A Yang–Mills field on the extremal Reissner–Nordström black hole

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Abstract
We consider a spherically symmetric (magnetic) SU(2) Yang–Mills field propagating on the exterior of the extremal Reissner–Nordström black hole. Taking advantage of the conformal symmetry, we reduce the problem to the study of the Yang–Mills equation in a geodesically complete spacetime with two asymptotically flat ends. We prove the existence of infinitely many static solutions (two of which are found in closed form) and determine the spectrum of their linear perturbations and quasinormal modes. Finally, using the hyperboloidal approach to the initial value problem, we describe the process of relaxation to the static endstates of evolution, both stable (for generic initial data) and unstable (for codimension-one initial data).

Keywords: Yang–Mills, Reissner–Nordström, hyperboloidal foliations

(Some figures may appear in colour only in the online journal)

1. Introduction and setup

The global dynamics of a Yang–Mills field propagating in a four-dimensional Minkowski spacetime is well understood: all solutions starting from smooth initial data at $t = 0$ remain smooth for all times [1] and decay to zero as $t \to \pm \infty$ [2, 3]. The global-in-time regularity holds true in any globally hyperbolic four-dimensional curved spacetime [4], however the phase portrait can be richer due to existence of non-trivial stationary solutions which play the role of unstable attractors, as was shown for the Schwarzschild background [5].

In this paper we consider the evolution of a Yang–Mills field on the exterior of the extremal Reissner–Nordström black hole. Our study is motivated by the expectation that an
interplay between the conformal structure of the extremal Reissner–Nordström black hole and the conformal invariance of the Yang–Mills equations may lead to an interesting behavior of solutions. This is a continuation of our studies of how the dissipation-by-dispersion phenomena, responsible for the relaxation to a stationary equilibrium in extended Hamiltonian systems, depend on the geometry of the underlying spacetime [6, 7].

The exterior (i.e., the domain of outer communication) of the extremal Reissner–Nordström black hole is a globally hyperbolic static spacetime \( \mathcal{M} \) whose metric, in coordinates \((\theta, \phi) \in S^2\), reads

\[
\hat{g} = -\left(1 - \frac{M}{r}\right)^2 \, dt^2 + \left(1 - \frac{M}{r}\right)^{-2} \, dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2),
\]

where \( M \) is positive constant. The metric \( \hat{g} \) is a spherically symmetric solution of the Einstein–Maxwell equations with mass \( M \) and charge \( Q = \sqrt{M} \). In order to better see the global properties of this spacetime, it is convenient to use dimensionless variables \((\tau, x) \in \mathbb{R}^2\):

\[
\tau = \frac{t}{4M}, \quad \text{and} \quad x = \ln\left(\frac{r}{M} - 1\right),
\]

in terms of which the metric (1) takes the form

\[
\hat{g} = \frac{16M^2}{(1 + e^{-\tau})^2} \, g,
\]

where

\[
g = -dt^2 + C^4 \, (dx^2 + d\theta^2 + \sin^2\theta \, d\phi^2).
\]

Hereafter, for typographical convenience we use the abbreviation \( C = C(x) = \cosh \frac{x}{2} \). We note that the Ricci scalar vanishes both for \( \hat{g} \) and \( g \). In contrast to \((\mathcal{M}, \hat{g})\), the spacetime \((\mathcal{M}, g)\) is geodesically complete. It has two asymptotically flat ends at \( x = \pm \infty \) (see figure 1). Asymptotic flatness is easily seen in terms of the coordinate \( \rho = C^2(x) \) for which we have

\[
g = -dt^2 + \left(1 - \frac{1}{\rho}\right)^{-1} \, d\rho^2 + \rho^2 (d\theta^2 + \sin^2\theta \, d\phi^2).
\]

Note that the reflection \( x \mapsto -x \) is the isometry of the metric \( g \) but only the conformal isometry for the metric \( \hat{g} \) [8]. On \((\mathcal{M}, g)\) the reflection interchanges the ‘left’ and the ‘right’ future null infinities, \( \mathcal{J}^+_{\text{L}} \) and \( \mathcal{J}^+_{\text{R}} \), while on \((\mathcal{M}, \hat{g})\) it interchanges the event horizon and \( \mathcal{J}^+_{\text{R}} \).

