SOLUTIONS OF THE HEXAGON EQUATION FOR ABELIAN ANYONS

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ABSTRACT. We address the problem of determining the obstruction to existence of solutions of the hexagon equation for abelian fusion rules and the classification of prime abelian anyons.

1. Introduction

Anyons are two-dimensional particles which in contrast to boson or fermions satisfy exotic statistics. The exchange of two identical anyons can in general be described by either abelian or non-abelian statistics. In the abelian case an exchange of two particles gives rise to a complex phase $e^{2\pi i \theta}$. Bosons and fermions correspond only to the phase changes $+1$ and $-1$ respectively. Particles with non-real phase change are considered anyons. In general, the statistics of anyons is described by unitary operators acting on a finite dimensional degenerate ground-state manifold, [12].

There has been increased interest in non-abelian anyons since they possess the ability to store, protect and manipulate quantum information [12,13,10,22,18]. In contrast, abelian anyons only seem good as quantum memory. Moreover, abelian anyons are interesting for two reasons. First, they have simpler physical realizations than non-abelian anyons; and second, gauging a finite group of topological symmetries of an abelian anyon theory, when it possible, leads to a new anyon theory that is in general non-abelian, [4]. Moreover, all concrete known examples of non-abelian anyon theories with integer global dimension are constructed from a gauging of an abelian anyon theory.

Mathematically speaking, an abelian anyon theory is a modular pointed category ([5,9]), and the latter comprise are the class of modular categories which are best understood. Abelian anyons correspond to triples $(A, \omega, c)$, where $A$ is a finite abelian group and $(\omega, c) \in Z^3_{ab}(A, U(1))$ is an abelian 3-cocycle. The set of modular categories up to gauge equivalence with a fixed abelian group $A$ forms an abelian group denoted by $H^3_{ab}(A, U(1))$ and called the third abelian cohomology group of $A$. The groups $H^3_{ab}(A,B)$ were defined and studied by Eilenberg and MacLane in [7,8] for any pair of abelian groups [7,8]. In this work, we address the problem of determining for an ordinary 3-cocycle $\omega \in Z^3(A,B)$ the obstruction to the existence of a map $c : A \times A \to B$ such that $(\omega, c) \in Z^3_{ab}(A,B)$. To that end, we construct a double complex associated to a finite abelian group and a map from...

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the ordinary group cohomology to the total cohomology of the double complex. We find several exact sequences involving $H^3_{ab}(A, B)$ and provide an explicit method for the construction of all possible abelian 3-cocycles. We finish the note with a reformulation of an old result of Wall [21] and Durfee [6] on the classification of indecomposable symmetric forms on finite abelian groups in terms of classification of prime abelian anyons.

The paper is organized as follows. In Section 2 we recall the definitions of group cohomology and abelian group cohomology. Section 3 contains a brief introduction to fusion algebras and the pentagon and hexagon equation. In section 4 we present the main results of the paper. We recall a theorem of Eilenberg and MacLane about the isomorphism between $H^3_{ab}(A, B)$ and Quad$(A, B)$ (the group of all quadratic forms from $A$ to $B$). We also show that Quad$(A, \mathbb{R}/\mathbb{Z})$ can be computed inductively from a decomposition of $A$ as direct sum of cyclic groups. In this section we also define the obstruction for the existence of solutions of the hexagon equation. Section 5 contains the classification of prime abelian anyon theories.

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2. Preliminaries

In this section we present some basic definitions of group cohomology and abelian group cohomology. A lot of this material can be found in [7] and [8].

We will denote by $U(1)$ the group of complex numbers of modulus 1, which we will often write additively through the identification with $\mathbb{R}/\mathbb{Z}$.

Given an abelian group $A$ we will denote by $S^2(A), \wedge^2 A$ and $A \otimes^2$ the second symmetric power, second exterior power and second tensor power of $A$, respectively. Here, we see $A$ as a $\mathbb{Z}$-module.

Given a group $G$ we will denote by $\hat{G}$ to the abelian group of all linear character of $G$, that is

$$\hat{G} = \text{Hom}(G, U(1)) = \text{Hom}(G, \mathbb{R}/\mathbb{Z}).$$

2.1. Group cohomology. We will recall the usual cocycle description of group cohomology associated to the normalized bar resolution of $\mathbb{Z}$, see [7] for more details. Let $G$ be a discrete group and let $A$ be a $\mathbb{Z}[G]$-module. Let $C^0(G, A) = A$, and let

$$C^n(G, A) = \{ f : \underbrace{G \times \cdots \times G}_{n\text{-times}} \to A | f(x_1, \ldots, x_n) = 0, \text{ if } x_i = 1_G \text{ for some } i \},$$

for $n \geq 1$.

Consider the cochain complex

$$0 \to C^0(G, A) \overset{\delta_0}{\to} C^1(G, A) \overset{\delta_1}{\to} C^2(G, A) \cdots C^n(G, A) \overset{\delta_n}{\to} C^{n+1}(G, A) \cdots$$
where
\[ \delta_n(f)(x_1, x_2, \ldots, x_{n+1}) = x_1 \cdot f(x_2, \ldots, x_{n+1}) \]
\[ + \sum_{i=1}^{n} (-1)^i f(x_1, \ldots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \ldots, x_{n+1}) \]
\[ + (-1)^{n+1} f(x_1, \ldots, x_n). \]

We denote,
\[ Z^n(G, A) := \ker(\delta_n) \text{ (n-cocycles)}, \]
\[ B^n(G, A) := \text{Im}(\delta_{n-1}) \text{ (n-coboundaries)} \]
and
\[ H^n(G, A) := Z^n(G, A) / B^n(G, A) \quad (n \geq 1), \]
the cohomology of \( G \) with coefficients in \( A \).

2.2. Eilenberg-MacLane cohomology theory of abelian groups. Let \( A \) be an abelian group. A space \( X \) having only one nontrivial homotopy group \( \pi_n(X) = A \) is called the Eilenberg-MacLane space \( K(A, n) \). Such space can be constructed as a CW complex or using the Dold-Kan correspondence between chain complexes and simplicial abelian groups. If \( A[n] \) is the chain complex which is \( A \) in dimension \( n \) and trivial elsewhere; the geometric realization of the corresponding simplicial abelian group is a \( K(A, n) \) space.

The abelian cohomology theory of the abelian group \( M \) with coefficients in the abelian group \( N \) is defined as
\[ H^n_{ab}(M, N) := \{ \text{Homotopy classes } K(M, 2) \to K(N, n+1) \} \]

In [7, 8] Eilenberg and MacLane defined a chain complex associated to any abelian group \( M \) to compute the abelian cohomology groups of the space \( K(M, 2) \).

We use the following notations for \( X, Y \) any two groups:
- \( X^p[Y^q = \{ x|y = (x_1, \ldots, x_p|y_1, \ldots, y_q), x_i \in X, y_j \in Y \}, \ p, q \geq 0 \).
- \( \text{Shuff}(p, q) \) the set of \( (p, q) \)-shuffles, i.e. an element in the symmetric group \( S_{p+q} \) such that \( \lambda(i) < \lambda(j) \) whenever \( 1 \leq i < j \leq p \) or \( p + 1 \leq i < j \leq p + q \).
- Any \( \pi \in \text{Shuff}(p, q) \) defines a map
\[ \pi : X^{p+q} \to X^{p+q} \]
\[ (x_1, \ldots, x_{p+q}) \mapsto (x_{\pi(1)}, \ldots, x_{\pi(p+q)}) \]

Let \( M \) and \( N \) be abelian groups. Define the abelian group \( C^0_{ab}(M, N) = 0 \) and for \( n > 0 \)
\[ C^n_{ab}(M, N) = \bigoplus_{p_1, \ldots, p_r \geq 1; \sum_{i=1}^r p_i = n+1} \text{Maps}(M^{p_1} \cdots |M_{p_r}, N), \]
where \( \text{Maps}(M^{p_1} \cdots |M_{p_r}, N) \) denotes the abelian group of all maps from \( M^{p_1} \cdots |M_{p_r} \) to \( N \).

