whether it exhibits any additional universal features.

2 Pure Lovelock gravity

We focus on Pure Lovelock gravity of order $p$ in $(d + 1)$-dimensions, whose action consists of the unique Lovelock term, $L_p$, and the cosmological constant $L_0$,

$$I[g] = -\kappa \int d^{d+1}x \sqrt{-g} \left( \frac{1}{2p} \delta^{\mu_1 \ldots \mu_{2p}}_{\nu_1 \ldots \nu_{2p}} R_{\mu_1 \nu_1}^{\mu_2 \nu_2} \cdots R_{\mu_{2p-1} \nu_{2p-1}}^{\mu_{2p} \nu_{2p}} - 2\Lambda \right) , \tag{2.1}$$

where $\alpha_p = -\kappa$ and $\alpha_0 = 2\kappa\Lambda$. The gravitational constant $\kappa$ has dimension $(\text{length})^{d+1-2p}$ and the cosmological constant has dimension of $(\text{length})^{-2p}$, and not $(\text{length})^{-2}$ as in General Relativity. Varying the action with respect to the metric $g_{\mu\nu}(x)$, one obtains equations of motion in the form

$$(p)G^\mu_\nu + \Lambda \delta^\mu_\nu = 0 , \tag{2.2}$$

where $\Lambda = 0$ or $\Lambda = \frac{(\pm 1)^p d!}{2(d - 2p)!^{1/2p}}$, and generalized Einstein tensor is symmetric of $p$-th order in the curvature,

$$(p)G^\mu_\nu = -\frac{1}{2p+1} \delta^{\mu_1 \ldots \mu_{2p}}_{\nu_1 \ldots \nu_{2p}} R^{\mu_1 \nu_1}_{\mu_2 \nu_2} \cdots R_{\mu_{2p-1} \nu_{2p-1}}^{\mu_{2p} \nu_{2p}} . \tag{2.3}$$
The fundamental fields in the first order formalism, vielbein $e^a_\mu (x)$ and the Riemann curvature tensor $R_{\mu\nu}^{ab}$, are related to the fields in the tensorial formalism through the relations $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$ and $\Gamma^\lambda_{\mu\nu} = \omega_{\mu}^{ab} e^a_\lambda e^b_\mu + e^2_\lambda \partial^a_{\mu} e^a_\nu - e^a_\lambda \partial^b_{\mu} e^a_\nu$, where $a, b = 0, 1, \ldots, d$ are the Lorentz indices. Note that the change of variables $(g, \Gamma) \to (e, \omega)$ is not unique, but is determined up to Lorentz rotations. With the new fields, we obtain Riemann curvature tensor $R_{\mu\nu}^{ab}$ and torsion tensor $T_{\mu\nu}^{a}$ as

$$R_{\mu\nu}^{ab} = \partial_a \omega_{\mu}^{b} - \partial_b \omega_{\mu}^{a} + \omega_{\nu}^{c} \omega_{\mu}^{bc} - \omega_{\nu}^{c} \omega_{\mu}^{bc} - \omega_{\mu}^{ab} \omega_{\nu}^{bc},$$

$$T_{\mu\nu}^{a} = D_{\mu} e^{a}_{\nu} - D_{\nu} e^{a}_{\mu},$$

where $D = D(\omega)$ is a covariant derivative with respect to the spin connection acting on the Lorentz indices only, e.g., $D_{\mu} e^{a}_{\nu} = \partial_{\mu} e^{a}_{\nu} + \omega^{a}_{\mu} e^{b}_{\nu}$. 

Naively, the first order PL action can be cast in the form

$$\bar{I}[e, \omega] = \int d^{d+1} x \left( \alpha_0 \mathcal{L}_0 + \alpha_p \mathcal{L}_p \right),$$

where we rescaled $\alpha_k \to -\alpha_k \frac{1}{(d+1-2k)!}$ and $\mathcal{L}_k \to -(d+1-2k)! \mathcal{L}_k$, and the Euler densities now become polynomials in $R$ and $e$,

$$\mathcal{L}_0 = \varepsilon_{a_1\cdots a_{d+1}} e^{a_1}_{\mu_1} \cdots e^{a_{d+1}}_{\mu_{d+1}} \sim e^{d+1},$$

$$\mathcal{L}_p = \frac{1}{2^p} \varepsilon_{a_1\cdots a_{d+1}} \varepsilon^{a_1\cdots a_{d+1}}_{\mu_1\mu_2} R^{a_1 a_2}_{\mu_1 \mu_2} \cdots R^{a_2 p+1}_{\mu_2 \mu_{2p}} e^{a_{2p+1}}_{\mu_{2p+1}} \cdots e^{a_{d+1}}_{\mu_{d+1}} \sim R^p e^{d+1-2p}.$$

Notation for the Levi-Civita symbol $\varepsilon^{a_1\cdots a_{d+1}}$ is given in Appendix A. The coupling constants become

$$\alpha_0 = \frac{2\Lambda \kappa}{(d+1)!}, \quad \alpha_p = -\frac{\kappa}{(d+1-2p)!}.$$
However, the field equations obtained from the action (2.10) after varying it in $e^a_\mu$ and $\omega^{ab}_\mu$ are, respectively,

$$
0 = \epsilon_{\alpha_1 \ldots \alpha_d} e^{\mu_1 \ldots \mu_d} \left( \frac{1}{2p} R^{\alpha_1 \alpha_2}_{\mu_1 \mu_2} \ldots R^{\alpha_{2p-1} \alpha_{2p}}_{\mu_{2p-1} \mu_{2p}} e^{\alpha_{2p+1} \ldots \alpha_d}_{\mu_{2p+1} \ldots \mu_d} + \frac{\alpha_0 (d+1)}{\alpha_p (d+1-2p)} e^{\alpha_1 \ldots \alpha_d}_{\mu_1 \ldots \mu_d} \right),
$$

(2.8)

$$
0 = \epsilon_{\alpha_1 \ldots \alpha_d} e^{\mu_1 \ldots \mu_d} \left( \frac{1}{2p} R^{\mu_2 \mu_3}_{\alpha_2 \alpha_3} \ldots R^{\mu_{2p-1} \mu_{2p} \mu_{2p+1} \ldots \mu_d}_{\alpha_{2p+1} \ldots \alpha_d} T^a_{\mu_2 \mu_3} e^{\alpha_{2p+1} \ldots \alpha_d}_{\mu_{2p+1} \ldots \mu_d} \right),
$$

(2.9)

These equations are not equivalent to the PL field Eqs. (2.2) because the Riemann spaces for which $T^a_{\mu \nu} = 0$ are not the only solutions of Eqs. (2.9) when $d+1 > 4$ and $p > 1$. Thus, treating $(e^a_\mu, \omega^{ab}_\mu)$ as independent fields changes the dynamics of the system. In order to use first order formalism and, at the same time, obtain field equations of Pure Lovelock gravity where $T^a_{\mu \nu} = 0$ is the unique solution, we introduce a Lagrange multiplier $\lambda^\mu_\nu$ that forces the torsion tensor to vanish through a constraint. The new action reads

$$
I[e, \omega, \lambda] = \int d^{d+1}x \left( \alpha_0 L_0 + \alpha_p L_p + \frac{1}{2} T^a_{\mu \nu} \lambda^\mu_\nu \right).
$$

(2.10)

The field $\lambda^\mu_\nu (x)$ is antisymmetric in the indices $[\mu \nu]$.

Although proposed action is explicitly torsionless, it does not imply that the equations of motion give the dynamics equivalent to the PL one. An example of the system where an addition of the constraint $T^a_\lambda a$ modifies the dynamics of a theory is Topologically Massive Gravity, where it introduces a term involving the Cotton tensor $^{[22][23][24]}$. There, the term with the multiplier has nontrivial implications on derivation of conserved charges $^{[25]}$. An influence of the multiplier, therefore, has to be well-understood on the level of the field equations.

The action (2.10) reaches an extremum on the equations of motion,

$$
\delta e^a_\mu : 0 = \epsilon_{\alpha_1 \ldots \alpha_d} e^{\mu_1 \ldots \mu_d} \left( \frac{\alpha_p}{2p} (d+1-2p) R^{\alpha_1 \alpha_2}_{\mu_1 \mu_2} \ldots R^{\alpha_{2p-1} \alpha_{2p}}_{\mu_{2p-1} \mu_{2p}} e^{\alpha_{2p+1} \ldots \alpha_d}_{\mu_{2p+1} \ldots \mu_d} \
+ \alpha_0 (d+1) e^{\alpha_1 \ldots \alpha_d}_{\mu_1 \ldots \mu_d} \right) + D_\nu \lambda^\mu_\nu,
$$

(2.11)

$$
\delta \omega^{ab}_\mu : 0 = \frac{1}{2p} \epsilon_{\alpha_1 \ldots \alpha_d} e^{\mu_1 \ldots \mu_d} R^{\alpha_1 \alpha_2}_{\mu_1 \mu_2} \ldots R^{\alpha_{2p-1} \alpha_{2p}}_{\mu_{2p-1} \mu_{2p}} e^{\alpha_{2p+1} \ldots \alpha_d}_{\mu_{2p+1} \ldots \mu_d} \
+ \frac{1}{2} \left( \epsilon_{\beta_0} \lambda^\mu_\nu - \epsilon_{\alpha_0} \lambda^\nu_\mu \right),
$$

(2.12)

$$
\delta \lambda^\mu_\nu : 0 = T^a_{\mu \nu}.
$$

(2.13)

