MODELLING AND COMPUTATION OF OPTIMAL MULTIPLE INVESTMENT TIMING IN MULTI-STAGE CAPACITY EXPANSION INFRASTRUCTURE PROJECTS

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(Communicated by Song Wang)

Abstract. So far, the optimal investment timing to maximize the total profit of multi-stage capacity expansion infrastructure projects is not clear. In the case of uncertain demands, the optimal multiple stopping time theory is adopted to model the optimal decision-making of investment timing for multi-stage expansion infrastructure projects in a finite time horizon. In this context, the first-stage of the project involves a dedicated asset investment for later expansion, and the capacity of the project at each stage is constrained, which makes the cash flow of the project exhibit the characteristic of bull call spread. The upwind finite difference method and multi-least squares Monte Carlo simulation are combined to solve the project value and determine the optimal exercise boundaries at all stages described by a sequence of demand thresholds. A multi-stage power plant project is taken as an example to validate the model. Through the example, the optimal investment strategies and the value of the multi-stage project are provided; the effects of the dedicated asset and capacity constraint are illustrated. This study novelly reveals the effect of the capacity constraints on the project value using the bull call spread theory.

1. Introduction. Investment in infrastructure projects is usually characterized by high cost and high uncertainty associated with the future environment. A method which can adapt to these characteristics is through multi-stage investment in small scale construction at the start with options to expand. The multi-stage capacity expansion investments can be found in many infrastructure projects, such as waste water treatment [14, 23], power plant [24], and airport [22]. The advantages of multi-stage capacity expansion investments lie in two aspects. First, it can effectively reduce project risk and improve project value [5]. If carried out in stages, it may create value for those projects that generate negative net present value (NPV) [13]. Second, it is cost-effective because it can keep the initial investment cost low [5],

2020 Mathematics Subject Classification. Primary: 49J20, 65C05, 65M06.
Key words and phrases. Multi-stage infrastructure projects, investment timing, limited capacity, optimal multiple stopping time, finite difference, multi-least squares Monte Carlo simulation.

This project was supported by the Scientific Research Plan of Tianjin Municipal Education Commission (2017SK076).

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and the good operation of early-stage projects may reduce the financing cost of later-stage project [21].

An important characteristic of multi-stage capacity expansion projects is that investment in dedicated assets is prevailing [14]. Here, dedicated assets refer to up-front or intermediate investments used for future expansions [30]. Unless future expansions are implemented, these investments remain idle [15]. A question is that how the dedicated assets influence the subsequent expansion decisions and project value. Huang and Pi [14] investigate the impact of dedicated asset investments on the value of multi-stage build-operate-transfer (BOT) projects, suggesting that dedicated asset investments may or may not create values depending on whether future expansions could be justified and realized. Zhao and Tseng [31] devote their study on the foundation of a multiple level parking garage, which is an up-front cost for later expansion and can be viewed as a dedicated asset, and it is claimed that the foundation can bring a flexibility to future expansion.

Another basic feature of multi-stage capacity expansion projects is the flexibility of investment decision-making, such as abandoning the option of expansion, the flexibility of investment time and capacity [25]. Marzouk and Ali [23] discuss two cases of multi-stage wastewater treatment plants with or without expansion obligations, and explore the effect of abandoned expansion on the minimum flow guarantee value. Under the right to abandon later investments, Huang and Pi [14] investigate the pricing of multi-stage BOT projects using a sequential compound option method. By applying real option model, Zhao and Tseng [31] evaluate the value of flexible capacity expansion of public parking garage. Neufville et al. [5] focus on flexible capacity expansion for a multistory parking garage and value the expansion options through a spread sheet approach. For an existing two-lane road, Krüger [16] analyzes the optimal expansion option from two possible capacity and four investment times in 20 years, and examine the real options of the three-lane intermediate road. In summary, the current research in flexible investment decision-making mainly focuses on flexible capacity expansion or abandonment rights and their effects on multi-stage investment value. To the best of our knowledge, few research is devoted to examine the timing of flexible investment from the perspective of maximizing the total revenue at all stages.

In practice, the forecast demand in infrastructure projects is often inaccurate and uncertain [10, 28]. To cope with this uncertainty, it is crucial to determine a suitable investment timing in infrastructure [7, 11]. Although the rationality that the investment timing should be optimized according to the changing demand has been widely recognized, previous studies on investment timing seem to focus on single investment.

Dahlgren and Leung [3] novelly investigate multiple optimal investment timing over an infinite time horizon for multi-stage small-scale infrastructure with short lifespan. In Dahlgren and Leung’s paper, two consecutive investments are separated by the capital lifetime, in this setting, only one stage project is in operation process at any given moment. Li et al. [18] explore the optimal expansion timing for multi-stage PPP (public private partnership) projects in which the government subsidy is involved; in this work, the timing of investment at the first stage of the project is fixed beforehand, and the operational period of PPP projects is short run. This paper extends the work of Li et al. [18] to consider the general infrastructure projects in which the operational period is long run and the timing of investment
at each stage needs to be optimized. Besides, the extension provides new insights in the impact of capacity constraint on the project value.

In this paper, we study optimal investment timing of multi-stage capacity expansion infrastructure projects over a finite time planning horizon under uncertain demands. It is assumed that the first-stage project involves a dedicated asset for later expansion, and that the capacity of each stage is limited. In addition, the investor can flexibly abandon future investments according to demand conditions. In this study, we attempt to answer the following research questions: (i) to maximize the project value at multiple stages, what is the optimal multiple investment timing in a finite time horizon? (ii) how does the dedicated asset in the first-stage project influence the subsequent investment strategies and the project value? (iii) what is the impact of capacity constraints on project value? (iv) what is the impact of uncertainty in volatility measurement on project value?

First, the optimal multiple stopping time theory driven by multi-exercise American options is adopted to establish a decision-making model for investment timing of multi-stage capacity expansion infrastructure projects. Then, the cash flow of the project and the expected return of each stage are analyzed theoretically, showing that the cash flow of the project accords with the characteristics of bull call spread, and the expected returns are governed by differential equations. Secondly, the finite difference method is used to calculate the expected return at each stage and then the multi-least squares Monte Carlo simulation (MLSM) is introduced to determine the optimal exercise boundaries at all stages and the project value. Finally, a hypothetical multi-stage power plant project is taken as an example to validate the model. Through the example, the project value obtained from multiple flexible investment timing is compared with the project value of single investment, and the optimal exercise boundaries described by a sequence of critical demand thresholds are given. In addition, the impact of dedicated assets on investment strategy and project value is also examined. Furthermore, the influence of capacity constraints is demonstrated. It is found that, due to the capacity constraint, in the case of sufficient demand the project value decreases with the increase of the demand volatility. This result is different from general one, and is further analyzed using the option spread theory.