The coboundary maps are defined as
\[ \partial : C^n_{ab}(M, N) \to C^{n+1}_{ab}(M, N) \]
\[ \partial(f)(x^1|x^2|\ldots|x^r) = \sum_{1 \leq i \leq r} \sum_{0 \leq j \leq p_i} (-1)^{j+i-1} f(x^1|\ldots|d_j x^i|\ldots|x^r) \]

\[ + \sum_{\pi \in \text{Shuf}(p_i, p_{i+1})} (-1)^{\epsilon + \epsilon(\pi)} f(x^1|\ldots|\pi(x^i|x^{i+1})|\ldots|x^r) \]

where

\[ d_j : M^{p_i} \to M^{p_i-1} \]

\[ (x_1, \ldots, x_{p_i}) \mapsto (x_1, \ldots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \ldots, x_{p_i}) \]

are the face operators; \( \epsilon_i = p_1 + \cdots + p_i + i \) and \( \epsilon(\pi) \) is the sign of the shuffle \( \pi \).

We denote, \( Z^0_{ab}(M, N) := \ker(\partial_n) \) (called abelian n-cocycles), \( B^0_{ab}(M, N) := \text{Im}(\partial_{n-1}) \) (called abelian n-coboundaries) and

\[ H^0_{ab}(M, N) := Z^0_{ab}(M, N)/B^0_{ab}(M, N) \quad (n \geq 1), \]

the abelian cohomology of \( M \) with coefficients in \( N \).

Let us write the first cochains groups and their coboundaries.

- \( C^0_{ab}(M, N) = 0 \).
- \( C^1_{ab}(M, N) = \text{Maps}(M, N) \).
- \( C^2_{ab}(M, N) = \text{Maps}(M^2, N) \).
- \( C^3_{ab}(M, N) = \text{Maps}(M^3, N) \oplus \text{Maps}(M|M, N) \).
- \( C^4_{ab}(M, N) = \text{Maps}(M^4, N) \oplus \text{Maps}(M^2|M, N) \oplus \text{Maps}(M|M^2, N) \).

Thus

- Since \( C^0_{ab}(M, N) = 0 \), \( H^1_{ab}(M, N) = Z^1_{ab}(M, N) = \text{Hom}(M, N) \).
- For \( f \in C^2_{ab}(M, N) \), we have

\[ \partial(f)(x, y, z) = f(y, z) - f(x y, z) + f(y, z) - f(x, y z), \quad \partial(f)(x|y) = f(x, y) - f(y, x). \]

Then \( H^2_{ab}(M, N) \cong \text{Ext}^1_{ \mathbb{Z}^2}(M, N) \) the group of abelian extensions of \( M \) by \( N \).

- Finally, for \( (\omega, c) \in C^3_{ab}(M, N) \) we have

\[ \partial(\omega)(x, y, z, t) = \omega(y, z, t) - \omega(x + y, z, t) + \omega(x, y + z, t) - \omega(x, y, z + t) + \omega(x, y, z), \]

\[ \partial(c)(x|y, z) = c(x|z) - c(x|y + z) + c(x|y) + c(x|y) - \omega(y, x, z) + \omega(y, z, x), \]

\[ \partial(c)(x, y|z) = c|y|z - c(x + y|z) + c(x|z) - \omega(x, y, z) + \omega(x, z, y) - \omega(z, x, y). \]

3. Fusion algebras

A fusion algebra is based on a finite set \( A \) (where elements will be called anyonic particles or simply particles). The elements in \( A \) will be denoted by \( a, b, c, \ldots \).

For every particle \( a \) there exists a unique anti-particle, that we denote by \( \overline{a} \). There is a unique trivial “vacuum” particle denoted by \( 1 \) (or sometimes \( 0 \)).

The fusion algebra has \textit{fusion rules}

\[ a \times b = \sum_c N^c_{ab} c \]

where \( N^c_{ab} \in \mathbb{Z}^{\geq 0} \) that count the number of ways the particles \( a \) and \( b \) fuse into \( c \).

The fusion rules obey the following relations
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• associativity \((a \times b) \times c = a \times (b \times c)\),
• commutativity \(a \times b = b \times a\),
• the vacuum is the identity for the fusion product, \(a \times 1 = a\),
• the rule \(a \mapsto \overline{a}\) defines an involution of the fusion rules, that is,
\[
\overline{a} = 1, \quad \overline{a} = a, \quad \overline{a \times b} = a \times \overline{b},
\]
where
\[
a \times b = \sum_c N_{ab}^c \overline{c}.
\]
• The fusion of \(a\) with its antiparticle \(\overline{a}\) contains the vacuum with multiplicity one, that is
\[
N_{a\overline{a}}^1 = 1.
\]

A fusion algebra is called **abelian** if
\[
\sum_c N_{ab}^c = 1
\]
for every \(a\) and \(b\). This is if the fusion of two particles \(a \times b = c\), is again one of the particles in \(A\). If \(A\) is an abelian fusion algebra, then the fusion product defines a structure of abelian group on \(A\) and conversely every finite abelian group defines a set of abelian fusion rules.

If we have a fusion algebra on the set \(A\) with \(n\) particles, we can assign to each particle \(b\) the matrix \(N_b\) whose entries are exactly \(N_{ab}^c\) in the position \((b, c)\). This is an \(n \times n\) integer matrix that contains all the information about the fusion rules of \(a\). It satisfies the equation
\[
N_a N_b = \sum_c N_{ab}^c N_c.
\]

### 3.1. The Pentagon equation for abelian anyons

Throughout this section, we will follow the notation of [20], slightly modified for our purposes. For further reading on these topics we direct the reader to [11, 24, 13].

Let \(A\) be a fusion algebra. Assign to each fusion product a vector space \([c]_{a,b}\) of dimension \(N_{a,b}^c\). If \(N_{a,b}^c = 0\) then \([c]_{a,b} = 0\). The vector spaces \([c]_{a,b}\) are called the **fusion spaces** of \(A\). The fusion space takes in account the ways in which the anyons \(a\) and \(b\) can fuse together to give \(c\).

