Inversion of the Dual Dunkl–Sonine Transform on $\mathbb{R}$ Using Dunkl Wavelets

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Abstract. We prove a Calderón reproducing formula for the Dunkl continuous wavelet transform on $\mathbb{R}$. We apply this result to derive new inversion formulas for the dual Dunkl–Sonine integral transform.

Key words: Dunkl continuous wavelet transform; Calderón reproducing formula; dual Dunkl–Sonine integral transform

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1 Introduction

The one-dimensional Dunkl kernel $e_{\gamma}$, $\gamma > -1/2$, is defined by

$$e_{\gamma}(z) = j_{\gamma}(iz) + \frac{z}{2(\gamma + 1)}j_{\gamma + 1}(iz), \quad z \in \mathbb{C},$$

where

$$j_{\gamma}(z) = \Gamma(\gamma + 1) \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n! \Gamma(n + \gamma + 1)}$$

is the normalized spherical Bessel function of index $\gamma$. It is well-known (see [3]) that the functions $e_{\gamma}(\lambda \cdot)$, $\lambda \in \mathbb{C}$, are solutions of the differential-difference equation

$$\Lambda_{\gamma}u = \lambda u, \quad u(0) = 1,$$

where

$$\Lambda_{\gamma}f(x) = f'(x) + \left(\gamma + \frac{1}{2}\right) \frac{f(x) - f(-x)}{x}$$

is the Dunkl operator with parameter $\gamma + 1/2$ associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. Those operators were introduced and studied by Dunkl [2, 3, 4] in connection with a generalization of the classical theory of spherical harmonics. Besides its mathematical interest, the Dunkl operator $\Lambda_{\alpha}$ has quantum-mechanical applications; it is naturally involved in the study of one-dimensional harmonic oscillators governed by Wigner’s commutation rules [6, 11, 16].

It is known, see for example [13, 14], that the Dunkl kernels on $\mathbb{R}$ possess the following Sonine type integral representation

$$e_{\beta}(\lambda x) = \int_{-|x|}^{|x|} K_{\alpha,\beta}(x, y) e_{\alpha}(\lambda y) |y|^{2\alpha + 1} dy, \quad \lambda \in \mathbb{C}, \quad x \neq 0,$$  \hspace{1cm} (1.1)
where
\[
K_{\alpha,\beta}(x, y) := \begin{cases} 
  a_{\alpha,\beta} \text{sgn}(x) \frac{(x^2 - y^2)^{\beta-\alpha-1}}{|x|^{2\beta+1}} & \text{if } |y| < |x|, \\
  0 & \text{if } |y| \geq |x|,
\end{cases}
\]
with \(\beta > \alpha > -\frac{1}{2}\), and
\[
a_{\alpha,\beta} := \frac{\Gamma(\beta + 1)}{\Gamma(\alpha + 1) \Gamma(\beta - \alpha)}.
\]

Define the Dunkl–Sonine integral transform \(X_{\alpha,\beta}\) and its dual \(\mathcal{t}X_{\alpha,\beta}\), respectively, by
\[
X_{\alpha,\beta} f(x) = \int_{|x|}^{|x|} K_{\alpha,\beta}(x, y) f(y) |y|^{2\alpha+1} dy,
\]
\[
\mathcal{t}X_{\alpha,\beta} f(y) = \int_{|x| \geq |y|} K_{\alpha,\beta}(x, y) f(x) |x|^{2\beta+1} dx.
\]

Soltani has showed in [14] that the dual Dunkl–Sonine integral transform \(\mathcal{t}X_{\alpha,\beta}\) is a transmutation operator between \(\Lambda_\alpha\) and \(\Lambda_\beta\) on the Schwartz space \(S(\mathbb{R})\), i.e., it is an automorphism of \(S(\mathbb{R})\) satisfying the intertwining relation
\[
\mathcal{t}X_{\alpha,\beta} \Lambda_\beta f = \Lambda_\alpha \mathcal{t}X_{\alpha,\beta} f, \quad f \in S(\mathbb{R}).
\]

The same author [14] has obtained inversion formulas for the transform \(\mathcal{t}X_{\alpha,\beta}\) involving pseudodifferential-difference operators and only valid on a restricted subspace of \(S(\mathbb{R})\).

The purpose of this paper is to investigate the use of Dunkl wavelets (see [5]) to derive a new inversion of the dual Dunkl–Sonine transform on some Lebesgue spaces. For other applications of wavelet type transforms to inverse problems we refer the reader to [7, 8] and the references therein.

The content of this article is as follows. In Section 2 we recall some basic harmonic analysis results related to the Dunkl operator. In Section 3 we list some basic properties of the Dunkl–Sonine integral transform and its dual. In Section 4 we give the definition of the Dunkl continuous wavelet transform and we establish for this transform a Calderón formula. By combining the results of the two previous sections, we obtain in Section 5 two new inversion formulas for the dual Dunkl–Sonine integral transform.

## 2 Preliminaries

**Note 2.1.** Throughout this section assume \(\gamma > -1/2\). Define \(L^p(\mathbb{R}, |x|^{2\gamma+1} dx), 1 \leq p \leq \infty\), as the class of measurable functions \(f\) on \(\mathbb{R}\) for which \(\|f\|_{p,\gamma} < \infty\), where
\[
\|f\|_{p,\gamma} = \left( \int_{\mathbb{R}} |f(x)|^p |x|^{2\gamma+1} dx \right)^{1/p}, \quad \text{if } p < \infty,
\]
and \(\|f\|_{\infty,\gamma} = \|f\|_\infty = \text{ess sup}_{x \in \mathbb{R}} |f(x)|\). \(S(\mathbb{R})\) stands for the usual Schwartz space.