Our model has three difficulties in calculation. First, the expected returns at all stages are path dependent, and governed by a system of differential equations. Second, the investment timing is in a finite time horizon, meaning that the pricing of the investments is not given by the analysis characterization in a perpetual American option, but by a numerical algorithm. Third, since the investment decision-making belongs to a multiple stopping time problem, the optimal strategies are more intricate than that of single stopping time.

The contributions of this study are mainly comprised of three aspects. First, the flexible investment timing of multi-stage capacity expansion infrastructure projects is investigated in a finite horizon time framework. So far, this topic has seldom been studied; therefore, investigation on it can enrich the investment theory of multi-stage infrastructure projects. Second, for limited capacity projects, this paper reveals the impact of capacity constraints on the variation of project value with respect to volatility, and uses the bull spread theory to explain the abnormal characteristics of the fluctuation of project value. Third, theoretical analysis and numerical algorithms are presented for complicated calculations.
The rest of the paper is as follows. In Section 2, the model for multi-stage investment timing decision-making is developed by using the optimal multiple stopping time theory, and the cash flow of the project and the expected return at each stage are analyzed. In Section 3, the finite difference method is used to solve the system of difference equations describing the expected return at each stage, and the least squares Monte Carlo simulation is presented to solve the optimal investment decision. To verify the constructed model, a hypothetical example of three-stage power plant is considered in Section 4. Finally, Section 5 concludes the paper with some concluding remarks.

2. The model.

2.1. Model formulation. An $N$-stage infrastructure project that produces a single good is scheduled to be invested in the $[0, T]$ period. After the completion of the construction of each stage, the corresponding stage of the project will enter into the operating period, during which the investor obtains income to recover investment cost. If the construction of the $i$-th ($i = 1, 2, \ldots, N$) stage project begins at time $t_i$, the construction period of each stage is assumed to be $\nu$, then the operation begins from $t + \nu$.

During the operating period, the demand for the output faces high uncertainty. Since the geometric Brownian motion (GBM) has been successfully used to model the evolution in the demand of infrastructure projects [11, 1], hence it works for the current study. The GBM describing the stochastic demand $X_s$ is shown as follows.

\begin{equation}
\forall t \in [0, T], s \geq t:
\begin{cases}
    dX_s = \alpha X_s ds + \sigma X_s dB_s \\
    X_t = x
\end{cases}
\end{equation}

where $\alpha$ is the drift, $\sigma$ is the volatility, $dB_s$ is an increment of the standard Brownian motion, which means that $dB_s \sim N(0, ds)$, and $x$ is the demand at time $t$.

The capacity at each stage of the multi-stage project is limited, and the planned capacity of the $i$-th item is denoted by $m_i$. The operation load of the $i$-th stage is assumed to be the remaining demand after the full-load operation of the previous projects. In other words, the operation load of the $i$-th stage at time $s$, denoted by $\Omega_i(X_s)$, is the difference between the demand $X_s$ and the total loads of the previous projects, and capped by the capacity $m_i$, that is,

\begin{equation}
\Omega_i(X_s) = \min\left\{ (X_s - \sum_{k=1}^{i-1} m_k)^+, m_i \right\}
\end{equation}

where $(\cdot)^+ = \max\{\cdot, 0\}$.

Let $p$ and $c$ denote the output price and the unit operation cost, respectively. The cash flow of the $i$-th stage at time $s$, represented by $f_i(X_s)$, is given by:

\begin{equation}
f_i(X_s) = (p - c) \cdot \Omega_i(X_s)
\end{equation}

The cumulative cash flow in the long run is the expected return. Then, the expected return of the $i$-th project implemented with the given demand $x$ at time $t$, expressed by $u_i(x)$, is:

\begin{equation}
u_i(x) = E^{t,x} \left[ \int_{t+\nu}^{+\infty} f_i(X_s)e^{-\gamma(s-t)} ds \right]
\end{equation}
Here, \( r \) refers to the discount rate, and \( E^{t,x}[\cdot] \) represents the conditional expectation given the demand \( x \) at time \( t \), which means that its value depends on the demand \( x \).

The investment cost is associated with the capacity of the project. Following the research of Dangl [4], the capacity cost at the \( i \)-th stage, which is the cost being matched with the project capacity of the \( i \)-th stage, \( I(m_i) \), is given by:

\[
I(m_i) = \lambda m_i^\beta
\]  

(5)

Here, \( \lambda \) and \( \beta \) are constants, and \( \lambda > 0, 0 < \beta \leq 1 \), and \( \beta < 1 \) means that the marginal costs are decreasing when the installed capacity is increasing, and it can be interpreted as an economy of scale. Some empirical studies suggest that many infrastructure assets, such as port, power plant, do exhibit economies of scale [9, 26].

It is assumed that there exists a dedicated asset investment in the first project. This means that the investment cost of the first stage includes two parts: one is the capacity cost of the first stage, and the other is the cost of the dedicated assets for later stages, which is represented by \( I_d \). That is to say, the investment cost of the first stage, \( K_1 \), is the sum of \( I(m_1) \) and \( I_d \): \( K_1 = I(m_1) + I_d \). Since a portion of investment of later expansions has been implemented as dedicated assets in the first stage, the investment cost of a later expansion is reduced. Suppose that the cost for the dedicated assets can be decomposed, and allocated proportionally to the capacity into the subsequent stages. The investment cost for a later expansion, denoted by \( K_i(i = 2, 3, \cdots, N) \), is the difference between the capacity cost and the dedicated asset allocated this stage, namely, \( K_i = I(m_i) - \frac{m_i}{\sum_{k=2}^{m} m_k} I_d \). In summary, the investment cost at each stage, \( K_i(i = 1, 2, \cdots, N) \), is:

\[
K_i = \begin{cases} 
I(m_i) + I_d, & i = 1 \\
I(m_i) - \frac{m_i}{\sum_{k=2}^{m} m_k} I_d, & i = 2, 3, \cdots, N 
\end{cases}
\]  