Now, consider the fusion of the particles \(a, b\) and \(c\). The associativity of the fusion rules ensures that \((a \times b) \times c = a \times (b \times c)\), but with the fusion spaces there are two different objects that can do this. The first being
\[
\bigoplus_{i \in A} \begin{bmatrix} i \\ a,b \end{bmatrix} \otimes \begin{bmatrix} d \\ i,c \end{bmatrix},
\]
and the second being
\[
\bigoplus_{i \in A} \begin{bmatrix} i \\ b,c \end{bmatrix} \otimes \begin{bmatrix} d \\ a,i \end{bmatrix}.
\]

We would like a family of linear isomorphisms that takes in account the distinct ways of "associating" fusion spaces in this context, thus we have the following definition:

An \(F\)-matrix for a fusion algebra \(A\) is a family of linear isomorphisms
For all \( a, b, c \) the pentagon equation is satisfied, where

\[
\bigoplus_{i,j} \begin{bmatrix} i & j \\ a,b,c \end{bmatrix} \otimes \begin{bmatrix} i & d \\ i,c \end{bmatrix} \rightarrow \bigoplus_{j \in A} \begin{bmatrix} j & d \\ b,c \end{bmatrix} \otimes \begin{bmatrix} d \\ a,j \end{bmatrix},
\]

which satisfies the pentagon equation:

\[
\begin{align*}
\bigoplus_{i,j} \begin{bmatrix} i & j & e \\ a,b,c \end{bmatrix} & \xrightarrow{F} \bigoplus_{i,j} \begin{bmatrix} i & j & e \\ b,c \end{bmatrix}, \\
\bigoplus_{i,j} \begin{bmatrix} i & j & e \\ a,b,c \end{bmatrix} & \xrightarrow{F} \bigoplus_{i,j} \begin{bmatrix} i & j & e \\ a,i,d \end{bmatrix},
\end{align*}
\]

or simply

\[
(3.1) \quad \sum_{i,j \in A} F \begin{bmatrix} j & e \\ b,c,d \end{bmatrix} F \begin{bmatrix} e & i \\ a,i,d \end{bmatrix} F \begin{bmatrix} e & i \\ a,b,c \end{bmatrix} = \sum_{i,j \in A} F \begin{bmatrix} e & i \\ i,c,d \end{bmatrix} F \begin{bmatrix} e & i \\ a,b,j \end{bmatrix}.
\]

In the diagram above,

\[
\tau : \bigoplus_{i,j} \begin{bmatrix} i & j \\ a,b \end{bmatrix} \rightarrow \bigoplus_{i,j} \begin{bmatrix} i & i \\ c,d \end{bmatrix}
\]

is the operator that swaps the components of \( \begin{bmatrix} i & j \\ a,b \end{bmatrix} \) and \( \begin{bmatrix} j & e \\ c,d \end{bmatrix} \). We also omit the tensor products and identity operators for simplicity.

We want that any transformation through the \( F \)-matrix starting and ending in the same spaces to be the same. Equation \( (3.1) \) ensures this.

Let us assume that \( A \) is an abelian fusion algebra. Then we must have that each fusion space is either one or zero dimensional and an \( F \)-matrix for \( A \) is determined by a family of scalars \( \omega \) such that

\[
\omega(a,b,c) := F \begin{bmatrix} d \\ a,b,c \end{bmatrix} \in \mathbb{C}^* \quad \text{ for all } a,b,c \in A.
\]

such that

\[
(3.2) \quad \omega(a_1 a_2, a_3, a_4) \omega(a_1, a_2 a_3 a_4) = \omega(a_1, a_2, a_3) \omega(a_1, a_2 a_3, a_4) \omega(a_2, a_3, a_4),
\]

for all \( a_1, a_2, a_3, a_4 \in A \).

A function \( \omega : A \times A \times A \rightarrow \mathbb{U}(1) \) satisfying equation \( (3.2) \) is just a standard 3-cocycle. Thus, the set of all solutions of the pentagon equation of an abelian fusion algebra is exactly \( Z^3(A, \mathbb{U}(1)) \).

A gauge transformation between two solutions of pentagon equation \( \omega, \omega' \in Z^3(A, \mathbb{U}(1)) \) is determined by a family of non zero scalars \( \{u(a, b)\}_{a,b \in A} \) such that

\[
\omega'(a,b,c) = \frac{u(ab, c)}{u(a, bc)u(b, c)} \omega(a,b,c),
\]

for all \( a, b, c \in A \). Thus, the set of gauge equivalence classes of solutions of the pentagon equation is the \( H^3(A, \mathbb{U}(1)) \).
3.2. The hexagon equation. In this section we will assume that $A$ is an abelian group and $\omega \in Z^3(A, U(1))$ is a 3-cocycle.

In the previous section, we extended the associativity of the fusion rules to the associativity of the fusion spaces through a family of linear operators called $F$-matrix. Now, we want to extend the commutativity as well.

In order to do this, we need a family of unitary operators $R_{a,b}^c : \begin{bmatrix} c \\ a,b \end{bmatrix} \rightarrow \begin{bmatrix} c \\ b,a \end{bmatrix}$ that satisfy

$$R_{a,1}^a = \text{Id} = R_{1,a}^a$$

and the hexagon equations

$$\sum_{i,j,k} R_{a,c}^i R_{b,a}^{ij} F_{\begin{bmatrix} j & b, a, c \end{bmatrix}} = \sum_{i,j,k} F_{\begin{bmatrix} i & a,c,b \end{bmatrix}} R_{b,c}^j F_{\begin{bmatrix} k & a, b \end{bmatrix}}$$

(3.3)

$$\sum_{i,j,k} (R_{a,c}^i)^{-1} F_{\begin{bmatrix} j & b, a, c \end{bmatrix}} (R_{a,b}^{ij})^{-1} = \sum_{i,j,k} F_{\begin{bmatrix} i & a,c,b \end{bmatrix}} (R_{b,c}^j)^{-1} F_{\begin{bmatrix} k & a, b \end{bmatrix}}$$

(3.4)

We will call such family an $R$-matrix, or a braiding, for $A$. As before, these equations imply that any transformation within the $R$ and the $F$-matrices are independent of the path.

In the case where $A$ is an abelian theory with an associated 3-cocycle $\omega$, a braiding is determined by a family of scalars $\{c_{a,b}\}_{a,b \in A}$ that satisfy the equations

$$\frac{\omega(b,a,c)}{\omega(a,b,c)\omega(b,c,a)} = \frac{c(a,bc)}{c(a,b)c(a,c)}$$

$$\frac{\omega(a,b,c)\omega(c,a,b)}{\omega(a,c,b)} = \frac{c(ab,c)}{c(a,c)c(b,c)}$$

Thus, $(\omega, c)$ is an abelian 3-cocycle. The solutions of the hexagon up to gauge equivalence is the group $H^3_{ab}(A, U(1))$.

4. Computing $H^3_{ab}(M, N)$

4.1. Quadratic forms and $H^3_{ab}(A, B)$. Let $A$ and $B$ be abelian groups. A quadratic form from $A$ to $B$ is a function $\gamma : A \rightarrow B$ such that

$$\gamma(a) = \gamma(-a)$$

(4.1)

$$\gamma(a + b + c) - \gamma(b + c) - \gamma(a + c) - \gamma(a + b) + \gamma(a) + \gamma(b) + \gamma(c) = 0,$$

(4.2)

for any $a, b, c \in A$. A map $\gamma : A \rightarrow B$ such that $\gamma(a) = \gamma(-a)$ satisfies (4.2) if and only if the map

$$b_\gamma : A \times A \rightarrow B$$

$$(a_1, a_2) \mapsto \gamma(a_1 + a_2) - \gamma(a_1) - \gamma(a_2)$$

is a symmetric bilinear form. It follows by induction that $\gamma(na) = n^2\gamma(a)$ for any positive integer $n$. 

We will denote by $\text{Quad}(A, B)$, the group of all quadratic forms from $A$ to $B$. Eilenberg and MacLane proved in [8, Theorem 26.1] that for any two abelian groups $A, B$, the map

$$\text{Tr} : H^3_{ab}(A, B) \to \text{Quad}(A, B)$$

$$(\omega, c) \mapsto [a \mapsto c(a, a)]$$

is a group isomorphism.