The Dunkl transform of order \(\gamma + 1/2\) on \(\mathbb{R}\) is defined for a function \(f\) in \(L^1(\mathbb{R}, |x|^{2\gamma+1} dx)\) by
\[
\mathcal{F}_\gamma f(\lambda) = \int_{\mathbb{R}} f(x) e^{\gamma(-i\lambda x)} |x|^{2\gamma+1} dx, \quad \lambda \in \mathbb{R}.
\]
Remark 2.2. It is known that the Dunkl transform $F_\gamma$ maps continuously and injectively $L^1(\mathbb{R}, |x|^{2\gamma+1}dx)$ into the space $C_0(\mathbb{R})$ (of continuous functions on $\mathbb{R}$ vanishing at infinity).

Two standard results about the Dunkl transform $F_\gamma$ are as follows.

Theorem 2.3 (see [1]).

(i) For every $f \in L^1 \cap L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ we have the Plancherel formula

$$\int_{\mathbb{R}} |f(x)|^2 |x|^{2\gamma+1}dx = m_\gamma \int_{\mathbb{R}} |F_\gamma f(\lambda)|^2 |\lambda|^{2\gamma+1}d\lambda,$$

where

$$m_\gamma = \frac{1}{2^{2\gamma+2}(\Gamma(\gamma + 1))^2}. \quad (2.2)$$

(ii) The Dunkl transform $F_\alpha$ extends uniquely to an isometric isomorphism from $L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ onto $L^2(\mathbb{R}, m_\gamma |\lambda|^{2\gamma+1}d\lambda)$. The inverse transform is given by

$$F_\gamma^{-1}g(x) = m_\gamma \int_{\mathbb{R}} g(\lambda)e_{\gamma}(i\lambda x)|\lambda|^{2\gamma+1}d\lambda,$$

where the integral converges in $L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$.

Theorem 2.4 (see [1]). The Dunkl transform $F_\alpha$ is an automorphism of $S(\mathbb{R})$.

An outstanding result about Dunkl kernels on $\mathbb{R}$ (see [12]) is the product formula

$$e_{\gamma}(\lambda x)e_{\gamma}(\lambda y) = T_\gamma^x(e_{\gamma}(\lambda \cdot))(y), \quad \lambda \in \mathbb{C}, \quad x, y \in \mathbb{R},$$

where $T_\gamma^x$ stand for the Dunkl translation operators defined by

$$T_\gamma^x f(y) = \frac{1}{2} \int_{-1}^{1} f \left( \sqrt{x^2 + y^2 - 2xyt} \right) \left( 1 + \frac{x-y}{\sqrt{x^2 + y^2 - 2xyt}} \right) W_\gamma(t)dt,$$

$$\quad + \frac{1}{2} \int_{-1}^{1} f \left( -\sqrt{x^2 + y^2 - 2xyt} \right) \left( 1 - \frac{x-y}{\sqrt{x^2 + y^2 - 2xyt}} \right) W_\gamma(t)dt, \quad (2.3)$$

with

$$W_\gamma(t) = \frac{\Gamma(\gamma + 1)}{\sqrt{\pi} \Gamma(\gamma + 1/2)} (1 + t) \left( 1 - t^2 \right)^{\gamma-1/2}.$$

The Dunkl convolution of two functions $f, g$ on $\mathbb{R}$ is defined by the relation

$$f \ast_\gamma g(x) = \int_{\mathbb{R}} T_\gamma^x f(-y)g(y)|y|^{2\gamma+1}dy. \quad (2.4)$$

Proposition 2.5 (see [13]).

(i) Let $p, q, r \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1}{r}$. If $f \in L^p(\mathbb{R}, |x|^{2\gamma+1}dx)$ and $g \in L^q(\mathbb{R}, |x|^{2\gamma+1}dx)$, then $f \ast_\gamma g \in L^r(\mathbb{R}, |x|^{2\gamma+1}dx)$ and

$$||f \ast_\gamma g||_{r, \gamma} \leq 4 ||f||_{p, \gamma} ||g||_{q, \gamma}. \quad (2.5)$$

(ii) For $f \in L^1(\mathbb{R}, |x|^{2\gamma+1}dx)$ and $g \in L^p(\mathbb{R}, |x|^{2\gamma+1}dx)$, $p = 1$ or 2, we have

$$F_\gamma(f \ast_\gamma g) = F_\gamma f F_\gamma g. \quad (2.6)$$

For more details about harmonic analysis related to the Dunkl operator on $\mathbb{R}$ the reader is referred, for example, to [9, 10].
3 The dual Dunkl–Sonine integral transform

Throughout this section assume $\beta > \alpha > -1/2$.

**Definition 3.1** (see [14]). The dual Dunkl–Sonine integral transform $t^\Lambda_{\alpha,\beta}$ is defined for smooth functions on $\mathbb{R}$ by

$$t^\Lambda_{\alpha,\beta}f(y) := \int_{|x| \geq |y|} K_{\alpha,\beta}(x,y) f(x)|x|^{2\beta+1} \, dx, \quad y \in \mathbb{R},$$

(3.1)

where $K_{\alpha,\beta}$ is the kernel given by (1.2).

**Remark 3.2.** Clearly, if $\text{supp}(f) \subset [-a,a]$ then $\text{supp}(t^\Lambda_{\alpha,\beta}f) \subset [-a,a]$.

The next statement provides formulas relating harmonic analysis tools tied to $\Lambda_{\alpha}$ with those tied to $\Lambda_{\beta}$, and involving the operator $t^\Lambda_{\alpha,\beta}$.

**Proposition 3.3.**

(i) The dual Dunkl–Sonine integral transform $t^\Lambda_{\alpha,\beta}$ maps $L^1(\mathbb{R}, |x|^{2\beta+1} \, dx)$ continuously into $L^1(\mathbb{R}, |x|^{2\alpha+1} \, dx)$.

