Classification of genus-1 holomorphic Lefschetz pencils

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Abstract: In this paper, we classify relatively minimal genus-1 holomorphic Lefschetz pencils up to smooth isomorphism. We first show that such a pencil is isomorphic to either the pencil on \(\mathbb{P}^1 \times \mathbb{P}^1\) of bidegree \((2, 2)\) or a blow-up of the pencil on \(\mathbb{P}^2\) of degree 3, provided that no fiber of a pencil contains an embedded sphere (note that one can easily classify genus-1 Lefschetz pencils with an embedded sphere in a fiber). We further determine the monodromy factorizations of these pencils and show that the isomorphism class of a blow-up of the pencil on \(\mathbb{P}^2\) of degree 3 does not depend on the choice of blown-up base points. We also show that the genus-1 Lefschetz pencils constructed by Korkmaz-Ozbagci (with nine base points) and Tanaka (with eight base points) are respectively isomorphic to the pencils on \(\mathbb{P}^2\) and \(\mathbb{P}^1 \times \mathbb{P}^1\) above, in particular these are both holomorphic.

Key words: Lefschetz pencil, monodromy factorization, holed torus relation, braid monodromy

1. Introduction

Classification problems of Lefschetz fibrations up to smooth isomorphism have attracted a lot of interest since around 1980. The first result concerning the problems was given in [13, 18], in which Kas and Moishezon independently classified genus-1 Lefschetz fibrations over the 2-sphere. This classification result was extended to more general genus-1 fibrations: those with general base spaces and achiral singularities [12, 15, 16]. Furthermore, Siebert and Tian [28] classified genus-2 Lefschetz fibrations over the 2-sphere with transitive monodromies and no reducible fibers by showing that such fibrations are always holomorphic. Classifications up to stabilizations by fiber sums were also studied in [1–5].

Whereas there are various results on classifications of Lefschetz fibrations, very little is known about the corresponding results on Lefschetz pencils, aside from the classification of genus-0 pencils implicitly given in [26]. In this paper, we will deal with the classification problem of genus-1 Lefschetz pencils. We first show that a genus-1 holomorphic Lefschetz pencil is isomorphic to either of the standard ones given below:

Theorem 1.1 Let \(f : X \to \mathbb{P}^1\) be a genus-1 relatively minimal holomorphic Lefschetz pencil. Suppose that no fibers of \(f\) contain an embedded sphere. Then either of the following holds:

- \(f\) is smoothly isomorphic to the one obtained by blowing-up the Lefschetz pencil \(f_n : \mathbb{P}^2 \to \mathbb{P}^1\), which is the composition of the Veronese embedding \(v_3 : \mathbb{P}^2 \hookrightarrow \mathbb{P}^9\) of degree 3 and a generic projection \(\mathbb{P}^9 \to \mathbb{P}^1\).\(^\ast\)

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The subscripts "n" and "s" for the Lefschetz pencils $f_n$ and $f_s$ represent the properties "nonspin" and "spin", respectively. Note that, needless to say, the blow-ups of $f_n$ also give Lefschetz pencils. Theorem 1.1 implies that such pencils are isomorphic to the blow-ups of $f_n$. The assumption of relative minimality and the additional requirement that no fibers contain an embedded sphere with any self-intersection number should not be confused. The latter is required to exclude inessential Lefschetz pencils. For more detail, see Remark 3.2.

Although the isomorphism classes of the pencils $f_n$ and $f_s$ do not depend on the choice of generic projections $\mathbb{P}^9 \to \mathbb{P}^1$ (cf. Remark 2.2), one cannot deduce immediately from Theorem 1.1 that the isomorphism class of a blow-up of $f_n$ does not depend on the choice of blown-up base points (one can indeed find in [10] examples of a pair of nonisomorphic pencils that are obtained by blowing-up a common pencil at the same number but different combinations of base points). We next address this issue by examining the monodromies of $f_n$ and $f_s$.

It is a standard fact in the literature that there is one-to-one correspondence between the isomorphism classes of genus- $g$ Lefschetz pencils with $m$ critical points and $k$ base points and the Hurwitz equivalence classes of factorizations

$$t_{c_m} \cdots t_{c_1} = t_{\delta_1} \cdots t_{\delta_k}$$

of the boundary multitwist $t_{\delta_1} \cdots t_{\delta_k}$ as products of positive Dehn twists in the mapping class group of a $k$-holed surface of genus $g$. Here, $\delta_i$ stands for a simple closed curve parallel to the $i$-th boundary component. Such a factorization is called a monodromy factorization in general, or also a $k$-holed torus relation when $g = 1$. Relying on the theory of braid monodromies due to Moishezon-Teicher [19–23] we determine the monodromy factorizations of $f_n$ and $f_s$. We further analyze the Hurwitz equivalence classes of the factorizations, and eventually show the following:

**Theorem 1.2** Let $f : X \to \mathbb{P}^1$ be a relatively minimal genus-1 holomorphic Lefschetz pencil without embedded spheres in fibers. The monodromy factorization of $f$ is Hurwitz equivalent to that of one of the pencils in Table 1 (the curves in the table are given in Figure 1). In particular, the isomorphism class of a blow-up of $f_n$ does not depend on the choice of blown-up base points.

Note that according to the aforementioned works of Kas and Moishezon [13, 18] the only genus-1 Lefschetz fibration that admits a $(−1)$-section is the well-known rational elliptic fibration $E(1) \to \mathbb{P}^1$, whose monodromy factorization is $(ta_1tb)^6 = 1$, where $a$ is the meridian and $b$ is the longitude of the torus. Thus, any genus-1 Lefschetz pencil, even a nonholomorphic one (if exists), must descend to this fibration after blowing-up all the base points. This is clearly reflected in Table 1, where once more blowing-up of the pencil $f_n \# 8\mathbb{P}^2$ results in $E(1) = \mathbb{P}^2 \# 9\mathbb{P}^2$ and $(ta_1tb)^6 = 1$.

Examples of explicit $k$-holed torus relations have also been discovered in purely topological and combinatorial ways, without relying on the knowledge from complex geometry. Korkmaz and Ozbagci [14] systematically constructed $k$-holed torus relations up to the largest possible $k = 9$ for the first time. Then an 8-holed torus relation was constructed by Tanaka [31]. In both of the studies, the authors obtained those relations by combining certain known relations (i.e. the 2-chain relation and the lantern relation) in the mapping class groups.
Table 1. Classification of the genus-1 holomorphic Lefschetz pencils. The boundary multitwist \( t_{\delta_1} \cdots t_{\delta_k} \) is represented by \( \partial_k \).

| Pencil | Number of base points | Monodromy factorization | Total space |
|--------|-----------------------|-------------------------|-------------|
| \( f_n \) | 9 | \( t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8 = \partial_9 \) | \( \mathbb{P}^2 \) |
| \( f_s \) | 8 | \( t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8 = \partial_8 \) | \( \mathbb{P}^1 \times \mathbb{P}^1 \) |
| \( f_n \# \mathbb{P}^2 \) | 8 | \( t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8 = \partial_8 \) | \( \mathbb{P}^2 \# \mathbb{P}^2 \) |
| \( f_n \# 2 \mathbb{P}^2 \) | 7 | \( t_1 t_2 t_3 t_4 t_5 t_6 t_7 = \partial_7 \) | \( \mathbb{P}^2 \# 2 \mathbb{P}^2 \) |
| \( f_n \# 3 \mathbb{P}^2 \) | 6 | \( t_1 t_2 t_3 t_4 t_5 t_6 = \partial_6 \) | \( \mathbb{P}^2 \# 3 \mathbb{P}^2 \) |
| \( f_n \# 4 \mathbb{P}^2 \) | 5 | \( t_1 t_2 t_3 t_4 t_5 t_6 = \partial_5 \) | \( \mathbb{P}^2 \# 4 \mathbb{P}^2 \) |
| \( f_n \# 5 \mathbb{P}^2 \) | 4 | \( t_1 t_2 t_3 t_4 t_5 t_6 = \partial_4 \) | \( \mathbb{P}^2 \# 5 \mathbb{P}^2 \) |
| \( f_n \# 6 \mathbb{P}^2 \) | 3 | \( t_1 t_2 t_3 t_4 t_5 t_6 = \partial_3 \) | \( \mathbb{P}^2 \# 6 \mathbb{P}^2 \) |
| \( f_n \# 7 \mathbb{P}^2 \) | 2 | \( t_1 t_2 t_3 t_4 t_5 t_6 = \partial_2 \) | \( \mathbb{P}^2 \# 7 \mathbb{P}^2 \) |
| \( f_n \# 8 \mathbb{P}^2 \) | 1 | \( t_1 t_2 t_3 t_4 t_5 t_6 = \partial_1 \) | \( \mathbb{P}^2 \# 8 \mathbb{P}^2 \) |

Figure 1. The curves on the \( k \)-holed torus \( \Sigma_k^k \).

Ozbagci [25] later observed that the two 8-holed torus relations by Korkmaz-Ozbagci and by Tanaka are not Hurwitz equivalent. Some other 8-holed torus relations are also constructed in [9, 10], though it turns out that they are Hurwitz equivalent to either Korkmaz-Ozbagci’s or Tanaka’s. We will show that the \( k \)-holed torus relations of Korkmaz-Ozbagci and Tanaka are Hurwitz equivalent to the monodromy factorizations in Table 1, in particular we conclude that the Lefschetz pencils associated with their relations are holomorphic (Theorems 5.1 and 5.2).

The virtue of our presentations of the \( k \)-holed torus relations in Table 1 is that the curves involved are remarkably simple as they are well-organized lifts of the meridian and longitude of a closed torus. As the
$k$-holed torus relations are fundamentally important to construct relations in the mapping class groups of even higher genera, having simpler expressions may help those who try to use them.

As our results in the present paper take care of holomorphic pencils, the next step shall be the ultimate classification of genus-1 smooth Lefschetz pencils. Although we speculate that any genus-1 Lefschetz pencil is isomorphic to one of the holomorphic ones, we do not have the machinery to prove this. This leaves the following open question.

**Question 1.3** Is there a nonholomorphic genus-1 Lefschetz pencil? In other words, is there a $k$-holed torus relation that is not Hurwitz equivalent to any of the $k$-holed torus relations in Table 1?

The paper is organized as follows. In Section 2, we briefly review basic properties of holomorphic Lefschetz pencils and monodromy factorizations. Section 3 is devoted to proving Theroem 1.1. In Section 4, we determine monodromy factorizations of the pencils $f_n$ and $f_s$. We analyze combinatorial properties of the monodromy factorizations of $f_n$ and $f_s$ in Section 5, completing the proof of Theorem 1.2. Finally in Section 6, we discuss some additional topics related to the pencils and $k$-holed torus relations.

## 2. Preliminaries

Throughout this paper, we will assume that manifolds are smooth, connected, oriented, and closed unless otherwise noted. We denote the $n$-dimensional complex projective space by $\mathbb{P}^n$. Let $X$ be a 4-manifold. A Lefschetz pencil on $X$ is a smooth mapping $f : X \setminus B \to \mathbb{P}^1$ defined on the complement of a nonempty finite subset $B \subset X$ satisfying the following conditions:

- for any critical point $p \in X$ of $f$, there exists a complex coordinate neighborhood $(U, \varphi : U \to \mathbb{C}^2)$ (resp. $(V, \psi : V \to \mathbb{C})$) at $p$ (resp. $f(p)$) compatible with the orientation such that $\psi \circ f \circ \varphi^{-1}(z, w)$ is equal to $z^2 + w^2$,

- for any $b \in B$, there exists a complex coordinate neighborhood $(U, \varphi)$ of $b$ compatible with the orientation and an orientation preserving diffeomorphism $\xi : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\xi \circ f \circ \varphi^{-1}(z, w)$ is equal to $[z : w]$,

- the restriction $f|_{\text{Crit}(f)}$ is injective.

The set $B$ is called the base point set of $f$. In this paper we will use the dashed arrow $\dashrightarrow$ to represent Lefschetz pencils, e.g., $f : X \dashrightarrow \mathbb{P}^1$, when we do not need to represent the base point set explicitly (note that this symbol will be also used to represent meromorphic mappings). For a Lefschetz pencil $f$, the genus of the closure of a regular fiber is called the genus of $f$.

A Lefschetz pencil $f$ is said to be relatively minimal if no fiber of $f$ contains a $(-1)$-sphere. Let $f : X \dashrightarrow \mathbb{P}^1$ be a Lefschetz pencil, $\tilde{X}$ be a blow-up of $X$ at a point and $\pi : \tilde{X} \to X$ be the blow-down mapping. One can construct a Lefschetz pencil $\tilde{f} : \tilde{X} \dashrightarrow \mathbb{P}^1$ so that $\tilde{f} = f \circ \pi$ on the complement of the exceptional sphere. Conversely, any relatively nonminimal Lefschetz pencil can be obtained from a relatively minimal one by this construction. In particular, relatively nonminimal Lefschetz pencils are inessential in the context of classification, and thus, we will assume that Lefschetz pencils are relatively minimal unless otherwise noted.

1082
2.1. Holomorphic Lefschetz pencils

A Lefschetz pencil $f : X \to \mathbb{P}^1$ is said to be holomorphic if there exists a complex structure of $X$ such that $f$ is holomorphic and we can take biholomorphic $\varphi, \psi$, and $\xi$ in the conditions in the definition above. A Lefschetz pencil on a complex surface $S$ is said to be holomorphic if it is holomorphic with respect to the given complex structure. For a complex surface $S$, it is well-known that a divisor $D \in \text{Div}(S)$ gives rise to a line bundle over $S$, which we denote by $[D]$ (see [8] for details).

**Proposition 2.1** Let $S$ be a complex surface, $f : S \to \mathbb{P}^1$ be a holomorphic Lefschetz pencil, and $F \subset S$ be the closure of a fiber of $f$.

1. The genus of $f$ is equal to $(2 + F^2 + K_S(F))/2$, where $K_S \in H^2(S; \mathbb{Z})$ is the canonical class of $S$ and $F \in H_2(S; \mathbb{Z})$ is the homology class represented by $F$.