In addition, the curvature and torsion tensors satisfy the First and Second Bianchi identities,

$$
D_\mu T^a_{\rho \sigma} + D_\rho T^a_{\sigma \mu} + D_\sigma T^a_{\mu \rho} = R^{ab}_{\mu \rho} \epsilon_{b \sigma} + R^{ab}_{\rho \sigma} \epsilon_{b \mu} + R^{ab}_{\sigma \mu} \epsilon_{b \rho},
$$

(2.14)

$$
D_\mu R^{ab}_{\rho \sigma} + D_\rho R^{ab}_{\sigma \mu} + D_\sigma R^{ab}_{\mu \rho} = 0.
$$

When the torsion tensor vanishes, the field equation (2.12) becomes

$$
0 = \epsilon_{\beta_0} \lambda^\mu_\nu - \epsilon_{\alpha_0} \lambda^\nu_\mu,
$$

(2.15)

from where $d(d+1)^2/2$ components of $\lambda^\mu_\nu$ can be solved as

$$
\lambda^\mu_\nu = 0.
$$

(2.16)
This result is obtained by rewriting (2.15) with the Lorentz indices as \( \lambda_{a, bc} - \lambda_{c, ba} = 0 \), and combining it with two other expressions obtained by performing the permutation of indices, which directly leads to \( \lambda_{a, bc} = 0 \) and therefore (2.16). Using (2.16), the last equation (2.11) is indeed equivalent to the Lovelock field equations in Riemann space. The Bianchi identities (2.14) in that case read

\[
R^a_{(\sigma \mu \rho)} = 0, \quad D(\mu)R^{ab}_{\rho \sigma} = 0.
\]

### 3 Action in the time-like foliation

Hamiltonian formalism is not explicitly covariant because it presents all the quantities in the time-like foliation \( x^\mu = (t, x^i) \), where \( x^0 = t \in \mathbb{R} \) is the temporal coordinate and \( x^i \) (\( i = 1, \ldots, d \)) are local coordinates at the spatial section \( \Sigma \).

In the tangent space, we decompose the indices as \( a = (0, \bar{a}) \). The vielbein \( e_a^i \) is invertible on \( \mathbb{R} \times \Sigma \) and its inverse is \( e_i^a \). We require that \( e_0^i \neq 0 \) and that the \( d \)-dimensional vielbein \( e_i^a \) is also invertible with the inverse

\[
(d) e_i^a = e_0^i - e_0^a e_i^0.
\]

In order to introduce canonical variables in the action (2.10), we have to define the action in configurational space, that is, in terms of the fields \( \lambda_a^\mu \), \( \omega_{ab}^\mu \) and its velocities \( \dot{\lambda}_a^\mu \), \( \dot{\omega}_{ab}^\mu \). To this end, we have the splitting of the fields in the time-like foliation

\[
e_\mu^a \rightarrow (e_\mu^a, e_\mu^0), \quad \omega_{ab}^\mu \rightarrow (\omega_{ab}^t, \omega_{ab}^i),
\]

and similarly for the multiplier \( \lambda_a^{\mu \nu} \rightarrow (\lambda_{ai}^t \equiv \lambda_{ai}^t) \). It is worthwhile noticing that \( \omega_{ab}^\mu \) transforms as a tensor of rank 2 under local Lorentz transformations on \( \Sigma \) and \( \omega_{ab}^\mu \) as the Lorentz gauge connection.

Since \( L = \int d^d x \mathcal{L} \), the Lagrangian scalar density of (2.11) can be written in a compact way,

\[
\mathcal{L} = \frac{1}{2} \dot{\omega}_{ab}^t S_{ab}^i + \dot{\omega}_{ab}^i S_a + \frac{1}{2} T_{ij}^a \lambda_a^{ij}.
\]

We neglect all boundary terms. In the action above, we introduce the quantities which do not depend on velocities and time-like components,

\[
\mathcal{L}_{ab} = \frac{p \alpha_0}{2p-2} \epsilon_{ab \cdots a_d} \epsilon_{t i_2 \cdots i_d} R_{i_2 i_3}^{a_1 \cdots a_d} \cdots R_{i_{2p-2} i_{2p-1}}^{a_2 \cdots a_d} e_{i_{2p}} e_{i_{2p+1}} \cdots e_{i_d},
\]

\[
S_a = \mathcal{H}_a = \mathcal{D}_t \lambda_a^0,
\]

\[
S_{ab} = \mathcal{H}_{ab} + e_{bi} \lambda_a^i - e_{ai} \lambda_b^i,
\]

where

\[
\mathcal{H}_a = \epsilon_{a a_1 \cdots a_d} \epsilon_{t i_1 \cdots i_d} (d + 1) \alpha_0 \epsilon_{i_1} \cdots \epsilon_{i_d} + \frac{p \alpha_0}{2p-2} (d + 1 - 2p) R_{i_1 i_2}^{a_1 a_2} \cdots R_{i_{2p-2} i_{2p-1}}^{a_{2p-1} a_{2p}} e_{i_{2p+1}} e_{i_{2p+2}} \cdots e_{i_d},
\]

\[
\mathcal{H}_{ab} = \frac{p \alpha_0}{2p-2} (d + 1 - 2p) \epsilon_{ab \cdots a_d} \epsilon_{t i_2 \cdots i_d} R_{i_2 i_3}^{a_1 \cdots a_d} \cdots R_{i_{2p-2} i_{2p-1}}^{a_2 \cdots a_d} T_{i_{2p+1}}^{a_{2p+1}} e_{i_{2p+2}} e_{i_{2p+3}} \cdots e_{i_d}.
\]

The Lagrangian (3.2) is similar to the one of Chern-Simons theory, whose Hamiltonian analysis was studied in Ref. [18].
4 Hamiltonian analysis in five dimensions

Let us start with the simplest case of five-dimensional Pure Gauss-Bonnet action \((d = 4, p = 2)\),
\[ I = \int d^5x \left[ \epsilon_{abcde} \epsilon^{\mu
u\rho\sigma\gamma} \left( \alpha_0 \epsilon_{\mu
u}^{ab} \epsilon_{\rho\sigma}^{ce} d^e_{\gamma} + \frac{\alpha^2}{4} R_{\mu
u}^{ab} R_{\rho\sigma}^{cd} e^e_{\gamma} \right) + \frac{1}{2} T^a_{\mu\nu} \lambda_a^{\mu
u} \right]. \] (4.1)
The Lagrangian has the form \((5.2)\) with particular tensors
\begin{align*}
L_{ab} &= 2\alpha_2 \epsilon^{ijkl} \epsilon_{abcde} R^{cd}_{jkl} e^e_i, \\
S_{ab} &= \mathcal{H}_{ab} + \epsilon_{bi} \lambda^i_a - \epsilon_{ai} \lambda^i_b, \\
S_a &= \mathcal{H}_a + D_t \lambda^i_a, \\
\mathcal{H}_{ab} &= \alpha_2 \epsilon_{abcde} \epsilon^{ijkl} R^{cd}_{jkl}, \\
\mathcal{H}_a &= \epsilon_{abcde} \epsilon^{ijkl} \left( 5\alpha_0 \epsilon^b_{ie} \epsilon^c_{de} d^e_{\gamma} + \frac{\alpha^2}{4} R^{bc}_{ij} R^{de}_{kl} \right),
\end{align*}
and the multipliers are conveniently written as
\[ \lambda^i_a = \frac{1}{3!} \epsilon^{ijkl} \lambda_{a, jkl}, \quad \lambda^i_a = \frac{1}{2!} \epsilon^{ijkl} \lambda_{a, tkl}. \] (4.3)

If we denote the generalized coordinates by \(q^M(x)\) and the corresponding conjugated momenta by \(\pi_M(x)\),
\[ q^M = \{ \epsilon^i_t, \epsilon^i_t, \omega^a_b, \omega^b_a, \lambda^i_a, \lambda^i_a \}, \quad \pi_M = \{ \pi^i_a, \pi^i_a, \pi^i_ab, \pi^i_ab, \pi^i_ab, \pi^i_ab \}, \] (4.4)
we can use the definition \(\pi_M = \frac{\partial L}{\partial \dot{q}^M}\) to find \(\pi_{ab} = L_{ab}^i\) and \(\pi_a^i = \lambda_a^i\), while all other momenta are zero. Thus, the Hessian matrix \(\frac{\partial^2 L}{\partial q^M \partial \dot{q}^M}\) is not invertible and we cannot express all velocities in terms of the momenta. In turn, we get the constraints, called

**Primary constraints:** \[ \Phi_M = \{ \phi^i_a, \phi^i_ab, \phi^i_ab, \pi^i_ab, \pi^i_ab \}. \] (4.5)
They are defined on the phase space as
\begin{align*}
\phi^i_a &= \pi^i_a \approx 0, \\
\phi^i_ab &= \pi^i_ab \approx 0, \\
p^i_ab &= \pi^i_ab \approx 0, \\
p^i_ab &= \pi^i_ab \approx 0.
\end{align*}
(4.6)
The surface \(\Phi_M \approx 0\) in the phase space is called the primary constraint surface, \(\Gamma_P\). The weak equality \(f(q, \pi) \approx 0\) on \(\Gamma_P\) implies that a phase space function \(f\) vanish on \(\Gamma_P\), but its derivatives (variations) are non-vanishing. This is different than the strong equality, \(f(q, \pi) = 0\), where both \(f\) and its variations vanish on \(\Gamma_P\). This distinction is relevant for definition of Poisson brackets, since \(f \approx 0\) does not imply \(\{f, \cdots\} \approx 0\).

