Information geometry of $U(1)$ instantons on self-dual manifolds

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Abstract

We use the information metric to investigate the moduli space of a $U(1)$ instanton on (anti)self-dual manifolds, finding an $AdS$ geometry similar to that for the moduli space of a Yang-Mills instanton on flat space. We discuss our results from the perspective of gauge/gravity duality.
1 Introduction

The Anti-de Sitter/conformal field theory conjecture of Maldacena [1] relates gravity on a $d + 1$ dimensional spacetime to a $d$ dimensional gauge theory, thus being a concrete realisation of the holographic idea [2] whereby the gauge theory lives on the boundary of the bulk AdS manifold [3, 4]. This duality has been extensively tested and has provided a fruitful alternative understanding of various aspects of gauge theories, both qualitatively and quantitatively. The large body of work is cited in some reviews, for example [5].

One important aspect of the duality concerns the role of instantons [6] in the boundary gauge theory. It was shown in Ref.[7] that for a $SU(N)$ instanton, in the large $N$ limit, the moduli space equipped with the usual $L^2$-metric was equivalent to the bulk geometry. On the other hand, the authors of Ref.[8] argued that the bulk spacetime corresponding to a boundary $SU(N)$ theory could be identified precisely with the one $SU(2)$-instanton moduli space equipped with an alternative metric. That is, in this latter proposal, one does not need the limits taken in [7] but one does need to adopt the information metric rather than the $L^2$-metric. This is an intriguing result, for it suggests that one may uncover, in a relatively simple way, the higher dimensional holographic bulk geometry corresponding to a boundary theory.

The information metric $G_{AB}$ is commonly used in probability theory and it associates a geometry to probability distributions [9],

$$G_{AB} = \int d^d x \sqrt{g} P(x^\mu; \xi^A)(\partial_A \ln P(x^\mu; \xi^A))(\partial_B \ln P(x^\mu; \xi^A))$$

(1)

where $P(x^\mu; \xi^A)$ is a probability density of $x^\mu$ while $\xi^A$, $A = 1, 2, ..., k$ are some continuous parameters. Hitchin [10] suggested the use of this metric for instanton studies through the identification $P \to F^2$, with $F$ the instanton field strength. The advantage of the information metric over the $L^2$-metric is its manifest gauge invariance and preservation of existent spacetime symmetries.

For the usual $SU(2)$ Yang-Mills instanton in flat space, one has

$$F^2 = \frac{\rho^4}{[(x-x_0)^2 + \rho^2]^4}$$

(2)

and so the information metric is [11, 8]

$$ds^2 = G_{AB}d\xi^A d\xi^B \sim \frac{d\rho^2 + d(x_0^\mu)^2}{\rho^2},$$

(3)

giving the moduli space an $AdS_5$ geometry. Deformations of the flat base space of the gauge theory were studied in [8] giving rise to corrections to the $AdS_5$ geometry. Another type of deformation of the base space that has been studied within this approach is the use of non-commutative space in the limit of small non-commutativity [12]. There have also been studies of the information geometry of supersymmetric theories [13], lower
dimensional field theories [14], and at finite temperature [15]. For earlier applications of
information geometry in statistical physics, see [16].

In this paper we wish to study the information geometry of gauge instantons living on
a manifold which is not a small deformation of flat space. We choose two of the simplest
possibilities for our investigation, a $U(1)$ instanton on a Eguchi-Hanson or Taub-NUT
manifold. In the next two sections we review the pertinent aspects of Ref.[17] and then
obtain the $U(1)$ instanton information geometry. In Section (4) we discuss the holographic
aspects of our results and in the final section we discuss the implications.

