A STUDY OF SUBGROUPS OF RIGHT-ANGLED COXETER GROUPS VIA STALLINGS-LIKE TECHNIQUES

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Abstract. We associate a cube complex to any given finitely generated subgroup of a right-angled Coxeter group, called the completion of the subgroup. A completion characterizes many properties of the subgroup such as whether it is quasiconvex, normal, finite-index or torsion-free. We use completions to show that reflection subgroups are quasiconvex, as are one-ended Coxeter subgroups of a 2-dimensional right-angled Coxeter group. We provide an algorithm that determines whether a given one-ended, 2-dimensional right-angled Coxeter group is isomorphic to some finite-index subgroup of another given right-angled Coxeter group. In addition, we answer several algorithmic questions regarding quasiconvex subgroups. Finally, we give a new proof of Haglund’s result that quasiconvex subgroups of right-angled Coxeter groups are separable.

1. Introduction

In the highly influential article [Sta83], Stallings introduced new tools to study subgroups of free groups. A crucial idea in Stallings’ work is that given a finite set of words in a free group, one can associate a labeled graph to this set, and perform a sequence of operations, now known as “Stallings folds,” to this graph. The resulting graph is, in some sense, a canonical object associated to the subgroup generated by the given words. This topological viewpoint provided clean new proofs for many theorems regarding subgroups of free groups. In [KM02], Kapovich–Miasnikov use Stallings’ ideas, re-cast in a more combinatorial form, to systematically study the subgroup structure of free groups, and to answer a number of algorithmic questions about such subgroups. The main goal of this article is to better understand subgroups of right-angled Coxeter groups through generalizations of these authors’ techniques.

Given a finite simplicial graph $\Gamma$, the associated right-angled Coxeter group $W_\Gamma$ is generated by order two elements corresponding to vertices of $\Gamma$, with the additional relations that two such generators commute if there is an edge in $\Gamma$ between the corresponding vertices. Right-angled Coxeter groups form a wide class of groups which have become central objects in geometric group theory. We refer to [Dan18] for a survey of recent work on these groups. One interesting feature of right-angled Coxeter groups is that they have a rich variety of subgroups, which includes all free groups, right-angled Artin groups [DJ00] and surface groups. Incredibly, all hyperbolic 3-manifold groups [Ago13] [Wis11] and Coxeter groups [HW10] are virtually subgroups of right-angled Coxeter groups as well.

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Given a subgroup $G$ of a right-angled Coxeter group $W_{\Gamma}$, we abstractly define edge-labeled cube complexes associated to $G$, called completions of $G$. If $G$ is additionally finitely generated, we explicitly build a standard completion for $G$ by the following procedure. We first build a subdivided “rose” graph whose petals are labeled by the words generating $G$. Next, we perform a sequence of operations of three possible types: fold, cube attachment, and cube identification. A completion can always be obtained as the direct limit of the complexes in this sequence.

As a completion $\Omega$ has the structure of a cube complex (and not just a graph), one may use techniques from cubical geometry to study it. Many properties of the subgroup $G$ can be characterized in terms of properties of $\Omega$. Additionally, loops in $\Omega$ correspond to elements in $G$. The following theorem summarizes our results relating properties of $G$ to properties of $\Omega$.

**Theorem A.** Let $G$ be a subgroup of the right-angled Coxeter group $W_{\Gamma}$. Then

1. $G$ is quasiconvex in $W_{\Gamma}$ if and only if $G$ is finitely generated and every (equivalently, some) standard completion for $G$ is finite (Theorem 8.4).
2. There exist characterizations of $G$ having finite index (Theorem 6.9), $G$ being torsion-free (Proposition 4.6), and $G$ being normal (Theorem 5.5) in terms of properties of a completion.
3. If $G$ is torsion-free, then any completion is non-positively curved (Proposition 7.3) and has fundamental group isomorphic to $G$ (Theorem 4.7).

We remark that characterizations similar to those above could also be obtained by considering the covering spaces perspective and applying results of Haglund [Hag08]. Theorem A also has some overlap with the work of Kharlampovich–Miasnikov–Weil [KMW17], who construct “Stallings graphs” for automatic groups. We further discuss the relation of our results to these and other works in Section 1.1.

Our proof strategies for Theorem A seem to be novel and are independent of the results mentioned above. Additionally, our procedure for constructing $\Omega$ is new, and this is what allows us to prove structural results regarding subgroups of right-angled Coxeter groups. Furthermore, the approach using completions is particularly well suited for addressing algorithmic questions.

A given subgroup may have multiple completions. Indeed two completions for a given subgroup need not even be homotopy equivalent (see Example 3.7). Nevertheless, most of our structural results do not depend on the specific completions chosen. Despite the non-uniqueness of completions, every completion for $G$ has a 1-dimensional subcomplex called its core graph, and any two core graphs for $G$ are isomorphic (see Proposition 5.3).

Theorem A provides a tool to show that subgroups of a right-angled Coxeter group are quasiconvex, by showing that their associated completions must be finite. For instance, we prove that finitely generated reflection subgroups, i.e. subgroups generated by reflections, must be quasiconvex:

**Theorem B.** (Theorem 10.5) Every finitely generated reflection subgroup of a right-angled Coxeter group is quasiconvex.

We next turn our attention to Coxeter subgroups, i.e. subgroups that are themselves isomorphic to some abstract finitely generated Coxeter group. A result of [Dye90] and [Deo89] shows that every reflection subgroup of a right-angled Coxeter group is a Coxeter subgroup. The converse to this statement is not true in general (see Remark 11.5.1), but it holds under certain hypotheses:
Theorem C (Theorem 11.4, Corollary 11.5). Every one-ended Coxeter subgroup of a 2-dimensional right-angled Coxeter group is a reflection subgroup. Consequently, every such Coxeter subgroup is quasiconvex by Theorem B.

Completions can be used to answer several algorithmic questions about subgroups of right-angled Coxeter groups. For instance, we consider the problem of determining, given two right-angled Coxeter groups, whether one can be embedded as a finite-index subgroup of the other:

Theorem D (Theorem 12.11). There is an algorithm which, given a one-ended, 2-dimensional right-angled Coxeter group $W_\Gamma$, and any right-angled Coxeter group $W_\Gamma'$, determines whether or not $W_\Gamma'$ is isomorphic to a finite-index subgroup of $W_\Gamma$. Moreover, the time-complexity of the algorithm is bounded by a function of the number of vertices of $\Gamma$ and $\Gamma'$.

The above theorem gives an algorithm that can often determine when two right-angled Coxeter groups are commensurable; thus, it provides a tool for studying commensurability classification. A few specific families of right-angled Coxeter groups have been classified up to commensurability (see [CP08, DST18, HST17]), but not much is known in general. We note that the precise statements of Theorem C and Theorem D use a significantly weaker hypothesis than one-endedness.

When $G$ is quasiconvex, Theorem A implies that every completion of $G$ is finite. This makes it possible to provide finite-time algorithms to check a number of basic properties concerning $G$, as described in Theorem E below. We remark that the existence of algorithms for (1) and (4) was already known [Kap96, KMW17] (see the discussion in Section 1.1).

Theorem E (Theorem 13.1). Let $G$ be a quasiconvex subgroup of a right-angled Coxeter group $W_\Gamma$ given by a finite generating set of words in $W_\Gamma$. Then there exist finite-time algorithms to solve the following problems.

1. (Membership Problem) Given $g \in W_\Gamma$, determine whether or not $g \in G$.
2. Given $g \in W_\Gamma$, determine whether or not a positive power of $g$ is in $G$.
3. Determine whether or not $G$ is torsion-free.
4. Determine the index of $G$ in $W_\Gamma$ (even if infinite).
5. Determine whether or not $G$ is normal.

In particular, the above theorem may be applied to reflection subgroups as these are quasiconvex by Theorem B. If the ambient right-angled Coxeter group $W_\Gamma$ is additionally 2-dimensional, then the time-complexity of these algorithms is bounded only by the length of generators of the reflection subgroup and the number of vertices of $\Gamma$ (see Theorem 10.7).

As another application, we give new proofs for some known results. We give a proof using completions of the following result of Haglund (see Theorem 9.4):

Theorem F ([Hag08, Theorem A]). Every quasiconvex subgroup of a right-angled Coxeter group is separable and is a virtual retract.

Theorem F can be thought of as a generalization of Marshall Hall’s Theorem from the free group setting to the setting of right-angled Coxeter groups. Also following Stallings’ approach, we give a simple proof, using completions, of the well-known result that right-angled Coxeter groups are residually finite (see Theorem 9.3).
Finally, we note that much of the work presented here can be used to study right-angled Artin groups as well. Given any right-angled Artin group $A$, Davis–Januszkiewicz construct a right-angled Coxeter group $W$ which contains $A$ as a finite-index subgroup [DJ00]. Thus, given a subgroup $G$ of a right-angled Artin group $A$, one can first embed $A$ as a finite-index subgroup of $W$ and construct a completion for $G$ considered as a subgroup of $W$. Now Theorem [A] and Theorem [E] can be used to understand properties of $G$ as a subgroup of $A$.

1.1. Relation to other works. The ideas in Stallings’ paper have led to many applications which are too numerous to list here. We discuss some results in the literature that are more closely related to ours.

Given a quasiconvex subgroup $G$ of a right-angled Coxeter group, Haglund shows that $G$ acts cocompactly on the combinatorial convex hull $\Sigma(G)$ of $G$ in the Davis complex of $W$ [Hag08]. It turns out that the completion $\Omega$ is very close to being equal to the quotient of $\Sigma(G)/G$ (one has to take care, as this quotient is an orbifold in general and the generating set for $G$ used in constructing $\Omega$ may not consist of reduced words). If one further developed this point of view, characterizations similar to those in Theorem [A] could be obtained from results in Haglund’s article.

Beeker–Lazarovich use a version of Stallings folds for cube complexes to give a characterization of quasiconvex subgroups of hyperbolic groups that act properly and cocompactly on CAT(0) cube complexes in terms of their hyperplane stabilizers [BL18]. Similar ideas were also used in Brown’s thesis in the setting of hyperbolic VH square complexes [Bro]. Sageev–Wise show that relatively quasiconvex subgroups of relatively hyperbolic groups which act on a finite-dimensional, locally finite CAT(0) cube complex admit a convex core [SW15]. This generalizes ideas of Haglund [Hag08]. We note that the results mentioned in this paragraph use some form of hyperbolicity, whereas, in our case, many right-angled Coxeter groups are not even relatively hyperbolic (see [BHS17]).

Kharlampovich–Miasnikov–Weil [KMW17] show that Stallings graphs (we refer the reader to their paper for a definition) can always be constructed for quasiconvex subgroups of automatic groups. They use this to address several algorithmic problems for quasiconvex subgroups of automatic groups, including the membership problem, deciding whether the subgroup is finite-index or finite, and computing intersections of subgroups. Since right-angled Coxeter groups are automatic [BH93], these results apply, thus providing algorithms for (1) and (4) in our Theorem [E]. Kapovich had already previously shown that the membership problem is solvable for automatic groups [Kap90]. Schupp uses Stallings graphs to show that certain classes of extra-large type Coxeter groups are locally quasiconvex and to answer algorithmic questions regarding their subgroups [Sch03]. We note that right-angled Coxeter groups are not of extra-large type. We refer the reader to [KMW17] for a detailed summary on the literature regarding constructing Stallings graphs for solving algorithmic problems.

In the spirit of Theorem [D] Kim–Koberda consider the question of understanding when one right-angled Artin group can be realized as a (not necessarily finite-index) subgroup of another [KK13]. They relate the existence of such embeddings to properties of certain associated graphs, called extension graphs and clique graphs. Using their work, Casals-Ruiz proves a number of algorithmic results about embeddings between right-angled Artin groups [CRT15]. In particular, she shows that there is an algorithm which, given a 2-dimensional right-angled Artin group $A_T$ and any
right-angled Artin group $A_\Gamma$, determines whether or not $A_{\Gamma'}$ is isomorphic to a subgroup of $A_\Gamma$.

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2. Preliminaries

Given a graph $\Gamma$, we will always denote the vertex and edge sets of $\Gamma$ by $V(\Gamma)$ and $E(\Gamma)$ respectively.

2.1. Right-angled Coxeter groups. We summarize some well-known facts regarding right-angled Coxeter groups which we will need throughout this article. We refer the reader to [Dan18] for a survey on right-angled Coxeter groups and to both [Dav08] and [BB05] as references on Coxeter groups.

Let $\Gamma$ be a simplicial graph with finite vertex set $S = V(\Gamma)$ and edge set $E = E(\Gamma)$. The right-angled Coxeter group $W_\Gamma$ associated to $\Gamma$ is the group given by the presentation:

$$W_\Gamma = \langle S \mid s^2 = 1 \text{ for } s \in S, \text{ } st = ts \text{ for } (s, t) \in E \rangle$$

We say that $S$ is a standard Coxeter generating set for $W_\Gamma$. Given $s, t \in V(\Gamma)$, we write $m(s, t) = 1$ if $s = t$, $m(s, t) = 2$ if $s$ is adjacent to $t$ and $m(s, t) = \infty$ otherwise.

We refer to the elements of $S$ as letters. A word $w$ in $W_\Gamma$ is a (possibly empty) sequence of letters in $S$. Let $w = s_1 \ldots s_n$ be a word in $W_\Gamma$, where $s_i \in S$ for $1 \leq i \leq n$. We let $|w| = n$ denote the length of $w$. If $w'$ is another word in $W_\Gamma$ such that $w$ and $w'$ are equal as elements of $W_\Gamma$, then we say that $w'$ is an expression for $w$. We say that $w$ is reduced if $|w| \leq |w'|$ for any expression $w'$ for $w$. Finally, we define the support of $w$, denoted by $\text{Support}(w)$, to be the set of vertices of $\Gamma$ which appear as a letter in $w$.

Given a vertex $v$ of $\Gamma$, the link of $v$, denoted by $\text{link}(v)$, is the set of all vertices of $\Gamma$ which are adjacent to $v$. The star of $v$, denoted by $\text{star}(v)$, is the set $\text{link}(v) \cup \{v\}$. Many times throughout this article, we will consider the special subgroup of $W_\Gamma$ generated by the link or star of a vertex.

We now recall some classes of graphs and their corresponding right-angled Coxeter groups. Recall that a graph is a clique if any pair of distinct vertices of the graph are adjacent. A right-angled Coxeter group $W_\Gamma$ is finite if and only if $\Gamma$ is a clique.
We say a graph is triangle-free if it does not contain a subgraph that is a clique with three vertices, i.e. a triangle. If \( \Gamma \) is triangle-free, we say that the right-angled Coxeter group \( W_\Gamma \) is 2-dimensional.

A graph \( \Gamma \) decomposes as a join graph \( \Gamma = \Gamma_1 \ast \Gamma_2 \) if there are induced subgraphs \( \Gamma_1 \) and \( \Gamma_2 \) of \( \Gamma \) such that \( V(\Gamma) = V(\Gamma_1) \cup V(\Gamma_2) \), and if \( v_1 \in V(\Gamma_1) \) and \( v_2 \in V(\Gamma_2) \), then \( v_1 \) and \( v_2 \) are adjacent in \( \Gamma \). The graph \( \Gamma \) decomposes as a join \( \Gamma = \Gamma_1 \ast \Gamma_2 \) if and only if \( W_\Gamma = W_{\Gamma_1} \times W_{\Gamma_2} \).

We describe a characterization of the number of ends of a right-angled Coxeter group in terms of the defining graph. These results were first proven in [MT09]. The group \( W_\Gamma \) is 2-ended if and only if either \( \Gamma \) consists of two non-adjacent vertices or there exists two non-adjacent vertices \( s, t \in V(\Gamma) \) and a clique subgraph \( K \subset \Gamma \) such that \( \Gamma = K \ast \{s, t\} \). Now suppose that \( W_\Gamma \) is not finite or 2-ended (each a case we already described in terms of \( \Gamma \)), then \( W_\Gamma \) has infinitely many ends if and only if \( \Gamma \) has a separating clique, i.e. there is some clique subgraph \( K \subset \Gamma \) such that \( \Gamma \setminus K \) has more than one component.

Finally, we say that a subgroup of a Coxeter group is a Coxeter subgroup if it is isomorphic to some finitely generated Coxeter group. We refer the reader to [BB05] for a definition of Coxeter groups. We will actually not need this definition, as we show below (the likely well-known fact) that every Coxeter subgroup of a right-angled Coxeter group is itself a right-angled Coxeter group.

**Proposition 2.1.** Let \( G \) be a Coxeter subgroup of a right-angled Coxeter group, then \( G \) is a right-angled Coxeter group.

**Proof.** As any finite subgroup of a Coxeter group is conjugate to a subgroup of a finite special subgroup (see [Dav08], Theorem 12.3.4) for instance), it follows that all finite order elements in a right-angled Coxeter group have order 2.

We note that Coxeter groups which are not right-angled must contain elements of order larger than 2. This is the case since such a group must necessarily contain generators \( s \) and \( t \) which satisfy the relation \( (st)^m \) for some \( m > 2 \) and do not satisfy the relation \( (st)^k \) for all \( 0 < k < m \) [BB05 Proposition 1.1.1]. Thus, \( G \) must indeed be a right-angled Coxeter group. \( \square \)

### 2.2. The word problem in right-angled Coxeter groups.

We discuss Tits’ solution to the word problem in right-angled Coxeter groups. We again refer the reader to [Dav08] or [BB05] for proofs of these facts.

Let \( w = s_1 \ldots s_n \) be a word in the right-angled Coxeter group \( W_\Gamma \). Suppose that \( m(s_i, s_{i+1}) = 2 \) for some \( 1 \leq i \leq n \). Then we may “swap” the letters \( s_i \) and \( s_{i+1} \) to obtain another expression \( w' = s_1 \ldots s_{i-1}s_{i+1}s_is_{i+2}\ldots s_n \) for \( w \). We say that \( w' \) is obtained from \( w \) by a swap move or by swapping \( s_i \) and \( s_{i+1} \). On the other hand, suppose that \( s_i = s_{i+1} \) (as vertices of \( \Gamma \)) for some \( 1 \leq i \leq n \). We can then obtain an expression \( w' = s_1 \ldots s_{i-1}s_{i+2}\ldots s_n \) for \( w \) by cancelling \( s_i \) and \( s_{i+1} \).

Let \( w \) be a word in \( W_\Gamma \), and let \( w' \) be a reduced expression for \( w \). There exists a sequence of words \( w = w_1, \ldots, w_m = w' \) such that \( w_{i+1} \) is obtained from \( w_i \) by either a swap move or a cancellation. This is known as Tits’ solution to the word problem. We call such a sequence of expressions for \( w \) a sequence of Tits moves.

We now discuss an alternative way to obtain a reduced expression for a word \( w = s_1 \ldots s_n \) in a right-angled Coxeter group \( W_\Gamma \). Suppose \( s_i = s_{i'} = s \) for some \( 1 \leq i < i' \leq n \) and that \( m(s, s_j) = 2 \) for all \( i < j < i' \). It follows that the word
\( w' = s_1 \cdots s_{i-1} s_{i+1} \cdots s_{i'} s_{i'+1} \cdots s_n \) is an expression for \( w \). We say that \( w' \) is obtained from \( w \) by a \textit{deletion} (as the two occurrences of \( s \) have been deleted).

We remark that our definition of a deletion, defined in the setting of right-angled Coxeter groups, is stated in a slightly stronger form than its classical statement for (not necessarily right-angled) Coxeter groups. It is a basic fact that given any word \( w \) in a right-angled Coxeter group, a reduced expression for \( w \) can be obtained by performing a sequence of deletions. As we could not find a statement of this in the literature using our exact definition of deletion, we later provide a proof of this fact (see Proposition 2.2).

2.3. \textbf{Cube complexes.} A \textit{cube complex} is a cell complex whose cells are Euclidean unit cubes, \( [-\frac{1}{2}, \frac{1}{2}]^n \), of varying dimension. We refer the reader to [CS11] and to [Wis12] for a background on cube complexes.

Let \( \Gamma \) be a simplicial graph. A cube complex is \( \Gamma \)-labeled if every edge in its 1-skeleton is labeled by a vertex of \( \Gamma \). The cube complexes we consider in this article will all be \( \Gamma \)-labeled. Given a simplicial path \( \alpha \) in the 1-skeleton of a \( \Gamma \)-labeled complex, the \textit{label} of \( \alpha \) is the word formed by the sequence of labels of consecutive edges in \( \alpha \).

Let \( \Omega \) be a cube complex. We say \( \Omega \) is \textit{non-positively curved} if the (simplicial) link of each vertex in \( \Omega \) is a flag simplicial complex. If \( \Omega \) is both non-positively curved and simply connected then we say that \( \Omega \) is a CAT(0) cube complex.

A \textit{path} in the cube complex \( \Omega \) is a simplicial path in its 1-skeleton. Given a path \( p \), we denote by \( |p| \) the number of edges in \( p \). Given two paths, \( p \) and \( p' \), such that the endpoint of \( p \) is equal to the startpoint of \( p' \), we let \( pp' \) denote their concatenation. A \textit{loop} in a complex is defined to be a closed path (possibly with backtracking). We define a \textit{graph-loop} to be an edge in a complex that connects a vertex to itself.

We will work with the combinatorial path metric on cube complexes. Namely, given two vertices of \( \Omega \), we define their distance to be the length of a shortest path in \( \Omega \) between them.

A \textit{midcube} of a cube \( c = [-\frac{1}{2}, \frac{1}{2}]^n \) is the restriction of one of the coordinates of \( c \) to 0. A \textit{hyperplane} \( H \) in \( \Omega \) is a maximal collection of midcubes in \( \Omega \), such that for any two midcubes \( m \) and \( m' \) in \( H \), it follows there is a sequence of midcubes \( m = m_1, \ldots, m_n = m' \) in \( H \) such that \( m_i \cap m_{i+1} \) is a midcube in \( \Omega \) for all \( 1 \leq i < n \).

We remark that for the cube complexes considered in this article, it will be the case that hyperplanes do not self-intersect. Thus, it can be checked that in this setting a hyperplane can also be defined to be a connected collection of midcubes in \( \Omega \), such that for any cube \( c \) in \( \Omega \) either \( H \cap c \) is a midcube or \( H \cap c = \emptyset \).

The \textit{carrier} of a hyperplane \( H \), denoted by \( N(H) \), is the set of all cubes which have non-empty intersection with \( H \). If \( H \) intersects an edge \( e \), then we say that \( e \) is dual to \( H \).

Let \( \Omega \) be a CAT(0) cube complex and \( H \) be a hyperplane in \( \Omega \). We will need the following well-known facts (see for instance [Wis12] Chapters 3.2 and 3.3):

1. \( \Omega \setminus H \) contains exactly two components
2. \( N(H) \) is convex in the combinatorial path metric
3. A path \( \gamma \) in \( \Omega \) is geodesic if and only if every hyperplane is dual to at most one edge of \( \gamma \). Thus, if \( \gamma \) is geodesic then \( |\gamma| \) is equal to the number of hyperplanes which intersect \( \gamma \).
2.4. Disk diagrams in cube complexes. We now recall some basic facts about disk diagrams, which are a useful tool for studying cube complexes. We refer to [Wis11] and [Wis12] for further details.

A disk diagram \( D \) is a contractible, finite, 2-dimensional cube complex (i.e., a square complex) that is equipped with a given planar embedding \( \Psi : D \to \mathbb{R}^2 \). The map \( \Psi \) gives a natural cellulation of the 2-sphere \( S^2 = \mathbb{R}^2 \cup \infty \). We call the path traced out by an attaching map of the cell containing \( \infty \) the boundary of \( D \) and denote it by \( \partial D \).

Given a cube complex \( \Omega \), a disk diagram in \( \Omega \) is a disk diagram \( D \) which admits a map to \( \Phi : D \to \Omega \) mapping \( n \)-cubes isometrically onto \( n \)-cubes (i.e. a combinatorial map). As the edges of the cube complexes we consider in this article will be labeled, we accordingly further require the edges of \( D \) to be labeled and the map from \( D \) to \( \Omega \) to respect this labeling.

Given a cube complex \( \Omega \) and a closed null-homotopic loop \( p : S^1 \to \Omega \), by a lemma of van Kampen, there always exists a disk diagram \( D \) in \( \Omega \) with combinatorial map \( \Phi : D \to \Omega \) and an identification of \( \partial D \) with \( S^2 \) such that \( \Phi \) restricted to \( \partial D \) is equal, as a map, to \( p \) (see for instance [Wis12, Lemma 3.1]).

Given a disk diagram \( D \) in \( \Omega \) and an edge \( e \) of \( D \), a dual curve dual to \( e \) is a hyperplane in \( D \) dual to \( e \). As \( D \) is planar, a dual curve in \( D \) can be dual to at most two edges along \( \partial D \).

