GUP-based and Snyder Non-Commutative Algebras, Relativistic Particle models, Deformed Symmetries and Interaction: A Unified Approach

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Abstract: We have developed a unified scheme for studying Non-Commutative algebras based on Generalized Uncertainty Principle (GUP) and Snyder form in a relativistically covariant point particle Lagrangian (or symplectic) framework. Even though the GUP based algebra and Snyder algebra are very distinct, the more involved latter algebra emerges from an approximation of the Lagrangian model of the former algebra. Deformed Poincare generators for the systems that keep space-time symmetries of the relativistic particle models have been studied thoroughly.

From a purely constrained dynamical analysis perspective the models studied here are very rich and provide insights on how to consistently construct approximate models from the exact ones when non-linear constraints are present in the system.

We also study dynamics of the GUP particle in presence of external electromagnetic field.

1 Introduction:

Operatorial forms of Non-Commutative (NC) phase space structures, of the generic form,

\[
\{x_i, p_j\} = \delta_{ij}(1 + f_1(p^2)) + f_2(p^2)p_i p_j, \\
\{x_i, x_j\} = f_{ij}(p), \quad \{x_i, p_j\} = g_{ij}(p); \quad i = 1, 2, 3,
\]

have created a lot of interest in recent years due to their potential application in generating Generalized Uncertainty Principle (GUP), the latter being compatible with String Theory or Quantum Gravity expectations of the presence of a minimum length scale or a maximum momentum scale or both. \(\beta\) is treated as a small parameter with \(\sqrt{\beta}\) being the measure of a minimum length scale. The argument goes as follows: resolution of position coordinate to an arbitrary precision can lead, via (Heisenberg) canonical uncertainty principle, to such a large accumulation of momentum or energy density that the latter can appreciably alter the space-time metric. Calculations from String Theory perspective \[\text{[1]}\] also suggest a minimum length scale in the form of minimum position uncertainty. This is the possible origin of GUP. The pioneering works

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proposing consistent NC algebras with a similar motivation, in a non-covariant framework, were by Kempf
\[\{x_i, p_j\} = \delta_{ij}(1 + \beta p^2) + \beta' p_i p_j, \quad \{x_i, x_j\} = (\beta' - 2\beta)(x_i p_j - x_j p_i), \quad \{p_i, p_j\} = 0. \tag{2}\]
by Kempf, Mangano and Mann \[3\]
\[\{x_i, p_j\} = \delta_{ij}(1 + \beta p^2), \quad \{x_i, x_j\} = -2\beta(x_i p_j - x_j p_i), \quad \{p_i, p_j\} = 0, \tag{3}\]
and by Kempf and Mangano \[4\],
\[\{x_i, p_j\} = \frac{\beta p^2 \delta_{ij}}{\sqrt{(1 + 2\beta p^2)} - 1} + \beta p_i p_j, \quad \{x_i, x_j\} = 0, \quad \{p_i, p_j\} = 0. \tag{4}\]
We are restricting ourselves to the classical counterpart of the commutators but the results derived here are applied to quantum commutators as well. The first work on operatorial NC algebra was by Snyder \[5\] with the structure same as that of \[3\]. In fact \[2\] \[2\] and \[3\] \[3\] are a generalized form of \[5\] and can be reduced to the Snyder form of NC \[5\] by suitable choice of parameters, as discussed by \[6\]. However, we will focus principally on the algebra \[4\] \[4\], (subsequently referred as KM) since it is structurally the simplest as the coordinates and momenta commute among themselves respectively.

The work reported in the present paper can be divided in to three parts which are inter connected. We work with a relativistically covariant generalization of the algebra \[4\]. In the first part we derive a generalized point particle Lagrangian with a non-canonical symplectic structure that is equivalent to \[4\]. Clearly from a physics point of view this type of an intuitive particle picture is very useful and appealing since we can see how it differs from the conventional relativistic point particle. Primarily it is essential in studying the dynamics of such particles (in Hamiltonian framework) that can reveal unique features of such particles. This can act as a precursor to field theories in such non-canonical space. It is not possible to obtain such information only from the phase space algebra which is essentially kinematical in nature. Similar point particle symplectic formalisms have been adopted for other forms of operatorial NC algebras such as \(\kappa\)-Minkowski algebra \[7, 8\], relevant in Doubly Special Relativity framework \[9\] or Very Special Relativity algebra \[10\], proposed in \[11\].

The point particle scheme is crucial for the second part of our work. This deals with the richness and intricacies of Dirac formalism \[13\] when applied to non-linear constraints, (i.e. constraints consisting of non-linear terms in phase space variables), which are necessary to induce operatorial phase space algebras as Dirac Brackets. In an explicit way we will show that the simpler algebra \[4\] can be “reduced” to the more complicated Snyder form Snyder \[4, 5\]. “Simple” and “complicated” refer to \(\{x_\mu, x_\nu\}\) being zero or non-zero respectively. It is clearly revealed how approximate forms of the Lagrangian, related to \[4\] \[4\], with terms up to at least \(O(\beta^2)\) can reproduce the algebra \[3, 2, 3\]. Furthermore, a simpler Lagrangian
with two parameters $\beta$ and $\beta'$ is also provided that gives rise to the Snyder algebra. This strongly brings in to fore a point that it is always advisable to impose approximations in a theory at the Lagrangian level and then go on to compute the symplectic structure rather than truncate the exact symplectic structure directly. In the latter procedure it is natural to encounter consistency problems in the algebra, (such as violation of Jacobi identity), leading to incorrect conclusions. Furthermore, the elegant connection between qualitatively different algebras, (such as \[4\] and \[2, 3, 5\], derived at different stages of approximation of the Lagrangian, will be lost. The comment below is relevant in this context.