We consider an \( SU(2) \) Yang–Mills field propagating in the spacetime \((\mathcal{M}, g)\). The gauge potential \( A_{\mu} = A^a_{\mu} \tau_a \) takes values in the Lie algebra \( su(2) \), where the generators \( \tau_a \) satisfy \([\tau_a, \tau_b] = i\epsilon_{abc} \tau_c\). In terms of the Yang–Mills field strength \( F_{\mu\nu} = \nabla_{\mu} A_{\nu} - \nabla_{\nu} A_{\mu} + [A_{\mu}, A_{\nu}]\), the Lagrangian density reads

\[
\mathcal{L} = \text{Tr} (F_{\mu\beta} F^{\alpha\beta} g^{\alpha\nu} g^{\beta\rho}) \sqrt{-\det(g_{\mu\nu})}.
\]

In four-dimensions the quantity \( g^{\alpha\nu} g^{\beta\rho} \sqrt{-\det(g_{\mu\nu})} \) is invariant under a conformal transformation \( g_{\mu\nu} \mapsto \Omega^2 g_{\mu\nu} \), hence if \( A_{\mu} \) solves the Yang–Mills equations in one metric, so does it in any conformally related metric. Taking advantage of this conformal invariance of the Yang–Mills equations, in the following we consider only the spacetime \((\mathcal{M}, g)\).

For the Yang–Mills potential we assume the spherically symmetric purely magnetic ansatz
which gives
\[
F = \partial_\tau W d\tau \wedge \omega + \partial_x W dx \wedge \omega - (1 - W^2) \tau_2 \, d\theta \wedge \sin \theta \, d\varphi.
\] (8)
Note that the vacuum is given by \(W = \pm 1\), while \(W = 0\) corresponds to the magnetic monopole with unit charge. Inserting the ansatz (8) into (6) we get the reduced Lagrangian density
\[
\mathcal{L} = -\frac{1}{2} (\partial_\tau W)^2 C^2 + \frac{1}{2} (\partial_x W)^2 C^{-2} + \frac{1}{4} (1 - W^2)^2 C^{-2},
\] (9)
and the corresponding Euler–Lagrange equation
\[
\partial_\tau W = C^{-2} \partial_x (C^{-2} \partial_x W) + C^{-4} W (1 - W^2).
\] (10)
We know from Chruściel and Shatah [4] that solutions of equation (10) starting at \(\tau = 0\) from smooth initial data remain smooth for all future times. The goal of this paper is to describe their asymptotic behavior for \(\tau \rightarrow \infty\). For physical reasons our analysis is restricted to solutions with finite (conserved) energy
\[
E = \frac{1}{2} \int_{-\infty}^{\infty} \left[ C^2 (\partial_\tau W)^2 + C^{-2} \left( (\partial_x W)^2 + \frac{(1 - W^2)^2}{2} \right) \right] dx < \infty.
\] (11)
Due to dissipation of energy by dispersion, such solutions are expected to settle down to critical points of the potential energy, i.e. static solutions of equation (10).

2. Static solutions

Time-independent solutions \(W = W(x)\) of equation (10) satisfy the ordinary differential equation
\[
W'' - \tanh \left( \frac{x}{2} \right) W' + W (1 - W^2) = 0.
\] (12)
We claim that, besides the constant solution \(W_0 = 1\), equation (12) has a countable family of smooth finite energy solutions \(W_n(x)\) \((n \in \mathbb{N})\) with the following properties (which, not very surprisingly, bear remarkable similarities to the Bartnik–McKinnon solutions of Einstein–Yang–Mills equations [9]):

\[Figure 1.\] Penrose diagram for \((M, g)\).
• $W_n(x)$ has $n$ zeros,
• $|W_n(x)| < 1$ for all finite $x$ and $\lim_{|x|\to\infty}|W_n(x)| = 1$,
• $W_n(x)$ is an even (respectively odd) function for even (respectively odd) $n$,
• As $n \to \infty$, $W_n(x)$ tend pointwise to $W_\infty = 0$ for any finite $x$.