To simplify notation, we write the arguments of the phase space functions symbolically, assuming that all quantities are defined at the same instant, \(x^0 = x'^0 = t\),
\begin{align*}
A &= A(x), \quad B' = B(x'), \quad \partial_i = \frac{\partial}{\partial x^i}, \quad \partial'_i = \frac{\partial}{\partial x'^i}, \\
\delta &= \delta (\vec{x} - \vec{x}'), \quad \delta_{ab}^{cd} = \delta^a_c \delta^d_b - \delta^a_d \delta^c_b. \quad (4.7)
\end{align*}
The fundamental Poisson brackets (PBs) different than zero are

\[
\{e^{a}_{\mu}, \pi^{\nu}_{b}\} = \delta^{a}_{b} \delta^{\nu}_{\mu} \delta, \\
\{\omega^{ab}_{\mu}, \pi^{\nu}_{cd}\} = \delta^{a}_{c} \delta^{b}_{d} \delta^{\nu}_{\mu} \delta, \\
\{\lambda^{i}_{a}, p^{b}_{j}\} = \delta^{a}_{b} \delta^{i}_{j} \delta, \\
\{\lambda^{ij}_{a}, p^{b}_{kl}\} = \delta^{a}_{b} \delta^{ij}_{kl} \delta.
\] (4.8)

The symplectic matrix $\Omega_{MN}$ of the primary constraints reads

\[
\{\Phi_{M}, \Phi'_{N}\} = \Omega_{MN} \delta,
\] (4.9)

and it is antisymmetric, $\Omega_{MN} = -\Omega_{NM}$. The only (independent) submatrices of the symplectic matrix different than zero are

\[
\{\phi^{ij}_{ab}, \phi'_{cd}\} = \Omega^{ij}_{abcd} \delta = -8\alpha^{2} \epsilon_{ijkl} \epsilon_{abcde} T^{e}_{kl} \delta, \\
\{\phi^{i}_{ab}, \phi'_{c}\} = \Omega^{i}_{abc} \delta = -2\alpha^{2} \epsilon_{ijkl} \epsilon_{abcde} R^{de}_{kl} \delta, \\
\{\phi^{i}_{a}, p'_{b}\} = -\delta^{i}_{b} \delta_{a} \delta.
\] (4.10)

The canonical Hamiltonian, $H_{C} = \pi_{M} q^{M} - \mathcal{L}$, defined on $\Gamma_{P}$ is

\[
H_{C}(p, q) = -\frac{1}{2} \omega^{ab}_{t} S_{ab} - e^{a}_{t} S_{a} - \frac{1}{2} T^{i}_{ij} \lambda_{ij}^{a},
\] (4.11)

and the total Hamiltonian, defined on the full phase space $\Gamma$, is obtained by introducing the indefinite multipliers $u^{M}(x)$,

\[
H_{T}(p, q, u) = H_{C}(p, q) + u^{M} \Phi_{M}(p, q),
\] (4.12)

where $u^{M} = \{u^{a}_{t}, u^{a}_{c}, u^{ab}_{t}, u^{ab}_{c}, v^{i}_{a}, v^{ij}_{a}\}$. Evolution of any quantity $A(q(x), \pi(x)) = A(x)$ in the phase space is given by

\[
\dot{A} = \int d\vec{x}\left(\{A, H_{C}'\} + u^{M} \{A, \Phi'_{M}\}\right) \\
\approx \int d\vec{x} \left\{A, H_{T}'\right\}.
\] (4.13)

This allows us to identify some field velocities with the Hamiltonian multipliers,

\[
\dot{\omega}^{ab}_{t} = u^{ab}_{t}, \quad \dot{e}^{a}_{t} = u^{a}_{t}, \\
\dot{\lambda}^{ij}_{a} = v^{ij}_{a}, \quad \dot{\lambda}^{i}_{a} = v^{i}_{a}.
\] (4.14)

Consistency of the theory requires that the primary constraints remain on the constraint surface during their evolution, that is,

\[
\dot{\Phi} = \int d\vec{x}\{\Phi, H_{C}'\} + \Omega_{MN} u^{N} \approx 0.
\] (4.15)

These consistency conditions will either solve some multipliers, or lead to the secondary constraints, or will be identically satisfied.
When the symplectic matrix has zero modes and \( \{ \Phi_M, \mathcal{H}_C \} \neq 0 \), the consistency conditions lead to the secondary constraints,

\[
\dot{\phi}_a^i = S_a \approx 0, \tag{4.16}
\]

\[
\dot{\phi}_a^{ab} = S_{ab} \approx 0, \tag{4.17}
\]

\[
\dot{p}_{ij}^a = T_{ij}^a \approx 0. \tag{4.18}
\]

Other consistency conditions solve Hamiltonian multipliers, such as \( \dot{p}_i^a \approx 0 \), which gives

\[
u_i^a = D_t \epsilon_i^a - \omega_i^{ab} e_{bi}. \tag{4.19}\]

On the other hand, from \( \dot{\phi}_a^{ij} \approx 0 \) we solve the multiplier,

\[
v_a^i = -\epsilon_{abcd} \epsilon^{ijkl} \left[ 20 \alpha_0 \epsilon^b_i \epsilon^c_j \epsilon^d_k \epsilon^e_l + \alpha_2 R_{kl}^{de} \left( u_j^b \cdot D_j^c \omega_i^e \right) + \omega_i^b \lambda_j^a + D_j \lambda_i^{aj} \right]. \tag{4.20}\]

Using the Bianchi identities, \( D_j \epsilon_{abcd} = 0 \) and the property that any totally antisymmetric tensor of rank 6 defined in five dimensions must vanish, that is,

\[-\epsilon_{abcd} \epsilon_{efij} + \epsilon_{defa} \epsilon_{eij} \approx \epsilon_{efab} \epsilon_{cij} + \epsilon_{eabc} \epsilon_{efij} = 0, \]

the last consistency condition for \( \dot{\phi}_a^{ij} \) becomes

\[0 \approx \dot{\phi}_a^{ij} \approx n_{at} \lambda_a^i - \lambda_{bi} \lambda_a^j + \lambda_{aj} e_{bi}. \tag{4.21}\]

One can show, in a similar way as for Eq. (4.16), that the constraints (4.17) and (4.21) are now equivalent to zero multipliers \( \lambda_i^a \approx 0 \) and \( \lambda_{ij}^a \approx 0 \).

So far, we have found the following

Secondary constraints: \( S_a \approx 0, \quad T_{ij}^a \approx 0, \quad \lambda_i^a \approx 0, \quad \lambda_{ij}^a \approx 0, \tag{4.22}\)

and we determined the multipliers \( u_i^a \) and \( v_i^a \). The functions \( \{ u_i^a, u_i^{ab}, u_i^{ij}, v_i^a \} \) remain arbitrary. A submanifold \( \Gamma_S \subset \Gamma \) defines the secondary constraint surface, where all constraints discovered until now vanish.

To ensure that the secondary constraints \( \lambda \) evolve on the constraint surface \( \Gamma_S \), we require that \( \dot{\lambda}_i^a = v_i^a \) and \( \dot{\lambda}_{ij}^a = v_{ij}^a \) vanish. It leads to \( v_i^a = 0 \), which by Eq. (4.20) can be equivalently expressed as

\[
\lambda_i^a = -\epsilon_{abcd} \epsilon^{ijkl} \left[ 20 \alpha_0 \epsilon^b_i \epsilon^c_j \epsilon^d_k \epsilon^e_l + \alpha_2 R_{kl}^{de} \left( u_j^b \cdot D_j^c \omega_i^e \right) \right] \approx 0 \quad \text{and} \quad v_i^a = 0. \tag{4.23}\]

Before we continue, we can notice that the pairs of conjugated variables \( (\lambda, p) \), all being the constraints, have their PB’s whose r.h.s (symplectic form) is invertible on \( \Gamma_S \). Thus, they are second class constraints that do not generate any symmetry, but represent redundant, non-physical quantities. They can be eliminated by defining the reduced phase space \( \Gamma^* \) with the Poisson brackets replaced by the Dirac brackets,

\[
\{ A, B' \}^* = \{ A, B' \} + \int dy \left[ \{ A, \lambda_i^a(y) \} \{ p_i^a(y), B' \} - \{ A, p_i^a(y) \} \{ \lambda_i^a(y), B' \} \right.
\]

\[+ \frac{1}{2} \{ A, \lambda_{ij}^a(y) \} \{ p_{ij}^a(y), B' \} - \frac{1}{2} \{ A, p_{ij}^a(y) \} \{ \lambda_{ij}^a(y), B' \} \]. \tag{4.24}

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It is straightforward to check (and it is a general property of the Dirac brackets) that the use of \( \{,\}^* \) turns the weak equality into the strong equality on \( \Gamma^* \),

\[
\begin{align*}
\lambda^i_\alpha &= 0, \quad p^a_i = 0, \quad \text{on } \Gamma^*, \\
\lambda^a_\alpha &= 0, \quad p^a_{ij} = 0, \quad \text{on } \Gamma^*,
\end{align*}
\]

because \( \{\lambda^a_\alpha, p^a_{ij}\}^* = 0 \) and \( \{\lambda^a_{ij}, p^a_{ikl}\}^* = 0 \) on \( \Gamma^* \). The remaining generalized coordinates of the space \( \Gamma^* \) are (\( e^a_\mu, \omega^{ab}_\mu, \pi^a_\alpha, \pi^{ab}_\mu \)) and their Dirac brackets remain unmodified (they are equal to the Poisson brackets). From now on we drop the star from the Dirac brackets.