Another very important aspect of point particle framework is that one can introduce interactions in a consistent way. This is the topic of the third part. It is possible that interactions can bring out certain interesting or even unphysical features (if there are any) of the generalization which do not show up in the free particle context. (For a recent work in this regard see for example \[12\] where $\kappa$-Minkowski particles are subjected to electrodynamical interactions.)

A covariantization of \[3, 5\] was performed in \[6\] for the \[3\] case:

\[
\{X^\mu, P^\nu\} = \delta^{\mu\nu}(1 + \beta P^2) + \beta' P^\mu P^\nu, \quad \{P^\mu, P^\nu\} = 0,
\]

\[
\{X^\mu, X^\nu\} = (2\beta - \beta') + (2\beta + \beta')\beta P^2 \left(\frac{P^\mu X^\nu - P^\nu X^\mu}{1 + \beta P^2}\right) \tag{5}
\]

where $\mu, \nu = 1, 2, 3$. However, the subsequent analysis to $O(\beta)$ is doubtful to say the least since as the authors themselves admit in \[6\] the Jacobi identity is maintained by the linearized algebra,

\[
\{X_\mu, P_\nu\} = (\delta_{\mu\nu}(1 + \beta P^2) + 2\beta P_\mu P_\nu),
\]

\[
\{P_\mu, P_\nu\} = \{X_\mu, X_\nu\} = 0 \tag{6}
\]

only to $O(\beta)$. Since the violation of Jacobi is of the following operatorial form,

\[
J(X_\mu, X_\nu, P_\lambda) = \{X_\mu, \{X_\nu, P_\lambda\}\} + \{P_\lambda, \{X_\mu, X_\nu\}\} + \{X_\nu, \{P_\lambda, X_\mu\}\},
\]

which implies

\[
J(X_\mu, X_\nu, P_\lambda) = 4\beta^2 P^2(\delta_{\nu\lambda}P_\mu - \delta_{\mu\lambda}P_\nu), \tag{7}
\]

it is possible that the expectation value of the RHS of \[14\] becomes large rendering the claim, that $O(\beta^2)$ contribution is always small, meaningless. Exact validity of Jacobi identity is imperative for the phase space algebra. Furthermore, due to this violation of Jacobi, there can not be any point particle interpretation of this NC symplectic structure since, (indeed, from our perspective), the NC structures appear as Dirac Brackets which always preserve Jacobi identity \[13\]. We have also constructed deformed Poincare generators that generate proper translations and rotations of the variables.
The paper is organized as follows. In Section 2 we propose the covariant GUP and develop the point particle Lagrangian corresponding to it and study the space-time symmetry properties of this novel particle model. In Section 3 we show how different forms of covariantized Snyder algebra can be generated from approximations of GUP particle model. As a bonus we also obtain point particle Lagrangians for these Snyder algebras. In Section 4 we discuss the GUP particle interacting with $U(1)$ gauge fields. The paper is concluded in Section 5 with a summary of our work and future directions. Some computational details are provided in the appendices at the end.

2 Covariantized GUP and the Point Particle

We begin by positing covariantized form of the NC algebra proposed in [4] in 3+1-dimensions, with a Minkowski metric $g_{00} = -g_{ii} = 1$,

$$\{x_\mu, p_\nu\} = -\frac{\beta p^2 g_{\mu\nu}}{\sqrt{(1 + 2\beta p^2)} - 1} - \beta p_\mu p_\nu \equiv -\Lambda g_{\mu\nu} - \beta p_\mu p_\nu,$$

$$\{x_\mu, x_\nu\} = 0, \quad \{p_\mu, p_\nu\} = 0,$$

where $\Lambda = \frac{\beta p^2}{\sqrt{(1 + 2\beta p^2)} - 1}$. For the spatial sector this reduces to

$$\{x_i, p^j\} = \Lambda \delta^j_i + \beta p^i p^j,$$

similar as in [4] with a mismatch in the value of $\Lambda$ (since $p^2$ has been replaced by $p^2$). We would like to interpret the above relations (8) as Dirac Brackets derived from a constrained symplectic structure. In some sense we are actually moving in the opposite direction of the conventional analysis where the computational steps are

$$\text{Lagrangian} \rightarrow \text{Constraints} \rightarrow \text{Dirac Brackets}$$

or equivalently

$$\text{Symplectic Structure} \rightarrow \text{Symplectic Matrix} \rightarrow \text{Symplectic Brackets}.$$  

The Dirac brackets and symplectic brackets turn out to be same. In our case the situation is reversed and our path of analysis will be

$$\text{Dirac Brackets} \rightarrow \text{Constraints} \rightarrow \text{Lagrangian}$$

or

$$\text{Symplectic Brackets} \rightarrow \text{Symplectic Matrix} \rightarrow \text{Lagrangian.}$$
The procedure is the following. The generic form of a Symplectic Bracket (SB) is of the form

$$\{f, g\}_{SB} = \Gamma_{ab} \partial_{a, \mu} f \partial_{b, \nu} g,$$

(9)

where $\partial_{a, \mu} = \frac{\partial}{\partial \eta^a \mu}$, $\eta^1_\mu = x^{\mu}$, $\eta^2_\mu = p^{\mu}$. This Symplectic Matrix also appears in the Dirac Brackets as

$$\{f, g\}_{DB} = \{f, g\} - \{f, \Phi^{\mu}_a \} \Gamma_{ab} \{\Phi^\nu_b, g\},$$

(10)

where $\Phi^{\mu}_a$ are a set of Second Class Constraints [13] (see appendix for a brief description of the Dirac procedure). Inverse of the $\Gamma$-matrix provides the constraint algebra

$$\Gamma^{ab}_{\mu \nu} = \{\Phi^a_\mu, \Phi^\nu_b\}.$$  

(11)

Indeed there is no unique way but from the nature of the constraint matrix one can make a judicious choice of the constraints and subsequently guess a form of the Lagrangian. We do not claim the Lagrangian derived in this way is unique, (in fact there might be more than one Lagrangians generating identical Dirac Brackets), but at least one can easily check that the derived Lagrangian yields the Dirac Brackets that one posited at the beginning.

