ON SUMS OF SQUARES OF THE RIEMANN ZETA-FUNCTION ON THE CRITICAL LINE

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Abstract. A discussion involving the evaluation of the sum \( \sum_{0 < \gamma \leq T} |\zeta(\frac{1}{2} + i\gamma)|^2 \) is presented, where \( \gamma \) denotes imaginary parts of complex zeros of \( \zeta(s) \). Three theorems involving certain integrals related to this sum are proved, and the sum is unconditionally shown to be \( \ll T \log^2 T \log \log T \).

1. Introduction

The aim of this note is to discuss the evaluation of the sum

\[
\sum_{0 < \gamma \leq T} |\zeta(\frac{1}{2} + i\gamma)|^2
\]

and some related integrals, where \( \gamma \) denotes imaginary parts of complex zeros of \( \zeta(s) \), and where every zero is counted with its multiplicity (see also [5] and [7]). The interest is in obtaining unconditional bounds for the above sum, since assuming the Riemann Hypothesis (RH) the sum trivially vanishes.

A more general sum than the one in (1.1) was treated by S.M. Gonek [3]. He proved, under the RH, that

\[
\sum_{0 < \gamma \leq T} \left| \zeta\left(\frac{1}{2} + i\left(\gamma + \frac{\alpha}{L}\right)\right) \right|^2 = \left(1 - \left(\frac{\sin \pi \alpha}{\pi \alpha}\right)^2\right) \frac{T}{2\pi} \log^2 T + O(T \log T)
\]

holds uniformly for \( |\alpha| \leq \frac{1}{2}L \), where \( L = \frac{1}{2\pi} \log(\frac{T}{\pi}) \). It would be interesting to recover this result unconditionally, but our method of proof does not seem capable of achieving this. To evaluate the sum in (1.1) we begin by considering the Stieltjes integral

\[
I(T) := \int_{T_0}^T |\zeta(\frac{1}{2} + it)|^2 \, dS(t),
\]

where \( T_0 \) is a suitable positive constant, \( T > T_0 \) and clearly it may be assumed that both \( T_0 \) and \( T \) are not an ordinate of a zeta-zero. As usual (see [2, Chapter 15] or [9, Section 9.3])

\[
S(t) = \frac{1}{\pi} \arg \zeta(\frac{1}{2} + it) = \frac{1}{\pi} 3 \text{m} \{ \log \zeta(\frac{1}{2} + it) \} \ll t.
\]
Here for \( t \neq \gamma \) the argument of \( \zeta(\frac{1}{2} + it) \) is obtained by continuous variation along the straight lines joining the points 2, 2 + it, \( \frac{1}{2} + it \), starting with the value 0. If \( t \) is an ordinate of a zeta-zero, then we define \( S(t) = S(t + 0) \). As is customary

\[
N(T) = \sum_{0 < \gamma \leq T} 1
\]

counts (with multiplicities) the number of positive imaginary parts of all complex zeros which do not exceed \( T \). We have (see [4, Chapter 1] or [10, Chapter 9])

\[
N(T) = \sum_{0 < \gamma \leq T} 1 = \frac{1}{\pi} \vartheta(T) + 1 + S(T), \tag{1.5}
\]

\[
\vartheta(T) := \Im \left\{ \log \Gamma\left(\frac{1}{4} + \frac{1}{2}iT\right) \right\} - \frac{1}{2}T \log \pi,
\]

whence \( \vartheta(T) \) is continuously differentiable. In fact, by using Stirling’s formula for the gamma-function it is found that

\[
\vartheta(T) = \frac{T}{2} \log \frac{T}{2\pi} - \frac{T}{2} - \frac{\pi}{8} + O\left(\frac{1}{T}\right).
\]

Thus when \( t \neq \gamma \) we can differentiate \( S(t) \) by using (1.4). If \( t = \gamma \), then by (1.5) it is seen that \( S(t) \) has a jump discontinuity which counts the number of zeros \( \rho \) with \( \gamma = \Im \rho = t \). Let \( \mathcal{J}_1(T, \varepsilon) \) denote the union of all subintervals \((\bar{\gamma} - \varepsilon, \bar{\gamma} + \varepsilon)\) lying in \([T_0, T] \) such that \( \bar{\gamma} \) denotes distinct ordinates of zeta-zeros, and \( \varepsilon > 0 \) is so small that these intervals are disjoint, and let

\[
\mathcal{J}_2(T, \varepsilon) := [T_0, T] \setminus \mathcal{J}_1(T, \varepsilon).
\]

Then from (1.4) and (1.5) we infer that

\[
I(T) = \lim_{\varepsilon \to 0} \int_{\mathcal{J}_1(T, \varepsilon)} |\zeta(\frac{1}{2} + it)|^2 d \left( N(t) - \frac{1}{\pi} \vartheta(t) - 1 \right)
\]

\[
+ \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\mathcal{J}_2(T, \varepsilon)} |\zeta(\frac{1}{2} + it)|^2 \Im d \log \zeta(\frac{1}{2} + it)
\]

\[
= \sum_{T_0 < \gamma \leq T} |\zeta(\frac{1}{2} + i\gamma)|^2 + \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\mathcal{J}_2(T, \varepsilon)} |\zeta(\frac{1}{2} + it)|^2 \Im \left( \frac{\zeta'(\frac{1}{2} + it)}{\zeta(\frac{1}{2} + it)} \right) dt
\]

\[
= \sum_{T_0 < \gamma \leq T} |\zeta(\frac{1}{2} + i\gamma)|^2 + \frac{1}{\pi} \int_{T_0}^{T} |\zeta(\frac{1}{2} + it)|^2 \Im \left( \frac{\zeta'(\frac{1}{2} + it)}{\zeta(\frac{1}{2} + it)} \right) dt,
\]

since in the last integral the zeros of \( \zeta(\frac{1}{2} + it) \) in the denominator are cancelled by the zeros of \( |\zeta(\frac{1}{2} + it)|^2 = \zeta(\frac{1}{2} + it)\zeta(\frac{1}{2} - it) \), and the integral in question in fact equals

\[
\Re \int_{T_0}^{T} \zeta'(\frac{1}{2} + it)\zeta(\frac{1}{2} - it) dt.
\]
Therefore we obtain the basic formula
\[
\sum_{T_0 < \gamma \leq T} |\zeta(\frac{1}{2} + i\gamma)|^2 \\
= \int_{T_0}^{T} |\zeta(\frac{1}{2} + it)|^2 \left( \frac{1}{\pi} \Im \left\{ \frac{\zeta'(\frac{1}{2} + it)}{\zeta(\frac{1}{2} + it)} \right\} \right) dt.
\] (1.6)