If $A$ is a finite abelian group, the group $\text{Quad}(A, \mathbb{R}/\mathbb{Z})$ can be computed using the following results.

**Proposition 4.1.** If $n$ is odd, then $\text{Quad}(\mathbb{Z}/n, \mathbb{R}/\mathbb{Z})$ is a cyclic group of order $n$, with generator given by

$$q_n : \mathbb{Z}/n \to \mathbb{R}/\mathbb{Z}$$

$$m \mapsto m^2/n.$$

If $n$ is even, $\text{Quad}(\mathbb{Z}/n, \mathbb{R}/\mathbb{Z})$ is a cyclic group of order $2n$, with generator given by

$$q_{2n} : \mathbb{Z}/n \to \mathbb{R}/\mathbb{Z}$$

$$m \mapsto m^2/2n.$$

**Proof.** Let $\gamma : \mathbb{Z}/n \to \mathbb{R}/\mathbb{Z}$ be a quadratic form. Since $\gamma(m) = m^2\gamma(1)$, the quadratic form is completely determined by $\gamma(1) \in \mathbb{Q}/\mathbb{Z}$. Since $q(n) = 0$, $n^2q(1) = 0$, and since $q(1) = q(-1)$, $2nq(1) = 0$.

If $n$ is odd. Then $nq(1) = 0$, so $q(1) \in \{1/n, 2/n, \ldots, 0\} \subset \mathbb{Q}/\mathbb{Z}$ define all possible quadratic forms. If $n$ is even, $q(1) \in \{1/2n, 2/2n, \ldots, 0\} \subset \mathbb{Q}/\mathbb{Z}$ define the possible quadratic forms. \hfill $\square$

**Remark 4.2.** Let $n$ be an even positive integer. An abelian 3-cocycle $(\omega, c) \in Z^3(\mathbb{Z}/n\mathbb{Z}, \mathbb{R}/\mathbb{Z})$ representing the cohomology class of the quadratic form $q_{2n}$ is given by

$$c(a, b) = \frac{ab}{2n}, \quad \omega(a, b, c) = \begin{cases} \frac{2}{n}, & \text{if } b + c \geq n, \\ 0, & \text{other case}. \end{cases}$$

**Proposition 4.3.** Let $A$ and $B$ be abelian group, then the map

$$T : \text{Hom}(A \otimes B, \mathbb{R}/\mathbb{Z}) \oplus \text{Quad}(A, \mathbb{R}/\mathbb{Z}) \oplus \text{Quad}(B, \mathbb{R}/\mathbb{Z}) \to \text{Quad}(A \oplus B, \mathbb{R}/\mathbb{Z})$$

$$f \oplus \gamma_A \oplus \gamma_B \mapsto [(a, b) \mapsto f(a \oplus b) + \gamma_A(a) + \gamma_B(b)],$$

is a group isomorphisms.

**Proof.** We will see that

$$W : \text{Quad}(A \oplus B, \mathbb{R}/\mathbb{Z}) \to \text{Hom}(A \otimes B, \mathbb{R}/\mathbb{Z}) \oplus \text{Quad}(A, \mathbb{R}/\mathbb{Z}) \oplus \text{Quad}(B, \mathbb{R}/\mathbb{Z})$$

$$\gamma \mapsto \gamma_A + \gamma_B + b_\gamma|_{(A \oplus 0) \times (0) \oplus B},$$

is the inverse of $T$. In fact,

$$T \circ W(\gamma)(a \oplus b) = \gamma(a) + \gamma(b) + (\gamma(a \otimes b) - \gamma(a) - \gamma(b))$$

$$= \gamma(a \oplus b),$$

and
Proof. Recall that Corollary 4.4. If \( A \) is a finite abelian group, then

\[
\text{Quad}(A, \mathbb{R}/\mathbb{Z}) = |A/2A||S^2(A)|.
\]

Proof. Recall that \( S^2(A \oplus B) \cong S^2(A) \oplus S^2(B) \oplus A \otimes B \) for any pair of abelian groups. In particular,

(4.4) \[ |S^2(A \oplus B)| = |S^2(A)||S^2(B)||A \otimes B|. \]

If \( A = B \oplus C \), by Proposition 4.3 we have

\[
\text{Quad}(A, \mathbb{R}/\mathbb{Z}) = |B/2B||S^2(C)||C/2C||S^2(B)||B \otimes C|
\]

\[
= ([B/2B][C/2C])(|S^2(B)||S^2(B)||B \otimes C))
\]

\[
= |A/2A||S^2(A)|.
\]

\[\square\]

4.2. A double complex for an abelian group. To describe the obstruction to the existence of a solution of the hexagon equation of a 3-cocycle \( \omega \in Z^3(A, U(1)) \), in this section we will define a double complex associated to an abelian group.

Let \( A \) and \( N \) be abelian groups. We define a double complex by \( D^{p,q}(A, N) = 0 \) if \( p \) or \( q \) are zero and

\[
D^{p,q}(A, N) := \text{Maps}(A^p|A^q; N), \quad p, q > 0
\]

with horizontal and vertical differentials the standard differentials, that is,

\[
\delta_h : D^{p,q}(A, N) = C^p(A, C^q(A, N)) \to D^{p+1,q}(A, N) = C^{p+1}(A, C^q(A, N))
\]

and

\[
\delta_v : D^{p,q}(A, N) = C^q(A, C^p(A, N)) \to D^{p,q+1}(A, N) = C^{q+1}(A, C^p(A, N))
\]

defined by the equations

\[
(\delta_h F)(g_1, ..., g_{p+1}|k_1, ..., k_q) = F(g_2, ..., g_{p+1}|k_1, ..., k_q)
\]

\[
+ \sum_{i=1}^p (-1)^i F(g_1, ..., g_ig_{i+1}, ..., g_{p+1}|k_1, ..., k_q)
\]

\[
+ (-1)^{p+1} F(g_1, ..., g_p|k_1, ..., k_q)
\]

\[
(\delta_v F)(g_1, ..., g_p|k_1, ..., k_{q+1}) = F(g_1, ..., g_p|k_2, ..., k_{q+1})
\]

\[
+ \sum_{j=1}^q (-1)^j F(g_1, ..., g_p|k_1, ..., k_jk_{j+1}, ..., k_{q+1})
\]

\[
+ (-1)^{q+1} F(g_1, ..., g_p|k_1, ..., k_q).
\]

For future reference it will be useful to describe the equations that define a 2-cocycle and the coboundary of a 1-cochains:
the obstruction to the hexagon equation.

ι induces a group homomorphism

For every \( n \in \alpha \) for all \( x, a, b, c \in \beta \),

Thus, let us describe the elements

\[
\delta_h(f)(x, y||z) = f(y||z) - f(x + y||z) + f(x||z) \\
\delta_v(f)(x||y, z) = f(x||z) - f(x||y + z) + f(x||y)
\]

The following result shows that the shuffle homomorphism can be interpreted as

\[
\text{Hom}(A \otimes^2 N, N) ightarrow Z^3_{ab}(A, N)
\]

for all \( x, y, z \in A, a, b, c \in B \).

