GLOBAL EXISTENCE AND STABILITY IN A TWO-SPECIES CHEMOTAXIS SYSTEM

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Abstract. This paper deals with the following two-species chemotaxis system
\[
\begin{align*}
    u_t &= \Delta u - \chi_1 \nabla \cdot (u \nabla v) + \mu_1 u (1 - u - a_1 w), & x \in \Omega, t > 0, \\
    v_t &= \Delta v - v + h(w), & x \in \Omega, t > 0, \\
    w_t &= \Delta w - \chi_2 \nabla \cdot (w \nabla z) + \mu_2 w (1 - w - a_2 u), & x \in \Omega, t > 0, \\
    z_t &= \Delta z - z + h(u), & x \in \Omega, t > 0,
\end{align*}
\]
under homogeneous Neumann boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary. The parameters in the system are positive and the signal production function $h$ is a prescribed $C^1$-regular function. The main objectives of this paper are two-fold: One is the existence and boundedness of global solutions, the other is the large time behavior of the global bounded solutions in three competition cases (i.e., a weak competition case, a partially strong competition case and a fully strong competition case). It is shown that the unique positive spatially homogeneous equilibrium $(u^*, v^*, w^*, z^*)$ may be globally attractive in the weak competition case (i.e., $0 < a_1, a_2 < 1$), while the constant stationary solution $(0, h(1), 1, 0)$ may be globally attractive and globally stable in the partially strong competition case (i.e., $a_1 > 1 > a_2 > 0$). In the fully strong competition case (i.e., $a_1, a_2 > 1$), however, we can only obtain the local stability of the two semi-trivial stationary solutions $(0, h(1), 1, 0)$ and $(1, 0, 0, h(1))$ and the instability of the positive spatially homogeneous $(u^*, v^*, w^*, z^*)$. The matter which species ultimately wins out depends crucially on the starting advantage each species has.

1. Introduction. Chemotaxis refers to the movement of cells in response to a chemical signal, a process by which cells change their movement state in the presence of chemical concentration gradient, approaching the chemical favorable environment and avoiding the adverse ones. This phenomenon plays a crucial role in morphogenesis and self-organization of various biological coherent and enthralled structures. Keller and Segel [11] first introduced the following mathematical model to describe the aggregation phase of Dictyostelium discoideum.

\[
\begin{align*}
    u_t &= \nabla \cdot (\nabla u - \chi u \nabla v), \\
    v_t &= \Delta v - v + u.
\end{align*}
\]
Since then, many authors have studied several different models in mathematical biology. And a large body of works have been devoted to determining when blow-up occurs or whether globally existing solutions exist.

Hillen and Painter [8] studied the formulation from a biological point of view. In contrast with the patterning properties, some key results on the analytical properties are summarised and the solution forms are classified. Horstmann [9] summarized various aspects and results for some general formulations of Keller-Segel models. Naturally, more effects, such as population growth and competition among species, incorporate life forms into complex ecosystems. As such, chemotaxis is not the only effect on the behavior of the cell populations. Often the growth of the population must be taken into account. A typical option to achieve this goal is the addition of logistic growth $\kappa u - \mu u^2$ to the first equation (see e.g. [10], [12], [13], [15], [16], [25], [30]). That is,

$$
\begin{cases}
  u_t = \nabla \cdot (\nabla u - \chi u \nabla v) + \kappa u - \mu u^2, \\
  \varepsilon v_t = \Delta v - v + u.
\end{cases}
$$

(2)

In fact, the presence of such logistic terms, particularly the quadratic term $-\mu u^2$, is sufficient to suppress any blow-up in many relevant situations. In the case where $\varepsilon = 0$, Tello and Winkler [30] proved the existence of global bounded classical solutions under the assumption that either the space dimension does not exceed two, or that the logistic damping effect is strong enough, and also established some multiplicity and bifurcation results for small logistic terms. In presence of certain sub-quadratic degradation terms, even finite-time blow-up is possible, as detected in [34] and [36]. In the fully parabolic case $\varepsilon > 0$, Osaki [24] indicated that in two-dimensional case where $x \in \mathbb{R}^2$, any blow-up phenomenon can be completely suppressed for $\mu > 0$. This result is extended by Winkler [33] to arbitrary space dimensions, who proved the existence and uniqueness of global, smooth, bounded solutions to (2) with large enough $\mu$. Lankeit [13] proved the existence of global weak solutions and showed that in the three-dimensional setting, there exist $\kappa_0 > 0$ and $T > 0$ such that the weak solutions become classical solutions when $\kappa < \kappa_0$ and $t > T$. Lankeit [13] also proved the attractivity of the trivial steady state when $\kappa \leq 0$ and the existence of an absorbing set when $\kappa > 0$ is sufficiently small.

However, if the effect of the logistic term is not strong enough, then finite-time blow-up is possible. In the one dimensional case, Winkler [35] proved that if $0 < \mu < 1$, then there is some criterion on the initial data that ensures the existence of some time up to which any threshold of the population density will be surpassed. Lankeit [14] obtained a criterion guaranteeing some kind of structure formation, which is an extension of the result of Winkler [35] to the higher dimensional case. In [25], a positive Lyapunov exponent together with a rich bifurcation structure was indicated by numerical experiments.

In view of various biological processes, researchers revised the second equation in (2) to $\varepsilon v_t = \Delta v - v + h(u)$ and obtained some researches for different representations of $h$. In the case where $h(u) = u(1 + u)^{\beta - 1}$, Nakaguchi and Osaki [21, 22] showed the global existence of solutions in $L^p$ space under certain relations between the degradation and production orders when $0 < \beta \leq 2$ and the source term given by $\mu u(1 - u^{\alpha - 1})$ satisfies $\alpha > 1$. While for a more general function $h$, Chaplain and Tello [3] studied the asymptotic behavior of solutions under the assumption that $2\chi|h'(x)| < \mu$. 
However, the situation of single population and single chemoattractant is rare in the real biological systems. So the discussion for a number of populations under the action of variety chemical substances has biological significance. Considering the competition of two species for resources or space (see, e.g. [6, 7, 37, 39]), in this paper we will consider the following boundary value problem with chemotaxis and Lotka-Volterra competition pattern:

\[
\begin{align*}
    & u_t = \Delta u - \chi_1 \nabla \cdot (u \nabla v) + \mu_1 u (1 - u - a_1 w), & x \in \Omega, t > 0, \\
    & v_t = \Delta v - v + h(w), & x \in \Omega, t > 0, \\
    & w_t = \Delta w - \chi_2 \nabla \cdot (w \nabla z) + \mu_2 w (1 - w - a_2 u), & x \in \Omega, t > 0, \\
    & z_t = \Delta z - z + h(u), & x \in \Omega, t > 0, \\
    & \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, & x \in \partial \Omega, t > 0,
\end{align*}
\]

in a bounded domain \( \Omega \subset \mathbb{R}^n \) \((n \geq 1)\) with smooth boundary \( \partial \Omega \), where \( \chi_1, \chi_2, \mu_1, \mu_2, a_1 \) and \( a_2 \) are positive parameters, \( u = u(x, t) \) and \( w = w(x, t) \) denote the densities of two cell population, whereas \( v = v(x, t) \) and \( z = z(x, t) \) stand for the concentration of the chemials produced by \( w(x, t) \) and \( u(x, t) \), respectively. In model (3), \( h(s) \) is a prescribed function on \([0, \infty)\), which represents the production of the chemical substance by the cells.