2. There exist sections $s_0, s_1$ of the line bundle $[F]$ such that $f$ is equal to $[s_0 : s_1] : S \to \mathbb{P}^1$.

3. Let $C \subset S$ be an irreducible curve. The intersection number $C \cdot F$ is greater than or equal to 0. Furthermore, it is equal to 0 if and only if $C$ is a component of a fiber of $f$ without base points.

**Proof** (1) is merely a consequence of the adjunction formula, and we can prove (2) in the same way as that for [11, Lemma 3.1]. In what follows we will prove (3). Let $\tilde{S}$ be the complex surface obtained by blowing-up $S$ at all the base points of $f$ and $\tilde{C} \subset \tilde{S}$ (resp. $\tilde{F} \subset \tilde{S}$) be the proper transform of $C$ (resp. $F$). The pencil $f$ induces a fibration $\tilde{f} : \tilde{S} \to \mathbb{P}^1$. Without loss of generality we can assume that $\tilde{F}$ does not contain any singular point of $\tilde{C}$ and any critical point of $\tilde{f}|_{\tilde{C}}$. Since $\tilde{F}$ is a fiber of $\tilde{f}$, the intersection number $\tilde{F} \cdot \tilde{C}$ is equal to $\tilde{f}(\tilde{F} \cap \tilde{C})$. Hence, we obtain:

$$F \cdot C = \tilde{F} \cdot \tilde{C} + \#(C \cap B) = \#(\tilde{F} \cap \tilde{C}) + \#(C \cap B) \geq 0.$$ 

Moreover, the equality holds only if $C \cap B = \emptyset$ and $\tilde{f}|_{\tilde{C}}$ is a constant map. The latter condition implies that $C$ is contained in a fiber of $f$. \qed

**Remark 2.2** For a line bundle $L$ over a complex surface $S$ with sections, we can define a meromorphic mapping $\varphi_L : S \to \mathbb{P}^{d-1}$ as follows:

$$\varphi_L(x) = [s_1(x) : \cdots : s_l(x)],$$

where $s_1, \ldots, s_l$ is a basis of $H^0(S; L)$. The statement (2) of Proposition 2.1 implies that $f$ is the composition of $\varphi_L : S \to \mathbb{P}^{m-1}$ (where $m = \text{dim} H^0(S; [F])$) and a projection $\mathbb{P}^{m-1} \to \mathbb{P}^1$. Note that the composition of $\varphi_L$ and a projection $\mathbb{P}^d \to \mathbb{P}^1$ is not always a Lefschetz pencil. It is known, however, that the composition is a Lefschetz pencil provided that $L$ is very ample and the projection is generic. Moreover, the smooth isomorphism class of the Lefschetz pencil does not depend on the choice of this projection (see [11, 32]).

2.2. Monodromy factorizations

For a compact oriented connected surface $\Sigma$ (possibly with boundaries), we denote by $\text{Diff}(\Sigma)$ the set of self-diffeomorphisms of $\Sigma$ preserving the boundary pointwise, endowed with the Whitney $C^\infty$-topology. Let $\text{MCG}(\Sigma) = \pi_0(\text{Diff}(\Sigma))$, which has the group structure defined by the composition of representatives.
Let $f : X \setminus B \to \mathbb{P}^1$ be a genus-$g$ Lefschetz pencil with $k$ base points and $Q = \{q_1, \ldots, q_m\} \subset \mathbb{P}^1$ be the set of critical values of $f$. We take a point $q_0 \in \mathbb{P}^1 \setminus Q$ and a path $\alpha_i \subset \mathbb{P}^1$ ($i = 1, \ldots, m$) from $q_0$ to $q_i$ satisfying the following conditions:

- $\alpha_1, \ldots, \alpha_m$ are mutually distinct except at the common initial point $q_0$.
- $\alpha_1, \ldots, \alpha_m$ appear in this order when we go around $q_0$ counterclockwise.

The system of paths $(\alpha_1, \ldots, \alpha_m)$ satisfying the conditions above is called a Hurwitz path system of $f$. Let $\gamma_i \subset \mathbb{P}^1$ be a based loop with the base point $q_0$ obtained by connecting $q_0$ with a small circle oriented counterclockwise by $\alpha_i$. It is known that the monodromy along $\gamma_i$ is the Dehn twist along some simple closed curve $c_i \subset \overline{f^{-1}(q_0)}$, called a vanishing cycle of $f$ with respect to the path $\alpha_i$. Furthermore, we can obtain the following relation in $\text{MCG}(\overline{f^{-1}(q_0)} \setminus \nu B)$:

$$t_{c_m} \cdots t_{c_1} = t_{\delta_1} \cdots t_{\delta_k}, \quad (2.1)$$

where $\nu B$ is a tubular neighborhood of $B \subset \overline{f^{-1}(q_0)}$ and $\delta_1, \ldots, \delta_k \subset \overline{f^{-1}(q_0)} \setminus \nu B$ are simple closed curves parallel to the boundary components. We call this relation a monodromy factorization of $f$. Conversely, let $\Sigma_g^k$ be a genus-$g$ compact surface with $k$ boundary components, and $c_1, \ldots, c_m \subset \Sigma_g^k$ be simple closed curves satisfying the relation (2.1) in $\text{MCG}(\Sigma_g^k)$. We can construct a genus-$g$ Lefschetz pencil $f : X \setminus B \to \mathbb{P}^1$ with $k$ base points and vanishing cycles $c_1, \ldots, c_m$, under some identification of the complement $\overline{f^{-1}(q_0)} \setminus \nu B$ of the closure of a regular fiber with $\Sigma_g^k$.

### 3. Complex surfaces admitting genus-1 Lefschetz pencils

This section is devoted to proving Theorem 1.1, which one can easily deduce from the following theorem.

**Theorem 3.1** Let $S$ be a complex surface, $f : S \to \mathbb{P}^1$ be a genus-1 holomorphic Lefschetz pencil, and $F$ be the closure of a fiber of $f$. Suppose that no fibers of $f$ contain an embedded sphere. Then either of the following holds:

- the complex surface $S$ can be obtained by blowing-up $\mathbb{P}^2$ at $l \leq 8$ points and $F$ is linearly equivalent to $3H - \sum_{i=1}^{l} E_i$, where $H$ is the total transform of a projective line $H'$ in $\mathbb{P}^2$ and $E_1, \ldots, E_l$ are the exceptional spheres.
- the complex surface $S$ is $\mathbb{P}^1 \times \mathbb{P}^1$ and $F$ is linearly equivalent to $2F_1 + 2F_2$, where $F_i$ is a fiber of the projection $\pi_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ onto the $i$-th component.

**Remark 3.2** This remark concerns the assumption that no fibers of a pencil contain an embedded sphere, which is required in not only the theorem above but also the main theorems in the paper. Even if a Lefschetz pencil is relatively minimal, a fiber of it might contain an embedded sphere. For example, let us consider the Lefschetz pencil $f_{g,k} : X_{g,k} \to \mathbb{P}^1$ with the following monodromy factorization:

$$t_{\delta_1} \cdots t_{\delta_k} = t_{\delta_1} \cdots t_{\delta_k} \text{ in } \text{MCG}(\Sigma_g^k), \quad (3.1)$$
The total space \( X_{g,k} \) is a ruled surface and the pencil \( f_{g,k} \) has \( k \) critical (resp. base) points corresponding to the twists in the left-hand (resp. right-hand) side of (3.1). This pencil is relatively minimal but each singular fiber of it contains a sphere. Furthermore, there exist other types of such Lefschetz pencils with genus-0: the pencils of degree 1 and 2 curves in \( \mathbb{P}^2 \). The former (resp. the latter) gives rise to the trivial relation \( 1 = \tau_\delta \) in \( \text{MCG}(D^2) \) (resp. the lantern relation) as the monodromy factorization. Such pencils, however, are not important in the context of classification; if a (not necessarily holomorphic) relatively minimal Lefschetz pencil has an embedded sphere in a fiber, it is isomorphic to one of the examples given here. This follows from the observation in [26, Remark 2.4] for genus-0 and the lemma below for higher genera.

**Lemma 3.3** Let \( f : X \to \mathbb{P}^1 \) be a relatively minimal Lefschetz pencil with genus-\( g \geq 1 \). Suppose that there exists an embedded sphere in a fiber of \( f \). Then a monodromy factorization of \( f \) is \( \tau_{\delta_1} \cdots \tau_{\delta_k} = \tau_{t_1} \cdots \tau_{t_m} \).

**Proof** [Proof of Lemma 3.3] Let \( m \) and \( k \) be the numbers of critical points and base points of \( f \), respectively, and \( \tau_{c_1} \cdots \tau_{c_m} = \tau_{t_1} \cdots \tau_{t_m} \) be a monodromy factorization of \( f \), where \( c_1, \ldots, c_m \subset \Sigma^k_g \) be simple closed curves in \( \Sigma^k_g \). By capping the boundary of \( \Sigma^k_g \) by disks, we can regard \( \Sigma^k_g \) as a subsurface of the closed surface \( \Sigma_g \). By the assumption, one of the vanishing cycles of \( f \), say \( c_1 \), is not essential in \( \Sigma_g \). Let \( S \) be the closure of the genus-0 component of the complement \( \Sigma^k_g \setminus c_1 \). Since \( f \) is relatively minimal, \( S \) contains a boundary component of \( \Sigma^k_g \). By capping all the boundary components of \( \Sigma^k_g \) except for one in \( S \), we obtain the following relation in \( \text{MCG}(\Sigma_g^1) \):

\[
t_\delta \cdot t_{c_2} \cdots t_{c_m} = t_\delta \Rightarrow t_{c_2} \cdots t_{c_m} = 1
\]

If one of the curves \( c_2, \ldots, c_m \) is essential in \( \Sigma^1_g \), the equality above implies that there exists a relatively minimal nontrivial genus-\( g \) Lefschetz fibration with a square-zero section, contradicting [29, Proposition 3.3] and [30, Lemma 2.1]. Thus, all the curves \( c_1, \ldots, c_m \) bounds a genus-0 subsurface in \( \Sigma^k_g \). We can also deduce from the observation above that the fundamental group of \( X \) is isomorphic to that of \( \Sigma_g \).

Let \( S_i \) be the genus-0 component of \( \Sigma^k_g \setminus c_i \). Suppose that \( S_i \) contains more than one component of \( \partial \Sigma^k_g \) for some \( i \). Then \( X \) has a symplectic structure such that there exists an embedded symplectic sphere \( C \) with positive square. Since \( X \) is not rational, one can verify in the same way as that in the proof of [17, Theorem 1.4 (ii)] that \( X \) is an irrational ruled surface, \( C \) is away from a maximal disjoint family of exceptional spheres, and after blow-down \( C \) becomes a fiber of a \( \mathbb{P}^1 \)-bundle. However, this contradicts that \( C \) has positive square. We can eventually show that \( S_i \) contains only one component of \( \partial \Sigma^k_g \) for each \( i = 1, \ldots, m \), in particular each \( c_i \) is isotopic to some \( \delta_j \). The lemma then follows from the fact that the subgroup of \( \text{MCG}(\Sigma^k_g) \) generated by \( t_{\delta_1}, \ldots, t_{\delta_k} \) is isomorphic to the free abelian group \( \mathbb{Z}^k \).

**Proof** [Proof of Theorem 3.1] Let \( f : S \to \mathbb{P}^1 \) be a genus-1 holomorphic Lefschetz pencil and \( \tilde{f} : \tilde{S} \to \mathbb{P}^1 \) be the Lefschetz fibration obtained by blowing-up all the base points of \( f \). By Lemma 3.3 and the assumption, \( \tilde{f} \) is relatively minimal. We can deduce from the classification of genus-1 Lefschetz fibrations in the smooth category (given in [18]) that \( \tilde{S} \) is diffeomorphic to \( \mathbb{P}^2 \# 9 \mathbb{P}^2 \). Thus, applying [6, Corollary 2], we can show that \( S \) is a rational surface, in particular \( S \) is either \( \mathbb{P}^2 \) or a blow-up of the Hirzebruch surface \( S_n \) for some \( n \geq 0 \) (for the definition of \( S_n \), see [8, Chap. 4, §3]). Suppose that \( S \) is the projective plane. Then the number of the
base points of \( f \) is equal to 9; thus, the self-intersection of \( F \) is also equal to 9. Since the line bundle \([F]\) has at least two linearly independent sections by (2) of Proposition 2.1, \( F \) is linearly equivalent to \( aH \) for some \( a > 0 \) (note that the linear equivalence class of a divisor of a simply connected Kähler manifold is uniquely determined by the corresponding second cohomology class). The self-intersection of \( aH \) is equal to \( a^2 \). Hence, we can conclude that \( F \) is linearly equivalent to \( 3H \).