2 $U(1)$ instanton on the Eguchi-Hanson manifold

The Euclidean gravity solutions with finite action and a self-dual curvature are manifolds
that are generally classified as gravitational instantons. This is because the gravitational
fields are localised in space, the metric becoming asymptotically locally flat at infinity. In
the case of the Eguchi-Hanson (EH) manifold the metric is [17]

$$ds^2_{EH} = (1 - \frac{a^4}{r^4})^{-1}dr^2 + \frac{r^2}{4}(d\theta^2 + \sin^2 \theta d\phi^2) + \frac{r^2}{4}(1 - \frac{a^4}{r^4})(d\psi + \cos \theta d\phi)^2$$

(4)

where $r^2 = t^2 + x^2 + y^2 + z^2$ and the coordinate ranges are

$$a \leq r \leq \infty, \ 0 \leq \theta \leq \pi, \ 0 \leq \phi \leq 2\pi, \ 0 \leq \psi \leq 2\pi.$$ 

(5)

Note that at $r = a$ the metric is completely regular with the topology of the manifold
being $R^2 \times S^2$. Let us briefly review some aspects of the EH solution. The relevant
spin-connections

$$\omega^1_0 = \omega^2_3 = (1 - \frac{a^4}{r^4})^{1/2}\sigma_x, \ \omega^2_0 = \omega^3_1 = (1 - \frac{a^4}{r^4})^{1/2}\sigma_y, \ \omega^3_0 = \omega^1_2 = (1 + \frac{a^4}{r^4})^{1/2}\sigma_z$$

(6)

are anti-self-dual. The $\sigma_i$’s in the above expressions are the Cartan-Maurer 1-forms and
obey the cyclic relation $d\sigma_x = 2\sigma_y \wedge \sigma_z$ [17]. It then follows that the curvature components
are also anti-self-dual $R^1_0 = R^2_3$, $R^2_0 = R^3_1$, $R^3_0 = R^1_2$.

As shown in [17], a solution of the coupled Einstein-Maxwell field equations exists
where the metric is taken to be the same as above and the 1-form gauge potential is locally

$$A_{(1)} = \frac{a^2}{r^2}\sigma_z = \frac{a^2}{2r^4}(d\psi + \cos \theta d\phi).$$

(7)

Correspondingly one finds that the Maxwell field strength

$$F_{(2)} = \frac{2a^2}{r^4}(e^3 \wedge e^0 + e^1 \wedge e^2)$$

(8)

1The vierbeins are $e^0 = (1 - \frac{a^4}{r^4})^{-1/2}dr, \ e^1 = r\sigma_x, \ e^2 = r\sigma_y, \ e^3 = r(1 - \frac{a^4}{r^4})^{1/2}\sigma_z$. 

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is anti-self-dual. Hence it is harmonic and the Euclidean energy-momentum tensor vanishes. Thus the Einstein equations retain their empty-space form,

$$R_{\mu\nu} = 0.$$  
(9)

Using the above facts, we get for a $U(1)$ instanton

$$(F_{\mu\nu})^2 = \frac{16a^4}{r^8}. \quad (10)$$

Thus we observe $F \sim 1/r^4$ and so the asymptotic behaviour of this $U(1)$ instanton is similar to that of a Yang-Mills instanton on flat Euclidean space.

It is possible to express the $U(1)$ result in a form that is closer to the $SU(2)$ result. To achieve this one makes the following coordinate change [17],

$$\rho^4 = r^4 - a^4 \quad (11)$$

so that the EH metric (4) in the new coordinates becomes

$$ds_{EH}^2 = (1 + \frac{a^4}{\rho^4})^{-\frac{1}{2}}(d\rho^2 + \rho^2\sigma_z^2) + (1 + \frac{a^4}{\rho^4})^{\frac{1}{2}}\frac{\rho^2}{4}(d\theta^2 + \sin^2 \theta d\phi^2) \quad (12)$$

with new coordinate ranges

$$0 \leq \rho \leq \infty, \ 0 \leq \theta \leq \pi, \ 0 \leq \phi \leq 2\pi, \ 0 \leq \psi \leq 2\pi. \quad (13)$$

The gauge potential and field strength are now given by

$$A_{(1)} = \frac{a^2}{\sqrt{\rho^4 + a^4}}\sigma_z,$$

$$F_{(2)} = \frac{2a^2}{\rho^4 + a^4}(e^3 \wedge e^0 + e^1 \wedge e^2). \quad (14)$$

