Complete conversion of the single photon frequency
in an optical microcavity

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We describe an optical cavity resonantly coupled to $N$ atoms which performs the complete conversion of the single photon frequency in the absence of external driving field. Two states of this cavity couple to two transitions between three atomic levels and two input-output waveguides. The problem is reduced to the consideration of coupled $2N+1$ localized and $N+1$ waveguide collective states of a photon and atoms and solved exactly using the Mahaux-Weidenmüller formalism. The conditions of complete frequency conversion are determined and the effect of cumulative action of atoms is analyzed.

1. Introduction

The amplitude of resonant transmission of a quantum particle, which propagates though a cavity surrounded by potential barriers, can approach unity. This well-known physical effect [1, 2] has been explored to describe a wide range of phenomena in nuclear, atomic, and molecular scattering [1-4], propagation of electrons in superlattices and microstructures [5-10], and propagation of photons in cavity quantum electrodynamics (QED) [11-13].

The resonant transmission amplitude of unity assumes that the motion of a particle is purely elastic and no energy is transferred to the environment. In a realistic experimental situation, when a particle interacts with the environment, its energy is not conserved. With a better precision, the total energy is conserved in an expanded system described by collective states of this particle together with the particles and collective excitations interacting with it. For example, in waveguide-cavity QED, which considers the propagation of photons interacting with atoms inside waveguides and optical cavities, it is practical to introduce the collective states of photons and atoms [14]. A system of collective input-output waveguide states $|\Lambda_p\rangle$ and collective localized states $|\Psi_n\rangle$ coupled to each other is illustrated in Fig. 1. In the simplest case of two input-output states, $|\Lambda_1\rangle$ and $|\Lambda_2\rangle$, and a single resonant cavity state, $|\Psi_1\rangle$, the resonant transmission amplitude of this system is determined by the Breit-Wigner formula [1, 2]:

$$S_{12} = \frac{2\pi i W_{11} W_{12}}{\Omega - \Omega_i + i(\pi W_{11}^2 + \pi W_{12}^2 + \frac{1}{2} \Gamma_1)}.$$  \hspace{1cm} (1)

The transmission probability $P = |S_{12}|^2$ is equal to unity if the energy $\hbar \Omega$ of waveguide state $|\Lambda_i\rangle$ is equal to the eigenenergy $\hbar \Omega_i$ of the cavity state $|\Psi_i\rangle$, the waveguide-cavity coupling parameters are equal, $W_{11} = W_{12}$, and the internal cavity losses are absent, $\Gamma_1 = 0$. The latter condition is satisfied for an ideal closed system. The resonant transmission through such system assumes the conservation of energy $\hbar \Omega = \hbar \omega + \hbar \omega_{\text{env}}^0$ which is the sum of the initial energy of the physical particle under consideration, $\hbar \omega$, and the total initial energy of other particles and collective excitations, $\hbar \omega_{\text{env}}^0$. Since the energy $\hbar \Omega$ can be redistributed in the process of transmission, the physical particle can be
transmitted inelastically and, as result, resonantly acquire or release (respectively, the environment can release or acquire) an energy $h\Delta \omega$ with the amplitude equal to unity [15, 16]. As an example, Eq. (1) describes the inelastic resonant transmission of a single photon interacting with a three-level atom in an optical waveguide [17, 18]. In this case, the collective input-output states $|\Lambda_1\rangle$, $|\Lambda_2\rangle$ and localized state $|\Psi_1\rangle$ are defined as follows: $|\Lambda_1\rangle$ is a state of a photon with initial energy $h\omega$ and an atom in the initial state $|g_1\rangle$ with energy $h\omega^{(g)}_1$; $|\Psi_1\rangle$ is the state of the atom in the excited state $|e\rangle$ with energy $h\omega^{(e)}_1$ which has acquired this photon; $|\Lambda_2\rangle$ is the state of the output photon with energy $h(\omega + \omega^{(g)}_1 - \omega^{(e)}_1)$ emitted by the atom which is transmitted to the final state $|g_2\rangle$ with energy $h\omega^{(g)}_2$.

However, the realization of the described system enabling complete inelastic transition of a single photon is challenging. In particular, while the input photon frequency $\omega$ can be set in resonance with frequency $\omega^{(g)}_1 - \omega^{(e)}_1$ of atomic transition $|g_1\rangle \rightarrow |e\rangle$, the atomic transition $|e\rangle \rightarrow |g_2\rangle$ with converted frequency $\omega^{(e)}_2 - \omega^{(g)}_2$ is not resonant. Therefore, transitions from the excited state $|e\rangle$ to states with frequencies other than $\omega^{(g)}_1$ and $\omega^{(g)}_2$ cannot be ignored. In practice, this excludes the possibility of the complete frequency conversion for a single photon in the described system. To the best of our knowledge, the question if, in the absence of an external driving field, the complete conversion of the single photon frequency is indeed possible in more complex QED systems has not been answered to date.

In this paper, we give the answer to this question and describe a simplest system of resonant optical cavity and atoms, which performs the complete conversion of the single photon frequency. Our solution is based on the fact that the states of an optical cavity can be set in resonance with the selected atomic transitions such that all other possible transitions are not resonant and can be ignored. In the case when a photon state of the optical cavity is set in resonance with a single atomic transition, the two-level model of atoms is justified [19-22]. However, such system cannot change the photon frequency due to the energy conservation. In order to enable the frequency conversion of a single photon, it is necessary to consider its interaction with atoms having more than two resonant levels and, respectively, an optical cavity having more than one resonant state.