(ii) For every $f \in L^1(\mathbb{R}, |x|^{2\beta+1} \, dx)$ we have the identity

$$F_{\beta}(f) = F_{\alpha} \circ t^\Lambda_{\alpha,\beta}(f).$$

(3.2)

(iii) Let $f, g \in L^1(\mathbb{R}, |x|^{2\beta+1} \, dx)$. Then

$$t^\Lambda_{\alpha,\beta}(f *_{\beta} g) = t^\Lambda_{\alpha,\beta}f *_{\alpha} t^\Lambda_{\alpha,\beta}g.$$  

(3.3)

**Proof.** Let $f \in L^1(\mathbb{R}, |x|^{2\beta+1} \, dx)$. By Fubini’s theorem we have

$$\int_{\mathbb{R}} t^\Lambda_{\alpha,\beta}(|f|)(y)|y|^{2\alpha+1} \, dy = \int_{\mathbb{R}} \left( \int_{|x| \geq |y|} K_{\alpha,\beta}(x,y)|f(x)||x|^{2\beta+1} \, dx \right) |y|^{2\alpha+1} \, dy$$

$$= \int_{\mathbb{R}} |f(x)| \left( \int_{|x|}^{\infty} K_{\alpha,\beta}(x,y)|y|^{2\alpha+1} \, dy \right) |x|^{2\beta+1} \, dx.$$

But by (1.1),

$$\int_{|x|}^{\infty} K_{\alpha,\beta}(x,y)|y|^{2\alpha+1} \, dy = e_{\beta}(0) = 1.$$  

(3.4)

Hence, $t^\Lambda_{\alpha,\beta}f$ is almost everywhere defined on $\mathbb{R}$, belongs to $L^1(\mathbb{R}, |x|^{2\alpha+1} \, dx)$ and $||t^\Lambda_{\alpha,\beta}f||_{1,\alpha} \leq ||f||_{1,\beta}$, which proves (i). Identity (3.2) follows by using (1.1), (2.1), (3.1), and Fubini’s theorem. Identity (3.3) follows by applying the Dunkl transform $F_{\alpha}$ to both its sides and by using (2.6), (3.2) and Remark 2.2. ■

**Remark 3.4.** From (3.2) and Remark 2.2, we deduce that the transform $t^\Lambda_{\alpha,\beta}$ maps $L^1(\mathbb{R}, |x|^{2\beta+1} \, dx)$ injectively into $L^1(\mathbb{R}, |x|^{2\alpha+1} \, dx)$.

From [14] we have the following result.
Theorem 3.5. The dual Dunkl–Sonine integral transform \( \mathcal{X}_{\alpha,\beta} \) is an automorphism of \( \mathcal{S}(\mathbb{R}) \) satisfying the intertwining relation
\[
\mathcal{X}_{\alpha,\beta} \Lambda f = \Lambda \mathcal{X}_{\alpha,\beta} f, \quad f \in \mathcal{S}(\mathbb{R}).
\]
Moreover \( \mathcal{X}_{\alpha,\beta} \) admits the factorization
\[
\mathcal{X}_{\alpha,\beta} f = \mathcal{V}_\gamma^{-1} \mathcal{V}_\gamma f \quad \text{for all } f \in \mathcal{S}(\mathbb{R}),
\]
where for \( \gamma > -1/2 \), \( \mathcal{V}_\gamma \) denotes the dual Dunkl intertwining operator given by
\[
\mathcal{V}_\gamma f(y) = \frac{\Gamma(\gamma + 1)}{\sqrt{\pi} \Gamma(\gamma + 1/2)} \int_{|x| \geq |y|} \text{sgn}(x) (x + y) (x^2 - y^2)^{-1/2} f(x) \, dx.
\]

Definition 3.6 (see [14]). The Dunkl–Sonine integral transform \( \mathcal{X}_{\alpha,\beta} \) is defined for a locally bounded function \( f \) on \( \mathbb{R} \) by
\[
\mathcal{X}_{\alpha,\beta} f(x) = \begin{cases} 
\int_{|x|} \mathcal{K}_{\alpha,\beta}(x, y) f(y) |y|^{2\alpha+1} \, dy & \text{if } x \neq 0, \\
 f(0) & \text{if } x = 0.
\end{cases}
\]

Remark 3.7.

(i) Notice that by [3.4], \( ||\mathcal{X}_{\alpha,\beta} f||_\infty \leq ||f||_\infty \) if \( f \in L^\infty(\mathbb{R}) \).

(ii) It follows from (1.1) that
\[
e_\beta(\lambda x) = \mathcal{X}_{\alpha,\beta}(e_\alpha(\lambda \cdot))(x)
\]
for all \( \lambda \in \mathbb{C} \) and \( x \in \mathbb{R} \).

Proposition 3.8.