Comparison with (8) allows us to identify

$$\{x^{\mu}, x^{\nu}\} \equiv \Gamma^{\mu \nu}_{11} = 0, \quad \{p^{\mu}, p^{\nu}\} \equiv \Gamma^{\mu \nu}_{22} = 0,$$

$$\{x^{\mu}, p^{\nu}\} \equiv \Gamma^{\mu \nu}_{12} = -(\Lambda g^{\mu \nu} + \beta p^{\mu} p^{\nu}) \equiv -\Gamma^{\mu \nu}_{12}. $$

(12)

The Symplectic Matrix is

$$\Gamma^{\mu \nu}_{ab} = \begin{bmatrix} 0 & -(\Lambda g^{\mu \nu} + \beta p^{\mu} p^{\nu}) \\ (\Lambda g^{\mu \nu} + \beta p^{\mu} p^{\nu}) & 0 \end{bmatrix}. $$

(13)

The inverse matrix is computed and it turns out to be the commutator matrix,

$$\Gamma^{\mu \nu}_{cd} = \begin{bmatrix} 0 & \left(\frac{2 \beta}{\Lambda} \right) \left(\alpha^a - \frac{\beta p_a p_b}{\Lambda^2 \sqrt{1 + 2 \beta p^2}}\right) \\ -\left(\frac{2 \beta}{\Lambda} \right) \left(\alpha^a - \frac{\beta p_a p_b}{\Lambda^2 \sqrt{1 + 2 \beta p^2}}\right) & 0 \end{bmatrix}. $$

(14)

It is convenient to work in the first order formalism where both $x_\mu$ and $p_\mu$ are treated as independent variables with the conjugate momenta, $\pi^x_\mu = \frac{\partial L}{\partial \dot{x}^{\mu}}$, $\pi^p_\mu = \frac{\partial L}{\partial \dot{p}^{\mu}}$ with two decoupled canonical algebra

$$\{x_\mu, \pi^x_\mu\} = -g_{\mu \nu}, \quad \{p_\mu, \pi^p_\mu\} = -g_{\mu \nu}. $$

We propose the following set of constraints:

$$\Phi^{1}_\mu = \pi^x_\mu \approx 0, $$

(15)
Thus we recover explicit expressions for the momenta:

\[
\begin{align*}
\pi^x & = 0, \quad \pi^p = -\frac{x \Lambda}{\Lambda^2 \sqrt{1 + 2 \beta p^2}}.
\end{align*}
\]  

Finally we can write down the cherished form of the point particle Lagrangian in the first order form as,

\[
L = -\frac{(xp)}{\Lambda} + \frac{\beta(xp)(p^2)}{\Lambda^2 \sqrt{1 + 2 \beta p^2}} + \lambda (f(p^2) - m^2),
\]  

where \( \lambda \) is a Lagrange multiplier. We have included a mass-shell condition \( f(p^2) - m^2 = 0 \) where \( f(p^2) \) denotes an arbitrary function that needs to fixed. This is done from hindsight since we will show below that this structure will be invariant under modified Lorentz generators. This particle model is one of our major results.

First of all it is straightforward to check that from the Lagrangian, through conventional Dirac Hamiltonian analysis of non-commuting constraints (or Second Class Constraints, as they are termed in literature), the algebra \( \mathfrak{S} \) can be derived as Dirac Brackets.

Let us check how the new Lagrangian fares as regards the conventional space-time symmetries, in particular translation and generalized rotation (i.e. spatial rotation and boosts). From the NC algebra \( \mathfrak{S} \) it is clear that the momentum \( p^\mu \) can not play the role of Translation generator because of the anomalous translation of \( x^\mu \),

\[
\delta x^\mu = \{x^\mu, (\sigma p)\} = -(\Lambda \sigma^\mu + \beta \sigma p_\mu),
\]  

where \( \sigma^\mu \) is the translation parameter. But a straightforward generalization of a transformation of variables proposed in [4] shows that \( \frac{p^\mu}{\Lambda} \) can act as the true Translation generator,

\[
\delta x^\mu = \{x^\mu, \sigma^\mu (\frac{p^\mu}{\Lambda})\} = -\sigma^\mu.
\]  

In a similar way the Lorentz generators also get modified to

\[
j_{\mu \nu} = \frac{1}{\Lambda} (x_\mu p_\nu - x_\nu p_\mu),
\]

such that correct transformation of the degrees of freedom are reproduced,

\[
[j_{\mu \nu}, p_\lambda] = g_{\mu \lambda p_\nu - g_{\nu \lambda p_\mu}, \ [j_{\mu \nu}, x_\lambda] = g_{\mu \lambda x_\nu - g_{\nu \lambda x_\nu}.
\]  

Indeed \( j_{\mu \nu} \) obeys the correct Lorentz algebra,

\[
\{j_{\mu \nu}, j_{\alpha \beta}\} = g_{\mu \alpha} j_{\nu \beta} - g_{\mu \beta} j_{\nu \alpha} - g_{\nu \beta} j_{\alpha \mu} + g_{\nu \alpha} j_{\beta \mu}.
\]
Let us fix the function $f(p^2)$ in the mass shell condition. Since $\{j_{\mu\nu}, p^2\} = 0$ any function of $p^2$ is Lorentz invariant but keeping Translation invariance in mind, a more natural choice would be $f(p^2) \rightarrow (p^2)^2$ leading to a modified mass shell condition $(p^2)^2 - m^2 = 0$. However this actually simplifies to $p^2 = M^2, M = m/(1 - \beta^2/m^2)$.