This formula depends implicitly on the functional equation for \( \zeta(s) \) (e.g., see [4, Chapter 1]), namely
\[
\pi^{-\frac{s}{2}} \Gamma(\frac{1}{2} s) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma(\frac{1}{2}(1-s)) \zeta(1-s).
\]

To see this note that we have, on using (1.5),
\[
\sum_{T_0 < \gamma \leq T} |\zeta(\frac{1}{2} + i\gamma)|^2 = \int_{T_0}^{T} |\zeta(\frac{1}{2} + it)|^2 dN(t)
\]
\[
= \int_{T_0}^{T} |\zeta(\frac{1}{2} + it)|^2 \left( \frac{1}{\pi} \Im \left\{ \frac{\Gamma'(\frac{1}{2} + \frac{1}{2}it)}{\Gamma(\frac{1}{2} + \frac{1}{2}it)} \right\} \right) dt + \text{d}S(t)
\]
\[
- \frac{1}{\pi} \int_{T_0}^{T} |\zeta(\frac{1}{2} + it)|^2 \frac{\log \pi}{2} dt
\]
\[
= \frac{1}{\pi} \int_{T_0}^{T} |\zeta(\frac{1}{2} + it)|^2 \text{Re} \{J(t)\} dt + \int_{T_0}^{T} |\zeta(\frac{1}{2} + it)|^2 \text{d}S(t)
\]
\[
- \frac{1}{\pi} \text{Re} \left( \int_{T_0}^{T} |\zeta(\frac{1}{2} + it)|^2 \frac{\zeta'(\frac{1}{2} + it)}{\zeta(\frac{1}{2} + it)} dt \right),
\]

say, where
\[
J(t) := \frac{1}{\pi} \frac{\Gamma'(\frac{1}{2} + \frac{1}{2}it)}{\Gamma(\frac{1}{2} + \frac{1}{2}it)} + \frac{\zeta'(\frac{1}{2} + it)}{\zeta(\frac{1}{2} + it)} - \frac{1}{\pi} \log \pi.
\] (1.7)

In the functional equation we set \( s = \frac{1}{2} + it \). Logarithmic differentiation gives then
\[
-\frac{1}{\pi} \log \pi + \frac{\Gamma'(\frac{1}{2} + \frac{1}{2}it)}{\Gamma(\frac{1}{2} + \frac{1}{2}it)} + \frac{\zeta'(\frac{1}{2} + it)}{\zeta(\frac{1}{2} + it)} = \frac{1}{\pi} \log \pi - \frac{1}{\pi} \frac{\Gamma'(\frac{1}{2} - \frac{1}{2}it)}{\Gamma(\frac{1}{2} - \frac{1}{2}it)} - \frac{\zeta'(\frac{1}{2} - it)}{\zeta(\frac{1}{2} + it)}.
\]

This implies that \( J(t) = -J(t) \). Hence \( J(t) \) is purely imaginary and \( \text{Re} J(t) = 0 \), which yields another proof of (1.6).

Clearly the problem of the estimation of the sum in (1.1) reduces, on integration by parts of the integral in (1.6) with \( S(t) \), to the estimation of the integral
\[
\int_{T_0}^{T} Z(t)Z'(t)S(t) dt.
\]

Namely we have
\[
|\zeta(\frac{1}{2} + it)|^2 = Z^2(t), \quad Z(t) := \chi^{-\frac{1}{2}}(\frac{1}{2} + it)\zeta(\frac{1}{2} + it), \quad \chi(s) = \frac{\zeta(s)}{\zeta(1-s)},
\]
where \( Z(t) \in C^\infty \) is so-called Hardy’s function. The integral above, without the oscillating factor \( S(t) \) is trivial, namely it equals
\[
\frac{1}{2} \int_{T_0}^T (|\zeta(\frac{1}{2} + it)|^2)' \, dt \ll T^{1/3}
\]
on using the crude bound \( \zeta(\frac{1}{2} + it) \ll t^{1/6} \). It turns out that this integral, with the oscillating factor \( S(t) \), is more difficult. Unconditionally the author [5] has proved
\[
\int_{T_0}^T Z(t)Z'(t)S(t) \, dt \ll \epsilon T \log^2 T (\log \log T)^{3/2+\epsilon},
\] (1.8)
while K. Ramachandra [7] used a different method to obtain a result which easily implies the right-hand side of (1.8) with \( T \log^2 T \log \log T \). The same bound holds then for the sum in (1.1). In this text we shall obtain another proof of Ramachandra’s result by a method that can be used to estimate a variety of integrals involving \( S(t) \). We shall prove

**THEOREM 1.** If \( C_0 \) is Euler’s constant, then unconditionally
\[
\Re \int_{T_0}^T \zeta'(\frac{1}{2} + it)\zeta(\frac{1}{2} - it) \, dt
= -\frac{T}{2} \log^2 \left( \frac{T}{2\pi} \right) + (1 - C_0) \log \left( \frac{T}{2\pi} \right) + (C_0 - 1)T + O(T^{1/3}).
\] (1.9)

**THEOREM 2.** If the Riemann Hypothesis is true, then
\[
\int_{T_0}^T Z(t)Z'(t)S(t) \, dt
= \frac{T}{4\pi} \log^2 \left( \frac{T}{2\pi} \right) + \frac{C_0 - 1}{2\pi} T \log \left( \frac{T}{2\pi} \right) + \frac{1 - C_0}{2\pi} T + O(T^{1/3}).
\] (1.10)

**THEOREM 3.** Unconditionally we have
\[
\sum_{0<\gamma \leq T} |\zeta(\frac{1}{2} + i\gamma)|^2 \ll T \log^2 T \log \log T.
\] (1.11)

**THEOREM 4.** Unconditionally we have
\[
\int_0^T |\zeta(\frac{1}{2} + it)|^2 S(t) \, dt \ll T \log T \log \log T,
\] (1.12)
while under the Riemann Hypothesis
\[
\int_0^T |\zeta(\frac{1}{2} + it)|^2 S(t) \, dt \ll T \log T.
\] (1.13)
2. Use of the explicit formula for $S(T)$