4.3. Obstruction. Consider the group homomorphism

\[
\tau_n : H^n(A, N) \rightarrow H^n(\text{Tot}^*(D^{*\cdot\cdot})(A, N))
\]

induced by the cochain map

\[
\tau : C^*(A, N) \rightarrow C^n(\text{Tot}^*(D^{*\cdot\cdot})(A, N))
\]

\[
\alpha \mapsto \otimes_{p=1}^{n-1} \alpha_p,
\]

where \( \alpha_p \in \text{Maps}(A^p|A^{n-p}, N) \) is defined by

\[
\alpha_p(a_1, \ldots, a_p|a_{p+1}, \ldots, a_n) = \sum_{\pi \in \text{Shuff}(p, a-n)} (-1)^{\epsilon(\pi)} \alpha(a_{\lambda(1)}, \ldots, a_{\lambda(n)}).
\]

For every \( n \in \mathbb{Z}^{\geq 2} \), we define the suspension homomorphism from

\[
s_n : H^n_{ab}(A, N) \rightarrow H^n(A, N)
\]

\[
\bigoplus_{p_1, \ldots, p_r \geq 1; \sum_{i=1}^r p_i = n+1} \alpha_{p_1, \ldots, p_r} \mapsto \alpha_n.
\]

The group homomorphism

\[
H^2(\text{Tot}^*(D^{*\cdot\cdot})(A, N))) = \text{Hom}(A \otimes^2 N, N) \rightarrow Z^3_{ab}(A, N)
\]

\[
c \mapsto (0, c),
\]

induces a group homomorphism \( \iota : H^2(\text{Tot}^*(D^{*\cdot\cdot})(A, N)) \rightarrow H^3_{ab}(A, N) \).

The following result shows that the shuffle homomorphism can be interpreted as

the obstruction to the hexagon equation.
Theorem 4.5. Let $A$ and $N$ be abelian groups. Then, the sequence

$$0 \to H^2_\text{ab}(A, N) \xrightarrow{s_2} H^2(A, N) \xrightarrow{\tau_2} H^2(\text{Tot}^*(D^{*, \ast}(A, N)) \xrightarrow{\iota} H^3_{\text{ab}}(A, N)$$

is exact.

Proof. The shuffle homomorphism $\tau_2 : H^2(A, N) \to H^2(\text{Tot}^*(D^{*, \ast}(A, N))$ is given by $\tau(\alpha(x, y) = \alpha(x, y) - \alpha(y, x)$, thus it is clear that the sequence is exact in $H^2(A, N)$.

An abelian 3-cocycle is in the kernel of the suspension map if it is cohomologous to an abelian 3-cocycle of the form $(0, c)$. But then $c \in \text{Hom}(A^{\otimes 2}, N) = H^2(\text{Tot}^*(D^{*, \ast}(A, N))$, hence the sequence is exact in $H^3_{\text{ab}}(A, N)$. Finally, if $\omega \in Z^3(A, N)$, then $\tau(\omega) = (\alpha_{\omega}, \beta_{\omega})$, where

$$\alpha_{\omega}(x, y, z) = \omega(x, y, z) - \omega(y, x, z) + \omega(y, z, x) \quad \beta_{\omega}(x, y, z) = \omega(x, y, z) - \omega(x, z, y) + \omega(z, x, y).$$

Thus, $[(\alpha_{\omega}, \beta_{\omega})] = 0$, if and only if there is $c : A \times A \to N$ such that

$$\delta_{\omega}(c) = \alpha_{\omega}, \quad \delta_{\beta}(c) = \beta_{\omega},$$

that is, $[\tau(\omega)] = 0$ if and only if there is $c : A \times A \to N$ such that $(\omega, c) \in Z^3_{\text{ab}}(A, N)$.

Thus, the sequence is exact in $H^3(A, N)$. \qed

Corollary 4.6 (Total obstruction). A gauge class of a solution of the pentagon equation $\omega \in H^3(A, \mathbb{R}/\mathbb{Z})$ admits a solution of the hexagon equation if and only if $\tau(\omega) = 0$ in $H^3(\text{Tot}^*(D^{*, \ast}(A, \mathbb{R}/\mathbb{Z})))$. \qed

Proposition 4.7. Let $A$ be a finite abelian group.

1. $\ker(s_3) \cong S^2(A)$.

2. Under the isomorphism $\text{Tr} : H^2_\text{ab}(A, \mathbb{R}/\mathbb{Z}) \to \text{Quad}(A, \mathbb{R}/\mathbb{Z})$ (see (4.3)), $\ker(s_3)$ corresponds to the subgroup

$$\text{Quad}_A(A, \mathbb{R}/\mathbb{Z}) = \{ q \in \text{Quad}(A, \mathbb{R}/\mathbb{Z}) : o(a)q(a) = 0, \forall a \in A \},$$

where $o(a)$ denotes the order of $a \in A$.

Proof. Since $\mathbb{R}/\mathbb{Z}$ is divisible and $A$ is finite, the group $H^2_\text{ab}(A, \mathbb{R}/\mathbb{Z}) = \text{Ext}_1^*(A, \mathbb{R}/\mathbb{Z})$ is null. Thus, by Theorem 4.5 we have an exact sequence

$$0 \to H^2(A, \mathbb{R}/\mathbb{Z}) \xrightarrow{\tau} \text{Hom}(A^{\otimes 2}, \mathbb{R}/\mathbb{Z}) \to \ker(s_3) \to 0.$$

The image of $\tau$ is $\text{Hom}(\wedge^2 A, \mathbb{R}/\mathbb{Z})$, hence

$$\ker(s_3) \cong \text{Hom}(A^{\otimes 2}, \mathbb{R}/\mathbb{Z}) / \text{Hom}(\wedge^2 A, \mathbb{R}/\mathbb{Z}) \cong \text{Hom}(A^{\otimes 2} / \wedge^2 A, \mathbb{R}/\mathbb{Z}) \cong \text{Hom}(S^2(A), \mathbb{R}/\mathbb{Z}).$$
where the last isomorphism is defined using the exact sequence
\[ 0 \to A^2 \to A^\oplus 2 \to S^2 A \to 0. \]

Now we will prove the second part. If \( A \) is cyclic the proposition follows by Remark 4.12. The general case follows from Proposition 4.3, since the image of \( \text{Hom}(A \otimes B, \mathbb{R}/\mathbb{Z}) \) by \( T \) lies in \( \text{Quad}_0(A \oplus B, \mathbb{R}/\mathbb{Z}) \).

We will denote by \( \mu_2 = \{1, -1\} \subset U(1) \cong \mathbb{R}/\mathbb{Z} \).

**Theorem 4.8.** Let \( A \) be an abelian finite group. The canonical projection \( \pi : A \to A/2A \) induces an isomorphism between the images of the respective suspension maps of \( H_{ab}^3(A, \mathbb{R}/\mathbb{Z}) \) and \( H_{ab}^3(A/2A, \mathbb{R}/\mathbb{Z}) \).

Moreover, for an elementary abelian 2-group \( (\mathbb{Z}/2\mathbb{Z})^\oplus n \),
\[ \text{Im}(s_3) \cong H^3((\mathbb{Z}/2\mathbb{Z}), \mu_2)^\oplus n = (\mathbb{Z}/2\mathbb{Z})^\oplus n. \]

**Proof.** By Corollary 4.4, Proposition 4.7, and Theorem 4.5, we have that \( |\text{Im}(s_3)| = |A/2A| \). In particular the size of the image of the suspension maps of \( H_{ab}^3(A, \mathbb{R}/\mathbb{Z}) \) and \( H_{ab}^3(A/2A, \mathbb{R}/\mathbb{Z}) \) are equal. Since \( \pi^* \) is an injective map between the image of the suspension maps it is an isomorphisms.