(6)

The net revenue of the \( i \)-th \((i = 1, 2, \cdots, N)\) stage invested at time \( t \), \( \pi_i(x) \), is:

\[
\pi_i(x) = u_i(x) - K_i
\]  

(7)

The total net revenues of the \( N \)-stage investments is the project value. Obviously, different investment strategies create different project values. The objective is to find \( N \) optimal investment opportunities that could maximize the project value. The optimal investment timing and the project value of the \( N \)-stage investments can be expressed as the following optimal multiple stopping time problem (8):

\[
V = \sup_{\mathcal{T} \in \mathcal{S}(N)} E \left[ \sum_{i=1}^{N} e^{-r\tau_i} \pi_i(X_{\tau_i}) \right]
\]  

(8)

where \( E[\cdot] \) denotes the expectation with respect to a probability measure, \( \mathcal{T} = (\tau_1, \tau_2, \cdots, \tau_N) \) is a stopping time vector, \( \tau_i (i = 1, 2, \cdots, N) \) represents the stopping time of the \( i \)-th stage project, meaning the exercise time of the investment, and \( \mathcal{S}(N) \) is the set of admissible stopping time vectors. It is natural that two consecutive investments needs to be separated by a certain time, that is, a refraction time, denoted by \( \delta \). Therefore, the set of admissible stopping time vectors, \( \mathcal{S}(N) \), is defined by:

\[
\mathcal{S}(N) = \{ \mathcal{T} \in (\tau_1, \tau_2, \cdots, \tau_N) \mid \tau_{i+1} - \tau_i \geq \delta (i = 1, 2, \cdots, N - 1) \}
\]  

(9)
2.2. Model analysis.

2.2.1. Capacity constraint and project cash flow. To elaborate the correlation between the capacity constraint and the cash flow of the investment, we rewrite $\Omega_i(X_s)$ and $f_i(X_s)$ in (2) and (3). Denote $\sum_{k=1}^{i} m_k = M_i$, then the operation load of $i$-th stage project, $\Omega_i(X_s)$, is rewritten as:

$$\Omega_i(X_s) = [(X_s - M_{i-1})^+ - (X_s - M_i)^+]$$

(10)

and the cash flow, $f_i(X_s)$, is rearranged as:

$$f_i(X_s) = (p - c) \cdot [(X_s - M_{i-1})^+ - (X_s - M_i)^+]$$

(11)

where

$$(X_s - M_{i-1})^+ - (X_s - M_i)^+ = \begin{cases} 0, & X_s \leq M_{i-1} \\ X_s - M_{i-1}, & M_{i-1} < X_s < M_i \\ m_i, & X_s \geq M_i \end{cases}$$

Figure 1 is the display of $\Omega_i(X_s)$. It is found that $\Omega_i(X_s)$ is equivalently a payoff of bull call spread, which is well known from most literature on options or derivatives [12]. In options trading, a bull call spread is designed by buying a call option at a lower strike ($M_{i-1}$ in (10)), and selling a call option with the same expiry at a higher strike ($M_i$ in (10)). Compared with buying a simple call option, the bull spread buyer reduces the cost of the spread if the price of the underlying security does not rise as anticipated; whereas the buyer also gives up the additional profits if the underlying security price rises beyond the higher strike. In other words, a bull spread strategy is created to profit from a moderate rise in the price of the underlying security.

In the current study, due to the constrained capacity of each stage, $m_i$, the output of each stage is capped and therefore the cash flow is limited. This means that the investor faces a cash flow with the characteristic of bull call spread. The cap of the cash flow derived by the capacity constraint makes the current work different from general investment literature. In previous studies, the cash flow or the asset value is generally assumed to follow a stochastic process without a ceiling.
2.2.2. Expected return. From equation (4), it is shown that the expected return of the \( i \)-th project invested at time \( t \), \( u_i(x) \), depends on the stochastic demand in the future, thus it is path dependent. By means of the principle of dynamic programming and Itô formula, it can be concluded that \( u_i(x)(i = 1, 2, \cdots, N) \) satisfies the differential equation presented in the following theorem.

**Theorem 2.1.** The expected return of the \( i \)-th project, \( u_i(x)(i = 1, 2, \cdots, N) \), satisfies the differential equation, \( \forall x \in (0, +\infty) \):

\[
\frac{1}{2} \sigma^2 x^2 \frac{d^2 u_i}{dx^2} + \alpha x \frac{du_i}{dx} - ru_i + R_i(x) = 0 \tag{12}
\]

with the boundary conditions:

\[
u_i(0) = 0, \quad u_i(+\infty) = \frac{(p - c)m_i}{r}e^{-rv} \tag{13}\]

where

\[
R_i(x) = (p - c) \cdot \left\{ xe^{-(r - \alpha)v} \Phi(g(\Sigma_{k=1}^{i-1}m_k)) - \Phi(g(\Sigma_{k=1}^{i}m_k)) \right\} - e^{-rv} \left[ (\Sigma_{k=1}^{i-1}m_k) \cdot \Phi(h(\Sigma_{k=1}^{i-1}m_k)) - (\Sigma_{k=1}^{i}m_k) \cdot \Phi(h(\Sigma_{k=1}^{i}m_k)) \right]
\]

\( \Phi \) is the cumulative distribution function of the standard normal distribution, and

\[
g(z) = \frac{\ln(x/z)}{\sigma \sqrt{v}} + \frac{\alpha \sqrt{v}}{\sigma} + \frac{1}{2} \frac{\alpha \sqrt{v}}{\sigma} - \frac{1}{2 \sigma \sqrt{v}}.
\]

**Proof.** By applying the principle of dynamic programming and Itô formula, the differential equations for \( u_i(x)(i = 1, 2, \cdots, N) \) can be easily obtained:

\[
\frac{1}{2} \sigma^2 x^2 \frac{d^2 u_i}{dx^2} + \alpha x \frac{du_i}{dx} - ru_i + E^{t,x} \left[ f_i(X_{t+\nu})e^{-rv} \right] = 0
\]