In the rest of the proof, we assume that \( S \) can be obtained by blowing-up the Hirzebruch surface \( S_n \) \((n \geq 0)\) \( l' \) times \((0 \leq l' \leq 7)\). Let \( E'_\infty \subset S_n \) be a section of \( S_n \) (as a \( \mathbb{P}^1 \)-bundle) with self-intersection \(-n\) (which is unique when \( n > 0 \)), and \( C' \subset S_n \) be a fiber of the same \( \mathbb{P}^1 \)-bundle on \( S_n \). We denote the total transforms of \( E'_\infty \) and \( C' \) by \( E_\infty \) and \( C \), respectively. Let \( \hat{E}_i \subset S \) be the total transform of the exceptional sphere appearing in the \( i \)-th blow-up of \( S_n \). Since \( S \) is simply connected and Kähler, we can assume that \( F \) is linearly equivalent to the following divisor:

\[
aE_\infty + bC - \sum_{i=1}^{l'} c_i \hat{E}_i \quad (a, b, c_i \in \mathbb{Z}).
\]

All the components of \( E_\infty, C \) and \( \hat{E}_i \) are spheres. Since no fiber of \( f \) contains a sphere, we can deduce the following inequality from (3) of Proposition 2.1:

\[
a > 0, \quad b > na, \text{ and } c_i > 0 \quad (i = 1, \ldots, l').
\]

(3.2)

Since the number of base points of \( f \) is equal to the self-intersection of \( F \), we obtain the following equality:

\[
8 - l' = -na^2 + 2ab - \sum_{i=1}^{l'} c_i^2.
\]

(3.3)

The canonical class of \( S_n \) is represented by the divisor \(-2E_\infty - (2+n)C\) (see [8, Chap. 4, §.3]). Thus, we can deduce the following equality from (1) of Proposition 2.1:

\[
8 - l' + (n - 2)a - 2b + \sum_{i=1}^{l'} c_i = 0 \iff b = \frac{1}{2} \left(8 - l' + (n - 2)a + \sum_{i=1}^{l'} c_i \right).
\]

(3.4)

Combining the equalities (3.3) and (3.4), we obtain:

\[
- na^2 + a \left((n - 2)a + 8 - l' + \sum_{i=1}^{l'} c_i \right) - \sum_{i=1}^{l'} c_i^2 - (8 - l') = 0
\]

\[
\iff -2a^2 + \left(8 - l' + \sum_{i=1}^{l'} c_i \right) a - \sum_{i=1}^{l'} c_i^2 - (8 - l') = 0.
\]

(3.5)

We can regard (3.5) as a quadratic equation on \( a \), whose discriminant must be nonnegative. Thus, the following inequality holds:

\[
\left(8 - l' + \sum_{i=1}^{l'} c_i \right)^2 - 8 \left(\sum_{i=1}^{l'} c_i^2 + (8 - l') \right) \geq 0
\]
\[
\left( \sum_{i=1}^{l'} c_i \right)^2 + 2(8 - l') \sum_{i=1}^{l'} c_i - 8 \sum_{i=1}^{l'} c_i^2 - l(8 - l') \geq 0. \tag{3.6}
\]

Applying the Cauchy-Schwarz inequality to the vectors \( \left( \sum_{i=1}^{l'} c_i, \ldots, \sum_{i=1}^{l'} c_i \right) \) and \( (c_1, \ldots, c_{l'}) \), we obtain the following inequality:

\[
\left( \sum_{i=1}^{l'} c_i \right)^2 \leq \sqrt{l'} \left( \sum_{i=1}^{l'} c_i \right) \cdot \sqrt{\sum_{i=1}^{l'} c_i^2} \Rightarrow \sum_{i=1}^{l'} c_i \leq \sqrt{l'} \sum_{i=1}^{l'} c_i^2. \tag{3.7}
\]

Combining the inequalities (3.6) and (3.7), we eventually obtain:

\[
l' \sum_{i=1}^{l'} c_i^2 + 2\sqrt{l'}(8 - l') \sqrt{\sum_{i=1}^{l'} c_i^2} - 8 \sum_{i=1}^{l'} c_i^2 - l'(8 - l') \geq 0
\]

\[
\Rightarrow -(8 - l') \left( \sqrt{\sum_{i=1}^{l'} c_i^2} - \sqrt{l'} \right)^2 \geq 0.
\]

Since \( l' \) is less than 8, we can deduce from this inequality that the sum \( \sum_{i=1}^{l'} c_i^2 \) is equal to \( l' \). This equality together with the inequality in (3.2) implies that \( c_1, \ldots, c_{l'} \) are all equal to 1. By substituting 1 for all the \( c_i \)'s in (3.5), we obtain:

\[-2a^2 + 8a - 8 = 0 \Rightarrow a = 2. \]

We can further deduce from (3.4) that \( b \) is equal to \( n + 2 \). Since \( b \) is greater than \( na \), \( n \) is equal to 0 or 1.

If \( n \) is equal to 1, the complex surface \( S \) is a blow-up of \( S_1 \), which is a blow-up of \( \mathbb{P}^2 \) at a single point. In other words, there is a sequence of blow-up from \( \mathbb{P}^2 \) to \( S \):

\[
S^{(0)} := \mathbb{P}^2 \leftarrow S^{(1)} := S_1 \leftarrow S^{(2)} \leftarrow \cdots \leftarrow S^{(l)} := S, \text{ where } l = l' + 1. \tag{3.8}
\]

We denote the exceptional sphere in \( \Sigma_1 \) by \( \hat{E}_0' \). The divisors \( E'_\infty \) and \( C' \) are respectively linearly equivalent to \( \hat{E}_0' \) and \( H' - \hat{E}_0' \). Let \( \hat{E}_0 \subset S \) be the total transform of \( \hat{E}_0' \). The closure of a fiber \( F \) of \( f \) is linearly equivalent to \( aE_\infty + bC - \sum_{i=1}^{l'} \hat{E}_i = 3H - \sum_{i=1}^{l'} E_i \), where \( E_i = \hat{E}_{i-1} \). Suppose that the \( j \)-th blow-up in the sequence (3.8) is applied at a point on the exceptional sphere appearing in the \( i \)-th blow-up for some \( i < j \). Then the divisor \( E_i - E_j \) would be linearly equivalent to a positive linear combination of spheres. By (3) of Proposition 2.1, the self-intersection \( (E_i - E_j) \cdot F \) would be positive, but this is not the case since \( F \) is linearly equivalent to \( 3H - \sum_{i=1}^{l'} E_i \). We can eventually conclude that \( S \) is obtained from \( \mathbb{P}^2 \) by blowing-up \( l \) points.

Finally, suppose that \( n \) is equal to 0. The complex surface \( S \) is \( S_0 = \mathbb{P}^1 \times \mathbb{P}^1 \), and \( E'_\infty \) and \( C' \) are respectively equal to \( F_1 \) and \( F_2 \). Since the blow-up of \( \mathbb{P}^1 \times \mathbb{P}^1 \) at a single point is biholomorphic to the surface obtained by blowing-up \( \mathbb{P}^2 \) at two points, we can assume that \( l \) is equal to 0 without loss of generality. The closure of a fiber \( F \) is then linearly equivalent to \( aF_1 + bF_2 = 2F_2 + 2F_2 \). This completes the proof of Theorem 3.1. \( \square \)

**Proof** [Proof of Theorem 1.1] We first observe that the Veronese embedding \( v_3 \) and the composition \( v_2 \circ \sigma \) are embeddings corresponding to the very ample line bundles \( [3H] \) and \( [2F_1 + 2F_2] \), respectively. Thus, according
to Remark 2.2, the corollary holds if \( S \) is either \( \mathbb{P}^2 \) or \( \mathbb{P}^1 \times \mathbb{P}^1 \). Suppose that \( S \) is obtained by blowing up \( \mathbb{P}^2 \) \( l \) times. We can regard the Lefschetz pencil \( f \) as a projective line in the complete linear system \( \mathbb{P}(H^0(S;\langle F \rangle)) \).

Let \( E = \sum_{i=1}^l E_i \) be the exceptional divisor, \( \pi : S \to \mathbb{P}^2 \) be the blow-down mapping and \( s \in H^0(S;\langle E \rangle) \) a nontrivial section. We can then define the following linear mapping:

\[
\xi : H^0(S;\langle F \rangle) \to H^0(S;\langle F \rangle \otimes \langle E \rangle) = H^0(S;\pi^*\langle 3H \rangle), \quad \xi(\tau) = \tau \otimes s.
\]

Since the blow-down mapping \( \pi \) is birational, we can identify \( H^0(S;\langle F \rangle \otimes \langle E \rangle) \) with \( H^0(S;\pi^*\langle 3H \rangle) \) via \( \pi \). Under this identification, the image \( \xi(f) \) is a genus-1 Lefschetz pencil defined on \( \mathbb{P}^2 \), which is smoothly isomorphic to \( f_n \) by Remark 2.2, and \( f \) can be obtained by blowing-up \( \xi(f) \). \( \square \)

4. Vanishing cycles of genus-1 Lefschetz pencils

As we have shown, any genus-1 Lefschetz pencil can be obtained by blowing-up either of the pencils \( f_n \) or \( f_s \) in Theroem 1.1. In this section we will determine vanishing cycles of these pencils relying on the theory of braid monodromies due to Moishezon and Téicher. Throughout this section, we denote the projective varieties \( v_3(\mathbb{P}^2) \) and \( v_2 \circ \sigma(\mathbb{P}^1 \times \mathbb{P}^1) \) by \( U_n \) and \( U_s \), respectively.

4.1. Braid monodromy techniques

In this subsection, we will give a brief review on the theory of braid monodromies. We will first explain how the theory is related with vanishing cycles of Lefschetz pencils appearing as generic pencils of very ample line bundles, and then recall several facts we need to obtain monodromies of \( f_n \) and \( f_s \). The reader can refer to [20-23] for more details on this subject.

Let \( V \subset \mathbb{P}^n \) be a nonsingular projective surface. Restricting a generic projection \( \mathbb{P}^n \dashrightarrow \mathbb{P}^2 \), we obtain a regular mapping \( \pi : V \to \mathbb{P}^2 \) whose critical value set \( C \) is a curve with nodes and cusps. We further take a generic projection \( \pi' : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \) with base point \( p_0 \in \mathbb{P}^2 \) so that the composition \( f := \pi' \circ \pi : V \dashrightarrow \mathbb{P}^1 \) is a Lefschetz pencil. The critical point set of \( f \) is equal to the set of critical points of \( \pi \) whose image by \( \pi \) is a branch point of the restriction \( \pi'|_C \). We can obtain the monodromy (or equivalently, vanishing cycles) of \( f \) from the braid monodromy of \( C \) (around branch points of \( \pi'|_C \)) explained below.

Let \( Q := \{q_1, \ldots, q_m\} \subset \mathbb{P}^1 \) be the set of images (by \( \pi' \)) of branch points of \( \pi'|_C \). Take a reference point \( q_0 \in \mathbb{P}^1 \setminus Q \). The closure of the preimage \( \pi'^{-1}(q_0) \) (which is equal to \( \pi^{-1}(q_0) \cup \{p_0\} \)) is a line in \( \mathbb{P}^2 \) intersecting \( C \) at \( d := \deg C \) points. We take \( d + 1 \) points \( v_0, v_1, \ldots, v_d \in \mathbb{P}^2 \) and fix an identification of the triple \( (\pi'^{-1}(q_0), \pi'^{-1}(q_0) \cap C, \{p_0\}) \) with \( (\mathbb{P}^2, \{v_1, \ldots, v_d\}, \{v_0\}) \). Note that we can also identify the restriction \( \pi|_{f^{-1}(q_0)} : f^{-1}(q_0) \to \pi'^{-1}(q_0) \) with a simple branched covering \( \theta : \Sigma \to \mathbb{P}^2 \) branched at \( v_1, \ldots, v_d \) (where a simple branched covering is a branched covering such that all the branched points have degree 2). We next take a Hurwitz path system \( (\alpha_1, \ldots, \alpha_m) \) of \( f \) with the base point \( q_0 \), and the corresponding loops \( \gamma_i \) for \( i = 1, \ldots, m \) as we took in Section 2.2. Taking the isotopy class of a parallel transport along \( \gamma_i \) preserving \( C \), we obtain a sequence of elements \( \tau_1, \ldots, \tau_m \) of the braid group \( B_d \) defined as follows:

\[
B_d := \pi_0(\text{Diff}(\mathbb{P}^2, \{v_1, \ldots, v_d\}, v_0)),
\]

where we denote by \( \text{Diff}(\mathbb{P}^2, \{v_1, \ldots, v_d\}, v_0) \) the group of orientation-preserving self-diffeomorphisms of \( \mathbb{P}^2 \) preserving \( v_0 \) and the set \( \{v_1, \ldots, v_d\} \). It is easy to see that each element \( \tau_i \) is a half twist along some
path $\beta_i \subset S^2$ between two points in $\{v_1, \ldots, v_d\}$. The path $\beta_i$ is called a Lefschetz vanishing cycle of the corresponding branched point of $\pi'_C$. The preimage $\theta^{-1}(\beta_i)$ has the unique circle component $c_i \subset \Sigma$, and this circle is a vanishing cycle of a Lefschetz singularity of $f$ in $f^{-1}(q_i)$ with respect to the path $\alpha_i$.

Remark 4.1 In the series of papers of Moishezon–Teicher, a Lefschetz vanishing cycle and a braid monodromy are defined not only for branched points of the restriction of a projection on the critical value set, but also for multiple points and cusps of a general curve in $\mathbb{P}^2$. The reader can refer to [21], for example, for details of this subject. Note that we will deal with braid monodromies of multiple points (which is a Dehn twist along some simple closed curve in a punctured sphere) in order to determine vanishing cycles of $f_n$ and $f_s$.

Remark 4.2 Although the product $t_{c_m} \cdots t_{c_1}$ in $\text{MCG}(f^{-1}(q_0))$ is equal to the unit, the product $\tau_m \cdots \tau_1$ is not equal to the unit since we do not consider braid monodromies of nodes and cusps of the critical value set $C$.

In summary, we can get vanishing cycles of the Lefschetz pencils $f_n$ and $f_s$ in Theorem 1.1 once we obtain Lefschetz vanishing cycles of the branch points of the critical value sets of generic projections from $U_n$ and $U_s$ to $\mathbb{P}^2$ (and the monodromies of simple branched coverings defined over a line in $\mathbb{P}^2$, which can be obtained easily in our situations). Moishezon and Teicher [23] obtained the Lefschetz vanishing cycles for $U_n$ by giving a projective degeneration of $U_n$ to a union of planes, and then analyzing how Lefschetz vanishing cycles are changed in the regeneration (the opposite deformation of the degeneration). As we will observe below, the Lefschetz vanishing cycles for $U_s$ can also be obtained in the same way. In what follows, we will review the definition of a projective degeneration and those for $U_n$ and $U_s$ given in [22] and [19], respectively.

