EDGE-CRITICAL SUBGRAPHS OF SCHRIJVER GRAPHS

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Abstract. For $k \geq 1$ and $n \geq 2k$, the Kneser graph $KG(n, k)$ has all $k$-element subsets of an $n$-element set as vertices; two such subsets are adjacent if they are disjoint. It was first proved by Lovász that the chromatic number of $KG(n, k)$ is $n - 2k + 2$. Schrijver constructed a vertex-critical subgraph $SG(n, k)$ of $KG(n, k)$ with the same chromatic number. For the stronger notion of criticality defined in terms of removing edges, however, no analogous construction is known except in trivial cases. We provide such a construction for $k = 2$ and arbitrary $n \geq 4$ by means of a nice explicit combinatorial definition.

1. Introduction

For positive integers $n, k$, where $n \geq 2k$, the Kneser graph $KG(n, k)$ has all $k$-element subsets of the set $[n] = \{1, \ldots, n\}$ as its vertices, with edges joining disjoint pairs of subsets. It was conjectured by Kneser [3] and proved by Lovász [4] that the chromatic number of $KG(n, k)$ is $n - 2k + 2$. Schrijver [6] proved that there is a subgraph of $KG(n, k)$ that is in general much smaller and still has chromatic number $n - 2k + 2$. This is the Schrijver graph $SG(n, k)$, defined as the induced subgraph of $KG(n, k)$ on the set of all stable $k$-subsets of $[n]$, that is, those that contain no pair of consecutive elements nor the pair $1, n$. In fact, Schrijver proved that $SG(n, k)$ is vertex-critical, i.e., the removal of any vertex of $SG(n, k)$ decreases the chromatic number.

It is natural to ask whether $SG(n, k)$ satisfies the stronger condition of criticality defined in terms of removing edges. A graph $G$ is said to be edge-critical if $\chi(G - e) < \chi(G)$ for each edge $e$ of $G$, where $\chi$ denotes the chromatic number. Equivalently, $G$ is edge-critical if none of its proper subgraphs has the same chromatic number. For instance, the graph $SG(2k + 1, k)$ is edge-critical (being isomorphic to a cycle of length $2k + 1$) and so is $SG(n, 1)$ (the complete graph $K_n$), but this is not the case for $SG(n, k)$ with $n \geq 2k + 2$ and $k \geq 2$.

In this paper, we give a simple combinatorial description of an edge-critical spanning subgraph of the graph $SG(n, 2)$ (for any $n \geq 4$) that was discovered in the course of our work on colouring quadrangulations of projective spaces [1] [2]. This is the first step to a description of such edge-critical subgraphs in $SG(n, k)$ for general $k$, which is currently work in progress.

For every integer $n \geq 4$, we define the graph $G_n$ as follows. The vertex set of $G_n$ is the set of all stable 2-subsets of $[n]$. A stable subset $\{a, b\}$, where...
a < b, is denoted by $ab$. Edges in $G_n$ only join disjoint pairs of 2-subsets of $[n]$. Let $ab$ and $cd$ be such a pair, where $a < c$. The vertices $ab$ and $cd$ are adjacent in $G_n$ if and only if one of the following holds:

- $a < c < b < d$ (a crossing pair), or
- $1 < a < c < d < b$ (a transverse pair).

See Figure 1 for an illustration of the definition. In this figure, vertices of $G_n$ are visualised as chords of the cycle $C_n$. Accordingly, we sometimes refer to vertices of $G_n$ as chords of $C_n$.

The main result of this paper is the following.

**Theorem 1.1.** For every $n \geq 4$, the graph $G_n$ is $(n - 2)$-chromatic and edge-critical.

Since $G_n$ is an edge-critical spanning subgraph of $SG(n, 2)$, it is interesting to compare the number of edges in $G_n$ to the number of edges in $SG(n, 2)$. The only case where $G_n$ is not a proper subgraph of $SG(n, 2)$ is when $n \leq 5$: $G_4$ and $SG(4, 2)$ are both isomorphic to $K_2$, while $G_5$ and $SG(5, 2)$ are both isomorphic to $C_5$. For $n > 5$, $G_n$ is a proper subgraph of $SG(n, 2)$, and the following proposition determines the asymptotic ratio of their sizes.

**Proposition 1.2.** As $n \to \infty$, the ratio $|E(G_n)|/|E(SG(n, 2))|$ tends to $2/3$.

**Proof.** Each edge of $G_n$ corresponds to either a crossing pair or a transverse pair of chords of the cycle $C_n$. Let us call the pairs of disjoint chords of $C_n$ not corresponding to any of these types lateral. (That is, chords $ab$ and $cd$ form a lateral pair if $a < b < c < d$ or $c < d < a < b$.)