Quantization of this GUP particle can be carried through in the conventional way by gauge fixing the reparameterization invariance.

The above analysis demonstrates that we have developed a consistent and relativistically covariant framework to represent a generalized point particle living in an NC phase space compatible with GUP.

3 Approximations leading to other algebras

As we have explained at the beginning, approximating the full NC algebra directly is not the proper way to derive an effective $O(\beta)$ corrected dynamical system since, in particular with operatorial NC algebras, there is always a drawback that Jacobi identities might be violated. The correct way is to approximate the system at the level of the Lagrangian because then we are assured that the $O(\beta)$ corrected NC brackets will also satisfy the Jacobi identities. (Details of the Dirac Bracket calculations are provided in the appendices.)

$O(\beta)$ results: With $\Lambda = 1 + \frac{1}{2} \beta p^2 + O(\beta^2)$ the Lagrangian \[\text{(18)}\] yields $L_{(1)}$ (without the mass-shell condition) to $O(\beta)$:

$$
L_{(1)} = -(x\dot{p})(1 - \frac{1}{2} \beta p^2) + \beta(xp)(p\dot{p}) + O(\beta^2)
$$

with the momenta,

$$
\pi^x_\mu = 0, \quad \pi^p_\mu = -x_\mu(1 - \frac{1}{2} \beta p^2) + \beta(xp)p_\mu
$$

leading to the constraints,

$$
\phi^1_\mu = \pi^x_\mu \approx 0, \quad \phi^2_\mu = \pi^p_\mu + x_\mu(1 - \frac{1}{2} \beta p^2) - \beta(xp)p_\mu \approx 0.
$$

We find the Dirac Brackets to be, (with details in Appendix A),

$$
\{x^\mu, p^\nu\} = -\left[ \frac{g_{\mu\nu}}{(1 - \frac{\beta p^2}{2})} + \frac{\beta p_\mu p_\nu}{(1 - \frac{3\beta p^2}{2})(1 - \frac{\beta p^2}{2})} \right],
$$

$$
\{x^\mu, x^\nu\} = \{p^\mu, p^\nu\} = 0.
$$

Notice that the algebra is still structurally similar as the exact one and the Snyder form with non-zero $\{x_\mu, x_\nu\}$ has not appeared. This agrees with previous results that the Snyder form is present only in $O(\beta^2)$ or when more than one $\beta$-like parameters are present \[\text{(8)}\ \text{[3, 6]}\]. However, linearizing this algebra to $O(\beta)$ is
once again problematic as it clashes with the Jacobi identity. We will see that the Snyder form is necessary
in the linearized system in order to exactly satisfy the Jacobi identity.

The combination \( x_\mu, (1 - \beta^2 p^2) p_\nu \) constitute a canonical pair, \( \{ x_\mu, (1 - \beta^2 p^2) p_\nu \} = -g_{\mu\nu} \). The operator
\( j_{\mu\nu} = (1 - \beta^2 p^2)(x_\mu p_\nu - x_\nu p_\mu) \) transforms \( x_\mu \) and \( p_\mu \) correctly and satisfies the correct Lorentz algebra (23).

\( O(\beta^2) \) results: With \( \Lambda \approx 1 + \frac{\beta^2 p^2}{2} - \left( \frac{\beta^2 p^2}{2} \right)^2 \) the Lagrangian \( L_{(2)} \) (without the mass-shell condition) becomes,
\[
L_{(2)} = -(x\dot{p}) \left( 1 - \frac{\beta p^2}{2} + \left( \frac{\beta^2 p^2}{2} \right)^2 \right) + \beta(xp)(p\dot{p}) \left( 1 - \frac{3\beta^2 p^2}{2} \right)
\] (28)
yielding the constraints,
\[
\phi^1_\mu = \Pi^\nu_\mu, \quad \phi^2_\mu = \Pi^\nu_\mu + x_\mu \left( 1 - \frac{\beta p^2}{2} + \left( \frac{\beta^2 p^2}{2} \right)^2 \right) - \beta p_\mu(xp) \left( 1 - \frac{3\beta^2 p^2}{2} \right).
\] (29)

The Dirac Brackets are, (with details in Appendix B),
\[
\{ x_\mu, x_\nu \} = D(x_\mu p_\nu - x_\nu p_\mu), \quad \{ p_\mu, p_\nu \} = 0,
\]
\[
\{ x_\mu, p_\nu \} = -\frac{g_{\mu\nu}}{\left( 1 - \beta^2 p^2 + \left( \frac{\beta^2 p^2}{2} \right)^2 \right)} - C p_\mu p_\nu
\] (30)
where
\[
C = \frac{\beta \left( 1 - \frac{3\beta^2 p^2}{2} \right)}{\left( 1 - \frac{3\beta^2 p^2}{2} + \frac{7\beta^2 p^4}{4} \right)}, \quad D = \frac{C \beta^2 p^2}{2 \left( 1 - \frac{3\beta^2 p^2}{2} \right)}.
\]

We notice that the Snyder form has been recovered once \( O(\beta^2) \) contributions are introduced. This GUP based algebra - Snyder algebra connection constitutes the other major result. It is possible to construct the deformed Poincare generators but the expressions are quite involved and not very illuminating.