Thus for a $U(1)$ instanton located away from the origin,

$$(F_{\mu\nu})^2 = \frac{16a^4}{(|x - x_0|^4 + a^4)^2} \quad (15)$$

where we have taken $\rho^2 = (x^i - x^i_0)^2 \ (i = 1, 2, 3, 4)$ everywhere including in the background (12). (In the EH background the base space is $R^4$, so we have been able to introduce four new position parameters $x_0^i$.) Equation (15) is strikingly similar to the Yang-Mills gauge invariant field strength in (2), the only difference being the distribution of powers in the denominator.

Now, since the scalar curvature is zero the on-shell Einstein action vanishes, so the entire gravitational contribution to the action comes from the surface term which also vanishes asymptotically [17]. That is, the total gravitational action vanishes,

$$S[g] = 0. \quad (16)$$
However, the Maxwell action is finite,\(^2\)

\[
S[A] = \frac{1}{2.2!} \int d^4x \sqrt{g} (F_{\mu\nu})^2 = \pi^2. \tag{17}
\]

Thus we may take

\[
P(x^\mu; \xi^A) \equiv \frac{1}{\pi^2} F^2 \tag{18}
\]

for our probability measure in computing the information metric, with coordinates \(\xi^A \equiv (\xi^0; \xi^a) = (a; x_0^1, \cdots, x_0^4)\) parametrizing various directions in instanton moduli space. We find

\[
\partial_{\xi^0} \log P = 4 \left( \frac{1}{\xi^0} - \frac{2(\xi^0)^3}{|x - \xi|^4 + (\xi^0)^4} \right)
\]

\[
\partial_{\xi^a} \log P = \frac{8(x - \xi)^2(x - \xi)_{\alpha}}{|x - \xi|^4 + (\xi^0)^4}. \tag{19}
\]

Using (11) and (19) we get the information metric as

\[
d_{\text{ADS}_{EH}}^2 \sim \frac{(d\xi^0)^2 + (d\xi^a)^2}{(\xi^0)^2}, \quad \alpha = 1, 2, 3, 4. \tag{20}
\]

That is, the information geometry corresponding to the moduli space of a \(U(1)\) instanton on a four-dimensional EH manifold is a five-dimensional Euclidean anti-de Sitter space, just as for the \(SU(2)\) instanton on flat space.

### 3 Instantons on Taub-NUT space

We would like to study another example of a \(U(1)\) instanton with self-dual curvature. The four-dimensional Taub-NUT Euclidean geometry is \[18, 19, 17\]

\[
d_{\text{NUT}}^2 = \frac{r + m}{4(r - m)} dr^2 + \frac{(r^2 - m^2)}{4} (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{m^2(r - m)}{r + m} (d\psi + \cos \theta d\phi)^2 \tag{21}
\]

with the coordinates having the ranges

\[
m \leq r \leq \infty, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi, \quad 0 \leq \psi \leq 4\pi. \tag{22}
\]

For large \(r\) the topology of the manifold is \(S^1 \times R^3\), while near \(r = m\), the apparent coordinate singularity which is removable, the topology becomes flat \(R^4\) \[17\]. The gauge potential in the background of (21) is

\[
A_{(1)} = \frac{r - m}{2(r + m)} (d\psi + \cos \theta d\phi). \tag{23}
\]

\(^2\) Note that \(\sqrt{\text{Det}(g)} = \frac{\xi^0}{8} \sin \theta\), and \(\frac{1}{8\pi} \int \sin \theta d\theta d\phi d\psi = 1\).
With that, we get
\[(F_{\mu\nu})^2 = \frac{16}{(r + m)^4}\] (24)
which is normalised as
\[
\frac{1}{4} \int_{m}^{\infty} dr (mr^2 - m^3) (F_{\mu\nu})^2 = 1.
\] (25)