An algebraic set $U_0 \subset \mathbb{P}^{n_0}$ is said to be equivalent to another algebraic set $U_1 \subset \mathbb{P}^{n_1}$ if there exist an algebraic set $W \subset \mathbb{P}^N$ and projections $\pi_0: \mathbb{P}^N \to \mathbb{P}^{n_0}$ and $\pi_1: \mathbb{P}^N \to \mathbb{P}^{n_1}$ such that the restriction $\pi_i|_W: W \to U_i$ is an isomorphism for $i = 0, 1$. An algebraic set $U' \subset \mathbb{P}^n$ is a projective degeneration of $U \subset \mathbb{P}^n$ if there exists an algebraic set $W \subset \mathbb{P}^N \times \mathbb{C}$ such that $W \cap (\mathbb{P}^N \times \{0\})$ is equivalent to $U'$ and $W \cap (\mathbb{P}^N \times \{\varepsilon\})$ is equivalent to $U$ for any $\varepsilon$ with sufficiently small $|\varepsilon| > 0$. In this paper, we mean by $U \leadsto U'$ that $U'$ is the result of a projective degenerations from $U$. Following the notations in [22], we will describe components of algebraic sets as follows:

- A surface equivalent to the image of the Veronese embedding of degree $d$ on $\mathbb{P}^2$ is denoted by $V_d$, and described by a triangle in figures.
- A surface equivalent to the image of the embedding $\varphi_{[E_{\infty}+tC]}$ on $S_1$ $(l > 1)$ is denoted by $T_l$, and described by a trapezoid in figures.
- A surface equivalent to the image of the embedding $\varphi_{[aF_1+bF_2]}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ $(a, b > 0)$ is denoted by $U_{a,b}$, and described by a square in figures.

Theorem 4.3 ([22]. A projective degeneration of $U_n$.) There exists a sequence of projective degenerations $U_n = Y^{(0)} \leadsto Y^{(1)} \leadsto \cdots \leadsto Y^{(5)}$ from $U_n$ to a union of 9 planes $Y^{(5)}$. The intermediate algebraic sets are described in Figure 2.
Theorem 4.4 ([19]. A projective degeneration of $U_s$.) There exists a sequence of projective degenerations $U_s =: Z^{(0)} \rightsquigarrow Z^{(1)} \rightsquigarrow Z^{(2)} \rightsquigarrow Z^{(3)}$ from $U_s$ to a union of $8$ planes $Z^{(3)}$. The intermediate algebraic sets are described in Figure 3.

Remark 4.5 In each of the intermediate algebraic sets in Figure 2 and 3, any two components adjacent to each other intersect on a rational curve, and the configuration of these curves are the same as that of the segments between two regions in the figures. For example, in the algebraic set $Y^{(5)}$, there are $9$ lines appearing as intersections of two adjacent components, and $7$ multiple points in the line arrangement (see Figure 4).

According to the observation in Remark 4.5, the sets of singular points of the algebraic sets $Y^{(5)}$ and $Z^{(3)}$ are unions of lines, and so are the images of them by projections to $\mathbb{P}^2$. The braid monodromies of these line arrangements in $\mathbb{P}^2$ are completely determined in [20, Theorem IX.2.1]. We will next review the relation between these braid monodromies and those for the original varieties $U_n$ and $U_s$ discussed in [21, 23].
In the sequences of regenerations given in Theorems 4.3 and 4.4, the line arrangement in \( Y^{(5)} \) (resp. \( Z^{(3)} \)) is also regenerated to the critical value set of the restriction of a generic projection on \( U_n \) (resp. \( U_s \)). In this regeneration process, each line in the arrangement is “doubled” in the following sense: if some small disk \( D \) intersects a line in the arrangement at the center of \( D \) transversely, this disk intersects the critical value set of the restriction of a generic projection at two points (note that, without loss of generality, we can assume that the critical value set is sufficiently close to the line arrangement. See \([19, \S.1]\)). Furthermore, taking account of the plane arrangement and its regeneration, we can observe that each of the multiple points of the line arrangement has either of the following two properties:

- Three planes \( P_1, P_2, P_3 \) go through this point. Among the three planes, \( P_i \) and \( P_{i+1} \) (\( i = 1, 2 \)) intersect on a line, while \( P_1 \) and \( P_3 \) intersect only at the point, in particular two lines \( P_1 \cap P_2 \) and \( P_2 \cap P_3 \) go through the point. In the regeneration process the line \( P_1 \cap P_2 \) regenerates before the regeneration of \( P_2 \cap P_3 \).

- Six planes \( P_1, \ldots, P_6 \) go through this point. Among the six planes, \( P_i \) and \( P_j \) intersect on a line if \( |j - k| = 1 \) modulo 6, or intersect only at the point otherwise. Among six lines \( P_1 \cap P_2, \ldots, P_6 \cap P_1, P_1 \cap P_2 \) and \( P_4 \cap P_5 \) regenerate first at the same time, \( P_2 \cap P_3 \) and \( P_5 \cap P_6 \) then regenerate at the same time, and lastly \( P_3 \cap P_4 \) and \( P_6 \cap P_1 \) regenerate at the same time.

In [19], the former multiple point is called a 2-point, while the latter one is called a type M 6-point. Following the rules below, we can determine the braid monodromies of branch points appearing around these points after the regeneration:

**Theorem 4.6** ([23, Lemma 1]) One branch point appears around a 2-point after the regeneration. Suppose that the Lefschetz vanishing cycle of the 2-point is a path \( \beta \) shown in Figure 5a, where \( v_i \) is the intersection of the reference fiber and the line \( P_i \cap P_{i+1} \) (\( i = 1, 2 \)). Then the Lefschetz vanishing cycle of the branch point appearing after the regeneration is the path \( \beta' \) shown in Figure 5b.

![Figure 5](image-url)
Theorem 4.7 ([23, Lemmas 5, 6, 7 and 8]) Six branch points appear around a type M 6-point after the regeneration. Suppose that the Lefschetz vanishing cycle of the type M 6-point is a system of paths shown in Figure 6a, where \( v_i \) is the intersection of the reference fiber and the line \( P_i \cap P_{i+1} \) (taking indeces modulo 6). Then there exists a system of reference paths \( \alpha_1, \ldots, \alpha_6 \) for the six branch points, which appear in this order when we go around the reference point counterclockwise, such that the Lefschetz vanishing cycle associated with \( \alpha_i \) is the path \( \beta_i \), where \( \beta_1 \) and \( \beta_6 \) are shown in Figures 6b and 6c, while \( \beta_2, \beta_3, \beta_4, \beta_5 \) are defined as:

\[
\begin{align*}
\beta_2 &= \beta, \quad \beta_3 &= \tau_{\gamma_3}^{-1} \tau_{\gamma_4}^{-1}(\beta), \quad \beta_4 &= \tau_{\gamma_1}^{-1} \tau_{\gamma_2}^{-1}(\beta), \quad \beta_5 &= \tau_{\gamma_1}^{-1} \tau_{\gamma_2}^{-1} \tau_{\gamma_3}^{-1} \tau_{\gamma_4}^{-1}(\beta).
\end{align*}
\]

(Here we denote the positive half twist along \( \gamma_i \) by \( \tau_{\gamma_i} \).

Remark 4.8 In general, a generic projection on a regenerated surface to \( \mathbb{P}^2 \) might have branch points which do not appear around multiple points of the original line arrangement. Such branch points are called extra branch points in [19, 27]. According to Proposition 3.3.4 in [27], there are no extra branch points in \( U_n \), while there are two extra branch points in \( U_s \) (cf. [19, Proposition 5.2.4]). We will explain how to determine the braid monodromies of these branch points in Section 4.3.

4.2. Vanishing cycles of a pencil of degree-3 curves in \( \mathbb{P}^2 \)

We will first calculate vanishing cycles of the Lefschetz pencil \( f_n : U_n \to \mathbb{P}^1 \) given in Theroem 1.1. As shown in Theorem 4.3, we can take a sequence of projective degenerations from \( U_n \) to a union of 9 planes \( Y^{(5)} \). Let \( C_n \) be the union of all the lines in \( Y^{(5)} \) appearing as intersections of two planes in \( Y^{(5)} \). We denote the planes in \( Y^{(5)} \), the lines in \( C_n \) and the multiple points in \( C_n \) by \( \{ P_i \}_{i=1}^9 \), \( \{ l_j \}_{j=1}^9 \) and \( \{ a_k \}_{k=1}^7 \), respectively, as shown in Figure 4. Note that all the multiple points in \( C_n \) are 2-points except for the unique type M 6-point \( a_4 \).
We can assume that $Y^{(5)}$ and $Y^{(0)} = U_n$ are both contained in $\mathbb{P}^N$ and these are sufficiently close (cf. [19, §.1]). Take generic projections $\pi: \mathbb{P}^N \rightarrow \mathbb{P}^2$ and $\pi': \mathbb{P}^2 \rightarrow \mathbb{P}^1$. Let $\tilde{\pi}: U_n \rightarrow \mathbb{P}^2$ be the restriction of $\pi$ on $U_n$ and $a_i' = \pi' \circ \pi(a_i)$. As observed in [23], we can regard $C_n$ as a subarrangement of a line arrangement dual to generic introduced in [20, Section IX]. By [20, Theorem IX.2.1], we can take a point $a_0' \in \mathbb{P}^1$ away from $a_1', \ldots, a_7'$ and a simple path $\alpha_i'$ from $a_0'$ to $a_i'$ so that $\alpha_i'$'s are mutually disjoint except at the common initial point $a_0'$, the paths $\alpha_1', \ldots, \alpha_7'$ appear in this order when we go around $a_0'$ counterclockwise, and the Lefschetz vanishing cycles associated with the paths $\alpha_1', \ldots, \alpha_7'$ are as shown in Figure 7, where the points labeled with $i$ is the intersection between the fiber $\pi'^{-1}(a_0') \subset \mathbb{P}^2$ and the line $(l_i)$ ($i = 1, \ldots, 9$). We next apply Theorems 4.6 and 4.7 in order to obtain the braid monodromies of the branch points of the restriction $\tilde{\pi}|_{\widetilde{C_n}}: \widetilde{C_n} \rightarrow \mathbb{P}^1$, where $\widetilde{C_n}$ is the critical value set of $\tilde{\pi}: U_n \rightarrow \mathbb{P}^2$. We eventually obtain a Hurwitz path system $(\alpha_1, \ldots, \alpha_{12})$ of $f_n$ ($= \pi' \circ \tilde{\pi}$) such that the Lefschetz vanishing cycles of the branch points associated with $\alpha_1, \ldots, \alpha_4, \alpha_9, \ldots, \alpha_{12}$ are as shown in Figure 8, while those associated with $\alpha_5, \ldots, \alpha_8$ are respectively equal to $\beta, \tau_{\gamma_3}^{-1}\tau_{\gamma_4}^{-1}(\beta), \tau_{\gamma_1}\tau_{\gamma_2}^{-1}(\beta), \tau_{\gamma_1}\tau_{\gamma_2}\tau_{\gamma_3}\tau_{\gamma_4}^{-1}(\beta)$, where the paths $\beta, \gamma_1, \ldots, \gamma_4$ are given in Figure 9 and $\tau_{\gamma_i}$ is the half twist along $\gamma_i$.

Figure 7. The Lefschetz vanishing cycles (LVC) of a line arrangement dual to points in general position.

In order to obtain vanishing cycles of $f_n$, we have to take the circle components of the preimages of the Lefschetz vanishing cycles under the branched covering

$$\tilde{\pi}|_{\pi'^{-1}(a_0')} : f_n^{-1}(a_0') \rightarrow \pi'^{-1}(a_0')$$

(4.1)

1093
(a) The Lefschetz vanishing cycles associated with $\alpha_1$ and $\alpha_2$.

(b) The Lefschetz vanishing cycle associated with $\alpha_3$.

(c) The Lefschetz vanishing cycle associated with $\alpha_4$.

(d) The Lefschetz vanishing cycle associated with $\alpha_9$.

(e) The Lefschetz vanishing cycles associated with $\alpha_{10}$ and $\alpha_{11}$.

(f) The Lefschetz vanishing cycle associated with $\alpha_{12}$.

Figure 8. The Lefschetz vanishing cycles of branch points of the restriction $\pi^{'|_{G_n}}$ : $\widetilde{C_n}$ → $\mathbb{P}^1$.

(a) The path $\beta$.

(b) The paths $\gamma_1, \gamma_2, \gamma_3, \gamma_4$.

Figure 9. Paths in $\pi^{'|_{G_n}}(a_0')$. 

1094
branched at $\pi^{-1}(a'_0) \cap C_n$. We denote the closure $\overline{\pi^{-1}(a'_0)}$ by $S$, the intersection $\overline{\pi^{-1}(a'_0)} \cap \widetilde{C}_n$ by $Q$. We take a point $q_0 \in S \setminus Q$ and regard an element $\sigma$ in the symmetry group $\mathfrak{S}_9$ as a self-bijections of $\pi^{-1}(q_0)$ sending the point in $\pi^{-1}(q_0)$ close to $P_i$ to that close to $P_{\sigma(i)}$ for each $i = 1, \ldots, 9$ (note that we assumed that $U_n$ is sufficiently close to $Y^{(5)}$). Let $\varphi : \pi_1(S \setminus Q, q_0) \to \mathfrak{S}_9$ be the monodromy representation of the branched covering (4.1). As shown in Figure 10, we take a system of oriented paths $\eta_1, \eta_1', \ldots, \eta_9, \eta_9'$ such that the common initial point of them is $q_0$ and the end point of $\eta_i$ (resp. $\eta_i'$) is the points labeled with $i$ (resp. $i'$).

Figure 10. The paths $\eta_1, \eta_1', \ldots, \eta_9, \eta_9'$ in $S$. In this figure, these paths appear in this order when we go around $q_0$ clockwise.

Let $\tilde{\eta}_i$ be a based loop in $S \setminus Q$ with the base point $q_0$ which can be obtained by connecting $q_0$ with a small clockwise circle around the point label with $i$ using $\eta_i$. We also take a based loop $\tilde{\eta}_i'$ in a similar manner.