Let us estimate the number of pairs of each of these three types. Any pair of chords determines a 4-tuple of elements of $[n]$, namely the endvertices of the chords. For crossing pairs, this is in fact a 1–1 correspondence, since any 4-element subset of $[n]$ determines precisely one crossing pair. Thus, the number of crossing pairs is $\binom{n}{4}$.

For transverse and lateral pairs, the correspondence is no longer one-to-one, but it is not hard to show that the number of pairs of each of these
types is \( \binom{n}{4} - O(n^3) \). It follows that
\[
|E(G_n)| = 2\binom{n}{4} - O(n^3),
\]
\[
|E(SG(n, 2))| = 3\binom{n}{4} - O(n^3),
\]
so the asymptotic ratio of these two quantities is 2/3 as claimed. □

2. Proof of Theorem 1.1

The Mycielski construction \[5\] is one of the earliest and arguably simplest constructions of triangle-free graphs of arbitrarily high chromatic number. Given a graph \( G = (V, E) \), we let \( M(G) \) be the graph with vertex set \( V \cup \{u : u \in V\} \cup \{\ast\} \), where there are edges \( \{u, v\} \) and \( \{u, \ast\} \) whenever \( \{u, v\} \in E \), and an edge \( \{u, \ast\} \) for all \( u \in V \). For each \( u \in V \), the vertex \( u \) is referred to as the clone of \( u \) in \( M(G) \).

It is an easy exercise to show that the chromatic number increases with each iteration of \( M(\cdot) \). Let \( M_k \) be the graph obtained from \( K_2 \) by iterating the Mycielski construction \( k - 2 \) times. It is easy to see that \( M_k \) is \( k \)-chromatic (in fact, \( M_k \) is \( k \)-edge-critical).

Theorem 1.1 follows immediately from the following two lemmas.

**Lemma 2.1.** For \( n \geq 5 \), there is a homomorphism
\[
h : M(G_{n-1}) \to G_n.
\]
In particular, \( \chi(G_n) \geq n - 2 \).

**Proof.** Let \( ab \) be a vertex of \( G_{n-1} \); recall that this means \( a < b \). The clone of \( ab \) in \( M(G_{n-1}) \) is denoted by \( \overline{ab} \). We define
\[
h(ab) = ab,
\]
\[
h(ab) = \begin{cases} an & \text{if } a \neq 1, \\ bn & \text{if } a = 1, \end{cases}
\]
\[
h(\ast) = \{1, n - 1\}.
\]

Observe first that in all cases, the value of the mapping \( h \) is a vertex of \( G_n \). The only case that needs an explanation is that of \( \overline{ab} \). Here, if \( a \neq 1 \), then \( an \) is stable, since \( a < b \leq n - 1 \). On the other hand, if \( a = 1 \), then \( 2 < b < n - 1 \) (since \( ab \) is a vertex of \( G_{n-1} \)), so \( bn \) is stable.

Let us show that \( h \) is a homomorphism. We consider an edge \( e \) of \( M(G_{n-1}) \) and prove, in each of the following cases, that the image of \( e \) under \( h \) is an edge of \( G_n \).

**Case 1:** \( e \) is an edge of \( G_{n-1} \). Note that \( h(ab) = ab \) and \( h(cd) = cd \). Since a crossing pair of vertices of \( G_{n-1} \) is also crossing in \( G_n \), and similarly for a transverse pair, \( e \) is an edge of \( G_n \).

**Case 2:** \( e \) has endvertices \( ab \) and \( \overline{cd} \), where \( ab \) and \( cd \) form an edge of \( G_{n-1} \). We have \( h(ab) = ab \). For \( h(\overline{cd}) \), we have
\[
h(\overline{cd}) = \begin{cases} cn & \text{if } c > 1, \\ dn & \text{if } c = 1. \end{cases}
\]
Suppose first that the pair $ab, cd$ is crossing in $G_{n-1}$. (This subcase is illustrated in Figure 2.) If $a < c < b < d$, then $h(cd) = cn$ and $a < c < b < n$, so the pair $ab, cn$ is crossing in $G_n$. (In particular, it is disjoint, which will not be repeated in the following subcases.) If $1 < c < a < d < b$, then again $h(cd) = cn$ and $c < a < b < n$, so the pair $ab, cn$ is transverse in $G_n$. Finally, if $c = 1$ (and $1 < a < d < b$), then $h(cd) = dn$ and $a < d < b < n$, so the pair $ab, dn$ is crossing.