In the Taub-NUT case the radial coordinate \(r\) is actually defined over a three-dimensional Euclidean base. Thus we take the radial coordinate \(r^2 = (x^i - x_0^i)^2\) in the Cartesian coordinates, where \(x_0^i\) \((i = 1, 2, 3)\) are three position variables. Like in the EH case we can define the information probability density as
\[P(x^\mu; \xi^A) \equiv \frac{c}{\pi^2} F^2\] (26)
where \(\xi^A \equiv (\xi^0; \xi^\alpha) = (m; x_0^1, \cdots, x_3^0)\) parametrize various directions in moduli space and \(c\) is an appropriate normalisation constant such that \(\int d^4 \sqrt{\gamma} P = 1\). We now find
\[
\partial_{\xi^0} \log P = -\frac{4}{|x - \xi| + \xi^0}, \quad \partial_{\xi^\alpha} \log P = \frac{4}{(|x - \xi| + \xi^0)} \frac{(x - \xi)_\alpha}{|x - \xi|}.
\] (27)
Using (21) and (27) we get the information metric
\[
ds^2_{ADSNUT} \sim \frac{(d\xi^0)^2 + (d\xi^\alpha)^2}{(\xi^0)^2}, \quad \alpha = 1, 2, 3.
\] (28)
which is a four-dimensional anti-de Sitter space. Note the difference: the information geometry corresponding to the moduli space of a \(U(1)\) instanton on a EH manifold was a five-dimensional anti-de Sitter space (20). Nevertheless the information metric (28) has a constant negative curvature which is striking.

## 4 Holography

Here we wish to investigate the holographic aspects of the information metrics obtained above. Let us consider first the EH space and define the function
\[
\Phi := \frac{u^4}{(|\xi|^4 + u^4)^2}
\] (29)
where we have set \(u \equiv \xi^0\) for simplicity. Then we obtain the following result for the Laplacian defined over the anti-de Sitter space (20)
\[
\nabla_{ADS} \Phi = u^2 [\partial_u \partial_u - \frac{3}{u} \partial_u + \nabla_{\xi^\alpha}] \Phi = \frac{u^4}{(\xi^4 + u^4)^2} \left( (32u^4 - 48\xi^2 u^2)(u^4 - \xi^4) + 32u^4 \xi^4 \right)
\] (30)
where $\nabla_{\xi^\alpha}$ is the Laplacian over flat $\xi^\alpha$ four-space. From this we deduce that $F^2$ may not be interpreted simply as a boundary-to-bulk propagator as was the case with YM instantons in Ref. [8]. However, consider the case of EH manifold in the asymptotic region where $\rho^2 \gg a^2$. In this region the EH metric (12) becomes a locally flat (ALE) space

$$ds^2 \sim d\rho^2 + \rho^2 (\sigma_x^2 + \sigma_y^2 + \sigma_z^2).$$

(31)

We can equivalently consider the small instanton-size limit $a \to 0$ of (12). In the holographic AdS coordinates, $\xi^A$, it would correspond to $u \to 0$ (or $|(x - \xi)|^2 \gg u^2$),\(^3\) so we get from (15)

$$F^2 \sim \frac{u^4}{|x - \xi|^6}$$

(32)

and in this approximation we get the bulk Laplacian equation

$$\nabla_{AdS}(F^2)^p = 4p(4p - 4)(F^2)^p + 8p(8p - 2)\frac{u^2}{\xi^2}(F^2)^p.$$

(33)

So, if we work up to the leading order ($u \to 0$) we can write (33) as

$$\nabla_{AdS}(F^2)^p \approx 4p(4p - 4)(F^2)^p + \mathcal{O}\left(\frac{u^2}{\xi^2}\right).$$

(34)

Also since

$$\lim_{u \to 0} F^2 \sim \delta(x - \xi),$$

(35)

this means that if we concentrate near the AdS boundary, $u \to 0$, then $(F^2)^p$ does indeed act like a boundary-to-bulk propagator for a massive scalar field. Specially for $p = 1$ eq. (34) defines massless scalar field propagation in the bulk, just as in Ref. [8].