The images $\varphi([\tilde{\eta}_i])$ and $\varphi([\tilde{\eta}_i'])$ are easily calculated as follows:

$$\varphi([\tilde{\eta}_1]) = \varphi([\tilde{\eta}_1']) = (12), \quad \varphi([\tilde{\eta}_2]) = \varphi([\tilde{\eta}_2']) = (23), \quad \varphi([\tilde{\eta}_3]) = \varphi([\tilde{\eta}_3']) = (24),$$

$$\varphi([\tilde{\eta}_4]) = \varphi([\tilde{\eta}_4']) = (35), \quad \varphi([\tilde{\eta}_5]) = \varphi([\tilde{\eta}_5']) = (47), \quad \varphi([\tilde{\eta}_6]) = \varphi([\tilde{\eta}_6']) = (56),$$

$$\varphi([\tilde{\eta}_7]) = \varphi([\tilde{\eta}_7']) = (58), \quad \varphi([\tilde{\eta}_8]) = \varphi([\tilde{\eta}_8']) = (78), \quad \varphi([\tilde{\eta}_9]) = \varphi([\tilde{\eta}_9']) = (79).$$

Note that all of these images are transpositions. We can thus describe the branched covering (4.1) as shown in Figure 11. In this figure, the red circles in the upper surface $\overline{f_n^{-1}(a'_0)}$ are the circle components of the preimages of the red paths between branch points in the lower sphere $\overline{\pi^{-1}(a'_0)}$. The point represented by $\times$ in the lower sphere is the base point of $\pi'$, while those in the upper surface are the preimages of it. As described in Figure 12, the complement of small disk neighborhoods of the base points of $f_n$ in the closure $\overline{f_n^{-1}(a'_0)}$ is a nine-holed torus. One can shrink the subsurfaces of the surface in Figure 12a labeled with 1, 6, and 9 to obtain that in Figure 12b. Those in Figures 12b and 12c are homeomorphic to each other. Taking the preimages of the paths described in Figure 8 and 9 under the branched covering described in Figure 11, we can eventually obtain a monodromy factorization $t_{c_{12}} \circ \cdots \circ t_{c_1} = t_{d_1} \circ \cdots \circ t_{d_9}$ of $f_n$, where the simple closed curves $c_1, \ldots, c_9, c_9, \ldots, c_{12}$ are given in Figure 13, while $c_6, c_7, c_8$ are respectively equal to $t_{d_1}^{-1}t_{d_1}^{-1}(c_5), t_{d_4}^{-1}t_{d_2}^{-1}t_4(c_5), t_{d_4}^{-1}t_4^{-1}t_2^{-1}t_4^{-1}(c_5)$, where the simple closed curves $d_1, d_2, d_3$ and $d_4$ are given in Figure 13d.

4.3. Vanishing cycles of a pencil of curves with bi-degree-$(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$

We will next calculate vanishing cycles of the Lefschetz pencil $f_s : U_s \to \mathbb{P}^1$. Again, let $C_s$ be the union of all the lines in $Z^{(3)}$ appearing as intersections of two plane components, and denote the planes in $Z^{(3)}$, the lines in $C_s$ and the multiple points in $C_s$ by $\{P_{i}\}_{i=1}^8$, $\{l_j\}_{j=1}^8$ and $\{a_k\}_{k=2}^6$, respectively, as shown in Figure 14. We further take points $a_1$ and $a_7$ on $l_8$ and $l_1$, respectively. Suppose that $Z^{(3)}$ and $U_s = Z^{(0)}$ are both
Figure 11. The branched covering given in Equation 4.1.

Figure 12. The complement of neighborhoods of the base points in $f^{-1}(\alpha'_0)$.

contained in $\mathbb{P}^N$ and these are sufficiently close. Moreover, without loss of generality, we can assume that the line arrangement $C_s$ is the same as that given in [20, Theorem IX.2.1] and the order of the lines in $C_s$ (given by indices) is the same as that in [20, Theorem IX.2.1] (meaning that the order of the vertices in $C_s$ is opposite to that in [20, Theorem IX.2.1]). As in the previous subsection, let $\pi : \mathbb{P}^N \rightarrow \mathbb{P}^2$ and $\pi' : \mathbb{P}^2 \rightarrow \mathbb{P}^1$ be generic projections, $\tilde{\pi} : U_s \rightarrow \mathbb{P}^2$ be the restriction of $\pi$, $\widetilde{C_s}$ be the critical value set of $\tilde{\pi}$ and $a'_i = \pi' \circ \pi(a_i)$. Applying [20, Theorem IX.2.1], we take a reference point $a'_0 \in \mathbb{P}^1$ and reference paths $\alpha'_i$ from $a'_0$ to $a'_i$ ($i = 2, \ldots, 6$) so that the the corresponding Lefschetz vanishing cycles are as shown in Figure 15.

As observed in Remark 4.8, there are branch points of $\tilde{\pi}|_{\widetilde{C_s}} : \widetilde{C_s} \rightarrow \mathbb{P}^1$ which are not close to multiple points of $C_s$. We take the regeneration from $Z^{(3)}$ to $Z^{(2)}$ so that the planes $P_4$ and $P_7$ (resp. $P_2$ and $P_5$) are
Figure 13. Vanishing cycles of the Lefschetz pencil $f_n$. 
Figure 14. Planes, lines and multiple points in $\mathbb{Z}^{(3)}$.

(a) The LVCs associated with $\alpha_2, \alpha_3, \alpha_5$ and $\alpha_6$.

(b) The LVC associated with $\alpha_4$.

Figure 15. The Lefschetz vanishing cycles (LVC) of a line arrangement dual to points in general position.

regenerated to $U_{1,1}$ going through the points $a_1, a_2, a_3, a_4$ (resp. $a_4, a_5, a_6, a_7$). (See [22, §3.5] for the detail of this regeneration.) Analyzing the model of such a regeneration given in the proof of [22, Proposition 14], we can verify that the two extra branch points of $\tilde{\pi}|_{\tilde{C}_s}$ appear around $a_1$ and $a_7$. We can further show that, for suitable reference paths $\alpha_1$ and $\alpha_{12}$ from $a_0'$ to the images of the branch points near $a_1$ and $a_7$, respectively, the Lefschetz vanishing cycles of the two extra branch points associated with $\alpha_1$ and $\alpha_{12}$ are as shown in Figure 16 (cf. [27, §3.3]). By applying Theorems 4.6 and 4.7, we can take reference paths $\alpha_i$ ($i = 2, \ldots, 11$) so that $(\alpha_1, \ldots, \alpha_{12})$ is a Hurwitz path system of $f_s$, and the Lefschetz vanishing cycles of branch points of $\tilde{\pi}|_{\tilde{C}_s}$ associated with $\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_8, \alpha_{11}$ are as shown in Figure 17, while those associated with $\alpha_5, \ldots, \alpha_8$ are respectively equal to $\beta, \tau_{\gamma_1}^{-1}\tau_{\gamma_4}^{-1}(\beta), \tau_{\gamma_1}^{-1}\tau_{\gamma_2}^{-1}(\beta), \tau_{\gamma_1}^{-1}\tau_{\gamma_2}^{-1}\tau_{\gamma_3}^{-1}\tau_{\gamma_4}^{-1}(\beta)$, where the paths $\beta, \gamma_1, \ldots, \gamma_4$ are given in Figures 17f and 17g. As in the previous subsection, we next consider the following branched covering:

Figure 16. The Lefschetz vanishing cycles associated with $\alpha_1$ and $\alpha_{12}$.

$$\tilde{\pi}|_{f_s^{-1}(a_0')} : f_s^{-1}(a_0') \to \pi'^{-1}(a_0').$$

Figure 17. The LVCs associated with $\alpha_1, \alpha_{12}$. By calculating the monodromy representation of this covering, we can describe this branched covering as shown in Figures 18 and 19. Taking the preimages of the paths described in Figure 17 under the branched covering described in Figure 18, we can obtain a monodromy factorization $t_{c_{12}} \cdots t_{c_1} = t_{d_8} \cdots t_{d_8}$ of $f_s$, where
the simple closed curves \(c_1, \ldots, c_5, c_9, \ldots, c_{12}\) are given in Figure 20, while \(c_6, c_7, c_8\) are respectively equal to 
\[t_{d_3}^{-1}t_{d_4}^{-1}(c_5), t_{d_1}^{-1}t_{d_2}^{-1}(c_5), t_{d_1}^{-1}t_{d_2}^{-1}t_{d_3}^{-1}t_{d_4}^{-1}(c_5),\]
where the simple closed curves \(d_1, d_2, d_3,\) and \(d_4\) are given in Figure 20e.

\[\begin{array}{cccccccccc}
1 & 1' & 2 & 2' & 3 & 3' & 4 & 4' & 5 & 5' & 6 & 6' & 7 & 7' & 8 & 8'
\end{array}\]

(a) The Lefschetz vanishing cycles associated with \(a_2\) and \(a_{11}\).

\[\begin{array}{cccccccccc}
1 & 1' & 2 & 2' & 3 & 3' & 4 & 4' & 5 & 5' & 6 & 6' & 7 & 7' & 8 & 8'
\end{array}\]

(b) The Lefschetz vanishing cycle associated with \(a_3\).

\[\begin{array}{cccccccccc}
1 & 1' & 2 & 2' & 3 & 3' & 4 & 4' & 5 & 5' & 6 & 6' & 7 & 7' & 8 & 8'
\end{array}\]

(c) The Lefschetz vanishing cycle associated with \(a_4\).

\[\begin{array}{cccccccccc}
1 & 1' & 2 & 2' & 3 & 3' & 4 & 4' & 5 & 5' & 6 & 6' & 7 & 7' & 8 & 8'
\end{array}\]

(d) The Lefschetz vanishing cycle associated with \(a_9\).

\[\begin{array}{cccccccccc}
1 & 1' & 2 & 2' & 3 & 3' & 4 & 4' & 5 & 5' & 6 & 6' & 7 & 7' & 8 & 8'
\end{array}\]

(e) The Lefschetz vanishing cycle associated with \(a_{10}\).

\[\begin{array}{cccccccccc}
1 & 1' & 2 & 2' & 3 & 3' & 4 & 4' & 5 & 5' & 6 & 6' & 7 & 7' & 8 & 8'
\end{array}\]

(f) The path \(\beta\).

\[\begin{array}{cccccccccc}
1 & 1' & 2 & 2' & 3 & 3' & 4 & 4' & 5 & 5' & 6 & 6' & 7 & 7' & 8 & 8'
\end{array}\]

(g) The paths \(\gamma_1, \gamma_2, \gamma_3, \gamma_4\).

**Figure 17.** The Lefschetz vanishing cycles of branch points of the restriction \(\pi'|_{\tilde{\mathcal{C}}_t} : \tilde{\mathcal{C}}_t \to \mathbb{P}^1\).
5. Combinatorial structures of genus-1 pencils

In this section we study the combinatorial structures of the monodromy factorizations associated with the genus-1 holomorphic Lefschetz pencils. We will simplify those factorizations and show that they are Hurwitz equivalent to the known $k$-holed torus relations, which were combinatorially constructed by Korkmaz-Ozbagci [14] and Tanaka [31]. In particular, we will see that a genus-1 holomorphic Lefschetz pencil obtained by blowing-up another holomorphic pencil is uniquely determined by the number of the blown-up points and independent of particular choices of such points. Thus, we complete the classification of genus-1 holomorphic Lefschetz pencils in the smooth category.
Figure 20. Vanishing cycles of the Lefschetz pencil $f_s$. 

(a) The vanishing cycles $c_1$ and $c_{12}$. 
(b) The vanishing cycles $c_2$ and $c_{11}$. 
(c) The vanishing cycle $c_3$. 
(d) The vanishing cycle $c_4$. 
(e) The vanishing cycle $c_5$. 
(f) The vanishing cycle $c_6$. 
(g) The vanishing cycle $c_{10}$. 

Vanishing cycles of the Lefschetz pencil $f_s$. 

Figure 20.
In the remainder of the paper, we simplify the notations regarding Dehn twists as follows. We will denote the right-handed Dehn twist along a curve $\alpha$ also by $\alpha$, and its inverse, i.e. the left-handed Dehn twist along $\alpha$, by $\bar{\alpha}$. We continue to use the functional notation for multiplication; $\beta\alpha$ means we first apply $\alpha$ and then $\beta$. In addition, we denote the conjugation $\alpha\beta\bar{\alpha}$ by $\alpha(\beta)$, which is the Dehn twist along the curve $t_\alpha(\beta)$. Finally, we use the symbol $\partial_k$ to denote the boundary multitwist $\delta_1\delta_2\cdots\delta_k$.

5.1. Monodromies of the minimal pencils

We first deal with the minimal holomorphic Lefschetz pencils $f_n$ and $f_s$ as they are the base cases in the sense that the other holomorphic pencils are obtained by blowing-up those two pencils.