![Fig. 1. A system of localized collective states $|\Psi_n\rangle$ and input-output waveguide collective states $|\Lambda_n\rangle$. Here $V_{nm}$ are coupling parameters between localized states and $W_{np}$ are coupling parameters between localized and waveguide states. Generally, the states are multiparticle and include collective excitations. In this paper, we consider localized and waveguide states composed of a single photon and atoms.](image-url)
We consider a system of an optical cavity resonantly coupled to $N$ atoms. It is assumed that two eigenfrequencies of the cavity $\omega^{(c)}_1$ and $\omega^{(c)}_2$ are in resonance with two electronic transitions between three levels $\omega^{(a)}_1$, $\omega^{(a)}_2$, and $\omega^{(a)}_3$ of $N$ identical atoms so that $\omega^{(c)}_m$ is close to $\omega^{(a)}_m - \omega^{(a)}_{m+1}$. A photon, which initial frequency $\omega$ is close to $\omega^{(c)}_1$ is launched into the cavity from the input waveguide. Inside the cavity, the photon can be acquired by one of these atoms, which, in turn, can emit a photon with different frequency $\omega + \omega^{(a)}_1 - \omega^{(a)}_2$. Below we determine the amplitude of inelastic transmission of a photon through the system described and find conditions when this amplitude approaches unity. To this end, we present the Hamiltonian of the resonant system illustrated in Fig. 1 as a sum of ket-bra products of collective states of a photon and atoms interacting with it. Once the coupling parameters between these states are known, the S-matrix of transmission and reflection amplitudes is determined exactly by the Mahaux-Weidenmüller formula [23, 24].

2. Mahaux-Weidenmüller formalism

The Mahaux-Weidenmüller theory [23, 24] considers a general multiparticle system which includes resonantly coupled $N$ localized collective states and $P$ input-output collective states illustrated in Fig. 1. The localized states $|\Psi_n\rangle$, $n=1,2,...,N$, if assumed uncoupled, have complex eigenfrequencies $\Omega^{(0)}_n$. The waveguide states $|\Lambda_{\rho,\Omega}\rangle$ with numbers $p=1,2,...,P$ and energy $\hbar\Omega$ can be presented as a sum of input states $|\Lambda^{(in)}_{\rho,\Omega}\rangle$ and output states $|\Lambda^{(out)}_{\rho,\Omega}\rangle$. These states are confined along all directions in the multiparticle configuration space except for the those which correspond to the propagation of particles along the waveguide $\rho$. The Hamiltonian of this system is

$$H = \hbar [\sum_n \Omega^{(0)}_n |\Psi_n\rangle \langle \Psi_n| + \sum_{\rho,\Omega} \Omega^{(in)}_{\rho,\Omega} |\Lambda^{(in)}_{\rho,\Omega}\rangle \langle \Lambda^{(in)}_{\rho,\Omega}| + \sum_{\rho,\Omega} \Omega^{(out)}_{\rho,\Omega} |\Lambda^{(out)}_{\rho,\Omega}\rangle \langle \Lambda^{(out)}_{\rho,\Omega}|$$

$$+ \sum_{m,n} V_{mn} |\Psi_m\rangle \langle \Psi_n| + \sum_{n,\rho,\Omega} (W_{\rho n} |\Lambda^{(out)}_{\rho,\Omega}\rangle \langle \Psi_n| + W_{\rho n} |\Psi_n\rangle \langle \Lambda^{(out)}_{\rho,\Omega}|)$$

$$]$$

where the coupling between states and waveguides are assumed to be real and direct coupling between the input-output states is neglected. The S-matrix $S = \{S_{\rho p}\}$ of transmission amplitudes from input states $|\Lambda^{(in)}_{\rho,\Omega}\rangle$ to output states $|\Lambda^{(out)}_{\rho,\Omega}\rangle$ is expressed through the matrix of eigenfrequencies $\Omega^{(0)} = \{\Omega^{(0)}_n, \delta_{mn}\}$ of uncoupled localized states $|\Psi_n\rangle$ and matrices of coupling parameters between localized states, $V = \{V_{mn}\}$, and between localized states and waveguides, $W = \{W_{\rho n}\}$, by the Mahaux-Weidenmüller formula [23, 24]:

$$S = I - 2i\pi W^\dagger (\Omega I - \Omega^{(0)} - V + i\pi WW^\dagger)^{-1} W,$$
microelectronic and nanoelectronic devices [26-28], and transmission of photonic microstructures [29-32]. In applications to waveguide and cavity QED, this approach often allows one to arrive at general expressions for the scattering S-matrix much easier, without cumbersome calculations based on more detailed models formulated with the second quantization formalism (see, e.g., an example of resonant propagation of a single photon through an optical cavity coupled to two-level atoms [19-22, 33-36] described in the Supplemental Material [37]).

![Fig. 2](image)

**Fig. 2.** (a) Possible configurations of cavities with two states, $|u_1\rangle$ and $|u_2\rangle$, which resonantly interact with transitions of atoms situated in the region where these states overlap in space. (b) Collective localized states and semi-infinite collective waveguide states which describe the propagation of a single photon resonantly coupled to two states of an optical cavity and $N$ three-level atoms positioned inside the cavity. (c) Schematics of transition between resonant frequencies of collective states considered, $\Omega_1^{(c)} \leftrightarrow \Omega_1^{(a)} \leftrightarrow \Omega_2^{(c)}$.