Two parameter \((\beta, \beta')\) results: We now provide a considerably simpler Lagrangian with two parameters \( \beta \) and \( \beta' \) that can induce the Snyder algebra. Note that \textit{ab initio} it would have been hard to guess this result as well as the explicit expressions for the algebra but in our constraint framework this is quite straightforward. From the constraint analysis that generates the Dirac Brackets it is clear that we need a non-vanishing \( \{ \phi_2^\mu, \phi_2^\nu \} \) to reproduce a non-vanishing \( \{ x_\mu, x_\nu \} \) bracket. Let us go back to (24) and Appendix A. It is now clear that the two non-canonical terms in \( L_{(1)} \) must have different \( \beta \)-factors to produce the desired effect. Hence we consider the Lagrangian \( L_{(\beta,\beta')} \) (without the mass-shell condition),
\[
L_{(\beta,\beta')} = -(x\dot{p}) \left( 1 - \frac{\beta p^2}{2} \right) + \beta'(xp)(p\dot{p}).
\] (31)
The constraints of the model are,
\[
\phi^1_\mu = \Pi^\nu_\mu, \quad \phi^2_\mu = \Pi^\nu_\mu + x_\mu \left( 1 - \frac{\beta^2 p^2}{2} \right) - \beta'(xp)p_\mu,
\] (32)
giving rise to the Dirac Brackets, (with details in Appendix C),

\[
\{x_\mu, x_\nu\} = D \frac{(\beta - \beta')}{\beta'} (x_\mu p_\nu - x_\nu p_\mu), \quad \{p_\mu, p_\nu\} = 0, \\
\{x_\mu, p_\nu\} = -\frac{g_{\mu\nu}}{(1 - \frac{\beta p^2}{2})} - D p_\mu p_\nu,
\]

(33)

where

\[
D = \frac{\beta'}{(1 - \frac{\beta p^2}{2} - \beta' p^2) \left(1 - \frac{\beta p^2}{2}\right)}.
\]

Clearly for \(\beta = \beta' \rightarrow \{x_\mu, x_\nu\} = 0\) leaving a GUP like algebra. We have not shown the deformed Poincaré generators which are quite complicated.

Incidentally there is a simple linear model with Snyder algebra obtainable from the previous one by putting \(\beta' = 0\) and considering terms up to \(O(\beta)\) only. The Lagrangian \(L_S\)

\[
L = -(xp) \left(1 - \frac{\beta p^2}{2}\right) + \frac{e}{2}(p^2 - m^2),
\]

(34)

yields the Snyder algebra,

\[
\{x_\mu, x_\nu\} = \beta \frac{(x_\mu p_\nu - x_\nu p_\mu)}{(1 - \frac{\beta p^2}{2})^2}, \quad \{p_\mu, p_\nu\} = -\frac{g_{\mu\nu}}{(1 - \frac{\beta p^2}{2})}, \quad \{p_\mu, x_\nu\} = 0.
\]

(35)

Interestingly, its’ \(O(\beta)\) linearized version,

\[
\{x_\mu, x_\nu\} = \beta(x_\mu p_\nu - x_\nu p_\mu) , \quad \{x_\mu, p_\nu\} = -g_{\mu\nu} \left(1 + \frac{\beta p^2}{2}\right) , \quad \{p_\mu, p_\nu\} = 0.
\]

(36)

also satisfies the Jacobi identity. One can check that \(\left(\frac{x_\mu}{(1 + \frac{\beta p^2}{2})}, p_\mu\right)\) constitutes a canonical pair with \(j_{\mu\nu} = \frac{1}{(1 + \frac{\beta p^2}{2})} (x_\mu p_\nu - x_\nu p_\mu)\) being the deformed Lorentz generator.

4 GUP particle in external electromagnetic field

We introduce minimally coupled \(U(1)\) gauge interaction to the free GUP particle Lagrangian \([18]\),

\[
L = -\frac{(xp)}{\Lambda} + \frac{\beta(xp)(pp)}{\Lambda^2 \sqrt{1 + 2\beta p^2}} + \lambda(f(p^2) - m^2) + e(A\dot{x}).
\]

(37)

The symplectic structure is changed and we need to compute the new Dirac algebra. The conjugate momenta follows are \([37]\) are

\[
\pi^x_\mu = \frac{\partial L}{\partial \dot{x}_\mu} = eA_\mu(x) , \quad \pi^p_\mu = \frac{\partial L}{\partial \dot{p}_\mu} = -\frac{x_\mu}{\Lambda} + \frac{\beta(xp)p_\mu}{\Lambda^2 \sqrt{1 + 2\beta p^2}}.
\]

(38)

The constraints follow:

\[
\phi^1_\mu = \pi^x_\mu - eA_\mu(x) , \quad \phi^2_\mu = \pi^p_\mu + \frac{x_\mu}{\Lambda} - \frac{\beta(xp)p_\mu}{\Lambda^2 \sqrt{1 + 2\beta p^2}}.
\]