The cube complexes we consider in this article will have the additional property that edges are labeled by elements of a simplicial graph \( \Gamma \). Furthermore, any square in such a cube complex will have opposite sides labeled by the same vertex of \( \Gamma \) and adjacent sides labeled by distinct adjacent vertices of \( \Gamma \). Thus, given a disk diagram in such a cube complex we can define the type of a dual curve to be the label of an edge (equivalently, all edges) dual to the dual curve. Furthermore, if two dual curves intersect, then their types must be adjacent vertices of \( \Gamma \).

Let \( W_\Gamma \) be a right-angled Coxeter group. There exists a CAT(0) cube complex \( \Sigma_\Gamma \) associated to \( W_\Gamma \), known as the Davis complex, whose 1-skeleton is the Cayley graph of \( W_\Gamma \) and whose edges are naturally labeled by vertices of \( \Gamma \) (see for instance [Davi98]). Thus, given any word \( w \) in \( W_\Gamma \) which is an expression for the identity element, it follows that there exists a disk diagram \( D \) that has edges labeled by vertices of \( \Gamma \) and whose boundary has label \( w \). Furthermore, \( D \) has the properties listed in the previous paragraph. We will often use these facts without mention.

Below we use disk diagrams to give a proof of a version of the so-called deletion property. We remark that a similar proof is used in [Bah05] for proving the standard version of the deletion property in Coxeter groups.

**Proposition 2.2.** If \( w = s_1 \ldots s_n \) is a word in a right-angled Coxeter group \( W_\Gamma \) which is not reduced, then a deletion can be applied to \( w \). In other words, for some \( 1 \leq i < i' \leq n \), we have that \( s_i = s_{i'} = s \), and \( m(s, s_j) = 2 \) for all \( i < j < i' \). Consequently \( w' = s_1 \ldots s_{i-1}s_{i+1} \ldots s_{i'-1}s_{i'+1} \ldots s_n \) is an expression for \( w \).

**Proof.** Let \( r \) be a reduced expression for \( w \). As \( rw^{-1} \) is equal to the identity element in \( W_\Gamma \), it follows that there is a disk diagram \( D \) with boundary \( rw^{-1} \). Let \( \alpha \) be the path along the boundary of \( D \) with label \( r \), and let \( \beta \) be the path along the boundary of \( D \) with label \( w \).

As \( r \) is reduced, no dual curve is dual to two edges of \( \alpha \). As \( w \) is not reduced, we have that \( |w| > |r| \). From these two facts we deduce there must be some dual
curve $H$ dual to two distinct edges, say $e$ and $e'$, of $\beta$. Furthermore, as $e$ and $e'$ are dual to the same dual curve, these edges are labeled by the same letter $s \in \Gamma$.

Let $\beta'$ be the subpath of $\beta$ from $e$ to $e'$. Without loss of generality, we can assume that no dual curve is dual to two edges of $\beta'$ (or else we could replace $H$ with such a dual curve). It follows that every dual curve dual to an edge of $\beta'$ must intersect $H$. Thus, the label $t_1 \ldots t_m$ of $\beta'$ has the property that $m(s, t_i) = 2$ for all $1 \leq i \leq m$. Hence, the claim follows as $w$ must contain the subword $st_1 \ldots t_m s$. Note that if $e$ and $e'$ were actually adjacent edges of $D$ then $w$ has a subword of the form $ss$ and the claim still follows.

The following lemma, which is required later, easily follows from Proposition 2.2.

**Lemma 2.3.** Let $h$ and $k$ be reduced words in a right-angled Coxeter group $W_\Gamma$. Then there is a reduced expression $\hat{h} \hat{k}$ for the word $hk$ such that $\hat{h}s_1 \ldots s_m$ is a reduced expression for $h$ and $s_m \ldots s_1 k$ is a reduced expression for $k$ where $s_i \in V(\Gamma)$ for $1 \leq i \leq m$.

We will need the following lemma which lets us deduce properties of words from properties of a disk diagram with the same words as labels of its boundary. We remark that any word considered in the lemma below could be the empty word.

**Lemma 2.4.** Let $w$ and $z$ be words that are equal as elements of a right-angled Coxeter group $W_\Gamma$. Suppose $w = w'w''$ and $z = z'z''$ where $w'$, $w''$, $z'$, $z''$ are reduced words in $W_\Gamma$. Let $D$ be a disk diagram with boundary label $wz^{-1}$, and let $\alpha_{w'}$ and $\alpha_{z'}$ be the paths in the boundary of $D$ with labels $w'$ and $z'$ respectively. Suppose further that every dual curve dual to $\alpha_{w'}$ is also dual to $\alpha_{z'}$. Then $z'$ has a reduced expression $z' = w'x$ where $x$ is some word in $W_\Gamma$.

**Proof.** By [Wis11] Lemma 2.3], there is a minimal area disk diagram $D'$ with the same boundary label and property as in the statement. By the proof of [Wis11] Lemma 2.6], if two dual curves dual to $\alpha_{z'}$ intersect, then there is a smaller area disk diagram $D''$ with boundary label $w(z'')^{-1}y^{-1}$ where $y$ is a word equal to $z'$ in $W_\Gamma$. Additionally, dual curves dual to the path in the boundary of $D''$ with label $w'$ are also dual to the path in the boundary of $D''$ labeled by $y$.

By repeatedly applying this argument, we eventually get a disk diagram $\hat{D}$ with boundary label $wz'^{-1}\hat{x}^{-1}$ where $\hat{x}$ is equal to $z'$ in $W_\Gamma$. Furthermore, no pair of curves dual to the path in the boundary of $\hat{D}$ labeled by $\hat{x}$ intersect, and every curve dual to the path in the boundary of $\hat{D}$ labeled by $w'$ is also dual to the path in the boundary of $\hat{D}$ labeled by $\hat{x}$. It follows that $w'$ is a prefix of $\hat{x}$, and the claim follows.

---

3. A complex for subgroups of a right-angled Coxeter group

The main goal of this section is to define a completion of a subgroup of a right-angled Coxeter group, and to construct completions for finitely generated subgroups. We begin by defining a completion of a $\Gamma$-labeled complex as the direct limit of a certain sequence of $\Gamma$-labeled complexes. We then show that there is a natural labeled graph associated to any finite generating set, such that a completion of this graph is also a completion of the group generated by the set.
3.1. **Completion of a complex.** Let $\Gamma$ denote a simplicial graph. In this paper, we only consider $\Gamma$-labeled cube complexes whose labeled cubes have two additional properties. Firstly, any pair of edges dual to a common mid-cube have the same label. As a result, hyperplanes in the cube complex have a well-defined label. Secondly, given a cube in such a complex and a set of edges of this cube which are all incident to a common vertex, no two edges in this set have the same label, and the full subgraph of $\Gamma$ induced by the vertices of $\Gamma$ corresponding to the labels of the edges in this set is a clique. When we mention a $\Gamma$-labeled cube complex, it will be implicit that the labeling has these additional properties.

Let $C$ be a $\Gamma$-labeled cube complex. We describe three operations that can be applied to $C$ to produce a new $\Gamma$-labeled cube complex.

**Fold operation:** A fold operation corresponds to collapsing a pair of adjacent edges with the same label into a single edge. More precisely, for $i = 1, 2$, let $e_i$ be an edge in $C$ with endpoints $v$ and $v_i$, where $e_1 \neq e_2$, but two or more of the vertices $v, v_1, v_2$ could be equal. Furthermore, suppose that $e_1$ and $e_2$ have the same label. Temporarily orient the edge $e_i$ from $v$ to $v_i$ (choosing the orientation arbitrarily if $v = v_i$, i.e. if $e_i$ is a graph-loop). Then the fold operation consists of forming a quotient of $C$ by identifying $e_1$ and $e_2$ so that their orientations agree, and then forgetting the orientation.

We remark that although the fold map corresponding to $e_1$ and $e_2$ is not unique when one of these edges is a graph-loop, this does not affect any of our applications.

**Cube identification operation:** Consider a collection of two or more distinct $i$-cubes in $C$, with $i \geq 2$, whose boundaries are equal. A cube identification operation consists of forming the quotient of $C$ in which all of the $i$-cubes in the collection have been identified to a single cube. Note that the 1-skeleton does not change in this process.

**Cube attachment operation:** Consider an $i$-tuple $e_1, \ldots, e_i$ of edges in $C$, with labels $s_1, \ldots, s_i$, which are all incident to a single vertex $v$. Suppose furthermore, that the vertices corresponding to their labels form an $i$-clique in $\Gamma$. A cube attachment operation consists of adding an $i$-cube $c$ to $C$ by identifying the edges $e_1, \ldots, e_i$ to $i$ edges in $c$ which are all incident to a single vertex of $c$. In the process, we end up adding some vertices and edges to $C$. Each new edge added is dual to a mid-cube of $c$ which is also dual to a unique edge in the set $\{e_1, \ldots, e_i\}$. This induces a labeling on the newly added edges, making the resultant complex $\Gamma$-labeled.

We say a complex is *folded* if no fold operation or cube identification operation can be performed to the complex. As fold operations and cube identification operations reduce the number of cells, any finite complex $C$ can be transformed into a folded complex through finitely many such operations.

We say a complex is *cube-full* if for any $i$-tuple of edges all incident to the same vertex such that the vertices corresponding to their labels form an $i$-clique in $\Gamma$, there exists an $i$-cube of $C$ whose boundary contains these $i$ edges.

Given a connected finite $\Gamma$-labeled complex $X$, consider a possibly infinite sequence:

$$\Omega_0 = X \xrightarrow{f_0} \Omega_1 \xrightarrow{f_1} \Omega_2 \cdots$$
where for each \( i \), the map \( f_i : \Omega_i \to \Omega_{i+1} \) is either a fold, cube identification or cube attachment operation. Let \( \Omega_X \) be the direct limit of this sequence. If \( \Omega_X \) is folded and cube-full, we call \( \Omega_X \) a completion of \( X \). We say

\[
\Omega_0 = X \xrightarrow{f_0} \Omega_1 \xrightarrow{f_1} \Omega_2 \cdots \to \Omega_X
\]

is a completion sequence for \( X \). We sometimes leave the maps \( f_i \) out of the notation when these maps are not relevant. We also set \( \hat{f} : X \to \Omega_X \) as the direct limit of the maps \( \{ f_i \} \).

**Example 3.1.** Let \( \Gamma_1 \) be the graph in Figure 1. The bottom of the figure shows a completion sequence for the \( \Gamma_1 \)-labeled complex \( X \). The completion \( \Omega \) is obtained from \( X \) by a fold operation followed by a cube attachment operation. Note that not all labels of \( \Omega \) are shown.

\[
\begin{align*}
\Gamma_1 &= \begin{array}{c}
\text{b} \\
\text{a} \quad \text{c} \\
\text{d} \quad \text{e}
\end{array} \\
\text{X} &= \begin{array}{c}
\text{d} \\
\text{b} \quad \text{a} \quad \text{a} \\
\text{c} \quad \text{e}
\end{array} \\
\Omega_1 &= \begin{array}{c}
\text{b} \\
\text{a} \quad \text{c} \\
\text{d} \quad \text{e}
\end{array} \\
\Omega_2 &= \Omega
\end{align*}
\]

**Figure 1.** A completion \( \Omega \) for the \( \Gamma_1 \)-labeled complex \( X \).

**Example 3.2.** Figure 2 shows a graph \( \Gamma_2 \) and a \( \Gamma_2 \)-labeled complex \( X \). A standard completion \( \Omega' \) of \( X \) (see Definition 3.4) is shown on the right (the labels of \( \Omega' \) are omitted). The cube complex \( \Omega' \) is topologically a bi-infinite cylinder.

\[
\begin{align*}
\Gamma_2 &= \begin{array}{c}
\text{a} \\
\text{d} \quad \text{c} \\
\text{b}
\end{array} \\
\text{X} &= \begin{array}{c}
\text{a} \\
\text{b} \quad \text{b} \\
\text{c}
\end{array} \\
\Omega' &= \begin{array}{c}
\ldots \\
\ldots \ldots \ldots
\end{array}
\end{align*}
\]

**Figure 2.** A completion \( \Omega' \) for the \( \Gamma_2 \)-labeled complex \( X \).

**Proposition 3.3** (Existence of a completion). Given any finite \( \Gamma \)-labeled complex \( X \), there exists a completion \( \Omega_X \) of \( X \).
Proof. We set $\Omega_0 = X$ and build $\Omega_i$ inductively. Suppose a finite complex $\Omega_i$ was obtained from $\Omega_0$ by a sequence of fold, cube identification and cube attachment operations. We iteratively perform fold and cube identification operations to $\Omega_i$ to obtain the complexes $\Omega_{i+1}, \Omega_{i+2}, \ldots, \Omega_{i+j}$, where $\Omega_{i+j}$ is folded. (This includes the case $i = 0$.) As $\Omega_i$ is finite, we conclude that $j$ (and hence $\Omega_{i+j}$) is finite as well.

Next we describe a sequence of operations to be performed to the finite folded complex $\Omega_{i+j}$. Choose a vertex $v$ of $\Omega_{i+j}$. Consider a maximal tuple of edges incident to $v$ such that their labels form a clique in $\Gamma$. If there is no cube in $\Omega_{i+j}$ whose boundary contains the tuple of edges, then attach an appropriately labeled cube of the appropriate dimension along the tuple of edges. Do this for each such maximal tuple at $v$, and then proceed to do the same for all the vertices of $\Omega_{i+j}$. Thus, the image of $\Omega_{i+j}$ is folded. (This includes the case $j = 0$.) As $\Omega_{i+j}$ is finite we conclude that $k$ (and hence $\Omega_{i+j+k}$) is finite. We then repeat the above procedure starting with the finite complex $\Omega_{i+j+k}$.

Let $\Omega_X$ be the direct limit of these complexes. Consider a pair of edges, say $e$ and $f$, in $\Omega_X$ incident to the same vertex $v$. It follows that some $\Omega_i$ contains preimages of $e$ and $f$ which are incident to the same vertex. Consequently, there is some folded $\Omega_{i'}$, with $i' \geq i$, which contains preimages of $e$ and $f$ that are incident to the same vertex. Thus, if $e$ and $f$ have the same label, then their preimages in $\Omega_{i'}$ must be identified. A similar argument shows that two cubes with the same boundary in $\Omega_X$ must be identified. It follows that $\Omega_X$ is folded.

Let $e_1, \ldots, e_n$ be edges all incident to a common vertex $v \in \Omega_X$ whose labels form an $n$-clique in $\Gamma$. There is some folded $\Omega_{i'}$ which contains a preimage $v'$ of $v$ and preimages $e'_1, \ldots, e'_n$ of $e_1, \ldots, e_n$ so that $e'_1, \ldots, e'_n$ are all incident to $v'$. As a result of the procedure described above, some $\Omega_{i+j}$ is the complex resulting from a cube attachment operation where an $n$-cube, say $c$, is attached to the image of the edges $e'_1, \ldots, e'_n$ in $\Omega_{i+j-1}$. Thus, the image of $c$ in $\Omega_X$ is an $n$-cube containing the edges $e_1, \ldots, e_n$. This shows $\Omega_X$ is cube-full. 

**Definition 3.4** (Standard Completion). We call the completion algorithm given in the proof of Lemma 3.3 a *standard completion* and the associated sequence:

$$\Omega_0 \to \Omega_1 \to \cdots \to \Omega_X$$

a standard completion sequence. We recall that this is a completion sequence obtained by alternately performing all possible fold and cube identification operations to a complex, then performing all possible cube attachments to the resulting folded complex, and iteratively repeating this procedure whenever possible.

In the next proposition, we show that if a completion is finite, then there is indeed a finite algorithm to obtain the completion.

**Proposition 3.5.** Let $X$ be a $\Gamma$-labeled complex. Let

$$X = \Omega_0 \to \Omega_1 \to \cdots \to \Omega$$

be a standard completion sequence. If $\Omega$ is finite then the completion sequence is finite, i.e. $\Omega = \Omega_N$ for some $N$.

Proof. As $\Omega$ is finite, by the definition of a direct limit, for some $M$ and all $n \geq M$, $\Omega_n$ contains an isometrically embedded subcomplex $Y_n$ isometric to $\Omega$, and the
natural map \( f : \Omega_n \to \Omega \) is a label-preserving isometry when restricted to \( Y_n \). As we have a standard completion, there exists an \( N \geq M \) such that \( \Omega_N \) is folded.

We claim that \( \Omega_N = \Omega \). Suppose for a contradiction that \( Y_N \subsetneq \Omega_N \). If \( \Omega_N \setminus Y_N \) contains a vertex or an edge, then since \( \Omega_N \) is connected, it follows that some vertex \( v \in Y_N \) is incident to an edge \( e \) that is not contained in \( Y_N \). Let \( s \) be the label of \( e \). As \( \Omega_N \) is folded, no edge in \( Y_N \) that is incident to \( v \) has label \( s \). However, the continuous map \( f : \Omega_N \to \Omega \) sends \( e \) to an edge incident to \( f(v) \) labeled by \( s \). This is a contradiction. Thus \( Y_N \) and \( \Omega_N \) have the same 1-skeleton.

Suppose there is a 2-cell \( c \) in \( \Omega_N \setminus Y_N \). Then the boundary \( \partial c \) of \( c \) is contained in \( Y_N \). Since \( \Omega_N \) folded, there is no cube in \( Y_N \) with boundary \( \partial c \). However, \( f \) sends \( \partial c \) to the boundary of a cube in \( \Omega \), and \( f(c) \) is a cube in \( \Omega \) with boundary \( f(\partial c) \), leading to a contradiction. Thus \( \Omega_N = \Omega \).

\[ \square \]

### 3.2. Completion of a subgroup

In this subsection, we define the completion of a subgroup of a right-angled Coxeter group and show a completion is guaranteed to exist for any finitely generated subgroup.

**Definition 3.6 (Completion of a subgroup).** Let \( G \) be a subgroup of a right-angled Coxeter group \( W_\Gamma \), and let \( \Omega \) be a connected \( \Gamma \)-labeled cube complex with basepoint the vertex \( B \in \Omega \). We say that \((\Omega, B)\) is a completion of \( G \) if:

1. \( \Omega \) is folded and cube-full.
2. Given any loop in \( \Omega \) based at \( B \), its label is a word which represents an element of \( G \).
3. For any reduced word \( w \) in \( W_\Gamma \) which represents an element of \( G \), there is a loop \( l \) based at \( B \) with label \( w \).

**Remark 3.6.1.** We sometimes say \( \Omega \) is a completion of \( G \), dropping the basepoint \( B \) from the notation, if the basepoint is not relevant.

**Example 3.7.** Let \( G \) be a finite subgroup of \( W_\Gamma \) generated by adjacent vertices \( a \) and \( b \) in \( \Gamma \). Let \( X \) be the rose graph consisting of one vertex, one graph-loop labeled by \( a \) and one graph-loop labeled by \( b \). Let \( \Omega_1 \) and \( \Omega_2 \) respectively be the torus and Klein bottle obtained by attaching a 2-cell to \( X \) (where we view the 2-cell to be a square). Then both \( \Omega_1 \) and \( \Omega_2 \) are completions for \( G \).

When the group \( G < W_\Gamma \) is finitely generated, we can construct a completion for \( G \) as the completion of a certain \( \Gamma \)-labeled graph associated to \( G \), which we now describe. Let \( G \) be generated by the finite generating set of words

\[ S_G = \{ w_i = s_{i_1} s_{i_2} \ldots s_{i_{m_i}} \mid 1 \leq i \leq n \} \]

where \( s_{ij} \in V(\Gamma) \) for each \( i, j \).

We associate to \( S_G \) the following \( \Gamma \)-labeled complex. We begin with a single base vertex \( B \). For each generator \( w_i \), we attach a circle subdivided to have \( m_i \) edges, such that edges of this circle are sequentially labeled, beginning at \( B \), by the letters \( s_{ij} \) for \( 1 \leq j \leq m_i \). We denote this resulting based complex by \((X(S_G), B)\) and call it the \( S_G \)-complex. Note that the label of a circle based at \( B \) in \( X(S_G) \) always corresponds to a generator in \( S_G \).

Let

\[ X(S_G) = \Omega_0 \to \Omega_1 \to \Omega_2 \cdots \to \Omega \]
be a completion of \( X(S_G) \). By a slight abuse of notation, we denote by \( B \) the vertex \( B \in X(S_G) \), its image in \( \Omega \), for any \( i \), and its image in \( \hat{\Omega} \). The next few lemmas show that \( \Omega \) is a completion of \( G \).

**Lemma 3.8.** Let \( G \) be a subgroup of a right-angled Coxeter group \( W_\Gamma \), given by a finite generating set \( S_G \). Let \( \Omega \) be any completion of \( X(S_G) \) where \( (X(S_G), B) \) is the \( S_G \)-complex. If \( w \) is a reduced word in \( W_\Gamma \) which represents an element of \( G \), then \( w \) is the label of some loop in \( \Omega \) based at \( B \).

**Proof.** Let \( X(S_G) = \Omega_0 \rightarrow \Omega_1 \rightarrow \Omega_2 \rightarrow \cdots \rightarrow \Omega \) be a completion sequence for \( X(S_G) \), and let \( w \) be a reduced word in \( W_\Gamma \) which represents an element of \( G \).

As \( S_G \) is a generating set of \( G \), it follows that \( w \) is equal in \( W_\Gamma \) to a word \( w' = h_1 \ldots h_k \) where \( h_i \in S_G \) for each \( i \). By construction, for each \( 1 \leq i \leq k \), there is a loop \( l_i \) based at \( B \) in \( \Omega_0 \), with label \( h_i \). Let \( l \) be the loop in \( \Omega_0 \) formed as a concatenation of loops: \( l_1 l_2 \ldots l_k \). Let \( \hat{l} \) be the image of \( l \) in \( \Omega \). Then \( \hat{l} \) has the same label as \( l \).

As \( w' \) and \( w \) are equal as elements of \( W_\Gamma \), the word \( w \) can be obtained from \( w' \) through a sequence of Tits moves. Suppose the first Tits move in this sequence is a swap performed to \( w' \) to obtain a new word \( w'' \).

We claim \( w'' \) is the label of a loop in \( \Omega \) as well. Note that there are adjacent edges \( e \) and \( f \) of \( \hat{l} \), labeled by \( s \) and \( t \) where \( s, t \in V(\Gamma) \) and \( m(s, t) = 2 \), such that \( w' = a_1 \ldots a_i sta_{i+1} \ldots a_m \) and \( w'' = a_1 \ldots a_i tsa_{i+1} \ldots a_m \), with \( a_j \in V(\Gamma) \). As \( m(s, t) = 2 \) and \( \Omega \) is cube-full, there must be a square \( Q \) in \( \Omega \) whose boundary contains \( ef \). We now obtain the desired loop by replacing \( ef \) in \( \hat{l} \) with the opposite path in the boundary of \( Q \).

On the other hand, suppose the first Tits move is a cancellation. In other words, \( w' = a_1 \ldots a_i ssa_{i+1} \ldots a_m \) is replaced by \( w'' = a_1 \ldots a_m \), where \( s \in V(\Gamma) \) and \( a_i \in V(\Gamma) \) for each \( i \). As \( \Omega \) is folded, \( \hat{l} \) must traverse an edge \( e \), labeled by \( s \), twice consecutively. It follows that either \( e \) is a graph-loop or that \( \hat{l} \) traverses \( e \) in one direction and immediately backtracks in the other direction. In either case, we can simply remove these two occurrences of the edge \( e \) from \( \hat{l} \) to obtain a new loop based at \( B \) with label \( w'' \).

By repeating this procedure for each Tits move, we obtain a loop in \( \Omega \) with label equal to \( w \). \( \square \)

**Lemma 3.9.** Let \( \hat{\Omega} \) be a \( \Gamma \)-labeled complex obtained by applying either a fold, cube identification or cube attachment operation to the \( \Gamma \)-labeled complex \( \Omega \). Let \( F : \hat{\Omega} \rightarrow \Omega \) be the natural map. Let \( B \) be a vertex of \( \hat{\Omega} \) and let \( \hat{B} \) be a vertex that is in the preimage under \( F \) of \( B \). Let \( \hat{w} \) be the label of a loop \( \hat{l} \) in \( \hat{\Omega} \) based at \( \hat{B} \). Then there exists a loop \( l \) in \( \Omega \) based at \( B \), with label \( \hat{w} \), such that \( \hat{w} \) and \( w \) represent the same element of the right-angled Coxeter group \( W_\Gamma \).

**Proof.** We analyze each type of operation separately:

**Cube identification operation:** If \( \Omega \) is obtained from \( \hat{\Omega} \) by a cube identification operation, then \( \Omega \) and \( \hat{\Omega} \) have the same 1-skeleton. Thus \( l \) is the image of a loop \( \hat{l} \) in \( \hat{\Omega} \) with the same label as \( l \).
Fold operation: Suppose that $\Omega$ is obtained from $\bar{\Omega}$ by a fold operation. Let $B = u_1, \ldots, u_m = B$ be the vertices of $l$ listed sequentially by the orientation of $l$.