The proof of existence of solutions $W_n$ and their properties is a straightforward adaptation of the proof given in [10] in the case of harmonic maps between 3-spheres (which satisfy the same equation as (12) with the nonlinearity $\sin(2W)$ instead of $W(1 - W^2)$). Key to the proof is the fact that equation (12) is asymptotically autonomous with the limiting equations for $x \to \pm \infty$, $W'' + W' + W(1 - W^2) = 0$, having saddle points at $W = \pm 1$ and a spiral at $W = 0$ (stable at $-\infty$ and unstable at $+\infty$). Using a shooting method one can show that there exist infinitely many homoclinic and heteroclinic orbits connecting the saddle points. The solutions are parametrized by the coefficients of the stable directions of the saddle points

$$W_n(x) = 1 - a_n e^{-x} + \mathcal{O}(e^{-3x}).$$

We refer the interested reader to [10] for the details. Note that due to the reflection symmetry $W \mapsto -W$, each solution $W_n$ has a copy $-W_n$. We adopt the convention that $W_n(\infty) = 1$. The parameters and energies of the first few solutions are given in table 1 and their profiles are depicted in figure 2.

Rather unexpectedly, with the help of Maple, we found the first two non-trivial solutions in closed form

$$W_1(x) = \tanh \left( \frac{x}{2} \right), \quad W_2(x) = \frac{2 \cosh x - 2 - \sqrt{6}}{2 \cosh x + 4 + 3 \sqrt{6}}. \tag{14}$$

We turn now to the linear stability analysis of the static solutions $W_n(x)$. This analysis is essential in understanding the role these solutions may play in the evolution. Following the standard procedure, we substitute $W(\tau, x) = W_n(x) + \delta W(\tau, x)$ into equation (10), linearize and separate the time dependence $\delta W(\tau, x) = e^{\lambda \tau} v(x)$. This yields the eigenvalue problem

$$L_n v := -\frac{1}{C^2} \frac{d}{dx} \left( \frac{1}{C^2} \frac{dv}{dx} \right) + \frac{3W_n^2(x) - 1}{C^4(x)} v = -\lambda^2 v. \tag{15}$$

Note that due to the reflection symmetry $x \mapsto -x$, the eigenfunctions are alternately even and odd. We denote them by $v^\text{even}(x)$ and the corresponding eigenvalues by $-(\lambda_{k}^{\text{even}})^2$ ($k \in \mathbb{N}$). We claim that the operator $L_n$ has exactly $n$ negative eigenvalues. For $n = 0$ this is evident because the potential in (15) is everywhere positive. For $n = \infty$ this follows from the fact that the $\lambda = 0$ solution is oscillating at infinity. For any finite $n \geq 1$, one can obtain a lower bound as follows. Consider the function $u_n(x) := W_n'(x)$. Differentiating equation (12), one finds that

$$\tilde{L}_n u_n = 0, \quad \text{where} \quad \tilde{L}_n = L_n + \frac{1}{2C^6(x)}. \tag{16}$$

By construction $u_n(x)$ has $(n - 1)$ zeros, hence from the Sturm oscillation theorem it follows that the operator $\tilde{L}_n$ has exactly $(n - 1)$ negative eigenvalues. Consequently, the operator $L_n$, which is the exponentially localized negative perturbation of $\tilde{L}_n$, has a least $(n - 1) + 1 = n$ negative eigenvalues (where $'+1'$ stands for the zero eigenvalue of $\tilde{L}_n$ going negative.

Numerics (see table 2) shows that this lower bound is sharp but it seems hard to prove this

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4 We point out in passing that in terms of the variable $y = \frac{1}{2}(\sinh x + x)$, defined by $dy/dx = C^2(x)$, equation (15) takes the form of the one-dimensional Schrödinger equation $\frac{d^2v}{dy^2} + V_n(x(y))v(y) = -\lambda^2 v(y)$, however this form is not very helpful because the function $x(y)$ is given only implicitly.
fact rigorously for general \( n \). In the case \( n = 1 \), non-existence of the second negative eigenvalue follows by the Sturm oscillation theorem from an easy to prove fact that the odd solution of equation (15) with \( \lambda = 0 \) is monotone.