Let us analyze the consistency condition of \( S_\alpha \). Using \( \bar{R}^{abc}_{ij} = D_i u^b_j - D_j u^b_i \) and \( \dot{\epsilon}_i^a = u_i^a \), we get

\[
\dot{S}_\alpha = \epsilon_{abcde} \varepsilon^{ijkl} \left[ 20 \alpha_0 D_i e^{b}_{j} e^{c}_{k} e^{d}_{l} + \alpha_2 D_i U^b_i e^{c}_{k} e^{d}_{l} + \omega^{ef}_{i} \left( 20 \alpha_0 e^{f}_{j} e^{e}_{k} e^{d}_{l} + \alpha_2 R^{f}_{ijkl} R^{de}_{ijkl} \right) \right],
\]

where we denoted

\[
U^a_i = u^a_i - D_i \omega^a_i,
\]

and used that \( [D_i, D_j] \omega^b_{ij} = R^b_{ijkl} \omega^c_{kl} - R^f_{ijkl} \omega^f_{ij} \). It can be recognized from the Lagrangian formalism that \( U^a_i = R^a_i \), because the Hamiltonian prescription treats all time derivatives as new functions. Next, we use a combinatorial identity, valid for any completely antisymmetric tensor \( \Sigma^{cdef} \),

\[
0 = D_t \epsilon_{acdef} \Sigma^{cdef} = \left( \epsilon_{acdef} \omega^b_{ta} + 2 \epsilon_{abcdef} \omega^b_{te} + 2 \epsilon_{acdef} \omega^b_{te} \right) \Sigma^{cdef}.
\]

For a particular choice of \( \Sigma^{fcd} = 20 \alpha_0 \epsilon^{f}_{j} e^{e}_{k} e^{d}_{l} + \alpha_2 R^{f}_{ijkl} R^{de}_{ijkl} \), we obtain that \( S_\alpha \) does not leave the surface \( \Gamma_S \) during its evolution,

\[
\dot{S}_\alpha = -D_i \dot{\lambda}^i_\alpha - \dot{\omega}^a_i \dot{S}_a \approx 0.
\]

Furthermore, we also have to require the same for the torsion tensor,

\[
\dot{T}^a_{ij} = D_i u^a_j - D_j u^a_i + u_i^b e_{bij} - u_j^b e_{bi} \approx 0.
\]

With the help of Eq. (4.19), we rewrite the last equation as

\[
0 \approx \dot{T}^a_{ij} \approx R^{ab}_{ij} e_{bt} + U^a_{ji} - U^a_{ij}.
\]

Here the vielbein projects the Lorentz indices to the spacetime ones, \( U^a_{ji} = U^a_{ij} e_{bij} \). The above equation gives 30 algebraic equations in 40 unknown functions \( U^a_{ij} \), which can be decomposed into 16 + 4 components \( (U^0_{ij}, U^a_{ij}) \). The final solution is

\[
U^a_{ij} = \frac{1}{2} R^{ab}_{ij} e_{bt} = \frac{1}{2} R^a_{ti} \quad \Rightarrow \quad U^\mu_{[ij]} = \left( 0, \frac{1}{2} R^k_{tij} \right).
\]

In that way, the 6+4 coefficients \( (U^0_{ij}, U^a_{ij}) \) become completely determined by the consistency of \( \dot{T}^a_{ij} \) and \( \dot{T}_{[kij]} \). Since \( U^a_{ij} = R^a_{itij} \), the above relation just represents the first Bianchi identity for the components \( (tij) \) rederived in the Hamiltonian way.
The 20 components of $\dot{T}_{(k)}^{ij}$ that do not solve the corresponding multipliers are exactly the ones symmetric in first two indices,

$$
\dot{T}_{(k)}^{ij} \approx \frac{1}{2} \left( e_{ak} \dot{T}_{i}^{aj} + e_{ai} \dot{T}_{k}^{aj} \right) \\
\approx \frac{1}{2} R_{kij} + U_{k[ji]} + \frac{1}{2} R_{tkj} + U_{i[jk]} = 0,
$$

that vanish due to the known $U_{i[jk]}$. Thus, these components do not lead to new conditions. We conclude that 30 equations $\dot{T}_{(k)}^{ij} = 0$ solve only 10 antisymmetric components $u_{i}^{ab}$ and remaining 20 equations do not give anything new – they are automatically satisfied.

Thanks to the relation (4.31) and because the curvature $R_{ijkl}^{ab}$ satisfies the First Bianchi identity, we can collect all First Bianchi identities in a covariant way,

$$
B_{\mu \nu \alpha \beta} = R_{\mu \nu \alpha \beta}^{a} = 0,
$$

where the components of the tensor $B$ are

$$
0 = B_{ijk}^{a} \equiv R_{ijk}^{a}, \\
0 = B_{tij}^{a} \equiv e_{bt} R_{tij}^{ab} - 2 U_{[ij]}^{a}.
$$

This has an important consequence on the number of linearly independent multipliers $U_{i}^{ab}$. Namely, we can prove that

$$
R_{\mu \nu \alpha \beta} - R_{\alpha \beta \mu \nu} = \frac{1}{2} \left( B_{\mu \nu \alpha \beta} + B_{\beta \mu \nu \alpha} - B_{\alpha \beta \mu \nu} - B_{\nu \alpha \beta \mu} \right) = 0,
$$

so the Riemann curvature is symmetric, $R_{\mu \nu \alpha \beta} = R_{\alpha \beta \mu \nu}$, or

$$
R_{tij} = R_{tji}, \quad R_{tijk} = R_{jkti}, \quad R_{ijkt} = R_{kijt}.
$$

The last relation in (4.35) does not give any further information because it is just the Bianchi identity on $\Sigma$. The first one, instead, shows that not all coefficients $U_{ij} = e_{ai} U_{i}^{a}$, are independent because $U_{tij}$ are symmetric, $U_{tij} = U_{tji} = e_{at} e_{bi} R_{tij}^{ab}$. The second condition in (4.35) is equivalent to

$$
U_{jki} = e_{at} e_{bi} R_{jk}^{ab},
$$

in a way consistent with (4.31). The only remaining unknown multipliers are 10 symmetric components $U_{t(ij)}$, leading to the final expression for $U_{i}^{ab}$ as

$$
U_{i}^{ab} = U_{t(ij)} e_{a}^{\mu} e_{b}^{\nu} = U_{t(ij)} \left( e_{ia} e_{jb} - e_{ib} e_{ja} \right) + e_{at} e_{di} R_{jk}^{ed} e_{ia} e_{kb}, \quad U_{tij} = U_{tji}.
$$

From the point of view of the irreducible components of $U_{i}^{ab}$, we can see the 10 components of $U_{t(ij)}$ as the only unsolved part in the table below,

| Multiplier $U_{\mu ij}$ | $U_{t(ij)}$ | $U_{tk}^{k}$ | $SU_{tij}$ | $AU_{kij}$ | $SU_{kij}$ | $TU_{kij}$ |
|-------------------------|-------------|--------------|------------|------------|------------|------------|
| 40 components           | 6           | 1            | 9          | 4          | 4          | 16         |
| Solved by               | $\dot{T}_{ij}^{0}$ | arbitrary | arbitrary | $\dot{T}_{[ijk]}^{1}$ | Bianchi | Bianchi. |

As it is well-known, the irreducible components of the rank 2 tensor $U_{tij}$ are: its antisymmetric part $U_{t[ij]}$, the trace $U_{tk}^{k}$ and the symmetric traceless component $SU_{tij} = U_{t(ij)} - \frac{1}{4} g_{ij} U_{tk}^{k}$. 

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On the other hand, the irreducible components of the rank 3 tensor $U_{i(jk)}$ are: its vectorial component (trace) $U_i \equiv U_{ij} j$, also written as $SU_{ijk} = g_{ij} U_k + g_{ik} U_j$, the axial-vector component $A U_{ijk} = U_{[ijk]}$ and the tensorial one $T U = U - A U - SU$.

The last equation to analyze is $\chi_a^i \approx 0$. It can be combined together with $H_a \approx 0$ into

$$\chi_a^i = \left( H_a, \chi_a^i \right) = \epsilon_{abcd}\epsilon^{\lambda\mu\nu\alpha\beta} \left( 5\alpha_0 \epsilon_{\mu}^{\beta} \epsilon_{\nu}^{c} \epsilon_{\alpha}^{d} + \frac{\alpha_2}{4} R_{\mu
u}^{bc} R_{\alpha\beta}^{de} \right) \approx 0 \ ,$$

in which we recognize the generalized Einstein equations with cosmological constant \( \Omega_a \). In contrast to the Einstein-Hilbert case, the multipliers in Eq. (4.23) cannot be fully solved because they are non-linear in the fields, causing ambiguities. In fact, if we write it as

$$2\alpha_2 \epsilon_{abcd} \epsilon^{ijkl} R_{kl}^{de} U_j^{bc} = -4\alpha_0 \epsilon_{abcd} \epsilon^{ijkl} \epsilon_{l}^{b} \epsilon_{k}^{e} \epsilon_{l}^{e},$$

then the rank of the matrix $\Omega_{abc} = -2\alpha_2 \epsilon_{abcd} \epsilon^{ijkl} R_{kl}^{de}$ explicitly depends on a considered background. More concretely, replacing the solution for the multipliers (4.37) in (4.39), we obtain a set of algebraic equations

$$M_{a(ijm)} U_{tij} = A_a^i \ , \quad \text{or} \quad MU = A \ ,$$

(4.40)

where the matrix of the system is obtained by symmetrization of

$$M_{\mu}^{ijm} = -\Omega_{abc} \epsilon_{\mu}^{a} \epsilon_{\tau}^{b} \epsilon_{mc} = \frac{2\alpha_2}{|e|} g_{\mu\tau} \epsilon_{mn\nu\lambda} \epsilon^{ijkl} R_{kl}^{n\lambda} R_{ij}^{m\nu} .$$

(4.41)

The non-homogeneous part of the system is

$$A_{\mu}^i = |e| \left( 120\alpha_0 \delta_{\mu}^i - \alpha_2 \epsilon_{\mu\nu\lambda} \epsilon^{ijkl} R_{kl}^{\nu\lambda} R_{ij}^{\mu\nu} \right) .$$

(4.42)

In the context of the equation (4.40), $M$ is the $20 \times 10$ matrix that acts on the 10-component column $U$. When the rank of $M$ is maximal, that is 10, then all components of $U$ can be determined. This is the case of the AdS space, as we will show below. A situation is completely different in the flat space, where the equation becomes homogeneous ($A = 0$) and the rank of $M = 0$ is zero. In that case, all 10 components of the vector $U_{t(ijm)}$ remain arbitrary. In EH case this matrix always has maximal rank because it does not depend on the curvature. In higher-dimensional PL gravity, $M$ is again polynomial in the curvature and may have different ranks. It is a generic feature of the Lovelock gravity, and it has already been noted for the Lovelock-Chern-Simons case [18].