A natural method to try to evaluate the integral in (1.6) is to use Lemmas 1-3 of Bombieri-Hejhal [1]. This has the advantage over other similar explicit expressions involving $S(T)$ since it incorporates smooth functions, which is particularly satisfactory in handling the error terms. It will be used in proving (1.13), but in evaluating the integral (1.6) (i.e., the sum in (1.11)) this approach seems ineffective. Specified to $\zeta(s)$ ($\Delta = 2$, eq. (5.4) with the $O(1)$-term written explicitly) [1] gives, for $|t| \geq 2, t \neq \gamma$,

\[
-\frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)} = \sum_{n=2}^{\infty} \Lambda(n)n^{-\sigma - it}v\left(e^{\log n/\log X}\right)
+ \sum_{\rho} \frac{\bar{u}(1 + (\bar{\rho} - \sigma - it)\log X)}{\rho - \sigma - it} + \frac{\bar{u}(1 + (1 - \sigma - it)\log X)}{\sigma + it - 1},
\]

\[
\log \zeta\left(\frac{1}{2} + it\right) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-\frac{1}{2} - it}v\left(e^{\log n/\log X}\right)
+ \sum_{\rho} \int_{\frac{1}{2}}^{\infty} \frac{\bar{u}(1 + (\bar{\rho} - \sigma - it)\log X)}{\rho - \sigma - it} \, d\sigma + \int_{\frac{1}{2}}^{\infty} \frac{\bar{u}(1 + (1 - \sigma - it)\log X)}{\sigma + it - 1} \, d\sigma.
\]

In (2.1) and (2.2) $\bar{\rho}$ denotes all zeros of $\zeta(s)$ (complex zeros denoted by $\rho$ and real, or “trivial zeros” at $-2, -4, \ldots$), while $X$ is a parameter satisfying $2 \leq X \leq t^2$ and $\Lambda(n)$ is the von Mangoldt function. We have

\[
\bar{u}(s) = \int_{0}^{\infty} u(x)x^{s-1} \, dx, \quad v(x) = \int_{x}^{\infty} u(t) \, dt,
\]

and $v(0) = 1$ with proper normalization, where $u(x) \in C^\infty$ is a real positive function with compact support in $[1, e]$. One has, for every integer $k \geq 0$,

\[
|\bar{u}(s)| \leq \max_{x} |u^{(k)}(x)|e^{\max(\Re s, 0)+4k(1+|s|)^{-k}},
\]

and for every fixed integer $K > 0$, and complex zeros $\rho = \beta + i\gamma$ and $t \geq 2$,

\[
\sum_{\rho} \int_{\frac{1}{2}}^{\infty} \frac{\bar{u}(1 + (\bar{\rho} - \sigma - it)\log X)}{\rho - \sigma - it} \, d\sigma
\ll 1 + \sum_{|\gamma - t| \leq 1/\log X} \log \left(1 + \frac{1}{|\gamma - t|\log X}\right) + \sum_{\rho} \frac{X^{\max(\beta - \frac{4}{5}, 0)}}{(1 + |\gamma - t|\log X)^K},
\]

and for $2 \leq X \leq T^{3/8}, T \geq 2, K \geq 3$ (for $\zeta(s)$ one can take $a = 1 - \varepsilon$ in [1, Lemma 3] by a zero-density result of M. Jutila [6] near the line $\sigma = \frac{1}{2}$)

\[
\int_{T}^{2T} \sum_{\rho} \frac{X^{\max(\beta - \frac{4}{5}, 0)}}{(1 + |\gamma - t|\log X)^K} \, dt \ll T \frac{\log T}{\log X},
\]
\[ \int_{T}^{2T} \left( 1 + \sum_{|\gamma - t| \leq 1/\log X} \log \left( 1 + \frac{1}{|\gamma - t| \log X} \right) \right) \, dt \ll T \log T. \tag{2.6} \]

Although the sum on the left-hand side of (2.4) is undefined when \( t = \gamma \), its absolute value is majorized, by (2.5) and (2.6), by an integrable expression. This enables us to deal effectively with integrals containing \( S(t) \).

At this point we specify \( X = 3 \), \( T_0 = 20 \), noting that (2.4)-(2.6) will hold for \( t \geq T_0 \) and \( T \geq T_0 \). In (1.6) we integrate the portion with \( dS(t) \) by parts, using (1.4) and (2.2). The integrated terms are then well defined, since \( T_0 \neq \gamma \), (i.e., \( \zeta(\sigma + 20i) \neq 0 \), \( T \neq \gamma \). Therefore

\[ \int_{T_0}^{T} |\zeta(\frac{1}{2} + it)|^2 dS(t) = \frac{1}{\pi} |\zeta(\frac{1}{2} + it)|^2 \Im f(t) \bigg|_{T_0}^T - \frac{2}{\pi} \int_{T_0}^{T} Z(t)Z'(t)\Im f(t) \, dt, \]

where \( f(t) \) denotes the right-hand side of (2.2), since the integral on the right hand-side of the above expression exists because of (2.4)-(2.6). In fact, the integral in question is \( \ll T^{4/3} \log T \), since (see [4]) \( Z(t) \ll t^{1/6} \), \( Z'(t) \ll t^{1/6} \). We now integrate back by parts the above expression and obtain from (1.6)

\[ \sum_{T_0 < \gamma \leq T} |\zeta(\frac{1}{2} + i\gamma)|^2 = \frac{1}{\pi} \Im \left\{ \int_{T_0}^{T} |\zeta(\frac{1}{2} + it)|^2 \left( df(t) - i \frac{\zeta'(\frac{1}{2} + it)}{\zeta(\frac{1}{2} + it)} \, dt \right) \right\}. \tag{2.7} \]