### 3. Hyperboloidal formulation

We will use the method of hyperboloidal foliations and Scri-fixing as developed by Zenginoğlu [11] on the basis of concepts introduced by Friedrich [12]. To implement this method we define a new time coordinate

\[
s = \tau - \frac{1}{2} (\cosh x + \ln(2\cosh x))
\]

and foliate the spacetime by hyperboloidal hypersurfaces \( \Sigma_s \) of constant \( s \). These are spacelike hypersurfaces that approach the ‘left’ future null infinity along outgoing null cones of constant advanced time \( \nu = \tau + \frac{1}{2} (\sinh x + x) \) and the ‘right’ future null infinity along outgoing null cones of constant retarded time \( \mu = \tau - \frac{1}{2} (\sinh x + x) \). To see this, note that

![Figure 2. The first four odd and even static solutions.](image.png)

| \( n \) | \( \alpha_n \) | \( \beta_n \) | \( \gamma_n \) | \( \delta_n \) |
|-------|-------|-------|-------|-------|
| 1     | 2.0   | 0.8   | 0.54089 |       |
| 2     | 15.798 | 0.9664 | 0.69937 | 0.17161 |
| 3     | 101.108 | 0.9945 | 0.72553 | 0.21010 | 0.033792 |
| 4     | 624.538 | 0.9991 | 0.72934 | 0.21772 | 0.040983 | 0.0057005 |
| 5     | 3835.14 | 0.99985 | 0.73015 | 0.21884 | 0.042775 | 0.0072103 |
| 6     | 23528 | 0.999976 |       |       |       |     |

![Table 1. The parameters of the first few static solutions \( W_n(x) \).](image.png)

| \( n \) | \( \lambda_1^{(0)} \) | \( \lambda_2^{(0)} \) | \( \lambda_3^{(0)} \) | \( \lambda_4^{(0)} \) |
|-------|-------|-------|-------|-------|
| 1     | 0.54089 |       |       |       |
| 2     | 0.69937 | 0.17161 |       |       |
| 3     | 0.72553 | 0.21010 | 0.033792 |       |
| 4     | 0.72934 | 0.21772 | 0.040983 | 0.0057005 |
| \( \infty \) | 0.73015 | 0.21884 | 0.042775 | 0.0072103 |

![Table 2. Lyapunov exponents of the unstable modes of \( W_n \).](image.png)
\[ (\partial_{s}s)(\partial_{s}s)g^{\alpha\beta} = \frac{1}{\cosh^2 x} \rightarrow 0 \quad \text{as} \quad |x| \rightarrow \infty, \]

and

\[ v_{1a} = s + \frac{1}{2} (\cosh x + \ln(2 \cosh x)) + \frac{1}{2} (\sinh x + x) \rightarrow s \quad \text{as} \quad x \rightarrow -\infty, \]
\[ u_{1a} = s + \frac{1}{2} (\cosh x + \ln(2 \cosh x)) - \frac{1}{2} (\sinh x + x) \rightarrow s \quad \text{as} \quad x \rightarrow \infty. \]

There is plenty of freedom in choosing a hyperboloidal foliation; the particular choice (17) is motivated by computational convenience. In terms of the coordinates \((s, x)\), equation (10) takes the form

\[ \frac{C^2}{\cosh^2 x} \partial_{s}W + 2 \tanh x \partial_{s}W + \frac{1}{\cosh^2 x} \partial_{s}W = \partial_{s}(C^{-2} \partial_{s}W) + C^{-2}W(1 - W^2). \quad (18) \]

Next, we compactify the real line \(-\infty < x < \infty\) to the interval \([-1, 1]\) by the coordinate transformation \(z = \tanh \frac{x}{2}\). This fixes \(J^+_1\) at \(z = -1\) and \(J^+_R\) at \(z = 1\). Equation (18) now becomes

\[ \frac{1}{(1 + z^2)^2} \partial_{s}W + \frac{2z}{1 + z^2} \partial_{z}W + \frac{1 - z^2}{(1 + z^2)^2} \partial_{z}W = \partial_{z}\left(\frac{(1 - z^2)^2}{4} \partial_{z}W\right) + W(1 - W^2). \quad (19) \]