When \( h(s) = s \), the questions of global existence and large time behavior has been addressed in [2] for weakly competitive species case and for the partially strong competition setting. But there is no relevant work for a general function \( h \) yet. Throughout this paper, we always assume that the signal production function \( h \) is \( C^1 \)-regular and satisfies

\[
h(s) \geq 0, \quad h(0) = 0 \quad \text{and} \quad 0 \leq h'(s) \leq L_h,
\]

where \( L_h = \sup_{s \geq 0} \{h'(s)\} > 0 \) is a constant. Moreover, we will consider (3) with the initial condition

\[
(u, v, w, z)(\cdot, 0) = (u_0, v_0, w_0, z_0) \in C^0(\bar{\Omega}) \times W^{1,\infty}(\Omega) \times C^0(\bar{\Omega}) \times W^{1,\infty}(\Omega) .
\]

We shall show that the solution of (3) with the initial condition (5) is global and bounded provided that

\[
L_h < \min \left\{ \sqrt{\frac{\mu_1}{7} - \frac{\chi_1^2}{2}}, \sqrt{\frac{\mu_2}{7} - \frac{\chi_2^2}{2}} \right\} .
\]

This means that the smallness condition on \( h' \) suppresses blow-up effects, which is in keeping with the results in one-species Keller-Segel chemotaxis systems; see, for example, [21, 22, 23]. Furthermore, the long time behavior of the bounded solutions is discussed in three cases. In the weak competition case (that is, \( a_1, a_2 < 1 \)), if

\[
L_h < \min \left\{ \frac{8\mu_1 a_1 (1 - a_2)(1 - a_1 a_2)}{\chi_1^2 a_2 (1 - a_1)}, \frac{8\mu_2 a_2 (1 - a_1)(1 - a_1 a_2)}{\chi_2^2 a_1 (1 - a_2)} \right\},
\]

then any global bounded solution stabilizes to the unique positive spatially homogeneous equilibrium \( (u_*, v_*, w_*, z_*) \), where

\[
    u_* = \frac{1 - a_1}{1 - a_1 a_2}, \quad v_* = h(w_*), \quad w_* = \frac{1 - a_2}{1 - a_1 a_2}, \quad z_* = h(u_*).
\]

Moreover, there is a positive constant \( L(\mu_1, \mu_2, \chi_1, \chi_2) \) depending on \( (\mu_1, \mu_2, \chi_1, \chi_2) \) such that \( (u_*, v_*, w_*, z_*) \) is globally asymptotical stable when

\[
L_h < L(\mu_1, \mu_2, \chi_1, \chi_2).
\]
In the case where \(a_2 < 1 < a_1\), if
\[
L_h \chi_2 < 2\sqrt{\mu_2 a_2 (1 - a_2)},
\]
then \((0, h(1), 1, 0)\) is globally asymptotically stable and every global solution converges towards the constant stationary solution \((0, h(1), 1, 0)\) as \(t \to \infty\). In the case where \(a_1, a_2 > 1\), however, we can only obtain the local stability of the two semi-trivial stationary solutions \((0, h(1), 1, 0)\) and \((1, 0, 0, h(1))\) and the instability of the positive spatially homogeneous \((u_*, v_*, w_*, z_*)\). The matter which species ultimately wins out depends crucially on the starting advantage each species has.

The organization of the remaining part of the paper is as follows. In section 2 we investigate the global existence and boundedness of solutions to (3)-(5). Moreover, we obtain some a priori estimates of \(u, v, w, z\) and a classical solution \((u, v, w, z)\) satisfies (3) are nonnegative \((0, h(1), 1, 0)\) and \((u_*, v_*, w_*, z_*)\). The matter which species ultimately wins out depends crucially on the starting advantage each species has.

2. Global existence. We first give the following result on the global existence and boundedness. We refer to [27] for the following local existence and extensibility result for classical solutions of (3).

**Lemma 2.1.** Suppose that the parameters \(\mu_1, \mu_2\) are positive and \(\chi_1, \chi_2, a_1, a_2\) are nonnegative. Furthermore, assume that \(h \in C^1([0, \infty))\) and the initial data \((u_0, v_0, w_0, z_0)\) satisfying (5) are nonnegative and \(u_0 \neq 0 \neq w_0\). Then there exist \(T_{\text{max}} \in (0, \infty]\) and a classical solution \((u, v, w, z)\) of (3) in \(\Omega \times (0, T_{\text{max}}]\) such that \((u, v, w, z) \in (C^0(\Omega \times [0, T_{\text{max}}]\) \cap C^\infty(\Omega \times (0, T_{\text{max}})))^4\) and \(u, v, w, z > 0\) in \(\Omega \times (0, T_{\text{max}}]\). Moreover,

\[
\text{either } T_{\text{max}} < \infty \text{ or } \lim_{t \to T_{\text{max}}} \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{10}
\]

Furthermore, we can obtain the global existence of solutions to (3)-(5).

**Theorem 2.2.** Suppose that the parameters \(\mu_1, \mu_2\) are positive and \(\chi_1, \chi_2, a_1, a_2\) are nonnegative. Assume that
\[
\mu_1 \geq \frac{7}{2} \chi_1^2 \quad \text{and} \quad \mu_2 \geq \frac{7}{2} \chi_2^2 \tag{11}
\]
and \(h\) satisfies (4) and (6). Then for any given nonnegative initial values \((u_0, v_0, w_0, z_0)\) with \(u_0 \neq 0 \neq w_0\), the problem (3)-(5) possesses a global classical solution \((u, v, w, z) \in (C^0(\Omega \times [0, \infty)) \cap C^\infty(\Omega \times (0, \infty)))^4\), which is bounded in \(\Omega \times (0, \infty)\) in the sense that there exists \(C > 0\) such that
\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} + \|z(\cdot, t)\|_{L^\infty(\Omega)} \leq C
\]
for all \(t > 0\).

In order to prove this theorem, we shall need the following auxiliary lemma to derive some time-independent estimates for \(v\) and \(z\). We refer the details of the proof to [29].

**Lemma 2.3.** Let \(T > 0\), \(\tau \in (0, T)\), \(a, b > 0\). Suppose that \(y: [0, T) \to [0, \infty)\) is absolutely continuous. If
\[
y'(t) + ay(t) \leq \psi(t)
\]
then
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for a.e. \( t \in (0, T) \) with some nonnegative function \( \psi \in L^1_{\text{loc}}([0, T)) \) satisfying \( \int_t^{t+\tau} \psi(s) \, ds \leq b \) for all \( t \in [0, T-\tau) \). Then

\[
y(t) \leq \max \left\{ y(0) + b, \frac{b}{a\tau} + 2b \right\}
\]

for all \( t \in (0, T) \).

In order to show the global existence of the solution, it is sufficient to derive the boundedness of \( u, w, \nabla v \) and \( \nabla z \). The following lemma is some basic properties of \( u \) and \( w \).