Denote \( R_i(x) = E^{t,x} \left[ f_i(X_{t+\nu})e^{-rv} \right] \). Then, by the definition of conditional expectation, it can be obtained:

\[
R_i(x) = (p - c) \cdot \left\{ xe^{-(r - \alpha)v} \Phi(g(\Sigma_{k=1}^{i-1}m_k)) - \Phi(g(\Sigma_{k=1}^{i}m_k)) \right\} - e^{-rv} \left[ (\Sigma_{k=1}^{i-1}m_k) \cdot \Phi(h(\Sigma_{k=1}^{i-1}m_k)) - (\Sigma_{k=1}^{i}m_k) \cdot \Phi(h(\Sigma_{k=1}^{i}m_k)) \right]
\]

The detail in the computation of \( R_i(x) \) can be referred to [18]. Thus, the validity of equation (12) is proved, and the boundary conditions (13) can be achieved from the expressions (4) and (11).

\[\square\]

3. Numerical algorithms. We give the detailed computational procedures for solving the optimal multiple stopping time problem (8). First, the finite difference method is used to solve the differential equations (12)-(13) describing \( u_i(x)(i = 1, 2, \cdots, N) \). Then the numerical solution of \( \pi_i(x)(i = 1, 2, \cdots, N) \) for equation (7) can be obtained. On this basis, the multi-least squares Monte Carlo simulation is introduced to solve the optimal multiple stopping time problem (8).

Certainly, the finite difference method and multi-least squares Monte Carlo simulation may be compared with other valid numerical algorithms. For instance, lattice methods, which are usually adopted to price real option for projects [27], can be compared with our numerical methods. This study does not involve the algorithm comparison since it is beyond the scope of research.
3.1. **Upwind finite difference method.** The upwind finite difference scheme [29, 19] is proposed to solve the PDE (12)-(13). Set \( X_{\text{max}} \) is large enough, and let the interval \([0, X_{\text{max}}]\) for the demand be divided uniformly into \( N + 1 \) sub-intervals with \( 0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = X_{\text{max}} \) and interval length \( \Delta x \). For \( \forall n = 1, 2, \cdots, N \), the discrete scheme of (12) is:

\[
    b_{n,n-1} u_i(x_{n-1}) + b_{n,n} u_i(x_n) + b_{n,n+1} u_i(x_{n+1}) = R_i(x_n) \tag{14}
\]

where

\[
    b_{n,n-1} = -\frac{1}{2} \frac{\sigma^2 x^2}{\Delta x^2}
\]

\[
    b_{n,n} = r + \frac{\sigma^2}{\Delta x} + \frac{\sigma^2 x^2}{(\Delta x)^2}
\]

\[
    b_{n,n+1} = -\frac{\alpha x}{\Delta x} - \frac{1}{2} \frac{\sigma^2 x^2}{(\Delta x)^2}
\]

These form an \( N \times N \) linear system for \( u_i = (u_i(x_1), u_i(x_2), \cdots, u_i(x_N))^T \) and \( R_i = (R_i(x_1) - b_{1,0} \cdot u_i(x_0), R_i(x_2), \cdots, R_i(x_{N-1}), R_i(x_N) - b_{N,N+1} \cdot u_i(x_{N+1}))^T \).

Here, \( u_i(x_0) \) and \( u_i(x_{N+1}) \) in (14) and \( R_i \) are the given boundary conditions in (13), and \( u_i(x_0) = 0 \) and \( u_i(x_{N+1}) \) is approximated by \( u_i(\infty) \).

Let \( B_n \ (n = 1, 2, \cdots, N) \) be \( 1 \times N \) row vectors defined by

\[
    B_1 = (b_{1,1}, b_{1,2}, 0, \ldots, 0),
\]

\[
    B_n = (0, \ldots, 0, b_{n,n-1}, b_{n,n}, b_{n,n+1}, 0, \ldots, 0), n = 2, 3, \ldots, N - 1,
\]

\[
    B_N = (0, \ldots, 0, b_{N,N-1}, b_{N,N}),
\]

where \( b_{n,n-1}, b_{n,n}, b_{n,n+1} \ (n = 1, 2, \ldots, N) \) are defined by Equations (15a)-(15c) and those entries which are not defined are zero. Set

\[
    \mathbf{B} = (B_1, B_2, \ldots, B_N)^T
\]

Then, the discrete scheme (14) can be rewritten as a system of linear algebra equations:

\[
    \mathbf{B} u_i = R_i \tag{16}
\]

By analyzing the matrix \( \mathbf{B} \), we can find that the matrix \( \mathbf{B} \) is an \( M \) matrix if the discount rate \( r > 0 \). This means that the inverse matrix of \( \mathbf{B} \) is non-negative if \( r > 0 \).

3.2. **Multi-least squares Monte Carlo simulation.** The multi-least squares Monte Carlo method (MLSM), which was expanded from the least squares Monte Carlo simulation (LSM) [20], has been successfully used to optimal multiple stopping time problems [18, 8]. A difference between our problem and most optimal multiple stopping time problems solved by MLSM is that there is a refraction time between two consecutive investment opportunities in our study. We modify the MLSM by constructing two types of continuation values to suit our problem. In addition, in order to reduce the variance of Monte Carlo methods, we follow the control variate method proposed by Lai et al. [17]. The calculations of the MLSM mainly comprise three steps as follows.

1. Using the stochastic process (1), generate \( L \) sample paths \( \{\omega_l\}_{l=1,2,\ldots,L} \) through the Monte Carlo simulation. The resulting \( X_j(\omega_l) \) is the demand value of the process at time \( s_j \) along the \( l \)th simulated path. Here, \( \{s_j = j\Delta\}^{L}_{j=0} \) be an equi-partition of \([0, T]\) with interval length \( \Delta \), and \( \{s_1, s_2, \ldots, s_J\} \) is the set of the discretized investment opportunities.
2. For each sample path $\omega_j$, go back in time to determine the exercise strategy by calculating the continuation value and the execution value at each timestep $s_j$ ($j = J, J-1, \ldots, 1$).