(39)
Since we have the Poission brackets
\begin{equation}
\{x_\mu, \pi^\nu_\nu\} = -g_{\mu\nu}, \quad \{p_\mu, \pi^\nu_\nu\} = -g_{\mu\nu},
\end{equation}
the constraint algebra is,
\begin{equation}
\{\phi^1_\mu, \phi^1_\nu\} = \{\pi^\mu_\mu - eA_\mu(x), \pi^\nu_\nu - eA_\nu(x)\} = -eF_{\mu\nu}(x),
\end{equation}
\begin{align}
\{\phi^1_\mu, \phi^2_\nu\} &= \{\pi^\mu_\mu - eA_\mu(x), \pi^\nu_\nu + x_\nu \frac{\beta(x)p_\nu}{\Lambda^2 \sqrt{1 + 2\beta p^2}} - \frac{\beta(x)p_\nu p_\nu}{\Lambda^2 \sqrt{1 + 2\beta p^2}}\} = \frac{1}{\Lambda} g_{\mu\nu} - \frac{\beta p_\mu p_\nu}{\Lambda^2 \sqrt{1 + 2\beta p^2}}, \\
\{\phi^2_\mu, \phi^2_\nu\} &= \{\pi^\mu_\mu + x_\mu \frac{\beta(x)p_\mu}{\Lambda^2 \sqrt{1 + 2\beta p^2}}, \pi^\nu_\nu - x_\nu \frac{\beta(x)p_\nu}{\Lambda^2 \sqrt{1 + 2\beta p^2}}\} = 0.
\end{align}
This gives the constraint matrix as
\begin{equation}
\{\phi^i_\mu, \phi^j_\nu\} = \begin{bmatrix}
-eF_{\mu\nu} & g_{\mu\nu} - \frac{\beta p_\mu p_\nu}{\Lambda^2 \sqrt{1 + 2\beta p^2}} \\
-g_{\mu\nu} + \frac{\beta p_\mu p_\nu}{\Lambda^2 \sqrt{1 + 2\beta p^2}} & 0
\end{bmatrix} = A + eB
\end{equation}
where
\begin{align}
A &= \begin{bmatrix}
0 & \frac{g_{\mu\nu}}{\Lambda^2} - \frac{\beta p_\mu p_\nu}{\Lambda^2 \sqrt{1 + 2\beta p^2}} \\
\frac{g_{\mu\nu}}{\Lambda^2} + \frac{\beta p_\mu p_\nu}{\Lambda^2 \sqrt{1 + 2\beta p^2}} & 0
\end{bmatrix}
, \quad B = \begin{bmatrix}
-F_{\mu\nu} & 0 \\
0 & 0
\end{bmatrix}
\end{align}
To $O(\epsilon)$ the inverse is
\begin{equation}
(A + eB)^{-1} = A^{-1} - eA^{-1}BA^{-1}.
\end{equation}
In the present case, the inverse of \([44]\) to first order of $\epsilon$ is
\begin{equation}
\{\phi^i_\mu, \phi^j_\nu\}^{-1} = \begin{bmatrix}
0 & -\Lambda g_{\mu\nu} - \beta p_\mu p_\nu \\
\Lambda g_{\mu\nu} + \beta p_\mu p_\nu & -\epsilon \Lambda(\Lambda F_{\mu\nu} + \beta p_\alpha(F_{\mu\alpha p_\nu} - F_{\nu\alpha p_\mu}))
\end{bmatrix}
\end{equation}
Therefore the Dirac brackets, modified by the $U(1)$ interaction, are
\begin{align}
\{x_\alpha, x_\gamma\}^* &= \{x_\alpha, x_\gamma\} - \{x_\alpha, \phi^1_\mu\} \{\phi^1_\mu, \phi^1_\nu\}^{-1} \{\phi^1_\nu, x_\gamma\} = 0,
\end{align}
\begin{equation}
\{x_\alpha, p_\gamma\}^* = \{x_\alpha, p_\gamma\} - \{x_\alpha, \phi^1_\mu\} \{\phi^1_\mu, \phi^2_\nu\}^{-1} \{\phi^2_\nu, p_\gamma\} = -(\Lambda g_{\alpha\gamma} + \beta p_\alpha p_\gamma),
\end{equation}
\begin{equation}
\{p_\alpha, p_\gamma\}^* = \{p_\alpha, p_\gamma\} - \{p_\alpha, \phi^2_\mu\} \{\phi^2_\mu, \phi^2_\nu\}^{-1} \{\phi^2_\nu, p_\gamma\} = -\epsilon \Lambda(\Lambda F_{\alpha\gamma} + \beta p_\alpha(F_{\alpha\alpha p_\gamma} - F_{\gamma\alpha p_\alpha})),
\end{equation}
The relativistic Hamiltonian is,
\begin{equation}
H = \frac{p^2}{m} - \sqrt{p^2}.
\end{equation}
Using the Dirac brackets (47-49) and (50), the Hamiltonian equations of motion are,

\[ \dot{x}_\alpha = \{x_\alpha, \frac{p_\gamma p_\gamma}{m} - \sqrt{p_\gamma p_\gamma}\}^* = -\frac{1}{m} \Lambda^2 \sqrt{1 + 2\beta p^2} \ p_\alpha, \]  
(51)

and

\[ \dot{p}_\alpha = \{p_\alpha, \frac{p_\gamma p_\gamma}{m} - \sqrt{p_\gamma p_\gamma}\}^* = -\frac{e}{m} \Lambda^2 \sqrt{1 + 2\beta p^2} \ p_\alpha F_{\alpha\gamma}. \]  
(52)

Keeping only \( O(e) \) terms we can eliminate \( p \), to get the modified Newton’s law,

\[ \ddot{x}_\alpha = -\frac{e}{m} \Lambda^2 \sqrt{1 + 2\beta p^2} \ \dot{x}_\alpha F_{\alpha\gamma} = \frac{e}{m} \Lambda^2 \sqrt{1 + 2\beta m^2} \ \dot{x}_\gamma F_{\alpha\gamma}. \]  
(53)

It is important to note that the dynamics in (52) and (53) is exact for the GUP parameter \( \beta \) although it is to the first order of \( e \). Hence the dynamics remains qualitatively unchanged with a renormalization of the charge. The \( O(\beta) \) equation of motion is

\[ \ddot{x}_\alpha = -\frac{e}{m} (1 + 2\beta m^2) \dot{x}_\gamma F_{\alpha\gamma}. \]  
(54)

### 5 Summary and Conclusion

The present paper deals with an extension of the Generalized Uncertainty Principle in a covariant setting.