**Lemma 2.4.** Under the assumptions of Theorem 2.2, we have the following basic estimates on the solution of (3):

\[
\int_{\Omega} u(\cdot, t) \leq C_1 \quad \text{and} \quad \int_{\Omega} w(\cdot, t) \leq C_1 \quad \text{for all} \quad t \in (0, T_{\text{max}})
\]

and

\[
\int_t^{t+\tau} \int_{\Omega} u^2 \leq C_2 \quad \text{and} \quad \int_t^{t+\tau} \int_{\Omega} w^2 \leq C_2 \quad \text{for all} \quad t \in (0, T_{\text{max}} - \tau)
\]

with \( C_1, C_2 > 0 \) and

\[
\tau := \min\{1, \frac{1}{2}T_{\text{max}}\}
\]

**Proof.** Integrating the first equation of (3) over \( \Omega \) implies that

\[
\frac{d}{dt} \int_{\Omega} u = \mu_1 \int_{\Omega} (u - u^2 - a_1 w).
\]

Since \( u, w \) are nonnegative, by the Cauchy-Schwarz inequality we have

\[
\frac{d}{dt} \int_{\Omega} u \leq \mu_1 \int_{\Omega} u - \frac{\mu_1}{|\Omega|} \left( \int_{\Omega} u \right)^2
\]

for all \( t \in (0, T_{\text{max}}) \). This implies

\[
\int_{\Omega} u(\cdot, t) \leq C_1
\]

for all \( t \in (0, T_{\text{max}}) \). Upon a time integration we have

\[
\int_t^{t+\tau} \int_{\Omega} u^2 \leq \left( 1 + \frac{1}{\mu_1} \right) C_1
\]

for all \( t \in (0, T_{\text{max}} - \tau) \). By applying the similar steps to the third equation of (3) we obtain the estimates of \( w \) and \( w^2 \). \( \square \)

**Lemma 2.5.** Under the assumptions of Theorem 2.2, we have

\[
\int_{\Omega} v(\cdot, t) \leq K_1 \quad \text{and} \quad \int_{\Omega} z(\cdot, t) \leq K_1 \quad \text{for all} \quad t \in (0, T_{\text{max}}),
\]

(14)

\[
\int_{\Omega} |\nabla v(\cdot, t)|^2 \leq K_2 \quad \text{and} \quad \int_{\Omega} |\nabla z(\cdot, t)|^2 \leq K_2 \quad \text{for all} \quad t \in (0, T_{\text{max}}),
\]

(15)

\[
\int_t^{t+\tau} \int_{\Omega} |\Delta v|^2 \leq K_3 \quad \text{and} \quad \int_t^{t+\tau} \int_{\Omega} |\Delta z|^2 \leq K_3 \quad \text{for all} \quad t \in (0, T_{\text{max}} - \tau),
\]

(16)

and

\[
\int_t^{t+\tau} \int_{\Omega} |\nabla v|^4 \leq K_4 \quad \text{and} \quad \int_t^{t+\tau} \int_{\Omega} |\nabla z|^4 \leq K_4 \quad \text{for all} \quad t \in (0, T_{\text{max}} - \tau),
\]

(17)
where \( \tau = \min\{1, \frac{1}{2}T_{\text{max}}\} \) and \( K_1, K_2, K_3, K_4 > 0 \).

Proof. It follows from (4) that
\[
0 \leq h(s) = h(s) - h(0) = h'(\eta)s \leq L_h s
\]
for some \( \eta \in [0, s] \). Then by integrating the second equation of (3) and recalling (12), we have
\[
\frac{d}{dt} \int_{\Omega} v + \int_{\Omega} v = \int_{\Omega} h(w) \leq L_h \int_{\Omega} w \leq L_h C_1.
\]
Obviously, we have
\[
\int_{\Omega} v(\cdot, t) \leq K_1.
\]
By multiplying the second equation of (3) with \(-\Delta v\) and integrating by parts we see that
\[
\frac{1}{2} \int_{\Omega} |\nabla v|^2 = - \int_{\Omega} |\Delta v|^2 - \int_{\Omega} |\nabla v|^2 - \int_{\Omega} h(w) \Delta v.
\]
By using Young’s inequality and recalling (18) we have
\[
\frac{1}{2} \int_{\Omega} |\nabla v|^2 + \int_{\Omega} |\nabla v|^2 \leq - \frac{1}{2} \int_{\Omega} |\Delta v|^2 + \frac{L_h^2}{2} \int_{\Omega} w^2 \leq \frac{L_h^2}{2} \int_{\Omega} w^2. \tag{19}
\]
For \( t \in (0, T_{\text{max}}) \), set \( y(t) \triangleq \int_{\Omega} |\nabla v(\cdot, t)|^2 \). Obviously, \( y(t) \) satisfies the condition stated in Lemma 2.3 with \( a = 2 \) and \( \psi(t) = L_h^2 \int_{\Omega} w^2(\cdot, t)^2, \ t \in (0, T_{\text{max}}) \) due to (13). Then we have \( \int_{\Omega} |\nabla v(\cdot, t)|^2 \leq K_2 \). By integrating (19) with respect to time, we immediately have
\[
\int_{t}^{t+\tau} \int_{\Omega} |\Delta v|^2 \leq \frac{L_h^2}{2} \int_{t}^{t+\tau} \int_{\Omega} w^2 \leq C_2 L_h^2
\]
for all \( t \in (0, T_{\text{max}} - \tau) \).

According to the Gagliardo-Nirenberg inequality combined with Theorem 3.4 in [26], there exists \( K \) such that for all \( t \in (0, T_{\text{max}}) \),
\[
\|\nabla v(\cdot, t)\|_{L^4(\Omega)} \leq K \|\Delta v(\cdot, t)\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla v(\cdot, t)\|_{L^2(\Omega)}^{\frac{1}{2}} + \|\nabla v(\cdot, t)\|_{L^2(\Omega)}^{\frac{1}{2}}.
\]
From the time integration and the fact that \( \tau \leq 1 \), we obtain
\[
\int_{t}^{t+\tau} \|\nabla v(\cdot, t)\|_{L^4(\Omega)}^{\frac{1}{2}} \leq K' \left[ \int_{t}^{t+\tau} \|\Delta v(\cdot, t)\|_{L^2(\Omega)}^{\frac{1}{2}} + \tau \right] \leq K''
\]
for all \( t \in (0, T_{\text{max}} - \tau) \).

Similarly, we can obtain the conclusions about \( z \), and so omit the details here. \( \square \)

**Lemma 2.6.** The solution of (3)-(5) satisfies
\[
\frac{d}{dt} \int_{\Omega} u^2 + \int_{\Omega} |\nabla u|^2 \leq \chi_1 \int_{\Omega} u^2 |\nabla v|^2 + 2\mu_1 \int_{\Omega} u^2 - 2\mu_1 \int_{\Omega} u^2(w + a_1 w), \tag{20}
\]
\[
\frac{d}{dt} \int_{\Omega} w^2 + \int_{\Omega} |\nabla w|^2 \leq \chi_2 \int_{\Omega} w^2 |\nabla z|^2 + 2\mu_2 \int_{\Omega} w^2 - 2\mu_2 \int_{\Omega} w^2(w + a_2 w), \tag{21}
\]
\[
\frac{d}{dt} \int_{\Omega} |\nabla v|^4 + \int_{\Omega} |\nabla |\nabla v|^2| \leq 4 \int_{\Omega} |\nabla v|^4 \leq 6L_h^2 \int_{\Omega} w^2 |\nabla v|^2, \tag{22}
\]
and
\[
\frac{d}{dt} \int_{\Omega} |\nabla z|^4 + \int_{\Omega} |\nabla |\nabla z|^2| \leq 4 \int_{\Omega} |\nabla z|^4 \leq 6L_h^2 \int_{\Omega} u^2 |\nabla z|^2. \tag{23}
\]
Lemma 2.7. Suppose that $h$ satisfies (6). Then there exist $C_1 > 0$ and $C_2 > 0$ such that the solution of (3)-(5) satisfies
\[
\int_{\Omega} u^2(\cdot,t) \leq C_1 \text{ and } \int_{\Omega} w^2(\cdot,t) \leq C_1 \text{ for all } t \in (0,T_{\text{max}}),
\]
and
\[
\int_{\Omega} |\nabla v(\cdot,t)|^4 \leq C_2 \text{ and } \int_{\Omega} |\nabla z(\cdot,t)|^4 \leq C_2 \text{ for all } t \in (0,T_{\text{max}}).
\]
Proof. By a straightforward computation and integration by parts we have