The project value with $q$ remaining investment rights at time $s_j$ is given by:

$$V^q_{j} (\omega_l) = \max\{C^q_{j}(\omega_l), E^q_{j}(\omega_l) + D^q_{j-1}(\omega_l)\}$$

where $C^q_{j}(\omega_l)$ is the continuation value, which is the project value created if an investment is not exerted at $s_j$ and continued in the future; $E^q_{j}(\omega_l)$ is the execution value. The execution value equals to the sum of the payoff of the investment, $E^q_{j}(\omega_l)$ (it can be obtained from the numerical solution of $\pi_{N-q+1}(x)$ through equation (7), that is, $E^q_{j}(\omega_l) = \pi_{N-q+1}(X_j(\omega_l))$), and the continuation value after executing the investment with one investment right less than the original one, $D^q_{j-1}(\omega_l)$.

There are two types of continuation value depending on whether or not to perform the investment at the current time. If the investment is not executed at time $s_j$, the next investment opportunity is $s_{j+1}$, then the continuation value is the conditional expectation on the information $\mathcal{F}^{(l)}_j$ available at time $s_j$ of the project value at time $s_{j+1}$, discounted to the time $s_j$, that is, $C^q_{j}(\omega_l) = E[e^{-r\delta}V^q_{j+1} | \mathcal{F}^{(l)}_j]$. $V^q_{j}$ is used to represent the project value for $q$ remaining investment rights at time $s_j$. If the investment is exerted at time $s_j$, the next investment should be at least separated by the refraction period $\delta$. Let $\theta = \lfloor \frac{1}{\Delta} \rfloor$, where $\lfloor \cdot \rfloor$ is the ceiling function, then $\theta \Delta \geq \delta$. The refraction period of two investments is set to be $\theta \Delta$ for the numerical calculations. Subsequently, the continuation value under this case is the conditional expectation on the information $\mathcal{F}^{(l)}_j$ of the project value at time $s_{j+\theta}$, discounted to the time $s_j$, that is, $D^q_{j}(\omega_l) = E[e^{-r\theta\Delta}V^q_{j+\theta} | \mathcal{F}^{(l)}_j]$.

The conditional expectations in the two continuation values are approximated by a polynomial of the current demand $X_j(\omega_l)$, shown as:

$$C^q_{j}(\omega_l) = E[e^{-r\Delta}V^q_{j+1} | \mathcal{F}_j] \approx \sum_k a^q_{j,k} L_k(X_j(\omega_l))$$

$$D^q_{j}(\omega_l) = E[e^{-r\theta\Delta}V^q_{j+\theta} | \mathcal{F}_j] \approx \sum_k a^q_{j,k} L_k(X_j(\omega_l))$$

where $L_k(\cdot)$ are basis functions expressed in the form of polynomials of the underlying risks, which are standard in the LSM literature. It can be proved that a limited number of base functions can approximate conditional expectations very well [20, 2]. The coefficients $a^q_{j,k}(\hat{a}^q_{j,k})$ are obtained by regressing the project value vector $e^{-r\Delta} \hat{V}^q_{j+1} (e^{-r\theta\Delta} \hat{V}^q_{j+\theta})$, onto basis vector $\{L_k(X_j)\}$. Here, $\hat{V}^q_{j} = (V^q_{j}(\omega_1), V^q_{j}(\omega_2), \cdots , V^q_{j}(\omega_L))^T$ is the project value vector for $q$ remaining investment rights at time $s_j$, $V^q_{j}(\omega_l)$ is the project value of the sample path $\omega_l$, and $X_j = (X_j(\omega_1), X_j(\omega_2), \cdots , X_j(\omega_L))^T$ is the demand vector at time $s_j$. The exercise strategy is made by comparing the execution value, $E^q_{j}(\omega_l) + D^q_{j-1}(\omega_l)$, with the continuation value $C^q_{j}(\omega_l)$. Obviously, the exercise condition is:

$$E^q_{j}(\omega_l) + D^q_{j-1}(\omega_l) > C^q_{j}(\omega_l).$$

When the exercise condition is satisfied, the investor will exercise the investment. The optimal stopping time of the $j$th stage project along the $l$th sample path,
\( \tau^\omega_l (i = 1, 2, \ldots, N) \), is yielded by
\[
\{ \tau^\omega_l \}_{i=1,2,\ldots,N} = \inf \{ \{ s_j \}_{j=1,2,\ldots,J} : E_j^{N-i+1} (\omega_l) + D_j^{N-i} (\omega_l) > C_j^{N-i+1} (\omega_l) \}
\]

3. Obtain the multi-stage project value and optimal investment strategy.

Based on the optimal stopping time set of each sample path, the \( N \)-stage project value is estimated by averaging the sum of the net revenue associated with the optimal stopping time, discounted to time 0:

\[
V = \frac{1}{L} \sum_{l=1}^{L} \sum_{i=1}^{N} e^{-r \tau^\omega_l i} \pi_i (X_{\tau_l} (\omega_l))
\]

The optimal investment strategies for multiple investments are described by a sequence of critical demand thresholds. To demonstrate the optimal investment strategies, a strategy matrix \( \Lambda \) is introduced. For \( N \) investment rights and \( J \) investment opportunities, the matrix \( \Lambda \) has \( N \) rows and \( J \) columns. The element \( \lambda_{ij} \) \( (i = 1, 2, \ldots, N; j = 1, 2, \ldots, J) \) in \( \Lambda \) represents the threshold of the demand for the \( i \)-th investment at the \( j \)-th opportunity. At time \( s_j \) \( (j = 1, 2, \ldots, J) \), the investor exercises the \( i \)-th investment when the actual demand exceeds \( \lambda_{ij} \). The values \( \lambda_{ij} \) can be extracted by finding the smallest samples relative to the optimal stopping time of \( i \)-th investment at time \( s_j \). By this means, it can be proved that the errors of the optimal thresholds are relatively small if the number of sample paths is sufficient [8].

4. Model application. In this section, the proposed model is applied to a hypothetical three-stage power plant project. Since uncertain demand is a distinct feature of the electricity market, this type of project is usually constructed in phases. Assume that a thermal power plant project is planned to be constructed in three stages over 10 years, the project capacity for each stage is 20 MW, and the annual power generation time is 3000 h, meaning that the annual maximum power generation per stage is \( 6 \times 10^4 \text{MW-h} \). The electricity price is assumed to be \( 5 \times 10^{-4} \) million CNY/MW-h, and the operating cost per unit of electricity is \( 4 \times 10^{-5} \) million CNY/MW-h. The construction period of each stage is 1.5 years, and the refractor time between two consecutive stages is at least 2 years. The parameters of the cost matching the capacity in (5) is set to be \( \lambda = 0.01 \) million CNY/MW-h and \( \beta = 0.9 \).