Suppose some vertex, say $v$, of $\Omega$ has preimage $F^{-1}(v) = \{\bar{v}_1, \bar{v}_2\}$. As only a single edge is folded in a fold operation, there is at most one such vertex. Let $\bar{f}_1$ and $\bar{f}_2$ be the two edges in $\bar{\Omega}$ which are folded and let $f$ be the edge in $\Omega$ which is their image. Let $s$ be the label of $f$. The endpoints of $f$ must be $v$ and some vertex $v'$ (which is possibly equal to $v$).

We say a vertex or edge of $l$ has unique preimage if its preimage under $F$ is a single vertex or edge. It is straightforward to check that $l$ can be subdivided into subpaths of the five types described below (though not all the types may be used):

1. An edge $p$ from $u = u_i$ to $u' = u_{i+1}$ where $u$ and $u'$ each have unique preimage.
2. A path $p$ from $u = u_i$ to $u' = u_{i'}$, where $u$ and $u'$ each have unique preimage, and $u_j = v$ for every $i < j < i'$.
3. A path $p$ from $B = u_1$ to $u = u_i$, where $u$ has unique preimage and $u_j = v$ for every $j < i$.
4. A path $p$ from $u = u_i$ to $B = u_m$, where $u$ has unique preimage and $u_j = v$ for every $j > i$.
5. $p = l$ and $u_i = v$ for all $i$.

We claim that for each path $p$ of a type described above, there is a path $\bar{p}$ in $\bar{\Omega}$ such that the label of $p$ and the label of $\bar{p}$ are equal as elements of $W_\Gamma$. Additionally, the image under $F$ of the endpoints of $\bar{p}$ are equal to the endpoints of $p$. Finally, if $p$ is of type 3, then $\bar{p}$ begins at $\bar{B}$, if $p$ is of type 4 then $\bar{p}$ ends at $\bar{B}$ and if $p$ is of type 5 then $\bar{p}$ begins and ends at $\bar{B}$. The lemma clearly follows from this claim.

We proceed to prove the claim for each type of subpath of $l$.

Type 1: Let $p = e$ be the edge in $l$ between $u = u_i$ and $u' = u_{i+1}$. Let $\bar{u}$ and $\bar{u}'$ be the unique preimages of $u$ and $u'$ under $F$. The preimage $F^{-1}(e)$ is either a single edge between $\bar{u}$ and $\bar{u}'$ or is a pair of edges between $\bar{u}$ and $\bar{u}'$ (in this case the fold operation identifies this pair of edges to get $e$). Let $\bar{e}$ be a choice of edge in $F^{-1}(e)$. We define $\bar{p}$ to the path that traverses $\bar{e}$. Clearly $\bar{p}$ and $p$ have the same label.

Type 2: In this case $p$ consists of an edge $e_1$ from $u$ to $v$, followed by a collection of graph-loops $q_1, \ldots, q_k$ based at $v$, followed by an edge $e_2$ from $v$ to $u'$.

Let $\bar{u}$ and $\bar{u}'$ be the unique preimages of $u$ and $u'$. Let $\bar{e}_1$ and $\bar{e}_2$ be edges (not necessarily unique) in the preimage of $e_1$ and $e_2$ respectively. Let $\bar{q}_1, \ldots, \bar{q}_k$ each be a choice of edge in the preimages of $q_1, \ldots, q_k$. For each $1 \leq i \leq k$, the edge $\bar{q}_i$ is either a graph-loop at $\bar{v}_1$, a graph-loop at $\bar{v}_2$ or an edge between $\bar{v}_1$ and $\bar{v}_2$.

Let $\bar{z}$ be the path from $\bar{v}_1$ to $\bar{v}_2$ obtained by traversing the edge $\bar{f}_1$ then the edge $\bar{f}_2$. Note that the label of $\bar{z}$ is equal to the identity element of $W_\Gamma$ as $\bar{f}_1$ and $\bar{f}_2$ have the same label. Form the path

$$\bar{p} = \bar{e}_1 \bar{p}_0 \bar{q}_1 \bar{p}_1 \bar{q}_2 \bar{p}_2 \ldots \bar{q}_k \bar{p}_k \bar{e}_2$$

Where for $1 \leq i < k$, we define $\bar{p}_i$ to either be $\bar{z}$, $\bar{z}^{-1}$ or the empty word in order for the endpoint of $\bar{q}_i$ to coincide with the startpoint of $\bar{q}_{i+1}$. The paths $\bar{p}_0$ and $\bar{p}_k$ are defined similarly in order for the endpoint of $\bar{e}_1$ to coincide with the startpoint of $\bar{q}_1$ and in order for the endpoint of $\bar{q}_k$ to coincide with the startpoint of $\bar{e}_2$.

The claim then follows for this case as the label of $\bar{p}$ is equal as an element of $W_\Gamma$ to the label of $p$. 
Type 5: In this case $v = B$ necessarily, and $l$ consists of a sequence of graph-loops $q_1, \ldots, q_k$. Let $\tilde{q}_1, \ldots, \tilde{q}_k$ be a choice of edges in the preimages of $q_1, \ldots, q_n$. As above, these preimages consist of graph-loops at $\tilde{v}_1$, graph-loops at $\tilde{v}_2$ and edges between $\tilde{v}_1$ and $\tilde{v}_2$. We may define the path

$$\tilde{p} = \tilde{p}_0 \tilde{q}_1 \tilde{q}_1 \ldots \tilde{q}_k \tilde{p}_k$$

where $\tilde{p}_i$, for $1 \leq i < k$, is defined similarly as in the previous case. Note that $\tilde{B}$ is either equal to $\tilde{v}_1$ or $\tilde{v}_2$. We then define $\tilde{p}_0$ to either be $\tilde{z} = \bar{f}_1 \bar{f}_2, \bar{z}^{-1}$ or the empty path in order to guarantee $\tilde{p}$ begins at $\tilde{B}$. Similarly define $\tilde{p}_k$ to guarantee that $\tilde{p}$ ends at $\tilde{B}$.

Type 3 and 4: The analysis of these cases is very similar to the cases done above and are omitted.

Cube attachment operation: Suppose $\Omega$ is obtained by attaching a cube $c$ to the union $e_1 \cup \cdots \cup e_n \subset \Omega$, where for $1 \leq i \leq n$, $e_i$ is an edge labeled $s_i$ from a vertex $v$ to a vertex $v_i$ (with $v = v_i$ if $e_i$ is a graph-loop), and the vertices of $\Gamma$ corresponding to the labels $s_1, \ldots, s_n$ form an $n$-clique.

If $l$ does not intersect $c \setminus \{e_1, \ldots, e_n\}$ then $l$ is clearly the image of a loop in $\Omega$ with the same label. Otherwise let $q$ be the closure of a maximal connected subpath of $l$ that is contained in $c \setminus \{e_1, \ldots, e_n\}$. In particular, $q$ is a path between

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**Figure 3.** The graph on the top demonstrates a path of type 2 in $\Omega$. The two graphs below it show the two possible choices of preimages for $p$. The graph-loops $\tilde{q}_1, \ldots, \tilde{q}_k$ is a subsequence of the graph-loops $\bar{q}_1, \ldots, \bar{q}_k$ that consist of graph-loops based at $\bar{v}_1$. Similarly, $\bar{q}_k, \ldots, \bar{q}_k'$ are edges between $\bar{v}_1$ and $\bar{v}_2$ and $\bar{q}_k', \ldots, \bar{q}_k''$ are graph-loops based at $\bar{v}_2$. Note that some of the vertices and edges shown may actually be equal $\bar{\Omega}$. For instance, it could be that $\bar{f}_1 = \bar{e}_1$. 
Let $h$ be the label of $q$. Since the vertices $s_1, \ldots, s_n$ form a clique in $\Gamma$, there is a reduced expression for $h$ given by $h' = s_1^\epsilon_1 \cdots s_n^\epsilon_n$ where for each $1 \leq i \leq n$, $\epsilon_i = 1$ if there is an odd number of occurrences of the generator $s_i$ in $h$ and $\epsilon_i = 0$ otherwise.

First assume $k \neq k'$. After renaming if necessary, we may assume $k = 1$ and $k' = n$. We claim that if $1 < j < n$ and $\epsilon_j = 1$, then $e_j$ is a graph-loop. To prove this, note that if $1 < j < n$ and $\epsilon_j = 0$, then $e_j$ and $v_n$ are on the same side of the midcube of $c$ dual to $e_j$. Thus $q$ crosses this midcube an even number of times, and therefore $\epsilon_j = 0$.

Consider the union of edges $q' = e_1^{\epsilon_1} \cup e_2^{\epsilon_2} \cup \cdots \cup e_n^{\epsilon_n}$, where $e_i^{\epsilon_i}$ is interpreted as empty if $\epsilon_i = 0$. The claim in the previous paragraph implies that this is in fact a path $q'$ with the same endpoints as $q$ (regardless of whether or not $e_1$ and $e_n$ are graph-loops). Observe that the label of $q'$ is $h'$.

By a similar argument, if $k = k' = 1$ and $\epsilon_1$ is a graph-loop, then $q' = e_1^{\epsilon_1} \cup e_2^{\epsilon_2} \cup \cdots \cup e_n^{\epsilon_n}$ is a path with label $h'$ and the same endpoints as $q$. Finally, suppose $k = k' = 1$ and $\epsilon_1$ is not a graph-loop. As before, if $j \neq 1$ and $\epsilon_j = 1$ then $e_j$ is a graph-loop. Moreover, $\epsilon_1 = 0$, because $q$ crosses the mid-cube dual to $e_1$ an even number of times. Thus in this case we define $q'$ to be the concatenation $e_1^{\epsilon_1} e_2^{\epsilon_2} \cup \cdots \cup e_n^{\epsilon_n} e_1$, and note that this is a continuous path with the same endpoints as $q$, and with label $s_1 h' s_1$, which is equal in $W_T$ to $h'$.

In each case we have produced a path $q'$ in $F(\Omega)$ with the same endpoints as $q$ and whose label is a word equal in $W_T$ to $h$. We replace $q$ with $q'$ in $l$. By performing all possible replacements of this sort, we obtain a loop in $F(\Omega)$ whose label is a word equal to $w$ in $W_T$. Thus the lemma follows for this case. □

**Lemma 3.10.** Let $G$ be a subgroup of a right-angled Coxeter group $W_T$ given by a finite generating set $S_G$. Let $\Omega$ be any completion of $X(S_G)$ where $(X(S_G), B)$ is the $S_G$-complex. Given a loop in $\Omega$ based at $B$, its label is a word representing an element of $G$.

**Proof.** Let

$$X(S_G) = \Omega_0 \xrightarrow{\Omega_1} \Omega_1 \xrightarrow{\Omega_2} \cdots \xrightarrow{} \Omega$$

be a completion sequence for $X(S_G)$. Let $B$ denote the basepoint of $X(S_G)$ as well as all its images in this sequence.

Consider a loop $l$ based at $B$ in $\Omega$, with label $w$. Then there exists $n$ such that $l = f(l')$ for some loop $l'$ based at $B$ in $\Omega_n$ which also has label $w$, where $f : \Omega_n \to \Omega$ is the natural map. By iteratively applying Lemma 3.9 starting with $l$, it follows there is a loop in $\Omega_0$ based at $B$ whose label is a word equal to $w$ in $W_T$. As the label of any loop based at $B$ in $\Omega_0$ represents an element of $G$, the lemma follows. □

The existence of completions for subgroups is now an immediate consequence of Lemma 3.8 and Lemma 3.10.

**Theorem 3.11.** Let $G$ be a subgroup of a right-angled Coxeter group $W_T$ given by a finite generating set $S_G$. Let $\Omega$ be any completion of $X(S_G)$, where $(X(S_G), B)$ is the $S_G$-complex. Then $(\Omega, B)$ is a completion of $G$. □

**Definition 3.12** (Standard Completion of a Subgroup). Let $G$ be a subgroup the right-angled Coxeter group $W_T$ generated by a finite generating set $S_G$. We call
a completion $\Omega$ of $G$ obtained by Theorem 3.11 a standard completion of $G$ with respect to $S_G$. In cases where it is understood that there is a finite generating set for $G$, we simply say $\Omega$ is a standard completion of $G$.

**Example 3.13.** The $\Gamma_1$-labeled cube complex $\Omega$ in Example 3.1 is a standard completion of the subgroup of $\Gamma_1$ generated by the words $w_1 = adb$ and $w_2 = acc$. Similarly, the $\Gamma_2$-labeled cube complex $\Omega'$ in Example 3.2 is a standard completion of the subgroup of $\Gamma_2$ generated by $w = abcd$.

**Remark 3.13.1.** Although every reduced word in $W_{\Gamma}$ representing an element of $G$ appears as a loop in the completion $\Omega$, it is not true that every word in $W_{\Gamma}$ representing an element of $G$ appears in $\Omega$ as a loop. For instance let $s$ be a vertex in $\Gamma$. Let $G$ be a subgroup of $W_{\Gamma}$ which is generated by a set of words in $W_{\Gamma}$, none of which contains the letter $s$. It follows that no edge in $\Omega$ is labeled $s$. Then the word $ss$, which is equal to the identity element, cannot be the label of any path in $\Omega$.

4. Basic properties of completions

We prove a few facts regarding completions that will be used throughout the rest of the paper. Recall from Section 2.2 that a deletion performed to a word $w$ in a right-angled Coxeter group produces an expression for $w$ with a pair of generators of the same type removed.

**Lemma 4.1.** Let $\Omega$ be a folded, cube-full, $\Gamma$-labeled complex. Let $p$ be a path in $\Omega$ with label $w$. Let $w'$ be an expression for $w$ obtained by performing $k$ deletions to $w$. Then there exists a path $p'$ in $\Omega$ such that the following properties hold.

1. The path $p'$ has label $w'$.
2. The paths $p$ and $p'$ have the same endpoints.
3. The Hausdorff distance between $p$ and $p'$ is at most $k$.
4. If $p$ does not traverse any graph-loops, then $p$ and $p'$ are homotopic relative to their endpoints.

**Proof.** Let $w_1$ be the word obtained by performing the first deletion to $w$. If $w = s_1 \ldots s_n$, with $s_i \in V(\Gamma)$, then $w_1 = s_1 \ldots s_{i-1}s_{i+1} \ldots s_{i'-1}s_{i'+1} \ldots s_n$ where $s_i = s_i' = s$ for some $1 \leq i < i' \leq n$ and $m(s, s_j) = 2$ for all $i < j < i'$. Let $\alpha$ be the subpath of $p$ labeled by $s_is_{i+1} \ldots s_{i'}$.

Suppose first that $i' - i > 1$. As $\Omega$ is cube-full, there exists a sequence of squares in $\Omega$ such that Figure 4 holds (although there may be additional edge or vertex identifications that are not shown).

![Figure 4](image)

Thus the subpath $\alpha$ of $p$, which runs along the bottom of Figure 4, can be replaced with the path which runs along the top of Figure 4 to obtain a new path $p_1$ with label $w_1$. Then $p_1$ is homotopic relative to endpoints to $p$ and is at Hausdorff distance at most 1 from $p$. 

On the other hand, if \( i' - i = 1 \), then as \( \Omega \) is folded, either \( \alpha \) traverses an edge labeled by \( s \) twice in opposite directions, or \( \alpha \) traverses a graph-loop labeled by \( s \) twice consecutively. In either case we can simply remove both occurrences of \( \alpha \) from \( p \) to obtain a new path \( p_1 \) with the same endpoints as \( p \), such that \( p_1 \) is labeled by \( w_1 \) and is at Hausdorff distance at most 1 from \( p \). If \( p \) does not traverse any graph-loops, then \( p_1 \) is homotopic relative to endpoints to \( p \). (However, in graph-loop case the new path is not necessarily homotopic relative to endpoints to \( p \).)

Finally, if \( p \) does not traverse any graph-loops, then \( p_1 \) does not either. This is clear when \( i' - i = 1 \). Now suppose \( i' - i > 1 \), and suppose \( p \) has no graph-loops, but \( p_1 \) has a graph-loop \( e \). Then \( e \) is an edge of one of the squares in Figure 4 and since \( \Omega \) is folded, the two edges of the square incident to \( e \) are identified in \( \Omega \). It follows that the edge opposite to \( e \) in the square, which is a part of \( p \), is also a graph-loop. This is a contradiction.

By repeating this process of obtaining \( p_1 \) from \( p \) inductively, we obtain the result.

\[ \Box \]

**Lemma 4.2.** Let \( \Omega \) be a folded, cube-full, \( \Gamma \)-labeled complex. Let \( p \) be a path in \( \Omega \) with endpoints the vertices \( u \) and \( v \), and let \( w \) be its label.

1. Any reduced word \( w' \) equal to \( w \) in \( W_\Gamma \) is the label of some path \( p' \) in \( \Omega \) from \( u \) to \( v \). Furthermore, if \( p \) does not traverse any graph-loops, then \( p \) and \( p' \) are homotopic relative to their endpoints.
2. If \( p \) has minimal length then \( w \) is reduced.
3. Let \( p \) and \( p' \) be paths in \( \Omega \) with the same endpoints. If \( p \) and \( p' \) are homotopic relative endpoints, then their labels are equal as elements of \( W_\Gamma \).

**Proof.** Let \( w'' \) be a reduced word equal to \( w \) in \( W_\Gamma \) which is obtained by a sequence of deletion operations. By Lemma 4.1 there is a path \( p'' \) with label \( w'' \) and with the same endpoints as \( p \). Furthermore, if \( p \) does not traverse any graph-loops, then \( p'' \) is homotopic relative endpoints to \( p \).

By Tits’ solution to the word problem, there is a sequence of words \( w'' = w_0, w_1, \ldots, w_n = w' \) so that \( w_{i+1} \) is obtained from \( w_i \) by swapping a pair of consecutive generators which commute.

Suppose that \( w_0 = a_1 \ldots a_s t s a_{i+1} \ldots a_n \) and \( w_1 = a_1 \ldots a_t t s a_{i+1} \), where \( s, t \in V(\Gamma) \), with \( m(s, t) = 2 \), and \( a_j \in V(\Gamma) \) for each \( j \). It follows there are consecutive edges of \( p'' \) with labels \( s \) and \( t \). As \( \Omega \) is cube-full, these edges are in a square with label \( stst \). By replacing these two edges of \( p'' \) with the other edges in this square, we obtain a new path whose label corresponds to swapping \( s \) and \( t \). Furthermore, this new path is homotopic, relative to endpoints, to \( p'' \). By iteratively repeating this process, we obtain the desired path \( p' \). This proves (1).

Let \( p \) be a path with minimal length and label \( w \). If \( w \) is not reduced, let \( w' \) be a reduced expression for \( w \). By (1) there is a path \( p' \) having the same endpoints as \( p \), with label \( w' \). However, \( |p'| < |p| \), a contradiction. This proves (2).

Let \( p \) and \( p' \) be as in (3). As the concatenation \( pp'^{-1} \) is null-homotopic, there exists a disk diagram \( D \) with boundary \( pp'^{-1} \). It readily follows that a homotopy of \( p \) to \( p' \) in \( D \) induces a sequence of Tits moves which show that \( p = p' \). This proves (3). \[ \Box \]

The following definition and proposition allow us to “go backwards” and obtain a subgroup from a \( \Gamma \)-labeled complex.
Definition 4.3 (Associated subgroup). Let $\Omega$ be a connected, $\Gamma$-labeled complex with base vertex $B$. Consider the set of all $g \in W_\Gamma$ such that there exists a loop in $\Omega$ based at $B$ whose label is a word in $W_\Gamma$ that represents of $g$. This set is easily seen to be a subgroup of $W_\Gamma$, and is called the \textit{subgroup of $W_\Gamma$ associated to} $(\Omega, B)$.

Proposition 4.4. Let $(\Omega, B)$ be a connected, folded, cube-full, $\Gamma$-labeled complex, and let $G$ be the subgroup of $W_\Gamma$ associated to $(\Omega, B)$. Then $(\Omega, B)$ is a completion of $G$.

Proof. Properties (1) and (2) in the definition of a completion of a subgroup (Definition 3.6) are immediate. To check property (3), let $w$ be a reduced word representing an element $g$ of $G$. By the definition of $G$, there exists a loop in $\Omega$ based at $B$ whose label, say $w'$, is a representative of $g$. Since $w$ is a reduced representative of $w'$, Lemma 4.2 (1) implies that there is a loop in $\Omega$ based at $B$ with label $w$. \qed

The following lemma describes the effect of changing the basepoint in a $\Gamma$-labeled complex on the associated subgroup of $W_\Gamma$.

Lemma 4.5. Let $\Omega$ be a connected, folded, cube-full, $\Gamma$-labeled complex. Let $B_1$ and $B_2$ be vertices of $\Omega$, and for $i = 1, 2$, let $G_i$ be the subgroup of $W_\Gamma$ associated to $(\Omega, B_i)$. Then $G_2 = w^{-1}G_1 w$, where $w$ is the label of some path from $B_1$ to $B_2$.

Proof. Let $\alpha$ be a path from $B_1$ to $B_2$ with label $w$. If $\beta$ is a loop in $\Omega$ based at $B_1$ representing an element of $G_1$, then the concatenation $\alpha^{-1}\beta\alpha$ represents an element of $G_2$. It follows that $w^{-1}G_1w \subseteq G_2$. Similarly $wG_2w^{-1} \subseteq G_1$. \qed

It is easy to detect torsion in subgroups of right-angled Coxeter groups using completions:

Proposition 4.6. Let $G$ be a subgroup of a right-angled Coxeter group $W_\Gamma$ and let $(\Omega, B)$ be a completion for $G$. Then $G$ has torsion if and only if there exists a loop in $\Omega$ (not necessarily passing through $B$) whose label is a reduced word representing an element in a finite special subgroup of $W_\Gamma$.

Proof. If $g \in G$ has finite order, then $g$ is conjugate into a finite special subgroup of $W_\Gamma$ (see [Dav08] Theorem 12.3.4 for instance). Write $g = wuw^{-1}$, where $u$ and $w$ are reduced and $w$ is the shortest word for which such an expression for $g$ exists. We claim $wuw^{-1}$ is reduced. If not, then a deletion is possible. It follows from our choices that some letter, say $s$, occurring in $w$ or $w^{-1}$ cancels with an occurrence of $s$ in $u$. Since $u$ belongs to a finite special subgroup, $s$ commutes with $u$. Thus we can write $g = w_1uw_1^{-1}$, where $w = w_1s$, a contradiction.

Since $wuw^{-1}$ is reduced, there is a loop $\alpha$ in $\Omega$ based at $B$ with label $wuw^{-1}$. Let $v$ be the vertex along $\alpha$ such that the label of $\alpha$ between $B$ and $v$ is $w$. As $\Omega$ is folded, the subpaths of $\alpha$ with labels $w$ and $w^{-1}$ are identified, and there is a loop based at $v$ with label $u$. This proves one direction of the claim.

For the other direction, suppose that there is a loop based at some vertex $x$ of $\Omega$ with label a reduced word $r$ representing an element in a finite special subgroup of $W_\Gamma$. As finite special subgroups of $W_\Gamma$ correspond to clique subgraphs of $\Gamma$, it follows that the support of $r$ is contained in a clique of $\Gamma$. Let $s \in V(\Gamma)$ be a letter in $r$. As $r$ is reduced, there is exactly one occurrence of $s$ in $r$. Let $h$ be the label of a path from $B$ to $x$ in $\Omega$. It follows that there is a loop in $\Omega$ based at $B$ with label $k = hrh^{-1}$. Furthermore, as there are an odd number of occurrences of the
For torsion-free subgroups, the following holds:

**Theorem 4.7.** Suppose \( G \) is a torsion-free subgroup of the right-angled Coxeter group \( W_\Gamma \). Then the fundamental group of any completion \( \Omega \) of \( G \) is isomorphic to \( G \).

**Proof.** Let \( B \) be a vertex of \( \Omega \). We define the isomorphism \( \phi : \pi_1(\Omega, B) \to G \) as follows. Let \( \alpha \) be a loop in \( \Omega \) based at \( B \) representing an element of \( \pi_1(\Omega, B) \). We may assume that \( \alpha \) is contained in the 1-skeleton of \( \Omega \). By property (2) of the definition of a completion, the label of \( \alpha \) represents an element of \( G \), and we define \( \phi(\alpha) \) to be this element. To see that \( \phi \) is well-defined, let \( \alpha \) and \( \alpha' \) be loops based at \( B \in \Omega \) that are homotopic relative basepoint. Then by Lemma 4.2(3), the labels of \( \alpha \) and \( \alpha' \) are equal as elements of \( G \).

It is clear that \( \phi \) is a surjective homomorphism. To check that \( \phi \) is injective, suppose that \( \phi(\alpha) \) is a word in \( W_\Gamma \) equal in \( G \) to the identity element. Note that \( \Omega \) cannot contain a graph-loop by Proposition 4.6. Thus, we can apply Lemma 4.2(1) to conclude that \( \alpha \) is null-homotopic. \( \square \)

5. Core graphs

In general, a given subgroup \( G \) of \( W_\Gamma \) does not have a unique completion. However, we now use completions to define a certain graph associated to \( G \) called a core, and we prove that it is unique. This is used in Theorem 5.5 to obtain a characterization for normality of a subgroup.