Let \( (\mathbb{Z}/2\mathbb{Z})^\oplus n \) be an elementary abelian 2-group. Recall that \( H^3((\mathbb{Z}/2\mathbb{Z}), \mathbb{R}/\mathbb{Z}) = H^3((\mathbb{Z}/2\mathbb{Z}, \mu_2) \cong \mathbb{Z}/2\mathbb{Z} \). Using the Remark 4.2 we have an injective group homomorphisms \( H^3((\mathbb{Z}/2\mathbb{Z}, \mu_2)^\oplus n \to \text{Im}(s_3) \), which is an isomorphism because both groups have the same order.

**Corollary 4.9.** For any abelian group we have an exact sequence
\[ (4.5) \quad 0 \to S^2(A) \to H_{ab}^3(A, \mathbb{R}/\mathbb{Z}) \to A/2A \to 0. \]

**Remark 4.10.** A related result is established by Mason and Ng in [15, Lemma 6.2].

### 4.4. Explicit abelian 3-cocycles

Abelian 3-cocycles for the group \( \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) were study in detail in [3], for \( \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} \) in [11] and in full generality in [19].

Let \( A \) be a finite abelian group and \( A = A_1 \oplus A_2 \) the canonical decomposition, where \( A_1 \) has order a power of 2 and \( A_2 \) has odd order. Since \( H_{ab}^3(A, \mathbb{R}/\mathbb{Z}) = H_{ab}^3(A_1, \mathbb{R}/\mathbb{Z}) \oplus H_{ab}^3(A_2, \mathbb{R}/\mathbb{Z}) \) and \( H_{ab}^3(A_2, \mathbb{R}/\mathbb{Z}) \cong S^2(A_2) \), the problem of a general description can be divided in the case of group of odd order and the case of abelian 2-groups.

#### 4.4.1. Case of \( A \) an odd abelian group

If \( A \) is an odd abelian group then map \( A \to A, a \mapsto 2a \) is a group automorphism of \( A \). Hence, given \( q \in \text{Quad}(A, \mathbb{R}/\mathbb{Z}) \) the symmetric bilinear form \( c := \frac{1}{2}b_q \in \text{Hom}(A^\oplus 2, \mathbb{R}/\mathbb{Z}) \), defines an abelian 3-cocycle \( (0, c) \in Z_{ab}^3(A, \mathbb{R}/\mathbb{Z}) \) such that \( \text{Tr}(c) = q \).

#### 4.4.2. Case of \( A \) an abelian 2-group

Let \( A = \bigoplus_{i=1}^n \mathbb{Z}/2^m \mathbb{Z} \). Then by Corollary 4.4 and Proposition 4.7, we have a commutative diagram
\[ (4.6) \quad \begin{array}{ccc}
0 & \to & \text{Hom}(S^2(A), \mathbb{R}/\mathbb{Z}) \\
\downarrow & & \downarrow \\
0 & \to & \text{Quad}_0(A, \mathbb{R}/\mathbb{Z}) \\
\end{array} \quad \begin{array}{ccc}
H_{ab}^3(A, \mathbb{R}/\mathbb{Z}) & \to & (\mathbb{Z}/2\mathbb{Z})^\oplus n \\
\text{Tr} & & \cong \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 0 \\
\end{array} \quad \begin{array}{ccc}
0 & \to & \text{Quad}_0(A, \mathbb{R}/\mathbb{Z}) \\
\downarrow & & \downarrow \\
0 & \to & \text{Quad}(A, \mathbb{R}/\mathbb{Z}) \\
\end{array} \quad \begin{array}{ccc}
\pi & \to & (\mathbb{Z}/2\mathbb{Z})^\oplus n \\
& & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 0 \\
\end{array} \]
where the horizontal sequences are exact and the vertical morphisms are isomorphisms.

Let \( q \in \text{Quad}_q(A, \mathbb{R}/\mathbb{Z}) \), then \( c \in \text{Hom}(A^{\otimes 3}, \mathbb{R}/\mathbb{Z}) \) defined by

\[
c(\vec{x}, \vec{y}) = \sum_i x_i y_i q(\vec{e}_i) + \sum_{i<j} x_i y_j b_i(\vec{e}_i, \vec{e}_j)
\]

is a such that \((0, c) \in \mathbb{Z}^3_{ab}(A, \mathbb{R}/\mathbb{Z}) \) represents \( q \).

A set-theoretical section \( j : (\mathbb{Z}/2\mathbb{Z})^{\oplus n} \to \text{Quad}(A, \mathbb{R}/\mathbb{Z}) \) of \( \pi \) in the exact sequence \([4.0]\) is defined easily as follows

\[
j(\vec{y})(\vec{x}) = \sum_i y_i x_i^2/2^{m_i} + 1.
\]

Abelian 3-cocycles representing the \( j(\vec{e}_j)/s \) are constructed as the pullback by the projection \( \pi_j : \bigoplus_{i=1}^n \mathbb{Z}/2^{m_i} \to \mathbb{Z}/2^{m_j} \) of the abelian 3-cocycle \((w, c) \in \mathbb{Z}^3(\mathbb{Z}/2^{m_j}, \mathbb{R}/\mathbb{Z}) \) defined in Remark \([4.2]\).

As a consequence of the previous discussion we have the following result.

**Proposition 4.11.** Let \( A \) be an abelian group. Every abelian 3-cohomology class has a representative 3-cocycle \((\omega, c) \in \mathbb{Z}^3_{ab}(A, \mathbb{R}/\mathbb{Z}) \) where \( \omega(a, b, c) \in \{0, \frac{1}{2}\} \subset \mathbb{R}/\mathbb{Z} \).

**Remark 4.12.** Proposition \([4.11]\) implies that the cohomology class of the square power of an abelian 3-cocycle is zero. This result was established in \([14]\) Lemma 4.4 (ii).

4.5. Partial obstructions. Since the obstruction of the existence of a solution of the hexagon equation is an element in the total cohomology of a double complex, we can analyze the obstruction by partial obstructions as follows.

**Proposition 4.13** (Partial obstruction 1). Let \( A \) and \( N \) be abelian groups and \( \omega \in \mathbb{Z}^3(A, N) \). If \( \tau_3(\omega) = (\alpha_\omega, \beta_\omega) \), then the cohomology class of \( \alpha_\omega(a|-, -) \in \mathbb{Z}^2(A, N) \) only depends on the cohomology class of \( \omega \). If \( \omega \) is in the image of the suspension map, then

\[
0 = [\alpha_\omega(a|-, -)] \in H^2(A, N)
\]

for all \( a \in A \).

**Proof.** For \((\omega, c) \in H^3(A, N) \) we have that \( \delta_v(c) = \alpha_\omega \), then \([\alpha_\omega(a|-, -)] = 0 \) for all \( a \in A \).

Let \( u : A \times A \to N \), and

\[
w'(a, b, c) = w(a, b, c) + u(a + b, c) + u(a, b + c) - u(a, b, c).
\]

Then

\[
\alpha_\omega'(a|c, d) = \alpha_\omega(a|b, c) + l_a(b) + l_a(c) - l_a(b + c),
\]

where \( l_a(b) = u(a, b) - u(b, a) \).