In the first stage, some investments, including land acquisition, road construction and auxiliary facilities installation, should accommodate future expansions, with reserve capacity investments being considered as dedicated assets. The cost of dedicated assets investment in the first stage, \( I_d \), is assumed to be a portion of the costs at later stages:

\[
I_d = \eta (I(m_2) + I(m_3)) \quad (\eta \in (0, 1))
\]

where \( \eta \) is the dedicated asset ratio, and \( I(m_2) \) and \( I(m_3) \) are, respectively, the cost matching the capacity of the second and third stages.

All parameters involved in the model are displayed in Table 1. Unless otherwise specified, in our analysis, we assume the parameter values are as given in Table 1. In real situation, parameters of \( c, \lambda, \beta, \alpha, \sigma \), need to be estimated from the historic data of similar projects using statistical methods. Discount rate \( r \) can be adopted by the rate of securities return. Other parameters are decided by the decision-maker by evaluating the features of the project.
Table 1. Default parameters used in the calculations

| Parameter                        | Symbol | Value       | Unit       |
|----------------------------------|--------|-------------|------------|
| Investment period                | $T$    | 10          | Year       |
| Planned investment times         | $N$    | 3           | time       |
| Construction period              | $\nu$  | 1.5         | Year       |
| Refraction time                  | $\delta$ | 2          | Year       |
| Capacity of i-th stage           | $m_i$  | $6 \times 10^4$ | MW·h/year |
| Unit price                       | $p$    | $5 \times 10^{-4}$ | million CNY/MW·h |
| Unit operational cost            | $c$    | $4 \times 10^{-5}$ | million CNY/MW·h |
| Construction cost parameter      | $\lambda$ | $1 \times 10^{-2}$ | million CNY/MW·h |
| Construction cost parameter      | $\beta$ | 0.9         |            |
| Drift                            | $\alpha$ | 6%         |            |
| Volatility rate                  | $\sigma$ | 15%        |            |
| Discount rate                    | $r$    | 8%          |            |
| Dedicated asset ratio            | $\eta$ | 0.1         |            |

4.1. Project value. In order to demonstrate the superiority of multi-stage investment, the project value of three-stage investment with the capacity of $6 \times 10^4$ MW·h per stage is compared with that of single period investment with the capacity of $18 \times 10^4$ MW·h, shown in Figure 2. From Figure 2(a), it is found that the project value of three-stage investment is higher than the project value of single period investment. The reason is that multi-stage investment has more flexible options, including flexible investment opportunity or abandoning future expansion, and these flexible rights can adapt to the changing environment and improve the value of the project. Under different level of volatility, Figure 2(b) depicts the increase in the relative value of three-stage investment over single period investment. Here, the increase in the relative value=$\frac{(three-stage project value - single period project value)\times 100}{single period project value}$. It is shown that the lower the demand, the higher the increase in the relative value, that is, flexibility in the multi-stage investment is more valuable. This is intuitive, because in the case of low demand, the future expansions in multi-stage investment can be cancelled, and the abandon right can create value. In addition, when the demand is low, the lower the level of the volatility, the higher the increase in the relative value, i.e., the worse the single period investment. This is because under low level of volatility, it is difficult for initial low demand to reach a level sufficient to support bigger capacity of single period investment.

4.2. Investment strategy. With the abandon right, the investor will give up an investment when the net present value of the investment is less than zero. From the viewpoint of option, the abandon right is a put option and can create value for the project. Under different demand levels of volatility, the distribution of investment times obtained from one million sample paths is shown in Figure 3, where the investment times of 0, 1, 2 mean that the investor exert the abandon right for the three-stage project. Further comparing different volatility, it is found that, under a high volatility, there is a larger probability that the investment times is distributed at both ends (that is, 0 or 3 times). This is because under high volatility, the demand is more likely to be at both ends (high demand or low demand), so is the
Figure 2. The comparisons between multi-stage and single period investments.

For high revenue, the number of investment times is more likely to be 3, and for low revenue, the number of investment times is more likely to be 0.

Figure 3. The distribution of investment times from one million sample path.

The optimal investment strategy is described by a series of demand thresholds between the exercise and the continuation of the investment rights. These demand thresholds are called optimal exercise boundaries, which are depicted in Figure 4. Here, $i$ refers to the $i$-th investment of the three investment rights. For instance, the exercise boundary of $i=1$ is the threshold of the first investment, and the investor should exercise the first investment when this value is exceeded by the actual demand. From this figure, it can be seen that the second opportunity of investment begins after the second year, and the third implementation is only possible after the fourth year. This is not difficult to understand since two consecutive investments are required to be separated by two years.

From Figure 4, it is shown that the boundaries of the first and second investments fluctuate at time 8, with a decreasing demand on the left side and an increasing
demand on the right side. The reason behind this is that time 8 is a critical point for investment, that is, if the investment is not executed at this time, later investment opportunities will be worthless during the later period because of the refraction time being two years. Consequently, near the eighth year, the investor will reduce his or her earning expectations and implement the investment at a lower level of demand. However, if the investment is not implemented at time 8, afterwards, there is only one chance to invest; in this case, the optimal strategy is to maximize the benefit of the only investment opportunity, and hence the exercise boundary will rise again. Unlike the first two investment boundaries, the boundary for the third investment changes smoothly except near the expiration date. This is because the third investment is the last one, and the investor may be less concerned about the worthlessness of the investment right except near the maturity.

In addition, Figure 4 also shows that the exercise boundary of the first stage is flat over time except near time 8, while the exercise boundaries of later two stages are relatively steep. The reason for the flat exercise boundary at first stage can be explained from the long operating period. Because the operating period of the project is long run, the project values at different investment opportunities over a finite time horizon (10 years in this example) have no significant differences. The steep exercise boundaries at later two stages can be interpreted from the dedicated assets. The dedicated assets are invested at the first stage and used to later expansions. If later expansions are not exerted, the dedicated assets will become idle. Investment in dedicated assets is accounted for in total costs; thus, in order to maximize the total profits at all stages, the investor would perform later expansions as long as the net profit of expansion is larger than 0. Accordingly, as time goes by, the investor will reduce the earning expectations. This leads that expansions are executed at a lower and lower level of demand over time, and the change of the exercise boundary is steep.