**Definition 5.1 (Core graph).** Given a \( \Gamma \)-labeled complex \( (\Omega, B) \), define its **core graph at** \( B \), denoted \( C(\Omega, B) \), to be the 1-dimensional subcomplex consisting of the union of all the loops in \( \Omega \) based at \( B \) whose labels are reduced words in \( W_\Gamma \).

**Remark 5.1.1.** The core graph of a completion is not necessarily its entire 1-skeleton. For instance, it is straightforward to check that the core graph for the complex \( \Omega \) of Figure 1 is not the whole 1-skeleton of \( \Omega \).

Note that \( C(\Omega, B) \) is a connected, \( \Gamma \)-labeled complex, and so it has an associated subgroup as in Definition 4.3. Moreover, the following holds:

**Lemma 5.2.** Let \( \Omega \) be a connected, folded, cube-full, \( \Gamma \)-labeled complex, and let \( C = C(\Omega, B) \). Then the subgroup of \( \Gamma \) associated to \( (C, B) \) is the same as the subgroup associated to \( (\Omega, B) \).

**Proof.** By Proposition 4.4, every element of the subgroup of \( W_\Gamma \) associated to \( \Omega \) at \( B \) is represented by a loop based at \( B \) labeled by a reduced word. The lemma follows easily from this observation. \( \square \)

Core graphs are unique in the following sense:

**Proposition 5.3.** Let \( G \) be a subgroup of \( W_\Gamma \), and let \( (\Omega_1, B_1) \) and \( (\Omega_2, B_2) \) be completions of \( G \). Then there is an isomorphism \( f : C_1(\Omega_1, B_1) \to C_2(\Omega_1, B_2) \), such that \( f(B_1) = B_2 \).

Proposition 5.3 is a consequence of the following lemma:
Lemma 5.4. Let $G_1 \leq G_2 \leq W_\Gamma$, and let $(\Omega_1, B_1)$ and $(\Omega_2, B_2)$ be completions of $G_1$ and $G_2$ respectively. Then there is a label preserving combinatorial map $f : C(\Omega_1, B_1) \to C(\Omega_2, B_2)$ such that $f(B_1) = B_2$.

Proof. Let $C_i$ denote $C(\Omega_i, B_i)$ for $i = 1, 2$. We first define $f$ on the vertices of $C_1$. Define $f(B_1) = B_2$. Now let $v_1 \neq B_1$ be a vertex in $C_1$. By the definition of a core graph, there is a loop in $\Omega_1$ based at $B_1$ which passes through $v_1$, and whose label $w$ is a reduced word in $W_\Gamma$. Let $\alpha_1$ be the part of this (oriented) loop which goes from $B_1$ to $v_1$, and let $w_\alpha$ be the subword of $w$ which labels $\alpha_1$. Since $w$ labels a loop in $\Omega_1$ based at $B_1$, by the definition of completion, it represents an element of $G_1$, which by hypothesis is a subgroup of $G_2$. Now since $w$ is reduced, again by the definition of a completion, there is a loop in $\Omega_2$ based at $B_2$ with label $w$, and by the definition of a core graph, this loop is contained in $C_2$. Let $\alpha_2$ be the initial segment of this loop which is labeled by $w_\alpha$, and let $v_2 \in C_2$ be the endpoint of $\alpha_2$. Define $f(v_1) = v_2$.

We must show that $f$ is well-defined on vertices, i.e., that it does not depend on the choice of loop containing $v_1$. Suppose we run the construction in the previous paragraph on a different loop containing $v_1$, and thus obtain the path $\beta_1$ from $B_1$ to $v_1$ with label $w_\beta$, and corresponding path $\beta_2$ in $C_2$ with label $w_\beta$, from $B_2$ to some vertex $v'_2$. Then we wish to show $v_2 = v'_2$.

Observe that the concatenation $\alpha_1 \beta_1^{-1}$ is a loop based at $B_1$. Thus its label $w_\alpha w_\beta^{-1}$ represents an element of $G_1 < G_2$. Let $u$ be a reduced word in $W_\Gamma$ representing $w_\alpha w_\beta^{-1}$. By the definition of completion, there is a loop $\gamma_2$ in $\Omega_2$ based at $B_2$ with label $u$. Now the concatenation $\gamma_2 \beta_2$ is a path in $\Omega_2$ from $B_2$ to $v'_2$ with label $w_\beta$. Note that $w_\alpha$ is a reduced word which represents $uw_\beta$. It follows from Lemma 4.2 that there is a path from $B_2$ to $v'_2$ with label $w_\alpha$. On the other hand, $\alpha_2$ is a path in $\Omega_2$ from $B_2$ to $v_2$ with label $w_\alpha$. However, since $\Omega_2$ is folded, there is at most one path starting at $B_2$ with a given label. It follows that the two paths are the same, and hence $v_2 = v'_2$ are required.

Let $e$ be an edge in $\Omega_1$ between (possibly non-distinct) vertices $u_1$ and $u_2$, and let $s$ be the label of $e$. In order to extend $f$ to edges, we show there is a unique edge $e'$ in $\Omega_2$ between $f(u_1)$ and $f(u_2)$. Let $\alpha$ be a loop in $\Omega_1$ which contains the edge $e$ and has reduced label $h$. As before, there must be a loop $\alpha'$ in $C_2$ based at $B_2$ with label $h$. By the construction of the map $f$ on vertices, it readily follows that $\alpha'$ contains the vertices $f(u_1)$, $f(u_2)$ and an edge labeled by $s$ between these vertices. This edge is unique since $\Omega_2$ is folded.

We now use Lemma 5.4 to establish the uniqueness of core graphs:

Proof of Proposition 5.3. Let $C_i$ denote $C(\Omega_i, B_i)$ for $i = 1, 2$. Lemma 5.4 implies that there are label preserving combinatorial maps $f_1 : C_1 \to C_2$ with $f_1(B_1) = B_2$, and $f_2 : C_2 \to C_1$ with $f_2(B_2) = B_1$. Then $f_2 \circ f_1$ is a label-preserving combinatorial map from $C_1$ to itself which fixes $B_1$. Since $C_1$ is folded and connected, there is exactly one label-preserving combinatorial map which fixes $B_1$, namely the identity. Thus $f_2 \circ f_1$ is the identity map of $C_1$, and similarly $f_1 \circ f_2$ is the identity map of $C_2$. It follows that $f_1$ is an isomorphism as required.

Remark 5.4.1. We remark that if $C$ is the core graph of a completion $(\Omega, B)$ of some $G < W_\Gamma$, then the complex induced by $C$ in $\Omega$ (by including cubes of $\Omega$ whose boundaries are contained in $C$) is a connected, folded, $\Gamma$-labeled complex which
satisfies properties (2) and (3) of the definition of a completion. However, it may not be cube-full. For instance, this is the case for the completion \( \Omega \) of Figure 1.

Next we characterize normal subgroups of right-angled Coxeter groups in terms of core graphs. In Section 13, we give a different version of this theorem that is used to give an algorithm that checks whether a quasiconvex subgroup is normal.

**Theorem 5.5.** Let \( G \) be a subgroup of \( W_\Gamma \), and let \( (\Omega, B) \) be a completion of \( G \). Consider the following subset of \( V(\Gamma) \):

\[
\Delta = \{ s \in V(\Gamma) \mid s \text{ commutes with every element of } G \}
\]

Then \( G \) is normal if and only if the following conditions are satisfied.

1. (N1) Given any \( s \in V(\Gamma) \setminus \Delta \), there is an edge in \( \Omega \) incident to \( B \) with label \( s \).
2. (N2) For every vertex \( v \) of \( \Omega \), there is an isomorphism from \( C(\Omega, B) \) to \( C(\Omega, v) \) which takes \( B \) to \( v \).

**Proof.** First suppose N1 and N2 are satisfied. To show that \( G \) is normal, it is enough to show that \( sGs \subset G \) for all \( s \in V(\Gamma) \). This is obvious when \( s \in \Delta \), so consider \( s \in V(\Gamma) \setminus \Delta \). By N1 there is an edge incident to \( B \) with label \( s \). Let \( v \) be its other endpoint. Then by Lemma 4.5, \( (\Omega, v) \) is a completion for \( sGs \). Thus \( sGs = G \).

Now suppose \( G \) is normal. We first show N1 is satisfied. Let \( s \in V(\Gamma) \setminus \Delta \), and let \( w \) be a reduced word representing an element of \( G \) which does not commute with \( s \). If \( w \) has a reduced expression \( w' \) which either begins or ends with \( s \), then since \( \Omega \) is a completion, there is a loop based at \( B \) with label \( w' \). It follows that there is an edge incident to \( B \) in \( \Omega \) labeled \( s \). On the other hand, if no expression for \( w \) begins or ends with \( s \), then \( sws \) is reduced by Lemma 2.3. Moreover, since \( G \) is normal, \( sws \in G \), and consequently there is a loop in \( \Omega \) based at \( B \) with label \( sws \). Once again, \( B \) is incident to an edge labeled by \( s \). Thus N1 holds in all cases.

To prove N2, let \( v \neq B \) be a vertex of \( C(\Omega, B) \), and let \( \alpha \) be a path in \( C(\Omega, B) \) from \( B \) to \( v \) with label \( w \). Then by Lemma 4.5, we know that \( (\Omega, v) \) is a completion for \( w^{-1}Gw = G \) (since \( G \) is normal). Then by Proposition 5.3, there is an isomorphism from \( C(\Omega, B) \) to \( C(\Omega, v) \) which takes \( B \) to \( v \).  

\( \square \)

6. **Index of a subgroup**

The main result of this section is Theorem 6.9 which states that the index of a subgroup can be computed from a completion. In order to correctly state Theorem 6.9 we define resolved completions and resolved generating sets which address a slight technical issue that arises when \( \Gamma \) contains a vertex that is adjacent to every other vertex.

**Definition 6.1** (Resolved Completion). Let \( \Omega \) be a completion of a subgroup \( G \) of \( W_\Gamma \). We say that \( \Omega \) is a **resolved completion** if given any \( s \in \Gamma \) such that \( V(\Gamma) = \text{star}(s) \), it follows that some edge of \( \Omega \) is labeled by \( s \).

**Lemma 6.2.** Let \( G \) be a subgroup of \( W_\Gamma \), and let \( \Omega \) be a resolved completion of \( G \). Let \( s \in \Gamma \) be such that \( V(\Gamma) = \text{star}(s) \). Then every vertex of \( \Omega \) is incident to an edge labeled by \( s \).
Proof. As $\Omega$ is resolved, let $e$ be an edge of $\Omega$ labeled by $s$. Let $v$ be a vertex of $\Omega$ incident to $e$. Let $u$ be any vertex adjacent to $v$, and let $t$ be the label of the edge $e'$ between $u$ and $v$. Either $t = s$ or $t$ is adjacent to $s$ in $\Gamma$. As $\Omega$ is cube-full, in the latter case there must be a square with label $sts$ in $\Omega$ that contains both the edge $e$ and $e'$. In either case $u$ is incident to an edge labeled by $s$ as well. Proceeding in this manner, since $\Omega$ is connected, we conclude every vertex in $\Omega$ is incident to an edge labeled by $s$. \qed

**Definition 6.3 (Full valence).** We say a vertex $v$ of a $\Gamma$-labeled complex has full valence if for each $s \in V(\Gamma)$ there is an edge with label $s$ incident to $v$. We say a $\Gamma$-labeled complex $\Omega$ is full valence if every vertex of $\Omega$ has full valence.

**Lemma 6.4.** Let $G$ be a subgroup of $W_T$, and let $(\Omega, B)$ be a resolved completion of $G$. If $\Omega$ is not full valence, then $G$ has infinite index in $W_T$.

**Proof.** Suppose there exists a vertex $v$ in $\Omega$ that is not incident to an edge labeled by $s$, for some $s \in V(\Gamma)$. Let $\alpha$ be a minimal length path in $\Omega$ from the base vertex $B$ to $v$, and let $w$ be the label of this path. We assume $|w|$ is minimal among the possible choices for $w$ and $v$. By Lemma 4.2(2), the word $w$ is reduced.

We begin by establishing a few facts, which will be used later in the proof:

(i) The word $ws$ is reduced. For otherwise by Proposition 2.2 there exists a reduced word $w'$, ending with $s$, which is an expression for $w$. By Lemma 4.2(1) there is a corresponding path in $\Omega$ from $B$ to $v$ with label $w'$. However, this is not possible as $v$ is not incident to an edge labeled by $s$.

(ii) No reduced expression for $w$ ends in a generator that commutes with $s$. For suppose $w' = s_1 \ldots s_n$, with $s_i \in \Gamma$, is a reduced word such that $s_n$ commutes with $s$ and $w$ is equal to $w'$ in $W_T$. Let $\alpha'$ be the path from $B$ to $v$ with label $w'$ and let $\hat{\alpha}$ be the subpath of $\alpha'$ with label $s_1 \ldots s_{n-1}$. Such paths exist by Lemma 4.2(1). Let $\hat{v}$ be the endpoint of $\hat{\alpha}$. No edge incident to $\hat{v}$ is labeled by $s$. For if there were such an edge, the fact that $\Omega$ is cube-full would imply that there is a square with label $ss_nss_n$ containing both $v$ and $\hat{v}$, contradicting the fact that $v$ is not incident to an edge labeled by $s$. However, it now follows that $\hat{v}$ is a vertex that is not incident to an edge labeled by $s$ and $|\hat{w}| < |w|$, contradicting the minimality of our choice of $w$. Thus no expression for $w$ can end with a generator that commutes with $s$.

(iii) No reduced word representing an element of $G$ begins with the label $ws$. To see this, note that every reduced word representing an element of $G$ labels a loop in $\Omega$ based at $B$. As $\Omega$ is folded, $\alpha$ is the only path beginning at $B$ with label $w$. The claim follows, since the endpoint $v$ of $\alpha$ is not incident to an edge labeled by $s$.

We now proceed with the proof. As $\Omega$ is resolved and by Lemma 6.2 there exists a vertex $t$ of $\Gamma$ that is not adjacent to $s$ in $\Gamma$. By Tits’ solution to the word problem, it follows that $(st)^n$ is reduced for all integers $n \geq 1$. Similarly, $w(st)^n$ is reduced for all integers $n \geq 1$, since $ws$ is reduced by (ii) above.

Suppose now, for a contradiction, that $G$ is a finite-index subgroup of $W_T$. In particular, as a set we have $W_T = Gg_1 \sqcup Gg_2 \ldots \sqcup Gg_n$ for finitely many elements $g_1, \ldots, g_n \in W_T$. Let $w_1, \ldots, w_n$ be reduced words representing $g_1, \ldots, g_n$. Let $M = \max\{|w_1|, \ldots, |w_n|\}$, and let $k = (st)^M$. Consider the word $h = wk$ which we know to be reduced. It follows that $h$ is equal to $h'h''$ in $W_T$, where $h'' = w_i$ for some $i$ and $h'$ is a reduced word in $G$. Form a disk diagram $D$ with boundary label
that the type of such a dual curve commutes with $s$ and $t$ do not commute. Let $C$ be the dual curve dual to the first edge of $p_k$. Note that $C$ is of type $s$.

Thus, we conclude that the dual curve intersect one another as $s$ and $t$. Furthermore, a pair of dual curves which are each dual to $p_k$ cannot intersect $h$. Let $C$ be the dual curve dual to the boundary of $D$. As $h$ is reduced, every dual curve dual to $p_k$ must necessarily intersect either $p_{h'}$ or $p_{h''}$. For if it did, every curve dual to $p_k$ would intersect $p_{h''}$ as well. However, as $|h''| \leq M$ and $|k| = 2M$, this is not possible. Thus, we conclude $C$ intersects $p_{h'}$.

Additionally, no dual curve dual to $p_w$ intersects $C$. For suppose there is such a dual curve, and suppose that it is the furthest such dual curve along $p_w$. It follows that the type of such a dual curve commutes with $s$ and commutes with every label of an edge appearing further along $p_w$. However, this implies that $w$ has an expression ending with a generator that commutes with $s$, which contradicts (ii) above. Thus, every dual curve dual to $p_w$ intersects $p_{h'}$ at an edge occurring before (in the orientation of $p_{h'}$) the edge of $p_{h'}$ dual to $C$.

By Lemma 6.6 there is a reduced word equal to $h'$ in $W_T$ which has $w$ as a prefix, which contradicts (iii) above. We remark that the above argument holds in the case when $w$ is the empty word, i.e. when $B = v$.

We now define resolved sets. Lemma 6.5 below, whose proof is immediate, states that any generating set can be easily extended to a resolved generating set.

**Definition 6.5** (Resolved set). A set $\{w_1, w_2, \ldots\}$ of words in $W_T$ is resolved if given any $s \in V(\Gamma)$ such that $V(\Gamma) = \text{star}(s)$, it follows that $s \in \text{Support}(w_i)$ for some $i$.

**Lemma 6.6.** Every set of words in $W_T$ which generate a subgroup $G$ can be extended to a resolved generating set of words for $G$. More specifically, suppose $T = \{w_1, w_2, \ldots\}$ is a generating set of words for a subgroup $G$ of $W_T$. Then $T' = T \cup \{s^2 \mid s \in \Gamma \text{ and } V(\Gamma) = \text{star}(s)\}$ is a resolved generating set for $G$.

Lemma 6.7 guarantees that a resolved completion can always be constructed for a finitely generated subgroup.

**Lemma 6.7.** Let $G$ be a finitely generated subgroup of $W_T$ and let $S_G$ be a finite resolved generating for $G$ (which exists by Lemma 6.6). If $\Omega$ is a standard completion for $G$ with respect to $S_G$, then $\Omega$ is resolved.

**Proof.** Let $\Omega_0 \rightarrow \Omega_1 \rightarrow \cdots \rightarrow \Omega$ be a standard completion of $G$ with respect to $S_G$. Let $s \in V(\Gamma)$ be such that $V(\Gamma) = \text{star}(s)$. As $S_G$ is resolved and as $\Omega_0$ is the “rose graph” of words in $S_G$, it follows that some edge of $\Omega_0$ is labeled by $s$. Thus, some edge in $\Omega$ is labeled by $s$.

**Lemma 6.8.** Let $G$ be a subgroup of $W_T$, and let $(\Omega, B)$ be a completion of $G$ which is full valence. Then the index of $G$ in $W_T$ is equal to the number of vertices in $\Omega_G$ (which could be infinite).

**Proof.** Let $B = v_1, v_2, \ldots$ be an enumeration of the vertices of $\Omega$. For each $i$, choose a minimal length path $\alpha_i$ from $B$ to $v_i$ and let $w_i$ be its label. We will show that the words $w_1, w_2, \ldots$ are expressions for right coset representatives for $G$. 

\[ h(h'h'')^{-1} = wkh''h'^{-1}. \]

Let $p_w, p_k, p_{h'}$ and $p_{h''}$ be the paths along the boundary of $D$ with labels respectively $w, k, h'$ and $h''$. Thus, $p_wp_kp_{h'}^{-1}p_{h''}^{-1}$ is a path tracing the boundary of $D$. 

As $h$ is reduced, every dual curve dual to $p_w$ must necessarily intersect either $p_{h'}$ or $p_{h''}$. For if it did, every curve dual to $p_k$ would intersect $p_{h''}$ as well. However, as $|h''| \leq M$ and $|k| = 2M$, this is not possible. Thus, we conclude $C$ intersects $p_{h'}$.
Let $w$ be a reduced word in $W\Gamma$. As every vertex of $\Omega$ has full valence, there is a path $\alpha$ in $\Omega$ beginning at the vertex $B$ with label $w$. Then, for some $i$, the concatenation $\alpha \alpha_i^{-1}$ is a loop based at $B$, and its label $ww_i^{-1}$ represents an element of $G$. Thus, $w$ can be represented by the coset $(ww_i^{-1})w_i$.

Let $n$ be the number of vertices in $\Omega$ (where $n$ could be infinite). We have established that the index of $G$ is at most $n$. We now show it is exactly $n$. Suppose, to the contrary, that there exist words $h$ and $h'$ representing elements of $G$ such that $hw_i$ is an expression for $h'w_j$, for some $1 \leq i < j \leq n$. It follows that $w_iw_j^{-1}$ is an expression for an element of $G$. Now consider the path $\beta$ in $\Omega$ with initial vertex $B$ and label $w_iw_j^{-1}$. This path exists and is unique as $\Omega$ is full valence.

We claim $\beta$ is a loop. For suppose not. Then $\beta$ ends in some vertex $v_k \neq B$ and it follows that $w_iw_j^{-1}w_k^{-1}$ represents an element of $G$. Consequently, as $w_iw_j^{-1}$ is a word representing an element of $G$, we conclude that $w_k$ represents an element of $G$ as well. However, this contradicts $\Omega$ being a completion, as $w_k$ is a reduced word and is not the label of a loop in $\Omega$ based at $B$. Thus $\beta$ must be a loop. However, since $\beta$ is labeled by $w_iw_j^{-1}$, this implies that $v_i = v_j$, a contradiction.

**Theorem 6.9.** Let $G$ be a subgroup of $W\Gamma$, and let $\Omega$ be a resolved completion of $G$. The subgroup $G$ has finite index in $W\Gamma$ if and only if $\Omega$ is finite and full valence.

**Proof.** First suppose $G$ has finite index in $W\Gamma$. Then by Lemma 6.4, $\Omega$ is full valence. Furthermore, $\Omega$ is finite by Lemma 6.8. The other direction is an immediate consequence of Lemma 6.8. □

The following corollary for finitely generated subgroups immediately follows from Theorem 6.9 and Lemma 6.7.

**Corollary 6.10.** Let $G$ be a finitely generated subgroup of $W\Gamma$ and let $\Omega$ be a resolved completion of $G$. The subgroup $G$ has finite index in $W\Gamma$ if and only if it is finite and full valence.

**7. Nonpositive curvature**

This section establishes criteria which guarantee that a completion is non-positively curved or a CAT(0) cube complex.

We begin with the following proposition, which shows that given a completion sequence of a $\Gamma$-labeled complex $X$, any hyperplane in any complex of this sequence is an extension of a hyperplane of $X$.

**Proposition 7.1.** Let $X$ be a $\Gamma$-labeled complex. Let

$$X = \Omega_0 \to \Omega_1 \to \cdots \to \Omega$$

be a completion sequence for $X$. Then, for all $i \geq 0$, every hyperplane in $\Omega_i$ intersects the image of $X$ in $\Omega_i$. Consequently, every hyperplane in $\Omega$ intersects the image of $X$ in $\Omega$.

**Proof.** The claim that hyperplanes in $\Omega_i$ intersect the image of $X$ will be proven by induction on $n$. The base case for $\Omega_0$ is trivially true.

Suppose every hyperplane in $\Omega_{n-1}$ intersects the image of $X$. Suppose first that $\Omega_n$ is obtained by attaching an $n$-cube $c$ to $\Omega_{n-1}$ along edges $e_1, \ldots, e_n$, all incident
to a common vertex of $\Omega_{n-1}$. Since each midcube of $c$ extends a hyperplane dual to one of the $e_i$'s, it follows that no new hyperplanes are created in $\Omega_n$. It is also clear that cube identification and fold operations do not produce new hyperplanes. Hence, the claim also holds for $\Omega_n$.

The claim follows for the completion $\Omega$ as any hyperplane in $\Omega$ contains the image of some hyperplane in $\Omega_n$ for some $n$. □

Recall that a graph-loop is an edge that connects a vertex to itself. A bigon in a CW complex is a pair of edges $e_1$ and $e_2$, such that the set of vertices that are endpoints of $e_1$ is the same as the set of vertices that are endpoints of $e_2$. Note that a bigon could consist of two graph-loops based at the same vertex. A commuting bigon in a $\Gamma$-labeled complex is a bigon whose edges are labeled by adjacent vertices of $\Gamma$.

Next we show that the presence of a commuting bigon is the only obstruction to a $\Gamma$-labeled complex being non-positively curved.

**Proposition 7.2.** Let $\Omega$ be a folded $\Gamma$-labeled cube complex which does not contain a commuting bigon. Suppose further that $\Omega$ is either cube-full or that $\Gamma$ is triangle-free. Then $\Omega$ is non-positively curved.

**Proof.** Let $v$ be a vertex of $\Omega$, and let $\Delta$ denote the link of $v$ in $\Omega$. We verify that $\Delta$ is non-positively curved by checking that $\Delta$ is a flag simplicial complex. We first check that $\Delta$ is simplicial. This part of the proof does not require that $\Omega$ is cube-full or that $\Gamma$ is triangle-free.

The complex $\Delta$ cannot contain a graph-loop. For if it did, then adjacent sides of some square would be identified in $\Omega$. However, adjacent sides of squares in $\Omega$ are always labeled by distinct elements of $\Gamma$, so such an identification cannot happen.