3. **Double eigenfrequency optical microcavity coupled to $N$ three-level atoms**

We consider a system consisting of an optical microcavity with two states, $|u_1\rangle$ and $|u_2\rangle$, and $N$ three-level atoms inside. The optical states are coupled to two input-output waveguides so that state $|u_m\rangle$ is coupled to waveguide $m$ only. Possible models of such cavities are illustrated in Fig. 2(a). While the geometry of two cavity states illustrated at the top of Fig. 2(a) is apparent, a model of one-dimensional configuration of the cavity states shown at the bottom of Fig. 2(a) is presented in the Supplemental Material. For the application of our concern, these microcavity structures can be fabricated of Fabry-Perot [38-41], photonic crystal [42, 43], toroidal [44], bottle [45], and SNAP [46, 47] microresonators. Atoms are identical and have two ground states $|g_m\rangle$, $m = 1,2$, with energies $\hbar \omega^{(a)}_{g_m}$ and one excited state $|e\rangle$ with complex energy $\hbar (\omega^{(e)}_e - i \Gamma^{(a)}_e)$ where $\Gamma^{(a)}_e$ determines the state dissipation. The complex eigenfrequencies of optical cavity states $\hbar (\omega^{(c)}_m - i \Gamma^{(c)}_m)$, where $\Gamma^{(c)}_m$ determines the internal cavity loss, are assumed to be in resonance with atomic transition
frequencies, i.e., $\omega^{(c)}_m$ is close to $\omega^{(a)}_e - \omega^{(a)}_u$. A single photon with energy $\hbar \omega$, which close to energy $\hbar \omega^{(c)}_0$ of state $|u_i\rangle$, enters the system through waveguide 1. After entering the cavity state $|u_i\rangle$ through waveguide 1, the photon can be resonantly absorbed by one of atoms into excited state $|e\rangle$ and bounce between these states and cavity states $|u_m\rangle$ prior to exiting into one of the waveguides.

The described system can be presented as a system of $2N+1$ localized collective eigenstates, $|c_i\rangle$, and $|a_i\rangle$, $|\Lambda_1\rangle$, $|\Lambda_2\rangle$, ..., $|\Lambda_{2N}\rangle$ (Fig. 2(b)). Collective state $|c_i\rangle$ has a photon localized in the cavity state $|u_1\rangle$ with energy $\hbar \omega^{(c)}_1$ and all atoms in the ground state $|g_1\rangle$ with energy $\hbar \omega^{(a)}_1$. Collective state $|a_n\rangle$ has no photon in the cavity, a single atom with number $n$ in the excited state $|e\rangle$ and all other atoms in their ground states $|g_1\rangle$. All eigenenergies of states $|a_n\rangle$ have the same value $\hbar \Omega^{(a)} = \hbar[\omega^{(e)} + (N-1)\omega^{(a)}_1 - \frac{1}{2} \Gamma^{(c)}_1]$. The waveguide states $|\Lambda_1\rangle$ and $|\Lambda_2\rangle$ have equal total energies $\hbar \Omega = \hbar[\omega + N\omega^{(a)}_1]$.

For the cavity structure and resonance condition considered, a photon can either exit from cavity state $|u_i\rangle$ through waveguide 1 with the amplitude $S_{1,1}$ and conserve its original frequency $\omega$ or exit through atom $n$ and cavity state $|u_2\rangle$ through waveguide 2 with the amplitude $S_{1,2}$ and frequency conversion $\omega \rightarrow \omega + \omega^{(a)}_2 - \omega^{(a)}_1$. Direct application of Eq. (3) to the system shown in Fig. 2(b) yields (see Supplemental material [37]):

$$S_{1,2} = \frac{4i\pi W_1 W_2 V_1 V_2}{2 \left( \Delta \omega_1 - \Delta \omega_2 \sum_{n=1}^{N+1} \frac{V_1^2}{\Delta \omega_1^2 + \Delta \omega_2^2} \right)},$$

$$\Delta \omega_1 = \omega - \omega^{(e)}_1 + i(\pi W_1^2 + \frac{1}{2} \Gamma^{(c)}_1),$$

$$\Delta \omega_2 = \omega - \omega^{(e)}_2 + \omega^{(a)}_1 - \omega^{(a)}_2 + i(\pi W_2^2 + \frac{1}{2} \Gamma^{(c)}_2),$$

$$\Delta \omega_2 = \omega - \omega^{(e)}_2 + \omega^{(a)}_1 - \omega^{(a)}_2 + i(\pi W_2^2 + \frac{1}{2} \Gamma^{(c)}_2),$$

where $V_1$ is the coupling between states $|c_i\rangle$ and $|a_n\rangle$. $V_2$ is the coupling between states $|c_2n\rangle$ and $|a_2n\rangle$. $W_1$ is the coupling between states $|\Lambda_1\rangle$ and $|c_i\rangle$, and $W_2$ is the coupling between states $|\Lambda_2\rangle$ and $|c_2n\rangle$. All couplings between states $|\Lambda_2\rangle$ and $|c_2n\rangle$ are equal because they are determined by the coupling between photon state $|u_2\rangle$ and waveguide 2. The full inelastic transmission probability is determined from Eq. (4) as

$$P = \sum_{n=1}^{N} |S_{1,2n}|^2.$$
4. Complete frequency conversion in the absence of losses