(i) For any \( f \in L^\infty(\mathbb{R}) \) and \( g \in L^1(\mathbb{R}, |x|^{2\beta+1} \, dx) \) we have the duality relation
\[
\int_{\mathbb{R}} \mathcal{X}_{\alpha,\beta} f(x) g(x) |x|^{2\beta+1} \, dx = \int_{\mathbb{R}} f(y) \mathcal{X}_{\alpha,\beta} g(y) |y|^{2\alpha+1} \, dy.
\]

(ii) Let \( f \in L^1(\mathbb{R}, |x|^{2\beta+1} \, dx) \) and \( g \in L^\infty(\mathbb{R}) \). Then
\[
\mathcal{X}_{\alpha,\beta}(\mathcal{X}_{\alpha,\beta} f \ast_\alpha g) = f \ast_\beta \mathcal{X}_{\alpha,\beta} g.
\]

Proof. Identity (3.7) follows by using (3.1), (3.5) and Fubini’s theorem. Let us check (3.8). Let \( \psi \in \mathcal{S}(\mathbb{R}) \). By using (3.4), (3.7) and Fubini’s theorem, we have
\[
\int_{\mathbb{R}} f \ast_\beta \mathcal{X}_{\alpha,\beta} g(x) \psi(x) |x|^{2\beta+1} \, dx = \int_{\mathbb{R}} \mathcal{X}_{\alpha,\beta} g(x) \psi \ast_\beta f^-(x) |x|^{2\beta+1} \, dx
\]
\[
= \int_{\mathbb{R}} g(y) \mathcal{X}_{\alpha,\beta} \psi \ast_\beta f^-(y) |y|^{2\alpha+1} \, dy = \int_{\mathbb{R}} g(y) (\mathcal{X}_{\alpha,\beta} \psi \ast_\alpha (\mathcal{X}_{\alpha,\beta} f^-))(y) |y|^{2\alpha+1} \, dy,
\]
where \( f^-(x) = f(-x), \ x \in \mathbb{R} \). But an easy computation shows that \( \mathcal{X}_{\alpha,\beta} f^- = (\mathcal{X}_{\alpha,\beta} f)^- \). Hence,
\[
\int_{\mathbb{R}} f \ast_\beta \mathcal{X}_{\alpha,\beta} g(x) \psi(x) |x|^{2\beta+1} \, dx = \int_{\mathbb{R}} g(y) \mathcal{X}_{\alpha,\beta} \psi \ast_\alpha (\mathcal{X}_{\alpha,\beta} f^-)(y) |y|^{2\alpha+1} \, dy
\]
\[
= \int_{\mathbb{R}} \mathcal{X}_{\alpha,\beta} f \ast_\alpha g(y) \mathcal{X}_{\alpha,\beta} \psi(y) |y|^{2\alpha+1} \, dy = \int_{\mathbb{R}} \mathcal{X}_{\alpha,\beta} (\mathcal{X}_{\alpha,\beta} f \ast_\alpha g)(x) \psi(x) |x|^{2\beta+1} \, dx.
\]
This clearly yields the result. ■
4 Calderón’s formula for the Dunkl continuous wavelet transform

Throughout this section assume $\gamma > -1/2$.

**Definition 4.1.** We say that a function $g \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ is a Dunkl wavelet of order $\gamma$, if it satisfies the admissibility condition

$$0 < C^\gamma_g := \int_0^\infty |\mathcal{F}_\gamma g(\lambda)|^2 \frac{d\lambda}{\lambda} = \int_0^\infty |\mathcal{F}_\gamma g(-\lambda)|^2 \frac{d\lambda}{\lambda} < \infty. \quad (4.1)$$

**Remark 4.2.**

(i) If $g$ is real-valued we have $\mathcal{F}_\gamma g(-\lambda) = \overline{\mathcal{F}_\gamma g(\lambda)}$, so (4.1) reduces to

$$0 < C^\gamma_g := \int_0^\infty |\mathcal{F}_\gamma g(\lambda)|^2 \frac{d\lambda}{\lambda} < \infty.$$

(ii) If $0 \neq g \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ is real-valued and satisfies

$$\exists \eta > 0 \quad \text{such that } \mathcal{F}_\gamma g(\lambda) - \mathcal{F}_\gamma g(0) = O(\lambda^\eta) \quad \text{as } \lambda \to 0^+$$

then (4.1) is equivalent to $\mathcal{F}_\gamma g(0) = 0$.

**Note 4.3.** For a function $g$ in $L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ and for $(a, b) \in (0, \infty) \times \mathbb{R}$ we write

$$g_{a,b}(x) := \frac{1}{a^{2\gamma+2}} T^{-b}_{\gamma} g_{a}(x),$$

where $T^{-b}_{\gamma}$ are the generalized translation operators given by (2.3), and $g_{a}(x) := g(x/a), x \in \mathbb{R}$.

**Remark 4.4.** Let $g \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ and $a > 0$. Then it is easily checked that $g_{a} \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$, $\|g_{a}\|_{2,\gamma} = a^{\gamma+1} \|g\|_{2,\gamma}$, and $\mathcal{F}_\gamma(g_{a})(\lambda) = a^{2\gamma+2} \mathcal{F}_\gamma(g)(a\lambda)$.

**Definition 4.5.** Let $g \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ be a Dunkl wavelet of order $\gamma$. We define for regular functions $f$ on $\mathbb{R}$, the Dunkl continuous wavelet transform by

$$\Phi^\gamma_g(f)(a,b) := \int_{\mathbb{R}} f(x) \overline{g_{a,b}(x)} |x|^{2\gamma+1}dx$$

which can also be written in the form

$$\Phi^\gamma_g(f)(a,b) = \frac{1}{a^{2\gamma+2}} f \ast_{\gamma} \tilde{g}_{a}(b),$$

where $\ast_{\gamma}$ is the generalized convolution product given by (2.4), and $\tilde{g}_{a}(x) := g(-x/a), x \in \mathbb{R}$.

The Dunkl continuous wavelet transform has been investigated in depth in [5] in which precise definitions, examples, and a more complete discussion of its properties can be found. We look here for a Calderón formula for this transform. We start with some technical lemmas.