In (2.7) we substitute (2.1) (with \( \sigma = \frac{1}{2} \) for \( \frac{\zeta'}{\zeta} \) and (2.2). Note that the terms coming from the series with \( \Lambda(n) \), the term with \( \sigma + it - 1 \) and the sums over trivial zeros \(-2, -4, \ldots\), being continuously differentiable and well defined for any \( t \in [T_0, T] \), will cancel out. This follows on using

\[ \frac{\partial \varphi(\sigma + it)}{\partial t} = i \frac{\partial \varphi(\sigma + it)}{\partial \sigma}, \tag{2.8} \]

which holds for any holomorphic function \( \varphi \). The remaining terms in (2.7), which have discontinuities at \( t = \gamma \), come from the sums over complex zeros \( \rho \). Hence from (2.7) it follows that

\[ \sum_{T_0 < \gamma \leq T} |\zeta(\frac{1}{2} + i\gamma)|^2 = \frac{1}{\pi} \left( I_1(T) + I_2(T) \right), \tag{2.9} \]

where

\[ I_1(T) := \int_{T_0}^{T} |\zeta(\frac{1}{2} + it)|^2 \, dg(t), \]

\[ g(t) := \Im \sum_{\rho} \int_{\frac{1}{2}}^{\infty} \frac{\hat{\nu}(1 + (\rho - \sigma - it) \log X)}{\rho - \sigma - it} \, d\sigma \tag{2.10} \]

for \( t \neq \gamma \) and \( t \geq T_0 \), and in general (see (1.4) and (2.2)) for \( t \geq T_0 

\[ g(t) = \pi S(t) - \Im \left\{ \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{\frac{1}{2} - it} \nu \left( e^{\log n/\log X} \right) \right. \]

\[ - \sum_{k=1}^{\infty} \int_{\frac{1}{2}}^{\infty} \frac{\hat{\nu}(1 - (2k + \sigma + it) \log X)}{2k + \sigma + it} \, d\sigma + \sum_{k=1}^{\infty} \int_{\frac{1}{2}}^{\infty} \frac{\hat{\nu}(1 - (\sigma - it) \log X)}{\sigma - it - 1} \, d\sigma \right\}. \tag{2.11} \]
We also have

\[ I_2(T) := \int_{T_0}^{T} |\zeta(\frac{1}{2} + it)|^2 h(t) \, dt, \]

\[ h(t) := 3 \mathrm{Im} \left( i \sum_{\rho} \frac{\tilde{a}(1 + (\rho - \frac{1}{2} - it) \log X)}{\rho - \frac{1}{2} - it} \right). \] (2.12)

The integral \( I_2(T) \) converges absolutely, since the poles of the sum over \( \rho \), namely the zeros \( \rho = \frac{1}{2} + it \), are cancelled by the corresponding zeros of \( \zeta(\frac{1}{2} + it) \). The integral \( I_1(T) \) in (2.10) is an improper Stieltjes integral, which will be handled in the following way. An integration by parts yields

\[ I_1(T) = |\zeta(\frac{1}{2} + it)|^2 g(t) \bigg|_{T_0}^{T} - 2 \int_{T_0}^{T} Z(t)Z'(t)g(t) \, dt. \] (2.13)

In the integrated terms we can use the expression (2.10) for \( g(t) \), since \( T_0 \) and \( T \) are not ordinates of any zeta-zero. In the integral on the right-hand side of (2.13) \( g(t) \) is given by (2.11). However, by using (2.4)-(2.6) we see that this integral converges absolutely if \( g(t) \) is given by (2.10). Hence by properties of Stieltjes integrals it is seen that the integration by parts in (2.13) is justified, and that this formula holds in fact when \( g(t) \) is given by (2.10).

To transform \( I_2(T) \) in (2.12), let \( \bar{\gamma}_1 < \ldots < \bar{\gamma}_n \) denote distinct ordinates of zeta-zeros lying in \([T_0, T]\), and let \( \varepsilon > 0 \) be so small that all intervals \( (\bar{\gamma}_j - \varepsilon, \bar{\gamma}_j + \varepsilon) \) \((j = 1, \ldots, n)\) are disjoint and lie in \([T_0, T]\). Then we have

\[ I_2(T) = \left( \int_{\bar{\gamma}_1 - \varepsilon}^{\bar{\gamma}_1 + \varepsilon} + \ldots + \int_{\bar{\gamma}_n - \varepsilon}^{\bar{\gamma}_n + \varepsilon} \right) |\zeta(\frac{1}{2} + it)|^2 h(t) \, dt \]

\[ + \left( \int_{T_0}^{\bar{\gamma}_1 - \varepsilon} + \int_{\bar{\gamma}_1 + \varepsilon}^{\bar{\gamma}_2 - \varepsilon} + \ldots + \int_{\bar{\gamma}_{n-1} + \varepsilon}^{\bar{\gamma}_n - \varepsilon} + \int_{\bar{\gamma}_n + \varepsilon}^{T} \right) |\zeta(\frac{1}{2} + it)|^2 h(t) \, dt \]

\[ = I_{21}(T, \varepsilon) + I_{22}(T, \varepsilon), \]

say. Hence

\[ I_2(T) = \lim_{\varepsilon \to 0} (I_{21}(T, \varepsilon) + I_{22}(T, \varepsilon)) = \lim_{\varepsilon \to 0} I_{22}(T, \varepsilon), \] (2.14)

since \(|\zeta(\frac{1}{2} + it)|^2 h(t)\) is in fact continuous on \([T_0, T]\). Let now, for \( t \neq \gamma \),

\[ k(t) := -3 \mathrm{Im} \left( \sum_{\rho} \int_{\frac{1}{2}}^{\infty} \frac{\tilde{a}(1 + (\rho - \sigma - it) \log X)}{\rho - \sigma - it} \, d\sigma \right), \] (2.15)

Then using (2.8) we have \( k'(t) = h(t) \), and integrating by parts we obtain

\[ I_{22}(T, \varepsilon) = |\zeta(\frac{1}{2} + i(\bar{\gamma}_1 - \varepsilon))|^2 k(\bar{\gamma}_1 - \varepsilon) - |\zeta(\frac{1}{2} + iT_0)|^2 k(T_0) \]

\[ + \ldots + |\zeta(\frac{1}{2} + iT)|^2 k(T) - |\zeta(\frac{1}{2} + i(\bar{\gamma}_n + \varepsilon))|^2 k(\bar{\gamma}_n + \varepsilon) \]

\[ - 2 \left( \int_{T_0}^{\bar{\gamma}_1 - \varepsilon} + \int_{\bar{\gamma}_1 + \varepsilon}^{\bar{\gamma}_2 - \varepsilon} + \ldots + \int_{\bar{\gamma}_{n-1} + \varepsilon}^{\bar{\gamma}_n - \varepsilon} + \int_{\bar{\gamma}_n + \varepsilon}^{T} \right) Z(t)Z'(t)k(t) \, dt. \]
Therefore from (2.9), (2.13) and (2.14) it follows, on integrating by parts and using $k(t) = -g(t)$ $(t \neq \gamma)$, that
\[
\pi \sum_{T_0 < \gamma \leq T} |\zeta(\frac{1}{2} + i\gamma)|^2 \\
= \lim_{\varepsilon \to 0} \sum_{T_0 < \gamma \leq T} \left( |\zeta(\frac{1}{2} + i(\gamma - \varepsilon))|^2 k(\gamma - \varepsilon) - |\zeta(\frac{1}{2} + i(\gamma + \varepsilon))|^2 k(\gamma + \varepsilon) \right),
\]
(2.16)
where $\gamma$ denotes distinct ordinates of zeros of $\zeta(s)$.