We next check that the 1-skeleton of $\Delta$ does not contain a bigon. Suppose there is such a bigon. It follows that there is a pair of edges $e_1$ and $e_2$ incident to a vertex $v$ and there exist (possibly non-distinct) squares $c_1$ and $c_2$ in $\Omega$ each containing both $e_1$ and $e_2$. If $c_1$ and $c_2$ are distinct squares, then by the given identifications, their boundary-labels, read starting from $v$ in the direction of $e_1$, must be the same. However, this is not possible as $\Omega$ is folded.

On the other hand, suppose $c_1$ and $c_2$ are the same square, $c = c_1 = c_2$. As $c$ is labeled by $stst$ for some adjacent vertices $s, t \in \Gamma$, the opposite edges of $c$ must be identified in $\Omega$. It is straightforward to check that the only possible such identification producing a bigon in $\Delta$ is that of $\mathbb{R}P^2$. In this case, opposite edges of $c$ are identified “with a flip.”

It follows that the image of the boundary of $c$ under the attaching map of $c$ is a bigon in $\Omega$ whose two edges are labeled $s$ and $t$. Since $s$ and $t$ are adjacent in $\Gamma$, this is a commuting bigon. This contradicts the assumption that $\Omega$ does not contain commuting bigons. Hence the 1-skeleton of $\Delta$ cannot contain a bigon. We have thus verified that $\Delta$ is simplicial.

We now check that $\Delta$ is flag. Let $u_1, \ldots, u_n$ be the vertices of a complete graph contained in the 1-skeleton of $\Delta$. For $1 \leq i \leq n$, let $e_i$ be the edge of $\Omega$ incident to $v$ which $u_i$ lies on, and let $s_i$ be the label of $e_i$. For each $1 \leq i < j \leq n$, we know that $s_i$ and $s_j$ are adjacent in $\Gamma$ since $u_i$ is adjacent to $u_j$ in $\Delta$. If $\Omega$ is cube-full, it follows that there is some cube in $\Omega$ containing $v \cup \bigcup_{i=1}^n e_i$. Thus, $\Delta$ is flag. On the other hand suppose that $\Gamma$ is triangle-free. As $\Omega$ is folded, the labels $s_1, \ldots, s_n$ are distinct. Furthermore, as $\Gamma$ is triangle-free and $s_1, \ldots, s_n$ as vertices of $\Gamma$ form
a complete graph, we have that $n \leq 2$. Thus the flag condition holds under the triangle-free assumption as well. \hfill \square

If $G$ is a torsion-free subgroup of $W_{\Gamma}$, then by Proposition 4.6, a completion of $G$ cannot have commuting bigons. Then Proposition 7.2 immediately implies:

**Proposition 7.3.** Any completion of a torsion-free subgroup of a right-angled Coxeter group is non-positively curved. \hfill \square

Our next objective is to show that a completion of a finite $\Gamma$-labeled tree is a finite CAT(0) cube complex.

**Lemma 7.4.** Let $X$ be a finite $\Gamma$-labeled tree, and let

$X \to \Omega_1 \to \Omega_2 \to \cdots \to \Omega$

be a completion sequence for $X$. Then $\Omega_i$ is simply connected for all $i$, and $\Omega$ is a CAT(0) cube complex.

**Proof.** We first show that $\Omega_n$ does not contain any graph-loops for $n \geq 0$. Note that the label of every loop in $\Omega_0$ based at $B$ represents the trivial element in $W_{\Gamma}$. For a contradiction, suppose that for some $n$, $\Omega_n$ contains a graph-loop $l$ with label $s$. Suppose $l$ is incident to a vertex $v \in \Omega_n$. Let $p$ be a geodesic in $\Omega_n$ from $B$ to $v$, and let $w$ be the label of $p$. It follows that the loop $plp^{-1}$ in $\Omega_n$ has label $wsww^{-1}$. As $wsww^{-1}$ has an odd number of occurrences of the letter $s$, it represents a nontrivial element of $W_{\Gamma}$. However, by iteratively applying Lemma 3.9 we conclude that the label of some loop in $\Omega_0$ based at $B$ represents a non-trivial element of $W_{\Gamma}$. This is a contradiction. Thus, $\Omega_n$ does not contain a graph-loop for any $n$.

Next, we show by induction that $\Omega_n$ is simply connected for all $n \geq 0$. The base case is true by hypothesis. Now assume that $\Omega_n$ is simply connected.

Suppose $\Omega_{n+1}$ is obtained from $\Omega_n$ by attaching a $k$-cube $c$ to the edges $e_1, \ldots, e_k$ of $\Omega_n$ which are all incident to the same vertex $v$. Then $\Omega_{n+1}$ can be homotoped onto $\Omega_n$ by homotoping $c$ onto $v \cup \bigcup_{i=1}^{k} e_i$. Hence $\Omega_{n+1}$ is simply connected.

If $\Omega_{n+1}$ is obtained from $\Omega_n$ by identifying a collection $\{e_i\}$ of $k$-cubes ($k \geq 2$) with identical boundary to a single cube $c$, then any null homotopy using the $e_i$’s can be replaced with one that only uses $c$, and hence $\Omega_{n+1}$ is simply connected.

Now suppose $\Omega_{n+1}$ is obtained from $\Omega_n$ by a fold operation. Specifically, suppose that the edges $e_i$ (with endpoints $v$ and $v_i$, for $i = 1, 2$) in $\Omega_n$ are identified to get the edge $e$ in $\Omega_{n+1}$. By the first paragraph these edges are not graph-loops.

If $v_1 = v_2$, then $e_1 \cup e_2$ is a loop, which is null homotopic because $\Omega_n$ is simply connected. Since identifying $e_1$ and $e_2$ is equivalent to attaching a disk to this loop, it follows that $\Omega_{n+1}$ is simply connected.

Finally, if $v_1 \neq v_2$, then let $\Omega'_n$ and $\Omega'_{n+1}$ be the complexes obtained by collapsing the contractible subspaces $e_1 \cup e_2$ and $e$ in $\Omega_n$ and $\Omega_{n+1}$ respectively to points. Then $\Omega'_n$ is homotopy equivalent to $\Omega_n$, $\Omega'_{n+1}$ is homotopy equivalent to $\Omega_{n+1}$, and $\Omega'_n$ is homeomorphic to $\Omega'_{n+1}$. Again, we conclude that $\Omega_{n+1}$ is simply connected.

We have established that $\Omega_n$ is simply connected for all $n \geq 0$, and it readily follows that $\Omega$ is simply-connected as well.

In order to show $\Omega$ is non-positively curved, by Lemma 7.2 it is enough to show that $\Omega$ does not contain a commuting bigon. Since $\Omega$ is simply connected, Lemma 4.2(3) implies that the group associated to $(\Omega, B)$ is trivial, and therefore torsion-free. By Proposition 4.6 there are no commuting bigons.
Finally, \( \Omega \) is a CAT(0) cube complex as it is a simply-connected and non-positively curved cube complex.

**Proposition 7.5.** Let \( X \) be a \( \Gamma \)-labeled finite tree and let

\[
X = \Omega_0 \to \Omega_1 \to \cdots \to \Omega
\]

be a completion sequence for \( X \). Then \( \Omega \) is a finite CAT(0) cube complex. Furthermore, there is a finite bound on the length of the completion sequence.

**Proof.** By Lemma 7.4, we know that \( \Omega \) is CAT(0). We claim that the diameter of \( \Omega \) is at most \( E \), where \( E \) is the number of edges of \( X \). For consider a geodesic \( \alpha \) in \( \Omega \).

By Lemma 7.1, every hyperplane that intersects \( \alpha \) must also intersect the image of \( X \) in \( \Omega \). Furthermore, as \( \Omega \) is CAT(0) (and not just non-positively curved), no hyperplane intersects \( \alpha \) twice. Thus, the length of \( \alpha \) is at most \( E \), and, as \( \alpha \) was an arbitrary geodesic, the diameter of \( \Omega \) is also at most \( E \). It follows that \( \Omega \) is finite, as it is locally finite (since it is folded) and has finite diameter.

The second part of the claim now follows from Proposition 3.5. \( \square \)

When \( \Gamma \) is triangle-free, we get the following more precise bound on the length of a standard completion sequence:

**Proposition 7.6.** Let \( X \) be a \( \Gamma \)-labeled finite tree where \( \Gamma \) is triangle-free. Let

\[
X = \Omega_0 \to \Omega_1 \to \cdots \to \Omega
\]

be a standard completion sequence for \( X \). Then \( \Omega \) is a finite CAT(0) cube complex. Furthermore, there is a finite bound on the length of the completion sequence depending only on the number of edges of \( X \) and on \( |V(\Gamma)| \).

**Proof.** By Proposition 7.5, \( \Omega = \Omega_N \) for some \( N \) and \( \Omega \) is a finite CAT(0) cube complex. Let \( E \) be the number of edges of \( X \). We are left to prove that \( N \) only depends on \( E \) and on \( |V(\Gamma)| \). Consider the subsequence of all folded complexes of the given standard completion:

\[
\Theta_1 = \Omega_{i_1}, \Theta_2 = \Omega_{i_2}, \ldots, \Theta_n = \Omega_{i_n}
\]

By Proposition 7.2, we know that \( \Theta_i \) is a CAT(0) cube complex. Furthermore, \( \Theta_i \) has diameter at most \( E \), by the proof of Proposition 7.6.

We claim that the complex \( \Theta_j \) is not isometric to \( \Theta_k \) for all \( k > i \). Suppose otherwise for a contradiction. Consider the sequence of operations performed to \( \Theta_j \) in order to obtain \( \Theta_k \). We can repeat this same sequence of operations to \( \Theta_k \) in order to obtain another folded complex isometric to \( \Theta_j \). By iteratively repeating this process, we obtain a standard completion sequence which is infinite. Furthermore, the direct limit \( \Omega' \) of this new standard completion sequence must be a finite complex. To see this, note that given \( m \) distinct cells in \( \Omega' \), there is some complex isometric to \( \Theta_j \) in the completion sequence, which contains \( m \) distinct preimages of the cells (since there are infinitely many such complexes in the sequence). Thus the size of \( \Omega' \) is bounded by the size of \( \Theta_j \). However, this contradicts Proposition 3.5.

Let \( F \) be the number of all possible CAT(0) cube complexes of diameter at most \( E \) and with at most \( |V(\Gamma)| \) edges incident to each vertex. As \( \Theta_j \) is not isometric to \( \Theta_k \) for all \( j \neq k \), it follows that \( n \leq F \).

For each \( 1 \leq j \leq n \), the number of cube attachments that can be applied to \( \Theta_j \), and the number of fold and cube identification operations that can be applied to
the resulting complex is bounded by a number \( K \) which depends only on \( E \). Thus, 
\[ N \leq KF \] where \( K \) and \( F \) depend only on \( E \) and on \( |V(\Gamma)| \).

8. QUASICONVEXITY

Let \( H \) be a group with fixed generating set. Recall that a subgroup \( G \) of \( H \) is \( M \)-quasiconvex, for \( M \geq 0 \), if any geodesic path in the Cayley graph of \( H \) with endpoints in \( G \) lies in the \( M \)-neighborhood of \( G \). We say \( G \) is quasiconvex if it is \( M \)-quasiconvex for some \( M \). In general, \( G \) may be quasiconvex with respect to one generating set for \( G \) and may not be quasiconvex with respect to some other generating set for \( G \). However, if a subgroup is quasiconvex with respect to some generating set, then it is quasi-isometrically embedded with respect to any generating set [BH99 Chapter III.11, Lemma 3.5].

When we say a subgroup is quasiconvex in a right-angled Coxeter group, we will always mean with respect to the standard generating set. The main result of this section is that a subgroup of a right-angled Coxeter group is quasiconvex if and only if any completion of the subgroup is finite.

We first prove a lemma relating distances in a completion of a subgroup to distances in the Cayley graph of the right-angled Coxeter group. Note that whenever we consider a Cayley graph of a right-angled Coxeter group, it is the Cayley graph associated to the standard Coxeter generating set.

**Lemma 8.1.** Let \( G \) be a subgroup of the right-angled Coxeter group \( W_{\Gamma} \) and let \((\Omega, B)\) be a completion of \( G \). Let \( w \) be the label of a path in \( \Omega \) from the basepoint \( B \) to some vertex \( v \in \Omega \). Let \( C \) be the Cayley graph of \( W_{\Gamma} \), and let \( v_w \) be the vertex in \( C \) which represents the element of \( W_{\Gamma} \) corresponding to \( w \). Then \( d_{\Omega}(B, v) = d_C(G, v_w) \). (Here \( G \) is naturally identified with the vertices in \( C \) which represent elements of \( G \).)

**Proof.** By Lemma [4.12] there is a path in \( \Omega \) from \( B \) to \( v \) with label a reduced expression for \( w \). Thus, without loss of generality, we may assume that \( w \) is reduced. Let \( \alpha \) be a geodesic in \( C \) from \( v_{id} \) to \( v_w \) with label \( w \), where \( v_{id} \) is the vertex in \( C \) labeled by the identity element. Let \( \beta \) be a geodesic in \( C \) from \( v_w \) to \( G \) which realizes the distance from \( v_w \) to \( G \). Let \( h \) be the label of \( \beta \). It follows that \( h \) is a reduced word. As \( \alpha \beta \) is a path from \( v_{id} \) to a vertex of \( G \), the word \( k = wh \) represents an element of \( G \). By Lemma [4.23] there is a reduced expression \( k = \tilde{w}h \) for \( k \) in \( W_{\Gamma} \) such that \( w' = \tilde{w}s_1\ldots s_m \) is a reduced expression for \( w \) and \( h' = s_m\ldots s_1h \) is a reduced expression for \( h \), where \( s_i \in V(\Gamma) \) for \( 1 \leq i \leq m \).

By Lemma [4.22], there is a path \( \alpha' \) with label \( w' = \tilde{w}s_1\ldots s_m \) from \( B \) to \( v \) in \( \Omega \). Furthermore, by the definition of the completion of a subgroup, there is a loop \( l \) with label \( \tilde{w}h \) in \( \Omega \) based at \( B \). Since \( \Omega \) is folded, \( \alpha' \) and \( l \) overlap on the part labeled \( \tilde{w} \). It follows that there is a path from \( B \) to \( v \) labeled by \( h^{-1} = h^{-1}s_1\ldots s_m \).

Let \( \gamma \) be a geodesic in \( \Omega \) from \( v \) to \( B \) with label \( z \). Note that \( z \) must be a reduced word, and that \( |z| \leq |h| = |h| \). As \( wz \) is the label of a loop in \( \Omega \), it follows by the definition of a completion that \( wz \) represents an element of \( G \). Thus there is a path in \( C \) from \( v_w \) to \( G \) with label \( z \). By the minimality of \( \beta \), we have that \( |h| \leq |z| \). Hence, \( |z| = |h| \). It now follows that:

\[ d_{\Omega}(v_w, G) = |\beta| = |h| = |z| = |\gamma| = d_{\Omega}(B, v_w) \]

\( \square \)
Lemma 8.2. Let $G$ be a subgroup of the right-angled Coxeter group $W_{Γ}$. If some completion $(Ω, B)$ of $G$ is finite, then $G$ is $M$-quasiconvex in $W_{Γ}$, where $M$ is the maximal distance of a vertex in $Ω$ from $B$.

Proof. Let $α$ be a geodesic in the Cayley graph of $W_{Γ}$ between two elements of $G$. Without loss of generality, we may assume that the endpoints of $α$ are the identity vertex $v_{id}$, and some vertex labeled by an element $g$ of $G$. It follows that the label $w$ of $α$ is a minimal length word representing $g$. Let $v$ be any vertex along $α$. Let $w'$ be the label of the subpath of $α$ from $v_{id}$ to $v$.

By the definition of a completion, there is a loop $l$ in $Ω$ based at $B$ with label $w$. Consequently, there is an initial subpath, $l'$, of $l$ with label $w'$. Let $u$ be the vertex of $Ω$ that is the endpoint of $l'$. By Lemma 8.1, $d_{Ω}(u, B) = d_{C}(v, G) \leq M$. □

Lemma 8.3. If $G$ is a quasiconvex subgroup of the right-angled Coxeter group $W_{Γ}$, then $G$ is finitely generated and every standard completion of $G$ is finite.

Proof. Suppose $G$ is $M$-quasiconvex in $W_{Γ}$. Then $G$ must be finitely generated as it is a quasiconvex subgroup of a finitely generated group [BH99 Chapter III.Γ Lemma 3.5]. Let $(Ω, B)$ be a standard completion of $G$. Let $Y$ be the subset of vertices of $Ω$ that are contained in a loop based at $B$ which is labeled by a reduced word, i.e., $Y$ consists of vertices belonging to the core $C(Ω, B) \subset Ω$. By Lemma 8.1, $d_{Ω}(v, B) \leq M$ for every vertex $v \in Y$.

Suppose $Ω$ is not finite. We will obtain a contradiction by finding a loop based at $B$ in $Ω$ which is labeled by a reduced word, but contains a vertex that has distance greater than $M$ from $B$.

By Proposition 7.1 and since $Ω$ is a standard completion, there are only finitely many hyperplanes in $Ω$. Thus, there exists an integer $N$ such that any geodesic in $Ω$ of length $N$ has at least $M + 3$ of its edges dual to the same hyperplane. Let $v$ be a vertex in $Ω$ that is at distance $N$ from the basepoint $B$. Let $α$ be a geodesic from $B$ to $v$, and let $w$ be the label of $α$. As $α$ is geodesic, $w$ is a reduced word. Let $H$ be an hyperplane that is dual to at least $M + 3$ edges of $α$, and let $s \in Γ$ be the type of $H$. Let $e_{1}$ and $e_{2}$ be the last two edges of $α$ (read from $B$ to $v$) that are dual to $H$.

Let $β_{1}$ be the smallest subpath of $α$ that contains both $e_{1}$ and $e_{2}$, and let $β_{2}$ be the geodesic along the carrier of $H$ from the endpoint of $β_{1}$ to the startpoint of $β_{1}$. Let $b_{1}$ be the label of $β_{1}$, and let $b_{2}$ be the label of $β_{2}$. Note that $b_{2}$ is a word in link($s$) and, in particular, has no occurrences of $s$.

Let $β$ be the subpath of $α$ from $B$ to the startpoint of $e_{1}$, and let $b$ be its label. There is some generator $t \in V(Γ)$ in the word $b$ which does not commute with $s$ and appears after every occurrence of $s$ in $b$. This follows since $w = bb_{1}$ is reduced, $b_{1}$ begins with the letter $s$ and $b$ contains the letter $s$.

Let $l$ be the loop $ββ_{1}β_{2}β^{-1}$ with label $h = bb_{1}b_{2}b^{-1}$. Let $h$ be a reduced expression for $h$ obtained by a series of deletions. We claim that $h$ and $h$ have the same number of occurrences of the generator $s$. For suppose that a deletion of an $s$ generator occurs at some point. Such a deletion must be between an $s$ generator in $bb_{1}$ and one in $b^{-1}$. This follows since $b_{2}$ does not contain any occurrences of the generator $s$, $bb_{1}$ is a reduced word, and $b^{-1}$ is a reduced word. Additionally for this to be possible, the occurrence of $t$ in $b^{-1}$ must first be deleted as well. As $t$ does not commute with $s$, there is no occurrence of $t$ in $b_{2}$. Thus, if this occurrence of $t$ were deleted, there must be an occurrence of $t$ also deleted in $bb_{1}$. However,
this is not possible as $bb_1$ ends with $s$ and no $s$ occurrence can be deleted before $t$ is deleted.

We will now construct a reduced expression for $h$. First obtain a reduced expression $\hat{bb}_1\hat{b}_2$ for $bb_1b_2$ by a series of deletion operations (here the hat notation indicates that some generators in the original words have been deleted). By Lemma 4.1, there is a path $p$ with the same endpoints as $\beta\beta_1\beta_2$ and with label $\hat{bb}_1\hat{b}_2$. In particular, the endpoint vertex of $p$ is the same as the endpoint vertex of $\beta$. Call this vertex $v_1$. The loop $p\beta^{-1}$ has label $\hat{bb}_1\hat{b}_2b^{-1}$ which is an expression for $h$.

Now let $\hat{h}$ be a reduced expression for $h$ obtained by performing deletion operations to the word $\hat{bb}_1\hat{b}_2b^{-1}$. As $\hat{bb}_1\hat{b}_2^{-1}$ and $b^{-1}$ are each reduced, each deletion operation must delete a pair of generators: one in $b^{-1}$ and one in $\hat{bb}_1\hat{b}_2^{-1}$. As no occurrence of $s$ can be deleted and as $b^{-1}$ has at least $M + 1$ occurrences of $s$, no more than $|b^{-1}| - (M + 1) = |\beta| - (M + 1)$ deletion operations can occur.

By the definition of a completion of a subgroup, there is a loop $\hat{l}$ in $\Omega$ with label $\hat{h}$ based at $B$. By Lemma 4.1(3), some vertex $u$ of $\hat{l}$ satisfies $d(u, v_1) \leq |\beta| - (M + 1)$. As $d(v_1, B) = |\beta|$, it follows that $u$ is distance at least $M + 1$ from $B$. Now $\hat{l}$ is a loop labeled by a reduced word $s$ by definition, $u \in Y$. However this contradicts the fact that all vertices in $Y$ have distance at most $M$ from $B$. $\square$

We immediately obtain the following theorem from Lemmas 8.2 and 8.3.

**Theorem 8.4.** Let $G$ be a subgroup of a right-angled Coxeter group. The following are equivalent:

1. $G$ is quasiconvex.
2. Some completion for $G$ is finite.
3. $G$ is finitely generated and every standard completion for $G$ is finite.

9. Residual finiteness and separability

In this section we give a new proof of a result of Haglund, which states that quasiconvex subgroups of right-angled Coxeter groups are separable and are virtual retracts. We additionally give a short proof of the well-known result that right-angled Coxeter groups are residually finite.

We begin with a preliminary result which is also used in later sections. Specifically, in a few arguments throughout the rest of this article, we will have a finite, cube-full, folded $\Gamma$-labeled complex, and we will want to attach certain additional graph-loops to this complex. We will then need the original complex to be isometrically embedded in the completion of the new complex. The following lemma guarantees this property.

**Lemma 9.1.** Let $\Omega$ be a finite, cube-full, folded $\Gamma$-labeled complex. Let $\Omega'$ be a complex obtained by attaching a set $L$ of labeled graph-loops to vertices of $\Omega$. Further suppose that the label of an attached graph-loop is distinct from the labels of every other edge incident to the vertex it is attached to, i.e., $\Omega'$ is a folded complex. Then there exists a completion $\Omega''$ of $\Omega'$ such that

1. The natural inclusion $i : \Omega \hookrightarrow \Omega''$ is an isometry.
2. Every edge of $\Omega''$ that is not in $i(\Omega')$ is a graph-loop attached to a vertex $v \in i(\Omega)$. Let $l$ be such a graph-loop and let $s$ be its label. Then there exists a graph-loop in $L$ with label $s$ attached to a vertex $u \in \Omega$ and a path in $i(\Omega)$ from $i(u)$ to $v$ whose label is a word in link$(s) \subset V(\Gamma)$.
The number of operations performed to obtain $\Omega''$ from $\Omega'$ is finite and only depends on the number of edges of $\Omega'$.

Proof. We build the completion $\Omega''$ by alternately performing a single cube attachment operation followed by all possible fold and cube identification operations.

By assumption, $\Omega'$ is a folded complex. Thus, each cube attachment operation is only done to a folded complex. We show that each folded complex in this completion sequence satisfies the conclusion of the lemma.

Let $v$ be a vertex of $\Omega'$ incident to edges $e_1, \ldots, e_n$ with distinct, pairwise commuting labels $s_1, \ldots, s_n$. Further suppose that $v \cup e_1 \cup \cdots \cup e_n$ are not all contained in a common $n$-cube and that $n$ is maximal out of such possible choices. Consider the corresponding cube attachment operation which glues a labeled cube to $v \cup e_1 \cup \cdots \cup e_n$. Let $c$ denote the image of this cube in the resulting complex.

By possibly relabeling, we may assume that the following holds for some $0 \leq k \leq n$: if $i \leq k$, then $e_i$ is not a graph-loop (and is therefore necessarily in $\Omega$), while if $i > k$, then $e_i$ is a graph-loop (and may or may not be in $\Omega$). Note that if $k = 0$, then $e_i$ is a graph-loop for all $i$.

If $k > 0$, it follows (since $\Omega$ is cube-full) that $e_1, \ldots, e_k$ are contained in a common $k$-cube $q$ of $\Omega$. We perform fold and cube identification operations to identify $q$ to a face of $c$. Otherwise if $k = 0$, define $q$ to be the 0-cube $v$.

Next, starting at $v$, we perform all possible fold operations to pairs of edges which are both in $c$. It readily follows that the 1-skeleton of this resulting complex will consist of $q$ and a graph-loop with label $s_i$, for each $k + 1 \leq i \leq n$ and each vertex of $q$. By a slight abuse of notation, we call this resulting complex $c$ as well.

We now check what other fold operations are possible. As $\Omega'$ is folded, the only type of possible additional fold operation would consist of an edge $e$ in $\Omega'$ and a graph-loop $f$ in $c$ such that $e$ and $f$ have the same label, $s$, and share an endpoint $u \in q$.