We assume that couplings to atoms are independent of their number as, e.g., for atoms situated near an antinode of the optical cavity (see, e.g., [33-35]) and set \( V_{1n} = V_1 \) and \( V_{2n} = V_2 \). Then Eq. (4) is simplified:

\[
P = \frac{4\pi^2 N_{11}^2 W_2^2 V_1^2 V_2^2}{\Delta \omega_1 \left( \Delta \omega_1 - V_2^2 \right) - N \Delta \omega_2 V_1^2},
\]

Eq. (6)

Rescaling \( N^{1/2}V_1 \rightarrow V_1 \) transfers this equation to the case of a single atom, \( N = 1 \) and reduces the problem to the well-known problem of resonant propagation through three successively coupled collective localized states illustrated in Fig. 3(a) (see e.g., [30]). Remarkably, Eq. (6) shows that coupling \( V_1 \) can be effectively enhanced by the increasing the number of atoms \( N \). This result manifests the partial cumulative action of atoms when only \( V_1 \) rather than both \( V_1 \) and \( V_2 \) are enhanced. The latter fact has a simple physical explanation. While the amplitude of transmission of a photon from the cavity state \( |u_i\rangle \) into one of the excited atomic states increases with the number of atoms \( N \), the amplitude of transmission from this atomic state into cavity state \( |u_i\rangle \) does not depend on the number of atoms. This situation is different from the cumulative action of two-level atoms (see Supplemental Material [37]) where the photon emitted from an excited atom can return back to the same collective state and be acquired by other atoms.

It follows from Eqs. (6) and (4a) that the cavity losses and atomic dissipation can be ignored if

\[
\Gamma_1^{(c)} \ll 2\pi W_{11}^2, \quad \Gamma_2^{(c)} \ll 2\pi W_{22}^2
\]

(7a) and

\[
\Gamma_1^{(a)} \ll 2\pi(W_{11}^2 + W_{22}^2), \quad \Gamma_2^{(a)} \ll \frac{2(NV_1^2 + V_2^2 + \pi W_{11}^2 W_{22}^2)}{\pi(W_{11}^2 + W_{22}^2)}.
\]

(7b)

Eq. (7a) is similar to the condition of complete resonant transparency of an empty optical cavity discussed in the introduction, while Eq. (7b) requires that the atomic dissipation was relatively small compared to the cavity-atom and/or cavity-waveguide couplings. This means that the dissipation time of atoms should be relatively large compared to the characteristic time of inelastic transition.

In the absence of losses, \( \Gamma_1^{(c)} = \Gamma_2^{(c)} = 0 \), and for the exact resonance between the frequencies of the cavity and atoms, \( \omega_1^{(c)} + \omega_{a_1}^{(c)} = \omega_2^{(c)} + \omega_{a_2}^{(c)} = \omega^{(a)} \), the inelastic transmission probability of a photon \( P \) determined by Eq. (6) can be expressed through four dimensionless parameters: relative cavity-atom couplings \( \gamma_1 = N^{1/2}V_1 / \pi W_{11}^2 \) and \( \gamma_2 = V_2 / \pi W_{22}^2 \), relative cavity-waveguide coupling \( \xi = W_{22}^2 / W_{11}^2 \), and relative frequency deviation \( \Delta = (\omega - \omega_1^{(c)}) / \pi W_{11}^2 \). Figs. 3(b)-(f) show characteristic spectrograms of inelastic transmission probability \( P = P(\gamma_1, \gamma_2, \xi, \Delta) \) corresponding to particular relations between \( \gamma_1 \) and \( \xi \) when \( P \) can achieve unity. Generally, satisfaction of equality \( \gamma_2^2 = \gamma_1^2 \xi \) is sufficient for \( P \) to achieve unity at \( \Delta = 0 \) as illustrated in Figs. 3(b) and (c) which show \( P(1, 1, \gamma^2, 0, \Delta) \) and \( P(1, 0, \gamma^2, \Delta) \), respectively. In particular, for \( \gamma_1 = \gamma_2 \) and \( \xi = 1 \) the value of \( P \) achieves unity at \( \Delta = 0 \) and also at two other values of \( \Delta \) symmetrically positioned along the lines \( \Delta = \pm \gamma \) as shown in the spectrogram of \( P(1, \gamma, 1, \Delta) \) in Fig. 3(d). The ultra-flat behavior of inelastic transmission
probability approaching unity is achieved along the white horizontal line of Fig. 3(d) which corresponds to

\[ P(0.5, 0.5, 1, \Delta) = \frac{1}{1 + \Delta^6} \]  

(8)

known as the Butterworth filter profile in the theory of signal processing (see, e.g., [32]). An alternative sufficient condition for achieving the unity probability of frequency conversion, \( \gamma_1^2 = \gamma_2^2 \Xi \) (for \( \Xi < 1 \) only) is illustrated in Fig. 3(e) showing \( P(\gamma_1^{1/2}, \gamma, \gamma_2^{1/2}, \Delta) \). Finally, the sufficient condition \( \gamma_1^2 = \gamma_2^2 = \Xi \) for unity probability \( P \) is illustrated in Fig. 3(f) which shows \( P(\gamma, \gamma, \gamma_2^2, \Delta) \).

Experimental realization of complete frequency conversion of a single photon illustrated in Fig. 3 is possible provided that the cavity loss and atomic dissipation are small enough. Specifically, it is required that the conditions of Eqs. (7) for the system losses are satisfied in the regions of surface plots in Figs. 3(b)-(f) where \( P \) approaches unity. For example, assuming that the introduced dimensionless parameters of the system have the same order of magnitude equal to unity, \( \gamma_1 \sim \gamma_2 \sim \Xi \sim 1 \), these conditions simply require that \( \Gamma_{a(e)}^{(a)}, \Gamma_{m(e)}^{(e)} \ll \pi W_1^2 \). Note that the condition \( \gamma_1 \sim 1 \) can be satisfied for sufficiently large number of atoms \( N \) even if \( V_1 \ll V_2 \).