**Lemma 4.6.** For all $f, g \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ and all $\psi \in \mathcal{S}(\mathbb{R})$ we have the identity

$$\int_{\mathbb{R}} f \ast_{\gamma} g(x) \mathcal{F}^{-1}\psi(x) |x|^{2\gamma+1}dx = m_{\gamma} \int_{\mathbb{R}} \mathcal{F}_\gamma f(\lambda) \mathcal{F}_\gamma g(\lambda) \psi^{-}(\lambda) |\lambda|^{2\gamma+1}d\lambda,$$

where $m_{\gamma}$ is given by (2.2).
Proof. Fix $g \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ and $\psi \in \mathcal{S}(\mathbb{R})$. For $f \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ put

$$S_1(f) := \int_{\mathbb{R}} f \ast_{\gamma} g(x)F^{-1}_\gamma \psi(x)|x|^{2\gamma+1}dx$$

and

$$S_2(f) := m_\gamma \int_{\mathbb{R}} F_\gamma f(\lambda)F_\gamma g(\lambda)\psi^-|\lambda|^{2\gamma+1}d\lambda.$$ 

By (2.5), (2.6) and Theorem 2.3, we see that $S_1(f) = S_2(f)$ for each $f \in L^1 \cap L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$. Moreover, by using (2.5), Hölder’s inequality and Theorem 2.3 we have

$$|S_1(f)| \leq ||f \ast_{\gamma} g||_\infty ||F^{-1}_\gamma \psi||_{1,\gamma} \leq 4||f||_{2,\gamma}||g||_{2,\gamma}||F^{-1}_\gamma \psi||_{1,\gamma}$$

and

$$|S_2(f)| \leq m_\gamma ||F_\gamma fF_\gamma g||_{1,\gamma} ||\psi||_\infty \leq m_\gamma ||F_\gamma f||_{2,\gamma}||F_\gamma g||_{2,\gamma} ||\psi||_\infty = ||f||_{2,\gamma}||g||_{2,\gamma} ||\psi||_\infty,$$

which shows that the linear functionals $S_1$ and $S_2$ are bounded on $L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$. Therefore $S_1 \equiv S_2$, and the lemma is proved. \[\square\]

Lemma 4.7. Let $f_1, f_2 \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$. Then $f_1 \ast_{\gamma} f_2 \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ if and only if $F_\gamma f_1F_\gamma f_2 \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ and we have

$$F_\gamma(f_1 \ast_{\gamma} f_2) = F_\gamma f_1F_\gamma f_2$$ 

in the $L^2$-case.

Proof. Suppose $f_1 \ast_{\gamma} f_2 \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$. By Lemma 4.6 and Theorem 2.3 we have for any $\psi \in \mathcal{S}(\mathbb{R})$,

$$m_\gamma \int_{\mathbb{R}} F_\gamma f_1(\lambda)F_\gamma f_2(\lambda)\psi(\lambda)|\lambda|^{2\gamma+1}d\lambda = \int_{\mathbb{R}} f_1 \ast_{\gamma} f_2(x)F^{-1}_\gamma \psi^-(-x)|x|^{2\gamma+1}dx$$

$$= \int_{\mathbb{R}} f_1 \ast_{\gamma} f_2(x)(F^{-1}_\gamma \psi^-)(x)|x|^{2\gamma+1}dx = m_\gamma \int_{\mathbb{R}} F_\gamma(f_1 \ast_{\gamma} f_2)(\lambda)\psi(\lambda)|\lambda|^{2\gamma+1}d\lambda,$$

which shows that $F_\gamma f_1F_\gamma f_2 = F_\gamma (f_1 \ast_{\gamma} f_2)$. Conversely, if $F_\gamma f_1F_\gamma f_2 \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$, then by Lemma 4.6 and Theorem 2.3, we have for any $\psi \in \mathcal{S}(\mathbb{R})$,

$$\int_{\mathbb{R}} f_1 \ast_{\gamma} f_2(x)F^{-1}_\gamma \psi(x)|x|^{2\gamma+1}dx = m_\gamma \int_{\mathbb{R}} F_\gamma f_1(\lambda)F_\gamma f_2(\lambda)\psi(\lambda)|\lambda|^{2\gamma+1}d\lambda$$

$$= \int_{\mathbb{R}} F^{-1}_\gamma(F_\gamma f_1F_\gamma f_2)(x)F^{-1}_\gamma \psi(x)|x|^{2\gamma+1}dx,$$

which shows, in view of Theorem 2.4, that $f_1 \ast_{\gamma} f_2 = F^{-1}_\gamma(F_\gamma f_1F_\gamma f_2)$. This achieves the proof of Lemma 4.7. \[\square\]

A combination of Lemma 4.7 and Theorem 2.3 gives us the following.

Lemma 4.8. Let $f_1, f_2 \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$. Then

$$\int_{\mathbb{R}} |f_1 \ast_{\gamma} f_2(x)|^2|x|^{2\gamma+1}dx = m_\gamma \int_{\mathbb{R}} |F_\gamma f_1(\lambda)|^2|F_\gamma f_2(\lambda)|^2|\lambda|^{2\gamma+1}d\lambda,$$

where both sides are finite or infinite.
Lemma 4.9. Let \( g \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx) \) be a Dunkl wavelet of order \( \gamma \) such that \( \mathcal{F}_\gamma g \in L^\infty(\mathbb{R}) \).