4.3. Effects of dedicated asset. The impact of the dedicated assets in the first stage on the project value and later investment strategy is shown in Figure 5(a) and Figure 5(b), respectively. The amount of the dedicated assets is represented by the dedicated asset ratio $\eta$ in (19). Figure 5(a) reveals that the project value decreases
with an increasing dedicated asset ratio. The reason is as follows. As the dedicated asset ratio increases, the investment cost in the dedicated assets rises; this makes the cost of the first stage is increased and the costs of later stages are decreased, which can be seen from expression (6). In notional amount, the increased cost in the first stage is equal to the decreased cost in later stages. Considering the time value of money, the discounted cost for the same notional amount is greater in the first stage than in later stages. This leads to a higher total cost, consequently reducing the project value. The implication is clear: reducing the investment amount of the dedicated assets as much as possible can increase the project value.

Turn to the exercise boundary, from Figure 5(b), it is found that the demand levels that trigger an investment are decreasing with the increase in the dedicated assets ratio. Here, $\eta$ of 0, 0.1, and 0.2 are used for illustration. When $\eta = 0$, it corresponds to the case without dedicated asset. As the dedicated assets ratio increases, the costs of the second and third stages decrease, therefore the investments will be triggered at lower demand levels. Here, for the sake of simplicity, only the investment strategy of the second stage is demonstrated. There are similar results for the third stage.

![Graphs](image.png)

**Figure 5.** The influences of the dedicated asset ratio.

### 4.4. Effects of capacity constraint

In this study, each stage of the project is installed with a certain capacity. When the demand during the operating period becomes unexpected, there is no possibility to adjust the capacity. Due to the capacity constraints, the output and consequently the revenue of the project are capped. The net revenue of the $i$-th ($i=1, 2, 3$) stage project under different demand levels, $\pi_i(x)$, is shown in Figure 6. It can be seen from Figure 6, after the demand reaches a certain level, the net revenue tends to a fixed level, which implies that the project operates at full capacity.

In order to further explore the effect of capacity constraints, the project values under different levels of demand volatility are displayed in Figure 7. From Figure 7, we can see that, when the demand is low, the project value increases with the increasing volatility $\sigma$. On the other hand, when the demand is high, the project value decreases with an increase in the volatility. The case under high demand is inconsistent with classical investment results represented by Dixit and Pindyck [6],...
where the project value increases with the increasing volatility no matter what the level of demand is. The reason is because the capacity constraints are taken into account in this study. Although the demand may be high, the revenue is constrained by limited capacity. For the study of [6], there is no cap on revenue.

To reveal the relationship between the capacity constraints and the variation of the project value with respect to the volatility, we explore the ‘Vega’ (the derivative of the option value with respect to the volatility) of the project value. Because the project value in this work is derived by the cash flow with the characteristic of bull call spread mentioned earlier, we analyse the ‘Vega’ of bull call spread mathematically. Since the closed form solution of European bull call spread can be obtained, we focus on the ‘Vega’ of the European bull call spread corresponding to our problem. Using similar symbols as in expression (10), the payoff of European bull call spread for the i-th investment, $\Omega_i^E(X_s)$, is expressed as:

$$\Omega_i^E(X_s) = (X_s - M_{i-1})^+ - (X_s - M_i)^+$$  \hspace{1cm} (20)
The ‘Vega’ for the bull call spread, \( \frac{\partial V}{\partial \sigma} \), can be easily obtained from the Black-Scholes model:

\[
\frac{\partial V^E}{\partial \sigma} = X_s \sqrt{T^E - s}(n(d_1) - n(d'_1))
\]

(21)

where \( T^E \) is the expiration date of the European call spread, \( n(\cdot) \) is the probability density function (PDF) of the standard normal distribution, \( d_1 = \frac{\ln X_s/(T^E - s)}{\sigma \sqrt{T^E - s}} + (r^* + \frac{\sigma^2}{2})(T^E - s) \), \( d'_1 = \frac{\ln X_s/(T^E - s)}{\sigma \sqrt{T^E - s}} \). \( r^* \) of \( d_1 \) and \( d'_1 \) is the risk-free discount rate.

It is obvious that the sign of \( \frac{\partial V}{\partial \sigma} \) depends on \( n(d_1) - n(d'_1) \), which is related to the demand \( X_s \) and the capacity \( m_i \) (Here, \( M_i = \sum_{k=1}^{t} M_k \)). Since the PDF of the standard normal distribution, \( n(\cdot) \), is symmetric around zero, we analyze the relation between \( n(d_1) \) and \( n(d'_1) \) in three situations: 1) \( d_1 > d'_1 \geq 0 \); 2) \( 0 \geq d_1 > d'_1 \); 3) \( d_1 > d'_1 > 0 \) (Note that \( d_1 > d'_1 \) is always true since \( M_i - 1 < M_i \)).

The outcomes are as follows:

- \( \frac{\partial V}{\partial \sigma} > 0 \) when \( X_s < \sqrt{M_{i-1} M_i e^{-(r^* + \frac{\sigma^2}{2})(T^E - s)}} \),
- \( \frac{\partial V}{\partial \sigma} < 0 \) when \( X_s > \sqrt{M_{i-1} M_i e^{-(r^* + \frac{\sigma^2}{2})(T^E - s)}} \), and
- \( \frac{\partial V}{\partial \sigma} = 0 \) occurs when \( X_s = \sqrt{M_{i-1} M_i e^{-(r^* + \frac{\sigma^2}{2})(T^E - s)}} \). In short, the ‘Vega’ is positive when the demand is low, meaning that the project value increases with the increasing volatility; the ‘Vega’ is negative when the demand is high, meaning that the project value decreases with the increasing volatility. From these outcomes, it is not difficult to understand the variation of the project value with respect to the volatility.