5.1.1. Monodromy of $f_n$

In Section 4.2 we obtained a monodromy factorization of $f_n$,

$$c_{12}c_{11}c_9c_8c_7c_6c_5c_3c_2c_1 = \delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8\delta_9$$

with the vanishing cycles computed in Figure 13 where $c_6 = \bar{d}_3\bar{d}_4(c_5)$, $c_7 = \bar{d}_1\bar{d}_2(c_5)$, and $c_8 = \bar{d}_1\bar{d}_2\bar{d}_3\bar{d}_4(c_5)$. The curves are redrawn on a standardly positioned torus in Figure 21. We further reposition the surface by pushing the boundary components as indicated in Figure 22; we first swap $\delta_1$ and $\delta_3$, also $\delta_7$ and $\delta_9$, then push the boundary components except for $\delta_2$ and $\delta_3$ along the meridian in the indicated directions. Accordingly, the vanishing cycles are now configured as in Figure 23. We further modify the factorization by Hurwitz moves.
Figure 22. Pushing of boundary components.

\[ \delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_7 \delta_8 \delta_9 = c_{12}c_{11}c_{10}c_9c_8c_7c_6c_5c_4c_3c_2c_1 \]
\[ \sim c_{11}c_9c_8c_7c_6c_5c_4c_3c_2c_1c_{10} \]
\[ \sim c_{11}c_9c_8c_7c_6c_5c_4c_3c_2c_10 \]
\[ \sim c_{11}c_9c_8c_7c_6c_5c_4c_3c_2c_10 \]
\[ \sim c_{11}c_9c_8c_7c_6c_5c_4c_3c_2c_10 \]
\[ \sim c_{11}c_9c_8c_7c_6c_5c_4c_3c_2c_10 \]
\[ \sim c_{11}c_9c_8c_7c_6c_5c_4c_3c_2c_10 \]
\[ \sim c_{11}c_9c_8c_7c_6c_5c_4c_3c_2c_10 \]
\[ \sim c_{11}c_9c_8c_7c_6c_5c_4c_3c_2c_10 \]

Figure 23. Vanishing cycles after repositioning the surface \( \Sigma^9_1 \).

\[ \delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_7 \delta_8 \delta_9 = c_{12}c_{11}c_{10}c_9c_8c_7c_6c_5c_4c_3c_2c_1 \]
\[ \sim c_{11}c_9c_8c_7c_6c_5c_4c_3c_2c_1c_{10} \]
\[ \sim c_{11}c_9c_8c_7c_6c_5c_4c_3c_2c_10 \]
\[ \sim c_{11}c_9c_8c_7c_6c_5c_4c_3c_2c_10 \]
\[ \sim c_{11}c_9c_8c_7c_6c_5c_4c_3c_2c_10 \]
\[ \sim c_{11}c_9c_8c_7c_6c_5c_4c_3c_2c_10 \]
\[ \sim c_{11}c_9c_8c_7c_6c_5c_4c_3c_2c_10 \]
\[ \sim c_{11}c_9c_8c_7c_6c_5c_4c_3c_2c_10 \]
\[ \sim c_{11}c_9c_8c_7c_6c_5c_4c_3c_2c_10 \]
where \( c_8' = \varepsilon_8(c_8), \ c_5' = \varepsilon_5(c_5), \ c_4' = \varepsilon_4(c_4), \ c_6' = \varepsilon_6(c_6), \ c_1' = \varepsilon_1(c_1) \), \( c_9' = \varepsilon_9(c_9) \), \( c_4'' = \varepsilon_4(c_4') \), \( c_5'' = \varepsilon_5(c_5') \). It is routine to observe that the resulting curves are as depicted in Figure 24a and the last expression is \( a_1 b_1 b_2 a_4 b_4 b_5 b_6 a_7 b_7 b_9 \) up to labeling and a permutation. Thus, we obtain the simpler monodromy factorization of \( f_n \):

![Diagram](image_url)

(a) The simplified relation \( \Sigma = \partial \).  
(b) The original relation.

**Figure 24.** The curves for Korkmaz-Ozbagci's 9-holed torus relation.

\[
\alpha_1 \beta_1 \beta_2 \beta_3 \beta_4 \beta_5 \beta_6 \beta_7 \beta_8 \beta_9 = \delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 \delta_8 \delta_9, \tag{5.1}
\]

We refer to this relation as \( \Sigma = \partial \).

We are now ready for proving the following:

**Theorem 5.1** The monodromy factorization \( \Sigma = \partial \) is Hurwitz equivalent to Korkmaz-Ozbagci’s 9-holed torus relation given in [14].

**Proof** With the curves shown in Figure 24b, Korkmaz and Ozbagci [14] gave the 9-holed torus relation

\[
\beta_4 \sigma_3 \sigma_6 \alpha_5 \sigma_1 \sigma_7 \sigma_2 \sigma_7 \sigma_5 \sigma_8 \alpha_8 = \delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 \delta_8 \delta_9, \tag{5.2}
\]

where \( \beta_4 = \alpha_4(\beta) \), \( \beta_1 = \alpha_1(\beta) \) and \( \beta_7 = \alpha_7(\beta) \). We modify this relation as follows:

\[
\delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 \delta_8 \delta_9 = \beta_4 \sigma_3 \sigma_6 \alpha_5 \beta_1 \sigma_4 \sigma_7 \sigma_2 \beta_7 \sigma_5 \sigma_8 \alpha_8 \\
\sim \beta_4(\sigma_3)\beta_1(\sigma_6)\beta_4 \sigma_5 \beta_1 \sigma_4 \beta_7 \sigma_5 \beta_8 \sigma_8 \\
\sim \beta_4(\sigma_3)\beta_1(\sigma_6)\alpha_5 \beta_1(\sigma_4)\beta_7 \sigma_5 \beta_8 \sigma_8 \\
\sim \alpha_5 \beta_4(\sigma_3)\beta_1(\sigma_4)\alpha_5 \beta_1(\sigma_6). 
\]

It is straightforward to see that the last expression coincides with the factorization \( \Sigma_9 \).  \( \square \)
5.1.2. Monodromy of $f_s$

A monodromy factorization of $f_s$ was computed in Section 4.3 as

$$c_{12}c_{11}c_9c_8c_7c_6c_5c_4c_3c_2c_1 = \delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8$$

with the vanishing cycles found in Figure 20 where $c_6 = \tilde{d}_3\tilde{d}_4(c_5)$, $c_7 = \tilde{d}_1\tilde{d}_2(c_5)$, and $c_8 = \tilde{d}_1\tilde{d}_2\tilde{d}_3\tilde{d}_4(c_5)$. The curves are redrawn on a standardly positioned torus in Figure 25. We perform the global conjugation by $\tilde{d}_1\tilde{d}_4\tilde{d}_2$ to put the vanishing cycles as in Figure 26. For simplicity, we keep using the same labeling $c_i$ for the resulting curves. We then transform the factorization as follows.

\begin{align*}
\text{Figure 25. Redrawing of the vanishing cycles of } f_s \text{ on a standard 8-holed torus } \Sigma_1^8. \\
\text{Figure 26. Vanishing cycles after the global conjugation by } \tilde{d}_1\tilde{d}_4\tilde{d}_2. 
\end{align*}
\[\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8 = c_1c_2c_3c_4c_5c_6c_7c_8c_9c_{10}c_{11}c_{12}c_{13}\]

The resulting curves are as depicted in Figure 27a and the last expression is \(a_1b_2a_3b_3b_4a_5b_5b_6b_7b_8\) up to labeling and a permutation. Thus, we obtain the simpler monodromy factorization of \(f_s\):

\[a_1b_1b_2a_3b_3b_4a_5b_5b_6a_7b_7b_8 = \delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8, \quad (5.3)\]

We refer to this relation as \(S_8 = \partial_8\).

**Theorem 5.2** The monodromy factorization \(S_8 = \partial_8\) is Hurwitz equivalent to Tanaka’s 8-holed torus relation given in [31].

**Proof** With the curves shown in Figure 27b, Tanaka [31] gave the 8-holed torus relation

\[\alpha_5\alpha_7\beta_6\beta_2\sigma_2\sigma_1\alpha_1\alpha_3\beta_6\sigma_4\sigma_7 = \delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8, \quad (5.4)\]
where \( \beta_6 = \alpha_6(\beta) \), \( \beta_2 = \alpha_2(\beta) \), \( \beta_2 = \alpha_2(\beta) \), and \( \beta_6 = \alpha_6(\beta) \). We modify this relation as follows:

\[
\delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 \delta_8 = \alpha_5 \alpha_7 \beta_6 \beta_2 \sigma_2 \sigma_1 \alpha_3 \beta_2 \beta_6 \sigma_4 \sigma_7 \\
\sim \alpha_7 \alpha_5 \beta_6 \beta_2 \sigma_2 \sigma_1 \alpha_3 \beta_2 \beta_6 \sigma_4 \sigma_7 \\
\sim \alpha_7 \alpha_5 (\beta_6) \alpha_5 \beta_2 \sigma_2 \sigma_1 \alpha_3 (\beta_2) \alpha_1 \beta_6 \sigma_4 \sigma_7 \\
\sim \alpha_7 \alpha_5 (\beta_6) \alpha_5 \beta_2 \beta_6 (\sigma_2) \beta_2 (\sigma_1) \beta_2 \alpha_3 \beta_2 (\sigma_4) \beta_6 (\sigma_7) \beta_6 \\
\sim \alpha_7 \alpha_5 (\beta_2) \beta_2 (\sigma_2) \beta_2 (\sigma_1) \beta_2 \alpha_3 \beta_2 (\sigma_4) \beta_6 (\sigma_7) \beta_6 \alpha_7 \alpha_5 (\beta_6) \\
\sim \alpha_7 \alpha_5 (\beta_2) \beta_2 (\sigma_2) \beta_2 (\sigma_1) \alpha_3 \beta_2 (\sigma_4) \alpha_1 \beta_6 (\sigma_7) \alpha_7 \alpha_5 (\beta_6). 
\]

The last expression coincides with \( S_8 \).

\( \square \)

5.2. Monodromy and uniqueness of the nonminimal pencils

By blowing-up some of the base points of \( f_n \) or \( f_s \) we obtain a nonminimal holomorphic Lefschetz pencil. In terms of monodromy factorization this corresponds to capping boundary components of \( N_3 = \partial_3 \) or \( S_8 = \partial_8 \) with disks and obtaining a \( k \)-holed torus relation with smaller \( k \). The question is whether the resulting pencil is (smoothly) determined only by the number of blow-ups and independent of a particular set of base points that we blow-up. We prove that the answer is affirmative by providing a “standard” \( k \)-holed torus relation \( N_k = \partial_k \) for each \( k \leq 8 \) and showing the blow-up of any one base point of \( N_k = \partial_k \), or additionally \( S_8 = \partial_8 \) when \( k = 8 \), is Hurwitz equivalent to \( N_{k-1} \).

The next lemma summarizes the techniques that we will repeatedly use in the Hurwitz equivalence computations.

**Lemma 5.3** Consider the curves \( a_i, b_i, b \) in the \( k \)-holed torus \( \Sigma_k^b \) as in Figure 1. Then the following relations between Dehn twists in \( \text{MCG}(\Sigma_k^b) \) are achieved by Hurwitz moves.

1. \( bb_i \sim b_i b \), \( a_i a_j \sim a_j a_i \).
2. \( a_i b a_i \sim b a_i b \).
3. \( b a_i b_i \sim a_i b_i a_{i+1} \sim b_i a_{i+1} b \sim a_{i+1} b a_i. \)

*Here the indices are taken modulo \( k \).*

The verification is easy.

5.2.1. One-time blow-up of \( f_n \) and the 8-holed torus relation \( N_8 = \partial_8 \)

We consider the 9-holed torus relation \( N_9 = \partial_9 \) with the curves in Figure 24a (or Figure 1) and cap one of the boundary components.

**Case 1:** Capping \( \delta_9 \). This yields the relation

\[
a_1 b_1 b_2 b_3 a_4 b_4 b_5 b_6 a_7 b_7 b_8 b = \delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 \delta_8 \quad (5.5)
\]

in \( \text{MCG}(\Sigma_9^b) \) where the curves are now understood to lie in \( \Sigma_7^b \) as in Figure 1 with \( k = 8 \) (see also Figure 28). Notice that the curve \( b_9 \) becomes the central longitude \( b \) as the boundary \( \delta_9 \) disappears. We modify the
The relation as follows.

\[
\delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 \delta_8 = a_1 b_1 b_2 b_3 a_4 b_4 b_5 a_7 b_7 b_8 b \\
\sim b a_1 b_1 b_2 b_3 a_4 b_4 b_5 b_6 a_7 b_7 b_8 \\
\sim a_1 b_1 a_2 b_2 b_3 a_4 b_4 b_5 b_6 a_7 b_7 b_8.
\]

Thus, we have the 8-holed torus relation

\[
a_1 b_1 a_2 b_2 b_3 a_4 b_4 b_5 b_6 a_7 b_7 b_8 = \delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 \delta_8,
\]

(5.6)
to which we refer as \( N_8 = \partial_8 \).

**Case 2:** Capping \( \delta_8 \) or \( \delta_7 \). Instead of \( \delta_9 \), we now cap \( \delta_8 \) or \( \delta_7 \) of \( N_9 = \partial_9 \). Then, after relabeling the curves so that they match the curves in Figure 1 with \( k = 8 \), we have

\[
a_1 b_1 b_2 b_3 a_4 b_4 b_5 b_6 a_7 b_7 b_8 = \delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 \delta_8,
\]

\[
a_1 b_1 b_2 b_3 a_4 b_4 b_5 b_6 a_7 b_7 b_8 = \delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 \delta_8.
\]

Both of them are clearly equivalent to the relation (5.5), and hence to \( N_8 = \partial_8 \), since \( b \) commutes with \( b_7 \) and \( b_8 \).

**Case 3:** The other boundary components \( \delta_1, \delta_2, \ldots, \delta_6 \). We can take advantage of the symmetry that the relation \( N_9 = \partial_9 \) possesses and reduce to the cases we have already discussed. In Figure 24a, consider the clockwise rotation \( r \) by \( 2\pi / 3 \) about the axis perpendicular to the page and through the center of the figure. This diffeomorphism maps \((a_i, b_i, \delta_i)\) to \((a_{i+3}, b_{i+3}, \delta_{i+3})\), where the indices are taken modulo 9. Then, via the rotation \( r \) the relation \( N_9 = \partial_9 \) becomes

\[
\delta_4 \delta_5 \delta_6 \delta_7 \delta_8 \delta_9 \delta_1 \delta_2 \delta_3 = a_4 b_4 b_5 b_6 a_7 b_7 b_8 a_1 b_1 b_2 b_3,
\]

which is just a permutation of \( N_9 = \partial_9 \). Therefore, capping \( \delta_6 \) of \( N_9 = \partial_9 \) is the same as capping \( \delta_9 \) of \( N_9 = \partial_9 \) after applying \( r \); hence, it results in the relation \( N_8 = \partial_8 \). In the same way, capping \( \delta_4 \) or \( \delta_5 \) reduces to capping \( \delta_7 \) or \( \delta_8 \), respectively. If we consider the counterclockwise rotation \( r^{-1} \) we can reduce the cases of \( \delta_1, \delta_2, \) and \( \delta_3 \) to the cases of \( \delta_7, \delta_8, \) or \( \delta_9 \), respectively.
5.2.2. Two-time blow-up of \( f_n \) and the 7-holed torus relation \( N_7 = \partial_7 \)

We take the 8-holed torus relations \( N_8 = \partial_8 \) and \( S_8 = \partial_8 \) and cap each one of the boundary components.