Fig. 3. (a) A system of coupled collective states of a two-level cavity and a single three-level atom which is equivalent to the system of \( N \) atoms with equal couplings. (b)-(f) Spectrograms of inelastic transmission probability \( P(\gamma_1, \gamma_2, \Xi, \Delta) \) for particular relations between \( \gamma_1, \gamma_2, \) and \( \Xi \) indicated on the figures. Horizontal white dashed line in (d) corresponds to ultra-flat behavior of \( P \) described by Eq. (8).
5. Summary

We have described the simplest cavity QED system, which enables the complete conversion of a single photon frequency in the absence of an external driving field. This system consists of an optical cavity having two states, which resonantly couple to two electronic transitions of atoms positioned inside the cavity. Using the Mahaux-Weidenmüller formalism, the general expression for the inelastic resonant transmission amplitudes of a single photon through the cavity with \( N \) atoms distributed inside is derived. For identical atoms having equal couplings \( V_1 \) to the input cavity state \(|u_1\rangle\) and equal couplings \( V_2 \) to the output cavity state \(|u_2\rangle\), the action of \( N \) atoms is reduced to the action of a single atom with couplings \( N^{1/2}V_1 \) and \( V_2 \). Thus, while the coupling of atoms to the input cavity state \(|u_1\rangle\) is cumulatively enhanced with the number of atoms \( N \), the effect of coupling to the second cavity state \(|u_2\rangle\) does not depend on the number of atoms. The contribution of cavity losses and atomic dissipation to the probability of the frequency conversion of a single photon is estimated. It is shown that the complete frequency conversion is possible provided that the dissipation time of a photon in the cavity and excited electronic state in atoms is small compared to conversion time. Overall, the Mahaux-Weidenmüller formalism allows one to consider systems consisting of a large number of quantum particles and collective excitations distributed inside microcavities. It is of a great interest to develop a similar approach for the input-output cavity QED problems with more than one photon and more complex configurations of optical microcavities and quantum excitations.

The author acknowledges the Royal Society Wolfson Research Merit Award (WM130110) and funding from the Horizon 2020 Framework Programme (H2020) (H2020-EU.1.3.3, 691011), Engineering and Physical Sciences Research Council (EPSRC) (EP/P006183/1), and US Army Research Laboratory (W911NF-17-2-0048). Discussions with M. Brodsky, A. Fotiadi, V. Malinovsky, A. Sukhorukov, and I. Yurkevich are greatly appreciated.
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The described exchange of energy between the propagating particle and environment within the collective state is similar to the exchange of energy between degrees of freedom of a wave propagating in a nonuniform waveguide which can be performed adiabatically or, as in our case, resonantly. Consider the propagation of a wave through a waveguide with a resonant state inside, which is described by Eq. (1). If the output part of the waveguide is thinner or wider than the input part then the wave can resonantly switch its propagation constant. While the total energy of the wave is conserved, the energy of its transverse degrees of freedom is transferred to or acquired by its longitudinal degree of freedom.

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1. Inverse of a partitioned matrix

Our calculations of S-matrix determined by the Mahaux-Weidenmüller formula, Eq. (3), are based on the expression for the inverse of a partitioned matrix [S1]:

$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} F & -FB^{-1}D^{-1}CF \\ -D^{-1}CFD^{-1} & D^{-1}+D^{-1}CFBD^{-1} \end{pmatrix}$, \quad F = (A - BD^{-1}C)^{-1}$  \quad (S1)

![Fig. S1.](image)

**Fig. S1.** (a) Illustration of an optical cavity with atoms inside coupled to the input and output waveguides. (b) Collective localized states and semi-infinite collective waveguide states which describe the propagation of a single photon resonantly coupled to a single state of an optical cavity and $N$ two-level atoms positioned inside the cavity. (c) Schematics of transition between resonant frequencies of collective states considered, $\Omega_{1}^{(c)} \leftrightarrow \Omega_{n}^{(a)} \leftrightarrow \Omega_{1}^{(c)}$.

2. Single eigenfrequency optical cavity coupled to $N$ two-level atoms

It is instructive to demonstrate the application of the Mahaux-Weidenmüller formalism to the classical problem of the resonant propagation of a single photon through a single eigenfrequency optical microcavity coupled to $N$ two-level atoms illustrated in Fig. S1(a) [S2, S3]. This system can be modelled by $N+1$ localized collective eigenstates, $|c_i\rangle$ and $|a_n\rangle$, $n=1,2,\ldots,N$, and two waveguide states, $|\Lambda_1\rangle$ and $|\Lambda_2\rangle$ shown in Fig. S1(b). The collective localized state $|c_i\rangle$ is composed of a single photon state $|c_i\rangle$ localized in the cavity and ground states $|g_n\rangle$ of $N$ atoms. The total eigenenergy of this state, if assumed uncoupled to atoms and waveguides, is $\hbar \Omega_{i}^{(c)} = \hbar (\omega_i^{(c)} + N\omega_{0}^{(a)} - \frac{1}{2} \Gamma_i^{(c)})$ where
\( \omega^{(c)} \) and \( \Gamma^{(c)} \) is the eigenfrequency and resonance width of uncoupled optical cavity. Localized states \( |a_n\rangle \) are the collective states of atom \( n \) in its excited state \( |e_n\rangle \) with complex eigenenergy \( h(\omega^{(a)}_e - \frac{i}{2} \Gamma^{(a)}_e) \) (here \( \Gamma^{(a)}_e \) is the atom dissipation rate), other atoms remaining in their ground states, and no photon. In the resonance approximation, the direct coupling of states \( |a_n\rangle \) to the waveguides is ignored. For uncoupled localized states \( (V_{in}^{(a)} = 0) \), the energy of localized state \( |a_n\rangle \) is \( h\Omega^{(a)} = h[\omega^{(a)}_e + (N-1)\omega^{(a)}_g - \frac{i}{2} \Gamma^{(a)}_e] \). The collective input and output states \( \{\Lambda_p\} \) are composed of the waveguide state of a photon with energy \( h\omega \) and localized ground states \( |g_n\rangle \) of atoms with eigenenergies \( h\omega^{(a)}_g \), \( n = 1, 2, ..., N \), which are distributed inside the cavity. The total energy of these states is \( h\Omega = h(\omega + N\omega^{(a)}_g) \).