We have established a Lagrangian (or symplectic) framework where the Generalized Uncertainty Principle and extended Snyder Algebra are studied in a unified way. The above non-commutative structures appear as Dirac Brackets. It is instructive to see how the Lagrangian model for Generalized Uncertainty Principle reduces to the Snyder model at different level of approximation. Quite interestingly, the more complicated Snyder algebra, (since the coordinates do not commute among themselves), emerges from an approximation of the model for the simpler algebra compatible to the Generalized Uncertainty Principle. Our analysis explicitly shows that, while considering approximations, it is not always pertinent to truncate a symplectic algebra directly as this approximation may invalidate Jacobi identities. On the other hand it is always legitimate to start from a Lagrangian (or symplectic structure), approximate at this level and subsequently compute the algebra as Dirac Brackets, especially when non-linear constraints are involved. But this requires construction of the Lagrangian for a given Non-Commutative algebra as we have done here for the algebra based on Generalized Uncertainty Principle. These generalized particle models and the connection between Generalized Uncertainty Principle based algebra and Snyder algebra are new results.

Finally we have studied behavior of the GUP particle in presence of external \( U(1) \) gauge interaction. We find that to lowest order in \( e \) (the gauge coupling) and to all orders in \( \beta \) (Non-Commutative GUP parameter), the charge gets modified without any qualitative change in dynamics.
Apart from the above algebraic consistency arguments in favor of a Lagrangian framework, from the physics point of view it is indeed appealing and worthwhile to have a point particle picturisation of the noncommutative algebra in question as it clearly shows how it differs from the conventional particle.

6 Appendices:

In the Dirac Hamiltonian scheme of constraint analysis, non-commuting constraints are termed as Second Class Constraints. Consider a set of Second Class Constraints $\phi^i_i \approx 0$ where the constraint commutator matrix $\{\phi^i_i, \phi^j_j\} = \Gamma^{i\mu}_{j\nu}$ in non-singular with an inverse $\Gamma^{\mu\nu}_{ij}$ such that

$$\Gamma^{ij}_{\mu\nu} \Gamma^{\nu\lambda}_{jk} = \delta^i_j \delta^\mu_\lambda.$$

The Dirac Bracket between two generic variables is defined as,

$$\{A, B\}_{\text{Dirac Bracket}} = \{A, B\} - \{A, \phi^i\} \{\phi^i, B\} = \{A, B\} - \{A, \phi^i\} \Gamma^{\mu\nu}_{ij} \{\phi^j, B\}. \quad (55)$$

This makes $\{\phi^i, A\}_{\text{Dirac Bracket}} = 0$ and one can exploit $\phi^i = 0$ strongly provided one uses Dirac Brackets. Upon quantization the Dirac Brackets are elevated to quantum commutators. Throughout our work we have dropped the subscript Dirac Bracket.

Appendix A: In the linearized model, $\Lambda = \frac{\beta p^2}{\sqrt{1+2\beta p^2-1}} \simeq 1 + \frac{1}{2} \beta p^2$ and

$$L(1) \simeq -(xp) \left( 1 - \frac{\beta p^2}{2} \right) + \beta(xp)p.$$

The momenta $\Pi^x_\mu = 0$, $\Pi^p_\mu = -x_\mu \left( 1 - \frac{\beta p^2}{2} \right) + \beta(xp)p_\mu$ gives the constraints $\phi^1_\mu = \Pi^x_\mu$, $\phi^2_\mu = \Pi^p_\mu + x_\mu \left( 1 - \frac{\beta p^2}{2} \right) - \beta(xp)p_\mu$. The constraint matrix

$$\Gamma^{i\mu}_{\nu\nu} = \begin{bmatrix} 0 & g_{\mu\nu} \left( 1 - \frac{\beta p^2}{2} \right) - \beta p_\mu p_\nu \\ -g_{\mu\nu} \left( 1 - \frac{\beta p^2}{2} \right) + \beta p_\mu p_\nu & 0 \end{bmatrix} \quad (57)$$

has the inverse,

$$\Gamma_{ij\mu\nu} = \begin{bmatrix} 0 & \left( \frac{g_{\mu\nu}}{1 - \frac{\beta p^2}{2}} + \frac{\beta p_\mu p_\nu}{1 - \frac{\beta p^2}{2}} \right) \left( 1 - \frac{3\beta p^2}{2} \right) \\ -\left( \frac{g_{\mu\nu}}{1 - \frac{\beta p^2}{2}} + \frac{\beta p_\mu p_\nu}{1 - \frac{\beta p^2}{2}} \right) \left( 1 - \frac{3\beta p^2}{2} \right) & 0 \end{bmatrix}. \quad (58)$$

Using the definition given above in (55) we find the Dirac Bracket

$$\{x^\mu, p^\nu\} = - \left[ \frac{g_{\mu\nu}}{1 - \frac{\beta p^2}{2}} + \frac{\beta p_\mu p_\nu}{1 - \frac{\beta p^2}{2}} \right], \quad \{x^\mu, x^\nu\} = \{p^\mu, p^\nu\} = 0. \quad (59)$$