**Case 1:** Capping \( \delta_8 \) or \( \delta_7 \) of \( N_8 = \partial_8 \). They give

\[
\begin{align*}
\delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 &= a_1 b_1 a_2 b_2 b_3 a_4 b_4 b_5 a_7 b_7 b_7, \\
\delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 &= a_1 b_1 a_2 b_2 b_3 a_4 b_4 b_5 a_7 b_7 b_7,
\end{align*}
\]

which are clearly equivalent as \( b \) commutes with \( b_7 \). Then, from the second

\[
\begin{align*}
\delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 &= a_1 b_1 a_2 b_2 b_3 a_4 b_4 b_5 a_7 b_7 b_7, \\
&\sim a_1 b_1 a_2 b_2 b_3 a_4 b_4 b_5 a_7 b_7,
\end{align*}
\]

which is the same as the one in Case 1 right before applying the rotation.

**Case 2:** Capping \( \delta_6, \delta_5, \) or \( \delta_4 \) of \( N_8 = \partial_8 \). They respectively give

\[
\begin{align*}
\delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 &= a_1 b_1 a_2 b_2 b_3 a_4 b_4 b_5 a_6 b_6 b_7, \\
\delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 &= a_1 b_1 a_2 b_2 b_3 a_4 b_4 b_5 a_6 b_6 b_7,
\end{align*}
\]

which are equivalent to each other. From the first,

\[
\begin{align*}
\delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 &= a_1 b_1 a_2 b_2 b_3 a_4 b_4 b_5 a_6 b_6 b_7, \\
&\sim a_1 b_1 a_2 b_2 b_3 a_4 b_4 b_5 a_6 b_6 b_7,
\end{align*}
\]

which is the same as the one in Case 1 right before applying the rotation.

**Case 3:** Capping \( \delta_2 \) or \( \delta_3 \) of \( N_8 = \partial_8 \). They give the equivalent relations

\[
\begin{align*}
\delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 &= a_1 b_1 a_2 b_2 b_3 b_4 b_5 a_6 b_6 b_7, \\
\delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 &= a_1 b_1 a_2 b_2 b_3 b_4 b_5 a_6 b_6 b_7.
\end{align*}
\]

From the first,

\[
\begin{align*}
\delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 &= a_1 b_1 a_2 b_2 b_3 b_4 b_5 a_6 b_6 b_7, \\
&\sim a_1 b_1 a_2 b_2 b_3 b_4 b_5 a_6 b_6 b_7.
\end{align*}
\]

By the counterclockwise rotation by \( 2\pi/7 \) we can shift the indices by \(-1\), which results in \( N_7 = \partial_7 \) up to a permutation.

**Case 4:** Capping \( \delta_1 \) of \( N_8 = \partial_8 \). This yields

\[
\begin{align*}
\delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 &= a_1 b_1 a_2 b_2 b_3 b_4 b_5 a_6 b_6 b_7, \\
&\sim b_1 b_1 b_2 b_3 b_4 b_5 a_6 b_6 b_7, \\
&\sim a_1 b_1 b_2 b_3 b_4 b_5 a_6 b_6 b_7, \\
&\sim a_1 b_1 b_2 b_3 b_4 b_5 a_6 b_6 b_7.
\end{align*}
\]
Shifting the indices by \(-3\) (or \(+4\)) by rotation we see that this is equivalent to \(N_7 = \partial_7\).

**Case 5:** Capping \(\delta_8\) or \(\delta_7\) of \(S_8 = \partial_8\). They yield the equivalent relations

\[
\begin{align*}
\delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 &= a_1 b_1 b_2 b_3 b_4 b_5 b_6 a_7 b_7 b_7, \\
\delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 &= a_1 b_1 b_2 b_3 b_4 b_5 b_6 a_7 b_7
\end{align*}
\]

From the first,

\[
\begin{align*}
\delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 &= a_1 b_1 b_2 b_3 b_4 b_5 b_6 a_7 b_7 \\
\sim &= a_1 b_1 a_2 b_2 a_3 b_3 a_4 b_4 a_5 b_5 a_6 b_6
\end{align*}
\]

which is exactly the expression \(N_7\).

**Case 6:** The other boundary components \(\delta_1, \delta_2, \ldots, \delta_8\) of \(S_8 = \partial_8\). Observe that the relation \(S_8 = \partial_8\) is symmetric with respect to the rotation by \(2\pi/4\). Therefore, in the similar way as Case 3 in Section 5.2.1, we can reduce to the cases of capping \(\delta_8\) or \(\delta_7\).

Note that from the argument so far we deduce that the blow-up of any two base points of \(f_n\) and the blow-up of any one base point of \(f_8\) are isomorphic.

### 5.2.3. Three-time blow-up of \(f_n\) and the 6-holed torus relation \(N_6 = \partial_6\)

We cap each one of the boundary components of \(N_7 = \partial_7\).

**Case 1:** Capping \(\delta_7\). We get

\[
\begin{align*}
\delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 &= a_1 b_1 a_2 b_2 a_3 b_3 b_4 a_5 b_5 b_6 a_7 b_7 \\
\sim &= a_1 b_1 a_2 b_2 a_3 b_3 b_4 a_5 b_5 a_6 b_6 \\
\sim &= a_1 b_1 a_2 b_2 a_3 b_3 b_4 a_5 b_5 a_6 b_6 \\
\sim &= a_1 b_1 a_2 b_2 a_3 b_3 a_4 a_5 b_5 a_6 b_6 \\
\sim &= a_1 b_1 a_2 b_2 a_3 b_3 a_4 a_5 b_5 a_6 b_6.
\end{align*}
\]

Thus, we obtain the following 6-holed torus relation \(N_6 = \partial_6\):

\[
a_1 b_1 a_2 b_2 a_3 b_3 a_4 b_4 a_5 b_5 a_6 b_6 = \delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6.
\]

**Case 2:** Capping \(\delta_6\) or \(\delta_5\). They give the equivalent relations

\[
\begin{align*}
\delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 &= a_1 b_1 a_2 b_2 a_3 b_3 b_4 a_5 b_5 a_6 b_6, \\
\delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 &= a_1 b_1 a_2 b_2 a_3 b_3 b_4 a_5 b_5 a_6 b_6.
\end{align*}
\]

From the second,

\[
\begin{align*}
\delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 &= a_1 b_1 a_2 b_2 a_3 b_3 b_4 a_5 b_5 a_6 b_6 \\
\sim &= a_1 b_1 a_2 b_2 a_3 b_3 a_4 b_4 a_5 b_5 a_6 b_6 = N_6.
\end{align*}
\]

**Case 3:** Capping \(\delta_4\) or \(\delta_3\). They give the equivalent relations

\[
\begin{align*}
\delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 &= a_1 b_1 a_2 b_2 a_3 b_3 b_4 a_4 b_4 b_5 a_6 b_6, \\
\delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 &= a_1 b_1 a_2 b_2 a_3 b_3 b_4 a_4 b_4 b_5 a_6 b_6.
\end{align*}
\]
From the first,
\[ \delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 = a_1 b_1 a_2 b_2 a_3 b_3 a_4 b_4 b_5 a_6 b_6, \]
\[ \sim a_1 b_1 a_2 b_2 a_3 b_3 a_4 a_5 b_5 a_6 b_6 = N_6. \]

Case 4: Capping \( \delta_2 \). We get
\[ \delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 = a_1 b_1 a_2 b_2 a_3 a_4 b_4 b_5 a_6 b_6 \]
\[ \sim a_1 b_1 a_2 b_2 b_3 a_4 b_4 b_5 a_6 b_6 \]
\[ \sim a_1 b_1 a_2 b_2 b_3 a_4 b_4 a_5 b_5 a_6 b_6 \]
\[ \sim a_1 b_1 a_2 b_2 a_3 b_3 a_4 a_5 b_5 a_6 b_6 = N_6. \]

Case 5: Capping \( \delta_1 \). We get
\[ \delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 = a_1 b_1 a_1 a_2 b_2 b_3 a_4 b_4 b_5 a_6 b_6 \]
\[ \sim b_1 b_1 a_2 b_2 a_3 a_4 b_4 b_5 a_6 b_6 \]
\[ \sim a_1 b_1 a_2 a_2 b_3 a_4 b_4 a_5 b_5 a_6 b_6 \]
\[ \sim a_1 b_1 a_2 b_2 a_3 b_3 a_4 a_5 b_5 a_6 b_6 = N_6. \]

5.2.4. Four-time blow-up of \( f_n \) and the 5-holed torus relation \( N_5 = \partial_5 \)
We cap each one of the boundary components of \( N_6 = \partial_6 \).
Case 1: Capping \( \delta_6 \). We get
\[ \delta_1 \delta_2 \delta_3 \delta_4 \delta_5 = a_1 b_1 a_2 b_2 a_3 b_3 a_4 a_5 b_5 a_1 b \]
\[ \sim a_1 b_1 a_2 b_2 a_3 a_4 b_4 a_5 b_5 \]
\[ \sim a_1 a_1 a_2 a_2 b_2 a_3 b_3 a_4 a_5 b_5. \]
We denote the resulting 5-holed torus relation by \( N_5 = \partial_5 \):
\[ a_1 a_1 b_1 a_2 b_2 a_3 b_3 a_4 a_5 b_5 = \delta_1 \delta_2 \delta_3 \delta_4 \delta_5. \] (5.9)
Case 2: The other boundary components \( \delta_1, \delta_2, \ldots, \delta_5 \). Observe that the relation \( N_6 = \partial_6 \) is symmetric with respect to the rotation by \( 2\pi/6 \). Therefore, we can reduce all the other cases to Case 1.

5.2.5. Five-time blow-up of \( f_n \) and the 4-holed torus relation \( N_4 = \partial_4 \)
We cap each one of the boundary components of \( N_5 = \partial_5 \).
Case 1: Capping \( \delta_5 \). We get
\[ \delta_1 \delta_2 \delta_3 \delta_4 = a_1 a_1 b_1 a_2 b_2 a_3 b_3 a_4 a_5 b_4 b_4 a_1 b \]
\[ \sim a_1 b a_1 a_1 a_1 b_1 a_2 b_2 b_3 a_4 b_4 \]
\[ \sim b a_1 b a_1 b_1 a_2 b_2 a_3 b_3 a_4 b_4 \]
\[ \sim a_1 b a_1 b_1 a_2 a_2 b_2 a_3 a_3 b_3 a_4 b_4 \]
\[ \sim a_1 a_1 b_1 a_2 a_2 b_2 a_3 b_3 a_4 a_4 b_4. \]
We take the last expression as the 4-holed torus relation $N_4 = \partial_4$:
\begin{equation}
\begin{aligned}
a_1a_1b_1a_2a_2b_2a_3a_3b_3a_4a_4b_4 &= \delta_1\delta_2\delta_3\delta_4.
\end{aligned}
\end{equation}

**Case 2**: Capping $\delta_4$. We get
\begin{equation}
\begin{aligned}
\delta_1\delta_2\delta_3\delta_4 &= a_1a_1b_1a_2a_2b_2a_3b_3b_3a_4b_4b_4 \\
&\sim a_1a_1b_1a_2a_2b_2a_3b_3a_4a_4b_4 = N_4.
\end{aligned}
\end{equation}

**Case 3**: Capping $\delta_3$. We get
\begin{equation}
\begin{aligned}
\delta_1\delta_2\delta_3\delta_4 &= a_1a_1b_1a_2a_2b_2a_3b_3a_3a_4b_4 \\
&\sim a_1a_1b_1a_2a_2b_2a_3b_3a_4a_4b_4 = N_4.
\end{aligned}
\end{equation}

**Case 4**: Capping $\delta_2$. We get
\begin{equation}
\begin{aligned}
\delta_1\delta_2\delta_3\delta_4 &= a_1a_1b_1a_2a_2b_2a_3b_3a_3a_4b_4 \\
&\sim a_1a_1b_1a_2b_2a_2a_3b_3a_3a_4b_4 \\
&\sim a_1a_1b_1a_2b_2a_2a_3b_3a_4a_4b_4 \\
&\sim a_1a_1b_1a_2a_2b_2a_3a_3b_3a_4a_4b_4 = N_4.
\end{aligned}
\end{equation}

**Case 5**: Capping $\delta_1$. We get
\begin{equation}
\begin{aligned}
\delta_1\delta_2\delta_3\delta_4 &= a_1b_1a_1b_1a_2a_2b_2a_3b_3a_3a_4b_4 \\
&\sim a_1b_1a_1b_1a_2b_2a_3b_3a_3a_4b_4 \\
&\sim a_1b_1a_1b_1a_2b_2a_3b_3a_3a_4b_4 \\
&\sim a_1b_1a_1b_1a_2b_2a_3a_3b_3a_4a_4b_4 \\
&\sim a_1a_1b_1a_2a_2b_2a_3a_3b_3a_4a_4b_4 = N_4.
\end{aligned}
\end{equation}

**Remark 5.4** The 4-holed torus relation $N_4 = \partial_4$ has a different but equally symmetric expression, which is given in [14]. We can relate the two relations as follows.

\begin{equation}
\begin{aligned}
\delta_1\delta_2\delta_3\delta_4 = N_4 &= a_1a_1b_1a_2a_2b_2a_3a_3b_3a_4a_4b_4 \\
&\sim a_1b_1a_2b_2a_3b_3a_3a_4a_4b_4 \\
&\sim b_1a_2b_2b_2a_3a_3b_3a_4b_4 \\
&\sim a_2b_1a_3b_3a_3a_4a_3a_1b_4 \\
&\sim (a_1a_3b_2a_4b)^2.
\end{aligned}
\end{equation}

The last expression is Korkmaz-Ozbagci’s 4-holed torus relation.