5. Conclusions. This paper models and calculates the optimal investment timing of multi-stage capacity expansion infrastructure projects in a finite time horizon. The upwind finite difference method is used to solve differential equations describing the expected returns at all stages, and the multi-least squares Monte Carlo simulation is introduced to solve optimal multiple stopping time model. Through an example of three-stage power plant project, both the total project value at all stages and the optimal investment strategy are presented, and meanwhile the validity of the numerical algorithms developed in this study is verified. This study novelty reveals the effect the capacity constraints on the project value using bull call spread theory.

This paper provides insights for the potential investor with the idea of when to exercise the multi-stage investments in a finite time horizon, maximizing profits under uncertain demand. However, there are still some limitations that deserve further study. First, the capacity at each stage of the project is assumed to be fixed beforehand. In future study, optimizing timing and capacity of multi-stage projects will be an interesting topic, and this is a type of optimal impulse control problem. Second, in addition to demand, other uncertainties can be considered. For instance, uncertainty in construction costs is also important. Third, the comparison of numerical algorithms can be implemented in future work and hence it can enrich the algorithm theory of the optimal multiple stopping time problem.

REFERENCES

[1] L. E. Brandao and E. Saraiva, The option value of government guarantees in infrastructure projects, Constr. Manage. Econ., 26 (2008), 1171–1180.

[2] G. Cortazar, M. Gravet and J. Urzua, The valuation of multidimensional American real options using the LSM simulation method, Comput. Oper. Res., 35 (2008), 113–129.
E. Dahlgren and T. Leung, An optimal multiple stopping approach to infrastructure investment decisions, *J. Econ. Dyn. Control*, 53 (2015), 251–267.

T. Dangl, Investment and capacity choice under uncertain demand, *Eur. J. Oper. Res.*, 117 (1999), 415–428.

R. De Neufville, S. Scholtes and T. Wang, Real options by spreadsheet: Parking garage case example, *J. Infrastruct. Syst.*, 12 (2006), 107–111.

A. K. Dixit and R. S. Pindyck, *Investment under Uncertainty*, Princeton University Press, Princeton, NJ, 1994.

P. Doan and K. Menyah, Impact of irreversibility and uncertainty on the timing of infrastructure projects, *J. Constr. Eng. Manage.*, 139 (2013), 331–338.

U. Dör, *Valuation of Swing Options and Examination of Exercise Strategies by Monte Carlo Techniques*, Master Dissertation, Christ Church College, University of Oxford, 2003.

C. F. Fisher, J. S. Paik and W. R. Schriver, *Power Plant Economy of Scale and Cost Trends—Further Analyses and Review of Empirical Studies*, University of Tennessee, 1986.

B. Flyvbjerg, M. Holm and S. Buhl, How common and how large are cost overruns in transport infrastructure projects? *Transp. Rev.*, 23 (2003), 71–88.

C. C. Gkochari, Optimal investment timing in the dry bulk shipping sector, *Transp. Res. Part E*, 79 (2015), 102–109.

K. C. Han and A. Heinemann, A bull call spread as a strategy for small investors, *J. of Pers. Fin.*, 6 (2008), 108–127.

H. B. Herath and C. Park, Multi-stage capital investment opportunities as compound real options, *Eng. Econ.*, 47 (2002), 1–27.

Y. L. Huang, C. C. Pi, Valuation of multi-stage BOT projects involving dedicated asset investments: A sequential compound option approach, *Constr. Manage. Econ.*, 27 (2009), 653–666.

B. Klein and K. B. Leffler, The role of market forces in assuring contractual performance, *J. Polit. Econ.*, 89 (1981), 615–641.

N. A. Krüger, To kill a real option - Incomplete contracts, real options and PPP, *Transp. Res. Part A*, 46 (2012), 1359–1371.

Y. Lai, Z. Li and Y. Zeng, Control variate methods and applications to Asian and basket options pricing under jump-diffusion models, *IMA J. Manage. Math.*, 26 (2015), 11–37.

J. Li, Y. Li and S. Zhang, Optimal expansion timing decisions in multi-stage PPP projects involving dedicated asset and government subsidies, *J. Ind. Manage. Optim.*, 16 (2020), 2065–2086.

Y. P. Lin, Upwind finite difference schemes for linear conservation law with memory, *Numer. Meth. Part. Diff. Equ.*, 10 (1994), 475–489.

F. A. Longstaff and E. S. Schwartz, Valuing American options by simulation: A simple least squares approach, *Rev. Financ. Stud.*, 14 (2001), 113–147.

E. Lukas, S. Mölls and A. Welling, Venture capital, staged financing and optimal funding policies under uncertainty, *Eur. J. Oper. Res.*, 250 (2016), 305–313.

J. Martins, R. C. Marques and C. O. Cruz, Maximizing the value for money of PPP arrangements through flexibility: An application to airports, *J. Air Transp. Manage.*, 39 (2014), 72–80.

M. Marzouk and M. Ali, Mitigating risks in wastewater treatment plant PPPs using minimum revenue guarantee and real options, *Util. Policy*, 53 (2017), 121–133.

J. Paslawski, Flexible approach for construction process management under risk and uncertainty, *Proc. Eng.*, 208 (2017), 114–124.

P. C. Pendharkar, Valuing interdependent multi-stage IT investments: A real options approach, *Eur. J. Oper. Res.*, 201 (2010), 847–859.

A. M. P. Santos, J. P. Mendes and C. Guedes Soares, A dynamic model for marginal cost pricing of port infrastructures, *Marit. Policy Manage.*, 43 (2016), 812–829.

N. Song, Y. Xie, W. Ching, et al., A real option approach for investment opportunity valuation, *J. Ind. Manage. Optim.*, 13 (2017), 1213–1235.

Z. Tan and H. Yang, Flexible build-operate-transfer contracts for road franchising under demand uncertainty, *Transp. Res. Part B*, 46 (2012), 1419–1439.

S. Wang, L. S. Jennings and K. L. Teo, An upwind finite-difference method for the approximation of viscosity solutions to Hamilton-Jacobi-Bellman equations, *IMA J. Math. Control Inf.*, 17 (2000), 167–178.

O. E. Williamson, *The Economic Institutions of Capitalism*, The Free Press, New York, 1985.
[31] T. Zhao and C. L. Tseng, Valuing flexibility in infrastructure expansion, *J. Infrastruct. Syst.*, **9** (2003), 89–97.

Received January 2020; revised August 2020.

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