For \( 0 < \varepsilon < \delta < \infty \) define

\[
G_{\varepsilon, \delta}(x) := \frac{1}{C_g} \int_\varepsilon^\delta g_a *_\gamma \bar{g}_a(x) \frac{da}{a^{4\gamma+5}}
\]  

(4.2)

and

\[
K_{\varepsilon, \delta}(\lambda) := \frac{1}{C_g} \int_\varepsilon^\delta |\mathcal{F}_\gamma g(a\lambda)|^2 \frac{da}{a}.
\]  

(4.3)

Then

\[
G_{\varepsilon, \delta} \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx), \quad K_{\varepsilon, \delta} \in (L^1 \cap L^2)(\mathbb{R}, |x|^{2\gamma+1}dx),
\]

(4.4)

and

\[
\mathcal{F}_\gamma(G_{\varepsilon, \delta}) = K_{\varepsilon, \delta}.
\]

Proof. Using Schwarz inequality for the measure \( \frac{da}{a^{4\gamma+5}} \) we obtain

\[
|G_{\varepsilon, \delta}(x)|^2 \leq \frac{1}{(C_g)^2} \left( \int_\varepsilon^\delta \frac{da}{a^{4\gamma+5}} \right)^2 \int_\varepsilon^\delta |g_a *_\gamma \bar{g}_a(x)|^2 \frac{da}{a^{4\gamma+5}}.
\]

so

\[
\int_\mathbb{R} |G_{\varepsilon, \delta}(x)|^2 |x|^{2\gamma+1}dx \leq \frac{1}{(C_g)^2} \left( \int_\varepsilon^\delta \frac{da}{a^{4\gamma+5}} \right)^2 \int_\varepsilon^\delta \int_\mathbb{R} |g_a *_\gamma \bar{g}_a(x)|^2 |x|^{2\gamma+1}dx \frac{da}{a^{4\gamma+5}}.
\]

By Theorem 2.3, Lemma 4.8, and Remark 4.4, we have

\[
\int_\mathbb{R} |g_a *_\gamma \bar{g}_a(x)|^2 |x|^{2\gamma+1}dx = m_\gamma \int_\mathbb{R} |\mathcal{F}_\gamma(g_a)\lambda|^4 |\lambda|^{2\gamma+1}d\lambda
\]

\[
\leq m_\gamma \|\mathcal{F}_\gamma(g_a)\|_2^2 \int_\mathbb{R} |\mathcal{F}_\gamma(g_a)\lambda|^2 |\lambda|^{2\gamma+1}d\lambda
\]

\[
= \|\mathcal{F}_\gamma(g_a)\|_2^2 \|g_a\|_{2,\gamma}^2 = a^{6\gamma+6} \|\mathcal{F}_\gamma g\|_\infty^2 \|g\|_{2,\gamma}^2.
\]

Hence

\[
\int_\mathbb{R} |G_{\varepsilon, \delta}(x)|^2 |x|^{2\gamma+1}dx \leq \frac{\|\mathcal{F}_\gamma g\|_\infty^2 \|g\|_{2,\gamma}^2}{(C_g)^2} \left( \int_\varepsilon^\delta a^{2\gamma+1}da \right) \left( \int_\varepsilon^\delta \frac{da}{a^{4\gamma+5}} \right) < \infty.
\]

The second assertion in (4.4) is easily checked. Let us calculate \( \mathcal{F}_\gamma(G_{\varepsilon, \delta}) \). Fix \( x \in \mathbb{R} \). From Theorem 2.3 and Lemma 4.7, we get

\[
g_a *_\gamma \bar{g}_a(x) = m_\gamma \int_\mathbb{R} |\mathcal{F}_\gamma(g_a)\lambda|^2 e_\gamma(i\lambda x) |\lambda|^{2\gamma+1}d\lambda,
\]

so

\[
G_{\varepsilon, \delta}(x) = \frac{m_\gamma}{C_g} \int_\varepsilon^\delta \left( \int_\mathbb{R} |\mathcal{F}_\gamma(g_a)\lambda|^2 e_\gamma(i\lambda x) |\lambda|^{2\gamma+1}d\lambda \right) \frac{da}{a^{4\gamma+5}}.
\]

As \( |e_\gamma(iz)| \leq 1 \) for all \( z \in \mathbb{R} \) (see [12]), we deduce by Theorem 2.3 that

\[
m_\gamma \int_\varepsilon^\delta \int_\mathbb{R} |\mathcal{F}_\gamma(g_a)\lambda|^2 |e_\gamma(i\lambda x)||\lambda|^{2\gamma+1}d\lambda \frac{da}{a^{4\gamma+5}}
\]
\[
\leq \int_\varepsilon^\delta \|g_a\|^2 \frac{da}{a^{4\gamma+5}} = \|g\|^2 \int_\varepsilon^\delta \frac{da}{a^{2\gamma+3}} < \infty.
\]

Hence, applying Fubini’s theorem, we find that
\[
G_{\varepsilon,\delta}(x) = m_\gamma \int_\Re \left( \frac{1}{C g} \int_\varepsilon^\delta |\mathcal{F}_\gamma g(a\lambda)|^2 \frac{da}{a} \right) e_\gamma(i\lambda x)|\lambda|^{2\gamma+1} d\lambda \\
= m_\gamma \int_\Re K_{\varepsilon,\delta}(\lambda) e_\gamma(i\lambda x)|\lambda|^{2\gamma+1} d\lambda
\]
which completes the proof.

We can now state the main result of this section.

**Theorem 4.10** (Calderón’s formula). Let \( g \in L^2(\Re, |x|^{2\gamma+1} dx) \) be a Dunkl wavelet of order \( \gamma \) such that \( \mathcal{F}_\gamma g \in L^\infty(\Re) \). Then for \( f \in L^2(\Re, |x|^{2\gamma+1} dx) \) and \( 0 < \varepsilon < \delta < \infty \), the function
\[
f^{\varepsilon,\delta}(x) := \frac{1}{C_g} \int_\varepsilon^\delta \int_\Re \Phi_\gamma(f)(a,b) g_a(b) |b|^{2\gamma+1} db \frac{da}{a}
\]
belongs to \( L^2(\Re, |x|^{2\gamma+1} dx) \) and satisfies
\[
\lim_{\varepsilon \to 0, \delta \to \infty} \|f^{\varepsilon,\delta} - f\|_{2,\gamma} = 0. \tag{4.5}
\]