If the RH holds, then $|\zeta(\frac{1}{2} + it)|^2 k(t)$ is continuous at $t = \gamma$, since the zero of $\frac{1}{2} + i\gamma - \sigma - it$ $(\sigma = \frac{1}{2}, t = \gamma)$ in the denominator of $k(t)$ is cancelled by the corresponding zero $\zeta(\frac{1}{2} + i\gamma)$. Thus the limit in (2.16) is equal to zero, and so is then the left-hand side, which is trivial anyway. If the RH is not true, then we can say that
\[
\pi \sum_{T_0 < \gamma \leq T} |\zeta(\frac{1}{2} + i\gamma)|^2 = \sum_{T_0 < \gamma \leq T} L(\gamma),
\]
(2.17)
where
\[
L(\gamma) := \lim_{\varepsilon \to 0} \left( |\zeta(\frac{1}{2} + i(\gamma - \varepsilon))|^2 k(\gamma - \varepsilon) - |\zeta(\frac{1}{2} + i(\gamma + \varepsilon))|^2 k(\gamma + \varepsilon) \right).
\]
We have
\[
L(\gamma) := \sum_{j=1}^{r(\gamma)} \ell_{\beta_j \gamma}(\gamma),
\]
where $\frac{1}{2} < \beta_1 \gamma < \beta_2 \gamma < \ldots \beta_r \gamma < 1$, $r = r(\gamma)$ are the distinct abscissas of zeros with imaginary part $\gamma$ for a given $\gamma$ and, for $\beta$ one of the $\beta_j \gamma$'s, we set
\[
\ell_{\beta}(\gamma) = m(\beta + i\gamma) \lim_{\varepsilon \to 0} \left( |\zeta(\frac{1}{2} + i(\gamma + \varepsilon))|^2 \Im \int_{\frac{1}{2}}^{\infty} \frac{\hat{u}(1 + (\beta - \sigma - i\varepsilon) \log X)}{\beta - \sigma - i\varepsilon} \, d\sigma \right) \\
- |\zeta(\frac{1}{2} + i(\gamma - \varepsilon))|^2 \Im \int_{\frac{1}{2}}^{\infty} \frac{\hat{u}(1 + (\beta - \sigma + i\varepsilon) \log X)}{\beta - \sigma + i\varepsilon} \, d\sigma,
\]
(2.19)
where $m(\rho)$ is the multiplicity of the zero $\rho$. Namely looking at the definition (2.15) of $k(t)$ it follows that in the sum over $\rho$, only the terms with $\rho = \beta_j \gamma + i\gamma$ will be discontinuous at $t = \gamma \pm \varepsilon$ as $\varepsilon \to 0$. If we develop $|\zeta(\frac{1}{2} + i(\gamma \pm \varepsilon))|^2$ by using three terms in the Taylor formula and apply to each term the reasoning that follows, we shall obtain without difficulty that
\[
\ell_{\beta}(\gamma) = m(\beta + i\gamma)|\zeta(\frac{1}{2} + i\gamma)|^2 \lim_{\varepsilon \to 0} \left\{ \int_{\frac{1}{2}}^{2} \Im \left( \frac{\hat{u}(1 + (\beta - \sigma - i\varepsilon) \log X)}{\beta - \sigma - i\varepsilon} \right) \, d\sigma \right\},
\]
(2.20)
since the portion for $\sigma \geq 2$ is continuous in $\sigma$. In view of $\hat{u}(\hat{s}) = \overline{u(s)}$ it follows that
\[
\ell_{\beta}(\gamma) = m(\beta + i\gamma)|\zeta(\frac{1}{2} + i\gamma)|^2 \times
\lim_{\varepsilon \to 0} \int_{\frac{1}{2}}^{2} 2\Re \hat{u}(1 + (\beta - \sigma - i\varepsilon) \log X) + 2(\beta - \sigma) \Im \hat{u}(1 + (\beta - \sigma - i\varepsilon) \log X) \, d\sigma.
\]
Noting that

$$\Im \tilde{u}(1 + (\beta - \sigma - i\varepsilon) \log X) = \Im \tilde{u}(1 + (\beta - \sigma) \log X) + O(\varepsilon) = O(\varepsilon),$$
$$\Re \tilde{u}(1 + (\beta - \sigma - i\varepsilon) \log X) = \tilde{u}(1 + (\beta - \sigma) \log X) + O(\varepsilon),$$

we obtain

$$\ell_\beta(\tilde{\gamma}) = 2m(\beta + i\tilde{\gamma})|\zeta(\frac{1}{2} + i\tilde{\gamma})|^2 \lim_{\varepsilon \to 0} \varepsilon \int_\frac{1}{2}^2 \tilde{u}(1 + (\beta - \sigma) \log X) \frac{1}{(\beta - \sigma)^2 + \varepsilon^2} d\sigma.$$  

By using $$\tilde{u}(1) = 1$$ (this is equivalent to $$v(0) = 1$$) and

$$\lim_{\varepsilon \to 0} \varepsilon \int_\frac{1}{2}^2 \frac{d\sigma}{(\sigma - \beta)^2 + \varepsilon^2} = \lim_{\varepsilon \to 0} \left( \arctan \frac{2 - \beta}{\varepsilon} - \arctan \frac{1 - \beta}{\varepsilon} \right) = \left\{ \begin{array}{ll}
\frac{\pi}{2} & (\beta = \frac{1}{2}), \\
\pi & (\frac{1}{2} < \beta \leq 1), \end{array} \right.$$  

it follows that

$$\ell_\beta(\tilde{\gamma}) = \left\{ \begin{array}{ll}
\pi m(\frac{1}{2} + i\tilde{\gamma})|\zeta(\frac{1}{2} + i\tilde{\gamma})|^2 (= 0) & (\beta = \frac{1}{2}), \\
2\pi m(\beta + i\tilde{\gamma})|\zeta(\beta + i\tilde{\gamma})|^2 & (\beta > \frac{1}{2}). \end{array} \right.$$  

(2.21)

Therefore from (2.16), (2.18) and (2.21) we obtain, since $$m(\beta + i\gamma) = m(1 - \beta + i\gamma),$$

$$\pi \sum_{T_0 < \gamma \leq T} |\zeta(\frac{1}{2} + i\gamma)|^2 = \sum_{T_0 < \gamma \leq T} L(\tilde{\gamma})$$

$$= \sum_{T_0 < \gamma \leq T, \beta > \frac{1}{2}} 2\pi m(\beta + i\tilde{\gamma})|\zeta(\beta + i\tilde{\gamma})|^2 = \pi \sum_{T_0 < \gamma \leq T} |\zeta(\frac{1}{2} + i\gamma)|^2,$$

which is unfortunately trivial. Thus other approaches are to be sought if one wishes to bound non-trivially the sum in (1.1) (cf. Theorem 3).