The \( S \)-matrix of the considered structure can be found by setting in Eq. (3)

\[
\Omega I - \Omega^{(0)} - V + i\pi W^T W = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]  

with

\[
A = \left( \Omega - \Omega^{(c)}_1 + i\pi(W_{11}^2 + W_{12}^2) \right),
\]

\[
B = \begin{pmatrix} V_{11} & V_{12} & \ldots & V_{1N} \end{pmatrix},
\]

\[
C = B^T,
\]

\[
D = \begin{pmatrix} \Omega - \Omega^{(a)}_1 & 0 & \ldots & 0 \\ 0 & \Omega - \Omega^{(a)}_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \Omega - \Omega^{(a)}_N \end{pmatrix},
\]

where \( \Omega^{(c)}_1 = \omega^{(c)}_1 + \sum_{n=1}^{N} \omega^{(a)}_g - \frac{i}{2} \Gamma^{(c)}_c \), \( \Omega^{(a)}_n = \omega^{(a)}_e - \omega^{(a)}_g + \sum_{m=1}^{N} \omega^{(a)}_g - \frac{i}{2} \Gamma^{(a)}_e \), and \( h\omega^{(a)}_g \) and \( h\omega^{(a)}_e - \frac{i}{2} \Gamma^{(a)}_e \) are the eigenfrequencies of two-level atom \( n \). After substitution of Eqs. (S3) into Eq. (S1) and simple calculations, Eq. (3) is reduced to:

\[
S_{11} = 1 - 2i\pi \frac{W_{11}^2}{\Xi}, \quad S_{12} = 2i\pi \frac{W_{11}W_{12}}{\Xi},
\]

\[
\Xi = \Omega - \Omega^{(c)}_1 - \sum_{n=1}^{N} \frac{(V_{1n}^{(a)})^2}{\Omega - \Omega^{(a)}_n}.
\]

For identical atoms, we have \( \Omega^{(a)}_n = \Omega^{(a)} \) and Eq. (S4) can be written as:
\[ S_{11} = 1 - 2i\pi \frac{W_{11}^2}{\Xi}, \quad S_{12} = 2i\pi \frac{W_{11}W_{12}}{\Xi}, \]

\[ \Xi = \omega - \omega_1^{(c)} + i\left(\frac{1}{2} \Gamma_1^{(c)} + \pi W_{11}^2 + \pi W_{12}^2\right) - \frac{N\bar{V}_1^2}{\omega + \omega^{(a)}_g - \omega^{(a)}_e + \frac{1}{2} \Gamma^{(a)}_e}, \] (S5)

\[ \bar{V}_1 = \left(\frac{1}{N} \sum_{n=1}^{N} (V_{in})^2\right)^{1/2}, \]

**Fig. S2.** Spectrogram of transmission probability \( P = |S_{12}|^2 \) as a function of dimensionless frequency deviation \( (\omega - \omega_1^{(c)}) / \pi W_{11}^2 \) and cumulative cavity-atom coupling \( N^{1/2}\bar{V}_1 / \pi W_{11}^2 \) for symmetric waveguide coupling \( W_{11} = W_{12} \), and negligible cavity and atomic losses.

where \( \bar{V}_1 \) is the root-mean-square of cavity-atom couplings \( V_{in} \). For the case of a single atom, \( N = 1 \), and symmetric waveguide-cavity coupling, \( W_{11} = W_{12} \), this result coincides with that found previously [S5]. It follows from Eq. (S5) that the contribution of \( N \) identical atoms to the scattering matrix is the same as that of a single atom with the cavity-atom coupling \( N^{1/2}\bar{V}_1 \) [S2, S3]. For negligible interactions with atoms, \( \bar{V}_1 = 0 \), Eq. (S5) coincides with Eq. (1). For relatively large \( N^{1/2}\bar{V}_1 \), the transmission amplitude has two resonances separated by Rabi frequency approximately equal to \( 2N^{1/2}\bar{V}_1 \). Remarkably, Eq. (S5) shows that transmission amplitude \( |S_{12}| \) approaches unity at these resonances if, similar to Eq. (1), the cavity waveguide couplings are equal, \( W_{11} = W_{12} \), the internal losses are small compared to the losses due to the coupling to waveguides, \( \Gamma_1^{(c)}, \Gamma_e^{(a)} \ll 4\pi W_{11}^2 \), and

\[ \Gamma_e^{(a)} \ll \frac{N\bar{V}_1^2}{\pi W_{11}^2}. \] (S6)