**Proof.** It is easily seen that
\[
f^{\varepsilon,\delta} = f *_\gamma G_{\varepsilon,\delta},
\]
where \( G_{\varepsilon,\delta} \) is given by (4.2). It follows by Lemmas 4.7 and 4.9 that \( f^{\varepsilon,\delta} \in L^2(\Re, |x|^{2\gamma+1} dx) \) and \( \mathcal{F}_\gamma(f^{\varepsilon,\delta}) = \mathcal{F}_\gamma(f) K_{\varepsilon,\delta} \), where \( K_{\varepsilon,\delta} \) is as in (4.3). From this and Theorem 2.3 we obtain
\[
\|f^{\varepsilon,\delta} - f\|^2_{2,\gamma} = m_\gamma \int_\Re |\mathcal{F}_\gamma(f^{\varepsilon,\delta} - f)(\lambda)|^2 |\lambda|^{2\gamma+1} d\lambda \\
= m_\gamma \int_\Re |\mathcal{F}_\gamma f(\lambda)|^2 (1 - K_{\varepsilon,\delta}(\lambda))^2 |\lambda|^{2\gamma+1} d\lambda.
\]
But by (4.1) we have
\[
\lim_{\varepsilon \to 0, \delta \to \infty} K_{\varepsilon,\delta}(\lambda) = 1, \quad \text{for almost all } \lambda \in \Re.
\]
So (4.5) follows from the dominated convergence theorem.

Another pointwise inversion formula for the Dunkl wavelet transform, proved in [5], is as follows.

**Theorem 4.11.** Let \( g \in L^2(\Re, |x|^{2\gamma+1} dx) \) be a Dunkl wavelet of order \( \gamma \). If both \( f \) and \( \mathcal{F}_\gamma f \) are in \( L^1(\Re, |x|^{2\gamma+1} dx) \) then we have
\[
f(x) = \frac{1}{C_g} \int_0^\infty \left( \int_\Re \Phi_\gamma(f)(a,b) g_a(b) |b|^{2\gamma+1} db \right) \frac{da}{a}, \quad \text{a.e.},
\]
where, for each \( x \in \Re \), both the inner integral and the outer integral are absolutely convergent, but possibly not the double integral.
5 Inversion of the dual Dunkl–Sonine transform using Dunkl wavelets

From now on assume $\beta > \alpha > -1/2$. In order to invert the dual Dunkl–Sonine transform, we need the following two technical lemmas.

Lemma 5.1. Let $0 \neq g \in L^1 \cap L^2(\mathbb{R}, |x|^{2\alpha+1} dx)$ such that $\mathcal{F}_\alpha g \in L^1(\mathbb{R}, |x|^{2\alpha+1} dx)$ and satisfying

$$\exists \eta > \beta - 2\alpha - 1 \quad \text{such that} \quad \mathcal{F}_\alpha g(\lambda) = \mathcal{O}(|\lambda|^\eta) \quad \text{as} \quad \lambda \to 0. \quad (5.1)$$

Then $X_{\alpha,\beta}g \in L^2(\mathbb{R}, |x|^{2\beta+1} dx)$ and

$$\mathcal{F}_\beta(X_{\alpha,\beta}g)(\lambda) = \frac{m_\alpha}{m_\beta} \frac{\mathcal{F}_\alpha g(\lambda)}{|\lambda|^{2(\beta-\alpha)}}.$$  

Proof. By Theorem 2.3 we have

$$g(x) = m_\alpha \int_\mathbb{R} \mathcal{F}_\alpha g(\lambda) e_{\alpha}(i\lambda x) |\lambda|^{2\alpha+1} d\lambda, \quad \text{a.e.}$$

So using (3.6), we find that

$$X_{\alpha,\beta}g(x) = m_\beta \int_\mathbb{R} h_{\alpha,\beta}(\lambda) e_{\beta}(i\lambda x) |\lambda|^{2\beta+1} d\lambda, \quad \text{a.e.} \quad (5.2)$$

with

$$h_{\alpha,\beta}(\lambda) := \frac{m_\alpha}{m_\beta} \frac{\mathcal{F}_\alpha g(\lambda)}{|\lambda|^{2(\beta-\alpha)}}.$$  

Clearly, $h_{\alpha,\beta} \in L^1(\mathbb{R}, |x|^{2\beta+1} dx)$. So it suffices, in view of (5.2) and Theorem 2.3 to prove that $h_{\alpha,\beta}$ belongs to $L^2(\mathbb{R}, |x|^{2\beta+1} dx)$. We have

$$\int_\mathbb{R} |h_{\alpha,\beta}(\lambda)|^2 |\lambda|^{2\beta+1} d\lambda = \left( \frac{m_\alpha}{m_\beta} \right)^2 \int_\mathbb{R} |\lambda|^{4\alpha-2\beta+1} |\mathcal{F}_\alpha g(\lambda)|^2 d\lambda$$

$$= \left( \frac{m_\alpha}{m_\beta} \right)^2 \left( \int_{|\lambda| \leq 1} + \int_{|\lambda| \geq 1} \right) |\lambda|^{4\alpha-2\beta+1} |\mathcal{F}_\alpha g(\lambda)|^2 d\lambda := I_1 + I_2.$$

By (5.1) there is a positive constant $k$ such that

$$I_1 \leq k \int_{|\lambda| \leq 1} |\lambda|^{2\eta+4\alpha-2\beta+1} d\lambda = \frac{k}{\eta + 2\alpha - \beta + 1} < \infty.$$