\[
\frac{d}{dt} \int_{\Omega} u|\nabla v|^2
\begin{align*}
&= \int_{\Omega} |\nabla v|^2 (\Delta u - \chi_1 \nabla \cdot (u \nabla v) + \mu_1 u (1 - u - a_1 w)) \\
&\quad + 2 \int_{\Omega} u \nabla v \cdot \nabla \{\Delta u - v + h(w)\}
\end{align*}
\begin{align*}
&= \int_{\Omega} |\nabla v|^2 \Delta u - \chi_1 \int_{\Omega} |\nabla v|^2 \nabla \cdot (u \nabla v) + \mu_1 \int_{\Omega} u |\nabla v|^2 - \mu_1 \int_{\Omega} u^2 |\nabla v|^2 \\
&\quad + \int_{\Omega} u \Delta |\nabla v|^2 - 2 \int_{\Omega} u |D^2 v|^2 - 2 \int_{\Omega} u |\nabla v|^2 + 2 \int_{\Omega} u h'(w) \nabla u \cdot \nabla v
\end{align*}
\begin{align*}
&= -2 \int_{\Omega} \nabla u \cdot \nabla |\nabla v|^2 + \chi_1 \int_{\Omega} u \nabla v \cdot \nabla |\nabla v|^2 + (\mu_1 - 2) \int_{\Omega} u |\nabla v|^2 \\
&\quad - \mu_1 \int_{\Omega} u^2 |\nabla v|^2 + \int_{\Omega} \frac{\partial |\nabla v|^2}{\partial v} - 2 \int_{\Omega} u |D^2 v|^2 + 2 \int_{\Omega} u h'(w) \nabla u \cdot \nabla v
\end{align*}
\begin{align*}
&\leq 3 \int_{\Omega} |\nabla u|^2 - (\mu_1 - \frac{\chi_1^2}{2} - L_h^2) \int_{\Omega} u^2 |\nabla v|^2 \\
&\quad + \int_{\Omega} |\nabla |\nabla v|^2|^2 + (\mu_1 - 2) \int_{\Omega} u |\nabla v|^2.
\end{align*}
\]

(27)

The last inequality is obtained by Young’s inequality. Similarly, we have

\[
\frac{d}{dt} \int_{\Omega} w|\nabla z|^2 \leq 3 \int_{\Omega} |\nabla w|^2 - (\mu_2 - \frac{\chi_2^2}{2} - L_h^2) \int_{\Omega} w^2 |\nabla z|^2 + \int_{\Omega} |\nabla |\nabla z|^2|^2 \\
&\quad + (\mu_2 - 2) \int_{\Omega} w |\nabla z|^2.
\]

(28)

By a simple linear combination of (20)-(23), (27) and (28), we have

\[
\frac{d}{dt} \left\{ \int_{\Omega} 3u^2 + \int_{\Omega} 3w^2 + \int_{\Omega} |\nabla v|^4 + \int_{\Omega} |\nabla z|^4 + \int_{\Omega} u |\nabla v|^2 + \int_{\Omega} w |\nabla z|^2 \right\}
\begin{align*}
&\leq 6\mu_1 \int_{\Omega} u^2 + 6\mu_2 \int_{\Omega} w^2 + 6L_h^2 \int_{\Omega} w^2 |\nabla v|^2 + 6L_h^2 \int_{\Omega} u^2 |\nabla z|^2 - 4 \int_{\Omega} |\nabla v|^4 \\
&\quad - 4 \int_{\Omega} |\nabla z|^4 - (\mu_1 - \frac{7}{2} \chi_1^2 - L_h^2) \int_{\Omega} u^2 |\nabla v|^2 - (\mu_2 - \frac{7}{2} \chi_2^2 - L_h^2) \int_{\Omega} w^2 |\nabla z|^2 \\
&\quad + (\mu_1 - 2) \int_{\Omega} u |\nabla v|^2 + (\mu_2 - 2) \int_{\Omega} w |\nabla z|^2.
\end{align*}
\]

(29)

This implies that for \( t \in (0, T_{\text{max}}) \),

\[
y(t) \triangleq \int_{\Omega} 3u^2 + \int_{\Omega} 3w^2 + \int_{\Omega} |\nabla v|^4 + \int_{\Omega} |\nabla z|^4 + \int_{\Omega} u |\nabla v|^2 + \int_{\Omega} w |\nabla z|^2
\]

(30)

satisfies

\[
y'(t) + y(t)
\begin{align*}
&\leq (6\mu_1 + 3) \int_{\Omega} u^2 + (6\mu_2 + 3) \int_{\Omega} w^2 + 6L_h^2 \int_{\Omega} w^2 |\nabla v|^2 + 6L_h^2 \int_{\Omega} u^2 |\nabla z|^2 \\
&\quad - (\mu_1 - \frac{7}{2} \chi_1^2 - L_h^2) \int_{\Omega} u^2 |\nabla v|^2 - (\mu_2 - \frac{7}{2} \chi_2^2 - L_h^2) \int_{\Omega} w^2 |\nabla z|^2
\end{align*}
\]

This completes the proof.
Proof. (of Theorem 2.2) Considering the first equation of (3), it follows from
for all \(s\), satisfies \(t\geq 0\) and Young’s inequality that

\[
-(\mu_1 - 1) \int_\Omega u|\nabla v|^2 - (\mu_2 - 1) \int_\Omega w|\nabla z|^2.
\]

By Young’s inequality and (6) we have

\[
6L_n^2 \int_\Omega w^2|\nabla v|^2 + 6L_n^2 \int_\Omega u^2|\nabla z|^2 - (\mu_1 \\
- \frac{7}{2} \lambda_1^2 - L_n^2) \int_\Omega u^2|\nabla v|^2 - (\mu_2 - \frac{7}{2} \lambda_2^2 - L_n^2) \int_\Omega w^2|\nabla z|^2
\]

\[
\leq - \left( \mu_1 - \frac{7}{2} \lambda_1^2 - 7L_n^2 \right) \int_\Omega u^4 - \left( \mu_2 - \frac{7}{2} \lambda_2^2 - 7L_n^2 \right) \int_\Omega |\nabla v|^4 - \left( \mu_2 - \frac{7}{2} \lambda_2^2 - 7L_n^2 \right) \int_\Omega w^4
\]

Thus we have

\[
y'(t) + y(t) \leq (6\mu_1 + 3) \int_\Omega u^2 + (6\mu_2 + 3) \int_\Omega w^2.
\]

In view of (13) and Lemma 2.3 we have \(y(t) \leq c_1\), which proves (25) and (26). \(\square\)

Combining with the previous results, now we can prove Theorem 2.2.