We are interested in the $\Lambda \neq 0$ backgrounds, where there are black hole solutions. We restrict our theory to the part of the phase space where the rectangular $20 \times 10$ matrix $M_{a(ijm)}(x)$ has maximal rank, 10, for all $x$. In that case, there exists only the left inverse of $M$, which is the $10 \times 20$ matrix $\Delta_{a(kl)i}(x)$ of the rank 10 defined by

$$\frac{1}{2} \Delta_{a(kl)i} M_{a(ijm)} = \delta_{k}^{i} \delta_{l}^{m} + \delta_{l}^{i} \delta_{k}^{m} .$$

(4.43)

The matrix $\Delta$ depends on $\epsilon_{\mu}^{a}$ and $\omega_{\mu}^{ab}$. Then the equation (4.40) can be solved and the multipliers are

$$U_{tij} = \Delta_{a(ijk)} A_{a}^{k} .$$

(4.44)
To show that the chosen subspace contains a non-empty set of solutions, we consider the AdS background

$$\hat{R}_{\mu\nu\alpha\beta} = -\frac{1}{\ell^2} \left( \tilde{g}_{\mu\alpha} \tilde{g}_{\nu\beta} - \tilde{g}_{\nu\alpha} \tilde{g}_{\mu\beta} \right),$$

(4.45)

with $\alpha_0 = \frac{1}{5\ell^4}$ and $\alpha_2 = -1$. Then

$$\bar{M}^{i(jm)}_{\mu} = \frac{4}{\ell^2 |\epsilon|} \tilde{g}^{\mu} \tilde{g}_{\mu} \left( \tilde{g}^{jm} \tilde{g}^{mn} + \tilde{g}^{ij} \tilde{g}^{nm} - 2 \tilde{g}^{in} \tilde{g}^{jm} \right),$$

(4.46)

where $\tilde{g} = \det[\tilde{g}_{kl}]$ is the determinant of the spatial background induced metric $\tilde{g}_{ij} = \tilde{g}_{ij}$ and its inverse is $\tilde{g}^{ij} = \tilde{g}^{ij} - \tilde{g}^{is} \tilde{g}^{sj} / \tilde{g}^{tt}$.

Now we linearize Eq. (4.40) around this background, i.e., $(\bar{M} + \delta M)(\bar{U} + V) = \bar{A} + \delta A$, where $\delta U_{i(jm)} = V_{jm}$. We multiply the zero order, $\bar{M} \bar{U} = \bar{A}$, by $\bar{e}_a^a$ and obtain

$$\bar{U}_{ij} = \frac{5\alpha_0 \ell^2 \tilde{g} \tilde{g}_{ij}}{\alpha_2} = -\frac{1}{\ell^2} \tilde{g} \tilde{g}_{ij},$$

(4.47)

where we replaced the values of the constants $\alpha_k$. We also used the identity $|\epsilon|^2 = -\tilde{g} = -\tilde{g}_{tt}$.

Projecting $M \bar{U} = \bar{A}$ by $\bar{e}_a^a$, we find that (4.47) is satisfied. One can obtain the same result from the definition $\bar{U}_{ij} = \bar{R}_{ij}$ coming from Eqs. (4.27) and (4.35).

The linear order equation, $M \bar{U} + \delta M \bar{A} = \delta A$, projected by $\bar{e}_a^a$ reads

$$M^{i(jm)}_{\mu} V_{jm} = C_{\mu}^i,$$

(4.48)

where we defined $C_{\mu}^i = -\delta M^{i(jm)}_{\mu} \bar{U}_{jm} + \bar{e}_a^a \delta \bar{A}_a^j$. After replacing the matrix $M$, see Eq. (4.46), we find

$$V_{i}^j - \delta_{ij} V_{j}^i = \frac{|\epsilon| \ell^2}{2 (\tilde{g})} C_{\mu}^i.$$  

(4.49)

In that way, all 10 symmetric multipliers $U = \bar{U} + V$ are uniquely solved in the AdS background with

$$V_{i}^j = \frac{|\epsilon| \ell^2}{2 (\tilde{g})} \left( C_{ij}^{\mu} - \frac{1}{3} \delta_{ij} C_{k}^{k} \right).$$

(4.50)

It is straightforward to check that the remaining equations, $\bar{M}^{i(jm)}_{\mu} V_{jm} = C_{\mu}^i$, are automatically satisfied. Therefore, we explicitly found the matrix $\Delta_{(kl)i}^{a}$ in the AdS background.

It is easy to prove in a similar way that the static black holes also belong to the chosen region of the phase space where the left inverse $\Delta_{(kl)i}^{a}$ exists. Namely, the same as for the AdS space, the black hole curvature $R_{\alpha\beta}^{\mu\nu}$ has each component proportional to $\delta_{\alpha\beta}^{\mu\nu}$, with different factors. An explicit check confirms that $M$ has maximal rank for static Pure Gauss-Bonnet black hole.

With all constraints identified and the Hamiltonian multipliers solved, we can obtain the information about the degrees of freedom and local symmetries in the theory in a particular class of backgrounds, where $M$ has maximal rank during the whole evolution of the fields.
5 Degrees of freedom and symmetries

Next step in the Hamiltonian analysis is to separate first and second class constraints. The first class constraints generate local symmetries and the second class constraints eliminate non-physical fields not related to the symmetries. If there are \( N_1 \) first and \( N_2 \) second class constraints in the phase space with \( N \) generalized coordinates, then a physical number of degrees of freedom is given by the Dirac formula

\[
N^* = N - N_1 - \frac{1}{2} N_2 .
\]

Thus, determination of a class of constraints is of essential importance for identification of the physical fields living on the reduced phase space \( \Gamma^* \). Furthermore, first class constraints are related to the existence of indefinite multipliers in a theory and their numbers should match since each first class constraint appearing in the Hamiltonian is multiplied by an arbitrary function. Let us recall from the previous section that the solved multipliers are \( \{ u^a_i, U^{ab}_i, v^i_a = 0, v^{ij}_a = 0 \} \), and the unsolved ones \( u^{ab}_i \) and \( u^a_i \) are related to the local symmetries, Lorentz transformations and diffeomorphisms. In addition, we do not know the explicit form of all multipliers \( U^{ab}_i \), because \( U_{(ij)} \) depends on the background. It is then expected that we will not be able to obtain a closed, background-independent form of all generators.

To find first class constraints, it is helpful to write the total Hamiltonian density \( \mathcal{H}_T \) with solved multipliers because it is known that this is first class quantity (it commutes with all constraints) and, therefore, only first class constraints will naturally appear there as a combination of other constraints. Thus, replacing the solutions \( \{4.19\}, \{4.23\} \) and \( \{4.27\} \) in \( \mathcal{H}_T \), we obtain the Hamiltonian density

\[
\mathcal{H} = \frac{1}{2} \omega^a_{t} J_{ab} - e^a_{t} J_{a} + u^a_{t} a^t_a + \frac{1}{2} u^{ab}_{t} a^t_{ab} + \frac{1}{2} U^{ab}_{i} \phi^{i}_{ab} + \partial_{i} D^{i} ,
\]

where the constraints \( (\mathcal{H}_a, \mathcal{H}_{ab}) \) are replaced by the new ones \( (J_{a}, J_{ab}) \),

\[
\begin{align*}
J_{ab} &= \mathcal{H}_{ab} - e_{ai} \pi^{i}_b + e_{bi} \pi^{i}_a + D_{i} \phi^{i}_{ab} \\
J_{a} &= \mathcal{H}_{a} + D_{i} \pi^{i}_a .
\end{align*}
\]

The total divergence \( D^{i} = e^a_{t} a^t_a + \frac{1}{2} \omega^a_{t} \phi^{i}_{ab} \) can be neglected, as it contributes only to a boundary term in the total Hamiltonian.