Appendix B: For $O(\beta^2)$ contribution, $\Lambda = 1 + \frac{\beta p^2}{2} - \left( \frac{\beta p^2}{2} \right)^2$,
\[ L_{(2)} = -(x\dot{p}) \left( 1 - \frac{\beta p^2}{2} + \left( \frac{\beta p^2}{2} \right)^2 \right) + \beta(xp)(p\dot{p}) \left( 1 - \frac{3\beta p^2}{2} \right) \]  

(60)

with the constraints,
\[ \phi_\mu^1 = \Pi_\mu^x, \quad \phi_\mu^2 = \Pi_\mu^p + x_\mu \left( 1 - \frac{\beta p^2}{2} + \left( \frac{\beta p^2}{2} \right)^2 \right) - \beta p_\mu(xp) \left( 1 - \frac{3\beta p^2}{2} \right) . \]  

(61)

The constraint matrix is
\[ \Gamma_{ij}^{\mu\nu} = \{ \phi_i^\mu, \phi_j^\nu \} = \begin{bmatrix} 0 & g_{\mu\nu} \left( 1 - \frac{\beta p^2}{2} + \left( \frac{\beta p^2}{2} \right)^2 \right) - \beta p_\mu p_\nu \left( 1 - \frac{3\beta p^2}{2} \right) \\
-g_{\mu\nu} \left( 1 - \frac{\beta p^2}{2} + \left( \frac{\beta p^2}{2} \right)^2 \right) + \beta p_\mu p_\nu \left( 1 - \frac{3\beta p^2}{2} \right) & \frac{1}{2} \beta^2 p^2 (x_\mu p_\nu - x_\nu p_\mu) \end{bmatrix} \]  

(62)

has the inverse,
\[ \Gamma_{ij}^{\mu\nu} = \{ \phi_i^\mu, \phi_j^\nu \}^{-1} = \begin{bmatrix} D(x_\mu p_\nu - x_\nu p_\mu) & -\frac{g_{\mu\nu}}{1 - \frac{\beta p^2}{2} + \left( \frac{\beta p^2}{2} \right)^2} - C p_\mu p_\nu \\
\frac{g_{\mu\nu}}{1 - \frac{\beta p^2}{2} + \left( \frac{\beta p^2}{2} \right)^2} + C p_\mu p_\nu & 0 \end{bmatrix} \]  

(63)

where
\[ C = \frac{\beta \left( 1 - \frac{3\beta p^2}{2} \right)}{\left( 1 - \frac{3\beta p^2}{2} + \frac{7\beta^2 p^4}{4} \right) \left( 1 - \frac{\beta p^2}{2} + \frac{\beta^2 p^4}{4} \right)} \]

and
\[ D = \frac{C\beta p^2}{2 \left( 1 - \frac{3\beta p^2}{2} \right)} . \]

Then the Dirac Brackets are
\[ \{x_\mu, x_\nu\} = D(x_\mu p_\nu - x_\nu p_\mu), \quad \{p_\mu, p_\nu\} = 0, \quad \{x_\mu, p_\nu\} = -\frac{g_{\mu\nu}}{1 - \frac{\beta p^2}{2} + \left( \frac{\beta p^2}{2} \right)^2} - C p_\mu p_\nu. \]  

(64)

**Appendix C:** For two parameters \( \beta \) and \( \beta' \) we have
\[ L_{(\beta,\beta')} = -(x\dot{p}) \left( 1 - \frac{\beta p^2}{2} \right) + \beta'(xp)(p\dot{p}) \]  

(65)

The constraints are
\[ \phi_\mu^1 = \Pi_\mu^x, \quad \phi_\mu^2 = \Pi_\mu^p + x_\mu \left( 1 - \frac{\beta p^2}{2} \right) - \beta' (xp)p_\mu \]  

(66)

The constraint matrix is
\[ \Gamma_{ij}^{\mu\nu} = \{ \phi_i^\mu, \phi_j^\nu \} = \begin{bmatrix} 0 & g_{\mu\nu} \left( 1 - \frac{\beta p^2}{2} \right) - \beta' p_\mu p_\nu \\
-g_{\mu\nu} \left( 1 - \frac{\beta p^2}{2} \right) + \beta' p_\mu p_\nu & (\beta - \beta')(x_\mu p_\nu - x_\nu p_\mu) \end{bmatrix} \]  

(67)
with the inverse,

\[ \Gamma_{ij\mu\nu} = \{ \phi^i, \phi^j \}^{-1} = \begin{bmatrix} \frac{D}{\beta'}(\beta - \beta')(x_\mu p_\nu - x_\nu p_\mu) & -\frac{g_{\mu\nu}}{(1-\frac{p^2}{2})} - Dp_\mu p_\nu \\ \frac{g_{\mu\nu}}{(1-\frac{p^2}{2})} + Dp_\mu p_\nu & 0 \end{bmatrix}, \]  

\[ (68) \]

where

\[ D = \frac{\beta'}{\left(1 - \beta p^2/2 - \beta' p^2 \right) \left(1 - \frac{\beta p^2}{2} \right)} \]

The Dirac Brackets are

\[ \{ x_\mu, x_\nu \} = \frac{D}{\beta'}(\beta - \beta')(x_\mu p_\nu - x_\nu p_\mu), \quad \{ p_\mu, p_\nu \} = 0, \quad \{ x_\mu, p_\nu \} = -\frac{g_{\mu\nu}}{(1-\frac{p^2}{2})} - Dp_\mu p_\nu. \]  

\[ (69) \]
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