Multiplying this equation by \(\partial_{z}W\) we obtain the local conservation law

\[ \partial_{s}e + \partial_{s}f = 0, \quad (20) \]

where

\[ e(s, z) = \frac{1}{(1 + z^2)^2} (\partial_{z}W)^2 + \frac{(1 - z^2)^2}{4} (\partial_{z}W)^2 + \frac{(1 - W^2)^2}{2}, \]
\[ f(s, z) = \frac{(1 - z^2)^2}{2} \partial_{z}W \partial_{z}W - \frac{2z}{1 + z^2} (\partial_{z}W)^2. \]

Integrating (20) over a hypersurface \(\Sigma\), we get the energy balance

\[ \frac{dE}{ds} = -\left(\partial_{s}W(s, 1)\right)^2 - \left(\partial_{s}W(s, -1)\right)^2, \quad (21) \]

where

\[ E(s) = \int_{-1}^{1} e(s, z) \, dz \quad (22) \]

is the Bondi-type energy. The formula (21) expresses the radiative loss of energy through the future null infinities. Since the energy \(E(s)\) is positive and monotone decreasing, it has a non-negative limit for \(s \rightarrow \infty\). For this reason the hyperboloidal formulation is very natural in analyzing relaxation processes that are due to the dispersive dissipation of energy. In the remainder of the paper we describe in detail the convergence to one of the static solutions \(W_n(z)\). We focus our attention on the first two static solutions \(W_0\) and \(W_1\) because, as follows

\[ ^5 \text{We slightly abuse notation and use the same letter for the function \(W(\tau, x)\) and any other function obtained from it by changing the variables.} \]
from the linear stability analysis in section 2, only these solutions may participate in the evolution of generic and codimension-one initial data.

We return now to the linear perturbation analysis and compute the quasinormal modes for \( W_0 \) and \( W_1 \). Substituting

\[
W(s, z) = W_0(z) + (1 + z^2)^{1/2}e^{\lambda z}u(z)
\]

into (19) and linearizing, we obtain the quadratic eigenvalue problem (the purpose of the factor \((1 + z^2)^{1/2}\) is to simplify the resulting equation)

\[
(1 - z^2)^2 u'' + 2z(\lambda(z^2 - 3) + 2z^2 + 2)u' + (\lambda^2(z^2 - 4) + 3\lambda z^2 - 1) + 4 - 12W_0^2)u = 0.
\]  

(23)

We shall compute the quasinormal modes for the first two static solutions \( W = W_0 \) and \( W = W_1 \) using Leaver’s method\(^6\) [14]. We seek solutions in the form of a power series around the ordinary point \( z = 0 \)

\[
u(z) = \sum_{j=0}^\infty a_j z^j.
\]  

(24)

This series converges for \( |z| < 1 \). The discrete values of \( \lambda \) for which the function defined by the series (24) is analytic at \( z = \pm 1 \) correspond to the eigenvalues (for real \( \lambda > 0 \) and quasinormal modes (for \( \Im(\lambda) < 0 \)). To find those values we need to determine the asymptotics of the coefficients \( a_j \) for large \( j \). To this end we substitute the series (24) into equation (23) for \( n = 0, 1 \) and get the three-term recurrence relation

\[
\begin{align*}
\alpha_0 a_2 + \beta_0 a_0 &= 0, \\
\alpha_1 a_3 + \beta_1 a_1 &= 0, \\
\alpha_j a_{j+2} + \beta_j a_j + \gamma_j a_{j-2} &= 0, \quad j \geq 2
\end{align*}
\]  

(25)

with

\[
\begin{align*}
\alpha_j &= j^2 + 3j + 2, \\
\beta_j &= -2j^2 - (6\lambda + 2)j - 4\lambda^2 - 3\lambda + 4 - 12b_0^2, \\
\gamma_j &= j^2 + (2\lambda - 1)j + \lambda^2 - \lambda - 2 - 12b_1^2,
\end{align*}
\]

where \( b_0 \) and \( b_1 \) are the first two coefficients of the Taylor expansion of \( W_0(z) \) around \( z = 0 \). For \( n = 0 \) we have \( b_0 = 1, b_1 = 0 \) and for \( n = 1 \) we have \( b_0 = 0, b_1 = 1 \). Even modes satisfy the boundary condition \( u(0) = 1, u'(0) = 0, \) hence \( a_0 = 1 \) and \( a_1 = 0 \), while for odd modes we have \( u(0) = 0, u'(0) = 1, \) hence \( a_0 = 0 \) and \( a_1 = 1 \).