Suppose then that $ab, cd$ is a transverse pair in $G_n$. Thus, $1 \notin \{a, b, c, d\}$ and in particular $h(cd) = cn$. If $a < c < d < b$, then $a < c < b < n$ and the pair $ab, cn$ is crossing. If $c < a < b < d$, then $c < a < b < n$ and $ab, cn$ is a transverse pair.

**Case 3:** $e$ has endvertices $cd$ and $*$, where $cd$ is a vertex of $G_{n-1}$. We have $h(*) = \{1, n - 1\}$. Since $n \in h(cd)$, the pair $h(cd), h(*)$ must be crossing.

This concludes the proof that $h$ is a homomorphism. It follows that $\chi(G_n) \geq \chi(M(G_{n-1})$. The statement that $\chi(G_n) \geq n - 2$ follows by induction with base case $\chi(G_5) = 3$: if we know that $\chi(G_{n-1}) \geq n - 3$, then $\chi(M(G_{n-1})) \geq n - 2$ since — as remarked above — an application of $M(\cdot)$ increases the chromatic number, and therefore $\chi(G_n) \geq n - 2$. \qed
Lemma 2.2. For every $n \geq 4$ and every edge $e \in E(G_n)$, the graph $G_n - e$ is $(n - 3)$-colourable.

Proof. Let $e$ be an edge of $G_n$ with endvertices $ab$ and $cd$. We will show that $G - e$ is $(n - 3)$-colourable.

Case 1: $ab$ and $cd$ are a crossing pair and $1 \in \{a, b, c, d\}$. Without loss of generality, assume that $a = 1$ and $1 < c < b < d$. Let $A = \{a, b, c, d\}$. We first colour all stable 2-subsets of $[n]$ not contained in $A$ using the $n - 4$ colours from the set $[n] \setminus A$: any such stable 2-subset $xy$ is coloured by $\min(\{x, y\} \setminus \{a, b, c, d\})$. Observe that this partial colouring is proper in $G_n$. (We will call it the min-based colouring with respect to $A$.)

Having used $n - 4$ colours, we have one colour left for the stable 2-subsets of $A$. Since $a = 1$, no pair of these subsets is transverse. Additionally, there is only one crossing pair, namely $ab$ and $cd$. Assign a new colour $\ell_1$ to each stable 2-subset of $A$; we obtain a proper $(n - 3)$-colouring of $G_n - e$. (See Figure 3 for an illustration of this and the following cases.)

Case 2: $ab$ and $cd$ are a crossing pair and $1 \notin \{a, b, c, d\}$. Let $A = \{1, a, b, c, d\}$ and start with the min-based colouring with respect to $A$. This uses $n - 5$ colours.

We will colour stable 2-subsets of $A$ using two new colours $\ell_1$ and $\ell_2$. Without loss of generality, assume that $a < c < b < d$. Assigning colour $\ell_1$ to all the stable 2-subsets in the set $\{1a, 1b, 1c, 1d, bc, bd\}$ and colour $\ell_2$ to those in $\{ab, ac, ad, cd\}$, it is easy to check that the 2-colouring of the induced subgraph of $G_n - e$ on the set of stable 2-subsets of $A$ is proper. Consequently, we obtain a proper $(n - 3)$-colouring of $G_n - e$.

Case 3: $ab$ and $cd$ are a transverse pair. By the assumption, $1 \notin \{a, b, c, d\}$. Without loss of generality, assume that $1 < a < c < d < b$. Since $cd$ is stable, there is $x \in [n]$ such that $c < x < d$. Let $A = \{1, a, b, c, d, x\}$ and start with a min-based colouring with respect to $A$ using $n - 6$ colours.

To colour the uncoloured stable 2-subsets of $[n]$ using new colours $\ell_1$, $\ell_2$, and $\ell_3$, we use the following rule:

- $\ell_1$ is assigned to stable 2-subsets in $\{1a, 1x, 1d, ax, dx\}$,
- $\ell_2$ is assigned to those in $\{1b, 1c, bc, bx, cx\}$,
- $\ell_3$ is assigned to those in $\{ab, ac, ad, cd, bd\}$.

We obtain a proper $(n - 3)$-colouring of $G_n - e$ since no crossing pair gets the same colour, and the only monochromatic transverse pair is $ab, cd$.

\[\Box\]

References

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Figure 3. Colouring of chords in the three cases of the proof of Lemma 2.2. The colouring uses one colour (left), two colours (middle) and three colours (right) represented by shades of grey.

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