This equation can be satisfied for large \( N \) even if the average cavity-atom coupling \( \bar{V}_1 \) is small. Since the process of interaction of a photon with atoms and microcavity is inelastic and dissipative, the possibility to reach the unity value of the transmission amplitude of a photon by increasing the number of atoms is not obvious. Fig. S2 shows the behavior of transmission probability \( P = |S_{12}|^2 \) as a
function of dimensionless frequency deviation \( (\omega - \omega^{(c)}) / \pi W^{2}_{11} \) and cumulative coupling \( N^{1/2} v_{i} / \pi W^{2}_{11} \) for symmetric waveguide coupling \( W_{11} = W_{12} \), and negligible cavity and atomic losses. In this figure, the resonances of transmission amplitude equal to unity at frequencies are separated by Rabi frequency. The width of these resonances \( \sim \pi W^{2}_{11} \) is independent of the number of atoms, while their separation grows proportionally to \( N^{1/2} v_{i} \).

3. Double eigenfrequency optical cavity coupled to \( N \) three-level atoms

Let us consider now the case of three-level atoms coupled to two states of optical cavity illustrated in Fig. 2 of the main text. The expression for the S-matrix can be found from Eq. (3) using Eqs. (S1) and (S2) with submatrices:

\[
A = \begin{pmatrix}
\Omega - \Omega^{(c)}_{1} + i \pi W^{2}_{11} & V_{11} & V_{12} & \ldots & V_{1N} \\
V_{11} & \Omega - \Omega^{(a)}_{1} + \frac{i}{2} \Gamma^{(a)}_{e} & 0 & \ldots & 0 \\
V_{12} & 0 & \Omega - \Omega^{(a)}_{1} + \frac{i}{2} \Gamma^{(a)}_{e} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
V_{1N} & 0 & 0 & \ldots & \Omega - \Omega^{(a)}_{1} + \frac{i}{2} \Gamma^{(a)}_{e}
\end{pmatrix},
\]

\[
B = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 \\
V_{21} & 0 & 0 & \ldots & 0 \\
0 & V_{22} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & V_{2N}
\end{pmatrix},
\]

\[
C = B^{T},
\]

\[
D = \begin{pmatrix}
\Omega - \Omega^{(c)}_{2} + i \pi W^{2}_{22} & 0 & \ldots & 0 \\
0 & \Omega - \Omega^{(a)}_{2} + i \pi W^{2}_{22} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \Omega - \Omega^{(a)}_{2} + i \pi W^{2}_{22}
\end{pmatrix},
\]

where

\[
\Omega = \omega + N \omega^{(a)}_{g_{1}} \\
\Omega^{(c)}_{1} = \omega^{(c)}_{1} + N \omega^{(a)}_{g_{1e}} - \frac{i}{2} \Gamma^{(c)}_{1} \\
\Omega^{(c)}_{2} = \omega^{(c)}_{2} + \omega^{(a)}_{g_{2e}} + (N - 1) \omega^{(a)}_{g_{1}} - \frac{i}{2} \Gamma^{(c)}_{2} \\
\Omega^{(a)}_{1} = \omega^{(a)}_{g_{1}} + (N - 1) \omega^{(a)}_{g_{1}} - \frac{i}{2} \Gamma^{(a)}_{e},
\]

and \( \hbar \omega^{(a)}_{g_{1}}, \hbar \omega^{(a)}_{g_{2}}, \) and \( \hbar \omega^{(a)}_{g_{1e}} - \frac{i}{2} \Gamma^{(a)}_{e} \) are the eigenfrequencies of three-level atom \( n \). Direct calculations yield the following expression for matrix \( F \) in Eq. (S1):
In order to find the inverse of matrix in Eq. (S9), we partition it as

$$F = \left( A - BD^{-1}C \right)^{-1} =$$

$$\begin{pmatrix}
\Omega - \Omega_1^{(c)} + i\pi W_{11}^2 & V_{11} & V_{12} & \cdots & V_{1N} \\
V_{11} & \Omega - \Omega_1^{(a)} + \frac{i}{2} \Gamma_e^{(a)} - Q_1 & 0 & \cdots & 0 \\
V_{12} & 0 & \Omega - \Omega_1^{(a)} + \frac{i}{2} \Gamma_e^{(a)} - Q_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
V_{1N} & 0 & 0 & \cdots & \Omega - \Omega_1^{(a)} + \frac{i}{2} \Gamma_e^{(a)} - Q_N
\end{pmatrix}^{-1}, \quad (S9)$$

$$Q_n = \frac{V_{2n}^2}{\Omega - \Omega_2^{(c)} + i\pi W_{22}^2}.$$ 

In order to find the inverse of matrix in Eq. (S9), we partition it as

$$F^{-1} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \quad (S10)$$

where

$$A_1 = \left( \Omega - \Omega_1^{(c)} + i\pi W_{11}^2 \right),$$

$$B_1 = \begin{pmatrix} V_{11} & V_{12} & \cdots & V_{1N} \end{pmatrix}, \quad C_1 = B_1^T,$$

$$D_1 = \begin{pmatrix}
\Omega - \Omega_1^{(a)} + \frac{i}{2} \Gamma_e^{(a)} - Q_1 & 0 & \cdots & 0 \\
0 & \Omega - \Omega_1^{(a)} + \frac{i}{2} \Gamma_e^{(a)} - Q_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \Omega - \Omega_1^{(a)} + \frac{i}{2} \Gamma_e^{(a)} - Q_N
\end{pmatrix},$$