We claim that $e$ must be a graph-loop. This is clear if $e$ is in $\Omega' \setminus \Omega$. Suppose $e \in \Omega$. There is a path $p$ from $u$ to $v$ in $q$ with label a word consisting only of generators that are distinct from and commute with $s$. Let $e_i$ be the edge at $v$ with label $s$. Then $e_i$ must be a graph-loop since otherwise $e$ would have already been folded onto $c$. As $\Omega$ is cube-full and $p \cup e \subset \Omega$, we conclude that $e_i \in \Omega$. Thus since $e_i$ is a graph-loop, $e$ is a graph-loop as well.

Thus, we simply fold $e$ onto $f$. After performing all such folds to $\Omega' \cup c$ we obtain the complex $\Omega'_1$. After possibly applying some cube identification operations, it follows that $\Omega'_1$ is folded. Furthermore, the 1-skeleton of $\Omega'_1$ is the same as the 1-skeleton $\Omega'$ with the possible addition of some new graph-loops. Thus, $\Omega$ is isometrically embedded in this new complex. The second conclusion of the lemma also readily follows from the construction.

We then iteratively repeat such cube attachments followed by such a sequence of fold and cube identification operations. After each iteration we have a folded complex satisfying the first two claims of the lemma. As the number of such operations is bounded by a function of the number of edges of $\Omega'$, the third claim of the lemma follows.

Let $G$ be a finitely generated subgroup of a right-angled Coxeter group $W_\Gamma$, and suppose that $G$ has a finite completion, $(\Omega, B)$. We define a construction that can be performed to $\Omega$ to produce a new complex.
Proposition 9.2. Let $s$ be a relation between natural inclusion of $\Omega$ into $E$. By Proposition 4.4 and Lemma 6.8, we have that $H$ has finite index in $W_\Gamma$, and suppose that $G$ has a finite completion $(\Omega, B)$. Let $(E, B)$ be the full valence extension of $(\Omega, B)$. Then $H$ is a finite-index subgroup of $W_\Gamma$, and there is a retraction from $H$ to $G$.

Proof. By Proposition 4.4 and Lemma 6.8, $H$ has finite index in $W_\Gamma$. We are left to prove that $G$ is a retract of $H$.

Let $L$ be the set of graph-loops in $E$ that are not in $\Omega$, i.e. the graph-loops added in the construction of $E_0$ or in the completion process. We define a map $\phi : H \to G$ as follows. Given an element $h \in H$, since $E$ is a completion of $H$ by Proposition 4.4, there is a loop $l$ in $E$ based at $B$ whose label $w$ is a word representing $h$. We remove from $l$ all graph-loops it traverses which are in $L$. Let $l'$ be the resulting loop in $\Omega$ based at $B$, and let $w'$ be its label. It follows that $w'$ represents an element $g \in G$. We set $\phi(h) = g$.

We first check that $\phi$ is well-defined. Let $l_1$ and $l_2$ be loops in $E$ based at $B$ with labels $w_1$ and $w_2$, such that $w_1$ and $w_2$ are distinct words, each representing the same element $h \in H$. Let $w$ be a reduced word representing $h$ in $W_\Gamma$. Let $l_1', l_2'$ and $l'$ be the loops obtained by removing graph-loops in $L$ from $l_1, l_2$ and $l$ respectively. Let $w_1', w_2'$ and $w'$ be the labels of $l_1', l_2'$ and $l'$ respectively. We must show that $w_1'$ and $w_2'$ represent the same element of $W_\Gamma$. To do so, we will show that $w'$ represents the same element in $W_\Gamma$ as both $w_1'$ and $w_2'$.

By Tits’ solution to the word problem, there is a sequence of Tits moves that can be performed to $w_1$ to obtain $w$. This sequence of Tits moves naturally produces a sequence of corresponding loops $l_1 = q_1, q_2, \ldots, q_n = l$ in $E$ whose labels are the corresponding words obtained by the Tits moves. Furthermore, if a cancellation move is performed to $q_i$ in order to obtain $q_{i+1}$, then as $E$ is folded, it readily follows that the edges involved in this cancellation move are either both in $L$ or both not in $L$. Thus, by forgetting the Tits moves performed to generators which are labels of graph-loops in $L$, this sequence of Tits moves induces a sequence of Tits moves performed to $w_1'$ to produce $w'$. Hence, $w_1'$ and $w'$ represent the same element of $W_\Gamma$. By the same argument, $w_2'$ and $w'$ represent the same element of $W_\Gamma$. Consequently, $\phi$ is well-defined.

It is clear that $\phi$ is a homomorphism. Furthermore, given an element $g \in G$ and a loop $l$ in $E$ based at $B$ with label a reduced word representing $g$, we have that $l$ is contained in the subcomplex $\Omega \subset E$. Thus, $l$ does not traverse any graph-loops in $L$. It follows that $\phi$ restricted to elements of $G$ is the identity. Hence, $\phi$ provides the desired retraction.

We give a proof using completions that right-angled Coxeter groups are residually finite. This is a well known result. In fact these groups are linear (see for
Every right-angled Coxeter group is residually finite.

Proof. Let $W_\Gamma$ be a right-angled Coxeter group. Let $g$ be a nontrivial element in $W_\Gamma$, and let $w$ be a reduced word representing $g$. Let $G$ be the trivial subgroup of $W_\Gamma$ given by the generating set $S_G = \{ww^{-1}\}$. The $S_G$-complex $\Omega_0 = X(S_G)$ consists of a circle, subdivided into labeled edges, whose label, read from a base vertex $B$, is $ww^{-1}$. We can iteratively perform fold operations to $\Omega_0$ and obtain a complex $\Omega_N$ that is a path labeled by $w$.

By Proposition 7.5, there is a completion $(\Omega, B)$ of $\Omega_N$ that is a finite CAT(0) cube complex. The image of $\Omega_N$ in $\Omega$ is a path, $p'$, based at $B$ and labeled by $w$. Furthermore, the path $p'$ is not a loop in $\Omega$. This follows since $\Omega$ is a completion of the trivial subgroup, and consequently every loop in $\Omega$ based at $B$ must have as label a word that is trivial in $W_\Gamma$.

Let $(E, B)$ be the full valence extension of $(\Omega, B)$. As $\Omega$ is isometrically embedded in $E$ by Lemma 9.4, $w$ is still not the label of a loop in $E$ based at $B$. Let $H$ be the subgroup of $W_\Gamma$ associated to $(E, B)$. By Proposition 9.2, $H$ has finite index in $W_\Gamma$. Furthermore, $g \not\in H$ as $w$ is a reduced word representing $g$ which is not the label of a loop in $E$.

A subgroup $G$ of a group $K$ is a virtual retract if $G$ is a retract of a finite index subgroup of $K$. The next theorem and corollary show that quasiconvex subgroups of a right-angled Coxeter group are virtual retracts and are separable. These results were first proven by Haglund in [Hag08].

Theorem 9.4. Let $G$ be a quasiconvex subgroup of a right-angled Coxeter group $W_\Gamma$. Then $G$ is a virtual retract of $W_\Gamma$.

Proof. By Theorem 8.4 there is a finite completion $(\Omega, B)$ of $G$. Let $(E, B)$ be the full valence extension of $(\Omega, B)$. Let $H$ be the subgroup of $W_\Gamma$ associated to $(E, B)$. By Proposition 9.2, $H$ is a finite-index subgroup of $W_\Gamma$, and $G$ is a retract of $H$.

It is well-known that a virtual retract of a residually finite group is separable. We refer the reader to [Hag08] Proposition 3.8 for a proof. The following corollary thus immediately follows.

Corollary 9.5. Every quasiconvex subgroup of a right-angled Coxeter group is separable.

10. Reflection Subgroups

Given a right-angled Coxeter group $W_\Gamma$, a reflection is an element of $W_\Gamma$ represented by a word of the form $wsus^{-1}$ where $s$ is a generator in $V(\Gamma)$ and $w$ is a word in $W_\Gamma$. In this section, we consider a subgroup $G$ generated by a finite set of reflections. We show there is a constructive algorithm to build a finite completion of $G$. In particular, such subgroups are always quasiconvex and their index can be computed. We will use these results to study Coxeter subgroups of 2-dimensional right-angled Coxeter groups in the next section.
10.1. **Trimmed sets of reflections.** The following observation guarantees that we can always use a nice generating set of reflections:

**Lemma 10.1.** Let $W_{\Gamma}$ be a right-angled Coxeter group, and let $G$ be a subgroup generated by a finite set of reflections $R'$. Then $G$ is generated by a the set of reduced reflections of the form:

$$R = \{w_is_iw_i^{-1} \mid w_i \in W_{\Gamma} \text{ and } s_i \in V(\Gamma), 1 \leq i \leq m\}$$

where for all $i \neq j$, no reduced expression for $w_j$ begins with $w_is_i$. Furthermore, there is a constructive algorithm to obtain $R$ from $R'$, whose time-complexity only depends on the number $\sum_{r \in R'} |r|$.

**Proof.** Without loss of generality, we may assume elements in $R'$ are reduced. Let $g = wsw^{-1}$ be a reflection in $R'$ so that $w$ has an expression, $w = w's'q$ where $h = w's'w'^{-1}$ is another reflection in $R'$ and $q$ is a word in $W_{\Gamma}$.

In $R'$, we replace $g$ with a reduced representative of the shorter length reflection $g' = hgh^{-1} = (w'q)s(w'q)^{-1}$, to obtain a new set $R''$. The set $R''$ still generates $G$, as $g = h^{-1}g'h$. By iteratively performing such replacements, we obtain the desired generating set $R$. This process must end since at each step we obtain a set of generators whose lengths sum to a strictly smaller number than those in the previous step. \[\square\]

**Definition 10.2** (Trimmed reflection set). We say that a set $R$ of reduced reflections is **trimmed** if it satisfies the conclusion of Lemma 10.1.

10.2. **A completion for reflection subgroups.** Throughout this subsection, we fix the notation in the discussion below. This notation is also used in Section 12.

Let $G$ be a subgroup of the right-angled Coxeter group $W_{\Gamma}$, generated by a finite set of reflections:

$$R = \{w_is_iw_i^{-1} \mid w_i \in W_{\Gamma} \text{ and } s_i \in V(\Gamma), 1 \leq i \leq m\}$$

By Lemma 10.1 we may assume without loss of generality that $R$ is trimmed.

Our goal is to give a finite completion $(\Omega_G, B)$ of $G$. We begin by describing the first complex $\Omega_0$ in this completion. For each $1 \leq i \leq m$, we attach a subdivided circle to the base vertex $B$ with label $w_is_iw_i^{-1}$. Next, for each $i$, we fold the two copies of $w_i$ onto one another, and we call this resulting graph $\Omega_0$. Thus the graph $\Omega_0$ has, for each $1 \leq i \leq m$, a path emanating from $B$ and labeled by $w_i$, with a graph-loop labeled by $s_i$ attached at its endpoint. By Theorem 3.11 any completion of $\Omega_0$ is a standard completion of $G$.

Let $T$ denote the tree obtained by removing the graph-loops from $\Omega_0$. Let $\mathcal{F}T$ be the folded tree obtained by iteratively performing fold operations to $T$. Let $\Omega_{\mathcal{F}T}$ be a standard completion of $\mathcal{F}T$. By Proposition 7.5, we know that $\Omega_{\mathcal{F}T}$ is a finite CAT(0) cube complex. Furthermore, $\Omega_{\mathcal{F}T}$ is also a completion of $T$ by construction. Let $f : T \rightarrow \mathcal{F}T \rightarrow \Omega_{\mathcal{F}T}$ be the natural map. By a slight abuse of notation, we also denote by $f$ the natural map $f : \mathcal{F}T \rightarrow \Omega_{\mathcal{F}T}$. Let $\hat{T} := f(T) = f(\mathcal{F}T)$.

Given a vertex $\hat{v}$ in $\Omega_{\mathcal{F}T}$, let

$$L_{\hat{v}} = \left\{ s \in V(\Gamma) \mid \exists v \in V(\Omega_0) \text{ incident to a graph-loop labeled } s, \text{ such that } \hat{v} = f(v) \right\}$$

Note that the set of vertices $\{\hat{v} \in \Omega_{\mathcal{F}T} \mid L_{\hat{v}} \neq \emptyset\}$ is exactly the set of vertices of $\Omega_{\mathcal{F}T}$ whose preimage in $T \subset \Omega_0$ contains a vertex incident to a graph-loop.
We would like to build the completion $\Omega_{\mathcal{FT}}$ by “adding back” the graph-loops to $\Omega_{\mathcal{FT}}$ and applying Lemma 9.1. However, there is a technical issue: when adding back a graph-loop labeled $s$ to a vertex of $\hat{v}$ of $\Omega_{\mathcal{FT}}$, a priori $\hat{v}$ might already be incident to an edge labeled by $s$. If this were true, then the hypothesis of Lemma 9.1 would not be satisfied. The next two lemmas show this situation is not possible. They are the technical results required to prove the main results of this section.

**Lemma 10.3.** Let $ks$ be a reduced word in $W_{\mathcal{F}}$ such that $s \in V(\Gamma)$ and $k$ is a (possibly empty) word consisting only of generators that are adjacent to $s$ in $\Gamma$. Then given any $v \in \Omega_0$ incident to a graph-loop labeled by $s$, no path in $\mathcal{T} \subset \Omega_0$ starting at $v$ is labeled by a word which is an expression for $ks$.

**Proof.** For a contradiction, suppose there exists such a path $\alpha$. As $\mathcal{T}$ is a tree, if $\alpha$ were not geodesic then some generator would be consecutively repeated in the label of $\alpha$. Thus, by possibly passing to a homotopic path, we may assume that $\alpha$ is geodesic in $\mathcal{T}$. Let $u$ be the endpoint of $\alpha$.

Let $\beta_1$ be the geodesic from the base vertex $B$ to $v$, with label $h_1$. Then there is an element $r_1 = h_1s\alpha^{-1}$ in $R$. We first show that $u$ does not lie on $\beta_1$. Suppose it does. Then $h_1$ has a suffix which is a reduced word equal in $W_{\mathcal{F}}$ to $sk^{-1}$. Since $k$ commutes with $s$, it follows that the expression $h_1s\alpha^{-1}$ is not reduced, a contradiction.

Now let $\beta_2$ be the geodesic from $B$ to $u$, with label $h_2$. Since $u$ does not lie on $\beta_1$, it follows that $h_2$ is non-empty. Moreover, there is a reflection $r_2 \in R$, given by $(h_2h'_{2})s'(h'_{2}^{-1}h_{2}^{-1})$, where $h'_{2}$ could be empty. Next, we claim that $h_1$ is non-empty. For if not, then $r_1 = s$, and in $W_{\mathcal{F}}$, we have $h_2 = ks$ and $r_2 = (ks)(h'_{2}s'h'_{2}^{-1})sk^{-1}$. However, in this case $r_2$ has a reduced expression that begins with $s$, which is not possible as $R$ is trimmed.

Thus $h_1$ and $h_2$ are non-empty and $ks$ is a reduced expression for $h_1^{-1}h_2$. By Lemma 2.3, there exist (possibly empty) words $x$, $k'$ and $k''$ such that either $k'x$ and $x^{-1}k''s$ are reduced expressions in $W_{\mathcal{F}}$ for respectively $h_1^{-1}$ and $h_2$, or alternatively $sk'x$ and $x^{-1}k''$ are reduced expressions in $W_{\mathcal{F}}$ for respectively $h_1^{-1}$ and $h_2$. Moreover, $k'k''$ is equal to $k$ in $W_{\mathcal{F}}$. The latter case implies that the reflection $r_1$ has another reduced expression $(sk'x)^{-1}s(sk'x)$. However, this is a contradiction as this word is clearly not reduced. In the former case, we have that $r_1$ has reduced expression $(k'x)^{-1}s(k'x)$ and $r_2$ has reduced expression $(x^{-1}k''s)h_{2}^{-1}$. As the given expression for $r_1$ is reduced and as $s$ commutes with $k'$, it must be that $k'$ is the empty word. In particular, $w_1 = x^{-1}$. Furthermore, since $s$ commutes with $k''$, $w_2$ has reduced expression $x^{-1}sk''h_{2}$. However, this contradicts our choice of $R$ since some expression for $w_2$ begins with $w_1s$. The claim follows. \[\square\]

**Lemma 10.4.** For every vertex $\hat{v}$ of $\Omega_{\mathcal{FT}}$ and every $s \in L_0$, $\hat{v}$ is not incident to an edge labeled by $s$ in $\Omega_{\mathcal{FT}}$.

**Proof.** For a contradiction, suppose that there exists some $\hat{v}$ of $\Omega_{\mathcal{FT}}$, $s \in L_0$ and an edge $d$ incident to $\hat{v}$ in $\Omega_{\mathcal{FT}}$ which is labeled by $s$. Note that $\hat{v} \in \mathcal{T}$.

Let $H$ be the hyperplane (recall that $\Omega_{\mathcal{FT}}$ is a CAT(0) cube complex) dual to $d$. In particular, $H$ is of type $s$. By Lemma 4.1, $H$ intersects $\mathcal{T}$ at some edge $\hat{e}$. Let $e$ be an edge of $\mathcal{T}$, such that $\hat{f}(e) = \hat{e}$. Note that the label of $e$ must be $s$. Let $v \in \mathcal{T}$ be such that $\hat{f}(v) = \hat{v}$. Let $\beta$ be a geodesic in $\mathcal{T}$ from $v$ to $e$. Let $\hat{\beta} = \hat{f}(\beta)$. It follows that $\hat{\beta}$ is a path in $\Omega_{\mathcal{FT}}$ from $\hat{v}$ to $\hat{e}$. By Lemma 4.2, there exists a geodesic
\[ \hat{\beta}' \] with the same endpoints as \( \hat{\beta} \) and with label a reduced expression for the label of \( \hat{\beta} \). Finally, let \( \gamma \) be a path in the carrier of \( H \) from \( \hat{v} \) to the endpoint of \( \hat{e} \). Note that the label of \( \gamma \) only consists of vertices in \( \text{link}(s) \subset \Gamma \).

As \( \hat{\beta}' \) is geodesic, any hyperplane intersects it at most once. Thus, as hyperplanes separate \( \Omega_{FT} \) into two components, it follows that any hyperplane that intersects \( \hat{\beta}' \) must also intersect \( \gamma \). It follows that the label of \( \hat{\beta}' \) consists only of vertices in \( \text{link}(s) \). However, the label of \( \hat{\beta}' \) and the label of \( \beta \) are expressions for the same element of \( W_{\Gamma} \). This implies that \( \beta \cup e \) is a path in \( T \) based at \( v \) whose label is of the form \( ks \), where every generator in \( k \) is adjacent to \( s \) in \( \Gamma \). This contradicts Lemma 10.3.

We are now ready to prove the main results of this section.

**Theorem 10.5.** Let \( G \) be a finitely generated reflection subgroup of a right-angled Coxeter group. Then there exists a finite completion of \( G \).

**Proof.** As previously discussed, we first obtain the completion \( \Omega_{FT} \) of \( T \) using Proposition 7.5. For each vertex \( \hat{v} \) of \( \Omega_{FT} \) and \( s \in L_{\hat{v}} \), we attach a graph-loop to \( \hat{v} \) labeled by \( s \). Let \( \Omega'_{FT} \) be the resulting complex. Note that by Lemma 10.4 such a graph-loop is never attached to a vertex that is incident to an edge with the same label as the graph-loop. Furthermore, note that \( \Omega'_{FT} \) can be obtained from \( \Omega_0 \) by applying the same completion sequence that was applied to \( T \) to obtain \( \Omega_{FT} \), while “ignoring” the graph-loops. We now get a finite completion \( \Omega_G \) of \( \Omega'_{FT} \) by applying Lemma 9.1. It follows that \( \Omega_G \) is a finite completion for \( G \).

The following corollary immediately follows from Theorem 10.5 and Theorem 8.4.

**Corollary 10.6.** Every finitely generated reflection subgroup of a right-angled Coxeter group is quasiconvex.

For 2-dimensional right-angled Coxeter groups, we obtain the following stronger result which shows that the time-complexity of the algorithm which builds the completion of a reflection subgroup is bounded by the size of words in the generating set of reflections. This result will be important in Section 12.

**Theorem 10.7.** Let \( W_{\Gamma} \) be a 2-dimensional right-angled Coxeter group. Let \( G \) be a subgroup of \( W_{\Gamma} \) generated by a finite set of reflection words \( R \). Then there is a finite completion sequence for \( G \) whose length only depends on the numbers \( \sum_{r \in R} |r| \) and \( |V(\Gamma)| \).

**Proof.** By Lemma 10.1, we may assume without loss of generality that \( R \) is trimmed. As before, we first obtain the completion \( \Omega_{FT} \) of \( T \). However, this time we use the more refined Proposition 7.6 which guarantees that the number of steps in this completion sequence only depends on \( |\Omega_0| \) and \( |V(\Gamma)| \). The rest of the proof follows by repeating the proof of Theorem 10.5 and noting that Lemma 9.1(3) guarantees the bound.

**Remark 10.7.1.** Note that by Lemma 9.1 the completion \( \Omega_G \) of the reflection group \( G \) given by Theorem 10.5 and by Theorem 10.7 contains the complex \( \Omega_{FT} \) as an isometrically embedded subcomplex. Moreover, the inclusion of \( \Omega_{FT} \) into \( \Omega_G \) satisfies the additional properties given by Lemma 9.1. These facts will be important in Section 12.
11. Coxeter subgroups of 2-dimensional right-angled Coxeter groups

In this section we study Coxeter subgroups of 2-dimensional right-angled Coxeter groups. Recall that such subgroups, by our definition, are always finitely generated.

It is clear that a Coxeter subgroup of a right-angled Coxeter group \( W_\Gamma \) must be generated by a set of involutions (order two elements) in \( W_\Gamma \). We show in Theorem 11.4 that under mild hypotheses these subgroups are generated by reflections. Consequently, Theorem 10.5 applies to these subgroups.

We first prove three lemmas involving the structure of particular types of words in a right-angled Coxeter group. The first is well-known and addresses involutions:

**Lemma 11.1.** Let \( g \) be an involution in the right-angled Coxeter group \( W_\Gamma \). Then there is an expression \( wkw^{-1} \) for \( g \), such that every generator in the word \( k \) is in a common clique of \( \Gamma \).

**Proof.** It is well known that every finite subgroup of a right-angled Coxeter group is contained in a conjugate of a special finite subgroup (see [Dav08, Theorem 12.3.4] for instance). Furthermore, a special subgroup of a right-angled Coxeter group is finite if and only if its defining graph is a clique. The lemma then follows. \( \square \)

The next lemma concerns the structure of particular types of commuting words.

**Lemma 11.2.** Let \( w = s_1 \ldots s_m \) and \( k = k_1 \ldots k_n \) be reduced commuting words in the right-angled Coxeter group \( W_\Gamma \) and suppose that the vertices \( k_1, \ldots, k_n \in V(\Gamma) \) are all contained in a common clique of \( \Gamma \). Then for each \( 1 \leq i \leq m \), either:

1. \( s_i = k_r \) for some \( 1 \leq r \leq n \) and \( m(s_i, s_j) = 2 \) for all \( j \neq i \) or
2. \( m(s_i, k_j) = 2 \) for all \( 1 \leq j \leq n \)

**Proof.** Note that as \( k \) is reduced and \( k_1, \ldots, k_n \) pairwise commute, it follows that \( k_i \neq k_j \) for all \( i \neq j \). As \( w \) and \( k \) commute, there exists a disk diagram \( R \) with boundary label \( wkw^{-1}k^{-1} \). We think of \( R \) as a rectangle. The vertical sides are labeled, read from bottom to top, by \( w \), and the horizontal sides of \( R \) are labeled, read from left to right, by \( k \). As \( w \) and \( k \) are reduced, no dual curve intersects the same side of \( R \) twice.

Fix \( s \in \Gamma \) such that \( s = s_i \) for some \( 1 \leq i \leq m \). Let \( e_1, \ldots, e_l \) be the set of edges labeled by \( s \) on the left side of \( R \), ordered from bottom to top. Let \( e'_1, \ldots, e'_l \) be the edges labeled by \( s \) on the right side of \( R \), ordered from bottom to top. We think of the edge \( e_i \) as lying “directly across” from \( e'_i \) in \( R \). Consider the set \( \mathcal{H} \) of all dual curves in \( R \) of type \( s \).

Suppose first that \( s \neq k_r \) for all \( 1 \leq r \leq n \). As dual curves of the same type do not intersect, it follows that for each \( 1 \leq j \leq l \), there is a curve in \( \mathcal{H} \) dual to both \( e_j \) and \( e'_j \). Let \( H \) be the dual curve in \( \mathcal{H} \) that is bottom-most in \( R \), i.e., that is dual to \( e_1 \) and \( e'_1 \). Let \( \alpha \) be the path along the boundary of \( R \) from the bottom of \( e_1 \) to the bottom of \( e'_1 \). Let \( t = k_r \) for any \( 1 \leq r \leq n \). Observe that the label of \( \alpha \) has an odd number of occurrences of the letter \( t \). Hence some curve dual to an edge in \( \alpha \) labeled by \( t \) must intersect \( H \). Thus, \( m(s, t) = 2 \) and item (2) in the statement of the lemma holds in this case.