The recurrence relation (25) has two linearly independent asymptotic solutions for \( j \to \infty \) (in the theory of finite difference equations, such solutions are called Birkhoff’s solutions, see chapter 8.6 in [16])

\[
a_j^\pm \sim j^\beta \exp(\pm 2\sqrt{\lambda} \sqrt{j}), \quad \text{where} \quad \beta = \frac{\lambda}{2} - \frac{3}{4},
\]  

(26)

(the leading order behavior does not depend on \( n \)), thus asymptotically

\[
a_j \sim c_+(\lambda)a_j^+ + c_-(\lambda)a_j^-.
\]  

(27)

\(^6\) An alternative (black box) method is to exploit the fact that equation (23) for \( n = 0, 1 \) has the form of a double confluent Heun equation and get Maple to do the rest of the job [6, 15], however we prefer a more transparent approach.
The series (24) converges at \( z = \pm 1 \) if and only if the coefficient of an exponentially growing term in (27) vanishes. The corresponding solution is then called a minimal solution of the recurrence relation and denoted \( a_j^{\text{min}} \). To find the minimal solution we define

\[
A_j = \frac{\beta_j}{\alpha_j}, \quad B_j = \frac{\gamma_j}{\alpha_j}, \quad r_j = \frac{a_j}{a_{j-2}},
\]

and rewrite (25) in the form

\[
r_j = -\frac{B_j}{A_j + r_{j+2}}.
\]

Iterating this formula we get the continued fraction representation

\[
r_j = -\frac{B_j}{A_j - \frac{B_{j+2}}{A_{j+2} - \frac{B_{j+4}}{A_{j+4} - \ldots}}}.
\]

According to Pincherle’s theorem [16], this continued fraction converges if and only if the recurrence relation (25) has a minimal solution \( a_j^{\text{min}} \) and then \( r_j = a_j^{\text{min}}/a_{j-2}^{\text{min}} \) for each \( j \geq 2 \). Using Pincherle’s theorem we obtain the following quantization conditions

\[
\frac{a_2}{a_0} = -\frac{B_0}{A_0} - \frac{B_2}{A_2} - \frac{B_4}{A_4} - \ldots \quad \text{for even modes},
\]

\[
\frac{a_3}{a_1} = -\frac{B_1}{A_1} - \frac{B_3}{A_3} - \frac{B_5}{A_5} - \ldots \quad \text{for odd modes}.
\]

The continued fractions above can be computed to any desired precision by downward recursion starting from some large \( j_{\text{max}} \) and an arbitrary initial value \( r_{j_{\text{max}}} \). Finally, the roots of the quantization conditions (30) and (31) are determined numerically (see table 3).

4. Numerical results

Following [6, 13], we define the auxiliary variables

\[
\Psi = \frac{1}{2} \partial_t W, \quad \text{and} \quad \Pi = \frac{1}{(1 + z^2)^2} \partial_z W + \frac{z}{1 + z^2} \partial_t W,
\]

and rewrite equation (19) as the first order symmetric hyperbolic system

\[
\partial_j W = (1 + z^2)^2 \Pi - 2z(1 + z^2)\Psi,
\]

\[
\partial_j \Psi = \partial_t \left( \frac{1}{2} (1 + z^2)^2 \Pi - z(1 + z^2)\Psi \right),
\]

\[
\partial_j \Pi = \partial_t \left( \frac{1}{2} (1 + z^2)^2 \Psi - z(1 + z^2)\Pi \right) + W(1 - W^2).
\]
We solve this system numerically using the method of lines with an 8th-order finite difference scheme in space and 4th-order Runge-Kutta integration in time. At the boundaries we use one-sided stencils. Note that there are no ingoing characteristics at the boundaries, hence no boundary conditions need, or can, be imposed.

As expected, for generic initial data the solution tends to one of the vacuum solutions $\pm W_0$. For intermediate times we observe ringdown along the fundamental quasinormal mode, while for later times the polynomial tail is seen to dominate (see figure 3).