The functions \( (J_{a}, J_{ab}) \) are not guaranteed yet to be first class because we still have to replace \( U^{ab}_{i} \). But to evaluate \( U \cdot \phi \), we have to choose a particular background for \( U_{(ij)} \), so we will not write it explicitly, as we prefer to keep the background-independent expressions. A more detailed analysis shows that after using Eqs. \( \{4.37\} \) and \( \{4.44\} \) the multiplies can be written as

\[
\begin{align*}
\frac{1}{2} U^{ab}_{i} \phi^{i}_{ab} &= \Delta^{c}_{(ij)k} A^{k}_{i} e^{j}_{b} e^{d}_{a} e^{i}_{b} \phi^{j}_{a} - e^a_{t} \left( \frac{1}{2} R_{acjk} e^{kb} e^{c}_{i} e^{d} \phi^{j}_{d} - \Delta^{c}_{(ij)k} A^{k}_{i} e^{j}_{b} e^{d} \phi^{j}_{a} \right) \\
&= -\frac{1}{2} \omega^{ab}_{t} \Delta J_{ab} - e^a_{t} \Delta J_{a} ,
\end{align*}
\]

so in general this expression can affect the generators \( J_{a} \) and \( J_{ab} \) because \( \Delta^{c}_{(ij)k} \) is a function of \( e^a_{t} \) and \( \omega^{ab}_{t} \). The first class generators that appear in the Hamiltonian \( \{5.2\} \) are \( J_{ab} = J_{ab} + \Delta J_{ab} \).
and \( J_a = J_a + \Delta J_a \). Note that these corrections contain the non-linear \( R^2 \) terms and the background-dependent \( \Delta(e, \omega) \). This is similar to what happens in the \( R + T^2 + R^2 \) theory \[28\]. Because of the complexity of the problem, in the next step we will not account for the \( U \cdot J \) term.

The temporal components of the fields, \( \omega^{ab} \) and \( e^a_i \), are Lagrangian multipliers because they are not dynamical, and in the Hamiltonian notation they are arbitrary functions multiplying the constraints. Therefore, the Hamiltonian \[52\] can be seen as the extended Hamiltonian, which contains constraints of all generations, both primary and secondary. Furthermore, since only first class constraints are associated to indefinite multipliers, we can identify them as

First class constraints : \( J_a, J_{ab}, \pi^i_a, \pi^i_{ab} \),

and there are \( N_1 = (5 + 10) \times 2 = 30 \) of them. With respect to the second class constraints, from \[4.22\] we know that \( T^a_{ij} \approx 0 \) is satisfied, but some components of the torsion tensor are first class and some are second class constraints. They cannot be separated explicitly. For example, 10 functions \( H_{ab} \) are linear combinations of \( T^a_{ij} \). This means, in order to define \( J_{ab} \) in terms of \( H_{ab} \), we had to change a basis of the constraints. In doing so, it is important that the\( regularity conditions \) are satisfied, ensuring that all constraints are linearly independent on the phase space because they have the maximal rank of the Jacobian with respect to the phase space variables.

In our case, we replaced the initial set of 30 constraints \( T^a_{ij} \) by a new one \( (H_{ab}, T_z) \). Then, the regularity conditions require that \( \text{rank} \left[ \frac{\partial (H_{ab}, T_z)}{\partial (q^M, p^N)} \right] = 30 \), what means that there must be 20 second class constraints \( T_z \). We shall denote them by \( T_z = \{ \tilde{T}^a_{ij} \} \) regardless their tensorial properties, to remember that they are redundant torsional components which do not generate any local symmetry. Thus, we represented \( T^a_{ij} \) by an equivalent set of the 10 + 20 constraints \( (H_{ab}, \tilde{T}_{ij}) \). Then we can identify the remaining set of the constraint as

Second class constraints : \( \tilde{T}^a_{ij}, \phi^i_u, \phi^i_{ab} \),

and there are \( N_2 = 20 + 20 + 40 = 80 \) of them. Then the count of the degrees of freedom is straightforward: for \( N = 25 + 50 = 75 \) dynamical fields \( (e^a_\mu, \omega^{ab}_\mu) \), the Dirac formula \[5.1\] gives the number degrees of freedom

\[
N^* = 5.
\]  

This is the same number as in the five-dimensional Einstein-Hilbert theory, and a maximal number that a PL gravity can contain. One of these degrees of freedom is the radial one. This can be proved by performing the Hamiltonian analysis of the action in minisuperspace approximation, which involves only the relevant degrees of freedom, similarly as in Ref. \[20\]. Fundamental fields in this approximation are the most general ones among \( g_{\mu\nu} \) and \( T_{\mu\nu\lambda} \) that have the same isometries. The identified radial degree of freedom corresponds to the metric component \( g_{tt} = -1/g_{rr} \).

If the rank of \( M \) in \[4.40\] is smaller than maximal, then some functions \( U_{(ij)} \) remain arbitrary, reflecting the fact that there are more local symmetries in the theory and less degrees of freedom. In the extreme case, when the rank of \( M \) is zero, all \( U_{(ij)} \) are indefinite, so there are 10 additional local symmetries because second class constraints are converted into the first class, thus \( N_1 \rightarrow N_1 + 10 = 40 \) and \( N_2 \rightarrow N_2 - 10 = 70 \). This implies that in the flat background...
the theory has $N^* - 10 + \frac{1}{2}10 = 0$ degrees of freedom. In general, a number of the degrees of freedom in five-dimensional PL gravity varies in the range

$$0 \leq N^* \leq 5.$$  \hspace{1cm}(5.6)

Let us analyze the local symmetries and their generators. The first class constraint $G \approx 0$ acts on the fundamental field $q$ through the smeared generator $G[\lambda] = \int d^4x \lambda G$, and the field transforms as $\delta q = \{q, G[\lambda]\}$. In our case, the generator for all first class constraints is

$$G[\Lambda, \dot{\Lambda}, \epsilon, \dot{\epsilon}] = \int d^4x \left[ \frac{1}{2} \Lambda^{ab} \left( J_{ab} - e_{at} \pi^t_b + e_{bt} \pi^t_a \right) - \frac{1}{2} D_t \Lambda^{ab} \pi^t_{ab} + \epsilon^a J_a - D_t \epsilon^a \pi^t_a \right]$$

$$= \int d^4x \left( \frac{1}{2} \dot{\Lambda}^{ab} \dot{J}_{ab} - \frac{1}{2} \dot{\Lambda}^{ab} \dot{\pi}^t_{ab} + \epsilon^a J_a - \dot{\epsilon}^a \pi^t_a \right),$$  \hspace{1cm}(5.7)

where we redefined $J_{ab} \rightarrow J_{ab} - e_{at} \pi^t_b + e_{bt} \pi^t_a$ and $J_a \rightarrow J_a + \omega_{ta} \pi^t_b$ in order to covariantize the $e \cdot \pi$ term. It is equivalent to redefinition of the multipliers, so that

$$\dot{J}_{ab} = J_{ab} - e_{at} \pi^t_b + e_{bt} \pi^t_a + \omega_{ta} \epsilon^t_c \pi^t_{cb} - \omega_{tb} \epsilon^t_a \pi^t_{ac},$$

$$\dot{J}_a = J_a + \omega_{ta} \pi^t_b.$$  \hspace{1cm}(5.8)

The parameters $\dot{\Lambda}^{ab}$ and $\dot{\epsilon}^a$ are required by the Castellani’s construction of the generators \cite{29} (for an alternative method, see \cite{30} \cite{31}), to replace the independent parameters by the first class constraints $\pi^t_{ab}$ and $\pi^t_a$. A reason for this is that in the Hamiltonian formalism all PB are taken at the same time and the time derivatives of parameters are treated as the new, independent functions, for example $D_t \Lambda^{ab}$ is linearly independent of $\Lambda^{ab}$. In addition, the Castellani method gives a procedure to determine these parameters in a way that recovers covariance of the Lagrangian theory. Direct calculation shows that, up to the background-dependent term $U \cdot \phi$, the given generators indeed satisfy the Castellani’s conditions.

The gauge transformations generated by $G[\Lambda, \dot{\Lambda}, \epsilon, \dot{\epsilon}]$ have the form

$$\delta e^a_\mu = \Lambda^{ab} e_{b\mu} - D_\mu \epsilon^a,$$

$$\delta \omega^{ab}_\mu = -D_\mu \Lambda^{ab}.$$  \hspace{1cm}(5.9)

The $\Lambda^{ab}(x)$ is recognized as a Lorentz gauge parameter. The local transformations with the parameter $\epsilon^a(x)$ are related to the diffeomorphisms on-shell and their explicit form cannot be written because it depends on the background.

The non-vanishing brackets between the constraints $\{\tilde{J}_{ab}, \tilde{J}_a, \pi^t_{ab}, \pi^t_a, T_{ij}^a, \phi^i_a, \phi^i_{ab}\}$ contain the Lorentz algebra

$$\{\tilde{J}_{ab}, \tilde{J}'_{cd} \} = \left( \eta_{ad} \tilde{J}_{bc} + \eta_{bc} \tilde{J}_{ad} - \eta_{ac} \tilde{J}_{bd} - \eta_{bd} \tilde{J}_{ac} \right) \delta,$$  \hspace{1cm}(5.10)

where the brackets with weakly $\tilde{J}_{ab}$ vanish with all other constraints, so they are explicitly first
class,
\[
\{ J_{ab}, \pi_b^c \} = (\eta_{bc} \pi_a^c - \eta_{ac} \pi_b^c) \delta ,
\]
\[
\{ J_{ab}, \pi_{cd}^e \} = (\eta_{ad} \pi_{bc}^e + \eta_{bc} \pi_{ad}^e - \eta_{ac} \pi_{bd}^e - \eta_{bd} \pi_{ac}^e) \delta ,
\]
\[
\{ J_{ab}, J_c^d \} = (\eta_{bc} J_a^d - \eta_{ac} J_b^d) \delta ,
\]
\[
\{ J_{ab}, \phi_c^i \} = (\eta_{ic} \phi_a^i - \eta_{ia} \phi_c^i) \delta ,
\]
\[
\{ J_{ab}, \phi_{cd}^i \} = (\eta_{ad} \phi_{bc}^i + \eta_{bc} \phi_{ad}^i - \eta_{ac} \phi_{bd}^i - \eta_{bd} \phi_{ac}^i) \delta ,
\]
\[
\{ J_{ab}, T_{i}^{cd} \} = (\delta_{ic} T_{a}^{jd} - \delta_{id} T_{a}^{jc} ) \delta .
\]