Proof. (of Theorem 2.2) Considering the first equation of (3), it follows from \(\mu_1, a_1, w > 0\) and Young’s inequality that

\[
u_t \leq \Delta u - \chi_1 \nabla \cdot (u\nabla v) + \mu_1 u(1 - u) \leq \Delta u - \chi_1 \nabla \cdot (u\nabla v) + \frac{\mu_1}{4}
\]

in \(\Omega \times (0, T')\). In view of the order-preserving property of Neumann heat semigroup \(e^{t\Delta}\), we see that

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq f_1(t) + f_2(t) + f_3(t),
\]

where

\[
f_1(t) = \|e^{(t-t_0)\Delta}u(\cdot, t_0)\|_{L^\infty(\Omega)},
\]

\[
f_2(t) = - \chi_1 \int_{t_0}^t \|e^{(t-s)\Delta} \nabla \cdot (u(\cdot, s)\nabla v(\cdot, s))\|_{L^\infty(\Omega)} ds,
\]

\[
f_3(t) = \frac{\mu_1}{4} \int_{t_0}^t \|e^{(t-s)\Delta}\|_{L^\infty(\Omega)} ds
\]

with fixed \(t_0 \in [0, t]\). Invoking the well-known smoothing estimate of \(e^{t\Delta}\), we have \(f_3(t) \leq \frac{\mu_1}{4}(t - t_0)\). Moreover, there exists \(m_1 > 0\) such that \(f_1(t) \leq m_1 C_1 (1 + (t - t_0)^{-1})\), where \(C_1\) is the constant given by Lemma 2.4. From the Hölder inequality we see that there exists \(m_2 > 0\) such that

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq m_2 \text{ for all } t \in (0, T_{max}).
\]

According to Lemma 1.3 in [32], for \(p > 3\) and suitably large \(m_3, m_4\) we have

\[
f_2(t) \leq m_3 (t - s)^{-1/p} \|e^{\frac{p-2}{2} \Delta} \nabla \cdot (u(\cdot, s) \nabla v(\cdot, s))\|_{L^p(\Omega)}
\]

\[
\leq m_4 (t - s)^{-1/p} \|u(\cdot, s) \nabla v(\cdot, s)\|
\]

\[
\leq m_4 m_2 (t - s)^{-\frac{3}{2}}
\]

for all \(s \in [t_0, t]\) with \(s \geq t - 1\). Thus we conclude from (33) that if \(t_0 \in [0, t]\) satisfies \(t_0 \geq t - 1\), then

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq m_5 \{(t - t_0)^{-1/2} + (t - t_0)^{-5/6} + (t - t_0)\}
\]

(34)
with the constant $m_5 > 0$. We can choose $t_0 = \max\{0, t-1\}$ in (34) and then obtain

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq m_6(1 + t^{-1})$$

for all $t \in (0, T_{\text{max}})$.

Similarly we can obtain the boundedness of $w$ in $\Omega \times (0, T_{\text{max}})$. In view of (10), this implies that $T_{\text{max}} = \infty$ and $u, w$ are bounded in $\Omega \times (0, \infty)$. By applying the standard parabolic theory to the second and fourth equations in (3), we see that $v, z$ are bounded in $\Omega \times (0, \infty)$. Thus the proof of Theorem 2.2 is completed. 

3. Global attractivity. As a preparation, we first establish higher regularity of the solution.

Lemma 3.1. Let $(u, v, w, z)$ be a global bounded solution of the boundary value problem (3)-(5). Then

$$\|u\|_{C^0(\overline{\Omega} \times [t, t+1])} + \|v\|_{C^0(\overline{\Omega} \times [t, t+1])} + \|w\|_{C^0(\overline{\Omega} \times [t, t+1])} + \|z\|_{C^0(\overline{\Omega} \times [t, t+1])} \leq C$$

for all $t > 1$ and some $\theta \in (0, 1), C > 0$.

The proof of Lemma 3.1 is similar to [1] and hence is omitted. To obtain some weak convergence information, we need to construct a Lyapunov functional and also need to establish the following differential inequality as preparation.

Lemma 3.2. Assume that $u_0 \neq 0 \neq w_0$, then for all $t > 0$,

$$\frac{d}{dt} \int_{\Omega} \ln u \geq -\frac{\chi_1^2}{4} \int_{\Omega} |\nabla v|^2 + \mu_1(1 - u - a_1 w)$$

(36)

and

$$\frac{d}{dt} \int_{\Omega} \ln w \geq -\frac{\chi_2^2}{4} \int_{\Omega} |\nabla z|^2 + \mu_2(1 - w - a_2 u).$$

(37)

Proof. According to the assumption that $u_0 \neq 0$ and the strong maximum principle, we have $u > 0$ in $\overline{\Omega} \times (0, \infty)$. By testing the first equation of (3) against $\frac{1}{u}$ and integrating by parts, we obtain

$$\frac{d}{dt} \int_{\Omega} \ln u = \int_{\Omega} \frac{|\nabla u|^2}{u^2} - \chi_1 \int_{\Omega} \frac{\nabla u \cdot \nabla v}{u} + \mu_1(1 - u - a_1 w)$$

for all $t > 0$. A direct application of Young’s inequality to the above formula implies (36). We obtain (37) by applying similar steps to $w$. 

3.1. Case $a_1 < 1$ and $a_2 < 1$. We shall see that $(u_*, v_*, w_*, z_*)$ is attractive to all nontrivial nonnegative solutions of (3)-(5) if the parameter coefficients $L_h$, $\chi_1$, and $\chi_2$ are large enough and satisfy (7). For this purpose, we start with some preparations.

Lemma 3.3.

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} (v - v_*)^2 \leq - \int_{\Omega} |\nabla v|^2 - \frac{1}{2} \int_{\Omega} (v - v_*)^2 + \frac{L_h^2}{2} \int_{\Omega} (w - w_*)^2$$

(38)

and

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} (z - z_*)^2 \leq - \int_{\Omega} |\nabla z|^2 - \frac{1}{2} \int_{\Omega} (z - z_*)^2 + \frac{L_h^2}{2} \int_{\Omega} (u - u_*)^2$$

(39)

for all $t > 0$. 

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Proof. By testing the second equation of (3) against \((v - v_*)\), we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} (v - v_*)^2 = -\int_{\Omega} |\nabla v|^2 - \int_{\Omega} (v - v_*)(v - h(w))
\]
\[
= -\int_{\Omega} |\nabla v|^2 - \int_{\Omega} (v - v_*)^2 + \int_{\Omega} (v - v_*)(h(w) - h(w_*))
\]
for all \(t > 0\). According to the assumption of \(h\) in (4) and Cauchy-Schwarz inequality we have
\[
\int_{\Omega} (v - v_*)(h(w) - h(w_*)) \leq \frac{1}{2} \int_{\Omega} (v - v_*)^2 + \frac{1}{2} \int_{\Omega} (h(w) - h(w_*))^2
\]
\[
\leq \frac{1}{2} \int_{\Omega} (v - v_*)^2 + \frac{L^2_h}{2} \int_{\Omega} (w - w_*)^2
\]
for all \(t > 0\), which yields (38). By applying similar steps to \(z\) we obtain (39).

Next we construct a Lyapunov functional to obtain the weak convergence.