Assume that \([\alpha_\omega(a|-, -)] = 0 \) for all \( a \in A \). Thus, there exists \( \eta \in C(A|A, N) \) such that \( \delta_v(\eta) = \alpha_\omega \). We have that \((0, \delta_h(\eta) + \beta_\omega) \in \mathbb{Z}^3(\text{Tot}^*(D^{\ast,*}(A, N))) \), thus

\[
\theta(\omega, \eta) := \delta_h(\eta) + \beta_\omega \in \mathbb{Z}^2_{ab}(A, \text{Hom}(A, N)).
\]

In fact,
\[
\delta_v (\delta_h (\eta) + \beta_\omega) = \delta_v (\delta_h (\eta)) + \delta_v (\beta_\omega) \\
= \delta_h (\delta_v (\eta)) + \delta_v (\beta_\omega) \\
= \delta_h (\alpha_\omega) - \delta_h (\alpha_\omega) = 0,
\]
that is, \(\theta(\omega, \eta)(a, b, x + y) = \theta(\omega, \eta)(a, b, x) + \theta(\omega, \eta)(a, b, y)\).

The cohomology of \(\theta(\omega, \eta)\) does not depend on the choice of \(\eta\). In fact, if \(\eta \in C(A, N)\) such that \(\delta_v (\eta) = \alpha_\omega\), then \(\mu := \eta - \eta' \in C^1(A, \text{Hom}(A, U(1)))\) and
\[
\theta(\omega, \eta) - \theta(\omega, \eta') = \delta_h (\eta - \eta') = \delta_h (\mu).
\]

Hence, we have defined a second obstruction
\[
\theta(\omega) \in \text{Ext}_2^Z(A, \text{Hom}(A, N)).
\]

**Corollary 4.14 (Obstruction 2).** Let \(\omega \in Z^2(A, N)\), such that \(0 = [\alpha_\omega (a) - , - ] \in Z^2(A, N)\) for all \(a \in A\). Then there exists \(c : A \times A \to N\) such that \((\omega, c) \in Z^3_{ab}(A, N)\) if and only if \(\theta(\omega) = 0\).

**Proof.** If \((\omega, c) \in Z^3_{ab}(A, N)\), then \(\tau(\omega) = 0\), that implies \(\theta(\omega) = 0\). Now, let \(\omega \in Z^3(A, N)\) and \(\eta \in C(A, N)\) such that \(\delta_v (\eta) = \alpha_\omega\), that is,
\[
\theta(\omega, \eta) = \delta_h (\eta) + \beta_\omega \in Z^2_{ab}(A, \text{Hom}(A, N)).
\]
If \([\theta(\omega, \eta)] = 0\), there is \(l : A \to \text{Hom}(A, N)\) such that \(\delta_h (l) = \delta_h (\eta) + \beta_\omega\). Thus \(c := \eta - l\) is such that \((\omega, c) \in Z^3_{ab}(A, N)\), since
\[
\delta_v (c) = \delta_v (\eta) - \delta_v (l) = \alpha_\omega
\]
and
\[
\delta_h (c) = \delta_h (\eta) - (\delta_h (\eta) + \beta_\omega) \\
= -\beta_\omega.
\]

\(\square\)

5. **Abelian anyons**

In this last section we will present the classification of all possible prime or indecomposable abelian anyon theories.

5.1. **S and T matrices of abelian theories.** By an abelian theory we will mean a triple \((A, \omega, c)\), where \(A\) is abelian group (or equivalently an abelian fusion rules) and \((\omega, c) \in Z^3(A, \mathbb{R}/\mathbb{Z})\) an abelian 3-cocycle (or equivalently a solution of the hexagon equation).

Let \((A, \omega, c)\) be an abelian theory. Recall that the associated quadratic form \(q : A \to \mathbb{R}/\mathbb{Z}\) is defined by \(q(a) = c(a, a)\). The topological spin of \(a \in A\) is defined as the phase
\[
\theta_a = e^{2\pi i q(a)},
\]
thus, the topological spin is exactly the associated quadratic form and by the Eilenberg and MacLane theorem [5, Theorem 26.1] it determines up to gauge equivalence the abelian theory.
Recall that the symmetric bilinear form
\[ b_q : A \times A \to \mathbb{R}/\mathbb{Z} \]
associated to the quadratic for \( q : A \to \mathbb{R}/\mathbb{Z} \) is defined by
\[ b_q(a, b) = q(a + b) - q(a) - q(b). \]
Since \( q(a) = c(a, a) \), we also have that
\[ b_q(a, b) = c(a, b) - c(b, a) \]
for all \( a, b \in A \). The map
\[ H_3^{ab}(A, \mathbb{R}/\mathbb{Z}) \to \text{Hom}(S^2(A)) \]
that associates the symmetric bilinear form \( b_q \) to an abelian 3-cocycle is a group homomorphism.

**Definition 5.1.** An anyonic abelian theory is an abelian theory \((A, \omega, c)\) such that one of the following equivalent conditions holds:

1. The \( S \)-matrix \( S_{ab} = |A|^{-1/2} e^{2\pi i b_q(a, b)} \) is non-singular.
2. The symmetric bilinear form \( b_q \) is non-degenerated.

We will say that an abelian theory \((A, \omega, c)\) is symmetric if its \( b_q \) is trivial, or equivalently, if \(-c(a, b) = c(b, a)\) for all \( a, b \in A \). We will denote by \( H_3^{ab}(A, \mathbb{R}/\mathbb{Z}) \) the subgroup of all equivalence class of symmetric abelian 3-cocycles.

The \( T \)-matrix of an abelian anyonic theory is the diagonal matrix of the topological spins, that is,
\[ T_{ab} = \delta_{a,b} e^{2\pi i q(a)}. \]
Thus, for abelian anyons the \( T \)-matrix completely determines the theory. On the contrary, the \( S \)-matrix does not always determine the theory, however the following result say that two abelian anyons with the same \( S \)-matrix only differ by a symmetric abelian 3-cocycle and their \( T \)-matrices by a linear character \( \chi : A \to \{1, -1\} \).

**Proposition 5.2.** Let \( A \) be an abelian group. Then the diagram
\[
\begin{array}{ccccccc}
0 & \to & H_3^{ab}(A, \mathbb{R}/\mathbb{Z}) & \to & H_3^{ab}(A, \mathbb{R}/\mathbb{Z}) & \xrightarrow{b} & \text{Hom}(S^2(A), \mathbb{R}/\mathbb{Z}) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & \text{Hom}(A, \frac{1}{2}\mathbb{Z}/\mathbb{Z}) & \to & \text{Quad}(A, \mathbb{R}/\mathbb{Z}) & \xrightarrow{b} & \text{Hom}(S^2(A), \mathbb{R}/\mathbb{Z}) & \to & 0
\end{array}
\]
commutes, the vertical morphisms are isomorphisms and the horizontal sequences are exact.

**Proof.** Clearly the kernel of \( b : \text{Quad}(A, \mathbb{R}/\mathbb{Z}) \to \text{Hom}(S^2(A), \mathbb{R}/\mathbb{Z}) \) is \( \text{Hom}(A, \frac{1}{2}\mathbb{Z}/\mathbb{Z}) \). Thus, by Corollary 4.4, the sequence
\[ 0 \to \text{Hom}(A, \frac{1}{2}\mathbb{Z}/\mathbb{Z}) \to \text{Quad}(A, \mathbb{R}/\mathbb{Z}) \to \text{Hom}(S^2(A), \mathbb{R}/\mathbb{Z}) \to 0, \]
is exact. \( \square \)

**Remark 5.3.** A related result was established in [15, Lemma 6.2 (ii), (iii)].
5.2. Prime abelian anyons. If \((A, \omega, c)\) and \((A', \omega', c')\) are abelian anyons theory, their direct sum is defined as the anyon theory \((A \oplus A', \omega \times \omega', c \times c')\), where \(\omega \times \omega'((a, a'), (b, b'), (c, c')) = \omega(a, b, c)\omega'(a', b', c')\) and similarly for \(c \times c'\).