**Figure 3.** Evolution of time-symmetric initial data $W(0, z) = 1 + \frac{1}{16}\exp(-2z^2)$ at a sample point $z = \frac{1}{2}$. The ringdown to $W_0$ for intermediate times is governed by the quasinormal mode with frequency $\lambda \approx -0.337 + 1.341i$. For late times $|\partial_s W(s, \frac{1}{2})| \sim s^{-5}$.

**Figure 4.** Evolution of time-symmetric initial data $W(0, z) = 10^{-5}\exp(-16 \tanh^2 z)$ at a sample point $z = \frac{1}{2}$. For early times we see the exponential growth along the principal unstable mode around $W_\infty$ with the Lyapunov exponent $\lambda^{\infty} \approx 0.73$. Afterwards, the evolution proceeds as in figure 3.
If initial data are close to an unstable static solution, then for early times we observe an exponentially fast departure from this solution along its principal unstable mode. This behavior, in the case of a small perturbation of $W_\infty$, is shown in figure 4.

Since parity is preserved in evolution, solutions starting from odd initial data cannot tend to $W_0$. Generically, odd solutions converge to $W_1$ (whose single unstable mode is even and therefore not excited). If we add a small even admixture of size $\varepsilon$ to odd initial data, then $W_1$ appears as an intermediate attractor with lifetime $\sim \frac{1}{\lambda_1(1)} \log \left( \frac{1}{\varepsilon} \right)$, where $\lambda_1(1) \approx 0.54$ is the Lyapunov exponent of the unstable mode of $W_1$ (see figure 5).

In all cases, the dynamics in the vicinity of static solutions is in excellent quantitative agreement with the results of linear stability analysis displayed in tables 2 and 3.

Finally, we discuss the late time polynomial tails. In contrast to the ringdown, which is sensitive to the interior structure of the spacetime and depends on the final attractor, the tails for the Yang–Mills field on an asymptotically flat spacetime are universal. They decay as $s^{-4}$ in the interior ($|z| < 1$) and as $s^{-2}$ along future null infinities ($z = \pm 1$) [2, 3]. Moreover, the spatial profiles of tails also appear to be universal. This can be seen by the following heuristic argument. Consider a solution tending asymptotically to $W_0$. Substituting

$$W(s, z) = 1 + s^{-2} f(y), \quad \text{where} \quad y = s(1 - z),$$

into equation (19) and linearizing, we get

$$y(y + 1)f'' + (2y - 1)f' - 2f = 0.$$  \hspace{1cm} (37)

Discarding the growing solution, we obtain

$$f(y) = \frac{A_R}{(1 + y)^2}.$$  \hspace{1cm} (38)
hence for $0 < z \leq 1$ and large $s$

$$W(s, z) - 1 \sim \frac{1}{s^2} \frac{A_R}{(1 + s(1 - z))^2}, \quad (39)$$

where the amplitude $A_R$ is the only trace of initial data. By an analogous argument, for $-1 \leq z < 0$ and large $s$

$$W(s, z) - 1 \sim \frac{1}{s^2} \frac{A_L}{(1 + s(1 + z))^2}. \quad (40)$$

In general, $A_R = A_L$. It follows from (39) that for large $s$

$$\frac{\partial^n W}{\partial z^n}\bigg|_{z=1} \sim A_R (n + 1)! s^{n-2}, \quad (41)$$

hence higher derivatives grow in time. This result can also be derived by inserting the power series expansion near $\mathcal{J}_R^+$,

$$W(s, z) = 1 + \sum_{n=0}^{\infty} c_n(s)(1 - z)^n, \quad (42)$$

into equation (19) and solving the resulting system of ordinary differential equations iteratively starting from $c_0(s) = A_R s^{-2}$. The formula (41) reflects the fact that the decay of the Yang–Mills field is not uniform in space (because the decay along $\mathcal{J}_R^+$ is slower than the decay in the interior). By the conformal invariance of the problem at hand, the formula (41) holds for the derivatives of Yang–Mills field at the future horizon of the extremal Reissner–Nordström black hole. This type of instability for solutions of the wave equation on the extremal Reissner–Nordström black hole was discovered by Aretakis [17, 18] (see also [19, 20] for a discussion of the relationship between the behavior of fields near the future (degenerate) horizon and the future null infinity).