**Lemma 3.4.** Let \((u, v, w, z)\) be a global bounded classical solution of (3) with \(u \neq 0 \neq w\). Assume that \(0 < a_1, a_2 < 1, \mu_1, \mu_2, \chi_1\) and \(\chi_2\) satisfy (7). Then there exist \(\delta_1 > 0, \delta_2 > 0\) and \(\varepsilon > 0\) such that the functions \(E_1\) and \(F_1\) defined by
\[
E_1(t) \triangleq \int_{\Omega} \left[ u(\cdot, t) - u_* - u_* \ln \left( \frac{u(\cdot, t)}{u_*} \right) \right] + \frac{\mu_1 a_1}{\mu_2 a_2} \int_{\Omega} \left[ \frac{1}{w_*} \right] \left[ \frac{w(\cdot, t) - w_* - w_* \ln \left( \frac{w(\cdot, t)}{w_*} \right)}{w_*} \right] + \frac{\delta_1}{2} \int_{\Omega} (v(\cdot, t) - v_*)^2 + \frac{\delta_2}{2} \int_{\Omega} (z(\cdot, t) - z_*)^2
\]
(41)
and
\[
F_1(t) \triangleq \int_{\Omega} (u(\cdot, t) - u_*)^2 + \int_{\Omega} (v(\cdot, t) - v_*)^2 + \int_{\Omega} (w(\cdot, t) - w_*)^2 + \int_{\Omega} (z(\cdot, t) - z_*)^2
\]
(42)
for \(t > 0\) satisfy
\[
E_1(t) \geq 0
\]
(43)
and
\[
\frac{d}{dt} E_1(t) \leq -\varepsilon F_1(t)
\]
(44)
for all \(t > 0\), where \((u_*, v_*, w_*, z_*)\) is given by (8).

**Proof.** According to (7) and (8), we see that there exist \(\delta_1, \delta_2 > 0\) such that \(\delta_1 \in I_1\), and \(\delta_2 \in I_2\), where
\[
I_1 \triangleq \left( \frac{\chi^2_1 u_*}{4}, \frac{2\mu_1 a_1}{a_2 L^2_h} (1 - a_2) \right), \quad I_2 \triangleq \left( \frac{\mu_1 a_1 \chi^2_2 w_*}{4\mu_2 a_2}, \frac{2\mu_1 (1 - a_1)}{L^2_h} \right).
\]
With these fixed \(\delta_1\) and \(\delta_2\), we shall study the property of the function \(E_1(t)\) defined in (41). Set
\[
H(s) \triangleq s - u_* - u_* \ln \frac{s}{u_*}.
\]
Obviously, \(H(s) \geq H(u_*) = 0\) for all \(s > 0\). Thus we obtain the nonnegativity of \(E_1(t)\). In view of the differential inequalities established in Lemmas 3.2 and 3.3 we
Theorem 3.5. Assume that\(0 < a_1, a_2 < 1, \mu_1, \mu_2, \chi_1\) and \(\chi_2\) satisfy (7). Then any nonnegative global bounded classical solution \((u, v, w, z)\) of (3)-(5) with \(u \not\equiv 0 \not\equiv w\) have the following property:

\[
\|u(\cdot, t) - u_\ast\|_{L^\infty(\Omega)} + \|v(\cdot, t) - v_\ast\|_{L^\infty(\Omega)} + \|w(\cdot, t) - w_\ast\|_{L^\infty(\Omega)} + \|z(\cdot, t) - z_\ast\|_{L^\infty(\Omega)} \to 0
\]

as \(t \to \infty\).

Proof. From (43) and (44) we infer that

\[
\int_1^\infty F_1(t) \leq \frac{E(1)}{\varepsilon} < \infty.
\]

According to (35), we know that \(u, v, w\) and \(z\) are Hölder continuous, uniformly with respect to \(t > 1\), in \(\Omega \times [t, t + 1]\). Thus by combining the uniform continuity in \((1, \infty)\) with the integrability property of \(F_1(t)\), we have

\[
F_1(t) \to 0 \text{ as } t \to \infty.
\]

By the Gagliardo-Nirenberg inequality there exists \(c_1 > 0\) such that

\[
\|\phi\|_{L^\infty(\Omega)} \leq c_1 \|\phi\|_{W^{1,\infty}(\Omega)}^{n/(n+2)} \|\phi\|_{L^2(\Omega)}^{2/(n+2)}
\]

for all \(\phi \in W^{1,\infty}(\Omega)\). Applying this to \(u(\cdot, t) - u_\ast\) for \(t > 0\), and using the boundedness of \((u(\cdot, t))_{t>1}\) in \(W^{1,\infty}(\Omega)\) established in Lemma 3.1, we conclude from (47) that \(u(\cdot, t) \to u_\ast\) in \(L^\infty(\Omega)\) as \(t \to \infty\). Repeating this argument for \(v, w\) and \(z\) yields (46).
In what follows, we shall find some sufficient conditions ensuring the global asymptotical stability of the unique spatially homogeneous steady state \((u_*, v_*, w_*, z_*)\). In view of the global attractivity discussed above, we only need to analyze its local stability. By applying some abstract stability results based on analytic semigroup theories (see [5, 19]), to get the stability of the spatially homogeneous steady state \((u_*, v_*, w_*, z_*)\), it suffices to prove that the steady state is spectrally stable, i.e., the linearized operator has only eigenvalues with nonnegative real parts (or two solutions) \(\rho\) of conjugate complex numbers with positive real parts. This means system (3) is locally asymptotically stable. Combining with Theorem 3.5, we obtain the following result.

\[
\begin{aligned}
&\quad = \Delta u - \chi_1 u \Delta v - \mu_1 u u - \mu_1 a_1 u w, \\
&v_t = \Delta v - v + h'(w) w, \\
w_t = \Delta w - \chi_2 w \Delta z - \mu_2 w w - \mu_2 a_2 w u, \\
z_t = \Delta z - z + h'(w) u.
\end{aligned}
\]

The corresponding characteristic equation is

\[
\prod_{n=0}^{\infty} \det(\rho I - A_n) = 0,
\]

where

\[
A_n = \begin{bmatrix}
-(\lambda_n + \mu_1 u_*) & \chi_1 u* \lambda_n & -\mu_1 a_1 u_* & 0 \\
0 & -(\lambda_n + 1) & h'(w_*) & 0 \\
-\mu_2 w_0 & 0 & -(\lambda_n + \mu_2 w_0) & \chi_2 w_0 \lambda_n \\
h'(u_*) & 0 & 0 & -(\lambda_n + 1)
\end{bmatrix},
\]

and \(0 = \lambda_1 < \lambda_2 < \cdots\) are the eigenvalues of the operator \(-\Delta\) on \(\Omega\) under the homogeneous Neumann boundary condition.

If \(h'(u_*) = h'(w_*) = 0\) or \(\chi_1 = \chi_2 = 0\), we have

\[
\det(\rho I - A_n) = (\rho + \lambda_n + 1)^2 \left[ (\rho + \lambda_n + \mu_1 u_*)(\rho + \lambda_n + \mu_2 w_0) - \mu_1 a_1 a_2 w_0 u_1 \right] = (\rho + \lambda_n + 1)^2 \left[ (\rho + \lambda_n)^2 + (\mu_1 u_0 + \mu_2 w_0)(\rho + \lambda_n) + (1 - a_1 a_2) \mu_1 \mu_2 w_0 u_1 \right].
\]

Since \(a_1 a_2 < 1\), thus all eigenvalues of \(A_n\) have negative real parts. If one of \(h'(u_*)\), \(h'(w_*)\), \(\chi_1\), and \(\chi_2\) is not equal to zero, however, it’s really a challenge to solve (49) for \(\rho\) and even to analyze the sign of the real parts of \(\rho\). But according to the continuous dependence of eigenvalues with respect to \(h'(u_*)\) and \(h'(w_*)\), there exists a positive constant \(L(\mu_1, \mu_2, \chi_1, \chi_2)\) such that all eigenvalues of \(A_n\) \(n \in \mathbb{N} \cup \{0\}\) have negative real parts when \(L_h < L(\mu_1, \mu_2, \chi_1, \chi_2)\). That is, \((u_*, v_*, w_*, z_*)\) is locally asymptotically stable. Combining with Theorem 3.5, we obtain the following result.