and, again, apply Eq. (S1). As the result we find:

$$S_{1,2m} = \frac{2i\pi W_{11}W_{22}V_{1n}V_{2m}}{\left( \left( \Omega - \Omega_1^{(c)} + i\pi W_{11}^2 \right) \left( \Omega - \Omega_1^{(a)} + \frac{i}{2} \Gamma_e^{(a)} - V_{1n}^2 \right) \left( \Omega - \Omega_1^{(c)} + i\pi W_{22}^2 \right) \right)},$$

$$\Sigma = \sum_{n=1}^{N} \left( \Omega - \Omega_2^{(c)} + i\pi W_{22}^2 \right) \left( \Omega - \Omega_2^{(a)} + \frac{i}{2} \Gamma_e^{(a)} - \frac{1}{2} \Gamma_e^{(a)} \right) - V_{2n}^2.$$ 

After substitution of notations from Eq. (S8), this equation coincides with Eq. (4) of the main text.

4. Model of an optical microcavity enabling complete frequency conversion

In the absence of losses, \( \Gamma_1^{(c)} = \Gamma_2^{(c)} = \Gamma_e^{(a)} = 0 \), and for exact resonant condition, \( \Delta \omega_1 = \Delta \omega_2 = \Delta \omega_e = 0 \), Eq. (4) of the main text yields:
This equation shows that the complete frequency conversion takes place if

\[ \sum_{n=1}^{N} \frac{V_{ln}^2}{\nu_{2n}^2} = \frac{W_{11}^2}{W_{22}^2} \sum_{n=1}^{N} \frac{V_{ln}^2}{\nu_{2n}^2}. \]  

(S14)

Remarkably, provided that waveguide-cavity couplings \( W_{11} \) and \( W_{22} \) satisfy this equation, random positions of atoms (leading to different \( V_{mn} \) for different \( n \)) will not reduce the effect of complete frequency conversion. If the field distribution of cavity modes \( |u_1\rangle \) and \( |u_2\rangle \) are proportional to each other in the region where atoms are situated so that \( u_1(\mathbf{r}) = \beta u_2(\mathbf{r}) \), then the ratio \( V_{1n} / V_{2n} = \beta \) does not depend on \( n \) and Eq. (S14) is reduced to \( W_{11}^2 = N\beta^2 W_{22}^2 \).

The top of Fig. 2(a) of the main text shows the configuration of two localized optical states and atoms positioned in the region where these states overlap. Each of these states is coupled to only one input-output waveguide. It apparent that the mirrors of this cavity can be designed so that the relation \( u_1(\mathbf{r}) = \beta u_2(\mathbf{r}) \) is accurately satisfied near the antinodes of these states. The design of an alternative one-dimensional configuration of optical states shown in the bottom of Fig. 2(a) is less obvious. Here we present a model of such microcavity. We compose it of three weakly coupled short-range cavities illustrated in Fig. S3(a) and (b). The optical states localized in this cavity are defined by the model wave equation:

\[ \frac{d^2 u}{dx^2} + \left( \kappa^2 + \alpha_1 \delta(x - d_1) + \alpha_2 \delta(x) + \alpha_3 \delta(x + d_2) \right) u = 0 \]  

(S15)

where \( \alpha_j \) are the strengths of cavities. Eq. (S15) possesses three localized states. We optimize the parameters of cavities \( \alpha_j \) and their separations \( d_j \) so that two of these states, \( u_1(x) \) and \( u_2(x) \) satisfy the conditions of our interest. First, we require that states \( u_1(x) \) and \( u_2(x) \) and, consequently, the corresponding collective states \( |c_1\rangle \) and \( |c_2n\rangle \), are coupled to a single waveguide only and maximized in the region of atoms as illustrated in Figs. 2(a) and S3(a). To this end, state \( u_1(x) \) (\( u_2(x) \)) is designed to be finite and state \( u_1(x) \) (\( u_2(x) \)) is designed to vanish near the input (output) waveguide (Fig. S3(c) and (d)). For the dipole atom-field interaction, we have \( V_{pn} \sim D_p u_p(x_n) \), where \( u_p(x) \) is the normalized photon state, \( x_n \) is the position of atom \( n \), and \( D_p \) is the dipole matrix element between atomic states [S4]. In Fig. S3(c), normalized functions \( u_1(x) \) and \( u_2(x) \) are approximately equal in the center area (region of atoms) which corresponds to \( u_1(x) / u_2(x) = 1 \) and couplings \( V_{1n} / V_{2n} = D_1 / D_2 \) independent of the atomic position (see Eq. (S12)). In Fig. S3(d) the values of normalized \( u_1(x) \) and \( u_2(x) \) are proportional in the region of atoms with a factor of 1.7 which corresponds to \( V_{1n} / V_{2n} = 1.7 D_1 / D_2 \).
**Fig. S3.** One-dimensional optical microcavity enabling full conversion of the single photon frequency. (a) Illustration of the distribution of localized optical states and atoms. Three coupled short-range cavities with strengths $\alpha_j$ and separations $d_j$ which are designed to resemble the distributions of localized states illustrated in the bottom of Fig. 2(a) and proportional to each other in the region of atoms. (b) Localized states $u_1(x)$ and $u_2(x)$ which are equal to each other in the region of atoms. (c) Localized states $u_1(x)$ and $u_2(x)$ proportional to each other in the region of atoms with the factor 1.7.

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