On the other hand, suppose that \( s = k_r \) for some \( 1 \leq r \leq n \). Let \( d \) be the edge on the bottom of \( R \) labeled by \( k_r \), and let \( d' \) be the edge on the top of \( R \) labeled by \( k_r \). The dual curves in \( \mathcal{H} \) must take one of two possible forms. The first possibility is that there are curves in \( \mathcal{H} \) dual to the following pairs of edges: \( d \) and \( e_1 \), \( e'_1 \) and
\( e_{j+1} \) for \( 1 \leq j \leq l - 1 \), and \( e_j' \) and \( d' \). Otherwise, there are curves in \( \mathcal{H} \) dual to the pairs \( d \) and \( e_1', e_j \) and \( e_{j+1}' \) for \( 1 \leq j \leq l - 1 \), and \( e_2 \) and \( d' \).

Let \( t \) be a letter in \( w \) such that \( t \neq s \). If \( t \) also appears in \( k \), then \( m(s, t) = 2 \). Otherwise, the dual curves labeled by \( t \) all go across \( R \), such that for all \( j \), the \( j \)th edge labeled \( t \) on the left side is paired with the \( j \)th edge labeled \( t \) on the right side. Now the structure of dual curves in \( \mathcal{H} \) (in either case) forces each dual curve labeled \( t \) to intersect a curve in \( \mathcal{H} \). It follows that \( m(s, t) = 2 \). Thus, \( s \) commutes with every generator of \( w \) that is not equal to \( s \). All that is left to show is that there is only one occurrence of the generator \( s \) in \( w \), i.e., that \( l = 1 \). However, if there were two occurrences of \( s \), these occurrences can be deleted (as \( s \) commutes with every generator in \( w \)). This is not possible, however, as \( w \) is reduced. Thus, \( m(s_i, s_j) = 2 \) for all \( 1 \leq j \leq m \) such that \( j \neq i \), and item (1) in the statement of the lemma holds. \( \square \)

The following lemma about commuting words will be used in the proof of Theorem 11.4.

**Lemma 11.3.** Let \( b \) and \( x = zs_1s_2z^{-1} \) be reduced commuting words in a right-angled Coxeter group \( W \), where \( s_1, s_2 \in V(\Gamma) \) and \( z \) is a word in \( W \). Suppose also that \( s_1 \) commutes with both \( s_2 \) and \( z \). Then \( b \) commutes with \( \hat{x} = zs_2z^{-1} \)

**Proof.** As in the previous lemma, we form a “rectangular” disk diagram \( R \) with boundary label \( bxb^{-1}x^{-1} \). The vertical sides of \( R \) are labeled, read from bottom to top, by \( b \), and the horizontal sides of \( R \) are labeled, read from left to right, by \( x \). As \( b \) and \( x \) are reduced, no dual curve intersects the same side of \( R \) twice. Let \( \mathcal{H} \) be all dual curves in \( R \) of type \( s_1 \).

As \( x \) is reduced and \( s_1 \) commutes with both \( s_2 \) and \( z \), it readily follows that there is only one occurrence of \( s_1 \) in \( x \). Let \( e_l \) be the unique edge on the top of \( R \) labeled by \( s_1 \), and let \( e_l \) be the unique edge on the bottom of \( R \) labeled by \( s_1 \).

First suppose that \( b \) does not contain any occurrences of the generator \( s_1 \). Then \( \mathcal{H} \) consists of a single dual curve \( H \) which is dual to both \( e_l \) and \( e_r \). Let \( N(H) \) be the set of cells in \( R \) that contain an edge dual to \( H \). It follows that the boundary \( \partial N(H) \) of \( N(H) \) has label \( s_1ys_1y^{-1} \) for some word \( y \). We can then excise \( \langle N(H) \backslash \partial N(H) \rangle \cup e_l \cup e_r \) from \( R \) and then glue back together the resulting components along their boundary paths labeled by \( y \). What results is a new disk diagram with boundary label \( bxb^{-1}x^{-1} \). Thus, \( b \) commutes with \( \hat{x} \).

On the other hand, suppose that \( b \) has one or more occurrence of \( s_1 \). We consider two cases. The first case is that \( s_1 \) commutes with every generator of \( b \). As \( b \) is reduced, it readily follows that there is an unique occurrence of \( s_1 \) in \( b \). Furthermore \( s_1b \) and \( bs_1 \) are both expressions for \( b \), where \( \hat{b} \) is the word obtained from \( b \) by removing the generator \( s_1 \). Now we have the following equalities in \( W \):

\[
\hat{b} \hat{x} = \hat{b}s_1s_1 \hat{x} = bx = xb = \hat{x}s_1 \hat{b} = \hat{b} \hat{x}
\]

Thus, \( \hat{b} \) and \( s_1 \) both commute with \( \hat{x} \), so \( b \) does as well, and the lemma follows for this case.

For the second case, suppose there are generators in \( b \) that do not commute with \( s_1 \). We will show that this case is actually not possible as we obtain a contradiction.

Let \( e_1, \ldots, e_l \) be the set of edges labeled by \( s_1 \) on the left side of \( R \) ordered from bottom to top, and let \( e'_1, \ldots, e'_l \) be the set of edges labeled by \( s_1 \) on the right side of \( R \) ordered from bottom to top. As in the previous lemma, the dual curves in \( \mathcal{H} \)
are either dual to the pairs $e_b$ and $e_1$, $e'_j$ and $e_{j+1}$ for $1 \leq j \leq l-1$, and $e'_l$ and $e_l$, or alternatively are dual to the pairs $e_b$ and $e'_1$, $e_j$ and $e'_{j+1}$ for $1 \leq j \leq l-1$, and $e_l$ and $e_{e_l}$. We assume that the dual curves in $H$ has the first configuration described (the proof in the other possible configuration same).

Consider the first occurrence in $b$ of a generator $t$ that does not commute with $s_1$, and let $d$ be the corresponding edge on the left side of $R$ labeled by $t$. Note that $d$ is the “bottom-most” edge on the left side of $R$ with label $t$. Let $T$ be the curve in $R$ that is dual to $d$. Note that $T$ cannot intersect any dual curve in $H$ as $s_1$ and $d$ do not commute. Furthermore, $T$ cannot be dual to an edge on the top or bottom of $R$ as the label of every such edge commutes with $s_1$. Thus, it readily follows by the structure of dual curves in $H$ that $d$ cannot lie before $e_1$. Similarly, $d$ cannot occur after $e_1$, as then the edge on the right side of $R$ labeled by $t$ occurs after $e'_1$, which is again not possible by the structure of $H$.

Thus $d$ must lie between $e_r$ and $e_{r+1}$ for some $1 \leq r < l$. However, as $T$ cannot intersect a dual curve in $H$ and cannot intersect the bottom of $R$, it follows that $T$ is dual to an edge on the right side of $R$ that lies before $e'_r$. Correspondingly, there is an edge on the left side of $R$ lying below $e_r$ with label $t$. This contradicts the fact that $d$ is the bottom-most such edge on the left side of $R$. The lemma follows. □

An isolated vertex of a graph is a vertex that is not adjacent to any other vertex. If $W_{\Gamma}$ is a 2-dimensional right-angled Coxeter group, we show:

**Theorem 11.4.** Let $G$ be a subgroup of a right-angled Coxeter group $W_{\Gamma}$, where $\Gamma$ is triangle-free. Suppose that $G$ is isomorphic to the Coxeter group $W_{\Gamma'}$ (which is right-angled by Proposition [2.7]), where $\Gamma'$ does not have an isolated vertex. Then $G$ is generated by a finite set of reflections in $W_{\Gamma}$.

**Proof.** As $G$ is a right-angled Coxeter group, it is generated by the standard Coxeter generating set corresponding to $\Gamma'$. In particular, there exists a finite generating set $I_G$ for $G$ consisting of reduced words representing involutions in $W_{\Gamma'}$, such that

1. There is a bijective map from $V(\Gamma')$ to the elements of $I_G$, and
2. If there is an edge between two vertices of $\Gamma'$ then the corresponding elements of $I_G$ commute.

We will inductively construct a sequence of generating sets for $G$, each of which consists of $|I_G|$ reduced words representing involutions in $W_{\Gamma}$ and satisfies properties (1) and (2). Furthermore, each generating set in the sequence will contain one more reflection than the previous one.

Let $r$ be the number of reflections in $I_G$. If $r = |I_G|$, then we are done. Otherwise, let $h \in I_G$ be such that $h$ is not a reflection. By properties (1) and (2) above, and since $\Gamma'$ does not have isolated vertices, there exists an $h' \in I_G$ which is distinct from and commutes with $h$. By Lemma [11.1] and since $\Gamma$ is triangle-free, we conclude that $h = ws_1s_2w^{-1}$ and $h' = w'k'w'^{-1}$, where $s_1$ and $s_2$ are adjacent vertices in $\Gamma$, $k'$ is a word of length at most two whose generators are in a common clique, and $w, w'$ are words in $W_{\Gamma}$.

Consider the subgroup $H = w'^{-1}Gw'$ of $W_{\Gamma}$. Note that $H$ is generated by $I_H = w'^{-1}I_Gw'$. Let $x$ be a reduced expression for $w'^{-1}hw'$ obtained by applying a sequence of deletions. Then $x$ must be of the form $x = zs_1s_2z^{-1}$ for some word $z$ in $W_{\Gamma}$. Note that $k' = w'^{-1}h'w' \in I_H$ and that $k'$ and $x$ commute.

We claim that either $k' = s_1$ or $k' = s_2$. First suppose for a contradiction that $k'$ is of length two, say $k' = k_1k_2$, for some distinct $k_1, k_2 \in V(\Gamma)$. If both $k_1 \neq s_1$ and
Lemma 11.2 further implies that similarly, this is not possible as $\Gamma$ is triangle-free. Thus $k_2$ is equal to either $s_1$ or $s_2$, and consequently $x = s_1 s_2 = k$, which is a contradiction since $h$ and $h'$ are distinct.

Suppose now that $k'$ has length one. Again by Lemma 11.2 and the fact that $\Gamma$ is triangle-free, $k'$ cannot consist of a generator distinct from $s_1$ and $s_2$. Thus $k' = s_1$ or $k' = s_2$. By possibly relabeling, we may assume that $k' = s_1$. Now Lemma 11.2 implies that $s_1$ commutes with $z$ and $s_2$. Let $y = k'x = zs_2z^{-1}$. We replace $x$ with $y$ in $I_H$ to form the new set $I_H'$. Note that $I_H'$ is still a generating set for $H$ as $z = k'y$.

Let $b \neq s_1$ be any element of $I_H'$ which commutes with $x$. Then since $x = zs_1s_2z^{-1}$ and $s_1$ commutes with $z$ and $s_2$, Lemma 11.3 implies that $b$ commutes with $y$. It follows that $w'w'^{-1}$ is a generating set for $G$ which satisfies properties (1) and (2) above. Finally, note that the number of reflections in $I_H'$, and hence in $w'w'^{-1}$ is exactly $r + 1$.

By repeating this process enough times, we are guaranteed a finite generating set for $G$ consisting only of reflections. \qed

Now it follows easily from Corollary 11.4 that subgroups of the kind in Theorem 11.4 are quasiconvex:

**Corollary 11.5.** Given a 2-dimensional right-angled Coxeter group, every Coxeter subgroup whose defining graph does not have an isolated vertex is quasiconvex. \qed

**Remark 11.5.1.** Note that the defining graph of a right-angled Coxeter group has an isolated vertex if and only if the group splits as a free product with a $\mathbb{Z}_2$ factor. Some version of the non-splitting hypothesis is required in Theorem 11.4 and Corollary 11.3. For consider the graph $\Gamma_2$ from Figure 2. The subgroup of $W_{\Gamma_2}$ generated by $ab$ and $cd$ is isomorphic to the infinite dihedral group. In particular, it is a right-angled Coxeter subgroup of $W_{\Gamma_2}$. However, it is straightforward to check that this subgroup cannot be generated by reflections and has an infinite completion (so is not quasiconvex by Theorem 8.4).

### 12. Deciding when a right-angled Coxeter group is a finite-index subgroup of a 2-dimensional right-angled Coxeter group

In this section, we provide an affirmative answer to the following question:

**Question 12.1.** Is there an algorithm which, given a 2-dimensional, one-ended right-angled Coxeter group $W_\Gamma$ and any right-angled Coxeter group $W_{\Gamma'}$, determines whether or not $W_\Gamma$ contains a finite-index subgroup isomorphic to $W_{\Gamma'}$?

The result we prove, Theorem 12.11 actually assumes a weaker hypothesis than one-endedness of $W_\Gamma$. We additionally show that the time-complexity of this algorithm only depends on the number of vertices of $\Gamma$ and $\Gamma'$. Furthermore, when such a subgroup does exist, the output of the algorithm is a set of words in $W_\Gamma$ which is a standard right-angled Coxeter group generating set for a subgroup isomorphic to $W_{\Gamma'}$.

Given a set of reflections $R$ which generate a finite-index subgroup of the right-angled Coxeter group $W_\Gamma$, under the right hypotheses, Proposition 12.2 below
bounds the sizes of elements in $\mathcal{R}$ as a function of $|V(\Gamma)|$ and $|\mathcal{R}|$. This proposition is a key step in the proof of the main theorem of this section, as it allows us to bound the number of sets of reflections that need to be investigated by our algorithm.

We define a certain class of graphs in order to state this proposition. Namely, we say a graph $\Delta$ is almost star if there exist vertices $s, t \in \Delta$ (possibly not distinct) such that $V(\Delta) = \text{star}(s) \cup \{t\}$.

**Proposition 12.2.** Let $W_{\Gamma}$ be a right-angled Coxeter group such that $\Gamma$ is triangle-free and is not almost star. Let

$$\mathcal{R} = \{w_is_iw_i^{-1} \mid w_i \in W_{\Gamma}, s_i \in V(\Gamma), 1 \leq i \leq N\}$$

be a trimmed set of reflections which generates a finite-index subgroup $G < W_{\Gamma}$. Then there exists a constant $M$, depending only on $|V(\Gamma)|$ and on $|\mathcal{R}|$ such that $|w_i| \leq M$ for all $w_is_iw_i^{-1} \in \mathcal{R}$.

In order to prove this proposition, we establish some notation and prove some preliminary lemmas. The notation in this discussion will be fixed until the proof of Proposition 12.2 is complete.

Let $M$ be the smallest integer such that if a reduced word $w$ in $W_{\Gamma}$ is longer than $M$, then $w$ contains $2N + 2$ occurrences of some letter of $V(\Gamma)$ (where $N = |\mathcal{R}|$). In particular, $M$ only depends on $|V(\Gamma)|$ and on $|\mathcal{R}|$. This $M$ will be the same as the constant in the proposition.

In order to establish a contradiction, we assume that $|w_1| > M$. By possibly relabeling, we assume that $|w_i| \geq |w_1|$ for all $1 < i \leq N$. By the previous paragraph, we may fix some vertex $\bar{s}$ of $\Gamma$ which occurs at least $2N + 2$ times in the word $w_1$.

We now choose convenient expressions for the elements of $\mathcal{R}$. Firstly, we assume $w_1$ is written in an expression where occurrences of $\bar{s}$ appear as far left as possible. More formally, if $w_1 = s_1\ldots s_m$ and $s_i = \bar{s}$, then for all $j < i$ there is no expression for $w_1$ in $W_{\Gamma}$ equal to the word $s_1\ldots s_{j-1}s_{j+1}\ldots s_is_{j+1}\ldots s_m$.

Given two words $w$ and $w'$ in $W_{\Gamma}$, let $\phi(w, w')$ denote the length of their largest common prefix. For each $2 \leq i \leq N$, we choose an expression for $w_i$ so that $\phi(w_1, w_i)$ is maximal out of all such possible choices for $w_i$. Clearly, there is no loss of generality in making these assumptions on $w_1, \ldots, w_N$.

We will now use the notation established in Section 10.2 associated to a reflection subgroup of a right-angled Coxeter group. As in that section, we have the based $\Gamma$-labeled complex $(\Omega_0, B)$, the labeled tree $T \subset \Omega_0$ and the associated folded based labeled tree $(\mathcal{F}T, B)$. Furthermore, $(\Omega_{\mathcal{FT}}, B)$ is a based finite CAT(0) cube complex which is a completion of $(\mathcal{F}T, B)$. By Theorem 10.7 there is a completion $(\Omega_G, B)$ of $(\Omega_0, B)$ (which is also a completion for $G$) whose associated completion sequence has length bounded by a function which depends only on $\sum_{\gamma \in \mathcal{R}} |\gamma|$ and $|V(\Gamma)|$. We again denote by $\hat{f}$ the natural map

$$\hat{f} : \mathcal{F}T \to \Omega_{\mathcal{FT}} \subset \Omega_G$$

and recall that $\Omega_{\mathcal{FT}}$ is isometrically embedded in $\Omega_G$.

Let $V$ be the set of vertices in $\mathcal{F}T$ that are the image of a vertex of $T \subset \Omega_0$ which has a graph-loop attached to it. Observe that $|V| \leq N$. Also note that at most $N$ vertices in $\mathcal{F}T$ have valence larger than 2.

Let $\alpha$ be the path in $\mathcal{F}T$ based at $B$ with label $w_1$. As there are at least $2N + 2$ occurrences of $\bar{s}$ in the word $w_1$, there must exist two edges of $\alpha$, say $e_1$ and $e_2$,
each with label \( \bar{s} \) such that every vertex between \( e_1 \) and \( e_2 \) has valence 2 and is not in \( \mathcal{V} \). Let \( \gamma \) be the geodesic in \( \mathcal{FT} \) between \( e_1 \) and \( e_2 \). By possibly passing to a subpath, we may assume that no edge in \( \gamma \) has label \( \bar{s} \). We also assume that \( e_1 \) is closer to \( B \) than \( e_2 \).

We now sketch how a contradiction will be established. As \( G \) is a finite-index subgroup of \( W_\Gamma \), it will follow that \( \Omega_G \) must be full-valence. We then focus on a specific vertex of \( \Omega_G \) which is contained in \( \hat{f}(\gamma) \). We show that the structure of edges and graph-loops incident to this vertex, together with the assumption that \( \Omega_G \) is full valence, must imply that \( \Gamma \) is an almost star graph, contradicting the hypotheses of Proposition 12.2. In order to carry out this argument, we must first gain a solid understanding of the structure of the subcomplex \( \hat{f}(\gamma) \). This is the purpose of the next four lemmas.

**Lemma 12.3.** Let \( v \) be a vertex of \( \gamma \), and let \( v' \) be any vertex in \( \mathcal{FT} \). If \( \hat{f}(v) = \hat{f}(v') \), then \( v = v' \).

*Proof.* Let \( \beta \) be a path in \( \mathcal{FT} \) from the base vertex \( B \) to \( v \), and let \( \beta' \) be a path in \( \mathcal{FT} \) from \( B \) to \( v' \). The label \( l \) of \( \beta \) is a prefix of \( w_1 \). Similarly, the label \( l' \) of \( \beta' \) is a prefix of \( w_i \), for some \( 1 \leq i \leq N \). Let \( \beta = \hat{f}(\beta) \) and \( \beta' = \hat{f}(\beta') \) be the images of these geodesics in \( \Omega_{\mathcal{FT}} \).

As \( \Omega_{\mathcal{FT}} \) is a CAT(0) cube complex, the loop \( \beta' \cup \beta^{-1} \) is homotopic, relative to basepoint, to \( B \). Thus, by Lemma 12.4, the label of \( \beta' \cup \beta^{-1} \) is equal to the identity element in \( W_\Gamma \). It follows that \( l' \) and \( l \) represent the same element of \( W_\Gamma \). However, our choice of \( w_i \) guarantees that \( w_i \) and \( w_1 \) share the largest possible prefix. It follows that \( l \) and \( l' \) are the same word. As \( \mathcal{FT} \) is folded, it follows that \( \beta = \beta' \) and that \( v = v' \). \( \square \)

**Lemma 12.4.** Let \( \beta \) be a path in \( \Omega_{\mathcal{FT}} \) whose label is a reduced word in \( W_\Gamma \). Then \( \beta \) is a geodesic.

*Proof.* Suppose \( \beta \) is not geodesic. As \( \Omega_{\mathcal{FT}} \) is a CAT(0) cube complex, it follows some hyperplane \( K \) is dual to two distinct edges \( k \) and \( k' \) of \( \beta \). Furthermore, we can choose \( K \), \( k \) and \( k' \) so that every hyperplane dual to an edge of \( \beta \) between \( k \) and \( k' \) intersects \( K \). However, from this it readily follows that the label of \( k \) (and of \( k' \)) commutes with the label of any edge of \( \beta \) between \( k \) and \( k' \). This implies that the label of \( \beta \) is not reduced, a contradiction. \( \square \)

For the next two lemmas, let \( \hat{e}_1 = \hat{f}(e_1) \) and \( \hat{e}_2 = \hat{f}(e_2) \). Furthermore, let \( H_1 \) and \( H_2 \) be the hyperplanes in \( \Omega_{\mathcal{FT}} \) that are dual respectively to \( \hat{e}_1 \) and \( \hat{e}_2 \).

**Lemma 12.5.** The edge \( \hat{e}_1 \) is the only edge of \( \hat{f}(\mathcal{FT}) \) that is dual to \( H_1 \). Similarly, the edge \( \hat{e}_2 \) is the only edge of \( \hat{f}(\mathcal{FT}) \) that is dual to \( H_2 \).

*Proof.* We prove the claim for \( H_1 \). The proof is analogous for \( H_2 \). Let \( \hat{e} \) be an edge in \( f(\mathcal{FT}) \subset \Omega_G \) dual to \( H_1 \). We will show that \( \hat{e} = \hat{e}_1 \).

As \( \mathcal{FT} \) is a tree, if the edge \( e_1 \) is removed (but its endpoints are not removed), there are exactly two resulting components. We let \( C_B \) denote the component that includes the vertex \( B \), and let \( C_B \) be the component which does not.

Let \( e \) be an edge of \( \mathcal{FT} \) such that \( \hat{f}(e) = \hat{e} \). We first claim that \( e \) cannot be in \( C_B \). For suppose otherwise. It follows that \( e \) and \( e_1 \) are contained in a common path \( \eta \) with reduced label \( w_j \) for some \( 1 \leq j \leq N \). By Lemma 12.4 \( \hat{f}(\eta) \) is a
geodesic in $\Omega_{FT}$. However, it now follows that the hyperplane $H_1$ is dual to two edges of a geodesic, contradicting the fact that a hyperplane in a CAT(0) cube complex is dual to at most one edge of a geodesic.

Thus, we may assume that either $e \in C_B$ or $e = e_1$. Let $\beta$ be a geodesic in $FT$ from $e_1$ to $e$, which includes these two edges. Due to the tree structure of $FT$, it follows that the label of $\beta$ is $k_1^{-1}k_2$, where $wk_1$ is a prefix of $w_1$ and $wk_2$ is a prefix of $w_j$ for some $1 \leq j \leq N$ and some reduced word $w$. Note that $w, k_1$ and $k_2$ could each be the empty word. Let $\hat{\beta} = \hat{f}(\beta)$ be the corresponding path in $\Omega_{FT}$.

Let $z$ be a geodesic along the carrier of $H_1$ from the endpoint to start point of $\hat{\beta}$. Let $z$ be the label of $\hat{\beta}$. Let $D$ be a disk diagram in $\Omega_{FT}$ with boundary $\hat{\beta} \cup \hat{\zeta}^{-1}$. The label of the boundary of $D$ is $k_1^{-1}k_2z$. Let $b_1$ and $b_2$ be the paths along $\partial D$ with labels respectively $k_1$ and $k_2$. Let $c$ the path along $\partial D$ labeled by $z$. Write the label of $b_1$ as $k_1 = s_1s_2\ldots s_m$, where $s_i \in V(\Gamma)$ for $1 \leq i \leq m$. Note that $s_m = \hat{s}$. For $1 \leq l \leq m$, let $d_l$ be the edge of $b_1$ with label $s_l$.

We claim that no dual curve is dual to both $b_1$ and $c$. For suppose there is such a dual curve $P$. Further suppose that $P$ intersects the edge $d_r$ such that $r$ is maximal out of such possible choices. Note that $r \neq m$ as $z$ only contains letters in $\text{link}(\hat{s})$, being the label of a geodesic in the carrier of a hyperplane of type $\hat{s}$. It follows that every dual curve dual to $d_l$, for $l > r$, intersects $P$. Let $p \in \Gamma$ be the type of $P$. As $P$ intersects $c$, and the label of $c$ is in $\text{link}(\hat{s})$, it follows that $p \in \text{link}(\hat{s}) \subset V(\Gamma)$. Additionally, $p$ commutes with $s_l$ for every $r < l \leq m$. However, this implies that $s_1s_{r-1}s_{r+1}\ldots s_ms_r$ is an expression for $k_1$. As $k_1$ is a subword of $w_1$, this contradicts our choice of $w_1$ having occurrences of $\hat{s}$ appear as "left-most" as possible. Thus, every dual curve dual to $b_1$ must intersect $b_2$.