**Theorem 3.6.** Assume that \(0 < a_1, a_2 < 1\), there exists a positive constant \(L(\mu_1, \mu_2, \chi_1, \chi_2)\) such that \((u_*, v_*, w_*, z_*)\) is globally asymptotically stable when \(L_h < L(\mu_1, \mu_2, \chi_1, \chi_2)\).

As either \(h'(u_*)\) or \(h'(w_*)\) increases and passes some critical values, one solution (or two solutions) \(\rho\) of (49) may vary from a negative number (respectively, a pair of conjugate complex numbers with negative real parts) to zero (respectively, a pair of purely imaginary numbers) and then to a positive number (respectively, a pair of conjugate complex numbers with positive real parts). This means system (3)
may undergo steady-state bifurcation (or Hopf bifurcation) and a branch of steady-state solutions (respectively, time-periodic solutions) emerge simultaneously from the endemic equilibrium \((u_*, v_*, w_*, z_*)\).

We end this subsection with an example. Take \(\mu_1 = \mu_2 = \mu, \ a_1 = a_2 = a, \ \chi_1 = \chi_2 = \chi\) and \(h'(u_*) = h'(w_*) = h'\), then we have \(u_* = w_* = \frac{1}{a + 1}\) and hence
\[
A_n \triangleq \begin{bmatrix} -\lambda_n - \mu u_* & \chi u_* h' & -\mu a u_* & 0 \\ -h' & -\lambda_n + 1 & 0 & 0 \\ -\mu a u_* & 0 & -\lambda_n + \mu u_* & \chi u_* h' \\ 0 & 0 & 0 & -\lambda_n + 1 \end{bmatrix}.
\]

By a direct calculation we obtain that all the eigenvalues of \(A_n, n \in \mathbb{N} \cup \{0\}\) have negative real parts provided that
\[
L_h < \frac{(1 + a)(\sqrt{\mu} + 1)^2}{\chi}.
\]

For an instance, take
\[
\Omega = (0, \pi), \quad h(u) = 1.5u, \quad a_1 = a_2 = 0.5, \quad \mu_1 = \mu_2 = 20, \quad \chi_1 = \chi_2 = 2,
\]
then it is easy to see that (52) is satisfied, which implies that the endemic equilibrium \((u_*, v_*, w_*, z_*) = (\frac{\pi}{3}, 1, \frac{\pi}{3}, 1)\) is globally asymptotical stable. Figure 1 describes the solution of model (3) with the following initial condition
\[
u(x, t) = 3.5 + \cos x, \quad v(x, t) = 3.5 + \cos x, \quad w(x, t) = 1 + \cos x, \quad z(x, t) = 3.5 + \sin x
\]
for \(t > 0\), and illustrate that the endemic equilibrium is globally asymptotical stable.
3.2. Case $a_2 < 1 < a_1$. In this case, we shall show that every global solution converges towards the constant stationary solution $(0, h(1), 1, 0)$ when $\mu_2$ is large enough. For this purpose, we introduce the following lemmas.

**Lemma 3.7.**
\[
\frac{d}{dt} \int_\Omega (v - h(1))^2 \leq - \int_\Omega |\nabla v|^2 - \frac{1}{2} \int_\Omega (v - h(1))^2 + \frac{L_h^2}{2} \int_\Omega (w - 1)^2, \quad (55)
\]
and
\[
\frac{d}{dt} \int_\Omega z^2 \leq - \int_\Omega |\nabla z|^2 - \frac{1}{2} \int_\Omega z^2 + \frac{L_h^2}{2} \int_\Omega u^2. \quad (56)
\]

The proof of Lemma 3.7 is similar to that of Lemma 3.3 and so is omitted here.

**Lemma 3.8.** Let $0 < a_2 < 1 < a_1$, and $\mu_1, \chi_1 > 0$. Assume that there exist $a'_1 \in [1, a_1]$ satisfying $a'_1 a_2 < 1$ and $\eta \in (a_2, \frac{1}{a_1})$ such that
\[
\frac{\mu_2}{\chi_2^2} > \frac{a'_1 L_h^2}{8a_2 (1 - a'_1 \eta)}. \quad (57)
\]
Let $(u, v, w, z)$ be a global bounded classical solution of (3) with $w \not\equiv 0$. Then there exist positive constants $\delta_1, \delta_2$ and $\varepsilon$ such that the functions $E_2(t)$ and $F_2(t)$ defined by
\[
E_2(t) \triangleq \int_\Omega u(\cdot, t) + \frac{\delta_1}{2} \int_\Omega (v(\cdot, t) - h(1))^2 + \frac{\mu_1 a'_1}{\mu_2 a_2} \int_\Omega [w(\cdot, t) - 1 - \ln w(\cdot, t)] + \frac{\delta_2}{2} \int_\Omega z^2(\cdot, t) \quad (58)
\]
and
\[
F_2(t) \triangleq \int_\Omega u^2(\cdot, t) + \int_\Omega (v(\cdot, t) - h(1))^2 + \int_\Omega (w(\cdot, t) - 1)^2 + \int_\Omega z^2(\cdot, t) \quad (59)
\]
with $t > 0$ satisfy
\[
E_2(t) \geq 0 \quad (60)
\]
and
\[
\frac{d}{dt} E_2(t) \leq -\varepsilon F_2(t) \quad (61)
\]
for all $t > 0$.

**Proof.** There exists $\delta_1 \in (0, \frac{2\mu_1 a'_1}{a_2 L_h^2} (1 - \frac{a_2}{\eta}))$ due to $\eta > a_2$. According to (57), we choose $\delta_2 \in (\frac{\mu_1 a'_1 \chi_2^2}{4\mu_2 a_2}, \frac{2\mu_1 (1 - a'_1 \eta)}{L_h^2})$. Note that the function $E_2(t)$ defined in (58) depends on the choices of $\delta_1$ and $\delta_2$. By a similar argument in Lemma 3.4 we can easily obtain the nonnegativity of $E_2$. In view of Lemmas 3.2 and 3.7 we have
\[
\frac{d}{dt} E_2(t) = \int_\Omega u_t + \frac{\mu_1 a'_1}{\mu_2 a_2} \int_\Omega w_t - \mu_1 a'_1 \frac{d}{dt} \int_\Omega \ln w + \frac{\delta_1}{2} \int_\Omega (v(\cdot, t) - h(1))^2 + \frac{\delta_2}{2} \int_\Omega z^2(\cdot, t)
\]
\[
\leq -\mu_1 \int_\Omega (w^2 - 2\mu_1 a'_1) \int_\Omega u(w - 1) + \frac{\delta_1 L_h^2}{2} \int_\Omega (w - 1)^2 - \frac{\mu_1 a'_1}{a_2} \int_\Omega (w - 1)^2
\]
Theorem 3.9. Assume that 
\[ \lambda = 0, \]
for all \( t > 0 \). By the decision of \( \alpha_1' \), \( \delta_1 \), \( \delta_2 \), \( \eta \) and evident choice for \( \varepsilon \), we obtain (61).