The numerical evidence confirming the profile (39) and the growth of higher derivatives along the future null infinity (41) is shown in figures 6 and 7.
5. Final remarks

In this paper we showed that the conformal isometry of the extremal Reissner–Nordstöm spacetime, combined with the conformal invariance of the Yang–Mills equation, leads to a very simple semilinear dispersive wave equation with surprisingly rich structure of static solutions. We showed that, due to the dissipation of energy by radiation, all smooth finite energy solutions relax asymptotically to a static equilibrium. This result is in the spirit of the celebrated soliton resolution conjecture which asserts that global solutions of dispersive wave equations eventually resolve into a radiation component that disperses plus a coherent component that behaves like a soliton. Our additional motivation was to illustrate the advantages of the hyperboloidal formulation of the initial value problem in studying the dissipation-by-dispersion phenomena. Using this approach, we showed that the convergence to the attractor occurs pointwise on the entire spatial hypersurfaces (the leaves of the hyperboloidal foliation), including the null infinity. Moreover, we defined (and calculated) quasinormal modes as genuine eigenmodes of a certain non-self-adjoint linear operator which has both conceptual and computational advantages over the standard definitions involving outgoing-wave boundary conditions.

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References

[1] Eardley D and Moncrief V 1981 The global existence of Yang–Mills–Higgs fields in 4-dimensional Minkowski space Commun. Math. Phys. 83 171
[2] Christodoulou D 1981 Solutions globales des équations de champ de Yang–Mills C. R. Acad. Sci., Paris A 293 39
[3] Bizoń P, Chmaj T and Rostworowski A 2007 Late-time tails of a Yang–Mills field on Minkowski and Schwarzschild backgrounds Class. Quantum Grav. 24 F55
[4] Chruściel P T and Shatah J 1997 Global existence of solutions of the Yang–Mills equations on globally hyperbolic four-dimensional Lorentzian manifolds Asian J. Math. 1 530
[5] Bizoń P, Rostworowski A and Zenginoğlu A 2010 Saddle-point dynamics of a Yang–Mills field on the exterior Schwarzschild spacetime Class. Quantum Grav. 27 175003
[6] Bizoń P and Kahl M 2015 Wave maps on a wormhole Phys. Rev. D 91 065003
[7] Bizoń P and Mach P 2014 Global dynamics of a Yang–Mills field on an asymptotically hyperbolic space Trans. Am. Math. Soc. http://dx.doi.org/10.1090/tran/6807
[8] Couch W E and Torrence R J 1984 Conformal invariance under spatial inversion of extreme Reissner–Nordström black holes Gen. Relativ. Gravit. 16 789
[9] Bartnik R and McKinnon J 1988 Particle-like solutions of the Einstein–Yang–Mills equations Phys. Rev. Lett. 61 141
[10] Bizoń P 1995 Harmonic maps between three-spheres Proc. R. Soc. A 451 779
[11] Zenginoğlu A 2008 Hyperboloidal foliations and Scri-fixing Class. Quantum Grav. 25 145002
[12] Friedrich H 1983 Cauchy problems for the conformal vacuum field equations in general relativity Commun. Math. Phys. 91 445
[13] Zenginoğlu A 2008 A hyperboloidal study of tail decay rates for scalar and Yang–Mills fields Class. Quantum Grav. 25 175013
[14] Leaver E W 1985 An analytic representation for the quasinormal modes of Kerr black holes Proc. R. Soc. A 402 285
[15] Fiziev P P and Staicova D R 2012 Am. J. Comput. Math. 2 95
[16] Elaydi S N 1999 An Introduction to Difference Equations (Berlin: Springer)
[17] Aretakis S 2011 Stability and instability of extreme Reissner–Nordström black hole spacetimes for linear scalar perturbations: I Commun. Math. Phys. 307 17
[18] Aretakis S 2011 Stability and instability of extreme Reissner–Nordström black hole spacetimes for linear scalar perturbations: II Ann. Henri Poincare 8 1491
[19] Bizoń P and Friedrich H 2013 A remark about wave equations on the extreme Reissner–Nordström black hole exterior Class. Quantum Grav. 30 065001
[20] Lucietti J, Murata K, Reall H S and Tanahashi N 2013 On the horizon instability of an extreme Reissner–Nordström black hole J. High Energy Phys. JHEP03(2013)035