By Lemma 2.4, $k_2$ is an expression in $W_T$ for $k_1k$, where $k$ is a possibly empty word in $W_T$. However, it follows from our choice of expression for $w_j$ (where we chose expressions for words in $R$ to have maximal common prefix with $w_1$) that $k_2$ and $k_1k$ are actually equal as words. Consequently $wk_1$ is a prefix of both $w_1$ and $w_j$. Hence, it must be the case that $e = e_1$ and so $\hat{e} = \hat{e}_1$.

**Lemma 12.6.** Let $Y$ be the subcomplex of $\Omega_{FT}$ bounded by $H_1$ and $H_2$. Let $\hat{v}$ be a vertex in $Y$. Then the label of any graph-loop in $\Omega_G$ incident to $\hat{v}$ (where we think of $\hat{v} \in \Omega_{FT} \subset \Omega_G$) is in $\text{link}(\hat{s}) \subset V(\Gamma)$.

**Proof.** Let $u$ be any vertex of $\Omega_0$ that is incident to a graph-loop. By construction, the image of $u$ in $FT$ is not contained in $\gamma$. By Lemma 12.5 and the fact that every vertex of $\gamma$ has valence 2 by construction, $f(FT) \cap Y = \gamma$. Thus either $H_1$ or $H_2$ separates $\hat{v}$ from $\hat{f}(u)$.

Suppose the label of the graph-loop attached to $\hat{v}$ is $t$. By Lemma 9.1 and Remark 10.7.1 there is a path $\eta$ in $\Omega_{FT}$ from $\hat{v}$ to a vertex $\hat{u}'$ such that $\hat{u}' = \hat{f}(\hat{u}')$, where $\hat{u}' \in \Omega_0$ is a vertex that is incident to a graph-loop labeled by $t$. Furthermore, the label of $\eta$ is a word in $\text{link}(t)$. It follows that $\eta$ must intersect either $H_1$ or $H_2$. Hence, the label of $\eta$ contains the generator $\hat{s}$. Thus, $\hat{s} \in \text{link}(t)$ and so $t \in \text{link}(\hat{s})$.

We are now ready to prove the proposition.

**Proof of Proposition 12.2.** Let $e$ be the edge of $\gamma$ adjacent to $e_2$. Let $v$ be the vertex $e \cap e_2$. Let $t$ be the label of $e$. By our choice of $w_1$ (having $\hat{s}$ occurrences appear
“left-most”), \( t \) and \( \bar{s} \) are not adjacent vertices of \( \Gamma \). Set \( \hat{e} = \hat{f}(e), \hat{e}_2 = \hat{f}(v_2), \hat{v} = \hat{f}(v) \) and \( \hat{\gamma} = \hat{f}(\gamma) \).

As \( \Gamma \) is not almost star, it readily follows that \( \text{star}(s) \subseteq V(\Gamma) \) for any \( s \in V(\Gamma) \). In particular, any finite set of words in \( \Gamma \) is resolved. As \( G \) is a finite-index subgroup of \( W_T \) given by a resolved generating set, \( \Omega_G \) is full valence by Corollary 6.10.

Again, as \( \Gamma \) is not almost star, there exists a vertex \( a \in \Gamma \) such that \( a \neq t \) and \( a \notin \text{star}(\bar{s}) \). As \( \Omega_G \) is full valence, there must exist an edge \( \hat{d} \) adjacent to \( \hat{v} \) labeled by \( a \). By Lemma 12.6 \( \hat{d} \) is not a graph-loop. Let \( \hat{u}' \) be the vertex of \( \hat{d} \) which is not equal to \( \hat{v} \). As \( \Omega_G \) is full valence, there must exist an edge \( \hat{d}' \) adjacent to \( \hat{u}' \) with label \( \bar{s} \) which is not a graph-loop by Lemma 12.6. Let \( H \) be the hyperplane dual to \( \hat{d}' \).

First note that \( \hat{d}' \) cannot be dual to \( H_2 \), for otherwise it would follow from the convexity of \( N(H_2) \) that \( \hat{d} \subset N(H_2) \), contradicting the fact that \( a \) is not in \( \text{star}(\bar{s}) \). Furthermore, \( \hat{d}' \) cannot be dual to \( H_1 \) either. For otherwise, as \( \hat{\gamma} \) is geodesic (by Lemma 12.4), it follows that the hyperplane dual to \( \hat{e} \) must intersect \( H_1 \), contradicting the fact that \( t \) is not in \( \text{link}(\bar{s}) \). Thus, \( H \neq H_1 \) and \( H \neq H_2 \).

By Proposition 7.4, \( H \) must intersect \( \hat{f}(FT) \subseteq \Omega_G \). As \( \hat{\gamma} \) does not have any edges labeled by \( \bar{s} \), it follows that \( H \) cannot intersect \( \hat{\gamma} \). Thus, by Lemma 12.5 \( H \) must intersect either \( H_1 \) or \( H_2 \). However, this is a contradiction as \( H, H_1 \) and \( H_2 \) are all of type \( \bar{s} \).

Before proving the main theorem, we need to address a special case and recall some known results. The next lemma describes finite-index subgroups of \( W_T \) for the case where \( \Gamma \) is a triangle-free join graph.

Lemma 12.7. Suppose \( \Gamma \) is a triangle-free graph which splits as a join \( \Gamma = A \star B \). Let \( \mathcal{R} \) be a finite set of reduced reflection words in \( W_T \) which generates the subgroup \( G < W_T \). Then \( G \) is a finite-index subgroup of \( W_T \) if and only if \( \mathcal{R} = \mathcal{R}_A \cup \mathcal{R}_B \) such that

1. \( \mathcal{R}_A \) (resp. \( \mathcal{R}_B \)) consists only of words in \( W_A \) (resp. \( W_B \)).
2. \( \mathcal{R}_A \) (resp. \( \mathcal{R}_B \)) generates a finite-index subgroup of \( W_A \) (resp. \( W_B \)).

Proof. The “if” direction is immediate. For the other direction, suppose that \( G \) has finite index in \( W_T \). Let

\[
\mathcal{R}_A = \{ws^{-1}w^{-1} \in \mathcal{R} \mid s \in V(A)\}
\]

and let \( \mathcal{R}_B = \mathcal{R} \setminus \mathcal{R}_A \). Note that if \( ws^{-1} \in \mathcal{R}_A \), then \( w \) does not have a letter in \( V(B) \). For if it did, \( ws^{-1} \) would not be reduced. Similarly, every reflection in \( \mathcal{R}_B \) does not contain a letter of \( V(A) \). This shows (1).

Thus, the subgroup \( G_A \) generated by \( \mathcal{R}_A \) is a subgroup of \( W_A \), and the subgroup \( G_B \) generated by \( \mathcal{R}_B \) is a subgroup of \( W_B \). As \( G \) has finite index in \( W_T \), it must be that \( G_A \) and \( G_B \) are finite-index subgroups respectively of \( W_A \) and \( W_B \).

Next we recall a theorem proven independently by Deodhar and by Dyer which states that reflection subgroups of Coxeter groups are Coxeter groups themselves.

Theorem 12.8 (Dyer90, Deo89). Let \( W \) be a Coxeter group and \( G < W \) a subgroup generated by reflections. Then \( G \) is a Coxeter group.

The following corollary follows in the setting of right-angled Coxeter groups.
Corollary 12.9. Let \( W \) be a right-angled Coxeter group and \( G < W \) a subgroup generated by a set \( R \) of reflections. Then \( G \) is a right-angled Coxeter group. Furthermore, if \( R \) is trimmed, then it is a standard Coxeter generating set.

Proof. By Theorem 12.8 and Proposition 2.1, \( G \) is a right-angled Coxeter group.

The second claim follows from the main theorem of [Dye90] and [Dye90, Proposition 3.5]. We also refer the reader to [Dye90, page 69] for an algorithm to determine a standard generating set for a reflection subgroup of a Coxeter group. □

The next proposition describes a trick to produce an index two right-angled Coxeter subgroup of a right-angled Coxeter group. We will use this trick in one of the cases in the proof of Theorem 12.11. This proposition is well known to experts. However, we do not know of a proof in the literature, so we provide one here.

Before stating the proposition, we define a certain graph operation. Namely, given a graph \( \Gamma \) and a vertex \( v \in \Gamma \), let \( \Delta \) be the subgraph of \( \Gamma \) induced by \( V(\Gamma) \setminus v \). We define \( D(\Gamma, v) \) to be the graph consisting of the union of two copies of \( \Delta \) which are identified along the subgraph of \( \Delta \) induced by \( \text{link}(v) \).

Proposition 12.10. Given any graph \( \Gamma \) and vertex \( s \in V(\Gamma) \), let \( \phi_s : W_\Gamma \to \mathbb{Z}_2 = \{1, a\} \) be the homomorphism defined by \( \phi_s(s) = a \) and \( \phi_s(t) = 1 \) for all \( t \in V(\Gamma) \) such that \( t \neq s \). Let \( K \) be the kernel of \( \phi_s \). Then \( K \) is generated by reflections and is isomorphic to the right-angled Coxeter group \( W_{D(\Gamma, s)} \).

Proof. Fix a vertex \( s \in \Gamma \) and set \( \phi = \phi_s \). We compute the kernel \( K \) of the map \( \phi \). For any word \( w \) in \( W_\Gamma \), it follows that \( \phi(w) = 0 \) if and only if \( w \) contains an even number of occurrences of the letter \( s \). Let \( R_1 = V(\Gamma) \setminus s \) and \( R_2 = \{sts \mid t \in R_1\} \). It readily follows that \( K \) can be generated by the set of reflections \( R = \{R_1 \cup R_2\} \). Thus \( K \) is a right-angled Coxeter group by Corollary 12.9.

Let \( \Delta \) be the graph whose vertices correspond to reflections in \( R \) and whose edges correspond to commuting reflections. By the condition described in [Dye90, Introduction], \( K \) is isomorphic to \( W_\Delta \). We now leave to the reader to check that \( \Delta \) is isomorphic, as a graph, to \( D(\Gamma, v) \). □

We are now ready to prove the main theorem of the section.

Theorem 12.11. There is an algorithm which, given a 2-dimensional right-angled Coxeter group \( W_\Gamma \) and a right-angled Coxeter group \( W_{\Gamma'} \), such that \( \Gamma' \) does not have an isolated vertex, determines whether or not \( W_{\Gamma'} \) is isomorphic to a finite-index subgroup of \( W_\Gamma \). The algorithm takes as input the graphs \( \Gamma \) and \( \Gamma' \), and the time-complexity of this algorithm only depends on the number of vertices of \( \Gamma \) and of \( \Gamma' \). Furthermore, if \( W_{\Gamma'} \) is isomorphic to a finite-index subgroup of \( W_\Gamma \), then the algorithm outputs an explicit set of words in \( W_\Gamma \) which generate this subgroup.

Proof. First observe that if there does exist some subgroup \( G \) of \( W_\Gamma \) that is isomorphic to \( W_{\Gamma'} \), then by Theorem 11.4 and Lemma 10.4, \( G \) is generated by a trimmed set of reflections \( R \) and \( |R| = |V(\Gamma')| \).

We prove the theorem by analyzing a few different cases depending on the structure of the graph \( \Gamma \).

(i) \( \Gamma \) is not almost star

Let \( I \) be a trimmed set of reduced reflections in \( W_\Gamma \). We say \( I \) is \( M \)-admissible if \( |I| = |V(\Gamma')| \) and \( |w| \leq M \) for every reflection \( wsw^{-1} \in I \). Let \( I_M \) be the collection
of all $M$-admissible trimmed sets of reflections. Note that there is a bound on $|I_M|$ depending only on $M$ and $|V(\Gamma')|$.

Suppose $G$ is a finite-index subgroup of $W_\Gamma$ which is isomorphic to $W_\Gamma'$, and let $\mathcal{R}$ be a trimmed generating set for $G$ as described in the first paragraph of the proof. As $\Gamma$ is not almost star, Proposition $[12.2]$ guarantees that $\mathcal{R} \in I_M$ where $M$ is as in that proposition and depends only on $|V(\Gamma)|$ and $|V(\Delta)| = |\mathcal{R}|$. It follows that there exists a finite-index subgroup of $W_\Gamma$ isomorphic to $W_\Gamma'$ if and only if some $I \in I_M$ generates a finite-index subgroup that is isomorphic to $W_\Gamma'$.

Thus, to prove the theorem we only need to show that there is an algorithm to decide whether a given $I \in I_M$ generates a finite-index subgroup $G$ isomorphic to $W_\Gamma'$. By Theorem $[10.7]$ and Corollary $[6.10]$ there is an algorithm to decide whether or not such a $G$ is a finite-index subgroup, and the time-complexity of this algorithm only depends on $|V(\Gamma')|$ and $|V(\Gamma)|$. By Corollary $[12.9]$ $G$ is a right-angled Coxeter group and $I$ is a standard Coxeter generating set. As any right-angled Coxeter group is defined by a unique graph $[Rad03]$, it is straightforward to check whether $I$ generates a right-angled Coxeter group isomorphic to $W_\Gamma'$. The theorem then follows in this case.

(ii) $|V(\Gamma)| \leq 2$

It easily follows that $W_\Gamma$ is isomorphic to either $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 * \mathbb{Z}_2$. In each case, such a group contains a finite number of finite-index right-angled Coxeter subgroups, up to isomorphism, and one can easily list which these are.

(iii) $\Gamma = A * B$ where $|V(A)|, |V(B)| \geq 2$

Suppose $G$ is a subgroup of $W_\Gamma$ which is isomorphic to $W_\Gamma'$. Let $\mathcal{R}$ be a trimmed generating set of reflections for $G$ as in the first paragraph of this proof.

If $G$ is a finite-index subgroup, then Lemma $[12.7]$ tells us that $\mathcal{R} = \mathcal{R}_A \cup \mathcal{R}_B$ where every reflection in $\mathcal{R}_A$ only contains generators in $A$ and every reflection in $\mathcal{R}_B$ only contains generators in $B$. Furthermore, $\mathcal{R}_A$ generates a finite-index subgroup of $W_A$ and $\mathcal{R}_B$ generates a finite-index subgroup of $W_B$. As $|V(A)|, |V(B)| \geq 2$, both $\mathcal{R}_A$ and $\mathcal{R}_B$ are non-empty.

Let $\Delta = \Delta_A * \Delta_B$ be the triangle-free join graph such that vertices of $\Delta_A$ correspond to elements of $\mathcal{R}_A$ and vertices of $\Delta_B$ correspond to elements of $\mathcal{R}_B$. It readily follows that $W_\Gamma'$ is isomorphic to the right-angled Coxeter group $W_\Delta$.

Thus we can assume that $\Gamma'$ is a join graph, $\Gamma' = A' * B'$. To prove the claim, again by Lemma $[12.7]$ it is enough to check whether $W_A'$ is isomorphic to a finite-index subgroup of $W_A$ and whether $W_B'$ is isomorphic to a finite-index subgroup of $W_B$. However, as $\Gamma$ is triangle-free, $A$ does not contain any edges. It follows that either $A$ is not almost star or $A$ consists of at most two isolated vertices. The same holds for $B$. Thus, we are done by cases (i) and (ii).

(iv) $\Gamma$ is not as in (i), (ii) or (iii)

As $\Gamma$ is not as in (i), we may assume that $\Gamma$ is almost star.

Suppose first that $V(\Gamma) \subseteq \text{star}(v)$ for all $v \in V(\Gamma)$. Let $s$ and $t$ be vertices of $\Gamma$ such that $V(\Gamma) = \text{star}(s) \cup \{t\}$. Note that $s$ and $t$ must be distinct in this case.

As $\Gamma$ is triangle-free, is not a join as in (iii), is not the star of a vertex, and contains more than two vertices, there must be a vertex $u$ of $\Gamma$ that is adjacent to $s$ and is not adjacent to any other vertex of $\Gamma$. 

Let $G$ be a subgroup of $W_\Gamma$ generated by reflections. Let $\phi = \phi_u : W_\Gamma \to \mathbb{Z}_2$ be the homomorphism described in Proposition 12.10 let $\Delta = D(\Gamma, u)$ and let $K = \ker(\phi)$. Note that $\Delta$ is not almost star and that $K$ is generated by reflections.

Let $i_G : G \to W_\Gamma$ be the inclusion map. Let $K' = \ker(\phi') = \phi \circ i_G$. We get the diagram below where all maps labeled by $i$ are the obvious inclusion homomorphisms.

Recall that given any group $G_1$, a subgroup $G_2$ of $G_1$ and a subgroup $G_3$ of $G_2$ then their indices satisfy the formula $[G_1 : G_3] = [G_1 : G_2][G_2 : G_3]$ where infinite values are interpreted appropriately. If we apply this formula to the groups in the above diagram and note that $[G : K'] = [W_\Gamma : K] = 2$, we get:

$$2[W_\Gamma : G] = [W_\Gamma : G][G : K'] = [W_\Gamma : K'] = [W_\Gamma : K][K : K'] = 2[K : K']$$

Thus, $G$ is a finite-index subgroup of $W_\Gamma$ if and only if $K'$ is a finite-index subgroup of $K$. As $\Delta$ is not almost star, it follows by (i) that there is an algorithm to check whether $K'$ is a finite-index subgroup of $K$. The theorem now follows.

On the other hand, suppose that $V(\Gamma) = \text{star}(s)$ for some vertex $s \in V(\Gamma)$. In this case we apply the same argument as before but instead take the homomorphism $\phi = \phi_s$. In this case, $\Delta = D(\Gamma, s)$ must be a non-empty graph with no edges. It thus follows by either (i) or (ii) that there is an algorithm to check whether $K'$ is a finite-index subgroup of $K$, where $K'$ and $K$ are defined as before. □

13. Other algorithmic properties of quasiconvex subgroups

This section is dedicated to the proof of Theorem 13.1 which gives several algorithmic results for quasiconvex subgroups of right-angled Coxeter groups.

We remark that the existence of algorithms for (1) and (4) in Theorem 13.1 below for quasiconvex subgroups of automatic groups (which include right-angled Coxeter groups) are known by work of Kapovich [Kap96, Lemma 2] and Kharlampovich–Miasnikov–Weil [KMW17] respectively. We include proofs here anyway, as it also easily follows from what we have already shown.

**Theorem 13.1.** Let $G$ be a quasiconvex subgroup of a right-angled Coxeter group $W_\Gamma$ given by a finite generating set of words in $W_\Gamma$. Then there exist finite-time algorithms to solve the following problems.

1. (Membership Problem) Given a word $w$ representing an element $g \in W_\Gamma$, determine whether or not $g \in G$.
2. Given a word $w$ representing an element $g \in W_\Gamma$, determine whether or not a positive power of $g$ is in $G$.
3. Determine whether or not $G$ is torsion-free.
4. Determine the index of $G$ in $W_\Gamma$ (even if infinite).
5. Determine whether or not $G$ is normal.
Let $\Omega$ be a standard completion of $G$. By Theorem 8.3, $\Omega$ is finite and by Proposition 3.5 it can be computed in finite time. We fix this notation throughout the rest of the section.

The proof of (1), (3) and (4) follow from work we have already done:

**Proof of (1), (3) and (4):** Let $w$ be a reduced word representing some element $g \in W_G$. It follows from the definition of a completion, that $g \in G$ if and only if there exists a loop in $\Omega$ based at $B$ with label $w$. This shows (1). The claims (3) and (4) follow respectively by Proposition 4.6 and Corollary 6.10. □

Before proving (2), we prove that powers of an element of a right-angled Coxeter group can be represented by words of a special form.

**Lemma 13.2.** Let $w$ be a reduced word in the right-angled Coxeter group $W_G$. Then there exist reduced words $x, h$ and $k$, such that $xhkx^{-1}$ is a reduced expression for $w$ and $xh^nk^{(n \mod 2)}x^{-1}$ is a reduced expression for $w^n$ for all integers $n > 0$.

**Proof.** Write $w = xyz^{-1}$ where $x$ and $y$ are reduced words and $|x|$ is maximal out of all such possible expressions. Let $K = \{k_1, \ldots, k_n\}$ be the set of vertices in $\Gamma$ that appear as letters in the word $y$ and which commute with every other letter of $y$. As $w$ is reduced, each element of $K$ appears as a letter of $w$ exactly once. Define the word $k = k_1 \ldots k_n$. By our choice of $k$, it follows that there exists a reduced word $h$, such that $hk$ is a reduced expression for $y$. Note that $h$ has the property that any generator which appears as the last letter of some reduced expression for $h$, cannot also appear as the first letter in some reduced expression for $h$. This follows since otherwise either $x$ is not maximal or such a generator should have been in $K$. The word $xhkx^{-1}$ will be the desired expression for $w$.

The word $xh^nk^{(n \mod 2)}x^{-1}$ is clearly an expression for the word $w^n$. Furthermore, the word $xh^nk^{(n \mod 2)}x^{-1}$ must be reduced. For otherwise, it follows by Proposition 2.2 that either $h$ is not reduced or that some generator appears as the first letter in some reduced expression for $h$ and as the last letter in some reduced expression for $h$, which we know is not possible. □

**Proof of (2):** Without loss of generality, we may assume that $w$ is reduced. Let $N$ be the number of vertices of $\Omega$. We claim that $g^m \in G$ for some positive integer $m$ if and only if $w^l$ represents an element of $G$ for some $l \leq N$. The theorem clearly follows from this claim and the fact that the membership problem is solvable for $G$.

One direction of the claim is clear. On the other hand, suppose $g^m \in G$ for some positive integer $m$. By possibly taking a power, we may assume that $m$ is even. By Lemma 13.2, there is a reduced expression for $w^m$ of the form $z = xh^mx^{-1}$. Let $\beta$ be a loop in $\Omega$ based at $B$ with label $z$. For $1 \leq i \leq m$, let $\alpha_i$ be the first subpath of $\beta$ with label $h^i$, and let $v_i$ be the endpoint of $\alpha_i$. As $\Omega$ is folded, the two subpaths of $\beta$ labeled $x$ are identified. It follows that $\alpha_m$ is a loop based at $v_m$. As there are at most $N$ vertices of $\Omega$, it follows that the set $\{v_1, \ldots, v_m\}$ contains at most $N$ distinct vertices. There must then exist some loop $\alpha$ based at $v_m$ with label $h^l$ for some $l \leq N$. Thus, if we replace $\alpha_m$ with $\alpha\alpha$ in $\beta$, we conclude that there is a loop in $\Omega$ based at $B$ with label $xh^{2l}x^{-1}$. By the definition of a completion, the word $xh^{2l}x^{-1}$, which is an expression for $w^{2l}$, represents an element of $G$. This proves the claim. □

We would like to use Theorem 5.5 to prove (5). However, the core of a completion may be difficult to algorithmically compute in general. Thus, for finitely generated
subgroups, we can give a different characterization of normality which is better suited to the algorithmic approach.

**Proposition 13.3.** Let $G$ be a subgroup of $W_\Gamma$ generated by a finite set of reduced words $S_G$, and let $\Omega$ be a standard completion for $G$ with respect to $S_G$. Consider the following subset of $V(\Gamma)$:

$$\Delta = \{ s \in V(\Gamma) \mid s \text{ commutes with every element of } G \}$$

Then $G$ is normal if and only if the following hold:

(N1) Given any $s \in V(\Gamma) \setminus \Delta$, there is an edge in $\Omega$ incident to $B$ with label $s$.

(N2') For every generator $w \in S_G$ of $G$, and for every vertex $v$ of $\Omega$, there exists a loop based at $v$ with label $w$.

**Proof.** If $G$ is normal, then Theorem 5.5 implies N1 and N2'. On the other hand, suppose N1 and N2' hold. As in the proof of Theorem 5.5, to show that $G$ is normal it is enough to show that $sGs \subseteq G$ for $s \in V(\Gamma) \setminus \Delta$. Let $v$ be the vertex which is adjacent to $B$ via an edge labeled $s$, which exists due to N1. Then by Lemma 4.5, the subgroup of $W_\Gamma$ associated to $(\Omega, v)$ is $sGs$, and by N2', $G$ is contained in this subgroup. Conjugating by $s$, it follows that $sGs \subseteq G$. \qed

Finally, we are ready to show (5):

**Proof of (5):** Let $S_G$ be a finite set of reduced words in $W_\Gamma$ which generate $G$. Since $S_G$ consists of reduced words and generates $G$, the set $\Delta$ from Proposition 13.3 is equal to the following set:

$$\{ s \in V(\Gamma) \mid \forall w \in S_G, s \text{ commutes with every letter in the support of } w \}$$

Thus $\Delta$ can be computed in finite time. It now follows that the conditions N1 and N2' can be checked in finite time as well. \qed

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