The above discussion can be clearly generalized to Dirichlet series possessing a functional equation similar to the functional equation satisfied by $$\zeta(s)$$ (see e.g., [8]). Also it may be remarked that, if $$\gamma$$ is the ordinate of a zero, a similar analysis may be made by using the identity

$$\lim_{\varepsilon \to 0} (S(\gamma + \varepsilon) - S(\gamma - \varepsilon)) = \lim_{\varepsilon \to 0} (N(\gamma + \varepsilon) - N(\gamma - \varepsilon)),$$

but again we shall obtain (by the above method of proof) nothing more than an obvious identity.

3. PROOF OF THE THEOREMS

We use (1.6) to write

$$I(T) = \sum_{T_0 < \gamma \leq T} |\zeta(\frac{1}{2} + i\gamma)|^2 + \frac{1}{\pi} \Re \int_{T_0}^T \zeta'(\frac{1}{2} + it) \zeta(\frac{1}{2} + it) dt,$$
where \( I(T) \) is defined by (1.3). On the other hand, using (1.5) it is found that

\[
I(T) = \int_{T_0}^{T} |\zeta(\frac{1}{2} + it)|^2 \, d \left( N(t) - \frac{t}{2\pi} \log \frac{t}{2\pi} + \frac{t}{2\pi} + O \left( \frac{1}{T} \right) \right)
\]

\[
= \sum_{T_0 < \eta \leq T} |\zeta(\frac{1}{2} + i\gamma)|^2 - \frac{1}{2\pi} \int_{T_0}^{T} |\zeta(\frac{1}{2} + it)|^2 \log \left( \frac{t}{2\pi} \right) \, dt + O(1),
\]

From the two expressions for \( I(T) \) it follows that

\[
\Re \int_{T_0}^{T} \zeta'(\frac{1}{2} + it)\zeta(\frac{1}{2} - it) \, dt = -\frac{1}{4} \int_{T_0}^{T} |\zeta(\frac{1}{2} + it)|^2 \log \left( \frac{t}{2\pi} \right) \, dt + O(1). \quad (3.1)
\]

One can also obtain (3.1) by using that \( J(t) \), defined by (1.7), is purely imaginary. We have (see [4, Chapter 15])

\[
\int_{0}^{T} |\zeta(\frac{1}{2} + it)|^2 \, dt = T \log \left( \frac{T}{2\pi} \right) + (2C_0 - 1)T + E(T), \quad E(T) \lesssim T^c \quad (3.2)
\]

with some \( c < 1/3 \) (the optimal value of \( c \) is not of primary concern here). Hence differentiating (3.2) and inserting the resulting expression in (3.1) it is seen that

\[
-\frac{1}{4} \int_{T_0}^{T} \log^2 \left( \frac{t}{2\pi} \right) + 2C_0 \log \left( \frac{t}{2\pi} \right) \, dt + O(T^{1/3})
\]

\[
= -\pi \int_{T_0/2\pi}^{T/2\pi} (\log^2 u + 2C_0 \log u) \, du + O(T^{1/3}).
\]

But as

\[
\int \log u \, du = u(\log u - 1), \quad \int \log^2 u \, du = u(\log^2 u - 2 \log u + 2),
\]

it follows that

\[
-\pi \int_{T_0/2\pi}^{T/2\pi} (\log^2 u + 2C_0 \log u) \, du
\]

\[
= -\pi \left\{ \frac{T}{2\pi} \log^2 \left( \frac{T}{2\pi} \right) + (2C_0 - 2) \frac{T}{2\pi} \log \left( \frac{T}{2\pi} \right) + (2 - 2C_0) \frac{T}{2\pi} \right\} + O(1)
\]

\[
= -\left\{ \frac{T}{2} \log^2 \left( \frac{T}{2\pi} \right) + (C_0 - 1)T \log \left( \frac{T}{2\pi} \right) + (1 - C_0)T \right\} + O(1),
\]

and (1.9) easily follows from (3.1). Note that a direct proof of (1.9), by using approximate functional equations for \( \zeta(\frac{1}{2} + it) \) and \( \zeta'(\frac{1}{2} + it) \), seems quite difficult.

To obtain Theorem 2 note that, on the RH, the left-hand side of (1.6) vanishes. Integration by parts gives then, for any given \( \varepsilon > 0 \),

\[
\int_{T_0}^{T} Z(t)Z'(t)S(t) \, dt = -\frac{1}{2\pi} \Re \int_{T_0}^{T} \zeta'(\frac{1}{2} + it)\zeta(\frac{1}{2} - it) \, dt + O_{\varepsilon} (T^c), \quad (3.3)
\]
since both $|Z(t)| = |\zeta(\frac{1}{2} + it)|$ and $Z'(t)$ are $\ll t^\varepsilon$ on the RH. Hence (1.10) follows from (1.9) and (3.3).