### 5.2.6. Six-time blow-up of $f_n$ and the 3-holed torus relation $N_3 = \partial_3$

We need to cap each one of the boundary components of $N_4 = \partial_4$. However, noticing that $N_4 = \partial_4$ is symmetric with respect to the rotation by $2\pi/4$, it is clear that any capping gives an equivalent 3-holed torus relation.
For reference, we give a symmetric expression. By capping $\delta_4$, we get
\[
\delta_1 \delta_2 \delta_3 = a_1 a_1 b_1 a_2 a_2 b_2 a_3 a_3 b_3 a_1 a_1 b
\sim a_1 a_1 b a_1 b_1 a_2 a_2 b_2 a_3 a_3 b_3
\sim a_1 b a_1 b a_1 a_2 a_2 b_2 a_3 a_3 b_3
\sim a_1 a_1 b a_1 b a_2 a_2 b_2 a_3 a_3 b_3.
\]
We take the last expression as the 3-holed torus relation $N_3 = \partial_3$:
\[
a_1 a_1 a_1 b a_1 a_2 a_2 b_2 a_3 a_3 b_3 = \delta_1 \delta_2 \delta_3. \tag{5.11}
\]

**Remark 5.5** The 3-holed torus relation $N_3 = \partial_3$ also has an alternative nice expression, which is called the star relation [7]. Here we show the equivalence explicitly.
\[
\delta_1 \delta_2 \delta_3 = N_3 = a_1 a_1 a_1 b a_2 a_2 a_2 b_2 a_3 a_3 a_3 b_3
\sim a_1 a_1 b a_1 a_2 a_2 a_2 b_2 a_3 a_3 a_3 b_3 a_1
\sim a_1 a_2 b a_1 a_3 b a_2 a_3 a_1 a b_3
\sim (a_1 a_2 b a_3)^3.
\]
The last expression gives nothing but the star relation.

5.2.7. Seven-time blow-up of $f_n$ and the 2-holed torus relation $N_2 = \partial_2$

Since $N_3 = \partial_3$ is symmetric with respect to the rotation by $2\pi/3$ it is obvious that capping any one boundary component of $N_3 = \partial_3$ yields an equivalent 2-holed torus relation.

Capping $\delta_3$ of $N_3 = \partial_3$ gives
\[
\delta_1 \delta_2 = a_1 a_1 a_1 b_1 a_2 a_2 a_2 b_2 a_1 a_1 b
\sim a_1 a_1 b_1 a_2 b a_2 a_2 b_2 a_1 a_1 b
\sim b a_1 b_1 a_2 b a_2 a_1 b a_2 a_1 b a_1
\sim a_2 b a_1 a_2 b a_2 a_1 b a_2 a_1 b a_1
\sim (a_1 b a_2)^4.
\]
Thus, we get the 2-holed torus relation $N_2 = \partial_2$: \[(a_1 b a_2)^4 = \delta_1 \delta_2, \tag{5.12}\]
which is also known as the 3-chain relation.

5.2.8. Eight-time blow-up of $f_n$ and the 1-holed torus relation $N_1 = \partial_1$

Capping either $\delta_2$ or $\delta_1$ of $N_2 = \partial_2$ gives
\[
\delta_1 = (a_1 b a_1)^4
= a_1 b a_1 b a_1 a_1 b a_1 a_1 b a_1
\sim a_1 b a_1 b a_1 b a_1 b a_1 b = (a_1 b)^6.
\]
Writing $a = a_1$, we get the 1-holed torus relation $N_1 = \partial_1$:

$$(ab)^6 = \delta_1,$$  \hspace{1cm} (5.13)

which is also known as the 2-chain relation.

**Remark 5.6** Our nonspin $k$-holed torus relations $N_k = \partial_k$ are all Hurwitz equivalent to Korkmaz-Ozbagci’s $k$-holed torus relations. The latter were constructed in the way that the 9-holed torus relation is a lift of the smaller $k$-holed torus relations; hence, conversely, they can be obtained by capping boundary components of the 9-holed torus relation.

### 6. Final remarks

In this section, we discuss two additional topics related to the genus-1 holomorphic Lefschetz pencils and the $k$-holed torus relations. We first present handle diagrams associated with the genus-1 pencils using the simplified $k$-holed torus relations. We then summarize the contact topological aspect of the $k$-holed torus relations.

#### 6.1. Handle diagrams

As we have shown, the relations $N_9 = \partial_9$ and $S_8 = \partial_8$ correspond to the minimal holomorphic Lefschetz pencils $f_n$ on $\mathbb{P}^2$ and $f_s$ on $\mathbb{P}^1 \times \mathbb{P}^1$, respectively (while the others $N_k = \partial_k$ ($k < 9$) are just blow-ups of them). In Figure 29, we draw two handle diagrams of the elliptic Lefschetz fibration $E(1) = \mathbb{P}^2 \# 9 \mathbb{P}^2 \to \mathbb{P}^1$ and locate the $(-1)$-sections corresponding to $N_9 = \partial_9$ and $S_8 = \partial_8$. In each of the diagrams (with $k = 9$ or 8), we

![Diagram with labeled sections](image)

(a) The nine $(-1)$-sections corresponding to $N_9 = \partial_9$.

![Diagram with labeled sections](image)

(b) The eight $(-1)$-sections corresponding to $S_8 = \partial_8$.

**Figure 29.** Handle diagrams of the elliptic Lefschetz fibration $E(1) = \mathbb{P}^2 \# 9 \mathbb{P}^2 \to \mathbb{P}^1$ with configurations of $(-1)$-sections. The circled numbers indicate the location of the intersection points of the regular fiber and sections.
first construct the trivial $T^2$-fibration over $D^2$ by attaching two 1-handles to a 0-handle and then attaching a 2-handle with framing 0. We fix $k$ points on the fiber that correspond to the boundary components of the $k$-holed torus. On this fiber with $k$ fixed points, we attach twelve more 2-handles with framing $-1$ along the vanishing cycles of the respective monodromy factorization, which describes a Lefschetz fibration over a disk $D^2$. If we ignore the fixed points we can attach 2-, 3-, and 4-handles to build up the elliptic fibration $E(1)$. Blowing-down those sections must yield the 4-manifolds $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, respectively, and the exceptional spheres become the base points of the Lefschetz pencils $f_n$ and $f_s$.

6.2. Stein fillings and $k$-holed torus relations

Positive Dehn twist factorizations (with homologically nontrivial curves) of elements in mapping class groups of holed surfaces also provide positive allowable Lefschetz fibrations over $D^2$, which in turn represent Stein fillings of contact 3-manifolds. There is another elegant interpretation of the $k$-holed torus relations in this point of view. In this subsection, we summarize the role of the $k$-holed torus relations in contact topology as the intersection of various fields of study. More detailed discussion and relevant references can be found in [25].

A simple elliptic singularity of degree $k$ is an isolated singularity such that the exceptional divisor of its minimal resolution consists of a nonsingular elliptic curve of self-intersection number $-k$. Let $(M_k, \xi_k)$ be the link of the simple elliptic singularity of degree $k$ with the contact structure given by the maximal complex tangencies. The contact 3-manifold $(M_k, \xi_k)$ can also be viewed as the unit circle bundle of a complex hermitian line bundle $L_k \rightarrow T^2$ over a 2-torus with $c_1(L_k) = -k$ where the contact structure is the horizontal distribution given by a connection on the line bundle.

The contact 3-manifold $(M_k, \xi_k)$ is Stein fillable and we present some natural examples of a Stein filling of it as follows.

- The minimal resolution of the simple elliptic singularity of degree $k$ provides a Stein filling of $(M_k, \xi_k)$. This filling can also be described as the unit disk bundle of the line bundle $L_k \rightarrow T^2$. Let $D_k$ denote this (equivalent) Stein filling.

- For $0 < k \leq 9$, the singularity of degree $k$ has a smoothing that is simply connected. Let $A_k$ denote this Stein filling.

- When $k = 8$, the singularity has another smoothing that is not simply connected. Let $B_8$ denote this Stein filling.

In fact, the above examples exhaust all the minimal Stein fillings of $(M_k, \xi_k)$ up to diffeomorphism as the strong symplectic fillings (underlying structures of Stein fillings) of the link of the simple elliptic singularity are classified by Ohta and Ono [24].

Theorem 6.1 ([24]) Suppose that $X$ is a minimal Stein filling of $(M_k, \xi_k)$.

1. If $k \geq 10$ then $X$ is diffeomorphic to $D_k$.

2. If $k \leq 9$ but $k \neq 8$ then $X$ is diffeomorphic to either $D_k$ or $A_k$. 

1115
3. If $k = 8$ then $X$ is diffeomorphic to either one of $D_8$, $A_8$, or $B_8$.

In [24], the diffeomorphism types of the smoothings are also described as follows:

- Consider the blow-up of $\mathbb{P}^2$ at $(9-k)$-points on a nonsingular cubic curve. The filling $A_k$ is diffeomorphic to the complement of the proper transform of the cubic curve in $\mathbb{P}^2 \setminus (9-k)\mathbb{P}^2$.

- The filling $B_8$ is diffeomorphic to the complement of a holomorphically embedded 2-torus in $\mathbb{P}^1 \times \mathbb{P}^1$ which is linearly equivalent to $2F_1 + 2F_2$, where $F_i$ is a fiber of the projection $\pi_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ onto the $i$-th component.

Therefore, we can view the smoothings of the simple elliptic singularities as the complements of a regular fiber in the genus-1 pencils.

- $A_k$ is diffeomorphic to the complement of a regular fiber of $f_k \mathbb{P}^2(9-k)\mathbb{P}^2$ in $\mathbb{P}^2 \setminus (9-k)\mathbb{P}^2$.

- $B_8$ is diffeomorphic to the complement of a regular fiber of $f_8$ in $\mathbb{P}^1 \times \mathbb{P}^1$.

Now we turn to the description of $(M_k, \xi_k)$ and its Stein fillings in terms of an open book and positive allowable Lefschetz fibrations. The contact 3-manifold $(M_k, \xi_k)$ indeed has the open book decomposition whose page is a $k$-holed torus and monodromy is the boundary multitwist $\psi_k = t_{\delta_1} \cdots t_{\delta_k}$ in $\text{MCG}(\Sigma_k)$.

Then, the positive allowable Lefschetz fibration over $D^2$ associated with the obvious Dehn twist factorization $\psi_k = t_{\delta_1} \cdots t_{\delta_k}$ prescribes a Stein filling of $(M_k, \xi_k)$. This filling is nothing but the disk bundle $D_k$. If the open book monodromy $\psi_k$ has another factorization (i.e. a $k$-holed torus relation) it also gives a Stein filling of $(M_k, \xi_k)$. The relationship between those Stein fillings and the positive allowable Lefschetz fibrations associated with the $k$-holed torus relations is observed in [25] as follows (see also Table 2).

Table 2. Classification of the Stein fillings of the link of the simple elliptic singularity of degree $k$ and corresponding genus-1 positive allowable Lefschetz fibrations.

| Degree of singularity | Monodromy factorization | Stein filling |
|-----------------------|--------------------------|--------------|
| $k > 1$               | $\psi_k = t_{\delta_1} \cdots t_{\delta_k}$ | $D_k$        |
| 9                     | $\psi_9 = t_{a_1} t_{b_1} t_{a_2} t_{b_2} t_{a_3} t_{b_3} t_{a_4} t_{b_4} t_{a_5} t_{b_5} t_{a_6} t_{b_6}$ | $A_0$ Smoothing ($\pi_1 = 1$) |
| 8                     | $\psi_8 = t_{a_1} t_{b_1} t_{a_2} t_{b_2} t_{a_3} t_{b_3} t_{a_4} t_{b_4} t_{a_5} t_{b_5}$ | $A_8$ Smoothing ($\pi_1 = 1$) |
| 7                     | $\psi_7 = t_{a_1} t_{b_1} t_{a_2} t_{b_2} t_{a_3} t_{b_3} t_{a_4} t_{b_4} t_{a_5} t_{b_5}$ | $A_7$ Smoothing ($\pi_1 = 1$) |
| 6                     | $\psi_6 = t_{a_1} t_{b_1} t_{a_2} t_{b_2} t_{a_3} t_{b_3} t_{a_4} t_{b_4} t_{a_5} t_{b_5}$ | $A_6$ Smoothing ($\pi_1 = 1$) |
| 5                     | $\psi_5 = t_{a_1} t_{b_1} t_{a_2} t_{b_2} t_{a_3} t_{b_3} t_{a_4} t_{b_4}$ | $A_5$ Smoothing ($\pi_1 = 1$) |
| 4                     | $\psi_4 = t_{a_1} t_{b_1} t_{a_2} t_{b_2} t_{a_3} t_{b_3} t_{a_4} t_{b_4}$ | $A_4$ Smoothing ($\pi_1 = 1$) |
| 3                     | $\psi_3 = t_{a_1} t_{b_1} t_{a_2} t_{b_2} t_{a_3} t_{b_3}$ | $A_3$ Smoothing ($\pi_1 = 1$) |
| 2                     | $\psi_2 = (t_{a_1} t_{b_1})^4$ | $A_2$ Smoothing ($\pi_1 = 1$) |
| 1                     | $\psi_1 = (t_{a_1} t_{b_1})^6$ | $A_1$ Smoothing ($\pi_1 = 1$) |
Theorem 6.2 ([25]) The Stein fillings of the link of the simple elliptic singularity of degree $k$ are realized as genus-1 positive allowable Lefschetz fibrations except for $D_1$. More explicitly,

1. For $1 < k$, the positive allowable Lefschetz fibration associated with monodromy factorization $\psi_k = t_{\delta_1} \cdots t_{\delta_k}$ prescribes the Stein filling $D_k$.

2. For $0 < k \leq 9$, the positive allowable Lefschetz fibration associated with monodromy factorization $\psi_k = N_k$ prescribes the Stein filling $A_k$.

3. For $k = 8$, the positive allowable Lefschetz fibration associated with monodromy factorization $\psi_8 = S_8$ prescribes the Stein filling $B_8$.

Note that the factorization $\psi_1 = t_{\delta_1}$ does not provide an allowable Lefschetz fibration since the vanishing cycle $\delta_1$ is homologically trivial on the fiber. This is why the filling $D_1$ is excluded in Theorem 6.2. The fact that $(M_k, \xi_k)$ has a unique Stein filling for $k \geq 10$ reflects that there is no $k$-holed torus relation for $k \geq 10$. This can also be seen from the fact that $E(1) = \mathbb{P}^2 \# 9\mathbb{P}^2$ can admit no more than nine $(-1)$-sections. The distinction between $N_8$ and $S_8$ up to Hurwitz equivalence is once again verified in this Theorem. This can be independently confirmed by computing the first homology groups of the positive allowable Lefschetz fibrations associated with $N_8$ and $S_8$ as demonstrated in [25] (Ozbagci used Korkmaz-Ozbagci’s and Tanaka’s 8-holed torus relations but the computation would be more straightforward if one uses our simplified expressions).

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