From Theorem 2.3 it follows that

$$I_2 = \left( \frac{m_\alpha}{m_\beta} \right)^2 \int_{|\lambda| \geq 1} |\lambda|^{2(\alpha-\beta)} |\mathcal{F}_\alpha g(\lambda)|^2 |\lambda|^{2\alpha+1} d\lambda$$

$$\leq \left( \frac{m_\alpha}{m_\beta} \right)^2 \int_{|\lambda| \geq 1} |\mathcal{F}_\alpha g(\lambda)|^2 |\lambda|^{2\alpha+1} d\lambda \leq \left( \frac{m_\alpha}{m_\beta} \right)^2 ||\mathcal{F}_\alpha g||_{2,\alpha}^2 \leq \frac{m_\alpha}{(m_\beta)^2} ||g||_{2,\alpha}^2 < \infty$$

which ends the proof.  \[\square\]
Lemma 5.2. Let \( 0 \neq g \in L^1 \cap L^2(\mathbb{R}, |x|^{2\alpha+1} \, dx) \) be real-valued such that \( \mathcal{F}_a g \in L^1(\mathbb{R}, |x|^{2\alpha+1} \, dx) \) and satisfying

\[
\exists \eta > 2(\beta - \alpha) \quad \text{such that} \quad \mathcal{F}_a g(\lambda) = \mathcal{O}(\lambda^\eta) \quad \text{as} \quad \lambda \to 0^+.
\] (5.3)

Then \( \mathcal{X}_{\alpha,\beta} g \in L^2(\mathbb{R}, |x|^{2\beta+1} \, dx) \) is a Dunkl wavelet of order \( \beta \) and \( \mathcal{F}_\beta (\mathcal{X}_{\alpha,\beta} g) \in L^\infty(\mathbb{R}) \).

Proof. By combining (5.3) and Lemma 5.1 we see that the dual Dunkl–Sonine transform:

\[
\mathcal{F}_\beta (\mathcal{X}_{\alpha,\beta} g)(\lambda) = \mathcal{O}(\lambda^{\eta - 2(\beta - \alpha)}) \quad \text{as} \quad \lambda \to 0^+.
\]

Thus, in view of Remark 4.2, \( \mathcal{X}_{\alpha,\beta} g \) satisfies the admissibility condition (1.1) for \( \gamma = \beta \). □

Remark 5.3. In view of Remark 4.2 each function satisfying the conditions of Lemma 5.4 is a Dunkl wavelet of order \( \alpha \).

Lemma 5.4. Let \( g \) be as in Lemma 5.2 Then for all \( f \in L^1(\mathbb{R}, |x|^{2\beta+1} \, dx) \) we have

\[
\Phi_{\mathcal{X}_{\alpha,\beta} g}^\beta (f)(a,b) = \frac{1}{a^{2(\beta - \alpha)}} \mathcal{X}_{\alpha,\beta} \left[ \Phi_g^\alpha \left( \mathcal{X}_{\alpha,\beta} f \right)(a,\cdot) \right](b).
\]

Proof. By Definition 4.5 we have

\[
\Phi_{\mathcal{X}_{\alpha,\beta} g}^\beta (f)(a,b) = \frac{1}{a^{2\beta + 2}} f *_{\beta} (\mathcal{X}_{\alpha,\beta} g)_a(b).
\]

But \( (\mathcal{X}_{\alpha,\beta} g)_a = \mathcal{X}_{\alpha,\beta} (\tilde{g}_a) \) by virtue of (1.2) and (3.3). So using (3.3) we find that

\[
\Phi_{\mathcal{X}_{\alpha,\beta} g}^\beta (f)(a,b) = \frac{1}{a^{2\beta + 2}} f *_{\beta} [\mathcal{X}_{\alpha,\beta} (\tilde{g}_a)](b)
= \frac{1}{a^{2(\beta - \alpha)}} \mathcal{X}_{\alpha,\beta} \left[ \Phi_g^\alpha \left( \mathcal{X}_{\alpha,\beta} f \right)(a,\cdot) \right](b),
\]

which gives the desired result. □

Combining Theorems 4.10, 4.11 with Lemmas 5.2, 5.4 we get

Theorem 5.5. Let \( g \) be as in Lemma 5.2 Then we have the following inversion formulas for the dual Dunkl–Sonine transform:

(i) If both \( f \) and \( \mathcal{F}_\beta f \) are in \( L^1(\mathbb{R}, |x|^{2\beta+1} \, dx) \) then for almost all \( x \in \mathbb{R} \) we have

\[
f(x) = \frac{1}{C_{\mathcal{X}_{\alpha,\beta} g}^\beta} \int_0^\infty \left( \int_{\mathbb{R}} \mathcal{X}_{\alpha,\beta} \left[ \Phi_g^\alpha \left( \mathcal{X}_{\alpha,\beta} f \right)(a,\cdot) \right](b)(\mathcal{X}_{\alpha,\beta} g)_{a,b}(x) |b|^{2\beta+1} db \right) da a^{2(\beta - \alpha) + 1}.
\]

(ii) For \( f \in L^1 \cap L^2(\mathbb{R}, |x|^{2\beta+1} \, dx) \) and \( 0 < \varepsilon < \delta < \infty \), the function

\[
f^{\varepsilon,\delta}(x) := \frac{1}{C_{\mathcal{X}_{\alpha,\beta} g}^\beta} \int_\varepsilon^\delta \left( \int_{\mathbb{R}} \mathcal{X}_{\alpha,\beta} \left[ \Phi_g^\alpha \left( \mathcal{X}_{\alpha,\beta} f \right)(a,\cdot) \right](b)(\mathcal{X}_{\alpha,\beta} g)_{a,b}(x) |b|^{2\beta+1} db \right) da a^{2(\beta - \alpha) + 1}
\]

satisfies

\[
\lim_{\varepsilon \to 0, \delta \to \infty} \| f^{\varepsilon,\delta} - f \|_{2,\beta} = 0.
\]

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