To prove Theorem 3 note first that it suffices to prove that

$$J(T) := \int_T^{2T} |\zeta(\frac{1}{2} + it)|^2 \, dS(t) \ll T \log^2 T \log \log T. \quad (3.4)$$

We have

$$J(T) = -\frac{1}{\pi} \int_T^{2T} |\zeta(\frac{1}{2} + it)|^2 \sum_{p \leq T} p^{-1/2} \log p \cdot \cos(t \log p) \, dt$$

$$+ \int_T^{2T} |\zeta(\frac{1}{2} + it)|^2 \, dR(t)$$

$$= -\frac{1}{\pi} \int_T^{2T} |\zeta(\frac{1}{2} + it)|^2 \sum_{p \leq T} p^{-1/2} \log p \cdot \cos(t \log p) \, dt$$

$$+ i \int_T^{2T} (\zeta(\frac{1}{2} + it)\zeta'(\frac{1}{2} - it) - \zeta'(\frac{1}{2} + it)\zeta(\frac{1}{2} - it)) \, R(t) \, dt$$

$$+ O(T^{1/3}), \quad (3.5)$$

where

$$R(t) := S(t) + \frac{1}{\pi} \sum_{p \leq T} p^{-1/2} \sin(t \log p). \quad (3.6)$$

Here $\delta = 1/(40k), k \in \mathbb{N}$, and $p$ denotes primes. Note that by K.-M. Tsang’s result [9] we have, uniformly in $k$,

$$\int_T^{2T} R^{2k}(t) \, dt \ll (Tck)^{2k} \quad (3.7)$$

with some absolute constant $c > 0$. Let $V = V(T) \to \infty$ as $T \to \infty$ be a positive function, and

$$H(T, V) := \{ t : (T \leq t \leq 2T) \land |R(t)| \geq V \}.$$ 

Then by taking $k = [V/(10c)]$ it follows from (3.7) that ($\mu(\cdot)$ denotes measure)

$$\mu(H(T, V)) \ll T \exp(-c_1 V) \quad (c_1 > 0). \quad (3.8)$$

Take now $V = \frac{100}{c_1} \log \log T$. Then by using (3.8) and Hölder’s inequality we obtain

$$\int_{H(T, V)} (\zeta(\frac{1}{2} + it)\zeta'(\frac{1}{2} - it) - \zeta'(\frac{1}{2} + it)\zeta(\frac{1}{2} - it)) \, R(t) \, dt$$

$$\ll \left( \mu(H(T, V)) \int_T^{2T} |\zeta(\frac{1}{2} + it)|^4 \, dt \int_T^{2T} |\zeta'(\frac{1}{2} + it)|^4 \, dt \int_T^{2T} R^4(t) \, dt \right)^{1/4} \ll T,$$
since $\int_0^T |\zeta'(\frac{1}{2} + it)|^4 \, dt \ll T \log^6 T$. Next we have to estimate

$$\int_T^{2T} |\zeta'(\frac{1}{2} + it)|^2 \sum_{p \leq T^\delta} p^{-1/2} \log p \cdot \cos(t \log p) \, dt,$$

of which the relevant part is, on using the approximate functional equation for $\zeta^2(s)$ (see [4, Chapter 4]),

$$\sum_{n \leq 2T} d(n)n^{-1/2} \sum_{p \leq T^\delta} p^{-1/2} \log p \int_T^{2T} \cos \left( \frac{t \log \frac{t}{2\pi n} - t - \frac{\pi}{4}}{2} \right) \sin(t \log p) \, dt.$$

If we write the trigonometric functions as exponentials, then (by the first derivative test, namely [4, Lemma 2.1]) the above expression will be $O(T)$ plus two conjugate expressions, one of which is

$$\sum_{n \leq 2T} d(n)n^{-1/2} \sum_{p \leq T^\delta} p^{-1/2} \log p \int_T^{2T} \exp \left( \frac{it \log \frac{t}{2\pi np} - t}{2} \right) \, dt. \quad (3.9)$$

For $pn < C_1 T$ or $pn > C_2 T$ with $C_1$ sufficiently small and $C_2$ sufficiently large the contribution will be $O(T)$ by the first derivative test. For $C_1 T \leq pn \leq C_2 T$ we obtain, by the second derivative test (see [4]) that the contribution is

$$\ll \sum_{n \leq 2T} d(n)n^{-1/2} \left( \frac{T}{n} \right)^{1/2} \cdot T^{1/2} \ll T \log^2 T.$$

Finally there remains

$$\int_{[T:2T][H(T,V)]} (\zeta'(\frac{1}{2} + it)\zeta'(\frac{1}{2} - it) - \zeta'(\frac{1}{2} + it)\zeta'(\frac{1}{2} - it))R(t) \, dt$$

$$\ll \log \log T \int_T^{2T} |\zeta'(\frac{1}{2} + it)|^2 \, dt$$

$$\ll \log \log T \left( \int_T^{2T} |\zeta'(\frac{1}{2} + it)|^2 \, dt \int_T^{2T} |\zeta'(\frac{1}{2} + it)|^2 \, dt \right)^{1/2}$$

$$\ll T \log^2 T \log \log T,$$

so that the proof of Theorem 3 is complete.

To prove the first part of Theorem 4, namely the bound (1.12) unconditionally, it suffices to use the Cauchy-Schwarz inequality for integrals and the preceding method the proof. Then (1.12) reduces to proving that

$$\int_T^{2T} |\zeta'(\frac{1}{2} + it)|^2 \sum_{p \leq T^\delta} p^{-1/2} \log p \cdot \cos(t \log p) \, dt \ll T \log T (\log \log T)^2.$$
If we write \( \zeta(\frac{1}{2} + it) \) as a sum of Dirichlet polynomials of length \( \ll \sqrt{T} \) and apply the mean value theorem for Dirichlet polynomials (see e.g., [4, Chapter 4]) the proof reduces to showing that

\[
\sum_{m \leq X} \frac{\omega^2(m)}{m} \ll \log X (\log \log X)^2 \quad (\omega(m) = \sum_{p|m} 1),
\]

which follows by partial summation from the formula (see [4, Chapter 13])

\[
\sum_{m \leq X} \omega^2(m) = X (\log \log X)^2 + O(X \log \log X).
\]

The problem of proving, under the RH, the bound in (1.13) is more involved. We shall use (1.3), (2.2) and (2.4) with \( \beta = \frac{1}{2} \), and prove the bound in (1.13) for the integral over \([T, 2T]\), which is sufficient. The contribution of the trivial zeros in (2.2) as well as of the second integral is easily seen to be \( \ll T \log T \), and so is also the contribution of the first term on the right-hand side of (2.4). The contribution of the second term is

\[
\int_T^{2T} |\zeta(\frac{1}{2} + it)|^2 \sum_{|\gamma-t| \leq 1/\log X} \log \left( 1 + \frac{1}{|\gamma-t| \log X} \right) \, dt \\
= \sum_{T^{-1/\log X} \leq \gamma \leq 2T+1/\log X} \int_{\gamma-1/\log X}^{\gamma+1/\log X} |\zeta(\frac{1}{2} + it)|^2 \log \left( 1 + \frac{1}{|\gamma-t| \log X} \right) \, dt \\
= \frac{1}{\log X} \sum_{T^{-1/\log X} \leq \gamma \leq 2T+1/\log X} \int_{-1}^{1} \left| \zeta(\frac{1}{2} + i(\gamma + \frac{u}{\log X})) \right|^2 \log \left( 1 + \frac{1}{|u|} \right) \, du.
\]

If we exchange the order of integration and summation and use (1.2) (which is known to hold under the RH), then the above contribution is \( \ll T \log T \), since \( X = T^\delta \) and

\[
\int_{-1}^{1} \log \left( 1 + \frac{1}{|u|} \right) \, du \ll 1.
\]

A similar analysis holds for the contribution of the second sum in (2.4), if we split it into portions \( \gamma \leq T/3, T/3 < \gamma \leq 3T \) and \( \gamma > 3T \).