Now, we can state the global attractivity of the constant stationary solution \((0, h(1), 1, 0)\).

**Theorem 3.9.** Assume that \( a_1 > 1 > a_2 > 0 \), \( \mu_1, \chi_1 > 0 \), and (9) is satisfied. Then any non-negative global bounded classical solution \((u, v, w, z)\) of (3) with \( w \neq 0 \) has the following property
\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t) - h(1)\|_{L^\infty(\Omega)} + \|w(\cdot, t) - 1\|_{L^\infty(\Omega)} + \|z(\cdot, t)\|_{L^\infty(\Omega)} \to 0 \quad (63)
\]
as \( t \to \infty \).

**Proof.** Obviously, the inequity of (9) is a special case of (57) when \( \alpha_1' = 1 \) and \( \eta = \frac{\chi_1}{2} \). Then we complete the proof by using a similar argument to the proof of Theorem 3.5. \( \square \)

Linearizing (3) at \((0, h(1), 1, 0)\) yields
\[
\begin{align*}
u_t &= \Delta u + \mu_1 u - \mu_1 \alpha_1 u, \\
v_t &= \Delta v - v + h'(1)w, \\
w_t &= \Delta w - \chi_2 \Delta z - \mu_2 w - \mu_2 a_2 u, \\
z_t &= \Delta z - z + h'(0)u.
\end{align*}
\]
By a direct calculation, we see that the characteristic equation is as follows
\[
(\rho + \lambda_n + \mu_1 (\alpha_1 - 1)) (\rho + \lambda_n + 1)(\rho + \lambda_n + \mu_2)(\rho + \lambda_n + 1) = 0,
\]
where \( 0 = \lambda_1 < \lambda_2 < \cdots \) are the eigenvalues of the operator \(-\Delta\) on \( \Omega \) under the homogeneous Neumann boundary condition. Obviously, all solutions to (65) are negative. This implies that the constant stationary solution \((0, h(1), 1, 0)\) is locally asymptotically stable. This together with its global attractivity, implies that \((0, h(1), 1, 0)\) is globally asymptotically stable. Namely, we obtain the following result.

**Theorem 3.10.** Assume that \( a_1 > 1 > a_2 > 0 \), \( \mu_1, \chi_1 > 0 \), and the positive parameters \( \mu_2, \chi_2 \) satisfy (9). Then the constant stationary solution \((0, h(1), 1, 0)\) is globally asymptotically stable.

3.3. **Case** \( a_1, a_2 > 1 \). According to (50) and (65), we see that the spatially homogeneous equilibrium \((u_*, v_*, w_*, z_*)\) is unstable. The constant stationary solutions \((0, h(1), 1, 0)\) and \((1, 0, 0, h(1))\) are locally asymptotically stable and each of the two steady-states has a domain of attraction. The matter which species ultimately wins out depends crucially on the starting advantage each species has.
4. Conclusions and discussions. In this paper, we investigate a two-species chemotaxis system with two signals. The goal of the present work is to study the large time behavior for the multidimensional system (3) under the standing assumptions (4)-(5). In Section 2, the existence and boundedness of the global solutions are established. Comparing with the results obtained in [28], we know that the inclusion of logistic growth terms rules out the blow-up regardless of the size of initial masses. In the single-species case, some examples of high-dimensional blow-up phenomena despite logistic-type growth restrictions have been found by Winkler [34] for the space dimension \( n \geq 5 \), but blow-up never occurs when the space dimension \( n \leq 2 \) ([30]). This means that we need to focus on the case where \( n \geq 3 \) to detect the existence of finite-time blow-up. In this paper it is interesting to notice that our assumptions are completely independent of \( n \). Nevertheless, we impose a smallness condition (6) on \( h' \) throughout its entire domain of definition.

It is natural to ask whether this constraint can be weakened by just requiring an appropriate smallness of \( h' \) at large values of its argument, because such saturation effects are known to suppress blow-up effects also in one-species Keller-Segel-type chemotaxis systems; see, for example, [18, 21]. In Section 3, the long time behavior of global bounded solutions in two competition cases is obtained by constructing a suitable Lyapunov energy function. It’s shown that in the weak competition case (i.e., \( 0 < a_1, a_2 < 1 \)), the unique positive spatially homogeneous equilibrium \((u_*, v_*, w_*, z_*)\) is globally asymptotic stable when \( L_h \) is small enough. And in the partially strong competition case (i.e., \( a_1 > 1 > a_2 > 0 \)), the constant stationary solution \((0, h(1), 1, 0)\) is globally asymptotically stable when \( \chi_2 L_h < 2\sqrt{\mu_2 a_2 (1 - a_2)} \). While in the strong competition case (i.e., \( a_1 > 1 > a_2 \)), \((u_*, v_*, w_*, z_*)\) is unstable and the constant stationary solutions \((0, h(1), 1, 0)\) and \((1, 0, 0, h(1))\) are both locally asymptotically stable.

From a biological standpoint, the restrictive conditions of \( h' \) indicate that the growth rate of chemotactic factor can not be too fast. In the weak competition case, the two species finally survive together. In other words, the competition is not aggressive. And in the partially strong competition case, the stronger dimensionless interspecific competition of the \( w \)-species dominates and the other species, \( u \), dies out. In the strong competition case, the matter which species ultimately wins out depends crucially on the starting advantage each species has.

In [20], Murray introduced the corresponding ODE model of (3)

\[
\begin{align*}
  u_t &= u(1 - u - a_1 v), \\
  v_t &= \rho v(1 - v - a_2 u),
\end{align*}
\]

and discussed the stability and instability of the steady states in four cases:

(i) \( a_1 < 1, a_2 < 1 \),
(ii) \( a_1 > 1, a_2 > 1 \),
(iii) \( a_1 < 1 < a_2 \),
(iv) \( a_1 > 1, a_2 < 1 \).

It is observed that \((0, 0)\) is always unstable, \((1, 0)\) is stable (respectively, unstable) if \( a_2 > 1 \) (respectively, \( < 1 \)), \((0, 1)\) is stable (respectively, unstable) if \( a_1 > 1 \) (respectively, \( < 1 \)). And \((u_*, v_*)\) is stable in case (i) and unstable in case (ii).

In our paper, we obtain the global asymptotic stability of the unique positive spatially homogeneous equilibrium \((u_*, v_*, w_*, z_*)\) and the constant stationary solution \((0, h(1), 1, 0)\) in the cases (i) and (iv), respectively. Note that the cases (iii) and (iv) are symmetrical, thus the global asymptotic stability of \((1, 0, 0, h(1))\) is
evident in case (iii). In case (ii), we obtain the local stability of the two semi-trivial steady states \((0, h(1), 1, 0)\) and \((1, 0, 0, h(1))\), and the instability of \((u_*, v_*, u_*, z_*)\) by a routine linearization analysis. Compared with the results in [20], it’s shown that for restrictive chemotaxis coefficients, the stabilization of solutions depends on the dimensionless parameter groupings \(a_1\) and \(a_2\) as well as the initial conditions.

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