(5.11)

For completeness, we also list the other non-vanishing brackets among the constraints,
\[
\{ J_a, J_b^c \} = -\frac{15}{4d2} \Omega_{ab}^{ij} \delta ,
\]
where \( \Omega_{ab}^{ij} = \Omega_{abcd} e^c_i e^d_j \) and, with introduced \( K_{abc}^i = 4\alpha_2 e^{ijkl} \epsilon_{abcde} \omega_{ji} R_{f}^{ef} + \eta_{ab} \pi_{c}^i - \eta_{ac} \pi_{b}^i \),
\[
\{ J_a, \pi_b^c \} = (\eta_{ab} \pi_c^i - \eta_{ac} \pi_b^i) \delta ,
\]
\[
\{ J_a, \phi_b^i \} = -120\alpha_0 |e| (e_a^i e_b^i - e_b^i e_a^i ) \delta ,
\]
\[
\{ J_a, T_{ij}^b \} = R_{ai}^b \delta ,
\]
\[
\{ T_{ij}^a, \phi_{kl}^b \} = -\delta_{ik}^a \delta_{jl}^b \delta_k + \omega_{ia}^{a} e_{j}^i \delta_{j}^a - \omega_{ja}^{a} e_{i}^j \delta_{i}^a \delta ,
\]
\[
\{ \phi_{ab}, T_{ij}^c \} = (e_{a}^{c} \delta_{i}^{d} \delta_{j}^{d} + e_{d}^{c} \delta_{i}^{a} \delta_{j}^{a} ) \delta .
\]

(5.12)

As already mentioned, the symplectic form in PL gravity is non-linear in the curvature so its rank depends on the particular background. This implies that the second class constraints cannot be in general separated from the first class constraints. The constraints whose brackets do not vanish explicitly on the constraint surface are the ones given by Eq. (5.12).

6 Hamiltonian analysis of PL gravity \((d+1)\)-dimensions

In this section we only give the main results of the Hamiltonian analysis in \(d+1\) dimensions and point out the differences with respect to the five-dimensional case. The generalized coordinates \(q^M\), momenta \(\pi_M\) and primary constraints have the form \([4.1] - [4.3]\), where now the indices run in the wider range, \(i = 1, \ldots d\) and \(a = 0, \ldots d\). The symplectic matrix has the components
\[
\Omega_{abcd}^{ij} = 4(p - 1)\beta_p e^{ijkl} \delta_{abcd} \cdots \delta_{abcd} R_{ijkl}^{a_1 a_2} \cdots R_{ijkl}^{a_d a_2} \cdots T_{ijkl}^{a_2 a_2} e_{a_2}^{a_2} \cdots e_{a_d}^{a_d} ,
\]
\[
\Omega_{abc}^{ij} = \beta_p e^{ijkl} \delta_{abc} \cdots \delta_{abc} R_{ijkl}^{a_1 a_2} \cdots R_{ijkl}^{a_d a_2} \cdots e_{a_2}^{a_2} \cdots e_{a_d}^{a_d} ,
\]
(6.1)

where \(\beta_p = -2^2 - p (d + 1 - 2p) p \alpha_p\) is a real constant. The matrix \(\Omega_{abcd}^{ij}\) is identically zero only in the Einstein-Hilbert gravity \((p = 1)\). In general \((p > 1)\), the matrix \(\Omega_{abcd}^{ij}\) only weakly vanishes
for the PL gravity (6.10). The phase space functions $L^i_{ab}$, $S_a$ and $S_{ab}$ in higher dimensions become

$$L^i_{ab} = -\frac{1}{d + 1 - 2p} \Omega_{abc}^i e_j^c,$$

$$S_a = \mathcal{H}_a + D_t \lambda^i_a,$$

$$S_{ab} = \mathcal{H}_{ab} + e_{bi} \lambda^i_a - e_{ai} \lambda^i_b,$$  \hspace{1cm} (6.2)

where

$$\mathcal{H}_{ab} = -\frac{1}{2} \Omega_{abc}^i T^c_{ij},$$

$$\mathcal{H}_a = (d + 1) \alpha_0 \epsilon_{ab1\ldots ad} e^{i_1\ldots i_d} e_{i_1}^{a_1} \ldots e_{i_d}^{a_d} - \frac{1}{4p} \Omega_{abc}^i R^b_{ij}.$$  \hspace{1cm} (6.3)

Using the Hamiltonian (4.11)–(4.12), we find the following secondary constraints from the condition of vanishing $\dot{p}^a_{ij}$, $\dot{p}_i^a$ and $\dot{p}_{ab}$,

$$T^a_{ij} \approx 0, \quad S_a \approx 0,$$

$$S_{ab} \approx e_{bi} \lambda_a^i - e_{ai} \lambda_b^i \approx 0.$$  \hspace{1cm} (6.4)

The requirement of vanishing $\dot{p}_i^a$ and $\dot{p}^a_{ij}$ solve the multipliers $v^a_i$ and $u^a_i$,

$$v^a_i = \epsilon_{ab}^{i_1\ldots i_d} + \frac{1}{2} \Omega_{cda}^i U^c_{ij} + \omega_t^a \lambda^i_a + D_t \lambda^i_a \approx 0,$$

$$u^a_i = D_t e_i^a - \omega_t^b e_{bi}.$$  \hspace{1cm} (6.5)

where it was convenient to define $U^i_{ab} = u^a_i - D_t \omega_t^a$ and

$$\Sigma^i_{ab} = \epsilon_{ab1\ldots ad} e^{i_1\ldots i_d} \left[ -d(d + 1) \alpha_0 e^{a_2}_i \ldots e^{a_{2p+1}}_{i_{2p+1}} + \frac{d - 2p}{4p} \beta_p R^{a_2 a_3}_{i_2 i_3} R^{a_4 a_5}_{i_4 i_5} \ldots R^{a_{2p-2p+1}}_{i_{2p+1}} e^{a_{2p+2}}_{i_{2p+2}} \ldots e^{a_d}_{i_d} \right].$$  \hspace{1cm} (6.6)

In odd-dimensional spaces with $d = 2p$, the last line in $\Sigma^i_{ab}$ vanishes. This is the case of five-dimensional Pure Gauss-Bonnet gravity analyzed in previous sections.

We ask that the constraint $\phi_{ab}^i \approx 0$ vanishes during its time evolution in $(d + 1)$-dimensional spacetime, leading to

$$\dot{\phi}_{ab}^i \approx -\frac{1}{2} \omega_t^{cd} \left( \Omega_{cda}^i e_{bj} - \Omega_{cda}^i e_{aj} + \Omega_{abc}^i e_{dj} - \Omega_{abc,d}^i e_{cj} \right) + e_{ta} \lambda_{bj} - e_{tb} \lambda_{ai} \lambda_{aj}^i e_{bj} + \lambda_{aj}^i e_{aj}.$$  \hspace{1cm} (6.7)

However, the first line identically vanishes due to the combinatorial identity

$$\epsilon_{ba1\ldots a_d} e_{aj} - \epsilon_{aa1\ldots ad} e_{bj} + \epsilon_{aba2\ldots ad} e_{a1j} - \cdots + (-1)^{d+1} \epsilon_{abca1\ldots ad-1} e_{a_dj} = 0,$$  \hspace{1cm} (6.8)
and the second line of (6.9) with Eq. (6.5) can be equivalently written as \( \lambda_a^i \approx 0 \) and \( \lambda_a^{ij} \approx 0 \), so that we find

Secondary constraints: \[
S_a \approx 0, \quad T_{ij}^a \approx 0, \quad \lambda_a^i \approx 0, \quad \lambda_a^{ij} \approx 0. \quad (6.11)
\]

Next, we have to require that the secondary constraints also evolve on the constraint surface. Thus, the requirement of vanishing \( \dot{\lambda}_a^j \) and \( \dot{\lambda}_a^i \) solve the multipliers \( v_{ai}^j \approx 0 \) and \( v_{aj}^i \approx 0 \), but because the form of \( v_a^i \) is already known from Eq. (6.6), we obtain the algebraic equation for the multipliers \( U_{jcd} \),

\[
0 \approx \lambda_a^i = e_t^b \chi_{t}^{ij} + \frac{1}{2} \Omega_{cda}^{ij} U_{jcd}. \quad (6.12)
\]

By replacing \( U_{i}^{ab} = R_{ti}^{ab} \), we can prove that the above expression combined with \( H_a \) is equivalent to the Lagrangian equations,

\[
0 \approx \lambda_a^i = (H_a, \chi_a^{i}) = (d + 1) \alpha_0 \epsilon_{aa_1 \ldots a_d} e^\lambda_{\mu_1 \mu_2 \ldots \mu_d} e^a_{\mu_1} e_{a_2} \ldots e^a_{\mu_d} \\
+ \frac{\alpha_0}{2d} (d + 1 - 2p) e^\lambda_{\mu_1 \ldots \mu_d} e_{aa_1 \ldots a_d} R_{\mu_1 \mu_2}^{a_1 a_2} \ldots R_{\mu_2 \ldots \mu_d}^{a_2 a_3} e^a_{\mu_3} \ldots e^a_{\mu_d}. \quad (6.13)
\]

Further calculation can be simplified by observing that, as in five dimensions, the pairs of conjugated constraints, \( \left( \lambda_a^i, p_{ij}^b \right) \) and \( \left( \lambda_a^{ij}, p_{ij}^b \right) \), are second class. It means they can be eliminated from the phase space by defining the reduced phase space \( \Gamma^* \), where Poisson brackets are replaced by the Dirac brackets \( \{ \cdot, \cdot \} \). The coordinates of the space \( \Gamma^* \) are \( (e_{\mu}, \omega_{\mu}, \pi_{\mu}, \pi_{ab}) \) and their Dirac brackets are equal to the Poisson brackets. From now on, we shall drop writing the star in the Dirac brackets, and continue working on \( \Gamma^* \).