There remains in (2.2) the contribution of

\[
\Im \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-1/2-\nu} \left( e^{\log n/\log X} \right) = - \sum_{2 \leq n \leq X} \frac{\Lambda(n)}{\log n} n^{-1/2} \sin(t \log n) \\
+ \sum_{2 \leq n \leq X} \frac{\Lambda(n)}{\log n} n^{-1/2} \left( 1 - v \left( e^{\log n/\log X} \right) \right) \sin(t \log n). \tag{3.10}
\]

By construction \( v(0) = 1 \) and \( v'(x) = -u(x) = 0 \) for \( 0 \leq x \leq 1 \), so that \( v(x) = 1 \) for \( 0 \leq x \leq 1 \), and then \( v(x) \) falls off monotonically to \( v(e) = 0 \). We have

\[
1 - v \left( e^{\log n/\log X} \right) \ll \frac{\log n}{\log X} \quad (2 \leq n \leq X).
\]
By the analysis that follows, the contribution of the right-hand side of (3.10) containing the $v$-function will be $\ll T \log T$, so we may concentrate on the first sum the right-hand side of (3.10). Since $\Lambda(n) = \log p$ if $n = p^m$, $p$ a prime and zero otherwise, it seen by using the approximate functional equation for $\zeta^2(s)$ (see e.g., [4, Chapter 4]) the main contribution from this sum will be contained in a multiple of

$$
\sum_{n \leq T/\pi} d(n)n^{-1/2} \sum_{p \leq T^3} p^{-1/2} \int_T^{2T} \cos \left( t \log \frac{t}{2\pi n} - t - \frac{\pi}{4} \right) \sin(t \log p) \, dt.
$$

By writing the trigonometric functions as exponentials and using the first and second derivative test it is seen that the above expression equals $O(T \log T)$ plus

$$
\sum_{n \leq T/\pi} d(n)n^{-1/2} \sum_{p \leq T^3} p^{-1/2} \Im \left\{ \int_T^{2T} e^{iF(t)} \, dt \right\},
$$

(3.11)

where we have set

$$
F(t) := t \log \frac{t}{2\pi np} - t - \frac{\pi}{4},
$$

so that

$$
F'(t) = \log \frac{t}{2\pi np}, \quad F''(t) = \frac{1}{t}.
$$

There will be a saddle point $t_0$ (solution of $F'(t_0) = 0$) for $t_0 = 2\pi np$. We split the range of summation in (3.11) into the subranges: I) $2\pi np \leq T - H$, II) $T - H < 2\pi np \leq T + H$, III) $T + H < 2\pi np \leq 2T - H$, IV) $2T - H < 2\pi np \leq 2T + H$ and V) $2\pi np > 2T + H$. The choice for $H$ will be

$$
H = T^{2/3}.
$$

The contribution of the ranges I) and V) is estimated analogously. In the former we have

$$
F'(t) \geq \log \frac{T}{2\pi np} \geq \log \frac{T}{T - H} \sim \frac{H}{T} = T^{-1/3}.
$$

Hence by the first derivative test the contribution is

$$
\ll T^{1/3} \sum_{p \leq T^3} p^{-1/2} \sum_{n \leq T/\pi} d(n)n^{-1/2} \ll T^{5/6+\delta} \ll T \log T.
$$

The contribution of the ranges II) and IV) is also estimated analogously. In the former we estimate the integral as $\ll T^{1/2}$ by the second derivative test. The contribution is then

$$
\ll T^{1/2} \sum_{p \leq T^3} p^{-1/2} \sum_{\frac{T-H}{p} \leq n \leq \frac{T+H}{p}} d(n)n^{-1/2}
\ll T^{1/2} \sum_{p \leq T^3} p^{-1/2} (T/p)^{-1/2} H p^{-1} \log T \ll T^{2/3} \log T \log \log T.
$$
There remains the range III) in which the saddle point method is used in the form of \cite[Lemma 4.6]{10} with the first $O$-term there in the form
\[
O(\lambda_{2}^{-1} \lambda_{3}^{1/3}) = O \left( T \cdot \left( \frac{1}{T^{2}} \right)^{1/3} \right) = O(T^{1/3}).
\]

In view of the choice of $H$ we have that $(t_{0} = 2\pi np)$
\[
\Im \left\{ \int_{T}^{2T} e^{iF(t)} \, dt \right\} = \Im \sqrt{\frac{2\pi}{F''(t_{0})}} \cdot e^{iF(t_{0}) + \frac{T}{4\pi}i + O(T^{1/3})}
\]
\[
= \Im \frac{2\pi}{F''(t_{0})} \cdot e^{-2\pi inp} + O(T^{1/3}) = O(T^{1/3})
\]
since $e^{-2\pi inp} = 1$ is real. This contribution is then
\[
\ll T^{1/3} \sum_{p \leq T^\beta} p^{-1/2} \sum_{\frac{1}{4}T \leq n \leq \frac{1}{4}T} d(n)n^{-1/2} \ll T^{5/6} \log T \log \log T.
\]
Collecting the above estimates we obtain (1.13), and the proof of Theorem 4 is complete.

In the preceding proof the RH was used via Gonek’s formula (1.2) and (2.4) with $\beta = \frac{1}{2}$. Thus an unconditional proof of (1.13) would require an unconditional proof of (1.2) (or an adequate upper bound estimate) plus an estimation of the integral with $|\zeta(\frac{1}{2} + it)|^{2}$ and (2.4) without the simplifying condition $\beta = \frac{1}{2}$. 
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