By [16, Theorem 4.4] any abelian anyon theory is a direct sum of prime abelian anyons.

Theorem 5.4. The following is the list of all equivalence classes of prime abelian anyons theories:

(i) If \(\omega = \pm 1\), \(\omega_{p, k}'\) denotes the abelian anyon with fusion rules given by \(\mathbb{Z}/p^k\mathbb{Z}\) and abelian 3-cocycle \((0, c)\), where \(c(x, y) = \frac{wx}{p^k}\) (mod \(\mathbb{Z}\)), for some \(u \in \mathbb{Z}^{>0}\) with \((p, u) = 1\) and \(\left(\frac{2u}{p}\right) = \epsilon\).

(ii) If \(\epsilon \in (\mathbb{Z}/8\mathbb{Z})^*\), \(\omega_{2, k}'\) denotes the abelian anyon with fusion rules given by \(\mathbb{Z}/2^k\mathbb{Z}\) and abelian 3-cocycle

\[
c(x, y) = \frac{ux + vy}{2^{k+1}} \pmod{\mathbb{Z}}, \quad \omega(x, y, z) = \begin{cases} \frac{x}{2} \pmod{\mathbb{Z}}, & \text{if } y + z \geq 2^k, \\ 0 \pmod{\mathbb{Z}}, & \text{otherwise.} \end{cases}
\]

for some \(u \in \mathbb{Z}^{>0}\) with \(u \equiv \epsilon \) (mod 8). The abelian anyons \(w_{2, k}'^1\) and \(w_{2, k}'^{-1}\) are defined for all \(k \geq 1\) and \(w_{2, k}'^5\) and \(w_{2, k}'^{-5}\) for all \(k \geq 2\).

(iii) \(E_k\) denoted the abelian anyon with fusion rules given by \(\mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z}\) and abelian 3-cocycle \((0, c)\), where \(c\in \text{Hom}(\mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z}, \mathbb{R}/\mathbb{Z})\) is defined by

\[
c(\vec{e}_i, \vec{e}_j) = \begin{cases} 0 \pmod{\mathbb{Z}}, & \text{if } i = j, \text{ or } i = 2, j = 1, \\ 2^{-k} \pmod{\mathbb{Z}}, & \text{if } i = 1, j = 2. \end{cases}
\]

(iv) \(F_k\) denoted the abelian anyon with fusion rules given by \(\mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z}\) and abelian 3-cocycle \((0, c)\), where \(c\in \text{Hom}(\mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z}, \mathbb{R}/\mathbb{Z})\) is defined by

\[
c(\vec{e}_i, \vec{e}_j) = \begin{cases} 2^{-k} \pmod{\mathbb{Z}}, & \text{if } i = j, \\ 0 \pmod{\mathbb{Z}}, & \text{if } i = 2, j = 1, \\ -2^{-k} \pmod{\mathbb{Z}}, & \text{if } i = 1, j = 2. \end{cases}
\]
In the cases $E_k$ and $F_k$, we denote $e_1^*(1,0)$ and $e_2^* = (0,1)$.

Proof. We recall from [21] the basic structure of a non-degenerate finite quadratic form over an abelian groups $G$. Let $(A,q)$ be a non-degenerate finite quadratic abelian group. The Sylow decomposition $G = \bigoplus_p A_p$ is an orthogonal direct sum decomposition with respect to the form $q$. Moreover, by the results of [21] and [6] each Sylow subgroup $A_p$, admits an orthogonal direct sum decomposition into indecomposable quadratic group of the following type:

(a) If $p \neq 2$ and $\epsilon = \pm 1$, $\omega_{p,k}^\epsilon$ denotes the quadratic abelian group $\mathbb{Z}/p^k \mathbb{Z}$ and quadratic form determined by $q(1) = up^{2k} (\text{mod } \mathbb{Z})$ for some $u \in \mathbb{Z}^{>0}$ with $(p, u) = 1$ and $\left(\frac{2u}{p}\right) = \epsilon$.

(b) If $\epsilon \in (\mathbb{Z}/8\mathbb{Z})^\times$, $\omega_{2,k}^\epsilon$ denotes the quadratic abelian group $\mathbb{Z}/2^k \mathbb{Z}$ and quadratic form determined by $q(1) = 2^{(k-1)}u (\text{mod } \mathbb{Z})$ for some $u \in \mathbb{Z}^{>0}$ with $u \equiv \epsilon (\text{mod } 8)$. The quadratic groups $w_{2,k}^1$ and $w_{2,k}^{-1}$ are defined for all $k \geq 1$ and $w_{2,k}^5$ and $w_{2,k}^{-5}$ for all $k \geq 2$.

(c) $E_k$ denoted the the quadratic abelian group $\mathbb{Z}/2^k \mathbb{Z} \oplus \mathbb{Z}/2^k \mathbb{Z}$ and quadratic form determined by $q((1,0)) = q(0,1) = 0 (\text{mod } \mathbb{Z})$ and $q((1,1)) = 2^k (\text{mod } \mathbb{Z})$.

(d) $F_k$ denoted the the quadratic abelian group $\mathbb{Z}/2^k \mathbb{Z} \oplus \mathbb{Z}/2^k \mathbb{Z}$ and quadratic form determined by $q((1,0)) = q(0,1) = q((1,1)) = 2^{-k} (\text{mod } \mathbb{Z})$.

Thus, to proof the theorem we only need to see that the abelian 3-cocycles defined in (i)-(iv) have the corresponding quadratic forms (a)-(d). In fact,

\[
\text{Case } \omega_{p,k}^\epsilon : \quad c(1,1) = up^{2k}.
\]

\[
\text{Case } \omega_{2,k}^\epsilon : \quad c(1,1) = u2^{2k-1}.
\]

\[
\text{Case } E_k : \quad c(e_1^*, e_1^*) = 0, \quad c((1,1),(1,1)) = c(e_1^*, e_2^*) = 2^{-k}.
\]

\[
\text{Case } F_k : \quad c(e_1^*, e_1^*) = 2^{-k}, \quad c((1,1),(1,1)) = 2c(e_1^*, e_1^*) + c(e_1^*, e_2^*) = 2^{-k}.
\]

\[\square\]

We finish with the formulas of the modular data of the prime abelian anyons theories (using the convention of identify $U(1)$ with $\mathbb{R}/\mathbb{Z}$):

(i) $\omega_{p,k}^\epsilon$, $p \neq 2$, $\epsilon = \pm 1$
\[
S_{a,b} = 2uabp^{2k}, \quad T_{a,a} = u^2p^{2k},
\]

(ii) $\omega_{2,k}^\epsilon$, $\epsilon \in (\mathbb{Z}/8\mathbb{Z})^\times$
\[
S_{a,b} = uab2^{-2k}, \quad T_{a,a} = u^22^{-(k+1)}.
\]

(iii) $E_k$
\[
S_{(a_1,a_2),(b_1,b_2)} = (a_1b_2 + b_1a_2)2^{-2k}, \quad T_{(a,b),(a,b)} = ab2^{-2k}
\]

(iv) $F_k$
\[
S_{(a_1,a_2),(b_1,b_2)} = (a_1b_1 + a_2b_2)2^{-(k+1)} - (a_1b_2 + a_2b_1)2^{-2k}
\]
\[
T_{(a,b),(a,b)} = (a^2 + b^2 - ab)2^{-2k}.
\]
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