In the leading order, we also recover the identity

$$\frac{1}{16} G^{AB} \partial_A \log F^2 \partial_B \log F^2 = 1 + \mathcal{O}\left(\frac{u^2}{\xi^2}\right)$$

(36)

that was noted in Ref. [8].

For Taub-NUT space (21) we shall concentrate near the region $r = m$. One can define a new coordinate patch [17]

$$d\tau = \frac{1}{2} \sqrt{\frac{r + m}{r - m}} dr.$$

Then the metric (21) becomes that of a flat $R^4$ space

$$ds^2 \sim d\tau^2 + \tau^2 (\sigma_x^2 + \sigma_y^2 + \sigma_z^2).$$

(37)

\(^3\)Given our AdS metric (20), $\xi^0 \to 0$ would correspond to the UV region while $\xi^0 \to \infty$ would define the IR limit for the boundary theory. Note that $\xi^0 = 0$ is also the boundary of the AdS space.
Expanding as $r = m + \epsilon$, with $\epsilon \approx \frac{r^2}{2m}$, we also obtain from (24)

$$F^2 \sim \frac{16(2m)^4}{(\tau^2 + (2m)^2)^4}.$$ (38)

The last two equations are the same as for YM instantons (2) in flat space where the information metric is $AdS_5$. So it is obvious that for the Taub-NUT instantons the information metric is $AdS_5$ and holographic when $r \sim m$, but the bulk is $AdS_4$ when $r > m$. That means, the information metric for the Taub-NUT $U(1)$ instantons in the $r \to m$ limit flows from $AdS_4$ space to the $AdS_5$. This should not appear unusual because the Taub-NUT space has Type-IIA string (D6-brane) as well as M-theory Kaluza-Klein monopole interpretation depending upon the size of the fibered $\psi$-direction in (21).

5 Discussion

We have studied two of the simplest examples of gauge theory on nontrivial space. We found, quite remarkably, that for a $U(1)$ instanton on the EH manifold, the information geometry of the moduli space is not flat but instead has constant negative curvature just as for the bulk theories in the usual gauge-gravity correspondence [1]. However our result does not have exactly the same bulk-boundary link that was found for the usual case in [8]. There are two ways to see this. Firstly, the boundary of $AdS$ space is flat whereas our base theory (the supposedly boundary theory) is a curved (anti)self-dual manifold. Secondly, in our case we find that our $F^2$ may not be interpreted simply as a bulk-to-boundary scalar propagator as was the case with YM instantons. However, in the leading order approximation $O(\frac{u^2}{R^2})$ as $u \to 0$, we do recover the holography of Ref.[8]. The same is true for the Taub-NUT case: There also when we work in the region $r \to m$, we get to the usual holographic interpretation.

That is, near the flat regions of the curved manifolds we studied, we do have a holographic interpretation of the bulk information geometry similar to Ref.[3, 4, 8]. The novel aspect of our result is that the gauge fields in our case are abelian in contrast to the Yang-Mills fields in the usual studies.

Our results lead us to the natural suggestion that the idea of Ref.[8] may perhaps be extended: Any gauge theory on a manifold, not just non-abelian ones on nearly flat manifolds, is dual to a gravity theory defined by the information geometry of the instantons of the gauge theory. Admittedly we have no evidence for this, except for the special cases mentioned in Section(1) which fall within the Maldacena framework and the limiting cases mentioned above. The results of the previous section indicate that such an extended duality would need to be more general than the boundary-bulk mapping discussed in Refs. [3, 4, 8], reducing in some limit to the usual flat (or near flat) manifold form.

Finally, we remark that without the Maxwell field in the EH background we do not have an obvious suitable candidate for the probability density in (11) because the pure
gravity sector, although representing a nontrivial manifold, gives zero scalar curvature. But in the pure gravity case perhaps it might be interesting to study some other candidate, such as the density for the Euler characteristic ($\chi$). In this way one might be able to study the information geometry of the gravitational instantons.

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