The evolution of \( S_a \) can be obtained after the long, but straightforward calculation, with the help of the identity \( 0 = D e_{ab a_1 \ldots a_d} \), which implies

\[
\omega_{ab}^b R_{t_{ij}^{a_1 a_2}}^{a_1 a_2} \Omega_{c_{ij}^{a_1 a_2}}^{a_1 a_2} = -\omega_{ab}^b R_{t_{ij}^{a_1 a_2}}^{a_1 a_2} \Omega_{c_{ij}^{a_1 a_2}}^{a_1 a_2} + (d - 2p) e^\lambda_{\mu_1 \ldots \mu_d} e_{aa_1 a_2 a_3} R_{\mu_2 \ldots \mu_d}^{a_2 a_3} e_{\mu_3} \ldots e_{\mu_d}. \quad (6.14)
\]

Then we find that \( S_a \) never leave the constraint surface,

\[
\dot{S}_a \approx -D_i \lambda_a^i - \omega_{ab}^b H_b \approx 0. \quad (6.15)
\]

Finally, the consistency condition of \( T_{ij}^a \) gives \( \frac{(d+1)d(d-1)}{2} \) algebraic equations for \( \bar{U}^{ab}_{ij} \) unknown functions \( U_{ij}^a = U_{ij}^{ab} e_{bi} \),

\[
0 \approx \dot{T}_{ij}^a \approx R_{ij}^{ab} e_{bi} + U_{ij}^a - U_{ij}^{ab}. \quad (6.16)
\]

This form is the same as in five dimensions, so we skip the detailed analysis \( \{1.31\} - \{1.36\} \) and conclude that the antisymmetric parts \( U^{ab}_{ij} \) of the multipliers are solved, \( U_{ti}^{a} \) remain unknown and the other are not independent due to the Bianchi identity. The final expression for \( U_{ij}^{ab} \) is given by Eq. \( \{1.37\} \). The result for the coefficients \( U \) can be summarized as

| Multiplier \( U_{ij}^{ab} \): \( U_{t[ij]} \), \( U_{t[ij]} \), \( AU_{kij} \), \( SU_{kij} \), \( TU_{kij} \) |
|-------------------|-----------------|-------------------|-------------------|-------------------|
| Components : \( \frac{d^2(d+1)}{2} \) \( \frac{d(d-1)}{2} \) \( \frac{d(d+1)}{2} \) \( d \) \( d \) \( d \) \( \frac{d(d^2-d-4)}{2} \) |
| Solved by : \( \dot{T}_{ij}^0 \) arbitrary \( \dot{T}_{ijk} \) Bianchi Bianchi |

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Solutions of $U_{t(ij)}$ depend on the equation $\chi_a^i = 0$ given by Eq. (6.12), which after using (4.37) becomes
\[ M_a^{(jm)} U_{tjm} = A_a^i. \tag{6.17} \]

The tensor $M$ has the form (4.41). This is the $d(d+1) \times \frac{d(d+1)}{2}$ matrix of order $p-1$ in the Riemann tensor. The non-homogenous part of the equation, on the other hand, is
\[ A_a^i = \Sigma_{ab} e^b_t + \frac{1}{2} \Omega_{a^{ij}}^{bc} e^b_t e^c_t R_{tjk}. \tag{6.18} \]

Higher-order dependence of $M$ in the curvature means that its rank can change throughout the phase space. When $\Lambda \neq 0$, there is always the region of $\Gamma^*$ where the rank of $M$ is maximal, that is $\frac{d(d+1)}{2}$ which enables to solve all $\frac{d(d+1)}{2}$ coefficients $U_{t(ij)}$. This completes the constraint analysis, which has the same structure as in five dimensions. Arbitrary multipliers are associated with the first class constraints, and the rest are second class constraints.

Therefore, we have $N = \frac{(d+1)^2(d+2)}{2}$ fundamental fields $(e_a^\mu, \omega_{ab}^\mu)$ in the PL gravity on the reduced space, $N_1 = (d+1)(d+2)$ first class constraints $(J_a, J_{ab}, \pi_{t}^a, \pi_{t}^{ab})$ and $N_2 = d^2(d+1)$ second class constraints $(\tilde{T}_{ij}^a, \phi^a_t, \phi^a_{ab})$. Therefore, the number of physical fields in the bulk in this particular background is
\[ N^* = \frac{(d+1)(d-2)}{2}. \tag{6.19} \]

In other background we can have less degrees of freedom, so that the number of degrees of freedom in a higher-dimensional PL gravity is $0 \leq N^* \leq \frac{(d+1)(d-2)}{2}$. The first class constraints and gauge generators have the same form as before, only the matrix $M$ and the tensor $A_{a}^\mu$ that appear in (4.30) are of order $p-1$ in the curvature and we will not write them explicitly – it is straightforward to repeat the previous calculation here.

## 7 Discussion

We performed a Hamiltonian analysis of the Pure Lovelock (PL) gravity in any dimension $D \geq 5$. This Lovelock gravity is not a mere correction of the Einstein-Hilbert theory because it does not even contain the linear term in the scalar curvature. Instead, its kinetic term is described by a $p$-th order polynomial in the Riemann tensor such that the equations of motion remain of second order in the metric. When the cosmological constant is included, the PL gravity has the unique dS and/or AdS vacuum.

The first order formalism was used to deal with non-linearities involved in the theory. We ensured that space-time is Riemannian by introducing the constraint that forced the torsion to vanish.

The detailed analysis revealed that the number of symmetries and degrees of freedom in this theory depends on the background. In the generic case, which include (A)dS space and spherically symmetric, static black holes, the theory contains $D(D-3)/2$ degrees of freedom, which is the same as in General Relativity. But in contrast to Relativity, a change of the background can increase an amount of local symmetries in the theory and convert previously physical fields into nonphysical ones, leading even to a topological theory (with no degrees of freedom in the bulk). This is typical for Lovelock theories. In the PL case, this change of
degrees of freedom is kept under control through the matrix $M$, whose rank can be between 0 and $D(D - 1)/2$, what yields between 0 and $D(D - 3)/2$ degrees.

A constraint analysis probes a number of physical components of the metric field $g_{\mu\nu}$, which is directly related to the Riemann tensor. Its relation to the PL Riemann tensor is indirect and not anchored to any metric or connection in a straightforward way. What turns out is that the maximum possible number of physical fields does not depend on a particular Lovelock theory, as was pointed out earlier in Ref. [16]. It reflects the fact that so long as the equations of motion are second order, the metric degrees of freedom would be the same for Einstein as well as Lovelock theories.

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A Conventions

We use the signature of the Minkowski metric $\eta_{ab} = \text{diag}(-++\cdots+)$. The Levi-Civita symbols in $d + 1$ and $d$ dimensions are defined by

$$dx^\mu_1 \wedge \cdots \wedge dx^\mu_{d+1} = \varepsilon^\mu_1 \cdots \mu_{d+1} d^{d+1}x, \quad \varepsilon^{i_1 i_2 \cdots i_d} = \varepsilon^{i_1 i_2 \cdots i_d}.$$ (A.1)

The generalized Kronecker delta of rank $s$ is constructed as the determinant

$$\delta^{\nu_1 \cdots \nu_s}_{\mu_1 \cdots \mu_s} = \begin{vmatrix} \delta_{\mu_1}^{\nu_1} & \delta_{\mu_1}^{\nu_2} & \cdots & \delta_{\mu_1}^{\nu_s} \\ \delta_{\mu_2}^{\nu_1} & \delta_{\mu_2}^{\nu_2} & \cdots & \delta_{\mu_2}^{\nu_s} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{\mu_s}^{\nu_1} & \delta_{\mu_s}^{\nu_2} & \cdots & \delta_{\mu_s}^{\nu_s} \end{vmatrix}. $$ (A.2)

If the range of indices is $D$, a contraction of $k \leq s$ indices in the Kronecker delta of rank $s$ produces a delta of rank $s - k$,

$$\delta^{\nu_1 \cdots \nu_k \cdots \nu_s}_{\mu_1 \cdots \mu_k \cdots \mu_s} \delta^{\mu_1}_{\nu_1} \cdots \delta^{\mu_k}_{\nu_k} = \frac{(D - s + k)!}{(D - s)!} \delta^{\nu_{k+1} \cdots \nu_s}_{\mu_{k+1} \cdots \mu_s}. $$ (A.3)

Other identities involving the Levi-Civita symbol and the generalized Kronecker delta are

$$\varepsilon^{\nu_1 \cdots \nu_{d+1}}_{\mu_1 \cdots \mu_{d+1}} = -\delta^{\mu_1 \cdots \mu_{d+1}}_{\nu_1 \cdots \nu_{d+1}}, \quad \varepsilon_{\alpha_1 \cdots \alpha_{d+1}}^{\mu_1 \cdots \mu_{d+1}} = |e| \varepsilon_{\mu_1 \cdots \mu_{d+1}},$$ (A.4)

where $|e| = \det[e^a]$.
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